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The twisted forms of a semisimple group over an $\mathbb{F}_q$-curve

par Rony A. BITAN, Ralf KÖHL et Claudia SCHOEMANN

Résumé. Soit $C$ une courbe projective, lisse et connexe définie sur un corps fini $\mathbb{F}_q$. Étant donné un $C - S$-schéma en groupes semisimples où $S$ est un ensemble fini de points fermés de $C$, nous décrivons l’ensemble de $(\mathcal{O}_S$-classes de) formes tordues de $G$ en termes d’invariants géométriques de son groupe fondamental $F(G)$.

Abstract. Let $C$ be a smooth, projective and geometrically connected curve defined over a finite field $\mathbb{F}_q$. Given a semisimple $C - S$-group scheme $G$ where $S$ is a finite set of closed points of $C$, we describe the set of $(\mathcal{O}_S$-classes of) twisted forms of $G$ in terms of geometric invariants of its fundamental group $F(G)$.

1. Introduction

Let $C$ be a projective, smooth and geometrically connected curve defined over a finite field $\mathbb{F}_q$. Let $\Omega$ be the set of all closed points on $C$. For any $p \in \Omega$ let $v_p$ be the induced discrete valuation on the (global) function field $K = \mathbb{F}_q(C)$, $\mathcal{O}_p$ the ring of integers in the completion $\hat{K}_p$ of $K$ with respect to $v_p$, and $k_p$ the residue field. Any finite subset $S \subset \Omega$ gives rise to a Dedekind scheme, namely, a Noetherian integral scheme of dimension 1 whose local rings are regular; If $S$ is nonempty it will be the spectrum of the Dedekind domain

$$\mathcal{O}_S := \{x \in K : v_p(x) \geq 0 \ \forall \ p \notin S\}.$$ 

Otherwise, if $S = \emptyset$, the corresponding Dedekind scheme is the curve $C$ itself, and we denote by $\mathcal{O}_S$ the structural sheaf of $C$.

Throughout this paper $G$ is an $\mathcal{O}_S$-group scheme whose generic fiber $G := G \otimes_{\mathcal{O}_S} K$ is almost-simple, and whose fiber $G_p = G \otimes_{\mathcal{O}_S} \mathcal{O}_p$ at any $p \in \Omega - S$ is semisimple, namely, (connected) reductive over $k_p$, and the rank of its root system equals that of its lattice of weights ([12, Exp. XIX Def. 2.7, Exp. XXI Def. 1.1.1]). Let $G^{sc}$ be the universal (central) cover (being simply-connected) of $G$, and suppose that its fundamental group $F(G) := \ker[G^{sc} \twoheadrightarrow G]$ (cf. [10, p. 40]) is of order prime to $\text{char}(K)$.

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A twisted form of $G$ is an $O_S$-group that is isomorphic to $G$ over some finite étale cover of $O_S$. We aim to describe explicitly (in terms of some invariants of $F(G)$ and the group of outer automorphisms of $G$) the finite set of all twisted forms of $G$, modulo $O_S$-isomorphisms. This is done first in Section 2 for forms arising from the torsors of the adjoint group $G^{ad}$, and then in Section 3, through the action of the outer automorphisms of $G$ on its Dynkin diagram, for all twisted forms. More concrete computations are provided in Sections 4, 5 and 6. The case of type A deserves a special consideration, this is done in Section 7. The Zariski topology is treated in Section 8.

Before we start may we quote B. Conrad in the abstract of [11]: “The study of such $\mathbb{Z}$-groups provides concrete applications of many facets of the theory of reductive groups over rings (scheme of Borel subgroups, automorphism scheme, relative non-abelian cohomology, etc.), and it highlights the role of number theory (class field theory, mass formulas, strong approximation, point-counting over finite fields, etc.) in analyzing the possibilities”.

2. Torsors

A $G$-torsor $P$ in the étale topology is a sheaf of sets on $O_S$ equipped with a (right) $G$-action, which is locally trivial in the étale topology, namely, locally for the étale topology on $O_S$, this action is isomorphic to the action of $G$ on itself by translation. The associated $O_S$-group scheme $\tilde{P}_G = G'$, being an inner form of $G$, is called the twist of $G$ by $P$ (e.g., [24, §2.2, Lem. 2.2.3, Exs. 1, 2]). We define $H^1_{\text{ét}}(O_S, G)$ to be the set of isomorphism classes of $G$-torsors relative to the étale topology (or the flat one; these two cohomology sets coincide when $G$ is smooth; cf. [2, VIII Cor. 2.3]). This set is finite ([5, Prop. 3.9]). The sets $H^1(K, G)$ (denoting the Galois cohomology) and $H^1_{\text{ét}}(\hat{O}_p, G_p)$ are defined similarly.

There exists a canonical map of pointed-sets:

$$\lambda : H^1_{\text{ét}}(O_S, G) \to H^1(K, G) \times \prod_{p \notin S} H^1_{\text{ét}}(\hat{O}_p, G_p).$$

defined by $[X] \mapsto [(X \otimes_{O_S} \text{Sp} K) \times \prod_{p \notin S} X \otimes_{O_S} \text{Sp} \hat{O}_p]$. Let $[\xi_0] := \lambda([G])$. The principal genus of $G$ is then $\ker(\lambda) = \lambda^{-1}([\xi_0])$, namely, the classes of $G$-torsors that are generically and locally trivial at all points of $O_S$. More generally, a genus of $G$ is any fiber $\lambda^{-1}([\xi])$ where $[\xi] \in \text{Im}(\lambda)$. The set of genera of $G$ is then:

$$\text{gen}(G) := \{\lambda^{-1}([\xi]) : [\xi] \in \text{Im}(\lambda)\},$$

hence $H^1_{\text{ét}}(O_S, G)$ is a disjoint union of all genera.

The ring of $S$-integral adèles $A_S := \prod_{p \in S} \hat{K}_p \times \prod_{p \notin S} \hat{O}_p$ is a subring of the adèles $A$. A $G$-torsor $P = \text{Iso}(G, G')$ belongs to the principal genus of
Twisted forms of a semisimple group

$G$ if it is both $A_S$- and $K$-trivial, hence the principal genus bijects as a pointed-set to the $S$-class set of $G$ (see [21, Thm. I.3.5]):

$$\text{Cl}_S(G) := G(A_S) \backslash G(A) / G(K).$$

Being finite ([5, Prop. 3.9]), its cardinality, called the $S$-class number of $G$, is denoted $h_S(G)$. As $G$ is assumed to have connected fibers, by Lang’s Theorem (recall that all residue fields are finite) all $H^1_{\text{ét}}(O_S,G_p)$ vanish, which indicates that any two $G$-torsors share the same genus if and only if they are $K$-isomorphic.

The universal cover of $G$ forms a short exact sequence of étale $O_S$-groups (cf. [10, p. 40]):

$$1 \to F(G) \to G^{\text{sc}} \to G \to 1.$$  

This gives rise by étale cohomology to the co-boundary map of pointed sets:

$$\delta_G : H^1_{\text{ét}}(O_S,G) \to H^2_{\text{ét}}(O_S,F(G))$$

which is surjective by ([13, Cor. 1]) as $O_S$ is of Douai-type (see [17, Def. 5.2 and Exam. 5.4(iii),(v)]). It follows from the fact that $H^2_{\text{ét}}(O_S,G^{\text{sc}})$ (resp., $H^2_{\text{ét}}(O_S,G^{\text{sc}})$) has only trivial classes and in finite number ([13, Thm. 1.1]).

A representation $\rho : G^{\text{sc}} \to \text{GL}_1(A)$ where $A$ is an Azumaya $O_S$-algebra, is said to be center-preserving if $\rho(Z(G^{\text{sc}})) \subseteq Z(\text{GL}_1(A))$. The restriction of $\rho$ to $F(G) \subseteq Z(G^{\text{sc}})$, composed with the natural isomorphism $Z(\text{GL}_1(A)) \cong \mathbb{G}_m$, is a map $\Lambda_\rho : F(G) \to \mathbb{G}_m$, thus inducing a map: $(\Lambda_\rho)_* : H^2_{\text{ét}}(O_S,F(G)) \to H^2_{\text{ét}}(O_S,\mathbb{G}_m) \cong \text{Br}(O_S)$. Together with the preceding map $\delta_G$ we get the map of pointed-sets:

$$(\Lambda_\rho)_* \circ \delta_G : H^1_{\text{ét}}(O_S,G) \to \text{Br}(O_S),$$

which associates any class of $G$-torsors with a class of Azumaya $O_S$-algebras in $\text{Br}(O_S)$.

When $F(G) = \mu_m$, the following composition is surjective:

$$(\Lambda_\rho)_* \circ \delta_G : H^1_{\text{ét}}(O_S,G) \to \text{Br}(O_S) \to m \text{Br}(O_S),$$

and coincides with $(\Lambda_\rho)_* \circ \delta_G$.

The original Tits algebras introduced in [26], are central simple algebras defined over a field, associated to algebraic groups defined over that field. This construction was generalized to group-schemes over rings as shown in [22, Thm. 1]. We briefly recall it here over $O_S$: Being semisimple, $G$ admits an inner form $G_0$ which is quasi-split (in the sense of [12, XXIV, 3.9], namely, not only requiring a Borel subgroup to be defined over $C-S$ but some additional data involving the scheme of Dynkin diagrams, see [10, Def. 5.2.10.]).
Definition 1. Any center-preserving representation \(\rho_0 : G_0 \to \text{GL}(V)\) gives rise to a “twisted” center-preserving representation: \(\rho : G \to \text{GL}_1(A_\rho)\), where \(A_\rho\) is an Azumaya \(O_S\)-algebra, called the Tits algebra corresponding to the representation \(\rho\), and its class in \(\text{Br}(O_S)\), is its Tits class.

