A Note on Minimal Hypersurfaces of an Odd Dimensional Sphere

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Abstract: We obtain the Wang-type integral inequalities for compact minimal hypersurfaces in the unit sphere $S^{2n+1}$ with Sasakian structure and use these inequalities to find two characterizations of minimal Clifford hypersurfaces in the unit sphere $S^{2n+1}$.

Keywords: clifford minimal hypersurfaces; sasakian structure; integral inequalities; reeb function; contact vector field

MSC: 53C40; 53C42; 53C25

1. Introduction

Let $M$ be a compact minimal hypersurface of the unit sphere $S^{n+1}$ with shape operator $A$. In his pioneering work, Simons [1] has shown that on a compact minimal hypersurface $M$ of the unit sphere $S^{n+1}$ either $A = 0$ (totally geodesic), or $\|A\|^2 = n$, or $\|A\|^2 (p) > n$ for some point $p \in M$, where $\|A\|$ is the length of the shape operator. This work was further extended in [2] and for compact constant mean curvature hypersurfaces in [3]. If for every point $p$ in $M$, the square of the length of the second fundamental form of $M$ is $n$, then it is known that $M$ must be a subset of a Clifford minimal hypersurface

$$S^l \left( \sqrt{\frac{l}{n}} \right) \times S^m \left( \sqrt{\frac{m}{n}} \right),$$

where $l, m$ are positive integers, $l + m = n$ (cf. Theorem 3 in [4]). Note that this result was independently proven by Lawson [2] and Chern, do Carmo, and Kobayashi [5]. One of the interesting questions in differential geometry of minimal hypersurfaces of the unit sphere $S^{n+1}$ is to characterize minimal Clifford hypersurfaces. Minimal hypersurfaces have also been studied in (cf. [6–8]). In [2], bounds on Ricci curvature are used to find a characterization of the minimal Clifford hypersurfaces in the unit sphere $S^4$. Similarly in [3,9–11], the authors have characterized minimal Clifford hypersurfaces in the odd-dimensional unit spheres $S^3$ and $S^5$ using constant contact angle. Wang [12] studied compact minimal hypersurfaces in the unit sphere $S^{n+1}$ with two distinct principal curvatures, one of them being simple and obtained the following integral inequality,

$$\int_M \|A\|^2 \leq n\text{Vol}(M),$$
where \( Vol(M) \) is the volume of \( M \). Moreover, he proved that equality in the above inequality holds if and only if \( M \) is the Clifford hypersurface,

\[
S^1 \left( \sqrt{\frac{T}{n}} \right) \times S^m \left( \sqrt{\frac{n - 1}{n}} \right).
\]

In this paper, we are interested in studying compact minimal hypersurfaces of the unit sphere \( S^{2n+1} \) using the Sasakian structure \((\varphi, \xi, \eta, g)\) (cf. [13]) and finding characterizations of minimal Clifford hypersurface of \( S^{2n+1} \). On a compact minimal hypersurface \( M \) of the unit sphere \( S^{2n+1} \), we denote by \( N \) the unit normal vector field and define a smooth function \( f = g(\xi, N) \), which we call the Reeb function of the minimal hypersurface \( M \). Also, on the hypersurface \( M \), we have a smooth vector field \( v = \varphi(N) \), which we call the contact vector field of the hypersurface (\( v \) being orthogonal to \( \xi \) belongs to contact distribution). Instead of demanding two distinct principal curvatures one being simple, we ask the contact vector field \( v \) of the minimal hypersurface in \( S^{2n+1} \) to be conformal vector field and obtain an inequality similar to Wang’s inequality and show that the equality holds if and only if \( M \) is isometric to a Clifford hypersurface. Indeed we prove

**Theorem 1.** Let \( M \) be a compact minimal hypersurface of the unit sphere \( S^{2n+1} \) with Reeb function \( f \) and contact vector field \( v \) a conformal vector field on \( M \). Then,

\[
\int_M (1 - f^2) \|A\|^2 \leq 2n \int_M (1 - f^2)
\]

and the equality holds if and only if \( M \) is isometric to the Clifford hypersurface \( S^l \left( \sqrt{\frac{l}{2n}} \right) \times S^m \left( \sqrt{\frac{m}{2n}} \right) \),

where \( l + m = 2n \).

