Some discretizations of geometric evolution equations and the Ricci iteration on the space of Kähler metrics, I

Yanir A. Rubinstein

Abstract. In this article and in its sequel we propose the study of certain discretizations of geometric evolution equations as an approach to the study of the existence problem of some elliptic partial differential equations of a geometric nature as well as a means to obtain interesting dynamics on certain infinite-dimensional spaces. We illustrate the fruitfulness of this approach in the context of the Ricci flow, as well as another flow, in Kähler geometry. We introduce and study dynamical systems related to the Ricci operator on the space of Kähler metrics that arise as discretizations of these flows. We pose some problems regarding their dynamics. We point out a number of applications to well-studied objects in Kähler and conformal geometry such as constant scalar curvature metrics, Kähler-Ricci solitons, Nadel-type multiplier ideal sheaves, balanced metrics, the Moser-Trudinger-Onofri inequality, energy functionals and the geometry and structure of the space of Kähler metrics. E.g., we obtain a new sharp inequality strengthening the classical Moser-Trudinger-Onofri inequality on the two-sphere.

Contents

1. Introduction ................................................... 2
2. Constructing canonical metrics in Kähler geometry ...................... 4
3. The Ricci iteration ................................................... 6
4. Some energy functionals on the space of Kähler metrics .............. 10
5. The Ricci iteration for negative and zero first Chern class ........ 14
6. The Ricci iteration for positive first Chern class ........................ 15
7. The Kähler-Ricci flow and the Ricci iteration for a general Kähler class 18
8. Another flow and the inverse Ricci operator for a general Kähler class . 19
9. The twisted Ricci iteration and a twisted inverse Ricci operator .... 22

* Massachusetts Institute of Technology. Email: yanir@member.ams.org
September 6th, 2007.
Mathematics Subject Classification (2000): Primary 32W20. Secondary 14J45, 26D15, 32M25, 32Q20, 39A12, 53C25, 58E11.
1 Introduction. Our main purpose in this article and in its sequel [R5] is to propose
the systematic use of certain discretizations of geometric evolution equations as an
approach to the study of the existence problem of certain elliptic partial differential
equations of a geometric nature as well as a means to obtain interesting dynamics
on certain infinite-dimensional spaces. We illustrate the fruitfulness of this approach
in the context of the Ricci flow, as well as another flow, in Kähler geometry. We
describe how this approach gives a new method for the construction of canonical
Kähler metrics. We also introduce a number of canonical dynamical systems on the
space of Kähler metrics that we believe merit further study. Some of the results and
constructions described here were announced previously [R3].

Given an elliptic partial differential equation, several classical methods are avail-
able to approach the problem of existence of solutions. In essence, standard elliptic
theory reduces the existence problem to the demonstration of certain a priori esti-
mates for solutions. The main difficulty lies therefore in devising methods to obtain
these estimates.

One common method, that goes back at least to Bernstein and Poincaré, is the con-
tinuity method. In this approach one continuously deforms the given elliptic operator
to another (oftentimes in a linear fashion), for which the existence problem is known
to have solutions. Ellipticity provides for existence of solutions for small perturba-
tions of this easier problem. In order to prove existence for the whole deformation
path one then seeks to establish a priori estimates, uniform along the deformation,
for solutions of the family of elliptic problems.

Another approach, drawing some of its motivation from Physics, is the heat flow
method, going back to Fourier. Here the idea is study a deformation of the elliptic
problem according to a parabolic heat equation whose equilibrium state is precisely
a solution to the original elliptic equation. Much of the standard elliptic theory has a
Some discretizations of geometric evolution equations

parabolic counterpart. First, one makes use of the latter in order to establish short-
time existence. Long-time existence and convergence then hinge upon establishing a
priori estimates, as before.

A third approach, going back, among others, to Euler and Cauchy, is the discretiza-
tion method, that can be considered as a blend of the two above. Here the idea is to
replace an evolution equation (or “flow”) by a countable set of elliptic equations that
arise by repeatedly solving a difference equation corresponding to discretizing the
flow equation in the time variable. This approach provides common and elementary
numerical algorithms, the Euler method and its variants, and is widely used in the
“real world”, for example to numerically integrate differential equations.

In this article we wish to explore this third approach in the context of certain
geometric evolution equations. To the best of our knowledge, it seems that it has not
been used before in a systematic manner in this context.

We would like to emphasize that when a particular elliptic equation has a solution
one morally expects all three methods to converge towards such a solution. Therefore
one should not take as a surprise the fact that the discretization method converges in
some of the cases we consider. The crux is thus not the convergence itself but rather
the new point of view and insights that this method provides; both to the study of
the original elliptic problem as well as to the understanding of the evolution equation,
the continuity method and the relation between the two. In addition, one may obtain
in this way non-trivial canonical discrete dynamical systems on infinite-dimensional
spaces that may be of some interest in their own right.

Let us consider as a simple illustration the Laplace equation on a bounded smooth
domain $\Omega$ in $\mathbb{R}^n$. The elliptic problem is then to find a function $u$ satisfying

$$\begin{align*}
\Delta u &= 0, \quad \text{on } \Omega, \\
u &= \psi, \quad \text{on } \partial \Omega.
\end{align*}$$

Consider then the difference equations

$$\begin{align*}
u_k - \nu_{k-1} &= \Delta \nu_k, \quad \text{on } \Omega, \\
u_k &= \psi, \quad \text{on } \partial \Omega, \\
u_0 &= u,
\end{align*}$$

where $u$ is any smooth function that agrees with the smooth function $\psi$ on the
boundary. Thus, one may write

$$\nu_k = (-\Delta + 1)^{-1} \circ \cdots \circ (-\Delta + 1)^{-1} u.$$  

One may then readily show that the sequence $\{\nu_k\}_{k \geq 0}$ exists for each $k \in \mathbb{N}$ and
converges exponentially fast in $k$ to the unique solution of (1).
To give further intuition as to why this method works we consider the following finite-dimensional problem: Given a positive semi-definite matrix $A$ and a vector $v$, find the projection of $v$ onto the zero eigenspace of $A$. One possible solution is to consider the sequence of vectors defined iteratively by

$$v_k = (A + I)^{-1}v_{k-1},$$
$$v_0 = v.$$ 

Then $\lim_{k \to \infty} v_k$ exists and is the required projection. This algorithm is nothing but the discretization of the flow

$$\frac{dv(t)}{dt} = -Av(t),$$
$$v(0) = v.$$ 

Discretizations corresponding to different time steps will produce equivalent dynamical systems

$$v_k = (\tau A + I)^{-1}v_{k-1},$$
$$v_0 = v,$$

whose convergence is faster the larger the time-step $\tau \in (0, \infty)$, with $\tau$ and the first non-zero eigenvalue of $A$ controlling the exponential factor of the speed of convergence.

2 Constructing canonical metrics in Kähler geometry. In this article we wish to apply the method described in the previous section towards the study of canonical Kähler metrics and the space of Kähler metrics. In this section we very briefly describe the problem and some background. We refer to [A3,Bes,F1,Si,T6] for more background.

The search for a canonical metric representative of a fixed Kähler class has been at the heart of Kähler geometry since its birth. Indeed, in his visionary article Kähler defined the eponymous manifold motivated by the fact that in this setting Einstein’s equation simplifies considerably and reduces to a second order partial differential equation for a single function $K$, namely, the local potential $u$ which represents the Kähler form $\omega$ on the open domain $U$ via $\omega|_U = \sqrt{-1} \partial \bar{\partial} u$ must satisfy

$$\det \left[ \frac{\partial^2 u}{\partial z^i \partial \bar{z}^j} \right] = e^{-\mu u}, \text{ on } U,$$

where $\mu$ is the Einstein constant. Two decades later, following Chern’s fundamental work on characteristic classes, Calabi introduced the concept of the space of Kähler metrics in a fixed cohomology class and formulated the problem on a compact closed manifold as an equation for a global smooth function (Kähler potential) $\varphi$,

$$\omega^n = \omega^n e^{\varphi - \mu \varphi},$$
Some discretizations of geometric evolution equations

where $f_\omega$ satisfies $\sqrt{-1}\partial \bar{\partial} f_\omega = \text{Ric}_\omega - \mu \omega$. This showed that a necessary condition for the existence of solutions is that the first Chern class be definite or zero. Calabi proposed that this equation should always admit a unique solution in each Kähler class when $\mu = 0$. In addition he suggested the study of a more general notion, that of an extremal metric [C1,C2]. Since then much progress has been made towards understanding when such metrics exist. Regarding Kähler-Einstein metrics, the most general result in this direction is given by the work of Aubin in the negative Ricci curvature case [A] and by Yau in the case of nonpositive Ricci curvature which provided a solution to Calabi’s conjecture [C1,Y]. Following this much work has gone into understanding the positive case, notably by Tian who provided a complete solution for complex surfaces, in addition to establishing an analytic characterization of Kähler-Einstein manifolds and a theory of stability [T1,T2]. For general extremal metrics however a general existence theory is not presently available although a conjectural picture, the so-called Yau-Tian-Donaldson conjecture, suggests that it should be intimately related with notions of stability in algebraic geometry [Th].

The principal tool in the study of Kähler-Einstein metrics has been the continuity method, as suggested initially by Calabi [C2], and later studied by Aubin and Yau. In the remaining case ($\mu > 0$) Bando and Mabuchi showed that the continuity method will converge to a Kähler-Einstein metric when one exists [BM]. Another important tool has been the Ricci flow introduced by Hamilton [H] in the more general setting of Riemannian manifolds. Cao has shown that the continuity method proofs for the cases $\mu \leq 0$ may be phrased in terms of the convergence of the Ricci flow [Ca]. Later, much work has gone into understanding the Ricci flow on Fano manifolds and recently Perelman and Tian and Zhu proved that the analogous convergence result holds in this case [TZ4].

The idea that there might be another way of approaching canonical metrics, in the form of a discrete infinite-dimensional iterative dynamical system, was suggested by Nadel [N2]. More recently, Donaldson has proposed a program for the construction of constant scalar curvature Kähler metrics on projective manifolds using finite-dimensional iteration schemes and balanced metrics which are in essence computable [Do3].

The main motivation for our work came from trying to approach Nadel’s basic problem: Find an (infinite-dimensional) iterative dynamical system on the space of Kähler metrics that converges to a Kähler-Einstein metric. In his note Nadel suggested one such dynamical system on Fano manifolds which, as we explain below, is not suited to the problem (see §§10.6). Nevertheless his idea is related to the right answer, and it is our purpose in this article and its sequel to describe our approach to Nadel’s problem and some of its consequences.

Setup and notation. Let $(M, J, g)$ be a connected compact closed Kähler manifold of complex dimension $n$ and let $\Omega \in H^2(M, \mathbb{R}) \cap H^{1,1}(M, \mathbb{C})$ be a Kähler class with
Let $\partial = \partial + \bar{\partial}$. Define the Laplacian $\Delta = -\bar{\partial} \circ \bar{\partial}^* - \partial^* \circ \bar{\partial}$ with respect to a Riemannian metric $g$ and assume that $J$ is compatible with $g$ and parallel with respect to its Levi-Civita connection. Let $g_{\text{form}} = 1/\pi \cdot g_{ij}(z)dz^i \otimes d\bar{z}^j$ be the associated Kähler metric, that is the induced Hermitian metric on $(T^{1,0}M, J)$, and let $\omega := \omega_g = \sqrt{-1/2\pi} \cdot g_{ij}(z)dz^i \wedge d\bar{z}^j$ denote its corresponding Kähler form, a closed positive $(1,1)$-form on $(M, J)$ such that $g_{\text{form}} = \frac{1}{2} g - \frac{1}{2} \omega$. Similarly denote by $g_\omega$ the Riemannian metric induced from $\omega$ by $g_\omega(\cdot, \cdot) = \omega(\cdot, J \cdot)$.

For any Kähler form we let $\text{Ric}(\omega) = -\sqrt{-1/2\pi} \cdot \partial \bar{\partial} \log \det(g_{ij})$ denote the Ricci form of $\omega$. It is well-defined globally and represents the first Chern class $c_1 := c_1(T^{1,0}M, J) \in H^2(M, \mathbb{Z})$. Alternatively it may be viewed as minus the curvature form of the canonical line bundle $K_M$, the top exterior product of the holomorphic cotangent bundle $T^{1,0}M$. One calls $\omega$ Kähler-Einstein if $\text{Ric} \omega = a \omega$ for some real $a$. The trace of the Ricci form with respect to $\omega$ is called the scalar curvature and is denoted by $s(\omega)$. The average of the scalar curvature is denoted by $s_0$ and does not depend on the choice of $\omega \in \mathcal{H}_\Omega$. Nor does the volume $V = \Omega^n([M])$.

Let $H_\omega$ denote the Hodge projection operator from the space of closed forms onto the kernel of $\Delta$. Denote by $D_\Omega$ the space of all closed $(1,1)$-forms cohomologous to $\Omega$, by $\mathcal{H}_\Omega$ the subspace of Kähler forms, and by $\mathcal{H}_{\Omega}^+ \subset \mathcal{H}_\Omega$ the subspace of Kähler forms whose Ricci curvature is positive (nonempty if and only if $c_1 > 0$).

For a Kähler form $\omega$ with $[\omega] = \Omega$ we will consider the space of smooth strictly $\omega$-plurisubharmonic functions (Kähler potentials)

$$\mathcal{H}_\omega = \{ \varphi \in C^\infty(M) : \omega_\varphi := \omega + \sqrt{-1} \partial \bar{\partial} \varphi > 0 \}.$$ 

Let $\text{Aut}(M, J)$ denote the complex Lie group of automorphisms (biholomorphisms) of $(M, J)$ and denote by $\text{aut}(M, J)$ its Lie algebra of infinitesimal automorphisms composed of real vector fields $X$ satisfying $L_X J = 0$. Let $G$ be any compact real Lie subgroup of $\text{Aut}(M, J)$, and let $\text{Aut}(M, J)_0$ denote the identity component of $\text{Aut}(M, J)$. We denote by $\mathcal{H}_{\Omega}(G) \subset \mathcal{H}_{\Omega}$ and $\mathcal{H}_\omega(G) \subset \mathcal{H}_\omega$ the corresponding subspaces of $G$-invariant elements.

### 3 The Ricci iteration

In this section we introduce the Ricci iteration and describe some of its elementary properties.

Hamilton’s Ricci flow on a Kähler manifold of definite or zero first Chern class is defined as the the set $\{\omega(t)\}_{t \in \mathbb{R}^+}$ satisfying the evolution equations

$$\begin{align*}
\frac{\partial \omega(t)}{\partial t} &= -\text{Ric} \omega(t) + \mu \omega(t), \quad t \in \mathbb{R}^+, \\
\omega(0) &= \omega \in \mathcal{H}_\Omega,
\end{align*}$$

where $\Omega$ is a Kähler class satisfying $\mu \Omega = c_1$ for some $\mu \in \mathbb{R}$ (see, e.g., [Cho]). We will sometimes refer to the equations (2) themselves as the Ricci flow.
Some discretizations of geometric evolution equations

We introduce the following dynamical system that is our main object of study in this article. It is a discrete version of this flow.

