Gravity and the Electroweak Theory

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Abstract

This work shows that gravity creates inertia for the electron. The Lagrangian directly couples the gravitational field with the electron spinor field. It does so via the covariant spinor derivative. The Lagrangian is invariant with respect to the electroweak gauge group $U(1) \otimes SU(2)_L$.

The field equations are solved for an electron in uniform motion. The solution is the Dirac spinor for a massive electron. In effect, gravitational coupling takes the place of the Dirac mass term.
1. Metrical Gravity.

The formalism of time-dependent, three-dimensional geometry was derived in [1] and reviewed in [2]. A brief summary follows. We introduce a scalar, 3-vector basis $e_{\mu} = (e_0, e_i)$ and define inner products

$$g_{\mu\nu} = e_{\mu} \cdot e_{\nu} = \begin{pmatrix} g_{00} & 0 & 0 \\ 0 & 0 & g_{ij} \\ 0 & 0 & \end{pmatrix}$$

(1)

The basis $e_{\mu}$ changes from point to point in the manifold according to the formula

$$\nabla_\nu e_{\mu} = e_{\lambda} Q_{\mu\nu}^\lambda$$

(2)

This separates into scalar and 3-vector parts

$$\nabla_\nu e_0 = e_0 Q_{0\nu}^0$$
$$\nabla_\nu e_i = e_j Q_{i\nu}^j$$

(3) (4)

By definition $Q_{0\nu}^0 = Q_{i\nu}^i \equiv 0$. The 18 coefficients $Q_{jk}^i$ would suffice in the case of ordinary static, three-dimensional geometry. Here, the geometry will be time-dependent, in general, and clock rates may differ from point to point. Hence, the additional coefficients $Q_{j0}^i$ and $Q_{0\nu}^0$, respectively. All 28 independent $Q_{\nu\lambda}^\mu$ are derivable from the metric

$$Q_{0\nu}^0 = \Gamma_{0\nu}^0 = \frac{1}{2} g_{00} \partial_\nu g_{00}$$
$$Q_{j0}^i = \Gamma_{j0}^i = \frac{1}{2} g^{ij} \partial_0 g_{lj}$$
$$Q_{jk}^i = \Gamma_{jk}^i = \frac{1}{2} g^{ij} (\partial_k g_{jl} + \partial_j g_{lk} - \partial_l g_{jk})$$

(5) (6) (7)

where

$$\Gamma_{\nu\lambda}^\mu = \frac{1}{2} g^{\mu\rho} (\partial_\lambda g_{\nu\rho} + \partial_\nu g_{\rho\lambda} - \partial_\rho g_{\nu\lambda})$$

(8)

are the Christofel coefficients. The symbols $\Gamma_{\nu\lambda}^\mu$ are symmetric in $\nu\lambda$, while the $Q_{\nu\lambda}^\mu$ are not. The following formula holds good
\[ Q^\mu_{\nu\lambda} = \Gamma^\mu_{\nu\lambda} + g^{\mu\rho} g_{\lambda\eta} Q^\eta_{[\nu\rho]} \] (9)

where

\[ Q^\mu_{[\nu\lambda]} \equiv Q^\mu_{\nu\lambda} - Q^\mu_{\lambda\nu} \] (10)

The gravitational field is introduced by means of the Einstein-Hilbert action

\[ \frac{\kappa}{2} \int R \sqrt{-g} \, d^4x = \frac{\kappa}{2} \int g^{\mu\nu} R_{\mu\nu} \sqrt{-g} \, d^4x \] (11)

where \( \kappa = c^4/8\pi G \), and the Ricci tensor is

\[ R_{\mu\nu} = \partial_\nu \Gamma^\lambda_{\mu\lambda} - \partial_\lambda \Gamma^\lambda_{\mu\nu} + \Gamma^\lambda_{\rho\nu} \Gamma^\rho_{\mu\lambda} - \Gamma^\lambda_{\lambda\rho} \Gamma^\rho_{\mu\nu} \] (12)

Variation of the gravitational action gives

\[ \delta \int \frac{\kappa}{2} g^{\mu\nu} R_{\mu\nu} \sqrt{-g} \, d^4x = \frac{\kappa}{2} \int \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) \delta g^{\mu\nu} \sqrt{-g} \, d^4x \] (13)

The source of gravitation is expressed in terms of a matter Lagrangian \( L_m \). Variation with respect to \( \delta g^{\mu\nu} \) defines the energy tensor

\[ \delta \int L_m \, d^4x = \frac{1}{2} \int T_{\mu\nu} \delta g^{\mu\nu} \sqrt{-g} \, d^4x \] (14)

Setting the sum of (13) and (14) equal to zero, we obtain the gravitational field equations

\[ \kappa \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) + T_{\mu\nu} = 0 \] (15)

There are seven field equations corresponding to the seven variations \( \delta g^{00} \) and \( \delta g^{ij} \).

The Lagrangian for gravitation may be derived by partial integration of the action [3]

\[ \frac{\kappa}{2} \int g^{\mu\nu} R_{\mu\nu} \sqrt{-g} \, d^4x = \frac{\kappa}{2} \int \left\{ \partial_\nu \left( \sqrt{-g} g^{\mu\nu} \Gamma^\lambda_{\mu\lambda} \right) - \partial_\lambda \left( \sqrt{-g} g^{\mu\nu} \Gamma^\lambda_{\mu\nu} \right) \\
+ \partial_\lambda \left( \sqrt{-g} g^{\mu\nu} \Gamma^\lambda_{\mu\nu} \right) - \partial_\nu \left( \sqrt{-g} g^{\mu\nu} \right) \Gamma^\lambda_{\mu\lambda} \\
+ g^{\mu\nu} \left( \Gamma^\lambda_{\rho\nu} \Gamma^\rho_{\mu\lambda} - \Gamma^\lambda_{\lambda\rho} \Gamma^\rho_{\mu\nu} \right) \sqrt{-g} \right\} d^4x \] (16)
The first two terms, when converted to boundary integrals, do not contribute to the field equations. With some rearranging, this leaves

\[ \frac{\kappa}{2} \int g^{\mu\nu} R_{\mu\nu} \sqrt{-g} \, d^4x = \frac{\kappa}{2} \int g^{\mu\nu} \left( \Gamma^\lambda_{\mu\nu} \Gamma^\rho_{\rho\lambda} - \Gamma^\lambda_{\rho\nu} \Gamma^\rho_{\mu\lambda} \right) \sqrt{-g} \, d^4x \]  

(17)

The Lagrangian

\[ \mathcal{L}_g = \frac{\kappa}{2} g^{\mu\nu} \left( \Gamma^\lambda_{\mu\nu} \Gamma^\rho_{\rho\lambda} - \Gamma^\lambda_{\rho\nu} \Gamma^\rho_{\mu\lambda} \right) \sqrt{-g} \]  

