SYMPLECTIC AND HYPERKÄHLER STRUCTURES IN A
DIMENSIONAL REDUCTION OF THE SEIBERG-WITTEN
EQUATIONS WITH A HIGGS FIELD

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ABSTRACT. In this paper we show that the dimensionally reduced Seiberg-
Witten equations lead to a Higgs field and study the resulting moduli spaces.
The moduli space arising out of a subset of the equations, shown to be non-
empty for a compact Riemann surface of genus $g \geq 1$, gives rise to a family of
moduli spaces carrying a hyperkähler structure. For the full set of equations
the corresponding moduli space does not have the aforementioned hyperkähler
structure but has a natural symplectic structure. For the case of the torus,
$g = 1$, we show that the full set of equations has a solution, different from the
“vortex solutions”.

1. INTRODUCTION

Dimensional reductions of various gauge theories from four dimensions to two
dimensions have proved to be geometrically very rich. For example, the moduli
space of solutions to the dimensional reduction of the self-dual Yang-Mills equations
over a Riemann surface, \cite{14}, exhibits, among other things, beautiful hyperkähler
structures.

It is important to study the analogous questions for the Seiberg-Witten equa-
tions. Though the main context of Seiberg-Witten theory is in four dimensions,
the 2-dimensional reduction also seems worth exploring. A reduction which gives
the “vortex” equations have been studied extensively e.g. Taubes \cite{30}, Bradlow,
Garcia-Prada \cite{5}, Olsen \cite{26}, Nergiz and SACLIOGLU \cite{23}, \cite{24} and others. This reduc-
tion does not have a Higgs field.

In the present paper we study a more general dimensional reduction of the
Seiberg-Witten equations which gives three equations. The novel feature is the
presence of a Higgs field, which in our case is an imaginary valued 1-form. In the
first few sections we consider the solutions to a subset of the equations, namely,
\[(2.1), (2.2)\] for genus $g \geq 1$ compact Riemann surfaces. The resulting moduli
spaces which we denote by $\mathcal{M}$ and $\Sigma_\Psi$ are hyperkähler. This is partly due to the
presence of the Higgs field, $\Phi$.

In parallel we show that the full set of equations \[(2.1) - (2.3)\] has a solution
with $\Phi \neq 0$ in the case of genus $g = 1$ compact Riemann surface. We assume that
solutions exist for genus $g > 1$ as well. The corresponding moduli space which
we denote by $\mathcal{N}$ has a symplectic and almost complex structure. The hyperkähler
structure, however, does not descend to $\mathcal{N}$. Setting $\Phi = 0$ results in “vortex”
equations. We calculate the “virtual” dimensions of the moduli spaces.
2. Dimensional Reductions of the Seiberg-Witten Equations

In this section we dimensionally reduce the Seiberg-Witten equations on $\mathbb{R}^4$ to $\mathbb{R}^2$ and define them over a compact Riemann surface $M$.

2.1. The Seiberg-Witten equations on $\mathbb{R}^4$: This is a brief description of the Seiberg-Witten equations on $\mathbb{R}^4$, [28, [1], [2], [21].

Identify $\mathbb{R}^4$ with the quaternions $\mathbb{H}$, coordinates $x = (x_1, x_2, x_3, x_4)$ identified with $\zeta = (\zeta_1, \zeta_2, \zeta_3, \zeta_4)$ and let $\{e_i, i = 1, 2, 3, 4\}$ be a basis for $\mathbb{H}$. Fix the constant spin structure $\Gamma : \mathbb{H} = T_z \mathbb{H} \to \mathbb{C}^{4 \times 4}$, given by $\Gamma(\zeta) = \begin{bmatrix} 0 & \gamma(\zeta)^* \\ \gamma(\zeta) & 0 \end{bmatrix}$, where

$$
\gamma(\zeta) = \begin{bmatrix} \zeta_1 + i\zeta_2 & -\zeta_3 - i\zeta_4 \\ \zeta_3 - i\zeta_4 & \zeta_1 - i\zeta_2 \end{bmatrix}.
$$

Thus $\gamma(e_1) = Id, \gamma(e_2) = J, \gamma(e_3) = J, \gamma(e_4) = K$.

where

$$
I = \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, J = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, K = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \end{bmatrix},
$$

so that $IJ = K, JK = J, KJ = I$ and $I^2 = J^2 = K^2 = -Id$.

Recall that $Spin^c(\mathbb{R}^4) = (Spin(\mathbb{R}^4) \times S^1)/\mathbb{Z}_2$. Since $Spin(\mathbb{R}^4)$ is a double cover of $SO(4)$, a spin$^c$ - connection involves a connection $\omega$ on $\mathbb{T}^\mathbb{H}$ and a connection $A = i \sum_{j=1}^4 A_j dx_j \in \Omega^1(\mathbb{H}, i\mathbb{R})$ on the characteristic line bundle $\mathbb{H} \times \mathbb{C}$ which arises from the $S^1$ factor (see [28, [21], [1] for more details). We set $\omega = 0$, which is equivalent to choosing the covariant derivative on the trivial tangent bundle to be $d$. This is legitimate since we are on $\mathbb{R}^4$. The curvature 2-form of the connection $A$ is given by $F(A) = dA \in \Omega^2(\mathbb{H}, i\mathbb{R})$. Consider the covariant derivative acting on $\Psi \in C^\infty(\mathbb{H}, \mathbb{C}^2)$ (the positive spinor on $\mathbb{R}^4$) induced by the connection $A$ on $\mathbb{H} \times \mathbb{C} : \nabla_j \Psi = (\frac{\partial}{\partial x_j} + iA_j)\Psi$. Then according to [28], the Seiberg-Witten equations for $(A, \Psi)$ on $\mathbb{R}^4$ are equivalent to the equations:

$$(SW1) : \nabla_1 \Psi = i\nabla_2 \Psi + J\nabla_3 \Psi + K\nabla_4 \Psi,$$

$$(SW2a) : F_{12} + F_{34} = \frac{1}{2} \Psi^* I \Psi - \frac{1}{2} (|\psi_1|^2 - |\psi_2|^2) \equiv \frac{1}{2} \eta_1,$$

$$(SW2b) : F_{13} + F_{42} = \frac{1}{2} \Psi^* J \Psi = i(Im \psi_1 \bar{\psi}_2) \equiv \frac{1}{2} \eta_2,$$

$$(SW2c) : F_{14} + F_{23} = \frac{1}{2} \Psi^* K \Psi = -i(Re \psi_1 \bar{\psi}_2) \equiv \frac{1}{2} \eta_3,$$

where $\Psi = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$.

