LIMITING DYNAMICS FOR STOCHASTIC NONCLASSICAL DIFFUSION EQUATIONS

PENG GAO

School of Mathematics and Statistics, and Center for Mathematics and Interdisciplinary Sciences, Northeast Normal University
Changchun 130024, China

(Communicated by Björn Schmalfuß)

Abstract. In this paper, we are concerned with the dynamical behavior of the stochastic nonclassical parabolic equation, more precisely, it is shown that the inviscid limits of the stochastic nonclassical diffusion equations reduces to the stochastic heat equations. The key points in the proof of our convergence results are establishing some uniform estimates and the regularity theory for the solutions of the stochastic nonclassical diffusion equations which are independent of the parameter. Based on the uniform estimates, the tightness of distributions of the solutions can be obtained.

1. Introduction.

1.1. Motivation. Nonclassical parabolic equation

\[ u_t - \Delta u_t - \Delta u + u^3 - u = 0 \]

arises as a model to describe physical phenomena such as non-Newtonian flow, soil mechanics and heat conduction, etc.; see [1, 5, 22, 30, 31] and references therein. Aifantis [1] provides a quite general approach for obtaining these equations.

In a number of applications, the systems are subject to stochastic fluctuations arising as a result of either uncertain forcing (stochastic external forcing) or uncertainty of the governing laws of the system. The need for taking random effects into account in modeling, analyzing, simulating and predicting complex phenomena has been widely recognized in geophysical and climate dynamics, materials science, chemistry, biology and other areas. Stochastic partial differential equations (SPDEs or stochastic PDEs) are appropriate mathematical models for complex systems under random influences [34]. The fact that in physical experiments there are always small irregularities which give birth to a new random phenomenon justifies the study of equations with noise. In this paper, we investigate

\[
\begin{align*}
&d(u^\varepsilon - \varepsilon u^\varepsilon_{xx}) + (-u^\varepsilon_{xx} + u^\varepsilon^3 - u^\varepsilon)dt = g(u^\varepsilon)dB &\text{in } I \times (0, T) \\
&u^\varepsilon(0, t) = 0 = u^\varepsilon(1, t) &\text{in } (0, T) \\
&w^\varepsilon(0) = u_0 &\text{in } I,
\end{align*}
\]

2020 Mathematics Subject Classification. Primary: 60H15, 35K70; Secondary: 35Q35, 35A01.

Key words and phrases. Inviscid limits, Singular perturbation, Stochastic nonclassical diffusion equation, Stochastic heat equation.

Peng Gao is supported by the Fundamental Research Funds for the Central Universities (2412020FZ022).
where \( \varepsilon \in (0, \frac{1}{2}] \), \( I = (0, 1), T > 0 \). This paper is concerned with the asymptotic behavior of solutions of (1) as \( \varepsilon \to 0 \).

In recent years, the nonclassical diffusion equation has attracted much attention [2, 3, 32, 36, 35]. Many efforts have been devoted to studying the singularly perturbed nonlinear SPDEs, see [6, 7, 8, 9, 10, 14, 21, 20, 24, 13, 33]. Motivated by previous research and from both physical and mathematical standpoints, the following mathematical questions arise naturally which are important from the point of view of dynamical systems:

- Does the solution \( u^\varepsilon \) for (1) converge as \( \varepsilon \to 0 \)?
- If \( u^\varepsilon \) converges as \( \varepsilon \to 0 \), what is the limit of \( u^\varepsilon \)?

In this paper we will answer the above problems. The question of asymptotic analysis of partial differential equations when some physical parameters converge to some limit has always been of great interest.

1.2. Mathematical setting. Through this paper, we make the following assumptions:

H1) Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\) be a complete filtered probability space on which a one-dimensional standard Brownian motion \( \{B(t)\}_{t \geq 0} \) is defined such that \( \{\mathcal{F}_t\}_{t \geq 0} \) is the natural filtration generated by \( w(\cdot) \), augmented by all the \( \mathcal{P} \)-null sets in \( \mathcal{F} \). Let \( H \) be a Banach space, and let \( C([0, T]; H) \) be the Banach space of all \( H \)-valued strongly continuous functions defined on \([0, T]\). We denote by \( L_p^p(0, T; H) \) the Banach space consisting of all \( H \)-valued \( \{\mathcal{F}_t\}_{t \geq 0} \)-adapted processes \( X(\cdot) \) such that \( E(\|X(\cdot)\|_{L_p^p(0, T; H)}^2) < \infty \); by \( L^\infty_p(0, T; H) \) the Banach space consisting of all \( H \)-valued \( \{\mathcal{F}_t\}_{t \geq 0} \)-adapted bounded processes; by \( L^p_{\mathbb{P}}(\Omega; C([0, T]; H)) \) the Banach space consisting of all \( H \)-valued \( \{\mathcal{F}_t\}_{t \geq 0} \)-adapted continuous processes \( X(\cdot) \) such that \( E(\|X(\cdot)\|_{L^\infty_p(0, T; H)}^2) < \infty \). All the above spaces are endowed with the canonical norm.

H2) For a random variable \( \xi \), we denote by \( L(\xi) \) its distribution.

H3) \((\cdot, \cdot)\) stands for the inner product in \( L^2(I) \).

H4) The letter \( C \) with or without subscripts denotes positive constants whose value may change in different occasions. We will write the dependence of constant on parameters explicitly if it is essential.

We make the the two different assumptions on \( g \).

(A) \( g \in C(\mathbb{R}) \) and there exists a constant \( L > 0 \) such that
\[
\|g(u)\|_{L^2(I)} \leq L(1 + \|u\|_{L^2(I)}) \quad \forall u \in L^2(I),
\|g(u_1) - g(u_2)\|_{L^2(I)} \leq L\|u_1 - u_2\|_{L^2(I)} \quad \forall u_1, u_2 \in L^2(I).
\]

(B) \( g \in C(\mathbb{R}) \) and there exists a constant \( L > 0 \) such that
\[
\|g(u)\|_{L^2(I)} \leq L(1 + \|u\|_{L^2(I)}) \quad \forall u \in L^2(I),
\|g(u)\|_{H^1(I)} \leq L(1 + \|u\|_{H^1(I)}) \quad \forall u \in H^1(I),
\|g(u_1) - g(u_2)\|_{H^1(I)} \leq L\|u_1 - u_2\|_{H^1(I)} \quad \forall u_1, u_2 \in H^1(I).
\]

1.3. Weak martingale solution.

**Definition 1.1.** A weak martingale solution of (1) is a system \( \{(\Omega, \mathcal{F}, \mathbb{P}), (\mathcal{F}_t)_{0 \leq t \leq T}, u, B\} \), where

1. \((\Omega, \mathcal{F}, \mathbb{P})\) is a complete probability space,
2. \((\mathcal{F}_t)_{0 \leq t \leq T}\) is a filtration satisfying the usual condition on \((\Omega, \mathcal{F}, \mathbb{P})\),
3. \(B\) is a \(\mathcal{F}_t\)-adapted \(\mathbb{R}\)-valued Wiener process,
(4) \( u \in L^p(\Omega, L^\infty(0, T; L^2(I))) \cap L^p(\Omega, L^2(0, T; H^1(I))) \cap L^{2p}(\Omega, L^4(0, T; L^4(I))) \), for every \( 1 \leq p \leq \infty \).

(5) For all \( \varphi \in H^1_0(I) \),
\[
[(u(t), \varphi) + \varepsilon(u_x(t), \varphi_x)] - [(u_0, \varphi) + \varepsilon(u_{0x}, \varphi_x)] + \int_0^t ((u_x, \varphi_x) + (u^3 - u, \varphi)) \, ds
= \int_0^t (g(u), \varphi) \, dB
\]
hold \( dt \otimes dP \)-almost everywhere.

(6) The function \( u(t) \) take values in \( L^2(I) \) and is continuous with respect to \( t \) \( P \)-almost surely.

The first main result of this paper is given in the next statement.

**Theorem 1.2.** Let assumption (A) be satisfied, \( T > 0 \) and \( u_0 \in H^1_0(I) \). For any \( \varepsilon \in (0, \frac{1}{2}] \), there exists a weak martingale solution \( \{(\Omega^\varepsilon, \mathcal{F}^\varepsilon, \mathbb{P}^\varepsilon), (\mathcal{F}^\varepsilon_t)_{0 \leq t \leq T}, u^\varepsilon, B^\varepsilon\} \) of problem (1) such that the following estimates hold for any \( 1 \leq p < \infty \):

\[
\mathbb{E} \sup_{0 \leq t \leq T} \|u^\varepsilon(t)\|_{L^p(I)}^p \leq C(p, T),
\]

(2) \[
\mathbb{E} \left( \int_0^T \|u^\varepsilon_x(t)\|_{L^p(I)}^p + \|u^\varepsilon\|_{L^p(I)}^p \, dt \right)^{\frac{p}{2}} \leq C(p, T),
\]

(3) \[
\mathbb{E} \sup_{0 \leq \theta \leq 1} \int_0^T \|u^\varepsilon(t + \theta) - u^\varepsilon(t)\|_{H^{-1}(I)}^2 \, dt \leq C(p, T)\delta,
\]

(4)

where \( C(p, T) \) is a constant independent of \( \varepsilon \).

Moreover, let \( u_1 \) and \( u_2 \) be two weak martingale solutions of problem (1) defined on the same prescribed stochastic basis \( \{(\Omega, \mathcal{F}, \mathbb{P}), (\mathcal{F}_t)_{0 \leq t \leq T}, B\} \) starting with the same initial condition \( u_0 \), then

\[
u_1 = u_2 \quad \mathbb{P} \text{-a.s. for all } t \in [0, T].
\]

**Remark 1.** If we replace \( g(u) \) in (1) by \( g(t, u) \) and assume that \( g(t, u) \) is nonlinear measurable mapping defined on \( [0, T] \times L^2(I) \) taking values on \( L^2(I) \), it is continuous with respect to \( u \) and there exists a constant \( C \) such that

\[
\|g(t, u)\|_{L^2(I)} \leq C(1 + \|u\|_{L^2(I)}) \quad \forall t \in [0, T] \quad \forall u \in L^2(I),
\]

\[
\|g(t, u_1) - g(t, u_2)\|_{L^2(I)} \leq C\|u_1 - u_2\|_{L^2(I)} \quad \forall u_1, u_2 \in L^2(I),
\]

the conclusion in Theorem 1.2 also holds.

**Remark 2.** Theorem 1.2 is established by the compactness method combines the Galerkin approximation scheme with sharp compactness results in function spaces of Sobolev type due to Simon and some celebrated probabilistic compactness results of Prokhorov and Skorokhod.

Asymptotic behavior of the weak martingale solutions for the stochastic nonclassical diffusion equations as \( \varepsilon \to 0 \) can be described by the following results.

**Theorem 1.3.** Let assumption (A) be satisfied, \( T > 0 \) and \( u_0 \in H^1_0(I) \). If
\[
\{(\Omega^\varepsilon, \mathcal{F}^\varepsilon, \mathbb{P}^\varepsilon), (\mathcal{F}^\varepsilon_t)_{0 \leq t \leq T}, u^\varepsilon, B^\varepsilon\}_{\varepsilon \in (0, \frac{1}{2}]} \text{ are the weak martingale solutions of problem } (1),
\]

there exists a subsequence \( \{\varepsilon_i\} \subset (0, \frac{1}{2}] \) with \( \varepsilon_i \to 0 \) as \( i \to \infty \), a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) and random variables \( (\bar{u}^\varepsilon, \bar{B}^\varepsilon) \), \( (u, B) \) on \( (\Omega, \mathcal{F}, \mathbb{P}) \) with values in \( L^2(0, T; L^2(I)) \times C([0, T]; \mathbb{R}) \) such that

\[
\mathcal{L}((\bar{u}^\varepsilon, \bar{B}^\varepsilon)) = \mathcal{L}(u^\varepsilon, B^\varepsilon)
\]
and the following convergences hold for any $1 \leq p < \infty$:

\[ \tilde{u}^i \to u \text{ strongly in } L^2(\Omega, L^2(0, T; L^2(I))), \]
\[ \tilde{u}^i \to u \text{ weakly in } L^p(\Omega, L^2(0, T; H^1(I))), \]
\[ \tilde{u}^i \to u \text{ weakly star in } L^p(\Omega, L^\infty(0, T; L^2(I))), \]
\[ \tilde{B}^i \to B \text{ in } C([0, T]; \mathbb{R}^1) \ P - a.s., \]

as $i \to \infty$ and \( \{ (\Omega, \mathcal{F}, \mathbb{P}), (\mathcal{F}_t)_{0 \leq t \leq T}, u, B \} \) is a weak martingale solution of problem

\[ \begin{cases} \frac{du}{dt} + (-u_{xx} + u^3 - u)dt = g(u)dB & \text{in } I \times (0, T) \\ u(0, t) = 0 = u(1, t) & \text{in } (0, T) \\ u(0) = u_0 & \text{in } I. \end{cases} \] (5)

**Remark 3.** If we replace \( g(u) \) in (1) by \( g(t, u) \) and assume that \( g(t, u) \) is nonlinear measurable mapping defined on \([0, T] \times L^2(I)\) taking values on \(L^2(I)\), it is continuous with respect to \( u \) and there exists a constant \( C \) such that

\[
\begin{align*}
\|g(t, u)\|_{L^2(I)} & \leq C(1 + \|u\|_{L^2(I)}) \quad \forall t \in [0, T] \ \forall u \in L^2(I), \\
\|g(t, u_1) - g(t, u_2)\|_{L^2(I)} & \leq C\|u_1 - u_2\|_{L^2(I)} \quad \forall u_1, u_2 \in L^2(I),
\end{align*}
\]

the conclusion in Theorem 1.3 also holds.

**1.4. Weak solution.** Next, we consider another kind of solution to (1).

**Definition 1.4.** A stochastic process \( u \) is said to be a weak solution of (1) if

\[
\begin{align*}
u & \in L^2(I) \text{-valued and } \mathcal{F}_t \text{-measurable for each } t \in [0, T], \\
u & \in L^2(\Omega; L^2([0, T]; L^2(I))), \\
u(0) & = u_0
\end{align*}
\]

and

\[
\begin{align*}
u(t, \varphi) - \varepsilon(u(t), \varphi_{xx}) & = (u_0, \varphi) - \varepsilon(u_0, \varphi_{xx}) + \int_0^t (u(s), \varphi_{xx})ds - \int_0^t (u^3 - u, \varphi)ds + \int_0^t (g(s), \varphi)dB(s)
\end{align*}
\]

holds for all \( t \in [0, T] \) and all \( \varphi \in H^2(I) \cap H^1_0(I) \), for almost all \( \omega \in \Omega \).

**Remark 4.** The weak solution of SPDEs has been discussed in [11].

**Theorem 1.5.** Let assumption (B) be satisfied, \( T > 0 \) and \( u_0 \in H^2(I) \cap H^1_0(I) \). For any \( \varepsilon \in (0, \frac{1}{4}] \), there exists a unique weak solution \( u^\varepsilon(t) \) to (1) in \( L^2(\Omega; C([0, T]; H^2(I) \cap H^1_0(I))) \) and for any \( 1 \leq p < \infty \), there exists a constant \( C(p, L, T, I, u_0) \) such that

\[
\begin{align*}
\mathbb{E} \sup_{0 \leq t \leq T} \|u^\varepsilon(t)\|_{L^2(I)}^2 + \mathbb{E} (\int_0^T \|u^\varepsilon_s\|_{L^2(I)}^2 dt)^p + \mathbb{E} (\int_0^T \int_I u^\varepsilon x dx dt)^p + \mathbb{E} (\int_0^T \varepsilon\|u^\varepsilon_x\|_{L^2(I)}^2 dt)^p \\
& \leq C(p, L, T, I, u_0).
\end{align*}
\] (7)

Moreover, there exists a constant \( C(L, T, I, u_0) \) such that

\[
\begin{align*}
\mathbb{E} \sup_{0 \leq t \leq T} (\|u^\varepsilon_x(t)\|^2_{L^2(I)} + \varepsilon\|u^\varepsilon_{xx}(t)\|^2_{L^2(I)}) + \mathbb{E} (\int_0^T \|u^\varepsilon_{xx}\|_{L^2(I)}^2 dt \leq C(L, T, I, u_0).
\end{align*}
\] (8)

**Remark 5.** Since nonlinear terms \( u^3 - u \) are not Lipschitz continuous, we will use a truncation argument which will lead to a local existence result. Then via some a priori estimates we obtain that the solution is also global.

Asymptotic behavior of the weak solutions for the stochastic nonclassical diffusion equations as \( \varepsilon \to 0 \) can be described by the following results.
Theorem 1.6. Let assumption (B) be satisfied, $T > 0$ and $u_0 \in H^2(I) \cap H^4_0(I)$. For any $\varepsilon \in (0, \frac{1}{2}]$, if $u^\varepsilon$ is the weak solution to (1) and $z$ is the weak solution to

\[
\begin{aligned}
dz + (-z_{xx} + z^3 - z)dt &= g(z)dB \\
|z(0, t)| &= 0 = z(1, t) \\
z(0) &= u_0
\end{aligned}
\]

then $u^\varepsilon$ converges in probability to $z$ in $L^2(0,T;H^1(I))$ as $\varepsilon \to 0$, namely, for any $\delta > 0$, we have

\[
\lim_{\varepsilon \to 0} P(\|u^\varepsilon - z\|_{L^2(0,T;H^1(I))} > \delta) = 0.
\]

This paper is organized as follows. In Section 2, we give some preliminaries and gather all the necessary tools. The existence of weak martingale solutions for (1) is discussed in Section 3, we introduce a Galerkin approximation scheme for the problem (1) and obtain a priori estimates for the approximating solutions, then we prove the crucial result of tightness of Galerkin’ s solutions and apply Prokhorov’s and Skorokhod’s compactness results to prove Theorem 1.2. Section 4 is concerned with the continuity of weak martingale solutions for (1) as $\varepsilon \to 0$. We derive the results of the tightness of the corresponding probability measures and perform the passage to the limit which establishes the convergence of weak martingale solutions. In Section 5, applying the Picard iteration method to the corresponding truncated equation, we give the local existence of weak solutions to (1). Then, the energy estimate shows that the weak solution is also global in time. Moreover, we obtain the uniform estimates for the solution of (1) which are independent of the parameter $\varepsilon$. Section 6 is concerned with the continuity of weak solutions for (1) as $\varepsilon \to 0$. We derive tightness property of weak solutions in $L^2(0,T;H^1(I))$ and perform the passage to the limit which establishes the convergence of weak solutions.

