KAC-MOODY GROUPS AND INTEGRABILITY OF SOLITON EQUATIONS

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Summary A new approach to integrability of affine Toda field theories and closely
related to them KdV hierarchies is proposed. The flows of a hierarchy are explicitly
identified with infinitesimal action of the principal abelian subalgebra of the nilpotent
part of the corresponding affine algebra on a homogeneous space.

1. Introduction

Soliton equations describe infinite-dimensional hamiltonian systems. They are
closely related to infinite-dimensional Lie groups and algebraic curves. These re-
lations account for complete integrability of soliton equations, i.e. the existence of
infinitely many integrals of motion in involution.

Recently a new insight has been brought into the theory by the observation that
these integrals of motion can be viewed as classical limits of quantum integrals of
motion of certain deformations of conformal field theories [Z, EY, HM].

In our previous works [FF1, FF2], using the technique of free field realization of
conformal field theories (cf. [F2] for a review), we gave a homological construction of
quantum integrals of motion, corresponding to particular deformations. This enabled
us to prove the existence of quantum integrals of motion for those deformations.
Tracing our construction back to the classical limit, we realized that it provides
a new approach to integrability of classical soliton equations, namely, affine Toda
equations. Here we will present this approach. We hope that it can be applied to
other soliton equations as well.

1.1. Recall that a Toda field theory can be associated to an arbitrary affine algebra
\( g \) [MOP]. Let \( a_i, i = 0, \ldots, l \), be the labels of the Dynkin diagram of \( g \), and \((\alpha_i, \alpha_j)\)
be the scalar product of the \( i \)th and \( j \)th simple roots of \( g \) [K2]. The system of Toda
equations corresponding to $g$ can be written in the form

$$\partial_\tau \partial_t \phi_i(t, \tau) = \sum_{j=0}^l (\alpha_i, \alpha_j) e^{-\phi_j(t, \tau)}, \quad i = 1, \ldots, l,$$

where each $\phi_i(t, \tau), i = 1, \ldots, l,$ is a family of functions in $t$, depending on the time variable $\tau$, and

$$\phi_0(t, \tau) = -\frac{1}{a_0^1} \sum_{i=1}^l a_i \phi_i(t, \tau).$$

We call local functionals in $u_i(t) = \partial_t \phi_i(t), i = 1, \ldots, l,$ which are preserved under the time evolution of the $\phi_i(t)$’s, the local integrals of motion of the system (1). Recall that a local functional is a functional of the form

$$F[u(t)] = \int P(u(t), \partial_t u(t), \ldots) dt,$$

where $P$ is a differential polynomial in $u(t) = (u_1(t), \ldots, u_l(t))$.

The equation (1) can be written in the Hamiltonian form:

$$\partial_\tau u(t, \tau) = \{H, u(t, \tau)\},$$

where $\{\cdot, \cdot\}$ is a certain Poisson bracket on the space of functionals in $u$, and $H$ is the Hamiltonian:

$$H = \sum_{i=0}^l \int e^{-\phi_i(t)} dt.$$

The space of local integrals of motion is by definition the kernel of the linear operator $\{H, \cdot\}$ on the space of local functionals. For brevity we will call these integrals of motion Toda integrals.

1.2. Instead of working with the space of local functionals in $u$, we will work with the algebra of differential polynomials in $u$, i.e. the polynomial algebra in variables $u_i^{(n)} = \partial^n u_i, i = 1, \ldots, l; n \geq 0$. We denote this algebra by $\pi_0$. The action of $\partial$ on $u_i^{(n)}$ can be extended to a derivation of $\pi_0$ by the Leibnitz rule. The space of local functionals in $u$ is then the quotient of $\pi_0$ by the image of $\partial$ (the subspace of total derivatives) and constants. We make $\pi_0$ into a $\mathbb{Z}$-graded algebra by putting $\deg u_i^{(n)} = -n - 1$. This induces a $\mathbb{Z}$-grading on the space of local functionals.

The Hamiltonian $H$ of the Toda equation is a sum of $l + 1$ terms $\int e^{-\phi_i(t)} dt$. It turns out that the operator of Poisson bracket with each of these terms gives rise to a certain derivation $Q_i$ of $\pi_0$.

The crucial observation, which enabled us to describe Toda integrals in [FF1, FF2] was that the operators $Q_i$ satisfy the Serre relations of the Lie algebra $g$. In other words, these operators generate an action of the nilpotent Lie subalgebra $n_+$ of $g$ on $\pi_0$. This allowed us to interpret the space of Toda integrals as the first cohomology of
with coefficients in the module $\pi_0$, $H^1(n_+, \pi_0)$, cf. [FF1, FF2]. We then computed this cohomology and found that it is spanned by elements of degrees $-m \in -I$, where $I$ is the set of exponents of $g$ modulo the Coxeter number. This agrees with the results previously established by other methods [MOP, DS1, DS2, KW, W1].

In the present work we will simplify and investigate further this construction.

1.3. Let $G$ be the Lie group corresponding to $g$. The upper nilpotent subgroup $N_+^c$ of $G$ can be considered as a big cell on the flag manifold $B_+ \backslash G$ of $g$, where $B_+$ is the lower Borel subgroup of $G$ (cf. Sect. 4 for precise definitions). On this big cell, $g$ infinitesimally acts from the right by vector fields.

The Lie algebra $g$ contains a principal Heisenberg Lie subalgebra $\hat{a} = a_+ \oplus a_- \oplus \mathbb{C}K$, where $a_+$ is the principal abelian subalgebra of $n_+$, and $a_-$ is its opposite with respect to a Cartan involution of $g$. The right action of $a_-$ on $N_+^c$ commutes with the right action of the Lie group $A_+ \subset N_+^c$. Therefore each element of $a_-$ gives rise to a vector field on $N_+^c/A_+$.

In the principal gradation, the Lie algebra $a_-$ is linearly spanned by elements $p_m$ of degrees $-m \in -I$. Denote by $\mu_m$ the corresponding vector fields on $N_+^c/A_+$. We will show that these vector fields coincide with the Hamiltonian vector fields defined by Toda integrals.

More precisely, consider $\pi_0$ as the algebra of regular functions on an infinite-dimensional affine space $U$ with coordinates $u_i^{(n)}$, $i = 1, \ldots, l$, $n \geq 0$. We will show, cf. Theorem 2, that $U$ is isomorphic to $N_+^c/A_+$ and the action of the operator $Q_i$ on $\pi_0$ coincides with the left infinitesimal action of the $i$th generator $e_i$ of $n_+$ on the space of regular functions on $N_+^c/A_+$.

The space $U$ is equipped with a generalized Hamiltonian structure in the sense of Gelfand-Dickey-Dorfman [GD1, GD2, GD3]. This structure allows us to associate to any local functional $F$ in $u$, a vector field $\xi_F$ on the space $U$. We will say that $\xi_F$ is the Hamiltonian vector field defined by $F$. On the other hand, we show that the space of Toda integrals is isomorphic to the dual space $a_-^*$ of $a_-$, cf. Theorem 1 below and [FF2]. Thus, Toda integrals have degrees $-m \in -I$. Denote by $\eta_m$ the vector field corresponding to the Toda integral of degree $-m \in -I$.

Theorem 3 states that under the isomorphism $N_+^c/A_+ \simeq U$, we have: $\eta_m = \mu_m$. In particular, this means that the vector fields $\eta_m$ commute with each other. Therefore Toda integrals are in involution with respect to the Poisson bracket $\{\cdot, \cdot\}$. This proves complete integrability of affine Toda equations.

1.4. In the course of proving this fact, we came across an interesting subalgebra of the Lie algebra of vector fields on $N_+$. On $N_+$, we have a right action of $g$ and a left action of $n_+$. Denote by $e_i^L$ the vector field on $N_+$, which corresponds to the left action of the generator $e_i$ of $n_+$. Let $L$ be the Lie algebra of vector fields $\alpha$ on $N_+$, such that $[e_i^L, \alpha] = -F_i(\alpha)e_i^L$ for all $i$, where $F_i(\alpha)$ is a function on $N_+$. Thus,
\( L \) preserves a certain foliation on \( N_+ \), and it seems that if \( g \neq sl_2 \), then \( L \) coincides with the Lie algebra of global vector fields on \( B_- \setminus G \).

The Lie algebra \( L \) can be defined for an arbitrary Kac-Moody algebra \( g \), and one can show that \( g \) (acting from the right) is contained in \( L \). In the Appendix we show that if \( g \) is finite-dimensional and \( g \neq sl_2 \), then \( L = g \), and if \( g \) is affine, then \( L \) is the semi-direct product of \( g \) and a Lie algebra of vector fields on the disc.

1.5. Let us now return to Toda integrals. Using vector fields \( \eta_m \) we can construct a hierarchy of differential equations on the space \( U \):

\[
\frac{\partial u^{(n)}_i(t)}{\partial t_m} = \eta_m \cdot u^{(n)}_i(t), \quad m \in I; \quad i = 1, \ldots, l; \quad n \geq 0,
\]

where \( t = (t_m)_{m \in I} \) are the times of the hierarchy. We will show that the vector field \( \xi_{H_1} \) coincides with \( \partial_t \). Therefore the equations with \( m = 1 \) read simply as \( \partial u^{(n)}_i / \partial t_1 = \partial_t u^{(n)}_i \), and hence \( t_1 \equiv t \).

It is known that Toda integrals coincide with integrals of motion of the generalized mKdV hierarchy associated to \( g \) \([DS1, DS2, KW, W1]\), so that equations (4) are equations of the mKdV hierarchy. This fact can be proved directly: using our geometric formalism we can rewrite equations (4) in the “zero-curvature” form and check that they coincide with the mKdV hierarchy. In the case \( g = \tilde{sl}_2 \) this follows from results of Enriquez \([E]\). Thus we obtain a new proof of integrability of generalized mKdV (and therefore KdV) equations.

We can also obtain solutions of the mKdV hierarchy using the (rational) action of the Lie group \( A_- \) of \( a_- \) on \( U \). Indeed, consider the following element of \( A_- \):

\[
g(t) = \exp \left( \sum_{m \in I} t_m \eta_m \right) = \exp \left( \sum_{m \in I} t_m P_m \right).
\]

This element operates on the manifold \( B_- \setminus G/A_+ \) from the right.

Now choose an initial condition \( x(0) \in U \subset B_- \setminus G/A_+ \) and put

\[
x(t) = x(0) \cdot g(t).
\]

One can show that \( x(t) \) lies in the “big cell” \( U \subset B_- \setminus G/A_+ \) for almost all values of the \( t_m \)’s (compare with \([W2, W3]\)). But then the \( u_i \)-coordinates \( u_i(t) \) of \( x(t) \in U \) give solutions of the mKdV hierarchy (4). It is evident that all solutions, for which the initial function \( u_i(t, 0, 0, \ldots) \) is smooth at least for one value of \( t \), can be obtained this way.

1.6. The connection between the KdV and mKdV hierarchies and affine groups has been studied before in the framework of tau-functions, infinite Grassmanians and dressing transformations, cf. \([DJKM, SW, W2, W3, C]\), and more recent papers \([KW, BB, HMi, AB]\).
However, previously it was established through the so-called Baker functions, of which our \(u_i\)'s are, roughly speaking, logarithmic derivatives (for a more precise statement, cf. [W2, W3]). The difference with our approach is that instead of using Baker functions, we identify the mKdV variables \(u_i^{(n)}\)'s directly with special coordinates on the big cell of the flag manifold modulo \(A_+\). We do that using the nilpotent action on \(U\) induced by the hamiltonian of the Toda equation.

1.7. As was mentioned above, the hamiltonian form of the mKdV and Toda hierarchies can be obtained from the generalized Hamiltonian structure on the space \(U\), which has been studied by Gelfand, Dickey and Dorfman [GD1, GD2, GD3]. This structure can be quantized, giving the vertex operator algebra structure on \(\pi_0\). This allows us to define the space of quantum Toda integrals. Generally, the dimension of the space of quantum integrals of a given degree could be smaller than the dimension of the space of classical integrals of the same degree. However, in [FF1, FF2] we proved that in our case these dimensions are the same in all degrees. In other words, we proved that all classical Toda integrals can be quantized. The quantum integrals are integrals of motion of certain deformations of conformal field theories, which were mentioned at the beginning of this Introduction.

It would be interesting to understand how the Gelfand-Dickey-Dorfman structure on \(U\) is related to the Lie-Poisson structure on the Borel group of \(G\) in the sense of Drinfeld [Dr1]. Quantization of Lie-Poisson structures leads to quantum groups [Dr2], and quantum groups indeed appeared in our construction of quantum integrals of motion, cf. [FF2]. Roughly speaking, the action of the Lie algebra \(n_+\) on \(\pi_0\), which plays a crucial role in our classical construction, becomes an action of the quantized universal enveloping algebra \(U_q(n_+)\). Thus we see that quantization of the geometric constructions given in this paper should involve both vertex operator algebras and quantum groups.

