QUANTUM LAYERS OVER SURFACES RULED OUTSIDE A COMPACT SET

CHRISTOPHER LIN AND ZHIQIN LU

Abstract. In this paper, we proved the quantum layer over a surface which is ruled outside a compact set, asymptotically flat but not totally geodesic admits ground states.

1. Introduction

The spectrum of the Laplacian on manifolds is a classic domain of research within geometric analysis. One of its least developed area is spectral analysis on noncompact, non-complete manifolds. An interesting paper by Duclos, Exner, and Krejčiřík [2], (2001) demonstrated the existence of discrete spectrum, or bound states, of the Dirichlet Laplacian acting on $L^2$ functions on a particular type of noncompact, non-complete manifold which they termed the “quantum layer”. In particular, the existence of bound states means that at least there exists the lowest bound state - the ground state. The main results in their paper were generalized in [5] and [6] by Lin and Lu to higher dimensions under more general geometric settings. The existence of discrete spectrum for the Laplacian is a non-trivial phenomenon on noncompact manifolds, even when the manifold is complete. The results in [2], [5], and [6] are rare instances in which the discrete spectrum is clearly known to exist on noncompact, non-complete manifolds.

We recall the definition of a quantum layer below, following [5].

**Definition 1.** Let $\Sigma \hookrightarrow \mathbb{R}^3$ be an isometrically immersed, oriented hypersurface. Let $N$ and $\bar{A}$ respectively be the unit normal vector field and the second fundamental form on $\Sigma$, and define the map

$$p : \Sigma \times (-a, a) \rightarrow \mathbb{R}^3$$

(1) by $(x, u) \mapsto x + uN$, where the number $a > 0$, which is called the thickness, is such that there is a constant $C_o$ such that $a\|\bar{A}\| < C_o < 1$ on $\Sigma$. We

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define a quantum layer to be the smooth manifold \( \Omega = \Sigma \times (-a, a) \) with the pull-back metric \( p^* (ds_E^2) \).

As one can see from the definition, a quantum layer is just a tubular neighborhood of a surface in \( \mathbb{R}^3 \). The relation between \( \vec{A} \) and the thickness \( a \) above is simply to ensure that the map (1) is also an immersion. The terminology of “quantum” simply eludes to the fact that when suitable geometric conditions are imposed on the tubular neighborhood, ground state exists.

Before we go any further, let us establish a consensus on the notions and terminologies from functional analysis used in this paper. A good source of this can, for example, be found in [7]. We will use \( \Delta \) to denote the Dirichlet Laplacian throughout the paper, which is a self-adjoint extension in \( L^2(\Omega) \) of the positive-definite Laplacian \( \Delta = -\text{div} \circ \text{grad} \) on smooth, compactly supported functions on \( \Omega \). The Dirichlet Laplacian \( \Delta \) is thus an unbounded operator with a dense domain \( \text{Dom}(\Delta) \subset W^{1,2}_{0}(\Omega) \subset L^2(\Omega) \).

The resolvent set \( \rho(\Delta) \) of the operator is the set of all complex numbers such that \( (\Delta - \lambda I)^{-1} \) exists as a bounded operator on \( L^2(\Omega) \), and the spectrum \( \sigma(\Delta) \) is defined as the set \( \mathbb{C} \setminus \rho(\Delta) \). The self-adjointness and positive-definiteness of \( \Delta \) imply that \( \sigma(\Delta) \subset [0, \infty) \). The spectrum can be decomposed as \( \sigma(\Delta) = \sigma_{\text{disc}}(\Delta) \sqcup \sigma_{\text{ess}}(\Delta) \), where the discrete spectrum \( \sigma_{\text{disc}}(\Delta) \) can simply be defined as all the isolated eigenvalues in \( \sigma(\Delta) \) with finite multiplicity and the essential spectrum \( \sigma_{\text{ess}}(\Delta) \) is simply the complement. The essential spectrum is stable under compact perturbations in the sense that \( \sigma_{\text{ess}}(\Delta + T) = \sigma_{\text{ess}}(\Delta) \) for any compact operator \( T \). One would like to think that compact perturbations of the metric on a manifold amounts to perturbing \( \Delta \) by a compact operator. This is a useful idealization. However, it is heuristic.

In light of the heuristic notion that the essential spectrum is invariant under compact perturbation of the metric, the following result (for which its less general version first appeared in [2]) is not surprising.

**Theorem 1** (see [2] or [5]). Let \( \Omega \) be a quantum layer over an isometrically immersed surface in \( \mathbb{R}^3 \), if we further assume that \( ||\vec{A}|| \to 0 \) at infinity, then the bottom of the essential spectrum of the Dirichlet Laplacian is bounded below as

\[
\inf \sigma_{\text{ess}}(\Delta) \geq \left( \frac{\pi}{2a} \right)^2.
\]

The variation principle says that

\[
\inf \sigma(\Delta) = \inf_{\phi \in C_0^\infty(\Omega)} \frac{\int_\Omega |\nabla \phi|^2}{\int_\Omega \phi^2}.
\]

\[\text{i.e., } \Delta \phi = \lambda \phi \text{ for some } \phi \in \text{Dom}(\Delta) \text{ and } (\lambda - \varepsilon, \lambda + \varepsilon) \cap \sigma(\Delta) = \emptyset.\]
Therefore assuming asymptotic flatness of the surface $\Sigma$, to prove that the quantum layer $\Omega$ has ground state it suffices to find a test function so that
\[
(2) \quad \inf \sigma(\Delta) < \left(\frac{\pi}{2a}\right)^2.
\]
This was achieved first in [2], and more generally in [5] (Theorem 1.1) by assuming $L^1$ Gauss curvature and non-positivity of the total Gauss curvature.

Thus there is the following conjecture:

**Conjecture 1.** Let $\Sigma$ be a complete, non-compact surface isometrically immersed in $\mathbb{R}^3$ and asymptotically flat. Consider the quantum layer built over it. If $\Sigma$ is not the plane and the Gaussian curvature $K$ is integrable on $\Sigma$, does the Dirichlet Laplacian on the quantum layer have non-empty discrete spectrum?

