A Non-Newtonian Noether’s Symmetry Theorem*

Delfim F. M. Torres
delfim@ua.pt
R&D Unit CIDMA, Department of Mathematics,
University of Aveiro, 3810-193 Aveiro, Portugal

Abstract

The universal principle obtained by Emmy Noether in 1918, asserts that the invariance of a variational problem with respect to a one-parameter family of symmetry transformations implies the existence of a conserved quantity along the Euler–Lagrange extremals. Here we prove Noether’s theorem for the recent non-Newtonian calculus of variations. The proof is based on a new necessary optimality condition of DuBois–Reymond type.

Keywords: non-Newtonian calculus of variations; Euler–Lagrange extremals; DuBois–Reymond condition; Erdmann condition; symmetry Noether’s theorem.

MSC 2020: 26A24; 49K05; 49S05.

1 Introduction

Recently, it has been shown that the non-Newtonian/multiplicative calculus introduced by Grossman and Katz in [12] is very useful in some problems of actuarial science, finance, economics, biology, demography, pattern recognition, signal processing, thermostatistics, and quantum information theory [2, 6, 16, 19, 21]. Additionally, a mathematical problem, which is difficult or impossible to solve in one calculus, can be easily revealed through another calculus [5, 32]. The literature on non-Newtonian calculus is rich and accessible, and we assume the reader to be familiar with it. If this is not the case, we refer, e.g., to [5, 11, 21]. Here we follow the notations and the results published open access in [31]. We just recall: the basic four operations, $x ⊕ y = x \cdot y;\quad x ⊖ y = x/y;\quad x ⊙ y = x \ln(y);\quad x ◊ y = x^{1/\ln(y)}, \quad y \neq 1$; the fact that $(\mathbb{R}^+, \oplus, \odot)$ is a field; and that with such arithmetic a real analysis is available, together with all fundamental topological properties for the non-Newtonian metric space and a full calculus, including non-Newtonian differential and integral equations [20, 25, 17, 13, 21, 24]. We also refer the reader to the recent book [4].

Motivated by applications in economics, physics, and biology, in which the variational functionals to be minimized are not of the standard form but given by a product or a quotient of integral functionals, the author has recently introduced the non-Newtonian calculus of variations, which allows to solve such multiplicative variational problems in a rather standard way, using non-Newtonian variations of the form $x ⊕ \epsilon \odot h$: see [31]. Let $(t, x, v) \mapsto L(t, x, v)$ be a given $C^2(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+; \mathbb{R}^+)$ function, called the Lagrangian. The fundamental problem of the non-Newtonian calculus of variations consists to minimize the integral functional

$$\mathcal{F}[x(\cdot)] = \int_{a}^{b} L(t, x(t), \dot{x}(t)) \, dt$$

over the class

$$\mathcal{X} := \left\{ x \in C^2(\mathbb{R}^+; \mathbb{R}^+) : x(a) = \alpha, \quad x(b) = \beta, \quad x(t) > 0 \quad \forall \quad t \in [a, b] \right\},$$

*This is a preprint of a paper whose final and definite form is published in 'Applicable Analysis', Print ISSN: 0003-6811, Online ISSN: 1563-504X. Submitted 02-Aug-2021, Accepted for publication 22-Nov-2021.
where \( \tilde{x}(t) \) is the non-Newtonian derivative of function \( x \) at (positive) time \( t \) and \( \int \) is the non-Newtonian (multiplicative) integral \([31]\). The problem is solved with the help of the Euler–Lagrange differential equation,

\[
\frac{d}{dt} \left[ \frac{\partial L}{\partial v}(t, x(t), \tilde{x}(t)) \right] = \frac{\partial L}{\partial x}(t, x(t), \tilde{x}(t)),
\]

which is a first-order necessary optimality condition \([31]\). Each solution of (3) is called an Euler–Lagrange extremal.

