A Technical Report on PLS-Completeness of Single-Swap for Unweighted Metric Facility Location and $K$-Means

Sascha Brauer
sascha.brauer@uni-paderborn.de
Department of Computer Science
Paderborn University
33098 Paderborn, Germany

Recently, [Bra17] showed that the single-swap heuristic for weighted metric uncapacitated facility location and $K$-Means is tightly PLS-complete. We build upon this work and present a stronger reduction, which proves tight PLS-completeness for the unweighted version of both problems.

1 Introduction

Metric facility location and $K$-Means are important problems in operations research and computational geometry. Both problems admit a fairly simple local search scheme called single-swap, which is known to compute a constant factor approximation of an optimal solution. Recently, [Bra17] showed that single-swap is tightly PLS-complete, which means that the local search algorithm requires exponentially many steps in the worst case and that given some initial solution it is PSPACE-complete to find the solution computed by the algorithm started on this initial solution. One shortcoming of the presented result is that it constructs a non-trivial weight function on the clients, hence, the reduction is not sufficient to classify the hardness of the problems on unweighted instances.

In this report, we modify the reduction presented in [Bra17] to obtain tight PLS-completeness of unweighted versions of metric uncapacitated facility location and discrete $K$-Means. This is an important extension of the result, since, besides being a formally stronger result, unweighted instances are significantly more relevant in practice. Furthermore, we present a lower bound on the number of dimensions required to embed the point set constructed by our reduction into Euclidean space.

2 Preliminaries and Notation

In an Uncapacitated Facility Location (UFL) problem we are given a set of clients $C$, a set of facilities $F$, an opening cost function $f : F \to \mathbb{R}$, and a distance function $d : C \times F \to \mathbb{R}$. The goal is to find a subset of facilities $O \subseteq F$ minimizing

$$\phi_{FL}(C, F, O) = \sum_{c \in C} \min_{o \in O} \{d(c, o)\} + \sum_{o \in O} f(o).$$

Metric Uncapacitated Facility Location is a special case of this problem, where we require the distance function $d$ to be a metric on $C \cup F$. The PLS problem MUFL/Swap consists of MUFL, where the so-called single-swap neighbourhood of a set of open facilities is given by sets of facilities obtained by newly opening a closed facility, closing an open facility, or doing both in one step (swapping two facilities).

In Discrete $K$-Means (DKM) we do not differentiate between clients and facilities, but are given a single set of points $C \subseteq \mathbb{R}^D$. We measure distance between points $p, q \in C$ as $d(p, q) = \|p - q\|^2$. Furthermore, instead of imposing an opening cost, we allow at most $K$ locations to be opened. Hence, the goal is to find $O \subseteq C$ with $|O| = K$ minimizing

$$\phi_{KM}(C, O) = \sum_{c \in C} \min_{o \in O} \{\|c - o\|^2\}.$$
The PLS problem DKM/Swap consists of DKM, where the single-swap neigbourhood is given by all subsets of points of \( C \) obtained by swapping two points. We forbid the open and close operations, as these leave the space of feasible solutions for DKM.

**Theorem 1 (SY91).** SAT/Flip is tightly PLS-complete.

For each clause set \( B \) and truth assignment \( T \) we denote the SAT cost of \( T \) with respect to \( B \) by \( w(B, T) \). For a literal \( x \) we denote the set of all clauses in \( B \) containing \( x \) by \( B(x) \). Further, we denote the set of all clauses in \( B \) satisfied by \( T \) by \( B_s(T) \) and let \( B_f(T) = B \setminus B_s(T) \). Finally, we set \( w_{\text{max}}^B = \max_{b \in B} \{ w(b) \} \) and \( w_{\text{min}}^B = \min_{b \in B} \{ w(b) \} \).

### 3 Facility Location

We show that unweighted MUFL/Swap is tightly PLS-complete.

