Gauge Invariance in Nonlocal Regularized QED

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Abstract

The existence of the invariant measure in nonlocal regularized actions is discussed. It is shown that the measure for nonlocally regularized QED, as presented in [1], exists to all orders, and is precisely what is required to maintain gauge invariance at one loop and guarantees perturbative unitarity. We also demonstrate how the given procedure breaks down in anomalous theories, and discuss its generalization to other actions.

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INTRODUCTION

The nonlocal regularization scheme introduced recently \cite{[1]} relies heavily on the existence of an invariant path integral measure to ensure that the extended nonlocal gauge invariance is respected in the quantum regime, thereby maintaining perturbative unitarity. Operationally, the measure is expected to correct a Lagrangian generated loop graph so that the process in question satisfies the Ward identities. It is not surprising then, that one can directly relate the measure contribution to these identities. This not only guarantees the existence of the measure (through existence of the Lagrangian generated graphs), but also produces an obvious minimal choice, which is necessary to ensure that the scheme does not produce arbitrary results.

In nonlocal regularization of QED, it has been argued previously that the invariant measure exists to one loop \cite{[1]}, but without any explicit results beyond second order. Comparisons have also been made between nonlocal regularized amplitudes and the corresponding dimensionally regulated ones, to infer the form of a possible measure contribution \cite{[2]}. The result has not been proven in general, and we feel that for a true understanding of nonlocal regularization, one should not have to resort to another regularization scheme. We will prove here the relationship between the measure and Ward identities in QED, and discuss how the result will also hold in other actions.

In section \[8\], we introduce the nonlocal regulated action and the measure consistency conditions. The Ward identities are developed in Sect. \[9\], followed by second, third and fourth order corrections in Sections \[10\], \[11\] and \[12\], respectively. Next we discuss gauge invariance of higher loop corrections, and explicitly demonstrate it for the two loop vacuum polarization corrections in Sect. \[13\]. Finally, in Section \[14\] possible barriers to consistent quantization are discussed. In an Appendix, we prove that the measure exists to all orders, and is exactly what is necessary in order to satisfy the Ward identities to one loop.
I. REGULARIZATION

A. Nonlocal Action

The standard gauge invariant Lagrangian for local QED is written as

\[
L = \bar{\psi}(i\partial - m)\psi - \frac{1}{4e^2}F^2 - \bar{\psi}\partial^\mu A_\mu \psi
\]

\[
\equiv \bar{\psi}S^{-1}\psi + \frac{1}{2e^2}A^\mu D^{-1}_{\mu\nu}A^\nu - \bar{\psi}\partial^\mu A_\mu \psi, \quad (1.1)
\]

which possesses the infinitesimal invariance:

\[
\delta A^\mu = \partial^\mu \theta, \quad \delta \psi = -i\theta \psi, \quad (1.2)
\]

where we have introduced the inverse propagators into the Lagrangian in order to clarify notation later. Gauge fixing is then implemented via the introduction of:

\[
L_{GF} = -\frac{1}{2\xi}(\partial^\mu A_\mu)^2, \quad (1.3)
\]

leading to the (now invertible) photon propagator:

\[
iD_{\mu\nu} = \frac{i}{\Box}(g_{\mu\nu} - (1 - \xi)\partial_\mu \partial_\nu). \quad (1.4)
\]

In order to produce a regulated action that is gauge invariant, we begin by introducing two types of smeared fermion propagators and give their Schwinger parameterized form (useful for explicit calculations but not used extensively in the present work):

\[
\hat{S}(p) = E_m^2(p)S(p) = -\int_1^\infty \frac{dx}{\Lambda^2} \exp(x\frac{p^2 - m^2}{\Lambda^2})(\hat{p} + m),
\]

\[
\bar{S}(p) = (1 - E_m^2(p))S(p) = -\int_0^1 \frac{dx}{\Lambda^2} \exp(x\frac{p^2 - m^2}{\Lambda^2})(\hat{p} + m), \quad (1.5)
\]

(note that \(\hat{S} + S = S\), the local propagator) and for the photon:

\[
\hat{D}_{\mu\nu} = E_0^2(P^2)D_{\mu\nu} = \int_1^\infty \frac{dx}{\Lambda^2} \exp(x\frac{p^2}{\Lambda^2})(g_{\mu\nu} - (1 - \xi)\frac{p_\mu p_\nu}{p^2}), \quad (1.6)
\]

where
We now construct the auxiliary ‘shadow field’ Lagrangian:

\[
L_{Sh} = \bar{\psi} \hat{S}^{-1} \psi + \bar{\phi} \hat{S}^{-1} \phi + \frac{1}{2e^2} A^\mu \hat{D}_\mu A^\nu - (\bar{\psi} + \bar{\phi}) A / (\psi + \phi). 
\]  

(1.8)

We remind the reader that the shadow fields \((\phi, \bar{\phi})\) are introduced merely as a device to generate the nonlocal action and symmetries in a compact form. They do not represent independent degrees of freedom, as they are constrained to obey their field equations at the classical level and are not integrated over in the generating functional. This is further demonstrated by the fact that their two-point function (the ‘barred’ propagator in (1.5)) does not have a pole and, hence, they are not propagating degrees of freedom, and should not be included in asymptotic states.

As discussed in [3], this particular choice of Lagrangian corresponds to a nonlocal regularization of QED, in which the classical theory retains the smearing on the internal photon lines, and internal fermion lines are ‘localized’. This guarantees decoupling of longitudinal photons from on-shell tree graphs [1], however in Section [V], we also develop the action that corresponds to localizing all the fields at the classical level. This is not necessary in order to guarantee gauge invariance but it treats all fields in a symmetrical way, and is another viable regularized QED action.

When performing quantum corrections, convergence is guaranteed by the presence of the ‘hatted’ propagator on at least one internal line. We have given the Schwinger parameterized form of the propagators in (1.5), since in practice one writes the local graph in Schwinger parameter form and restricts the range of parameter integrals appropriate for the process in question [1]. (For example, when one calculates single fermion loop graphs, the unit hypercube is removed from the volume of integration. This corresponds to the absence of the contribution from the graph with all ‘barred’ internal lines.) Clearly (1.8) is invariant under (the BRST generalization of):