Lemma 2.1. If \(G\) is adjoint, then for any center-preserving representation \(\rho\) of \(G_0^\text{sc}\), and a twisted \(G\)-form \(P G\) by a \(G\)-torsor \(P\), one has: \(((\Lambda_\rho)_* \circ \delta_G)([P G]) = [P A_\rho] - [A_\rho] \in \text{Br}(O_S)\) where \([P A_\rho]\) and \([A_\rho]\) are the Tits classes of \((P G)^\text{sc}\) and \(G^\text{sc}\) corresponding to \(\rho\), respectively.

Proof. By descent \(F(G_0) \cong F(G)\), so we may write the short exact sequences of \(O_S\)-groups:

\[
\begin{align*}
(2.5) \quad &1 \to F(G) \to G^\text{sc} \to G \to 1 \\
&1 \to F(G) \to G_0^\text{sc} \to G_0 \to 1
\end{align*}
\]

which yield the following commutative diagram of pointed sets (cf. [16, IV, Prop. 4.3.4]):

\[
\begin{array}{cc}
H^1_{\text{ét}}(O_S, G_0) & H^1_{\text{ét}}(O_S, G) \\
\downarrow \delta_0 & \downarrow \delta_G \\
H^2_{\text{ét}}(O_S, F(G)) & H^2_{\text{ét}}(O_S, F(G))
\end{array}
\]

in which \(r_G(x) := x - \delta_G([G])\), so that \(\delta_G = r_G \circ \delta_0\) maps \([G]\) to \([0]\). The image of any twisted form \(P G\) where \([P] \in H^1_{\text{ét}}(O_S, G)\) (see in Section 1), under the coboundary map

\[
\delta : H^1_{\text{ét}}(O_S, G_0) \to H^2_{\text{ét}}(O_S, Z(G_0^\text{sc}))
\]

induced by the universal covering of \(G_0\) corresponding to \(\rho\), is \([P A_\rho]\), where \(P A_\rho\) is the Tits-algebra of \((P G)^\text{sc}\) (see [22, Thm. 1]). But \(G_0\) is adjoint, so \(Z(G_0^\text{sc}) = F(G_0) \cong F(G)\), thus the images of \(\delta\) and \(\delta_0\) coincide in \(\text{Br}(O_S)\), whence:

\[
((\Lambda_\rho)_* (\delta_G([G]))) = ((\Lambda_\rho)_* (\delta_0([P G])) - \delta_0([G])) = [P A_\rho] - [A_\rho]. \quad \Box
\]

The fundamental group \(F(G)\) is a finite, of multiplicative type (cf. [12, XXII, Cor. 4.1.7]), commutative and smooth \(O_S\)-group (as its order is assumed prime to \(\text{char}(K)\)).

Lemma 2.2. If \(G\) is not of type A, or \(S = \emptyset\), then \(H^1_{\text{ét}}(O_S, G)\) is isomorphic to \(H^2_{\text{ét}}(O_S, F(G))\).

Proof. Applying étale cohomology to sequence (2.1) yields the exact sequence:

\[
H^1_{\text{ét}}(O_S, G^\text{sc}) \to H^1_{\text{ét}}(O_S, G) \xrightarrow{\delta_G} H^2_{\text{ét}}(O_S, F(G))
\]
in which $\delta_G$ is surjective (see (2.2)). If $G$ is not of absolute type $A$, it is locally isotropic everywhere ([6, 4.3 and 4.4]), in particular at $S$. This is of course redundant when $S = \emptyset$. Thus $H^1_{\text{ét}}(O_S, G^{sc})$ vanishes ([4, Lem. 2.3]). Changing the base-point in $H^1_{\text{ét}}(O_S, G)$ to any $G$-torsor $P$, it is bijective to $H^1_{\text{ét}}(O_S, P_G)$ where $P_G$ is an inner form of $G$ (see Section 1), thus an $O_S$-group of the same type. Similarly all fibers of $\delta_G$ vanish. This amounts to $\delta_G$ being injective thus an isomorphism. □

The following two invariants of $F(G)$ were defined in [4, Def. 1]:

**Definition 2.** Let $R$ be a finite étale extension of $O_S$. We define:

$$i(F(G)) := \begin{cases} m\text{Br}(R) & F(G) = \text{Res}_{R/O_S}(\mu_m) \\ \ker(m\text{Br}(R) \xrightarrow{N(2)} m\text{Br}(O_S)) & F(G) = \text{Res}_{R/O_S}(1) \mu_m \end{cases}$$

where for a group $\ast$, $m\ast$ stands for its $m$-torsion part, and $N(2)$ is induced by the norm map $N_{R/O_S}$.

For $F(G) = \prod_{k=1}^r F(G)_k$ where each $F(G)_k$ is one of the above, $i(F(G)) := \prod_{k=1}^r i(F(G)_k)$.

We also define for such $R$:

$$j(F(G)) := \begin{cases} \text{Pic}(R)/m & F(G) = \text{Res}_{R/O_S}(\mu_m) \\ \ker(\text{Pic}(R)/m \xrightarrow{N(1)/m} \text{Pic}(O_S)/m) & F(G) = \text{Res}_{R/O_S}(1) \mu_m \end{cases}$$

where $N(1)$ is induced by $N_{R/O_S}$, and again

$$j\left( \prod_{k=1}^r F(G)_k \right) := \prod_{k=1}^r j(F(G)_k).$$

**Definition 3.** We call $F(G)$ admissible if it is a finite direct product of factors of the form:

1. $\text{Res}_{R/O_S}(\mu_m)$,
2. $\text{Res}_{R/O_S}^{(1)}(\mu_m)$, $[R : O_S]$ is prime to $m$,

where $R$ is any finite étale extension of $O_S$.

**Lemma 2.3.** If $F(G)$ is admissible then there exists a short exact sequence of abelian groups:

$$1 \to j(F(G)) \to H^2_{\text{ét}}(O_S, F(G)) \xrightarrow{i} i(F(G)) \to 1.$$  

This sequence splits thus reads: $H^2_{\text{ét}}(O_S, F(G)) \cong j(F(G)) \times i(F(G)).$

**Proof.** This sequence was shown in [4, Cor. 2.9] for the case $S$ is nonempty. The proof based on applying étale cohomology to the related Kummer exact sequence is similar for $S = \emptyset$. The splitting when $F(G)$ is quasi-split was proved in [15, Thm. 1.1]. When $F(G) = \text{Res}_{R/O_S}^{(1)}(\mu_m)$, $[R : O_S]$ prime to
Consider the exact diagram obtained by étale cohomology applied to the Kummer exact sequences related to $\mu_m$ over $O_S$ and $R$:

$$
\begin{align*}
1 & \longrightarrow \text{Pic}(R)/m \longrightarrow H^2_{\text{ét}}(R, \mu_m) \xrightarrow{i^*} \text{Br}(R)[m] \longrightarrow 1 \\
1 & \longrightarrow \text{Pic}(O_S)/m \longrightarrow H^2_{\text{ét}}(O_S, \mu_m) \xrightarrow{i^*} \text{Br}(O_S)[m] \longrightarrow 1.
\end{align*}
$$

The splitting of the two rows then implies the one in the assertion.

As a result we have two bijections as pointed-sets: the first is $\text{gen}(G) \cong i(F(G))$; the affine case shown in [4, Cor. 3.2] holds as aforementioned for $S = \emptyset$ as well, in which case $\text{Br}(C)$ is trivial ([8, Thm. 4.5.1.(v)]) thus $G$ admits a single genus. The second bijection is $\text{Cl}_S(G) \cong i(F(G))$ unless $G$ is anisotropic at $S$, for which it does not have to be injective [4, Prop. 4.1]; hence when $S$ is empty this bijection is guaranteed. Combining Lemma 2.2 with Lemma 2.3 these form (unless $G$ is anisotropic at $S$) an isomorphism of finite abelian groups:

$$
H^1_{\text{ét}}(O_S, G) \cong j(F(G)) \times i(F(G)).
$$

### 3. Twisted-forms

Before continuing with the classification of $G$-forms, we would like to recall the following general construction due to Giraud and prove one related Lemma. Let $R$ be a unital commutative ring. A central exact sequence of étale $R$-group schemes:

$$
1 \rightarrow A \xrightarrow{i} B \xrightarrow{\pi} C \rightarrow 1
$$

induces by étale cohomology a long exact sequence of pointed-sets ([16, III, Lem. 3.3.1]):

$$
1 \rightarrow A(R) \rightarrow B(R) \rightarrow C(R) \rightarrow H^1_{\text{ét}}(R, A) \xrightarrow{i_*} H^1_{\text{ét}}(R, B) \rightarrow H^1_{\text{ét}}(R, C)
$$

in which $C(R)$ acts "diagonally" on the elements of $H^1_{\text{ét}}(R, A)$ in the following way: For $c \in C(R)$, a preimage $X$ of $c$ under $B \rightarrow C$ is a $A$-bitorsor, i.e., $X = bA = Ab$ for some $b \in B(R')$, where $R'$ is a finite étale extension of $R$ ([16, III, 3.3.3.2]). Then given $[P] \in H^1_{\text{ét}}(R, A)$:

$$
c \ast P = P \wedge X = (P \times X)/(p^a, a^{-1}x).
$$

The exactness of (3.2) implies that $B(R) \xrightarrow{\pi} C(R)$ is surjective if and only if $\ker(i_*) = 1$. This holds true starting with any twisted form $PB$ of $B$, $[P] \in H^1_{\text{ét}}(R, A)$. 