2. Preliminary. This section is devoted to some preliminaries for the proof of Theorem 1.2–Theorem 1.6.

2.1. Some tools. The following compactness results is important for tightness property of Galerkin solutions.

Lemma 2.1. (See [28, Theorem 5]) Let $X$, $B$ and $Y$ be some Banach spaces such that $X$ is compactly embedded into $B$ and let $B$ be a subset of $Y$. For any $1 \leq p, q \leq \infty$, let $V$ be a set bounded in $L^p(0,T;X)$ such that

\[
\lim_{\theta \to 0} \int_0^{T-\theta} \|v(t+\theta) - v(t)\|_Y^p dt = 0,
\]

uniformly for all $v \in V$. Then $V$ is relatively compact in $L^p(0,T;B)$.

According to Lemma 2.1, we can obtain the following compactness result.

Corollary 1. Let $X$, $B$ and $Y$ satisfy the same assumptions in Lemma 2.1 and $\mu_m, \nu_m$ be two sequences which converge to zero as $m \to \infty$. Then

\[
\mathcal{Z} = \left\{ q \in L^2(0,T;X) \cap L^\infty(0,T;B) \mid \sup_m \frac{1}{\nu_m} \sup_{|q| \leq \mu_m} \left( \int_0^{T-\theta} \|q(t+\theta) - q(t)\|_Y^2 dt \right)^{\frac{1}{2}} < +\infty \right\}
\]

in $L^2(0,T;B)$ is compact.

Remark 6. The above compactness result plays a crucial role in the proof of the tightness of the probability measures generated by the sequence $\{u^\varepsilon\}_{\varepsilon > 0}$.
Now we introduce several spaces which will be used in the next section. Let\( \mu_m, \nu_m \) be two sequences that defined in Corollary 1.

- The space \( X^1_{\mu_m, \nu_m} \) is a Banach space with the norm
  \[
  \| y \|_{X^1_{\mu_m, \nu_m}} = \sup_{0 \leq t \leq T} \| y(t) \|_{L^2(I)} + \left( \int_0^T \| y(t) \|_{H^1(I)}^2 dt \right)^{\frac{1}{2}} + \sup_m \frac{1}{\nu_m} \sup_{|\theta| \leq \mu_m} \int_0^{T-\theta} \| y(t + \theta) - y(t) \|_{H^{-1}(I)}^2 dt.
  \]
  \( X^1_{p,\mu_m, \nu_m} \) is a space consist of all random variables \( y \) on \( (\Omega, \mathcal{F}, \mathbb{P}) \) which satisfy
  \[
  \mathbb{E} \sup_{0 \leq t \leq T} \| y(t) \|_{L^2(I)}^{2p} < \infty, \quad \mathbb{E} \left( \int_0^T \| y(t) \|_{H^1(I)}^2 dt \right)^{\frac{2p}{2}} < \infty,
  \]
  \[
  \mathbb{E} \sup_m \frac{1}{\nu_m} \left( \sup_{|\theta| \leq \mu_m} \int_0^{T-\theta} \| y(t + \theta) - y(t) \|_{H^{-1}(I)}^2 dt \right)^{\frac{1}{2}},
  \]
  where \( \mathbb{E} \) denotes the mathematical expectation with respect to the probability measure \( \mathbb{P} \).

- The space \( Y^2_{p,\mu_m, \nu_m} \) is a Banach space.
  \[
  \| y \|_{Y^2_{\mu_m, \nu_m}} = \left( \mathbb{E} \sup_{0 \leq t \leq T} \| y(t) \|_{H^1(I)}^2 \right)^{\frac{1}{2p}} + \left( \mathbb{E} \left( \int_0^T \| y(t) \|_{H^2(I)}^2 dt \right)^{\frac{p}{2}} \right)^{\frac{1}{2}} + \mathbb{E} \sup_m \frac{1}{\nu_m} \left( \sup_{|\theta| \leq \mu_m} \int_0^{T-\theta} \| y(t + \theta) - y(t) \|_{L^2(I)}^2 dt \right)^{\frac{1}{2}},
  \]
  \( X^2_{p,\mu_m, \nu_m} \) is a space consist of all random variables \( y \) on \( (\Omega, \mathcal{F}, \mathbb{P}) \) which satisfy
  \[
  \mathbb{E} \sup_{0 \leq t \leq T} \| y(t) \|_{H^1(I)}^2 < \infty, \quad \mathbb{E} \left( \int_0^T \| y(t) \|_{H^2(I)}^2 dt \right)^{\frac{p}{2}} < \infty,
  \]
  \[
  \mathbb{E} \sup_m \frac{1}{\nu_m} \left( \sup_{|\theta| \leq \mu_m} \int_0^{T-\theta} \| y(t + \theta) - y(t) \|_{L^2(I)}^2 dt \right)^{\frac{1}{2}},
  \]
  where \( \mathbb{E} \) denotes the mathematical expectation with respect to the probability measure \( \mathbb{P} \).

\( X^2_{p,\mu_m, \nu_m} \) is a Banach space.
In order to pass from martingale to pathwise solutions we make essential use of an elementary but powerful characterization of convergence in probability as given in [17].

**Lemma 2.2. (Gyöngy-Krylov Theorem)** (See [17, Lemma 1.1], [23, Proposition 6.3])

Let \( E \) be a Polish space equipped with the Borel \( \sigma \)-algebra. A sequence of \( E \)-valued random element \( z_n \) converges in probability if and only if for every pair of subsequences \( z_{l}, z_{m} \) there exists a subsequence \( w_k = (z_{l(k)}, z_{m(k)}) \) converging weakly to a random element \( w \) supported on the diagonal \( \{(x, y) \in E \times E : x = y \} \).

Prokhorov’s Theorem and Skorohod’s Theorem will be used to establish the tightness of \( u^\varepsilon \). The following two lemmas will play crucial roles in the proof of Theorem 1.5.

**Lemma 2.3 (Prokhorov’s Theorem).** A sequence of measures \( \{\mu_n\} \) on \((E, B(E))\) is tight if and only if it is relatively compact, that is there exists a subsequence \( \{\mu_{n_k}\} \) which weakly converges to a probability measure \( \mu \).

**Lemma 2.4 (Skorohod’s Theorem).** For an arbitrary sequence of probability measures \( \{\mu_n\} \) on \((E, B(E))\) weakly converges to a probability measure \( \mu \), there exists a probability space \((\Omega, F, P)\) and random variables \( \xi, \xi_1, \ldots, \xi_n, \ldots \) with values in \( E \) such that the probability law of \( \xi_n \), \( L(A) = P\{\omega \in \Omega : \xi_n(\omega) \in A\} \), for all \( A \in F \), is \( \mu_n \), the probability law of \( \xi \) is \( \mu \), and \( \lim_{n \to \infty} \xi_n = \xi \), \( P \)-a.s.

**Lemma 2.5. (See [25, Theorem 8.94])** Let \( G \) be a bounded domain of \( \mathbb{R}^k (k \geq 1) \) with smooth boundary, there exists a set of positive real numbers \( \{\lambda_n\}_{n \in \mathbb{N}} \) such that the corresponding solutions \( \{e_n\}_{n \in \mathbb{N}} \) of the problem

\[
\begin{align*}
-\Delta e_n &= \lambda_n e_n \quad \text{in } G \\
  e_n(x) &= 0 \quad \text{on } \partial G
\end{align*}
\]

form a basis in \( H^2(G) \cap H^1_0(G) \), which is orthonormal in \( L^2(G) \).

**Lemma 2.6.** Let \( G \) be a bounded domain of \( \mathbb{R}^k (k \geq 1) \) with smooth boundary, if \( u \) is the solution to the following equation

\[
\begin{align*}
 u - \varepsilon \Delta u &= f \quad \text{in } G \\
  u &= 0 \quad \text{on } \partial G
\end{align*}
\]

where \( \varepsilon \in (0, 1) \). Then

\[
\|u\|_{H^{-1}(G)} \leq \|f\|_{H^{-1}(G)},
\]

Proof. Let

\[
u = \sum_{n=1}^{\infty} u_n e_n, \quad f = \sum_{n=1}^{\infty} f_n e_n,
\]

then we have

\[
u_n + \varepsilon \lambda_n u_n = f_n,
\]

according to [29, P57], we have

\[
\|u\|_{H^{-1}(G)} = \sum_{n=1}^{\infty} \lambda_n^{-1} u_n^2 = \sum_{n=1}^{\infty} \lambda_n^{-1} \left( \frac{1}{1 + \varepsilon \lambda_n} \right)^2 f_n^2 \leq \sum_{n=1}^{\infty} \lambda_n^{-1} f_n^2 = \|f\|_{H^{-1}(G)}^2.
\]

\[ \square \]
2.2. The linear stochastic nonclassical diffusion equations. This section is devoted to some preliminaries for the proof of Theorem 1.5. In this section, we let $G$ be a bounded domain of $\mathbb{R}^k (k \geq 1)$ with smooth boundary. We will use the results in this subsection with $k = 1$ in Section 5.

**Definition 2.7.** A stochastic process $u$ is said to be a solution of
\begin{equation}
\begin{cases}
d(u - \varepsilon \Delta u) + (-\Delta u + f)dt = gdB \\
\quad in \ G \times (0, T) \\
u(x, t) = 0 \\
u(0) = u_0
\end{cases}
\end{equation}
if
\begin{equation}
\begin{align*}
u(t, \varphi) = & (u_0, \varphi) - \varepsilon \int_0^t \langle u(s, \Delta \varphi) - \int_0^s (f(s), \varphi)ds + \int_0^s (g(s), \varphi)dB(s), \varphi \rangle ds + \int_0^t (f(s), \varphi)ds + \int_0^t (g(s), \varphi)dB(s) \\
& = (u_0, \varphi) - \varepsilon \int_0^t \langle \nabla u(s), \Delta \varphi \rangle ds - \int_0^t \langle f(s), \varphi \rangle ds + \int_0^t \langle g(s), \varphi \rangle dB(s)
\end{align*}
\end{equation}
holds for all $t \in [0, T]$ and all $\varphi \in H^2(G) \cap H^1_0(G)$, for almost all $\omega \in \Omega$.

**Proposition 1.** For any $\varepsilon \in (0, \frac{1}{2}]$, there exists a constant $C$ independent of $\varepsilon$.

1) If $u_0 \in L^2(\Omega; L^2(G))$, $f \in L^2(\Omega; L^2(0, T; H^{-1}(G)))$, $g \in L^2(\Omega; L^2(0, T; L^2(G)))$, then (11) has a unique solution $u \in L^2(\Omega; C([0, T]; L^2(G)))$ and
\begin{equation}
\mathbb{E} \sup_{0 \leq t \leq T} \|u(t)\|_{L^2(G)}^2 \leq C \mathbb{E} \|u_0\|_{L^2(G)}^2 + \mathbb{E} \int_0^T \|f(t)\|_{H^{-1}(G)}^2 dt + \mathbb{E} \int_0^T \|g(t)\|_{L^2(G)}^2 dt.
\end{equation}

2) If $u_0 \in L^2(\Omega; H^1_0(G))$, $f \in L^2(\Omega; L^2(0, T; H^{-1}(G)))$, $g \in L^2(\Omega; L^2(0, T; L^2(G)))$, then (11) has a unique solution $u \in L^2(\Omega; C([0, T]; H^1_0(G))) \cap L^2(\Omega, L^2(0, T, H^1(G)))$ and
\begin{equation}
\begin{align*}
\mathbb{E} \sup_{0 \leq t \leq T} (\|u(t)\|_{L^2(G)}^2 + \varepsilon \|\nabla u(t)\|_{L^2(G)}^2) + \mathbb{E} \int_0^T \|\nabla u(t)\|_{L^2(G)}^2 dt \\
& \leq C \mathbb{E} \|u_0\|_{L^2(G)}^2 + \|\nabla u_0\|_{L^2(G)}^2 + \mathbb{E} \int_0^T \|f(t)\|_{H^{-1}(G)}^2 dt + \mathbb{E} \int_0^T \|g(t)\|_{L^2(G)}^2 dt.
\end{align*}
\end{equation}
Moreover, it holds that
\begin{equation}
(u(t), \varphi) - \varepsilon \langle \Delta u(t), \varphi \rangle = (u_0, \varphi) - \varepsilon \int_0^t \langle \Delta u(s), \varphi \rangle ds - \int_0^t \langle f(s), \varphi \rangle ds + \int_0^t \langle g(s), \varphi \rangle dB(s)
\end{equation}
for all $t \in [0, T]$ and all $\varphi \in H^1_0(G)$, for almost all $\omega \in \Omega$.

3) If $u_0 \in L^2(\Omega; H^2(G) \cap H^1_0(G))$, $f \in L^2(\Omega; L^2(0, T; L^2(G)))$, $g \in L^2(\Omega; L^2(0, T; H^1(G)))$, then (11) has a unique solution $u \in L^2(\Omega; C([0, T]; H^2(G) \cap H^1_0(G))) \cap L^2(\Omega, L^2(0, T, H^2(G)))$ and
\begin{equation}
\begin{align*}
\mathbb{E} \sup_{0 \leq t \leq T} (\|\nabla u(t)\|_{L^2(G)}^2 + \varepsilon \|\Delta u(t)\|_{L^2(G)}^2) + \mathbb{E} \int_0^T \|\Delta u(t)\|_{L^2(G)}^2 dt \\
& \leq C \mathbb{E} \|\nabla u_0\|_{L^2(G)}^2 + \|\Delta u_0\|_{L^2(G)}^2 + \mathbb{E} \int_0^T \|f(t)\|_{L^2(G)}^2 dt + \mathbb{E} \int_0^T \|g(t)\|_{H^1(G)}^2 dt.
\end{align*}
\end{equation}
Moreover, it holds that
\begin{equation}
(u(t), \varphi) - \varepsilon \langle \Delta u(t), \varphi \rangle = (u_0, \varphi) - \varepsilon \int_0^t \langle \Delta u(s), \varphi \rangle ds - \int_0^t \langle f(s), \varphi \rangle ds + \int_0^t \langle g(s), \varphi \rangle dB(s)
\end{equation}
for all $t \in [0, T]$ and all $\varphi \in L^2(G)$, for almost all $\omega \in \Omega$.

The proof of Proposition 1 is in Appendix.

3. Proof of Theorem 1.2. If there is no danger of confusion, we shall omit the subscript $\varepsilon$, we use $\pi_n$ instead of $\pi_n^\varepsilon$ and $\pi_n$ instead of $\pi_n^\varepsilon$. The proof of the existence of the weak martingale solution is divided into several steps.

**Step 1. Construct the approximate solution.**

Let $\{(\Omega, F, \mathbb{P}), (\mathcal{F}_t)_{0 \leq t \leq T}, \mathcal{B}\}$ be a fixed stochastic basis and $\{\varepsilon_n\}_{n \in N}$ be an orthonormal basis of $L^2(I)$ which was obtained in Lemma 2.5. Set $H_n = \text{Span}\{e_1, e_2, ..., e_n\}$ and let $P_n$ be the $L^2-$orthogonal projection from $L^2(I)$ onto $H_n$.

We set

$$\pi_n(t) = \sum_{k=1}^n c_k^n(t)e_k$$

and it is the solution of the following system of stochastic differential equations

$$\begin{cases}
d(\pi_n - \varepsilon \pi_{nxx}) + (-\pi_{nxx} + P_n \pi_n^3 - \pi_n)dt = P_n g(\pi_n)dB & \text{in } Q \\
\pi_n(0, t) = 0 = \pi_n(1, t), & \text{in } (0, T) \\
\pi_n(x, 0) = P_n u_0 := u_{n0}(x) & \text{in } I
\end{cases}$$

defined on $\{(\Omega, F, \mathbb{P}), (\mathcal{F}_t)_{0 \leq t \leq T}, \mathcal{B}\}$. The mathematical expectation with respect to $\mathbb{P}$ is denoted by $\mathbb{E}$.

It is easy to see that $c_k^n$ satisfies the following system of stochastic differential equations

$$\begin{cases}
d(c_k^n) + \frac{1}{1+\varepsilon k}(\lambda_k c_k^n + (P_n \pi_n^3, e_k) - c_k^n)dt = \frac{1}{1+\varepsilon k}(P_n g(\pi_n), e_k)dB \\
c_k^n(0) = (u_0, e_k)
\end{cases}$$

(18)

By the classical theory of stochastic differential equations, there is a local $\pi_n$ defined on $[0, T_n]$. The following a priori estimates will enable us to prove that $T_n = T$.

**Step 2. A priori estimates.**

**Lemma 3.1.** There exists a positive constant $C$ independent of $\varepsilon$ such that

$$\mathbb{E} \sup_{0 \leq t \leq T} \|\pi_n(t)\|_{L^2(I)}^2 + \varepsilon \|\pi_{nxx}(t)\|_{L^2(I)}^2 + \mathbb{E} \int_0^T (\|\pi_{nxx}(t)\|_{L^2(I)}^2 + \|\pi_n(t)\|_{L^4(I)}^4)dt \leq C$$

(19)

for any $n \geq 1$.

