String automorphic motives of nondiagonal varieties

Rolf Schimmrigk

Indiana University South Bend
1700 Mishawaka Ave., South Bend, IN 46634

Abstract

In this paper automorphic motives are constructed and analyzed with a view toward the understanding of the geometry of compactification manifolds in string theory in terms of the modular structure of the worldsheet theory. The results described generalize a framework considered previously in two ways, first by relaxing the restriction to modular forms, and second by extending the construction of motives from diagonal varieties to nondiagonal spaces. The framework of automorphic forms and representations is described with a view toward applications, emphasizing the explicit structure of these objects.
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1 Introduction

The purpose of the present paper is to extend and generalize previous results on the construction of the compact "internal" varieties in string theory from first principles by using the worldsheet theory. The basic idea of this program is to associate pure or mixed motives to conformal field theoretic modular forms and to use these motives as the geometric building blocks of the geometry spanned the extra dimensions predicted by string theory. Gluing these motives together then determines the global structure of the varieties. Previous work in this direction was restricted in two ways. First, the class of varieties was restricted to diagonal hypersurfaces, with conformal field theories given by exactly solvable diagonal Gepner models. Within this class of spaces it is possible to consider deformation families that correspond to deformation of the conformal field theory along marginal operators. It can be shown that certain types of singular fibers in such deformation families that lead to phases that are modular, with forms that can be constructed from the $N = 2$ supersymmetric minimal factors. Second, the relation between the structure of the conformal field theory $T_\Sigma$ on the worldsheet $\Sigma$ and the motives $M(X)$ of the compact variety $X$ obtained was based purely on the identification of the motivic $L$-function $L(M, s)$ with $L$-functions $L(f_i, s)$ associated to modular cusp forms $f_i$ that are derived from the theory $\Sigma$. What is needed is a more conceptual picture that explains the experimental results based on $L$-function identities alone.

The main motivation of the work described here is to relax the condition of diagonality to admit configurations for which the moduli space does not necessarily contain diagonal fibers, and to relax the constraint of modularity. This establishes that the framework proposed for an emergent spacetime program via automorphic motives extends to more general spacetimes. An extension of modularity is provided by the more general framework of automorphic forms $\phi$, and their associated representations $\pi$, as considered by Harish-Chandra [1, 2], Langlands [3, 4], and many other mathematicians, in combination with the conjecture that all (pure) motives are automorphic. While in essence the relation between motives and automorphic forms is again mediated by the $L$-function, $L(\pi, s) = L(M, s)$, the structure of these automorphic objects allows to make the expected relation more precise. As a first step in this direction
the automorphic nature of some higher rank motives is analyzed. The strategy followed in
the present work is a generalization of the one considered first in [5], where the automorphy
of K3 fibered diagonal Calabi-Yau varieties was established by constructing the varieties via
the twist map introduced in [6, 7] (see also the independent work of Voisin [8] and later work
by Borcea [9]). Here this approach will be applied to motives of nondiagonal spaces.

This paper is organized to start with the geometry as encoded in motives, then continues to
the number theoretic structure of these motives, and then proceeds to automorphic forms and
representations. In the final part these ingredients are combined in the discussion of some
examples. More precisely, the structure is as follows. Section 2 introduces the varieties con-
sidered in this work and discusses their arithmetic structure. Section 3 outlines the concept
of motives, while Section 4 describes the construction of the motives relevant for nondiagonal
varieties. Section 5 introduces some necessary number theory that provides a link between
the motives and automorphic forms, while Section 6 describes the conceptual structure of au-
tomorphic forms necessary for understanding the link to motives in a conceptual way. Section
7 describes how these two different types of objects can be related in a transparent way with
the least amount of machinery and Section 8 makes the general framework concrete for the
class of Ω-motives. In Sections 9 and 10 applications are given in the context of modular and
automorphic motives for nondiagonal spacetimes.

2 Arithmetic of diagonal and nondiagonal varieties

The most important quantity associated to a motive is its $L$-function, an object obtained via
the local zeta functions of Artin and Schmidt

$$Z(X/\mathbb{F}_q, t) = \exp \left( \sum_{r \geq 1} \frac{N_{q,r}(X)}{r} t^r \right),$$

where $N_{q,r}(X)$ counts the number of points $\#(X/\mathbb{F}_{q^r})$ of the variety $X$ over the finite field $\mathbb{F}_{q^r}$. For curves and simple types of varieties it is easily seen that the zeta function decomposes
into factors that are associated to the cohomology groups $H^j(X)$ of $X$, and it was conjectured
by Weil [10] that this is the case more generally. This cohomological interpretation of the Artin-Schmidt zeta functions

\[ Z(X_n/F_p, t) = \prod_{j=0}^{n-j} \frac{\mathcal{P}_j^{2j}(X_n, t)}{\mathcal{P}_j^{2j-1}(X_n, t)}, \quad (2) \]

proven by Grothendieck [11], shows that only a finite amount of information has to be computed in order to determine this object. Here \( \mathcal{P}_0^{2j}(X_n, t) = 1 - t \), \( \mathcal{P}_2^{2j}(X_n, t) = 1 - p^n t \) and the remaining \( \mathcal{P}_j^{2j}(X_n, t) \) are polynomials whose degree is given by the Betti numbers \( b^i = \dim_{\mathbb{R}} H^i(X_n) \) of the variety. The global Hasse-Weil L-function is obtained by combining the local factors \( \mathcal{P}_j^{2j}(X, t) \) as

\[ L(H^j(X), s) = \prod_p \frac{1}{\mathcal{P}_j^{2j}(X, p^{-s})}. \quad (3) \]

Details about bad primes will not be of relevance in this paper.

For the case of \( \Omega \)-motives of hypersurfaces of weighted projective spaces this arithmetic structure can easily be computed directly, if very inefficiently. A more conceptual analysis of the cardinalities \( N_{q,r} \) is useful not only for calculational efficiency, but it is also necessary in order to extract from the cohomological zeta functions the local factors of the motives that are obtained via projectors acting on the cohomology. Both of these issues can be addressed as follows.

Consider a hypersurface \( X_n^d \) of dimension \( n \) and degree \( d \) embedded in weighted projective spaces \( \mathbb{P}_{(w_0, \ldots, w_{n+1})} \) with weight vectors denoted by \((w_0, \ldots, w_{n+1}) \in \mathbb{N}^{n+2}\). For smooth hypersurfaces of degree \( d \) the monomial part \( H_{\text{mon}}(X_n^d) \) of the cohomology of \( X_n^d \) is isomorphic to a subset \( U_{\text{coh}} \) of integral vectors defined as

\[ U_{n,d} = \left\{ u \in \mathbb{Z}^{n+2} \mid 0 \leq u_i \leq d_i - 1, \; d \mid \sum_i u_i w_i \right\}, \quad (4) \]

i.e. \( H_{\text{mon}}(X_n^d) \cong U_{\text{coh}} \subset U_{n,d} \).

Associated to the elements of \( U_{n,d} \) are products of Gauss sums, defined in terms of two characters associated to finite fields \( \mathbb{F}_p \). The first is a character on the multiplicative group \( \mathbb{F}_p^\times \) of
\[ F_p \] into the cyclic group \( \mu_{p-1} \) generated by the \((p-1)\)st root of unity

\[ \chi_p : \ F_p^\times \rightarrow \mu_{p-1}, \quad (5) \]
defined by \( \chi_p(v) = \xi_p^m \), where \( \xi_n = e^{2\pi i/n} \), and the integer \( m \) is determined by the generator \( g \in F_p^\times \) as \( v = g^m \). The second is an additive character

\[ \Psi_p : \ F_p \rightarrow \mu_p \]
can be defined as \( \Psi_p(v) = \xi_p^v \). With these ingredients the Gauss sums, defined as

\[ G_{n,p} = \sum_{u \in F_p^\times} \chi_p^n(u)\psi_p(u), \quad (7) \]
can be combined into the Gauss sum products defined as products

\[ \mathcal{G}_p^{(n)}(u) = \prod_{i=0}^{n+1} G_{-u_iw_i,k,p}, \quad (8) \]
were \( k = (p-1)/d \in \mathbb{Z} \). These objects can be generalized to finite extension \( F_q \) of \( F_p \) with \( q = p^r \) for some integer \( r \in \mathbb{N} \) via the trace operator from \( F_q \) to \( F_p \) (more details can be found e.g. in [12]).

