PROOF OF SOME CONGRUENCES CONJECTURED BY GUO AND LIU

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Abstract. Let \( n \) and \( r \) be positive integers and \( p \) a prime. Define the numbers \( S_n^{(r)} \) and \( T_n^{(r)} \) by
\[
S_n^{(r)} = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} (2k+1)^r , \\
T_n^{(r)} = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} (2k+1)^r (-1)^k .
\]
In this paper we first prove the following congruences:
\[
\sum_{k=0}^{n-1} S_k^{(2r)} \equiv 0 \pmod{n^2} , \\
\sum_{k=0}^{n-1} T_k^{(2r)} \equiv 0 \pmod{n^2}
\]
and
\[
\sum_{k=0}^{p-1} T_k^{(2)} = \frac{p^2}{2} \left( 5 - 3 \left( \frac{p}{5} \right) \right) \pmod{p^3} .
\]
We also show that there exist integers \( a_{2r-1} \) and \( b_r \), independent of \( n \), such that
\[
a_{2r-1} \sum_{k=0}^{n-1} S_k^{(2r-1)} \equiv 0 \pmod{n^2} , \\
b_r \sum_{k=0}^{n-1} k S_k^{(r)} \equiv 0 \pmod{n^2} .
\]
This confirms several recent conjectures of Guo and Liu.

1. Introduction

In [Su1], Z.-W. Sun defined two new kinds of numbers,
\[
S_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} (2k+1) \quad (n = 0, 1, 2, \ldots)
\]

Key words and phrases. Central binomial coefficients, congruences, Bernoulli numbers, Zeilberger algorithm.

2010 Mathematics Subject Classification. 11B65, 11B68, 05A10, 11A07.

This research was supported by the Natural Science Foundation (Grant No. 11171140) of China.
and
\[ R_n = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \frac{1}{2k-1} \quad (n = 0, 1, 2, \ldots), \]
he showed a lot of congruences involving these two numbers, like
\[ \frac{1}{n^2} \sum_{k=0}^{n-1} S_k = \sum_{k=0}^{n-1} \left( \frac{n-1}{k} \right)^2 C_k \in \mathbb{Z}, \]
where \( C_k = \frac{1}{k+1} \binom{2k}{k} \) is the k-th Catalan number, and
\[ \frac{1}{n} \sum_{k=0}^{n-1} S_k(x) \in \mathbb{Z}[x]. \]
And he gave lots of conjectures about these two numbers, such as [Su1, Conjecture 5.4]. For every \( n = 1, 2, 3, \ldots \), we have
\[ \frac{3}{n} \sum_{k=0}^{n-1} R_k^2 \in \mathbb{Z} \quad \text{and} \quad \frac{1}{n} \sum_{k=0}^{n-1} (2k+1)R_k^2 \in \mathbb{Z}. \]

**Conjecture 1.1.** [Su1, Conjecture 5.5, 5.6] For any positive integer \( n \), we have
\[ \frac{4}{n^2} \sum_{k=0}^{n-1} kS_k \in \mathbb{Z}, \quad \frac{1}{n^2} \sum_{k=0}^{n-1} S_k^{(2)} \in \mathbb{Z}. \]

Guo and Liu proved some conjectures(contain Conjecture 1.1) of professor Z.W. Sun about \( S_n \) and other two numbers similar to \( S_n \) in [GL], they also defined two numbers
\[ S_n^{(r)} = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} (2k+1)^r, \quad T_n^{(r)} = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} (2k+1)^r (-1)^k, \]
which extend \( S_n \) and \( S_n^- \) in [Su1], note that \( T_n^{(2)} \) is \( S_n^- \) introduced by Sun [Su1, Conjecture 5.6], Guo and Liu also gave few conjectures about \( S_n^{(r)} \) and \( T_n^{(r)} \) at the end of [GL], for example:

**Conjecture 1.2.** [GL, Conjecture 5.1] Let \( n \) and \( r \) be positive integers and \( p \) a prime. Then
\[ \sum_{k=0}^{n-1} S_k^{(2r)} \equiv 0 \pmod{n^2}, \]
\[ \sum_{k=0}^{n-1} T_k^{(2r)} \equiv 0 \pmod{n^2}, \]
\[
p^{-1} \sum_{k=0}^{p-1} T_{k}^{(2)} \equiv \frac{p^2}{2} \left( 5 - 3 \left( \frac{p}{5} \right) \right) \pmod{p^3}.
\]

