Turnpike Properties for Mean-Field Linear-Quadratic Optimal Control Problems

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Abstract. This paper is concerned with an optimal control problem for a mean-field linear stochastic differential equation with a quadratic functional in the infinite time horizon. Under suitable conditions, including the stabilizability, the (strong) exponential, integral, and mean-square turnpike properties for the optimal pair are established. The keys are to correctly formulate the corresponding static optimization problem and find the equations determining the correction processes. These have revealed the main feature of the stochastic problems which are significantly different from the deterministic version of the theory.

Key words. Strong turnpike property, mean-field, stochastic optimal control, linear-quadratic, static optimization, stabilizability, Riccati equation.

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1 Introduction

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete probability space on which a standard one-dimensional Brownian motion \(W = \{W(t); t \geq 0\}\) is defined and denote by \(\mathcal{F} = \{\mathcal{F}_t\}_{t \geq 0}\) the usual augmentation of the natural filtration generated by \(W\). Consider the following controlled linear mean-field stochastic differential equation (SDE, for short)

\[
\begin{aligned}
dX(t) = & \left\{ AX(t) + \bar{A}E[X(t)] + Bu(t) + \bar{B}E[u(t)] + b \right\} dt \\
& + \left\{ CX(t) + \bar{C}E[X(t)] + Du(t) + \bar{D}E[u(t)] + \sigma \right\} dW(t), \quad t \geq 0,
\end{aligned}
\]

and the quadratic cost functional

\[
J_T(x; u(\cdot)) = \mathbb{E} \int_0^T \left[ \left( \begin{array}{cc} Q & S^	op \\ S & R \end{array} \right) \left( \begin{array}{c} X(t) \\ u(t) \end{array} \right) \right] + 2 \left( \begin{array}{c} q \\ r \end{array} \right) \left( \begin{array}{c} X(t) \\ u(t) \end{array} \right) \\
+ \left( \begin{array}{cc} Q & S^	op \\ S & R \end{array} \right) \left( \begin{array}{c} E[X(t)] \\ E[u(t)] \end{array} \right) \left( \begin{array}{c} E[X(t)] \\ E[u(t)] \end{array} \right) \right] dt,
\]

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where $A, \bar{A}, C, \bar{C}, Q, \bar{Q}, R, \bar{R}, S, \bar{S}, R, \bar{R} \in \mathbb{R}^{n \times n}$, $B, \bar{B}, D, \bar{D} \in \mathbb{R}^{n \times m}$, and $R, \bar{R} \in \mathbb{R}^{m \times m}$ are constant matrices with $Q, \bar{Q}, R, \bar{R}$ being symmetric; the superscript $\top$ denotes the transpose of matrices; $\langle \cdot, \cdot \rangle$ denotes the inner product of two vectors; $b, \sigma, q \in \mathbb{R}^n$ and $r \in \mathbb{R}^m$ are constant vectors; $x \in \mathbb{R}^n$ is called an initial state, and the process $u(\cdot)$, called a control, is selected from the space

$$
\mathcal{U}[0,T] = \left\{ u : [0,T] \times \Omega \to \mathbb{R}^m \mid u \text{ is } \mathbb{F}\text{-progressively measurable, } \mathbb{E} \int_0^T |u(t)|^2 dt < \infty \right\}.
$$

We introduce the following problem.

**Problem (MFLQ)$_T$.** For a given initial state $x \in \mathbb{R}^n$, find a control $u_T(\cdot) \in \mathcal{U}[0,T]$ such that

$$
J_T(x; u_T(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}[0,T]} J_T(x; u(\cdot)) \equiv V_T(x).
$$

(1.3)

The above problem is referred to as a mean-field linear-quadratic (LQ, for short) optimal control problem over the finite time horizon $[0,T]$, whose homogenous version (i.e., the case of $b, \sigma, q = 0, r = 0$) was initially studied by Yong [20] (in which the coefficients of the state equation and the weighting matrices of the cost functional are allowed to be time-dependent) under the standard condition in LQ theory. The results of Yong [20] were later generalized by Sun [16] to the case of uniformly convex cost functionals with nonhomogenous terms and by Huang–Li–Yong [9] to the infinite time horizon case. Along this line, many researchers investigated the mean-field LQ problem from various points of view. For example, Basei–Pham [1] studied the mean-field LQ problem using a weak martingale approach; Li–Li–Yu [10] introduced the notion of relax compensators to deal with indefinite problems; and Lü [11] considered a mean-field LQ problem for stochastic evolution equations.

For a fixed time horizon, finite or infinite, Problem (MFLQ)$_T$ has been well studied in recent years, whose optimal control usually admits a closed-loop representation via the solutions of two related Riccati differential/algebra equations (see [17]). However, the limiting behavior of the optimal pair as the time horizon tends to infinity has not yet been fully discussed.

In the case that the diffusion term of the state equation (1.1) and the mean-field terms $\mathbb{E}[X(t)], \mathbb{E}[u(t)]$ are absent, Problem (MFLQ)$_T$ reduces to a deterministic LQ problem, denoted by Problem (LQ)$_T$, for which Porretta–Zuazua [14] and Trélat–Zuazua [19] obtained the following result: The optimal pair exponentially converges in the transient time (as $T \to \infty$) to the minimum point of the corresponding static optimization problem, under the controllability assumption on the state equation and the observability assumption on the cost functional. More precisely, it was shown in [14, 19] that there exist constants $K, \lambda > 0$, independent of $T$, such that the optimal pair $(X^*_T(\cdot), u^*_T(\cdot))$ of Problem (LQ)$_T$ satisfies

$$
|X^*_T(t) - x^*| + |u^*_T(t) - u^*| \leq K \left[ e^{-\lambda t} + e^{-\lambda(T-t)} \right], \quad \forall t \in [0,T],
$$

(1.4)

where $(x^*, u^*)$ is the solution to the following static optimization problem:

$$
\begin{align*}
\text{Minimize} \quad & F_0(x,u) \equiv \langle Qx, x \rangle + 2\langle Sx, u \rangle + \langle Ru, u \rangle + 2\langle q, x \rangle + 2\langle r, u \rangle, \\
\text{subject to} \quad & Ax + Bu + b = 0.
\end{align*}
$$

(1.5)
Clearly, (1.4) implies that
\[
|X_t^*(t) - x^*| + |u_t^*(t) - u^*| \leq 2Ke^{-\delta t}, \quad \forall t \in [\delta T, (1 - \delta)T],
\]
for any \( \delta \in (0, \frac{1}{2}) \). This means that in the major part of the time interval \([0, T]\), the optimal pair \((X_t^*(\cdot), u_t^*(\cdot))\) of Problem (LQ)\(_T\) stays close to the solution \((x^*, u^*)\) of the static optimization problem (1.5). Such a phenomenon is very similar to the turnpike in the US highway system. This is the main reason of the name “turnpike property” was made. Mathematically, we refer to (1.4) as the exponential turnpike property of the (deterministic) LQ optimal control problem.

The turnpike property was first realized by von Neumann \([13]\) for infinite time horizon deterministic optimal growth problems\(^1\). The name “turnpike” was first coined by Dorfman–Samuelson–Solow \([5]\) in 1958. See \([3]\) for an excellent survey. In the past several decades, the turnpike properties have attracted attentions of many researchers as such a property often gives people an essential picture of the optimal pair without solving it analytically and leads to a significant simplification in numerical methods for solving such kind of optimal control problems. A large number of papers have been published for finite and infinite dimensional problems in the context of discrete-time and continuous-time systems; see, for example, \([3, 22, 4, 24, 7, 12, 23, 2, 8, 15, 6]\) and the references therein.

The above mentioned works focus on deterministic optimal control problems. We now look at the stochastic case. To begin with, we first assume that
\[
\bar{A} = \bar{C} = 0, \quad \bar{B} = \bar{D} = 0, \quad \begin{pmatrix} Q & S^	op \\ S & \bar{R} \end{pmatrix} = 0,
\]
i.e., the problem does not involve mean-field terms. We denote the corresponding optimal control problem by Problem (SLQ)\(_T\). For such a problem, to discuss the turnpike properties of the optimal pair, the major difficulty is to correctly formulate the corresponding static optimization problem. Note that intuitively mimicking the situation of Problem (LQ)\(_T\) will lead to an incorrect static optimization problem. Recently, Sun–Wang–Yong \([18]\) investigated the turnpike properties for Problem (SLQ)\(_T\) and found that the correct static optimization problem takes the following form:
\[
\begin{aligned}
&\text{Minimize} \quad F_1(x, u) \equiv F_0(x, u) + \langle P(Cx + Du + \sigma), Cx + Du + \sigma \rangle, \\
&\text{subject to} \quad Ax + Bu + b = 0,
\end{aligned}
\]
where \( P > 0 \) is the solution to the following algebraic Riccati equation:
\[
PA + A^\top P + C^\top PC + Q \\
- (PB + C^\top PD + S^\top)(R + D^\top PD)^{-1}(B^\top P + D^\top C + S) = 0,
\]
whose solvability is guaranteed under some mild conditions. It was shown that the expectation of the optimal pair exhibits similar turnpike properties as the deterministic case, that is, for some positive constants \( K, \lambda > 0 \) independent of \( T \),
\[
|\mathbb{E}[X_t^*(t) - x^*]| + |\mathbb{E}[u_t^*(t) - u^*]| \leq K \left[ e^{-\lambda t} + e^{-\lambda(T-t)} \right], \quad \forall t \in [0, T].
\]
\(^1\)Later, these problems have been formulated as infinite time horizon optimal control problems.
It is worth noting that (1.8) only tells us \( \mathbb{E}[X^*_T(\cdot)], \mathbb{E}[u^*_T(\cdot)] \) converges exponentially in the transient time. In general, however, we could not expect the following:

\[
\mathbb{E}|X^*_T(t) - x^*| + \mathbb{E}|u^*_T(t) - u^*| \leq K \left[ e^{-\lambda t} + e^{-\lambda(T-t)} \right], \quad \forall t \in [0, T].
\]

(1.9)

But the behavior of the path \( t \mapsto (X^*_T(t), u^*_T(t)) \) seems to be more important and it is a crucial step to establish general nonlinear theory.

In this paper, we shall investigate the turnpike properties for Problem (MFLQ)\(_T\). The main novelty and contributions of the paper can be briefly summarized as follows.

- We are concerned with stochastic mean-field LQ problems. By including the expectations of the state and the control, our Problem (MFLQ)\(_T\) substantially extends the ones studied previously. The significance is that we have successfully find the correct formulation of the static optimization problem for the current case, merely under the stabilizability of the state equation, which covers that for Problem (SLQ)\(_T\) (without mean-field terms) presented in [18].

- As (1.9) could not be expected in general, it needs to be corrected. We find the proper replacement of \((x^*, u^*)\). It turns out that there exist stochastic processes \( X^*(\cdot) \) and \( u^*(\cdot) \) independent of \( T \), which can be determined explicitly, such that the optimal pair \((X^*_T(\cdot), u^*_T(\cdot))\) of Problem (MFLQ)\(_T\) satisfies the following strong turnpike property:

\[
\mathbb{E}|X^*_T(t) - X^*(t)|^2 + \mathbb{E}|u^*_T(t) - u^*(t)|^2 \leq K \left[ e^{-\lambda t} + e^{-\lambda(T-t)} \right], \quad \forall t \in [0, T],
\]

(1.10)

for some constants \( K, \lambda > 0 \) independent of \( T \). The processes \( X^*(\cdot) \) and \( u^*(\cdot) \) have time-invariant means \( x^* \) and \( u^* \), respectively, and \((x^*, u^*)\) is exactly the solution to the corresponding static optimization problem.

- We point out that for deterministic LQ problems, the controllability is normally assumed to establish the turnpike properties; see, for example, [14, 19, 12]. We find that it suffices to assume the stabilizability condition for the state equation, which is much weaker than the controllability condition. As a matter of fact, for controlled stochastic linear SDEs (with or without mean-field terms), the former is much natural and easy to check than the latter.