2.2. Dimensional Reduction to $\mathbb{R}^2$: Using the same method of dimensional reduction as in [14], we get the general form of the reduced equations which contain the so-called Higgs field. Namely, impose the condition that none of the $A_i$’s and $\Psi$ in $(SW1)$ and $(SW2)$ depend on $x_3$ and $x_4$, i.e. $A_i = A_i(x_1, x_2), \Psi = \Psi(x_1, x_2)$ and set $\phi_1 = -iA_3$ and $\phi_2 = -iA_4$. The $(SW2)$ equations reduce to the following system on $\mathbb{R}^2$, $F_{12} = \frac{i}{2} \eta_1$, and two other equations, which, after introducing complex coordinates $z = x_1 + ix_2$, can be rewritten as: $\frac{\partial}{\partial x_1} = \frac{\partial}{\partial x_2} = \frac{i}{2} \eta_2, \eta_3$. Setting $\phi_1, \phi_2 = \phi$ and $\omega = idz \wedge d\bar{z}$ we rewrite the reduction of $(SW2)$ as the following two equations,

$$(1) \ F(A) = \frac{i}{2} (|\psi_1|^2 - |\psi_2|^2) \omega,$$

$$(2) \ 2\bar{\Phi} = -i(\psi_1 \bar{\psi}_2) \omega,$$

where $\Phi = \phi dz - \bar{\omega} d\bar{z} \in \Omega^1(\mathbb{R}^2, i\mathbb{R})$ and $\psi_1, \psi_2 \in C^\infty(\mathbb{R}^2, \mathbb{C})$ are spinors on $\mathbb{R}^2$. Next consider the Dirac equation $(SW1)$:
\[ \nabla_1 \psi - J \nabla_2 \psi - J \nabla_3 \psi - K \nabla_4 \psi = 0 \]

which is rewritten as
\[
\begin{bmatrix}
\frac{\partial}{\partial x} + i A_1 - i \frac{\partial}{\partial y} + A_2 \\
\frac{\partial}{\partial y} + i A_3 + i \frac{\partial}{\partial x} - A_4 \\
- \frac{\partial}{\partial y} - i A_3 + i \frac{\partial}{\partial x} - A_4 \\
\frac{\partial}{\partial x} - i A_1 + i \frac{\partial}{\partial y} - A_2
\end{bmatrix}
\begin{bmatrix}
\psi_1 \\
\psi_2 \\
\psi_3 \\
\psi_4
\end{bmatrix} = 0.
\]

Introducing \( A = \frac{i}{2} (A_1 - i A_2) d\bar{z} \) and \( \bar{A} = \frac{i}{2} (A_1 + i A_2) d\bar{z} \) where the total connection \( A - \bar{A} = i (A_1 dx + A_2 dy) \) we can finally write it as
\[
\begin{bmatrix}
- \frac{i}{2} \partial d\bar{z} \\
(\partial - \bar{A}) \\
(\bar{\partial} + A) \\
- \frac{i}{2} \phi dz
\end{bmatrix}
\begin{bmatrix}
\psi_1 \\
\psi_2 \\
\psi_3 \\
\psi_4
\end{bmatrix} = 0.
\]

We call equations (1)–(3) as the dimensionally reduced Seiberg-Witten equations over \( \mathbb{C} \).

2.3. The Dimensionally Reduced Equations on a Riemann surface. Let \( M \) be a compact Riemann surface of genus \( g \geq 1 \) with a conformal metric \( ds^2 = h^2 dz \otimes d\bar{z} \) and let \( \omega = i h^2 dz \wedge d\bar{z} \) be a real form proportional to the induced Kähler form. Let \( L \) be a line bundle with a Hermitian metric \( H \). Let \( \psi_1, \psi_2 \) be sections of the line bundle \( L \) i.e., \( \psi_1, \psi_2 \in \Gamma(M, L) \). Then we have an inner product
\[ < \psi_1, \psi_2 > _H \]
and norm \( |\psi|_H \in \mathbb{C}^\infty \) of the sections of \( L \). Let \( A - \bar{A} \) be a unitary connection on \( L \) and \( \Phi = \partial d\bar{z} - \bar{\partial} dz \in \Omega^1(M, i\mathbb{R}) \). We will assume that \( \Psi = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \) is not identically zero. We can rewrite the equations (1) – (3) in an invariant form on \( M \) as follows:

\[ F(A) = i \frac{(|\psi_1|^2_H - |\psi_2|^2_H)}{2} \omega, \]

\[ 2 \bar{\partial} \Phi = -i < \psi_1, \psi_2 >_H \omega, \]

\[ \begin{bmatrix}
- \frac{i}{2} \partial d\bar{z} \\
(\partial - \bar{A}) \\
(\bar{\partial} + A) \\
- \frac{i}{2} \phi dz
\end{bmatrix}
\begin{bmatrix}
\psi_1 \\
\psi_2
\end{bmatrix} = 0. \]

Note: Equation (2.3) has two equations. One equation (2.3a) comes with \( -\bar{A} \) and the other one (2.3b) with \( A \). This is unitarity of the connection \( A - \bar{A} \) on the line bundle. Setting \( \Phi = 0 \) in equation (2.2) and (2.3) we obtain the usual vortex equations (where either \( \psi_1 \) or \( \psi_2 \) is zero).

