IRREDUCIBLE CHARACTERS OF GENERAL LINEAR SUPERALGEBRA AND SUPER DUALITY

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Abstract. We develop a new method to solve the irreducible character problem for a wide class of modules over the general linear superalgebra, including all the finite-dimensional modules, by directly relating the problem to the classical Kazhdan-Lusztig theory. Furthermore, we prove that certain parabolic BGG categories over the general linear algebra and over the general linear superalgebra are equivalent. We also verify a parabolic version of a conjecture of Brundan on the irreducible characters in the BGG category of the general linear superalgebra.

1. Introduction

The problem of finding the finite-dimensional irreducible characters of simple Lie superalgebras was first posed in [K1, K2]. This problem turned out to be one of the most challenging problems in the theory of Lie superalgebras, and in the type A case was first solved by Serganova [Sc]. Later on, inspired by [LLT], Brundan in [B] provided an elegant new solution of the problem. To be more precise, Brundan in [B, Conjecture 4.32 and (4.35)] gave a conjectural character formula for every irreducible highest weight \( gl(m|n) \)-module in the Bernstein-Gelfand-Gelfand category \( \mathcal{O} \) in terms of certain Brundan-Kazhdan-Lusztig polynomials. The validity of the conjecture would imply a remarkable formulation of the Kazhdan-Lusztig theory of \( \mathcal{O} \) in terms of canonical and dual basis on a certain Fock space. Brundan then solved the finite-dimensional irreducible character problem by verifying the conjecture for the subcategory of finite-dimensional \( gl(m|n) \)-modules, in which case the Fock space is \( \hat{E}^{m|n} \) (see Section 4.3).

One of the main purposes of the present paper is to establish Brundan’s conjecture for a substantially larger subcategory of \( \mathcal{O} \) of \( gl(m|n) \)-modules, which includes all the finite-dimensional ones. We note that a similar Fock space formulation is known among experts for modules of the general linear algebra \( gl(m + n) \) in the category \( \mathcal{O} \), and in particular for modules in the maximal parabolic subcategory corresponding to the Levi subalgebra \( gl(m) \oplus gl(n) \), in which case the Fock space is \( \hat{E}^{m+n} \) (see Section 4.3).

Let \( \mathfrak{g} \) and \( \tilde{\mathfrak{g}} \) denote direct limits of general linear algebras \( gl(m + n) \) and of general linear superalgebras \( gl(m|n) \), respectively, as \( n \to \infty \) (see Section 2.2 and Section 2.3). Motivated by [B] it was shown in [CWZ] that in the limit \( n \to \infty \) the Fock spaces \( \hat{E}^{m+n} \) and \( \hat{E}^{m|n} \) have compatible canonical and dual canonical bases, and that the Kazhdan-Lusztig polynomials in \( \hat{E}^{m+n} \) and \( \hat{E}^{m|n} \) can be identified. Now the Kazhdan-Lusztig

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polynomials of $\hat{E}^{m+\infty}$ describe the $\mathfrak{g}$-module category $\mathcal{O}_{[-m,-2]}$ whose objects are the direct limits of modules in the above-mentioned maximal parabolic subcategory, while those of $\hat{E}^{m\infty}$ describe the $\mathfrak{g}$-module category $\mathcal{O}_{[-m,-2]}$ whose objects are direct limits of finite-dimensional $\mathfrak{gl}(m|n)$-modules. From this and Brundan’s formulation it follows that the classical (parabolic) Kazhdan-Lusztig polynomials of the general linear algebra also give solution to the finite-dimensional irreducible character problem for the general linear superalgebra.

Motivated by [CWZ] Wang and the first author in [CW] compare a more general parabolic $\mathfrak{g}$-module category $\mathcal{O}_Y$ with a corresponding $\mathfrak{g}$-module category $\mathcal{O}_Y$, where $Y$ here is any subset of $[-m,-2]$ (see Sections 2.2, 2.3, and Remark 3.13). A precise statement of the parabolic Brundan conjecture ([B, Conjecture 4.32]) on the character of irreducible $\mathfrak{g}$-modules in $\mathcal{O}_Y$ was given in [CW, Conjecture 3.10]. The results in [CWZ] [CW] suggest a direct connection between the categories $\mathcal{O}_Y$ and $\mathcal{O}_Y$. In fact, the categories $\mathcal{O}_Y$ and $\mathcal{O}_Y$ are conjectured to be equivalent in [CW, Conjecture 4.18], which was referred to as super duality.

The purpose of the present paper is to establish this super duality. Our main idea is the introduction of a bigger Lie superalgebra $\tilde{\mathfrak{g}}$ (Section 2.1), which contains and interpolates $\mathfrak{g}$ and $\mathfrak{g}$. We then study a corresponding category $\tilde{\mathcal{O}}_Y$ of $\tilde{\mathfrak{g}}$-modules and define certain truncation functors $T: \tilde{\mathcal{O}}_Y \to \mathcal{O}_Y$ and $\tilde{T}: \tilde{\mathcal{O}}_Y \to \mathcal{O}_Y$ (Section 3.2). These functors are shown to send parabolic Verma $\tilde{\mathfrak{g}}$-modules to the respective parabolic Verma $\mathfrak{g}$- and $\mathfrak{g}$-modules, and furthermore irreducible $\tilde{\mathfrak{g}}$-modules to the respective irreducible $\mathfrak{g}$- and $\mathfrak{g}$-modules. From this we obtain in Theorem 3.16 a solution of the irreducible character problem for $\tilde{\mathfrak{g}}$-modules. The solution of the irreducible character problem then allows us to compare the Kazhdan-Lusztig polynomials in $\mathcal{O}_Y$ with those in $\mathcal{O}_Y$. This then enables us to prove in Theorem 5.1 that the functors $T$ and $\tilde{T}$ define equivalences of categories from which super duality follows.

Note that a special case of the super duality conjecture was already formulated for the categories $\mathcal{O}_{[-m,-2]}$ and $\mathcal{O}_{[-m,-2]}$ in [CWZ, Conjecture 6.10]. A proof of this special case was announced recently in [BS], with a proof to appear in a sequel of [BS]. Our method differs significantly from that of Brundan and Stroppel, as ours is independent of [B]. Furthermore our approach enables us to explicitly construct functors inducing this equivalence, and it is applicable to more general module categories.

We want to emphasize that, in contrast to [CWZ], the arguments presented in this article do not depend on [B] or [Se], and hence Theorem 3.16 also gives an independent solution to the finite-dimensional irreducible character problem for the general linear superalgebra as a special case. By directly relating the irreducible character problem of Lie superalgebras to that of Lie algebras our solution of the problem becomes surprisingly elementary. Our method is applicable to other finite and infinite-dimensional superalgebras, e.g. the ortho-symplectic Lie superalgebras [CLW].

This article is organized as follows. In Section 2 the Lie superalgebras $\tilde{\mathfrak{g}}$, $\mathfrak{g}$ and $\mathfrak{g}$ are defined, together with the module categories $\tilde{\mathcal{O}}_Y$, $\mathcal{O}_Y$ and $\mathcal{O}_Y$. In Section 3 the main tool, odd reflections [LSS], of making connections between these categories
is introduced, and the crucial Lemma 3.2 is proved. In Section 4 we show that the Kazhdan-Lusztig polynomials of these categories coincide, from which we then derive in Section 5 the equivalence of these categories.

We conclude this introduction by setting the notation to be used throughout this article. The symbols \( \mathbb{Z}, \mathbb{N}, \) and \( \mathbb{Z}_+ \) stand for the sets of all, positive and non-negative integers, respectively. For \( m \in \mathbb{Z} \) we set \( \langle m \rangle := m \), if \( m > 0 \), and \( \langle m \rangle := 0 \), otherwise.

For integers \( a < b \) we set \( [a, b] := \{a, a+1, \cdots, b\} \). Let \( \mathcal{P} \) denote the set of partitions. For \( \lambda \in \mathcal{P} \) we denote by \( \lambda' \) the transpose partition of \( \lambda \), by \( \ell(\lambda) \) the length of \( \lambda \) and by \( s_\lambda(y_1, y_2, \cdots) \) the Schur function in the indeterminates \( y_1, y_2, \cdots \) associated with \( \lambda \).

For a super space \( V = V_0 \oplus V_1 \) and a homogeneous element \( v \in V \), we use the notation \( |v| \) to denote the \( \mathbb{Z}_2 \)-degree of \( v \). Let \( U(g) \) denote the universal enveloping algebra of a Lie (super)algebra \( g \). Finally all vector spaces, algebras, tensor products, et cetera, are over the field of complex numbers \( \mathbb{C} \).

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2. THE LIE SUPERALGEBRAS \( g, \bar{g} \) AND \( \bar{g} \)

2.1. The Lie superalgebra \( \bar{g} \). For \( m \in \mathbb{N} \), let \( \bar{V} \) denote the complex super space with homogeneous basis \( \{v_r | r \in [-m, -1] \cup \frac{1}{2}\mathbb{N}\} \). The \( \mathbb{Z}_2 \)-gradation is determined by \( |v_r| = 1 \), for \( r \in \frac{1}{2} + \mathbb{Z}_+ \), and \( |v_i| = 0 \), for \( i \in [-m, -1] \cup \mathbb{N} \). We denote by \( \bar{g} \) the Lie superalgebra of endomorphisms of \( \bar{V} \) vanishing on all but finitely many \( v_r \). For \( r, s, p \in [-m, -1] \cup \frac{1}{2}\mathbb{N} \), let \( E_{rs} \) denote the endomorphism defined by \( E_{rs}(v_p) := \delta_{sp}v_r \). Then \( \bar{g} \) equals the Lie superalgebra spanned by these \( E_{rs} \). Let \( \bar{g}^{\leq 0} \) be the subalgebra isomorphic to the linear algebra \( \mathfrak{gl}(m) \) spanned by \( E_{ij}, i, j \in [-m, -1] \).

Let \( \bar{h} \) stand for the Cartan subalgebra spanned by the \( E_{rs} \) and let \( \{\epsilon_r \in \bar{h}^* | r \in [-m, -1] \cup \frac{1}{2}\mathbb{N}\} \) be the basis dual to \( \{E_{rs} \in \bar{h} | r \in [-m, -1] \cup \frac{1}{2}\mathbb{N}\} \). Let \( \bar{\Pi} \) denote the simple roots \( \{\alpha_{-m} := \epsilon_{-m} - \epsilon_{-m+1}, \cdots, \alpha_{-1} := \epsilon_{-1} - \epsilon_{1/2} \} \cup \{\alpha_r := \epsilon_r - \epsilon_{r+1/2} | r \in \frac{1}{2}\mathbb{N}\} \). The corresponding Dynkin diagram is

\[
\begin{array}{cccccccc}
\circ & - & \circ & - & \circ & - & \circ & - \\
\alpha_{-m} & \alpha_{-m+1} & \alpha_{-2} & \alpha_{-1} & \alpha_{1/2} & \alpha_n & \alpha & \cdots
\end{array}
\]

For \( \alpha \in \bar{\Pi} \) let \( \alpha^\vee \) denote the simple coroot corresponding to \( \alpha \). Explicitly, we have \( \alpha^\vee_{-1} = E_{-1,-1} + E_{1/2,1/2}, \alpha^\vee_{1/2} = -E_{1/2,1/2}, \alpha^\vee_{1} = E_{1} + E_{3/2,3/2}, \alpha^\vee_{3/2} = -E_{3/2,3/2} - E_{22} \) et cetera.

Given \( \alpha \in \bar{h}^* \), let \( \bar{g}_\alpha := \{x \in \bar{g} \mid [h, x] = \alpha(h)x, \forall h \in \bar{h}\} \) and let \( \bar{\Delta} \) denote the set of all roots. The positive roots with respect to \( \bar{\Pi} \) will be denoted by \( \bar{\Delta}_+ \), while \( \bar{\mathfrak{b}} \) denotes the Borel subalgebra with respect to \( \bar{\Delta}_+ \). Let \( \bar{\mathfrak{n}} := [\bar{\mathfrak{b}}, \bar{\mathfrak{b}}] \) and let \( \bar{\mathfrak{n}}_- \) be the opposite nilradical.
For any subset \( Y \subseteq [-m, -2] \) (including \( Y = \emptyset \)), define

\[
\tilde{I}_Y := \tilde{h} \oplus (\oplus_{\alpha \in \tilde{\Delta}_Y} \tilde{g}_\alpha),
\end{equation}

\[
\tilde{u}_Y := \oplus_{\alpha \in \tilde{\Delta}_+ \setminus (\tilde{\Delta}_Y)_+} \tilde{g}_\alpha,
\end{equation}

\[
(\tilde{u}_Y)_- := \oplus_{\alpha \in \tilde{\Delta}_+ \setminus (\tilde{\Delta}_Y)_+} \tilde{g}_{-\alpha},
\end{equation}

\[
\tilde{p}_Y := \tilde{I}_Y \oplus \tilde{u}_Y,
\end{equation}

where \( \tilde{\Delta}_Y := \tilde{\Delta} \cap (\oplus_{r \in Y} \oplus_{i \in \mathbb{Z}} \mathbb{Z} \alpha_r) \) and \( (\tilde{\Delta}_Y)_+ := \tilde{\Delta}_+ \cap \tilde{\Delta}_Y \). Then \( \tilde{p}_Y \) is a parabolic subalgebra of \( \tilde{\mathfrak{g}} \) with Levi subalgebra \( \tilde{I}_Y \) and nilpotent radical \( \tilde{u}_Y \). Set \( \tilde{p}_Y^{\leq 0} := \tilde{\mathfrak{g}}^{\leq 0} \cap \tilde{p}_Y \) and \( \tilde{I}_Y^{< 0} := \tilde{\mathfrak{g}}^{< 0} \cap \tilde{I}_Y \).

Given \( \lambda \in \tilde{h}^* \), we denote by \( L(\tilde{I}_Y, \lambda) \) the irreducible \( \tilde{I}_Y \)-module of highest weight \( \lambda \) with respect to \( \tilde{I}_Y \cap \tilde{h} \), which we may regard as an irreducible \( \tilde{p}_Y \)-module in the usual way. Define the parabolic Verma \( \tilde{\mathfrak{g}} \)-module

\[
\tilde{K}(\lambda) := \text{Ind}_{\tilde{p}_Y}^{\tilde{I}_Y} L(\tilde{I}_Y, \lambda).
\]

Let \( \tilde{L}(\lambda) \) be the irreducible \( \tilde{g} \)-module of highest weight \( \lambda \) with respect to \( \tilde{b} \).

Set

\[
\mathcal{P}_Y := \{ \lambda = (\lambda_{-m}, \lambda_{-m+1}, \ldots, \lambda_1, \lambda_2, \ldots) \mid \\
\lambda_i \in \mathbb{Z}, \forall i; \left( \sum_{i=-m}^{-1} \lambda_i \epsilon_i, \alpha^* \right) \in \mathbb{Z}_+, \forall j \in Y; (\lambda_1, \lambda_2, \ldots) \in \mathcal{P} \}.
\]

For \( \lambda \in \mathcal{P}_Y \) we let \( \lambda_+ := (\lambda_1, \lambda_2, \ldots) \in \mathcal{P} \) and \( \lambda^{< 0} := \sum_{i=-m}^{-1} \lambda_i \epsilon_i \).

Given a partition \( \mu = (\mu_1, \mu_2, \ldots) \) we set \( \theta(\mu) \) to be the sequence of integers

\[
\theta(\mu) := (\theta(\mu)_{1/2}, \theta(\mu)_{1}, \theta(\mu)_{3/2}, \theta(\mu)_{2}, \ldots),
\]

where \( \theta(\mu)_{i-1/2} := \langle \mu'_i - (i - 1) \rangle \) and \( \theta(\mu)_i := \langle \mu'_i - i \rangle \), for \( i \in \mathbb{N} \). We recall that \( \langle \cdot \rangle \) is defined at the end of the Introduction.

**Example 2.1.** For the partition \( \lambda = (7, 6, 3, 3, 1) \), we have \( \theta(\lambda) = (5, 6, 3, 4, 2, 0, 0, \ldots) \). This can be read off the Young diagram of \( \lambda \) as follows:

```
  6
 /|
6 3
 /|
5 4
 /|
5 3
 /|
5 4
```

Set

\[
\tilde{\mathcal{P}}_Y := \{ \lambda^{< 0} + \sum_{r \in \mathbb{Z}^N} \theta(\lambda_+)_r \epsilon_r \in \tilde{h}^* \mid \lambda = (\lambda_i) \in \mathcal{P}_Y \}.
\]
For a semisimple $\tilde{h}$-module $\tilde{M}$ and $\gamma \in \tilde{h}^*$, we define

$$\tilde{M}_\gamma := \{ m \in \tilde{M} | hm = \gamma(h)m, \forall h \in \tilde{h} \}.$$  

2.2. The subalgebra $g$. We denote by $g$ the subalgebra of $\tilde{g}$ generated by $E_{rs}, r, s \in [-m, -1] \cup \mathbb{N}$. The Cartan subalgebra $\tilde{h}$ has basis $\{ E_{ii} | i \in [-m, -1] \cup \mathbb{N} \}$, with dual basis $\{ \epsilon_i | i \in [-m, -1] \cup \mathbb{N} \}$. The corresponding Dynkin diagram is

$$\begin{array}{ccccccccc}
\alpha_{-\text{m}} & - & \alpha_{-\text{m+1}} & - & \alpha_{-2} & - & \beta_{-1} & - & \beta_1 & - & \beta_{n} & - & \beta_{n+1/2} & - & \beta_{\text{m+1/2}} & - \\
\end{array}$$

where

$$\beta_{i-1} := \epsilon_{i-1} - \epsilon_{i+1}, \quad \beta_i := \epsilon_i - \epsilon_{i+1}, \quad i \geq 1.$$  

Set $b := \tilde{b} \cap g$, $n := \tilde{n} \cap g$ and $n_\cdot := \tilde{n}_- \cap g$. Set $l_Y := \tilde{l}_Y \cap g$, $p_Y := \tilde{p}_Y \cap g$, $u_Y := \tilde{u}_Y \cap g$, and $(u_Y)_- := (\tilde{u}_Y)_- \cap g$.

