Sticky islands in stochastic webs and anomalous chaotic cross-field particle transport by $E \times B$ electron drift instability

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(Dated: July 7, 2020 ©The authors – version 0026dm – DRAFT in progress, not to be cited)

The $E \times B$ electron drift instability, present in many plasma devices, is an important agent in cross-field particle transport. In presence of a resulting low frequency electrostatic wave, the motion of a charged particle becomes chaotic and generates a stochastic web in phase space. We define a scaling exponent to characterise transport in phase space and we show that the transport is anomalous, of super-diffusive type. Given the values of the model parameters, the trajectories stick to different kinds of islands in phase space, and their different sticking time power-law statistics generate successive regimes of the super-diffusive transport.

Keywords: Stochastic web, ExB drift instability, Hall thruster, super-diffusive transport

PACS:
05.45.-a Nonlinear dynamics and chaos
52.20.Dq Particle orbits
52.25.Fi Transport properties
52.75.Di Ion and plasma propulsion
05.45.Pq Numerical simulations of chaotic systems

I. INTRODUCTION

The formation of stochastic web structures and the chaotic transport of charged particles in presence of electrostatic waves and magnetic field has been investigated for several decades [1][7]. In purely chaotic situations where a central limit theorem is valid, the transport process is random walk like, and the variance grows linearly with time [3][10]. But in the case of mixed phase space where both chaotic and regular trajectories coexist, the transport processes are not so clear [11][13]. Transport in such systems can be linked with Lévy flight type processes [14]. In presence of a magnetic field, due to the interaction with electrostatic waves, the dynamics of the charged particles become chaotic and, for certain parameter values, they form stochastic webs where chaotic sticky islands, inside which trajectories show regular features, coexist with a chaotic “sea” between islands. Large scale transport is possible through this chaotic domain [15][10]. These web structures exhibit different shapes which depend on the wave vectors $k$ and amplitudes of the electrostatic wave, and on the frequency ratios $\omega_k/\omega_{ce}$ of electrostatic waves frequencies to the cyclotron frequency $\omega_{ce}$.

The study of the particle transport in these web structures helps to understand the anomalous collisionless transport mechanism in magnetized plasmas. In most of the previous studies, the formation of stochastic webs and the associated transport were investigated for high wave frequency ($\omega_k \gg \omega_{ce}$) [17].

Here, we consider the collisionless transport mechanism of electrons due to the $E \times B$ electron drift instability. In magnetized Hall plasmas, the $E \times B$ electron drift, plasma density, temperature, magnetic field gradients and ion flow are the sources of the $E \times B$ drift instabilities or electron cyclotron drift instability [18]. This instability is observed in many magnetized plasma devices like magnetrons for material processing [19], magnetic filters [20], Penning gauges [21], linear magnetized plasma devices dedicated to study cross-field plasma instabilities [22], Hall thrusters for space propulsion and many fusion devices. In Hall thrusters and other devices, this $E \times B$ drift instability plays a dominant role for anomalous particle transport. In most of these devices, the electrostatic modes generated by $E \times B$ drift instability have very small frequencies compared to the electron cyclotron frequency ($\omega_k \ll \omega_{ce}$). Therefore, the resonance condition with the cyclotron harmonics, $\omega_k - \ell \omega_{ce} = \ell \omega_{ce}$, is not satisfied.

In our recent work [23], we present the anomalous transport of electrons due to wave particle interaction in Hall thruster using a three-dimensional test particle model.

In this paper, we mainly focus on the consequence of the $E \times B$ drift instability. We discuss the formation of stochastic web structures and characterize the associ-
lated transport properties using a reduced two-degrees-of-freedom Hamiltonian which helps to simplify the original dynamics complexity. In real thrusters, due to the presence of wall boundaries, particles can reflect from the boundary, a process which may destroy the wave formation. We found that, in presence of a single background electrostatic wave along with the uniform, static electric and magnetic fields, the trajectories generate web structures and, due to the formation of these web structures, the particle transport is super-diffusive.

Section II presents the model and its two descriptions (respectively time-dependent and time-independent). Sec. III indicates the numerical method used to integrate the evolution equations. Sec. IV discusses the chaotic web structures generated by the dynamics, Sec. V analyses the transport in these structures, and Sec. VI discusses the effect of sticking to invariant islands on transport. We conclude in Sec. VII.

II. REDUCED HAMILTONIAN DYNAMICS AND THE ELEMENTARY MODEL

A. Fields acting on an electron

We consider a Cartesian coordinate system for the numerical modelling, with x-direction along the magnetic field $B_0$, y-direction as $E_0 \times B_0$ drift direction and z-direction along the constant electric field $E_0$.

In Hall thruster geometry, unstable low frequency ($\omega \ll \omega_c$) electrostatic waves are generated due to $E \times B$ drift instability. A 3D dispersion relation of this instability for Hall thruster is derived by Cavalier et al. [24]. The most unstable mode [25] is given by $k_{\text{max}} \sim (\lambda_{\text{De}} \sqrt{2})^{-1}$ and $\omega_{\text{max}} \sim \omega_{\text{pi}} / \sqrt{3}$. Its propagation angle deviates by $\tan^{-1}(k_z/k_y) \sim 10 - 15^\circ$ from the y-direction near the thruster exit plane. Hence, the wave vector along the z-direction $k_z \approx 0.2 k_y$, and the electric field along the z-direction is dominated by the stronger constant axial electric field $E_0$. Therefore, for simplicity, we remove here the z-variation of the electric field.