The remaining case of the conjecture above, which is the $\int_{\Sigma} K > 0$ case, was answered partially through special examples of surfaces in both [2] and [5]. In particular, in [5] (Theorem 1.3) the example of quantum layers over a convex surface (graph of a convex function $f : \mathbb{R}^2 \to \mathbb{R}$ in $\mathbb{R}^3$) is demonstrated to have ground state (assuming asymptotic flatness). A convex surface has positive Gauss curvature everywhere.

We must emphasize that in the proof of [2] for the $\int_{\Sigma} K \leq 0$ case, there was no reference to the immersion of $\Sigma$ in the ambient Euclidean space. Thus the existence of ground state there is of an intrinsic nature. However, for the convex surface example, a careful analysis of the (integral of) mean curvature was used in an essential way. Therefore, we believe that in order to answer the conjecture for the $\int_{\Sigma} K > 0$ case, the mean curvature - or more generally - the second fundamental form of $\Sigma$, must always play a central role.

In this paper we will discuss another rather general example of surfaces whose layers possess bound states, and the mean curvature on such surfaces will be heavily involved in the analysis. We consider embedded surfaces in $\mathbb{R}^3$ that is ruled outside a compact subset. A ruled surface is such that for each point on the surface there passes an Euclidean straight line, called a ruling, lying also on the surface, and such that a local collection of such lines and the orthogonal flow lines through them constitute a local coordinate system on the surface. Since this description is local in nature (at least we can allow the situation where the rulings end somewhere on the surface), we can define surfaces which are potentially only ruled outside a compact subset of the surface. More details about ruled surfaces will be given in the next section.

Our main result is as follows:

**Theorem 2** (Main Theorem). Let $\Sigma$ be a surface in $\mathbb{R}^3$ that is ruled outside a compact subset. Then the bottom of the spectrum of the Dirichlet Laplacian
on a quantum layer $\Omega$ of thickness $a$ over $\Sigma$ has the upperbound

$$\inf \sigma(\Delta) < \left( \frac{\pi}{2a} \right)^2.$$ 

**Remark 1.** Note that for a surface which is ruled outside a compact set, the Gauss curvature is automatically integrable. The surface also can’t be flat outside a compact set, because otherwise the total Gauss curvature would be zero and in that case, the theorem follows from the main theorem in [1], or [3].

Using Theorem 1 we also obtain the following:

**Corollary 1.** The layer above, along with the assumption that the second fundamental form goes to zero at infinity, has Dirichlet ground state.

We end this section with an overview of the rest of the paper. In section 2 we discuss the (local) geometry of ruled surfaces. In section 3 we give relevant information about the topology of noncompact surfaces with integrable Gauss curvature. The discussion there will center around the generalized Gauss-Bonnet Theorem of Hartman [3]. In particular, we will make essential use of the theorem of B. White [8] in the proof of the main theorem. Section 4 contains the proof of the main theorem.

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2. The Geometry of Ruled Surfaces

Here we discuss some basic geometry about ruled surfaces.

**Definition 2.** A non-intersecting surface $\Sigma$ in $\mathbb{R}^3$ is called a ruled surface if every point lies in a coordinate chart of the form

$$x(s,v) = \beta(s) + v\delta(s),$$

where $\beta$ and $\delta$ are vectors in $\mathbb{R}^3$.

It is important to note that the coordinate charts described above are in general only local. We can always choose $\beta$ to be unit speed and $\delta$ to be a unit vector field. Furthermore, we may reparameterize the coordinate chart above so that

$$\begin{cases}
|\beta'| = 1; \\
|\delta| = 1; \\
(\beta', \delta) = 0.
\end{cases}$$
By product rule we also have $\langle \delta', \delta \rangle = 0$. From Definition 2 we see that at each point of a ruled surface $\Sigma$, there exists a straight line segment in $\mathbb{R}^3$ that also lies in $\Sigma$. Such a line segment, called a generator, is simply the $v$-parameter curve of the coordinate system provided by (5). Now, we see that $\langle x_s, x_v \rangle = \langle \beta', \delta \rangle + v \langle \delta', \delta \rangle = 0$, hence we have local orthogonal coordinate systems on a ruled surface $\Sigma$. Then letting $X = x_v$ and $Y = x_s/|x_s|$, we get local orthonormal frames on $\Sigma$. Denote by $f(v)$ and $h(t)$ the integral curves of $X$ and $Y$, respectively. Let $N$ denote the oriented unit normal on $\Sigma$, then the Gauss curvature on $\Sigma$ is

$$K = \langle \frac{d}{dv} N \circ f, X \rangle \langle \frac{d}{dt} N \circ h, Y \rangle - \langle \frac{d}{dv} N \circ f, Y \rangle^2.$$  

Since the ruling $f(v)$ is really a line segment, $f'' = 0$, so by product rule we see that

$$\langle \frac{d}{dv} N \circ f, X \rangle = -\langle N, f'' \rangle = 0.$$

Thus $K \leq 0$ on a ruled surface by (5). A surface $\Sigma$ is called a developable surface if it is a ruled surface such that its normal is parallel in $\mathbb{R}^3$ along any of its generators, i.e. $\frac{dv}{dt} N \circ f = 0$. Then again by (5) we see that $K \equiv 0$ on a developable surface. In fact the concept of a developable surface is inseparable from the zero Gauss curvature condition:

**Theorem 3** (Massey’s Theorem). (Corollary to Theorem 5, [4]) A complete, connected surface in $\mathbb{R}^3$ is a developable surface if and only if $K \equiv 0$.