\[
\delta A^\mu = \partial^\mu \theta
\]
\[ \delta \psi = -i E^2 \theta (\psi + \phi), \]
\[ \delta \phi = -i (1 - E^2) \theta (\psi + \phi), \]  
\[ \text{(1.9)} \]
and the conserved Noether current (generalizing the local vector current) is given by
\[ J^\mu = (\bar{\psi} + \bar{\phi}) \gamma^\mu (\psi + \phi). \]  
\[ \text{(1.10)} \]
To generate the action in terms of physical fields alone, we must remove them from the classical action by forcing them to obey their classical equations of motion. We have:
\[ \phi = \bar{S} \bar{A} (\psi + \phi) = (1 - \bar{S} \bar{A})^{-1} \bar{S} \bar{A} \psi, \]  
\[ \text{(1.11)} \]
and the Lagrangian, gauge transformations and Noether current are then given by:
\[ L_{NL} = \bar{\psi} \hat{S}^{-1} \psi + \frac{1}{2 e^2} A^\mu \hat{D}_\mu A^\nu - \bar{\psi} \hat{A} (1 - \bar{S} \bar{A})^{-1} \psi, \]
\[ = \bar{\psi} \hat{S}^{-1} \psi + \frac{1}{2 e^2} A^\mu \hat{D}_\mu^{-1} A^\nu - \bar{\psi} \hat{A} \psi, \]
\[ - \bar{\psi} \bar{A} \bar{S} A \psi - \bar{\psi} \bar{A} \bar{S} \bar{A} \bar{S} A \psi - \bar{\psi} \bar{A} \bar{S} \bar{A} \bar{S} \bar{A} \bar{S} A \psi - \ldots, \]  
\[ \text{(1.12)} \]
\[ \delta \psi = -i E^2 \theta (1 - \bar{S} \bar{A})^{-1} \psi \]
\[ = -i E^2 \theta (1 + \bar{S} \bar{A} + \bar{S} \bar{A} \bar{S} \bar{A} + \ldots) \psi, \]  
\[ \text{(1.13)} \]
\[ J^\mu = \bar{\psi} (1 - \bar{A} \bar{S})^{-1} \gamma^\mu (1 - \bar{S} \bar{A})^{-1} \psi \]
\[ = \bar{\psi} (\gamma^\mu + \bar{A} \bar{S} \gamma^\mu + \gamma^\mu \bar{S} \bar{A} + \ldots) \psi, \]  
\[ \text{(1.14)} \]
These results reproduce the classical theory described in [1] (up to a rescaling of the electromagnetic field strength \( A \to -eA \)).

**B. Generating the Measure**

Quantizing the theory described by (1.12) in the path integral formalism requires finding an invariant measure that respects the full nonlocal gauge invariance described by (1.13),
since the trivial measure is no longer invariant. We therefore require a method to generate consistency conditions on an invariant measure in order to retain the nonlocal invariance in the quantum regime, and thereby guarantee decoupling. The simplest way to do this is to consider how the trivial measure transforms under the nonlocal regularization gauge transformations, and require that there is a contribution from the measure that compensates.

Writing the full invariant measure as the product of the trivial measure and an exponentiated action term:

$$\mu_{\text{inv}}[\phi] = d\phi \exp(iS_{\text{meas}}[\phi]), \quad (1.15)$$

we find [3]:

$$\delta S_{\text{meas}} = i \text{Tr} \left[ \frac{\partial}{\partial \phi} \delta \phi \right]. \quad (1.16)$$

This procedure determines the measure up to arbitrary gauge invariant terms but, as we shall see, there is a natural minimal choice determined through relating the measure to the one loop graph it corrects, resulting in a unique (if it exists) longitudinal measure. We feel that any other invariant terms properly belong in the Lagrangian and should not be introduced into the measure. The measure is also constrained to be an entire function of the 4-momentum invariants for the particular process, ensuring that no additional degrees of freedom become excited in the quantum regime.

We then write the expectation of any operator as:

$$< T^*[O[A^\mu, \psi, \overline{\psi}]] > = \int d\mu_{\text{inv}} O[A^\mu, \psi, \overline{\psi}] \exp[i \int d^4x L_{NL}], \quad (1.17)$$

and the perturbative expansion is implemented as usual via the generating functional:

$$Z[S_\mu, \bar{\eta}, \eta] = \int d\mu_{\text{inv}} \exp[i \int d^4x (L_{NL} + S_\mu A^\mu + \bar{\eta} \psi + \psi \bar{\eta})]. \quad (1.18)$$

## II. WARD IDENTITIES

In order to generate identities on n-point functions, one transforms the fields as in Eq. (1.9), and sets the infinitesimal variation of the generating functional to zero:
\[ Z_0 = \frac{\delta}{\delta \omega} Z[S_\mu, \bar{\eta}, \eta] |_{\omega=0} = \int d\mu_{inv} \ K[A^\mu, S^\mu, \bar{\psi}, \psi, \bar{\eta}, \eta](x) \exp(iS[J]) = 0, \quad (2.1) \]

where \( K \) is given by:

\[ K = -\frac{1}{e^2 \xi} \left( \square \partial_\mu A^\mu - \partial_\mu S^\mu - i\bar{\eta}E^2(1 - \bar{S}A)^{-1}\psi + i\bar{\psi}(1 - A\bar{S})^{-1}E^2\eta. \]

(2.2)

Setting all sources to zero gives:

\[ Z_0 |_{J=0} = -\frac{1}{e^2 \xi} \left( \square \partial_\mu A^\mu(x) \right) < T^*[A^\mu(x)] > = 0, \quad (2.3) \]

and one derivative with respect to a photon source gives:

\[ \frac{1}{i} \frac{\delta}{\delta S_\alpha(y)} Z_0 |_{J=0} = -\frac{1}{e^2 \xi} \frac{(\square \partial_\mu)_x}{E_0^2} < T^*[A^\mu(x)A^\alpha(y)] > -\delta^\alpha \delta(x-y) = 0. \quad (2.4) \]

This relation is seen to hold to lowest order as the delta-function term cancels the longitudinal term in the bare propagator. It also provides a relation on the the irreducible corrections to the photon self energy (after truncating the external legs):

\[ p_\mu \Pi^\mu_\alpha = 0. \quad (2.5) \]

Higher derivatives with respect to photon sources then gives similar relations:

\[ \prod_j \frac{1}{i} \frac{\delta}{\delta S_{\alpha_j}(y_j)} Z_0 |_{J=0} = -\frac{1}{e^2 \xi} \frac{(\square \partial_\mu)_x}{E_0^2} < T^*[A^\mu(x)\prod_j A^\alpha(y_j)] > = 0, \quad (2.6) \]

which results in the identity on the n-point photon function:

\[ p_\mu N^{\mu_1...\mu_{n-1}} = 0. \quad (2.7) \]

Further identities are derived from taking functional derivatives of \( Z_0 \) with respect to the sources. For example, one derivative with respect to each of the fermion and antifermion sources provides (to lowest order):

\[ \frac{1}{i} \frac{\delta}{\delta \eta(z)} \frac{1}{i} \frac{\delta}{\delta \eta(y)} Z_0 |_{J=0} = -\frac{1}{e^2 \xi} \frac{(\square \partial_\mu)_x}{E_0^2} < T^*[A^\mu(x)\bar{\psi}(z)\psi(y)] > - (E_m^2 \delta^4(x-y)) < T^*[\bar{\psi}(z)\psi(x)] > + (E_m^2 \delta^4(x-z)) < T^*[\bar{\psi}(x)\psi(y)] > = 0, \quad (2.8) \]
leads to the usual identity on the vertex correction:

$$-i(p' - p)_\mu \Gamma^\mu(p, p') = -ie(S^{-1}(p) - S^{-1}(p')),$$

(2.9)

where we are referring to the fully corrected functions. Although the form of identities pertaining to higher point graphs are not simple in general, it is clear that they guarantee decoupling of photons from graphs with on-shell external fermions.