Lemma 3.1. The following are equivalent:

1. the push-forward map \( H^1_{\text{ét}}(R, A) \xrightarrow{i_*} H^1_{\text{ét}}(R, B) \) is injective,
2. the quotient map \( P B(R) \xrightarrow{\pi} C(R) \) is surjective for any \([P] \in H^1_{\text{ét}}(R, A)\),
3. the \( C(R) \)-action on \( H^1_{\text{ét}}(R, A) \) is trivial.

Proof. Consider the exact and commutative diagram (cf. [16, III, Lem. 3.3.4])

\[
\begin{array}{ccc}
B(R) & \xrightarrow{\pi} & C(R) & \xrightarrow{i_*} & H^1_{\text{ét}}(R, A) & \xrightarrow{\cong} H^1_{\text{ét}}(R, B) \\
& & & \downarrow{\theta_P} & & \downarrow{r} \\
PB(R) & \xrightarrow{\pi} & C(R) & \xrightarrow{i'_*} & H^1_{\text{ét}}(R, PA) & \xrightarrow{\cong} H^1_{\text{ét}}(R, PB),
\end{array}
\]

where the map \( i'_* \) is obtained by applying étale cohomology to the sequence (3.1) while replacing \( B \) by the twisted group scheme \( PB \), and \( \theta_P \) is the induced twisting bijection.

(1) ⇔ (2). The map \( i_* \) is injective if and only if \( \ker(i'_*) \) is trivial for any \( A \)-torsor \( P \). By exactness of the rows, this is condition (2).

(1) ⇔ (3). By [16, Prop. III.3.3.3(iv)], \( i_* \) induces an injection of \( H^1_{\text{ét}}(R, A)/C(R) \) into \( H^1_{\text{ét}}(R, B) \). Thus \( i_* : H^1_{\text{ét}}(R, A) \to H^1_{\text{ét}}(R, B) \) is injective if and only if \( C(R) \) acts on \( H^1_{\text{ét}}(R, A) \) trivially. \qed

Following B. Conrad in [10], we denote the group of outer automorphisms of \( G \) by \( \Theta \).

Proposition 3.2 ([10, Prop. 1.5.1]). Assume \( \Phi \) spans \( X_Q \) and that \((X_Q, \Phi)\) is reduced. The inclusion \( \Theta \subseteq \text{Aut}(\text{Dyn}(G)) \) is an equality, if the root datum is adjoint or simply-connected, or if \((X_Q, \Phi)\) is irreducible and \((\mathbb{Z}\Phi^\vee)^*/\mathbb{Z}\Phi \) is cyclic.

Remark 3.3. The only case of irreducible \( \Phi \) in which the non-cyclicity in Proposition 3.2 occurs, is of type \( D_{2n}(n \geq 2) \), in which \((\mathbb{Z}\Phi^\vee)^*/\mathbb{Z}\Phi \cong (\mathbb{Z}/2)^2 \) (cf. [10, Ex. 1.5.2]).

Remark 3.4. Since \( G \) is reductive, \( \text{Aut}(G) \) is representable as an \( \mathcal{O}_S \)-group and admits the short exact sequence of smooth \( \mathcal{O}_S \)-groups (see [12, XXIV, 3.10],[11, §3]):

\[
(3.4) \quad 1 \to G^{\text{ad}} \to \text{Aut}(G) \to \Theta \to 1.
\]

Applying étale cohomology we get the exact sequence of pointed-sets:

\[
(3.5) \quad \text{Aut}(G)(\mathcal{O}_S) \to \Theta(\mathcal{O}_S) \to H^1_{\text{ét}}(\mathcal{O}_S, G^{\text{ad}}) \xrightarrow{i_*} H^1_{\text{ét}}(\mathcal{O}_S, \text{Aut}(G)) \to H^1_{\text{ét}}(\mathcal{O}_S, \Theta)
\]
in which by Lemma 3.1 the $\Theta(\mathcal{O}_S)$-action is trivial on $H^1_{\text{ét}}(\mathcal{O}_S, G^{\text{ad}})$ if and only if $i_*$ is injective, being equivalent to the surjectivity of $$(P\text{Aut}(G))(\mathcal{O}_S) = \text{Aut}(P\mathcal{G})(\mathcal{O}_S) \to \Theta(\mathcal{O}_S)$$ for all $[P] \in H^1_{\text{ét}}(\mathcal{O}_S, \Theta)$ (this action is trivial inside each genus).

It is a classical fact that $H^1_{\text{ét}}(\mathcal{O}_S, \text{Aut}(G))$ is in bijection with twisted forms of $G$ up to isomorphism (for a general statement of this correspondence, see [7, §2.2.4]). Therefore this pointed-set shall be denoted from now and on by $\text{Twist}(G)$. This bijection is done by associating any twisted form $H$ of $G$ with the $\text{Aut}(G)$-torsor $\text{Iso}(G, H)$. If $H$ is an inner-form of $G$, then $[H]$ belongs to $\text{Im}(i_*)$ in (3.5).

Sequence (3.4) splits, provided that $G$ is quasi-split (as in Section 2). Recall that $G$ admits an inner form $G_0$ which is quasi-split. Then $\text{Aut}(G_0) \cong G_0^{\text{ad}} \rtimes \Theta$ (the outer automorphisms group of the two groups are canonically isomorphic). This implies by [14, Lem. 2.6.3] the decomposition

$$\text{Twist}(G_0) = H^1_{\text{ét}}(\mathcal{O}_S, \text{Aut}(G_0)) = \prod_{[P] \in H^1_{\text{ét}}(\mathcal{O}_S, \Theta)} H^1_{\text{ét}}(\mathcal{O}_S, P(\mathcal{G}^{\text{ad}}))/\Theta(\mathcal{O}_S)$$

where the quotients are taken modulo the action (3.3) of $\Theta(\mathcal{O}_S)$ on the $P(\mathcal{G}^{\text{ad}})$-torsors. But $\text{Twist}(G_0) = \text{Twist}(G)$ and as $G_0$ is inner:

$$H^1_{\text{ét}}(\mathcal{O}_S, P(\mathcal{G}^{\text{ad}})) = H^1_{\text{ét}}(\mathcal{O}_S, P(\mathcal{G}^{\text{ad}})),$$

hence (3.6) can be rewritten as:

$$\text{Twist}(G) = \prod_{[P] \in H^1_{\text{ét}}(\mathcal{O}_S, \Theta)} H^1_{\text{ét}}(\mathcal{O}_S, P(\mathcal{G}^{\text{ad}}))/\Theta(\mathcal{O}_S).$$

The pointed-set $H^1_{\text{ét}}(\mathcal{O}_S, \Theta)$ classifies étale extensions of $\mathcal{O}_S$ whose automorphism group embeds into $\Theta$. As all $H^1_{\text{ét}}(\mathcal{O}_S, P(\mathcal{G}^{\text{ad}}))$ are finite, $\text{Twist}(G)$ is finite. Together with Lemma 2.2 we get:

**Proposition 3.5.** If $G$ is not of type A then:

$$\text{Twist}(G) \cong \prod_{[P] \in H^1_{\text{ét}}(\mathcal{O}_S, \Theta)} H^2_{\text{ét}}(\mathcal{O}_S, F(P(\mathcal{G}^{\text{ad}})))/\Theta(\mathcal{O}_S),$$

the $\Theta(\mathcal{O}_S)$-action on each component is carried by Lemma 2.2 from the one on $H^1_{\text{ét}}(\mathcal{O}_S, P(\mathcal{G}^{\text{ad}}))$, cf. (3.3).

**Corollary 3.6.** When $S = \emptyset$, i.e., over $C$, any outer form of $G$ has a unique genus on which $\Theta(\mathcal{O}_S)$ acts trivially, hence one has (including for type A):

$$\text{Twist}(G) \cong \prod_{[P] \in H^1_{\text{ét}}(\mathcal{O}_S, \Theta)} j(F(P(\mathcal{G}^{\text{ad}}))).$$
Since \( C \) is smooth, \( \mathcal{O}_S \) is a Dedekind ring and any finite étale covering of it is the normalization of \( \mathcal{O}_S \) (or of \( C \) when \( S = \emptyset \)) in some finite separable extension of \( K \), which is unramified outside \( S \). So we may look on the fundamental groups over the according extension of fields; The following is the list of all types of absolutely almost-simple \( K \)-groups (e.g., [23, p. 333]):

| Type of \( G \) | \( F(\text{G}^{\text{ad}}) \) | \( \text{Aut}(\text{Dyn}(\text{G})) \) |
|-----------------|-----------------|-----------------|
| \( ^1A_{n-1}>0 \) | \( \mu_n \) | \( \mathbb{Z}/2 \) |
| \( ^2A_{n-1}>0 \) | \( R_{L/K}^{(1)}(\mu_n) \) | \( \mathbb{Z}/2 \) |
| \( B_n, C_n, E_7 \) | \( \mu_2 \) | 0 |
| \( ^1D_n \) | \( \mu_4, n = 2k + 1 \mu_2 \times \mu_2, n = 2k \) | \( \mathbb{Z}/2 \) |
| \( ^2D_n \) | \( R_{L/K}^{(1)}(\mu_4), n = 2k + 1 \) \( R_{L/K}(\mu_2), n = 2k \) | \( \mathbb{Z}/2 \) |
| \( ^3,6D_4 \) | \( R_{L/K}^{(1)}(\mu_2) \) | \( S_3 \) |
| \( ^1E_6 \) | \( \mu_3 \) | \( \mathbb{Z}/2 \) |
| \( ^2E_6 \) | \( R_{L/K}^{(1)}(\mu_3) \) | \( \mathbb{Z}/2 \) |
| \( E_8, F_4, G_2 \) | 1 | 0 |