(18)

depends upon the metric \( g^{\mu\nu} \) and its first derivatives \( \partial_\lambda g^{\mu\nu} \). Therefore, variation (13) now takes the form

\[
\delta \int \mathcal{L}_g \, d^4x = \int \left[ \frac{\partial \mathcal{L}_g}{\partial g^{\mu\nu}} \delta g^{\mu\nu} + \frac{\partial \mathcal{L}_g}{\partial \partial_\lambda g^{\mu\nu}} \delta (\partial_\lambda g^{\mu\nu}) \right] \, d^4x \\
= \int \left[ \frac{\partial \mathcal{L}_g}{\partial g^{\mu\nu}} - \partial_\lambda \frac{\partial \mathcal{L}_g}{\partial \partial_\lambda g^{\mu\nu}} \right] \delta g^{\mu\nu} \, d^4x 
\]  

(19)

Variation of the matter action (14) takes a similar form

\[
\delta \int \mathcal{L}_m \, d^4x = \int \left[ \frac{\partial \mathcal{L}_m}{\partial g^{\mu\nu}} - \partial_\lambda \frac{\partial \mathcal{L}_m}{\partial \partial_\lambda g^{\mu\nu}} \right] \delta g^{\mu\nu} \, d^4x 
\]  

(20)

Defining the total Lagrangian

\[ \mathcal{L} = \mathcal{L}_g + \mathcal{L}_m \]  

(21)

and setting the sum of (19) and (20) equal to zero, we obtain the gravitational field equations

\[ \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} - \partial_\lambda \frac{\partial \mathcal{L}}{\partial \partial_\lambda g^{\mu\nu}} = 0 \]  

(22)

The Lagrange equations have a direct bearing upon the question of energy, which is defined in terms of \( \mathcal{L} \) (appendix C).
2. Tetrads.

In order to accommodate spinors, the system $e_\mu$ can be expanded in terms of tetrads on a Pauli basis

$$e_\mu = e_\alpha^\mu (x) \sigma_\alpha \quad (23)$$

where

$$e_\alpha^\mu = \begin{pmatrix} e_0^0 & 0 & 0 & 0 \\ 0 & 0 & e^a_i & 0 \\ 0 & e^a_i \\ 0 & 0 \end{pmatrix} \quad (24)$$

The tetrads can be chosen such that

$$e_1^2 = e_2^1 \quad e_3^2 = e_2^3 \quad e_3^1 = e_3^1 \quad (25)$$

leaving seven independent functions $e_\alpha^\mu (x)$. The metric now depends upon the underlying tetrad field

$$g_{\mu\nu} = \eta_{\alpha\beta} e_\alpha^\mu e_\beta^\nu \quad (26)$$

As shown in appendix A, the covariant spinor derivative

$$D_\mu \psi = \partial_\mu \psi + iqA_\mu \psi + \Gamma_\mu \psi \quad (27)$$

yields the Lagrangian (adding the kinetic term for $A_\mu$)

$$L_e = i \hbar \bar{\psi} \gamma^\mu \partial_\mu \psi - (\partial_\mu \bar{\psi}) \gamma^\mu \psi + \frac{\hbar c}{4} \bar{\psi} \gamma_5 \tilde{\gamma}^\alpha \psi \epsilon^{\delta\alpha\beta\gamma} e^\lambda_\alpha e^\nu_\beta \partial_\nu e_{\beta\lambda} - \frac{\hbar c q}{4} \bar{\psi} \gamma^\mu A_\mu - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad (28)$$

$\tilde{\gamma}^\alpha$ are the constant Dirac matrices, while $\gamma^\mu (x) = e_\alpha^\mu (x) \tilde{\gamma}^\alpha$. The electron field equation is found by varying $\bar{\psi}$

$$\frac{\partial L_e}{\partial \dot{\psi}} - \partial_\mu \frac{\partial L_e}{\partial (\partial_\mu \psi)} = 0 \quad (29)$$
Substitution of

\[ \frac{\partial L_e}{\partial \psi} = \hbar c \left\{ \frac{i}{2} \gamma^\mu \partial_\mu \psi - q \gamma^\mu \psi A_\mu + \frac{1}{4} \gamma_5 \gamma^\delta \psi e^{\delta \alpha \beta \gamma} e^\lambda e^\nu \partial_\nu e_{\beta \lambda} \right\} \sqrt{-g} \]

(30)

\[ \frac{\partial L_e}{\partial (\partial_\mu \psi)} = -\frac{i}{2} \hbar c \sqrt{-g} \gamma^\mu \psi \]

(31)

gives

\[ i \gamma^\mu \partial_\mu \psi + \frac{i}{2} \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} \gamma^\mu) \psi - q \gamma^\mu \psi A_\mu + \frac{1}{4} \gamma_5 \gamma^\delta \psi e^{\delta \alpha \beta \gamma} e^\lambda e^\nu \partial_\nu e_{\beta \lambda} = 0 \]

(32)

A similar calculation with respect to \( \delta \psi \) yields the conjugate equation

\[ -i (\partial_\mu \bar{\psi}) \gamma^\mu - \frac{i}{2} \bar{\psi} \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} \gamma^\mu) \bar{\psi} - q \bar{\psi} \gamma^\mu A_\mu + \frac{1}{4} \gamma_5 \gamma^\delta \bar{\psi} e^{\delta \alpha \beta \gamma} e^\lambda e^\nu \partial_\nu e_{\beta \lambda} = 0 \]

(33)

Multiply the first equation by \( \bar{\psi} \) the second by \( \psi \) and subtract, to find conservation of charge

\[ \partial_\mu \left( \sqrt{-g} \bar{\psi} \gamma^\mu \psi \right) = 0 \]

(34)

In the following sections, we will be concerned with oscillating gravitational fields of very small amplitude. The coordinate system is taken to be nearly rectangular

\[ g_{\mu \nu} = \eta_{\mu \nu} + h_{\mu \nu} \]

(35)

\[ g^{\mu \nu} = \eta^{\mu \nu} - h^{\mu \nu} \]

(36)
The indices of $h_{\mu\nu}$ are raised by $\eta^{\mu\nu}$; for example, in lowest order

$$g_{\mu\lambda} g^{\lambda\nu} = (\eta_{\mu\lambda} + h_{\mu\lambda})(\eta^{\lambda\nu} - h^{\lambda\nu})$$

$$= \delta^\nu_\nu + h^{\nu}_\mu - h^\nu_\mu = \delta^\nu_\mu$$ \hspace{1cm} (37)

The tetrads are expressed in a similar fashion

$$e^{\alpha}_\mu = \delta^\alpha_\mu + \xi^{\alpha}_\mu$$ \hspace{1cm} (38)

$$e^{\mu}_\alpha = \delta^\mu_\alpha - \xi^\mu_\alpha$$ \hspace{1cm} (39)

so that

$$e^{\alpha}_\mu e^{\nu}_\alpha = \delta^\nu_\mu = (\delta^\alpha_\mu + \xi^{\alpha}_\mu)(\delta^\alpha_\nu - \xi^\alpha_\nu)$$

$$= \delta^\nu_\mu + \delta^\alpha_\nu \xi^{\alpha}_\mu - \delta^\alpha_\mu \xi^\alpha_\nu$$ \hspace{1cm} (40)

