On duality and reflection factors for the sinh-Gordon model
with a boundary

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Abstract

The sinh-Gordon model with integrable boundary conditions is considered in low order perturbation theory. It is pointed out that results obtained by Ghoshal for the sine-Gordon breather reflection factors suggest an interesting dual relationship between models with different boundary conditions. Ghoshal’s formula for the lightest breather is checked perturbatively to $O(\beta^2)$ in the special set of cases in which the $\phi \rightarrow -\phi$ symmetry is maintained. It is noted that the parametrisation of the boundary potential which is natural for the semi-classical approximation also provides a good parametrisation at the ‘free-fermion’ point.

1. Introduction

In recent years, some progress has been made concerning the question of integrability for two-dimensional field theories restricted to a half-line [1-10], or to an interval. (For a recent review of some aspects of this, see [11].) In this article it is intended to concentrate on the simplest example, other than free field theory, and to estimate using perturbation theory the reflection factors corresponding to those boundary conditions which preserve integrability. Thus, the model to be studied is the sinh-Gordon model restricted to the region $x < 0$. There are several reasons for doing this. Firstly, Ghoshal [2] has suggested a formula for the reflection factors for the breather states of the sine-Gordon model using
the reflection bootstrap equations and the soliton reflection factors suggested earlier by Ghoshal and Zamolodchikov [1]. It is plausible to suppose that the reflection factor for the lightest breather state, analytically continued to imaginary coupling, should provide a sensible hypothesis for the reflection factors for the sinh-Gordon particle. On the other hand, the reflection factors of the sinh-Gordon model are accessible perturbatively and so a start can be made towards checking Ghoshal’s formula. Unfortunately, the relationship between the two classical parameters which may be introduced via the boundary potential and the quantum reflection factors is unclear except in certain cases. Therefore, the perturbative calculation is potentially a useful way to suggest the missing connection. In those cases where the relationship between classical and quantum parameters is already known, perturbation theory should provide a verification. The second reason concerns weak-strong coupling duality.

The S-matrix describing the elastic scattering of a pair of sinh-Gordon particles is known [12] to have the form

\[ S(\Theta) = -\frac{1}{(B)(2-B)} \]  \hspace{1cm} (1.1)

where

\[ B = \frac{1}{2\pi} \frac{\beta^2}{1 + \beta^2/4\pi} \]  \hspace{1cm} (1.2)

and

\[ (x) = \frac{\sinh \left( \frac{\Theta}{2} + \frac{i\pi x}{4} \right)}{\sinh \left( \frac{\Theta}{2} - \frac{i\pi x}{4} \right)}. \]  \hspace{1cm} (1.3)

The rapidity difference of the two particles is represented by \( \Theta \). This is clearly invariant under the transformation

\[ \beta \to \frac{4\pi}{\beta}, \]  \hspace{1cm} (1.4)

and this invariance is referred to as the weak-strong coupling duality. This form of duality is shared by all the other affine Toda models based on the ade series of data [12][13], and a generalised form of it is enjoyed by those models based on the data derived from the other (ie not simply-laced) affine diagrams [14][15]. In some earlier works concerning the reflection factors [3][4] it was suggested that they should enjoy the same duality property. However, this is unlikely to be the case, as the following argument demonstrates.

When there is a boundary condition at \( x = 0 \), provided it is chosen to maintain integrability, then it is expected that a single particle approaching the boundary will be elastically reflected from it so that

\[ |\theta >_{\text{out}} = \mathcal{K}(\theta)| - \theta >_{\text{in}}, \]
where \( \theta \) is the rapidity of the particle. In the sinh-Gordon model there is only one such particle. Ghoshal’s formula \([4]\) for the sinh-Gordon reflection factors, after translation to the above notation, has the form:

\[
K(\theta) = \frac{(1) (2 - B/2) (1 + B/2)}{(1 - E(\beta)) (1 + E(\beta)) (1 - F(\beta)) (1 + F(\beta))},
\]

(1.5)

where \( E \) and \( F \) are two functions of \( \beta \) which also depend on the boundary parameters, but which are not yet fully determined in terms of those parameters.\(^1\) However, for the Neumann boundary condition Ghoshal requires

\[
F = 0, \quad E = 1 - B/2
\]

(1.6)

leading to a reflection factor \( K_N \), given by:

\[
K_N = \frac{(1 + B/2)}{(B/2)(1)}.
\]

(1.7)

Clearly, this is not self-dual; rather its dual partner is \( K^*_N \), given by:

\[
K^*_N = \frac{(2 - B/2)}{(1 - B/2)(1)}.
\]

(1.8)

What about a verification of (1.6) perturbatively? In effect this has been provided implicitly by Kim \([16,17]\) who has developed perturbation theory for affine Toda models satisfying the Neumann boundary condition, and has provided detailed calculations up to \( O(\beta^2) \). In other words, the \( O(\beta^2) \) check can be made merely by expanding (1.7) and comparing with the formulae given by Kim \([16]\). Explicitly, from (1.7),

\[
K_N \sim 1 - \frac{i\beta^2 \sinh \theta}{8} \left[ \frac{1}{\cosh \theta - 1} - \frac{1}{\cosh \theta} \right],
\]

(1.9)

which is in perfect agreement.\(^2\)

To discover which boundary condition is dual to the Neumann condition it will be enough to consider the classical limit of (1.8). However, first it is necessary to consider the classical limit of (1.5). That is, taking \( \beta \to 0 \),

\[
K_0(\theta) = -\frac{(1) ^2}{(1 - E(0)) (1 + E(0)) (1 - F(0)) (1 + F(0))}.
\]

(1.10)

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1 In Ghoshal’s notation \( E = B\eta/\pi, \ F = iB\vartheta/\pi \).

