Chaotic dynamics in collective models of nuclei

P Stránský\textsuperscript{1,2}, M Macek\textsuperscript{1}, P Cejnar\textsuperscript{1}, A Frank\textsuperscript{2}, R Fossion\textsuperscript{2} and E Landa\textsuperscript{2}

\textsuperscript{1} Institute of Particle and Nuclear Physics, Faculty of Mathematics and Physics, Charles University, V Holesovičkách 2, 180 00 Prague, Czech Republic
\textsuperscript{2} Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México, Apartado Postal 70-543, 04510 México, D.F., Mexico

E-mail: pavel.stransky@nucleares.unam.mx

Abstract. We present results of an extensive analysis of classical and quantum signatures of chaos in the geometric collective model (GCM) and the interacting boson model (IBM) of nuclei. Apart from comparing the regular fraction of the classical phase space and the Brody parameter for the nearest neighbor spacing distribution in the quantum case, we also adopt (i) the Peres lattices allowing one to distinguish ordered and disordered parts of spectra and to reveal main ordering principles of quantum states, (ii) the geometrical method to determine the position where the transition from order to chaos occurs, and (iii) we look for the $1/f^\alpha$ power law in the power spectrum of energy level fluctuations. The Peres method demonstrates the adiabatic separation of collective rotations in the IBM.

1. Introduction

It has been shown in recent years that simple models of nuclear collective dynamics—the geometric collective model (GCM) and the interacting boson model (IBM)—exhibit a high degree of variability in regular and chaotic features with energy and control parameters [1, 2, 3]. The complex dynamics encoded in relatively simple Hamiltonians well position these systems to be laboratories for detailed investigation into classical-quantum correspondence and for testing different approaches of measuring and visualizing chaos. In addition, it appears that the study of chaos can help in understanding the dynamical structure of the systems and allows for observation of new phenomena, such as the quantum phase transitions [4] and the quasi-dynamical symmetry (QDS) [5].

Classical chaos is commonly studied by means of the visual method of Poincaré sections and numerically characterized by quantities derived from Lyapunov exponents. This approach requires the tedious solution of differential equations of motion for a large set of trajectories in order to obtain an overall image of the degree of regularity. To bypass these difficulties, Horwitz et al [6] have proposed recently another way to distinguish between regular and chaotic dynamics. When the system, evolving in a flat Cartesian space with some potential, is transformed to be in a free motion on a special curved manifold, the Riemannian geometry gives a very simple condition of stability: a negative eigenvalue of a matrix obtained from the curvature tensor appearing inside the kinematically accessible area of the system implies instability, and thus chaoticity of the motion.

Quantum measures of chaos are based on certain statistical properties of the energy levels, for example on the nearest-neighbor-spacing distribution (and the related Brody parameter $\omega$) [7].
It is clear that the statistics describe only some bulk features of a sufficiently large portion of the spectrum. The method introduced by Peres [8], however, makes it possible to assign regular, chaotic, or a mixed type of dynamics to individual states. It uses lattices formed from the expectation values $A_i = \langle \psi_i | A | \psi_i \rangle$ of an arbitrary operator $A$ plotted against the energies $E_i = \langle \psi_i | H | \psi_i \rangle$. Due to arguments based on the semiclassical EBK quantization, the lattices of points $(E_i, A_i)$ show regular patterns in integrable systems, whereas in the chaotic case the resulting images are formed by a combination of ordered and disordered patterns, whose relative size depends on the degree of chaoticity. Note that the Peres method is entirely visual and can be considered as a quantum analogue of the classical Poincaré sections.

Long-range correlations of eigenlevels are usually studied by the spectral rigidity $\Delta_3$ or the number variance $\Sigma_2$, from which, however, it is difficult to quantify the degree of regularity of a system. Incorporating long-range correlations into quantitative studies has been done by Relaño et al [9] considering spectral fluctuation of a time series of deviations between the unfolded eigenenergies and uniformly distributed levels. The ensemble average of the power spectrum obeys the power law

$$\langle S(f) \rangle = \frac{1}{f^\alpha}$$

where $\alpha$, bounded between 1 and 2 (for a totally chaotic and an integrable system, respectively), can serve as a measure of regularity.

