Stabilized convex symplectic manifolds are Weinstein

Yakov Eliashberg, Noboru Ogawa, and Toru Yoshiyasu

Abstract We show that a stabilized convex symplectic (or Liouville) manifold with the homotopy type of a half-dimensional CW-complex is symplectomorphic to a flexible Weinstein manifold.

1. Introduction

1.1. Convex symplectic manifolds
Recall that a primitive $\lambda$ of a symplectic form $\omega$, $d\lambda = \omega$, is called a Liouville form, and the vector field $Z$ which is $\omega$-dual to $\lambda$, $\iota(Z)\omega = \lambda$, is called a Liouville vector field. The equation $\iota(Z)\omega = \lambda$ is equivalent to the equation $L_Z\omega = \lambda$, where $L_Z$ is the Lie derivative. That is, Liouville vector fields are conformally symplectic.

An open symplectic manifold $(V, \omega)$ with an exact symplectic form $\omega$ is called symplectically convex (see [4]), or Liouville, if there exists a Liouville form $\lambda$ such that the corresponding Liouville vector field $Z$ is complete and there exists an exhaustion $\bigcup_{j=1}^{\infty} V_j$, $V_j \subset V_{j+1}$, by compact domains $V_j$ with smooth boundaries $\partial V_j$ such that $Z$ is outward transverse to $\partial V_j$. The domains $V_j$ with this property are called Liouville domains.

Given a Liouville domain $(V_1, \lambda)$, the attractor $\text{Core}(V_1, \lambda) := \bigcap_{t>0} Z^{-t}(V_1)$ of the field $-Z$ is called the core of the Liouville domain. For a convex symplectic manifold $V = \bigcup_{j=1}^{\infty} V_j$ with a fixed Liouville form $\lambda$, we define its core as $\text{Core}(V, \lambda) := \bigcup_{j=1}^{\infty} \text{Core}(V_j, \lambda)$. Equivalently, we can define $\text{Core}(V, \lambda)$ as

$$\text{Core}(V, \lambda) = \bigcup_{K \subset V, \text{compact}} \bigcap_{t>0} Z^{-t}(K),$$

and this definition shows the independence of $\text{Core}(V, \lambda)$ from the choice of exhausting Liouville domains. Of course, the core does depend on the choice of the Liouville form $\lambda$. 

*Kyoto Journal of Mathematics, Vol. 61, No. 2 (2021), 323–337
DOI 10.1215/21562261-2021-0004, © 2021 by Kyoto University
Received March 26, 2020. Accepted August 18, 2020.
First published online March 29, 2021.
2020 Mathematics Subject Classification: Primary 53D35; Secondary 57R17.*
An important class of convex symplectic manifolds is formed by convex symplectic manifolds of finite type, or as they are also called convex symplectic manifolds with cylindrical ends. One says that \((V, \omega)\) is a convex symplectic manifold of finite type if it admits a Liouville form \(\lambda\) with a Liouville vector field \(Z\) and a compact Liouville subdomain \(V_1 \subset V\), that is, a domain with boundary \(\partial V_1\) transverse to \(Z\), such that each point of \(V \setminus V_1\) belongs to a \(Z\) trajectory originating from a point of \(\partial V_1\). The manifold \((V, \omega)\) can be identified with the completion of the Liouville domain \(V_1\), that is, attaching to \(V_1\) the cylindrical end \(([0, \infty) \times \partial V_1, d(e^s(\lambda|_{\partial V_1})))\), where \(s\) is the coordinate corresponding to the factor \([0, \infty)\).

Note that the definition of a convex symplectic manifold of finite type fits into the definition of a general convex symplectic manifold by taking the translates \(V_n := Z^{n-1}(V_1)\), \(n \geq 1\), as the required exhausting sequence. For a finite type \((V, \lambda)\), its core is compact: Core\((V, \lambda) = \text{Core}(V_1, \lambda)\), and conversely finite-type convex symplectic manifolds can be characterized among convex symplectic manifolds as those which have a compact core for some choice of the Liouville form.

By fixing a cylindrical end structure, a contact structure is induced on the ideal boundary \(\partial_{\infty} V \cong \partial V_1\). However, this contact boundary is not determined by the symplectomorphism type of \(V\). In fact, as was shown by Sylvain Courte in [2], even the diffeomorphism type of \(\partial_{\infty} V\) can depend on the choice of the cylindrical end structure on a given convex symplectic manifold of finite type.

### 1.2. Weinstein manifolds

We say that a convex symplectic manifold \(V\) is of Weinstein type, or simply Weinstein, if the corresponding Liouville vector field \(Z\) can be chosen to admit a Lyapunov function \(\phi: V \to \mathbb{R}\) which is Morse or generalized Morse (i.e., possibly with death-birth singularities). The Lyapunov condition means that \(|d\phi(Z)| \geq c\|Z\|^2\) for a positive function \(c > 0\) and some choice of a Riemannian metric on \(V\). We note that the Lyapunov function \(\phi\) can always be modified to be exhausting (i.e., proper and bounded below) and constant on boundaries \(\partial V_j\) of domains \(V_j\) implied by the definition of symplectic convexity. The core of a Weinstein manifold \((V, \lambda)\) is stratified by \(Z\)-stable manifolds of zeros of \(Z\), which are isotropic (see [4], [1]). Hence, the critical points of a Lyapunov Morse function for a Liouville field have index at most \(n = \frac{1}{2}\dim V\), and therefore, any Weinstein manifold admits an exhausting Morse function with critical points of index at most \(n\); that is, it has Morse type at most \(n\) and, in particular, is homotopy equivalent to an \(n\)-dimensional CW-complex.

Not every convex symplectic manifold is Weinstein. Indeed, it may have Morse type greater than \(n\). The first example of this type, a 4-dimensional convex symplectic manifold of Morse type 3, was constructed by Dusa McDuff in [12]. More examples were constructed in [8], [13], and [10].

The product \((V, \omega) \times (V', \omega')\) of two symplectically convex manifolds is symplectically convex, and the product of two Weinstein manifolds is Weinstein. If
Stabilized convex symplectic manifolds are Weinstein

(V', ω') = (R^{2k}, ω_{st}), then the product (V, ω) × (R^{2k}, ω_{st}) is called the stabilization, or k-stabilization, of (V, ω).

1.3. Main results
We prove the following theorem.

THEOREM 1.1
Let V be a (2n − 2)-dimensional convex symplectic manifold of Morse type at most n. Then its 1-stabilization X is Weinstein, and moreover, if n ≥ 3, then it is flexible Weinstein (see Section 2 below for the definition and discussion of flexibility). In particular, the 1-stabilization of McDuff’s example in [12] is Weinstein.

REMARK 1.2
(1) It was proved in [4] that, for any two tangentially homotopy equivalent convex symplectic manifolds, their 2-stabilizations are symplectomorphic. Moreover, it was shown that for Weinstein manifolds 1-stabilization is sufficient. This implies that a 2-stabilization of a (2n − 2)-dimensional convex symplectic manifold of Morse type at most n is Weinstein. The improvement in this paper became possible thanks to the development of the theory of flexible Weinstein manifolds (see [1]).

