SECOND PHASE TRANSITION LINE

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Abstract. We study the phase transition line of the almost Mathieu operator, that separates arithmetic regions corresponding to singular continuous and a.e. pure point regimes, and prove that both purely singular continuous and a.e. pure point spectrum occur for dense sets of frequencies.

1. Main results

In systems with phase transitions the interface between the two phases often exhibits the critical phenomena and is the most difficult set of parameters to study. At the same time, the insights on the critical case often shed light on the creation, dissipation, and the mechanism behind both phases. In this paper we study the critical regime for the hyperbolic almost Mathieu operator.

It has been known since the work of Aubry-Andre [1] that the almost Mathieu family:

\[(H_{\lambda,\alpha,\theta}u)_n = u_{n+1} + u_{n-1} + 2\lambda \cos 2\pi(n\alpha + \theta)u_n,\]

where \(\theta \in \mathbb{R}\) is the phase, \(\alpha \in \mathbb{R}\backslash \mathbb{Q}\) is the frequency and \(\lambda \in \mathbb{R}\) is the coupling constant, undergoes a phase transition at \(\lambda = 1\), where the Lyapunov exponent changes from zero everywhere on the spectrum [11] to positive everywhere on the spectrum [13]. Aubry-Andre conjectured [1], that at \(\lambda = 1\) the spectrum changes from absolutely continuous for \(\lambda < 1\) to pure point for \(\lambda > 1\). This has since been proved, for all \(\alpha, \theta\) for \(\lambda < 1\) [2, 4, 6, 24] and for Diophantine \(\alpha, \theta\) (so a.e.) for \(\lambda > 1\) [16]. The “a.e.” cannot be removed as in the hyperbolic regime, while there is no absolutely continuous spectrum, the distinction between singular continuous and pure point depends in an interesting way on the arithmetics of \(\alpha, \theta\). The relevant issue is an interplay between the rate of exponential growth of the transfer-matrix cocycle and the depth of the small denominators: the exponential rate of approximation of \(\alpha\) by the rationals. A conjecture dating back to 1994 [15] (see also [17]) was that for a.e. \(\theta\) there is another transition, from singular continuous to pure point spectrum, precisely where the two rates compensate each other. Namely, for \(\alpha \in \mathbb{R}\backslash \mathbb{Q}\) with continued fraction approximants \(\frac{p_n}{q_n}\), let

\[\beta(\alpha) := \limsup_{n \to \infty} \frac{\ln q_{n+1}}{q_n}.\]
\( \beta(\alpha) \) measures how exponentially Liouvillean \( \alpha \) is. Then at \( \lambda = e^\beta \) there is, for a.e. \( \theta \), a transition from singular continuous to pure point spectrum.\(^1\) This was proved recently in [9] by an improvement of the Gordon-type method for the singular continuous region and reducibility as a corollary of subcriticality and duality for the pure point one. Moreover, the arithmetic version for the pure point region (specifying the a.e. \( \theta \) as phases \( \theta \) that are \( \alpha \)-Diophantine) was established in [19], through a constructive proof of localization, that demonstrated also the continued fraction expansion driven universal hierarchical structure of corresponding eigenfunctions. Finally, for \( \lambda = 1 \) there is singular continuous spectrum except for an explicit full measure set of \( \theta \) (namely, for \( \theta \)'s which are not rational with respect to \( \alpha \)) [7, 8, 12, 25]. We can summarize those known results in

**Theorem 1.1.** Let \( \alpha \in \mathbb{R}\setminus\mathbb{Q} \), then we have the following:

1. If \( |\lambda| < 1 \), then \( H_{\lambda,\alpha,\theta} \) has purely absolutely continuous spectrum for all \( \theta \).
2. If \( |\lambda| = 1 \), then \( H_{\lambda,\alpha,\theta} \) has purely singular continuous spectrum for (explicit) a.e. \( \theta \).
3. If \( 1 \leq |\lambda| < e^\beta \), then \( H_{\lambda,\alpha,\theta} \) has purely singular continuous spectrum for all \( \theta \).
4. If \( |\lambda| > e^\beta \), then \( H_{\lambda,\alpha,\theta} \) has purely point spectrum with exponentially decaying eigenfunctions for (explicit) a.e. \( \theta \).

Therefore, as far as a.e. \( \theta \) is concerned only the phase transition case \( |\lambda| = e^\beta, 0 < \beta < \infty \), is missing. This is what we want to address in this paper. We will show

**Theorem 1.2.** Let \( 0 < \beta < \infty \). Then

1. There exists a dense set of \( \alpha \) with \( \beta(\alpha) = \beta \) such that \( H_{\lambda,\alpha,\theta} \) with \( \lambda = e^{\beta(\alpha)} \), has purely singular continuous spectrum for all \( \theta \).
2. There exists a dense set of \( \alpha \) with \( \beta(\alpha) = \beta \) such that \( H_{\lambda,\alpha,\theta} \) with \( \lambda = e^{\beta(\alpha)} \), has pure point spectrum for a.e. \( \theta \).