In 1918, Emmy Noether (1882–1935) established a general theorem asserting that the invariance of a variational integral functional under a family of transformations depending smoothly on a real parameter \( s \), implies the existence of a conserved quantity along the Euler–Lagrange extremals \([17, 18]\). As corollaries, all the conservation laws known to classical mechanics are easily obtained. For a survey of Noether’s theorem and its generalizations, see \([1, 8, 14, 23]\). In the present article, we show that Noether’s theorem is still valid in the non-Newtonian setting (Theorem 5). In order to prove such result, we first prove a non-Newtonian DuBois–Reymond condition (Theorem 1). In the particular case when the Lagrangian is autonomous, that is, when the Lagrangian \( L \) does not depend on the independent variable \( t \) (i.e., \( L(t, x, v) = L(x, v) \)), we obtain a second Erdmann type equation (Corollary 2). Two illustrative examples of application of the obtained non-Newtonian Noether symmetry theorem are given: non-Newtonian conservation of energy, when the problem is invariant under multiplicative space translations \( t \mapsto t \oplus s = s \cdot t \) (Example 6); non-Newtonian conservation of momentum, when the problem is invariant under multiplicative space translations \( x \mapsto x \oplus s = s \cdot x \) (Example 7).

2 The DuBois–Reymond Condition

The importance of DuBois–Reymond optimality condition, and its relation with Noether’s symmetry theorem, is well known in the literature of the calculus of variations \([22, 29]\). We begin by proving a non-Newtonian DuBois–Reymond necessary optimality condition.

**Theorem 1** (DuBois–Reymond condition). If \( x(t), t \in [a, b] \), is a solution to problem

\[
\mathcal{F}[x] = \int_a^b L(t, x(t), \tilde{x}(t)) \, dt \longrightarrow \min_{x \in \mathcal{X}}
\]

with \( \mathcal{X} \) as in \([2]\), then \( x(t) \) satisfies the DuBois–Reymond condition

\[
\frac{\partial L}{\partial t}(t, x(t), \tilde{x}(t)) = \frac{d}{dt} \left\{ L(t, x(t), \tilde{x}(t)) \oplus \frac{\partial L}{\partial v}(t, x(t), \tilde{x}(t)) \ominus \tilde{x}(t) \right\}
\]

for all \( t \in [a, b] \).

**Proof.** We show the DuBois–Reymond first-order necessary optimality condition \([4]\) as a consequence of the Euler–Lagrange equation \([3]\). Let \( x(t), t \in [a, b] \), be a minimizer to problem \([7]\). For simplicity, in what follows we omit the arguments, being clear that all partial derivatives of the Lagrangian \( L \) are evaluated at \((t, x(t), \tilde{x}(t))\). A direct computation shows that the right-hand side of \([4]\) is given by

\[
\frac{d}{dt} \left\{ L \ominus \frac{\partial L}{\partial v} \oplus \tilde{x}(t) \right\} = \frac{\partial L}{\partial t} \ominus \frac{\partial L}{\partial v} \circ \tilde{x}(t) \ominus \frac{\partial L}{\partial v} \oplus \tilde{x}(t) \ominus \frac{\partial L}{\partial v} \oplus \tilde{x}(t)
\]

\[
= \frac{\partial L}{\partial t} \oplus \left[ \frac{\partial L}{\partial t} \ominus \frac{\partial L}{\partial v} \right] \oplus \tilde{x}(t).
\]

Since \( x(\cdot) \) is a solution of \([7]\), it must satisfy the Euler–Lagrange equation \( \frac{\partial L}{\partial x} \ominus \frac{d}{dt} \left( \frac{\partial L}{\partial v} \right) = 1 \), and \([4]\) simplifies to

\[
\frac{d}{dt} \left\{ L \ominus \frac{\partial L}{\partial v} \oplus \tilde{x}(t) \right\} = \frac{\partial L}{\partial t}.
\]
which proves the intended result.

In the autonomous case, when the Lagrangian \( L \) does not depend on the independent variable \( t \), that is, \( L(t, x, v) = L(x, v) \), we get, as an immediate corollary of our Theorem \( \text{[1]} \) a non-Newtonian Erdmann necessary optimality condition.

**Corollary 2** (Erdmann condition). If \( x(t), \ t \in [a, b] \), is a solution to the autonomous variational problem

\[
F[x] = \int_a^b L(x(t), \overline{x}(t)) \, dt \longrightarrow \min_{x \in \mathcal{X}},
\]

where \( \mathcal{X} \) is defined in \( [2] \), then \( x(t) \) satisfies the Erdmann condition

\[
L(x(t), \overline{x}(t)) \odot \frac{\partial L}{\partial v}(x(t), \overline{x}(t)) \odot \overline{x}(t) = \text{constant}
\]

for all \( t \in [a, b] \).