Let \( C = A \times \Gamma(M, L \oplus L) \times \mathcal{H} \), where \( A \) is the space of connections on a line bundle \( L \), \( \Gamma(M, L \oplus L) \) the space of sections of the bundle \( L \oplus L \) and \( \mathcal{H} \) be \( \Omega^1(M, i\mathbb{R}) \), the space of Higgs fields. The gauge group \( G = Maps(M, U(1)) \) acts on \( B \) as \( (A, \Psi, \Phi) \rightarrow (A + u^{-1} du, u^{-1} \Psi, \Phi) \) and leaves the space of solutions to (2.1)–(2.3) invariant. There are no fixed points of this action. Because a fixed point would mean that there is a connection \( A_0 \) such that \( A_0 + u^{-1} du = A_0 \) for all \( u \) in the gauge group. This is not possible. We assume throughout that \( \Psi \) is not identically zero.

Taking the quotient by the gauge group of the solutions to (2.1)–(2.3) we obtain a moduli space which we denote by \( \mathcal{M} \). Let us denote the moduli space of solutions to (2.1) – (2.2) as \( \mathcal{M} \) where we let the equivalence class of \( \Psi \) vary. We define a new moduli space \( \Sigma_{[\Psi]} \), by fixing an equivalence class of \( \Psi \) as follows. Choose an appropriate \( \Psi \) such that \( < \psi_1, \psi_2 > \omega \) is \( \bar{\partial} \)-exact and let \( W = A \times \{G \cdot \Psi\} \times \mathcal{H} \subset \mathcal{E} \) where \( \{G \cdot \Psi\} \) is the orbit of \( \Psi \) due to action of the gauge group. Let \( S_1 = W \cap \bar{S} \), where \( \bar{S} \) is the solution space to equations (2.1) and (2.2) on \( C \). Define \( \Sigma_{[\Psi]} = S_1 / G \).
Any point $p \in \Sigma[\Phi]$ is given by $p = ([A, \Psi, \Phi])$ where $\Psi$ is now fixed, $[\cdot, \cdot, \cdot]$ denotes the gauge equivalence class and $(A, \Psi, \Phi)$ satisfy equations (2.1) and (2.2). $S_1$ essentially consists of $(A, \Phi) \in A \times H$ such that $dA = \frac{i}{2} f_1 \omega$ and $2\bar{\Phi} = f_2 \omega$, where $f_1 = \psi_1^2 - |\psi_2|_H^2$ and $f_2 = -i < \psi_1, \psi_2 >_H \in C^\infty(M)$. We will see that if $(A, \Phi) \in \Sigma[\Phi]$, then if one changes $A \to A' = A + \alpha$ such that $d\alpha = 0$, $\alpha$ unique up to exact forms, and $\Phi^{0,1} \to \Phi^{0,1} = \Phi^{0,1} + \eta^{0,1}$ such that $\bar{\partial}\eta^{0,1} = 0$ then, the $(A', \Phi') \in \Sigma[\Phi]$. Thus $\Sigma[\Phi]$ is an affine space.

**Proposition 2.1.** If $L$ is a trivial line bundle on a compact Riemann surface of genus $g = 1$ then (2.1) – (2.2) have a solution (with $\Psi \neq 0$, $\Phi \neq 0$). If $L = K^{-1}$ on a compact Riemann surface of genus $> 1$ then (2.1) – (2.2) has a solution (with $\Psi \neq 0$, $\Phi \neq 0$). Thus $\mathcal{M}, \Sigma[\Phi]$ is non-empty.

**Proof.** For genus $g = 1$ let us take a metric of the form $ds^2 = dz \otimes d\bar{z}$. Let $\Psi_1 = 1$ and $\Psi_2 = e^{i\bar{z}} = e^{i(kz + k\bar{z})}$ be an eigenfunction of the Laplacian with eigenvalue $-1$, i.e. $|k| = 1$. Let $A$ be any flat connection and $\partial\bar{\partial}z = \frac{\partial(e^{-\psi})}{\partial z}$. They satisfy (2.1) and (2.2). The reason for this kind of solution will be clear in proposition (2.2).

For $g > 1$, $L = K^{-1}$ has a metric same as the metric on the surface $ds^2 = h^2 dz \otimes d\bar{z}$, so that we can write $h$ instead of $H$. We take $A - A$ to be the usual connection induced by the metric i.e. $A = \frac{\partial \log h}{\partial z}$ and $\bar{A} = \frac{\partial \log h}{\partial \bar{z}}[12]$. Then $F(A) = -(\frac{\partial \log h}{\partial z}) dz \wedge d\bar{z} = -iK\omega$ where $K = \Delta_{\omega} \frac{\partial \log h}{\partial z}$ is the Gaussian curvature of the metric $h$ and $\omega = \frac{i}{2} h^2 dz \wedge d\bar{z}$ is the Kähler form corresponding to the metric. Thus the equation (2.1) just reduces to $K = \frac{|\psi_1|^2 - |\psi_2|^2}{|\psi_1|^2 + |\psi_2|^2}$. The right hand side is a smooth function on the Riemann surface. We know that as long as $|\psi_2|^2 < |\psi_1|^2$, there is a solution when the genus is $> 1$. In fact there exists a metric, in every conformal class, such that any arbitrary negative definite function can be admitted as a Gaussian curvature of a Riemann surface of genus $> 1[3, 4, 13]$. Let $\Phi = \partial \bar{w}$. Also let $\psi_1 = v\psi_2$, where $v$ is a function. This is possible since $\psi_1, \psi_2$ are both sections of the same line bundle. Then equation (2.2) becomes

$\Delta_h w = \frac{1}{\rho^2} \frac{\partial^2 w}{\partial z \partial \bar{z}} = \frac{\tau}{\rho^2} |\psi_2|^2 = \tau$ where $\Delta_h$ is the Laplacian induced by the metric $h$. We choose $\psi_2$ arbitrarily. Now we choose $v$ to be such that $\int_M \frac{\tau}{\rho^2} |\psi_2|^2 h^2 dz d\bar{z} = 0$ and $|\psi_2|^2 < |\psi_1|^2$ hold. By Hodge theory [12] there exists a Green’s operator $G$ for the Laplacian such that $w = G\tau$ is a solution to equation (2.2).

**Proposition 2.2.** Let $L$ be a trivial line bundle on a compact Riemann surface of genus $g = 1$ then (2.1) – (2.3) have a solution with $\Psi \neq 0, \Phi \neq 0$. Thus $\mathcal{N}$ is non-empty.

