ORBIT INEQUIVALENT ACTIONS OF NON-AMENABLE GROUPS

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ABSTRACT. Consider two free measure preserving group actions $\Gamma \curvearrowright (X, \mu), \Delta \curvearrowright (X, \mu)$, and a measure preserving action $\Delta \curvearrowright (Z, \nu)$ where $(X, \mu), (Z, \nu)$ are standard probability spaces. We show how to construct free measure preserving actions $\Gamma \curvearrowright (Y, m), \Delta \curvearrowright (Y, m)$ on a standard probability space such that $E^\Delta \subset E^\Gamma$ and $d$ has $a$ as a factor. This generalizes the standard notion of co-induction of actions of groups from actions of subgroups. We then use this construction to show that if $\Gamma$ is a countable non-amenable group, then $\Gamma$ admits continuum many orbit inequivalent free, measure preserving, ergodic actions on a standard probability space.

1. INTRODUCTION

Let us consider a standard probability space $(X, \mu)$ with a countable group $\Gamma$ acting on $(X, \mu)$ in a Borel measure preserving manner. This gives rise to the orbit equivalence relation $E_\Gamma = \{(\gamma \cdot x, x) \mid x \in X\}$. Two such actions $\Gamma \curvearrowright (X, \mu), \Delta \curvearrowright (Y, \nu)$ are orbit equivalent if there exist conull subsets $A \subset X, B \subset Y$ and a measurable, measure preserving bijection $f : A \to B$ such that for any $x, y \in A$, we have $xE_\Gamma y$ if and only if $f(x)E_\Delta f(y)$.

The theory of orbit equivalence was originally motivated by its connections to operator algebras. Orbit equivalence first appeared in a paper by Murray and von Neumann [MvN36] via the “group measure space” construction. One may from a measure preserving free ergodic action of an infinite countable group obtain a type II$_1$ von Neumann factor with an abelian Cartan subalgebra. Two von Neumann algebras obtained in this fashion are isomorphic via an isomorphism preserving the Cartan subalgebras if and only if the corresponding actions are orbit equivalent (see [EM77]).

The first orbit equivalence result is due to Dye [Dye65], who showed that all ergodic measure preserving actions of $\mathbb{Z}$ are orbit equivalent. Later, the work of Ornstein, Weiss, Connes and Feldman (see [CFW81], [OW80]) provided a complete classification of ergodic measure preserving actions of amenable groups. In particular, it was established that all such actions are orbit equivalent to a $\mathbb{Z}$-action and, consequently, the orbit equivalence relation remembers only that the group is amenable.
For non-amenable groups, the situation is quite different. Connes, Weiss [CW80] and Schmidt [Sch81] showed that all non-amenable groups without Kazhdan’s property (T) admit at least two orbit inequivalent free, measure preserving ergodic actions. Bezuglyi and Golodets [BG81] showed that there exists a non-amenable group with continuum many orbit inequivalent such actions. Results concerning classes of groups exhibiting this phenomenon of continuum many actions gradually increased throughout the years. Zimmer [Zim84] showed that this holds for a number of specific groups with property (T).

Recently, Hjorth [Hjo05] showed that actually all groups with property (T) admit continuum many orbit inequivalent free, measure preserving, ergodic actions. Gaboriau and Popa [GP05] then used relative property (T) to show this for all non-cyclic free groups while Ioana [Ioa07] showed this for all groups that admit $F_2$ as a subgroup. The question of which groups admit continuum many orbit inequivalent actions has also been implicitly or explicitly considered as well as answered for classes of certain groups in the papers Monod-Shalom [MS06], Popa [Pop06], Kechris [Kec], Tornquist [Tör05], Fernos [Fer06].

The main goal of this paper is to present the proof of the following theorem:

**Theorem 1.1.** Let $\Gamma$ be a countable, non-amenable group. Suppose that there are free, measure preserving actions $\Gamma \actson (X, \mu)$, $F_2 \actson (X, \mu)$ on a standard probability space $(X, \mu)$ such that $\Gamma$ acts ergodically and $E_{F_2} \subseteq E_{\Gamma}$. Then $\Gamma$ admits continuum many orbit inequivalent free, measure preserving, ergodic actions.

Gaboriau and Lyons [GL] showed that every countable, non-amenable group admits a free, measure preserving, ergodic action on a standard probability space $(X, \mu)$ so that the orbit equivalence relation induced by the action contains the orbit equivalence relation induced by a free, measure preserving action of $F_2$ on $(X, \mu)$. From this and Theorem 1.1, we obtain the following corollary:

**Corollary 1.2.** Suppose that $\Gamma$ is a countable, non-amenable group. Then $\Gamma$ induces continuum many orbit inequivalent free, measure preserving, ergodic actions.