**Proof.** Since the Lagrangian \( L \) does not depend explicitly on \( t \), one has \( \frac{\partial L}{\partial t} = 1 \) and \( \text{[1]} \) reduces to \( \text{[7]} \).

Corollary \( \text{[2]} \) already shows the importance of DuBois–Reymond condition to establish **constants of motion**, that is, quantities, like the one given by the left-hand side of equality \( \text{[7]} \), that are conserved along the solutions of a problem of the calculus of variations. The constant of motion given by the Erdmann condition is obtained under the assumption that the Lagrangian is autonomous, that is, time-invariant (i.e., there exists a symmetry of the problem under time translations). Such relation between the invariance of the problem (or the existence of symmetry transformations) and the existence of a constant of motion is the subject of our next section.

## 3 Noether’s Symmetry Theorem

Before formulating the Noether theorem for non-Newtonian extremals of the calculus of variations, we need to introduce a notion of invariance. We require the symmetry transformation to leave the problem invariant up to first order terms in the parameter \( s \) and up to exact differentials. Such exact differentials are known in physics as **gauge-terms** \( \text{[23, 28]} \).

**Definition 3** (invariance). Consider a \( s \)-parameter family of \( C^1 \)-transformations

\[
(t, x) \longrightarrow (T(t, x, \overline{x}, s), X(t, x, \overline{x}, s))
\]

that reduce to the identity for \( s = 1 \), that is,

\[
T(t, x, v, 1) = t, \quad X(t, x, v, 1) = x,
\]

for any \( t, x, v \in \mathbb{R}^+ \). We say that the integral functional \( \text{[1]} \) is invariant under the one-parameter family of \( C^1 \)-transformations \( \text{[8]} \) up to the gauge-term \( \Phi(t, x, \overline{x}) \) if, and only if,

\[
\frac{d}{dt} \Phi(t, x(t), \overline{x}(t)) = \left\{ \frac{d}{ds} \left\{ \begin{split} T(t, x(t), \overline{x}(t), s), X(t, x(t), \overline{x}(t), s), \\
\frac{dX}{dt}(t, x(t), \overline{x}(t), s) \odot \frac{dT}{dt}(t, x(t), \overline{x}(t), s) \end{split} \right\} \right\}_{s=1}
\]

for all \( x(\cdot) \in C^2([a, b]; \mathbb{R}^+) \).
Remark 4. Similarly to the 1918 paper of Emmy Noether \[17, 18\], in our Definition 3 we consider that the derivatives of the trajectories \(x\) may also occur in the parameter family of transformations. This possibility has been widely forgotten in the literature of the calculus of variations, being, however, very powerful from the point of view of optimal control \[20, 27, 30\]. The subject of non-Newtonian optimal control will be addressed in a forthcoming publication.

Theorem 5 (Noether’s theorem). If (1) is invariant under the one-parameter family of time-space transformations

\[
(t, x) \rightarrow (T(t, x, \bar{x}, s), X(t, x, \bar{x}, s))
\]

up to the gauge-term \(\Phi(t, x, \bar{x})\), then

\[
\left[ \frac{\partial L}{\partial \dot{x}} (t, x(t), \bar{x}(t)) \otimes \frac{\partial}{\partial s} T(t, x(t), \bar{x}(t), s) \right]_{s=1} \otimes \Phi(t, x(t), \bar{x}(t)) = 0
\]

is constant in \(t \in [a, b]\) along any solution \(x(\cdot) \in C^2([a, b]; \mathbb{R}^+\) of the variational problem (1).