In order to simplify the notation, we mix indices obtaining

$$\xi^{\nu}_\mu = \xi^{\nu}_\mu$$ \hspace{1cm} (41)

In terms of tetrads, the metric is

$$g_{\mu\nu} = \eta_{\alpha\beta} e^{\alpha}_\mu e^{\beta}_\nu = \eta_{\mu\nu} + \eta_{\alpha\beta}(\delta^\alpha_\mu \xi^{\beta}_\nu + \delta^\alpha_\nu \xi^{\beta}_\mu)$$ \hspace{1cm} (42)

so that

$$h_{\mu\nu} = \eta_{\alpha\beta}(\delta^\alpha_\mu \xi^{\beta}_\nu + \delta^\alpha_\nu \xi^{\beta}_\mu)$$

$$= 2\xi_{\mu\nu}$$ \hspace{1cm} (43)

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3. $U(1) \otimes SU(2)_L$ Gauge Invariance. [4, 5, 6]

The gravitational coupling term in $L_e$ (28) contains the factor $\bar{\psi} \gamma_5 \gamma_8 \psi$. This factor does not mix right- and left-handed spinor components, $\psi_R$ and $\psi_L$. In order to prove this, set

$$\psi = \psi_R + \psi_L = \frac{1 + \gamma_5}{2} \psi + \frac{1 - \gamma_5}{2} \psi$$

(44)

where $(\gamma_5)^2 = 1$ and $\gamma_5 \gamma_8 = -\gamma_8 \gamma_5$. Also, $\bar{\psi}_R = \bar{\psi} \frac{1 - \gamma_5}{2}$ and $\bar{\psi}_L = \bar{\psi} \frac{1 + \gamma_5}{2}$.

In the expansion

$$\bar{\psi} \gamma_5 \gamma_8 \psi = (\bar{\psi}_R + \bar{\psi}_L) \gamma_5 \gamma_8 (\psi_R + \psi_L)$$

$$= \bar{\psi}_R \gamma_5 \gamma_8 \psi_R + \bar{\psi}_L \gamma_5 \gamma_8 \psi_L + \bar{\psi}_R \gamma_5 \gamma_8 \psi_L + \bar{\psi}_L \gamma_5 \gamma_8 \psi_R$$

(45)

the mixed terms are identically zero. For example,

$$\bar{\psi}_R \gamma_5 \gamma_8 \psi_L = \psi \frac{1 - \gamma_5}{2} \gamma_5 \gamma_8 \frac{1 - \gamma_5}{2} \psi = \psi \gamma_5 \gamma_8 \frac{1 - (\gamma_5)^2}{4} \psi = 0$$

(46)

Therefore,

$$\bar{\psi} \gamma_5 \gamma_8 \psi = \bar{\psi}_R \gamma_5 \gamma_8 \psi_R + \bar{\psi}_L \gamma_5 \gamma_8 \psi_L$$

(47)

An expression of this type will be invariant under $U(1) \otimes SU(2)_L$ gauge transformations. Introduce the right-handed singlet $\psi_R = e_R$ and left-handed doublet $\psi_L = \left( \begin{array}{c} \nu_L \\ e_L \end{array} \right)$ in order to form the Lagrangian

$$L_{e-w} = \frac{i}{2} \hbar c [\bar{\psi}_R \gamma_\mu \partial_\mu \psi_R + \bar{\psi}_L \gamma_\mu \partial_\mu \psi_L] + \text{h.c.}$$

$$+ \frac{\hbar c}{4} [\bar{\psi}_R \gamma_5 \gamma_8 \psi_R + \bar{\psi}_L \gamma_5 \gamma_8 \psi_L] \epsilon^{\delta \alpha \beta \gamma} e_\alpha ^\lambda e_\gamma ^\nu \partial_\nu e_\beta \partial_\lambda L_{\text{int}}$$

(48)

$L_{\text{int}}$ contains the electroweak interaction terms as well as kinetic terms for $A_\mu, W_\mu^\pm$, and $Z_0^\mu$.

\footnote{The Dirac mass term $m \bar{\psi} \psi = m (\bar{\psi}_R \psi_R + \bar{\psi}_L \psi_L)$ mixes right- and left-handed spinors and cannot appear in the electroweak Lagrangian.}
4. An Electron in Uniform Motion.

In the previous article [2], the gravitational field was found for a spin up electron at rest

\[
\psi = \frac{1}{\sqrt{V}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \exp(-i\omega t) \tag{49}
\]

which took the form

\[
e^\alpha_\mu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^1_1 & e^1_2 & 0 \\ 0 & e^2_1 & e^2_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{50}
\]

Here, we seek the solution for an electron in uniform motion along \(x^3\)

\[
\psi = \frac{N}{\sqrt{V}} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \exp(-ik_\mu x^\mu) \quad k_\mu = (k_0, k_3) \tag{51}
\]

The second term in the electron equation (32) is zero, which leaves

\[
i\gamma^\mu \partial_\mu \psi + \frac{1}{4} \gamma^5 \gamma^\beta \psi \epsilon^{\alpha\beta\gamma} e^\lambda_\alpha e_\gamma \epsilon_\beta \partial_\mu e_\beta = 0 \tag{52}
\]

In the present case,

\[
\partial_0 \psi = -ik_0 \psi \quad \text{and} \quad \partial_3 \psi = -ik_3 \psi
\]

so that

\[
(k_0 \gamma^0 + k_3 \gamma^3)\psi + \frac{1}{4} \gamma^5 \gamma^\beta \psi \epsilon^{\alpha\beta\lambda} e^\lambda_\alpha e_\beta \partial_0 e_\beta + \frac{1}{4} \gamma^5 \gamma^0 \psi \epsilon^{\alpha\beta\lambda} e^\lambda_\alpha e_\beta \partial_3 e_\beta = 0 \tag{54}
\]

where the tetrad assumes the form (50). The Dirac matrix representation is

\[
\gamma^0 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix} \quad \gamma^a = \begin{pmatrix} 0 & \sigma_a \\ -\sigma_a & 0 \end{pmatrix} \quad \gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{55}
\]

\[2\text{The trial solution (51) gives } \partial_\mu \psi = -ik_\mu \psi \text{ and } \partial_\mu \bar{\psi} = ik_\mu \bar{\psi}. \text{ From the conservation law (34), it follows that } \psi \partial_\mu (\sqrt{-g} \gamma^\mu) \psi = 0.\]
while \( \epsilon^{0123} = -1 \). Substitution yields four equations

\[
\begin{align*}
k_0 u_1 + k_3 u_3 + \frac{1}{4} u_1 (e_1^n \partial_0 e_{2n} - e_2^n \partial_0 e_{1n}) + \frac{1}{4} u_3 (e_1^n \partial_3 e_{2n} - e_2^n \partial_3 e_{1n}) &= 0 \\
k_0 u_2 - k_3 u_4 - \frac{1}{4} u_2 (e_1^n \partial_0 e_{2n} - e_2^n \partial_0 e_{1n}) + \frac{1}{4} u_4 (e_1^n \partial_3 e_{2n} - e_2^n \partial_3 e_{1n}) &= 0 \\
k_3 u_1 + k_0 u_3 + \frac{1}{4} u_3 (e_1^n \partial_0 e_{2n} - e_2^n \partial_0 e_{1n}) + \frac{1}{4} u_1 (e_1^n \partial_3 e_{2n} - e_2^n \partial_3 e_{1n}) &= 0 \\
k_3 u_2 - k_0 u_4 + \frac{1}{4} u_4 (e_1^n \partial_0 e_{2n} - e_2^n \partial_0 e_{1n}) - \frac{1}{4} u_2 (e_1^n \partial_3 e_{2n} - e_2^n \partial_3 e_{1n}) &= 0
\end{align*}
\]

