Evaluations of some Euler-Apéry-type series

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Received: 28 June 2021 / Accepted: 23 September 2021 / Published online: 28 September 2021
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Abstract In this paper, we use the methods of contour integration and generating function involving Fuss-Catalan numbers to study some Euler-Apéry-type series. In particular, we obtain some explicit formulas for some Euler-Apéry-type series. Based on these formulas, we further show that some series are reducible to logarithms (such as: $\log(2)$, $\log(3)$, $\log(5)$ etc.), zeta values and multiple polylogarithms. Moreover, we establish a recurrence relation for general Euler-Apéry-type series involving multiple harmonic star sum. Furthermore, some interesting new consequences and illustrative examples are considered.

Keywords Euler-Apéry-type series · Contour integration · Fuss-Catalan numbers · Generating function

Mathematics Subject Classification 65B10 · 11B65 · 11M32

1 Introduction

We begin with some basic notations. A finite sequence $k = (k_1, \ldots, k_r)$ of positive integers is called an index. As usual, we put

$|k| := k_1 + \cdots + k_r, \quad d(k) := r,$

and call them the weight and the depth of $k$, respectively. If $k_1 > 1$, $k$ is called admissible.

For an index $k = (k_1, \ldots, k_r)$ and positive integer $n$, the multiple harmonic sums (MHSs for short) and multiple harmonic star sums (MHSSs for short) are defined by

$\zeta_n(k) \equiv \zeta_n(k_1, k_2, \cdots, k_r) := \sum_{n \geq n_1 > n_2 > \cdots > n_r \geq 1} \frac{1}{n_1^{k_1} n_2^{k_2} \cdots n_r^{k_r}}, \quad (1.1)$

$\zeta_n^*(k) \equiv \zeta_n^*(k_1, k_2, \ldots, k_r) := \sum_{n \geq n_1 \geq n_2 \geq \cdots \geq n_r \geq 1} \frac{1}{n_1^{k_1} n_2^{k_2} \cdots n_r^{k_r}}, \quad (1.2)$
We set \( \zeta_n(\emptyset) = \zeta_n^*(\emptyset) := 1 \), and \( \zeta_n(k) := 0 \) if \( n < k \). When taking the limit \( n \to \infty \) in \((1.1)\) and \((1.2)\), we get the so-called the multiple zeta values (MZVs for short) and the multiple zeta star values (MZSVs for short), respectively:
\[
\zeta(k) := \lim_{n \to \infty} \zeta_n(k), \quad (1.3)
\]
and
\[
\zeta^*(k) := \lim_{n \to \infty} \zeta_n^*(k), \quad (1.4)
\]
defined for an admissible index \( k \) to ensure convergence of the series. The study of multiple zeta values began in the early 1990s with the works of Hoffman \([13]\) and Zagier \([19]\). For an admissible index \( k \), Hoffman \([13]\) called \((1.3)\) multiple harmonic series. Zagier \([19]\) called \((1.3)\) multiple zeta values since for \( r = 1 \) they generalize the usual Riemann zeta values \( \zeta(k) \). Due to their surprising and sometimes mysterious appearance in the study of many branches of mathematics and theoretical physics, these special values have attracted a lot of attention and interest in the past three decades (see, for example, the book by Zhao \([21]\)).

In general, for any index \( k = (k_1, \ldots, k_r) \) and \( N \)th roots of unity \( z_1, \ldots, z_r \), the colored MZVs (CMZVs for short) of level \( N \) are defined as
\[
L_{k_1, \ldots, k_r}(z_1, \ldots, z_r) := \sum_{n_1 > \cdots > n_r > 0} \frac{z_1^{n_1} \cdots z_r^{n_r}}{n_1^{k_1} \cdots n_r^{k_r}},
\]
which converges if \((k_1, z_1) \neq (1, 1)\) (see \([21, \text{Ch. 15}]\)), in which case we call \((s; z)\) admissible. The level two colored MZVs are called alternating MZVs (AMZVs for short). In this case, namely, when \((\eta_1, \ldots, \eta_r) \in [\pm 1]^r\) and \((k_1, \eta_1) \neq (1, 1)\), we set \( \zeta(k; \eta) = L_{k_1}(\eta) \). Further, we put a bar on top of \( k_j \) if \( \eta_j = -1 \). For example,
\[
\zeta(2, 3, \bar{1}, 4) = \zeta(2, 3, 1, 4; -1, 1, -1, 1).
\]

The subject of this paper is the Euler-Apéry-type series whose general terms is a product of central binomial coefficients, generalized harmonic numbers and \((n4^n)^{-1}\). Euler-Apéry-type series play an important role in many fields, such as analysis of algorithms, combinatorics, number theory and elementary particle physics. Therefore, they have attracted wide attention for a long time. For example, in 1979, Apéry \([1]\) proved the irrationality of \( \zeta(3) \) by using the series involving the central binomial coefficients:
\[
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3 \binom{2n}{n}} = \frac{2}{3} \zeta(3).
\]
He also considered the following analogous one:
\[
\sum_{n=1}^{\infty} \frac{1}{n^2 \binom{2n}{n}} = \frac{1}{3} \zeta(2),
\]
which has been known since the nineteenth century. In particular, in recent years, many relations among Euler-Apéry-type series, alternating MZVs and colored MZVs have been found. See \([2,7,15,17,22]\) and the references therein for recent results on Euler-Apéry-type series. For example, Au \([2]\) proved that
\[
\sum_{n=1}^{\infty} \frac{\zeta_n(k_2, \ldots, k_r)}{n^{k_1}} \cdot \frac{\binom{2n}{n}}{4^n} \in \mathbb{Q}[\text{AMZVs of weight } |k|],
\]
and
\[
\sum_{n=1}^{\infty} \frac{\zeta_n(k_2, \ldots, k_r)}{n^{k_1}} \cdot \frac{\binom{2n}{n}^2}{4^{2n}} \in \mathbb{Q}[\text{CMZVs of level } \leq 4 \text{ and weight } |k|]
\]
for any index \( k \). In fact, we conjecture that the following conclusions hold:
\[
\sum_{n=1}^{\infty} \frac{\zeta_n(k_2, \ldots, k_r)}{n^{k_1}} \cdot \frac{\binom{2n}{n}^q}{4^{qn}} \in \mathbb{Q}[\text{CMZVs of level } \leq 2q \text{ and weight } |k|],
\]
for positive integer $q$, but authors’ limited knowledge leads to no rigorous proof. We also have not discussed this issue in detail in this paper.

