MEROMORPHIC PROPERTIES OF THE RESOLVENT ON ASYMPTOTICALLY HYPERBOLIC MANIFOLDS

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Abstract. On an asymptotically hyperbolic manifold \((X^{n+1}, g)\), Mazzeo and Melrose have constructed the meromorphic extension of the resolvent \(R(\lambda) := (\Delta_g - \lambda(n - \lambda))^{-1}\) for the Laplacian. However, there are special points on \(\frac{1}{2}(n - N)\) that they did not deal with. We show that the points of \(\frac{1}{2} - N\) are at most some poles of finite multiplicity, and that the same property holds for the points of \(\frac{n+1}{2} - N\) if and only if the metric is ‘even’. On the other hand, there exist some metrics for which \(R(\lambda)\) has an essential singularity on \(\frac{n+1}{2} - N\) and these cases are generic. At last, to illustrate them, we give some examples with a sequence of poles of \(R(\lambda)\) approaching an essential singularity.

1. Introduction and statements of the results

The purpose of this work is to analyze, near the points \((\frac{n-k}{2})_{k \in \mathbb{N}}\), the meromorphically continued resolvent for the Laplacian

\[
R(\lambda) := (\Delta_g - \lambda(n - \lambda))^{-1}
\]

on some non-compact spaces \((X^{n+1}, g)\) called asymptotically hyperbolic manifolds. This meromorphic extension in \(\mathbb{C} \setminus \frac{1}{2}(n - N)\) with finite rank poles, proved by Mazzeo and Melrose \([17]\), is a beautiful application of Melrose’s pseudodifferential calculus on manifolds with corners, which generalizes some well-known results on hyperbolic spaces. Meromorphic extensions of resolvents have been studied in many frameworks and their finite rank poles, called resonances, serve in a sense as discrete data similar in character to eigenvalues of a compact manifold. As far as we are concerned, the construction of \([17]\) does not treat the special points \((\frac{n-k}{2})_{k \in \mathbb{N}}\) and, as Borthwick and Perry noticed in their article \([4]\), it seems possible that these points are poles with infinite rank residues, or even essential singularities of \(R(\lambda)\).

However, if the manifold has constant negative sectional curvature away from a compact, Guillopé and Zworski \([11]\) did show the meromorphic continuation of the resolvent to \(\mathbb{C}\) with finite rank poles. The key to analyze the points of \(\frac{1}{2}(n - N)\) is the special structure of the metric near infinity, in the sense that its Laplacian is locally the hyperbolic Laplacian, whose coefficients remain smooth at \(z = 0\) in the new coordinates \((x_1, \ldots, x_n, z = y^2)\) on \(\mathbb{H}^{n+1}\). We could then follow the construction of \([17]\) and search the good conditions to set on the metric in order to use the same kind of arguments: we would find that the natural assumption is to take a metric with an even asymptotic expansion at infinity, in a sense we will explain later.

In fact, our philosophy will be to use the properties of the scattering operator, whose poles are essentially the resonances (cf. \([4]\)). The recent work of Graham and Zworski \([8]\) gives indeed a simple and explicit presentation of the scattering operator \(S(\lambda)\) on asymptotically hyperbolic manifolds which allows us to study the nature of \(S(\lambda)\) near \(\frac{1}{2}(n + N)\). Thanks to their calculus and the formula \(S(n - \lambda) = S(\lambda)^{-1}\), we detail the behavior of \(S(\lambda)\) near \(\frac{1}{2}(n - N)\) and the relations between \(R(\lambda)\) and \(S(\lambda)\) provide a good analysis of the resolvent in a neighbourhood of the points \((\frac{n-k}{2})_{k \in \mathbb{N}}\).

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Firstly, let us recall some basic definitions and results to understand the problem. Let \( \bar{X} = X \cup \partial X \) a \( n+1 \)-dimensional smooth compact manifold with boundary and \( x \) a defining function for the boundary, that is a smooth function \( x \) on \( \bar{X} \) such that

\[
x \geq 0, \quad \partial X = \{ m \in \bar{X}, x(m) = 0 \}, \quad dx|_{\partial X} \neq 0.
\]

We say that a smooth metric \( g \) on the interior \( X \) of \( \bar{X} \) is conformally compact if \( x^2 g \) extends smoothly as a metric to \( \bar{X} \). An asymptotically hyperbolic manifold is a conformally compact manifold such that for all \( y \in \partial X \), all sectional curvatures at \( m \in X \) converge to \(-1\) as \( m \to y \).

Such a manifold is necessarily complete and the spectrum of its Laplacian \( \Delta_y \) consists of absolutely continuous spectrum

\[
\text{smoothly as a metric to } \bar{X}.
\]

We say that a smooth metric \((0, x, x')\) of \( x \) exists a neighbourhood \( M \) of \( \bar{X} \) and \( \Delta_y \) consists of \( x \) sections: for \( B \in \mathbb{R} \) and \( f \in B \), let \( \|B\|_2 \) be some Banach spaces, we denote \( \mathcal{L}(B, B) \) (or \( \mathcal{L}(B) \) if \( B = B \)) the space of bounded linear operators from \( B \) to \( B \). If \( U \) is an open domain of \( \mathbb{C} \), \( \mathcal{K}(U, B) \) (resp. \( \mathcal{K}(U, B) \)) is the set of holomorphic (resp. meromorphic) functions on \( U \) with values in \( B \).

\( M(\lambda) \) is said meromorphic in \( \mathcal{K}(U, B) \) with values in the Banach space \( B \) if for each \( \lambda_0 \in U \), there exists a neighbourhood \( V_{\lambda_0} \) of \( \lambda_0 \), an integer \( p > 0 \) and some \( (M_i)_{i=1,...,p} \) in \( B \) such that for all \( \lambda \in V_{\lambda_0} \setminus \{\lambda_0\} \) we have the finite Laurent expansion

\[
M(\lambda) = \sum_{i=1}^{p} M_i (\lambda - \lambda_0)^{-i} + \mathcal{H}(\lambda), \quad \mathcal{H}(\lambda) \in \mathcal{K}(V_{\lambda_0}, B).
\]

(1.1)

It is easy to see that \( M(\lambda) \) is holomorphic in \( U \setminus S \) where \( S \) is a discrete set of \( U \) whose elements are the poles of \( M(\lambda) \). \( p \) is the order of the pole, \( M_1 = \text{Res}_{\lambda_0} M(\lambda) \) is the residue of \( M(\lambda) \) at \( \lambda_0 \) and if \( B_0 = \mathcal{L}(B_1, B_2) \) is a space of continuous linear maps, \( m_{\lambda_0}(M(\lambda)) := \text{rank} M_1 \) is called the multiplicity of \( \lambda_0 \) and \( \text{Rank}_{\lambda_0} M(\lambda) := \sum_{i=1}^{p} \text{rank} M_i \) the total polar rank of \( M(\lambda) \) at \( \lambda_0 \). If now the total polar rank is finite at each pole of \( M(\lambda) \) we say that \( M(\lambda) \) is finite-meromorphic and \( \mathcal{M}_f(U, B) \) denotes the space of finite-meromorphic functions in \( U \) with values in \( B \). At last, if \( \lambda_0 \in U \) and \( M(\lambda) \) is meromorphic in \( U \setminus \{\lambda_0\} \) but not in \( U \), we will say that \( \lambda_0 \) is an essential singularity of \( M(\lambda) \).

At last, note that all these definitions extend to locally convex vector spaces (see for instance Bunke-Olbrich [4]).

Here is an interpretation of the result of Mazzeo and Melrose [17 Th. 7.1]:

**Theorem 1.1.** Let \((X, g)\) be an asymptotically hyperbolic manifold, \( \Delta_g \) its Laplacian acting on functions and \( x \) a boundary defining function on \( \bar{X} \). The modified resolvent

\[
R(\lambda) := (\Delta_g - \lambda(n - \lambda))^{-1} \in \mathcal{M}_f \left( (0_0, \mathcal{L}(L^2(X))) \right)
\]

with poles at points \( \lambda \in (0_0) \) such that \( \lambda(n - \lambda) \in \sigma_{pp}(P) \), extends to a finite-meromorphic family

\[
R(\lambda) \in \mathcal{M}_f \left( (0_N \setminus (Z^+ \cup Z^-), \mathcal{L}(x^N L^2(X), x^{-N} L^2(X))) \right), \quad \forall N \geq 0
\]

where \( \mathcal{L}(L^2(X)) \) is the space of bounded linear operators from \( L^2(X) \) to \( L^2(X) \).
The poles of this extension are called resonances and they do not depend on \( N \) (neither the multiplicity \( m_{\lambda_0}(n-2\lambda)R(\lambda) \) of a resonance \( \lambda_0 \)), they form a discrete set \( \mathcal{R} \subset \mathbb{C} \setminus \{Z^1_+ \cup Z^2_+\} \). At each \( \lambda \in (Z^1_+ \cup Z^2_+) \), the behavior of \( R(\lambda) \) is not clear, it could be a pole of infinite multiplicity or an essential singularity. Observe that the equation \((\Delta_g - \lambda(n-\lambda))R(\lambda) = 1\) implies that for a pole \( \lambda_0 \) of \( R(\lambda) \), \( \text{Rank}_{\lambda_0} R(\lambda) < \infty \) is equivalent to \( m_{\lambda_0}(R(\lambda)) < \infty \). Let us now give a definition which will be essential and will be explained in Section 2:

**Definition 1.2.** Let \((X,g)\) be an asymptotically hyperbolic manifold and \( k \in \mathbb{N} \cup \{\infty\} \). We say that \( g \) is even modulo \( O(x^{2k+1}) \) if there exists \( \epsilon > 0 \), a boundary defining function \( x \) and some tensors \((h_{2i})_{i=0,...,k}\) on \( \partial X \) such that

\[
\phi^*(x^2g) = dt^2 + \sum_{i=0}^{k} h_{2i}t^{2i} + O(t^{2k+1})
\]

where \( \phi \) is the diffeomorphism induced by the flow \( \phi_t \) of the gradient \( \text{grad}_x g(x) \):

\[
\phi : \begin{cases} 
(0, \epsilon) \times \partial X & \rightarrow \phi([0, \epsilon) \times \partial X) \subset X \\
(t, y) & \rightarrow \phi_t(y)
\end{cases}
\]

Using the relations between resolvent and scattering operator in a way similar to \([6, 9, 19]\) and the calculus of the residues of \( S(\lambda) \) by Graham-Zworski \([8]\) we find a necessary and sufficient condition on the metric to have a finite-meromorphic extension of the resolvent to \( \mathbb{C} \).

**Proposition 1.3.** Under the assumptions of Theorem \([7]\) the modified resolvent extends to a finite-meromorphic family

\[
R(\lambda) \in \text{Mer}_f \left( \mathcal{O}_N \setminus Z^1_+, \mathcal{L}(x^N L^2(X), x^{-N} L^2(X)) \right), \quad \forall N \geq 0
\]

and if \( g \) is even modulo \( O(x^{2k+1}) \), this extension satisfies

\[
R(\lambda) \in \text{Mer}_f \left( \mathcal{O}_N, \mathcal{L}(x^N L^2(X), x^{-N} L^2(X)) \right), \quad \forall N \in [0, k + \frac{1}{2}]
\]

Conversely if \((1.3)\) holds true for \( k \geq 2 \) then \( g \) is even modulo \( O(x^{2k-1}) \).

We then deduce the following

**Theorem 1.4.** Let \((X,g)\) be an asymptotically hyperbolic manifold, then \( R(\lambda) \) admits a finite-meromorphic extension to \( \mathbb{C} \) if and only if \( g \) is even modulo \( O(x^\infty) \).

**Remark:** as a matter of fact, the usual examples are some particular cases of even metrics: the hyperbolic metrics perturbed on a compact \([10, 11, 12, 19]\), the De Sitter-Schwarzschild model \([23]\), the almost-product type metrics \([14]\). The asymptotically Einstein manifolds of dimension \( n + 1 \) are only even modulo \( O(x^n) \) in general \([7, 8]\).

Let us denote by \( \mathcal{M}_{ah}(X) \) the space of asymptotically hyperbolic metrics on \( X \) with the topology inherited from \( x^{-2} C^\infty(X, T^*X \otimes T^*X) \). If the metric is not even, there is at least a point of \( Z^1_+ \) which is either a pole of infinite multiplicity or an essential singularity of \( R(\lambda) \). The following results show that an essential singularity appears generically.

**Theorem 1.5.** Let \( \bar{X} \) be a compact manifold with boundary of dimension \( n + 1 > 2 \). Then the set of metrics in \( \mathcal{M}_{ah}(X) \) for which \( \frac{2k}{n-1} \) is an essential singularity of \( R(\lambda) \) contains an open and dense set in \( \mathcal{M}_{ah}(X) \).

Note that the proof of Theorem \([6]\) implies a little more general result including the case \( n = 1 \): the same is indeed satisfied for the point \( \frac{2k}{n-1} - k \) if we consider the set of even metrics modulo \( O(x^{2k+1}) \) instead of \( \mathcal{M}_{ah}(X) \) and if \( \frac{2k}{n-1} - k \neq 0 \).

In dimension \( n + 1 = 2 \), we will see that for \( g \in \mathcal{M}_{ah}(X) \) analytic near the boundary such that \( \partial X \) is a connected geodesic of \( (\bar{X}, x^2g) \), the resolvent is meromorphic if and only if \( g \) is even.
Finally, to illustrate these results, we give some examples with a sequence of resonances approaching an essential singularity of $R(\lambda)$.

**Proposition 1.6.** For all $k \in \mathbb{N}_0$ such that $2k \neq n - 1$, there exists a $n + 1$-dimensional asymptotically hyperbolic manifold such that the extension (1.3) has a sequence of poles which converges to $\frac{n-1}{2} - k$.

These cases are the first examples (as far as we know) of essential singularities coming from the meromorphic extension of the resolvent. If the resonances are interpreted as eigenvalues of an operator (see [1] when the extension is finite-meromorphic), we could think that these essential singularities are some isolated points in the essential spectrum of this operator.

