An Effective Lagrangian with Broken Scale 
and Chiral Symmetry III: 
Mesons at Finite Temperature 

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Abstract 
We investigate the finite temperature behavior of the meson sector of an effective Lagrangian which describes nuclear matter. A method is developed for evaluating the logarithmic terms in the effective potential which involves expansion and resummation; the result is written in terms of the exponential integral. In the absence of explicit chiral symmetry breaking, a phase transition restores the symmetry at a temperature of 190 MeV; when the pion has a mass the transition is smooth. At a much higher temperature a first order phase transition restores scale symmetry. 
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1 Introduction

Spontaneous chiral symmetry breaking has long been studied in the linear sigma model of Gell-Mann and Lévy [1]. At finite temperature the restoration of chiral symmetry in both the linear and non-linear versions of the model has been discussed in, for example, Refs. [2, 3] (see also references therein). If the potential of that model, \( \frac{1}{4}\lambda(\sigma^2 + \pi^2 - f^2)^2 \), is used in calculations of nuclear matter it leads to compression moduli which are much larger than the observed value; furthermore the binding energies predicted for finite nuclei are much too small [4, 5]. In previous work [4, 6], hereinafter referred to as I and II respectively, we have therefore replaced this potential with a form which incorporates broken scale symmetry in addition to spontaneously broken chiral symmetry, as suggested by quantum chromodynamics (QCD). In particular the potential contains logarithmic terms involving the glueball field \( \phi \) and the \( \sigma \) and \( \pi \) fields. At temperature \( T = 0 \) this led to a good description of nuclear matter and finite nuclei at the mean field level. In II the Lagrangian was extended to include explicit chiral symmetry breaking, an additional chiral-invariant term, and the isotriplet vector mesons, and satisfactory agreement with low energy \( \pi N \) scattering data was obtained.

The purpose of the present paper is to examine the finite temperature, \( T > 0 \), properties of our Lagrangian. Previous studies [4, 8] of models of this general type at \( T > 0 \) have simply included temperature effects for the nucleons. Clearly thermal effects for the mesons are also needed, particularly those due to the pion which will be dominant at low temperatures. However, while the use of logarithmic potentials is straightforward at \( T = 0 \), it is far
from obvious how to proceed at $T > 0$, even at the mean field level. Since the analysis is quite complicated, we will focus here on the mesonic part of our Lagrangian which contains the $\phi$, $\sigma$ and $\pi$ fields. The Lagrangian and our thermal analysis is discussed in Section 2. We give our numerical results in Section 3 and Section 4 contains our conclusions.

## 2 Theory

### 2.1 Equations of Motion

As mentioned, we simplify the Lagrangian by excluding nucleons, as well as the $\omega$ meson which couples to them. We also take the simplest form for the mesonic contributions from I, augmented by the explicit chiral symmetry breaking discussed in II so as to endow the pion with a mass. Then our effective Lagrangian involves the glueball field $\phi$ and the chiral partner fields $\sigma$ and $\pi$ and takes the form

$$
\mathcal{L}_M = \frac{1}{2} \partial_{\mu} \sigma \partial^{\mu} \sigma + \frac{1}{2} \partial_{\mu} \pi \cdot \partial^{\mu} \pi + \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - \mathcal{V}
$$

$$
\mathcal{V} = B\phi^4 \left( \ln \frac{\phi}{\phi_0} - \frac{1}{4} \right) - \frac{1}{2} B \delta \phi^4 \ln \frac{\sigma^2 + \pi^2}{\sigma_0^2} + \frac{1}{2} B \delta \zeta^2 \frac{\phi^2}{2 \zeta^2} \left[ \sigma^2 + \pi^2 - \frac{\phi^2}{2 \zeta^2} \right]$$

$$
- \frac{1}{4} \epsilon_1 \left( \frac{\phi}{\phi_0} \right)^2 \left[ \frac{4 \sigma}{\sigma_0} - 2 \left( \frac{\sigma^2 + \pi^2}{\sigma_0^2} \right) - \left( \frac{\phi}{\phi_0} \right)^2 \right] - \frac{3}{4} \epsilon_1'.
$$

(1)