In this paper, we shall study some families of variations of Euler-Apéry-type series by using the tools developed in Wang and the second named author’s previous papers [15]. In particular, we will use the method of contour integration to prove the result that the Euler-Apéry-type series

$$\sum_{n=1}^{\infty} \frac{n}{(n-1/2)q} \cdot \frac{\binom{2n}{n}}{4^n} \in \mathbb{Q}[\pi, \log(2), \zeta(3), \zeta(5), \zeta(7), \ldots]$$

and the combined series

$$\sum_{n=1}^{\infty} \frac{n}{(n-1/2)q} \cdot \frac{\binom{2n}{n}}{4^n} (-1)^{n-1} + \frac{\binom{2n}{n}}{4^n} (-1)^{n-1} \in \mathbb{Q}[\pi, \log(2), \zeta(3), \zeta(5), \zeta(7), \ldots].$$

for positive integer $p$, and give explicit formulas. Moreover, we also give an explicit evaluation for the Euler-Apéry-type series

$$\Theta_{m,p}(x) := \sum_{n=1}^{\infty} \left( \frac{mn}{n} \right) x^n n^{-p},$$

for positive integers $m$ and $p$, by using the method of generating function involving Fuss-Catalan numbers. Further, we establish a recurrence formula of the Euler-Apéry-type series

$$\sum_{n=1}^{\infty} \frac{\zeta^*(m_1, \ldots, m_p)}{n^k 4^n} \binom{2n}{n}$$

for positive integers $k, m_1, \ldots, m_p$ with $m_p \geq 2$.

2 Preliminaries

2.1 Residue theorem and expansions of Gamma function

The contour integration is an efficient method to evaluate infinite series by reducing them to a finite number of residue computations. Flajolet and Salvy [9] used this method to compute the classical Euler sums, and found many interesting results. In this section, we will use the method to consider the Euler-Apéry-type series. Next, we will give several lemmas. First, we define a complex kernel function $\xi(s)$ with two requirements: (i). $\xi(s)$ is meromorphic in the whole complex plane. (ii). $\xi(s)$ satisfies $\xi(s) = o(s)$ over an infinite collection of circles $|s| = \rho_k$ with $\rho_k \to \infty$. Applying these two conditions of kernel function $\xi(s)$, Flajolet and Salvy discovered the following residue lemma.

Lemma 2.1 ([9, Lem. 2.1]) Let $\xi(z)$ be a kernel function and let $r(z)$ be a rational function which is $O(z^{-\alpha})$ at infinity. Then

$$\sum_{\alpha \in E} \text{Res}(r(z)\xi(z), \alpha) + \sum_{\beta \in S} \text{Res}(r(z)\xi(z), \beta) = 0,$$

where $S$ is the set of poles of $r(z)$ and $E$ is the set of poles of $\xi(z)$ that are not poles of $r(z)$. Here $\text{Res}(h(z), \lambda)$ denotes the residue of $h(z)$ at $z = \lambda$.

It is clear that the formula (2.1) is also true if $r(z)\xi(z) = O(z^{-\alpha})$ ($\alpha > 1$) over an infinite collection of circles $|z| = \rho_k$ with $\rho_k \to +\infty$.

Lemma 2.2 [15, Lem. 3.2] For $|z| < 1$, the following identities hold:

$$\Gamma(z+1)e^{\gamma z} = \sum_{n=0}^{\infty} C_n \frac{z^n}{n!} \quad \text{and} \quad \{\Gamma(z+1)e^{\gamma z}\}^{-1} = \sum_{n=0}^{\infty} D_n \frac{z^n}{n!},$$
where \( \gamma := \lim_{n \to \infty} (H_n - \log n) \) is the Euler-Mascheroni constant (\( H_n \) is classical harmonic number defined by \( H_n := \sum_{k=1}^{n} \frac{1}{k} \)), and

\[
C_n := Y_n(0, 0! \xi(2), -2! \xi(3), \ldots, (-1)^n (n - 1)! \xi(n)) ,
\]

\[
D_n := Y_n(0, -! \xi(2), 2! \xi(3), \ldots, (-1)^n-1 (n - 1)! \xi(n)) .
\]

The exponential complete Bell polynomials \( Y_n \) are defined by

\[
\exp \left( \sum_{k=1}^{\infty} \frac{x_k}{k!} t^k \right) = \sum_{n=0}^{\infty} Y_n(x_1, x_2, \ldots, x_n) \frac{t^n}{n!} ,
\]

and satisfy the recurrence

\[
Y_0 = 1 , \quad Y_n(x_1, x_2, \ldots, x_n) = \sum_{j=0}^{n-1} \binom{n-1}{j} x_{n-j} Y_j(x_1, x_2, \ldots, x_j) , \quad n \geq 1 ;
\]

see [8, Section 3.3] and [14, Section 2.8].

**Lemma 2.3** [15, Lem. 3.3] For nonnegative integer \( n \), when \( z \to -n \), we have

\[
\frac{1}{\Gamma(z) e^{\gamma(z-1)}} = (-1)^n n! e^{-\gamma(n+1)} \sum_{k=0}^{\infty} A_k(n)(z+n)^k-1,
\]

\[
\frac{1}{\Gamma(z) e^{\gamma(z-1)}} = (-1)^n n! e^{\gamma(n+1)} \sum_{k=0}^{\infty} B_k(n)(z+n)^k+1,
\]

where

\[
A_k(n) := \sum_{k_1 + k_2 = k \atop k_1, k_2 \geq 0} e_n^*(\{1\}_{k_1}) \frac{C_{k_2}}{k_2!}, \quad B_k(n) := \sum_{k_1 + k_2 = k \atop k_1, k_2 \geq 0} (-1)^{k_1} \xi_n(\{1\}_{k_1}) \frac{D_{k_2}}{k_2!} .
\]

### 2.2 Fuss-Catalan numbers

The **Fuss-Catalan numbers** ([3,10,11]) are defined by

\[
F_m(n) := \frac{1}{(m-1)n+1} \binom{mn}{n} \quad (m \geq 1, n \geq 0)
\]

and satisfy the generating function relation [11, Eq. (2.9)]

\[
G_m(x) = 1 + x G_m(x)^m, \quad G_m(0) := 1,
\]

where

\[
G_m(x) := \sum_{n=0}^{\infty} F_m(n) x^n \left( |x| \leq \frac{(m-1)^{m-1}}{m^m} \right) .
\]

Now, we will prove the function \( G_m(x) \) is strictly increasing function on the open interval \((- (m - 1)^{m-1}/m^m, (m - 1)^{m-1}/m^m))\). First, we consider the map:

\[
f : \left[ 0, \frac{m}{m-1} \right] \to \mathbb{R}, \quad y \mapsto \frac{y - 1}{y^m}
\]

Obviously, on the open interval \((0, m/(m - 1))\) we have

\[
f'(y) = \frac{m - (m - 1)y}{y^{m+1}} > 0.
\]
Hence, the function \( f \) is strictly increasing. Also,
\[
\lim_{y \to 0^+} f(y) = -\infty, \quad f \left( \frac{m}{m-1} \right) = \frac{(m-1)^{m-1}}{m^m}.
\]

It follows that \( f \) is a bijective increasing mapping from \((0, m/(m-1))\) to \((-\infty, (m-1)^{m-1}/m^m)\). Therefore, the inverse function \( g = f^{-1} \) is defined on \((-\infty, (m-1)^{m-1}/m^m)\) and satisfies
\[
f(g(x)) = x = \frac{g(x) - 1}{g(x)^m}.
\]
Thus, we obtain
\[
g(x) = 1 + xg(x)^m
\]
and \( g(x) \) is strictly increasing function on the open interval \((-\infty, (m-1)^{m-1}/m^m)\).