III. VACUUM POLARIZATION

We will begin with a brief review of the results derived elsewhere [1] on vacuum polarization to one loop. The nontrivial contributions to the measure come from:

$$\delta \psi = -iE^2 \theta \bar{S}A\psi,$$

(3.1)

and, as given in the original paper, produce the necessary contribution to vacuum polarization in order to satisfy the Ward identity and keep the photon transverse:

$$S^{(2)}_{\text{meas}} = -\frac{\Lambda^2}{4\pi^2} \int \frac{dp}{(2\pi)^4} \frac{dq}{(2\pi)^4} \delta^4(p + q) M_v(p) A^\mu(p) A_\mu(q),$$

(3.2)

$$M_v(p) = \int_0^1 dt t^2 \Lambda^2 \exp\left(t \frac{p^2}{\Lambda^2} - \frac{1}{1 - t} \frac{m^2}{\Lambda^2}\right).$$

(3.3)

However, we will now rewrite this result in a form that makes it more obvious what is happening. First consider the one loop contribution to vacuum polarization (Fig. [2]):

$$-i\Pi^{\mu\nu} = -\int \frac{dk}{(2\pi)^4} \sum_{Ev} Tr[S(k)\gamma^\nu S(p + k)\gamma^\mu].$$

(3.4)

where the sum represents all terms coming from (1.12):

$$\sum_{Ev} = E_m^2(k)E_m^2(p + k) + (1 - E_m^2(k))E_m^2(p + k) + E_m^2(k)(1 - E_m^2(p + k))$$

$$= E_m^2(p + k)(1 - E_m^2(k)) + E_m^2(k)$$

$$= E_m^2(k)(1 - E_m^2(p + k)) + E_m^2(p + k).$$

(3.5)
FIGURES

We now ‘dot’ $p_\mu$ into this and use the relation (which will receive heavy use in this paper);

$$\phi = S^{-1}(p + k) - S^{-1}(k). \quad (3.6)$$

Then:

$$-i p_\mu \Pi^{\mu\nu} = -\int \frac{dk}{(2\pi)^4} \frac{T}{E_\nu} \left[ Tr[S(k)\gamma^\nu] - Tr[S(p + k)\gamma^\nu] \right]. \quad (3.7)$$

Re-writing the sums as the last two terms in (3.5), one each for the traces:

$$-i p_\mu \Pi^{\mu\nu} = -\int \frac{dk}{(2\pi)^4} \frac{T}{E_\nu} \left[ E_m^2(p + k) Tr[S(k)\gamma^\nu] + Tr[S(p + k)\gamma^\nu] \right]$$

$$+ \int \frac{dk}{(2\pi)^4} \frac{T}{E_\nu} \left[ E_m^2(k) Tr[S(p + k)\gamma^\nu] + Tr[S(p + k)\gamma^\nu] \right], \quad (3.8)$$

where the second and fourth terms cancel by a simple shift of loop momentum. Note that we have been careful to only consider partitioning the sum into separately convergent terms.

We are then left with:

$$-i p_\mu \Pi^{\mu\nu} = -\int \frac{dk}{(2\pi)^4} \frac{T}{E_\nu} \left[ E_m^2(p + k) Tr[S(k)\gamma^\nu] - E_m^2(k) Tr[S(p + k)\gamma^\nu] \right]$$

$$\overset{df}{=} -i p_\mu \Pi^{\mu\nu}_L, \quad (3.9)$$

where:

$$\Pi^{\mu\nu} = (g^{\mu\nu} - \frac{p^{\mu}p^{\nu}}{p^2}) \Pi_T + \frac{p^{\mu}p^{\nu}}{p^2} \Pi_L, \quad \Pi^{\mu\nu}_L \overset{df}{=} \Pi_L g^{\mu\nu}. \quad (3.10)$$

From the above gauge transformation, we find a condition on the measure that is easily identifiable with (3.9):
\[ \delta S_{\text{meas}}^{(2)} = - \int \frac{dp \, dq}{(2\pi)^4} \delta^4(p + q) \theta(p) A_\nu(q) \]
\[ \int \frac{dk}{(2\pi)^4} \{ E^2(p + k) Tr[S(k) \gamma'] - E^2(k) Tr[S(p + k) \gamma'] \} \]
\[ = \int \frac{dp \, dq}{(2\pi)^4} \delta^4(p + q) \theta(p) A_\nu(q)(-i)p_\mu \Pi_{L}^{\mu\nu}(p), \quad (3.11) \]

which leads to:

\[ S_{\text{meas}}^{(2)} = \frac{1}{2} \int \frac{dp \, dq}{(2\pi)^4} \delta^4(p + q) A_\mu(p) A_\nu(q) \Pi_{L}^{\mu\nu}(p). \quad (3.12) \]

where we have used the fact that the two point function is symmetrized in the external fields, even after the longitudinal projection is performed. A simple calculation will reproduces (3.12) and we have thus reduced the existence of the measure to the existence of the longitudinal projection of the graph. (i.e. The measure is just what is required in order for the process in question to satisfy the Ward identities.) Note that the measure (3.12) is an entire function of the finite complex \( p^2 \) plane and we are therefore sure that we are not introducing additional degrees of freedom at the quantum level through the measure. We could also see this directly from the gauge transformations leading to (3.11) since the barred propagators do not have a pole, so that when the resulting measure term is analytically continued to Minkowski space, we will not pick up any imaginary parts.

IV. THIRD ORDER CORRECTIONS

We now check the Ward identity (2.9) on the vertex correction shown in Fig. 2:

\[ -i \Gamma^\mu(p, p') = \int \frac{dk}{(2\pi)^4} \sum_{E \Gamma} \gamma^\alpha S(p' + k) \gamma^\mu S(p + k) \gamma^\beta e^2 D_{\alpha\beta}(k), \quad (4.1) \]

where the sum is over all terms in the Lagrangian (1.12):

\[ \sum_{E \Gamma} = \left[ E_0^2(k)[E_m^2(p + k)E_m^2(p' + k) + E_m^2(p + k)(1 - E_m^2(p' + k))] \right. \]
\[ \left. + (1 - E_m^2(p + k))E_m^2(p' + k) + (1 - E_m^2(p + k))(1 - E_m^2(p' + k)) \right] = E_0^2(k), \quad (4.2) \]

so that the fermion line is fully localized. (In this regularization, it is easy to see that in any process throughtgoing fermion lines are always fully localized.)
FIG. 2. Vertex correction

