Analytical approximation for Landau’s constants by using BPES method

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Abstract
In this study, approximation formulas for evaluating Landau constants are elaborated by using the Boubaker Polynomials Expansion Scheme (BPES). Results are compared to some referred studies.

Keywords: Landau constants; Approximation theory; Complex analysis; Boubaker Polynomials Expansion Scheme (BPES); Error analysis.

1 Introduction

Landau’s constants are defined [1]-[4] for all positive integers n, by:

$$G_n = \sum_{k=0}^{n} \frac{1}{16^k} \binom{2k}{k}^2 = 1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1\cdot3}{2\cdot4}\right)^2 + \cdots + \left(\frac{(2n-1)!!}{(2n)!!}\right)^2.$$ (1.1)

These constants were defined in relation with a set F of complex analytic functions f defined on an open region containing the closure of the unit disk $D (D = \{z : |z| < 1\})$
satisfying the conditions:
\[ f(z)|_{z=0} = 0 \quad \text{and} \quad \frac{d}{dz}f(z)|_{z=0} = 1. \] (1.2)

If \( \ell(f) \) is the supremum of all numbers such that \( f(D) \) contains a disk of radius 1, and that \( f \) verifies the additional conditions:
\[ |f(z)| < 1(|z| < 1) \quad \text{and} \quad f(z) = \sum_{k=0}^{\infty} a_k z^k \] (1.3)

then:
\[ L = \inf \{ \ell(f) : f \in F \} = \left| \sum_{k=0}^{\infty} a_k \right| \leq G_n \] (1.4)

In this paper, a convergent protocol is proposed in order to give analytical expressions to the Landau’s constants.

2 Resolution process

In concordance of the approximations performed by Watson [2], Zhao [4] and Popa [3], a sequence \( \{\omega_n\}_{n \geq 0} \) is defined as:
\[ \omega_n = G_n - \frac{1}{\pi} \ln(n + A) - \frac{1}{\pi} (\gamma + \ln 16) + \frac{B}{n + C} \] (2.1)

where \( \gamma \) is Euler’s constant and \( A, B \) and \( C \) are unknown constants.

The resolution process aims to find accurate approximation to the values of the parameters \( A, B \) and \( C \) such that \( \{\omega_n\}_{n \geq 0} \) is the fastest sequence which would converge to zero.

3 Approximation using The Boubaker Polynomials Expansion Scheme (BPES)

3.1 Presentation

The Boubaker Polynomials Expansion Scheme BPES [5]-[24] is a resolution protocol which has been successfully applied to several applied-physics and mathematics problems. The BPES protocol ensures the validity of the related boundary conditions regardless main equation features. The BPES is mainly based on Boubaker polynomials first derivatives properties:
\[ \sum_{k=1}^{N} B_{4k}(x) \big|_{x=0} = -2N \neq 0, \quad \sum_{k=1}^{N} B_{4k}(r) = 0 \] (3.1)

and

\[ \sum_{k=1}^{N} \frac{dB_{4k}(x)}{dx} \big|_{x=0} = 0, \quad \sum_{k=1}^{N} \frac{dB_{4k}(x)}{dx} \big|_{x=0} = \sum_{k=1}^{N} H_{k} \] (3.2)

with

\[ H_{n} = B_{4n}(r_{n}) = \frac{4\alpha_{n}[2 - r_{n}^{2}]\sum_{k=1}^{n} B_{4k}^{2}(r_{n})}{B_{4(n+1)}(r_{n})} + 4r_{n}^{3}. \]

Several solution have been proposed through the BPES in many fields such as numerical analysis \[5\]-\[8\], theoretical physics \[9\]-\[11\], mathematical algorithms \[12\], heat transfer \[13\], homodynamic \[14, 15\], material characterization \[16\], fuzzy systems modeling \[17\]-\[22\] and biology \[23, 24\].

### 3.2 Application

The resolution protocol is based on setting \( \hat{A}, \hat{B} \) and \( \hat{C} \) as estimators to the constants \( A, B \) and \( C \), respectively:

\[
\begin{align*}
\hat{A} &= \frac{1}{2N_{0}} \sum_{k=1}^{N_{0}} \hat{\alpha}_{k} B_{4k}(x_{r_{k}}), \\
\hat{B} &= \frac{1}{2N_{0}} \sum_{k=1}^{N_{0}} \hat{\beta}_{k} B_{4k}(x_{r_{k}}), \\
\hat{C} &= \frac{1}{2N_{0}} \sum_{k=1}^{N_{0}} \hat{\gamma}_{k} B_{4k}(x_{r_{k}})
\end{align*}
\] (3.3)

where \( B_{4k} \) are the \( 4k \)-order Boubaker polynomials \[15\]-\[22\], \( r_{k} \) are \( B_{4k} \) minimal positive roots, \( N_{0} \) is a prefixed integer, and \( a_{k}|_{k=1,\ldots,N_{0}} \) are unknown pondering real coefficients.

