ON THE EHRHART POLYNOMIAL OF SCHUBERT MATROIDS

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Abstract. In this paper, we give a formula for the number of lattice points in the dilations of Schubert matroid polytopes. As applications, we obtain the Ehrhart polynomials of uniform and minimal matroids as special cases, and give a recursive formula for the Ehrhart polynomials of \((a, b)\)-Catalan matroids. Ferroni showed that uniform and minimal matroids are Ehrhart positive. We show that all sparse paving Schubert matroids are Ehrhart positive and their Ehrhart polynomials are coefficient-wisely bounded by those of minimal and uniform matroids. This confirms a conjecture of Ferroni for the case of sparse paving Schubert matroids. Furthermore, we introduce notched rectangle matroids, which include minimal matroids, sparse paving Schubert matroids and panhandle matroids. We show that three subfamilies of notched rectangle matroids are Ehrhart positive, and conjecture that all notched rectangle matroids are Ehrhart positive.

1 Introduction

Let \(S\) be a subset of \([n] := \{1, 2, \ldots, n\}\). The Schubert matroid \(SM_n(S)\) is the matroid with ground set \([n]\) and bases 
\[
\{T \subseteq [n] : T \leq S\},
\]
where \(T \leq S\) means that: \(|T| = |S|\) and the \(i\)-th smallest element of \(T\) does not exceed that of \(S\) for \(1 \leq i \leq |T|\). Schubert matroids were first studied by Crapo [8] under the name of nested matroids and rediscovered in various contexts. They have been called shifted matroids [1], freedom matroids [9], generalized Catalan matroids [6] and PI-matroids [3] in the literature. In particular, uniform matroids, minimal matroids [16], \((a, b)\)-Catalan matroids [5] and panhandle matroids [20] are subclasses of Schubert matroids. It is also worth mentioning that Schubert matroids are subfamilies of lattice path matroids [5, 6, 22], or more generally transversal matroids [1] and positroids [28].

It follows from Derksen and Fink [12, Theorem 5.4] that essentially Schubert matroids form a basis of the indicator function space of all matroids. In a more elementary language, the Ehrhart polynomial of an arbitrary matroid polytope is an integer linear combination of Ehrhart polynomials of Schubert matroid polytopes. Moreover, Schubert matroid polytopes are the Minkowski summands of the Newton polytopes of key polynomials and Schubert polynomials, see Fink, Mészáros and St. Dizier [19].

Suppose that \(M = ([n], B)\) is a matroid with ground set \([n]\) and base set \(B\). The matroid polytope \(\mathcal{P}(M)\) associated to \(M\) is the convex hull
\[
\mathcal{P}(M) = \text{conv}\{e_B : B \in B\},
\]
where \(e_B = e_{b_1} + \cdots + e_{b_k}\) for \(B = \{b_1, \ldots, b_k\} \subseteq [n]\) and \(\{e_i : 1 \leq i \leq n\}\) is the standard basis of \(\mathbb{R}^n\). Given a polytope \(\mathcal{P}\) and a positive integer \(t\), the \(t\)-dilation \(t\mathcal{P}\) of \(\mathcal{P}\) is defined as \(t\mathcal{P} = \{ta : a \in \mathcal{P}\}\). Let \(i(\mathcal{P}, t) = |t\mathcal{P} \cap \mathbb{Z}^n|\) denote the number of lattice points in \(t\mathcal{P}\). It
is well known that for integral polytopes, $i(P, t)$ is a polynomial in $t$, called the Ehrhart polynomial of $P$. For simplicity, write $i(M, t)$ for $i(P(M), t)$.

It was conjectured by De Loera, Haws and Köppe [25] that all matroids are Ehrhart positive, i.e., the Ehrhart polynomial of any matroid polytope has positive coefficients. Moreover, since matroid polytopes are specific families of generalized permutahedra, Castillo and Liu [24] further conjectured that generalized permutahedra are also Ehrhart positive. For the study of Ehrhart positivity of various polytopes, see the survey of Liu [24]. Recently, Ferroni [15,16] showed that hypersimplices and minimal matroids are Ehrhart positive. For the study of Ehrhart positivity of various polytopes, see the survey of Castillo and Liu [24] further conjectured that generalized permutohedra are also Ehrhart positive. In [17], Ferroni showed that all sparse paving matroids of rank 2 are Ehrhart positive, but provided counterexamples to both aforementioned conjectures of Ehrhart positivity. In [18], Ferroni, Jochemko and Schröter further showed that all matroids of rank 2 are Ehrhart positive and are coefficient-wisely bounded by minimal and uniform matroids.

In this paper, we consider the Ehrhart polynomials of Schubert matroid polytopes. We provide a formula for the number of lattice points in the $t$-dilation $tP(\mathrm{SM}_n(S))$ of $P(\mathrm{SM}_n(S))$. To this end, we first show that $tP(\mathrm{SM}_n(S))$ is in fact the Newton polytope of the key polynomial $\kappa_{\alpha}(x)$, where $\alpha$ is the indicator vector of $S$. It follows from Fink, Mészáros and St. Dizier [19] that each lattice point in the Newton polytope of $\kappa_{\alpha}(x)$ is an exponent vector of $\kappa_{\alpha}(x)$. Then we use Kohnert algorithm to generate all the different monomials of $\kappa_{\alpha}(x)$ and thus obtain a formula for the number of lattice points in $tP(\mathrm{SM}_n(S))$.

As applications, we obtain the Ehrhart polynomials of hypersimplices [21] and minimal matroids [16] as simple special cases, and give a recursive formula for the Ehrhart polynomials of $(a, b)$-Catalan matroids. We also show that all sparse paving Schubert matroids are Ehrhart positive by proving that they are coefficient-wisely bounded by the minimal and uniform matroids. Ferroni [16] conjectured that all matroids are coefficient-wisely bounded by the minimal and uniform matroids, which was disproved by Ferroni [17] later on. We confirm this conjecture for the case of sparse paving Schubert matroids. Moreover, we introduce notched rectangle matroids, and show that three subfamilies of notched rectangle matroids are Ehrhart positive. We conjecture that all notched rectangle matroids are Ehrhart positive.

To describe our results, we need some notations. Assume that $S \subseteq [n]$ is a finite set of positive integers. Since we only consider Schubert matroids $\mathrm{SM}_n(S)$, it suffices to let $n$ be the maximal element of $S$. The indicator vector $\mathbb{I}(S)$ of $S$ is the 0-1 vector $\mathbb{I}(S) = (i_1, \ldots, i_n)$, where $i_j = 1$ if $j \in S$, and 0 otherwise. Clearly, $i_n = 1$. For simplicity, write $\mathbb{I}(S) = (0^{r_1}, 1^{r_2}, \ldots, 0^{r_{2m-1}}, 1^{r_{2m}})$, where $0^{r_1}$ represents $r_1$ copies of 0’s, $1^{r_2}$ represents $r_2$ copies of 1’s, etc. Thus $S$ can be written as an integer sequence $r(S) = (r_1, r_2, \ldots, r_{2m})$ of length $2m$, where $r_1 \geq 0$ and $r_i > 0$ for $i \geq 2$. It is easy to see that given such an integer sequence $r$, there is a unique set $S$ whose indicator vector $\mathbb{I}(S)$ can be written in this way. We will use $S$, $r$ or $r(S)$ interchangeably with no further clarification. For example, let $S = \{2, 6, 7, 10\} \subseteq [10]$, then $\mathbb{I}(S) = (0, 1, 0^3, 1^2, 0^2, 1)$ and $r(S) = (1, 1, 3, 2, 2, 1)$.

Given $r = (r_1, r_2, \ldots, r_{2m})$, define two integer sequences $u = (u_1, \ldots, u_m)$ and $v = \ldots, v_m)$.
(v_1, \ldots, v_m) as follows. For 1 \leq i \leq m, let
\[
u_i = \min \left\{ r_{2i-1}, \sum_{j=i+1}^{m} r_{2j} \right\} \quad \text{and} \quad u_i = \min \left\{ r_{2i}, \sum_{j=1}^{i-1} r_{2j-1} \right\},
\]
where empty sums are interpreted as 0. Assume that a, b, t \geq 0 and c \in \mathbb{Z} are all integers, define
\[
F(a, b, c, t) = \sum_{j=0}^{a+b} (-1)^j \binom{a+b}{j} \left( \binom{t+1}{2}(b-j) + a + c - 1 \right) \frac{1}{a+b-1}.
\]
By convention, \binom{0}{0} = 1 and \binom{n}{k} = 0 if k < 0 or n < k. Notice that if j > \frac{bt+c}{t+1} in (1.3), then \((t+1)(b-j) + a + c - 1 < a + b - 1\), and thus \(\binom{(t+1)(b-j)+a+c-1}{a+b-1} = 0\).

**Theorem 1.1.** Let \(S \subseteq [n]\) with \(r(S) = (r_1, \ldots, r_{2m})\). We have
\[
i(SM_a(S), t) = \sum_{(c_1, \ldots, c_m)} \prod_{j=1}^{m} F(r_{2j-1}, r_{2j}, c_j, t),
\]
where \(c_1 + \cdots + c_m = 0\), and for \(1 \leq j \leq m\),
\[-tv_j \leq c_j \leq tu_j \quad \text{and} \quad c_1 + \cdots + c_j \geq 0.
\]
Since the variable t appears as the upper limit of the sum, (1.4) is not a legitimate polynomial. Nevertheless, there are still many applications. For example, let \(S = \{n-k+1, \ldots, n\}\), where \(n > k \geq 1\). Then we obtain the uniform matroid \(U_{k,n}\). In this case, \(r = (n-k, k), m = 1, c_1 = 0\), by Theorem 1.1
\[
i(U_{k,n}, t) = F(n-k, k, 0, t).
\]
The Ehrhart polynomial \(i(U_{k,n}, t)\) was first obtained by Katzman [21] and then shown to have positive coefficients by Ferroni [15].

**Corollary 1.2** (Katzman [21]). We have
\[
i(U_{k,n}, t) = F(n-k, k, 0, t) = \sum_{i=0}^{k-1} (-1)^i \binom{n}{i} \left( (k-i)t - i + n - 1 \right) \frac{1}{n-1}.
\]
Let \(S = \{2, 3, \ldots, k, n\}\), where \(n > k \geq 2\), we are lead to the minimal matroid \(T_{k,n}\). Minimal matroids were first studied independently by Dinolt [13] and Murty [27]. Ferroni [10] showed that \(T_{k,n}\) is the graphic matroid of a \((k+1)\)-cycle with one edge replaced by \(n-k\) parallel copies. In this case, \(r(S) = (1, k-1, n-k-1, 1), u = (1, 0), v = (0, 1)\) and \((c_1, c_2) = (j, -j)\) for \(0 \leq j \leq t\), thus by Theorem 1.1
\[
i(T_{k,n}, t) = \sum_{j=0}^{t} F(1, k-1, j, t)F(n-k-1, 1, -j, t).
\]
Since both \(F(1, k-1, j, t)\) and \(F(n-k-1, 1, -j, t)\) are binomials, we can re-obtain the following closed formula of \(i(T_{k,n}, t)\).
Corollary 1.3 (Ferroni [16]). We have
\[ i(T_{k,n}, t) = \frac{1}{\binom{n-1}{k-1}} \left( \frac{t + n - k}{n - k} \right)^{k-1} \sum_{j=0}^{k-1} \binom{n - k + j - 1}{j} \left( \frac{t + j}{j} \right). \] (1.5)

It is apparent that both \( i(T_{k,n}, t) \) and \( i(T_{k,n}, t - 1) \) have positive coefficients. We proceed to consider some further applications of Theorem 1.1.

1.1 Notched rectangle matroids

Given a lattice path \( P \) from \((0, 0)\) to \((m, n)\) consisting of East steps \( E = (1, 0) \) and North steps \( N = (0, 1) \), label the steps of \( P \) by \( 1, 2, \ldots, m + n \). Let \( B(P) \) denote the set of labels of North steps of \( P \). For example, in Figure 1.1, the dashed lattice path \( P \) has \( B(P) = \{2, 4, 5, 6, 10\} \). Let \( U, L \) be two lattice paths from \((0, 0)\) to \((m, n)\), such that \( L \) never goes above \( U \). The lattice path matroid \( M[U, L] \) is the matroid on the ground set \([m + n] \) with base consisting of \( B(P) \), where \( P \) is a lattice path from \((0, 0)\) to \((m, n)\) never going below \( L \) and never going above \( U \). For the study of lattice path matroids, see [3, 6, 22].

It is easy to see that a Schubert matroid \( SM_n(S) \) is a lattice path matroid \( M(U, L) \), where
\[ B(U) = \{1, 2, \ldots, |S|\} \text{ and } B(L) = S. \]

Bonin and de Mier [6] Definition 8.1] introduced notch matroids, which are lattice path matroids of the form \( M[U, E^m N^n] \) or \( M[U, E^{m-1}NEN^{n-1}] \). Recently, Hanley et al. [20] studied another subfamily of lattice path matroids they called panhandle matroids, and conjectured that panhandle matroids are Ehrhart positive. Notice that notch matroids are not Schubert matroids in general, but panhandle matroids are in fact Schubert matroids \( SM_n(S) \) with \( r(S) = (a, b, c, 1) \).

We introduce a more general family of matroids, which includes minimal matroids, sparse paving Schubert matroids and panhandle matroids.

Definition 1.4. A notched rectangle matroid is a Schubert matroid \( SM_n(S) \) with \( r(S) = (a, b, c, d) \), where \( a, b, c, d \) are positive integers.

Figure 1.1 is an illustration of the notched rectangle matroid with parameters \((3, 2, 3, 3)\), which is exactly the Schubert matroid \( SM_n(S) \) with \( r(S) = (3, 2, 3, 3) \), or equivalently, \( S = \{4, 5, 9, 10, 11\} \).

We express the Ehrhart polynomials of three subfamilies of notched rectangle matroids as positive combinations of \( i(U_{a+b}, t) \), which imply Ehrhart positivity of these matroids. For convenience, let \( i(r(S), t) \) denote \( i(SM_n(S), t) \).

