THE CUP SUBALGEBRA HAS THE ABSORBING AMENABILITY PROPERTY

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ABSTRACT. Consider an inclusion of diffuse von Neumann algebras $A \subset M$. We say that $A \subset M$ has the absorbing amenability property if for any diffuse subalgebra $B \subset A$ and any amenable intermediate algebra $B \subset D \subset M$ we have that $D$ is contained in $A$. We prove that the cup subalgebra associated to any subfactor planar algebra has the absorbing amenability property.

INTRODUCTION AND MAIN RESULTS

Amenability is a fundamental concept in various area of mathematics. Connes proved the striking result that a von Neumann algebra is amenable if and only if it is hyperfinite [Con76]. In this article, we study maximal amenable subalgebras. Fuglede and Kadison showed that any II$_1$ factor contains a maximal amenable subfactor [FK51]. Popa exhibited the first example of an abelian maximal amenable subalgebra of a II$_1$ factor, thus giving a counter-example to a question of Kadison [Pop83]. He defined the notion of asymptotic orthogonality property (AOP) and showed that a singular maximal abelian subalgebra (masa) with the AOP is maximal amenable.
Many other examples have been given using the same strategy [Ge96, She06, CFRW10, Bro14, Hou14a, BCa]. A completely new strategy in proving maximal amenability has been given in [BCb]. Peterson conjectured that any maximal amenable subalgebra of a free group factor is the unique maximal amenable extension of any of its diffuse subalgebra. Inspired by this question and the work of Houdayer on maximal Gamma extensions [Hou14b, Hou15], we consider the notion of absorbing amenability property (AAP). An inclusion of von Neumann algebras $A \subset M$ has the AAP if for any diffuse subalgebra $B \subset A$ and any amenable intermediate algebra $B \subset D \subset M$ we have that $D$ is contained in $A$. In particular, if $A$ is amenable, then it is maximal amenable. Houdayer proved that the generator masa has the AAP [Hou15]. The second author showed that the radial masa has the AAP [Wen].

In this article, we present a new class of examples that have the AAP. Those examples are constructed with Jones planar algebras [Jon]. If $\mathcal{P}$ is a subfactor planar algebra, then we can associate to it a II$_1$ factor $M$ [GJS10]. This II$_1$ factor is isomorphic to an interpolated free group factor $L(F_t)$ where $t$ is a linear combination of the index and the global index of $\mathcal{P}$ [Dyk94, Rad94, GJS11, Har13]. This factor admits a generic abelian subalgebra $A \subset M$ that we call the cup subalgebra. The first author previously proved that the cup subalgebra is maximal amenable [Bro14]. We prove the following theorem:

\textbf{Theorem A.} The cup subalgebra associated to any subfactor planar algebra has the absorbing amenability property.

This provides many examples of subalgebras of interpolated free group factors with the AAP. Note, it is still unknown if there exists a subfactor planar algebra such that its associated cup subalgebra is isomorphic to the generator or the radial masa.

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1. Preliminaries

1.1. Planar algebras. A planar algebra is a collection of complex $*$-algebras $\mathcal{P} = (\mathcal{P}_n^\pm : n \geq 0)$ on which the set of shaded planar tangles acts. See [Jon, Jon12] for more details. We follow similar conventions that was used in [CJS14] for drawing a shaded planar tangle. We decorate strings with natural numbers to indicate that they represent a given number of parallel strings. The distinguished interval of a box is decorated by a dollar sign if it is not at the top left corner. We do not draw the outside box and will omit unnecessary decorations. The left and right traces of a planar algebra are the maps $\tau_1: \mathcal{P}_n^+ \to \mathcal{P}_0^+$ and $\tau_r: \mathcal{P}_n^- \to \mathcal{P}_0^-$ defined for any $n \geq 0$ such that

$$\tau_l(x) = \left( \begin{array}{c} x \end{array} \right) \text{ and } \tau_r(x) = \left( \begin{array}{c} y \end{array} \right) \text{ for any } x \in \mathcal{P}_n^\pm.$$ 

Suppose that $\mathcal{P}_0^\pm = \mathbb{C}$. The planar algebra is called spherical if the two traces agree on each element of $\mathcal{P}$. We say that $\mathcal{P}$ is non-degenerate if the sesquilinear forms $(x, y) \mapsto \tau_l(xy^*)$ and $(x, y) \mapsto \tau_r(xy^*)$ are positive definite. A subfactor planar algebra is a planar algebra such that each space $\mathcal{P}_n^\pm$ is finite dimensional, $\mathcal{P}_0^\pm = \mathbb{C}$, $\mathcal{P}$ is spherical and non-degenerate. The modulus of a subfactor planar algebra is the value of a closed loop.