2 Kim did not comment on this agreement. He chose to assume instead that the \( a^{(1)}_1 \) reflection factor should be self-dual and proposed an alternative formula for \( K_N \), disagreeing with Ghoshal, but nevertheless in agreement with perturbation theory to \( O(\beta^2) \).
Fortunately, this may be calculated independently and the result was already reported some time ago [5].

In order to state the result of the ‘classical’ calculation further details of the model are required. Its equation of motion and boundary condition are:

\begin{align}
\partial^2 \phi &= -\frac{\sqrt{2}}{\beta} \left( e^{\sqrt{2} \beta \phi} - e^{-\sqrt{2} \beta \phi} \right) & x < 0 \\
\frac{\partial \phi}{\partial x} &= -\frac{\sqrt{2}}{\beta} \left( \sigma_1 e^{\beta \phi / \sqrt{2}} - \sigma_0 e^{-\beta \phi / \sqrt{2}} \right) & x = 0,
\end{align}

where the $\sqrt{2}$ is a conventional normalisation in Toda theory. The two constants $\sigma_0$ and $\sigma_1$ are essentially free and represent the degrees of freedom permitted at the boundary [1]. The constraints on the boundary parameters are discussed in [5] and by Fujii and Sasaki [18]. The boundary condition is required to have the given form as a consequence of maintaining integrability on the half-line. It is also convenient to write

$$\sigma_i = \cos a_i \pi.$$ (1.12)

Then, the classical reflection factor is determined by first finding the lowest energy static solution to (1.11) (the ‘vacuum configuration’) and then solving the linearised scattering problem in this static background. Further details of this are given below and the result for the reflection factor calculated in this approximation (and for $|a_i| \leq 1$) is:

$$K_0(\theta) = -\frac{(1)^2}{(1-a_0-a_1)(1+a_0+a_1)(1-a_0+a_1)(1+a_0-a_1)}. \quad (1.13)$$

The formula (1.13) clearly has the same form as the classical limit of Ghoshal’s formula, and is unity when $a_0 = a_1 = 1/2$ (the Neumann condition), in agreement with the classical limit of $K_N$, eq(1.7).

On the other hand, the classical limit of $K_N^*$ is not unity. Rather, it is $-1/(1)^2$, and is in agreement with (1.13) for the choice $a_0 = a_1 = 0$. In other words, it is tempting to conclude that the two boundary conditions

$$\frac{\partial \phi}{\partial x} = 0 \quad \text{and} \quad \frac{\partial \phi}{\partial x} = -\frac{\sqrt{2}}{\beta} \left( e^{\beta \phi / \sqrt{2}} - e^{-\beta \phi / \sqrt{2}} \right) \quad \text{at} \quad x = 0,$$ (1.14)

are dual, in the sense that their reflection factors are transformed into each other under the transformation (1.4).

It should now be clear that it is the functional dependence of $E$ and $F$ on the three parameters $a_0$, $a_1$ and $\beta$ which must be found together with a proper understanding of which pairs of boundary data are dual to each other, and which (if any) are self-dual. The rest of the paper is concerned with these questions.
2. The propagator in a general background

To calculate the propagator it will be necessary to consider the linear perturbation around the static background solution to (1.11). The static solutions have the form

$$e^{\beta \phi_0 / \sqrt{2}} = \frac{1 + e^{2(x-x_0)}}{1 - e^{2(x-x_0)}}$$

(2.1)

where the parameter $x_0$ is determined by the boundary condition and is given by:

$$\coth x_0 = \sqrt{1 + \sigma_0 / (1 + \sigma_1)}.$$ 

(2.2)

Note that provided $\sigma_0 \geq \sigma_1$ it is guaranteed that $x_0 \geq 0$; if on the other hand $\sigma_1 \geq \sigma_0$ it would be necessary to take the background provided by the inverse of the right hand side of (2.1). In the calculations which follow it will be assumed $\sigma_0 \geq \sigma_1$ for definiteness.

Linearising (1.11) around the static background leads to a pair of equations for the first order correction to $\phi$:

$$\partial^2 \phi_1 + 4 \left(1 + \frac{2}{\sinh^2 2(x-x_0)}\right) \phi_1 = 0 \quad x < 0$$

$$\frac{\partial \phi_1}{\partial x} + \left(\sigma_1 \sqrt{\frac{1 + \sigma_0}{1 + \sigma_1}} + \sigma_0 \sqrt{\frac{1 + \sigma_1}{1 + \sigma_0}}\right) \phi_1 = 0 \quad x = 0.$$ 

(2.3)

Fortunately, these equations may be solved exactly and the required Green’s function can be written down explicitly once the eigenfunctions of the second order differential operator in the first of eqs(2.3) have been determined. It is convenient to write $\phi_{k,\omega}$ to denote the eigenfunction corresponding to the eigenvalue $\omega^2 - k^2 - 4$, and then

$$\phi_{k,\omega} = i e^{-i\omega t} r(k) \left(F(k,x) e^{ikx} + F(-k,x) e^{-ikx}\right),$$