Also in [12], Wang studied embedded compact minimal non-totally geodesic hypersurfaces in \( S^{n+1} \) those are symmetric with respect to \( n + 2 \) pair-wise orthogonal hyperplanes of \( R^{n+2} \). If \( M \) is such a hypersurface, then it is proved that

\[
\int_M \|A\|^2 \geq nVol(M),
\]

and the equality holds precisely if \( M \) is a Clifford hypersurface. Note that compact embedded hypersurface has huge advantage over the compact immersed hypersurface, as it divides the ambient unit sphere \( S^n \) into two connected components.

In our next result, we consider compact immersed minimal hypersurface \( M \) of the unit sphere \( S^{2n+1} \) such that the Reeb function \( f \) is a constant along the integral curves of the contact vector field \( v \) and show that above inequality of Wang holds, and we get another characterization of minimal Clifford hypersurface in the unit sphere \( S^{2n+1} \). Precisely, we prove the following.

**Theorem 2.** Let \( M \) be a compact minimal hypersurface of the unit sphere \( S^{2n+1} \) with Reeb function \( f \) a constant along the integral curves of the contact vector field \( v \). Then,

\[
\int_M \|A\|^2 \geq 2nVol(M)
\]

and the equality holds if and only if \( M \) is isometric to the Clifford hypersurface \( S^l \left( \sqrt{\frac{l}{2n}} \right) \times S^m \left( \sqrt{\frac{m}{2n}} \right) \),

where \( l + m = 2n \).
2. Preliminaries

Recall that conformal vector fields play an important role in the geometry of a Riemannian manifold. A conformal vector field $v$ on a Riemannian manifold $(M, g)$ has local flow consisting of conformal transformations, which is equivalent to

$$L_v g = 2\rho g. \quad (1)$$

The smooth function $\rho$ appearing in Equation (1) defined on $M$ is called the potential function of the conformal vector field $v$. We denote by $(\varphi, \xi, \eta, g)$ the Sasakian structure on the unit sphere $S^{2n+1}$ as a totally umbilical real hypersurface of the complex space form $(\mathbb{C}^{n+1}, \mathcal{J}, \langle \cdot, \cdot \rangle)$, where $\mathcal{J}$ is the complex structure and $\langle \cdot, \cdot \rangle$ is the Euclidean Hermitian metric. The Sasakian structure $(\varphi, \xi, \eta, g)$ on $S^{2n+1}$ consists of a $(1,1)$ skew symmetric tensor field $\varphi$, a smooth unit vector field $\xi$, a smooth 1-form $\eta$ dual to $\xi$, and the induced metric $g$ on $S^{2n+1}$ as real hypersurface of $C^{n+1}$ and they satisfy (cf. [13])

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta \circ \varphi = 0, \quad \eta(\xi) = 1, \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad (2)$$

and

$$(\nabla \varphi) (X, Y) = g(X, Y)\xi - \eta(Y)X, \quad \nabla_X \xi = -\varphi X, \quad (3)$$

where $X, Y$ are smooth vector fields, $\nabla$ is Riemannian connection on $S^{2n+1}$ and the covariant derivative

$$(\nabla \varphi)(X, Y) = \nabla_X \varphi Y - \varphi (\nabla_X Y).$$

We denote by $N$ and $A$ the unit normal and the shape operator of the hypersurface $M$ of the unit sphere $S^{2n+1}$. We denote the induced metric on the hypersurface $M$ by the same letter $g$ and denote by $\nabla$ the Riemannian connection on the hypersurface $M$ with respect to the induced metric $g$. Then, the fundamental equations of hypersurface are given by (cf. [14])

$$\nabla_X Y = \nabla_X Y + g(AX, Y), \quad \nabla_X N = -AX, \quad X, Y \in \mathcal{X}(M), \quad (4)$$