Finally, let us remark that one can associate analogues of mKdV hierarchies to arbitrary Heisenberg subalgebras of an affine algebra, cf. [BDHM, HMi, AB]. In this work we only consider the case of principal subalgebras. Generalization of our approach to arbitrary Heisenberg subalgebras is straightforward, but we will leave the details for a separate publication.

1.8. The paper is organized as follows. In Sect. 2 we recall the hamiltonian formalism of affine Toda field theories, following Gelfand-Dickey-Dorfman and Kuperschmidt-Wilson. Then in Sect. 3 we show that the Toda hamiltonian (3) gives rise to an action of the nilpotent subalgebra \(n_+\) of \(g\) on the algebra of differential polynomials \(\pi_0\). In Sect. 4 we give a geometric construction of modules contragradient to Verma modules over \(g\) in the space of functions on \(N_+\) and homomorphisms between them, following [Ko2]. We also prove an important Proposition 3. Using these results, we prove in Sect. 5 that the algebra \(\pi_0\) is isomorphic to the algebra of regular functions on the space \(N_+/A_+\). In Sect. 6 we use the Bernstein-Gelfand-Gelfand resolution and
results of Sect. 4 to show that the space of Toda integrals is isomorphic to the first cohomology of a complex $F^*(g)$ and to the dual space of $a_+$. We explain in Sect. 7 how to attach explicitly a Toda integral to a class in the first cohomology of $F^*(g)$, and then obtain a crucial formula (22) for the commutator of the vector fields $\eta_m$ and $Q_i$. Using this formula we prove in Sect. 8 that the hamiltonian vector field $\eta_m$ defined by a Toda integral coincides with the vector field $\mu_m$, which corresponds to the right action of an element of the Lie algebra $a_-$ on $N_+/A_+$. In the Appendix we study the subalgebra $L$ of the Lie algebra of vector fields on $N$.

2. Hamiltonian formalism

Let $g$ be an affine algebra, twisted or non-twisted, of rank $l+1$. To $g$ one canonically associates a finite-dimensional Lie algebra $\bar{g}$, whose Dynkin diagram is obtained from the Dynkin diagram of $g$ by deleting its 0th node. We have the Cartan decomposition: $\bar{g} = \bar{n}_- \oplus \bar{h} \oplus \bar{n}_+$, where $\bar{h}$ is the Cartan subalgebra and $\bar{n}_\pm$ are the nilpotent subalgebras of $\bar{g}$.

We consider $g$ as defined over the formal Laurent power series $\mathbb{C}((t))$, with the topology of inverse limit. The Lie algebra $g$ is the universal central extension of a loop algebra $g'$ adjoined with a $\mathbb{Z}$-grading operator $d$. If $g$ is non-twisted, $g' = \bar{g} \otimes \mathbb{C}((t))$; if $g$ is twisted, $g'$ consists of Laurent series, which have special properties with respect to an automorphism of $\bar{g}$, cf. [K2].

Recall that the Lie algebra $g$ has the Cartan decomposition: $g = n_+ \oplus h \oplus n_-$, where $n_+$ and $n_-$ are the nilpotent subalgebras, and $h$ is the Cartan subalgebra of $g$. For a non-twisted affine algebra $g$, $n_+ = (\bar{n}_+ \otimes 1) \oplus (\bar{g} \otimes \mathbb{C}[t])$.

We have: $h = \bar{h} \otimes 1 \oplus \mathbb{C}d \oplus \mathbb{C}K$, $K$ being a generator of the center. We have a linear basis $h_1, \ldots, h_l, d$ in $h$. The upper and lower nilpotent Lie subalgebras $n_+$ and $n_-$ are generated by elements $e_i, i = 0, \ldots, l$, and $f_i,i = 0, \ldots, l$, which satisfy the Serre relations [K2]:

$$(\text{ad } e_i)^{-a_{ij}+1} \cdot e_j = 0, \quad (\text{ad } f_i)^{-a_{ij}+1} \cdot f_j = 0.$$ 

The dual space $h^*$ to $h$ is linearly spanned by functionals $\alpha_0, \ldots, \alpha_l$, and $\Lambda_0$ [K2]. The value of $\alpha_j$ on the $i$th generator $h_i$ of the Cartan subalgebra of $g$ is equal to the element $a_{ij}$ of the Cartan matrix of $g$. Denote by $(\cdot,\cdot)$ the invariant scalar product on $h^*$, normalized as in [K2]. Let $a_i, i = 0, \ldots, l$, be the labels of the Dynkin diagram of $g$ [K2]. They satisfy

$$\sum_{0 \leq j \leq l} a_ja_{ij} = 0, \quad \forall i = 0, \ldots, l.$$ 

Let $U$ be the inverse limit of finite-dimensional linear spaces with coordinates $u^{(n)}_i, 1 \leq i \leq l; 0 \leq n \leq N$. Denote by $\pi_0$ the algebra of algebraic functions on $U$; this is a free polynomial algebra with generators $u^{(n)}_i, i = 1, \ldots, l, n \geq 0$. Define a
derivation $\partial$ on $\pi_0$ by putting $\partial u_i^{(n)} = u_i^{(n+1)}$, and extending it to arbitrary elements of $\pi_0$ by the Leibnitz rule. We can consider $\partial$ as a vector field on $U$:

$$
\partial = \sum_{1 \leq i \leq l, n \geq 0} u_i^{(n+1)} \frac{\partial}{\partial u_i^{(n)}}.
$$

We call $\pi_0$ the algebra of differential polynomials. Introduce a $\mathbb{Z}$-grading on $\pi_0$ by putting $\deg u_i^{(n)} = -n - 1$. With respect to this grading, the derivative $\partial$ is a homogeneous linear operator of degree $-1$.

We are going to introduce a hamiltonian structure on $U$ in the generalized sense of Gelfand, Dickey and Dorfman [GD1, GD2, GD3]. We now briefly recall their general definition of hamiltonian operator.

Let $\mathfrak{a}$ be a Lie algebra. A complex $\Omega = \bigoplus_{j \geq 0} \Omega^j$ with a differential $d : \Omega^j \to \Omega^{j+1}$, $j \geq 0$, is called an $\mathfrak{a}$-complex, if for any $a \in \mathfrak{a}$ there is a linear map $i_a : \Omega^j \to \Omega^{j-1}$, $j > 0$, such that $i_a i_b + i_b i_a = 0$ and $[i_a d + d i_a, i_b] = [a, b]$.

Given an $\mathfrak{a}$-complex, a linear operator $H : \Omega^1 \to \mathfrak{a}$ is called hamiltonian operator if (1) $H$ is antisymmetric: $i_H(\omega_1)(\omega_2) = -i_H(\omega_2)(\omega_1)$ for any $\omega_1, \omega_2 \in \Omega^1$; (2) $[H, H] = 0$, where $[\cdot, \cdot]$ is the Schoutens bracket, cf. [GD3].

The standard example of an $\mathfrak{a}$-complex is the de Rham complex of a manifold $M$ with the Lie algebra of vector fields on $M$ as $\mathfrak{a}$. Then the notion of hamiltonian operator is equivalent to the standard notion of hamiltonian structure on $M$.

Now fix a set $Z$ of vector fields on $M$ and define $\mathfrak{a}_Z$ to be the stabilizer of $Z$ in $\mathfrak{a}$. Denote by $\Omega_0$ the image of $L_z = i_z d + d i_z$, $z \in Z$, in $\Omega$, and put $\Omega_Z = \Omega / \Omega_0$. Then $\Omega_Z$ is an $\mathfrak{a}_Z$-complex.

Such a complex naturally arises in our context. We take as $M$ the space $U$. Then $\Omega^j$ consists of differential forms, which can be represented as finite sums

$$
\omega = \sum_{1 \leq i_1 \leq \cdots \leq i_j} \omega_{i_1, \ldots, i_j}^{n_1, \ldots, n_j} du_{i_1}^{(n_1)} \wedge \cdots \wedge du_{i_j}^{(n_j)},
$$

where $\omega_{i_1, \ldots, i_j}^{n_1, \ldots, n_j}$ is a polynomial in $u_i^{(n)}$'s. In particular, $\Omega^0 = \pi_0$. The Lie algebra $\mathfrak{a}$ consists of all vector fields

$$
\sum_{n \geq 0} \sum_{1 \leq i \leq l} X_{i, n} \frac{\partial}{\partial u_i^{(n)}},
$$

where $X_{i, n} \in \pi_0$ and the sum may be infinite. The map $i_a$ is defined as the usual contraction of a differential form with vector field $a$.

Now put $Z = \{\partial\}$. The Lie algebra $\mathfrak{a}_Z$ consists of all vector fields, which commute with $\partial$. We have:

$$
[\partial, \partial_i^{(n)}] = -\partial_i^{(n-1)}.
$$

Therefore such vector fields have the form

$$
\sum_{n \geq 0} \sum_{1 \leq i \leq l} (\partial^n X_i) \frac{\partial}{\partial u_i^{(n)}}
$$
for some $X_i \in \pi_0$, $i = 1, \ldots, l$.

The space $\Omega^0_0$ is the quotient of $\Omega^0 = \pi_0$ by the action of $\partial$. Let $\mathcal{F}_0$ be the quotient of $\Omega^0_0$ by the constants. We can introduce a $\mathbb{Z}$-grading on $\mathcal{F}_0$ by adding 1 to the grading induced from $\pi_0$. The space $\mathcal{F}_0$ can be interpreted as the space of local functionals of the form (2), since the integral of a total derivative or a constant is equal to 0. Denote by $\mathcal{F}$ the projection $\pi_0 \to \mathcal{F}_0$.

The space $\Omega^1_0$ is the quotient of $\Omega^1$ by the action of $\partial$. Therefore any element of this space can be uniquely represented in the form

$$\sum_{1 \leq i \leq l} Y_i du_i^{(0)}.$$ 

The differential $d : \Omega^0_0 \to \Omega^1_0$ is given by the formula

$$d\bar{P} = \sum_{1 \leq i \leq l} \frac{\delta P}{\delta u_i} du_i^{(0)} \mod \text{Im } \partial,$$

where $\bar{P}$ is the projection of $P \in \pi_0$ on $\Omega^0_0$ and

$$\frac{\delta P}{\delta u_i} = \sum_{n \geq 0} \sum_{1 \leq i \leq l} (-\partial)^n \frac{\partial P}{\partial u_i^{(n)}}$$

is the variational derivative with respect to $u_i$. This map is well-defined, because the variational derivative of a total derivative is 0.

Now consider the operator $H : \Omega^1_0 \to \mathfrak{a}_0$, which sends

$$\sum_{1 \leq i \leq l} Y_i du_i^{(0)} \mod \text{Im } \partial$$

to

$$\sum_{n \geq 0} \sum_{1 \leq i, j \leq l} (\alpha_i, \alpha_j) (\partial^{n+1}Y_i) \frac{\partial}{\partial u_j^{(n)}}.$$

This operator is hamiltonian, cf. [GD3]. Using this operator we can now assign to any element $P \in \pi_0$ a vector field $H(d\bar{P})$ on $U$. Denote the corresponding derivation of $\pi_0$ by $\xi_P$. Since constants are annihilated by $d$, $\xi_P$ depends only on the image of $P$ in $\mathcal{F}_0$, $fP$. Sometimes we will write $\xi_fP$ instead of $\xi_P$. We easily find that

$$(6) \quad \xi_P = \sum_{1 \leq i \leq l, n \geq 0} (\partial^{n+1} \cdot \delta_i P) \frac{\partial}{\partial u_i^{(n)}},$$

where we put

$$\delta_i P = \sum_{n \geq 0} \sum_{1 \leq j \leq l} (\alpha_i, \alpha_j) (-\partial)^n \frac{\partial P}{\partial u_j^{(n)}}, \quad i = 1, \ldots, l.$$
In particular,
\[
\partial = \sum_{1 \leq i \leq l, n \geq 0} u_i^{(n+1)} \frac{\partial}{\partial u_i^{(n)}} = \xi_P, \quad P = \frac{1}{2} \sum_{1 \leq i \leq l} u_i^{(0)} u_i^{(0)},
\]
where \(u_i^{(0)}, i = 1 \ldots, l,\) are vectors dual to \(u_i^{(0)}, i = 1 \ldots, l,\) with respect to the scalar product defined by \((\cdot, \cdot)\).

It is proved in [GD3] that if \(H\) is a hamiltonian operator, then one can define a structure of Lie algebra on \(\Omega^0\) by the formula
\[
\{F_1, F_2\} = H(dF_1) \cdot F_2.
\]
Moreover, the map \(H \circ d : \Omega^0 \rightarrow \mathfrak{a}\) becomes a homomorphism of Lie algebras. If \(\Omega\) is the de Rham complex on a manifold, this is of course the standard definition of Poisson bracket.