Here $\zeta = \frac{\phi_0}{\sigma_0}$ and in the vacuum $\phi = \phi_0$, $\sigma = \sigma_0$ and $\pi = 0$, regardless of whether or not the explicit symmetry breaking term $\epsilon_1'$ is present (an additional term, $\epsilon_2'$, was unfavored in II and is omitted here). Thus we have spontaneous, as well as explicit, chiral symmetry breaking. The quantities $B$
and $\delta$ are parameters. For the latter, guided by the QCD beta function, we take $\delta = 4/33$ as in I and II. The logarithmic terms here contribute to the trace anomaly: in addition to the standard contribution from the glueball field [9, 10] there is also a contribution from the $\sigma$ field. Specifically the trace of the “improved” energy-momentum tensor is

$$\theta^\mu_\mu = 4\mathcal{V}^i(\Phi_i) - \sum_i \Phi_i \frac{\delta\mathcal{V}}{\delta\Phi_i} = 4\epsilon_{\text{vac}} \left( \frac{\phi}{\phi_0} \right)^4,$$

(2)

where $\Phi_i$ runs over the scalar fields $\{\phi, \sigma, \pi\}$ and the vacuum energy, $\epsilon_{\text{vac}} = -\frac{1}{4}B\phi_0^4(1 - \delta) - \epsilon'_1$.

We take the vacuum glueball mass to be approximately 1.6 GeV in view of QCD sum rule estimates [11] of 1.5 GeV and recent lattice estimates [12] of 1.7 GeV. Since the mass is large in comparison to the temperatures of interest we shall neglect thermal effects for the glueball. We define the ratio of the mean field to the vacuum value to be $\chi = \phi/\phi_0$. Then Lagrange’s equations for the glueball and $\sigma$ fields in infinite matter are:

$$0 = 4B_0\chi^3 \ln \chi - B_0\delta\chi \left[ 2\chi^2 \ln \left( \frac{\sigma^2 + \pi^2}{\sigma_0^2} \right) - \left( \frac{\sigma^2 + \pi^2}{\sigma_0^2} \right) + \chi^2 \right] - \epsilon'_1 \chi \left[ \frac{2\sigma}{\sigma_0} - \left( \frac{\sigma^2 + \pi^2}{\sigma_0^2} \right) - \chi^2 \right],$$

$$0 = -B_0\sigma_0^2\delta\chi^4 \left( \frac{\sigma}{\sigma^2 + \pi^2} \right) + B_0\delta\chi^2\sigma - \epsilon'_1\chi^2(\sigma_0 - \sigma),$$

(3)

where we have defined $B_0 = B\phi_0^4$.

We wish to take into account thermal effects for the $\sigma$ and $\pi$ fields. To that end we break $\sigma$ into a mean field part $\bar{\sigma}$ and a fluctuation $\Delta\sigma$ with mean value $\langle \Delta\sigma \rangle = 0$. The mean value of the pion field $\langle \pi \rangle$ is, of course, zero. We
write
\[
\frac{\sigma^2 + \pi^2}{\sigma_0^2} = \frac{1}{\sigma_0^2}(\sigma^2 + 2\sigma \Delta \sigma + \Delta \sigma^2 + \pi^2) = \nu^2 + 2\nu \Delta \nu + \psi^2 ,
\] (4)

where, as in I, \( \nu = \bar{\sigma}/\sigma_0 \) and here \( \Delta \nu = \Delta \sigma/\sigma_0 \) and \( \psi^2 = (\Delta \sigma^2 + \pi^2)/\sigma_0^2 . \)

The treatment of the formal equations (3) at finite temperature is far from obvious. Simply expanding the fluctuations out to lowest order, as in Ref. [13], will not properly treat \( \sigma^2 + \pi^2 \) when it occurs in the denominator or in the logarithm in Eqs. (3). (It is known that approximating the logarithm at zero temperature gives poor results.) We shall proceed in two steps. For the first step it is useful to bear in mind that at low temperatures \( \nu \sim 1 \) and the thermal average \( \langle \psi^2 \rangle \) is small, while at high temperatures \( \nu \) is small but the thermal fluctuations are large. This means that the cross term \( 2\nu \Delta \nu \) of Eq. (4) is small in both limits. Thus we expand out this term; odd powers can be dropped since the thermal average gives zero. This yields