(2.4)

where $r(k)$ is a real, even function of $k$ (which will be determined by normalising the Green’s function properly), and where $F(k,x)$ is given by:

$$F(k,x) = P(k) \left(ik - 2 \coth 2(x-x_0)\right),$$

with

$$P(k) = (ik)^2 - 2ik \sqrt{1 + \sigma_0} \sqrt{1 + \sigma_1} + 2(\sigma_0 + \sigma_1).$$

The classical reflection factor (1.13) follows immediately from (2.4) on setting $\omega = 2 \cosh \theta$, $k = 2 \sinh \theta$, calculating $F(-k, -\infty)/F(k, -\infty)$, and using the convenient parametrisation (1.12).
The Green’s function, or configuration space propagator, constructed from these eigenfunctions is not hard to find. It is:

\[
G(x, t; x', t') = i \int \frac{d\omega}{2\pi} \int \frac{dk}{2\pi} \mathcal{G}(\omega, k; x, t; x', t')
\]

where

\[
\mathcal{G}(\omega, k; x, t; x', t') = e^{-i\omega(t-t')} \frac{f(k, x)f(-k, x')e^{ik(x-x')}}{\omega^2 - k^2 - 4 + i\epsilon} + \mathcal{K}_0 f(-k, x)f(-k, x')e^{-ik(x+x')}
\]

\[
f(k, x) = \frac{ik - 2 \coth 2(x - x_0)}{ik + 2},
\]

and

\[
\mathcal{K}_0 = \frac{P(-k)}{P(k)} \frac{ik - 2}{ik + 2}.
\]

When \(k = 2 \sinh \theta\) this expression for \(\mathcal{K}_0\) reduces to the previous one, eq(1.13). To check that (2.5) is normalised correctly it is enough to note that it is in the limit \(x, x' \to -\infty\).

If \(\sigma_0 = \sigma_1 = \sigma\), eq(2.5) simplifies because in that case \(f(k, x) = 1\) and the factor \(\mathcal{K}_0\) reduces to

\[
\mathcal{K}_\sigma = \frac{ik + 2\sigma}{ik - 2\sigma}.
\]

In other words, in this special set of cases,

\[
\mathcal{G}(\omega, k; x, t; x', t') = e^{-i\omega(t-t')} \frac{e^{ik(x-x')}}{\omega^2 - k^2 - 4 + i\epsilon} + \mathcal{K}_\sigma e^{-ik(x+x')}
\]

In this situation, the static background is simply \(\phi_0 = 0\).

In this article a pragmatic approach will be taken [16]. The classical reflection factor is an integral part of the free field propagator calculated within the classical background. The full two-point function may then be calculated perturbatively (once subtractions are made to remove infinities), and provides a working definition of the quantum reflection factor in the sense that it will be the coefficient of \(e^{-ik(x+x')}\) in the residue of the on-shell pole in the asymptotic region \(x, x' \to -\infty\).
3. Low order perturbation theory

Since perturbation theory will be developed around an $x$-dependent background field configuration, it is possible to use a more-or-less standard Feynman expansion in configuration space, using the above propagator but recognising that the 'vertices' will be position dependent. One consequence of this is that although the sinh-Gordon theory on the whole line has only vertices at which an even number of lines join, the theory on the half-line must also, in general, contain vertices at which an odd number of lines meet. In other words, the symmetry $\phi \rightarrow -\phi$, enjoyed by the bulk theory, is broken by most boundary conditions. The exceptions to this are precisely the boundary conditions for which $\sigma_0 = \sigma_1$. In those cases, the classical background is $\phi_0 = 0$ and there will be no additional vertices compared with the bulk theory other than those arising specifically from the boundary term itself. Moreover, the boundary terms will only generate vertices at which an even number of lines meet. In this article, it is intended only to perform detailed calculations in the special case $\sigma_0 = \sigma_1 = \sigma$. The more general situation will be treated elsewhere since the computations are more lengthy and intricate.

To $O(\beta^2)$, there are two contributions which need to be calculated and each of them is a single loop diagram. The first comes from the boundary and is described by:

$$B_1(x, t; x', t') = -i\sigma\beta^2C_1\int_{-\infty}^{\infty} dt''G(x, t; 0, t'')G(0, t''; 0, t'')G(0, t''; x', t'),$$  \hspace{1cm} (3.1)

where $C_1$ is a combinatorial factor. The second comes from the bulk potential and has the form

$$B_2(x, t; x', t') = -8i\beta^2C_2\int_{-\infty}^{\infty} dt''\int_{0}^{\infty} dx''G(x, t; x'', t'')G(x'', t''; x'', t'')G(x'', t''; x', t'),$$  \hspace{1cm} (3.2)

where $C_2$ is another combinatorial factor. The relative factor of 8 between the two expressions arises from the different factors of $\sqrt{2}$ in the bulk and boundary potential terms. The expression (3.1) is relatively simple to analyse and it will be considered first.

The integral over $t''$ in expression (3.1) gives a delta function allowing the $\omega''$, say, integral to be done leaving the momentum integrals for all three propagators, and the frequency integral for the loop (middle propagator) to be performed.