(56) \hspace{1cm} (57) \hspace{1cm} (58) \hspace{1cm} (59)

These equations pair \((u_1, u_3)\) or \((u_2, u_4)\). Consider the case \(u_2 = u_4 = 0\): the equations will be satisfied for all values of \(u_1\) and \(u_3\), if the coefficients are zero \[^3\]

\[
\begin{align*}
4k_0 + e_1^n \partial_0 e_{2n} - e_2^n \partial_0 e_{1n} &= 0 \\
4k_3 + e_1^n \partial_3 e_{2n} - e_2^n \partial_3 e_{1n} &= 0
\end{align*}
\]

(60) \hspace{1cm} (61)

When expressed in terms of the \(\xi_\mu (38)\), these equations read

\[
\begin{align*}
4k_0 + (\xi_1^1 - \xi_2^1) \partial_0 \xi_2^1 - \xi_1^2 \partial_0 (\xi_1^1 - \xi_2^1) &= 0 \\
4k_3 + (\xi_1^1 - \xi_2^1) \partial_3 \xi_2^1 - \xi_1^2 \partial_3 (\xi_1^1 - \xi_2^1) &= 0
\end{align*}
\]

(62) \hspace{1cm} (63)

The first order terms vanish, leaving only second order terms. A solution is given by

\[
\begin{align*}
(\xi_1^1 - \xi_2^1) &= 2a \cos(-k'_\mu x^\mu) \\
\xi_1^2 &= a \sin(-k'_\mu x^\mu)
\end{align*}
\]

(64) \hspace{1cm} (65)

\[^3\text{For the case } u_1 = u_3 = 0, \text{ the coefficients of } u_2 \text{ and } u_4 \text{ must be zero}\]

\[
\begin{align*}
4k_0 - e_1^n \partial_0 e_{2n} + e_2^n \partial_0 e_{1n} &= 0 \\
4k_3 - e_1^n \partial_3 e_{2n} + e_2^n \partial_3 e_{1n} &= 0
\end{align*}
\]

Thus, the two cases are mutually exclusive.
where the amplitude \( a \) is small compared with 1. Substitution yields

\[
\begin{align*}
    k_0 &= \frac{1}{2}a^2 k'_0 \ll k'_0 \\
    k_3 &= \frac{1}{2}a^2 k'_3 \ll k'_3
\end{align*}
\] (66)

Therefore, the electron’s frequency and wave vector are much smaller than those of the gravitational field. Their phase velocities are equal, \( k_0/k_3 = k'_0/k'_3 \).

The spinor

\[
\psi = \frac{N}{\sqrt{V}} \begin{pmatrix} u_1 \\ 0 \\ u_3 \\ 0 \end{pmatrix} \exp (-i k_\mu x^\mu) \quad k_\mu = (k_0, k_3)
\] (68)

has the form of a Dirac spinor for a positive energy electron of mass \( m \), moving along \( x^3 \). Therefore, it will satisfy the Dirac equation

\[
i \gamma^\mu \partial_\mu \psi - \frac{mc}{\hbar} \psi = 0
\] (69)

By giving rise to such a solution, gravity creates inertia for the electron. Components \( u_1 \) and \( u_3 \) now satisfy

\[
\begin{align*}
    k_0 u_1 - k_3 u_3 - \frac{mc}{\hbar} u_1 &= 0 \\
    k_3 u_1 + k_0 u_3 + \frac{mc}{\hbar} u_3 &= 0
\end{align*}
\] (70)

so that

\[
u_3 = \frac{-\hbar k_3 u_1}{(\hbar k_0 + mc)} \quad \text{and} \quad \hbar^2 k_0^2 = \hbar^2 k_3^2 + m^2 c^2
\] (72)

The final spinor is

\[
\psi = \frac{N}{\sqrt{V}} \begin{pmatrix} 1 \\ 0 \\ -\frac{\hbar k_0}{(\hbar k_0 + mc)} \\ 0 \end{pmatrix} \exp (-i k_\mu x^\mu) \quad k_\mu = (k_0, k_3)
\] (73)

where

\[
N^2 = \frac{\hbar k_0 + mc}{2\hbar k_0}
\] (74)
We turn now to the gravitational field equations (15), retaining only the largest terms in the Ricci tensor

\[ R_{\mu \nu} = \partial_\nu \Gamma^\lambda_{\mu \lambda} - \partial_\lambda \Gamma^\lambda_{\mu \nu} \]  

(75)

\[ \partial_\rho g_{\mu \nu} = \eta_{\alpha \beta} \left( \varepsilon^\alpha_{\mu} \partial_\rho \varepsilon^\beta_{\nu} + \varepsilon^\alpha_{\nu} \partial_\rho \varepsilon^\beta_{\mu} \right) \]

\[ = \eta_{\alpha \beta} \left( \delta^\alpha_{\mu} \partial_\rho \xi^\beta_{\nu} + \delta^\alpha_{\nu} \partial_\rho \xi^\beta_{\mu} \right) \]

\[ = 2 \partial_\rho \xi_{\mu \nu} \]  

(76)

It follows that

\[ R_{\mu \nu} = \eta^{\lambda \rho} \{ \partial_\lambda \partial_\rho \xi_{\mu \nu} + \partial_\mu \partial_\nu \xi_{\lambda \rho} - \partial_\mu \partial_\lambda \xi_{\rho \nu} - \partial_\nu \partial_\lambda \xi_{\mu \rho} \} \]  

(77)

The field equations are

\[ \frac{\kappa \lambda^3}{V} \partial_3 \partial_3 (\xi^1_1 + \xi^2_2) + T_{00} = 0 \]  

(78)

\[ \frac{\kappa \lambda^3}{V} (\partial_0 \partial_0 - \partial_3 \partial_3) \xi^2_2 + T_{11} = 0 \]  

(79)

\[ \frac{\kappa \lambda^3}{V} (\partial_0 \partial_0 - \partial_3 \partial_3) \xi^1_1 + T_{22} = 0 \]  

(80)

\[ \frac{\kappa \lambda^3}{V} \partial_0 \partial_0 (\xi^1_1 + \xi^2_2) + T_{33} = 0 \]  

(81)

\[ \frac{\kappa \lambda^3}{V} (\partial_0 \partial_0 - \partial_3 \partial_3) \xi^1_2 - T_{12} = 0 \]  