Proof. We know that if \(x\) is a solution of (1), then it satisfies the Euler–Lagrange equation (3) and the DuBois–Reymond condition (4). Having in mind that for \(s = 1\) we have the identity transformation, \(T(t, x, \bar{x}, 1) = t, X(t, x, \bar{x}, 1) = x\), condition (3) yields

\[
\frac{d}{dt} \Phi(t, x(t), \bar{x}(t)) = \frac{\partial L}{\partial t} (t, x(t), \bar{x}(t)) \otimes \frac{\partial}{\partial s} T(t, x(t), \bar{x}(t), s) \bigg|_{s=1} \tag{11}
\]

\[
\otimes \frac{\partial L}{\partial x} (t, x(t), \bar{x}(t)) \otimes \frac{\partial}{\partial s} X(t, x(t), \bar{x}(t), s) \bigg|_{s=1}
\]

\[
\otimes \frac{\partial L}{\partial v} (t, x(t), \bar{x}(t)) \otimes \left( \frac{d}{dt} \frac{\partial}{\partial s} X(t, x(t), \bar{x}(t), s) \bigg|_{s=1} \right)
\]

\[
\otimes L(t, x(t), \bar{x}(t)) \otimes \frac{d}{dt} \frac{\partial}{\partial s} T(t, x(t), \bar{x}(t), s) \bigg|_{s=1} .
\]

From (3) one can write

\[
\frac{\partial L}{\partial x} (t, x(t), \bar{x}(t)) \otimes \frac{\partial}{\partial s} X(t, x(t), \bar{x}(t), s) \bigg|_{s=1} \tag{12}
\]

while from (1) one gets

\[
\frac{\partial L}{\partial t} (t, x(t), \bar{x}(t)) \otimes \frac{\partial}{\partial s} T(t, x(t), \bar{x}(t), s) \bigg|_{s=1} \tag{13}
\]

\[
\otimes \frac{\partial L}{\partial v} (t, x(t), \bar{x}(t)) \otimes \bar{x}(t) \otimes \frac{d}{dt} \frac{\partial}{\partial s} T(t, x(t), \bar{x}(t), s) \bigg|_{s=1}
\]

\[
= \frac{d}{dt} \left( L(t, x(t), \bar{x}(t)) \otimes \frac{\partial L}{\partial v} (t, x(t), \bar{x}(t)) \otimes \bar{x}(t) \right) \otimes \frac{\partial}{\partial s} T(t, x(t), \bar{x}(t), s) \bigg|_{s=1} .
\]
Substituting (12) and (13) into (11),

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial v}(t, x(t), \bar{x}(t)) \otimes \frac{\partial}{\partial s} X(t, x(t), \bar{x}(t), s) \right) \bigg|_{s=1} \otimes \Phi(t, x(t), \bar{x}(t))
\]

\[
\oplus \left( L(t, x(t), \bar{x}(t)) \otimes \frac{\partial L}{\partial v}(t, x(t), \bar{x}(t)) \otimes \bar{x}(t) \right) \bigg|_{s=1} \otimes \frac{\partial}{\partial s} T(t, x(t), \bar{x}(t), s)
\]

\[
= 1
\]

and the intended conclusion is obtained.

4 Illustrative Examples

Our first example is the analog in classical mechanics to conservation of energy and gives an alternative proof to the constant of motion (7).

Example 6 (conservation of energy). Let us consider the autonomous problem of the calculus of variations (6). It is easy to see that this problem is invariant, in the sense of Definition 3, under the symmetry transformation \((t, x) \rightarrow (T(t), X(x))\) given by \(T = t \oplus s\) and \(X = x\), which for \(s = 1\) reduces to the identity transformation, with \(\Phi = \text{constant}\). Indeed, in this case the invariance condition (9) is clearly satisfied:

\[
1 = \frac{d}{ds} \{ L(x(t), \bar{x}(t)) \} \iff 1 = \frac{d}{ds} \{ L(x(t), \bar{x}(t)) \} \iff 1 = 1.
\]

It follows from Noether’s theorem (Theorem 5) that (10) is a constant of motion, that is,

\[
L(x(t), \bar{x}(t)) \otimes \frac{\partial L}{\partial v}(x(t), \bar{x}(t)) \otimes \bar{x}(t)
\]

is constant in \(t \in [a, b]\) along any solution of the variational problem (6).

Our second example is the non-Newtonian analog to conservation of momentum in classical mechanics.