**Proposition 2.** SAT/Flip \( \leq_{\text{PLS}} \) MUFL/Swap and this reduction is tight.

In the following, we present our PLS-reduction \((\Phi, \Psi)\) and prove its correctness. We omit the tightness proof, as is can easily be obtained by incorporating the arguments presented here into the tightness proof from [Bra17].

#### 3.1 Construction of \( \Phi \) and \( \Psi \)

First, we construct the function \( \Phi \) mapping a Max 2-Sat instance to a Metric Uncapacitated Facility Location instance. Let \((B, w)\) be a Max 2-Sat instance over the variables \( \{x_n\}_{n \in [N]} \). Let further \( M := |B| \) and \( W := M \cdot w_{\text{max}}^B \). We assume that \( M \geq 2 \). In the following, we construct an instance \((C, F, f, d) \in \text{Metric Uncapacitated Facility Location}\). We set \( F = \{x_n, \bar{x}_n\}_{n \in [N]} \) and locate a client at each facility and a client corresponding to each clause, so \( C = F \cup B \). We set \( d : C \cup F \times C \cup F \rightarrow \mathbb{R} \) to

\[
d(p, q) = d(q, p) = \begin{cases} 
0 & \text{if } p = q \\
1 & \text{if } p = x_n \wedge q = \bar{x}_n \\
1 + \frac{w(b_m)}{W} & \text{if } (p = x_n \lor p = \bar{x}_n) \land q = b_m \land p \in b_m \\
1 + \frac{c \cdot w(b_m)}{W} & \text{if } (p = x_n \lor p = \bar{x}_n) \land q = b_m \land \bar{p} \in b_m \\
2 & \text{else,}
\end{cases}
\]

where \( 1 < c < 2 \).

It is easy to see that \( d \) is a metric and that the points closest to each other are literals and their negation, that clauses are closer to literals the contain, then to the literal’s negation, and that all other point pairs have the same, even larger, distance to each other.

The opening cost function is constant \( f \equiv 2 \).

Second, we construct the function \( \Psi \) mapping solutions of \( \Phi(B, w) \) back to solutions of \( (B, w) \). Given a set \( O \subset F \) we let each variable \( x_n \) be true if \( x_n \in F \) and let it be false otherwise.

In the following, we denote \( \Phi(B, w) = (C, F, 2, d) \) and \( \Psi(B, w, O) = T_D \).

#### 3.2 \((\Phi, \Psi)\) is a PLS-reduction

We need to argue that \( T_D \) is locally optimal for \((B, w)\) if \( O \) is locally optimal for \( \Phi(B, w) \). To prove this, we define a particular subset of solutions for \( \Phi(B, w) \) we call reasonable solutions.

**Definition 3.** Let \( O \subset F \). We call \( O \) reasonable if \(|O| = N \) and

\[
\forall n \in [N] : x_n \in O \lor \bar{x}_n \in O.
\]
To prove correctness of our reduction, we observe the following crucial properties of reasonable solutions. The restriction of $\Psi$ to reasonable solutions is a bijection, there is close relation of the MUFL cost of a reasonable solution and the cost of its image under $\Psi$, and all locally optimal solutions of $(C, F; 2, d)$ are reasonable.

**Lemma 4.** If $O \subset F$ is reasonable, then

$$\phi_{FL}(C, F, O) = 3N + M + \frac{1}{W} \sum_{b_m \in B} w(b_m) + \frac{c - 1}{W} \sum_{b_m \in B(T_O)} w(b_m).$$

**Proof.** Observe, that reasonable solutions incur opening cost of $2N$. By definition of $\Psi$ and reasonable solutions, we have that a variable $x_n$ is assigned true if and only if $x_n \in O$ and is assigned false if and only if $\bar{x}_n \in O$. We obtain that the service cost of clients corresponding to literals is $0$, or its negated facility at distance $1$ has cost $1$, or $c \cdot w(b_m)/W$ and the rest at distance $2$. We obtain

$$\phi_{FL}(C, F, O) = 2N + N + \sum_{b_m \in B(T_O)} \left(1 + \frac{w(b_m)}{W}\right) + \sum_{b_m \in B(T_O)} \frac{c \cdot w(b_m)}{W}$$

$$= 3N + M + \sum_{b_m \in B(T_O)} \frac{w(b_m)}{W} + \sum_{b_m \in B(T_O)} \frac{c \cdot w(b_m)}{W}$$

$$= 3N + M + \frac{1}{W} \sum_{b_m \in B} w(b_m) + \frac{c - 1}{W} \sum_{b_m \in B(T_O)} w(b_m).$$