\[
0 = B_0 \delta \chi \left( -2\chi^2 \ln(\nu^2 + \psi^2) + \frac{4\chi^2 \nu^2 \Delta \nu^2}{(\nu^2 + \psi^2)^2} + \frac{8\chi^2 \nu^4 \Delta \nu^4}{(\nu^2 + \psi^2)^4} + \nu^2 + \psi^2 \right) + B_0 \chi^3(4 \ln \chi - \delta) - \epsilon'_1 \chi(2\nu - \nu - \psi^2 - \langle \psi^2 \rangle - \chi^2) ,
\]

\[
0 = B_0 \delta \chi^4 \nu \left( -\frac{1}{\nu^2 + \psi^2} + \frac{2\nu^2 \Delta \nu^2}{(\nu^2 + \psi^2)^2} - \frac{4\nu^4 \Delta \nu^2}{(\nu^2 + \psi^2)^3} + \frac{8\nu^2 \Delta \nu^4}{(\nu^2 + \psi^2)^4} \right) + B_0 \delta \chi^2 \nu - \epsilon'_1 \chi^2(1 - \nu). \]

(5)

Here the angle brackets indicate that we have taken a thermal average. For most purposes it is sufficient to truncate at \( O(\nu^2 + \psi^2)^{-3} \), however in the absence of explicit symmetry breaking, \( \epsilon'_1 = 0 \), there is a small region near the chiral phase transition where solutions cannot be obtained. This difficulty is
alleviated by going one power higher. The expansion parameter is of order 
$4\nu^2(\Delta \nu^2)/\langle \nu^2 + \langle \psi^2 \rangle \rangle^2$ which \textit{a posteriori} we find to be $< 0.05$ – this is satisfactorily small. For the thermal average of the square of a fluctuating field, for example for a component $\pi_a$ of the pion field, one has the standard result

$$\langle \pi_a^2 \rangle = \frac{1}{2\pi^2} \int_0^{\infty} dk \frac{k^2}{\omega_{\pi}} \frac{1}{e^{\beta \omega_{\pi}} - 1}.$$  

(6)

Here $\beta = 1/T$ is the inverse temperature and $\omega_{\pi}^2 = k^2 + m_{\pi}^*2$, with $m_{\pi}^*$ the effective pion mass which will be discussed in the next subsection. In calculating thermodynamic integrals, such as this, we find it convenient to make use of the numerical approximation scheme of Ref. [14].

In order to evaluate Eq. (5) we take the second step, which is to make a formal expansion in $\psi^2$. It is sufficient to consider the logarithm in Eq. (5) which gives

$$\langle \ln(\nu^2 + \psi^2) \rangle = \ln \nu^2 + \left\langle \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n \left( \frac{\psi^2}{\nu^2} \right)^n}{n} \right\rangle$$

$$= \ln \nu^2 + \sum_{n=1}^{\infty} (-1)^{n+1} (n+1)(n-1)! y^{-n}.$$  

(7)

Here we have defined

$$y^{-1} = \frac{\langle \psi^2 \rangle}{2\nu^2} \equiv \frac{\langle \Delta \sigma^2 + \pi^2 \rangle}{2\nu^2 \sigma_0^2}.$$  

(8)

In obtaining Eq. (7) we have used the counting factors described in the Appendix. These require that $\langle \Delta \sigma^2 \rangle = \langle \pi^2 \rangle / 3$. This will be exact in the high temperature regime when chiral symmetry has been restored. At lower temperatures where we use the appropriate values for the thermal expectation
values in Eq. (8), this will be approximate. However we do obtain the correct low temperature limit. We also remark that if the fluctuations in the $\sigma$ field are arbitrarily ignored, the qualitative behavior is similar to the present case. Thus we believe the approximation is reasonable, although we would wish to have a more quantitative assessment of the errors involved. It would seem to be essential to take into account in some reasonable fashion the vertices in Eq. (7) with large numbers of fields attached.