**Remark 1.1.**

1. We note that \( \lambda = e^{\beta(\alpha)} \) implies that there are no exponentially decaying eigenfunctions [19]. Thus part (2) provides an example of nonexponentially decaying eigenfunctions in the regime of positive Lyapunov exponents.
2. The “a.e.” in part (2) cannot be improved to “all” [20].
3. As usual, \( \ln \lambda \) can be viewed as a shortcut for Lyapunov exponent \( L(E) \) on the spectrum of the almost Mathieu operator, so it is natural to conjecture that for general analytic potentials there will be a transition at \( L(E) = \beta(\alpha) \). Indeed, singular continuous spectrum

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\(^1\)The exclusion of a measure zero set is not needed for the singular continuous part, but is necessary for the pure point part [20].
does hold throughout the $L(E) < \beta(\alpha)$ regime even for Lipschitz potentials. \(^2\) However, the localization result for general analytic case does not currently exist even for the Diophantine case (nor under any other arithmetic condition). Moreover, by Avila’s global theory \(^3\) Lyapunov exponent is a stratified analytic function with finitely many strata. Thus the set $A_{cr} := \{ E : L(E) = \beta(\alpha) \}$ will only be uncountable if $L$ is constant on one of the strata. However, even for the potentials where the set $A_{cr}$ is a small subset of the spectrum, the study of what happens at those energies is still interesting as they represent the border between two different behaviors.

2. Preliminaries

Let $T = \mathbb{R}/\mathbb{Z}$. For a bounded analytic (possibly matrix valued) function $F$ defined on $\{ \theta | |\Im \theta| < h \}$, let $\| F \|_h = \sup_{|\Im \theta| < h} \| F(\theta) \|$. Let $C^\omega_h(T, \ast)$ be the set of all these $\ast$-valued functions ($\ast$ will usually denote $\mathbb{R}$, $SL(2, \mathbb{R})$). Also we denote $C^\omega(\mathbb{T}, \ast) = \cup_{h>0} C^\omega_h(\mathbb{T}, \ast)$.

2.1. Continued Fraction Expansion. Let $\alpha \in (0, 1)$ be irrational. Define $a_0 = 0$, $a_0 = \alpha$, and inductively for $k \geq 1$,

$$ a_k = \lfloor \alpha_{k-1}^{-1} \rfloor, \quad \alpha_k = \alpha_{k-1}^{-1} - a_k = G(\alpha_{k-1}) := \left\{ \frac{1}{\alpha_{k-1}} \right\}. $$

Let $p_0 = 0, p_1 = 1, q_0 = 1, q_1 = a_1$, and we define inductively

$$ p_k = a_k p_{k-1} + p_{k-2}, \quad q_k = a_k q_{k-1} + q_{k-2}. $$

Then $(q_n)$ is the sequence of denominators of the best rational approximations of $\alpha$ and we have

$$ \forall 1 \leq k < q_n, \quad \| k\alpha \|_T \geq \| q_{n-1} \alpha \|_T, \quad (2.1) $$

and

$$ \frac{1}{2q_{n+1}} \leq \| q_n \alpha \|_T \leq \frac{1}{q_{n+1}}. \quad (2.2) $$

As a direct consequence of (2.1) and (2.2), we have the following:

**Lemma 2.1.** Suppose that there exists $p \in \mathbb{N}$, such that $a_i = 1$ if $i \geq p$. Then we have

$$ \| k\alpha \|_T \geq \frac{1}{4|k|}, \quad \forall |k| \geq q_{p-1}. \quad (2.3) $$

For any $\alpha, \alpha' \in \mathbb{R}\setminus\mathbb{Q}$, let $n$ be the first index for which the continued fraction expansions of $\alpha$ and $\alpha'$ differ. Define $d_H(\alpha, \alpha') = \frac{1}{n+1}$. Then $(\mathbb{R}\setminus\mathbb{Q}, d_H)$ is a complete metric space. Also, by (2.2), if $d_H(\alpha, \alpha') = \frac{1}{n+1}$, we have that $|\alpha - \alpha'| < \frac{1}{q_{n-1}(\alpha)}$.

\(^2\)This is essentially contained in [9]. Additionally, it follows from a more recent theorem of [21] where singularities are also allowed.
Finally, we introduce the set of \( \alpha \)-Diophantine phases. For \( \tau > 1, \gamma > 0 \), set
\[
DC_\alpha(\gamma, \tau) = \{ \phi \in \mathbb{R} \mid \| 2\phi - m\alpha \|_{\mathbb{R}/\mathbb{Z}} \geq \frac{\gamma}{(|m|+1)\tau}, m \in \mathbb{Z} \}.
\]
Clearly, \( \cup_{\gamma > 0} DC_\alpha(\gamma, \tau) \) is a full measure set.

### 2.2. Cocycle, reducibility, rotation number.

Let \( \alpha \in \mathbb{R} \setminus \mathbb{Q}, A \in C^\omega(\mathbb{T}, \text{SL}(2, \mathbb{R})) \). We define the quasi-periodic \( \text{SL}(2, \mathbb{R}) \)-cocycle \((\alpha, A)\) as an action on \( \mathbb{T} \times \mathbb{R}^2 \) by \((x, v) \mapsto (x+\alpha, A(x)v)\). Recall that two cocycles \((\alpha, A^i), i = 1, 2, \) are called \( C^k \) \((k = \infty, \omega)\) conjugated if there exists \( B \in C^k(\mathbb{T}, \text{PSL}(2, \mathbb{R})) \) such that \( A^1(\theta) = B(\theta + \alpha)A^2(\theta)B(\theta)^{-1} \). If \((\alpha, A)\) is \( C^k \) conjugated to a constant cocycle, then it is called \( C^k \) reducible. A cocycle \((\alpha, A)\) is said to be \textit{almost reducible} if the closure of its analytical conjugacy class contains a constant.