**Corollary 5.** If $O, O' \subset C$ are reasonable solutions for $\Phi(B, w)$, then

$$w(B, T_O) < w(B, T_{O'}) \Leftrightarrow \phi_{FL}(C, F, O) > \phi_{FL}(C, F, O').$$

**Proof.** Observe that

$$w(B, T_O) = \sum_{b_m \in B(T_O)} w(b_m) \leq \sum_{b_m \in B(T_O)} w(b_m) = w(B, T_{O'})$$

$$\Leftrightarrow \sum_{b_m \in B(T_O)} w(b_m) > \sum_{b_m \in B(T_{O'})} w(b_m).$$

Observe, that by Lemma 4 the only summand of the cost $\phi_{FL}(C, F, O)$ of a reasonable solution $O$ actually depending on $O$ is $(c - 1)/W \sum_{b_m \in B(T_O)} w(b_m)$. Since $c > 1$, we obtain

$$\sum_{b_m \in B(T_O)} w(b_m) > \sum_{b_m \in B(T_{O'})} w(b_m)$$

$$\Leftrightarrow \phi_{FL}(C, F, O) > \phi_{FL}(C, F, O').$$

**Lemma 6.** If $O \subset F$ is locally optimal for $\Phi(B, W)$, then $O$ is reasonable.

**Proof.** The following proof is presented in two steps. First, we argue that no locally optimal solution can contain both a literal and its negation. Second, we show that every locally optimal solution contains a facility corresponding to each of the variables.

Assume, there is an $n$ such that $x_n, \bar{x}_n \in O$. Observe, that $B(x_n) \leq M$ and that no client in $C \setminus (B(x_n) \cup \{x_n\})$ is closer to $x_n$ than it is to $\bar{x}_n$. Since $1 < c < 2$ and by definition of $W$, we obtain

$$\phi_{FL}(C, F, O) = \sum_{c \in C \setminus (B(x_n) \cup \{x_n\})} d(c, O) + \sum_{b_m \in B(x_n)} \left(1 + \frac{w(b_m)}{W}\right) + |O| 2.$$
Now assume, that $x_n, \bar{x}_n \notin O$. Connecting the clients located at $x_n$ and $\bar{x}_n$ to a facility newly opened at $x_n$ is sufficient to reduce the overall cost. We obtain

$$\phi_{FL}(C, F, O) = \sum_{c \in C \setminus \{x_n, \bar{x}_n\}} d(c, O) + \sum_{b_m \in B(x_n)} \left(1 + \frac{c \cdot w(b_m)}{W}\right) + \frac{1}{d(x_n, x_n)} + |O|/2$$

$\geq \phi_{FL}(C, F, O \cup \{x_n\})$

$\square$

**Corollary 7.** If $O$ is locally optimal for $\Psi(B, w)$, then $T_O$ is locally optimal for $(B, w)$.

**Proof.** Combine Corollary 5 and Lemma 6.

### 4 K-Means

We complement our results by showing that we can obtain tight PLS-completeness for DKM/Swap, as well.

**Proposition 8.** SAT/Flip $\leq_{PLS}$ DKM/Swap and this reduction is tight.

This reduction is similar to the previously presented reduction for MUFL. We mostly have to change some of the constants involved, and finally argue that there is a point set in $\mathbb{R}^D$ exhibiting the required interpoint distances.