**Definition 3.1.** Let $\Omega$ be a Kähler class satisfying $\mu\Omega = c_1$ for some $\mu \in \mathbb{R}$. Given a Kähler form $\omega \in \mathcal{H}_\Omega$ and a number $\tau > 0$ define the time $\tau$ Ricci iteration to be the sequence of forms $\{\omega_{k\tau}\}_{k \geq 0} \subseteq \mathcal{H}_\Omega$, satisfying the equations

$$
\omega_{k\tau} = \omega_{(k-1)\tau} + \tau \mu \omega_{k\tau} - \tau \text{Ric} \omega_{k\tau}, \quad k \in \mathbb{N},
$$

$$
\omega_0 = \omega, \tag{3}
$$

for each $k \in \mathbb{N}$ for which a solution exists in $\mathcal{H}_\Omega$.

We pose the following elementary conjecture concerning the limiting behavior of the Ricci iteration in the presence of fixed points.

**Conjecture 3.2.** Let $(M,J)$ be a compact Kähler manifold admitting a Kähler-Einstein metric. Let $\Omega$ be a Kähler class such that $\mu\Omega = c_1$ with $\mu \in \mathbb{R}$. Then for any $\omega \in \mathcal{H}_\Omega$ and for any $\tau > 0$, the time $\tau$ Ricci iteration exists for all $k \in \mathbb{N}$ and converges in the sense of Cheeger-Gromov to a Kähler-Einstein metric.

Regarding this conjecture we prove in this article the following simple result.¹

**Proposition 3.3.** Let $(M,J)$ be a compact Kähler manifold admitting a unique Kähler-Einstein metric. Let $\Omega$ be a Kähler class such that $\mu\Omega = c_1$ with $\mu \in \mathbb{R}$. Then for any $\omega \in \mathcal{H}_\Omega$ and for any $\tau > 1/\mu$ (when $\mu > 0$) or $\tau > 0$ (when $\mu \leq 0$), the time $\tau$ Ricci iteration exists for all $k \in \mathbb{N}$ and converges to a Kähler-Einstein metric.

Now, let the Ricci potential $f : \varphi \in \mathcal{H}_\omega \to f_{\varphi} \in C^\infty(M)$ be the vector field on $\mathcal{H}_\omega$ satisfying

$$
\sqrt{-1} \partial \bar{\partial} f_{\varphi} = \text{Ric} \varphi - \omega_{\varphi}, \quad \frac{1}{V} \int_M e^{f_{\varphi}} \omega_{\varphi}^n = 1. \tag{4}
$$

For each $k$ write

$$
\omega_{k\tau} = \omega_{\psi_{k\tau}}, \quad \text{with} \quad \psi_{k\tau} = \sum_{l=1}^k \varphi_{l\tau}.
$$

¹ We refer the reader to the sequel [R5] where we intend to discuss some of the remaining cases. These cases that we also find of significant geometric interest require more involved and detailed analysis. Also we wished to keep this first article’s length reasonable as well as not to further delay this first part that introduces our main constructions and ideas regarding discrete dynamical systems on the space of Kähler metrics.
The iteration (3) on $\mathcal{H}_\Omega$ can be written as the following system of complex Monge-Ampère equations on $\mathcal{H}_\omega$,

$$\omega^n_{\psi_{(k)}} = \omega^n e^{f_\omega + \frac{\partial}{\partial t} \psi_{(k)}} = \omega^n_{\psi_{(k-1)}} e^{\frac{\partial}{\partial t} \mu \psi_{(k)} - \frac{\partial}{\partial t} \psi_{(k-1)}}, \quad k \in \mathbb{N}$$

(implicit in this equation is also a normalization for $\varphi_{k\tau}$ that eliminates the ambiguity in passing from an equation on $\mathcal{H}_\Omega$ to one on $\mathcal{H}_\omega$).

We now mention some basic features of the iteration.

The most elementary one is that at each step one gains regularity (two derivatives). This is a discrete version of the infinite smoothing property of heat equations.

Another distinctive feature of the iteration is that it turns the solution of each type of Monge-Ampère equation into the next simplest one. Indeed, to find a Kähler-Einstein metric of negative scalar curvature $-n$ one needs to solve the equation

$$\omega^n = \omega^n e^{f_\omega + \varphi}.$$  

The corresponding time one Ricci iteration requires solving at each step the equation

$$\omega^n = \omega^n e^{f_\omega + 2\varphi}.$$  

Similarly, the Calabi-Yau equation

$$\omega^n = \omega^n e^{f_\omega},$$  

is traded for a sequence of equations of the previous type (6), and finally, the most difficult equation,

$$\omega^n = \omega^n e^{f_\omega - \varphi},$$  

for a Kähler-Einstein metric of positive scalar curvature $n$, is turned into a sequence of Calabi-Yau equations (8) via the time one iteration, or to a sequence of equations of the type (6) for smaller time steps.

We now discuss the link the iteration creates between classical continuity method paths and the heat equation. In several places in the literature on canonical metrics it is mentioned that the continuity method and the heat flow method are morally equivalent (see, e.g., [J,Si]). The following discussion comes to make this statement somewhat more explicit.

Indeed, note that one may also consider the time step of the iteration as a dynamical parameter and study the continuity method path it defines. More precisely, the time $\tau$ Ricci iteration is given by a sequence $\{\omega_{k\tau}\}_{k \geq 1}$ of Kähler forms in $\mathcal{H}_\Omega$ satisfying

$$\omega_{k\tau} = \omega_{(k-1)\tau} + \tau \mu \omega_{k\tau} - \tau \text{Ric} \omega_{k\tau}, \quad \omega_0 = \omega,$$  

(10)
Some discretizations of geometric evolution equations

for each $k \in \mathbb{N}$ for which a solution exists. Now set $k = 1$ and consider the path
$
\{ \omega_\tau := \omega_{\varphi_\tau} \}_{\tau \geq 0} \text{ in } \mathcal{H}_\Omega \text{ (for each } \tau \text{ for which it exists)}. \text{ In } \mathcal{H}_\omega \text{ we obtain the path}
$$
\omega^n_{\varphi_\tau} = \omega^n e^{f_\omega + (\frac{1}{\tau} - \mu) \varphi_\tau}, \quad \tau \in (0, \infty).
$$
(11)

Let us compare this path to others that appeared previously in the literature.

In the case $\mu = 1$, when restricted to the segment $\tau \geq 1$ this is just a reparametrization of Aubin’s path [A2] given by
$$
\omega^n_{\varphi_s} = \omega^n e^{f_\omega - s \varphi_s}, \quad s \in [0, 1],
$$
(12)
via $s = 1 - \frac{1}{\tau}$. Here the solution for (12) at $s = 0$ is given by the Calabi-Yau Theorem. Namely, one typically first solves the family of equations introduced by Calabi [C2, (11); Y]
$$
\omega^n_{\varphi_s} = \omega^n e^{(s + 1) f_\omega + c_s}, \quad s \in [-1, 0],
$$
(13)
and then continues to work with the path (12).

For the path (11) we still need to invoke the Calabi-Yau Theorem to show closedness at $\tau \to 1^-$. However, this path can be viewed as a continuity version of the Ricci flow and has various monotonicity properties that (13) does not. When studying the Ricci iteration for the case $\Omega = c_1$ this will be useful (note also that this path may be used in place of (13) to prove the Calabi-Yau Theorem).

Remark 3.4. We note that Calabi’s path (13) can in fact be interpreted as a continuity path arising from a flow, however not the Ricci flow, see (42) below.

In light of this relation to the continuity method, the Ricci iteration is seen to interpolate between the continuity method ($\tau = \infty$) and the Ricci flow ($\tau = 0$). Cao and Perelman proved that when a Kähler-Einstein metric exists, the flow will converge to it in the sense of Cheeger-Gromov. Aubin, Bando-Mabuchi and Yau proved the analogous result for the continuity method. These results are the main motivation for Conjecture 3.2.

Next, in the case $\mu = -1$, the continuity path (11) that arises from the Ricci iteration equation is the same as the continuity path considered by Tian and Yau in their study of Kähler-Einstein metrics of negative Ricci curvature on some non-compact manifolds [TY, p. 586]. The innovative idea of Tian and Yau was to consider a continuity parameter “starting from infinity” observing that along this path one has a uniform lower bound for the Calabi-Yau Theorem.

Now an open problem concerning the flow equation (2) with $\mu > 0$ is whether one has a uniform lower bound for the Ricci curvature depending only on the initial data in the absence of a Kähler-Einstein metric. Along the time $\tau$ iteration one does have such a bound, depending on $\tau$, namely, $\operatorname{Ric} \omega_{k\tau} > \frac{\mu - 1}{\tau} \omega_{k\tau}$. One possible approach to this problem might be to show that the Ricci flow stays asymptotically close to the time $\tau$ Ricci iteration for some range of time steps $\tau$. 

9
Remark 3.5. We remark that when $\mu = 1$ another path has been considered previously by Demailly and Kollár [DK, (6.2.3)], given by $\omega_n^{\phi t} = \omega^n e^t \omega^{-t \phi}$, $t \in [0, 1]$. As written, this path also does not require to start from a solution to a Calabi-Yau equation. Yet in order to get openness for it one assumes $\text{Ric} \omega > 0$ and this involves solving a Calabi-Yau equation (indeed there is no way to produce a Kähler-Einstein metric without entering $\mathcal{H}^+_c$, and the Calabi-Yau Theorem amounts to $\mathcal{H}^+_c \neq \emptyset$). This path can also be explained in terms of a discretization; see (43).

Remark 3.6. Another relation between a continuity path, defined by Tian and Zhu, and a discretized flow, this time a modified Ricci flow, will be discussed in Section 9.

4 Some energy functionals on the space of Kähler metrics. In this section we will obtain a monotonicity result along the Ricci iteration for a family of energy functionals. This result has independent interest, and it seems interesting to compare it with corresponding studies for the Ricci flow (see Remark 4.4).

We briefly recall the pertinent definitions and properties of these energy functionals. For more details on this subject we refer to a previous article [R2] and the references therein.

We call a real-valued function $A$ defined on a subset $\text{Dom}(A)$ of $\mathcal{D}_\omega \times \mathcal{D}_\Omega$ an energy functional if it is zero on the diagonal restricted to $\text{Dom}(A)$. By a Donaldson-type functional, or exact energy functional, we will mean an energy functional that satisfies the cocycle condition $A(\omega_1, \omega_2) + A(\omega_2, \omega_3) = A(\omega_1, \omega_3)$ with each of the pairs appearing in the formula belonging to $\text{Dom}(A)$ [Do1,M1,T6]. We will occasionally refer to both of these simply as functionals and exact functionals, respectively.

The functionals $I, J$, introduced by Aubin [A2], are defined for each pair $(\omega, \omega^\phi := \omega + \sqrt{-1} \partial \bar{\partial} \phi) \in \mathcal{D}_\omega \times \mathcal{D}_\Omega$ by

$$I(\omega, \omega^\phi) = V^{-1} \int_M \sqrt{-1} \partial \bar{\partial} \phi \wedge \bar{\partial} \phi \wedge \sum_{l=0}^{n-1} \omega^{n-1-l} \wedge \omega^\phi_l = V^{-1} \int_M \varphi(\omega^n - \omega^\phi_n),$$

$$J(\omega, \omega^\phi) = \frac{V^{-1}}{n+1} \int_M \sqrt{-1} \partial \bar{\partial} \phi \wedge \partial \phi \wedge \sum_{l=0}^{n-1} (n-l) \omega^{n-l-1} \wedge \omega^\phi_l.$$  (15)

We say that an exact functional $A$ is bounded from below on $U \subseteq \mathcal{H}_\omega$ if for every $\omega$ such that $(\omega, \omega^\phi) \in \text{Dom}(A)$ and $\omega^\phi \in U$ holds $A(\omega, \omega^\phi) \geq C_\omega$ with $C_\omega$ independent of $\omega^\phi$. We say it is proper (in the sense of Tian) on a set $U \subseteq \mathcal{H}_G(G)$ if for each $\omega \in \mathcal{H}_G(G)$ there exists a smooth function $\nu_\omega : \mathbb{R} \to \mathbb{R}$ satisfying $\lim_{s \to \infty} \nu_\omega(s) = \infty$ such that $A(\omega, \omega^\phi) \geq \nu_\omega((I - J)(\omega, \omega^\phi))$ for every $\omega^\phi \in U$ [T4, Definition 5.1; T6]. This is well-defined, in other words depends only on $[\omega]$. Properness of a functional implies it has a lower bound.
Some discretizations of geometric evolution equations

Previously [R2] we introduced the following collection of energy functionals for each \( k \in \{0, \ldots, n\}, \)

\[
I_k(\omega, \omega_\varphi) = \frac{1}{V} \int_M \sqrt{-1} \partial \varphi \wedge \bar{\partial} \varphi \wedge \sum_{l=0}^{k-1} \frac{k-l}{k+1} \omega^{n-1-l} \wedge \omega_\varphi^l.
\]

\[
= \frac{V^{-1}}{k+1} \int_M \varphi (k \omega^n - \sum_{l=1}^{k} \omega^{n-l} \wedge \omega_\varphi^l).
\]  

(16)

Note that \( I_n = J, I_{n-1} = ((n+1) J - I)/n. \)

For each \( \mu \in \mathbb{R} \) define the Ding functional

\[
F_\mu(\omega, \omega_\varphi) = \begin{cases} 
-\frac{1}{n+1} \frac{1}{V} \int_M \varphi \sum_{l=0}^{n} \omega^{n-l} \wedge \omega_\varphi^l - \frac{1}{\mu} \log \frac{1}{V} \int_M e^{f_\omega - \mu \varphi} \omega^n, & \text{for } \mu \neq 0, \\
-\frac{1}{n+1} \frac{1}{V} \int_M \varphi \sum_{l=0}^{n} \omega^{n-l} \wedge \omega_\varphi^l + \frac{1}{V} \int_M \varphi e^{f_\omega} \omega^n, & \text{for } \mu = 0.
\end{cases}
\]

(17)

The critical points of these functionals are Kähler-Einstein metrics [D]. However, there is an important difference between the two cases in (17). While for the first the functional is exact, for \( \mu = 0 \) this is not true. This is because the second term of \( F_0 \) is not exact on the space \( \mathcal{H}_\omega \), while the first is. This rather peculiar phenomenon is reflected also by a property of the generalized Ding functional (see the end of §§10.3).

Define the Chen-Tian functionals

\[
E_k(\omega, \omega_\varphi) = \frac{1}{k+1} \frac{1}{V} \int_{M \times [0,1]} \left[ \mu \omega^{k+1} - (\text{Ric} \omega_\varphi - \Delta \varphi)^{k+1} \right] \wedge (\omega_\varphi + \varphi)^{n-k} \wedge dt.
\]

(18)

These are well-defined independently of the choice of path and exact [CT]. Moreover, in the case \( \mu \neq 0 \), the functionals \( E_k \) are related to the K-energy \( E_0 \) introduced by Mabuchi [M1] via the following relation:

**Proposition 4.1.** Let \( \mu \neq 0 \). Let \( k \in \{0, \ldots, n\} \). For every \( (\omega, \omega_\varphi) \in \mathcal{H}_{\mu c_1} \times \mathcal{H}_{\mu c_1} \),

\[
\mu^{k+1} E_k(\omega, \omega_\varphi) = \mu E_0(\omega, \omega_\varphi) + I_k(\omega_\varphi, \mu \text{Ric} \omega_\varphi) - I_k(\omega, \mu \text{Ric} \omega).
\]

(19)

**Proof.** This relation has been previously demonstrated for the case \( \mu = 1 \) [R2, Proposition 2.6] (recall that the case \( \mu = 1, k = n \) is a result of Bando and Mabuchi). In general the same proof goes through by keeping track of the constant \( \mu \). \( \Box \)
Finally, recall the definitions of the following subsets of the space of Kähler forms on a Fano manifold [R2]:

\[ A_k(\omega) = \{ \omega_\varphi \in H_{c_1} : E_k(\omega, \omega_\varphi) \geq 0 \} \]

\[ B_k = \{ \omega_\varphi \in H_{c_1} : I_k(\omega_\varphi, \text{Ric}_{\omega_\varphi}) \geq 0 \} \]

When a Kähler-Einstein form \( \omega \) exists we denote \( A_k := A_k(\omega) \). This is well-defined and does not depend on the choice of the Kähler-Einstein form. We recall that both the sets \( A_k \) and \( B_k \) strictly contain the set \( H_{c_1}^+ \), of Kähler forms of positive Ricci curvature (see Theorem 10.6 below for a more precise statement).