![Figure 1](image-url)

**Figure 1.** The resonances of $\Delta_g$ in a case where $g$ is not even

The paper is organized as follows: Section 2 recalls some basic geometric facts on asymptotically hyperbolic manifolds and explains Definition 1.2. Section 3 shows how to use the scattering operator instead of the resolvent; we give in Section 4 the proofs of the main results and Section 5 contains the examples.

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2. **Geometry of $(X, g)$, even metric**

Let $(X, g)$ an asymptotically hyperbolic manifold, $x_0$ a boundary defining function and $H := x_0^2 g$
the induced metric by \( g \) and \( x_0 \) on \( \bar{X} \). We can easily check that neither the conformal class \([H|_{\partial \bar{X}}]\) of the metric \( H|_{\partial \bar{X}} \) on \( \partial \bar{X} \), nor the value at the boundary \(|dx_0|_H|_{\partial \bar{X}}\) depends on the choice of the function \( x_0 \). Moreover, Mazzeo and Melrose \cite{MM} remarked that the sectional curvatures of \( g \) at \( m \in X \) approach \(-|dx_0|_H^2(y)\) when \( m \to y \in \partial \bar{X} \), so we can summarize the property ‘asymptotically hyperbolic’ with the identity \(|dx_0|_H = 1 \) on \( \partial \bar{X} \) (which does not depend on the choice of \( x_0 \)).

If \((X, g)\) is asymptotically hyperbolic, it is shown by Graham \cite{G} that there exists, for each metric \( h_0 \in [H|_{\partial \bar{X}}] \), a unique boundary defining function \( x \) such that \(|dx|_{x^2g} = 1 \) in an open neighbourhood \( V_x \subset \bar{X} \) of \( \partial \bar{X} \) and \((x^2g)|_{\partial \bar{X}} = h_0 \). So there is a collar \( U_x := [0, \epsilon_x] \times \partial \bar{X} \) linked to \( x \) by the diffeomorphism

\[
\phi : \begin{cases}
U_x & \to \phi(U_x) \subset V_x \\
(t, y) & \to \phi_t(y)
\end{cases}
\]

where \( \phi_t \) is the flow of the gradient \( \text{grad} x^2g(x) \). Note that \( t = x \) as functions on \( \phi(U_x) \). In the open collar \((0, \epsilon_x) \times \partial \bar{X} \), the metric \( g \) can be expressed by

\[
\phi^*g = \frac{dt^2 + h(t, y, dy)}{t^2}, \quad h(0, y, dy) = h_0(y, dy), \quad h \in C^\infty(U_x, S^2(T^*U_x))
\]

\( S^2(T^*U_x) \subset T^*U_x \otimes T^*U_x \) being the bundle over \( U_x \) of symmetric 2-tensors. This form is called a model form and we shall write \( g \) and \( x \) instead of \( \phi^*g \) and \( t \) in \((2.2)\) for simplicity. Let us now define the set of boundary defining functions that induce a model form of the metric:

\[
Z(\partial \bar{X}) := \{ x \in C^\infty(\bar{X}); x \geq 0, \partial \bar{X} = x^{-1}(0), \exists \epsilon_x > 0, \forall m \in x^{-1}([0, \epsilon_x]), |dx|_{x^2g}(m) = 1 \}
\]

for which Graham \cite{G} has constructed a bijection

\[
[H|_{\partial \bar{X}}] \leftrightarrow Z(\partial \bar{X}).
\]

The symmetric tensor \( h(t, y, dy) \) in \((2.2)\) defines a family of metrics \( h(t) \) on the hypersurfaces \( \{x = t\} \), and it depends on the choice of the function \( x \in Z(\partial \bar{X}) \). We can easily check that, for a fixed \( k \in \mathbb{N} \), the vanishing condition modulo \( O(x^k) \) at \( \partial \bar{X} \)

\[
h(x) - h(0) = O(x^k)
\]

is invariant with respect to the boundary defining function \( x \in Z(\partial \bar{X}) \). But to treat our problem it is more natural to choose the weaker condition introduced in Definition \ref{Def1} there exists \( x \in Z(\partial \bar{X}) \) and \( k \in \mathbb{N} \) such that the Taylor expansion of \( x^2g \) at \( x = 0 \) consists only of even powers of \( x \) up through the \( x^{2k+1} \) term

\[
x^2g = dx^2 + l(x^2, y, dy) + O(x^{2k+1}), \quad l \in C^\infty(U_x, S^2(T^*U_x))
\]

in the collar \( U_x \) linked to \( x \) by \((2.1)\). As a matter of fact, this property is not associated to a particular defining function, as we could think, but to the set \( Z(\partial \bar{X}) \) or equivalently to the conformal class \([H|_{\partial \bar{X}}]\): indeed, if the property is satisfied for one function of \( Z(\partial \bar{X}) \), it is satisfied for all functions of \( Z(\partial \bar{X}) \). The metric is then said to be even modulo \( O(x^{2k+1}) \).

**Lemma 2.1.** Let \((X, g)\) be an asymptotically hyperbolic manifold. Suppose that there exists a function \( x \in Z(\partial \bar{X}) \) and \( k \in \mathbb{N} \) such that the metric \( x^2g \) can be expressed by

\[
x^2g = dx^2 + l(x^2, y, dy) + O(x^{2k+1}), \quad l \in C^\infty(U_x, S^2(T^*U_x))
\]

in the collar \( U_x = [0, \epsilon_x] \times \partial \bar{X} \) linked to \( x \) by \((2.1)\). Then, for all function \( t \in Z(\partial \bar{X}) \), the metric \( t^2g \) can be expressed by

\[
t^2g = dt^2 + p(t^2, z, dz) + O(t^{2k+1}), \quad p \in C^\infty(U_t, S^2(T^*U_t))
\]

in the collar \( U_t = [0, \epsilon_t] \times \partial \bar{X} \) linked to \( t \) by \((2.1)\).

**Proof:** firstly, we recall Graham calculus in \cite{G} Lem. 2.1 and 2.2. Let \( x \in Z(\partial \bar{X}) \) such that \((2.4)\) holds and \( t \in Z(\partial \bar{X}) \). Let \( h \) defined as in \((2.2)\) and set

\[
t = e^\omega x, \quad \omega \in C^\infty(U_x)
\]
According to [7], \( \omega \) is a solution of the non-linear equation
\[
2\partial_t x + x \left( (\partial_t x)^2 + \sum_{ij} h^{ij}(x) \partial_{y_i} \omega \partial_{y_j} \omega \right) = 0.
\]

We easily obtain \( \partial_t \omega \big|_{t=0} = 0 \) and by differentiating an even number of times with respect to \( x \), it can be shown by induction that \( \partial_t^{2j+1} \omega \big|_{t=0} = 0 \) for \( j \leq k \) (cf. [7] for details). Recall now that the collar linked to \( t \) is constructed by the diffeomorphism
\[
\phi' : \begin{cases} 
U_t := [0, \epsilon_t) \times \partial X 
\quad \rightarrow \quad \phi'(U_t) \\
(t, z) \quad \rightarrow \quad \phi'_t(z)
\end{cases}
\]
where \( \phi'_t(z) \) is the flow of \( \partial_t \omega \big|_{t=0} = 0 \). We deduce that
\[
(2.5) \quad 2\partial_t x + x \left( (\partial_t x)^2 + \sum_{ij} h^{ij}(x) \partial_{y_i} \omega \partial_{y_j} \omega \right) = 0.
\]

To begin, note that
\[
\text{grad}_{\omega} t = t^{-2} \text{grad}_g t = e^{-\omega} \text{grad}_{\omega} x + e^{-\omega} x \text{grad}_{\omega} \omega
\]
and we will show by induction that we have for all \( m \leq 2k + 2 \)
\[
(2.6) \quad \partial_t^{2j} x(t, z) \big|_{t=0} = 0 \quad \forall j, 0 \leq 2j \leq m \\
\partial_t^{2j+1} y(t, z) \big|_{t=0} = 0 \quad \forall j, 0 \leq 2j + 1 \leq m.
\]

Suppose now that (2.6) is satisfied for the integer \( m \) (with \( m \leq 2k + 1 \)).
If \( m + 1 \) is even and \( m + 1 \leq 2k + 2 \), we obtain from (2.7)
\[
\partial_t^{m+1} x(0, z) = \partial_t^m e^{-\omega} \big|_{t=0} + m \partial_t^{m-1} (e^{-2\omega} \partial_x \omega) \big|_{t=0}.
\]

Remark that if \( f(x, y) \) is an arbitrary even function (resp. odd) in \( x \) modulo \( O(x^{2l+1}) \) (resp. modulo \( O(x^{2l}) \)), then the composition of \( f(x, y) \) with
\[
(t, z) \rightarrow (x(t, z), y(t, z))
\]
is even (resp. odd) in \( t \) modulo \( O(l^{\min(2l+1, m+2)}) \) (resp. modulo \( O(l^{\min(2l, m+1)}) \)). Therefore, we deduce that
\[
\partial_t^m e^{-\omega} \big|_{t=0} = 0.
\]
because \( m \) is odd and \( e^{-\omega} \) is even modulo \( O(x^{2k+3}) \). Moreover, \( m - 1 \) is even and the derivatives \( \partial_t^{m-1} (e^{-2\omega} \partial_x \omega) \big|_{t=0} \) split into a sum of products of derivatives of \( e^{-2\omega} \) and \( \partial_x \omega \). In each product, if we differentiate an odd number of times one of the terms at \( t = 0 \), the number of derivatives in the other term must be odd too and the previous argument shows that the product vanishes because \( e^{-2\omega} \) is even in \( x \) modulo \( O(x^{2k+3}) \). If the number of derivatives for one term of the product is even, the number for the other term is even too and the product vanishes because \( \partial_x \omega \) is odd in \( x \) modulo \( O(x^{2k+2}) \). We then deduce that
\[
\partial_t^{m+1} x(0, z) = 0.
\]
On the other hand, if \( m + 1 \) is odd and \( m + 1 \leq 2k + 1 \), we use the same trick for equation (2.3) and we just have to differentiate an odd (= \( m \)) number of times a product of even functions in \( t \) modulo \( O(t^{\min(2k+1, m+1)}) \), which proves that

\[
\partial_t^{m+1} y_i(0, z) = 0
\]

and we conclude by induction that (2.6) is true for all \( m \leq 2k + 2 \).

We finally have to show that for all \( \xi \in T_x \partial \bar{X} \)

\[
t^2 g(\xi, \xi) = e^{2\omega} \left( (\partial_x dz(\xi))^2 + h(x, y, \partial_x y, dz(\xi)) \right)
\]

is an even function in \( t \) modulo \( O(t^{2k+1}) \), which is a simple consequence of the odd-even properties of \( \omega, x, y \) and \( h \). \( \square \)

Let us denote by \( \bar{X}^2 := (\bar{X} \cup X)/\partial \bar{X} \) the double of \( X \), which is firstly a topological space. Choose \( x \) a boundary defining function of \( \partial \bar{X} \). From the diffeomorphism (2.1), we can construct a \( C^\infty \) atlas on \( \bar{X}^2 \), using the fact that \( \partial \bar{X} \subset \bar{X}^2 \) is contained in an open set \( V_x^2 := (V_1 \cup V_2)/\partial \bar{X} \) (with \( V_1 = V_2 = \phi(U_x) \)) diffeomorphic to \( (\epsilon_x, \epsilon_x) \times \partial \bar{X} \) via

\[
(-\epsilon_x, \epsilon_x) \times \partial \bar{X} \cong \left\{ \begin{array}{lll}
[t, y) & \quad [\phi_t(y)], \quad \phi_t(y) \in V_1 \quad \text{if } t \leq 0 \\
\phi_t(y), \quad \phi_t(y) \in V_2 \quad \text{if } t \geq 0
\end{array} \right.
\]

The remaining charts come easily from the charts of the interior \( X \) of \( \bar{X} \). Remark that this \( C^\infty \) structure on \( \bar{X}^2 \) depends on the choice of \( x \). If now we denote this structure by \( \bar{X}^2_x \), there is a global diffeomorphism

\[
\bar{X}^2_x \cong \bar{X}^2_x
\]

for two different boundary defining functions \( x \) and \( x' \), but it is not the case that any one of these structures is natural with respect to \( C^\infty (\bar{X}) \). By the even functions of \( x \) modulo \( O(x^{2k+1}) \) on \( \bar{X} \), we shall mean the smooth functions on \( X \) which admit a \( C^{2k} \) continuation to \( \bar{X}^2 \) and are invariant with respect to the natural involution exchanging the factors on \( \bar{X}^2 \). The result is a class of functions which depends on the choice of \( x \): a function whose Taylor expansion in \( x \) at \( x = 0 \) is even modulo \( O(x^{2k+1}) \) does not necessarily have an even Taylor expansion in \( x' \) modulo \( O(x'^{2k+1}) \) at \( x' = 0 \), if \( x, x' \) are two different boundary defining functions of \( \bar{X} \). In the proof of Lemma 2.1 it is shown that if the metric can be expressed by (2.3) for one boundary defining function \( x \), then the coordinate changes \( (x, y) \to (x', y') \) on \( [0, \epsilon) \times \partial \bar{X} \) which leave the metric under a model form have local expansions of the form

\[
x' = x \sum_{j=0}^{k+1} a_j(y) x^{2j} + O(x^{2k+4}), \quad y' = \sum_{j=0}^{k+1} b_j(y) x^{2j} + O(x^{2k+3})
\]

with some smooth functions \( a_j \) and \( b_j \), thus they induce some \( C^{2k+2} \) compatible charts on \( \bar{X}^2_x \). As a conclusion, if \( x \in Z(\partial \bar{X}) \) the structure \( \bar{X}^2_x \) does not depend on the choice of \( x \) as a \( C^{2k+2} \) structure on \( \bar{X}^2 \) and we obtain a natural choice (with respect to \( g \)) of \( C^{2k+2} \) structure on \( \bar{X}^2 \) induced by the functions in \( Z(\partial \bar{X}) \). Moreover, it admits a \( C^{2k} \) conformal class of metrics which are invariant with respect to the natural involution exchanging the factors on \( \bar{X}^2 \): this is obtained by extending by symmetry the metrics \( x^2 g \) for each \( x \in Z(\partial \bar{X}) \).