**Conjecture 1.3.** [GL, Conjecture 5.2] Let \( n \) and \( r \) be positive integers. Then there exist integers \( a_{2r-1} \) and \( b_r \), independent of \( n \), such that

\[
a_{2r-1} \sum_{k=0}^{n-1} S_k^{(2r-1)} \equiv 0 \pmod{n^2},
\]

\[
b_r \sum_{k=0}^{n-1} kS_k^{(r)} \equiv 0 \pmod{n^2}.
\]

**Conjecture 1.4.** [GL, Conjecture 5.3] Let

\[
a_3 = 3, \ a_5 = 15, \ a_7 = 21, \ a_9 = 15, \ a_{11} = 33, \ a_{13} = 1365, \ a_{15} = 3.
\]

\[
b_2 = 12, \ b_3 = 4, \ b_4 = 60, \ b_5 = 20, \ b_6 = 84, \ b_7 = 28, \ b_8 = 60, \ b_9 = 20.
\]

Then the conjecture 1.3 holds.

In this paper we mainly prove Conjecture 1.2 and Conjecture 1.3 , and partially answer the Conjecture 1.4.

We know that the Euler numbers \( \{E_n\} \) and Euler polynomials \( \{E_n(x)\} \) are defined by

\[
\frac{2e^t}{e^{2t} + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!} \left( |t| < \frac{\pi}{2} \right) \text{ and } \frac{2e^{xt}}{e^{t} + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} \left( |t| < \pi \right).
\]

And we also have the property of Euler polynomials

\[
E_n(x + y) = \sum_{k=0}^{n} \binom{n}{k} E_k(x) y^{n-k}.
\]  

(1.1)

Recall that the Bernoulli numbers given by

\[
\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \quad (0 < |x| < 2\pi).
\]

Motivated by the above work, we mainly obtain the following results in this paper.

**Theorem 1.1.** For any positive integers \( n \) and \( r \), we have

\[
\sum_{k=0}^{n-1} S_k^{(2r)} \equiv 0 \pmod{n^2}.
\]  

(1.2)
This result can detrude one conjecture in [Su1, Conjecture 5.6], that is
\[ \frac{1}{n^2} \sum_{k=0}^{n-1} S_k^+ \in \mathbb{Z}, \]
which has been solved by Guo and Liu in [GL].

**Theorem 1.2.** Let \( n \) and \( r \) be positive integers, we can obtain
\[ \sum_{k=0}^{n-1} T_k^{(2r)} \equiv 0 \pmod{n^2}. \] (1.3)

This result also can push out one conjecture in [Su1, Conjecture 5.6], that is
\[ \frac{1}{n^2} \sum_{k=0}^{n-1} S_k^- \in \mathbb{Z}, \] where \( S_k^- = T_k^{(2)} \).

The above two are divisibility results, following we give a congruence result:

**Theorem 1.3.** Let \( p \) be a prime, then we have
\[ \sum_{k=0}^{n-1} T_k^{(2)} \equiv \frac{p^2}{2} \left( 5 - 3 \left( \frac{5}{p} \right) \right) \pmod{p^3}. \] (1.4)

The following theorem is more challenging than the above three Theorems, because it is more difficult when the index number of \( S_k \) is odd.

**Theorem 1.4.** Let \( n \) and \( r \) be positive integers, then there exist integers \( a_{2r-1} \) and \( b_r \), independent of \( n \), such that
\[ a_{2r-1} \sum_{k=0}^{n-1} S_k^{(2r-1)} \equiv 0 \pmod{n^2}, \] (1.5)
\[ b_r \sum_{k=0}^{n-1} k S_k^{(r)} \equiv 0 \pmod{n^2}. \] (1.6)

We are going to prove Theorem 1.1 in section 2, then prove Theorem 1.2 in section 3, and at last we will prove Theorem 1.3 and Theorem 1.4 in section 4 and section 5 respectively. Our proofs make use of some sophisticated combinatorial identities and the Zeilberger algorithm.

We give two lemmas here, which will be used to prove the above four Theorems.
Lemma 1.1. For any nonnegative integers \( m \) and \( n \), we have
\[
\sum_{k=0}^{n} \binom{x+k}{m} = \binom{n+x+1}{m+1} - \binom{x}{m+1},
\] (1.7)
and
\[
\sum_{k=0}^{n} \binom{n}{k}^2 \binom{x+k}{2n+1} = \binom{x}{n}^2.
\] (1.8)

Remark 1.1. Both (1.5) and (1.6) can be found in [G, (1.48) and (6.30)].