In our case we obtain a Lie bracket on \(\Omega^0\) and hence on \(\mathcal{F}_0\) by putting
\[
\{\int P, \int R\} = \int (\xi_P \cdot R).
\]

**Remark 1.** If we view elements of \(\mathcal{F}_0\) as functionals on the space of functions on the circle with values in the Cartan subalgebra of \(\tilde{\mathfrak{g}},\) \(u(t) = (u_1(t), \ldots, u_l(t))\), then we can interpret formula (8) as a Poisson bracket between such functionals, cf. [GD1, GD2, FF2].

We remark that conventions in this paper differ slightly from those in [FF2].

Now we extend the map \(\xi\) from \(\pi_0\) to a larger space, following Kuperschmidt and Wilson [KW, W1].

Denote by \(\Lambda\) the root lattice in \(\mathfrak{h}^*\), which is spanned by \(\alpha_0, \ldots, \alpha_l\).

Let us formally introduce variables \(\phi_i, i = 0, \ldots, l\). For any element \(\lambda = \sum_{0 \leq i \leq l} \lambda_i \alpha_i\) of \(\Lambda\), define the linear space \(\pi_\lambda = \pi_0 \otimes e^\lambda\), where \(\tilde{\lambda} = \sum_{0 \leq i \leq l} \lambda_i \phi_i\), equipped with an action of \(\partial\) by the formula
\[
\partial \cdot (P \otimes e^\tilde{\lambda}) = (\partial P) \otimes e^\lambda + \left(\sum_{0 \leq i \leq l} \lambda_i u_i^{(0)} P\right) \otimes e^\lambda,
\]
where we put
\[
u_0^{(n)} = -\frac{1}{a_0} \sum_{1 \leq i \leq l} a_i u_i^{(n)}.
\]

This formula means that we put \(\partial \phi_i = u_i^{(0)}\).

Introduce a \(\mathbb{Z}\)-grading on \(\pi_\lambda\) by putting \(\deg e^\lambda = (\rho^\vee, \lambda)\), where \(\rho^\vee \in \mathfrak{h}^*\) is such that \((\rho^\vee, \alpha_i) = 1, i = 0, \ldots, l\).

Let \(\mathcal{F}_\lambda\) be the quotient of \(\pi_\lambda\) by the subspace of total derivatives and \(f\) be the projection \(\pi_\lambda \rightarrow \mathcal{F}_\lambda\). We define a \(\mathbb{Z}\)-grading on \(\mathcal{F}_\lambda\) by adding 1 to the grading induced from \(\pi_\lambda\).
For any \( P \in \mathcal{F}_0 \) the derivation \( \xi_P : \pi_0 \to \pi_0 \) can be extended to a linear operator on \( \oplus_{\lambda \in \Lambda} \pi_\lambda \) by the formula

\[
\xi_P = \sum_{1 \leq i \leq l, n \geq 0} (\partial^{n+1} \cdot \delta_i P) \frac{\partial}{\partial u_i^{(n)}} + \sum_{1 \leq i \leq l} \delta_i P \frac{\partial}{\partial \phi_i},
\]

where \( \partial/\partial \phi_i \cdot (Se^\lambda) = \lambda_i Se^\lambda \). This defines a structure of \( \mathcal{F}_0 \)-module on \( \pi_\lambda \).

For any \( P \in \pi_0 \) the operator \( \xi_P \) commutes with the action of derivative. Hence we obtain the structure of an \( \mathcal{F}_0 \)-module on \( \pi_\lambda \). This defines a structure of \( \mathcal{F}_0 \)-module on \( \pi_\lambda \).

Similarly, any element \( R \in \pi_\lambda \) defines a linear operator \( \xi_R \), acting from \( \pi_0 \) to \( \pi_\lambda \) and commuting with \( \partial \):

\[
\xi_{Se^\lambda} = \sum_{1 \leq i \leq l, n \geq 0} \partial^n \left( \partial(\delta_i S \cdot e^\lambda) - S \frac{\partial e^\lambda}{\partial \phi_i} \right) \frac{\partial}{\partial u_i^{(n)}}.
\]

The operator \( \xi_R \) depends only on the image of \( R \) in \( \mathcal{F}_\lambda \). Therefore it gives rise to a map \( \{ \cdot, \cdot \} : \mathcal{F}_\lambda \times \mathcal{F}_0 \to \mathcal{F}_\lambda \). We have for any \( P \in \mathcal{F}_0, R \in \mathcal{F}_\lambda \):

\[
\int \xi_R \cdot P = -\int \xi_P \cdot R.
\]

Therefore our bracket \( \{ \cdot, \cdot \} \) is antisymmetric.

We also have for any \( P \in \mathcal{F}_0, R \in \oplus_{\lambda \in \Lambda} \mathcal{F}_\lambda \):

\[
\xi_{\{P,R\}} = [\xi_P, \xi_R].
\]

This can be proved in the same way as in the case when \( R \in \mathcal{F}_0 \).

### 3. Toda Integrals and Nilpotent Action

Denote by \( \tilde{Q}_i, i = 0, \ldots, l \), the linear operators \( -\xi_{\int e^{-\phi_i}} : \pi_0 \to \pi_{-\alpha_i} \). We have:

\[
\tilde{Q}_i = -\sum_{n \geq 0} (\partial^n e^{-\phi_i}) \partial_i^{(n)},
\]

where

\[
\partial_i^{(n)} = \sum_{1 \leq j \leq l} (\alpha_i, \alpha_j) \frac{\partial}{\partial u_j^{(n)}}.
\]

Clearly, \( \partial^n e^{-\phi_i} = B_i^{(n)} e^{-\phi_i} \), where \( B_i^{(n)} \)'s are certain polynomials in \( u_i^{(m)} \)'s. These polynomials, which are closely related to Schur’s polynomials, are called Faà di Bruno polynomials. More precisely, Faà di Bruno polynomials are \( B_i^{(n)} \) with \( u_i^{(m)} \)'s replaced...
These polynomials appear ubiquitously in the theory of solitons, cf., e.g., [D]. They satisfy a recurrence relation:

\[
B_i^{(n)} = -u_i^{(0)} B_i^{(n-1)} + \partial B_i^{(n-1)},
\]

with the initial condition \( B_i^{(0)} = 1 \).

We can now define derivations \( Q_0, \ldots, Q_l \) of \( \pi_0 \) by the formula

\[
Q_i = -\sum_{n \geq 0} e^{\phi_i} (\partial^n e^{-\phi_i}) \partial_i^{(n)} = -\sum_{n \geq 0} B_i^{(n)} \partial_i^{(n)}.
\]

The following statement was proved in [FF2], Proposition 2.2.8. It also follows from Theorem 2 below.

**Proposition 1.** The operators \( Q_i \) satisfy the Serre relations of the Lie algebra \( g \):

\[
(ad Q_i)^{-a_{ij}+1} \cdot Q_j = 0,
\]

where \( \|a_{ij}\| \) is the Cartan matrix of \( g \).

According to Proposition 1, we obtain a structure of \( n_+ \)-module on \( \pi_0 \) by assigning to each generator \( e_i \) of the nilpotent Lie subalgebra \( n_+ \) of \( g \) the operator \( Q_i : \pi_0 \to \pi_0, i = 0, \ldots, l \).

Each operator \( \bar{Q}_i \) commutes with the action of \( \partial \) and induces a linear operator

\[
\{H, \cdot\} : F_0 \to F_{-\alpha_i}.
\]

Thus, up to a sign, the sum of the operators \( \bar{Q}_i \) coincides with the operator of the bracket

\[
\{H, \cdot\} : F_0 \to \bigoplus_{0 \leq i \leq l} F_{-\alpha_i}
\]

with the Hamiltonian (3) of the Toda field theory associated to \( g \). The elements of the space of local functionals, \( F_0 \), which lie in the kernel of this operator, are called local integrals of motion of the corresponding Toda theory. They are conserved with respect to the Toda equation. For brevity, we will call them Toda integrals.

Thus, by definition, the space of Toda integrals is the intersection of kernels of the operators \( \bar{Q}_i : F_0 \to F_{-\alpha_i}, i = 0, \ldots, l \).

Denote by \( I \) the set of exponents of \( g \) modulo the Coxeter number. In this paper we will give a new proof of the following result.

**Theorem 1.** The space of Toda integrals is linearly spanned by elements \( H_m, m \in I \), where \( \deg H_m = -m \).

Theorem 1 will be proved in Sect. 6. We will show that the space of Toda integrals is isomorphic to the first cohomology of a certain complex \( F^\ast(g) \) constructed from the dual of the Bernstein-Gelfand-Gelfand (BGG) resolution of \( g \). In order to do that we will realize modules contragradient to Verma modules over \( g \) in the space of functions on the nilpotent group \( N_+ \), cf. Sect. 4, and establish an isomorphism between the space \( \pi_0 \) and the space of functions on \( N_+/A_+ \), cf. Theorem 2.
Remark 2. The map $\xi : \pi_0 \to \text{End} \pi_0$ can be quantized in the following sense. Put $\pi^\hbar_\lambda = \pi_\lambda \otimes \mathbb{C}[[\hbar]]$. There exists a map $\xi^\hbar : \pi^\hbar_0 \to \text{End} \pi^\hbar_0$ such that (1) $\xi^\hbar$ factors through $\mathcal{F}^\hbar_0 = \mathcal{F}_0 \otimes \mathbb{C}[[\hbar]] = \pi^\hbar_0 / \partial \cdot \pi^\hbar_0$, (2) the bracket $\{ \cdot, \cdot \}_h : \mathcal{F}^\hbar_0 \times \mathcal{F}^\hbar_0 \to \mathcal{F}^\hbar_0$, defined by the formula
\[
\{ \int P, \int R \} = \int \xi_P \cdot R,
\]
where $\int$ denotes the projection $\pi^\hbar_0 \to \mathcal{F}^\hbar_0$, is a Lie bracket; (3) $\xi^\hbar = h \xi^{(1)} + h^2 (\ldots)$, and the map $\pi_0 \to \text{End} \pi_0$ induced by $\xi^{(1)}$ coincides with $\xi$.

Such a map $\xi^\hbar$ can be defined using the vertex operator algebra structure on $\pi_0$. This is explained in [FF2], Sect. 4 (where $\hbar$ is denoted by $\beta^2$). Here we will give an explicit formula for $\xi^\hbar : \pi_0 \to \text{End} \pi_0$.

Introduce a Heisenberg Lie algebra with generators $b_i(n), i = 1, \ldots, l; n \in \mathbb{Z}$, and 1, and relations
\[
[b_i(n), b_j(m)] = h \cdot n \cdot (\alpha_i, \alpha_j) \delta_{n,-m} 1, \quad [b_i(n), 1] = 0.
\]
We can make $\pi_0$ into a module over this Lie algebra by defining the following action of the generators:
\[
b_i(n) = \begin{cases} 
\frac{u_i^{(-n-1)}}{(-n-1)!}, & n < 0, \\
\sum_{1 \leq j \leq i} h \cdot n! \cdot (\alpha_i, \alpha_j) \frac{\partial}{\partial u_j^{(n-1)}} , & n > 0,
\end{cases}
\]
and
\[
b_i(0) = 0, \quad 1 = \text{Id}.
\]
Denote
\[
b_i(z) = \sum_{n \in \mathbb{Z}} b_i(n) z^{-n-1}.
\]
For a monomial $u_{i_1}^{(n_1)} \ldots u_{i_m}^{(n_m)} \in \pi_0$ define the following formal power series
\[
: \partial_z^{-n_1} b(z) \ldots \partial_z^{-n_m} b(z) :,
\]
where the dots stand for the normal ordering [FF2]. The Fourier coefficients of this series are linear operators acting on $\pi^\hbar_0$. By linearity we obtain a map $Y(\cdot, z) : \pi^\hbar_0 \to \text{End} \pi^\hbar_0[[z, z^{-1}]]$. This map defines a structure of vertex operator algebra on $\pi_0$, which depends on $h$ [FF2]. For $P \in \pi^\hbar_0$ denote by $\xi^\hbar_P$ the linear endomorphism of $\pi^\hbar_0$ given by the residue, i.e. the $(-1)$st Fourier component, of $Y(P, z)$. This gives us a map $\xi^\hbar : \pi^\hbar_0 \to \text{End} \pi^\hbar_0$, which satisfies the conditions above.

