An experimental study of the modified accelerated overrelaxation (MAOR) scheme on stationary helmholtz equation

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Abstract. This research aims to experiment the Modified Accelerated Overrelaxation (MAOR) scheme on second order iterative method for solving two dimensional (2D) Helmholtz Equation. The equation is discretized using the standard second order (Full Sweep) finite difference method. Previous well known relaxation schemes includes the Gauss-Seidel (GS), Successive Overrelaxation (SOR), Modified SOR (MSOR) and Accelerated Overrelaxation (AOR) schemes. These schemes has at most two relaxation parameter while the MAOR scheme have more than two. Several numerical experiments were conducted on two different equations to test the feasibility and the superiority of the MAOR scheme compared to the previous schemes on different mesh size.

1. Introduction

This research considers the model problem of the 2D Helmholtz equation, which is represented as

\[
\begin{align*}
\mathbf{u}_{xx}(x,y) + \mathbf{u}_{yy}(x,y) - \alpha \mathbf{u} &= f(x,y), \quad (x,y) \in \mathbb{R}^2 \\
\end{align*}
\]

such that the solution domain \( \mathbb{R}^2 = (0,1) \times (0,1) \) which is subject to Dirichlet boundary conditions and satisfies the exact solution \( \mathbf{u}(x,y) = e(x,y) \) where \( \alpha \) is a non-negative constant. The domain \( \mathbb{R}^2 \) is uniformly discretized in direction of \( x \) and \( y \) with the mesh size of \( h = \frac{1}{n+1} \), where \( x_i = ih, y_j = jh \) for all \( i,j = 0,1,2,\cdots,n,n+1 \). The notation \( u_{i,j} \) represents the computed solution of \( u(x_i,y_j) \).

In order to solve Eq.(1), the equation is discretized using the finite difference approximation with second order central difference scheme, that is derived as:

\[
\begin{align*}
\mathbf{u}_{xx} &= \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} \\
\mathbf{u}_{yy} &= \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta y^2}
\end{align*}
\]
By substituting Eq. (2) and (3) into Eq. (1) and let $h^2 = \Delta x^2 = \Delta y^2$, the Helmholtz equation can be rewritten as,

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2} - \alpha u_{i,j} = f_{i,j} \quad (4)$$

which can be simplified into,

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - \rho u_{i,j} = h^2 f_{i,j} \quad (5)$$

where $\rho = 4 + \alpha h^2$. Eq. (5) yields the second order five-point or Full Sweep (FS) iterative method.

In order to solve Eq. (1), a linear system can be formed from the approximation of Eq. (5), which is defined as:

$$Au = b \quad (6)$$

where $A$ is a large and sparse square matrix while $u$ and $b$ are column vectors. Matrix $A$ can then be decomposed into,

$$A = D - L - U \quad (7)$$

where $D$ is the diagonal matrix, $L$ is the strictly lower triangular matrix and $U$ is the strictly upper triangular matrix. Next, by substituting Eq. (7) into Eq. (6) and manipulating it, the general scheme for Gauss-Seidel (GS) is defined as,

$$u^{(k+1)} = (D - L)^{-1} \left[ Uu^{(k)} + b \right] \quad (8)$$

which yields,

$$u^{(k+1)}_{i,j} = \frac{1}{\rho} \left( u^{(k)}_{i+1,j} + u^{(k+1)}_{i-1,j} + u^{(k)}_{i,j+1} + u^{(k+1)}_{i,j-1} - h^2 f_{i,j} \right). \quad (9)$$

2. Related Works on Well-known Relaxation Schemes

Throughout the years, many researches have been done on developing relaxation schemes to quicken the convergent rate of iterative methods. To assist in explaining these scheme, Figures 1 and 2 shows the domain on Natural and Red-Black ordering system respectively.
The Successive Overrelaxation (SOR) scheme has been done by [2,3] such that its general formula can be defined as,
\[
\mathbf{u}^{(k+1)} = (D - \omega L)^{-1} \left[ ((1 - \omega)D + \omega U) \mathbf{u}^{(k)} + \omega \mathbf{b} \right]
\]  
(10)
and by referring to the stencil in Figure 1 (b) yields,

\[ u^{(k+1)}_{i,j} = \omega \rho \left( u^{(k)}_{i+1,j} + u^{(k)}_{i-1,j} + u^{(k)}_{i,j+1} + u^{(k)}_{i,j-1} - h^2 f_{i,j} \right) + (1 - \omega) u^{(k)}_{i,j} \]  

(11)

where \( \omega \) is the relaxation parameter.

