ON THE QUERY COMPLEXITY OF ESTIMATING THE DISTANCE TO HEREDITARY GRAPH PROPERTIES

CARLOS HOPPEN, YOSHIHARU KOHAYAKAWA, RICHARD LANG, HANNO LEFMANN, AND HENRIQUE STAGNI

Abstract. Given a family of graphs \( F \), we prove that the normalized edit distance of any given graph \( \Gamma \) to being induced \( F \)-free is estimable with a query complexity that depends only on the bounds of the Friese–Kannan Regularity Lemma and on a Removal Lemma for \( F \).

1. INTRODUCTION AND MAIN RESULTS

Property testing is concerned with very fast (randomized) algorithms for approximate decisions, where the aim is to distinguish between graphs that satisfy a given property and graphs that are ‘far’ from satisfying this property. Research in this area has achieved tremendous success since its systematic study was initiated by Goldreich, Goldwasser and Ron [16].

In this paper, we consider randomized algorithms that have the ability to query whether any desired pair of vertices in the input graph is adjacent or not. Let \( \mathcal{G} \) be the set of finite simple graphs, and let \( \mathcal{G}(V) \) be the set of such graphs with vertex set \( V \). We shall consider subsets \( \mathcal{P} \) of \( \mathcal{G} \) that are closed under isomorphism, which are called graph properties. To avoid technicalities, we restrict ourselves to graph properties \( \mathcal{P} \) such that \( \mathcal{P} \cap \mathcal{G}(V) \neq \emptyset \) for \( V \neq \emptyset \). This includes all nontrivial monotone and hereditary graph properties, which are properties that are inherited by subgraphs and by induced subgraphs, respectively.

Here, we shall focus on hereditary properties. As it turns out, all hereditary graph properties are given by a set \( \text{Forb}(\mathcal{F}) \) containing all graphs that do not have an induced copy of an element of a fixed graph family \( \mathcal{F} \).

A graph property \( \mathcal{P} \) is said to be testable if, for every \( \varepsilon > 0 \), there exist a positive integer \( q_\mathcal{P} = q_\mathcal{P}(\varepsilon) \), called the query complexity, and a randomized algorithm \( T_\mathcal{P} \), called a tester, which may perform at most \( q_\mathcal{P} \) queries in the input graph, satisfying the following property. For an \( n \)-vertex input graph \( \Gamma \), the algorithm \( T_\mathcal{P} \) distinguishes with probability at least \( 2/3 \) between the cases in which \( \Gamma \) satisfies \( \mathcal{P} \) and in which no graph obtained from \( \Gamma \) by the addition or removal of at most \( \varepsilon n^2 \) edges satisfies \( \mathcal{P} \). Alon and Shapira [5] proved that every hereditary graph property is testable. Moreover, in joint work with Fischer and Newman [2], they found a combinatorial characterization of testable graph properties. Recently, such a characterization has also been obtained for uniform hypergraphs by Joos, Kim, Kühn and Osthus [20].
Property testing may be stated in terms of graph distances: given two graphs \( \Gamma \) and \( \Gamma' \) on the same vertex set \( V(\Gamma) = V(\Gamma') \), we define the normalized edit distance between \( \Gamma \) and \( \Gamma' \) by
\[
d_1(\Gamma, \Gamma') = \frac{|E(\Gamma) \triangle E(\Gamma')|}{|V|^2},
\]
where \( E(\Gamma) \triangle E(\Gamma') \) denotes the symmetric difference of their edge sets. If \( \mathcal{P} \) is a graph property, let the distance between a graph \( \Gamma \) and \( \mathcal{P} \) be
\[
d_1(\Gamma, \mathcal{P}) = \min\{d_1(\Gamma, \Gamma') : V(\Gamma') = V(\Gamma) \text{ and } \Gamma' \in \mathcal{P}\}.
\]
Thus a graph property \( \mathcal{P} \) is testable if there is a tester with bounded query complexity that distinguishes with probability at least \( 2/3 \) between the cases \( d_1(\Gamma, \mathcal{P}) = 0 \) and \( d_1(\Gamma, \mathcal{P}) > \varepsilon \).

Similarly, a function \( z : \mathcal{G} \to \mathbb{R} \) is called a graph parameter if it is invariant under relabeling of vertices. A graph parameter \( z : \mathcal{G} \to \mathbb{R} \) is estimable (or testable) if, for every \( \varepsilon > 0 \) and every large enough graph \( \Gamma \), with probability at least \( 2/3 \), the value of \( z(\Gamma) \) can be approximated up to an additive error of \( \varepsilon \) by an algorithm that only has access to a subgraph of \( \Gamma \) induced by a set of vertices of size \( s_z = s_z(\varepsilon) \), chosen uniformly at random. The query complexity of such an algorithm is \( \left( s_z \right)^2 \) and the size \( s_z \) is called its sample complexity. Estimable parameters have been introduced by Fischer and Newman [10]. Borgs, Chayes, Lovász, Sós and Vesztergombi [7] later gave a complete characterization of estimable graph parameters which, in particular, also implies that the distance from hereditary graph properties is estimable. However, their approach does not provide explicit bounds on the sample complexity. With a different strategy, Fischer and Newman [10] proved that the distance to every testable property is estimable, providing Wowzer-type bounds for the sample complexity. This strategy has been improved by Alon, Shapira and Sudakov [6], but the query complexity still depends on constants given by strong versions of the Regularity Lemma.

Note that, by definition, a hereditary property \( \text{Forb}(\mathcal{F}) \) being testable is equivalent to the existence of an induced Removal Lemma for \( \mathcal{F} \). Our results rely on the following version, which was proved by Alon and Shapira [5]. For graphs \( F \) and \( \Gamma \), let \( \text{hom}(F, \Gamma) \) be the probability that a random mapping \( \varphi : V(F) \to V(\Gamma) \) is an induced homomorphism, i.e., a function preserving adjacency and non-adjacency between \( F \) and \( \Gamma \).