In [Ioa07], Ioana considered groups $\Gamma$ such that $F_2 \leq \Gamma$. Given $\Delta \leq \Gamma$ and an action $a$ of $\Delta$, there is a way to co-induce from this an action of $\Gamma$ so that the resulting action of $\Gamma$ restricted to $\Delta$ has the original action by $\Delta$ as a factor. Ioana then used an action of $F_2$ on $T^2$ as well as continuum many actions of $F_2$ obtained from irreducible non-isomorphic representation of $F_2$ and showed that co-inducing actions of $\Gamma$ from these actions yields continuum many orbit inequivalent actions of $\Gamma$. This result uses the fact that $(F_2 \times \mathbb{Z}^2, \mathbb{Z}^2)$ has relative property (T) and the fact that the co-induced action of $\Gamma$ has a strong connection to the action of $F_2$ on $T^2$. Here, the semidirect product $F_2 \times \mathbb{Z}^2$ is formed by letting $\text{SL}_2(\mathbb{Z})$ act on $\mathbb{Z}^2$ and viewing $F_2$ as a finite index subgroup of $\text{SL}_2(\mathbb{Z})$. In Section 2, we generalize the notion of a co-induction. In particular, given free measure preserving actions $\Gamma, \Delta \actson (X, \mu)$ such that $E_\Delta \subseteq E_{\Gamma}$ and a measure preserving action $\Delta \actson (Z, \nu)$, we show how to construct actions $\Gamma \actson (Y, m)$, $\Delta \actson (Y, m)$ so that $E_\Delta \subseteq E_{\Gamma}$ and $d$ has $a$ as a factor. In the special case when $\Delta$ is a subgroup of $\Gamma$, this reduces to the standard induction. In Section 3, we fit our actions from Section 2 into a theorem of Ioana.
and provide the proof of Theorem 1.1.

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2. THE ACTIONS OF $\Gamma$

The aim of this section is to generalize the notion of a co-induced action.

Throughout this paper, we will use the following notation.

Let $a$ and $b$ be two measure preserving actions of a group $\Gamma$ on $(X, \mu)$ and $(Y, \nu)$, respectively. $b$ is a factor of $a$, written $b \sqsubseteq a$, if there is a Borel measure-preserving map $p : X \to Y$ such that for $\gamma \in \Gamma$,

$$p(\gamma^a \cdot x) = \gamma^b \cdot p(x).$$

$E_a^\Gamma$ will denote the orbit equivalence relation induced by $a$ where

$$E_a^\Gamma = \{ (x, \gamma^a \cdot x) \mid x \in X, \gamma \in \Gamma \}.$$

The diagonal action $\Gamma \wr a \times b$ $(X \times Y, \mu \times \nu)$ is given by

$$\gamma^{a \times b} \cdot (x, y) = (\gamma^a \cdot x, \gamma^b \cdot y).$$

Let $L_0^2(X, \mu) = \{ f \in L^2(X) \mid \int_X f \, d\mu = 0 \}$. This is the orthogonal complement of the constant functions in $L^2(X)$. The Koopman representation $\kappa_0^a$ of $\Gamma \wr a$ $(X, \mu)$ on $L_0^2(X)$ is defined by

$$\gamma_0^a \cdot f(x) = f((\gamma^{-1})^a \cdot x).$$

If $\pi_1$ and $\pi_2$ are unitary representations of $\Gamma$, then $\pi_1 \leq \pi_2$ if $\pi_1$ is isomorphic to a subrepresentation of $\pi_2$. For a Borel space $X$, $B(X)$ will denote the Borel $\sigma$-algebra on $X$ and $P(X)$ will denote the space of probability measures on $X$. We will often drop the superscript and write $\gamma \cdot x$ as opposed to $\gamma^a \cdot x$ when it is clear which action is being used.

First recall the construction of a co-induced action of $\Gamma$ from an action of a subgroup $\Delta$. This first appeared in [DGRS08] and can also be found in [Gab05]. Suppose that $\Delta \wr a$ $(Z, \nu)$ in a Borel measure preserving manner where $(Z, \nu)$ is a standard probability space. Let $T \subset \Gamma$ be a left transversal of the cosets of $\Delta$ in $\Gamma$. Then the space

$$Y = \{ f : \Gamma \to Z \mid f(\gamma_0 \cdot \gamma) = \gamma_0^a \cdot f(\gamma) \}$$

has a natural identification with the space $Z^T$. The co-induced action of $\Gamma$ on $Z^T$ is obtained by identifying the action of $\Gamma$ on $Y$ given by

$$\gamma_0 \cdot f(\gamma) = f(\gamma_0^{-1} \gamma)$$

with the action on $Z^T$ given by

$$\gamma \cdot f(t) = \gamma_0^{-1} f(s)$$
where \( s \in T \) and \( \gamma_0 \in \Delta \) are such that \( s\gamma_0 = \gamma^{-1}_t \). This action is then measure preserving on the standard probability space \((T^T, \mu^T)\).

For our generalization, instead of letting \( \Delta \leq \Gamma \), we assume that the two groups \( \Delta \) and \( \Gamma \) admit free, measure preserving actions so that the orbit equivalence relation of the former is contained in the orbit equivalence relation of the latter.

**Theorem 2.1.** Suppose that there are free measure preserving actions \( \Delta \curlyequivalence^a_0 (X, \mu) \) and \( \Gamma \curlyequivalence^{b_0} (X, \mu) \) such that \( b_0 \) is ergodic and \( E^a_\Delta \subseteq E^{b_0}_\Gamma \). Also, let \( \Delta \curlyequivalence^a (Z, \nu) \) be measure preserving. Then there is a standard probability space \((Y, \mu)\) and actions \( c, d \) and map \( p \) so that the following hold:

1. \( \Gamma \curlyequivalence^c (Y, \mu) \) is free, measure preserving;
2. \( \Delta \curlyequivalence^d (Y, \mu) \) is free, measure preserving;
3. \( p : Y \to Z \) is measure preserving;
4. \( E^a_d \subseteq E^c_\Gamma \);
5. \( p \) witnesses that \( a \sqsubseteq d \).

Moreover, if \( a \) is ergodic, then \( c \) can be made ergodic as well.

