OPTIMAL ANTIBLOCKING SYSTEMS OF INFORMATION SETS FOR THE BINARY CODES RELATED TO TRIANGULAR GRAPHS

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Dedicated to Professor Helmut Karzel on the occasion of his 92nd birthday.

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Abstract. We present AI-systems for the binary codes obtained from the adjacency relation of the triangular graphs $T(n)$ for any $n \geq 5$. These AI-systems are optimal and have for $n$ odd the full error-correcting capability.

1. Introduction

An antiblocking system of information sets (for short AI-system) for a $t$-error correcting code $C$ is a set $\mathfrak{A}$ of information sets for the code such that for every possible error vector $e$ of weight $t$ or less there exists an information set $I$ in $\mathfrak{A}$ with $e_i = 0$ for all $i \in I$. Clearly, an AI-system can be used for decoding. The method is described fully in [2] (cf. also Section 2). This antiblocking decoding is comparable with permutation decoding which uses a so-called PD-set of automorphisms of the code (cf. [1], [4]). But while for every $t$-error correcting linear code an AI-system exists (for example the system of all information sets) the existence of PD-sets is not guaranteed (cf. [3]). The relation between PD-sets and AI-systems results from the fact that for a PD-set $\Sigma$ with respect to an information set $I$ the set $\mathfrak{A} = \{\sigma^{-1}(I) \mid \sigma \in \Sigma\}$ is an AI-system.

Both algorithms, permutation decoding as well as antiblocking decoding, are the more efficient the smaller the PD-set and the AI-system, respectively. Therefore it is of special interest to find small PD-sets and small AI-systems.

In 2004 J.D. Key, J. Moori, and B.G. Rodrigues [1] presented PD-sets for the binary codes $C_n$ from triangular graphs $T(n)$.

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$^1$ $\sigma_\pi$ denotes the permutation part of $\sigma$. 
The code $C_n$ is a linear $\binom{n}{2}, n-1, n-1 \}$-code and $\binom{n}{2}, n-2, 2(n-2)$-code for $n$ odd and $n$ even, respectively (cf. [6], [1]). For $n$ odd and $n$ even they found a PD-set $S$ with $|S| = n$ and $|S| = n^2 - 2n + 2$, respectively.

In this paper we construct AI-systems $\mathfrak{A}$ for these codes consisting of pairwise disjoint information sets with $|\mathfrak{A}| = \lfloor \frac{n}{2} \rfloor$ (cf. Theorem 1). The codes $C_n$ themselves are not the best ones in terms of error-correcting capability, unfortunately. But the found AI-systems are optimal in contrast to the AI-systems belonging to the PD-sets of [1]. While the AI-system $\mathfrak{A}$ for $n$ odd has the full error-correcting capability, the AI-system $\mathfrak{A}$ for $n$ even yields only a partial antiblocking decoding algorithm as it can correct only $\frac{n}{2} - 1$ errors instead of $n - 3$. In the last part of Section 3 we enlarge the AI-system $\mathfrak{A}$ for $n$ even such that $\frac{n}{2}$ errors can be corrected (cf. Proposition 3). This enlarged system is optimal for $n = 6$ and $n = 8$. In view of the found AI-systems we determine in the last section the standard generator matrices for the codes $C_n$. As examples we present for $n = 5, 6, 7, 8, 9, 10$ the AI-systems as well as the standard generator matrices.

2. Preliminaries

For the convenience of the reader, and in order to establish our notations we recall the relevant definitions and results of [3] and [2].

Let $P$ be a finite set, $k, t$ positive integers with $k + t \leq |P|$, and let $\mathfrak{A}$ be a subset of the power set of $P$. In order to avoid trivial cases we assume $|P| \geq 3$.

$\mathfrak{A}$ is called a $t$-antiblocking system on $P$ of order $k$ if for all $A \in \mathfrak{A}$ the cardinality $|A| = k$ and if the following holds

(AB) For every $B \subset P$ with $|B| = t$ there exists $A \in \mathfrak{A}$ such that $B \cap A = \emptyset$.

For the size of a $t$-antiblocking system $\mathfrak{A}$ of order $k$ on an $n$-set $P$ there is a lower bound, the Schönheim bound: namely the complementary set family $\mathcal{C} = \{C_A \mid A \in \mathfrak{A}\}$ forms a $t$-covering and therefore with $r = n - k$

$$|\mathfrak{A}| = |\mathcal{C}| \geq \left\lceil \frac{n}{r} \left( \frac{n-1}{r-1} \left( \frac{n-2}{r-2} \cdots \frac{n-t+1}{r-t+1} \right) \cdots \right) \right\rceil =: g(n, k, t)$$

by Schönheim [5, Theorem 1].

Proposition 1. (cf. [3]). If $t < \lfloor \frac{n}{2} \rfloor$, then there exists up to permutations a unique $t$-antiblocking system $\mathfrak{A}$ of order $k$ on $P$ with $|\mathfrak{A}| = t + 1 = g(n, k, t)$. Each $t$-antiblocking system of order $k$ on $P$ of cardinality $t+1$ consists of pairwise disjoint $k$-sets.

Let $F$ be a finite field, and let $n$ be a positive integer. Let $C$ be a linear $[n, k, d]$-code, i.e. $C$ is a $k$-dimensional vector subspace of the vector space $F^n$ with minimum Hamming weight $d$ (for the basic concepts of coding theory see [4].). For every positive integer $t \leq \frac{d-1}{2}$ the code $C$ is a $t$-error-correcting code.

Let $w_t : F^n \to \mathbb{Z}; \ x = (x_1, \cdots, x_n) \mapsto |\{i \mid x_i \neq 0\}|$ denote the Hamming weight. For $I \subset \{1, \ldots, n\}$ let $\mathbb{C}I := \{1, \ldots, n\} \setminus I$ and let

$$p_I : F^n \to F^I, \ x \mapsto x|_I : \begin{cases} I & \to F \\ i & \mapsto x_i \end{cases}$$

be the $I$-projection of $F^n$.

$I$ is called an information set for $C$ if $|I| = k$ and $p_I(C) = F^I$, i.e. the restriction $p_I|_{C}$ of $p_I$ on $C$ is a bijection. For an information set $I$ for $C$, let $\gamma_I := (p_I|_C)^{-1}p_I :$
is called adjacency mapping code and hence syn
Then a system instead of a PD-set in the permutation decoding algorithm; for details see [2].