Next we compute the mean curvature on a ruled surface $\Sigma$. Let $\times$ denote the cross product of vectors in $\mathbb{R}^3$. Moreover, since $|x_v| = |\delta| = 1$ we have $|x_s \times x_v| = |x_s|$. The mean curvature on $\Sigma$ is computed as

$$H = \langle -\frac{d}{dv} N \circ f, X \rangle + \langle -\frac{d}{dt} N \circ h, Y \rangle = \frac{\langle x_s \times x_v, x_{ss} \rangle}{|x_s \times x_v|^3}. $$

3. The Topology of Noncompact Surfaces

The famous Gauss-Bonnet Theorem asserts that the total Gauss curvature of a compact 2-dimensional manifold without boundary is a constant multiple of its Euler Characteristic. If a surface is complete, non-compact, but has integrable Gauss curvature, then the total Gauss curvature is no longer completely topological. In 1957, A. Huber showed that a complete, noncompact surface $\Sigma$ of integrable curvature is conformally equivalent to

It is an elementary exercise to show that a ruled surface is developable if and only if it has zero Gauss curvature. The more general theorem of Massey, on the other hand, is quite non-trivial.
a compact Riemann surface with finitely many punctures. In particular, Huber also showed that in this case
\[ \int_{\Sigma} K \leq 2\pi \chi(\Sigma), \]
where \( \chi(\Sigma) \) is the Euler characteristic of \( \Sigma \). The finitely many punctures correspond to the ends of \( \Sigma \). Let us denote the ends by \( \{E_1, \ldots, E_k\} \), and define the corresponding isoperimetric constants
\[ \lambda_i = \lim_{r \to \infty} \frac{\text{Area}(B(r) \cap E_i)}{\pi r^2}, \]
relative to any fixed point \( p \in \Sigma \) with respect to which the geodesic distance \( r \) is measured. The ends contribute to the deficit in (7) via the following wonderful result.

**Theorem 4** (Hartman [3] ’64). Let \( \Sigma \) be a complete, noncompact surface with integrable Gauss curvature. Then
\[ \frac{1}{2\pi} \int_{\Sigma} K = \chi(\Sigma) - \sum_{i=1}^{k} \lambda_i, \]
where \( \chi(\Sigma) \) is the Euler characteristic of the surface.

The Euler Characteristic is defined by \( \chi(\Sigma) = \sum (-1)^i b_i \), where \( b_i \) is the \( i \)-th Betti number. Now, we always assume implicitly that manifolds are connected, hence path-connected. Thus for a noncompact (path-connected) surface, \( \chi(\Sigma) = 1 - b_1 \), and in the case that \( \Sigma \) also has integrable Gauss curvature we see that in fact
\[ \int_{\Sigma} K \leq 2\pi. \]
Moreover, if we assume that \( \int_{\Sigma} K > 0 \), then by (8) we must have \( b_1 = 0 \) as well. This means \( \chi(\Sigma) = 1 \), which via the uniformization theorem for surfaces implies that \( \Sigma \) is conformally equivalent to \( \mathbb{R}^2 \). Therefore we see that positive total Gauss curvature surfaces are topologically very simple (it is just \( \mathbb{R}^2 \)). However, ironically the analysis which are hopeful towards deducing the existence of ground state is significantly less straightforward. Contrasting this with the non-positive total Gaussian curvature case, where via Hartman’s formula, if we start with any surface and add sufficient many handles we could obtain a ground state eventually.[3]

For later use, we shall need the following result of B. White, which depends on the way in which a surface sits in \( \mathbb{R}^3 \).

**Theorem 5** (B. White [8] ’87). Let \( \Sigma \) be a surface immersed in \( \mathbb{R}^3 \). If \( \int_{\Sigma} |\vec{A}|^2 < \infty \), then \( K \) is integrable and \( \int_{\Sigma} K = 4\pi n \) for some \( n \in \mathbb{Z} \).

[3]This was first discussed in a subsequent paper [2] to [1].
4. Proof of Main Result

We assume that the surface $\Sigma$ is ruled outside $\overline{B(R_0)}$, where we have suppressed the reference point $x_0$ with respect to which the geodesic distance $R_0$ is measured. From now on if the center of a geodesic ball of certain radius is suppressed, it is implicit that the center is the point $x_0$. Thus each point in $\Sigma \setminus \overline{B(R_0)}$ is contained in a local coordinate chart given by $x(s,v) = \beta(s) + v\delta(s)$ for a curve $\beta$ in $\mathbb{R}^3$ and a nonzero outward-pointing vector field $\delta$ along $\beta$.

We would like to have a finite cover of $\Sigma \setminus \overline{B(R_0)}$ with ruled coordinate charts satisfying property (4). Let $R_1 > R_0$. For any point $p \in \partial B(R_1)$, there exists a local ruled coordinate chart with property (4). However, we must pay attention to the possibility that in the reparametrization described above, $t(s)$ may be so negative for some $s$ that $\gamma(s)$ is no longer on the surface $\Sigma$. Such a possibility requires consideration since our surface is not entirely ruled. Note that however, (10)\

$$|t(s)| \leq \int_0^s |\langle \beta', \delta \rangle| \, du \leq \int_0^s du = l(\beta).$$

Therefore for each point $p \in \partial B(R_1)$, without loss of generality let $x(s,v) : (-\varepsilon, \varepsilon) \times (b, \infty) \to \Sigma$ be a ruled coordinate chart satisfying property (4) such that $p = x(0,0)$. If we further require that $\varepsilon < l(\beta)$, then the image of $x$ will definitely lie in $\Sigma$. Then since $\partial B(R_1)$ is compact, there is a finite collection of such coordinate charts covering $\Sigma \setminus \overline{B(R_1)}$, assuming that we also take $R_1$ large enough.

By (9) and using property (4), the mean curvature has the expression (11)\

$$H = \frac{\langle x_s \times x_v, x_{ss} \rangle}{|x_s \times x_v|^3} = \frac{\langle \beta' \times \delta, \beta'' \rangle + v\left(\langle \delta' \times \delta, \beta'' \rangle + \langle \beta' \times \delta, \delta'' \rangle\right) + v^2\langle \delta' \times \delta, \delta'' \rangle}{\left(1 + 2v\langle \beta', \delta' \rangle + v^2|\delta'|^2\right)^{3/2}}.$$\

Let us denote the numerator of $H$ above by $P$, and the term inside the $3/2$ power in the denominator by $Q$. We will simplify matters by using the notations:

$$A = 2\langle \beta', \delta' \rangle ; \quad B = |\delta'|^2 ; \quad D = \langle \beta' \times \delta, \beta'' \rangle ;$$

$$E = \langle \delta' \times \delta, \beta'' \rangle + \langle \beta' \times \delta, \delta'' \rangle ; \quad J = \langle \delta' \times \delta, \delta'' \rangle.$$\

Note that since we assume $\Sigma$ is non-flat on $\Sigma \setminus \overline{B(R_0)}$, there must be a point $p \in \Sigma \setminus \overline{B(R_0)}$ such that $H(p) \neq 0$. Moreover, by (11) if we assume $v$ is large.
enough, \( H \neq 0 \) along the ruling line for \( s = 0 \). Switching the orientation of \( \Sigma \) if necessarily, we can then take \( \varepsilon \) small enough and \( v_0 \) large enough for all \( s \in (-\varepsilon, \varepsilon) \) such that \( H > 0 \) for all \( v \geq v_0 \) on \( x((-\varepsilon, \varepsilon) \times (v_0, \infty)) \). This assumption on the positive sign of \( H \) will be made implicitly throughout the remaining arguments.