For this first investigation, we consider only the fastest growing mode. The total electric field acting on the particle is

$$E(x, y, z, t) = \phi_t \mathbf{k} \sin \alpha(x, y, t) + E_0 \mathbf{e}_z,$$  

with the local phase $\alpha(x, y, t) := k_x x + k_y y - \omega_t t$, where the wave vector $\mathbf{k} = k_x \mathbf{e}_x + k_y \mathbf{e}_y$ and angular frequency $\omega_t$ follow the dispersion relation of the $E \times B$ drift instability [24] and $k_z = 0$. The origin of time is such that $\alpha = 0$ for $x = y = 0$, $t = 0$. The position $\mathbf{r} = (x, y, z)$, velocity $\mathbf{v}$, time $t$ and the potential $\phi_t$ are normalized with Debye length $\lambda_{\text{De}}$, thermal velocity $v_{\text{th}}$, electron plasma frequency $\omega_{\text{pe}}$, and $m_e v_{\text{th}}^2 / |q|$, respectively. We choose the amplitude $\phi_t$ equal to the saturation potential [25] at the exit plane of the thruster $|\phi_t|_{\text{rms}} = T_e / (6\sqrt{2}) = 0.056 v_{\text{th}}^2$. We consider a single mode with $(k_x, k_y, \omega_t) = (0.001, 0.754, 1.23 \cdot 10^{-3})$.

From here on, we write $\omega_c$ for $\omega_{\text{ce}}$. In normalized units, $\omega_c = (|q| B_0) / m_e = 0.1 \omega_{\text{pe}}$, $|q| E_0 | / m_e = 0.04 \omega_{\text{pe}} v_{\text{th}}$, and $v_{\text{d}} = E_0 / B_0 = 0.4 v_{\text{th}}$. Therefore, the y-component of the mode phase velocity $\omega_t / k_y \ll v_{\text{d}}$.

As a result, in the Lorentz equation of motion of a particle with mass $m$ and charge $q$

$$\mathbf{\dot{r}} = \frac{q}{m} (\mathbf{E}(\mathbf{r}, t) + \mathbf{r} \times \mathbf{B}),$$  

the electric field $\mathbf{E}(\mathbf{r}, t)$ has a constant part $E_0$ along z-direction and a slowly time varying part in $x - y$ plane. Eq. (2) can be written componentwise, using Eq. (1), as

$$\begin{align*}
\dot{x} & = \frac{q E_{1x}}{m} \sin(k_x x + k_y y - \omega t), \\
\dot{y} & = \frac{q E_{1y}}{m} \sin(k_x x + k_y y - \omega t) + \omega_c \dot{z}, \\
\dot{z} & = \frac{q E_0}{m} - \omega_c y,
\end{align*}$$

where $E_{1x} = k_x \phi_0$ and $E_{1y} = k_y \phi_0$ are the amplitude of the x- and y-components of electric field, respectively, while $\omega_c = q B_0 / m$ and $\omega_t$ are the cyclotron and wave frequency, respectively. Eq. (5) can be integrated:

$$\dot{z} + \omega_c y = \frac{q E_0}{m} t + a,$$

where $a = v_{\text{d}0} + \omega_c y_0$ is a constant of integration, $v_{\text{d}0}$ and $y_0$ are the particle’s initial z-component velocity and position along y-direction, respectively. Substituting $\dot{z}$ in Eq. (4), and recalling the drift velocity $v_{\text{d}} = E_0 / B_0$, we reduce the equation of motion of the particle to a system of two equations,

$$\begin{align*}
\dot{y} + \omega_c^2 y & = \frac{q E_{1y}}{m} \sin(k_x x + k_y y - \omega_t t) + v_{\text{d}} \omega_c^2 t + w_c a, \\
\dot{x} & = \frac{q E_{1x}}{m} \sin(k_x x + k_y y - \omega_t t).
\end{align*}$$

B. Time-dependent Hamiltonian

System (7) derives from the Hamiltonian

$$H(x, p_x, y, p_y, t)$$

with

$$H = \frac{p_x^2 + p_y^2}{2m} + \frac{m}{2} \omega_c^2 y^2 - (t + A) m v_{\text{d}} \omega_c^2 y$$

$$+ q \phi_0 \cos(k_x x + k_y y - \omega_t t),$$

where $A = (v_{\text{d}0} + \omega_c y_0) / (\omega_c v_{\text{d}})$ is a constant. By means of the generating function

$$F(x, p_x, y, p_y, t) = P_x x + (P_y + v_{\text{d}}) (y - (t + A) v_{\text{d}}),$$

we change to new variables $(P_x, P_y, X, Y)$ in a frame moving with a constant velocity $v_{\text{d}}$ along the y-direction...
(which we call a “drifted frame” for figures),
\[
X = \frac{\partial F}{\partial P_x} = x,
\]
\[
Y = \frac{\partial F}{\partial P_y} = y - (t + A) v_d,
\]
\[
p_x = \frac{\partial F}{\partial x} = P_x,
\]
\[
p_y = \frac{\partial F}{\partial y} = P_y + v_d,
\]
\[
\frac{\partial F}{\partial t} = -(P_y + v_d) v_d.
\]
Using these new coordinates (10), the new Hamiltonian (after removing terms irrelevant to the motion) and the equations of motion read
\[
K(P_x, P_y, X, Y, t) = \frac{P_x^2 + P_y^2}{2m} + \frac{m}{2} \omega_r^2 Y^2 + q \phi_1 \cos \alpha,
\]
\[
\ddot{X} = \frac{q \phi_1}{m} k_x \sin \alpha,
\]
\[
\ddot{Y} + \omega_r^2 Y = \frac{q \phi_1}{m} k_y \sin \alpha,
\]
with \( \alpha = k_x X + k_y Y + (v_d k_y - \omega_1) t + \zeta \) and where \( \zeta = k_x v_d A \) is constant.

The dimensionless equations of motion are obtained using the dimensionless variables \( X' = k_x X + \zeta, \ Y' = k_y Y, \ t' = \omega_r t \). Introducing the new notation \( \beta = k_x/k_y, \ v_1 = (v_d k_y - \omega_1)/\omega_r \), we obtain the dimensionless equations
\[
\frac{d^2 X'}{dt'^2} = \varepsilon \beta^2 \sin (X' + Y' + v_1 t'),
\]
\[
\frac{d^2 Y'}{dt'^2} + Y' = \varepsilon \sin (X' + Y' + v_1 t').
\]

In this paper, we solve Eqs. (12) numerically using a second order symplectic scheme. The dynamics involves two degrees of freedom with a time-periodic dynamics (with period \( 2\pi/\nu_1 \)), so that the effective phase space is 5-dimensional. The coordinate \( X' \) admits periodic boundary condition (with period \( 2\pi \)), whereas \( Y' \) runs over the real line.