(2) Even if V is of finite type, we do not know whether the ideal contact boundary of its stabilization is isomorphic to the ideal contact boundary of the corresponding flexible Weinstein manifold.

Theorem 1.1 is a corollary of a more general theorem which we formulate below.

Given a manifold V and a closed subset A ⊂ V, we say that each point of A has an access to infinity if every compact subset B ⊂ A has an arbitrarily small open neighborhood U ∋ B such that each connected component C of V \ U is not compact.

For instance, if V is a noncompact connected manifold and A ⊂ V is a closed (as a subset) submanifold of V of codimension greater than 1, then each point of A has an access to infinity. For codimension 1 connected submanifolds, the condition is violated only for compact submanifolds homological to 0.

If V is a symplectic manifold and A ⊂ V is a locally closed subset, then we say that A admits a symplectic extension of positive codimension if for each point a ∈ A there exist a neighborhood U_a ∋ a in V and a closed (as a subset) symplectic submanifold Σ_a ⊂ U_a of positive codimension such that U_a ∩ A ⊂ Σ_a, and each point of U_a ∩ A has an access to infinity in Σ_a.

THEOREM 1.3
Let (X, ω) be a 2n-dimensional, n ≥ 3, convex symplectic manifold of Morse type at most n. Suppose that, for an appropriate choice of a Liouville form λ, its core C := Core(X, λ) can be presented as a finite or countable union C = ∪_{i≥1} C_i of
disjoint sets $C_i$ which admits a symplectic extension of positive codimension and such that $\bigcup_{i \leq j} C_i$ is compact for all $j \geq 1$. Then $X$ is symplectomorphic to a flexible Weinstein manifold.

The core of any stabilized convex symplectic manifold clearly admits a symplectic extension of positive codimension, and hence, Theorem 1.1 is a special case of Theorem 1.3.

Here is another corollary of Theorem 1.3. We say that a submanifold $A$ of a symplectic manifold $V$ is nowhere coisotropic if each tangent plane $T_x A \subset T_x V$ is not coisotropic, that is, $(T_x A)^{\perp \omega} \not\subset T_x A$.

**THEOREM 1.4**

Let $X$ be a $2n$-dimensional, $n \geq 3$, convex symplectic manifold of Morse type at most $n$. Suppose that, for an appropriate choice of a Liouville form $\lambda$, the core of $X$ admits a stratification $\text{Core}(X, \lambda) = \bigcup_{i \geq 1} S_i$ of codimension at least 3 such that each stratum is nowhere coisotropic. Then the convex symplectic manifold $X$ is Weinstein and, moreover, flexible Weinstein.

We use above the term stratification in a weak sense. A stratified closed subset $A \subset V$ is a closed set presented as a finite or countable union of locally closed submanifolds $A_i$, called strata, $A := \bigcup_{i \geq 1} A_i$ such that all unions $\bigcup_{i \leq j} A_i$ are compact.

**Proof of Theorem 1.4**

It is sufficient to prove that each stratum $S_i$ admits a symplectic extension of positive codimension. Note that any non-coisotropic subspace $A$ of codimension at least 3 in a symplectic vector space $(B, \omega)$ is contained in a symplectic subspace $C \subset B$ such that $\dim A < \dim C < \dim B$. Indeed, take a vector $v \in A^{\perp \omega} \setminus A$, and consider its $\omega$-orthogonal complement subspace $v^{\perp \omega} \subset B$. Note that $A \subset v^{\perp \omega}$.

Then any codimension 1 subspace $C \subset v^{\perp \omega}$ which is transverse to $v$ and contains $A$ is a codimension 2 symplectic subspace of $B$. Given $x \in S_i$, we therefore can find a $(2n-2)$-dimensional symplectic subspace $C_x$ so that $T_x S_i \subset C_x \subset T_x X$. Let us choose a complementary subspace $\theta_x \subset C_x$ such that $\theta_x \oplus T_x S_i = C_x$, and extend it continuously to a field $\theta$ of planes transverse to $S_i$ on a neighborhood $U_x$ of $x$ in $S_i$. If the neighborhood $U_x$ is small enough, then the space $C_y := \text{Span}(\theta_y, T_y S_i)$ is symplectic for each $y \in U_x$, and so is the codimension 2 symplectic hypersurface $\Sigma$ containing $U_x$ and tangent to the plane field $\theta$. □

**Organization**

In Sections 2 and 3, we review Weinstein flexibility (see [1], [7], [5]) and Gromov’s $h$-principle for exact symplectic embeddings of open symplectic manifolds (see [9], [6]), respectively. Section 4 is a remark on the existence of a Liouville homotopy and a symplectomorphism. With the help of the above tools, we first
prove Theorem 1.1 in Section 5 to make the main ideas more transparent, and then we prove Theorem 1.3 in Section 6.

2. Recollection of Weinstein flexibility

2.1. Loose Legendrian knots

We recall that an \((n - 1)\)-dimensional submanifold \(\Lambda\) of a \((2n - 1)\)-dimensional contact manifold \((M, \xi)\) is called Legendrian if it is tangent to \(\xi\). The contact plane field \(\xi\) carries a canonical conformal symplectic structure, and tangent planes to a Legendrian are Lagrangian subspaces of \(\xi\) for that conformal structure. A formal Legendrian submanifold \(\Lambda \rightarrow M\) is an \((n - 1)\)-dimensional smooth submanifold together with a homotopy of its tangent planes to a field of Lagrangian subspaces of \(\xi\).

In [14], Emmy Murphy introduced a class of so-called loose Legendrian submanifolds in contact manifolds of dimension at least 5 for which a formal Legendrian isotopy between Legendrian embeddings yields a genuine Legendrian isotopy. We define this class below. We begin with an operation of stabilization of a Legendrian submanifold which was first introduced in [3] (see also [14], [1]).

In \(\mathbb{R}^{2n-1}\), \(n \geq 3\), with the standard contact form \(\alpha = dz - \sum_{i=1}^{n-1} y_i \, dx_i\) consider a Legendrian submanifold \(\Lambda_0\) with the front \(F_0 = \{z^2 = x_1^3\}\), that is, \(\Lambda_0 = \{z^2 = x_1^3, 4y_1^2 = 9x_1, y_2 = \cdots = y_{n-1} = 0\}\). Let \(\mathbb{R}^{n-1}_+ = \{z = 0, y = 0\}\) be the \(x\)-coordinate subspace, and let \(\mathbb{R}^{n-1}_+ = \{x_1 > 0\} \cap \mathbb{R}^{n-1}\). Choose open domains \(U\) and \(U' \subset U\) with smooth boundaries, and let \(\theta: \mathbb{R}^{n-1}_+ \rightarrow \mathbb{R}\) be a function supported in \(U\) such that \(U' = \{x \in \mathbb{R}^{n-1}_+ | \theta(x) > 2x_1^{3/2}\}\). Let \(\Lambda_U \subset \mathbb{R}^{2n-1}\) be a Legendrian submanifold whose front is obtained from \(F_0\) by replacing the branch \(z = -x_1^{3/2}\) by the graph \(z = -x_1^{3/2} + \theta(x)\). The Legendrian submanifold \(\Lambda_U\) is called the \(U\)-stabilization of \(\Lambda_0\). Given any Legendrian submanifold \(\Lambda\) in a contact manifold \((M, \xi)\), one can find Darboux coordinates in an arbitrarily small neighborhood \(U\) of a point \(a \in \Lambda\) such that the pair \((U, \Lambda \cap U)\) is contactomorphic to \((\mathbb{R}^{2n-1}, \Lambda_0)\). Hence, the \(U\)-stabilization operation can be performed on \(\Lambda\) in a neighborhood \(U \ni a\). We will keep the notation \(\Lambda_U\) for the stabilized Legendrian. As was shown in [3], the Legendrians \(\Lambda\) and \(\Lambda_U\) are always smoothly isotopic, and if \(\chi(U) = 0\), then they are formally Legendrian isotopic.