Assume now that \( A : \mathbb{T} \rightarrow \text{SL}(2, \mathbb{R}) \) is homotopic to the identity. Then there exist \( \psi : \mathbb{T} \times \mathbb{T} \rightarrow \mathbb{R} \) and \( u : \mathbb{T} \times \mathbb{T} \rightarrow \mathbb{R}^+ \) such that
\[
A(x) \cdot \begin{pmatrix}
\cos 2\pi y \\
\sin 2\pi y
\end{pmatrix} = u(x, y) \begin{pmatrix}
\cos 2\pi (y + \psi(x, y)) \\
\sin 2\pi (y + \psi(x, y))
\end{pmatrix}.
\]
The function \( \psi \) is called a \textit{lift} of \( A \). Let \( \mu \) be any probability measure on \( \mathbb{T} \times \mathbb{T} \) which is invariant under the continuous map \( \mathbb{T} : (x, y) \mapsto (x+\alpha, y+\psi(x, y)) \), projecting over Lebesgue measure on the first coordinate. Then the number
\[
\rho(\alpha, A) = \int \psi d\mu \mod \mathbb{Z}
\]
does not depend on the choices of \( \psi \) and \( \mu \), and is called the \textit{fibered rotation number} of \((\alpha, A)\), see [13] and [22].

Rotation number plays a fundamental role in the reducibility theory:

**Theorem 2.1.** [5, 14] Let \((\alpha, A) \in \mathbb{R} \setminus \mathbb{Q} \times C^\omega(\mathbb{T}, \text{SL}(2, \mathbb{R})) \) with \( h > h' > 0, R \in \text{SL}(2, \mathbb{R}) \). Then for every \( \tau > 1, \gamma > 0 \), if \( \rho(\alpha, A) \in DC_\alpha(\gamma, \tau) \), then there exists \( \varepsilon = \varepsilon(\tau, \gamma, h - h') \), such that if \( \| A(\theta) - R \|_h < \varepsilon(\tau, \gamma, h - h') \), then there exist \( B \in C^\omega_0(\mathbb{T}, \text{SL}(2, \mathbb{R})) \), \( \varphi \in C^\omega_{h'}(\mathbb{T}, \mathbb{R}) \), such that
\[
B(\theta + \alpha)A(\theta)B(\theta)^{-1} = R_{\varphi(\theta)}.
\]
Moreover, we have the following estimates
\[
(1) \| B - \text{id} \|_{h'} \leq \| A(\theta) - R \|_h^{\frac{1}{2}},
\]
\[
(2) \| \varphi(\theta) - \varphi(0) \|_{h'} \leq 2\| A(\theta) - R \|_h.
\]

### 3. Singular continuous spectrum

Before giving the proof of Theorem 1.2(1), we introduce some notations. Denote the spectrum of \( H_{\lambda, \alpha, \theta} \) by \( \Sigma(\lambda, \alpha) \). It doesn’t depend on \( \theta \), since \( \theta \rightarrow \theta + \alpha \) is minimal. Let
\[
A(E, \theta) = S_E^\lambda(\theta) = \begin{pmatrix}
E - 2\lambda \cos 2\pi(\theta) & -1 \\
1 & 0
\end{pmatrix},
\]
A_0 = I, and for k \geq 1 we set
\[ A_k(E, \theta) = A(E, \theta + (k-1)\alpha) \cdots A(E, \theta + \alpha)A(E, \theta), \]
\[ A_{-k}(E, \theta) = A_k(E, \theta - k\alpha)^{-1}. \]
In case both \( \alpha \) and \( \alpha' \) are involved in the argument, we will use
\[ A'_{k}(E, \theta) = A(E, \theta + (k-1)\alpha') \cdots A(E, \theta + \alpha')A(E, \theta) \]
and \( A'_{-k}(E, \theta) = A'_k(E, \theta - k\alpha')^{-1} \).

A formal solution \( u_{\theta E}^n \) of (3.1)
\[ H_{\lambda, \alpha, \theta} u_{\theta E}^n = Eu_{\theta E}^n \]
is said to be normalized if
\[ \sum_{|k| \leq N} |u_{\theta E}^n(k)|^2 \geq C^2. \]
We then have

**Lemma 3.1.** Let \( \lambda > 1 \). If for any \( C > 0 \) there exists \( N \in \mathbb{N} \) such that \((\lambda, \alpha)\) is \((C, N)\) bad, then \( H_{\lambda, \alpha, \theta} \) has purely singular continuous spectrum for any \( \theta \in \mathbb{R} \).

**Proof.** The assumption of Lemma 3.1 implies absence of \( \ell^2 \) solutions. Since Lyapunov exponents are positive for all \( E \) for \( \lambda > 1 \) [13], there is no absolutely continuous spectrum either [23].

Of course the notion of \((C, N)\) badness is very general and can be defined for an arbitrary potential with obvious modifications. Similarly, Lemma 3.1 is also a very general statement, requiring only positivity of the Lyapunov exponent. The converse is not true in general: purely singular continuous spectrum (or absence of point spectrum) for all phases doesn’t necessarily imply \((C, N)\) badness for any \( C \). However in our case, we do have the following:

**Proposition 3.1.** If \( 1 \leq |\lambda| < e^\beta \), then for any \( C > 0 \), there exists \( N \in \mathbb{N} \) such that \((\lambda, \alpha)\) is \((C, N)\) bad.