Theorem 1.5. Let \( a, b \) be positive integers. Then
\[ i((a, b, a, b), t) = \frac{1}{2} i(U_{2b, 2a+2b}, t) + \frac{1}{2} i(U_{b,a+b}, t)^2, \] (1.6)
\[ i((a, a, b, b), t) = \frac{1}{2} i(U_{a+b, 2a+2b}, t) + \frac{1}{2} i(U_{a,2a}, t)i(U_{b,2b}, t), \] (1.7)
Figure 1.1: A notched rectangle matroid with parameters $(3, 2, 3, 3)$. 

\[ i((1, 1, a, a + 1), t) = i((a + 1, a, 1, 1), t) = \frac{1}{2}(t + 2)i(U_{a+1, 2a+2}, t). \] (1.8)

**Conjecture 1.6.** All notched rectangle matroids are Ehrhart positive.

### 1.2 $(a, b)$-Catalan matroids

Let \( r = \overbrace{(a, b, a, b, \ldots, a, b)}^{2n} \), where \( a, b, n \geq 1 \), we obtain the \((a, b)\)-Catalan matroid \( C_{n}^{a, b} \) introduced by Bonin, de Mier and Noy [5, Definition 3.7]. In particular, when \( a = b = 1 \), we obtain the Schubert matroid \( C_{n}^{1, 1} \), which is equivalent to \( SM_{2n}(S) \) with \( S = \{2, 4, \ldots, 2n\} \).

In [1], Ardila studied the Catalan matroid \( C_{n} \), which is the Schubert matroid \( SM_{2n-1}(S) \) with \( S = \{1, 3, \ldots, 2n-1\} \) and an additional loop \( 2n \). It is easy to see that \( C_{n}^{1, 1} \) is isomorphic to \( C_{n+1} \).

A composition \( \sigma = (\sigma_1, \ldots, \sigma_s) \) of \( n \) is an ordered nonnegative integer sequence such that \( \sigma_1 + \cdots + \sigma_s = n \). Let \( \ell(\sigma) = s \) denote the number of parts of \( \sigma \). Given two compositions \( \sigma \) and \( \sigma' \), we say that \( \sigma \) and \( \sigma' \) are equivalent, denoted as \( \sigma \sim \sigma' \), if \( \sigma' \) can be obtained from \( \sigma \) by cyclic shifting, i.e., \( \sigma' = (\sigma_j, \ldots, \sigma_s, \sigma_1, \ldots, \sigma_{j-1}) \) for some \( 2 \leq j \leq s \). Let \( d(\sigma) \) denote the cardinality of the equivalence class of \( \sigma \). Denote \( \Gamma_n \) by a transversal of the equivalence classes consisting of compositions of \( n \) with at least two parts and minimal parts larger than 1. That is, if \( \sigma \in \Gamma_n \), then \( \min\{\sigma_1, \ldots, \sigma_s\} > 1, \ell(\sigma) > 1 \), and if \( \sigma, \sigma' \in \Gamma_n \), then \( \sigma' \) and \( \sigma \) are not equivalent.

**Theorem 1.7.** For \( a, b \geq 1 \) and \( n \geq 2 \), we have

\[
i(C_{n}^{a, b}, t) = \frac{1}{n}i(U_{nb, na+nb}, t) - \frac{1}{n}i(U_{b, a+b}, t)^n + i(U_{b, a+b}, t) \cdot i(C_{n-1}^{a, b}, t) + \sum_{\sigma \in \Gamma_n} (-1)^{\ell(\sigma)} \frac{d(\sigma)}{\ell(\sigma)} \cdot i(C_{\sigma}^{a, b}, t),
\] (1.9)

where \( i(C_{\sigma}^{a, b}, t) = \prod_{j=1}^{\ell(\sigma)} i(C_{\sigma_j}^{a, b}, t) \) and

\[
i(C_{\sigma_j}^{a, b}, t) = i(C_{\sigma_j}^{a, b}, t) - i(U_{b, a+b}, t) \cdot i(C_{\sigma_j-1}^{a, b}, t),
\] (1.10)
and \(i(C^{a,b}_1, t) = i(U_{b,a+b}, t)\).

For example, since \(\Gamma_2 = \Gamma_3 = \emptyset\) and \(\Gamma_4 = \{(2, 2)\}\), we have
\[
\begin{align*}
i(C^{a,b}_2, t) &= \frac{1}{2}i(U_{2b,2a+2b}, t) + \frac{1}{2}i(U_{b,a+b}, t)^2 \\
i(C^{a,b}_3, t) &= \frac{1}{3}i(U_{3b,3a+3b}, t) - \frac{1}{3}i(U_{b,a+b}, t)^3 + i(U_{b,a+b}, t) \cdot i(C^{a,b}_2, t) \\
i(C^{a,b}_4, t) &= \frac{1}{4}i(U_{4b,4a+4b}, t) - \frac{1}{4}i(U_{b,a+b}, t)^4 + i(U_{b,a+b}, t) \cdot i(C^{a,b}_3, t) + \frac{1}{2}i(C^{a,b}_2, t).
\end{align*}
\]

For \(n = 9\), let \(\Gamma_9 = \{(7, 2), (6, 3), (5, 4), (5, 2, 2), (4, 3, 2), (4, 2, 3), (3, 3, 3), (3, 2, 2, 2)\}\). Thus
\[
i(C^{a,b}_9, t) = \frac{1}{9}i(U_{9b,9a+9b}, t) - \frac{1}{9}i(U_{b,a+b}, t)^9 + i(U_{b,a+b}, t) \cdot i(C^{a,b}_8, t) \\
+ i(C^{a,b}_{(7,2)}, t) + i(C^{a,b}_{(6,3)}, t) + i(C^{a,b}_{(5,4)}, t) - i(C^{a,b}_{(5,2,2)}, t) \\
- i(C^{a,b}_{(4,3,2)}, t) - i(C^{a,b}_{(4,2,3)}, t) - \frac{1}{3}i(C^{a,b}_{(3,3,3)}, t) + i(C^{a,b}_{(3,2,2,2)}, t),
\]
where \(i(C^{a,b}_{(4,3,2)}, t) = i(C^{a,b}_{(4,2,3)}, t)\).

Computational experiments suggest the following two conjectures.

**Conjecture 1.8.** For integers \(a, b, n \geq 1\), \(i(U_{nb,na+nb}, t) - i(U_{b,a+b}, t)^n\) has positive coefficients.

**Conjecture 1.9.** For integers \(a, b, n \geq 1\), \(i(C^{a,b}_n, t)\) has positive coefficients.

Notice that the positivity of \(i(C^{a,b}_n, t)\) implies the positivity of \(i(C^{a,b}_n, t)\).

### 1.3 Sparse paving Schubert matroids

Let \(r = (k-1, 1, 1, n-k-1)\), where \(n > k \geq 2\), we obtain a special Schubert matroid, denoted as \(\text{Sp}_{k,n}\). In fact, as will be shown in Proposition 6.1, \(\text{Sp}_{k,n}\) is a sparse paving matroid, and a Schubert matroid \(\text{SM}_n(S)\) is sparse paving if and only if \(r(S) = (n-k, k)\) or \(r(S) = (k-1, 1, 1, n-k-1)\), namely, \(\text{SM}_n(S)\) is a uniform matroid or
\[S = \{k, k+2, \ldots, n\}.
\]

**Theorem 1.10.** Sparse paving Schubert matroids are Ehrhart positive and are coefficient-wisely bounded by minimal and uniform matroids. That is, we have the coefficient-wise inequality
\[
i(T_{k,n}, t) \leq i(\text{Sp}_{k,n}, t) \leq i(U_{k,n}, t).
\] (1.11)

The organization of this paper is as follows. In Section 2, we recall basic definitions and notations of matroids and key polynomials. In Section 3, we give a proof of Theorem 1.1. In Section 4, we explore some further properties of \(F(a, b, c, t)\) and prove Corollary 1.3 and Theorem 1.5. Section 5 is devoted to prove Theorem 1.7. Finally, we show that sparse paving Schubert matroids are Ehrhart positive in Section 6.
2 Preliminaries

A matroid is a pair $M = (E, \mathcal{I})$ consisting of a finite set $E$, called the ground set, and a collection $\mathcal{I}$ of subsets of $E$, called independent sets, such that:

1. $\emptyset \in \mathcal{I}$;
2. If $J \in \mathcal{I}$ and $I \subseteq J$, then $I \in \mathcal{I}$;
3. If $I, J \in \mathcal{I}$ and $|I| < |J|$, then there exists $j \in J \setminus I$ such that $I \cup \{j\} \in \mathcal{I}$.

By (2), a matroid $M$ is determined by the collection $\mathcal{B}$ of maximal independent sets, called the bases of $M$. By (3), all the bases have the same cardinality, called the rank of $M$, denoted as $\text{rk}(M)$. So we can write $M = (E, \mathcal{B})$. The dual of $M$ is the matroid $M^* = (E, \mathcal{B}^*)$, where $\mathcal{B}^* = \{E \setminus B : B \in \mathcal{B}\}$. It is easy to check that the dual of a Schubert matroid $\text{SM}_n(S)$ is isomorphic to $\text{SM}_n(S')$, where $r(S')$ is the reverse of $r(S)$.

A subset $I$ of $E$ is called dependent if it is not an independent set. If $C \subseteq E$ is dependent but every proper subset of $C$ is independent, we say that $C$ is a circuit. A subset $F$ of $E$ is called a flat if $\text{rk}_M(F \cup \{a\}) > \text{rk}_M(F)$ for every $a \notin F$. A hyperplane $H$ is a flat such that $\text{rk}_M(H) = \text{rk}(M) - 1$.

We say that $M$ is paving if every circuit of $M$ has cardinality at least $\text{rk}(M)$. A matroid $M$ is sparse paving if both $M$ and its dual are paving. A matroid is sparse paving if and only if every subset of cardinality $\text{rk}(M)$ is either a basis or a circuit-hyperplane, see, for example, Bonin [3] or Ferroni [17, Lemma 2.7].

The rank function $\text{rk}_M : 2^E \to \mathbb{Z}$ of $M$ is defined by

$$\text{rk}_M(T) = \max\{|T \cap B| : B \in \mathcal{B}\}, \text{ for } T \subseteq E.$$ 

Let $\text{rk}_S$ denote the rank function of a Schubert matroid $\text{SM}_n(S)$. Fan and Guo [14, Theorem 3.3] provided an efficient algorithm to compute $\text{rk}_S(T)$ for any $T \subseteq [n]$. It is well known that the matroid polytope $P(M)$ defined in (1.1) associated to a matroid $M = ([n], \mathcal{B})$ is a generalized permutohedron parametrized by the rank function of $M$, see, for example, Fink, Mészáros and St. Dizier [19]. To be specific,

$$P(M) = \left\{ x \in \mathbb{R}^n : \sum_{i \in [n]} x_i = \text{rk}_M([n]) \text{ and } \sum_{i \in T} x_i \leq \text{rk}_M(T) \text{ for } T \subseteq [n] \right\}. \quad (2.1)$$

The key polynomials $\kappa_\alpha(x)$ associated to compositions $\alpha \in \mathbb{Z}_{\geq 0}^n$ can be defined recursively as below. If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a partition (i.e., weakly decreasing), then set $\kappa_\alpha(x) = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$. Otherwise, choose an index $i$ such that $\alpha_i < \alpha_{i+1}$, and let $\alpha'$ be obtained from $\alpha$ by interchanging $\alpha_i$ and $\alpha_{i+1}$. Set

$$\kappa_\alpha(x) = \partial_i(x, \kappa_{\alpha'}(x)).$$

Here $\partial_i$ is the divided difference operator sending a polynomial $f(x) \in \mathbb{R}[x_1, \ldots, x_n]$ to

$$\partial_i(f(x)) = \frac{f(x) - s_i f(x)}{x_i - x_{i+1}},$$
where \( s_i f(x) \) is obtained from \( f(x) \) by interchanging \( x_i \) and \( x_{i+1} \). Key polynomials are also called Demazure characters, they are characters of the Demazure modules for the general linear groups, see Demazure [10, 11].

Kohnert [23] found that the key polynomial \( \kappa_\alpha(x) \) can be generated by applying the Kohnert algorithm to the skyline diagram of \( \alpha \), see also Reiner and Shimozono [29]. Recall that the skyline diagram \( D(\alpha) \) of a composition \( \alpha = (\alpha_1, \ldots, \alpha_n) \) is a diagram consisting of the first \( \alpha_i \) boxes in row \( i \). For example, Figure 2.2 is the skyline diagram of \( \alpha = (1, 3, 0, 2) \).

![Figure 2.2: The skyline diagram \( D(\alpha) \) for \( \alpha = (1, 3, 0, 2) \).](image)

The Kohnert algorithm is defined based on Kohnert moves on diagrams. A diagram \( D \) is a finite collection of boxes in \( \mathbb{Z}_{\geq 0}^2 \). A box in row \( i \) and column \( j \) of the grid is denoted \( (i, j) \). Here, the rows (respectively, columns) are labeled increasingly from top to bottom (respectively, from left to right). A Kohnert move on \( D \) selects the rightmost box in a row of \( D \) and moves it within its column up to the first available position. To be specific, a box \( (i, j) \) of \( D \) can be moved up to a position \( (i', j) \) by a Kohnert move whenever: (i) the box \( (i, j) \) is the rightmost box in the \( i \)-th row of \( D \) and moves it within its column up to the first available position. To be specific, a box \( (i, j) \) of \( D \) can be moved up to a position \( (i', j) \) by a Kohnert move whenever: (i) the box \( (i, j) \) is the rightmost box in the \( i \)-th row of \( D \), (ii) the box \( (i', j) \) does not belong to \( D \), and (iii) for any \( i < r < i' \), the box \( (r, j) \) belongs to \( D \).

A Kohnert diagram for \( D(\alpha) \) is the diagram obtained from \( D(\alpha) \) by applying a sequence of Kohnert moves. For a diagram \( D \), let \( x^D = \prod_{(i,j) \in D} x_i \). Kohnert [23] showed that

\[
\kappa_\alpha(x) = \sum_D x^D,
\]

where the sum takes over all the Kohnert diagrams for \( D(\alpha) \). For example, Figure 2.3 displays all the Kohnert diagrams for \( \alpha = (0, 2, 1) \). Thus \( \kappa_{(0,2,1)}(x) = x_2^2x_3 + x_1x_2x_3 + x_1x_2^2 + x_1^2x_3 + x_1^2x_2 \).

![Figure 2.3: Kohnert diagrams for \( \alpha = (0, 2, 1) \).](image)

### 3 Proof of Theorem 1.1

In order to give a proof of Theorem 1.1 we first show that \( tP(\text{SM}_n(S)) \) is in fact the Newton polytope of a key polynomial.
Recall that for a polynomial
\[ f = \sum_{\alpha \in \mathbb{Z}_{\geq 0}} c_{\alpha} x^\alpha \in \mathbb{R}[x_1, \ldots, x_n], \]
the Newton polytope of \( f \) is the convex hull of the exponent vectors of \( f \):
\[ \text{Newton}(f) = \text{conv}\{\alpha : c_{\alpha} \neq 0\}. \]

It is obvious that each exponent vector of \( f \) is a lattice point in \( \text{Newton}(f) \). Monical, Tokcan and Yong [26] introduced the notion of saturated Newton polytope (SNP) of a polynomial \( f \), i.e., \( f \) has saturated Newton polytope (SNP) if every lattice point in \( \text{Newton}(f) \) is also an exponent vector of \( f \). It was conjectured by Monical, Tokcan and Yong [26] and proved by Fink, Mészáros and St. Dizier [19] that key polynomials \( \kappa_{\alpha}(x) \) have SNP.