1.2. Construction of a II$_1$ factor. We recall a construction due to Jones et al. [JJSW10]. Consider the direct sum $Gr\mathcal{P} = \bigoplus_{n \geq 0} \mathcal{P}_n^+$ that we equipped with the following Bacher product and involution:

$$xy = \sum_{a=0}^{\min(2n,2m)} \begin{array}{c} x \vline a \vline y \end{array} \text{ and } x^\dagger = \left( \begin{array}{c} x^* \end{array} \right), \text{where } x \in \mathcal{P}_n^+ \text{ and } y \in \mathcal{P}_m^+.$$ 

Consider the linear form $\tau: Gr\mathcal{P} \to \mathbb{C}$ that sends $x \in \mathcal{P}_0^\pm$ to itself and 0 to any element in $\mathcal{P}_n^\pm$ if $n \neq 0$. The vector space $Gr\mathcal{P}$ endowed with those operation is an associative $*$-algebra with a faithful tracial state. Let $H$ be the completion of $Gr\mathcal{P}$ for the inner product $(x, y) \mapsto \tau(xy^*)$. The left multiplication of $Gr\mathcal{P}$ on $H$ is bounded and defines a $*$-representation [GJS10, JJSW10]. Let $M$ be the von Neumann algebra generated by $Gr\mathcal{P}$ inside $B(H)$. It is an interpolated free group factor [GJS10, Har13]. We define another multiplication on $Gr\mathcal{P}$ by requiring that if $x \in \mathcal{P}_n$ and $y \in \mathcal{P}_m$, then

$$x \bullet y = \begin{array}{c} x \vline y \end{array} \in \mathcal{P}_{n+m}^+.$$ 

Denote by $x^n$ the $n$-th power of $x$ for this multiplication. Remark, $\|a \bullet b\|_2 = \|a\|_2 \|b\|_2$, for all $a \in \mathcal{P}_n$ and $b \in \mathcal{P}_m$. Therefore, this multiplication is a continuous bilinear form for the $L^2$-norm $\| \cdot \|_2$ of $M$. We extend this operation on $L^2(M) \times L^2(M)$ and still denote it by $\bullet$.

1.3. The cup subalgebra. Let $\cup$ be the unity of the $*$-algebra $\mathcal{P}_1^+$, viewed as an element of $M$ [GJS10]. Let $A \subset M$ be the von Neumann subalgebra generated by $\cup$. We call it the cup subalgebra.

1.4. Strong asymptotical orthogonal property. Popa introduced the notion of asymptotic orthogonality property (AOP) in [Pop83]. We consider a strengthening of this notion which was used by Houdayer and the second author [Hou14b, Wen].

Definition 1.1. Let $A \subset M$ be a diffuse subalgebra of a tracial von Neumann algebra. This inclusion has the strong asymptotic orthogonality property (SAOP) if for any free ultrafilter $\omega$ and any diffuse subalgebra $B \subset A$ we have

$$xy \perp yx \text{ for any } x \in (M^\omega \cap B') \ominus A^\omega \text{ and } y \in M \ominus A.$$
Note, a diffuse subalgebra \( A \subset M \) has the SAOP if and only if it has the AOP relative to all of its diffuse subalgebras in the sense of Houdayer [Hou14b, Definition 5.1].

The following theorem is an extension of a theorem of Popa [Pop83].

**Theorem 1.2.** [Hou14b, Theorem 8.1] If \( A \subset M \) is a diffuse subalgebra with the SAOP such that \( L^2(M) \otimes L^2(A) \) is a mixing \( A \)-bimodule (e.g. is a direct sum of the coarse bimodule \( L^2(A) \otimes L^2(A) \)), then it has the AAP.

See also [Wen, Proposition 2.1].