**Proof.** This is a Gordon-type statement that is essentially contained in the proof of Theorem 1.1 [9]. We just give a short argument here for completeness. If \( 1 \leq |\lambda| < e^\beta \), then for any \( \epsilon > 0 \), by uniform upper semicontinuity and telescoping (see e.g. Proposition 3.1 of [9]), one has the following: there exists \( K = K(\lambda, \alpha, \epsilon) \) which doesn’t depend on \( \theta, E \), such that if \( n \geq K \), we have
\[ \sup_{\theta \in \mathbb{T}} \| A_{q_n}(E, \theta + q_n\alpha) - A_{q_n}(E, \theta) \| \leq e^{-(\beta - \ln |\lambda| - \epsilon)q_n}. \]
\[ \sup_{\theta \in \mathbb{T}} \| A_{-q_n}(E, \theta + q_n\alpha) - A_{-q_n}(E, \theta) \| \leq e^{-(\beta - \ln |\lambda| - \epsilon)q_n}. \]
The following lemma essentially completes the proof:

**Lemma 3.2.** Suppose that (3.2) and (3.3) hold, and let \( u_E^\theta(k) \) be a normalized solution of (3.1). Then we have

\[
\max \{ \| A_n(\theta)u_E^\theta \|, \| A_{-n}(\theta)u_E^\theta \|, \| A_{2n}(\theta)u_E^\theta \| \} \geq \frac{1}{4},
\]

where \( u_E^\theta = \begin{pmatrix} u_E^\theta(0) \\ u_E^\theta(1) \end{pmatrix} \).

**Proof.** For any \( M \in \text{SL}(2, \mathbb{R}) \), one has

\[
M + M^{-1} = \text{tr}M \cdot \text{Id}.
\]

Taking \( M = A_n(\theta) \), we distinguish two cases.

**Case 1:** If \( |\text{tr}A_n(\theta)| > 1 \), (3.5) gives

\[
\max \{ \| A_n(\theta)u_E^\theta \|, \| A_{n}(\theta)^{-1}u_E^\theta \| \} \geq \frac{1}{2}.
\]

Then (3.4) follows from (3.6) and (3.3).

**Case 2:** If \( |\text{tr}A_n(\theta)| < 1 \), then by (3.5), one has

\[
\| A_{2n}(\theta)u_E^\theta \| = \| A_{2n}(\theta)u_E^\theta - A_{n}(\theta)^2u_E^\theta - trA_n(\theta) \cdot A_{n}(\theta)u_E^\theta \| \\
\geq 1 - (1 + \| A_n(\theta + q_n\alpha) - A_n(\theta, \theta) \|)\| A_n(\theta)u_E^\theta \|.
\]

Then (3.4) follows from (3.2). \(\square\)

This directly implies that for any \( C > 0 \), \((\lambda, \alpha)\) is \((C, N)\) bad for sufficiently large \( N \). \(\square\)

The notion of \((C, N)\)-badness is robust with respect to perturbations, in the sense that

**Lemma 3.3.** If \((\lambda, \alpha)\) is \((C, N)\) bad, then for any \( C' < C \), there exists \( \varepsilon = \varepsilon(\lambda, \alpha, C - C', N) > 0 \) such that if \( |\alpha' - \alpha| < \varepsilon \), then \((\lambda, \alpha')\) is \((C', N)\) bad.

**Proof.** By Hölder continuity of the spectrum in Hausdorff topology [10], for any \( E' \in \Sigma(\lambda, \alpha') \), there exists \( E \in \Sigma(\lambda, \alpha) \), with \( |E - E'| < C|\alpha' - \alpha|^\frac{1}{2} < Ce^\frac{1}{4} \). Hence with estimate

\[
|E - 2\lambda \cos 2\pi(\theta + n\alpha) - E' + 2\lambda \cos 2\pi(\theta + n\alpha')| \leq C(n\varepsilon + e^\frac{1}{2}),
\]

by a telescoping argument, we obtain that for any \( \delta > 0 \), if \(|m| > m_0(\lambda, \alpha, \delta)\) is large enough, we have

\[
\sup_{\theta \in \mathbb{R}} \| A_m(\theta) - A_m(E', \theta) \| \leq e^{|m|(|\ln \lambda + \delta|)(e^\frac{1}{2} + m\varepsilon)}.
\]
This is a standard argument; see e.g. section 3 of [9] for details. Let $u^\theta_{E'}(1) = u^\theta_E(1)$, $u^\theta_{E'}(0) = u^\theta_E(0)$, where $u^\theta_E(k)$ is a normalized solution of (3.1). We have

$$|u^\theta_{E'}(k) - u^\theta_E(k)| \leq (\epsilon^\frac{1}{2} + N \epsilon)e^N(\ln\lambda + \delta) \quad \forall |k| \leq N.$$ 

Therefore we can select $\epsilon$ small enough, not depending on $\theta$, such that

$$\left( \sum_{|k| \leq N} |u^\theta_{E'}(k)|^2 \right)^{\frac{1}{2}} \geq \left( \sum_{|k| \leq N} |u^\theta_E(k)|^2 \right)^{\frac{1}{2}} - \left( \sum_{|k| \leq N} |u^\theta_{E'}(k) - u^\theta_E(k)|^2 \right)^{\frac{1}{2}} > C'.$$