Let \( \mathfrak{A} \) be a \( t \)-antiblocking system of order \( k \) on the set of the coordinate positions. Then \( \mathfrak{A} \) is called a \( t \)-AI-system for \( C \) if every \( A \in \mathfrak{A} \) is an information set. With this notation we can establish the Antiblocking Decoding Algorithm. It uses a \( t \)-AI-system instead of a PD-set in the permutation decoding algorithm; for details see [2].

**Antiblocking Decoding Algorithm (cf. [2]):** Let \( \mathfrak{A} \) be a \( t \)-AI-system for the linear \([n, k, d]\)-code \( C \).

1. For a received senseword \( w \in F^n \) compute \( \gamma_A(w) \) and \( wt(syn_A(w)) \) for \( A \in \mathfrak{A} \) until an \( A' \) is found with \( wt(syn_A(w)) \leq t \).
2. \( w \) is decoded as \( c = \gamma_A(w) \in C \).
3. If \( wt(syn_A(w)) > t \) for all \( A \in \mathfrak{A} \) then there is no \( c \in C \) with \( wt(w - c) \leq t \).

Clearly the antiblocking decoding algorithm is the more efficient the smaller the AI-system.

A \( t \)-antiblocking system \( \mathfrak{A}_0 \) is called optimal if \( |\mathfrak{A}_0| = \min \{|\mathfrak{A}| \mid \mathfrak{A} \text{ is a } t\text{-antiblocking system on } P \text{ of order } k \} = b(n, k, t) \). Clearly, \( b(n, k, t) \geq g(n, k, t) \).

In the next section we will use the following

**Lemma 1.** Let \( K \) be a commutative field, \( u \) a positive integer and \( A \in K^{n\times u} \) a symmetric matrix of rank \( k \). Denote by \( a_{11}, \ldots, a_u \) the rows of \( A \). Further, let \( j_1, \ldots, j_k \in \{1, \ldots, u\} \) such that \( a_{j_1}, \ldots, a_{j_k} \) are linearly independent. Denote by \( a_{1h}', \ldots, a_{uh}' \) the rows of the matrix \( A' \) formed by the columns \( a_{j_1}'r_h, (h = 1, \ldots, k) \). Then \( a_{j_1}', \ldots, a_{j_k}' \) are linearly independent. In other words, the principal \( k \times k \) submatrix of \( A \) indexed by \( j_1, \ldots, j_k \) has rank \( k \).

**Proof.** Note that \( A' \) is a submatrix of \( A \) since \( A \) is symmetric. The columns \( a_{j_h}'r_h \) \((h = 1, \ldots, k)\) are linearly independent, hence the rank of \( A' \) is \( k \). Therefore \( \dim(a_{j_h}'r_h \mid j = 1, \ldots, u) = k \). Since the rank of \( A \) is \( k \) there exist for every row \( a_{j_0}'r_h \) of \( A' \) elements \( \lambda_h \in K \) with \( a_{j_0}'r_h = \sum_{h=1}^{k} \lambda_h a_{j_h}'r_h \), hence \( a_{j_0}'r_h = \sum_{h=1}^{k} \lambda_h a_{j_h}'r_h \). Therefore \( \langle a_{j_h}'r_h \mid h = 1, \ldots, k \rangle = \langle a_{j_h}'r_h \mid j = 1, \ldots, u \rangle \) implying the assertion as \( \dim(a_{j_h}'r_h \mid j = 1, \ldots, u) = k \).

3. **AI-systems for the binary codes related to triangular graphs**

Let \( n \) be a positive integer, \( N(n) := \{1, 2, \ldots, n\} \) and \( P(n) := \{X \mid X \subset N(n), |X| = 2\} \) the set of all 2-subsets of \( N(n) \). The set \( P(n) \) is the set of vertices of the triangular graph \( T(n) \); two vertices \( X, Y \in P(n) \) are adjacent iff \( |X \cap Y| = 1 \) (cf. [1], [6]).

Denote by \( F_2 \) the prime field of order 2. The mapping

\[
\text{ad} : P(n) \times P(n) \to F_2, \quad (X, Y) \mapsto X^\text{ad}Y \defeq \begin{cases} 1 & \text{if } |X \cap Y| = 1, \\ 0 & \text{otherwise} \end{cases}
\]

is called adjacency mapping.
For $X \in P(n)$ we consider the mapping $X^{ad} : P(n) \rightarrow F_2$, $Y \mapsto X^{ad}Y$. The set $P(n)^{ad} := \{X^{ad} \mid X \in P(n)\}$ is a subset of the vector space $F_2^{P(n)}$ of all mappings $P(n) \rightarrow F_2$.

The linear code $C_n := \langle P(n)^{ad} \rangle \subset F_2^{P(n)}$ is called the code related to $T(n)$.

Some of the following results can be found already in [1]. Anyway we prove them again as we want to avoid the 1-designs related to the triangular graphs and in order to make this paper self-contained.

**Remark 2.** For $n = 4$ we have $\{1,2\}^{ad} = \{3,4\}^{ad}$.

**Lemma 2.** Let $n \geq 5$. Then for $X,Y \in P(n)$, $X \neq Y$ it holds $X^{ad} \neq Y^{ad}$.

**Proof.** Because of $n \geq 5$ there exists $a \in N(n)$, $a \notin X \cup Y$. Since $X \neq Y$ there is $x \in X \setminus Y$. Then $X \cap \{x,a\} = \{x\}$ and $Y \cap \{x,a\} = \emptyset$. Therefore $X^{ad}\{x,a\} = 1$ and $Y^{ad}\{x,a\} = 0$, thus $X^{ad} \neq Y^{ad}$.

Referring to Remark 2 and Lemma 2 we assume in the following that $n \geq 5$.

As we are interested in AI-systems we look for information sets of $C_n$.

**Proposition 2.** Let $B = \{B_1, \ldots, B_k\} \subset P(n)$. If $B^{ad} = \{B_1^{ad}, \ldots, B_k^{ad}\}$ is a basis of $C_n$ then $B$ is an information set.

**Proof.** Let $B^{ad} = \{B_1^{ad}, \ldots, B_k^{ad}\}$ be a basis of $C_n$. Denote by $A := (X^{ad}Y)_{X,Y \in P(n)}$ the adjacency matrix and $A'$ the submatrix of the columns indexed by $B$. The rows of $A$ correspond to the mappings $X^{ad}$ for $X \in P(n)$, hence the rank of $A$ is $k = \dim C_n$. The rows of the matrix $A'$ correspond to the restrictions of the mappings $X^{ad}$ to $B$. Therefore $\{X^{ad}|_B \mid X \in B\}$ is linearly independent by Lemma 1. Since $B^{ad}$ is a basis of $C_n$ we obtain that

$$p_B|_{C_n} = \text{Rest}_B : C_n \rightarrow F_2^B, \quad c \mapsto c_B := \begin{cases} \{B \rightarrow F_2 \\ X \mapsto c(X) \end{cases}$$

is a bijection.

