Series expansions for Maass forms on the full modular group from the Farey transfer operators

Claudio Bonanno*  Stefano Isola†

Abstract

We analyze the relations previously established by Mayer, Lewis-Zagier and the authors, among the eigenfunctions of the transfer operators of the Gauss and the Farey maps, the solutions of the Lewis-Zagier three-term functional equation and the Maass forms on the modular surface $SL(2,\mathbb{Z})\backslash \mathcal{H}$. As main result, we establish new series expansions for the Maass cusp forms and the non-holomorphic Eisenstein series restricted to the imaginary axis.

1 Introduction

One of the most interesting objects in the mathematics literature are certainly the Maass forms on the full modular group $\Gamma = PSL(2,\mathbb{Z})$, that is smooth $\Gamma$-invariant complex functions $u$ defined on the upper half-plane $\mathbb{H} = \{z = x + iy : y > 0\}$, increasing less than exponentially as $y \to \infty$, and satisfying $\Delta u = \lambda u$ for some $\lambda \in \mathbb{C}$, where $\Delta = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$ is the hyperbolic Laplacian. Maass forms divide into cusp and non-cusp forms according to their behaviour at the cusp of the modular surface $\Gamma\backslash \mathcal{H}$, and into even and odd forms according to whether $u(-x + iy) = \pm u(x + iy)$.

Despite their importance, Maass cusp forms remain a mysterious object. No explicit construction exists and all basic information about their existence comes from the Selberg trace formula. For this reason there has been some interest in finding correspondence with other spaces of functions. One approach is due to the contributions by Mayer [10] and Lewis and Zagier [8, 9]. Maass non-cusp forms are instead well known, being generated by the non-holomorphic Eisenstein series.

In the paper [10] Mayer used the relation between the length of the closed geodesics on $\Gamma\backslash \mathcal{H}$, the so-called length spectrum of $\Gamma\backslash \mathcal{H}$, as encoded in the associated Selberg zeta function $Z(q)$, and the spectral parameters of the group $\Gamma$, that is the complex numbers of the form $q(1-q)$ for which there exists a Maass cusp form satisfying $\Delta u = \lambda u$ with $\lambda = q(1-q)$, to obtain the equality

$$Z(q) = \det (1 - \mathcal{L}_q) \det (1 + \mathcal{L}_q), \quad q \in \mathbb{C}. \quad (1.1)$$

Here “det” indicates the determinant in the sense of Fredholm, and $\mathcal{L}_q$ denotes the meromorphic extension to $q \in \mathbb{C}$ of the family of nuclear of order zero endomorphisms defined by

$$(\mathcal{L}_q h)(z) = \sum_{n=1}^{\infty} \frac{1}{(z+n)^{2q}} h \left( \frac{1}{z+n} \right)$$

for $\Re(q) > \frac{1}{2}$, on the space $H(D)$ of holomorphic functions in the disk $D = \{ z \in \mathbb{C} : |z-1| < \frac{3}{2} \}$. The connection [10] comes from the arithmetic properties of the length spectrum of $\Gamma\backslash \mathcal{H}$ and the fact that the

* Dipartimento di Matematica, Università di Pisa, Largo Bruno Pontecorvo 5, I-56127 Pisa, Italy. Email: claudio.bonanno@unipi.it
† Dipartimento di Matematica e Informatica, Università di Camerino, via Madonna delle Carceri, I-62032 Camerino, Italy. Email: stefano.isola@unicam.it

1 Up to our knowledge.
endomorphisms $L_q$ are the transfer operators of the Gauss map, which is a dynamical system related to the continued fractions expansion of a real number. Combining Mayer’s equality \((1.1)\) sharpened by Efrat in \([4]\) with the known positions of the zeroes of $Z(q)$ as implied by the Selberg trace formula, one can state the following

**Theorem A** (\([4],[10]\)). Let $q = \xi + i\eta$ be a complex number with $\xi > 0$ and $q \neq \frac{1}{2}$. Then:

(i) there exists a nonzero $h \in H(D)$ such that $L_q h = h$ if and only if $q$ is either an even spectral parameter of $\Gamma$, that is there exists an even Maass cusp form $u$ such that $\Delta u = q(1-q)u$, or $2q$ is a non-trivial zero of the Riemann zeta function, or $q = 1$;

(ii) there exists a nonzero $h \in H(D)$ such that $L_q h = -h$ if and only if $q$ is an odd spectral parameter of $\Gamma$, that is there exists an odd Maass cusp form $u$ such that $\Delta u = q(1-q)u$.

In the papers \([8],[9]\) Lewis and Zagier introduced a three-term functional equation whose solutions are in one-to-one correspondence with the Maass cusp and non-cusp forms. Using the results for the spectral parameters of $\Gamma \backslash H$ and for the Maass non-cusp forms, they proved

**Theorem B** (\([9]\)). There is an isomorphism between the Maass cusp forms with eigenvalue $q(1-q)$ and the space of real-analytic solutions of the three-term functional equation

$$\psi(x) = \psi(x+1) + (x+1)^{-2q} \psi\left(\frac{x}{x+1}\right), \quad x \in \mathbb{R}^+ \quad (1.2)$$

with the conditions

$$\psi(x) = O(1) \text{ as } x \to 0^+, \quad \psi(x) = O(1/x) \text{ as } x \to +\infty \quad (1.3)$$

Moreover the Maass non-cusp forms, which for a given $q$ are a one-dimensional space generated by the non-holomorphic Eisenstein series $E(z,q)$, are in one-to-one correspondence with the functions

$$\psi_q^+(x) = \frac{\zeta(2q)}{2} (1 + x^{-2q}) + \sum_{m,n \geq 1} \frac{1}{D(m^2 + n)^{2q}}, \quad \Re(q) > 1 \quad (1.4)$$

where $\zeta(s)$ is the Riemann zeta function, which when multiplied by $\frac{\Gamma(2q)}{q(1-q)}$ can be analytically continued to $q \in \mathbb{C}$ as solutions of \((1.2)\).

In \([9]\) the solutions of equation \((1.2)\) are called *period functions* because of an analogy, explored in the paper, with the classical Eichler-Shimura-Manin period polynomials of the holomorphic cusp forms. Moreover, the period functions associated to a Maass forms are divided into even and odd functions.

Putting together Theorems A and B we have the following situation for the zeroes of the Selberg zeta function $Z(q)$:

- if $q$ is an even spectral parameter with $\xi = \frac{1}{2}$, then there exist a nonzero $h \in H(D)$ such that $L_q h = h$ and an even real-analytic function $\psi(x)$ which satisfies \((1.2)\) with conditions \((1.3)\);

- if $q$ is an odd spectral parameter with $\xi = \frac{1}{2}$, then there exist a nonzero $h \in H(D)$ such that $L_q h = -h$ and an odd real-analytic function $\psi(x)$ which satisfies \((1.2)\) with conditions \((1.3)\);

- if $2q$ is a non-trivial zero of the Riemann zeta function, then there exist a nonzero $h \in H(D)$ such that $L_q h = h$ and \((1.2)\) has solutions given by multiples of the analytic continuation of the function $\psi_q^+$;

- if $q = 1$ then there exist a nonzero $h \in H(D)$ such that $L_q h = h$, in fact we have $h(x) = \frac{x}{x+1}$, and \((1.2)\) has solutions given by multiples of the function $\psi_1^+(x) = \frac{1}{2}$.
Moreover there is an explicit relation between the eigenfunctions of the operator $L_q$ and the period functions relative to the same $q$. Namely $h(x) = \psi(x+1)$, and the same hold on $D$ where $\psi(z+1)$ is the holomorphic extension of $\psi$ to $\mathbb{C} \setminus (-\infty, 0]$.

The beauty of Mayer’s result lies in the displaying of the power of the theory of transfer operators for dynamical systems, but the spectral properties of the operators $L_q$ turned out to be difficult to study, see [3] and [1]. On the other side, Lewis and Zagier approach has the advantage of introducing a relation of Maass forms with solutions of an equation with a finite number of terms, which might be easier to handle.

These two aspects are combined in our paper [2], where we used a family of signed transfer operators for the Farey map, a “parent” of the Gauss map, defined for $\xi = \Re(q) > 0$ by

\[(P_q^\pm f)(z) = (P_{0,q} f)(z) \pm (P_{1,q} f)(z) := \left(\frac{1}{z+1}\right)^{2q} f\left(\frac{z}{z+1}\right) \pm \left(\frac{1}{z+1}\right)^{2q} f\left(\frac{1}{z+1}\right)\]  

where $z \in B = \{z \in \mathbb{C} : |z - \frac{1}{2}| < \frac{1}{2}\}$ and $f \in H(B)$. We studied the problem of existence of eigenfunctions for $P_q^\pm$ and proved

**Theorem C** ([2]). (a) If $f \in H(B)$ satisfies $P_q^\pm f = \lambda f$ with $\lambda \neq 0$ then $f \in H(\{\Re(z) > 0\})$ and we call it even in the sense that $L_q f = f$, where

\[(L_q f)(z) := \frac{1}{z^{2q}} f\left(\frac{1}{z}\right)\]

Moreover it satisfies

\[\lambda f(z) = f(z+1) + (z+1)^{-2q} f\left(\frac{z}{z+1}\right), \quad \Re(z) > 0.\]

(b) If $f \in H(B)$ satisfies $P_q^- f = \lambda f$ with $\lambda \neq 0$ then $f \in H(\{\Re(z) > 0\})$ and we call it odd in the sense that $L_q f = -f$. Moreover it satisfies

\[(L_q f)(z) := \frac{1}{z^{2q}} f\left(\frac{1}{z}\right)\]

Moreover it satisfies

\[\lambda f(z) = f(z+1) + (z+1)^{-2q} f\left(\frac{z}{z+1}\right), \quad \Re(z) > 0.\]

(c) If $f \in H(\{\Re(z) > 0\})$ satisfies $P_q^- f = \lambda f$ for $\lambda \neq 0$, then $\varphi_\pm := \frac{1}{2}(f \pm L_q f)$ satisfies $P_q^\pm \varphi_\pm = \lambda \varphi_\pm$.

(d) If $f \in H(B)$ satisfies $P_q^+ f = \lambda f$ with $\lambda \in [0,1)$ then there exists $\phi \in L^2((0, +\infty), t^{2q-1} e^{-t} dt)$ such that $f$ can be written as

\[f(z) = c \frac{\lambda^+}{z^{2q}} + b \frac{\Gamma(2q-1)}{\Gamma(2q)} \frac{1}{z} + \frac{1}{z^{2q}} \int_0^\infty e^{-\frac{t}{z}} \phi(t) t^{2q-1} dt, \quad \Re(z) > 0,\]

where $c,b \in \mathbb{C}, \phi(0) is finite and $\phi(t) - \phi(0) = O(t)$ as $t \to 0^+$, and the last term is bounded as $\Re(z) \to 0$. Moreover if $\lambda \neq 1$ then $b = 0$.

(e) If $f \in H(B)$ satisfies $P_q^- f = \lambda f$ with $\lambda \in [0,1)$ then there exists $\phi \in L^2((0, +\infty), t^{2q-1} e^{-t} dt)$ such that $f$ can be written as

\[f(z) = c \frac{\lambda^-}{z^{2q}} + \frac{1}{z^{2q}} \int_0^\infty e^{-\frac{t}{z}} \phi(t) t^{2q-1} dt, \quad \Re(z) > 0,\]

where $c \in \mathbb{C}, \phi(0) is finite and $\phi(t) - \phi(0) = O(t)$ as $t \to 0^+$, and the last term is bounded as $\Re(z) \to 0$.

Using the operators $P_q^\pm$ we introduced a generalization of the transfer operators $L_q$, namely the two variable operators $L_{q,w}$ formally defined as

\[L_{q,w} = w P_1 (1-w P_{0,q})^{-1}\]

We proved that as operators acting on the Banach space $H_{\infty}(D_q)$ of functions holomorphic on $D_q = \{z \in \mathbb{C} : |z - 1| < \frac{1}{2} - \varepsilon\}$ and bounded on $\overline{D_q}$, they are nuclear of order zero for $\Re(q) > 0$ and $w \in \mathbb{C} \setminus (1, \infty)$. Moreover the function $q \mapsto L_{q,w}$ is analytic in $\Re(q) > 0$ for any $w \in \mathbb{C} \setminus [1, \infty)$ and is meromorphic in $\Re(q) > 0$ for $w = 1$ with a simple pole at $q = \frac{1}{2}$. Analogously the function $w \mapsto L_{q,w}$ is analytic in $w \in \mathbb{C} \setminus [1, \infty)$.
for any \( q \) with \( \Re(q) > 0 \). Hence we can compute the Fredholm determinants of the operators \((1 \pm L_{q,w})\) and define the two-variable Selberg zeta function

\[
Z(q, w) := \det(1 - L_{q,w}) \det(1 + L_{q,w})
\]

(1.10)

for \( \Re(q) > 0 \) and \( w \in \mathbb{C} \setminus (1, \infty) \). For \( w = 1 \) the function \( Z(q, 1) \) is meromorphic in \( \Re(q) > 0 \) with a simple pole at \( q = \frac{1}{2} \) and coincides with the Selberg zeta function \( Z(q) \).

