REPRESENTING THE BIG TILTING SHEAVES AS
HOLOMORPHIC MORSE BRANES

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Abstract. We introduce Morse branes in the Fukaya category of a holomorphic symplectic manifold, with the goal of constructing tilting objects in the category. We give a construction of a class of Morse branes in the cotangent bundles, and apply it to give the holomorphic branes that represent the big tilting sheaves on flag varieties.

1. Introduction

For a complex semisimple Lie group $G$ and a Borel subgroup $B \subset G$ with its unipotent radical $N$, the category of $N$-equivariant perverse sheaves on $\mathcal{B} = G/B$ corresponds to the principal block of the BGG Category $\mathcal{O}$. The indecomposable tilting perverse sheaves form a natural basis for the category, and they are in bijection with the Schubert cells. One can also view the tilting sheaves from other perspectives, i.e., as $\mathcal{D}$-modules via the Riemann-Hilbert correspondence or as Lagrangian branes in the Fukaya category $F(T^*B)$ via the Nadler-Zaslow correspondence. There have been several constructions of tilting objects as sheaves or $\mathcal{D}$-modules, including certain averaging or limiting process, i.e., taking nearby cycles (c.f. [7], [14], [3], [5]). In this paper, we construct the tilting object corresponding to the open Schubert cell, often referred as the big tilting, as a holomorphic Lagrangian brane in the Fukaya category $F(T^*B)$.

The construction is simple. Consider the moment map for the Hamiltonian $N$-action on $T^*B$, $\mu_N : T^*B \to n^*$, where $n$ is the Lie algebra of $N$. Take a non-degenerate character $\bar{\epsilon}$ of $n$ in $n^*$, then $L_{\bar{\epsilon}} = \mu_N^{-1}(\bar{\epsilon})$ is a closed (smooth) holomorphic Lagrangian in $T^*B$. It is just an $N$-orbit and we can equip it with a canonical brane structure to make it correspond to a perverse sheaf (c.f. [11]).

Theorem 1.1. The brane $L_{\bar{\epsilon}}$ corresponds to the big tilting sheaf on $\mathcal{B}$, via the Nadler-Zaslow correspondence.

The construction fits into a more general setting as Morse branes in holomorphic symplectic manifolds, which we will introduce below. The consideration of Morse branes is largely motivated from the approach by Nadler in [14] to construct tilting sheaves.

Key words and phrases. Tilting sheaves, Morse branes, holomorphic branes, Fukaya categories.

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1.1. Morse branes in holomorphic symplectic manifolds. We will work in the setting that an exact holomorphic symplectic manifold \((M, \omega_C)\) is endowed with two commuting \(\mathbb{C}^*\)-actions: one is Hamiltonian and is denoted as \(\mathbb{C}^*_X\), and the other, denoted as \(\mathbb{C}^*_Z\), scales \(\omega_C\) by a positive weight and it contracts \(M\) to a compact core as \(t \to 0\). We also assume that the \(\mathbb{C}^*_X\)-action has finitely many fixed points, and we will denote the union of their ascending (resp. descending) manifolds as \(\Lambda_X\) (resp. \(\Lambda_X^{opp}\)). Both \(\Lambda_X\) and \(\Lambda_X^{opp}\) are holomorphic conical Lagrangians with respect to the \(\mathbb{C}^*_Z\)-action, by the commutativity condition of the two actions. We assume that \(\Lambda_X\) and \(\Lambda_X^{opp}\) are disjoint away from the compact core of \(M\).

Consider the Fukaya category \(F_{\Lambda_X}(M)\), whose objects are (closed) Lagrangian branes in \(M\) that are dilated towards \(\Lambda_X\) by \(\mathbb{C}^*_Z\) as \(t \to 0\). We call a brane \(L \in F_{\Lambda_X}(M)\) a Morse brane if it intersects \(\Lambda_X^{opp}\) uniquely and transversely at a point in the smooth portion of \(\Lambda_X^{opp}\). The name comes from the principle that it plays the role of calculating the “microlocal stalk” in \(F_{\Lambda_X^{opp}}(M)\) at the intersection point (c.f. [17] and [11]).

We give a natural construction of a class of Morse branes in the situation when \(M\) is the cotangent bundle of a complex projective variety with a contracting \(\mathbb{C}^*_Z\)-action on the fibers (of weight 1). The specialty of cotangent bundles is that if \(k_0\) is the minimum of the positive weights of the \(\mathbb{C}^*_X\)-action on the tangent spaces of the fixed points, then we can use the flow of \(\mathbb{C}^*_X - k_0 Z\) to construct holomorphic Morse branes. We expect the construction to be generalized to some other holomorphic symplectic manifolds (e.g. hypertoric varieties, the resolution of the Slodowy slices) with more careful investigation of the weights of the two \(\mathbb{C}^*\)-actions, and we leave this for a future work.

The construction goes as follows. Take a point \(x\) in the fixed loci of \(\mathbb{C}^*_X - k_0 Z\), and take the ascending manifold of \(x\) with respect to the \(\mathbb{C}^*_X - k_0 Z\)-action. By the weights condition, this is a (not necessarily closed) holomorphic Lagrangian submanifold and we denote it by \(L_x\). The main theorem we get is the following.

**Theorem 1.2.** If \(x \in (\Lambda_X^{opp})^{sm}\), then \(L_x\) is a holomorphic Morse brane in \(F_{\Lambda_X}(M)\).

1.2. Application to the construction of tilting objects. In the case of a cotangent bundle, we have a \(\mathbb{C}^*_X\)-action on the base \(K\) which induces the Hamiltonian \(\mathbb{C}^*_X\)-action on \(T^*K\), and the Lagrangian \(\Lambda_X\) (resp. \(\Lambda_X^{opp}\)) is the conormal variety to the stratification \(S\) (resp. \(S^{opp}\)) defined by the ascending (resp. descending) manifolds of the fixed points in \(K\).