Lemma 1.2. [MS, (Lemma 3.2)] For any nonnegative integer \( n \), we have
\[
\sum_{k=0}^{n} \binom{n}{k}^2 \binom{x+k}{2n} = \frac{1}{(4n+2)(2n)} \sum_{k=0}^{n} (2x-3k) \binom{x}{k}^2 \binom{2k}{k}.
\] (1.9)

The identity (1.9) is crucial to our proofs of Theorems 1.1-1.4.

2. Proof of Theorem 1.1

Lemma 2.1. For any positive integer \( n \) and \( r \), we have
\[
\sum_{k=0}^{n-1} (2k+1)^{2r-1} \equiv 0 \pmod{n}.
\]

Proof: We know that
\[
\sum_{k=0}^{n-1} (2k+1)^{2r-1} = \sum_{j=0}^{n-1} (2n-2j-1)^{2r-1} \equiv -\sum_{k=0}^{n-1} (2k+1)^{2r-1} \pmod{2n},
\]
then
\[
2 \sum_{k=0}^{n-1} (2k+1)^{2r-1} \equiv 0 \pmod{2n},
\]
so we can deduce that
\[
\sum_{k=0}^{n-1} (2k+1)^{2r-1} \equiv 0 \pmod{n}.
\]
Thus we have done the proof of Lemma 2.1.

Remark 2.1. We can see that in the proof of this lemma, we need the index number of \( S_k \) to be odd, if it is even, we can not get the result like this, so when the index number of \( S_k \) is even, the case is more difficult.
Then via Lemma 1.2 and Lemma 2.1 we can obtain

$$\sum_{k=0}^{n-1} S_k^{(2r)} = \sum_{k=0}^{n-1} \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{2j} (2j + 1)^{2r}$$

$$= \sum_{j=0}^{n-1} \binom{2j}{j} (2j + 1)^{2r} \sum_{k=j}^{n-1} \sum_{l=0}^{j} \binom{j}{l}^2 \binom{k + l}{2j}$$

$$= \sum_{j=0}^{n-1} \binom{2j}{j} (2j + 1)^{2r} \sum_{l=0}^{j} \binom{j}{l}^2 \sum_{k=j}^{n-1} \binom{k + l}{2j}$$

$$\equiv \sum_{j=0}^{n-1} \binom{2j}{j} (2j + 1)^{2r} \sum_{l=0}^{j} \binom{j}{l}^2 \binom{n + l}{2j + 1}.$$ 

Proof of Theorem 1.1: By Lemma 1.1 we can deduce that

$$\sum_{k=0}^{n-1} S_k^{(2r)} = \frac{1}{2} \sum_{j=0}^{n-1} (2j + 1)^{2r-1} \sum_{l=0}^{j} (2n - 3l) \binom{n}{l}^2 \binom{2l}{l}$$

$$= n \sum_{j=0}^{n-1} (2j + 1)^{2r-1} + \frac{1}{2} \sum_{j=0}^{n-1} (2j + 1)^{2r-1} \sum_{l=1}^{j} (2n - 3l) \binom{n}{l}^2 \binom{2l}{l}$$

$$= n \sum_{j=0}^{n-1} (2j + 1)^{2r-1} + n^2 \sum_{j=1}^{n-1} (2n - 3l) \binom{n - 1}{l - 1}^2 \binom{2l-1}{l-1} \sum_{j=0}^{n-1} (2j + 1)^{2r-1}$$

$$\equiv n^2 \sum_{j=1}^{n-1} (2n - 3l) \binom{n - 1}{l - 1}^2 \binom{2l-1}{l-1} \sum_{j=0}^{n-1} (2j + 1)^{2r-1} \pmod{n^2}.$$