Finally we managed to obtain a relation between the eigenfunctions of \( L_{q,w} \) and of \( P_q^\pm \), thus, thanks to Theorem C-(a,b), obtaining a relations between the solutions of the generalized three-term functional equation (1.7) and the zeroes of the function \( Z(q) \).

**Theorem D** (2). (a) Let \( w = 1 \). Then:

- \( q \) is an even spectral parameter with \( \xi = \frac{1}{2} \) if and only if there exists an even \( f \in H(B) \) such that \( P_q^+ f = f \) satisfies (1.7) with \( \lambda = 1 \) (or (1.2)) and it can be written as in (1.8) with \( c = b = 0 \);
- \( q \) is an odd spectral parameter with \( \xi = \frac{1}{2} \) if and only if there exists an odd \( f \in H(B) \) such that \( P_q^- f = f \) satisfies (1.7) with \( \lambda = 1 \) (or (1.2)) and it can be written as in (1.9) with \( c = 0 \);
- \( 2q \) is a non-trivial zero of the Riemann zeta function if and only if there exists an even \( f \in H(B) \) such that \( P_q^+ f = f \) satisfies (1.7) with \( \lambda = 1 \) (or (1.2)) and it can be written as in (1.8) with \( c = 0 \) and \( b \neq 0 \);
- \( q = 1 \) is a zero of \( Z(q,1) \) since \( f(z) = \frac{1}{z} \) satisfies \( P_1^+ f = f \).

(b) Let \( w \in \mathbb{C} \setminus [1, \infty) \). Then:

- \( q \) is an “even” zero of \( Z(q,w) \) if and only if there exists an even \( f \in H(B) \) such that \( P_q^+ f = \frac{1}{w} f \) satisfies (1.7) with \( \lambda = \frac{1}{w} \) and it can be written as in (1.8) with \( c = b = 0 \);
- \( q \) is an “odd” zero of \( Z(q,w) \) if and only if there exists an odd \( f \in H(B) \) such that \( P_q^- f = \frac{1}{w} f \) satisfies (1.7) with \( \lambda = \frac{1}{w} \) and it can be written as in (1.9) with \( c = 0 \).

Since by Theorem C-(a,b), eigenfunctions of \( P_q^\pm \) satisfy a three-term equation which is a generalization of the Lewis-Zagier equation (1.2), we call the functions \( f \) of Theorem D generalized period functions (gpf) associated to the zeroes of the zeta function \( Z(q,w) \), even and odd according to whether they correspond to even or odd zeroes. Moreover we distinguish the two classes of gpf of \( f \) with \( b = 0 \), which we call 0-gpf, and \( b \neq 0 \), which we call b-gpf. In the \( w = 1 \) case the 0-gpf correspond to the Maass cusp forms and the b-gpf to the non-cusp forms. Moreover in the \( w \neq 1 \) case the set of b-gpf is empty.

In this paper we use the integral correspondence proved in [3] between the period functions of Theorem B and the Maass forms, to obtain our results, new series expansions for the Maass forms restricted to the imaginary axis in terms of Legendre functions in Section 4 and as formal power series in Section 3.

## 2 Notations for special functions and integral transforms

We use the standard notation \( \mathbb{F}_1(a, b; c; x) \) for the hypergeometric function, \( J_\nu(z) \) for the Bessel functions of first kind, \( K_\nu(z) \) for the modified Bessel functions of the third kind, \( L_\nu^n(t) \) for the generalized Laguerre polynomials, \( \Gamma(\nu) \) for the Gamma function, \( \zeta(\nu, a) \) for the Hurwitz zeta function and \( \Phi(z, \nu, a) \) for the Lerch zeta function (see [3]), \( P_\nu^n \) for the Legendre functions in the real interval \((-1, 1)\).

In the following we use the following integral transforms (see [6] and [7]):

- **Laplace transform**

\[
\mathcal{L}[\varphi](z) := \int_0^\infty e^{-zt} \varphi(t) \, dt
\]
• **Symmetric Hankel transform**

\[
\mathcal{H}_\nu [\varphi](z) := \int_0^\infty J_\nu (tz) \sqrt{tz} \varphi(t) \, dt
\]

• **Asymmetric Hankel transform**

\[
\mathcal{J}_\nu [\varphi](z) := \int_0^\infty J_{2\nu}(2\sqrt{tz}) \left( \frac{t}{z} \right)^\nu \varphi(t) \, dt
\]

• **Borel generalized transform**

\[
\mathcal{B}_\nu [\varphi](z) := \frac{1}{z^{2\nu}} \int_0^\infty e^{-\frac{t}{z}} t^{2\nu-1} \varphi(t) \, dt
\]

• **Mellin transform**

\[
\mathcal{M}[\varphi](\rho) := \int_0^\infty \varphi(t) t^{\rho-1} \, dt
\]

We also use the notation

\[
\chi_\alpha(t) := t^\alpha \quad \text{and} \quad \exp_\alpha(t) := e^{\alpha t}, \ \alpha \in \mathbb{C}
\]

and write \( q = \xi + i\eta \), with \( \xi > 0 \) and \( \eta \in \mathbb{R} \). Moreover we write \( f(z) \cong g(z) \) for two functions \( f, g \) which coincides up to a non-vanishing multiplication constant possibly depending only on \( q \).

### 3 From gpf to Maass forms on the imaginary axis

To study the set of gpf, we used in [2] the integral transform \( \mathcal{B}_q \) on the spaces of functions \( L^p(m_q) \) in \( \mathbb{R}^+ \) with \( m_q(dt) = t^{2\xi-1} e^{-t} \, dt \). Letting

\[
L^p(m_q) := \left\{ \phi : \mathbb{R}^+ \to \mathbb{C} : \int_0^\infty |\phi(t)|^p t^{2\xi-1} e^{-t} \, dt < \infty \right\}
\]

with the norm

\[
\|\phi\|_p := \left( \int_0^\infty |\phi(t)|^p t^{2\xi-1} e^{-t} \, dt \right)^{\frac{1}{p}}
\]

it is immediate to check that

\[
L^1(m_q) \ni \phi \mapsto \mathcal{B}_q[\phi] \in \mathcal{H}(B)
\]

and that \( \mathcal{B}_q \) is continuous on \( L^1(m_q) \) with values on \( \mathcal{H}(B) \) with the standard topology induced by the family of supremum norms on compact subsets of \( B \). Moreover, since \( m_q(0, \infty) = \Gamma(2\xi) \), one has \( L^p(m_q) \subset L^1(m_q) \) for all \( p \in [1, \infty] \).

Putting together Proposition 2.5, Theorem 2.8 and Corollary 2.10 in [2], we have

**Proposition 3.1** ([2]). *If \( f \) is a generalized period function associated to a zero \( q \) and to the eigenvalue \( \lambda = \frac{1}{w} \), then there exist \( b \in \mathbb{C} \) and a function \( \varphi \in L^2(m_q) \) such that

\[
f(z) = \mathcal{B}_q \left[ \frac{b}{\Gamma(2q)} \chi_1 + \varphi \right](z)
\]

and

\[
(M \pm N_q) \left( \frac{b}{\Gamma(2q)} \chi_1 + \varphi \right) = \lambda \left( \frac{b}{\Gamma(2q)} \chi_1 + \varphi \right),
\]

where \( \lambda = \frac{1}{w} \) and \( \pm \) denotes the sign of \( q \).*
where \( M \) and \( N_q \) are linear operators defined by

\[
M(\phi)(t) := e^{-t} \phi(t) \tag{3.4}
\]

\[
N_q(\phi)(t) := \mathcal{J}_q \left( \exp_{-1} \phi \right)(t) = \int_0^\infty J_{2q-1}(2\sqrt{st}) \left( \frac{s}{t} \right)^{q-\frac{1}{2}} e^{-s} \phi(s) \, ds \tag{3.5}
\]

and the signs “+” and “−” correspond to the case of even or odd gpf respectively. Moreover, there exists a sequence \( \{a_{n,q}\}_{n \geq 0} \) with \( \limsup_n |a_{n,q}|^{1/n} \leq 1 \) such that: in the even case, for \( w = 1 \)

\[
\phi(t) = \frac{e^{-t}}{1 - e^{-t}} \sum_{n=1}^\infty \frac{(-1)^n a_{n,q} t^n}{\Gamma(n+2q)} + \frac{a_{0,q}}{\Gamma(2q)} \left( \frac{e^{-t}}{1 - e^{-t}} - \frac{1}{t} \right) \tag{3.6}
\]

with \( a_{0,q} = b \), and for \( w \in \mathbb{C} \setminus [1, \infty) \),

\[
\phi(t) = \frac{w e^{-t}}{1 - w e^{-t}} \sum_{n=0}^\infty \frac{(-1)^n a_{n,q} t^n}{\Gamma(n+2q)} ; \tag{3.7}
\]

in the odd case, for all \( w \in \mathbb{C} \setminus (1, \infty) \), the constant \( b \) in \( (3.2) \) vanishes and the function \( \phi \) can be written as in \( (3.7) \).

Finally, the invariance under the involution \( I_q \) defined in \( (1.6) \), implies that if \( b = 0 \)

\[
\mathcal{B}_q[\varphi] = \pm \mathcal{L} [\chi_{2q-1}\varphi] \tag{3.8}
\]

where again the signs “+” and “−” correspond to the case of even or odd gpf respectively.

### 3.1 The even case for 0-gpf

In \([8]\) and \([9]\) it is proved that the set of even period functions, that is even gpf with \( w = 1 \), is in one-to-one correspondence with the set of even Maass cusp forms. This correspondence is proved using the Fourier series expansions of the even cusp forms as

\[
u_m(x + iy) = y^{\frac{1}{2}} \sum_{n \geq 1} c_{n,q} K_{q-\frac{1}{2}}(2\pi ny) \cos(2\pi nx) \tag{3.9}
\]

where the coefficients \( c_{n,q} \) have at most polynomial growth. In particular the correspondence is given in \([8]\) in terms of the Laplace and Hankel transforms as

\[
\psi(z) = \mathcal{L} \left[ \chi_q \mathcal{H}_{q-\frac{1}{2}}[u_m(iy)] \right](z) \tag{3.10}
\]

Since the gpf \( f(z) \) of Proposition 3.1 coincides with \( \psi(z) \) up to a multiplication constant, using \( (3.2) \) with \( b = 0 \) and \( (3.8) \), we obtain

\[
\mathcal{L} \left[ \chi_q \mathcal{H}_{q-\frac{1}{2}}[u_m(iy)] \right](z) = \mathcal{L} [\chi_{2q-1}\varphi](z) \tag{3.11}
\]

from which we obtain an integral correspondence between cusp forms and the eigenfunctions \( \varphi \) of Proposition 3.1, namely

\[
\varphi(t) = t^{1-q} \mathcal{H}_{q-\frac{1}{2}}[u_m(iy)](t) \tag{3.11}
\]

see \([9]\) equation (2.27)]. Using \( (3.9) \), one obtains (see \([9]\) equation (2.28))

\[
\varphi(t) = \sum_{n \geq 1} n^{\frac{1}{2}-q} c_{n,q} \frac{t}{(2\pi n)^2} + \sum_{k=0}^\infty \frac{(-1)^k}{(4\pi^2)^{k+1}} L_u \left( q + 2k + \frac{3}{2} \right) t^{2k+1} \tag{3.12}
\]
where $L_u(\rho)$ is the Dirichlet $L$-series associated to $u_m$, namely

$$L_u(\rho) := \sum_{n \geq 1} c_{n,q} n^{-\rho}. \quad (3.13)$$

This shows that $\varphi$ is an odd function, and gives a relation between the coefficients $\{a_{n,q}\}$ of (3.6) and $\{c_{n,q}\}$ of (3.9). Indeed, recalling that

$$\frac{e^{-t}}{1 - e^{-t}} = \sum_{n \geq 0} B_n \frac{t^{n-1}}{n!}$$

where $\{B_n\}$ are the Bernoulli numbers, we find from (3.6) with $a_{0,q} = 0$

$$\sum_{i \geq -1, j \geq 1, i+j=n} (-1)^j \frac{B_{i+1} a_{i,q}}{(i+1)! \Gamma(j+2q)} = \begin{cases} \frac{(-1)^{n-1}}{(2\pi)^{n+1}} L_u(q+n + \frac{1}{2}), & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases} \quad (3.14)$$

up to multiplication by a constant depending possibly on $q$.