$$R(X, Y)Z = g(Y, Z)X - g(X, Z)Y + g(AY, Z)AX - g(AX, Z)AY, \quad (5)$$

$$(\nabla A)(X, Y) = (\nabla A)(Y, X), \quad X, Y \in \mathcal{X}(M), \quad (6)$$

where $\mathcal{X}(M)$ is the Lie algebra of smooth vector fields and $R(X, Y)Z$ is the curvature tensor field of the hypersurface $M$. The Ricci tensor of the minimal hypersurface $M$ of the unit sphere $S^{2n+1}$ is given by

$$\text{Ric}(X, Y) = (2n - 1)g(X, Y) - g(AX, AY), \quad X, Y \in \mathcal{X}(M) \quad (7)$$

and

$$\sum_{i=1}^{2n} (\nabla A)(e_i, e_i) = 0 \quad (8)$$

holds for a local orthonormal frame $\{e_1, \ldots, e_{2n}\}$ on the minimal hypersurface $M$.

Using the Sasakian structure $(\varphi, \xi, \eta, g)$ on the unit sphere $S^{2n+1}$, we analyze the induced structure on a hypersurface $M$ of $S^{2n+1}$. First, we have a smooth function $f$ on the hypersurface $M$ defined by $f = g(\xi, N)$, which we call the Reeb function of the hypersurface $M$, where $N$ is the unit normal vector field. As the operator $\varphi$ is skew symmetric, we get a vector field $v = \varphi N$ defined on $M$, which we call the contact vector field of the hypersurface $M$. Note that the vector field $v$ is orthogonal to $\xi$, and therefore lies in the contact distribution of the Sasakian manifold $S^{2n+1}$. We denote by $u = \xi^T$ the tangential component of $\xi$ to the hypersurface $M$ and, consequently, we have $\xi = u + fN$. Let $\alpha$ and $\beta$ be smooth 1-forms on $M$ dual to the vector fields $u$ and $v$, respectively, that is, $\alpha(X) = g(X, u)$ and $\beta(X) = g(X, v), X \in \mathcal{X}(M)$. For $X \in \mathcal{X}(M)$, we set $JX = (\varphi X)^T$ the tangential component of $\varphi X$ to the hypersurface, which gives a skew symmetric $(1,1)$ tensor field $J$ on the hypersurface $M$. It follows
that $\varphi X = JX - \beta(X)N$. Thus, we get a structure $(J, u, v, \alpha, \beta, f, g)$ on the hypersurface $M$ and using properties in Equations (2) and (3) of the Sasakian structure $(\varphi, \xi, \eta, g)$ on the unit sphere $S^{2n+1}$ and Equation (4), it is straightforward to see that the structure $(J, u, v, \alpha, \beta, f, g)$ on the hypersurface $M$ has the properties described in the following Lemma.

**Lemma 1.** Let $M$ be a hypersurface of the unit sphere $S^{2n+1}$. Then, $M$ admits the structure $(J, u, v, \alpha, \beta, f, g)$ satisfying

(i) $f^2 = -1 + \alpha \otimes u + \beta \otimes v,$

(ii) $Ju = -f v,$ $Jv = fu,$

(iii) $g(JX, JY) = g(X, Y) - \alpha(X)\alpha(Y) - \beta(X)\beta(Y),$ 

(iv) $\nabla_X u = -fX + fAX,$ $\nabla_X v = -fX - fAX,$

(v) $(\nabla f)(X, Y) = g(X, Y)u - \alpha(Y)X + g(AX, Y)v - \beta(Y)AX,$

(vi) $\nabla f = -Au + v,$

(vii) $\|u\|^2 = \|v\|^2 = (1 - f^2),$ $g(u, v) = 0,$

where $\nabla f$ is the gradient of the Reeb function $f$.