The dynamics depends on three parameters, \( \varepsilon, \beta \) and \( \nu_1 \). For \( \varepsilon = 0 \), \( X' \) is ballistic and \( Y' \) is a harmonic oscillator, in agreement with the well-known solutions for particle motion in stationary, uniform fields \( B_0 \) and \( B_0 \).

Note that, in Hall thrusters, \( B_0 \) is radial and \( B_0 \) is axial, so that the drift is azimuthal. The coordinates \( y \) and \( Y \) are thus defined on circles, while \( x \) and \( X \) are actually bounded by the inner and outer cylindrical chamber walls. The origin for \( Y \) and \( X \) are determined by the initial conditions \((y_0, v_{y0})\) and the phase convention for the electrostatic mode, respectively.

C. Time-independent Hamiltonian

A time-independent Hamiltonian can be derived by means of a Galileo transformation along \( X \) with velocity \( \nu_1 \omega_r/k_x \). With the generating function and change of variables
\[
F = \left( P_x - \frac{\nu_1 \omega_r}{k_x} \right) \left( X + \frac{\nu_1 \omega_r^2 + \zeta}{k_x} \right) + P_y Y,
\]
\[
X = X + \frac{\nu_1 \omega_r}{k_x} t + \frac{\zeta}{k_x},
\]
\[
P_x = P_x - \frac{\nu_1 \omega_r}{k_x},
\]
\[
Y = Y, \quad P_y = P_y,
\]
\[
\frac{\partial F}{\partial t} = \frac{\nu_1 \omega_r}{k_x} \left( P_x - \frac{\nu_1 \omega_r}{k_x} \right),
\]
the Hamiltonian (up to terms irrelevant to the motion) and the equations of motion can be written as
\[
K(P_x, P_y, X', Y') = \frac{P_x^2 + P_y^2}{2m} + \frac{m}{2} \omega_r^2 Y'^2 + q \phi_1 \cos \alpha,
\]
\[
\ddot{X'} = \frac{q \phi_1}{m} k_x \sin \alpha,
\]
\[
\ddot{Y'} + \omega_r^2 Y' = \frac{q \phi_1}{m} k_y \sin \alpha,
\]
Setting \( X' = k_x X, Y' = k_y Y \) and \( t' = \omega_r t \), the equations of motion (14) reduce to
\[
\frac{d^2 X'}{dt'^2} = \varepsilon \beta^2 \sin (X' + Y') ,
\]
\[
\frac{d^2 Y'}{dt'^2} + Y' = \varepsilon \sin (X' + Y') .
\]
Eq. (15) is solved numerically for various parameters and initial conditions.

This dynamics depends on two parameters only, \( \varepsilon \) and \( \beta \). It involves two degrees of freedom, with the coordinate \( X' \) obeying periodic boundary condition (with period \( 2\pi \)), whereas \( Y' \) runs over the real line. As the dynamics is autonomous, it preserves the “energy” \( K \). Therefore, trajectories stay on smooth 3-dimensional surfaces, and they may be visualised by means of 2-dimensional Poincaré sections.

While coordinates \((x, y)\) and \((X, Y)\) are related with the Hall thruster chamber, coordinates \((X', Y')\) simplify further the dynamics, provided one does not worry about boundary conditions. Therefore, we use both the time-independent and the time-dependent representations in the following discussions.

For \( \varepsilon = 0 \), viz. in absence of electrostatic wave, the dynamics is integrable. The dimensionless actions are the linear momentum \( dX'/dt' \) with angle the position \( X' \) periodic with period \( 2\pi \) in agreement with the boundary condition, and the gyration energy \( R'^2/2 = (Y')^2 + \)
In presence of the electrostatic wave, for small $\varepsilon$, these actions generate two adiabatic invariants. For $\beta$ also small, the actions evolve on different time scales (in terms of the dimensionless $t'$), namely $\varepsilon^{-1}$ for the oscillations of $\mathcal{Y}'$ and $\varepsilon^{-1} \beta^{-2}$ for the nearly-ballistic motion of $\mathcal{X}'$.

III. NUMERICAL METHOD

Because the right hand sides of Eqs (12) and (15) depend on space, the infinitesimal generators for both velocity and position equations do not commute, and one uses a time-splitting numerical integration scheme. Since the dynamics is Hamiltonian and we are interested in long-time evolution, we choose a symplectic scheme [20].

The positions are advanced with the map $r(t + \Delta t) = T_{v,\Delta t}(r(t)) = r(t) + \mathbf{v} \Delta t$, and the velocities are advanced in the form $\mathbf{v}(t + \Delta t) = T_{E,\Delta t}(\mathbf{v}(t)) = \mathbf{v}(t) + \varepsilon \beta^2 \sin(\mathcal{X} + \mathcal{Y}) \Delta t$ and $v_y(t + \Delta t) = T_{E,\Delta t}(v_y(t)) = v_y(t) + (\varepsilon \sin(\mathcal{X} + \mathcal{Y}) - \mathcal{Y}) \Delta t$. As a result, we use a second-order symmetric leapfrog scheme, which evolves (15) as the map

$$
\begin{pmatrix}
r(t + \Delta t) \\
v(t + \Delta t)
\end{pmatrix} = A \begin{pmatrix}
r(t) \\
v(t)
\end{pmatrix},
$$

(16)

$$
A = T_{E,\Delta t/2} \circ T_{E,\Delta t} \circ T_{v,\Delta t/2}.
$$

(17)

We evolve Eqs (12) similarly, with $T_{E,\Delta t}$ evaluated at midpoint $t + \Delta t/2$.

IV. STOCHASTIC WEB STRUCTURE

The values of $(\varepsilon, \beta^2, \nu_1)$ in a Hall thruster device for the fastest growing mode $(k_x = 0.001, k_y = 0.754, \omega = 1.23 \cdot 10^{-3})$ are $\varepsilon = 3.21, \beta^2 = 1.75 \cdot 10^{-6}$ and $\nu_1 = 3$.