**Lemma 1.** Let \( t > 0 \) and \( \alpha, \beta \in (-\varepsilon, \varepsilon) \) such that \( \alpha < \beta \). Suppose that \( \deg(P) \) and \( \deg(Q) \) remain unchanged on \( (\alpha, \beta) \). Then the integral

\[
\int_{t}^{\beta} \int_{\alpha}^{t+\tau} H|s \times v| \, dv \, ds = o(t_0)
\]

for large enough \( t \) if and only if \( \deg(P) < \deg(Q) \) for all \( s \in (\alpha, \beta) \).

**Proof.** First we suppose \( \deg(P) < \deg(Q) \), which consists of only two cases: \( (\deg(P), \deg(Q)) = (1, 2) \), and \( (0, 2) \). For the first case, since \( |s \times v| = \sqrt{1 + Av + Bv^2} \), we see that

\[
\int_{t}^{\beta} \int_{\alpha}^{t+\tau} H|s \times v| \, dv \, ds = \int_{t}^{\beta} \int_{t}^{t+\tau} P \, dv \, ds < C \log \left( \frac{t + t_0}{t} \right)
\]

for some constant \( C > 0 \), hence \( o(t_0) \). For the second case, similarly we can find constants \( C_1, C_2 > 0 \) such that for large enough \( v \),

\[
P < C_1 \quad \text{and} \quad Q > C_2 v^2
\]

for all \( s \in (-\varepsilon, \varepsilon) \). Then by (11), since \( |s \times v| = \sqrt{1 + Av + Bv^2} \), we see that

\[
(12) \quad \int_{t}^{\beta} \int_{\alpha}^{t+\tau} H|s \times v| \, dv \, ds < C \log \left( \frac{t + t_0}{t} \right)
\]

for some constant \( C > 0 \), hence \( o(t_0) \). For the second case, similarly we can find constants \( C_1, C_2 > 0 \) such that for large enough \( v \),

\[
P < C_1 \quad \text{and} \quad Q > C_2 v^2
\]

for all \( s \in (\alpha, \beta) \). Then we see that for large enough \( t \),

\[
(13) \quad \int_{t}^{\beta} \int_{\alpha}^{t+\tau} H|s \times v| \, dv \, ds < C \left( \frac{1}{t} - \frac{1}{t + t_0} \right),
\]

which is clearly \( o(t_0) \).

Next suppose \( \deg(P) \geq \deg(Q) \), which consists of three cases: \( (\deg(P), \deg(Q)) = (2, 2) \), \( (0, 0) \), and \( (1, 0) \). In any of these cases, we can long divide (for each \( s \)) and get

\[
(14) \quad \frac{P}{Q} = \tilde{P} + \frac{R}{Q}
\]

where \( \deg(\tilde{P}) = \deg(P) - \deg(Q) \), and \( \deg(R) < \deg(Q) \). For large enough \( t \), the integral over \( (\alpha, \beta) \times (t, t + t_0) \) of the second term in (14) is \( o(t_0) \) by applying the previous argument. However, integrating the first term in (14) we see that it becomes a polynomial in \( t_0 \) of degree at least 1. To be more precise, for large enough \( t \) we have

\[
\int_{t}^{\beta} \int_{\alpha}^{t+\tau} H|s \times v| \, dv \, ds = f(t_0) + o(t_0),
\]
where \( f(t_0) \) is a polynomial of degree either 1 (corresponding to the (2,2) and (0,0) cases listed above) or 2 (corresponding to the (1,0) case) with coefficients in \( t \). Hence it cannot be \( o(t_0) \).

Lemma 1 is an important lemma that will allow us to deduce the existence of ground states. In particular, we will need the integral of \( H \) over a sector-like region in \( \Sigma \setminus B(R_0) \) to grow at least linearly in \( t_0 \) (i.e., in the ruling direction). In view of this, we shall need to prove the following result.

**Proposition 1.** Let \( \Sigma \) be a surface embedded in \( \mathbb{R}^3 \) that is ruled, but non-flat outside \( B(R_0) \) for some \( R_0 > 0 \). Furthermore, assume the total Gauss curvature \( \int_{\Sigma} K > 0 \). Then there exists \( p \in \Sigma \setminus B(R_0) \), \( H(p) \neq 0 \), and a ruled coordinate chart \( x : (-\varepsilon, \varepsilon) \times (b, \infty) \rightarrow \Sigma \) satisfying (4) such that \( p = x(0,0) \) and \( \deg(P) \geq \deg(Q) \) at \( s = 0 \in (-\varepsilon, \varepsilon) \).

Before we proceed with the proof of Proposition 1, let us remark on its nature. Any point \( p \in \Sigma \setminus B(R_0) \) such that \( H(p) \neq 0 \) lies in a ruled coordinate chart satisfying (4), and we can always reparameterize the chart so that \( x(0,0) = p \) by translation in the \( v \)-direction. Moreover, the degrees of \( P \) and \( Q \) certainly do not change under translation in \( v \). Therefore, the proposition is really a statement about the ruled coordinate charts covering \( \Sigma \setminus B(R_0) \).