Given $\lambda \in \tilde{h}^*$, we denote by $L(l_Y, \lambda)$ the irreducible $l_Y$-module of highest weight $\lambda$ with respect to $l_Y \cap b$, which we may regard as an irreducible $p_Y$-module in the usual way. Define the parabolic Verma $g$-module

$$K(\lambda) := \text{Ind}_p^g L(l_Y, \lambda).$$  

Let $L(\lambda)$ be the irreducible $g$-module of highest weight $\lambda$ with respect to $b$.

As usual, we identify $P_Y$ with the following set of weights in $h^*$:

$$P_Y = \{ \lambda < 0 + \sum_{j \in \mathbb{N}} \lambda_i \epsilon_j \in h^* | \lambda = (\lambda_i) \in P_Y \}.$$  

For a semisimple $h$-module $M$ and $\gamma \in h^*$, we define

$$M_\gamma := \{ m \in M | hm = \gamma(h)m, \forall h \in h \}.$$  

2.3. The subalgebra $\tilde{g}$. We denote by $\tilde{g}$ the subalgebra of $\tilde{g}$ generated by $E_{rs}, r, s \in [-m, -1] \cup \frac{1}{2} + \mathbb{Z}_+$. The Cartan subalgebra $\tilde{h}$ has basis $\{ E_{rr} | r \in [-m, -1] \cup \frac{1}{2} + \mathbb{Z}_+ \}$, with dual basis $\{ \epsilon_r | r \in [-m, -1] \cup \frac{1}{2} + \mathbb{Z}_+ \}$. The corresponding Dynkin diagram is

$$\begin{array}{ccccccccc}
\alpha_{-\text{m}} & - & \alpha_{-\text{m+1}} & - & \alpha_{-2} & - & \beta_{1/2} & - & \beta_{\text{m+1/2}} & - & \beta_{\text{m+1/2}} & - \\
\end{array}$$

where

$$\beta_{i-1/2} := \epsilon_{i-1/2} - \epsilon_{i+1/2}, \quad i \geq 1.$$  

Set $b := \tilde{b} \cap \tilde{g}$, $n := \tilde{n} \cap \tilde{g}$ and $n_\cdot := \tilde{n}_- \cap \tilde{g}$. Set $l_Y := \tilde{l}_Y \cap \tilde{g}$, $p_Y := \tilde{p}_Y \cap \tilde{g}$, $u_Y := \tilde{u}_Y \cap \tilde{g}$, and $(u_Y)_- := (\tilde{u}_Y)_- \cap \tilde{g}$.

Given $\lambda \in \tilde{h}^*$ let $L(l_Y, \lambda)$ be the irreducible $l_Y$-module of highest weight $\lambda$ with respect to $l_Y \cap \tilde{b}$, which we may regard as an irreducible $\tilde{p}_Y$-module. Define the parabolic Verma $\tilde{g}$-module

$$\tilde{K}(\lambda) := \text{Ind}_{\tilde{p}_Y}^{\tilde{g}} L(l_Y, \lambda).$$  

Let $\tilde{L}(\lambda)$ be the irreducible $\tilde{g}$-module of highest weight $\lambda$ with respect to $\tilde{b}$. 
Let
\[ \overline{\mathcal{P}}_{\gamma} := \{ \lambda < 0 + \sum_{i \in \mathbb{N}} (\lambda_+)_i \epsilon_{i-1/2} \in \mathfrak{h}^* \mid \lambda = (\lambda_i) \in \mathcal{P}_{\gamma} \}. \]

For a semisimple \( \mathfrak{h} \)-module \( \overline{M} \) and \( \gamma \in \mathfrak{h}^* \), we define
\[ \overline{M}_\gamma := \{ m \in M \mid hm = \gamma(h)m, \forall h \in \mathfrak{h} \}. \]

2.4. Parametrization for \( \mathcal{P}_{\gamma}, \overline{\mathcal{P}}_{\gamma} \) and \( \overline{\mathcal{P}}_{\gamma} \). The set \( \mathcal{P}_{\gamma} \) parameterizes the sets \( \mathcal{P}_{\gamma}, \overline{\mathcal{P}}_{\gamma} \) and \( \overline{\mathcal{P}}_{\gamma} \). From now on we will use the following notation. For \( \lambda = (\lambda_i) \in \mathcal{P}_{\gamma} \), let
\[ \lambda := \lambda < 0 + \sum_{i=1}^{\infty} \lambda_i \epsilon_i \in \mathcal{P}_{\gamma}, \]
\[ \lambda^2 := \lambda < 0 + \sum_{i=1}^{\infty} (\lambda_+)_i \epsilon_{i-1/2} \in \overline{\mathcal{P}}_{\gamma}, \]
\[ \lambda^\theta := \lambda < 0 + \sum_{r \in \frac{1}{2}\mathbb{N}} \theta(\lambda_+)_i \epsilon_r \in \overline{\mathcal{P}}_{\gamma}. \]

Example 2.2. Let \( m = 3, Y = \emptyset \) and \( \lambda = (-5, 2, -3, 7, 6, 3, 1) \) (cf. Example 2.1). We have
\[ \lambda = -5\epsilon_{-3} + 2\epsilon_{-2} - 3\epsilon_{-1} + 7\epsilon_1 + 6\epsilon_2 + 3\epsilon_3 + 3\epsilon_4 + 1\epsilon_5, \]
\[ \lambda^2 = -5\epsilon_{-3} + 2\epsilon_{-2} - 3\epsilon_{-1} + 5\epsilon_{\frac{3}{2}} + 4\epsilon_{\frac{1}{2}} + 4\epsilon_\frac{1}{2} + 2\epsilon_\frac{3}{2} + 2\epsilon_\frac{3}{2} + 2\epsilon_\frac{1}{2} + 1\epsilon_1, \]
\[ \lambda^\theta = -5\epsilon_{-3} + 2\epsilon_{-2} - 3\epsilon_{-1} + 5\epsilon_{\frac{3}{2}} + 6\epsilon_1 + 3\epsilon_2 + 4\epsilon_2 + 2\epsilon_3. \]

2.5. Categories of \( \mathfrak{g}^\# \)-, \( \mathfrak{g}^\# \)-, and \( \overline{\mathfrak{g}} \)-modules. Let \( \overline{\mathcal{O}}_{\gamma} \) (respectively \( \mathcal{O}_{\gamma}, \overline{\mathcal{O}}_{\gamma} \)) be the category of \( \mathfrak{g}^\# \)- (respectively \( \mathfrak{g}^\#, \overline{\mathfrak{g}} \))-modules \( M \) such that \( M \) is a semisimple \( \mathfrak{g}^\# \)- (respectively \( \mathfrak{h}^*, \overline{\mathfrak{h}}^* \))-module and \( \dim M_\gamma < \infty \) for each \( \gamma \in \mathfrak{h}^* \) (respectively \( \mathfrak{h}^*, \overline{\mathfrak{h}}^* \)) satisfying
(i) \( M \) decomposes over \( \overline{I}_{\gamma} \) (respectively \( I_{\gamma}, \overline{I}_{\gamma} \)) into a direct sum of \( L(I_{\gamma}, \mu^\theta) \) (respectively \( L(I_{\gamma}, \mu), L(\overline{I}_{\gamma}, \mu^\theta) \)), \( \mu \in \mathcal{P}_{\gamma} \),
(ii) \( M \) has a filtration of \( \mathfrak{g}^\# \)- (respectively \( \mathfrak{g}, \overline{\mathfrak{g}} \))-modules \( M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots \) such that for all \( i \geq 0 \) \( M_i/M_{i+1} \cong L(\nu_i^\theta) \) (respectively \( L(\nu_i), L(\overline{\nu}_i^\theta) \)), for some \( \nu_i \in \mathcal{P}_{\gamma} \).

The morphisms in \( \mathcal{O}_{\gamma} \) are \( \mathfrak{g} \)-homomorphisms. The morphisms in \( \overline{\mathcal{O}}_{\gamma} \) (respectively \( \mathcal{O}_{\gamma}, \overline{\mathcal{O}}_{\gamma} \)) are (not necessarily even) \( \overline{\mathfrak{g}} \)- (respectively \( \overline{\mathfrak{g}}, \overline{\mathfrak{g}} \))-homomorphisms. Clearly \( K(\lambda) \), for \( \lambda \in \mathcal{P}_{\gamma} \), lies in \( \mathcal{O}_{\gamma} \). By [CK] Theorem 3.2 and [CK] Theorem 3.1, \( \overline{K}(\lambda^\theta) \) and \( \overline{K}(\lambda^2) \), for \( \lambda \in \mathcal{P}_{\gamma} \), lie in \( \overline{\mathcal{O}}_{\gamma} \) and \( \overline{\mathcal{O}}_{\gamma} \), respectively.

Let \( \overline{\mathcal{O}}_{\gamma} \) (respectively \( \mathcal{O}_{\gamma}, \overline{\mathcal{O}}_{\gamma} \)) denote the full subcategory of \( \overline{\mathcal{O}}_{\gamma} \) (respectively \( \mathcal{O}_{\gamma}, \overline{\mathcal{O}}_{\gamma} \)) consisting of objects having finite composition series.

Set \( \overline{N} := \sum_{j=-m}^{1} \mathbb{Z} \epsilon_j + \sum_{r \in \frac{1}{2}\mathbb{N}} \mathbb{Z} \epsilon_r \). Let \( \overline{V} \) be a semisimple \( \mathfrak{h} \)-module such that \( \overline{M} = \bigoplus_{\gamma \in \overline{N}} \overline{M}_\gamma \). Then \( \overline{V} \) is a \( \mathbb{Z}_2 \)-graded vector space \( \overline{V} = \overline{V}_0 \bigoplus \overline{V}_1 \) such that
\[ \overline{V}_0 := \bigoplus_{\mu \in \overline{N}_0} \overline{V}_\mu \quad \text{and} \quad \overline{V}_1 := \bigoplus_{\mu \in \overline{N}_1} \overline{V}_\mu, \] where \( \overline{N}_0 \) and \( \overline{N}_1 \) are the even and odd parts of \( \overline{N} \).\]
where \( \overline{\Gamma}_\epsilon := \{ \mu \in \overline{\Gamma} | \sum_{r \in 1/2 + \mathbb{Z}_+} \mu(E_{r,r}) \equiv \epsilon \mod 2 \} \).

Set \( \overline{\Gamma} := \bigoplus_{j=-\infty}^{\epsilon} \mathbb{Z} \epsilon_j + \sum_{r \in 1/2 + \mathbb{Z}_+} \mathbb{Z} \epsilon_r \). Let \( \overline{\mathcal{V}} \) be a semisimple \( \overline{\mathfrak{g}} \)-module such that \( \overline{\mathcal{M}} = \bigoplus_{\gamma \in \overline{\Gamma}} \overline{\mathcal{M}}_{\gamma} \). Then \( \overline{\mathcal{V}} \) is an \( \mathbb{Z}_2 \)-graded vector space \( \overline{\mathcal{V}} = \overline{\mathcal{V}}_0 \bigoplus \overline{\mathcal{V}}_1 \) such that
\[
\overline{\mathcal{V}}_0 := \bigoplus_{\mu \in \overline{\Gamma}_0} \overline{\mathcal{V}}_{\mu} \quad \text{and} \quad \overline{\mathcal{V}}_1 := \bigoplus_{\mu \in \overline{\Gamma}_1} \overline{\mathcal{V}}_{\mu},
\]
where \( \overline{\Gamma}_\epsilon := \{ \mu \in \overline{\Gamma} | \sum_{r \in 1/2 + \mathbb{Z}_+} \mu(E_{r,r}) \equiv \epsilon \mod 2 \} \).

It is clear that \( \mathcal{O}_Y \) is an abelian category. For \( \tilde{M} \in \mathcal{O}_Y \), let \( \tilde{M} \in \hat{\mathcal{O}}_Y \) denote the \( \mathfrak{g} \)-module \( \tilde{M} \) equipped with the \( \mathbb{Z}_2 \)-gradation given by (2.1). The \( \mathbb{Z}_2 \)-gradation given by (2.2) is compatible with the \( \mathfrak{g} \)-action. Therefore, for \( M, N \in \mathcal{O}_Y \), and \( \tilde{\varphi} \in \text{Hom}_{\mathcal{O}_Y}(M, N) \), the kernel and the cokernel of \( \tilde{\varphi} \) have structures of \( \mathbb{Z}_2 \)-graded vector spaces defined by (2.1). The induced \( \mathfrak{g} \)-actions on the kernel and cokernel are compatible with this \( \mathbb{Z}_2 \)-gradation. Thus the kernel and the cokernel of \( \tilde{\varphi} \) belong to \( \mathcal{O}_Y \), and hence \( \mathcal{O}_Y \) and \( \mathcal{O}^0_Y \) are abelian categories. Note that the homomorphic image of \( \tilde{\varphi} \) may not be a \( \mathfrak{g} \)-submodule of \( N \). Similarly, the \( \mathbb{Z}_2 \)-gradation given by (2.2) of any object \( \tilde{M} \in \mathcal{O}_Y \) is compatible with its \( \mathfrak{g} \)-action. Moreover, \( \mathcal{O}_Y \) and \( \mathcal{O}^0_Y \) are abelian categories.

We define \( \mathcal{O}^0_Y \) and \( \mathcal{O}^0_Y' \) to be the full subcategories of \( \hat{\mathcal{O}}_Y \) and \( \mathcal{O}^0_Y \), respectively, consisting of objects with \( \mathbb{Z}_2 \)-gradations given by (2.1) (c.f. [B, §4-e]). Note that the morphisms in \( \mathcal{O}^0_Y \) and \( \mathcal{O}^0_Y' \) are of degree 0. For \( \tilde{M} \in \mathcal{O}_Y \), it is clear that \( \tilde{M} \) is isomorphic to \( \tilde{M} \) in \( \mathcal{O}_Y \). Thus \( \mathcal{O}_Y \) and \( \mathcal{O}^0_Y \) have isomorphic skeletons and hence they are equivalent categories. Similarly, \( \mathcal{O}^0_Y \) and \( \mathcal{O}^0_Y' \) are equivalent categories.

Analogously define \( \mathcal{O}^0_Y \) and \( \mathcal{O}^0_Y' \) to be the respective full subcategories of \( \mathcal{O}_Y \) and \( \mathcal{O}^0_Y \) consisting of objects with \( \mathbb{Z}_2 \)-gradations given by (2.2). Similarly the morphisms in \( \mathcal{O}^0_Y \) and \( \mathcal{O}^0_Y' \) are of degree 0. Moreover, \( \mathcal{O}^0_Y \) and \( \mathcal{O}^0_Y \) are equivalent categories, and \( \mathcal{O}^0_Y \) and \( \mathcal{O}^0_Y' \) are equivalent categories.

### 3. Odd Reflection and Character Formulae

We shall briefly explain the effect of an odd reflection on the highest weight of an irreducible module that was studied in [PS, Lemma 1] (see also [KW, Lemma 1.4]). Fix a Borel subalgebra \( \mathcal{B} \) with corresponding set of positive roots \( \Delta_+(\mathcal{B}) \). Let \( \alpha \) be an isotropic odd simple root and \( \alpha^\vee \) be its corresponding coroot. Applying the odd reflection with respect to \( \alpha \) changes the Borel subalgebra \( \mathcal{B} \) into a new Borel subalgebra \( \mathcal{B}(\alpha) \) with corresponding set of positive roots \( \Delta_+(\mathcal{B}(\alpha)) = \{-\alpha \} \cup \Delta_+(\mathcal{B}) \setminus \{\alpha\} \). Now let \( \lambda \) be the highest weight with respect to \( \mathcal{B} \) of an irreducible module. If \( \langle \lambda, \alpha^\vee \rangle \neq 0 \), then the highest weight of this irreducible module with respect to \( \mathcal{B}(\alpha) \) is \( \lambda - \alpha \). If \( \langle \lambda, \alpha^\vee \rangle = 0 \), then the highest weight remains unchanged. In the sequel we will sometimes refer to the highest weight with respect to the new Borel subalgebra as the new highest weight.