Proof. Indeed, it follows from Itô’s rule that

$$d(c_k^n)^2 = 2c_k^n dc_k^n + (dc_k^n)^2 = 2c_k^n \left( -\lambda_k c_k^n + (P_n \pi_n^3, e_k) + c_k^n \right)dt + (P_n g(\pi_n), e_k)dB + \frac{1}{(1+\varepsilon k)^2} \left| (P_n g(\pi_n), e_k) \right|^2 dt,$$

namely, we have

$$\frac{(1+\varepsilon k)c_k^n}{1+\varepsilon k)^2} \left| (P_n g(\pi_n), e_k) \right|^2 dt.$$

Taking the sum on $k$ in (20), following [19, P28] or [15], we get

$$d(\|\pi_n(t)\|_{L^2(I)}^2 + \varepsilon \|\pi_{nxx}(t)\|_{L^2(I)}^2 + 2\|\pi_n\|_{L^4(I)}^4)dt = (2\|\pi_n\|_{L^2(I)}^2 + \sum_{k=1}^n \frac{1}{1+\varepsilon k} \left| (P_n g(\pi_n), e_k) \right|^2 )dt + 2\varepsilon \|\pi_{nxx}\|_{L^2(I)}^2 dt$$

(21)

namely,

$$\|\pi_n(t)\|_{L^2(I)}^2 + \varepsilon \|\pi_{nxx}(t)\|_{L^2(I)}^2 + 2\|\pi_n\|_{L^4(I)}^4 = \|\pi_n(t)\|_{L^2(I)}^2 + \varepsilon \|\pi_{nxx}(t)\|_{L^2(I)}^2 + 2\|\pi_n\|_{L^4(I)}^4 dt + \sum_{k=1}^n \frac{1}{1+\varepsilon k} \left| (P_n g(\pi_n), e_k) \right|^2 ds$$

$$+ 2 \int_0^T (\pi_n, P_n g(\pi_n))dB,$$

(22)
It is easy to see
\[
\mathbb{E} \left| \int_0^t \sum_{k=1}^n \frac{1}{1+\varepsilon\lambda_k} |(P_n g(\overline{\pi}_n), e_k)|^2 \, ds \right| \leq \mathbb{E} \left| \int_0^t \sum_{k=1}^n |(P_n g(\overline{\pi}_n), e_k)|^2 \, ds \right| \\
\leq \mathbb{E} \left| \int_0^t \|P_n g(\overline{\pi}_n)\|^2_{L^2(I)} \, ds \right| \\
\leq C \mathbb{E} \int_0^t \left( 1 + \|\overline{\pi}_n(s)\|^2_{L^2(I)} \right) \, ds.
\]

By the Burkholder-Davis-Gundy inequality and Cauchy inequality, we can obtain that for any \( \delta > 0 \),
\[
\mathbb{E} \sup_{0 \leq s \leq t} \left| \int_0^s (\overline{\pi}_n, P_n g(\overline{\pi}_n)) \, dB \right| = \mathbb{E} \sup_{0 \leq s \leq t} \left| \int_0^s (P_n \overline{\pi}_n, g(\overline{\pi}_n)) \, dB \right| \\
= \mathbb{E} \sup_{0 \leq s \leq t} \left| \int_0^s (\overline{\pi}_n, g(\overline{\pi}_n)) \, dB \right| \\
\leq \delta \mathbb{E} \sup_{0 \leq s \leq t} \|\overline{\pi}_n(s)\|^2_{L^2(I)} + C(\delta) \mathbb{E} \int_0^t \|g(\overline{\pi}_n(s))\|^2_{L^2(I)} \, ds \\
\leq \delta \mathbb{E} \sup_{0 \leq s \leq t} \|\overline{\pi}_n(s)\|^2_{L^2(I)} + C(\delta) \mathbb{E} \int_0^t (1 + \|\overline{\pi}_n(s)\|^2_{L^2(I)}) \, ds.
\]

It follows from (22) that
\[
\mathbb{E} \sup_{0 \leq s \leq t} (\|\overline{\pi}_n(s)\|^2_{L^2(I)} + \varepsilon \|\overline{\pi}_{nx}(s)\|^2_{L^2(I)}) + 2 \mathbb{E} \int_0^t (\|\overline{\pi}_{nx}\|^2_{L^2(I)} + \|\overline{\pi}_n\|^4_{L^4(I)}) \, ds \\
\leq \delta \mathbb{E} \sup_{0 \leq s \leq t} \|\overline{\pi}_n(s)\|^2_{L^2(I)} + C \left( \mathbb{E} \|u_{n0}\|^2_{H^1(I)} + \mathbb{E} \int_0^t (1 + \|\overline{\pi}_n(s)\|^2_{L^2(I)}) \, ds \right).
\]

By choosing \( \delta > 0 \) small enough, yields
\[
\mathbb{E} \sup_{0 \leq s \leq t} (\|\overline{\pi}_n(s)\|^2_{L^2(I)} + \varepsilon \|\overline{\pi}_{nx}(s)\|^2_{L^2(I)}) + \mathbb{E} \int_0^t (\|\overline{\pi}_{nx}\|^2_{L^2(I)} + \|\overline{\pi}_n\|^4_{L^4(I)}) \, ds \\
\leq C \left( \mathbb{E} \|u_{n0}\|^2_{H^1(I)} + \mathbb{E} \int_0^t (1 + \|\overline{\pi}_n(s)\|^2_{L^2(I)}) \, ds \right).
\]

According to Gronwall’s lemma, we obtain that
\[
\mathbb{E} \sup_{0 \leq t \leq T} (\|\overline{\pi}_n(s)\|^2_{L^2(I)} + \|\overline{\pi}_{nx}(s)\|^2_{L^2(I)}) \leq C, \\
\mathbb{E} \int_0^T (\|\overline{\pi}_{nx}\|^2_{L^2(I)} + \|\overline{\pi}_n\|^4_{L^4(I)}) \, dt \leq C.
\]

□

The following result is related to the higher integrability of \( \overline{\pi}_n \).

**Lemma 3.2.** For any \( 1 \leq p < \infty \), there exists a constant \( C_p \), independent of \( \varepsilon \) such that
\[
\mathbb{E} \sup_{0 \leq t \leq T} \left( \|\overline{\pi}_n(s)\|^2_{L^2(I)} + \|\overline{\pi}_{nx}(s)\|^2_{L^2(I)} \right)^{\frac{p}{2}} \leq C_p, \\
\mathbb{E} \left( \int_0^T \left( \|\overline{\pi}_{nx}\|^2_{L^2(I)} + \|\overline{\pi}_n\|^4_{L^4(I)} \right) dt \right)^{\frac{p}{2}} \leq C_p
\]
for any \( n \geq 1 \).
By Itô’s rule, we obtain that
\[ \phi_n(t) = \|\pi_n(t)\|^2_{L^2(I)} + \varepsilon \|\pi_{nx}(t)\|^2_{L^2(I)}, \]
\[ K = (2\|\pi_n\|^2_{L^2(I)} + \sum_{k=1}^{n} \frac{1}{1+\varepsilon\lambda_k} (P_n g(t, \pi_n), e_k)^2) - 2(\|\pi_{nx}\|^2_{L^2(I)} + \|\pi_n\|^4_{L^4(I)}), \]
\[ L = 2(\pi_n, P_n g(t, \pi_n)). \]

Thus we can rewrite (21) as
\[ d\phi_n = Kdt + Ld\mathcal{B}. \]

By Itô’s rule, we obtain that
\[ d\phi_n = \frac{p}{2} \phi_n^{\frac{p-2}{2}} (K + \frac{p-2}{4} \phi_n^{-1} L^2) dt + \frac{\sqrt{p}}{2} \phi_n^{\frac{p-2}{2}} Ld\mathcal{B}, \]
for any \( 2 \leq p < \infty \). Namely, we have
\[ \phi_n(t) = \phi_n^0(t) + \int_0^t \frac{p}{2} \phi_n^{\frac{p-2}{2}} (K + \frac{p-2}{4} \phi_n^{-1} L^2) ds + \int_0^t \frac{\sqrt{p}}{2} \phi_n^{\frac{p-2}{2}} Ld\mathcal{B}. \quad (25) \]

Using the properties of \( g \) and Young’s inequality, we have
\[ \frac{\phi_n^{\frac{p}{2}}}{\phi_n^{\frac{p-2}{2}} K} \leq \frac{\phi_n^{\frac{p}{2}}}{\phi_n^{\frac{p-2}{2}} (2\|\pi_n\|^2_{L^2(I)} + \sum_{k=1}^{n} \frac{1}{1+\varepsilon\lambda_k} (P_n g(\pi_n), e_k)^2) - 2(\|\pi_{nx}\|^2_{L^2(I)} + \|\pi_n\|^4_{L^4(I)}))} \]
\[ \leq \frac{\phi_n^{\frac{p}{2}}}{\phi_n^{\frac{p-2}{2}} (2\|\pi_n\|^2_{L^2(I)} + \|P_n g(\pi_n)\|^2_{L^2(I)})} \]
\[ \leq \frac{\phi_n^{\frac{p}{2}}}{\phi_n^{\frac{p-2}{2}} (1 + \|\pi_n\|^2_{L^2(I)})} \]
\[ \leq C(1 + \phi_n^{\frac{p}{2}}), \]
\[ \phi_n^{\frac{p-2}{2}} \phi_n^{-1} L^2 = C\phi_n^{\frac{p-2}{2}} (\pi_n, P_n g(\pi_n))^2 \]
\[ \leq C\phi_n^{\frac{p-2}{2}} \|\pi_n\|^2_{L^2(I)} \|P_n g(\pi_n)\|^2_{L^2(I)} \]
\[ \leq C\phi_n^{\frac{p-2}{2}} \|\pi_n\|^2_{L^2(I)} (1 + \|\pi_n\|^2_{L^2(I)}) \]
\[ \leq C(1 + \phi_n^{\frac{p}{2}}). \]

According to the Burkholder-Davis-Gundy inequality and Young’s inequality, it can be deduced that
\[ \mathbb{E} \sup_{0 \leq s \leq t} \left| \int_0^s \frac{p}{2} \phi_n^{\frac{p-2}{2}} Ld\mathcal{B} \right| = 2\mathbb{E} \sup_{0 \leq s \leq t} \left| \int_0^s \frac{p}{2} \phi_n^{\frac{p-2}{2}} (\pi_n, P_n g(\pi_n))d\mathcal{B} \right| \]
\[ \leq C\mathbb{E} \left( \int_0^t \phi_n^{\frac{p}{2}} (\pi_n, P_n g(\pi_n))^2 ds \right)^{\frac{1}{2}} \]
\[ \leq C\mathbb{E} \left( \int_0^t \phi_n^{\frac{p}{2}} \|\pi_n\|^2_{L^2(I)} \|P_n g(\pi_n)\|^2_{L^2(I)} ds \right)^{\frac{1}{2}} \]
\[ \leq C\mathbb{E} \left( \int_0^t \phi_n^{\frac{p}{2}} \|\pi_n\|^2_{L^2(I)} \|g(\pi_n)\|^2_{L^2(I)} ds \right)^{\frac{1}{2}} \]
\[ \leq C\mathbb{E} \left( \int_0^t \phi_n^{\frac{p}{2}} \|\pi_n\|^2_{L^2(I)} (1 + \|\pi_n\|^2_{L^2(I)}) ds \right)^{\frac{1}{2}} \]
From the above estimates and (25), by choosing $\delta > 0$ small enough, it holds that

$$\mathbb{E} \sup_{0 \leq s \leq t} \phi_n^p \leq C + C\mathbb{E} \int_0^t \phi_n^2 ds.$$  

According to Gronwall’s lemma and the definition of $\phi_n$, we obtain that

$$\mathbb{E} \sup_{0 \leq s \leq T} (\|\pi_n(s)\|_{L^2(I)}^2 + \varepsilon \|\pi_n(x)\|_{L^2(I)}^2)^{\frac{p}{2}} \leq C_p. \tag{26}$$

In view of (22), there holds

$$\|u_{n0}\|_{H^1(I)}^2 + \int_0^t (\|\pi_n\|_{L^2(I)}^2 + \|\pi_n\|_{L^2(I)}^4) ds$$

$$= \|u_{n0}\|_{H^1(I)}^2 + \int_0^t (\|\pi_n\|_{L^2(I)}^2 + \|\pi_n\|_{L^2(I)}^4) ds + 2 \int_0^t \|\pi_n\|_{L^2(I)}^2 + \sum_{k=1}^n \frac{1}{1 + \varepsilon \lambda_k} |(P_n g(\pi_n), e_k)|^2 ds$$

$$+ 2 \int_0^t (\pi_n, P_n g(\pi_n)) d\mathbb{B}$$

$$\leq \|u_{n0}\|_{H^1(I)}^2 + \int_0^t (\|\pi_n\|_{L^2(I)}^2 + \|P_n g(\pi_n)\|_{L^2(I)}^2) ds$$

$$+ 2 \int_0^t (\pi_n, P_n g(\pi_n)) d\mathbb{B}.$$  

Thus, we have

$$\int_0^T (\|\pi_n\|_{L^2(I)}^2 + \|\pi_n\|_{L^2(I)}^4) dt \leq C \left( \|u_{n0}\|_{H^1(I)}^2 + \int_0^T (1 + \|\pi_n\|_{L^2(I)}^2) dt + (\int_0^T (\pi_n, P_n g(\pi_n)) d\mathbb{B}) \right).$$

then, for any $2 \leq p < \infty$, it holds that

$$\left( \int_0^T \|\pi_n\|_{L^2(I)}^2 + \|\pi_n\|_{L^2(I)}^4 \right)^{\frac{p}{2}}$$

$$\leq C_p \left( \|u_{n0}\|_{H^1(I)}^p + \left( \int_0^T (1 + \|\pi_n(s)\|_{L^2(I)}^2) ds \right)^{\frac{p}{2}} + \int_0^T (\pi_n, P_n g(\pi_n)) d\mathbb{B} \right)^{\frac{p}{2}}.$$
Proof. We set Remark 7. In the above lemma, for any \( \{n\} \), the Galerkin solution is extended to 0 outside \([0, T]\). According to (26), it holds that

\[
\mathbb{E} \left[ \int_0^T (\pi_n, P_n g(\pi_n)) d\mathcal{B} \right] \leq \mathbb{E} \sup_{0 \leq t \leq T} \left| \int_0^t (\pi_n, P_n g(\pi_n)) d\mathcal{B} \right|^\frac{p}{2} \leq C \mathbb{E} \left( \int_0^T (\pi_n, P_n g(\pi_n))^2 dt \right)^{\frac{p}{4}} \leq C \mathbb{E} \left( \int_0^T (1 + \phi_n^2) dt \right)^{\frac{p}{4}}.
\]

Thus

\[
\mathbb{E} \left( \int_0^T \left( \|\pi_n\|_{L^2(I)}^2 + \|\pi_n\|_{L^4(I)}^4 \right) dt \right)^{\frac{p}{2}} \leq C \mathbb{E} |u_{n_0}|_{H^1(I)}^p + C \mathbb{E} \left( \int_0^T (1 + \phi_n) dt \right)^{\frac{p}{2}} + C \mathbb{E} \left( \int_0^T (1 + \phi_n^2) dt \right)^{\frac{p}{4}} \leq C \mathbb{E} |u_{n_0}|_{H^1(I)}^p + C T^{\frac{p}{4}} \mathbb{E} \left( 1 + \sup_{0 \leq t \leq T} \phi_n \right)^{\frac{p}{4}} + C T^{\frac{p}{4}} \mathbb{E} \left( 1 + \sup_{0 \leq t \leq T} \phi_n^2 \right) \leq C (1 + \mathbb{E} |u_{n_0}|_{H^1(I)}^p + \mathbb{E} \sup_{0 \leq t \leq T} \phi_n^2).
\]

According to (26), it holds that

\[
\mathbb{E} \left( \int_0^T (\|\pi_{nx}\|_{L^2(I)}^2 + \|\pi_n\|_{L^4(I)}^4) dt \right)^{\frac{p}{2}} \leq C_p.
\]

Case II: \( 1 \leq p < 2 \). This case can be obtained from Case I and the Young inequality.

The next estimate is very important for the proof of the tightness of the law of the Galerkin solution \( \{\pi_n\}_{n \geq 1} \).

**Lemma 3.3.** There exists a positive constant \( C \) independent of \( \varepsilon \) such that

\[
\mathbb{E} \sup_{0 \leq |\theta| \leq \delta} \left( \int_0^{T-\theta} \|\pi_n(t + \theta) - \pi_n(t)\|_{H^{-1}(I)}^2 dt \right) \leq C \delta,
\]

for any \( 0 < \delta \leq 1 \).