4. Split fundamental group

In the following we show Proposition 3.5. We start with the simple case in which \( \Theta = 0 \):

**Corollary 4.1.** If \( \mathcal{G} \) is of the type \( B_{n>1}, C_{n>1}, E_7, E_8, F_4, G_2 \), for which \( F(\text{G}^{\text{ad}}) \cong \mu_m \) there exists an isomorphism of finite abelian groups\n
\[
\text{Twist}(\mathcal{G}) \cong \text{Pic}(\mathcal{O}_S)/m \times m \text{Br}(\mathcal{O}_S).
\]

**Proof.** As \( \mathcal{G} \) is not of type A this derives from Proposition 3.5, together with the fact that \( \Theta(\mathcal{O}_S) = 0 \) whence there is a single component \( H^1_{\text{et}}(\mathcal{O}_S, \mathcal{G}^{\text{ad}}) \) on which the action of \( \Theta(\mathcal{O}_S) \) is trivial, and the description of the isomorphic group \( H^2_{\text{et}}(\mathcal{O}_S, F(\mathcal{G}^{\text{ad}})) \) is as in the split case in Lemma 2.3. \( \square \)

**Example 4.2.** Given a regular quadratic \( \mathcal{O}_S \)-form \( Q \) of rank \( 2n + 1 \), its special orthogonal group \( \mathcal{G} = SO_Q \) is smooth and connected of type \( B_n \) ([9, Thm. 1.7]). Since \( F(\mathcal{G}) = \mu_2 \) we assume \( \text{char}(K) \) is odd. According to Corollary 4.1 we then get

\[
\text{Twist}(\mathcal{G}) \cong \text{Pic}(\mathcal{O}_S)/2 \times 2 \text{Br}(\mathcal{O}_S).
\]

In case \( |S| = 1 \) and \( Q \) is split by an hyperbolic plane, an algorithm producing explicitly the inner forms of \( Q \) is provided in [3, Algorithm 1].
5. Quasi-split fundamental group

Unless $G$ is of absolute type $D_4$, $\Theta$ is either trivial or equals \{id, $\tau: A \mapsto (A^{-1})^t$\}. In the latter case, $\tau$ acts on the $G^{\text{ad}}$-torsors via $X = G^{\text{ad}}b$, where $b$ is an outer automorphism of $G$, defined over some finite étale extension of $\mathcal{O}_S$ (see (3.3)). In particular:

$$\tau \ast G^{\text{ad}} = (G^{\text{ad}} \times X)/(ga, a^{-1}x),$$

which is the opposite group $(G^{\text{ad}})^{\text{op}}$, as the action is via $a^{-1}x = x(a^t)$, ($a$ is viewed as an element of $G^{\text{ad}}$, not as an inner automorphism). Now if $\tau$ is defined over $\mathcal{O}_S$, then $\text{Aut}(G)(\mathcal{O}_S) \to \Theta(\mathcal{O}_S)$ is surjective and $(G^{\text{ad}})^{\text{op}}$ is $\mathcal{O}_S$-isomorphic to $G^{\text{ad}}$, hence as $\tau$ is the only non-trivial element in $\Theta(\mathcal{O}_S)$, the map $H^1_{\text{ét}}(\mathcal{O}_S, P\text{Aut}(G)) \to H^1_{\text{ét}}(\mathcal{O}_S, \Theta)$ is surjective for all $[P] \in H^1_{\text{ét}}(\mathcal{O}_S, \Theta)$. This implies by Remark 3.4 that $\Theta(\mathcal{O}_S)$ acts trivially on $H^1_{\text{ét}}(\mathcal{O}_S, G^{\text{ad}})$. Otherwise, $G^{\text{ad}}$ and $(G^{\text{ad}})^{\text{op}}$ represent two distinct classes in $H^1_{\text{ét}}(\mathcal{O}_S, G^{\text{ad}})$, being identified by $\Theta(\mathcal{O}_S)$.

For any extension $R$ of $\mathcal{O}_S$ and $L$ of $K$, we denote $G_R := G \otimes_{\mathcal{O}_S} R$ and $G_L := G \otimes_K L$, respectively. Let $[A_G]$ be the Tits class of the universal covering $G^{\text{sc}}$ of $G$ (see Definition 1). This class does not depend on the choice of the representation $\rho$ of $G^{\text{sc}}$, thus its notation is omitted. Recall that when $F(G)$ splits $w_G^{\text{ad}}$ defined in (2.4) coincides with $\Lambda_* \circ \delta_G$. Similarly, when $F(G) = \text{Res}_{R/\mathcal{O}_S}(\mu_m)$ (quasi-split) where $R/\mathcal{O}_S$ is finite étale, $\Lambda_* \circ \delta^R_G$ and $w_G^{\text{ad}}$ defined over $R$, coincide.

**Proposition 5.1.** Suppose $\Theta \cong \mathbb{Z}/2$ and that $F(G^{\text{ad}}) = \text{Res}_{R/\mathcal{O}_S}(\mu_m)$, $R$ is finite étale over $\mathcal{O}_S$. Then TFAE:

1. $G_R$ admits an outer automorphism,
2. $[A_G]$ is 2-torsion in $\text{Br}(R)$,
3. $\Theta(R)$ acts trivially on $H^1_{\text{ét}}(R, G^{\text{ad}}_R)$.

If, furthermore, $G$ is not of type A, or $S = \emptyset$, then these facts are also equivalent to:

4. $G$ admits an outer automorphism,
5. $[A_G]$ is 2-torsion in $\text{Br}(R)$,
6. $\Theta(\mathcal{O}_S)$ acts trivially on $H^1_{\text{ét}}(\mathcal{O}_S, G^{\text{ad}})$.

**Proof.** By Lemma 2.1 the map $\Lambda_* \circ \delta^{\text{ad}}_G : H^1_{\text{ét}}(R, G^{\text{ad}}_R) \to \text{Br}(R)$ maps $[H^{\text{ad}}]$ to $[A_R] - [A_{G_R}]$, where $[A_R]$ is the Tits class of $H^{\text{sc}}$ for a $G^{\text{ad}}_R$-torsor $H^{\text{ad}}_R$. Consider this combined with the long exact sequence obtained by applying
étale cohomology to the sequence (3.4) tensored with $R$:

\[ \text{Cl}_R(G_{fr}) \]

\[ \text{Aut}(G_R)(R) \longrightarrow \Theta_R \longrightarrow H^1_{\text{ét}}(R, G_{fr}) \xrightarrow{i_*} \text{Twist}(G_R) \]

(5.1)

\[ \text{Cl}_R(G_{fr}) \]

where \( \text{Cl}_R(G_{fr}) \) is the principal genus of \( G_{fr} \) (see [4, Prop. 3.1]) noting that \( F(G_{fr}) = \mu_m \). Being an inner form of \( G_{fr} \), \( (G_{fr})^{op} \) is obtained by a representative in \( H^1_{\text{ét}}(R, G_{fr}) \). Its \( w_{G_{fr}} \)-image: \( [A_{G_{fr}}^{op}] - [A_{G_{fr}}] \) is trivial if and only if \( A_{G_{fr}} \) is of order \( \leq 2 \) in \( m \text{Br}(R) \), which is equivalent to \( \text{Aut}(G_R)(R) \) surjecting on \( \Theta(R) \), and \( \Theta(R) \) acting trivially on \( H^1_{\text{ét}}(R, G_{fr}) \) (see at the beginning of Section 5).

If, furthermore, \( G \) is not of type \( A \) or \( S = \emptyset \), then by Lemma 2.2, together with the Shapiro Lemma we get the isomorphisms of abelian groups:

\[ H^1_{\text{ét}}(O_S, G_{fr}) \cong H^2_{\text{ét}}(O_S, F(G_{fr})) \cong H^2_{\text{ét}}(R, \mu_m) \cong H^1_{\text{ét}}(R, G_{fr}) \]

(5.2)

So if \( \Theta(R) \) acts trivially on \( H^1_{\text{ét}}(R, G_{fr}) \), then so does \( \Theta(O_S) \) on \( H^1_{\text{ét}}(O_S, G_{fr}) \). On the other hand if it does not, this implies that \( \text{Aut}(G_R)(R) \rightarrow \Theta(R) \cong \mathbb{Z}/2 \) is not surjective, thus neither is \( \text{Aut}(G)(O_S) \rightarrow \Theta(O_S) \), which is equivalent to \( \Theta(O_S) \) acting non-trivially on \( H^1_{\text{ét}}(O_S, G_{fr}) \) by Remark 3.4. Moreover, since \( i(F(G_{fr})) = i(F(G_{fr})) = m \text{Br}(R) \) (Definition 2), the identification (5.2) shows that \( \text{Cl}_R(G_{fr}) \) bijects to \( \text{Cl}_S(G_{fr}) \), whence \( [A_{G_{fr}}] \) is 2-torsion in \( m \text{Br}(R) \) if and only if \( [A_{G}] \) is.

If we wish to interpret a \( G \)-torsor as a twisted form of some basic form, we shall need to describe \( G \) first as the automorphism group of such an \( O_S \)-form.