We know that

$$\frac{(n-1)(2n-3l)}{l} = 2 \binom{n}{l} - 3 \binom{n - 1}{l - 1} \in \mathbb{Z} \quad \text{and} \quad \frac{n}{l} \binom{n-1}{l-1} = \binom{n}{l} \in \mathbb{Z},$$

so with Lemma 2.1 we have

$$\sum_{j=0}^{n-1} (2j + 1)^{2r-1} \equiv 0 \pmod{n},$$

then

$$\frac{n^2 (n-1)^2 (2n-3l)}{l^2} \sum_{j=0}^{n-1} (2j + 1)^{2r-1} \equiv 0 \pmod{n^2},$$

and

$$\sum_{j=0}^{l-1} (2j + 1)^{2r-1} \equiv 0 \pmod{l},$$

then

$$\frac{n^2 (n-1)^2 (2n-3l)}{l^2} \sum_{j=0}^{l-1} (2j + 1)^{2r-1} \equiv 0 \pmod{n^2}.$$
Hence
\[ n^2(2n - 3l) \binom{n-1}{l-1}^2 \frac{(2l-1)}{l^2} \sum_{j=l}^{n-1} (2j+1)^{2r-1} \equiv 0 \pmod{n^2}. \]

Therefore
\[ \sum_{k=0}^{n-1} S_k^{(2r)} \equiv 0 \pmod{n^2}. \]

So we finish the proof of Theorem 1.1. \( \square \)

3. Proof of Theorem 1.2

Lemma 3.1. For any positive integer \( n \) and \( r \), we have
\[ \sum_{k=0}^{n-1} (2k+1)^{2r-1}(-1)^k \equiv 0 \pmod{n}. \]

Proof: By \( E_n(x) + E_n(x+1) = 2x^n \), \( E_n\left(\frac{1}{2}\right) = E_{2n} \) and (1.1), we have
\[
\begin{align*}
\sum_{k=0}^{n-1} (-1)^k(2k+1)^{2r-1} & = 2^{2r-2} \sum_{k=0}^{n-1} \left((-1)^k E_{2r-1}(k + \frac{1}{2}) - (-1)^{k+1} E_{2r-1}(k + \frac{3}{2})\right) \\
& = 2^{2r-2} \left(E_{2r-1}\left(\frac{1}{2}\right) - (-1)^n E_{2r-1}(n + \frac{1}{2})\right) \\
& = \frac{E_{2r-1}}{2} - (-1)^n \frac{E_{2r-1}}{2} - (-1)^n 2^{2r-2} \sum_{k=0}^{2r-2} \binom{2r-1}{k} E_k \frac{2^k}{k!} 2^{2r-2-k} \\
& \equiv 0 \pmod{n}.
\end{align*}
\]

(Note that \( E_{2n-1} = 0 \) and \( E_n \in \mathbb{Z} \) for all \( n \geq 1 \), \( \frac{22r-2}{2^r} \in \mathbb{Z} \) for all \( k = 0, 1, \cdots, 2r - 2 \).

So we complete the proof of Lemma 3.1.

Remark 3.1. We know that \( E_k = 0 \) when \( k \) is odd and \( E_n \in \mathbb{Z} \) for all \( n \geq 1 \), so we can prove this lemma by the above method. If the index number of \( 2k + 1 \) is not \( 2r - 1 \), it is an even number \( 2r \), then we can not prove it like this.
Proof of Theorem 1.2: By Lemma 1.1 we can deduce that

\[ \sum_{k=0}^{n-1} T_k^{(2r)} = \sum_{k=0}^{n-1} \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j} (2j+1)^{2r} (-1)^j \]

\[ = \sum_{j=0}^{n-1} \binom{2j}{j} (2j+1)^{2r} (-1)^j \sum_{l=0}^{j} \binom{j}{l}^2 \binom{k+l}{2j} \]

\[ = \sum_{j=0}^{n-1} \binom{2j}{j} (2j+1)^{2r} (-1)^j \sum_{l=0}^{j} \binom{j}{l} 2^{n-1} \binom{k+l}{2j} \]

\[ \equiv \sum_{j=0}^{n-1} \binom{2j}{j} (2j+1)^{2r} (-1)^j \sum_{l=0}^{j} \binom{j}{l} 2^{n} \binom{n+l}{2j+1}. \]