In addition, Kincaid and Young [4] invented the Modified SOR (MSOR) scheme which has been such that it is identical to the SOR scheme with the Red-Black ordering system, which can be defined as,

\[ u^{(k+1)} = (D - \omega_r L)^{-1} \left[ ((1 - \omega_r)D + \omega_r U) u^{(k)} + \omega_r b \right] \]  

(12)

\[ u^{(k+1)} = (D - \omega_b L)^{-1} \left[ ((1 - \omega_b)D + \omega_b U) u^{(k)} + \omega_b b \right] \]  

(13)

and by referring to the stencil in Figure 2 (b) for all \( \bullet \) points and Figure 2 (c) for all \( \bullet \) points yields,

\[ u^{(k+1)}_{i,j} = \frac{\omega_r}{\rho} \left( u^{(k)}_{i+1,j} + u^{(k)}_{i-1,j} + u^{(k)}_{i,j+1} + u^{(k)}_{i,j-1} - h^2 f_{i,j} \right) + (1 - \omega_r) u^{(k)}_{i,j} \]  

(14)

\[ u^{(k+1)}_{i,j} = \frac{\omega_b}{\rho} \left( u^{(k+1)}_{i+1,j} + u^{(k+1)}_{i-1,j} + u^{(k+1)}_{i,j+1} + u^{(k+1)}_{i,j-1} - h^2 f_{i,j} \right) + (1 - \omega_b) u^{(k)}_{i,j} \]  

(15)

where \( \omega_r \) and \( \omega_b \) are the preconditioner parameters on \( \bullet \) and \( \bullet \) points respectively.

Furthermore, Hadjidimos [5] invented the Accelerated Overrelaxation (AOR) scheme which is a two-parameter generalization of the SOR scheme and defined as,

\[ u^{(k+1)} = (D - \theta L)^{-1} \left[ ((1 - \omega)D + (\omega - \theta) L + \omega U) u^{(k)} + \omega b \right] \]  

(16)

and by also referring to the stencil in Figure 1 (b) yields,

\[ u^{(k+1)}_{i,j} = \frac{\theta}{\rho} \left( u^{(k+1)}_{i+1,j} - u^{(k)}_{i+1,j} + u^{(k+1)}_{i,j-1} - u^{(k+1)}_{i,j-1} \right) \]

\[ + \frac{\omega}{\rho} \left( u^{(k)}_{i+1,j} + u^{(k)}_{i-1,j} + u^{(k)}_{i,j+1} + u^{(k)}_{i,j-1} - h^2 f_{i,j} \right) + (1 - \omega) u^{(k)}_{i,j} \]  

(17)

where \( \theta \) and \( \omega \) are the preconditioner parameters.

From these previous researches, the SOR scheme has a better convergent rate than the GS scheme. Similarly, the MSOR [6] and AOR [7] scheme both have a better convergent rate than the SOR scheme. However, there has been no comparison between the MSOR and AOR scheme. The next section will discuss a relaxation scheme that has a better convergent rate than the above mentioned schemes.
3. The Full Sweep Modified Accelerated Overrelaxation (FSMAOR) Iterative Method

The MAOR scheme has been theoretically developed by Hadjidimos et al. [1] where they defined \( \theta \) and \( \omega \) as,

\[
\theta = \begin{bmatrix} \theta_r I_r & 0 \\ 0 & \theta_b I_b \end{bmatrix}, \quad \omega = \begin{bmatrix} \omega_r I_r & 0 \\ 0 & \omega_b I_b \end{bmatrix}
\]

with \( \theta_r \) and \( \omega_r \) denoted as the preconditioner parameters on the \( \bullet \) points, \( \theta_b \) and \( \omega_b \) denoted as the preconditioner parameters on the \( \bigcirc \) points, and \( I_r \) and \( I_b \) are identity matrices. Hence, the general scheme for MAOR can be defined as,

\[
\begin{align*}
\text{MAOR}_r: & \quad u^{(k+1)} = (D - \theta_r L)^{-1} \left[ (1 - \omega_r) D + (\omega_r - \theta_r) L + \omega_r U \right] u^{(k)} + \omega_r b \\
\text{MAOR}_b: & \quad u^{(k+1)} = (D - \theta_b L)^{-1} \left[ (1 - \omega_b) D + (\omega_b - \theta_b) L + \omega_b U \right] u^{(k)} + \omega_b b
\end{align*}
\]

