GRID POLYGONS FROM PERMUTATIONS AND THEIR ENUMERATION BY THE KERNEL METHOD

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Abstract. A grid polygon is a polygon whose vertices are points of a grid. We define an injective map between permutations of length \( n \) and a subset of grid polygons on \( n \) vertices, which we call consecutive-minima polygons. By the kernel method, we enumerate sets of permutations whose consecutive-minima polygons satisfy specific geometric conditions. We deal with 2-variate and 3-variate generating functions involving derivatives, cases which are not routinely solved by the kernel method.

Key words. Grid polygons, kernel method, permutations.

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1. Introduction and examples. The enumeration of permutations that satisfy certain constraints has recently attracted interest (see e.g. [1, 3, 4, 7, 8, 12, 13, 14, 15]). In particular, permutation patterns have been extensively studied over the last decade (see for instance [4] and references therein). The tools involved in these works include generating trees (with either one or two labels), combinatorial approaches, recurrences relations, enumeration schemes, scanning elements algorithms, etc.

Permutations are traditionally associated to a number of combinatorial and algebraic objects, like matrices, trees, posets, graphs, etc. (see e.g. [5, 11]). The purpose of this work is to begin a study of the interplay between permutations and polygons. A practical motivation comes from computational geometry, where the complexity of algorithms for polygons is an important subject [6]. Of course, it is intuitive that imposing combinatorial constraints on geometric objects commonly reduces generality; on the other side, techniques from combinatorics may provide a fertile background for the design of algorithms, even if the analysis is restricted to toy-cases.

Here we associate permutations to a subset of the grid polygons and enumerate sets of permutations whose polygons satisfy specific geometric conditions. Clearly, there are many potential ways to associate permutations to polygons. Each ways presumably has a special feature which helps to underline some particular property of the permutations. If we want to keep a one-to-one correspondence, this arbitrariness is materialized in two points:

• Of all possible polygons associated to a given permutation, we choose the one with a fixed extremal property, for example, the polygon with minimum area or perimeter.

• We decide how to construct a polygon according to some chosen rule. The rule should guarantee the association of each permutation to a single polygon, unequivocally.

We opt here for the second approach, as it is formalized in what follows. A grid of side \( n \) is an \( n \times n \) array containing \( n^2 \) points, \( n \) in each row and each column. The distance between two closest points in the same row or in the same column is usually taken to be 1 unit. A permutation of length \( n \) is a complete ordering of the elements of the set \( [n] = \{1, ..., n\} \). We associate a grid of side \( n \), denoted by \( L_\pi \), to a permutation \( \pi \) of length \( n \). If the permutation takes \( i \) to \( j = \pi_i \), we mark the point \((i, j)\) of the grid, that is the point in the row \( i \) and the column \( j \). For example, the grid \( L_\pi \) represented in Figure 1.1 is associated to the permutation \( \pi = 4523176 \).

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A grid polygon on $n$ vertices is a polygon whose vertices are $n$ points of a grid. A permutation polygon on $n$ vertices is a grid polygon with the following properties: the side of the grid is $n$; in every row and every column of the grid there is one and only one vertex of the polygon. It is intuitive to observe that a permutation can be associated to more than one polygon depending on how we connect the marked points of the grid. We need to fix some terminology. Let $\pi$ be a grid of side $n$ of a permutation $\pi$.

- A point $(i, j)$ is said to be a left-right minimum of $L_\pi$ if there is no point $(i', j')$ of $L_\pi$ such that $i' < i$ and $j' < j$.
- A point $(i, j)$ is said to be a right-left minimum of $L_\pi$ if there is no point $(i', j')$ of $L_\pi$ such that $i' > i$ and $j' < j$.
- A point $(i, j)$ is said to be a source of $L_\pi$ if either $i = 1$, $i = n$, or $(i, j)$ is not a left-right minimum or a right-left minimum.

We say that two points $(i, j)$ and $(i', j')$ of the grid $L_\pi$ (resp. of the set of left-right minima, right-left minima, sources) are consecutive if there is no point $(a, b)$ in the set of (resp. left-right minima’s, right-left minima’s, sources) such that $i < a < i'$ or $i' < a < i$. For example, the left-right-minima of $L_{4523176}$ are $(1, 4)$, $(3, 2)$ and $(5, 1)$; the right-left-minima are $(7, 6)$ and $(5, 1)$; the sources are $(1, 4)$, $(2, 5)$, $(4, 3)$, $(6, 7)$, and $(7, 6)$ (see Figure 1.1).

**Definition 1.1.** A consecutive-minima polygon (in what follows just polygon) of a permutation $\pi$, denoted by $P_\pi$, is a permutation polygon in which two vertices $a = (i, j)$ and $b = (i', j')$, $i < i'$, are connected if one of the following conditions is satisfied:

- $a$ and $b$ are consecutive left-right minima of $L_\pi$;
- $a$ and $b$ are consecutive right-left minima of $L_\pi$;
- $a$ and $b$ are consecutive sources of $L_\pi$.

In such a context, $(a, b)$ is called an edge of $P_\pi$.

For example, the polygon $P_\pi$ for all $\pi \in S_4$ is represented in Figure 1.2. In the next sections, we will deal with several questions about the number of polygons on $n$ vertices that satisfy a certain set of conditions. In order to do so, we first need to give some further definition. Let $P$ be a polygon, an edge $((i, j), (i', j'))$, $i < i'$, of $P$ is said to be increasing (resp. decreasing) if $j < j'$ (resp. $j > j'$). A path of $P$ is a sequence $(a_0, a_1), (a_1, a_2), \ldots, (a_{n-1}, a_n)$ of edges of $P$. A face of $P$ is either a maximal path of increasing edges or a maximal path of decreasing edges. For example, there are exactly 2, 2, 16, 4 polygons on 4 vertices of exactly one, two, three, and four faces, respectively. This can be observed in Figure 1.2.

**Definition 1.2.** A polygon is said to be $k$-faces if it has exactly $k$ faces. In particular, in the case $k = 3$, the polygon is called triangular, and in case $k = 4$ the polygon is called square.

We will present an explicit formula for the number of $k$-faces polygons on $n$, where $k = 2, 3, 4$. It seems to be a challenging question to find an explicit formula for any $k$.

The technique considered in this paper makes use of generating functions to convert recurrence rela-
tions to functional equations. These are then solved by the kernel method as described in [2]. It may be interesting to remark that the kernel method is a routine approach when dealing with 2-variate generating functions. However, for functional equations with more than two variables there is no systematic approach. Bousquet-Mélou [3] enumerates four different pattern avoiding classes of permutations, by using the kernel method with 3-variate generating functions. We suggest here another class (namely, the square permutations), to which corresponds a functional equation defining 3-variate generating functions. Interestingly, such permutations are not immediately related to pattern avoidance. Among the other techniques, we remark the use of a two variable functional equation involving a derivative (see Theorem 4, below), something that does not appear to be common in enumerative combinatorics.

The remainder of the paper is composed of five sections. In Section 2, we make some general observations about consecutive-minima polygons. We characterize convex polygons and enumerate polygons on \( n \) vertices with maximum number of faces. In Section 3, 4 and 5, we enumerate 2-faces, 3-faces and 4-faces polygons, respectively. Section 6 is a list of open problems.