The following monotonicity result will be useful later. For \( E_0, F_\mu \) and \( E_1 \) with \( 1/\mu = \tau = 1 \) this was proven before [B,DT,SW].

**Proposition 4.2.**  
(i) The functional \( E_0 \) is monotonically decreasing along the time \( \tau \) iteration \((\tau > 0)\) whenever the initial point is not Kähler-Einstein. 
(ii) When \( 1/\mu = \tau = 1 \) the same is true for \( F_1, E_1 \), and, when the initial metric lies in \( B_k \supset H_{c_1}^+ \), also for \( E_k, k \geq 2 \).

**Proof.** By exactness, it suffices to show monotonicity at each step of the iteration.  
(i) For concreteness we derive the result only for \( \tau = 1 \) but explicitly compute the energy decrease along the iteration (in general see Lemma 9.3, stated for \( \mu = 1 \), that works for all \( \mu \)). Consider the equation \( \omega_\varphi^n = \omega^n e^{\int_0^\omega - (\mu - 1)\varphi_1} \). One has [Che1; T4, p. 254; T5, (5.14)]

\[
E_0(\omega, \omega_\varphi) = \frac{1}{V} \int_M \log \frac{\omega_\varphi^n}{\omega^n} \omega_\varphi^n - \mu (I - J)(\omega, \omega_\varphi) + \frac{1}{V} \int_M f_\omega (\omega^n - \omega_\varphi^n).
\]

First, let \( \mu = 1 \). One has,

\[
E_0(\omega, \omega_\varphi) = -(I - J)(\omega, \omega_\varphi) + \frac{1}{V} \int_M f_\omega \omega^n.
\]  

This is nonpositive by the definition of \( f_\omega \) and Jensen’s inequality.

When \( \mu = -1 \) one has

\[
E_0(\omega, \omega_\varphi) = \frac{1}{V} \int_M 2\varphi_1 \omega_\varphi^n + (I - J)(\omega, \omega_\varphi) + \frac{1}{V} \int_M f_\omega \omega^n
\]

\[
= \frac{1}{V} \int_M 2\varphi_1 \omega_\varphi^n + \frac{1}{V} \int_M \varphi_1 (\omega^n - \omega_\varphi^n) - J(\omega, \omega_\varphi) + \frac{1}{V} \int_M f_\omega \omega^n
\]

\[
= -(I + J)(\omega, \omega_\varphi) + \frac{1}{V} \int_M (f_\omega + 2\varphi_1) \omega^n.
\]

Each term is nonpositive, once again by using the normalization inherent in (7).
Some discretizations of geometric evolution equations

When $\mu = 0$ one has

$$E_0(\omega, \omega_{\varphi_1}) = \frac{1}{V} \int_M \varphi_1 \omega_{\varphi_1}^n + \frac{1}{V} \int_M f_\omega \omega^n$$

$$= -I(\omega, \omega_{\varphi_1}) + \frac{1}{V} \int_M (f_\omega + \varphi_1) \omega^n,$$

and we may argue as before.

(ii) Using (14)-(15) and (17) one has

$$F_1(\omega, \omega_{\varphi_1}) = -(I - J)(\omega, \omega_{\varphi_1}) - \frac{1}{V} \int_M \varphi_1 \omega_{\varphi_1}^n - \log \frac{1}{V} \int_M e^{f_\omega - \varphi_1} \omega^n \quad (23)$$

$$= -(I - J)(\omega, \omega_{\varphi_1}) - \frac{1}{V} \int_M \varphi_1 \omega_{\varphi_1}^n - \log \frac{1}{V} \int_M e^{-\varphi_1} \omega_{\varphi_1}^n. \quad (24)$$

By Jensen’s inequality the last two terms combined are nonpositive, and so we conclude by the positivity of $I - J$. (Alternatively, the iteration stays on the submanifold of $\mathcal{H}_\omega$ defined by the equation $\frac{1}{V} \int_M e^{f_\omega - \varphi_1} \omega^n = 1$ and hence the third term is identically zero, while the second one is negative, again by a special case of Jensen’s inequality: $1 - \frac{1}{V} \int_M \varphi_1 \omega^n \leq \frac{1}{V} \int_M e^{-\varphi_1} \omega^n = \frac{1}{V} \int_M \omega_2^n = 1$.)

Next, when $1/\mu = \tau = 1$ Proposition 4.1 gives

$$E_k(\omega, \omega_{\varphi_1}) = E_0(\omega, \omega_{\varphi_1}) + I_k(\omega_{\varphi_1}, \text{Ric} \omega_{\varphi_1}) - I_k(\omega, \text{Ric} \omega). \quad (25)$$

Next, recall the inequality $I_k \leq J \quad [R2, (9)]$. Since in our case $\text{Ric} \omega_{\varphi_1} = \omega$ we deduce from (25) and (22) that

$$E_k(\omega, \omega_{\varphi_1}) \leq -(I - J)(\omega, \omega_{\varphi_1}) + J(\omega_{\varphi_1}, \omega) - I_k(\omega, \text{Ric} \omega).$$

Since $J(\omega, \omega_{\varphi_1}) + J(\omega_{\varphi_1}, \omega) = I(\omega, \omega_{\varphi_1})$, one has $E_k(\omega, \omega_{\varphi_1}) \leq 0$ if $I_k(\omega, \text{Ric} \omega) \geq 0$, or in other words, if $\omega \in B_k$. Finally, note that the subspace $B_k$ is preserved under the iteration since, in fact, after the first step the iteration will stay in $\mathcal{H}_{c_1}^+$. □

Remark 4.3. An alternative derivation of the second part of (ii) could be to choose a particular path connecting $\omega$ and $\omega_1$ and note that each of the contributions has a preferred sign. For example, choosing Calabi’s continuity path (13) produces three terms of which two are evidently nonnegative. However then one still needs to manipulate the third term which comes up, $-\frac{1}{V^2} \int_M \sum_{i=1}^k (i+1) f_\omega (\sqrt{-1} \partial \bar{\partial} f_\omega)^i \wedge \omega^{n-i}$, and argue that it equals precisely $I_k(\omega, \text{Ric} \omega)$ and then use the results of [R2] as above. However to derive (i) for all time steps one needs to use instead the path (11) along which $E_0$ is monotonic, as alluded to after (13) above.
Remark 4.4. Here it is interesting to compare with the Ricci flow. One knows that $F_1, E_0$ are monotonically decreasing along the flow and that as long as $\text{Ric} \omega > -\omega$ the same is true for $E_k$ [CT, §§3.3, Proposition 4.9]. However an analogous result is not known along the Ricci flow for $E_k$, $k \geq 2$. In the case that a Kähler-Einstein metric exists one knows that the flow will converge. One also knows that when restricted to the space $B_k$ the functional $E_k$ attains a minimum precisely on the space of Kähler-Einstein metrics, however that outside this space it is not true that these functionals are bounded from below on $H_{c_1} [R^2, §5]$ (see also §§10.2 below). Thus all that is apparent at the present moment is that once the flow stays in $B_k$, the functional $E_k$ will eventually decrease, however even then we do not know whether this will happen monotonically.

5 The Ricci iteration for negative and zero first Chern class. In this section we prove the existence and convergence of the Ricci iteration in the case that either $c_1 < 0$ and $\Omega = -c_1$, or that $c_1 = 0$ and $\Omega$ is an arbitrary Kähler class.

We start with a result that is a simple consequence of the theory of elliptic complex Monge-Ampère equations. This result is the existence part of Proposition 3.3 in the cases under consideration.

Lemma 5.1. Let $(M, J)$ be a compact Kähler manifold whose first Chern class is negative or zero. When $c_1 = 0$ denote by $\Omega$ a Kähler class; otherwise let $\Omega = \mu c_1$ denote a Kähler class with $\mu < 0$. Then for any $\omega \in H_{\Omega}$, the time $\tau$ Ricci iteration exists for all $k \in \mathbb{N}$ and all $\tau \in (0, \infty)$.

Proof. It is enough to show existence for one step of the iteration in order to show the iteration exists for each $k \in \mathbb{N}$ (by repeating the argument at each step).

Let $\omega_1 = \omega_0 + \tau \mu \omega_1 - \tau \text{Ric} \omega_1$. Fix $\tau \in (0, \infty)$. The existence of $\omega_1$ amounts to solving the equation

$\omega_1 = \omega_0 + \tau \mu \omega_1 - \tau \text{Ric} \omega_1.$

Let $\omega_{\varphi_1} = \omega_1$ with $\varphi_1 \in H_{\omega}$. This can be written as a complex Monge-Ampère equation:

$\omega_{\varphi_1}^n = \omega^n e^{f_\omega + (\frac{1}{\tau} - \mu) \varphi_1}, \quad \int_M \omega^n e^{f_\omega + (\frac{1}{\tau} - \mu) \varphi_1} = V. \quad (26)$

Under the assumption $\mu < 1/\tau$, and hence in particular if $\mu \leq 0$, the maximum principle gives an a priori $L^\infty$ estimate on $\varphi_1$. Then the work of Aubin and Yau [A1,Y] immediately applies to give higher-order estimates. We conclude that a unique solution $\omega_{\varphi_1} \in H_{\Omega}$ exists.

We now turn to the proof of the convergence statement of Proposition 3.3 in the cases under consideration.
Some discretizations of geometric evolution equations

Proof of Proposition 3.3 ($\mu \leq 0$). Assume first that $c_1 < 0$ and let $\Omega = -c_1$. We have the following system of Monge-Ampère equations:

$$\omega_{\psi, \tau}^n = \omega^n e^{f_\tau + \psi_{k \tau} + \frac{1}{\tau} \varphi_{k \tau}}, \quad k \in \mathbb{N}.$$ 

We first prove an a priori uniform bound, independent of $k$ in an inductive manner. The first equation reads $\omega_{\psi, \tau}^n = \omega^n e^{f_\tau + (1 + \frac{1}{\tau}) \varphi_{k \tau}}$. At the maximum of $\varphi_{k \tau}$ we have $\omega_{\psi, \tau}^n \leq \omega$ and thus $(1 + \frac{1}{\tau}) \sup \varphi_{k \tau} \leq -\inf f_\omega$. A similar argument at the minimum of $\varphi_{k \tau}$ gives $-(1 + \frac{1}{\tau}) \inf \varphi_{k \tau} \leq \sup f_\omega$. The second equation reads $\omega_{\varphi, \varphi_{k \tau}}^n = \omega_{\psi, \tau}^n e^{-\frac{1}{\tau} \varphi_{k \tau} + (1 + \frac{1}{\tau}) \varphi_{2 \tau}}$. The maximum/minimum principle now gives $(1 + \frac{1}{\tau}) \sup \varphi_{2 \tau} \leq \frac{1}{\tau} \sup \varphi_{\tau}$ and $-(1 + \frac{1}{\tau}) \inf \varphi_{2 \tau} \leq -\frac{1}{\tau} \inf \varphi_{\tau}$ or $\sup \varphi_{2 \tau} \leq -\frac{\tau}{(1 + \tau)^2} \sup f_\omega$ and $-\inf \varphi_{2 \tau} \leq -\frac{\tau}{(1 + \tau)^2} \sup f_\omega$. We then have $\sup \psi_{k \tau} \leq -\inf f_\omega$, $-\inf \psi_{k \tau} \leq \sup f_\omega$. This uniform bound implies the existence of an a priori $C^{2, \alpha}$ bound on $\psi_{k \tau}$, independently of $k$. As a result, by elliptic regularity theory, a subsequence converges to a smooth solution which we denote by $\psi_\infty$. In fact the convergence is exponentially fast and there is no need to take a subsequence: $||\psi_{k \tau} - \psi_{(k-1) \tau}||_{C^{2, \alpha}} \leq C \tau (1 + \tau)^{-k}$.

Now, by Proposition 4.2, we notice that unless $\omega_0$ is itself Kähler-Einstein, the functional $E_0$ is strictly decreasing along the iteration. In particular, since $\omega_\infty$ is a fixed point of the iteration it must be Kähler-Einstein.

We now consider the case $\mu = 0$, for which we have the following system of equations,

$$\omega_{\psi_{k \tau}}^n = \omega^n e^{f_\tau + \frac{1}{\varphi_{k \tau}}}, \quad k \in \mathbb{N}.$$ 

We may rewrite this as $\omega_{\psi_{k \tau}}^n = \omega^n e^{-\frac{1}{\tau} \varphi_{(k-1) \tau} + \frac{1}{\tau} \varphi_{k \tau}}$ from which we have $\sup \varphi_{k \tau} \leq \sup \varphi_{(k-1) \tau} \leq \ldots \leq -\tau \inf f_\omega$. Therefore we have

$$||e^{f_\tau + \frac{1}{\tau} \varphi_{k \tau}}||_{L^\infty(M)} \leq e^{\text{osc} f_\omega}, \quad \forall k \in \mathbb{N}.$$ 

Now, by Yau’s work it follows that there exists an a priori $C^{2, \alpha}$ bound on $\psi_{k \tau}$, independently of $k$. Combined with the monotonicity result it follows, as before, that a subsequence converges to a Kähler potential of a Kähler-Einstein metric. Moreover, since the Kähler-Einstein metric is unique (in each fixed Kähler class) [C2] any converging subsequence will necessarily converge to the same limit point. This then implies that our original sequence converges to this limit. 

6 The Ricci iteration for positive first Chern class. We turn to the study of the iteration on Fano manifolds that, as noted in the Introduction, is our main motivation for introducing the Ricci iteration. Most of the applications described in Section 10 are for this class of manifolds.

We first introduce an operator that arises very naturally although it seems to have not been defined previously in the literature. It exists and is well-defined by the Calabi-Yau Theorem [Y].
Definition 6.1. Define the inverse Ricci operator \( \text{Ric}^{-1} : \mathcal{D}_{c_1} \to \mathcal{H}_{c_1} \) by letting
\[
\text{Ric}^{-1} \omega := \omega_{\varphi}
\]
with \( \omega_{\varphi} \) the unique Kähler form in \( \mathcal{H}_{c_1} \) satisfying \( \text{Ric} \omega_{\varphi} = \omega \). Similarly denote higher order iterates of this operator by \( \text{Ric}^{-l} \) for each \( l \in \mathbb{N} \). Let \( \text{Ric}^0 := \text{Id} \) denote the identity operator.

There exists a generalization of this operator to any Kähler manifold (Definition 8.1). For another direction in which this operator may be generalized see Definition 9.2.