Moreover, Fink, Mészáros, St. Dizier [19] also showed that the Newton polytopes of key polynomials \( \kappa_{\alpha}(x) \) are the Minkowski sum of Schubert matroid polytopes associated to the columns of \( D(\alpha) \). More precisely, let \( D(\alpha) = (D_1, \ldots, D_n) \), where \( D_j \) is the \( j \)-th column of \( D(\alpha) \). View \( D_j \) as a subset of \([n]\):
\[ D_j = \{1 \leq i \leq n: (i, j) \in D_j\}. \]
Then the column \( D_j \) defines a Schubert matroid \( \text{SM}_n(D_j) \). Let \( \text{rk}_j \) denote the rank function of \( \text{SM}_n(D_j) \). Then
\[ \text{Newton}(\kappa_{\alpha}) = \mathcal{P}(\text{SM}_n(D_1)) + \cdots + \mathcal{P}(\text{SM}_n(D_n)) \]
\[ = \left\{ x \in \mathbb{R}^n : \sum_{i \in [n]} x_i = |D(\alpha)| \quad \text{and} \quad \sum_{i \in T} x_i \leq \text{rk}_\alpha(T) \quad \text{for} \quad T \subseteq [n] \right\}, \quad (3.1) \]
where \( |D(\alpha)| \) denotes the number of boxes in \( D(\alpha) \) and
\[ \text{rk}_\alpha(T) = \text{rk}_1(T) + \cdots + \text{rk}_n(T). \]

**Lemma 3.1.** Let \( S \) be a subset of \([n]\) and \( \alpha = I(S) \) be the indicator vector of \( S \). Given any positive integer \( t \), we have
\[ t\mathcal{P}(\text{SM}_n(S)) = \text{Newton}(\kappa_{t\alpha}). \]

**Proof.** It is easy to see that \( \text{rk}_S([n]) = |S| \) is the number of elements in \( S \). By (2.1), we find that
\[ t\mathcal{P}(\text{SM}_n(S)) = \left\{ x \in \mathbb{R}^n : \sum_{i \in [n]} x_i = t \cdot |S| \quad \text{and} \quad \sum_{i \in T} x_i \leq t \cdot \text{rk}_S(T) \quad \text{for} \quad T \subseteq [n] \right\}. \]
On the other hand, since now \( \alpha = I(S) \) is a 0-1 vector, \( D(t\alpha) \) has exactly \( t \) columns, every column determines the same Schubert matroid, which is exactly \( \text{SM}_n(S) \). Moreover, \( |D(t\alpha)| = t \cdot |S| \) and \( \text{rk}_{t\alpha}(T) = t \cdot \text{rk}_S(T) \). Thus by (3.1), we conclude that
\[ \text{Newton}(\kappa_{t\alpha}) = t\mathcal{P}(\text{SM}_n(S)). \]
This completes the proof.

Now we are in a position to give a proof of Theorem 1.1.

**Proof of Theorem 1.1**  By Lemma 3.1, the number of lattice points in \( tP(SM_n(S)) \) is the same as that in Newton(\( \kappa_{t\alpha} \)). Since key polynomials have saturated Newton polytopes, \( i(P(SM_n(S)), t) \) is equal to the number of different monomials in \( \kappa_{t\alpha}(x) \).

Now we enumerate all the different monomials in \( \kappa_{t\alpha}(x) \) by Kohnert algorithm. Let \( D(t\alpha) \) be the skyline diagram of \( t\alpha \). Let \( D \) be a Kohnert diagram obtained from \( D(t\alpha) \) by applying a sequence of Kohnert moves. Let \( n = r_1 + r_2 + \cdots + r_{2m} \) denote the number of parts of \( \alpha \), or equivalently, the number of rows of \( D \), and denote \( \beta = (\beta_1, \ldots, \beta_n) \), where \( \beta_i \) is the number of boxes in the \( i \)-th row of \( D \). Clearly, we have \( 0 \leq \beta_i \leq t \). For \( 1 \leq j \leq m \), let
\[
d_j = r_1 + r_2 + \cdots + r_{2j}
\]
and
\[
c_j = \sum_{i=d_{j-1}+1}^{d_j} \beta_i - r_{2j}t,
\]
where \( d_0 = 0 \). Since the \( \beta_1 + \cdots + \beta_n = (r_2 + r_4 + \cdots + r_{2m})t \), we have
\[
c_1 + c_2 + \cdots + c_m = (\beta_1 + \cdots + \beta_n) - (r_2 + r_4 + \cdots + r_{2m})t = 0.
\]
It is also easy to see that the number of boxes in the top \( d_j \) rows of \( D \) is larger than or equal to that of \( D(t\alpha) \), and the number of boxes in the bottom \( d_m - d_j \) rows of \( D \) is smaller than that of \( D(t\alpha) \). That is,
\[
\beta_1 + \cdots + \beta_{d_j} \geq (\alpha_1 + \cdots + \alpha_{d_j})t = (r_2 + r_4 + \cdots + r_{2j})t.
\]
and
\[
\beta_{d_j+1} + \cdots + \beta_{d_m} \leq (\alpha_{d_j+1} + \cdots + \alpha_{d_m})t = (r_{2j+2} + \cdots + r_{2m})t.
\]
Thus we have
\[
c_1 + c_2 + \cdots + c_j = (\beta_1 + \cdots + \beta_{d_j}) - (r_2 + r_4 + \cdots + r_{2j})t \geq 0.
\]
Moreover, we have
\[
c_j = \sum_{i=d_{j-1}+1}^{d_j} \beta_i - r_{2j}t \leq \sum_{i=d_{j-1}+1}^{d_m} \beta_i - r_{2j}t
\]
\[
\leq \sum_{i=d_{j-1}+1}^{d_m} t\alpha_i - r_{2j}t = (r_{2j+2} + \cdots + r_{2m})t.
\]
And
\[
c_j = \sum_{i=d_{j-1}+1}^{d_j} \beta_i - r_{2j}t \leq \sum_{i=d_{j-1}+1}^{d_j} t - r_{2j}t = (r_{2j-1} + r_{2j})t - r_{2j}t = r_{2j-1}t.
\]
Thus we have

\[
c_j \leq \min\{r_{2j-1}, r_{2j+2} + \cdots + r_{2m}\} t = tu_j. \tag{3.2}
\]

Similarly, we have

\[
c_j = \sum_{i=1}^{d_j} \beta_i - \sum_{i=1}^{d_{j-1}} \beta_i - r_{2j} t \geq \sum_{i=1}^{d_j} t \alpha_i - \sum_{i=1}^{d_{j-1}} \beta_i - r_{2j} t
\]

\[
\geq \sum_{i=1}^{d_j} t \alpha_i - \sum_{i=1}^{d_{j-1}} t - r_{2j} t = \left( \sum_{i=1}^{j} r_{2i} \right) t - \sum_{i=1}^{d_{j-1}} t - r_{2j} t
\]

\[
= -(r_1 + r_3 + \cdots + r_{2j-3}) t.
\]

And

\[
c_j = \sum_{i=d_{j-1}+1}^{d_j} \beta_i - r_{2j} t \geq -r_{2j} t.
\]

Then we find

\[
c_j \geq -\min\{r_{2j}, r_1 + r_3 + \cdots + r_{2j-3}\} t = -tv_j. \tag{3.3}
\]

Therefore, \( \beta = (\beta_1, \ldots, \beta_n) \) satisfies the following system of equations

\[
\begin{aligned}
\sum_{i=1}^{d_1} \beta_i &= r_2 t + c_1, \\
\sum_{i=d_1+1}^{d_2} \beta_i &= r_4 t + c_2, \\
&\quad \vdots \\
\sum_{i=d_{m-1}+1}^{d_m} \beta_i &= r_{2m} t + c_m,
\end{aligned}
\]

(3.4)

where \( c_1 + c_2 + \cdots + c_m = 0 \), and for \( 1 \leq j \leq m \),

\[-tv_j \leq c_j \leq tu_j \quad \text{and} \quad c_1 + c_2 + \cdots + c_j \geq 0.\]

Now we enumerate the number of nonnegative integer solutions of the equation

\[
\sum_{i=d_{j-1}+1}^{d_j} \beta_i = r_{2j} t + c_j, \quad (0 \leq \beta_i \leq t). \tag{3.5}
\]

Since \( 0 \leq \beta_i \leq t \), it is easy to see that the number of solutions of equation (3.5) is the coefficient of \( x^{r_{2j} t + c_j} \) in

\[
(1 + x + \cdots + x^t)^{d_j-d_{j-1}}
\]

\[
= (1 + x + \cdots + x^{r_{2j-1}+r_{2j}})
\]
\[(1 - x^{i+1})^{r_{2j-1}+r_{2j}} \cdot (1 - x)^{-r_{2j-1}+r_{2j}}
\]
\[= \sum_{i'=0}^{r_{2j-1}+r_{2j}} (-1)^{i'} \left( r_{2j-1} + r_{2j} \right)^{x^i(t+1)i'} \left( \sum_{j'=0}^{\infty} \left( r_{2j-1} + r_{2j} + j' - 1 \right) x^{j'} \right)
\]
\[= \sum_{j'=0}^{\infty} \sum_{i'=0}^{r_{2j-1}+r_{2j}} (-1)^{i'} \left( r_{2j-1} + r_{2j} \right) \left( r_{2j-1} + r_{2j} + j' - 1 \right) x^{j'+(t+1)i'}.
\] (3.6)

Let \( j' = r_{2j}t + c_j - (t+1)i' \) in (3.6), we see that the coefficient of \( x^{r_{2j}t+c_j} \) is
\[F(r_{2j-1}, r_{2j}, c_j, t) := \sum_{i'=0}^{r_{2j-1} \pm r_{2j}} (-1)^{i'} \left( r_{2j-1} + r_{2j} \right) \left( t+1 \right) \left( r_{2j} - i' \right) + r_{2j-1} + c_j - 1.
\] (3.7)

Consequently, the number of different monomials in \( \kappa_{ta}(x) \) is
\[\sum_{(c_1, \ldots, c_m)} \prod_{j=1}^{m} F(r_{2j-1}, r_{2j}, c_j, t).
\]

Conversely, suppose that \( (\beta_1, \ldots, \beta_n) \) is an integer sequence such that \( 0 \leq \beta_i \leq t \) and \( (\beta_1, \ldots, \beta_n) \) satisfies the system of equations (3.3), we shall show that there is a diagram \( D \) whose \( i \)-th row has \( \beta_i \) boxes and \( D \) can be obtained from \( D(ta) \) by applying Kohnert moves.

First of all, by adding all the equations in (3.3) together and combing the condition \( c_1 + \cdots + c_m = 0 \), we have
\[\beta_1 + \beta_2 + \cdots + \beta_n = (r_2 + r_4 + \cdots + r_{2m})t.
\]
That is, \( \beta_1 + \beta_2 + \cdots + \beta_n \) is equal to the total number of boxes in \( D(ta) \). We construct \( D \) as follows. Fill the sequence of integers \( 1^{\beta_1}, 2^{\beta_2}, \ldots, n^{\beta_n} \) into the boxes of \( D(ta) \) along the rows from top to bottom and from right to left. Then move the box \((i, j)\) filled with \( k \) to \((k, j)\). Denote the resulting diagram by \( D \). For example, Figure 3.4 displays the construction of \( D \) for \( ta = (0, 0, 3, 0, 3, 3) \) and \( \beta = (2, 2, 1, 3, 0, 1) \).

![Figure 3.4: An illustration of the construction of D.](image)

We aim to show that \( D \) is indeed a Kohnert diagram. Since \( 0 \leq \beta_i \leq t \), by the construction of \( D \), it is easy to see that there do not exist two boxes in the same column of \( D(ta) \) that are filled with the same integer. By the definition of Kohnert moves, to show that \( D \) is indeed a Kohnert diagram, it suffices to show that there does not exist a box of \( D(ta) \) which is filled with an integer larger than its row index.
There are three cases.

Case 1. \( l = k \), that is,
\[
r_1 + r_2 + \cdots + r_{2l-1} < i < s \leq r_1 + r_2 + \cdots + r_{2l} = d_l.
\]
By (3.4), we have
\[
\beta_1 + \cdots + \beta_{d_l} = (r_2 + r_4 + \cdots + r_{2l})t + c_1 + \cdots + c_l.
\]
Since \( c_1 + \cdots + c_l \geq 0 \), we find that
\[
\beta_1 + \cdots + \beta_{d_l} \geq (r_2 + r_4 + \cdots + r_{2l})t.
\]
That is to say, the integers \( 1, 2, \ldots, d_l \) must occupy at least the top \( d_l = r_1 + r_2 + \cdots + r_{2l} \) rows (including empty rows) of \( D(t \alpha) \). On the other hand, since the box \((i, j)\) is filled with \( s \) and \( s > i \), we see that the rows \( i + 1, \ldots, s, s + 1, \ldots, d_l \) of \( D(t \alpha) \) are occupied by some of the integers among \( s, s + 1, \ldots, d_l \). In particular, the integers \( s, s + 1, \ldots, d_l \) must occupy at least \( d_l - s + 1 \) rows of \( D(t \alpha) \). Therefore,
\[
\beta_s + \beta_{s+1} + \cdots + \beta_{d_l} > (d_l - s + 1)t.
\]
Thus there must exist some \( \beta_j > t \), which contradicts with the assumption \( 0 \leq \beta_j \leq t \).

Case 2. \( l \geq k + 1 \) and \( i = d_k \). In this case, we have
\[
s > r_1 + r_2 + \cdots + r_{2l-1} \geq r_1 + r_2 + \cdots + r_{2k+1} = i + r_{2k+1}.
\]
Similar to Case 1, we see that the integers \( s, s + 1, \ldots, d_l \) must occupy all the boxes of \( D(t \alpha) \) in the rows \( s, s + 1, \ldots, d_l \). Thus
\[
\beta_s + \beta_{s+1} + \cdots + \beta_{d_l} > (d_l - s + 1)t,
\]
which is a contradiction.