A connected Legendrian submanifold \(\Lambda \subset M\) is called loose if it can be destabilized, that is, it is Legendrian isotopic to a stabilization of another Legendrian knot. A possibly disconnected Legendrian is called loose if each of its components is loose in the complement of the others.

THEOREM 2.1 ([14, Theorem 1.3])

For any contact manifold of dimension at least 5, the inclusion of the space of loose Legendrian embeddings into the space of formal Legendrian embeddings is a homotopy equivalence.
2.2. Flexible Weinstein manifolds

The notion of Weinstein flexibility, introduced in [1], is based on the theory of loose Legendrians. Let $(V, \lambda, \phi)$ be a $2n$-dimensional Weinstein manifold. Consider its partition into elementary cobordisms: $V = W_1 \cup \cdots \cup W_m \cup \cdots$, where $W_i = \phi^{-1}([c_{i-1}, c_i])$ for regular values $c_i$ of $\phi$ separating the critical values $a_i$ of $\phi$, that is, $c_0 < a_0 < c_1 < a_1 < c_2 < \cdots$. Each cobordism $W_i$ deformation retracts onto the union of the stable disks (with respect to the Liouville vector field $Z$) of its critical points and the stable disk of an index $k$ critical point of value $a_i$ intersects the level set $M_i = \phi^{-1}(c_i)$ in the $(k-1)$-dimensional isotropic attaching sphere for the contact structure $\{\lambda|_{M_i} = 0\}$.

The Weinstein structure $(V, \lambda, \phi)$ is called flexible if for each cobordism $W_i$ the attaching Legendrian spheres of critical points of index $n$ on the level $a_i$ form a loose Legendrian link in the contact level set $M_i$. In particular, subcritical Weinstein manifolds (i.e., those for which $\phi$ has no critical points of index $n$) are flexible.

The following $h$-principle-type result clarifies the term “flexible.”

THEOREM 2.2 ([1, Sections 13.1])

For Weinstein structures on a fixed manifold or domain $V$ of dimension $2n \geq 6$, the following statements hold.

1. (Existence) Given a nondegenerate 2-form $\eta$ and an exhausting Morse function $\phi: V \to \mathbb{R}$ without critical points of index greater than $n$, there exists a flexible Weinstein structure $(\lambda, \phi)$ (with the same function $\phi$) such that $\eta$ and $d\lambda$ are homotopic as nondegenerate 2-forms.

2. (Homotopy) Two flexible Weinstein structures $(\lambda_0, \phi_0)$ and $(\lambda_1, \phi_1)$ are Weinstein homotopic if and only if $d\lambda_0$ and $d\lambda_1$ are homotopic as nondegenerate 2-forms.

3. (Morse–Smale theory for Lyapunov functions) Given a flexible Weinstein structure $(\lambda, \phi)$ and any Morse function $\psi: V \to \mathbb{R}$ without critical points of index greater than $n$, there exists a Weinstein homotopy $(\lambda_t, \phi_t)$ with $(\lambda_0, \phi_0) = (\lambda, \phi)$ and $\phi_1 = \psi$.

The definition of flexibility naturally extends to Weinstein cobordisms.

It is important to point out that the flexibility property is not invariant under Weinstein homotopy (see [15]). When calling a symplectic manifold flexible Weinstein, we always mean the existence of a flexible Weinstein structure for the given symplectic form.

2.3. Symplectic embeddings of flexible Weinstein manifolds

For two symplectic manifolds $(W, \omega)$ and $(X, \eta)$, a formal symplectic embedding of $W$ into $X$ is a smooth embedding $f: W \to X$ together with a homotopy $\Phi_t: TW \to f^*TX$, $t \in [0, 1]$, of injective bundle homomorphisms such that $\Phi_0 = df$ and $\Phi_1^* \eta = \omega$. Any genuine symplectic embedding $f: X \to W$ can be considered formal by setting $\Phi_1 \equiv df$. 
A symplectic embedding \( f : (W, d\lambda) \to (X, d\mu) \) between two exact symplectic manifolds with fixed Liouville forms is called exact if \( f^* \mu = \lambda + dH \) for some function \( H \) on \( W \). Note that if \( W \) is compact, then given an exact symplectic isotopy \( f_t : (W, d\lambda) \to (X, d\mu) \) there exists an ambient Hamiltonian isotopy \( F_t : X \to X \) such that \( F_t|_{f_0(W)} = f_t, \ t \in [0, 1] \). Hence, we will refer in this paper to an exact symplectic isotopy as a Hamiltonian isotopy. //

**Theorem 2.3 ([7, Corollary 6.3])**

Let \((W, \lambda, \phi)\) be a \(2n\)-dimensional Weinstein domain, and let \(W_0\) be its Weinstein subdomain. Suppose that the Weinstein cobordism \((W \setminus \text{Int} W_0, \lambda, \phi)\) is flexible. Let \((X, d\mu)\) be a convex symplectic manifold of the same dimension \(2n\) such that the Liouville vector field \(Z\) dual to \(\mu\) is forward complete.

(1) Any formal symplectic embedding \( f : (W, d\lambda) \to (X, d\mu) \) which is a genuine exact symplectic embedding on \(W_0\) is formally isotopic relative to \(W_0\) (rel. \(W_0\)) to a genuine exact symplectic embedding.

(2) Any two exact symplectic embeddings \( f_0, f_1 : (W, d\lambda) \to (X, d\mu) \) which coincide on \(W_0\) and are formally isotopic rel. \(W_0\) can be connected by a Hamiltonian isotopy \( f_t : (W, d\lambda) \to (X, d\mu), \ t \in [0, 1], \) fixed on \(W_0\). The nonparametric part (1) is proved in [7]. The parametric part (2) will appear in [5].

### 3. \(h\)-Principle for symplectic embeddings

In this section, we review Gromov’s \(h\)-principle for symplectic embeddings of open symplectic manifolds (see [9]; see also [6]). We continue to use in this paper the term “symplectic” rather than isometric as in [9, Section 3.4.2(B)] or isosymplectic as in [6, Theorem 12.1.1].