#### 3.1. Odd reflection and a fundamental lemma.
3.1.1. A sequence of odd reflections and $\tilde{\Pi}^c(n)$. Starting with the Dynkin diagram (D1) of Section 2.1 and given a positive integer $n$ we apply the following sequence of $\frac{n(n+1)}{2}$ odd reflections. First we apply one odd reflection corresponding to $\epsilon_1 - \epsilon_1$, then we apply two odd reflections corresponding to $\epsilon_3/2 - \epsilon_2$ and $\epsilon_1 - \epsilon_2$. After that we apply three odd reflections corresponding to $\epsilon_5/2 - \epsilon_3$, $\epsilon_3/2 - \epsilon_3$, and $\epsilon_1/2 - \epsilon_3$, et cetera, until finally we apply $n$ odd reflections corresponding to $\epsilon_{n-1/2} - \epsilon_n$, $\epsilon_{n-3/2} - \epsilon_n$, $\ldots$, $\epsilon_{1/2} - \epsilon_n$. The resulting new Borel subalgebra for $\tilde{g}$ will be denoted by $\tilde{b}^c(n)$ and the corresponding simple roots are

$$\tilde{\Pi}^c(n) := \{\alpha_{-m}, \ldots, \alpha_{-2}; \beta_1, \ldots, \beta_{n-1}; \epsilon_n - \epsilon_1/2; \beta_1/2, \ldots, \beta_{n-1}/2; \alpha_{n+1}, \alpha_{n+1}, \ldots\}.$$ 

A Borel subalgebra is completely determined by giving an ordered homogeneous basis for the standard module. The ordered basis corresponding to the Borel subalgebra of $\tilde{\Pi}^c(n)$ is given below.

\[
\{v_{-m}, \ldots, v_1, v_3/2, \ldots, v_{n-1}, v_{n+1/2}, v_{n+3/2}, v_{n+2}, v_{n+5/2}, \ldots\}.
\]

3.1.2. A sequence of odd reflections and $\tilde{\Pi}^s(n)$. On the other hand given (D1) and $n$ we can also apply the following different sequence of $\frac{n(n+1)}{2}$ odd reflections. First we apply one odd reflection corresponding to $\epsilon_1 - \epsilon_3/2$, then we apply two odd reflections corresponding to $\epsilon_2 - \epsilon_5/2$ and $\epsilon_1 - \epsilon_5/2$. After that we apply three odd reflections corresponding to $\epsilon_3 - \epsilon_7/2$, $\epsilon_2 - \epsilon_7/2$, and $\epsilon_1 - \epsilon_7/2$, et cetera, until finally we apply $n$ odd reflections corresponding to $\epsilon_n - \epsilon_{n+1/2}$, $\epsilon_{n-1} - \epsilon_{n+1/2}$, $\ldots$, $\epsilon_{1} - \epsilon_{n+1/2}$. The resulting new Borel subalgebra for $\tilde{g}$ will be denoted by $\tilde{b}^s(n)$ and the corresponding simple roots are

$$\tilde{\Pi}^s(n) := \{\alpha_{-m}, \ldots, \alpha_{-2}; \beta_1, \ldots, \beta_{n-1}; \epsilon_n + \epsilon_1/2 - \epsilon_1; \beta_1/2, \ldots, \beta_{n-1}/2; \alpha_{n+1}, \alpha_{n+1}, \ldots\}.$$ 

The ordered basis corresponding to the Borel subalgebra of $\tilde{\Pi}^s(n)$ is given below.

\[
\{v_{-m}, \ldots, v_1, v_3/2, \ldots, v_{n-1}, v_{n+1/2}, v_1, v_2, \ldots, v_n, v_{n+1}, v_{n+3/2}, v_{n+2}, v_{n+5/2}, \ldots\}.
\]

**Remark 3.1.** We note that the simple roots used in the above two sequences of odd reflections are all roots of $\tilde{k}_Y$ and hence these sequences of odd reflections leave the set of roots of $\tilde{u}_Y$ invariant.

We denote by $\tilde{b}^c_Y(n)$ and $\tilde{b}^s_Y(n)$ the Borel subalgebras of $\tilde{k}_Y$ corresponding to the sets of simple roots $\tilde{\Pi}^c(n) \cap \Delta_Y$ and $\tilde{\Pi}^s(n) \cap \Delta_Y$, respectively.
3.1.3. A fundamental lemma.

**Lemma 3.2.** Given $\lambda \in \mathcal{P}_Y$, let $n \in \mathbb{N}$.

(i) Suppose that $\ell(\lambda_+) \leq n$. Then the highest weight of $L(\tilde{\mathcal{Y}}, \lambda^0)$ with respect to the Borel subalgebra $\mathfrak{b}^\ast_Y(n)$ is $\lambda$, regarded as an element in $\tilde{\mathcal{P}}_Y$.

(ii) Suppose that $\ell(\lambda_+) \leq n$. Then the highest weight of $L(\tilde{\mathcal{Y}}, \lambda^0)$ with respect to the Borel subalgebra $\mathfrak{b}^\ast_Y(n)$ is $\lambda^\natural$, regarded as an element in $\tilde{\mathcal{P}}_Y$.

**Proof.** We shall only give the proof for (i), as (ii) is analogous.

Certainly $\lambda^{<0}$ is unaffected by the sequence of odd reflections in Section 3.1.1. We will show more generally by induction on $k$ that after applying the first $k(k + 1)/2$ odd reflections in Section 3.1.1 this weight becomes

\[ \lambda_{[k]} = \lambda^{<0} + \sum_{i=1}^{k} \lambda_i \epsilon_i + \sum_{i=1}^{k} \left( (\lambda_+)_i - k \right) \epsilon_{i-1/2} + \sum_{j \geq k+1} \langle (\lambda_+)_j - j + 1 \rangle \epsilon_{j-1/2} + \sum_{j \geq k+1} \langle \lambda_j - j \rangle \epsilon_j. \]

From (3.1) the lemma follows.

Suppose that $k = 1$. If $\ell(\lambda_+) < 1$, then $\lambda^0 = \lambda^{<0}$ and in particular $\langle \lambda^0, E_{1/2,1/2} + E_{11} \rangle = 0$, and thus the new highest weight is $\lambda_{[1]} = \lambda^{<0} = \lambda^0$. If $\ell(\lambda_+) \geq 1$, then $(\lambda_+)_1 \geq 1$ and $\lambda_1 \geq 1$ and thus

\[ \lambda^0 = \lambda^{<0} + (\lambda_+)_1 \epsilon_{1/2} + (\lambda_1 - 1) \epsilon_1 + \cdots. \]

Now $\langle \lambda^0, E_{1/2,1/2} + E_{11} \rangle > 0$, and hence the highest weight after the odd reflection with respect to $\epsilon_{1/2} - \epsilon_1$ is

\[ \lambda_{[1]} = \lambda^{<0} + \lambda_1 \epsilon_1 + ((\lambda_+)_1 - 1) \epsilon_{1/2} + \cdots, \]

proving (3.1) in the case $k = 1$.

Now suppose that (3.1) is true for $k$. We shall derive the formula for $k+1$. If $\ell(\lambda_+) \leq k$, then $\lambda_{[k]} = \lambda^{<0} + \sum_{i=1}^{k} \lambda_i \epsilon_i$. Therefore we have $\langle \lambda_{[k]}, E_{i-1/2,i-1/2} + E_{k+1,k+1} \rangle = 0$, for $1 \leq i \leq k + 1$. So the odd reflections with respect to $\epsilon_{i-1/2} - \epsilon_{k+1}$ do not affect $\lambda_{[k]}$. Thus we have $\lambda_{[k+1]} = \lambda_{[k]}$. So in this case we are done.

Now assume that $\ell(\lambda_+) \geq k + 1$. Let $s = \lambda_{k+1}$. We distinguish two cases.

First suppose that $\lambda_{k+1} \geq k + 1$. Then $(\lambda_+''_{|k+1})_{k+1} \geq k + 1$ and hence (3.1) becomes

\[ \lambda_{[k]} = \lambda^{<0} + \sum_{i=1}^{k} \lambda_i \epsilon_i + \sum_{i=1}^{k} ((\lambda_+''_i)_i - k) \epsilon_{i-1/2} + ((\lambda_+''_{k+1})_{k+1} - k) \epsilon_{k+1/2} + (\lambda_{k+1} - k - 1) \epsilon_{k+1} + \sum_{j \geq k+2} \langle (\lambda_+''_j)_j - j + 1 \rangle \epsilon_{j-1/2} + \sum_{j \geq k+2} \langle \lambda_j - j \rangle \epsilon_j. \]
Now \( \langle \lambda_{[k]}, E_{k+1/2} + E_{k+1,k+1} \rangle > 0 \) so that after the odd reflection with respect to \( \epsilon_{k+1/2} - \epsilon_{k+1} \) the new weight becomes

\[
\lambda_{[k,1]} = \lambda^{<0} + \sum_{i=1}^{k} \lambda_i \epsilon_i + \sum_{i=1}^{k-1} ((\lambda_+)^i - k) \epsilon_{i-1/2} + ((\lambda_+)^{i+1} - k - 1) \epsilon_{k+1/2} \\
+ (\lambda_{k+1} - k) \epsilon_{k+1} + \sum_{j \geq k+2} \langle (\lambda_+)^j - j + 1 \rangle \epsilon_{j-1/2} + \sum_{j \geq k+2} \langle \lambda_j - j \rangle \epsilon_j.
\]

Now \( \langle \lambda_{[k,1]}, E_{k-1/2} + E_{k+1,k+1} \rangle > 0 \) so after the odd reflection with respect to \( \epsilon_{k-1/2} - \epsilon_{k+1} \) we get

\[
\lambda_{[k,2]} = \lambda^{<0} + \sum_{i=1}^{k} \lambda_i \epsilon_i + \sum_{i=1}^{k-1} ((\lambda_+)^i - k) \epsilon_{i-1/2} + ((\lambda_+)^{i+1} - k - 1) \epsilon_{k-1/2} \\
+ ((\lambda_+)^{k+1} - k - 1) \epsilon_{k+1/2} + (\lambda_{k+1} - k + 1) \epsilon_{k+1} \\
+ \sum_{j \geq k+2} \langle (\lambda_+)^j - j + 1 \rangle \epsilon_{j-1/2} + \sum_{j \geq k+2} \langle \lambda_j - j \rangle \epsilon_j.
\]

Finally after a total of \( k + 1 \) odd reflections we end up with

\[
\lambda_{[k,k+1]} = \lambda^{<0} + \sum_{i=1}^{k+1} \lambda_i \epsilon_i + \sum_{i=1}^{k+1} ((\lambda_+)^i - k) \epsilon_{i-1/2} \\
+ \sum_{j \geq k+2} \langle (\lambda_+)^j - j + 1 \rangle \epsilon_{j-1/2} + \sum_{j \geq k+2} \langle \lambda_j - j \rangle \epsilon_j,
\]

which equals \( \lambda_{[k+1]} \).

Now consider the case \( \lambda_{k+1} = s < k + 1 \). We have \( (\lambda_+)^j \geq k + 1 \), for \( j \leq s \) and \( (\lambda_+)^j < k + 1 \), for \( j > s \). Thus (3.1) becomes

\[
\lambda_{[k]} = \lambda^{<0} + \sum_{i=1}^{k} \lambda_i \epsilon_i + \sum_{i=1}^{s} ((\lambda_+)^i - k) \epsilon_{i-1/2},
\]

where \( ((\lambda_+)^i - k) > 0 \), for \( i \leq s \). It follows that odd reflections with respect to \( \epsilon_{k+1/2} - \epsilon_{k+1}, \cdots, \epsilon_{s+1/2} - \epsilon_{k+1} \) do not affect \( \lambda_{[k]} \), while odd reflections with respect to \( \epsilon_{s-1/2} - \epsilon_{k+1}, \cdots, \epsilon_{1/2} - \epsilon_{k+1} \) affect \( \lambda_{[k]} \). From this we obtain

\[
\lambda_{[k+1]} = \lambda_{[k,k+1]} = \lambda^{<0} + \sum_{i=1}^{k+1} \lambda_i \epsilon_i + \sum_{i=1}^{s} ((\lambda_+)^i - k - 1) \epsilon_{i-1/2},
\]

which concludes the proof. \( \square \)

**Corollary 3.3.** Let \( \lambda \in P_Y \) and \( n \in \mathbb{N} \).

(i) Suppose that \( \ell(\lambda_+) \leq n \). Then \( \tilde{K}(\lambda^0) \) is a highest weight module with respect to the Borel subalgebra \( \tilde{b}^c(n) \) with highest weight \( \lambda \), regarded as an element in \( \tilde{P}_Y \). Also the highest weight of \( \tilde{L}(\lambda^0) \) with respect to the Borel subalgebra \( \tilde{b}^c(n) \) is \( \lambda \), regarded as an element in \( \tilde{P}_Y \).
(ii) Suppose that \( \ell(\lambda^\ell) \leq n \). Then \( \widetilde{K}(\lambda^\theta) \) is a highest weight module with respect to the Borel subalgebra \( \tilde{\mathfrak{h}}^s(n) \) with highest weight \( \lambda^2 \), regarded as an element in \( \widetilde{P}_Y \). Also the highest weight of \( \tilde{L}(\lambda^\theta) \) with respect to the Borel subalgebra \( \tilde{\mathfrak{b}}^s(n) \) is \( \lambda^3 \), regarded as an element in \( \widetilde{P}_Y \).

**Proof.** As an \( \tilde{I}_Y \)-module \( \widetilde{K}(\lambda^\theta) \) contains a unique copy of \( L(\tilde{I}_Y, \lambda^\theta) \) that is annihilated by \( \tilde{u}_Y \). By Lemma 3.2 with respect to \( \tilde{b}_Y^c(n) \) the highest weight of \( L(\tilde{I}_Y, \lambda^\theta) \) is \( \lambda \). Now by Remark 3.1 \( \tilde{b}_Y^c(n) + \tilde{u}_Y = \tilde{b}^c(n) \). Thus \( \widetilde{K}(\lambda^\theta) \) has a non-zero vector of weight \( \lambda \) annihilated by \( \tilde{b}^c(n) \). This vector clearly generates \( \widetilde{K}(\lambda^\theta) \) over \( \tilde{g} \), proving the first statement of (i). A verbatim argument proves the second statement as well.

Part (ii) is similar and so its proof is omitted. \( \square \)

### 3.2. The Functors \( T \) and \( \mathcal{T} \)

Recall that \( \tilde{I} := \sum_{j=-m}^{-1} \mathbb{Z} e_j + \sum_{r \in \mathbb{N}_0} \mathbb{Z} e_r \) and \( \mathcal{T} := \sum_{j=-m}^{-1} \mathbb{Z} e_j + \sum_{r \in \mathbb{Z}_+} \mathbb{Z} e_r \). Set \( \Gamma := \sum_{j=-m}^{-1} \mathbb{Z} e_j + \sum_{i \in \mathbb{N}} \mathbb{Z} e_i \). Given a semisimple \( \tilde{h} \)-module \( M \) such that \( M = \bigoplus_{\gamma \in \Gamma} \tilde{M}_\gamma \), we define

\[
T(M) := \bigoplus_{\gamma \in \Gamma} \tilde{M}_\gamma, \quad \text{and} \quad \mathcal{T}(M) := \bigoplus_{\gamma \in \Gamma} \tilde{M}_\gamma.
\]

Note that \( T(M) \) is an \( \tilde{h} \)-submodule of \( M \) (regarded as an \( \tilde{h} \)-module), and \( \mathcal{T}(M) \) is an \( \tilde{h} \)-submodule of \( M \) (regarded as an \( \tilde{h} \)-module). Also if \( M \) is also an \( \tilde{I}_Y \)-module, then \( T(M) \) is an \( \tilde{I}_Y \)-submodule of \( M \) (regarded as an \( \tilde{I}_Y \)-module), and \( \mathcal{T}(M) \) is an \( \tilde{I}_Y \)-submodule of \( M \) (regarded as an \( \tilde{I}_Y \)-module). Furthermore if \( M \in \tilde{O}_Y \), then \( T(M) \) is a \( g \)-submodule of \( M \) (regarded as a \( g \)-module), and \( \mathcal{T}(M) \) is a \( \tilde{g} \)-submodule of \( M \) (regarded as a \( \tilde{g} \)-module).

