Ergodicity and uniform in time truncation bounds for inhomogeneous birth and death processes with additional transitions from and to origin

Alexander Zeifman*, Anna Korotysheva, Yacov Satin†, Rostislav Razumchik‡, Victor Korolev§, Sergey Shorgin¶

Abstract. In this paper one presents the extension of the transient analysis of the class of continuous-time birth and death processes defined on non-negative integers with special transitions from and to the origin. From the origin transitions can occur to any state. But being in any other state, besides ordinary transitions to neighbouring states, a transition to the origin can occur. All possible transition intensities are assumed to be non-random functions of time and may depend on the state of the process. We improve previously known ergodicity and truncation bounds for this class of processes which were known only for the case when transitions from the origin decay exponentially (other intensities must have unique uniform upper bound). We show how the bounds can be obtained the decay rate is slower than exponential. Numerical results are also provided.

Keyword: inhomogeneous process, birth and death process, truncation, ergodicity, bounds.

*Vologda State University, Institute of Informatics Problems of the FRC CSC RAS, ISEDT RAS, corresponding author, telephone/fax +78172721632, e-mail a.zeifman@mail.ru
†Vologda State University, Institute of Informatics Problems of the FRC CSC RAS
‡Institute of Informatics Problems of the FRC CSC RAS, Peoples’ Friendship University
§Moscow State University, Institute of Informatics Problems of the FRC CSC RAS, Moscow, Russia
¶Institute of Informatics Problems of the FRC CSC RAS, Moscow, Russia
1 Introduction

In this paper consideration is given to one subclass of continuous-time Markov chains – inhomogeneous birth and death processes with additional transitions from and to origin. More strictly speaking, one considers an inhomogeneous continuous-time Markov chain \( \{X(t), \ t \geq 0\} \) with state space \( \mathcal{X} = \{0, 1, 2, \ldots\} \). All possible transition intensities are assumed to be non-random functions of time and may depend on the state of the process. From state 0 the chain can jump to any state \( i > 0 \) with transition intensity \( r_i(t) \). But transitions from each state \( i > 0 \) happen with intensities \( \mu_i(t), \lambda_i(t) \) and \( \beta_i(t) \) and can be either to state \( (i - 1) \) or \( (i + 1) \) or 0, respectively.

Such subclass of processes finds its application in the study of queueing systems with catastrophes and bulk arrivals (see, for example, [7, 1, 3, 2, 8, 5, 9, 11, 4, 10]). For more details, concerning possible applications, one can refer to [11] and references therein. Note that in the cited papers authors call transitions from and to origin (governed by intensities \( r_i(t) \) and \( \beta_i(t) \)) as “mass arrivals when empty” and “mass exodus” transitions, respectively. In order to keep the connection with the previously obtained results in what follows we use these term as well.

The motivation for this research was given by papers [11]–[14], where authors studied transient behaviour of various inhomogeneous birth and death processes being a subclass of \( X(t) \). Specifically their results concerned ergodicity and perturbation bounds, and bounds of truncation \(^1\) of the processes (for example, probability of being in a particular state at time \( t \), or expected value of the process at time \( t \), which started from any given state). It is well-known that exact computation of transient state probability distribution is not very appealing way to analyse the behaviour of system described by processes with time-dependent rates. A good alternative is to have bounds for performance characteristics of interest, which can be computed fast and are tight enough to make results meaningful. With respect to this observation, the direction of research, indicated by papers [11]–[14], looks promising.

Clearly ergodicity, perturbation and truncation bounds heavily depend on the restrictions imposed on the transition intensities of the process. The harder restrictions are, the better bounds one can obtain. But hard restrictions make the results inapplicable in most cases which are interesting for applications. Thus the main effort is usually made toward the loosening the restrictions and improving bounds. In [11] the authors considered the special case of the process \( X(t) \) in which transitions to state 0 cannot depend on the current state of the process (i.e. \( \beta_i(t) = \beta(t) \) for each \( i \)), and obtained the

\(^1\)Which allow the calculation of the limiting characteristics with given accuracy.
first ergodicity, perturbation and truncation bounds. In this paper by applying a different method we demonstrate that truncation bounds obtained in papers [11] can be improved. Specifically, it is shown that one can obtain uniform in time error bounds of truncation for the general process $X(t)$. Uniform error bounds of truncation have already been obtained for several processes in [12] and [13]. In [13] consideration was given to the process $X(t)$ with $r_i(t) = \beta_i(t) \equiv 0$ (i.e. pure inhomogeneous birth and death processes). In [12] one managed to obtain uniform truncation bounds for the process $X(t)$ in the case when $\beta_i(t) \equiv 0$ and intensities $r_i(t)$ decrease exponentially with $i$ i.e. there exists $q > 1$ such that $r_i(t) \leq q^{-i}$ for any $i > 0$. It is worth noticing that using results from [12] and [13], one can construct truncation bounds for $X(t)$ with minor restrictions on intensities $r_i(t)$, $\mu_i(t)$, $\lambda_i(t)$ and $\beta_i(t)$, but these bounds are uninformative because they tend to infinity as $t \to \infty$.