Let $\Delta f$ be the Laplacian of the Reeb function $f$ of the minimal hypersurface $M$ of the unit sphere $S^{2n+1}$ defined by $\Delta f = \text{div} \nabla f$. Then using Lemma 1 and $\frac{1}{2} \Delta f^2 = f \Delta f + \|\nabla f\|^2$ and Equations (6) and (8), we get the following:

**Lemma 2.** Let $M$ be a minimal hypersurface of the unit sphere $S^{2n+1}$. Then, the Reeb function $f$ satisfies

(i) $\Delta f = -\left(2n + \|A\|^2\right)f,$

(ii) $\frac{1}{2} \Delta f^2 = -\left(2n + \|A\|^2\right)f^2 + \|\nabla f\|^2.$

On the hypersurface $M$ of the unit sphere $S^{2n+1}$, we define a $(1, 1)$ tensor field $\Psi = JA - AJ$, then it follows that $g(\Psi X, Y) = g(X, \Psi Y), X, Y \in \mathcal{X}(M)$, that is, $\Psi$ is symmetric and that $tr \Psi = 0$. Next, we prove the following:

**Lemma 3.** Let $M$ be a compact minimal hypersurface of the unit sphere $S^{2n+1}$. Then,

$$\int_M \left(1 - f^2\right) \|A\|^2 = \int_M \left(2n - 2n(2n + 1)f^2 + \frac{1}{2} \|\Psi\|^2\right).$$

**Proof.** Using Equation (7), we have $Ric(v, v) = (2n - 1) \|v\|^2 - \|Av\|^2$. Now, using Lemma 1, we get

$$(\mathcal{L}_0 \Psi)(X, Y) = -2f g(X, Y) - g(\Psi X, Y),$$

which on using the fact that $tr \Psi = 0$, gives

$$|\mathcal{L}_0 \Psi|^2 = 8nf^2 + \|\Psi\|^2.$$

Also, using (iii) of Lemma 1, we have

$$\|JA\|^2 = \|A\|^2 - \|Au\|^2 - \|Av\|^2,$$

which together with second equation in (iv) of Lemma 1 and the fact that $tr JA = 0$, implies

$$\|\nabla v\|^2 = 2nf^2 + \|A\|^2 - \|Au\|^2 - \|Av\|^2.$$
Note that second equation in (iv) of Lemma 1 also gives
\[
\text{div} v = -2nf.
\]

Now, inserting above values in the following Yano’s integral formula (cf. [15])
\[
\int_M \left( Ric(v,v) + \frac{1}{2} |\mathcal{L}_v g|^2 - ||\nabla v||^2 - (\text{div} v)^2 \right) = 0,
\]
we get
\[
\int_M \left( (2n-1) ||v||^2 + 2nf^2 + \frac{1}{2} ||\Psi||^2 - ||A||^2 + ||Au||^2 - 4n^2 f^2 \right) = 0. \tag{9}
\]

Also, (vi) of Lemma 1, gives \(Au = v - \nabla f\), that is, \(||Au||^2 = ||v||^2 + ||\nabla f||^2 - 2v(f)\), which on using \(\text{div}(fv) = v(f) + f\text{div} v = v(f) - 2nf\), gives
\[
||Au||^2 = ||v||^2 + ||\nabla f||^2 - 2\text{div}(fv) - 4nf^2.
\]

Inserting above value of \(||Au||^2\) in Equation (9), yields
\[
\int_M \left( 2n ||v||^2 - 2nf^2 + \frac{1}{2} ||\Psi||^2 - ||A||^2 + ||\nabla f||^2 - 4n^2 f^2 \right) = 0. \tag{10}
\]

Integrating (ii) of Lemma 2, we get
\[
\int_M ||\nabla f||^2 = \int_M \left( 2n + ||A||^2 \right) f^2,
\]
which together with \(||v||^2 = 1 - f^2\) and Equation (10) proves the integral formula. \(\square\)

**Lemma 4.** Let \(M\) be a minimal hypersurface of the unit sphere \(S^{2n+1}\). Then, the contact vector field \(v\) is a conformal vector field if and only if \(JA = AJ\).