**Remark 7.** In the above lemma, \( \pi_n \) is extended to 0 outside \([0, T]\).

**Proof.** We set \( \pi_n(t) = (\pi_n - \varepsilon \pi_{nxx})(t) \), it is easy to see that

\[
\pi_n(t + \theta) - \pi_n(t) = \int_t^{t+\theta} \pi_{nxx}(s) ds - \int_t^{t+\theta} (P_n \pi_n^3 - \pi_n)(s) ds + \int_t^{t+\theta} P_n g(\pi_n(s)) d\mathcal{B},
\]

which implies

\[
\|\pi_n(t + \theta) - \pi_n(t)\|_{H^{-1}(I)} \leq \|\int_t^{t+\theta} \pi_{nxx}(s) ds\|_{H^{-1}(I)} + \|\int_t^{t+\theta} (P_n \pi_n^3 - \pi_n)(s) ds\|_{H^{-1}(I)} + \|\int_t^{t+\theta} P_n g(\pi_n(s)) d\mathcal{B}\|_{H^{-1}(I)} \leq \int_t^{t+\theta} \|\pi_{nxx}(s)\|_{H^{-1}(I)} ds + \int_t^{t+\theta} \|\pi_n^3(s) - \pi_n(s)\|_{H^{-1}(I)} ds + \|\int_t^{t+\theta} P_n g(\pi_n(s)) d\mathcal{B}\|_{H^{-1}(I)}.
\]

(28)
Taking the square in both side of (28), we have
\[
\|\pi(t + \theta) - \pi(t)\|_{H^{1-1}(I)}^2 \\
\leq (\int_{t}^{t+\theta} \|\pi_{xx}(s)\|_{H^{1-1}(I)} ds + \int_{t+\theta}^{t} \|\pi_{xx}(s)\|_{H^{1-1}(I)} ds )^2 + C \theta \int_{t}^{t+\theta} \|\pi_{xx}(s)\|_{H^{1-1}(I)}^2 ds \\
\leq C \theta \int_{t}^{t+\theta} \|\pi_{xx}(s)\|_{H^{1-1}(I)}^2 ds + C \|\pi(t + \theta) - \pi(t)\|_{H^{1-1}(I)}^2.
\]

We can infer from (23) and (24) that
\[
E \int_{0}^{T} \int_{t}^{t+\delta} \|\pi_{xx}(s)\|_{H^{1-1}(I)} ds dt \leq \delta E \int_{0}^{T} \|\pi_{xx}(t)\|_{L^2(I)}^2 dt \leq C \delta,
\]
\[
E \int_{t}^{T} \int_{t}^{t+\delta} \|\pi_{xx}(s)\|_{H^{1-1}(I)} ds dt \leq E \int_{t}^{T} \int_{t}^{t+\delta} \|\pi_{xx}(s)\|_{H^{1-1}(I)} ds dt \leq \delta E \int_{t}^{T} \|\pi_{xx}(s)\|_{H^{1-1}(I)}^2 ds dt \\
\leq C \delta \sqrt{E \int_{0}^{T} \|\pi_{xx}\|_{L^2(I)}^2 dt}^2 + E \sup_{0 \leq t \leq T} \|\pi_{xx}\|_{L^2(I)}^2 + E \int_{0}^{T} \|\pi_{xx}\|_{L^2(I)}^2 dt \\
\leq C \delta.
\]

By the Burkholder-Davis-Gundy inequality and Young’s inequality, we have
\[
E \sup_{0 \leq |\theta| \leq \delta} \int_{t}^{T} \|P_{n} g(\pi(s)) d\theta\|_{L^2(I)}^2 dt \leq E \sup_{0 \leq |\theta| \leq \delta} \int_{t}^{T} \|P_{n} g(\pi(s)) d\theta\|_{L^2(I)}^2 dt \\
\leq E \int_{t}^{T} \sup_{0 \leq |\theta| \leq \delta} \int_{t}^{t+\theta} \|P_{n} g(\pi(s)) d\theta\|_{L^2(I)}^2 dt \\
= \int_{t}^{T} E \sup_{0 \leq |\theta| \leq \delta} \|P_{n} g(\pi(s)) d\theta\|_{L^2(I)}^2 dt \\
\leq C E \int_{t}^{T} \|P_{n} g(\pi(s)) \|_{L^2(I)}^2 ds dt \\
\leq C \delta E \int_{t}^{T} \|P_{n} g(\pi(t)) \|_{L^2(I)}^2 dt \\
\leq C \delta E \int_{t}^{T} \|g(\pi(t)) \|_{L^2(I)}^2 dt \\
\leq C \delta E \int_{t}^{T} (1 + \|\pi_{xx}\|_{L^2(I)}^2) dt \\
\leq C \delta.
\]

It follows from (28)-(30) that
\[
E \sup_{0 \leq |\theta| \leq \delta} \int_{t}^{T} \|\pi(t+\theta) - \pi(t)\|_{H^{1-1}(I)}^2 dt \leq E \sup_{0 \leq |\theta| \leq \delta} \int_{t}^{T} \|\pi(t+\theta) - \pi(t)\|_{H^{1-1}(I)}^2 dt \leq C \delta.
\]

By Lemma 2.6, we have
\[
\|\pi_{n}(t)\|_{H^{1-1}(I)} \leq \|\pi_{n}(t)\|_{H^{1-1}(I)},
\]

thus, we have (27).

**Step 3. Tightness property of Galerkin solutions.**

We may rewrite Lemma 2.1 in the following more convenient form.
By the same way as in [26, P919], according to the priori estimates (19)(23)(24)(27), we obtain that

**Lemma 3.4.** For any $1 \leq p < \infty$ and for any sequences $\mu_m, \nu_m$ converging to 0 such that the series $\sum_{m=1}^{\infty} \frac{\mu_m^2}{\nu_m}$ converges, $\{\pi_n : n \in \mathbb{N}\}$ is bounded in $X_{p,\mu_m,\nu_m}^1$ (the explicit definition of the space $X_{p,\mu_m,\nu_m}^1$ can be found in Section 2) for any $m$.

Let

$$X = C([0,T];\mathbb{R}^1) \times L^2(0,T;L^2(I))$$

and $\mathcal{B}(X)$ be the $\sigma$–algebra of the Borel sets of $X$.

For each $n$, let $\Phi_n$ be the map

$$\Phi_n : \Omega \to X$$

$$\omega \to (\mathcal{B}(\omega), \pi_n(\omega)),$$

and $\Pi_n$ be a probability measure on $(X, \mathcal{B}(X))$ defined by

$$\Pi_n(A) = \mathbb{P}(\Phi_n^{-1}(A)), A \in \mathcal{B}(X).$$

**Proposition 2.** The family of probability measures $\{\Pi_n : n = 1, 2, 3, \ldots\}$ is tight in $X$.

**Proof.** For any $\rho > 0$, we should find the compact subsets

$$\Sigma_\rho \subset C([0,T];\mathbb{R}^1), Y_\rho \subset L^2(0,T;L^2(I)),$$

such that

$$\mathbb{P}(\omega : \mathcal{B}(\omega, \cdot) \notin \Sigma_\rho) \leq \frac{\rho}{2},$$

$$\mathbb{P}(\omega : \pi_n(\omega, \cdot) \notin Y_\rho) \leq \frac{\rho}{2}.$$  (31)

(32)

Noting the formula

$$\mathbb{E}|\mathcal{B}(t_2) - \mathcal{B}(t_1)|^{2i} = (2i - 1)! (t_2 - t_1)^i, i = 1, 2, ...$$

we define

$$\Sigma_\rho := \left\{ B(\cdot) \in C([0,T];\mathbb{R}^1) : \sup_{t_1, t_2 \in [0,T],|t_2 - t_1| \leq \frac{1}{n^6}} n|B(t_2) - B(t_1)| \leq \rho \right\}$$

where $n \in \mathbb{N}$, $\rho$ is a constant depending on $\rho$ and will be chosen later.

By the Chebyshev inequality, we get

$$\mathbb{P}(\omega : \mathcal{B}(\omega, \cdot) \notin \Sigma_\rho) \leq \mathbb{P}\left( \bigcup_{n} \left\{ \omega : \sup_{t_1, t_2 \in [0,T],|t_2 - t_1| \leq \frac{1}{n^6}} |\mathcal{B}(t_2) - \mathcal{B}(t_1)| > \frac{L_\rho}{n} \right\} \right)$$

$$\leq \sum_{n=1}^{\infty} \sum_{i=0}^{n^6-1} \left( \frac{n}{L_\rho} \right)^4 \mathbb{E} \sup_{\frac{t}{n^6} \leq |t| \leq \frac{(i+1)}{n^6}} |\mathcal{B}(t) - \mathcal{B}(\frac{t}{n^6})|^4$$

$$\leq C \sum_{n=1}^{\infty} \left( \frac{n}{L_\rho} \right)^4 (Tn^{-6})^2 n^6$$

$$= \frac{C}{L_\rho} \sum_{n=1}^{\infty} \frac{1}{n^2}.$$
we choose \( L_ρ^4 = 2Cρ^{-1} \sum_{n=1}^{∞} \frac{1}{n^2} \) to get (31).

Let \( Y_ρ \) be a ball of radius \( M_ρ \) in \( Y_{1,τ}^1 \) (the explicit definition of the space \( Y_{1,τ}^1 \) can be found in Section 2), centered at zero, namely
\[
Y_ρ = \{ u \in Y_{1,τ}^1 \mid ||u||_{Y_{1,τ}^1} \leq M_ρ \}.
\]

From Corollary 1, \( Y_ρ \) is a compact subset of \( L^2(0, T; L^2(I)) \), and
\[
\mathbb{P}(ω : ω(\cdot) \notin Y_ρ) \leq \mathbb{P}(ω : ||ω||_{Y_{1,τ}^1} > M_ρ) \leq \frac{1}{M_ρ} \mathbb{E}[||ω||_{Y_{1,τ}^1}^2] \leq \frac{C}{M_ρ},
\]
choosing \( M_ρ = 2Cρ^{-1} \), we get (32).

It follows from (31) and (32) that
\[
Π_n(Σ_ρ × Y_ρ) \geq 1 - ρ,
\]
for any \( n \geq 1 \).

Thus, the family of probability measures \( \{ Π_n : n = 1, 2, 3, \ldots \} \) is tight in \( X \).  □

**Step 4. Applications of Prokhorov Theorem and Skorokhod Theorem.**

By Lemma 2.3, we can find a probability measure \( Π \) and extract a subsequence from \( Π_n \) such that
\[
Π_n(0) → Π
\]
weakly in \( X \).

By Lemma 2.4, there exists a probability space \( (Ω, F, P) \) and random variables \( (u_n, B_{n,i}) \), \( (u, B) \) on \( (Ω, F, P) \) with values in \( X \) such that the probability law of \( (u_n, B_{n,i}) \) is \( Π_n \). Furthermore,
\[
(u_n, B_{n,i}) → (u, B) \quad \text{in} \quad X \quad \text{P-a.s.}
\]
and the probability law of \( (u, B) \) is \( Π \).

Set
\[
F_t = \sigma\{u(s, B(s)) : s ∈ [0, t]\}.
\]
By the idea in [26, 27], we can know \( B(t) \) is a \( F_t \)-standard Wiener process.

We claim that \( (u_n, B_{n,i}) \) verifies the following \( dt \otimes dP \)-almost everywhere:
\[
[(u_n(t), ϕ) + ε(u_n, t, ϕ_x)] - [(u_n, 0, ϕ) + ε(u_n, 0, ϕ_x)] = \int_0^t (u_n, 0, ϕ_x + P_n(u_n - u_n, ϕ)) ds
\]
for all \( ϕ \in \mathcal{H}_1^1(I) \).

Indeed, we set
\[
\xi_n(t) = [\pi_n(t) - ε\pi_n xx(t)] - [u_n0 - εu_n0xx] + \int_0^t (-\pi_n xx + P_n u_n^3 - u_n) ds - \int_0^t P_n g(\pi_n) dB,
\]
\[
η_n(t) = [u_n(t) - εu_n, xx(t)] - [u_n, 0 - εu_n, 0xx] + \int_0^t (-u_n, xx + P_n u_n^3 - u_n) ds - \int_0^t P_n g(\eta_n) dB_n,
\]
and
\[
X_n = \int_0^T \|\xi_n(t)\|_{H^{-1}(I)} dt,
\]
\[
Y_n = \int_0^T \|\eta_n(t)\|_{H^{-1}(I)} dt.
\]

It is easy to see almost surely \( X_n = 0 \), hence, in particular, \( \mathbb{E}\left[\frac{X_n}{1 + X_n}\right] = 0 \).

Next, we show that
\[
\mathbb{E}\left[\frac{Y_n}{1 + Y_n}\right] = 0,
\]
which will imply (33). Indeed, motivated by [26], we introduce a regularization of $g$, given by
\[ g^\rho(y(t)) = \frac{1}{\rho} \int_0^t \beta \left(\frac{t-s}{\rho}\right) g(y(s)) ds, \]
where $\beta$ is a mollifier. It is easy to check that
\[ \mathbb{E} \int_0^T \|g^\rho(y(t))\|_{L^2(I)}^2 dt \leq \mathbb{E} \int_0^T \|g(y(t))\|_{L^2(I)}^2 dt \]
and
\[ g^\rho(y(\cdot)) \rightarrow g(y(\cdot)) \text{ in } L^2(\Omega, L^2(0, T; L^2(I))). \]
Then we denote by $X_{n,\rho}$ and $Y_{n,\rho}$ the analog of $X_n$ and $Y_n$ with $g$ replaced by $g^\rho$.
Introduce the mapping
\[ \Phi_{n,\rho}(B, u_n) = \frac{X_{n,\rho}}{1 + X_{n,\rho}}, \]
onlyi owing to the definition of $X_{n,\rho}$, it is easy to see that $\Phi_{n,\rho}$ is bounded and continuous on $C([0, T], \mathbb{R}^1) \times L^2(0, T; L^2(I))$. Similarly, set
\[ \Psi_{n,\rho}(B_n, u_n) = \frac{Y_{n,\rho}}{1 + Y_{n,\rho}}. \]
According to Lemma 2.4, we have
\[ \mathbb{E} \frac{Y_{n,\rho}}{1 + Y_{n,\rho}} = \mathbb{E} \Psi_{n,\rho}(B_n, u_n) = \int \Psi_{n,\rho} d\Pi_n = \mathbb{E} \Phi_{n,\rho}(B, u_n) = \mathbb{E} \frac{X_{n,\rho}}{1 + X_{n,\rho}}, \]
therefore,
\[ \mathbb{E} \frac{Y_{n,\rho}}{1 + Y_{n,\rho}} - \mathbb{E} \frac{X_{n,\rho}}{1 + X_{n,\rho}} = \mathbb{E} \left( \frac{Y_{n,\rho}}{1 + Y_{n,\rho}} - \frac{X_{n,\rho}}{1 + X_{n,\rho}} \right) \]
\[ = \mathbb{E} \left( \frac{Y_{n,\rho}}{1 + Y_{n,\rho}} \right) - \mathbb{E} \left( \frac{X_{n,\rho}}{1 + X_{n,\rho}} \right) \]
It is clear that
\[ \left| \mathbb{E} \frac{Y_{n,\rho}}{1 + Y_{n,\rho}} \right| \leq \left| \mathbb{E} \frac{Y_{n,\rho}}{1 + Y_{n,\rho}} - \mathbb{E} \frac{X_{n,\rho}}{1 + X_{n,\rho}} \right| \]
\[ \leq \mathbb{E} \left( \frac{Y_{n,\rho}}{1 + Y_{n,\rho}} - \frac{X_{n,\rho}}{1 + X_{n,\rho}} \right) \]
\[ \leq C \left( \mathbb{E} \int_0^T \|g^\rho(u_n(t)) - g(u_n(t))\|_{L^2(I)}^2 dt \right)^{\frac{1}{2}}. \]
As $\rho \rightarrow 0$, it follows that
\[ \left| \mathbb{E} \frac{Y_{n,\rho}}{1 + Y_{n,\rho}} \right| = \mathbb{E} \frac{X_{n,\rho}}{1 + X_{n,\rho}} = 0. \]
It follows that (33) holds.

**Step 5. Passage to the limit.**

From (33), it follows that $u_n$ satisfies the results of (19)(23)(24)(27), we can extract from $u_n$ a subsequence still denoted with the same fashion and a function $u$ such that
\[ u_n \rightarrow u \text{ weakly } \ast \text{ in } L^p(\Omega, L^\infty(0, T; L^2(I))), \]
\[ u_n \rightarrow u \text{ weakly in } L^p(\Omega, L^2(0, T; H^1(I))), \]
\[ u_n \rightarrow u \text{ weakly in } L^4(\Omega, L^4(0, T; L^4(I))), \]
\[ u_n \rightarrow u \text{ strongly in } L^2(0, T; L^2(I)) \ P-a.s. \]
By Vitali’s convergence theorem, we have
\[ u_{n_i} \to u \text{ strongly in } L^2(\Omega, L^2(0, T; L^2(I))). \]

It follows from these facts that we can extract again from \( u_{n_i} \), a subsequence still denoted by the same symbols such that
\[ u_{n_i} \to u \text{ almost everywhere } dt \otimes d\mathbb{P} \quad \text{in } L^2(I), \]
\[ u_{n_i} \to u \text{ almost everywhere } dt \otimes dx \otimes d\mathbb{P} \quad \text{in } [0, T] \times I \times \Omega. \]

It follows from (35) that for any \( t \in [0, T] \),
\[ u_{n_i} \to u \text{ almost everywhere } dt \otimes dx \otimes d\mathbb{P} \quad \text{in } [0, t] \times I \times \Omega. \] (36)

Since \( u_{n_i} \) is bounded in \( L^4(\Omega, L^4(0, T; L^4(I))) \), we have \( u_{n_i}^3 \) is bounded in \( L^2([0, T] \times I \times \Omega) \). Combining this and (36), we deduce that
\[ u_{n_i}^3 \to u^3 \text{ weakly in } L^2([0, T] \times I \times \Omega). \] (37)

By (34), the continuity of \( g \), and the applicability of Vitali’s convergence theorem we have
\[ P_{n_i} g(u_{n_i}) \to g(u) \text{ strongly in } L^2(\Omega, L^2(0, T; L^2(I))). \] (38)

By the idea in [4, P284] and [26, P922], we can know
\[ \int^t_0 P_{n_i} g(u_{n_i}) dB_{n_i} \to \int^t_0 g(u) dB \text{ weakly in } L^2(\Omega, L^2(I)) \] (39)
for any \( t \in [0, T] \).