In this paper it is shown that one can find uniform in time error bounds of truncation for the general process $X(t)$ under less stringent conditions on $r_i(t)$ that were used in other papers. The restriction of exponentially decay of $r_i(t)$ with $i$ (required in [11]) is replaced by much weaker condition of convergence of the series $\sum_i g(i)r_i(t)$, where the function $g(i)$ depends on the characteristic which has to be calculated using the truncated process (for example, the expected value of the process). With respect to other intensities $\mu_i(t)$, $\lambda_i(t)$ and $\beta_i(t)$ the only necessary assumption needed is that any their linear combination has a single uniform upper bound. In what follows we heavily rely on methodology developed in [6, 16, 17, 15], which is based on the logarithmic norm of linear operators and special transformations of the intensity matrix governing the behaviour of the considered Markov process.

The article is organized as follows. In the next section the subclass of the birth and death processes under consideration and auxiliary results are introduced. In section 3 it is explained how one can obtain the ergodicity bounds. Section 4 is devoted to method of truncation that allows the calculation of the limiting characteristics. In section 5 one shows several examples of how the obtained results can be applied for the calculation of performance characteristics of a specific queueing model. Conclusion gives directions of future research.

2 Description of the birth and death process

Let the process $X(t)$, $t \geq 0$, be an inhomogeneous continuous-time Markov chain with state space $\mathcal{X} = \{0, 1, 2, \ldots\}$. Transition, whenever it occurs from state 0 can be to any state $i > 0$, $i \in \mathcal{X}$. Transition from state $i > 0$
can be either to neighbouring state \((i - 1)\) or \((i + 1)\), or to state 0. All possible transition intensities are assumed to be non-random functions of time, and may depend (except for transition to the 0) on the state of the process. Denote by \(p_{ij}(s, t) = \Pr \{X(t) = j | X(s) = i\}, i, j \geq 0, 0 \leq s \leq t\) transition probabilities of \(X(t)\) and by \(p_i(t) = \Pr \{X(t) = i\}\) probability that Markov chain \(X(t)\) is in state \(i\) at time \(t\). Let \(\mathbf{p}(t) = (p_0(t), p_1(t), \ldots)^T\) be probability distribution vector at time \(t\). Throughout the paper we assume that for \(j \neq i\)

\[
\Pr (X(t + h) = j | X(t) = i) = \begin{cases} 
\lambda_i(t) h + \alpha_{ij}(t, h), & \text{if } j = i + 1, \ i > 0, \\
\mu_i(t) h + \alpha_{ij}(t, h), & \text{if } j = i - 1, \ i > 1, \\
\beta_i(t) h + \alpha_{ij}(t, h), & \text{if } j = 0, \ i > 1, \\
r_j(t) h + \alpha_{ij}(t, h), & \text{if } j \geq 1, \ i = 0, \\
(\mu_1(t) + \beta_1(t)) h + \alpha_{ij}(t, h), & \text{if } j = i - 1, \ i = 1, \\
\alpha_{ij}(t, h) & \text{otherwise,}
\end{cases}
\]

(1)

where all \(\alpha_i(t, h)\) are \(o(h)\) uniformly in \(i\), i. e., \(\sup_i |\alpha_i(t, h)| = o(h)\). Intensity functions \(\beta_j(t)\) and \(r_j(t), j \geq 1\), are henceforth called mass exodus and mass arrivals intensities.

We assume that all intensity functions are linear combinations of a finite number of locally integrable on \([0, \infty)\) nonnegative functions. Then the corresponding intensity matrix is

\[
Q(t) = \begin{pmatrix}
a_{00}(t) & r_1(t) & r_2(t) & r_3(t) & r_4(t) & \ldots & \ldots \\
\beta_1(t) + \mu_1(t) & a_{11}(t) & \lambda_1(t) & 0 & 0 & \ldots & \ldots \\
\beta_2(t) & \mu_2(t) & 0 & \lambda_2(t) & 0 & \ldots & \ldots \\
& & \ldots & \ldots & \ldots & \ldots & \ldots \\
\beta_j(t) & 0 & \ldots & \mu_j(t) & a_{jj}(t) & \lambda_j(t) & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},
\]

Let \(a_{ij}(t) = q_{ji}(t)\) for \(j \neq i\) and

\[
a_{ii}(t) = -\sum_{j \neq i} a_{ji}(t) = -\sum_{j \neq i} q_{ij}(t).
\]

(2)

According to standard approach, which was applied in [6, 16, 17, 15], we assume that the intensity matrix is essentially bounded, i. e.

\[
|a_{ii}(t)| \leq L < \infty,
\]

(3)

for almost all \(t \geq 0\).
Probabilistic dynamics of the considered process \( X(t) \) is given by the forward Kolmogorov system

\[
\frac{dp(t)}{dt} = A(t) p(t),
\]