We then see that the dynamical system corresponding to the time one Ricci iteration on a Fano manifold with \( \mu = 1 \) is nothing but the evolution of iterates of the inverse Ricci operator,
\[
\omega_l = \text{Ric}^{-l} \omega_0.
\]

The following result concerns the “allowed” time steps in the iteration for any Fano manifold and is well-known. Note that unlike in the previous, unobstructed, cases, the allowed range for the time step is restricted unless an analytic “semi-stability” condition holds.

Define
\[
\tau_M(G) = \sup \{ t : (11) \text{ has a solution for each } \tau \in (0, t) \text{ and } \omega \in \mathcal{H}_{c_1}(G) \}. \quad (27)
\]
By definition this is a holomorphic invariant. Recall also the definition of Tian’s invariants \([T1,T3]\)
\[
\alpha_M(G) = \sup \{ a : \sup_{\varphi \in \mathcal{H}_\omega(G)} \frac{1}{V} \int_M e^{-a(\varphi - \sup \varphi)} \omega^n < \infty \}, \quad (28)
\]
\[
\beta_M(G) = \sup \{ b : \text{Ric} \omega \geq b \omega, \ \omega \in \mathcal{H}_\Omega(G) \}, \quad (29)
\]
where in \((28)\) \( \omega \) is any element of \( \mathcal{H}_{c_1}(G) \).

Lemma 6.2. Let \((M,J)\) be a Fano manifold and let \( G \) be a compact subgroup of \( \text{Aut}(M,J) \).

(i) For any \( \omega \in \mathcal{H}_{c_1}(G) \), the time \( \tau \) Ricci iteration exists for all \( k \in \mathbb{N} \) and all \( \tau \in [0, \tau_M(G)] \). One has
\[
\frac{1}{1-\beta_M(G)} \geq \tau_M(G) \geq \left| \frac{1}{\max \{1-(n+1)\alpha_M(G)/n,0\}} \right| > 1.
\]

(ii) Assume that \( E_0 \) is bounded from below on \( \mathcal{H}_{c_1}^+(G) \). Then \( \tau_M(G) = \infty \).

Proof. (i) By the Calabi-Yau Theorem \( \tau_M(G) \geq 1 \). According to Tian \([T1]\) the path \((12)\) exists for each \( s \in [0, (n+1)\alpha_M(G)/n] \cap [0,1] \) whenever \( \omega \in \mathcal{H}_{c_1}(G) \). Note that \( \tau = 1/(1-s) \) and that \( \alpha_M(G) > 0 \).

(ii) This is equivalent to a result of Bando and Mabuchi \([BM, \text{Theorem 5.7}]\). \qed

Combined with Theorem 10.7 we therefore obtain the existence part of Proposition 3.3 for \( \mu > 0 \).
Corollary 6.3. Let \((M, J)\) be a Kähler-Einstein Fano manifold. Then for any \(\omega \in \mathcal{H}_{c_1}\), the time \(\tau\) Ricci iteration exists for all \(k \in \mathbb{N}\) and all \(\tau \in (0, \infty)\).

We now turn to the proof of the remaining part of Proposition 3.3. The argument is essentially due to Bando and Mabuchi [BM]. We assume for simplicity, as in the statement of the theorem, that \(\text{aut}(M, J) = \{0\}\) (for the additional details necessary for the general case we refer to [R5] (see also §9)). For the case \(\mu = 1\) we are solving the system of equations

\[
\omega^n_{\psi_{k\tau}} = \omega^n e^{f - \psi_{k\tau} + \frac{1}{\tau} \varphi_{k\tau}}, \quad k \in \mathbb{N}.
\]

(30)

Let \(G_{k\tau}\) be a Green function for \(-\Delta_{k\tau} = -\Delta_{\partial, \omega_{\psi_{k\tau}}}\) satisfying \(\int_M G_{k\tau}(\cdot, y) \omega^n_{\psi_{k\tau}}(y) = 0\). Set \(A_{k\tau} = -\inf_{M \times M} G_{k\tau}\). Since \(-n < \Delta_0 \psi_{k\tau}\) and \(n > \Delta_{k\tau} \psi_{k\tau}\) the Green formula gives

\[
\psi_{k\tau}(x) - \frac{1}{V} \int_M \psi_{k\tau} \omega^n_0 = -\frac{1}{V} \int_M G_0(x, y) \Delta_0 \psi_{k\tau}(y) \omega^n_0(y) \leq nA_0,
\]

\[
\psi_{k\tau}(x) - \frac{1}{V} \int_M \psi_{k\tau} \omega^n_{\psi_{k\tau}} = -\frac{1}{V} \int_M G_{k\tau}(x, y) \Delta_{k\tau} \psi_{k\tau}(y) \omega^n_{\psi_{k\tau}}(y) \geq -nA_{k\tau}.
\]

Hence

\[
\text{osc} \psi_{k\tau} \leq n(A_0 + A_{k\tau}) + I(\omega_0, \omega_{\psi_{k\tau}}).
\]

(31)

Since by Theorem 10.7 (ii) \(E_0\) is proper on \(\mathcal{H}_{c_1}\) in the sense of Tian, if \(E_0(\omega, \cdot)\) is uniformly bounded from above on a subset of \(\mathcal{H}_{c_1}\) so is \(I(\omega, \cdot)\). By the monotonicity of \(E_0\) along the iteration we conclude that \(I(\omega, \omega_{\psi_{k\tau}})\) is uniformly bounded independently of \(k\).

It remains to bound \(A_{k\tau}\). This can be done using a special case of Bando and Mabuchi’s Green’s function estimate that we now state.

Theorem 6.4. [BM, Theorem 3.2] Let \((N, h)\) be a connected compact closed Riemannian manifold of nonnegative Ricci curvature. Let \(G_h\) denote the Green function of \(d^{*h} \circ d + d \circ d^{*h}\) satisfying \(\int_N G(x, y) dV_h(y) = 0\) for each \(x \in N\) and let \(A_h = -\inf_{M \times M} G_h\). Then

\[
A_h \leq c_n \frac{\text{diam}(N, h)^2}{\text{Vol}(N, h)},
\]

with \(c_n\) depending only on \(n\).

Now, along the iteration it holds \(\text{Ric} \omega_k > (\tau - 1) \omega_k > 0\). By Myers’ Theorem [P, p. 245] then

\[
\text{diam}(M, \omega_k)^2 \leq \pi^2 (2n - 1) / (\tau - 1).
\]

(32)
Set $\tilde{\psi}_{k\tau} := \psi_{k\tau} - \frac{1}{k} \int_M \psi_{k\tau} \omega^n$. Now Theorem 6.4 and equations (31)-(32) yield the estimate $||\tilde{\psi}_{k\tau}||_{L^\infty} \leq C$. As in Section 5, the general theory of Monge-Ampère equations now provides for uniform higher derivative estimates. We may thus extract a converging subsequence from $\{\tilde{\psi}_{k\tau}\}_{k \geq 0}$. Thanks to the monotonicity of $E_0$ it must converge to a Kähler potential for a Kähler-Einstein metric. Since such a metric is unique [BM, Remark 9.3] the same argument as before gives the convergence of the full orbit of the Ricci iteration.

7 The Kähler-Ricci flow and the Ricci iteration for a general Kähler class.

A natural question is whether on an arbitrary Kähler manifold one may define an iteration scheme generalizing the Ricci iteration. To answer this question of course one first needs to generalize the Ricci flow itself. In this section we recall one such possibility. We end with a conjecture regarding the convergence of this iteration.

A flow on the space of Kähler forms $\mathcal{H}_\Omega$ can be considered as an integral curve of a vector field on this space. A vector field $\chi$ on $\mathcal{H}_\Omega$ is an assignment $\omega \mapsto \chi_\omega \in C^\infty(M)/\mathbb{R}$. The Ricci flow describes the dynamics of minus the Ricci potential vector field $-f$. Recall that the vector field $f$ is the assignment $\omega \mapsto f_\omega$ with $f_\omega$ defined by $\text{Ric} \omega - \mu_\omega = \sqrt{-1} \partial \bar{\partial} f_\omega$, $\mu \in \mathbb{R}$, where $\mu_\Omega = c_1$.

The Ricci iteration in turn can be thought of as a piecewise linear trajectory in $\mathcal{H}_\Omega$ induced from the Ricci potential vector field $-f$ and approximating its integral curves.

Motivated by this one is naturally led to extend the definition of the Ricci flow (2) to an arbitrary Kähler manifold, simply by defining the flow lines to be integral curves of minus the Ricci potential vector field $-f$ on $\mathcal{H}_\Omega$, with $\Omega$ an arbitrary Kähler class. Recall that the Ricci potential is defined in general by $\text{Ric} \omega - H_\omega \text{Ric} \omega = \sqrt{-1} \partial \bar{\partial} f_\omega$.

The resulting flow equation can be written as

$$
\frac{\partial \omega(t)}{\partial t} = -\text{Ric} \omega(t) + H_t \text{Ric} \omega(t), \quad t \in \mathbb{R}_+,
$$

$$
\omega(0) = \omega,
$$

for each $t$ for which a solution exists in $\mathcal{H}_\Omega$ (throughout subscripts are meant to indicate that the relevant object corresponds to the metric indexed by that subscript).

This flow, introduced by Guan, is part of the folklore in the field although it has not been much studied.\(^2\)

\(^2\) It seems that Guan first considered this flow in unpublished work in the 90’s (see references to [G1]). After completing this article I also became aware, thanks to G. Székelyhidi, of a recent preprint [G2] posted by Guan on his webpage in which this flow is studied. We hope that the elementary discussion in this section is still of some interest even though it was written before learning of [G1,G2]. For a different but related flow see [S].
Corresponding to this flow we introduce the following dynamical system on \( \mathcal{H}_\Omega \) which generalizes Definition 3.1.

**Definition 7.1.** Given a Kähler form \( \omega \in \mathcal{H}_\Omega \) let the time \( \tau \) Ricci iteration be the sequence of forms \( \{\omega_{k\tau}\}_{k \geq 0} \), satisfying the equations

\[
\omega_{k\tau} = \omega_{(k-1)\tau} + \tau H_{k\tau} \text{Ric} \omega_{k\tau} - \tau \text{Ric} \omega_{k\tau}, \quad k \in \mathbb{N},
\]

for each \( k \in \mathbb{N} \) for which a solution exists in \( \mathcal{H}_\Omega \).

As in Section 3, setting \( k = 1 \) and varying \( \tau \) defines a continuity path that is of independent interest.

An observation that goes back to Calabi characterizes the equilibrium state of the flow and the iteration.

**Lemma 7.2.** [C1, Theorem 1] The Ricci form of a Kähler metric is a harmonic representative of \( c_1 \) with respect to the metric if and only if its scalar curvature is constant.

**Proof.** One has

\[
n \text{Ric} \omega \wedge \omega^{n-1} = \text{tr}_\omega \text{Ric} \omega \omega^n = s(\omega)\omega^n.
\]

Since \( \omega \) is a harmonic representative of its class, we see that \( s(\omega) \) is harmonic, i.e., constant, if and only if \( \text{Ric} \omega \) is.

An infinitesimal automorphism \( X \in \text{aut}(M,J) \) naturally induces a vector field \( \psi^X \) on \( \mathcal{H}_\Omega \) given by

\[
\psi^X : \omega \mapsto \psi^X \omega \in C^\infty(M)/\mathbb{R}, \quad \text{where } \mathcal{L}_X \omega = \sqrt{-1} \partial \bar{\partial} \psi^X \omega.
\]

Recall the following generalization of the notion of a constant scalar curvature Kähler metric, due to Guan. Alternatively it may be seen as a generalization of the notion of a Kähler-Ricci soliton to an arbitrary class.

**Definition 7.3.** [G1] Let \( X \in \text{aut}(M,J) \). A Kähler metric \( \omega \) will be called a Kähler-Ricci soliton if it satisfies

\[
\text{Ric} \omega - H_\omega \text{Ric} \omega = \mathcal{L}_X \omega.
\]

Equivalently, if the vector field \( \psi^X - f \) on \( \mathcal{H}_\omega \) has a zero at \( \omega \).

Motivated by the results for Kähler-Einstein manifolds we believe the following conjecture should hold.
Conjecture 7.4. Let \((M,J)\) be a compact closed Kähler manifold, and assume that there exists a constant scalar curvature Kähler metric representing the class \(\Omega\). Then for any \(\omega \in H_{\Omega}\), the Kähler-Ricci flow (33) and the Ricci iteration (34) exist and converge in an appropriate sense to a constant scalar curvature metric.

Similarly, we believe an analogous result should hold for Kähler-Ricci solitons (36) using the twisted constructions of Section 9.

8 Another flow and the inverse Ricci operator for a general Kähler class.

Our purpose in this section is to explain why the inverse Ricci operator— that appeared as a very singular iterative construction for anticanonically polarized Fano manifolds—is in fact a special case of a more general construction on any Kähler manifold. This gives another application of our approach explained in the Introduction since it involves a discretization of another geometric flow equation.

To that end, given a Kähler form \(\omega\) let us consider the flow equations

\[
\frac{\partial \text{Ric} \omega(t)}{\partial t} = -\text{Ric} \omega(t) + H_t \text{Ric} \omega(t), \quad t \in \mathbb{R}^+, \quad \omega(0) = \omega, \tag{37}
\]

for each \(t\) for which a solution exists.

The following brief and informal discussion comes to motivate this definition. Consider the case when the first Chern class is definite \((\mu \in \mathbb{R} \setminus \{0\}\) with \(\Omega = \mu c_1\)), or zero \((\Omega\) is arbitrary), and take \(\omega \in H_{\Omega}\). The evolution equation then becomes

\[
\frac{\partial \text{Ric} \omega(t)}{\partial t} = -\text{Ric} \omega(t) + \mu \omega(t). \tag{38}
\]

Assume momentarily that the flow preserves the Kähler class and that it exists on some time interval \([0,T]\). Then on the level of potentials it can be written as

\[-\Delta_t \phi_t = \log \frac{\omega^n_{\phi_t}}{\omega^n} + \mu \phi_t - f_\omega + a_t, \quad \phi_0 = \text{const}, \tag{39}\]

or as a Monge-Ampère equation

\[
\omega^n_{\phi_t} = \omega^n e^{f_\omega - \mu \phi_t - \Delta_t \phi_t - a_t}, \tag{40}
\]

with \(a_t\) a certain normalizing constant. Set \(u := \Delta_t \phi_t\). A time derivative of (39) gives

\[
\frac{du}{dt} = -u + \mu G_t u + b_t,
\]

with
Some discretizations of geometric evolution equations

with \( b_t \) another normalizing constant. One may show that

\[
|u|_{L^\infty(M \times [0,T])} < Ce^{-t},
\]

when \( \mu \leq 0 \) and that \( |u|_{L^\infty(M \times [0,T])} < Ce^{(\mu/\lambda_1(t)-1)t} \), when \( \mu > 0 \), where \( \lambda_1(t) \) is the first nonzero eigenvalue of \(-\Delta_t\). The constant \( C \) depends a priori on \( t \). Going back to (40) one may show an a priori estimate

\[
|\phi_t|_{L^\infty(M \times [0,T])} < C_1,
\]

with \( C_1 \) depending only on \( \omega \), whenever \( \mu \leq 0 \). This then implies a priori estimates on higher order derivatives. Finally, take a converging subsequence. Along this subsequence \( \lambda_1 \) is uniformly bounded away from zero. Going back to the exponential decay of \( u \) we apply uniform Schauder estimates to conclude that \( \phi_t \) is uniformly decaying. It then follows that the limit is a Kähler-Einstein metric. By uniqueness of the metric one then argues, as earlier on, that the flow itself converges exponentially fast to a Kähler-Einstein metric. On the other hand, the case \( \mu > 0 \) would require more work, quite likely in the spirit of the corresponding result for the Ricci flow [Ca,TZ4] (cf. also [A2,BM]).