Cardinality problems of algebraic varieties are most often formulated in the \( p \)-adic framework, following the work of Dwork in the 1960s. For the understanding of the nondiagonal motives constructed below it is of advantage to consider the complex framework instead. A computation outlined in [12] shows that the multiplicative affine cardinalities \( N_p^\times(X_n^d)_{\text{aff}} \) of Brieskorn-Pham hypersurfaces of arbitrary type

\[ X_{n,\text{BP}}^d = \left\{ \sum_{i=0}^{n+1} z_i^d = 0 \right\} \subset \mathbb{P}(w_0,...,w_{n+1}) \quad (9) \]
is given by

\[ N_p^\times(X_{n,\text{BP}}^d)_{\text{aff}} = \left( \frac{(p-1)^{n+2}}{p} \right) + \left( \frac{(p-1)}{p} \right) \sum_{u_i \in U_{n,d}} \mathcal{G}_p^{(n)}(u). \quad (10) \]

An extension of this result can be obtained with similar methods as in [12] for varieties that are nondiagonal. In the present paper the focus is on weighted hypersurfaces of the type

\[ X_{n,\text{ND}}^d = \left\{ \sum_{i=0}^{n} z_i^d + z_n z_{n+1}^d = 0 \right\} \subset \mathbb{P}(w_0,...,w_{n+1}), \quad (11) \]
These spaces are of interest because they contain the class of all nondiagonal Gepner models \cite{13} (constructed in \cite{14, 15}) and they overlap with the more general class of Kazama-Suzuki models.

The formula that gives the cardinalities for these spaces can be written in the form

\[
N_p^X(X_{n,ND}^d) = \frac{(p-1)^{n+2}}{p} + \frac{(p-1)}{p} \sum_{u_i \in U_{n,d}} \mathbb{G}_p^{(n-2)}(u)G_{-w_{i_u}(d_u-1, d_u+1)} \frac{k}{d_u+1} p G_{-w_{i_u+1} \frac{k}{d_u+1}}. 
\]

These multiplicative affine cardinalities can be used to compute the projective motivic cardinalities that play a central role in the present work.

3 Motives

Motives can be thought of in two different ways, either in terms of Grothendieck’s formulation involving correspondences, or as Galois representations. A discussion of both the original Grothendieck notion as well as the Galois theoretic picture of these objects in a physical context can be found in \cite{5}. An in-depth discussion of many aspects of motives can be found in the illuminating collection of articles in ref. \cite{16}. In this section the general structure of motives is briefly described while in the next section the motives relevant for the present work are constructed.

3.1 Cohomological realizations of motives

The motives of interest in the present work can be thought of as realized by subspaces of the intermediate cohomology of a variety \(X_n\) of complex dimension \(n\)

\[
H(M) \subset H^n(X_n). 
\]

For such motives their weight \(w_M\), defined as the degree of the cohomology \(w_M = \text{deg} H(M)\), is given by the dimension of the variety \(w_M = \text{dim}_C X_n\). More precisely, the Hodge decomposition
of the variety induces a decomposition of the realization $H(M)$ as
\[
\bigoplus_{r_i+s_i=w_M}^{\text{rk}(M)} H^{r_i,s_i}(M) \subset H^n(X_n),
\] (14)
where $\text{rk}(M)$ is the rank of the motive. The Hodge decomposition admits an action of the multiplicative group $\mathbb{C}^\times$ via the characters $\chi^{r,s}$ defined by
\[
\chi^{r,s}(z) = z^{-r}z^{-s}.
\] (15)

### 3.2 Tensor product of motives

One of the fundamental reasons why motives are more useful than monolithic varieties in the context of the emergent spacetime program via automorphic forms and representations is that they can be viewed as the simplest building blocks which can be used to build more complicated structures. One of the constructions that facilitate this process is the tensor product. This product is reminiscent of the tensor product of conformal field theories, motivating the picture that perhaps both types of objects form structured sets (categories) that eventually might be shown to be isomorphic. More concretely tensor products are useful because when present they allow to associate lower rank automorphic objects to higher rank irreducible motives. For pure motives $M_i$ of rank $\text{rk}(M_i)$ and weights $w_M(M_i)$ corresponding rank and weight of the tensor product are given by
\[
\text{rk } M_1 \otimes M_2 = \text{rk } M_1 \cdot \text{rk } M_2
\]
\[
w_M(M_1 \otimes M_2) = w_M(M_1) + w_M(M_2).
\] (16)

### 3.3 $L$-functions of motives

The concrete functional relation between the worldsheet theory $T_\Sigma$ and the variety $X$ is provided by an identity of the $L$-functions that are associated to the different kinds of objects in these models. As noted above, $L$-functions of a variety $X$ are obtained via the Weil-Grothendieck factorization of the Artin-Schmidt zeta functions as cohomological functions
that encode the information obtained by probing a variety with varying, but finite, resolutions
that are given by the size of the primes \( p \in \mathbb{N} \) that determine the order of the finite fields \( \mathbb{F}_p \). By using Grothendieck’s notion of a motive \( M \) as an object defined by a correspondence, one can think of its cohomological realization \( H(M) \) as given by a projector acting on the cohomology groups of the variety. The resulting three-fold factorization allows to construct the global \( L \)-function of a motive \( M \) as the product

\[
L(M, s) = \prod_p L_p(M, s),
\]

where the local factors \( L_p(M, s) = \mathcal{P}_p(M, p^{-s})^{-1} \) are given for motives \( M \) with \( H(M) \subset H^n(X_n) \) for varieties of dimension \( n \) by

\[
\mathcal{P}_p(M, t) = \det(1 + H_p(M)t).
\]

Here \( t \) is a formal variable, which is set to \( t = p^{-s}, s \in \mathbb{C} \), and \( H_p(M) \) is a matrix whose rank is equal to the rank of the motive, computable in the context of weighted hypersurfaces in terms of Gauss sums. This will be made more explicit further below. One of the key issues in the theory of motivic \( L \)-functions is to establish continuation to the whole complex plane.

4 \( \Omega \)-motives

4.1 General concept of the \( \Omega \)-motive

The general notion of the \( \Omega \)-motive is based on the idea to associate a number field \( K_X \) to the compact variety and to define the motive by orbit of the Galois group \( \text{Gal}(K_X/\mathbb{Q}) \) of \( K_X \) over \( \mathbb{Q} \) on a distinguished cohomology class \( \Omega \) [5]. This concept applies to all Calabi-Yau varieties, and more generally to Fano varieties of special type. This section briefly outlines the general framework before applying it to diagonal and nondiagonal varieties.

The conceptual basis of the field \( K_X \) is motivated by the Weil conjectures for varieties restricted to finite fields \( \mathbb{F}_p \), for any finite prime \( p \). Denote this restriction for varieties of
complex dimension $n$ by $X_n/\mathbb{F}_p$. The Weil-Grothendieck factorization of the local zeta functions leads to the natural question of what the nature is of the complex numbers that arise in the factorization these polynomials

$$P_p^i(t) = \prod_j (1 + \delta_j^i(p)t).$$

(19)

For special classes of manifolds it is easy to see that the $\delta_j^i(p)$ are algebraic numbers, and it was Weil’s supposition that this always the case and that for smooth manifolds the norm of these numbers is determined by the primes $p$ as

$$|\delta_j^i(p)| = p^{i/2}.$$  

(20)

This conjecture was one of the main motivations for Grothendieck’s development of arithmetic algebraic geometry before Deligne’s proof using Grothendieck’s framework [17].

Given the algebraic nature of the coefficients $\delta_j^i(p)$ provided by the proof of the Weil conjectures as given by Grothendieck and Deligne it makes sense to define a number field $K_{X_n}$ associated to an algebraic variety $X_n$ of dimension $n$ as

$$K_{X_n} := \mathbb{Q}\{\delta_j^i(p)\}.$$  

(21)

The $\Omega$-motive for any Calabi-Yau variety (CY) or Fano variety of special type (SF) is then defined as the orbits of the $\Omega$-form under the Galois group $\text{Gal}(K_X/\mathbb{Q})$ of the field $K_X$$

$$H(M_\Omega) = \langle\text{Gal}(K_X/\mathbb{Q}), \Omega\rangle.$$  

(22)

For Calabi-Yau $n$-folds $X_n$ one has $\Omega \in H^{n,0}(X_n)$, while for Fano varieties of special type $\Omega \in H^{n+1-Q-1}(X_n)$. The Fano varieties specialize to Calabi-Yau spaces for $Q = 1$ (more details of these varieties are discussed below).