3. From the resolvent to the scattering operator

3.1. Stretched products. To begin, let us introduce a few notations and recall some basic things on stretched products (the reader can refer to Mazzeo-Melrose [17], Mazzeo [16] or Melrose [18] for details). Let \( \bar{X} \) a smooth compact manifold with boundary and \( x \) a boundary defining function. The manifold \( \bar{X} \times X \) is a smooth manifold with corners, whose boundary hypersurfaces are diffeomorphic to \( \partial \bar{X} \times X \) and \( \bar{X} \times \partial X \), and defined by the functions \( \pi_L^x, \pi_R^x \) (\( \pi_L \) and \( \pi_R \))
being the left and right projections from \( \tilde{X} \times \tilde{X} \) onto \( \tilde{X} \). For notational simplicity, we now write \( x, x' \) instead of \( \pi_L^* x, \pi_R^* x \) and let
\[
\delta_{\tilde{X}} := \{ (m, m) \in \partial \tilde{X} \times \partial \tilde{X}; m \in \partial \tilde{X} \}.
\]

Remark the elementary embeddings
\[
\delta_{\partial \tilde{X}} \to \partial \tilde{X} \times \partial \tilde{X} \to \partial \tilde{X} \times X \to \tilde{X} \times \tilde{X}
\]
which will often be regarded as inclusions. The blow up of \( \tilde{X} \times \tilde{X} \) along the ‘diagonal’ \( \delta_{\partial \tilde{X}} \) of \( \partial \tilde{X} \times \partial \tilde{X} \) will be denoted by \( \tilde{X} \times \tilde{X} \) and the blow-down map
\[
\beta : \tilde{X} \times \tilde{X} \to \tilde{X} \times \tilde{X}.
\]

This manifold with corners has three boundary hypersurfaces \( \mathcal{T}, \mathcal{B}, \mathcal{F} \) defined by some functions \( \rho, \rho', R \) such that \( \beta^*(x) = R \rho, \beta^*(x') = R \rho' \). Globally, \( \delta_{\partial \tilde{X}} \) is replaced by a larger manifold, namely by its doubly inward-pointing spherical normal bundle of \( \delta_{\partial \tilde{X}} \), whose each fiber is a quarter of sphere. From local coordinates \((x, y, x', y')\) on \( \tilde{X} \times \tilde{X} \), this amounts to introducing polar coordinates \((R, \rho, \rho', \omega, y)\) around \( \delta_{\partial \tilde{X}} \):
\[
R := \left( x^2 + x'^2 + |y - y'|^2 \right)^{\frac{1}{2}}, \quad (\rho, \rho', \omega) := \left( \frac{x}{R}, \frac{x'}{R}, \frac{y - y'}{R} \right)
\]
with \( R, \rho, \rho' \in [0, \infty) \). These polar coordinates are useful to describe the singularities of the Schwartz kernel of \( R(\lambda) \).

![Figure 2. The blow-down map](image)

Similarly, we denote by \( \partial \tilde{X} \times_0 \tilde{X} \) the blow-up of \( \partial \tilde{X} \times \tilde{X} \) along \( \delta_{\partial \tilde{X}} \). It can be naturally embedded in \( \tilde{X} \times_0 \tilde{X} \) with respect to (3.1):
\[
\partial \tilde{X} \times_0 \tilde{X} \simeq (\tilde{X} \times_0 \tilde{X}) \cap \mathcal{F}
\]
using \( \beta(\mathcal{T}) = \{ x = 0 \} \simeq \partial \tilde{X} \times \tilde{X} \). With these identifications, \( \bar{\beta} := \beta|_\mathcal{T} \) is the blow-down map
\[
\bar{\beta} : \partial \tilde{X} \times_0 \tilde{X} \to \partial \tilde{X} \times \tilde{X}.
\]

This manifold with corners has two boundary hypersurfaces \( \bar{\mathcal{B}}, \bar{\mathcal{F}} \) defined by the functions \( \bar{\rho}' := \rho'|_\mathcal{T} \) and \( \bar{R} := R|_\mathcal{T} \).

Finally, the blow-up \( \partial \tilde{X} \times \partial \tilde{X} \) of \( \partial \tilde{X} \times \partial \tilde{X} \) along \( \delta_{\partial \tilde{X}} \) can be naturally embedded in \( \partial \tilde{X} \times_0 \tilde{X} \) with respect to (3.1):
\[
\partial \tilde{X} \times_0 \partial \tilde{X} \simeq (\partial \tilde{X} \times_0 \tilde{X}) \cap \bar{\mathcal{B}}
\]
3.2. Half densities. Let $\Gamma^\frac{1}{2}_0(\bar{X})$ the line bundle of singular half-densities on $\bar{X}$, trivialized by $\nu := \frac{dvol_g}{r^2}$, and $\Gamma^\frac{1}{2}(\partial \bar{X})$ the bundle of half densities on $\partial \bar{X}$, trivialized by $v_0 := \frac{dvol_{h_0}}{r^2}$ (where $h_0 = x^2 g|_{\partial \bar{X}}$). From these bundles, one can construct the bundles $\Gamma^\frac{1}{2}_0(\bar{X} \times \bar{X})$, $\Gamma^\frac{1}{2}_0(\partial \bar{X} \times \bar{X})$ and $\Gamma^\frac{1}{2}_0(\bar{X} \times \partial \bar{X})$ by tensor products, whose sections are respectively $\nu \otimes \nu$, $v_0 \otimes \nu$ and $\nu \otimes v_0$. Finally, let $\Gamma^\frac{1}{2}_0(\bar{X} \times_0 \bar{X})$, $\Gamma^\frac{1}{2}_0(\partial \bar{X} \times_0 \bar{X})$ and $\Gamma^\frac{1}{2}_0(\bar{X} \times_0 \partial \bar{X})$ the bundles obtained by lifting under $\beta$, $\beta$ and $\beta_0$ the three previous bundles. If $M$ denotes $\bar{X}$, $\bar{X} \times \bar{X}$ or $\partial \bar{X} \times \bar{X}$, we write $C^\infty(M, \Gamma^\frac{1}{2}_0(M))$ the space of smooth sections of $\Gamma^\frac{1}{2}_0(M)$ that vanish to all order at all the boundary hypersurfaces of $M$, and $C^{-\infty}(M, \Gamma^\frac{1}{2}_0(M))$ its topological dual, whose elements are the extendible distributional half densities. The Hilbert space $L^2(\bar{X}, \Gamma^\frac{1}{2}_0(\bar{X}))$ constructed by completing $C^\infty(\bar{X}, \Gamma^\frac{1}{2}_0(\bar{X}))$ with respect to the norm

$$
||f||_2 = \left( \int_{\bar{X}} f \bar{f} \right)^\frac{1}{2}
$$

is isomorphic to $L^2(X, dvol_g)$ and will be denoted by $L^2(X)$. Similarly it will be more practical to write $H^R(\partial \bar{X})$ for the Sobolev space of order $R$ on $\partial \bar{X}$ with values in half densities. At last, $\Gamma^\frac{1}{2}_0(M)$ and $\Gamma^\frac{1}{2}_0(M)$ will be replaced by $\Gamma^\frac{1}{2}_0$ and $\Gamma^\frac{1}{2}$, where $M$ is one of the previously introduced manifolds.

Set $(.,.)$ the symmetric non-degenerate products on $L^2(\bar{X})$ and $L^2(\partial \bar{X})$

$$(u,v) := \int_X uv, \quad (u,v) := \int_{\partial \bar{X}} uv.$$ 

For $\alpha \in \mathbb{R}$, we can check by using the first product that the dual space of $x^\alpha L^2(X)$ is isomorphic to $x^{-\alpha} L^2(X)$. We shall also use the following tensorial notation for $E = x^\alpha L^2(X)$ (resp. $E = L^2(\partial \bar{X})$), $\psi, \phi \in E'$

$$\phi \otimes \psi : \begin{cases} E & \to E' \\
\phi & \to \psi, f \end{cases}.$$ 

3.3. Resolvent. The meromorphically continued resolvent $R(\lambda)$ on $\mathbb{C} \setminus (Z^+_1 \cup Z^+_2)$ is a continuous operator from $C^\infty(\bar{X}, \Gamma^\frac{1}{2}_0)$ to $C^{-\infty}(\bar{X}, \Gamma^\frac{1}{2}_0)$, its associated Schwartz kernel being

$$r(\lambda) \in C^{-\infty}(\bar{X} \times \bar{X}, \Gamma^\frac{1}{2}_0)$$

whose properties, studied in [17], are recalled for instance in [14, Th. 2.1], namely

$$r(\lambda) = r_0(\lambda) + r_1(\lambda) + r_2(\lambda)$$

$$\beta^*(r_0(\lambda)) \in I^{-2}(\bar{X} \times_0 \bar{X}, \Gamma^\frac{1}{2}_0)$$

$$\beta^*(r_1(\lambda)) \in x^\lambda \rho^\lambda C^\infty(\bar{X} \times_0 \bar{X}, \Gamma^\frac{1}{2}_0)$$

$$r_2(\lambda) \in x^{-\lambda} x^\lambda C^\infty(\bar{X} \times_0 \bar{X}, \Gamma^\frac{1}{2}_0)$$

where $I^{-2}(\bar{X} \times_0 \bar{X}, \Gamma^\frac{1}{2}_0)$ denotes the set of conormal distributions of degree $-2$ on $\bar{X} \times_0 \bar{X}$ associated to the closure of the lifted interior diagonal (see Fig. 2)

$$\mathcal{D}_l := \beta^{-1}(\{(m,m) \in X \times X; m \in \bar{X}\})$$
Finally, the only poles of $E_{\rm Barreto}$ work [14] that 
\( \lambda \in \mathbb{C} \) \( \setminus (Z^1 \cup Z^2) \) and \( r_0(\lambda) \) is the kernel of a holomorphic family of operators

\[
R_0(\lambda) \in \mathcal{K}o\text{r}(\mathbb{C}, \mathcal{L}(x^{\alpha}L^2(\bar{X}), x^{-\omega}L^2(\bar{X}))), \quad \forall \alpha \geq 0
\]

Note also that Patterson-Perry arguments [19] Lem. 4.9] prove that \( R(\lambda) \) does not have poles on the line \( \{ \Re(\lambda) = \frac{m}{2} \} \), except maybe \( \lambda = \frac{m}{2} \), it is a consequence of the absence of embedded eigenvalues (see Mazzeo [18]). The only poles of \( R(\lambda) \) in the half plane \( \{ \Re(\lambda) > \frac{m}{2} \} \) is the finite set of \( \lambda \) such that \( \lambda \in (\mathbb{N} + \mathbb{N}) \cap \mathbb{R} \) and \( x \) a fixed boundary defining function of \( \bar{X} \), the Poisson operator is the unique continuous operator

\[
\mathcal{P}(\lambda) : \mathcal{C}^\infty(\partial \bar{X}, \Gamma^\perp) \to x^\lambda \mathcal{C}^\infty(\bar{X}, \Gamma^\perp) + x^{n-\lambda} \mathcal{C}^\infty(\bar{X}, \Gamma^\perp)
\]

such that

\[
\begin{cases}
(\Delta_g - \lambda(n - \lambda))\mathcal{P}(\lambda) = 0 \\
\mathcal{P}(\lambda)f = x^n - \lambda F_1(\lambda) + x^\lambda F_2(\lambda) \\
F_1(\lambda), F_2(\lambda) \in \mathcal{C}^\infty(\bar{X}, \Gamma^\perp) \\
(\mathcal{P}(\lambda))_{|_{\partial X}} = f
\end{cases}
\]

Graham and Zworski [3] gave a simple construction of \( \mathcal{P}(\lambda) \) and Joshi-Sá Barreto [14] proved that its Schwartz kernel is a ‘weighted restriction’ of the resolvent kernel \( r(\lambda) \) on the boundary \( \{ x' = 0 \} \), which implies that \( \mathcal{P}(\lambda) \) can be meromorphically continued to \( \mathbb{C} \setminus (Z^1 \cup Z^2) \).

For what follows, we now define the operator \( E(\lambda) \) whose Schwartz kernel \( e(\lambda) \) is the weighted restriction of \( r(\lambda) \) on the boundary \( \{ x = 0 \} \):

\[
e(\lambda) := \tilde{\beta}_+ \left( \beta^* (x^{-\frac{m}{2}} r(\lambda)) \right) \in \mathcal{C}^\infty(\partial \bar{X} \times \bar{X}, \Gamma^\perp).
\]

According to [32] and [33], the distribution \( e(\lambda) \) has conormal singularities on the boundaries described by

\[
\tilde{\beta}^*(e(\lambda)) \in \rho^M \mathcal{R}^{-\lambda + \frac{m}{2}} \mathcal{C}^\infty(\partial \bar{X} \times_0 \bar{X}, \Gamma^\perp) + \tilde{\beta}^*(x^\lambda \mathcal{C}^\infty(\partial \bar{X} \times \bar{X}, \Gamma^\perp)).
\]

We then easily deduce that \( E(\lambda) \) is continuous from \( \mathcal{C}^\infty(\bar{X}, \Gamma^\perp) \) to \( \mathcal{C}^\infty(\partial \bar{X}, \Gamma^\perp) \), and its transpose is well defined from \( \mathcal{C}^\infty(\partial \bar{X}, \Gamma^\perp) \) to \( \mathcal{C}^\infty(\bar{X}, \Gamma^\perp) \). As a matter of fact, it follows from Joshi-Sá Barreto work [14] that

\[
\mathcal{P}(\lambda) = \mathcal{E}(\lambda)
\]

on \( \mathcal{C}^\infty(\partial \bar{X}, \Gamma^\perp) \). Following the holomorphic-meromorphic properties of [32] and [33], we obtain

\[
\tilde{\beta}^* (e(\lambda)) \in \mathcal{K}o\text{r}(U_N, L^2(\partial \bar{X} \times_0 \bar{X}, \Gamma^\perp))
\]

\[
U_N := \{ \lambda \in \mathbb{C} : \frac{n}{2} - N < \Re(\lambda) < N, \lambda \notin (R \cup Z^1 \cup Z^2) \}
\]

and using the isomorphism \( \tilde{\beta}^* \) between \( L^2(\partial \bar{X} \times_0 \bar{X}, \Gamma^\perp) \) and \( L^2(\partial \bar{X} \times_0 \bar{X}, \Gamma^\perp) \), we see that \( E(\lambda) \) is Hilbert-Schmidt from \( x^N L^2(X) \) to \( L^2(\partial \bar{X}) \), and

\[
E(\lambda) \in \mathcal{K}o\text{r}(U_N, \mathcal{L}(x^N L^2(X), L^2(\partial \bar{X}))).
\]

It also implies that its transpose \( \mathcal{E}(\lambda) \) is holomorphic in \( U_N \) with values in \( \mathcal{L}(L^2(\partial \bar{X}), x^{-N} L^2(X)) \). Finally, the only poles of \( E(\lambda) \) and \( \mathcal{E}(\lambda) \) in \( \{ \Re(\lambda) > \frac{m}{2} \} \) are the complex numbers \( \lambda_n \) such that
$\lambda_c(n - \lambda_c) \in \sigma_{pp}(\Delta_g)$, proving that they are of order 1 with finite multiplicity.

