Complex Energies and Beginnings of Time
Suggest a Theory of Scattering and Decay

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Abstract
Many useful concepts for a quantum theory of scattering and decay (like Lippmann-Schwinger kets, purely outgoing boundary conditions, exponentially decaying Gamow vectors, causality) are not well defined in the mathematical frame set by the conventional (Hilbert space) axioms of quantum mechanics. Using the Lippmann-Schwinger equations as the takeoff point and aiming for a theory that unites resonances and decay, we conjecture a new axiom for quantum mechanics that distinguishes mathematically between prepared states and detected observables. Suggested by the two signs \( \pm \imath \epsilon \) of the Lippmann-Schwinger equations, this axiom replaces the one Hilbert space of conventional quantum mechanics by two Hardy spaces. The new Hardy
space theory automatically provides Gamow kets with exponential
time evolution derived from the complex poles of the $S$-matrix. It
solves the causality problem since it results in a semigroup evolution.
But this semigroup brings into quantum physics a new concept of the
semigroup time $t = 0$, a beginning of time. Its interpretation and
observations are discussed in the last section.

1 Introduction

Quantum theory falls into, roughly, two categories [1]:

I. The description of spectra and structures of micro-physical systems

II. Scattering and decay phenomena

The distinction between the two categories is primarily one between two
ways one looks at the physical objects, rather than a separation of physics
into two different areas. The first is used for stable states and also for slowly
decaying states when the finiteness of their lifetime is ignored. The second is
used for rapidly decaying states and resonance phenomena. The notions of
slow and fast are not defined by a time scale in nature but by the capabilities
of the experimental apparatuses that we choose or are forced to use in a
particular experiment. For instance, the singly excited states of atoms and
molecules are mostly treated like stable states whereas the doubly excited
states (Auger states) are mostly treated as resonances or decaying states.
However, when one does the calculations of the energies of the Auger states
(e.g., of He) one ignores that they decay [1].

The same holds in nuclear physics and in high energy physics. When one
is interested only in the spectra and the structure of relativistic particles, one
ignores their lifetimes even though the different states of the same multiplet
can have lifetimes that are orders of magnitudes apart. (E.g., one can mea-
sure the lifetime of $\Omega^-$ but one cannot measure the lifetime of $\Delta$ [2]. The
existence and properties of $\Delta$ are determined from lineshape measurements
and lifetime was chosen as the inverse of the lineshape width on the basis of
some theoretical ideas/approximations for which a theory did not exist [3].)

For category I (spectra and structure), one uses a theory of stationary
states and time symmetric (reversible) evolutions. The energy values are dis-
crete and the time evolution is unitary and the superpositions are effectively
finite. Such systems are well described by conventional quantum mechanics
in the Hilbert space $\mathcal{H}$. Infinite superpositions are handled by perturbative methods (of discrete spectra).

The second category (scattering and decay) deals with continuous energy spectra and predominantly asymmetric time evolutions. If one wants to use energy eigenstates, the continuous energy values already require more than what the conventional axioms of quantum mechanics are able to accommodate. This has been overcome by introducing the Dirac kets $|E\rangle$, which -if they are mathematically defined at all- are defined as functionals on the Schwartz space. With this definition, energy wave functions $\psi(E) = \langle E|\psi \rangle$ do not constitute the entire Hilbert space of (Lebesgue) square-integrable functions, but only the subspace of infinitely differentiable, rapidly decreasing functions, i.e., Schwartz space functions.

The introduction of Dirac kets augments the conventional axiomatic framework of quantum mechanics based on the Hilbert space and leads to the Gelfand triplet $\Phi \subset \mathcal{H} \subset \Phi^*$, where $\Phi$ is the Schwartz space and $|E\rangle \in \Phi^*$ [1]. However, the Gelfand triplet based on the Schwartz space is not sufficient to obtain a theory that includes scattering and decay. The reason is that the dynamical (Schrödinger or Heisenberg) equations, when defined as differential equations in the Schwartz space of wavefunctions, integrate to a continuous group of evolution operators, much like the unitary group solution of these equations in the Hilbert space.

In contrast, resonances and decaying states have been intuitively associated to an asymmetric “irreversible” time evolution [4]. Thus, they require a time asymmetric theory, and in the absence of such a mathematical theory, their description can only be approximate and must contain contradictions. If one is guided by the Hilbert space mathematics, one always runs into problems with a quantum theory of resonances and decay; in particular, Gamow vectors with exponential decay do not exist in the Hilbert space. Therefore, in the heuristic treatment of scattering theory, one just ignored the mathematical subtleties of the Hilbert space. In particular, one worked with mathematically undefined kets $|E^{\pm}\rangle$, used an infinitesimal imaginary energy part $\pm \imath \epsilon$ to obtain, respectively, the incoming and outgoing solutions of the Lippmann-Schwinger equations [5], and distinguished between “states at time $t' < t_0 =$ time defined by preparation” and “states characteristic of the experiment”, observed at $t'' > t_0$ [6]. One restricted by fiat the time $t$ in $e^{iHt}$ to $t \geq 0$ [7], and for decaying states, one postulated purely outgoing boundary conditions [8], undisturbed by the fact that it was in conflict with the unitary group evolution $-\infty < t < \infty$, a direct consequence of
the conventional Hilbert space axioms of quantum mechanics (by the Stone-von Neumann theorem [9, 10]). These heuristic methods were successful for physical applications, but when one compared them with the mathematical consequences of the Hilbert space axiom, one had contradictions. Examples of these are: the exponential catastrophe in which Gamow vectors and unitary time evolution were mutually contradictory [11] and references therein; deviations from the exponential decay law [12]; and problems with (Einstein) causality [13].

It is thus clear that one has to go beyond the mathematical theory which has worked for Category I problems. But many of the empirical notions, like Gamow states and Lippmann-Schwinger kets, have been very successful for the descriptions of scattering and decay, and their successful features need to be preserved when they are incorporated into the new rigorous theory. However, other mathematical consequences of the conventional axioms need to be eliminated. This means we require a new hypothesis which preserve the successful features and alter the conflicting fallouts from the conventional theory. New mathematical entities will have to be defined, which we will call again by their old names, like Lippmann-Schwinger kets or Gamow kets, but they will now have new features and are constituents of a consistent theory of resonance scattering and decay. The new mathematical hypothesis will be conjectured taking the useful features of these heuristic notions as the starting point.

2 Conventional Quantum Theory Conflicts with the Lippmann-Schwinger Equations

By conventional quantum theory, we hereon mean not only the usual axioms [10] in terms of the Hilbert space mathematics, but also the Dirac formalism mathematically justified by, as stated above in the Introduction, a Gelfand triplet of the Schwartz space. The axiomatic framework of conventional quantum mechanics consists of the following:

(A1) One distinguishes (physically) between observables represented by self-adjoint operators (e.g., $A, \Lambda$ (positive operators), or vectors $\psi$ if $\Lambda = |\psi\rangle\langle\psi|$) and states represented by trace class operators (e.g., $W$ or vectors $\phi$ if $W = |\phi\rangle\langle\phi|$).

The quantities compared with experimental data are the Born prob-
abilities $\mathcal{P}_{W(t)}(\Lambda) = \text{Tr}(W(t)\Lambda) = \text{Tr}(W\Lambda(t))$, or, in the special case $W = |\phi\rangle\langle\phi|$ and $\Lambda = |\psi\rangle\langle\psi|$, $\mathcal{P}_{\phi(t)}(\psi) = |\langle\psi|\phi(t)\rangle|^2 = |\langle\psi(t)|\phi\rangle|^2$. That is,

$$\mathcal{P}_{\phi(t)}(\psi) = |\langle\psi|\phi(t)\rangle|^2 = |\langle\psi(t)|\phi\rangle|^2 \simeq \frac{N_1(t)}{N}.$$ 

The experimental quantities $\frac{N_1(t)}{N}$ are the ratios of large integers (detector counts which necessarily change in time in discrete steps). On the other hand, every mathematical theory is an idealization and thus quantum theory also idealizes to continuous time translations, in consequence of which the calculated Born probabilities $\mathcal{P}_{\phi(t)}(\psi)$ change continuously in time in a particular way. The equality between the two quantities $\mathcal{P}_{\phi(t)}(\psi)$ and $\frac{N_1(t)}{N}$ is approximate—and the sign $\simeq$ expresses this aspect of the statistical character of quantum mechanical predictions—and the meaning of the continuity for $\phi(t)$ or $\mathcal{P}_{\phi(t)}(\psi)$ as a function of $t$ is a mathematical choice.