As
\[ u_{n_i} \to u \text{ weakly in } L^p(\Omega, L^2(0, T; H^1(I))), \]
then
\[ u_{n_i;x} \to u_{x} \text{ weakly in } L^2(\Omega, L^2(0, T; H^{-1}(I))). \] (40)

Collecting all the convergence results (34)-(40), we deduce that \((u, B)\) verifies the following equation \( dt \otimes d\mathbb{P} \)–almost everywhere:
\[ \frac{1}{2} u_{t}(t) \varphi + \varepsilon u_{x}(t) \varphi_x - \frac{1}{2} \int^t_0 \left( (u_0, \varphi) + \varepsilon (u_{0x}, \varphi_x) + \int^t_0 ((u_x, \varphi_x) + (u^3 - u, \varphi))ds \right) dB_{n_i} \]
for all \( \varphi \in H^1_0(I) \).

Estimates (2)-(4) follow from passing to the limits in (23), (24) and (27).

4. Proof of Theorem 1.3. This section is motivated by [13]. It follows from Theorem 1.2 that there exists a sequence of weak martingale solutions
\[ \{(\Omega^\varepsilon, \mathcal{F}^\varepsilon, \mathbb{P}^\varepsilon), (\mathcal{F}^\varepsilon_t)_{0 \leq t \leq T}, u^\varepsilon, B^\varepsilon \} \]
satisfy the inequalities
\[ \mathbb{E} \sup_{0 \leq t \leq T} (\|u^\varepsilon(t)\|^2_{L^2(I)} + \varepsilon \|u^\varepsilon_x(t)\|^2_{L^2(I)})^{\frac{1}{2}} \leq C(p, T), \]
\[ \mathbb{E} \left( \int^T_0 (\|u^\varepsilon_x(t)\|^2_{L^2(I)} + \|u^\varepsilon\|^4_{L^4(I)})dt \right)^{\frac{1}{2}} \leq C(p, T), \]
\[ \mathbb{E} \sup_{0 \leq \theta \leq \delta \leq 1} \int^T_0 \|u^\varepsilon(t + \theta) - u^\varepsilon(t)\|^2_{H^{-1}(I)} dt \leq C(p, T) \delta, \]
where \( C(p, T) \) is a constant independent of \( \varepsilon \).

By the same way as in [26, P919] and [13, P2237], according to the priori estimates (41), we obtain that
Lemma 4.1. For any $1 \leq p < \infty$ and for any sequences $\mu_m, \nu_m$ converging to 0 such that the series $\sum_{m=1}^{\infty} \frac{\mu_m}{\nu_m}$ converges, \{u^\varepsilon\}_{\varepsilon \in (0, \frac{1}{2}]} is bounded in $X^1_{\mu_m, \nu_m}$ (the explicit definition of the space $X^1_{\mu_m, \nu_m}$ can be found in Section 2) for any $m$.

Let
\[
X = C([0,T]; \mathbb{R}^1) \times L^2(0,T; L^2(I))
\]
and $\mathcal{B}(X)$ be the $\sigma$-algebra of the Borel sets of $X$.

For each $\varepsilon$, let $\Phi_{\varepsilon}$ be the map
\[
\Phi_{\varepsilon} : \Omega^\varepsilon \to X
\]
\[
\omega \to (B^\varepsilon(\omega), u^\varepsilon(\omega)),
\]
and $\Pi_{\varepsilon}$ be a probability measure on $(X, \mathcal{B}(X))$ defined by
\[
\Pi_{\varepsilon}(A) = \mathbb{P}(\Phi_{\varepsilon}^{-1}(A)), A \in \mathcal{B}(X).
\]

Proposition 3. The family of probability measures \{\Pi_{\varepsilon} : \varepsilon \in (0, \frac{1}{2}]\} is tight in $X$.

Proof. We use the same method as in Proposition 2.

For any $\rho > 0$, we should find the compact subsets
\[
\Sigma_{\rho} \subset C([0,T]; \mathbb{R}^1), Y^1_{\rho} \subset L^2(0,T; L^2(I)),
\]
such that
\[
\mathbb{P}(\omega : B^\varepsilon(\omega, \cdot) \notin \Sigma_{\rho}) \leq \frac{\rho}{2}, \quad (42)
\]
\[
\mathbb{P}(\omega : u^\varepsilon(\omega, \cdot) \notin Y^1_{\rho}) \leq \frac{\rho}{2}. \quad (43)
\]

Noting the formula
\[
\mathbb{E}[|B^\varepsilon(t_2) - B^\varepsilon(t_1)|^2] = (2i - 1)! (t_2 - t_1)^i, i = 1, 2, ...
\]
we define
\[
\Sigma_{\rho} := \left\{ B(\cdot) \in C([0,T]; \mathbb{R}^1) : \sup_{t_1, t_2 \in [0,T], |t_2 - t_1| \leq \frac{1}{n}} n|B(t_2) - B(t_1)| \leq \rho \right\},
\]
\[
Y^1_{\rho} = \left\{ u \in Y^1_{\mu_m, \nu_m} : \|u\|_{Y^1_{\mu_m, \nu_m}} \leq \rho \right\},
\]
where $n \in \mathbb{N}$, $L_\rho$, $M_\rho$ two constants depending on $\rho$ and will be chosen later.

By the Chebyshev inequality and the same argument as in Proposition 2, we get
\[
\mathbb{P}(\omega : B^\varepsilon(\omega, \cdot) \notin \Sigma_{\rho}) \leq \frac{C}{\rho} \sum_{n=1}^{\infty} \frac{1}{n^3},
\]
\[
\mathbb{P}(\omega : u^\varepsilon(\omega, \cdot) \notin Y^1_{\rho}) \leq \frac{C}{M_\rho},
\]
we choose $L_\rho^4 = 2C \rho^{-1} \sum_{n=1}^{\infty} \frac{1}{n^7}$, $M_\rho = 2C \rho^{-1}$, to get (42) and (43).

It follows from (42) and (43) that
\[
\Pi_{\varepsilon}(\Sigma_{\rho} \times Y^1_{\rho}) \geq 1 - \rho,
\]
for any $\varepsilon \in (0, \frac{1}{2}]$.

Thus, the family of probability measures \{\Pi_{\varepsilon} : \varepsilon \in (0, \frac{1}{2}]\} is tight in $X$. \qed
From the tightness of \( \{\Pi_{\varepsilon} : \varepsilon \in (0, \frac{1}{T}]\} \) in the Polish space \( X \) and Prokhorov’s theorem, we infer the existence of a subsequence \( \Pi_{\varepsilon_i} \) of probability measures and a probability measure \( \Pi \) such that \( \Pi_{\varepsilon_i} \rightarrow \Pi \) weakly as \( i \rightarrow \infty \).

By Lemma 2.4, there exists a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) and random variables \((\tilde{u}^{\varepsilon_i}, \tilde{B}^{\varepsilon_i}), (u, B) \) on \( (\Omega, \mathcal{F}, \mathbb{P}) \) with values in \( X \) such that
\[
\mathcal{L}(\tilde{u}^{\varepsilon_i}, \tilde{B}^{\varepsilon_i}) = \Pi_{\varepsilon_i},
\]
\[
\mathcal{L}(u, B) = \Pi,
\]
\[
(\tilde{u}^{\varepsilon_i}, \tilde{B}^{\varepsilon_i}) \rightarrow (u, B) \text{ in } X \; P - a.s.
\]

By the same argument as in (33), we have
\[
[(\tilde{u}^{\varepsilon_i}(t), \varphi) + \varepsilon(\tilde{u}^{\varepsilon_i}_s(t), \varphi_x)] - [(\tilde{u}_0^{\varepsilon_i}, \varphi) + \varepsilon(\tilde{u}_0^{\varepsilon_i}, \varphi_x)] + \int_0^t (((\tilde{u}^{\varepsilon_i}), \varphi_x) + (\tilde{u}^{\varepsilon_i}_x, \varphi))) ds
= \int_0^t (g(\tilde{u}^{\varepsilon_i}), \varphi) dB^{\varepsilon_i},
\]
for all \( \varphi \in H^1_0(I) \).

From (44), it follows that \( \tilde{u}^{\varepsilon_i} \) satisfies the results of (19)(23)(24)(27), we can extract from \( \tilde{u}^{\varepsilon_i} \) a subsequence still denoted with the same fashion and a function \( u \) such that
\[
\tilde{u}^{\varepsilon_i} \rightarrow u \text{ weakly * in } L^p(\Omega, L^\infty(0, T; L^2(I))),
\]
\[
\tilde{u}^{\varepsilon_i} \rightarrow u \text{ weakly in } L^p(\Omega, L^2(0, T; H^1(I))),
\]
\[
\tilde{u}^{\varepsilon_i} \rightarrow u \text{ weakly in } L^4(\Omega, L^4(0, T; L^4(I))),
\]
\[
\tilde{u}^{\varepsilon_i} \rightarrow u \text{ strongly in } L^2(0, T; L^2(I)) \; P - a.s.
\]

By Vitali’s convergence theorem, we have
\[
\lim_{i \rightarrow \infty} E\|\tilde{u}^{\varepsilon_i} - u\|^2_{L^2(0, T; L^2(I))} = 0,
\]
according to this equality, Theorem 1.5, [4, P284], [12, P1126, Lemma 2.1] and [17, P151, Lemma 3.1], it is easy to see that for any \( \delta > 0 \), we have
\[
\lim_{i \rightarrow \infty} P(\|((\tilde{u}^{\varepsilon_i}(t), \varphi) - (u(t), \varphi))\|_{L^2(0, T)} > \delta) = 0,
\]
\[
\lim_{i \rightarrow \infty} P(\|\int_0^t (\tilde{u}^{\varepsilon_i}_x(s), \varphi_x) ds - \int_0^t (u_s, \varphi_x) ds\|_{L^2(0, T)} > \delta) = 0,
\]
\[
\lim_{i \rightarrow \infty} P(\|\int_0^t (\tilde{u}^{\varepsilon_i}_x - \tilde{u}^{\varepsilon_i}, \varphi) ds - \int_0^t (u^x - u, \varphi) ds\|_{L^2(0, T)} > \delta) = 0,
\]
\[
\lim_{i \rightarrow \infty} P(\|\int_0^t (g(\tilde{u}^{\varepsilon_i}(s), \varphi) dB^{\varepsilon_i}(s) - \int_0^t (g(u(s), \varphi) dB(s)\|_{L^2(0, T)} > \delta) = 0.
\]

Since
\[
E \sup_{0 \leq t \leq T} |\varepsilon_i(\tilde{u}^{\varepsilon_i}_x(t), \varphi_x)|^2 \leq E \sup_{0 \leq t \leq T} \varepsilon_i^2\|\tilde{u}^{\varepsilon_i}_x(t)\|^2_{L^2(I)}\|\varphi_x\|^2_{L^2(I)}
\leq \varepsilon_i\|\varphi_x\|^2_{L^2(I)}E \sup_{0 \leq t \leq T} \varepsilon_i\|\tilde{u}^{\varepsilon_i}_x(t)\|^2_{L^2(I)},
\]
we have
\[
\lim_{i \rightarrow \infty} E \sup_{0 \leq t \leq T} |\varepsilon_i(\tilde{u}^{\varepsilon_i}_x(t), \varphi_x)|^2 = 0.
\]

By taking the limit in probability as \( i \) goes to infinity in (44), we deduce that \( (u, B) \) verifies the following equation \( dt \otimes dB \)-almost everywhere:
\[
(u(t), \varphi) - (u_0, \varphi) + \int_0^t (((u_s, \varphi_x) + (u^x - u, \varphi)) ds = \int_0^t g(u(t), \varphi) dB
\]
for all \( \varphi \in H^1_0(I) \). Namely, \( \{((\Omega, \mathcal{F}, \mathbb{P}), (\mathcal{F}_t)_{0 \leq t \leq T}, u, B) \) is a weak martingale solution of problem (5).

5. Proof of Theorem 1.5. If there is no danger of confusion, we shall omit the subscript \( \varepsilon \), we use \( u \) instead of \( u^\varepsilon \) and \( v \) instead of \( v^\varepsilon \). The proof is divided into several steps.
5.1. Local existence. Based on Proposition 1, we can obtain the following result.

Proposition 4. For any \( \varepsilon \in (0, \frac{1}{2}) \), \( T > 0 \). If

\[
\begin{align*}
    &u_0 \in L^2(\Omega; H^2(I) \cap H^1_0(I)), \\
    &\|f(u_1) - f(u_2)\|_{L^2(I)} \leq L\|u_1 - u_2\|_{H^1(I)}, \\
    &\|f(u)\|_{L^2(I)} \leq L(1 + \|u\|_{H^1(I)}),
\end{align*}
\]

then equation

\[
\begin{cases}
    d(u^\varepsilon - \varepsilon u_{xx}^\varepsilon) + (-u_{xx}^\varepsilon + f(u^\varepsilon))dt = g(u^\varepsilon)dB & \text{in } I \times (0,T) \\
    u^\varepsilon(0,t) = 0 = u^\varepsilon(1,t) & \text{in } (0,T) \\
    u^\varepsilon(0) = u_0 & \text{in } I,
\end{cases}
\]

has a unique solution \( u^\varepsilon \in L^2(\Omega; C([0,T]; H^2(I) \cap H^1_0(I))) \) and

\[
\mathbb{E} \sup_{0 \leq t \leq T} \|u^\varepsilon(t)\|_{L^2(I)}^2 + \varepsilon \|u^\varepsilon_{xx}(t)\|_{L^2(I)}^2 + \mathbb{E} \int_0^T \|u^\varepsilon_{xx}(t)\|_{L^2(I)}^2 dt 
\leq C\mathbb{E} \sup_{0 \leq t \leq T} \|u_0\|_{L^2(I)}^2 + \|u_{0xx}\|_{L^2(I)}^2,
\]

where \( C = C(L,T,I) \).

Proof. The main idea in this part comes from \([18]\). We set \( u_0(t) = u_0 \), and let \( u_{n+1}(t) \) be the solution of

\[
\begin{cases}
    d(u - \varepsilon u_{xx}) + (-u_{xx} + f(u_0(t)))dt = g(u_0(t))dB & \text{in } I \times (0,T) \\
    u(0,t) = 0 = u(1,t) & \text{in } (0,T) \\
    u(0) = u_0 & \text{in } I.
\end{cases}
\]

Then,

\[
\begin{align*}
    &\mathbb{E} \sup_{0 \leq t \leq T} \|(u_{n+1} - u_n)(t)\|_{L^2(I)}^2 + \varepsilon \|(u_{n+1} - u_n)(t)\|_{H^1(I)}^2 \\
    &\leq C \mathbb{E} \int_0^T \|f(u_0(t)) - f(u_{n-1}(t))\|_{L^2(I)}^2 dt + \mathbb{E} \int_0^T \|\varepsilon u_{xx}(t) - u_{n-1}(t)\|_{H^1(I)}^2 ds \\
    &\leq C \mathbb{E} \int_0^T L^2 \|u_0(t) - u_{n-1}(t)\|_{L^2(I)}^2 dt + \mathbb{E} \int_0^T \|\varepsilon u_{xx}(t) - u_{n-1}(t)\|_{H^1(I)}^2 ds \\
    &\leq CL^2 \mathbb{E} \int_0^T \sup_{0 \leq \tau \leq t} \|u_n(\tau) - u_{n-1}(\tau)\|_{H^1(I)}^2 ds \\
    &\leq CL^2 \mathbb{E} \int_0^T \sup_{0 \leq \tau \leq t} \|u_n(\tau) - u_{n-1}(\tau)\|_{L^2(I)}^2 ds \\
    &\leq CL^2 \mathbb{E} \int_0^T \sup_{0 \leq \tau \leq t} \|u_n(\tau) - u_{n-1}(\tau)\|_{L^2(I)}^2 ds.
\end{align*}
\]

We define

\[
Q_n(t) = \mathbb{E} \sup_{0 \leq s \leq t} \|(u_{n+1} - u_n)(s)\|_{L^2(I)}^2 + \varepsilon \|(u_{n+1} - u_n)_{xx}(s)\|_{L^2(I)}^2,
\]

then, we have

\[
Q_n(t) \leq CL^2 \int_0^t Q_{n-1}(s) ds.
\]

It is easy to see that

\[
Q_1(t) \leq C_0, \\
Q_n(t) \leq C_0^n t^{2n},
\]

where \( C_0 \) is a constant.
which yields
\[ \sum_{n=1}^{+\infty} \sqrt{Q_n(T)} < +\infty. \] (50)

Consequently, \( \{u_n\}_{n=1}^{+\infty} \) is a Cauchy sequence in \( L^2(\Omega, C([0, T]; H^2(I))) \). Then it is easy to see that the limit gives a solution of (45).

According to Proposition 1 (3), we have
\[
\begin{align*}
\mathbb{E} \sup_{0 \leq s \leq T} (\|u_x(t)\|_{L^2(I)}^2 + \varepsilon \|u_{xx}(t)\|_{L^2(I)}^2) + \mathbb{E} \int_0^T \|u_{xx}(t)\|_{L^2(I)}^2 \, ds \\
\leq C \mathbb{E} \left( \|u_0\|_{L^2(I)}^2 + \|u_{xx}(0)\|_{L^2(I)}^2 \right) + \mathbb{E} \int_0^T \|f(u(s))\|_{L^2(I)}^2 \, ds + \mathbb{E} \int_0^T \|g(u(s))\|_{L^2(I)}^2 \, ds \\
\leq C(L) \mathbb{E} \left( \|u_0\|_{L^2(I)}^2 + \|u_{xx}(0)\|_{L^2(I)}^2 \right) + \mathbb{E} \int_0^T (1 + \|u(s)\|_{H^1(I)}^2) \, ds \\
\leq C(L) \mathbb{E} \left( \|u_0\|_{L^2(I)}^2 + \|u_{xx}(0)\|_{L^2(I)}^2 \right) + T + \int_0^T \mathbb{E} \sup_{0 \leq \tau \leq s} \|u_x(\tau)\|_{L^2(I)}^2 \, ds,
\end{align*}
\]

the Gronwall inequality now implies (46).