While the series in Eq. (7) is divergent, we regard it as a formal expansion which must be resummed before it can be evaluated. The counting factors for an odd number of field components lead to expressions involving the error function. For an even number of field components, four in the present case, the resummation requires the exponential integrals $E_n(y) = \int_1^\infty dt \frac{e^{-yt}}{t^n}$. (9)

Matching the series of Eq. (7) to the asymptotic expansion of the exponential integrals for large $y$, we obtain

$$\langle \ln(\nu^2 + \psi^2) \rangle = \ln \nu^2 + e^y[E_1(y) + E_2(y)]$$
$$= \ln \nu^2 + (1 - y)e^y E_1(y) + 1. \quad (10)$$

In the limit of low temperature ($y \to \infty$) or high temperature ($y \to 0$) we obtain

$$\langle \ln(\nu^2 + \psi^2) \rangle \to \ln \nu^2 + \frac{\langle \psi^2 \rangle}{\nu^2} \quad (y \to \infty)$$
$$\to \ln(0.7631 \langle \psi^2 \rangle) \quad (y \to 0). \quad (11)$$
The low temperature expression simply corresponds to expanding the logarithm to lowest order, as it should. In the high temperature expression the numerical factor is $0.7631 = e^{1-\gamma}/2$, where $\gamma$ is Euler’s constant.

The remaining quantities needed in Eq. (5) can be evaluated in analogous fashion or, more conveniently, by differentiating Eq. (10) with respect to $\nu^2$. In order to consistently evaluate the counting we replace $\Delta \nu^2$ in Eq. (5) by $\frac{1}{4} \psi^2$. The terms involving a fourth power of the fluctuations are evaluated in analogous fashion using the approximation

$$
\left< \frac{\Delta \nu^4}{(\nu^2 + \psi^2)^4} \right> = 3 \left< \frac{\Delta \nu^2 \pi^2}{\sigma^2_0 (\nu^2 + \psi^2)^4} \right> = \left< \frac{(\psi^2)^2}{8(\nu^2 + \psi^2)^4} \right>.
$$

(12)

Then Lagrange’s equations (5) can be written in final form:

$$
0 = B_0 \delta \chi^3 \left[ \left( -2 + 2y - y^2 + 2y^3 + \frac{3}{2} y^4 + \frac{1}{4} y^5 \right) e^y E_1(y) - 3 + y - \frac{5}{2} y^2 
- \frac{4}{3} y^3 - \frac{1}{12} y^4 - 2 \ln \nu^2 \right] + (B_0 \delta + \epsilon'_1) \chi \nu^2 \left( 1 + \frac{2}{y} \right) + 4B_0 \chi^3 \ln \chi 
- \epsilon'_1 \chi (2\nu - \chi^2),
$$

(13)

$$
0 = \frac{B_0 \delta \chi^4 y}{6\nu} \left[ (3y^2 + 6y^3 + y^4) e^y E_1(y) - 3 + y - 5y^2 - y^3 \right] 
+ B_0 \delta \chi^2 \nu - \epsilon'_1 \chi^2 (1 - \nu).
$$

(14)

Exact chiral symmetry restoration, $\nu = 0$, will only be obtained from Eq. (13) when there is no explicit symmetry breaking, $\epsilon'_1 = 0$. We also note that Eqs. (13) and (14) permit a solution $\chi = 0$ corresponding to scale restoration, in which case $\nu$ is undefined.
2.2 Masses

For each field we define the effective mass at finite temperature as the thermal average of the second derivative of the potential. This means that we only consider contributions arising from a single interaction vertex. Since the mixing between the glueball and the $\sigma$ meson is small, we neglect it here for simplicity. Specifically

$$\sigma_0^2 m_{\sigma}^2 = \left< \frac{\partial^2 V}{\partial \Delta \sigma^2} \right> = (B_0 \delta + \epsilon'_1) \chi^2 + \left< \frac{-B_0 \delta \sigma_0^2 \chi^4}{\sigma^2 + \pi^2} + \frac{2B_0 \delta \sigma_0^2 \chi^4 \sigma^2}{(\sigma^2 + \pi^2)^2} \right>,$$

$$\sigma_0^2 m_{\pi}^2 = \left< \frac{\partial^2 V}{\partial \pi_a^2} \right> = (B_0 \delta + \epsilon'_1) \chi^2 + \left< \frac{B_0 \delta \sigma_0^2 \chi^4}{\sigma^2 + \pi^2} + \frac{2B_0 \delta \sigma_0^2 \chi^4 \pi_a^2}{(\sigma^2 + \pi^2)^2} \right>,$$

$$\phi_0^2 m_{\phi}^2 = \left< \frac{\partial^2 V}{\partial \phi^2} \right> = 4B_0 \chi^2 (3 \ln \chi + 1) + 3(\epsilon'_1 - B_0 \delta) \chi^2 + (B_0 \delta + \epsilon'_1) \nu^2 - 2\epsilon'_1 \nu + \left< -6B_0 \delta \chi^2 \ln \left( \frac{\sigma^2 + \pi^2}{\sigma_0^2} \right) + (B_0 \delta + \epsilon'_1) \psi^2 \right>.$$