**Proof.** By ergodicity of \( \Gamma \curlyequivalence X \), we may assume that the number of \( \Delta \)-equivalence classes in each \( \Gamma \)-equivalence class is uniform. In fact, without loss of generality, we will suppose that each \( \Gamma \)-equivalence class consists of infinitely many \( \Delta \)-equivalence classes.

Consider the space
\[
Y = \{(x, f) \mid f : [x]_\Gamma \to Z, f(\gamma \cdot x_0) = \gamma^{a \times a_0} \cdot f(x_0) \ \forall \gamma \in \Delta\},
\]
which we intend to represent as the standard probability space \((X \times \mathbb{Z}^\mathbb{N}, \mu \times \nu^\mathbb{N})\).

In the context of the original co-induced action, \([x]_\Gamma\) takes the place of \(\Gamma\). The following lemma is an adaptation of a coset transversal to our situation.

**Lemma 2.2.** There exists a sequence of functions \(\{g_i\}_{i \in \mathbb{N}}\) from \(X\) to \(X\) so that the following conditions hold:

1. each \(g_i\) is Borel;
2. \(g_0(x) = x\) for each \(x \in X\);
3. given \(x \in X\), \(\{g_i(x)\}_{i \in \mathbb{N}}\) enumerates a transversal for the \(\Delta\)-equivalence classes in \([x]_\Gamma\);
4. if \(i \neq j\) and \(x \in X\), then \(g_i(x) \neq g_j(x)\).

**Proof.** Let \(\{\gamma_n\}_{n \in \mathbb{N}}\) enumerate the elements of \(\Gamma\) so that \(\gamma_0 = e\). We define \(g_i : X \to X\) inductively on \(i\). First, define
\[
h_i(x) = \text{least } k \in \mathbb{N} \text{ such that}
\]
\[
\forall l < k, \exists j < i \left( ((\gamma_l \cdot x, \gamma_{h_j}(x)) \in E^a_\Delta) \right)
\]
\[
\wedge \ \forall j < i, \left( (\gamma_k \cdot x, \gamma_{h_j}(x)) \notin E^a_\Delta \right).
\]

That is, we want to take the least \(k\) such that the \(\Delta\)-equivalence class of \(\gamma_k \cdot x\) has not already appeared for a previous \(h_j\) and the \(\Delta\)-equivalence class of each \(\gamma_l \cdot x\)
for each \( l < k \) has already appeared. Then let
\[
g_i(x) = \gamma h_i(x) \cdot x.
\]
Conditions (2), (3) and (4) are clearly satisfied by our construction. It suffices to check that \( h_i \) is Borel since then \( g_i \) is a composition of two Borel maps.

Note that
\[
h_{i-1}(k) = \bigcup_{l<k} \bigcap_{j<i} \{ x \in X \mid (\gamma l \cdot x, g_j(x)) \in E_{a0} \}
\]
and projections of sets with countable sections are Borel. \( \square \)

Thus, we have an isomorphism \( F: Y \to X \times Z^N \) given by
\[
F(x, f) = (x, f(g_0(x)), f(g_1(x)), ...)
\]
and we may let \( m \) be the product measure \( \mu \times \nu^N \).

As for the actions, let \( \Gamma \rtimes^c (Y, m) \) be defined by
\[
\gamma^c \cdot (x, f) = (\gamma \cdot x, f) \quad \forall \gamma \in \Gamma
\]
and \( \Delta \rtimes^d (Y, m) \) be defined analogously by
\[
\gamma_0^d \cdot (x, f) = (\gamma_0 \cdot x, f) \quad \forall \gamma_0 \in \Delta.
\]

We will write the action \( c \) as a skew-product action on \( X \times Z^N \) consistent with the above representation. For this purpose, let \( S_\infty \) act on \( \Delta^N \) by shift, i.e., for \( \alpha \in S_\infty \), \( \delta \in \Delta^N \)
\[
\alpha \cdot \delta(k) = \delta(\alpha^{-1}(k))
\]
and consider the semidirect product \( S_\infty \ltimes \Delta^N \). Then define the cocycles
\[
\alpha: \Gamma \times X \to S_\infty, \quad \delta: \Gamma \times X \to \Delta^N, \quad \beta: \Gamma \times X \to S_\infty \ltimes \Delta^N
\]
by
\[
\alpha(\gamma, x)(k) = n \iff g_k(x) = g_n(\gamma \cdot x) \quad \delta(\gamma, x)(k)g_\alpha(\gamma, x)^{-1}(k)(x) = g_k(\gamma \cdot x) \quad \beta = (\alpha, \delta).
\]
Then let \( S_\infty \ltimes \Delta^N \rtimes Z^N \) by
\[
(\alpha, \delta) \cdot f(k) = \delta(k) \cdot f(\alpha^{-1}(k)).
\]

Lemma 2.3. The following hold:
1. \( \beta \) is a cocyle;
2. \( S_\infty \ltimes \Delta^N \rtimes Z^N \) defines an action;
(3) for all $\gamma \in \Gamma$ and $(x, f) \in X \times Z^N$,
$$\gamma^c \cdot (x, f) = (\gamma \cdot x, \beta(\gamma, x) \cdot f).$$

Proof. (1) Let $\gamma_1, \gamma_2 \in \Gamma$ and $x \in X$. Observe that $\alpha$ is a cocycle, i.e.,
$$\alpha(\gamma_1 \gamma_2, x) = \alpha(\gamma_1, \gamma_2 \cdot x) \alpha(\gamma_2, x).$$
Indeed, if $\alpha(\gamma_2, x)(k) = l$ and $\alpha(\gamma_1, \gamma_2 \cdot x)(l) = n$, then
$$g_k(x) E_{\Delta} g_l(\gamma_2 \cdot x), \quad g_l(\gamma_2 \cdot x) E_{\Delta} g_n(\gamma_1 \gamma_2 \cdot x).$$
Consequently, $g_k(x) E_{\Delta} g_n(\gamma_1 \gamma_2 \cdot x)$ and, by the definition of $\alpha$, $\alpha(\gamma_1 \gamma_2, x)(k) = n$.