The uniqueness can also be obtained from the Gronwall inequality. \( \square \)

5.2. **Global existence.** Let \( \rho \in C_c^\infty(\mathbb{R}) \) be a cut-off function such that \( \rho(r) = 1 \) for \( r \in [0, 1] \) and \( \rho(r) = 0 \) for \( r \geq 2 \). For any \( R > 0, y \in H^1(I) \), we set
\[
\rho_R(y) = \rho \left( \frac{\|y\|_{H^1(I)}}{R} \right), \quad f_R(y) = \rho_R(y) y^3.
\]

It is easy to see
\[
\|f_R(y_1) - f_R(y_2)\|_{L^2(I)} \leq CR^2 \|y_1 - y_2\|_{H^1(I)}.
\]

The truncated equation corresponding to (1) is the following stochastic partial differential equation:
\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{d(u - \varepsilon u_{xx})}{dt} + (-u_{xx} + f_R(u) - u) = g(u) dB \\
u(0, t) = 0 = u(1, t) \\
u(0) = u_0.
\end{array} \right.
\end{align*}
\] (51)

It follows from Proposition 4 that (51) has a unique solution \( u_R \in L^2(\Omega; C([0, T]; H^2(I) \cap H_0^1(I))) \). We define
\[
\tau_R = \inf \{ t \geq 0 \mid \|u_R(t)\|_{H^2(I)} \geq R \}
\]
with the usual convention that \( \inf \emptyset = +\infty \).

Since the sequence of stopping times \( \tau_R \) is non-decreasing on \( R \), we can put
\[
\tau^* = \lim_{R \to \infty} \tau_R.
\]

We can define a local solution to (51) as
\[
u(t) = u_R(t)
\]
on \([0, \tau_R]\), which is well defined since
\[
u_{R_1}(t) = u_{R_2}(t)
\]
on \([0, \tau_{R_1} \wedge \tau_{R_2}]\).

Indeed, \( u_{R_1}(t) - u_{R_2}(t) \) is the solution of
\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{d(h - \varepsilon h_{xx})}{dt} + (-h_{xx} + f_{R_1}(u_{R_1}) - f_{R_2}(u_{R_2}) - h) = g(u_{R_1}) - g(u_{R_2}) dB \\
h(0, t) = 0 = h(1, t) \\
h(0) = 0,
\end{array} \right.
\end{align*}
\]
for \( t \leq [0, \tau_{R_1} \land \tau_{R_2}] \) with \( R_1 \leq R_2 \), it follows from Proposition 1 that

\[
\mathbb{E} \sup_{0 \leq s \leq t} (\|h_x(s)\|_{L^2(I)}^2 + \varepsilon \|h_{xx}(s)\|_{L^2(I)}^2)
\]

\[
\leq C\mathbb{E} \int_0^t \|f_{R_2}(u_{R_1}) - f_{R_2}(u_{R_2})\|_{L^2(I)}^2 ds + \mathbb{E}\int_0^t \|g(u_{R_1}) - g(u_{R_2})\|_{L^2(I)}^2 ds
\]

\[
= C\mathbb{E} \int_0^t \|f_{R_2}(u_{R_1}) - f_{R_2}(u_{R_2})\|_{L^2(I)}^2 ds + \mathbb{E}\int_0^t \|g(u_{R_1}) - g(u_{R_2})\|_{L^2(I)}^2 ds
\]

\[
\leq \beta(t) \mathbb{E} \sup_{0 \leq s \leq t} \|h_x(s)\|_{L^2(I)}^2,
\]

where \( \beta(t) \) is a continuous increasing function with \( \beta(0) = 0 \). If we take \( t \) sufficiently small, we have \( u_{R_1} = u_{R_2} \) on \([0, t]\). Repeating the same argument in the interval \([t, 2t]\) and so on yields

\[ u_{R_1} = u_{R_2} \]

in the whole interval \([0, \tau_{R_1} \land \tau_{R_2}]\).

At the end, if \( \tau^* < +\infty \), the definition of \( u \) yields

\[ \lim_{t \to \tau^*} \|u(t)\|_{H^2(I)} = +\infty, \]

which shows that \( u \) is a unique local solution to (51) on the interval \([0, \tau^*]\), and thus completes the proof.

We will exploit an energy inequality.

For any \( T > 0 \), set \( \tau = \inf\{\tau^*, T\} \) and \( t < \tau \).

**Step 1.** We first prove (7).

Set

\[ v(t) = (u - \varepsilon u_{xx})(t). \]

It follows from Itô’s rule that

\[
dv^2 = 2u dv + (dt)^2
\]

\[
= 2(u - \varepsilon u_{xx})[(u_{xx} - u^3 + u)dt + g(u)dB + g^2(u)dt
\]

\[
= (2u u_{xx} - 2u^3 + 2u^2 - 2u u_{xx}^2 + 2u u_{xx} u^3 - 2u u_{xx} u)dt + 2v u dB + g^2(u)dt,
\]

namely, we have

\[
\|v(t)\|_{L^2(I)}^2 + \int_0^t [2(1 - \varepsilon)]\|u_{xx}\|_{L^2(I)}^2 + 2f_t u^4 dx + 2\|u_{xx}\|_{L^2(I)}^2 ds
\]

\[
= \|v(0)\|_{L^2(I)}^2 + \int_0^t [2(1 - \varepsilon)]\|u_{xx}\|_{L^2(I)}^2 + 2f_t u^4 dx + 2\|u_{xx}\|_{L^2(I)}^2 ds + 2f_t (v, g(u))dB + \|f_t g(u)\|_{L^2(I)}^2 dt
\]

\[
= \|v(0)\|_{L^2(I)}^2 + \int_0^t [2(1 - \varepsilon)]\|u_{xx}\|_{L^2(I)}^2 + 6f_t u^2 u^2 dx + 2\|u_{xx}\|_{L^2(I)}^2 ds + 2f_t (v, g(u))dB + \|f_t g(u)\|_{L^2(I)}^2 dt
\]

\[
\leq \|v(0)\|_{L^2(I)}^2 + \int_0^t [2(1 - \varepsilon)]\|u_{xx}\|_{L^2(I)}^2 ds + 2f_t (v, g(u))dB + \|f_t g(u)\|_{L^2(I)}^2 ds.
\]

After some calculation, we obtain

\[
(\sup_{0 \leq t \leq \tau} \|v(t)\|_{L^2(I)}^2 + \int_0^t [2(1 - \varepsilon)]\|u_{xx}\|_{L^2(I)}^2 + 2f_t u^4 dx + 2\|u_{xx}\|_{L^2(I)}^2 dt)^p
\]

\[
\leq C(p)[\|v(0)\|_{L^2(I)}^2 + \int_0^\tau \|u\|_{L^2(I)}^2 dt]^p + \sup_{0 \leq t \leq \tau} \int_0^t (v, g(u))dB^p + \int_0^t \|g(u)\|_{L^2(I)}^2 dt)^p.
\]

by the Burkholder-Davis-Gundy inequality, we have

\[
\mathbb{E} \sup_{0 \leq t \leq \tau} \|v(t)\|_{L^2(I)}^2 + \mathbb{E} \int_0^\tau \|u_{xx}\|_{L^2(I)}^2 dt)^p + \mathbb{E} \int_0^\tau \varepsilon \|u_{xx}\|_{L^2(I)}^2 dt)^p
\]

\[
\leq C(p)\mathbb{E} \|v(0)\|_{L^2(I)}^2 + \mathbb{E} \int_0^\tau \|u\|_{L^2(I)}^2 dt)^p + \mathbb{E} \sup_{0 \leq t \leq \tau} \int_0^t (v, g(u))dB^p + \mathbb{E} \int_0^t \|g(u)\|_{L^2(I)}^2 dt)^p
\]

\[
\leq C(p)\mathbb{E} \|v(0)\|_{L^2(I)}^2 + \mathbb{E} \int_0^\tau \|u\|_{L^2(I)}^2 dt)^p + \mathbb{E} \sup_{0 \leq t \leq \tau} \int_0^t (v, g(u))dB^p + \mathbb{E} \int_0^\tau \|g(u)\|_{L^2(I)}^2 dt)^p
\]

\[
\leq C(p)\mathbb{E} \|v(0)\|_{L^2(I)}^2 + \mathbb{E} \int_0^\tau \|u\|_{L^2(I)}^2 dt)^p + \mathbb{E} \|v(t)\|_{L^2(I)}^2 dt)^p + C(p)\mathbb{E} \sup_{0 \leq t \leq \tau} \|v(t)\|_{L^2(I)}^2 dt)^p
\]

\[
\leq C(p, \alpha, L, T)[1 + \mathbb{E} \|v(0)\|_{L^2(I)}^2 + \mathbb{E} \int_0^\tau \|u\|_{L^2(I)}^2 dt)^p + C(\sigma, T)] + \mathbb{E} \|v(t)\|_{L^2(I)}^2 dt)^p.
\]
By taking $0 < \sigma \ll 1, 0 < \rho \ll 1$, we have

\[ \mathbb{E} \sup_{0 \leq t \leq \tau} \|v(t)\|_{L^2(I)}^{2p} + \mathbb{E}(\int_0^\tau \|u_x\|_{L^2(I)}^2 dt)^p + \mathbb{E}(\int_0^\tau u^4 dx dt)^p + \mathbb{E}(\int_0^\tau \|u_x\|_{L^2(I)}^2 dt)^p \leq C(p, \rho, L, \sigma, T)[\mathbb{E}\|v(0)\|_{L^2(I)}^{2p} + 1] \leq C(p, L, T, I, u_0). \]

By the regularity theory of elliptic equation

\[ \begin{cases} u - \varepsilon u_{xx} = v & \text{in } I \\ u(0, t) = u(1, t) = 0, \end{cases} \]

we have

\[ \|u(t)\|_{L^2(I)} \leq \|v(t)\|_{L^2(I)}. \]

This implies that (7) holds.

**Step 2.** We shall prove (8).

According to Gagliardo-Nirenberg inequality, we have

\[ \|u\|_{L^p(I)} \leq C\|u\|_{H^1(I)}^{2/3}\|u\|_{L^8(I)}^{4/3}, \]

thus,

\[ \mathbb{E}\int_0^\tau \|u_x\|_{L^2(I)}^2 dt = \mathbb{E}\int_0^\tau \|u\|_{H^1(I)}^2 dt \leq CE\int_0^\tau \|u\|_{H^1(I)}^2 dt \cdot \sup_{0 \leq t \leq \tau} \|u\|_{L^2(I)}^4 \]

\[ \leq CE\left(\int_0^\tau \|u\|_{H^1(I)}^2 dt \cdot \sup_{0 \leq t \leq \tau} \|u\|_{L^2(I)}^4 \right) \]

\[ \leq CE\left(\int_0^\tau \|u_x\|_{L^2(I)}^2 dt \cdot \sup_{0 \leq t \leq \tau} \|u\|_{L^2(I)}^4 \right) \]

\[ \leq C\mathbb{E}\left[\int_0^\tau \|u_x\|_{L^2(I)}^2 dt \right]^2 + \mathbb{E}\sup_{0 \leq t \leq \tau} \|u\|_{L^2(I)}^8. \]

In view of (7) and (52), there holds that $u^3 - u \in L^2(\Omega; L^2(0, T; L^2(I)))$, moreover, $g(u) \in L^2(\Omega; L^2(0, T; H^1(I)))$, according to (3) in Proposition 1, we have

\[ \mathbb{E}\sup_{0 \leq t \leq \tau} \left(\|u_x(t)\|_{L^2(I)}^2 + \varepsilon \|u_x(t)\|_{L^2(I)}^2 \right) + \mathbb{E}\int_0^\tau \|u_x(t)\|_{L^2(I)}^2 dt \leq C\mathbb{E}\left[\|u_{xx}\|_{L^2(I)}^2 + \|u_{xxx}\|_{L^2(I)}^2 + \mathbb{E}\int_0^\tau \|u^3 - u(t)\|_{L^2(I)}^2 dt + \mathbb{E}\int_0^\tau \|g(u)\|_{H^1(I)}^2 dt \right]. \]

With the help of (7) and (52), one finds that

\[ \mathbb{E}\sup_{0 \leq t \leq \tau} \left(\|u_x(t)\|_{L^2(I)}^2 + \varepsilon \|u_x(t)\|_{L^2(I)}^2 \right) + \mathbb{E}\int_0^\tau \|u_x(t)\|_{L^2(I)}^2 dt \leq C\left[\mathbb{E}\|u_{xx}\|_{L^2(I)}^2 + \mathbb{E}\int_0^\tau \|u_{xxx}\|_{L^2(I)}^2 dt \right]^2 + \mathbb{E}\sup_{0 \leq t \leq \tau} \|u\|_{L^2(I)}^4 + \mathbb{E}\int_0^\tau \|u\|_{H^1(I)}^2 dt + C(T) \]

\[ \leq C(u_0, T, I). \]

Namely, we prove (8).

**Step 3.** We shall prove $\mathbb{P}\{\omega \in \Omega \mid \tau^*(\omega) = +\infty\} = 1$. 
Indeed, by the Chebyshev inequality, (8) and the definition of \( u \), we have

\[
\mathbb{P} \left( \omega \in \Omega | \tau^*(\omega) < +\infty \right) = \lim_{T \to +\infty} \mathbb{P} \left( \omega \in \Omega | \tau^*(\omega) \leq T \right) = \lim_{T \to +\infty} \mathbb{P} \left( \omega \in \Omega | \tau(\omega) = \tau^*(\omega) \right) = \lim_{T \to +\infty} \lim_{R \to +\infty} \mathbb{P} \left( \omega \in \Omega | \tau_R(\omega) \leq \tau(\omega) \right) = \lim_{T \to +\infty} \lim_{R \to +\infty} \mathbb{P} \left( \omega \in \Omega | \sup_{0 \leq t \leq T} \| u(t) \|_{H^2(t)} \geq \sup_{0 \leq t \leq \tau_R} \| u(t) \|_{H^2(t)} \right) = \lim_{T \to +\infty} \mathbb{P} \left( \omega \in \Omega | \sup_{0 \leq t \leq T} \| u(t) \|_{H^2(t)} \geq R \right) = C \theta (T)}

this show that

\[
\mathbb{P} \left( \{ \omega \in \Omega | \tau^*(\omega) = +\infty \} \right) = 1,
\]

namely, \( \tau_\infty = +\infty \) P-a.s.

6. Proof of Theorem 1.6.

6.1. A priori estimate of \( \{u^\varepsilon\}_{\varepsilon \in (0, \frac{1}{2})} \). In this section, we will establish the following estimate

\[
\mathbb{E} \sup_{0 \leq \theta \leq \delta} \int_{0}^{T} \| u^\varepsilon(t + \theta) - u^\varepsilon(t) \|_{L^2(t)}^2 dt \leq C \delta. \tag{53}
\]

Establishing this estimate directly for \( u^\varepsilon \) is very difficult, motivated by Section 2, we should establish estimate for \( v^\varepsilon \), then by applying the regularity theory of elliptic equation, we can obtain the estimate for \( u^\varepsilon \).