In good situations, \(S = \{S_\alpha\}\) and \(S^{opp} = \{S^{opp}_\alpha\}\) are transverse to each other, and \(S^{opp}\) is simple (see Definition 2.4). Then Theorem 1.1 is a special case of a more general result.

**Theorem 1.3.** If \(x \in (\Lambda_X^{opp})^{sm}\), then \(L_x\) corresponds to a tilting sheaf on \(K\) under the Nadler-Zaslow correspondence.

Once we have obtained Theorem 1.2, the proof of Theorem 1.3 follows from a similar argument as in [14]. Namely, the stalk (resp. costalk) of the corresponding
sheaf on $S_{opp}$ can be calculated by the microlocal stalk of the costandard (resp. standard) sheaf for $S_{opp}$ at $x$, therefore they are concentrated in the right degrees.

We expect Morse branes to give tilting objects in the Fukaya category of a wide class of holomorphic symplectic manifolds. In the case of symplectic resolutions, the Fukaya categories are expected to be equivalent to the category of modules over certain quantizations of the manifolds. Therefore the tilting branes are expected to correspond to tilting objects in certain representation categories.

1.3. Organization. The paper is organized as follows. In Section 2 we recall some basic definitions and facts about constructible sheaves, perverse sheaves and tilting sheaves. In Section 3 we make the basic set-up for the Fukaya category of a holomorphic symplectic manifolds, and we also briefly review the definition of Fukaya categories and the Nadler-Zaslow correspondence. Next, we give the construction of a class of holomorphic Morse branes and the proof of Thm 1.2 in Section 4. Lastly, we give the big tilting brane in $T^*B$ and prove Theorem 1.1 in Section 5. The exactly same proof applies to Theorem 1.3.

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2. Tilting perverse sheaves

2.1. Constructible sheaves. This subsection reviews some basic definitions and properties of constructible sheaves with the main purpose of introducing notations. We recommend [13] for an introduction to the theory of constructible sheaves. We will keep working in the subanalytic setting.

Let $M$ be a real analytic manifold. Fix a Whitney stratification $\mathcal{S} = \{S_\alpha\}$ on $M$. A sheaf $\mathcal{F}$ of $\mathbb{C}$-vector spaces on $M$ is said to be constructible with respect to $\mathcal{S}$, if its pull-back to each stratum $i_{\alpha*}\mathcal{F}$ is locally constant. Let $D_S(M)$ (resp. $D(M)$) be the bounded derived category of complexes of sheaves whose cohomology sheaves are all constructible with respect to $\mathcal{S}$ (resp. with respect to some stratification). Let $Sh_S(M)$ (resp. $Sh(M)$) be the natural dg-enhancement of $D_S(M)$ (resp. $D(M)$). We will always refer to an object in $Sh(M)$ a sheaf rather than a complex of sheaves.

For any map $f : M_1 \to M_2$ between two analytic manifolds, there are standard operations $f_*, f_! : Sh(M_1) \to Sh(M_2)$, $f^*, f^! : Sh(M_2) \to Sh(M_1)$, where all of our functors have been derived and we always omit the derived notation. There is also the Verdier duality $\mathbb{D} : Sh(M) \cong Sh(M)^{op}$, which intertwines the $*,!$ functors, i.e. $f_! = \mathbb{D} f_* \mathbb{D}$ and $f^! = \mathbb{D} f^* \mathbb{D}$. 
For any open embedding $i: U \hookrightarrow M$ and closed embedding of the complement $j: Z \hookrightarrow M$, there are the standard triangles

$$i_!i^*F \to F \to j_*j^*F, \quad j_!j^*F \to F \to i_*i^*F,$$

from which it is not hard to deduce that $Sh_S(M)$ is generated by $i_{S_\alpha *}L_{S_\alpha}, S_\alpha \in S$, where $L_{S_\alpha}$ ranges in the set of irreducible local systems on $S_\alpha$.

### 2.2 Perverse sheaves and tilting sheaves.

Here we recall the basic definitions and properties of perverse sheaves and tilting sheaves. We refer the reader to [9], [12] for more discussions on perverse sheaves and [2] on tilting sheaves.

#### 2.2.1 Perverse sheaves.

The most natural definition of perverse sheaves may be through the Riemann-Hilbert correspondence. For a complex analytic manifold $M$, the Riemann-Hilbert correspondence gives an equivalence between the bounded derived category of regular holonomic $D$-modules and $D(M)$. The obvious $t$-structure on the $D$-module side induces an interesting $t$-structure on $D(M)$, which is called the perverse $t$-structure. The perverse sheaves are the objects in the heart of the $t$-structure. In other words, a perverse sheaf corresponds to a single regular holonomic $D$-module. There are other characterizations of perverse sheaves. A commonly used one is the following definition through the degrees of cohomological (co)stalks of sheaves.

**Definition 2.1.** A sheaf $F$ is perverse if the followings hold for all $S_\alpha \in S$:

1. $H^\bullet(i_{S_\alpha *}^*\mathcal{F}) = 0$ for all $\bullet > -\dim_S S_\alpha$;
2. $H^\bullet(i_{S_\alpha *}^*\mathcal{F}) = 0$ for all $\bullet < -\dim_S S_\alpha$.