It remains to show that
$$\beta(\gamma_1 \gamma_2, x) = \beta(\gamma_1, \gamma_2 \cdot x) \beta(\gamma_2, x).$$

By our definitions of $\alpha$ and $\delta$,
$$\begin{align*}
\delta(\gamma_1 \gamma_2, x)(k) g_{\alpha(\gamma_1 \gamma_2, x)^{-1}(k)}(x) \\
= g_k(\gamma_1 \gamma_2 \cdot x) \\
= \delta(\gamma_1, \gamma_2 \cdot x)(k) \left[ g_{\alpha(\gamma_1, \gamma_2 \cdot x)^{-1}(k)}(\gamma_2 \cdot x) \right] \\
= \delta(\gamma_1, \gamma_2 \cdot x)(k) \left[ \delta(\gamma_2, x)(\alpha(\gamma_1, \gamma_2 \cdot x)^{-1}(k)) g_{\alpha(\gamma_1, \gamma_2 \cdot x)^{-1}(k)}(x) \right] \\
= \left[ \delta(\gamma_1, \gamma_2 \cdot x)(k) \delta(\gamma_2, x)(\alpha(\gamma_1, \gamma_2 \cdot x)^{-1}(k)) \right] g_{\alpha(\gamma_1, \gamma_2 \cdot x)^{-1}(k)}(x)
\end{align*}$$
and, as a result,
$$\delta(\gamma_1 \gamma_2, x)(k) = \delta(\gamma_1, \gamma_2 \cdot x)(k) \delta(\gamma_2, x)(\alpha(\gamma_1, \gamma_2 \cdot x)^{-1}(k)).$$

Finally, from the above calculations,
$$\begin{align*}
(\alpha(\gamma_1, \gamma_2 \cdot x), \delta(\gamma_1, \gamma_2 x))(\alpha(\gamma_2, x) \delta(\gamma_2, x)) \\
= (\alpha(\gamma_1, \gamma_2 \cdot x), \delta(\gamma_1, \gamma_2 x))(\alpha(\gamma_2, x) \delta(\gamma_2, x)) \\
= (\alpha(\gamma_1 \gamma_2, x), \delta(\gamma_1, \gamma_2 x)[\alpha(\gamma_1, \gamma_2 x) \cdot \delta(\gamma_2, x)]) \\
= (\alpha(\gamma_1 \gamma_2, x), \delta(\gamma_1 \gamma_2 x))
\end{align*}$$
establishing that $\beta$ is a cocycle.

(2) Let $(\alpha_1, \delta_1), (\alpha_2, \delta_2) \in S_{\infty} \times \Delta^N$ and $f \in Z^N$. Then
$$\begin{align*}
(\alpha_1, \delta_1) \cdot (\alpha_2, \delta_2) f(k) &= (\alpha_1, \delta_1) \cdot \delta_2(k) f(\alpha_2^{-1} \cdot \alpha_1)(k) \\
&= \delta_1(k) \delta_2(\alpha_1^{-1} \cdot \alpha_2^{-1})(k) f(\alpha_2^{-1} \cdot \alpha_1^{-1})(k) \\
&= (\alpha_1 \alpha_2, \delta_1(\alpha_1 \cdot \delta_2)) \cdot f(k).
\end{align*}$$

(3) Given $(x, f) \in Y$ and $\gamma \in \Gamma$,
$$\begin{align*}
f(g_k(\gamma \cdot x)) &= f(\delta(\gamma, x)(k) g_{\alpha(\gamma, x)^{-1}(k)}(x)) \\
&= \delta(\gamma, x)(k) f(\alpha(\gamma, x)^{-1}(k)(x)).
\end{align*}$$
Thus, $\gamma^c(x, f) = (\gamma \cdot x, \beta(\gamma, x) \cdot f)$.

Since the actions $\Gamma \acts (X, \mu)$ and $S_\infty \times \Delta^\mathbb{N} \acts (Z^\mathbb{N}, \nu^\mathbb{N})$ are measure preserving, the action

$\Gamma \acts^c (X \times Z^\mathbb{N}, \mu \times \nu^\mathbb{N})$

formed by a skew-product is measure preserving as well (see [Gla03]).

For the action $\Delta \acts^d (Y, m)$, define the cocycle $\sigma: \Delta \times X \to \Gamma$ by

$\sigma(\gamma_0, x) = \gamma \iff \gamma_0 \cdot x = \gamma \cdot x$

for any $\gamma_0 \in \Delta$. Then define $\Delta \acts^d X \times Z^\mathbb{N}$ by

$\gamma_0 \cdot (x, f) = \sigma(\gamma_0, x)^c \cdot (x, f)$.

At this point, we can see that our particular construction mandates the freeness of the actions $a_0$ and $b_0$ to define the cocycles $\delta$ and $\sigma$, respectively.

