Dirac operator normality and chiral properties on the lattice

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Abstract

Normality in connection with $\gamma_5$-hermiticity determines the basic chiral properties and rules. The Ginsparg-Wilson (GW) relation is one of the allowed constraints on the spectrum. Interrelations between features of the spectrum, the sum rule for chiral differences of real modes and contributions to the Ward identity are pointed out. The alternative chiral transformation of Lüschener gives the same Ward identity as the usual one, in the global and in the local case. Imposing normality on a general function of the hermitean Wilson-Dirac (HWD) operator, inevitably leads at the same time to the Neuberger operator and to the GW relation. In this context also the case with zero eigenvalues of the HWD operator is handled. The eigenvalue flows of the HWD operator obey a differential equation the characteristic features of the solutions of which are discussed.
1. Introduction and overview

Recently considerable progress in the description of chiral fermions has been initiated by works of Neuberger [1] from the point of view of the overlap formalism [2] and of Hasenfratz et al. [3] in the context of fixed point actions. This has also revived considerations of the Ginsparg-Wilson (GW) relation [4] which in both cases turned out to be satisfied. On the basis of the GW relation Lüscher [5] proposed an alternative chiral transformation providing chiral symmetry at the classical level. The finite form of this transformation has been given by Chiu [6].

Neuberger [1], in particular, was able to derive an explicit form of the massless Dirac operator based on the hermitean Wilson-Dirac (HWD) operator, which also plays a major rôle in the overlap formalism [2]. These developments have then given rise to several numerical studies of eigenvalue flows of the HWD operator with the mass parameter [7], relying on the fact that this operator has well defined spectral properties.

Despite the many publications which followed the mentioned works, there are clearly still many questions open. In the present paper we address a number of them which are related to a more precise understanding and to basic properties of these new developments.

We start from the observation that the Dirac operator must be normal in order that reliable conclusions become possible. This follows from two theorems of the spectral theory of operators in unitary spaces. The first one of them says that normality is necessary and sufficient in order that the eigenvectors form a complete system. Apart from the fact that otherwise very little is known on spectral properties, this implies that without normality there are necessarily defects which can hardly be tolerated in making predictions.

The second one of these theorems more specifically concerns chiral properties. It states that normality guarantees that an eigenvector of the operator is at same time an eigenvector of the adjoint operator. This together with $\gamma_5$-hermiticity (i.e. hermiticity of the operator multiplied by $\gamma_5$) is exactly what provides the basis for chiral behaviors.

We point out that, given normality and $\gamma_5$-hermiticity, one already obtains the basic rules and properties. Apart from the general structure of the eigenvector system, the fundamental sum rule for chiral differences of real modes emerges. The relations between the operator and such modes, and in particular those to the index, as needed in various contexts, immediately follow.

A further consequence of normality is that the operator decomposes into commuting operators related to the real part and to the imaginary part of its eigenvalues. This allows to study possible constraints on the spectrum. In particular, restricting it to a one-dimensional set, we find that, in addition to zero, the curve must meet the real axis at least at one further point, in order that the sum rule mentioned above allows a nonzero index. From this point of view the general nature of the GW relation becomes clear. It
is just one of such constraints which satisfies the requirement of a further real eigenvalue in a minimal way.

The symmetry at the classical level provided by the alternative chiral transformation makes things similar to what one is accustomed to in continuum theory, however, at the price of complications due to the action dependence of this transformation. At the quantum level, where there is anyway no such symmetry, the question arises, what the precise difference to the usual chiral transformation is.

In order to have a basis for a general comparison of transformations, we derive the Ward identities in an appropriately general way. It is seen that by the normality of the operator the global chiral transformation leads just to the sum rule for chiral differences of real modes. The correspondence of the contributions to terms familiar in continuum theory is pointed out. The results of the local chiral transformation are similarly identified.

It is shown that the alternative chiral transformation leads exactly to the same result as the usual chiral one, even without assuming the GW relation. Imposing the GW relation has the only effect to specialize the results to that case with only two real eigenvalues. We also give the local version of the alternative transformation. The resulting Ward identity is again seen to agree with that of the usual local chiral one.

The operator of Neuberger is the only explicit form of a massless Dirac operator on the lattice presently known. In order to get more insight into the possibilities of the construction of such operators it appears desirable to have also a derivation of it which does not rely on the overlap formalism. Further, there is the somewhat unsatisfactory point that zero eigenvalues of the HWD operator so far had to be excluded.

In our derivation of the indicated operator the requirement of normality is central. To avoid doublers and at the same time to deal with well defined mathematical properties, at present the only possibility is to start from the HWD operator. Therefore we consider a general function of this operator and impose the necessary conditions on it. Doing this it turns out that the requirement of normality is an extremely strong one, leading at the same time to the Neuberger form of the Dirac operator and to the GW constraint on the spectrum. In addition the inclusion of zeros of the HWD operator gets nonstraightforward. Nevertheless, also for this problem a way out is found.

For the explicit Dirac operator the relations of its eigenvectors to that of the HWD operator become transparent. The apparent importance of the HWD operator suggests to study its eigenvalue flows with the mass parameter also analytically. We show that they satisfy a differential equation and give a complete overview of the characteristic properties of its solutions.

Section 2 is devoted to the basic chiral properties. In Section 3 possible constraints on the spectrum are discussed. Section 4 contains the general derivation of Ward identities. In Section 5 the results for particular transformations are analyzed. Section 6 gives the
systematic construction of a normal operator. In Section 7 the spectral flows of the HWD operator are investigated.