### 4.2 $\Omega$-motives of diagonal and nondiagonal hypersurfaces

The concept of the $\Omega$-motive given above can be made more concrete and computable for hypersurfaces in toric or weighted projective spaces. For diagonal weighted Calabi-Yau hypersurfaces $X_n^d$ of complex dimension $n$ and degree $d$ it reduces to the motive defined as the
Galois orbit of the holomorphic \((n, 0)\)-form with respect to the Galois group \(\text{Gal}(\mathbb{Q}(\mu_d)/\mathbb{Q})\) defined by the cyclotomic field \(K_X = \mathbb{Q}(\mu_d)\), where \(\mu_d = \langle \xi_d \rangle\) is the cyclic group generated by \(\xi_d = e^{2\pi i/d}\). The degree of such varieties is given by the lowest common multiple of the degrees \(d_i\) of the monomials \(z_i^{d_i}\) and it would seem natural to adopt the same definition for nondiagonal hypersurfaces. However, while the orbits so constructed can be used to extract the motive, they are not irreducible. To obtain irreducible motives for varieties of type \((11)\) define an integer \(v \in \mathbb{Z}\) as the lowest common multiple of the degrees \(d_i\) of the variables \(z_i\) for \(i \neq n\) of the monomials of the defining polynomial

\[
v = \text{lcm}\{d_i\}_{i \neq n}.
\]

The field \(K_X\) associated to the hypersurface \(X\) is then the abelian field \(K_X = \mathbb{Q}(\mu_v)\) and the motive is given by the orbit of the \(\Omega\) form, which now can be written in a more explicit form because of the lattice representation of the monomial cohomology given by \(U_{n,d}\) in Section 2. In this realization of the differential form \(\Omega\) for both CYs and SFs corresponds to the unit vector \(u_\Omega := (1, ..., 1) \in \mathbb{Z}^{n+2}\) for a variety of complex dimension \(n\). The explicit form of the \(\Omega\)-form makes it possible to define the action of the Galois group \(\text{Gal}(K_X/\mathbb{Q})\) more concretely. The precise form of this action is different in the diagonal and the nondiagonal cases: while in the diagonal case the modding is by \(d_i\) in all the components this does not hold for nondiagonal hypersurfaces. In this case the form of the action is determined by the same modding condition for all components except the variable that enters linear in the coupling

\[
\sigma_r(u_{\Omega,i}) \equiv ru_{\Omega,i} \pmod{d_i}, \quad i \neq n,
\]

while the \(n^{\text{th}}\) component is determined by the constraint \(d\sum_i w_i u_i\), leading to a unique result for \(\sigma_r(u_{\Omega})\). In the following the resulting orbit of vectors obtained from the Galois group is denoted by

\[
U_\Omega = \left\{ \sigma_r(u_{\Omega}) \mid \sigma_r \in \text{Gal}(K_X/\mathbb{Q}) \right\} \subset U_{\text{coh}} \subset U_{n,d}.
\]

The cohomological representation of the \(\Omega\)-motive can now be written as

\[
H(M_\Omega) \cong \langle \text{Gal}(\mathbb{Q}(\mu_v)/\mathbb{Q}), u_\Omega \rangle,
\]
where \( u_\Omega = (1, 1, \ldots, 1) \in U_n,d \). The \( \Omega \)-motives of both diagonal and nondiagonal varieties (11) are of CM type, as implied by the cardinality formulae (10) and (12).

### 4.3 Cardinalities and \( L \)-functions of \( \Omega \)-motives

The projective motivic cardinalities can be obtained by considering the Galois orbit of vectors \( U_\Omega \subset U \) spanned by the vector \( u_\Omega \). Viewing this orbit as a realization of the \( \Omega \)-motive \( H(M_\Omega) = \langle \text{Gal}(K_X/Q), u_\Omega \rangle \), as described above, the motivic cardinalities can be obtained from the cardinality formula given above as

\[
N_p(M_\Omega) = \frac{1}{p} \sum_{u \in U_\Omega} \mathbb{G}_p(u).
\]

There are different conventions for the local polynomials \( P_p(M,t) \) of the motivic part of the zeta functions. In the present paper these factors are written as \( P_p(M_\Omega, t) = \det (1 + H_p(M_\Omega)t) \), following the notation used earlier. With this notation the matrices \( H_p(M_\Omega) \) can be expressed in terms of the Gauss sum products of the Galois orbit as

\[
H_p(M_\Omega) = (-1)^{w + 1} \frac{1}{p} \left( \begin{array}{c}
\mathbb{G}_p(\sigma_1(u_\Omega)) \\
\vdots \\
\mathbb{G}_p(\sigma_r(u_\Omega))
\end{array} \right),
\]

leads with \( \text{tr} \ H_p(M_\Omega) = (-1)^{w + 1} N_p(M_\Omega) \) to the \( L \)-function coefficients.

Defining the local factors of the \( L \)-function at the good primes essentially as the inverse of the polynomials \( P_p(M_\Omega, t) \) leads to

\[
L(M_\Omega, s) = \prod_p \frac{1}{P(M_\Omega, p^{-s})},
\]

hence the coefficients of the expansion

\[
L(M_\Omega, s) = \sum_n \frac{a_n(M_\Omega)}{n^s}
\]

are determined as

\[
a_p(M_\Omega) = - \text{tr} \ H_p(M_\Omega) = (-1)^w N_p(M_\Omega).
\]
With these local factors the $L$-functions can be computed explicitly and the comparison with automorphic $L$-functions $L(\phi, s)$ can be attempted. This is the strategy followed in the automorphic spacetime program in the context of diagonal varieties in [5] and references therein.

5 Algebraic Hecke characters

Algebraic Hecke characters provide the key ingredients that allow to link motives with complex multiplication to automorphic objects. They are also useful because they provide the simplest context in which algebraicity can be made explicit.

5.1 Infinity type

To gain a more conceptual picture it is useful to note that up to factors of $p$ the Gauss sum products $\mathbb{G}_p$ are essentially Jacobi sums and therefore define algebraic Hecke characters $\Psi$, as first observed by in ref. [18]. Weil’s notion of Hecke characters of type $A_0$ has been reformulated by Deligne [19] into the notion of an algebraic Hecke character as follows. Hecke characters can be associated to arbitrary number fields $K$ and are defined as maps on the group $I_m(\mathcal{O}_K)$ of fractional ideals prime to the modulus ideal $m$. They are characterized by their behavior on the principal ideals $a = (z)$, $z \in K$ where the algebraic characters $\chi_{\text{alg}} : K^\times \rightarrow \mathbb{E}$ are defined by elements of the group ring

$$S = \sum_\ell n^\ell \sigma_\ell \in \mathbb{Z}[\text{Hom}(K, \overline{\mathbb{Q}})]$$

as

$$\chi_{\text{alg}}(z) = z^S = \prod_\ell \sigma_\ell(z)^{n^\ell}. \quad (33)$$

Definition. Let $K$ be a number field and $\mathcal{O}_K$ its ring of integers. For $c \subset \mathcal{O}_K$ an integral ideal denote by $\mathcal{I}_c(K)$ the group of fractional ideals of $K$ that are prime to $c$ and by $\mathcal{D}_c(K)$ the principal ideals $(z)$ of $K$ such that $z \equiv 1(\text{mod } c)$. An algebraic Hecke character $\Psi$ modulo $c$ is a multiplicative function $\chi$ on $\mathcal{I}_c(K)$ whose structure on $(z) \in \mathcal{D}_c(K)$ is determined by an
algebraic character \( \chi_{\text{alg}} \)

\[
\Psi((z)) = \chi_{\text{alg}}(z) = z^S
\]  

in terms of the infinity type \( S \). The integer \( w = n_\ell + n_{c\ell} \) for all \( \ell \), with \( \sigma_{c\ell} = \sigma_\ell \), is called the weight of the character \( \Psi \).

5.2 \( L \)-functions of Hecke characters

Associated to a Hecke character \( \Psi \) with modulus \( m \) of a number field \( K \) with a ring of integers \( \mathcal{O}_K \) is an \( L \)-function, denoted by \( L(\Psi, s) \), which is defined for all ideals \( n \in \mathcal{I}_m(\mathcal{O}_K) \) prime to the modulus \( m \) as

\[
L(\Psi, s) = \sum_{n \in \mathcal{I}_m(\mathcal{O}_K)} \frac{\Psi(n)}{Nn^s},
\]

where \( Nn \) is the norm of the ideal. Part of the motivation for introducing these objects is the fact, established by Hecke, that like in the case of Dirichlet characters the \( L \)-functions of Hecke characters admit a product formulation

\[
L(\Psi, s) = \prod_{p \in \mathcal{I}_m(\mathcal{O}_K)} (1 - \Psi(p)Np^{-s})^{-1}
\]

For modular forms with complex multiplication in the sense of Ribet [20] there exist algebraic Hecke characters \( \Psi_f \) such that the \( L \)-function of the modular form \( f \) agrees with that of the character \( L(f, s) = L(\Psi_f, s) \).

5.3 Hodge type of the motives \( M(\Psi) \)

Associated to Hecke characters are motives \( M(\Psi) \) with complex multiplication such that their \( L \)-functions agree \( L(M(\Psi), s) = L(\Psi, s) \), where the rhs is the Hecke \( L \)-function associated to \( \Psi \). This construction is usually formulated in the context of motives associated to abelian varieties, enhanced to a set (category) of motives by including Artin motives (see e.g Deligne [22]). In the present case the inversion is of more interest, i.e. the recovery of algebraic Hecke
characters from the \( \Omega \)-motives. This will be described further below for algebraic Hecke characters constructed from Gauss sum products.