2. Some general observations.

2.1. Convexity. A polygon is convex if the internal angle formed at each vertex is smaller than 180°. Give a sequence \( a_1, a_2, \ldots, a_n \), we say that the subsequence \( a_{i_1}, \ldots, a_{i_m} \) with \( i_1 < i_2 < \cdots < i_m \) is fast-growing if

\[
\frac{a_{i_j} - a_{i_{j+1}}}{i_j - i_{j+1}} < \frac{a_{i_{j+1}} - a_{i_{j+2}}}{i_{j+1} - i_{j+2}}.
\]
for any \( j = 1, 2, \ldots, m - 2 \), and slow-growing if
\[
\frac{a_{i_j} - a_{i_{j+1}}}{i_j - i_{j+1}} > \frac{a_{i_{j+1}} - a_{i_{j+2}}}{i_{j+1} - i_{j+2}},
\]
for any \( j = 1, 2, \ldots, m - 2 \).

The consecutive-minima polygon \( P_\pi \) is convex if and only if
- the subsequence of left-right-minima of \( \pi \) is fast-growing;
- the subsequence of right-left-maxima of \( \pi \) is fast-growing;
- the subsequence \( L_\pi \) of the sources of \( \pi \) is slow-growing.

2.2. Number of faces. The number of 1-face polygons on \( n \) vertices is exactly 2, that is the polygons corresponding to the two permutations \( 12 \ldots n \) and \( n \ldots 1 \). The number of different shapes of consecutive-minima polygon on \( n \) vertices is exactly \( n \). To clarify this observation, let \( P \) be any consecutive-minima polygon on \( n \) vertices, such that \( k \) is maximal if each face is a segment connecting two vertices, and thus \( k \leq n \). It is not difficult to show that there exists at least one \( k \)-face consecutive-minima polygon for any \( k = 1, 2, \ldots, n \). Let \( \phi^{2k} = 2436587 \ldots (2k-2)(2k-3)(2k)(2k+1)(2k+2) \ldots n1(2k-1) \) and \( \phi^{2k+1} = 2436587 \ldots (2k)(2k-1)(2k+1)(2k+2) \ldots n \) be two permutations of length \( n \) for all \( k \geq 1 \), then we can see that \( P_{\phi^k} \) has exactly \( k \)-faces. Hence, for any \( n \geq 1 \) we have \( n \) different shapes of consecutive-minima polygons on \( n \) vertices.

What can be said about permutations with maximum number of faces? Let \( \pi \) be a permutation of length \( n \geq 3 \). Since the maximum number of faces of \( P_\pi \) is \( n \), one of the following holds:
1. \( \pi \) is an alternating permutation (\( \pi \) is said to be alternating if either \( \pi_1 > \pi_2 < \pi_3 > \pi_4 < \cdots \pi_n \) or \( \pi_1 < \pi_2 > \pi_3 < \pi_4 > \cdots \pi_n \) such that \( \pi_1 = 1 \) and \( \pi_n = 2 \);
2. \( \pi \) is an alternating permutation such that \( \pi_1 = 2 \) and \( \pi_n = 1 \);
3. Removing the letter \( \pi_i = 1, 2 \leq i \leq n-1 \), from \( \pi \) then
\[
(\pi_1 - 1) \ldots (\pi_{i-1} - 1)(\pi_{i+1} - 1) \ldots (\pi_n - 1)
\]
is a permutation satisfying either (1) or (2).

It is not hard to see that the number of alternating permutations of length \( n \) (see \( \text{A000111} \) and references therein) satisfying either (1) or (2) is exactly \( E_{n-2} \) if \( n \) is odd, otherwise it is 0 \( (E_n \) is the number of alternating permutations on length \( n \)). Hence, we can state the following result.

**Proposition 2.1.** The number of polygons on \( n \) vertices with maximum number of faces \( (n \) faces \) is given by \( 2E_n \) if \( n \) odd, and \( 2(n-2)E_{n-3} \) if \( n \) even.

3. Enumeration of two faces polygons. A permutation is said to be parallel if its polygon has exactly two faces. For instance, there are 2 parallel permutations of length 4, namely 1324 and 4231. We denote the set of all parallel permutations of length \( n \) by \( P_n \). Given \( a_1, a_2, \ldots, a_d \in \mathbb{N} \), we define
\[
p_{n;a_1,a_2,\ldots,a_d} = \#\{\pi_1\pi_2 \ldots \pi_n \in P_n \mid \pi_1\pi_2 \ldots \pi_d = a_1a_2 \ldots a_d\},
\]
and we denote the cardinality of the set \( P_n \) by \( p_n \). The main result of this section can be formulated as follows.

**Theorem 3.1.** The number of parallel permutations of length \( n \) is
\[
\frac{2}{n-1} \left(\frac{2n-4}{n-2}\right)^2 - 2,
\]
for all \( n \geq 2 \).
Proof. First, let us enumerate the permutations of length $n$ that begin at letter 1, having polygon with at most two faces. From the definitions we have that
\[ p_{n;1} = p_{n;1,2} + p_{n;1,3} + \cdots + p_{n;1,n}. \]

Besides, for all $a = 3, 4, \ldots, n$,
\[ p_{n;1,a} = p_{n;1,a,2} + \sum_{j=a+1}^{n} p_{n;1,a,j} = p_{n-1;1,a-1} + \sum_{j=a+1}^{n} p_{n-1;1,j-1} = \sum_{j=a-1}^{n-1} p_{n-1;1,j}, \]

with the initial conditions $p_{n;1,2} = p_{n-1;1}$. To solve the recurrence relation of the sequence $p_{n;1,a}$, we need to define $p_{n}(v) = \sum_{a=2}^{n} p_{n;1,a}v^{a-2}$. Thus, multiplying the above recurrence relation by $v^{a-2}$ and summing over $a = 3, 4, \ldots, n$ we obtain that
\[ p_{n}(v) = p_{n-1}(1) + \sum_{a=3}^{n} \sum_{j=a-1}^{n-1} p_{n-1;1,j}v^{a-2}, \]

which is equivalent to
\[ p_{n}(v) = p_{n-1}(1) + \frac{v}{1-v}(p_{n-1}(1) - vp_{n-1}(v)), \]

for $n \geq 4$. Let $p(v; x) = \sum_{n \geq 3} p_{n}(v)x^{n}$. Multiplying the above recurrence relation with $x^{n}$ and summing over all possible $n \geq 4$, by using the initial condition $p_{3}(v) = 1$, we obtain the following functional equation:
\[ p(v; x) = x^{3} + \frac{x}{1-v}(p(1; x) - v^{2}p(v; x)). \]

This type of equation can be solved using the kernel method [2]. Substitute $v = C(x) = \frac{1-\sqrt{1-4x}}{2x}$ in the above functional equation to get $p(1; x) = x^{3}C^{2}(x) = x^{2}(C(x) - 1)$. So, the number of permutations of length $n$ that begin at letter 1 and whose polygon has at most two faces is exactly $c_{n-2}$, for $n \geq 3$, where $c_{m} = \frac{1}{m+1} \binom{2m}{m}$ is the $m$-th Catalan number. Hence, from the fact there exists only one permutation of length $n$ whose polygon has exactly one face, namely $12 \ldots n$, we get that the number of parallel permutations of length $n$ that begin at letter 1 is $c_{n-2}$, for all $n \geq 2$. By making use of the fact that each parallel permutation of length $n$ can begin at either 1 or $n$, we obtain that the number of parallel permutations of length $n$ is $2(c_{n-2} - 1)$, for all $n \geq 2$. \[ \square \]

We now generalize the above enumeration as follows. A polygon is said to be $m$-isolated polygon if it is a grid polygon contains exactly $m$ isolated points. A permutation $\pi$ is said to be $m$-parallel if begins at letter 1 and by deleting $m$ vertices from $L_{\pi}$ gives a polygon with at most two faces. Table \[ \ref{table1} \] includes the number of $m$-parallel permutations of length $n$ for $m \leq 5$ and $n \leq 12$.

