Effective potential and mass behavior of a self-interacting scalar field theory due to thermal and external electric and magnetic fields effects

M. Loewe\textsuperscript{1,2,3}, D. Valenzuela\textsuperscript{4}, R. Zamora\textsuperscript{5,6,a}

\textsuperscript{1} Facultad de Ingeniería, Arquitectura y Diseño, Universidad San Sebastián, Santiago, Chile
\textsuperscript{2} Centre for Theoretical and Mathematical Physics, and Department of Physics, University of Cape Town, Rondebosch 7700, South Africa
\textsuperscript{3} Centro Científico-Tecnológico de Valparaíso CCTVAL, Universidad Técnica Federico Santa María, Casilla 110-V, Valparaíso, Chile
\textsuperscript{4} Universidad Técnica Federico Santa María, Casilla 110-V, Valparaíso, Chile
\textsuperscript{5} Instituto de Ciencias Básicas, Universidad Diego Portales, Casilla 298-V, Santiago, Chile
\textsuperscript{6} Centro de Investigación y Desarrollo en Ciencias Aeroespaciales (CIDCA), Academia Politécnica Aeronáutica, Fuerza Aérea de Chile, Casilla, 8020744 Santiago, Chile

Received: 6 April 2023 / Accepted: 26 July 2023 / Published online: 8 August 2023
© The Author(s), under exclusive licence to Società Italiana di Fisica and Springer-Verlag GmbH Germany, part of Springer Nature 2023
Communicated by André Peshier

Abstract In this article we address the subject of finding the behavior of a charged scalar field $\phi$ under the influence of external constant magnetic and electric fields, perpendicular to each other, including also thermal effects. For this purpose we derive an expression for the corresponding bosonic propagator. As an application, we explore, in the weak field sector, the mass correction for the self interacting $\lambda \phi^4$ theory when only one type of field, including finite temperature, is present and the case where both fields are included in the analysis. We also analyze the phase diagram associated to spontaneous symmetry breaking of the theory finding inverse magnetic catalysis (IMC) or inverse electric catalysis (IEC) for the cases where only a magnetic field or only an electric field are present, respectively. In both cases, taken separately, we have a scenario where the critical temperature associate to symmetry restoration diminishes as function of the corresponding field strengths. A similar situation happens when both type of fields are simultaneously present. We have dubbed this case as inverse magnetic-electric catalysis (IMEC). In this situation, both fields cooperate for the occurrence of IMEC.

1 Introduction

In this article we derive a propagator for a charged scalar field in the presence of constant external magnetic and electric fields. This propagator is used then to explore the mass behavior of the $\phi$ field, in the frame of a self-interacting $\lambda \phi^4$ theory, as function of temperature and the external magnetic and electric field strengths, taking both fields simultaneously. As a second application, we analyze the phase diagram of this theory starting from the broken phase. For this purpose we find the effective potential, including the resummation of ring diagrams, finding the dependence of the critical temperature on the external electromagnetic field strengths.

In the existing literature, different aspects concerning the behavior of hadronic parameters and the properties of the QCD phase diagram in the presence of thermal and external magnetic effects have been considered during the last years [1–28]. In [29] we might find general review articles concerning the role of magnetic effects in strongly interacting matter. In particular, it is interesting to mention that it is possible to find inverse magnetic catalysis in the frame of effective models, in agreement with the well known results from the lattice community [30], i.e. the fact that the critical temperature diminishes with the magnetic field strength. We have analyzed this point by a careful discussion of symmetry restoration both in a self-interacting scalar field model as well as in the linear sigma field coupled to quarks, dealing in the last case with chiral symmetry restoration. It turns out that this can only be achieved by going beyond the mean field approximation, through a ring diagram resummation. [31,32]. A recent review about these facts, including a discussion of thermo-magnetic effects in the Nambu-Jona-Lasinio model, is presented in [33]. A discussion of the effective potential, in the case where we have a pure external electric field, shows the occurrence of electric anticatalysis for small values of the electric field, whereas catalysis appears
for high values of the electric field strength [34,35]. We also want to mention, in the context of the influence of external fields, a complete different aspect, namely the evolution of the residues of renormalons in this theory under the influence of an electric field which has also recently been addressed by some of us. [36].

In this article we would like to consider the situation or scenario where the effects of both kind of external fields, electric and magnetic, are taken into account. The physical scenario where such a case appears corresponds to peripheral collisions of asymmetric heavy ions. For example, in Au-Cu collisions an electric dipole type field is generated due to the imbalance of the number of charged particles (protons) in both nuclei. This field will basically be perpendicular to the magnetic field generated also under such conditions. In [37] we might find an analysis when both types of fields are present, but in a parallel configuration, which corresponds actually to the chiral magnetic effect. For the purpose of analyzing collisions between asymmetric nuclei, however, the configuration where both fields are perpendicular to each other is the relevant one.

This article is organized as follows. In Sect. 2 we derive the bosonic propagator in the presence of external constant magnetic and electric fields, in a perpendicular configuration relative to each other. We also present the propagator’s expansion for the case of weak fields. In Sect. 3 we make use of the propagator to find the electromagnetic mass correction of the charged scalar field. In Sect. 4 we proceed with the discussion of the effective potential at the one loop level, including also ring corrections, finding the behavior of the critical temperature where the symmetry is restored as function of the magnetic and electric field strengths. Finally, in Sect. 5 we present our conclusions.

2 The boson propagator in the presence of an electric and magnetic field

We begin with the proper time representation for a charged boson propagator in the presence of an external electromagnetic field encoded in the $F^{\mu\nu}$ tensor [38,39] which is given by

$$D(p) = \int_0^\infty dt \, e^{-m^2t} e^{ip\left(\frac{\tan(Z)}{2}\right)} e^{ip\nu v} \sqrt{\det(\cos(Z))},$$  

(1)

where $Z^{\mu\nu} \equiv q F^{\mu\nu} t$, with $q$ the electric charge and $p$ the euclidean four-momentum defined as $p = (p_1, p_2, p_3, p_4)$. Our analysis is presented in the Euclidean formulation.