3.5. **Scattering operator.** For $\mathcal{R}(\lambda) \geq \frac{n}{2}$, $\lambda \notin \frac{1}{2}(n + \mathbb{N}) \cup \mathbb{R}$, the scattering operator $S(\lambda)$ is defined by

$$S(\lambda) : C^\infty(\partial X, \Gamma^\perp) \to C^\infty(\partial \tilde{X}, \Gamma^\perp)$$

using the notations \[\text{(3.9)}\]. Using the meromorphic continuation of $R(\lambda)$ to $\mathbb{C} \setminus (Z^1_+ \cup Z^2_+)$, Joshi-Sá Barreto \[\text{[14]}\] proved that $S(\lambda)$ can be meromorphically continued (weakly) to the same set with Schwartz kernel $s(\lambda)$

$$s(\lambda) := (2\lambda - n)(\beta_\partial)_* (\beta^* \big( x^{-\lambda+\frac{n}{2}} x^i - \lambda^i + \frac{n}{2} r(\lambda) \big) |_{\mathcal{R} \cap \mathcal{B}})$$

that is, in view of \[\text{(3.5)}\] and \[\text{(3.8)}\],

$$s(\lambda) = (\beta_\partial)_* (r^{-2\lambda}k_1(\lambda) + k_2(\lambda))$$

with $k_1(\lambda)$ and $k_2(\lambda)$ respectively holomorphic and meromorphic in $\lambda \in \mathbb{C} \setminus (Z^1_+ \cup Z^2_+)$. To understand the distribution $r^{-2\lambda}k_1(\lambda)$ on $\partial \tilde{X} \times_0 \partial \tilde{X}$, we remark that, up to a smooth half-density section, it is a $L^1$ function for $\mathcal{R}(\lambda) < \frac{n}{2}$, $\lambda \notin \mathcal{R} \cup Z^1_+ \cup Z^2_+$ which can be meromorphically continued to $\mathbb{C} \setminus (Z^1_+ \cup Z^2_+)$ in the distribution sense (see \[\text{[13]}\] Th. 3.2.4 or \[\text{[12]}\] for instance). This continuation makes appear some poles in the physical sheet at $Z^2_+$ and possibly at $Z^1_+$ though $k_1(\lambda)$ is holomorphic at these points. The recent work of Graham-Zworski \[\text{[8]}\] gives a nice description of these poles, these are first order poles for $S(\lambda)$ whose residues can be calculated explicitly. For $k \in \mathbb{N}$, the residue of $S(\lambda)$ at $\frac{n}{2}$ is the sum of a differential operator on $\partial \tilde{X}$ and of a smoothing finite-rank operator which only appears when $\frac{n+k}{2} \in \sigma_{pp}(\Delta_g)$. This differential operator only depends on the $k_1$ first derivatives of the metric at the boundary and it is never zero for $k$ even but can be zero for $k$ odd according to whether the metric is even or not (it will be detailed later).

For $\lambda \in \mathbb{C} \setminus (\mathcal{R} \cup \frac{1}{2}\mathbb{Z})$, $s(\lambda)$ is a polyhomogeneous conormal distribution of order $-2\lambda$ associated to $\delta_{\partial \tilde{X}}$, thus $S(\lambda)$ is a one-step pseudodifferential operator of order $2\lambda - n$ on $\partial \tilde{X}$. Following Shubin’s definition \[\text{[24]}\] Def. 11.2, $S(\lambda)$ is a holomorphic family in

$$\left\{ \mathcal{R}(\lambda) < \frac{n}{2} \right\} \setminus (\mathcal{R} \cup Z^1_+ \cup Z^2_-)$$

of zeroth order pseudodifferential operators on the compact manifold $\partial \tilde{X}$. Therefore, $S(\lambda)$ is holomorphic in the same open of $\mathbb{C}$ with values in $\mathcal{L}(L^2(\partial \tilde{X}))$.

If $h_0 := x^2 y_{\mathcal{R} \partial \tilde{X}}$, the principal symbol of $S(\lambda)$ is given by Joshi and Sá Barreto \[\text{[13]}\]:

$$\sigma_0 (S(\lambda)) = c(\lambda)\sigma_0 (\Lambda^{2\lambda - n})$$

$$\Lambda := (1 + \Delta_{h_0})^{\frac{n}{2}}, \quad c(\lambda) := 2^{n-2\lambda}\frac{\Gamma(\frac{n}{2} - \lambda)}{\Gamma(\lambda - \frac{n}{2})}$$

which leads us to set the factorization (see \[\text{[21]}\] [\text{12}] [\text{19}] for a similar approach)

$$\tilde{S}(\lambda) := c(n - \lambda)\Lambda^{-\lambda+\frac{n}{2}} S(\lambda)\Lambda^{-\lambda+\frac{n}{2}}.$$

So $\tilde{S}(\lambda)$ can be expressed by

$$\tilde{S}(\lambda) = 1 + K(\lambda)$$

where $K(\lambda)$ is a compact operator for $\lambda \in \mathbb{C} \setminus (\mathcal{R} \cup \frac{1}{2}\mathbb{Z})$. Notice that the poles of $S(\lambda)$ on $Z^2_+$ already appear on the principal symbol \[\text{[3.10]}\], but $K(\lambda)$ is regular on $Z^2_+$ in view of the factorization by the Gamma factor.

Recall the functional equations satisfied by $S(\lambda)$ and $\tilde{S}(\lambda)$ on $\{ \mathcal{R}(\lambda) = \frac{n}{2} \} \setminus \{ \frac{n}{2} \}$, (cf. \[\text{[8]}\])

$$S^{-1}(\lambda) = S(n - \lambda) = S(\lambda)^*, \quad \tilde{S}^{-1}(\lambda) = \tilde{S}(n - \lambda) = \tilde{S}(\lambda)^*$$

(3.11)
which show that $S(\lambda)$ is regular on the critical line $\{ \Re(\lambda) = \frac{n}{2} \}$.

In order to use analytic Fredholm theorem to invert $1 + K(\lambda)$, we give the meromorphic properties of $K(\lambda)$, naturally inherited from $S(\lambda)$, in a neighbourhood of the physical sheet $\mathcal{O}_0$:

**Lemma 3.1.** For all $\epsilon \in (0, \frac{1}{2})$ and $\alpha > 0$:

$$
K(\lambda) \in \mathcal{H}ol\left( \mathcal{O}_z \setminus (Z^1_+ \cup \mathbb{R}), \mathcal{L}(L^2(\partial \mathbb{X}), H^{1-\alpha}(\partial \mathbb{X})) \right)
$$

At each $\lambda_j := \frac{n+1}{2} + j \in \mathbb{Z}_+$ with $j \in \mathbb{N}_0$, $K(\lambda)$ is either regular or has a first order pole.

**Proof:** let $\Psi^m(\partial \mathbb{X}, \Gamma^\pm)$ the space of pseudodifferential operator of order $m$ on $\partial \mathbb{X}$, acting on half densities. As said before, the distribution $(\beta_0)_+(r^{-2\lambda}k_1(\lambda))$ is holomorphic in $\mathcal{O}_z \setminus (Z^1_+ \cup \mathbb{Z}_+ \cup \mathbb{R})$ with at most some first order poles at $Z^1_+ \cup \mathbb{Z}_+$. We then kill the poles of $S(\lambda)$ at $\mathbb{Z}_+$ by dividing by $\Gamma(\frac{n}{2} - \lambda)$ and according to Shubin’s definition [24, Def. 11.2], the operators \( (\Gamma(\frac{n}{2} - \lambda))^{-1}S(\lambda) \) form a holomorphic family

$$
\frac{S(\lambda)}{\Gamma(\frac{n}{2} - \lambda)} \in \mathcal{H}ol\left( \mathcal{O}_z \setminus (Z^1_+ \cup \mathbb{R}), \Psi^{2\Re(\lambda)+n+\epsilon}(\partial \mathbb{X}, \Gamma^\pm) \right), \ \forall \alpha > 0.
$$

It can easily be checked in charts by studying its total local symbol defined by local Fourier transformation of its Schwartz kernel, moreover $S(\lambda)$ is a one-step pseudodifferential operator of order $2\lambda - n$.

Therefore, the holomorphic properties of $\lambda^{-\lambda+\frac{n}{2}}$ (cf. [24 Th. 1.11]), the calculus \( \mathfrak{B}_1 \) and the composition properties of holomorphic pseudodifferential operators imply that

$$
K(\lambda) \in \mathcal{H}ol\left( \mathcal{O}_z \setminus (Z^1_+ \cup \mathbb{R}), \Psi^{-1+\alpha}(\partial \mathbb{X}, \Gamma^\pm) \right), \ \forall \alpha > 0.
$$

At last, \( 3.12 \) follows immediately by using the Sobolev continuity properties of holomorphic pseudodifferential operators on compact manifolds. \]
Proof: we first show that \( M(\lambda) \in \mathcal{H}o(U \setminus \{\lambda_0\}, \mathcal{B}_0) \); it is sufficient to prove that for all \( \lambda_1 \in U \setminus \{\lambda_0\} \) there exists \( \epsilon > 0 \) such that

\[
\int_T M(\lambda)d\lambda = 0
\]

for all triangle \( T \) included in the open disc \( \{ |\lambda - \lambda_1| < \epsilon \} \). Note that (3.14) is satisfied when we replace \( M(\lambda) \) by \( j \circ M(\lambda) \). But since \( j \) is continuous, the integral of \( j \circ M(\lambda) \) on \( T \), defined as a limit of a Riemann sum, is exactly

\[
j \circ \int_T M(\lambda)d\lambda = \int_T j \circ M(\lambda)d\lambda.
\]

We then have the desired identity (3.14) since \( j \) is injective. Using the same arguments and the meromorphic assumption on \( j \circ M(\lambda) \) we obtain the following Laurent expansion

\[
j \circ M(\lambda) = \sum_{k=-p}^{-1} j \circ M_k(\lambda - \lambda_0)^k + j \circ H(\lambda)
\]

\[
M_k := \frac{1}{2\pi i} \int_T (\lambda - \lambda_0)^{-k-1} M(\lambda)d\lambda, \quad H(\lambda) := \frac{1}{2\pi i} \int_T (z - \lambda)^{-1} M(z)dz
\]

for \( \lambda \) near \( \lambda_0 \) and \( T \) a triangle around \( \lambda_0 \). Since \( j \) is injective we have

\[
M(\lambda) = \sum_{k=-p}^{-1} M_k(\lambda - \lambda_0)^k + H(\lambda)
\]

and \( H(\lambda) \) is holomorphic near \( \lambda_0 \) with values in \( \mathcal{B}_0 \). It remains to remark that if \( j \circ M_k \) has finite rank then it is the same for \( M_k \), \( j \) being injective.

Let us study the relations between the meromorphic properties of \( R(\lambda) \), \( S(\lambda) \) and \( \tilde{S}(\lambda) \).

**Proposition 3.3.** Let \( U \subset \{ \Re(\lambda) < \frac{n}{2} \} \) an open set in \( \mathbb{C} \), the following assertions are equivalent:

1. \( R(\lambda) \) is meromorphic in \( U \).
2. \( S(\lambda) \) is meromorphic in \( U \).
3. \( \tilde{S}(\lambda) \) is meromorphic in \( U \).
4. \( R(\lambda) \) is finite-meromorphic in \( U \).
5. \( S(\lambda) \) is finite-meromorphic in \( U \).
6. \( \tilde{S}(\lambda) \) is finite-meromorphic in \( U \).
7. \( \tilde{S}(n - \lambda) \) is finite-meromorphic in \( U \).