**Example 5.2.** Let \( A \) be a division \( O_S \)-algebra of degree \( n > 2 \). Then \( G = SL(A) \) of type \( A_{n-1} > 1 \) is smooth and connected ([10, Lem. 3.3.1]). It admits a non-trivial outer automorphism \( \tau \). If the transpose anti-automorphism \( A \cong A^{op} \) is defined over \( O_S \) (extending \( \tau \) by inverting again), then \( \tau \in \text{Aut}(G)(O_S) \). Otherwise, as \( (G_{fr})^{op} \) is not \( O_S \)-isomorphic by some conjugation to \( G_{fr} = \text{PGL}(A) \), it represents a non-trivial class in \( H^1_{\text{ét}}(O_S, G_{fr}) = \text{Im}(G) \), whilst its image in \( \text{Twist}(G) \) is trivial by the inverse isomorphism \( x \mapsto x^{-1} \) defined over \( O_S \) (say, by the Cramer rule). So finally \( \Theta(O_S) \) acts trivially on \( H^1_{\text{ét}}(O_S, \text{PGL}(A)) \) if and only if \( \text{ord}(A) \leq 2 \) in \( \text{Br}(O_S) \), as Proposition 5.1 predicts.
5.1. Type $D_{2k}$. Let $A$ be an Azumaya $\mathcal{O}_S$-algebra (char($K$) $\neq 2$) of degree $2n$ and let $(f, \sigma)$ be a quadratic pair on $A$, namely, $\sigma$ is an involution on $A$ and $f : \text{Sym}(A, \sigma) = \{x \in A : \sigma(x) = x\} \to \mathcal{O}_S$ is a linear map. The scalar $\mu(a) := \sigma(a) \cdot a$ is called the multiplier of $a$. For $a \in A^\times$ we denote by $\text{Int}(a)$ the induced inner automorphism. If $\sigma$ is orthogonal, the associated similitude group is:

$$
\text{GO}(A, f, \sigma) := \{a \in A^\times : \mu(a) \in \mathcal{O}_S^\times, \ f \circ \text{Int}(a) = f\},
$$

and the map $a \mapsto \text{Int}(a)$ is an isomorphism of the projective similitude group $\text{PGO}(A, f, \sigma) := \text{GO}(A, f, \sigma)/\mathcal{O}_S^\times$ with the group of rational points $\text{Aut}(A, f, \sigma)$. Such a similitude is said to be proper if the induced automorphism of the Clifford algebra $C(A, f, \sigma)$ is the identity on the center; otherwise it is said to be improper. The subgroup $G = \text{PGO}^+(A, f, \sigma)$ of these proper similitudes is connected and adjoint, called the projective special similitude group. If the discriminant of $\sigma$ is a square in $\mathcal{O}_S^\times$, then $G$ is of type $1^1D_n$. Otherwise of type $2^1D_n$.

When $n = 2k$, in order that $\Theta$ captures the full structure of $\text{Aut}(\text{Dyn}(G))$, we would have to restrict ourselves to the two edges of simply-connected and adjoint groups (see Remark 3.3).

**Corollary 5.3.** Let $G$ be of type $2^1D_{2k}, k \neq 2$, simply-connected or adjoint. For any $[P] \in H^1_{\text{et}}(\mathcal{O}_S, \mathbb{Z}/2)$ let $R_P$ be the corresponding quadratic étale extension of $\mathcal{O}_S$. Then:

$$
\text{Twist}(G) \cong \prod_{[P] \in H^1_{\text{et}}(\mathcal{O}_S, \mathbb{Z}/2)} \text{Pic}(R_P)/2 \times \text{Br}(R_P).
$$

**Proof.** Any form $P(G^{\text{ad}})$ has Tits class $[A_{P_G}]$ of order $\leq 2$ in $2\text{Br}(R_P)$. Hence as $\Theta \cong \mathbb{Z}/2$ and $G$ is not of type A, by Proposition 5.1 $\Theta(\mathcal{O}_S)$ acts trivially on $H^1_{\text{et}}(\mathcal{O}_S, P(G^{\text{ad}}))$ for all $P$ in $\Theta(\mathcal{O}_S)$. All fundamental groups are admissible, so the Corollary statement is Proposition 3.5 together with the description of each $H^2_{\text{et}}(\mathcal{O}_S, F(P(G^{\text{ad}})))$ as in Lemma 2.3. \hfill $\square$

6. Non quasi-split fundamental group

When $F(G^{\text{ad}})$ is not quasi-split, we cannot apply the Shapiro Lemma as in (5.2) to gain control on the action of $\Theta(\mathcal{O}_S)$ on $H^1_{\text{et}}(\mathcal{O}_S, F(G^{\text{ad}}))$. Still under some conditions this action is provided to be trivial.

**Remark 6.1.** As opposed to $m\text{Br}(K)$ which is infinite for any integer $m > 1$, $m\text{Br}(\mathcal{O}_S)$ is finite. To be more precise, if $S \neq \emptyset$, $\text{Sp}(\mathcal{O}_S)$ is obtained by removing $|S|$ points from the projective curve $C$, hence $|m\text{Br}(\mathcal{O}_S)| = m^{|S|−1}$ (see the proof of [4, Cor. 3.2]). When $S = \emptyset$ we have $\text{Br}(C) = 1$. In particular, if $G$ is not of absolute type A and $F(G^{\text{ad}})$ splits over an extension $R$ such that the number of places in $\text{Frac}(R)$ which lie above
places in $S$ is 1, or when $S = \emptyset$, then $G^\text{ad}$ can posses only one genus and consequently the $\Theta(O_S)$-action on $H^2_{\text{ét}}(O_S, G^\text{ad})$ is trivial.

E. Artin in [1] calls a Galois extension $L$ of $K$ imaginary if no prime of $K$ is decomposed into distinct primes in $L$. We shall similarly call a finite étale extension of $O_S$ imaginary if no prime of $O_S$ is decomposed into distinct primes in it.

**Lemma 6.2.** If $R$ is imaginary over $O_S$ and $m$ is prime to $[R : O_S]$, then $m \text{Br}(R) = m \text{Br}(O_S)$.

**Proof.** If $S = \emptyset$ and $R/C$ is imaginary then $\text{Br}(R) = \text{Br}(O_S) = 1$. Otherwise, the composition of the induced norm $N_{R/O_S}$ with the diagonal morphism coming from the Weil restriction

$$\mathbb{G}_m, O_S \to \text{Res}_{R/O_S} (\mathbb{G}_m, R) \xrightarrow{N_{R/O_S}} \mathbb{G}_m, O_S$$

is the multiplication by $n := [R : O_S]$. It induces together with the Shapiro Lemma the maps:

$$H^2_{\text{ét}}(O_S, \mathbb{G}_m, O_S) \to H^2_{\text{ét}}(R, \mathbb{G}_m, R) \xrightarrow{N^{(2)}} H^2_{\text{ét}}(O_S, \mathbb{G}_m, O_S)$$

whose composition is the multiplication by $n$ on $H^2_{\text{ét}}(O_S, \mathbb{G}_m, O_S)$. Identifying $H^2_{\text{ét}}(*, \mathbb{G}_m)$ with $\text{Br}(*)$ and restricting to the $m$-torsion subgroups gives the composition

$$m \text{Br}(O_S) \to m \text{Br}(R) \xrightarrow{N^{(2)}} m \text{Br}(O_S)$$

being still multiplication by $n$, thus an automorphism when $n$ is prime to $m$. This means that $m \text{Br}(O_S)$ is a subgroup of $m \text{Br}(R)$. As $R$ is imaginary over $O_S$, it is obtained by removing $|S|$ points from the projective curve defining its fraction field, so $|m \text{Br}(R)| = |m \text{Br}(O_S)| = m^{|S| - 1}$ by Remark 6.1, and the assertion follows.

**Corollary 6.3.** If $F(G) = \text{Res}_{R/O_S}^{(1)}(\mu_m)$ is admissible and $R/O_S$ is imaginary, then $i(F(G)) = \ker(m \text{Br}(R) \to m \text{Br}(O_S))$ (see Definition 2) is trivial, hence $G$ admits a single genus (cf. [4, Cor. 3.2]).

### 6.1. Type $E_6$

A hermitian Jordan triple over $O_S$ is a triple $(A, \mathfrak{X}, U)$ consisting of a quadratic étale $O_S$-algebra $A$ with conjugation $\sigma$, a free of finite rank $O_S$-module $\mathfrak{X}$, and a quadratic map $U : \mathfrak{X} \to \text{Hom}_A(\mathfrak{X}^\sigma, \mathfrak{X}) : x \mapsto U_x$, where $\mathfrak{X}^\sigma$ is $\mathfrak{X}$ with scalar multiplication twisted by $\sigma$, such that $(\mathfrak{X}, U)$ is an (ordinary) Jordan triple as in [20]. In particular if $\mathfrak{X}$ is an $Albert$ $O_S$-algebra, then it is called an hermitian Albert triple. In that case the associated trace form $T : A \times A \to O_S$ is symmetric non-degenerate and it follows that the structure group of $\mathfrak{X}$ agrees with its group of norm similarities. Viewed as an $O_S$-group, it is reductive with center of rank 1 and its semisimple part, which we shortly denote $G(A, \mathfrak{X})$, is simply connected.
of type $E_6$. It is of relative type $^{1}E_6$ if $A \cong O_S \times O_S$ and of type $^{2}E_6$ otherwise.

Groups of type $^{1}E_6$ are classified by four relative types, among them only $^{1}E_{6,2}'$ has a non-commutative Tits algebra, thus being the only type in which $\Theta(O_S) \cong \mathbb{Z}/2$ may act non-trivially on $H^1_{\text{et}}(O_S, \mathcal{G}_{\text{ad}})$. More precisely, the Tits-algebra in that case is a division algebra $D$ of degree 3 (cf. [25, p. 58]) and the $\Theta(O_S)$-action is trivial if and only if $\text{ord}(D) \leq 2$ in $\text{Br}(O_S)$. But $\text{ord}(D)$ is odd, thus this action is trivial if and only if $D$ is a matrix $O_S$-algebra.