As $\varepsilon$ is not small, the gyration action is definitely not conserved, as will be seen in the phase space plots. However, $\varepsilon \beta^2 \sim 5 \cdot 10^{-6}$ so that the ballistic action is almost conserved over times $t \sim 10^5 \omega^{-1} \sim 10^6 \omega_{pe}^{-1}$.

Here we first focus on transport for web structures with three-fold rotational symmetry ($\nu_1 = 3$) as in the Hall thruster geometry, and its harmonic the six-fold rotational symmetry ($\nu_1 = 6$). To assess the importance of having an integer value for the forcing frequency $\nu_1$, we also consider the non-resonant value $\nu_1 = 1.39$. For the time-independent description, the initial velocity $\mathcal{X}^\prime_0$ plays a similar role, and we also contrast the integer values 3 and 6 with the rational value $3.5$.

We evolve the dynamics of 1024 particles having initial Gaussian velocity distribution with unit standard deviation along the $y$-directions and $\langle \mathcal{X}^\prime_0 \mathcal{Y}^{\prime}_0 \rangle = 0$. Since the web trajectory drifts rapidly for high value of $\mathcal{X}^\prime_0$, precluding the presence of islands, we consider a very small standard deviation $\sigma_x = 0.001$ along $x$-direction. Therefore, we first consider the time-dependent dynamics in Sec. IV.A then the time-independent cases in Sec. IV.B.

A. Time-dependent Hamiltonian

We evolve Eqs (12) numerically for three different sets of parameters $(\varepsilon, \beta^2, \nu_1) = (3.21, 1.75 \cdot 10^{-6}, 6.0), (3.21, 1.75 \cdot 10^{-6}, 3.0)$ and $(0.69, 1.83 \cdot 10^{-5}, 1.39)$, respectively.

We first plot the stroboscopic Poincaré section in the $(\mathcal{Y}^{\prime}, \mathcal{P}^{\prime}_y)$ plane of a single particle trajectory at times $t = 2n\pi/\nu_1$, where $n = 0, 1, 2, ...$. The parameter $\varepsilon$, which
is the ratio of the bounce frequency to the cyclotron frequency, determines the radius of the stochastic web. The value of $\nu_1$ determines the shape of the web structure. For integer $\nu_1$, we observe a web structure with $\nu_1$-fold rotational symmetry (Figs 1 and 2). For parameters $(\varepsilon, \beta^2, \nu_1) = (0.69, 1.83 \cdot 10^{-5}, 1.39)$ with non-integer $\nu_1$, the dynamics generates a Halloween mask-like, deformed three-fold web structure (Fig. 3).

For the time-dependent Hamiltonian, the dynamics depends on $(\varepsilon, \beta^2, \nu_1)$. Parameter $\varepsilon$ expresses the ratio of bounce frequency to cyclotron frequency, $\beta^2$ expresses the ratio of the parallel and perpendicular components of the wave electric field, and $\nu_1$ is the normalized frequency of the electrostatic wave in the drifted frame.

A large value of $\beta^2$ or $\dot{X}_0'$ causes the dynamics detuning from the longitudinal resonance condition, $k_y \dot{X}' = \ell \omega_c$, where $\ell$ is an integer. Therefore, the orbits in the web structure drift more rapidly, covering the entire phase space inside the web and destroying islands. In our present simulation, $\beta^2 \sim 10^{-6}$ and $|\dot{X}_0'| \ll 1$ which induces a slow drift of the trajectory.

Along $Y'$, the dynamics Eq. (12) has two time scales, one associated with the electrostatic wave (with period $2\pi/\nu_1$ in the drifting frame) and the other associated with a simple harmonic oscillator with period $2\pi$. Therefore, an integer value of $\nu_1$ causes resonance between these two time scales and one can eliminate the time dependence by taking the Poincaré section at a regular time interval, $nT = 2\pi n / \nu_1$, to generate the stochastic web structures in the Poincaré section plot. The reduced frequency $\nu_1$ will determine the shape of the web structure.

For fixed values of $\varepsilon$, $\beta^2$ and $\nu_1$, any initial condition $(X_0', Y_0', \dot{X}_0', \dot{Y}_0')$, within the chaotic domain of the stochastic web, generates a similar web structure, and the particles with initial conditions outside the web structure and well inside the sticky islands (regions with no points in the web structures) generate regular trajectories.

**B. Time-independent Hamiltonian**

In the time-independent Hamiltonian, $\dot{X}' = \dot{X}' - \nu_1$ and the dynamics depends on $(\varepsilon, \beta^2)$ only. In this case, for any initial condition $(X_0', Y_0', \dot{X}_0', \dot{Y}_0')$ within the chaotic domain of the stochastic web, the shape of the web structure depends on the initial $X_0'$ values. For different $X_0'$ values, the trajectory lies on different energy surfaces $K = \text{constant}$. Thus, particles with different ini-
tial conditions generate different web structures. Since \( \beta^2 \sim 10^{-6} \) in this study, the motion along the \( X' \) direction is almost ballistic, \( X' \approx \text{constant} \). Therefore, we can generate Poincaré section plots by taking sections at \( X' = n \pi \), where \( n \) is an integer. For integer values of \( X' \), it generates Poincaré sections similar to those generated for the time-dependent Hamiltonian with the same integer values of \( \nu' \).

Figures 4 and 5 display the stroboscopic plot of the time-independent dynamics Eq. (15) with \( (\varepsilon, \beta^2) = (3.21, 1.75 \cdot 10^{-6}) \) and two different initial velocities along the \( X' \) direction, \( X'_0 = 3 \) and \( 3.5 \) respectively. In the stroboscopic plots, each point is taken at a time when \( X' \) is an integer multiple of \( 2\pi \). For integer values of \( X'_0 \), the dynamics generates web structures similar to those generated in the Poincaré section plot for the cases of time-dependent dynamics \( \rho_c \) with same integer values of \( \nu' \). Fig. 4 presents the stroboscopic plot of a particle with \( v_{0x} = 3.5 \), which corresponds to a higher-order resonance \((7/2)\). For fractional values of \( X'_0 \), the stroboscopic plot generates different structures, because each of the different initial conditions lies on a different energy \((K = \text{constant})\) surface. Therefore, the web structures in the time-independent dynamics highly depend on the initial conditions of the particle.