The middle propagator is clearly divergent and a minimal subtraction will be made to yield a finite part. That is, the logarithmically divergent integral

$$\int \frac{d\omega''}{2\pi} \int \frac{dk''}{2\pi} \frac{i}{\omega''^2 - k''^2 - 4 + i\epsilon} \left(1 + \frac{ik'' + 2\sigma}{ik'' - 2\sigma}\right)$$

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will be replaced by the finite part
\[ \int \frac{d\omega''}{2\pi} \int \frac{dk''}{2\pi} \frac{i}{\omega''^2 - k''^2 - 4 + i\epsilon} \left( \frac{4\sigma}{ik'' - 2\sigma} \right) \]
which can be evaluated to obtain first
\[ \frac{1}{2} \int \frac{dk''}{2\pi} \frac{1}{\sqrt{k''^2 + 4}} \left( \frac{4\sigma}{ik'' - 2\sigma} \right), \]
and then finally, using (1.12), to obtain
\[ -(a \mod 1) \frac{\cos a\pi}{\sin a\pi}. \tag{3.3} \]
The expression (3.3) is invariant under the transformation \( a \to a + 1 \).

The momenta of the other two propagators may be integrated out by closing the contours in the upper half plane (first setting \( k \to -k \) in the first term of the first propagator). Provided \( \sigma > 0 \), the reflection factor term does not contribute a pole. On the other hand, if \( \sigma < 0 \), there is an extra pole but its contribution may be discounted in the limit \( x, x' \to -\infty \), because it will be exponentially damped. Thus, the two integrals reduce to
\[ \int \frac{dk}{2\pi} \int \frac{dk'}{2\pi} \frac{i}{\omega^2 - k^2 - 4 + i\epsilon} \frac{i}{\omega'^2 - k'^2 - 4 + i\epsilon} e^{-i(kx + k'x')} \left( \frac{2ik}{ik - 2\sigma} \right) \left( \frac{2ik'}{ik' - 2\sigma} \right) \]
and hence to,
\[ \left( \frac{-i}{2} \right)^2 \left( \frac{1}{k} \right)^2 e^{-i\hat{k}(x+x')} \left( \frac{2i\hat{k}}{ik - 2\sigma} \right)^2, \tag{3.4} \]
where \( \hat{k} = \sqrt{\omega^2 - 4} \).

Combining the results contained in (3.3) and (3.4), the contribution (3.1) becomes
\[ B_1 = \frac{ia\beta^2}{8\tan^2 a\pi} C_1 \int d\omega \frac{\kappa_0(\hat{k}) e^{-i\omega(t-t')} e^{-i\hat{k}(x+x')}}{\cosh \theta - \sin a\pi - \cosh \theta + \sin a\pi}, \tag{3.5} \]
where the integral over \( \omega \) is to be understood as a positive energy integral, and it is convenient to set \( \hat{k} = 2\sinh \theta \) in the last terms of the integrand. The overall factor of \( a \) should be interpreted modulo unity. Summarising, the correction to the reflection factor from the boundary piece is
\[ \frac{ia\beta^2 C_1}{2\tan^2 a\pi} \sinh \theta \left( \frac{1}{\cosh \theta - \sin a\pi} - \frac{1}{\cosh \theta + \sin a\pi} \right). \tag{3.6} \]
The evaluation of (3.2) proves to be more testing. The integral over \( t'' \) is straightforward again and provides a delta function allowing one of the non-loop energy integrals (\( \omega \) or \( \omega' \)) to be performed immediately. The loop integral is

\[
\int \frac{d\omega'' \, dk''}{2\pi} \frac{i}{\omega''^2 - k''^2 - 4 + i\epsilon} \left( 1 + K_{\sigma}(k'') \, e^{-2ik''x''} \right),
\]

which is logarithmically divergent. However, a minimal subtraction of the term without the exponential dependence on a position coordinate leaves a finite integral. This subtraction is natural, coinciding with what would be achieved by normal-ordering in the usual bulk theory on the full line, as has been emphasised by Kim [16]. What remains is:

\[
\int_{-\infty}^{0} dx'' \int \frac{d\omega \, dk}{2\pi} \, e^{-i\omega(t-t')} \frac{i}{\omega^2 - k^2 - 4 + i\epsilon} \left( e^{-ik(x-x'')} + K_{\sigma}(k) e^{-ik(x+x'')} \right)
\]

\[
\int \frac{d\omega'' \, dk''}{2\pi} \frac{i}{\omega''^2 - k''^2 - 4 + i\epsilon} \, K_{\sigma}(k'') \, e^{-2ik''x''}
\]

\[
\int \frac{d\omega' \, dk'}{2\pi} \frac{i}{\omega'^2 - k'^2 - 4 + i\epsilon} \left( e^{ik'(x''-x')} + K_{\sigma}(k') e^{-ik'(x'+x'')} \right) \tag{3.7}
\]

The integration over \( x'' \) is achieved using the device

\[
\int_{-\infty}^{0} dx'' \, e^{(i\lambda + \rho)x''} = -\frac{i}{\lambda - i\rho},
\]

where \( \rho \) is a positive constant which will be taken to zero at the end of the calculation. This introduces a collection of poles for the \( k'' \) integration which need to be dealt with separately.