Example 7 (conservation of momentum). Let us now consider the following non-Newtonian problem of the calculus of variations:

\[
\mathcal{F}[x] = \int_a^b L(t, \bar{x}(t)) \, dt \rightarrow \min_{x \in \mathcal{X}}.
\]

In this case, since the Lagrangian \(L\) does not depend on \(x\), it is trivial to see that the functional \(\mathcal{F}[x]\) of (14) is invariant under the symmetry transformation \((t, x) \rightarrow (T(t), X(x, s))\) given by \(T = t \oplus s\) and \(X = x\), which for \(s = 1\) reduces to the identity transformation, with \(\Phi = \text{constant}\). It follows from Theorem 5 that

\[
\frac{\partial L}{\partial v}(t, \bar{x}(t))
\]

is a constant of motion.

From Examples 6 and 7 we are motivated to define the generalized momentum \(p(t)\) as

\[
p(t) = \frac{\partial L}{\partial v}(t, x(t), \bar{x}(t))
\]

and the Hamiltonian function \(H\) by

\[
H(t, x, v, p) = L(t, x, v) \otimes p \otimes v.
\]
With definitions (15) and (16), the DuBois–Reymond condition (4) can be written as
\[ \frac{d}{dt} [H(t, x(t), \bar{x}(t), p(t))] = \frac{\partial H}{\partial t}(t, x(t), \bar{x}(t), p(t)) \]
and the Erdmann condition (7) as \( H(x(t), \bar{x}(t), p(t)) = \text{constant} \). Such Hamiltonian perspective will be our starting point to develop a non-Newtonian optimal control theory. This is under development and will be addressed elsewhere.

5 Discussion

In 1918, Felix Klein (1849–1925) presented the paper “Invariant variation problems” by Emmy Noether (1882–1935) [17, 18] at a session of the Royal Society of Sciences in Göttingen. This important paper is considered as a milestone in the relation between symmetry transformations and conservation laws in physics [15]. Here we have proved such correspondence, between the existence of symmetry transformations and constants of motion, in the context of the non-Newtonian calculus of variations recently introduced by the author in [31]. The main difficulty is to obtain such invariance transformations. For that, one needs to solve a PDE that arises from condition (9) of invariance. While the resolution of such PDE has been automatized in the framework of the classical calculus of variations and optimal control with the help of a computer algebra system [9, 10], the automatic computation of conservation laws for non-Newtonian variational problems is still under investigation.

Acknowledgement

This research was supported by FCT and the Center for Research and Development in Mathematics and Applications (CIDMA), project UIDB/04106/2020.