So, the generating function of Fuss-Catalan numbers \( G_m(x) \) is strictly increasing function on the open interval \((- (m - 1)^{m-1}/m^m, (m - 1)^{m-1}/m^m)\), and we have \(0 < G_m(x) \leq \frac{m}{m-1}\) and
\[
G_m \left( \frac{(m-1)^{m-1}}{m^m} \right) = \frac{m}{m-1}.
\]

**3 Main Results**

In this section, we will use the methods of contour integration and generating function to establish some explicit relations of some Euler-Apéry-type series. Specially, we establish an explicit connection between \( \Theta_{m,p}(x) \) and the generating function of Fuss-Catalan numbers.

**3.1 Evaluations via contour integration**

**Theorem 3.1** For positive integers \( m, p \) and \( q > 1 \),

\[
(-1)^{m+q} \sum_{k_1+k_2+\cdots+k_5=m-1 \atop k_1,k_2,k_3 \geq 0} \frac{2^{k_3+k_5}k_5!}{k_5!} \left( \frac{k_4+q-1}{q-1} \right) \log^{k_1}(2)
\]

\[
\times \sum_{n=1}^{\infty} A_{k_1}^{(m)}(n-1)A_{k_2}^{(m)}(n-1)B_{k_3}^{(m)}(2n-1) \frac{\binom{2n}{n}^m}{n^{k_4+q-m\zeta_3}} = 0,
\]

where
\[
C_n^{(m)} := Y_n(0, 1m\zeta(2), -2!m\zeta(3), \ldots, (-1)^{n-1}(n-1)!m\zeta(n)), \quad C_0^{(m)} = 1,
\]

\[
D_n^{(m)} := Y_n(0, -1m\zeta(2), 2!m\zeta(3), \ldots, (-1)^{n-1}(n-1)!m\zeta(n)), \quad D_0^{(m)} = 1,
\]

\[
A_k^{(m)}(n) := \sum_{k_1+k_2=n \atop k_1,k_2 \geq 0} \tilde{C}_k^{(m)}(n), \quad \tilde{C}_k^{(m)}(n) := \sum_{k_1+\cdots+k_m=n \atop k_1,\ldots,k_m \geq 0} \zeta_n^{(1\{1\zeta_1\})} \cdots \zeta_n^{(1\{1\zeta_m\})},
\]

\[
B_k^{(m)}(n) := \sum_{k_1+k_2=n \atop k_1,k_2 \geq 0} \tilde{D}_k^{(m)}(n), \quad \tilde{D}_k^{(m)}(n) := \sum_{k_1+\cdots+k_m=n \atop k_1,\ldots,k_m \geq 0} \zeta_n^{(1\{1\zeta_1\})} \cdots \zeta_n^{(1\{1\zeta_m\})}.
\]
Proof The proof of Theorem 3.1 is completely similar to the proof of Theorem [15, Thm. 3.4]. From Lemma 2.2, we have
\[ \{ \Gamma(z + 1) e^{\gamma z} \}^m = \sum_{n=0}^{\infty} C_n^{(m)} \frac{z^n}{n!} \quad \text{and} \quad \{ \Gamma(z + 1) e^{\gamma z} \}^{-m} = \sum_{n=0}^{\infty} D_n^{(m)} \frac{z^n}{n!}. \]

By elementary calculations give
\[ \left\{ \frac{\Gamma(z - n)}{\Gamma(z + 1)} \right\}^m \left\{ \prod_{k=0}^{n} \frac{1}{z - k} \right\}^m = \left\{ \frac{-1)^n \frac{1}{n!} \sum_{k=0}^{\infty} \frac{\zeta_n((1)_k) z^k}{k} \right\}^m \]
\[ = \frac{(-1)^{mn}}{(n!)^m} \sum_{k=0}^{\infty} \zeta_k^{(m)} (n) z^{k+m} \]
\[ \left\{ \frac{\Gamma(z + 1)}{\Gamma(z - n)} \right\}^m \left\{ \prod_{k=0}^{n} (z - k) \right\}^m = \left\{ \frac{-1)^n n! \sum_{k=0}^{\infty} (-1)^k \zeta_n((1)_k) z^k \right\}^m \]
\[ = (-1)^{mn} (n!)^m \sum_{k=0}^{\infty} (-1)^k \eta_k^{(m)} (n) z^{k+m}, \]
for \( n \geq 0 \). Further, we obtain the following relations
\[ \left\{ \frac{\Gamma(z - n) e^{\gamma z}}{\Gamma(z + 1)} \right\}^m \left\{ \prod_{k=0}^{n} \frac{1}{z - k} \right\}^m = \left\{ \frac{-1)^n \frac{1}{n!} \sum_{k=0}^{\infty} \frac{\zeta_k^{(m)} (n) C_k^{(m)} k_2^{(m)} k_2^{(m)} z^{k+m}}{k_2^{(m)} k_2^{(m)}} \right\}^m \]
\[ = (-1)^{mn} (n!)^m \sum_{k=0}^{\infty} A_k^{(m)} (n) z^{k+m} \]
\[ \left\{ \frac{\Gamma(z - n) e^{\gamma z}}{\Gamma(z - n)} \right\}^m \left\{ \prod_{k=0}^{n} (z - k) \right\}^m = (-1)^{mn} (n!)^m \sum_{k=0}^{\infty} \sum_{k_1 + k_2 = k} (-1)^{k_1} \eta_k^{(m)} (n) \frac{D_k^{(m)} k_2^{(m)} k_2^{(m)} z^{k+m}}{k_2^{(m)} k_2^{(m)}} \]
\[ = (-1)^{mn} (n!)^m \sum_{k=0}^{\infty} B_k^{(m)} (n) z^{k+m}. \]

Hence, by direct calculations, we find that for positive integer \( n \), if \( z \to -n \), then
\[ \{ \Gamma(z + 1) e^{\gamma z} \}^m = (-1)^{m(n-1)} (n-1)!^m e^{-\gamma mn} \sum_{k=0}^{\infty} A_k^{(m)} (n-1) (z + n)^{k+m}, \quad (3.2) \]
\[ \{ \Gamma(z + 1) e^{\gamma z} \}^{-m} = (-1)^{m(n-1)} (n-1)!^m e^{\gamma mn} \sum_{k=0}^{\infty} B_k^{(m)} (n-1) (z + n)^{k+m}. \quad (3.3) \]

It is obvious that if \( m = 1 \), then (3.2) and (3.3) become the two formulas in Lemma 2.3.