(82)

A length parameter \( \lambda \) is introduced together with the arbitrary volume of integration, \( V \). The energy tensor is derived in appendix B (128). Make use of the spinor (73) and

\[ (u^*_1 u_1 + u^*_3 u_3) = \frac{2hk_0}{hk_0 + mc} \]  

(83)

\[ (u^*_1 u_3 + u^*_3 u_1) = \frac{-2hk_3}{hk_0 + mc} \]  

(84)

to find
\[ T_{00} = \frac{\hbar c k_0}{V} + \frac{\hbar c}{4V}(e_1^ne_2n - e_2^ne_1n) \]  
(85)

\[ T_{11} = -\frac{\hbar c}{2V}\partial_0 e_{21} + \frac{\hbar c k_3}{2k_0 V}\partial_3 e_{21} \]  
(86)

\[ T_{22} = \frac{\hbar c}{2V}\partial_0 e_{12} - \frac{\hbar c k_3}{2k_0 V}\partial_3 e_{12} \]  
(87)

\[ T_{33} = \frac{\hbar c k_3^2}{k_0 V} + \frac{\hbar c k_3}{4k_0 V}(e_1^ne_2n - e_2^ne_1n) \]  
(88)

\[ T_{12} = -\frac{\hbar c}{4V}\partial_0 (e_{11} - e_{22}) + \frac{\hbar c k_3}{4k_0 V}\partial_3 (e_{11} - e_{22}) \]  
(89)

Comparison with (60, 61) shows that \( T_{00} = T_{33} = 0 \); also, \( T_{11} = -T_{22} \).

The field equations then give \( \xi_1 = -\xi_2 \) so that two independent equations remain

\[ \kappa \lambda^3(\partial_0 \partial_0 - \partial_3 \partial_3)\xi_1 - \frac{\hbar c}{2}\partial_0 \xi_1 + \frac{\hbar c k_3}{2k_0}\partial_3 \xi_1 = 0 \]  
(90)

\[ \kappa \lambda^3(\partial_0 \partial_0 - \partial_3 \partial_3)\xi_2 + \frac{\hbar c}{2}\partial_0 \xi_2 - \frac{\hbar c k_3}{2k_0}\partial_3 \xi_1 = 0 \]  
(91)

These equations are satisfied by (64, 65), if

\[ \kappa \lambda^3 = \frac{\hbar c}{2k_0'} \]  
(92)

The gravitational field is given by

\[ e_1^1 = 1 + a \cos(-k'_\mu x^\mu) \]  
(93)

\[ e_2^1 = 1 - a \cos(-k'_\mu x^\mu) \]  
(94)

\[ e_1^2 = e_2^1 = a \sin(-k'_\mu x^\mu) \]  
(95)

where \( k'_\mu = (k'_0, k'_3) \). The metrical determinant is constant \( \sqrt{-g} = 1 - a^2 \) so that the condition \( \partial_\mu(\sqrt{-g} \gamma^\mu) = 0 \) is satisfied (footnote 2).
Finally, we calculate the total energy (153, appendix C)

\[ \mathcal{H} = \frac{\kappa}{2} \left( \eta^{00} \Gamma_{m0}^l \Gamma_{l0}^m + \eta^{lm} \Gamma_{m0}^l \Gamma_{l0}^m \right) - \frac{i}{2} \hbar c [\bar{\psi} \gamma^3 \partial_3 \psi - (\partial_3 \bar{\psi}) \gamma^3 \psi] - \frac{\hbar c}{4} \bar{\psi} \gamma_5 \gamma_0 \psi \epsilon^{0ab3} \epsilon_{c}^{3} \epsilon_{am} \partial_3 \epsilon_{b}^{m} \]  

(96)

The last two terms are equal to \( T_{33} = 0 \). Therefore, the energy is determined by the gravitational field

\[ \mathcal{H} = \frac{\kappa \lambda^3}{V} \left\{ (\partial_0 \xi_1^1)^2 + (\partial_3 \xi_1^1)^2 + (\partial_0 \xi_2^1)^2 + (\partial_3 \xi_2^1)^2 \right\} = \frac{\hbar c}{2k_0^\prime V} a^2 \left( k_0^\prime 2 + k_3^2 \right) \]  

(97)

Integrate over all of space to find

\[ E = \int \mathcal{H} \, d^3 x = \hbar \omega + \frac{\hbar c k_3^2}{k_0} \]  

(98)

A constant energy must be subtracted in order to obtain \( E = \hbar \omega \).
Appendix A: Electron Lagrangian.

The scalar, 3-vector basis changes according to the formula
\[ \nabla_\nu e_\mu = e_\lambda Q^\lambda_{\mu\nu} \] (99)
Expanding \( e_\mu \) in terms of tetrads
\[ e_\mu = e^\alpha_{\mu} \sigma_\alpha \] (100)
we have
\[ \nabla_\nu e_\mu = \sigma_\alpha \partial_\nu e^\alpha_{\mu} + e^\alpha_{\mu} \nabla_\nu \sigma_\alpha \]
\[ = \sigma_\alpha \left( \partial_\nu e^\alpha_{\mu} + e^\beta_{\mu} \omega^\alpha_{\beta\nu} \right) \] (101)
where, by definition, \[ 4 \nabla_\nu \sigma_\alpha = \sigma_\beta \omega^\beta_{\alpha\nu} \] (102)
Equate the two expressions (99) and (101) to find
\[ e^\alpha_{\lambda} Q^\lambda_{\mu\nu} = \partial_\nu e^\alpha_{\mu} + e^\beta_{\mu} \omega^\alpha_{\beta\nu} \] (103)
Contract this equation with \( e_\alpha \rho \) and form the tensor
\[ g_{\rho\lambda} Q^\lambda_{\mu\nu} = e_\alpha \rho (\partial_\nu e^\alpha_{\mu} - \partial_\mu e^\alpha_{\nu}) + e_\alpha \rho (e^\beta_{\mu} \omega^\alpha_{\beta\nu} - e^\beta_{\nu} \omega^\alpha_{\beta\mu}) \]
\[ = e_\alpha \rho (\partial_\nu e^\alpha_{\mu} - \partial_\mu e^\alpha_{\nu}) + \omega_{\rho\mu\nu} - \omega_{\rho\nu\mu} \] (104)
From (9), it follows that the totally anti-symmetric tensor
\[ Q_{[\mu\nu\lambda]} = g_{\mu\rho} Q^\rho_{[\nu\lambda]} + g_{\nu\rho} Q^\rho_{[\lambda\mu]} + g_{\lambda\rho} Q^\rho_{[\mu\nu]} = 0 \] (105)
Therefore,
\[ 0 = e^\alpha_{[\rho} \partial_{\nu} e^{\alpha_{\mu]} + \omega_{[\rho\mu\nu]} \] (106)