**Proof.** Suppose that \(JA = AJ\). Then, using Lemma 1 and symmetry of shape operator \(A\) and skew symmetry of the operator \(J\), we have
\[
(\mathcal{L}_v g)(X,Y) = g(\nabla_X v, Y) + g(\nabla_Y v, X) = -2f g(X,Y), \quad X \in \mathfrak{X}(M),
\]
which proves that \(v\) is a conformal vector field with potential function \(-f\). Conversely, suppose \(v\) is conformal vector field with potential function \(\rho\). Then, using Equation (1), we have
\[
(\mathcal{L}_v g)(X,Y) = g(\nabla_X v, Y) + g(\nabla_Y v, X) = 2\rho g(X,Y),
\]
which on using Lemma 1, gives
\[
g(-JAX - fX,Y) + g(-JAY - fY,X) = 2\rho g(X,Y),
\]
that is,
\[
g(AJX - JAX,Y) = 2(\rho + f) g(X,Y).
\]

Choosing a local orthonormal frame \(\{e_1, \ldots, e_{2n}\}\) on the minimal hypersurface \(M\) and taking \(X = Y = e_i\) in above equation and summing, we get \(\rho = -f\). This gives \(g(AJX - JAX,Y) = 0\), \(X, Y \in \mathfrak{X}(M)\), that is, \(JA = AJ\). \(\square\)
Lemma 5. Let $M$ be a minimal hypersurface of the unit sphere $S^{2n+1}$. If the contact vector field $v$ is a conformal vector field on $M$, then

$$A_H = \frac{\|A\|^2}{2n}v.$$

Proof. Suppose $v$ is a conformal vector field. Then, by Lemma 4, we have $fA = Af$. Note that for the Hessian operator $A_f$ of the Reeb function $f$ using Lemma 1, we have

$$A_f(X) = \nabla_X \nabla f = \nabla_X(v - Au) = -fX - \nabla_X Au, \quad X \in \mathfrak{X}(M),$$

which on using (vi) of Lemma 1, gives

$$A_f(X) = -f(X + A^2X) - (\nabla A)(X, u).$$

Taking covariant derivative in above equation gives

$$\left(\nabla A_f\right)(X, Y) = -X(f)((Y + A^2Y) - f \left(\nabla A^2\right)(X, Y) - \left(\nabla^2 A\right)(X, Y, u) + (\nabla A)(Y, AX) - f (\nabla A)(Y, AX),$$

where we used (iv) of Lemma 1. Now, on taking a local orthonormal frame $\{e_1, \ldots, e_{2n}\}$ on the minimal hypersurface $M$ and taking $X = Y = e_i$ in above equation and summing, we get

$$\sum_{i=1}^{2n} \left(\nabla A_f\right)(e_i, e_i) = -\nabla f - A^2 \nabla f - f \sum_{i=1}^{2n} \left(\nabla A^2\right)(e_i, e_i) - \sum_{i=1}^{2n} \left(\nabla^2 A\right)(e_i, e_i, u) + \sum_{i=1}^{2n} \left(\nabla A\right)(e_i, Ae_i).$$

Note that for the minimal hypersurface, we have

$$\sum_{i=1}^{2n} \left(\nabla A\right)(e_i, Ae_i) = \sum_{i=1}^{2n} \left(\nabla A^2\right)(e_i, e_i) - \sum_{i=1}^{2n} \left(\nabla^2 A\right)(e_i, e_i, u) + \sum_{i=1}^{2n} \left(\nabla A\right)(e_i, J e_i).$$

Thus, the previous equation takes the form

$$\sum_{i=1}^{2n} \left(\nabla A_f\right)(e_i, e_i) = -\nabla f - A^2 \nabla f - 2f \sum_{i=1}^{2n} \left(\nabla A^2\right)(e_i, e_i) - \sum_{i=1}^{2n} \left(\nabla^2 A\right)(e_i, e_i, u) + \sum_{i=1}^{2n} \left(\nabla A\right)(e_i, J e_i). \quad (11)$$