Thus, the Gelfand-Dickey-Dorfman structure on $\pi_0$ can be viewed as a classical limit of the structure of vertex operator algebra on $\pi_0$. Note that one can also define classical limits of other Fourier components of $Y(P, z)$ for any $P \in \pi_0$. Altogether they satisfy certain properties, which are corollaries of axioms of vertex operator algebra.
In [FF2] we also defined quantum deformations of the maps \( \xi : \pi_0 \to \text{End} \pi_\lambda \) and \( \pi_\lambda \to \text{Hom}(\pi_0, \pi_\lambda) \). This enabled us to quantize the operators \( \tilde{Q}_i : \pi_0 \to \pi_{-\alpha_i} \) and \( \tilde{Q}_i : \mathcal{F}_0 \to \mathcal{F}_{-\alpha_i} \), and define the space of quantum Toda integrals as the intersection of kernels of the quantum operators \( \tilde{Q}_0, \ldots, \tilde{Q}_l \). For these quantum operators one has an analogue of Proposition 1: in a certain sense they satisfy the Serre relations \( U \) of the quantized universal enveloping algebra \( (at \ least \ in \ the \ case \ when \ all \ exponents \ of \ g \ are \ odd \ and \ the \ Coxeter \ number \ is \ even). \)

4. Geometric construction

Let \( G \) be the Lie group of \( g \). This group is the universal central extension of the loop group of \( \tilde{G} \) or its subgroup, if \( g \) is twisted. The group \( G \) is also defined over formal Laurent series, and it is also equipped with the topology of inverse limit. We consider the induced topology on Lie subgroups of \( G \).

Let \( B_+ \) and \( B_- \) be the Borel subgroups of \( G \). Their projections on the loop group of \( \tilde{G} \) consist of its \( \mathbb{C}[[t]] \)-points (respectively, \( \mathbb{C}[t^{-1}] \)-points), whose image in the constant Lie subgroup \( \tilde{G} \) of \( G \) belongs to the finite-dimensional Borel subgroup \( \tilde{B}_+ \) (respectively, \( \tilde{B}_- \)), or its invariant part with respect to an automorphism, if \( g \) is twisted. Let \( H = B_+ \cap B_- \) be the Cartan subgroup of \( G \).

Let \( N_+ \) be radical of \( B_+ \); this is the Lie group of \( n_+ \). The group \( N_+ \) is a prounipotent proalgebraic group, which is the inverse limit of finite-dimensional algebraic groups \( N_+^{(n)} \setminus N_+, n > 0 \), isomorphic to finite-dimensional affine spaces. Here \( N_+^{(n)} \) denotes the subgroup of \( N_+ \), which consists of elements, which are equal to 1 modulo \( t^n \). The exponential map \( \exp : n_+ \to N_+ \) is an isomorphism. By the space of functions on \( N_+ \) we will always mean the space of algebraic functions, which is the inductive limit of the spaces of algebraic functions on \( N_+^{(n)} \setminus N_+, n > 0 \).

Consider \( p = \sum_{0 \leq i \leq l} a_i e_i \), a principal element of \( n_+ \), where \( e_i, i = 0, \ldots, l \), are the generators of \( n_+ \), and \( a_i, i = 0, \ldots, l \), are the labels of the Dynkin diagram of \( g \). Denote by \( a_+ \) the principal commutative Lie subalgebra of \( a_i \), which is the centralizer of \( p \) in \( a_i \).

The Lie algebra \( g \) is graded by weights of the Cartan subalgebra \( h \). We can also introduce the principal \( \mathbb{Z} \)-grading on \( g \) by putting \( \deg h_i = \deg d = \deg K = 0, \deg e_i = -\deg f_i = 1, i = 0, \ldots, l \). It is known that with respect to this grading the Lie algebra \( n_+ \) is linearly spanned by elements of degrees equal to the exponents of \( g \) modulo the Coxeter number. It is also known that the Lie algebra \( n_+ \) splits into the direct sum \( n_+ = \text{Ker} p \oplus \text{Im} p \), where \( \text{Ker} p = a_+ \) and \( \text{Im} p = \oplus_{j \geq 0} n_+^j \) is the direct sum of homogeneous components \( n_+^j \) of degree \( j \geq 0 \), each having the same dimension \( l \). For a proof of these facts, cf. [K1], Proposition 3.8 (b).
The Lie subalgebra \( \mathfrak{a}_+ \) is the positive half of the principal Heisenberg subalgebra \( \hat{\mathfrak{a}} \) of \( \mathfrak{g} \). This Heisenberg algebra is the central extension by \( \mathbb{C} \hat{K} \) of the commutative subalgebra \( \mathfrak{a} \) of the loop algebra \( \mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]/\mathbb{C} \hat{K} \), which is the centralizer of \( p \) in \( \mathfrak{g}' \).

We have \( \mathfrak{a} = \mathfrak{a}_+ \oplus \mathfrak{a}_- \), where \( \mathfrak{a}_- \) is a commutative Lie subalgebra of \( \mathfrak{n}_- \). The Lie algebra \( \mathfrak{a}_- \) is linearly spanned by generators of degrees equal to minus the exponents of \( \mathfrak{g} \) modulo the Coxeter number. Let us choose for each \( m \in I \) a linear generator \( p_m \) of \( \mathfrak{a}_- \) degree \( -m \). We choose

\[
p_1 = \sum_{0 \leq i \leq l} \frac{(\alpha_i, \alpha_i)}{2} f_i.
\]

The group \( N_+ \) is isomorphic to the big cell \( X \) of the flag manifold \( F = B_- \backslash G \), which is the orbit of the image of \( 1 \in G \) under the action of \( N_+ \). Flag manifolds of \( G \) have been studied, e.g., in [KL, PK, KP, PS, Ka].

The group \( G \) acts on the flag manifold from the right: \((g, f) \rightarrow fg^{-1}, g \in G, f \in F\). This induces an infinitesimal action of \( \mathfrak{g} \) on the big cell \( X \simeq N_+ \) by vector fields. The Lie algebra of vector fields on the circle \( \text{Vect} = \mathbb{C}((t^k))t\partial_t \), where \( k \) is the order of the associated automorphism of \( \hat{\mathfrak{g}} \), infinitesimally acts on the group \( G \), cf., e.g., [PS]. Its Lie subalgebra \( \text{Vect}_- = \mathbb{C}[t^{-k}]t\partial_t \) preserves the group \( B_- \) and hence it maps to vector fields on \( F \) and on \( X \simeq N_+ \). Thus, the Lie algebra \( \mathfrak{g} \times \text{Vect}_- \) maps to the Lie algebra \( \mathcal{V} \) of vector fields on \( N_+ \). Note that the central element \( K \) maps to 0, and the image of \( d \) coincides with the image of \( t\partial_t \in \text{Vect}_- \). Therefore this map factors through the Lie algebra \( \tilde{\mathfrak{g}} = \mathfrak{g}' \times \text{Vect}_- \). Denote the vector field corresponding to an element \( \alpha \in \tilde{\mathfrak{g}} \) by \( \alpha^R \).

The Lie group \( N_+ \) acts on itself from the left: \((n_1, n_2) \rightarrow n_1n_2 \) and from the right: \((n_1, n_2) \rightarrow n_2n_1^{-1} \). These actions induce two infinitesimal actions of the Lie algebra \( \mathfrak{n}_+ \) on \( N_+ \), and hence two embeddings of \( \mathfrak{n}_+ \) into the Lie algebra \( \mathcal{V} \). Denote by \( \mathfrak{n}_+^L \) and \( \mathfrak{n}_+^R \) the images of these embeddings. For \( \beta \in \mathfrak{n}_+ \) we denote by \( \beta^L \) and \( \beta^R \) the images of \( \beta \) in \( \mathfrak{n}_+^L \) and \( \mathfrak{n}_+^R \), respectively. The Lie algebra \( \mathfrak{n}_+^R \) coincides with the image of \( \mathfrak{n}_+ \subset \tilde{\mathfrak{g}} \) in \( \mathcal{V} \). Note that \( \mathfrak{n}_+^L \) and \( \mathfrak{n}_+^R \) commute with each other.

We will now describe a geometric construction of modules contragredient to the Verma modules over \( \mathfrak{g} \) and homomorphisms between them. This construction is an affine analogue of Kostant’s construction [Ko2] (cf. also [Zh, KV, BMP]) in the case of simple Lie algebras, though our argument is somewhat different from his. Our treatment can be easily generalized to arbitrary Kac-Moody algebras.

For \( \lambda \in \mathfrak{h}^* \), denote by \( \mathcal{C}_\lambda \) the one-dimensional representation of \( \mathfrak{b}_+ \), on which \( \mathfrak{h} \subset \mathfrak{b}_+ \) acts according to its character \( \lambda \), and \( \mathfrak{n}_+ \subset \mathfrak{b}_+ \) acts trivially. Let \( M_\lambda \) be the Verma module over \( \mathfrak{g} \) of highest weight \( \lambda \):

\[
M_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{b}_+)} \mathcal{C}_\lambda.
\]

Denote by \( \langle \cdot, \cdot \rangle \) the pairing \( M_\lambda^* \times M_\lambda \rightarrow \mathbb{C} \). Let \( \omega \) be the Cartan anti-involution on \( \mathfrak{g} \), which maps generators \( e_0, \ldots, e_l \) to \( f_0, \ldots, f_l \) and vice versa and preserves \( \mathfrak{h} \).
[K2]. It extends to an anti-involution of $U(\mathfrak{g})$. Let $M_\lambda^*$ be the module contragradient to $M_\lambda$. As a linear space, $M_\lambda^*$ is the restricted dual of $M_\lambda$. The action of $x \in \mathfrak{g}$ on $y \in M_\lambda^*$ is defined as follows:

$$\langle x \cdot y, z \rangle = \langle y, \omega(x) \cdot z \rangle, \quad z \in M_\lambda.$$

Suppose first that the highest weight $\lambda$ is integral. Then the module $\mathbb{C}_\lambda$ over $\mathfrak{b}_-$ can be integrated to a module over $B_-$. To this module we associate in the standard way an equivariant line bundle on $F$. The module $M_\lambda^*$ can be realized as the space of sections of the restriction of this line bundle to the big cell $X$. But this line bundle can be trivialized over $X$. Therefore the module $M_\lambda^*$ is isomorphic to the space of algebraic functions on $X$ with respect to the twisted action of $\mathfrak{g}$ by first order differential operators. For an element $\beta$ of $\mathfrak{g}$ this differential operator is equal to $\beta^R + F_\lambda(\beta)$, where $F_\lambda(\beta)$ is an algebraic function on $X$. Note that if $\beta$ is homogeneous, $F_\lambda(\beta)$ is also homogeneous of the same weight.

We can interpret the functions $F_\lambda(\beta)$ as elements of the group $H^1(\mathfrak{g}, \mathbb{C}[X])$, where $\mathbb{C}[X]$ is the space of algebraic functions on $X$ considered as a $\mathfrak{g}$-module with respect to the right infinitesimal action. Indeed, in the standard complex of Lie algebra cohomology an element of $H^1(\mathfrak{g}, \mathbb{C}[X])$ is realized as a linear functional $f$ on $\mathfrak{g}$ with values in $\mathbb{C}[X]$. Such an element defines a deformation of the $\mathfrak{g}$-module $\mathbb{C}[X]$: the deformed action of $\beta \in \mathfrak{g}$ is obtained by adding to the old action the operator of multiplication by $f(\beta)$.

Note that as a $\mathfrak{g}$-module, $\mathbb{C}[X] = M_0^*$ is coinduced from the trivial representation of $\mathfrak{b}_-$. By Shapiro's lemma (cf., e.g., [Fu], Sect. 1.5.4, [G], Sect. II.7), $H^1(\mathfrak{g}, \mathbb{C}[X]) \cong H^1(\mathfrak{b}_-^*, \mathbb{C}) \cong (\mathfrak{b}_-/[\mathfrak{b}_-, \mathfrak{b}_-])^* = \mathfrak{h}^*$. We see that all elements of $H^1(\mathfrak{g}, \mathbb{C}[X])$ have weight 0. On the other hand, functions on $X$ can only have negative or 0 weights and the only functions, which have weight 0 are constants, which are invariant with respect to the action of $\mathfrak{g}$. Therefore the coboundary of any element of the 0th group of the complex, $\mathbb{C}[X]$, has a non-zero weight. Hence any cohomology class from $H^1(\mathfrak{g}, \mathbb{C}[X])$ canonically defines a one-cocycle $f$, that is a map $\mathfrak{g} \to \mathbb{C}[X]$. Thus, having identified the space of deformations with $\mathfrak{h}^*$, we can assign to each $\lambda \in \mathfrak{h}^*$ and each $\beta \in \mathfrak{g}$, a function on $X$ — this is our $F_\lambda(\beta)$.

By linearity, if $\lambda = \sum_{i=0}^l \lambda_i \alpha_i$, then $F_\lambda(\beta) = \sum_{i=0}^l \lambda_i F_i(\beta)$, where we put $F_i(\beta) = F_{\alpha_i}(\beta)$. Thus, this construction works for arbitrary, not only integral, values of $\lambda$. In particular, we have: $F_\lambda(\beta) = 0$ for any $\lambda$, if $\beta \in \mathfrak{n}_+$, and $F_\lambda(\beta) = \lambda(\beta)$, if $\beta \in \mathfrak{h}$.