Recall that given a manifold \(V\) and a closed subset \(A \subset V\) we say that each point of \(A\) has an access to infinity if every compact subset \(B \subset A\) has an arbitrarily small open neighborhood \(U \supset B\) such that each connected component \(C\) of \(V \setminus U\) is not compact. Slightly reformulating Gromov’s \(h\)-principle for symplectic embeddings from [9], we have the following theorem. We use below Gromov’s notation \(\mathcal{O}p A\) for an unspecified neighborhood of a closed subset \(A\).

**Theorem 3.1 ([9, Section 3.4.2(B)], see also [6, Theorem 12.1.1])**

Let \((W, \omega)\) and \((X, \eta)\) be symplectic manifolds of dimension \(2n\) and \(2m\), respectively. Suppose that \(X\) is an open manifold and that \(m < n\). Then the following statements hold.

(1) For a formal symplectic embedding \((\varphi, \Phi)\) of \((X, \eta)\) into \((W, \omega)\), there exists a symplectic embedding \(f : (X, \eta) \to (W, \omega)\) formally isotopic to \((\varphi, \Phi)\).

(2) Any symplectic embeddings \(f_0, f_1 : (X, \eta) \to (W, \omega)\) which are formally isotopic can be connected by a symplectic isotopy \(f_t : (X, \eta) \to (W, \omega), \ t \in [0, 1]\).
(3) Let \( A \subset X \) be a closed subset such that each of its points has an access to infinity, and let \((\varphi, \Phi_t)\) be a formal symplectic embedding of \((X, \eta)\) into \((W, \omega)\) which is a genuine symplectic embedding on \(\mathcal{O}_p A\). Then, there exists a symplectic embedding \( f: (X, \eta) \to (W, \omega) \) formally isotopic to \((\varphi, \Phi_t)\) rel. \( A \).

(4) Let \( A \) be as in (3). Then for any two symplectic embeddings \( f_0, f_1: X \to W \) which coincide on \(\mathcal{O}_p A\) and are formally isotopic rel. \( A \), there exists a symplectic isotopy \( f_t: (X, \eta) \to (W, \omega) \), \( t \in [0, 1] \), fixed on \(\mathcal{O}_p A\).

If symplectic forms \( \omega = d\lambda \) and \( \eta = d\mu \) are exact, then one can talk about exact symplectic embeddings \( f: X \to W \) which satisfy the condition \( f^* \lambda = \mu + dH \) (see Section 2.3 above).

**Proposition 3.2**

Let \((X, \eta = d\mu)\) be a convex symplectic manifold. Then for any symplectomorphism \( f_0: (X, \eta) \to (X, \eta) \), there exists a symplectic diffeotopy \( f_t: (X, \eta) \to (X, \eta) \) such that \( f_1 \) is exact, that is, \( f_1^* \mu = \mu + dH \) for a smooth function \( H: X \to \mathbb{R} \).

For the case when \((X, \eta)\) is a finite-type convex symplectic manifold, this was proven in [1, Lemma 11.2]. To prove the statement in the general case we will need the following two lemmas.

**Lemma 3.3**

Let \(((0, \infty) \times \Sigma, \omega = d(s\alpha))\) be the symplectization of a contact manifold \((\Sigma, \xi = \ker \alpha)\) with a fixed contact form \(\alpha\). Let \( R \) be the Reeb vector field of \(\alpha\) (i.e., \( \iota(R)d\alpha = 0 \) and \( \alpha(R) = 1 \)). For a closed 1-form \( \theta \) on \(\Sigma\), denote \( \hat{\theta} := \pi^* \theta \), and let \( Y \) be the symplectic vector field \(\omega\)-dual to \(\hat{\theta}\), that is, \( \iota(Y)\omega = \hat{\theta} \). Here \( \pi: (0, \infty) \times \Sigma \to \Sigma \) is the projection to the second factor. Then one has the equality \( ds(Y) = \theta(R) \).

**Proof**

Let us write \( Y = aR + b \frac{\partial}{\partial s} + Y_\xi \), where \( Y_\xi \in \xi \) and \( a, b: (0, \infty) \times \Sigma \to \mathbb{R} \). Then

\[
\theta(R) = (\iota(Y)(ds \wedge \alpha + s d\alpha))(R) = ds \wedge \alpha \left( aR + b \frac{\partial}{\partial s} + Y_\xi, R \right) + s d\alpha(Y, R) = b = ds(Y).
\]

**Lemma 3.4**

Let \((X, d\mu)\) be a convex symplectic manifold, and let \( Z \) be the Liouville field corresponding to \(\mu\). Then there exists an exhaustion \( X = \bigcup_{j=1}^{\infty} X_j \) by compact domains with smooth boundaries transverse to \( Z \) such that the following condition is satisfied: for any \( j \geq 1 \) one has

\[
Z^{ln(1+2T_j)}(\partial X_j) \subset \text{Int} \ X_{j+1}, \quad j \geq 1,
\]

where \( T_j := \max \{1, \max_{x \in \partial X_j} |\theta(R_j(x))|\} \) and \( R_j \) is the Reeb vector field of the contact form \(\mu|_{\partial X_j} \).
Proof
We begin with any exhaustion $\bigcup_{j=1}^{\infty} X_j^0$ by compact domains with smooth boundaries transverse to $Z$ and then inductively modify it to ensure the property (1) by using the following procedure. Set $X_1 := X_1^0$ and $T_1 := \max\{1, \max_{x \in \partial X_1} |\theta(R_1(x))|\}$, where $R_1$ is the Reeb vector field of the contact form $\mu|_{\partial X_1}$. Define $X_j^1 := Z^{\ln(1+3T_1)}(X_j^0)$, $j \geq 2$, and denote $X_2 := X_2^0$ and $T_2 := \max\{1, \max_{x \in \partial X_2} |\theta(R_2(x))|\}$, where $R_2$ is the Reeb vector field of the contact form $\mu|_{\partial X_2}$. Define $X_j^2 := Z^{\ln(1+3T_2)}(X_j^1)$, $j \geq 3$, and denote $X_3 := X_3^2$. Continuing this process we construct the required exhaustion.

Proof of Proposition 3.2
We have $f_0^*\mu = \mu - \theta$ for a closed 1-form $\theta$. Let $Z$ be the Liouville field corresponding to the Liouville form $\mu$. Choose an exhaustion $X = \bigcup_{j=1}^{\infty} X_j$ which satisfies property (1).