There is another natural characterization of perverse sheaves through microlocal stalks (also called local Morse groups or vanishing cycles). Let’s first briefly review the definition of microlocal stalks. Microlocal stalks are well defined in the real setting (c.f. [9]), however, we will restrict ourselves to the complex setting for simplicity. For any covector $(x, \xi) \in \Lambda_S^{sm}$, we choose a generic germ of holomorphic function $F$ near $x$ such that $F(x) = 0$ and $dF = \xi$. Here the genericity condition can be interpreted as that the graph of $dF$ as a germ of Lagrangian in $T^*M$ is transverse to $\Lambda_S$ at $(x, \xi)$.

**Definition 2.2.** The microlocal stalk of $F \in Sh_S(M)$ at $(x, \xi)$, denoted as $M_{x, \xi}(\mathcal{F})$ is defined to be

$$M_{x, \xi}(\mathcal{F}) = \Gamma(B_\epsilon(x), B_\epsilon(x) \cap \{\Re F < 0\}; \mathcal{F}),$$

for $\epsilon > 0$ sufficiently small.

Now we can define the singular support of a sheaf $\mathcal{F} \in Sh_S(M)$ to be

$$SS(\mathcal{F}) = \{(x, \xi) \in \Lambda_S^{sm} : M_{x, \xi}(\mathcal{F}) \neq 0\}.$$

One important feature about microlocal stalk is that it is perverse $t$-exact. Moreover, we have the following microlocal characterization of perverse sheaves.
Proposition 2.3. A sheaf $\mathcal{F}$ is perverse if and only if all of its microlocal stalks are concentrated in degree 0.

2.2.2. Tilting sheaves. Tilting sheaves form a special kind of perverse sheaves. Under some natural assumptions on the stratification $\mathcal{S}$, the indecomposable tilting sheaves form a natural basis for the category of perverse sheaves.

Definition 2.4. A complex stratification $\mathcal{S} = \{S_\alpha\}$ is called simple if the frontier of each stratum $\overline{S_\alpha} - S_\alpha$ is a Cartier divisor in $\overline{S_\alpha}$.

It is a standard fact that the Schubert stratification on a flag variety $\mathcal{B} = G/B$ is simple. If $\mathcal{S}$ is simple, then the standard and costandard sheaves $i_* \mathcal{L}_{S_\alpha}[-\dim S_\alpha]$, $i! \mathcal{L}_{S_\alpha}[-\dim S_\alpha]$ are both perverse sheaves, for any local system $\mathcal{L}_{S_\alpha}$ on $S_\alpha$.

Definition 2.5. A sheaf $\mathcal{F} \in Sh_{\mathcal{S}}(M)$ is tilting if for all $S_\alpha \in \mathcal{S}$, we have

1. $H^\bullet(i^*_S,\mathcal{F}) = 0$ for all $\bullet \neq -\dim C_{S_\alpha}$;
2. $H^\bullet(i_{S_\alpha}^! \mathcal{F}) = 0$ for all $\bullet \neq -\dim C_{S_\alpha}$.

Proposition 2.6. If $\mathcal{S}$ is simple and $\pi_1(S_\alpha) = \pi_2(S_\alpha) = 0$ for every $S_\alpha \in \mathcal{S}$, then there is a unique indecomposable tilting perverse sheaf supported on each $\overline{S_\alpha}$, and this gives a bijection between indecomposable tilting perverse sheaves and the strata in $\mathcal{S}$.

3. Fukaya categories on holomorphic symplectic varieties

Let $M$ be a (quasi-projective) holomorphic symplectic variety with an exact holomorphic symplectic form $\omega_C$.

3.1. Two $\mathbb{C}^*$-actions. We assume that $M$ is equipped with two commuting (algebraic) $\mathbb{C}^*$-actions: $\mathbb{C}^*_X$ and $\mathbb{C}^*_Z$, where $X$ and $Z$ denote the integral vector fields of the corresponding $U(1)$-actions respectively.

The $\mathbb{C}^*_X$-action should be Hamiltonian with respect to $\omega_C$, and it should have finitely many fixed points. We index the fixed points by $x_\alpha, \alpha \in I$, and use $\mathcal{G}_X(x_\alpha)$ (resp. $\mathcal{U}_X(x_\alpha)$) to denote the ascending manifold (resp. descending manifold) of $x_\alpha$. There is a natural partial ordering on the fixed point set $I$, namely $x_\alpha < x_\beta$ if $x_\alpha \in \mathcal{S}_X(x_\beta)$. The ascending manifold of each fixed point is a holomorphic Lagrangian manifold in $M$, and we will denote the union of them by $\Lambda_X$.

The $\mathbb{C}^*_Z$-action contracts $M$ to a compact core, denoted as Core$(M)$, and it acts on $\omega_C$ by weight $k$, for some integer $k \geq 1$. By the commutativity assumption, $\Lambda_X$ is conical with respect to the $\mathbb{C}^*_Z$-action.

3.2. Examples. A class of interesting examples of holomorphic symplectic manifolds are the conical symplectic resolutions. We refer the readers to the definition and a list of examples in Section 2 of [4].

In this paper we will mostly focus on the case when $M = T^*K$ is the cotangent bundle of a complex projective variety $K$, the $\mathbb{C}^*_X$-action will be the induced Hamiltonian action from a given $\mathbb{C}^*_X$-action on $K$ (with isolated fixed points), and the $\mathbb{C}^*_Z$-action will be the contraction on the cotangent fibers. In particular, we have $k = 1$. 
3.3. The Fukaya category $F_\Lambda(M)$.