Then via Lemma 1.2 and Lemma 3.1 we can obtain

\[ \sum_{k=0}^{n-1} T_k^{(2r)} = \frac{1}{2} \sum_{j=0}^{n-1} (2j+1)^{2r-1} (-1)^j \sum_{l=0}^{j} (2n-3l) \binom{n}{l}^2 \binom{2l}{l} \]

\[ = n \sum_{j=0}^{n-1} (2j+1)^{2r-1} (-1)^j + \frac{1}{2} \sum_{j=1}^{n-1} (2j+1)^{2r-1} (-1)^j \sum_{l=1}^{j} (2n-3l) \binom{n}{l}^2 \binom{2l}{l} \]

\[ = n \sum_{j=0}^{n-1} (2j+1)^{2r-1} (-1)^j + n^2 \sum_{l=1}^{n-1} (2n-3l) \binom{n-1}{l-1}^2 \binom{2l-1}{l-1} \sum_{j=l}^{n-1} (2j+1)^{2r-1} (-1)^j \]

\[ \equiv n^2 \sum_{l=1}^{n-1} (2n-3l) \binom{n-1}{l-1}^2 \binom{2l-1}{l-1} \sum_{j=l}^{n-1} (2j+1)^{2r-1} (-1)^j \pmod{n^2}. \]

We know that

\[ \frac{(n-1)(2n-3l)}{l} = 2 \binom{n}{l} - 3 \binom{n-1}{l-1} \in \mathbb{Z} \text{ and } \frac{n}{l} \binom{n-1}{l-1} = \binom{n}{l} \in \mathbb{Z}, \]

so with Lemma 3.1 we have \( \sum_{j=0}^{n-1} (2j+1)^{2r-1} (-1)^j \equiv 0 \pmod{n} \), then

\[ \frac{n^2(n-1)^2(2n-3l)}{l^2} \sum_{j=0}^{n-1} (2j+1)^{2r-1} (-1)^j \equiv 0 \pmod{n^2}, \]

and \( \sum_{j=0}^{l-1}(2j+1)^{2r-1} (-1)^j \equiv 0 \pmod{l} \), then

\[ \frac{n^2(n-1)^2(2n-3l)}{l^2} \sum_{j=0}^{l-1}(2j+1)^{2r-1} (-1)^j \equiv 0 \pmod{n^2}. \]
So we can get that

\[ n^2(2n - 3l) \left( \frac{n - 1}{l - 1} \right)^2 \left( \frac{2l - 1}{l} \right) \sum_{j=l}^{n-1} (2j + 1)^{2r-1} (-1)^j \equiv 0 \pmod{n^2}. \]

Then we have completed the proof of Theorem 1.2. \( \square \)

4. Proof of Theorem 1.3

Proof of Theorem 1.3: By Lemma 1.1 we can deduce that

\[
\sum_{k=0}^{p-1} T_k^{(2)} = \sum_{k=0}^{p-1} \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j} (2j + 1)^2 (-1)^j \\
= \sum_{j=0}^{p-1} \binom{2j}{j} (2j + 1)^2 (-1)^j \sum_{k=j}^{p-1} \sum_{l=0}^{j} \binom{j}{l}^2 \binom{k+l}{2j} \\
= \sum_{j=0}^{p-1} \binom{2j}{j} (2j + 1)^2 (-1)^j \sum_{l=0}^{j} \sum_{k=j}^{p-1} \binom{k+l}{2j} \\
= \sum_{j=0}^{p-1} \binom{2j}{j} (2j + 1)^2 (-1)^j \sum_{l=0}^{j} \binom{j}{l}^2 \binom{p+l}{2j+1}.
\]

Then via Lemma 1.2 we can obtain

\[
\sum_{k=0}^{p-1} T_k^{(2)} = \frac{1}{2} \sum_{j=0}^{p-1} (2j + 1)(-1)^j \sum_{l=0}^{j} (2p - 3l) \binom{p}{l}^2 \binom{2l}{l} \\
= p \sum_{j=0}^{p-1} (2j + 1)(-1)^j + \frac{1}{2} \sum_{j=1}^{p-1} (2j + 1)(-1)^j \sum_{l=0}^{j} (2p - 3l) \binom{p}{l}^2 \binom{2l}{l} \\
= p \sum_{j=0}^{p-1} (2j + 1)(-1)^j + \frac{p^2}{2} \sum_{l=1}^{p-1} (2p - 3l) \binom{p-1}{l-1}^2 \binom{2l}{l} \sum_{j=l}^{p-1} (2j + 1)(-1)^j \\
= p^2 - \frac{3p^2}{2} \sum_{l=1}^{p-1} \binom{2l}{l} \binom{p-1}{l-1}^2 \sum_{j=l}^{p-1} (2j + 1)(-1)^j \\
= p^2 - \frac{3p^2}{2} \sum_{l=1}^{p-1} \binom{2l}{l} (-1)^l \pmod{p^3},
\]