Thus, for any weight $\lambda \in \mathfrak{h}^*$, we realize the module $M_\lambda^*$ as the space of functions on $X$ equipped with the twisted action of $\mathfrak{g}$ by first order differential operators.

Vector $y$ from the Verma module $M_\lambda$ is called singular vector of weight $\mu \in \mathfrak{h}^*$, if $\mathfrak{n}_+ \cdot y = 0$ and $x \cdot y = \mu(x) y$ for any $x \in \mathfrak{h}$. We have $M_\lambda \simeq U(\mathfrak{n}_-) \cdot v_\lambda$, where $v_\lambda$, which is called the highest weight vector, is a generator of the space $\mathbb{C}_\lambda$. This vector is a singular vector of weight $\lambda$. Any singular vector of $M_\lambda$ of weight $\mu$ can be uniquely represented as $P \cdot v_\lambda$ for some element $P \in U(\mathfrak{n}_-)$. This singular vector
canonically defines a homomorphism of $\mathfrak{g}$-modules $i_P : M_\mu \to M_\lambda$, which sends $u \cdot v_\mu$ to $(uP) \cdot v_\lambda$ for any $u \in U(n_-)$. Denote by $i_P^*$ the dual homomorphism $M_\lambda^* \to M_\mu^*$.

There is an isomorphism $U(n_-) \to U(n_+)$, which maps the generators $f_0, \ldots, f_l$ to $-e_0, \ldots, -e_l$. Denote by $\bar{P}$ the image of $P \in U(n_-)$ under this isomorphism.

The homomorphism $n_+ \to \mathcal{V}$, mapping $\alpha \in n_+$ to $\alpha^L$, can be extended in a unique way to a homomorphism from $U(n_+)$ to the algebra of differential operators on $X$. Denote the image of $u \in U(n_+)$ under this homomorphism by $u^L$.

It turns out that using the left infinitesimal action of $n_+$ on $X$, one can realize the dual homomorphisms $i_P^* : M_\lambda^* \to M_\mu^*$ via differential operators on $X$.

**Proposition 2.** If $P \cdot v_\lambda$ is a singular vector in $M_\lambda$ of weight $\mu$, then the homomorphism $i_P^* : M_\lambda^* \to M_\mu^*$ is given by the differential operator $\bar{P}^L$.

**Proof.** As an $n_-$-module, the module $M_\lambda$ is isomorphic to $U(n_-)$. Hence in addition to the left action of $\mathfrak{g}$, we have a right action of $n_-$ on $M_\lambda$:

\[
(n, u \cdot v_\lambda) \mapsto -(un) \cdot v_\lambda, \quad n \in n_-, \quad u \in U(n_-).
\]

If we realize $M_\lambda^*$ as the space of functions on $X$, this right action corresponds to the action of $n_+^L$ on $\mathbb{C}[X]$.

More precisely,

\[
\langle \beta^L \cdot x, u \cdot v_\lambda \rangle = -\langle x, (u\omega(\beta)) \cdot v_\lambda \rangle, \quad \forall \beta \in n_+, \quad u \in U(n_-).
\]

Therefore

\[
\langle \bar{P}^L \cdot x, u \cdot v_\lambda \rangle = \langle x, (uP) \cdot v_\lambda \rangle.
\]

On the other hand, by definition,

\[
\langle i_P^* \cdot x, u \cdot v_\lambda \rangle = \langle x, i_P(\mu) \cdot (u \cdot v_\lambda) \rangle = \langle x, (uP) \cdot v_\lambda \rangle,
\]

and Proposition follows. \(\square\)

We can now derive the following important result.

**Proposition 3.**

(a) If $\alpha \in \mathcal{V}$ is such that for any $\beta \in n_+^L$ (respectively, $\beta \in n_+^R$) $[\alpha, \beta] = 0$, then $\alpha \in n_+^R$ (respectively, $\alpha \in n_+^L$).

(b) For any $\beta \in \mathfrak{g}$ we have:

\[
[e_i^L, \beta^R] = -F_i(\beta)e_i^L, \quad i = 0, \ldots, l.
\]

**Proof.** Part (a) is clear, because such a vector field is $n_+^L$- or $n_+^R$-invariant, and hence is uniquely defined by its value at the origin of the group $N_+$.

To prove part (b), consider the particular case of Proposition 2, when $\lambda = 0$ and $\mu = -\alpha_i$. It is known that vector $f_i \cdot v_0$ is a singular vector of $M_0$ of weight $-\alpha_i$. By Proposition 2, $-e_i^L$ is a $\mathfrak{g}$-homomorphism from $M_0^*$ to $M_{\alpha_i}^*$. It is therefore a $\mathfrak{g}$-homomorphism, since the action of $\mathcal{V}ect_-$ on any $M_\lambda^*$ (and any module from the category $\mathcal{O}$ of $\mathfrak{g}$-modules) can be expressed in terms of the action of $\mathfrak{g}$ via the Sugawara construction.
Thus the operator \( e_i^L \) intertwines the actions of \( \tilde{\mathfrak{g}} \) on \( M^*_0 \) and \( M^*_{-\alpha_i} \). But \( \beta \in \tilde{\mathfrak{g}} \) acts on \( M^*_0 \) as \( \beta^R \), and on \( M^*_{-\alpha_i} \) as \( \beta^R - F_i(\beta) \). This gives us formula (16).

It is clear that the Cartan subalgebra \( \mathfrak{h} \) of \( \mathfrak{g} \) acts diagonally on \( \mathcal{V} \), and thus defines a grading of \( \mathcal{V} \) by weights of \( \mathfrak{g} \). In particular, vector fields \( e_i^L \) and \( e_i^R \) both have weight \( \alpha_i \). Therefore we have \( [e_i^L, h_j^R] = -a_{ji}e_i^L \), and so \( F_i(h_j) = a_{ji} \).

Now consider \( F_i(f_j) \). Let us apply \( e_k^R \) to the left and right hand sides of formula (16) for \( \beta = f_j \). Since \( [e_k^R, e_i^L] = 0 \) by Proposition 3 (a), we obtain:

\[
[e_k^R, [e_i^L, f_j^R]] = [[e_k^R, e_i^L], f_j^R] + [e_i^L, [e_k^R, f_j^R]] = \delta_{k,j}[e_i^L, h_j^R] = -a_{ji}\delta_{k,j}e_i^L
\]

Therefore \( F_i(f_j) \) satisfies:

\[
(e_k^R \cdot F_i(f_j)) = a_{ji}\delta_{k,j}.
\]

There are unique functions \( x_i, i = 0, \ldots, l \), on \( N_+ \), which have the property

\[
e_k^R \cdot x_j = -e_k^L \cdot x_j = \delta_{k,j}.
\]

We see that \( F_i(f_j) = a_{ji}x_j \).

5. Isomorphism between \( \pi_0 \) and \( \mathbb{C}[N_+/A_+] \)

Now denote by \( A_+ \) the image of \( a_+ \) in \( N_+ \) under the exponential map. Let \( Y \) be the homogeneous space \( N_+/A_+ \). On \( Y \) we have the left infinitesimal action of \( n_+ \). Since the right infinitesimal action of the Lie algebra \( a_- \) commutes with the right action of \( A_+ \) on \( N_+ \), we obtain a homomorphism from \( a_- \) to the Lie algebra of vector fields on the space \( Y \).

There exists \( \Delta \in \mathfrak{h} \), such that \( [\Delta, e_i] = 1 \) for \( i = 0, \ldots, l \). The action of \( \Delta \) on \( \mathfrak{g} \) coincides with the action of the principal \( \mathbb{Z} \)-grading. Therefore it preserves the Lie algebra \( a_+ \), and hence \( \Delta^R \) defines a vector field on \( Y \), for which we will use the same notation. This gives us a \( \mathbb{Z} \)-grading on the space of functions on \( Y \).

Let \( y \) be an element of \( a_- \) of degree \( m \). Then the functions \( F_i(y) \) are \( a_+^R \)-invariant algebraic functions on \( N_+ \) of degree \( m \). Indeed, the action of \( y \) on \( M_{a_+^*} \) is given by \( y^R + F_i(y) \) and the action of \( x \in a_+^L \subset n_+^R \) is given by \( x^R \). These actions must commute (note that the central element maps to 0 in \( \mathcal{V} \)). But since \( [y^R, x^R] = 0 \), we obtain \( x^R \cdot F_i(y) = 0 \). In particular, \( F_i(p_1) \) are \( a_+^R \)-invariant algebraic functions on \( N_+ \) of degree \( -1 \).

Denote by \( u_i \) the algebraic function on \( Y = N_+/A_+ \), corresponding to \( F_i(p_1), i = 1, \ldots, l \). Formula (15) gives:

\[
u_i = \sum_{0 \leq j \leq l} (\alpha_i, \alpha_j)x_j,
\]

because \( a_{ji} = 2(\alpha_i, \alpha_j)/(\alpha_j, \alpha_j) \). Let \( u_i^{(m)} \), \( i = 1, \ldots, l, m \geq 0 \), be the algebraic function \( p_1^m \cdot u_i^{(0)} \). Denote, as in the previous section, by \( U^{(m)} \) the linear space with
coordinates $u_i^{(j)}$, $j = 1, \ldots, l$, $i = 0, \ldots, m$, and by $U$ the inverse limit of the spaces $U^{(m)}$.

**Proposition 4.** The homogeneous space $Y = N_+/A_+$ is isomorphic to the space $U$.

**Proof.** Consider the values of the differentials $du_i^{(n)}$ of the functions $u_i^{(n)}$ at the image $\tilde{1} \in Y$ of $1 \in N_+$. They are vectors in the cotangent space to $Y$ at $\tilde{1}$, which is naturally isomorphic to $(n_+/a_+)^*$. The covectors $du_i^{(0)}$, $i = 1, \ldots, l$, form a linear basis in $(n_+^1)^*$. The element $p_1$ of $a_-$, which sends $du_i^{(m)}$ to $du_i^{(m+1)}$, maps $(n_+^m)^*$ isomorphically to $(n_+^{m+1})^*$. Hence the vectors $du_i^{(m)}$ are linearly independent. Therefore the functions $u_i^{(n)}$, $i = 1, \ldots, l, n \geq 0$, are algebraically independent and hence $\mathbb{C}[U]$ embeds into $\mathbb{C}[Y]$. By definition, the function $u_i^{(n)}$ has degree $-n$ with respect to the $\mathbb{Z}$-grading defined by the vector field $\Delta$. Hence the character of $\mathbb{C}[U]$ is given by

$$\text{ch } \mathbb{C}[U] = \text{tr } q^{\Delta} = \prod_{n>0} (1 - q^n)^{-l}.$$

On the other hand, the image of the adjoint action of $a_+$ coincides with $\text{Im } p \simeq n_+/a_+$. Recall that the exponential map $\exp : n_+ \rightarrow N_+$ is an isomorphism. From Campbell-Baker-Hausdorff formula we derive that any element of $N_+$ can be uniquely presented as the product of an element of $A_+$ and an element $\exp x$ of $N_+$, where $x \in \text{Im } p \subset n_+$. Therefore $Y$ is isomorphic to $n_+/a_+$. Hence with respect to the $\mathbb{Z}$-grading defined by the vector field $\Delta$, the space $\mathbb{C}[Y]$ of algebraic functions on $Y$ is a free polynomial algebra with $l$ generators of each negative degree. Hence its character coincides with the character of $\mathbb{C}[U]$. But since $\mathbb{C}[U]$ is embedded into $\mathbb{C}[Y]$, $\mathbb{C}[U] \simeq \mathbb{C}[Y]$, and Proposition follows.

We can now identify the algebra $\mathbb{C}[U]$ with the algebra of differential polynomials $\pi_0$ from the previous section; the operator $p_1$ gets identified with $\partial$.

**Theorem 2.** $\mathbb{C}[Y]$ and $\pi_0$ are isomorphic as $n_+$-modules.

**Proof.** We have

$$e_i = \sum_{1 \leq j \leq l, n \geq 0} C_{i,j}^{(m)} \frac{\partial}{\partial u_i^{(m)}} ,$$

where $C_{i,j}^{(m)}$ are certain polynomials in $u_i^{(n)}$. By definition of $u_i^{(0)}$ and formula (16), we have

$$[e_i, p_1] = -u_i^{(0)} e_i .$$

Since

$$p_1 = \sum_{1 \leq i \leq l, n \geq 0} u_i^{(m+1)} \frac{\partial}{\partial u_i^{(m)}},$$

we have

$$[e_i, p_1] = -u_i^{(0)} e_i .$$
from formula (19) we find the recurrence relations for the coefficient of $\partial/\partial u_j^{(m-1)}$ in the vector field $e_i$:

$$C_{i,j}^{(m)} = -u_i^{(0)} C_{i,j}^{(m-1)} + p_1 \cdot C_{i,j}^{(m-1)}.$$  

These recurrence relations coincide with the recurrence relations for the coefficients of the vector fields $Q_i$ on $U$. We also have, according to formulas (17) and (18), $e_i \cdot u_j^{(0)} = -(\alpha_i, \alpha_j)$, therefore $C_{i,j}^{(0)} = -(\alpha_i, \alpha_j)$. Formulas (14) and (13) then show that the vector field $e_i$ coincides with $Q_i$ and Proposition follows.