The last inequality holds since $(\lambda, \alpha)$ is $(C, N)$ bad. The smallness of $\epsilon$ is independent of $E' \in \Sigma(\lambda, \alpha')$, since $N$ doesn’t depend on $E \in \Sigma(\lambda, \alpha)$. \hfill \Box

For any given $\alpha \in \mathbb{R} \setminus Q$, $\varepsilon > 0$, $\beta > \ln \lambda > 0$, $K > 0$, let $S(\alpha, \varepsilon, \beta', K)$ be the set of $\alpha' \in \mathbb{R} \setminus Q$ that satisfy the following properties: $d_H(\alpha, \alpha') < \varepsilon$, $a_K(\alpha') = e^{\beta q_{K-1}(\alpha')}$, $\beta(\alpha') = \beta'$. The fundamental construction for our proof is the following:

**Lemma 3.4.** Assume $\beta(\alpha) > \ln \lambda > 0$. Then for any $C > 0$, $\beta' > \ln \lambda > 0$, $\varepsilon > 0$, $M > 0$, there exists $N = N(\lambda, \alpha) > 0$, $K > M$ and $\alpha' \in S(\alpha, \varepsilon, \beta', K)$ such that $(\lambda, \alpha')$ is $(C, N)$ bad.

**Proof.** Let $\alpha = [a_1, a_2, \ldots]$. Since $\beta(\alpha) > \ln \lambda > 0$, then by Proposition 3.1, for any given $C > 0$, there exists $N \in \mathbb{N}$ such that $(\lambda, \alpha)$ is $(C + 1, N)$ bad. We now construct $\alpha'$ as follows. Let $\epsilon(\lambda, \alpha, C - C', N)$ be as defined in Lemma 3.3. For any $\beta > \ln \lambda > 0$, $\varepsilon > 0$, $M > 0$, find $K$ so that $K > \max\{\frac{1}{\varepsilon}, M\}$ and $q_{K-1}^2 < \epsilon(\lambda, \alpha, 1, N)$. We define $\alpha' = [a_1', a_2', \ldots]$, where

$$a_n' = \begin{cases} a_n, & n < K - 1 \\ e^{\beta q_{k2K-2}(\alpha')}, & n = k2K \\ 1, & k2K + 1 \leq n \leq (k + 1)2K - 1 \end{cases}$$

By Lemma 3.3, we have $(\lambda, \alpha')$ is $(C, N)$ bad, and it is straightforward to check that $\alpha' \in S(\alpha, \varepsilon, \beta', K)$.

We now finish the proof of Theorem 1.2(1). For any $\alpha \in \mathbb{R} \setminus Q$ with $\lambda = e^{\beta(\alpha)}$, we write $\alpha = [a_1, a_2, \ldots]$. For any $\varepsilon > 0$, by the construction of Lemma 3.4, one can find $\alpha^1$ such that $\alpha^1 \in S(\alpha, \frac{\varepsilon}{2}, \ln \lambda + \frac{1}{2}, N^1)$ with $N^1 = [\frac{2}{\varepsilon}]$. This implies that $\beta(\alpha^1) = \ln \lambda + \frac{1}{2}$. Then by Proposition 3.1, for given $C_1 > 0$, there exists $N_1 \in \mathbb{N}$ such that $(\lambda, \alpha^1)$ is $(C_1, N_1)$ bad. We now construct $\alpha^k$ by induction. Given $\alpha^k$, $C_k = kC_1$, by Lemma 3.4, there exist $\alpha^{k+1} \in S(\alpha^k, kC_k, \ln \lambda + \frac{1}{k}, N^{k+1})$, $N_{k+1} = N(\lambda, \alpha^k)$, $N^{k+1} > 2N^k$ such that $(\lambda, \alpha^{k+1})$ is $(C_{k+1}, N_{k+1})$ bad. Let $\alpha_{\infty} = \lim \alpha^k$. The limit exists since $d_H(\alpha^k, \alpha^{k+1}) < \frac{\varepsilon}{2^k}$. Also by the construction, we have

$$a_{N^k}(\alpha_{\infty}) = e^{(\ln \lambda + \frac{1}{2^k})q_{N^k-1}(\alpha_{\infty})},$$

$$|\alpha_{\infty} - \alpha^k| < \left(\frac{1}{q_{N^k-1}(\alpha_{\infty})}\right)^2 < \epsilon(\lambda, \alpha^k, 1, N_k).$$
Therefore (3.7) implies that $\beta(\alpha_\infty) = \ln \lambda$. By Lemma 3.3 and (3.8), we have $(\lambda, \alpha_\infty)$ is $(C_k - 1, N_k)$ bad, so $H_{\lambda, \alpha_\infty, \theta}$ has purely singular continuous spectrum for all $\theta$ by Lemma 3.1.

4. Pure point spectrum

The pure point spectrum will be a corollary of the following full measure reducibility result.

**Theorem 4.1.** Let $0 < \beta(\alpha) < \infty$. There exists a dense set of $\alpha$ with $\lambda = e^{\beta(\alpha)}$, such that for a full measure set of $E \in \Sigma(H_{\lambda - 1, \alpha})$, the almost Mathieu cocycle $(\alpha, S_E^{\lambda^{-1}})$ is $C^\infty$ reducible.