We denote the set of all $m$-parallel permutations of length $n$ by $P_{n}^{m}$. Given $a_{1}, a_{2}, \ldots, a_{d} \in \mathbb{N}$, we define
\[ p_{n,m;a_{1},a_{2},\ldots,a_{d}} = \#\{\pi_{1}\pi_{2}\ldots\pi_{n} \in P_{n}^{m} \mid \pi_{1}\pi_{2}\ldots\pi_{d} = a_{1}a_{2}\ldots a_{d}\}. \]

The cardinality of the set $P_{n}^{m}$ is denoted by $p_{n,m}$. \[ \ref{th3.2} \]

**Theorem 3.2.** Let $p(v; q, x) = \sum_{n \geq 1} \sum_{a=2}^{n} \sum_{m=0}^{a} p_{n,m;1,a}v^{m-2}x^{n}$ be the ordinary generating function for the sequence $p_{n,m;1,a}$. Then
\[ \left( 1 + \frac{xv^{2}}{1-v} \right) r(v; q, x) - v^{2}xq \frac{d}{dv}r(v; q, x) = x^{3} + \frac{x}{1-v}r(1; q, x). \]
Proof. Define $p_{n; a_1, a_2, \ldots, a_d} = \sum_{m \geq 0} p_{n, m; a_1, \ldots, a_d} q^m$ for any $n$ and $d$. From the definitions, we have that

$$p_n = p_{n; 1} = p_{n; 1, 2} + p_{n; 1, 3} + \cdots + p_{n; 1, n}.$$  

Besides, for all $a = 3, 4, \ldots, n - 1$,

$$p_{n; 1, a} = p_{n; 1, a, 2} + \sum_{j=a+1}^{n} p_{n; 1, a, j} + \sum_{j=3}^{a-1} p_{n; 1, a, j}$$

$$= p_{n-1; 1, a-1} + \sum_{j=a+1}^{n} p_{n-1; 1, j-1} + q(a-3)p_{n-1; 1, a-1} = q(a-3)p_{n-1; 1, a-1} + \sum_{j=a-1}^{n-1} p_{n-1; 1, j},$$

with the initial conditions $p_{n; 1, 2} = p_{n-1; 1}$ and $p_{n; 1, n} = 0$. To solve the recurrence relation of the sequence $p_{n; 1, a}$, we need to define the generating function $p_n(v; q) = \sum_{a=2}^{n} p_{n; 1, a} v^{a-2}$. Thus, multiplying the above recurrence relation by $v^{a-2}$ and summing over $a = 3, 4, \ldots, n - 1$ we obtain that for all $n \geq 5$,

$$p_n(v; q) = p_{n-1}(1) + \frac{v}{1-v} (p_{n-1}(1; q) - vp_{n-1}(v; q)) + v^2q \frac{d}{dv} (p_{n-1}(v; q)),$$

with the initial condition $p_3(v; q) = 1$. Let $p(v; q, x) = \sum_{n \geq 3} p_n(v; q) x^n$. If we multiply the above recurrence relation by $x^n$ and we sum over all $n \geq 5$, we then obtain the requested functional equation. \qed

**Theorem 3.3.** Let $p_m(v; x)$ be the coefficient of $q^m$ in the ordinary generating function $p(v; q, x)$, that is, $p_m(v; x) = \sum_{n \geq 3} p_{m; n} v^{m-2} x^n$ (define $p_{-1}(v; x) = 0$). Then

$$\left(1 + \frac{xv^2}{1-v}\right) p_m(v; x) = v^2 x \frac{d}{dv} p_{m-1}(v; x) + \frac{x}{1-v} p_m(1; x),$$

where $\delta_{m,0} = 1$ if $m = 0$ and $\delta_{m,0} = 0$ otherwise.

**Theorem 3.4.** For any $m \geq 0$, the ordinary generating function $p_m(v; x)$ can be written as

$$\frac{p'_m(v; x)}{(1 - v + xv^2)^{2m+1}}$$

\[\text{Table 3.1}\]

| $m/n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|-------|---|---|---|---|---|---|---|---|---|---|----|----|----|
| 0     | 1 | 1 | 1 | 1 | 2 | 5 | 14 | 42 | 132 | 429 | 1430 | 4862 | 16796 |
| 1     | 0 | 0 | 0 | 0 | 0 | 1 | 8  | 46 | 232 | 1093 | 4944  | 21778 | 94184 |
| 2     | 0 | 0 | 0 | 0 | 0 | 0 | 0  | 2  | 26  | 220  | 1527  | 9436  | 54004 |
| 3     | 0 | 0 | 0 | 0 | 0 | 0 | 0  | 0  | 6   | 112   | 1275   | 11384  | 87556  |
| 4     | 0 | 0 | 0 | 0 | 0 | 0 | 0  | 0  | 0   | 24    | 596    | 8638    | 95126   |
| 5     | 0 | 0 | 0 | 0 | 0 | 0 | 0  | 0  | 0   | 0     | 120    | 3768    | 66938   |

Number of $m$-parallel permutations of length $n$. 


such that \( p'_m(v; x) \) is a power series, where \( v_0 = C(x) = \frac{1 - \sqrt{1 - 4x}}{2x} \).

Proof. We prove this result by induction on \( m \). For \( m = 0 \), Theorem 3.3 for \( m = 0 \) gives that

\[
\left(1 + \frac{x^2}{1 - v}\right) p_0(v; x) = x^3 + \frac{x}{1 - v} p_0(1; x).
\]

Again, this type of equation can be solved with the kernel method. Substituting \( v = C(x) = \frac{1 - \sqrt{1 - 4x}}{2x} \) in the above equation, we get that \( p_0(1; x) = x^3 C(x) \). That is the number of 0-parallel permutations on \( n \) letters is given by \( \frac{1}{n!} \binom{2n - 3}{n - 2} \), for all \( n \geq 3 \). Moreover, the ordinary generating function \( p_0(v; x) \) is given by

\[
p_0(v; x) = \frac{(1 - v)x^3 + x^4 C^2(x)}{1 - v + x^2},
\]

hence the theorem holds for \( m = 0 \). Let \( p_m(v; x) = \frac{p'_m(v; x)}{(1 - v + x^2)^{2m+2}} \), where \( p'_m(C(x); x) \) is a power series. Then Theorem 3.3 gives that

\[
\left(1 + \frac{x^2}{1 - v}\right) p_{m+1}(v; x) = v^2 \frac{d}{dv} p_m(v; x) + \frac{x}{1 - v} p_{m+1}(1; x),
\]

which is equivalent to

\[
(1 - v + x^2) p_{m+1}(v; x) = v^2 (1 - v)x \frac{d}{dv} p_m(v; x) - (2m + 1) v^2 (1 - v)(1 - 2vx) x p'_m(v; x)
\]

\[
+ x(1 - v + x^2)^{2m+2} p_{m+1}(1; x).
\]

Multiplying by \( (1 - v + x^2)^{2m+2} \), we get that

\[
\left(1 - v + x^2\right)^{2m+3} p_{m+1}(v; x)
\]

\[
= v^2 (1 - v)(1 - v + x^2) x \frac{d}{dv} p'_m(v; x) - (2m + 1) v^2 (1 - v)(1 - 2vx) x p'_m(v; x)
\]

\[
+ x(1 - v + x^2)^{2m+2} p_{m+1}(1; x).
\]