2. Proof of the main theorem

**Proposition 2.1.** Let \((A, \tau)\) be a tracial von Neumann algebra and \(B \subset A\) a diffuse subalgebra. Denote by \(L^2(A)\) the Gelfand-Naimark-Segal completion of \(A\) for the trace \(\tau\). Consider a sequence \(\xi = (\xi_n : n \geq 0)\) of unit vectors of the coarse bimodule \(L^2(A) \otimes L^2(A)\). Suppose that for any \(b \in B\) we have \(\lim_{n \to \infty} \|b \cdot \xi_n - \xi_n \cdot b\| = 0\). Then, if \(p \in B(L^2(A))\) is a finite rank projection, \(\lim_{n \to \infty} \|(p \otimes 1)\xi_n\| = \lim_{n \to \infty} \|(1 \otimes p)\xi_n\| = 0\).

**Proof.** Let \(A, B, \xi, \) and \(p\) as above. It is sufficient to prove the proposition when \(p\) is a rank one projection. Let \(\eta \in L^2(A)\) be a unit vector such that \(p = p\eta\) is the rank one projection onto \(\mathbb{C}\eta\). Consider \(0 < \varepsilon < 1\) and a natural number \(I\) such that \(16/(I + 1) < \varepsilon\). Since \(B\) is diffuse, there exists a sequence of unitaries \((u_n)_n \in B\) such that \(\lim_{n \to \infty} (u_n \cdot \xi_1, \xi_2) = 0\) for any \(\xi_1, \xi_2 \in L^2(A)\). Consider the quantity \(\delta = \max(|(u_n \cdot \eta, u_m \cdot \eta)| : n \neq m, n, m \leq I)\). By [Hou14a, Proposition 2.3], we have that

\[
\sum_{i=0}^{I} \|(p_{u_i \cdot \eta} \otimes 1)\xi_n\|^2 \leq g(\delta)\|\xi_n\|^2 \text{ for any } n \geq 0,
\]

where \(g\) is a positive function satisfying \(\lim_{\delta \to 0} g(\delta) = 1\). Hence, there exists a subsequence \((v_n)_n\) such that

\[
\sum_{i=0}^{I} \|(p_{v_i \cdot \eta} \otimes 1)\xi_n\|^2 \leq 2\|\xi_n\|^2 = 2 \text{ for any } n \geq 0.
\]

Let \(\lambda : B \to B(L^2(A) \otimes L^2(A))\) be the left action of \(B\) on the coarse bimodule \(L^2(A) \otimes L^2(A)\). Observe, \(p_{v_i \cdot \eta} \otimes 1 = \lambda(v_i) \circ (p_{v_i \otimes 1} \otimes 1) \circ \lambda(v_i)^*\) and \(v_i\) is a unitary, for any \(i \geq 0\). Therefore, \(\|\lambda(v_i)\xi_n\| = \|(p_{v_i \otimes 1}v_i^* \cdot \xi_n)\|\) for any \(n, i \geq 0\). By assumption, there exists \(N > 0\) such that for any \(n \geq N\) and \(i \leq I\) we have \(|v_i^* \cdot \xi_n - \xi_n \cdot v_i^*| < \varepsilon/4\). Therefore,

\[
\|(p_{v_i \cdot \eta} \otimes 1)\xi_n\| = \|(p_{v_i \cdot \eta} \otimes 1)(\xi_n \cdot v_i^*)\| \
\leq \|(p_{v_i \cdot \eta} \otimes 1)(v_i^* \cdot \xi_n - \xi_n \cdot v_i^*)\| + \|(p_{v_i \cdot \eta} \otimes 1)(v_i^* \cdot \xi_n)\| \leq \varepsilon/4 + \|(p_{v_i \cdot \eta} \otimes 1)\xi_n\| \text{ for any } n \geq N, i \leq I.
\]

We obtain

\[
\sum_{i=0}^{I} \|(p_{v_i \cdot \eta} \otimes 1)\xi_n\|^2 \leq \sum_{i=0}^{I} (\varepsilon^2/16 + \varepsilon/2) \|(p_{v_i \cdot \eta} \otimes 1)\xi_n\| + \|(p_{v_i \cdot \eta} \otimes 1)\xi_n\|^2 \leq (I + 1) (\varepsilon^2/16 + \varepsilon/2) + 2 \text{ for any } n \geq N.
\]

Therefore, \(\|(p_{v_i \cdot \eta} \otimes 1)\xi_n\|^2 \leq \varepsilon/16 + \varepsilon/2 + 2\varepsilon/16 \leq \varepsilon\) for any \(n \geq N\). The same proof shows that there exists \(M > 0\) such that for any \(n \geq M\) we have \(\|\lambda(p\eta)\xi_n\|^2 \leq \varepsilon\). This proves the proposition. \(\square\)