The aim of this contribution is to establish the standard methods of classical and quantum chaos in the GCM and IBM and then make a step beyond: we use Peres lattices to present the manifestation of QDS by well-pronounced rotational bands for high-energy states in the IBM with symmetry breaking interaction switched on.

2. Geometric collective model

We use the GCM in the nonrotating regime, i.e. considering only the vibrations with zero angular momentum $J$. The Hamiltonian $H = T + V$ describes quadrupole motions of nuclei. Quantizing the system and separating the vibrational and rotational degrees of freedom by moving into the intrinsic frame we obtain the kinetic term in the form

$$T^{5D} = -\frac{\hbar^2}{2K} \left( \frac{1}{\beta^4} \frac{\partial}{\partial \beta} \beta^4 \frac{\partial}{\partial \beta} + \frac{1}{\beta^2 \sin 3\gamma} \frac{\partial}{\partial \gamma} \sin 3\gamma \frac{\partial}{\partial \gamma} \right) + T_{\text{rot}}. \quad (1)$$

The system has in total 5 degrees of freedom which is indicated by the label 5D. $K$ is a mass parameter and $(\beta, \gamma)$ are the well-known Bohr (shape) coordinates. For states with $J = 0$ the rotational part $T_{\text{rot}}$ vanishes. The potential (with adjustable parameters $A, B, C$) reads as

$$V = A\beta^2 + B\beta^3 \cos 3\gamma + C\beta^4. \quad (2)$$

There is another way how to deal with the kinetic term in the case of the principal frame at rest. If the 3 rotational degrees of freedom are frozen before the quantization is carried out, we get the familiar 2D kinetic term in polar coordinates $(\beta, \gamma)$:

$$T^{2D} = -\frac{\hbar^2}{2K} \left( \frac{1}{\beta} \frac{\partial}{\partial \beta} \beta \frac{\partial}{\partial \beta} + \frac{1}{\beta^2} \frac{\partial^2}{\partial \gamma^2} \right). \quad (3)$$

Both expressions (1) and (3) have the same classical limit and offer a possibility to study the influence of the quantization method on the quantum chaotic features of the spectrum. Let us emphasize that only the 5D case represents the standard (nuclear) GCM.

We diagonalize both types of the GCM Hamiltonian in the respective 2D or 5D oscillator bases. Due to scaling properties, only one of the parameters $(A, B, C, K)$ in the classical case, or two in the quantum case, determine the qualitative features of the system. We fix $C = 1$ and move along the path of three connected lines (each described by only one principal control parameter), passing all possible configurations of the system: (i) $B \in (0, 1)$ with $A = -1,$
Figure 1. Complete map of the classical regular fraction $f_{\text{reg}}$ as a function of control parameters and energy $E$. The degree of chaos is coded in shades of gray, with light gray (black) corresponding to complete order (chaos). The plot is divided by the thick white lines into three regions of different scaling ($A = -1$, $B = 1$, and $A = 1$, respectively) with free parameters being varied along the horizontal axis. The white dashed line indicates the deformed-spherical shape phase transition. The black dashed line corresponds to the convex-concave transition of the border of kinematically accessible area.

(ii) $A \in (-1, 1)$ with $B = 1$ where the deformed-spherical ground-state shape phase transition is crossed at $(A, B) = (1/4, 1)$, and (iii) $B \in (1, 0)$ with $A = 1$. In the quantum case we consider the classicality constant $\kappa = \hbar/K$ as the second fundamental parameter, which influences only the density of energy levels [10]. We choose two independent Peres operators identified with (i) the square of the angular momentum operator $L^2$ connected with rotations about angle $\gamma$, which is the $J = 0$ projected $O(5)$ invariant (the seniority operator) in the 5D case or the $O(2)$ invariant in the 2D case, and (ii) the nonintegrable perturbation $H' = \beta^3 \cos 3\gamma$ (the Hamiltonian is integrable for $B = 0$ with $L^2$ being the second integral of motion).