We now use the involution property of the $\mathcal{H}$-transform to define the inverse transform of (3.11) for general $w \in \mathbb{C} \setminus (1, \infty)$.

**Definition 3.2.** For any $q$ with $\Re(q) > 0$ and $w \in \mathbb{C} \setminus (1, \infty)$, define the one-parameter family of functions

$$u_\beta(iy) := \mathcal{H}_{q-\frac{1}{2}}[\exp_{-\beta} \chi_{q-1} \varphi](y), \quad \Re(\beta) > 0 \quad (3.15)$$

for functions $\varphi : (0, +\infty) \to \mathbb{C}$ which make the integral converge.

The integral in (3.13) is absolutely convergent if $\varphi$ is as in Proposition 3.1 that is $\varphi$ is in $L^2(m_q)$, satisfies $(M + N_q)\varphi = \frac{1}{w} \varphi$ and can be written as in (3.6) with $a_0 = 0$ for $w = 1$ and as in (3.7) for $w \in \mathbb{C} \setminus [1, \infty)$. Indeed by definition $\varphi$ satisfies $\varphi(t) = O(1)$ as $t \to 0^+$ and $\varphi(t) = O(e^{\epsilon t})$ as $t \to \infty$ for all $\epsilon > 0$, and the Bessel function $J_{\nu}(t)$ satisfies the estimates $J_{\nu}(t) = O(t^\nu)$ as $t \to 0^+$, and $J_{\nu}(t) = O(t^{-\frac{1}{2}})$ as $t \to \infty$ (see [3] vol. II).

**Theorem 3.3.** For any $q$ with $\Re(q) > 0$ and any $w \in \mathbb{C} \setminus (1, \infty)$, the function $u_\beta(iy)$ with $\varphi$ as in Proposition 3.1 with $a_{0,q} = 0$ for $w = 1$, can be extended as an analytic function of $\beta$ to a small domain containing the origin for all $y > 0$, and $u(iy) := u_0(iy)$ satisfies

$$u(iy) = w \left[ g(y) + \frac{1}{y} \right], \quad \forall y > 0 \quad (3.16)$$

where

$$g(y) = \mathcal{H}_{q-\frac{1}{2}}[\exp_{-1} \chi_{q-1} \varphi](y) = \sum_{n=0}^{\infty} (-1)^n a_{n,q} \frac{y^{n+q}}{(1+y^2)^{\frac{n+q+1}{2}}} P_{n+q+\frac{1}{2}} \left( \frac{y}{(1+y^2)^{\frac{1}{2}}} \right),$$

and $\{a_{n,q}\}$ is given in (3.6) with $a_{0,q} = 0$ for $w = 1$, and in (3.7) for $w \in \mathbb{C} \setminus [1, \infty)$.

**Proof.** Let us fix $y > 0$. Using the functional equation $(M + N_q)\varphi = \frac{1}{w} \varphi$, we can write

$$u_\beta(iy) = w \mathcal{H}_{q-\frac{1}{2}}[\exp_{-\beta} \chi_{q-1} M\varphi](y) + w \mathcal{H}_{q-\frac{1}{2}}[\exp_{-\beta} \chi_{q-1} N_q\varphi](y) \quad (3.17)$$

since the first integral on the right hand side is absolutely convergent. Moreover we can change the order of integration in the second integral, that is

$$\mathcal{H}_{q-\frac{1}{2}}[\exp_{-\beta} \chi_{q-1} N_q\varphi](y) = \int_0^\infty J_{q-\frac{1}{2}}(ty) \sqrt{ty} e^{-\beta t} t^{-q-1} \int_0^\infty J_{2q-1}(2\sqrt{s}t) \left( \frac{s}{t} \right)^{q+\frac{1}{2}} e^{-s} \varphi(s) \, ds \, dt = \int_0^\infty J_{q-\frac{1}{2}}(ty) \sqrt{ty} e^{-\beta t} t^{-q-1} \int_0^\infty J_{2q-1}(2\sqrt{s}t) \left( \frac{s}{t} \right)^{q+\frac{1}{2}} e^{-s} \varphi(s) \, ds \, dt = \int_0^\infty \int_0^\infty \cdots$$
The integral on the right hand side is absolutely convergent if 

\[ \Re \left( 1 + \frac{\beta}{y^2 + \beta^2} \right) > 0 \]

hence the left hand side can be extended as an analytic function of \( \beta \) to a small domain containing the origin. In particular we find that

\[
\mathcal{H}_{q, -1/2}^{(2, 0)}[\exp_{-\beta} \chi_{q-1} N_q \varphi](y) = \mathcal{H}_{q, -1/2}^{(2, 0)}[\exp_{-\beta} \chi_{q-1} \varphi] \left( \frac{1}{y} \right)
\]

(3.18)

Coming back to (3.17), the first term

\[
\mathcal{H}_{q, -1/2}^{(2, 0)}[\exp_{-\beta} \chi_{q-1} M \varphi](y) = \int_{0}^{\infty} J_{q, -1/2}(ty) \sqrt{t} e^{-\beta t} e^{-\frac{t}{y} \varphi(t)} dt
\]

is absolutely convergent for \( \Re(\beta) > -1 \), hence again can be extended as an analytic function of \( \beta \) to \( \Re(\beta) > -1 \), satisfying

\[
\mathcal{H}_{q, -1/2}^{(2, 0)}[\exp_{-\beta} \chi_{q-1} M \varphi] \bigg|_{\beta=0} (y) = \mathcal{H}_{q, -1/2}^{(2, 0)}[\exp_{-\beta} \chi_{q-1} \varphi] (y)
\]

(3.19)

Hence, putting together (3.19) and (3.18), we have proved that \( u_\beta(iy) \) can be extended, as an analytic function of \( \beta \), to a small domain containing the origin for all \( y > 0 \), and

\[
u(iy) := u_0(iy) = u \mathcal{H}_{q, -1/2}^{(2, 0)}[\exp_{-\beta} \chi_{q-1} \varphi] (y) + w \mathcal{H}_{q, -1/2}^{(2, 0)}[\exp_{-\beta} \chi_{q-1} \varphi] \left( \frac{1}{y} \right)
\]

(3.20)

This establishes (3.14) with \( g = \mathcal{H}_{q, -1/2}^{(2, 0)}[\exp_{-\beta} \chi_{q-1} \varphi] \). We now use the power series expansion for \( \varphi \) to obtain the series representations for \( g \).

First we write \( g(y) = G(y, \beta) \bigg|_{\beta=0} \) where

\[
G(y, \beta) := \mathcal{H}_{q, -1/2}^{(2, 0)}[\exp_{-(1+\beta)} \chi_{q-1} \varphi] (y)
\]

for \( y > 0 \) and \( \Re(\beta) > -1 \), the integral on the right hand side being absolutely convergent by the estimates used to justify the convergence in (3.15). Then we use the identity

\[
\frac{we^{-t}}{1 - we^{-t}} = \frac{1}{1 - we^{-t}} - 1
\]

in the definition of \( G(y, \beta) \) to obtain

\[
G(y, \beta) = \int_{0}^{\infty} J_{q, -1/2}(ty) \sqrt{y} e^{-(1+\beta)t} \frac{we^{-t}}{1 - we^{-t}} \sum_{n=0}^{\infty} \frac{(-1)^n a_{n,q} \Gamma(n+2q)}{\Gamma(n+2q)} dt = 
\]

\[
\frac{1}{w} \int_{0}^{\infty} J_{q, -1/2}(ty) \sqrt{y} \frac{we^{-(1+\beta)t}}{1 - we^{-t}} \sum_{n=0}^{\infty} \frac{(-1)^n a_{n,q} \Gamma(n+2q)}{\Gamma(n+2q)} dt - \int_{0}^{\infty} J_{q, -1/2}(ty) \sqrt{y} e^{-(1+\beta)t} \sum_{n=0}^{\infty} \frac{(-1)^n a_{n,q} \Gamma(n+2q)}{\Gamma(n+2q)} dt = 
\]
Using now the proved analytic extension for $u$ and the similar one for $P$

Moreover, using [6, vol. II, eq. 8.6.(6) page 29], we obtain

and

in (0, $y$)

Moreover, letting

Using [5, vol II, pag. 14], it holds

where we also used that lim sup

Moreover, in (3.22), we get

Hence the convergence in (3.22) is uniform on any compact interval contained in (0, $y$). Hence we can write

Using now the proved analytic extension for $u$, we can write for the second term on the right hand side

Moreover, using [6] vol. II, eq. 8.6.(6) page 29], we obtain

and the proof is complete.

We have thus proved the validity of the following expansion for $y \in (0, \infty)$

Moreover, letting $y = \tan \vartheta$ with $\vartheta \in (0, \frac{\pi}{2})$ in \([3,22]\), we get

and from the integral representation valid for $\xi > 0$ (see [5] vol. I, eq. (27) page 159])

and the similar one for $P_{n+q-\frac{1}{2}}^{-q+\frac{1}{2}}(\sin \vartheta)$, we see that the convergence in \([3,22]\) is uniform on any compact interval contained in $(0, \frac{\pi}{2})$. Hence the convergence in \([3,22]\) is uniform on any compact interval contained in $(0, \infty)$. 

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Corollary 3.4. Letting $w = 1$ and $\{a_{n,q}\}$ as in (3.6) with $a_{0,q} = 0$, the function $u(iy)$ in (3.22) is the restriction to the imaginary axis of an even Maass cusp form.

Proof. It follows from the fundamental theorem of Maass (see [12, Theorem 2, p. 234] and [8, Proposition 2.1]) that even Maass cusp forms are uniquely determined by their restriction on the imaginary axis, and correspond to coefficients $\{c_{n,q}\}$ which make the series (3.3) satisfy $u_{m}(iy) = u_{m}(i \frac{1}{y})$.

By definition we have that the function $u(iy)$ in (3.10) satisfies $u(iy) = u(i \frac{1}{y})$. Then the proof is finished by (3.10), (3.11) and Proposition 3.1.

3.2 The even case for b-gpf

We now extend Theorem 3.3 to the case of even b-gpf, which do exist only for $w = 1$. We recall that Maass non-cusp forms are a one-dimensional subspace in the space of $\Gamma$-invariant solutions to $\Delta u = \lambda u$, which is spanned by the non-holomorphic Eisenstein series defined for $\xi > 0$ as

$$E(z, q) = \zeta(2q) y^q \left(1 + \frac{1}{|z|^{2q}}\right) + \sum_{c,d \geq 1} \left(\frac{y}{cz+d} \right)^q, \quad z = x + iy, \quad (3.24)$$

and extended to $\mathbb{C}$ as a meromorphic function with a simple pole at $q = 1$, by the Fourier series expansions

$$E(x + iy, q) = \zeta(2q) y^q + \frac{\pi^{\frac{q}{2}} \Gamma(q - \frac{1}{2})}{\Gamma(q)} \zeta(2q-1) y^{1-q} + y^\frac{1}{2} \sum_{n \geq 1} \tilde{c}_{n,q} K_{q-\frac{1}{2}}(2\pi ny) \cos(2\pi nx) \quad (3.25)$$

where

$$\tilde{c}_{n,q} = \frac{4\pi^q}{\Gamma(q)} n^{\frac{1}{2} - q} \sum_{d \mid n} d^{2q-1}.$$  

Moreover it is proved in [3] and [9] (see the proof of equation (2.30) and page 243) that the function $\psi_q^+$ defined in (3.1) for $\xi > 1$, which is an eigenfunction of $\mathcal{P}_q^+$ with eigenvalue $\lambda = 1$, satisfies

$$\psi_q^+(z) = \frac{\zeta(2q)}{2} \left(1 + z^{-2q}\right) + \frac{2^{-q+\frac{1}{2}}}{\Gamma(q + \frac{1}{2})} \mathcal{L}^q \left[\chi_q \mathcal{H}_{q-\frac{1}{2}} \left(E(iy, q)\right)\right] (z) \quad (3.26)$$

where

$$\tilde{E}(iy, q) = 2 \sum_{c,d \geq 1} \left(\frac{y}{c^2y^2 + d^2}\right)^q. \quad (3.27)$$

It is shown in [9] that the function $\frac{1}{\Gamma(q-1)} \psi_q^+$ can be analytically continued to $\mathbb{C}$, we give here a proof of this fact for $\{\xi > 0\}$ using the $\mathcal{B}_q$ transform.