3.3.1. A brief review of the Fukaya category in the real setting. For any real exact symplectic manifold $(M, \omega)$ with a conical end (with respect to the Liouville flow for a preferred primitive of $\omega$), one can define its infinitesimal Fukaya category $F(M)$, denoted by $F(M)$. Roughly speaking, an object in the Fukaya category is a (complex of) Lagrangian brane(s) $(L, \Phi, P)$ consisting of the data of a properly embedded Lagrangian submanifold $L$, a grading $\Phi : L \to \mathbb{R}$, and a relative Pin-structure on $L$. In the following, to make the notations simple, we usually denote a brane only by its underlying Lagrangian submanifold when there is no cause of confusion. Moreover, one compactifies $M$ by the conical structure on the ends to $\overline{M} = M \cup M^\infty$, where $M^\infty$ is the contact boundary of $M$ which is also referred as the infinity of $M$. We also require that $L$ is well-behaved near the infinity of $M$ in the sense that $L^\infty = L \cap M^\infty$ is a Legendrian subset of $M^\infty$, which can be equivalently described as $\lim_{t \to 0^+} t \cdot L$ is contained in a conical Lagrangian.

The morphism between two objects $(L_1, \Phi_1, P_1)$ and $(L_2, \Phi_2, P_2)$ is the Floer complex $CF(L_1, L_2) = \bigoplus_{p \in L_1 \cap L_2} \mathbb{C} \cdot [\deg p]$, where $\mu_1$ is defined by counting pseudo-holomorphic discs bounded by the two Lagrangians. The degree of $p$, denoted as $\deg p$, depends on the gradings $\Phi_1$ and $\Phi_2$. The relative Pin-structures also enter into the story because these are needed to give an orientation of the (0-dimensional) moduli spaces of pseudo-holomorphic strips, so that one can count the points. Of course, implicit in the definition is the transversality between $L_1$ and $L_2$ and certain standard treatment of $L_1^\infty$ and $L_2^\infty$ if they overlap.

The composition of morphisms $\mu_2 : CF(L_2, L_3) \otimes CF(L_1, L_2) \to CF(L_1, L_3)$ is defined by counting pseudo-holomorphic triangles bounded by the three Lagrangians. There are also higher compositions $\mu_n, n \geq 3$ which are defined by counting pseudo-holomorphic polygons. The sequence $\{\mu_n\}_{n \geq 1}$ satisfies the $A_\infty$-relation, which makes the Fukaya category into an $A_\infty$-category.

Since we will only use a short list of theorems or facts about the Fukaya categories, we find it not necessary to go through the long story of the subject. We will review the statements we need in the next subsection and refer the reader to [13, 11] and [16] for more details on the definition of Fukaya categories.

3.3.2. The subcategory $F_\Lambda(M)$. Continuing on the real setting, for any conical Lagrangian $\Lambda \subset M$, we define the full subcategory $F_\Lambda(M)^{naive}$ to be generated by objects $L$ with $L^\infty \subset \Lambda^\infty$. We put the superscript “naive” because the actual definition of $F_\Lambda(M)$ is defined microlocally, which corresponds to $Sh_\Lambda(K)$ when $M = T^*K$. Given an $L \in F(M)$, for any $\xi \in (L^\infty)^{sm}$, one can construct a
Lagrangian disc $L_\xi$ (which is also an object in $F(M)$) whose infinity is disjoint from $L^\infty$ and which intersects the cone over $L^\infty$ transversely at a unique point in the ray pointing to $\xi$. For more details of the construction of $L_\xi$, we refer the reader to Section 3.7 in [17] and Section 4 in [11] (in the cotangent bundle case). Once such a brane $L_\xi$ can be constructed for every $\xi$, we can define the \textit{microlocal support} of a brane $L$, which is a conical Lagrangian. Then $F_\Lambda(M)$ is the full subcategory generated by branes whose microlocal support is contained in $\Lambda^\infty$.

In this paper, we will be mostly interested in the objects in $F_\Lambda(M)^{\text{naive}}$, so it is not harmful to keep that as an intuitive replacement of $F_\Lambda(M)$.

3.3.3. $F(M, \omega_C)$ and $F_{\Lambda\times}(M)$. In the holomorphic symplectic setting, as we started with, we take the real part of $\omega_C$ and the $\mathbb{R}^+$-factor in $\mathbb{C}^*_\mathbb{Z}$ to serve as the Liouville flow, then these fit into the real setting, and give us the Fukaya category $F(M, \omega_C)$. Similarly, we can define $F_{\Lambda\times}(M)$ to be the subcategory of $F(M, \omega_C)$ in the real setting.

There are some special features about the Fukaya category of a holomorphic symplectic manifold. For example, one can do a projective compactification $\overline{M}_C = M \cup M^\infty$ of $M$ using the $\mathbb{C}^*_\mathbb{Z}$-action, so that $M^\infty = (M - \text{Core}(M))/\mathbb{C}^*_\mathbb{Z}$ (we will omit the subscript $C$ from now on) is a divisor in $\overline{M}$. Moreover, there is a specific class of Lagrangians—the holomorphic Lagrangians. In [11], it is proved that any holomorphic Lagrangian brane in $M = T^*K$ represents a perverse sheaf on $K$, under the Nadler-Zaslow correspondence. Hence one could roughly think of the class of the holomorphic branes as the heart of a $t$-structure on the Fukaya category.

3.4. The Nadler-Zaslow correspondence. Given a compact real analytic manifold $K$, the Nadler-Zaslow correspondence gives a quasi-equivalence between the Fukaya category $F(T^*K)$ and the dg-category $\text{Sh}(K)$ of constructible sheaves on $K$. The theorem also holds for a given microlocal support condition, i.e. given a conical Lagrangian $\Lambda \subset T^*K$ (containing the zero-section), we have $F_\Lambda(T^*K) \simeq \text{Sh}_\Lambda(K)$, where $\text{Sh}_\Lambda(K)$ denotes for the full subcategory consisting of sheaves whose singular support is contained in $\Lambda$.