In conventional quantum mechanics one makes this choice by identifying

(A2) The set of states $\{\phi\} = \text{The set of observables } \{\psi\} = \mathcal{H} = \text{Hilbert space}$

In Dirac’s formalism one assumes in addition that

(A3) for every observable, e.g., $H$, one has a complete set of eigenkets $|E\rangle$ such that

(3a) $H|E\rangle = E|E\rangle$ and

(3b) Every vector, state $\phi$ or observable $\psi$, is a continuous superposition of the eigenkets extending over all “physical values” $0 \leq E < \infty$:

$$\phi = \sum_{j,j_3,\eta} \int dE |E,j,j_3,\eta\rangle \langle E,j,j_3,\eta|\phi\rangle$$

(here $j,j_3$ and $\eta$ are some additional quantum numbers representing the degeneracy of the eigenkets with energy $E$.)

Nearly everyone discussing the foundations of quantum mechanics [14] distinguishes between states and observables as asserted by (A1) above. The Hilbert space axiom (A2) is already in conflict with this hypothesis (A1) because the content of (A1) is a basic distinction between a state and an observable. Also, the hypothesis (A3), the Dirac formalism, is not possible
within the framework of the Hilbert space axiom (A2) since neither (3a) nor
(3b) is well defined as a vector identity in the Hilbert space when \( E \) is a
continuous parameter.

One can overcome this difficulty and make (A3) mathematically tenable
by restricting the vectors \( \{ \phi \} \) and \( \{ \psi \} \) to a subspace \( \Phi \) of the Hilbert space
and constructing a Gelfand triplet, \( \{ \phi \} = \{ \psi \} = \Phi \subset \mathcal{H} \subset \Phi^* \). With this
choice of \( \Phi \), the eigenkets \( |E\rangle \) can be defined as the elements of the dual
space \( \Phi^* \) and (3b) can be proved as the nuclear spectral theorem. As stat
above, if the Schwartz space is chosen for \( \Phi \) so that energy wavef
ctions \( \phi(E) = \langle E|\phi \rangle = \langle \phi|E \rangle \) are smooth and rapidly decreasing at infinity, then
the dual space \( \Phi^* \), which consists of continuous anti-linear functionals on
\( \Phi \), is realized by the space of tempered distributions. Therefore, in this
representation, the eigenkets \( |E\rangle \) find realization as tempered distributions.

In scattering theory, one has in-states \( \{ \phi^+ \} \) and out-observables \( \{ \psi^- \} \n(\text{which are usually called out-states}). An in-state \( \phi^+ \) is prepared at \( t \to -\infty \n in the asymptotic region as the interaction-free in-states \( \phi^{\text{in}} \) such that
\[ \phi^{\text{in}} \to \phi^+ \]

Similarly, for \( t \to \infty \), the out-observable \( \psi^- \) becomes the interaction free
out-observable \( \psi^{\text{out}} \) which describes a measurable property in the asymptotic region:
\[ \psi^- \to \psi^{\text{out}} \]

The superscripts \( \pm \) of state vectors \( \phi^+ \) and \( \psi^- \) have their origins in the
labels of the eigenkets \( |E^\pm\rangle \) of the full Hamiltonian \( H = H_0 + V \),
\[ H|E^\pm\rangle = E|E^\pm\rangle \quad \text{(2.1)} \]
The Dirac basis vector expansion of (3b) above holds for every \( \phi^+ \) and every
\( \psi^- \) in terms of the eigenkets \( |E^+\rangle \) and \( |E^-\rangle \), respectively:
\[ \{ \phi^+ \} \ni \phi^+ = \sum_{jj\eta} \int_0^\infty \ dE |Ejj\eta^+\rangle \langle Ejj\eta^+|\phi^+ \rangle \quad \text{(2.2+)} \]
\[ \{ \psi^- \} \ni \psi^- = \sum_{jj\eta} \int_0^\infty \ dE |Ejj\eta^-\rangle \langle Ejj\eta^-|\psi^- \rangle \quad \text{(2.2-)} \]
The eigenkets \( |E^\pm\rangle \) of the full Hamiltonian in (2.1) are also assumed to be
the plane-wave solutions to the Lippmann-Schwinger equations
\[ |E^\pm\rangle = |E\rangle + \lim_{\epsilon \to 0} \frac{1}{E - H_0 \pm i\epsilon} V|E^\pm\rangle = \Omega^\pm |E\rangle \quad \text{(2.3\pm)} \]
where $|E\rangle$ fulfill the eigenvalue equation $H_0|E\rangle = E|E\rangle$ for the “free Hamiltonian” $H_0$ of (2.1).

As seen from (2.1), the eigenkets $|E^+\rangle$ and $|E^-\rangle$ both correspond to the same eigenvalue $E$, but (2.3$\pm$) shows that they fulfill different boundary conditions expressed by $+i0$ and $-i0$.

In scattering theory, the set of functions that are admitted to serve as energy wave functions in (2.2$\pm$),

$$\phi^+(E) = \langle Ejj_3\eta^+|\phi^+\rangle = \langle E|\phi^{in}\rangle \quad (2.4+)$$

and

$$\psi^-(E) = \langle Ejj_3\eta^-|\psi^-\rangle = \langle E|\psi^{out}\rangle \quad (2.4-)$$

are usually assumed to be the same set of smooth functions as the functions $\langle Ejj_3\eta|\phi\rangle$ that appear in the basis vector expansion hypothesis (A3b). That is,

$$\{\phi^+(E)\} = \{\psi^-(E)\} = \{\phi(E)\} = \text{Schwartz function space} \quad (2.5a)$$

For the vectors, this means

$$\{\phi^+\} = \{\psi^-\} = \Phi \subset \mathcal{H} \subset \Phi^\times \quad (2.5b)$$

(where $\Phi$ is dense in $\mathcal{H}$). The assumption $\{\phi^+\} = \{\psi^-\} = \Phi$ (or, the version $\mathcal{H}^{in} = \mathcal{H}^{out} = \mathcal{H}$) is known in scattering theory and quantum field theory as the assumption of asymptotic completeness.

The time evolution of the state $\phi^+(t)$ is given by the Schrödinger equation

$$i\hbar \frac{d\phi^+(t)}{dt} = H\phi^+(t) \quad (2.6+)$$

The solution to this equation under the Hilbert space boundary condition of assumption (A2) above is

$$\phi^+(t) = e^{-iHt}\phi^+, \text{ with } -\infty < t < \infty \quad (2.7+)$$

The time evolution of the observable $\Lambda(t) = |\psi^-(t)\rangle\langle\psi^-(t)|$ is given by the Heisenberg equation of dynamical motion

$$\frac{d\Lambda(t)}{dt} = -\frac{i}{\hbar} [\Lambda(t), H], \text{ or by } i\hbar \frac{d\psi^-(t)}{dt} = -H\psi^-(t) \quad (2.6-)$$

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The solution of this equation under the Hilbert space boundary condition of assumption (A2) is

$$\Lambda(t) = e^{iHt}e^{-iHt}, \quad \text{or} \quad \psi^-(t) = e^{iHt}\psi^- \quad \text{with} \quad -\infty < t < \infty \quad (2.7-)$$

If $\{\phi^+\}$ and $\{\psi^-\}$ are assumed to be a Hilbert space and if the Hamiltonian $H$ is a self-adjoint operator, then, by the well-known Stone-von Neumann theorem [9], (2.7+) and (2.7−) are necessarily the unique solutions to the dynamical equations in the Schrödinger and Heisenberg pictures, (2.6+) and (2.6−). Moreover, this theorem asserts that the operators $e^{-iHt}$ and $e^{iHt}$ are unitary for each $-\infty < t < \infty$ and that the mappings $t \to e^{-iHt}\phi^+$ and $t \to e^{iHt}\psi^-$ are continuous. It is noteworthy that Stone’s theorem requires the (norm complete) Hilbert space $\{\phi^+\} = \{\psi^-\} = \mathcal{H}$, in contrast to, say, (2.5) above. However, it is possible to show that the solutions (2.7±) hold for all $-\infty < t < \infty$ also for the Schwartz space completion of (2.5), although there are subtle mathematical differences between the two cases (A2) and (2.5) [15].