Theorem 3.5. The equation

$$\psi_q^+(z) = \mathcal{B}_q \left[\zeta(2q) \frac{\delta_0(t)}{t^{2q-1}} + \frac{e^{-t}}{1 - e^{-t}} \sum_{n=0}^{\infty} \frac{(-1)^n a_{n,q} t^n}{\Gamma(n + 2q)}\right] (z) \quad (3.28)$$

where

$$\begin{cases}
    a_{0,q} = \zeta(2q-1) \\
    a_{1,q} = -\frac{\Gamma(2q+1)}{2} \zeta(2q-1) \\
    a_{n,q} = (-1)^n \frac{\Gamma(n+2q)}{(n+1)! \Gamma(2q)} \left(\zeta(2q-1) + \frac{n+1}{2} + \sum_{i=2}^{n} \left(\begin{array}{c} n+1 \\ i \end{array}\right) B_i \left(\zeta(2q-1+i) - 1\right)\right), \quad n \geq 2
\end{cases}$$

defines a meromorphic extension of $\psi_q^+(z)$ to $q \in \{\xi > 0\}$ with simple pole at $q = 1$ and residue the function $\frac{\zeta}{t^2}$, which is the density of the invariant measure of the Farey map.
Proof. We first use [2, Remark 2.6] and in particular
\[
\mathcal{B}_q \left( \frac{\zeta(2q) \, \delta_q(t)}{2^{2q-1}} \right)(z) = \frac{\zeta(2q)}{2} z^{-2q}
\]  
(3.29)
to obtain the second term on the right hand side of (1.4). The first term is obtained by
\[
\frac{\zeta(2q)}{2} = \mathcal{B}_q \left( \frac{\zeta(2q)}{2 \Gamma(2q)} \right)(z) = \mathcal{B}_q \left[ \frac{\zeta(2q)}{2 \Gamma(2q)} \frac{e^{-t}}{1 - e^{-t}} \sum_{n=1}^{\infty} \frac{t^n}{n!} \right](z)
\]  
(3.30)
For the other terms we argue as follows
\[
\sum_{m,n \geq 1} \frac{1}{(mz+n)^{2q}} = \frac{1}{z^{2q}} \sum_{m,n \geq 1} \frac{1}{n^{2q} \left( \frac{m}{n} + \frac{1}{2} \right)^{2q}} = \frac{1}{\Gamma(2q) z^{2q}} \sum_{m,n \geq 1} \frac{1}{n^{2q}} \mathcal{L} \left[ y^{2q-1} e^{-\frac{t}{t^{-z}}} \right] \left( \frac{1}{z} \right) = 
\]  
\[
= \frac{1}{\Gamma(2q)} \sum_{m,n \geq 1} \frac{1}{n^{2q}} \mathcal{B}_q \left[ e^{-\frac{\pi}{n}\pi} \right](z) = \mathcal{B}_q \left[ \frac{1}{\Gamma(2q)} \sum_{m,n \geq 1} \frac{e^{-\frac{\pi}{n}\pi}}{n^{2q}} \right](z) = \mathcal{B}_q \left[ \frac{1}{\Gamma(2q)} \sum_{n \geq 1} \frac{1}{n^{2q}} \frac{e^{-\frac{\pi}{n}\pi}}{1 - e^{-\frac{\pi}{n}\pi}} \right](z)
\]
Let \( \xi > 1 \), then
\[
\sum_{n \geq 1} \frac{1}{n^{2q}} \frac{e^{-\frac{\pi}{n}\pi}}{1 - e^{-\frac{\pi}{n}\pi}} = \frac{1}{1 - e^{-\xi}} \sum_{n \geq 1} \frac{1}{n^{2q}} \frac{e^{-\frac{\pi}{n}\pi} - 1}{\xi - 1} = \frac{1}{1 - e^{-\xi}} \sum_{n \geq 1} \frac{1}{n^{2q}} \sum_{j=0}^{n-1} (\xi)^j = 
\]  
\[
= \frac{1}{1 - e^{-\xi}} \left[ \sum_{n \geq 1} \frac{1}{n^{2q}} + \sum_{k \geq 0} \left( \sum_{n \geq 2} \frac{\zeta(2q)}{n^{2q+k}} \frac{t^k}{k!} \right) \right] = \frac{1}{1 - e^{-\xi}} \sum_{k \geq 0} A_{k,q} \frac{t^k}{k!}
\]
with
\[
A_{0,q} = \zeta(2q) + \sum_{n \geq 2} \frac{n-1}{n^{2q}} = \zeta(2q - 1)
\]
and in general
\[
A_{k,q} = \sum_{n \geq 2} \frac{S_k(n-1)}{n^{2q+k}} , \quad k \geq 1
\]
where \( S_k(n-1) = \sum_{j=1}^{n-1} j^k \). Notice that \( S_k(n-1) \leq n^{k+1} \), whence for \( \xi > 1 \) the sum defining \( A_{k,q} \) is convergent and \( |A_{k,q}| \leq \zeta(2\xi - 1) \) for all \( k \geq 1 \). Hence the series \( \sum_{k \geq 0} A_{k,q} \frac{t^k}{k!} \) converges for \( t \in \mathbb{R} \) and
\[
\sum_{m,n \geq 1} \frac{1}{(mz+n)^{2q}} = \mathcal{B}_q \left[ \frac{1}{\Gamma(2q)} \frac{e^{-t}}{1 - e^{-t}} \sum_{k \geq 0} A_{k,q} \frac{t^k}{k!} \right](z)
\]  
(3.31)
Moreover, we recall that
\[
S_k(n) = \frac{1}{k+1} n^{k+1} + \frac{1}{2} n^k + \frac{1}{k+1} \sum_{i=2}^{k} \binom{k+1}{i} \sum_{i=2}^{k} \binom{k+1}{i} B_i \frac{n^{k+1-i}}{2}\]
where \( B_i \) are the Bernoulli numbers. Hence for \( \xi > 1 \) we can write
\[
A_{1,q} = \sum_{n \geq 2} \frac{S_1(n-1)}{n^{2q+1}} = \frac{1}{2} \sum_{n \geq 2} \frac{n(n-1)}{n^{2q+1}} = \frac{1}{2} \left( \zeta(2q-1) - 1 \right) - \frac{1}{2} \left( \zeta(2q) - 1 \right) = \frac{1}{2} \zeta(2q-1) - \frac{1}{2} \zeta(2q)
\]
and for all $k \geq 2$

$$A_{k,q} = \sum_{n \geq 2} \frac{S_k(n) - n^k}{n^{2q+k}} = \sum_{n \geq 2} \frac{S_k(n) - \frac{1}{k+1} n^{k+1} - \frac{1}{2} n^k}{n^{2q+k}} + \frac{1}{k+1} \left( \zeta(2q-1) - 1 \right) + \frac{1}{2} \left( \zeta(2q) - 1 \right) - \left( \zeta(2q) - 1 \right) =$$

$$= \frac{1}{k+1} \zeta(2q-1) - \frac{1}{2} \zeta(2q) + \frac{k-1}{2(k+1)} + \frac{1}{k+1} \sum_{i=2}^{k} \binom{k+1}{i} B_i \left( \zeta(2q - 1 + i) - 1 \right)$$

These expressions for $A_{k,q}$ are holomorphic in \{ $\xi > 0$ \} for all $k \geq 0$ except for simple poles at $q = \frac{1}{2}$ and $q = 1$. Moreover, using

$$|S_k(n) - \frac{1}{k+1} n^{k+1} - \frac{1}{2} n^k| \leq k^2 n^{k-1}, \quad k \geq 1$$

which is proved in the Appendix A we have that

$$\left| \sum_{n \geq 2} \frac{S_k(n) - \frac{1}{k+1} n^{k+1} - \frac{1}{2} n^k}{n^{2q+k}} \right| \leq k^2 \sum_{n \geq 2} \frac{n^{k-1}}{n^{2q+k}} = k^2 \zeta(2\xi + 1)$$

for all $q \in \{ \xi > 0 \}$, whence $|A_{k,q}| = O(k^2)$ for all $q \in \{ \xi > 0 \}$. This implies that (3.31) is valid for $\xi > 0$, and putting together (3.29), (3.30) and (3.31), we get for $\xi > 0$

$$\psi^+_q(z) = \mathcal{B}_q \left[ \frac{\zeta(2q)}{2} \frac{\delta_0(t)}{t^{2q-1}} + \frac{e^{-t}}{1 - e^{-t}} \sum_{n=0}^{\infty} \frac{(-1)^n a_{n,q} t^n}{\Gamma(n+2q)} \right](z)$$

where

$$\begin{cases}
  a_{0,q} = \zeta(2q - 1) \\
  a_{n,q} = (-1)^n \frac{\Gamma(n+2q)}{n! \Gamma(2q)} \left( \frac{\zeta(2q)}{2} + A_{n,q} \right), \quad n \geq 1
\end{cases}$$

which are holomorphic except for a simple pole at $q = 1$.

There is also a pole at $q = \frac{1}{2}$ in the coefficient $\frac{\zeta(2q)}{2}$ of the first term in the argument of the $\mathcal{B}_q$ transform. However, when applying the $\mathcal{B}_q$, we obtain that $\psi^+_q$ can be written as in (1.3) with $c = \frac{\zeta(2q)}{2}$ and $b = \zeta(2q - 1)$, so the first two terms are given by

$$\frac{\zeta(2q)}{2} \frac{1}{z^{2q}} + \frac{\zeta(2q - 1) \Gamma(2q - 1)}{\Gamma(2q)} \frac{1}{z}$$

so that there is no pole at $q = \frac{1}{2}$, as it happens for the Eisenstein series in (3.28).

Finally we can compute the residue for $\psi^+_q$ at $q = 1$ using (3.28). The only contributing terms are those containing $\zeta(2q - 1)$, which has residue $\frac{1}{2}$. Hence $\text{Res}_{q=1}(a_{n,q}) = \frac{(-1)^n}{2}$ and

$$\text{Res}_{q=1} \left[ \frac{\zeta(2q)}{2} \frac{\delta_0(t)}{t^{2q-1}} + \frac{e^{-t}}{1 - e^{-t}} \sum_{n=0}^{\infty} \frac{(-1)^n a_{n,q} t^n}{\Gamma(n+2q)} \right] = \frac{e^{-t}}{1 - e^{-t}} \sum_{n=0}^{\infty} \frac{t^n}{2 \Gamma(n+2)}$$

which gives

$$\text{Res}_{q=1}(\psi^+_q) = \mathcal{R}_1 \left[ \frac{e^{-t}}{1 - e^{-t}} \sum_{n=0}^{\infty} \frac{t^n}{2 \Gamma(n+2)} \right](z) = \mathcal{R}_1 \left[ \frac{1}{2z} \right](z) = \frac{1}{2z}.$$

This concludes the proof.
By Theorem C-(a), the function $\psi_q^+$ satisfies the equation $\mathcal{I}_q \psi_q^+ = \psi_q^+$, as is easily verified using the definition (1.4). Then, using (3.31) and (3.20) it follows that the function

$$\hat{\varphi}(t) := \frac{1}{\Gamma(2q)} \frac{e^{-t}}{1 - e^{-t}} \sum_{k \geq 0} A_{k,q} \frac{k!}{k!}$$

(3.33)

satisfies

$$\mathcal{B}_q[\hat{\varphi}] = \mathcal{L}[\chi_{2q-1} \hat{\varphi}] = \frac{2^{-q-\frac{1}{2}}}{\Gamma(q + \frac{1}{2})} \mathcal{L} \left[ \chi_q \mathcal{K}_{q-\frac{1}{2}} [\tilde{E}(iy, q)] \right],$$

from which we get the analogous of (3.11)

$$\hat{\varphi}(t) = \frac{2^{-q-\frac{1}{2}}}{\Gamma(q + \frac{1}{2})} t^{1-q} \mathcal{K}_{q-\frac{1}{2}} [\tilde{E}(iy, q)](t),$$

(3.34)

for $\tilde{E}(iy, q)$ defined in (3.27). From this we get an analytic continuation of $E(iy, q)$ different from the Fourier series expansion (3.26).