Then, considering the contour integral
\[ \lim_{R \to \infty} \oint_{C_R} F(z) dz \lim_{R \to \infty} \oint_{C_R} \frac{1}{2\pi i} \left\{ \frac{\Gamma(z + 1) z^2}{(2z + 1)} \right\}^m \]
\[ \frac{dz}{\Gamma(2z + 1)} = \lim_{R \to \infty} \oint_{C_R} \frac{1}{2\pi i} \left\{ \frac{\Gamma(z + 1) z^2}{(2z + 1)} \right\}^m \]
where \( \lim_{R \to \infty} \oint_{C_R} \) denotes integration along large circles, that is, the limit of integrals \( \lim_{R \to \infty} \oint_{|z|=R} \). Clearly, the function \( F(z) \) has poles only at all non-positive integers. For a negative integer \( -n \), by (3.2) and (3.3), if \( z \to -n \), we have
\[ F(z) = (-1)^n \left\{ \frac{(2n - 1)!}{((n-1)!)^2} \right\}^m \left\{ \sum_{k=0}^{\infty} A_k^{(m)} (n-1) A_k^{(m)} (n-1) B_k^{(m)} (2n - 1) 2^{k_1 + k_2 + k_3 - m} (z + n)^{k_1 + k_2 + k_3 - m} \right\} \]
Hence, the residue is
\[
\text{Res}(F(z), -n) = (-1)^{m+q} \sum_{k_1+k_2+\ldots+k_m=m-1 \atop k_1, k_2, \ldots, k_m \geq 0} \frac{2^{k_1+k_2+m}k_3}{k_4!} \left( \frac{k_4+q-1}{q-1} \right) \log^k(2)
\]
\[
\times A_k^{(m)}(n-1)A_k^{(m)}(n-1)B_k^{(m)}(2n-1) \frac{(2n)^m}{n^{k_4+q-m4m}}.
\]

Similarly, if \( z \to 0 \), we may easily deduce that
\[
F(z) = \frac{4^{m+q}}{z^q} \sum_{k_1, k_2, k_3 = 0}^{\infty} \frac{c_k^{(m)}c_2^{(m)}d_3^{(m)}2k}{k_1!k_2!k_3!} z^{k_1+k_2+k_3},
\]
and the residue of the pole of order \( q \) at 0 is
\[
\text{Res}(F(z), 0) = \sum_{k_1+\ldots+k_q=m-1 \atop k_1, \ldots, k_q \geq 0} \frac{c_k^{(m)}c_2^{(m)}d_3^{(m)}2k}{k_1!k_2!k_3!} \log^k(2).
\]

Summing these two contributions gives the statement of the theorem. This completes the proof of the theorem.

Therefore, form Theorem 3.1, we can find many explicit relations of Euler-Apéry-type series. In particular, letting \( m = 1 \) in Theorem 3.1 yields the following well-known result [15, Thm. 3.4]:
\[
\sum_{n=1}^{\infty} \frac{(2n)!}{4^n n^{q-1}} = (-1)^q \sum_{k_1+\ldots+k_q=m-1 \atop k_1, \ldots, k_q \geq 0} \frac{C_k c_k^2 d_k}{k_1!k_2!k_3!k_4!} 2^{k_1+k_2+k_4} \log^k(2).
\]

By a similar argument as in the proof of Theorem 3.1, we also establish the following explicit evaluation of Euler-Apéry-type series involving generalized harmonic numbers and central binomial coefficients.

**Theorem 3.2** For positive integers \( p, q \geq 2 \),
\[
\sum_{i+j+k+n=p-1 \atop i, j, k \geq 0} \frac{2^i(-1)^j}{i!k!} \left( \frac{q+j-1}{j} \right) \log^j(2) \sum_{n=1}^{\infty} \sum_{m=1}^{n} \frac{4^n}{n^{q+j}} \lim_{z \to n} \frac{d^k}{dz^k} \Gamma(z+1)^2
\]
\[
+ (-1)^{p+q+1} \sum_{n=1}^{\infty} \frac{\zeta(p) - H_n^{(p)}}{4^n n^{q-1}} \left( \frac{2n}{n} \right)
\]
\[
+ \sum_{k_1+k_2+k_3+k_4=p-1 \atop k_1, k_2, k_3, k_4 \geq 0} \frac{C_k c_k^2 d_k}{k_1!k_2!k_3!k_4!} 2^{k_1+k_2+k_3+k_4} \log^k(2) \]
\[
+ \sum_{k_1+\ldots+k_q=m-1 \atop k_1, \ldots, k_q \geq 0} \frac{C_k c_k^2 d_k}{k_1!k_2!k_3!k_4!} 2^{k_1+k_2+k_3+k_4} \log^k(2) \zeta(k_4+p) \left( \frac{k_4+p-1}{k_4} \right)
\]
\[
= 0,
\]

where \( H_n^{(p)} \) is generalized nth harmonic number of order \( p \) defined by
\[
H_n^{(p)} := \sum_{k=1}^{n} \frac{1}{k^p}, \quad H_n \equiv H_n^{(1)} \quad \text{and} \quad H_0^{(p)} := 0.
\]
Proof The proof is based on the function

\[ F(z) := \psi(p-1)(-z) \frac{\Gamma(z + 1)^2}{\Gamma(2z + 1)} 4^z \]

and the usual residue computation (\( \psi(z) \) is digamma function). We leave the detail to the interested reader. \( \square \)

**Corollary 3.3** For positive integer \( q \geq 2 \),

\[
2 \log(2) \sum_{n=1}^{\infty} \frac{4^n}{n^q (2n^n)} - q \sum_{n=1}^{\infty} \frac{4^n}{n^q (2n^n)} + 2 \sum_{n=1}^{\infty} \frac{H_n - H_{2n}}{n^q (2n^n)} 4^n - (-1)^q \sum_{n=1}^{\infty} \frac{\zeta(2) - H_{n-1}^{(2)}}{4^n n^{q-1}} (2n^n) \\
\]

\[
= 0. \tag{3.6}
\]

**Proof** The result immediately follows from the fact

\[
\lim_{z \to n} \frac{d}{dz} \frac{\Gamma(z + 1)^2}{\Gamma(2z + 1)} = \lim_{z \to n} 2 \frac{\Gamma(z + 1)^2}{\Gamma(2z + 1)} (\psi(z + 1) - \psi(2z + 1)) = 2 \frac{H_n - H_{2n}}{(2n^n)}
\]

and Theorem 3.2 with \( p = 2 \). \( \square \)

Now, we use the contour integration to evaluate the explicit formulas of the series

\[
\sum_{n=1}^{\infty} \frac{n}{(n-1/2)^q} \left( \frac{2n}{n} \right) 4^n
\]

and the combined series

\[
\sum_{n=1}^{\infty} \frac{n}{(n-1/2)^q} \left( \frac{2n}{n} \right) (-1)^{n-1} + (-1)^{q+1} \sum_{n=1}^{\infty} \frac{\left( \frac{2n}{n} \right)}{4^n n^{q-1}} (-1)^{n-1}.
\]

**Theorem 3.4** For positive integer \( q > 1 \),

\[
\sum_{n=1}^{\infty} \frac{n}{(n-1/2)^q} \left( \frac{2n}{n} \right) 4^n = \frac{2}{\pi} \sum_{k_1 + k_2 + 2k_5 = 2} \frac{C_{k_1} C_{k_2} D_{k_1} 2^{k_1+k_2} (\log(2))^{k_1} \tilde{\gamma}(2k_5 + 2)}{k_1! k_2! k_3! k_4!}, \tag{3.7}
\]

where \( \tilde{\gamma}(k) := 2^k \sum_{n=1}^{\infty} \frac{1}{(2n-1)^k} = (2^k - 1) \zeta(k) \) for \( k > 1 \).