We will collect some of the results involved in the Nadler-Zaslow correspondence that we will use in later sections without proof. We refer the interested reader to [16] and [15] for more details. In the following, we will fix a Whitney stratification $\mathcal{S} = \{S_\alpha\}$ on $K$ such that each stratum is connected and is a cell, and we will always work in the subanalytic setting.

- \textit{(Co)Standard branes.}
  For each stratum $S_\alpha \in \mathcal{S}$, one can define a \textit{standard brane} on it, denoted as $L_{S_\alpha}$ as follows. Pick a function $m_\alpha : K \rightarrow \mathbb{R}$ such that $m_\alpha > 0$ on $S_\alpha$ and $m_\alpha = 0$ on $K - S_\alpha$. Now define $L_{S_\alpha}$ to be $\Gamma_{d\log m_\alpha + T_{S_\alpha}^* K}$. It is shown in [16] that $L_{S_\alpha}$ can be equipped with a canonical grading and a canonical

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3 This is not a precise statement, since not every perverse sheaf can be represented by a holomorphic brane.
Pin-structure, so we will refer $L_{S_\alpha}$ as the standard brane on $S_\alpha$. Note that $L_{S_\alpha}$ as an object in $F(T^*K)$ doesn’t depend on the choices of $m_\alpha$. The involution on $T^*K$ that negates the cotangent vectors correspond to the Verdier duality on $\text{Sh}(K)$. We will call the involution of $L_{S_\alpha}$ a costandard brane.

- Generators of $F_{\Lambda S}(T^*K)$.
  Under the Nadler-Zaslow correspondence, each standard brane $L_{S_\alpha}$ goes to the standard sheaf $i_{S_\alpha}^! C_{S_\alpha}$, and the involution of $L_{S_\alpha}$ goes to the costandard sheaf $i_{S_\alpha} C_{S_\alpha}$, where $i_{S_\alpha} : S_\alpha \hookrightarrow K$ is the embedding. If we put a standard or costandard sheaf (resp. brane) for each stratum, then they will generate $\text{Sh}_{\Lambda S}(K)$ (resp. $F_{\Lambda S}(T^*K)$) by taking shifts and iterated cones.

4. Holomorphic Morse branes in $F_{\Lambda X}(M)$

We will continue on the set-up for the Fukaya category of a holomorphic symplectic manifold in Section 3.

4.1. Definition of Morse branes in $F_{\Lambda X}(M)$.

Let $\Lambda^\text{opp}_X$ be the union of the descending manifolds of $\mathbb{C}_X^*$. We assume that $\Lambda_X$ and $\Lambda^\text{opp}_X$ are disjoint away from the compact core of $M$.

**Definition 4.1.** A Lagrangian brane $L$ in $F_{\Lambda X}(M)$ is called a Morse brane, if it satisfies that $L$ intersects $\Lambda^\text{opp}_X$ in a single point that is contained in the smooth part of $\Lambda^\text{opp}_X$, and the intersection is transverse.

The consideration of Morse branes is largely motivated by the results in [14], in which the author constructed tilting perverse sheaves on the flag variety $\mathcal{B}$ by means of Morse theory. We will see the applications of the notion of Morse branes in the construction of big tilting sheaves in Section 5.

4.2. Construction of a class of holomorphic Morse branes in cotangent bundles.

In this section, we assume that $M$ is the cotangent bundle of a smooth projective variety. The action by $\mathbb{C}_Z^*$ is dilating the fibers with weight 1, and we assume that

\begin{equation}
\text{(4.1) the minimum of the positive weights of } \mathbb{C}_X^* \text{ on the tangent spaces at the fixed points is } k_0.
\end{equation}

We will use $a_X$, $a_Z$, $a_{X-k_0Z} : \mathbb{C}^* \to \text{Aut}(M)$ to denote the action of $\mathbb{C}_X$, $\mathbb{C}_Z$ and $\mathbb{C}_{X-k_0Z}$ on $M$, respectively. Again, we index the fixed points of $\mathbb{C}_X^*$ by $x_\alpha$, $\alpha \in I$.

We will denote each fixed locus of $\mathbb{C}_{X-k_0Z}$ containing a $\mathbb{C}_X$-fixed point $x_\alpha$ by $E_\alpha$.

**Lemma 4.2.** For any $x$ in the fixed loci of $\mathbb{C}_{X-k_0Z}$, the ascending manifold $\mathcal{G}_{X-k_0Z}(x)$ is a holomorphic Lagrangian submanifold (not necessarily closed).

**Proof.** First, $x$ must lie in the descending manifold of the $\mathbb{C}_X$-fixed point

\[ y = \lim_{t \to 0} a_Z(t) \cdot x = \lim_{t \to 0} a_X(t) \cdot x, \]
therefore it belongs to $E_\alpha$ for some $\alpha \in I$.