2. Basic chiral operator properties

We require $D$ to be normal

$$[D, D^\dagger] = 0 \quad (2.1)$$

and $\gamma_5$-hermitean

$$D^\dagger = \gamma_5 D \gamma_5 \quad . \quad (2.2)$$

Because of (2.1) the solutions $f_k$ of the eigenequation

$$Df_k = \lambda_k f_k \quad (2.3)$$

form a complete orthonormal set, on the basis of which general conclusions become possible. By (2.1) simultaneous eigenvectors of $D$ and $D^\dagger$ exist. In the present context this has the important consequence that one also has

$$D^\dagger f_k = \lambda_k^* f_k \quad (2.4)$$

which together with (2.2) leads to

$$D\gamma_5 f_k = \lambda_k^* \gamma_5 f_k \quad . \quad (2.5)$$

The comparison of (2.3) multiplied by $\gamma_5$ with (2.5) then gives

$$[\gamma_5, D]f_k = 0 \quad \text{if } \lambda_k \text{ real } , \quad (2.6)$$

which tells that in the subspace of real eigenvalues of $D$ one can introduce simultaneous eigenvectors of $D$ and of $\gamma_5$, i.e. ones with chirality.

Multiplying (2.3) from the left by $f_l^\dagger \gamma_5$ and its adjoint $f_l^\dagger D^\dagger = f_l^\dagger \lambda_l^*$ from the right by $\gamma_5 f_k$ one obtains the relation

$$f_l^\dagger \gamma_5 f_k = 0 \quad \text{for } \lambda_l^* \neq \lambda_k \quad , \quad (2.7)$$

which actually reflects the orthogonality of eigenvectors related to different eigenvalues of $D$. In view of (2.6) and of the comparison of (2.5) with (2.3), respectively, it is convenient to introduce in more detail

$$f_k = \begin{cases} 
  f_k^{(5)} & \text{for } \text{Im}\lambda_k = 0 \quad \text{with } \gamma_5 f_k^{(5)} = c_k f_k^{(5)} \\
  f_k^{(1)} & \text{for } \text{Im}\lambda_k > 0 \quad \text{with } \gamma_5 f_k^{(1)} = f_k^{(2)} \\
  f_k^{(2)} & \text{for } \text{Im}\lambda_k < 0 \quad \text{with } \gamma_5 f_k^{(2)} = f_k^{(1)} 
\end{cases} \quad (2.8)$$
where for the chirality $c_r$ possible values are $+1$ and $-1$. Obviously eigenvectors related to complex eigenvalues always come in pairs, while those related to real ones need not to do so. Of course, how many of each type occur depends on the particular $D$, however, by (2.1) and (2.2), the structure of the eigenvector system is in any case the one described.

For the numbers of modes related to a real eigenvalue $\lambda$ of $D$

$$N_\pm(\lambda) = \sum_{\lambda_k = \lambda \text{ real}} \frac{1 \pm c_k}{2} \quad (2.9)$$

from (2.7), (2.8) and $\text{Tr}(\gamma_5) = 0$ one obtains the sum rule for the chiral differences of real modes

$$\sum_{\lambda \text{ real}} \left( N_+(\lambda) - N_-(\lambda) \right) = 0 \quad (2.10)$$

It in particular implies that the difference for eigenvalue zero $N_-(0) - N_+(0)$, the index of $D$, can only be nonvanishing if a corresponding difference from nonzero eigenvalues exists.

From (2.7), (2.8) and (2.9) one also readily gets the useful relations

$$\varepsilon \text{Tr}((D + \varepsilon)^{-1}\gamma_5) \rightarrow N_+(0) - N_-(0) \quad \text{for} \quad \varepsilon \rightarrow 0 \quad (2.11)$$

$$\text{Tr}((D + \varepsilon)^{-1}\gamma_5 D) \rightarrow \sum_{\lambda \neq 0 \text{ real}} \left( N_+(\lambda) - N_-(\lambda) \right) \quad \text{for} \quad \varepsilon \rightarrow 0 \quad (2.12)$$

$$\text{Tr}(\gamma_5 D) = \sum_{\lambda \neq 0 \text{ real}} \lambda \left( N_+(\lambda) - N_-(\lambda) \right) \quad (2.13)$$

between $D$ and $N_+(\lambda) - N_-(\lambda)$. It will be discussed later that (2.11) and (2.12) are related to the index term and the topological charge ($F \tilde{F}$) term, respectively, of continuum theory. They obviously add up to the sum rule (2.10). If only one nonzero value for real eigenvalues occurs, one can also use (2.13) instead of (2.12) to relate $D$ to the corresponding chiral difference.