Thus, the aim of this research is to experiment the MAOR scheme as theorized by Hadjidimos. To achieve this, the scheme is implemented on the second order iterative methods for solving the Helmholtz equation. Since the Red-Black ordering system is applied in this scheme, the two stencils for updating a grid point on iteration \( k + 1 \) can be shown in the Figure 3. Table 1 shows the updated points for the second order iterative methods on Red-Black ordering system.

![Figure 3. The five-point stencil in variable \( k \) for (a) \( \bullet \) points and (b) \( \bigcirc \) points.](image)

| Points | Second Order (5-point) |
|--------|-----------------------|
| \( k \) | \( k + 1 \) |
| \( \bullet \) | 4 | 0 |
| \( \bigcirc \) | 0 | 4 |

With the stencil in Figure 3 and the general scheme for MAOR in Eq. (18) and Eq. (19), the FSMAOR iterative can be derived as follows:

\[
\begin{align*}
\text{FSMAOR}_r: & \quad u^{(k+1)}_{i,j} = \frac{\omega_r}{\rho} \left( u^{(k)}_{i+1,j} + u^{(k)}_{i-1,j} + u^{(k)}_{i,j+1} + u^{(k)}_{i,j-1} - h^2 f_{i,j} \right) + (1 - \omega_r) u^{(k)}_{i,j} \\
\text{FSMAOR}_b: & \quad u^{(k+1)}_{i,j} = \frac{\omega_b}{\rho} \left[ u^{(k+1)}_{i-1,j} + u^{(k+1)}_{i+1,j} + u^{(k+1)}_{i,j+1} + u^{(k+1)}_{i,j-1} - \left( u^{(k)}_{i-1,j} + u^{(k)}_{i+1,j} + u^{(k)}_{i,j+1} + u^{(k)}_{i,j-1} \right) \right] \\
& \quad + \frac{\omega_b}{\rho} \left( u^{(k)}_{i+1,j} + u^{(k)}_{i-1,j} + u^{(k)}_{i,j+1} + u^{(k)}_{i,j-1} - h^2 f_{i,j} \right) + (1 - \omega_b) u^{(k)}_{i,j}
\end{align*}
\]

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All the points in the solution domain are divided into ● and ○ points as shown in Figure 2 (a). Using Eq. (20) and Eq. (21), iterations on these points are generated until a certain convergence criteria is met. The full algorithm for the iterative process for MAOR scheme as shown in Algorithm 1.

Algorithm 1: Full Algorithm for FSMAOR scheme.

4. Numerical Results
Two numerical test were conducted to validate the superiority of the MAOR scheme. Three criteria were examine for comparing the GS, SOR, MSOR, AOR and MAOR schemes on second order FS iterative methods which includes the number of iterations, execution time and maximum absolute error. In this research, the tolerance is set as $\varepsilon = 10^{-12}$. The tests were carried out on a Laptop with Intel(R) Core(TM) i5-7200U COU 2.50GHz, 4.00GB RAM running Windows 10 using Code::Blocks C++ programming language. All five schemes were run with several mesh sizes of 32, 64, 128 and 256 to solve the following examples:

| Example | Equation | Boundary Condition | Exact Solution |
|---------|----------|--------------------|---------------|
| 1 [8]   | $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - 10u = 6 - 10(2x^2 + y^2)$ | Dirichlet | $u(x, y) = 2x^2 + y^2$ |
| 2 [9–11]| $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = (x^2 + y^2)e^{xy}$ | Dirichlet | $u(x, y) = e^{xy}$ |

The results for Example 1 and 2 are shown in Tables 2 and 3 and respectively. The percentage reductions of number of iterations and execution time for the GS, SOR, MSOR and AOR when compared to the MAOR scheme is shown in Table 4.
Table 2. Number of iterations, execution time (seconds) and average absolute error for the iterative methods for Example 1.