If \( U \cap \mathbb{Z}^2 = \emptyset \) we just have

\[
(7) \iff (6) \Rightarrow (5) \iff (4).
\]

Proof: (2) \( \Rightarrow \) (1) and (5) \( \Rightarrow \) (4): let \( N > \frac{n}{2} \), we first show that in the open set

\[
\{ \lambda \in \mathbb{C}; n - N < \Re(\lambda) < \frac{n}{2} \text{ and } \lambda, n - \lambda \notin (\mathbb{Z}^1 \cup \mathbb{Z}^2 \cup \mathbb{R}) \}
\]

we have the following holomorphic identity on \( \mathcal{L}(x^N L^2(X), x^{-N} L^2(X)); \)

\[
R(\lambda) - R(n - \lambda) = (2\lambda - n)^dE(n - \lambda)S(\lambda)E(n - \lambda).
\]

Observe that the proof of Green’s formula obtained by Agmon [2], Perry [20] or Guillopé [9] for hyperbolic quotients remains true in our framework (see also Borthwick [3] Prop. 4.5] in our setting): for \( \lambda, n - \lambda \notin (\mathbb{R} \cup \mathbb{Z}^1 \cup \mathbb{Z}^2), m, m' \in X \) and \( m \neq m' \)

\[
r(\lambda; m, m') - r(n - \lambda; m, m') = (n - 2\lambda) \int_{\partial X} e(\lambda; m)e(n - \lambda; m')
\]

which can be reformulated by

\[
R(\lambda) - R(n - \lambda) = (n - 2\lambda)^dE(\lambda)E(n - \lambda)
\]
considered as continuous operators from $\mathcal{C}^\infty(\tilde{X}, \Gamma^+_0)$ to $C^{-\infty}(\tilde{X}, \Gamma^+_0)$. From \((3.10)\), \((3.8)\) and \((3.10)\) it is straightforward to check that
\[
e(\lambda; y, m') = -\int_{\partial \tilde{X}} s(\lambda; y) e(n - \lambda; m')
\]
where $y \in \partial \tilde{X}$ and $m' \in X$. This identity can be expressed by
\[
(3.18) \quad E(\lambda) = -iS(\lambda)E(n - \lambda)
\]
considered as continuous operators from $\mathcal{C}^\infty(\tilde{X}, \Gamma^+_0)$ to $C^{-\infty}(\partial \tilde{X}, \Gamma^+_0)$. We deduce from \((3.17)\) and \((3.18)\) the weak identity \((3.16)\) in the open set of $\mathbb{C}$
\[
\{ \lambda \in \mathbb{C}; \Re(\lambda) < \frac{\mu}{2} \text{ and } \lambda, n - \lambda \notin (Z^1 \cup Z^2 \cup \Re) \}
\]
Moreover, according to \((3.17)\), the fact that $S(\lambda)$ is holomorphic on $\mathcal{L}(L^2(\partial \tilde{X}))$ in this open set and \((3.13)\), we have the desired holomorphic identity \((3.15)\).

Let $\lambda_0 \in \{ \Re(\lambda) < \frac{\mu}{2} \}$ and $N > |\Re(\lambda_0)| + n$, the identity \((3.14)\) holds near $\lambda_0$ with values in $\mathcal{L}(x^{-N}L^2(X), x^{-N}L^2(X))$. On $\Re(\lambda) < \frac{\mu}{2}$, we have seen that $R(n - \lambda)$, $E(n - \lambda)$ and $iE(n - \lambda)$ are finite-meromorphic with only poles the points $\lambda_0 \in \mathbb{C}$ such that $\lambda_0 - (n - \lambda_0) \in \sigma_{pp}(\Delta_y)$. One deduces that \((3.13)\) and \((3.15)\) prove \((2) \Rightarrow (1)\) and \((5) \Rightarrow (4)\).

\((1) \Rightarrow (2)\): let $\lambda_0 \in \{ \Re(\lambda) < \frac{\mu}{2} \}$ be a pole of $R(\lambda)$ and $U := B(\lambda_0, \epsilon) \subset \{ \Re(\lambda) < \frac{\mu}{2} \}$ be an open disc of $\mathbb{C}$ around $\lambda_0$ with radius $\epsilon$ taken sufficiently small to avoid other poles of $R(\lambda)$.

As claimed before, $S(\lambda)$ is holomorphic in $U$ with values in $\mathcal{L}(L^2(\partial \tilde{X}))$, more precisely it is a holomorphic family of pseudodifferential operators of negative order. In $U$, $\lambda_0$ is the only pole of $x^{-\lambda+\frac{\mu}{2}}R(\lambda)x^{-\lambda+\frac{\mu}{2}}$ defined as an operator of $\mathcal{L}(x^{2s}L^2(X), x^{-2s}L^2(X))$. We then have the expansion in $U$
\[
(3.19) \quad x^{-\lambda+\frac{\mu}{2}}R(\lambda)x^{-\lambda+\frac{\mu}{2}} = \sum_{i=-p}^{-1} (\lambda - \lambda_0)^i A_i + H(\lambda),
\]
\[
A_i \in \mathcal{L}(x^{2s}L^2(X), x^{-2s}L^2(X)), \quad H(\lambda) \in \mathcal{O}(U, \mathcal{L}(x^{2s}L^2(X), x^{-2s}L^2(X))).
\]

Moreover the Schwartz kernels $a_k$ of $A_k$ and $H(\lambda)$ can be described by an integral on the circle $C(\lambda_0, \frac{\mu}{2})$
\[
a_k = \frac{1}{2\pi i} \int_{C(\lambda_0, \frac{\mu}{2})} (z - \lambda_0)^{-k-1}(xx')^{-z+\frac{\mu}{2}}r(z)dz,
\]
\[
h(\lambda) = \frac{1}{2\pi i} \int_{C(\lambda_0, \frac{\mu}{2})} (\lambda - \lambda_0)^{-1}(xx')^{-z+\frac{\mu}{2}}r(z)dz, \quad |\lambda - \lambda_0| < \frac{\mu}{2}
\]

The structure of $r(\lambda)$ implies that $\beta^*(a_k)$ is the sum of a section in $\beta^*((xx')^{\frac{\mu}{2}}C^\infty(\tilde{X} \times \tilde{X}, \Gamma^+_0))$ and of smooth section on $(\tilde{X} \times \tilde{X}) \setminus \mathcal{F}$ which has a conormal singularity (not necessarily polyhomogeneous) on $\mathcal{F}$ of order $-2\Re(\lambda_0) + n - \epsilon$. Likewise $h(\lambda)$ is the sum of $(xx')^{-\lambda+\frac{\mu}{2}}r_0(\lambda)$ and a distribution $h_1(\lambda)$ whose lift $\beta^*(h_1(\lambda))$ has the same structure than $\beta^*(a_k)$ ($h_1(\lambda)$ is an integral like \((3.20)\) with $r_1(\lambda) + r_2(\lambda)$ instead of $r(\lambda)$).

Using the representation \((3.3)\) of $S(\lambda)$ by its Schwartz kernel $s(\lambda)$ and the fact that $s(\lambda)$ is, up to a smooth half-density section, a $L^1$ function on $\partial \tilde{X} \times \partial \tilde{X}$ for $\lambda \in U$, we find
\[
s(\lambda) = \sum_{k=-p}^{-1} \frac{(\lambda - \lambda_0)^k}{2\pi i} \int_{C(\lambda_0, \frac{\mu}{2})} \frac{s(z)}{(2z - n)(z - \lambda_0)^{k+1}}dz + \frac{1}{2\pi i} \int_{C(\lambda_0, \frac{\mu}{2})} \frac{s(z)}{(2z - n)(z - \lambda_0)}dz.
\]

Since $S(\lambda)$ is holomorphic on $\{ \lambda \in \mathbb{C}; 0 < |\lambda - \lambda_0| < \frac{\mu}{2} \}$ with values in $\mathcal{L}(L^2(\partial \tilde{X}))$, we have
\[
S(\lambda) = \sum_{k=-p}^{-1} \frac{(\lambda - \lambda_0)^k}{2\pi i} \int_{C(\lambda_0, \frac{\mu}{2})} \frac{S(z)}{(2z - n)(z - \lambda_0)^{k+1}}dz + \frac{1}{2\pi i} \int_{C(\lambda_0, \frac{\mu}{2})} \frac{S(z)}{(2z - n)(z - \lambda_0)}dz
\]
on the same set with values in $L(L^2(\partial \bar{X}))$. The second integral being holomorphic near $\lambda_0$, we conclude that $S(\lambda)$ admits a finite Laurent expansion at $\lambda_0$. (1) $\Rightarrow$ (2) is then proved.

(4) $\Rightarrow$ (5): following what we did before, it suffices to show that if the polar part of $R(\lambda)$ has a finite total rank then it is the same for $S(\lambda)$. Suppose where $A_i$ are some finite rank operators. The Schwartz kernel of $A$ is

$$a_i(x, y, x', y') = \sum_{j=1}^{r_i} \psi_{ij}(x, y) \varphi_{ij}(x', y') \left( \frac{dxdydx'dy'}{x^{n+1}x'^{n+1}} \right)^{\frac{1}{2}}, \quad \psi_{ij}, \varphi_{ij} \in x^{-2\epsilon}L^2(X, dvol_0)$$

$$\dim \text{Vect}\{\varphi_{ij}; j = 1, \ldots, r_i\} = \dim \text{Vect}\{\psi_{ij}; j = 1, \ldots, r_i\} = r_i = \text{rank } A_i,$$

Note that elliptic regularity implies that $\psi_{ij}$ and $\varphi_{ij}$ are smooth in $X$. Since $(\psi_{ij})_j$ are independent, one can easily see that there exist $r_i$ points $m_1, \ldots, m_{r_i} \in X$ such that the matrix $(M_{jk})_{j,k} := (\psi_{ij}(m_k))_{j,k}$ has rank $r_i$. Moreover

$$\phi_{ij}(x, y) := \sum_{j=1}^{r_i} \psi_{ij}(m_k) \varphi_{ij}(x, y) \in x^\frac{n}{2}C^\infty(\bar{X})$$

since $a_i \in (x^{n})^{\frac{n}{2}}C^\infty(\bar{X} \times \bar{X} \setminus \delta \bar{X}, \Gamma_0^\infty)$. But $(\phi_{ij})_{j=1, \ldots, r_i}$ is a basis of $\text{Vect}\{\varphi_{ij}; j = 1, \ldots, r_i\}$, hence

$$\varphi_{ij} \in x^\frac{n}{2}C^\infty(\bar{X}), \quad j = 1, \ldots, r_i$$

By the same arguments, the same result holds about $\psi_{ij}$ but with others $m_k \in X$. The restriction of $a_i$ on $x = x' = 0$ is then explicit and $S(\lambda)$ can be expressed by

$$S(\lambda) = (2\lambda - n) \sum_{i=-p}^{-1} (\lambda - \lambda_0)^i \sum_{j=1}^{r_i} \psi_{ij}^r \otimes \varphi_{ij}^r + H_1^r(\lambda)$$

where $\psi_{ij}^r, \varphi_{ij}^r \in C^\infty(\partial \bar{X}, \Gamma_0^\infty)$ are defined by

$$\psi_{ij}^r = \left( \psi_{ij} \left| \frac{dxdy}{x^{n+1}} \right|^{\frac{1}{2}} \right)_{x=0}, \quad \varphi_{ij}^r = \left( \varphi_{ij} \left| \frac{dxdy}{x^{n+1}} \right|^{\frac{1}{2}} \right)_{x=0}$$

and $H_1^r(\lambda)$ is holomorphic near $\lambda_0$ in $L(L^2(\partial \bar{X}))$.

(2) $\iff$ (3): it is sufficient to observe that $c(n - \lambda)\Lambda^{-\frac{n}{2}}$ and $\Lambda^{-\frac{n}{2}}$ with their inverse are meromorphic in $L(H^p(\partial \bar{X}), H^{p-N}(\partial \bar{X}))$ for all $p \in \mathbb{R}$ and $N > -\Re(\lambda) + \frac{n}{2}$, and conclude with (4.13) and Lemma 3.2.

(6) $\iff$ (7): assume that $\widetilde{S}(\lambda) = 1 + K(\lambda)$ is finite-meromorphic in $U$ with $K(\lambda)$ compact, analytic Fredholm theorem then proves that $\widetilde{S}^{-1}(\lambda) = \widetilde{S}(n - \lambda)$ is finite-meromorphic in $U$. The reverse is identical.

(6) $\Rightarrow$ (5): observe that $c(\lambda)\Lambda^{-\frac{n}{2}}$ and $\Lambda^{-\frac{n}{2}}$ are holomorphic with values in $L(L^2(\partial \bar{X}))$ and use (6.13).

(5) $\Rightarrow$ (6): if $U \cap Z^2 = \emptyset$, $c(n - \lambda)\Lambda^{-\frac{n}{2}}$ and $\Lambda^{-\frac{n}{2}}$ are holomorphic in $U$ with values in Sobolev spaces, so it remains to apply (6.13) and Lemma 3.2. \qed
Proof of Proposition 3.3 using Proposition 3.3 it suffices to study the meromorphic properties of $\tilde{S}(\lambda)$. Note that \((4.11)\) implies that $\tilde{S}(\lambda)$ is unitary on \(\{\Re(\lambda) = \frac{n}{2}\}\) (and $\lambda \neq \frac{n}{2}$), so $1 + K(\lambda)$ is invertible at one point of $O_\e$. The analytic Fredholm theorem allows to prove that
\[
\tilde{S}^{-1}(\lambda) = (1 + K(\lambda))^{-1}
\]
is a well defined finite-meromorphic family of operator in $O_\e \setminus (Z_+^2 \cup \Re)$. Moreover since $\tilde{S}(n-\lambda) = \tilde{S}(\lambda)^{-1}$ on \(\{\Re(\lambda) = \frac{n}{2}\}\), the meromorphic extension of $\tilde{S}(\lambda)$ is given by
\[
\tilde{S}(\lambda) := (1 + K(n-\lambda))^{-1} \in \text{Mer}_f \left\{ \{\Re(\lambda) < \frac{n}{2} + \epsilon\} \setminus (Z_+^2 \cup (n - \Re)), \mathcal{L}(L^2(\partial\tilde{X})) \right\}.
\]
It is clear that $(n - \Re) \cap \{\Re(\lambda) < \frac{n}{2}\}$ is the discrete spectrum, i.e. the points $\lambda_\e$ such that $\lambda_\e = \lambda_{n-\lambda_\e}$ is in $\sigma_{pp}(\Delta_g)$. If $\lambda_\e$ is one of these `eigenvalues' which is not in $Z_+^2$, then $n - \lambda_\e$ is a first order pole of finite multiplicity of $R(\lambda)$ and at most a first order pole of finite multiplicity of $K(\lambda)$ in view of $(3.11)$ and the fact that $S(\lambda)$ has at most some first order poles on $Z_+^2$. One concludes that $S(\lambda)$ is finite-meromorphic in $\{\Re(\lambda) < \frac{n}{2}\} \setminus Z_+^2$, hence by Proposition 3.3, $\tilde{R}(\lambda)$ is finite-meromorphic in the same open set.