**Proof of Theorem 1.2(2)** From [9], one knows that if $(\alpha, S_E^{\lambda^{-1}})$ is $C^\infty$ reducible for a full measure set of $E \in \Sigma(H_{\lambda - 1, \alpha})$, then for almost every $\phi$, $H_{\lambda, \alpha, \phi}$ has pure point spectrum. See also Theorem 3.1 of [18] for a more general result of this nature. Thus Theorem 1.2(2) follows from Theorem 4.1.

To prove Theorem 4.1, we start with

**Lemma 4.1.** Let $\lambda > 1$, $0 < \beta < \infty$, $0 < \delta_j < \beta/2$, $\tau > 1$, $\gamma_1, \gamma_2 > 0$. Suppose that $\alpha_j = [a_1, \ldots, a_j, 1, 1, \ldots]$ and let $\alpha_{j,n} = [\tilde{a}_1, \tilde{a}_2, \ldots]$, where

$$
\tilde{a}_i = \begin{cases} a_i, & i \leq n - 1 \\ e^{(\beta - 2\delta_j)\ln(1/\alpha_j)}, & i = n \\ 1, & i \geq n + 1 \end{cases}
$$

Suppose that $\rho(\alpha_j, S_E^{\lambda^{-1}}) \in DC_{\alpha_j}(\gamma_1, \tau)$, $\rho(\alpha_{j,n}, S_E^{\lambda^{-1}}) \in DC_{\alpha_{j,n}}(\gamma_2, \tau)$. Then there exist $B_j \in C^*(\mathbb{T}, SL(2, \mathbb{R}))$ and $B_{j,n} \in C^*(\mathbb{T}, SL(2, \mathbb{R}))$, with $* = \omega$ if $\ln \lambda > \beta$ and $\infty$ if $\ln \lambda = \beta$, so that $(\alpha_j, S_E^{\lambda^{-1}})$ is reducible by $B_j(\theta)$ and $(\alpha_{j,n}, S_E^{\lambda^{-1}})$ is reducible by $B_{j,n}(\theta)$. Moreover, for any $\epsilon > 0$, there exists $N = N(\alpha_j, \delta_j, \tau, \gamma_1, \gamma_2, \epsilon)$, such that for $n \geq N$, we have the following:

1. if $\lambda > e^\beta$, then $\|B_{j,n} - B_j\|_{\ln 1 - \beta} \leq \epsilon$.
2. if $\lambda = e^\beta$, then $\text{dist}_{C^\infty}(B_{j,n}, B_j) \leq \epsilon$.

**Proof.** We first recall

**Theorem 4.2.** [16] Suppose that $\alpha$ is Diophantine, $\theta$ is Diophantine w.r.t $\alpha$, $\lambda > 1$. Then $H_{\lambda, \alpha, \theta}$ has pure point spectrum with exponentially decaying eigenfunctions. Moreover, each eigenfunction $u(n)$ satisfies

$$
(4.1) \quad \lim_{|n| \to \infty} \frac{\ln(u^2(n) + u^2(n + 1))}{2|n|} = -\ln \lambda.
$$

By the definition of $\alpha_j$, we know it is Diophantine. Then by the assumption that $\lambda > 1$, $\rho(\alpha_j, S_E^{\lambda^{-1}}) \in DC_{\alpha_j}(\tau, \gamma_1)$, using Theorem 4.2 and Aubry
duality (e.g. [6]), we have the following result: for any given \( \delta_j > 0 \), there exist \( T_j = T_j(\alpha_j, \delta_j, \gamma_1, \tau) \), \( B_j(\theta) \in C_{\infty}^{\omega}(T, SL(2, \mathbb{R})) \), such that

\[
(4.2) \quad B_j(\theta + \alpha_j)S_E^{\lambda^{-1}}(\theta)B_j(\theta)^{-1} = R_{\rho(\alpha_j, S_E^{\lambda^{-1}})},
\]

\( \deg B_j(\theta) = 0 \), and

\[
(4.3) \quad \|B_j\|_{\ln \lambda - \delta_j/4} \leq T_j.
\]

Theorem 2.5 of [6] contains the proof of this standard result; we just point out that the fact that \( B_j(\theta) \in C_{\infty}^{\omega}(T, SL(2, \mathbb{R})) \) follows from (4.1).

By (4.2) and the Cauchy estimate, we have

\[
B_j(\theta + \alpha_{j,n})S_E^{\lambda^{-1}}(\theta)B_j(\theta)^{-1} = e^{F_j(\theta)}R_{\rho(\alpha_j, S_E^{\lambda^{-1}})},
\]

with estimate

\[
\|F_j\|_{\ln \lambda - \delta_j/2} \leq |\alpha_j - \alpha_{j,n}|\|\partial B_j\|_{\ln \lambda - \delta_j/2}\|B_j\|_{\ln \lambda - \delta_j/4} \leq \frac{4T_j^2}{q_{n-1}^2 \delta_j}.
\]

Then there exists \( \bar{N} = \bar{N}(\alpha_j, \delta_j, \tau, \gamma_1, \gamma_2) \), such that if \( n \geq \bar{N} \), then we have

\[
(4.4) \quad \|F_j\|_{\ln \lambda - \delta_j/2} \leq \frac{4T_j^2}{q_{n-1}^2 \delta_j} \leq \varepsilon(\tau, \gamma_2, \delta_j/2),
\]

where \( \varepsilon = \varepsilon(\tau, \gamma, h - h') \) is defined in Theorem 2.1.