Motivated by this discussion, we introduce the following dynamical system on \( \mathcal{H}_\Omega \) obtained as the time one Euler method for this flow:

\[
\text{Ric} \omega_{k+1} = H_k \text{Ric} \omega_k, \quad k \in \mathbb{N},
\]

\[
\omega_0 = \omega.
\]

(41)

It can be thought of as describing the dynamics of a generalized inverse Ricci operator. This motivates the following definition, generalizing Definition 6.1 to an arbitrary Kähler manifold.

**Definition 8.1.** Define the inverse Ricci operator \( \text{Ric}^{-1}_\Omega : \mathcal{H}_\Omega \to \mathcal{H}_\Omega \) by letting \( \text{Ric}^{-1}_\Omega \omega := \omega_\phi \) with \( \omega_\phi \) the unique Kähler form in \( \mathcal{H}_\Omega \) satisfying \( \text{Ric} \omega_\phi = H_\omega \text{Ric} \omega \).

Similarly we denote higher order iterates of this operator by \( \text{Ric}^{-l}_\Omega \) for each \( l \in \mathbb{N} \).

Calabi-Yau manifolds are singled-out as those manifolds for which this operator is a constant map. In general the dynamics of this operator seem intriguing.

We end this section with two remarks regarding continuity method paths induced from the flow (38), directly continuing the discussion in Section 3. First, it is interesting to note that discretizing this flow for time steps \( \tau \in [0,1] \) gives rise to the well-known continuity path of the Calabi-Yau Theorem (here \( \mu = 0 \)) introduced by Calabi [C2, (11)],

\[
\text{Ric} \omega_{\phi_{\tau}} - \text{Ric} \omega = -\tau \text{Ric} \omega \quad \Rightarrow \quad e^{\tau f_{\omega} + d_{\tau} \wedge} \omega^n = \omega_{\phi_{\tau}},
\]

(42)

\[3\] To obtain this path in the equivalent setting of the search for a Kähler metric with prescribed Ricci form, one considers the flow obtained by replacing the harmonic projection term in (37) by a prescribed form representing \( c_1 \).
The K-energy decreases along this path, however not monotonically in general. This is in contrast to the continuity path arising from the Ricci iteration and fits in well with what we would expect: the former arises from the Euler method (as opposed to the backwards Euler method) and so one does not expect monotonicity, nor convergence for large enough time steps.

Also, we remark that the backwards Euler method of the same evolution equation (38) yields the continuity path

\[ \omega_{n+1} = \omega^n e^{\frac{\tau}{2} (f \omega - \mu \varphi)}, \quad \tau \geq 0, \]

that coincides in the case \( \mu = 1 \), after reparametrization, with the continuity path used by Demailly and Kollár alluded to earlier (Remark 3.5).

9 The twisted Ricci iteration and a twisted inverse Ricci operator. When searching for canonical metrics, the presence of continuous symmetries has traditionally required additional analysis. Although the arguments are very similar to the previous sections, there are certain differences. In this section we merely introduce some of the dynamical constructions relevant to this case which will be further used and studied in the sequel [R5] in the setting of convergence towards Kähler-Einstein metrics with continuous symmetries and Kähler-Ricci solitons (or multiplier Hermitian structures). We also state a monotonicity result that will be used in §§10.6.

In the presence of holomorphic vector fields one oftentimes modifies the flow equation by a time-dependent family of automorphisms [CT,TZ4]. More generally, one may study the dynamics of a perturbation of the vector field \(-f\) by an arbitrary vector field \(\chi\). Adapting the point of view of either Section 7 or Section 8 yields two ways to obtain discrete dynamics. The following definition corresponds to the former.

**Definition 9.1.** Given a vector field \(\chi : \omega \mapsto \chi \omega \in C^\infty(M)/\mathbb{R} \) on \(\mathcal{H}_\Omega\) define the \(\chi\)-twisted time \(\tau\) Ricci iteration to be the sequence of forms \(\{\omega_k\}_{k \geq 0}\) satisfying the equations

\[
\begin{align*}
\omega_{k+1} &= \omega_k + \tau \Pi \omega_k + \tau \sqrt{-1} \partial \bar{\partial} \chi \omega_k, \\
\omega_0 &= \omega,
\end{align*}
\]

for each \(k \in \mathbb{N}\) for which a solution exists in \(\mathcal{H}_\Omega\).

The construction in Section 7 corresponds to the zero vector field. The case \(\chi = \psi^X\), with \(X\) an infinitesimal automorphism, will be useful when studying convergence towards solitons.
When $\Omega = c_1, \tau = 1$ this iteration takes on a special form, giving a certain generalized inverse Ricci operator (cf. Definition 6.1).

**Definition 9.2.** Given a vector field $\chi : \omega \mapsto \chi_\omega \in \mathcal{C}^\infty(M)/\mathbb{R}$ on $\mathcal{H}_{c_1}$ define the $\chi$-twisted Ricci operator $\text{Ric}_\chi : \mathcal{H}_{c_1} \rightarrow \mathcal{D}_{c_1}$ by letting $\text{Ric}_\chi \omega := \text{Ric} \omega - \sqrt{-1} \partial \bar{\partial} \chi_\omega$.

Define the $\chi$-twisted inverse Ricci operator $\text{Ric}_{\chi}^{-1} : \mathcal{H}_{c_1} \rightarrow \mathcal{H}_{c_1}$ by letting $\text{Ric}_{\chi}^{-1} \omega := \omega_\varphi$ whenever there exists a unique Kähler form $\omega_\varphi$ in $\mathcal{H}_{c_1}$ satisfying $\text{Ric}_\chi \omega_\varphi = \omega$.

Denote higher-order iterates of these operators by $\text{Ric}_\chi^l$ for $l \in \mathbb{Z}$, setting $\text{Ric}_\chi^0 := \text{Id}$.

Recall that the Bakry-Émery Ricci form associated to a pair $(\omega, a) \in \mathcal{H}_\Omega \times \mathcal{C}^\infty(M)$ is the form $\text{Ric} \omega - \sqrt{-1} \partial \bar{\partial} a$, that can viewed as the Ricci form of the Kähler manifold $(M, J)$ equipped with a Kähler form whose top exterior product equals $e^{2\pi a} \omega^n$ [BE, (4b)]. The twisted Ricci operator is thus an assignment of a Bakry-Émery Ricci form to each Kähler form determined by a vector field on $\mathcal{H}_\Omega$. The simplest examples include the zero vector field and the Ricci potential vector field that yield the Ricci operator and the identity operator, respectively. Note that the fixed points of the twisted Ricci operator are for certain choices of $\chi$ the multiplier Hermitian structures defined by Mabuchi [M3]. The twisted inverse Ricci operator is not defined for general $\chi$, however it is for some geometrically significant vector fields. Assume that $X$ belongs to a reductive Lie subalgebra of $\text{aut}(M, J)$ and that the one-parameter subgroup $T_{1X}$ generated by $JX$ is a compact torus in $\text{Aut}(M, J)$. When $\chi = \psi^X$ the operator $\text{Ric}_{\chi}^{-1}$ restricted to $\mathcal{H}_{c_1}(T_{1X})$ exists and is well-defined according to a theorem of Zhu [Z]. More generally, this is still true when $\chi$ is a smooth function of $\psi^X$ under some assumptions [M3].

First, continuing the discussion of Section 3 (Remark 3.6) observe that when $\Omega = c_1$ and $\chi = \psi^X$ the continuity method path obtained by setting $k = 1$ and letting $\tau$ vary in the segment $[1, \infty)$ coincides with the Tian-Zhu continuity path [TZ2, (1.4)]

$$\omega(t) = e^{\int_{\varphi_s} - \psi^X \varphi_s - s \varphi_s} \omega_\varphi, \quad s \in [0, 1],$$  \hspace{1cm} (45)

via the reparametrization $s = 1 - \frac{1}{t}$, discretizing the $\psi^X$-twisted Kähler-Ricci flow [TZ4, (4.4)]

$$\frac{d\omega(t)}{dt} = -\text{Ric} \omega(t) + \omega(t) + \mathcal{L}_X \omega(t), \quad t \in \mathbb{R}_+, \hspace{1cm} (46)$$

$$\omega(0) = \omega \in \mathcal{H}_{c_1}(T_{1X}).$$

In fact, more generally Mabuchi’s continuity path [M3, (5.1.4)] in the context of multiplier Hermitian structures is obtained in the same manner from (44) as a result of discretizing the corresponding twisted Kähler-Ricci flow.

We now discuss briefly the special case of Kähler-Ricci solitons. This is mainly done for the sake of concreteness since, due to the work of Mabuchi, the relevant computations go through also for general multiplier Hermitian structures.
In their study of Kähler-Ricci solitons on Fano manifolds Tian and Zhu introduced a twisted version of the functional $E_0$ [TZ3]. To define it we first recall some relevant facts [F1, §2.4; M3, TZ2]. Given $X \in \text{aut}(M,J)$, let $L^X_\omega$ denote the elliptic operator $L^X_\omega \phi := \Delta_\omega \phi + X \phi$. This operator is self-adjoint with respect to the $L^2(M, e^{\psi^X_\omega} \omega^n)$ inner product denoted by $\langle \cdot, \cdot \rangle_{\psi^X}$. The vector field $\psi^X$ on $\mathcal{H}_{c_1}$ induces a vector field on the space of Kähler potentials (that we still denote by the same notation) by decreeing that $\int_{M} e^{\psi^X_\omega} \omega^n = 1$ for each $\omega \in \mathcal{H}_{c_1}$. One then has $\psi^X_\omega \phi = \psi^X_\omega + X \phi$ since $\frac{d}{dt} \frac{1}{V} \int_{M} e^{\psi^X_\omega} + X(t^\nu) \omega^n = \langle L^X_\omega \nu, 1 \rangle_{\psi^X} = 0$. Define a functional on $\mathcal{H}_{c_1}(T_{1,X}) \times \mathcal{H}_{c_1}(T_{1,X})$ by

$$E^X_0(\omega, \omega) = \frac{1}{V} \int_{[0,1]} \langle \hat{\phi}_t, L^X_{\omega_\tau} (\psi^X_\omega - f_{\omega_\tau}) \rangle_{\psi^X} dt$$

(47)

This functional is well-defined independently of a choice of path and exact. Its critical points are Kähler-Ricci solitons.

**Lemma 9.3.** Assume that $X$ belongs to a reductive Lie subalgebra of $\text{aut}(M,J)$ and that the one-parameter subgroup $T_{1,X}$ generated by $JX$ is a compact torus in $\text{Aut}(M,J)$. The functional $E^X_0$ is monotonically decreasing along the $\psi^X$-twisted time $\tau$ Ricci iteration for each $\tau > 0$ for which the iteration exists.

**Proof.** Let $\omega \in \mathcal{H}_{c_1}(T_{1,X})$. Whenever the $\psi^X$-twisted time $\tau$ Ricci iteration exists the same is true for smaller time step iterations. Hence the continuity path

$$\omega_\tau^n = \omega^n e^{f_{\omega_\tau} - \psi^X_\omega_\tau + \frac{1}{t} \phi_\tau}, \quad t \in [0, \tau],$$

(48)

exists. Differentiating equation (48) gives $(L^X_{\omega_\tau} + 1 - \frac{1}{t}) \hat{\phi}_\tau = -\frac{1}{t^2} \phi_\tau$. Hence one has

$$E^X_0(\omega_0, \omega_\tau) = \frac{1}{V} \int_{[0,\tau]} \frac{1}{t} \langle \hat{\phi}_t, L^X_{\omega_\tau} \phi_\tau \rangle_{\psi^X} dt$$

$$= -\frac{1}{V} \int_{[0,\tau]} t \langle L^X_{\omega_\tau} \phi_\tau, (L^X_{\omega_\tau} + 1 - \frac{1}{t}) \phi_\tau \rangle_{\psi^X} dt \leq 0.$$

When $\tau \leq 1$ the last inequality is a just a consequence of the ellipticity of $L^X_{\cdot}$. When $\tau > 1$ it follows since $L^X_{\omega_\tau} + 1 - \frac{1}{t}$ is still elliptic [TZ2, Lemma 2.2 (ii)].

10 Some applications. In this section we describe several applications of the Ricci iteration and the inverse Ricci operator to some classical objects and problems in Kähler and conformal geometry.
10.1 The Moser-Trudinger-Onofri inequality on the Riemann sphere and its higher dimensional analogues. We recall some notions from \([R2]\) and explain how the results there on the Moser-Trudinger-Onofri inequality can be rephrased in terms of the inverse Ricci operator. This sheds new light on our discussion there and at the same time expands it (this was omitted from \([R2]\) for the sake of brevity).

Let \(\omega_{FS,c}\) denote the Fubini-Study form of constant Ricci curvature \(c\) on \((S^2,J)\), the Riemann sphere, given locally by

\[
\omega_{FS,c} = \sqrt{-1} \frac{dz \wedge d\bar{z}}{c \pi (1 + |z|^2)^2}.
\]

Here \(V = \int_{S^2} \omega_{FS,c} = c_1([M])/c = 2/c\). For \(c = 1/2\pi\) it is induced from restricting the Euclidean metric on \(\mathbb{R}^3\) to the radius 1 sphere. Denote by \(W^{1,2}(S^2)\) the space of functions on \(S^2\) that are square-summable and so is their gradient (with respect to some Riemannian metric). The Moser-Trudinger-Onofri inequality states:

**Theorem 10.1.** \([Mo,O,Tr]\) For \(\omega = \omega_{FS,2/V}\) and any function \(\varphi\) on \(S^2\) in \(W^{1,2}(S^2)\) one has

\[
\frac{1}{V} \int_{S^2} e^{-\varphi + \frac{1}{2} \int_{S^2} \varphi \omega} \omega \leq \left( e^{\frac{1}{V} \int_{S^2} \sqrt{-1} \partial \varphi \wedge \bar{\partial} \varphi} - 1 \right) \int_{S^2} \varphi \omega \log \left( \frac{1}{V} \int_{S^2} e^{-\varphi} \omega \right) \geq 0.
\]

Equality holds if and only if \(\varphi\) is the pull-back of \(\omega\) by a Möbius transformation.

Several proofs of this classical result have been given in the literature, and we list here the ones we are aware of, chronologically: Onofri \([O]\), Hong \([H]\), Osgood-Phillips-Sarnak \([OPS]\), Beckner \([Be]\), Carlen and Loss \([CL1,CL2]\), Ghigi \([Gh]\) (for more background we refer to Chang \([Ch]\)). All of these proofs use crucially some symmetrization/rearrangement arguments that reduce the problem to a single dimension. Previously we gave a new proof of this inequality coming from Kähler geometry \([R2]\). At the same time we also formulated an optimal (in a sense to be clarified below) extension of it to higher-dimensional Kähler-Einstein manifolds of positive scalar curvature, extending the work of Ding and Tian.