In the case of CM motives the Hodge type \( \oplus H^{r,s}(M(\Psi)) \) can be made explicit because it is determined by the infinity type of the character \( \Psi \). If the field \( K \) has degree \( \deg K \) the infinity type is parametrized by integers \( S \cong (n_1, \ldots, n_{\deg K}) \). The degree of a CM field is even, hence the integers \( n_i \) can be arranged as \( (n_1, \ldots, n_{\deg K}, n_{\deg K}, \ldots, n_{c_1}) \), where \( n_{c_\ell} \) is associated to the complex conjugate \( \overline{\sigma}_\ell \) in the infinity type. The weight of the infinity type is defined as

\[
\omega = n_\ell + n_{c_\ell},
\]

and it is assumed to be independent of \( \ell \). The embeddings \( \tau^m \) of the coefficient field \( E \) act on \( S \), leading to transformed infinity types

\[
\tau^m \circ S = (n_1^m, \ldots, n_{\deg K}^m, n_{c_1}^m), \quad m = 1, \ldots, \deg E.
\]

The Hodge type of the associated motive is given by

\[
H^{n_\ell^m, n_{c_\ell}^m}(M(\Psi)) = H^{n_\ell^m, \omega - n_\ell^m}(M(\Psi)).
\]

Over \( \mathbb{Q} \) the motive is given by

\[
H_{\sigma_\ell}(M(\Psi)) = \bigoplus_{m=1}^{\deg E} H^{n_\ell^m, \omega - n_\ell^m}(M(\Psi)).
\]

This general picture of algebraic Hecke characters and their associated motives arises in the context of weighted hypersurfaces via characters that are induced by the finite field Jacobi sums. The resulting characters are particularly simple in those cases where the Jacobi sums lead to characters associated to imaginary quadratic fields because in this special case it was shown already by Hecke that the associated \( L \)-functions are modular.

### 5.4 Jacobi sum Hecke characters

For Hecke characters of Jacobi sum type the first step is to view the Jacobi sums \( j_p(\alpha) \) associated to finite fields \( \mathbb{F}_p \) with values in a cyclotomic field \( K_d = \mathbb{Q}(\mu_d) \) as characters on the
ideals of this field, leading to cyclotomic characters $J_a(p)$ defined on prime ideal $p|p$ in terms of the finite field Jacobi sums $j_p(\alpha)$

$$J_a(p) \leftrightarrow j_p(\alpha), \quad a = d\alpha. \quad (41)$$

Here $\alpha \in \mathbb{Q}^{n+1}$ is related to the vectors $u \in U_{n,d}$ as $\alpha_i = \frac{u_i u}{d}$. For the vector $u \in U_\Omega$ and its Galois images $a_r = \sigma_r(u_\Omega)$ with components $a^i_r$, one thus obtains algebraic Hecke characters $\Psi_r := J_{a_r}$ whose structure encoded in the characters

$$\chi^r_{\text{alg}}(z) = z^{S_r} = \prod_{\sigma_r \in \text{Gal}(K_X/\mathbb{Q})} \sigma_r^{-1}(z)^{n^\ell_r} \quad (42)$$

has been determined by Weil [10] as

$$n^\ell_r = \left\lfloor \frac{1}{\ell} \sum_{j=1}^{n+1} \left\langle \frac{\ell a_r^j}{d} \right\rangle \right\rfloor, \quad (43)$$

where $n + 2$ is the rank of the Jacobi sum, $\lfloor \cdot \rfloor$ is the integral part, and $\langle x \rangle = x - \lfloor x \rfloor$ is the rational part.

This explicit form of the infinity type for Jacobi sum Hecke characters provides one way to correlate the $u$-vectors to their Hodge type.

6 Algebraic cuspidal automorphic forms and representations

Conformal invariance of string theory makes it reasonable to ask whether the modular forms expected to appear in the intermediate cohomology of the compact varieties can be derived from the modular forms that arise on the worldsheet. This question leads to a modular realization of the notion of an emergent spacetime in string theory [5]. Modularity is too narrow a framework, however. While all elliptic curves are modular and modular motives also occur in higher dimensions, most motives are not modular, necessitating the generalization to automorphic forms and representations. While all pure motives are expected to be automorphic, not all automorphic forms are thought to be motivic. The class of forms that are believed to admit a motivic interpretation is comprised of those automorphic objects that are algebraic.
6.1 Automorphic forms

The focus in this paper will be on automorphic objects associated to GL(n), leading to the standard $L$-functions, which are conjectured by Langlands to account for all $L$-functions. While several instances of such relations have been established, the general proof of functoriality seems out of reach at present. The general notion of an automorphic form is motivated to a large extent by the idea to generalize the group theoretic lift of classical modular forms to GL(2) to higher rank groups. This provides a more conceptual view already for modular forms and places these objects into a more coherent framework. Background discussions of automorphic forms and representations can be found in [21].

Roughly, automorphic forms are complex valued functions $\phi$ on a group $G$ that satisfy a number of constraints. First, the covariance of modular forms with respect to some congruence subgroup $\Gamma \subset SL(2, \mathbb{Z})$ is traded for the invariance of $\phi$ with respect to an arithmetic subgroup $\Gamma \subset G(\mathbb{Q})$. Next, the invariance of GL(2) forms with respect to the maximal compact group $K = SO(2, \mathbb{R}) \subset SL(2, \mathbb{R})$ is generalized to $C$-finiteness for a maximal compact subgroup $C(\mathbb{R}) \subset G(\mathbb{R})$, i.e. the right translations with respect to $C$ given by $\{\phi(gk) \mid g \in G, k \in C\}$ span a finite dimensional space. The penultimate constraint requires that $\phi$ is annihilated by an ideal in the center $Z(U(\mathfrak{g}))$ of the universal enveloping algebra $U(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ of $G$, and finally it is assumed that $\phi$ is of bounded growth. This structure is made more explicit for the GL(2)-automorphic lift of holomorphic modular forms in Section 7 and examples are discussed in Sections 9 and 10.

Associated to automorphic forms are representations $\pi$, induced by the right regular representation acting on the space of square integrable function $L^2(G)$, that are related to $\phi$ by the fact that the form $\phi$ is an element in an invariant subspace associated to $\pi$. The key that turns automorphic representations and forms into manageable objects is the fact that they are characterized by conjugacy classes in the dual group $\hat{G}$. The simplification that appears in the context of GL(n) automorphic objects of interest in the present paper is that the dual group is again GL(n) and that it is not necessary the problem of L-packets.
6.2 Local factors $\phi_v$ and $\pi_v$ and their Langlands parameters

Similarly to the factorization of the motivic $L$-functions the $L$-functions of Hecke eigenforms decompose into local factors. The analogous localization of automorphic structures is given by the tensor product $\pi = \otimes_v \pi_v$ for the automorphic representation, and the corresponding decomposition $\phi = \otimes_v \phi_v$ for the automorphic forms. Here $v$ runs through the finite primes $p$ as well as the so-called infinite primes $v|\infty$ associated with the archimedean fields $\mathbb{R}$ and $\mathbb{C}$.

In general the local Langlands parameters associated to $\pi_v$ or $\phi_v$ are (equivalence classes of) maps from the Weil-Deligne group $\text{WD}(K_v)$ of the local field $K_v$ to the $L$-group $L_G$, which has as a factor the dual group $\hat{G}$

$$r_v : \text{WD}(K_v) \longrightarrow L_G. \quad (44)$$

These maps are of quite different structure, depending on whether $v$ is a finite prime or whether $v|\infty$.

6.3 Infinity type of $\pi$

The conjectural relation between pure motives and automorphic forms and representations becomes most transparent when expressed in terms of the infinity type of the algebraic automorphic representations. This structure was first emphasized in [23], which can serve as a reference for a more detailed discussion of these objects. For the archimedean case the Weil-Deligne group is just the Weil group $W_{K_v}$ and $L_G = \hat{G}$. According to Langlands [24] the archimedean components $\pi_v$, $v|\infty$, are determined by the representations $r_v$ of the Weil groups $W_{K_v}$

$$r_v : W_{K_v} \longrightarrow \hat{G} = \text{GL}(r, \mathbb{C}) \quad (45)$$

if $G$ is the general linear group. In both cases $v \cong \mathbb{R}, \mathbb{C}$ the component $\pi_v$ is obtained from a representation of $\mathbb{C}^\times \subset W_v$, $v|\infty$, where $\mathbb{C}^\times = W_{\mathbb{C}}$. Denoting the restriction of $r_v$ to $\mathbb{C}^\times$ by

$$r_v|_{\mathbb{C}^\times} = r_\infty \quad (46)$$
leads to the map

$$r_\infty(\pi) : \quad W_C = \mathbb{C}^\times \longrightarrow \text{GL}(r, \mathbb{C})$$  \hspace{1cm} (47)

which characterizes the archimedean component $\pi_\infty$. Since $\mathbb{C}^\times$ is an abelian group all its representations decompose into 1-dimensional representations

$$r_\infty(\pi) = \bigoplus_{i=1}^{r} \chi^i_\infty,$$  \hspace{1cm} (48)

where the precise form of the characters $\chi^i_\infty$ depends on the structure of $\pi$.