If the solutions (2.7±) hold for the vectors $\phi^+$ and $\psi^-$, then it follows, by duality, that the eigenkets $|E^+\rangle$ and $|E^-\rangle$ behave much like $\psi^-$ and $\phi^+$, respectively. That is,

$$\langle \phi(t)|E^+\rangle = \langle e^{-iHt}\phi^+|E^+\rangle = \langle \phi^+|e^{iH^\times t}|E^+\rangle = e^{iEt}\langle \phi^+|E^+\rangle \quad (2.8a)$$

Or, as an eigenvalue equation between functionals,

$$e^{iH^\times t}|E^+\rangle = e^{iEt}|E^+\rangle, \quad -\infty < t < \infty \quad (2.8b)$$

Likewise,

$$\langle \psi^-(t)|E^-\rangle = \langle e^{iHt}\psi^-|E^-\rangle = \langle \psi^-|e^{-iH^\times t}|E^-\rangle = e^{-iEt}\langle \psi^-|E^-\rangle \quad (2.9a)$$

Or, as an eigenvalue equation between functionals,

$$e^{-iH^\times t}|E^-\rangle = e^{-iEt}|E^-\rangle, \quad -\infty < t < \infty \quad (2.9b)$$

In (2.9a) and (2.9b), $H^\times$ is the uniquely defined extension of $\bar{H} = H^\dagger$ to the space $\Phi^\times$. It is clear that (2.8b) and (2.9b) depend on the time evolution of $\phi^+$ and $\psi^-$, given by (2.7+) and (2.7−). The latter equations depend on the assumption that $\phi^+$ and $\psi^-$ are elements of the Schwartz space $\Phi$ of (2.5). Therefore, if (2.8b) and (2.9b) hold, then $|E^\pm\rangle$ must be Schwartz space kets,
i.e., functionals on the Schwartz space, meaning that \( \phi^+(E) = \langle +E | \phi^+ \rangle \) and \( \psi^-(E) = \langle -E | \psi^- \rangle \) are infinitely differentiable, rapidly decreasing functions on the real (and positive) energy axis.

This requirement on \( |E^\pm\rangle \), however, is in contradiction with the requirement that \( |E^\pm\rangle \) be solutions of the Lippmann-Schwinger equations (2.3±) which contain the complex energies \( E \pm i\epsilon \). As already mentioned, there is a physical distinction between the vectors \( \phi^+ \) and \( \psi^- \) as being related to experimentally accessible \( \phi^\text{in} \) and \( \psi^\text{out} \) for \( t \to -\infty \) and \( t \to \infty \), respectively. 

As we shall see in the next section, these asymmetric boundary conditions in time are what give rise to the limits \( \epsilon \to 0^+ \) and \( \epsilon \to 0^- \) in (2.3±) that define the \( \pm \) signs in the kets \( |E^\pm\rangle \).

### 3 What the Lippmann-Schwinger Equations Suggest

It is the term \( \pm i\epsilon \) in (2.3±) which tells us that the Lippmann-Schwinger kets \( |E^\pm\rangle = \lim_{\epsilon \to 0} |E \pm i\epsilon\rangle \) cannot be ordinary Dirac kets (Schwartz space functionals). The infinitesimals \( \pm i\epsilon \) indicate that the energy wave functions \( \langle \phi^+ | E^+ \rangle \) and \( \langle \psi^- | E^- \rangle \) must not only be Schwartz space functions of the real variable \( E \), as asserted by the axiom (2.5), but they must also be limits of functions defined on some region of the upper and lower complex plane of \( E \). It is simplest to assume that \( \langle \phi^+ | E^+ \rangle \) and \( \langle \psi^- | E^- \rangle \) are boundary values of analytic functions defined on such a region in the (open) upper complex half-plane \( \mathbb{C}_+ \) and lower complex half-plane \( \mathbb{C}_- \), respectively. As the complex semi-plane in energy, one takes the second (or higher) Riemann surface of the analytic \( S \)-matrix. Thus, we have the following basic hypothesis which replaces (2.5):

**Functions**

\[
\text{Functions } \phi^+(E) = \langle +E | \phi^+ \rangle = \overline{\langle \phi^+ | E^+ \rangle} \text{ have analytic extensions into } \mathbb{C}_-. \tag{3.1+}
\]

and

\[
\text{Functions } \psi^-(E) = \langle -E | \psi^- \rangle = \overline{\langle \psi^- | E^- \rangle} \text{ have analytic extensions into } \mathbb{C}_+. \tag{3.1-}
\]

To make (2.3±) possible, the analytic extensions of (3.1+) and (3.1−) must exist at least on a small strip below and above on the real energy axis (i.e., the physical scattering energies). We shall generalize this to the
hypothesis that the analytic extensions of the energy wave functions should exist on the entire upper and lower energy half-planes.

The requirement \((3.1\pm)\) is not inconsistent with the Schwartz space hypothesis of \((2.5)\). Rather, \((3.1\pm)\) strengthens \((2.5)\). However, the stronger condition \((3.1\pm)\) is not consistent with the solutions \((2.7\pm)\) of the dynamical equations \((2.6\pm)\), obtained as consequences of the weaker condition \((2.5)\). Likewise, the time evolutions equations \((2.8b)\) and \((2.9b)\), which one universally assumes for (all) energy eigenkets, also do not hold under the hypothesis \((3.1\pm)\).

As stated above, the requirements of \((3.1)\) are supplementary to the usual hypothesis of quantum mechanics. Thus, the wave functions \(\phi^+(E)\) and \(\psi^-(E)\) are still assumed to be, for instance, smooth, rapidly decreasing and square integrable. The simultaneous requirements of analyticity and square integrability introduces certain (unexpected) restrictions into the theory. For instance, it can be shown [17, 19] that these requirements can be met for the time translated functions \((2.8a)\) and \((2.9a)\) only if \(t \geq 0\).

Since the time translation equations \((2.8b)\) and \((2.9b)\) are derived from \((2.8a)\) and \((2.9a)\), the conclusion \(t \geq 0\) also holds for the kets \(\langle E^+ \mid \phi^+ \rangle\).

Thus, the first conclusion that we draw from the Lippmann-Schwinger equations \((2.3\pm)\) is that the time evolution of the vectors \(\phi^+\) and \(\psi^-\) in \((2.2\pm)\) should not be given by the unitary group solution of the dynamical equations \((2.6\pm)\), but by the semigroup solution:

\[
\phi^+(t) = e^{-iHt} \phi^+ \quad \text{for } 0 \leq t < \infty \text{ only.} \tag{3.2+}
\]

\[
\psi^-(t) = e^{iHt} \psi^- \quad \text{for } 0 \leq t < \infty \text{ only.} \tag{3.2-}
\]

From this we see that as a consequence of the \(\pm i\epsilon\) in the Lippmann-Schwinger equations \((2.3\pm)\), the \(\{\phi^+\}\) and \(\{\psi^-\}\) given by the Dirac basis vector expansion \((2.2\pm)\) are in general different mathematical quantities with different (“conjugate”) semigroups \((3.2\pm)\) of time evolution. The unitary group evolution \((2.7\pm)\) which follows from \((2.5)\) is in conflict with the Lippmann-Schwinger equations.