The rest of the paper is organized as follows. In Section 2, we make some necessary preliminaries by introducing some notation, assumptions, and a number of basic results that will be needed later. In Section 3, we introduce the static optimization problem associated with Problem (MFLQ)\(_T\) and state the main results of the paper. Section 4 is devoted to the stability of Riccati equations, which plays a central role in studying the turnpike properties. In Section 5, we give the proofs of the main results stated in Section 3.

2 Preliminaries

In this section, we introduce the basic notation and assumptions which will be used throughout this paper. We also collect a number of basic results on stochastic LQ optimal control problems and prove a simple matrix inequality for later use.
For notational simplicity, we let
\[ \hat{A} = A + \hat{A}, \quad \hat{B} = B + \hat{B}, \quad \hat{C} = C + \hat{C}, \quad \hat{D} = D + \hat{D}, \]
\[ \hat{Q} = Q + \hat{Q}, \quad \hat{S} = S + \hat{S}, \quad \hat{R} = R + \hat{R}. \]  
(2.1)

Denote by \( S^n \) the space of symmetric \( n \times n \) real matrices. For \( P, \Pi \in S^n \), we let
\[ Q(P) = PA + A^T P + C^T PC + Q, \quad \hat{Q}(P, \Pi) = \Pi \hat{A} + \hat{A}^T \Pi + \hat{C}^T P \hat{C} + \hat{Q}, \]
\[ S(P) = B^T P + D^T PC + S, \quad \hat{S}(P, \Pi) = B^T \Pi + \hat{D}^T P \hat{C} + \hat{S}, \]
\[ R(P) = R + D^T PD, \quad \hat{R}(P) = \hat{R} + \hat{D}^T P \hat{D}. \]
(2.2)

For matrices \( M, N \in S^n \), we will write \( M \geq N \) (respectively, \( M > N \)) if \( M - N \) is positive semi-definite (respectively, positive definite). The following basic assumptions will be imposed throughout the paper.

(A1) The weighting matrices in (1.2) satisfy
\[ R, \hat{R} > 0, \quad Q - S^T R^{-1} S > 0, \quad \hat{Q} - \hat{S}^T \hat{R}^{-1} \hat{S} > 0. \]

(A2) The controlled ordinary differential equation (ODE, for short)
\[ \dot{X}(t) = \hat{A}X(t) + \hat{B}u(t), \quad t \geq 0 \]  
(2.3)
is stabilizable, i.e., there exists a \( \hat{\Theta} \in \mathbb{R}^{m \times n} \) such that all the eigenvalues of \( \hat{A} + \hat{B} \hat{\Theta} \) have negative real parts. In this case, \( \hat{\Theta} \) is called a stabilizer of (2.3). The controlled SDE
\[ dX(t) = [AX(t) + Bu(t)]dt + [CX(t) + Du(t)]dW(t), \quad t \geq 0 \]  
(2.4)
is \( L^2 \)-stabilizable, i.e., there exists a \( \Theta \in \mathbb{R}^{m \times n} \) such that for every initial state \( x \), the solution \( X(\cdot) \) of
\[
\begin{cases}
    dX(t) = (A + B \Theta)X(t)dt + (C + D \Theta)X(t)dW(t), & t \geq 0, \\
    X(0) = x
\end{cases}
\]
satisfies \( \mathbb{E} \int_0^\infty |X(t)|^2 dt < \infty \). In this case, \( \Theta \) is called a stabilizer of (2.4).

Let \( C([0, T]; S^n) \) be the space of \( S^n \)-valued continuous functions on \([0, T]\). With the notation (2.1) and (2.2), the differential Riccati equations associated with Problem (MFLQ)_T can be respectively written as
\[
\begin{align*}
    \dot{P}_T(t) + Q(P_T(t)) - S(P_T(t))^T R(P_T(t))^{-1} S(P_T(t)) &= 0, \\
    P_T(T) &= 0,
\end{align*}
\]  
(2.5)

and
\[
\begin{align*}
    \dot{H}_T(t) + \hat{Q}(P_T(t), \Pi_T(t)) - \hat{S}(P_T(t), \Pi_T(t))^T \hat{R}(P_T(t))^{-1} \hat{S}(P_T(t), \Pi_T(t)) &= 0, \\
    \Pi_T(T) &= 0.
\end{align*}
\]  
(2.6)

We present the following result concerning the existence and uniqueness of solutions to (2.5) and (2.6). For a proof, we refer to Yong [20].
Lemma 2.1. Let (A1) hold. Then for any $T > 0$, the differential Riccati equations (2.5) and (2.6) admit a unique solution pair $(P_T(\cdot), \Pi_T(\cdot)) \in C([0, T]; \mathbb{S}^n) \times C([0, T]; \mathbb{S}^n)$ satisfying $P_T(t), \Pi_T(t) \geq 0$ for all $t \in [0, T]$.

When considering mean-field LQ optimal control problems over an infinite time horizon, we encounter algebraic Riccati equations (AREs, for short) instead of differential ones. The following result establishes the unique solvability of the associated AREs, whose proof can be found in [9]; see also [17, Chapter 3].

Lemma 2.2. Let (A1)–(A2) hold. Then the AREs
\begin{align}
Q(P) - S(P)^T R(P)^{-1} S(P) &= 0, \\
\hat{Q}(P, \Pi) - \hat{S}(P, \Pi)^T \hat{R}(P)^{-1} \hat{S}(P, \Pi) &= 0
\end{align}
admit a unique solution pair $(P, \Pi)$ satisfying $P, \Pi > 0$. Moreover, the matrix
$$\hat{\Theta} \triangleq -\hat{R}(P)^{-1} \hat{S}(P, \Pi)$$
is a stabilizer of system (2.3), and the matrix
$$\Theta \triangleq -R(P)^{-1} S(P)$$
is a stabilizer of system (2.4).

Comparing (2.7) with (2.5), and (2.8) with (2.6), we may guess that there are some connections between $P$ and $P_T(\cdot)$, and between $\Pi$ and $\Pi_T(\cdot)$. The following result, proved in [18], establishes the connection between $P$ and $P_T(\cdot)$. In Section 4, we will establish the connection between $\Pi$ and $\Pi_T(\cdot)$.

Lemma 2.3. Let (A1)–(A2) hold. Let $P_T(\cdot)$ and $P$ be the unique solutions of (2.5) and (2.7), respectively. Then there exist constants $K, \lambda > 0$, independent of $T$, such that
$$|P_T(t) - P| \leq Ke^{-\lambda(T-t)}, \quad \forall t \in [0, T].$$

We conclude this section with a simple matrix inequality that will be needed later.

Lemma 2.4. Let $M \in \mathbb{R}^{n \times m}$ and $K \in \mathbb{S}^m$. If $K > 0$, then
$$M(K + M^T M)^{-1} M^T \leq I.$$

Proof. Let $L \in \mathbb{R}^{n \times m}$ be an invertible matrix such that $K = L^T L$. Then
$$M(K + M^T M)^{-1} M^T = ML^{-1}[I + (ML^{-1})^T (ML^{-1})]^{-1} (ML^{-1})^T.$$
Thus, without loss of generality we may assume $K = I$. Observe that
$$M(I + M^T M)^{-1} M^T = MM^T (I + MM^T)^{-1}.$$
For any $\varepsilon > 0$, $N_\varepsilon \triangleq \varepsilon I + MM^T > 0$ and hence
$$N_\varepsilon (I + N_\varepsilon)^{-1} = (N_\varepsilon^{-1} + I)^{-1} \leq I.$$
Letting $\varepsilon \to 0$ yields the desired result. ■
3 Main Results

We present the main results of the paper in this section. The rigorous proofs are deferred to the subsequent sections.

First, let us introduce the static optimization problem associated with Problem \((\text{MFLQ})_T\). From the previous section we know (recall Lemma 2.2 and the notation of (2.1) and (2.2)) that under \((A1)\) and \((A2)\), the ARE (2.7) has a unique positive definite solution \(P\). Let

\[ V \triangleq \{ (x, u) \in \mathbb{R}^n \times \mathbb{R}^m \mid \hat{A}x + \hat{B}u + b = 0 \}, \]

(3.1)

and define a continuous function \(F : V \to \mathbb{R}\) by

\[
F(x, u) = \left\langle \hat{Q}x, x \right\rangle + \left\langle \hat{R}u, u \right\rangle + 2\left(\hat{S}x, u\right) + 2\langle q, x \rangle + 2\langle r, u \rangle + \left\langle P(\hat{C}x + \hat{D}u + \sigma), \hat{C}x + \hat{D}u + \sigma \right\rangle.
\]

(3.2)

The static optimization problem associated with Problem \((\text{MFLQ})_T\) can be stated as follows.

**Problem (O).** Find a pair \((x^*, u^*) \in V\) such that

\[
F(x^*, u^*) = \min_{(x, u) \in V} F(x, u) \equiv V.
\]

For the solvability of Problem (O), we have the following result.

**Proposition 3.1.** Let \((A1)\)–\((A2)\) hold. Then Problem (O) has a unique solution. Moreover, \((x^*, u^*)\) is the solution of Problem (O) if and only if there exists a unique \(\lambda^* \in \mathbb{R}^n\) such that

\[
\begin{align*}
\hat{A}x^* + \hat{B}u^* + b &= 0, \\
\hat{A}^\top \lambda^* + \hat{Q}x^* + \hat{C}^\top P(\hat{C}x^* + \hat{D}u^* + \sigma) + \hat{S}^\top u^* + q &= 0, \\
\hat{B}^\top \lambda^* + \hat{R}u^* + \hat{D}^\top P(\hat{C}x^* + \hat{D}u^* + \sigma) + \hat{S}x^* + r &= 0.
\end{align*}
\]

(3.3)

The proof of Proposition 3.1 is similar to the case without mean-field terms. We omit it here and refer the reader to [18] (or [21]) for details.

Let \((x^*, u^*)\) be the solution of Problem (O) and \(\lambda^* \in \mathbb{R}^n\) the vector in (3.3). Set

\[
\sigma^* = \hat{C}x^* + \hat{D}u^* + \sigma, \quad \Theta = -\mathcal{R}(P)^{-1}\mathcal{S}(P),
\]

(3.4)

where \(P\) is the solution to the ARE (2.7). Let \(X^*(\cdot)\) be the solution to the SDE

\[
\begin{align*}
\frac{dX^*(t)}{dt} &= (A + B\Theta)X^*(t)dt + [(C + D\Theta)X^*(t) + \sigma^*]dW(t), \quad t \geq 0, \\
X^*(0) &= 0,
\end{align*}
\]

(3.5)

and define

\[
X^*(t) \triangleq X^*(t) + x^*, \quad u^*(t) \triangleq \Theta X^*(t) + u^*.
\]

(3.6)

Clearly, \(\mathbb{E}[X^*(t)] = 0\). Hence, one has \(\mathbb{E}[X^*(t)] = x^*\) and \(\mathbb{E}[u^*(t)] = u^*\) for all \(t \geq 0\).

We now state the main result of the paper, which establishes the exponential turnpike property of Problem \((\text{MFLQ})_T\).
**Theorem 3.2.** Let (A1)–(A2) hold. Let \((X^*_T(\cdot), u^*_T(\cdot))\) be the optimal pair of Problem (MFLQ)_T for the initial state \(x\). Then there exist constants \(K, \lambda > 0\), independent of \(T\), such that

\[
\mathbb{E}|X^*_T(t) - X^*(t)|^2 + \mathbb{E}|u^*_T(t) - u^*(t)|^2 \leq K \left[ e^{-\lambda t} + e^{-\lambda(T-t)} \right], \quad \forall t \in [0, T].
\]  

(3.7)

We call (3.7) the strong exponential turnpike property for the optimal pair \((X^*_T(\cdot), u^*_T(\cdot))\). The above result has several consequences. The first corollary establishes the strong integral and the mean-square turnpike properties for Problem (MFLQ)_T, whose proof is direct.