As we said in the introduction, we are interested in a configuration where $B \perp \hat{E}$. Without loss of generality, we assume that $\hat{B} = B_0 \hat{z}$ and $\hat{E} = E_0 \hat{z}$ that is

$$F^{\mu\nu} = \begin{bmatrix} 0 & B_0 & 0 & iE_0 \\ -B_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -iE_0 & 0 & 0 & 0 \end{bmatrix}. \quad (2)$$

Thus, we can decompose $F^{\mu\nu}$ as

$$F^{\mu\nu} = iE_0 E^{\mu\nu} + B_0 B^{\mu\nu}, \quad (3)$$

where

$$E^{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$ \quad (4)

and

$$B^{\mu\nu} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (5)$$

Let us compute the square of $F^{\mu\nu}$

$$(F^{2})^{\mu\nu} = E_0^2 (E^{\mu\nu})^2 + B_0^2 (B^{\mu\nu})^2 + iE_0 B_0 S^{\mu\nu}, \quad (6)$$

with

$$E_0^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (7)$$

$$B_0^2 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (8)$$

and

$$S^{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}. \quad (9)$$

We find the following relations for the powers of $F^{\mu\nu}$

$$(F^{2n})^{\mu\nu} = \left[- \left(-E_0^2 + B_0^2\right)\right]^{n-1} (F^{2})^{\mu\nu}, \quad (10)$$

$$(F^{2n+1})^{\mu\nu} = \left[- \left(-E_0^2 + B_0^2\right)\right]^{n} F^{\mu\nu} \forall n \in \mathbb{N}^0. \quad (11)$$

Summarizing, we have

$$(F^{0})^{\mu\nu} \equiv \delta^{\mu\nu}, \quad (12)$$

$$(F^{2n})^{\mu\nu} = (-\lambda^2)^{n-1} (F^{2})^{\mu\nu}, \quad (13)$$

$$(F^{2n+1})^{\mu\nu} = (-\lambda^2)^{n} F^{\mu\nu} \forall n \in \mathbb{N}^0, \quad (14)$$

where we have defined

$$\lambda^2 \equiv -E_0^2 + B_0^2. \quad (15)$$
With these relations, we obtain

$$\left[\cos(Z)\right]^{\mu\nu} = \delta^{\mu\nu} + 2\frac{\sin^2(\frac{1}{2}q\lambda t)}{\lambda^2} \left(F^2\right)^{\mu\nu}$$

$$\Rightarrow \det[\cos(Z)] = \cos^2(q\lambda t), \quad (16)$$

and

$$\left[\frac{\tan(Z)}{Z}\right]^{\mu\nu} = \delta^{\mu\nu} - \left[\frac{\tan(q\lambda t)}{q\lambda t} - 1\right] \left(F^2\right)^{\mu\nu} \frac{1}{\lambda^2} \quad (17)$$

Finally, inserting Eqs. (16) and (17) in Eq. (1), we get

$$D(p) = \int_0^\infty dt e^{-i\left(p_3^2 + \frac{1}{\sqrt{2}}(B_0 p_1 - i E_0 p_2)^2 + m^2\right)}$$

$$\times e^{-\frac{\tan(q\lambda t)}{\sqrt{q\lambda t}}} \left(p_3^2 + \frac{1}{\sqrt{2}}(B_0 p_1 + i E_0 p_2)^2\right)$$

$$\times \cos(q\lambda t)$$

$$= \int_0^\infty dt e^{-i\left(p_3^2 + \frac{1}{\sqrt{2}}(B_0 p_1 + i E_0 p_2)^2 + m^2\right)}$$

$$\times e^{-\frac{\tanh\left(\frac{q\lambda t}{\sqrt{q\lambda t}}\right)}{\sqrt{q\lambda t}}} \left(p_3^2 + \frac{1}{\sqrt{2}}(B_0 p_1 + i E_0 p_2)^2\right)$$

$$\times \cosh(q\lambda t). \quad (18)$$

2.1 Weak field expansion of the propagator

In order to find a weak field expansion for the propagator, we follow the ideas presented in [40]. Our computation, in [40] also revealed that the dominant region for the integration over the proper-time parameter is the small $t$ region and that, in order to obtain analytical results, this region can be extended to cover the whole original domain $0 \leq t \leq \infty$. Therefore, we can perform a Taylor expansion for small $E_0$ and $B_0$ values in Eq. (18), using the expressions

$$\frac{\tanh(x)}{x} \approx 1 - \frac{1}{3}x^2 + \frac{2}{15}x^4$$

$$x \approx 1 - \frac{1}{2}x^2 \quad (19)$$

$$e^x \approx 1 + x. \quad (20)$$

After integrating the above expression in the proper time $t$, we obtain the expansion of the propagator given in Eq. (18) up to the order $O(E^2)$ and $O(B^2)$. We get

$$D(p) \approx \frac{1}{p^2 + m^2} + \frac{4i p_4 p_2 (qB)(qE)}{(p^2 + m^2)^4}$$

$$+ \frac{(qE)^2 (m^2 + p^2 - 2(p_3^2 + p_1^2))}{(p^2 + m^2)^4}$$

$$- \frac{(qB)^2 (m^2 + p^2 - 2p_3^2)}{(p^2 + m^2)^4}, \quad (22)$$

where $p_\perp = (0, p_1, p_2, 0)$. It is important to avoid $p_\perp^4$ terms in the numerator of the above expression. When going into a finite temperature scenario, those terms are cumbersome to be handled. We can write the previous expression as