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1Actually, this feature of time evolution can be seen from a simple heuristic argument that goes as follows. If the time translated function \(\langle \phi^+(t)|E^+ \rangle\), just like the function \(\langle \phi^+|E^+ \rangle\) is the square integrable boundary value function of an analytic function defined in the upper half-plane, then for \(E = E + i\epsilon\), we have \(\langle \phi^+(t)|E + i\epsilon \rangle = e^{i(E+i\epsilon)t} \langle \phi^+|E + i\epsilon \rangle\). Since \(\epsilon\) is positive, \(e^{i(E+i\epsilon)t} \langle \phi^+|E + i\epsilon \rangle\) is bounded for arbitrary values of \(\epsilon\) only if \(t\) is positive. A similar argument holds for the time translation of the observable wave functions \(\langle \psi^-(t)|E^- \rangle\) of \((2.9a)\). The rigorous proof is given in text following (3.8).
Schwinger equations. Time evolutions which are not in conflict with the Lippmann-Schwinger equations (2.3±) are (3.2±).

Thus, on the basis of (3.1±), we identify two different vector spaces \( \{ \phi^+ \} \neq \{ \psi^- \} \), one for the states and the other for the observables. The operators \( e^{-iHt} \) and \( H \) in (3.2+) are operators defined in the vector space \( \{ \phi^+ \} \). Likewise, operators \( e^{iHt} \) and \( H \) in (3.2−) are operators defined in the vector space \( \{ \psi^- \} \). Now, from (3.1+) we know that the wave functions \( \phi^+(E) = \langle +E | \phi^+ \rangle \) corresponding to the vectors \( \phi^+ \) are analytic in \( \mathbb{C}_- \). Therefore, we call the vector space \( \{ \phi^+ \} \equiv \Phi_- \). Similarly, from (3.1−), the wave functions \( \psi^−(E) = \langle \psi^- | E^- \rangle \) are analytic in \( \mathbb{C}_+ \), and for this reason we call the vector space \( \{ \psi^- \} \equiv \Phi_+ \). The two vector spaces \( \Phi_\pm \) are then two different subspaces of the Hilbert space \( \mathcal{H} \) (and also of the Schwartz space \( \Phi \)):

\[
\begin{align*}
\phi^+ & \in \Phi_- \subset \mathcal{H} \quad (3.3+) \\
\psi^- & \in \Phi_+ \subset \mathcal{H} \quad (3.3-)
\end{align*}
\]

What remains now is to put additional conditions on the analytic functions (3.1±) such that the spaces \( \Phi_\pm \) become nuclear spaces. Then, the triplet of spaces

\[
\Phi_- \subset \mathcal{H} \subset \Phi_-^* \quad (3.4+)
\]

\[
\Phi_+ \subset \mathcal{H} \subset \Phi_+^* \quad (3.4-)
\]

become Gelfand triplets, also known as Rigged Hilbert Spaces. The ordinary Dirac kets require one RHS (2.5b). However, if the kets are also to fulfill the Lippmann-Schwinger equations (2.3±), one needs the pair of RHS’s, (3.4±).

The \( \Phi_\pm^* \) in (3.4±) are the dual spaces, consisting of continuous anti-linear functionals on \( \Phi_\pm \). The new kets \( |E^\pm\rangle \) have then a well defined meaning as elements of the dual spaces \( \Phi_\pm^* \), and the nuclear property of (3.4±) allows Dirac’s basis vector expansion (2.2±) to be established as the nuclear spectral theorem of Gelfand et al and Maurin [20]. The pair of Gelfand triplets (3.4) have been constructed by Gadella [19] by choosing for the spaces of wave functions (3.1) particular subspaces of Hardy functions [21] \(^3\)

\(^2\)To be precise in notation, one should distinguish between \( H = H_- \), the restriction of the Hilbert space operator \( \bar{H} \) to \( \Phi_- = \{ \phi^+ \} \) and \( H = H_+ \), the restriction of the Hilbert space operator \( \bar{H} \) to \( \Phi_+ = \{ \psi^- \} \). For the sake of notational simplicity we will avoid this distinction whenever it does not lead to misunderstanding.

\(^3\)This choice is the following:

\[
\phi^+(E) \in \mathcal{H}_-^2 \cap S|_{R_+} \quad (3.5+)
\]
Associated with an operator defined in the Hilbert space $\mathcal{H}$, there exist two triplets of operators corresponding to the two triplets of spaces in (3.5). For instance, for the Hamiltonian $H$,

$$H_- \subset \bar{H} = H^\dagger \subset H_-^\times$$  \hspace{1cm} (3.6+)

$$H_+ \subset \bar{H} = H^\dagger \subset H_+^\times$$  \hspace{1cm} (3.6−)

where $H_\mp$ are the uniquely defined restrictions $\bar{H}\big|_{\Phi_\mp}$ of the self-adjoint Hamiltonian $\bar{H}$ to the dense subspace $\Phi_\mp$ of $\mathcal{H}$. The operators $H_\mp^\times$ are the conjugate operators of $H_\mp$, which are uniquely defined extensions of $H^\dagger$ to $\Phi_\mp^\times$. When their meaning is clear from the context, we usually omit the subscripts $\mp$ and superscript $\times$ in these various operators and denote all of them simply by $H$.

Defining the Lippmann-Schwinger kets now as functionals on $\Phi_\mp$, the $|E j \beta \xi \rangle \in \Phi_\mp^\times$ have analytic extensions into the whole complex semi-plane

$$\psi^-(E) \in \mathcal{H}_2^\mp \cap \mathcal{S}|_{\mathbb{R}_+}$$  \hspace{1cm} (3.5−)

Here, $\mathcal{H}_2^\mp$ denote Hardy class functions. $\mathcal{S}$ stands for the Schwartz space, and the symbol $|_{\mathbb{R}_+}$ represents the restriction of the domains of functions in $\mathcal{H}_2^\mp \cap \mathcal{S}$ to the positive real line, $\mathbb{R}_+$, assumed to be the range of scattering energy values. Loosely speaking, Hardy class functions $f^\mp \in \mathcal{H}_2^\mp$ are functions defined on the real line fulfilling the following two properties [17, 19, 21]:

1. $f^\pm(x)$ are point-wise limits of analytic functions $F^\pm(z)$ on $\mathbb{C}_\pm$, i.e., $f^\pm(x) = \lim_{y \to 0} F^\pm(x \pm iy)$

2. The $f^\pm$ are square integrable, $\int_{-\infty}^{\infty} |f^\pm(x)|^2 \, dx < \infty$

The intersections $\mathcal{H}_2^\mp \cap \mathcal{S}$ ensure that the functions $\phi^+(E)$ and $\psi^-(E)$, in addition to having the desired analyticity properties for complex energies, are, for real energy values, infinitely differentiable and rapidly decreasing at infinity. Equally importantly, when defined as in 3.5, the nuclearity of the Schwartz space $\mathcal{S}$ can be used to define a topology for $\Phi_\pm$ so that these spaces are nuclear. The one-to-one association of smooth Hardy functions for the energy wave functions in (3.5±) is more restrictive than the analyticity of the wave functions in the small strip above or below the real axis, the weakest condition demanded by the Lippmann-Schwinger equations (2.3). It is a mathematical idealization, like the idealization to Lebesgue square integrable functions in Hilbert space quantum mechanics. The Hardy space idealization, a refinement of the Hilbert space idealization, is better suited for quantum physics because it provides a mathematical distinction between states $\phi^+ \in \Phi_-$ and observables $\psi^- \in \Phi_+$. It also provides a mathematical basis for the Lippmann-Schwinger integral equations, which incorporate the in-coming and out-going boundary conditions.
\( \mathbb{C}_\pm \) of the second sheet of the \( S \)-matrix. This property has turned out to be very important for the unified theory of resonances and decay.