As we have remarked, we do not consider thermal fluctuations in the glueball field and so its mass does not enter the equations. However it will be useful to display the mass in Sec. 3. The $\sigma$ and $\pi$ masses are needed in evaluating $\langle \psi^2 \rangle$ as indicated in Eq. (6). As a result the equations of motion and the expressions for the masses must be evaluated self-consistently. Treating Eq. (15) as discussed in the previous subsection and using the equations of motion (13) and (14) for the case where $\nu \neq 0$ and $\chi \neq 0$ we obtain

$$\sigma_0^2 m_{\sigma}^2 = 2B_0 \delta \chi^4 \nu^2 \left< \frac{1}{(\nu^2 + \psi^2)^2} - \frac{8\Delta \nu^2}{(\nu^2 + \psi^2)^3} + \frac{4(3\nu^2 \Delta \nu^2 + 2\Delta \nu^4)}{(\nu^2 + \psi^2)^4} \right>$$

$$+ \left< \frac{\epsilon'_1 \chi^2}{\nu} \right>$$

$$= \frac{B_0 \delta \chi^4 y^2}{3\nu^2} \left[ -(18y + 15y^2 + 2y^3)e^y E_1(y) + 7 + 13y + 2y^2 \right]$$
\[ \sigma_0^2 m_{\pi}^2 = \frac{\epsilon'_1 \chi^2}{\nu}, \quad (16) \]

\[ \sigma_0^2 m_{\sigma}^2 = \frac{\epsilon'_1 \chi^2}{\nu}, \quad (17) \]

\[ \phi_0^2 m_{\phi}^2 = 4(B_0 \chi^2 + \epsilon'_1 \nu) - 2(B_0 \delta + \epsilon'_1) (\nu^2 + \langle \psi^2 \rangle). \quad (18) \]

It is straightforward to verify by taking the \( y \to \infty \) limit that the zero temperature results of II are obtained. In the case when \( \nu, y \to 0 \) the \( \sigma \) and \( \pi \) masses become equal:

\[ \sigma_0^2 m_{\sigma}^2 \to \sigma_0^2 m_{\pi}^2 = \frac{\epsilon'_1 \chi^2}{\nu} \to B_0 \delta \chi^2 \left( 1 - \frac{\chi^2}{\langle \psi^2 \rangle} \right), \quad (19) \]

where for the last expression we have used Eq. (14). In the case where there is no explicit symmetry breaking, \( \epsilon'_1 = 0 \), at sufficiently high temperature \( \nu = 0 \) and the masses are precisely given by the last expression in Eq. (19).

### 2.3 Thermodynamics

The grand potential per unit volume can easily be written down:

\[ \frac{\Omega}{V} = \langle V \rangle + \frac{T}{2 \pi^2} \int dk k^2 \left[ \ln(1 - e^{-\beta \omega_{\sigma}}) + 3 \ln(1 - e^{-\beta \omega_{\pi}}) \right] 
- \frac{1}{2} m_{\sigma}^2 \langle \Delta \sigma^2 \rangle - \frac{1}{2} m_{\pi}^2 \langle \pi^2 \rangle, \quad (20) \]

where \( \omega_{\sigma}^2 = k^2 + m_{\sigma}^2 \) and \( \omega_{\pi}^2 = k^2 + m_{\pi}^2 \). The subtraction of the last two terms in Eq. (20) is necessary to avoid double counting [16]. The evaluation of the thermal average of the potential follows the discussion in Subsec. 2.1 and we simply quote the result.