(4)

where \( A(t) = Q^T(t) \) is the transposed intensity matrix of the process. Throughout the paper by \( \| \cdot \| \) we denote the \( l_1 \)-norm, i.e., \( \|x\| = \sum_i |x_i| \), and \( \|B\| = \sup_j \sum_i |b_{ij}| \) for a matrix \( B = (b_{ij})_{i,j=0}^\infty \). Let \( \Omega \) be the set all stochastic vectors, i.e. \( l_1 \) – vectors with non-negative coordinates and unit norm. Then we have \( \|A(t)\| = 2 \sup_k |a_{kk}(t)| \leq 2L \) for almost all \( t \geq 0 \). Hence, the operator function \( A(t) \) from \( l_1 \) into itself is bounded for almost all \( t \geq 0 \) and locally integrable on \([0;\infty)\). Therefore, we can consider (4) as a system of differential equations in the space \( l_1 \) with bounded operator.

It is well known (see [4]) that the Cauchy problem for such system as (4) has a unique solution for arbitrary initial condition and \( p(s) \in \Omega \) implies \( p(t) \in \Omega \) for \( t \geq s \geq 0 \).

Denote by \( E(t,k) = E\{X(t) | X(0) = k\} \) the expected value (mean) of the process \( X(t) \) at moment \( t \) under initial condition \( X(0) = k \).

Recall that process \( X(t) \) is called weakly ergodic, if \( \|p^*(t) - p^{**}(t)\| \to 0 \) as \( t \to \infty \) for any initial conditions \( p^*(0), p^{**}(0) \), where \( p^*(t) \) and \( p^{**}(t) \) are the corresponding solutions of (4). Process \( X(t) \) has the limiting mean \( \varphi(t) \), if \( \lim_{t \to \infty} (\varphi(t) - E(t,k)) = 0 \) for any \( k \).

### 3 Ergodicity bounds

In order to obtain new ergodicity bounds we apply the approach from [11]. If for each instant \( t \) one denotes by \( \beta_*(t) \) the greatest lower bound of \( \beta_i(t) \) with respect to \( i \), i.e.

\[
\beta_*(t) = \inf_i \beta_i(t),
\]

(5)

then the forward Kolmogorov system (4) can be rewritten in the following form:

\[
\frac{dp}{dt} = A^*(t) p + g(t), \quad t \geq 0.
\]

(6)

Here \( g(t) = (\beta_*(t), 0, 0, \ldots)^T, A^*(t) = (a_{ij}^*(t))_{i,j=0}^\infty \), and

\[
a_{ij}^*(t) = \begin{cases} a_{ij}(t) - \beta_*(t), & \text{if } i = 0, \\ a_{ij}(t), & \text{otherwise}. \end{cases}
\]

(7)

Further results heavily rely of the notion of the logarithmic norm of an operator function. Let \( B(t), t \geq 0 \) be a one-parameter family of bounded
linear operators on a Banach space $\mathcal{B}$ and let $I$ denote the identity operator. For a given $t \geq 0$, the number
\[
\gamma(B(t)) = \lim_{h \to +0} \frac{\|I + hB(t)\| - 1}{h}
\]
is called the logarithmic norm of the operator $B(t)$. If $\mathcal{B}$ is an $(N+1)-$dimensional vector space with $l_1-$ norm such that the operator $B(t)$ is given by the matrix $B(t) = (b_{ij}(t))_{i,j=0}^N$, $t \geq 0$, then the logarithmic norm of $B(t)$ can be found explicitly:
\[
\gamma(B(t)) = \sup_j \left(b_{jj}(t) + \sum_{i \neq j} |b_{ij}(t)|\right), \quad t \geq 0.
\]
On the other hand, the logarithmic norm of the operator $B(t)$ is related to the Cauchy operator $V(t,s)$ of the system
\[
\frac{dx}{dt} = B(t)x, \quad t \geq 0
\]
in the following way:
\[
\gamma(B(t)) = \lim_{h \to +0} \frac{\|V(t+h,t)\| - 1}{h}, \quad t \geq 0.
\]
From the latter relation one can deduce the following bounds of the Cauchy operator $V(t,s)$:
\[
\|V(t,s)\| \leq e^{\int_s^t \gamma(B(\tau)) d\tau}, \quad 0 \leq s \leq t.
\]
Here we can find the exact value of the logarithmic norm of operator function $A^*(t)$, namely
\[
\gamma(A^*(t)) = \sup_i \left(a_{ii}^*(t) + \sum_{j \neq i} |a_{ji}^*(t)|\right) = -\beta^*(t).
\]
Let $U^*(t,s)$ be the Cauchy operator for equation (6). Then $\|U^*(t,s)\| \leq e^{-\int_s^t \beta^* (\tau) d\tau}$, and hence for any initial conditions $p^*(0), p^{**}(0)$ and any $t \geq 0$ we have
\[
\|p^*(t) - p^{**}(t)\| \leq e^{-\int_0^t \beta^* (\tau) d\tau} \|p^*(0) - p^{**}(0)\|,
\]
Thus the following statement holds.
\footnote{For further details on the logarithmic norm one can refer, for example, to \textsuperscript{6, 10, 15}.}
Theorem 1. Let the catastrophe rates be essential, i.e.,
\[
\int_0^\infty \beta_*(t) \, dt = \infty.
\] (11)

Then the process \( X(t) \) is weakly ergodic (in the uniform operator topology) and the following bound for the rate of convergence holds:
\[
\|p^*(t) - p^{**}(t)\| \leq 2e^{-\int_0^t \beta_*(\tau) \, d\tau},
\] (12)
for any initial conditions \( p^*(0), p^{**}(0) \) and any \( t \geq 0 \).