In sum, we have conjectured the new hypothesis which distinguishes mathematically between states and observables:

Set of prepared states defined by preparation apparatus (accelerator), e.g., in-states
\[ \{ \phi^+ \} = \Phi_- \subset \mathcal{H} \subset \Phi^\times_- \tag{3.7+} \]

Set of registered observables defined by registration apparatus (detector), e.g., out-states
\[ \{ \psi^- \} = \Phi_+ \subset \mathcal{H} \subset \Phi_+^\times \tag{3.7−} \]

We take (3.7) as a fundamental axiom which replaces the Hilbert space axiom (A2) of Section 2.

The spaces \( \Phi_\pm \) are two different dense subspaces of the same Hilbert space \( \mathcal{H} \). As stated above, the spaces \( \Phi_\pm \) can be understood as the abstract vector spaces whose realizations in terms of energy wave functions have the smooth Hardy space property (3.5±). In other words, the space \( \Phi_- \) is given by the set of vectors \( \{ \phi^+ \} \) whose Dirac vector expansion is given by (2.2+), where the “coordinates \( \langle {}^+E_{jj}\eta | \phi^+ \rangle \) with the continuous label” \( E \) (the analogue of the label \( i = 1, 2, 3 \) in the basis vector expansion \( \vec{x} = \sum_{i=1}^3 \vec{e}_i x^i \)) are the smooth Hardy functions \( \phi^+(E) = \langle {}^+E_{jj}\eta | \phi^+ \rangle \) with the property (3.5+). Similarly, the space \( \Phi_+ \) is the set of vectors \( \{ \psi^- \} \) whose “coordinates” with respect to the continuous basis \( \{ E_{jj}\eta^- \} \) are the smooth Hardy functions (2.4−) with the property (3.5−). An immediate mathematical consequence of the Hardy space axiom (3.5±) is that the solutions of the dynamical equations (2.6±) have the important (semigroup) property (3.2):

For \( \phi^+(t) \) fulfilling Schrödinger’s Eq., \( \phi^+(t) = e^{-iH^-t} \phi^+ \) for \( t \geq 0 \) (3.8+)

For \( \psi^−(t) \) fulfilling Heisenberg’s Eq., \( \psi^−(t) = e^{iH^+t} \psi^− \) for \( t \geq 0 \) (3.8−)

This semigroup time evolution (3.8±) is a consequence of a theorem of Paley and Wiener [22] (See also the appendix of [18]) for Hardy class functions. The theorem states that if \( G_-(E) \) is a Hardy class function, then its Fourier transform
\[
\hat{G}_-(\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dE e^{iE\tau} G_-(E) \tag{3.9}\]

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must fulfill the condition
\[ \check{G}_-(\tau) = 0 \quad \text{for} \quad -\infty < \tau < 0 \quad (3.9b) \]

It further follows from the theorem that for any positive value of \( \tau \), say \( |\tau_0| \), there exists a Hardy function \( G^{\tau_0}(E) \in \mathcal{H} \cap \mathcal{S} \) such that
\[ \check{G}^{\tau_0}(\tau) \neq 0 \quad \text{for} \quad 0 < \tau < |\tau_0| \quad (3.10) \]

Now, consider the Hardy space function \( \langle +E|\phi^+ \rangle \) and the Hardy space function \( \langle +E|\phi^+(t) \rangle \) of the time translated state \( \phi^+(t) \). Since \( \phi^+(t) \) fulfills the Schrödinger equation (2.6+), \( \phi^+(t) = e^{-iHt}\phi^+ \) and its expansion coefficients \( e^{-iEt}\langle +E|\phi^+ \rangle \) in the basis vector expansion
\[
\phi^+(t) = \int dE|E^+\rangle\langle +E|\phi^+(t) \rangle = \int dE|E^+\rangle\langle +E|e^{-iHt}\phi^+ \rangle = \int dE|E^+\rangle (e^{-iEt}\langle +E|\phi^+ \rangle)
\]
as well as the expansion coefficient \( \langle +E|\phi^+ \rangle \) in (2.2+) must, according to (3.7+), be a Hardy function of the lower half-plane \( \mathbb{C}_- \) if both \( \phi^+ \) and \( \phi^+(t) \) are to represent prepared states. That is,
\[ G_-(E) \equiv \langle +E|\phi^+ \rangle \in \mathcal{H}_- \cap \mathcal{S} \quad (3.11a) \]
as well as
\[ G^t_-(E) \equiv e^{-iEt}\langle +E|\phi^+ \rangle \in \mathcal{H}_- \cap \mathcal{S} \quad (3.11b) \]

It is an elementary property that the Fourier transform \( \check{G}^t_-(\tau) \) of the function (3.11b) is related to the Fourier transform \( \check{G}_- \) of the function (3.11a):
\[ \check{G}^t_-(\tau) \equiv \frac{1}{\sqrt{2\pi}} \int dE e^{iE\tau} G^t_-(E) = \frac{1}{\sqrt{2\pi}} \int dE e^{iE(\tau-t)} G_-(E) = \check{G}_-(\tau - t) \quad (3.12) \]

Now, if we want both \( G_-(E) \) and \( G^t_-(E) \) to be Hardy space functions as in (3.11a) and (3.11b), then it follows from the Paley-Wiener theorem (3.9) that
\[ \check{G}_-(\tau) = 0 \quad \text{for} \quad -\infty < \tau < 0 \quad (3.13a) \]
and
\[ \check{G}^t_-(\tau) = 0 \quad \text{for} \quad -\infty < \tau < 0 \quad (3.13b) \]
But, because of (3.12), we also have
\[ \hat{G}_-(\tau - t) = 0 \quad \text{for} \quad -\infty < \tau - t < 0 \] (3.13c)

From (3.13a) and (3.13c), we have the simultaneous conditions \(-\infty < \tau < 0\) and \(-\infty < \tau < t\). These two requirements on \(\tau\) are clearly satisfied for positive values of \(t\). If \(t\) is negative, say \(t = -|t|\), then the property \(\hat{G}''_-(\tau) = \hat{G}_-(\tau - t) = 0\) is ensured only for \(-\infty < \tau < -|t|\), not for \(-\infty < \tau < 0\) as required by (3.13b). In fact, from (3.10), we see that there is at least one function in the space \(H^-_2 \cap S\) for which the condition (3.13b) is not fulfilled for \(-|t| < \tau < 0\). Therefore, \(t \geq 0\) must hold, and the time evolution for the states \(\phi^+(t)\) can only be defined for the semigroup (3.8+). A similar argument using the Hardy functions \(H^+_2 \cap S\) leads to the conclusion (3.8−).

The conjugate operators\(^4\) of \(U_\pm(t)\), defined by the identities \(\langle \phi^+ | U^\times_- t | F^+ \rangle = \langle \phi^+ | U^\times_- | F^+ \rangle\) for every \(\phi^+ \in \Phi_-, \ F^+ \in \Phi^\times_-\) and \(\langle U_+ \psi^- | F^- \rangle = \langle \psi^- | U^\times_+ F^- \rangle\) for every \(\psi^- \in \Phi_-, \ F^- \in \Phi^\times_+\), give the time evolutions in the dual spaces \(\Phi^\times_-\):

\[ U^\times_-(t) | F^+ \rangle = e^{iH^\times_- t} | F^+ \rangle, \quad t \geq 0, \quad F^+ \in \Phi^\times_- \] (3.14+)

\[ U^\times_+(t) | F^- \rangle = e^{-iH^\times_+ t} | F^- \rangle, \quad t \geq 0, \quad F^- \in \Phi^\times_+ \] (3.14−)

For the special case \(F = |E_{jj3}\eta^-\rangle\) where \(H^\times_+ |E_{jj3}\eta^-\rangle = E|E_{jj3}\eta^-\rangle\),

\[ U^\times_+(t) |E_{jj3}\eta^-\rangle = e^{-iH^\times_+ t} |E_{jj3}\eta^-\rangle = e^{-iEt} |E_{jj3}\eta^-\rangle \quad \text{for} \quad t \geq 0 \text{ only.} \] (3.15)