**Corollary 3.3.** Let (A1)–(A2) hold. Then,

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \mathbb{E} \left[ |X^*_T(t) - X^*(t)|^2 + |u^*_T(t) - u^*(t)|^2 \right] dt = 0.
\]

Consequently,

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \left[ |\mathbb{E}[X^*_T(t)] - x^*|^2 + |\mathbb{E}[u^*_T(t)] - u^*|^2 \right] dt = 0.
\]

The second corollary shows that for any initial state \(x\), the value \(V_T(x)\) of Problem (MFLQ)_T converges to the minimum of Problem (O) in the time-average sense.

**Corollary 3.4.** Let (A1)–(A2) hold. Then

\[
\lim_{T \to \infty} \frac{1}{T} V_T(x) = V, \quad \forall x \in \mathbb{R}^n.
\]

Let \((X^*_T(\cdot), u^*_T(\cdot))\) be the optimal pair of Problem (MFLQ)_T for the initial state \(x\), and let \((Y^*_T(\cdot), Z^*_T(\cdot))\) be the adjoint process, i.e., the adapted solution to the following mean-field backward SDE:

\[
\begin{align*}
\begin{cases}
\frac{dY^*_T(t)}{dt} &= - \{ A^\top Y^*_T(t) + \bar{A}^\top \mathbb{E}[Y^*_T(t)] + C^\top Z^*_T(t) + \bar{C}^\top \mathbb{E}[Z^*_T(t)] + QX^*_T(t) \\
&\quad + Q \mathbb{E}[X^*_T(t)] + S^\top u^*_T(t) + S^\top \mathbb{E}[u^*_T(t)] + q \} dt + Z^*_T(t)dW(t), \\
Y^*_T(T) &= 0.
\end{cases}
\end{align*}
\]

(3.8)

The next corollary shows that a strong exponential turnpike property also holds for the adjoint process.

**Corollary 3.5.** Let (A1)–(A2) hold. Let \(Y^*(t) \triangleq PX^*(t) + \lambda^*, \quad Z^*(t) \triangleq P(C + D \Theta)X^*(t) + P\sigma^*\). Then for some constants \(K, \lambda > 0\) independent of \(T\),

\[
\mathbb{E}|Y^*_T(t) - Y^*(t)|^2 + \mathbb{E}|Z^*_T(t) - Z^*(t)|^2 \leq K \left[ e^{-\lambda t} + e^{-\lambda(T-t)} \right], \quad \forall t \in [0, T].
\]

(3.10)
4 Stability of Riccati Equations

As mentioned in Section 2, there is a certain connection between the solutions to the differential Riccati equation (2.6) and the ARE (2.8). We now examine this connection. The main result of this section is as follows, which plays a central role in the study of turnpike properties.

**Theorem 4.1.** Let (A1)–(A2) hold. Let $\Pi_T(\cdot)$ and $\Pi(\cdot)$ be the unique solutions of (2.6) and (2.8), respectively. Then there exist positive constants $K, \lambda > 0$ such that

$$|\Pi_T(t) - \Pi| \leq Ke^{-\lambda(T-t)}, \quad \forall 0 \leq t \leq T < \infty.$$  \hfill(4.1)

Before going further, let us point out that in order to prove Theorem 4.1, we can assume without loss of generality that

$$S = 0, \quad \bar{S} = 0.$$  \hfill(4.2)

Indeed, let

$$A = A - BR^{-1}S, \quad C = C - DR^{-1}S, \quad Q = Q - S^T R^{-1}S,$$  \hfill(4.3)

and let $\mathcal{P}(\cdot)$ be the positive semi-definite solution to the Riccati equation

$$\begin{cases}
\dot{\mathcal{P}} + \mathcal{P}A + A^T \mathcal{P} + C^T \mathcal{P}C + Q \\
-(\mathcal{P}B + C^T \mathcal{P}D)(R + D^T \mathcal{P}D)^{-1}(B^T \mathcal{P} + D^T \mathcal{P}C) = 0,
\end{cases}$$  \hfill(4.4)

We have

$$\mathcal{P}A + A^T \mathcal{P} + C^T \mathcal{P}C + Q = \mathcal{P}A + A^T \mathcal{P} + C^T \mathcal{P}C + Q - (\mathcal{P}B + C^T \mathcal{P}D + S^T)R^{-1}S$$
$$- S^T R^{-1}(B^T P + D^T \mathcal{P}C + S) + S^T R^{-1}(R + D^T \mathcal{P}D)R^{-1}S.$$  \hfill(4.5)

On the other hand,

$$\mathcal{P}B + C^T \mathcal{P}D = (\mathcal{P}B + C^T \mathcal{P}D + S^T) - S^T R^{-1}(R + D^T \mathcal{P}D),$$

and hence

$$(\mathcal{P}B + C^T \mathcal{P}D)(R + D^T \mathcal{P}D)^{-1}(B^T \mathcal{P} + D^T \mathcal{P}C)$$
$$= (\mathcal{P}B + C^T \mathcal{P}D + S^T)(R + D^T \mathcal{P}D)^{-1}(B^T \mathcal{P} + D^T \mathcal{P}C + S)$$
$$- (\mathcal{P}B + C^T \mathcal{P}D + S^T)R^{-1}S - S^T R^{-1}(B^T \mathcal{P} + D^T \mathcal{P}C + S)$$
$$+ S^T R^{-1}(R + D^T \mathcal{P}D)R^{-1}S.$$  \hfill(4.6)

Substituting (4.5) and (4.6) into (4.4), we see that $\mathcal{P}(\cdot)$ satisfies the same equation as $P_T(\cdot)$, which implies $\mathcal{P}(\cdot) = P_T(\cdot)$. By a similar argument, we can show that $\Pi_T(\cdot)$ satisfies

$$\begin{cases}
\Pi_T(t) + \Pi_T \hat{A} + \hat{A}^T \Pi_T + \hat{C}^T P_T \hat{C} + \hat{Q} \\
-(\Pi_T \hat{B} + \hat{C}^T P_T \hat{D})(\hat{R} + \hat{D}^T P_T \hat{C})^{-1}(\hat{B}^T \Pi_T + \hat{D}^T P_T \hat{C}) = 0,
\end{cases}$$  \hfill(4.7)

$$\Pi_T(T) = 0,$$
where
\[
\hat{A} = \hat{A} - \hat{B}\hat{R}^{-1}\hat{S}, \quad \hat{C} = \hat{C} - \hat{D}\hat{R}^{-1}\hat{S}, \quad \hat{Q} = \hat{Q} - \hat{S}^\top\hat{R}^{-1}\hat{S}.
\] (4.8)

Thus, by making transformations (4.3) and (4.8), we can assume without loss of generality that (4.2) holds.

To prove Theorem 4.1, we need the following lemma.

**Lemma 4.2.** Let (A1) hold. Then for any positive semi-definite matrix \( \Delta \geq 0 \),
\[
\begin{pmatrix}
\hat{C}^\top \Delta \hat{C} + \hat{Q} & \hat{C}^\top \Delta \hat{D} \\
\hat{D}^\top \Delta \hat{C} & \hat{R}(\Delta)
\end{pmatrix} \geq 0,
\]
or equivalently,
\[
\hat{C}^\top \Delta \hat{C} + \hat{Q} - \left(\hat{C}^\top \Delta \hat{D}\right)\hat{R}(\Delta)^{-1}\left(\hat{D}^\top \Delta \hat{C}\right) \geq 0.
\]

**Proof.** Since \( \hat{R} > 0 \) and \( \Delta \geq 0 \), we have
\[
\hat{R}(\Delta) = \hat{R} + \hat{D}^\top \Delta \hat{D} \geq \hat{R} > 0.
\] (4.9)

Let \( M \in \mathbb{R}^{n \times n} \) be such that \( \Delta = M^\top M \). Then by Lemma 2.4,
\[
I - \left(M\hat{D}\right)\hat{R}(\Delta)^{-1}\left(\hat{D}^\top M^\top\right) = I - \left(M\hat{D}\right)\left[\hat{R} + \left(M\hat{D}\right)^\top\left(M\hat{D}\right)\right]^{-1}\left(M\hat{D}\right)^\top \geq 0.
\]
Consequently,
\[
\hat{C}^\top \Delta \hat{C} + \hat{Q} - \left(\hat{C}^\top \Delta \hat{D}\right)\hat{R}(\Delta)^{-1}\left(\hat{D}^\top \Delta \hat{C}\right) = \hat{Q} + \hat{C}^\top \left[\Delta - \left(\Delta \hat{D}\right)\hat{R}(\Delta)^{-1}\left(\hat{D}^\top \Delta\right)\right] \hat{C}
\]
\[
= \hat{Q} + \left(M\hat{C}\right)^\top \left[I - \left(M\hat{D}\right)\hat{R}(\Delta)^{-1}\left(\hat{D}^\top M^\top\right)\right] \left(M\hat{C}\right) \geq 0.
\]
The proof is complete. \( \blacksquare \)

In the following proof and in the sequel, we shall denote by \( K \) and \( \lambda \) two generic positive constants, which do not depend on \( T \) and may vary from line to line.

**Proof of Theorem 4.1.** As mentioned previously, we may assume \( S = 0 \) and \( \hat{S} = 0 \) to simplify the proof. Define for \( t \in [0, \infty) \),
\[
\Sigma(t) \triangleq P_T(T - t), \quad \Upsilon(t) \triangleq \Pi_T(T - t); \quad \text{if } t \leq T.
\]
It is shown in [18] that the definition of \( \Sigma(t) \) does not depend on the choice of \( T \geq t \), and
\[
0 < \Sigma(s) \leq \Sigma(t), \quad |\Sigma(t) - P| \leq Ke^{-\lambda t}, \quad \forall 0 \leq s \leq t < \infty, \tag{4.10}
\]
for some positive constants \( K, \lambda > 0 \). A similar argument shows that the definition of \( \Upsilon(t) \) does not depend on the choice of \( T \geq t \) either, and \( \Upsilon(\cdot) \) satisfies the following ODE:
\[
\begin{cases}
\Upsilon(t) = \hat{Q}(\Sigma(t), \Upsilon(t)) - \hat{S}(\Sigma(t), \Upsilon(t))^\top \hat{R}(\Sigma(t))^{-1}\hat{S}(\Sigma(t), \Upsilon(t)), \\
\Upsilon(0) = 0.
\end{cases} \tag{4.11}
\]
Step 1. We first show that $\lim_{t \to \infty} \mathcal{T}(t) = \Pi$.