$$D(p) \approx \frac{1}{p^2 + m^2} + \frac{4i p_4 p_2 (qB)(qE)}{(p^2 + m^2)^4}$$

$$- \frac{(qE)^2 + (qB)^2}{(p^2 + m^2)^4}$$

$$+ 2(qE)^2 p_\perp^2 + (qE)^2 (p_3^2 + p_1^2)$$

$$+ \frac{2(qE)^2 m^2}{(p^2 + m^2)^4}. \quad (23)$$

3 Mass correction

There are several quantities that could be analyzed using the propagator derived in the previous section. Perhaps, the most simple situation is the analysis of the electromagnetic mass correction, including also temperature effects valid for the whole range of temperature. Next, we will consider the effective potential, including ring contributions. We remember that we are dealing here with the $\lambda \phi^4$ theory,

$$L = (D_\mu \phi)^\dagger D^\mu \phi + \mu^2 \phi^\dagger \phi - \frac{\lambda}{4} (\phi^\dagger \phi)^2, \quad (24)$$

where $\phi$ is a charged scalar field and

$$D_\mu = \partial_\mu + i q A_\mu. \quad (25)$$

The squared mass parameter $\mu^2$ and the self-coupling $\lambda$ are taken to be positive. Notice that we are in the broken phase. We can write the complex field $\phi$ in terms of real components $\sigma$ and $\chi$,

$$\phi(x) = \frac{1}{\sqrt{2}}[\sigma(x) + i \chi(x)],$$

$$\phi^\dagger(x) = \frac{1}{\sqrt{2}}[\sigma(x) - i \chi(x)]. \quad (26)$$

Following the usual procedure that allows the occurrence of spontaneous symmetry breaking, the $\sigma$ field develops a vacuum expectation value $v$

$$\sigma \rightarrow \sigma + v. \quad (27)$$
which can later be taken as the order parameter of the theory. After this shift, the Lagrangian can be rewritten as

$$\mathcal{L} = -\frac{1}{2}[\sigma(\partial_\mu + iqA_\mu)^2\sigma] - \frac{1}{2}\left(\frac{3\lambda v^2}{4} - \mu^2\right)\sigma^2 - \frac{1}{2}[\chi(\partial_\mu + iqA_\mu)^2\chi] - \frac{1}{2}\left(\frac{\lambda v^2}{4} - \mu^2\right)\chi^2$$

$$+ \frac{\mu^2}{2}v^2 - \frac{\lambda}{16}v^4 + \mathcal{L}_I,$$

where $\mathcal{L}_I$ is given by

$$\mathcal{L}_I = -\frac{\lambda}{16}(\sigma^4 + \chi^4 + 2\sigma^2\chi^2),$$

From Eq. (28) we see that the $\sigma$ and $\chi$ masses are given by

$$m_{\sigma}^2 = \frac{3}{4}\lambda v^2 - \mu^2,$$

$$m_{\chi}^2 = \frac{1}{4}\lambda v^2 - \mu^2.$$  

(30)

For the discussion of the propagator, we will consider only the $\sigma$ boson mass correction which is given by

$$m_\sigma(T, B, E) = m_0 + \Pi_\sigma(T, B, E),$$

(31)

where $m_0$ is the bare mass of the $\sigma$ field, being $m_\sigma(T, B, E)$ the mass at the one loop level, corrected by temperature and the external electromagnetic fields. The Feynman diagrams that contribute to the self-energy of the $\sigma$ field are shown in figure Fig. (1).

Therefore, the self-energy $\Pi_\sigma(T, B, E)$ corresponds to

$$\Pi_\sigma(T, B, E) = \frac{\lambda}{4}(12\Pi(m_\sigma) + 2\Pi(m_\chi)),$$

(32)

where

$$\Pi(m_i) \equiv \Pi = T \sum_n \int \frac{d^3p}{(2\pi)^3} D(\omega_n, p, m_i).$$

(33)

being $m_i$ the $\sigma$ or the $\chi$ mass. Finite temperature effects are handled in the imaginary time formalism in the usual way, i.e.

$$p_4 \rightarrow \omega_n = 2\pi n T, \ n \in \mathbb{Z}.$$  

(34)

where the integral in $p_4$ converts into a sum over Matsubara frequencies according to,

$$\int \frac{d^4p}{(2\pi)^4} f(p) = T \sum_{n \in \mathbb{Z}} \int \frac{d^3p}{(2\pi)^3} f(\omega_n, p),$$

(35)

where we have introduced the notation $p = (p_1, p_2, p_3)$. To calculate Eq. (33) in an analytic way, we proceed through the weak field expansion of the propagator, up to a quadratic order of Eq. (23), getting

$$\Pi = T \sum_n \int \frac{d^3p}{(2\pi)^3} \left[ \frac{1}{\omega_n^2 + p^2 + m^2} + \frac{4ip_2\omega_n(qB)(qE)}{(\omega_n^2 + p^2 + m^2)^2} \right.$$  

$$\left. - \frac{(qE)^2 + (qB)^2}{(\omega_n^2 + p^2 + m^2)^2} \right. + \frac{2(qE)m^2}{(\omega_n^2 + p^2 + m^2)^3}$$

$$\left. \equiv \Pi_1 + \Pi_{\Pi} + \Pi_{III} + \Pi_{IV} + \Pi_V, \right.$$  

(36)

where

$$\Pi_1 = T \sum_n \int \frac{d^3p}{(2\pi)^3} \frac{1}{\omega_n^2 + p^2 + m^2},$$

$$\Pi_{\Pi} = T \sum_n \int \frac{d^3p}{(2\pi)^3} \frac{4ip_2\omega_n(qB)(qE)}{(\omega_n^2 + p^2 + m^2)^2},$$

$$\Pi_{III} = -T \sum_n \int \frac{d^3p}{(2\pi)^3} \frac{(qE)^2 + (qB)^2}{(\omega_n^2 + p^2 + m^2)^3},$$

$$\Pi_{IV} = T \sum_n \int \frac{d^3p}{(2\pi)^3} \frac{2(qE)m^2}{(\omega_n^2 + p^2 + m^2)^3},$$

$$\Pi_V = T \sum_n \int \frac{d^3p}{(2\pi)^3} \frac{2(qE)m^2}{(\omega_n^2 + p^2 + m^2)^4}.$$  

(37)