Case 3. \( l \geq k + 1 \) and \( i < d_k \). In this case, we have \( i + 1 \leq d_k \) and
\[
s > r_1 + r_2 + \cdots + r_{2l-1} \geq r_1 + r_2 + \cdots + r_{2k+1} \geq i + r_{2k+1} \geq i + 1.
\]
By the choice of \((i, j)\), \( i \) is the largest index such that \( s > i \) and \((i, j)\) is filled with \( s \), we see that \( s \) can not appear in the \((i + 1)\)-st row of \( D(t \alpha) \). Thus we have
\[
\beta_1 + \beta_2 + \cdots + \beta_s \leq (\alpha_1 + \alpha_2 + \cdots + \alpha_i)t \leq (r_2 + r_4 + \cdots + r_{2k})t.
\]
Moreover, since \( s > r_1 + r_2 + \cdots + r_{2l-1} = d_{l-1} + r_{2l-1} \geq d_{l-1} \) and \( \beta_s > 0 \), we derive that
\[
\beta_1 + \cdots + \beta_s \geq \beta_1 + \cdots + \beta_{d_{l-1}} = (r_2 + r_4 + \cdots + r_{2l-2})t + c_1 + \cdots + c_{l-1},
\]
Combining (3.10) and (3.11), we get
\[
(r_2 + r_4 + \cdots + r_{2l-2})t + c_1 + \cdots + c_{l-1} < (r_2 + r_4 + \cdots + r_{2k})t.
\]
Since \( c_1 + \cdots + c_{l-1} \geq 0 \), we must have \( 2l - 2 < 2k \), that is, \( l < k + 1 \). This is a contradiction.
4 Notched rectangle matroids

In this section, we consider Ehrhart polynomials of notched rectangle matroids. We first explore some further properties of $F(a, b, c, t)$ as defined in (1.3), and then prove Corollary 1.3 and Theorem 1.5.

By (3.5) and (3.7) in the proof of Theorem 1.1, $F(a, b, c, t)$ is the number of integer solutions of the equation

$$\begin{align*}
x_1 + x_2 + \cdots + x_{a+b} &= bt + c, \\
0 \leq x_i &\leq t, \text{ for } 1 \leq i \leq a+b.
\end{align*}$$

We proceed to develop some further properties of the polynomial $F(a, b, c, t)$.

**Lemma 4.1.** We have

$$
F(a, b, c, t) = F(b, a, -c, t) \quad (4.2)
$$

$$
F(a, b, c, t) = F(a+1, b-1, c+t, t) \quad (4.3)
$$

$$
F(a+1, b, 0, t) = \sum_{i=0}^{t} F(a, b, -i, t). \quad (4.4)
$$

**Proof.** Let $y_i = t - x_i$ in the equation (4.1). Then $y_1 + y_2 + \cdots + y_{a+b} = (a+b)t - (bt + c) = at - c$, where $0 \leq y_i \leq t$. It is easy to see that the number of integer solutions of this equation is $F(b, a, -c, t)$. Thus (4.2) holds.

Since both $F(a, b, c, t)$ and $F(a+1, b-1, c+t, t)$ are the number of solutions of $x_1 + \cdots + x_{a+b} = bt + c = (b-1)t + t + c$, where $0 \leq x_i \leq t$, we obtain (4.3).

Similarly, since $F(a+1, b, 0, t)$ is the number of solutions of the equation $x_1 + \cdots + x_{a+b} = bt - x_{a+b+1}$, where $0 \leq x_i \leq t$ for $i = 1, \ldots, a+b+1$, we see that (4.4) follows.

Let $a, b, c, d$ be nonnegative integers, and $r(S) = (a, b, c, d)$. Then $u = (\min\{a, d\}, 0), v = (0, \min\{a, d\})$ and by (4.2),

$$
i((a, b, c, d), t) = i((d, c, b, a), t) = \sum_{j=0}^{t-\min\{a, d\}} F(a, b, j, t)F(c, d, -j, t). \quad (4.5)
$$

**Theorem 4.2.** For any nonnegative integers $a, b, c, d$, we have

$$
i((a, b, c, d), t) + i((b, a, d, c), t) = i(U_{b+d,a+b+c+d}, t) + i(U_{b,a+b}, t)i(U_{d,c+d}, t). \quad (4.6)
$$

**Proof.** Since $F(a+c, b+d, 0, t)$ is the number of solutions of $x_1 + \cdots + x_{a+b+c+d} = (b+d)t$, where $0 \leq x_i \leq t$, which is equal to the sum of number of solutions of

$$
\begin{align*}
x_1 + \cdots + x_{a+b} &= bt + j, \\
x_{a+b+1} + \cdots + x_{a+b+c+d} &= dt - j, \\
0 \leq x_i &\leq t, \text{ for } 1 \leq i \leq a+b+c+d,
\end{align*}
$$

...
Similarly, one can check that $j$ of matroid subdivisions. More precisely, we split the uniform matroid polytope into two pieces:

$$F(a+c, b+d, 0, t)$$

for all possible integers $j$. It is clear that if $j < -bt$, then the first equation of (4.7) has no solutions. If $j < -ct$, then the second equation has no solution. Thus $j \geq -c \cdot \min\{b, d\}$. Therefore,

$$F(a+c, b+d, 0, t) = \sum_{j=-c \cdot \min\{b, d\}}^{t-\min\{a,d\}} F(a, b, j, t) F(c, d, -j, t)$$

Remark. As pointed out by the referee, the statement of Theorem 4.2 is actually a fact of matroid subdivisions. More precisely, we split the uniform matroid polytope

$$\mathcal{P}(U_{b+d,a+c+b+d}) = \left\{ x \in \mathbb{R}_{\geq 0}^{a+b+c+d} : \sum_{i=1}^{a+b+c+d} x_i = b + d \right\}$$

into two pieces:

$$\mathcal{P}_1 = \left\{ x \in \mathbb{R}_{\geq 0}^{a+b+c+d} : \sum_{i=1}^{a+b+c+d} x_{a+b+i} \leq d, \sum_{i=1}^{a+b+c+d} x_i = b + d \right\}$$

and

$$\mathcal{P}_2 = \left\{ x \in \mathbb{R}_{\geq 0}^{a+b+c+d} : \sum_{i=1}^{a+b} x_i \leq b, \sum_{i=1}^{a+b+c+d} x_i = b + d \right\}.$$

And their intersection is a common facet

$$H = \left\{ x \in \mathbb{R}_{\geq 0}^{a+b+c+d} : \sum_{i=1}^{a+b} x_i = b, \sum_{i=1}^{a+b+c+d} x_{a+b+i} = d \right\}.$$

One can check that $\mathcal{P}_1 = \mathcal{P}(\text{SM}_a(a, b, c, d))$ and $\mathcal{P}_2$ can be obtained by a rotation of the Schubert matroid polytope

$$\mathcal{P}(\text{SM}_a(c, d, a, b)) = \left\{ x \in \mathbb{R}_{\geq 0}^{a+b+c+d} : \sum_{i=1}^{a+b} x_{c+d+i} \leq b, \sum_{i=1}^{a+b+c+d} x_i = b + d \right\}.$$

Thus $i(\mathcal{P}_2, t) = i((c, d, a, b), t) = i((b, a, d, c), t)$. Moreover, it is easy to see that $i(H, t) = i(U_{b,a+b}, t)i(U_{d,c+d}, t)$. Therefore, we have

$$i(U_{b+d,a+b+c+d}, t) = i((b, c, d), t) + i((b, a, d, c), t) - i(U_{b,a+b}, t)i(U_{d,c+d}, t).$$
Corollary 4.3. We have
\[ i((1, 1, a, b), t) + i((1, 1, b - 1, a + 1), t) = (t + 2)i(U_{b,a+b+1}, t). \] (4.8)

Proof. Since \( F(1, i, t) = t + 1 - i \) for \( 0 \leq i \leq t \), by (4.5), we find
\[
i((1, 1, a, b), t) + i((1, 1, b - 1, a + 1), t)
= \sum_{i=0}^{t} ((t - i + 1)F(a, b, t - i, t) + (t - i + 1)F(b - 1, a + 1, -i, t))
= \sum_{i=0}^{t} ((t - i + 1)F(a, b, -i, t) + (t - i + 1)F(a, b, i - t, t))
= \sum_{i=0}^{t} ((t - i + 1)F(a, b, -i, t) + (i + 1)F(a, b, -i, t))
= \sum_{i=0}^{t} (t + 2)F(a, b, -i, t)
= (t + 2)i(U_{b,a+b+1}, t),
\]
where the second step holds by (4.2) and (4.3), and the last step follows from (4.4). \( \square \)

Proof of Theorem 1.5. By (4.5), we find
\[
i((a, b, a, b), t) = i((b, a, b, a), t).
\]
The equations (1.6) and (1.7) follow directly from (4.6). And (1.8) is a special case of (4.8). \( \square \)

In the rest of this section, we give a proof of Corollary 1.3.

Recall that the minimal matroid \( T_{k,n} \) is the Schubert matroid \( SM_n(S) \) with
\[ S = \{2, 3, \ldots, k, n\}, \]
where \( n > k \geq 2 \).

Proof of Corollary 1.3. Since \( r = (1, k - 1, n - k - 1, 1) \), by (4.5), we have
\[ i(T_{k,n}, t) = \sum_{j=0}^{t} F(1, k - 1, j, t)F(n - k - 1, 1, -j, t). \] (4.9)

By (4.2) and (4.3), we have
\[
F(1, k - 1, j, t) = F(k - 1, 1, -j, t) = F(k, 0, t - j, t) = \binom{k + t - j - 1}{t - j}
\]
and
\[
F(n - k - 1, 1, -j, t) = F(n - k, 0, t - j, t) = \binom{t - j + n - k - 1}{n - k - 1}.
\]
Then
\[
i(T_{k,n}, t) = \sum_{j=0}^{t} \binom{t - j + n - k - 1}{n - k - 1} \binom{k + t - j - 1}{t - j}
\]

\[ 16 \]
Thus we need to show that
\[
\sum_{j=0}^{t} \binom{j+n-k-1}{n-k-1} \binom{k+j-1}{j} = \frac{1}{(n-k-1)!} \left( t + n - k \right) \sum_{j=0}^{k-1} \binom{n-k+j-1}{j} \binom{t+j}{j}.
\] (4.10)

Let \( s = n-k \) in (4.11), then we aim to show that
\[
\binom{s+k-1}{s} \sum_{j=0}^{t} \binom{j+s-1}{s-1} \binom{j+k-1}{k-1} = \binom{t+s}{s} \sum_{j=0}^{k-1} \binom{j+s-1}{s-1} \binom{t+j}{j}.
\] (4.11)

It is easy to see that the left hand side of (4.12) is the coefficient of \( y^{s-1}x^{k-1} \) in
\[
\binom{s+k-1}{s} \sum_{j=0}^{t} (1+y)^{s+j-1}(1+x)^{k+j-1} = \binom{k-1+s}{s} (1+y)^{s-1}(1+x)^{k-1} \cdot \frac{1-((1+y)(1+x))^{t+1}}{-x-y-xy}. \] (4.13)

Similarly, the right hand side of (4.12) is the coefficient of \( y^{s-1}x^{t} \) in
\[
\binom{t+s}{s} \sum_{j=0}^{k-1} (1+y)^{s+j-1}(1+x)^{t+j} = \binom{t+s}{s} (1+y)^{s-1}(1+x)^{t} \cdot \frac{1-((1+y)(1+x))^{k}}{-x-y-xy}. \] (4.14)

One can check that the coefficient of \( y^{s-1}x^{k-1} \) in (4.13) is equal to the coefficient of \( y^{s-1}x^{t} \) in (4.14). Thus (4.12) follows. This completes the proof.

To conclude this section, we remark that since \( F(a,b,0,t) = i(U_{b,a+b}, t) \) has positive coefficients for any \( a,b \geq 1 \), it is natural to ask whether \( F(a,b,c,t) \) defined in (1.3) has positive coefficients or not for any \( c \). The following conjecture was verified for \( a,b,c \leq 10 \).

**Conjecture 4.4.** \( F(a,b,c,t) \) has positive coefficients for any \( a,b \geq 1 \) if and only if \( c = 0, \pm 1 \).

Since \( F(1,1,c,t) = t+1-|c| \), we see that if \( |c| > 1 \), then \( F(1,1,c,t) \) has negative coefficients. Thus to prove Conjecture 4.4 it is enough to show that if \( c = \pm 1 \), then \( F(a,b,c,t) \) is a positive polynomial in \( t \) for any \( a,b \geq 1 \).

\[\boxed{5} \quad (a,b)\text{-Catalan matroids}\]

In this section, we give a proof of Theorem 1.7. Recall that the \((a,b)\)-Catalan matroid \( \mathcal{C}_{n}^{a,b} \) is the Schubert matroid \( SM_{(a+b)n}(S) \), where \( r(S) = (a,b,a,b,\ldots,a,b) \).
Given a composition $\sigma$ of $n$, $\ell(\sigma)$ denotes the number of parts of $\sigma$, $d(\sigma)$ denotes the cardinality of the equivalent class containing $\sigma$. And $\Gamma_n$ is the set of pairwise non-equivalent compositions of $n$ with at least two parts and minimal parts larger than 1.

It can be seen that (1.9) is equivalent to

$$F(na, nb, 0, t) = n \cdot i(C_{\sigma}^{a,b}, t) + i(U_{b,a+b}, t) + \sum_{\sigma \in \Gamma_n} (-1)^{\ell(\sigma)-1} \frac{n d(\sigma)}{\ell(\sigma)} \cdot i(C_{\sigma}^{a,b}, t),$$

(5.1)

where $i(C_{\sigma}^{a,b}, t) = \prod_{j=1}^{\ell(\sigma)} i(C_{\sigma_j}^{a,b}, t)$ and

$$i(C_{\sigma_j}^{a,b}, t) = i(C_{\sigma_j}^{a,b}, t) - i(U_{b,a+b}, t) \cdot i(C_{\sigma_{j-1}}^{a,b}, t).$$

We shall prove (5.1) by interpreting both sides in terms of weighted enumerations of certain lattice paths.

Let us begin with interpreting $i(C_{\sigma}^{a,b}, t)$ and $F(na, nb, 0, t)$ separately. By Theorem 1.1, since $r = (a, b, \ldots, a, b)$, for $1 \leq j \leq n$, we have

$$u_j = \min\{a, (n-j)b\}, \quad v_j = \min\{b, (j-1)a\}. \quad (5.2)$$

Thus

$$i(C_{\sigma}^{a,b}, t) = \sum_{(c_1, \ldots, c_n) \in \mathbb{N}^n} \prod_{j=1}^{n} F(a, b, c_j, t),$$

where $c_1 + \cdots + c_n = 0, c_1 + \cdots + c_j \geq 0$ and $-tv_j \leq c_j \leq tu_j$, for $1 \leq j \leq n$.

