Soft supersymmetry breaking terms and lepton flavor violations in modular flavor models

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Abstract
We study the soft supersymmetry (SUSY) breaking terms due to the modulus F-term in the modular flavor models of leptons. It is found that the soft SUSY breaking terms are constrained by the modular forms, and specific patterns are derived. Those phenomenological implications are discussed in such as the lepton flavor violation $\mu \rightarrow e + \gamma$ and $\mu \rightarrow 3e$ decays and $\mu \rightarrow e$ conversion in nuclei. In order to examine numerically, two modular flavor $A_4$ models are taken. The SUSY breaking scale is significantly constrained by inputting the observed upper bound of the $\mu \rightarrow e + \gamma$ decay. The SUSY mass scale is larger than around 8 TeV and 5 TeV for the two $A_4$ models, respectively. Therefore, the current experimental upper bound for the $\mu \rightarrow e + \gamma$ decay corresponds to the new physics of the SUSY particle at the 5 – 10 TeV scale in the modular flavor models. The SUSY scale will be explored by future experiments of lepton flavor violation up to 8 – 17 TeV. The predicted branching ratio depends on a modulus $\tau$ significantly. It decreases of one order at the large $\text{Im} \tau$. We also calculate the branching ratios of tauon decays to $e + \gamma$ and $\mu + \gamma$. Predicted ones are at most $O(10^{-15})$, which are much below the current experimental bounds.
1 Introduction

The origin of flavor is one of the important issues in particle physics. Non-Abelian flavor symmetries are interesting approaches among various approaches to understand the flavor origin. Indeed, a lot of works have been presented by using various non-Abelian discrete groups for flavors to understand the flavor structures of quarks and leptons. Those are motivated by the precise observation of flavor mixing angles of leptons [1–10]. In particular, the $A_4$ flavor models are attractive because the $A_4$ group is the minimal one including a triplet irreducible representation, which allows for a natural explanation of the existence of three families of quarks and leptons [11–17]. In spite of such a theoretical effort, we have not yet the fundamental theory of flavor.

Flavor symmetries control not only the flavor structure of quarks and leptons, but also the flavor structure of their superpartners and lead to specific patterns in soft supersymmetry (SUSY) breaking terms. Soft SUSY breaking terms were studied in several models with non-Abelian flavor symmetries [18–21], and they are different from patterns of soft SUSY breaking terms in other flavor models. (See e.g. [22].) Such structure can be observed directly and/or indirectly if the mass scale of superpartners is light enough. For example, flavor changing processes are important to test the flavor structure of superpartners different from the flavor structure of quarks and leptons.

A new direction to flavor symmetry, modular invariance has been proposed in the lepton sector [23]. The modular symmetry arises from the compactification of a higher dimensional theory on a torus or an orbifold as well as low-energy effective field theory of superstring theory [24–29]. The shape of the compact space is parametrized by a modulus $\tau$ living in the upper-half complex plane, up to modular transformations. The finite groups $S_3$, $A_4$, $S_4$, and $A_5$ are isomorphic to the finite modular groups $\Gamma_N$ for $N = 2, 3, 4, 5$, respectively [30].

In this approach, fermion matrices are written in terms of modular forms which are holomorphic functions of the modulus $\tau$. The lepton mass matrices have given successfully in terms of $A_4$ modular forms [23]. Modular invariant flavor models have been also proposed on the $\Gamma_2 \simeq S_3$ [31], $\Gamma_4 \simeq S_4$ [32] and $\Gamma_5 \simeq A_5$ [33]. Based on these modular forms, flavor mixing of quarks/leptons have been discussed intensively in these years.

The vacuum expectation value (VEV) of the modulus $\tau$ plays a role in modular flavor symmetric models, in particular realization of quark and lepton masses and their mixing angles. The modulus VEV is fixed as the potential minimum of the modulus potential. (See for the modulus stabilization in modular flavor models, e.g. [34–37].) At such a minimum, the F-term of the modulus $F_\tau$ may be non-vanishing, and lead to SUSY breaking, the so-called moduli-mediated SUSY breaking [38–41], although there may be other sources of SUSY breaking. That leads to specific patterns of soft SUSY breaking terms. Thus, our purpose in this paper is to study such specific patterns of soft SUSY breaking terms due to $F_\tau$ and its phenomenological implications such as the lepton flavor violations.