#### 4.1 Constructing $\Phi$ and $\Psi$

First, we construct the function $\Phi$ mapping a MAX 2-Sat instance to a DISCRETE K-MEANS instance. Let $(B, w)$ be a MAX 2-SAT instance over the variables $\{x_n\}_{n \in [N]}$. Let further $M := |B|$ and $W := M \cdot \max w$. We assume that $M \geq 2$. In the following, we construct an instance $(C, K) \in$ DISCRETE K-MEANS. Abstractly define the point set $C = \{x_n, \bar{x}_n\}_{n \in [N]} \cup B$. The distance function $d : C \times C \to \mathbb{R}$ is defined as

$$d(p, q) = d(q, p) = \begin{cases} 0 & \text{if } p = q \\ 1 & \text{if } p = x_n \land q = \bar{x}_n \\ 1 + \epsilon \left(\frac{3}{2} + \frac{w(b_m)}{2W}\right) & \text{if } (p = x_n \lor p = \bar{x}_n) \land q = b_m \land p \in b_m \\ 1 + 2\epsilon & \text{if } (p = x_n \lor p = \bar{x}_n) \land q = b_m \land p \in b_m \\ 1 + 2\epsilon & \text{else,} \end{cases}$$

where $1 < c < 2$ and

$$\epsilon = \frac{1}{4N + 2M}.$$
4.2 \((\Phi, \Psi)\) is a PLS-reduction

Recall the definition of reasonable solutions and observe that all reasonable solutions are also feasible solutions for the DKM instance \((C, N)\).

Lemma 9. If \(O \subset C\) is reasonable, then

\[
\phi_{KM}(C, O) = N + M \left(1 + \epsilon \frac{3}{2}\right) + \frac{\epsilon}{2W} \sum_{b_m \in B} w(b_m) + \frac{\epsilon}{2W} (c - 1) \sum_{b_m \in B_f(T_O)} w(b_m).
\]

Proof. By definition of \(\Psi\) and reasonable solutions, we have that a variable \(x_n\) is assigned true if and only if \(x_n \in O\) and is assigned false if and only if \(x_n \notin O\). We obtain that \(\phi_{KM}(\{x_n, \bar{x}_n\}_{n \in [N]}, O) = N\), since each point corresponding to a literal is either in \(O\) and has cost 0, or its negated literal at distance 1 is in \(O\) and it has cost 1. It is easy to see, by definition of the point set and \(\Psi\), that a point corresponding to a clause \(b_m \in B_f(T_O)\) has at least one mean at distance \(1 + \epsilon (3/2 + w(b_m)/(2W))\) and that a point corresponding to a clause \(b_m \in B_f(T_O)\) has two means at distance \(1 + \epsilon (3/2 + c \cdot w(b_m)/(2W))\) and the rest at distance \(1 + 2\epsilon\). We obtain

\[
\phi_{KM}(C, O) = \phi_{KM}(\{x_n, \bar{x}_n\}_{n \in [N]}, O) + \phi_{KM}(B, O) = N + M \left(1 + \frac{3}{2}\right) + \frac{\epsilon}{2W} \sum_{b_m \in B_f(T_O)} w(b_m) + \frac{\epsilon}{2W} (c - 1) \sum_{b_m \in B_f(T_O)} w(b_m)
\]

Corollary 10. If \(O, O' \subset C\) are reasonable solutions for \(\Phi(B, w)\), then

\[
w(B, T_O) < w(B, T_{O'}) \iff \phi_{KM}(C, O) > \phi_{KM}(C, O')\]

Proof. Analogous to the proof of Corollary[5]

Lemma 11. If \(O \subset C\) is locally optimal for \(\Phi(B, w)\), then \(O\) is reasonable.

Proof. Recall, that each point \(b_m \in C\) has exactly two points at distance \(1 + \epsilon (3/2 + w(b_m)/(2W))\) and two points at distance \(1 + \epsilon (3/3 + c \cdot w(b_m)/(2W))\) (the points corresponding to the literals in the clause \(b_m\) and their negations, respectively). In the following, we call these four points adjacent to \(b_m\). All the other points have distance \(1 + 2\epsilon\) to \(b_m\) and are hence strictly farther away. Assume to the contrary that there exists an \(n \in [N]\), such that \(x_n, \bar{x}_n \notin O\).