The pole contributions after integrating out \( \omega'' \) are

\[
-\frac{i}{2} \int \frac{d\omega \, dk \, dk'}{2\pi \cdot 2\pi} \, e^{-i\omega(t-t')} e^{-i(kx-x')} \frac{i}{\omega^2 - k^2 - 4 + i\epsilon} \frac{i}{\omega'^2 - k'^2 - 4 + i\epsilon}
\]

\[
\int \frac{dk''}{2\pi} \frac{1}{\sqrt{k''^2 + 4}} K_{\sigma}(k'') \left[ \frac{1}{k - 2k'' + k' - i\rho} + \frac{K_{\sigma}(k)}{k - 2k'' - k' - i\rho} \right] + \frac{K_{\sigma}(k)}{-k - 2k'' + k' - i\rho} + \frac{K_{\sigma}(k) K_{\sigma}(k')}{-k - 2k'' - k' - i\rho},
\]

and the \( k'' \) integrations may be done by collapsing the contour into the upper half-plane and onto the branch cut running from \( k'' = 2i \) to infinity along the imaginary axis, avoiding all the poles (the ‘\( i\rho \)’ effectively indicates the sense in which the \( k'' \) contours must be interpreted), with the possible exception of the pole in \( K_{\sigma}(k'') \). If \( \sigma > 0 \) this pole is
avoided automatically; if \( \sigma < 0 \), it cannot be avoided but its residue integrated over \( k \) and \( k' \) yields exponentially decreasing terms as \( x, x' \to -\infty \).

The integrals along the cut are all reducible to integrals of the following form

\[
\int_2^\infty \frac{dy}{\sqrt{y^2 - 4}} \frac{1}{y + 2\zeta} = \frac{1}{\sqrt{1 - \zeta^2}} \left( \frac{\pi}{2} - \tan^{-1} \sqrt{\frac{1 + \zeta}{1 - \zeta}} \right). \tag{3.8}
\]

Thus, for example, with a little manipulation,

\[
\int \frac{dk''}{2\pi} \frac{1}{\sqrt{k''^2 + 4}} \mathcal{K}_\sigma(k'') \frac{1}{k'' - 2k'' + k' - i\rho} = \frac{1}{\pi} \int_2^\infty \frac{dy}{\sqrt{y^2 - 4}} \mathcal{K}_\sigma(iy) \frac{1}{k + k' - 2iy}
\]

\[
= \frac{i}{2\pi} \int_2^\infty \frac{dy}{\sqrt{y^2 - 4}} \left( \frac{\mathcal{K}_\sigma \left( \frac{k + k'}{2} \right)}{y + i(k + k')/2} + \frac{1 - \mathcal{K}_\sigma \left( \frac{k + k'}{2} \right)}{y + 2\sigma} \right),
\]

which can be evaluated using (3.8), to yield:

\[
\frac{i}{2\pi} \left[ \frac{2\mathcal{K}_\sigma \left( \frac{k + k'}{2} \right)}{\sqrt{4 + (k + k')^2}/4} \left( \frac{\pi}{2} - \tan^{-1} \sqrt{\frac{4 + i(k + k')}{4 - i(k + k')}} \right)
\]

\[
+ \frac{1 - \mathcal{K}_\sigma \left( \frac{k + k'}{2} \right)}{\sqrt{1 - \sigma^2}} \left( \frac{\pi}{2} - \tan^{-1} \sqrt{\frac{1 + \sigma}{1 - \sigma}} \right) \right]. \tag{3.9}
\]

Thus, for example, with a little manipulation,

\[
\int \frac{dk''}{2\pi} \frac{1}{\sqrt{k''^2 + 4}} \mathcal{K}_\sigma(k'') \frac{1}{k'' - 2k'' + k' - i\rho} = \frac{1}{\pi} \int_2^\infty \frac{dy}{\sqrt{y^2 - 4}} \mathcal{K}_\sigma(iy) \frac{1}{k + k' - 2iy}
\]

\[
= \frac{i}{2\pi} \int_2^\infty \frac{dy}{\sqrt{y^2 - 4}} \left( \frac{\mathcal{K}_\sigma \left( \frac{k + k'}{2} \right)}{y + i(k + k')/2} + \frac{1 - \mathcal{K}_\sigma \left( \frac{k + k'}{2} \right)}{y + 2\sigma} \right),
\]

which can be evaluated using (3.8), to yield:

\[
\frac{i}{2\pi} \left[ \frac{2\mathcal{K}_\sigma \left( \frac{k + k'}{2} \right)}{\sqrt{4 + (k + k')^2}/4} \left( \frac{\pi}{2} - \tan^{-1} \sqrt{\frac{4 + i(k + k')}{4 - i(k + k')}} \right)
\]

\[
+ \frac{1 - \mathcal{K}_\sigma \left( \frac{k + k'}{2} \right)}{\sqrt{1 - \sigma^2}} \left( \frac{\pi}{2} - \tan^{-1} \sqrt{\frac{1 + \sigma}{1 - \sigma}} \right) \right]. \tag{3.9}
\]

The \( k, k' \) integrals can be performed next and, up to vanishingly small factors as before, one obtains as far as (3.9) is concerned a relatively simple result,

\[
\frac{i}{2\pi} \left[ \frac{2\mathcal{K}_\sigma \left( \frac{k + k'}{2} \right)}{\sqrt{4 + k^2}} \left( \frac{\pi}{2} - i\theta \right) + \left( 1 - \mathcal{K}_\sigma \left( \frac{k + k'}{2} \right) \right) \frac{a\pi}{2\sin a\pi} \right]. \tag{3.10}
\]