\[ \langle V \rangle = \chi^4 \left[ B_0 \ln \chi - \frac{1}{4} B_0 (1 + \delta) + \frac{1}{4} \epsilon'_1 \right] + \left( B_0 \delta + \epsilon'_1 \right) \chi^2 \nu^2 \left( \frac{1}{2} + \frac{1}{y} \right) \]
\[-\epsilon_1' \chi^2 \nu - \frac{1}{2} B_0 \delta \chi^4 \left[(1 - y + \frac{1}{2} y^2 - y^3 - \frac{3}{4} y^4 - \frac{1}{12} y^5) e^y E_1(y) \right. \\
+ 1 - \frac{1}{2} y + \frac{5}{12} y^2 + \frac{2}{5} y^3 + \frac{1}{12} y^4 + \ln \nu^2 \right] + \frac{1}{4} \left[ B_0 (1 - \delta) + \epsilon_1' \right]. \tag{21} \]

We have added a constant term here so that $\langle V \rangle$ is zero in the vacuum. The pressure $P$ is of course $-\Omega/V$.

Now if one takes the partial derivative of $\Omega/V$ with respect to $\chi$ or $\nu$ the equations of motion (13) and (14) ought to be obtained. This is true if one ignores the dependence of the masses on these variables. If this is taken into account derivatives of the explicit $m^2$ terms in (20) cancel with derivatives of the Bose partition functions. Derivatives of $\langle \psi^2 \rangle$ do not cancel precisely. They would do so if all the terms arising from derivatives of the original logarithm of Eq. (1) were retained in the subsequent equations. However this gives additional contributions to the pion mass so that in the absence of explicit symmetry breaking the mass is no longer zero at low temperatures in violation of Goldstone’s theorem. As it is necessary to approximate, one cannot have it both ways. We prefer to truncate the equations of motion and the mass expressions at a given order and approximate the grand potential— we have verified that the additional terms needed to produce the exact equations of motion are small in comparison to the terms retained.

In this spirit we ignore the derivatives of $\langle \psi^2 \rangle$ in deriving the energy density which takes the simple form

\[
\frac{E}{V} = \langle V \rangle - \frac{1}{2} m_\sigma^2 \langle \Delta \sigma^2 \rangle - \frac{1}{2} m_\pi^2 \langle \pi^2 \rangle \\
+ \frac{1}{2 \pi^2} \int dk \, k^2 \left[ \frac{\omega_\sigma}{e^{\beta \omega_\sigma} - 1} + \frac{3 \omega_\pi}{e^{\beta \omega_\pi} - 1} \right]. \tag{22} \]
3 Results

The explicit symmetry breaking parameter, $\epsilon_1'$, is chosen to yield the vacuum pion mass ($\epsilon_1' = \sigma_0^2 m_{\pi}^2$) for one set of calculations and to be zero for another set. In order to fit nuclear matter saturation in these two cases we find $B_0$ to be $(334.9 \text{ MeV})^4$ and $(342.6 \text{ MeV})^4$, respectively. Based on the results of II we take $\sigma_0 = 110 \text{ MeV}$ and $\phi_0 = 140.9 \text{ MeV}$. We stress that in evaluating $\langle \psi^2 \rangle$ the values of the sigma and pion masses from Eqs. (16) and (17) are used.

In Figs. 1 and 2 we display the sigma and glueball mean fields as a function of temperature. Here, and in the subsequent figures, the dashed line refers to $\epsilon_1' > 0$ and the solid curve gives the $\epsilon_1' = 0$ results. In Fig. 1 the solid line shows a chiral phase transition and the dotted curve indicates a region of instability where the pressure is not maximized. Similarly the dotted and dash-dotted parts of the curves in Fig. 2 indicate regions of instability for the scale restoration phase transition. The pion and sigma masses are displayed in Figs. 3 and 4, respectively.

Consider first the $\epsilon_1' = 0$ results (solid curves) where $\nu$ and $m_\sigma^*$ become zero at the chiral phase transition temperature $T_c$. Below $T_c$ the $\nu = 0$ solution results in unphysical (imaginary) masses. The value of $T_c$ can be obtained by observing where the masses in Eq. (19) become zero, giving $T_c = \sqrt{3} \chi \sigma_0$. The effect of the glueball field is rather small here since $\chi$ is only slightly less than unity at these temperatures (Fig. 2). With our parameters $T_c = 187 \text{ MeV}$. Fig. 1 indicates a weakly first order phase transition, however we do not believe that our approximation scheme is sufficiently accurate to
determine the order of the transition. As one would expect, the behavior in the transition region is sensitive to the prescription used to define the masses. For example, we have remarked that if the equations are truncated at a lower order no solution is obtained for a small region in this vicinity. For $T > T_c$ the field $\nu = 0$ and the pion and sigma masses are equal.