Figure 1 presents the complete map of the classical regularity quantified by the regular fraction of the classical phase space $f_{\text{reg}}$. Leaving the leftmost and the rightmost $B = 0$ integrable limits, we observe a progressive retreat of regularity with increase of the perturbation $B$. In the intermediate domain the structure is extremely complex, forming sharp branching valleys and curved edges. At $B \approx 0.6$ the regularity achieves a well-pronounced maximum, which has been shown to be tightly connected with the IBM semiregular arc [11]. The shape-phase transition region (the white vertical dashed line) is characterized by accentuated chaoticity. The dashed black line encircles the region where the kinematically accessible region in the $(x \equiv \beta \cos \gamma, y \equiv \beta \sin \gamma)$ plane has concave shape. The lower part of the line corresponds exactly with the curve obtained by the Horwitz’s geometrical method, which should separate regions with regular and chaotic dynamics. However, it is evident from the figure that this criterion provides only an estimation.

The decay of regular pattern in the quantum Peres lattices when introducing nonintegrability is demonstrated in Figure 2. Starting from a totally ordered case (panel a), small nonintegrable
Figure 2. Peres lattices of the 2D GCM for $\kappa = 2.5 \cdot 10^{-5}$ and $A = -1$ with 3000 levels plotted. The transition between integrable (panel a) and nonintegrable (panels b–d) regimes are observed.

Perturbation causes disturbances only in a part of a lattice (panel b). Maximal disorder is reached at $B = 0.24$. For higher $B$ the progression reverses and new regular structures start to appear, which is observed in both $\langle L^2 \rangle$ and $\langle H' \rangle$ lattices. The regular islands can be identified with remnants of classical tori.

Figure 3 (a) presents the power spectrum for the method of the $1/f^\alpha$ noise in an integrable case of the model. The value of the exponent $\alpha = 2$ is in full agreement with the prediction of the theory [9].

In the quantum case we use two measures of regularity: (i) the adjunct $1 - \omega$ of the Brody parameter, which is estimated by the $\chi^2$ fit applied to the nearest-neighbor spacing distribution of the unfolded levels, and (ii) the quantity $\alpha - 1$ of the $1/f^\alpha$ noise in the averaged power spectrum of energy level fluctuation. In order to obtain the energetic dependence of the regularity, we divide the calculated spectra (usually 30 000 well-converging levels) into windows of 1000 levels. Both quantum measures can then be directly compared with $f_{\text{reg}}$ [see Figure 3(b)]. The

Figure 3. (a) Average power spectrum of the $1/f^\alpha$ noise method in the integrable $\gamma$-soft case ($A = -1, B = 0$) of the 2D GCM with $\kappa = 4 \cdot 10^{-6}$, using 500 sets of 64 levels, starting from level 8000. We use the simplified notation $x \equiv \log f$ and $y \equiv \log \langle S(f) \rangle$. (b) Comparison of the classical $f_{\text{reg}}$ and quantum measures of regularity $1 - \omega$ and $\alpha - 1$ for the 2D GCM in the mixed dynamics configuration with $A = 0.25, B = 1$ and $\kappa = 1 \cdot 10^{-6}$. The standard errors of the measures are 3%, 5% and 20%, respectively.
Figure 4. First column: Classical and quantum chaos of the GCM for $B = 0.62$. Peres lattices for $\kappa = 1 \cdot 10^{-4}$ are compared for 2D (a) and 5D (b) quantization. (c) Classical regular fraction of the phase space $f_{\text{reg}}$ and (d) corresponding quantum counterpart—Brody parameter $\omega$ calculated for $\kappa = 2.5 \cdot 10^{-5}$. Second column: IBM dynamics at $\chi = -0.9$ and $\eta = 0.5$. (e) Peres lattices of the $n_d$ operator for different angular momenta ($N = 30$ bosons), and (f) corresponding fraction of regularity $f_{\text{reg}}$.

correspondence is apparent, though the quantum measures tend to overemphasize the regularity.