**Proof of Proposition 1** Let \( p \in \Sigma \setminus B(R_0) \) with \( H(p) \neq 0 \). Consider a ruled coordinate chart \( x : (-\varepsilon, \varepsilon) \times (b, \infty) \rightarrow \Sigma \) containing \( p \) such that \( x(0,0) = p \). Suppose \( \deg(P) < \deg(Q) \) for all \( s \in (-\varepsilon, \varepsilon) \) on such a chart, we will prove that

\[
\int_{-\varepsilon}^{\varepsilon} \int_{0}^{\infty} H^2 |x_s \times x_v| \, dv \, ds < \infty.
\]

Now suppose \( \deg(P) < \deg(Q) \) at \( s = 0 \) for all points \( p = x(0,0) \) with \( H(p) \neq 0 \). By continuity, we may assume \( \varepsilon \) is small enough so that \( H(s,0) \neq 0 \) for all \( s \in (-\varepsilon, \varepsilon) \). For such a chart \( x \), \( \deg(P) < \deg(Q) \) for all \( s \in (-\varepsilon, \varepsilon) \). To see this, suppose \( \deg(P) \geq \deg(Q) \) at some \( s_0 \in (-\varepsilon, \varepsilon) \). Then we can make the reparametrization

\[
\beta(s) \rightarrow \beta(s - s_0) \quad \text{and} \quad \delta(s) \rightarrow \delta(s - s_0)
\]

to obtain a new ruled coordinate chart \( \tilde{x} \) with \( \tilde{x}(0,0) = x(s_0,0) \). Note that property (4) is preserved for \( \tilde{x} \), and moreover the coefficients \( A, B, D, E, J \) are also invariant under the reparametrization above. Thus \( \deg(\tilde{P}) = \deg(P) \) and \( \deg(\tilde{Q}) = \deg(Q) \), and we would get a contradiction. Then via (15) and by our earlier remark on the existence of a finite cover of \( \Sigma \setminus B(R_0) \) by ruled coordinate charts, it would imply that \( H \in L^2(\Sigma) \). By the elementary formula \( H^2 = |A|^2 + 2K \) and the fact that \( K \in L^1(\Sigma) \), we must then
have $\int_\Sigma \|A\|^2 < \infty$, which by B. White’s theorem (Theorem 5) means that $\int_\Sigma K = 4\pi n$ for some $n \in \mathbb{Z}$. However, the assumption of connectedness and noncompactness of $\Sigma$ implies we must have Euler characteristic $\chi(\Sigma) \leq 1$. Then by Hartman’s formula and our hypothesis we must have

$$0 < \int_\Sigma K \leq 2\pi,$$

which would give a contradiction.

Now we prove (15) assuming $\deg(P) < \deg(Q)$ for all $s \in (-\varepsilon, \varepsilon)$ on a chart. First we see that

$$\int_0^t H^2 |x_s \times x_v| \, dv = \int_0^t \frac{P^2}{Q^{5/2}} \, dv. \quad (17)$$

First consider all $s \in (-\varepsilon, \varepsilon)$ such that $(\deg(P), \deg(Q)) = (1, 2)$. Since the coefficients of $P$ and $Q$ are functions of $s$ bounded on $(-\varepsilon, \varepsilon)$, we can choose $c > 0$ such that $Q > 1$, $Q^2 > Cv^4$ for some constant $C > 0$, and $P^2 < Cv^2$ for some constant $C > 0$, for all $v > c$ and for all such $s$. Then for any such $s$, we have

$$\int_c^t \frac{P^2}{Q^{5/2}} \, dv \leq \int_c^t \frac{P^2}{Q^{2}} \, dv$$

$$< C \int_c^t \frac{1}{v^2} \, dv$$

$$= C \left( \frac{1}{c} - \frac{1}{t} \right), \quad (18)$$

which converges as $t \to \infty$. Similarly, for any $s \in (-\varepsilon, \varepsilon)$ at which $(\deg(P), \deg(Q)) = (0, 2)$ we have

$$\int_c^t \frac{P^2}{Q^{5/2}} \, dv \leq P^2 \int_c^t \frac{1}{Q^2} \, dv$$

$$< C P^2 \int_c^t \frac{1}{v^4} \, dv$$

$$= C P^2 \left( \frac{1}{c^3} - \frac{1}{t^3} \right), \quad (19)$$

which again converges as $t \to \infty$. Combining (18) and (19) proves (15) and hence the proposition.

**Remark 2.** Although this is not necessary for the proof of Proposition 1, it is in good spirit to check that $\deg(P) \geq \deg(Q)$ at $p = x(0, 0)$ does not render $H^2$ unintegrable over $(-\varepsilon, \varepsilon) \times (0, \infty)$ for a small enough $\varepsilon > 0$ where $\deg(P)$ and $\deg(Q)$ are unchanged. As in the previous case of $\deg(P) < \deg(Q)$, we can narrow down to three cases. The first two consist of $\deg(P)$
= 1, 0 coupled with \( \deg(Q) = 0 \). It is clear in these two cases that
\[
\int_{-\varepsilon}^{\varepsilon} \int_0^t H^2 |x_s \times x_v| \, dv \, ds
\]
does not converge as \( t \to \infty \). For the last case of \( \deg(P) = \deg(Q) = 2 \), one can verify that there exists \( c, k > 0 \) such that
\[
\int_0^t \frac{P^2}{Q^{5/2}} \, dv \geq k \int_0^t \frac{1}{v} \, dv
\]
(20)
as \( t \to \infty \).

**Corollary 2.** There exists a point \( p \in \Sigma \setminus B(R_0) \), \( H(p) \neq 0 \), and a ruled coordinate chart \( x : (-\varepsilon, \varepsilon) \times (b, \infty) \to \Sigma \) satisfying \( \deg(P) \geq \deg(Q) \) for all \( s \in (-\varepsilon, \varepsilon) \). Moreover, we can choose the chart so that \( \deg(P) \) and \( \deg(Q) \) are fixed for all \( s \in (-\varepsilon, \varepsilon) \).

**Proof.** By Proposition 1, there exists at least one point \( p \in \Sigma \setminus B(R_0) \), \( H(p) \neq 0 \), and a ruled coordinate chart \( x : (-\varepsilon, \varepsilon) \to \Sigma \) such that \( x(0, 0) = p \) and \( \deg(P) \geq \deg(Q) \) at \( s = 0 \). Suppose at such a point \( p = x(0, 0) \), the first assertion of the corollary is false. Then by the smoothness of the coefficients of \( P \) and \( Q \) in \( s \), there must exist an \( \varepsilon > 0 \), such that \( \deg(P) \geq \deg(Q) \) at \( s = 0 \) and \( \deg(P) < \deg(Q) \) for all \( s \in (-\varepsilon, \varepsilon) \). Thus the ruling lines that pass through these points must comprise a set of discrete lines, and hence of measure zero. Then applying the integration of the \( H^2 \) argument in Proposition 1 we would get the same contradiction as we did there.