**Lemma 2.4.** If $A \subset Y$ is $\Delta$-invariant with $\Gamma$-invariant probability measure $m'$, then the action $\Delta \acts^d \{A, m'\}$ is measure preserving.

**Proof.** Let $B \subset A$ be Borel and let $\gamma_0 \in \Delta$. Then

$m'(\gamma_0 \cdot B) = m'(\bigcup_{\gamma \in \Gamma} \{ (x, f) \in B \mid \gamma_0 \cdot (x, f) = \gamma \cdot (x, f) \})$

$= m'\bigg(\bigcup_{\gamma \in \Gamma} \gamma \cdot \{ (x, f) \in B \mid \gamma_0 \cdot (x, f) = \gamma \cdot (x, f) \} \bigg)$

$= \sum_{\gamma \in \Gamma} m'\bigg(\{ (x, f) \in B \mid \gamma_0 \cdot (x, f) = \gamma \cdot (x, f) \} \bigg) = m'(B)$.

Define the map $p: Y \to Z$ by $p(x, f) = f(x) = g_0(x)$. Since for each $(x, f) \in Y$, $f$ is $\Delta$-equivariant, it is clear that $p$ is also a $\Delta$-equivariant map. To see that $p$ is measure preserving and does, in fact, witness that $a \subseteq d$, let $A \subset Z$ be arbitrary. Then

$p_*(\mu \times \nu^\mathbb{N})(A) = \mu \times \nu^\mathbb{N}\left(\{(x, f) \in Y \mid f(1) \in A\}\right) = \nu(A)$.

We will show how to obtain ergodicity of $c$ in the proof of Lemma 2.6 since we will use some facts concerning ergodic decomposition we have yet to prove.

We now need a general lemma concerning ergodic decompositions (see [KM04]).

Let $\Gamma \acts (X, \mu), (Y, \nu)$ be Borel and measure preserving where $X$ and $Y$ are standard probability spaces and $\nu$ is ergodic. Suppose that $p: X \to Y$ is a $\Gamma$-equivariant map, i.e.,

$\forall \gamma \in \Gamma \quad p(\gamma \cdot x) = \gamma \cdot p(x)$. 
Consider the ergodic decomposition of $X$ with respect to the action $\Gamma \curvearrowright X$. This is given by a $\Gamma$-invariant Borel map $\Phi: X \to I$ where $I$ is a standard Borel space and a Borel map $i \in I \mapsto \mu_i \in P(X)$ such that the following hold:

1. for each $i \in I$, if we let
   \[ X_i = \{ x \in X \mid \Phi(x) = i \}, \]
   then $X_i$ is $\Gamma$-invariant and $\mu_i$ is the unique ergodic $\Gamma$-invariant measure on $X_i$;
2. $\mu = \int_I \mu_i \, d\eta(i)$ where $\eta = \Phi_*\mu$.

**Lemma 2.5.** The following hold:

1. If $A \subset X$ is a $\Gamma$-invariant subset and $B \subset Y$, then
   \[ \mu(A \cap p^{-1}(B)) = \nu(B)\mu(A). \]
2. If $\Delta \curvearrowright (X, \mu)$ is another Borel measure preserving action such that for any $\Delta$-invariant set $A \subset X$ and any $B \subset Y$, we have
   \[ \mu(A \cap p^{-1}(B)) = \nu(B)\mu(A), \]
   then for $\Phi_*\mu$-almost every $i \in I$, if $A \subset X$ is $\Delta$-invariant, we have
   \[ \mu_i(A \cap p^{-1}(B)) = \nu(B)\mu_i(A). \]

**Proof.** (1) It suffices to show that for some subset $I_0 \subset I$ such that $\Phi_*\mu(I_0) = 1$, we have $p_*\mu_i = \nu$ for all $i \in I_0$. Granted this, we may finish the proof. Indeed, since $A$ is $\Gamma$-invariant, then up to null sets, $A = \Phi^{-1}(I)$ for some subset $I \subset I_0$. Thus,

\[
\mu(A \cap p^{-1}(B)) = \int_I \mu_i(p^{-1}(B)) \, d\Phi_*\mu(i) \\
= \int_I \nu(B) \, d\Phi_*\mu(i) \\
= \nu(B)\mu(A).
\]

Since the measure $\nu$ on $Y$ is ergodic and $\Gamma$-invariant, we may let $C \subset Y$ be such that $\nu(C) = 1$ and $\nu$ is the unique $\Gamma$-invariant probability measure on $C$. By the fact that $p$ is measure-preserving, we have that $\mu(p^{-1}(C)) = 1$. Then for $\Phi_*\mu$-null many $i \in I$, we have $p_*\mu_i(C) = 1$. Also, by equivariance of the map $p$, $p_*\mu_i$ is a $\Gamma$-invariant measure on $C$. By uniqueness of $\nu$, it must be that $p_*\mu_i = \nu$.

(2) Suppose that this fails on a set of $\Phi_*\mu$-positive measure. Then, without loss of generality, we may find a subset $D \subset I$ of $\Phi_*\mu$-positive measure such that for each $i \in D$, there is a $\Delta$-invariant $F_\sigma$ subset $A_i \subset X$ such that

\[ \mu_i(p^{-1}(B) \cap A_i) < \nu(B)\mu_i(A_i). \]

We show that there is a $\Phi_*\mu$-measurable assignment $\psi$ from $D$ to the $F_\sigma$ subsets of $X$ so that for each $i \in D$, the above inequality holds where $A_i = \psi(i)$.