Now, differentiating the above recurrence relation \( 2m + 2 \) times respect to \( v \), we can write

\[
\frac{d^{2m+2}}{dv^{2m+2}} \left[(1 - v + x^2)^{2m+3} p_{m+1}(v; x)\right]
\]

\[
= \frac{d^{2m+2}}{dv^{2m+2}} \left[v^2 (1 - v)(1 - v + x^2) x \frac{d}{dv} p'_m(v; x) - (2m + 1) v^2 (1 - v)(1 - 2vx) x p'_m(v; x)\right]
\]

\[
+ x(1 - v + x^2)^{2m+2} \left[p_{m+1}(1; x)\right].
\]

Substituting \( v = C(x) \),

\[
- (2m + 2)x (1 - 2xC(x))^{2m+2} p_{m+1}(1; x)
\]

\[
= \frac{d^{2m+2}}{dv^{2m+2}} \left[v^2 (1 - v)(1 - v + x^2) x \frac{d}{dv} p'_m(v; x) - (2m + 1) v^2 (1 - v)(1 - 2vx) x p'_m(v; x)\right]_{v = C(x)}.
\]

Since \( p'_m(v; x) \) is a generating function defined at \( v = C(x) \) then any derivative of \( p'_m(v; x) \) respect to \( v \) is a generating function defined at \( v = C(x) \), which gives an explicit formula for \( p_{m+1}(1; x) \). If we substitute the formulas of \( p_{m+1}(1; x) \) and \( p_m(v; x) = \frac{p'_m(v; x)}{(1 - v + x^2)^{2m+2}} \) in the functional equation

\[
\left(1 + \frac{x^2}{1 - v}\right) p_{m+1}(v; x) = v^2 v^{2m+2} p_m(v; x) + \frac{x}{1 - v} p_{m+1}(1; x)
\]
we see that the generating function $p_{m+1}(v; x)$ can be written as 
\[
p'_{m+1}(C(x); x)
\]
such that $p'_{m+1}(C(x); x)$ is a power series. Hence, the theorem is proved by induction on $m$. \(\square\)

**Theorem 3.5.** For $m = 0, 1, 2, 3, 4, 5$ the ordinary generating function $p_m(1; x)$ is given by

\[
\begin{align*}
p_0(1; x) &= \frac{x(1-2x)}{2} - \frac{x}{2}\sqrt{1-4x} \\
p_1(1; x) &= \frac{x(2x^2-4x+1)}{2(1-4x^2)} + \frac{x(2x-1)}{2\sqrt{1-4x}} \\
p_2(1; x) &= \frac{x(1-2x)(4x^2-6x+1)}{2(1-4x^2)} - \frac{x(6x^2-28x^3+30x^2-10x+1)}{2\sqrt{1-4x}} \\
p_3(1; x) &= \frac{x(24x^6-152x^5+300x^4-256x^3+96x^2-16x+1)}{2(1-4x)^4} + \frac{x(2x-1)(18x^4-48x^3+46x^2-12x+1)}{2\sqrt{1-4x}} \\
p_4(1; x) &= \frac{x(1-2x)(96x^6-328x^5+496x^4-392x^3+124x^2-18x+1)}{2(1-4x)^6} - \frac{x(126x^8-888x^7+2268x^6-3068x^5+2310x^4-924x^3+198x^2-22x+1)}{2\sqrt{1-4x}^3} \\
p_5(1; x) &= \frac{x(864x^{10}-6048x^9+17264x^8-28736x^7+30984x^6-21504x^5+8960x^4-2240x^3+336x^2-28x+1)}{2(1-4x)^7} + \frac{x(2x-1)(630x^8-2352x^7+4404x^6-4960x^5+3526x^4-1240x^3+238x^2-24x+1)}{2\sqrt{1-4x}^3}
\end{align*}
\]

We remark that it is not hard to prove by induction, as the proof of Theorem 3.4 that our generating function $p_m(1; x)$ is a rational function in the variables $x$ and $\sqrt{1-4x}$.

**4. Enumeration of three faces polygons.** A permutation $\pi$ is said to be triangular if it begins at letter 1 and its polygon $P_\pi$ has at most 3 faces. For example, there exist 1, 1, 6, 20 triangular permutations of length 1, 2, 3, 4, respectively. We denote the set of all triangular permutations of length $n$ by $n$. Given $a_1, a_2, \ldots, a_d \in \mathbb{N}$, we define

\[
t_{n;a_1, a_2, \ldots, a_d} = \#\{\pi_1 \pi_2 \ldots \pi_n \in_n | \pi_1 \pi_2 \ldots \pi_d = a_1 a_2 \ldots a_d\},
\]

The cardinality of the set $n$ is denoted by $t_n$.

**Theorem 4.1.** The number of triangular permutations of length $n+2$ is $(\frac{2^n}{n})$. Moreover, the ordinary generating function $t(v; x) = \sum_{n\geq2} t_{n; a_1, a_2, \ldots, a_d} x^n$ is given by

\[
\frac{x^2(1-v)(1-2xv)}{(1-2v)(1-v+2xv)} + \frac{x^3}{1-v+xv^2}, \quad \frac{1}{\sqrt{1-4x}}
\]

**Proof.** From the definitions, we have that $t_n = t_{n;1} = t_{n;1,2} + t_{n;1,3} + \cdots + t_{n;1,n}$. For all $a = 3, 4, \ldots, n-1$,

\[
t_{n;1,a} = t_{n;1,a,2} + \sum_{j=a+1}^n t_{n;1,a,j} = t_{n-1;1,2} + \sum_{j=a+1}^n t_{n-1;1,j-1} = t_{n-1;1,j},
\]

with the initial conditions $t_{n;1,2} = t_{n-1;1}$ and $t_{n;1,1} = 2^{n-3}$. To see that

\[
t_{n;1,n} = 2^{n-3}
\]

we consider the following equation $t_{n;1,n} = t_{n;1,n,2} + t_{n;1,n,n-1} = t_{n-1;1,n-1} + t_{n-1;1,n-1} = 2t_{n-1;1,n-1}$ for all $n \geq 4$, and $t_{3;1,3} = 1$ which implies that $t_{n;1,n} = 2^{n-3}$ as claimed.
To solve the recurrence relation of the sequence \( t_{n,1,a} \), we need to define \( t_n(v) = \sum_{a=2}^{n} t_{n,1,a}v^{a-2} \). Thus, multiplying the above recurrence relation by \( v^{a-2} \) and summing over \( a = 3, 4, \ldots, n-1 \) we obtain that

\[
t_n(v) = t_{n-1}(1) + 2n-3v^{n-2} + \sum_{a=3}^{n-1} \sum_{j=a-1}^{n-1} t_{n-1;1,j}v^{a-2},
\]

which is equivalent to

\[
t_n(v) = t_{n-1}(1) + 2n-4v^{n-2} + \frac{v}{1-v}(t_{n-1}(1) - vt_{n-1}(v)),
\]

for \( n \geq 4 \). Let \( t(v;x) = \sum_{n \geq 2} t_n(v)x^n \). If multiplying the above recurrence relation with \( x^n \) and summing over all possibly \( n \geq 4 \) by using the initial conditions \( t_2(v) = 1 \) and \( t_3(v) = 1 + v \), we then obtain the following functional equation

\[
t(v;x) = \frac{vx}{1-v}(t(1;x) - vt(v;x)) + xt(1;x) - \frac{(1-xv)^2x^2}{1-2xv},
\]

which is equivalent to

\[
(1 + \frac{vx^2}{1-v})t(v;x) = \frac{x}{1-v}t(1;x) - \frac{(1-xv)^2x^2}{1-2xv}.
\]

This type of equation can be solved using the kernel method. Substitute \( v = \frac{1 - \sqrt{1-4x}}{2x} \) in the above functional equation to get \( t(1;x) = \frac{x^2}{\sqrt{1-4x}} \), that is, the number of triangular permutations of length \( n \) is exactly \( \binom{2n-4}{n-2} \), as required. Moreover, substituting the expression of \( t(1;x) \) in the functional equation, we get an explicit formula for \( t(v;x) \), as claimed. □

As a corollary of Theorem 4.1 and Theorem 4.1, we get the following.