Dotting the photon momentum into this gives:

\[-i(p' - p)_\mu \Gamma^\mu(p, p') = \int \frac{dk}{(2\pi)^4} \gamma^\alpha S(p' + k)(p' - \not{p})S(p + k)\gamma^\beta e^2 \hat{D}_{\alpha\beta}(k), \quad (4.3)\]

and using the identity: \( p' - \not{p} = S^{-1}(p' + k) - S^{-1}(p + k) \), we immediately recognize:

\[-i(p' - p)_\mu \Gamma^\mu(p, p') = \int \frac{dk}{(2\pi)^4} \left[ \gamma^\alpha S(p + k)\gamma^\beta - \gamma^\alpha S(p' + k)\gamma^\beta \right] e^2 \hat{D}_{\alpha\beta}(k) \]

\[= -i [\Sigma(p) - \Sigma(p')], \quad (4.4)\]

where

\[-i\Sigma(p) = \int dk(2\pi)^4 \sum_{E\Sigma} \gamma^\alpha S(p + k)\gamma^\beta e^2 D_{\alpha\beta}(k), \quad (4.5)\]

with now

\[\sum_{E\Sigma} = E_0^2(k)[E_m^2(p + k) + (1 - E_m^2(p + k))] = E_0^2(k). \quad (4.6)\]

We therefore have to third order:

\[-i(p' - p)\Gamma(p, p') = ie[S^{-1}(p) - \Sigma(p)] - ie[S^{-1}(p') - \Sigma(p')], \quad (4.7)\]

and the identity (2.9) is satisfied.

That there is no measure contribution in odd orders was already demonstrated in [1], and here we sketch the result for this particular case. The relevant transformations are:
\[ \delta \psi = -i E_m^2 \theta S \bar{A} \bar{A} \psi, \quad \delta \bar{\psi} = \bar{\psi} A \bar{A} \bar{S} \theta E_m^2 i, \quad (4.8) \]

leading to the condition on the measure:

\[
\delta S_{\text{meas}}^{(3)} = - \int \frac{dp \ dq_1 \ dq_2}{(2\pi)^8} (2\pi)^4 \delta^4(p + q_1 + q_2) \int \frac{dk}{(2\pi)^4} \theta(p) \\
\times \left\{ E_m^2(k + p) Tr[\bar{S}(k) A(q_1) \bar{S}(k - q_1) A(q_2)] \right\} \\
- E_m^2(k) Tr[A(q_1) \bar{S}(k - q_1) A(q_2) \bar{S}(k + p)], \quad (4.9) \]

and it is not hard to see that the surviving terms from the traces are of opposite sign and therefore cancel. This is consistent with the fact that the related triangle graph disappears by Furry’s theorem, and hence the measure can be related to a projection of which is zero.

Instead of the regulated Lagrangian (4.8) in which only the smearing of the internal fermion lines is removed at the classical level, we could also remove the smearing from the internal photon lines. This is accomplished by introducing a shadow field for the photon as well as the fermions:

\[
L_{Sh} = \bar{\psi} \bar{S}^{-1} \psi + \bar{\phi} \bar{S}^{-1} \phi + \frac{1}{2e^2} A^\mu \bar{D}^{-1}_\mu A^\nu + \frac{1}{2e^2} B^\mu \bar{D}^{-1}_\mu B^\nu \\
- (\bar{\psi} + \bar{\phi}) A(\psi + \phi). \quad (4.10) \]

Then the field equations that are used to remove the shadow fields are:

\[
\phi = \bar{S}(A + B)(\psi + \phi), \\
B_\mu = e^2 \bar{D}_\mu (\bar{\psi} + \bar{\phi}) \gamma^\nu (\psi + \phi). \quad (4.11) \]

Although we have not been able to give the full nonlocal Lagrangian in closed form in terms of the physical fields alone, it should be clear that one can generate it to any order by iterating (4.11) in (4.10).

The unitary gauge Lagrangian posessese the gauge invariance:

\[
\delta \psi = -i E_m^2 \theta (\psi + \phi), \quad \delta \phi = -i (1 - E_m^2) \theta (\psi + \phi), \\
\delta A^\mu = E_0^2 \partial^\mu \theta, \quad \delta B^\mu = (1 - E_0^2) \partial^\mu \theta, \quad (4.12) \]
as guaranteed by the nonlocal construction [2]. It also has the ‘dynamically trivial’ invar-
iance:

$$\delta A^\mu = (1 - E_0^2) \partial^\mu \theta, \quad \delta B^\mu = -(1 - E_0^2) \partial^\mu \theta,$$

which allows one to instead write in (4.12):

$$\delta A^\mu = \partial^\mu \theta, \quad \delta B^\mu = 0,$$  (4.14)

so that it is clear that longitudinal photons should still decouple. Indeed, it is not hard to
test that the Ward identity is unchanged in this case, even though the range of parameter
integrals is now different in the calculation of the vertex correction. We shall see, however,
that there is now a contribution from the measure that ‘corrects’ the vertex further, allowing
consistency with the Ward identity.

Repeating the above calculation of the vertex correction merely involves additional terms
in the sum (4.12), which we will now write as:

$$\sum_{E\Sigma} = E_m^2(k)(1 - E_m^2(p + k))(1 - E_m^2(p' + k)) + \sum_{E\Sigma}$$  (4.15)

where the second term refers to the sum corresponding to the region in the fermion correction:

$$\sum_{E\Sigma} = \sum_{E\Sigma} + (1 - E_0^2(k))E_m^2(p + k),$$  (4.16)

(we denote all quantities calculated in this ‘extended’ regularization of QED by primes).

Calculating the vertex correction gives:

$$-i \Gamma^{\mu\nu} = \int \frac{dk}{(2\pi)^4} \sum_{E\Sigma} \gamma^\alpha S(p' + k)\gamma^\beta S(p + k)D_{\alpha\beta}(k) - \sum_{E\Sigma} \gamma^\beta S(p' + k)\gamma^\alpha D_{\alpha\beta}(k)$$

$$+ \int \frac{dk}{(2\pi)^4} D_{\alpha\beta}(k) \{ E_m^2(p' + k)\gamma^\alpha S(p + k)\gamma^\beta - E_m^2(p + k)\gamma^\alpha S(p' + k)\gamma^\beta \}$$

$$= -i[\Sigma(p) - \Sigma(p')]$$

$$+ \int \frac{dk}{(2\pi)^4} \tilde{D}_{\alpha\beta}(k) \{ E_m^2(p' + k)\gamma^\alpha \tilde{S}(p + k)\gamma^\beta - E_m^2(p + k)\gamma^\alpha \tilde{S}(p' + k)\gamma^\beta \}$$

$$= -iq^\mu \Gamma^{\mu\nu}_L.$$  (4.17)
We determine the measure by first iterating the shadow field equations into the transformations to find the additional terms at third order;

\[ \delta \psi = -i E^2_m \theta \bar{S} \gamma^\mu \psi e^2 \bar{D}_{\mu\nu} \bar{\gamma}^\nu \psi, \]

\[ \delta \bar{\psi} = \bar{\psi} \gamma^\nu \psi e^2 \bar{D}_{\mu\nu} \bar{\gamma}^\mu \bar{S} \theta E^2_m i. \] (4.18)