A function satisfies (49) if and only if

\[
F_1(\omega, \omega_\varphi) = \frac{1}{V} \int_{S^2} \frac{1}{2} \sqrt{-1} \partial \varphi \wedge \bar{\partial} \varphi - \frac{1}{V} \int_{S^2} \varphi \omega - \log \frac{1}{V} \int_{S^2} e^{-\varphi} \omega \geq 0.
\]

This functional was studied already by Berger and Moser \([Ber,Mo]\). Moser, extending work of Trudinger, was able to show that \(F_1(\omega, \omega_\varphi) \geq -C\). Then, Onofri showed that \(C = 0\) and characterized the cases of equality.

Aubin first suggested a connection between the classical inequality (49) and Kähler-Einstein manifolds \([A2, (4)]\). Following this, Ding \([D]\) showed how to generalize the functional \(F_1\) to higher-dimensional Fano manifolds—see Equation (17)—using
Aubin’s functional $J$. Using this observation, and modifying the proof of a fundamental result of Bando and Mabuchi concerning the boundedness of the K-energy, Ding and Tian proved Theorem 10.1 for those functions that belong to the subspace $\mathcal{H}_\omega \subset W^{1,2}(S^2)$. We state both results and their corollary. The corollary is Ding and Tian’s restricted generalization$^4$ of the Moser-Trudinger-Onofri inequality to higher-dimensional Kähler-Einstein manifolds.

**Theorem 10.2.** [BM, Theorem A; B, Theorem 1; DT, Theorem 1.1] Let $(M, J, \omega)$ be a Kähler-Einstein Fano manifold. Then $E_0(\omega, \omega_\varphi), F_1(\omega, \omega_\varphi) \geq 0$ for all $\omega_\varphi \in \mathcal{H}_{c_1}$ with equality if and only if $\omega_\varphi = h^*\omega$ with $h \in \text{Aut}(M, J)_0$.

**Corollary 10.3.** [DT] Let $(M, J, \omega)$ be a Kähler-Einstein Fano manifold with $\omega \in \mathcal{H}_{c_1}$. Then for each $\varphi \in \mathcal{H}_\omega$ holds

$$
\frac{1}{V} \int_M e^{-\varphi} + \int_M \varphi \omega^n \omega_n \leq e^{J(\omega, \omega_\varphi)}. 
$$

(50)

Equality holds if and only if $\omega_\varphi$ is the pull-back of $\omega$ by a holomorphic transformation.

One should note that the subspace of Kähler potentials can be considered as a rather small “ball” sitting inside $C^\infty(S^2) \subset W^{1,2}(S^2)$ since in general a large enough multiple of an element of $\mathcal{H}_\omega$ will no longer belong to $\mathcal{H}_\omega$. Following the work of Ding and Tian it remained an open problem how to extend their techniques and provide a complex-geometric proof of the Moser-Trudinger-Onofri inequality. The key hurdle in proving Theorem 10.1 is to extend the argument to the set $C^\infty(M) \setminus \mathcal{H}_\omega$ which a priori has no clear geometric significance as it represents indefinite forms rather than Kähler forms.

Alternatively, what is missing is a geometric interpretation of the Berger-Moser-Ding functional $F_1$. The following result is the key ingredient in our proof of (49) [R2, Lemma 2.4].

**Proposition 10.4.** Let $\Omega = c_1$. The following relation holds

$$(\text{Ric}^{-1})^* E_n = F_1, \text{ on } \mathcal{H}_{c_1} \times \mathcal{D}_{c_1}.$$ 

This provides a geometric interpretation for $F_1$. Indeed, the functional $E_n$ is the potential for the Laplacian of the determinant of the Ricci tensor, considered as a 1-form on $\mathcal{H}_\Omega$, i.e., $dE_n(\omega, \omega_\varphi) = \Delta_{\omega_\varphi} \left( \left( \frac{\text{Ric} \omega_\varphi}{\omega_\varphi} \right)^n \right) \omega_\varphi^n$.

---

$^4$ This term is meant to emphasize that this generalized a restricted version of the classical inequality.
Hence, this result explains the geometric meaning the set $C^\infty(M) \setminus \mathcal{H}_\omega$ plays in the Moser-Trudinger-Onofri inequality. Namely, a function will satisfy this inequality if and only if it represents the Ricci form of a Kähler metric whose Ricci energy $E_n$ is nonnegative with respect to a Kähler-Einstein metric. It now becomes important to understand the sets $A_n = \{\omega_\varphi \in \mathcal{H}_{c_1} : E_n(\omega,\omega_\varphi) \geq 0\}$, defined in Section 4. Naturally, we introduce the following definition.

**Definition 10.5.** Let the Moser-Trudinger-Onofri neighborhood of $\mathcal{H}_\omega$ be the subset

$$MTO_n = \{\varphi \in C^\infty(M) : \varphi \text{ satisfies (50) on the Fano manifold } (M,J), \dim \mathbb{C} M = n\}. \tag{51}$$

We are now in a position to state our generalization of Corollary 10.3 that is optimal in higher dimensions as well as some practical bounds. The result says that the Moser-Trudinger-Onofri inequality holds in higher dimensions on a canonically defined set $MTO_n$ that is strictly larger than the space of Kähler potentials $\mathcal{H}_\omega$ and is geometrically related to Ricci curvature.

**Theorem 10.6.** Let $(M,J,\omega)$ be a Kähler-Einstein Fano manifold.

(i) The generalized Moser-Trudinger-Onofri inequality (50) holds precisely on the set $MTO_n = \text{Ric} (A_n)$. Furthermore, $\mathcal{H}_{c_1} \subsetneq MTO_n \subsetneq \mathcal{D}_{c_1}$.

(ii) Define the sets $B_k := \{\omega_\varphi \in \mathcal{H}_{c_1} : I_k(\omega_\varphi,\text{Ric} \omega_\varphi) \geq 0\}$. Then one has $\mathcal{H}_{c_1} \subsetneq B_k \subsetneq A_k$.

(iii) One has $A_1 = B_1 = \mathcal{H}_{c_1}$, $B_2 \supseteq \{\omega_\varphi \in \mathcal{H}_{c_1} : \text{Ric} \omega_\varphi + 2\omega_\varphi \geq 0\}$, $B_3 \supseteq \{\omega_\varphi \in \mathcal{H}_{c_1} : \text{Ric} \omega_\varphi + \omega_\varphi \geq 0\}$, and for each $k$ one may readily obtain an explicit bound on the set $B_k$, hence on $A_k$, in terms of a lower bound on the Ricci curvature using (16). In particular there exist $c_n > 0$ depending only on $n$ such that,

$$MTO_n \supseteq \{\varphi \in C^\infty(M) : \omega_\varphi \geq -c_n \text{Ric}^{-1} \omega_\varphi\},$$

and, e.g., $c_1 = \infty, c_2 \geq 2, c_3 \geq 1$.

As a corollary we are now able to provide a complex-geometric proof of the classical Moser-Trudinger-Onofri inequality.

**Proof of Theorem 10.1.** Observe that $MTO_1 = \text{Ric} (A_1) = \text{Ric} (\mathcal{H}_{c_1}) = \mathcal{D}_{c_1}$. The last equality requires solving the equation$^5 \text{Ric} \omega_\varphi = \omega_\psi$ for $\varphi$, equivalently Poisson’s equation $\Delta \omega_\varphi = e^{f_\omega - \psi} - 1$. $\square$

---

$^5$ This is the classical $n=1$ version of the Calabi-Yau Theorem whose proof goes back at least to Wallach and Warner [WW].
10.2 An analytic characterization of Kähler-Einstein manifolds and an analytic criterion for almost-Kähler-Einstein manifolds. In the first part of this subsection we explain how the inverse Ricci operator can be used to solve a problem concerning energy functionals on the space of Kähler forms. We hope this sheds new light on the solution of this problem that we gave previously [R2].

Chen and Tian’s generalization of Mabuchi’s Kähler energy, $E_0$, and of Bando and Mabuchi’s Ricci energy, $E_n$, to a family of functionals $\{E_k\}_{k=0}^n$ (see Section 4 for definitions) naturally raised the question of whether Tian’s analytic characterization of Kähler-Einstein manifolds in terms of $E_0$ generalizes to these functionals. In addition it raised the question whether Bando and Mabuchi’s criterion for almost-Kähler-Einstein manifolds in terms of $E_0$ generalizes to these functionals. These questions were also independently raised by Chen [Che2, p. 37; CLW, §1.3]. We now recall both of these fundamental results and explain how to generalize them. This provides an answer to these questions. It shows that the answer is both “yes” and “no”: these criteria extend to the other functionals $\{E_k\}$, however they fail to extend in an identical manner. The subtlety comes from the appearance of the inverse Ricci operator as we will see below.

**Theorem 10.7.** Let $(M, J)$ be a Fano manifold.

(i) [B,BM,DT] If either $F_1$ or $E_0$ is bounded from below on $\mathcal{H}_{c_1}$ then for each $\epsilon > 0$ there exists a Kähler metric $\omega_\epsilon \in \mathcal{H}_{c_1}$ satisfying $\text{Ric}_{\omega_\epsilon} > (1 - \epsilon)\omega_\epsilon$.

(ii) [T5,T6,TZ1] Assume that $\text{Aut}(M, J)$ is finite. Then the properness of $F_1$ (or $E_0$) on $\mathcal{H}_{c_1}$ is equivalent to the existence of a Kähler-Einstein metric.

Our strategy in extending these results to the functionals $\{E_k\}_{k=0}^n$ was: (a) first prove a new formula that expresses $E_k$ in terms of the sum of $E_0$ and another new exact energy functional $(\omega, \omega_\varphi) \mapsto I_k(\omega_\varphi, \text{Ric}_{\omega_\varphi}) - I_k(\omega, \text{Ric}_{\omega})$ (Proposition 4.1) and use it to show

\[
F_1 \text{ bounded from below on } \mathcal{H}_{c_1} \Rightarrow E_0 \text{ bounded from below on } \mathcal{H}_{c_1} \\
\Rightarrow E_1 \text{ bounded from below on } \mathcal{H}_{c_1} \\
\Rightarrow E_2 \text{ bounded from below on } \mathcal{H}_{c_1}^+ \\
\vdots \\
\Rightarrow E_n \text{ bounded from below on } \mathcal{H}_{c_1}^+.
\]

(b) Next use Proposition 10.4 to conclude:

\[
E_n \text{ bounded from below on } \mathcal{H}_{c_1}^+ \Rightarrow F_1 \text{ bounded from below on } \mathcal{H}_{c_1}.
\]

\[6\] In the general case a slightly more involved statement holds (see [R2] for details).
Finally, some additional arguments were needed in order to prove that the proper-
ness of $E_n$ on $\mathcal{H}^+_{c_1}$ implies the existence of a Kähler-Einstein metric.

We can now state the extension of the theorems of Bando-Mabuchi and Tian to
the energy functionals $\{E_k\}$. The case $k = 1$ was proven before by Chen-Li-Wang
and Song-Weinkove in a different manner [CLW,SW].

**Theorem 10.8.** Let $(M,J)$ be a Fano manifold.

(i) If either $F_1$ or $E_k$ (for some $k \in \{0, \ldots, n\}$) is bounded from below on $\mathcal{H}^+_{c_1}$ then
for each $\epsilon > 0$ there exists a Kähler metric $\omega \in \mathcal{H}_{c_1}$ satisfying $\text{Ric} \omega > (1 - \epsilon)\omega$.

(ii) Assume that $\text{Aut}(M,J)$ is finite. Then the properness of $F_1$ or of $E_k$ (for some
$k \in \{0, \ldots, n\}$) on $\mathcal{H}^+_{c_1}$ is equivalent to the existence of a Kähler-Einstein metric.

It is important to note that the appearance of the inverse Ricci operator in step
(b) was a crucial ingredient. The discrepancy between the behavior of $F_1, E_0, E_1$
and that of the functionals $E_2, \ldots, E_n$ can be explained using the time one Ricci
iteration: the first three are unconditionally monotone along the iteration, while for
the latter $n - 1$ this is true if and only if one assumes that the initial point lies in $B_k$,
and $\mathcal{H}^+_{c_1} \subset B_k \subset \mathcal{A}_k \subset \mathcal{H}_{c_1}$ (Proposition 4.2). Furthermore, along the first step of the
iteration the functionals $E_k$ may increase by an arbitrary amount! To be precise, for
any Fano manifold (Kähler-Einstein or not) we have the following result [R2]:

The Ricci energy $E_n$ is bounded from below on $\mathcal{H}_{c_1}$ if and only if $n = 1$. We conclude
that the assumption in Theorem 10.8 (ii) is essential and cannot be weakened from
$\mathcal{H}^+_{c_1}$ to $\mathcal{H}_{c_1}$. This explains our remark earlier on the subtlety present when $k \geq 2$.

Previously, several authors (for references see [R2]) have proven the four implications
on the equivalence of the boundedness from below of $F_1, E_0$ and $E_1$. Often
they appealed to results on the Ricci flow. This suggests that in this context the
Ricci iteration rather than the flow is perhaps more suited.

**Remark 10.9.** In light of the discussion above it would be interesting to know
whether there exist initial conditions in $\mathcal{H}_{c_1}$ for which $E_k$, $k \geq 2$, increases by an
arbitrary amount along the Ricci flow restricted to the time interval $[0, 1]$.

10.3 A new Moser-Trudinger-Onofri inequality on the Riemann sphere and a family
of energy functionals. In the first part of this subsection we prove results
that improve on the restricted generalized Moser-Trudinger-Onofri inequality (Corol-
larly 10.3) in a different direction than that explored in §§10.1 (Theorem 10.6).
Namely, we show that the inequality holds even when one adds certain negative
terms to the exponent on the right hand side. This is done by expressing the excess

---

7 See footnote to Theorem 10.7.
8 We believe that the same result should hold for $E_k$ for each $2 \leq k \leq n$. 

29
in the inequality in geometric terms, namely in terms of the inverse Ricci operator. This is different from Tian’s approach to a strengthened inequality on Kähler-Einstein manifolds [T6, Theorem 6.21] and in particular involves sharp constants and a precise characterization of the case of equality. In the future we hope to address the relation between these two approaches. In the second part we introduce a family of energy functionals and explain their relation to the improved inequality.

We now state the main result of this subsection.

**Theorem 10.10.** Let \((M, J, \omega)\) be a Fano Kähler-Einstein manifold. Then for each \(\varphi \in MTO_n\) holds

\[
\frac{1}{V} \int_M e^{-(\varphi - \frac{1}{V} \int_M \varphi)} \omega^n \leq e^{J(\omega, \omega_\varphi) - \sum_{j=1}^\infty J(\text{Ric}^{-j} \omega_\varphi, \text{Ric}^{-j+1} \omega_\varphi)}.
\]

(52)

Each of the terms in the sum is nonnegative precisely when \(\omega_\varphi \in \text{Ric}(B_n) \supsetneq \mathcal{H}_{c_1}.^{9}\) Equality holds if and only if \(\omega_\varphi\) is the pull-back of \(\omega\) by a holomorphic transformation.

Recall that Theorem 10.6 strengthened the restricted generalized Moser-Trudinger-Onofri inequality (Corollary 10.3) by optimally enlarging the set of functions on which it holds to a set strictly containing \(\mathcal{H}_\omega\). Theorem 10.10 further strengthens Theorem 10.6: it shows that the sets \(MTO_n\) (see (51)) are characterized by an inequality stronger than (50). A version of this result holds also under the assumption that the K-energy is bounded from below. For simplicity we only state the result in the Kähler-Einstein setting.