In the case of type $^{2}E_6$, one has six relative types (cf. [25, p. 59]), among which only $^{2}E_{6,2}$ has a non-commutative Tits algebra (cf. [26, p. 211]). Its Tits algebra is a division algebra of degree 3 over $R$, and its Brauer class has trivial corestriction in $\text{Br}(O_S)$. By Albert and Riehm, this is equivalent to $D$ possessing an $R/O_S$-involution.

From now on $\sim$ denotes the equivalence relation on the Brauer group which identifies the class of an Azumaya algebra with the class of its opposite.

**Corollary 6.4.** Let $G$ be of (absolute) type $E_6$. For any $[P] \in H^1_{\text{et}}(O_S, \mathbb{Z}/2)$ let $R_P$ be the corresponding quadratic étale extension of $O_S$. Then

$$\text{Twist}(G) \cong \text{Pic}(O_S)/3 \times 3 \text{Br}(O_S)/\sim \prod_{1 \neq [P]} \ker(\text{Pic}(R_P)/3 \to \text{Pic}(O_S)/3) \times (\ker(3\text{Br}(R_P) \to 3\text{Br}(O_S)))/\sim,$$

where $[P]$ runs over $H^1_{\text{et}}(O_S, \mathbb{Z}/2)$. The relation $\sim$ is trivial in the first component unless $\mathcal{G}_{\text{ad}}$ is of type $^{1}E_{6,2}'$ and is trivial in the other components unless $P(G_{\text{ad}})$ is of type $^{2}E_{6,2}''$.

**Proof.** The group $\Theta(O_S)$ acts trivially on members of the same genus, so it is sufficient to check its action on the set of genera for each type. Since $F(P(G_{\text{ad}}))$ is admissible for any $[P] \in H^1_{\text{et}}(O_S, \Theta)$, by [4, Cor. 3.2] the set of genera of each $P(G_{\text{ad}})$ bijects as a pointed-set to $i(F(P(G_{\text{ad}})))$, so the assertion is Proposition 3.5 together with Lemma 2.3. The last claims are retrieved from the above discussion on the trivial action of $\Theta(O_S)$ when $P(G_{\text{ad}})$ is not of type $^{1}E_{6,2}'$ or $^{2}E_{6,2}''$. \qed
Example 6.5. Let $C$ be the elliptic curve $Y^2Z = X^3 + ZX^2 + Z^3$ defined over $\mathbb{F}_3$. Then:

$$C(\mathbb{F}_3) = \{(1 : 0 : 1), (0 : 1 : 2), (0 : 1 : 1), (0 : 1 : 0)\}.$$  

Removing the $\mathbb{F}_3$-point $\infty = (0 : 1 : 0)$ the obtained smooth affine curve $C^\text{af}$ is $y^2 = x^3 + x + 1$. Letting $O_{(\infty)} = \mathbb{F}_3[C^\text{af}]$ we have $\text{Pic}(O_{(\infty)}) \cong C(\mathbb{F}_3)$ (e.g., [3, Ex. 4.8]). Among the affine supports of points in $C(\mathbb{F}_3) - \{\infty\}$:

$$\{(1, 0), (0, 1/2) = (0, 2), (0, 1)\},$$

only $(1, 0)$ has a trivial $y$-coordinate thus being of order 2 (according to the group law there), to which corresponds the fractional ideal $P = (x - 1, y)$ of order 2 in $\text{Pic}(O_{(\infty)})$. As $\text{Pic}(O_{(\infty)})/3 = 1$ and $\text{Br}(O_{(\infty)}) = 1$, a form of type $^1E_6$ has no non-isomorphic inner form, while a form of type $^2E_6$ may have more; for example $R = O_{(\infty)} \oplus P$ being geometric and étale cannot be imaginary over $O_{(\infty)}$, which means it is obtained by removing two points from a projective curve, thus

$$\text{ker}(3\text{Br}(R) \to 3\text{Br}(O_{(\infty)}))/\sim = 3\text{Br}(R)/\sim$$

$$= \{[R], [A], [A^{op}]\}/\sim = \{[R], [A]\},$$

hence an $O_{(\infty)}$-group of type $E_6$ splitting over $R$ admits a non-isomorphic inner form.

6.2. Type $D_{2k+1}$. Recall from Section 5.1 that an adjoint $O_S$-group $G$ of absolute type $D_n$ can be realized as $\text{PGO}^+(A, \sigma)$ where $A$ is Azumaya of degree $2n$ and $\sigma$ is an orthogonal involution on $A$. Suppose $n$ is odd. If $G$ is of relative type $^1D_n$ then $F(G) = \mu_4$ is admissible, thus not being of absolute type $A$, $\text{Cl}_S(G)$ bijects to $j(\mu_4) = \text{Pic}(O_S)/4$ and $\text{gen}(G)$ bijects to $i(\mu_4) = 4\text{Br}(O_S)$. Otherwise, when $G$ is of type $^2D_n$, then $F(G) = \text{Res}_{R/O_S}^{(1)}(\mu_4)$ where $R/O_S$ is quadratic. Again not being of absolute type $A$, $\text{Cl}_S(G) \cong j(F(G)) = \ker(\text{Pic}(R)/4 \to \text{Pic}(O_S)/4)$, but here, as $F(G)$ is not admissible, by [4, Cor. 3.2] $\text{gen}(G)$ only injects in $i(F(G)) = \ker(4\text{Br}(R) \to 4\text{Br}(O_S))$. If $R/O_S$ is imaginary, then by Lemma 6.2 $i(F(G)) = 1$. Altogether by Proposition 3.5 we get:

Corollary 6.6. Let $G$ be of (absolute) type $D_{2k+1}$. For any $[P] \in H^1_{\text{ét}}(O_S, \mathbb{Z}/2)$ let $R_P$ be the corresponding quadratic étale extension of $O_S$. Then there exists an exact sequence of pointed-sets

$$\text{Twist}(G) \hookrightarrow \text{Pic}(O_S)/4 \times \prod_{1 \neq [P]} \ker(\text{Pic}(R_P)/4 \to \text{Pic}(O_S)/4)$$

$$\times \left(4\text{Br}(O_S)/\sim \times \prod_{1 \neq [P]} (\ker(4\text{Br}(R_P) \to 4\text{Br}(O_S)))/\sim \right),$$
where $[P]$ runs over $H^1_0(\mathcal{O}_S, \mathbb{Z}/2)$ and $[A] \sim [A^{op}]$. This map surjects onto the first component. Whenever $R_P/\mathcal{O}_S$ is imaginary $\ker(4\text{Br}(R_P) \to 4\text{Br}(\mathcal{O}_S)) = 1$ and this map is a bijection.

**Example 6.7.** Let $\mathcal{O}_\{\infty\} = \mathbb{F}_q[x]$ ($q$ is odd) obtained by removing $\infty = (1/x)$ from the projective line over $\mathbb{F}_q$. Suppose $q \in 4\mathbb{N} - 1$ so $-1 \not\in \mathbb{F}_q^2$, and let $G = \text{SO}_{10}$ be defined over $\mathcal{O}_\{\infty\}$. The discriminant of an orthogonal form $Q_B$ induced by an $n \times n$ matrix $B$ is $\text{disc}(Q_B) = (-1)^{\frac{n(n-1)}{2}} \det(B)$. As $\text{disc}(Q_{110}) = -1$ is not a square in $\mathcal{O}_\{\infty\}$, $G$ is considered of type $^2D_5$. It admits a maximal torus $T$ containing five $2 \times 2$ rotations blocks $(a \ b \ c \ d)$: $a^2 + b^2 = 1$ on the diagonal. Over $R = \mathcal{O}_\{\infty\}[i]$ such block is diagonalizable; it becomes $\text{diag}(t, t^{-1})$. The obtained diagonal torus $T'_s = PT_sP^{-1}$ where $T_s = T \otimes R$ and $P$ is some invertible $10 \times 10$ matrix over $R$, is split and 5-dimensional, so may be identified with the $5 \times 5$ diagonal torus, whose positive roots are:

$$\alpha_1 = \varepsilon_1 - \varepsilon_2, \ \alpha_2 = \varepsilon_2 - \varepsilon_3, \ \alpha_3 = \varepsilon_3 - \varepsilon_4, \ \alpha_4 = \varepsilon_4 - \varepsilon_5, \ \alpha_5 = \varepsilon_4 + \varepsilon_5.$$ 

Let $g$ be the matrix differing from the $10 \times 10$ unit only at the last $2 \times 2$ block, being $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then $\det(g) = -1$ thus $\text{disc}(Q_g) = 1$ where $Q_g$ is the induced quadratic form. This means that $G' = \text{SO}(Q_g)$ of type $^1D_5$ is the unique outer form of $G$ (up to $\mathcal{O}_S$-isomorphism). Then $\Theta = \text{Aut}(\text{Dyn}(G))$ acts on $\text{Lie}(gT'_s g^{-1})$ by mapping the last block $\begin{pmatrix} 0 & \ln(t) \\ -\ln(t) & 0 \end{pmatrix}$ to $\begin{pmatrix} 0 & -\ln(t) \\ \ln(t) & 0 \end{pmatrix}$ and so swapping the above two roots $\alpha_4$ and $\alpha_5$. Since $\mathcal{O}_\{\infty\}$ and $R$ are PIDs, their Picard groups are trivial. As only one point was removed in both domains also $\text{Br}(\mathcal{O}_\{\infty\}) = \text{Br}(R) = 1$. We remain with only the two above forms, i.e., $\text{Twist}(G) = \{ [G], [G'] \}$.