| Schemes | Mesh Size | Number of Iterations | Execution Time (in seconds) | Average Absolute Errors |
|---------|-----------|----------------------|----------------------------|-------------------------|
|         | 32       | 64                   | 128 | 256 | 32       | 64       | 128 | 256 | 32       | 64       | 128 | 256 |
| GS      | 1743     | 6364                | 23504 | 87116 |
| SOR     | 146      | 274                 | 596   | 1120  |
| MSOR    | 125      | 243                 | 492   | 1015  |
| AOR     | 146      | 273                 | 591   | 1120  |
| MAOR    | 122      | 234                 | 466   | 902   |
| GS      | 0.031    | 0.360               | 5.234 | 76.969 |
| SOR     | 0.016    | 0.031               | 0.203 | 1.562 |
| MSOR    | 0.015    | 0.031               | 0.156 | 1.047 |
| AOR     | 0.015    | 0.031               | 0.171 | 1.218 |
| MAOR    | 0.015    | 0.031               | 0.125 | 1.000 |

Table 3. Number of iterations, execution time (seconds) and average absolute error for the iterative methods for Example 2.

| Schemes | Mesh Size | Number of Iterations | Execution Time (in seconds) | Average Absolute Errors |
|---------|-----------|----------------------|----------------------------|-------------------------|
|         | 32       | 64                   | 128 | 256 | 32       | 64       | 128 | 256 | 32       | 64       | 128 | 256 |
| GS      | 2613     | 9557                | 35328 | 130979 |
| SOR     | 172      | 336                 | 773   | 1545  |
| MSOR    | 153      | 298                 | 604   | 1353  |
| AOR     | 169      | 333                 | 734   | 1454  |
| MAOR    | 149      | 288                 | 563   | 1117  |
| GS      | 0.031    | 0.516               | 7.500 | 110.907 |
| SOR     | 0.016    | 0.031               | 0.235 | 1.969 |
| MSOR    | 0.016    | 0.031               | 0.156 | 1.328 |
| AOR     | 0.016    | 0.031               | 0.203 | 1.656 |
| MAOR    | 0.015    | 0.031               | 0.141 | 1.203 |

| GS      | 1.2418E-06 | 3.1175E-07 | 7.7384E-08 | 1.6793E-08 |
| SOR     | 1.2419E-06 | 3.1193E-07 | 7.8068E-08 | 1.9525E-08 |
| MSOR    | 1.2419E-06 | 3.1193E-07 | 7.8078E-08 | 1.9504E-08 |
| AOR     | 1.2419E-06 | 3.1193E-07 | 7.8077E-08 | 1.9525E-08 |
| MAOR    | 1.2419E-06 | 3.1193E-07 | 7.8076E-08 | 1.9528E-08 |
Table 4. The percentage reduction of the Number of Iterations and Execution Time for all schemes compared to the MAOR scheme.

| Example | Schemes | Numbers of iterations (%) | Execution time (%) |
|---------|---------|---------------------------|-------------------|
| 1       | GS      | 93.00 - 98.96             | 51.61 - 98.70     |
|         | SOR     | 14.60 - 21.81             | 0.00 - 38.42      |
|         | MSOR    | 2.40 - 11.13              | 0.00 - 19.87      |
|         | AOR     | 14.28 - 21.15             | 0.00 - 26.90      |
| 2       | GS      | 94.30 - 99.15             | 51.61 - 98.92     |
|         | SOR     | 13.37 - 27.70             | 0.00 - 40.00      |
|         | MSOR    | 2.61 - 17.44              | 0.00 - 9.62       |
|         | AOR     | 11.83 - 23.30             | 0.00 - 30.54      |

5. Conclusion and Future Works
From the experiments above, it shows the superiority of the MAOR scheme on second order iterative methods for solving 2D Helmholtz equation. Through the surveillance in Tables 2, 3 and 4, it is shown that the MAOR scheme have the best convergent rate when compared with GS, SOR, MSOR and AOR scheme respectively for both Examples. For future works, this study will be extended to different points and group iterative methods which will be reported seperately in the future.

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