**Proof.** By definition \(F_1(\omega, \omega_\varphi) \geq 0\) for each \(\varphi \in MTO_n\). Observe that by Theorem 10.6 (i) it follows that \(\text{Ric}^{-1}\) preserves \(MTO_n\). Therefore for each \(l \in \mathbb{N}\),

\[
F_1(\omega, \text{Ric}^{-l} \omega_\varphi) \geq 0, \quad \forall \varphi \in MTO_n.
\]

By exactness of \(F\) we obtain

\[
F_1(\omega, \omega_\varphi) + F_1(\omega_\varphi, \text{Ric}^{-1} \omega_\varphi) + \ldots + F_1(\text{Ric}^{-l+1} \omega_\varphi, \text{Ric}^{-l} \omega_\varphi) \geq 0,
\]

that is,

\[
F_1(\omega, \omega_\varphi) \geq \sum_{j=1}^l F_1(\text{Ric}^{-j} \omega_\varphi, \text{Ric}^{-j+1} \omega_\varphi).
\]

(53)

\footnote{More precisely, each of the terms with \(j \geq 2\) is nonnegative for all \(\varphi \in MTO_n\), while the term with \(j = 1\) is nonnegative precisely when \(\omega_\varphi \in \text{Ric}(B_n)\). Therefore, after possibly omitting the first term in the sum, (52) is an improvement over (50) for all \(\varphi \in MTO_n\) and not just for the subset \(\text{Ric}(B_n) \subset MTO_n\) (both strictly contain \(\mathcal{H}_{c_1}\)).}
Now, using (14)-(15) and (24) one has

$$F_1(\omega, \omega) = J(\omega, \omega) - \frac{1}{V} \int_M \varphi \omega^n - \log \frac{1}{V} \int_M e^{\varphi - \varphi} \omega^n.$$  \hspace{1cm} (54)

It follows that for any $\alpha \in H_{c_1}$ holds

$$F_1(\alpha, \text{Ric} \alpha) = J(\alpha, \text{Ric} \alpha) - \frac{1}{V} \int_M f_\alpha \alpha^n.$$  \hspace{1cm} (55)

Combining (53)-(55), and letting $t$ tend to infinity, yields

$$\frac{1}{V} \int_M e^{-(\varphi - \varphi) \int_M \varphi} \omega^n \leq e^{J(\omega, \omega) - \sum_{j=1}^{\infty} J(\text{Ric}^{-j} \omega, \text{Ric}^{-j+1} \omega) \cdot e^{\sum_{j=1}^{\infty} f^{(j)}(\text{Ric}^{-j} \omega^n)},}$$  \hspace{1cm} (56)

where $f^{(j)}$ is the push-forward of the vector field $f$ under $\text{Ric}^{-j}$. Since by (4) the second term in (55) is nonnegative the desired inequality now follows from (56).

The last statement follows from the fact that $I_n = J$, Proposition 10.4, and the definition of $B_n$ (21).

In the case of the Riemann sphere $S^2$, Theorem 10.6 (iii) implies $\text{Ric}(B_1) = \text{Ric}(A_1) = MTO_1 = C^\infty(S^2)$. Therefore we have the following improvement of the classical Moser-Trudinger-Onofri inequality (Theorem 10.1). For notation we refer to §§10.1.

**Corollary 10.11.** Denote by $(S^2, J, \omega = \omega_{FS, 2/V})$ a round sphere of volume $V$. For any function $\varphi$ on $S^2$ in $W^{1,2}(S^2)$ one has

$$\frac{1}{V} \int_{S^2} e^{-\varphi + \frac{1}{2} \int_{S^2} \varphi} \omega \leq e^{\frac{1}{2} \int_{S^2} \Delta \varphi - \sum_{j=1}^{\infty} f^{(j)}(\text{Ric}^{-j} \omega^n),}$$  \hspace{1cm} (57)

Each of the terms in the sum is nonnegative, and equals zero if and only if $\omega_\varphi$ is obtained from $\omega$ by a Möbius transformation. This also characterizes when equality holds in (57).

Note that the smoothing property of the iteration (see page 8) implies that the extra terms in the sum are meaningful under the assumption $\varphi \in W^{1,2}(S^2)$.

Motivated by Proposition 10.4 we define the following family of energy functionals. For each $k \in \{0, \ldots, n\}$ and $l \in \mathbb{N} \cup \{0\}$ let $E_{k,l}$ denote the pull-back by $\text{Ric}^{-l}$ of the Chen-Tian functional $E_k$ (see (18)). That is

$$E_{k,l}(\omega, \omega) = E_k(\text{Ric}^{-l} \omega, \text{Ric}^{-l} \omega).$$  \hspace{1cm} (58)
For example, \( E_{n,1} = F_1 \), and
\[
E_{k,1}(\omega, \omega \varphi) = F_1(\omega, \omega \varphi) - (J - I_k)(\text{Ric}^{-1} \omega, \omega \varphi) + (J - I_k)(\text{Ric}^{-1} \omega, \omega) , \quad k = 0, \ldots, n.
\]

In light of this, Theorem 10.10 is seen to be a corollary of the following inequality:
\[
E_{n,l+1}(\omega, \cdot) = (\text{Ric}^{-1})^* F_1 \big|_{\{\omega\} \times \text{MTO}_n} \geq 0, \quad \forall l \in \mathbb{N}.
\]

Finally, we remark that in light of Definition 8.1 and Proposition 10.4 one may also extend the definition of Ding’s functional to an arbitrary Kähler manifold and class. This might have some future applications.

10.4 Construction of Nadel-type obstruction sheaves. Up until this point we have scarcely concerned ourselves with the behavior of the various dynamical systems constructed in the absence of a fixed point. In this subsection we show that in this situation, and in the Fano setting, the Ricci iteration will produce Nadel-type obstruction sheaves, similarly to the continuity method and the Ricci flow. The basic references for this subsection are Demailly-Kollár [DK] and Nadel [N1].

Let \( \text{PSH}(M, J, \omega) \subseteq L^1_{\text{loc}}(M) \) denote the set of \( \omega \)-plurisubharmonic functions. For \( \varphi \in \text{PSH}(M, J, \omega) \) define the multiplier ideal sheaf associated to \( \varphi \) as the sheaf \( \mathcal{I}(\varphi) \) defined for each open set \( U \subseteq M \) by local sections
\[
\mathcal{I}(\varphi)(U) = \{ h \in \mathcal{O}_M(U) : |h|^2 e^{-\varphi} \in L^1_{\text{loc}}(M) \}.
\]
Such sheaves are coherent. Such a sheaf is called proper if it is neither zero nor the structure sheaf \( \mathcal{O}_M \).

Nadel showed that in the absence of a Kähler-Einstein metric the continuity method (12) will produce a certain family of multiplier ideal sheaves. Phong, Sësum and Sturm showed that certain multiplier ideal sheaves can be obtained also from the Ricci flow
\[
\omega^n = \omega^n e^{\varphi - \varphi + \hat{\varphi}}, \quad \varphi(0) = \text{const}.
\]
Denote by \([x]\) the largest integer not larger than \(x\).

**Theorem 10.12.** [PSS] Let \((M, J)\) be a Fano manifold not admitting a Kähler-Einstein metric. Let \( \gamma \in (1, \infty) \) and let \( \omega \in \mathcal{H}_{c_1} \). Then there exists an initial condition \( \varphi(0) \) and a subsequence \( \{ \varphi_{t_j} \}_{j \geq 0} \) of solutions of (62) such that \( \lim_{j \to \infty} \varphi_{t_j} = \varphi_\infty \in \text{PSH}(M, J, \omega) \) and \( \mathcal{I}(\gamma \varphi_\infty) \) is a proper multiplier ideal sheaf satisfying
\[
H^r(M, \mathcal{I}(\gamma \varphi_\infty) \otimes K_M^{-[\gamma]}) = 0, \quad \forall r \geq 1.
\]

Their proof relies on some of Perelman’s estimates for the Ricci flow as well as the following theorem of Kołodziej.
Some discretizations of geometric evolution equations

Theorem 10.13. [Ko] Let \( F \in L^p(M, \omega) \), \( p > 1 \) be a positive continuous function with \( \int_M F \omega^n = 1 \). There exists a bounded solution \( \varphi \) to the equation \( \omega^n_{\varphi} = F \omega^n \) on \( M \) which satisfies \( \text{osc} \varphi \leq C \) with \( C \) depending only on \( \|F\|_{L^p(M, \omega), p} \) and \( (M, \omega) \).

Let \( \tau = 1/\mu = 1 \). The following simple result is a discrete analogue of Theorem 10.12. Its very simple proof compared to that of the analogous result for the Ricci flow is our main motivation for including it here. Moreover, the sheaves produced in this way are essentially computable (see the next subsection).

Theorem 10.14. Let \( (M, J) \) be a Fano manifold not admitting a Kähler-Einstein metric. Let \( \gamma \in (1, \infty) \) and let \( \omega \in \mathcal{H}_{c1} \). Then there exists a subsequence \( \{\psi_{jk}\}_{k \geq 1} \) of solutions of (5) such that \( \lim_{k \to \infty} \psi_{jk} = \psi_{\infty} \in PSH(M, J, \omega) \) and \( \mathcal{I}(\gamma \psi_{\infty}) \) is a proper multiplier ideal sheaf satisfying (63).

Proof. Indeed, since the iteration takes the form
\[
\omega^n_{\psi_{l+1}} = \omega^n e^{f - \psi_l}, \quad l \in \mathbb{N},
\]
Theorem 10.13 can be directly applied (observe that from (4) an estimate on \( \text{osc} \psi_l \) implies one on \( \|\psi_l\|_{L^\infty(M)} \)) to construct sheaves with \( \gamma > 1 \), making use of Proposition 4.2 (for more details see [R4, §2 (iv)]).

Remark 10.15. One may also construct multiplier ideal sheaves for the Ricci iteration with other time steps and for exponents in the range \((n/(n+1), 1)\) much the same as the continuity method sheaves constructed by Nadel as well as the analogous ones constructed in [R4] for the Ricci flow (we hope to discuss this in more detail in the sequel; note that the latter construction strengthened Theorem 10.12).

We remark that in the context of this section, it is also interesting to study the limiting behavior of the inverse Ricci operator (Section 8) under iteration.

10.5 Relation to balanced metrics. In this paragraph we describe an immediate corollary of the work of Donaldson. It was pointed out to me by J. Keller. It gives with no further work an algorithm for computing Kähler-Einstein metrics using balanced metrics: Given a polarized Hodge manifold \((X, L)\) and a volume form \( \nu \) Donaldson [Do3] constructs a sequence of pull-backs of Fubini-Study metrics in \( \mathcal{H}_{c1}(L) \) induced from Kodaira embeddings that converge to a solution of the Calabi-Yau equation \( \omega^n_{\varphi} = \nu \). Since in the Fano case our time one Ricci iteration consists precisely of solving a Calabi-Yau equation at each iteration we see that repeated application of Donaldson’s constructions approximates the Ricci iteration and in this sense provides a quantization of the Ricci flow.

Another consequence is the possibility to numerically construct Nadel-type sheaves on Fano manifolds admitting no Kähler-Einstein metrics, by §§10.4.
Note that more generally one may approximate in the same manner the orbits of the iteration given by the inverse Ricci operator (Definition 8.1) on an arbitrary Kähler manifold with $\Omega \in H^2(M, \mathbb{Z})$.

Finally, it would be interesting to find more relations between discretizations of other geometric flows and iteration schemes involving Bergman metrics.

10.6 A question of Nadel. As explained in Section 2 one of the original motivations for our work was a question raised by Nadel \cite{N2}: Given $\omega \in \mathcal{H}_{c_1}$ define a sequence of metrics $\omega, \text{Ric} \omega, \text{Ric} (\text{Ric} \omega), \ldots$, as long as positivity is preserved; what are the periodic orbits of this dynamical system? The cases $k = 2, 3$ in the following theorem are due to Nadel.

**Theorem 10.16.** Let $(M, J, \omega)$ be a Fano manifold and assume that $\text{Ric}^l \omega = \omega$ for some $l \in \mathbb{Z}$. Then $\omega$ is Kähler-Einstein.

**Proof.** The theorem follows from Proposition 4.2. Indeed, note that the nonexistence of periodic fixed points of negative order implies that of positive order, and vice versa. Therefore assume that for some $\omega \in \mathcal{H}_{c_1}$ and some $l \in \mathbb{N}$ one has $\text{Ric}^{-l} \omega = \omega$. By the cocycle condition we thus have

$$0 = E_0(\omega, \text{Ric}^{-l} \omega) = \sum_{i=0}^{l-1} E_0(\text{Ric}^{-i} \omega, \text{Ric}^{-i-1} \omega).$$

(64)

By Proposition 4.2 one has

$$E_0(\text{Ric}^{-i} \omega, \text{Ric}^{-i-1} \omega) < 0,$$

unless $\text{Ric}^{-i} \omega = \text{Ric}^{-i-1} \omega$. Therefore each of the terms in (64) vanishes and $\omega$ is Kähler-Einstein.

Moreover, from the proof we have the following stronger conclusion:

**Corollary 10.17.** Let $(M, J, \omega)$ be a Fano manifold with trivial Futaki character. Assume that $\text{Ric}^l \omega = h^* \omega$ for some $l \in \mathbb{Z}$ and some $h \in \text{Aut}(M, J)$. Then $h = \text{id}$ and $\omega$ is Kähler-Einstein.

Lemma 9.3 implies the following natural generalization of Theorem 10.16 to the setting of solitons.

**Corollary 10.18.** Let $(M, J, \omega)$ be a Fano manifold and assume that $X$ belongs to a reductive Lie subalgebra of $\text{Aut}(M, J)$ and that the one-parameter subgroup $T_{JX}$ generated by $JX$ is a compact torus in $\text{Aut}(M, J)$. Let $\omega \in \mathcal{H}_{c_1}(T_{JX})$. Assume that $\text{Ric}^l_{JX} \omega = \omega$ for some $l \in \mathbb{Z}$. Then $\omega$ is a Kähler-Ricci soliton.
In addition, under the assumption that the Tian-Zhu character \([TZ3]\) is trivial one has a statement analogous to Corollary 10.17. Also, as noted in Section 9, and using a generalized character introduced by Futaki \([F2]\), this result extends to the setting of multiplier Hermitian structures.

To conclude this subsection we remark that what now becomes apparent is that Nadel’s iteration scheme is precisely the Euler method for the conjugate Ricci flow and is thus dual to our iteration that corresponds to the backwards Euler method for the Ricci flow.

**Remark 10.19.** In light of Theorem 10.16 perhaps it would be interesting to re-examine Nadel’s generalized maximum principle which was used to provide a completely different proof for the cases \(k = 2, 3\).

### 10.7 The Ricci index and a canonical nested structure on the space of Kähler metrics

In this subsection we describe a new canonical structure inherent in the space of Kähler forms determined by the complex structure and the Kähler class alone.

Consider first the case of a Fano manifold. As we saw earlier the iteration of the inverse Ricci operator on \(\mathcal{H}_{c_1}\) has the advantage of possessing infinite orbits starting at any initial points. The Ricci operator on the other hand lacks this property, according to the Calabi-Yau theorem. This motivates the following definition.

**Definition 10.20.** Let \((M, J)\) be a Fano manifold. For each \(l \in \mathbb{N} \cup \{0\}\) denote by \(\mathcal{H}_{c_1}^{(l)}\) denote the domain of definition of \(\text{Ric}^l\).