**Proof** Similarly to Theorem 3.1, we consider the contour integral

\[
\lim_{R \to \infty} \oint_{C_R} F(z) dz = \lim_{R \to \infty} \oint_{C_R} \frac{\pi \tan(\pi z) \Gamma(z + 1)^2 4^z}{z^q \Gamma(2z + 1)} dz.
\]

The function \( F(z) \) has poles only at 0 and \( n - 1/2 \) (\( n = 1, 2, 3, \ldots \)). By straightforward calculations, we obtain

\[
\text{Res}(F(z), n - 1/2) = -\pi \frac{n}{(n-1/2)^q} 4^n
\]

and

\[
\text{Res}(F(z), 0) = \frac{2}{\pi} \sum_{k_1 + k_2 + 2k_5 = 2} \frac{C_{k_1} C_{k_2} D_{k_1} 2^{k_1+k_2} (\log(2))^{k_1} \tilde{\gamma}(2k_5 + 2)}{k_1! k_2! k_3! k_4!}.
\]

Hence, we complete the proof of Theorem 3.4. \( \square \)
As an example, setting \( q = 2 \) gives
\[
\sum_{n=1}^{\infty} \frac{n}{(n-1/2)^2} \cdot \frac{(2n)}{4^n} = \pi.
\]

**Theorem 3.5** For positive integer \( q > 1 \),
\[
\sum_{n=1}^{\infty} \frac{n}{(n-1/2)^q} \cdot \frac{(2n)}{4^n} (-1)^{n-1} + (-1)^{q+1} \sum_{n=1}^{\infty} \frac{(2n)}{4^n n^{q-1}} (-1)^{n-1}
\]
\[
= \frac{2}{\pi} \sum_{k_1,\ldots,k_4 \geq 0, k_1+\cdots+k_4 = q-1} C_{k_1} C_{k_2} D_{k_3} 2^{k_1+k_4} \log(2)^{k_3} \tilde{\iota}(2k_5+1),
\]
where \( \tilde{\iota}(k) := 2^k \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^k} \) for \( k \geq 1 \).

**Proof** In this case, consider
\[
\lim_{R \to \infty} \oint_{C_R} F(z) \, dz = \lim_{R \to \infty} \oint_{C_R} \frac{\pi \Gamma(z+1) 2^z}{\cos(\pi z) z^q \Gamma(2z+1)} \, dz.
\]

Based on residue computation and Lemma 2.3, we obtain this formula. \( \square \)

It is well-known that \( \tilde{\iota}(2k) \) is related to the Bernoulli number \( B_{2k} \) by
\[
\tilde{\iota}(2k) = \frac{(-1)^{k-1} 2^{2k-1} (2^{2k} - 1) B_{2k} \pi^{2k}}{(2k)!} \quad (k \geq 1),
\]
where
\[
\pi \tan(\pi z) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{2^{2k} (2^{2k} - 1) B_{2k} \pi^{2k}}{(2k)!} z^{2k-1},
\]
and \( \tilde{\iota}(2k+1) \) is related to the Euler number \( E_{2k} \) by
\[
\tilde{\iota}(2k+1) = \frac{(-1)^k E_{2k} \pi^{2k+1}}{2(2k)!} \quad (k \geq 0), \quad \text{where} \quad \sec(z) = \sum_{k=0}^{\infty} \frac{(-1)^k E_{2k}}{(2k)!} z^{2k}.
\]

In particular, we have \( E_0 = 1, E_2 = -1, E_4 = 5, E_6 = -61 \) and \( E_8 = 1385 \). Hence, from Theorems 3.4 and 3.5, we know that the series
\[
\sum_{n=1}^{\infty} \frac{n}{(n-1/2)^q} \cdot \frac{(2n)}{4^n}
\]
and the combined series
\[
\sum_{n=1}^{\infty} \frac{n}{(n-1/2)^q} \cdot \frac{(2n)}{4^n} (-1)^{n-1} + (-1)^{q+1} \sum_{n=1}^{\infty} \frac{(2n)}{4^n n^{q-1}} (-1)^{n-1}
\]
can be expressed as a \( \mathbb{Q} \)-linear combination of \( \pi, \log(2) \) and odd Riemann zeta values \( \zeta(2k+1) \) (\( k \geq 1 \)).

Noting that from \([4, 12]\), we can find the following well-known formula
\[
\sum_{n=1}^{\infty} \frac{(2n)}{4^n n} \pi^n = 2 \log \left( \frac{2}{1 + \sqrt{1 - z}} \right) \quad (z \in [-1, 1]).
\]
Setting $z = -1$ yields
\[
\sum_{n=1}^{\infty} \frac{(2n)}{4^n n} (-1)^{n-1} = 2 \log(1 + \sqrt{2}) - 2 \log(2).
\]
Hence, letting $q = 2$ in Theorem 3.5, we obtain
\[
\sum_{n=1}^{\infty} \frac{n}{(n - 1/2)^2} \cdot \frac{2n}{4^n} (-1)^{n-1} - \sum_{n=1}^{\infty} \frac{(2n)}{4^n n} (-1)^{n-1} = 2 \log(2).
\]
Therefore,
\[
\sum_{n=1}^{\infty} \frac{n}{(n - 1/2)^2} \cdot \frac{2n}{4^n} (-1)^{n-1} = 2 \log(1 + \sqrt{2}).
\]
Furthermore, we also deduce the following results
\[
\sum_{n=1}^{\infty} \frac{(2n)}{4^n} (-1)^{n-1} = 1 - \frac{1}{\sqrt{2}},
\]
\[
\sum_{n=1}^{\infty} \frac{n}{n - 1/2} \cdot \frac{(2n)}{4^n} (-1)^{n-1} = \frac{1}{2} \sqrt{2},
\]
\[
\sum_{n=1}^{\infty} \frac{1}{n - 1/2} \cdot \frac{(2n)}{4^n} (-1)^{n-1} = 2 \sqrt{2} - 2,
\]
\[
\sum_{n=1}^{\infty} \frac{1}{(n - 1/2)^2} \cdot \frac{(2n)}{4^n} (-1)^{n-1} = 4 \log(1 + \sqrt{2}) + 4 - 4 \sqrt{2}.
\]

**Theorem 3.6** For positive integer $q > 1$,
\[
\sum_{n=1}^{\infty} \frac{(3n)}{n^q} \left(\frac{4}{27}\right)^n - 2^{q-1} \sum_{n=1}^{\infty} \frac{\Gamma \left(\frac{3}{2} - n\right)}{\Gamma \left(\frac{5}{2} - n\right)} (2n - 1)! (2n - 1)^{q-1} = (-1)^q \sum_{k_1+k_2+k_3+k_4=q-1}^{\infty} C_{k_1} C_{k_2} D_{k_3} 3^{k_4} (3 \log(3) - 2 \log(2)) k_1 k_2 k_3 k_4!.
\]