Since the action of $\mathbb{C}^*_X - k_0 Z$ is Morse-Bott, the decomposition of the tangent space at $x$ into weight spaces is the same as that at $x_\alpha$. By the assumption \[(\ref{4.1})\], we know that the ascending manifold of $x_\alpha$ with respect to $C_{X - k_0 Z}$ is the same as the ascending manifold with respect to $C_X^*$, thus the negative weight space of $C_{X - k_0 Z}$ has the dimension of a Lagrangian. Now at $x$, we only need to show that in a small neighborhood, the ascending manifold $\mathfrak{g}_{X - k_0 Z}(x)$ is isotropic, since $C_{X - k_0 Z}$ scales $\omega_C$ with weight $-k = -k_0$. First, the tangent space at $x$ is isotropic by a similar reason of weights: the negative weights for $X - Z$ are at most $-2k_0$. To show that near $x$ we have $\mathfrak{g}_{X - k_0 Z}(x)$ locally be a Lagrangian, we identify a neighborhood of $0$ in $T_x M$ with a neighborhood of $x$ in $M$ by an $a_{X - k_0 Z}(\mathbb{R})$-equivariant diffeomorphism, and use the equivariant version of Moser’s argument to modify the diffeomorphism into a local equivariant symplectomorphism.

\[\square\]

**Remark 4.3.** It is easy to see that $C(L) := \lim_{t \to 0} a_Z(t) \cdot L$ is both $\mathbb{C}^*_Z$ and $\mathbb{C}^*_X$-invariant. However, we cannot conclude that $C(L)$ is contained in the conical Lagrangian $\Lambda_X$.

We will denote every Lagrangian constructed in Lemma \[4.2\] by $L_{\alpha,x}$, for $x \in E_\alpha$. Now we work with the projective compactification of $M$ with respect to the action of $\mathbb{C}^*_Z$, defined by

$$\overline{M} = (M \times \mathbb{C} - \text{Core}(M) \times \{0\}) / \mathbb{C}^*_Z.$$  

Since our $M$ is the cotangent bundle of a projective variety, $\overline{M}$ is again projective. The action of $\mathbb{C}^*_X$ and $\mathbb{C}^*_Z$ both extend to $\overline{M}$ by keeping their actions on $M$ and acting trivially on the extra factor $\mathbb{C}$. In particular, they will preserve $M^\infty = \overline{M} - M$. We will denote the projectivization of a conical line $\mathbb{C}^*_Z \cdot v$ in $M$ by $[v] \in M^\infty$.

Now by basic properties of algebraic $\mathbb{C}^*$-actions on smooth projective varieties and its relations to Morse theory (c.f. \[4\] Section 2.4), we can deduce the following.

**Theorem 4.4.** If $E_\alpha \not\subset \bigcup_{x_\beta \prec x_\alpha} \overline{\Xi}_X(x_\beta)$, then for any $x \in E_\alpha - \bigcup_{x_\beta \prec x_\alpha} \overline{\Xi}_X(x_\beta)$, $L_{\alpha,x}$ is a Morse brane in $F_{\Lambda_X}(M)$ with $L_{\alpha,x} \cap \overline{\Lambda_X}^{\text{proj}} = \{x\}$.

**Proof.** First, we have $\overline{\Xi}(x_\beta) = \overline{\Xi}_{X - k_0 Z}(E_\beta)$ by Assumption \[(\ref{4.1})\]. Next, we claim that the boundary of $L_{\alpha,x}$ consists of points in $\overline{M}$ that can be connected to $x$ by piecewise flow lines, which are usually called broken flow lines. This follows from the properties of finite volume flow in \[10\], and can be argued in the same way as Lemma 3.4 in loc. cit.

Now by the assumption that $E_\alpha \not\subset \bigcup_{x_\beta \prec x_\alpha} \overline{\Xi}_X(x_\beta)$, there is no flow line of $X - k_0 Z$ that travels from $E_\beta$ to $x$. Therefore, for any broken flow line ending on $x$, the last portion must start from a point $[v]$ on a critical manifold inside $M^\infty$. We claim that $[v]$ is lying in $\Lambda_X^\infty$. Note that the critical manifolds in $M^\infty$ are
exactly the projectivization of the conical lines in \( M \) that are fixed (pointwise) by \( C^*_X + k Z \) for some nonzero integer \( k \). This means the conical line corresponding to \([v] \in \Lambda^\infty_X\) is fixed by \( C^*_X + k[v] Z \) for some positive integer \( k[v] \). Suppose the contrary, we have \( k[v] < 0 \), this would imply the descending manifold of \([v] \) under the flow of \( C^*_X - k[v] Z \) is contained in \( M^\infty \), which cannot be true, so the claim follows. Since \( \Lambda^\infty_X \cap (\Lambda^\infty_X)^{opp} = \emptyset \) by assumption, we can conclude that the broken line is contained in \( M^\infty \) except for the last portion.

Now we can model the piece of flow line in \( M^\infty \) ending at \([v] \) by a flow of \( C^*_X + k[v] Z \) ending at a point \( x_0 \in C^*_X \), \( v \) in \( M \), which means that the projectivization of the latter flow line in \( M^\infty \) will be equal to that piece of flow line (here we have used again that \( C^*_X + k[v] Z \) gives rise to a Morse-Bott flow on \( M \)). If the starting point of the flow line modeled on is away from the zero section, then by rescaling it with \( a_{-k[v]}(t) \) with respect to some parametrization (so that we get a flow line of \( C^*_X \)), it is clear that the whole flow line at infinity lies in \( \Lambda^\infty_X \). On the other hand, if the flow line starts at some fixed point \( x_\beta \) of \( C^*_X \), then there are two cases after rescaling the flow line of \( C^*_X + k[v] Z \) in \( M \) by \( a_{-k[v]}(t) \): one is at 0 the flow line approaches something away from the compact core, the other is at 0 it remains to be at \( x_\beta \). The first case directly implies that the flow line in \( M^\infty \) is contained in \( \Lambda^\infty_X \), and the second implies that \( \Lambda^\infty_X \cap (\Lambda^\infty_X)^{opp} \neq \emptyset \), which is a contradiction.

By induction on the pieces of the broken flow line (from \( \infty \) to 0), we get that the whole broken line is lying in \( \Lambda^\infty_X \) except for the first piece, and the theorem follows.