Let now \( D \) be such a diagonal matrix (\( D = \text{diag}(d_0, d_1, d_2, \ldots) \)) such that the inequalities \( 1 = d_0 \leq d_1 \leq \ldots \) hold. Consider the corresponding space of sequences \( l_{1D} = \{z = (p_0, p_1, p_2, \ldots)\} \) such that \( \|z\|_{1D} = \|Dz\|_1 < \infty \). Then we can obtain the following estimate for the logarithmic norm of the operator function \( A^*(t) \) in the \( l_{1D} \)-norm:
\[
\gamma (A^*(t))_{1D} = \gamma (DA^*(t)D^{-1}) = \sup_i \left( a_{ii}^*(t) + \sum_{j \neq i} \frac{d_j}{d_i} a_{ji}^*(t) \right) = -\beta^{**}(t),
\] (13)
where \( \beta^{**}(t) = \inf_i \left( |a_{ii}^*(t)| - \sum_{j \neq i} \left| \frac{d_j}{d_i} a_{ji}^*(t) \right| \right) \).

Theorem 2. Let there exist a sequence \( \{d_i\} \) such that
\[
\int_0^\infty \beta^{**}(t) \, dt = \infty.
\] (14)

Then the following bound of the rate of convergence holds:
\[
\|p^*(t) - p^{**}(t)\|_{1D} \leq e^{-\int_0^t \beta^{**}(\tau) \, d\tau} \|p^*(0) - p^{**}(0)\|_{1D},
\] (15)
for any initial conditions \( p^*(0), p^{**}(0) \) and any \( t \geq 0 \).

Let \( l_{1E} = \{z = (p_0, p_1, p_2, \ldots)\} \) be a set of sequences such that \( \|z\|_{1E} = \sum_{k \geq 1} k|p_k| < \infty \). Put \( W = \inf_{k \geq 1} \frac{d_k}{k} \). Then \( W \|z\|_{1E} \leq \|z\|_{1D} \), and we have the following statement.

Corollary 1. Let the conditions of Theorem 2 hold and \( W > 0 \). Then the process \( X(t) \) has the limiting mean, say \( \phi(t) = E(t, 0) \), and the following bound of the rate of convergence holds:
\[
|E(t, j) - E(t, 0)| \leq 1 + \frac{d_j}{W} e^{-\int_0^t \beta^{**}(\tau) \, d\tau}.
\] (16)
for any initial condition \( j \) and any \( t \geq 0 \).
4 Truncation bounds

Consider the “truncated” process $X_N(t)$ on the state space $E_N = \{0, 1, \ldots, N\}$ with the corresponding reduced intensity matrix $A_N(t)$. Below we will identify the finite vector with entries, say $(a_0, a_1, \ldots, a_N)^T$ and the infinite vector with the same first $N$ coordinates and others equal to zero. Let us rewrite the system (6) as

$$\frac{dy_N}{dt} = A^*(t)y_N + g(t) + (A_N^*(t) - A^*(t))y_N. \quad (17)$$

and consider the corresponding “truncated” system

$$\frac{dy_N}{dt} = A_N^*(t)y_N + g_N(t). \quad (18)$$

The solutions to the system (18) and (6) have the following form respectively

$$p(t) = U^*(t, 0)p(0) + \int_0^t U^*(t, \tau)g(\tau)\,d\tau. \quad (19)$$

$$y_N(t) = U^*(t, 0)y_N(0) + \int_0^t U^*(t, \tau)g(\tau)\,d\tau + \int_0^t U^*(t, \tau)(A_N^*(\tau) - A^*(\tau))y_N(\tau)\,d\tau. \quad (20)$$

Let the initial condition for both processes coincide, i.e. $y_N(0) = p(0)$. Then in any norm one can write

$$\|p(t) - y_N(t)\| \leq \int_0^t \|U^*(t, \tau)\| \|(A_N^*(\tau) - A^*(\tau))y_N(\tau)\|\,d\tau. \quad (21)$$

To evaluate the Cauchy matrix we use the logarithmic norm (8), and we additionally assume that there are constants $M$ and $a$ such that

$$\|U^*(t, \tau)\| \leq e^{-\int_\tau^t \beta^*(u)\,du} \leq Me^{-a(t-\tau)}. \quad (22)$$