**Proof.** Let us solve for the case of the case of the torus, $g = 1$. Let our torus be thought of as $0 \leq x \leq 2\pi$ and $0 \leq y \leq 2\pi$ with the endpoints identified. We take the metric on the torus to be $ds^2 = dz \otimes d\bar{z}$, i.e. $h = 1$. The equations are then as follows

(2.1) $F(A) = -\frac{i}{2} |\psi_1|^2 - |\psi_2|^2 dz \wedge d\bar{z} = 0$

(2.2) $\partial \bar{\partial}z\Phi = \frac{1}{2} \Psi_1 \Psi_2 d\bar{z} \wedge dz$

(2.3a) $\bar{\partial}w = A - \frac{i}{2} (\bar{\partial}z) \frac{\psi_2}{\psi_1} = 0$

(2.3b) $\partial \psi_1 = A - \frac{i}{2} \partial \bar{z} \frac{\psi_2}{\psi_1} = 0$

where $\Phi = \partial \bar{w}$. Let $\bar{\omega}$ be any flat connection and $\bar{\partial}w = \frac{\partial(e^{-\psi})}{\partial z}$. They satisfy (2.1) and (2.2). The reason for this kind of solution will be clear in proposition (2.2).

Thus the equation (2.1) just reduces to $K = \frac{|\psi_1|^2 - |\psi_2|^2}{|\psi_1|^2 + |\psi_2|^2}$. The right hand side is a smooth function on the Riemann surface. We know that as long as $|\psi_2|^2 < |\psi_1|^2$, there is a solution when the genus is $> 1$. In fact there exists a metric, in every conformal class, such that any arbitrary negative definite function can be admitted as a Gaussian curvature of a Riemann surface of genus $> 1[3, 4, 13]$. Let $\Phi = \partial \bar{w}$. Also let $\psi_1 = v\psi_2$, where $v$ is a function. This is possible since $\psi_1, \psi_2$ are both sections of the same line bundle. Then equation (2.2) becomes

$\Delta_h w = \frac{1}{\rho^2} \frac{\partial^2 w}{\partial z \partial \bar{z}} = \frac{\tau}{\rho^2} |\psi_2|^2 = \tau$ where $\Delta_h$ is the Laplacian induced by the metric $h$. We choose $\psi_2$ arbitrarily. Now we choose $v$ to be such that $\int_M \frac{\tau}{\rho^2} |\psi_2|^2 h^2 dz d\bar{z} = 0$ and $|\psi_2|^2 < |\psi_1|^2$ hold. By Hodge theory [12] there exists a Green’s operator $G$ for the Laplacian such that $w = G\tau$ is a solution to equation (2.2).
Since we took the line bundle to be trivial, one solution would be to take \( \Psi_1 = c_1, \Psi_2 = c_1e^{i\alpha_2(z+\bar{z})} \), \( \phi dz = -ic_2e^{-i\alpha_2(z+\bar{z})}dz \), \( A = -\frac{dz}{\alpha} \) where \( c_1 \) is a complex constant and \( c_2 \) is a real constant satisfying \(|c_1| = \sqrt{2c_2}\).

Near this solution there is a 4-dimensional moduli space, see proposition \((2.3)\).

**Proposition 2.3.** Let us consider the moduli spaces \( \Sigma_\Psi, \mathcal{M}, \mathcal{N} \), respectively. Suppose \((A, \Psi, \Phi)\) is a point on the moduli space such that \( \Psi \) is not identically 0. The (virtual) dimension of \( \Sigma_\Psi \) is \( 4g \), and \( \mathcal{M} \) is infinite dimensional. The (virtual) dimension of \( \mathcal{N} \) is \( 2g + 2 \). If either \( \Phi = 0 \) and \( \psi_1 \) or \( \psi_2 \) is zero (the vortext case) then the dimension of \( \mathcal{N} \) is \( c_1(L) + g + 1 \) or \( c_1(\bar{L}) + g + 1 \), respectively.

**Proof.** To calculate the dimension of \( \Sigma_\Psi \), we linearize equations (2.1) and (2.2) with equivalence class of \( \Psi \) fixed to obtain:

\[
\begin{align*}
(I) & \quad d\alpha = 0 \\
(II) & \quad \bar{\partial}\eta^{1,0} = 0,
\end{align*}
\]

where \((\alpha, \beta, \eta) \in T_p\mathcal{W} \) and \( \eta = \eta^{1,0} + \eta^{0,1} \). Taking into account the gauge group action, we get \( \dim\{\alpha \in \Omega^1(M, i\mathbb{R})|d\alpha = 0\}/\{\alpha = df\} = 2g \). Also, \( \dim\{\eta \in \Omega^{1,0}(M, \mathbb{C})|\bar{\partial}\eta = 0\} = 2g \). Thus the \( \dim[T_p\Sigma_\Psi] = 4g \).

\( \mathcal{M} \) is infinite dimensional since \( \Psi \) is not fixed.

To calculate the dimension of \( \mathcal{N} \) let \( \mathcal{S} \) be the solution space to (2.1) – (2.3). Consider the tangent space \( T_p\mathcal{S} \) at a point \( p = (A, \Psi, \Phi) \in \mathcal{S} \), which is defined by the linearization of equations (2.1) – (2.3). Let \( X = (\alpha, \beta, \gamma) \in T_p\mathcal{S} \), where \( \alpha \in \Omega^1(M, i\mathbb{R}) \) and \( \beta = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \in \Gamma(M, L \oplus L) \), and \( \gamma \in \mathcal{H} \). The linearizations of the equations are as follows

\[
(2.1)' \quad d\alpha = \frac{i}{2}(\beta_1 \bar{\psi}_1 + \psi_1 \bar{\beta}_1 - \beta_2 \bar{\psi}_2 - \psi_2 \bar{\beta}_2)\omega,
\]

\[
(2.2)' \quad \bar{\partial}\eta^{1,0} = -\frac{1}{2}(\psi_1 \bar{\psi}_2 + \beta_1 \bar{\beta}_2)\omega,
\]

\[
(2.3)' \quad \left[ \begin{array}{c} -\frac{1}{2}\phi dz \\ \bar{\partial} + A \\ -\frac{1}{2}\phi dz \\ \bar{\partial} + A \\ -\frac{1}{2}\phi dz \end{array} \right] \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} + \begin{bmatrix} -\frac{1}{2}\gamma^{0,1} \\ \alpha \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = 0.
\]