On the other hand, since $F(na, nb, 0, t)$ is the number of solutions of

$$\begin{cases}
    x_1 + x_2 + \cdots + x_{(a+b)n} = bnt, \\
    0 \leq x_i \leq t, \text{ for } 1 \leq i \leq (a+b)n,
\end{cases}$$

which is equivalent to the system of equations

$$\begin{cases}
    x_{1,1} + \cdots + x_{1,a+b} = bt + c'_1, \\
    x_{2,1} + \cdots + x_{2,a+b} = bt + c'_2, \\
    \vdots \\
    x_{n,1} + \cdots + x_{n,a+b} = bt + c'_n,
\end{cases} \quad (5.3)$$

for all possible integers $c'_1, \ldots, c'_n$, where $0 \leq x_{i,j} \leq t$ for $1 \leq i \leq n$ and $1 \leq j \leq a+b$, and $c'_1 + \cdots + c'_n = 0$. It is easy to see that we can require $-bt \leq c'_j \leq at$ for $1 \leq j \leq n$. Thus

$$F(na, nb, 0, t) = \sum_{(c'_1, \ldots, c'_n) \in \mathbb{N}^n} \prod_{j=1}^{n} F(a, b, c'_j, t),$$

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where $c'_1 + \cdots + c'_n = 0$, and $-bt \leq c'_j \leq at$ for $1 \leq j \leq n$.

Let

$$C^{a,b}_n = \left\{ (c_1, \ldots, c_n) \mid \sum_{i=1}^n c_i = 0, \sum_j c_j \geq 0 \text{ and } -tv_j \leq c_j \leq tu_j, \forall 1 \leq j \leq n \right\}$$

(5.4)

and

$$F^{a,b}_n = \left\{ (c_1, \ldots, c_n) \mid \sum_{i=1}^n c_i = 0, -bt \leq c_j \leq at, \forall 1 \leq j \leq n \right\}. \quad (5.5)$$

Clearly, $C^{a,b}_n \subseteq F^{a,b}_n$. We can view each sequence $c = (c_1, \ldots, c_n) \in F^{a,b}_n$ as a lattice path from $(0, 0)$ to $(n, 0)$ such that $c_j$ represents: an up step $(0, 0) \rightarrow (1, c_j)$ if $c_j > 0$, a down step $(0, 0) \rightarrow (1, -|c_j|)$ if $c_j < 0$, or a horizontal step $(0, 0) \rightarrow (1, 0)$ if $c_j = 0$. Assign a weight to $c$ as

$$\text{wt}(c) = \prod_{j=1}^n F(a, b, c_j, t).$$

In particular, $F(a, b, 0, t)^n$ is the weight of the path $(0, 0, \ldots, 0)$. Then

$$i(C^{a,b}_n, t) = \sum_{c \in C^{a,b}_n} \text{wt}(c) \quad \text{and} \quad F(na, nb, 0, t) = \sum_{c \in F^{a,b}_n} \text{wt}(c)$$

can be viewed as weighted enumerations of lattice paths in $C^{a,b}_n$ and $F^{a,b}_n$, respectively.

Let

$$\overline{C}^{a,b}_n = \{ (c_1, \ldots, c_n) \in C^{a,b}_n \mid c_n \neq 0 \}.$$

(5.6)

Then we find

$$i(\overline{C}^{a,b}_n, t) = \sum_{c \in \overline{C}^{a,b}_n} \text{wt}(c). \quad (5.7)$$

Moreover, given a composition $\sigma = (\sigma_1, \ldots, \sigma_s)$ of $n$, denote

$$\overline{C}^{a,b}_\sigma = \{ (c_1, \ldots, c_s) \mid c_j \in \overline{C}^{a,b}_{\sigma_j}, \text{ for } 1 \leq j \leq s \},$$

(5.8)

then

$$i(\overline{C}^{a,b}_\sigma, t) = \prod_{j=1}^{\ell(\sigma)} i(\overline{C}^{a,b}_{\sigma_j}, t) = \sum_{c \in \overline{C}^{a,b}_\sigma} \text{wt}(c).$$

Therefore, (5.1) is equivalent to

$$\sum_{c \in F^{a,b}_n} \text{wt}(c) = n \cdot \sum_{c \in \overline{C}^{a,b}_n} \text{wt}(c) + \text{wt}((0, \ldots, 0)) + \sum_{\sigma \in \Gamma_n} (-1)^{\ell(\sigma)-1} \frac{nd(\sigma)}{\ell(\sigma)} \sum_{c \in \overline{C}^{a,b}_\sigma} \text{wt}(c). \quad (5.9)$$

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To prove (5.9), we shall enumerate the number of appearances of each \( c \in F_{n}^{a,b} \) in the right hand side of (5.9) by inclusion-exclusion. To this end, define two shifting operators \( R \) and \( L \) on each \( c = (c_{1}, \ldots , c_{n}) \in F_{n}^{a,b} \) as follows

\[
R(c) = (c_{2}, \ldots , c_{n}, c_{1}), \\
L(c) = (c_{n}, c_{1}, \ldots , c_{n-1}),
\]

and let \( R^{m}(c) = (c_{m+1}, \ldots , c_{n}, c_{1}, \ldots , c_{m}) \) denote the effect of applying \( R \) to \( c \) \( m \) times, where \( m \geq 1 \). Similar \( L^{m}(c) \) denote applying \( L \) to \( c \) \( m \) times. It is clear that \( R^{m}(c) = L^{m}(c) = c \). Denote \( L_{e}^{m} \) by the list of paths obtained from \( c \) by applying \( L \) to \( c \) \( m-1 \) times, i.e.,

\[
L_{e}^{m} = (c, L(c), L^{2}(c), \ldots , L^{m-1}(c)).
\]

It is quite possible that \( L_{e}^{m} \) contains repeated paths. Similarly, let

\[
R_{e}^{m} = (c, R(c), R^{2}(c), \ldots , R^{m-1}(c)).
\]

Obviously, each path in \( L_{e}^{m} \) or \( R_{e}^{m} \) has the same weight.

Denote \( n * \overline{C}_{n}^{a,b} \) by the list of paths that each \( c \) of \( \overline{C}_{n}^{a,b} \) are replaced by the \( n \) paths in \( \Gamma_{n} \). Similarly, denote \( \frac{nd(\sigma)}{\ell(\sigma)} * \overline{C}_{\sigma}^{a,b} \) by the list of paths that each \( c \) of \( \overline{C}_{\sigma}^{a,b} \) are replaced by the \( \frac{nd(\sigma)}{\ell(\sigma)} \) paths in \( \overline{C}_{\sigma}^{a,b} \) for any \( \sigma \in \Gamma_{n} \). If there is a minus sign in front of some \( \frac{nd(\sigma)}{\ell(\sigma)} * \overline{C}_{\sigma}^{a,b} \), then we delete the number of appearances of each \( c \in \frac{nd(\sigma)}{\ell(\sigma)} * \overline{C}_{\sigma}^{a,b} \) in the enumeration. Assume that \( \Gamma_{n} \) has \( \gamma_{n} \) different compositions, i.e., \( \Gamma_{n} = \{ \sigma^{1}, \sigma^{2}, \ldots , \sigma^{\gamma_{n}} \} \). Let

\[
\left( n * \overline{C}_{n}^{a,b}, (1) \ell(\sigma^{1})-1 \frac{nd(\sigma^{1})}{\ell(\sigma^{1})} * \overline{C}_{\sigma^{1}}^{a,b}, \ldots , (1) \ell(\sigma^{\gamma_{n}})-1 \frac{nd(\sigma^{\gamma_{n}})}{\ell(\sigma^{\gamma_{n}})} * \overline{C}_{\sigma^{\gamma_{n}}}^{a,b} \right).
\]

(5.10)

To prove (5.9), we aim to show that, after cancellations, each \( c \in F_{n}^{a,b} \), \( c \neq (0, \ldots , 0) \) appears exactly once in (5.10).

For example, let \( n = 6, a = 2, b = 3, t = 1 \) and \( c = (1, -1, 2, -2, 1, -1) \in \overline{F}_{6}^{2,3} \). Then \( \Gamma_{6} = \{ (4, 2), (3, 3), (2, 2, 2) \} \). We aim to enumerate the number of appearances of \( c \) in

\[
(6 * \overline{C}_{6}^{2,3}, -6 * \overline{C}_{(4,2)}^{2,3}, -3 * \overline{C}_{(3,3)}^{2,3}, 2 * \overline{C}_{(2,2,2)}^{2,3}).
\]

(5.11)

One can check that there are 3 appearances of \( c \) in \( 6 * \overline{C}_{6}^{2,3} \). That is, for the 3 paths \( c_{1} = c, c_{2} = (2, -2, 1, -1, 1, -1), c_{3} = (1, -1, 1, -1, 2, -2) \) in \( \overline{C}_{6}^{2,3} \), \( c \) appears in each of \( \Gamma_{6}^{c_{1}}, \Gamma_{6}^{c_{2}}, \Gamma_{6}^{c_{3}} \) exactly once. Similarly, \( c \) appears in \( 6 * \overline{C}_{(4,2)}^{2,3} \) 3 times with a minus sign. That is, for the 3 paths \( c_{1}^{'}, c_{2}^{'}, c_{3}^{'} = c_{3} \) in \( \overline{C}_{(4,2)}^{2,3} \), \( c \) appears in each of \( \Gamma_{6}^{c_{1}^{'}} \), \( \Gamma_{6}^{c_{2}^{'}} \), \( \Gamma_{6}^{c_{3}^{'}} \) exactly once. Moreover, \( c \) appears in \( 3 * \overline{C}_{(3,3)}^{2,3} \) 0 times, and appears in \( 2 * \overline{C}_{(2,2,2)}^{2,3} \) exactly once. That is, \( c \in \overline{C}_{(2,2,2)}^{2,3} \) and \( c \) only appears in \( \Gamma_{2}^{c} = (c, (-1, 2, -2, 1, -1, -1)) \) once. Therefore, the total number of appearances of \( c \) in (5.11) is 1.

Given a path \( c \in F_{n}^{a,b} \), if \( c \) does not go below the x-axis, then we write \( c \geq 0 \) for simplicity, and say \( c \) is nonnegative. Otherwise, write \( c < 0 \) and say \( c \) is negative. Obviously, if \( c \geq 0 \), then \( c_{1} + \cdots + c_{j} \geq 0 \) for any \( 1 \leq j \leq n \).
It is clear that $C_{a,b} \subseteq F_{a,b}$ and $C_{a,b} \subseteq F_{a,b}$ for any $\sigma \in \Gamma_n$. By the definitions of $C_{a,b}$ and $F_{a,b}$ in (5.4) and (5.5), it seems possible that there exists $c \in F_{a,b}$ and $c \geq 0$, but $c \not\in C_{a,b}$. However, we show that this situation cannot happen.

For $c = (c_1, \ldots, c_n) \in F_{a,b}$, let

$$
E_c = \{(c_j, \ldots, c_n, c_1, \ldots, c_{j-1}) | 1 \leq j \leq n\}
$$

denote the set of paths that can be obtained from $c$ by cyclic shifting.

**Lemma 5.1.** Each $E_c$ contains a nonnegative path.

**Proof.** Given $c = (c_1, \ldots, c_n)$, define $d_j = c_1 + \cdots + c_j$ for $1 \leq j \leq n$. If $d_1, \ldots, d_n \geq 0$, then it is clear that $c$ is nonnegative. If $d_j < 0$ for some $j$, then let $k$ be the smallest index such that $d_k = \min\{d_1, \ldots, d_n\}$. Let $c' = (c_{k+1}, \ldots, c_n, c_1, \ldots, c_k) \in E_c$. It is easy to see that $c'$ is nonnegative. \qed

**Lemma 5.2.** We have

$$
F_{a,b} = \bigcup_{c \in C_{a,b}} E_c.
$$

(5.12)

**Proof.** By Lemma [5.1]

$$
F_{a,b} = \bigcup_{c \in F_{a,b}, c \geq 0} E_c.
$$

Clearly, $C_{a,b} \subseteq \{c \in F_{a,b} | c \geq 0\}$. We aim to show that

$$
\{c \in F_{a,b} | c \geq 0\} \subseteq C_{a,b}.
$$

Given $c = (c_1, \ldots, c_n) \in F_{a,b}$ such that $c \geq 0$, we need to show that $-tv_j \leq c_j \leq tu_j$ for any $1 \leq j \leq n$. Since $c \geq 0$, we have $c_1 + \cdots + c_{j-1} \geq 0$. Adding the first $j$ equations in (5.3) together, we obtain

$$
jbt + c_1 + \cdots + c_{j-1} + c_j = \sum_{i=1}^{j} \sum_{i'=1}^{a+b} x_{i,i'} \leq bmt,
$$

(5.13)

thus $c_j \leq (n - j)bt$. Combing the fact $c_j \leq at$, we arrive at

$$
c_j \leq \min\{(n - j)bt, at\} = tu_j.
$$

On the other hand, by (5.13), we obtain

$$
(j - 1)bt + c_1 + \cdots + c_{j-1} = \sum_{i=1}^{j-1} \sum_{i'=1}^{a+b} x_{i,i'} \leq (j - 1)(a + b)t,
$$

thus

$$
(n - j + 1)bt + c_j + \cdots + c_n = \sum_{i=j}^{n} \sum_{i'=1}^{a+b} x_{i,i'} \geq bmt - (j - 1)(a + b)t,
$$

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so we see that
\[ c_j \geq -(j-1)at - (c_{j+1} + \cdots + c_n) \geq -(j-1)at, \] (5.14)
where \( c_{j+1} + \cdots + c_n \leq 0 \) since \( c_1 + \cdots + c_n = 0 \) and \( c_1 + \cdots + c_j \geq 0 \). Combining the fact \( c_j \geq -bt \) and (5.14), we obtain
\[ c_j \geq \min \{-bt, -(j-1)at\} = -tv_j, \]
as desired. \[\Box\]

By Lemma 5.1 and Lemma 5.2, we can divide the discussions into three cases, that is, \( c \geq 0, c_n \neq 0 \), or \( c \geq 0, c_n = 0 \), or \( c < 0 \).

**Proposition 5.3.** Let \( c = (c_1, \ldots, c_n) \geq 0 \) such that \( c_n \neq 0 \). Then \( c \) appears exactly once in (5.10).