Consider the deterministic LQ optimal control problem with state equation

$$
\begin{cases}
\dot{X}(t) = \hat{A}X(t) + \hat{B}u(t), & t \in [0, T], \\
X(0) = x,
\end{cases}
$$

and cost functional

$$
\mathcal{J}_T(x; u(\cdot)) = \int_0^T \left\langle \begin{pmatrix} \hat{C}^\top P_T(t)\hat{C} + \hat{Q} & \hat{C}^\top P_T(t)\hat{D} \\ \hat{D}^\top P_T(t)\hat{C} & \hat{R}(P_T(t)) \end{pmatrix} \begin{pmatrix} X(t) \\ u(t) \end{pmatrix}, \begin{pmatrix} X(t) \\ u(t) \end{pmatrix} \right\rangle dt.
$$

Since $P_T(t) > 0$,

$$
\hat{R}(P_T(t)) = \hat{R} + \hat{D}^\top P_T(t)\hat{D} \geq \hat{R} > 0,
$$

and by Lemma 4.2,

$$
\left( \begin{array}{cc} \hat{C}^\top P_T(t)\hat{C} + \hat{Q} & \hat{C}^\top P_T(t)\hat{D} \\ \hat{D}^\top P_T(t)\hat{C} & \hat{R}(P_T(t)) \end{array} \right) \geq 0.
$$

Because of (4.12) and (4.13), the above deterministic LQ problem is uniquely solvable with value function given by $\langle \Pi_T(0)x, x \rangle$. Note that $\mathcal{J}_T(x; u(\cdot))$ can be written as

$$
\mathcal{J}_T(x; u(\cdot)) = \int_0^T \left\langle \begin{pmatrix} \hat{C}^\top \Sigma(T - t)\hat{C} + \hat{Q} & \hat{C}^\top \Sigma(T - t)\hat{D} \\ \hat{D}^\top \Sigma(T - t)\hat{C} & \hat{R}(\Sigma(T - t)) \end{pmatrix} \begin{pmatrix} X(t) \\ u(t) \end{pmatrix}, \begin{pmatrix} X(t) \\ u(t) \end{pmatrix} \right\rangle dt.
$$

For $T_2 \geq T_1 > 0$, set

$$
\Delta(t) = \Sigma(T_2 - t) - \Sigma(T_1 - t), \quad 0 \leq t \leq T_1.
$$

Since $\Sigma(\cdot)$ is nondecreasing, $\Delta(t) \geq 0$ and hence by Lemma 4.2,

$$
\left( \begin{array}{cc} \hat{C}^\top \Delta(t)\hat{C} + \hat{Q} & \hat{C}^\top \Delta(t)\hat{D} \\ \hat{D}^\top \Delta(t)\hat{C} & \hat{R}(\Delta(t)) \end{array} \right) \geq 0.
$$

Thus, for any square-integrable $u : [0, T_2] \to \mathbb{R}^n$,

$$
\mathcal{J}_{T_2}(x; u(\cdot)) - \mathcal{J}_{T_1}(x; u(\cdot)) \\
\geq \int_0^{T_1} \left\langle \begin{pmatrix} \hat{C}^\top \Sigma(T_2 - t)\hat{C} + \hat{Q} & \hat{C}^\top \Sigma(T_2 - t)\hat{D} \\ \hat{D}^\top \Sigma(T_2 - t)\hat{C} & \hat{R}(\Sigma(T_2 - t)) \end{pmatrix} \begin{pmatrix} X(t) \\ u(t) \end{pmatrix}, \begin{pmatrix} X(t) \\ u(t) \end{pmatrix} \right\rangle dt \\
- \int_0^{T_1} \left\langle \begin{pmatrix} \hat{C}^\top \Sigma(T_1 - t)\hat{C} + \hat{Q} & \hat{C}^\top \Sigma(T_1 - t)\hat{D} \\ \hat{D}^\top \Sigma(T_1 - t)\hat{C} & \hat{R}(\Sigma(T_1 - t)) \end{pmatrix} \begin{pmatrix} X(t) \\ u(t) \end{pmatrix}, \begin{pmatrix} X(t) \\ u(t) \end{pmatrix} \right\rangle dt \\
= \int_0^{T_1} \left\langle \begin{pmatrix} \hat{C}^\top \Delta(t)\hat{C} + \hat{Q} & \hat{C}^\top \Delta(t)\hat{D} \\ \hat{D}^\top \Delta(t)\hat{C} & \hat{R}(\Delta(t)) \end{pmatrix} \begin{pmatrix} X(t) \\ u(t) \end{pmatrix}, \begin{pmatrix} X(t) \\ u(t) \end{pmatrix} \right\rangle dt \\
\geq 0.
$$

This implies that for $0 \leq T_1 \leq T_2$,

$$
\langle \Pi_{T_1}(0)x, x \rangle \leq \langle \Pi_{T_2}(0)x, x \rangle, \quad \forall x \in \mathbb{R}^n.
$$
Since $x$ is arbitrary, $\mathcal{Y}(t) = \Pi_t(0)$ is nondecreasing in $t$. On the other hand, take $\hat{\Theta}$ to be a stabilizer of the system (2.3). By (4.10), $P_t(t) \leq P$ and hence (recalling Lemma 4.2)

\[
\langle \Pi_t(0) x, x \rangle \leq \int_0^T \left( \begin{bmatrix} \hat{C}^T P_t(t) \hat{C} + \hat{Q} & \hat{C}^T P_t(t) \hat{D} \\ \hat{D}^T P_t(t) \hat{C} & \hat{R}(P_t(t)) \end{bmatrix} \begin{bmatrix} X(t) \\ \hat{R}X(t) \end{bmatrix} \right) dt \\
\leq \int_0^T \left( \begin{bmatrix} \hat{C}^T P \hat{C} + \hat{Q} & \hat{C}^T \hat{D} \\ \hat{D}^T \hat{C} & \hat{R}(P) \end{bmatrix} \begin{bmatrix} X(t) \\ \hat{R}X(t) \end{bmatrix} \right) dt \\
\leq \int_0^\infty \left( \begin{bmatrix} \hat{C}^T P \hat{C} + \hat{Q} & \hat{C}^T \hat{D} \\ \hat{D}^T \hat{C} & \hat{R}(P) \end{bmatrix} \begin{bmatrix} X(t) \\ \hat{R}X(t) \end{bmatrix} \right) dt < \infty.
\]

Thus, $\mathcal{Y}(t) = \Pi_t(0)$ is bounded above, which, together with the monotonicity of $\mathcal{Y}(t)$, implies that

\[
\mathcal{Y}_\infty \equiv \lim_{t \to \infty} \mathcal{Y}(t)
\]

exists and is finite. From (4.11) we have

\[
\mathcal{Y}(t + 1) - \mathcal{Y}(t) = \int_t^{t+1} \hat{Q}(\Sigma(s), \mathcal{Y}(s)) - \hat{S}(\Sigma(s), \mathcal{Y}(s))^T \hat{R}(\Sigma(s))^{-1} \hat{S}(\Sigma(s), \mathcal{Y}(s)) ds.
\]

Letting $t \to \infty$ yields

\[
\hat{Q}(P, \mathcal{Y}_\infty) - \hat{S}(P, \mathcal{Y}_\infty)^T \hat{R}(P)^{-1} \hat{S}(P, \mathcal{Y}_\infty) = 0.
\]

By uniqueness, $\mathcal{Y}_\infty = \Pi$.

**Step 2.** Set $\Delta(t) \equiv \mathcal{Y}(t) - \Pi$ and $\Lambda(t) \equiv \Sigma(t) - P$. We show that $\Delta(\cdot)$ satisfies the following ODE:

\[
\Delta(t) = \Delta(t) \hat{A} + \hat{A}^T \Delta(t) + f(\Delta(t), \Lambda(t)) + g(\Lambda(t)), \quad (4.14)
\]

where

\[
\hat{A} \equiv \hat{A} + B \hat{\Theta} \quad \text{with} \quad \hat{\Theta} \equiv -\hat{R}(P)^{-1} \hat{S}(P, \Pi),
\]

\[
f(\Delta(t), \Lambda(t)) \equiv -\hat{S}(\Lambda(t), \Delta(t))^T \hat{R}(\Sigma(t))^{-1} \hat{S}(\Lambda(t), \Delta(t))
\-
\left( (\hat{D} \hat{\Theta})^T \Lambda(t) \hat{D} \hat{R}(\Sigma(t))^{-1} \hat{B}^T \Delta(t) \right)^T
\-
\left( (\hat{D} \hat{\Theta})^T \Lambda(t) \hat{D} \hat{R}(\Sigma(t))^{-1} \hat{B}^T \Delta(t) \right),
\]

\[
g(\Lambda(t)) \equiv \hat{C}^T \Lambda(t) \hat{C} - (\hat{D} \hat{\Theta})^T \Lambda(t) \hat{D} \hat{R}(\Sigma(t))^{-1} \hat{S}(P, \Pi)
\-
\left( \hat{S}(P, \Pi)^T \hat{R}(\Sigma(t))^{-1} \hat{D}^T \Lambda(t) \hat{C} \right)^T
\-
\left( \hat{S}(P, \Pi)^T \hat{R}(\Sigma(t))^{-1} \hat{D}^T \Lambda(t) \hat{C} \right).
\]

To verify (4.14), we first observe that

\[
\dot{\Delta}(t) = \Delta(t) \hat{A} + \hat{A}^T \Delta(t) + \hat{C}^T \Lambda(t) \hat{C}
\-
\left[ \hat{B}^T \mathcal{Y}(t) + \hat{D}^T \Sigma(t) \hat{C} \right]^T \left[ \hat{R} + \hat{D}^T \Sigma(t) \hat{D} \right]^{-1} \left[ \hat{B}^T \mathcal{Y}(t) + \hat{D}^T \Sigma(t) \hat{C} \right], \quad (4.15)
\]

\[
+ \left[ \hat{B}^T \Pi + \hat{D}^T P \hat{C} \right]^T \left[ \hat{R} + \hat{D}^T P \hat{D} \right]^{-1} \left[ \hat{B}^T \Pi + \hat{D}^T P \hat{C} \right].
\]

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Noting that
\[ \hat{B}^\top T + \hat{D}^\top \Sigma(t)\hat{C} = \left[ \hat{B}^\top \Delta(t) + \hat{D}^\top \Lambda(t)\hat{C} \right] + \left[ \hat{B}^\top \Pi + \hat{D}^\top P\hat{C} \right], \]
we have
\[
\left[ \hat{B}^\top Y(t) + \hat{D}^\top \Sigma(t)\hat{C} \right]^\top \left[ \hat{R} + \hat{D}^\top \Sigma(t)\hat{D} \right]^{-1} \left[ \hat{B}^\top Y(t) + \hat{D}^\top \Sigma(t)\hat{C} \right] \\
= \left[ \hat{B}^\top \Delta(t) + \hat{D}^\top \Lambda(t)\hat{C} \right]^\top \left[ \hat{R} + \hat{D}^\top \Sigma(t)\hat{D} \right]^{-1} \left[ \hat{B}^\top \Delta(t) + \hat{D}^\top \Lambda(t)\hat{C} \right] \\
+ \left[ \hat{B}^\top \Pi + \hat{D}^\top P\hat{C} \right]^\top \left[ \hat{R} + \hat{D}^\top \Sigma(t)\hat{D} \right]^{-1} \left[ \hat{B}^\top \Pi + \hat{D}^\top P\hat{C} \right] \\
= \left[ \hat{R} + \hat{D}^\top \Sigma(t)\hat{D} \right]^{-1} \hat{D}^\top \Lambda(t)\hat{D} \left[ \hat{R} + \hat{D}^\top \Sigma(t)\hat{D} \right]^{-1}.
\]