We study the soft SUSY breaking terms in the modular flavor models of leptons. It is found that the soft SUSY breaking terms are constrained by the modular forms and there appears a specific pattern of soft SUSY breaking terms due to the modulus F-term in the modular flavor symmetric models. In order to discuss the soft SUSY breaking terms in the lepton flavor violation (LFV), we examine numerically $\mu \rightarrow e + \gamma$ and $\mu \rightarrow 3e$ decays and $\mu \rightarrow e$ conversion in nuclei in the modular

\footnote{Recently, in Ref. [42], SUSY breaking phenomenology was studied in the modular flavor $S_3$ invariant SU(5) GUT model [43] by assuming the F-term of 24 chiral field.}
flavor $A_4$ model. The SUSY breaking scale is significantly constrained by inputting the observed upper bound of the $\mu \to e + \gamma$ decay [44].

In section 2, we give a brief review on the modular symmetry. In section 3, we present the soft SUSY breaking terms in the modular flavor models. In section 4, we calculate LFV, e.g., the $\mu \to e + \gamma$ decay and $\mu \to 3e$ decays and $\mu \to e$ conversion in nuclei in terms of the soft SUSY breaking masses in the modular flavor $A_4$ models, and present numerical discussions. Section 5 is devoted to a summary. In Appendix A, the tensor product of the $A_4$ group is presented.

2 Modular group and modular forms

The modular group $\bar{\Gamma}$ is the group of linear fractional transformations $\gamma$ acting on the modulus $\tau$, belonging to the upper-half complex plane as:

$$\tau \longrightarrow \gamma \tau = \frac{a \tau + b}{c \tau + d}, \quad \text{where } a, b, c, d \in \mathbb{Z} \text{ and } ad - bc = 1, \quad \text{Im}[\tau] > 0,$$

which is isomorphic to $PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z})/\{I, -I\}$ transformation. This modular transformation is generated by $S$ and $T$,

$$S : \tau \longrightarrow -\frac{1}{\tau}, \quad T : \tau \longrightarrow \tau + 1,$$

which satisfy the following algebraic relations,

$$S^2 = I, \quad (ST)^3 = I.$$

We introduce the series of groups $\Gamma(N)$ ($N = 1, 2, 3, \ldots$), called principal congruence subgroups, defined by

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \mid (a \ b) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}. \quad (4)$$

For $N = 2$, we define $\bar{\Gamma}(2) \equiv \Gamma(2)/\{I, -I\}$. Since the element $-I$ does not belong to $\Gamma(N)$ for $N > 2$, we have $\bar{\Gamma}(N) \equiv \Gamma(N)$. The quotient groups defined as $\Gamma_N \equiv \bar{\Gamma}/\bar{\Gamma}(N)$ are finite modular groups. In this finite groups $\Gamma_N$, $T^N = I$ is imposed. The groups $\Gamma_N$ with $N = 2, 3, 4, 5$ are isomorphic to $S_3$, $A_4$, $S_4$ and $A_5$, respectively [30].

Modular forms of level $N$ are holomorphic functions $f(\tau)$ transforming under $\Gamma(N)$ as:

$$f(\gamma \tau) = (c\tau + d)^k f(\tau), \quad \gamma \in \Gamma(N),$$

where $k$ is the so-called modular weight.

The low-energy effective field theory derived from superstring theory has also the modular symmetry. Under the modular transformation of Eq.(1), chiral superfields $\phi$ transform as [45],

$$\phi_i \rightarrow (c\tau + d)^{-k_i} \rho_{ij}(\gamma)\phi_j,$$

where $-k_i$ is the modular weight and $\rho_{ij}(\gamma)$ denotes a unitary representation matrix of $\gamma \in \bar{\Gamma}$.

We study global supersymmetric models, e.g., the minimal supersymmetric standard model. The superpotential, which is built from matter fields and modular forms, is assumed to be modular.
invariant, i.e., to have a vanishing modular weight. For given modular forms this can be achieved by assigning appropriate weights to the matter superfields.

The kinetic terms are derived from a Kähler potential. The Kähler potential of chiral matter fields $\phi_i$ with the modular weight $-k_i$ is given simply by

$$K_{\text{matter}} = K_{\bar{i}i}|\phi_i|^2, \quad K_{\bar{i}i} = \frac{1}{i(\bar{\tau} - \tau)^{|k_i|}},$$

where the superfield and its scalar component are denoted by the same letter, and $\bar{\tau} = \tau^*$ after taking the VEV. Therefore, the canonical form of the kinetic terms is obtained by changing the normalization.