3. **Constraints on location of spectrum**

In order to study possible constraints on the spectrum of $D$ we use the decomposition

$$D = u + iv \quad \text{with} \quad u = u^\dagger = \frac{1}{2}(D + D^\dagger) \quad , \quad v = v^\dagger = \frac{1}{2i}(D - D^\dagger) \quad (3.1)$$

for which (2.1), the normality of $D$, implies

$$[u, v] = 0 \quad . \quad (3.2)$$

By (2.2), the $\gamma_5$-hermiticity of $D$, it follows that (3.1) is at the same time the decomposition into the parts commuting and anticommuting with $\gamma_5$ and that $[\gamma_5, u] = 0$ and
\{\gamma_5, v\} = 0 \text{ hold. According to (3.2) } u, v \text{ and } D \text{ have simultaneous eigenvectors. The real eigenvalues of } u \text{ and } v \text{ are simply the real and imaginary parts, respectively, of those of } D. \text{ Therefore we can specify the location of the spectrum of } D \text{ by constraining } u \text{ and } v.\\

A particular simplification arises if the spectrum is located on a one-dimensional set. This can be realized by imposing the condition

$$\mathcal{F}(u, v) = 0$$  \hspace{1cm} (3.3)$$

with a suitably chosen function \(\mathcal{F}(u, v)\) which, as a function of commuting hermitean operators, is well defined. In the notation of (2.3), it satisfies the eigenequation \(\mathcal{F}(u, v)f_k = \mathcal{F}(\text{Re}\lambda_k, \text{Im}\lambda_k)f_k\) and has the spectral representation \(\mathcal{F}(u, v) = \sum_k \mathcal{F}(\text{Re}\lambda_k, \text{Im}\lambda_k)f_k f_k^\dagger\). Because we wish to allow for the eigenvalue 0 of \(D\), the function \(\mathcal{F}\) considered as a function of real numbers should have the property \(\mathcal{F}(0, 0) = 0\). Since we in addition want that the index of \(D\) can be nonzero, according to (2.10) also at least one real eigenvalue different from zero must be possible. Therefore in addition one has to require \(\mathcal{F}(\beta, 0) = 0\) for at least one real \(\beta \neq 0\). Thus the function must have the particular properties

$$\mathcal{F}(0, 0) = 0 \text{ and } \mathcal{F}(\beta, 0) = 0 \text{ for some } \beta \neq 0$$  \hspace{1cm} (3.4)$$
in order that the curve specified by (3.3) meets the real axis at zero and at least at one further point.

A simple possibility which satisfies condition (3.4) is a circle through zero with center on the real axis

$$\mathcal{F}(u, v) = (u - \rho)^2 + v^2 - \rho^2 = 0.$$  \hspace{1cm} (3.5)$$

Using (3.1) and (3.2) one can write (3.3) as \(\rho(D + D^\dagger) = D^\dagger D\) which by (2.2) is seen to be just the GW relation

$$\{\gamma_5, D\} = \rho^{-1}D\gamma_5D.$$  \hspace{1cm} (3.6)$$

In contrast to the original form [4], however, (3.6) does not involve a further operator sandwiched in addition to \(\gamma_5\) into its r.h.s. because this would spoil the normality of \(D\). Therefore here only a real constant \(\rho^{-1}\) remains. Obviously (3.3) meets the requirement (3.4) in a minimal way, admitting only 2\(\rho\) in addition to 0. The sum rule (2.10) then simplifies to two terms. This conforms with the observation of Chiu [8] that in the case of the GW relation \(\{\gamma_5, D\} = D\gamma_5D\) the chiral differences obtained at 0 and at 2 add up to zero.

By (3.4) the choice \(\mathcal{F} = u\), corresponding to \(\{\gamma_5, D\} = 0\), is excluded because there is only one real eigenvalue (on the finite lattice with \(D\) being bounded a second one can also not occur at \(\infty\)). This choice may, however, be approached in the continuum limit. In fact, considering the stereographic projection of the circle (3.5) on the sphere of complex numbers it is seen that for increasing radius it approaches the circle through \(\infty\).
on this sphere which corresponds to the imaginary axis in the plane. With \(\rho \sim 1/a\) for decreasing lattice spacing \(a\) the envisaged approach indeed occurs. This suggests that in the continuum the sum rule for chiral differences could possibly be satisfied by eigenvalues at 0 and at \(\infty\). Of course, the subtleties of the respective limit remain to be investigated.

The form of the constraint on the spectrum depends on the particular properties of the operator \(D\) considered. The Neuberger operator is tied to the GW constraint. For other constructions in any case also \((1.4)\) is to be required. In addition, also the appropriate behavior in the limit should be guaranteed.

\section{General form of the Ward identity}

In dealing with Ward identities it should be remembered that the expectation values

\[ \langle \mathcal{O} \rangle = \frac{\int [dU][d\bar{\psi}d\psi]e^{-S_U-S_i}\mathcal{O}}{\int [dU][d\bar{\psi}d\psi]e^{-S_U-S_i}} \quad \text{with} \quad S_i = \bar{\psi}M\psi \]  

(4.1)

involve integrals \(\int [d\bar{\psi}d\psi]e^{-S_i} = \det M\) and

\[ \int [d\bar{\psi}d\psi]e^{-S_i}\psi_{j_1}\ldots\psi_{j_s}\bar{\psi}_{k_1}\ldots\bar{\psi}_{k_s} = \sum \epsilon_{l_1\ldots l_s}^{k_1\ldots k_s} M_{j_1 l_1}^{-1}\ldots M_{j_s l_s}^{-1} \det M \]  

(4.2)

(where \(\epsilon_{l_1\ldots l_s}^{k_1\ldots k_s} = +1, -1\) or 0 if \(k_1\ldots k_s\) is an even, odd or no permutation of \(l_1\ldots l_s\)). Thus, in order that the expectation values \((4.1)\) are properly defined, \(M^{-1}\) must exist and \(\det M\) be nonzero. Therefore, to be able to proceed in the presence of zero modes of \(D\) one has to put \(M = D + \varepsilon\) and let \(\varepsilon\) go to zero in the final result.