Thus,
\[
\text{(2)} = \left[ \hat{B}^\top \Pi + \hat{D}^\top P\hat{C} \right]^\top \left[ \hat{R} + \hat{D}^\top \Sigma(t)\hat{D} \right]^{-1} \left[ \hat{B}^\top \Pi + \hat{D}^\top P\hat{C} \right] \\
- \left[ \hat{B}^\top \Pi + \hat{D}^\top P\hat{C} \right]^\top \left[ \hat{R} + \hat{D}^\top \Sigma(t)\hat{D} \right]^{-1} \hat{D}^\top \Lambda(t)\hat{D} \left[ \hat{R} + \hat{D}^\top \Sigma(t)\hat{D} \right]^{-1} \left[ \hat{B}^\top \Pi + \hat{D}^\top P\hat{C} \right] \\
= \left[ \hat{B}^\top \Pi + \hat{D}^\top P\hat{C} \right]^\top \left[ \hat{R} + \hat{D}^\top \Sigma(t)\hat{D} \right]^{-1} \left[ \hat{B}^\top \Pi + \hat{D}^\top P\hat{C} \right] \\
+ (\hat{D}\hat{\Theta})^\top \Lambda(t)\hat{D} \hat{R}(\Sigma(t))^{-1} \hat{S}(P, \Pi),
\]
and
\[
\text{(4)} = \left[ \hat{B}^\top \Pi + \hat{D}^\top P\hat{C} \right]^\top \left[ \hat{R} + \hat{D}^\top \Sigma(t)\hat{D} \right]^{-1} \hat{D}^\top \Lambda(t)\hat{C} \\
+ \left[ \hat{B}^\top \Pi + \hat{D}^\top P\hat{C} \right]^\top \left[ \hat{R} + \hat{D}^\top \Sigma(t)\hat{D} \right]^{-1} \hat{B}^\top \Delta(t) \\
= \left[ \hat{B}^\top \Pi + \hat{D}^\top P\hat{C} \right]^\top \left[ \hat{R} + \hat{D}^\top \Sigma(t)\hat{D} \right]^{-1} \hat{D}^\top \Lambda(t)\hat{C} \\
+ \left[ \hat{B}^\top \Pi + \hat{D}^\top P\hat{C} \right]^\top \left[ \hat{R} + \hat{D}^\top \Sigma(t)\hat{D} \right]^{-1} \hat{B}^\top \Delta(t) \\
- \left[ \hat{B}^\top \Pi + \hat{D}^\top P\hat{C} \right]^\top \left[ \hat{R} + \hat{D}^\top \Sigma(t)\hat{D} \right]^{-1} \hat{D}^\top \Lambda(t)\hat{D} \left[ \hat{R} + \hat{D}^\top \Sigma(t)\hat{D} \right]^{-1} \hat{B}^\top \Delta(t) \\
= \hat{S}(P, \Pi)^\top \hat{R}(\Sigma(t))^{-1} \hat{D}^\top \Lambda(t)\hat{C} - (\hat{B}\hat{\Theta})^\top \Delta(t) + (\hat{D}\hat{\Theta})^\top \Lambda(t)\hat{D} \hat{R}(\Sigma(t))^{-1} \hat{B}^\top \Delta(t).
Substituting these into (4.15) and noting that \( \gamma = 4^T \), we obtain

\[
\Delta(t) = \Delta(t)\hat{A} + \hat{A}^T \Delta(t) + \hat{C}^T \Lambda(t)\hat{C} \\
- \left[ \hat{B}^T \Delta(t) + \hat{D}^T \Lambda(t)\hat{C} \right]^T \left[ \hat{R} + \hat{D}^T \Sigma(t)\hat{D} \right]^{-1} \left[ \hat{B}^T \Delta(t) + \hat{D}^T \Lambda(t)\hat{C} \right] \\
- (\hat{D}\hat{\theta})^T \Lambda(t)\hat{D}\hat{R}(\Sigma(t))^{-1} \hat{S}(P, \Pi) \\
- \left[ \hat{S}(P, \Pi)^T \hat{R}(\Sigma(t))^{-1} \hat{D}^T \Lambda(t)\hat{C} - (\hat{B}\hat{\theta})^T \Delta(t) + (\hat{D}\hat{\theta})^T \Lambda(t)\hat{D}\hat{R}(\Sigma(t))^{-1} \hat{B}^T \Delta(t) \right]^T \\
= \Delta(t) \left( \hat{A} + \hat{B}\hat{\theta} \right) + \left( \hat{A} + \hat{B}\hat{\theta} \right)^T \Delta(t) + \hat{C}^T \Lambda(t)\hat{C} \\
- \left[ \hat{B}^T \Delta(t) + \hat{D}^T \Lambda(t)\hat{C} \right]^T \left[ \hat{R} + \hat{D}^T \Sigma(t)\hat{D} \right]^{-1} \left[ \hat{B}^T \Delta(t) + \hat{D}^T \Lambda(t)\hat{C} \right] \\
- \left[ (\hat{D}\hat{\theta})^T \Lambda(t)\hat{D}\hat{R}(\Sigma(t))^{-1} \hat{B}^T \Delta(t) \right]^T - \left[ (\hat{D}\hat{\theta})^T \Lambda(t)\hat{D}\hat{R}(\Sigma(t))^{-1} \hat{B}^T \Delta(t) \right] \\
- (\hat{D}\hat{\theta})^T \Lambda(t)\hat{D}\hat{R}(\Sigma(t))^{-1} \hat{S}(P, \Pi) \\
- \left[ \hat{S}(P, \Pi)^T \hat{R}(\Sigma(t))^{-1} \hat{D}^T \Lambda(t)\hat{C} \right] - \left[ \hat{S}(P, \Pi)^T \hat{R}(\Sigma(t))^{-1} \hat{D}^T \Lambda(t)\hat{C} \right],
\]

which is exactly (4.14).

**Step 3.** Next we show that there exist positive constants \( K, \lambda > 0 \) such that

\[
|Y(t) - \Pi| \leq Ke^{-\lambda t}, \quad \forall 0 \leq t \leq T < \infty,
\]

which is equivalent to (4.1).

By the variation of constants formula, for \( t \geq s \geq 0 \),

\[
\Delta(t) = e^{\hat{A}(t-s)} \left\{ \Delta(s) + \int_s^t e^{\hat{A}(s-r)} \left[ f(\Delta(r), \Lambda(r)) + g(\Lambda(r)) \right] e^{\hat{A}(s-r)} dr \right\} e^{\hat{A}(t-s)}.
\]

Clearly, there exists a constant \( \rho > 0 \) such that (noting that \( |\Lambda(t)| \) is bounded so that \( |\Lambda(t)|^2 \leq K|\Lambda(t)|) \n
\[
|f(\Delta(t), \Lambda(t))| + |g(\Lambda(t))| \leq \rho \left[ |\Delta(t)|^2 + |\Lambda(t)| \right].
\]

By Lemma 2.2, \( \Lambda \) is stable, so

\[
|e^{\hat{A}t}| \leq \alpha_1 e^{-\beta_1 t}, \quad \forall t \geq 0
\]

for some constants \( \alpha_1, \beta_1 > 0 \). Also, by Lemma 2.3,

\[
|\Lambda(t)| = |\Sigma(t) - P| \leq \alpha_2 e^{-\beta_2 t}, \quad \forall t \geq 0
\]

for some positive constants \( \alpha_2, \beta_2 > 0 \). Take \( \alpha = \alpha_1 + \alpha_2 \) and \( \beta = \beta_1 + \beta_2 \) so that

\[
|e^{\hat{A}t}| \leq \alpha e^{-\beta t}, \quad |\Lambda(t)| \leq \alpha e^{-\beta t}, \quad \forall t \geq 0.
\]

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From (4.17), (4.18), and (4.19) we get
\[
|\Delta(t)| \leq \alpha^2 e^{-2\beta(t-s)} \left\{ |\Delta(s)| + \rho \int_s^t e^{2\beta(r-s)} \left[ |\Delta(r)|^2 + |\Delta(r)| \right] dr \right\}
\]
\[
\leq \alpha^2 e^{-2\beta(t-s)} \left\{ |\Delta(s)| + \rho \int_s^t e^{2\beta(r-s)} \left[ |\Delta(r)|^2 + \alpha e^{-3\beta r} \right] dr \right\}
\]
\[
= \alpha^2 e^{-2\beta(t-s)} \left\{ |\Delta(s)| + \rho \alpha e^{-2\beta s} \beta^{-1} \left[ e^{-\beta s} - e^{-\beta t} \right] + \rho \int_s^t e^{2\beta(r-s)} |\Delta(r)|^2 dr \right\}
\]
\[
\leq \alpha^2 e^{-2\beta(t-s)} \left\{ |\Delta(s)| + \rho \alpha e^{-2\beta s} \beta^{-1} + \rho \int_s^t e^{2\beta(r-s)} |\Delta(r)|^2 dr \right\}.
\]

For fixed \( s \geq 0 \), set
\[
h(t) = e^{2\beta(t-s)} |\Delta(t)|, \quad k(s) = \alpha^2 |\Delta(s)| + \rho \alpha^3 e^{-2\beta s} \beta^{-1}.
\]
Then
\[
h(t) \leq k(s) + \rho \alpha^2 \int_s^t e^{-2\beta(r-s)} |h(r)|^2 dr, \quad t \geq s.
\]
Set
\[
\theta(t) = k(s) + \rho \alpha^2 \int_s^t e^{-2\beta(r-s)} |h(r)|^2 dr, \quad p(t) = \frac{1}{\theta(t)}; \quad t \geq s.
\]
Then
\[
p'(t) = -\frac{\theta'(t)}{[\theta(t)]^2} = -\frac{\rho \alpha^2 e^{-2\beta(t-s)} |h(t)|^2}{[\theta(t)]^2} \geq -\rho \alpha^2 e^{-2\beta(t-s)}.
\]

Since \( \lim_{s \to \infty} k(s) = 0 \), for \( s \) large enough such that
\[
p(s) = \frac{1}{\theta(s)} = \frac{1}{k(s)} \geq 1 + \frac{\rho \alpha^2}{2\beta},
\]
we have
\[
p(t) \geq p(s) - \rho \alpha^2 \int_s^t e^{-2\beta(r-s)} dr = p(s) - \frac{\rho \alpha^2}{2\beta} \left[ 1 - e^{-2\beta(s-t)} \right] \geq 1
\]
and hence
\[
h(t) \leq \theta(t) = \frac{1}{p(t)} \leq 1, \quad \forall t \geq s.
\]

It follows that
\[
|\Delta(t)| = e^{-2\beta(t-s)} h(t) \leq Ke^{-\lambda t}, \quad \forall t \geq 0,
\]
for some constants \( K, \lambda > 0 \).

\section{The Turnpike Property}

In this section we prove the main results stated in Section 3. According to [16, Theorem 2.3], under the assumption (A1), Problem (MFLQ) admits a unique optimal control for
every initial state \( x \). Moreover, a pair \( (X^*_t(\cdot), u^*_t(\cdot)) \) is optimal for \( x \) if and only if it satisfies the following optimality system:

\[
\begin{aligned}
&\begin{cases}
  dX^*_t(t) = \{ AX^*_t(t) + \bar{A}E[X^*_t(t)] + Bu^*_t(t) + \bar{B}E[u^*_t(t)] + b \} \, dt \\
  \quad + \{ CX^*_t(t) + \bar{C}E[X^*_t(t)] + Du^*_t(t) + \bar{D}E[u^*_t(t)] + \sigma \} \, dW(t),
\end{cases}

\end{aligned}
\]

\[
\begin{aligned}
&\begin{cases}
  dY^*_t(t) = -\{ A^T Y^*_t(t) + \bar{A}^T E[Y^*_t(t)] + \bar{C}^T Z^*_t(t) + \bar{C}^T E[Z^*_t(t)] + Q X^*_t(t) \\
  \quad + \bar{Q} E[X^*_t(t)] + \bar{S}^T E[u^*_t(t)] + \bar{q} \} \, dt + Z^*_t(t) \, dW(t),
\end{cases}

\end{aligned}
\]

\[
\begin{aligned}
X^*_t(0) = x, \quad Y^*_t(T) = 0,
\end{aligned}
\]

\[
\begin{aligned}
B^T Y^*_t(t) + \bar{B}^T E[Y^*_t(t)] + D^T Z^*_t(t) + \bar{D}^T E[Z^*_t(t)] + SX^*_t(t) + \bar{S} E[X^*_t(t)]
\end{aligned}
\]

\[
\begin{aligned}
+ Ru^*_t(t) + \bar{R} E[u^*_t(t)] + r = 0.
\end{aligned}
\]

(5.1)

Let \( (X^*_t(\cdot), u^*_t(\cdot)) \) be the optimal pair of Problem (MFLQ)_T for the initial state \( x \) and \((Y^*_t(\cdot), Z^*_t(\cdot)) \) the adapted solution to the corresponding adjoint equation in (5.1). Let \((x^*, u^*)\) be the unique solution of Problem (O) and \( \lambda^* \in \mathbb{R}^n \) the vector in (3.3). Define

\[
\tilde{X}_T(\cdot) = X^*_T(\cdot) - x^*, \quad \tilde{u}_T(\cdot) = u^*_T(\cdot) - u^*, \quad \tilde{Y}_T(\cdot) = Y^*_T(\cdot) - \lambda^*.
\]