It is easy to see that

\[
v^\varepsilon(t + \theta) - v^\varepsilon(t) = \int_{t}^{t + \theta} u^\varepsilon_{xx}(s) ds - \int_{t}^{t + \theta} (u^\varepsilon - u^\varepsilon)(s) ds + \int_{t}^{t + \theta} g(u^\varepsilon(s)) dB,
\]

which implies

\[
\| v^\varepsilon(t + \theta) - v^\varepsilon(t) \|_{L^2(t)} \leq \| \int_{t}^{t + \theta} u^\varepsilon_{xx}(s) ds \|_{L^2(t)} + \| \int_{t}^{t + \theta} (u^\varepsilon - u^\varepsilon)(s) ds \|_{L^2(t)} + \| \int_{t}^{t + \theta} g(u^\varepsilon(s)) dB \|_{L^2(t)} \leq \int_{t}^{t + \theta} \| u^\varepsilon_{xx}(s) \|_{L^2(t)} ds + \int_{t}^{t + \theta} \| (u^\varepsilon - u^\varepsilon)(s) \|_{L^2(t)} ds + \| \int_{t}^{t + \theta} g(u^\varepsilon(s)) dB \|_{L^2(t)}. \tag{54}
\]

Taking the square in both side of (54), we have

\[
\| v^\varepsilon(t + \theta) - v^\varepsilon(t) \|_{L^2(t)}^2 \leq \left( \int_{t}^{t + \theta} \| u^\varepsilon_{xx}(s) \|_{L^2(t)} ds + \int_{t}^{t + \theta} \| (u^\varepsilon - u^\varepsilon)(s) \|_{L^2(t)} ds + \| \int_{t}^{t + \theta} g(u^\varepsilon(s)) dB \|_{L^2(t)} \right)^2 \leq C \theta \int_{t}^{t + \theta} \| u^\varepsilon_{xx} \|_{L^2(t)}^2 dt + \| (u^\varepsilon - u^\varepsilon) \|_{L^2(t)}^2 ds + C \| \int_{t}^{t + \theta} g(u^\varepsilon(s)) dB \|_{L^2(t)}^2
\]
We can infer from (8) and (52) that
\[
E \int_0^T \int_t^{t+\theta} \| u_{xx}^\varepsilon(t) \|_{L^2(I)}^2 ds dt \leq \delta E \int_0^T \| u_{xx}^\varepsilon(t) \|_{L^2(I)}^2 dt \leq C \delta,
\]
\[
E \int_0^T \int_t^{t+\theta} \| u \varepsilon^3 - u^\varepsilon \|_{L^2(I)}^2 ds dt = \delta E \int_0^T \| u \varepsilon^3 - u^\varepsilon \|_{L^2(I)}^2 dt
\leq C \delta [E (\int_0^T \| u_{xx}^\varepsilon(t) \|_{L^2(I)}^2 dt)^2 + E \sup_{0 \leq t \leq T} \| u^\varepsilon \|_{L^2(I)}^8]
+ E \int_0^T \| u^\varepsilon \|_{L^2(I)}^2 dt \leq C \delta.
\]
\tag{55}

By the Burkholder-Davis-Gundy inequality and Young’s inequality, we have
\[
E \sup_{0 \leq t \leq \theta} \int_t^{t+\theta} g(u^\varepsilon(s)) dB \|_{L^2(I)}^2 dt \leq E \sup_{0 \leq t \leq \delta} \int_t^{t+\theta} g(u^\varepsilon(s)) dB \|_{L^2(I)}^2 dt
\leq E \int_0^T E \sup_{0 \leq t \leq \delta} \int_t^{t+\theta} g(u^\varepsilon(s)) dB \|_{L^2(I)}^2 dt
\leq C \delta E \int_0^T \| g(u^\varepsilon(s)) \|_{L^2(I)}^2 ds dt
\leq C \delta E \int_0^T (1 + \| u^\varepsilon \|_{L^2(I)}^2) dt \leq C \delta.
\tag{56}

It follows from (55)-(56) that
\[
E \sup_{0 \leq t \leq \theta} \int_0^{T} \| u^\varepsilon(t + \theta) - u^\varepsilon(t) \|_{L^2(I)}^2 dt \leq C \delta.
\]

By the regularity theory of elliptic equation
\[
\begin{aligned}
\begin{cases}
u^\varepsilon - \varepsilon u_{xx}^\varepsilon = 0 & \text{in } I \\
u^\varepsilon(0, t) = 0 = u^\varepsilon(1, t),
\end{cases}
\end{aligned}
\]
we have
\[
\| u^\varepsilon(t) \|_{L^2(I)} \leq \| u^\varepsilon(t) \|_{L^2(I)}
\]
thus, we have (53).

6.2. **Tightness property of \( \{u^\varepsilon\}_{\varepsilon \in (0, \frac{1}{2})} \) in \( L^2(0, T; H^1(I)) \).** We may rewrite Lemma 2.1 in the following more convenient form. By the same way as in [26, P919], according to the priori estimates (7)(8) and (53), we obtain that

**Lemma 6.1.** For any \( 1 \leq p < \infty \) and for any sequences \( \mu_m, \nu_m \) converging to 0 such that the series \( \sum_{m=1}^{\infty} \frac{\mu_m^p}{\nu_m} \) converges, \( \{u^\varepsilon\}_{\varepsilon \in (0, \frac{1}{2})} \) is bounded in \( X^2_{p, \mu_m, \nu_m} \) (the explicit definition of the space \( X^2_{p, \mu_m, \nu_m} \) can be found in Section 2) for any \( m \).
Set $S = L^2(0,T;H^1(I))$ and $\mathcal{B}(S)$ the $\sigma$-algebra of the Borel sets of $S$.

For any $\varepsilon \in (0,\frac{1}{2}]$, let $\Phi_\varepsilon$ be the map

$$\Phi_\varepsilon : \Omega \rightarrow S$$

$$\omega \rightarrow u^\varepsilon(\omega),$$

and $\Pi_\varepsilon$ be a probability measure on $(S, \mathcal{B}(S))$ defined by

$$\Pi_\varepsilon(A) = \mathbb{P}(\Phi_\varepsilon^{-1}(A)), A \in \mathcal{B}(S).$$

**Proposition 5.** The family of probability measures $\{\Pi_\varepsilon : \varepsilon \in (0,\frac{1}{2}]\}$ is tight in $S$.

**Proof.** For any $\rho > 0$, we should find the compact subsets

$$Y^1_\rho \subset L^2(0,T;H^1(I)),$$

such that

$$\mathbb{P}(\omega : u^\varepsilon(\omega, \cdot) \notin Y^1_\rho) \leq \rho. \quad (57)$$

Indeed, let $Y^2_\rho$ be a ball of radius $M_\rho$ in $Y^2_{\mu_m,\nu_m}$ (the explicit definition of the space $Y^2_{\mu_m,\nu_m}$ can be found in Section 2), centered at zero and with sequences $\mu_m, \nu_m$ independent of $\varepsilon$, converging to 0 and such that the series $\sum_{m=1}^{\infty} \nu_m^2 \mu_m \varepsilon$ converges. From Corollary 1, $Y^2_\rho$ is a compact subset of $L^2(0,T;H^1(I))$, and

$$\mathbb{P}(\omega : u^\varepsilon(\omega, \cdot) \notin Y^2_\rho) \leq \mathbb{P}(\omega : \|u^\varepsilon\|_{Y^2_{\mu_m,\nu_m}} > M_\rho) \leq \frac{1}{M_\rho} \mathbb{E}\|u^\varepsilon\|_{Y^2_{\mu_m,\nu_m}} \leq \frac{C}{M_\rho},$$

choosing $M_\rho = C\rho^{-1}$, we get (57).

This proves that

$$\Pi_\varepsilon(Y^2_\rho) \geq 1 - \rho,$$

for any $\varepsilon \in (0,\frac{1}{2}]$. \qed

**6.3. The convergence result.** The main idea in this part comes from [6, 7]. The proof of Theorem 1.6 is divided into several steps.

**Step 1.** We prove that $u^\varepsilon$ converges in probability to some random variable $z \in L^2(0,T;H^1(I))$.

As proved in Proposition 5, the family $\mathcal{L}(u^\varepsilon)$ is tight in $L^2(0,T;H^1(I))$. Then, due to the Skorokhod theorem for any two sequences $\{\varepsilon_n\}_{n \in N}$ and $\{\varepsilon_m\}_{m \in N}$ converging to zero, there exist subsequences $\{\varepsilon_{n(k)}\}_{k \in N}$ and $\{\varepsilon_{m(k)}\}_{k \in N}$ and a sequence of random elements

$$\{\rho_k\}_{k \in N} := \{(u_{1,k}^\varepsilon, u_{2,k}^\varepsilon, \hat{B}_k)\}_{k \in N}$$

in $L^2(0,T;H^1(I)) \times L^2(0,T;H^1(I)) \times C([0,T];\mathbb{R})$, defined on some probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$, such that

$$\mathcal{L}(\rho_k) = \mathcal{L}(u_{n(k)}^\varepsilon, u_{m(k)}^\varepsilon, B),$$

namely,

$$\mathcal{L}(u_{1,k}^\varepsilon, u_{2,k}^\varepsilon, \hat{B}_k) = \mathcal{L}(u_{n(k)}^\varepsilon, u_{m(k)}^\varepsilon, B),$$

for each $k \in N$, and $\rho_k$ converges $\bar{\mathbb{P}}$-a.s. to some random element $\rho := (u_1, u_2, \hat{B}) \in L^2(0,T;H^1(I)) \times L^2(0,T;H^1(I)) \times C([0,T];\mathbb{R})$.

We now prove $u_1 = u_2$. 

Indeed, according to the fact that $u_1^k$ and $u_2^k$ solve (1) with $B$ replaced by $\hat{B}_k$, namely, we have
\[
\begin{aligned}
&\left\{\begin{array}{ll}
d(u_1^k - \varepsilon_m(k)u_{1xx}^k) + (-u_1^{k3} + u_1^k)dt = g(u_1^k)d\hat{B}_k & \text{in } I \times (0, T) \\
u_1^k(0, t) = 0 = u_1^k(1, t) & \text{in } (0, T) \quad (58) \\
u_1^k(0) = u_0 & \text{in } I
\end{array}\right. \\
&\left\{\begin{array}{ll}
d(u_2^k - \varepsilon_m(k)u_{2xx}^k) + (-u_2^{k3} + u_2^k)dt = g(u_2^k)d\hat{B}_k & \text{in } I \times (0, T) \\
u_2^k(0, t) = 0 = u_2^k(1, t) & \text{in } (0, T) \quad (59) \\
u_2^k(0) = u_0 & \text{in } I,
\end{array}\right.
\end{aligned}
\]

it holds that
\[
(u_1^k(t), \varphi) + \varepsilon_m(k)(u_{1x}^k(t), \varphi_x) = (u_0, \varphi) + \varepsilon_m(k)(u_{0x}, \varphi_x) + \int_0^t(u_{1x}(s), \varphi_x)ds + \int_0^t(u_1^{k3} - u_1^k, \varphi)ds + \int_0^t(g(u_1^k), \varphi)d\hat{B}_k(s),
\]
and
\[
(u_2^k(t), \varphi) + \varepsilon_m(k)(u_{2x}^k(t), \varphi_x) = (u_0, \varphi) + \varepsilon_m(k)(u_{0x}, \varphi_x) + \int_0^t(u_{2x}(s), \varphi_x)ds + \int_0^t(u_2^{k3} - u_2^k, \varphi)ds + \int_0^t(g(u_2^k), \varphi)d\hat{B}_k(s).
\]

It follows from Vitali’s convergence theorem that
\[
\lim_{k \to \infty} E\|u_1^k - u_1\|_{L^2(0, T; H^1(I))} = 0,
\]
according to this equality, Theorem 1.5, [4, P284], [12, P1126, Lemma 2.1] and [17, P151, Lemma 3.1], it is easy to see for any $\delta > 0$ and any $\varphi \in H_0^1(I)$, we have
\[
\begin{aligned}
&\lim_{k \to \infty} \mathbb{P}(\|u_1^k(t), \varphi - (u_1(t), \varphi)\|_{L^2(0, T)} > \delta) = 0, \\
&\lim_{k \to \infty} \mathbb{P}(\|\int_0^t(u_{1x}(s), \varphi_x)ds - \int_0^t(u_{1x}(s), \varphi_x)ds\|_{L^2(0, T)} > \delta) = 0, \\
&\lim_{k \to \infty} \mathbb{P}(\|\int_0^t(u_1^{k3} - u_1^k, \varphi)ds - \int_0^t(u_1^{k3} - u_1^k, \varphi)ds\|_{L^2(0, T)} > \delta) = 0, \\
&\lim_{k \to \infty} \mathbb{P}(\|\int_0^t(g(u_1^k), \varphi)d\hat{B}(s) - \int_0^t(g(u_1), \varphi)d\hat{B}(s)\|_{L^2(0, T)} > \delta) = 0.
\end{aligned}
\]

By the same way, we have
\[
\begin{aligned}
&\lim_{k \to \infty} \mathbb{P}(\|u_2^k(t), \varphi - (u_2(t), \varphi)\|_{L^2(0, T)} > \delta) = 0, \\
&\lim_{k \to \infty} \mathbb{P}(\|\int_0^t(u_{2x}(s), \varphi_x)ds - \int_0^t(u_{2x}(s), \varphi_x)ds\|_{L^2(0, T)} > \delta) = 0, \\
&\lim_{k \to \infty} \mathbb{P}(\|\int_0^t(u_2^{k3} - u_2^k, \varphi)ds - \int_0^t(u_2^{k3} - u_2^k, \varphi)ds\|_{L^2(0, T)} > \delta) = 0, \\
&\lim_{k \to \infty} \mathbb{P}(\|\int_0^t(g(u_2^k), \varphi)d\hat{B}(s) - \int_0^t(g(u_2), \varphi)d\hat{B}(s)\|_{L^2(0, T)} > \delta) = 0.
\end{aligned}
\]

By taking the limit in probability as $k$ goes to infinity, we have
\[
(u_1(t), \varphi) = (u_0, \varphi) + \int_0^t(u_{1x}(s), \varphi_x)ds + \int_0^t(u_1^{k3} - u_1^k, \varphi)ds + \int_0^t(g(u_1), \varphi)d\hat{B}(s),
\]
and
\[
(u_2(t), \varphi) = (u_0, \varphi) + \int_0^t(u_{2x}(s), \varphi_x)ds + \int_0^t(u_2^{k3} - u_2^k, \varphi)ds + \int_0^t(g(u_2), \varphi)d\hat{B}(s).
\]

Then, $u_1, u_2$ coincide with the unique solution of heat equation perturbed by the noise $\hat{B}$, thus $u_1 = u_2$.

It follows from Lemma 2.2 that $u^\varepsilon$ converges in probability to some random variable $z \in L^2(0, T; H^1(I))$.

**Step 2.** We prove that $z$ is the solution of (9).

It follows from
\[
\lim_{\varepsilon \to 0} \mathbb{P}(\|u^\varepsilon - z\|_{L^2(0, T; H^1(I))} > \delta) = 0
\]
that
\[ \lim_{\varepsilon \to 0} \mathbb{P}(\|u^\varepsilon(t) - (z(t), \varphi)\|_{L^2(0,T)} > \delta) = 0, \]
\[ \lim_{\varepsilon \to 0} \mathbb{P}(\|f_t^\varepsilon(u^\varepsilon_x(s), \varphi_x)ds - f_0^\varepsilon(z_x(s), \varphi_x)ds\|_{L^2(0,T)} > \delta) = 0, \]
\[ \lim_{\varepsilon \to 0} \mathbb{P}(\|f^\varepsilon - u^\varepsilon\|_{L^2(0,T)} > \delta) = 0, \]
\[ \lim_{\varepsilon \to 0} \mathbb{P}(\|f^\varepsilon\|_{L^2(0,T)} > \delta) = 0. \]
Noting that
\[ E \sup_{0 \leq t \leq T} |\varepsilon(u_x^\varepsilon(t), \varphi_x)|^2 \leq E \sup_{0 \leq t \leq T} \varepsilon^2\|u^\varepsilon_x(t)\|_{L^2(I)}^2 \varphi_x^2 \leq \varepsilon\|\varphi_x\|^2_{L^2(I)} E \sup_{0 \leq t \leq T} \varepsilon\|u_x^\varepsilon(t)\|^2_{L^2(I)}, \]
we have
\[ \lim_{\varepsilon \to 0} E \sup_{0 \leq t \leq T} |\varepsilon(u_x^\varepsilon(t), \varphi_x)|^2 = 0. \]

By taking the limit in probability as \( \varepsilon \) goes to zero in
\[ (u^\varepsilon(t), \varphi) + \varepsilon(u_x^\varepsilon(t), \varphi_x) = (u_0, \varphi) + \varepsilon(u_0x^\varepsilon, \varphi_x) + \int_0^t (u_x^\varepsilon(s), \varphi_x)ds + \int_0^t (u^\varepsilon - u^\varepsilon, \varphi)ds + \int_0^t (g(u^\varepsilon), \varphi)dB(s), \]
we deduce that \( z \) verifies the following equation \( dt \otimes d\mathbb{P} \)-almost everywhere:
\[ (z(t), \varphi) = (u_0, \varphi) + \int_0^t (z_x(s), \varphi_x)ds + \int_0^t (\varepsilon^3 - z, \varphi)ds + \int_0^t (g(z), \varphi)dB(s), \]
that is \( z \) is the solution of (9).