Fermionic Ward identities arise from the condition that \(\int [d\bar{\psi}d\psi]e^{-S_i}\mathcal{O}\) must not change under the transformation

\[ \psi' = \exp(i\eta \Gamma)\psi \quad , \quad \bar{\psi}' = \bar{\psi} \exp(i\eta \bar{\Gamma}) \]  

(4.3)

where \(\eta\) is a parameter. This means that one gets the identity

\[ \frac{d}{d\eta} \int [d\bar{\psi}'d\psi']e^{-S_i}\mathcal{O}'|_{\eta=0} = 0 \]  

(4.4)

with three contributions, one from the derivative of the integration measure, one from that of the action and one from that of \(\mathcal{O}\). For the measure contribution within \([d\bar{\psi}'d\psi'] = [d\bar{\psi}d\psi]\left(\det \exp(i\eta \Gamma)\det \exp(i\eta \bar{\Gamma})\right)^{-1}\) one obtains

\[ \frac{d}{d\eta} \left(\det \exp(i\eta \Gamma)\right)|_{\eta=0} = i \text{Tr} \bar{\Gamma} \quad , \quad \frac{d}{d\eta} \left(\det \exp(i\eta \bar{\Gamma})\right)|_{\eta=0} = i \text{Tr} \Gamma \]  

(4.5)
With respect to the derivative of $\mathcal{O}$ we note that for products $\mathcal{P}$ of $\psi$ and $\bar{\psi}$ fields, since the Grassmann-even combinations $\psi_j \frac{\partial}{\partial \psi_k}$ and $\bar{\psi}_j \frac{\partial}{\partial \bar{\psi}_k}$ can be readily shifted to the appropriate place, one can relate

$$\frac{d \mathcal{P}'}{d \eta} \bigg|_{\eta=0} = i \sum_l \left( (\Gamma_l \psi)_l \frac{\partial \mathcal{P}}{\partial \psi_l} + (\bar{\psi}_l \bar{\Gamma})_l \frac{\partial \mathcal{P}}{\partial \bar{\psi}_l} \right).$$

(4.6)

The fermionic part of $\mathcal{O}$ in general is made up of such products and of sums thereof. More specifically it can even be considered to be made up of products of equal numbers of $\psi$ and $\bar{\psi}$ fields because only such products contribute to the integrals; for the same reason $\mathcal{O}$ can also be considered to be Grassmann-even. Thus in any case (4.6) applies to $\mathcal{O}$ and (4.4) becomes

$$i \int [d \bar{\psi} d\psi] e^{-S_f} \left( \bar{\psi}(\bar{\Gamma}M + M\Gamma)\psi \mathcal{O} + \bar{\psi} \bar{\Gamma} \frac{\partial \mathcal{O}}{\partial \bar{\psi}} - \frac{\partial \mathcal{O}}{\partial \psi} \Gamma \psi \right) = 0.$$

(4.7)

This generalizes the relation [3] which, with suitable choices of $\mathcal{O}$, is used in many applications to the case where the integration measure is not invariant.

In studies of the singlet axial vector current and its relation to anomaly, index and topological charge it usually suffices to consider the case $\mathcal{O} = 1$, as is e.g. also done in Ref. [3]. To keep things general we avoid this here, integrating out the $\bar{\psi}$ and $\psi$ fields in the second term of (4.7) without specifying $\mathcal{O}$. Then at the same time that term gets on equal footing with the first one, as is desirable for a convenient comparison of transformations. To integrate out the indicated fields we use the identity

$$0 = \frac{1}{2} \int [d \bar{\psi} d\psi] \left( \left( \frac{\partial}{\partial \psi} M^{-1} \right)_j (e^{-S_i} \psi_k \mathcal{O}) + \left( M^{-1} \frac{\partial}{\partial \bar{\psi}} \right)_k (e^{-S_i} \bar{\psi}_j \mathcal{O}) \right)$$

$$= \int [d \bar{\psi} d\psi] e^{-S_i} \left( \bar{\psi}_j \psi_k \mathcal{O} + M^{-1} \bar{\psi}_j \mathcal{O} + \frac{1}{2} (\frac{\partial \mathcal{O}}{\partial \psi} M^{-1})_j \psi_k - \frac{1}{2} \bar{\psi}_j (M^{-1} \frac{\partial \mathcal{O}}{\partial \bar{\psi}})_k \right).$$

(4.8)

which relies on the fact that $\int [d \bar{\psi} d\psi] (\partial / \partial \psi_l) G = 0$ and $\int [d \bar{\psi} d\psi] (\partial / \partial \bar{\psi}_l) G = 0$ for any function $G$. Then (4.7) becomes

$$iW \int [d \bar{\psi} d\psi] e^{-S_i} \mathcal{O} + \frac{i}{2} \int [d \bar{\psi} d\psi] e^{-S_i} \left( \frac{\partial \mathcal{O}}{\partial \psi} M^{-1} R \psi + \bar{\psi} R M^{-1} \frac{\partial \mathcal{O}}{\partial \bar{\psi}} \right) = 0.$$

(4.9)

where $R = \bar{\Gamma}M - M\Gamma$ and

$$W = \text{Tr} \left( - \bar{\Gamma} - \Gamma + M^{-1} (\bar{\Gamma}M + M\Gamma) \right).$$

(4.10)