Since \( \rho(\alpha_{j,n}, S_E^{\lambda^{-1}}) \in DC_{\alpha_{j,n}}(\tau, \gamma_2) \), \( \deg B_j(\theta) = 0 \), we have

\[
\rho(\alpha_{j,n}, e^{F_j(\theta)}R_{\rho(\alpha_j, S_E^{\lambda^{-1}})}) = \rho(\alpha_{j,n}, S_E^{\lambda^{-1}}) \in DC_{\alpha_{j,n}}(\tau, \gamma_2),
\]

so by (4.4), one can apply Theorem 2.1, getting \( \bar{B}_n(\theta) \in C_{\infty}^{\omega}(T, SL(2, \mathbb{R})) \), \( \varphi_n \in C_{\infty}^{\omega}(T, \mathbb{R}) \), such that

\[
\bar{B}_n(\theta + \alpha_{j,n})e^{F_j(\theta)}R_{\rho(\alpha_j, S_E^{\lambda^{-1}})}\bar{B}_n(\theta)^{-1} = R_{\varphi_n(\theta)}.
\]

Moreover, we have the estimates

\[
\|\bar{B}_n - \text{id}\|_{\ln \lambda - \delta_j} \leq \frac{cT_j}{q_{n-1}^2 \delta_j^{1/2}},
\]

\[
(4.5) \quad \|\varphi_n(\theta) - \hat{\varphi}_n(0)\|_{\ln \lambda - \delta_j} \leq \frac{cT_j^2}{q_{n-1}^2 \delta_j}.
\]

Now we let

\[
\psi_n(\theta) - \psi_n(\theta + \alpha_{j,n}) = \varphi_n(\theta) - \hat{\varphi}_n(0),
\]

so we have

\[
(4.6) \quad \hat{\psi}_n(k) = \frac{\hat{\varphi}_n(k)}{1 - e^{2\pi i \alpha_{j,n}}}.\]
Set $\tilde{q}_n = q_n(\alpha_{j,n})$. Then by (4.5),(4.6),(2.2), and Lemma 2.1, we have the following estimates:

$$\|\psi_n\|_{\ln \lambda - \beta} \leq \left( \sum_{0 < |k| < q_n-1} + \sum_{q_n-1 \leq |k| < \tilde{q}_n} + \sum_{|k| \geq \tilde{q}_n} \right) \frac{|\tilde{\phi}_n(k)|}{\|k\alpha_{j,n}\|_T} e^{\|k\|_T (\ln \lambda - \beta)}$$

$$\leq c \left( q_n - 1 + q_n e^{q_n - 1 (\beta - 2\delta_j)} \right) \sum_{q_n - 1 \leq |k| < \tilde{q}_n} e^{-|k| (\beta - \delta_j)}$$

$$+ \sum_{|k| \geq \tilde{q}_n} 4|k| e^{-|k| (\beta - \delta_j)} \frac{T_j^2}{q_n - 1 \delta_j} \leq \frac{cT_j^2}{q_n - 1 \delta_j}$$

which implies that $\psi_n \in C_{\ln \lambda - \beta}(\mathbb{T}, \mathbb{R})$. Let $B_{j,n}(\theta) = R_{\psi_n}(\theta) \tilde{B}_n(\theta)B_j(\theta)$. Then by the definition of $\psi_n(\theta)$, we have

$$B_{j,n}(\theta + \alpha_{j,n}) S_{E}^{\lambda - 1}(\theta) B_{j,n}(\theta)^{-1} = R_{\tilde{\phi}_n}(0).$$

Moreover, for any $\epsilon > 0$, there exists $N = N(\alpha_j, \delta_j, \tau, \gamma_1, \gamma_2, \epsilon)$, such that if $n \geq N$, we have

$$\|B_j - B_{j,n}\|_{\ln \lambda - \beta} \leq 2\|B_j\|_{\ln \lambda - \beta / 4} (\|\tilde{B}_n - id\|_{\ln \lambda - \beta} + \|R_{\psi_n} - id\|_{\ln \lambda - \beta})$$

$$\leq \frac{cT_j^3}{q_n - 1 \delta_j} \leq \epsilon,$$

which establishes the first part of the lemma.

For the second part, note that for $s \geq 0$, we have

$$\|f\|_{C^s} \leq \sum_{j=0}^{s} \sum_{k \in \mathbb{Z}} |k|^j |\hat{f}(k)|.$$

Then, similar as before, for any $s \in \mathbb{Z}$, we have

$$\|\psi_n\|_{C^s} \leq \left( \sum_{0 < |k| < q_n-1} + \sum_{q_n-1 \leq |k| < \tilde{q}_n} + \sum_{|k| \geq \tilde{q}_n} \right) \frac{|k|^{s+1} |\tilde{\phi}_n(k)|}{\|k\alpha_{j,n}\|_T}$$

$$\leq \left( q_n - 1 + \sum_{0 < |k| < q_n-1} |k|^{s+1} e^{-|k| (\beta - \delta_j)} + q_n e^{q_n - 1 (\beta - 2\delta_j)} \sum_{q_n - 1 \leq |k| < \tilde{q}_n} |k|^{s+1} e^{-|k| (\beta - \delta_j)}$$

$$+ \sum_{|k| \geq \tilde{q}_n} 4|k|^{s+2} e^{-|k| (\beta - \delta_j)} \frac{T_j^2}{q_n - 1 \delta_j} \right) \leq \frac{cT_j^2}{q_n - 1 \delta_j}.$$