We use the equality (15), and also we will assume that there are constants $M_1$ and $a_1$ such that

$$\|U^*(t, \tau)\|_{1D} \leq e^{-\int_\tau^t \beta^{**}(u)\,du} \leq M_1e^{-a_1(t-\tau)}. \quad (23)$$
Choose any constant $\theta$ such that $\theta \geq \sup_t \beta^* (t)$. Then one can bound the solution of the system (6) in the $l_{1D}$-norm, introduced above, as follows

$$ \| p (t) \|_{1D} = \| U^* (t, 0) p (0) + \int_0^t U^* (t, \tau) g (\tau) \, d\tau \| \leq $$

$$ \leq \| U^* (t, 0) \|_{1D} \| p (0) \|_{1D} + \int_0^t \| U^* (t, \tau) \|_{1D} \| g (\tau) \|_{1D} d\tau \leq $$

$$ \leq M_1 e^{-\alpha_1 t} \| p (0) \|_{1D} + \frac{M_1 \theta}{a_1}, \quad (24) $$

Noticing that $\| U_N^* (t, s) \|_{1D} \leq \| U^* (t, 0) \|_{1D}$ and $g_N (\tau) = g (\tau)$ the solution of the system (18) in the $l_{1D}$-norm can be bounded as

$$ \| y_N (t) \|_{1D} \leq \| U_N^* (t, 0) y_N (0) \| + \int_0^t \| U_N^* (t, \tau) g_N (\tau) \| d\tau \leq $$

$$ \leq \| U^* (t, 0) \|_{1D} \| p (0) \|_{1D} + \int_0^t \| U^* (t, \tau) \|_{1D} \| g (\tau) \|_{1D} d\tau \leq $$

$$ \leq M_1 e^{-\alpha_1 t} \| p (0) \|_{1D} + \frac{M_1 \theta}{a_1}. \quad (25) $$

Therefore

$$ p_N = \frac{d_N p_N}{d_N} \leq \frac{\| y_N (t) \|_{1D}}{d_N} \leq \frac{M_1 \theta}{a_1 d_N} + \frac{M_1 e^{-\alpha_1 t}}{d_N} \| p (0) \|_{1D}. \quad (26) $$

Next, noticing that the difference $A^* (\tau) - A_N^* (\tau)$ is equal to

$$ A^* (\tau) - A_N^* (\tau) = \left( \begin{array}{cccccccc}
- \sum_{n \geq N+1} r_n & 0 & 0 & 0 & 0 & \cdots & \beta_{N+1} - \beta^* & \cdots & \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & \\
0 & \cdots & \cdots & \cdots & 0 & -\lambda_N & \mu_{N+1} & 0 & \cdots \\
r_{N+1} & 0 & 0 & \cdots & \cdots & \lambda_N & a_{N+1,N+1} & \mu_{N+2} & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \lambda_{N+1} & \cdots & \cdots
\end{array} \right), \quad (27) $$

and for the $(A^* (\tau) - A_N^* (\tau)) y_N$ we have

$$ (A^* (\tau) - A_N^* (\tau)) y_N = \left( \begin{array}{cccc}
- \sum_{n \geq N+1} r_n y_0 & 0 \\
\cdots & -\lambda_N y_N \\
r_{N+1} y_0 + \lambda_N y_N & \cdots \\
r_{N+2} y_0 & \cdots
\end{array} \right). \quad (28) $$
then, using bound (26), we have the following bound:

\[
\| (A^*_N (\tau) - A^* (\tau)) y_N (\tau) \| \leq 2 \sum_{n \geq N+1} r_n p_0 + 2 \lambda_N p_N \leq (29)
\]

\[
\leq R_{N+1} + \frac{2M1L}{d_N} \left( \frac{\theta}{a_1} + e^{-a_1 t} \|p(0)\|_{1D} \right),
\]

where \( R_{N+1} \) is the remainder of the convergent series \( \sum_{n}^\infty r_n \).

Therefore, under assumptions (21), (22), one has the following bound of truncation from (29):

\[
\| p(t) - y_N (t) \| \leq \left( M R_{N+1} + \frac{2MM1L}{d_N} \left( \frac{\theta}{a_1} + e^{-a_1 t} \|p(0)\|_{1D} \right) \right)^t \int_0^t e^{-a(t-\tau)} d\tau \leq \frac{M R_{N+1}}{a} + \frac{2MM1L}{ad_N} \left( \frac{\theta}{a_1} + e^{-a_1 t} \|p(0)\|_{1D} \right).
\]

**Theorem 3**. Let the conditions of Theorems 1 and 2 hold. Then the following bound of truncation holds for any initial conditions \( p(0) = y_N (0) \) and any \( t \geq 0 \)

\[
\| p(t) - y_N (t) \| \leq M R_{N+1} + \frac{2MM1L}{ad_N} \left( \frac{\theta}{a_1} + e^{-a_1 t} \|p(0)\|_{1D} \right). \quad (30)
\]