Let \( \tilde{M} = \bigoplus_{\gamma \in \Gamma} \tilde{M}_\gamma \) and \( \tilde{N} = \bigoplus_{\gamma \in \Gamma} \tilde{N}_\gamma \) be two semisimple \( \tilde{h} \)-modules. We let

\[
T_{\tilde{M}} : \tilde{M} \longrightarrow T(\tilde{M}) \quad \text{and} \quad \mathcal{T}_{\tilde{M}} : \tilde{M} \longrightarrow \mathcal{T}(\tilde{M})
\]

be the natural projections. If \( \tilde{f} : \tilde{M} \longrightarrow \tilde{N} \) is an \( \tilde{h} \)-homomorphism, we let

\[
T[\tilde{f}] : T(\tilde{M}) \longrightarrow T(\tilde{N}) \quad \text{and} \quad \mathcal{T}[\tilde{f}] : \mathcal{T}(\tilde{M}) \longrightarrow \mathcal{T}(\tilde{N})
\]

be the corresponding restriction maps. Note that \( T_{\tilde{M}} \) and \( T[\tilde{f}] \) (respectively, \( \mathcal{T}_{\tilde{M}} \) and \( \mathcal{T}[\tilde{f}] \)) are \( \tilde{h} \)- (respectively, \( \tilde{h} \)-) homomorphisms. Also if \( \tilde{f} \) is also \( \tilde{I}_Y \)-homomorphism of \( \tilde{I}_Y \)-modules, then \( T_{\tilde{M}} \) and \( T[\tilde{f}] \) (respectively, \( \mathcal{T}_{\tilde{M}} \) and \( \mathcal{T}[\tilde{f}] \)) are \( \tilde{I}_Y \)- (respectively, \( \tilde{I}_Y \)-) homomorphisms. Furthermore if \( \tilde{f} \) is also \( \tilde{g} \)-homomorphism of \( \tilde{g} \)-modules, then \( T_{\tilde{M}} \) and \( T[\tilde{f}] \) (respectively, \( \mathcal{T}_{\tilde{M}} \) and \( \mathcal{T}[\tilde{f}] \)) are \( g \)- (respectively, \( \tilde{g} \)-) homomorphisms. It is easy to see that we have the following commutative diagrams.

\[
\begin{array}{ccc}
\tilde{M} & \xrightarrow{\tilde{f}} & \tilde{N} \\
\downarrow T_{\tilde{M}} & & \downarrow T_{\tilde{N}} \\
T(\tilde{M}) & \xrightarrow{T[\tilde{f}]} & T(\tilde{N})
\end{array}
\]

\[
\begin{array}{ccc}
\tilde{M} & \xrightarrow{\tilde{f}} & \tilde{N} \\
\downarrow \mathcal{T}_{\tilde{M}} & & \downarrow \mathcal{T}_{\tilde{N}} \\
\mathcal{T}(\tilde{M}) & \xrightarrow{\mathcal{T}[\tilde{f}]} & \mathcal{T}(\tilde{N})
\end{array}
\]
For an indeterminate $e$ we let $x_r := e^{r_i}, r \in [-m, -1] \cup \frac{1}{2} \mathbb{N}$. The formal character of an object in $\mathcal{O}_Y, \mathcal{O}'_Y$, and $\mathcal{O}_Y$ is then an element in $\mathbb{Z}[[x_{-m}^\pm, \ldots, x_{-1}^\pm]] \otimes \mathbb{Z}[x_{1/2}, x_{1}, \bar{x}_{3/2}, x_{3/2}, \ldots], Z[[x_{-m}^\pm, \ldots, x_{-1}^\pm]] \otimes \mathbb{Z}[x_{1}, x_{2}, \bar{x}_{3}, x_{3}, \ldots]$, and $Z[[x_{-m}^\pm, \ldots, x_{-1}^\pm]] \otimes \mathbb{Z}[x_{1}, x_{2}, \ldots]$, respectively. For $\lambda \in \mathcal{P}_Y$, we remark that the character of $L(\tilde{\mathfrak{g}}^\theta, \lambda^\theta)$ is given by [CK] Section 3.2.3

\begin{equation}
chL(\tilde{\mathfrak{g}}^\theta, \lambda^\theta) = chL(\tilde{\mathfrak{g}}^<0, \lambda^<0)HS_{\lambda^\theta}(x_{1/2}, x_{1}, x_{3/2}, x_{2}, \ldots),
\end{equation}

where (c.f. [S, BR])

$$HS_\eta(x_{1/2}, x_{1}, x_{3/2}, x_{2}, \ldots) := \sum_{\mu \leq \eta} s_\mu(x_{1/2}, x_{3/2}, \ldots)s(\eta/\mu)(x_{1}, x_{2}, \ldots), \quad \eta \in \mathcal{P}.$$}

Here and below $L(\tilde{\mathfrak{g}}^<0, \lambda^<0)$ stands for the irreducible $\tilde{\mathfrak{g}}^<0$-module of highest weight $\lambda^<0$ and so $chL(\tilde{\mathfrak{g}}^<0, \lambda^<0)$ is a product of Schur Laurent polynomials in $x_{-m}, \ldots, x_{-1}$ depending on $Y$ and $\lambda^<0$.

**Lemma 3.4.** For $\lambda \in \mathcal{P}_Y$, we have

(i) $T(L(\tilde{\mathfrak{g}}^\theta, \lambda^\theta)) = L(\mathfrak{g}^\theta, \lambda),$

(ii) $T(L(\tilde{\mathfrak{g}}^\theta, \lambda^\theta)) = L(\tilde{\mathfrak{g}}^\theta, \lambda^\theta).$

**Proof.** Since $L(\mathfrak{g}^\theta, \lambda)$ is irreducible, it is enough to prove that $T(L(\tilde{\mathfrak{g}}^\theta, \lambda^\theta))$ and $L(\mathfrak{g}^\theta, \lambda)$ have the same character.

Applying $T$ to $L(\mathfrak{g}^\theta, \lambda^\theta)$ has the effect of setting $x_{j-1/2} = 0, j \in \mathbb{N}$, in the character. Thus we have

$$chT(L(\tilde{\mathfrak{g}}^\theta, \lambda^\theta)) = chL(\tilde{\mathfrak{g}}^<0, \lambda^<0)s_{\lambda^\theta}(x_{1}, x_{2}, \ldots),$$

which is the character of $L(\mathfrak{g}^\theta, \lambda)$. This proves (i).

The proof for (ii) is analogous and hence omitted. \qed

**Lemma 3.5.** If $\tilde{M}$ is a highest weight $\tilde{\mathfrak{g}}$-module of highest weight $\lambda^\theta$ with $\lambda \in \mathcal{P}_Y$, then $T(\tilde{M})$ and $\overline{T}(\tilde{M})$ are highest weight $\mathfrak{g}$- and $\overline{\mathfrak{g}}$-modules of highest weight $\lambda \in \mathcal{P}_Y$ and $\lambda^\theta \in \overline{\mathcal{P}}_Y$, respectively.

**Proof.** We will only show this for $T(\tilde{M})$, as the case of $\overline{T}(\tilde{M})$ is analogous. Let $v$ be a nonzero vector in $\tilde{M}$ of weight $\lambda$ obtained from a non-zero vector of weight $\lambda^\theta$ by applying the sequence of odd reflections of Section 3.1.1. Such a vector by Corollary 3.3 is a $\tilde{\mathfrak{g}}^\theta(n)$-highest weight vector of the $\tilde{\mathfrak{g}}$-module $M$, for $n \gg 0$. Evidently $v \in T(M)$ and, since $\mathfrak{b} = \tilde{\mathfrak{g}}^\theta(n) \cap \mathfrak{g}$, $v$ is a $\mathfrak{b}$-singular vector. The $\mathfrak{g}$-module $T(M)$, regarded as an $\mathfrak{I}_Y$-module, is completely reducible by Lemma 3.3. Thus to prove the lemma it is enough to show that every vector $w \in T(\tilde{M})$ of weight $\mu \in \mathcal{P}_Y$ lies in $\mathfrak{u}(n_-)v$. To see this, choose $n$ so that $\ell(\lambda^\theta) < n$ and $\ell(\mu^\theta) < n$. Then with respect to $\tilde{\mathfrak{g}}^\theta(n)$, $v$ is a highest weight vector of $\tilde{M}$ and hence $w \in \mathfrak{u}(\widetilde{\mathfrak{n}}^\theta(n))v$, where $\widetilde{\mathfrak{n}}^\theta(n)$ is the opposite nilradical of $\tilde{\mathfrak{g}}^\theta(n)$. Now the conditions $\ell(\lambda^\theta) < n$ and $\ell(\mu^\theta) < n$ imply that

\begin{equation}
\lambda - \mu = \sum_{i=-m}^{-1} a_i \epsilon_i + \sum_{j=1}^{n-1} b_j \epsilon_j, \quad a_i, b_j \in \mathbb{Z}
\end{equation}
But $\lambda - \mu$ is also a finite $\mathbb{Z}_+$-linear combination of simple roots from $\tilde{\Phi}^c(n)$. So we can write

$$\lambda - \mu = \sum_{\alpha \in \tilde{\Phi}^c(n)} a_\alpha \alpha, \quad a_\alpha \in \mathbb{Z}_+. $$

If there were some $\alpha \in \{\epsilon_n - \epsilon_1/2; \beta_{1/2}, \cdots, \beta_{n-1/2}; \alpha_{n+1/2}, \alpha_{n+1}, \cdots\}$ with $a_\alpha \neq 0$, then it is easy to see that $\langle \lambda - \mu, E_{rr} \rangle \neq 0$, for $r \notin [-m, -1] \cup [1, n - 1]$. It contradicts (3.4). Therefore $\lambda - \mu$ is a $\mathbb{Z}_+$-linear combination of $\{\alpha_m, \cdots, \alpha_{-2}; \beta_1, \cdots, \beta_{n-1}\}$, and hence $w \in \mathcal{U}(\mathfrak{n}_-)v).

**Theorem 3.6.** For $\lambda \in \mathcal{P}_Y$, we have

$$T(\tilde{K}(\lambda^\theta)) = K(\lambda), \quad T(\tilde{L}(\lambda^\theta)) = L(\lambda);$$

$$\mathcal{T}(\tilde{K}(\lambda^\theta)) = \mathcal{K}(\lambda^\theta), \quad \mathcal{T}(\tilde{L}(\lambda^\theta)) = \mathcal{L}(\lambda^\theta).$$

**Proof.** We will show this for $T$. The argument for $\mathcal{T}$ is analogous. Computing the character of $\tilde{K}(\lambda^\theta)$ we have (see (3.3))

$$\text{ch}\tilde{K}(\lambda^\theta) = \prod_{i<0,j \in \mathbb{N}, r \in \frac{1}{2} + \mathbb{Z}_+} \frac{(1 + x_r^{-1}x_j)}{(1 - x_i^{-1}x_j)} \text{ch(Ind}_{\tilde{\mathfrak{g}}^{<0}} L(\tilde{\mathfrak{g}}^{<0}, \lambda^<0)) \text{HS}_{\lambda^\theta}(x_1^+, x_1, x_2^+, x_2, \cdots).$$

Application of $T$ amounts to setting the variables $x_r = 0, r \in \frac{1}{2} + \mathbb{N}$. Thus

$$\text{ch}T(\tilde{K}(\lambda^\theta)) = \prod_{i<0,j \in \mathbb{N}} \frac{1}{(1 - x_i^{-1}x_j)} \text{ch(Ind}_{\tilde{\mathfrak{g}}^{<0}} L(\tilde{\mathfrak{g}}^{<0}, \lambda^<0)) \text{hs}_{\lambda^\theta}(x_1, x_2, \cdots),$$

which equals $\text{ch}K(\lambda)$. Since $T(\tilde{K}(\lambda^\theta))$ is a highest weight module by Lemma 3.3 we see that $T(\tilde{K}(\lambda^\theta)) = K(\lambda).

Let $\tilde{M} := \tilde{L}(\lambda^\theta)$ with $\lambda \in \mathcal{P}_Y$. Suppose that $M := T(\tilde{M})$ is not irreducible. Since by Lemma 3.3 the $\mathfrak{g}$-module $M$ is a highest weight module, it must have a $b$-singular vector inside $M$ that is not a highest weight vector. Suppose that $w$ is such a $b$-singular vector of weight $\mu \in \mathcal{P}_Y$. We can choose $n \gg 0$ such that $\lambda$ is the highest weight of $\tilde{M}$ with respect to $\mathfrak{b}^c(n)$, and $\ell(\lambda_-) < n$ and $\ell(\mu_+) < n$. By Corollary 3.3 there exists a $\tilde{\mathfrak{b}}^c(n)$-highest weight vector $v_\lambda$ of the $\mathfrak{g}$-module $\tilde{M}$ of weight $\lambda$. It is clear that $v_\lambda$ is a $b$-highest weight vector of $\tilde{M}$ and hence $w \in \mathcal{U}(\mathfrak{a})v_\lambda$, where $\mathfrak{a}$ is the subalgebra of $\mathfrak{n}_-$ generated by root vectors in $\mathfrak{n}_-$ corresponding to the roots $-\alpha_m, \cdots, -\alpha_{-2}, -\beta_1$ and $-\beta_j, 1 \leq j \leq k$, for some $k$.

Choose $q \in \mathbb{N}$ such that $q \geq n$ and $q > k + 1$. Note that $v_\lambda$ is also a $\tilde{\mathfrak{b}}^c(q)$-highest weight vector of the $\mathfrak{g}$-module $\tilde{M}$ of weight $\lambda$. Since $w$ is $b$-singular it is annihilated by the root vectors corresponding to the root $\alpha_m, \cdots, \alpha_{-2}, -\beta_1, -\beta_j$, for all $j \in \mathbb{N}$. Also $w$ is annihilated by the root vectors corresponding to the root in $\tilde{\Phi}^c(q) \setminus \{\alpha_m, \cdots, \alpha_{-2}, \beta_1, \beta_j, \cdots, \beta_{q-1}\}$ since $w \in \mathcal{U}(\mathfrak{a})v_\lambda$ and these root vectors commute with $\mathfrak{a}$. It follows that $w$, regarded as in $\tilde{M}$, is then a $\tilde{\mathfrak{b}}^c(q)$-singular vector, contradicting the irreducibility of $\tilde{M}$. $\square$

**Proposition 3.7.** $T$ and $\mathcal{T}$ define exact functors from $\tilde{\mathcal{O}}_Y$ to $\mathcal{O}_Y$ and from $\tilde{\mathcal{O}}_Y$ to $\mathcal{O}_Y$, respectively. Furthermore, $T$ and $\mathcal{T}$ send $\tilde{\mathcal{O}}^f_Y$ to $\mathcal{O}^f_Y$ and $\tilde{\mathcal{O}}^f_Y$ to $\mathcal{O}^f_Y$, respectively.
Proof. Exactness is clear from the definitions.

It remains to prove that if \( \widetilde{M} \in \mathcal{O}_Y \), then \( T(\widetilde{M}) \in \mathcal{O}_Y \) and \( \mathcal{T}(\widetilde{M}) \in \mathcal{O}_Y \). We will only prove \( T(\widetilde{M}) \in \mathcal{O}_Y \), as the proof of \( \mathcal{T}(\widetilde{M}) \in \mathcal{O}_Y \) is analogous.

Clearly \( \dim T(\widetilde{M}) \gamma < \infty \), for all \( \gamma \in h^* \). Now if \( \widetilde{M} \cong \bigoplus_{\mu \in \mathcal{P}_Y} L(\widetilde{\iota}_Y, \mu^\theta)^{m(\mu)} \), then by Lemma 3.3 \( T(\widetilde{M}) \cong \bigoplus_{\mu \in \mathcal{P}_Y} L(\iota_Y, \mu)^{m(\mu)} \). (Here and below \( m(\mu) \) stands for the multiplicity of \( L(\iota_Y, \mu^\theta) \) in \( \widetilde{M} \).)

Finally by Theorem 3.6 \( T(\widetilde{L}(\lambda^\theta)) = L(\lambda) \). By exactness of \( T \) it follows that a downward filtration for \( \widetilde{M} \) gives rise to a corresponding downward filtration of \( T(\widetilde{M}) \) with a one-to-one correspondence between the composition factors. Hence \( T(\widetilde{M}) \in \mathcal{O}_Y \).

The second part of the proposition is clear. \( \square \)

3.3. Some consequences. Let \( M \in \mathcal{O}_Y \). We may regard \( \text{ch}M \) as an element in \( \mathbb{Z}[[x_{-m}^\pm, \cdots, x_{-1}^\pm]] \otimes \Lambda_Z(x_1, x_2, \cdots) \), where \( \Lambda_Z(x_1, x_2, \cdots) \) denotes the space of (completed) symmetric functions in the variables \( x_1, x_2, \cdots \). Similarly, for \( \widetilde{M} \in \mathcal{O}_Y \) and \( \widetilde{M} \in \mathcal{O}_Y \), \( \text{ch}M \) and \( \text{ch}M \) may be viewed as elements in \( \mathbb{Z}[[x_{-m}^\pm, \cdots, x_{-1}^\pm]] \otimes \Lambda_Z(x_1, x_2, \cdots) \) and \( \mathbb{Z}[[x_{-m}^\pm, \cdots, x_{-1}^\pm]] \otimes \Lambda_Z(x_1, x_2, \cdots) \), respectively.

Let \( \overline{\omega} : \Lambda_Z(x_1, x_2, \cdots) \rightarrow \Lambda_Z(x_1, x_2, \cdots) \) be the ring homomorphism that sends the \( n \)th complete symmetric function in \( x_1, x_2, \cdots \) to the \( n \)th elementary symmetric function in \( x_1, x_2, \cdots \). Let \( \overline{\omega} : \Lambda_Z(x_1, x_2, \cdots) \rightarrow \Lambda_Z(x_1, x_2, \cdots) \) be the ring homomorphism defined by sending the \( n \)th complete symmetric function in \( x_2, x_3, \cdots \) (respectively in \( x_1, x_3, \cdots \)) to the \( n \)th complete symmetric function in \( x_1, x_2, \cdots \) (respectively to the \( n \)th elementary symmetric function in \( x_1, x_2, \cdots \)).

**Corollary 3.8.** Let \( \lambda \in \mathcal{P}_Y \). We have

(i) \( \overline{\omega}(\text{ch}L(\lambda)) = \text{ch}L(\lambda) \).

(ii) \( \overline{\omega}(\text{ch}L(\lambda)) = \text{ch}L(\lambda) \).