The same holds for $\mathcal{O}_S = \mathbb{F}_q[x, x^{-1}]$: again it is a UFD thus $G = \text{SO}_{10}$ defined over it still possesses only one non-isomorphic outer form. As $\mathcal{O}_S$ is obtained by removing two points from the projective $\mathbb{F}_q$-line, this time $4\text{Br}(\mathcal{O}_S)$ is not trivial, but still equals $4\text{Br}(\mathcal{O}_S)$, so: $\ker(4\text{Br}(R) \to 4\text{Br}(\mathcal{O}_S)) = 1$.

**6.3. Type $D_4$.** This case deserves a special regard as $\Theta$ is the symmetric group $S_3$ when $G$ is adjoint or simply-connected (cf. Proposition 3.2). Suppose $C$ is an Octonion $\mathcal{O}_S$-algebra with norm $N$. For any similitude $t$ of $N$ (see Section 5.1) there exist similitudes $t_2$ and $t_3$ such that

$$t_1(xy) = t_2(x) \cdot t_3(y) \ \forall x, y \in C.$$ 

Then the mappings:

$$\alpha : [t_1] \mapsto [t_2], \ \beta : [t_1] \mapsto [t_3]$$

(6.1)
where $\hat{t}(x) := \mu(t)^{-1} \cdot t(x)$, satisfy $\alpha^2 = \beta^3 = \text{id}$ and generate $\Theta = \text{Out}(\text{PGO}^+(N)) \cong S_3$.

$$1^1D_4$$

Having three conjugacy classes, there are three classes of outer forms of $G$ (cf. [10, p. 253]), which we denote as usual by $1^1D_4$, $2^2D_4$ and $3^3D_4$. The groups in the following table are the generic fibers of these outer forms, $L/K$ is the splitting extension of $F(G^{\text{ad}})$ (note that in the case $6^2D_4$ $L/K$ is not Galois):

| Type of $G$ | $F(G^{\text{ad}})$ | $[L : K]$ |
|-------------|-------------------|----------|
| $1^1D_4$    | $\mu_2 \times \mu_2$ | 1        |
| $2^2D_4$    | $R_{L/K}(\mu_2)$   | 2        |
| $3^3D_4$    | $F_{L/K}^{(1)}(\mu_2)$ | 3        |

Starting with $G$ of type $1^1D_4$, one sees that $F(P(G^{\text{ad}}))$ (splitting over some corresponding extension $R/O_S$) is admissible for any $[P] \in H^1_{\text{ét}}(O_S, \Theta)$, thus according to Lemma 2.3

$$\forall [P] \in H^1_{\text{ét}}(O_S, \Theta) : H^2_{\text{ét}}(O_S, F(P(G^{\text{ad}}))) \cong j(F(P(G^{\text{ad}}))) \times i(F(P(G^{\text{ad}}))).$$

The action of $\Theta(O_S)$ is trivial on the first factor, classifying torsors of the same genus, so we concentrate on its action on $i(F(P(G^{\text{ad}})))$. Since $\Theta \neq \mathbb{Z}/2$ we cannot use Proposition 5.1, but we may still imitate its arguments:

The group $\Theta(O_S)$ acts non-trivially on $H^1_{\text{ét}}(O_S, P(G^{\text{ad}}))$ for some $[P] \in H^1_{\text{ét}}(O_S, \Theta)$ if it identifies two non isomorphic torsors of $P(G^{\text{ad}})$. The Tits algebras of their universal coverings lie in $(2\text{Br}(O_S))^2$ if $P(G^{\text{ad}})$ is of type $1^1D_4$, i.e., if $P$ belongs to the trivial class in $H^1_{\text{ét}}(O_S, \Theta)$, in $2\text{Br}(R)$ for $R$ quadratic étale over $O_S$ if $P(G^{\text{ad}})$ is of type $1^1D_4$, i.e., if $[P] \in 2H^1_{\text{ét}}(O_S, \Theta)$, and in $\ker(2\text{Br}(R) \to 2\text{Br}(O_S))$ for a cubic étale extension $R$ of $O_S$ if $P(G^{\text{ad}})$ is one of the types $3^3D_4$, i.e., if $[P] \in 3H^1_{\text{ét}}(O_S, \Theta)$. Therefore these Tits algebras must be 2-torsion, which means that the two torsors are $O_S$-isomorphic in the first case and $R$-isomorphic in the latter three. If $F(P(G^{\text{ad}}))$ is quasi-split this means (by the Shapiro Lemma) that $\Theta(O_S)$ acts trivially on $H^1_{\text{ét}}(O_S, P(G^{\text{ad}}))$. If $F(P(G^{\text{ad}}))$ is not quasi-split, according to Corollary 6.3 if $R$ is imaginary over $O_S$ then $i(F(P(G^{\text{ad}}))) = 1$.

If a quadratic form $Q$ has a trivial discriminant on a vector space $V$, the Tits algebras of the group are $\text{End}(V)$ and the two components $C_+(Q)$, $C_-(Q)$ of the even Clifford algebra of $Q$, and the triality automorphism cyclically permutes those three. More generally, if the group is represented as $\text{PGO}^+(A, \sigma)$ for some orthogonal involution of trivial discriminant on a central simple algebra $A$ of degree 8, triality automorphisms permute $A$.
and the two components of the Clifford algebra $\mathbf{C}(A, \sigma)$; Altogether we finally get:

**Corollary 6.8.** Let $G$ be of (absolute) type $D_4$ being simply-connected or adjoint. For any $[P] \in H^1_{\text{ét}}(\mathcal{O}_S, \Theta)$ let $R_P$ be the corresponding étale extension of $\mathcal{O}_S$. Then:

$$\text{Twist}(G) \cong \left( \text{Pic}(\mathcal{O}_S)/2 \times 2\text{Br}(\mathcal{O}_S) \right)^2 \coprod_{1 \neq [P] \in 2H^1_{\text{ét}}(\mathcal{O}_S, \Theta)} \text{Pic}(R_P)/2 \times 2\text{Br}(R_P) \coprod_{1 \neq [P] \in 3H^1_{\text{ét}}(\mathcal{O}_S, \Theta)} \ker(\text{Pic}(R_P)/2 \to \text{Pic}(\mathcal{O}_S)/2) \times (\ker(2\text{Br}(R_P) \to 2\text{Br}(\mathcal{O}_S)))/\Theta(\mathcal{O}_S).$$

If $R_P$ is imaginary over $\mathcal{O}_S$, then $\ker(2\text{Br}(R_P) \to 2\text{Br}(\mathcal{O}_S)) = 1$.

7. The anisotropic case

Now suppose that $G$ does admit a twisted form such that the generic fiber of its universal covering is anisotropic at $S$. As previously mentioned, such group must be of absolute type $\mathbf{A}$ and $S \neq \emptyset$. Over a local field $k$, an outer form of a group of type $\mathbf{A}$ which is anisotropic, must be the special unitary group arising by some hermitian form $h$ in $r$ variables over a quadratic extension of $k$ or over a quaternion $k$-algebra ([27, §4.4]).

A unitary $\mathcal{O}_S$-group is $U(B, \sigma) := \text{Iso}(B, \sigma)$ where $B$ is a non-split quaternion Azumaya defined over an étale quadratic extension $R$ of $\mathcal{O}_S$ and $\sigma$ is a unitary involution on $B$, i.e., whose restriction to the center $R$ is not the identity. The special unitary group is the kernel of the reduced norm:

$$\text{SU}(B, \sigma) := \ker(\text{Nrd} : U(B, \tau) \to GL_1(R)).$$

These are of relative type $2\mathbf{C}_2m$ ($m \geq 2$) ([27, §4.4]) and isomorphic over $R$ to type $\mathbf{A}_{2m-1}$.

So in order to determine exactly when $H^1_{\text{ét}}(\mathcal{O}_S, GL_1(A))$ does not vanish, we may restrict ourselves to $\mathcal{O}_S$-groups whose universal covering is either $\mathbf{SL}_1(A)$ or $\mathbf{SU}(B, \sigma)$. In the first case, the reduced norm applied to the units of $A$ forms the short exact sequence of smooth $\mathcal{O}_S$-groups:

$$1 \to \mathbf{SL}_1(A) \to \mathbf{GL}_1(A) \xrightarrow{\text{Nrd}} \mathbb{G}_m \to 1.$$ (7.1)

Then étale cohomology gives rise to the long exact sequence:

$$1 \to \mathcal{O}_S^\times / \text{Nrd}(A^\times) \to H^1_{\text{ét}}(\mathcal{O}_S, \mathbf{SL}_1(A)) \xrightarrow{\iota} H^1_{\text{ét}}(\mathcal{O}_S, \mathbf{GL}_1(A)) \xrightarrow{\text{Nrd}} H^1_{\text{ét}}(\mathcal{O}_S, \mathbb{G}_m) \cong \text{Pic}(\mathcal{O}_S)$$ (7.2)
in which \( \text{Nrd}_* \) is surjective since \( \text{SL}_1(A) \) is simply-connected and \( \mathcal{O}_S \) is of Douai-type (see above).

**Definition 4.** We say that the local-global Hasse principle holds for \( G \) if 
\[ h_S(G) = |\text{Cl}_S(G)| = 1. \]

Thus the Hasse principle says that an \( \mathcal{O}_S \)-group is \( \mathcal{O}_S \)-isomorphic to \( G \) if and only if it is \( K \)-isomorphic to it. This is automatic for simply-connected groups which are not of type A or when \( S = \emptyset \) for which by Lemma 2.2 \( H^1_{\text{et}}(\mathcal{O}_S, G) \cong H^2_{\text{et}}(\mathcal{O}_S, F(G)) \) is trivial.