7. Appendix.

Proof of Proposition 1. The main idea in this part comes from [18, 15, 16]. We consider the stochastic differential equation
\[ \begin{cases} 
(1 + \varepsilon \lambda_k)dc_k + (\lambda_k c_k + f_k)dt = g_k dB \\
\phantom{\text{\( f_k(t) = (f(t), e_k), \) \( g_k(t) = (g(t), e_k). \)}} c_k(0) = (u_0, e_k), 
\end{cases} \]
(60)
where
\[ f_k(t) = (f(t), e_k), \quad g_k(t) = (g(t), e_k). \]
We set
\[ u^m = \sum_{k=1}^m c_k(t)e_k, \quad u_{0m} = \sum_{k=1}^m c_k(0)e_k = \sum_{k=1}^m (u_0, e_k)e_k, \]
\[ f^m = \sum_{k=1}^m f_k(t)e_k, \quad g^m = \sum_{k=1}^m g_k(t)e_k, \]
it holds that
\[ \|u_{0m} - u_0\|_{L^2(\Omega; L^2(G))} \to 0, \]
\[ \|f^m - f\|_{L^2(\Omega; L^2(0,T;H^{-1}(\Theta)))} \to 0, \]
\[ \|g^m - g\|_{L^2(\Omega; L^2(0,T;L^2(G)))} \to 0, \]
as \( m \to \infty. \)
1) We have
\[ \|u^m(t)\|_{L^2(G)}^2 = \sum_{k=1}^m c_k^2(t), \]
it follows from Itô’s rule that
\[ dc_k^2 = 2c_k dc_k + (dc_k)^2 \]
\[ = 2c_k \frac{1}{1 + \varepsilon \lambda_k} (-\lambda_k c_k dt - f_k dt + g_k dB) + \frac{1}{(1 + \varepsilon \lambda_k)^2} u_k^2 dt \]
\[ = -\frac{2\lambda_k c_k^2}{1 + \varepsilon \lambda_k} dt - \frac{2c_k f_k}{1 + \varepsilon \lambda_k} dt + \frac{2c_k g_k}{1 + \varepsilon \lambda_k} dB + \frac{1}{(1 + \varepsilon \lambda_k)^2} g_k^2 dt, \]
thus,
\[
c_k^2(t) + \int_0^t \frac{2\lambda_k c_k^2}{1 + \varepsilon \lambda_k} ds
\]
\[
\leq c_k^2(0) + \int_0^t \frac{2c_k g_k}{1 + \varepsilon \lambda_k} dB + \int_0^t \frac{1}{(1 + \varepsilon \lambda_k)^2} g_k^2 ds.
\]

namely, we have
\[
c_k^2(t) + \int_0^t \frac{\lambda_k c_k^2}{1 + \varepsilon \lambda_k} ds \leq c_k^2(0) + \int_0^t \frac{2c_k g_k}{1 + \varepsilon \lambda_k} dB + \int_0^t \frac{1}{(1 + \varepsilon \lambda_k)^2} g_k^2 ds,
\]

Taking mathematical expectation from both sides of the above inequality, we have
\[
\mathbb{E} \int_0^T \frac{\lambda_k c_k^2}{1 + \varepsilon \lambda_k} dt \leq \mathbb{E} c_k^2(0) + \mathbb{E} \int_0^T \frac{f_k^2}{(1 + \varepsilon \lambda_k) \lambda_k} dt + \mathbb{E} \int_0^T \frac{1}{(1 + \varepsilon \lambda_k)^2} g_k^2 dt.
\]

By the Burkholder-Davis-Gundy inequality, we have
\[
\mathbb{E} \sup_{0 \leq t \leq T} c_k^2(t) \leq \mathbb{E} c_k^2(0) + \mathbb{E} \int_0^T \frac{f_k^2}{(1 + \varepsilon \lambda_k) \lambda_k} dt + \mathbb{E} \sup_{0 \leq t \leq T} | f_k \int_0^t \frac{g_k}{1 + \varepsilon \lambda_k} dB | + \mathbb{E} \int_0^T \frac{1}{(1 + \varepsilon \lambda_k)^2} g_k^2 dt
\]
\[
\leq \mathbb{E} c_k^2(0) + \mathbb{E} \int_0^T \frac{f_k^2}{(1 + \varepsilon \lambda_k) \lambda_k} dt + \frac{1}{2} \mathbb{E} \sup_{0 \leq t \leq T} c_k^2(t) + C \mathbb{E} \int_0^T \frac{1}{(1 + \varepsilon \lambda_k)^2} g_k^2 dt + \mathbb{E} \int_0^T \frac{1}{(1 + \varepsilon \lambda_k)^2} g_k^2 dt,
\]

thus,
\[
\mathbb{E} \sup_{0 \leq t \leq T} c_k^2(t) \leq C \mathbb{E} c_k^2(0) + \mathbb{E} \int_0^T \frac{f_k^2}{(1 + \varepsilon \lambda_k) \lambda_k} dt + \mathbb{E} \int_0^T \frac{1}{(1 + \varepsilon \lambda_k)^2} g_k^2 dt.
\]

According to (61) and (62), we have
\[
\mathbb{E} \sup_{0 \leq t \leq T} c_k^2(t) + \mathbb{E} \int_0^T \frac{\lambda_k c_k^2}{1 + \varepsilon \lambda_k} dt \leq C \mathbb{E} c_k^2(0) + \mathbb{E} \int_0^T \frac{f_k^2}{(1 + \varepsilon \lambda_k) \lambda_k} dt + \mathbb{E} \int_0^T \frac{1}{(1 + \varepsilon \lambda_k)^2} g_k^2 dt.
\]

Taking the sum on \( k \) in (63), we get
\[
\mathbb{E} \sup_{0 \leq t \leq T} ||u_m(t)||_{L^2(G)}^2 \leq C \mathbb{E} ||u_{0m}||_{L^2(G)}^2 + \mathbb{E} \int_0^T \|f'(t)||_{H^{-1}(G)} dt + \mathbb{E} \int_0^T \|g^m(t)||_{L^2(G)} dt
\]
\[
\leq C \mathbb{E} ||u_{0m} - u_0||_{L^2(G)}^2 + \mathbb{E} \int_0^T \|f' - f(t)||_{H^{-1}(G)} dt + \mathbb{E} \int_0^T \|g^m - g(t)||_{L^2(G)} dt,
\]

where \( C \) denotes a positive constant independent of \( n, m \) and \( T \).

Next we observe that the right-hand side of (65) converges to zero as \( n, m \to \infty \). Hence, it follows that \( \{u_m\}_{m=1}^\infty \) is a Cauchy sequence that converges strongly in \( L^2(\Omega, C([0, T]; L^2(G))) \). Let \( u \) be the limit, namely, we have
\[
||u_m - u||_{L^2(\Omega, C([0, T]; L^2(G)))} \to 0,
\]
as \( m \to \infty \).

Also, it follows from (60) that
\[
(u_m(t), e_k) - \varepsilon(u(t), \triangle e_k)
\]
\[
= (u_{0m}, e_k) - \varepsilon(u_{0m}, \triangle e_k) + \int_0^t (u_m(s), \Delta e_k) ds - \int_0^t f'(m(s), e_k) ds + \int_0^t (g^m(s), e_k) dB(s)
\]
for all \( k = 1, 2, 3 \cdots \), and all \( t \in [0, T] \), for almost all \( \omega \in \Omega \).

By taking the limit in above equality as \( m \) goes to infinity, it holds that
\[
(u(t), e_k) - \varepsilon(u(t), \triangle e_k)
\]
\[
= (u_0, e_k) - \varepsilon(u_0, \Delta e_k) + \int_0^t (u(s), \Delta e_k) ds - \int_0^t f'(s, e_k) ds + \int_0^t (g(s), e_k) dB(s)
\]
for all \( k = 1, 2, 3 \cdots \), and all \( t \in [0, T] \), for almost all \( \omega \in \Omega \). Thus, we have
\[
(u(t), \varphi) - \varepsilon(u(t), \Delta \varphi)
\]
\[
= (u_0, \varphi) - \varepsilon(u_0, \Delta \varphi) + \int_0^t (u(s), \Delta \varphi) ds - \int_0^t f'(s, \varphi) ds + \int_0^t (g(s), \varphi) dB(s)
\]
holds for all $t \in [0, T]$ and all $\varphi \in H^2(G) \cap H^1_0(G)$, for almost all $\omega \in \Omega$.

Namely, $u$ is a solution to (11). By taking the limit in (64) as $m$ goes to infinity, we can obtain (13).

Now, we prove the uniqueness of the solution for (11). Indeed, if $u_1$ and $u_2$ are the solutions for (11), according to (13), we have

$$
E \sup_{0 \leq t \leq T} \| (u_1 - u_2)(t) \|^2_{L^2(G)} \leq 0,
$$

thus, $u_1 \equiv u_2$.

2) Let $h_k = (1 + \varepsilon \lambda_k) c_k^2$, following [19, P28] or [15], we have

$$
\| u^n(u) \|^2_{L^2(G)} + \varepsilon \| \nabla u^n(u) \|^2_{L^2(G)} = \sum_{k=1}^{m}(1 + \varepsilon \lambda_k)c_k^2(t) = \sum_{k=1}^{m} h_k.
$$

By multiplying (63) by $1 + \varepsilon \lambda_k$, we have

$$
E \sup_{0 \leq t \leq T} h_k(t) + E \int_0^T \lambda_k c_k^2 dt \leq C(E h_k(0) + E \int_0^T \int_0^T \frac{1}{1 + \varepsilon \lambda_k} g_k^2 dt dt)
$$

Taking the sum on $k$ in (66), we get

$$
E \sup_{0 \leq t \leq T} \| (u^n(u))^2_{L^2(G)} + \varepsilon \| \nabla u^n(u) \|^2_{L^2(G)} + E \int_0^T \| \nabla u^n(u) \|^2_{L^2(G)} dt
$$

Thus,

$$
E \sup_{0 \leq t \leq T} \| (u^n(u) - u^n(u))^2_{L^2(G)} + \varepsilon \| \nabla (u^n(u) - u^n(u)) \|^2_{L^2(G)} + E \int_0^T \| \nabla (u^n(u) - u^n(u)) \|^2_{L^2(G)} dt
$$

Thus, we have

$$
\| u^n(u) - u^n(u) \|^2_{L^2(\Omega,C([0, T]; H^1(G)))} \cap L^2(\Omega,L^2(0, T; H^1(G))) \to 0,
$$

as $m \to \infty$.

Also, it follows from (60) that

$$(u^n(t), e_k) + \varepsilon (\nabla u^n(t), \nabla e_k) = (u_0, e_k) + \varepsilon (\nabla u_0, \nabla e_k) - \int_0^t (L^2(\Omega, C([0, T]; H^1(G))) \cap L^2(\Omega, L^2(0, T; H^1(G))) \to 0,
$$

for all $k = 1, 2, 3, \cdots$, and all $t \in [0, T]$, for almost all $\omega \in \Omega$.

By taking the limit in above equality as $m$ goes to infinity, it holds that

$$(u(t), e_k) + \varepsilon (\nabla u(t), \nabla e_k) = (u_0, e_k) + \varepsilon (\nabla u_0, \nabla e_k) - \int_0^t (L^2(\Omega, C([0, T]; H^1(G))) \cap L^2(\Omega, L^2(0, T; H^1(G))) \to 0,
$$

for all $k = 1, 2, 3, \cdots$, and all $t \in [0, T]$, for almost all $\omega \in \Omega$.

Thus, it holds that

$$(u(t), \varphi) + \varepsilon (\nabla u(t), \nabla \varphi) = (u_0, \varphi) + \varepsilon (\nabla u_0, \nabla \varphi) + \int_0^t (L^2(\Omega, C([0, T]; H^1(G))) \cap L^2(\Omega, L^2(0, T; H^1(G))) \to 0,
$$

holds for all $t \in [0, T]$ and all $\varphi \in H^1_0(G)$, for almost all $\omega \in \Omega$.

By taking the limit in (67) as $m$ goes to infinity, we can obtain (14).

3) We have

$$
\| \nabla u^n(t) \|^2_{L^2(G)} + \varepsilon \| \Delta u^n(t) \|^2_{L^2(G)} = \sum_{k=1}^{m}(1 + \varepsilon \lambda_k)c_k^2(t) = \sum_{k=1}^{m} \lambda_k h_k.
$$
Multiplying (63) by \((1 + \varepsilon \lambda_k)\lambda_k\), we have
\[
\mathbb{E} \sup_{0 \leq t \leq T} (\lambda_k h_k(t)) + \mathbb{E} \int_0^T \lambda_k^2 \varepsilon_k^2 dt \leq C(\mathbb{E}(\lambda_k h_k(0)) + \mathbb{E} \int_0^T f_k^2 dt + \mathbb{E} \int_0^T \lambda_k g_k^2 dt).
\]
Taking the sum on \(k\) in (69), we get
\[
\mathbb{E} \sup_{0 \leq t \leq T} \left( \|\nabla u^m(t)\|_{L^2(G)}^2 + \varepsilon \|\nabla u^m(t)\|_{L^2(G)}^2 \right) + \mathbb{E} \int_0^T \|\Delta u^m(t)\|_{L^2(G)}^2 dt \leq C\mathbb{E}(\|\nabla u_0\|_{L^2(G)}^2 + \varepsilon \|\nabla u_0\|_{L^2(G)}^2) + \mathbb{E} \int_0^T \|f^m(t)\|_{L^2(G)}^2 dt + \mathbb{E} \int_0^T \|g^m(t)\|_{H^1(G)}^2 dt,
\]
thus,
\[
\mathbb{E} \sup_{0 \leq t \leq T} \left( \|\nabla (u^m - u^n)(t)\|_{L^2(G)}^2 + \varepsilon \|\Delta (u^m - u^n)(t)\|_{L^2(G)}^2 \right) + \mathbb{E} \int_0^T \|\Delta (u^m - u^n)(t)\|_{L^2(G)}^2 dt \leq C\mathbb{E}(\|\nabla u_{0m} - \nabla u_{0n}\|_{L^2(G)}^2 + \varepsilon \|\nabla u_{0m} - \nabla u_{0n}\|_{L^2(G)}^2) + \mathbb{E} \int_0^T \|(f^m - f^n)(t)\|_{L^2(G)}^2 dt + \mathbb{E} \int_0^T \|(g^m - g^n)(t)\|_{H^1(G)}^2 dt.
\]
where \(C\) denotes a positive constant independent of \(n, m, T\). Next we observe that the right-hand side of (70) converges to zero as \(n, m \to \infty\). Hence, it follows that \(\{u^m\}_{m=1}^\infty\) is a Cauchy sequence that converges strongly in \(L^2(\Omega, C([0, T]; H^2(G))) \cap L^2(\Omega, L^2(0, T; H^2(G)))\). Let \(u\) be the limit.

By the same argument as in 1) and 2), \(u\) is the solution of (11).

**Acknowledgments.** This work is supported by the Fundamental Research Funds for the Central Universities (2412020FZ022). Peng Gao would like to thank the financial support of the China Scholarship Council (No. 201806025036) and the hospitality of CNRS and IMJ, Université Paris Diderot-Paris 7 during his visit from December 2018 to November 2019. Peng Gao would like to thank the referees and the editor for their careful comments and useful suggestions. Peng Gao sincerely thank Professor Yong Li for many useful suggestions and help.

**REFERENCES**

[1] E. C. Aifantis, On the problem of diffusion in solids, *Acta Mechanica*, 37 (1980), 265–296.
[2] C. T. Anh and T. Q. Bao, Pullback attractors for a class of non-autonomous nonclassical diffusion equations, *Nonlinear Anal.*, 73 (2010), 399–412.
[3] L. Bai and F. Zhang, Existence of random attractors for 2D-stochastic nonclassical diffusion equations on unbounded domains, *Results Math.*, 69 (2016), 129–160.
[4] A. Bensoussan, Stochastic Navier-Stokes equations, *Acta Appl. Math.*, 38 (1995), 267–304.
[5] R. Camassa and D. D. Holm, An integrable shallow water equation with peaked solitons, *Phys. Rev. Lett.*, 71 (1993), 1661–1664.
[6] S. Cerrai and M. Freidlin, On the Smoluchowski-Kramers approximation for a system with an infinite number of degrees of freedom, *Probab. Theory Related Fields*, 135 (2006), 363–394.
[7] S. Cerrai and M. Freidlin, Smoluchowski-Kramers approximation for a general class of SPDEs, *J. Evol. Equ.*, 6 (2006), 657–689.
[8] S. Cerrai and M. Salins, On the Smoluchowski-Kramers approximation for a system with infinite degrees of freedom exposed to a magnetic field, *Stochastic Process. Appl.*, 127 (2017), 273–303.
[9] S. Cerrai and M. Salins, Smoluchowski-Kramers approximation and large deviations for infinite-dimensional nongradient systems with applications to the exit problem, *Ann. Probab.*, 44 (2016), 2591–2642.
[10] S. Cerrai and M. Salins, Smoluchowski-Kramers approximation and large deviations for infinite dimensional gradient systems, *Asymptot. Anal.*, 88 (2014), 201–215.
[11] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, 2nd edition, Cambridge University Press, 2014.
[12] A. Debussche, N. Glatt-Holtz and R. Temam, Local martingale and pathwise solutions for an abstract fluids model, Phys. D, 240 (2011), 1123–1144.

[13] G. Deugoué, P. A. Razafimandimby and M. Sango, On the 3-D stochastic magnetohydrodynamical-α model, Stochastic Process. Appl., 122 (2012), 2211–2248.

[14] G. Deugoue and M. Sango, Weak solutions to stochastic 3D Navier-Stokes-α model of turbulence: α-asymptotic behavior, J. Math. Anal. Appl., 384 (2011), 49–62.

[15] P. Gao, Carleman estimate and unique continuation property for the linear stochastic Korteweg-de Vries equation, Bull. Aust. Math. Soc., 90 (2014), 283–294.

[16] P. Gao, Global Carleman estimates for linear stochastic Kawahara equation and their applications, Math. Control Signals Systems, 28 (2016), 1–22.

[17] I. Gyöngy and N. Krylov, Existence of strong solutions for Itô’s stochastic equations via approximations, Probab. Theory Related Fields, 105 (1996), 143–158.

[18] J. U. Kim, Approximate controllability of a stochastic wave equation, Appl. Math. Optim., 49 (2004), 81–98.

[19] J. U. Kim, Periodic and invariant measures for stochastic wave equations, Electron. J. Differential Equations, (2004), 1–30.

[20] Y. Lv and A. J. Roberts, Averaging approximation to singularly perturbed nonlinear stochastic wave equations, J. Math. Phys., 53 (2012), 062702.

[21] Y. Lv and W. Wang, Limiting dynamics for stochastic wave equations, J. Differential Equations, 244 (2008), 1–23.

[22] J. C. Peter and M. E. Gurtin, On a theory of heat conduction involving two temperatures, Z. Angew. Math. Phys., 19 (1968), 614–627.

[23] M. Renardy and R. C. Rogers, An Introduction to Partial Differential Equations, Applied Mathematical Sciences, 68. Springer-Verlag, New York, 1988.

[24] W. Wang and Y. Lv, Limit behavior of nonlinear stochastic wave equations with singular perturbation, Discrete Contin. Dyn. Syst. Ser. B, 13 (2010), 175–193.

[25] E. Waymire and J. Duan, Probability and Partial Differential Equations in Modern Applied Mathematics, Springer-Verlag, New York, 2005.

[26] F. H. Zhang and W. Han, Pullback attractors for nonclassical diffusion delay equations on unbounded domains with non-autonomous deterministic and stochastic forcing terms, Electron. J. Differential Equations, 2016 (2016), Paper No. 139, 28 pp.

[27] W. Zhao and S. Song, Dynamics of stochastic nonclassical diffusion equations on unbounded domains, Electron. J. Differential Equations, 282 (2015), 1–22.

Received June 2021; revised October 2021; early access December 2021.

E-mail address: gaopengjilindaxue@126.com