Taking into account the quotient by the gauge group \( \mathcal{G} \), we arrive at the following sequence \( \mathcal{C} \)

\[
0 \to \Omega^0(M, i\mathbb{R}) \xrightarrow{d_1} \Omega^1(M, i\mathbb{R}) \oplus \Gamma(M, \mathcal{L}) \oplus \mathcal{H} \xrightarrow{d_2} \Omega^2(M, i\mathbb{R}) \oplus \Omega^2(M, \mathbb{C}) \oplus V \to 0,
\]

where \( \mathcal{L} = L \oplus L, \quad V = (L \otimes \Omega^{0,1}(M)) \oplus (L \otimes \Omega^{1,0}(M)) \),

\[
\begin{align*}
d_1 f &= (df, -f \Psi, 0), \quad d_2(\alpha, \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}, \gamma) = (A, B, C), \\
A &= d\alpha - \frac{i}{2}(|(\psi_1 \bar{\beta}_1 + \beta_1 \bar{\psi}_1) - (\psi_2 \bar{\beta}_2 + \beta_2 \bar{\psi}_2)|\omega) \in \Omega^2(M, i\mathbb{R}) \\
B &= \bar{\partial}\gamma^{1,0} + \frac{1}{2}(\psi_1 \bar{\beta}_2 + \beta_1 \bar{\psi}_2)\omega \in \Omega^2(M, \mathbb{C}) \\
C &= \left[ \begin{array}{c} -\frac{1}{2}\phi dz \\ \bar{\partial} + A \\ -\frac{1}{2}\phi dz \end{array} \right] \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} + \begin{bmatrix} -\frac{1}{2}\gamma^{0,1} \\ \alpha \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \in V.
\end{align*}
\]
It is easy to check that \(d_2d_1 = 0\), so that this is a complex. Clearly, \(H^0(C) = 0\), because if \(f \in \ker(d_1)\), then \(df = 0\) and \(f \Psi = 0\), which implies \(f = 0\) since we are in the neighbourhood of a point where \(\Psi \neq 0\).

The Zariski dimension of the moduli space is \(\dim H^1(C)\) while the virtual dimension is \(\dim H^1(C) - \dim H^2(C)\), and coincides with the Zariski dimension whenever \(\dim H^2(C)\) is zero (namely the smooth points of the solution space \([2]\), page 66). The virtual dimension is \(= \dim H^1(C) - \dim H^2(C) = \text{index of } C\).

To calculate the index of \(C\), we consider the family of complexes \((\mathcal{C}', d')\), \(0 \leq t \leq 1\), where

\[
d_{1t} = (df, -tf \Psi, 0), \quad d_2(\alpha, \beta, \gamma) = (A_t, B_t, C_t),
\]

\[
A_t = da - \frac{t^2}{2}[(\Psi \bar{\beta}_1 + \beta_1 \bar{\psi}_1) - (\psi_2 \bar{\beta}_2 + \beta_2 \bar{\psi}_2)]\omega,
\]

\[
B_t = \tilde{\partial} \gamma^{1,0} + \frac{t}{2}(\bar{\psi}_1 \bar{\beta}_2 + \beta_1 \bar{\psi}_2)\omega,
\]

\[
C_t = \begin{cases} -\frac{t}{2} \bar{\psi} dz \quad (\tilde{\partial} - \bar{A}) \\ \bar{\partial} + A \\ \frac{t}{2} \bar{\psi} dz \end{cases}, \quad \gamma = \begin{bmatrix} \beta_1 \\ \beta_2 \\ -\frac{t}{2} \bar{\psi} \end{bmatrix}, \quad t \begin{bmatrix} -\frac{1}{2} \gamma^{0,1} \\ \alpha \\ -\frac{1}{2} \gamma^{1,0} \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}.
\]

Clearly, \(\text{ind}(\mathcal{C}')\) does not depend on \(t\). The complex \(C^0\) (for \(t = 0\)) is

\[
0 \to \Omega^0(M, i \mathbb{R}) \xrightarrow{d_t} \Omega^1(M, i \mathbb{R}) \oplus \Gamma(M, \mathcal{L}) \oplus \mathcal{H}^0 \Omega^2(M, i \mathbb{R}) \oplus \Omega^2(M, \mathcal{C}) \oplus \mathcal{V} \to 0
\]

where

\[
d_t f = (df, 0, 0, d'_{1t}(\alpha, \beta, \gamma)
\]

\[
= (da, \tilde{\partial} \gamma^{1,0}, \mathcal{D}_A \beta).
\]

Here \(\mathcal{D}_A = \begin{bmatrix} 0 & \partial - \bar{A} \\ \partial + A & 0 \end{bmatrix}\).

\(C^0\) decomposes into a direct sum of three complexes

(a) \(0 \to \Omega^0(X, i \mathbb{R}) \xrightarrow{d_t} \Omega^1(M, i \mathbb{R}) \to 0\),

(b) \(0 \to \Omega^1(M, i \mathbb{R}) \xrightarrow{\tilde{\partial}} \Omega^{1,1}(M, i \mathbb{R}) \to 0\),

(c) \(0 \to \Gamma(M, S) \xrightarrow{D^\partial} \Gamma(M, S') \to 0\), where \(S = L \oplus L, S' = (L \otimes K) \oplus (L \otimes \bar{K})\).

\(\dim H^1(\text{complex (a)}) = 2g, \dim H^1(\text{complex (b)}) = 2g\).

The complex (c) breaks into two complexes as follows

(c1) \(0 \to \Gamma(M, L) \xrightarrow{\tilde{\partial} + \bar{A}} \Gamma(M, L \otimes \bar{K}) \to 0\).

(c2) \(0 \to \Gamma(M, L) \xrightarrow{\bar{\partial} - \bar{A}} \Gamma(M, L \otimes K) \to 0\).