**Proof** Considering the contour integral
\[
\lim_{R \to \infty} \oint_{|z| = \rho_k} F(z) \, dz = \lim_{R \to \infty} \oint_{|z| = \rho_k} \frac{\Gamma(1+z) \Gamma(1+2z)}{\Gamma(1+3z)z^q} \left(\frac{27}{4}\right)^z \, dz
\]
and using residue theorem, we may easily deduce the desired result. \qed

**Remark 3.7** We note that the formula (2.1) in Lemma 1 is also true if $r(z) = O(z^{-\alpha}) (\alpha > 1)$ over an infinite collection of circles $|z| = \rho_k$ with $\rho_k \to +\infty$. On the other hand, using the duplication formulas of Gamma function
\[
\prod_{j=1}^{n} \Gamma \left(z + \frac{j - 1}{n}\right) = (2\pi)^{(n-1)/2} n^{1/2-nz} \Gamma(nz)
\]
and the asymptotic expansion for the ratio of two gamma functions
\[
\frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b} \left(1 + O \left(1/z\right)\right) \quad (|\arg(a+z)| < \pi, |z| \to \infty),
\]
Hence, if \( q \geq 2 \), then the functions \( F(z) \) in the proof of Thms. 3.4–3.6 satisfy the condition \( F(z) = O(z^{-\alpha}) \) \( (\alpha > 1) \).

### 3.2 Evaluations via Fuss-Catalan numbers

In this subsection, we will establish the explicit evaluation of \( \Theta_{m,p}(x) \) by using the relation of generating function of Fuss-Catalan numbers.

According to the relation of generating function of Fuss-Catalan numbers, we have

\[
H_m(x) := \sum_{n=0}^{\infty} \binom{mn}{n} x^n = G_m(x) + (m-1)x G'_m(x) = \frac{G_m(x)}{m-(m-1)G_m(x)}. \tag{3.10}
\]

Hence, by an elementary calculation yields

\[
\Theta_{m,1}(x) = \sum_{n=1}^{\infty} \binom{mn}{n} x^n = \int_0^x \frac{H_m(t) - 1}{t} dt = m \log(G_m(x)). \tag{3.11}
\]

In particular, if setting \( x = \frac{(m-1)^{m-1}}{m^m} \) yields

\[
\sum_{n=1}^{\infty} \binom{mn}{n} \left( \frac{(m-1)^{m-1}}{m^m} \right)^n = m \log \left( \frac{m}{m-1} \right).
\]

Further, using (3.11) we can get the following more general result.

**Theorem 3.8** For positive integers \( m \) and \( p \),

\[
\Theta_{m,p+1}(x) = \sum_{n=1}^{\infty} \binom{mn}{n} x^n \left( \frac{x^{n_1}}{n_{t_1}^{p_1}} \cdots \frac{x^{n_r}}{n_{t_r}^{p_r}} \right), \tag{3.13}
\]

where \( G_m(x)^{-1} = 1/G_m(x) \) and \( |x| \leq (m-1)^{m-1}/m^m \), and for any index \( k = (k_1, \ldots, k_r) \in \mathbb{N}^r \), the classical multiple polylogarithm function are defined by

\[
\text{Li}_k(x) := \sum_{n_1 > \cdots > n_t > 0} \frac{x^{n_1}}{n_{t_1}^{k_1} \cdots n_{t_r}^{k_r}}, \tag{3.14}
\]

which converges if \( |x_j \cdots x_r| < 1 \) for all \( j = 1, \ldots, r \). It can be analytically continued to a multi-valued meromorphic function on \( \mathbb{C}^r \) (see [20]).
ProofReplacing \( x \) by \( t_p \) in (3.11), then applying the iterated integral
\[
\int_{0 < t_p < \cdots < t_1 < x} t_p^n dt_1 \cdots dt_p
\]
gives
\[
\sum_{n=1}^{\infty} \binom{mn}{n} \int_{0 < t_p < \cdots < t_1 < x} t_p^n dt_1 \cdots dt_p = \sum_{n=1}^{\infty} \binom{mn}{n} \frac{x^n}{n^{p+1}}
\]
\[
=m \int_{0 < t_p < \cdots < t_1 < x} \log(G_m(t_p)) dt_1 \cdots dt_p
\]
\[
= \frac{m}{(p-1)!} \int_{0}^{x} \frac{\log^{p-1}(x/t) \log(G_m(t_p))}{t} dt
\]
\[
= \frac{m}{(p-1)!} \sum_{k=0}^{p-1} (-1)^k \binom{p-1}{k} \log^{p-1-k}(x) \int_{0}^{x} \frac{\log^k(t) \log(G_m(t_p))}{t} dt.
\]

From Section 2 we know that the function is strictly increasing on \(|x| \leq (m-1)^{n-1}/m^n\). Hence, letting \(w = G_m(t)\) and \(u = G_m(x)\), by a straightforward calculation we have

\[
\int_{0}^{x} \frac{\log^k(t) \log(G_m(t_p))}{t} dt
\]
\[
= \int_{1}^{u} \left( m \log^k\left( \frac{w^{m-1}}{w-1} \right) \log w - m \log^k\left( \frac{w^{m-1}}{w-1} \right) \log w \right) dw
\]
\[
= \int_{1/u}^{1} \left( \log^k\left( \frac{w^{-1}}{1-w} \right) \log(1-w) - m \log^k\left( \frac{w^{-1}}{1-w} \right) \log(1-w) \right) dw
\]
\[
= \sum_{j=0}^{k} (-1)^j \binom{k}{j} \int_{0}^{1-1/u} \left( \log^{k-j}(w) \log^{j+1}(1-w) - (m-1) \log^{k-j}(w) \log^{j+1}(1-w) \right) dw.
\]

Then applying the two well-known identities [16]

\[
\log^k(1-x) = (-1)^k k! \sum_{n=1}^{\infty} \frac{x^n}{n^{k-1}} \binom{n}{k-1} \ (x \in [-1, 1])
\]

and

\[
\int_{0}^{x} t^n \log^m(t) dt = \sum_{l=0}^{m} \binom{m}{l} \frac{(-1)^l}{(n+1)^{l+1}} x^{n+1} \log^{m-l}(x),
\]

and according to the definition of multiple polylogarithm function \( \text{Li}_{k_1, k_2, \ldots, k_r}(x) \) with an elementary calculation, we obtain the desired result. \( \square \)

**Corollary 3.9** For positive integer \( m \),

\[
\Theta_{m, 2}(x) = -\frac{m(m-1)}{2} \log^2(G_m(x)) + mL_2(1 - G_m(x)). \tag{3.15}
\]