The set of operators \(\{U_-(t) = e^{-iH_- t} : 0 \leq t < \infty\}\) do not form a group because there is no inverse operator \((U_-(t))^{-1}\) for every element of this set as required by the group axioms. In contrast, for the set of unitary operators \(\{U(t) = e^{-iH t} : -\infty < t < \infty\}\) in the Hilbert space \(H\) there is an inverse operator \((U(t))^{-1} = U(-t)\) for every \(U(t)\) so that the set constitutes a group. Aside from the absence of inverse operators, the set of operators \(\{U_-(t) = e^{-iH_- t} : 0 \leq t < \infty\}\) fulfills all other defining axioms of a group, and is called a semigroup. Therefore, there are two different representations of the time translation semigroup \(0 \leq t < \infty\) given by the operators \(U_+(t) = e^{\mp iH_+ t}\) of

\(^4\)Note that the operators acting on the spaces \(\Phi_\mp\) are labeled by the \(\mp\) signs, e.g., \(U_\mp, \ H_\mp\). The \(\mp\) signs labeling the spaces follow from the mathematicians' convention for the lower and upper Hardy class. The signs that label the vectors, on the other hand, follow from most physicists' notation of scattering theory and are opposite to those that label the spaces: \(\phi^+ \in \Phi_-, \ \psi^- \in \Phi_+, \ |E^\pm\rangle \in \Phi^\times_\pm\).
(3.2±) in the two spaces \( \Phi_\mp \). Likewise, the conjugate operators defined above in (3.14±) also furnish two representations of the time translation semigroup \( 0 \leq t < \infty \) in the dual spaces \( \Phi_\mp^* \). In both of these cases, we have the condition \( t \geq 0 \) (because of the difference in sign on the right hand side of the dynamical equations (2.6±)).

The semigroup time evolution is an important consequence of the axiom (3.7±). This axiom makes it possible for the Hamiltonians \( H_\mp \) to have eigenkets with complex eigenvalues. The semigroup character of time evolution makes the probability densities for complex energy eigenstates finite. If one would force the unitary time evolution (2.7±) on these eigenstates with complex energy, one would obtain infinite probabilities, which is the well-known “exponential catastrophe” for the original Gamow wave functions [11].

Under the new axiom (3.7±), the Gamow state vector is derived from the \( S \)-matrix pole at complex energy value \( z_R = E_R - i\Gamma/2 \) as an eigenket (functional) \( |z_R^j\eta\rangle \in \Phi_\mp \) with generalized eigenvalue \( z_R \) [16, 17, 18]. In the construction of these Gamow kets, the eigenvalue \( z_R \) is the complex position of the \( S \)-matrix pole. Under the new axiom (3.7±), eigenkets of essentially self-adjoint Hamiltonians with complex energy are now well defined as functionals on the spaces \( \Phi_\mp \): the Lippmann-Schwinger kets \( |E \mp i\epsilon, jj\eta\rangle \) can be analytically extended into the complex semi-plane \( \mathbb{C}_\mp \) (this means the bra \( \langle \mp E + i\epsilon, jj\eta | \) and the ket \( |E - i\epsilon, jj\eta \rangle \) as well as the integrand in the scalar product \( (\psi^-,\phi^+) \) can be analytically extended into the lower semiplane \( \mathbb{C}_- \) of the second sheet of the \( S \)-matrix \( S_j(E) \) except at singularities). The Gamow vectors are the evaluation of the analytically extended kets \( |z_R^j\eta\rangle \) in the lower half plane at the position \( z_R = E_R - i\Gamma/2 \) of the first order \( S \)-matrix pole. (Gamow-Jordan vectors belong to the higher order poles [23].) Then, from (3.15), the time evolution of the Gamow vectors is given by

\[
e^{-iH_\mp^* t}|z_R^j\eta\rangle = e^{-iz_R t}|z_R^j\eta\rangle = e^{-iE_R t}e^{-\frac{t}{2}\Gamma}|z_R^j\eta\rangle \quad \text{for} \quad t \geq 0 \quad \text{only}. \tag{3.16}
\]

This means there is an association between the the resonance pole of the \( j \)-th partial scattering amplitude \( a_j(E) \) and the Gamow vectors:

\[
\text{Resonance pole at } z_R = E_R - i\frac{\Gamma}{2} \quad \text{described by } a_j(E) = \frac{\Gamma}{E - z_R} \quad \text{is equivalent to} \quad \text{Space of states of Gamow vectors spanned by } |z_R^j\eta\rangle \tag{3.17}
\]

The resonance is defined by a pole of the \( S \)-matrix element of angular momentum \( j \) at the complex energy \( z_R = E_R - i\frac{\Gamma}{2} \) and is measured as a
Lorentzian (Breit-Wigner) bump with maximum at $E_R$ and full width at half-maximum $\Gamma$:

$$|a_j(E)|^2 = \frac{|r|}{(E - E_R)^2 + \left(\frac{\Gamma}{2}\right)^2} \tag{3.18}$$

To this resonance corresponds a ket which is defined by the Cauchy integral around the $S$-matrix pole $z_R$

$$|z_R j j_3 \eta^-\rangle = \frac{1}{2\pi i} \oint dz \frac{z j j_3 \eta^-\rangle}{z - z_R} = \frac{i}{2\pi} \int_{-\infty}^{\infty} dE \frac{|E j j_3 \eta^-\rangle}{E - z_R} \tag{3.19}$$

The second equality of (3.19) is the Titchmarsh theorem for Hardy functions (written here for functionals). This equality and the association (3.17) between Breit-Wigner resonance amplitude and Gamow state therefore require the new axiom (3.7±). (3.19) expresses the new ket $|z_R j j_3 \eta^-\rangle$ by a Dirac basis vector expansion as in (A3), except that the continuous summation extends over all real energy values $-\infty < E < \infty$, where $-\infty < E$ means that for the “unphysical” values $E < 0$, the energy $E$ is on the second Riemann sheet. We call the ket (3.19) with the energy wave function given by the Breit-Wigner amplitude (3.17) a Gamow vector because one can prove (again, using axiom (3.7±) that it fulfills (3.16). This Gamow vector (3.19) provides a state vector description to the Breit-Wigner resonance (3.18). The semigroup time evolution (3.16) of this state vector shows that this state is exponentially decaying with a lifetime $\tau = \frac{1}{\Gamma}$, where $\Gamma = -2\Im(z_R)$.

Unstable particles that are characterized by their lifetime are called decaying states, and they are conceptually and experimentally different from resonances, which are characterized by the resonance energy and width. From (3.16) and the fact that $z_R$ is the $S$-matrix pole, we see that the Gamow vector provides a unified description of decaying states and resonances, which can now be collectively called quasistable states. They elevate the heuristic lifetime-width relation $\tau = \frac{1}{\Gamma}$ to an exact and universal identity between two quantities that are observationally and mathematically different.

The time evolution equations (3.2), (3.14), (3.15) and (3.16) imply a particular finite value $t = 0$ at which time begins. What is the physical meaning of this initial moment of time? To answer the question, notice that under the axiom (3.7), the Born probabilities $\mathcal{P}_{\phi^+}(\psi^-(t))$ are defined, due to (3.8–), only for $t \geq 0$:

$$\mathcal{P}_{\phi^+}(\psi^-(t)) = |\langle \psi^-(t)|\phi^+\rangle|^2 = |\langle \psi^-|\phi^+(t)\rangle|^2 \text{ for } t \geq 0 \text{ only.} \tag{3.20}$$
For a resonance or decaying state represented by a Gamow vector $|z^{-}_R\rangle$, we have, using (3.16),

$$
P_{|z_R\rangle}(\psi^{-}(t)) = |\langle\psi^{-}(t)|z^{-}_R\rangle|^2 = |\langle\psi^{-}|z^{-}_R(t)\rangle|^2 = e^{-\Gamma t} |\langle\psi^{-}|z^{-}_R\rangle|^2 \text{ for } t \geq 0 \text{ only.}$$

Equations (3.20) and (3.21) tell us that a time independent observable $\psi^{-}$ can be measured in a time dependent state $\phi^{+}(t)$ only after a particular instant $t = 0$. (or, equivalently, the time dependent observable $\psi^{-}(t)$ can be measured in a time independent state $\phi^{+}$ only after the same instant $t = 0$).