Let $F: \mathbb{N}^\mathbb{N} \to F_\sigma(X)$ (where $F_\sigma(X)$ is the set of $F_\sigma$ subsets of $X$) be a Borel bijection in the sense that

\[ \{(w, x) \mid x \in F(w)\} \subset \mathbb{N}^\mathbb{N} \times Y \]
is Borel. Then let $F_1: \mathbb{N}^\mathbb{N} \rightarrow \mathcal{B}(X)$ be defined by $F_1(w) = F(w) \cap p^{-1}(B)$.

Consider the subset $D_0 \subset \mathbb{N}^\mathbb{N} \times P(X)$ defined by

$$D_0 = \{(w, \lambda) \in \mathbb{N}^\mathbb{N} \times P(X) \mid \lambda(p^{-1}(B) \cap F(w)) < \nu(B)\lambda(F(w))\}.$$

We observe that $D_0$ is Borel. Indeed, by 17.25 of [Kec95], the maps

$$(w, \lambda) \in \mathbb{N}^\mathbb{N} \times P(X) \mapsto \nu(B)\lambda(F(w)),$$

are Borel and, as a result, the map

$$(w, \lambda) \in \mathbb{N}^\mathbb{N} \times P(X) \mapsto \lambda(p^{-1}(B) \cap F(w)) - \nu(B)\lambda(F(w))$$

is also Borel. This establishes that $D_0$ is Borel. Now let

$$D_1 = \{(w, i) \in \mathbb{N}^\mathbb{N} \times \mathcal{I} \mid (w, \mu_i) \in D_0\}.$$

$D_1$ is Borel as well since the assignment $i \in \mathcal{I} \mapsto \mu_i \in P(X)$ is Borel.

Finally, $D = \text{proj}_2(D_1)$ and $D$ is analytic. Then $\mathbb{N}^\mathbb{N} \times D \subset \mathbb{N}^\mathbb{N} \times \mathcal{I}$ is analytic as well and, by 18.1 of [Kec95], there is a $\Phi_*\mu$-measurable assignment $\psi: D \rightarrow \mathbb{N}^\mathbb{N}$ such that

$$(\psi(i), i) \in D_1 \quad \forall i \in D.$$

Note that $\bigcup_{i \in D} A_{\psi(i)}$ is a measurable $\Gamma$-invariant subset of $X$ of $\mu$-positive measure so we aim to obtain a contradiction to the fact that

$$\mu\left(\bigcup_{i \in D} A_{\psi(i)} \cap p^{-1}(B)\right) = \nu(B)\mu\left(\bigcup_{i \in D} A_{\psi(i)}\right).$$

We have

$$\mu(p^{-1}(B) \cap \bigcup_{i \in D} A_{\psi(i)}) = \int_D \mu_i(p^{-1}(B) \cap A_{\psi(i)}) \, d\Phi_*\mu(i)$$

$$< \int_D \nu(B)\mu_i(A_{\psi(i)}) \, d\Phi_*\mu(i)$$

$$= \nu(B)\mu\left(\bigcup_{i \in D} A_{\psi(i)}\right).$$

We are now ready to specifically consider an action of $\Gamma$ induced from an action of $\mathbb{F}_2$. Fix actions $\Gamma \curvearrowleft h_0$ $(X, \mu)$ and $\mathbb{F}_2 \curvearrowleft a_0$ $(X, \mu)$ satisfying the hypotheses of Theorem 14. Let $\text{SL}_2(\mathbb{Z}) \curvearrowleft (\mathbb{T}^2, h)$ where $h$ is the Haar measure as follows:

$$A \cdot (t_1, t_2) = (A^{-1})^t \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}.$$

Fix a copy of $\mathbb{F}_2$ in $\text{SL}_2(\mathbb{Z})$ with finite index and let $a$ be the action of $\mathbb{F}_2$ on $(\mathbb{T}^2, h)$ given by restricting the action of $\text{SL}_2(\mathbb{Z})$ on $(\mathbb{T}^2, h)$. This action is then free, measure preserving and weakly mixing. For more on this, see Section 16 of [Kec95] or [Tör05].
Lemma 2.6. Given $a$ as specified above, suppose we have the following:

1. $F_2 \curvearrowleft a^\pi (Z, \nu)$ is a weakly mixing action;
2. $F_2 \curvearrowleft a \times a^\pi T^2 \times Z$ is the diagonal action obtained from $a$ and $a^\pi$;
3. $q: T^2 \times Z \to T^2$ is given by $q(t, z) = t$.

Then there is a standard probability space $(Y, m)$ and actions $c, d$ and map $p$ so that the following hold:

1. $\Gamma \curvearrowleft c (Y, m)$ is free, measure preserving, ergodic;
2. $F_2 \curvearrowleft d (Y, m)$ is free, measure preserving;
3. $q: T^2 \times Z \to T^2$ is measure preserving;
4. $E_d \subset E_c$;
5. for any non-null $d$-invariant subset $Y_0 \subset Y$, $p|Y_0$ witnesses that $a \times a^\pi \subseteq d|Y_0$;
6. $\forall \gamma \in \Gamma \setminus \{e\}$, $m\left(\left\{y \in Y \mid q \circ p(\gamma \cdot y) = q \circ p(y)\right\}\right) = 0$.

Proof. Consider the space $(Y, m) = \left(\mathbb{X} \times (T^2 \times Z)^\mathbb{N}, \mu \times (h \times \nu)^\mathbb{N}\right)$.