**Lemma 3.** If $a,x,y \in N(n)$, $a \neq x \neq y \neq a$ then $\{x,a\}^{ad} + \{y,a\}^{ad} = \{x,y\}^{ad}$.

**Proof.** Let $Z \in P(n)$. Put $X = \{x,a\}$, $Y = \{y,a\}$. If $x, y, a \notin Z$ then $X \cap Z = \emptyset = Y \cap Z = \{x,y\} \cap Z$, hence $X^{ad}Z = 0 = Y^{ad}Z = \{x,y\}^{ad}Z$ and therefore $(X^{ad} + Y^{ad})(Z) = X^{ad}Z + Y^{ad}Z = 0 = \{x,y\}^{ad}Z$.

(2) If $x,y \notin Z$, $a \in Z$ then $X^{ad}Z = 1 = Y^{ad}Z$ and $\{x,y\}^{ad}Z = 0$, hence $(X^{ad} + Y^{ad})(Z) = 1 + 1 = 0 = \{x,y\}^{ad}Z$.

(3) If $x \in Z$, $y \notin Z$ then $\{x,y\}^{ad}Z = 1$ and $X^{ad}Z = \begin{cases} 0 & \text{if } a \in Z \\ 1 & \text{if } a \notin Z \end{cases}$, hence $(X^{ad} + Y^{ad})(Z) = 1 = \{x,y\}^{ad}Z$.

As well, if $y \in Z$, $x \notin Z$ then $(X^{ad} + Y^{ad})(Z) = 1 = \{x,y\}^{ad}Z$.

(4) If $x,y \in Z$ then $a \notin Z$ and $X^{ad}Z = 1 = Y^{ad}Z$ and $\{x,y\}^{ad}Z = 0$, thus $(X^{ad} + Y^{ad})(Z) = 0 = \{x,y\}^{ad}Z$.

(1), (2), (3), and (4) imply $\{x,a\}^{ad} + \{y,a\}^{ad} = \{x,y\}^{ad}$. 

Note that Lemma 3 corresponds to formula (3) in [1].
Lemma 4. Let $A \subset P(n)$ such that $\bigcup_{X \in A} X = N(n)$ is a disjoint union. Then $n$ is even and $f := \sum_{X \in A} X^{ad} = 0$.

Proof. Clearly, $n$ is even, as $N(n)$ is a disjoint union of 2-sets. Let $Y = \{y_1, y_2\} \subset P(n)$. Then there exists exactly one $X_i \in A$ with $y_i \in X_i$ for $i = 1, 2$. For all $X \in A \setminus \{X_1, X_2\}$ it holds $X \cap Y = \emptyset$, hence $X^{ad}Y = 0$.

If $X_1 = X_2$ then $X_1^{ad}Y = 0$ and thus $\sum_{X \in A} X^{ad}Y = 0$. If $X_1 \neq X_2$ then $X_1^{ad}Y = 1 = X_2^{ad}Y$ and $\sum_{X \in A} X^{ad}Y = 1 + 1 = 0$. Therefore $f = 0$. \qed

Lemma 5. Let $c, d \in N(n)$, $c \neq d$ and $B_c := \{(i, c)^{ad} \mid i \in N(n), i \neq c\}$, $dB_c := B_c \setminus \{(d, c)^{ad}\}$. Then

(1) For $n$ odd, $B_c$ is a basis of $C_n$.
(2) For $n$ even, $dB_c$ is a basis of $C_n$.

Proof. (0) Let $\{x_1, x_2\} \in P(n)$, $x_1, x_2 \neq c$. Then $\{x_1, c\}^{ad} + \{x_2, c\}^{ad} = \{x_1, x_2\}^{ad}$ by Lemma 3, hence $\{x_1, x_2\}^{ad} \in \langle B_c \rangle$. Therefore $P(n)^{ad} \subset \langle B_c \rangle \subset C_n$ and thus $\langle B_c \rangle = C_n$.

(1) Let $\emptyset \neq X \subset N(n), c \notin X$ and $f_X := \sum_{x \in X} \{x, c\}^{ad}$.
If $|X|$ is odd then $X \neq N(n) \setminus \{c\}$ and there exists $y \in N(n), y \neq c, y \notin X$. Then $\{x, c\}^{ad} \{y, c\} = 1$ for all $x \in X$, hence $f_X(\{y, c\}) = 1$.
If $|X|$ is even choose $x_0 \in X$. Then $\{x_0, c\}^{ad} \{x_0, c\} = 0$ and $\{x, c\}^{ad} \{x_0, c\} = 1$ for all $x \in X \setminus \{x_0\}$. Since $|X \setminus \{x_0\}|$ is odd it follows $f_X(\{x_0, c\}) = 1$.
Therefore $f_X \neq 0$, i.e. only the trivial linear combination of mappings of $B_c$ yields the zero map. Hence $B_c$ is linearly independent, thus $B_c$ is a basis by (0).

(2) Let $\emptyset \neq X \subset N(n), c, d \notin X$ and $f_X := \sum_{x \in X} \{x, c\}^{ad}$.
Let $x_0 \in X$. Then $\{x_0, c\} \cap \{x_0, d\} = \emptyset$ and $\{x, c\} \cap \{x_0, d\} = \emptyset$ for all $x \in X$, $x \neq x_0$, hence $\{x_0, c\}^{ad} \{x_0, d\} = 1$ and $\{x, c\}^{ad} \{x_0, d\} = 0$ for all $x \in X$, $x \neq x_0$, thus $f_X(\{x_0, d\}) = 1$. Therefore $f_X \neq 0$, i.e. only the trivial linear combination of mappings of $dB_c$ yields the zero map. Hence $dB_c$ is linearly independent.

Let $m := \frac{n}{2}$ and $X_i := (2i - 1, 2i)$ for $i = 1, ..., m$. Then $\bigcup_{i=1,...,m} X_i = N(n)$ is a disjoint union of $N(n)$ and thus $\sum_{i=1}^m X_i^{ad} = 0$ by Lemma 4. Hence $\sum_{i=1}^{m-1} X_i^{ad} = \{2m - 1, 2m\}^{ad} = X_m^{ad}$.