**Remark 3.** This result was proved by other methods in [FF2], Propositions 3.1.10 and 3.2.5. Note that Proposition 1 follows from it.

**Remark 4.** Consider the homogeneous space $\tilde{Y} = B_+/A_+$. Since $B_+ = N_+ \times H$, we have: $\tilde{Y} = Y \times H$. The space of regular functions on $H$ can be identified with $\bigoplus_{\Lambda \in \Lambda} C e^\Lambda$ with multiplication $e^\lambda, e^\mu = e^{\lambda+\mu}$. Here $\Lambda$ is the root lattice in $h^*$. Therefore, by Proposition 4, the algebra $C[\tilde{Y}]$ of regular functions on $\tilde{Y}$ is isomorphic to $\bigoplus_{\lambda \in \Lambda} \pi_\lambda$.

Define an action of the generator $h_i$ of the Cartan subalgebra $\mathfrak{h}$ of $g$ on $\bigoplus_{\lambda \in \Lambda} \pi_\lambda$ as $-\partial/\partial \phi_i$ and the action of the generator $e_i$ of $\mathfrak{n}_+ \in g$ on $\bigoplus_{\lambda \in \Lambda} \pi_\lambda$ as $Q_i$ given by formula (12). One easily checks that this defines an action of the Borel subalgebra $\mathfrak{b}_+ = \mathfrak{h} \oplus \mathfrak{n}_+$ of $g$ on $\bigoplus_{\lambda \in \Lambda} \pi_\lambda$. From Theorem 2 we derive the following result.

**Corollary 1.** $C[\tilde{Y}]$ and $\bigoplus_{\lambda \in \Lambda} \pi_\lambda$ are isomorphic as $\mathfrak{b}_+$-modules.

It is known that the group $B_+$ is equipped with a Lie-Poisson structure [Dr1, Dr2]. On the other hand, via the isomorphism of Corollary 1, $B_+/A_+$ is equipped with a generalized Hamiltonian structure described in Sect. 2. It would be interesting to connect these two structures.

### 6. BGG Resolution and Proof of Theorem 1

In this section we will show that the space of Toda integrals is isomorphic to $\mathfrak{a}_+^*$. For that we will need the dual of the Bernstein-Gelfand-Gelfand (BGG) resolution [BGG, RW] and results of Sect. 4.

Recall that the dual of the BGG resolution is a complex $B^*(g) = \bigoplus_{j \geq 0} B_j(g)$, where $B_j(g) = \bigoplus_{l(w) = j} M_{w(\rho) - \rho}^*$. Here $M_{\lambda}^*$ is the module contragradient to the Verma module of highest weight $\lambda$, and the differentials of the resolution commute with the action of $g$. The 0th cohomology of $B^*(g)$ is one-dimensional and all higher cohomologies of $B^*(g)$ vanish, so that $B^*(g)$ is an injective resolution of the trivial representation of $\mathfrak{n}_+$.

Using Proposition 2, one can explicitly construct the differentials of the dual BGG resolution.

It is known that for each pair of elements of the Weyl group, such that $w \prec w'$, there is a singular vector $P_{w, w'} \cdot v_{w(\rho) - \rho}$ in $M_{w(\rho) - \rho}$ of weight $w'(\rho) - \rho$. By Proposition 2, this vector defines the homomorphism $P_{w, w'} : M_{w(\rho) - \rho}^* \rightarrow M_{w'(\rho) - \rho}^*$.
It is possible to normalize all $P_{w,w'}$'s in such a way that $P_{w_1',w''} P_{w,w_1'} = P_{w_2',w''} P_{w,w_2'}$ for any quadruple of elements of the Weyl group, satisfying $w < w_1', w_2' < w''$. Then we obtain: $\bar{P}_{w_1',w''}^L \bar{P}_{w,w_1'}^L = \bar{P}_{w_2',w''}^L \bar{P}_{w,w_2'}^L$.

The differential $\delta^j : B^j(\mathfrak{g}) \to B^{j+1}(\mathfrak{g})$ of the BGG complex can be written as follows:

$$\delta^j = \sum_{l(w)=j, l(w')=j+1, w < w'} \epsilon_{w,w'} \bar{P}_{w,w'}^L,$$

where $\epsilon_{w,w'} = \pm 1$ are chosen as in [BGG, RW].

By construction the right action of the Lie algebra $\mathfrak{g}$ on this complex commutes with the differentials. Therefore we can take the subcomplex of invariants $F^*(\mathfrak{g}) = B^*(\mathfrak{g})^a_+^R$ with respect to the action of the Lie algebra $a_+^R$.

**Lemma 1.** The cohomology of the complex $F^*(\mathfrak{g})$ is isomorphic to the exterior algebra $\Lambda^*(a_+^*)$.

**Proof.** We have: $F^*(\mathfrak{g}) = \oplus_{j \geq 0} F^j(\mathfrak{g})$, with

$$F^j(\mathfrak{g}) = \oplus_{l(w)=j} \pi^{(w(\rho) - \rho)},$$

where $\pi^{(w(\rho) - \rho)}$ denotes the space of $a_+$-invariants of the module $M^*_w(\rho) - \rho$. As $\mathfrak{n}_+$-modules all $\pi^{(w(\rho) - \rho)}$ are isomorphic to $\pi_0$. Since $B^*(\mathfrak{g})$ is an injective resolution of the trivial representation of $\mathfrak{n}_+$, it is also an injective resolution of the trivial representation of $\mathfrak{a}_+$. Therefore the cohomology of the complex $F^*(\mathfrak{g})$ is nothing but $H^*(\mathfrak{a}_+, \mathbb{C})$, which is $\Lambda^*(\mathfrak{a}_+^*)$, since $\mathfrak{a}_+$ is abelian. □

**Remark 5.** The cohomology of the complex $F^*(\mathfrak{g})$ is also isomorphic to $H^*(\mathfrak{n}_+, \pi_0)$, cf. [FF2], Proposition 2.4.5. The isomorphism $H^*(\mathfrak{n}_+, \pi_0) \simeq H^*(\mathfrak{a}_+, \mathbb{C})$ follows from Shapiro’s lemma. □

The action of the Lie algebra $a_+^R$ on $B^*(\mathfrak{g})$ gives rise to an $a_-$-action on the complex $F^*(\mathfrak{g})$. We know that if $x$ is an element of $a_-$, then each of the functions $F_i(x)$ is invariant under the $a_+^R$-action. Denote by $\bar{F}_i(x)$ the corresponding function on $N_+/A_+$, $\bar{F}_i(x) \in \pi_0$. The action of $x$ on $\pi^{(\lambda)}$, where $\lambda = \sum_{0 \leq i \leq l} \lambda_i \alpha_i$, is given by the first order differential operator

$$x^R + \sum_{0 \leq i \leq l} \lambda_i \bar{F}_i(x).$$

By construction, this action commutes with the differentials of the complex $F^*(\mathfrak{g})$.

Hence this action defines an action of $a_-$ on the cohomologies of the complex $F^*(\mathfrak{g})$.

**Lemma 2.** The action of $a_-$ on the cohomologies of the complex $F^*(\mathfrak{g})$ is trivial.
Proof. Let $\Omega^*(n_+)$ be the de Rham complex of the big cell $X$ of the flag manifold. This complex is an injective resolution of the trivial representation of $n_+$, which is isomorphic to the tensor product of the space of functions on $X$ and the exterior algebra $\Lambda^*(n_+)$. The Lie algebra $\mathfrak{g}$ infinitesimally acts on $\Omega^*(n_+)$ from the right by vector fields, and this action commutes with the differentials of the complex.

Let $\Omega^*(a_+)$ be the de Rham complex of the Lie group $A_+$. This is an injective resolution of the trivial representation of $a_+$, which is isomorphic to the tensor product of the space of functions on $A_+$ and the exterior algebra $\Lambda^*(a_+)$. The embedding $a_+ \rightarrow n_+$ induces a surjective homomorphism $\rho : \Omega^*(n_+) \rightarrow \Omega^*(a_+)$. The corresponding homomorphism of $a_+$-invariants $\bar{\rho} : \Omega^*(n_+)^{a_+} \rightarrow \Omega^*(a_+)^{a_+} = \Lambda^*(a_+)$ induces an isomorphism of the cohomologies.

The Lie algebra $a_- \subset \mathfrak{g}$ acts on $\Omega^*(n_+)$. Let $a_-$ act trivially on $\Omega^*(a_+)$. Since $a_-$ commutes with $a_+$, the map $\rho$ commutes with the action of $a_-$. Hence the map $\bar{\rho}$ also commutes with the action of $a_-$. Therefore the action of $a_-$ on the cohomologies of $\Omega^*(n_+)^{a_+}$ coincides with its action on the cohomologies of $\Omega^*(a_+)^{a_+}$ and hence is trivial.

The complex $B^*(\mathfrak{g})$ is a subcomplex of $\Omega^*(n_+)$ [BGG, RW], and the embedding $B^*(\mathfrak{g}) \rightarrow \Omega^*(n_+)$ commutes with the action of $\mathfrak{g}$. Since both $B^*(\mathfrak{g})$ and $\Omega^*(n_+)$ are injective resolutions of the trivial representation of $a_+$, the map $B^*(\mathfrak{g})^{a_+} \rightarrow \Omega^*(n_+)^{a_+}$ induces an isomorphism on cohomologies. But it is also an $a_-$-homomorphism, therefore the action of $a_-$ on the cohomologies of the complex $B^*(\mathfrak{g})^{a_+} = F^*(\mathfrak{g})$ is trivial. □

According to the Lemma, $p_1 \in a_-$ acts trivially on the cohomologies. We already know that its action on $\pi_0$ coincides with the action of $\partial$. Consider now the action of $p_1$ on $\pi^{(\lambda)}$. According to formula (20), it is given by $\partial + \sum_{0 \leq i \leq l} \lambda_i u_i^{(0)}$. But this coincides with the action of $\partial$ on $\pi_\lambda$, given by (9). Therefore $\pi^{(\lambda)} \simeq \pi_\lambda$ with respect to the action of $n_+$ and with respect to the action of $p_1 \equiv \partial$.

Proof of Theorem 1. Since $\partial = p_1$ commutes with the differentials of the complex $F^*(\mathfrak{g})$, we can consider the double complex

$$\begin{align*}
C & \rightarrow F^*(\mathfrak{g}) \xrightarrow{\pm p_1} F^*(\mathfrak{g}) \rightarrow C.
\end{align*}$$

Here $C \rightarrow \pi_0 \subset F^*(\mathfrak{g})$ and $F^*(\mathfrak{g}) \rightarrow \pi_0 \rightarrow C$ are the embedding of constants and the projection on constants, respectively. We place $C$ in dimensions $-1$ and $2$ of our complex, and $F^*(\mathfrak{g})$ in dimensions $0$ and $1$.

In the spectral sequence, in which $\pm p_1$ is the 0th differential, the first term is the complex $F^*(\mathfrak{g})[-1]$, where

$$F^j(\mathfrak{g}) \simeq \bigoplus_{l(w) = j} F_w(\rho) - \rho.$$
Indeed, if \( \lambda \neq 0 \), then in the complex

\[
\pi_\lambda \xrightarrow{p_1} \pi_\lambda
\]

the 0th cohomology is 0, and the first cohomology is, by definition, the space \( \mathcal{F}_\lambda \). If \( \lambda = 0 \), then in the complex

\[
\mathbb{C} \rightarrow \pi_0 \xrightarrow{p_1} \pi_0 \rightarrow \mathbb{C}
\]

the 0th cohomology is 0 and the first cohomology is, by definition, the space \( \mathcal{F}_0 \).

In particular, \( \mathcal{F}^0(\mathfrak{g}) = \mathcal{F}_0 \), \( \mathcal{F}^1(\mathfrak{g}) = \bigoplus_{0 \leq i \leq l} \mathcal{F}_{-\alpha_i} \), and the differential \( \delta^0 : F^0(\mathfrak{g}) \rightarrow F^1(\mathfrak{g}) \) is given by \( \delta^0 = \sum_{0 \leq i \leq l} \bar{Q}_i \). By definition, the 0th cohomology of the complex \( \mathcal{F}^*(\mathfrak{g}) \) and hence the 1st cohomology of the double complex (21) is isomorphic to the space of Toda integrals.