We may obtain the actions $c$ and $d$ on $(Y, m)$ from the construction in Theorem 2.1. Note that since $a_0$ and $b_0$ are free, the actions $c$ and $d$ are free as well. We just need to select a measure on $Y$ so that the action of $\Gamma$ on $Y$ with respect to this measure is measure preserving and ergodic. For this purpose, we take an ergodic decomposition of $Y$ with respect to the action $c$ and let $\Phi: Y \to I$ and $i \in I \mapsto m_i \in \mathcal{P}(Y)$ be the corresponding Borel assignments. Our remaining goal is to show that $\Phi^* m$-almost every measure in $I$ satisfies our conditions.

Lemma 2.7. The following hold:

1. If $B \subset T^2 \times Z$, then for $\Phi^* m$-almost every $i \in I$, for every $F_2$-invariant $A \subset Y$, $m_i(A \cap p^{-1}(B)) = h \times \nu(B)m_i(A)$.
2. If $\gamma \in \Gamma \setminus \{e\}$, then for almost every $i \in I$,

Proof. (1) By Lemma 2.5, if $A \subset Y$ is $F_2$-invariant, then $m(A \cap p^{-1}(B)) = h \times \nu(B)m(A)$.

Moreover, for $\Phi^* m$-almost every $i \in I$,

$m_i(A \cap p^{-1}(B)) = h \times \nu(B)m_i(A)$.

(2) We first show that for any $\gamma \in \Gamma \setminus \{e\}$,

$m\left(\left\{(x, f) \in Y \mid q \circ p(x, f) = q \circ p(\gamma \cdot (x, f))\right\}\right) = 0$. 
Fix $\gamma \in \Gamma \setminus \{e\}$. Define

$$Y_\gamma = \left\{ (x, f) \in Y \mid q \circ p(x, f) = q \circ p(\gamma \cdot (x, f)) \right\}$$

and suppose that $m(Y_\gamma) > 0$. Without loss of generality, we may assume that for all $(x, f) \in Y_\gamma$,

$$g_0(x) = \gamma_0 \cdot g_{k}(\gamma \cdot x)$$

for some fixed $k \in \mathbb{N}$ and $\gamma_0 \in F_2$. If $k = 0$, then $\gamma_0 \neq e$ and since the action of $F_2$ is free on $T^2$, it is impossible that $q(f(x)) = q(f(\gamma \cdot x))$. On the other hand, if $k \neq 0$,

$$Y_\gamma = \left\{ (x, f) \in Y \mid f(g_0(x)) = \gamma_0 \cdot f(g_k(x)) \right\}.$$

The measure $h \times \nu$ is non-atomic and, hence, $m(Y_\gamma) = 0$.

Let $D \subset I$ be a set of $\Phi_\ast m$-positive measure so that for $i \in D$,

$$m_i \left( \left\{ (x, f) \in Y_i \mid q \circ p(x, f) = q \circ p(\gamma \cdot (x, f)) \right\} \right) > \delta$$

for some $\delta > 0$. We’ve established $Y_\gamma$ has measure zero. Thus,

$$0 = m(Y_\gamma \cap \bigcup_{i \in D} Y_i)$$

$$= \int_D m_i(Y_\gamma \cap Y_i) \, d\Phi_\ast m(i)$$

$$> \delta \int_D d\Phi_\ast m(i)$$

$$= \delta \cdot m(\bigcup_{i \in D} Y_i)$$

which is impossible by our choice of $D$.

□

Let $B = \{B_n\}_{n \in \mathbb{N}}$ generate the Borel $\sigma$-algebra on $T^2 \times Z$. Without loss of generality, we may assume that $B$ is clopen, invariant under the action of $F_2$ and closed under Boolean operations. Let $i \in I$ be such that conditions (1) and (2) of Lemma 2.7 hold for all $\gamma \in \Gamma \setminus \{e\}$ and $B \in B$. Since $Y_i$ is $\Gamma$-invariant, it is $F_2$-invariant as well and, as a result, $E_{d|Y_i} \subset E_{c|Y_i} \cap \text{d}^d Y_i$ ($Y_i, m_i$) is measure preserving by Lemma 2.4 conditions (5) and (6) follow from Lemma 2.7 (1) and Lemma 2.7 (2), respectively. Thus, $(Y_i, m_i)$ with actions $\Gamma \acts c Y_i$ ($Y_i, m_i$) and $F_2 \acts d Y_i$ ($Y_i, m_i$) and factor map $p$ are as desired.

Looking back at Theorem 2.1, note that Lemma 2.5 makes no assumptions on the action of $\Delta$ on $(Y, \nu)$ except ergodicity and neither does Lemma 2.7 (1) on the action of $F_2$ on $(T^2 \times Z, h \times \nu)$. Thus, the proof here shows that in Theorem 2.1, the action of $c$ can be made ergodic when $a$ is ergodic.

□
3. PROOF OF THE MAIN THEOREM

We now proceed as in Ioana [Ioa07] with the action described in Section 2 replacing the co-induced action and making a change to the order of operations in constructing actions.