Next we argue that we can fix \( \deg(P) \) and \( \deg(Q) \) in a small enough interval of \( s \). Observe that due to the smoothness of the coefficient functions in \( s \), the degrees of \( P \) and \( Q \) cannot decrease in an arbitrarily small neighborhood of \( s \), but it can certainly increase. Now, if \( (\deg(P), \deg(Q)) = (2, 2) \) at \( s = 0 \), then since this is the case of the largest possible degrees, we can certainly find an \( \varepsilon > 0 \) small enough so that the degrees remain 2 for all \( s \in (-\varepsilon, \varepsilon) \). If we are in the \( (1, 0) \) case at \( s = 0 \), then either the degrees remain as \( (1, 0) \) in a small interval about \( s = 0 \), or there exists an \( s_0 \) near \( s \) at which the degrees increase to \( (2, 2) \) and to which we can apply the preceding argument upon reparametrization of the chart. For the last case of \( (0, 0) \), if the degrees increase near \( s = 0 \) then we simply apply the arguments in the previous two cases.

**Proof of Theorem 2.** Now, the surface is not totally-geodesic (non-planar) by our hypothesis. Therefore if \( \int_{\Sigma} K \leq 0 \), the conclusion of the theorem follows as a special case of the result in [5]. For the remaining of the proof we will assume that \( \int_{\Sigma} K > 0 \).
By the variation principle, it suffices to find a test function \( \phi \in W^{1,2}_0(\Omega) \) such that
\[
Q(\phi, \phi) = \int_{\Omega} |\nabla \phi|^2 - \left( \frac{\pi}{2a} \right)^2 \int_{\Omega} \phi^2 < 0.
\]
We define a test function (family of test functions) of the form \( \phi_\varepsilon = \chi \psi + \varepsilon \chi_1 j \),
where \( \varepsilon \) is some non-zero number to be determined, \( \chi = \cos \frac{\pi}{2a} u \), \( \chi_1 = u \cos \frac{\pi}{2a} u \), and \( \psi \) and \( j \) are defined below.

The assumption of integrable Gauss curvature implies that \( \Sigma \) is parabolic, and hence for any \( R_1 > 0 \) there exists an \( R_2 > R_1 \) for which
\[
\int_{\Sigma} |\nabla \psi_{R_1, R_2}|^2 < \frac{\varepsilon_0}{2},
\]
where \( \psi_{R_1, R_2} \) is the unique solution to the boundary value problem
\[
\begin{cases}
\Delta \psi = 0 & \text{on } B(R_2) \setminus B(R_1); \\
\psi|_{B_p(R_1)} \equiv 1; \\
\psi|_{\Sigma \setminus B_p(R_2)} \equiv 0.
\end{cases}
\]
We will let \( R_1 > R_0 \). Then for \( R_1 < R_2 < R_3 < R_4 \), we let
\[
\psi = \psi_{R_3, R_4} - \psi_{R_1, R_2}.
\]

We want \( j \) to be a \( W^{1,2} \) function on \( \Sigma \) with support in \( \{ \psi \equiv 1 \} \) and \( j \leq 1 \). Before defining \( j \) precisely, we proceed with some preliminary estimates. By our choices of \( \chi \) and \( \chi_1 \), the fact that \( K \leq 0 \) on \( \Sigma \setminus B(R_0) \), and the requirement that \( j \leq 1 \) and \( \text{supp} j \subset \{ \psi \equiv 1 \} \), we get
\[
Q(\phi_\varepsilon, \phi_\varepsilon) = Q(\chi \psi, \chi \psi) + 2\varepsilon Q(\chi \psi, \chi_1 j) + \varepsilon^2 Q(\chi_1 j, \chi_1 j)
\leq C_1 \int_{\Sigma} |\nabla \psi|^2 + \varepsilon a \int_{\Sigma} jH + \varepsilon^2 C_2 \|j\|_{W^{1,2}}^2
\]
for \( C_1, C_2 > 0 \) depending only on the geometry of \( \Omega \). We will choose \( j \) so that \( \|j\|_{W^{1,2}} \neq 0 \). Then viewing the right-hand side of the inequality in (24) as a quadratic polynomial in the variable \( \varepsilon \), it will be negative for some \( \varepsilon \) if and only if its discriminant is positive, which is equivalent to the condition
\[
\frac{\left( \int_{\Sigma} jH \right)^2}{\|j\|_{W^{1,2}}^2} > C_1 \int_{\Sigma} |\nabla \psi|^2,
\]
\([	ext{This form of test function first appeared in [2], and was also used extensively in [5]. However, the argument we will give here for (21) is different in an essential way.}]
\[
[\text{For a proof of this, see [5].}]
\]
where we absorbed all the geometric constants into a single constant $C_1$. Now, by our choice of $\psi$ along with the parabolicity of $\Sigma$, for $R_1 > R_0$ fixed we can choose $R_2$ and then $R_3 < R_4$ big enough so that

$$\int_{\Sigma} |\nabla \psi|^2 = \int_{\Sigma} |\nabla \psi_{R_1, R_2}|^2 + \int_{\Sigma} |\nabla \psi_{R_3, R_4}|^2$$

$$< \frac{\varepsilon_0}{2} + \frac{\varepsilon_0}{2} = \varepsilon_0.$$  

(26)

Observe that what is essential is the choice of $R_2$, as $R_3 < R_4$ can always be chosen after $R_2$ so that (26) holds. By our requirement on $j$, the choice of $R_2$ may affect the ratio on the left-hand side of (25). In view of this consideration, we seek a (family of) $j$ such that

$$\left( \int_{\Sigma} jH \right)^2 \geq C$$

(27)

for a constant $C$ independent of $R_2$, as long as $R_2$ is large enough. Inequality (27) is a sufficient condition for (25) since we can then choose $R_2$ large enough for a small enough $\varepsilon_0$ satisfying (26) and

$$C_1 \varepsilon_0 < C.$$  

In a nutshell, the proof will be complete if we construct a (family of) $j$ so that (27) holds for some constant $C$ independent of $R_2$, for $R_2$ large enough.