There are numerous estimates of $T_c$ in the literature. The standard chiral model \[ \text{model} \] yields $T_c = \sqrt{2} f_\pi$ which with the pion decay constant $f_\pi = 93$ MeV gives $T_c = 132$ MeV. Apart from the numerical factor, our value is larger because $\sigma_0$ has to be greater than $f_\pi$ in order to fit nuclei; see the discussion in I and II. We remark that, to a good approximation, the effect of modifying $\sigma_0$ is simply to scale the temperatures on the abscissae of the figures. As with the standard model our $\sigma$ mass is zero at the critical temperature. Gerber and Leutwyler \[ \text{Gerber and Leutwyler} \] have studied two-flavor quantum chromodynamics using an effective chiral Lagrangian to three loop order and estimate a value of 190 MeV for $T_c$ with massless quarks. Finally we mention that the two-flavor lattice QCD results of Brown \textit{et al.} \[ \text{Brown et al.} \] show a second order phase transition for massless quarks, but this is washed out by any finite mass. The latter seems to be in line with our dashed curves with $\epsilon'_1 > 0$ where there is no phase transition and $\nu$ smoothly decreases to a small value. Symmetry restoration takes place at somewhat higher temperatures when the pion mass is finite, as would be expected. Figures 3 and 4 show that the pion mass smoothly increases from the vacuum value and the sigma mass initially drops, but then starts to increase at $T \sim 350$ MeV to become degenerate with the pion mass.

There is a further interesting feature which occurs at high temperature, although we caution that the model may well not be reliable in this region.
Also, physically, deconfinement may have taken place since lattice calculations suggest that chiral restoration and deconfinement occur at similar temperatures \[19\]. Indeed with a crude lowest order treatment of gluons and massless quarks, taking the bag constant to be \(|\epsilon_{\text{vac}}|\), we find that the deconfined phase is preferred for \(T > 170\) MeV. In spite of these caveats we show (Fig. 2) that at a rather high temperature the solution with \(\chi \sim 1\) becomes unstable and \(\chi\) drops to zero. We interpret this as a first order phase transition which restores scale symmetry. With the present approximation the transition temperature is \(\sim 550\) MeV. The physics behind this is indicated in Fig. 5 by a qualitative plot of \(\Omega/V\) as a function of \(\chi\) with \(\epsilon'_1 > 0\) (note that each curve is normalized to zero at \(\chi = 0\)). At low temperatures, while there is a \(\chi = 0\) solution, the minimum corresponds to \(\chi \sim 1\). At \(T = 530\) MeV the depth of these two minima become equal and beyond this the stable solution is \(\chi = 0\). The unstable solution is indicated in Fig. 2 by the dotted \((\epsilon'_1 = 0)\) or dash-dotted \((\epsilon'_1 > 0)\) curve out to the point where the minimum and maximum of the potential, marked by dots in Fig. 5, coalesce. The \(\chi = 0\) regime should not be taken too seriously since all that remains of \(\Omega\) are the thermal partition functions for massless \(\sigma\) and \(\pi\) mesons. Nevertheless we believe that a first order phase transition of this type is a general feature of the model. The critical temperature is rather high here because of the coupling of the glueball to the \(\sigma\) and \(\pi\) mesons; Agasyan \[20\] has discussed the pure glue sector where the estimated critical temperature is lower.

In Fig. 6 we plot the glueball mass as a function of temperature. (Note that when \(\chi = 0\) we have set \(\nu = 0\) since it is undefined and is small prior to scale restoration.) The glueball mass begins to drop significantly with
decreasing $\chi$ for $T > 400$ MeV, but remains much larger than $m^*_\sigma$ and $m^*_\pi$. Thermal fluctuations are not expected to be important, except, possibly, for the highest temperatures discussed here.

Finally in Fig. 7 we show the pressure versus the energy density with the points labelled by the corresponding temperatures. At low temperatures our pressure is slightly negative since our approximation to the grand potential results in inaccuracies at the 1% level in the delicate cancellation of large terms. The chiral phase transition is just visible for the solid curve in Fig. 7, and beyond this temperature interactions become less important and the massless gas result $E/V = 3P$ is rapidly approached.