**Theorem 3.6.** The function $U(iy)$ defined by

$$U(iy) := \zeta(2q) \left( y^q + y^{-q} \right) - 2 \zeta(2q) \left( \frac{y}{1 + y^2} \right)^q +$$

$$+ 2^{q+\frac{1}{2}} \Gamma \left( q + \frac{1}{2} \right) \sum_{n=0}^{\infty} (-1)^n b_{n,q} \frac{y^{\frac{n}{2}} P^{-\frac{n}{q} + \frac{1}{2}} \left( \frac{1}{(1+y^2)^{\frac{n}{2}}} \right) + y^{n+q} P^{-\frac{n}{q} + \frac{1}{2}} \left( \frac{y}{(1+y^2)^{\frac{n}{2}}} \right)}{(1+y^2)^{\frac{n}{2} + \frac{1}{2} + \frac{q}{2}}}$$

(3.35)

with

$$\begin{cases} b_{0,q} = \zeta(2q - 1) \\
 b_{1,q} = -\frac{\Gamma(2q+1)}{2 \Gamma(2q)} \left( \zeta(2q - 1) - \zeta(2q) \right) \\
 b_{n,q} = (-1)^n \frac{\Gamma(n+2q)}{(n+1)! \Gamma(2q)} \left( \zeta(2q - 1) - \frac{n+1}{2} \zeta(2q) + \frac{n+1}{2} + \sum_{i=2}^{n} \binom{n+1}{i} B_i \left( \zeta(2q - 1 + i) - 1 \right) \right), n \geq 2
\end{cases}$$

gives an analytic continuation of the Eisenstein series $E(iy, q)$ in (3.24) to $q \in \mathbb{C}$ with a simple pole at $q = 1$ with residue the constant function $\frac{\pi}{q}^2$.

*Proof.* Writing the Eisenstein series $E(iy, q)$ as in (3.24)

$$E(iy, q) = \zeta(2q) \left( y^q + y^{-q} \right) + \tilde{E}(iy, q),$$

we proceed as in Theorem 3.3 to invert the relation (3.33).

The proof follows the same lines as that of Theorem 5.3 with some modifications. The first is that the function $\hat{\varphi}$ satisfies the functional equation

$$((M + N_q)\hat{\varphi})(t) = \hat{\varphi}(t) - \frac{\zeta(2q)}{\Gamma(2q)} \hat{\varphi}(t) = -t$$

(3.36)

This follows by applying $\mathcal{P}_q^+$ to $\psi_q^+(z) = \frac{\zeta(2q)}{2} (1 + z^{-2q}) + \mathcal{B}_q[\hat{\varphi}](z)$. Indeed

$$\psi_q^+(z) = (\mathcal{P}_q^+ \psi_q^+)(z) = \mathcal{P}_q^+ \left( \frac{\zeta(2q)}{2} (1 + z^{-2q}) \right) + (\mathcal{P}_q^+ \mathcal{B}_q[\hat{\varphi}](z)) =$$
Using
\[
(1 + z)^{-2q} = \frac{1}{\Gamma(2q)} \int_0^\infty e^{-t(1+z)} t^{2q-1} dt = \frac{1}{\Gamma(2q)} \mathcal{B}_q[e^{-t}](z)
\]
we obtain (3.36).

Letting now
\[
\tilde{U}_\beta(zy) := 2^{q+\frac{1}{2}} \Gamma \left( q + \frac{1}{2} \right) \mathcal{H}_{q-\frac{1}{2}}[\exp_{-\beta} \chi_{q-1} \tilde{\varphi}](y), \quad \Re(\beta) > 0
\]
we get from (3.36)
\[
\tilde{U}_\beta(zy) = 2^{q+\frac{1}{2}} \Gamma \left( q + \frac{1}{2} \right) \left( \mathcal{H}_{q-\frac{1}{2}}[\exp_{-\beta} \chi_{q-1} (M + N_q) \tilde{\varphi}](y) + \frac{\zeta(2q)}{\Gamma(2q)} \mathcal{H}_{q-\frac{1}{2}}[\exp_{-\beta - 1} \chi_{q-1} \tilde{\varphi}](y) \right)
\]
For the first term on the right hand side, for \( \xi > \frac{1}{2} \) we can repeat the arguments of the proof of Theorem 3.3 leading to (3.20), to get
\[
\mathcal{H}_{q-\frac{1}{2}}[\exp_{-\beta} \chi_{q-1} (M + N_q) \tilde{\varphi}](y) = \mathcal{H}_{q-\frac{1}{2}}[\exp_{-\beta} \chi_{q-1} \tilde{\varphi}](y) + \mathcal{H}_{q-\frac{1}{2}}[\exp_{-\beta - 1} \chi_{q-1} \tilde{\varphi}](y)
\]
whereas the second term is absolutely convergent for \( \beta = 0 \), whence we simply have
\[
\mathcal{H}_{q-\frac{1}{2}}[\exp_{-\beta - 1} \chi_{q-1} \tilde{\varphi}](y) = \mathcal{H}_{q-\frac{1}{2}}[\exp_{-\beta} \chi_{q-1} \tilde{\varphi}](y).
\]
Hence we obtain the continuation of \( \tilde{U}_\beta \) to a neighborhood of \( \beta = 0 \), and define \( \tilde{U}(zy) := \tilde{U}_0(zy) \) by
\[
\tilde{U}(zy) = 2^{q+\frac{1}{2}} \Gamma \left( q + \frac{1}{2} \right) \left( \mathcal{H}_{q-\frac{1}{2}}[\exp_{-\beta} \chi_{q-1} \tilde{\varphi}](y) + \mathcal{H}_{q-\frac{1}{2}}[\exp_{-\beta - 1} \chi_{q-1} \tilde{\varphi}](y) \right)
\]
To finish the proof, we use (3.33) to write \( \tilde{\varphi} \) as
\[
\tilde{\varphi}(t) = \frac{e^{-t}}{1 - e^{-t}} \sum_{n=0}^{\infty} \frac{(-1)^n b_{n,q} t^n}{\Gamma(n + 2q)}
\]
with \( b_{n,q} = (-1)^n \frac{\Gamma(n + 2q)}{\Gamma(2q)} a_{n,q} \), whence
\[
\left\{
\begin{array}{l}
b_{0,q} = \zeta(2q - 1) \\
b_{1,q} = -\frac{\Gamma(2q + 1)}{2 \Gamma(2q)} \left( \zeta(2q - 1) - \zeta(2q) \right) \\
b_{n,q} = (-1)^n \frac{\Gamma(n + 2q)}{(n + 1)! \Gamma(2q)} \left( \zeta(2q - 1) - \frac{n + 1}{2} \zeta(2q) + \frac{n - 1}{2} \sum_{i=2}^{n+1} \binom{n+1}{i} B_i \left( \zeta(2q - 1 + i) - 1 \right) \right), n \geq 2 
\end{array}
\right.
\]
Then we define as in the proof of Theorem 3.3 the function \( \tilde{g}(y) = G(y, \beta) \big|_{\beta=0} \) with
\[
\tilde{G}(y, \beta) := \mathcal{H}_{q-\frac{1}{2}}[\exp_{(1+\beta)} \chi_{q-1} \tilde{\varphi}](y)
\]
and repeat the same argument as above to show that
\[
\sum_{n=0}^{\infty} \frac{(-1)^n b_{n,q}}{\Gamma(n + 2q)} \mathcal{H}_{q-\frac{1}{2}}[\exp_{-1} \chi_{n+q-1} \tilde{\varphi}](y) = \frac{2^{-q+\frac{1}{2}}}{\Gamma(q+\frac{1}{2})} \tilde{U}(iy) - \tilde{G}(y, 0) = \tilde{g} \left( \frac{1}{y} \right) + R(y),
\]
with \( R(y) := \frac{\zeta(2q)}{\Gamma(2q)} \mathcal{H}_{q-\frac{1}{2}}^{\alpha, \beta} \{ \exp -1 \chi_{q-1} \} (y) \). The above equation can be used to obtain an expression for \( \tilde{g}(\frac{1}{y}) \) and the analogous for \( \tilde{g}(y) \), that when substituted in (3.33) finally give

\[
\frac{2^{-q-\frac{1}{2}}}{\Gamma(q + \frac{1}{2})} \tilde{U}(iy) = \sum_{n=0}^{\infty} \frac{(-1)^n b_{n,q}}{\Gamma(n + 2q)} \mathcal{H}_{q-\frac{1}{2}}^{\alpha, \beta} \{ \exp -1 \chi_{n+q-1} \} (y) + \sum_{n=0}^{\infty} \frac{(-1)^n b_{n,q}}{\Gamma(n + 2q)} \mathcal{H}_{q-\frac{1}{2}}^{\alpha, \beta} \{ \exp -1 \chi_{n+q-1} \} \left( \frac{1}{y} \right) - R \left( \frac{1}{y} \right)
\]

The last step of the proof consists of the calculations of the Hankel transforms. The first one is the same as in Theorem 3.3, that is

\[
\sum_{n=0}^{\infty} \frac{(-1)^n b_{n,q}}{\Gamma(n + 2q)} \mathcal{H}_{q-\frac{1}{2}}^{\alpha, \beta} \{ \exp -1 \chi_{n+q-1} \} (y) = \sum_{n=0}^{\infty} (-1)^n b_{n,q} \frac{y^{\frac{1}{2}}}{(1 + y^2)^{\frac{1}{2} + \frac{1}{2}}} P_{n+q-\frac{1}{2}} \left( \frac{1}{(1 + y^2)^{\frac{1}{2}}} \right),
\]

and the second one is

\[
R(y) = \frac{\zeta(2q)}{\Gamma(2q)} 2^{-q-\frac{1}{2}} \Gamma(q) \left( \frac{y}{1 + y^2} \right)^q = \frac{\zeta(2q)}{\Gamma(q + \frac{1}{2})} \left( \frac{y}{1 + y^2} \right)^q
\]

where we have used [3] vol. II, eq. 8.6.(5), p. 29] and \( \Gamma(2q) = \pi^{-\frac{q}{2}} 2^{2q-1} \Gamma(q) \Gamma(q + \frac{1}{2}) \).

In the proof of Theorem 3.3 we proved that |\( A_{n,q} | = O(n^2) \), whence arguing as in (3.23), we obtain that the expansion (3.35) is well defined for all \( q \in \mathbb{C} \), except \( q = \frac{1}{2} \) and \( q = 1 \), and for all \( y > 0 \). Moreover it is uniformly convergent in \( y \) on any compact interval contained in \((0, \infty)\).

We now first show that the expression (3.35) has no pole at \( q = \frac{1}{2} \). It is enough to show that the term multiplying \( \zeta(2q) \) vanishes at \( q = \frac{1}{2} \), indeed

\[
\lim_{q \to \frac{1}{2}} (2q - 1) U(iy) = y^{\frac{1}{2}} + y^{-\frac{1}{2}} - 2 \left( \frac{y}{1 + y^2} \right)^{\frac{1}{2}} - \sum_{n=1}^{\infty} \frac{y^{\frac{1}{2}} P_n \left( \frac{1}{(1+y^2)^{\frac{1}{2}}} \right) + y^{n+\frac{1}{2}} P_n \left( \frac{y}{(1+y^2)^{\frac{1}{2}}} \right)}{(1+y^2)^{\frac{n+1}{2}}},
\]

where \( P_n \) are the Legendre polynomials. Using the equation (see [11] eq. 18.12.11, pag. 449])

\[
\sum_{n=0}^{\infty} P_n(\alpha) \beta^n = (1 - 2\alpha \beta + \beta^2)^{-\frac{1}{2}}
\]

for \( \alpha \in (0, 1) \) and \( |\beta| < 1 \), we obtain

\[
\sum_{n=1}^{\infty} \frac{y^{\frac{1}{2}} P_n \left( \frac{1}{(1+y^2)^{\frac{1}{2}}} \right) + y^{n+\frac{1}{2}} P_n \left( \frac{y}{(1+y^2)^{\frac{1}{2}}} \right)}{(1+y^2)^{\frac{n+1}{2}}} = \left( \frac{y}{1+y^2} \right)^{\frac{1}{2}} \left[ (1+y^2)^{\frac{1}{2}} + (1+y^2)^{\frac{1}{2}} - 2 \right],
\]

hence

\[
\lim_{q \to \frac{1}{2}} (2q - 1) U(iy) = 0.
\]

At \( q = 1 \), the expression (3.35) has instead a pole with a residue that can be computed using \( \text{Res}_{q=1}(b_{n,q}) = \frac{(-1)^n}{n!} \). Letting \( y = \tan \theta \) as above we find

\[
\text{Res}_{q=1}(U)(iy) = 2^{\frac{1}{2}} \Gamma \left( \frac{3}{2} \right) (\sin \theta \cos \theta)^{\frac{1}{2}} \sum_{n=0}^{\infty} \left[ (\cos \theta)^{n+\frac{1}{2}} P_{n+q-\frac{1}{2}}^{-1} (\cos \theta) + (\sin \theta)^{n+\frac{1}{2}} P_{n+q-\frac{1}{2}}^{-\frac{1}{2}} (\sin \theta) \right]
\]