We can compute this cohomology, using the other spectral sequence associated to our double complex. Since \( H^*(F^*(\mathfrak{g})) \simeq \wedge^*(\mathfrak{a}_+^*) \), we obtain in the first term the following complex

\[
\mathbb{C} \rightarrow \wedge^*(\mathfrak{a}_+^*) \xrightarrow{\pm p_1} \wedge^*(\mathfrak{a}_+^*) \rightarrow \mathbb{C}.
\]

By Lemma 2, the action of \( p_1 \) on \( \wedge^*(\mathfrak{a}_+^*) \) is trivial and hence the cohomology of the double complex (21) is isomorphic to \( \wedge^*(\mathfrak{a}_+^*)/\mathbb{C} \oplus \wedge^*(\mathfrak{a}_+^*)/\mathbb{C}[-1] \). In particular, we see that the space of Toda integrals is isomorphic to \( \mathfrak{a}_+^* \).

With respect to the \( \mathbb{Z} \)-grading on \( \mathcal{F}_\lambda \), introduced in Sect. 2, the differentials of the complex are homogeneous of degree 0. Moreover, the corresponding \( \mathbb{Z} \)-grading on cohomology coincides with the one induced by the principal grading on \( \mathfrak{a}_+ \). Therefore the space of Toda integrals is linearly spanned by elements \( H_m \) of degrees \(-m \in \mathbb{Z}\).

**Remark 6.** In [FF2], Theorems 3.1.11 and 3.2.6, we proved Theorem 1 in the case when all exponents of \( \mathfrak{g} \) are odd and the Coxeter number is even (this excludes \( D^{(1)}_{2n} \), \( E^{(1)}_6 \), and \( E^{(1)}_8 \)). In this case the degrees of all elements \( p_m \) are odd. The statement of Lemma 2 follows from simple degree counting in this case, since the image of a cohomology class of odd degree under the action of an operator of odd degree should be of even degree and hence should vanish. In particular, it follows that \( \partial = p_1 \) acts trivially on cohomologies, and we can apply the proof of Theorem 1 above.

**7. Vector fields corresponding to Toda integrals**

Let us explain how to construct a Toda integral starting from a class in the first cohomology of the complex \( \mathcal{F}^*(\mathfrak{g}) \).

Consider such a class \( \mathcal{H} \in \bigoplus_{0 \leq i \leq l} \pi_{-\alpha_i} \). Since \( \partial = p_1 \) acts trivially on cohomologies of the complex \( \mathcal{F}^*(\mathfrak{g}) \), \( \partial \mathcal{H} \) is a coboundary, i.e. there exists such \( h \in \pi_0 \) that \( \delta^0 \cdot h = \partial \mathcal{H} \).
By construction, the element $h$ has the property that $\tilde{Q}_i \cdot h \in \pi_{-\alpha_i}$ is a total derivative for $i = 0, \ldots, l$. But it itself is not a total derivative, because otherwise $H$ would also be a trivial cocycle. Therefore, $\int h \neq 0$. But then $\int h$ is a KdV hamiltonian, because by construction $\tilde{\delta}^0 \cdot \int h = \int \tilde{\delta}^0 \cdot h = 0$ and hence $\tilde{Q}_i \cdot \int h = 0$ for any $i = 0, \ldots, l$.

For $m \in I$ denote by $H_m \in F_0$ the Toda integral, corresponding to an element of $a_+^* \subset g$ of degree $-m$. Denote by $\eta_m$ the derivation $\xi_{H_m}$. In particular, simple calculation shows that we can choose as $H_1$ vector $\sum_{0 \leq i \leq l} e^{-\phi_i}$. Then $\partial H_1 = -\sum_{0 \leq i \leq l} u_i^{(0)} e^{-\phi_i}$ and $h_1 = \frac{1}{2} \sum_{1 \leq i \leq l} u_i^{(0)} u_i^{(0)i}$. Hence $\eta_1 = \partial$, by (7).

Now $\eta_m$ is a vector field on $Y \simeq U$. On the other hand the right infinitesimal action of the generator $p_m$ of the Lie algebra $a_- \subset g$ on $U$ also defines a vector field on $U$, which we denote by $\mu_m$.

**Theorem 3.** The vector field $\eta_m$ coincides with the vector field $\mu_m$ up to a non-zero constant multiple for any $m \in I$.

Note that we have already established this for $m = 1$. Indeed, we have just shown that $\eta_1 = \partial$, and we already know that the action of $\partial$ coincides with the action of $p_1$.

**Corollary 2.** The Toda integrals commute with each other:

$$\{H_n, H_m\} = 0$$

in $F_0$ for any $n, m \in I$.

**Proof.** Since $p_m, m \in I$, lie in a commutative Lie algebra, they commute with each other. So do the corresponding vector fields: $[\mu_n, \mu_m] = 0$. By Theorem 3, the same holds for the vector fields $\eta_m, m \in I$: $[\eta_n, \eta_m] = 0$. By formula (11), injectivity of the map $\xi$ on $F_0$, and the definiton of the vector fields $\eta_m$, the corresponding Toda integrals also commute with each other. 

Let us now compute the commutator $[Q_j, \eta_m]$. From the definition of the Toda integrals and (11) we obtain:

$$[\tilde{Q}_j, \eta_m] = [-\xi \int e^{-\phi_j}, \xi_{H_m}] = -\xi \int e^{-\phi_j, H_m} = 0,$$

since $\{ e^{-\phi_j}, H_m \} = \int (Q_j, H_m) = 0$. Therefore

$$[Q_j, \eta_m] = -(\delta_j H_m) Q_j. \tag{22}$$

Indeed, in contrast to the operator $\tilde{Q}_j$, which acts from $\pi_0$ to $\pi_{-\alpha_j}$, the operator $Q_j$ acts from $\pi_0$ to itself. Hence this commutator should be equal to $\Delta_j^* \cdot Q_j$, where
\( \Delta^j_m \) is the difference between the actions of the operator \( \eta_m \) on \( \pi_{-\alpha_j} \) and \( \pi_0 \). This difference is equal to
\[
\sum_{1 \leq i \leq l} \delta_i H_m \frac{\partial e^{-\phi_j}}{\partial \phi_i} = -\delta_j H_m.
\]

8. Proof of Theorem 2

Our proof will be based on formula (22), which turns out to be a defining property for vector fields \( \eta_m \).

We will say that a vector field \( \alpha \) on \( N_+ \) satisfies property (P), if it satisfies formula
\[
[e^L_i, \alpha] = -F_i(\alpha)e^L_i, \quad i = 0, \ldots, l,
\]
where \( F_i(\alpha) \) are certain functions on \( N_+ \). Clearly, if \( \alpha \) and \( \beta \) satisfy property (P), so does their commutator. Thus, vector fields, which satisfy property (P), form a Lie subalgebra of \( \mathcal{V} \), which we denote by \( \mathcal{L} \). According to Proposition 3, \( \tilde{g} \) is a Lie subalgebra of \( \mathcal{L} \) and we prove in Proposition 6 of the Appendix that in fact \( \mathcal{L} \cong \tilde{g} \).

Consider now the vector field \( \eta_m \) corresponding to the Toda integral \( H_m \), for some \( m \in I \). By formula (22), it satisfies property (P) on the homogeneous space \( N_+/A_+ \).

We want to show that there exists an \( \mathfrak{a}_+^R \)-invariant vector field \( \tilde{\eta}_m \) on \( N_+ \) satisfying property (P), such that its projection to \( N_+/A_+ \) coincides with \( \eta_m \).

Let us define a trivial one-cocycle \( f_m \) on the Lie algebra \( \mathfrak{n}_+ \) with coefficients in vector fields on \( N_+/A_+ \) by putting \( f_m(x) = [x, \eta_m] \). Any one-cocycle \( f \) satisfies the relation
\[
f([x, y]) = [x, f(y)] - [y, f(x)].
\]
Since \( \mathfrak{n}_+ \) is generated by \( e_i, i = 0, \ldots, l \), \( f \) is uniquely determined by its values on \( f_m(e_i) \).

Now suppose that we have assigned to each \( e_i \) an element of our module. These elements are values of a one-cocycle \( f \) on the \( e_i \)'s, if and only if the value of \( f \) on any of the Serre relations, inductively constructed using (24), vanishes. For example, if we have the relation \([e_i, e_j] = 0 \) in \( \mathfrak{n}_+ \) and \( f \) is a one-cocycle, then the relation \([e_i, f(e_j)] - [e_j, f(e_i)] = 0 \) should hold.

Our cocycle \( f_m \) has a specific form due to the formula (22): the value of \( f_m \) on each \( e_i \) is proportional to \( e_i \), i.e. equal to \( e_i \) multiplied by a function. By induction, one can show that in this case the value of \( f_m \) on the Serre relation
\[
(ad e_i)^{-a_{ij}+1} e_j = 0
\]
is a linear combination
\[
h_i e_i + h_j e_j + h_{ij} [e_i, e_j] + \ldots + h_{i\ldots ij} (ad e_i)^{-a_{ij} \cdot e_j},
\]
where \( h_\bullet \) are certain functions on \( N_+/A_+ \).
For example, if we have $f_m(e_i) = g_ie_i$, $f_m(e_j) = g_je_j$, and the relation $[e_i, [e_i, e_j]] = 0$, then

$$f_m([e_i, [e_i, e_j]]) = (e_je_i g_i - 2 e_ie_j g_i) e_i + (e_i^2 g_j) e_j + (e_i g_i + 2 e_i g_j) [e_i, e_j].$$

The linear term of the vector field $(ad e_i)^k \cdot e_j$ is non-zero and has degree $k+1$ (cf. [FF2], proofs of Propositions 3.1.10 and 3.2.5). Therefore the linear terms of these vector fields with $k > 0$ are linearly independent from each other and from the linear terms of the vector fields $e_i$ and $e_j$, which have degree 1. The linear terms of the latters are given by $-\partial_i^{(0)}$ and $-\partial_j^{(0)}$ and hence are also linearly independent, if $g$ is not of rank two. But if $g$ is of rank two, then one can check directly that $e_0$ and $e_1$ are linearly independent at a generic point of $Y$. Thus we see that the vector fields $e_i, e_j, [e_i, e_j], \ldots, (ad e_i)^{-m_j} \cdot e_j$, are linearly independent at a generic point of $Y$.

Vanishing of the linear combination (25) of these vector fields multiplied by certain functions then implies that each of the functions $h, \ldots, \tilde{h}$ vanishes identically on $N+/A_+$.

We now want to “lift” the one-cocycle $f_m$ of $n_+$ with coefficients in vector fields on $N+/A_+$ to a one-cocycle $\tilde{f}_m$ of $n_+$ with coefficients in vector fields on $N_+$ with respect to the left action. For the one-cocycle $f_m$ we have:

$$f_m(e_i) = -(\delta_i H_m) e^L_i,$$

by (22).

So we want to put

$$\tilde{f}_m(e_i) = -\epsilon^*(\delta_i H_m) e^L_i, \quad i = 0, \ldots, l,$$

where $\epsilon$ is the projection $N_+ \to N+/A_+$. For $\tilde{f}_m$ to be a one-cocycle, the value of $\tilde{f}_m$ on any of the Serre relations, inductively constructed using (24), must vanish. But this value is given by formula (25), where we should replace each of the functions $h, \ldots, \tilde{h}$ by $\epsilon^*(h, \ldots, \tilde{h})$. Therefore the value on a Serre relation is equal to 0. Hence there exists a one-cocycle $\tilde{f}_m$ of $n_+$ with coefficients in vector fields on $N_+$, which satisfies (26).

The cohomology $H^i(n^L_R, V)$ vanishes for $i > 0$, because as a module over $n^L_R$, $V$ is dual to a free module, with $n^R_+$ as the space of invariants. In particular, $H^1(n^L_R, V) = 0$, and therefore there exists a vector field $\bar{\eta}_m$ on $N_+$, such that $\tilde{f}_m(x) = [x, \bar{\eta}_m]$. In particular, we obtain:

$$[e^L_i, \bar{\eta}_m] = -\epsilon^*(\delta_i H_m) e^L_i, \quad i = 0, \ldots, l.$$

Thus, the vector field $\bar{\eta}_m$ satisfies property (P). By Proposition 6, it lies in $\tilde{g}$. Let $p^R$ be the vector field of the right infinitesimal action of an element $p \in a_+ \subset n_+$ on $N_+$. We have

$$[e^L_i, [p^R, \bar{\eta}_m]] = [[e^L_i, p^R], \bar{\eta}_m] = [p^R, [e^L_i, \bar{\eta}_m]] = 0,$$

for any $i = 0, \ldots, l$, because $[e^L_i, p^R] = 0$ and

$$[p^R, [e^L_i, \bar{\eta}_m]] = -\left(p^R \cdot \epsilon^*(\delta_i H_m)\right) e^L_i - \epsilon^*(\delta_i H_m) [e^L_i, p^R] = 0,$$
since by definition
\[ x \cdot \epsilon^\ast(\delta_i H_m) = 0 \quad \forall x \in a^R. \]

Therefore, by Proposition 3, (a), \[[p^R, \tilde{\eta}_m] \in n^R. \] But the degree of \( \tilde{\eta}_m \) is equal to \(-m < 0\), so that the degree of the commutator with \( p \) is equal to \(-m+1 \leq 0\), whereas the degree of any element of \( n^R \) should be positive. We conclude that \([p^R, \tilde{\eta}_m] = 0\).