(5.2)

Subtracting (3.3) from (5.1) yields

\[
\begin{aligned}
&\begin{cases}
  d\tilde{X}_T(t) = \{ A\tilde{X}_T(t) + \bar{A}E[\tilde{X}_T(t)] + B\tilde{u}_T(t) + \bar{B}E[\tilde{u}_T(t)] \} \, dt \\
  \quad + \{ C\tilde{X}_T(t) + \bar{C}E[\tilde{X}_T(t)] + D\tilde{u}_T(t) + \bar{D}E[\tilde{u}_T(t)] + \sigma^* \} \, dW(t),
\end{cases}

\end{aligned}
\]

\[
\begin{aligned}
&\begin{cases}
  d\tilde{Y}_T(t) = -\{ A^T \tilde{Y}_T(t) + \bar{A}^T E[\tilde{Y}_T(t)] + \bar{C}^T Z^*_T(t) + \bar{C}^T E[Z^*_T(t)] + Q \tilde{X}_T(t) \\
  \quad + \bar{Q} E[\tilde{X}_T(t)] + \bar{S}^T E[\tilde{u}_T(t)] - \bar{C}^T P \sigma^* \} \, dt + Z^*_T(t) \, dW(t),
\end{cases}

\end{aligned}
\]

\[
\begin{aligned}
\tilde{X}_T(0) = x - x^*, \quad \tilde{Y}_T(T) = -\lambda^*,
\end{aligned}
\]

\[
\begin{aligned}
B^T \tilde{Y}_T(t) + \bar{B}^T E[\tilde{Y}_T(t)] + D^T Z^*_T(t) + \bar{D}^T E[Z^*_T(t)] + SX^*_T(t) + \bar{S} E[\tilde{X}_T(t)]
\end{aligned}
\]

\[
\begin{aligned}
+ Ru^*_T(t) + \bar{R} E[u^*_T(t)] - \bar{D}^T P \sigma^* = 0,
\end{aligned}
\]

(5.3)

where \( \sigma^* = \bar{C}x^* + \bar{D}u^* + \sigma \). From (5.3) we see that \((\tilde{X}_T(\cdot), \tilde{u}_T(\cdot))\) is an optimal pair of the mean-field stochastic LQ problem with state equation

\[
\begin{aligned}
&\begin{cases}
  dX(t) = \{ AX(t) + \bar{A}E[X(t)] + Bu(t) + \bar{B}E[u(t)] \} \, dt \\
  \quad + \{ CX(t) + \bar{C}E[X(t)] + Du(t) + \bar{D}E[u(t)] + \sigma^* \} \, dW(t),
\end{cases}

\end{aligned}
\]

\[
\begin{aligned}
X(0) = x - x^*,
\end{aligned}
\]

(5.4)

and cost functional

\[
J(x; u) = \mathbb{E}\left\{ -2\langle \lambda^*, X(T) \rangle + \int_0^T \left( \begin{array}{c}
Q & S \\
S & R
\end{array} \right) \left( \begin{array}{c}
X(t) \\
u(t)
\end{array} \right) \left( \begin{array}{c}
X(t) \\
u(t)
\end{array} \right) - 2\langle \begin{array}{c}
C^T P \sigma^* \\
D^T P \sigma^*
\end{array} \left( \begin{array}{c}
X(t) \\
u(t)
\end{array} \right) \right) \right\} dt.
\]

Thus, by [16, Theorem 5.2] we have the following result.
and let \( \varphi_t(\cdot) \) and \( \hat{\varphi}_t(\cdot) \) be the solutions to
\[
\begin{aligned}
\begin{cases}
\dot{\varphi}_t(t) + [A + B\Theta_t(t)]^\top \varphi_t(t) + [C + D\Theta_t(t)]^\top [P_t(t) - P]\sigma^*, \\
\varphi(T) = -\lambda^*,
\end{cases}
\end{aligned}
\tag{5.7}
\]

and
\[
\begin{aligned}
\begin{cases}
\dot{\hat{\varphi}}_t(t) + [\hat{A} + \hat{B}\hat{\Theta}_t(t)]^\top \hat{\varphi}_t(t) + [\hat{C} + \hat{D}\hat{\Theta}_t(t)]^\top [P_t(t) - P]\sigma^*, \\
\hat{\varphi}(T) = -\lambda^*,
\end{cases}
\end{aligned}
\tag{5.8}
\]

respectively. Then the process \( \hat{u}_t(\cdot) \) defined in (5.2) is given by
\[
\hat{u}_t(t) = \Theta_t(t)[\hat{X}_t(t) - \mathbb{E}[\hat{X}_t(t)]] + \hat{\Theta}_t(t)\mathbb{E}[\hat{X}_t(t)] + \theta_t(t) + \hat{\theta}_t(t),
\tag{5.9}
\]

where
\[
\begin{aligned}
\theta(t) &= -\mathcal{R}(P_t(t))^{-1}[B^\top \varphi_t(t) + D^\top (P_t(t) - P)\sigma^*], \\
\hat{\theta}(t) &= -\hat{\mathcal{R}}(P_t(t))^{-1}[\hat{B}^\top \hat{\varphi}_t(t) + \hat{D}^\top (P_t(t) - P)\sigma^*].
\end{aligned}
\tag{5.10, 5.11}
\]

**Lemma 5.2.** Let (A1)–(A2) hold. Then there exist positive constants \( K, \lambda > 0 \), independent of \( T \), such that
\[
|\varphi_t(t)| + |\theta_t(t)| + |\hat{\theta}_t(t)| \leq Ke^{-\lambda(T-t)}, \quad \forall t \in [0, T].
\]

**Proof.** It is shown in [18] that \( \varphi_t(\cdot) \), the solution of (5.7), satisfies
\[
|\varphi_t(t)| \leq Ke^{-\lambda(T-t)}, \quad \forall t \in [0, T]
\]
for some positive constants \( K, \lambda > 0 \) independent of \( T \). In a similar manner, we can show that the solution \( \hat{\varphi}_t(\cdot) \) of (5.8) satisfies the same inequality with possibly different constants \( K \) and \( \lambda \). The desired result then follows from the fact that \( \mathcal{R}(P_t(t)) \geq R > 0 \), \( \hat{\mathcal{R}}(P_t(t)) \geq \hat{R} > 0 \) and Lemma 2.3. \( \blacksquare \)

If we substitute (5.9) into (5.4), we see that the process \( \hat{X}_t(\cdot) \) defined in (5.2) satisfies the closed-loop system
\[
\begin{aligned}
d\hat{X}_t(t) &= \{(A + B\Theta_t)(\hat{X}_t(t) - \mathbb{E}[\hat{X}_t(t)]) + (\hat{A} + \hat{B}\hat{\Theta}_t)\mathbb{E}[\hat{X}_t(t)] + \hat{B}(\theta(t) + \hat{\theta}_t(t))\}dt \\
&+ \{(C + D\Theta_t)(\hat{X}_t(t) - \mathbb{E}[\hat{X}_t(t)]) + (\hat{C} + \hat{D}\hat{\Theta}_t)\mathbb{E}[\hat{X}_t(t)] + \hat{D}(\theta(t) + \hat{\theta}_t(t)) + \sigma^*\}dW, \\
\hat{X}_t(0) &= x - x^*.
\end{aligned}
\tag{5.12}
\]

We now provide an estimate for \( \mathbb{E}[\hat{X}_t(t)] \).
Proposition 5.3. Let (A1)–(A2) hold. Then there exist constants $K, \lambda > 0$, independent of $T$, such that

$$
|\mathbb{E}[\tilde{X}_T(t)]| \leq K \left[ e^{-\lambda t} + e^{-\lambda(T-t)} \right], \quad \forall t \in [0, T].
$$

(5.13)

Proof. The function $t \mapsto \mathbb{E}[\tilde{X}_T(t)]$ satisfies the following ODE:

$$
\begin{align*}
\frac{d}{dt} \mathbb{E}[\tilde{X}_T(t)] &= [\hat{A} + \hat{B} \hat{\theta}_T(t)] \mathbb{E}[\tilde{X}_T(t)] + \hat{B} [\theta_T(t) + \hat{\theta}_T(t)], \\
\mathbb{E}[\tilde{X}_T(0)] &= x - x^* \equiv \tilde{x}.
\end{align*}
$$

(5.14)

Using the notation

$$
\hat{A}_T(t) = \hat{A} + \hat{B} \hat{\theta}_T(t), \quad \hat{\theta} = -\tilde{\mathcal{R}}(P)^{-1} \mathcal{S}(P, \Pi), \quad \hat{A} = \hat{A} + \hat{B} \hat{\theta},
$$

we have by the variation of constants formula that

$$
\mathbb{E}[\tilde{X}_T(t)] = e^{\hat{A}t} \tilde{x} + \int_0^t e^{\hat{A}(t-s)} \left[ (\hat{A}_T(s) - \hat{A}) \mathbb{E}[\tilde{X}_T(s)] + \hat{B} (\theta_T(s) + \hat{\theta}_T(s)) \right] ds.
$$

By Lemma 2.2, $\hat{A}$ is stable, so

$$
|e^{\hat{A}t}| \leq Ke^{-\lambda t}, \quad \forall t \geq 0
$$

for some constants $K, \lambda > 0$ independent of $T$. Since

$$
\hat{\theta} - \hat{\theta}_T(t) = \left( \hat{R} + \hat{D}^\top P_T(t) \hat{D} \right)^{-1} \left( \hat{B}^\top P_T(t) + \hat{D}^\top P_T(t) \hat{C} + \hat{S} \right)
$$

$$
- \left( \hat{R} + \hat{D}^\top P \hat{D} \right)^{-1} \left( \hat{B}^\top \Pi + \hat{D}^\top P \hat{C} + \hat{S} \right)
$$

$$
= \left( \hat{R} + \hat{D}^\top P_T(t) \hat{D} \right)^{-1} \left( \hat{B}^\top [\Pi_T(t) - \Pi] + \hat{D}^\top [P_T(t) - P] \hat{C} \right)
$$

$$
+ \left[ \left( \hat{R} + \hat{D}^\top P_T(t) \hat{D} \right)^{-1} - \left( \hat{R} + \hat{D}^\top P \hat{D} \right)^{-1} \right] \left( \hat{B}^\top \Pi + \hat{D}^\top P \hat{C} + \hat{S} \right)
$$

$$
= \left( \hat{R} + \hat{D}^\top P_T(t) \hat{D} \right)^{-1} \left( \hat{B}^\top [\Pi_T(t) - \Pi] + \hat{D}^\top [P_T(t) - P] \hat{C} \right)
$$

$$
+ \left( \hat{R} + \hat{D}^\top P_T(t) \hat{D} \right)^{-1} \hat{D}^\top [P_T(t) - P] \hat{D} \hat{\theta},
$$

we have by Lemma 2.3 and Theorem 4.1,

$$
|\hat{A}_T(t) - \hat{A}| \leq K |\hat{\theta}_T(t) - \hat{\theta}| \leq K \left[ |\Pi_T(t) - \Pi| + |P_T(t) - P| \right]
$$

$$
\leq Ke^{-\lambda(T-t)}, \quad \forall t \in [0, T].
$$

(5.15)

Also recalling Lemma 5.2, we obtain

$$
|\mathbb{E}[\tilde{X}_T(t)]| \leq Ke^{-\lambda t} + K \int_0^t e^{-\lambda(t-s)} e^{-\lambda(T-s)} \mathbb{E}[\tilde{X}_T(s)] + 1 ds
$$

$$
\leq K \left[ e^{-\lambda t} + e^{-\lambda(T-t)} \right] + K \int_0^t e^{-\lambda(T+s-2t)} \mathbb{E}[\tilde{X}_T(s)] ds
$$

$$
\leq K \left[ e^{-\lambda t} + e^{-\lambda(T-t)} \right] + K \int_0^t e^{-\lambda(T-s)} \mathbb{E}[\tilde{X}_T(s)] ds.
$$

The desired result then follows from Gronwall’s inequality.
Recall the SDE (3.5). We have the following result.