The modular forms of weight $k$ span the linear space $\mathcal{M}_k(\Gamma(N))$. For example, for $\Gamma_3 \simeq A_4$, the dimension of the linear space $\mathcal{M}_k(\Gamma(3))$ is $k + 1$ [46, 48], i.e., there are three linearly independent modular forms of the lowest non-trivial weight 2. These forms have been explicitly obtained [23] in terms of the Dedekind eta-function $\eta(\tau)$:

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad q = \exp(i2\pi\tau),$$

where $\eta(\tau)$ is a so called modular form of weight $1/2$. In what follows we will use the following basis of the $A_4$ generators $S$ and $T$ in the triplet representation:

$$S = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix},$$

where $\omega = \exp(i\frac{2}{3}\pi)$. The modular forms of weight 2 transforming as a triplet of $A_4$ can be written in terms of $\eta(\tau)$ and its derivative [23]:

$$Y_1 = \frac{i}{2\pi} \left( \frac{\eta'(\tau/3)}{\eta(\tau/3)} + \frac{\eta'((\tau + 1)/3)}{\eta((\tau + 1)/3)} + \frac{\eta'((\tau + 2)/3)}{\eta((\tau + 2)/3)} - \frac{27\eta'(3\tau)}{\eta(3\tau)} \right),$$

$$Y_2 = -\frac{i}{\pi} \left( \frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega^2 \frac{\eta'((\tau + 1)/3)}{\eta((\tau + 1)/3)} + \omega \frac{\eta'((\tau + 2)/3)}{\eta((\tau + 2)/3)} \right),$$

$$Y_3 = -\frac{i}{\pi} \left( \frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega \frac{\eta'((\tau + 1)/3)}{\eta((\tau + 1)/3)} + \omega^2 \frac{\eta'((\tau + 2)/3)}{\eta((\tau + 2)/3)} \right).$$

The overall coefficient in Eq. (10) is one possible choice. It cannot be uniquely determined. The triplet modular forms of weight 2 have the following $q$-expansions:

$$Y_3^{(2)} = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} = \begin{pmatrix} 1 + 12q + 36q^2 + 12q^3 + \ldots \\ -6q^{1/3}(1 + 7q + 8q^2 + \ldots) \\ -18q^{2/3}(1 + 2q + 5q^2 + \ldots) \end{pmatrix}.$$

They also satisfy the constraint [23]:

$$Y_2^2 + 2Y_1Y_3 = 0.$$

The modular forms of the higher weight, $k = 4, 6, 8 \ldots$, can be obtained by the $A_4$ tensor products of the modular forms with weight 2, $Y_3^{(2)}$, as given in Appendix A.
3 Soft SUSY breaking terms

We study soft SUSY breaking terms due to the modus F-term within the framework of supergravity theory, using the unit $M_P = 1$, where $M_P$ denotes the reduced Planck scale. The full Kähler potential is given as:

$$
K = K_0(\tau, M) + K^\text{matter},
$$

$$
K_0(\tau, M) = -\ln(i(\bar{\tau} - \tau)) + K(M, \bar{M}),
$$

(13)

where $M$ denotes moduli other than $\tau$. The superpotential is given as:

$$
W = Y_{ijk}(\tau)\Phi_i\Phi_j\Phi_k + M_{ij}(\tau)\Phi_i\Phi_j \cdots.
$$

(14)

We assume that the gauge kinetic function is independent of the modulus $\tau$, i.e. $f(M)$.

Let us consider the case that the SUSY breaking is occurred by some F-terms of moduli $X$, $F^X$ ($F^X \neq 0$). In the canonical normalization, the soft masses $\tilde{m}_i$ and the A-term are given as [38]:

$$
\tilde{m}_i^2 = m_{3/2}^2 - \sum_X |F^X|^2 \partial_X \partial_{\bar{X}} \ln K_{ii},
$$

(15)

and

$$
A_{ijk} = A_i + A_j + A_k - \sum_X F^X \partial_X Y_{ijk},
$$

$$
A_i = \sum_X F^X \partial_X \ln e^{-K_0/3} K_{ii},
$$

(16)

where $i$, $j$ and $k$ denote flavors. Here, Yukawa couplings $\tilde{Y}_{ijk}$ in global SUSY superpotential are related with Yukawa couplings $Y_{ijk}$ in the supergravity superpotential as follows:

$$
|\tilde{Y}_{ijk}|^2 = e^{K_0/3} |Y_{ijk}|^2.
$$

(17)

That is, the global SUSY superpotential has vanishing modular weight, while the supergravity superpotential has the modular weight $-1$. Some modular flavor models are studied in global SUSY basis. At any rate, we can realize the same results of quark and lepton mass ratios and mixing angles by properly shifting assignment of modular weights for matter fields.