Using Proposition 3.3 we deduce that $R(\lambda)$ is finite-meromorphic on $O_N$ if and only if the points of $Z_+^2 \cap O_N$ are some poles of finite total polar rank of $\tilde{S}(n-\lambda)$, or equivalently if and only if the points of $Z_+^2 \cap (n - O_N)$ are some poles of finite total polar rank of $K(\lambda)$. Let us show the following lemma which is a direct consequence of the results of Graham and Zworski [8].

**Lemma 4.1.** Let $k \in \mathbb{N}$, $(X, g)$ an asymptotically hyperbolic manifold and suppose that $g$ is an even metric modulo $O(x^{2k+1})$ which we write in the collar $U := (0, \epsilon_x) \times \partial\tilde{X}$
\[
g = x^{-2} \left( dx^2 + \sum_{i=0}^{k} h_{2i}x^{2i} + h_{2k+1}x^{2k+1} + O(x^{2k+2}) \right)
\]
with $(h_{2i})_{i=0,\ldots,k}$ and $h_{2k+1}$ some symmetric tensors on $\partial\tilde{X}$. Then for $j = 0,\ldots,k+1$ the points $\lambda_j := \frac{k+1}{2} + j$ are at most some first order poles of $S(\lambda)$ whose residue is given by
\[
\text{Res}_{\lambda_j} S(\lambda) = \Pi_{\lambda_j}, \quad j = 0,\ldots,k+1
\]
(4.2) \[
\text{Res}_{\lambda_k} S(\lambda) = \Pi_{\lambda_k} - \frac{(n - \lambda_k)}{4} \text{Tr}(h_0^{-1}h_{2k+1})
\]
\[
\text{Res}_{\lambda_{k+1}} S(\lambda) = \Pi_{\lambda_{k+1}} - p_{2k+3}
\]
where $\Pi_{\lambda_j}$ is the finite rank operator whose Schwartz kernel is
\[
\pi_{\lambda_j} := (2\lambda_j - n) \left((xx')^{-\lambda_j + \frac{1}{2}} \text{Res}_{\lambda_j}(\lambda) \right)_{|\partial\tilde{X} \times \partial\tilde{X}}, \quad j = 0,\ldots,k
\]
h$^{-1}_0$ is the metric induced by $h_0$ on $T^*\partial\tilde{X}$ and $p_{2k+3}$ is a differential operator of order 2 on $\partial\tilde{X}$ whose principal symbol vanishes identically only if
\[
(2 - n(n - \lambda_\e)) \text{Tr}(h_0^{-1}h_{2k+1}) = 0
\]
(4.3) \[
\text{Proof:} \text{ first note that for } m \in \partial\tilde{X} \text{ the tensors } h_i(m) \text{ will be considered as symmetric matrices in } \mathbb{R}^n \text{ via the Euclidean scalar product of the chart, } \text{Tr}(h_0^{-1}h_{2k+1}) \text{ can be understood in that way or by the trace of the linear operator associated to } h_{2k+1} \text{ via the scalar product } h_0 \text{ on } T\partial\tilde{X}. \text{ Now we use the construction of the Poisson operator according to [8]. The first step is to construct, for a given } f_0 \in C^\infty(\partial\tilde{X}), \text{ a solution } F_{\infty} \in C^\infty(\tilde{X}) \text{ of}
\]
\[
(\Delta_g - \lambda(n - \lambda))x^{n-\lambda}F_{\infty} = O(x^{\infty}), \quad F_{\infty}|_{x=0} = f_0
\]
(4.4) it is then clear that a solution in the collar $U_\e$ is sufficient (we can always multiply it by a smooth cut-off function with support near $\partial\tilde{X}$ and which is equal to 1 in a neighbourhood of $\partial\tilde{X}$). Let
we first have the identity
\[ (\Delta_y - \lambda(n - \lambda))x^{n-\lambda} = x^{n-\lambda}\mathcal{D}_\lambda \]
\[ \mathcal{D}_\lambda := -x^2\partial_x^2 + \left(2\lambda - n - 1 - \frac{x}{2}\text{Tr}(h^{-1}(x)\partial_x h(x))\right)x\partial_x - \frac{(n - \lambda)x}{2}\text{Tr}(h^{-1}(x)\partial_x h(x)) + x^2\Delta_h(x) \]
and for \( f \in C^\infty(\partial X) \) and \( j \in \mathbb{N}_0 \)
\[ \mathcal{D}_\lambda(f x^j) = j(2\lambda - n - j)f x^j + x^j\mathcal{G}(\lambda - j)f, \]
\[ (G(z)f)(x, y) := x^2\Delta_h(x)f(y) - \frac{(n - z)x}{2}\text{Tr}(h^{-1}(x)\partial_x h(x))f(y). \]

Suppose now that \( F \in C^\infty(X) \) is a function such that \( \mathcal{D}_\lambda F = O(x^j) \) (with \( j \geq 1 \)), then since \( G(z)f = O(x^j) \) for all \( f \in C^\infty(\partial X) \), (4.5) ensures that
\[ \mathcal{D}_\lambda F - \mathcal{D}_\lambda \left( \frac{x^{j}(x^{-j}\mathcal{D}_\lambda(F))|_{x=0}}{j(2\lambda - n - j)} \right) = O(x^{j+1}). \]

Let \( f_0 \in C^\infty(\partial X) \) fixed; since (4.5) implies that \( \mathcal{D}_\lambda f_0 = O(x) \), the previous remark allows us to construct the functions \( F_j \in C^\infty(X) \) (for \( j \geq 0 \)) and \( f_j \in C^\infty(\partial X) \) (for \( j \geq 1 \)) by the induction formula
\[ F_0 = f_0, \quad f_j = \frac{-(x^{-j}\mathcal{D}_\lambda(F_{j-1}))|_{x=0}}{j(2\lambda - n - j)}, \quad F_j = F_{j-1} + f_j x^j, \quad j \geq 1. \]

By construction, we obtain
\[ \mathcal{D}_\lambda(F_{j-1}) = O(x^j) \]
and according to Borel lemma, there exists a function \( F_\infty \in C^\infty(X) \) whose Taylor coefficients at \( x = 0 \) are \( f_j \), which gives that \( F_\infty \) is a solution for (4.4). In other respects, we can express \( f_j \) by
\[ f_j = p_{j, \lambda} f_0 \]
where \( p_{j, \lambda} \) is a differential operator of order \( \leq 2[j/2] \) on \( \partial X \).

Recall that Proposition 3.5 of \( \mathbb{R} \) proves that the residues of \( S(\lambda) \) at \( (\lambda_l)_{l=0,\ldots,k+1} \) are
\[ \text{Res}_{\lambda_l} S(\lambda) = \Pi_{\lambda_l} - p_{2l+1}, \quad p_{2l+1} := \text{Res}_{\lambda_l}(p_{2l+1, \lambda}). \]

Consequently, it remains to calculate \( (p_{2l+1})_{l=0,\ldots,k+1} \).

We will denote by \( D^j \) the set of differential operators of order \( j \) on \( \partial X \) and for notational simplicity \( D^j \) also means all differential operator of order \( j \) on \( \partial X \) that we do not need to know explicitly. Let us now set
\[ K := \text{Tr}(h_0^{-1}h_{2k+1}). \]

In the Taylor expansion of \( G(z) \) at \( x = 0 \), we use the assumption (4.11) and group the even powers of \( x \) together in \( G_2(z) \) and the odd powers of \( x \) together in \( G_1(z) \) to obtain \( G(z) = G_1(z) + G_2(z) \) with
\[ G_1(z) = -x^{2k+1}\left(\frac{(n - z)(2k + 1)}{2}\right)K + x^{2k+3} Q + O(x^{2k+5}), \]
\[ G_2(z) = x^2\Delta_h^0 + x^2D^0 + O(x^4), \]
\[ Q \in D^2, \quad \sigma_0(Q)(\xi) = \langle h_0^{-1}h_{2k+1}h_0^{-1}\xi, \xi \rangle, \]
where \( \sigma_0(Q) \) is the principal symbol (of order 2) of \( Q \). Hence, a first application is that for all \( f \in C^\infty(\partial X) \)
\[ \mathcal{D}_\lambda(x^{2j} f) \text{ is even modulo } O(x^{2k+2j+1}). \]

We then show by induction that \( F_j \) is even in \( x \) for \( j = 0, \ldots, 2k \). \( F_0 \) is even, suppose now that \( F_{j-1} \) is even for a fixed \( j \leq 2k - 1 \). If \( j \) is even, (4.6) clearly implies that \( F_j \) is even. On the other hand, \( F_{j-1} \) being even, (4.10) shows that \( \mathcal{D}_\lambda(F_{j-1}) \) is even modulo \( O(x^{2k+1}) \). So if \( j \) is odd, \( x^{-j}\mathcal{D}_\lambda(F_{j-1}) \) is odd modulo \( O(x^{2k+1-j}) \) and it vanishes at \( x = 0 \) since \( 2k+1-j \geq 2 \) by
assumption on $j$. From the definitions of $f_j$ and $F_j$ in (4.9), we finally obtain that $f_j = 0$ and that $F_j = F_{j-1}$ is even. As a conclusion $p_{2l+1, \lambda} = p_{2l+1} = 0$ for $l = 0, \ldots, k - 1$.

Now, to construct $f_{2k+1}$, remark (4.10) shows that the coefficient of order $2k + 1$ of $\mathcal{D}_\lambda F_{2k}$ is exactly the coefficient of order $2k + 1$ of $\mathcal{D}_\lambda f_0$, namely

$$-\frac{(n - \lambda)(2k + 1)}{2} K x^{2k+1} f_0.$$ 

One concludes that $p_{2k+1, \lambda} = \frac{1}{4}(n - \lambda)(\lambda - \lambda_k)^{-1} K$, hence its residue at $\lambda = \frac{n+1}{2} + k$ is

$$p_{2k+1} = \frac{(n - \lambda_k)}{4} K$$

and (4.7) is obtained using (4.10).

For the term $p_{2k+3, \lambda}$, we shall only study its principal symbol. To obtain $f_{2k+3}$, we need to evaluate the coefficient before $x^{2k+3}$ in $\mathcal{D}_\lambda\left(\sum_{i=0}^{2k+2} x^i f_i\right)$. But since $f_{2l+1} = 0$ for $l < k$ we can use (4.10) to check that the only terms having a non zero coefficient before $x^{2k+3}$ in $\mathcal{D}_\lambda(F_{2k+2})$ come from $\mathcal{D}_\lambda f_i$ with $i \in \{0, 2, 2k + 1\}$.

Consider now the three cases. According to (4.8), the term of order $x^{2k+3}$ in $\mathcal{D}_\lambda f_0$ is

$$Q f_0 x^{2k+3}.$$ 

The one in $\mathcal{D}_\lambda f_2$ is

$$-\frac{1}{2} (n + \lambda + 2)(2k + 1) K f_2 x^{2k+3} = \frac{(n - \lambda + 2)(2k + 1)}{2(2\lambda - n - 2)} K(\Delta_{h_0} + D^0) f_0 x^{2k+3}.\] Finally for $\mathcal{D}_\lambda f_{2k+1}$, the term of order $x^{2k+3}$ comes from $x^{2k+3} G_2(\lambda - 2k - 1) f_{2k+1}$, it is

$$(\Delta_{h_0} + D^0) f_{2k+1} x^{2k+3} = \frac{(n - \lambda)}{4(\lambda - \lambda_k)} (\Delta_{h_0} + D^0) f_0.$$ 

As before, let us set $\lambda_j := \frac{n+1}{2} + j$ for $j \in \mathbb{N}_0$. We then deduce that

$$f_{2k+3} = -\frac{1}{2(2k + 3)(\lambda - \lambda_{k+1})} \left( Q + \frac{(n - \lambda + 2)(2k + 1)}{2(2\lambda - n - 2)} K\Delta_{h_0} + \frac{(n - \lambda - \lambda_k)}{4(\lambda - \lambda_k)} \Delta_{h_0} K + D^1 \right) f_0.$$ 

Now taking the residue at $\lambda_{k+1}$ we find

$$p_{2k+3} = \frac{1}{2(2k + 3)} \left( Q + \frac{(n - \lambda_k)}{2} K\Delta_{h_0} + D^1 \right)$$

which is a differential operator with principal symbol

$$\sigma_0(p_{2k+3})(\xi) = -\frac{1}{2(2k + 3)} \left( -h_0^{-1} h_{2k+1}^{-1} \xi + \frac{(n - \lambda_k)}{2} K h_0^{-1} \xi, \xi \right).$$

If $\sigma_0(p_{2k+3}) = 0$ we have

$$h_0^{-1} h_{2k+1}^{-1} = \frac{(n - \lambda_k)}{2} K = 0,$$

so it remains to take the trace of this identity and find (4.9). \hfill \square

If $g$ is even modulo $O(x^{2k+1})$, the residue of $S(\lambda)$ at $\lambda_j$ has finite rank for $j = 0, \ldots, k - 1$ according to Lemma 3.3 so it is the very same thing for $\tilde{S}(\lambda)$ and Proposition 3.3 then proves that $R(\lambda)$ is finite-meromorphic near the points $(\frac{n+1}{2} - j)_{j=0, \ldots, k-1}$.