To give a proof of Proposition 5.3, we need to enumerate how many copies of \( c \) appearing in each \( \frac{n \mathcal{d}(\sigma)}{\ell(\sigma)} \ast \mathcal{T}_{a,b}^{\sigma} \). To this end, we first give a combinatorial interpretation of the coefficient \( \frac{n \mathcal{d}(\sigma)}{\ell(\sigma)} \).

For a composition \( \sigma = (\sigma_1, \ldots, \sigma_s) \) of \( n \), denote \( p(\sigma) \) by the least period of \( \sigma \), that is, \( p(\sigma) \) is the smallest integer such that \( \sigma_i = \sigma_{i+p(\sigma)} \) for all \( i \). If \( \sigma \) has no period, then we define \( p(\sigma) = \ell(\sigma) \). It is easy to see that \( p(\sigma) = d(\sigma) \) is the cardinality of the equivalent class containing \( \sigma \). Let
\[ T(\sigma) = \sigma_1 + \cdots + \sigma_{p(\sigma)} \] (5.15)
be the sum of elements in a least period of \( \sigma \). Since \( p(\sigma) = d(\sigma) \) and \( \frac{n}{T(\sigma)} = \frac{\ell(\sigma)}{p(\sigma)} \), we have
\[ T(\sigma) = \frac{n d(\sigma)}{\ell(\sigma)}. \] (5.16)
For example, if \( \sigma = (4, 3, 4, 3, 4, 3, 4, 3) \), then \( n = 28, p(\sigma) = d(\sigma) = 2, \ell(\sigma) = 8 \), and \( T(\sigma) = 4 + 3 = 7 \). If \( \sigma = (3, 3, 2, 2) \), then \( n = 10, p(\sigma) = d(\sigma) = \ell(\sigma) = 4 \) and \( T(\sigma) = 3 + 3 + 2 + 2 = 10 \).

For \( c = (c_1, \ldots, c_n) \geq 0 \) and \( c_n \neq 0 \), we associate a unique composition
\[ \pi(c) = (\pi_1, \ldots, \pi_\ell) \] (5.17)
to \( c \) as follows. Assume that \( \ell_1 \) is the smallest index such that \( c_1 + \cdots + c_{\ell_1} = 0 \) and \( c_{\ell_1} < 0 \). Then let \( \pi_1 = \ell_1 \). Let \( \ell_2 \) be the smallest index such that \( c_{\ell_1+1} + \cdots + c_{\ell_2} = 0 \) and \( c_{\ell_2} < 0 \). Then let \( \pi_2 = \ell_2 - \ell_1 \). Continue this process, we can obtain \( \pi(c) \) eventually. Since \( c \geq 0 \) and \( c_n \neq 0 \), it is easy to see that \( \pi_i \geq 2 \) for each \( 1 \leq i \leq \ell \) and \( c \in \mathcal{T}_{a,b}^{\pi(c)} \).
Moreover,
\[ \{\pi(c) \mid c \geq 0, c_n \neq 0\} = \Gamma_n \cup \{(n)\}. \] (5.18)
For example, let \( c = (0, 0, 3, -1, 0, -2, 0, 1, -1, 0, 1, 1, -2, 1, -1) \). Then \( \pi(c) = (6, 3, 4, 2) \), see Figure 5.3 for an illustration of the path \( c \).
Given a composition \( \sigma = (\sigma_1, \ldots, \sigma_s) \), arrange \( \sigma_1, \ldots, \sigma_s \) on a directed circle, such that there is a directed edge from \( \sigma_i \) to \( \sigma_{i+1} \) for \( 1 \leq i \leq s - 1 \), and a directed edge from \( \sigma_s \) to \( \sigma_1 \). If \( \sigma = (n) \) has only one part, then there is a directed loop on the node \( n \). We call such a configuration the circle representation of \( \sigma \), denoted as \( G(\sigma) \). We view all the edges in \( G(\sigma) \) different, even if they have the same nodes and directed edges.

For example, Figure 5.6 displays the circle representations of \((8)\), \((4, 4)\), \((4, 3, 1)\), \((4, 1, 3)\), respectively. There are two different edges in Figure 5.6(b).

By contracting a directed edge, say \( \sigma_i \rightarrow \sigma_{i+1} \), of \( G(\sigma) \), we mean delete this edge and form a new node labeled by \( \sigma_i + \sigma_{i+1} \), and keep all the other edges unchanged. Since all the edges in \( G(\sigma) \) are viewed different, it is quite possible that different ways of contracting the edges lead to the same circle representation. For example, Figure 5.6(b) can be viewed as obtained by contracting the directed edge \( 3 \rightarrow 1 \) in Figure 5.6(c), or contracting the directed edge \( 1 \rightarrow 3 \) in Figure 5.6(d). Moreover, we can obtain Figure 5.6(a) twice by contracting the two different edges of Figure 5.6(b).

If \( \sigma \) and \( \sigma' \) have the same circle representation, then \( \sigma \sim \sigma' \) are equivalent. It is easy to see that \( T(\sigma) \ast \mathcal{C}_{c, b}^n \) and \( T(\sigma') \ast \mathcal{C}_{c', b}^n \) contain the same number of appearances of each \( c \in \mathcal{F}_{n}^{a, b} \). Given a composition \( \tau \), after contracting some edges of \( G(\tau) \), we obtain a new circle representation, which is \( G(\sigma) \) for some composition \( \sigma \). To read off a specific \( \sigma \), we can choose any node in \( G(\sigma) \) as the first element \( \sigma_1 \), and then read off \( \sigma_2, \sigma_3, \ldots \) of \( \sigma \) from \( G(\sigma) \) clock-wisely.

**Proposition 5.4.** Let \( c \geq 0 \) and \( c_n \neq 0 \). Then the number of appearances of \( c \) in \( T(\sigma) \ast \mathcal{C}_{c, b}^n(\sigma) \) is equal to the number of ways that \( G(\sigma) \) can be obtained by contracting edges in \( G(\pi(c)) \).

**Proof.** Let \( c = (c_1, \ldots, c_n) \) and \( \pi(c) = (\pi_1, \ldots, \pi_\ell) \). We first show that if there is a way of contracting edges of \( G(\pi(c)) \) to obtain \( G(\sigma) \), then \( c \) appears at least once in \( T(\sigma) \ast \mathcal{C}_{c, b}^n(\sigma) \). Then we show that if \( c \) appears once in \( T(\sigma) \ast \mathcal{C}_{c, b}^n(\sigma) \), then there is a way of contracting edges of \( G(\pi(c)) \) to obtain \( G(\sigma) \).

A contracting of edges of \( G(\pi(c)) \) is equivalent to adding consecutive elements of \( \pi(c) \) together, where we arrange \( \pi(c) \) on a circle, thus \( \pi_1 \) and \( \pi_\ell \) can be added together. After
contracting edges of $G(\pi(c))$, we obtain $G(\sigma)$. Since $\sigma$ may have a period, to read off $\sigma$, we need to locate a position of $\sigma_1$, and then read off $\sigma_2, \sigma_3$, etc. from $G(\sigma)$ clock-wisely. There are two cases, depending on whether $\pi_1$ and $\pi_\ell$ are added together or not.

Case 1. There exist $1 \leq j < i \leq \ell$ such that $\pi_i, \ldots, \pi_\ell, \pi_1, \ldots, \pi_j$ are added together. Let $\sigma_1 = \pi_i + \cdots + \pi_\ell + \pi_1 + \cdots + \pi_j$.

Case 2. There exists $1 \leq i \leq \ell$ such that $\pi_i, \ldots, \pi_\ell$ are added together. Let $\sigma_1 = \pi_i + \cdots + \pi_\ell$.

For both cases, let

$$c' = L^{\pi_i + \cdots + \pi_\ell}(c).$$

One can check that $c' \in \overline{C}_\alpha$. Since $\pi_i + \cdots + \pi_\ell \leq T(\sigma)$ and $c = R^{\pi_i + \cdots + \pi_\ell}(c')$, we find that $c$ will appear in $R_{c'}^{T(\sigma)} = (c', R(c'), \ldots, R^{T(\sigma)-1}(c'))$ at least once.

In the following, we show that if $c$ appears in $T(\sigma) * \overline{C}_\alpha$ once, then there is a way of contracting edges of $G(\pi(c))$ to obtain $G(\sigma)$.

Suppose that there exists $c' \in \overline{C}_\alpha$ such that $R_{c'}^{T(\sigma)} = (c', R(c'), \ldots, R^{T(\sigma)-1}(c'))$ contains $k_0$ copies of $c$. We aim to construct $k_0$ different ways of contracting edges of $G(\pi(c))$ to obtain $G(\sigma)$. Let $0 \leq i_1 < i_2 < \cdots < i_{k_0} \leq T(\sigma) - 1$ such that

$$R^{i_1}(c') = R^{i_2}(c') = \cdots = R^{i_{k_0}}(c') = c.$$

Then $c' = L^{i_1}(c) = \cdots = L^{i_{k_0}}(c)$ and there exist $i'_1 < i'_2 < \cdots < i'_{k_0}$ such that

$$R^{i'_1}(\pi(c')) = R^{i'_2}(\pi(c')) = \cdots = R^{i'_{k_0}}(\pi(c')) = \pi(c).$$

Since $c' \in \overline{C}_\alpha$, we can add consecutive elements of $\pi(c') = (\pi'_1, \ldots, \pi'_z)$ to obtain $\sigma = (\sigma_1, \ldots, \sigma_z)$. If we require that $\pi'_1$ and $\pi'_z$ can not be added together, then there are integers $j_1 < j_2 < \cdots < j_s = z$ such that

$$\sigma_1 = \pi'_1 + \cdots + \pi'_{j_1}, \sigma_2 = \pi'_{j_1+1} + \cdots + \pi'_{j_2}, \ldots, \sigma_s = \pi'_{j_{s-1}+1} + \cdots + \pi'_{j_s}.$$

For the $k_0$ appearances of $c$ in $R_{c'}^{T(\sigma)}$, we can construct $k_0$ ways of contracting edges of $G(\pi(c))$ as following. For each $j \in \{i'_1, \ldots, i'_{k_0}\}$, we can add the elements in $L^j(\pi(c))$ with the same positions of elements in $\pi(c')$. More precisely, for $j \in \{i'_1, \ldots, i'_{k_0}\}$, let

$$\sigma_1 = \pi_{1-j} + \cdots + \pi_{j_1-j}, \sigma_2 = \pi_{j_1+1-j} + \cdots + \pi_{j_2-j}, \ldots, \sigma_s = \pi_{j_{s-1}+1-j} + \cdots + \pi_{j_s-j},$$

where the indices are taken modulo $j_s = z$. It is easy to see that these $k_0$ constructions correspond to $k_0$ different ways of contracting edges of $G(\pi(c))$.

For example, let $\pi(c) = (2, 3, 2, 2, 3, 2, 2, 3, 2, 3, 2, 3, 2, 3, 2, 3, 2, 3, 2, 3, 2, 3, 2, 3, 2, 3, 2, 3)$, $\sigma = (14, 14)$ and $\pi(c') = (2, 2, 3, 2, 2, 3, 2, 2, 3, 2, 3, 2, 3, 2, 3, 2, 3, 2, 3, 2, 3, 2, 3, 2, 3, 2, 3)$. Then $i_1 = 2, i_2 = 9$, i.e., $R^9(c') = R^9(c') = c$ and $i'_1 = 1, i'_2 = 4$. Since $j_1 = 6, j_2 = 12$, for $j = i'_1 = 1$, let $\sigma_1 = \pi_{12} + \pi_1 + \cdots + \pi_5$ and $\sigma_2 = \pi_6 + \cdots + \pi_{11}$. For $j = i'_2 = 4$, let $\sigma_1 = \pi_9 + \cdots + \pi_{12} + \pi_1 + \pi_2$ and $\sigma_2 = \pi_3 + \cdots + \pi_8$.

Moreover, if there is another $c'' \in \overline{C}_\alpha$ such that $R_{c''}^{T(\sigma)} = (c'', R(c''), \ldots, R^{T(\sigma)-1}(c''))$ contains $m_0$ copies of $c$. Let

$$R^{i'_1}(c'') = R^{i'_2}(c'') = \cdots = R^{i'_{m_0}}(c'') = c.$$
Then we must have \( \{i_1, \ldots, i_{k_0}\} \cap \{i_1', \ldots, i_{m_0}'\} = \emptyset \). In fact, if there exists \( j \in \{i_1, \ldots, i_{k_0}\} \cap \{i_1', \ldots, i_{m_0}'\} \), then \( R^j(c') = R^j(c'') = c \), this implies \( c' = c'' \). By the same constructions for \( c' \), we can obtain \( m_0 \) ways of contracting edges of \( G(\pi(c)) \), different from the above \( k_0 \) ways. This completes the proof. 

Proof of Proposition 5.3 Assume that \( \pi(c) \) has \( m = \ell(\pi(c)) \) parts, i.e., \( G(\pi(c)) \) has \( m \) edges. It is clear that the number of appearances of \( c \) in \( (-1)^{m-1} T(\pi(c)) * \overline{\sigma}_{\pi(c)} \) is \( (-1)^{m-1} \), corresponding to contracting 0 edges in \( G(\pi(c)) \). By Proposition 5.4 if we contract one edge of \( G(\pi(c)) \), then \( c \) will be enumerated by \( m \) times, with sign \( (-1)^m \). Similarly, if we contract any two edges of \( G(\pi(c)) \), \( c \) will be enumerated by \( \binom{m}{2} \) times with sign \( (-1)^{m+1} \), etc. Therefore, the total number of appearances of \( c \) is
\[
(-1)^{m-1} \binom{m}{0} + (-1)^m \binom{m}{1} + (-1)^{m+1} \binom{m}{2} + \cdots + (-1)^{m-1} \binom{m}{m-1} = 1,
\]
as required. 

For the running example in (5.11), given \( c = (1, -1, 2, -2, 1, -1) \), we have \( \pi(c) = (2, 2, 2) \). We need to contract edges of \( G((2, 2, 2)) \) to obtain circle representations of \((6), (4, 2), (3, 3), (2, 2, 2) \). For \( \sigma = (6) \), there are 3 ways of contracting 2 edges among all 3 edges in \( G((2, 2, 2)) \) to obtain \( G((6)) \), so \( c \) is counted \( \binom{3}{2} \) times in \( 6 * C^2_3 \). For \( \sigma = (4, 2) \), there are 3 ways to contract one edge of \( G((2, 2, 2)) \) to obtain \( G((4, 2)) \), so \( c \) is counted \( \binom{3}{1} \) times in \( 6 * C^2_3 \) with a minus sign. Similarly, we can contract 0 edges in \( G((2, 2, 2)) \) to obtain \( G((2, 2, 2)) \), which means \( c \) is counted \( \binom{3}{0} \) times in \( 2 * C^2_3 \). We can not obtain \( G((3, 3)) \) by contracting any edges of \( G((2, 2, 2)) \). Consequently, the total number of appearances of \( c \) in (5.11) is
\[
\binom{3}{2} - \binom{3}{1} + \binom{3}{0} = 1.
\]

In the following, we consider the case \( c = (c_1, \ldots, c_n) < 0 \).