**Proof** This follows immediately from the Theorem 3.8 with \( p = 1 \). \( \square \)
Example 3.10 We have

\[
\sum_{n=1}^{\infty} \left( \frac{3n}{n} \right) \left( \frac{4}{27} \right)^n = 3 \log(3/2),
\]

\[
\sum_{n=1}^{\infty} \left( \frac{2n}{n^4} \right) (-1)^{n+1} = \log^2(2\sqrt{2} - 2) + 2 \text{Li}_2 \left( \frac{1 - \sqrt{2}}{2} \right),
\]

\[
\sum_{n=1}^{\infty} \left( \frac{3n}{n^2} \right) \left( \frac{4}{27} \right)^n = -3 \log^2(3/2) + 3 \text{Li}_2(1/3),
\]

\[
\sum_{n=1}^{\infty} \left( \frac{1}{n} \right) \left( \frac{1}{8} \right)^n = -3 \log^2(\sqrt{3} - 1) + 3 \text{Li}_2 \left( \frac{3 - \sqrt{3}}{4} \right),
\]

\[
\sum_{n=1}^{\infty} \left( \frac{81}{256} \right)^n = -6 \log^2(4/3) + 4 \text{Li}_2(1/4).
\]

Clearly, from Theorem 3.8 and (3.11), we can evaluate the explicit evaluations of \( \Theta_{2,p}(-1/4) \) and \( \Theta_{3,p}(4/27) \). Then applying Theorem 3.5 and 3.6 we obtain the explicit formulas of the two infinite series

\[
\sum_{n=1}^{\infty} \frac{n}{(n - 1/2)^p} \left( \frac{2n}{4^n} \right) (-1)^{n+1} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\Gamma \left( \frac{3-2n}{2} \right) \left( \frac{4}{27} \right)^n}{\Gamma \left( \frac{3-6n}{2} \right)(2n-1)!(2n-1)^p}.
\]

For example, setting \( q = 2 \) and \( 3 \) in Theorem 3.6 yield

\[
\sum_{n=1}^{\infty} \frac{3n}{n^2} \left( \frac{4}{27} \right)^n - 2 \sum_{n=1}^{\infty} \frac{\Gamma \left( \frac{3-2n}{2} \right) \left( \frac{4}{27} \right)^n}{\Gamma \left( \frac{3-6n}{2} \right)(2n-1)!(2n-1)} = 3 \log(3) - 2 \log(2),
\]

\[
\sum_{n=1}^{\infty} \frac{3n}{n^2} \left( \frac{4}{27} \right)^n - 4 \sum_{n=1}^{\infty} \frac{\Gamma \left( \frac{3-2n}{2} \right) \left( \frac{4}{27} \right)^n}{\Gamma \left( \frac{3-6n}{2} \right)(2n-1)!(2n-1)^2} = -\frac{9}{2} \log^2(3) + 6 \log(2) \log(3) - 2 \log^2(2) + \frac{\pi^2}{3}.
\]

Then, applying the Example 3.10, we get the two cases

\[
\sum_{n=1}^{\infty} \frac{\Gamma \left( \frac{3-2n}{2} \right) \left( \frac{4}{27} \right)^n}{\Gamma \left( \frac{3-6n}{2} \right)(2n-1)!(2n-1)} = -\frac{1}{2} \log(2),
\]

\[
\sum_{n=1}^{\infty} \frac{\Gamma \left( \frac{3-2n}{2} \right) \left( \frac{4}{27} \right)^n}{\Gamma \left( \frac{3-6n}{2} \right)(2n-1)!(2n-1)^2} = \frac{3}{4} \text{Li}_2(1/3) + \frac{3}{8} \log^2(3) - \frac{1}{4} \log^2(2) - \frac{\pi^2}{12}.
\]

4 A Recurrence Relation of Euler-Apéry-Type Series

In this section, we will establish a recurrence formula for Euler-Apéry-Type series by iterated integral of multiple polylogarithm. The theory iterated integrals was developed first by K.T. Chen in the 1960’s [5,6]. It has played important roles in the study of algebraic topology and algebraic geometry in past half century. Its simplest form over \( \mathbb{R} \) is

\[
\int_{a}^{b} f_p(t) f_{p-1}(t) f_{p-2}(t) \cdots f_1(t) dt := \int_{a < t_p < \cdots < t_1 < b} f_p(t_p) f_{p-1}(t_{p-1}) \cdots f_1(t_1) dt_1 dt_2 \cdots dt_p.
\]
which can be easily extended to iterated path integrals over \( C \). In particular, let \( k = (k_1, \ldots, k_r) \in \mathbb{N}^r \), from [18, Eq. (2.1)], we have
\[
\text{Li}_k \left( \frac{x_1, x_2, \ldots, x_r}{x_1, x_2, \ldots, x_{r-1}} \right) = \int_0^1 \left( \frac{x_r \, dt}{1-x_r t} \right) \left( \frac{dt}{t} \right)^{k_r-1} \cdots \left( \frac{x_1 \, dt}{1-x_1 t} \right) \left( \frac{dt}{t} \right)^{k_1-1}. \tag{4.1}
\]
For an index \( m = (m_1, \ldots, m_p) \), its Hoffman dual is the index \( m^\vee = (m'_1, \ldots, m'_p) \) determined by \( |m| := m_1 + \cdots + m_p = m'_1 + \cdots + m'_p \) and
\[
\{1, 2, \ldots, |m| - 1\} = \{m_1, m_1 + m_2, \ldots, m_1 + \cdots + m_{p-1}\} \cup \{m'_1, m'_1 + m'_2, \ldots, m'_1 + \cdots + m'_{p-1}\}.
\]
For example, we have
\[
(1, 1, 2, 1)^\vee = (3, 2) \quad \text{and} \quad (1, 2, 1, 1)^\vee = (2, 3).
\]

**Theorem 4.1** For positive integers \( k, m_1, \ldots, m_p \) with \( m_p \geq 2 \),
\[
(-1)^p \sum_{n=1}^{\infty} \xi_n^\ast (m_1, \ldots, m_p) \frac{2n}{n^{k+4}} \sum_{j=1}^{p} (-1)^{j-1} \xi(m_p, \ldots, m_j) \sum_{n=1}^{\infty} \frac{\xi_n^\ast (m_1, \ldots, m_{j-1})}{n^{k+4}} \frac{2n}{n^{k+4}}
\]
\[
= -2^{p+1} \sum_{j=1,2,\ldots,|m|+k-1} \text{Li}_{k+1, m_1, m_1 + m_2, \ldots, m_1 + \cdots + m_{p-1}} (-1, -\sigma_1, \sigma_1 \sigma_2, \ldots, \sigma \sigma_p^{j+1}, \sigma \sigma_p^{j+1}, \ldots, \sigma \sigma_p^{j+1}). \tag{4.2}
\]
where \( |\tilde{m}| := m_1 + m_2 + \cdots + m_j - j \).