For $i = 1, ..., m - 1$ it holds $\{2i - 1, 2i\}^{ad} = \{1, 2i - 1\}^{ad} + \{1, 2i\}^{ad}$ by Lemma 3. Hence $\{X_i^{ad} \mid i = 1, ..., m - 1\} \subset \langle N B_1 \rangle$ and thus $\{2m - 1, 2m\}^{ad} = X_m^{ad} = \sum_{i=1,...,m-1} X_i^{ad} \in \langle N B_1 \rangle$.

Therefore $\{1, n\}^{ad} = \{1, 2m\}^{ad} = \{1, 2m - 1\}^{ad} + \{2m - 1, 2m\}^{ad} \in \langle N B_1 \rangle$, hence $\langle N B_1 \rangle = \langle B_1 \rangle = C_n$ by (0). Therefore, since $|d B_c| = |n B_1|$ and $d B_c$ is linearly independent, $d B_c$ is a basis of $C_n$. \qed

For $c = n$ in Lemma 5 the set $B_n$ corresponds to (6) in [1].

Since $|B_1| = n - 1$ and $|n B_1| = n - 2$ we have as a consequence of Lemma 5

Corollary 1. (cf. [6])

(1) If $n$ is odd then $\dim C_n = n - 1$.
(2) If $n$ is even then $\dim C_n = n - 2$.

We are going to look for $t$-AI-systems on $P(n)$ of order $k = \begin{cases} n - 1 & \text{for } n \text{ odd} \\ n - 2 & \text{for } n \text{ even} \end{cases}$ for $t < \lfloor \frac{|P(n)|}{k} \rfloor$. By Proposition 1 in [3] such $t$-AI-systems consist of pairwise disjoint information sets. Unfortunately the information sets corresponding via Proposition
Lemma 6. Let \( n = 2m \). For \( l \in \{1, \ldots, m\} \) the set \( A^\text{ad}_l := \{X^\text{ad} \mid X \in A_l\} \) is linearly independent.

Proof. The proof is divided into three steps.

(1) \( A^\text{ad}_l \) is linearly independent (cf. [1] Lemma 3.5). Let \( X \subset \{1, \ldots, 2m - 2\} \), \( X \neq \emptyset \) and \( x_0 := \min X \). We show that \( f := \sum_{x \in X} \{x, x + 1\}^\text{ad} \neq 0 \).

For all \( x \in X \), \( x \neq x_0 \) it holds \( \{x, x + 1\} \cap \{x_0, 2m\} = \emptyset \) since \( x_0 := \min X \) and \( x < 2m - 1 \), hence \( \{x, x + 1\}^\text{ad} \{x_0, 2m\} = 0 \). Furthermore \( \{x_0, x_0 + 1\} \cap \{x_0, 2m\} = \{x_0\} \) since \( x_0 \neq 2m - 1 \), hence \( \{x_0, x_0 + 1\}^\text{ad} \{x_0, 2m\} = 1 \).

Therefore \( f(\{x_0, 2m\}) = 1 \), thus \( f \neq 0 \).

(2) Let \( l > 1 \). Let \( V_l := \{(m-l+1, j) \mid j = m+1, \ldots, m+l\} \) and \( V_0 := \{i, m+l\} \) with \( i = m-l+2, \ldots, m-1 \) and \( H := V_l \cup V_0 \). Then \( H^\text{ad} \) is linearly independent.

Let \( X \subset \{m-l+2, \ldots, m-1\} \) and \( Y \subset \{m+1, \ldots, m+l\} \), such that \( X \cup Y \neq \emptyset \).

We show that \( f := \sum_{x \in X} \{x, m+l\}^\text{ad} + \sum_{y \in Y} \{m-l+1, y\}^\text{ad} \neq 0 \).
With $c = m - l + 1$ and $d = m$ it follows that $V_{c,d}^m$ is linearly independent by Lemma 5 (2). Therefore we may assume $X \neq \emptyset$. Hence there is $x_0 \in X$, $m - l + 2 \leq x_0 \leq m - 1$. Then $\{m - l + 1, y\}^d \{x_0, m\} = 0$ for all $y \in Y$, $\{x, m + l\}^d \{x_0, m\} = 0$ for all $x \in X$, $x \neq x_0$ and $\{x_0, m + l\}^d \{x_0, m\} = 1$. Therefore $f(\{x_0, m\}) = 1$, thus $f \neq 0$.

(3) For $1 < l \leq m$ we have $A_{l,d}^d$ is linearly independent. Let $A \subset A_l$ and $f := \sum_{a \in A} a^d$. We show $f \neq 0$.

Let $D_l := \{\{i, i + l\} \mid i \in \{1, ..., m - l + 1, l, ..., 2m - l - 1\}\}$ and $W_l := \{\{m - l + 1, j\} \mid j = m + 2, ..., m + l\} \cup \{\{i, m + l\} \mid i = m - l + 2, ..., m - 1\}$. Then $A_l = D_l \cup W_l$. Since $W_l$ is linearly independent by (2) we may assume $A \cap D_l \neq \emptyset$. Let $X := \bigcup_{a \in A} a$ and $x_0 := \min X$, $x_1 := \max X$. Since $A \cap D_l \neq \emptyset$ it holds $\{x_0, x_0 + l\} \in D_l$ or $\{x_1 - l, x_1\} \in D_l$.

Case 1. $\{x_0, x_0 + l\} \in D_l$. Then $\{x_0, x_0 + l\} \cap \{x_0, 2m\} = \{x_0\}$ as $x_0 \neq 2m$, hence $\{x_0, x_0 + l\}^d \{x_0, 2m\} = 1$. Let $a \in A$, $a \neq \{x_0, x_0 + l\}$. Then $a \cap \{x_0, 2m\} = \emptyset$ as $\max a \leq x_1$ and $\max a = x_1$ would imply $a = \{x_1 - l, x_1\}$. Hence $a^d \{x_1 - l, x_1\} = 1$. Therefore $f(\{x_1 - l, x_1\}) = 1$.

Case 2. $\{x_1 - l, x_1\} \in D_l$. Then $\{x_1 - l, x_1\} \cap \{x_1, 2m\} = \{x_1\}$. Let $a \in A$, $a \neq \{x_1 - l, x_1\}$. Then $a \cap \{x_1, 2m\} = \emptyset$ as $\max a \leq x_1$ and $\max a = x_1$ would imply $a = \{x_1 - l, x_1\}$. Hence $a^d \{x_1, 2m\} = 1$. Therefore $f(\{x_1, 2m\}) = 1$.