The first term in the above expression corresponds just to the vacuum plus a thermal contribution. The second term vanishes, due to symmetry reasons, when doing the integral in $p_2$. Then we have a sequence of a pure thermal term followed by thermomagnetic and thermoelastic contributions. Let us first calculate $\Pi_1$,

$$\Pi_1 = T \sum_n \int \frac{d^3p}{(2\pi)^3} \frac{1}{\omega_n^2 + p^2 + m^2} \times \frac{1}{2\sqrt{p^2 + m^2}} \int_0^\infty \frac{p^2}{2\sqrt{p^2 + m^2}} \left( 1 + 2n_B\left(\sqrt{p^2 + m^2}\right) \right).$$

(38)

We do first the sum over Matsubara frequencies according to

$$T \sum_n \frac{1}{\omega_n^2 + p^2 + m^2} = \frac{1}{2\sqrt{p^2 + m^2}} \times \left( 1 + 2n_B\left(\sqrt{p^2 + m^2}\right) \right).$$

(39)

where $n_B$ is the Bose-Einstein distribution

$$n_B(x) = \frac{1}{e^{x/T} - 1}. \tag{40}$$
The first term in Eq. (38) is handled through the standard renormalization procedure in the $\overline{\text{MS}}$ scheme, and we obtain
\begin{equation}
\frac{1}{2\pi^2} \int_0^\infty \frac{p^2}{2\sqrt{p^2 + m^2}} = -\frac{m^2}{16\pi^2} \left( \ln \left( \frac{\mu^2}{m^2} \right) - \frac{1}{2} \right),
\end{equation}
where $\tilde{\mu}$ is the ultraviolet renormalization scale. The second term in Eq. (38) is
\begin{equation}
\frac{1}{2\pi^2} \int_0^\infty dp \frac{p^2}{\sqrt{p^2 + m^2}} n_B \left( \sqrt{p^2 + m^2} \right) = \frac{1}{2\pi^2} \sum_{n=0}^\infty \int_0^\infty \frac{p^2 e^{-(n+1)\sqrt{p^2 + m^2}/T}}{\sqrt{p^2 + m^2}} = mT \frac{\sum_{n=1}^\infty K_1(nm/T)}{n},
\end{equation}
where the modified Bessel function $K_n(x)$ has the form
\begin{equation}
K_n(x) = \int_0^\infty du e^{-x \cosh(u)} \cosh(au).
\end{equation}
It is easy to see that $\Pi_{II}$ vanishes. Let us proceed now to calculate $\Pi_{III}$.
\begin{equation}
\Pi_{III} = -\frac{1}{2} \left( \frac{\partial}{\partial m^2} \right)^2 T \sum_n \int \frac{d^3p}{(2\pi)^3} \frac{\omega_n^2 + p^2 + m^2}{(\omega_n^2 + p^2 + m^2)}.
\end{equation}
where we have employed
\begin{equation}
\frac{(-1)^n}{n!} \left( \frac{\partial}{\partial m^2} \right)^n \frac{1}{\omega_n^2 + p^2 + m^2} = \frac{1}{(\omega_n^2 + p^2 + m^2)^{n+1}}.
\end{equation}
Making use of the Matsubara frequencies according to Eq. (39), we obtain
\begin{equation}
\Pi_{III} = \frac{(qE)^2 + (qB)^2}{2} \left( \frac{\partial}{\partial m^2} \right)^2 \frac{1}{2\pi^2} \int_0^\infty dp \frac{p^2}{2\sqrt{p^2 + m^2}} \times \left( 1 + 2n_B \left( \sqrt{p^2 + m^2} \right) \right).
\end{equation}
We notice that the first term in the previous expression does not correspond to the vacuum, as it was the case in $\Pi_I$. Here we have pure electric and magnetic contributions. The integral can be done using Eq. (43) getting
\begin{equation}
\Pi_{III} = -\frac{(qE)^2 + (qB)^2}{32\pi^2 m^2} \left[ \frac{1}{32\pi^2 m^2} \right] + \frac{1}{16\pi^2 m T} \sum_{n=1}^\infty nK_1(nm/T).
\end{equation}
In order to calculate $\Pi_{IV}$ we use first Eq. (45) with $n = 3$, finding
\begin{equation}
\Pi_{IV} = -\frac{1}{6} \left( \frac{\partial}{\partial m^2} \right)^3 T \sum_n \int \frac{d^3p}{(2\pi)^3} \frac{(qB)^2 p_2^3 + (qE)^2 (p_3^2 + p_2^2)}{(\omega_n^2 + p^2 + m^2)^{3/2}}.
\end{equation}
Due to the symmetry of the integral we can replace $p_2^3 = 2p/3$ and $p_3^2 + p_2^2 = 2p/3$, having then
\begin{equation}
\Pi_{IV} = -\frac{2((qB)^2 + (qE)^2)}{9} \left( \frac{\partial}{\partial m^2} \right)^3 \int_0^\infty dp \frac{p^2}{(2\pi)^3} (\omega_n^2 + p^2 + m^2) \times (1 + 2n_B(\sqrt{p^2 + m^2})).
\end{equation}
After the sum over Matsubara frequencies we get
\begin{equation}
\Pi_{IV} = -\frac{((qB)^2 + (qE)^2)}{9\pi^2} \left( \frac{\partial}{\partial m^2} \right)^3 \int_0^\infty dp \frac{p^4}{2\sqrt{p^2 + m^2}} \times (1 + 2n_B(\sqrt{p^2 + m^2})).
\end{equation}
In an analogous way to the case $\Pi_{III}$, the purely electric and magnetic cases are obtained by calculating the integrals whereas for the temperature dependent case we use Eq. (43), obtaining
\begin{equation}
\Pi_{IV} = \frac{(qB)^2 + (qE)^2}{48\pi^2 m^2} \left[ \frac{1}{48\pi^2 m^2} \right] - \frac{1}{24\pi^2 m T} \sum_{n=1}^\infty nK_1(nm/T).
\end{equation}
Finally, we calculate $\Pi_V$ by means of the same procedure as in the previous cases, i.e. using Eq. (45) proceeding then with the sum over Matsubara frequencies. We obtain
\begin{equation}
\Pi_V = \frac{(qE)^2 m^2}{6\pi^2} \left( \frac{\partial}{\partial m^2} \right)^3 \int_0^\infty dp \frac{p^2}{2\sqrt{p^2 + m^2}} \times \left( 1 + 2n_B \left( \sqrt{p^2 + m^2} \right) \right),
\end{equation}
and using Eq. (43), we have
\begin{equation}
\Pi_V = (qE)^2 \left[ \frac{1}{48\pi^2 m^4} \right] + \frac{1}{48\pi^2 m^2 T} \sum_{n=1}^\infty nK_2(nm/T).
\end{equation}
Putting together all different contributions, we finally obtain