Consider disjoint domains

$$U_j := \bigcup_{t \in [0, \ln(1+2T_j)]} Z^t(\partial X_j) \subset X,$$

and set $\alpha_j := \mu|_{\partial X_j}$, $j \geq 1$. These domains can be identified with the domains $[1, 1+2T_j] \times \partial X_j$ in the symplectizations $((0, \infty) \times \partial X_j, d(s\alpha_j))$ of the contact manifolds $(\partial X_j, \ker \alpha_j)$ via Liouville isomorphisms

$$\phi_j : ([1, 1+2T_j] \times \partial X_j, s\alpha_j) \to (U_j, \mu) : (s, x) \mapsto Z^{\ln s}(x),$$

where $s \in [1, 1+2T_j]$ and $x \in \partial X_j$. Let $\theta_j$ be the closed 1-form on $U_j$ defined by the formula

$$\theta_j := (\phi_j)^*(\pi_j^*(\theta|_{\partial X_j})),$$

where $\pi_j : [1, 1+2T_j] \times \partial X_j \to \partial X_j$ is the projection to the second factor. Note that $\theta|_{U_j} - \theta_j = dH_j$ for a smooth function $H_j : U_j \to \mathbb{R}$. Let $\delta_j : U_j \to \mathbb{R}$ be a cutoff function supported in $U_j$ and equal to 1 on $\tilde{U}_j := \phi_j([1+T_j/2, 1+3T_j/2] \times \partial X_j)$. Then the closed 1-form

$$\tilde{\theta} = \theta - dG,$$

where $G = \left(\sum_{j=1}^{\infty} \delta_j H_j\right)$, coincides with $\theta_j$ on $\tilde{U}_j$ for all $j \geq 1$. We claim that the symplectic vector field $Y$ which is $\eta$-dual to $\tilde{\theta}$ is complete, that is, its flow $Y^t$ is defined for all time $t \in \mathbb{R}$. Indeed, according to Lemma 3.3 any trajectory entering $U_i$ spends time at least 1 there, and hence, in time at most $T$ it can cross only finitely many domains $U_j$. Note that $L_Y \tilde{\theta} = \iota(Y) d\tilde{\theta} + d(\iota(Y) \tilde{\theta}) = 0$ and $L_Y \mu = \iota(Y) \eta + d(\mu(Y)) = \tilde{\theta} + dH'$, where $H' := \mu(Y)$. Hence, by defining $H'_j := \int_0^t (H' \circ Y^s) \, ds$, we compute $(Y^t)^* \tilde{\theta} = \tilde{\theta}$ and $(Y^t)^* \mu = \mu + t\tilde{\theta} + dH'_t$. We can now define the required isotopy by the formula $f_t := f_0 \circ Y^t$. Then
\[ f_1^* \mu = (Y^1)^* f_0^* \mu = (Y^1)^* (\mu - \theta) = (Y^1)^* (\mu - \tilde{\theta} - dG) = \mu + \tilde{\theta} + dH_1' - \tilde{\theta} - d(G \circ Y^1) = \mu + d(H_1' - G \circ Y^1). \]

Hence, if in Theorem 3.1 we assume that \( \omega = d\lambda, \eta = d\mu, \) and \( (X, d\mu) \) is symplectically convex, then we can arrange the constructed symplectic embeddings in (1) and (3) to be exact and the symplectic isotopies in (2) and (4) to be Hamiltonian.

4. Liouville homotopy versus symplectomorphism

The following notion of Liouville homotopy, which formalizes the concept of a smooth family of convex symplectic structures on a given manifold, was introduced in [1]. A smooth family \( \mu_s, s \in [0,1], \) of Liouville forms on a manifold \( X \) is called a simple Liouville homotopy if there exists a smooth family of exhaustions \( X = \bigcup_{k=1}^{\infty} X_k^k \) by compact domains \( X_k^k \subset X \) with smooth boundaries along which the corresponding Liouville field \( Z_s \) is outward pointing. A Liouville homotopy is a composition of finitely many simple Liouville homotopies. It was shown in [1, Proposition 11.8] that given a Liouville homotopy \( \mu_s \) there exists an isotopy \( \phi_s : X \to X, s \in [0,1], \) starting from \( \phi_0 = \text{id}_X \) such that \( \phi_s^* \mu_s = \mu_0 + dH_s, \) and in particular, the forms \( \tilde{\mu}_s := \phi_s^* \mu_s \) are Liouville for the same symplectic structure \( \omega = d\mu_0. \) The following proposition shows that the converse is also true.

**Proposition 4.1**

Let \( (X, \omega) \) and \( (X', \omega') \) be two symplectomorphic convex symplectic manifolds. Then there exist a symplectomorphism \( \varphi : (X, \omega) \to (X', \omega') \) and a Liouville homotopy \( \mu_s \) connecting \( \mu_0 = \mu \) and \( \mu_1 = \varphi^* \mu'. \)

**Proof**

According to Proposition 3.2 the symplectomorphism \( \varphi \) can be chosen to satisfy \( \varphi^* \mu' = \mu + dH \) for a smooth function \( H \) on \( X. \) Choose the exhaustions \( X = \bigcup_{j=1}^{\infty} X_j^0 \) and \( X = \bigcup_{j=1}^{\infty} X_j^1 \) defining convex structures for forms \( \mu_0 := \mu \) and \( \mu_1 := \mu + dH, \) respectively. We can arrange that

\[ X_j^1 \subset \text{Int} X_j^0 \subset X_j^0 \subset \text{Int} X_{j+1}^1 \]

for all \( j \geq 1. \) Let \( \tilde{H} \) be a smooth function which is equal to 0 on \( O \bigcap_{j=1}^{\infty} \partial X_j^0 \) and equal to \( H \) on \( O \bigcap_{j=1}^{\infty} \partial X_j^1. \) Then the required Liouville homotopy can be defined as the composition of two simple Liouville homotopies:

\[ \mu_s := \begin{cases} 
\mu_0 + 2s d\tilde{H}, & s \in [0,1/2], \\
\mu_0 + d\tilde{H} + (2s - 1) d(H - \tilde{H}), & s \in [1/2,1],
\end{cases} \]

with the constant exhaustions \( X = \bigcup_{j=1}^{\infty} X_j^0 \) and \( X = \bigcup_{j=1}^{\infty} X_j^1, \) respectively. \( \square \)

The notion of Weinstein homotopy can be defined in a similar way. However, it is unknown whether two Weinstein structures on the same symplectic manifold are homotopic.
5. Proof of Theorem 1.1

If \( n = 2 \), then \( \dim V = 2 \). Any 2-dimensional convex symplectic manifold is Weinstein and the theorem is trivially true. Hence, we assume that \( n \geq 3 \).

Choose a Liouville form \( \mu_V \) on the convex symplectic manifold \( V \), and denote by \( \mu \) the corresponding stabilized Liouville form \( \mu_V + \frac{1}{2}(x dy - y dx) \) on \( X = V \times \mathbb{R}^2 \). Denote \( \eta := d\mu \). By the assumption, \( X \) is of Morse type at most \( n \). Take an exhausting Morse function \( \phi: X \to \mathbb{R} \) without critical points of index greater than \( n \). Applying Theorem 2.2(1) to the pair \((\eta,\phi)\), we obtain a flexible Weinstein structure \( \mathfrak{M} = (\omega = d\lambda, \phi) \) on \( X \) such that symplectic forms \( \eta \) and \( \omega \) are homotopic as nondegenerate 2-forms. For the sake of convenience, the ambient space of \( \mathfrak{M} \) is denoted by \( W \) instead of \( X \). Thus, there exists a pair \((\varphi, \Phi_s)\), where \( \varphi: X \to W \) is the identity and \( \Phi_s: TX \to TW \), \( s \in [0,1] \), is a homotopy of bundle isomorphisms covering \( \varphi \) starting at \( \Phi_0 = d\varphi \) and ending at a symplectic isomorphism \( \Phi_1 = \Phi: (TX, \eta) \to (TW, \omega) \).