Lemma 7.

(1) Let $n = 2m$ be even. Then $A_l^d$ is a basis of $C_n$ for $l = 1, ..., m$.

(2) Let $n = 2m - 1$ be odd. Then $A_l^d$ is a basis of $C_n$ for $l = 1, ..., m - 1$.

Proof. For $l = 1, ..., m$ it holds $|A_l| = m - l + 1 + l - 1 + l - 2 + m - l = 2m - 2$, hence $|A_l^d| = 2m - 2$ by Lemma 2.

(1) By Corollary 1 it holds $\dim C_n = n - 2$. Since $|A_l^d| = 2m - 2 = n - 2$ and since $A_l^d$ is linearly independent by Lemma 6 it follows that $A_l^d$ is a basis for $C_n$.

(2) Let $C_{2m} = (P(2m)^d)$ and $C_{2m - 1} = (P(2m - 1)^d)$. By (1) $A_l^d$ generates $C_{2m}$. Therefore the restrictions of the mappings of $A_l^d \subset F_2^{P(2m)}$ to $P(2m - 1)$ generate $C_{2m - 1}$. Since $|A_l^d| = 2m - 2 = n - 1$ and since $\dim C_{2m - 1} = n - 1$ it follows that $A_l^d$ is a basis for $C_{2m - 1} = C_n$.

Theorem 1. Let $n \geq 5$ be an integer and $C_n$ the binary code related to the triangular graph $T(n)$. For $l = 1, ..., m := \lceil \frac{n}{2} \rceil$ let be

$$A_l = \{\{i, i + l\} \mid i = 1, ..., m - l + 1\} \cup$$
$$\{\{m - l + 1, j\} \mid j = m + 2, ..., m + l\} \cup$$
$$\{\{i, m + l\} \mid i = m - l + 2, ..., m - 1\} \cup$$
$$\{\{i, i + l\} \mid i = m, ..., 2m - l - 1\}.$$

(1) If $n = 2m - 1$ is odd then $\mathfrak{A}_n := \{A_l \mid l = 1, ..., m - 1\}$ is an optimal $(m - 2)$-AI-system for $C_n$.

(2) If $n = 2m$ is even then $\mathfrak{A}_n := \{A_l \mid l = 1, ..., m\}$ is an optimal $(m - 1)$-AI-system for $C_n$.

Proof. By Lemma 7 the set $A_l^d$ is a basis of $C_n$ for $l \in N(m - 1)$ and $l \in N(m)$, respectively. Hence $A_l$ is an information set by Proposition 2. Since the information sets $A_l$ are pairwise disjoint and since $|A| = m - 1$ and $|A| = m$, respectively, the
system $\mathfrak{M}$ is an $(m - 2)$-AI-system for $C_{2m-1}$ and an $(m - 1)$-AI-system for $C_{2m}$, respectively. For $P(n)$ and $k = \dim C_n$ it holds
\[
\left\lfloor \frac{|P(n)|}{k} \right\rfloor = \begin{cases} 
    m - 1 & \text{if } n = 2m - 1 \\
    m & \text{if } n = 2m
\end{cases}.
\]
Hence $\mathfrak{M}$ is optimal in both cases by Proposition 1.

Remark 3. For odd $n = 2m - 1$ the full error-correcting capability of the code $C_n$ is $t = m - 2$, and thus the $(m - 2)$-AI-system of Theorem 1 is suitable for the antiblocking decoding of the code $C_n$. On the other hand the full error-correcting capability of the code $C_{2m}$ is $t = 2m - 3$. Hence the $(m - 1)$-AI-system of Theorem 1 yields only a partial antiblocking decoding algorithm for the code $C_{2m}$.

In the last part of this section we will enlarge the $(m - 1)$-AI-system $\mathfrak{M}_2$ to an $m$-AI-system $\mathfrak{M}_2^m$ of size $m + \lceil \frac{m}{2} \rceil$. Put $m_1 := \lceil \frac{m}{2} \rceil$.

Let $Q_0 := \{ \{i, 2m\} \mid i = m, \ldots, 2m - 1\}$, $Y_1 := \{ \{i, 2m\} \mid i = 1, \ldots, m\}$, and for positive numbers $l$ with $1 < l < m_1 - 1$ let
\[Y_l := \{ \{i, 2m - l + 1\} \mid i = 1, \ldots, m - l\} \text{ and } Q_l := \{ \{l, 2m - l\}, \{m, 2m - l + 1\}\}.
\]
Finally put
\[Y_{m_1 - 1} := \begin{cases} 
    \{ \{m_1 - 1, 3m_1 + 2\}, \{m_1, 3m_1 + 2\} \} & \text{for } m = 2m_1 \\
    \{ \{m_1 - 1, 3m_1\} \} & \text{for } m = 2m_1 - 1
\end{cases},
\]
\[Q_{m_1 - 1} := \begin{cases} 
    \{ \{m_1 - 1, 3m_1 + 1\}, \{m_1 + 1, 3m_1 + 2\} \} & \text{for } m = 2m_1 \\
    \{ \{m_1 - 1, 3m_1 - 1\} \} & \text{for } m = 2m_1 - 1
\end{cases}
\]
and $Y_{m_1} := \emptyset$.

Note that $Q_l \cup Y_l \subset A_{m-l+1}$ for $l = 1, \ldots, m_1 - 1$.

We have $|Q_0| = m$, $|Y_l| = m - 2l$, $Q_l = 2$ for $l = 1, \ldots, m_1 - 2$ and $|Y_{m_1 - 1}| = |Q_{m_1 - 1}| = 2$ for $m = 2m_1$, $|Y_{l-1}| = |Q_{m_1 - 1}| = 1$ for $m = 2m_1 - 1$. For $l = 1, \ldots, m_1$ let $A_{m+l} := \bigcup_{i=0}^{l-1} Q_i \cup Y_i$. Then $|A_{m+l}| = 2m - 2$.

Lemma 8. For $l = 1, \ldots, m_1$ the set $A_{m+l}$ is an information set for the code $C_{2m}$.

Proof. (1) $A_{m+l}^\text{ad} = m-l B_{2m}$ is a basis by Lemma 5(2) and Corollary 1(2), hence $A_{m+l}$ is an information set by Proposition 2.