Suppose the case of $X = \tau$. The Kähler potential $K$ in Eq. (13) leads to the soft mass

$$
\tilde{m}_i^2 = m_{3/2}^2 - k_i |F^\tau|^2 (2\text{Im}\tau)^2.
$$

(18)

The A-term is written by

$$
A_{ijk} = A^0_{ijk} + A'_{ijk},
$$

$$
A^0_{ijk} = (1 - k_i - k_j - k_k) \frac{F^\tau}{(2\text{Im}(\tau))^2},
$$

$$
A'_{ijk} = \frac{F^\tau}{Y_{ijk}} \frac{dY_{ijk}(\tau)}{d\tau}.
$$

(19)
Note that in our convention $\tau$ is dimensionless, and $F^\tau$ has the dimension one. Gaugino masses can be generated by F-terms of other moduli, $F^M$, while $F^M$ have universal contributions on soft masses and A-terms.

Since we have common weights for three generations in the simple modular flavor model, the soft mass $\tilde{m}_i$ is flavor blind. Therefore, we have universal mass matrices

$$\tilde{m}_{Li}^2 = \tilde{m}_L^2, \quad \tilde{m}_{ei}^2 = \tilde{m}_e^2,$$

that is, they are proportional to the unit matrix.

The first term of $A_{ijk}$ term in Eq. (19), $A^0_{ijk}$ is also flavor blind. If there is another source of SUSY breaking, $A^0_{ijk}$ is shifted by $\Delta A$ as

$$A^0_{ijk} + \Delta A = A^0,$$

where $\Delta A$ is also flavor blind. Therefore, we write

$$A_{ijk} = A^0 + A'_{ijk},$$

where the second term of r.h.s. in Eq. (22) only depends on the flavor.

### 4 A-term in modular $A_4$ flavor model

We discuss the soft SUSY breaking terms in Eq. (22) in the modular $A_4$ models. In order to present the explicit form of the A-term, we consider successful lepton mass matrices to be consistent with observed lepton masses and flavor mixing angles. A simple global SUSY model is shown in Table I, where the three left-handed lepton doublets $L$ compose a $A_4$ triplet, and the right-handed charged leptons $e^c$, $\mu^c$ and $\tau^c$ are $A_4$ singlets. $H_u$ and $H_d$ are Higgs doublets. The weight $k$’s of the superfields of left-handed leptons and right-handed charged leptons is $-2$ and $0$, respectively, which are common for three generations. The charged lepton mass matrix is given in terms of modular forms of $A_4$ triplet with weight 2, $Y^{(2)}_3$ simply. The neutrinos are supposed to be Majorana particles in Table I. Since there are no right-handed neutrinos, neutrino mass matrix can be written by using the Weinberg operator. Then, the neutrino mass term is given in terms of modular forms of $A_4$ triplet $Y^{(4)}_3$ and $A_4$ singlets $Y^{(4)}_1$ and $Y^{(4)}_1'$ with weight 4. This model has been discussed focusing on the flavor mixing numerically.

| $SU(2)$ | $A_4$ | $k$ | $L$ | $(e^c, \mu^c, \tau^c)$ | $H_u$ | $H_d$ | $Y_r^{(2)}$ | $Y_r^{(4)}$ |
|--------|-------|-----|-----|----------------------|-----|-----|-----------|-----------|
| 2      | 3     | -2  | 1   | 1                    | 2   | 2   | 3, {3, 1, 1'} | 1         |
| 1      | 1''   | 0   | 0   | 0                    | 0   | 0   | 2, 4      |           |

Table 1: Representations of $SU(2)$, $A_4$ and weights $k$ for matter fields and modular forms of weight 2 and 4. The subscript $r$ represents the $A_4$ representation of modular forms.

\footnote{We can construct the model, where the same modular forms appear in the Yukawa couplings and the Weinberg operators in the supergravity superpotential, by properly shifting the assignment of modular weights for matter fields.}
For charged lepton sector, Yukawa couplings $Y_{ijk}$ are given in terms of modular forms in Eq. (11)

$$Y_{ijk} = \text{diag}[\alpha_e, \beta_e, \gamma_e]\begin{pmatrix} Y_1 & Y_3 & Y_2 \\ Y_2 & Y_1 & Y_3 \\ Y_3 & Y_2 & Y_1 \end{pmatrix}_{RL},$$

(23)

where coefficients $\alpha_e, \beta_e$ and $\gamma_e$ are taken to be real without loss of generality.