For algebraic automorphic representations the $\chi^i_\infty$ take the form

$$\chi^i_\infty(z) = z^{r_i} z^{s_i}, \quad r_i, s_i \in \mathbb{Z},$$  \hspace{1cm} (49)

in an appropriate normalization of $\pi$. (In the literature several different conventions are adopted, depending on whether the viewpoint is more motivically oriented or more number theoretically oriented. This leads to different conditions for the exponents $r_i, s_i$.) The collection of pairs of integers

$$I_\infty(\pi) = \{(r_i, s_i) \mid r_i, s_i \in \mathbb{Z}\}$$  \hspace{1cm} (50)

is the infinity type of the algebraic automorphic representation $\pi$.

### 6.4  The tensor product of automorphic representations

One of the main problems in the theory of automorphic motives is to establish the automorphic analog of the tensor product. For this purpose automorphic product was introduced by Langlands [4] some time ago in the context of isobaric representations. While the automorphy of this product has not been established in general, for certain classes of forms and representations proofs have been found. The most important of these results for the present paper is the result of Ramakrishnan for the product $\text{GL}(2) \times \text{GL}(2)$ [29].
6.5 Automorphic \( L \)-functions

In the case of modular motives \( M \) the relation between the spacetime geometry and the worldsheet theory \( T_\Sigma \) can be made functionally by relating the \( L \)-function \( L(M, s) \) of modular motives to the \( L \)-function \( L(f, s) \) of a modular form \( f \) construction from \( T_\Sigma \)

\[
L(M, s) = L(f, s).
\]  \( \text{(51)} \)

The notion of a modular \( L \)-function is immediate by considering its Fourier expansion \( f(q) = \sum_n a_n q^n \) and the inverse Mellin transform. Such relations have been established in the context of Gepner models and singular fibers of deformations in these models in refs. \([5, 12]\). The extension of these results to automorphic motives involves the notion of an automorphic \( L \)-function \( L(\pi, s) = L(\phi, s) \), leading to an expected relation of the form

\[
L(M, s) = L(\pi, s).
\]  \( \text{(52)} \)

The precise form of this relation depends on the conventions.

The definition of \( L(\pi, s) \) for a representation \( \pi \) is most transparent for irreducible admissible representations \( \pi \) of \( \text{GL}(n) \) because it is known that each such representation can be factored into a restricted tensor product \( \pi = \otimes_v \pi_v \) such that each \( \pi_v \) is an irreducible admissible representation of \( \text{GL}(n) \) over the local fields \( K_v \) \([25]\). Furthermore, for almost all \( v \) the components \( \pi_v \) belong to the unramified principal series representations, which give rise to semisimple conjugacy classes \( A_v(\pi) \) in \( \text{GL}(n, \mathbb{C}) \) (ref. \([26]\) is useful for background material of these concepts). In general the conjugacy classes belong to the dual group \( \hat{G} \) of \( G \). The eigenvalues of these matrices are often called Satake parameters, and by abuse of notation the matrices \( A_v(\pi) \) will be called Satake matrices. The precise form of \( A_v(\pi) \) depends on the normalization of the automorphic representation and the different conventions adopted in the literature result in shifts of the argument \( s \) of the \( L \)-function. Given the Satake matrices \( A_v(\pi) \), the local factors of the \( L \)-function of the automorphic representation are defined as

\[
P_v(\pi, t) = \det (1 - A_v t),
\]  \( \text{(53)} \)
leading to the global $L$-function

$$L(\pi, s) := \prod_v \frac{1}{\mathcal{P}_v(A_v(\pi), p^{-s})}. \quad (54)$$

For a representation $r$ of $\hat{G} = \text{GL}(n, \mathbb{C})$ the $L$-function associated to $(\pi, r)$ can be defined via

$$\mathcal{P}_v(\pi, r, t) = \det(1 - r(A_v)t).$$

An agreement of the $L$-function is thus obtained by an agreement of the Hecke eigenvalue matrix $H_p(M)$ of the motive and the automorphic Satake matrix $A_v$.

## 7 The special cases GL(2) and GL(4)

In the present paper the focus will be on automorphic objects of rank two and four.

### 7.1 GL(2) automorphic objects: the space AutF\textsubscript{hol}(GL(2))

The structure of automorphic objects can be made most explicit when they are obtained via lifts of modular forms. This case in particular allows to make transparent the implications of the different normalizations that exist in the literature. Since in the present paper the focus is on automorphic motives it is natural to choose a normalization of the automorphic representations that reflect this relation in the simplest possible way. In a later section examples of GL(2) string automorphic motives arising from nondiagonal Calabi-Yau varieties are discussed in some detail.

For modular motives the most interesting class of modular forms is that of holomorphic cusp forms of some level $N$ with respect to the Hecke congruence group $\Gamma_0$ and some character $\epsilon_N$, the space of which will be denoted by $S_w(\Gamma_0(N), \epsilon_N)$. For fixed $N$ and weight $w$ this space is finite dimensional and formulae can be found in Shimura [27]. The lift of a modular form $f \in S_w(\Gamma_0(N), \epsilon_N)$ to an automorphic form on $\text{GL}(2)$ is obtained by noting that a quotient of the group $\text{GL}(2)^+$ generates the upper half-plane $\mathcal{H}$. Define the action of $g \in \text{GL}(2, \mathbb{R})^+$ on
$i = \sqrt{-1}$ by

$$
    g i = \frac{ai + b}{ci + d}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
$$

Since the stabilizer of $i$ is given by the subgroup $SO(2, \mathbb{R})$ and the center $Z(\mathbb{R})$, the upper half-plane can be interpreted as

$$
    \mathcal{H} = GL(2, \mathbb{R})^+/Z(\mathbb{R})SO(2, \mathbb{R}).
$$

The lift $\phi_f$ of a modular form of weight $w$ is defined as

$$
    \phi_f(g) = \frac{(\deg(g))^{w/2}}{(ci + d)^w} f(g(i)), \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2)^+.
$$

Since $f(\gamma \tau) = \epsilon_N(d)(c\tau + d)^w f(\tau)$ for $\gamma \in \Gamma_0(N) \subset SL(2, \mathbb{C})$ the automorphic form transforms just with the character $\epsilon_N$ under the congruence group. This relation can be inverted to construct a modular form $f$ from an automorphic forms $\phi$ as

$$
    f(\tau) = f(g(i)) = \left( \frac{ci + d}{\sqrt{\deg g}} \right)^w \phi(g).
$$

It is possible to intrinsically identify within AutF(GL(2)) the subspace of cusp forms which is the image of the above lift map (see e.g. ref. [28]).

The infinite component $\pi_v$ for $v|\infty$ of GL(2) forms is of discrete series representation type, denoted by $D_w$ for integral $w$, with a Langlands parameter $r_\infty = r_v(D_w) = r_v(D_w)|_{\mathbb{C}^\times}, v|\infty$, given for $\pi = \pi_f$ with $f$ of weight $w$ by

$$
    r_\infty(z) = \begin{pmatrix} z^{w-1} & 0 \\ 0 & \overline{z}^{w-1} \end{pmatrix}.
$$

The local factors $L_p(\pi, s)$ of the $L$-function $L(\pi, s)$ of the automorphic representation $\pi = \otimes_v \pi_v$ at the finite primes $v = p$ are described by rank two Satake matrices $A_p(\pi)$ whose elements are determined by the Fourier coefficients of the modular form. If $f$ is a Hecke eigenform its Fourier series can be written as an Euler product

$$
    L(f, s) = \prod_p \frac{1}{1 - a_p p^{-s} + \epsilon_N(p)p^{w-1}p^{-2s}}.
$$
Factorizing the polynomial \( P_p(f, t) = 1 - a_p t + \epsilon_N p^w t^2 = (1 - \gamma_p^{(1)} t)(1 - \gamma_p^{(2)} t) \) leads to the Hecke matrix

\[
H_p(f) = \begin{pmatrix} \gamma_p^{(1)} & 0 \\ 0 & \gamma_p^{(2)} \end{pmatrix}
\]

in terms of which the \( L \)-function can be determined via the local factors \( P_p(f, t) = \det (1 + H_p(f)t) \).

Identifying the Satake matrix \( A_p(\pi) \) with the Hecke matrix

\[
A_p(\pi) = H_p(f)
\]

thus leads to the automorphic \( L \)-function as

\[
L(\pi, s) = L(f, s).
\]

As noted earlier, other normalizations are in use as well, resulting in shifts in \( s \).