Note that only the ‘1PI derivative’ need be taken, since the other terms will merely reproduce lower order measure contributions attached to barred propagators, serving to localize corrected tree graphs. This merely implies replacing (3.12) with:

\[ S^{(2)}_{\text{meas}} = \frac{1}{2} \int \frac{dp \ dq}{(2\pi)^4} \delta^4(p + q)(A_\mu(p) + B_\mu(p))(A_\nu(q) + B_\nu(q))\Pi_{\mu\nu}. \] (4.19)

The remaining contributions result in:

\[ \delta S^{(3)}_{\text{meas}} = \int \frac{dq \ dp \ dp'}{(2\pi)^6} (2\pi)^4 \delta^4(q + p - p') \frac{dk}{(2\pi)^4} \theta(q) \bar{D}_{\mu\nu}(k) \]

\[ \times \{ E^2_m(k + p')\bar{\psi}(-p')\gamma^\mu \bar{S}(k + p)\gamma^\nu \psi(p) - E^2_m(k + p)\bar{\psi}(-p')\gamma^\mu \bar{S}(k + p')\gamma^\nu \psi(p) \} \]

\[ = \int \frac{dq \ dp \ dp'}{(2\pi)^6} (2\pi)^4 \delta^4(q + p - p') \theta(q) \bar{\psi}(-p') \{ -iq_\mu \Gamma^\mu_L - i[\Sigma(p')' - \Sigma(p)] \} \psi(p). \] (4.20)

Note that we can write:

\[ \Sigma(p')' - \Sigma(p)' = q_\mu \Sigma^\mu, \] (4.21)

since the \( q \to 0 \) limit implies \( p = -p' \), and so we then have the required measure:

\[ S^{(3)}_{\text{meas}} = \int \frac{dq \ dp \ dp'}{(2\pi)^6} (2\pi)^4 \delta^4(q + p - p')A_\mu(q)\bar{\psi}(-p') \{ \Gamma^\mu_L + \Sigma^\mu \} \psi(p), \] (4.22)

and we see that the measure again ensures the validity of the Ward identity.

V. BOX GRAPH

Before turning to the general proof that each term in the measure is identically the longitudinal projection of the related one loop graph, we will explicitly demonstrate it for the box graph, which contains all of the essential features. We write the four point photon graph as:
\[ -iB^{\mu\alpha\beta\gamma} = -i(B_1^{\mu\alpha\beta\gamma} + \text{perms.}), \quad (5.1) \]

where:

\[ -iB_1^{\mu\alpha\beta\gamma} = - \int \frac{dk}{(2\pi)^4} \sum_{EB} Tr[S(k + p)\gamma^\mu S(k)\gamma^\alpha S(k - q_1)\gamma^\beta S(k + p + q_3)\gamma^\gamma], \quad (5.2) \]

is the contribution corresponding to Fig. 3.

**FIG. 3.** One contribution to the 4-point photon function. The others correspond to the six permutations of \((q_1, q_2, q_3)\).

In this case, there are six permutations of final legs. Dotting \(p_\mu\) into this and reducing the traces as in the previous Section, we get:

\[
- ip_\mu B_1^{\mu\alpha\beta\gamma} = - \int \frac{dk}{(2\pi)^4} \sum_{EB} Tr[S(p + k)\gamma^\mu S(k)\gamma^\alpha S(k - q_1)\gamma^\beta S(k + p + q_3)\gamma^\gamma] \\
= - \int \frac{dk}{(2\pi)^4} \sum_{EB} \{ Tr[S(k)\gamma^\alpha S(k - q_1)\gamma^\beta S(k + p + q_3)\gamma^\gamma] \\
- Tr[S(p + k)\gamma^\gamma S(k - q_1)\gamma^\beta S(k + p + q_3)\gamma^\gamma] \} \\
= - \int \frac{dk}{(2\pi)^4} \{ \sum_{ET} Tr[S(k)\gamma^\alpha S(k - q_1)\gamma^\beta S(k + p + q_3)\gamma^\gamma] \\
- \sum_{ET} Tr[S(p + k)\gamma^\gamma S(k - q_1)\gamma^\beta S(k + p + q_3)\gamma^\gamma] \\
+ E_{m_1}^2(p + k) Tr[S(k)\gamma^\gamma S(k - q_1)\gamma^\beta S(k + p + q_3)\gamma^\gamma] \\
- E_{m_2}^2(k) Tr[S(p + k)\gamma^\gamma S(k - q_1)\gamma^\beta S(k + p + q_3)\gamma^\gamma] \} \quad (5.3),
\]

where we have separated the sum over smearing regions into two terms as before.

One can now show that the first two terms will cancel other permuted terms via Furry’s theorem. The remaining piece is then:
\[ -i\mu \mathcal{B}_{\mu}^{\alpha\beta\gamma} = -\int \frac{dk}{(2\pi)^4} \left\{ E_m^2(p + k)Tr[\mathcal{S}(k)\gamma^\alpha \mathcal{S}(k - q_1)\gamma^\beta \mathcal{S}(k + p + q_3)\gamma^\gamma] \right. \]
\[ - E_m^2(k)Tr[\mathcal{S}(p + k)\gamma^\alpha \mathcal{S}(k - q_1)\gamma^\beta \mathcal{S}(k + p + q_3)\gamma^\gamma] \left\} \text{perms.} \]
\[ \equiv -i\mu \mathcal{B}_{L}^{\alpha\beta\gamma}. \quad (5.4) \]

There is also a contribution from the measure that will resurrect the identity, coming from the transformations:
\[ \delta\psi = -iE_m^2\theta \bar{\mathcal{S}}\mathcal{A}\bar{\mathcal{S}}\mathcal{A}\psi, \quad \delta\bar{\psi} = \bar{\psi}\mathcal{A}\bar{\mathcal{S}}\mathcal{A}\bar{\mathcal{S}}\theta E_m^2i, \quad (5.5) \]

which produces:
\[ \delta S_{\text{meas}}^{(4)} = -\int \frac{dp\ dq_1\ dq_2\ dq_3}{(2\pi)^8} (2\pi)^4\delta^4(p + q_1 + q_2 + q_3) \int \frac{dk}{(2\pi)^4}\theta(p) \times \left\{ E_m^2(k + p)Tr[\mathcal{S}(k)\mathcal{A}(q_1)\mathcal{S}(k - q_1)\mathcal{A}(q_2)\mathcal{S}(k + p + q_3)\mathcal{A}(q_3)] \right. \]
\[ - E_m^2(k)Tr[\mathcal{A}(q_3)\mathcal{S}(k - q_3)\mathcal{A}(q_2)\mathcal{S}(k - q_3 - q_2)\mathcal{A}(q_1)\mathcal{S}(k + p)] \left\} \text{perms.} \quad (5.6) \]