Since the A-term is given in Eq. (22), the soft mass term $h_{ijk} = Y_{ijk} A_{ijk}$ is given

$$h_{ijk} = A_0 \times \text{diag}[\alpha_e, \beta_e, \gamma_e]\begin{pmatrix} Y_1 & Y_3 & Y_2 \\ Y_2 & Y_1 & Y_3 \\ Y_3 & Y_2 & Y_1 \end{pmatrix}_{RL} + F^\tau \times \text{diag}[\alpha_e, \beta_e, \gamma_e]\begin{pmatrix} Y'_1 & Y'_3 & Y'_2 \\ Y'_2 & Y'_1 & Y'_3 \\ Y'_3 & Y'_2 & Y'_1 \end{pmatrix}_{RL},$$

(24)

where $Y'$ is the derivative of $Y$ with respect to $\tau$.

In the super-CKM (SCKM) basis, the first term of the r.h.s. in Eq. (24) is diagonal. Therefore, the second term of the r.h.s. only contributes to the LFV. Since the modulus $\tau$ and couplings $\alpha_e, \beta_e, \gamma_e$ are fixed by the experimental data of neutrino oscillations in the model of Table 1, we can estimate the magnitude of LFV if $F^\tau$ is given.

These expressions are given at high energy scale, for example, GUT scale. The effects of the renormalization group (RG) running on the soft mass terms should be taken into account at the electroweak (EW) scale. We consider the small tan $\beta$ scenario, where the Yukawa couplings of charged leptons and down-type quarks are small. Then, the largest contributions of the effect for off-diagonal elements of the A-term are those of gauge couplings. Then, we can estimate the running effects by

$$A_{ijk}(m_Z) = \exp \left[ \frac{-1}{16\pi^2} \int_{m_Z}^{m_{\text{GUT}}} dt \left( \frac{9}{5} g_1^2 + 3 g_2^2 \right) \right] A_{ijk}(m_{\text{GUT}}) \approx 1.5 \times A_{ijk}(m_{\text{GUT}}),$$

(25)

which is flavor independent. Thus, the RG effect does not change the flavor structure at the high energy scale. The RG effect can be absorbed in the averaged slepton mass at the low energy. On the other hand, we do not need to discuss the A-term of the neutrino sector because there are no right-handed neutrinos.

5 LFV in SUSY flavor violation

The SUSY flavor phenomena of LFV for the lepton sector were discussed by introducing gauge singlet scalars (flavons) in the non-Abelian discrete symmetry \[52,53\]. In contrast to previous works, our modular flavor $A_4$ models constrain the flavor changing processes significantly via modular forms as discussed in the previous section.

Let us define mass insertion parameters, $\delta_{\ell}^{LL}, \delta_{\ell}^{LR}, \delta_{\ell}^{RL}$ and $\delta_{\ell}^{RR}$ by

$$m^2_{\ell} \begin{pmatrix} \delta_{\ell}^{LL} & \delta_{\ell}^{LR} \\ \delta_{\ell}^{RL} & \delta_{\ell}^{RR} \end{pmatrix} = \begin{pmatrix} \tilde{m}^2_L & \tilde{m}^2_{LR} \\ \tilde{m}^2_{RL} & \tilde{m}^2_e \end{pmatrix} - \text{diag}(m^2_{\ell}),$$

(26)

where $m_{\ell}$ is an average slepton mass. Here, $\tilde{m}^2_L$ and $\tilde{m}^2_R$ are universal diagonal matrices as given in Eq. (20). By using $h_{ijk} = Y_{ijk} A_{ijk}$ in Eq. (24), we get $\tilde{m}^2_{RL} = v_d h_{ijk}$, where $v_d$ is the VEV of the neutral component of Higgs doublet $H_d$. We have also $\tilde{m}^2_{LR} = \tilde{m}^2_{RL}$. 
Let us examine the effect of the A-term on the LFV rare decay such as $\ell_i \to \ell_j + \gamma$, $\ell_i \to \ell_j \bar{\ell}_k$ and LFV conversion $\mu N \to eN$. Once non-vanishing off diagonal elements of the slepton mass matrices are generated in the SCKM basis, the LFV rare decays and conversion are naturally induced by one-loop diagrams with the exchange of gauginos and sleptons.