### 7.2 GL(4) automorphic objects

Automorphic forms and representations associated to higher rank groups behave quite differently from the automorphic forms associated to GL(2), in particular the subspace generated by holomorphic modular forms. The key here is that these forms are in general not induced by GL(2) forms, even the algebraic automorphic objects that are conjectured to arise from motivic. Part of the difference is that the archimedean components \( \pi_\infty \) of GL(\( n \)) automorphic forms are not of discrete series type for \( n > 2 \), but instead are tempered. Nevertheless, the infinity type takes a form that is of similar structure as the GL(2) infinity type, except that it involves more characters

\[
r_\infty(\pi) = \text{diag}(\chi_\infty^1, \ldots, \chi_\infty^4),
\]

with

\[
\chi_\infty^i(z) = z^{r_i} \bar{z}^{s_i} \quad r_i, s_i \in \mathbb{Z}
\]

where \( z \in \mathbb{C}^\times \).

The question thus becomes how for motivically induced automorphic forms and representations the infinity type is determined by the motive. In the applications below it will become
clear that the tempered representations of the archimedean components $\pi_\infty$ of $GL(n)$ can form a tensor product of the discrete series type representations $\pi_{\infty}^{(i)}$ of $GL(2)$ building blocks.

8 Automorphic motives

It is generally expected that pure motives are automorphic and that the class of automorphic objects obtained in this way coincides with those that are algebraic.

8.1 Hodge type and infinity type

The discussion of the cohomology type of the motives and the infinity type of algebraic automorphic forms has been summarized above in a form that emphasizes the similarity in structure: both types of objects can be represented by vector spaces that admit a filtration characterized by a vector of pairs of integers $(r_i, s_i) \in \mathbb{Z}^2$. If the rank $r$ of the automorphic representations is identified with the rank $\text{rk}(M)$ of the motive the above description makes it plausible to expect a relation of the form

$$H(M) \cong r_\infty(\pi) \otimes \chi, \quad (66)$$

where $\chi$ is a character that takes into account possible twists that may be chosen to implement different normalizations of the automorphic representation. If the infinity type $I_\infty(\pi)$ is chosen to be given by the Langlands normalization then the character $\chi$ of a $GL(n)$ representation is usually chosen to take the form $\chi = | \cdot |^{(1-n)/2} \mid_C$ (see e.g. [23]).

The decomposition of both the cohomology and the infinity type of the algebraic representation $\pi$ leads to the more precise relations

$$\bigoplus_{r_i+s_i=w_M} H^{r_i,s_i}(M) \cong \bigoplus_{i=1}^{\text{rk}(M)} \chi_\infty^i \otimes \chi. \quad (67)$$

The problem now is to put this conjectural relation into a computable form that can be tested.
In the following this will be considered in the context of Ω-motives, in particular for weighted Fermat hypersurfaces.

### 8.2 Structure of GL(2) automorphic motives

In this section the structure of GL(2) automorphic motives is made explicit. The key here is that the Hodge structure of the motive can take quite different values because the transition from the motive to the automorphic form involves the Tate twist. In the simplest case, relevant e.g. for Calabi-Yau varieties, the Hodge structure of an automorphic GL(2)-form of weight $w_\pi = w_f = w + 1$ e.g. is given by

$$H(M) \cong H^{w,0}(M) \oplus H^{0,w}(M).$$

(68)

In general this is not the case however. In the more general class of special Fano varieties, introduced in the context of mirror symmetry for rigid Calabi-Yau varieties [30, 31, 32], the Hodge structure of the modular motives (associated to GL(2) automorphic forms) takes the form

$$H(M) \cong H^{w+Q-1,Q-1} \oplus H^{Q-1,w+Q-1},$$

(69)

where $Q$ is a positive integer which can be thought of either as the total charge of the underlying Landau-Ginzburg model, or as the codimension of the critical variety associated to this Fano variety. More details about these spaces can be found in the above references.

Modular motives of such varieties and their mirror Calabi-Yau varieties have been discussed in [33]. The $L$-function that turns out to be relevant for the associated modular form is not determined by the motivic cardinalities $N_p(M_\Omega)$ per se, but by the renormalized cardinalities defined by

$$N^Q_p(M) := \frac{N_p(M)}{p^{Q-1}}.$$ 

(70)

The relation between the weight $w_\phi$ of the automorphic form and the weight $w_M$ of motive is

$$w_\phi = w_M + 1 - 2(Q - 1)$$

(71)
and the infinity type \( r_{\infty}(\pi^Q) \) of the automorphic form of the modular motive takes the form

\[
r_{\infty}(\pi^Q) = \begin{pmatrix} z^{w_M - 2(Q-1)} & 0 \\ 0 & \bar{z}^{w_M - 2(Q-1)} \end{pmatrix}.
\] (72)

**Remark.**

Historically, the construction of modular motives has focused on Kuga-Sato varieties, following the original work of Deligne [34], completed later by Scholl [35]. In string theory the focus is mostly on Calabi-Yau varieties, generalized to the class of special type Fano varieties in the context of mirror symmetry for rigid Calabi-Yau spaces.

### 9 Automorphic GL(2) motives in dimension two

In the case of rank two motives the automorphic structure can be linked to the more familiar framework of modular forms. This is useful not only because modular motives do appear in higher dimensional varieties, but because GL(2) automorphic representations also appear as building blocks of higher rank automorphic motives.

Perhaps the simplest nontrivial example of a nondiagonal modular motive is given by the \( \Omega \)-motive of the K3 surface

\[
X_2^{6D} = \left\{ z_0^6 + z_1^6 + z_2^3 + z_2 z_3^2 = 0 \right\} \subset \mathbb{P}(1,1,2,2).
\] (73)

This variety corresponds to the exactly solvable Gepner model given by the tensor product

\[
T_\Sigma^{6D} = (4_A^{\otimes 2} \otimes 4_D)_{\text{GSO}},
\] (74)

with central charge \( c = 6 \), and the subscript indicates the GSO projection. The Galois group \( \text{Gal}(K_X/\mathbb{Q}) \) is determined by \( v = \text{lcm}\{d_i\}_{i \neq n} = 6 \), hence leads to \( \text{Gal}(\mathbb{Q}(\mu_6)/\mathbb{Q}) = \{\sigma_1, \sigma_5\} \). The \( \Omega \)-motive of this surface is spanned by the Galois orbit of the vector \( u_\Omega \), leading to the realization

\[
H(M_\Omega) \cong \langle \text{Gal}(K_X/\mathbb{Q}), u_\Omega \rangle = u_\Omega \oplus \bar{u}_\Omega = (1,1,1,1) \oplus (5,5,0,1)
\] (75)
of the motive, where \( \bar{u}_\Omega \) is the vector dual to \( u_\Omega \).

Associated to the motive are Gauss sum products \( G_p(u) \) and \( G_p(\bar{u}_\Omega) \) which can be used to define the Hecke matrix

\[
H_p = -\frac{1}{p} \begin{pmatrix} G_p(u_\Omega) & 0 \\ 0 & G_p(\bar{u}_\Omega) \end{pmatrix}
\]

and the local factors \( P_p(t) = \det(1 + H_p(M)t) \). Here the Gauss sum product \( G_p(\bar{u}_\Omega) \) for the dual vector \( \bar{u}_\Omega \) is the complex conjugate of \( G_p(u_\Omega) \). The motivic cardinalities \( N_p(M_\Omega) \) lead to

\[
\text{tr } H_p(M_\Omega) = -N_p(M_\Omega),
\]

and the coefficients of the associated \( L \)-function are given by \( a_p(M_\Omega) = \text{tr } N_p(M_\Omega) \), as described in Section 4. The computation of the Gauss sum products \( G_p(u) \) leads to the \( L \)-function of the \( \Omega \)-motive

\[
L_\Omega(X^{6\text{D}}_2, s) = 1 - \frac{13}{7^s} - \frac{1}{13^s} + \frac{11}{19^s} - \frac{46}{31^s} + \frac{47}{37^s} + \cdots
\]

where \( \frac{1}{\zeta} \) denotes the incomplete \( L \)-function as usual. The question now is whether this \( L \)-function is modular and if so, what the associated form is.

One way to establish modularity and to explicitly determine the structure of the resulting modular form is by applying a motivic isomorphism between varieties of \( D \)-type and varieties of diagonal type [36]. The diagonal varieties will be called to be of \( A \)-type and denoted by \( X^A \), in reference to their affine \( A \)-type partition function invariants of the underlying exactly solvable conformal field theory. In the present case this motivic \( AD \)-isomorphism implies that the \( \Omega \)-motive \( M_\Omega(X^{6\text{D}}_2) \) of the \( D \)-type K3 surface is isomorphic to the \( \Omega \)-motive to the diagonal K3 surface given by

\[
X^{6_1A}_2 = \{ z_0^6 + z_1^6 + z_2^6 + z_3^2 = 0 \} \subset \mathbb{P}_{(1,1,1,3)}
\]

where the superscript superscript 6_1 is used because further below a second diagonal degree 6 K3 surface will appear.