**Proof.** Since \( \overline{\omega}(\lambda) = \lambda \), \( \overline{\omega}(\text{ch}L(\lambda)) = \text{ch}L(\lambda) \), the corollary follows directly from Lemma 3.3 and Theorem 3.6. \( \square \)

**Remark 3.9.** Corollary 3.8 (ii) is consistent with the prediction of the super duality conjecture, and in the case of irreducible polynomial representations (i.e. \( \lambda \in \mathcal{P}_{[-m,-2]} \)) with \( \lambda_1 \geq \lambda_2 \) gives \cite[Theorem 6.10]{BH}. In the case of \( Y = [m-2] \) it gives \cite[Corollary 6.15]{CWZ}. For infinite-dimensional unitary modules appearing in certain Howe dualities it also recovers \cite[Theorem 5.3]{CLZ}.

For \( n \in \mathbb{N} \), we recall the truncation functor \( \mathfrak{t}_n : \mathcal{O}_Y \rightarrow (\mathcal{O}_Y)_n \) of \cite[Definition 3.1]{CW}, where here and further we use a subscript \( n \) to indicate a corresponding truncated category of \( \mathfrak{gl}(m|n) \)-modules. For \( \gamma \in \sum_{i=m}^{n} \mathbb{Z} \epsilon_i + \sum_{j=1}^{n} \mathbb{Z} \epsilon_{j-1} \) let \( \mathcal{K}_n(\gamma) \) and \( \mathcal{T}_n(\gamma) \) be the parabolic Verma \( \mathfrak{gl}(m|n) \)-module and irreducible \( \mathfrak{gl}(m|n) \)-module of highest weight \( \gamma \) in the category \( (\mathcal{O}_Y)_n \), respectively. We recall the following.

**Lemma 3.10.** \cite[Corollary 3.3]{CW} Let \( \lambda \in \mathcal{P}_Y \). The truncation functor \( \mathfrak{t}_n \), for every \( n \in \mathbb{N} \), is exact and it sends \( \mathcal{K}(\lambda) \) and \( \mathcal{T}(\lambda) \) to \( \mathcal{K}_n(\lambda) \) and \( \mathcal{T}_n(\lambda) \), respectively, if \( \{\lambda, E_{n+1/2,n+1/2}\} = 0 \), and to zero otherwise.
Proposition 3.11. The module $\overline{K}(\lambda)$ lies in $\overline{O}_Y^f$, for all $\lambda \in P_Y$. Thus category $\overline{O}_Y^f$ is the category of finitely generated $\mathfrak{g}$-modules that as $\mathfrak{g}_Y$-modules are direct sums of $L(\tilde{I}_Y, \mu)$, $\mu \in P_Y$, with a locally nilpotent $\mathfrak{u}_Y$-action.

Proof. Consider a fixed $\lambda \in P_Y$. Choose $n \gg 0$ so that $\langle \lambda, E_{n+1/2, n+1/2} \rangle = 0$ and the degree of atypicality for $\lambda$ does not increase anymore with increasing $n$. Assume $\overline{L}(\mu)$ is a composition factor in $\overline{K}(\lambda)$. We have $\mu \in P_Y$. Choose $k \geq n$ such that $\text{tr}_k(\overline{L}(\mu)) \neq 0$. Then $\lambda$ and $\mu$ share the same central character in $\overline{O}_k$. Therefore our choice of $n$ together with $\mu \in P_Y$ implies that $\langle \mu, E_{n+1/2, n+1/2} \rangle = 0$. Thus by Lemma 3.10 the multiplicity of each $\overline{L}_n(\mu)$ inside each $\overline{K}_n(\lambda)$ is the same as that of $\overline{L}(\mu)$ in $\overline{K}(\lambda)$. Since the $\mathfrak{gl}(m|n)$-module $\overline{K}_n(\lambda)$ has finite composition series (because as a $\mathfrak{gl}(m|n)_0$-module it is isomorphic to the tensor product of a generalized Verma module and a finite-dimensional module), it follows that $\overline{K}(\lambda) \in \overline{O}_Y^f$.

By a standard argument a finitely generated $\mathfrak{g}$-module $\overline{M}$ that as $\mathfrak{g}_Y$-module is a direct sum of $L(\tilde{I}_Y, \mu)$, $\mu \in P_Y$, with a locally nilpotent $\mathfrak{u}_Y$-action, has a finite filtration by highest weight modules, which are quotients of $\overline{K}(\lambda)$, for $\lambda \in P_Y$. Thus $\overline{M} \in \overline{O}_Y^f$ and hence the proposition follows.

Theorem 3.6 and Proposition 3.11 give the following.

Corollary 3.12. (i) The module $\tilde{K}(\lambda) \in \tilde{O}_Y^f$, for all $\lambda \in P_Y$. Hence the category $\tilde{O}_Y^f$ is the category of finitely generated $\mathfrak{g}$-modules that as $\mathfrak{g}_Y$-modules are direct sums of $L(\tilde{I}_Y, \mu)$, $\mu \in P_Y$, with a locally nilpotent $\mathfrak{u}_Y$-action.

(ii) The module $\hat{K}(\lambda) \in \hat{O}_Y^f$, for all $\lambda \in P_Y$. Hence the category $\hat{O}_Y^f$ is the category of finitely generated $\mathfrak{g}$-modules that as $\mathfrak{g}_Y$-modules are direct sums of $L(\tilde{I}_Y, \mu)$, $\mu \in P_Y$, with a locally nilpotent $\mathfrak{u}_Y$-action.

Remark 3.13. Proposition 3.11 and its Corollary 3.12 imply that the categories $\overline{O}_Y^f$ and $\overline{O}_Y^f$ are the categories $\hat{O}_{m, \infty}^+$ and $\hat{O}_{m, \infty}^+$ of [CW], respectively. We note that the proof of Proposition 3.11 that we have presented above is elementary. In the proof above we have only used the rather easy Lemma 3.10.

By Theorem 3.6 and Proposition 3.7 and Corollary 3.12 we have the following.

Corollary 3.14. Let $\lambda, \mu \in P_Y$. The numbers of composition factors of $\tilde{K}(\lambda^\theta)$, $K(\lambda)$ and $\overline{K}(\lambda^\theta)$ that are isomorphic to $\tilde{L}(\mu^\theta)$, $L(\mu)$ and $\overline{L}(\mu^\theta)$, respectively, are the same.

Recall the super Bruhat ordering for weights in $\overline{h}^*$ (see e.g. [B, §2-b] or [CW, §2.3]) which we denote by $\preceq$. Let us denote by $\succeq$ the classical Bruhat ordering on $\hat{h}^*$. As a further application we present a super analogue of a classical theorem of BGG.

Corollary 3.15. Let $\lambda \in P_Y$ and $\gamma \in \overline{h}^*$. If $\overline{L}(\gamma)$ is a subquotient of $\overline{K}(\lambda^\theta)$, then $\lambda^\theta \preceq \gamma$.

Proof. Clearly $\gamma = \mu^\theta$ for some $\mu \in P_Y$. By Corollary 3.14 $\overline{L}(\mu^\theta)$ is a subquotient of $\overline{K}(\lambda^\theta)$ if and only if $L(\mu)$ is a subquotient of $K(\lambda)$. By the classical version of the BGG Theorem (e.g. [H, Section 5.1]) $\mu \preceq \lambda$. Now [CW, Lemma 4.6] implies that $\mu^\theta \preceq \lambda^\theta$. □
3.4. Irreducible characters.

**Theorem 3.16.** Let $\lambda \in \mathcal{P}_Y$. Let $\text{ch}L(\lambda) = \sum_{\mu \in \mathcal{P}_Y} a_{\mu \lambda} \text{ch}K(\mu)$. Then

(i) $\text{ch}L(\lambda^0) = \sum_{\mu \in \mathcal{P}_Y} a_{\mu \lambda} \text{ch}\overline{K}(\mu^0),$

(ii) $\text{ch}L(\lambda^2) = \sum_{\mu \in \mathcal{P}_Y} a_{\mu \lambda} \text{ch}\overline{K}(\mu^2).

**Proof.** Since $\overline{\omega}(\text{ch}K(\lambda)) = \text{ch}\overline{K}(\lambda^2)$ and $\overline{\omega}(\text{ch}K(\lambda)) = \text{ch}\overline{K}(\lambda^0)$, for $\lambda \in \mathcal{P}_Y$, the theorem follows directly from Corollary 3.8. □

**Remark 3.17.** By [HL] the coefficients $a_{\mu \lambda}$ in Theorem 3.16 equal $f_{\mu \lambda}(1)$, where $f_{\mu \lambda}(q)$ are the classical (parabolic) Kazhdan-Lusztig polynomials [DJ] [KL] (see also [CW, Proposition 4.4]). Since by [CW] Theorem 4.7 the polynomials $f_{\mu \lambda}(q)$ equal $\ell_{\mu \lambda}(q)$ (see [HL]) Theorem 3.16 verifies [CW, Conjecture 3.10], which is a parabolic version of a conjecture of Brundan [B, Conjecture 4.32]. In particular, Theorem 3.16 in the special case $Y = [-m, -2]$, together with Lemma 3.10 gives an independent new proof of the first part of [B, Theorem 4.37]. We note that our results do not rely on [B, Section 3.4].

Below we work out in more detail a character formula for the irreducible $\mathfrak{gl}(m|n)$-module $\mathcal{L}_n(\gamma)$, where $\gamma$ is a weight of the form

$$-1 \sum_{i=-m}^{n-1} \gamma_i \epsilon_i + \sum_{j=1}^{n} \gamma_j \epsilon_{j-1}/2, \quad \gamma_i, \gamma_j \in \mathbb{Z},$$

with $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n$. Recall that the one-dimensional determinant module $\text{det}^0$ has (highest) weight $1_{m|n} = - \sum_{i=-m}^{n-1} \epsilon_i - \sum_{j=1}^{n} \epsilon_{j-1}/2$. For $k \in \mathbb{Z}$ and an $\mathfrak{g}$-semisimple $\mathfrak{gl}(m|n)$-module $M$ with $\text{ch}M = \sum_{\eta} \dim M_\eta e^K$ we have

$$\text{ch}(M \otimes \text{det}^0) = \sum_{\eta} \dim M_\eta e^{\eta + k1_{m|n}}.$$

Clearly $\mathcal{L}_n(\gamma + k1_{m|n}) = \mathcal{L}_n(\gamma) \otimes \text{det}^0$. Thus taking tensor product with a suitable power of the determinant module, if necessary, we may assume that $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n \geq 0$ and so $\gamma \in \mathcal{P}_Y$.

Let $\lambda \in \mathcal{P}_Y$ (with $Y = \emptyset$) be such that $\lambda^2 = \gamma$. We have $\ell(\lambda^+) \leq n$. Now Theorem 3.16 (ii) (together with Lemma 3.10) implies that the character of the irreducible $\mathfrak{gl}(m|n)$-module of highest weight $\lambda^2$ equals to

$$\text{ch}\mathcal{L}_n(\gamma) = \sum_{\mu \in \mathcal{P}_Y, \ell(\mu^+) \leq n} a_{\mu \lambda} \text{ch}\mathcal{K}_n(\mu^2),$$

where $\mathcal{K}_n(\mu^2)$ is the parabolic Verma $\mathfrak{gl}(m|n)$-module corresponding to $Y = \emptyset$. As the coefficients $a_{\mu \lambda}$ are known by Remark 3.17, (3.5) gives the irreducible character for $\mathfrak{gl}(m|n)$-module $\mathcal{L}_n(\gamma)$. In the special case of $\gamma - m \geq \gamma - m + 1 \geq \cdots \geq \gamma - 1$ we obtain an irreducible character formula for finite-dimensional irreducible $\mathfrak{gl}(m|n)$-module. A formula (corresponding to our case $Y = [-m, -2]$) was obtained in [Se] [B]. Theorem 3.16 is obtained using an approach very different from [Se] and [B], and provides an independent solution of the irreducible character problem.
4. Kazhdan-Lusztig Polynomials

4.1. Homology of Lie superalgebras. Let \( L = L_0 \oplus L_1 \) be a Lie superalgebra and let \( \mathcal{T}(L) \) be the tensor algebra of \( L \). Then \( \mathcal{T}(L) = \bigoplus_{n=0}^{\infty} \mathcal{T}^n(L) \) is an associative superalgebra with a canonical \( \mathbb{Z} \)-gradation. For \( v \in L \), we let \( |v| := \epsilon, \epsilon \in \mathbb{Z}_2 \).

The exterior algebra of \( L \) is the quotient algebra \( \Lambda(L) := \mathcal{T}(L)/J \), where \( J \) is the homogeneous two-sided ideal of \( \mathcal{T}(L) \) generated by the elements of the form

\[
x \otimes y + (-1)^{|x||y|} y \otimes x,
\]

where \( x \) and \( y \) are homogeneous elements of \( L \). The \( \Lambda(L) \) is also an associative superalgebra with a \( \mathbb{Z} \)-gradation inherited from \( \mathcal{T}(L) \). More precisely, we have \( \Lambda(L) = \bigoplus_{n=0}^{\infty} \Lambda^nL \), where \( \Lambda^nL \) is the set of all homogeneous elements of \( \mathbb{Z} \)-degree \( n \) in \( \Lambda(L) \), for each \( n \geq 0 \). For \( \mathbb{Z}_2 \)-homogeneous elements \( x_1, x_2, \ldots, x_k \in L \), the image of the element \( x_1 \otimes x_2 \otimes \cdots \otimes x_k \) under the canonical quotient map from \( \mathcal{T}^k(L) \) to \( \Lambda^k(L) \) will be denoted by \( x_1x_2 \cdots x_k \).

For an \( L \)-module \( V \), the \( k \)th homology group \( H_k(L; V) \) of \( L \) with coefficient in \( V \) is defined to be the \( k \)th homology group of the following complex (see e.g. [11]):

\[
\cdots \xrightarrow{\partial} \Lambda^n(L) \otimes V \xrightarrow{\partial} \Lambda^{n-1}(L) \otimes V \xrightarrow{\partial} \cdots \xrightarrow{\partial} \Lambda^1(L) \otimes V \xrightarrow{\partial} \Lambda^0(L) \otimes V \longrightarrow 0,
\]

where the boundary operator \( \partial \) is given by

\[
(4.1) \quad \partial(x_1x_2 \cdots x_n \otimes v) := \sum_{1 \leq s < t \leq n} (-1)^{s+t+|x_s|+|x_t|} |x_s|+|x_t| \cdot x_s \cdot x_t \cdots x_n \otimes v
\]

\[
+ \sum_{s=1}^{n} (-1)^{s+|x_s|} |x_s| \cdot x_s \cdots x_n \otimes x_s v.
\]

Here the \( x_s \) are homogeneous elements in \( L \) and \( v \in V \). Furthermore \( [x_s, x_t] \in \Lambda(L) \) denotes the linear term corresponding to \( [x_s, x_t] \in L \) and, as usual, \( \tilde{y} \) indicates that the term \( y \) is omitted.

4.2. Comparison of homology groups. Here and further we shall suppress the subscript \( Y \) and denote \( (\tilde{u}_Y)_-, (\tilde{u}_Y)_+ \) and \( (\tilde{u}_Y)_- \) by \( \tilde{u}_- \), \( \tilde{u}_+ \) and \( \tilde{u}_- \), respectively.

For \( \tilde{M} \in \tilde{\mathcal{O}}_Y^\dagger \) we denote by \( M = T(\tilde{M}) \in \mathcal{O}_Y^\dagger \) and \( \overline{M} = \overline{T(\tilde{M})} \in \overline{\mathcal{O}}_Y^\dagger \). Let \( \tilde{d} : \Lambda(\tilde{u}_-) \otimes \tilde{M} \rightarrow \Lambda(\tilde{u}_-) \otimes \tilde{M}, d : \Lambda(\tilde{u}_-) \otimes M \rightarrow \Lambda(\tilde{u}_-) \otimes M \) and \( \overline{d} : \Lambda(\overline{u}_-) \otimes \overline{M} \rightarrow \Lambda(\overline{u}_-) \otimes \overline{M} \) be the boundary operator of the complex of \( \tilde{u}_- \)-homology with coefficients in \( \tilde{M} \), the boundary operator of the complex of \( u_- \)-homology with coefficients in \( M \) and the boundary operator of the complex of \( \overline{u}_- \)-homology with coefficients in \( \overline{M} \), respectively. Note that \( \tilde{d}, d, \) and \( \overline{d} \) are \( \mathfrak{l}_Y \)-homomorphism, \( \mathfrak{i}_Y \)-homomorphism and \( \overline{\mathfrak{l}}_Y \)-homomorphism, respectively.

The following lemma is easy.

Lemma 4.1. We have

(i) \( T(\Lambda(\tilde{u}_-)) = \Lambda(u_-) \),

(ii) \( \overline{T(\Lambda(\tilde{u}_-))} = \Lambda(\overline{u}_-) \).
The \( \Lambda \)-module \( \Lambda(u_-) \) is a direct sum of \( L(I_Y, \mu) \), \( \mu \in \mathcal{P}_Y \), each appearing with finite multiplicity. Using \([\mathbf{BR}]\) one can show that \( \Lambda(\pi_-) \), as an \( \tilde{I}_Y \)-module, is a direct sum of \( L(\tilde{I}_Y, \mu^\circ) \), \( \mu \in \mathcal{P}_Y \), each appearing with finite multiplicity (\([\mathbf{CK}] \) Lemma 3.2). Similarly, it follows that \( \Lambda(\tilde{u}_-) \), as an \( \tilde{I}_Y \)-module, is a also direct sum of \( L(\tilde{I}_Y, \mu^\circ) \), \( \mu \in \mathcal{P}_Y \), each appearing with finite multiplicity (\([\mathbf{CK}] \) Section 3.2.3).