In the case of the quasistable state of (3.21), the time $t = 0$ is interpreted as the time at which the state $|z^{-}_R(t)\rangle$ has been prepared, i.e., the quasistable particle is produced or formed. The observable $|\psi^{-}(t)\rangle\langle\psi^{-}(t)|$ representing the decay products can be detected only after this time, $t \geq 0$. From this point of view, the semigroup condition $t \geq 0$ expresses a simple causality condition: The observable $\psi^{-}$ can be measured only at times $t$ larger than the time $t = 0$ at which the state is prepared.

Such a particular moment cannot be singled out if we instead use the unitary group evolution of the Hilbert space, for which the probabilities (3.20) are necessarily defined for all $-\infty < t < +\infty$. It is well known that there are serious problems with accommodating causality into the conventional formalism of quantum mechanics [13]. Therefore, the causal time evolution that follows from the new Hardy space axiom is welcome. But it also poses a new question: what is the meaning of the semigroup time $t = 0$ and how can we observe it? This will be discussed in the following section.

4 Observing the Semigroup Time of Causal Evolution

The causal quantum mechanical semigroup $(3.2\pm)$ introduces a new concept, the semigroup time $t = t_0$. In the mathematical description, we call this $t_0 = 0$, but physically $t_0$ could be any finite time ($\neq -\infty$). This concept of a beginning of time is foreign to the conventional mathematical theory of quantum physics based on the Hilbert space axiom (A2) (or its slightly strengthened version (2.5)), in consequence of which follow the time evolution equations $(2.7\pm)$ with $-\infty < t < +\infty$. Nevertheless, a beginning of time $t_0$ has been mentioned before by Gell-Mann and Hartle in their quantum theory of the universe [7], where $t_0$ was chosen as the big bang time and
where the restriction of the unitary group evolution (2.7−) to \((t - t_{\text{bigbang}}) \geq 0\) was introduced by fiat, in contradiction to the prediction (2.7±) of the Hilbert space axiom (A2). In our theory presented in this paper, the time asymmetry (3.2±) is a consequence of our Hardy space axiom (3.7±) which was demanded by the heuristic \((\mp i\epsilon)\) in scattering theory (and also in the propagator of field theory).

We now want to answer the questions: what is the meaning of this beginning of time \(t_0\) for quantum systems in experiments in the laboratory, and why have we not been more aware of its existence before?

In the usual experiments with quantum systems one works with a large ensemble. For example, the preparation time of an excited state of an atom or ion corresponds to the many different laboratory clock times at which each individual atom or ion of the ensemble is created. The situation is different if one can work with single quantum systems. By now, there are several experiments that use single, laser-cooled ions [24, 25]. The original experiments used \(Ba^+\) in a Paul-Straubel trap, Fig. 1. This is one of the simplest cases that nature provides with the most suitable arrangements for resonance energy levels and lifetimes, as depicted in Fig. 2.

![Figure 1: Schematics of the experimental setup used in [24, 25]](image-url)
Figure 2: Simplified energy-level scheme of $Ba^+$. 

In these experiments a single laser-cooled $Ba^+$ ion in a trap undergoes two laser driven transitions. First, driven by the 493-nm dye laser (Fig. 1), the ion goes from the ground state $6S_{1/2}$ into the excited state $6P_{1/2}$ from where it almost instantaneously (8 ns) decays into state $5D_{3/2}$. Second, from state $5D_{3/2}$ the ion is driven back to state $6P_{1/2}$ by the 650-nm dye laser (Fig. 1), from where it decays into the ground state, emitting 493-nm fluorescence radiation. This fluorescence radiation is monitored by the photo multiplier tube (PMT) in Fig. 1. Initially, the intensity of the fluorescence radiation shown in Fig. 3 is essentially a constant at about 16,000 counts/sec. Then, at the time “lamp on”, a 455-nm filtered Barium lamp (Fig. 1) is turned on. After this “lamp-on” time, the fluorescence radiation changes rapidly at random times from the initial value of 16,000 counts/sec to the background value of no fluorescence. The explanation is the following: The Barium lamp occasionally excites the $Ba^+$ into the state $6P_{3/2}$ from where it makes a fast transition into the state $5D_{5/2}$. This is a metastable state described by the Gamow vector $|z_R\ 5D_{5/2}^-\rangle \equiv \psi^G$. Since there is only one $Ba^+$ atom, it can either go through the transition levels $6S_{1/2} \leftrightarrow 6P_{1/2} \leftrightarrow 5D_{3/2}$ or be “shelved” in the metastable state $5D_{5/2}$. While it is shelved there cannot be fluorescent radiation $6P_{1/2} \rightarrow 6S_{1/2}$, which results in a dark period.
Figure 3: Amplification of single quantum jumps by the fluorescence of $S_{1/2} \leftrightarrow P_{1/2}$. The 203 onset times $t_i^0$ (three shown) of dark fluorescence are the preparation times of single “$D_{5/2}$”–quantum systems. The ensemble $\{t_i^0\}$ is the preparation time $t_0 = 0$ of the decaying quantum state $| 5 D_{5/2} \rangle$ which represents this ensemble of “$D_{5/2}$”–quantum systems. Usually the quantum mechanical ensemble is thought of as a large number of micro-objects at one and the same time. Here, the quantum mechanical ensemble is one and the same micro object prepared at a large number of times $t_i^0$, which is the same time $t = 0$ of the state $| 5 D_{5/2} \rangle$ describing the ensemble (this figure is taken from Ref. [24].)

The experiment [24] reported 203 dark periods, of which three are shown in Fig. 3. The state vector $\psi^G$ represents the ensemble of these 203 single quantum systems. (The superscript $-$ in $\psi^G = | z_R \ 5D_{5/2} \rangle$ indicates that this is an eigenstate of the total Hamiltonian $H = H_0 + H_I$, including the interaction $H_I$ and thus not an eigenstate of the orbital angular momentum with $(L = 2) = D_5$.) The state $\psi^G$ evolves in time according to (3.16) and decays exponentially in time according to (3.21).) Fig. 3 shows that each of the single systems making up the ensemble described by the state vector $| z_R \ 5D_{5/2} \rangle = \psi^G$ is individually produced by the resonance production
process
\[ \gamma(455\text{-nm}) + 6S_{1/2} \rightarrow 6P_{3/2} \rightarrow \gamma(615\text{-nm}) + 5D_{5/2} \]  
(4.1)

at particular laboratory times \( t^1_0, t^2_0, t^3_0, \cdots, t^{203}_0 \). (Of these, \( t^1_0, t^2_0 \) and \( t^3_0 \) are shown in Fig. 3 as the onset time of the first three dark periods.) These excited ions in \( 5D_{5/2} \) then decay according to
\[ 5D_{5/2} \rightarrow 6S_{1/2} + \gamma(1.76\mu\text{m}) \]  
(4.2)
at times \( t^1_1, t^2_1, t^3_1, \cdots, t^{203}_1 \), the instances at which the fluorescence returns to its pre-“lamp-on” levels. The duration of the dark period \( \Delta t^i = t^i_1 - t^i_0, \ i = 1, 2, 3, \cdots, 203 \), is the time which the \( i \)-th individual quantum system \( 5D_{5/2} \) “lives”. That is, at every onset time \( t^i_0 \) of the \( i \)-th dark period, the accuracy of which is determined by the short production time of (4.1), an individual \( 5D_{5/2} \) is “created”. It “lives” for the duration \( \Delta t^i = t^i_1 - t^i_0 \) and decays at \( t^i_1 \), the end of the \( i \)-th dark period.