The \( \Omega \)-motive of the surface (79) has been analyzed in detail in [37], where it was shown that its \( L \)-function is modular in terms of a form \( f_{3,27} \) of weight three and level 27

\[
L_\Omega(X^{6_1A}_2, s) = L(f_{3,27}, s).
\]
The form \( f_{3,27} \) is given in closed form by

\[
f_{3,27}(q) := \eta^2(q^3)\eta^2(q^9)\vartheta(q^3),
\]

where \( \vartheta(q) \) is a theta series \( \vartheta(q) = \sum_{z \in \mathcal{O}_K} q^{Nz} \) associated to the imaginary quadratic Eisenstein field \( K = \mathbb{Q}(\sqrt{-3}) \) with \( \mathcal{O}_K \) the ring of integers in this field. The string theoretic interpretation of \( f_{3,27} \) is most transparent by noting that it is the symmetric square of its \( \eta \)-product factor, which defines a modular form of weight two and level 27. This factor can be expressed in terms of the Hecke indefinite modular forms \( \Theta_{k,\ell,m} \) as

\[
f_{2,27}(\tau) = \Theta_{1,1}^1(3\tau)\Theta_{1,1}^1(q^3),
\]

a form that is associated to the elliptic diagonal curve \( E^3 \subset \mathbb{P}_2 \). The weight one forms

\[
\Theta_{k,\ell,m}^1(\tau) = \eta^3(\tau)c_{k,\ell,m}^1(\tau)
\]

are obtained in terms of the string functions \( c_{k,\ell,m}^1(\tau) \) of Kac-Peterson [38], associated to the algebra \( A^{(1)}_1 \). This affine Lie algebra is the basic building block of the \( N = 2 \) superconformal models of the Gepner models, hence provides a direct link between the worldsheet theory \( T_{\Sigma} \) and the motive of the compact variety. More details about this diagonal model can be found in [37].

The motivic \( AD \)-isomorphism implies that the \( L \)-function of the \( \Omega \)-motive of the nondiagonal K3 surface \( X_2^6D \) is equal to the \( L \)-function of the diagonal surface

\[
L_\Omega(X_2^{6D}, s) = L_\Omega(X_2^{6A}, s)
\]

hence modular in terms of \( f_{3,27} \). With this result the automorphic form \( \phi \) is given by the lift \( \phi_{3,27} = \phi_{f_{3,27}} \) and with the appropriate normalization the identity \( L_\Omega(X_2^{6D}, s) = L(f_{3,27}, s) \) translates into an identity between the motivic \( L \)-function and the automorphic \( L \)-function \( L(\phi_{3,27}, s) \).

The motivic \( AD \)-isomorphism implies furthermore that the Gauss sum products \( G_p(u) \) for \( u \in \{ u^\Omega, \overline{u}^\Omega \} \) are directly given in terms of the finite field Jacobi sums \( j_p(\alpha_\Omega) \) that describe the
diagonal surface, where \( \alpha_\Omega = \frac{\mu}{\Gamma} \). This shows that the infinity type of \( X_2^{6D} \) can be computed directly from the Weil formula, leading for \( a_\Omega = (1, 1, 1, 3) \) and \( \bar{\omega}_\Omega = (5, 5, 5, 3) \) to
\[
S(a_\Omega) = 2 \cdot \sigma_1 + 0 \sigma_5 \\
S(\bar{\omega}_\Omega) = 0 \sigma_1 + 2 \cdot \sigma_5.
\] (85)

Thus the infinity type
\[
\rho_\infty = \left( \begin{array}{cc}
\chi_\infty^1 & 0 \\
0 & \chi_\infty^2
\end{array} \right)
\] (86)
is given by \( \chi_\infty^1(z) = z^{S(\omega)} \) and \( \chi_\infty^2(z) = z^{S(\bar{\omega})} \), leading to the Hodge type of the corresponding motive \( M \) of the nondiagonal variety whose weight is given by \( w_M = n_1 + n_5 = 2 \)
\[
H(M) = H^{2,0}(M) \oplus H^{0,2}(M),
\] (87)
in the present case the motive is pure and the weight of the motive is given by the degree of the cohomology.

The key structure of such \( \text{GL}(2) \) automorphic representation is that the space of functions is 1-dimensional and that the corresponding automorphic form \( \phi = \phi_\pi \) is determined by the lift of a cusp modular form \( f(\tau) \) on the complex upper half-plane of some level \( N \) and a weight \( w_f \) that is determined by the weight \( w_M \) of the motive as \( w_f = w_M + 1 \).

The infinite component \( \pi_\infty \) of the automorphic representation \( \pi = \otimes_v \pi_v \) only determines the infinite factor of the completed \( L \)-function of the motive. The main arithmetic information is contained in the local factors \( \pi_p \) at the finite primes, similar to the local factors \( L_p(M_\Omega, s) \) of the motive, obtained from the local zeta functions \( Z(X/\mathbb{F}_p, t) \). Identifying the Satake matrix \( A_p(\pi) \) is identified with the Hecke matrix
\[
A_p(\pi) = H_p(M_\Omega)
\] (88)
then leads to the above relation between the motive and the automorphic representation determine via \( \phi_{3,27} \) as a lift of a weight 3 form with respect to the Hecke congruence group \( \Gamma_0(27) \).
Other nondiagonal K3 surfaces of $D$-type or of more general type can be discussed in a similar way. An example that does not correspond to an exactly solvable model of Gepner type but is string modular nevertheless is given by the K3 surface

$$X_2^{12\text{ND}} = \{ z_0^6 + z_1^3 + z_2^4 + z_2z_3^3 = 0 \} \subset \mathbb{P}_{(2,4,3,3)}. \tag{89}$$

In this case the field $K_X$ of the variety is given by $\mathbb{Q}(\mu_v)$ with (here $n = 2$)

$$v = \text{lcm}\{d_i\}_{i \neq n} = 6, \tag{90}$$

hence the rank of the motive is $\text{rk} \, M_{\Omega}(X_2^{12\text{ND}}) = 2$ and therefore is modular because the motive is of CM type. The modular form $f_{3,48}$ such that

$$L_{\Omega}(X_2^{12\text{ND}}, s) = L(f_{3,48}, s). \tag{91}$$

can be shown to be of weight three and level $N = 48$, with the closed form expression

$$f_{3,48}(q) = \eta^3(q^2)\eta^3(q^6) \otimes \chi_3, \tag{92}$$

where $\chi_3$ is a Legendre character of the type $\chi_n(p) = \left(\frac{n}{p}\right)$.

The form (92) was shown in [37] to be the symmetric square of the elliptic modular form $f_{2,144} \in S_2(\Gamma_0(144))$ which has a string interpretation in terms of the Hecke indefinite modular form $\Theta^1_{1,1}(\tau)$ as

$$f_{2,144}(q) = \Theta^1_{1,1}(q^6)^2 \otimes \chi_3. \tag{93}$$

This reflects the fact that the K3 surface $X_2^{12\text{ND}}$ is an elliptic fibration with a generic elliptic fiber in the configuration $E^6(\lambda) \subset \mathbb{P}_{(1,2,3)}$. The $L$-function of the diagonal curve $E^6$ was computed in [39] and is given by the weight two form above, leading to

$$L(E^6, s) = L(f_{2,144}, s). \tag{94}$$

It would be interesting to find an exactly solvable conformal field theory construction of this K3 surface that explains the $L$-function identity (91).
10 An automorphic GL(4)-motive in dimension three

In this section the automorphic structure of a rank four motive of the nondiagonal Calabi-Yau threefold

\[ X_{3}^{12D} := \{ z_0^{12} + z_1^{12} + z_2^6 + z_3^3 + z_3 z_2^2 = 0 \} \subset \mathbb{P}_{(1,1,2,4,4)} \]  

is considered. This variety is associated to an exactly solvable tensor model of the form

\[ T_{\Sigma}^{12D} = \left( 10^{\otimes 2} \otimes 4_A \otimes 6_D \right)_{GSO} \]

on the worldsheet.