Conversely, if $R(\lambda)$ is finite-meromorphic near these points (with $k \geq 2$), Proposition 3.3 tells us that $\tilde{S}(\lambda)$ is finite-meromorphic near the points $(\lambda_j)_{j=0, \ldots, k-1}$. Assume that $g$ is even modulo $O(x^{2l+1})$ with $l \leq k - 2$, the residues of $\tilde{S}(\lambda)$ at $\lambda_l$ and $\lambda_{l+1}$ must be some finite rank operators. Following Lemma 4.11 this implies that $(n - \lambda_l)\text{Tr}(h_0^{-1} h_{2l+1}) = 0$ and $(2 - n(n - \lambda_l))\text{Tr}(h_0^{-1} h_{2l+1}) = 0$, thus $\text{Tr}(h_0^{-1} h_{2l+1}) = 0$. Taking the expression of the principal symbol of $p_{2l+3}$ in (4.11) yields $h_{2l+1} = 0$, so $g$ must be even modulo $O(x^{2l+3})$. Since $g$ is always even
modulo $O(x)$, an easy induction proves that $g$ is even modulo $O(x^{2k-1})$ and the proof of Proposition 1.3 is achieved.

**Remark**: notice that the same kind of arguments (or using Joshi-Sá Barreto [14] Th. 1.2 formula) prove that the set of residues of $S(\lambda)$ on $(\Delta_{2k})_{k \in \mathbb{N}}$ determines the Taylor expansion of the metric at the boundary.

Observe that a choice of even metric modulo $O(x^{2k+1})$ satisfying
\[
\text{meas}\{\text{Tr}(h_0^{-1}h_{2k+1}) = 0\} = 0, \quad \lambda_k \neq n
\]
with the notation of (4.11) gives a residue of $S(\lambda)$ (and $\tilde{S}(\lambda)$) at $\lambda_k$ which is injective modulo the projection on the $L^2$-eigenspace. Let us show that $\tilde{S}^{-1}(\lambda)$ must have an essential singularity at $\lambda_k$ in this case.

**Lemma 4.2.** Let $\mathcal{B}$ a Banach space of infinite dimension, $\lambda_0 \in \mathbb{C}$ and $U$ a neighbourhood of $\lambda_0$. Let $M(\lambda) \in \mathcal{H}o\ell(U \setminus \{\lambda_0\}, \mathcal{L}(\mathcal{B}))$ a meromorphic family of bounded operators in $U$ which satisfies
\[
(4.12) \quad M(\lambda) = 1 + \frac{K_{-1}}{\lambda - \lambda_0} + K(\lambda), \quad K(\lambda) \in \mathcal{H}o\ell(U, \mathcal{L}(\mathcal{B})),
\]
where $K_{-1}$ and $K(\lambda)$ are compact and
\[
\dim \ker K_{-1} < \infty.
\]
If there exists $z \in U$ such that $M(z)$ is invertible, then $M(\lambda)$ is invertible for almost every $\lambda \in U$ with inverse $M^{-1}(\lambda)$ finite-meromorphic in $U \setminus \{\lambda_0\}$ and $\lambda_0$ is an essential singularity of $M^{-1}(\lambda)$.

**Proof**: to simplify, we take $\lambda_0 = 0$. The fact that $M(\lambda)$ is invertible almost everywhere in $U$ with inverse finite-meromorphic in $U \setminus \{0\}$ is a consequence of analytic Fredholm theorem. Assume now that $M^{-1}(\lambda)$ has a finite Laurent expansion at 0
\[
M^{-1}(\lambda) = \sum_{i=-p}^{\infty} N_i \lambda^i, \quad p \geq 0.
\]
We can take the Laurent expansion of $M(\lambda)$ at 0
\[
M(\lambda) = \sum_{i=-1}^{\infty} M_i \lambda^i = K_{-1} \lambda^{-1} + 1 + K_0 + \sum_{i=1}^{\infty} K_i \lambda^i,
\]
where $K_i$ is compact, and make the product
\[
M(\lambda)M^{-1}(\lambda) = \sum_{i=-p-1}^{\infty} \lambda^i \sum_{j+k=i} M_j N_k = 1,
\]
which leads to the system
\[
(4.13) \quad \sum_{j=-1}^{i+p} M_j N_{i-j} = \delta_{0i}, \quad i \geq -p - 1.
\]
Let us show by induction that $N_i$ has finite rank for $i \leq 0$. Taking equation (4.13) with $i = -p - 1$ yields $K_{-1} N_{-p} = 0$, and by assumption on $K_{-1}$ we find that $N_{-p}$ has finite rank. Let $I \leq -1$, suppose now that $N_i$ has finite rank for all $i \leq I$ and let us prove that $N_{I+1}$ has finite rank. For $i = I$, equation (4.13) implies that
\[
K_{-1} N_{I+1} = - \sum_{j=0}^{I+p} M_j N_{I-j}
\]
has finite rank by induction assumptions (since \(I - j \leq I\)). Let \(r\) an integer such that

\[
    r > \dim \ker K_{-1} + \dim \text{Im} K_{-1} N_{t+1}
\]

If \(N_{t+1}\) has infinite rank, there exists a family of independent vectors \((\varphi_i)_{i=1,\ldots,r}\) in \(\text{Im}(N_{t+1})\) and the restriction of \(K_{-1}\) on \(E := \bigoplus_{i=1}^{r} \mathbb{C}\varphi_i\) is a linear map on a vector space of finite dimension satisfying

\[
    \dim \ker K_{-1}|_E + \text{rank } K_{-1}|_E < \dim E = r
\]

which is not possible. We deduce that \(N_{t+1}\) has finite rank.

Take \(I_{\lambda_k}\) with \(i = 0\): \(K_{-1}\) and \((N_i)_{i \leq 0}\) being compact, it is clear that

\[
    1 = K_{-1} N_1 + \sum_{j=0}^{p} M_j N_{-j}
\]

is compact, which is not possible. \(\square\)

**Corollary 4.3.** Let \(k \in \mathbb{N}_0\) and \((X, g)\) an asymptotically hyperbolic manifold and \(g\) even modulo \(O(x^{2k+1})\) that we write as in (4.14). If \(k \neq \frac{n-1}{2}\) and

\[
    \text{meas}\{ \text{Tr}(h_0^{-1} h_{2k+1}) = 0\} = 0,
\]

then \(n - \lambda_k = \frac{n-1}{2} - k\) is an essential singularity of \(R(\lambda)\).

**Proof:** it suffices to combine Lemma 4.1 and Lemma 4.2 and remark that the multiplication by a smooth function on \(L^2(\partial X)\) is injective if the measure of its zeros vanishes. To show that the residue of \(S(\lambda)\) at \(\lambda_k\) has a kernel of finite dimension, we use that the sum of a bounded injective operator and a finite rank operator has a finite dimensional kernel. \(\square\)

For \(k = 0\), \(m(x, g) := (2n)^{-1} \text{Tr}(h_0^{-1} h_1)\) is exactly the mean curvature of \(\partial X\) in \((\bar{X}, x^2 g)\), it is a smooth function on \(\partial X\) which depends on \(x\). However it can be defined invariantly with respect to \(x\) as a smooth section \(m(g)\) of the conormal bundle \(|N^* \partial X|\). In other words, a new choice of boundary defining function \(t = e^{\omega} x\) (with \(\omega \in C^\infty(\bar{X})\)) gives \(m(t, g) = e^{-\omega} m(x, g)\) where \(\omega_0 := \omega|_{\partial X}\). If the mean curvature is almost everywhere non zero and \(n \neq 1\), the corollary claims that \(\frac{n-1}{2}\) is an essential singularity of \(R(\lambda)\).

Recall that \(\mathcal{M}_{ah}(X)\) is the space of asymptotically hyperbolic metrics on \(X\). If \(x_0\) is a fixed boundary defining function, the map

\[
\begin{align*}
    \{ M_{ah}(X) & \to \{ G \in C^\infty(\bar{X}, S^2_1(T^* \bar{X})); |dx_0|_{G} = 1 \text{ on } \partial X \} \\
    g & \to G_0 \} = x_0^2 g
\end{align*}
\]

is bijective and we shall identify these two spaces. \(\mathcal{M}_{ah}(X)\) inherits its \(C^\infty\) topology from \(C^\infty(\bar{X}, T^* \bar{X} \otimes T^* \bar{X})\) which is defined as usual by semi-norms \((N_i)_{i \in \mathbb{N}}\) measuring the derivatives of the tensors (\(\bar{X}\) is compact). It is not difficult to see that the mean curvature \(m(\cdot)\) is continuous from \(\mathcal{M}_{ah}(X)\) to \(C^\infty(\partial X, |N^* \partial X|)\).

**Proof of Theorem 1.5** according to Thom theorem, the set of Morse functions on \(\partial \bar{X}\) is open and dense in \(C^\infty(\partial \bar{X})\) and it is exactly the same for the Morse sections of \(C^\infty(\partial X, |N^* \partial X|)\). Let us denote by \(V \) this subset of \(C^\infty(\partial \bar{X}, |N^* \partial X|)\). If \(s \in V\) then \(\text{meas}(s^{-1}(0)) = 0\), hence \(m^{-1}(V)\) is an open set of \(\mathcal{M}_{ah}(X)\) contained in the set of metrics in \(\mathcal{M}_{ah}(X)\) for which \(R(\lambda)\) has an essential singularity at \(\frac{n-1}{2}\). It remains to show that \(m^{-1}(V)\) is dense in \(\mathcal{M}_{ah}(X)\).

We first check that \(m(\cdot)\) is a surjective map. Let \(g_0 \in \mathcal{M}_{ah}(X)\) and take its model form

\[
    g_0 = x^{-2}((dx^2 + h_0 + h_1 x + O(x^2)),
\]

where \(x \in Z(\partial \bar{X})\). Let

\[
    g_{x, \varphi} := g_0 + x^{-1} h_0 \varphi(\epsilon^{-1} x),
\]

for \(\epsilon > 0\) and \(\varphi \in C^\infty(\bar{X})\) with \(\varphi(0) = 0\).
where \( \epsilon > 0, \varphi \in C^\infty(\partial \bar{X}) \) and \( \chi \in C^\infty_0([0,2]) \) such that \( \chi(t) = 1 \) if \( t \leq \frac{1}{2} \) and \( \chi(t) = 0 \) if \( t \geq 1 \). \( g_{r,\varphi} \) is an asymptotically hyperbolic metric if \( \epsilon \) is taken sufficiently small (but depending on \( \sup_{\partial \bar{X}} |\varphi| \)). Nevertheless \( \epsilon \) can be chosen independent with respect to \( \varphi \) if \( |\varphi| \leq 1 \), we will denote it \( \epsilon_0 \). It is straightforward to see that

\[
m(g_{r,\varphi}) = m(g_0) + \varphi|dx|,
\]

hence each section \( f|dx| \) of \( C^\infty(\partial \bar{X}, [N^* \partial \bar{X}]) \) can be written \( m(g_{r,\varphi}) \) by taking \( \varphi := f - m(g_0)|dx|^{-1} \) (with notation \( (\ref{eq:4.15}) \)).

Let \( g_0 \in M_{ah}(\bar{X}), \psi := m(g_0), \epsilon_0 \) defined as before and \( B(g_0) := \cap_{i \in I} B_i(g_0, r_i) \) a finite intersection in \( M_{ah}(\bar{X}) \) of ‘open balls’ around \( g_0 \) with radius \( r_i \) for the semi-norms \( N_i \). Let \( I_0 \) be the largest number of derivatives of the metric measured by the semi-norms \( (N_i)_{i \in I} \). We set \( W(\psi_0) \) a finite intersection of ‘open balls’ around \( \psi_0 \) for some semi-norms of \( C^\infty(\partial \bar{X}, [N^* \partial \bar{X}]) \) which control the \( I_0 \) first derivatives of the section on \( \partial \bar{X} \). The radius of these balls can be chosen sufficiently small (depending on \( (r_i)_{i \in I} \)) such that for all section \( s \in W(\psi_0) \) the tensor \( g_{r_0,\varphi} \) defined in \( (\ref{eq:4.15}) \) with \( \varphi := (s - \psi_0)|dx|^{-1} \) lies in \( B(g_0) \), in other words the function \( \varphi \to g_{r_0,\varphi} \) is continuous in a neighbourhood of \( 0 \). Note that \( g_{r_0,\varphi} \) is a metric because we are in the case where \( \sup_{\partial \bar{X}} |\varphi| \) approaches \( 0 \) and we can suppose \( |\varphi| \leq 1 \). Since \( V \) is dense in \( C^\infty(\partial \bar{X}, [N^* \partial \bar{X}]) \), one can find a Morse section \( s \) in the neighbourhood \( W(\psi_0) \) of \( \psi_0 \) such that \( m(g_{r_0,\varphi}) = s \), there exists a metric in \( m^{-1}(V) \cap B(g_0) \). We conclude that \( m^{-1}(V) \) is dense in \( M_{ah}(\bar{X}) \) and the proof is achieved.

Concerning \( n - \lambda_k = \frac{4\pi}{\omega} - k \) with \( k > 0 \), we can show the same result by using the residue calculus of \( S(\lambda) \) at \( \lambda_k \) in Lemma \( (\ref{lem:4.1}) \) for even metrics modulo \( O(x^{2k+1}) \).

**Remark:** assume that \( \partial \bar{X} \) is connected and that there exists an analytic neighbourhood of \( \partial \bar{X} \) in \( \bar{X} \), hence we can take a boundary defining function \( x \) which is analytic near \( \partial \bar{X} \). If \( x^2g \) is analytic for \( g \in M_{ah}(\bar{X}) \), then \( m(g) \) is analytic on \( \partial \bar{X} \) and

\[
\text{meas}\{ m(g) = 0 \} > 0 \iff m(g) = 0.
\]

If \( n \geq 2 \) and \( m(g) \) is not identically zero, \( n - \lambda_0 = \frac{n-1}{2} \) is an essential singularity of \( R(\lambda) \) according to Corollary \( (\ref{cor:4.3}) \). On the other hand if \( m(g) = 0 \), the arguments preceding Lemma \( (\ref{lem:4.1}) \) prove that \( R(\lambda) \) is finite-meromorphic near \( n - \lambda_0 \). Consequently, we obtain that

\[
R(\lambda) \text{ is meromorphic near } n - \lambda_0 \iff \partial \bar{X} \text{ is minimal in } (\bar{X}, x^2g),
\]

where the minimality of \( \partial \bar{X} \) does not depend on the choice of \( x \).