(2) Let $l \in \{2, \ldots, m_1\}$. For $X \in A_{m+l} \setminus Q_0$ it holds max $X \geq 2m - l \geq 3m_1 - 1 > m$. Because of $l \geq 2$ we have $\{m-1, 2m\} \in Q_1 \subset A_{m+l}$. For $j \in \{1, \ldots, m_1\}$, $j \neq m - l$ there is an $X \in A_{m+l} \setminus Q_0$ with $j \in X$ and $2m - 1 \geq \max X > m$, thus $\{\max X, 2m\} \in Q_0$, hence $\{j, 2m\}^\text{ad} = \{j, \max X\}^\text{ad} + \{\max X, 2m\}^\text{ad} \in (A_{m+l}^\text{ad})$ by Lemma 3. Therefore $m-l B_{2m} \subset (A_{m+l}^\text{ad})$ and thus $C_{2m} = (A_{m+l}^\text{ad})$. Hence $A_{m+l}$ is an information set by Proposition 2.

Proposition 3. $\mathfrak{M}_{2m} := \{A_i \mid i = 1, \ldots, m + m_1\}$ is an $m$-AI-system for the code $C_{2m}$.

Proof. Since $\mathfrak{M}_{2m}$ consists of $m + m_1$ information sets for $C_{2m}$ we have only to show that $\mathfrak{M}$ is an antiblocking system. Let $B \subset P(n)$ with $|B| = m$. We may assume $A_i \cap B \neq \emptyset$ for $i = 1, \ldots, m + m_1 - 1$, and therefore we may assume $B = \{T_i \mid i = 1, \ldots, m\}$ and $T_i \in A_i$ for $i = 1, \ldots, m$.

Since $Q_0 \cap A_i = \emptyset$ for $i = 1, \ldots, m$ it holds $Q_0 \cap B = \emptyset$. 

If $\bigcup_{i=0}^{l-1} Q_i \cap B = \emptyset$ for $l < m_1$ then $Y_l \cap B \neq \emptyset$ since $\mathcal{A}_{m+l} \cap B \neq \emptyset$, hence $T_{m-l+1} \in Y_l$ as $Y_l \subset \mathcal{A}_{m-l+1}$ and $\mathcal{A}_{m-l+1} \cap B = \{T_{m-l+1}\}$, thus $Q_l \cap B = \emptyset$ as $Q_l \subset \mathcal{A}_{m-l-1} \cap Y_l$.

Therefore, since $Q_0 \cap B = \emptyset$ we obtain in succession $Q_l \cap B = \emptyset$ for $l = 1, \ldots, m_1 - 1$. Hence $\mathcal{A}_{m+m_1} \cap B = \emptyset$.

**Remark 4.** The Schönheim bound for an $m$-AI-system for the code $C_{2m}$ is $g(2m^2 - m, 2m - 2, m) = m + 2$ and $|\mathfrak{X}_{2m}| = m + \lceil \frac{m}{2} \rceil$, hence the difference is $\lceil \frac{m}{2} \rceil - 2$, but it is not known if there exists an $m$-AI-system $\mathfrak{X}''$ with $|\mathfrak{X}''| = m + 2$.

For $n = 6$ and $n = 8$ we have $m_1 = 2$. Thus $\mathfrak{X}_6'$ is an optimal 3-AI-system for the code $C_6$ and $\mathfrak{X}_8'$ is an optimal 4-AI-system for the code $C_8$. Note that for $n = 6$ the full error-correcting capability is $t = 3$.

4. **Standard generator matrix for $C_n$**

Let $n \geq 5$ be an integer, $C_n$ the code related to the triangular graph $T(n)$ and $\mathfrak{A}_n$ the AI-system for $C_n$ presented in Theorem 1.

In order to derive the code $C_n$ in an explicit form by a generator matrix we have to order the set of the coordinate positions $P(n)$. For this purpose we order firstly each information set $A_i \in \mathfrak{A}_n$ and also the remaining set $R := P(n) \ \bigcup_{i=1}^{\lceil \frac{m}{2} \rceil} A_i$ lexicographic. Then, as the $A_i$'s are pairwise disjoint, we order the set of the coordinate positions $P(n)$ by $A_1, A_2, \ldots, A_{\lceil \frac{m}{2} \rceil}, R$.

For $m = \lceil \frac{n}{2} \rceil$ and $i = 1, \ldots, m - 1$ put

$$g_{2i} = \sum_{j=1}^{i} \{2j - 1, 2j\}^{ad} \quad \text{and} \quad g_{2i-1} = \sum_{j=i}^{m-1} \{2j, 2j+1\}^{ad}$$

We are going to show that

$$G_n := (g_{h}(x))_{x \in P(n)}$$

is the standard generator matrix.

Note

(a) $g_2 = \{1, 2\}^{ad}$ and $g_{2i} = g_{2(i-1)} + \{2i - 1, 2i\}^{ad}$ for $i = 2, \ldots, m - 1$,

(b) $g_{2m-3} = \{2(m - 1), 2m - 1\}^{ad}$ and

$$g_{2m-3} = \{2(i - 1), 2i - 1\}^{ad} + g_{2i-1} \text{ for } i = m - 1, \ldots, 2.$$

(c) For $x = \{j, j+1\} \in A_1$ we have

$$i, j+1\}^{ad}(x) = \begin{cases} 1 & \text{for } j = i - 1 \text{ and } j = i + 1 \\ 0 & \text{else} \end{cases}$$

In particular

(d) $g_2(x) = \begin{cases} 1 & \text{for } x = \{2, 3\} \\ 0 & \text{for } x \in A_1 \ \bigcup \{\{2, 3\}\} \end{cases}$

(e) $g_{2m-3}(x) = \begin{cases} 1 & \text{for } x = \{2m - 3, 2m - 2\} \\ 0 & \text{for } x \in A_1 \ \bigcup \{\{2m - 3, 2m - 2\}\} \end{cases}$.
By (a), (d) and (c) we obtain for $i, j = 1, ..., m - 1$
\[
g_{2i}(\{j, j + 1\}) = \begin{cases} 
1 & \text{for } j = 2i \\
0 & \text{for } j \neq 2i 
\end{cases}.
\]

By (b), (e) and (c) we obtain for $i, j = 1, ..., m - 1$
\[
g_{2i-1}(\{j, j + 1\}) = \begin{cases} 
1 & \text{for } j = 2i - 1 \\
0 & \text{for } j \neq 2i - 1 
\end{cases}.
\]

Therefore the columns of $G_n$ indexed by $A_1$ form the identity matrix $I_k$ for $k = 2m - 1$, thus $G_n$ is the standard generator matrix for $C_n$.