The number field associated to this variety is the cyclotomic field \( K_X = \mathbb{Q}(\mu_{12}) \), which leads to the realization of the \( \Omega \)-motive as

\[ H(M_{\Omega}) = \langle \text{Gal}(K_X/\mathbb{Q}), u_\Omega \rangle = (1, 1, 1, 1) \oplus (5, 5, 5, 0, 1) \oplus (7, 7, 1, 1, 1) \oplus (11, 11, 5, 0, 1). \]  

The local factors \( L_p(M_{\Omega}, s) \) of the \( L \)-function of the \( \Omega \)-motive are therefore given by the Gauss sum products associated to the motivic vectors \( \sigma_{\ell}(u_{\Omega}) \in U_{\Omega} \)

\[ H_p(M_{\Omega}) = \frac{1}{p} \begin{pmatrix} \mathbb{G}_p(\sigma_1(u_{\Omega})) & \mathbb{G}_p(\sigma_5(u_{\Omega})) \\ \mathbb{G}_p(\sigma_7(u_{\Omega})) & \mathbb{G}_p(\sigma_{11}(u_{\Omega})) \end{pmatrix} \]  

via the polynomials

\[ \mathcal{P}_p(M_{\Omega}, t) = \det (1 + H_p(M_{\Omega})t) = \prod_{\ell \in (\mathbb{Z}/12\mathbb{Z})^\times} \left( 1 + \frac{1}{p} \mathbb{G}_p(\sigma_{\ell}(u_{\Omega}))t \right). \]

The resulting projective motivic cardinalities lead to the \( L \)-function coefficients

\[ a_p(M_{\Omega}) = -\frac{1}{p} \sum_{u \in \Omega} \mathbb{G}_p(u), \]

whose computation leads to the \( L \)-function

\[ L_{\Omega}(X_{3}^{12D}, s) = 1 - \frac{132}{13^s} + \frac{52}{37^s} + \frac{740}{61^s} - \frac{276}{73^s} - \frac{36}{97^s} - \frac{1284}{109^s} + \cdots \]
with \( \dot{\;} \) indicating the incomplete \( L \)-function as before.

The question is again whether this \( L \)-function is automorphic, and if so, what its structure is. The structure of this \( L \)-function can be understood by proceeding in a way analogous of the strategy adopted in [5] for rank four automorphic motives derived from diagonal varieties. The idea is to find modular motivic building blocks and to build the \( L \)-function of the \( \Omega \)-motive of the threefold via the Rankin-Selberg convolution of the lower-weight modular forms. In the present example it can be checked that the \( L \)-function of the modular form (92) of weight three and level 48 factors into the \( L \)-function (101). It was shown in [37] that this modular form is \( \Omega \)-motivic, with the motive coming from the K3 hypersurface

\[
X_2^{62A} = \{ z_0^6 + z_1^6 + z_2^3 + z_3^3 = 0 \} \subset \mathbb{P}_{(1,1,2,2)}.
\]  

As noted above, the string theoretic nature of the modular form (92) becomes apparently by noting that it is the symmetric square of a modular form of weight two and level 144 which is given in terms of the Hecke indefinite modular forms \( \Theta^{k}_{k,m} \) as \( f_{2,144}(q) = \Theta^{1}_{1,1}(q^6) \otimes \chi_3 \).

The quotient series obtained from the two \( L \)-series \( L_{\Omega}(X_3^{12D}) \) and \( L_{\Omega}(X_2^{62A}, s) \) is modular of weight two and level 256, hence the threefold \( L \)-function is the Rankin-Selberg convolution of two modular forms, much like in the case of the automorphic rank four motives discussed in ref. [5].

The proof of the automorphy of the rank four motive of the CY threefold can be completed by applying again the motivic isomorphism of [36] between motives of \( D \)-varieties and motives of \( A \)-varieties. In the present case this isomorphism implies that the \( \Omega \)-motive of the nondiagonal variety \( X_3^{12D} \) is identical to the \( \Omega \)-motive of the diagonal Calabi-Yau threefold

\[
X_3^{12A} := \{ z_0^{12} + z_1^{12} + z_2^6 + z_3^6 + z_4^2 = 0 \} \subset \mathbb{P}_{(1,1,2,2,6)},
\]  

which is known to be automorphic. General discussions aimed at nonexplicit automorphy obtained by base change for cyclic extension fields can be found in [40, 41] for extensions of prime degree and for non-prime degree in [42]. Automorphy of the Rankin-Selberg convolution of \( GL(2) \)-forms was established in ref. [29].
This result allows to compute the infinity type of the algebraic Hecke characters of the non-diagonal CM motive

\[ S(\sigma_r(a)) = \sum_{\ell \in \mathbb{Z}/d \mathbb{Z}} n^s_\ell \sigma_\ell \]

via the characters \( J_{\sigma_r(a)} \) associated to \( \mathbb{Q} (\mu_d) \) with \( d = 12 \). Weil’s formulae (42) and (43) lead to

\[ S(\sigma_1(a)) = 3\sigma_1 + \sigma_5 + 2\sigma_7 + 0\sigma_{11} \]
\[ S(\sigma_5(a)) = \sigma_1 + 3\sigma_5 + 0\sigma_7 + 2\sigma_{11} \]
\[ S(\sigma_7(a)) = 2\sigma_1 + 0\sigma_5 + 3\sigma_7 + \sigma_{11} \]
\[ S(\sigma_{11}(a)) = 0\sigma_1 + 2\sigma_5 + \sigma_7 + 3\sigma_{11}. \]

(105)

Put slightly different, the infinity type associated to the \( \Omega \)-motive \( M_\Omega \) can be viewed as an infinity type matrix \( S = (n^s_\ell) \), denoted by the same symbol.

The infinity matrix \( S \) determines the Hodge type of the motive via the motivic weight \( w_M = n^s_\ell + n^{cs}_r = 3 \) as

\[ H_{\sigma_r}(M_\Omega) = \bigoplus_{s=1}^{\deg E} H^{n^s_{\ell}, w-n^s_{\ell}} = H^{3,0} \oplus H^{1,2} \oplus H^{2,1} \oplus H^{0,3}. \]

(106)

In the present case the automorphy of the diagonal hypersurface implies that modular Hecke matrices coming from the \( \Omega \)-motive of the K3 surface \( X^6_{2B} \)

\[ H_p(M_\Omega(X^6_{2B})) = \frac{1}{p} \begin{pmatrix} \mathbb{G}_p(u^{(2)}_\Omega) & 0 \\ 0 & \mathbb{G}_p(w^{(2)}_\Omega) \end{pmatrix}, \]

(107)

and the Hecke matrix of the modular form \( f_{2,256} \), given by its Fourier coefficients \( f_{2,256}(q) = \sum_n a_n(f)q^n \) via

\[ a_p(f) = \alpha_p(f) + \beta_p(f) \]
\[ p = \alpha_p(f)\beta_p(f) \]

(108)

as

\[ H_p(f) = \begin{pmatrix} \alpha_p(f) & 0 \\ 0 & \beta_p(f) \end{pmatrix}. \]

(109)
lead to the rank four Hecke matrix of the threefold by considering the tensor product

$$H_p(M_\Omega(X_3^{12D})) = H_p(M_\Omega(X_2^{6B})) \otimes H_p(f).$$ (110)

The infinity types of the algebraic Hecke characters then lead to the following Langlands parameter of the archimedean component

$$r_\infty(z) = \begin{pmatrix} z^3 & z \bar{z}^2 & z^2 \bar{z} & z^6 \end{pmatrix}. \quad (111)$$

11 Outlook

In the present paper the focus has been on the automorphic structure of nondiagonal space-times whose underlying conformal field theory structure is given by exactly solvable theories that are of nondiagonal Gepner type, or Kazama-Suzuki type. Within these classes of varieties one can consider family spaces, which in the underlying conformal field theory correspond to deformations along marginal operators. Such deformations have previously been considered for diagonal Gepner models, with a particular focus on fibers in the families that are modular. For such points in the moduli space it is possible to ask whether the motivic modular forms are again related to the modular forms on the worldsheet theory $T_\Sigma$, hence whether the geometric forms admit a string theoretic interpretation. In ref. [12] such string theoretic modularity was shown to exist for mixed motives that arise from singular fibers in deformation families of diagonal varieties. This shows that the phase transitions between topologically distinct Calabi-Yau varieties described by modular mixed motives encode at least part of the conformal field theoretic structure of the rational points. This is a strong indication that the modular phase transitions that arise in these families are string theoretically consistent.

The results described in the present paper on automorphic motives of nondiagonal rational theories suggest a two-fold extension of the family analysis in [12]. It would be interesting to
consider fibers in the deformation families diagonal Gepner models that give rise to automorphic forms and representations rather than modular forms and to relate these automorphic structures to the conformal field theory on the worldsheet. For smooth fibers this involves automorphic pure motives, but in the case of singular fibers describing phase transitions this involves the concrete construction of automorphic representations associated to mixed motives. Such automorphic mixed motives are at present not understood, but conjecturally provide the most general framework that exists at present time.

In the context of the worldsheet theory work remains to be done in particular in the context of $G/H$ exactly solvable theories. Missing in these constructions extending the $SU(2)/U(1)$ theory underlying the Gepner models is an understanding of the string functions of Kac and Peterson that play a pivotal role in the Gepner models. The explicit construction of these objects along the lines of Kac-Peterson [38] would provide an important step in extending the automorphic spacetime program to a more comprehensive framework.

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