---

4 The \( \omega^\alpha_{\beta\nu} \) give the change of orientation of the orthonormal basis \( \sigma_\alpha \) from point to point. Since \( \eta_{\alpha\beta} = \frac{1}{4} (\sigma_\alpha \bar{\sigma}_\beta + \sigma_\beta \bar{\sigma}_\alpha) \) where \( \bar{\sigma}_\alpha = (\sigma_\alpha, -\sigma_a) \) and \( \nabla_\nu \eta_{\alpha\beta} = \partial_\nu \eta_{\alpha\beta} = 0 \), we have \( \omega_{\alpha\beta\nu} = -\omega_{\beta\alpha\nu} \). Moreover, \( \sigma_0 \sigma_a = \sigma_a \) implies that \( \nabla_\nu \sigma_0 = 0 \) or \( \omega_{0\beta\nu} = 0 \). This leaves 12 parameters \( \omega_{\alpha0\nu} \). They comprise 3 rotation parameters along each of the four coordinates \( x^\nu \).
or

\[ \omega_{[\mu\nu\lambda]} = e_\alpha^\gamma \partial_{\nu} e_{\alpha\lambda} \]  

(107)

where

\[ \omega_{[\mu\nu\lambda]} \equiv \frac{1}{6} \left( \omega_{\mu\nu\lambda} + \omega_{\nu\lambda\mu} + \omega_{\lambda\mu\nu} - \omega_{\nu\mu\lambda} - \omega_{\mu\lambda\nu} - \omega_{\lambda\nu\mu} \right) \]  

(108)

The covariant spinor derivative is [7, 8]

\[ D_\mu \psi = \partial_\mu \psi + iqA_\mu \psi + \Gamma_\mu \psi \]  

(109)

where

\[ \Gamma_\mu = \frac{1}{8} \left( \gamma^\alpha \gamma^\beta - \gamma^\beta \gamma^\alpha \right) \omega_{\alpha\beta\mu} = \frac{1}{4} \gamma^{[\alpha} \gamma^{\beta]} \omega_{\alpha\beta\mu} \]  

(110)

We have included the U(1) term \( iqA_\mu \psi \). The conjugate expression is

\[ D_\mu \overline{\psi} = \partial_\mu \overline{\psi} - iqA_\mu \overline{\psi} - \overline{\psi} \Gamma_\mu \]  

(111)

giving the electron Lagrangian

\[ L_e = \frac{i}{2} \hbar c \left( \gamma^\mu D_\mu \psi - (D_\mu \overline{\psi}) \gamma^\mu \psi \right) 
= \frac{i}{2} \hbar c \left( \gamma^\mu \partial_\mu \psi - (\partial_\mu \overline{\psi}) \gamma^\mu \psi \right) - \hbar \alpha A_\mu \gamma^\mu \psi 
+ \frac{i}{2} \hbar c \psi \left( \gamma^\mu \Gamma_\mu + \Gamma_\mu \gamma^\mu \right) \psi \]  

(112)

The gravitational coupling term can be expressed in terms of the tetrad field as follows:

\[ \gamma^\mu \Gamma_\mu + \Gamma_\mu \gamma^\mu = \frac{1}{4} e_\gamma^\mu \left( \gamma^\gamma \gamma^{[\alpha} \gamma^{\beta]} + \gamma^{[\alpha} \gamma^{\beta]} \gamma^\gamma \right) \omega_{\alpha\beta\gamma} \]  

\[ = \frac{1}{2} \gamma^{[\alpha} \gamma^{\beta]} \gamma^\gamma \omega_{\alpha\beta\gamma} \]  

(113)
where we have used the identity [8]

\[ \hat{\gamma}^\gamma_\alpha \hat{\gamma}^\beta_\beta \equiv \frac{1}{2} (\hat{\gamma}^\gamma_\alpha \hat{\gamma}^\gamma_\beta + \hat{\gamma}^\beta_\alpha \hat{\gamma}^\gamma_\gamma) \]  

(114)

The identity [8]

\[ \hat{\gamma}^\gamma_\alpha \hat{\gamma}^\beta_\beta \hat{\gamma}^\gamma_\gamma \equiv -i\gamma_5 \hat{\gamma}_5 \epsilon^{\delta\alpha\beta\gamma} \]  

(115)

yields

\[ \gamma^\mu \Gamma_\mu + \Gamma_\mu \gamma^\mu = -\frac{i}{2} \gamma_5 \hat{\gamma}_\delta \epsilon^{\delta\alpha\beta\gamma} \omega_{\alpha\beta\gamma} \]  

(116)

since \( \epsilon^{\delta\alpha\beta\gamma} \) is totally anti-symmetric. Substitute (107) in order to obtain

\[ \gamma^\mu \Gamma_\mu + \Gamma_\mu \gamma^\mu = -\frac{i}{2} \gamma_5 \hat{\gamma}_\delta \epsilon^{\delta\alpha\beta\gamma} e_\alpha e_\beta e_\gamma \partial_\nu e_\beta \lambda \]  

(117)

and the Lagrangian

\[ L_e = \frac{i}{2} \hbar c [\bar{\psi} \gamma^\mu \partial_\mu \psi - (\partial_\mu \bar{\psi}) \gamma^\mu \psi] - \hbar c q \bar{\psi} \gamma^5 \partial_\mu A_\mu \] 

\[ + \frac{\hbar c}{4} \psi \gamma_5 \hat{\gamma}_5 \epsilon^{\delta\alpha\beta\gamma} e_\alpha \gamma_\nu \partial_\nu e_\beta \lambda \]  

(118)
Appendix B: Electron Energy Tensor.

The electron energy tensor is found by varying the tetrad field $e_{\alpha}^{\mu}$

$$\delta \int L_e \, d^4x = \int \left( \frac{\partial L_e}{\partial e_{\alpha}^{\mu}} \delta e_{\alpha}^{\mu} + \frac{\partial L_e}{\partial (\partial_{\lambda} e_{\alpha}^{\mu})} \delta (\partial_{\lambda} e_{\alpha}^{\mu}) \right) d^4x$$

$$= \int \left[ \frac{\partial L_e}{\partial e_{\alpha}^{\mu}} - \frac{\partial}{\partial (\partial_{\lambda} e_{\alpha}^{\mu})} \right] \delta e_{\alpha}^{\mu} \, d^4x$$

$$= \int \sqrt{-g} A_{\mu\nu} e^{\beta\nu} \delta e_{\beta}^{\mu} \, d^4x$$

where

$$\sqrt{-g} A_{\mu\nu} \equiv e_{\alpha\nu} \left[ \frac{\partial L_e}{\partial e_{\alpha}^{\mu}} - \frac{\partial}{\partial (\partial_{\lambda} e_{\alpha}^{\mu})} \right]$$

The action is invariant under arbitrary rotations of the tetrad \cite{7}. An infinitesimal rotation takes the form

$$\delta e_0 = 0 \quad \delta e_i = e_a^i \delta e_a^i \quad e^{ab} = -e^{ba}$$

Thus,

$$\delta \int L_e \, d^4x = \int \sqrt{-g} A_{ij} e_a^i e_b^j \delta e^a^b \, d^4x = 0$$