The \( \tilde{I}_Y \)-module \( \Lambda(\tilde{u}_-) \otimes \tilde{M} \) is of course completely reducible. The \( \tilde{I}_Y \)-module \( \Lambda(\tilde{u}_-) \otimes \tilde{M} \) is also direct sum of \( L(\tilde{I}_Y, \mu^\circ) \), \( \mu \in \mathcal{P}_Y \).

**Lemma 4.2.** For \( \tilde{M} \in \tilde{\mathcal{O}}_Y^\dagger \) and \( \lambda \in \mathcal{P}_Y \), we have

(i) \( T(\Lambda(\tilde{u}_-) \otimes \tilde{M}) = \Lambda(u_-) \otimes M \), and thus \( T(\Lambda(\tilde{u}_-) \otimes \tilde{L}(\lambda^\circ)) = \Lambda(u_-) \otimes L(\lambda) \).

Moreover, \( T[\tilde{d}] = d \).

(ii) \( \tilde{T}(\Lambda(\tilde{u}_-) \otimes \tilde{M}) = \Lambda(\tilde{u}_-) \otimes \tilde{M} \), and thus \( \tilde{T}(\Lambda(\tilde{u}_-) \otimes \tilde{L}(\lambda^\circ)) = \Lambda(\tilde{u}_-) \otimes \tilde{L}(\lambda^\circ) \).

Moreover, \( \tilde{T}[\tilde{d}] = \tilde{d} \).

**Proof.** By Lemma 4.1 Theorem 3.6 and the compatibility of \( T \) and \( \tilde{T} \) under tensor product we have the first part of (i) and (ii). Using the definitions \([4.1]\) of \( \tilde{d} \), \( d \) and \( \tilde{d} \), we have \( \tilde{d}(v) = d(v) \) for all \( v \in \Lambda(u_-) \otimes M \) and \( \tilde{d}(w) = \tilde{d}(w) \) for all \( v \in \Lambda(\tilde{u}_-) \otimes \tilde{M} \).

Hence we have \( T[\tilde{d}] = d \) and \( \tilde{T}[\tilde{d}] = \tilde{d} \).

\( \square \)

Lemmas 3.4 and 4.2 now imply the following.

**Lemma 4.3.** Suppose \( \Lambda(\tilde{u}_-) \otimes \tilde{M} \cong \bigoplus_{\mu \in \mathcal{P}_Y} L(I_Y, \mu)^{m(\mu)} \), as \( \tilde{I}_Y \)-modules. Then

(i) \( \Lambda(u_-) \otimes M \cong \bigoplus_{\mu \in \mathcal{P}_Y} L(I_Y, \mu)^{m(\mu)} \), as \( I_Y \)-modules.

(ii) \( \Lambda(\pi_-) \otimes \tilde{M} \cong \bigoplus_{\mu \in \mathcal{P}_Y} L(\tilde{I}_Y, \mu^\circ)^{m(\mu)} \), as \( \tilde{I}_Y \)-modules.

By Lemma 4.2 and 3.2, we have the following commutative diagram.

\[
\begin{array}{cccccc}
\cdots & \longrightarrow & \Lambda^{n+1}(\tilde{u}_-) \otimes \tilde{M} & \overset{d}{\longrightarrow} & \Lambda^n(\tilde{u}_-) \otimes \tilde{M} & \overset{d}{\longrightarrow} & \Lambda^{n-1}(\tilde{u}_-) \otimes \tilde{M} & \overset{d}{\longrightarrow} & \cdots \\
\text{\( T \)-} & & \downarrow{T_{\Lambda^{n+1}(\tilde{u}_-) \otimes \tilde{M}}} & & \downarrow{T_{\Lambda^n(\tilde{u}_-) \otimes \tilde{M}}} & & \downarrow{T_{\Lambda^{n-1}(\tilde{u}_-) \otimes \tilde{M}}} & & \\
\cdots & \longrightarrow & \Lambda^{n+1}(u_-) \otimes M & \overset{d}{\longrightarrow} & \Lambda^n(u_-) \otimes M & \overset{d}{\longrightarrow} & \Lambda^{n-1}(u_-) \otimes M & \overset{d}{\longrightarrow} & \cdots 
\end{array}
\]

Thus \( T \) induces an \( I_Y \)-homomorphism from \( H_n(\tilde{u}_-; \tilde{M}) \) to \( H_n(u_-; M) \). Similarly, \( \tilde{T} \) induces an \( \tilde{I}_Y \)-homomorphism from \( H_n(\tilde{u}_-; \tilde{M}) \) to \( H_n(\tilde{u}_-; \tilde{M}) \). Moreover, we have the following.

**Theorem 4.4.** We have for \( n \geq 0 \)

(i) \( T(H_n(\tilde{u}_-; \tilde{M})) \cong H_n(u_-; M) \), as \( I_Y \)-modules.

(ii) \( \tilde{T}(H_n(\tilde{u}_-; \tilde{M})) \cong H_n(\tilde{u}_-; \tilde{M}) \), as \( \tilde{I}_Y \)-modules.

**Proof.** We shall only prove (i), as the argument for (ii) is parallel. By Lemma 4.2 and 4.2, we have

(\( T(\text{Ker}(\tilde{d})) = \text{Ker}(d) \cap (\Lambda(u_-) \otimes M) = \text{Ker}(d) \))

and

(\( T(\text{Im}(\tilde{d})) = \text{Im}(\tilde{d}) \cap (\Lambda(u_-) \otimes M) = \text{Im}(d) \)).
Since $T$ is an exact functor, we have
\[ T(\bigoplus_{n \geq 0} H_n(\bar{u}_-; \bar{M})) = T(\ker(d))/T(\text{Im}(d)) = \ker(d)/\text{Im}(d) = \bigoplus_{n \geq 0} H_n(\bar{u}_-; M). \]

This completes the proof of the theorem. \(\square\)

Theorem 3.6 implies the following.

**Corollary 4.5.** For $\lambda \in P_Y$ and $n \geq 0$, we have

1. $T(H_n(\bar{u}_-; \bar{L}(\lambda^0))) \cong H_n(\bar{u}_-; L(\lambda))$, as $\mathfrak{l}_Y$-modules.
2. $T(H_n(\bar{u}_-; \bar{L}(\lambda^0))) \cong H_n(\bar{u}_-; \bar{L}(\lambda))$, as $\hat{\mathfrak{l}}_Y$-modules.

### 4.3. Kazhdan-Lusztig polynomials

Let $\mathfrak{g}_{t_\infty}$ be the infinite-dimensional general linear algebra with basis consisting of elementary matrices $E_{ij}$, $i, j \in \mathbb{Z}$. Let $U_q(\mathfrak{g}_{t_\infty})$ be its quantum group acting on the natural module $\mathbb{V}$ (see [B §2-c] or [CW §2.1] for precise definition). Let $\mathcal{W}$ be the restricted dual of $\mathbb{V}$ ([B §2-d], [CW §2.3]). Let $m_1, \ldots, m_s \in \mathbb{N}$ with $\sum_{i=1}^s m_i = m$ and $\bar{L}_Y^0 \cong \bigoplus_{i=1}^s \mathfrak{g}_0(m_i)$. Consider a certain topological completion $\hat{\mathcal{E}}^{m|\infty}$ of the Fock space ([B §2-d], [CW §2.3])

\[ \hat{\mathcal{E}}^{m|\infty} := \Lambda^{m_1}(\mathbb{V}) \otimes \Lambda^{m_2}(\mathbb{V}) \otimes \cdots \Lambda^{m_s}(\mathbb{V}) \otimes \Lambda^\infty(\mathcal{W}). \]

By arguments essentially going back to [KL] (c.f. [B Theorem 2.17]) $\hat{\mathcal{E}}^{m|\infty}$ has three sets of distinguished basis, namely the standard, canonical and dual canonical basis, parameterized by $P_Y$, denoted respectively by $\{K_{f_{\lambda}}|\lambda \in P_Y\}$, $\{U_{f_{\lambda}}|\lambda \in P_Y\}$, and $\{T_{f_{\lambda}}|\lambda \in P_Y\}$. Furthermore one has

\[ (4.3) \quad U_{f_{\lambda}} = \sum_{\mu \in P_Y} u_{\mu, \lambda}(q) K_{f_{\mu}}, \quad T_{f_{\lambda}} = \sum_{\mu \in P_Y} \ell_{\mu, \lambda}(q) K_{f_{\mu}}, \]

where $u_{\mu, \lambda}(q) \in \mathbb{Z}[q]$ and $\ell_{\mu, \lambda}(q) \in \mathbb{Z}[q^{-1}]$ [CW (2.3)], which are parabolic versions of ([B (2.18)]).

The following theorem is an analogue of Vogan’s cohomological interpretation of the Kazhdan-Lusztig polynomials.

**Theorem 4.6.** We have for $\lambda, \mu \in P_Y$ 

\[ \ell_{\mu, \lambda}(q^{-1}) = \sum_{n=0}^{\infty} \dim \text{Hom}_{\mathbb{V}}(L(\bar{u}_Y, \mu^\circ), H_n(\bar{u}_-; L(\lambda^\circ))) q^n. \]

**Proof.** Consider a topological completion $\hat{\mathcal{E}}^{m+\infty}$ of the Fock space ([CW §2.2])

\[ \hat{\mathcal{E}}^{m+\infty} := \Lambda^{m_1}(\mathbb{V}) \otimes \Lambda^{m_2}(\mathbb{V}) \otimes \cdots \Lambda^{m_s}(\mathbb{V}) \otimes \Lambda^\infty(\mathbb{V}). \]

$\hat{\mathcal{E}}^{m+\infty}$ has the standard, canonical and dual canonical basis, parameterized by $P_Y$, denoted respectively by $\{K_{f_{\lambda}}|\lambda \in P_Y\}$, $\{U_{f_{\lambda}}|\lambda \in P_Y\}$, and $\{L_{f_{\lambda}}|\lambda \in P_Y\}$. Similarly to ([L3]) one has

\[ U_{f_{\lambda}} = \sum_{\mu \in P_Y} u_{\mu, \lambda}(q) K_{f_{\mu}}, \quad L_{f_{\lambda}} = \sum_{\mu \in P_Y} \ell_{\mu, \lambda}(q) K_{f_{\mu}}, \]

\[ \ell_{\mu, \lambda}(q^{-1}) = \sum_{n=0}^{\infty} \dim \text{Hom}_{\mathbb{V}}(L(\bar{u}_Y, \mu^\circ), H_n(\bar{u}_-; L(\lambda^\circ))) q^n. \]
where $u_{\mu \lambda}(q) \in \mathbb{Z}[q]$ and $l_{\mu \lambda}(q) \in \mathbb{Z}[q^{-1}]$. It is folklore that (c.f. [CW, Theorems 4.15 and 4.16])

\begin{equation}
chL(\lambda) = \sum_{\mu \in \mathcal{P}_Y} l_{\mu \lambda}(1)chK(\mu).
\end{equation}

From [V, Conjecture 3.4] and the Kazhdan-Lusztig conjecture proved in [BB, BK] we conclude

\[ l_{\mu \lambda}(-q^{-1}) = \sum_{n=0}^{\infty} \dim \mathbb{C} Hom_{\mathcal{O}_Y}(L(l_Y, \mu), H_n(u_{-}; L(\lambda))) q^n. \]

Now by [CW, Theorem 4.7] we have

\[ \ell_{\mu^\natural \lambda^\natural}(q) = l_{\mu \lambda}(q). \]

By Corollary 4.5 we have

\[ \dim \mathbb{C} Hom_{\mathcal{O}_Y}(L(l_Y, \mu), H_n(u_{-}; L(\lambda))) = \dim \mathbb{C} Hom_{\tilde{\mathcal{O}}_Y}(L(l_Y, \mu^\natural), H_n(u_{-}; L(\lambda^\natural))), \]

and hence the theorem follows. \qed

**Remark 4.7.** Let $\tilde{U}(\lambda^\natural)$ denote the tilting module in $\tilde{\mathcal{O}}_Y$ corresponding to $\lambda^\natural \in \mathcal{P}_Y$. By Remark [3.17] we have $ch\tilde{U}(\lambda^\natural) = \sum_{\mu \in \mathcal{P}_Y} u_{\mu^\natural \lambda^\natural}(1)ch\tilde{K}(\mu^\natural)$. This, together with the remark following [CW, Conjecture 3.10], implies that

\[ ch\tilde{U}(\lambda^\natural) = \sum_{\mu \in \mathcal{P}_Y} u_{\mu^\natural \lambda^\natural}(1)ch\tilde{K}(\mu^\natural). \]

### 5. Super Duality

#### 5.1. Equivalence of the categories $\mathcal{O}_Y\textsuperscript{f}$ and $\mathcal{O}_Y\textsuperscript{f}$

The goal of this section is to establish the following.

**Theorem 5.1.** Recall $T$ and $\tilde{T}$ from Section 3.2. We have the following.

(i) $T : \tilde{\mathcal{O}}_Y = \mathcal{O}_Y$ is an equivalence of categories.

(ii) $\tilde{T} : \mathcal{O}_Y = \mathcal{O}_Y$ is an equivalence of categories.

(iii) The categories $\mathcal{O}_Y\textsuperscript{f}$ and $\tilde{\mathcal{O}}_Y\textsuperscript{f}$ are equivalent.

Since by Section 2.5 $\tilde{\mathcal{O}}_Y\textsuperscript{0} = \mathcal{O}_Y\textsuperscript{0}$ and $\tilde{\mathcal{O}}_Y\textsuperscript{0} = \tilde{\mathcal{O}}_Y\textsuperscript{0}$ it is enough to prove Theorem 5.1 for $\mathcal{O}_Y\textsuperscript{0}$ and $\tilde{\mathcal{O}}_Y\textsuperscript{0}$. In order to keep notation simple we will from now on drop the superscript 0 and use $\mathcal{O}_Y$ and $\tilde{\mathcal{O}}_Y$ to denote the respective categories $\mathcal{O}_Y\textsuperscript{0}$ and $\tilde{\mathcal{O}}_Y\textsuperscript{0}$ for the remainder of the article. Henceforth, when we write $\tilde{K}(\lambda^0) \in \tilde{\mathcal{O}}_Y$ and $\tilde{L}(\lambda^0) \in \tilde{\mathcal{O}}_Y$, $\lambda \in \mathcal{P}_Y$, we will mean the corresponding modules equipped with the $\mathbb{Z}_2$-gradation (2.1).

Similar convention applies to $\tilde{K}(\lambda^0) \in \tilde{\mathcal{O}}_Y$ and $\tilde{L}(\lambda^0) \in \tilde{\mathcal{O}}_Y$.

For $M, N \in \mathcal{O}_Y$, and $i \in \mathbb{N}$ the $i$th extension $Ext_{\mathcal{O}_Y}(M, N)$ can be understood in the sense of Baer-Yoneda (see e.g. [M Chapter VII]) and $Ext_{\mathcal{O}_Y}(M, N) := Hom_{\mathcal{O}_Y}(M, N)$.

In a similar way extensions in $\mathcal{O}_Y$ and $\tilde{\mathcal{O}}_Y$ can be interpreted. From this viewpoint the
exact functors $T$ and $\overline{T}$ induce natural maps on extensions by taking the projection of the corresponding exact sequences.

For $\tilde{M} \in \mathcal{O}_Y^f$, we let $M = T(M)$ and $\overline{M} = \overline{T}(\tilde{M})$. Since all the proofs in this section for the functors $T$ and $\overline{T}$ are parallel, we shall only give proofs for the functor $T$ without further explanation.

**Lemma 5.2.** Let

$$0 \to A \to B \to C \to 0$$

be an exact sequence of $\tilde{g}$-modules (respectively, $g$-modules, $\overline{g}$-modules) such that $A$, $C \in \mathcal{O}_Y^f$ (respectively, $\mathcal{O}_Y^f$, $\overline{\mathcal{O}}_Y^f$). Then $B$ also belongs to $\mathcal{O}_Y^f$ (respectively, $\mathcal{O}_Y^f$, $\overline{\mathcal{O}}_Y^f$).

**Proof.** The statement for $\mathcal{O}_Y^f$ is clear. The statements for the categories $\overline{\mathcal{O}}_Y^f$ and $\overline{\mathcal{O}}_Y^f$ follow, for example, from [CK, Theorems 3.1 and 3.2]. \square

For $\tilde{g}$-(respectively, $\tilde{\mathfrak{R}}$, and $g$)-modules $A$ and $C$, let $\mathcal{E}xt^i_{(\overline{u}(g),\overline{u}(\tilde{\mathfrak{R}}))}(C, A)$ (respectively, $\mathcal{E}xt^i_{(\overline{u}(g),\overline{u}(g))}(C, A)$ and $\mathcal{E}xt^i_{(\overline{u}(g),\overline{u}(g))}(C, A))$ denote the $i$th relative extension group of $A$ by $C$ (see e.g. [Ku, Appendix D]).

Let $C$ be a $\tilde{u}_Y$-(respectively, $\tilde{\Pi}_Y$-, and $u_Y$-)modules. Let $H^i(\tilde{u}_Y; C)$ (respectively, $H^i(\tilde{\Pi}_Y; C)$ and $H^i(u_Y; C)$) denote the $i$th $\tilde{u}_Y$-(respectively, $\tilde{\Pi}_Y$- and $u_Y$-)cohomology group with coefficients in $C$. Let $\mathcal{H}^i(\tilde{u}_Y; C)$ (respectively, $\mathcal{H}^i(\tilde{\Pi}_Y; C)$ and $\mathcal{H}^i(u_Y; C)$) denote the $i$th restricted (in the sense of [LI, Section 4]) $\tilde{u}_Y$-(respectively, $\tilde{\Pi}_Y$- and $u_Y$-)cohomology group with coefficients in the $C$.