**Lemma 1.1 (Induced Removal Lemma).** For every \( \varepsilon > 0 \) and every (possibly infinite) family \( \mathcal{F} \) of graphs, there exist \( M = M(\varepsilon, \mathcal{F}) \), \( \delta = \delta(\varepsilon, \mathcal{F}) > 0 \) and \( n_0 = n_0(\varepsilon, \mathcal{F}) \) such that the following holds. If a graph \( \Gamma \) on \( n \geq n_0 \) vertices satisfies \( d_1(\Gamma, \text{Forb}(\mathcal{F})) \geq \varepsilon \), then there is a graph \( F \in \mathcal{F} \) with \( |V(F)| \leq M \) such that \( \text{hom}(F, \Gamma) \geq \delta \).

The first induced Removal Lemma was proved using a strong version of the Regularity Lemma and therefore had upper bounds on \( 1/\delta \) and \( n_0 \) of size \text{Wowzer}(\text{poly}(1/\varepsilon)) [1]. The best known upper bound on the induced Removal Lemma is due to Conlon and Fox [8], but is still of tower type. Alon and Shapira [11] characterized all graphs \( F \), with the possible exception of \( F \in \{P_4, C_4\} \), such that the induced Removal Lemma for \( \mathcal{F} = \{F\} \) holds for \( \delta(\varepsilon, F) = \text{poly}(\varepsilon) \). The case \( F = P_4 \) was shown to satisfy this property by Alon and Fox [3], while Gishboliner and Shapira [13] made progress in the case \( F = C_4 \). The question of deciding which hereditary properties admit a Removal Lemma that may be proven without the use of the Regularity Lemma was raised by Goldreich [14], and by Alon and Fox [3], among others, and is currently under research.

*This function is one level higher in the Wainer hierarchy than the tower function.*
The goal of this work is to relate hereditary parameter estimation directly to the bounds of Removal Lemmas by avoiding Szemerédi’s Regularity Lemma. In [17, 18], the current authors proved a similar result for monotone properties. To this end, the concept of recoverable graph properties was introduced. Roughly speaking, for a function $f : [0, 1] \rightarrow \mathbb{N}$, a graph property $P$ is $f$-recoverable if every large graph $G \in P$ is $\varepsilon$-close to admitting a partition $\mathcal{V}$ of its vertex set into at most $f(\varepsilon)$ classes that witnesses membership in $P$ (i.e., such that any graph that can be partitioned in the same way must be in $P$). It was shown that every monotone graph property $\text{Forb}_{\text{mon}}(\mathcal{F})$ is $f$-recoverable for some function $f$ that depends only on the bounds of a ‘weighted’ graph Removal Lemma for the family $\mathcal{F}$. This improved the required sample complexity for estimating $d_1(\cdot, \text{Forb}_{\text{mon}}(\mathcal{F}))$ for families $\mathcal{F}$ that admit Removal Lemmas with better bounds. Owing to recent improvements by Fox [11] on the bounds for the Removal Lemma, this resulted in a better upper bound for distance estimation to monotone properties.

Our main result here is an analogue of this for hereditary properties. For every graph property $P$, let $d_P$ be the graph parameter defined by $d_P(\Gamma) = d_1(\Gamma, P)$ for graphs $\Gamma$, which determines the distance between a graph and $P$.

**Theorem 1.2.** Let $P = \text{Forb}(\mathcal{F})$ be any hereditary property. Then the graph parameter $d_P$ is estimable with sample complexity $s_{\text{\textit{1.2}}} = \exp\{\text{poly}(\delta^{-M^2}, M, \log n_0)\}$, where $M$, $\delta$ and $n_0$ are as in Lemma [11] with input $\varepsilon/3$ and $P$.

Theorem 1.2 provides an upper bound on the sample complexity of estimating the distance to a hereditary property $\text{Forb}(\mathcal{F})$, which solely depends on the upper bounds for the associated Removal Lemma. In particular, for families $\mathcal{F}$ that admit a Removal Lemma with sample complexity polynomial in $1/\varepsilon$, our result states that the distance to $\text{Forb}(\mathcal{F})$ can be estimated with a sample complexity that is exponential in a polynomial in $1/\varepsilon$. Such families are currently actively sought after. Recent findings include the family consisting of a path on three edges [3], finite families containing a bipartite, a co-bipartite and a split graph [14] and the family of induced cycles of length at least four [9] (for $\mathcal{F}$ as in the last example, $\text{Forb}(\mathcal{F})$ is the set of chordal graphs). This is a substantial improvement over previous approaches, like [10] and [6], which rely on Szemerédi’s Regularity Lemma and therefore provide bounds which are at least a tower of height polynomial in $1/\varepsilon$.

As briefly mentioned before, the approach in our previous work [17, 18] is based on a removal lemma for weighted graphs. Since we wished to arrive at a result involving classical (unweighted) removal lemmas, it was necessary to relate our weighted removal lemma with the classical removal lemma. In this paper, so that we can use the classical (induced) removal lemma and its bounds directly, we take an alternative approach: instead of recoverable properties, we consider the notion of ‘attestable’ properties. Roughly speaking, a graph property $P$ is $f$-attestable if every large graph $G \in P$ is $\varepsilon$-close to admitting a partition $\mathcal{V}$ of its vertex set into at most $f(\varepsilon)$ classes that witnesses closeness to $P$ (i.e., such that any graph that can be partitioned in the same way must be close to $P$). Recall that, in contrast, recoverable refers to membership in $P$. The proof of Theorem 1.2 consists of two steps. First we prove that $f$-attestable properties are estimable with sample complexity polynomial in $f$ (see Theorem 2.4). Then we show that

---

1As for hereditary properties, it is well-known that every monotone graph property $P$ is the set $\text{Forb}_{\text{mon}}(\mathcal{F})$ of all graphs that do not contain a copy (not necessarily induced) of a graph in a family $\mathcal{F}$.  

3
hereditary properties are $f$-attestable, where $f$ is exponential in the bound given by Lemma 1.1 (see Theorem 3.3).