Therefore one can find $N = N(\alpha_j, \delta_j, \tau, \gamma_1, \gamma_2, \epsilon, s)$, such that if $n \geq N$, one reaches $\|B_j - B_{j,n}\|_{C^s} \leq 2\|B_j\|_{\ln \lambda - \delta_j / 4} (\|\tilde{B}_n - id\|_{\ln \lambda - \delta_j / 2} + \|R_{\psi_n} - id\|_{C^s}) \leq \epsilon / 2$. Choosing $N = \max_{s \leq C \ln \epsilon^{-1}} N(\alpha_j, \delta_j, \tau, \gamma_1, \gamma_2, \epsilon, s)$, we obtain the desired result.
Lemma 4.2. Let \( \lambda > 1, \alpha \in \mathbb{R} \setminus \mathbb{Q} \). For a full measure set of \( E \in \Sigma(H_{\lambda^{-1}}, \alpha) \), there exists \( B_E \in C^\omega(\mathbb{T}, SL(2, \mathbb{R})) \) such that
\[
B_E(\theta + \alpha)S_{E}^{\lambda^{-1}}(\theta)B_E(\theta)^{-1} \in SO(2, \mathbb{R}).
\]
Furthermore, we have
\[
\frac{d\rho(\alpha, S_{E}^{\lambda^{-1}})}{dE} = -\frac{1}{8\pi} \int_{\mathbb{T}} \|B_E(\theta)\|_{HS}^2 d\theta \leq -\frac{1}{4\pi}.
\]
Here \( \| \cdot \|_{HS} \) denotes the Hilbert-Schmidt norm.

Proof. This is a combination of full measure rotations reducibility \([5, 26]\) and formula (1.5) of \([5]\).

Proof of Theorem 4.1 Let \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \) with \( \lambda = e^{\beta(\alpha)} \). We write \( \alpha = [a_1, a_2, \cdots] \). Assume \( \beta = \beta(\alpha) = \ln \lambda \). For any \( \varepsilon > 0 \), we first perturb \( \alpha \) to \( \alpha_0 = [a_1, \cdots, a_{n_0}, 1, 1, \cdots] \) so that \( d_H(\alpha, \alpha_0) = \frac{1}{a_{n_0} + 1} \leq \frac{\varepsilon}{4} \). Fix \( 4\varepsilon_0 < \beta, \tau > 1, \gamma > 0 \). We now proceed by induction. Given \( n_j, j \geq 1 \), find
\[
n_j = \max\{N(\alpha_j^{-1}, \frac{\varepsilon_0}{2j-1}, \tau, \frac{\gamma}{2j}, \frac{\varepsilon}{2j}), n_{j-1} + 1, \frac{2j}{\epsilon}\},
\]
where \( N(\alpha_j, \delta, \tau, \gamma, \varepsilon) \) is as defined in Lemma 4.1. Then define \( \alpha_j = [a_j^1, a_j^2, \cdots] \), where
\[
a_j^i = \begin{cases} 
  a_j^{i-1}, & i < n_j \\
  e^{\frac{\beta}{2j}q_{n_j-1}(\alpha_j-1)}, & i = n_j \\
  1, & i \geq n_j + 1 
\end{cases}
\]
Set \( \alpha_\infty = \lim \alpha_j \). The limit exists since \( d_H(\alpha_j, \alpha_j^{-1}) < \frac{\varepsilon}{2j} \). Also by the construction, for \( j \geq 1 \), we have
\[
a_j(\alpha_\infty) = e^{(\beta - \frac{2\lambda_0}{2j})q_{n_j-1}(\alpha_\infty)},
\]
which implies that \( \beta(\alpha_\infty) = \beta \). For these \( \alpha_j \), we define
\[
B = \bigcup_{n=1}^{\infty} \bigcap_{j=n}^{\infty} \{E | \rho(\alpha_j, S_{E}^{\lambda^{-1}}) \in DC_{\alpha_j}(\frac{\gamma}{2j}, \tau)\}.
\]
By Lemma 4.2, we have \( \text{Leb}(\{E | \rho(\alpha_j, S_{E}^{\lambda^{-1}}) \notin DC_{\alpha_j}(\frac{\gamma}{2j}, \tau)\}) = O(\frac{1}{2j}) \) uniformly in \( j \), thus \( B \) is a full measure set by the Borel-Cantelli Lemma. For any fixed \( E \in B \), there exists \( n_E \in \mathbb{Z} \) such that \( \rho(\alpha_j, S_{E}^{\lambda^{-1}}) \in DC_{\alpha_j}(\frac{\gamma}{2j}, \tau) \) for any \( j \geq n_E \). By the definition of \( \alpha_j \) and Lemma 4.1, we know \( (\alpha_j, S_{E}^{\lambda^{-1}}) \) is reducible by \( B_j \in C^\infty(\mathbb{T}, SL(2, \mathbb{R})) \), and \( \text{dist}_{C^\infty}(B_j, B_{j-1}) < \frac{\varepsilon}{2j} \). Let \( B_\infty(\theta) = \lim B_j(\theta) \), then \( B_\infty \in C^\infty(\mathbb{T}, SL(2, \mathbb{R})) \) and \( (\alpha_\infty, S_{E}^{\lambda^{-1}}) \) is reducible by \( B_\infty \), so \( C^\infty \) reducible. \( \square \)
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