The following proposition is an analogue of [RW] §7 Theorem 2] (cf. [Ku, Lemma 9.1.8]).

**Proposition 5.3.** Let $\lambda \in \mathcal{P}_Y$ and $\tilde{N} \in \tilde{\mathcal{O}}_Y$. For $i \geq 0$, we have

(i) $$\mathcal{E}xt^i_{(\overline{u}(g),\overline{u}(\tilde{\mathfrak{R}}))}(K(\lambda^0), \tilde{N}) \cong \text{Hom}_{\tilde{\mathfrak{R}}_Y}(L(\tilde{\mathfrak{R}}, \lambda^0), H^i(\tilde{\Pi}_Y; \tilde{N})) \cong \text{Hom}_{\tilde{\mathfrak{R}}_Y}(L(\tilde{\mathfrak{R}}, \lambda^0), \mathcal{H}^i(\tilde{\Pi}_Y; \tilde{N})).$$

(ii) $$\mathcal{E}xt^i_{(\overline{u}(g),\overline{u}(\tilde{\mathfrak{R}}))}(K(\lambda^1), \tilde{N}) \cong \text{Hom}_{\tilde{\mathfrak{R}}_Y}(L(\tilde{\mathfrak{R}}, \lambda^1), H^i(\tilde{\Pi}_Y; \tilde{N})) \cong \text{Hom}_{\tilde{\mathfrak{R}}_Y}(L(\tilde{\mathfrak{R}}, \lambda^1), \mathcal{H}^i(\tilde{\Pi}_Y; \tilde{N})).$$

(iii) $$\mathcal{E}xt^i_{(\overline{u}(g),\overline{u}(\tilde{\mathfrak{R}}))}(K(\lambda), N) \cong \text{Hom}_{\tilde{\mathfrak{R}}_Y}(L(\tilde{\mathfrak{R}}, \lambda), H^i(\mathfrak{R}_Y; N)) \cong \text{Hom}_{\tilde{\mathfrak{R}}_Y}(L(\tilde{\mathfrak{R}}, \lambda), \mathcal{H}^i(\mathfrak{R}_Y; N)).$$

**Proof.** We have the following relative version of Koszul resolution for the trivial module $\tilde{\mathfrak{R}}_Y$-modules (see e.g. [GL §1]):

$$\cdots \tilde{\partial}_{k+1} \tilde{C}_k \tilde{\partial}_k \tilde{C}_{k-1} \tilde{\partial}_{k-1} \cdots \tilde{\partial}_1 \tilde{C}_0 \tilde{\varepsilon} \tilde{C} \to 0,$$

$$\cdots \partial_{k+1} C_k \partial_k C_{k-1} \partial_{k-1} \cdots \partial_1 C_0 \varepsilon C \to 0, \tag{5.1}$$
where \( \tilde{C}_k := \mathfrak{U}(\tilde{p}_Y) \otimes_{\mathfrak{U}(\tilde{i}_Y)} \Lambda^k(\tilde{p}_Y/\tilde{i}_Y) \) is a \( \tilde{p}_Y \)-module with \( \tilde{p}_Y \) acting on the left of first factor for \( k \geq 0 \) and \( \tilde{e} \) is the augmentation map from \( \mathfrak{U}(\tilde{p}_Y) \) to \( \mathbb{C} \). The \( \tilde{p}_Y \)-homomorphism \( \tilde{\partial}_k \) is given by

\[
(\tilde{\partial}_k)(a \otimes \mathfrak{t}_1 \mathfrak{t}_2 \ldots \mathfrak{t}_k) = \sum_{1 \leq s < t \leq k} (-1)^{s+t}x_s \sum_{i=1}^{s-1} |x_i| + |x_t| + |x_t| + |x_s| + |x_t| x_t \otimes \mathfrak{t}_1 \mathfrak{t}_2 \ldots \mathfrak{t}_s \mathfrak{t}_t \mathfrak{t}_k,
\]

\[
+ \sum_{s=1}^k (-1)^{s+1} |x_s| \sum_{i=1}^{s-1} |x_i| a x_s \otimes \mathfrak{t}_1 \mathfrak{t}_2 \ldots \mathfrak{t}_s \mathfrak{t}_t \mathfrak{t}_k.
\]

Here \( a \in \mathfrak{U}(\tilde{p}_Y) \) and the \( x_is \) are homogeneous elements in \( \tilde{p}_Y \) and \( \mathfrak{t}_i \) denotes \( x_i + \tilde{i}_Y \) in \( \tilde{p}_Y/\tilde{i}_Y \). As usual, \( \hat{y} \) indicates that the term \( y \) is omitted. Since \( \tilde{C}_k \cong \mathfrak{U}(\hat{\mathfrak{u}}_Y) \otimes \Lambda^k \hat{\mathfrak{u}}_Y \) as \( \tilde{i}_Y \)-module, \( \tilde{C}_k \) is completely reducible as \( \tilde{i}_Y \)-module, and hence the image of \( \tilde{\partial}_k \) is a direct summand of \( \tilde{C}_{k-1} \).

Let \( L(\tilde{i}_Y, \lambda^\theta) \) also denote the irreducible \( \tilde{p}_Y \)-module on which \( \tilde{u}_Y \) acts trivially. For \( \lambda \in \mathfrak{P}_Y \) and \( k \geq 0 \), \( \tilde{D}_k := \tilde{C}_k \otimes L(\tilde{i}_Y, \lambda^\theta) \) is a \( \tilde{p}_Y \)-module. Tensoring (5.1) with \( L(\tilde{i}_Y, \lambda^\theta) \) we obtain an exact sequence of \( \tilde{p}_Y \)-modules

\[
\cdots \to \tilde{\partial}_{k+1} \tilde{D}_k \tilde{\partial}_k \tilde{D}_{k-1} \tilde{\partial}_{k-1} \cdots \tilde{\partial}_1 \tilde{D}_0 \tilde{\partial}_0 \to L(\tilde{i}_Y, \lambda^\theta) \to 0,
\]

where \( \tilde{\partial}_k := \tilde{\partial}_k \otimes 1 \) for \( k > 0 \) and \( \tilde{\partial}_0 := \tilde{e} \otimes 1 \).

For \( k \geq 0 \), let \( \tilde{E}_k := \mathfrak{U}(\hat{\mathfrak{g}}) \otimes_{\mathfrak{U}(\tilde{p}_Y)} \tilde{D}_k \). Tensoring (5.3) with \( \mathfrak{U}(\hat{\mathfrak{g}}) \otimes_{\mathfrak{U}(\tilde{p}_Y)} \) we obtain an exact sequence of \( \hat{\mathfrak{g}} \)-modules

\[
\cdots \to \tilde{\partial}_{k+1} \tilde{E}_k \tilde{\partial}_k \tilde{E}_{k-1} \tilde{\partial}_{k-1} \cdots \tilde{\partial}_1 \tilde{E}_0 \tilde{\partial}_0 \to K(\lambda^\theta) \to 0,
\]

where \( \tilde{\partial}_k := 1 \otimes \tilde{\partial}_k \) for \( k \geq 0 \). We observe that

\[
\tilde{E}_k = \mathfrak{U}(\hat{\mathfrak{g}}) \otimes_{\mathfrak{U}(\tilde{p}_Y)} \left( \tilde{C}_k \otimes L_0(\tilde{i}_Y, \lambda^\theta) \right) = \mathfrak{U}(\hat{\mathfrak{g}}) \otimes_{\mathfrak{U}(\tilde{p}_Y)} \left( \mathfrak{U}(\tilde{p}_Y) \otimes_{\mathfrak{U}(\tilde{i}_Y)} \left( \Lambda^k(\tilde{p}_Y/\tilde{i}_Y) \otimes L(\tilde{i}_Y, \lambda^\theta) \right) \right) = \mathfrak{U}(\hat{\mathfrak{g}}) \otimes_{\mathfrak{U}(\tilde{i}_Y)} \left( \Lambda^k(\tilde{p}_Y/\tilde{i}_Y) \otimes L(\tilde{i}_Y, \lambda^\theta) \right).
\]

By [Ku, Lemma 3.1.7] the \( \tilde{E}_k \)s are \( (\mathfrak{U}(\hat{\mathfrak{g}}), \mathfrak{U}(\tilde{i}_Y)) \)-projective modules. Since the image of \( \tilde{\partial}_k \) in (5.1) is a \( \mathfrak{U}(\tilde{i}_Y) \)-direct summand of \( \tilde{C}_{k-1} \), the image of \( \tilde{\rho}_k \) in (5.4) is also a \( \mathfrak{U}(\tilde{i}_Y) \)-direct summand of \( \tilde{E}_{k-1} \), for \( k \geq 1 \), and hence (5.4) is a \( (\mathfrak{U}(\hat{\mathfrak{g}}), \mathfrak{U}(\tilde{i}_Y)) \)-projective resolution of \( K(\lambda^\theta) \). It follows therefore that the relative extension group \( \mathcal{E}xt^1_{\mathfrak{U}(\hat{\mathfrak{g}}), \mathfrak{U}(\tilde{i}_Y)}(K(\lambda^\theta), \tilde{N}) \) equals the \( i \)th cohomology group of the following complex:

\[
0 \to \text{Hom}_{\mathfrak{U}(\hat{\mathfrak{g}})}(\tilde{E}_0, \tilde{N}) \overset{\tilde{\rho}_0}{\to} \text{Hom}_{\mathfrak{U}(\hat{\mathfrak{g}})}(\tilde{E}_1, \tilde{N}) \overset{\tilde{\rho}_1}{\to} \text{Hom}_{\mathfrak{U}(\hat{\mathfrak{g}})}(\tilde{E}_2, \tilde{N}) \overset{\tilde{\rho}_2}{\to} \cdots
\]

Since

\[
\text{Hom}_{\mathfrak{U}(\hat{\mathfrak{g}})}(\tilde{E}_i, \tilde{N}) \cong \text{Hom}_{\mathfrak{U}(\tilde{i}_Y)}(L(\tilde{i}_Y, \lambda^\theta), \text{Hom}_{\mathfrak{C}}(\Lambda^i \tilde{u}_Y, \tilde{N})),
\]

By
Lemma 5.5. By Proposition 5.3, we have
\[ \text{Ext}_i(L(\bar{\mathbb{I}}_Y, \lambda^\theta), H^i(\bar{u}_Y; \bar{N})) \]
and hence for each \( i \geq 0 \),
\[ \text{Ext}_i(L(\bar{\mathbb{I}}_Y, \lambda^\theta), H^i(\bar{u}_Y; \bar{N})) \cong \text{Hom}_{\bar{\mathbb{I}}_Y}(L(\bar{\mathbb{I}}_Y, \lambda^\theta), H^i(\bar{u}_Y; \bar{N})). \]

Since the \( \mathfrak{g} \)-semisimple submodule of \( H^i(\bar{u}_Y; \bar{N}) \) with weights in \( \bar{\Gamma} \) equals \( \mathcal{H}^i(\bar{u}_Y; \bar{N}) \) we have
\[ \text{Hom}_{\bar{\mathbb{I}}_Y}(L(\bar{\mathbb{I}}_Y, \lambda^\theta), H^i(\bar{u}_Y; \bar{N})) \cong \text{Hom}_{\bar{\mathbb{I}}_Y}(L(\bar{\mathbb{I}}_Y, \lambda^\theta), \mathcal{H}^i(\bar{u}_Y; \bar{N})). \]
This completes the proof of part (i). The proofs of (ii) and (iii) are analogous. \( \square \)

Corollary 5.4. Let \( \lambda \in \mathcal{P}_Y \) and \( \bar{N} \in \bar{\mathcal{O}}_Y \). For \( i \geq 0 \), we have
\[ \text{Ext}_i(L(\bar{\mathbb{I}}_Y, \lambda^\theta), H^i(\bar{u}_Y; \bar{N})) \cong \text{Ext}_i(L(\bar{\mathbb{I}}_Y, \lambda^\theta), \mathcal{H}^i(\bar{u}_Y; \bar{N})). \]

Proof. Using the arguments of the proof of Theorem 4.4 we can show that
\[ T(\mathcal{H}^i(\bar{u}_Y; \bar{N})) \cong \mathcal{H}^i(\bar{u}_Y; N), \]
and hence \( T \) induces an isomorphism \((\forall \lambda \in \mathcal{P}_Y, \forall i \in \mathbb{Z}_+)\)
\[ \text{Hom}_{\bar{\mathbb{I}}_Y}(L(\bar{\mathbb{I}}_Y, \lambda^\theta), \mathcal{H}^i(\bar{u}_Y; \bar{N})) \cong \text{Hom}_{\bar{\mathbb{I}}_Y}(L(\bar{\mathbb{I}}_Y, \lambda^\theta), \mathcal{H}^i(\bar{u}_Y; N)). \]
By Proposition 5.3, we have
\[ \text{Ext}_i(L(\bar{\mathbb{I}}_Y, \lambda^\theta), \mathcal{H}^i(\bar{u}_Y; \bar{N})) \cong \text{Ext}_i(L(\bar{\mathbb{I}}_Y, \lambda^\theta), \mathcal{H}^i(\bar{u}_Y; N)). \]
Similarly, we have
\[ \text{Ext}_i(L(\bar{\mathbb{I}}_Y, \lambda^\theta), \mathcal{H}^i(\bar{u}_Y; \bar{N})) \cong \text{Ext}_i(L(\bar{\mathbb{I}}_Y, \lambda^\theta), \mathcal{H}^i(\bar{u}_Y; N)). \]
\( \square \)

Lemma 5.5. Let \( \lambda \in \mathcal{P}_Y \) and \( \bar{N} \in \bar{\mathcal{O}}_Y \). For \( i = 0, 1 \), we have
(i) \( T : \text{Ext}_0^{\mathfrak{g} \mathcal{O}_Y}(\bar{K}(\lambda^\theta), \bar{N}) \to \text{Ext}_0^{\mathfrak{g} \mathcal{O}_Y}(K(\lambda), N) \) is an isomorphism,
(ii) \( T : \text{Ext}_1^{\mathfrak{g} \mathcal{O}_Y}(\bar{K}(\lambda^\theta), \bar{N}) \to \text{Ext}_1^{\mathfrak{g} \mathcal{O}_Y}(K(\lambda^\theta), N) \) is an isomorphism.

Proof. It is well known that \( \text{Ext}_1(L(\bar{\mathbb{I}}_Y, \lambda^\theta), \bar{N}) \) is isomorphic to the equivalence classes of \( \bar{\mathbb{I}}_Y \)-trivial extensions of \( \bar{N} \) by \( \bar{K}(\lambda^\theta) \) \([\text{Ho}] \ \text{§}2\) and
\[ \text{Ext}_0(L(\bar{\mathbb{I}}_Y, \lambda^\theta), \bar{N}) = \text{Hom}_{\bar{\mathbb{I}}_Y}(\bar{K}(\lambda^\theta), \bar{N}). \]
Hence we have, for \( i = 0, 1 \),
\[ \text{Ext}_i^{\mathfrak{g} \mathcal{O}_Y}(\bar{K}(\lambda^\theta), \bar{N}) \cong \text{Ext}_i(L(\bar{\mathbb{I}}_Y, \lambda^\theta), \bar{N}). \]
Similarly, for \( i = 0, 1 \),
\[ \text{Ext}_i^{\mathfrak{g} \mathcal{O}_Y}(K(\lambda), N) \cong \text{Ext}_i(L(\bar{\mathbb{I}}_Y, \lambda^\theta), K(\lambda), N). \]
By Corollary 5.4, we have, for \( i = 0, 1 \),
\[ \text{Ext}_i^{\mathfrak{g} \mathcal{O}_Y}(\bar{K}(\lambda^\theta), \bar{N}) \cong \text{Ext}_i^{\mathfrak{g} \mathcal{O}_Y}(K(\lambda), N). \]
Since all the isomorphisms involved are natural, it is not hard to see the isomorphism (5.6) is indeed induced by $T$. This completes the proof of (i). Part (ii) is analogous. □

**Lemma 5.6.** Let $\tilde{N} \in \mathcal{O}_Y^f$ and

\begin{equation}
0 \longrightarrow \tilde{M}' \xrightarrow{\tilde{i}} \tilde{M} \longrightarrow \tilde{M}'' \longrightarrow 0
\end{equation}

be an exact sequence of $\tilde{g}$-modules in $\mathcal{O}_Y^f$. Then

(i) The (5.8) induces the following commutative diagram with exact rows. (We will use subscripts to distinguish various maps induced by $T$.)

\[
\begin{array}{cccc}
0 & \longrightarrow & \text{Hom}_{\mathcal{O}_Y^f}(\tilde{M}'', \tilde{N}) & \longrightarrow & \text{Hom}_{\mathcal{O}_Y^f}(\tilde{M}, \tilde{N}) & \longrightarrow & \text{Hom}_{\mathcal{O}_Y^f}(\tilde{M}', \tilde{N}) & \xrightarrow{\tilde{\partial}} \\
& & \downarrow T_{\tilde{M}'', \tilde{N}} & & \downarrow T_{\tilde{M}, \tilde{N}} & & \downarrow T_{\tilde{M}', \tilde{N}} \\
0 & \longrightarrow & \text{Hom}_{\mathcal{O}_Y^f}(M'', N) & \longrightarrow & \text{Hom}_{\mathcal{O}_Y^f}(M, N) & \longrightarrow & \text{Hom}_{\mathcal{O}_Y^f}(M', N) & \xrightarrow{\partial} \\
& & \downarrow T_{1, M'', \tilde{N}} & & \downarrow T_{1, M, \tilde{N}} & & \downarrow T_{1, M', \tilde{N}} \\
& & \text{Ext}^1_{\mathcal{O}_Y^f}(\tilde{M}'', \tilde{N}) & \longrightarrow & \text{Ext}^1_{\mathcal{O}_Y^f}(\tilde{M}, \tilde{N}) & \longrightarrow & \text{Ext}^1_{\mathcal{O}_Y^f}(M', \tilde{N}) \\
& & \downarrow T_{\tilde{M}'', \tilde{N}} & & \downarrow T_{\tilde{M}, \tilde{N}} & & \downarrow T_{\tilde{M}', \tilde{N}} \\
& & \xrightarrow{\tilde{\partial}} & & \xrightarrow{\partial} & & \\
& & \text{Ext}^1_{\mathcal{O}_Y^f}(M'', N) & \longrightarrow & \text{Ext}^1_{\mathcal{O}_Y^f}(M, N) & \longrightarrow & \text{Ext}^1_{\mathcal{O}_Y^f}(M', N) \\
\end{array}
\]

(ii) The analogous statement holds replacing $T$ by $\mathcal{T}$ in (i), $M$ by $\tilde{M}$, et cetera.