2. Attestable properties

We denote the set of all complete weighted graphs (with loops and edge weights between 0 and 1) by $\mathcal{G}^*$. Let $\mathcal{G}^*(V)$ be the set of all weighted graphs on vertex set $V$. The distance between two weighted graphs $R, R' \in \mathcal{G}^*(V)$ is given by

$$d_1(R, R') = \frac{1}{|V|^2} \sum_{(i,j) \in V^2} |R(i,j) - R'(i,j)|.$$ 

As with graphs, a weighted graph property $\mathcal{P}^*$ is a subset of $\mathcal{G}^*$ that is closed under (weight-preserving) isomorphisms. For a weighted graph $R \in \mathcal{G}^*(V)$ and a weighted graph property $\mathcal{P}^*$, let

$$d_1(R, \mathcal{P}^*) = \min\{d_1(R, R') : R' \in \mathcal{G}^*(V) \cap \mathcal{P}^*\}.$$ 

An equipartition of a graph $\Gamma$ is a partition $\mathcal{V} = \{V_i\}_{i=1}^k$ of its vertex set $\Gamma(V)$, such that $|V_i| \leq |V_j| + 1$ for all $1 \leq i, j \leq k$. Since we will only consider equipartitions $\{V_i\}_{i=1}^k$ of graphs of size much larger than $k$, we will ignore divisibility issues and assume that every class has exactly $n/k$ vertices.

Given an equipartition $\mathcal{V} = \{V_i\}_{i=1}^k$ of a graph $\Gamma$, we write $\Gamma(V_i, V_j)$ for the number of edges $v_i v_j$ with $v_i \in V_i$ and $v_j \in V_j$. The reduced graph $\Gamma/\mathcal{V} \in \mathcal{G}^*$ of $\Gamma$ by $\mathcal{V}$ is a weighted graph with vertex set $[k] = \{1, \ldots, k\}$ and edge weights

$$\Gamma/\mathcal{V} (i, j) = \frac{\Gamma(V_i, V_j)}{|V_i||V_j|}$$

for all $1 \leq i, j \leq k$. As we will see, the reduced graph $\Gamma/\mathcal{V}$ can provide some information about $\Gamma$ if the number of classes of $\mathcal{V}$ is large enough (but still small with respect to the order of $\Gamma$), especially if $\mathcal{V}$ is a regular partition (in the sense of Frieze-Kannan). Given a set $\mathcal{V}$ and an integer $K \leq |\mathcal{V}|$ we denote the set of all equipartitions of $\mathcal{V}$ into at most $K$ classes by $\Pi_K(\mathcal{V})$. We also define the set

$$\Gamma/\Pi_K = \{\Gamma/\mathcal{V} : \mathcal{V} \in \Pi_K(\Gamma(V))\}$$

of all reduced graphs of $\Gamma$ with vertex size at most $K$.

The next theorem is a slight modification of Theorem 3.2 [18]. It states that if a graph parameter $z : \mathcal{G} \to \mathbb{R}$ can be expressed as the optimal value of a certain optimization problem over $\Gamma/\Pi_K$, then $z$ is estimable with sample complexity which is polynomial in $K$ and in the reciprocal of the error parameter.

Theorem 2.1 (Theorem 3.2 [18]). There are positive constants $a$ and $b$ satisfying the following. Let $z : \mathcal{G} \to \mathbb{R}$ be a graph parameter and suppose that there are a weighted graph parameter $z^* : \mathcal{G}^* \to \mathbb{R}$, an integer $K \geq 1$ and a constant $c \geq 1$ such that

1. $z(\Gamma) = \max_{R \in \Gamma/\Pi_K} z^*(R)$ for every $\Gamma \in \mathcal{G}$, and
2. $|z^*(R) - z^*(R')| \leq c \cdot d_1(R, R')$ for all weighted graphs $R, R' \in \mathcal{G}^*$ on the same vertex set.

Then $z$ is estimable with sample complexity $\varepsilon_{2.1}(\varepsilon) = K^a(2c/\varepsilon)^b$. 

4
The proof of Theorem 3.2 in [18] shows that the constants a and b in the statement do not depend on the graph parameters \( z \) and \( z^* \), but rather on a result about sampling. Since we will use Theorem 2.1 as a black box, let us briefly sketch its proof. Given a graph \( \Gamma \), let \( V \) be a partition of \( V(\Gamma) \) into at most \( K \) classes for which \( z(\Gamma) = z^*(\Gamma/V) \). Next, let \( \overline{\Gamma} \) denote a subgraph on \( q \) vertices chosen uniformly at random. A recent result of Shapira and the fifth author [23] guarantees that there are constants \( a \) and \( b \) such that if \( q \geq K^a(2\varepsilon/c)^b \), then with high probability there is a partition \( \overline{V} \) of \( V(\overline{\Gamma}) \) into at most \( K \) classes such that \( d_1(\Gamma/V, \overline{\Gamma}/\overline{V}) < \varepsilon/c \). It then follows quickly that \( z(\overline{\Gamma}) \leq z(\Gamma) + \varepsilon \). A symmetric argument yields \( z(\overline{\Gamma}) \leq z(\Gamma) + \varepsilon \), as desired.

Now, let \( P \) be a hereditary graph property and suppose that we want to estimate the parameter \( d_P(\Gamma) \) for a graph \( \Gamma \). Our aim is to show that \( d_P(\Gamma) \) can be approximated by an optimization parameter over the set \( \Gamma/\Pi_K \), for some positive integer \( K \). This will allow us to apply Theorem 2.1 to get sample complexity bounds for estimating \( d_P \).

For every weighted graph \( R \in \mathcal{G}^* \), we define the property of being reducible to a weighted graph that is very close to \( R \) by

\[
\mathcal{G}_R = \{ \Gamma \in \mathcal{G} : \text{there is an equipartition } V \text{ of } \Gamma \text{ for which } d_1(\Gamma/V, R) \leq 2/|V| \}. \tag{1}
\]

We could define \( \mathcal{G}_R \) by requiring that \( d_1(\Gamma/V, R) \) should be 0. However, this definition would not be ‘robust’. For instance, \( \mathcal{G}_R \) would be empty whenever \( R \) contains an edge of irrational weight. In fact, even if \( R \) is a reduced graph coming from a concrete graph \( G \), with this more restrictive definition, it could be that \( \mathcal{G}_R \) fails to contain graphs of arbitrarily large orders, because of trivial divisibility issues. The definition in (1) avoids such anomalies.