**Corollary 4.2.** The number of polygons on \( n \) vertices with exactly three faces is \( \frac{4(n-2)}{n-1} \binom{2n-4}{n-2} \), for all \( n \geq 2 \).

**Proof.** Theorem 4.1 and Theorem 4.1 give that the number of permutations \( \pi \) of length \( n \) that begin at letter 1 and its polygon \( P_\pi \) has exactly three faces is \( \frac{\binom{2n-4}{n-2}}{n-1} \). If \( P_\pi \) has exactly three faces, then also \( P_{\pi'} \) and \( P_{\pi''} \) have exactly three faces, where \( \pi' = \text{the complement of } \pi \) and \( \pi'' = \text{the reversal of } \pi \). (Recall that the reversal of a permutation \( \pi_1 \pi_2 \ldots \pi_n \) is \( \pi_n \ldots \pi_2 \pi_1 \); the complement of is the permutation \( n+1 - \pi_1(n+1 - \pi_2) \ldots (n+1 - \pi_n) \).) From this fact, we obtain that the number of polygons on \( n \) vertices with exactly three faces is four times the number of permutations \( \pi \) of length \( n \) that begin at letter 1 and whose polygon \( P_\pi \) has exactly three faces. This number is \( \frac{4(n-2)}{n-1} \binom{2n-4}{n-2} \), as required. □

Let us now generalize the above enumeration of polygons with exactly three faces. A polygon is said to be \( m \)-isolated polygon if its a grid polygon contains exactly \( m \) isolated points. A permutation \( \pi \) is said to be \( m \)-triangular if it begins at letter 1 and by deleting \( m \) vertices from \( L_\pi \) gives a polygon with exactly three faces. Clearly a 0-triangular permutation is a triangular permutation as defined above. In Table 4.1 we give the number of \( m \)-triangular permutations of length \( n \) for \( m \leq 5 \) and \( n \leq 13 \). We denote the set of all \( m \)-triangular permutations of length \( n \) by \( T_n^m \). Given \( a_1, a_2, \ldots, a_d \in \mathbb{N} \), we define

\[
r_{n,m,a_1,a_2,\ldots,a_d} = \# \{ \pi_1 \pi_2 \ldots \pi_n \in T_n^m \mid \pi_1 \pi_2 \ldots \pi_d = a_1 a_2 \ldots a_d \},
\]
It is well known that the unsigned Stirling numbers of ways to permute a list of $n$ items into $k$ cycles (these count the number of ways to permute a list of $n$ items into $k$ cycles [10, Sequence A008275]), satisfy $\prod_{j=1}^{n}(p+j) = \sum_{a=2}^{n} r_n(v;1,a) x^{a-2}$.
Theorem 4.6. This identity leads to that

\[ \sum_{j=0}^{n+1} s_{n+1,j} p^j. \]

Theorem 4.3 gives a recurrence relation for the ordinary generating function \( r_m(v; x) \) for the number of \( m \)-triangular permutations of length \( n \), as follows.

**Theorem 4.4.** Let \( r_m(v; x) \) be the coefficient of \( q^n \) in the ordinary generating function \( r(v; q, x) \), that is, \( r_m(v; x) = \sum_{n \geq 4} \sum_{a=2}^{n} r_{n,m; a} q^a v^n x^n \) (define \( r_{-1}(v; x) = 0 \)). Then

\[
\left( 1 + \frac{xv^2}{1 - v} \right) r_m(v; x) = v^2 \frac{d}{dv} r_{m-1}(v; x) + 2 \delta_{m,0} \frac{1 - v^3}{1 - v} x^4 + \frac{x}{1 - v} r_m(1; x) + x^2 (d_m(vx) - d_{m-1}(vx)),
\]

where \( \delta_{0,0} = 1, \delta_{m,0} = 0 \) with \( m \neq 0 \), and \( d_m(y) = \sum_{n \geq 3} 2^{n-2-m} y^n s_{n-2,n-2-m} \) where \( s_{n,k} \) are the unsigned Stirling numbers of the first kind.

Theorem 4.4 can be used to obtain an explicit formula for \( r_m(v; x) \) for given \( m \). First, we need the following lemma.

**Lemma 4.5.** For all \( n \geq 0 \),

\[
s_{n,n} = 1, \quad s_{n,n-1} = \frac{n(n-1)}{2}, \quad s_{n,n-2} = \frac{n(n-1)(n-2)(3n-1)}{24}.
\]

Moreover, the ordinary generating function \( d_k(x) = \sum_{n \geq 3} 2^{n-2-k} x^n s_{n-2,n-2-k} \) is a rational function with only one pole at \( x = \frac{1}{2} \).

**Proof.** It is well known that the unsigned Stirling numbers \( s_{n,k} \) of the first kind satisfy the recurrence relation \( s_{n,k} = s_{n-1,k-1} + (n-1) s_{n-1,k} \), for all \( k = 1, 2, \ldots, n-1 \) and with the initial conditions \( s_{n,n} = s_{n,0} = 1 \). We use the help of any scientific computing software one can obtain the requested result. Using the recurrence relation and induction, we get that the ordinary generating function \( d_k(x) \) has only one pole at \( x = \frac{1}{2} \), as required.

**Theorem 4.4** provides an algorithm for finding \( r_m(v; x) \) for any given \( m \geq 0 \), since we are dealing with a functional equation with one variable, and this type of functional equations can be solved using the kernel method. We remark that we cannot just substitute \( v = C(x) = \frac{1 - \sqrt{1 - 4x}}{2x} \) in the functional equation of the statement of Theorem 4.4 since it may well be that the generating function \( \frac{d}{dv} r_{m-1}(v; x) \) is not defined at \( v = C(x) \). This kind of recurrence relation for given \( m \) can be solved with the following result.

**Theorem 4.6.** For any \( m \geq 0 \), the ordinary generating function \( r_m(v; x) \) can be written as

\[
\frac{r_m'(v; x)}{(1 - v + xv^2)^{2m+1}}
\]

such that \( r_m'(v_0; x) \) is a power series, where \( v_0 = C(x) = \frac{1 - \sqrt{1 - 4x}}{2x} \).