As a first step, the coefficients, the coefficients \( \hat{\alpha}_{k}|_{k=1,\ldots,N_{0}} \) are determined through Falaleev approximation \[yy\]:

\[ G_{n} \approx \frac{1}{\pi} \left[ \ln \left( \frac{n+3}{4} \right) + \gamma + \ln 16 \right] \] (3.4)

The BPES solution for \( \hat{A} \) is obtained by determining the non-null set of coefficients \( \hat{\alpha}_{k}|_{k=1,\ldots,N_{0}} \) that minimizes the absolute difference \( \Delta_{N_{0}} \):

\[ \Delta_{N_{0}} = \left| \frac{1}{2N_{0}} \sum_{k=1}^{N_{0}} \hat{\alpha}_{k} \Lambda_{k} - \frac{1}{\pi}(\gamma + \ln 16) \right| \] (3.5)
with
\[ \Lambda_k = \frac{3}{4} \int_0^1 \sum_{k=1}^{N_0} x B_{4k}(x r_k) dx. \]

Values of \( \hat{B} \) and \( \hat{C} \) are consecutively deduced from coefficients \( \hat{\xi}_k \) \( k = 1, \ldots, N_0 \) and \( \hat{\xi}_k \) \( k = 1, \ldots, N_0 \) which minimize the absolute difference \( \Delta'_{N_0} \):

\[ \Delta'_{N_0} = \left| \frac{1}{2N_0} \sum_{k=1}^{N_0} \hat{\xi}_A X_k - \frac{1}{\pi} (\gamma + \ln 16) + \frac{1}{2N_0} \sum_{k=1}^{N_0} \hat{\xi}_B Y_k \left( 1 - \frac{1}{2N_0} \sum_{k=1}^{N_0} \hat{\xi}_C Z_k \right) \right| \quad (3.6) \]

with
\[ X_k = \frac{3}{4\pi} \int_0^1 \sum_{k=1}^{N_0} x B_{4k}(x r_k) dx \quad \text{and} \quad Y_k = Z_k = \int_0^1 \sum_{k=1}^{N_0} B_{4k}(x r_k) dx. \]

Hence the final solution is:
\[ G_n = \frac{1}{\pi} \ln(n + A) + \frac{1}{\pi} (\gamma + \ln 16) - \frac{B}{n + C} \quad (3.7) \]

with
\[
\begin{aligned}
A & = \hat{A} = \frac{1}{2N_0} \sum_{k=1}^{N_0} \xi_k B_{4k}(x r_k), \\
B & = \hat{B} = \frac{1}{2N_0} \sum_{k=1}^{N_0} \xi_k B_{4k}(x r_k), \\
C & = \hat{C} = \frac{1}{2N_0} \sum_{k=1}^{N_0} \xi_k B_{4k}(x r_k).
\end{aligned}
\]

### 4 Results, plots and discussion

Numerical solutions obtained by the given method are gathered in Table 1 along with precedent refereed approximations by Falaleev [25] and Brutman [26].

Plots of the BPES solution are presented in Fig. 1, along with referred solutions [25, 26].

Figure 2 displays the obtained values along with those recorded by Falaleev [25] and Brutman [26]. For accuracy purposes, error analysis has been carried out for the two referred datasets. Examination of the quadratic error plots (Fig. 2) shows that the amplitudes of the error are more exaggerated according to Falaleev approximation [25], particularly for low values of \( n \).

Moreover, Figure 2 monitors an obvious logarithmic profile in the range with a quadrature error which doesn’t exceed 3.5% on the whole range (Fig. 2). The obtained profile is in good agreement with the values recorded elsewhere [25]-[35].
| n  | Landau constants | Quadratic Error vs. Exact |
|----|------------------|--------------------------|
|    | Falaleev Approximation [25] | Brutman Approximation [26] | BPES Approximation |
| 0  | 0.97463516       | 0.97469795               | 0.97473411         | 9.7924E-9 | 1.30765E-9 |
| 1  | 1.24433844       | 1.24440124               | 1.24449240         | 2.37026E-8 | 8.31042E-9 |
| 2  | 1.38829778       | 1.38827257               | 1.38841873         | 4.36628E-8 | 2.13632E-8 |
| 3  | 1.48693516       | 1.48699795               | 1.48719911         | 6.96731E-8 | 4.04659E-8 |
| 4  | 1.56218004       | 1.56224284               | 1.56249900         | 1.01733E-7 | 6.56187E-8 |
| 5  | 1.62299481       | 1.62305761               | 1.62336877         | 1.39843E-7 | 9.68215E-8 |
| 6  | 1.67403346       | 1.67409626               | 1.67442424         | 1.84004E-7 | 1.34074E-7 |
| 7  | 1.71800808       | 1.71807088               | 1.71849204         | 2.34214E-7 | 1.77377E-7 |
| 8  | 1.75663844       | 1.75670124               | 1.7571740          | 2.90474E-7 | 2.2673E-7  |
| 9  | 1.79108390       | 1.79114669               | 1.79167785         | 3.52784E-7 | 2.82133E-7 |
| 10 | 1.82216319       | 1.82222598               | 1.82281214         | 4.21145E-7 | 3.43585E-7 |
| 11 | 1.85047605       | 1.85053884               | 1.85118001         | 4.95555E-7 | 4.11088E-7 |
| 12 | 1.87647497       | 1.87653777               | 1.87723393         | 5.76015E-7 | 4.84641E-7 |
| 13 | 1.90050977       | 1.90057257               | 1.90132373         | 6.62525E-7 | 5.64244E-7 |
| 14 | 1.92285648       | 1.92291928               | 1.92372544         | 7.58085E-7 | 6.49986E-7 |
| 15 | 1.94373675       | 1.94379954               | 1.94466070         | 8.53696E-7 | 7.41599E-7 |
| 16 | 1.96333123       | 1.96339403               | 1.96431019         | 9.58356E-7 | 8.39352E-7 |
| 17 | 1.98178915       | 1.98185195               | 1.98282311         | 1.06907E-6 | 9.43155E-7 |
| 18 | 1.99923515       | 1.99929795               | 2.00032411         | 1.18585E-6 | 1.05301E-6 |
| 19 | 2.01577445       | 2.01583725               | 2.01691841         | 1.30864E-6 | 1.16891E-6 |

Figure 1: Solution plots
5 Conclusion

In this study, approximation formulas for evaluating Landau constants have been presented and discussed. The proposed estimate has been compared with two other estimates which are of special importance in approximation theory. Results have been favorable for the performed method in terms of both convergence and accuracy.

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