In order to find the bound for the mean of the truncated process consider the coordinates of the vector \( p(t) \). We have

\[
p_{N+1} + 2p_{N+2} + 3p_{N+3} + \cdots \leq \frac{1}{d_{N+1}} d_{N+1} p_{N+1} + \frac{2}{d_{N+2}} d_{N+2} p_{N+2} + \frac{3}{d_{N+3}} d_{N+3} p_{N+3} + \cdots \leq \|p(t)\|_{1D} \sup_{i \geq 1} \frac{i}{d_{N+i}}. \quad (31)
\]

From \( (31) \) we get the bound for the difference in means of the original and the truncated process:

\[
|E(t, k) - E_N(t, k)| \leq \|p(t) - y_N (t)\|_{1E} < |p_0 - p_{N0}| + |p_1 - p_{N1}| + 2 |p_2 - p_{N2}| + \cdots +
\]

\[
+N |p_N - p_{NN}| + (N + 1) p_{N+1} + (N + 2) p_{N+2} + (N + 3) p_{N+3} + \cdots \leq N \|p(t) - y_N (t)\| + p_{N+1} + 2p_{N+2} + 3p_{N+3} + \cdots
\]

From relations (24) and (31) one can obtain the following corollary.
Corollary 2. Let the conditions of Theorem 3 hold and assume the series $\sum n r_n$ converges. Then for any initial conditions $X(0) = X_N(0) = k$ and any $t \geq 0$ the following truncation bound holds:

$$|E(t, k) - E_N(t, k)| \leq \frac{NMR_{N+1}}{a} + \left( \frac{2NM_1L}{ad_N} + \sup_{i \geq 1} \frac{i}{d_{N+i}} \right) \left( \frac{\theta}{a_1} + e^{-a_1t\|p(0)\|_D} \right),$$

where $\|p(0)\|_D = d_k$.

5 Examples

Efficiency of the bounds obtained for the process $X(t)$ in the previous sections will illustrated in the queueing theory context. Specifically we consider $M_t/M_t/S$ queue with catastrophes and bulk arrivals when empty when intensities are periodic functions of time which can be described by process $X(t)$. In each example it is shown how to find approximations for the limiting value of the mean number of customers in the system and limiting value of empty system with given error. For convenience we first give detailed description of the system and then proceed to examples. Queueing system consists of single infinite capacity queue and $S$ servers. Two flows of customers arrive at the system: flow of ordinary customers and flow of catastrophes. If at time $t$ there is at least one customer in the system then new arrivals of ordinary customers happen according to inhomogeneous Poisson process with intensity $\lambda(t)$. But if at time $t$ the system is empty ordinary customers arrive in bulk (or groups) in accordance with a inhomogeneous Poisson process of intensity $r(t)$. The size of arriving group is a random variable with probability distribution $g_n$, $n = 1, 2, \ldots$, having finite mean. The sizes and interarrival times of successive arriving groups are stochastically independent. Let $r_n(t) = g_n r(t)$. Each ordinary customer upon arrival occupies one place in the queue and waits for service. Whenever server becomes free customer from the queue (if there is any) enters server and get served according to exponential distribution with intensity $\mu(t)$ (service discipline is unimportant and for certainty one can consider that customer are served in FIFO manner). Additional inhomogeneous Poisson flow of catastrophes of intensity $\beta_n(t)$ arrives at the system. If arriving customer of this flow finds the system busy it removes all customers from the system and leaves it. Otherwise it has no effect on it.

In order to illustrate the behaviour of system’s performance characteristics we will consider several special cases: when there are few servers in the system and when there are many servers in the system. We will consider three
performance characteristics: the limiting probability $p_0(t)$ of the empty system, the limiting probability $\Pr(X(t) \leq S) = \sum_{i=0}^{S} p_i(t)$ of the empty queue, and the limiting mean $E(t, 0)$ if initially the system was empty.

### 5.1 Case when $S=3$

Let the number of servers in the system be equal to $S = 3$. We will consider two examples which differ in the values for the intensities $r_n(t)$.

**Example 1.** Let the intensities have the form

$$
\begin{align*}
\lambda_n(t) &= \lambda(t) = 1 + \sin 2\pi t, \\
\mu_n(t) &= \min(n, S) \mu(t) = \min(n, 3) (1 - \sin 2\pi t), \\
\beta_n(t) &= 2 + \cos 2\pi t + \frac{1}{n}, \\
r_n(t) &= \frac{1 + \sin 2\pi t}{4^n}.
\end{align*}
$$

This is almost the same example as example 1.1 in [11], except for the fact that here the intensities $\beta_n(t)$ are dependent on $n$, but in [11] such dependency was not allowed. Notice also that here the values of $r_n(t)$ decay exponentially. Due to the fact that (11) holds, then according to *Theorem* 1 the process $X(t)$ is exponentially weakly ergodic. Hence $X(t)$ has the limiting 1-periodic regime and the correspondent limiting 1-periodic mean, see details in [15, 13]. The following bound for the rate of convergence holds:

$$
\|p^*(t) - p^{**}(t)\| \leq 2e^{-\int_0^t \beta(\tau) d\tau} \leq 4e^{-2t},
$$

for any initial conditions $p^*(0), p^{**}(0)$ and any $t \geq 0$.