**Proposition 5.4.** Let (A1)–(A2) hold. Then there exists a constant \( K > 0 \) such that the solution \( X^*(\cdot) \) to the SDE (3.5) satisfies

\[
\mathbb{E}|X^*(t)|^2 \leq K, \quad \forall t \geq 0.
\]  

(5.16)

**Proof.** Let \( P \) be the solution to the ARE (2.7) and let \( \Theta = -R(P)^{-1}S(P) \). By Itô’s rule,

\[
\mathbb{E}(PX^*(t), X^*(t)) = \mathbb{E} \int_0^t \left\{ 2\langle PX^*(s), (A + B\Theta)X^*(s) \rangle + \langle P[(C + D\Theta)X^*(s) + \sigma^*], (C + D\Theta)X^*(s) + \sigma^* \rangle \right\} ds
\]

\[
= \mathbb{E} \int_0^t \left\{ \langle P(A + B\Theta) + (A + B\Theta)\top P + (C + D\Theta)\top P(C + D\Theta) \rangle X^*(s), X^*(s) \rangle
\]

\[
+ 2\langle P(C + D\Theta)X^*(s), \sigma^* \rangle + \langle P\sigma^*, \sigma^* \rangle \right\} ds
\]

\[
= \mathbb{E} \int_0^t \left[ -\left\langle (Q + \Theta\top R\Theta + S\top \Theta + \Theta\top S)X^*(s), X^*(s) \right\rangle + 2\langle P(C + D\Theta)X^*(s), \sigma^* \rangle + \langle P\sigma^*, \sigma^* \rangle \right] ds.
\]

Note that by (A1),

\[
Q + \Theta\top R\Theta + S\top \Theta + \Theta\top S = Q - S\top R^{-1}S + (\Theta + R^{-1}S)\top R(\Theta + R^{-1}S) > 0.
\]

Let \( \mu > 0 \) and \( \nu > 0 \) be the smallest eigenvalues of \( P \) and \( Q + \Theta\top R\Theta + S\top \Theta + \Theta\top S \), respectively. Then

\[
\mathbb{E}|X^*(t)|^2 \leq \frac{1}{\mu} \mathbb{E}(PX^*(t), X^*(t))
\]

\[
\leq \frac{1}{\mu} \mathbb{E} \int_0^t \left[ -\nu|X^*(s)|^2 + 2|X^*(s)| \cdot |(C + D\Theta)\top P\sigma^*| + \langle P\sigma^*, \sigma^* \rangle \right] ds
\]

\[
\leq \frac{1}{\mu} \mathbb{E} \int_0^t \left[ -\nu|X^*(s)|^2 + \frac{2}{\nu}(C + D\Theta)\top P\sigma^*|^2 + \langle P\sigma^*, \sigma^* \rangle \right] ds.
\]

It follows from the Gronwall inequality that

\[
\mathbb{E}|X^*(t)|^2 \leq \frac{1}{\mu} \left[ \frac{2}{\nu}(C + D\Theta)\top P\sigma^*|^2 + \langle P\sigma^*, \sigma^* \rangle \right] \int_0^t \exp \left[ \frac{\nu}{2\mu}(s - t) \right] ds
\]

\[
\leq \frac{2}{\nu} \left[ \frac{2}{\nu}(C + D\Theta)\top P\sigma^*|^2 + \langle P\sigma^*, \sigma^* \rangle \right].
\]

This completes the proof. \( \blacksquare \)

We now prove Theorem 3.2.

**Proof of Theorem 3.2.** Let \( \Theta_T(t) \) and \( \hat{\Theta}_T(t) \) be as in (5.5) and (5.6), respectively, and let

\[
\Theta = -R(P)^{-1}S(P), \quad \hat{\Theta} = -\hat{R}(P)^{-1}\hat{S}(P, \Pi).
\]

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For notational simplicity, we write
\[
A = A + B\Theta, \quad C = C + D\Theta, \quad \mathcal{A}_T(t) = A + B\Theta_T(t), \quad \mathcal{C}_T(t) = C + D\Theta_T(t),
\]
\[
\hat{A} = \hat{A} + \hat{B}\Theta, \quad \hat{C} = \hat{C} + \hat{D}\Theta, \quad \hat{\mathcal{A}}_T(t) = \hat{A} + \hat{B}\Theta_T(t), \quad \hat{\mathcal{C}}_T(t) = \hat{C} + \hat{D}\Theta_T(t).
\]
Then the process
\[
V_T(t) \triangleq \tilde{X}_T(t) - \mathbb{E}[\tilde{X}_T(t)] - X^*(t), \quad t \in [0, T]
\]
satisfies \(V_T(0) = 0\) and
\[
dV_T(t) = \{ A_T(t)V_T(t) + [A_T(t) - A]X^*(t) \} \, dt
\]
\[
\quad + \{ C_T(t)V_T(t) + [C_T(t) - C]X^*(t) + \hat{C}_T(t)\mathbb{E}[\tilde{X}_T(t)] + \hat{D}[\theta_T(t) + \hat{\theta}_T(t)] \} \, dW(t).
\]
Let \(P > 0\) be the solution to the ARE (2.7). Then Itô’s rule implies that
\[
\mathbb{E}(PV_T(t), V_T(t)) = \mathbb{E} \int_0^t \left\{ 2(PV_T, A_TV_T + (A_T - A)X^*) + \langle P[C_TV_T + (C_T - C)X^* + h_T], C_TV_T + (C_T - C)X^* + h_T \rangle \right\} \, ds,
\]
where we have suppressed \(s\) in the integrand, and
\[
h_T(s) = \hat{C}_T(s)\mathbb{E}[\tilde{X}_T(s)] + \hat{D}[\theta_T(s) + \hat{\theta}_T(s)].
\]
Note that
\[
\mathbb{E}[X^*(t)] = 0, \quad \mathbb{E}[V_T(t)] = 0, \quad \forall t \in [0, T].
\]
Thus,
\[
\mathbb{E}(P[C_TV_T + (C_T - C)X^* + h_T], C_TV_T + (C_T - C)X^* + h_T)
\]
\[
= \mathbb{E}(P[C_TV_T + (C_T - C)X^*], C_TV_T + (C_T - C)X^*) + \langle Ph_T, h_T \rangle.
\]
Substituting the above into (5.17) and expanding the integrand yields
\[
\mathbb{E}(PV_T(t), V_T(t)) = \int_0^t \mathbb{E} \left\{ \langle (P + \mathcal{A}_T^T P + \mathcal{C}_T^T P\mathcal{C}_T) V_T, V_T \rangle + \langle \hat{\mathcal{C}}_T^T \hat{P}\mathcal{C}_T \mathbb{E}[\tilde{X}_T], \mathbb{E}[\tilde{X}_T] \rangle + 2\langle [P(\mathcal{A}_T - A) + \mathcal{C}_T^T P(\mathcal{C}_T - C)]X^*, V_T \rangle + k_T \right\} \, ds,
\]
where
\[
k_T(s) = \langle P[C_T(s) - C]X^*(s), [C_T(s) - C]X^*(s) \rangle
\]
\[
\quad + \langle 2\hat{P}\hat{\mathcal{C}}_T(s)\mathbb{E}[\tilde{X}_T(s)] + P\hat{D}[\theta_T(s) + \hat{\theta}_T(s)], \hat{D}[\theta_T(s) + \hat{\theta}_T(s)] \rangle.
\]
Next, let \(II > 0\) be the solution to the ARE (2.8). Then integration by parts gives
\[
\langle II\mathbb{E}[\tilde{X}_T(t)], \mathbb{E}[\tilde{X}_T(t)] \rangle - \langle II\tilde{x}, \tilde{x} \rangle
\]
\[
= \int_0^t \left\{ \langle (II\hat{A}_T + \hat{A}_T^T II)\mathbb{E}[\tilde{X}_T], \mathbb{E}[\tilde{X}_T] \rangle + 2\langle II\mathbb{E}[\tilde{X}_T], \hat{D}[\theta_T + \hat{\theta}_T] \rangle \right\} \, ds.
\]
Adding (5.18) and (5.19), we obtain
\[
\mathbb{E}\langle PV_T(t), V_T(t) \rangle + \langle 
\Pi \mathbb{E}[\tilde{X}_T(t)], \mathbb{E}[\tilde{X}_T(t)] \rangle - \langle \Pi \tilde{x}, \tilde{x} \rangle
\]
\[
= \int_0^T \mathbb{E}\left\{ \langle (PA_T + A_T^T P + C_T^T P C_T) V_T, V_T \rangle + \langle \Pi \tilde{A}_T + \tilde{A}_T^T \Pi + \tilde{C}_T^T \tilde{P} \tilde{C}_T \rangle \mathbb{E}[\tilde{X}_T], \mathbb{E}[\tilde{X}_T] \rangle + 2\langle [P(A_T - A) + C_T^T P(C_T - C)]X^*, V_T \rangle + \phi_T \right\} ds, \tag{5.20}
\]
where
\[
\phi_T(s) = k_T(s) + 2\langle \Pi \mathbb{E}[\tilde{X}_T(s)], \tilde{B}[\theta_T(s) + \tilde{\theta}_T(s)] \rangle.
\]
We observe the following facts:

\[
P A_T(s) + A_T(s)^T P + C_T(s)^T P C_T(s)
\]
\[
= P A + A^T P + C^T P C + P[A_T(s) - A] + [A_T(s) - A]^T P + [C_T(s) - C]^T P C + C_T(s)^T P[C_T(s) - C],
\]
\[
P A + A^T P + C^T P C = -(Q + \Theta^T R \Theta + S^T \Theta + \Theta^T S) < 0,
\]
\[
\Pi \tilde{A}_T(s) + \tilde{A}_T(s)^T \Pi + \tilde{C}_T(s)^T \tilde{P} \tilde{C}_T(s)
\]
\[
= \Pi \tilde{A} + \tilde{A}^T \Pi + \tilde{C}^T \tilde{P} \tilde{C} + \Pi[\tilde{A}_T(s) - \tilde{A}] + [\tilde{A}_T(s) - \tilde{A}]^T \Pi + [\tilde{C}_T(s) - \tilde{C}]^T \tilde{P} \tilde{C} + \tilde{C}_T(s)^T \tilde{P}[\tilde{C}_T(s) - \tilde{C}],
\]
\[
\Pi \tilde{A} + \tilde{A}^T \Pi + \tilde{C}^T \tilde{P} \tilde{C} = -(\tilde{Q} + \tilde{\Theta}^T \tilde{R} \tilde{\Theta} + \tilde{S}^T \tilde{\Theta} + \tilde{\Theta}^T \tilde{S}) < 0.
\]