**Proof** We note that the formula (3.9) can be rewritten as the form
\[
\sum_{n=1}^{\infty} \frac{2n}{n^{4n}} x^n = 2 \log \left( \frac{2}{1 + \sqrt{1 - x}} \right) = \int_{0 < t < x} \frac{dt}{1 - t + \sqrt{1 - t}}.
\]
Hence, by a direct calculation, we can get the following result
\[
\int_0^x \frac{dt}{1 - t + \sqrt{1 - t}} \left( \frac{dt}{t} \right)^{k-1} \cdots \left( \frac{dt}{t} \right)^{m_p-1}
\]
\[
= \sum_{n_1>\cdots>n_p>0} \frac{x^{n_1}}{n_1^{m_p} \cdots n_p^{m_1}} 4^{n_1+1}
\]
\[
= \sum_{n=1}^{\infty} \frac{2n}{n^{4n}} \sum_{n_1>\cdots>n_p} \frac{x^{n_1}}{n_1^{m_p} \cdots n_p^{m_1}}
\]
\[
= \sum_{n=1}^{\infty} \frac{2n}{n^{4n}} \sum_{n_1>\cdots>n_p} \frac{x^{n_1}}{(n_1 + n)^{m_p} \cdots (n_p + n)^{m_1}}
\]\
\[
= \sum_{n=1}^{\infty} \frac{2n}{n^{4n}} \sum_{n_1>\cdots>n_p} \frac{x^{n_1}}{(n_1 + n)^{m_p} \cdots (n_p + n)^{m_1}}
\]
\[
= \sum_{n_1>\cdots>n_p} \frac{x^{n_1}}{(n_1 + n)^{m_p} \cdots (n_p + n)^{m_1}}. \tag{4.3}
\]
The iterated integral on left-hand side can be rewritten as
\[
\int_0^x \frac{dt}{1 - t + \sqrt{1 - t}} \left( \frac{dt}{t} \right)^{k-1} \cdots \left( \frac{dt}{t} \right)^{m_p-1}
\]
\[
= \int_{1-x}^1 \left( \frac{dt}{1 - t} \right)^{m_p-1} \left( \frac{dt}{1 - t} \right)^{m_1-1} \cdots \left( \frac{dt}{1 - t} \right)^{k-1} \frac{dt}{1 + \sqrt{1 - t}}
\]
\[
= 2^{p+1} \int_{1-x}^1 \frac{2tdt}{1 - t^2} \left( \frac{dt}{1 - t^2} \right)^{m_p-1} \left( \frac{dt}{1 - t^2} \right)^{m_1-1} \cdots \left( \frac{dt}{1 - t^2} \right)^{k-1} \frac{dt}{1 + \sqrt{1 - t}}.
\]
Letting \( x = 1 \) and noting the fact that
\[
\frac{2t dt}{1 - t^2} = \sum_{\sigma \in \{\pm 1\}} \frac{\sigma dt}{1 - \sigma t},
\]
with the help of (4.1), we obtain
\[
\int_0^1 \frac{dt}{1 - t + \sqrt{1-t}} \left( \frac{dt}{1 - t} \right)^{k-1} \frac{dt}{1 - t} \left( \frac{dt}{1 - t} \right)^{m_1-1} \cdots \frac{dt}{1 - t} \left( \frac{dt}{1 - t} \right)^{m_p-1} = -2p+1 \sum_{\sigma_j \in \{\pm 1\}, j=1,2,...,m_p} \text{Li}_{k+1,m_1,...,m_{p-1},m_p-1}(x) \zeta^*(m_1, \ldots, m_p, x).
\]

From [18, Thm. 4.2], first setting \( n \to \infty \), then replacing \((l, r)\) by \((n, p)\), following by letting \( k = (m_p, \ldots, m_1) \) and \( x = (x, \{1\}_{p-1}) \), we get
\[
\sum_{n_1 > n_2 > \cdots > n_p \geq n} x^{n_1+n} \frac{(n_1 + n)^{m_p} \cdots (n_p + n)^{m_1}}{(n_1 + n)^{m_p} \cdots (n_p + n)^{m_1}} = (-1)^p \sum_{j=1}^p (-1)^j \text{Li}_{m_p,...,m_{p+1-j}}(x) \zeta^*(m_1, \ldots, m_{p-j}) + (-1)^p \zeta^*(m_1, \ldots, m_p),
\]
where
\[
\zeta^*(m_1, \ldots, m_{p-1}, m_p; x) := \sum_{n \geq n_1 \geq \cdots \geq n_p \geq 1} x^{n_1+n} \frac{n_1^{m_1} \cdots n_{p-1}^{m_{p-1}} n_p^{m_p}}{n_1^{m_1} \cdots n_{p-1}^{m_{p-1}} n_p^{m_p}},
\]
and \( \zeta^*(\emptyset; x) := x^n \). Hence, setting \( x = 1 \) yields
\[
\sum_{n_1 > n_2 > \cdots > n_p \geq n} \frac{1}{(n_1 + n)^{m_p} \cdots (n_p + n)^{m_1}} = (-1)^p \sum_{j=1}^p (-1)^j \zeta(m_p, \ldots, m_{p+1-j}) \zeta^*(m_1, \ldots, m_{p-j}) + (-1)^p \zeta^*(m_1, \ldots, m_p). \tag{4.5}
\]

Thus, letting \( x = 1 \) in (4.3) and combining (4.4) and (4.5) yields the desired evaluation. \( \square \)

Setting \( p = 1, 2 \) give
\[
\sum_{n=1}^{\infty} \frac{\zeta_n^*(m)}{n^{k} 4^n} \frac{(2n)}{n} = \zeta(m) \sum_{n=1}^{\infty} \frac{(2n)}{n^{k} 4^n} - 4 \sum_{\sigma_j \in \{\pm 1\}, j=1,2,...,m+k-2} \text{Li}_{k+1,m-1}(x) \zeta^*(m_1, \ldots, m_{k-3}, \sigma_{m+k-2}) \tag{4.6}
\]
and
\[
\sum_{n=1}^{\infty} \frac{\zeta_n^*(m_1, m_2)}{n^{k} 4^n} \frac{(2n)}{n} = \zeta(m_2) \sum_{n=1}^{\infty} \frac{\zeta_n^*(m_1)}{n^{k} 4^n} \frac{(2n)}{n} - \zeta(m_1, m_1) \sum_{n=1}^{\infty} \frac{(2n)}{n^{k} 4^n} - 8 \sum_{\sigma_j \in \{\pm 1\}, j=1,2,...,m_1+m_2+k-3} \text{Li}_{k+1,m_1,m_2-1}(x) \zeta^*(m_1, \ldots, m_{1+m_2+k-3}). \tag{4.7}
\]

**Remark 4.2** The formula (4.6) is essentially equivalent to [15, Thm. 2.3].
Acknowledgements   The authors expresses their deep gratitude to Professors Masanobu Kaneko and Jianqiang Zhao for valuable discussions and comments. Both authors thank the anonymous referee for many invaluable comments and suggestions which have improved the paper greatly. The authors are supported by the Scientific Research Foundation for Scholars of Anhui Normal University. The Y. Wang is supported by Natural Science Foundation of Anhui Province (Grant No. 2008085QA06). The corresponding author C. Xu is supported by the National Natural Science Foundation of China (Grant No. 12101008), the Natural Science Foundation of Anhui Province (Grant No. 2108085QA01) and the University Natural Science Research Project of Anhui Province (Grant No. KJ2020A0057).

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