(Note that \( \sum_{j=0}^{p-1} (2j + 1)(-1)^j = p \), \( \sum_{j=1}^{p-1} (2j + 1)(-1)^j = p + (-1)^l \) and \( \binom{p-1}{l-1}^2 = \Pi_{j=1}^{p-1} (2j - 1)^2 \equiv 1 \pmod{p} \).
We know that \((\binom{2l}{l}) = (-\frac{3}{2})^l(-4)^l \equiv (\frac{p-1}{l})(-4)^l \pmod{p}\), so we have
\[
\sum_{k=0}^{p-1} T_k^{(2)} \equiv p^2 - \frac{3}{2}p^2 \sum_{l=1}^{(p-1)/2} \left(\frac{p-1}{l}\right)4^l \equiv p^2 - \frac{3}{2}p^2(5^{(p-1)/2} - 1) \equiv \frac{p^2}{2} \left(5 - 3\left(\frac{5}{p}\right)\right) \pmod{p^3}.
\]

Note that \(\left(\frac{5}{p}\right) = \left(\frac{p}{5}\right)\), so we finish the proof of Theorem 1.3. □

5. Proof of Theorem 1.4

In [KM] we know
\[
S_m(n) = 1^m + 2^m + \cdots + (n-1)^m
\]
and the m-th Bernoulli number can be written as
\[
B_m = \frac{U_m}{V_m}
\]
where \((U_m, V_m) = 1\) and \(V_m > 0\).

**Lemma 5.1.** [KM, Proposition 15.2.2] If integer \(m \geq 2\) is even, then for all \(n \geq 1\) we have
\[
V_m S_m(n) \equiv nU_m \pmod{n^2}.
\]

**Lemma 5.2.** Let \(n\) and \(r\) be positive integers and \(r > 1\), then we have
\[
V_{2r-2} \sum_{j=0}^{n-1} (2j + 1)^{2r-2} \equiv 0 \pmod{2n}.
\]

**Proof:** We know \(\sum_{j=0}^{n-1} (2j + 1)^{2r-2} = S_{2r-2}(2n) - 2^{2r-2}S_{2r-2}(n)\), and by Lemma 5.1 we have
\[
V_{2r-2} S_{2r-2}(2n) \equiv 2nU_{2r-2} \equiv 0 \pmod{2n}
\]
and
\[
V_{2r-2} S_{2r-2}(n) \equiv nU_{2r-2} \equiv 0 \pmod{n}
\]
for all \(r > 1\), so for all positive integer \(r > 1\) we have
\[
V_{2r-2} \sum_{j=0}^{n-1} (2j + 1)^{2r-2} \equiv 0 \pmod{2n}.
\]

Therefore we have done the proof of Lemma 5.2.

**Proof of (1.5):** For \(r = 1\) we know \(\sum_{k=0}^{n-1} S_k \equiv 0 \pmod{n^2}\), which is given by Z.W.Sun, so we just need to show it for all \(r > 1\). By Lemma
1.1 we can deduce that
\[
\sum_{k=0}^{n-1} S_k^{(2r-1)} = \sum_{k=0}^{n-1} \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j} (2j + 1)^{2r-1}
\]
\[
= \sum_{j=0}^{n-1} \binom{2j}{j} (2j + 1)^{2r-1} \sum_{k=j}^{n-1} \sum_{l=0}^{j} \binom{j}{l}^2 \binom{k+l}{2j}
\]
\[
= \sum_{j=0}^{n-1} \binom{2j}{j} (2j + 1)^{2r-1} \sum_{l=0}^{j} \binom{j}{l}^2 \sum_{k=j}^{n-1} \binom{k+l}{2j}
\]
\[
= \sum_{j=0}^{n-1} \binom{2j}{j} (2j + 1)^{2r-1} \sum_{l=0}^{j} \binom{j}{l}^2 \binom{n+l}{2j+1}.
\]