**Corollary 7.1.** Let \( G = \text{SL}_1(A) \) where \( A \) is a quaternion \( \mathcal{O}_S \)-algebra.

1. If \( \text{Nrd} : A^\times \to \mathcal{O}_S^\times \) is not surjective, then the Hasse principle does not hold for \( G \).
2. If the generic fiber \( G \) is isotropic at \( S \), then \( \text{Twist}(G) \) is in bijection as a pointed-set with the abelian group \( \text{Pic}(\mathcal{O}_S)/2 \times 2\text{Br}(\mathcal{O}_S) \).

**Proof.** (1). The generic fiber \( \text{SL}_1(A) \) is simply-connected thus due to Harder \( H^1(K, \text{SL}_1(A)) = 1 \), which indicates that \( \text{SL}_1(A) \) admits a single genus (cf. Section 2), i.e., \( H^1_{\text{et}}(\mathcal{O}_S, \text{SL}_1(A)) \) is equal to \( \text{Cl}_S(\text{SL}_1(A)) \). By the exactness of sequence (7.2), \( H^1_{\text{et}}(\mathcal{O}_S, \text{SL}_1(A)) \) cannot vanish if \( \text{Nrd}(A^\times) \neq \mathcal{O}_S^\times \).

(2). Being of type \( A_1 \), \( G = \text{SL}_1(A) \) does not admit a non-trivial outer form, which implies that \( \text{Twist}(G) = H^1_{\text{et}}(\mathcal{O}_S, G_{\text{ad}}) \). The short exact sequence of the universal covering of \( G_{\text{ad}} = \text{PGL}_1(A) \) with fundamental group \( \mu_2 \), induces the long exact sequence (cf. (2.2)):
\[ H^1_{\text{et}}(\mathcal{O}_S, \text{SL}_1(A)) \to H^1_{\text{et}}(\mathcal{O}_S, \text{PGL}_1(A)) \xrightarrow{\delta_{G_{\text{ad}}}} H^2_{\text{et}}(\mathcal{O}_S, \mu_2) \]
in which since \( H^1_{\text{et}}(\mathcal{O}_S, \text{SL}_1(A)) \) is trivial (due to strong approximation when \( G \) is isotropic at \( S \)), the rightmost term is isomorphic by Lemma 2.3 to \( \text{Pic}(\mathcal{O}_S)/2 \times 2\text{Br}(\mathcal{O}_S) \). \( \square \)

**Example 7.2.** Let \( C \) be the projective line defined over \( \mathbb{F}_3 \) and \( S = \{t, t^{-1}\} \). Then \( K = \mathbb{F}_3(t) \) and \( \mathcal{O}_S = \mathbb{F}_3[t, t^{-1}] \). For the quaternion \( \mathcal{O}_S \)-algebra \( A = (i^2 = -1, j^2 = -t)\mathcal{O}_S \) we get:
\[ \forall x, y, z, w \in \mathcal{O}_S : \text{Nrd}(x + yi + zj + wk) = x^2 + y^2 + t(z^2 + w^2) \]
which shows that \( \text{Nrd}(A^\times) = \mathcal{O}_S^\times = \mathbb{F}_3^\times \cdot t^n, n \in \mathbb{Z} \). As \( \mathcal{O}_S \) is a UFD, the Hasse principle holds for \( G = \text{SL}_1(A) \), though its generic fiber \( G \cong \text{Spin}_q \) for \( q(x, y, z) = x^2 + y^2 + tz^2 \) is anisotropic at \( S \) (cf. [19, Lemma 6]). We have two distinct classes in \( \text{Twist}(G) \), namely, \([G] \) and \([G_{\text{op}}] \). For \( A = (-1, -1)\mathcal{O}_S \), however, we get:
\[ \text{Nrd}(x + yi + zj + wk) = x^2 + y^2 + z^2 + w^2 \]
which does not surject on \( \mathcal{O}_S^\times \) as \( t \notin \text{Nrd}(A^\times) \), so the Hasse principle does not hold for \( \text{SL}_1(A) \).
Similarly, applying étale cohomology to the exact sequence of smooth \( \mathcal{O}_S \)-groups:

\[
1 \to \mathbf{SU}(B, \sigma) \to \mathbf{U}(B, \sigma) \xrightarrow{\text{Nrd}} \mathbf{GL}_1(R) \to 1
\]

induces the exactness of:

\[
1 \to R^\times / \text{Nrd}(\mathbf{U}(B, \sigma)(\mathcal{O}_S)) \to H^1_{\text{ét}}(\mathcal{O}_S, \mathbf{SU}(B, \sigma)) \\
\quad \to H^1_{\text{ét}}(\mathcal{O}_S, \mathbf{U}(B, \sigma)) \xrightarrow{\text{Nrd}} H^1_{\text{ét}}(\mathcal{O}_S, \text{Aut}(R)).
\]

Let \( A = D(\mathbf{B}, \sigma) \) be the discriminant algebra. If \( R \) splits, namely, \( R \cong \mathcal{O}_S \times \mathcal{O}_S \), then \( B \cong A \times A^{\text{op}} \) and \( \sigma \) is the exchange involution. Then \( \mathbf{U}(B, \sigma) \cong \mathbf{GL}_1(A) \) and \( \mathbf{SU}(B, \sigma) \cong \mathbf{SL}_1(A) \), so we are back in the previous situation.

**Corollary 7.3.** If \( \mathbf{U}(B, \sigma)(\mathcal{O}_S) \xrightarrow{\text{Nrd}} R^\times \) is not surjective then the Hasse-principle does not hold for \( \mathbf{SU}(B, \sigma) \).

8. In the Zariski topology

A \( \mathcal{G} \)-torsor \( P \) is Zariski, if the twisted form \( P^\mathcal{G} \) is generically and locally everywhere away of \( S \) isomorphic to \( \mathcal{G} \), i.e., if it belongs to the principal genus of \( \mathcal{G} \) (see Section 2). Let \( \mathcal{G}_0 \) be a quasi-split semisimple \( \mathcal{O}_S \)-group with an almost-simple generic fiber. The continuous morphism between the categories of open subsets of \( \mathcal{O}_S: (\mathcal{O}_S)_{\text{ét}} \to (\mathcal{O}_S)_{\text{Zar}} \) results, given a variety \( X \) defined over \( \mathcal{O}_S \), in the opposite inclusion of cohomology sets \( H^r_{\text{Zar}}(\mathcal{O}_S, X) \subseteq H^r_{\text{ét}}(\mathcal{O}_S, X) \) for all \( r > 0 \). The restriction of the decomposition (3.6)

\[
(8.1) \quad \text{Twist}(\mathcal{G}_0) \cong H^1_{\text{ét}}(\mathcal{O}_S, \text{Aut}(\mathcal{G}_0)) \cong \prod_{[P]} H^1_{\text{ét}}(\mathcal{O}_S, P(\mathcal{G}_0^{\text{ad}}))/\Theta(\mathcal{O}_S)
\]

to Zariski torsors gives (compare with [18, p. 181]):

\[
(8.2) \quad \text{Twist}_{\text{Zar}}(\mathcal{G}_0) \cong H^1_{\text{Zar}}(\mathcal{O}_S, \text{Aut}(\mathcal{G}_0)) \cong H^1_{\text{Zar}}(\mathcal{O}_S, \mathcal{G}_0^{\text{ad}})/\Theta(\mathcal{O}_S).
\]

But as aforementioned, \( H^1_{\text{Zar}}(\mathcal{O}_S, \mathcal{G}_0^{\text{ad}}) \) is equal to the principal genus of \( \mathcal{G}_0^{\text{ad}} \) on which the action of \( \Theta(\mathcal{O}_S) \) is trivial, hence (8.2) refines to:

\[
(8.3) \quad \text{Twist}_{\text{Zar}}(\mathcal{G}_0) \cong H^1_{\text{Zar}}(\mathcal{O}_S, \mathcal{G}_0^{\text{ad}}).
\]

Moreover, restricting the bijection \( H^1_{\text{ét}}(\mathcal{O}_S, \mathcal{G}_0^{\text{ad}}) \cong H^2_{\text{ét}}(\mathcal{O}_S, F(\mathcal{G}_0^{\text{ad}})) \) (Lemma 2.2) to the Zariski topology, \( H^1_{\text{Zar}}(\mathcal{O}_S, \mathcal{G}_0^{\text{ad}}) \) can be replaced with \( H^2_{\text{Zar}}(\mathcal{O}_S, F(\mathcal{G}_0^{\text{ad}})) \). All twisted forms of \( \mathcal{G}_0 \) in the Zariski topology being \( K \)-isomorphic are isotropic, so this time this includes groups of type A. Suppose \( F(\mathcal{G}_0^{\text{ad}}) \) is admissible, splitting over an étale extension \( R \) of \( \mathcal{O}_S \). Then \( H^2_{\text{Zar}}(\mathcal{O}_S, \mathcal{G}_m) \cong \text{Pic}(\mathcal{O}_S) \) while as \( R \) is locally factorial \( H^2_{\text{Zar}}(R, \mathcal{G}_m) \) is trivial ([8, Rem. 3.5.1]) thus \( i(F(\mathcal{G}_0)) \) as well (see Definition 2). Hence
similarly as was done for $H^2_{\text{ét}}(\mathcal{O}_S, F(G_0^{\text{ad}}))$ in Lemma 2.3, we get that $H^2_{\text{Zar}}(\mathcal{O}_S, F(G_0^{\text{ad}})) \cong j(F(G_0))$.

**Corollary 8.1.** Let $G_0$ be a semisimple $\mathcal{O}_S$-group with an almost-simple generic fiber and an admissible fundamental group. Then: $\text{Twist}_{\text{Zar}}(G_0) \cong j(F(G_0))$.

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