Therefore, the antisymmetric part of $A_{ij}$ must be zero

$$\frac{1}{2} (A_{ij} - A_{ji}) = 0$$

and we define the symmetric part to be

$$T_{\mu\nu} = \frac{1}{2} (A_{\mu\nu} + A_{\nu\mu})$$

It follows that, for an arbitrary variation,

$$\delta \int L_e \, d^4x = \int \sqrt{-g} A_{\mu\nu} \frac{1}{2} \left( e^{\beta\nu} \delta e_{\beta}^{\mu} + e^{\beta\nu} \delta e_{\mu}^{\beta} \right) \, d^4x$$

$$= \frac{1}{2} \int e_{\alpha\nu} \left[ \frac{\partial L_e}{\partial e_{\alpha}^{\mu}} - \frac{\partial}{\partial (\partial_{\lambda} e_{\alpha}^{\mu})} \right] \delta g^{\mu\nu} \, d^4x$$

$$= \frac{1}{2} \int T_{\mu\nu} \delta g_{\mu\nu} \sqrt{-g} \, d^4x$$
Explicitly,

\[
\frac{\partial \mathcal{L}_e}{\partial e^\mu_{\alpha}} = \frac{i}{2} \hbar c \{ \bar{\psi} \gamma_\lambda \partial_\mu \psi - (\partial_\mu \bar{\psi}) \gamma_\lambda \psi \} e^{\alpha \lambda} \sqrt{-g} - \hbar c q \bar{\psi} \gamma_\lambda \psi A_\mu e^{\alpha \lambda} \sqrt{-g} \\
+ \frac{\hbar c}{4} \psi \gamma_5 \hat{\gamma}_\beta \psi e^{\delta \alpha \beta \gamma} e^{\lambda \gamma} (\partial_\lambda e_{\beta \mu} - \partial_\mu e_{\beta \lambda}) \sqrt{-g} \tag{126}
\]

\[
\frac{\partial \mathcal{L}_g}{\partial (\partial_\lambda e^\mu_{\alpha})} = -\frac{\hbar c}{4} \psi \gamma_5 \hat{\gamma}_\beta \psi e^{\delta \alpha \beta \gamma} e^{\lambda \gamma} e_{\beta \mu} \sqrt{-g} \tag{127}
\]

which give

\[
T_{\mu \nu} = \frac{i}{4} \hbar c \{ \bar{\psi} \gamma_\mu \partial_\nu \psi + \bar{\psi} \gamma_\nu \partial_\mu \psi - (\partial_\mu \bar{\psi}) \gamma_\nu \psi - (\partial_\nu \bar{\psi}) \gamma_\mu \psi \} \\
- \frac{1}{2} \hbar c q \left( \bar{\psi} \gamma_\mu \psi A_\nu + \bar{\psi} \gamma_\nu \psi A_\mu \right) \\
+ \frac{\hbar c}{4} \psi \gamma_5 \hat{\gamma}_\beta \psi e^{\delta \alpha \beta \gamma} e^{\lambda \gamma} (e_{\alpha \mu} \partial_\lambda e_{\beta \nu} + e_{\alpha \nu} \partial_\lambda e_{\beta \mu}) \\
- \frac{1}{2} (e_{\alpha \mu} \partial_\nu e_{\beta \lambda} + e_{\alpha \nu} \partial_\mu e_{\beta \lambda}) \tag{128}
\]

The Lagrange form of the gravitational field equations can now be expressed in terms of tetrads. We first substitute \( g^{\mu \nu} = \eta^{\alpha \beta} e^\mu_{\alpha} e^\nu_{\beta} \) in \( \mathcal{L}_g \) to find [2]

\[
\frac{\partial \mathcal{L}_g}{\partial g^{\mu \nu}} - \partial_\lambda \left[ \frac{\partial \mathcal{L}_g}{\partial (\partial_\lambda g^{\mu \nu})} \right] = \frac{1}{2} e_{\alpha \nu} \left[ \frac{\partial \mathcal{L}_g}{\partial e^\mu_{\alpha}} - \partial_\lambda \left[ \frac{\partial \mathcal{L}_g}{\partial (\partial_\lambda e^\mu_{\alpha})} \right] \right] \tag{129}
\]

This, together with the energy tensor for the electron,

\[
T_{\mu \nu} = e_{\alpha \nu} \left[ \frac{\partial \mathcal{L}_e}{\partial e^\mu_{\alpha}} - \partial_\lambda \left[ \frac{\partial \mathcal{L}_e}{\partial (\partial_\lambda e^\mu_{\alpha})} \right] \right] \tag{130}
\]

give the field equations

\[
\frac{\partial \mathcal{L}}{\partial e^\mu_{\alpha}} - \partial_\lambda \frac{\partial \mathcal{L}}{\partial (\partial_\lambda e^\mu_{\alpha})} = 0 \tag{131}
\]
Appendix C: Energy Conservation.

The principle of energy conservation derives from the Lagrange equations of motion, by means of the Hamilton function

\[ \mathcal{H} = \sum_{\phi} c\pi \partial_0 \phi - \mathcal{L} \quad (132) \]

The fields \( \phi \) include \( e_{\alpha}^{\mu}, \psi, \overline{\psi}, \) and \( A_{\mu} \), while the momenta are defined by

\[ c\pi = \frac{\partial \mathcal{L}}{\partial (\partial_0 \phi)} \quad (133) \]

The temporal change of \( \mathcal{H} \) is

\[ \partial_0 \mathcal{H} = \sum_{\phi} \partial_0 \left[ \frac{\partial \mathcal{L}}{\partial (\partial_0 \phi)} \right] \partial_0 \phi + \frac{\partial \mathcal{L}}{\partial (\partial_0 \phi)} \partial_0 \partial_0 \phi - \frac{\partial \mathcal{L}}{\partial \phi} \partial_0 \phi - \frac{\partial \mathcal{L}}{\partial (\partial_\lambda \phi)} \partial_0 \partial_\lambda \phi \]

\[ = \sum_{\phi} \partial_0 \left[ \frac{\partial \mathcal{L}}{\partial (\partial_0 \phi)} \right] \partial_0 \phi - \frac{\partial \mathcal{L}}{\partial \phi} \partial_0 \phi - \frac{\partial \mathcal{L}}{\partial (\partial_\phi \phi)} \partial_0 \partial_\phi \quad (134) \]

Making use of

\[ \partial_n \left[ \frac{\partial \mathcal{L}}{\partial (\partial_n \phi)} \right] \partial_0 \phi = \partial_n \left[ \frac{\partial \mathcal{L}}{\partial (\partial_n \phi)} \right] \partial_0 \phi + \frac{\partial \mathcal{L}}{\partial (\partial_n \phi)} \partial_n \partial_0 \phi \quad (135) \]

we find

\[ \partial_0 \mathcal{H} = \sum_{\phi} \left[ \partial_\lambda \frac{\partial \mathcal{L}}{\partial (\partial_\lambda \phi)} \right] \partial_0 \phi - \partial_n \left[ \frac{\partial \mathcal{L}}{\partial (\partial_n \phi)} \right] \partial_0 \phi \]

\[ = \sum_{\phi} -\partial_n \left[ \frac{\partial \mathcal{L}}{\partial (\partial_n \phi)} \right] \partial_0 \phi \quad (136) \]