To evaluate the terms with derivatives of $\mathcal{O}$ in (4.9) further we remember that the fermionic part of $\mathcal{O}$ can be considered to be made up of products of type $\mathcal{P} = \psi_j \bar{\psi}_{k_1} \ldots \psi_{j_s} \bar{\psi}_{k_s}$.
for which by (4.2) we find
\[ -\int [d\bar{\psi}d\psi]e^{-S_f}R\psi = +\int [d\bar{\psi}d\psi]e^{-S_f}\bar{\psi}R^{-1}\frac{\partial P}{\partial \bar{\psi}} = \]
\[ s\sum_{l_1\ldots l_s}\epsilon^{k_1\ldots k_s}M^{-1}_{j_1l_1}\ldots M^{-1}_{j_sl_s}(M^{-1}RM^{-1})_{j_sl_s}\det M \]  
\( (4.11) \)

This shows that the terms in (4.9) with derivatives of \( O \) cancel and we remain with
\[ iW\int [d\bar{\psi}d\psi]e^{-S_f}O = 0 \]  
\( (4.12) \)

which inserted into (4.1) gives the Ward identity \( \langle WO \rangle = 0 \) or, if desired, also the one in a background gauge field \( W\langle O \rangle_U = 0 \).

5. Results for particular transformations

For the global chiral transformation, which in terms of (4.3) is given by
\[ \Gamma = \bar{\Gamma} = \gamma_5 \]  
\( (5.1) \)
the measure contribution \(-\text{Tr}(\bar{\Gamma} + \Gamma)\) vanishes and one obtains
\[ W = \text{Tr}(M^{-1}\{\gamma_5, M\}) \]  
\( (5.2) \)
or inserting \( M = D + \varepsilon \)
\[ W = \text{Tr}\left((D + \varepsilon)^{-1}\{\gamma_5, D\}\right) + 2\varepsilon\text{Tr}\left((D + \varepsilon)^{-1}\gamma_5\right) \]  
\( (5.3) \)
The first term in (5.3) by (2.12) is seen to become the sum over \( 2(N_+ (\lambda) - N_- (\lambda)) \) for nonvanishing real \( \lambda \) and the second one by (2.11) the difference \( 2(N_+(0) - N_-(0)) \). By (5.3) they add up to
\[ W \to 2\sum_{\lambda \text{ real}}\left(N_+(\lambda) - N_- (\lambda)\right) \quad \text{for} \quad \varepsilon \to 0 \]  
\( (5.4) \)

Thus we obviously arrive just at the sum rule for chiral differences of real modes (2.10).

Analogous to the features known in continuum theory for quite some time [11], the first term in (5.3) is the the topological charge \((F\tilde{F})\) term while the second one is the index term. The latter by (2.11) is obvious. For the first term the limit has been established long ago [11] for the Wilson-Dirac operator and recently [12] also for the Neuberger operator (for which the use of \( \text{Tr}(\gamma_5 D) \) by (2.12) and (2.13) with only one nonzero \( \lambda \) is equivalent to using \( \text{Tr}\left((D + \varepsilon)^{-1}\{\gamma_5, D\}\right) \)).
Next we consider the alternative transformation \[5, 6\] for which in our notation

\[
\Gamma = \gamma_5 (1 - (2\rho)^{-1} M) , \quad \bar{\Gamma} = (1 - (2\rho)^{-1} M) \gamma_5 .
\] (5.5)

With (5.3) one now gets \(-\text{Tr}(\bar{\Gamma} + \Gamma) = +\rho^{-1}\text{Tr}(\gamma_5 M)\) for the measure contribution and \(\text{Tr}(M^{-1}(\Gamma M + M\Gamma)) = \text{Tr}(M^{-1}\{\gamma_5, M\}) - \rho^{-1}\text{Tr}(\gamma_5 M)\) for the action contribution. Obviously the extra term of the latter cancels the measure term so that again the result (5.2) is obtained, and notably, even without assuming the GW relation.

If with the alternative transformation in addition the GW relation (3.6) is imposed, inserting \(M = D + \varepsilon\), for the action contribution one gets \(\text{Tr}((M^{-1}(\bar{\Gamma} M + M\Gamma)) = 2\varepsilon(1 + (2\rho)^{-1}\varepsilon)\text{Tr}((D + \varepsilon)^{-1}\gamma_5)\) which by (2.11) becomes \(2(N_+(0) - N_-(0))\). For the measure contribution one has \(-\text{Tr}(\bar{\Gamma} + \Gamma) \to \rho^{-1}\text{Tr}(\gamma_5 D)\) which by (2.13) equals \(\sum_{\lambda \neq 0,\text{ real}} \lambda(N_+(\lambda) - N_-(\lambda))\). In the GW case with only \(\lambda = 2\rho\) this becomes \(2(N_+(2\rho) - N_-(2\rho))\). Taking both contributions\(^1\) together it is obvious that one gets the same results as before now specialized to the case where only 0 and \(2\rho\) occur for real eigenvalues.