The decay $\ell_i \to \ell_j + \gamma$ is described by the dipole operator and the corresponding amplitude reads [54–60],

$$T = m_\ell e^3 q_j (p - q) [i q' \sigma_{\mu \nu} (A_{LL}^{ij} P_L + A_{LR}^{ij} P_R)] u_i (p),$$

where $p$ and $q$ are momenta of the initial lepton $\ell_i$ and of the photon, respectively, and $A_{LL}^{ij}$, $A_{LR}^{ij}$ are the two possible amplitudes in this process. The branching ratio of $\ell_i \to \ell_j + \gamma$ can be written as follows:

$$\frac{\text{BR}(\ell_i \to \ell_j \gamma)}{\text{BR}(\ell_i \to \ell_j \bar{\ell}_k \ell_k)} \simeq \frac{48 \pi^3 \alpha}{G_F^2} (|A_{LL}^{ij}|^2 + |A_{LR}^{ij}|^2),$$

where $\alpha$ is the electromagnetic fine-structure constant. In the mass insertion approximation, the A-term contribution is that

$$A_{LL}^{ij} \simeq \frac{\alpha_1}{4 \pi} \frac{\delta_{RL}^{ij}}{m_\ell^2} \left( \frac{M_1}{m_\ell} \right) \times 2 f_2 (x_1),$$
$$A_{LR}^{ij} \simeq \frac{\alpha_1}{4 \pi} \frac{\delta_{LR}^{ij}}{m_\ell^2} \left( \frac{M_1}{m_\ell} \right) \times 2 f_2 (x_1),$$

where $x_1 = M_1^2 / m_\ell^2$ and $\alpha_1 = g_1^2 / 4 \pi$. Mass parameters $M_1$ and $m_\ell$ are the $U(1)_Y$ gaugino mass and the charged lepton mass, respectively. The loop function $f_2 (x)$ is given explicitly as follows:

$$f_2 (x) = \frac{-5 x^2 + 4 x + 1 + 2 x (x + 2) \log x}{4 (1 - x)^4}.$$

The mass insertion parameters $\delta_{RL}^{ij}$ and $\delta_{LR}^{ij}$ are given in Eq. [26]. The contributions of $\delta_{LL}^{ij}$ and $\delta_{RR}^{ij}$ are neglected because the off diagonal components vanish as discussed in Eq. [20]. In SUSY models, the branching ratio of $\ell_i \to 3 \ell_j$ and conversion rate of $\mu N \to eN$ also can be related as

$$\frac{\text{BR}(\ell_i \to 3 \ell_j)}{\text{BR}(\ell_i \to \ell_j \gamma)} \simeq \frac{\alpha}{3 \pi} \left( 2 \log \frac{m_\ell}{m_\ell} - 3 \right),$$
$$\frac{\text{CR}(\mu N \to eN)}{\text{BR}(\ell_i \to \ell_j \gamma)} \simeq \alpha.$$

In numerical calculations of the $\mu \to e + \gamma$ ratio, we take a sample parameter set to be consistent with the observed lepton masses and flavor mixing angles in the model of Table 1 [49–51] as follows:

$$A : \tau = -0.0796 + 1.0065 i, \quad \alpha e / \gamma e = 6.82 \times 10^{-2}, \quad \beta e / \gamma e = 1.02 \times 10^{-3},$$

which is referred as the parameter set $A$. This model favors the modulus $\tau$ being close to $i$, where important physics such as $CP$ and the hierarchy of fermion masses are revealed [36, 61, 63]. In this sample, $\tan \beta = 5$ is taken in order to fit the lepton masses at hight energy scale [49–51]. On the
other hand, the SUSY mass parameters are the gaugino mass \( M_1 \) and the averaged slepton mass \( m_{\tilde{\ell}} \), which are low energy observables at the EW scale. The SUSY breaking parameter \( F^\tau \) is expected to be the same order compared with \( m_{\tilde{\ell}} \).

In order to see the SUSY mass scale dependence for the \( \mu \to e + \gamma \) branching ratio, we show them (red curves) by taking the averaged mass scale \( m_0 \equiv m_{\tilde{\ell}} = F^\tau \) with \( M_1 = 3 \) TeV (solid curve) and 5 TeV (dashed curve) for simplicity in Fig. 1. The SUSY mass scale \( m_0 \) should be larger than around 8 TeV to be consistent with the observed upper bound (black line). The predicted LFV branching ratio can be examined by future experiment such as MEG-II [64] (orange line) up to \( m_0 \approx 12 \) TeV. The magnitudes of the mass insertion parameters are proportional to \( F^\tau \). Those are given as

\[
| (\delta_{RL}^{\mu e}) | \approx 2.1 \times 10^{-5} \left( \frac{F^\tau}{10 \text{ TeV}} \right), \quad | (\delta_{LR}^{\mu e}) | \approx 9.7 \times 10^{-8} \left( \frac{F^\tau}{10 \text{ TeV}} \right). \tag{34}
\]

Therefore, the amplitude \( | A_{LR}^{\mu e} | \) is much larger than \( | A_{RL}^{\mu e} | \).