Using [11] eq. 14.5.12, pag. 359] we get

\[
(\sin \theta)^{\frac{1}{2}} (\cos \theta)^{n+1} P_{n+\frac{1}{2}}^{-\frac{1}{2}} (\cos \theta) = \frac{1}{n+1} \left( \frac{2}{\pi} \right)^{\frac{1}{2}} (\cos \theta)^{n+1} \sin ((n+1) \theta)
\]

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where $u_{\varphi}$ for a function namely properties of Maass forms. Using (3.40) and integrating term-by-term we get the analogous of (3.12), p. 130, and vol. II, eq. (6) p. 5 we finally have we obtain the analogous of (3.11) and [9, equation (2.27)] for odd Maass cusp forms. By [6, vol. I, eq. (9) 2 for (3.10) by applying [8, Proposition 4.3] to get period functions proved in [8] is extended to the odd case in [9, Section II.1]. We can formally proceed as We now repeat the approach of Section 3.1 to odd period functions, which in the case $0$-gpf recalling $\Gamma(\xi)$, $\varphi = \varphi$, and find by (3.8) that
\[ \varphi(t) = i^{1-2q} \int_0^t \tau^{q-1} \mathcal{H}_{q-\frac{1}{2}}(y(u_m)(\tau)) d\tau = t^{-q} \mathcal{H}_{q-\frac{1}{2}}[(u_m)(y)](t). \]
Notice that the Hankel transform in (3.42) is absolutely convergent for $\xi > 0$ thanks to the rapid decay properties of Maass forms. Using (3.40) and integrating term-by-term we get the analogous of (3.12), namely
\[ \varphi(t) = \sum_{n \geq 1} n c_{n,q} \mathcal{H}_{q-\frac{1}{2}} \left[ \chi_{\frac{1}{2}}(y) K_{q-\frac{1}{2}}(2\pi ny) \right](t) \]
\[ = \sum_{n \geq 1} n^{-q-\frac{1}{2}} c_{n,q} \left( \frac{1}{(2\pi n)^2 + 1} - 1 \right) + L_u \left( q + \frac{1}{2} \right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2\pi)^{2k}} L_u \left( q + 2k + \frac{1}{2} \right) t^{2k} \]
\[ \text{Proposition 4.3 in [8] can be applied only for } \xi > \frac{3}{2}. \]

3.3 The odd case for 0-gpf

We now repeat the approach of Section 3.1 to odd period functions, which in the case $w = 1$ are in one-to-one correspondence with the set of odd Maass cusp forms as shown in [9]. Also in this case it is fundamental to use the Fourier series expansion for the odd cusp forms given by
\[ u_m(x + iy) = y^{\frac{1}{2}} \sum_{n \geq 1} c_{n,q} K_{q-\frac{1}{2}}(2\pi ny) \sin(2\pi nx), \quad (3.40) \]
with $c_{n,q}$ with at most polynomial growth. The integral correspondence between even cusp forms and even period functions proved in [8] is extended to the odd case in [9, Section II.1]. We can formally proceed as for (3.10) by applying [8, Proposition 4.3] to get
\[ \psi(z) = \frac{1}{z} \mathcal{L} \left[ \chi_{q-1} \mathcal{H}_{q-\frac{1}{2}}[y(u_m)(iy)] \right](z), \quad (3.41) \]
where $(u_m)_x = \frac{\partial}{\partial x} u_m$, and find by (3.8) that
\[ \frac{1}{z} \mathcal{L} \left[ \chi_{q-1} \mathcal{H}_{q-\frac{1}{2}}[y(u_m)(iy)] \right](z) = \mathcal{L} \left[ \chi_{q-1} \varphi \right](z) \]
for a function $\varphi \in L^2(m_q)$ satisfying $(M - N_q)\varphi = \varphi$, with expansion as in (3.6) with $a_0,q = 0$. From this we obtain the analogous of (3.11) and [9, equation (2.27)] for odd Maass cusp forms. By [6, vol. I, eq. (9) p. 130, and vol. II, eq. (6) p. 5] we finally have
\[ \varphi(t) = t^{1-2q} \int_0^t \tau^{q-1} \mathcal{H}_{q-\frac{1}{2}}[y(u_m)(\tau)] d\tau = t^{-q} \mathcal{H}_{q-\frac{1}{2}}[(u_m)(y)](t). \]

Definition 3.7. For any \( q \) with \( \Re(q) > 0 \) and \( w \in \mathbb{C} \setminus (1, \infty) \), define the one-parameter family of functions

\[
(u_x)_\beta(iy) := \mathcal{H}_{q-\frac{1}{2}}[\exp_{-\beta} \chi_q \varphi](y), \quad \Re(\beta) > 0
\]

for functions \( \varphi : (0, +\infty) \to \mathbb{C} \) which make the integral converge.

Then we show that if \( \varphi \) is an eigenfunction of \( (M - N_q) \) then we can put \( \beta = 0 \) in (3.44).

Theorem 3.8. For any \( q \) with \( \Re(q) > 0 \) and any \( w \in \mathbb{C} \setminus (1, \infty) \), the function \((u_x)_\beta(iy)\) with \( \varphi \) as in Proposition 3.7 with \( a_{0,q} = 0 \), can be extended as an analytic function of \( \beta \) to a small domain containing the origin for all \( y > 0 \), and \( u_x(iy) := (u_x)_0(iy) \) satisfies

\[
u \frac{d}{d\beta} (u_x)_\beta(iy) + (M - N_q)(u_x)_\beta(iy) = 0.
\]

Proof. Let us fix \( y > 0 \). Using the functional equation \((M - N_q)\varphi = \frac{1}{q} \varphi \), we can write

\[
(u_x)_\beta(iy) = w \mathcal{H}_{q-\frac{1}{2}}[\exp_{-\beta} \chi_q M \varphi](y) - w \mathcal{H}_{q-\frac{1}{2}}[\exp_{-\beta} \chi_q N_q \varphi](y)
\]

since the first integral on the right hand side is absolutely convergent. Moreover we can change the order of integration in the second integral, that is

\[
\mathcal{H}_{q-\frac{1}{2}}[\exp_{-\beta} \chi_q N_q \varphi](y) = \int_0^\infty J_{q-\frac{1}{2}}(ty) \sqrt{t} e^{-\beta t} t^q \int_0^\infty J_{2q-1}(2\sqrt{st}) (s \beta)^{-\frac{q}{2}} e^{-s} \varphi(s) ds dt = \int_0^\infty e^{-s} s^{q-1} \sqrt{s} \varphi(s) \int_0^\infty J_{q-\frac{1}{2}}(ty) \sqrt{t} J_{2q-1}(2\sqrt{st}) e^{-\beta t} t^\frac{q}{2} dt ds
\]

since again the two-variable integral is absolutely convergent. Now we use \([6] \) vol. I, eq. 4.1.(6), p. 129, and eq. 4.14.(38), p. 186] to write

\[
\int_0^\infty J_{q-\frac{1}{2}}(ty) \sqrt{t} J_{2q-1}(2\sqrt{st}) e^{-\beta t} t^\frac{q}{2} dt = -\sqrt{y} \frac{d}{d\beta} \int_0^\infty J_{q-\frac{1}{2}}(ty) J_{2q-1}(2\sqrt{st}) e^{-\beta t} dt = -\sqrt{y} \frac{d}{d\beta} \left[ e^{-\sqrt{s} \beta} (y^2 + \beta^2)^{-\frac{1}{2}} J_{q-\frac{1}{2}} \left( \frac{sy}{y^2 + \beta^2} \right) \right],
\]

whence

\[
\mathcal{H}_{q-\frac{1}{2}}[\exp_{-\beta} \chi_q N_q \varphi](y) = -\int_0^\infty e^{-s} s^{q-1} \sqrt{s} \varphi(s) \sqrt{y} \frac{d}{d\beta} \left[ e^{-\sqrt{s} \beta} (y^2 + \beta^2)^{-\frac{1}{2}} J_{q-\frac{1}{2}} \left( \frac{sy}{y^2 + \beta^2} \right) \right] ds.
\]

Computing all the terms in the previous derivative, we see as in the proof of Theorem 3.3 that all the addends of the integral are absolutely convergent for

\[
\Re \left( \frac{1 + \frac{\beta}{y^2 + \beta^2}}{y^2 + \beta^2} \right) > 0.
\]
Hence we can again set $\beta = 0$ and it turns out that there is only one non-vanishing term, so
\[
\mathcal{H}_{q-\frac{1}{2}}[\exp_{-\beta} Nq^{\frac{1}{2}}\varphi]_{\beta = 0}(y) = \frac{1}{y^2} \mathcal{H}_{q-\frac{1}{2}}[\chi_q N\varphi]\left(\frac{1}{y}\right)
\]
So we argue as in Theorem 3.3 and from (3.46) we get (3.45) with
\[
g(y) = \mathcal{H}_{q-\frac{1}{2}}[\exp_{-1} \chi_q \varphi](y).
\]
The proof is finished as in Theorem 3.3 since we can write $g = \mathcal{H}_{q-\frac{1}{2}}[\exp_{-1} \chi_q-1 \varphi]$ with $\varphi(t) = t \varphi(t)$. Hence
\[
\varphi(t) = \frac{w e^{-t}}{1 - we^{-t}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} a_{n-1,q} t^n}{\Gamma(n + 2q - 1)}
\]
and at the end we get
\[
-\frac{1}{y^2} g\left(\frac{1}{y}\right) = \sum_{n=1}^{\infty} (-1)^{n-1} (n + 2q - 1) a_{n-1,q} \frac{y^{\frac{n}{2}}}{(1 + y^2)^{\frac{n}{2} + \frac{1}{4}}} \Gamma\left(q - \frac{1}{2}, \frac{y}{1 + y^2}\right) \frac{1}{(1 + y^2)^{\frac{1}{2}}}
\]
This finishes the proof.

As in the even case, one can show that the expansion for $u_x(iy)$ obtained in Theorem 3.8 is uniformly convergent on any compact interval of $(0, \infty)$. Moreover we have

**Corollary 3.9.** Letting $w = 1$, the function $u_x(iy)$ in (3.35) is the restriction to the imaginary axis of the $x$-derivative of an odd Maass cusp form.

**Proof.** It follows from the fundamental theorem of Maass (see [12] Theorem 2 and Exercise 6, p. 234) that odd Maass cusp forms are uniquely determined by their restriction on the imaginary axis, and correspond to coefficients $\{c_{n,q}\}$ which make the series (3.40) satisfy $(u_m)_x(iy) = -y^2 (u_m)_x(i\frac{1}{y})$.

By definition we have that the function $u_x(iy)$ in (3.45) satisfies $u_x(iy) = -y^2 u_x(i\frac{1}{y})$. Then the proof is finished by using (3.33) and [9] Chap. II, Section 3, to show that (3.42) is a bijection between odd Maass cusp forms and the eigenfunctions of $M - N_q$ as in Proposition 3.4.

## 4 Power series expansions for Maass forms on the imaginary axis

In (3.22) and (3.35) we have given series expansions for Maass forms in terms of the Legendre functions $P_{\mu}$. In particular for non-cusp forms we have explicit expressions for the coefficients $b_{n,q}$ of the series. We now use properties of the Legendre functions to obtain expansions in terms of rational functions.

We have used $P_{\nu}^\mu$ with $\nu = n + q - \frac{1}{2}$ and $\mu = -q + \frac{1}{2}$, in particular $\mu + \nu = n$ is an integer, whence using [11] eq. 14.3.11, p. 354 we get

\[
P_{\frac{n+\frac{1}{2}}{-q+\frac{1}{2}}} (t) = \begin{cases} 
-\binom{\frac{1}{2}}{\frac{1}{2}} \frac{2^{n+\frac{1}{2}} \Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2} + \frac{1}{4}\right)} (1 - t^2)^{-\frac{1}{2} + \frac{1}{2}} \mathbf{F}_1\left(-\frac{n}{2}, \frac{n}{2} + q; \frac{1}{2}; t^2\right), & \text{if } n \text{ is even} \\
-\binom{\frac{1}{2}}{\frac{1}{2}} \frac{2^{n+\frac{1}{2}} \Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2} + \frac{1}{4}\right)} t (1 - t^2)^{-\frac{1}{2} + \frac{1}{2}} \mathbf{F}_1\left(-\frac{n-1}{2}, \frac{n+1}{2} + q; \frac{3}{2}; t^2\right), & \text{if } n \text{ is odd}
\end{cases}
\]

where $\mathbf{F}_1$ is the hypergeometric function. Moreover, since the first variable of $\mathbf{F}_1$ is in both case a non-positive integer, then the hypergeometric function is a polynomial in $t^2$, more precisely for $k \in \mathbb{N}$ and $c \neq 0, -1, -2, \ldots$, it holds

\[
\mathbf{F}_1(-k, b; c; t^2) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{\Gamma(b + j)}{\Gamma(b) \Gamma(c + j)} t^{2j}
\]
where the coefficients

\[ \alpha \]

Notice that each term of the finite sums is invariant with respect to the transformation \( y \mapsto \frac{1}{y} \).