**Proof.** We prove the theorem by induction on \( m \). For \( m = 0 \), Theorem 4.4 together with Lemma 4.5 for \( m = 0 \) give that

\[
\left( 1 + \frac{xv^2}{1 - v} \right) r_0(v; x) = 2 \frac{1 - v^3}{1 - v} x^4 + \frac{2v^3x^5}{2vx - 1}.
\]
This type of equation can be solved using the kernel method. Substitute \( v = C(x) = \frac{1 - \sqrt{1 - 4x}}{2x} \) in the above functional equation to get that

\[
 r_0(1; x) = \frac{x^2}{\sqrt{1 - 4x}} - x^2(2x + 1).
\]

Then, the number of 0-triangular permutations on \( n \) letters is given by \( \binom{2n-4}{n-2} \), for all \( n \geq 2 \). Moreover, the ordinary generating function \( r_0(v; x) \) is given by

\[
 r_0(v; x) = \frac{x^3(1 - 2vx) + x^3(2v^4x^2 + 2v^3x^2 - 2vx^3 + 2vx - 1)\sqrt{1 - 4x}}{(1 - 2vx)(1 - v + x^2)\sqrt{1 - 4x}},
\]

hence the theorem holds for \( m = 0 \). Let \( r_m(v; x) = \frac{r'_m(v; x)}{(1 - v + x^2)^{2m+1}} \), where \( r'_m(C(x); x) \) is a power series. Then Theorem 4.4 gives that

\[
 \left(1 + \frac{xv^2}{1 - v}\right) r_{m+1}(v; x) = v^2 x \frac{d}{dv} r_m(v; x) + x \frac{1}{1 - v} r_{m+1}(1; x) + x^2 (d_{m+1}(vx) - d_m(vx)),
\]

which is equivalent to

\[
(1 - v + x^2)^{2m+3} r_{m+1}(v; x) = v^2(1 - v)(1 - v + x^2)^{2m+2} r'_m(v; x) - (2m + 1)v^2(1 - v)(1 - 2vx) x \frac{r'_m(v; x)}{(1 - v + x^2)^{2m+2}} + x r_{m+1}(1; x) + x^2(1 - v)(d_{m+1}(vx) - d_m(vx)).
\]

Multiplying by \((1 - v + x^2)^{2m+2}\), we obtain

\[
(1 - v + x^2)^{2m+3} r_{m+1}(v; x) = v^2(1 - v)(1 - v + x^2)^{2m+2} x \frac{d}{dv} r'_m(v; x) - (2m + 1)v^2(1 - v)(1 - 2vx) x r'_m(v; x) + x(1 - v + x^2)^{2m+2} r_{m+1}(1; x) + x^2(1 - v)(1 - v + x^2)^{2m+2}(d_{m+1}(vx) - d_m(vx)).
\]

Now, differentiating the above recurrence relation \( 2m + 2 \) times respect to \( v \) we get that

\[
\frac{d^{2m+2}}{dv^{2m+2}} \left[(1 - v + x^2)^{2m+3} r_{m+1}(v; x)\right] = \frac{d^{2m+2}}{dv^{2m+2}} \left[v^2(1 - v)(1 - v + x^2)^{2m+2} x \frac{d}{dv} r'_m(v; x) - (2m + 1)v^2(1 - v)(1 - 2vx) x r'_m(v; x)\right] + x \frac{d^{2m+2}}{dv^{2m+2}} \left[(1 - v + x^2)^{2m+2} r_{m+1}(1; x)\right] + x^2(1 - v)(1 - v + x^2)^{2m+2}(d_{m+1}(vx) - d_m(vx)).
\]

Substituting \( v = C(x) \),

\[
- (2m + 2)x(1 - 2x C(x))^{2m+2} r_{m+1}(1; x)
\]

\[
= \frac{d^{2m+2}}{dv^{2m+2}} \left[ v^2(1 - v)(1 - v + x^2)^{2m+2} x \frac{d}{dv} r'_m(v; x) - (2m + 1)v^2(1 - v)(1 - 2vx) x r'_m(v; x)\right] \bigg|_{v=C(x)} + x \frac{d^{2m+2}}{dv^{2m+2}} \left[x^2(1 - v)(1 - v + x^2)^{2m+2}(d_{m+1}(vx) - d_m(vx))\right] \bigg|_{v=C(x)}.
\]

Since \( r'_m(v; x) \) is a generating function defined at \( v = C(x) \) then any derivative of \( r'_m(v; x) \) respect to \( v \) is a generating function defined at \( v = C(x) \). Lemma 4.5 provides that the above recurrence
relation gives to an explicit formula for \( r_m(1; x) \). If we now substitute the formula of \( r_m(1; x) \) and \( r_m(v; x) = \frac{r'_m(v; x)}{(1-v+x^2)(1-v+2x^2)^{2m+1}} \) in the functional equation

\[
(1 + \frac{x^2}{1-v}) r_{m+1}(v; x) = v^2 p \frac{d}{dv} r_m(v; x) + \frac{x}{1-v} r_{m+1}(1; x) + x^2 (d_{m+1}(vx) - d_m(vx)),
\]

we obtain that the generating function \( r_{m+1}(v; x) \) can be written as \( \frac{r'_m(v; x)}{(1-v+x^2)(1-v+2x^2)^{2m+1}} \), such that \( r'_m(1)(C(x); x) \) is power series. Hence, the theorem is proved by induction on \( m \).

In conjunction with the kernel method, Theorem 4.7 provides an algorithm for finding \( r_m(v; x) \) for any given \( m \geq 0 \). From the proof of Theorem 4.6 using any scientific computing software, we can state the following.

**Theorem 4.7.** For \( m = 0, 1, 2, 3, 4, 5 \) the ordinary generating function \( r_m(1; x) \) is given by

\[
\begin{align*}
    r_0(1; x) &= -x^2 (2x + 1) + \frac{x^3}{\sqrt{1 - 4x}} \\
    r_1(1; x) &= -x^2 (2x^2 + 2x - 1) - \frac{1}{\sqrt{1 - 4x}} \\
    r_2(1; x) &= \frac{x^2 (4x^3 + 12x^2 - 6x - 1)}{(1 - 4x)^2} + \frac{x^2 (6x^4 + 4x^3 - 26x^2 + 4x + 1)}{\sqrt{1 - 4x}} \\
    r_3(1; x) &= \frac{x^2 (24x^6 - 56x^5 + 188x^4 + 136x^3 + 68x^2 - 22x - 1)}{(1 - 4x)^3} - \frac{x^2 (24x^5 + 18x^4 - 120x^3 - 18x^2 + 24x + 1)}{\sqrt{1 - 4x}} \\
    r_4(1; x) &= \frac{x^2 (144x^7 - 208x^6 - 720x^5 + 360x^4 + 1064x^3 - 204x^2 - 66x - 1)}{(1 - 4x)^4} \\
    &\quad + \frac{x^2 (120x^8 - 504x^7 - 132x^6 + 2064x^5 - 1064x^4 - 1608x^3 + 70x^2 + 64x + 1)}{\sqrt{1 - 4x}} \\
    r_5(1; x) &= \frac{x^2 (864x^{10} - 4128x^9 + 4688x^8 + 9536x^7 - 3304x^6 - 42736x^5 + 12584x^4 + 7304x^3 - 1444x^2 - 154x - 1)}{(1 - 4x)^5} \\
    &\quad + \frac{x^2 (1008x^9 - 2538x^8 + 336x^7 + 800x^6 + 1365x^5 - 15378x^4 + 3472x^3 + 1758x^2 + 156x + 1)}{\sqrt{1 - 4x}}.
\end{align*}
\]

We remark that it is not hard to prove by induction, as the proof of Theorem 4.6, that our generating function \( p_m(1; x) \) is a rational function in the variables \( x \) and \( \sqrt{1 - 4x} \).