The goal of this section is to construct an exact symplectomorphism \( F: (X,\mu) \to (W,\lambda) \). This will be given by the telescope construction, the so-called Mazur trick (see [11]), following the scheme of the proof in [4, Proposition 2.2.A]. Take exhaustions \( X_1 \subset X_2 \subset \cdots \subset X, \bigcup_{i=1}^{\infty} X_i = X \), by Liouville subdomains of \( X \) and \( W_1 \subset W_2 \subset \cdots \subset W, \bigcup_{i=1}^{\infty} W_i = W \), by Weinstein subdomains of \( W \).

Set \( \mu_i = \mu|_{X_i} \) and \( \lambda_i = \lambda|_{W_i} \) for \( i \geq 1 \). The construction is split into several steps.

**Lemma 5.1**

For each \( i \geq 1 \) there exists an exact symplectic embedding \( f_i: (X_i,\mu_i) \to (W,\lambda) \) which is formally isotopic to \((\varphi|_{X_i},\Phi|_{TX_i})\).

**Proof**

Since \( V \) is an open symplectic manifold of dimension \( 2n - 2 < \dim W \), we can apply Theorem 3.1(1) to \((\varphi|_V,\Phi_s|_{TV})\) and obtain an exact symplectic embedding \( V = V \times \{0\} \to W \). Moreover, it extends to an open neighborhood \( U \) of \( V \), and \( U \) and \( W \) are Weinstein subdomains of \( W \). Applying Theorem 2.2(1) to \((\eta,\mu)\), we obtain a flexible Weinstein structure \( \mathfrak{M} = (\omega = d\lambda, \phi) \) on \( X \) such that symplectic forms \( \eta \) and \( \omega \) are homotopic as nondegenerate 2-forms. For the sake of convenience, the ambient space of \( \mathfrak{M} \) is denoted by \( W \) instead of \( X \). Thus, there exists a pair \((\varphi, \Phi_s)\), where \( \varphi: X \to W \) is the identity and \( \Phi_s: TX \to TW \), \( s \in [0,1] \), is a homotopy of bundle isomorphisms covering \( \varphi \) starting at \( \Phi_0 = d\varphi \) and ending at a symplectic isomorphism \( \Phi_1 = \Phi: (TX, \eta) \to (TW, \omega) \).

The goal of this section is to construct an exact symplectomorphism \( F: (X,\mu) \to (W,\lambda) \). This will be given by the telescope construction, the so-called Mazur trick (see [11]), following the scheme of the proof in [4, Proposition 2.2.A]. Take exhaustions \( X_1 \subset X_2 \subset \cdots \subset X, \bigcup_{i=1}^{\infty} X_i = X \), by Liouville subdomains of \( X \) and \( W_1 \subset W_2 \subset \cdots \subset W, \bigcup_{i=1}^{\infty} W_i = W \), by Weinstein subdomains of \( W \).

Set \( \mu_i = \mu|_{X_i} \) and \( \lambda_i = \lambda|_{W_i} \) for \( i \geq 1 \). The construction is split into several steps.

**Lemma 5.1**

For each \( i \geq 1 \) there exists an exact symplectic embedding \( f_i: (X_i,\mu_i) \to (W,\lambda) \) which is formally isotopic to \((\varphi|_{X_i},\Phi|_{TX_i})\).

**Proof**

Since \( V \) is an open symplectic manifold of dimension \( 2n - 2 < \dim W \), we can apply Theorem 3.1(1) to \((\varphi|_V,\Phi_s|_{TV})\) and obtain an exact symplectic embedding \( V = V \times \{0\} \to W \). Moreover, it extends to an open neighborhood \( U \) of \( V \), and hence, we get an exact symplectic embedding \( h_U: (U,\mu|_U) \to (W,\lambda) \) which is formally isotopic to \((\varphi|_U,\Phi|_{TU})\).

We have \( \text{Core}(X_i,\mu_i) \subset V \subset U \), and therefore, there exists \( t_i > 0 \) such that \( Z^{-t_i}(X_i) \subset U \), where \( Z^t \) stands for the flow generated by the Liouville vector field of \( \mu \). Set \( h_i := h_U|_{Z^{-t_i}(X_i)} \). Using the flow \( Z^t \) of the Liouville field \( Z^t \) we construct an exact symplectic embedding \( f_i: (X_i,\mu_i) \to (W,\lambda) \) by the formula

\[
f_i = Z^{-t_i}_\lambda \circ h_i \circ Z^{-t_i}_\mu.
\]

Indeed,

\[
f_i^* \lambda = (Z^{-t_i}_\mu)^* \circ h_i^* \circ (Z^{-t_i}_\lambda)^*(\lambda) = (Z^{-t_i}_\mu)^* \circ h_i^*(e^{t_i}(\mu_i + dH)) = e^{t_i}(Z^{-t_i}_\mu)^*(\mu_i + dH)
\]

\[
= e^{t_i}(e^{-t_i}(\mu_i + d(H \circ Z^{-t_i}_\mu))) = \mu_i + d(e^{t_i}(H \circ Z^{-t_i}_\mu)).
\]

By the construction, \( f_i \) is formally isotopic to \((\varphi|_{X_i},\Phi|_{TX_i})\). \( \square \)
The next lemma is a special case of Theorem 2.3(1).

**LEMMA 5.2**
There exists an exact symplectic embedding $g_i : (W_i, \lambda_i) \to (X, \mu)$ which is formally isotopic to $(\varphi^{-1}|_{W_i}, \Phi^{-1}|_{TW_i})$.

There exist subfamilies $\{X_{ik}\}$ and $\{W_{jk}\}$ such that $f_{ik}(X_{ik}) \subset W_{jk}$, $\varphi(X_{ik}) \subset W_{jk}$, $g_{jk}(W_{jk}) \subset X_{ik+1}$, and $\varphi^{-1}(W_{jk}) \subset X_{ik+1}$. After renumbering, we have the following diagram:

$$
\begin{array}{c}
\begin{array}{ccc}
(X_1, \mu_1) & \overset{H_1}{\leftarrow} & (X_2, \mu_2) & \overset{H_2}{\leftarrow} & (X_3, \mu_3) & \ldots & \overset{H_\infty}{\leftarrow} & (X, \mu) \\
W_1 & & W_2 & & W_3 & & \ldots & & W
\end{array}
\end{array}
\end{array}
$$

where $\iota_{X_k}$ and $\iota_{W_k}$ are the inclusions.

**LEMMA 5.3**
The compositions $g_k \circ f_k : X_k \to X_{k+1}$ and $f_{k+1} \circ g_k : W_k \to W_{k+1}$ are Hamiltonian isotopic to the inclusions $\iota_{X_k} : X_k \to X_{k+1}$ and $\iota_{W_k} : W_k \to W_{k+1}$, respectively.