V. TRANSPORT PROPERTIES

To characterise the transport properties, we consider a simple observable. Previous studies \([7, 27]\) for time-dependent one-degree-of-freedom Hamiltonian systems focused on the norm of velocity \((\dot{p}, \dot{q})\) in phase space, where \( p, q \) are canonical co-ordinates. Here, we consider the arc length \( s \) of the trajectory in position space only, or, in dimensionless variables of Eqs (12),

\[
S'(t) = \int_0^t \sqrt{dX'^2 + dY'^2}.
\]

Numerically, we consider the global average speed along the trajectory of a typical particle \( i \)

\[
\bar{v}_i(n) = \frac{1}{n\Delta t} \sum_{k=0}^{n-1} \sqrt{[\Delta X'_i(t_k)]^2 + [\Delta Y'_i(t_k)]^2},
\]

where \( k \) is the timestep index, with coordinate increments

\[
\Delta X'_i(t_k) = X'_i(t_{k+1}) - X'_i(t_k), \quad \Delta Y'_i(t_k) = Y'_i(t_{k+1}) - Y'_i(t_k).
\]

We define \( \rho_n(\bar{v}) \) as the sampling density of the distribution of \( \bar{v}_i(n) \)'s. Good ergodic properties of the dynamics \( \rho_n \) would include the convergence of \( \rho_n \) towards a Dirac distribution for \( n \rightarrow \infty \), in which case the support \( \langle v \rangle \) of the limit is the ergodic average of the \( \bar{v}_i(n) \)'s,

\[
\lim_{n \rightarrow \infty} \rho_n(\bar{v}) = \delta(\bar{v} - \langle v \rangle),
\]

and, almost surely with respect to the initial condition \((\text{viz. index } i)\),

\[
\langle v \rangle := \lim_{n \rightarrow \infty} \bar{v}_i(n).
\]

One method to assess the convergence of \( \rho_n \) is to look at how fast its maximum value \( \rho_{\text{max}}(n) \) diverges towards \( +\infty \) with \( n \). If the dynamics is sufficiently chaotic in the sense that a scaling applies to increments in \( \langle v \rangle \), one may expect

\[
\rho_{\text{max}}(n) \sim n^\alpha,
\]

FIG. 6: Distribution of arc length for the stochastic web with \( \nu_1 = 3 \) at \( \omega_0 t = 8 \cdot 10^2, 13 \cdot 10^2, 21 \cdot 10^2, 85 \cdot 10^2, 34 \cdot 10^2, 13 \cdot 10^6 \) and \( 5.4 \cdot 10^7 \).

FIG. 7: Distribution of arc length for the stochastic web with \( \nu_1 = 6 \) at \( \omega_0 t = 8 \cdot 10^2, 6.7 \cdot 10^2, 1.0 \cdot 10^3, 4.2 \cdot 10^3, 1.7 \cdot 10^6, 6.9 \cdot 10^6 \) and \( 2.7 \cdot 10^7 \).
where the exponent $\alpha$ characterises the nature of the transport. If increments in $\langle \omega \rangle$ are quite independent and a central limit theorem applies, transport is diffusive and $\alpha = 1/2$. For $\alpha > 1/2$ it is sub-diffusive, and for $\alpha < 1/2$ it is super-diffusive.

A. Time-dependent Hamiltonian

In Figs 6 and 7 we plot the distribution of

$$u_{i,n} = \bar{v}_i(n) - \langle v \rangle$$

for two different web structures, with $\nu_1 = 3$ and $\nu_1 = 6$, respectively. One can calculate the arc length for a time-independent dynamics also, but in the time-dependent dynamics, the parameter $\nu_1 = (\nu_1 k_y - \omega_1)/\omega_c$ is important in Hall thrusters. It expresses the frequency of the electrostatic modes generated by the $\mathbf{E} \times \mathbf{B}$ instability, in a frame which moves with the drift velocity $\bar{v}_d$ along the $\mathbf{E}_0 \times \mathbf{B}_0$ direction. As our study is motivated by the anomalous chaotic transport in Hall thruster devices, we choose the time-dependent dynamics for characterising the transport for $\nu_1 = 3$ and $\nu_1 = 6$.

To characterize the transport, we consider two different stochastic webs corresponding to values (3.21, 1.75 $\cdot 10^{-6}$, 3) and (3.21, 1.75 $\cdot 10^{-6}$, 6) for parameters ($\varepsilon$, $\beta^2$, $\nu_1$). We evolve the equations of motion (12) for 1024 particles with all initial conditions inside the chaotic domain, we calculate the arc length for each particle trajectory for a long time evolution ($10^8 \omega_{pe}^{-1}$) using a timestep value $\Delta t = 3.33 \cdot 10^{-3} \omega_c^{-1}$ in simulation, and we generate the distribution of the arc length.

We plot the distribution of the arc length at different times in Figs 6 and 7 for both cases. To avoid nonphysical peaks in the distribution of $\rho_n$, the length $n$ of the time sequence should be sufficiently long for the dynamics to reach a saturation state, i.e. for the Poincaré section of each particle’s trajectory to sample the entire phase-space reach of the web. Here, we construct the distribution functions $\rho_n$ at times $t_n \geq 800 \omega_{pe}^{-1}$.

In the plot, the strong sharp peaks are associated with the stickiness phenomenon, by which a trajectory may remain for a long time close to the regular islands. The number of sharp peaks depends on the number of resonance generating sticky islands within the web structures, which we further discuss in the next section. We obtain more peaks for $\nu_1 = 6$ (Fig. 7) than for $\nu_1 = 3$ (Fig. 6). The relative magnitude of these sharp peaks decreases as $n \to \infty$ because the contribution from the chaotic domain becomes larger compared to the contribution from the sticky regular trajectories as we consider a longer time evolution.