For a graph property \( P \) and for every \( \varepsilon > 0 \), we define

\[
\mathcal{P}^*_\varepsilon = \{ R \in \mathcal{G}^* : d_P(\Gamma) \leq \varepsilon \text{ for all } \Gamma \in \mathcal{G}_R \}.
\]

In other words, \( \mathcal{P}^*_\varepsilon \) is the set of all reduced graphs \( R \) that attest \( \varepsilon \)-closeness to \( P \), in the sense that if a graph \( \Gamma \) admits a reduced graph close to \( R \), then \( \Gamma \) must be \( \varepsilon \)-close to \( P \). This motivates the following definition.

**Definition 2.2.** Given a function \( f : (0,1] \to \mathbb{N} \), we say that a graph property \( P \) is \( f \)-attestable if, for any \( \varepsilon > 0 \) and any graph \( \Gamma \in \mathcal{P} \) with \( |V(\Gamma)| \geq f(\varepsilon)^{3/2} \), there exists a reduced graph \( R \in \Gamma/\Pi_{f(\varepsilon)} \) of \( \Gamma \) for which \( R \in \mathcal{P}^*_\varepsilon \).

It is possible to connect attestable properties with parameter distances. For an integer \( K > 0 \) and \( \varepsilon > 0 \), we define the graph parameter \( d_p^{(K,\varepsilon)} : \mathcal{G} \to [0,1] \) such that

\[
d_p^{(K,\varepsilon)}(\Gamma) = \min_{R \in \Gamma/\Pi_K} d_1(\Gamma, \mathcal{P}^*_\varepsilon).
\]

So if \( P \) is \( f \)-attestable, then by definition \( d_p^{(K,\varepsilon)}(\Gamma) = 0 \) for \( K = f(\varepsilon) \) and all graphs \( \Gamma \in \mathcal{P} \) with \( |V(\Gamma)| \geq K^{3/2} \). The next lemma shows that \( d_p^{(K,\varepsilon)} \) is our desired optimization parameter.

**Lemma 2.3.** Let \( P \) be an \( f \)-attestable graph property for a function \( f : (0,1] \to \mathbb{R} \). Fix \( \varepsilon > 0 \) and let \( K = f(\varepsilon) \). Then every graph \( \Gamma \in \mathcal{G}(V) \) with \( |V(\Gamma)| \geq K^{3/2} \) satisfies

\[
|d_P(\Gamma) - d_p^{(K,\varepsilon)}(\Gamma)| \leq \varepsilon.
\]

Before we prove Lemma 2.3 let us see how it implies the main result of this section:

**Theorem 2.4.** Let \( P \) be an \( f \)-attestable graph property, for a function \( f : (0,1] \to \mathbb{R} \). Then, the graph parameter \( d_P \) is estimable with sample complexity

\[
s_p(\varepsilon) = \text{poly}(f(\varepsilon/2), 1/\varepsilon).
\]
Proof. Fix $\varepsilon > 0$ and let $K = f(\varepsilon/2)$. Consider a graph $\Gamma$ on at least $K^{3/2}$ vertices. By Lemma 2.3, we have
\[ |d_p(\Gamma) - d_p^{(K, \varepsilon/2)}(\Gamma)| \leq \frac{\varepsilon}{2}. \]
Define $z^*(R) = d_1(R, \mathcal{P}^*_\varepsilon)$ and note that $d_p^{(K, \varepsilon/2)}(\Gamma) = \min_{R \in \Gamma/\Pi_K} z^*(R)$. Moreover, $|z^*(R) - z^*(R')| \leq 1 \cdot d_1(R, R')$ for all $R, R' \in \mathcal{G}^*$. Therefore, we may apply Theorem 2.4 to conclude that $d_p^{(K, \varepsilon/2)}(\Gamma)$ may be approximated within error $\varepsilon/2$ with probability at least $2/3$ by randomly choosing a subgraph of $\Gamma$ of size $R^*/\varepsilon^2$.

Proof of Lemma 2.4. Fix $0 < \varepsilon < 1$, $K = f(\varepsilon)$. Let $V = [n]$ with $n \geq K^{3/2}$. We first show that $d_p^{(K, \varepsilon)}(\Gamma) \leq d_p(\Gamma)$. Let $G \in \mathcal{P}$ be a graph such that $d_1(\Gamma, G) = d_p(\Gamma)$. Since $\mathcal{P}$ is $f$-attestable, we can fix an equipartition $V = \{V_i\}_{i=1}^k$, with $k \leq K$, for which $G/\mathcal{V} \in \mathcal{P}^*_\varepsilon$. In particular, we have
\[ d_p^{(K, \varepsilon)}(\Gamma) \leq d_1(\Gamma/\mathcal{V}, G/\mathcal{V}) = \frac{1}{k^2} \sum_{(i,j) \in [k]^2} \frac{||\Gamma(V_i, V_j) - G(V_i, V_j)||}{|V_i||V_j|} \leq \frac{1}{n^2} \sum_{(i,j) \in [k]^2} \sum_{u \in V_i, v \in V_j} |\Gamma(u, v) - G(u, v)| = d_1(\Gamma, G) = d_p(\Gamma). \]