Proposition 5.5. If \( c < 0 \), then \( c \) appears in (5.10) exactly once.

Proof. For \( 1 \leq j \leq n \), let \( d_j = c_1 + \cdots + c_j \). Let \( k \) be the largest index such that \( d_k = \min\{d_1, \ldots, d_n\} \). Define
\[
c' = L^{n-k}(c).
\]
It is easy to see that \( c' \in \overline{C}^b_n \) and \( n-k \) is the smallest integer \( m \) such that \( L^m(c) \geq 0 \). By Proposition 5.3, the total number of appearances of \( c' \) in (5.10) is 1. For example, let \( c = (2, -4, 7, -7, 4, -4, 4, -2) \). Then \( k = 6 \) and \( c' = (4, -2, 2, -4, 7, -7, 4, -4) \). The paths \( c \) and \( c' \) are displayed in Figure 5.7.

For any path \( c'' \in \overline{C}^b_n \), let
\[
R^{T(\sigma)}(c'') = (c'', R(c''), R^2(c''), \ldots, R^{T(\sigma)-1}(c'')).
\]
We aim to show that the number of appearances of \( c \) and \( c' \) in \( R^{T(\sigma)}(c'') \) are the same. We first show that if there are two paths \( c \) in \( R^{T(\sigma)}(c'') \), then there is a path \( c' \) between them. Suppose that there exist \( i < j \) such that \( R^i(c''') = R^j(c'') = c \). Then
\[
c' = L^{n-k}(c) = L^{n-k}(R^i(c''')) = R^{j-(n-k)}(c''),
\]
(5.19)
Figure 5.7: The paths $c$ and $c'$.

Since $j - i > n - k$, we have $i < j - (n - k) < j$, which means that $c'$ appears between $R^j(c'')$ and $R^j(c''')$ at least once. Similarly, we can obtain that there is a $c$ between any two $c'$ in $R^{T(\sigma)}(c''')$.

At this moment, we can only conclude that the number of appearances of $c$ and $c'$ in $R^{T(\sigma)}(c''')$ are equal or differ by 1. Let $i_0$ be the smallest index such that $R^{i_0}(c') = c$ and $j_0$ be the largest index such that $R^{j_0}(c') = c$. Since $c'' = L^{i_0}(c) \geq 0$, we find that $i_0 \geq n - k$. Since $c' = L^{n-k}(c)$, we find that $c'$ must appear in $(c'', R(c''), \ldots, R^{i_0}(c'''))$. Similarly, we can show that there is a $c$ appearing to the right of the right-most $c'$. This completes the proof. 

Finally, we consider the case $c \geq 0, c_n = 0$.

**Proposition 5.6.** If $c \geq 0, c_n = 0$, then $c$ appears in (5.10) exactly once.

**Proof.** Let $q$ be the largest index such that $c_q \neq 0$ and $c_{q+1} = \cdots = c_n = 0$. Define 

$$c' = L^{n-q}(c) = (c_{q+1}, \ldots, c_n, c_1, \ldots, c_q).$$

Similar to the proof of Proposition 5.5, we can show that the number of appearances of $c$ and $c'$ are exactly the same in each $T(\sigma) * \mathcal{C}_{a,b}^\sigma$. 

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### 6 Sparse paving Schubert matroids

In this section, we study sparse paving Schubert matroids. Recall that a matroid $M$ is sparse paving if and only if every subset of cardinality $rk(M)$ is either a basis or a circuit-hyperplane.

Let $r = (k - 1, 1, 1, n - k - 1)$, where $n > k \geq 2$, we obtain a Schubert matroid, denoted as $Sp_{k,n}$. As will be shown, $Sp_{k,n}$ is sparse paving, and a Schubert matroid is sparse paving if it is either a uniform matroid or a uniform matroid with one basis removed. Equivalently, a sparse paving Schubert matroid is a notch matroid with the upper bounding path $U = N^{n-k}E^k$, see Bon and de Mier [3, Definition 8.1].

**Proposition 6.1.** A Schubert matroid $SM_n(S)$ of rank $n-k$ is sparse paving if and only if it is uniform or $r(S) = (k - 1, 1, 1, n - k - 1)$, i.e., 

$$S = \{k + 1, \ldots, n\} \text{ or } \{k, k+2, \ldots, n\}.$$

**Proof.** If $S = \{k + 1, \ldots, n\}$, then $SM_n(S)$ is the uniform matroid $U_{n-k,n}$, which is sparse paving by definition. Suppose that $S = \{k, k+2, \ldots, n\}$. Then we aim to show that
every \((n-k)\)-subset of \([n]\) is either a basis or a circuit-hyperplane. In fact, it is easy to see that there is exactly one \((n-k)\)-subset of \([n]\) which is not a basis of \(\text{SM}_n(S)\), i.e., \(T = \{k+1, \ldots, n\}\). It is also straightforward to check that \(T\) is both a circuit and a flat. Moreover, \(T\) is also a hyperplane since \(\text{rk}_S(T) = n - k - 1\). Thus \(\text{SM}_n(S)\) is sparse paving.

On the contrary, suppose that a Schubert matroid \(M = \text{SM}_n(S)\) is sparse paving and it is not uniform. Then every subset of cardinality \(n-k\) is either a basis or a circuit-hyperplane. Since \(M\) is not uniform, it has a circuit-hyperplane. It is easy to see that \(M\) is connected, by [6, Theorem 8.3], \(M\) is a notch matroid. Thus \(r(S) = (k-1, 1, 1, n-k-1)\).

By the proof of Proposition [6,1] if \(S = \{k, k+2, \ldots, n\}\), then its corresponding sparse paving Schubert matroid \(\text{Sp}_{k,n}\) has exactly one circuit-hyperplane, i.e., \(\{k+1, \ldots, n\}\). The following lemma follows from Ferroni [17, Corollary 4.6].

**Lemma 6.2.** We have

\[ i(\text{Sp}_{k,n}, t) = i(\text{Sp}_{n-k,n}, t) = i(U_{k,n}, t) - i(T_{k,n}, t-1). \]

By Lemma 6.2 we find that \(i(U_{k,n}, t) - i(\text{Sp}_{k,n}, t) = i(T_{k,n}, t-1)\). By Ferroni [16, Theorem 1.6], \(i(T_{k,n}, t-1)\) has positive coefficients. Therefore,

\[ i(\text{Sp}_{k,n}, t) \leq i(U_{k,n}, t). \]

To finish the proof of Theorem [17,10] we still need to show that \(i(T_{k,n}, t) \leq i(\text{Sp}_{k,n}, t)\), that is,

\[ i(U_{k,n}, t) - i(T_{k,n}, t-1) - i(T_{k,n}, t) \]

has positive coefficients. We aim to prove two slightly stronger statements, i.e.,

\[ i(U_{k,n}, t) \geq i(U_{2,n}, t) \]

and

\[ i(U_{2,n}, t) - i(T_{k,n}, t-1) - i(T_{k,n}, t) \geq 0. \]

In order to prove (6,1), we need the combinatorial interpretation of the coefficients of \(i(U_{k,n}, t)\). For any \(m \geq 0\), Ferroni [15, Theorem 4.3] showed that the coefficient of \(t^m\) in \(i(U_{k,n}, t)\) is

\[ [t^m]i(U_{k,n}, t) = \frac{1}{(n-1)!} \sum_{j=0}^{k-1} W(j, n, m+1) A(m, k-j-1), \]

where \(W(j, n, m+1)\) are the weighted Lah numbers and \(A(m, k-j-1)\) are the Eulerian numbers. In particular, \(W(0, n, k) = \left[\begin{array}{c} n \\ k \end{array}\right]\) is the unsigned Stirling number of the first kind. The following properties of \(\left[\begin{array}{c} n \\ k \end{array}\right]\) are well known,

\[ t(t+1) \cdots (t+n-1) = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right] t^k. \]
and
\[
\left[ \frac{n}{k} \right] = (n-1) \left[ \frac{n-1}{k} \right] + \left[ \frac{n-1}{k-1} \right],
\] (6.4)

see, for example, Stanley [30].

**Lemma 6.3.** For \(3 \leq k \leq \frac{n}{2}\), we have the coefficient-wise relation
\[
\iota(U_{2,n}, t) \leq \iota(U_{k,n}, t).
\]

**Proof.** By (6.3), we have
\[
[t^m]i(U_{2,n}, t) = \frac{1}{(n-1)!}(W(0, n, m+1)A(m, 1) + W(1, n, m+1)A(m, 0)).
\]

There are four cases to consider.

Case 1. \(k < m\). By (6.3),
\[
[t^m]i(U_{k,n}, t) \geq \frac{1}{(n-1)!}(W(0, n, m+1)A(m, k-1) + W(1, n, m+1)A(m, k-2)).
\]

Since the Eulerian numbers are unimodal and \(k-1 \geq 2\), we have \(A(m, k-1) \geq A(m, 1)\) and \(A(m, k-2) \geq A(m, 0)\). So \([t^m]i(U_{k,n}, t) \geq [t^m]i(U_{2,n}, t)\).

Case 2. \(k = m = 3\). When \(n = 4, 5\), the lemma holds obviously. When \(n \geq 6\), it is easy to see that
\[
W(0, n, 4) \leq W(1, n, 4) \text{ and } W(1, n, 4) \leq W(2, n, 4).
\]

By (6.3),
\[
[t^3]i(U_{3,n}, t) = \frac{1}{(n-1)!}(W(0, n, 4) + 4W(1, n, 4) + W(2, n, 4))
\]
\[
\geq \frac{1}{(n-1)!}(4W(1, n, 4) + W(2, n, 4))
\]
\[
\geq \frac{1}{(n-1)!}(4W(0, n, 4) + W(1, n, 4))
\]
\[
= [t^3]i(U_{2,n}, t).
\]

Case 3. \(k = m \geq 4\). By (6.3),
\[
[t^m]i(U_{k,n}, t) \geq \frac{1}{(n-1)!}(W(1, n, m+1)A(m, m-2) + W(m-2, n, m+1)A(m, 1)).
\]

Since \(A(m, m-2) \geq A(m, 0) = 1\), it is enough to show that
\[
W(m-2, n, m+1) \geq W(0, n, m+1).
\] (6.5)

We shall give a combinatorial proof of (6.5).
Denote \( \mathcal{W}(\ell, n, m) \) by the set of partitions of weight \( \ell \) of \([n]\) into \( m \) linearly ordered blocks, and let \( W(\ell, n, m) \) be the cardinality of \( \mathcal{W}(\ell, n, m) \), see Ferroni [15]. For a partition \( \pi \) with linearly ordered blocks, the weight of \( \pi \) is \( w(\pi) = \sum_{B \in \pi} w(B) \), where \( w(B) \) is the number of elements in \( B \) that are smaller than the first element in \( B \). In order to prove (6.5), we construct an injection from \( \mathcal{W}(0, n, m+1) \) to \( \mathcal{W}(m-2, n, m+1) \).

By definition, \( \mathcal{W}(0, n, m+1) \) is the set of partitions of \([n]\) into \( m+1 \) blocks, the elements of each block are arranged increasingly. Suppose that \( \tau \in \mathcal{W}(0, n, m+1) \). We aim to construct a partition \( \tau' \in \mathcal{W}(m-2, n, m+1) \) from \( \tau \). Let \( B_1, \ldots, B_j \) be the blocks of \( \tau \) having more than one element and the smallest element of \( B_i \) is smaller than the smallest element of \( B_{i+1} \) for \( 1 \leq i \leq j-1 \). Apparently,

\[
|B_1| + \cdots + |B_j| = n - (m + 1 - j).
\]

For any linearly ordered block \( B = (b_1, \ldots, b_s) \) with \( w(B) = 0 \), that is, \( b_1 \) is the smallest element of \( B \). Assume that \( b_m \) is the largest element of \( B \). Let \( B' \) be obtained from \( B \) by cyclically shifting \( b_m \) to the first position, i.e., \( B' = (b_m, \ldots, b_s, b_1, \ldots, b_{m-1}) \). Then \( w(B') = |B| - 1 \). Therefore,

\[
w(B'_1) + \cdots + w(B'_j) = |B_1| - 1 + \cdots + |B_j| - 1 = n - (m + 1).
\]

Since \( k = m \leq \frac{n}{2} \), we have \( n - (m + 1) \geq m - 1 \). Thus we can construct a partition \( \tau' \in \mathcal{W}(m-2, n, m+1) \) with weight \( m - 2 \) from \( \tau \) as follows. There exists some index \( i \) (\( 1 \leq i < j \)) such that

\[
w(B'_1) + \cdots + w(B'_i) \leq m - 2 \text{ and } w(B'_i) + \cdots + w(B'_{i+1}) > m - 2.
\]

We can cyclically shift a suitable element of \( B_{i+1} \) to the first position to obtain a new block \( B''_{i+1} \), such that

\[
w(B'_1) + \cdots + w(B'_i) + w(B''_{i+1}) = m - 2.
\]

Keep the other blocks of \( \tau' \) the same with those of \( \tau \). It is easy to see that this construction is an injection from \( \mathcal{W}(0, n, m+1) \) to \( \mathcal{W}(m-2, n, m+1) \). This completes the proof of (6.5).

Case 4. \( k > m \). By (6.3),

\[
[t^m][U_{k,n,t}] \geq \frac{1}{(n-1)!}(W(k-1, n, m+1)A(m, 0) + W(k-2, n, m+1)A(m, 1)).
\]

We aim to show that

\[
\varphi(n, k, m) := W(k-2, n, m+1) - W(0, n, m+1) \geq 0 \quad (6.6)
\]

and

\[
\psi(n, k, m) := W(k-1, n, m+1) - W(1, n, m+1) \geq 0. \quad (6.7)
\]

By Ferroni [15, Remark 3.9], \( W(\ell, n, m) = W(n - \ell, n, m) \), then we have

\[
\varphi(n, k, m) = W(n - m - k + 1, n, m+1) - W(0, n, m+1).
\]
Since $m < k \leq \frac{n}{2}$, we find $n - (m + 1) \geq n - m - k + 1 > 0$. By the same arguments in the proof of (6.5), we can conclude that $W(n - m - k + 1, n, m + 1) \geq W(0, n, m + 1)$. Thus $\phi(n, k, m) \geq 0$.