We now substitute (4.2) into (3.22) and (3.35) and get, with \( \alpha_{n,q} \) being equal to \( a_{n,q} \) and \( b_{n,q} \) respectively,

\[
\sum_{n=0}^{\infty} (-1)^n \alpha_{n,q} \frac{y^{q/2} \left( \frac{1}{1+y^2} \right)^{q}}{(1+y^2)^{q+\frac{1}{2}}} + \frac{y^{n+q+\frac{1}{2} \left( \frac{1}{1+y^2} \right)^{q}}}{(1+y^2)^{q+\frac{1}{2}}} =
\]

\[
\left\{ \begin{array}{ll}
(-1)^{\frac{n}{2}} \frac{2^{-q+\frac{1}{2} \Gamma \left( \frac{n+1}{2} \right)} \left( y^2 \right)^{q}}{\Gamma \left( \frac{n+1}{2} \right) \Gamma \left( \frac{n+1}{2} \right)} \sum_{j=0}^{\infty} (-1)^j \left( \frac{n}{2} \right) j \left( q \right) \frac{1+y^{n+2j}}{\Gamma \left( \frac{n+1}{2} \right)} + y^{n+q+\frac{1}{2} \left( \frac{1}{1+y^2} \right)^{q}} & \text{if } n \text{ is even} \\
(-1)^{\frac{n}{2}} \frac{2^{-q+\frac{1}{2} \Gamma \left( \frac{n+1}{2} \right)} \left( y^2 \right)^{q}}{\Gamma \left( \frac{n+1}{2} \right) \Gamma \left( \frac{n+1}{2} \right)} \sum_{j=0}^{\infty} (-1)^j \left( \frac{n}{2} \right) j \left( q \right) \frac{1+y^{n+2j}}{\Gamma \left( \frac{n+1}{2} \right)} & \text{if } n \text{ is odd}
\end{array} \right.
\]

(4.2)

where we have split the sum in one with even indices and one with odd indices. At this point we can group together all the coefficients multiplying terms of the form \( \frac{y^{q/2} \left( \frac{1}{1+y^2} \right)^{q}}{(1+y^2)^{q+\frac{1}{2}}} \) and get

\[
\sum_{n=0}^{\infty} (-1)^n \alpha_{n,q} \frac{y^{q/2} \left( \frac{1}{1+y^2} \right)^{q}}{(1+y^2)^{q+\frac{1}{2}}} + \frac{y^{n+q+\frac{1}{2} \left( \frac{1}{1+y^2} \right)^{q}}}{(1+y^2)^{q+\frac{1}{2}}} = 2^{-q+\frac{1}{2} \left( \frac{y}{1+y^2} \right)^{q}} \sum_{s=0}^{\infty} (-1)^s \eta_{s,q} \frac{1+y^{2s}}{(1+y^2)^s}
\]

(4.3)

where the coefficients \( \eta_{s,q} \) are given by a finite sum. In particular

\[
\eta_{s,q} := \begin{cases} 
\sum_{i=0}^{s} \alpha_{s+i,q} \gamma_{s+i,q} \beta_{s+i,q} & \text{if } s \text{ even} \\
\sum_{i=0}^{s} \alpha_{s+i,q} \gamma_{s+i,q} \beta_{s+i,q} & \text{if } s \text{ odd}
\end{cases}
\]

(4.4)

where

\[
\gamma_{2k,q} = \frac{\Gamma \left( \frac{2k+1}{2} \right)}{\Gamma \left( \frac{2k+1}{2} \right) \Gamma \left( k + q \right)} \quad \text{and} \quad \gamma_{2k+1,q} = \frac{\Gamma \left( \frac{2k+1}{2} + 1 \right)}{\Gamma \left( \frac{2k+1}{2} + 1 \right) \Gamma \left( k + q + 1 \right)}
\]

\[
\beta_{2k,q} = \frac{k \Gamma \left( k + q + j \right)}{\Gamma \left( \frac{1}{2} + j \right)} \quad \text{and} \quad \beta_{2k+1,q} = \frac{k \Gamma \left( k + q + j + 1 \right)}{\Gamma \left( \frac{1}{2} + j \right)}
\]

Letting \( \alpha_{n,q} = (-1)^n \frac{\Gamma(n+2q)}{n! \Gamma(n+2)} \beta_{n,q} \) in (4.4) we get

\[
\eta_{s,q} = (-1)^s \frac{2^s}{s! \Gamma \left( \frac{q}{2} \right) \Gamma \left( q + \frac{1}{2} \right)} \sum_{i=0}^{s} (-1)^i 2^{-i} \beta_{s+i,q} \left( \frac{s}{i} \right)
\]

(4.5)

We have thus proved
Proposition 4.1. An even Maass cusp form with eigenvalue $q(1-q)$, can be formally written when restricted to the imaginary axis as

$$u(iy) = 2^{-q+\frac{1}{2}} \left( \frac{y}{1+y^2} \right)^q \sum_{s=0}^{\infty} (-1)^s \eta_{s,q} \frac{1+y^{2s}}{(1+y^2)^s}$$

with $\eta_{s,q}$ as in (4.5) and $\beta_{n,q} = (-1)^n \frac{n! \Gamma(2q)}{\Gamma(n+2q)} a_{n,q}$, where $\{a_{n,q}\}$ is given in (3.36).

When writing the same expansion for the non-cusp forms $U(iy)$ as in (3.35), we can use the explicit expression for the coefficient $\{b_{n,q}\}$, which are defined in terms of the $\{A_{n,q}\}$ of Theorem 3.3. We first get

$$U(iy) = \zeta(2q) \left( y^q + y^{-q} \right) + 2 \left( \frac{y}{1+y^2} \right)^q \left[ -\zeta(2q) + \sum_{s=0}^{\infty} 2^s \frac{\Gamma(s+q)}{s! \Gamma(q)} \left( \sum_{i=0}^{s} \left( \begin{array}{c} s \\ i \end{array} \right) \frac{(-1)^i A_{s+i,q}}{2^i} \right) \frac{1+y^{2s}}{(1+y^2)^s} \right]$$

then recalling $A_{0,q} = \zeta(2q-1)$ and for $n \geq 1$

$$A_{n,q} = \frac{1}{n+1} \sum_{\ell=0}^{n} \left( \begin{array}{c} n+1 \\ \ell \end{array} \right) B_{\ell} \left( \frac{2q-1-\ell}{2} \right) = \frac{1}{n+1} \sum_{\ell=0}^{n} \left( \begin{array}{c} n+1 \\ \ell \end{array} \right) B_{\ell} \zeta(2q-1+\ell),$$

for the properties of Bernoulli numbers, we obtain for $s \geq 1$

$$\sum_{i=0}^{s} \left( \begin{array}{c} s \\ i \end{array} \right) \frac{(-1)^i A_{s+i,q}}{2^i} = \sum_{i=0}^{s} \sum_{\ell=0}^{s+i} B_{\ell} \zeta(2q-1+\ell) \left( \begin{array}{c} s+i+1 \\ \ell \end{array} \right) \left( \begin{array}{c} s \\ i \end{array} \right) \frac{(-1)^i}{2^i (s+i+1)} =$$

$$= \sum_{\ell=0}^{s} \left( \begin{array}{c} s+i+1 \\ \ell \end{array} \right) \left( \begin{array}{c} s \\ i \end{array} \right) \frac{(-1)^i}{2^i (s+i+1)} B_{\ell} \zeta(2q-1+\ell) + \sum_{\ell=s+1}^{2s} \left( \begin{array}{c} s+i+1 \\ \ell \end{array} \right) \left( \begin{array}{c} s \\ i \end{array} \right) \frac{(-1)^i}{2^i (s+i+1)} B_{\ell} \zeta(2q-1+\ell)$$

Moreover for $\ell \geq 2$

$$\sum_{i=0}^{s} \left( \begin{array}{c} s+i+1 \\ \ell \end{array} \right) \left( \begin{array}{c} s \\ i \end{array} \right) \frac{(-1)^i}{2^i (s+i+1)} = -\frac{\Gamma(s+1) \Gamma\left(\frac{3\ell}{2}\right)}{2^s (s+1+\ell) \Gamma\left(\frac{3\ell}{2}+s\right) \pi \ell}$$

which vanishes for $\ell$ even, and since $B_{\ell} = 0$ for $\ell \geq 2$ and odd, we obtain

$$\sum_{\ell=0}^{s} \left( \begin{array}{c} s+i+1 \\ \ell \end{array} \right) \left( \begin{array}{c} s \\ i \end{array} \right) \frac{(-1)^i}{2^i (s+i+1)} B_{\ell} \zeta(2q-1+\ell) =$$

$$= B_{0} \zeta(2q-1) \sum_{i=0}^{s} \left( \begin{array}{c} s+i+1 \\ i \end{array} \right) \frac{(-1)^i}{2^i (s+i+1)} + B_{1} \zeta(2q) \sum_{i=0}^{s} \left( \begin{array}{c} s \\ i \end{array} \right) \frac{(-1)^i}{2^i} = \frac{\pi^2}{2^{s+1}} \frac{\Gamma(s+1)}{\Gamma(s+\frac{3}{2})} \zeta(2q-1) - \frac{1}{2^{s+1}} \zeta(2q).$$

Also, letting $k = \ell - s \geq 1$, we have

$$\sum_{i=k}^{s} \left( \begin{array}{c} s+k+1 \\ s+i \end{array} \right) \left( \begin{array}{c} s \\ i \end{array} \right) \frac{(-1)^i}{2^i (s+i+1)} = \frac{(-1)^k}{2^{k-1} (s+k)} \left( \begin{array}{c} s \\ k-1 \end{array} \right) \frac{\Gamma(k+1)}{\Gamma\left(\frac{k-s}{2}\right) \Gamma\left(\frac{s+k+3}{2}\right)}$$

whose second term vanishes for $s-k$ even, that is for $\ell = s - k + 2k$ even, and since $B_{\ell} = 0$ for $\ell \geq 2$ and odd, we obtain for $s \geq 1$

$$\sum_{\ell=s+k}^{2s} \left( \begin{array}{c} s+i+1 \\ \ell \end{array} \right) \left( \begin{array}{c} s \\ i \end{array} \right) \frac{(-1)^i}{2^i (s+i+1)} B_{\ell} \zeta(2q-1+\ell) =$$

$$= \sum_{k=1}^{s} \sum_{i=k}^{s} \left( \begin{array}{c} s+i+1 \\ s+k \end{array} \right) \left( \begin{array}{c} s \\ i \end{array} \right) \frac{(-1)^i}{2^i (s+i+1)} B_{s+k} \zeta(2q-1+s+k) =$$

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Using the notation \( A \) Proof of (3.32) we obtain

\[
\sum_{k=0}^{s} \binom{s}{i} \frac{(-1)^i}{2^s} = \frac{\pi^2}{2s+1} \frac{\Gamma(s+1)}{\Gamma(s+\frac{3}{2})} \zeta(2q-1) - \frac{1}{2s+1} \zeta(2q) + \frac{\pi^2}{2s+1} \frac{\Gamma(s+1)}{\Gamma(s+\frac{3}{2})} \zeta(2q-1) + \frac{1}{2s+1} \zeta(2q)
\]

Substituting (4.8) in (4.6), and using the identities

\[
\sum_{s=0}^{\infty} \frac{\Gamma(s+q)}{s! \Gamma(q)} z^s = (1-z)^{-q} \quad \text{and} \quad \sum_{s=0}^{\infty} \frac{\Gamma(s+q)}{s! \Gamma(q)} z^s = \frac{2}{\pi^2} \text{F}_1 \left(1, q; \frac{3}{2}; z\right),
\]

we obtain

\textbf{Proposition 4.2.} The Eisenstein series \( E(iy, q) \) defined in (3.24) can be formally written as

\[
E(iy, q) = 2 \left( \frac{y}{1+y^2} \right)^q \left[ \zeta(2q-1) \text{F}_1 \left(1, q; \frac{3}{2}; \frac{1}{1+y^2} \right) + \zeta(2q-1) \text{F}_1 \left(1, q; \frac{3}{2}; \frac{y^2}{1+y^2} \right) \right] + 4 \left( \frac{y}{1+y^2} \right)^q \sum_{s=1}^{\infty} \frac{\Gamma(s+q)}{s! \Gamma(q)} \left( \sum_{k=1}^{s} \binom{s}{k-1} \frac{(-1)^k}{2k(s+k)} B_{s+k} \zeta(2q-1+s+k) \right) \frac{1+y^{2s}}{(1+y^2)^s}
\]

Notice that in the previous expression there is no dependence on \( \zeta(2q) \), hence it is clear that there is no pole at \( q = \frac{1}{2} \), and the only pole is at \( q = 1 \) which depends on the \( \zeta(2q-1) \) term.