References

[1] V. M. Alekseev, V. M. Tikhomirov, and S. V. Fomin. Optimal control. Contemporary Soviet Mathematics. Consultants Bureau, New York, 1987.
[2] A. E. Bashirov, E. Mısırlı, Y. Tandoğdu, and A. Özyapıcı. On modeling with multiplicative differential equations. Appl. Math. J. Chinese Univ. Ser. B, 26(4):425–438, 2011.
[3] D. Binbaşoğlu, S. Demiriz, and D. Türkoğlu. Fixed points of non-Newtonian contraction mappings on non-Newtonian metric spaces. J. Fixed Point Theory Appl., 18(1):213–224, 2016.
[4] M. Burgin and M. Czachor. Non-diophantine Arithmetics in Mathematics, Physics and Psychology. World Scientific, Singapore, 2021.
[5] W. Campillay-Llanos, F. Guevara, M. Pinto, and R. Torres. Differential and integral proportional calculus: how to find a primitive for \( f(x) = 1/\sqrt{2\pi}e^{-(1/2)x^2} \). Internat. J. Math. Ed. Sci. Tech., 52(3):463–476, 2021.
[6] M. Czachor. Unifying aspects of generalized calculus. Entropy, 22(10), 2020.
[7] C. Duyar, B. Sağır, and O. Oğur. Some basic topological properties on Non-Newtonian real line. British J. Math. Comput. Sci., 9(4):300–307, 2015.
[8] I. M. Gelfand and S. V. Fomin. Calculus of variations. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1963.
[9] P. D. F. Gouveia and D. F. M. Torres. Automatic computation of conservation laws in the calculus of variations and optimal control. Comput. Methods Appl. Math., 5(4):387–409, 2005. arXiv:math/0509140
[10] P. D. F. Gouveia, D. F. M. Torres, and E. A. M. Rocha. Symbolic computation of variational symmetries in optimal control. Control Cybernet., 35(4):831–849, 2006. arXiv:math/0604072
[11] M. Grossman. Bigeometric calculus. Archimedes Foundation, Rockport, Mass., 1983.
[12] M. Grossman and R. Katz. Non-Newtonian calculus. Lee Press, Pigeon Cove, Mass., 1972.
[13] N. Gungor. Some geometric properties of the non-Newtonian sequence spaces \( l_p(N) \). Math. Slovaca, 70(3):689–696, 2020.
[14] J. Jost and X. Li-Jost. Calculus of variations, volume 64 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1998.
[15] H. A. Kastrup. The contributions of Emmy Noether, Felix Klein and Sophus Lie to the modern concept of symmetries in physical systems. In Symmetries in physics (1600–1980) (San Feliu de Guixols, 1983), pages 113–163. Univ. Autonoma Barcelona, Barcelona, 1987.
[16] M. Mora, F. Cordova-Lepe, and R. Del-Valle. A non-Newtonian gradient for contour detection in images with multiplicative noise. Pattern Recognition Letters, 33(10):1245–1256, 2012.
[17] E. Noether. Invariante variationsprobleme. Nachr. Ges. Wiss. Göttingen, Math.-Phys. Kl., 1918:235–257, 1918.
[18] E. Noether. Invariant variation problems. Transport Theory Statist. Phys., 1(3):186–207, 1971.
[19] A. Ozyapici and B. Bilgehan. Finite product representation via multiplicative calculus and its applications to exponential signal processing. Numer. Algorithms, 71(2):475–489, 2016.
[20] E. Pap. Generalized real analysis and its applications. International Journal of Approximate Reasoning, 47(3):368–386, 2008.
[21] M. Pinto, R. Torres, W. Campillay-Llanos, and F. Guevara-Morales. Applications of proportional calculus and a non-Newtonian logistic growth model. Proyecciones, 39(6):1471–1513, 2020.
[22] S. P. S. Santos, N. Martins, and D. F. M. Torres. Variational problems of Herglotz type with time delay: DuBois-Reymond condition and Noether’s first theorem. Discrete Contin. Dyn. Syst., 35(9):4593–4610, 2015. arXiv:1501.04873
[23] W. Sarlet and F. Cantrijn. Generalizations of Noether’s theorem in classical mechanics. SIAM Rev., 23(4):467–494, 1981.
[24] B. Sağır and F. Erdoğan. On non-Newtonian power series and its applications. Konuralp J. Math., 8(2):294–303, 2020.
[25] S. Tekin and F. Başar. Certain sequence spaces over the non-Newtonian complex field. Abstr. Appl. Anal., pages Art. ID 739319, 11, 2013.
[26] D. F. M. Torres. Conservation laws in optimal control. In Dynamics, bifurcations, and control (Kloster Irsee, 2001), volume 273 of Lect. Notes Control Inf. Sci., pages 287–296. Springer, Berlin, 2002.
[27] D. F. M. Torres. On the Noether theorem for optimal control. Eur. J. Control, 8(1):56–63, 2002.
[28] D. F. M. Torres. Gauge symmetries and Noether currents in optimal control. Appl. Math. E-Notes, 3:49–57, 2003. arXiv:math/0301116
[29] D. F. M. Torres. Proper extensions of Noether’s symmetry theorem for nonsmooth extremals of the calculus of variations. Commun. Pure Appl. Anal., 3(3):491–500, 2004.
[30] D. F. M. Torres. Quasi-invariant optimal control problems. Port. Math. (N.S.), 61(1):97–114, 2004. arXiv:math/0302264
[31] D. F. M. Torres. On a non-Newtonian calculus of variations. Axioms, 10(3), Art. 171, 15 pp., 2021. arXiv:2107.14152
[32] M. Waseem, M. Aslam Noor, F. Ahmed Shah, and K. Inayat Noor. An efficient technique to solve nonlinear equations using multiplicative calculus. Turkish J. Math., 42(2):679–691, 2018.