The local chiral transformation in the present context is conveniently introduced by

\[
\Gamma = \bar{\Gamma} = \gamma_5 \hat{e}(n) \quad \text{with} \quad (\hat{e}(n))_{n''n'} = \delta_{n''n} \delta_{nn'}
\] (5.6)

for which (4.10) gives

\[
W = \text{Tr}\left((M^{-1}\{\gamma_5 \hat{e}(n), M\})\right) .
\] (5.7)

By inserting the decomposition \(M = \frac{1}{2}(M - \gamma_5 M \gamma_5) + \frac{1}{2}(M + \gamma_5 M \gamma_5)\) (into parts anti-commuting and commuting with \(\gamma_5\)) and also \(M = D + \varepsilon\) into \(\{\gamma_5 \hat{e}(n), M\}\) this becomes

\[
W = \frac{1}{2}\text{Tr}\left((M^{-1}[\hat{e}(n), [\gamma_5, D]])\right) + \frac{1}{2}\text{Tr}\left((M^{-1}\{\hat{e}(n), \{\gamma_5, D\}\})\right) + 2\varepsilon\text{Tr}\left((M^{-1}\gamma_5 \hat{e}(n))\right) .
\] (5.8)

The first term in (5.8) is seen to vanish upon summation over \(n\) and accordingly corresponds to the divergence of the singlet axial vector current. Summation over \(n\) in the rest, responsible for current nonconservation, leads to the results of the global transformation. The second term in (5.8) in the limit gives the FF-density of continuum theory \[11, 12\]. The third term in (5.8) is the local version of the index contribution. To visualize things in terms of current expressions one should remember that the \(M^{-1}\) factors in (5.8) correspond to the integrated out \(\bar{\psi}\) and \(\psi\) fields.

We note that the local transformation related to the alternative chiral transformation (5.3) can also be introduced. It is given by

\[
\Gamma = \gamma_5 \hat{e}(n)(1 - (2\rho)^{-1} M) , \quad \bar{\Gamma} = (1 - (2\rho)^{-1} M) \gamma_5 \hat{e}(n) .
\] (5.9)

\(^1\)In Ref. \[3\] the action contribution is missing. The evaluation of the measure contribution there is actually a use of the identity \(0 = \text{Tr} \gamma_5 = \varepsilon \text{Tr}((D + \varepsilon)^{-1}\gamma_5) + \text{Tr}((D + \varepsilon)^{-1}\gamma_5 D)\) which by inserting the GW relation (3.6) becomes \(0 = \varepsilon(1 + \varepsilon(2\rho)^{-1})\text{Tr}((D + \varepsilon)^{-1}\gamma_5) + (2\rho)^{-1}\text{Tr}(\gamma_5 D)\).
The calculation of $W$ with this transformation again leads to (5.7) so that it becomes obvious that also in the local case nothing new is obtained.

Of course, also other transformations could straightforwardly be considered along the present lines. For the nonsinglet chiral one with flavor operator $T_l$ one has $\Gamma = \bar{\Gamma} = \gamma_5 T_l$ in the global and $\Gamma = \bar{\Gamma} = \gamma_5 T_l \hat{e}(n)$ in the local case, with current conservation resulting from $\text{Tr} T_l = 0$. Conserved vector currents are related to $\Gamma = \bar{\Gamma} = \gamma_5 T_l \hat{e}(n)$ in the nonsinglet case and to $\Gamma = -\bar{\Gamma} = \hat{e}(n)$ in the singlet case.

6. Derivation of normal operator

The Wilson-Dirac operator $X/a$ (with hermitean $\gamma$-matrices in 4-dimensional euclidean space and $0 < r \leq 1$) is given by

$$X = \frac{r}{2} \sum_\mu \nabla_\mu^\dagger \nabla_\mu + m + \frac{1}{2} \sum_\mu \gamma_5 (\nabla_\mu - \nabla_\mu^\dagger)$$  \hspace{1cm} (6.1)

where $(\nabla_\mu)_{n'n} = \delta_{n'n} - U_{\mu n} \delta_{n',n+\hat{\mu}}$ (which implies $\nabla_\mu^\dagger \nabla_\mu = \nabla_\mu \nabla_\mu^\dagger = \nabla_\mu + \nabla_\mu^\dagger$). For the operator $X$ one has $\gamma_5$-hermiticity,

$$X^\dagger = \gamma_5 X \gamma_5$$ \hspace{1cm} (6.2)

however, in the presence of a gauge field (with $[\nabla_\mu, \nabla_\nu] \neq 0$ and $[\nabla_\mu^\dagger, \nabla_\nu] \neq 0$ for $\mu \neq \nu$ and thus $[X^\dagger, X] \neq 0$) $X$ is not normal.

To derive a normal and $\gamma_5$-hermitean operator $D$ one needs to start from

$$H = \gamma_5 X$$  \hspace{1cm} (6.3)

which, being hermitean, in contrast to $X$ has well defined spectral properties. The strategy then is, instead of $X = \gamma_5 H$, to consider

$$D = \gamma_5 E(H) + C$$  \hspace{1cm} (6.4)

with some general function $E(H)$ and some constant $C$, and to determine those quantities by imposing the necessary conditions. Requiring $\gamma_5$-hermiticity (2.2) of $D$ it follows that $E(H)$ must be hermitean and that $C$ must be real. Since with

$$H \phi_l = \alpha_l \phi_l$$  \hspace{1cm} (6.5)

(where $\alpha_l$ is real and the $\phi_l$ form a complete orthonormal set) one has the representation $E(H) = \sum_l E(\alpha_l) \phi_l^\dagger$, hermiticity of $E(H)$ simply means that $E(\alpha)$ considered as a function of a real parameter $\alpha$ must be a real function.
From the requirement of normality (2.1) of $D$ we obtain the condition

$$[\gamma_5, E(H)^2] = 0 \quad .$$

(6.6)

