Reduction of the gravitational lens equation to a one-dimensional non-linear form for the tilted Plummer model family

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ABSTRACT

The gravitational lens equation for the tilted Plummer family of models can be reduced to a one-dimensional non-linear equation. For certain values of the slope of the radial profile it can be reduced to a polynomial form. Both forms are advantageous to find the roots, i.e. the images for a given model. The critical curve equations can also be reduced to a non-linear or polynomial form, and therefore it is useful to find the caustics. This lens model family has ample use in gravitational lensing, and can produce up to five images.

Key words: gravitational lensing.

1 INTRODUCTION

Smooth non-singular isothermal sphere (SNIS) models are often used in gravitational lens theory. Including ellipticity, these models are called elliptical SNIS (ESNIS) (Blandford & Kochanek 1987) or tilted Plummer family (Kassiola & Kovner 1993). This class of models can be generated from elliptical mass distributions or elliptical potentials. The models obtained by means of mass distributions are realistic even for higher ellipticities. The models generated using elliptical potentials are restricted to small ellipticities (Kassiola & Kovner 1993), because the isodensity contours become dumbbell-shaped for higher ellipticities. Moreover, the density for these elliptical potentials could be negative for certain combinations of parameters (Kassiola & Kovner 1993). For more information about these models, the interested reader may consult the references at the end of this Letter.

For ESNIS models, the gravitational lens equation (GLE) becomes non-linear, and hard to solve if the external perturbations (density and shear) are taken into account. However, for ESNIS models obtained from elliptical potentials, it is possible to simplify this equation to a one-dimensional non-linear or polynomial form. We will show how to do it.

The advantage of reducing the GLE to a one-dimensional non-linear or polynomial form is that there are simple iterative methods to obtain the roots of this class of equations. The critical curve equation can also be reduced to a non-linear or polynomial form. This allows us to find the caustics easily. Examples of these models will be presented in this Letter.

2 ESNIS MODELS

The tilted Plummer family of elliptic potentials has the form

$$\psi(x) = \begin{cases} \frac{\kappa_i}{2} \left( x_0^2 + \epsilon_1 x_1^2 + \epsilon_2 x_2^2 \right)^{\beta} - \psi_0, & 0 < \beta < 1, \\ \frac{\kappa_i}{2} \ln \left( x_0^2 + \epsilon_1 x_1^2 + \epsilon_2 x_2^2 \right), & \beta = 0, \end{cases}$$

where $x = (x_1, x_2)$, $x_1$ and $x_2$ are measured in the direction of the principal axes of the ellipsoid (image position), $\beta$ is the slope of the radial profile of the potential, $x_0$ is the core radius (same units as $x_1$ and $x_2$), $\kappa_0$ is a constant, $\epsilon_i = 1 \pm \epsilon (i = 1, 2)$ with $\epsilon$ as the lens ellipticity, and $\psi_0$ is a constant potential reference, which will be taken as null.

The scaled deflection angle components for the ESNIS models have the following form:

$$\alpha_i(x) = \frac{\kappa_i x_i}{\left( x_0^2 + \epsilon_1 x_1^2 + \epsilon_2 x_2^2 \right)^{1/2}},$$

where $\kappa_i = \kappa_0 \epsilon_i$.

The surface mass density is given by

$$\kappa(x) = \kappa_0 \left( x_0^2 + \epsilon_1 x_1^2 + \epsilon_2 x_2^2 \right)^{-2/3} \left[ x_0^2 + \beta_1 x_1^2 + \beta_2 x_2^2 \right],$$

where $\beta_i = 1 + (\beta - 1) \epsilon_i$.

To avoid negative values of surface mass density and the dumbbell form of the isodensity contour shape, the ellipticity $\epsilon$ should be restricted to small values (Kassiola & Kovner 1993):

$$\epsilon \leq \frac{\beta}{3 - \beta}.$$
where \( \sigma \) and \( \gamma \) are the density and shear of external perturbations (nearby galaxies or a cluster contribution), and \( \phi \) is the shear angle.