(c1) comes from the equation \((\partial + A)\bar{\psi}_1 = 0\). Taking complex conjugate one gets \((\tilde{\partial} + \bar{A})\bar{\psi}_1 = 0\) which is the holomorphicity of a section of \(\bar{L}\). Thus the first complex in (c) can be rewritten as

\[(c1) 0 \to \Gamma(M, L) \xrightarrow{\tilde{\partial} + \bar{A}} \Gamma(M, \bar{L} \otimes K) \to 0\).

By Riemann Roch, the index of (c1) is \((c_1(\bar{L}) - g + 1)\) and that of (c2) is \((c_1(L) - g + 1)\) and thus the sum is \(2g + 2g + c_1(\bar{L}) - g + 1 + c_1(L) - g + 1\) or \(2g + 2\). If \(\psi_1 = 0\) then the dimension is \(2g + c_1(L) - g + 1 = c_1(L) + g + 1\). If \(\psi_2 = 0\) then the dimension is \(c_1(L) + g + 1\).

**Note:** For simplicity of the exposition (to avoid writing the norms explicitly in terms of the metric \(H\)) we choose \(\bar{L} = L^{-1}\), so that \(\|\psi_1\|^2, \|\psi_2\|^2, <\psi_1, \psi_2> \in C^\infty(M, \mathcal{C})\) are well defined. Thus throughout the rest of the paper, we shall drop the subscripts \(H\) in equation (2.1) – (2.3). We must mention that this assumption is not essential, but it is just to make the exposition simpler.
3. SYMPLECTIC AND ALMOST COMPLEX STRUCTURES

In the next theorem we discuss symplectic and complex structures on $\mathcal{N}$. For similar work on the vortex moduli space, see [4], [11].

Let $\mathcal{C} = \mathcal{A} \times \Gamma(M, L \oplus L) \times \mathcal{H}$ be the space on which equations (2.1) − (2.3) are imposed. Let $p = (A, \Psi, \Phi) \in \mathcal{C}, X = (\alpha_1, \beta_1, \gamma_1), Y = (\alpha_2, \eta, \gamma_2) \in T_p \mathcal{C}$. On $\mathcal{C}$ one can define a metric

$$g(X, Y) = \int_M \ast \alpha_1 \wedge \alpha_2 + \int_M \text{Re} \beta \eta \omega + \int_M \ast \gamma_1 \wedge \gamma_2$$

and an almost complex structure $I = \begin{bmatrix} * & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & * \end{bmatrix}: T_p \mathcal{C} \rightarrow T_p \mathcal{C}$ where $*$ : $\Omega^1 \rightarrow \Omega^1$ is the Hodge star operator on $M$ (which takes type $dx$ forms to type $dy$ and $dy$ to $-dx$, i.e $*(\eta dz) = -i \eta dz, * (\eta d\bar{z}) = i \eta d\bar{z}$). We define

$$\Omega(X, Y) = -\int_M \alpha_1 \wedge \alpha_2 + \int_M \text{Re} I \beta_1, \gamma_2 - \int_M \gamma_1 \wedge \gamma_2$$

where $I = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ such that $g(I X, Y) = \Omega(X, Y)$. Moreover, we have the following:

**Proposition 3.1.** The metrics $g$, the symplectic form $\Omega$, and the almost complex structure $I$ are invariant under the gauge group action on $\mathcal{C}$.

**Proof.** Let $p = (A, \Psi, \Phi) \in \mathcal{C}$ and $u \in G$, where $u \cdot p = (A + u^{-1}du, u^{-1}\Psi, \Phi)$.

Then $u_* : T_p \mathcal{C} \rightarrow T_{u \cdot p} \mathcal{C}$ is given by the mapping $(\text{Id}, u^{-1} \text{Id})$ and it is now easy to check that $g$ and $\Omega$ are invariant and $I$ commutes with $u_*$. \qed

**Proposition 3.2.** The equation (2.1) can be realised as a moment map $\mu = 0$ with respect to the action of the gauge group and the symplectic form $\Omega$.

**Proof.** Let $\zeta \in \Omega(M, i\mathbb{R})$ be the Lie algebra of the gauge group (the gauge group element being $u = e^\zeta$); It generates a vector field $X_\zeta$ on $\mathcal{C}$ as follows :

$$X_\zeta(A, \Psi, \Phi) = (d\zeta, -\zeta \Psi, 0) \in T_p \mathcal{C}, p = (A, \Psi, \Phi) \in \mathcal{C}.$$

We show next that $X_\zeta$ is Hamiltonian. Namely, define $H_\zeta : \mathcal{C} \rightarrow \mathbb{C}$ as follows:

$$H_\zeta(p) = \int_M \zeta \cdot (F_A - i \frac{(|\psi_1|^2 - |\psi_2|^2)}{2} \omega).$$

Then for $X = (\alpha, \beta, \gamma) \in T_p \mathcal{C}$,

$$dH_\zeta(X) = \int_M \zeta d\alpha - i \int_M \zeta \text{Re}(\psi_1 \beta_1 - \psi_2 \beta_2) \omega$$

$$= \int_M (-d\zeta) \wedge \alpha - \int_M \text{Re} I \zeta \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}, \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \wedge \omega$$

$$= \Omega(X_\zeta, X),$$

where we use that $\tilde{\zeta} = -\zeta$. \qed
Thus we can define the moment map $\mu : C \to \Omega^2(M, i\mathbb{R}) = \mathcal{G}^*$ (the dual of the Lie algebra of the gauge group) to be

$$
\mu(A, \Psi) = (F(A) - i\frac{(|\psi_1|^2 - |\psi_2|^2)}{2}\omega).
$$

Thus equation (2.1)) is $\mu = 0$.

\[\Box\]

**Lemma 3.3.** Let $S$ be the solution spaces to equation (2.1) – (2.3), $X \in T_p S$. Then $\mathcal{I}X \in T_p S$ if and only if $X$ is orthogonal to the gauge orbit $O_p = G \cdot p$.