Also, observe that for some constant \( K > 0 \),
\[|A_T(t)| + |\tilde{A}_T(t)| + |C_T(t)| + |\tilde{C}_T(t)| \leq K, \quad \forall 0 \leq t \leq T < \infty,\]
and similar to (5.15), we have
\[|A_T(t) - A| + |\tilde{A}_T(t) - \tilde{A}| + |C_T(t) - C| + |\tilde{C}_T(t) - \tilde{C}| \leq Ke^{-\lambda(T-t)}, \quad \forall 0 \leq t \leq T < \infty,\]
for some constants \( K, \lambda > 0 \). Then it follows that for some constant \( \alpha > 0 \),
\[
\mathbb{E}\{ [PA_T(s) + A_T(s)^T P + C_T(s)^T P C_T(s)] V_T(s), V_T(s) \}
\]
\[
\leq K \left[ -2\alpha + e^{-\lambda(T-s)} \right] \mathbb{E}[V_T(s)]^2, \tag{5.21}
\]
\[
\langle \Pi \tilde{A}_T(s) + \tilde{A}_T(s)^T \Pi + \tilde{C}_T(s)^T \tilde{P} \tilde{C}_T(s) \rangle \mathbb{E}[\tilde{X}_T(s)], \mathbb{E}[\tilde{X}_T(s)] \rangle
\]
\[
\leq K \left[ -2\alpha + e^{-\lambda(T-s)} \right] \mathbb{E}[\tilde{X}_T(s)]^2.
\]
Using Proposition 5.3, we can further conclude that
\[
\langle [\Pi \tilde{A}_T(s) + \tilde{A}_T(s)^T \Pi + \tilde{C}_T(s)^T \tilde{P} \tilde{C}_T(s)] \mathbb{E}[\tilde{X}_T(s)], \mathbb{E}[\tilde{X}_T(s)] \rangle \leq Ke^{-\lambda(T-s)}. \tag{5.22}
\]
Moreover, by the Cauchy–Schwarz inequality and (5.16),
\[
2\mathbb{E}\{ [P(A_T(s) - A) + C_T(s)^T P(C_T(s) - C)]X^*(s), V_T(s) \}
\]
and by Lemma 5.2, Proposition 5.3, and Proposition 5.4,

$$E|\phi_T(s)| \leq Ke^{-\lambda(T-s)}.$$  (5.24)

Substitution of (5.21)–(5.24) into (5.20), we obtain

$$E\langle PV_T(t), V_T(t) \rangle - \langle I\hat{x}, \hat{x} \rangle \leq E\langle PV_T(t), V_T(t) \rangle + \langle I\hat{X}_T(t), E[\hat{X}_T(t)] \rangle - \langle I\hat{x}, \hat{x} \rangle \leq K \int_0^T \left[ \left(-\alpha + e^{-\lambda(T-s)}\right)E|V_T(s)|^2 + e^{-\lambda(T-s)} \right] ds.$$  (5.23)

Since $P > 0$, the above implies that

$$E|V_T(t)|^2 \leq K + K \int_0^T \left[ \left(-\alpha + e^{-\lambda(T-s)}\right)E|V_T(s)|^2 + e^{-\lambda(T-s)} \right] ds.$$  (5.24)

Then, by Gronwall’s inequality, we have

$$E|V_T(t)|^2 \leq Ke^{-\lambda M} + e^{-\lambda(t-T)}, \quad \forall t \in [0, T],$$

with possibly different constants $K, \lambda > 0$. The desired (3.7) now follows from

$$X_T^*(t) - X^*(t) = V_T(t) + E[\hat{X}_T(t)]$$

and

$$u_T^*(t) - u^*(t) = \Theta_T(t)[X_T^*(t) - X^*(t)] + \left[\Theta_T(t) - \Theta\right]X^*(t) + \left[\hat{\Theta}_T(t) - \hat{\Theta}_T(t)\right]E[\hat{X}_T(t)] + \theta_T(t) + \hat{\theta}_T(t).$$

The proof is complete. \(\blacksquare\)

Next, we prove Corollary 3.4.

**Proof of Corollary 3.4.** By Proposition 5.4,

$$E|X^*(t)|^2 + E|u^*(t)|^2 \leq K, \quad \forall t \geq 0$$

for some constant $K > 0$, and hence by Theorem 3.2,

$$E|X_T^*(t)|^2 + E|u_T^*(t)|^2 \leq K, \quad \forall 0 \leq t \leq T < \infty,$$

for some possibly different constant $K > 0$. The above, together with Corollary 3.3, implies that

$$\lim_{T \to \infty} \frac{1}{T} V_T(x) = \lim_{T \to \infty} \frac{1}{T} J_T(x; u_T^*(\cdot))$$

$$= \lim_{T \to \infty} \frac{1}{T} \int_0^T \left\{ E\left[QX_T^*(t), X_T^*(t)\right] + 2\langle SX_T^*(t), u_T^*(t)\rangle + \langle Ru_T^*(t), u_T^*(t)\rangle \right\}$$

$$+ \langle \tilde{Q}E[X_T^*(t)], E[X_T^*(t)] \rangle + 2\langle \tilde{S}E[X_T^*(t)], E[u_T^*(t)] \rangle + \langle \tilde{R}E[u_T^*(t)], E[u_T^*(t)] \rangle$$

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Proposition 5.4, we obtain
\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \left\{ \mathbb{E}\left[ (Q X^*(t), X^*(t)) + 2\langle S X^*(t), u^*(t) \rangle + \langle R u^*(t), u^*(t) \rangle \right] \\
+ \langle \bar{Q} \mathbb{E}[X^*(t)], \mathbb{E}[X^*(t)] \rangle + 2\langle \mathbb{E}[X^*(t)], \mathbb{E}[u^*(t)] \rangle + \langle \bar{R} \mathbb{E}[u^*(t)], \mathbb{E}[u^*(t)] \rangle \\
+ 2\langle q, \mathbb{E}[X^*(t)] \rangle + 2\langle r, \mathbb{E}[u^*(t)] \rangle \right\} \, dt.
\]

Noting that
\[
\mathbb{E}[X^*(t)] = \mathbb{E}[X^*(t) + x^*] = x^*, \quad \mathbb{E}[u^*(t)] = \mathbb{E}[\Theta X^*(t) + u^*] = u^*,
\]
we further have
\[
\lim_{T \to \infty} \frac{1}{T} V_T(x) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \left\{ \mathbb{E}\left[ (Q X^*(t), X^*(t)) + \langle Q x^*, x^* \rangle \\
+ 2\langle S X^*(t), \Theta X^*(t) \rangle + 2\langle S x^*, u^* \rangle \\
+ \langle R \Theta X^*(t), \Theta X^*(t) \rangle + \langle R u^*, u^* \rangle \\
+ \langle \bar{Q} x^*, x^* \rangle + 2\langle \bar{S} x^*, u^* \rangle + \langle \bar{R} u^*, u^* \rangle + 2\langle q, x^* \rangle + 2\langle r, u^* \rangle \right\} \, dt
\]
\[
= \lim_{T \to \infty} \frac{1}{T} \int_0^T \mathbb{E}((Q + S^\top \Theta + \Theta^\top S + \Theta^\top R \Theta) X^*(t), X^*(t)) \, dt
\]
\[
+ \langle \bar{Q} x^*, x^* \rangle + 2\langle \bar{S} x^*, u^* \rangle + \langle \bar{R} u^*, u^* \rangle + 2\langle q, x^* \rangle + 2\langle r, u^* \rangle. \tag{5.25}
\]

On the other hand, with \( P > 0 \) being the solution to the ARE \((2.7)\) and noting that \( \mathbb{E}[X^*(t)] = 0 \), we have
\[
\mathbb{E}(PX^*(T), X^*(T)) = \mathbb{E} \int_0^T \left\{ 2\langle PX^*(t), (A + B \Theta) X^*(t) \rangle \\
+ \langle P[(C + D \Theta) X^*(t) + \sigma^*], (C + D \Theta) X^*(t) + \sigma^* \rangle \right\} \, dt
\]
\[
= \mathbb{E} \int_0^T \left\{ \langle P(A + B \Theta) + (A + B \Theta)^\top P \\
+ (C + D \Theta)^\top P(C + D \Theta) \rangle X^*(t), X^*(t) \rangle + \langle P \sigma^*, \sigma^* \rangle \right\} \, dt
\]
\[
= \mathbb{E} \int_0^T \left[ -\langle (Q + S^\top \Theta + \Theta^\top S + \Theta^\top R \Theta) X^*(t), X^*(t) \rangle + \langle P \sigma^*, \sigma^* \rangle \right] \, dt.
\]

Since \( \mathbb{E}[X^*(T)]^2 \) is bounded in \( T \) (see Proposition 5.4), we obtain
\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \mathbb{E}((Q + S^\top \Theta + \Theta^\top S + \Theta^\top R \Theta) X^*(t), X^*(t)) \, dt
\]
\[
= \lim_{T \to \infty} \frac{1}{T} \left[ -\mathbb{E}(PX^*(T), X^*(T)) + \int_0^T \langle P \sigma^*, \sigma^* \rangle \, dt \right]
\]
\[
= \langle P \sigma^*, \sigma^* \rangle. \tag{5.26}
\]

Combining \((5.25)\) and \((5.26)\), we get the desired result.

Finally, we prove Corollary 3.5.
Proof of Corollary 3.5. By a tedious but straightforward calculation we can verify that in (5.3), the adapted solution \((\tilde{Y}_T(\cdot), Z^*_T(\cdot))\) to the backward SDE has the following relation with \((\tilde{X}_T(\cdot), \tilde{u}_T(\cdot))\):

\[
\tilde{Y}_T(t) = P_T(t) \left\{ \tilde{X}_T(t) - \mathbb{E}[\tilde{X}_T(t)] + \Pi_T(t) \mathbb{E}[\tilde{X}_T(t)] + \hat{\varphi}_T(t), \right. \\
Z^*_T(t) = P_T(t) \left\{ C\tilde{X}_T(t) + D\tilde{u}_T(t) + \tilde{C}\mathbb{E}[\tilde{X}_T(t)] + \hat{D}\mathbb{E}[\tilde{u}_T(t)] + \sigma^* \right\}.
\]

Using the relation (5.9), we further obtain

\[
Z^*_T(t) = P_T(t) \left\{ [C + D\Theta_T(t)]\tilde{X}_T(t) + [\tilde{C} - D\Theta_T(t)]\mathbb{E}[\tilde{X}_T(t)] + \hat{D}\mathbb{E}[\tilde{u}_T(t)] + \sigma^* \right\}.
\]

For \(\mathbb{E}|Y^*_T(t) - Y^*(t)|^2\), we first observe that

\[
Y^*_T(t) - Y^*(t) = \tilde{Y}_T(t) - PX^*(t)
= P_T(t)\tilde{X}_T(t) - PX^*(t) + [\Pi_T(t) - P_T(t)]\mathbb{E}[\tilde{X}_T(t)] + \hat{\varphi}_T(t)
= P_T(t)[X^*_T(t) - X^*(t)] + [P_T(t) - P]X^*(t)
+ [\Pi_T(t) - P_T(t)]\mathbb{E}[\tilde{X}_T(t)] + \hat{\varphi}_T(t).
\]

By Lemma 2.3 and Theorem 4.1, \(P_T(\cdot)\) and \(\Pi_T(\cdot)\) are bounded uniformly in \(T\), and by Proposition 5.4, \(|X^*(\cdot)|\) is also bounded. The above then implies that

\[
\mathbb{E}|Y^*_T(t) - Y^*(t)|^2 \leq K \left\{ \mathbb{E}|X^*_T(t) - X^*(t)|^2 + |P_T(t) - P|^2 + |\mathbb{E}[\tilde{X}_T(t)]|^2 + |\hat{\varphi}_T(t)|^2 \right\},
\]

for some constant \(K > 0\). Recalling Lemma 2.3, Theorem 3.2, Lemma 5.2, and Proposition 5.3, we obtain that for some constants \(K, \lambda > 0\) independent of \(T\),

\[
\mathbb{E}|Y^*_T(t) - Y^*(t)|^2 \leq K \left[ e^{-\lambda t} + e^{-\lambda(T-t)} \right], \quad \forall t \in [0,T].
\]

In a similar manner, we can show that

\[
\mathbb{E}|Z^*_T(t) - Z^*(t)|^2 \leq K \left[ e^{-\lambda t} + e^{-\lambda(T-t)} \right], \quad \forall t \in [0,T].
\]

The proof is complete.

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