$$
\Pi = - \frac{m^2}{16\pi^2} \left( \ln \left( \frac{\tilde{\mu}^2}{m^2} \right) + 1 \right) + \frac{mT}{2\pi^2} \sum_{n=1}^{\infty} \frac{K_1(nm/T)}{n} + \frac{1}{96\pi^2 T^2} \frac{[(qE)^2 - (qB)^2]^2}{m^2} + \frac{(qE)^2}{48\pi^2 T^2} \sum_{n=1}^{\infty} nK_2(nm/T) - \frac{5}{48\pi^2 T^2} \frac{[(qE)^2 + (qB)^2]^2}{m^2} \sum_{n=1}^{\infty} nK_1(nm/T). \tag{54}
$$

Notice that we have separated the thermal part, the electromagnetic contribution, the electro-thermal contribution and the thermo-magnetic contribution, respectively. Using our result for the self-energy correction, we will proceed now to present graphs for Eq. (31). As a reference value for the bare mass we will take the pion mass, i.e. \( m_0 = 140 \text{ MeV} \).

The idea is to show a graph with the evolution of the sigma mass in terms of temperature and the electric and magnetic field intensities. Since in the self-energy calculation our results are valid for the whole range of temperature values, we will make a graph for the sigma mass as function of temperature (without any restrictions) for different strengths of the magnetic and electric field strengths, staying nevertheless always in the weak sector in both cases. This is shown in Fig. 2.

The blue curve represents the temperature mass evolution in absence of electric and magnetic fields. The orange curve corresponds to the case where the electric field vanishes but where the magnetic field has the value of 0.9\( m_0^2 \). The green curve corresponds to the opposite case where the magnetic field vanishes and where the electric field has the value of 0.9\( m_0^2 \). Finally, the red curve represents the case where both fields have an intensity of 0.9\( m_0^2 \). We observe that the mass grows, as we in fact expected, as function of temperature. In addition, we notice that when only the electric field is present the mass grows also with temperature. In the case where we only have a magnetic field, the mass diminishes for low temperature. When temperature becomes bigger than \( T \approx 65 \text{ MeV} \) the mass starts to grow. Finally, when both fields are present, the mass follows the same pattern. It diminishes for small temperature values, until \( T \approx 75 \text{ MeV} \), the fields induce an a less pronounces increasing behavior for the mass.

In order to get a different perspective, we show in Fig. 3 a graph for the mass evolution as function of the strengths of the fields in the \( T = 0 \) case. The blue curve represents the evolution of the sigma mass as function of the magnetic field when the electric field vanishes whereas the opposite case, the evolution of the mass as function of the electric field when the magnetic field vanishes, is represented by the orange curve. Finally, the green curve represents the mass evolution as function of both field intensities (growing the field strengths in the same proportion). We observe that for the zero temperature case, in the presence of only a magnetic field (blue curve) the mass diminishes. The opposite situation happens for a pure electric field (orange curve) where the mass grows with the field intensity. Finally, for the case when both kind of fields are present and growing in the same proportion, the mass remains constant (green curve).

### 4 Effective potential

As we said in the previous section, we want also to analyze the effective potential for our \( \lambda \phi^4 \) model, including the contribution of ring diagrams, in the presence of a weak electromagnetic external fields and temperature contributions, as well, being these effects valid for the whole range of temperature values.
In our model, the tree level potential is given by

\[ V^{(\text{tree})} = -\frac{1}{2} \mu^2 v^2 + \frac{1}{16} \lambda v^4, \quad (55) \]

The effective potential at the one loop level corresponds to [41,42]

\[ V^{(\text{1-loop})} = \sum_{i=\sigma,\chi} \left( \frac{T}{2} \sum_n \int \frac{d^3 p}{(2\pi)^3} \ln[D(\omega_n, p, m_i)]^{-1} \right) \]

\[ = \sum_{i=\sigma,\chi} \left( \frac{T}{2} \sum_n \int dm_i^2 \int \frac{d^3 p}{(2\pi)^3} D(\omega_n, p, m_i) \right) \]

\[ \equiv \frac{T}{2} \sum_n \int dm^2 \int \frac{d^3 p}{(2\pi)^3} D(\omega_n, p, m), \quad (56) \]

where in the last equality we denoted \( m_i \) as \( m \). It is understood that the result corresponds to the sum of both contributions. Using now the propagator according to Eq. (23), we get

\[ V^{(\text{1-loop})} = \frac{T}{2} \sum_n \int dm^2 \int \frac{d^3 p}{(2\pi)^3} \left[ \frac{1}{\omega_n^2 + p^2 + m^2} \right. \]

\[ + \frac{4i p_2 \omega_n (q B)(q E)}{(\omega_n^2 + p^2 + m^2)^2} - \left( \frac{q E^2}{\omega_n^2 + p^2 + m^2} \right)^2 \]

\[ + \frac{2(q B^2)p_1^2 + (q E)^2 (p_2^2 + p_3^2)}{(\omega_n^2 + p^2 + m^2)^4} \]

\[ + \frac{2(q E^2)^2 m^2}{(\omega_n^2 + p^2 + m^2)^4} \]