Let us define some variables for the sake of simplicity:

\[
\chi = x_0^2 + \varepsilon_1 x_1^2 + \varepsilon_2 x_2^2.
\]

\( f(\chi) = \chi^{1-\beta} \Rightarrow \chi(f) = f^{1/(1-\beta)}. \) \hfill (5)

With these definitions the GLE system takes the form

\[
y_1 = \left( M_{11} - \frac{\kappa_1}{f} \right) x_1 + M_{12} x_2,
\]

\( y_2 = \left( M_{22} - \frac{\kappa_2}{f} \right) x_2 + M_{21} x_1, \) \hfill (6)

Solving for \( x_1 \) and \( x_2 \), we get

\[
g x_1 = \left( M_{22} - \frac{\kappa_2}{f} \right) y_1 - M_{12} y_2 \Rightarrow f_1,
\]

\( g x_2 = \left( M_{11} - \frac{\kappa_1}{f} \right) y_2 - M_{21} y_1 \Rightarrow f_2, \) \hfill (8)

where

\[
g = \left( M_{11} - \frac{\kappa_1}{f} \right) \left( M_{22} - \frac{\kappa_2}{f} \right) - M_{12} M_{21}. \] \hfill (10)

Substituting (8) and (9) into (5) yields

\[
g^2 f(\chi) = x_0^2 g^2 + \varepsilon_1 f_1^2 + \varepsilon_2 f_2^2. \] \hfill (11)

By substitution of the right-hand side of equations (8), (9) and (10) into the last equation, we obtain

\[
\left( \chi(f) - x_0^2 \right) P_1(f) = P_2(f), \] \hfill (12)

where

\[
P_1(f) = [f^2 g]^2 = \left( (M_{11} - \kappa_1)(M_{22} - \kappa_2) - M_{12} M_{21} \right)^2 = \left[ N f^2 - (\kappa_1 M_{22} - \kappa_2 M_{11}) f + \kappa_1 \kappa_2 \right]^2 \] \hfill (13)

and

\[
P_2(f) = \left\{ \varepsilon_1 \left( (M_{22} - \kappa_2) y_1 - M_{12} y_2 \right) f \right\}^2 + \varepsilon_2 \left( (M_{11} - \kappa_1) y_2 - M_{21} y_1 \right) f \] \hfill (14)

with

\[
N = \det(M) = M_{11} M_{22} - M_{12} M_{21}, \]

\[
A_2 = \varepsilon_1 (M_{22} y_1 - M_{12} y_2)^2 + \varepsilon_2 (M_{11} y_2 - M_{21} y_1)^2, \]

\[
A_1 = 2 \left( \kappa_2 \varepsilon_1 y_1 (M_{22} y_1 - M_{12} y_2) + \kappa_1 \varepsilon_2 y_2 (M_{11} y_2 - M_{21} y_1) \right) \]

and

\[
A_0 = \varepsilon_1 (\kappa_2 y_2)^2 + \varepsilon_2 (\kappa_1 y_1)^2. \]

\( P_1(f) \) and \( P_2(f) \) are both fourth-order polynomials.

From (12) we get

\[
f P_1(f)^{1-\beta} = \left[ P_2(f) + x_0^2 P_1(f) \right]^{1-\beta}. \] \hfill (15)

The last equation is a non-linear one-dimensional equation for \( f \) that can be solved by means of iterative numerical methods: for example, the Brent method (Moré & Cosnard 1980).

Now, if we set \( \beta = m/n \), with \( m, n \) integers \( (n > m) \), then (15) becomes a polynomial form:

\[
f^n P_1(f)^{n-m} = \left[ P_2(f) + x_0^2 P_1(f) \right]^{n-m}. \] \hfill (16)

The image positions can be obtained as follows: first of all solve (15) or (16); now substituting the \( f \)-values into the equations (8), (9) and (10) yields \( f_1, f_2, \) and \( g \); finally use equations (8) and (9) again to get \( x_1 \) and \( x_2 \).

## 4 The Critical Curves and Caustics

The critical curve is the curve formed by all image positions on which the determinant of the Jacobian vanishes and the caustic curve is the projection of this critical curve to the source plane.

The components of the Jacobian \( J = \frac{\partial y}{\partial x} \) are

\[
J_{11} = \frac{\partial y_1}{\partial x_1} = \left( M_{11} - \frac{\kappa_1}{f} \right) + 2(1 - \beta) \kappa_1 \varepsilon_1 x_0^2 \chi^\eta, \]

\[
J_{12} = \frac{\partial y_1}{\partial x_2} = M_{12} + 2(1 - \beta) \kappa_1 \varepsilon_2 x_0^2 \chi^\eta, \]

\[
J_{21} = \frac{\partial y_2}{\partial x_1} = M_{21} + 2(1 - \beta) \kappa_2 \varepsilon_1 x_0^2 \chi^\eta, \]

\[
J_{22} = \frac{\partial y_2}{\partial x_2} = \left( M_{22} - \frac{\kappa_2}{f} \right) + 2(1 - \beta) \kappa_2 \varepsilon_2 x_0^2 \chi^\eta, \] \hfill (17)

where \( \eta = \beta - 2 \).