We can now symmetrize on the external photon fields to produce identical terms with different momentum labelings:
\[ \delta S_{\text{meas}}^{(4)} = \frac{-1}{3!} \int \frac{dp\ dq_1\ dq_2\ dq_3}{(2\pi)^8} (2\pi)^4\delta^4(p + q_1 + q_2 + q_3) \int \frac{dk}{(2\pi)^4}\theta(p) A_\alpha(q_1) A_\beta(q_2) A_\gamma(q_3) \times \left\{ E_m^2(k + p)Tr[\mathcal{S}(k)\gamma^\alpha \mathcal{S}(k - q_1)\gamma^\beta \mathcal{S}(k + p + q_3)\gamma^\gamma] \right. \]
\[ - E_m^2(k)Tr[\mathcal{S}(p + k)\gamma^\alpha \mathcal{S}(k - q_1)\gamma^\beta \mathcal{S}(k + p + q_3)\gamma^\gamma] \left\} + \text{perms.} \]
\[ = \frac{1}{3!} \int \frac{dp\ dq_1\ dq_2\ dq_3}{(2\pi)^8} (2\pi)^4\delta^4(p + q_1 + q_2 + q_3)\theta(p) A_\alpha(q_1) A_\beta(q_2) A_\gamma(q_3)(-i)\mu\mathcal{B}_{L}^{\alpha\beta\gamma}, \quad (5.7) \]

which then gives
\[ S_{\text{meas}}^{(4)} = \frac{1}{4!} \int \frac{dp\ dq_1\ dq_2\ dq_3}{(2\pi)^8} (2\pi)^4\delta^4(p + q_1 + q_2 + q_3) \]
\[ A_\mu(p) A_\alpha(q_1) A_\beta(q_2) A_\gamma(q_3) \mathcal{B}_{L}^{\mu\alpha\beta\gamma}, \quad (5.8) \]

which then leads to the fully corrected gauge invariant photon 4-point function:
\[ -i\mathcal{B}_{T}^{\mu\alpha\beta\gamma} = -i\mathcal{B}_{L}^{\mu\alpha\beta\gamma} + i\mathcal{B}_{L}^{\mu\alpha\beta\gamma}. \quad (5.9) \]

An identical construction is performed on the one loop contribution to a general photon n-point function in the Appendix.
VI. HIGHER LOOP CONSIDERATIONS

We will begin by showing that the two loop correction to vacuum polarization is gauge invariant as a consequence of the existence of the box graph contribution to the one loop measure. The measure corrected (transverse) box graph amplitude will be written as (5.9), then the three contributions to the two loop vacuum polarization correction can now be written as:

\[-i\Pi^{\mu\gamma}_2(p) = \frac{1}{2} \int \frac{dq_1}{(2\pi)^4} (2\pi)^4\delta^4(q_1 + q_2) i e \hat{D}_{\alpha\beta}(q_1)(-i)B_T^{\mu\alpha\beta\gamma}(p, q_1, q_2, -p - q_1 - q_2), \tag{6.1}\]

since we only have ‘hatted’ photon propagators. The transversality of the resulting two point function then trivially follows from the transversality of the full one loop four photon process.

It is not hard to see that this result persists for all photon n-point functions, namely that transversality follows from the transversality of the related single loop graph. This holds for internal loops as well, so that any number of fermion loops may contribute to a process, but all longitudinal photons decouple from each separately. Similarly, when one considers an on shell throughgoing fermion line, one can rewrite the process in an analogous manner (tree graph \times delta functions and photon propagators), and infer decoupling from decoupling of the related tree graph. Thus in proving the existence of the measure, we have shown that the regularization is consistent with gauge invariance, as was to be expected.

Although this result is fairly simple in this case, it is not so easy to implement in general. Even if one considers the ‘symmetric’ regularization discussed in Section [IV], one can not so easily break up a multiloop graph into lower loop pieces, since the parameter integral regions remain entangled in general. We feel that this only a technical problem and should not be impossible to resolve.
VII. ANOMALOUS THEORIES

There are two possible ways in which the construction here could fail. The first is that the longitudinally projected vertex function has symmetries that are destroyed when inserted into the exponentiated measure. This is the case in the (1 + 1)-chiral Schwinger model studied by Hand \[5\]. Briefly, the local Lagrangian (1.1) differs by the insertion of an axial component in the coupling: $-\bar{\psi} A_P \gamma^\nu \psi$, where $P_L = \frac{1}{2}(1 - \gamma^5)$. This changes the second order term in the nonlocal gauge transformation (3.1) to:

$$\delta \psi = -i E^2 \theta S \bar{A} P_L \psi,$$

and the measure consistency condition becomes:

$$\delta S_{(2)}^{meas} = \frac{1}{2} \int \frac{dp \ dq}{(2\pi)^4} (2\pi)^4 \delta^4(p + q) \theta(p) A_\nu(q) - \int \frac{dk}{(2\pi)^4} \left\{ E^2(p + k) T_r[S(k) \gamma^\nu P_L] - E^2(k) T_r[S(p + k) \gamma^\nu P_L] \right\}$$

$$= \int \frac{dp \ dq}{(2\pi)^4} (2\pi)^4 \delta^4(p + q) \theta(p) A_\nu(q) (-i) p_\mu \Pi_{\mu\nu}^L(p). \quad (7.2)$$

We have calculated the vacuum polarization result using the same conventions as in Fig. [1] and identified the longitudinal projection as in section (III).

This result however, is not consistent with writing:

$$S_{(2)}^{meas} = \frac{1}{2} \int \frac{dp \ dq}{(2\pi)^4} (2\pi)^4 \delta^4(p + q) A_\mu(p) A_\nu(q) \Pi_{\mu\nu}^L(p), \quad (7.3)$$

since a short calculation leads to:

$$\Pi_{\mu\nu}^L = -\frac{4}{(2\pi)^2} \int_0^1 \frac{dx}{(1 + x)^2} \exp\left[ \frac{x}{1 + x} \frac{p^2}{\Lambda^2} \right] (g_{\mu\nu} - \varepsilon_{\mu\nu}), \quad (7.4)$$

(the antisymmetric piece arising from the two dimensional trace over $\gamma^\mu \gamma^\nu \gamma^5$). Clearly the action is symmetric under exchange of the field variables and the $\varepsilon$ term is eliminated, making it impossible to satisfy the consistency condition (7.2).