As defined in the previous section, \( \alpha \) is the action of \( \mathbb{F}_2 \) on \( (\mathbb{T}^2, h) \). The following lemma is Theorem 1.3 of [Ioa07]:

**Lemma 3.1.** Let \( \Gamma \) be a group such that \( \mathbb{F}_2 \leq \Gamma \) is a subgroup and let \( \{c_i\}_{i \in I} \) be an uncountable set of orbit equivalent ergodic, free, measure preserving actions \( \Gamma \curvearrowright c_i (Y_i, \mu_i) \) so that the following conditions hold:

1. \( \alpha \) is a quotient of \( c_i|_{\mathbb{F}_2} \) with quotient map \( p_i : Y_i \to \mathbb{T}^2 \);
2. \( \forall i \in I \quad \forall \gamma \in \Gamma \setminus \{e\}, \quad m\left( \left\{ y \in Y_i \mid p_i(\gamma c_i : y) = p_i(y) \right\} \right) = 0. \)

Then there is an uncountable set \( J \subset I \) such that for every \( i, j \in J \), there are non-null \( c_i|_{\mathbb{F}_2} \)-invariant and \( c_j|_{\mathbb{F}_2} \)-invariant subsets \( Y'_i \) and \( Y'_j \) of \( Y_i \) and \( Y_j \), respectively, so that \( c_i|_{\mathbb{F}_2}\left|_{Y'_i} \right. \) is conjugate to \( c_j|_{\mathbb{F}_2}\left|_{Y'_j} \right. \).

We may obtain the following generalization by changing the requirement that \( \mathbb{F}_2 \) is a subgroup of \( \Gamma \) to a requirement \( E^d_i \mathbb{F}_2 \subset E^c_i \mathbb{F}_2 \) where \( d_i, c_i \) are actions of \( \mathbb{F}_2 \) and \( \Gamma \), respectively. However, only cosmetic alteration needs to be made to Ioana’s original proof.

**Lemma 3.2.** Let \( \{c_i\}_{i \in I} \) be an uncountable set of orbit equivalent ergodic, free, measure preserving actions \( \Gamma \curvearrowright c_i (Y_i, \mu_i) \) such that for each \( i \), there is a free measure preserving action \( \mathbb{F}_2 \curvearrowright d_i (Y_i, \mu_i) \) so that the following conditions hold:

1. \( E^{d_i}_{\mathbb{F}_2} \subset E^{c_i}_{\mathbb{F}_2} \);
2. \( \alpha \) is a quotient of \( d_i \) with quotient map \( p_i : Y_i \to \mathbb{T}^2 \);
3. \( \forall i \in I \quad \forall \gamma \in \Gamma \setminus \{e\}, \quad m\left( \left\{ y \in Y_i \mid p_i(\gamma c_i : y) = p_i(y) \right\} \right) = 0. \)

Then there is an uncountable set \( J \subset I \) such that for every \( i, j \in J \), there are non-null \( d_i \)-invariant and \( d_j \)-invariant subsets \( Y'_i \subset Y_i \) and \( Y'_j \subset Y_j \), respectively, so that \( d_i|_{Y'_i} \) is conjugate to \( d_j|_{Y'_j} \).

Let \( \{\pi_i\}_{i \in I} \) be a set of continuum many non-isomorphic, irreducible weakly mixing representations of \( \mathbb{F}_2 \). For each such representation, using Theorem E.1 of [Kec], obtain a Gaussian action \( \mathbb{F}_2 \curvearrowright \alpha_{\pi_i} (Z_i, \nu_i) \) such that \( \pi_i \cong \pi_j \implies \alpha_{\pi_i} \cong \alpha_{\pi_j} \). In addition, we will have \( \pi_i \leq \kappa_0^{a_{\pi_i}} \) and \( \alpha_{\pi_i} \) will be a weakly mixing action.

For each \( i \in I \), let the actions \( \Gamma \curvearrowright c_i (Y_i, m) \) and \( \mathbb{F}_2 \curvearrowright d_i (Y_i, m) \) and the map \( p_i : Y_i \to \mathbb{T}^2 \times Z_i \) be obtained from Lemma 2.6. Also, let \( q : \mathbb{T}^2 \times Z_i \to \mathbb{T}^2 \) be given by \( q(t, z) = t \). Then \( c_i, d_i \) satisfy the conditions of Lemma 3.2 with quotient map \( q \circ p_i \).
We claim that for each \( i \in I \), the set
\[
J_i = \{ j \in I \mid c_i \text{ is orbit equivalent to } c_j \}
\]
is countable. Otherwise, by Lemma 3.2, there is an uncountable set \( J \subset J_i \) such that for any \( i, j \in J \), there exist non-null \( \mathbb{F}_2 \)-invariant subsets \( Y_i', Y_j' \) of \( Y_i, Y_j \), respectively such that \( d_i|Y_i' \) is orbit equivalent to \( d_j|Y_j' \).

If we take the Koopman representation of \( d_i \) restricted to \( Y_i' \), then
\[
\kappa_{d_i|Y_i'} \leq \kappa_{d_i|Y_i}
\]
From our construction, we have
\[
\pi_i \leq \kappa_{\sigma_i} \leq \kappa_{d_i|Y_i'}
\]
since \( a_{\sigma_i} \leq a \times a_{\sigma_i} \) and
\[
\kappa_{d_i|Y_i} \leq \kappa_{d_j|Y_j'}
\]
since \( a \times a_{\sigma_i} \leq d_i|Y_i' \). As a result, \( \pi_i \leq \kappa_{d_i|Y_i'} \) and, finally,
\[
\pi_i \leq \kappa_{d_i|Y_i'} \leq \kappa_{d_j|Y_j'} \leq \kappa_{d_j}
\]
However, a separable unitary representation can only have countably many non-isomorphic irreducible subrepresentations and since the \( \pi_j \)'s are pairwise non-equivalent, it must be that each \( J_i \) is countable and this completes the proof.

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