Case 1: There exists an \(n \in [M]: b_m \in O\), such that \(b_m = \{x_o, x_p\}\) (where one or both of these literals might be negated). One important observation is that if we exchange \(b_m\) for some other location then only its own cost and the cost of its adjacent points can increase. All other points, which might be connected to \(b_m\), are at distance \(1 + 2\epsilon\) and can hence be connected to any other location for at most the same cost.

Case 1.1: \(x_o, \bar{x}_o, x_p, \bar{x}_p \notin O\). Each point adjacent to \(b_m\) has distance at least \(1 + \epsilon (3/2 + w_{min}/(2W))\) to every other points in \(P\). Hence, we have that \(\phi_{KM}(\{b_m, x_o, \bar{x}_o, x_p, \bar{x}_p\}, O) \geq 4 + 4\epsilon (3/2 + w_{min}/(2W)) > 4 + 6\epsilon\). However,

\[
\phi(\{b_m, x_o, \bar{x}_o, x_p, \bar{x}_p\}, \{x_o\}) = 1 + \epsilon (3/2 + w(b_m)/(2W)) + 1 + 2 + 4\epsilon < 4 + 6\epsilon,
\]

and hence \((O \setminus \{b_m\}) \cup \{x_o\}\) is in the neighbourhood of \(O\) and has strictly smaller cost.
Case 1.2: \( x_p \in O \lor \bar{x}_p \in O \) and \( x_o, \bar{x}_o \notin O \). In this case, removing \( b_m \) from \( O \) does not affect the cost of \( x_p \) and \( \bar{x}_p \). We obtain \( \phi(\{b_m, x_o, \bar{x}_o\}, O) \geq 2 + 2\epsilon(3/2 + w_{\text{min}}^B/(2W)) > 2 + 3\epsilon \). Observe, that
\[
\phi(\{b_m, x_o, \bar{x}_o\}, (O \setminus \{b_m\}) \cup \{x_o\}) \leq 1 + \epsilon(3/2 + w(b_m)/(2W)) + 1 < 2 + 2\epsilon.
\]

Case 1.3: \( x_p \in O \lor \bar{x}_p \in O \) and \( x_o, \bar{x}_o \in O \). Here we have that removing \( b_m \) from \( O \) does not affect the cost of its adjacent points at all. However, similar to before we have \( \phi(\{b_m, x_o, \bar{x}_o\}, O) \geq 2 + 2\epsilon(3/2 + w_{\text{min}}^B/(2W)) > 2 + 3\epsilon \). Again, we obtain
\[
\phi(\{b_m, x_o, \bar{x}_o\}, (C \setminus \{b_m\}) \cup \{x_o\}) \leq 1 + \epsilon(3/2 + c \cdot w(b_m)/(2W)) + 1 < 2 + 2\epsilon.
\]