Next, we proceed to show that $d_p(\Gamma) \leq d_p^{(K, \varepsilon)}(\Gamma) + \varepsilon$. Let $R \in \Gamma/\Pi_K$ and $S \in \mathcal{P}^*_\varepsilon$ be such that $d_1(R, S) = d_p^{(K, \varepsilon)}(\Gamma)$. Let $k = |V(R)|$ and fix an equipartition $\mathcal{V} = \{V_i\}_{i=1}^k$ of $\Gamma$ such that $R = \Gamma/\mathcal{V}$. Let us construct a graph $G \in \mathcal{G}_S$ by modifying $\Gamma$ as follows. For each $1 \leq i < j \leq k$ such that $R(i, j) > S(i, j)$, we remove exactly $||(R(i, j) - S(i, j)||V_i||V_j||$ edges from $\Gamma$ between $V_i$ and $V_j$; if $S(i, j) > R(i, j)$, we add exactly $||(S(i, j) - R(i, j)||V_i||V_j||$ edges between $V_i$ and $V_j$ to $\Gamma$. Indeed we have $G \in \mathcal{G}_S$ as
\[ d_1(G/\mathcal{V}, S) = \frac{1}{k^2} \sum_{(i,j) \in [k]^2} |G/\mathcal{V}(i,j) - S(i,j)| \leq \frac{1}{k^2} \left( k + \sum_{(i,j): i \neq j; R(i,j) > S(i,j)} \frac{|\Gamma(V_i, V_j)| - |(R(i,j) - S(i,j)||V_i||V_j||}{|V_i||V_j|} - S(i,j) \right) + \frac{1}{k^2} \left( k + \sum_{(i,j): i \neq j; R(i,j) < S(i,j)} \frac{|\Gamma(V_i, V_j)| + |(S(i,j) - R(i,j)||V_i||V_j||}{|V_i||V_j|} - S(i,j) \right) \leq \frac{1}{k^2} \left( k + \sum_{(i,j) \in [k]^2} \frac{1}{|V_i||V_j|} \right) = \frac{1}{k} + \frac{k^2}{n^2} \leq \frac{2}{k}. \]

Moreover,
\[ d_1(\Gamma, G) \leq \frac{1}{n^2} \sum_{(i,j) \in [k]^2} |R(i,j) - S(i,j)||V_i||V_j| = \frac{1}{k^2} \sum_{(i,j) \in [k]^2} |R(i,j) - S(i,j)| = d_1(R, S). \]
Since $S \in \mathcal{P}^*$, it follows that $d_1(G, \mathcal{P}) \leq \varepsilon$. Hence, by the triangle inequality, $d_\varphi(\Gamma) \leq d_1(\Gamma, G) + \varepsilon \leq d_\varphi^{(K, \varepsilon)}(\Gamma) + \varepsilon$, as required. \hfill \Box

3. Hereditary properties are attestable

Let $\varphi : V(F) \to V(R)$ be a function from the vertex set of a graph $F \in \mathcal{G}$ to the vertex set of a weighted graph $R \in \mathcal{G}^*$. The homomorphism weight is defined as

$$\text{hom}_\varphi(F, R) = \prod_{(i, j) \in E(F)} R(\varphi(i), \varphi(j)) \prod_{(i, j) \notin E(F)} \left(1 - R(\varphi(i), \varphi(j))\right).$$

We can interpret $\text{hom}_\varphi(F, R)$ as the probability that $\varphi$ is an induced homomorphism from $F$ to $H$, where $H \in \mathcal{G}(V(R))$ is a graph in which each edge $ij \in \left(V(R)\right)$ is independently present with probability $R(i, j)$. The homomorphism density $\text{hom}(F, R)$ of $F \in \mathcal{G}$ in $R \in \mathcal{G}^*$ is defined as the average homomorphism weight over all mappings $\varphi : V(F) \to V(R)$. Note that if $\Gamma$ is a graph, then $\text{hom}(F, \Gamma)$ is the probability that a random mapping $\varphi : V(F) \to V(\Gamma)$ is an induced homomorphism from $F$ to $\Gamma$. In particular, if $\Gamma \in \text{Forb}(\{F\})$, then

$$\text{hom}(F, \Gamma) \leq \left(\frac{|V(F)|}{2}\right) \cdot \frac{1}{|V(\Gamma)|}, \quad (2)$$

since a random mapping from $V(F)$ to $V(\Gamma)$ is not injective with probability at most $\left(\frac{|V(F)|}{2}\right)/|V(\Gamma)|$.

The next result shows that if $\text{hom}(F, \Gamma)$ is bounded away from zero, then so is $\text{hom}(F, \Gamma/\mathcal{V})$.

**Lemma 3.1.** Let $F$ and $\Gamma$ be graphs and $f = |V(F)|$. Then $\text{hom}(F, \Gamma/\mathcal{V}) \geq \text{hom}(F, \Gamma)^f$ holds for every equipartition $\mathcal{V} = \{V_i\}_{i=1}^f$ of $\Gamma$.

**Proof of Lemma 3.1.** Suppose that $\mathcal{V} = \{V_i\}_{i=1}^f$ is an equipartition of $\Gamma$. Let $\Phi = V(\Gamma)^V(F)$ be the set of all functions from $V(F)$ to $V(\Gamma)$. For a mapping $\alpha : V(F) \to [k]$ we set

$$\Phi_\alpha = \{\varphi \in \Phi : \varphi(u) \in V_\alpha(u) \text{ for all } u \in V(F)\}.$$

Let $\varphi \in \Phi$ be chosen uniformly at random. For all mappings $\alpha : V(F) \to [k]$ and edges $uv \in E(F)$, we have

$$\mathbb{P}(\text{hom}_\varphi(F, \Gamma) = 1 \mid \varphi \in \Phi_\alpha) \leq \mathbb{P}(\Gamma(\varphi(u), \varphi(v)) = 1 \mid \varphi \in \Phi_\alpha) = \Gamma/\mathcal{V}(\alpha(u), \alpha(v)),$$

since $\Gamma/\mathcal{V}(\varphi(u), \varphi(v))$ is the probability that $\Gamma(x, y) = 1$ when $x \in V_\alpha(u)$ and $y \in V_\alpha(v)$ are chosen uniformly (and independently) at random. Analogously, we also have

$$\mathbb{P}(\text{hom}_\varphi(F, \Gamma) = 1 \mid \varphi \in \Phi_\alpha) \leq 1 - \Gamma/\mathcal{V}(\alpha(u), \alpha(v))$$

for all mappings $\alpha : V(F) \to [k]$ and all non-edges $uv \notin E(F)$. We can apply the last two inequalities to bound the homomorphism density $\text{hom}(F, \Gamma/\mathcal{V})$ from below as follows

$$\text{hom}(F, \Gamma/\mathcal{V}) = \sum_{\alpha : V(F) \to [k]} \frac{1}{k^f} \cdot \left(\prod_{uv \in E(F)} \frac{\Gamma/\mathcal{V}(\alpha(u), \alpha(v))}{1 - \Gamma/\mathcal{V}(\alpha(u), \alpha(v))}\right)$$