The other three pole pieces may be treated in the same manner yielding contributions similar to (3.9), except that \( k + k' \) is replaced by one of \( k - k', -k + k' \) and \( -k - k' \), in turn, leading to corresponding differences in the analogues of (3.9). Assembling all four contributions, and including the numerical factors from the integrations leads to the following correction to the reflection factor:

\[
- \frac{i\beta^2 C_2}{2} \mathcal{K}_\sigma \left( \frac{k}{2} \right) \sinh \theta \left[ \frac{1}{2} \left( \frac{1}{\cosh \theta + 1} - \frac{1}{\cosh \theta} \right)
\]

\[
+ \frac{a}{\sin^2 a\pi} \left( \frac{1}{\cosh \theta - \sin a\pi} - \frac{1}{\cosh \theta + \sin a\pi} \right) \right]. \tag{3.11}
\]
In (3.11), the overall factor $a$ is restricted to the range $0 \leq a \leq 1/2$, in order that $\sigma$ be positive.

The combinatorial factors $C_1$ and $C_2$ are equal since they arise from four-point interactions which are in essence identical, differing only by constant factors. Moreover, the common factor is $1/2$, as is easily checked using, for example, the functional integral form of the Green’s function.

Finally, notice that the two contributions from the boundary summarised in (3.6) and (3.11) combine neatly (within the restricted $a$ range) to give the result:

$$
\delta K_\sigma(\mathbf{k}) = -\frac{i\beta^2}{8} K_\sigma(\mathbf{k}) \sinh \theta \left[ \frac{1}{\cosh \theta + 1} - \frac{1}{\cosh \theta} \right] 
+ 2a \left( \frac{1}{\cosh \theta - \sin a\pi} - \frac{1}{\cosh \theta + \sin a\pi} \right).$$

For the special case $a = 1/2$, (3.12) collapses to (1.9), as expected.

4. Comparison with Ghoshal’s formula

To make the comparison with Ghoshal’s formula up to $O(\beta^2)$ the expansion of (1.5) is required. It is:

$$
\mathcal{K}(\theta) \sim K_0(\theta) \left( 1 - \frac{i\beta^2}{8} \sinh \theta \ G(\theta) \right),
$$

where

$$
G(\theta) = \frac{1}{\cosh \theta + 1} - \frac{1}{\cosh \theta} 
+ \frac{e_1}{\cosh \theta + \sin(e_0\pi/2)} - \frac{e_1}{\cosh \theta - \sin(e_0\pi/2)}
+ \frac{f_1}{\cosh \theta + \sin(f_0\pi/2)} - \frac{f_1}{\cosh \theta - \sin(f_0\pi/2)},
$$

and $E \sim e_0 + e_1\beta^2/4\pi$, $F \sim f_0 + f_1\beta^2/4\pi$. Comparison with the classical reflection factor $K_0(\theta)$, eq(1.13) reveals that $e_0 = a_0 + a_1$, $f_0 = a_0 - a_1$; comparison with eq(3.12) reveals further that when $a_0 = a_1 = a$,

$$
E \sim a \left( 2 - \frac{\beta^2}{2\pi} \right).
$$

However, since $f_0$ vanishes, nothing can be concluded to this order concerning $F$. Nevertheless, it is satisfying that the correction (3.12) is of the same general form as (4.2).
Given the coupling constant always appears to enter in the combination $B(\beta)$, it is tempting to suppose on the basis of the above that

$$E(a, \beta) = 2a \left( 1 - \frac{B}{2} \right). \quad (4.3)$$

On the other hand, consider the weak-strong coupling transformation applied to (1.3). The numerator of (1.3) is invariant under a duality transformation and, if it is assumed there is another theory within the same class whose boundary condition implies the dual reflection factor, there must be an $a^*$ such that

either $E(a, 4\pi/\beta) = \pm E(a^*, \beta) \mod 4$ or $E(a, 4\pi/\beta) = \pm F(a^*, \beta) \mod 4$. \quad (4.4)

This is a plausible assumption given the invariance of the two-particle S-matrix under the duality transformation, and given each boundary condition within the special class $a_0 = a_1$ preserves the symmetry $\phi \to -\phi$. The first possibility within (4.4) cannot work given the form of (4.3)—there is simply no suitable $a^*$. On the other hand, the second alternative is possible and suggests

$$F(a, \beta) = (1 - 2a)\frac{B}{2} \quad \text{and} \quad a^* = \frac{1}{2} - a. \quad (4.5)$$

With $F$ given by (4.5), there would be no change to the perturbation theory result (3.12) up to the order given. On the other hand, the next order should provide a test of the duality argument.

However, the correct formulae cannot be so simple. Recall that a Dirichlet condition $\phi(0, t) = 0$ also belongs to the special class, on allowing $a$ to be pure imaginary and infinite. As before, the perturbative result may be obtained from (4.2). On taking the appropriate limit, all that remains is the first part, in perfect agreement with Ghoshal’s formula for the Dirichlet case, which is self-dual and given by

$$\mathcal{K}_D = \frac{(2 - B/2)(1 + B/2)}{(1)}.$$

Within the special cases, it would be obtained by taking $E \to i\infty$ and $F \to 0 \mod 4$. Clearly, the correct formula would not be obtained using (4.3) and (4.5): (4.3) is satisfactory but (4.5) is not.
Moreover, the special coupling constant for which $B = -2$, the ‘free-fermion’ point of the sine-Gordon model\(^3\) has been analysed before [19] and relations have been written down between the parameters $\eta$ and $\vartheta$ occurring in Ghoshal’s formula at this value. Explicitly, using the notation of [19] but in the normalisations of this paper, the ratio of the boundary to bulk coupling $\gamma$, and the parameter $\phi_0$ are given by

$$
\gamma = 4 \cos a_0 \pi \cos a_1 \pi, \quad \cosh \phi_0 = \frac{1}{2} \left( \frac{\cos a_0 \pi}{\cos a_1 \pi} + \frac{\cos a_1 \pi}{\cos a_0 \pi} \right).
$$