This is the case as well in the AAA sector of the $U(1)$ chiral invariant model studied in [3]. The VVA sector however, demonstrates the second possible inconsistency. The Ward identities on the triangle graph lead to two conditions:
\[ -i p_\mu \Gamma^{\mu\alpha\beta} = -i q_1 \Gamma^{\mu\alpha\beta} = 0, \tag{7.5} \]

the first showing decoupling of the longitudinal axial vector boson, and the second the vector boson. The problem is that the longitudinally projected vertex functions generated from each of these conditions, and that appear in the measure consistency conditions, are not identical, and are incompatible.

From the two contributions to the VVA sector in Fig. 4, we find the longitudinal projection coming from the axial vector sector:

\[
- i p_\mu \Gamma^{\mu\alpha\beta} = - \int \frac{dk}{(2\pi)^4} \times \left\{ E_m^2(p + k)Tr[\gamma^5 \bar{S}(k)\gamma^\alpha \bar{S}(k - q_1)\gamma^\beta] + E_m^2(p + k)Tr[\gamma^5 \bar{S}(k)\gamma^\beta \bar{S}(k - q_2)\gamma^\alpha] - E_m^2(k)Tr[\gamma^5 \bar{S}(k + p)\gamma^\alpha \bar{S}(k - q_1)\gamma^\beta] - E_m^2(k)Tr[\gamma^5 \bar{S}(k + p)\gamma^\beta \bar{S}(k - q_2)\gamma^\alpha] \right\} = - \frac{8p_\mu}{(4\pi)^2} \epsilon^{\mu\nu\alpha\beta} q_{1\nu} M(q_1; p, q_2) + \epsilon^{\mu\beta\nu\alpha} q_{2\nu} M(q_2; p, q_1), \tag{7.6} \]

where

\[
M(p; q_1, q_2) = \int \frac{dx \, dy}{(1 + x + y)^3} \exp \left[ \frac{xy}{1 + x + y} \frac{p^2}{\Lambda^2} + \frac{x}{1 + x + y} \frac{q_1^2}{\Lambda^2} + \frac{y}{1 + x + y} \frac{q_2^2}{\Lambda^2} \right], \tag{7.7} \]

whereas that from the vector projection is:

\[
- i q_1 \Gamma^{\mu\alpha\beta} = - \int \frac{dk}{(2\pi)^4} \times \left\{ E_m^2(k)Tr[\gamma^5 \bar{S}(k - q_1)\gamma^\beta \bar{S}(k + p)\gamma^\mu] + E_m^2(k - q_1)Tr[\gamma^5 \bar{S}(k + p)\gamma^\beta \bar{S}(k)\gamma^\mu] - E_m^2(k - q_1)Tr[\gamma^5 \bar{S}(k)\gamma^\beta \bar{S}(k + p)\gamma^\mu] - E_m^2(k - q_1)Tr[\gamma^5 \bar{S}(k + p)\gamma^\beta \bar{S}(k - q_1)\gamma^\mu] \right\} = - \frac{8q_{1\alpha}}{(4\pi)^2} \epsilon^{\mu\nu\alpha\beta} q_{2\nu} M(q_2; p, q_1) + \epsilon^{\mu\beta\nu\alpha} p_\nu M(p; q_1, q_2). \tag{7.8} \]

It is not hard to see from this that it is impossible to write a measure correction that will satisfy \(7.5\).
FIG. 4. The two contributions to the one-loop $VVA$ sector. Note that the axial vector field is denoted $B^\mu$, with coupling $-i\gamma^\mu\gamma^5$.

This shows that one need not attempt to calculate the measure directly from the nonlocal gauge transformations in order to determine whether a theory is anomalous or not. It is sufficient (and equivalent) to check the Ward-Takahashi identities on graphs where possible conflicts of this type may arise.

CONCLUSIONS

We have shown that nonlocal regulated QED has a one loop invariant measure to all orders through directly equating it to the related longitudinally projected vertex functions. This result then leads to decoupling of longitudinal photons from all processes with on-shell external fermions, and so we have proven that we will generate a gauge invariant perturbation series.

Clearly the same considerations will hold in other theories as well, and one may state with some confidence that if one can consistently introduce a measure into the generating functional solely on the basis of imposing the Ward-Takahashi identities, then this is indeed the required invariant measure.

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APPENDIX A: ONE LOOP QED MEASURE

Consider first the n-point photon graph, a portion of which is shown in Fig. 5:

\[-i N_{\mu_1, \alpha_2, \ldots, \alpha_{n-1}} = -i \sum_{\alpha, \beta} N_{\mu_1, \alpha_2, \ldots, \alpha_{n-1}}, \quad (A1)\]

where the sum is over permutations of \(\alpha_i\). (Note that we could just as easily permute say \(\mu, \alpha_2 \ldots \alpha_{n-1}\) and recover the same result. This means that \(p_\mu N_{\mu_1, \ldots, \alpha_{n-1}} = q_\alpha N_{\mu_1, \alpha_2, \ldots, \alpha_{n-1}}, \) a result that will be important later.)

![FIG. 5. A section of one contribution to the general n-point photon graph.](image)
We find for the identity permutation:

\[-ip_\mu N_1^{\mu_1...\mu_{n-1}} = - \int \frac{dk}{(2\pi)^4} \sum_{E(n)} \{ Tr[\hat{S}(k)\gamma^{\alpha_1}S(k-q_1)\gamma^{\alpha_2}\cdots\gamma^{\alpha_{n-2}}S(k+p+q_{n-1})\gamma^{\alpha_{n-1}}] \]

\[ - Tr[S(k+p)\gamma^{\alpha_1}S(k-q_1)\gamma^{\alpha_2}\cdots\gamma^{\alpha_{n-2}}S(k+p+q_{n-1})\gamma^{\alpha_{n-1}}]. \quad (A2)\]

First we note that the odd order graphs will disappear by Furry’s theorem (one can see this by considering the permutation that replaces \( q_m \) by \( q_{n-m} \), and noting that the two terms differ by the reversal of the fermion line, and hence must cancel). Then we expand the trace (as before) into a term that gives a contribution to an \( n-1 \) order graph and a term with barred propagators:

\[-ip_\mu N_1^{\mu_1...\mu_{n-1}} = - \int \frac{dk}{(2\pi)^4} \]

\[ \{ \sum_{E(n-1)} Tr[\hat{S}(k)\gamma^{\alpha_1}S(k-q_1)\gamma^{\alpha_2}\cdots\gamma^{\alpha_{n-2}}S(k+p+q_{n-1})\gamma^{\alpha_{n-1}}] \]

\[ - \sum_{E(n-1)} Tr[S(k+p)\gamma^{\alpha_1}S(k-q_1)\gamma^{\alpha_2}\cdots\gamma^{\alpha_{n-2}}\tilde{S}(k+p+q_{n-1})\gamma^{\alpha_{n-1}}] \]

\[ + E_{m}^{2}(k+p)Tr[\hat{S}(k)\gamma^{\alpha_1}\tilde{S}(k-q_1)\gamma^{\alpha_2}\cdots\gamma^{\alpha_{n-2}}\tilde{S}(k+p+q_{n-1})\gamma^{\alpha_{n-1}}] \]

\[ - E_{m}^{2}(k)Tr[\hat{S}(k+p)\gamma^{\alpha_1}\tilde{S}(k-q_1)\gamma^{\alpha_2}\cdots\gamma^{\alpha_{n-2}}\tilde{S}(k+p+q_{n-1})\gamma^{\alpha_{n-1}}]. \quad (A3)\]