5. The big tilting branes in \( T^*B \)

Let \( G \) be a semisimple Lie group over \( \mathbb{C} \), \( B \subset G \) be a Borel subgroup and \( \mathcal{B} \) be the flag variety \( G/B \). Fix a maximal torus \( H \subset B \). Let \( B^- \) be the opposite Borel subgroup, and \( N \subset B \) (resp. \( N^- \subset B^- \)) be the unipotent radical of \( B \) (resp. \( B^- \)). Let \( \mathfrak{g}, \mathfrak{b}, \mathfrak{b}^-, \mathfrak{n}, \mathfrak{n}^-, \mathfrak{h} \) be the Lie algebra of \( G, B, B^-, N, N^-, H \) respectively. For a general Borel \( \mathfrak{b}_w \), we will use \( \mathfrak{n}_w \) to denote its nilradical. Let \( \Delta, \Phi^+ \) and \( \Phi^- \) denote respectively the set of simple, positive and negative roots. Let \( W = N_G(H)/H \) be the Weyl group of \( G \). Let \( \mathcal{S} = \{ S_w \}_{w \in W} \) (resp. \( \mathcal{S}^- = \{ S_w^- \}_{w \in W} \) be the Schubert stratification (resp. opposite Schubert stratification) on \( \mathcal{B} \) determined by the orbits of \( N \) (resp. \( N^- \)). Fixing the coweight in \( \mathfrak{h} \) whose pairing with the simple roots are all \(-1 \), usually called \( \hat{\rho} \), its induced \( C^* \)-action on \( \mathcal{B} \) has fixed points naturally indexed by \( W \), denoted as \( p_w, w \in W \), and the ascending (resp. descending) manifolds of each of the fix points \( p_w \) coincide with \( S_w \) (resp. \( S_w^- \)). Let \( s_w : S_w \hookrightarrow \mathcal{B} \) and \( s_w^- : S_w^- \hookrightarrow \mathcal{B} \) (resp. \( i_{p_w} : p_w \hookrightarrow \mathcal{B} \)) be the embeddings of the strata (resp. fixed points).

The \( C^* \)-action on \( \mathcal{B} \) induces a Hamiltonian action on \( T^*\mathcal{B} \), which we will use as the \( C^*_X \)-action as in Section 3.1. It is easy to see that \( \Lambda_X \) coincides with the conormal variety of \( \mathcal{S} \), i.e. \( \Lambda_\mathcal{S} = \bigcup_{s_w \in \mathcal{S}} T^*_s \mathcal{B} \). The transversality between the Schubert stratification and the opposite one implies that \( \Lambda^\infty_X \cap (\Lambda^\infty_X)^{opp} = \emptyset \). The
\( \mathbb{C}_z \)-action on \( T^*B \) is the natural \( \mathbb{C}^* \)-action on the cotangent fibers with weight 1. It is clear that \( k_0 = 1 \) in (4.1) for this case.

Let \( w_0 \) be the longest element in \( W \). Let \( z_\alpha, \alpha \in w_0(\Delta) \) be the linear coordinates around \( p_{w_0} \) which correspond to the negative of the simple roots \( w_0(\Delta) \subset h^* \). Let \( F_{w_0} = \sum_{\alpha \in w_0(\Delta)} c_\alpha z_\alpha \) be a generic linear function on \( S_{w_0} \), i.e. \( \prod_{\alpha \in \mathcal{S}} c_\alpha \neq 0 \). Then \( L_{w_0, (dF_{w_0})_{p_{w_0}}} \) is the same as the Lagrangian graph \( \Gamma_{dF_{w_0}} \). For any Lagrangian graph, there is a natural brane structure one can put on it, similarly to the case of standard and costandard branes, and this will be the default brane structure on \( \Gamma_{dF_{w_0}} \).

For any \( w \in W \), let \( \mathfrak{b}_w \) denote for the Borel \( \text{Ad}_w B \), and \( \mathfrak{n}^{-}_{\mathfrak{b}_w} = \bigoplus_{\alpha \in w(\Phi^+)} \mathfrak{g}_\alpha \). Let \( N_w^- \) be the unipotent group whose Lie algebra is \( n^- \cap \mathfrak{n}^-_{\mathfrak{b}_w} \), then each \( S_w^- \) is the orbit of \( p_w \) under the action of \( N_w^- \). The conormal to \( S_w^- \) at \( \mathfrak{b}_w \in B \) is \( (n^- \cap \mathfrak{n}^-_{\mathfrak{b}_w})^\perp \cap \mathfrak{n}_{\mathfrak{b}_w} \approx n^- \cap \mathfrak{n}_{\mathfrak{b}_w} = \bigoplus_{\alpha \in w(\Phi^+)} \mathfrak{g}_\alpha \), with respect to the Killing form. Similarly, the conormal at any \( \mathfrak{b}_x \in S_w^- \) can be identified with \( n^- \cap \mathfrak{n}_{\mathfrak{b}_x} \subset \text{Ad}_{N^-}(n^- \cap \mathfrak{n}_{\mathfrak{b}_w}) \).

**Lemma 5.1** *(Lemma 5.17 [14])*. For any sheaf \( \mathcal{F} \in \text{Sh}_S(B) \), we have
\[
 i_{p_w}^* s_{w^*} \mathcal{F} \simeq \mathbb{D}(\text{Hom}(\mathcal{F}, s_{w^*} \mathcal{C}_{S^-}[\dim S^-])[- \dim S^-]),
\]
\[
 i_{p_w}^* s_{w^*} \mathcal{F} \simeq \mathbb{D}(\text{Hom}(\mathcal{F}, s_{w^*} \mathcal{C}_{S^-}[\dim S^-])[- \dim S^-]),
\]
for all \( w \in W \).