\[ \equiv V_I + V_{\Pi} + V_{\text{III}} + V_{IV} + V_V, \quad (57) \]

where

\[ V_I = \frac{T}{2} \sum_n \int dm^2 \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\omega_n^2 + p^2 + m^2} \]

\[ V_{\Pi} = \frac{T}{2} \sum_n \int dm^2 \int \frac{d^3 p}{(2\pi)^3} \frac{4i p_2 \omega_n (q B)(q E)}{(\omega_n^2 + p^2 + m^2)^2} \]

\[ V_{\text{III}} = -\frac{T}{2} \sum_n \int dm^2 \int \frac{d^3 p}{(2\pi)^3} \frac{q E^2}{(\omega_n^2 + p^2 + m^2)^2} \]

\[ V_{IV} = \frac{T}{2} \sum_n \int dm^2 \int \frac{d^3 p}{(2\pi)^3} \frac{2(q B^2)p_1^2 + (q E)^2 (p_2^2 + p_3^2)}{(\omega_n^2 + p^2 + m^2)^4} \]

\[ V_V = \frac{T}{2} \sum_n \int dm^2 \int \frac{d^3 p}{(2\pi)^3} \frac{2(q E^2)^2 m^2}{(\omega_n^2 + p^2 + m^2)^4}. \quad (58) \]

The procedure is analogous to the previous calculation. For the case \( V_I \) we have a vacuum contribution and an exclusively thermal part obtaining in the MS scheme,

\[ V_I = -\frac{m^4}{64\pi^2} \left( \ln \left( \frac{\mu^2}{m_i^2} \right) + \frac{3}{2} \right) - \frac{m^2 T^2}{2\pi^2} \sum_{n=1}^{\infty} K_2(nm_i/T). \]

The case \( V_{\Pi} = 0 \) due to the integral in \( dp_2 \) and for the case \( V_{\text{III}} \) we have

\[ V_{\text{III}} = (\frac{(q E)^2}{2} + (q B)^2) \]

\[ \times \left[ \frac{1}{64\pi^2} \ln \left( \frac{\mu^2}{m_i^2} \right) + \frac{1}{16\pi^2} \sum_{n=1}^{\infty} K_0(nm/T) \right]. \quad (60) \]

Using the same strategy employed for the term \( \Pi_{IV} \), for \( V_{IV} \) we obtain

\[ V_{IV} = -((q E)^2 + (q B)^2) \]

\[ \times \left[ \frac{1}{96\pi^2} \ln \left( \frac{\mu^2}{m_i^2} \right) - \frac{1}{144\pi^2} \right. \]

\[ \left. - \frac{1}{24\pi^2} \sum_{n=1}^{\infty} K_0(nm/T) \right]. \quad (61) \]

Finally, for \( V_{V} \), we get

\[ V_{V} = -(q E)^2 \left[ \frac{1}{96\pi^2} \ln \left( \frac{\mu^2}{m_i^2} \right) + \frac{1}{96\pi^2} \right. \]

\[ + \frac{1}{24\pi^2} \sum_{n=1}^{\infty} K_0(nm/T) \frac{m}{48\pi^2 T} \]

\[ \left. \sum_{n=1}^{\infty} K_1(nm/T) \right]. \quad (62) \]

The sum of all contributions gives us finally the one loop effective potential

\[ V^{(1\text{-loop})} = \sum_{i=\sigma,\chi} \left( -\frac{m_i^4}{64\pi^2} \left( \ln \left( \frac{\mu^2}{m_i^2} \right) + \frac{3}{2} \right) \right. \]

\[ - \frac{m_i^2 T^2}{2\pi^2} \sum_{n=1}^{\infty} K_2(nm_i/T) \]

\[ + \frac{(q B)^2 - (q E)^2}{192\pi^2} \ln \left( \frac{\mu^2}{m_i^2} \right) \]

\[ + \frac{5[(q E)^2 + (q B)^2]}{48\pi^2} \sum_{n=1}^{\infty} n K_0(nm_i/T) \]

\[ + (q E)^2 \left[ -\frac{1}{288\pi^2} - \frac{1}{24\pi^2} \sum_{n=1}^{\infty} \frac{K_0(nm_i/T)}{n} \right. \]

\[ \left. - \frac{m_i}{48\pi^2 T} \sum_{n=1}^{\infty} K_1(nm_i/T) \right] + (q B^2)^2 \right]. \quad (63) \]

4.1 The ring potential

The ring contribution is given by [41,42]

\[ V^{(\text{ring})} = \frac{T}{2} \sum_n \int \frac{d^3 p}{(2\pi)^3} \ln[1 + \Pi(\omega_n, p) D(\omega_n, p)]. \quad (64) \]
The previous expression can be written as
\[
V^{(\text{ring})} = \frac{T}{2} \sum_{n} \int \frac{d^3 p}{(2\pi)^3} \ln \left[ \left( D(\omega_n, p) \right)^{-1} + \Pi(\omega_n, p) \right] \\
\times \left( D(\omega_n, p) \right)
\]
\[
= \frac{T}{2} \sum_{n} \int \frac{d^3 p}{(2\pi)^3} \ln[D(\omega_n, p)] \\
+ \frac{T}{2} \sum_{n} \int \frac{d^3 p}{(2\pi)^3} \ln[D(\omega_n, p)]^{-1} + \Pi(\omega_n, p)].
\]
(65)

Notice that by considering the sum of the one loop and rings contributions, we get
\[
V^{(1\text{-loop})} + V^{(\text{ring})} = \frac{T}{2} \sum_{n} \int \frac{d^3 p}{(2\pi)^3} \ln \left[ D(\omega_n, p) \right]^{-1} \\
+ \Pi(\omega_n, p). 
\]
(66)