Then via Lemma 1.2 and Lemma 5.2 we can obtain
\[
V_{2r-2} \sum_{k=0}^{n-1} S_k^{(2r-1)} = \frac{V_{2r-2}}{2} \sum_{j=0}^{n-1} (2j + 1)^{2r-2} \sum_{l=0}^{j} (2n - 3l) \binom{n}{l}^2 \binom{2l}{l}
\]
\[
= V_{2r-2} \sum_{j=0}^{n-1} (2j + 1)^{2r-2} + \frac{V_{2r-2}}{2} \sum_{j=0}^{n-1} (2j + 1)^{2r-2} \sum_{l=1}^{j} (2n - 3l) \binom{n}{l}^2 \binom{2l}{l}
\]
\[
= V_{2r-2} \sum_{j=0}^{n-1} (2j + 1)^{2r-2} + n^2 \sum_{l=1}^{n-1} \frac{V_{2r-2}(2n - 3l)}{l^2} \binom{n-1}{l-1}^2 \binom{2l-1}{l-1} \sum_{j=l}^{n-1} (2j + 1)^{2r-2}
\]
\[
\equiv V_{2r-2} \sum_{l=1}^{n-1} (2n - 3l) \binom{n-1}{l-1}^2 \binom{2l-1}{l-1} \sum_{j=l}^{n-1} (2j + 1)^{2r-2} \pmod{2n^2}.
\]

We know \(\binom{n}{l}^2 = \binom{n}{l}^2 - 3 \binom{n}{l} \in \mathbb{Z}\) and \(\frac{n(n-1)}{l} = \binom{n}{l} \in \mathbb{Z}\), so with Lemma 5.2 we have
\[
\frac{V_{2r-2} n^2 \binom{n-1}{l-1}^2 (2n - 3l)}{l^2} \sum_{j=0}^{l-1} (2j + 1)^{2r-2} \equiv 0 \pmod{2n^2},
\]
and
\[
\frac{V_{2r-2} n^2 \binom{n-1}{l-1}^2 (2n - 3l)}{l^2} \sum_{j=0}^{l-1} (2j + 1)^{2r-2} \equiv 0 \pmod{2n^2}.
\]

So we can get that
\[
V_{2r-2} n^2 (2n - 3l) \binom{n-1}{l-1}^2 \frac{(2l-1)}{l^2} \sum_{j=l}^{n-1} (2j + 1)^{2r-2} \equiv 0 \pmod{2n^2}.
\]
Therefore
\[ V_{2r-2} \sum_{k=0}^{n-1} S_k^{(2r-1)} \equiv 0 \pmod{2n^2}, \]
so we just need to take
\[
\begin{cases}
    a_{2r-1} = 1 & \text{if } r=1, \\
    a_{2r-1} = \frac{1}{2} V_{2r-2} & \text{if } r>1.
\end{cases}
\]
Then we have finished the proof of (1.5).

**Lemma 5.3.** For any positive integer \( n \) and \( 0 \leq j < n \) we have
\[
(2j + 1) \binom{2j}{j} \sum_{k=j}^{n-1} (2k - j + 1) \binom{k}{j}^2 \equiv 0 \pmod{n^2}.
\]

**Proof:** Set \( u_j = (2j + 1) \binom{2j}{j} \sum_{k=j}^{n-1} (2k - j + 1) \binom{k}{j}^2 \), then by Zeilberger we can get the recurrence
\[
u_j = \frac{(1 + 2j)n^2 \binom{2j}{j} \binom{n-1}{j}^2}{j+1}
\]
for all \( 0 \leq j < n - 1 \), while \( \binom{2j}{j+1} \in \mathbb{Z} \), so we have \( u_j \equiv 0 \pmod{n^2} \) for all \( 0 \leq j < n - 1 \), and we know
\[
u_{n-1} = (2n - 1) \binom{2n - 2}{n - 1} n = n^2 \binom{2n - 1}{n - 1} \equiv 0 \pmod{n^2},
\]
so \( u_j \equiv 0 \pmod{n^2} \) for all \( 0 \leq j < n \), therefore we finish the proof of Lemma 5.3.

**Proof of (1.6):** First we know that
\[
\sum_{k=0}^{n-1} (4kS_k^{(r)} - S_k^{(r+1)} + 3S_k^{(r)})
\]
\[
= \sum_{k=0}^{n-1} \sum_{j=0}^{k} (4k - 2j + 2) \binom{k}{j}^2 \binom{2j}{j} (2j + 1)^r
\]
\[
= \sum_{j=0}^{n-1} \binom{2j}{j} (2j + 1)^r \sum_{k=j}^{n-1} (4k - 2j + 2) \binom{k}{j}^2 = 2 \sum_{j=0}^{n-1} (2j + 1)^{r-1} u_j
\]
then by Lemma 5.3 we can deduce that
\[
\sum_{k=0}^{n-1} (4kS_k^{(r)} - S_k^{(r+1)} + 3S_k^{(r)}) \equiv 0 \pmod{2n^2},
\]
we know when \( r = 1 \), \( \sum_{k=0}^{n-1} 4kS_k^{(r)} \equiv 0 \pmod{n^2} \), which is given by Guo and Liu [GL], so we just need to show it for all \( r > 1 \).