In this calculation, the model of lepton mass matrices is fixed as seen in Table 1. There are variant neutrino mass matrices in modular flavor \( A_4 \) model, where the charged lepton mass matrix is the same one in Eq. (23) [69–71]. In those models, the A-term of the neutrino sector appears due to right-handed neutrinos. However, this contribution is suppressed as far as the seesaw mechanism works at high energy.

A simple alternative model is presented in Table 2 where neutrino masses are generated via seesaw mechanism by introducing the right-handed neutrino \( \nu^c \). A sample parameter set, referred the parameter set B, to be consistent with the observed lepton masses and flavor mixing angles as
| $SU(2)$ | $A_4$ | $k$ | \(L\) | \((e^c, \mu^c, \tau^c)\) | \(\nu^c\) | \(H_u\) | \(H_d\) | \(Y^{(2)}\) |
|--------|------|-----|------|----------------|------|-----|-----|------|
| 2 | (1, 1', 1'') | 3 | -1 | 1 | 2 | 2 | 1 | 2 | 2 | 1 | 1 | 0 | 0 | 2 |

Table 2: Representations of $SU(2)$, $A_4$, and the modular weight in the type I seesaw model.

follows $^{69,70}$:

$$B: \quad \tau = 0.48151 + 1.30262 i, \quad \alpha_e/\gamma_e = 2.03 \times 10^2, \quad \beta_e/\gamma_e = 3.30 \times 10^3, \quad (35)$$

where $\tan \beta = 1$ is taken. The magnitude of modulus $\tau$ is larger than the one in Eq. $^{33}$. 

We also show the branching ratio (blue) versus the SUSY mass scale $m_0$ in Fig. 1. In this case, the SUSY mass scale $m_0$ should be larger than around 5 TeV to be consistent with the observed upper bound. The predicted LFV branching ratio can be examined by future experiment (orange line) up to $m_0 \simeq 8$ TeV. The magnitudes of the mass insertion parameters are given as

$$|(\delta^{RL}_{\mu e})_{\mu e}| \simeq 8.4 \times 10^{-6} \left( \frac{F^\tau}{10 \text{ TeV}} \right), \quad |(\delta^{LR}_{\mu e})_{\mu e}| \simeq 3.7 \times 10^{-8} \left( \frac{F^\tau}{10 \text{ TeV}} \right). \quad (36)$$

The amplitude $|A_{\mu e}^L|$ is much larger than $|A_{\mu e}^R|$ as well as the case of the parameter set A.

The predicted branching ratio apparently decreases as the magnitude of $\tau$ increase as seen in Fig. 1. In order to see the $\text{Im} \tau$ dependence of the branching ratio, we show the branching ratio versus $\text{Im} \tau$ in Fig. 2, where solid and dashed curves correspond to $F^\tau = m_\tilde{t}$ and $F^\tau = M_1$ with $m_\tilde{t} = 10 \text{ TeV}$ and $M_1 = 3 \text{ TeV}$, respectively. We choose $|\text{Re} \tau| = 0, 0.25, 0.5$ in the fundamental region $SL(2, Z)$. For each $\tau$ of Fig. 2, we do not take into account lepton mixing angles consistent with observed ones since they depend on the model of the neutrino mass matrix. The charged lepton mass matrix of Eq. $^{23}$ is completely determined by inputting observed charged lepton masses if $\tau$ is fixed. Therefore, we can see the $\text{Im} \tau$ dependence of the branching ratio for each $\text{Re} \tau$ as seen in Fig. 2. The branching ratio depends on both $\text{Im} \tau$ and $\text{Re} \tau$ significantly below $\text{Im} \tau \simeq 1.4$. Thus, the branching ratio changes more than one order depending on $\tau$.

Figures 3 and 4 show the SUSY mass scale dependence for the $\mu \to 3e$ branching ratio and $\mu N \to eN$ conversion rate in the same parameter sets in Fig. 1. We can see that the predicted branching ratio and conversion rate are enough below the current experimental bound for $m_0 > 3$ TeV. Future experiments will explore these predictions at the level of $10^{-16}$ $^{65,68}$, which corresponds to $m_0 \simeq 10 - 16$ TeV in $\mu \to 3e$ decay and $11 - 17$ TeV in $\mu \to e$ conversion.

In conclusion, the current experimental search for the $\mu \to e + \gamma$ decay provides a clue of the SUSY particles at the 5–10 TeV scale in the modular flavor models. The predictions of modular flavor models will be examined in future experimental searches up to 8–17 TeV scale. Lastly, we comment on the Higgs mass and SUSY particle masses. There exist many parameters to determine the soft masses unless SUSY breaking mechanism is specified. Adjusting those parameters, it will be possible to obtain the Higgs and SUSY spectrum which is consistent with the current LHC bounds. However, such analyses are beyond the scope of this paper and left for future studies.