In both Figs 6 and 7, the distribution after time $t_n \sim 10^7 \omega_{pe}^{-1}$ (yellow line) has almost zero relative strength of the sharp peaks, compared to the height of the smoother distribution. Stickiness generates a memory effect and Lévy flights [28]. In absence of these sticky trajectories, the transport is purely diffusive and the exponent $\alpha$ takes the value 1/2. In the presence of these sticky trajectories, the transport will be anomalous.

To measure $\alpha$, we find the value of $\rho_{\max}$ from the local maximum of the central smooth flat peak location. In Fig. 8 we plot the time evolution of $\rho_{\max}$ for both cases $\nu_1 = 3$ and $\nu_1 = 6$. From the curve fitting, we obtain two different values $\alpha_3 = 0.17$ and $\alpha_3 = 0.39$ for the case $\nu_1 = 3$ (magenta data dots and dashed line), and $\alpha_6 = 0.15$ and $0.33$ for $\nu_1 = 6$ (black). Both values of $\alpha$ in both cases are below 0.5. Thus, the diffusion is anomalous and super-diffusive. After a longer time evolution, most of the particles spend more time within the chaotic region of the stochastic web. Then, the contribution from the sticky islands decreases in comparison with the contribution from the chaotic domain. Therefore the diffusion rate increases at longer time ($t_n > 10^8 \omega_{pe}^{-1}$).

Note however that, even for large times, the exponent being smaller than 1/2 implies that the average speed fluctuations $\bar{v}_i - \langle v \rangle$ do not approach a Gaussian distribution, hence they do not obey the central limit theorem over this time scale.

B. Time-independent Hamiltonian

Similarly, we analyse transport for a stochastic web structure generated from the time-independent Hamiltonian with the corresponding arc length

$$S'(t) = \int_0^t \sqrt{d\mathcal{X}^2 + d\mathcal{Y}^2}.$$  

The trajectories of 1024 particles are computed numerically up to time $10^8 \omega_{pe}^{-1}$ with time step $\Delta t = 5.5 \cdot 10^{-3}$. All particles are initially randomly distributed in $\mathcal{X} = \mathcal{Y} = \pi$ plane within $-\pi \leq \mathcal{X}_0 \leq \pi$ and $-\pi \leq \mathcal{Y}_0 \leq \pi$. Their

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FIG. 8: Evolution of $\rho_{\max}$ versus $n$ for the time-dependent Hamiltonian with $\nu_1 = 3$ (magenta) and $\nu_1 = 6$ (black).
initial velocities along the \(Y'\) direction are drawn from a Gaussian distribution with unit standard deviation. Along the \(X'\) direction, we consider three different values \(X' = 3, 3.5\) and 6 in order to analyse the transport in three different web structures generated from the time-independent Hamiltonian \(H_{\text{max}}\). For all three cases, we consider \(\epsilon = 3.21\) and \(\sigma^2 = 1.75 \cdot 10^{-6}\).

Figs. 9 and 10 present the distribution of the arc length at different times for all three cases. Similarly to the time-dependent cases, sharp peaks in the distribution of arc lengths appear due to the presence of the sticky islands, and the number of sharp peaks is larger for the web structures with six-fold rotational symmetry, \(X' = 6\), than for the three-fold rotational symmetry, \(X' = 3\). In the case \(X' = 3.5\), as seen in Fig. 9, the number and area of sticky islands are smaller than for \(X' = 3\). Therefore, the height in the smooth part of distribution, due to the chaotic domain, is larger for \(X' = 3.5\).

To estimate the exponent values from Eq. (24), we plot in Fig. 12 the time evolution of \(\rho_{\text{max}}\) for all three cases. For \(X' = 3\) (magenta) and 6 (black), the plots are similar to the time-dependent cases. From the curve fitting, we obtain two different values of \(\alpha\) in two different regimes of the plots, \(\alpha_3 = (0.15, 0.25)\) and \(\alpha_6 = (0.18, 0.25)\). In both
cases, the transport is anomalous of super-diffusive type. The values for the shorter time span \((t < 5 \cdot 10^3 \omega_{pe}^{-1})\) are very close to the values that are recovered for the time-dependent dynamics.

For fractional values of \(\mathcal{X}_0\), the number and area of the sticky islands are smaller than for the other two cases. Most of the region within the stochastic webs is part of the chaotic domain, therefore the height of the distribution increases at a higher rate than in the other two cases, as we increase the value of \(n\). For \(\mathcal{X}_0 = 3.5 = 7/2\), we find higher exponent values, \(\alpha_{3.5} = (0.23, 0.46)\). At short time, the relative contribution from sticky trajectories is significantly large, which reduces the exponent value to \(\alpha_{3.5} = 0.23\); in contrast, for longer time \(t > 5 \cdot 10^4 \omega_{pe}^{-1}\), the contribution from the chaotic region dominates over the contribution from sticky islands, and sharp peaks almost disappear, which increases the exponent value to \(\alpha_{3.5} = 0.46\), so that the transport becomes closer to a diffusive type.

For \(\mathcal{X}_0 = 1.39\) (further away from a simple rational) for \((\varepsilon, \beta^2) = (0.69, 1.83 \cdot 10^{-5})\), we also draw 1024 initial conditions in the chaotic part of the domain defined by \(-\pi \leq \mathcal{X}_0 \leq \pi\), \(\mathcal{X}_0\) a Gaussian random number with expectation 0 and standard deviation 1, and \(\mathcal{X}_0\) outside the islands (typically, \(0.1 \pi \leq |\mathcal{X}_0| \leq 0.6 \pi\)). With these parameters and initial conditions, the same analysis applies, as seen from the peaks in the arc length distribution in Fig. 13 and from the slopes \(\alpha_{1.39} = (0.32, 0.42)\) in Fig. 14. In the next section, we discuss this change of exponent values more quantitatively.