One has

\[
\mathcal{D}_{c_1} = \mathcal{H}_{c_1}^{(0)} \supset \mathcal{H}_{c_1}^{(1)} \supset \mathcal{H}_{c_1}^{(2)} = \mathcal{H}_{c_1}^+ \supset \cdots \supset \mathcal{H}_{c_1}^{(l)} \supset \cdots.
\]

In other words, we may define on \(\mathcal{H}_{c_1}\) an integer-valued function

\[
\omega \mapsto r(\omega),
\]

where \(r(\omega)\) is the unique positive integer satisfying \(\omega \in \mathcal{H}_{c_1}^{(r(\omega))} \setminus \mathcal{H}_{c_1}^{(r(\omega)+1)}\). When no such number exists we set \(r(\omega) = \infty\). We call the function \(r : \mathcal{H}_{c_1} \to \mathbb{N}\) the Ricci index. The number \(r(\omega)\) is a Riemannian invariant of the manifold \((M, J, \omega)\). It may also be defined for general Riemannian manifolds however it seems hard to study in such generality.

One may extend such a construction to a general Kähler manifold in at least two ways, using either the Ricci iteration or the inverse Ricci operator. Choosing the latter we obtain the following extension of Definition 10.20.

**Definition 10.21.** Let \((M, J)\) be a Kähler manifold and let \(\Omega\) denote a Kähler class. For each \(l \in \mathbb{N}\) denote by \(\mathcal{H}_{c_1}^{(l)}\) the image of \(\mathcal{H}_\Omega\) under \(\text{Ric}^{-l+1}_\Omega\).
Several natural questions arise that we hope to touch upon in the future. What is $\mathcal{H}^{(\infty)}_\Omega := \bigcap_{l=1}^{\infty} \mathcal{H}^{(l)}_\Omega$? How to asymptotically relate the Ricci index to the time parameter of the Ricci flow? Also, how to relate the Ricci index, on the one hand, to the metric structure on the space $\mathcal{H}_\Omega [M2, Se, Do2]$ defined by $\langle \mu, \nu \rangle_\omega = \frac{1}{V} \int_M \mu \nu \omega^n$, $\forall \mu, \nu \in T_\omega \mathcal{H}_\Omega \cong C^\infty(M)/\mathbb{R}$, and, on the other hand, to sublevel sets of Calabi's energy and Mabuchi's K-energy? Finally, what is the relation between the Ricci index and positivity?

I would like to express my deep gratitude to my teacher, Gang Tian, for his advice, warm encouragement and for pointing out to me the relevance of [TY]. I am indebted to him as well as to J. Song for suggesting that the inverse Ricci operator I defined could be related to the Ricci flow. This had a decisive impact on the present work. I would also like to thank Curtis McMullen, whose class several years ago has been a source of great inspiration. The first version of this article [R1] was written while I was a Visiting Scholar at Peking University during Summer 2005. The present version was completed in Summer 2007 during a visit to the Technion. I thank both institutions for their hospitality and partial financial support. Much of the present work was presented in the past year at several places, including at Imperial College in December 2006. This material is based upon work supported under a National Science Foundation Graduate Research Fellowship.

Bibliography

[A1] Thierry Aubin, Équations du type Monge-Ampère sur les variétés kählériennes compactes, *Bulletin des Sciences Mathématiques* 102 (1978), 63–95.

[A2] _, Réduction du cas positif de l’équation de Monge-Ampère sur les variétés kählériennes compactes à la démonstration d’une inégalité, *Journal of Functional Analysis* 57 (1984), 143–153.

[A3] _, Some nonlinear problems in Riemannian Geometry, Springer, 1998.

[BE] Dominique Bakry, Michel Émery, Diffusions hypercontractives, in *Séminaire de probabilités XIX, 1983/84* (J. Azéma et al., Eds.), Lecture Notes in Mathematics 1123, Springer, 1985, 177–206.

[B] Shigetoshi Bando, The K-Energy Map, Almost Kähler-Einstein Metrics and an Inequality of the Miyaoka-Yau Type, *Tohoku Mathematical Journal* 39 (1987), 231–235.

[BM] Shigetoshi Bando, Toshiki Mabuchi, Uniqueness of Kähler-Einstein metrics modulo connected group actions, in *Algebraic Geometry, Sendai, 1985* (T. Oda, Ed.), Advanced Studies in Pure Mathematics 10, Kinokuniya, 1987, 11–40.
Some discretizations of geometric evolution equations

[Be] William Beckner, Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality, *Annals of Mathematics* **138** (1993), 213–242.

[Ber] Melvyn S. Berger, Riemannian structures of prescribed Gaussian curvature for compact 2-manifolds, *Journal of Differential Geometry* **5** (1971), 325–332.

[Bes] Alfred L. Besse, Einstein manifolds, Springer, 1987.

[C1] Eugenio Calabi, The variation of Kähler metrics. I. The structure of the space; II. A minimum problem, *Bulletin of the American Mathematical Society* **60** (1954), 167–168.

[C2] ______, On Kähler manifolds with vanishing canonical class, in *Algebraic geometry and topology. A symposium in honor of S. Lefschetz* (R. H. Fox, Ed.), Princeton University Press, 1957, 78–89.

[Ca] Huai-Dong Cao, Deformations of Kähler metrics to Kähler-Einstein metrics on compact Kähler manifolds, *Inventiones Mathematicae* **81** (1985), 359–372.

[CL1] ______, Competing symmetries of some functionals arising in mathematical physics, in *Stochastic processes, physics and geometry* (S. Albeverio et al., Eds.), World Scientific, 1990, 277–288.

[CL2] Eric A. Carlen, Michael Loss, Competing symmetries, the logarithmic HLS inequality and Onofri’s inequality on $S^n$, *Geometric and Functional Analysis* **2** (1992), 90–104.

[Ch] Sun-Yung A. Chang, Non-linear elliptic equations in conformal geometry, European Mathematical Society, 2004.

[Che1] Xiu-Xiong Chen, On the lower bound of the Mabuchi energy and its application, *International Mathematics Research Notices* (2000), 607–623.

[Che2] ______, On the lower bound of energy functional $E_1$ (I)—a stability theorem on the Kähler-Ricci flow, *The Journal of Geometric Analysis* **16** (2006), 23–38.

[CLW] Xiu-Xiong Chen, Hao-Zhao Li, Bing Wang, On the Kähler-Ricci flow with small initial $E_1$ energy (I), preprint, arxiv: math.DG/0609694 v2. To appear in *Geometric and Functional Analysis*.

[CT] Xiu-Xiong Chen, Gang Tian, Ricci flow on Kähler-Einstein surfaces, *Inventiones Mathematicae* **147** (2002), 487–544.

[Cho] Bennett Chow et al., The Ricci flow: Techniques and applications. Part I: Geometric aspects, American Mathematical Society, 2007.

[DK] Jean-Pierre Demailly, János Kollár, Semi-continuity of complex singularity exponents and Kähler-Einstein metrics on Fano orbifolds, *Annales Scientifiques de l'École Normale supérieure* **34** (2001), 525–556.

[D] Wei-Yue Ding, Remarks on the existence problem of positive Kähler-Einstein metrics, *Mathematische Annalen* **282** (1988), 463–471.

[DT] Wei-Yue Ding, Gang Tian, The generalized Moser-Trudinger inequality, in *Nonlinear Analysis and Microlocal Analysis: Proceedings of the International Conference*
at Nankai Institute of Mathematics (K.-C. Chang et al., Eds.), World Scientific, 1992, 57–70. ISBN 9810209134.

[Do1] Simon K. Donaldson, Anti self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles, *Proceedings of the London Mathematical Society* **50** (1985), 1–26.

[Do2] _____, Symmetric spaces, Kähler geometry and Hamiltonian dynamics, in *Northern California Symplectic Geometry Seminar* (Ya. Eliashberg et al., Eds.), American Mathematical Society Translations: Series 2 **196**, American Mathematical Society, 1999, 13–33.

[Do3] _____, Some numerical results in complex differential geometry, preprint, April 27th, 2006.

[F1] Akito Futaki, Kähler-Einstein metrics and integral invariants, Lecture Notes in Mathematics **1314**, Springer, 1988.

[F2] _____, Some invariant and equivariant cohomology classes of the space of Kähler metrics, *Proceedings of the Japan Academy. Series A* **78** (2002), 27–29.

[Gh] Alessandro Ghigi, On the Moser-Onofri and Prékopa-Leindler inequalities, *Collectanea Mathematica* **56** (2005), 143–156.

[G1] Daniel Z.-D. Guan, Quasi-Einstein metrics, *International Journal of Mathematics* **6** (1995), 371–379.

[G2] _____, Extremal-solitons and $C^\infty$ convergence of the modified Calabi flow on certain $\mathbb{CP}^1$ bundles, preprint, December 22nd, 2006.

[H] Richard S. Hamilton, Three-manifolds with positive Ricci curvature, *Journal of Differential Geometry* **17** (1982), 255–306.

[Ho] Chong-Wei Hong, A best constant and the Gaussian curvature, *Proceedings of the American Mathematical Society* **97** (1986), 737–747.

[J] Jürgen Jost, Nonlinear methods in Riemannian and Kählerian geometry, Birkhäuser, 1988.

[K] Erich Kähler, Über eine bemerkenswerte Hermitesche Metrik, *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg* **9** (1933), 173–186.

[Ko] Sławomir Kołodziej, The complex Monge-Ampère equation and pluripotential theory, *Memoirs of the American Mathematical Society* **178** (2005), no. 840.

[M1] Toshiki Mabuchi, K-energy maps integrating Futaki invariants, *Tôhoku Mathematical Journal* **38** (1986), 575–593.

[M2] _____, Some symplectic geometry on compact Kähler manifolds. I, *Osaka Journal of Mathematics* **24** (1987), 227–252.

[M3] _____, Multiplier Hermitian structures on Kähler manifolds, *Nagoya Mathematical Journal* **170** (2003), 73–115.

[Mo] Jürgen Moser, A sharp form of an inequality by N. Trudinger, *Indiana University Mathematics Journal* **20** (1971), 1077–1092.
Some discretizations of geometric evolution equations

[N1] ______, Multiplier ideal sheaves and Kähler-Einstein metrics of positive scalar curvature, *Annals of Mathematics* **132** (1990), 549–596.

[N2] Alan M. Nadel, On the absence of periodic points for the Ricci curvature operator acting on the space of Kähler metrics, in *Modern Methods in Complex Analysis: The Princeton Conference in Honor of Gunning and Kohn* (T. Bloom et al., Eds.), Annals of Mathematics Studies **137**, Princeton University Press, 1995, 273–282.

[O] Enrico Onofri, On the positivity of the effective action in a theory of random surfaces, *Communications in Mathematical Physics* **86** (1982), 321–326.

[OPS] Brad Osgood, Ralph Phillips, Peter Sarnak, Extremals of determinants of Laplacians, *Journal of Functional Analysis* **80** (1988), 148–211.

[P] Peter Petersen, Riemannian geometry, Springer, 1998.

[PSS] Duong-Hong Phong, Nataša Šešum, Jacob Sturm, Multiplier ideal sheaves and the Kähler-Ricci flow, preprint, arxiv: [math.DG/0611794](http://arxiv.org/abs/math.DG/0611794) v2.

[R1] Yanir A. Rubinstein, On iteration of the Ricci operator on the space of Kähler metrics, I, manuscript, August 14th, 2005, unpublished.

[R2] ______, On energy functionals and the existence of Kähler-Einstein metrics, preprint, arxiv: [math.DG/0612440](http://arxiv.org/abs/math.DG/0612440) v3.

[R3] ______, The Ricci iteration and its applications, preprint, arxiv: 0706.2777 v3 [math.DG].

[R4] ______, On the construction of Nadel multiplier ideal sheaves and the limiting behavior of the Ricci flow, preprint, arxiv: 0708.1590 v2 [math.DG].

[R5] ______, Some discretizations of geometric evolution equations and the Ricci iteration on the space of Kähler metrics, II, preprint, in preparation.

[Se] Stephen Semmes, Complex Monge-Ampère and symplectic manifolds, *American Journal of Mathematics* **114** (1992), 495–550.

[S] Santiago R. Simanca, Heat flows for extremal Kähler metrics, *Annali della Scuola Normale Superiore di Pisa* **4** (2005), 187–217.

[Si] Yum-Tong Siu, Lectures on Hermitian-Einstein metrics for stable bundles and Kähler-Einstein metrics, Birkhäuser, 1987.

[SW] Jian Song, Ben Weinkove, Energy functionals and canonical Kähler metrics, *Duke Mathematical Journal* **137** (2007), 159–184.

[Th] Richard P. Thomas, Notes on GIT and symplectic reduction for bundles and varieties, in *Surveys in Differential Geometry: Essays in memory of S.-S. Chern* (S.-T. Yau, Ed.), International Press, 2006, 221–273.

[T1] Gang Tian, On Kähler-Einstein metrics on certain Kähler manifolds with $C_1(M) > 0$, *Inventiones Mathematicae* **89** (1987), 225–246.

[T2] ______, On Calabi’s conjecture for complex surfaces with positive first Chern class, *Inventiones Mathematicae* **101** (1990), 101–172.
[T3] Y. A. Rubinstein, On stability of the tangent bundles of Fano varieties, *International Journal of Mathematics* **3** (1992), 401–413.

[T4] Y. A. Rubinstein, The K-energy on hypersurfaces and stability, *Communications in Analysis and Geometry* **2** (1994), 239–265.

[T5] Y. A. Rubinstein, Kähler-Einstein metrics with positive scalar curvature, *Inventiones Mathematicae* **130** (1997), 1–37.

[T6] Y. A. Rubinstein, Canonical Metrics in Kähler Geometry, Birkhäuser, 2000.

TZ1] Gang Tian, Xiao-Hua Zhu, A nonlinear inequality of Moser-Trudinger type, *Calculus of Variations* **10** (2000), 349–354.

TZ2] Gang Tian, Xiao-Hua Zhu, Uniqueness of Kähler-Ricci solitons, *Acta Mathematica* **184** (2000), 271–305.

TZ3] Gang Tian, Xiao-Hua Zhu, A new holomorphic invariant and uniqueness of Kähler-Ricci solitons, *Commentarii Mathematici Helvetici* **77** (2002), 297–325.

TZ4] Gang Tian, Shing-Tung Yau, Convergence of Kähler-Ricci flow, *Journal of the American Mathematical Society* **20** (2007), 675–699.

TY] Gang Tian, Shing-Tung Yau, Existence of Kähler-Einstein metrics on complete Kähler manifolds and their applications to algebraic geometry, in *Mathematical Aspects of String Theory* (S.-T. Yau, Ed.), Advanced Series in Mathematical Physics, 1, World Scientific, 1987, 574–628.

Tr] Neil S. Trudinger, On imbeddings into Orlicz spaces and some applications, *Journal of Mathematics and Mechanics* **17** (1967), 473–483.

WW] Nolan R. Wallach, Frank W. Warner, Curvature forms for 2-manifolds, *Proceedings of the American Mathematical Society* **25** (1970), 712–713.

Y] Shing-Tung Yau, On the Ricci curvature of a compact Kähler manifold and the Complex Monge-Ampère equation, I, *Communications in Pure and Applied Mathematics* **31** (1978), 339–411.

Z] Xiao-Hua Zhu, Kähler-Ricci soliton type equations on compact complex manifolds with $C_1(M) > 0$, *The Journal of Geometric Analysis* **10** (2000), 759–774.