Therefore, it is not necessary to calculate something else. In fact, we have just to replace the mass \( m^2 \) by \( m^2 + \Pi \) in Eq. (63). The expression for the effective potential up to ring order is given by
\[
V^{(\text{tree})} + V^{(1\text{-loop})} + V^{(\text{ring})} = \frac{T}{2} \sum_{n} \int \frac{d^3 p}{(2\pi)^3} \ln \left[ D(\omega_n, p) \right]^{-1} \\
+ \Pi(\omega_n, p) \\
- \frac{1}{2} \mu^2 v^2 + \frac{1}{16} \lambda v^4 \\
+ \sum_{i=\sigma,\lambda} \left( \frac{(m_i^2 + \Pi_i)^2}{64\pi^2} \ln \left( \frac{\sqrt{\mu^2}}{m_i^2 + \Pi_i} \right) \right) + \frac{3}{2} \\
- \frac{(m_i^2 + \Pi_i) T^2}{2\pi^2} \sum_{n=1}^{\infty} K_2(n\sqrt{m_i^2 + \Pi_i}/T) \\
+ \frac{(qB)^2 - (qE)^2}{192\pi^2} \ln \left( \frac{\sqrt{\mu^2}}{m_i^2 + \Pi_i} \right) \\
+ \frac{5(qE)^2 + (qB)^2}{48\pi^2} \sum_{n=1}^{\infty} nK_0(n\sqrt{m_i^2 + \Pi_i}/T) \\
+ (qE)^2 \left[ - \frac{1}{288\pi^2} - \frac{1}{24\pi^2} \sum_{n=1}^{\infty} K_0(n\sqrt{m_i^2 + \Pi_i}/T) \right] \\
- \frac{m_i}{48\pi^2 T} \sum_{n=1}^{\infty} K_1(n\sqrt{m_i^2 + \Pi_i}/T) + \frac{(qB)^2}{144\pi^2},
\]
(67)
where \( \Pi_i \) is given by Eq. (32). We stress that the inclusion of rings is crucial for the analyticity of the effective potential, avoiding the appearance of imaginary mass terms. If we want to determine the critical temperature where the symmetry is restored, we need to explore the behavior of the effective potential as function of the expectation value \( v \). It becomes more flat for a growing temperature as shown in Fig. 4. At the critical temperature the first and second derivatives, in fact, all derivatives, vanish becoming then convex for \( T > T_c \). Here we have to do with a second order phase transition since in the whole procedure no degenerated vacuums appear. We obtain, then, the behavior of the critical temperature as function of the strengths of the electromagnetic fields which is shown in Fig. 5. The blue curve corresponds to the case where we have only an electric field and the blue curve corresponds to the case where only a magnetic field is present.
5 Conclusions

The main novelty of this work is the derivation of a bosonic propagator in a proper time representation, when external constant magnetic an electric fields are simultaneously present in a perpendicular configuration. Since this propagator is quite cumbersome, in order to get analytical results we have presented a weak field approximation which makes possible to get analytic results for mass corrections and for the effective potential due to the electric and magnetic field effects, including also temperature in the discussion. We worked in the frame of a complex scalar self-interacting field theory. Our results for the mass correction were presented in Figs. 2 and 3. The mass grows with temperature when we also turn the electric field on. When the magnetic field is turned on, however, the mass diminishes for small temperature values (up to $T \approx 65 MeV$). For higher temperature values, it starts again to grow. For the situation where both fields are present, the mass also diminish for small temperatures (up to $T \approx 75 MeV$) having a less pronounced growing behavior for higher temperatures. From Fig. 3 When $T = 0$ we see that the mass diminishes as function of the magnetic field strength, growing with the intensity of the electric field. These opposite behaviors can also be observed in other scenarios as, for example, the determination of scattering lengths or the dependence of renormalon residues [44–47].

As an important conclusion about the effective potential, we want to stress the occurrence of IMC or IEC for the cases where only a magnetic field or only an electric field are present, respectively. IMEC was clearly found, when both fields or the data will not be deposited. [Authors' comment: There are no references for a bosonic propagator in the presence of an external magnetic and electric field. The existing literature, quotes in our article, refers only to the fermionic case.]

Acknowledgements M. Loewe and R. Zamora acknowledge support from ANID/CONICYT FONDECYT Regular (Chile) under Grants No. 1200483 and 1220035. ML acknowledges support from FONDECYT Regular under grant No. 1190192. ML acknowledges also support from ANID PIA/APOYO AFB 180002 (Chile) and from the Programa de de Financiamiento Basal FB210008 para Centros Científicos y Tecnológicos de excelencia de ANID.

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors' comment: There are no references for a bosonic propagator in the presence of an external magnetic and electric field. The existing literature, quotes in our article, refers only to the fermionic case.]