VI. EFFECT OF STICKY ISLAND ON ANOMALOUS TRANSPORT: CHANGE OF \(\alpha\)

A contrast between the law of large numbers and the central limit theorem (CLT) is evidenced by estimating the exponent \(\alpha\) : \(\alpha\) would be 0.5 when CLT can be applied, and if \(\alpha \neq 0.5\) then it is anomalous. The superdiffusive or subdiffusive nature of transport is measured by the weight of the tails in the distributions: (i) if there are fat tails (compared to the diffusive bulk of the distribution), the maximum \(\rho_{\text{max}}\) of the distribution will grow slower than \(1/2\) and we expect superdiffusion, while (ii) if there are thinner tails, the maximum \(\rho_{\text{max}}\) of the distribution will grow faster than \(1/2\) and we expect subdiffusion. The presence of the sharp peaks in the distribution of \(\rho_{\text{max}}(\vec{v})\), Figs. 6, 9, 10, 11 and 13 increases the effective weight of the tail of the distribution, which makes exponent \(\alpha < 1/2\).

Each of the sharp peaks is related to the presence of sticky islands in the phase space. Thus, one can select the portions of trajectories contributing to each peak and locate them in the Poincaré map. Specifically, given a trajectory (with initial condition labeled \(1 \leq i \leq N = 1024\)) over a long time span, \(z_i(t)\), and a sticking time \(T_s\), we fix a delay \(\tau\), and consider the arc lengths \(\Delta S_{i,m}^s(T_s) = S^s_i(T_s + m\tau) - S^s_i(m\tau)\) of the portion of \(z_i\) over \([m\tau, T_s + m\tau]\), for \(0 \leq m \leq M - 1\) for some large \(M\). For a moderate value of \(\tau\) (say, \(\sim 200 \omega_{pe}^{-1}\)), this method generates \(MN\) time sequences of length \(T_s\) which we analyse. In particular, when these portions keep sticking to the same island, we plot their Poincaré section in the \((\mathcal{X}', d\mathcal{X}'/dt')\) plane for integer \(\mathcal{X}'/(2\pi)\).

This is done in Fig. 15 for the web structures with three-fold symmetry for the trajectories with sticking duration \(T_s\) such that \(\log_{10}(\omega_{pe} T_s) = 2.92\). There are three peaks, \(P_{k1}\) at \(u_0 = 1.0\), \(P_{k2}\) at \(u_0 = 3.0\) and \(P_{k3}\) at \(u_0 = 5.0\), in Fig. 9 each with a finite width. For each peak, we identify the trajectories contributing to the peak, and plot their Poincaré section, whereby the sticky regions emerge. In Fig. 15, the sticky regions denoted by blue, red and black dots are associated with the
peaks $P_{k_1}$, $P_{k_2}$ and $P_{k_3}$, respectively.

In Fig. 15 the distribution of arc lengths for the web with six-fold symmetry has seven peaks, namely $P_{k_1}$ at $u_n = -3.6$, $P_{k_2}$ at $u_n = -1.5$, $P_{k_3}$ at $u_n = 0.5$, $P_{k_4}$ at $u_n = 2.3$, $P_{k_5}$ at $u_n = 4.2$, $P_{k_6}$ at $u_n = 6.2$ and $P_{k_7}$ at $u_n = 8.1$. In a similar way, we locate the sticky region in the phase space for each of these peaks for the same duration $T_s$, such that $\log_{10}(\omega_{pe}T_s) = 2.92$. In Fig. 16 the blue, red, green, magenta, cyan, yellow and black dots identify the sticky regions associated with the peaks $P_{k_1}$ to $P_{k_7}$, respectively. Thus, all peaks in the distribution plots are associated with different sticky sets.

In Fig. 16 we similarly identify the sticky sets associated with the peaks for the stochastic web with Halloween mask like structure of Eq. (16) for $\varepsilon = 0.09$, $\beta^3 = 1.83 \cdot 10^{-5}$ and $\lambda_0' = 1.39$.

Due to the Poincaré recurrence theorem, these sets should become empty as time goes on and contribute to increase the maximum of the average speed distribution, but some sticky sets may leak slower or faster than others, which can change the transport coefficient. To understand the influence of stickiness on anomalous transport, we now consider the web structure with three-fold symmetry, and investigate the change of each peak for increasing evolution time $t = n\Delta t$. This analysis makes it possible to see the time evolution of the particles trapped in the corresponding islands. From the distribution $\rho(t)\langle v \rangle$ of arc lengths, one can count the number of data points that contribute to each specific peak at different times $t$.

Then one estimates how long the particles are sticking to each specific island, by monitoring the change of area localized under each of those peaks as a function of $n$. Therefore, this area yields the weight of sticking to that particular island, until at least $\tau = n\Delta t$, which can be written as

$$w_{Pk}(\tau, T_{\text{max}}) = T_{\text{max}}^{-1} \int_0^{T_{\text{max}}} \rho_{Pk}(t)dt,$$  \hspace{1cm} (27)

where $P_{k}$ is the index for each peak and the statistics are gathered for a “very long” run $[0, T_{\text{max}}]$. Under an ergodic assumption \cite{28, 29}, this weight would enable to estimate the probability that a trajectory would stick to island $P_{k}$ for at least the duration $\tau$. For large sticking time, a self-similar behaviour in the small scales in phase space near the island will be associated with a power law decay with an exponent $\gamma$,

$$w_{Pk}(T_s, T_{\text{max}}) \sim T_s^{1-\gamma_P}.$$  \hspace{1cm} (28)