The last step is by virtue of the Lagrange equations of motion. Integrate over all 3-dimensional space and discard surface terms, in order to obtain conservation of total energy

\[ \frac{d}{dx} \int \mathcal{H} d^3x = 0 \quad (137) \]

We now derive an explicit expression for the energy density

\[ \mathcal{H} = c\pi^{\alpha}_{\mu} \partial_0 e_{\alpha}^{\mu} + c\pi \partial_0 \psi + \partial_0 \overline{\psi} c\pi + c\pi^{\mu} \partial_0 A_{\mu} - \mathcal{L} \quad (138) \]
The spinor momenta are

\[
\begin{align*}
    c\pi &= \frac{\partial L}{\partial (\partial_0 \psi)} = \frac{i}{2} \hbar c \sqrt{-g} \gamma^0 
    \psi \\
    c\pi &= \frac{\partial L}{\partial (\partial_0 \bar{\psi})} = -\frac{i}{2} \hbar c \sqrt{-g} \gamma^0 \bar{\psi}
\end{align*}
\] (139) (140)

The electromagnetic momenta follow from

\[
L_{e-m} = -\frac{1}{4} \left( 2g^{00}g^{ij} F_{i0} F_{j0} + g^{ii} g^{lm} F_{ij} F_{lm} \right)
\] (141)

\[
\frac{\partial F_{j0}}{\partial (\partial_0 A_i)} = -\delta^j_i
\] (142)

Therefore,

\[
\begin{align*}
    c\pi^0 &= \frac{\partial L}{\partial (\partial_0 A_0)} = 0 \\
    c\pi^i &= \frac{\partial L}{\partial (\partial_0 A_i)} = \sqrt{-g} F^{i0}
\end{align*}
\] (143) (144)

Turning now to the tetrad momenta, the gravitational Lagrangian

\[
L_g = \frac{\kappa}{2} g_{\mu\nu} \left( \Gamma^\lambda_{\mu\nu} \Gamma^\rho_{\rho\lambda} - \Gamma^\lambda_{\rho\nu} \Gamma^\rho_{\mu\lambda} \right)
\]

\[
= \frac{\kappa}{2} \left\{ g^{00} \left( \Gamma^l_{m0} \Gamma^m_{0l} - \Gamma^l_{00} \Gamma^m_{ml} \right) + g^{lm} \left( \Gamma^n_{lm} \Gamma^p_{0m} - \Gamma^p_{nl} \Gamma^m_{nm} \right) \\
+ g^{im} \left( \Gamma^n_{lm} \Gamma^p_{0m} - \Gamma^p_{nl} \Gamma^m_{nm} \right) \right\}
\] (145)

contains time derivatives in the first term

\[
\frac{\partial \Gamma^l_{m0}}{\partial (\partial_0 g^{ij})} = -\frac{1}{2} g_{im} \delta^l_j \\
\frac{\partial \Gamma^l_{00}}{\partial (\partial_0 g^{ij})} = -\frac{1}{2} g_{ij}
\] (146)

Therefore,

\[
\frac{\partial L_g}{\partial (\partial_0 g^{ij})} = -\frac{\kappa}{2} \sqrt{-g} g^{00} \left( g_{il} \Gamma^l_{j0} - g_{ij} \Gamma^l_{00} \right)
\] (147)

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The electron Lagrangian (28) contains time derivatives in the coupling term

\[ \frac{\partial \mathcal{L}_e}{\partial (\partial_0 e^\alpha)_{\beta}} = \frac{\hbar c}{4} \gamma_5 \hat{\gamma}_\beta \psi \epsilon^{\delta_\alpha \beta \gamma} e_\gamma^0 e_{\alpha \mu} \sqrt{-g} \]

which leaves

\[ \frac{\partial \mathcal{L}_e}{\partial (\partial_0 e^a_i)} = -\frac{\hbar c}{4} \gamma_5 \hat{\gamma}_d \psi \epsilon^{dab} e_0^0 e_{bi} \sqrt{-g} \]  

The tetrad momenta are given by

\[ c_{\pi}^{\alpha} = \frac{\partial \mathcal{L}}{\partial (\partial_0 e_\alpha)} = \frac{\partial \mathcal{L}_g}{\partial (\partial_0 g^{\alpha \lambda})} \frac{\partial (\partial_0 g^{\mu \lambda})}{\partial (\partial_0 e_\alpha)} + \frac{\partial \mathcal{L}_e}{\partial (\partial_0 e_\alpha)} \]

\[ = \frac{\partial \mathcal{L}_g}{\partial (\partial_0 g^{\alpha \lambda})} \left( \delta^{\nu}_\mu e_\alpha + \delta^{\lambda}_\mu e_{\alpha \nu} \right) + \frac{\partial \mathcal{L}_e}{\partial (\partial_0 e_\alpha)} \]  

It follows that

\[ c_{\pi}^0 = \frac{\partial \mathcal{L}}{\partial (\partial_0 e_0)} = 0 \]  

\[ c_{\pi}^a_i = \frac{\partial \mathcal{L}}{\partial (\partial_0 e^a_i)} \]

\[ = -\kappa \sqrt{-g} g^{00} \left( e_0^i \Gamma^l_{\alpha 0} - e^a_i \Gamma^l_{\alpha 0} \right) - \frac{\hbar c}{4} \gamma_5 \hat{\gamma}_d \psi \epsilon^{dab} e_0^0 e_{bi} \sqrt{-g} \]
Substituting the momenta into (138), we find the total energy density to be

\[ \mathcal{H} = \frac{\kappa}{2} \sqrt{-g} \left\{ g^{00} \left( \Gamma_{m0}^{\ell} \Gamma_{l0}^{m} - \Gamma_{l0}^{\ell} \Gamma_{m0}^{m} \right) - g^{lm} \left( \Gamma_{lm}^{n} \Gamma_{0n}^{0} - \Gamma_{0n}^{0} \Gamma_{pm}^{p} \right) \right. \\
- g^{lm} \left( \Gamma_{lm}^{n} \Gamma_{mp}^{p} - \Gamma_{nl}^{p} \Gamma_{mp}^{n} \right) \right\} \\
+ \frac{1}{2} \sqrt{-g} g^{00} g^{lm} (\partial_l A_0 \partial_m A_0 - \partial_0 A_l \partial_0 A_m) + \frac{1}{4} \sqrt{-g} g^{ij} g^{lm} F_{ij} F_{lm} \\
- \frac{i}{2} \hbar c [\bar{\psi} \gamma^l \partial_l \psi - (\partial_l \bar{\psi}) \gamma^l \psi] \sqrt{-g} + \hbar c \bar{\psi} \gamma^\mu \psi A_\mu \sqrt{-g} \\
- \frac{\hbar c}{4} \bar{\psi} \gamma^5 \gamma^0 \psi \epsilon^{oabe} e^l_{ \cam} \partial_l e^m_{ \bm} \sqrt{-g} \right) (153) \]
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