This is a rather remarkable observation because it means that the excited \( Ba^+ \) in the quasistable \( 5D_{5/2} \)-level lives for a precise time \( \Delta t^i \). However, these times \( \Delta t^i \) are not reproducible quantities, as seen from the different duration lengths of the dark fluorescence periods.

The reproducible quantity is the ensemble average of the time intervals \( \Delta t^i \), the lifetime of the state \( 5D_{5/2} \):
\[ \tau^{\text{exp}} = \sum_i \Delta t^i \frac{N_D(t: \Delta t^i > t)}{N_D}, \]  
(4.3)

Here, \( N_D(t: \Delta t^i > t) \) is the number of dark periods of duration \( \Delta t^i > t \) and \( N_D \) is the total number of dark periods (203 for this experiment). In the Gamow vector description of the quasistable state \( 5D_{5/2} \), a theoretical prediction of the quantity \( \tau^{\text{exp}} \) can be made in terms of the resonance width, as shown below. The individual times \( \Delta t^i \) are not predictable quantities in quantum mechanics.

Let us now turn to the description of the state \( 5D_{5/2} \) by the Gamow state \( \psi^G \) and the problem of the physical meaning of the beginning semigroup time \( t_0 \). As discussed above, the ensemble state \( 5D_{5/2} \) consists of a large number of individual quantum physical systems, each created at a different laboratory time \( t^i_0 \). These times depend on the preparation conditions such as the intensity of the barium lamp (in the present experiment, it is chosen such that a transition to \( P_{3/2} \) takes place once every 10 s). However, as seen
from (4.3), the reproducible experimental quantities depend only on the time intervals \( \Delta t^i \), and not on the individual creation times \( t^i_0 \) or the decay times \( t^i_1 \). The time interval \( \Delta t^i = t^i_1 - t^i_0 \) is clearly invariant under a translation by \( t^i \), i.e., \( \Delta t^i = (t^i_1 - t^i) - (t^i_0 - t^i) \). Now, a time \( t^i \) can be chosen for each laboratory creation time \( t^i_0 \) such that
\[
 t^i_0 - t^i = t_0 \tag{4.4}
\]
where the time \( t_0 \) is independent of the index \( i \). The particular choice \( t_0 = 0 \) (i.e., \( t^i = t^i_0 \)) corresponds to the beginning semigroup evolution time of the Gamow state \( \psi^G \).

What (4.4) shows, above all, is that the individual micro-physical systems that make up an ensemble described by a quantum mechanical state can be prepared at different times (and, for that matter, different points in space). The time \( t_0 = 0 \) of (4.4) provides a reference time for the entire ensemble of the creation times \( \{t^i_0\} \),
\[
\text{Ensemble of experimental preparation times}\{t^i_0\} = \begin{cases} \text{Theoretical semigroup time} \\ t_0 = 0 \text{ of the prepared state} \end{cases} \tag{4.5}
\]
Thus, the individual systems of the ensemble can be treated as if they were created at the same laboratory time and the duration that each micro system “lives” can simply be characterized by the time at which it decays. This feature makes it possible to describe the entire ensemble by a single Gamow state vector \( \psi^G \) and the time evolution of the entire ensemble by a single time variable \( t \geq t_0 \). Such a state vector description, in turn, makes it possible to use the standard probability interpretation also for an ensemble that consists of a large number of micro systems created at vastly different laboratory times. For instance, by using (3.16) for the Gamow vector \( \psi^G(t) = e^{-iHt}\lvert z_R j j_R \eta \rangle = e^{-iHt}\lvert z_R 5^D 5/2 \rangle \), the lifetime of the excited state \( 5^D 5/2 \) can be computed in analogy to (3.21) as:
\[
\tau_{\text{theor}} = \int_{t_0=0}^{\infty} dt e^{-\Gamma t} = \frac{1}{\Gamma} \tag{4.6}
\]
The experimental quantity of (4.3) is to be compared with this theoretical quantity.

New in these remarkable experiments of [24, 25] is that the different creation times \( t^i_0 \) and durations times \( \Delta t^i \) for the single quantum systems are precisely and individually measured as the onset and duration of the dark periods of Fig. 3. These onset times are an experimental demonstration of the semigroup time \( t_0 = 0 \) of time asymmetric quantum theory.
5 Summary

Many of the heuristic notions used in the description of scattering and decay phenomena, like the incoming and outgoing Lippmann-Schwinger kets $|E^\pm\rangle = |E \pm i\epsilon\rangle$ with infinitesimal $\epsilon$, purely outgoing boundary conditions, time asymmetry and causality are not well defined in the mathematical frame set by the conventional (Hilbert space) quantum mechanics. Combining these notions with the Hilbert space axiom leads to contradictions, like the exponential catastrophe in which Gamow vectors and unitary time evolution conflicted [11], the deviations from the exponential decay where the exponential time dependence for the experimental counting rates conflicted with the mathematical properties of Hilbert space vectors [12], and the problems with (Einstein) causality where stability of matter (semi-boundedness of the Hilbert space Hamiltonian) leads to instant propagation of probabilities [13]. The $\pm i\epsilon$ of the Lippmann-Schwinger kets (or, of the propagator in relativistic quantum field theory) overcomes many of these problems.

But the Lippmann-Schwinger kets are mathematically undefined kets; they are not vectors of the Hilbert space and they cannot be defined as Schwartz space functionals because of the $\pm i\epsilon$. Therefore one cannot derive their time evolution (or, in the relativistic case, their evolution under Poincaré transformations). Nevertheless, one assumes it to be a unitary time evolution (as one also had assumed for the ordinary Dirac kets) with time extending over $-\infty < t < \infty$. This however is in conflict with the infinitesimal imaginary part $\pm i\epsilon$ since it would lead to non-continuous and unbounded (non-unitary) operators for time evolution (or, in the relativistic case, non-unitary representations of the Poincaré group). Complex extensions of energy (or, in the relativistic case, the invariant square mass $s = p_\mu p^\mu$) away from the real axis requires that the energy wave functions be boundary values of analytic functions in the complex semi-planes, not just (Lebesgue) square-integrable or smooth functions of real energy.

Using the Lippmann-Schwinger equation as the takeoff point and attempting to accommodate as many of the heuristic notions of scattering and decay as possible, we conjectured in this paper the new hypothesis (3.7±). It replaces the Hilbert space boundary conditions (A2) for the solutions of the Schrödinger or Heisenberg equation by the Hardy space boundary conditions (3.7±). Many of the heuristic notions, such as Gamow’s wave functions, that had been introduced phenomenologically into the description of scattering and decay phenomena appear also in this new quantum theory, but
now they have a rigorous mathematical foundation. Furthermore, the new theory leads to important novel conclusions, salient among which is a basic, quantum mechanical time asymmetry, expressed by the semigroup evolution of $(3.8 \pm )$. This overcomes the causality problem and leads to exponential decay for certain kets with complex energy, the Gamow kets.

Gamow kets have been derived from the resonance poles of the S-matrix using the new axiom $(3.7 \pm )$. Their energy wave function is a Lorentzian (Breit-Wigner) energy distribution characterized by its central value $E_R$ and width $\Gamma$, and the lifetime of its exponential decay is exactly $\tau = \frac{\hbar}{\Gamma}$. The new axiom $(3.7 \pm )$ thus provides a unified theory of resonance scattering and exponential decay.

But the semigroup also introduces a beginning of time for quantum systems, which is represented by the mathematical semigroup time $t = 0$. Though such a time has been mentioned before as the big bang time for universes [7] and its idea is already contained in the classic paper [6], one has not been much aware of it in the usual experiments with quantum systems in the laboratory. In the final section 4, we therefore discussed an experiment with single laser-cooled $Ba^+$ ions in a trap [24] where the beginnings of time for single micro-systems have been observed.

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