We can also calculate the branching ratios of tauon decays, $\tau \to e + \gamma$ and $\tau \to \mu + \gamma$. Their predicted branching ratios are at most $\mathcal{O}(10^{-15})$, which are much below the current experimental bounds. The present and future bounds on these processes are summarized in Table $^{31}$ $^{44,72}$ and $^{64,68}$.
We have studied the soft SUSY breaking terms due to the modulus F-term in the modular flavor models of leptons. It is found that the soft SUSY breaking terms are constrained by the modular forms, and the specific pattern of soft SUSY breaking terms appears.

Those phenomenological implications have been discussed in such as the lepton flavor violation, $\mu \to e + \gamma$ and $\mu \to 3e$ decays, and $\mu \to e$ conversion. In order to examine numerically, parameter sets A and B are adopted in two modular flavor $A_4$ models. The SUSY mass scale is significantly constrained by inputting the observed upper bound of the $\mu \to e + \gamma$ decay. The SUSY mass scale is larger than around 8 TeV and 5 TeV for parameter sets A and B, respectively. Therefore, the current experimental upper bound for the $\mu \to e + \gamma$ decay corresponds to the new physics of the SUSY particles at the 5–10 TeV scale in the modular flavor $A_4$ models. The predicted branching ratio and conversion rate will be examined by future experiments for the SUSY scale up to 8–17 TeV. The branching ratio depends on $\tau$ significantly. It decreases of one order at the large Im $\tau$. We have also calculated the branching ratios of tauon decays, $\tau \to e + \gamma$ and $\tau \to \mu + \gamma$. Their predicted branching ratios are at most $O(10^{-15})$, which are much below the current experimental bounds.

It is important to perform similar analyses in other modular flavor models. These specific patterns of soft SUSY breaking terms of the modular flavor models can be tested in the future experiments of the lepton flavor violations.

## 6 Summary

Table 3: Present and future upper bounds of the lepton flavor violation for each process [44,72] and [64–68].
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Appendix

A  Tensor product of $A_4$ group

We take the generators of $A_4$ group for the triplet as follows:

\[
S = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix},
\]

(37)

where $\omega = e^{i\frac{2\pi}{3}}$ for a triplet. In this base, the multiplication rule is

\[
\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_3 \otimes \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}_3 = (a_1 b_1 + a_2 b_3 + a_3 b_2)_1 \oplus (a_3 b_3 + a_1 b_2 + a_2 b_1)_1',
\]

\[
\oplus \frac{1}{3} \begin{pmatrix} 2a_1 b_1 - a_2 b_3 - a_3 b_2 \\ 2a_3 b_3 - a_1 b_2 - a_2 b_1 \\ 2a_2 b_2 - a_1 b_3 - a_3 b_1 \end{pmatrix}_3 \oplus \frac{1}{2} \begin{pmatrix} a_2 b_2 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \\ a_3 b_1 - a_1 b_3 \end{pmatrix}_3,
\]

\[
1 \otimes 1 = 1, \quad 1' \otimes 1' = 1'', \quad 1'' \otimes 1'' = 1', \quad 1' \otimes 1'' = 1,
\]

(38)

where

\[
T(1') = \omega, \quad T(1'') = \omega^2.
\]

(39)

More details are shown in the review [2][3].

By using above tensor products of the modular forms with weight 2, $Y_3^{(2)}(\tau)$, the modular forms of the higher weight, $k$, are obtained. For weight 4, that is $k = 4$, there are five modular forms by the tensor product of $3 \otimes 3$ as:

\[
Y_1^{(4)}(\tau) = Y_1(\tau)^2 + 2Y_2(\tau)Y_3(\tau), \quad Y_1'(\tau) = Y_3(\tau)^2 + 2Y_1(\tau)Y_2(\tau),
\]

\[
Y_1''(\tau) = Y_2(\tau)^2 + 2Y_1(\tau)Y_3(\tau) = 0, \quad Y_3^{(4)}(\tau) = \begin{pmatrix} Y_1^{(4)}(\tau) \\ Y_1'^{(4)}(\tau) \\ Y_1''^{(4)}(\tau) \end{pmatrix} = \begin{pmatrix} Y_1(\tau)^2 - Y_2(\tau)Y_3(\tau) \\ Y_3(\tau)^2 - Y_1(\tau)Y_2(\tau) \\ Y_2(\tau)^2 - Y_1(\tau)Y_3(\tau) \end{pmatrix},
\]

(40)

where $Y_1''^{(4)}(\tau)$ vanishes due to the constraint of Eq. (12).

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