**Proof.** Let $X_\zeta \in T_p O_p$, where $\zeta \in \Omega^0(M, +i\mathbb{R})$, $g(X, X_\zeta) = -\Omega(\mathcal{I}X, X_\zeta) = -\int_M \zeta \cdot d\mu(\mathcal{I}X)$, and therefore $\mathcal{I}X$ satisfies the linearization of equation (2.1) iff $d\mu(\mathcal{I}X) = 0$, i.e., if $g(X, X_\zeta) = 0$ for all $\zeta$. Second, it is easy to check that $\mathcal{I}X$ satisfies the linearization of equation (2.2), (2.3) whenever $X$ does.

\[\Box\]

**Theorem 3.4.** $\mathcal{N}$ has a natural symplectic structure and an almost complex structure compatible with the symplectic form $\Omega$ and the metric $g$.

**Proof.** First we show that the almost complex structure descends to $\mathcal{N}$. Then using this and the symplectic quotient construction we will show that $\Omega$ gives a symplectic structure on $\mathcal{N}$.

(a) To show that $\mathcal{I}$ descends as an almost complex structure we let $pr : S \to S/G = \mathcal{N}$ be the projection map and set $[p] = pr(p)$. Then we can naturally identify $T_{[p]}\mathcal{N}$ with the quotient space $T_p S/T_p O_p$, where $O_p = G \cdot p$ is the gauge orbit. Using the metric $g$ on $S$ we can realize $T_{[p]}\mathcal{N}$ as a subspace in $T_p S$ orthogonal to $T_p O_p$. Then by lemma 3.3 this subspace is invariant under $\mathcal{I}$. Thus $I_{[p]} = \mathcal{I}|_{T_p (O_p)}$, gives the desired almost complex structure. This construction does not depend on the choice of $p$ since $\mathcal{I}$ is $G$-invariant.

(b) The symplectic structure $\Omega$ descends to $\mu^{-1}(0)/G$, (by proposition 3.2 and by the Marsden-Wienstein symplectic quotient construction), since the leaves of the characteristic foliation are the gauge orbits). Now, as a 2-form $\Omega$ descends to $\mathcal{N}$, due to proposition (3.1), so does the metric $g$. We check that equation (2.2), (2.3), does not give rise to new degeneracy of $\Omega$ (i.e. the only degeneracy of $\Omega$ is due to (2.1) but along gauge orbits). Thus $\Omega$ is symplectic on $\mathcal{N}$. Since $g$ and $\mathcal{I}$ descend to $\mathcal{N}$ the latter is symplectic and almost complex.

### 3.1. Hyperkähler structure in the moduli spaces $\mathcal{M}$ and $\Sigma_\Psi$. We recall that we realised equation (2.1) as a moment map. To realize the equation (2.2) as a moment map we first rewrite the second equation as

\[\text{(2.2)} \quad 2\partial \Phi' = -<\psi_1, \psi_2>_H \omega\]

where $\Phi' = -i\Phi = -i\partial \bar{\partial} z + i\partial \bar{\partial} \bar{z} \in \Omega^1(M, \mathbb{R})$. We rename $\Phi'$ as $\Phi$. This notation will be valid only in this section where we do not consider equation (2.3). We need to define another symplectic form $Q$ on $C$, which is complex-valued,

$$
Q(X, Y) = -2 \int_M \alpha_1 \wedge \gamma_2 + 2 \int_M \alpha_2 \wedge \gamma_1 - \int_M (\beta_1 \beta_2 - \beta_2 \beta_1) \omega
$$

where $X = (\alpha_1, \beta_1, \gamma_1), \ Y = (\alpha_2, \beta_2, \gamma_2) \in T_p \mathcal{E}$.

**Proposition 3.5.** The vector field $X_\zeta$ induced by the gauge action is Hamiltonian with respect to the symplectic form $Q$. 

Proof. Define the Hamiltonian to be $H_{\zeta}(A,\psi,\phi) = \int_M \zeta(2\bar{\partial}\Phi + \psi_1\bar{\psi}_2\omega)$, where $\zeta \in \Omega(M,i\mathbb{R})$. Then for $X = (\alpha,\beta_1,\gamma) \in T_p\mathcal{C}$,

$$dH_{\zeta}(X) = \int_M \zeta(2\bar{\partial}\gamma + (\beta_2^1\bar{\psi}_2 + \psi_1\bar{\beta}_2^2)\omega)$$

$$= 2\int M \bar{\partial}\zeta \wedge \gamma + \int M (\zeta\beta_1^1\bar{\psi}_2 + \zeta\psi_1\bar{\beta}_2^2)\omega$$

$$= 2\int M \bar{\partial}\zeta \wedge \gamma + \int M (-\beta_2^1\bar{\psi}_2 + \zeta\psi_1\bar{\beta}_2^2)\omega$$

$$= \mathcal{Q}(X_\zeta,X).$$

where $X_\zeta = (d\zeta,-\zeta\Phi,0)$. Thus we can define the moment map of the action with respect to the form $\mathcal{Q}$ to be: $\mu_{\mathcal{Q}} = 2\bar{\partial}\Phi + \langle \psi_1,\bar{\psi}_2 \rangle \omega$. Thus equation (2.2) is precisely $\mu_{\mathcal{Q}} = 0$. 

**Proposition 3.6.** The configuration space $\mathcal{C}$ has a Riemannian metric $g(X,Y) = \int_M *\alpha_1 \wedge \alpha_2 + \int_M \text{Re} <\beta_1,\beta_2> \omega + \int_M \gamma_1 \wedge \gamma_2$, where $X = (\alpha_1,\beta_1) = \left[\begin{array}{c} \beta_1^1 \\ \beta_1^2 \end{array}\right], \gamma_1, Y = (\alpha_2,\beta_2) = \left[\begin{array}{c} \beta_2^1 \\ \beta_2^2 \end{array}\right], \gamma_2) \in T_p\mathcal{C}$, and three complex structures

$I = \left[\begin{array}{ccc} * & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & * \end{array}\right], J = \left[\begin{array}{ccc} 0 & 0 & * \\ 0 & J & 0 \\ * & 0 & 0 \end{array}\right], K = \left[\begin{array}{ccc} 0 & 0 & -1 \\ 0 & K & 0 \\ 1 & 0 & 0 \end{array}\right]$

where $I = \left[\begin{array}{ccc} i & 0 \\ 0 & -i \end{array}\right], J = \left[\begin{array}{ccc} 0 & 1 \\ -1 & 0 \end{array}\right]$ and $K = \left[\begin{array}{ccc} 0 & i \\ i & 0 \end{array}\right]$, and $*: \Omega^1(M) \rightarrow \Omega^1(M)$ is the Hodge-star operator. They satisfy $I,J = J,K = I$, and $KJ = J$. The three symplectic structures $\omega_1(X,Y) = g(IX,Y), \omega_2(X,Y) = g(JX,Y), \omega_3(X,Y) = g(KX,Y)$ are such that $\omega_2 + i\omega_3 = \mathcal{Q}$.