$$= \sum_{\alpha : V(F) \to [k]} \mathbb{P}(\varphi \in \Phi_\alpha) \cdot \left(\prod_{uv \in E(F)} \frac{\Gamma/\mathcal{V}(\alpha(u), \alpha(v))}{1 - \Gamma/\mathcal{V}(\alpha(u), \alpha(v))}\right)$$

$$\geq \sum_{\alpha : V(F) \to [k]} \mathbb{P}(\varphi \in \Phi_\alpha) \cdot \mathbb{P}(\text{hom}_\varphi(F, \Gamma) = 1 \mid \varphi \in \Phi_\alpha)^f.$$
Since \( x \mapsto x^{(\frac{1}{2})} \) is convex for every \( x \geq 0 \), we get that

\[
\text{hom}(F, \Gamma / \nu) \geq \left( \sum_{\alpha : V(F) \to [k]} \mathbb{P}(\varphi \in \Phi_\alpha) \cdot \mathbb{P} (\text{hom}_\varphi (F, \Gamma) = 1 \mid \varphi \in \Phi_\alpha) \right)^{\left( \frac{1}{2} \right)} \geq \text{hom}(F, \Gamma)^{f^2},
\]

as desired.

Note that the converse of Lemma 3.1 does not hold in general. For instance, the complete bipartite graph \( \Gamma = K_{n,n} \) satisfies \( \text{hom}(K_3, K_{n,n}) = 0 \), but \( \text{hom}(K_3, K_{n,n}/\nu) \) is close to 1/2 if \( \nu \) is a random equipartition of large size. However, \( \text{hom}(F, \Gamma) \) and \( \text{hom}(F, \Gamma / \nu) \) are known to be close, provided \( \nu \) is a Frieze-Kannan-regular partition. To make this precise we need to set up some notation. We define the cut-distance between two weighted graphs \( R_1, R_2 \in G^*(V) \) to be

\[
d_{\square}(R_1, R_2) = \frac{1}{|V|^2} \max_{\alpha, \beta} \left| \sum_{x \in V} \alpha(x) \cdot (R_1(x, y) - R_2(x, y)) \cdot \beta(y) \right|
\]

where the maximum is over all functions \( \alpha, \beta : V \to [0, 1] \). Note that by taking \( \alpha(v) = \beta(v) = 1 \) for all \( v \in V \) we obtain \( d_{\square}(R_1, R_2) \leq d_1(R_1, R_2) \). Moreover, we can bound the homomorphism density of a graph \( F \) in \( R_1 \) and \( R_2 \) in terms of the cut-distance:

**Lemma 3.2.** Let \( R_1, R_2 \in G^*(V) \) be weighted graphs and \( F \) a graph on \( f \) vertices. Then

\[ | \text{hom}(F, R_1) - \text{hom}(F, R_2) | \leq f^2 d_{\square}(R_1, R_2) \]

In the proof of Lemma 3.2 we will use the following fact.

**Fact 3.3.** Let \( a_1, \ldots, a_t \) and \( b_1, \ldots, b_t \) be real numbers. Then

\[
\prod_{i=1}^t a_i - \prod_{i=1}^t b_i = \sum_{j=1}^t \left( \prod_{i=1}^j a_i \cdot (a_{j+1} - b_j) \cdot \prod_{i=j+1}^t b_i \right).
\]

**Proof.** Observe that

\[
\sum_{j=1}^t \left( \prod_{i=1}^j a_i \cdot (a_{j+1} - b_j) \cdot \prod_{i=j+1}^t b_i \right) = \sum_{j=1}^t \left( \prod_{i=1}^j a_i \cdot \prod_{i=j+1}^t b_i \right) - \sum_{j=1}^t \left( \prod_{i=1}^j a_i \cdot \prod_{i=j+1}^t b_i \right) = \prod_{i=1}^t a_i - \prod_{i=1}^t b_i,
\]

which is the desired result. \( \square \)

**Proof of Lemma 3.2.** Let \( R_1, R_2 \in G^*(V) \) be weighted graphs with \( n = |V| \) and \( F \in \mathcal{G} \) be a graph with \( V(F) = [f] \). For every pair \( uv \in \binom{[f]}{2} \), any vertices \( x, y \in V \) and \( i \in \{1, 2\} \), define

\[
g_i^{(u,v)}(x, y) = \begin{cases} R_i(x, y) & \text{if } uv \in E(F) \\ 1 - R_i(x, y) & \text{if } uv \notin E(F). \end{cases}
\]

We have

\[
n^f (\text{hom}(F, R_1) - \text{hom}(F, R_2)) = \sum_{(x_1, \ldots, x_f) \in \binom{[f]}{2}} \left( \prod_{uv \in \binom{[f]}{2}} g_i^{(u,v)}(x_u, x_v) - \prod_{uv \in \binom{[f]}{2}} g_2^{(u,v)}(x_u, x_v) \right),
\]

\( ^{\dagger} \)This is equivalent to the definition of cut-distance in [21 Theorem 8.10].
where the sum is over all \( n^f \) sequences of length \( f \) of vertices of \( R_1 \). By considering an arbitrary linear ordering \( < \) of the elements \( uv \in \binom{[f]}{2} \), we apply Fact 3.3 to get

\[
n^f (\hom(F, R_1) - \hom(F, R_2)) = \sum_{(x_1, \ldots, x_f) \in \binom{[f]}{2}} \sum_{uv \in \binom{[f]}{2}} \left( \prod_{ab < uv} g_1^{(a,b)}(x_a, x_b) \cdot (g_1^{(u,v)}(x_u, x_v) - g_2^{(u,v)}(x_u, x_v)) \cdot \prod_{ab > uv} g_2^{(a,b)}(x_a, x_b) \right)
\]

\[
= \sum_{uv \in \binom{[f]}{2}} \sum_{\bar{x} \in \mathcal{X}_{uv}} \left( \prod_{ab < uv} g_1^{(a,b)}(x_a, x_b) \cdot (g_1^{(u,v)}(x_u, x_v) - g_2^{(u,v)}(x_u, x_v)) \cdot \prod_{ab > uv} g_2^{(a,b)}(x_a, x_b) \right),
\]

where the sum \( \sum_{\bar{x}} \) is over all sequences \( \bar{x} = (x_{uv})_{uv \in \binom{[f]}{2} \setminus \{u,v\}} \) of \( f - 2 \) vertices of \( R_1 \), indexed by vertices \( w \in [f] \), with \( w \neq u, v \).