Note that the standard generator matrix for $C_{2m - 1}$ is the submatrix of $G_{2m}$ formed by the first $|P(2m - 1)| = (\frac{2m - 1}{2})$ columns of $G_{2m}$.

In order to apply the antiblocking decoding algorithm we need for every $A_l \in \mathcal{A}_n$ the syndrome $syn_l := syn_{\mathcal{A}_l}$ (cf. Section 2).

Let $A_l \in \mathcal{A}_n$. The columns indexed by $A_l$ form a $k \times k$ submatrix $S_l$ of $G_n$. Since $A_l$ is an information set, then the matrix $S_l$ is invertible. Therefore, by suitable row operations of $G_n$ we can achieve a generator matrix $iG_n$ such that the columns indexed by $A_l$ form the identity matrix $I_k$. Clearly, $iG_n = S_l^{-1}G_n$. Denote by $\gamma_l := \gamma_{A_l}$ (cf. Section 2) and $A_l$ the submatrix of $iG_n$ indexed by $\mathcal{L}(A_l)$. Then $\gamma_l(v) = p_{A_l}(v)S_l^{-1}G_n$ for $v \in F_2^n$ and $syn_l(v) = p_{\mathcal{L}A_l}(v) - p_{A_l}(v)A_l$ (cf. Remark 1).

As examples we write down the found $\text{AI}$-systems and standard generator matrices for $n = 5, 6, 7, 8, 9$ and $10$, i.e. $m = \lceil \frac{n}{2} \rceil$ is 3, 4 and 5.

**Example 1.** $m = 3$, i.e. $n = 5$ and $n = 6$.

In this case the sets $\mathcal{A}_l$ considered in Theorem 1 are
\[
\mathcal{A}_1 = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}\}, \\
\mathcal{A}_2 = \{\{1, 3\}, \{2, 4\}, \{2, 5\}, \{3, 5\}\}, \\
\mathcal{A}_3 = \{\{1, 4\}, \{1, 5\}, \{1, 6\}, \{2, 6\}\}.
\]

Furthermore,
\[
g_1 = \{2, 3\}^{ad} + \{4, 5\}^{ad}, \quad g_2 = \{1, 2\}^{ad}, \quad g_3 = \{4, 5\}^{ad}, \quad g_4 = \{1, 2\}^{ad} + \{3, 4\}^{ad}.
\]

1.a. $n = 5$.

By Theorem 1 the system $\mathcal{A}_5 = \{\mathcal{A}_1, \mathcal{A}_2\}$ is a $1$-$\text{AI}$-system for the code $C_5$.

The set of the coordinate positions is $P(5) = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \{\{1, 4\}, \{1, 5\}\}$.

For $C_5$ we obtain the standard generator matrix
\[
G_5 = \begin{pmatrix}
I_4 & \begin{pmatrix} 
1 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 1 
\end{pmatrix}
\end{pmatrix}.
\]

1.b. $n = 6$.

By Theorem 1 the system $\mathcal{A}_6 = \{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\}$ is a $2$-$\text{AI}$-system for the code $C_6$. 
With the additional information sets \( A_4 = \{1, 6\}, \{3, 6\}, \{4, 6\}, \{5, 6\} \) and \( A_5 = \{2, 6\}, \{3, 6\}, \{4, 6\}, \{5, 6\} \) the system \( \mathcal{A}_6' = \{A_1, A_2, A_3, A_4, A_5\} \) is a 3-AI-system for the code \( C_6 \) (cf. Proposition 3).

For \( C_6 \) we obtain the standard generator matrix

\[
G_6 = \begin{pmatrix}
I_4 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0
\end{pmatrix}.
\]

For this example we will compute the syndrome for each \( A_l \in \mathcal{A}_6' \), i.e. the matrices \( A_l \).

The inverses of the submatrices \( S_l \) of \( G_6 \) formed by the columns indexed by the information sets \( A_l \) are

\[
S_1^{-1} = I_4, \quad S_2^{-1} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0
\end{pmatrix}, \quad S_3^{-1} = \begin{pmatrix}
0 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0
\end{pmatrix}, \quad S_4^{-1} = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1
\end{pmatrix}, \quad S_5^{-1} = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 \\
1 & 0 & 0 & 1
\end{pmatrix}.
\]

Denote by \( R \) the submatrix of \( G_6 \) formed by the last three columns, and denote by \( R_4 \) and \( R_5 \) the submatrices of \( G_6 \) formed by the columns indexed by \( \{1, 4\}, \{1, 5\}, \{2, 6\} \) and \( \{1, 4\}, \{1, 5\}, \{1, 6\} \), respectively.

The submatrices \( A_l \) of \( S_l^{-1} G_6 \) formed by the columns indexed by \( \mathcal{L}A_l \) are

\[
A_1 = \begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0
\end{pmatrix},
\]

\[
A_2 = (S_2^{-1}|S_2^{-1} S_3|S_2^{-1} R) = \begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0
\end{pmatrix},
\]

\[
A_3 = (S_3^{-1}|S_3^{-1} S_2|S_3^{-1} R) = \begin{pmatrix}
0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0
\end{pmatrix},
\]

\[
A_4 = (S_4^{-1}|S_4^{-1} S_2|S_4^{-1} R_4) = \begin{pmatrix}
0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1
\end{pmatrix}.
\]
Example 2. $m = (0111 \ 0110 \ 1000 \ 010)$. 

1. $\text{syn}_1(w) = (0110 \ 1000 \ 010) - (0111)A_1$
   
2. $\text{syn}_2(w) = (0111 \ 1000 \ 010) - (0110)A_2$

For $l = 1, 2, 3, 4, 5$ and $v \in F_2^{P(n)}$ we have $\text{syn}_l(v) = p_{\mathcal{A}_l}(v) - p_{\mathcal{A}_l}(v)A_l$.

Let us apply the Antiblocking Decoding Algorithm to the received senseword $w = (0111 \ 0110 \ 1000 \ 010)$.

1. $\text{syn}_1(w) = (0110 \ 1000 \ 010) - (0111)A_1$
   
2. $\text{syn}_2(w) = (0111 \ 1000 \ 010) - (0110)A_2$

Example 2. $m = 4$, i.e. $n = 7$ and $n = 8$.