**5. Enumeration of four faces polygons.** A permutation \( \pi \) is said to be *square* if the subsequence of the sources of \( L_\pi \) lies on at most two faces of \( P_\pi \). For example, there exists 1, 2, 6, 24, 104, 464, 2088 square permutations of length 1, 2, 3, 4, 5, 6, 7, respectively. We denote the set of all square permutations of length \( n \) by \( Q_n \). Given \( a_1, a_2, \ldots, a_d \in \mathbb{N} \), we define

\[ q_{n; a_1, a_2, \ldots, a_d} = \# \{ \pi_1 \pi_2 \ldots \pi_n \in Q_n \mid \pi_1 \pi_2 \ldots \pi_d = a_1 a_2 \ldots a_d \}. \]

The cardinality of the set \( Q_n \) by \( q_n \). Clearly, a triangular permutation is a square permutation. We derive an explicit formula for the number of square permutations of length \( n \) as follows.

**Theorem 5.1.** The ordinary generating function for the number of square permutations of length \( n \) is given by

\[ 1 + x + \frac{2(1 - 3x)x^2}{(1 - 4x)^2} - \frac{4x^3}{(1 - 4x)^{3/2}}. \]

Moreover, the number of square permutation of length \( n \) is

\[ 2(n + 2)4^{n-3} - 4(2n - 5) \binom{2n - 6}{n - 3}, \]

Grid polygons from permutations
for all $n \geq 3$.

**Proof.** From the symmetry arising in the construction of square permutations we have that for all $n \geq a > b \geq 1$,

$$q_{n;a,b} = q_{n;n+1-a,n+1-b} \text{ and } q_{n;a,b} = q_{n;b,a}. \tag{5.1}$$

Define $Q_n(u, v) = \sum_{n=1}^\infty \sum_{b=1}^n q_{n;a,b} u^{n-1} v^{b-1}$, for all $n \geq 2$, and $Q(u, v; x) = \sum_{n=0}^\infty Q_n(v, u)x^n$ to be the ordinary generating function for the sequence $Q_n(u, v)$. Thus, (5.1) gives

$$Q(v, u; x) = Q'(v; u; x) + Q'(u; v; x), \tag{5.2}$$

where

$$Q'(u; v; x) = \sum_{n \geq 2} Q'_n(v, u)x^n = \sum_{n \geq 2} x^n \sum_{a=2}^{n-1} \sum_{b=1}^{n-1} q_{n;a,b} u^{a-1} v^{b-1}. \tag{5.3}$$

To find an explicit formula for $Q'(1, 1; x)$, which leads us to explicit formula for $Q(1, 1, \ldots, x)$, the ordinary generating function for the number of square permutations of length $n$, we need to divide the generating function $Q'(u; v; x)$ into three parts. For all $n \geq a > b \geq 1$, define

$$A(v; x) = \sum_{n \geq 2} A_n(v)x^n = \sum_{n \geq 2} x^n \sum_{a=2}^{n-1} q_{n;a,1} u^{a-1},$$

$$B(v; x) = \sum_{n \geq 2} B_n(v)x^n = \sum_{n \geq 2} x^n \sum_{b=2}^{n-1} q_{n;1,b} v^{b-1},$$

$$C(v; u; x) = \sum_{n \geq 2} C_n(u)x^n = \sum_{n \geq 4} x^n \sum_{a=3}^{n-1} \sum_{b=2}^{n-1} q_{n;a,b} u^{a-1} v^{b-1}. \tag{5.3}$$

Clearly, for all $n \geq 2$, $Q'_n(v, u) = C_n(v, u) + u^{a-1}B_n(u) + A_n(v)$ and then

$$Q'(u; v; x) = C(v; u; x) + \frac{1}{v}B(u; xv) + A(v; x). \tag{5.3}$$

**Expression for $A(v; x)$**: First, we find an explicit formula for the ordinary generating function $A(v; x)$. From the definitions and (5.1), we have that

$$q_{n;2,1} = q_{n-1;1} = \sum_{b=2}^{n-1} q_{n-1;1,b} = \sum_{b=2}^{n-1} q_{n-1;b,1} = A_{n-1}(1),$$

$$q_{n;a,1} = q_{n;a,1} + \sum_{b=a+1}^{n} q_{n,a,b} = q_{n-1;a-1,1} + \sum_{b=a+1}^{n} q_{n-1,b-1} = \sum_{b=a-1}^{n-1} q_{n-1;b,1},$$

$$q_{n;1,1} = q_{n;1,1} + q_{n;1,n-1} = 2q_{n-1;n-1,1}. \tag{5.4}$$

Using $q_{3;3,1} = 1$ and the recurrence relation for the sequence $q_{n;1,1}$, we obtain that, for all $n \geq 3$,

$$q_{n;1,1} = 2^{n-3}. \tag{5.4}$$

Multiplying by $v^{a-1}$ and summing over all $a = 3, 4, \ldots, n-1$, we obtain that

$$A_n(v) = vA_{n-1}(1) + \sum_{a=3}^{n-1} q_{n-1;a,1} u^{a-1},$$

$$= vA_{n-1}(1) + \sum_{a=2}^{n-2} q_{n-1;a,1} \frac{2^{a-1}}{1-v^{a-1}} + \frac{2^{n-1}}{1-v} q_{n-1;n-1,1} + q_{n,1,v^{n-1}}.$$
Then (5.4) leads us to
\[ A_n(v) = vA_{n-1}(v) + \frac{v^2}{1-v} (A_{n-1}(1) - 2^{n-4} - A_{n-1}(v) + 2^{n-4}v^{n-2}) + 2^{n-4}v^{n-1} = 0, \]
which is equivalent to
\[ A_n(v) = vA_{n-1}(v) + \frac{v^2}{1-v} (A_{n-1}(1) - A_{n-1}(v)) + 2^{n-4}v^{n-4}, \]
for all \( n \geq 4 \), with initial conditions \( A_2(v) = v \) and \( A_3(v) = v^2 \). Writing the above recurrence relation in terms of generating functions,
\[ A(v; x) - (v + v^2)x^3 - vx^2 = vx(A(1; x) - x^2) + \frac{xv^2}{1-v} (A(1; x) - x^2 - A(v; x) + vx^2) + \frac{v^3x^4}{1-2vx}. \]
Equivalently,
\[ \left(1 + \frac{v^2x}{1-v}\right) A(v; x) = vx^2 + \frac{v^3x^4}{1-2vx} + \frac{vx}{1-v} A(1; x). \]
This type of equation can be solved systematically using the kernel method. We substitute \( v = \frac{1-\sqrt{1-4x}}{2x} \) in the above functional equation to get \( A(1; x) = \frac{x^2}{1-4x} \) and then
\[ (5.5) \quad A(v; x) = \frac{1}{1-v + v^2x} \left( v(1-v)x^2 + \frac{v^3(1-v)x^4}{1-2vx} + \frac{vx^3}{\sqrt{1-4x}} \right). \]

**Expression for** \( B(v; x) \): Using the symmetry on the set of square permutations, see (5.1), we obtain that
\[ B(v; x) = \sum_{n \geq 3} x^n \sum_{j=2}^{n-1} q_{n,j,n} - n = \sum_{n \geq 3} x^n \sum_{j=2}^{n-1} q_{n,n+1-j,1} v^{j-1} = \sum_{n \geq 3} x^n \sum_{j=2}^{n-1} q_{n,j,1} v^{n-j}, \]
and from the definition of the generating function \( A(v; x) \) together with (5.4),
\[ B(v; x) = \frac{1}{v} A(1/v; vx) - \frac{x^2(1-x)}{1-2x}. \]