To satisfy this condition is the central point. Wishing to get a general solution, one must require $E(H)^2$ to be independent of $H$. Though being inevitable, this is clearly quite drastic. It means that $E(H)^2$ should be a multiple of the identity

$$E(H)^2 = \rho^2 \mathbb{1} \quad ,$$

(6.7)
or that $E(\alpha) = \pm \rho$ with some constant $\rho$ which, without restricting generality, we can take to be positive. In order to keep the properties of $E(H)$ as close as possible to those of $H$ we further require $E(\alpha)$ to be nondecreasing and odd. This fixes the signs and we end up with

$$E(\alpha) = \rho \epsilon(\alpha)$$

(6.8)

where $\epsilon(\alpha) = \pm 1$ for $\alpha \gtrless 0$. If all $\alpha_l \neq 0$ this is already the solution and $E(H)/\rho$ is just the function $H/\sqrt{H^2}$ of Neuberger $\mathbb{1}$. If $\alpha_l = 0$ occur we have to specify $\epsilon(0)$. It would be tempting to take the value zero for this, however, because of (6.7), i.e. the necessity to keep the procedure independent of $H$, this is definitely not possible. Thus one has to choose either $+1$ or $-1$ for $\epsilon(0)$, and one must decide for one of them since no $H$-independent criterion for selection appears available. The oddness of the function, which then is violated at $\alpha = 0$, can be recovered by doing independent calculations for each of the two choices and taking the mean of the final results. In Section 7 we will see that in terms of counting eigenvalue flows this procedure has a natural equivalent.

To fix the constant $C$ in (6.4) we note that, because $\rho^{-1} \gamma_5 E(H)$ is unitary, the spectrum of $\gamma_5 E(H)$ is on the circle with radius $\rho$ and center at zero. Thus to get the appropriate spectrum of $D$ we put $C = \rho$ and get

$$D = \rho (1 + \gamma_5 \epsilon(H)) \quad .$$

(6.9)

It appears important to emphasize that, by the necessity to satisfy (6.7), one cannot escape simultaneously arriving at the Ginsparg-Wilson constraint (3.5) on the spectrum and at the Neuberger form (6.9) of the Dirac operator. In addition (6.7) unavoidably produces the somewhat delicate situation with the choice of $\epsilon(0)$.

To complete the derivation the occurring parameters are to be fixed. The continuum limit in the case $U = \mathbb{1}$ with the representation $(\nabla_\mu)_{pp} = 1 - e^{-ip_\mu a}$ indicates that masslessness requires $m < 0$ and one gets $\rho = |m|/a$. Further, it is also known that to avoid effects of doublers on the finite lattice one needs $m > -2r$. A choice with major analytical simplifications is $-m = r = 1$. It should be noted that $X$ enters (6.9) only

2 Which, by the way, would lead to $F(u, v) = ((u-\rho)^2 + v^2)((u-\rho)^2 + v^2 - \rho^2) = 0$ instead of (3.3).
up to a positive constant factor, so that, for example, using $X/a$ instead of $X$ would not change anything.

Since real eigenvalues occur only at 0 and at $2\rho$, the sum rule (2.10) for the operator (6.9) reduces to $N_- (0) - N_+ (0) + N_+ (2\rho) - N_- (2\rho) = 0$. Similarly (2.13) becomes $\text{Tr} (\gamma_5 D) = 2\rho (N_+ (2\rho) - N_- (2\rho))$. Combining these two relations one has $N_- (0) - N_+ (0) = (2\rho)^{-1} \text{Tr} (\gamma_5 D)$ and inserting the particular form (6.9) of $D$ into this one gets

$$N_- (0) - N_+ (0) = \frac{1}{2} \text{Tr} (\epsilon (H))$$

(6.10)

for its index. Also the eigenvectors of $D$ and of $H$ can now be related in detail. For this we note that (2.3) in terms of $\gamma_5 \epsilon (H)$ becomes $\gamma_5 \epsilon (H) f_k = (\lambda_k / \rho - 1) f_k$, so that parametrizing complex eigenvalues as $\lambda_k = \rho (1 + e^{i\varphi_k})$ with $0 < \varphi_k < \pi$ and remembering (2.8) we get the eigenequations of $\epsilon (H)$

$$\epsilon (H) f_k^{(\pm)} = \mp c_k f_k^{(\pm)} \quad \text{for} \quad \lambda_k = \left\{ \begin{array}{ll} 0 \\ 2\rho \end{array} \right.$$  

$$\epsilon (H) f_k^{(\pm)} = \pm f_k^{(\pm)} \quad \text{with} \quad f_k^{(\pm)} = \frac{1}{\sqrt{2}} (e^{-i\varphi_k/2} f_k^{(1)} \pm e^{i\varphi_k/2} f_k^{(2)}) .$$

(6.11)

These equations are to be compared with

$$\epsilon (H) \phi_l = \epsilon (\alpha_l) \phi_l$$

(6.12)

which results from (5.5). Obviously the vectors in (6.11) are linear combinations $\tilde{\phi}_k^{\pm} = \sum_l \tilde{b}_{kl}^{(\pm)} \phi_l^{(\pm)}$ where $\phi_l^{(\pm)} = \phi_l$ for $\epsilon (\alpha_l) = \pm 1$. By the properties of $D$ derived here and the general structure obtained in Section 2 it is guaranteed that task to find the coefficients $\tilde{b}_{kl}^{(\pm)}$ has a solution.

7. Relations for spectral flows

The studies of the flows of eigenvalues of $H$ with $m$ can be justified on the basis of (5.10) which in terms of $N_+^H$ and $N_-^H$, the numbers of positive and negative eigenvalues of $H$, in the absence of eigenvalues zero of $H$ reads

$$N_- (0) - N_+ (0) = \frac{1}{2} (N_+^H - N_-^H) .$$

(7.1)

The crossing of zero of an eigenvalue which occurs at some $m$ is connected to a change of the difference of the numbers of positive and negative eigenvalues by $+2$ or $-2$, respectively, depending on the direction of the crossing. Therefore the net number of crossings is related to the index of $D$. 