The determinant of the Jacobian is

\[
\mathcal{J} = \det(J) = J_{11} J_{22} - J_{12} J_{21}
\]

\[
= \frac{1}{(f P_1(f))^2} \left[ \left( (M_{11} - \kappa_1) P_1(f) \right)^2 + 2(1 - \beta) \kappa_1 \varepsilon_1 x_0^2 \chi^\eta \left( C_1 f - \kappa_1 y_1 \right)^2 \chi^\eta \right]
\]

\[
\times \left( (M_{22} - \kappa_2) P_1(f) \right)^2 + 2(1 - \beta) \kappa_2 \varepsilon_2 x_0^2 \chi^\eta \left( C_2 f - \kappa_2 y_2 \right)^2 \chi^\eta
\]

\[
- \left( 2(1 - \beta) \kappa_1 \varepsilon_2 (C_1 f - \kappa_2 y_2) (C_2 f - \kappa_1 y_1) f \chi^\eta \right)
\]

\[
- M_{12} f \left( C_1 f - \kappa_2 y_2 \right) \left( C_2 f - \kappa_1 y_1 \right) f \chi^\eta
\]

\[
- M_{21} f \left( C_1 f - \kappa_2 y_2 \right) \left( C_2 f - \kappa_1 y_1 \right) f \chi^\eta, \] \hfill (18)

where

\[
C_1 = M_{22} y_1 - M_{12} y_2 \]

and

\[
C_2 = M_{11} y_2 - M_{21} y_1. \]

In (18) we have substituted (8) and (9). From (18), it is clear that the critical and caustic curves could also be found by solving a one-dimensional non-linear equation or a polynomial in \( f \).

## 5 Examples

### 5.1 Plummer Model

Let us take \( \beta = 0 \). We get from (15) the following polynomial:

\[
B_1 f^3 - B_1 f^4 + B_2 f^3 + B_2 f^2 + B_1 f - B_0 = 0, \] \hfill (19)

where

\[
B_5 = \mathcal{M}^2, \]

\[
B_4 = \mathcal{M}^2 x_0^2 + \varepsilon_1 C_1^2 + \varepsilon_2 C_2^2 + 2 C_3 \mathcal{M}, \]

\[
B_3 = 2 \kappa_1 \kappa_2 \mathcal{M} + (2x_0^2 \mathcal{M} + C_3) C_3 + 2 \kappa_2 \varepsilon_1 y_1 C_1 + 2 \kappa_1 \varepsilon_2 y_2 C_2, \]

\[
B_2 = 2 \kappa_1 \kappa_2 C_3 - \kappa_1 \kappa_2 y_1^2 - \kappa_1 \kappa_2 y_2^2 - x_0^2 \left( C_3^2 + 2 \kappa_1 \kappa_2 \mathcal{M} \right), \]

\[
B_1 = \kappa_1 \kappa_2 \left[ \kappa_1 \kappa_2 - 2x_0^2 C_3 \right]. \]
shown in Fig. 1. The values of the model parameters are \( \sigma = 0.05, \gamma = 0.2, \varepsilon = 0.2, \phi = 36^\circ, \kappa_0 = 1, x_0 = 1/2 \). The green dots are the positions of the images for a point source. The position of the extended source is \((0.95, 0.5)\). For this model, one gets up to five images. We have tried with other \( \beta \)-values (with different values for the other parameters) to find more than five images, but all of our ESNIS models always produce no more than five images.

We have designed XFGLENSES, an interactive program intended to visualize and model gravitational lenses (Frutos-Alfaro 2001). The modelling part of this program is not finished yet. Figs 1 and 2 were generated with this software. The interested reader can download it from http://www.tat.physik.uni-tuebingen.de/~frutto/. On this website, there is also extensive information about the software. To solve the GLEs, the Brent method was implemented in the program. To visualize the extended images, we use the Kayser–Schramm method (Schramm & Kayser 1987; Kayser & Schramm 1988).

### 6 CONCLUSIONS

Although the ESNIS models generated from elliptical potentials have an ellipticity restriction, these models are very useful in gravitational lens theory. The polynomial reduction presented here is useful not only to invert the GLE, but also to find the critical curves and caustics of a given ESNIS model. Moreover, it could be useful to model gravitational lenses with small ellipticities. This class of models always produces up to five images. XFGLENSES can be used to visualize the results of this class of models. To see the capabilities and models that this software has available, visit the website mentioned above.

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