**A particular case:** suppose now that \( n = 1 \), \( g \in M_{ah}(X) \) analytic near \( \partial \bar{X} \) with \( \partial \bar{X} \) connected. \( S(\lambda) \) is holomorphic at \( \lambda_0 = 1 \) according to Lemma \( (\ref{lem:4.1}) \) and the fact that \( 0 \notin \sigma_{pp}(\Delta_g) \). Furthermore, there exists an analytic boundary defining function \( x \) such that

\[
(4.16) \quad x^2g = dx^2 + \sum_{i=0}^{\infty} h_i x^i, \quad h_i \in C^\infty(\partial \bar{X}, S^2(T^* \partial \bar{X}))
\]

near \( \partial \bar{X} \). If \( g \) is even modulo \( O(x^{2k+1}) \) with \( k \geq 1 \), the previous remark and Lemma \( (\ref{lem:4.1}) \) imply that \( R(\lambda) \) is meromorphic on \( \mathbb{C} \setminus \cup_{i\geq k} \{n - \lambda_i\} \) if and only if \( h_{2k+1} = 0 \). If we consider the space of even metric modulo \( O(x^3) \) (i.e. such that \( \partial \bar{X} \) is a geodesic of \( (\bar{X}, x^2g) \)) we obtain by induction that \( R(\lambda) \) is meromorphic if and only if \( g \) is even and in that case the poles have finite multiplicity.

5. **Examples with accumulation of resonances**

Before giving the example of Proposition \( (\ref{prop:1.6}) \), it is useful to remark that we can easily find an asymptotically hyperbolic manifold such that the term \( (\ref{eq:4.2}) \) is a constant (not \( 0 \)). In that case, Lemma \( (\ref{lem:4.1}) \) implies that \( S(\lambda) \) has a first order pole of infinite multiplicity at \( \lambda_k = \frac{n-1}{2} + k \) and
the residue is a constant. Therefore, after renormalization, \( S(\lambda) \) can be expressed near \( \lambda_k \) by

\[
\tilde{S}(\lambda) := 1 + c_k \frac{\lambda^{1-2k}}{\lambda - \lambda_k} + H(\lambda)
\]

with \( c_k \neq 0 \) and \( H(\lambda) \) holomorphic compact. If \((\phi_j)_{j \in \mathbb{N}}\) is an orthonormal basis of eigenfunctions of \( \Delta \) on \( L^2(\partial X) \) and \((\alpha_j)_{j \in \mathbb{N}}\) the associated eigenvalues, \( H(\lambda)\phi_j \) converges strongly to 0 when \( j \to \infty \), thus for a large fixed \( j \), \( H(\lambda)\phi_j \) becomes insignificant in the expression of \( \tilde{S}(\lambda)\phi_j \) when \( \lambda \) is close to \( \lambda_k \). If \( H(\lambda) \) was null, we would have the following formula for the inverse of \( \tilde{S}(\lambda) \)

\[
(\tilde{S}(\lambda))^{-1} = \sum_{j \in \mathbb{N}} \frac{\lambda - \lambda_k}{\lambda - \lambda_k + c_k\alpha_j - \lambda^{1-2k}} \phi_j(.,\phi_j).\]

In other words, \((1 + c_k \frac{\lambda^{1-2k}}{\lambda - \lambda_k})^{-1}\) has a sequence of poles \( z_j = \lambda_k - c_k\alpha_j^{-1-2k} \) which converges to \( \lambda_k \). As a conclusion, the functional equation (3.1) could be used to argue that \( \tilde{S}(\lambda) \) has a sequence of poles which converge to \( n - \lambda_k \). The key to control the ‘little’ perturbation \( H(\lambda) \) will be Rouche’s theorem.

**Proof of Proposition 1.6** let \( k \in \mathbb{N}_0 \) such that \( 2k \neq n - 1 \) and the collar \( U := (0, 2) \times S^n \) which carries the metric

\[
g := x^{-2}(dx^2 + d(x)h_0),
\]

\[
d(x) := \left(1 - \frac{x^2}{4}\right)^2 + \chi(x)x^{2k+1},
\]

where \( h_0 := g_{S^n} \) is the canonical metric on the n-dimensional sphere \( S^n \), and \( \chi \in C^\infty([0, 2]) \) a non negative function such that \( \chi(x) = 1 \) for \( x \in [0, \frac{1}{2}] \) and \( \chi(x) = 0 \) for \( x \in [1, 2] \). Set \( B_{n+1} := \{ m \in \mathbb{R}^{n+1} : |m| < 1 \} \) and the diffeomorphism

\[
S^n \times (0,2) \rightarrow \psi^{-1} B_{n+1} \setminus \{0\}
\]

\[
(\omega, x) \rightarrow \frac{2 - x}{2 + x}.
\]

It is straightforward to see that \( \psi^*g \) can be extended smoothly on \( B_{n+1} \) (it is actually the hyperbolic metric \( 4|dm|^2(1 - |m|^2)^2 \) in \( \{|m| \leq \frac{1}{4}\} \)). Set \( (X,G) \) the obtained asymptotically hyperbolic manifold, which clearly satisfies the assumptions 4.1: we get \( \mathbb{H}^{n+1} \) perturbed by a \( O(x^{2k+1}) \) near the boundary \( S^n \).

We then obtain

\[
\text{Tr}(h_0^{-1}h_{2k+1}) = n
\]

and we have the following expression for the Laplacian in the collar \( U \)

\[
\Delta_g = -x^2\partial_x^2 + (n - 1)x\partial_x + \frac{x^2}{d(x)}\Delta_{h_0} - \frac{n}{2}\frac{d'(x)}{d(x)}x^2\partial_x.
\]

Lemma 4.1 implies that the scattering operator \( S(\lambda) \) associated to \( \Delta_G \) has a first order pole at \( \lambda_k \) with residue

\[
\text{Res}_{\lambda_k} S(\lambda) = -\frac{n(n - \lambda_k)}{4} + \Pi_{\lambda_k},
\]

where \( \Pi_{\lambda_k} \) is a finite rank operator on \( L^2(S^n) \) whose structure can be detailed

\[
\Pi_{\lambda_k} = \sum_{l=1}^{m} \varphi_l \otimes \varphi_l, \quad \varphi_l \in C^\infty(S^n) \subset L^2(S^n).
\]

Let \((v_j)_{j \in \mathbb{N}}\) the eigenvalues of \( \Delta_{h_0} \) (repeated with multiplicity) and \((\phi_j)_{j \in \mathbb{N}}\) the associated orthonormal basis of eigenfunctions. Since the metric \( G \) is radial on \( B_{n+1} \), the Laplace operator
We deduce that
\[ \Delta_G \simeq \bigoplus_{j \in \mathbb{N}} P_j, \quad P_j := -x^2 \partial_x^2 + (n - 1)x \partial_x + \frac{x^2}{d(x)} \nu_j - \frac{n \, d'(x)}{2 \, d(x)} x^2 \partial_x \]
onumber
on
\[ L^2(X, dvol_G) \simeq \ell^2 \left( \mathbb{N}, \left[ 0, 2 \right], \frac{d(x)^\frac{n}{2}}{x^{n+1}} \, dx \right) \]
onumber
with singular Dirichlet condition at \( x = 2 \). We deduce that the resolvent and the scattering operator can also be decomposed into a direct sum
\[ R(\lambda) = \bigoplus_{j \in \mathbb{N}} (P_j - \lambda(n - \lambda))^{-1}, \quad S(\lambda) = \bigoplus_{j \in \mathbb{N}} S_j(\lambda). \]
On the other hand, recall that the expression of the principal symbol of \( S(\lambda) \) given in (3.10) allows to factorize
\[ c(n - \lambda) \Lambda^{-\lambda + \frac{n}{2}} S(\lambda) \Lambda^{-\lambda + \frac{n}{2}} = 1 + K(\lambda), \]
where \( K(\lambda) \) is a meromorphic family of compact operators on \( L^2(S^n) \). Given the expression of the residue (5.1), we obtain for \( \lambda \) in a neighbourhood \( V_k \) of \( \lambda_k \)
\[ 1 + K(\lambda) = 1 + \frac{K_{\lambda_k}}{\lambda - \lambda_k} + H(\lambda), \]
\[ K_{\lambda_k} := c(n - \lambda_k) \left( -\frac{n(n - \lambda_k)}{4} \Lambda^{-1-2k} + \Lambda^{-\frac{n}{2}-k} \Pi_{\lambda_k} \Lambda^{-\frac{n}{2}-k} \right), \]
(5.2)
where \( H(\lambda) \in \mathcal{Hol}(V_k, \mathcal{L}(L^2(S^n), H^{1-\epsilon}(S^n))) \), \( \forall \epsilon > 0 \), using the decomposition of \( S(\lambda) \) on the orthonormal basis \( (\phi_j)_{j \in \mathbb{N}} \) of \( L^2(S^n) \), we have
\[ K(\lambda) \phi_j = K_j(\lambda) \phi_j, \quad K_j(\lambda) := \langle K(\lambda) \phi_j, \phi_j \rangle, \]
\[ H(\lambda) \phi_j = H_j(\lambda) \phi_j, \quad H_j(\lambda) := \langle H(\lambda) \phi_j, \phi_j \rangle, \]
with \( H_j(\lambda) \) holomorphic on \( V_k \) and satisfying on \( V_k \)
\[ |H_j(\lambda)| \leq \|\Lambda^{-\frac{n}{2}} \Lambda^{\frac{n}{2}} H(\lambda) \phi_j(\phi_j)\| \]
\[ \leq \|\Lambda^{\frac{n}{2}} H(\lambda) \phi_j\| \alpha_j^{\frac{n}{2}} \leq C \alpha_j^{\frac{n}{2}}, \]
where \( \alpha_j := (1 + v_j)^{-\frac{n}{2}} \) converge to 0 when \( j \to \infty \) and \( C > 0 \). Observe that (5.2) and the bound of \( \|\Lambda^{\frac{n}{2}} H(\lambda)\| \) are direct consequences of (3.12).

In the same way, \( \mathbb{C} \phi_j \) is globally fixed by \( \Pi_{\lambda_k} \)
\[ \Pi_{\lambda_k} \phi_j = \beta_j \phi_j, \quad \beta_j := \langle \Pi_{\lambda_k} \phi_j, \phi_j \rangle = \sum_{i=1}^{m} \langle \varphi_i, \phi_j \rangle \langle \varphi_i, \varphi_i \rangle_{j \to \infty} \to 0. \]
We deduce that
\[ 1 + K_j(\lambda) = 1 + \frac{m_k}{\lambda - \lambda_k} \alpha_j^{1+2k} + H_j(\lambda), \]
\[ m_k := c(n - \lambda_k) \left[ \frac{n(n - \lambda_k)}{4} + \beta_j \right]. \]
Remark that if \( k > \frac{n-1}{2} \), then \( \Pi_{\lambda_k} = 0 \) since \( \lambda_k(n - \lambda_k) = \frac{n^2}{4} - (\frac{1}{2} + k)^2 \notin \sigma_{pp}(\Delta_G) \). We check that if \( \beta_j = 0 \) then \( m_k \neq 0 \), thus there exists \( J \in \mathbb{N} \) such that for \( j \geq J \) we have \( m_k \neq 0 \) since \( \beta_j \to 0 \) when \( j \to \infty \). Consequently, we obtain an explicit expression to inverse \( S_j(\lambda) \) for \( j \geq J \) and \( \lambda \in V_k \)
\[ S_j(\lambda)^{-1} = c(n - \lambda) \alpha_j^{2\lambda-n} \frac{\lambda - \lambda_k}{(\lambda - \lambda_k)(1 + H_j(\lambda)) + m_k \alpha_j^{1+2k}} \]
if the denominator is not 0.
Let us choose \( \epsilon > 0 \) such that the disc with centre \( \lambda_k \) and radius \( \epsilon \) is included in \( V_k \) and set \( z_j := \lambda_k - m_k \alpha_j^{1+2k} \). There exists an integer \( J_0 \geq J \) such that the circle \( C(z_j, \frac{\epsilon}{2}) \) with centre \( z_j \)
and radius $\frac{1}{2}$ is included in $V_k$ since $z_j \to \lambda_k$ when $j \to \infty$. Since $J_0$ can be chosen as large as we want, it is not restrictive to suppose that $|m_k\alpha_j^{1+2k}| \leq \epsilon$, so set

$$\epsilon_j = \frac{|z_j - \lambda_k|}{2} = \frac{|m_k\alpha_j^{1+2k}|}{2}$$

and both following holomorphic functions in $V_k$

$$f_j(\lambda) := \lambda - z_j = \lambda - \lambda_k + m_k\alpha_j^{1+2k}, \quad g_j(\lambda) := (\lambda - \lambda_k)H_j(\lambda)$$

$f_j(\lambda)$ has a unique zero $z_j$, and we have on the circle $C(z_j, \epsilon_j)$ with centre $z_j$ and radius $\epsilon_j$

$$|f_j(\lambda)| = \epsilon_j, \quad \forall \lambda \in C(z_j, \epsilon_j)$$

with $C > 0$ which does not depend on $j$. We then deduce that there exists an integer $J_1 \geq J_0$ such that for all $j \geq J_1$

$$|g_j(\lambda)| < |f_j(\lambda)|, \quad \forall \lambda \in C(z_j, \epsilon_j)$$

what ensures, by Rouché’s theorem, that $f_j + g_j$ has exactly one zero in the disc whose boundary is $C(z_j, \epsilon_j)$. But $(f_j + g_j)(\lambda_k) = m_k\alpha_j^{1+2k} \neq 0$ thus $S_j(\lambda)^{-1}$ has a unique pole in the disc whose boundary is $C(z_j, \epsilon_j)$ and $S(\lambda)^{-1}$ has a sequence of poles converging to $\lambda_k$. By the inversion formula $S(n - \lambda) = S(\lambda)^{-1}$, we deduce that there exists a sequence of resonances approaching $\frac{n-1-k}{\alpha}$.

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