**Proof**
By Lemmas 5.1 and 5.2, $g_k \circ f_k$ is formally isotopic to $\iota_{X_k} = \varphi^{-1}|_{W_k} \circ \varphi|_{X_k}$. Applying Theorem 3.1(2) to this formal isotopy restricted on $V_k = V \cap X_k$, we get a Hamiltonian isotopy $h^s_k : V_k \to X_{k+1}$, $s \in [0, 1]$, such that $h^0_k = \iota_{X_k}|_{V_k}$ and $h^1_k = g_k \circ f_k|_{V_k}$. Arguing as in the proof of Lemma 5.1, we can define a Hamiltonian isotopy $\psi^s_k : X_k \to X_{k+1}$ between $\psi^0_k = \iota_{X_k}$ and $\psi^1_k = g_k \circ f_k$ by the formula

$$
\psi^s_k := Z^{t_k}_\mu \circ \tilde{h}^s_k \circ Z^{-t_k}_\mu.
$$

Here $\tilde{h}^s_k$ is an extension of $h^s_k$ to an open neighborhood $U_k$ of $V_k$ as in the proof of Lemma 5.1 and $t_k$ is a sufficiently large number so that $Z^{-t_k}_\mu (X_k) \subset U_k$. Similarly, we use Theorem 2.3(2) to construct a Hamiltonian isotopy connecting $f_{k+1} \circ g_k$ and $\iota_{W_k}$.

**Proof of Theorem 1.1**
We construct an exact symplectomorphism from $X$ to $W$ by induction over $k$. First, set $F_1 := f_1 : (X_1, \mu_1) \to (W_1, \lambda_1)$. Lemma 5.3 constructs for any $k \geq 1$ a Hamiltonian isotopy connecting the inclusion $\iota_{X_k} : X_k \to X_{k+1}$ with the composition $g_k \circ f_k$. Hence, cutting off this isotopy outside $X_k$ we get a Hamiltonian isotopy $G^s_k : (X, \mu) \to (X, \mu)$, $s \in [0, 1]$, such that $G^0_k = \text{id}_X$, $G^1_k|_{X_k} = g_k \circ f_k$, and $\text{supp}(G^s_k) \subset X_{k+1}$. Similarly, Lemma 5.3 allows us to construct a Hamiltonian isotopy $H^s_k : (W, \lambda) \to (W, \lambda)$, $s \in [0, 1]$, such that $H^0_k = \text{id}_W$, $H^1_k|_{W_k} = f_{k+1} \circ g_k$. 


and \( \text{supp}(H^k) \subset W_{k+1} \). Set \( G_k := G^k_1 \) and \( H_k := H^k_1 \). For \( k \geq 2 \), we define the exact symplectic embedding

\[
F_k := (H_{k-1} \circ \cdots \circ H_1)^{-1} \circ f_k \circ (G_{k-1} \circ \cdots \circ G_1)|_{X_k} : (X_k, \mu_k) \to (W_k, \lambda_k).
\]

Since \( \text{supp}(G_j) \subset X_{j+1} \) and \( H_{k-1}^{-1} \circ f_k \circ G_{k-1}|_{X_{k-1}} = f_{k-1} \), the restriction \( F_k|_{X_{k-1}} \) is equal to \( F_{k-1}, k \geq 2 \). Hence, we can define an exact symplectic embedding \( F : X \to W \) by setting \( F := F_k \) on \( X_k \) for \( k \geq 1 \). Let us show that \( F \) is surjective. Note that \( W_{k-1} \subset F_k(X_k) \) for \( k \geq 2 \). Indeed, we have

\[
H_{k-1} \circ H_{k-2} \circ \cdots \circ H_1(W_{k-1}) = H_{k-1}(W_{k-1})
\]

\[
= f_k(g_{k-1}(W_{k-1}))
\]

\[
\subset f_k(X_k)
\]

\[
= f_k(G_{k-1} \circ \cdots \circ G_1(X_k)),
\]

and thus, \( W_{k-1} \subset F_k(X_k) \). Hence,

\[
W = \bigcup_{k \geq 1} W_k = \bigcup_{k \geq 2} F_k(X_k) = F \left( \bigcup_{k \geq 2} X_k \right) = F(X).
\]

Therefore, \( F : (X, \mu) \to (W, \lambda) \) is an exact symplectomorphism. \( \square \)

### 6. Proof of Theorem 1.3

The proof follows the same scheme as the proof of Theorem 1.1. As in that proof let \( W \) be a flexible Weinstein manifold which is formally symplectomorphic to \( X \), let \( \{(X_i, \mu_i) \mid i \geq 1\} \) and \( \{(W_i, \lambda_i) \mid i \geq 1\} \) be exhaustions by Liouville and Weinstein subdomains of \( X \) and \( W \), respectively, and let \( (\varphi, \Phi_t) \) be the formal symplectic embeddings of \( (X, \mu) \) into \( (W, \lambda) \). We begin by proving Lemmas 5.1, 5.2, and 5.3 in the current context.

**Step 1:** Construction of exact symplectic embeddings \( f_i : (X_i, \mu_i) \to (W, \lambda), i \geq 1 \), in the formal isotopy class of \( (\varphi|_{X_i}, \Phi|_{T X_i}) \). Denote \( S_i := \bigcap_t Z_{\mu}^{-t}(X_i) \). The attractor \( S_i \) is compact, we have \( S_i \subset \text{Core}(X, \mu) \), and hence, there exists an integer \( N \) such that \( S_i \subset \bigcup_{j \leq N} C_j \).

Since the first subset \( C_1 \) is compact, there exists a finite open cover \( \{U_{p_1}, \ldots, U_{p_k}\} \) of \( C_1 \) such that each intersection \( U_{p_j} \cap C_1 \) admits a symplectic extension \( \Sigma_{p_j} \) of positive codimension. Choose also a cover \( \{U'_{p_1}, \ldots, U'_{p_k}\} \) such that \( U'_{p_j} \subset U_{p_j}, j = 1, \ldots, k_1 \). Applying Theorem 3.1(1) to \( (\varphi|_{\Sigma_{p_1}}, \Phi_t|_{\Sigma_{p_1}}) \), we obtain an exact symplectic embedding \( f^{(1)}_1 : \Sigma_{p_1} \to W \), and moreover, we can modify \( \Phi_t \) so that \( \Phi_1|_{T \Sigma_{p_1}} = df^{(1)}_1 \). The symplectic neighborhood theorem then allows us to extend \( f^{(1)}_1 \) to a neighborhood \( \Omega_1 \supset \Sigma_{p_1} \) in \( U_{p_1} \). We will keep the notation \( f^{(1)}_1 \) for this extension. Denote \( A_1 := C_1 \cap \bigcup_{p_1} U'_{p_1} \cap U_{p_2} \). Note that every point of \( A_1 \) has an access to infinity in \( \Sigma_{p_2} \). Hence, applying Theorem 3.1(3) we find a symplectic embedding \( f^{(1)}_2 : \Sigma_{p_2} \to W \) which coincides with \( f^{(1)}_1 \) on \( \mathcal{O} p A_1 \subset \Sigma_{p_2} \). We further modify \( \Phi_t \) so that \( \Phi_1|_{T \Sigma_{p_2}} = df^{(1)}_2 \) and then use the symplectic neighborhood theorem to extend the exact symplectic embedding
$f_2^{(1)}$ to a neighborhood $\Omega_2 \supset \Sigma_{p_2}$ in $U_{p_2}$ so that the extended embedding $f_2^{(1)}$ coincides with $f_1^{(1)}$ on $Op\, A_1$ in $U_{p_2}$. Continuing this process we construct an exact symplectic embedding $f^{(1)}: Op\, C_1 \to W$. Choosing a sufficiently small neighborhood $U_1 \supset C_1$, where $f^{(1)}$ is defined, we find a finite open cover $\{U_{p_2}^1, \ldots, U_{p_k}^2\}$ of the compact set $C_2 \setminus U_1$ such that each intersection $U_{p_j}^2 \cap C_2$ admits a symplectic extension $\Sigma_{p_j}$ of positive codimension. Repeating the above process inductively over elements of the cover $U_{p_j}^2$ of $C_2 \setminus U_1$ and then continuing a similar process for $C_3, \ldots, C_N$ we construct an exact symplectic embedding $h_i: Op\, (C_1 \cup \cdots \cup C_N) \to W$. By our construction $Z^-T(\mu_i)$ for a sufficiently large $T$ is contained in a neighborhood of $\bigcup_{j \leq N} C_j$ where $h_i$ is defined. Hence, the formula $f_i := Z^T \circ h_i \circ Z^-T$ defines the required exact symplectic embedding $f_i: X_i \to W$.