Let \( \mathcal{N} \) and \( \mathcal{N}^{\text{reg}} \) respectively be the nilpotent cone and the orbit of regular nilpotent elements in \( \mathfrak{g} \).

**Lemma 5.2**. For any \( w < w_0 \), \( \Gamma_{dF_{w_0}} \cap \text{Int}(p_w) = \emptyset \).

**Proof.** Consider the moment maps \( \mu_G : T^*B \to \mathcal{N} \) (the Springer resolution) and \( \mu_N : T^*B \to n^* \simeq n^- \) of the Hamiltonian \( G \)-action and \( N \)-action on \( T^*B \) respectively, then \( \Gamma_{dF_{w_0}} \) is nothing but \( \mu_N^{-1}(e) = \text{Ad}_G^{-1}(e) \), where \( e \) is the image \( \mu_G(dF_{w_0}|_{p_{w_0}}) \) whose projection \( \bar{e} \) is the character of \( \mathfrak{n} \) corresponding to the linear function \( \sum_{\alpha \in w(\Delta)} c_\alpha z_\alpha \). It follows from our assumption that \( e \) lies in \( \mathcal{N}^{\text{reg}} \). We only need to show that for any \( w < w_0 \), \( n^- \cap \mathfrak{n}_{\mathfrak{b}_w} \) are singular values of \( \mu_G \), or in other words, \( (n^- \cap \mathfrak{n}_{\mathfrak{b}_w}) \cap \mathcal{N}^{\text{reg}} = \emptyset \), because then \( \mu_G(\text{Int}(p_w)) = \text{Ad}_{N^-}(n^- \cap \mathfrak{n}_{\mathfrak{b}_w}) \) will not intersect \( \mathcal{N}^{\text{reg}} \).

Note that \( \mathcal{N}^{\text{reg}} \cap n^- = \text{Ad}_{B^-} e \). Therefore, if we decompose any element in \( \mathcal{N}^{\text{reg}} \cap n^- \) with respect to the weight decomposition, it will have a nonzero component in each negative simple root space. However, the elements in \( n^- \cap \mathfrak{n}_{\mathfrak{b}_w} \) cannot satisfy this property for \( w < w_0 \), hence we are done. \( \square \)

**Proposition 5.3**. The Lagrangian graph \( \Gamma_{dF_{w_0}} \) is a Morse brane in \( F_{\Lambda_X}(T^*B) \).
Proof. This directly follows from Theorem 4.4 and Lemma 5.2. Alternatively, one can directly use the fact that $\Gamma_{dF_{w_0}} = \mu_{N}^{-1}(\bar{e})$ to deduce that the Lagrangian is closed. Also from this one easily sees that $\lim_{t \to 0} aZ(t) \cdot \Gamma_{dF_{w_0}} \subset \mu_{N}^{-1}(0) = \Lambda_X$, so $\Gamma_{dF_{w_0}} \subset \Lambda_X^\infty$.

Theorem 5.4. The Lagrangian graph $\Gamma_{dF_{w_0}}[\text{dim}_C B]$ corresponds to the big (indecomposable) tilting perverse sheaf.

Proof. We first show that the sheaf corresponding to $\Gamma_{dF_{w_0}}$ plays the role of a Morse kernel on $Sh_{S^{-}}(B)$, i.e. calculating vanishing cycles. Since $S$ and $S^{-}$ are transverse, we can make a refinement of $S^{-}$, denoted as $\tilde{S}^{-}$, which is still transverse to $S$ and gives a triangulation of $B$. We can also make $\tilde{S}^{-}$ so that $dF_{w_0}|_{p_{w_0}} \in \Lambda_{S^{-}}^\text{sm}$. Now we only need to show that $F_{\Lambda_{\tilde{S}^{-}}}(T^*B)$ can be generated by the brane $T_{p_{w_0}} B$, which represents the skyscraper sheaf, and other branes whose intersection with $\Gamma_{dF_{w_0}}$ is empty, for $HF(\Gamma_{dF_{w_0}}, T_{p_{w_0}} B) \simeq \mathbb{C}$ by the general fact about the grading of the transverse intersection of two holomorphic Lagrangian branes (c.f. [11]).

From Section 3.4, we know that $F_{\Lambda_{\tilde{S}^{-}}}(T^*B)$ is generated by the standard or costandard branes on the strata. If the closure of a stratum doesn’t contain $p_{w_0}$, then its standard brane has no intersection with $\Gamma_{dF_{w_0}}$ after sufficient dilations. For those strata whose closure contains $p_{w_0}$ (but not $\{p_{w_0}\}$), it is easy to see that at least one of the standard brane and costandard brane doesn’t intersect $\Gamma_{dF_{w_0}}$ (after sufficient dilations). We collect these branes and they together with $T_{p_{w_0}} B$ will generate $F_{\Lambda_{\tilde{S}^{-}}}(T^*B)$.

Now by Lemma 5.1 the stalk and costalk of the sheaf corresponding to $\Gamma_{dF_{w_0}}[\text{dim}_C B]$ on $S_w$ are concentrated in the right degrees for being a tilting sheaf. It is easy to see this sheaf is exactly the tilting sheaf $T_{p,F}$ for some $p,F$ introduced in [14].

Lastly, by the multiplicity formula $\text{mult}_{T_w}(T_{p,F}) = \dim M_{p,F}(IC_{w})$ in [14], where $T_w$ is the minimal tilting sheaf on $S_w$, we see that $\Gamma_{dF_{w_0}}[\text{dim}_C B]$ corresponds to the big indecomposable tilting sheaf.

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