But \( \tilde{\eta}_m \in \mathfrak{g}, \) and the only elements of \( \mathfrak{g} \), which commute with \( p^R \), are elements of \( a^R \). Comparing degrees we see that \( \tilde{\eta}_m \) coincides with the vector field \( p^R \), corresponding to a generator \( p_m \) of \( a_\prec \) of degree \(-m\). This generator acts on \( N_+/A_+ \) by the vector field \( \mu_m \). By construction, \([\mu_m - \eta_m, e_i] = 0\) for any \( i = 0, \ldots, l \). Hence \( \mu_m - \eta_m \) is an \( n_\prec \)–invariant vector field on \( N_+/A_+ \).

The space of vector fields on \( N_+/A_+ \) as a module over \( n_\prec \) is coinduced from the \( a_\prec \)-module \( n_\prec/a_\prec \). Therefore the space of \( n_\prec \)-invariant vector fields on \( N_+/A_+ \) is isomorphic to the space of \( a_\prec \)-invariants of \( n_\prec/a_\prec \). The latter space is 0, and hence \( \mu_m - \eta_m = 0 \). Therefore the vector field \( \mu_m \) of the infinitesimal right action of the element \( p_m \in a_\prec \) on \( N_+/A_+ \) coincides with the vector field \( \eta_m \) of the \( m \)th Toda integral \( H_m \).

9. Appendix

The Lie algebra \( \mathcal{L} \) can be defined for an arbitrary Kac-Moody algebra \( \mathfrak{g} \) as the Lie algebra of vector fields on the big cell of the flag manifold, satisfying the relations (23).

The Lie algebra \( \mathcal{L} \) preserves a certain geometric structure on the flag manifold \( F \). Denote by \( P_i, i \in J \), the parabolic subgroup of \( G \), obtained by adjoining to the Borel subgroup \( B_+ \) the one-parameter subgroup of the negative simple root generator \( f_i \) of \( \mathfrak{g} \). Here \( J \) is the set of simple roots of \( \mathfrak{g} \). We have natural bundles: \( F \to G/P_i, i \in J \).

The fiber of such a bundle is a projective line. The tangent spaces to the fibers of these bundles defines \(|J|\) tangent directions at each point of \( F \). The Lie algebra \( \mathcal{L} \) consists of vector fields, which are infinitesimal symmetries of this structure. It seems plausible that all elements of \( \mathcal{L} \) for \( \mathfrak{g} \neq \mathfrak{sl}_2 \) are restrictions to \( N_+ \) of globally defined vector fields on the flag manifold \( B_- \backslash G \).

Clearly, \( \mathfrak{g} \) itself is a Lie subalgebra of \( \mathcal{L} \). The knowledge of \( \mathcal{L} \) is important for understanding representation theory of \( \mathfrak{g} \). Let \( \beta \) be an element of \( \mathcal{L} \) and \( \lambda = \sum_{i \in J} \lambda_i \alpha_i \) be a weight of the Cartan subalgebra of \( \mathfrak{g} \), where \( \alpha_i, i \in J \), are the simple roots of \( \mathfrak{g} \) and \( \lambda_i \)'s are complex numbers. Define a map from \( \mathcal{L} \) to the Lie algebra of differential operators of the first order on the big cell of \( F \), which sends \( \beta \in \mathcal{L} \) to the sum of the vector field, corresponding to \( \beta \), and the function \( \sum_{i \in J} \lambda_i F_i(\beta) \), where the \( F_i(\beta) \)'s are defined by the relations (23). This is clearly a homomorphism of Lie algebras. As we already mentioned above, the space of functions on the big cell of \( F \) with respect to this action of the Lie algebra \( \mathfrak{g} \subset \mathcal{L} \) coincides with the contragradient module to the Verma module with highest weight \( \lambda, M^\lambda \). Thus we obtain a structure of \( \mathcal{L} \)-module.
on $M^*_\lambda$. Since any irreducible module from the category $\mathcal{O}$ of $\mathfrak{g}$ can be realized as a submodule of some $M^*_\lambda$, we obtain a structure of $\mathcal{L}$-module on an arbitrary $\mathfrak{g}$-module.

We do not know a description of $\mathcal{L}$ for Kac-Moody algebras other than finite-dimensional or affine.

**Proposition 5.** If $\mathfrak{g}$ is a finite-dimensional simple Lie algebra other than $\mathfrak{sl}_2$, then the Lie algebra $\mathcal{L}$ coincides with $\mathfrak{g}$. For $\mathfrak{g} = \mathfrak{sl}_2$ the Lie algebra $\mathcal{L}$ coincides with the Lie algebra of vector fields on the line.

**Proof.** In the same way as in the proof of Proposition 6, we reduce the problem to the calculation of the cohomology $H^1(n_+, \mathfrak{g})$. Since $\mathfrak{g}$ is finite-dimensional, we can use the Borel-Weil-Bott-Kostant theorem [B, Ko1], which gives:

$$H^1(n_+, \mathfrak{g}) = \bigoplus_{i=1, \ldots, l} C_{s_i(\Lambda_{adj} + \rho) - \rho}.$$

Here $l$ is the rank of $\mathfrak{g}$, $\Lambda_{adj}$ is the highest weight of the adjoint representation, $\rho$ is the half-sum of the positive roots of $\mathfrak{g}$, and $s_i, i = 1, \ldots, l$, are the reflections from the Weyl group of $\mathfrak{g}$. We denote by $C_{\lambda}$ a one-dimensional representation of the Cartan subalgebra of $\mathfrak{g}$, on which it acts according to its character $\lambda$.

If $\mathcal{L} \neq \mathfrak{g}$, then there should exist a vector field $\beta \in \mathcal{L}$, whose weight and hence the weights of the corresponding functions $F_j(\beta)$ should be equal to $s_i(\Lambda_{adj} + \rho) - \rho$ for some $i = 1, \ldots, l$ (cf. the proof of Proposition 6). If $\mathfrak{g}$ is not $\mathfrak{sl}_2$, all of the weights $s_i(\Lambda_{adj} + \rho) - \rho$ are non-zero and non-negative. Therefore $\beta$ can not satisfy property (P), because functions on the big cell can only have negative or zero weights, and hence $\mathcal{L} = \mathfrak{g}$.

If $\mathfrak{g} = \mathfrak{sl}_2$, one can check by hand that all vector fields on the big cell satisfy property (P). □

**Proposition 6.** If $\mathfrak{g}$ is an affine algebra, $\mathcal{L}$ is isomorphic to $\tilde{\mathfrak{g}}$.

**Proof.** The Lie algebra $\mathfrak{g}$ acts on $\mathcal{V}$ by commutation. We have the exact sequence

$$0 \longrightarrow \tilde{\mathfrak{g}} \longrightarrow \mathcal{L} \longrightarrow M \longrightarrow 0$$

of $\mathfrak{g}$-modules. We will show that the module $M$ belongs to the category $\mathcal{O}$ of modules over $\mathfrak{g}$ [BGG, RW]. A module $L$ belongs to the category $\mathcal{O}$, if it satisfies two properties: (1) the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ acts on $L$ diagonally; (2) for any $x \in M$, $U(n_+) \cdot x$ is finite-dimensional. Let us show that these properties are satisfied for the $\mathfrak{g}$-module $M$.

But we have already seen that the first property is satisfied on the whole Lie algebra $\mathcal{V}$. Recall that the Cartan subalgebra $\mathfrak{h}$ maps to $\mathcal{V}$. One easily checks that the adjoint action of the image of $\mathfrak{h}$ defines a grading of $\mathcal{V}$ by the weights of $\mathfrak{h}$, and that $\mathcal{L}$, and hence $M$, are graded Lie subalgebras of $\mathcal{V}$. Note that in $M$ only negative or 0 weights can occur. Indeed, by definition, the commutation relations in $\mathcal{V}$ preserve the grading. Formula (23) then implies that the weight of a vector field $x \in \mathcal{L}$ is
equal to the weight of each of the functions $F_i(x)$. But functions on $X$ can only have negative or 0 weights, and if all $F_i(x)$’s vanish, then $x \in \mathfrak{n}_+^L \subset \bar{\mathfrak{g}}$ by Proposition 3, (a).

To show that the second property is satisfied, consider an element $x$ of $M$ of weight $\gamma$. If $y$ is an element of $\mathfrak{n}_+$, then, since $y^R$ commutes with $\mathfrak{n}_+^L$, we obtain:

$$[[y^R, x], e^L_i] = - \left( y^R \cdot F_i(x) \right) e^L_i.$$  

As an $\mathfrak{n}_+^R$-module, the space of functions on $X$ is isomorphic to the dual of the free module with one generator. The weight of each of $F_i(x)$ is equal to $\gamma$, therefore any element of $U(\mathfrak{n}_+^L)$ of weight greater than $-\gamma$ maps $x$ to a vector field, which commutes with all the $e^L_i$’s and hence lies in $\mathfrak{n}_+^R \subset \bar{\mathfrak{g}}$. The subspace of $U(\mathfrak{n}_+)$ of elements of weight less than or equal to $-\gamma$ is finite-dimensional. Therefore $U(\mathfrak{n}_+) \cdot x$ is finite-dimensional mod $\bar{\mathfrak{g}}$.

Suppose that $M \neq 0$. Then it should contain a highest weight vector, i.e. a vector field $\nu$, which satisfies $[\mathfrak{n}_+^R, \nu] \in \bar{\mathfrak{g}}$. But then $\mathfrak{C} \nu \oplus \bar{\mathfrak{g}}$ should be an extension of a trivial one-dimensional $\mathfrak{n}_+^R$-module by the $\mathfrak{n}_+^R$-module $\bar{\mathfrak{g}}$. This extension must be non-trivial, because otherwise we would be able to find $\nu' \in \mathfrak{C} \nu \oplus \bar{\mathfrak{g}}$, such that $[\mathfrak{n}_+^R, \nu'] = 0$. But then by Proposition 3, (a), $\nu' \in \mathfrak{n}_+^L$, and then $\nu'$ can not satisfy property (P).

Thus, this extension should define a non-zero element in the group $H^1(\mathfrak{n}_+, \bar{\mathfrak{g}})$. Recall that $\bar{\mathfrak{g}} = \mathfrak{g} \times \mathfrak{vect}_-$, where $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}] / \mathfrak{ck}$. The cohomology $H^4(\mathfrak{n}_+, \mathfrak{g}')$ was computed in [FFi]: $H^4(\mathfrak{n}_+, \mathfrak{g}') \simeq H^3(\mathfrak{n}_+, \mathfrak{C}) \otimes H^1(\mathfrak{n}_+, \mathfrak{g}')$. The space $H^4(\mathfrak{n}_+, \mathfrak{g}')$ is naturally identified with the Lie algebra $\mathfrak{vect} = \mathfrak{C}[t^k, t^{-k}] t \partial_t$ of vector fields on the circle. A vector field $\delta$ defines a 1-cocycle $f_\delta$ on $\mathfrak{n}_+$ with coefficients in $\mathfrak{g}'$ by the formula $f_\delta(x) = [\delta, x]$.

From the long exact sequence associated with the short exact sequence of $\mathfrak{n}_+$-modules

$$0 \longrightarrow \mathfrak{g}' \longrightarrow \bar{\mathfrak{g}} \longrightarrow \mathfrak{vect}_- \longrightarrow 0$$

we obtain: $H^4(\mathfrak{n}_+, \mathfrak{g}) \simeq H^4(\mathfrak{n}_+, \mathfrak{C}) \otimes \mathfrak{vect}_+$, where $\mathfrak{vect}_+ = t^k \mathfrak{C}[t^k] t \partial_t$. In particular, $H^4(\mathfrak{n}_+, \mathfrak{g}) \simeq \mathfrak{vect}_+$. But then the vector field $\nu$, defining the extension, should have a positive weight. Therefore the functions $F_i(\nu), i = 0, \ldots, l$, in formula (23) should also have positive weights and hence they must vanish. But then $\nu \in \mathfrak{n}_+^R \subset \bar{\mathfrak{g}}$, by Proposition 3, (a), and so $M = 0$.  

It is interesting to notice that according to Proposition 6, the flag manifold of an affine algebra “knows” that a half of the Lie algebra of vector fields on the circle acts on $\mathfrak{g}$ by exterior automorphisms and that this action lifts to any highest weight $\mathfrak{g}$-module. Thus, although the flag manifold comes from the Kac-Moody definition of $\mathfrak{g}$, it also contains some information about $\mathfrak{g}$ as the central extension of a loop group. Finding $\mathcal{L}$ for general Kac-Moody algebras may therefore shed light on their possible “hidden symmetries”.

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