Let $d_n = 2^n$. One has that $L = 12$, $M = e^{1/\pi}$, $a = 2$, $\beta_{**}(t) = \frac{3}{2} + \cos 2\pi t - \frac{3}{2} \sin 2\pi t$, $a_1 = 1.5$ and $M_1 = e^{2.5/\pi}$, and Corollary 1 implies the following bound

$$
|E(t,j) - E(t,0)| \leq 3(d_j + 1) e^{-1.5t}
$$

for any initial condition $j$ and any $t \geq 0$.

Then using *Theorem* 3 and *Corollary* 2 with truncation error $10^{-5}$ for $N = 30$ and $t \in [10, 11]$ one obtains the following bounds of truncation:

$$
\|p(t) - y_N(t)\| \leq 10^{-7},
$$

$$
|E(t,0) - E_N(t,0)| \leq 3 \cdot 10^{-6}.
$$
In Fig.1–3 one can see the behaviour of the limiting probabilities $p_0(t)$ and $\Pr(X(t) \leq S)$, and the limiting mean $E(t, 0)$ for different values of $t$.

**Example 2.** Let the intensities have the form

$$\begin{align*}
\lambda_n(t) &= \lambda(t) = 1 + \sin 2\pi t, \\
\mu_n(t) &= \min(n, S) \mu(t) = \min(n, 3) (1 - \sin 2\pi t), \\
\beta_n(t) &= 2 + \cos 2\pi t + \frac{1}{n}, \\
r_n(t) &= \frac{1 + \sin 2\pi t}{n^{10}}.
\end{align*}$$

Notice that though the series of $\sum r_n(t)$ though converges rapidly its $n$-th term does not decay exponentially because (from starting from a certain value) $r_n(t)$ is always greater than the corresponding term $r_n(t)$ in the previous example. Due to the fact that (11) holds, then according to Theorem 1 the process $X(t)$ is exponentially weakly ergodic. Hence $X(t)$ has the limiting $1-$periodic regime and the correspondent limiting $1-$periodic mean.

The following bound for the rate of convergence holds:

$$\|p^*(t) - p^{**}(t)\| \leq 2e^{-\int_0^t \frac{\beta(\tau)}{n} d\tau} \leq 4e^{-2t},$$

(37)

for any initial conditions $p^*(0), p^{**}(0)$ and any $t \geq 0$.

Let $d_n = \frac{3^n}{2}$, if $n < 100$, and $d_n = \frac{3^{100}n+1}{100}$, if $n \geq 100$. One has that $L = 12, M = e^{1/\pi}, a = 2, \beta^*(t) = \frac{5}{6} + \cos 2\pi t - \frac{5}{6} \sin 2\pi t, a_1 = 0.8$ and $M_1 = e^{1.8/\pi},$ and Corollary 1 implies the following bound

$$|E(t, j) - E(t, 0)| \leq 2(d_j + 1) e^{-0.8t}$$

(38)

for any initial condition $j$ and any $t \geq 0$.

Then using *Theorem 3* and *Corollary 2* with truncation error $10^{-5}$ for $N = 55$ and $t \in [17, 18]$ one obtains the following bounds of truncation:

$$\|p(t) - y_N(t)\| \leq 3 \cdot 10^{-8},$$

(39)

$$|E(t, 0) - E_N(t, 0)| \leq 2 \cdot 10^{-6}.$$

(40)

The behaviour of the limiting probabilities $p_0(t)$ and $\Pr(X(t) \leq S)$, and the limiting mean $E(t, 0)$ for different values of $t$ one can see in in Fig.4–6.
5.2 Case when S=20

The following two examples show what changes in the behaviour of system’s performance characteristics with the change in the number of servers, arrival intensities.

Example 3. Let the intensities have the form

\[
\begin{align*}
\lambda_n(t) &= \lambda(t) = 15 (1 + \sin 2\pi t), \\
\mu_n(t) &= \min(n, S) \mu(t) = \min(n, 20) (1 - \sin 2\pi t), \\
\beta_n(t) &= 2 + \cos 2\pi t + \frac{1}{n}, \\
r_n(t) &= \frac{1 + \sin 2\pi t}{4^n}.
\end{align*}
\]

Due to the fact that (11) holds, then according to Theorem 1 the process \(X(t)\) is exponentially weakly ergodic. Hence \(X(t)\) has the limiting 1–periodic regime and the correspondent limiting 1–periodic mean. The following bound for the rate of convergence holds:

\[
\|p^*(t) - p^{**}(t)\| \leq 2e^{-\int_0^t \beta(\tau)d\tau} \leq 4e^{-2t},
\]

for any initial conditions \(p^*(0), p^{**}(0)\) and any \(t \geq 0\).