By Corollary 2, let $p \in \Sigma \setminus \overline{B(R_0)}$ be a point such that $H(p) \neq 0$ and consider a ruled coordinate chart $x : (-\varepsilon, \varepsilon) \times (b, \infty) \rightarrow \Sigma$ satisfying (11) such that $p = x(0, 0)$, $\deg(P) \geq \deg(Q)$, with $\deg(P)$ and $\deg(Q)$ fixed for all $s = 0 \in (-\varepsilon, \varepsilon)$. Moreover, we let $R_1 = \text{dist}(x_0, p)$. For any $R_2 > R_1$ and $t_0 > 1$ let $\Gamma = \{(s, v) \in \mathbb{R}^2 | -\varepsilon \leq s \leq \varepsilon, v_0 \leq v \leq v_0 + t_0\}$ such that $x(\Gamma) \subset B_p(R_3) \setminus B_p(R_2)$. We define $j$ by $j = j_1(s)j_2(v)$, with cut-off functions

$$j_1(s) = \begin{cases} 1 & \alpha - \varepsilon < s < \varepsilon - \alpha; \\ 0 & |s| \geq \varepsilon \end{cases}$$

and

$$j_2(v) = \begin{cases} 1 & v_0 + \alpha < v < v_0 + t_0 - \alpha; \\ 0 & v \leq v_0, v \geq v_0 + t_0, \end{cases}$$

(28)

(29)

where $\alpha > 0$ is a fixed small number, $|j_1'(s)| \leq 1/\alpha$, and $|j_2'(v)| \leq 1/\alpha$.

Now, by the definition of $j$ above and (11), we have

$$\int_{\Sigma} jH > \int_{\{j \equiv 1\}} H = \int_{\alpha - \varepsilon}^{\varepsilon - \alpha} \int_{v_0 + t_0 - \alpha}^{v_0 + t_0 - \alpha} \frac{P}{Q} dv ds.$$
By our choice of the ruled coordinate chart \( x \) above, Lemma 1 implies

\[
\int_\Sigma jH > f(t_0) + o(t_0)
\]

for large enough \( v_0 \), where \( f(t_0) \) is a polynomial in \( t_0 \) of degree \( n = 1 \) or \( 2 \) and has coefficients in \( v_0 \).

Next we wish to give an upperbound estimate for \( \| j \|_{W^{1,2}} \). First we see that

\[
\| j \|_{W^{1,2}} = \int_\Sigma j^2 + \int_\Sigma |\nabla j|^2 
\leq (1 + \| \nabla j \|_\infty^2) \text{vol}(x(\Gamma)).
\]

By the metric on \( \Sigma \) given by the chart (3), we see that

\[
|\nabla j|^2 = j_2^2 G^{ss} \left| \frac{\partial j_1}{\partial s} \right|^2 + 2 j_1 j_2 G^{sv} \left| \frac{\partial j_1}{\partial s} \frac{\partial j_2}{\partial v} + j_1^2 G^{vv} \left| \frac{\partial j_2}{\partial v} \right|^2 
\leq \left( \frac{1}{1 + 2v \langle \beta', \delta' \rangle + v^2 |\delta'|^2} \right) \frac{1}{\alpha^2} + \frac{1}{\alpha^2}
\]

(32)

If the polynomial \( 1 + 2v \langle \beta', \delta' \rangle + v^2 |\delta'|^2 \) has degree 2 for all \( s \in (-\varepsilon, \varepsilon) \), then since its discriminant

\[
4\langle \beta', \delta' \rangle^2 - 4|\delta'|^2 \leq 4|\beta'|^2 |\delta'|^2 - 4|\delta'|^2 = 0,
\]

it is always positive for \( v \) large enough. The other possibility is that it is identically equal to 1. In any case, we can choose a \( v_0 \) large enough so that for all \( v \geq v_0 \)

(33)

\[ |\nabla j|^2 < C_3 \]

for all \( s \in (-\varepsilon, \varepsilon) \), for some constant \( C_3 > 0 \).

Next we will estimate the volume growth of \( x(\Gamma) \). The volume form in the ruled coordinate system is

\[ d\Sigma = \sqrt{1 + Av + Bv^2} \, ds \, dv. \]

There are two possibilities at \( s = 0 \), either \( B = 0 \) or \( B \neq 0 \). In the latter case, we can certainly assume that \( \varepsilon \) is small enough so that \( B \neq 0 \) for all \( s \in (-\varepsilon, \varepsilon) \). If \( B = 0 \) at \( s = 0 \), then by a similar argument as in Corollary 2 we can assume that \( B = 0 \) for all \( s \in (-\varepsilon, \varepsilon) \).

Suppose \( B \neq 0 \) for all \( s \in (-\varepsilon, \varepsilon) \). If \( A^2 - 4B < 0 \) at \( s = 0 \) we can always take \( \varepsilon \) small enough so that \( A^2 - 4B < 0 \) for all \( s \in (-\varepsilon, \varepsilon) \). Assuming so,
we have
\[\text{vol}(x(\Gamma)) = \int_{-\varepsilon}^{\varepsilon} \int_{v_0}^{v_0 + t_0} \sqrt{1 + Av + Bv^2} \, dv \, ds\]
\[= \int_{-\varepsilon}^{\varepsilon} \int_{v_0}^{v_0 + t_0} \sqrt{B(v + \frac{A}{2B})^2 + 1 - \frac{A^2}{4B}} \, dv \, ds\]
\[= \int_{-\varepsilon}^{\varepsilon} \int_{v_0 + \frac{A}{2B}}^{v_0 + t_0 + \frac{A}{2B}} \sqrt{Bx^2 + 1 - \frac{A^2}{4B}} \, dx \, ds\]
\[= \int_{-\varepsilon}^{\varepsilon} \int_{v_0 + \frac{A}{2B}}^{v_0 + t_0 + \frac{A}{2B}} \sqrt{\frac{4B^2}{4B - A^2}} x^2 + 1 \, dx \, ds\]
\[= \int_{-\varepsilon}^{\varepsilon} \int_{v_0 + \frac{A}{2B}}^{v_0 + t_0 + \frac{A}{2B}} \sqrt{\frac{4B^2}{4B - A^2}} \left(\frac{y + 1}{2}\right) \, dy \, ds\]
\[\leq \int_{-\varepsilon}^{\varepsilon} \sqrt{\frac{4B^2}{4B - A^2}} \left(y + 1\right) \, dy \, ds\]
(34)
\[= t_0 v_0 \int_{-\varepsilon}^{\varepsilon} \sqrt{B} \, ds + t_0^2 \int_{-\varepsilon}^{\varepsilon} \frac{\sqrt{B}}{2} \, ds + t_0 \int_{-\varepsilon}^{\varepsilon} \left(\frac{A}{2\sqrt{B}} + \sqrt{\frac{4B - A^2}{4B}}\right) \, ds.\]