Fix \( uv \in \binom{[f]}{2} \) and a sequence \( \bar{x} \) as above. Then there must be functions \( \alpha^{\bar{x}}, \beta^{\bar{x}} : V \to [0, 1] \) for which we can write

\[
\prod_{ab < uv} g_1^{(a,b)}(x_a, x_b) \cdot \prod_{ab > uv} g_2^{(a,b)}(x_a, x_b) = \alpha^{\bar{x}}(x_u) \cdot \beta^{\bar{x}}(x_v),
\]

since no term on the left side of the equation depends on both \( x_u \) and \( x_v \). Hence,

\[
n^f \left| \hom(F, R_1) - \hom(F, R_2) \right| \leq \sum_{uv \in \binom{[f]}{2}} \sum_{\bar{x} \in \mathcal{X}_{uv}} \left( \alpha^{\bar{x}}(x_u)(g_1^{(u,v)}(x_u, x_v) - g_2^{(u,v)}(x_u, x_v)) \beta^{\bar{x}}(x_v) \right)
\]

\[
\leq \sum_{uv \in E(F)} \sum_{\bar{x} \in \mathcal{X}_{uv}} \left( \alpha^{\bar{x}}(x_u)(R_1(x_u, x_v) - R_2(x_u, x_v)) \beta^{\bar{x}}(x_v) \right)
\]

\[
+ \sum_{uv \in \bar{E}(F)} \sum_{\bar{x} \in \mathcal{X}_{uv}} \left( \alpha^{\bar{x}}(x_u)(R_2(x_u, x_v) - R_1(x_u, x_v)) \beta^{\bar{x}}(x_v) \right).
\]

By the definition of the cut-distance, the absolute value of each of the sums over \( x_u, x_v \) can be bounded by \( d_{\square}(R_1, R_2)n^2 \). Therefore

\[
\left| \hom(F, R_1) - \hom(F, R_2) \right| \leq \frac{1}{n^f} \left( |E(F)|n^{f-2} d_{\square}(R_1, R_2)n^2 + (f^2 - |E(F)|)n^{f-2} d_{\square}(R_1, R_2)n^2 \right)
\]

\[
= f^2 \frac{d_{\square}(R_1, R_2)}{n},
\]

as required. \(\square\)

For an equipartition \( \mathcal{V} = \{V_i\}_{i=1}^k \) of a graph \( \Gamma \in \mathcal{G}(V) \), we define the \textit{blown-up reduced graph} \( \Gamma_{\mathcal{V}} \in \mathcal{G}^*(V) \) by setting for every \( 1 \leq i \leq j \leq k \), and vertices \( u \in V_i \) and \( v \in V_j \)

\[
\Gamma_{\mathcal{V}}(u, v) = \begin{cases} \frac{\Gamma(V_i, V_j)}{|V_i||V_j|} & \text{if } i \neq j \\ 0 & \text{otherwise.} \end{cases}
\]

The equipartition \( \mathcal{V} \) is \( \gamma \)-FK-regular, if \( d_{\square}(\Gamma, \Gamma_{\mathcal{V}}) \leq \gamma \). The Frieze–Kannan Regularity Lemma asserts that every sufficiently large graph admits an FK-regular partition.

**Lemma 3.4** (Frieze–Kannan Regularity Lemma [12]). For every \( \gamma > 0 \) and every \( k_0 > 0 \), there is \( K = k_0 \cdot 2^{\text{poly}(1/\gamma)} \) such that every graph \( \Gamma \) on \( n \geq K \) vertices admits a \( \gamma \)-FK-regular equipartition into \( k \) classes, where \( k_0 \leq k \leq K \).
Note that Lemma 3.4 is close to best possible, since Conlon and Fox found graph instances where the number of classes in any \( \gamma \)-FK-regular partition is at least \( \delta M \geq 2^{\frac{1}{2\log\gamma}} \) (for a previous result, see Lovász and Szegedy [22]). Now we are ready to show that hereditary graph properties are attestable.

**Theorem 3.5.** For every family \( \mathcal{F} \) of graphs, the property \( \text{Forb}(\mathcal{F}) \) is \( f \)-attestable for \( f(\varepsilon) = 2^{\text{poly}(\delta^{-M^2}, M, \log n_0)} \), where \( \delta \), \( M \) and \( n_0 \) are as in Lemma 1.1 with input \( \mathcal{F} \) and \( \varepsilon \).

**Proof.** Let \( \delta \), \( M \) and \( n_0 \) be as in Lemma 1.1 with inputs \( \mathcal{F} \) and \( \varepsilon \). Let \( K \) be as in Lemma 3.4 with input

\[
\begin{align*}
    k_0 &= \max \left\{ n_0, \frac{2M^2}{\delta M^2} \right\} \quad \text{and} \quad 
    \gamma = \frac{\delta M^2}{2M^2}.
\end{align*}
\]

Note that \( K = 2^{\text{poly}(\delta^{-M^2}, M, \log n_0)} \). Let \( G \in \text{Forb}(\mathcal{F}) \) be a graph with \( n \geq K \) vertices. We claim that if \( \mathcal{V} \) is a \( \gamma \)-FK-regular equipartition of \( G \) into \( k_0 \leq k \leq K \) classes, then \( R := G/\mathcal{V} \in \mathcal{P}(\varepsilon) \). This will prove the theorem for \( f(\varepsilon) = K \).