Fix a subfactor planar algebra \(\mathcal{P}\) with modulus \(\delta > 1\) and denote by \(A \subset M\) its associated cup subalgebra. Consider the subspace \(V_n \subset \mathcal{P}_n^+ : n \geq 0\) of elements that vanishes when they are capped off on the top left corner and vanished when they are capped off on the top right corner.
Let $V \subset L^2(M)$ be their orthogonal direct sum. By [JSW10, Theorem 4.9], the following map is an isomorphism of $A$-bimodules:

$$\phi : L^2(A) \oplus (L^2(A) \otimes V \otimes L^2(A)) \rightarrow L^2(M), a + b \otimes v \otimes c \mapsto a + b \cdot v \cdot c.$$ 

This implies that the $A$-bimodule $L^2(M) \otimes L^2(A)$ is isomorphic to an infinite direct sum of the coarse bimodule. We identify $L^2(M)$ with $\phi^{-1}(L^2(M))$.

Consider the finite dimensional subspace $L_m = \text{Span}(\cup^k : k \leq m) \subset A$ for $m \geq 0$, where $\cup^0 = 1 \in P_0^+$. Denote by $L_m^\perp$ the orthogonal complement of $L_m$ inside $L^2(A)$ for any $m \geq 0$.

**Lemma 2.2.** Let $m \geq 0$ and $x \in M \cap L_m^\perp \otimes V \otimes L_m^\perp$, $y \in M \cap L_m \otimes V \otimes L_m$. Then $xy \in L_m^\perp \otimes V \otimes L_m$ and $yx \in L_m \otimes V \otimes L_m^\perp$. In particular, $xy \perp yx$.

**Proof.** Consider $x = \cup^k \cdot w \cup^l$ and $y = \cup^s \cdot w \cup^t$, where $s, t < m+1 \leq k, l$ and $v, w \in V \cap \text{Gr} P$. We have that

$$xy = \sum_{i=0}^{s+1} \delta^{[i/2]} \cup^k \cdot v \cup^s \cdot l+i-s \cdot w \cup^t,$$

where $[i/2] = i/2$ if $i$ is even and $i/2 - 1/2$ if $i$ is odd. Observe, $L_m^\perp$ is equal to the closure of $\text{Span}(\cup^k : k \geq m+1)$. Therefore, $xy \in L_m^\perp \otimes V \otimes L_m$ and similarly $yx \in L_m \otimes V \otimes L_m^\perp$. The space $M \cap L_m^\perp \otimes V \otimes L_m$ (resp. $M \cap L_m \otimes V \otimes L_m^\perp$) is the weak closure of $\text{Span}(\cup^k \cdot v \cup^l : k, l \geq m+1, v \in V \cap \text{Gr} P)$ (resp. $\text{Span}(\cup^s \cdot w \cup^t : s, t \leq m, w \in V \cap \text{Gr} P)$). This concludes the proof by a density argument.

We are ready to prove the main theorem of the article.

**Proof of Theorem [A]**. Let $P$ be a subfactor planar algebra, $A \subset M$ its associated cup subalgebra, and $B \subset A$ a diffuse subalgebra. Consider $x \in M^\omega \cap A^\omega$ in the relative commutant of $B$ and $y \in M \cap A$, where $\omega$ is a free ultrafilter on $N$. Let us show that $xy \perp yx$. Observe, $GrP$ is a weakly dense subalgebra of $M$. Therefore, we can assume that $y \in GrP$ by Kaplansky density theorem. This implies that there exists $m \geq 0$ such that $y \in GrP \cap L_m \otimes V \otimes L_m$. Let $(x_n)_n$ be a representative of $x$ in the ultrapower $M^\omega$. We can assume that for any $n \geq 0$ we have $x_n \in L^2(M) \otimes L^2(A)$. Let $p \in B(L^2(A))$ be the orthogonal projection onto $L_m$. It is a finite rank projection. Therefore, by Proposition [2.1], $(p \otimes 1)x = (1 \otimes p)x = 0$. Hence, we can assume that $x_n \in L_m^\perp \otimes V \otimes L_m^\perp$ for any $n \geq 0$. Lemma [2.2] implies that $x_n y \perp y x_n$ for any $n \geq 0$. This implies that $xy \perp yx$.

Theorem [1.2] implies that $A \subset M$ has the AAP.

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