Case 2: There is no \( m \in [M] \), such that \( b_m \in O \). Consequently, there is an \( o \in [N], o \neq n \): \( x_o, \bar{x}_o \in O \). W.l.o.g. assume that \( |B(x_o)| < M \) (otherwise just exchange \( x_o \) for \( \bar{x}_o \) in the following argument). Observe that
\[
\phi(B(x_o) \cup \{x_o, x_n, \bar{x}_n\}, O) = 2 + 4\epsilon + \sum_{b_m \in B(x_o)} 1 + \epsilon(3/2 + w(b_m)/(2W)).
\]
The only points affected by removing \( x_o \) from \( O \) are \( x_o \) and the points corresponding to clauses in \( B(x_o) \). Hence,
\[
\phi(C \setminus (B(x_o) \cup \{x_o, x_n, \bar{x}_n\}), O) = \phi(C \setminus (B(x_o) \cup \{x_o, x_n, \bar{x}_n\}), O \setminus \{x_o\}) \\
\geq \phi(C \setminus (B(x_o) \cup \{x_o, x_n, \bar{x}_n\}), (O \setminus \{x_o\}) \cup \{x_n\}).
\]
However, recall that the points in \( B(x_o) \) are at distance \( 1 + \epsilon(3/2 + c \cdot w(b_m)/(2W)) \) from \( \bar{x}_o \in O \). We obtain
\[
\phi(B(x_o) \cup \{x_o, x_n, \bar{x}_n\}, (O \setminus \{x_o\}) \cup \{x_n\}) \\
\leq \phi(B(x_o) \cup \{x_o, \bar{x}_o\}, \{x_o, x_n\}) \\
= 2 + \sum_{b_m \in B(x_o)} 1 + \epsilon(3/2 + c \cdot w(b_m)/(2W)) \\
= 2 + \sum_{b_m \in B(x_o)} 1 + \epsilon(3/2 + w(b_m)/(2W)) + (c - 1)\epsilon/(2W) \sum_{b_m \in B(x_o)} w(b_m)_{\text{max}} \\
< 2 + \epsilon + \sum_{b_m \in B(x_o)} 1 + \epsilon(3/2 + w(b_m)/(2W)) \\
< \phi(B(x_o) \cup \{x_o, x_n, \bar{x}_n\}, O).
\]

4.3 Embedding \( C \) into \( \ell^2_2 \)

So far, we regarded \( C \) as an abstract point set, only given by fixed pairwise interpoint distances. Using the results presented in \[\text{Bra17}\] one can easily see that point set \( C \) presented here can be embedded isometrically into squared Euclidean space, as well. We complement this by showing a lower bound on the number of dimensions required to embed \( C \) which asymptotically matches the number of dimensions of a possible embedding.

**Theorem 12.** If \( X \subset \mathbb{R}^D \) is a set of pairwise equidistant points, then \( \|X\| \leq D + 1 \).

**Proof.** By prove this theorem by induction on the number of dimensions. The claim obviously holds for \( D = 1 \) (there are at most 2 pairwise equidistant points on a line). Assume that the claim holds for any fixed \( D \in \mathbb{N} \). Let \( c \in \mathbb{R}_{>0} \) be a constant, \( X \subset \mathbb{R}^{D+1} \) be a set of \( D + 1 \) equidistant points and let \( H \) be the \( D \)-dimensional hyperplane spanned by the points in \( X \). We want to find another point \( x_{D+2} \in \mathbb{R}^{D+1} \), such that \( \forall x \in X : \|x - x_{D+2}\|^2 = c \). By induction
Corollary 13. There is no isometric embedding of \( C \) into squared Euclidean space using less than \( \max\{N, M\} - 1 \) dimensions.

**Proof.** By definition of our distances, the sets \( B \) and \( \{x_n\}_{n \in [N]} \) form sets of pairwise equidistant points. Thus, we obtain the claim by applying Theorem 12.

5 Open Problems

By proving tight PLS-completeness for unweighted MUFL and DKM we solved one of the open problems posed by Bra17. We furthermore answered the question if it was possible to asymptotically reduce the dimensionality of the DKM reduction by presenting a lower bound on the number of dimensions required to embed the point set constructed by our reduction. Since we asymptotically match the number of dimensions used when actually embedding the point set, this shows that it is not possible the obtain a better result using the reduction presented here. Hence, to prove PLS-completeness of DKM in low dimensional space, we need to come up with a structurally new idea.
References

[Bra17] S. Brauer. Complexity of Single-Swap Heuristics for Metric Facility Location and Related Problems. In *Tenth International Conference on Algorithms and Complexity*, CIAC ’17, 2017. To appear.

[SY91] A. A. Schäffer and M. Yannakakis. Simple Local Search Problems that are Hard to Solve. *SIAM Journal on Computing*, 20(1), 1991.