In order to analyze the sticking-times statistics, we count the number of data points sticking to each island and plot them in logarithmic scale versus the duration. Each sticky set leaks with time and mixes into the more regular chaotic domain. On the one hand, in the distribution function, their sharp peaks are separated along the $u$ axis; on the other hand, in the
(\gamma', dY'/dt') section plane, those sticky sets overlap with each other. The number density within each sticky set decreases rapidly away from the surface \( R' = \text{constant} \), with the dimensionless radius \( R' = \sqrt{Y'^2 + (dY'/dt')^2} \) related to the gyration adiabatic invariant. Therefore, to count the dots belonging only to the sticky sets, we define the maximum and minimum value of \( R' \) by \( R_{\text{out}}' \) and \( R_{\text{in}}' \), respectively for each peak. We consider three different annular domains in phase space, one for each peak, using the radius \( R' \). In the \((\gamma', dY'/dt')\) section plane, \( Pk_1 \) is associated with the annulus with inner radius \( R_{\text{in}}' = 9.4 \) and outer radius \( R_{\text{out}}' = 13.5 \); similarly, \( Pk_2 \) with \( R_{\text{in}}' = 12.23 \) and \( R_{\text{out}}' = 16.76 \), and \( Pk_3 \) with \( R_{\text{in}}' = 15.72 \) and \( R_{\text{out}}' = 20.0 \). From the data set associated with each peak, we identify those points which satisfy \( R_{\text{in}}' \leq k_y \sqrt{Y'^2 + \dot{Y}'^2/\omega^2} \leq R_{\text{out}}' \).

We perform this counting for all the \( n \) values to obtain \( W_{pk}(n) \cong MN \dot{w}_{pk} \), we plot \( \log_{10}(W_{pk}) \) vs. \( \log_{10}(\omega_{pe}T_s) \), and read the exponent \( \gamma_{pk} \) from the slope according to Eq. (28).

Fig. 18 presents these results. The time evolutions of \( Pk_1, Pk_2 \) and \( Pk_3 \) are presented by blue, red and black dot respectively. Among all three peaks, \( Pk_1 \) is the strongest peak in the distribution. Initially, the exponents in Eq. (28) for all three peaks have very small values \(|1 - \gamma_{pk}| \ll 1\), namely \( \gamma_{pk1} = 1.23 \), \( \gamma_{pk2} = 1.17 \) and \( \gamma_{pk3} = 1.06 \). For \( \log_{10}(\omega_{pe}T_s) = 5.3 \), the exponent value for the strongest peak \( Pk_1 \) increases to \( \gamma_{pk1} = 2.11 \). The cross-over of the second strongest peak \( Pk_2 \) occurs for \( \log_{10}(\omega_{pe}T_s) = 4.2 \), when the exponent value changes to \( \gamma_{pk2} = 1.64 \), which still implies \(|1 - \gamma_{pk2}| < 1\). For the weakest peak \( Pk_3 \), the exponent changes to \( \gamma_{pk3} = 2.15 \) when \( \log_{10}(\omega_{pe}T_s) = 5.7 \). Since the exponent value for the strongest peak changes when \( \log_{10}(\omega_{pe}T_s) = 5.3 \), the strength of this peak starts to decrease (leak) faster, which helps to increase the maximum \( \rho_{\text{max}} \) of the average distribution \( \rho_n(i) \) at a faster rate. Therefore, the value of the transport exponent \( \alpha \), from Eq. (24), increases after \( \log_{10}(\omega_{pe}T_s) = 5.3 \), which is also observed in Fig. 12.

Fig. 19 presents the three sticky regions for the sticking time \( T_s \) such that \( \log_{10}(\omega_{pe}T_s) = 5.63 \). Comparing with Fig. 15 (with \( \log_{10}(\omega_{pe}T_s) = 2.92 \)) extracted from the same \( MN \) time sequences, we see that the strength (number of dots) of the sticky set associated with the peak \( Pk_1 \) decreases by a very large amount and starts to become empty.

VII. CONCLUSIONS

In this paper, we discuss the transport due to electrostatic waves generated by the \( \mathbf{E} \times \mathbf{B} \) drift instability. The original time-dependent 3-degrees-of-freedom problem is reduced to a 2-degrees-of-freedom time-dependent model and a 2-degrees-of-freedom autonomous model. Due to the wave-particle interaction, the dynamics become chaotic, and trajectories form stochastic web structures with different shape for different parameters, which we investigated for both the time-dependent and time-independent descriptions.

Along with each web structure, there occur sticky is-
lands where the trajectory spends more time compared to the purely chaotic domain, which affects the diffusion rate [23]. We use a scaling exponent for characterising the particle transport, and find that the transport is anomalous, of super-diffusive type. The presence of sticky islands generates sharp peaks in the distribution of arc-length (a phase-space observable) which increases the effective weight of the tail in the distribution. Considering the Poincaré recurrence theorem and Kac’ lemma for the sticky sets, we estimate a power law decay for the probability that a trajectory would stick to an island. With increasing duration of the time-evolution, sticky sets start to become empty and they decay with a higher exponent value. This change in the exponent \( \gamma \) values also affects the transport-coefficient exponent \( \alpha \) values.

In real Hall thrusters, the \( \mathbf{E} \times \mathbf{B} \) instability generates many unstable modes, with different frequencies and wavevectors. In this case, even for small amplitude waves, the dynamics cannot be reduced to a time-independent 2-degrees-of-freedom model. However, each wave will typically bear its own dimensionless parameters \((\varepsilon_i, \beta_i, k_{zi}/k_{yi})\), with small values for \( \beta_i \) and \( k_{zi}/k_{yi} \). Therefore, the several-wave dynamics will exhibit resonance overlap between the structures generated by these individual waves, resulting in smaller islands (if any survive [13–22]) and more regular transport [23].

Beside the effect of several waves, three issues must also be considered. First, the thruster chamber is a cylinder, where the intensity of the radial magnetic field decreases for larger radius (here \( x \)) and the azimuthal coordinate (here \( y \)) is periodic. Second, electrons do not stay forever in the chamber, which implies that tools of transient chaos [23] will be relevant. Third, the electrons charge and current generate electromagnetic fields, so that the system needs a self-consistent many-body description.

Acknowledgements

This work is part of IFCPRA project 5204-3. We acknowledge the financial support from CE-FIPRA/IFCPRA. The Centre de Calcul Intensif d’Aix-Marseille is acknowledged for granting access to its high performance computing resources. We are grateful to Professors Dominique Escande and Abhijit Sen for many fruitful discussions and their comments.

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