The first two terms are contributions to \( n-1 \) order graphs and will disappear when symmetrized, since \( n-1 \) is now odd. We are then left with:

\[-ip_\mu N_1^{\mu_1...\mu_{n-1}} = - \int \frac{dk}{(2\pi)^4} \]

\[ + E_{m}^{2}(k+p)Tr[\hat{S}(k)\gamma^{\alpha_1}\tilde{S}(k-q_1)\gamma^{\alpha_2}\cdots\gamma^{\alpha_{n-2}}\tilde{S}(k+p+q_{n-1})\gamma^{\alpha_{n-1}}] \]

\[ - E_{m}^{2}(k)Tr[\hat{S}(k+p)\gamma^{\alpha_1}\tilde{S}(k-q_1)\gamma^{\alpha_2}\cdots\gamma^{\alpha_{n-2}}\tilde{S}(k+p+q_{n-1})\gamma^{\alpha_{n-1}}], \quad (A4)\]

and so we define:

\[-ip_\mu N_L^{\mu_1...\mu_{n-1}} \equiv -ip_\mu \sum_{P} N_P^{\mu_1...\mu_{n-1}}. \quad (A5)\]

This is related to the calculation of the measure as follows. Consider the gauge transformations containing \( n-1 \) photon fields:

\[ \delta\psi = -iE_{m}^{2}\theta(\tilde{S}\bar{A})^{n-1}\psi. \quad (A6)\]
Calculating the measure consistency condition as in (1.16), we find that this gives:

\[
\delta S_{\text{meas}}^{(n)} = - \int \frac{dp \, dq_1 \ldots dq_{n-1}}{(2\pi)^{2n}} (2\pi)^4 \delta^4(p + q_1 + \ldots + q_{n-1}) \]

\[
\int \frac{dk}{(2\pi)^4} \theta(p) E_m^2(k + p) \]

\[
Tr[\bar{S}(k) A(\alpha_1) \bar{S}(k - q_1) A(\alpha_2) \ldots A(q_{n-2}) \bar{S}(k + p + q_{n-1}) A(\alpha_{n-1})] \]

\[
= - \frac{1}{(n - 1)!} \int \frac{dp \, dq_1 \ldots dq_{n-1}}{(2\pi)^{2n}} (2\pi)^4 \delta^4(p + q_1 + \ldots + q_{n-1}) \]

\[
A_{\alpha_1}(q_1) A_{\alpha_2}(q_2) \ldots A_{\alpha_{n-1}}(q_{n-1}) \sum_{p} \int \frac{dk}{(2\pi)^4} \theta(p) E_m^2(k + p) \]

\[
Tr[\bar{S}(k)\gamma^{\alpha_1} \bar{S}(k - q_1)\gamma^{\alpha_2} \ldots \gamma^{\alpha_{n-2}} \bar{S}(k + p + q_{n-1})\gamma^{\alpha_{n-1}}],
\]

and once the conjugate is added to this we find the full condition on the measure:

\[
\delta S_{\text{meas}}^{(n)} = - \frac{1}{(n - 1)!} \int \frac{dp \, dq_1 \ldots dq_{n-1}}{(2\pi)^{2n}} (2\pi)^4 \delta^4(p + q_1 + \ldots + q_{n-1}) \]

\[
A_{\alpha_1}(q_1) A_{\alpha_2}(q_2) \ldots A_{\alpha_{n-1}}(q_{n-1}) \sum_{p} \int \frac{dk}{(2\pi)^4} \theta(p) \]

\[
\{ E_m^2(k + p) Tr[\bar{S}(k)\gamma^{\alpha_1} \bar{S}(k - q_1)\gamma^{\alpha_2} \ldots \gamma^{\alpha_{n-2}} \bar{S}(k + p + q_{n-1})\gamma^{\alpha_{n-1}}] \}

\[
- E_m^2(k) Tr[\bar{S}(k + p)\gamma^{\alpha_1} \bar{S}(k - q_1)\gamma^{\alpha_2} \ldots \gamma^{\alpha_{n-2}} \bar{S}(k + p + q_{n-1})\gamma^{\alpha_{n-1}}] \}

\[
= \frac{1}{(n - 1)!} \int \frac{dp \, dq_1 \ldots dq_{n-1}}{(2\pi)^{2n}} (2\pi)^4 \delta^4(p + q_1 + \ldots + q_{n-1}) \]

\[
\theta(p) A_{\alpha_1}(q_1) A_{\alpha_2}(q_2) \ldots A_{\alpha_{n-1}}(q_{n-1})(-i)p_{\mu} N_{L}^{\mu_{\alpha_1} \ldots \alpha_{n-1}}.
\]

Due to the above stated symmetry, we can then immediately write:

\[
S_{\text{meas}}^{(n)} = \frac{1}{n!} \int \frac{dp \, dq_1 \ldots dq_{n-1}}{(2\pi)^{2n}} (2\pi)^4 \delta^4(p + q_1 + \ldots + q_{n-1}) \]

\[
\int \frac{dk}{(2\pi)^4} A_{\mu}(p) A_{\alpha_1}(q_1) A_{\alpha_2}(q_2) \ldots A_{\alpha_{n-1}}(q_{n-1}) N_{L}^{\mu_{\alpha_1} \ldots \alpha_{n-1}}.
\]

The resulting full n-point function:

\[
- i N_{T}^{\mu_{\alpha_1} \ldots \alpha_{n-1}} = - i N_{L}^{\mu_{\alpha_1} \ldots \alpha_{n-1}} + i N_{L}^{\mu_{\alpha_1} \ldots \alpha_{n-1}},
\]

is then transverse.

At order n, the local graph diverges as \((p^2)^{2-n/2}\) and so the measure contribution at the same order here will be proportional to \((\Lambda^2)^{2-n/2}\). This indicates that nothing beyond \(n = 4\)
will survive in the local limit, which is in accord with the fact that counterterms are not necessary beyond fourth order in local regularization schemes. Note that this result relates the measure directly to the one loop graph that it is required to ‘fix up’, hence only even orders appear in the measure. One may wonder about whether the resulting amplitude is an entire function of the 4-momentum invariants of the process in question, but since the term explicitly comes from a convergent integral over barred propagators (that have no pole), we will not encounter any singularities when passing to Minkowski spacetime. We have also not had to resort to putting any external fields on shell, so that longitudinal photons will also decouple from internal fermion loops as well.