\section{Proof of (3.32)\textbf{A}}

Using the notation \( |x| \) and \( \{x\} \) for the integer and fractional part of a real number, we have

\[
S_k(n) - \frac{1}{k+1} n^{k+1} - \frac{1}{2} n^k = \int_0^n \left[ (|x| + 1)^k - x^k - \frac{1}{2} k x^{k-1} \right] dx = \int_0^n \left[ \sum_{j=0}^{k} \binom{k}{j} |x|^j \{x\}^{k-j} - \frac{1}{2} k \sum_{j=0}^{k-1} \binom{k-1}{j} |x|^j \{x\}^{k-j-1} \right] dx = \int_0^n \left[ |x|^k (1 - \{x\}^{k-1}) + |x|^{k-1} \left( k - k|\{x\}| - \frac{1}{2} k \right) + \sum_{j=0}^{k-2} |x|^j \left( \binom{k}{j} (1 - \{x\}^{k-j}) - \frac{1}{2} k \left( \binom{k-1}{j} \{x\}^{k-j-1} \right) \right) \right] dx = \sum_{h=0}^{n-1} \int_h^{b+1} k |x|^{k-1} \left( \frac{1}{2} - \{x\} \right) dx + \int_0^n \left[ \sum_{j=0}^{k-2} \binom{k}{j} |x|^j \left( 1 - \{x\}^{k-j} - \frac{1}{2} k \left( \binom{k-1}{j} \{x\}^{k-j-1} \right) \right) \right] dx = \sum_{h=0}^{n-1} \int_h^{b+1} k h^{k-1} \left( \frac{1}{2} - x + h \right) dx + \int_0^n \left[ \sum_{j=0}^{k-2} \binom{k}{j} |x|^j \left( 1 - \{x\}^{k-j} - \frac{1}{2} k \left( \binom{k-1}{j} \{x\}^{k-j-1} \right) \right) \right] dx = \sum_{h=0}^{n-1} \int_h^{b+1} k h^{k-1} \left( \frac{1}{2} - x + h \right) dx + \int_0^n \left[ \sum_{j=0}^{k-2} \binom{k}{j} |x|^j \left( 1 - \{x\}^{k-j} - \frac{1}{2} k \left( \binom{k-1}{j} \{x\}^{k-j-1} \right) \right) \right] dx = \sum_{h=0}^{n-1} \int_h^{b+1} k h^{k-1} \left( \frac{1}{2} - x + h \right) dx + \int_0^n \left[ \sum_{j=0}^{k-2} \binom{k}{j} |x|^j \left( 1 - \{x\}^{k-j} - \frac{1}{2} k \left( \binom{k-1}{j} \{x\}^{k-j-1} \right) \right) \right] dx = \sum_{h=0}^{n-1} \int_h^{b+1} k h^{k-1} \left( \frac{1}{2} - x + h \right) dx + \int_0^n \left[ \sum_{j=0}^{k-2} \binom{k}{j} |x|^j \left( 1 - \{x\}^{k-j} - \frac{1}{2} k \left( \binom{k-1}{j} \{x\}^{k-j-1} \right) \right) \right] dx =
\]
\[
= \int_0^n \left[ \sum_{j=0}^{k-2} \binom{k}{j} x^j \left( 1 - \{x\}^{k-j} - \frac{k-j}{2} \{x\}^{k-j-1} \right) \right] dx \leq \\
\leq \int_0^n \sum_{j=0}^{k-2} \binom{k}{j} x^j dx = \int_0^n k(k-1) \sum_{j=0}^{k-2} \frac{1}{(k-j)(k-j-1)} \binom{k-2}{j} x^j dx \leq \\
\leq k^2 \int_0^n \sum_{j=0}^{k-2} \binom{k-2}{j} x^j dx = k^2 \int_0^n (|x|+1)^{k-2} dx = k^2 S_{k-2}(n) \leq k^2 n^{k-1}
\]

B Spectral properties of the terms with Legendre functions

We now give some properties of the functions used in the series (3.22) in terms of the hyperbolic Laplacian \( \Delta = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \). Using recurrence relations and the formulas for derivatives of the Legendre functions which can be found in [5, vol. I], using the notation
\[
F_q(n, y) := \frac{y^{n+q}}{(1+y^2)^{n+\frac{q+1}{2}}} P_{n+q-\frac{1}{2}} \left( \frac{y}{(1+y^2)^{\frac{1}{2}}} \right)
\]
we find that for all \( n \geq 0 \)
\[
-y^2 \frac{\partial^2}{\partial y^2} [F_q(n, y)] = \\
= -(n+q)(n+q-1) F_q(n, y) + 2(n+2q)(n+q) F_q(n+1, y) - (n+2q)(n+2q+1) F_q(n+2, y)
\]
Hence for a series
\[
F_q(y) := \sum_{n=0}^{\infty} (-1)^n a_n F_q(n, y), \quad y \in (0, \infty)
\]
with \( \limsup |a_n|^{\frac{1}{n}} \leq 1 \), we find
\[
-y^2 \frac{\partial^2}{\partial y^2} F_q(y) = \sum_{n=0}^{\infty} (-1)^n b_n F_q(n, y)
\]
where
\[
\begin{align*}
b_0 &= q(1-q) a_0 \\
b_1 &= -q(1+q) a_1 - 4q^2 a_0 \\
b_n &= -(n+q)(n+q-1)a_n - 2(n+2q-1)(n+q-1)a_{n-1} - (n+2q-2)(n+2q-1)a_{n-2}, \quad n \geq 2
\end{align*}
\]
It follows that

**Theorem B.1.** For any \( q \) the function
\[
E_q(y) = \sum_{n=0}^{\infty} \frac{\Gamma(n+2q)}{n!\Gamma(2q)} F_q(n, y)
\]
satisfies \( \Delta E_q(y) = q(1-q)E_q(y) \).
Proof. Using (B.3) and (B.4) we have to find a solution of the system

\[
\begin{align*}
  a_0 &\in \mathbb{C} \\
  a_1 &= -2qa_0 \\
  a_n &= -2 \frac{2+q-1}{n} a_{n-1} - n^{-2} 2^{q-2} \frac{a_{n-2}}{n}, \quad n \geq 2
\end{align*}
\]  

(B.5)

with \( \limsup |a_n|^\frac{1}{n} \leq 1. \)

Consider the generating function

\[ f(z) = \sum_{n \geq 0} a_n z^n \]

with \( a_0 = 1, \) of the solution of (B.5). From the recurrence relation of \( (a_n), \) it turns out that \( f \) satisfies

\[
\begin{align*}
  (1+z)^2 f'(z) + 2q(z+1) f(z) &= 0 \\
  f(0) &= 1
\end{align*}
\]  

(B.6)

We now want to show that the solution \( f \) of (B.6) is analytic for \( |z| < 1. \) Letting for any \( \alpha \in \mathbb{C} \)

\[ (1+z)^\alpha := \exp \left( \alpha \log |1+z| + i \arg(1+z) \right) \]

with \( \arg(1+z) \in (-\pi, \pi], \) we have that \( (1+z)^\alpha \) is well defined as a single-valued analytic function on the cut plane \( \mathbb{C} \setminus (-\infty, -1], \) hence in particular for \( |z| < 1. \) It follows that

\[ f(z) = (1+z)^{-2q} \]

is the solution of (B.6) and is analytic for \( |z| < 1. \) Hence for all \( n \geq 0 \)

\[ a_n = (-1)^n \frac{\Gamma(n+2q)}{n! \Gamma(2q)} \]

is the solution of (B.5) with \( a_0 = 1 \) and \( \limsup |a_n|^\frac{1}{n} \leq 1. \)

Remark B.2. For \( q = 1 \) we have

\[ E_1(y) = \sum_{n=0}^{\infty} (n+1) F_1(n, y) \]

Letting \( y = \tan \theta \) with \( \theta \in (0, \frac{\pi}{2}) \) as in (3.23), we get using [11] eq. 14.5.12, p. 359]

\[ F_1(n, \theta) = (\cos \theta)^{\frac{\theta}{2}} (\sin \theta)^{n+1} P_{n+\frac{1}{2}} (\sin \theta) = \frac{\Gamma(n+2q)}{\Gamma(2q)} \left( \frac{\pi}{2} \right)^{\frac{1}{2}} (\sin \theta)^{n+1} \sin \left( (n+1) \left( \frac{\pi}{2} - \theta \right) \right) \]

Hence

\[ \sum_{n=0}^{\infty} (n+1) F_1(n, \theta) = \left( \frac{2}{\pi} \right)^{\frac{1}{2}} \Re \left( \sum_{n=1}^{\infty} \frac{1}{i} \left( 1 - \frac{\exp(-2i\theta)}{2} \right)^n \right) = \left( \frac{2}{\pi} \right)^{\frac{1}{2}} \Re \left( \frac{1 - \exp(-2i\theta)}{1 + \exp(-2i\theta)} \right) = \left( \frac{2}{\pi} \right)^{\frac{1}{2}} \frac{\sin(2\theta)}{1 + \cos(2\theta)} \]

Hence, using \( y = \tan \theta, \)

\[ E_1(y) = \left( \frac{2}{\pi} \right)^{\frac{1}{2}} y \]

which satisfies \( \Delta E_1(y) = 0. \)

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For $q = \frac{1}{2}$ we have

$$E_{1/2}(y) = \sum_{n=0}^{\infty} F_{1/2}(n, y) = \left( \frac{y}{1 + y^2} \right)^{\frac{1}{2}} \sum_{n=0}^{\infty} \left( \frac{y}{(1 + y^2)^{\frac{1}{2}}} \right)^n P_n \left( \frac{y}{(1 + y^2)^{\frac{1}{2}}} \right)$$

where $P_n(x)$ are the Legendre polynomials. Using [11, eq. 18.12.11, p. 449], we have

$$E_{1/2}(y) = \frac{y^{1/2}}{4}$$

which satisfies $\Delta E_{1/2}(y) = \frac{1}{4} E_{1/2}(y)$.

**Conjecture.** For all $q$ we have

$$E_q(y) = \left( \frac{2}{\pi} \right)^{q-\frac{1}{2}} y^q$$

**Theorem B.3.** For $\Re(q) > 1$ it holds

$$E_q(z) := \sum_{(c,d)=1} E_q \left( \frac{y}{|cz + d|^2} \right) = \text{const}(q) E(z, q)$$

where $E(z, q)$ is the non-holomorphic Eisenstein series relative to the modular group $SL(2, \mathbb{Z})$.

**Proof.** From the asymptotic behaviour

$$P^{-q+\frac{1}{2}} \left( \frac{y}{(1+y)^{\frac{1}{2}}} \right) = \frac{2^{-q+\frac{1}{2}} \pi^{\frac{1}{2}}}{\Gamma(\frac{q}{2} + q + \frac{1}{2}) \Gamma(-\frac{q}{2} + \frac{1}{2})} + O(y), \quad \text{as } y \to 0$$

it follows that

$$|F_q(n, y)| \leq y^q \left( \frac{y}{\sqrt{1+y^2}} \right)^n \left| \frac{2^{-q+\frac{1}{2}} \pi^{\frac{1}{2}}}{\Gamma(\frac{q}{2} + q + \frac{1}{2}) \Gamma(-\frac{q}{2} + \frac{1}{2})} + O(y) \right|, \quad \text{as } y \to 0$$

it follows that $E_q(y) = O(y^q)$ as $y \to 0$. Hence the series defining $E_q(z)$ converges for $\Re(q) > 1$.

Moreover from Theorem B.1 it follows that $E_q(z)$ is an eigenfunction of the hyperbolic Laplacian with eigenvalue $q(1 - q)$.

Finally from harmonic analysis on the modular surface, it is well known that the space of eigenfunctions with eigenvalue $q(1 - q)$ is one-dimensional for $\Re(q) \neq \frac{1}{2}$.

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