It follows that
\[ (5.6) \quad B(v; x) = \frac{vx^3(1-x)}{(1-2x)(1-v-x)} - \frac{x^3v^2}{(1-v-x)\sqrt{1-4vx}}. \]

**Expression for** \( C(v,u;x) \): From the definitions and (5.1), for all \( n \geq a > b \geq 2 \),
\[ q_{n;a,b} = \sum_{j=1}^{b-1} q_{n;a,b,j} + \sum_{j=a+1}^{n} q_{n;a,b,j} = \sum_{j=1}^{b-1} q_{n-1;a-1,j} + \sum_{j=a+1}^{n} q_{n-1;j-1,1} = \sum_{j=1}^{b-1} q_{n-1;a-1,j} + \sum_{j=a}^{n-1} q_{n-1;j,b}. \]
Thus, for all \( n \geq 5 \),

\[
C_n(v, u) = \sum_{a=2}^{n-2} \sum_{b=1}^{a-1} \left( \sum_{j=1}^{b-1} q_{n-1; a-1,j} + \sum_{j=a}^{n-1} q_{n-1;j,b} \right) u^{a-1}v^{b-1} = \sum_{a=2}^{n-2} \sum_{b=1}^{a-1} q_{n-1;a,b} \frac{u^j - u^i}{1-v} + \sum_{a=3}^{n-1} \sum_{b=2}^{a-1} q_{n-1;a,b} \frac{u^j - u^i}{1-v} \frac{u^j}{1-v}. 
\]

Therefore, by the definition of the sequences \( A_n(v), B_n(v) \) and \( C_n(n, u) \) together with (5.1), for all \( n \geq 5 \),

\[
C_n(v, u) = \frac{vu}{1-u} (C_{n-1}(v, u) - C_{n-1}(vu, 1)) + \frac{v}{1-v}(C_{n-1}(1, vu) - C_{n-1}(v, u)) + \frac{v}{1-v}(B_{n-1}(vu) - v^n - 2B_{n-1}(u)) + \frac{uv}{1-u} (A_{n-1}(v) - A_{n-1}(vu)) - 2n-4u^{n-1}(1-u). 
\]

By converting the above recurrence relation in terms of generating functions with the use of the initial condition \( C_4(v, u) = 2uv^2 \) (this holds immediately from the definitions), we can write

\[
C(v, u; x) = 2uv^2x^4 + \frac{vx}{1-v}(C(v, u; x) - C(vu, 1; x)) + \frac{vx}{1-v}(C(1, vu; x) - C(v, u; x)) + \frac{vx}{1-v}(B(vu; x) - uvx^3) - \frac{v}{1-v}(B(u; vx) - v^2ux^3) + \frac{vx}{1-v} (A(v; x) - vx^2 - uv(1 + vx)x^3) - \frac{2v^ax^5}{(1-2vx)(1-v)} + \frac{2v^ax^5}{(1-2vx)(1-v)}. 
\]

It is well known that this type of functional equations with several variables are in general very hard to solve (see e.g. [3]). However, in our case, we are able to find an explicit formula for the ordinary generating function \( C(1, 1; x) \), as it is described below.

\textbf{Explicit formula for} \( C(1, 1; x) \): Substituting \( u = v^{-1} \) in the above functional equation gives

\[
C(v, v^{-1}; x) = 2vx^4 - \frac{vx}{1-v}(C(v, v^{-1}; x) - C(1, 1; x)) + \frac{vx}{1-v}(C(1, 1; x) - C(v, v^{-1}; x)) + \frac{vx}{1-v}(B(1; x) - x^3) - \frac{v}{1-v}(B(v^{-1}; vx) - vx^3) - \frac{vx}{1-v} (A(v; x) - vx^2 - v(1 + vx)x^3) + \frac{2v^ax^5}{(1-2vx)(1-v)} - \frac{2v^ax^5}{(1-2vx)(1-v)}. 
\]

This is equivalent to

\[
\left( 1 + \frac{2vx}{1-v} \right) C(v, v^{-1}; x) = -(1 + x + vx)x^3 + \frac{2vx}{1-v} C(1, 1; x) + \frac{vx}{1-v} B(1; x) - \frac{x}{1-v} B(v^{-1}; vx) - \frac{vx}{1-v} A(v; x) + \frac{vx}{1-v} A(1; x) + \frac{2v^ax^5}{(1-2vx)(1-v)} - \frac{2v^ax^5}{(1-2vx)(1-v)}. 
\]

By taking \( v = \frac{1}{1-2x} \) and using (5.5) and (5.6),

\[
C(1, 1; x) = \frac{2(3x - 1)x^2}{(1-4x)^{3/2}} + \frac{2x^2(1 - 7x + 15x^2 - 8x^3)}{(1-2x)/(1-4x^2)}. 
\]

\textbf{Explicit formula for} \( Q(1, 1; x) \): Equations (5.3), (5.4) and (5.6) give an explicit formula for \( Q'(1, 1; x) \), namely

\[
Q'(1, 1; x) = \frac{(1-3x)x^2}{(1-4x)^{3/2}} - \frac{2x^3}{(1-4x)^{3/2}}. 
\]
Hence, by (5.2), we obtain that \( Q(1, 1; x) = 2Q'(1, 1; x) \) and the ordinary generating function for the number of square permutations of length \( n \) is given by \( 1 + x + 2Q'(1, 1; x) \) (1 for the empty permutation and \( x \) for the permutation of length 1), as required.

**Corollary 5.2.** The number of polygons on \( n \) vertices with four faces such that the sources of the polygon lies on exactly two faces is given by

\[
2(n + 2)4^{n-3} - 2(n + 1)\binom{2n-4}{n-3}.
\]

**Proof.** The formula is obtained directly from Theorem 5.1 and Corollary 4.2.

6. **Open problems.** In this paper we have used a technique based on the kernel method to solve functional equations for enumerating \( k \)-faces polygons on \( n \) vertices, where \( k = 2, 3, 4 \). The results suggest the following problems:

- The most important question in our context is to find an explicit formula for the number of \( k \)-faces polygons on \( n \) vertices for any \( k \).
- We found that the number of 2-faces polygons on \( n \) vertices is given by \( \frac{2}{n-1} \binom{2n-4}{n-2} - 2 \) and that the number of polygons on \( n \) vertices with at most 2 faces equals \( \frac{2}{n-1} \binom{2n-4}{n-2} \), which is twice of the \( n-2 \)-th Catalan number (see [10, A000108]). This result can be explained combinatorially by considering the number of permutations of length \( n \) that can be put in increasing order on two parallel queues (see Exercise 6.19 (jj) in [11]). Also, we proved analytically that the number of triangular permutations of length \( n + 2 \) is given by \( \binom{2n}{n} \). This numbers appears frequently in mathematics (see [10, A000984]). One question is to find bijections between consecutive-minima polygons and other mathematical objects.
- Can we find a combinatorial interpretations for the formula \( 2(n + 2)4^{n-3} - 2(n + 1)\binom{2n-4}{n-3} \), the number of square permutations of length \( n \).
- Theorem 3.4 gives the generating function \( p_m(1; x) \) for the number of \( m \)-parallel permutations of length \( n \) when \( m \leq 5 \). Can we find an explicit formula for any \( m \). The same can be asked for the generating functions for the number of \( m \)-triangular permutations of length \( n \) (see Theorem 4.7).
- All questions about geometric properties of consecutive-minima polygons remain open (for example, maximal perimeter, maximal area, number of different polygons up to symmetries, etc.).

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