Figure 4 confronts the Peres lattices and the measures of chaos for both 2D and 5D quantization schemes. The lattices are displayed in panels (a) and (b), showing small differences in structure, but the distribution of regular and chaotic areas is equal. This is in agreement with the negligible difference between the 2D and 5D cases of $\omega(E)$ (panel d), which remains within the limits of the standard error in our calculations. Finally, observe the influence of the classicality parameter $\kappa$ on the level density by comparing panel (a) with Figure 2 (d).

3. Interacting Boson Model
To demonstrate a different application of the Peres method we consider the Hamiltonian [1]

$$H = \alpha \left[ \frac{\eta}{N} n_d - \frac{1 - \eta}{N^2} Q_x \cdot Q_x \right]$$

(4)
with \( n_d \) the \( d \)-boson number operator (\( N \) is a conserved total number of bosons), \( Q_\chi \) the quadrupole operator, and \((\eta, \chi)\) two external parameters defining the Casten triangle. Its vertices \((\eta, \chi) = (0, 0), (1, 0), (0, -\sqrt{7}/2)\) correspond to the \( O(6), U(5), \) and \( SU(3) \) dynamical symmetries, respectively. The system is integrable in the triangle vertices and along the side with \( \chi = 0 \). The Peres operator we can take any of the Casimir invariants connected with the symmetry groups, but our results are best demonstrated by the choice the \( U(5) \) operator \( n_d \).

Recently, some low-energy rotational bands populating the spectrum below the saddle point energy in the axially-deformed part of the Casten triangle were explained analytically using specific \( SU(3) \)-like boson condensates [12]. In this contribution we turn to the highly excited states. Figure 4(e) shows the eigenstates of IBM in the arc of regularity in the interior of the Casten triangle, distributed into 4 Peres lattices according to their angular momentum. Apart from a few ordered low-lying states (present also in other parts of the parametric triangle), we may observe two distinct regular regions: one at \( E \approx 0 \) and the second reaching the highest accessible energy. Surprisingly, the lattices for different angular momentum \( L \) form the same pattern there. The analyses based on wave function correlation, \( E2 \) transitions and moments of inertia reveal that these “regular” energies associate into bands, showing signatures of \( SU(3) \) QDS in the \( L(L+1) \) excitation energy pattern [13]. This behavior is strongly accentuated along the regular arc. While the low-energy dynamics are almost completely regular in the entire Casten triangle, the partially increased regularity of the arc (relative to the neighboring regions) is brought about essentially by the regular high-lying modes, which has already been shown [1] [this is also supported by the classical fraction of regularity \( f_{\text{reg}} \) for states with \( L = 0 \), drawn in the panel (f)]. It suggests a connection between quasi-dynamical symmetry and regularity.

4. Concluding remarks
We have calculated some measures of chaos for the classical and quantum versions of the geometric collective model (we dispose of two different quantum GCM with the same classical limit) and the interacting boson models in the wide range of external parameters and energy, observing good correspondence between the classical fraction of regularity and the Brody parameter that quantifies chaos in the quantum case. This correspondence is supported by the visual observation of the regular and chaotic structures in the Peres lattices.

In addition, we adopted the geometric method by Horwitz et al that estimates the position of regular-chaotic transition and follows the convex-concave shape change of the kinematically accessible area border in the GCM, which has been proven analytically [14]. We also adopted the method of \( 1/f^\alpha \) noise by Relaño et al, obtaining expected \( \alpha = 2 \) exponent in an integrable configuration and a correspondence in the energetic dependence of the measures \( 1 - \omega \) and \( \alpha - 1 \) in a configuration with mixed dynamics.

In IBM we have observed a strong correlation between the variation of classical and quantum degrees of regularity and the occurrence rate of the rotational bands. This suggests that the separation of rotational motion from intrinsic vibrational dynamics is enhanced by the intrinsic regularity, while on the other hand, the chaotic intrinsic states are more prone to mixing, if the rotation comes into play; the regular states seem to be protected by their symmetry, in contrast. Different methods of rotational band identification are described in [13], being in full agreement with the results obtained by Peres method.

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