In terms of these,

$$
k = (1 - \gamma \cosh \phi_0 + \gamma^2 / 4)^{-1/2}, \quad \cos \xi = k(1 - \gamma \cosh \phi_0 / 2),
$$

and the two equations [1],

$$
\cos \eta \cos i \vartheta = -(\cos \xi) / k, \quad \cos^2 \eta + \cos^2 i \vartheta = 1 + 1/k^2,
$$

are solved by

$$
\cos \eta = \pm \cos(a_0 + a_1) \pi, \quad \cos \vartheta = \pm \cos(a_1 - a_0) \pi.
$$

Hence, it is simple to deduce that at the free fermion point, and within the cases $a_0 = a_1$,

$$
\frac{B\eta}{\pi} = E = 4a \quad \frac{iB\vartheta}{\pi} = F = 0 \mod 4.
$$

Again, the formula (4.3) is adequate but the formula (4.5) is not.

A more general analysis might proceed along the following lines. Instead of (4.3) and (4.5), one could write ($q = \beta^2 / 4\pi$)

$$
E(a, q) = 2\frac{e(a, q)}{1 + q} \quad F(a, q) = 2\frac{f(a, q)}{1 + q},
$$

in which case, the duality transformation would require,

$$
e(a, 1/q) = \pm f(a^*, q) \mod 2 \left( \frac{1 + q}{q} \right) \quad f(a, 1/q) = \pm e(a^*, q) \mod 2(1 + q).
$$

Then, taking $e(a, q) \equiv a$ would be a consistent choice (as suggested by the known facts concerning $e(a, q)$), provided

$$
a = \epsilon_1 f(\epsilon_2 f(a, 1/q) \mod 2(1 + q), q) \mod 2 \left( \frac{1 + q}{q} \right),
$$

\(^3\) It is also a special point of the sinh-Gordon model in the sense that the S-matrix (1.1) is unity there.
where $\epsilon_1$ and $\epsilon_2$ are choices of sign. However, it has not yet proved possible to obtain a satisfactory conjecture based on this idea. For example, it is straightforward to see that choices of $f(a, q)$ of the form

$$f(a, q) = \frac{A(q) + aB(q)}{C(q) + aD(q)}$$

are inadequate once all the constraints are taken into account.

Alternatively, one might return to the first of (4.4) and seek a formula for $E(a, q)$ of the form

$$E(a, q) = \frac{n_0(a) + n_1(a)q + n_2(a)q^2 + \cdots + n_k(a)q^k}{d_0(a) + d_1(a)q + d_2(a)q^2 + \cdots + d_k(a)q^k}, \quad (4.9)$$

where the coefficients of the polynomials are constrained by the data described above and by the duality requirement. Neither linear nor quadratic polynomials appear to be satisfactory, while cubics and polynomials of higher degree are insufficiently constrained.

5. Discussion

Clearly, there is much left undone. Ghoshal’s formula has been verified to lowest order and it has proved possible to find a compact parametrisation for the free-fermion point. However, it has not yet allowed a consistent conjecture for the general parameter dependence even within the special class of models considered here. In the general case, $a_0 \neq a_1$, the calculations are more intricate for the reasons stated before.

It is intriguing to wonder about the supersymmetric sinh-Gordon theory. It has been noted [20] that only a few boundary conditions are compatible with both supersymmetry and integrability. Moreover, they belong to the special class considered here, at least as far as the bosonic part is concerned (and in the normalisation of this paper correspond to $\sigma_0 = \sigma_1 = \pm 1$). It would be interesting to understand the effect of the fermions on the duality transformation, since one would guess the two allowed points are either self-dual, or dual partners.

If and how duality works for the other affine Toda theories with a boundary will be interesting to discover. For the ade series of cases the S-matrix is self-dual, but the parameters which may be introduced at the boundary are highly constrained [21,22], and known merely to be choices of sign. Presumably, the duality transformation could relate the possibilities in pairs, although some may be self-dual, and the Neumann condition is likely, bearing in
mind the result for sinh-Gordon, to be related to the choice (+) for all boundary parameters. The other affine Toda models are arranged in pairs related by duality and it is known that for most of them there are one or possibly two extra parameters permitted at the boundary. It will certainly be interesting to understand how duality will work for these cases. Up to now, some of the classical backgrounds have been found \cite{5,21} but there has been no attempt to calculate the reflection factors perturbatively, except for the Neumann condition.

Finally, one could take a different point of view. There remains the (unwelcome?) possibility that the weak-strong coupling transformation does not relate pairs of boundary conditions at all, that

$$E(a, \beta) = 2a \left(1 - \frac{B}{2}\right), \quad F = 0,$$

is the correct parametrisation, that the Dirichlet condition and the condition with \(a = 0\) are self-dual, while all the rest within the given class have reflection factors whose large \(\beta\) limit is \(-1/(1)^2\).