On the other hand, if \(B \neq 0\) for all \(s \in (-\varepsilon, \varepsilon)\) and \(A^2 - 4B = 0\) at \(s = 0\), using the same argument in Corollary 2 we can assume that \(A^2 - 4B = 0\) for all \(s \in (-\varepsilon, \varepsilon)\). Assuming so, we have
\[\text{vol}(x(\Gamma)) = \int_{-\varepsilon}^{\varepsilon} \int_{v_0}^{v_0 + t_0} \sqrt{1 + Av + Bv^2} \, dv \, ds\]
\[= \int_{-\varepsilon}^{\varepsilon} \int_{v_0}^{v_0 + t_0} \sqrt{B(v + \frac{A}{2B})^2} + 1 \, dv \, ds\]
\[= \int_{-\varepsilon}^{\varepsilon} \int_{v_0 + \frac{A}{2B}}^{v_0 + t_0 + \frac{A}{2B}} \sqrt{Bx^2 + 1} \, dx \, ds\]
\[= \int_{-\varepsilon}^{\varepsilon} \int_{v_0 + \frac{A}{2B}}^{v_0 + t_0 + \frac{A}{2B}} \sqrt{4B^2} \, dx \, ds\]
\[= \int_{-\varepsilon}^{\varepsilon} \int_{v_0 + \frac{A}{2B}}^{v_0 + t_0 + \frac{A}{2B}} \sqrt{\frac{4B^2}{4B - A^2}} \left(y + 1\right) \, dy \, ds\]
(35)
\[= t_0 v_0 \int_{-\varepsilon}^{\varepsilon} \sqrt{B} \, ds + t_0^2 \int_{-\varepsilon}^{\varepsilon} \frac{\sqrt{B}}{2} \, ds + t_0 \int_{-\varepsilon}^{\varepsilon} \frac{A}{2\sqrt{B}} \, ds.\]

For brevity, we will use the following notations:

\[C_4 = \int_{-\varepsilon}^{\varepsilon} \sqrt{B} \, ds, C_5 = \int_{-\varepsilon}^{\varepsilon} \frac{\sqrt{B}}{2} \, ds,\]
\[C_6 = \int_{-\varepsilon}^{\varepsilon} \left(\frac{A}{2\sqrt{B}} + \sqrt{\frac{4B - A^2}{4B}}\right) \, ds \text{ or } \int_{-\varepsilon}^{\varepsilon} \frac{A}{2\sqrt{B}} \, ds.\]

If \(B = 0\) for all \(s \in (-\varepsilon, \varepsilon)\), then
\[\text{vol}(x(\Gamma)) = 2\varepsilon t_0.\]
Now, by (30), with any fixed $t_0 > 0$ there exists a $v_0$ large enough so that

\[
\int_{\Sigma} jH > C_7 t_0^n
\]

for some constant $C_7 > 0$ (which depends on $v_0$), with $n = 1$ for cases (34) or (35) and $n = 1$ or 2 for case (36) corresponding to $\deg(P) = 0$ or 1.

Renaming the square of $C_7$ as itself, by (31), (33), and (37) with a choice of a large enough $v_0$, we see that in either case (34) or case (35),

\[
\left( \int_{\Sigma} jH \right)^2 \frac{\|j\|_{W^{1,2}}} > C_7 t_0^2 \frac{C_3}{C_4 t_0 v_0 C_4 + t_0^2 C_5 + t_0 C_6}.
\]

Note that once $v_0$ is fixed, the constants $C_3, C_4, C_5, C_6,$ and $C_7$ depend only on the metric along the $s$-parameter curve $\beta(s): (-\varepsilon, \varepsilon) \rightarrow \Sigma$, which is fixed from the start. Then the right-hand side of (38) either converges to $C_7 t_0$ (when $n = 1$) or goes to infinity (when $n = 2$), as we take $t_0 \rightarrow \infty$ (by letting $R_3 \rightarrow \infty$). Therefore for a large enough $t_0$, there must be a constant $C > 0$ such that

\[
\left( \int_{\Sigma} jH \right)^2 \frac{\|j\|_{W^{1,2}}} > C.
\]

For the case of (36), the denominator of the right-hand side of (38) will always be linear in $t_0$, while the numerator is either quadratic or to the 4th power in $t_0$, hence (39) is readily achieved. The proof is now complete. □

An immediate consequence of Theorem 2 is the following result, which follows via Massey’s Theorem.

**Corollary 3.** Let $\Sigma$ be an embedded surface in $\mathbb{R}^3$ with zero Gauss curvature but non-flat outside a compact subset. Then for a layer $\Omega$ over $\Sigma$, we have

\[
\inf \sigma(\Delta) < \left( \frac{\pi}{2a} \right)^2.
\]

Moreover, if the second fundamental form $\tilde{A} \rightarrow 0$ at infinity on $\Sigma$ then ground states exist.

**Remark 3.** Our proof depends heavily on the fact that the surface is ruled outside a compact set. If we assume that the surface is asymptotically flat, then it is asymptotically ruled on any compact set. However, with only the asymptotic flatness, we don’t know the behavior of the surface at infinity. Thus the analysis in this paper is not enough to prove Conjecture 1. More careful estimates are needed but the motivation of the development of the techniques in this paper is clearly justified.
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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, IRVINE, IRVINE, CA 92697

_E-mail address_, Christopher Lin: _ccplin@msri.org_

_E-mail address_, Zhiqin Lu: _zlu@math.uci.edu_