4 Conclusions

We have discussed the finite temperature behavior of the meson sector of an effective Lagrangian with which we have successfully described nuclear matter and finite nuclei. This is non-trivial because of the $\ln(\sigma^2 + \pi^2)$ term in the effective potential. Our method of handling this term involved expansion and resummation of an infinite series with the final result cast in terms of the exponential integral.

Our results showed that at sufficiently high temperature the mean value of the $\sigma$ field became small, signalling chiral restoration. Hitherto it has not been appreciated that chiral symmetry is regained in this type of model. In the absence of explicit chiral symmetry breaking a phase transition was obtained at $T_c = 187$ MeV; we do not believe that our calculations are sufficiently accurate to determine the order of this transition. In the physical
case where explicit chiral symmetry breaking was present and the pion had a vacuum mass a smooth restoration of chiral symmetry was found, the onset of which occurred at a somewhat higher temperature. Well beyond this a first order phase transition took place in which the mean glueball field dropped to zero, implying restoration of scale invariance. This is an interesting physical feature, but deconfinement is expected to have taken place before this temperature is reached and the application of our model at such a high temperature should be taken with a grain of salt. One would like to know how these results, particularly chiral restoration, are affected when nucleon degrees of freedom are present in addition to the mesons. This will be the subject of future work.

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Appendix. Counting for a General Vertex

We need to evaluate the thermal average of $(\psi^2)^n$, where

$$\psi^2 = \sum_{i=1}^{N} \psi_i^2,$$

which involves the sum over the squares of $N$ fluctuating fields. This corresponds to evaluating diagrams of the type shown in Fig. 8 where, in general, the flower would have $n$ petals. We assume that the thermal average $\langle \psi_i^2 \rangle$
is independent of the label $i$, which amounts to assuming the masses of the particles involved are the same. Because of this assumption the result of taking the thermal average of each possible pair of fields at a general vertex can be written

$$\langle (\psi^2)^n \rangle = c_n (\psi^2)^n, \quad (24)$$

where $c_n$ is a number which gives the counting. Consider $(\psi^2)^{n+1}$. The two additional fields can be averaged together. Alternatively we can break one of the $n$ original pairs and combine the additional fields with them in 2 ways; this requires that the additional fields have the appropriate label $i$ leading to a factor of $1/N$. Thus

$$\langle (\psi^2)^{n+1} \rangle = \left(1 + \frac{2n}{N}\right) \langle \psi^2 \rangle \langle \psi^2 \rangle^n \quad \text{or}$$

$$c_{n+1} = \left(\frac{N + 2n}{N}\right) c_n. \quad (25)$$

Since $c_1 = 1$, we obtain

$$c_n = \frac{(N + 2n - 2)!!}{N!!N^{n-1}}. \quad (26)$$

For the case at hand, $N = 4$, $c_n = (n + 1)!/2^n$.

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**Figure Captions**

**Figure 1.** The mean sigma field, $\nu = \bar{\sigma}/\sigma_0$, as a function of temperature. The dashed (solid) line corresponds to the presence (absence) of explicit chiral symmetry breaking. The dotted curve indicates a thermodynamically unstable region.

**Figure 2.** The mean glueball field, $\chi = \phi/\phi_0$, as a function of temperature. The dotted and dash-dotted curves indicate thermodynamically unstable regions. See caption to Fig. 1.

**Figure 3.** The pion effective mass as a function of temperature. See caption to Fig. 1.

**Figure 4.** The sigma effective mass as a function of temperature. See caption to Fig. 1.

**Figure 5.** Schematic representation of the grand potential per unit volume, $\Omega/V$, as a function of the glueball field $\chi$ for various temperatures with $\epsilon'_1 > 0$. Note that for each case $\Omega$ is normalized to zero at $\chi = 0$.  

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**Figure 6.** The glueball effective mass as a function of temperature. See caption to Fig. 1.

**Figure 7.** Plot of the pressure versus energy density for the temperatures indicated. See caption to Fig. 1.

**Figure 8.** Representative diagram whose thermal part is evaluated to determine $\langle (\psi^2)^4 \rangle$. 
Figure 3
Figure 5
Figure 6
Figure 8