References

1. Sh. Fayazbakhsh, N. Sadooghi, Phys. Rev. D 88, 065030 (2013)
2. A. Ayala, J.L. Hernández, L. A. Hernández, R.L.S Farias and R. Zamora, Phys. Rev. D 103, 054038 (2021)
3. A. Ayala, J.L. Hernández, L. A. Hernández, R.L.S. Farias and R. Zamora, Phys. Rev. D 102, 114038 (2020)
4. C.A. Dominguez, L.A. Hernández, M. Loewe, C. Villavicencio, R. Zamora, Phys. Rev. D 102, 094007 (2020)
5. Yu.A. Simonov, Phys. At. Nucl. 79, 455 (2016)
6. R.M. Aguirre, Eur. Phys. J. A 55, 28 (2019)
7. T. Yoshida, K. Suzuki, Phys. Rev. D 94, 074043 (2016)
8. D. Dudal, T.G. Mertens, Phys. Rev. D 91, 086002 (2015)
9. K. Marasinghe, T. Tuchin, Phys. Rev. C 84, 044908 (2011)
10. P. Graber, H. Hattori, S.H. Lee, M. Oka, S. Ozaki and K. Suzuki, Phys. Rev. D 93, 054026 (2016)
11. C.S. Machado, S.I. Finazzo, R.D. Matheus, J. Noronha, Phys. Rev. D 89, 074027 (2014)
12. S. Cho, K. Hattori, S.H. Lee, K. Morita, S. Ozaki, Phys. Rev. Lett. 113, 172301 (2014)
13. A. Ayala, R. L. S. Farias, S. Hernández-Ortíz, L. A. Hernández, D. Manreza Paret and R. Zamora, Phys. Rev. D 98, 114008 (2018)
14. S. Cho, K. Hattori, S.H. Lee, K. Morita, S. Ozaki, Phys. Rev. D 91, 045025 (2015)
15. S. Ghosh, A. Mukherjee, M. Mandal, S. Sarkar, P. Roy, Phys. Rev. D 94, 094043 (2016)
16. A. Bandyopadhyay, S. Mallik, Eur. Phys. J. C 77, 771 (2017)
17. S.S. Avancini, W.R. Tavares, M.B. Pinto, Phys. Rev. D 93, 014010 (2016)
18. S.S. Avancini, R.L. Farias, M.B. Pinto, W.R. Tavares, V.S. Timoteo, Phys. Lett. B 767, 247 (2017)
19. A. Ayala, C.A. Dominguez, L.A. Hernández, M. Loewe, R. Zamora, Phys. Rev. D 92, 096011 (2015)
20. A. Ayala, S. Hernández-Ortíz, L.A. Hernández, V. Knapp-Perez, R. Zamora, Phys. Rev. D 201, 074023 (2020)
21. M. Ruggieri, G.X. Peng, Phys. Rev. D 93, 094021 (2016)
22. D. Cohen, D.A. McGady, E.S. Werbos, Phys. Rev. C 76, 055201 (2007)
23. W.R. Tavares, S.S. Avancini, Phys. Rev. D 97, 094001 (2018)
24. Pedro Costa, Márcio. Ferreira, Débora. P. Menezes, João. Moreira, Constança Providência, Phys. Rev. D 92, 036012 (2015)
25. Jens O. Andersen, Eur. Phys. J. A 57, 189 (2021)
26. A. Ahmad, A. Raya, J. Phys. G43, 065002 (2016)
27. M. Ferreira, P. Costa, O. Lourenço, T. Fredeto and C. Providência, Phys. Rev. D, 89, 116011 (2014)
28. A. Ayala, L. A. Hernández, M. Loewe, J. C. Rojas and R. Zamora, Eur. Phys. J. A56, (2020)
29. V. A. Miransky and I. A. Shovkovy, Phys. Rept. 576, 1 (2015)
30. D. Kharzeev, K. Landsteiner, A. Schmitt, Ho-Ung Yee, Editors, Strongly Interacting Matter in Magnetic Fields, Springer-Verlag Berlin Heidelberg 2013, and references therein. G. Baym.T. Hatsuda, T. Kojo, P.D. Powell, Y. Song, and T. Takasuka, Rept. Prog. Phys. 81 (2018) 5, 056902; D. Blaschke, A. Ayyarjan, and A. Friesen (Editors), “Compact Stars in the QCD Phase Diagram”, Universe, MpiipaG (2020)
31. G. S. Bali, F. Bruckmann, G. Endrödi, Z. Fodor, S. D. Katz and A. Schäfer, Phys. Rev. D 86, 071502 (2012); G. Bali, F. Bruckmann, G. Endrödi, Z. Fodor, S. Katz, S. Krieg, A. Schaefers, and K. K. Szabo, JHEP 1202, 044 (2012); G. Bali, F. Bruckmann, G. Endrödi, S. Katz, and A. Schäfer, JHEP 1408, 177 (2014)
32. A. Ayala, M. Loewe, R. Zamora, Phys. Rev. D 91, 016002 (2015)
33. A. Ayala, M. Loewe, A. Mizher, R. Zamora, Phys. Rev. D 90, 036001 (2014)
34. Alejandro Ayala, Luis. A. Hernández, Marcelo Loewe, and Cristian Villacicenio, Eur. Phys. J.A 57 (2021), 7, 234
35. M. Loewe, D. Valenzuela, R. Zamora, Phys. Rev. D 105, 036017 (2022)
36. W.R. Tavares, R.L.S. Farias, S.S. Avancini, Phys. Rev. D 101, 016017 (2020)
36. M. Loewe, R. Zamora, Phys. Rev. D 105, 076011 (2022)
37. G. Cao and X.G. Huang, Phys. Rev. D 93, 016007 (2016); M. Ruggeri, and G. X. Peng, Phys. Rev. D 93, 094021 (2016); M. Ruggeri, Z. Y. Lu and G. X. Peng, Phys. Rev. D 94, 116002 (2016)
38. Walter Dittrich and Martin Reuter, “Effective Lagrangian in Quantum Electrodynamics”. Springer-Verlag, Berlin Heidelberg New York, Tokyo (1985). See also Walter Dittrich and Holger Gies, “Probing the Quantum Vacuum: Perturbative effective action approach in Quantum Electrodynamics and its application”, Springer Tracts in Modern Physics, Volume 166 (2000)
39. A. Ahmad, N. Ahmadiniaz, O. Corradini, S. P. Kim and C. Schubert, Nuclear Physics B, 919 (2017)
40. A. Ayala, J. Castaño, L. A. Hernandez, J. Salinas, R. Zamora, Eur. Phys. J. A57 (2021)
41. M. Le Bellac, Thermal Field Theory (Cambridge University Press, Cambridge, 1996)
42. J.I. Kapusta, C. Gale, Finite-Temperature Field Theory Principles and Applications (Cambridge University Press, Cambridge, 2006)
43. M. Abramowitz, I. A. Stegun, (Eds.). Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing. New York: Dover, pp. 576–579 (1972)
44. M. Correa, M. Loewe, R. Zamora, Phys. Rev. D 99, 096024 (2019)
45. M. Loewe, L. Monje, R. Zamora, Phys. Rev. D 104, 016020 (2021)
46. M. Loewe, E. Muñoz, R. Zamora, Phys. Rev. D 100, 116006 (2019)
47. M. Loewe, L. Monje, E. Muñoz, A. Raya, R. Zamora, Phys. Rev. D 99, 056002 (2019)