In this case the sets $\mathcal{A}_l$ considered in Theorem 1 are

1. $\mathcal{A}_1 = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 6\}, \{6, 7\}\}$
2. $\mathcal{A}_2 = \{\{1, 3\}, \{2, 4\}, \{3, 5\}, \{3, 6\}, \{4, 6\}, \{5, 7\}\}$
3. $\mathcal{A}_3 = \{\{1, 4\}, \{2, 5\}, \{2, 6\}, \{2, 7\}, \{3, 7\}, \{4, 7\}\}$
4. $\mathcal{A}_4 = \{\{1, 5\}, \{1, 6\}, \{1, 7\}, \{1, 8\}, \{2, 8\}, \{3, 8\}\}$

Furthermore,

1. $g_1 = \{2, 3\}^{ad} + \{4, 5\}^{ad} + \{6, 7\}^{ad}$
2. $g_2 = \{1, 2\}^{ad}$
3. $g_3 = \{4, 5\}^{ad} + \{6, 7\}^{ad}$
4. $g_4 = \{1, 2\}^{ad} + \{3, 4\}^{ad}$
5. $g_5 = \{6, 7\}^{ad}$
6. $g_6 = \{1, 2\}^{ad} + \{3, 4\}^{ad} + \{5, 6\}^{ad}$

2.a. $n = 7$.

By Theorem 1 the system $\mathfrak{A}_7 = \{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\}$ is a 2-$1$-system for the code $C_7$. The set of the coordinate positions is $P(7) = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3 \cup \{\{1, 5\}, \{1, 6\}, \{1, 7\}\}$. For $C_7$ we obtain the standard generator matrix

$$G_7 = I_6 \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{pmatrix}.$$
2.b. \( n = 8 \).

By Theorem 1 the system \( \mathcal{A}_8 = \{A_1, A_2, A_3, A_4\} \) is a 3-AI-system for the code \( C_8 \). The set of the coordinate positions is \( P(8) = A_1 \cup A_2 \cup A_3 \cup A_4 \cup \{\{4,8\}, \{5,8\}, \{6,8\}, \{7,8\}\} \).

For \( C_8 \) we obtain the standard generator matrix

\[
G_8 = \begin{pmatrix}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0
\end{pmatrix}.
\]

With the additional information sets
\( \mathcal{A}_5 = \{\{1,8\}, \{2,8\}, \{4,8\}, \{5,8\}, \{6,8\}, \{7,8\}\} \) and
\( \mathcal{A}_6 = \{\{1,7\}, \{3,8\}, \{4,8\}, \{5,8\}, \{6,8\}, \{7,8\}\} \) the system
\( \mathcal{A}_8' = \{A_1, A_2, A_3, A_4, A_5, A_6\} \) is a 4-AI-system for the code \( C_8 \) (cf. Proposition 3).

Example 3. \( m = 5 \), i.e. \( n = 9 \) and \( n = 10 \).

In this case the sets \( \mathcal{A}_l \) considered in Theorem 1 are

\[
\begin{align*}
A_1 &= \{\{1,2\}, \{2,3\}, \{3,4\}, \{4,5\}, \{5,6\}, \{6,7\}, \{7,8\}, \{8,9\}\}, \\
A_2 &= \{\{1,3\}, \{2,4\}, \{3,5\}, \{4,6\}, \{4,7\}, \{5,7\}, \{6,8\}, \{7,9\}\}, \\
A_3 &= \{\{1,4\}, \{2,5\}, \{3,6\}, \{3,7\}, \{3,8\}, \{4,8\}, \{5,8\}, \{6,9\}\}, \\
A_4 &= \{\{1,5\}, \{2,6\}, \{2,7\}, \{2,8\}, \{2,9\}, \{3,9\}, \{4,9\}, \{5,9\}\}, \\
A_5 &= \{\{1,6\}, \{1,7\}, \{1,8\}, \{1,9\}, \{1,10\}, \{2,10\}, \{3,10\}, \{4,10\}\}.
\end{align*}
\]

Furthermore,

\[
\begin{align*}
g_1 &= \{2,3\}^{ad} + \{4,5\}^{ad} + \{6,7\}^{ad} + \{8,9\}^{ad}, \quad g_2 = \{1,2\}^{ad}, \\
g_3 &= \{4,5\}^{ad} + \{6,7\}^{ad} + \{8,9\}^{ad}, \quad g_4 = \{1,2\}^{ad} + \{3,4\}^{ad}, \\
g_5 &= \{6,7\}^{ad} + \{8,9\}^{ad}, \quad g_6 = \{1,2\}^{ad} + \{3,4\}^{ad} + \{5,6\}^{ad}, \\
g_7 &= \{8,9\}^{ad}, \quad g_8 = \{1,2\}^{ad} + \{3,4\}^{ad} + \{5,6\}^{ad} + \{7,8\}^{ad}.
\end{align*}
\]

3.a. \( n = 9 \).

By Theorem 1 the system \( \mathcal{A}_9 = \{A_1, A_2, A_3, A_4\} \) is a 3-AI-system for the code \( C_9 \). The set of the coordinate positions is \( P(9) = A_1 \cup A_2 \cup A_3 \cup A_4 \cup \{\{1,6\}, \{1,7\}, \{1,8\}, \{1,9\}\} \).

With the matrices
\[ S_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad S_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}. \]

the standard generator matrix for \( C_9 \) is \( G_9 = (I_8|S_2|S_3|S_4|R) \).

3.b. \( n = 10 \).

By Theorem 1 the system \( \mathcal{A}_{10} = \{ \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5 \} \) is a 4-AI-system for the code \( C_{10} \).

The set of the coordinate positions is \( P(10) = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3 \cup \mathcal{A}_4 \cup \mathcal{A}_5 \cup \{\{5,10\}, \{6,10\}, \{7,10\}, \{8,10\}, \{9,10\}\} \).

For \( C_{10} \) we obtain the standard generator matrix

\[ G_{10} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}. \]

With the additional information sets

\( \mathcal{A}_6 = \{\{1,10\}, \{2,10\}, \{3,10\}, \{5,10\}, \{6,10\}, \{7,10\}, \{8,10\}, \{9,10\}\} \),

\( \mathcal{A}_7 = \{\{1,9\}, \{2,9\}, \{4,10\}, \{5,10\}, \{6,10\}, \{7,10\}, \{8,10\}, \{9,10\}\} \),

\( \mathcal{A}_8 = \{\{1,9\}, \{2,8\}, \{4,10\}, \{5,10\}, \{6,10\}, \{7,10\}, \{8,10\}, \{9,10\}\} \),

the system \( \mathcal{A}'_{10} = \{ \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6, \mathcal{A}_7, \mathcal{A}_8 \} \) is a 5-AI-system for the code \( C_{10} \) (cf. Proposition 3).

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