Let \(d_n = (1 + \frac{1}{8})^n\). One has that \(L = 74, M = e^{1/\pi}, a = 2, \beta^{**}(t) = \frac{559}{2520} + \cos 2\pi t - \frac{143}{72} \sin 2\pi t, a_1 = 0.2\) and \(M_1 = e^{3/\pi}\), and Corollary 1 implies the bound

\[
|E(t,j) - E(t,0)| \leq 3(d_j + 1)e^{-0.2t}
\]

for any initial condition \(j\) and any \(t \geq 0\).

Then for exponentially decaying values of \(r_n(t)\) using Theorem 3 and corollary 2 with truncation error \(10^{-5}\) for \(N = 220\) and \(t \in [60, 61]\) one obtains the following bounds of truncation:

\[
\|p(t) - y_N(t)\| \leq 3 \cdot 10^{-8},
\]

\[
|E(t,0) - E_N(t,0)| \leq 6 \cdot 10^{-6}.
\]

Fig. 7–9 show the behaviour of \(p_0(t), \Pr(X(t) \leq S)\) and \(E(t,0)\) for different values of \(t\).
**Example 4.** Let the intensities have the form
\[
\lambda_n(t) = \lambda(t) = 15 \left( 1 + \sin 2\pi t \right), \\
\mu_n(t) = \min(n, S) \mu(t) = \min(n, 20) \left( 1 - \sin 2\pi t \right), \\
\beta_n(t) = 2 + \cos 2\pi t + \frac{1}{n}, \\
r_n(t) = \frac{1 + \sin 2\pi t}{n^{10}}.
\]

Due to the fact that (11) holds, then according to Theorem 1 the process \(X(t)\) is exponentially weakly ergodic. Hence \(X(t)\) has the limiting 1-periodic regime and the correspondent limiting 1-periodic mean. The following bound for the rate of convergence holds:
\[
\|p^*(t) - p^{**}(t)\| \leq 2e^{-t \int_0^t \beta(\tau) d\tau} \leq 4e^{-2t},
\]
for any initial conditions \(p^*(0), p^{**}(0)\) and any \(t \geq 0\).

Let \(d_n = \frac{9^n}{8}, \) if \(n < 200, \) and \(d_n = \frac{9^{200} n + 1}{200}, \) if \(n \geq 200. \) One has that \(L = 74, M = e^{1/\pi}, a = 2, \beta_+(t) = \frac{17}{72} + \cos 2\pi t - \frac{143}{72} \sin 2\pi t, \) \(a_1 = 0.2 \) and \(M_1 = e^{3/\pi}, \) and Corollary 1 implies the bound
\[
|E(t, j) - E(t, 0)| \leq 3(d_j + 1)e^{-0.2t}
\]
for any initial condition \(j\) and any \(t \geq 0.\)

Then using Theorem 3 and corollary 2 with truncation error \(10^{-5}\) for \(N = 220\) and \(t \in [56, 57]\) one obtains the following bounds of truncation:
\[
\|p(t) - y_N(t)\| \leq 3 \cdot 10^{-8},
\]
\[
|E(t, 0) - E_N(t, 0)| \leq 6 \cdot 10^{-6}.
\]

The behaviour of \(p_0(t), \Pr(X(t) \leq S)\) and \(E(t, 0)\) for different values of \(t\) is shown in Fig. 10–12.

### 6 Conclusion

In this paper we have obtained ergodicity and truncation bounds for a class of inhomogeneous birth and death process with an additional arrival from/to origin under relaxed conditions on the transitions from the origin. Previously these bounds were known only for exponentially decaying transitions from the origin. The obtained results are especially accurate in cases when the
arrival rate is not very high and become worse with its increase. The study of the boundaries for the arrival rate which lead to accurate estimates as well as the development of new approaches to deal with high load values is a promising direction of research.

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Figure 1: Case $S = 3$. $r_n(t)$ decay exponentially. Approximation of the limiting probability of empty queue $p_0(t)$ on [10, 11].

Figure 2: Example 1. Approximation of the limiting probability $\Pr(X(t) \leq S)$ on [10, 11].
Figure 3: Example 1. Approximation of the limiting mean $E(t, 0)$ on [10, 11].

Figure 4: Example 2. Approximation of the limiting probability of empty queue $Pr(X(t) = 0)$ on [17, 18].
Figure 5: Example 2. Approximation of the limiting probability \( \Pr(X(t) \leq S) \) on \([17, 18]\).

Figure 6: Example 2. Approximation of the limiting mean \( E(t, 0) \) on \([17, 18]\).
Figure 7: Example 3. Approximation of the limiting probability of empty queue $Pr (X(t) = 0)$ on [60, 61].

Figure 8: Example 3. Approximation of the limiting probability $Pr (X(t) \leq S)$ on [60, 61].
Figure 9: Example 3. Approximation of the limiting mean $E(t, 0)$ on $[60, 61]$.

Figure 10: Example 4. Approximation of the limiting probability of empty queue $\Pr(X(t) = 0)$ on $[56, 57]$. 
Figure 11: Example 4. Approximation of the limiting probability $\Pr(X(t) \leq S)$ on $[56, 57]$.

Figure 12: Example 4. Approximation of the limiting mean $E(t, 0)$ on $[56, 57]$. 