**Case 1.** If \( 2 \mid r \), then by (1.2) we have \( V_r \sum_{k=0}^{n-1} S_k^{(r)} \equiv 0 \pmod{2n^2} \) since 6 always divides the denominator of \( B_r \) [KM, Page 233, Corollary], and with (1.5) we have \( V_r \sum_{k=0}^{n-1} S_k^{(r+1)} \equiv 0 \pmod{2n^2} \), so we can get

\[
V_r \sum_{k=0}^{n-1} 4kS_k^{(r)} \equiv 0 \pmod{2n^2}.
\]

**Case 2.** If \( 2 \nmid r \) and \( r > 1 \), then by (1.2) we have \( V_{r-1} \sum_{k=0}^{n-1} S_k^{(r+1)} \equiv 0 \pmod{2n^2} \) since 6 always divides the denominator of \( B_{r-1} \) [KM, Page 233, Corollary], and with (1.5) we know \( V_{r-1} \sum_{k=0}^{n-1} S_k^{(r)} \equiv 0 \pmod{2n^2} \), thus we have

\[
V_{r-1} \sum_{k=0}^{n-1} 4kS_k^{(r)} \equiv 0 \pmod{2n^2}.
\]

So combining Case 1 and Case 2 we finally get that there exist integers \( b_r \) such that

\[
b_r \sum_{k=0}^{n-1} kS_k^{(r)} \equiv 0 \pmod{n^2},
\]

we just need to take

\[
\begin{cases}
  b_r = 2V_{r-1} & \text{if } r \equiv 1 \pmod{2}, \\
  b_r = 2V_r & \text{if } r \equiv 0 \pmod{2}.
\end{cases}
\]

Therefore we complete the proof of (1.6). \( \square \)

**Corollary 5.1.** We can deduce some special value of \( a_{2r-1} \), like

\[
a_3 = 3, a_5 = 15, a_7 = 21, a_9 = 15, a_{11} = 33, a_{13} = 1365, a_{15} = 3.
\]

**Proof:** We know

\[
B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}, B_6 = \frac{1}{42}, B_8 = -\frac{1}{30},
\]

\[
B_{10} = \frac{5}{66}, B_{12} = -\frac{691}{2730}, B_{14} = \frac{7}{6},
\]

so we have

\[
V_2 = 6, V_4 = 30, V_6 = 42, V_8 = 30, V_{10} = 66, V_{12} = 2730, V_{14} = 6,
\]

then by (1.5) we can deduce that

\[
a_3 = 3, a_5 = 15, a_7 = 21, a_9 = 15, a_{11} = 33, a_{13} = 1365, a_{15} = 3.
\]
Corollary 5.2. We can deduce some special value of $b_r$, such as

\[ b_2 = b_3 = 12, b_4 = b_5 = 60, b_6 = b_7 = 84, b_8 = b_9 = 60, \]

and

\[ b_{10} = b_{11} = 132, b_{12} = b_{13} = 5460, b_{14} = b_{15} = 12. \]

Proof: We know $b_{2r+1} = b_{2r}$ in the proof of (1.6), and by (1.6) and

\[ V_2 = 6, V_4 = 30, V_6 = 42, V_8 = 30, V_{10} = 66, V_{12} = 2730, V_{14} = 6. \]

we have

\[ b_2 = 12, b_4 = 60, b_6 = 84, b_8 = 60, b_{10} = 132, b_{12} = 5460, b_{14} = 12. \]

Thus

\[ b_2 = b_3 = 12, b_4 = b_5 = 60, b_6 = b_7 = 84, b_8 = b_9 = 60, \]

and

\[ b_{10} = b_{11} = 132, b_{12} = b_{13} = 5460, b_{14} = b_{15} = 12. \]

Remark 5.1. The corollary 5.1 and 5.2 partially answer the conjecture 5.3 of Guo and Liu in [GL].

Acknowledgment. The author would like to thank professors Z.-W. Sun and Hao Pan for helpful comments.

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