Suppose by contradiction that there is a graph \( H \in \mathcal{G}_R \) such that \( d_1(H, \text{Forb}(\mathcal{F})) > \varepsilon \). Since \( |V(H)| \geq k_0 \geq n_0 \), Lemma 1.1 asserts there must be a graph \( F \in \mathcal{F} \), with \( |V(F)| \leq M \), for which \( \text{hom}(F, H) \geq \delta \). As \( H \in \mathcal{G}_R \), there is a partition \( \mathcal{V}' \) of \( H \) into \( k \) classes for which \( d_1(H/\mathcal{V}', R) \leq 2/k \). It follows from Lemma 3.1 that \( \text{hom}(F, H/\mathcal{V}) \geq \delta M^2 \). Hence, by Lemma 3.2

\[
\text{hom}(F, R) \geq \delta M^2 - M^2 d_1(H/\mathcal{V}', R) \geq \delta M^2 - M^2 d_1(H/\mathcal{V}', R) \geq \delta M^2 - \frac{2M^2}{k} \geq \frac{\delta M^2}{2}.
\]

On the other hand, \( R \) is the reduced graph of \( G \) with respect to a \( \gamma \)-FK-regular partition. So by the above, Lemma 3.2 implies that \( \text{hom}(F, G) \geq \delta M^2 - M^2 \gamma \geq \frac{\delta M^2}{2} \). But this contradicts (2), which asserts that \( \text{hom}(F, G) \) is at most \( \left( \frac{M^2}{2} \right) \frac{1}{n} \leq \frac{M^2}{4} \). \( \square \)

Note that Theorem 1.2 follows from Theorem 2.4 and 3.5.

**References**

[1] N. Alon, E. Fischer, M. Krivelevich, and M. Szegedy. Efficient testing of large graphs. *Combinatorica*, 20(4):451–476, 2000.

[2] N. Alon, E. Fischer, I. Newman, and A. Shapira. A combinatorial characterization of the testable graph properties: it’s all about regularity. *SIAM J. Comput.*, 39(1):143–167, 2009.

[3] N. Alon and J. Fox. Easily testable graph properties. *Combin. Probab. Comput.*, 24:646–657, 2015.

[4] N. Alon and A. Shapira. A characterization of easily testable induced subgraphs. *Combin. Probab. Comput.*, 15(6):791–805, 2006.

[5] N. Alon and A. Shapira. A characterization of the (natural) graph properties testable with one-sided error. *SIAM Journal on Computing*, 37(6):1703–1727, 2008.

[6] N. Alon, A. Shapira, and B. Sudakov. Additive approximation for edge-deletion problems. *Annals of Mathematics*, 170(1):371–411, 2009.

[7] C. Borgs, J. T. Chayes, L. Lovász, V. T. Sós, and K. Vesztergombi. Convergent sequences of dense graphs. I. Subgraph frequencies, metric properties and testing. *Adv. Math.*, 219(6):1801–1851, 2008.

[8] D. Conlon and J. Fox. Bounds for graph regularity and removal lemmas. *Geom. Funct. Anal.*, 22(5):1191–1256, 2012.

[9] R. de Joannis de Verclos. Chordal graphs are easily testable. [https://arxiv.org/abs/1902.06135v1](https://arxiv.org/abs/1902.06135v1)

[10] E. Fischer and I. Newman. Testing versus estimation of graph properties. *SIAM J. Comput.*, 37(2):482–501, 2007.

[11] J. Fox. A new proof of the graph removal lemma. *Ann. of Math. (2)*, 174(1):561–579, 2011.
[12] A. Frieze and R. Kannan. Quick approximation to matrices and applications. *Combinatorica*, 19(2):175–220, 1999.

[13] L. Gishboliner and A. Shapira. Efficient removal without efficient regularity. *Combinatorica*. To appear.

[14] L. Gishboliner and A. Shapira. Removal lemmas with polynomial bounds. In *STOC*, pages 510–522. ACM, 2017.

[15] O. Goldreich. *Contemplations on Testing Graph Properties*, pages 547–554. LNCS 6650, Springer Berlin Heidelberg, Berlin, Heidelberg, 2011.

[16] O. Goldreich, S. Goldwasser, and D. Ron. Property testing and its connection to learning and approximation. *J. ACM*, 45(4):653–750, July 1998.

[17] C. Hoppen, Y. Kohayakawa, R. Lang, H. Lefmann, and H. Stagni. Estimating parameters associated with monotone properties. In *APPROX/RANDOM 2016: Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques*, volume 60 of LIPIcs. Leibniz Int. Proc. Inform., pages 35:1–35:13. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2016.

[18] C. Hoppen, Y. Kohayakawa, R. Lang, H. Lefmann, and H. Stagni. Estimating parameters associated with monotone properties. 2017. [https://arxiv.org/abs/1707.08225](https://arxiv.org/abs/1707.08225), submitted, pp. 19.

[19] C. Hoppen, Y. Kohayakawa, R. Lang, H. Lefmann, and H. Stagni. Estimating the distance to a hereditary graph property. *Electronic Notes in Discrete Mathematics*, 61:607 – 613, 2017. Proceedings of the European Conference on Combinatorics, Graph Theory and Applications (EUROCOMB ’17).

[20] F. Joos, J. Kim, D. Kühn, and D. Osthus. A characterization of testable hypergraph properties. 2017. [https://arxiv.org/abs/1707.03303](https://arxiv.org/abs/1707.03303) submitted, pp. 82.

[21] L. Lovász. *Large networks and graph limits*, volume 60 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2012.

[22] L. Lovász and B. Szegedy. Szemerédi’s lemma for the analyst. *Geom. Funct. Anal.*, 17(1):252–270, 2007.

[23] A. Shapira and H. Stagni. Partitions of a sample and applications. in preparation, 2019.

**Instituto de Matemática, UFRGS, Avenida Bento Gonçalves, 9500, 91501-970 Porto Alegre, RS, Brazil (C. Hoppen)**

E-mail address: choppen@ufrgs.br

**Instituto de Matemática e Estatística, Universidade de São Paulo, Rua do Matão 1010, 05508-090 São Paulo, Brazil (Y. Kohayakawa and H. Stagni)**

E-mail address: {yoshi|stagni}@ime.usp.br

**Combinatorics and Optimization Department, University of Waterloo, 200 University Avenue West, Waterloo N2L 3G1, Canada (R. Lang)**

E-mail address: r7lang@uwaterloo.ca

**Fakultät für Informatik, Technische Universität Chemnitz, Strasse der Nationen 62, 09111 Chemnitz, Germany (H. Lefmann)**

E-mail address: lefmann@informatik.tu-chemnitz.de