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References

[1] S. Ghoshal and A.B. Zamolodchikov, ‘Boundary S matrix and boundary state in two-dimensional integrable quantum field theory’, Int. J. Mod. Phys. A9 (1994) 3841.

[2] S. Ghoshal, ‘Boundary state boundary S-matrix of the sine-Gordon model’, Int. J. Mod. Phys. A9 (1994) 4801.

[3] A. Fring and R. Köberle, ‘Factorized scattering in the presence of reflecting boundaries’, Nucl. Phys. B421 (1994) 159;
A. Fring and R. Köberle, ‘Affine Toda field theory in the presence of reflecting boundaries’, Nucl. Phys. B419 (1994) 647.

[4] R. Sasaki, ‘Reflection bootstrap equations for Toda field theory’, in Interface between Physics and Mathematics, eds W. Nahm and J-M Shen, (World Scientific 1994) 201.

[5] E. Corrigan, P.E. Dorey, R.H. Rietdijk, ‘Aspects of affine Toda field theory on a half line’, Suppl. Prog. Theor. Phys. 118 (1995) 143.

[6] E. Corrigan, P.E. Dorey, R.H. Rietdijk and R. Sasaki, ‘Affine Toda field theory on a half line’, Phys. Lett. B333 (1994) 83.

[7] P. Bowcock, E. Corrigan, P.E. Dorey and R. H. Rietdijk, ‘Classically integrable boundary conditions for affine Toda field theories’, Nucl. Phys. B445 (1995) 469;
P. Bowcock, E. Corrigan and R. H. Rietdijk, ‘Background field boundary conditions for affine Toda field theories’, Nucl. Phys. B465 (1996) 350.

[8] E. K. Sklyanin, ‘Boundary conditions for integrable equations’, Funct. Anal. Appl. 21 (1987) 164;
E. K. Sklyanin, ‘Boundary conditions for integrable quantum systems’, J. Phys. A21 (1988) 2375;
A. MacIntyre, ‘Integrable boundary conditions for classical sine-Gordon theory’, J. Phys. A28 (1995) 1089.

[9] H. Saleur, S. Skorik and N.P. Warner, ‘The boundary sine-Gordon theory: classical and semi-classical analysis’, Nucl. Phys. B441 (1995) 421;
S. Skorik and H. Saleur, ‘Boundary bound states and boundary bootstrap in the sine-Gordon model with Dirichlet boundary conditions’, J. Phys. A28 (1995) 6605.

[10] S. Penati and D. Zanon, ‘Quantum integrability in two-dimensional systems with boundary’, Phys. Lett. B358 63.

[11] E. Corrigan, ‘Integrable field theory with boundary conditions’, DTP-96/49; hep-th/9612138.

[12] A.E. Arinshtein, V.A. Fateev and A.B. Zamolodchikov, ‘Quantum S-matrix of the 1+1 dimensional Toda chain’, Phys. Lett. B87 (1979) 389-392.

[13] H.W. Braden, E. Corrigan, P.E. Dorey and R. Sasaki, ‘Affine Toda field theory and exact S-matrices’, Nucl. Phys. B338 (1990) 689;
H.W. Braden, E. Corrigan, P.E. Dorey and R. Sasaki, ‘Multiple poles and other features of affine Toda field theory’, *Nucl. Phys. B356* (1991) 469-98.

[14] G.W. Delius, M.T. Grisaru and D. Zanon, ‘Exact S-matrices for non simply-laced affine Toda theories’, *Nucl. Phys. B382* (1992) 365-408.

[15] E. Corrigan, P.E. Dorey and R. Sasaki, ‘On a generalised bootstrap principle’, *Nucl. Phys. B408* (1993) 579-99;

P.E. Dorey, ‘A remark on the coupling dependence in affine Toda field theories’, *Phys. Lett. B312* (1993) 291.

[16] J.D. Kim, ‘Boundary reflection matrix in perturbative quantum field theory’, *Phys. Lett. B353* (1995) 213.

[17] J.D. Kim, ‘Boundary reflection matrix for A-D-E affine Toda field theory’, DTP-95-31; [hep-th/9506031](http://arxiv.org/abs/hep-th/9506031);

J.D. Kim, ‘Boundary reflection matrices for non simply-laced affine Toda field theories’, *Phys. Rev. D53* (1996) 4441;

J.D. Kim and I.G. Koh, ‘Square root singularity in boundary reflection matrix’, *Phys. Lett. B388* (1996) 550.

[18] A. Fujii and R. Sasaki, ‘Boundary effects in integrable field theory on a half line’, *Prog. Theor. Phys. 93* (1995) 1123.

[19] M. Ameduri, R. Konik and A. LeClair, ‘Boundary sine-Gordon interactions at the free fermion point’, *Phys. Lett. B354* (1995) 376.

[20] T. Inami, S. Odake and Y-Z Zhang, ‘Supersymmetric extension of the sine-Gordon theory with integrable boundary interactions’, *Phys. Lett. B359* (1995) 118;

M. Moriconi and K. Schoutens, ‘Reflection matrices for integrable N=1 supersymmetric theories’, *Nucl. Phys. B487* (1997) 756.

[21] P. Bowcock, ‘Classical backgrounds and scattering for affine Toda theory on a half line’, DTP-96-37; [hep-th/9609233](http://arxiv.org/abs/hep-th/9609233).