13
To include also zero eigenvalues of $H$ in these considerations, one has to note that in
the very moment of crossing a positive (negative) eigenvalue has disappeared, however, a
negative (positive) one has not yet appeared. Still using (7.1) then agrees with the notion
that the index in that moment has only changed by $\frac{1}{2}$. With this understanding (7.1)
is no longer equivalent to (6.10) in which $\epsilon(0) = 0$ is forbidden by (6.7). However, the
analogue of the procedure described in Section 6 (of working with the mean of the choices
$\epsilon(0) = +1$ and $\epsilon(0) = -1$), in the case of counting flows is seen to lead to the same result
as the counting at the crossing point mentioned above. Thus the latter appears valid and
natural.

To investigate properties of the flows of eigenvalues of $H$ analytically we first derive
some relations. Multiplying (6.5) by $\phi_l^\dagger \gamma_5$ one gets
$$\phi_l^\dagger \gamma_5 H \phi_l = \alpha_l \phi_l^\dagger \gamma_5 \phi_l$$
and summing this and its hermitian conjugate one has
$$\phi_l^\dagger \{\gamma_5, H\} \phi_l = 2 \alpha_l \phi_l^\dagger \gamma_5 \phi_l.$$ From this by inserting
(6.3) with (6.1) one obtains
$$\alpha_l \phi_l^\dagger \gamma_5 \phi_l = g_l(m) + m$$
with
$$g_l(m) = \frac{r}{2} \sum_\mu \|\nabla_\mu \phi_l\|^2 (7.2)$$
(7.2)

(7.4)

Combining (7.2) and (7.4) we get the differential equation
$$\dot{\alpha}_l \alpha_l = g_l(m) + m$$
which can be readily integrated to give
$$\alpha_l^2(m) = \alpha_l^2(m_b) + 2 \int_{m_b}^{m} d m' (m' + g_l(m')) = \alpha_l^2(m_b) + m^2 - m_b^2 + 2 \int_{m_b}^{m} d m' g_l(m') \quad (7.6)$$
in which particular solutions are determined by the choice of $\alpha_l^2(m_b)$.

To get an overview of the set of solutions we note that because of $H \rightarrow m \gamma_5$ for
$m \rightarrow \pm \infty$ one gets $g_l(m) \rightarrow 0$ for $m \rightarrow \pm \infty$. Therefore, since $g_l(m)$ is nonnegative, the equation
$m + g_l(m) = 0$ has at least one solution $m_0 \leq 0$. If this is the only one we choose
$m_b = m_0$. Because then $\int_{m_0}^{m} d m' (m' + g_l(m')) \geq 0$ for all $m$ it becomes obvious that we can freely
choose $\alpha_l^2(m_0) \geq 0$. In this way we get all solutions allowed by (7.4), which requires $\dot{\alpha}_l$ to be finite (the forbidden solutions can be conveniently seen by choosing $\alpha_l^2(m_b) = 0$
and $m_b \neq m_0$). The extension to the general case, where one has to deal with $2z + 1$
solutions of \( m + g(m) = 0 \) with \( m_{2z} \leq m_{2z-1} \leq \ldots \leq m_1 \leq m_0 \), is straightforward. Then among the \( m_y \) with even \( y \) one has to equate that to \( m_y \) which leads to the lowest value of \( \int_{m_y}^{m_y'} \! d m' (m' + g_l(m')) \) for some fixed \( \tilde{m} \) (or one of those in case of degeneracy).

We thus have a complete specification of the solutions. Clearly all solutions obtained show the asymptotic behaviors \( \alpha_l^2(m) \to m^2 \) for \( m^2 \to \infty \). It is seen that the points \( m_y \) with even \( y \) determine the characteristic features. If \( \alpha_l^2(m_y) > 0 \) there is a minimum of the solution \( +\sqrt{\alpha_l^2(m)} \) and a maximum of the solution \( -\sqrt{\alpha_l^2(m)} \) at that point. If \( \alpha_l^2(m_y) = 0 \) then \( +\sqrt{\alpha_l^2(m)} \) coming from above continues as \( -\sqrt{\alpha_l^2(m)} \) below the zero, and analogously \( -\sqrt{\alpha_l^2(m)} \) from above as \( +\sqrt{\alpha_l^2(m)} \) below, i.e. one gets two solutions which cross zero at that point. For the square of the derivative at the crossing point (using \( \dot{\alpha}_l^2 = (\dot{\alpha}_l\alpha_l)^2/\alpha_l^2 \) and \( (7.5) \)) one obtains

\[
\dot{\alpha}_l^2(m) \to 1 + \dot{g}_l(m) \quad \text{for} \quad m \to m_y \quad \text{and} \quad \alpha_l^2(m_y) = 0 \quad (7.7)
\]

which shows that \( \dot{g}_l(m) \) in general will cause deviations from the chiral value 1. The solutions of the differential equations describe the possibilities for flows which occur. Which values of \( \alpha_l^2(m_y) \) and which signs of \( \pm\sqrt{\alpha_l^2(m)} \) are selected and what the detailed properties of the function \( g_l(m) \) are depends on the eigenequation \( (6.5) \) which in the present context no longer appears directly.

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