**Step 2:** Construction of exact symplectic embeddings $g_i: (W_i, \lambda_i) \to (X, \mu)$, $i \geq 1$, in the formal isotopy class of $(\varphi^{-1}|_{W_i}, \Phi^{-1}|_{T_{W_i}})$. This is a corollary of Theorem 2.3(1), as in the case of Theorem 1.1.

**Step 3:** Proof that the compositions $g_k \circ f_k: X_k \to X_{k+1}$ and $f_{k+1} \circ g_k: W_k \to W_{k+1}$ are Hamiltonian isotopic to the inclusions $\iota_{X_k}: X_k \to X_{k+1}$ and $\iota_{W_k}: W_k \to W_{k+1}$, respectively (after readjusting the indices as in the proof of Theorem 1.1). Steps 1 and 2 imply that $g_k \circ f_k$ is formally isotopic to $\iota_{X_k} = \varphi^{-1}|_{W_k} \circ \varphi|_{X_k}$. To construct a genuine Hamiltonian isotopy connecting $g_k \circ f_k$ and $\iota_{X_k}$ we repeat the proof in Step 1, but using instead Theorems 3.1(2) and 3.1(4). The existence of a Hamiltonian isotopy connecting $f_{k+1} \circ g_k$ and $\iota_{W_k}$ is even more straightforward using Theorem 2.3(2).

**Step 4.** With the analogues of Lemmas 5.1, 5.2, and 5.3 established, we construct the required exact symplectomorphism $f: X \to W$ using the telescope construction exactly as in the proof of Theorem 1.1.

\[\square\]

**Acknowledgments.** Our collaboration on this article started during Kenji Fukaya’s 60th birthday conference “Geometry and Everything” at the Research Institute for Mathematical Sciences (RIMS) at Kyoto University and continued during the visit of the first author to RIMS. The authors thank the Institute for its hospitality.

Eliashberg’s work was partially supported by National Science Foundation grant DMS–1807270. Ogawa’s work was partially supported by Japan Society for the Promotion of Science KAKENHI grant JP17K05283.

**References**

[1] K. Cieliebak and Y. Eliashberg, *From Stein to Weinstein and Back: Symplectic Geometry of Affine Complex Manifolds*, Amer. Math. Soc. Colloq. Publ. 59, Amer. Math. Soc., Providence, 2012. MR 3012475. DOI 10.1090/coll/059.

[2] S. Courte, *Contact manifolds with symplectomorphic symplectizations*, Geom. Topol. 18 (2014), no. 1, 1–15. MR 3158770. DOI 10.2140/gt.2014.18.1.
Stabilized convex symplectic manifolds are Weinstein

[3] Y. Eliashberg, Topological characterization of Stein manifolds of dimension > 2, Internat. J. Math. 1 (1990), no. 1, 29–46. MR 1044658. DOI 10.1142/S0129167X90000034.

[4] Y. Eliashberg and M. Gromov, “Convex symplectic manifolds” in Several Complex Variables and Complex Geometry, Part 2 (Santa Cruz, 1989), Proc. Sympos. Pure. Math. 52, Amer. Math. Soc., Providence, 1991, 135–162. MR 1128541. DOI 10.1090/pspum/052.2/1128541.

[5] Y. Eliashberg, O. Lazarev, and E. Murphy, in preparation.

[6] Y. Eliashberg and N. Mishachev, Introduction to the h-Principle, Grad. Stud. Math. 48, Amer. Math. Soc., Providence, 2002. MR 1909245. DOI 10.1090/gsm/048.

[7] Y. Eliashberg and E. Murphy, Lagrangian caps, Geom. Funct. Anal. 23 (2013), no. 5, 1483–1514. MR 3102911. DOI 10.1007/s00039-013-0239-2.

[8] H. Geiges, Symplectic manifolds with disconnected boundary of contact type, Int. Math. Res. Not IMRN 1994, no. 1, 23–30. MR 1255250. DOI 10.1155/S1073792894000048.

[9] M. Gromov, Partial Differential Relations, Ergeb. Math. Grenzgeb. (3) 9, Springer, Berlin, 1986. MR 0864505. DOI 10.1007/978-3-662-02267-2.

[10] P. Massot, K. Niederkrüger, and C. Wendl, Weak and strong fillability of higher dimensional contact manifolds, Invent. Math. 192 (2013), no. 2, 287–373. MR 3044125. DOI 10.1007/s00222-012-0412-5.

[11] B. Mazur, Stable equivalence of differentiable manifolds, Bull. Amer. Math. Soc. 67 (1961), 377–384. MR 0130697. DOI 10.1090/S0002-9904-1961-10626-5.

[12] D. McDuff, Symplectic manifolds with contact type boundaries, Invent. Math. 103 (1991), no. 3, 651–671. MR 1091622. DOI 10.1007/BF01239530.

[13] Y. Mitsumatsu, Anosov flows and non-Stein symplectic manifolds, Ann. Inst. Fourier (Grenoble) 45 (1995), no. 5, 1407–1421. MR 1370752.

[14] E. Murphy, Loose Legendrian embeddings in high dimensional contact manifolds, preprint, arXiv:1201.2245v5 [math.SG].

[15] E. Murphy and K. Siegel, Subflexible symplectic manifolds, Geom. Topol. 22 (2018), no. 4, 2367–2401. MR 3784524. DOI 10.2140/gt.2018.22.2367.

Eliashberg: Department of Mathematics, Stanford University, Stanford, California, USA; eliash@stanford.edu

Ogawa: Department of Mathematics, Tokai University, Kanagawa, Japan; nogawa@tsc.u-tokai.ac.jp

Yoshiyasu: Center for Genomic Medicine, Graduate School of Medicine, Kyoto University, Kyoto, Japan; tyoshiyasu@genome.med.kyoto-u.ac.jp