Similarly, since $n - (m + 1) \geq k - 1 > 1$, we can also utilize the same arguments in the proof of (6.5) to show that $W(k - 1, n, m + 1) \geq W(1, n, m + 1)$. That is, $\psi(n, k, m) \geq 0$. This completes the proof.

**Lemma 6.4.** For positive integers $m, n$, we have

$$\left[ \frac{n+1}{m+1} \right] = \sum_{j=0}^{n} j! \left\lfloor \frac{n}{j} \right\rfloor!.$$

**Proof.** Recall that $\left[ \frac{n+1}{m+1} \right]$ represents the number of permutations on $[n+1]$ with $m+1$ cycles. Alternatively, we can first choose $j$ numbers from $1, 2, \ldots, n$ to form $m$ cycles, there are $\binom{n}{j} \left[ \frac{j}{m} \right]$ such ways, then the left $n-j$ numbers and the number $n+1$ form another cycle, there are $(n-j)!$ such ways.

Now we are ready to give a proof of Theorem 1.10.

**Proof of Theorem 1.10.** By Lemma 6.2 and Lemma 6.3, it suffices to show that

$$i(U_{2,n}, t) - i(T_{k,n}, t - 1) - i(T_{k,n}, t)$$

has positive coefficients. By Lemma 6.2 again, it is enough to only consider $k \leq \frac{n}{2}$.

We first simplify $[t^m]i(U_{2,n}, t)$, $[t^m]i(T_{k,n}, t)$ and $[t^m]i(T_{k,n}, t - 1)$ for $0 \leq m \leq n$, separately.

By Ferroni [15], Corollary 3.13,

$$W(\ell, n, m) = \sum_{j=0}^{\ell} \sum_{i=0}^{n-m} (-1)^{i+j} \binom{n}{j} \left[ \frac{m+\ell-j-1}{j-i} \right] \left[ \frac{n-j}{m+i-j} \right].$$

Then

$$W(1, n, m + 1) = (m + 1) \left\lfloor \frac{n}{m+1} \right\rfloor - n \left\lfloor \frac{n-1}{m} \right\rfloor. \quad (6.8)$$

Since $A(m, 1) = 2^n - m - 1$ and $A(m, 0) = 1$, by (6.3) and (6.8), we have

$$(n-1)! [t^m]i(U_{2,n}, t)$$

$$= W(0, n, m + 1)A(m, 1) + W(1, n, m + 1)A(m, 0)$$

$$= 2^m \left\lfloor \frac{n}{m+1} \right\rfloor - n \left\lfloor \frac{n-1}{m} \right\rfloor \quad (6.9)$$

$$= 2^m \left( (n-1) \left\lfloor \frac{n-1}{m+1} \right\rfloor + \left\lfloor \frac{n-1}{m} \right\rfloor \right) - n \left( (n-2) \left\lfloor \frac{n-2}{m} \right\rfloor + \left\lfloor \frac{n-2}{m-1} \right\rfloor \right)$$

$$= (n-2) \left( 2^m \left\lfloor \frac{n-1}{m+1} \right\rfloor - (n-1) \left\lfloor \frac{n-2}{m} \right\rfloor \right) + 2 \left( 2^m \left\lfloor \frac{n-1}{m} \right\rfloor - (n-2) \left\lfloor \frac{n-2}{m-1} \right\rfloor \right)$$
\[ + 2^m \left[ \frac{n-1}{m+1} \right] - (n-2) \left[ \frac{n-2}{m} \right] + (n-2) \left[ \frac{n-2}{m-1} \right] \]
\[ \geq (n-2) \left( 2^m \left[ \frac{n-1}{m+1} \right] - (n-1) \left[ \frac{n-2}{m} \right] \right) + 2 \left( 2^{m-1} \left[ \frac{n-1}{m} \right] - (n-1) \left[ \frac{n-2}{m-1} \right] \right) \]
\[ + 2^m \left[ \frac{n-1}{m+1} \right] - (n-2) \left[ \frac{n-2}{m} \right] . \]  
(6.10)

Replacing \( t \) with \( t - 1 \) in (L.5) and extracting the coefficient of \( t^m \), we find

\[
(n-1)! \cdot [t^m] i(T_{k,n}, t - 1) \\
= \sum_{j=0}^{k-1} \sum_{l=0}^{j} \frac{(k-1)!}{j!} \left( \frac{n-k-1+j}{n-k-1} \right) \left[ \frac{n-k}{m-l} \right] \\
= \sum_{j=0}^{k-1} \sum_{l=0}^{j} \frac{(k-1)!}{j!} \left( \frac{n-k-1+j}{n-k-1} \right) \left( \frac{n-k-2+j}{n-k-1} \right) \left[ \frac{n-k}{m-l} \right] .
\]
(6.11)

Since \( n-k-1+j \leq n-2 \) and \( \frac{n-k-1+j}{n-k-1} \leq \frac{n-2}{n-k-1} \leq 2 \), we have

\[
\frac{n-k-1+j}{n-k-1} \left[ \frac{n-k}{m-l} \right] = \frac{n-k-1+j}{n-k-1} \left( \frac{n-k-1}{m-l} \right) + \left[ \frac{n-k-1}{m-l-1} \right] \\
\leq (n-2) \left[ \frac{n-k-1}{m-l} \right] + 2 \left[ \frac{n-k-1}{m-l-1} \right] .
\]

Then we conclude that

\[
(n-1)! \cdot [t^m] i(T_{k,n}, t - 1) \\
\leq \sum_{j=0}^{k-1} \sum_{l=0}^{j} \frac{(k-1)!}{j!} \left( \frac{n-k-2+j}{n-k-1} \right) \left( n-2 \right) \left[ \frac{n-k-1}{m-l} \right] + 2 \left[ \frac{n-k-1}{m-l-1} \right] \left[ \frac{j}{l} \right] .
\]
(6.12)

Similarly, by (L.5),

\[
(n-1)! \cdot [t^m] i(T_{k,n}, t) \\
= \sum_{j=0}^{k-1} \sum_{l=0}^{j} \frac{(k-1)!}{j!} \left( \frac{n-k-1+j}{n-k-1} \right) \left[ \frac{j+1}{l+1} \right] \left[ \frac{n-k+1}{m-l+1} \right] \\
= \sum_{j=0}^{k-1} \sum_{l=0}^{j} \frac{(k-1)!}{j!} \left( \frac{n-k-1+j}{n-k-1} \right) \left( \frac{n-k-2+j}{n-k-1} \right) \left[ \frac{j+1}{l+1} \right] \left[ \frac{n-k+1}{m-l+1} \right] .
\]
(6.13)

Since \( \frac{n-k-1+j}{n-k-1} (n-k) = (n-2) + (j-k+2 + \frac{j}{n-k-1}) \) and \( \frac{n-k-1+j}{n-k-1} \leq 2 \), we have

\[
\frac{n-k-1+j}{n-k-1} \left[ \frac{n-k+1}{m-l+1} \right] = \frac{n-k-1+j}{n-k-1} \left( n-k \right) \left[ \frac{n-k}{m-l+1} \right] + \left[ \frac{n-k}{m-l} \right] .
\]
\[
\leq (n-2)\left[ \frac{n-k}{m-l+1} \right] + \left( j-k+2 + \frac{j}{n-k-1} \right) \left[ \frac{n-k}{m-l+1} \right] + 2\left[ \frac{n-k}{m-l} \right].
\]

Therefore, we find
\[
(n-1)! \cdot [t^m]i(T_{k,n}, t)
\leq \sum_{j=0}^{k-1} \sum_{l=0}^{j} \frac{(k-1)!}{j!} \left( n-k-2 + j \right) \left( n-2 \left[ \frac{n-k}{m-l+1} \right] + 2\left[ \frac{n-k}{m-l} \right] \right) \left[ \frac{n-k}{m-l+1} \right] + h(n, m),
\]

where
\[
h(n, m) = \sum_{j=0}^{k-1} \sum_{l=0}^{j} \frac{(k-1)!}{j!} \left( n-k-2 + j \right) \left( j-k+2 + \frac{j}{n-k-1} \right) \left[ \frac{n-k}{m-l+1} \right] + h(n, m),
\]

(6.14)

Consequently, by (6.9), (6.11) and (6.13), we have
\[
[t^m](i(U_{2,n}, t) - i(T_{k,n}, t-1) - i(T_{k,n}, t))
= \frac{1}{(n-1)!} \left( 2^m \left[ \frac{n}{m+1} \right] - n \left[ \frac{n-1}{m} \right] \right)
- \frac{1}{(n-1)!} \sum_{j=0}^{k-1} \sum_{l=0}^{j} \frac{(k-1)!}{j!} \left( n-k-1 + j \right) \left[ \frac{j}{l} \left[ \frac{n-k}{m-l} \right] + \left[ j+1 \right] \left[ \frac{n-k+1}{m-l+1} \right] \right).
\]

Denote by
\[
f(n, m) = 2^m \left[ \frac{n}{m+1} \right] - n \left[ \frac{n-1}{m} \right]
- \sum_{j=0}^{k-1} \sum_{l=0}^{j} \frac{(k-1)!}{j!} \left( n-k-1 + j \right) \left[ \frac{j}{l} \left[ \frac{n-k}{m-l} \right] + \left[ j+1 \right] \left[ \frac{n-k+1}{m-l+1} \right] \right).
\]

We aim to show that \( f(n, m) \geq 0 \) for \( n \geq 4 \) and \( m \geq 0 \) by induction on \( n \).

It is easy to check that \( f(4, m) \geq 0 \) for any \( m \geq 0 \). Moreover, since
\[
\sum_{j=0}^{k-1} \left( n-k-1 + j \right) = \frac{k}{n-k} \binom{n-1}{k},
\]
we have
\[
f(n, 0) = (n-1)! - (k-1)!(n-k)! - (n-k)!(k-1)! \sum_{j=1}^{k-1} \left( n-k-1 + j \right)
= (n-1)! - (k-1)!(n-k)! - (n-k)!(k-1)! \left( \frac{k}{n-k} \binom{n-1}{k} - 1 \right)
= 0.
\]
Assume that \( f(n-1, m) \geq 0 \) for \( n \geq 5 \) and \( m \geq 0 \). For \( m \geq 1 \), by (6.10), (6.12) and (6.14), we derive that

\[
f(n, m) \geq (n-2) \left( 2^m \binom{n-1}{m+1} - (n-1) \binom{n-2}{m} \right) + 2 \left( 2^{m-1} \binom{n-1}{m} - (n-1) \binom{n-2}{m-1} \right)
- \sum_{j=0}^{k-1} \sum_{l=0}^{j} \frac{(k-1)!}{j!} \left( j - k + 2 \right) \binom{n-k-2+j}{j} \binom{n-k-1}{l} \binom{n-k}{m-l+1} - m + 1
- \sum_{j=0}^{k-1} \sum_{l=0}^{j} \frac{(k-1)!}{j!} \left( j - k + 2 \right) \binom{n-k-2+j}{j} \binom{n-k-1}{l} \binom{n-k}{m-l+1} - m + 1
+ 2^m \binom{n-1}{m+1} - (n-2) \binom{n-2}{m} - h(n, m)
= (n-2) f(n-1, m) + 2 f(n-1, m-1) + 2^m \binom{n-1}{m+1} - (n-2) \binom{n-2}{m} - h(n, m)
\geq 2^m \binom{n-1}{m+1} - (n-2) \binom{n-2}{m} - h(n, m).
\]

To complete the proof, let

\[
g(n, m) = 2^m \binom{n-1}{m+1} - (n-2) \binom{n-2}{m} - h(n, m).
\]

We aim to prove that \( g(n, m) \geq 0 \) for any \( n \geq 4 \) and \( m \geq 0 \) by induction on \( n \).

It’s easy to check \( g(4, m) = 0 \) for any \( m \geq 0 \) and \( g(n, -1) = 0 \). Assume that \( g(n-1, m) \geq 0 \) for \( n \geq 5 \) and \( m \geq 0 \). Then we have

\[
g(n, m) = 2^m \binom{n-1}{m+1} - (n-2) \binom{n-2}{m}
- \sum_{j=0}^{k-1} \sum_{l=0}^{j} \frac{(k-1)!}{j!} \left( j - k + 2 \right) \binom{n-k-2+j}{j} \binom{n-k-1}{l} \binom{n-k}{m-l+1} - m + 1
= 2^m \left( (n-2) \binom{n-2}{m+1} + \binom{n-2}{m} \right) - (n-2) \left( (n-3) \binom{n-3}{m} + \binom{n-3}{m-1} \right)
- \sum_{j=0}^{k-1} \sum_{l=0}^{j} \frac{(k-1)!}{j!} \left( j - k + 2 \right) \binom{n-k-2+j}{j} \binom{n-k-1}{l} \binom{n-k}{m-l+1} - m + 1
\times \frac{n-k-2+j}{n-k-2} \binom{n-k-3+j}{j} \binom{n-k-1}{l} \binom{n-k}{m-l+1} - m + 1 - 1
\geq (n-2) \left( 2^m \binom{n-2}{m+1} - (n-3) \binom{n-3}{m} \right) + 2 \left( 2^{m-1} \binom{n-2}{m} - (n-3) \binom{n-3}{m-1} \right)
- \sum_{j=0}^{k-1} \sum_{l=0}^{j} \frac{(k-1)!}{j!} \left( j - k + 2 \right) \binom{n-k-3+j}{j} \binom{n-k-1}{l} \binom{n-k}{m-l+1} - m + 1
\times \frac{n-k-3+j}{n-k-2} \binom{n-k-1}{m-l+1} + 2 \left( 2^{m-1} \binom{n-2}{m} - (n-3) \binom{n-3}{m-1} \right) \binom{n-k-1}{l+1} + 1
\]
\[(n - 2)g(n - 1, m) + 2g(n - 1, m - 1) \geq 0,
\]

where the third step follows from the relations

\[
\left( j - k + 2 + \frac{j}{n - k - 1} \right) \frac{n - k - 2 + j}{n - k - 2} (n - k - 1) \leq \left( j - k + 2 + \frac{j}{n - k - 2} \right) (n - 2)
\]

and

\[
\left( j - k + 2 + \frac{j}{n - k - 1} \right) \frac{n - k - 2 + j}{n - k - 2} \leq 2 \left( j - k + 2 + \frac{j}{n - k - 2} \right).
\]

This completes the proof.

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