**Proof.** We shall only prove (i), as the argument for (ii) is analogous. By [M] Chapter VII Proposition 2.2], the rows are exact. We only need to show that the following diagram is commutative.

\[
\begin{array}{cccc}
\text{Hom}_{\mathcal{O}_Y^f}(\tilde{M}', \tilde{N}) & \xrightarrow{\tilde{\partial}} & \text{Ext}^1_{\mathcal{O}_Y^f}(\tilde{M}'', \tilde{N}) & \\
\downarrow T_{\tilde{M}', \tilde{N}} & & \downarrow T_{1, \tilde{M}'', \tilde{N}} \\
\text{Hom}_{\mathcal{O}_Y^f}(M', N) & \xrightarrow{\partial} & \text{Ext}^1_{\mathcal{O}_Y^f}(M'', N) & \\
\end{array}
\]

Let $\tilde{f} \in \text{Hom}_{\mathcal{O}_Y^f}(\tilde{M}', \tilde{N})$. Then $\tilde{\partial}(\tilde{f}) \in \text{Ext}^1_{\mathcal{O}_Y^f}(\tilde{M}'', \tilde{N})$ is bottom exact row of the following commutative diagram:

\[
\begin{array}{cccc}
0 & \longrightarrow & \tilde{M}' & \xrightarrow{\tilde{i}} & \tilde{M} & \longrightarrow & \tilde{M}'' & \longrightarrow & 0 \\
& & \downarrow \tilde{f} & & \downarrow & & \| & & \\
0 & \longrightarrow & \tilde{N} & \longrightarrow & \tilde{E} & \longrightarrow & \tilde{M}'' & \longrightarrow & 0
\end{array}
\]

Here $\tilde{E}$ is the pushout of $\tilde{f}$ and $\tilde{i}$, and all maps are the obvious ones. Let $f := T_{\tilde{M}', \tilde{N}}[\tilde{f}]$. Since $T$ is compatible with pushouts, $T_{\tilde{M}', \tilde{N}}[\tilde{\partial}(\tilde{f})] = \partial(f)$, which is the pushout of $f$ and $T[\tilde{f}]$. On the other hand, $T_{\tilde{M}', \tilde{N}}[\tilde{f}] = f$ and hence $\partial(T_{\tilde{M}', \tilde{N}}[\tilde{f}]) = \partial(f)$. This completes the proof. □
By Lemma 5.3 and the fact that $T(\tilde{\varphi}) \neq 0$ and $\overline{T}(\tilde{\varphi}) \neq 0$ for any nonzero $\tilde{\varphi}$-homomorphism $\tilde{\varphi}$ from $L(\tilde{\varphi}, \lambda^\theta)$ to itself with $\lambda \in \mathcal{P}_Y$, we have the following.

**Lemma 5.7.** Let $\tilde{M}, \tilde{N} \in \tilde{\mathcal{O}}_Y$. We have

(i) $T : \text{Hom}_{\tilde{\mathcal{O}}_Y}(\tilde{M}, \tilde{N}) \to \text{Hom}_{\mathcal{O}_Y}(M, N)$ is an injection,

(ii) $\overline{T} : \text{Hom}_{\tilde{\mathcal{O}}_Y}(\tilde{M}, \tilde{N}) \to \text{Hom}_{\mathcal{O}_Y}(\tilde{M}, \tilde{N})$ is an injection.

**Lemma 5.8.** Let $\lambda \in \mathcal{P}_Y$ and $\tilde{N} \in \tilde{\mathcal{O}}_Y$. We have

(i) $T : \text{Hom}_{\tilde{\mathcal{O}}_Y}(\tilde{L}(\lambda^\theta), \tilde{N}) \to \text{Hom}_{\mathcal{O}_Y}(L(\lambda), N)$ is an isomorphism.

(ii) $\overline{T} : \text{Hom}_{\tilde{\mathcal{O}}_Y}(\tilde{L}(\lambda^\theta), \tilde{N}) \to \text{Hom}_{\mathcal{O}_Y}(\overline{L}(\lambda^\theta), \overline{N})$ is an isomorphism.

**Proof.** Consider the commutative diagram with exact rows

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \tilde{M} & \longrightarrow & \tilde{K}(\lambda^\theta) & \longrightarrow & \tilde{L}(\lambda^\theta) & \longrightarrow & 0 \\
\quad & & \downarrow T_{\tilde{M}} & & \downarrow T_{\tilde{K}(\lambda^\theta)} & & \downarrow T_{\tilde{L}(\lambda^\theta)} & & \\
0 & \longrightarrow & M & \longrightarrow & K(\lambda) & \longrightarrow & L(\lambda) & \longrightarrow & 0
\end{array}
\]

We obtain the following commutative diagram with exact rows.

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \text{Hom}_{\tilde{\mathcal{O}}_Y}(\tilde{L}(\lambda^\theta), \tilde{N}) & \longrightarrow & \text{Hom}_{\mathcal{O}_Y}(\tilde{K}(\lambda^\theta), \tilde{N}) & \longrightarrow & \text{Hom}_{\mathcal{O}_Y}(M, \tilde{N}) \\
\quad & & \downarrow T_{L(\lambda^\theta), \tilde{N}} & & \downarrow T_{K(\lambda^\theta), \tilde{N}} & & \downarrow T_{\tilde{M}, \tilde{N}} & & \\
0 & \longrightarrow & \text{Hom}_{\mathcal{O}_Y}(L(\lambda), N) & \longrightarrow & \text{Hom}_{\mathcal{O}_Y}(K(\lambda), N) & \longrightarrow & \text{Hom}_{\mathcal{O}_Y}(M, N)
\end{array}
\]

By Lemma 5.5 $T_{\tilde{K}(\lambda^\theta), \tilde{N}}$ is an isomorphism and by Lemma 5.7 $T_{\tilde{M}, \tilde{N}}$ is an injection. This implies that $T_{L(\lambda^\theta), \tilde{N}}$ is an isomorphism. \qed

**Lemma 5.9.** Let $\lambda \in \mathcal{P}_Y$ and $\tilde{N} \in \tilde{\mathcal{O}}_Y$. We have

(i) $T : \text{Ext}^1_{\tilde{\mathcal{O}}_Y}(\tilde{L}(\lambda^\theta), \tilde{N}) \to \text{Ext}^1_{\mathcal{O}_Y}(L(\lambda), N)$ is an injection.

(ii) $\overline{T} : \text{Ext}^1_{\tilde{\mathcal{O}}_Y}(\tilde{L}(\lambda^\theta), \tilde{N}) \to \text{Ext}^1_{\mathcal{O}_Y}(\overline{L}(\lambda^\theta), \overline{N})$ is an injection.

**Proof.** Let

\[
(5.9) \quad 0 \longrightarrow \tilde{N} \longrightarrow \tilde{E} \xrightarrow{f} \tilde{L}(\lambda^\theta) \longrightarrow 0
\]

be an exact sequence of $\tilde{g}$-modules. Suppose that $(5.9)$ gives rise to a split exact sequence of $g$-modules

\[
0 \longrightarrow N \longrightarrow E \xrightarrow{T[\tilde{f}]} L(\lambda) \longrightarrow 0.
\]

Thus there exists $\psi \in \text{Hom}_{\mathcal{O}_Y}(L(\lambda), E)$ such that $T[\tilde{f}] \circ \psi = 1_{L(\lambda)}$. By Lemma 5.8 there exists $\tilde{\psi} \in \text{Hom}_{\tilde{\mathcal{O}}_Y}(\tilde{L}(\lambda^\theta), \tilde{E})$ such that $T[\tilde{\psi}] = \psi$. Thus $T[\tilde{f} \circ \tilde{\psi}] = T[\tilde{f}] \circ T[\tilde{\psi}] = 1_{L(\lambda)}$. By Lemma 5.8 we have $\tilde{f} \circ \tilde{\psi} = 1_{\tilde{L}(\lambda^\theta)}$, and hence $(5.9)$ is split. \qed

**Lemma 5.10.** Let $\tilde{M}, \tilde{N} \in \tilde{\mathcal{O}}_Y$. We have
Lemma 5.12. Let $\tilde{T} : \text{Hom}_{\tilde{O}_Y}(\tilde{M}, \tilde{N}) \rightarrow \text{Hom}_{\tilde{O}_Y}(M, N)$ is an isomorphism.

Proof. We proceed by induction on the length of a composition series of $\tilde{M}$. If $\tilde{M}$ is irreducible, then it is true by Lemma 5.8

Consider the following commutative diagram with exact top row of $\tilde{g}$-modules and exact bottom row of $g$-modules.

\[
\begin{array}{ccccccc}
0 & \longrightarrow & M' & \overset{i}{\longrightarrow} & M & \longrightarrow & \tilde{L}(\lambda^0) & \longrightarrow & 0 \\
\downarrow{T_{M'}} & & \downarrow{T_M} & & \downarrow{T_{\tilde{L}(\lambda^0)}} & & & \\
0 & \longrightarrow & M' & \overset{i}{\longrightarrow} & M & \longrightarrow & L(\lambda) & \longrightarrow & 0
\end{array}
\] (5.10)

The sequence (5.10) induces the following commutative diagram with exact rows.

\[
\begin{array}{ccccccc}
0 & \longrightarrow & \text{Hom}_{\tilde{O}_Y}(\tilde{L}(\lambda^0), \tilde{N}) & \longrightarrow & \text{Hom}_{\tilde{O}_Y}(\tilde{M}, \tilde{N}) & \overset{i^*}{\longrightarrow} & \\
\downarrow{T_{\tilde{L}(\lambda^0), \tilde{N}}} & & \downarrow{T_{\tilde{M}, \tilde{N}}} & & & & \\
0 & \longrightarrow & \text{Hom}_{O_Y}(L(\lambda), N) & \longrightarrow & \text{Hom}_{O_Y}(M, N) & \overset{i^*}{\longrightarrow} & \\
\end{array}
\]

The map $T_{M', \tilde{N}}$ is an isomorphism by induction. The map $T_{\tilde{L}(\lambda^0), \tilde{N}}$ is an injection by Lemma 5.9 Also $T_{\tilde{L}(\lambda^0), \tilde{N}}$ is an isomorphism by Lemma 5.8 This implies that $T_{\tilde{M}, \tilde{N}}$ is an isomorphism.

Lemma 5.11. Let $\tilde{M}, \tilde{N} \in \tilde{O}_Y$. We have

(i) $T : \text{Ext}^1_{\tilde{O}_Y}(\tilde{M}, \tilde{N}) \rightarrow \text{Ext}^1_{O_Y}(M, N)$ is an injection,

(ii) $\tilde{T} : \text{Ext}^1_{\tilde{O}_Y}(\tilde{M}, \tilde{N}) \rightarrow \text{Ext}^1_{O_Y}(M, N)$ is an injection.

Proof. The proof is virtually identical to the proof of Lemma 5.9. Here we use Lemma 5.10 instead of Lemma 5.8.

Lemma 5.12. Let $\lambda \in P_Y$ and $\tilde{N} \in \tilde{O}_Y$. We have

(i) $T : \text{Ext}^1_{\tilde{O}_Y}(\tilde{L}(\lambda^0), \tilde{N}) \rightarrow \text{Ext}^1_{O_Y}(L(\lambda), N)$ is an isomorphism.

(ii) $\tilde{T} : \text{Ext}^1_{\tilde{O}_Y}(\tilde{L}(\lambda^0), \tilde{N}) \rightarrow \text{Ext}^1_{O_Y}(L(\lambda^0), N)$ is an isomorphism.
Proof. Consider the following commutative diagram with exact rows.

$$
\begin{array}{cccccc}
0 & \longrightarrow & \tilde{M} & \longrightarrow & \tilde{K}(\lambda^\theta) & \longrightarrow & \tilde{L}(\lambda^\theta) & \longrightarrow & 0 \\
\downarrow T_{\tilde{M}} & & \downarrow T_{\tilde{K}(\lambda^\theta)} & & \downarrow T_{\tilde{L}(\lambda^\theta)} & & \\
0 & \longrightarrow & M & \longrightarrow & K(\lambda) & \longrightarrow & L(\lambda) & \longrightarrow & 0
\end{array}
$$

The sequence (5.11) induces the following commutative diagram with exact rows.

$$
\begin{array}{cccccc}
\text{Hom}_{\tilde{O}_V}(\tilde{K}(\lambda^\theta), \tilde{N}) & \longrightarrow & \text{Hom}_{\tilde{O}_V}(\tilde{M}, \tilde{N}) & \longrightarrow & \text{Ext}_{\tilde{O}_V}^1(\tilde{L}(\lambda^\theta), \tilde{N}) & \longrightarrow & \tilde{\pi}^* \\
\downarrow T_{\tilde{K}(\lambda^\theta), \tilde{N}} & & \downarrow T_{\tilde{M}, \tilde{N}} & & \downarrow T_{\tilde{L}(\lambda^\theta), \tilde{N}} & & \\
\text{Hom}_{O_V}(K(\lambda), N) & \longrightarrow & \text{Hom}_{O_V}(M, N) & \longrightarrow & \text{Ext}_{O_V}^1(L(\lambda), N) & \longrightarrow & \pi^* \\
& & & \downarrow \pi^* & & \downarrow \pi^* & & \pi^* \\
& & & \text{Ext}_{\tilde{O}_V}^1(\tilde{K}(\lambda^\theta), \tilde{N}) & \longrightarrow & \text{Ext}_{O_V}^1(\tilde{M}, \tilde{N}) & \longrightarrow & \text{Ext}_{O_V}^1(M, N)
\end{array}
$$

The map $T_{\tilde{M}, \tilde{N}}^1$ is an injection by Lemma 5.11. The map $T_{\tilde{K}(\lambda^\theta), \tilde{N}}^1$ is an isomorphism by Lemma 5.12. Also $T_{\tilde{K}(\lambda^\theta), \tilde{N}}$ and $T_{\tilde{M}, \tilde{N}}$ are isomorphisms by Lemma 5.10. Thus $T_{\tilde{L}(\lambda^\theta), \tilde{N}}^1$ is an isomorphism.

We have now all the ingredients to prove of Theorem 5.1.

Proof of Theorem 5.1. Lemma 5.12 implies that for every $M \in \tilde{O}_V^1$, there exists $\tilde{M} \in \tilde{O}_V^1$, such that $T(\tilde{M}) = M$, i.e. the functor $T$ is essentially surjective. Now Lemma 5.10 says that $T$ is full and faithful. It is well-known that an essentially surjective functor that is full and faithful is an equivalence of categories (see e.g. [P]), proving (i). (ii) is proved in an analogous fashion, while (iii) follows from combining (i) and (ii).

Remark 5.13. Theorem 5.1 (iii) was stated as a conjecture in [CW] Conjecture 4.18. [CW] Conjecture 4.18 in the special case of $Y = [-m, -2]$ was already formulated in [CWZ] Conjecture 6.10. A proof of Theorem 5.1 (iii) in the special case $Y = [-m, -2]$ was announced by Brundan and Stroppel in [BS]. The proof we have presented here is different from the one announced in [BS], as it constructs directly the functors inducing this equivalence and also does not rely on [B].

Remark 5.14. The classical BGG-type resolutions for the finite-dimensional modules of [Le] and for the unitarizable modules of [EW] in terms of parabolic Verma $g$-modules together with Theorem 5.1 imply the existence of BGG-type resolutions for the corresponding $\tilde{g}$- and $\tilde{\mathfrak{g}}$-modules in terms of parabolic Verma $\tilde{g}$- and $\tilde{\mathfrak{g}}$-modules, respectively. The case of $\tilde{\mathfrak{g}}$ and $Y = [-m, -2]$ was already established in [CKL].
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