**Proof.**

$$\omega_1(X,Y) = -\int M \alpha_1 \wedge \alpha_2 + \int_M \text{Re} <I\beta_1,\beta_2> \omega + \int_M \gamma_1 \wedge \gamma_2$$

$$\omega_2(X,Y) = \int_M -\gamma_1 \wedge \alpha_2 - \int_M \alpha_1 \wedge \gamma_2 + \int_M \text{Re}(\beta_2^1\bar{\beta}_2^2 - \beta_1^1\bar{\beta}_1^2)\omega$$

$$\omega_3(X,Y) = \int_M *\gamma_1 \wedge \alpha_2 + \int_M \text{Re} <K\beta_1,\beta_2> + \int_M *\alpha_1 \wedge \gamma_2$$

$$= \int_M *\gamma_1 \wedge \alpha_2 + \int_M \text{Re}(i\beta_1^1\bar{\beta}_2^1 + i\beta_1^1\bar{\beta}_2^1)\omega + \int_M *\alpha_1 \wedge \gamma_2$$
so that indeed

\[(\omega_2 + i\omega_3)(X,Y) = \int_M (-\gamma_1 - i*\gamma_1) \wedge \alpha_2 + \int_M [Re(\beta_1^2 \beta_2^1 - \overline{\beta_1^1 \beta_2^2})] + i Re(i(\beta_1^2 \beta_2^1 + i\beta_1^1 \beta_2^2)) \omega + \int (-\alpha_1 + i*\alpha_1) \wedge \gamma_2 = -2\int_M (\gamma_1)^{1,0} \wedge \alpha_2^{1,0} - \int_M (\beta_1^1 \beta_2^2 - \overline{\beta_1^2 \beta_2^1}) \omega - 2\int_M (\alpha_1)^{0,1} \wedge \gamma_2^{1,0} = \mathcal{Q}(X,Y).\]

Let \(\tilde{S} = \mu^{-1}(0) \cap \mu_Q^{-1}(0) \subset \mathcal{E}\) be the solution space to the equations (2.1) and (2.2), and denote by \(\mathcal{M} = \tilde{S}/G\) the corresponding moduli space.

**Theorem 3.7.** Let \(M\) be a compact Riemann surface of \(g \geq 1\). Let \(\mathcal{M}\) be the moduli space of solutions to equations (2.1) and (2.2). Then the Riemannian metric \(g\) induced by the metric on \(\mathcal{C}\) is hyperkählerian, and \(\mathcal{M}\) is hyperkähler.

**Proof.** Since \(\mathcal{I}, \mathcal{J}, \mathcal{K}, g\) and \(\omega_1, \mathcal{Q}\) are \(G\)-invariant, and \(\mathcal{M}\) comes from a symplectic reduction, it follows that the symplectic forms \(\omega_i, i = 1, 2, 3\), descend to \(\mathcal{M}\) as symplectic forms. Also, from the proof of theorem (3.4) and proposition (3.6) it follows that \(\mathcal{I}, \mathcal{J}, \mathcal{K}\) are well defined almost complex structures on \(\mathcal{M}\). To show that they are integrable, we use the following lemma of Hitchin (see [14]).

**Lemma 3.8.** Let \(g\) be an almost hyperkähler metric, with 2-forms \(\omega_1, \omega_2, \omega_3\) corresponding to almost complex structures \(\mathcal{I}, \mathcal{J}\) and \(\mathcal{K}\). Then \(g\) is hyperkähler if each \(\omega_i\) is closed.

**Theorem 3.9.** Let \(M\) be a Riemann surface of genus \(g \geq 1\). Fix the equivalence class of \(\Psi\) such that \(\psi_1\) and \(\psi_2\) are each not identically zero and such that \(\psi_1, \psi_2 > \omega\) is \(\hat{\partial}\)-exact. Then, \(\Sigma[\Psi]\) is hyperKähler affine manifold of dimension \(4g\). Also, \(\mathcal{M} = \mathcal{S}p \times \Sigma[\Psi]\) where \(\mathcal{S}p = \{\psi : \psi_1, \psi_2 > \omega\) is \(\hat{\partial}\)-exact.\}

**Proof.** On \(\mathcal{W}\) one defines the same symplectic forms \(\omega_1, \omega_2\)and \(\omega_3\) as in the previous section. On \(\Sigma[\Psi]\) these forms restrict to

\[
\omega_1[\Psi] = -\int_M \alpha_1 ^* \wedge \alpha_2 + \int_M \gamma_1 \wedge \gamma_2,
\]

\[
\omega_2[\Psi] = -\int_M \gamma_1 \wedge \alpha_2 - \int_M \alpha_1 \wedge \gamma_2
\]

\[
\omega_3[\Psi] = -\int_M * \gamma_1 \wedge \alpha_2 + \int_M * \alpha_1 \wedge \gamma_2
\]

which are, by arguments same as in the previous section, hyperkählerian with respect to the complex structures \(\mathcal{I}_1 = \begin{bmatrix} * & 0 \\ 0 & -* \end{bmatrix}, \mathcal{J}_1 = \begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix}, \mathcal{K}_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\)

and to the Riemannian metric \(g(X,Y) = \int_M * \alpha_1 \wedge \alpha_2 + \int_M * \gamma_1 \wedge \gamma_2\) where \(X = (\alpha_1, \gamma_1)\) and \(Y = (\alpha_2, \gamma_2) \in T_p \Sigma[\Psi].\)

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