Now, using the definition of Hessian operator, we have

$$R(X, Y)\nabla f = \left(\nabla A_f\right)(X, Y) - \left(\nabla A_f\right)(Y, X),$$

which gives

$$\text{Ric}(Y, \nabla f) = g \left( Y_i \sum_{i=1}^{2n} \left(\nabla A_f\right)(e_i, e_i) - Y(Af) \right)$$

and we conclude

$$Q(\nabla f) = -\nabla(\Delta f) + \sum_{i=1}^{2n} \left(\nabla A_f\right)(e_i, e_i), \quad (12)$$
where $Q$ is the Ricci operator defined by $Ric(X,Y) = g(QX,Y)$, $X,Y \in \mathfrak{X}(M)$. Using (i) of Lemma 2, we have
\[
\nabla (\Delta f) = -2n \nabla f - \|A\|^2 \nabla f - f \nabla \|A\|^2
\]
and, consequently, using $Q(X) = (2n - 1)X - A^2X$ (outcome of Equation (7)), the Equation (12) takes the form
\[
\sum_{i=1}^{2n} (\nabla A_f) (e_i, e_i) = (2n - 1) \nabla f - A^2 (\nabla f) - 2n \nabla f - \|A\|^2 \nabla f - f \nabla \|A\|^2,
\]
that is,
\[
\sum_{i=1}^{2n} (\nabla A_f) (e_i, e_i) = -\nabla f - A^2 (\nabla f) - \|A\|^2 \nabla f - f \nabla \|A\|^2. \quad (13)
\]
Also, note that
\[
X (\|A\|^2) = X \left( \sum_{i=1}^{2n} g(Ae_i, Ae_i) \right) = 2 \sum_{i=1}^{2n} g((\nabla A) (X, e_i), Ae_i) = 2 \sum_{i=1}^{2n} g(X, (\nabla A) (e_i, Ae_i)),
\]
where we have used Equation (6) and symmetry of the shape operator $A$. Therefore, the gradient of the function $\|A\|^2$ is
\[
\nabla \|A\|^2 = 2 \sum_{i=1}^{2n} (\nabla A) (e_i, Ae_i),
\]
and, consequently, Equation (13), takes the form
\[
\sum_{i=1}^{2n} (\nabla A_f) (e_i, e_i) = -\nabla f - A^2 (\nabla f) - \|A\|^2 \nabla f - 2f \sum_{i=1}^{2n} (\nabla A) (e_i, Ae_i). \quad (14)
\]
Using Equations (11) and (14), we conclude
\[
-\|A\|^2 \nabla f = -\sum_{i=1}^{2n} (\nabla^2 A) (e_i, e_i, u) + \sum_{i=1}^{2n} (\nabla A) (e_i, Je_i). \quad (15)
\]
Now, using Equations (6) and (8) and the Ricci identity, we have
\[
\sum_{i=1}^{2n} (\nabla^2 A) (e_i, e_i, u) = \sum_{i=1}^{2n} (\nabla^2 A) (e_i, u, e_i) = \sum_{i=1}^{2n} (R(e_i, u) Ae_i - AR(e_i, u)e_i),
\]
which on using Equation (5) and $trA = 0$ gives
\[
\sum_{i=1}^{2n} (\nabla^2 A) (e_i, e_i, u) = -\|A\|^2 Au + 2n Au. \quad (16)
\]
Also, using $JA = AJ$, we have
\[
\sum_{i=1}^{2n} (\nabla A) (e_i, je_i) = \sum_{i=1}^{2n} (\nabla e_i J Ae_i - A ((\nabla J) (e_i, e_i) + J (\nabla e_i e_i)) = \sum_{i=1}^{2n} ((\nabla J) (e_i, Ae_i) - A ((\nabla J) (e_i, e_i)),
\]
which on using (v) of Lemma 1, yields

$$\sum_{i=1}^{2n} (\nabla A) (e_i, Je_i) = \|A\|^2 v - 2nAu. \quad (17)$$

Finally, using (vi) of Lemma 1 and Equations (16) and (17) in Equation (15), we get

$$- \|A\|^2 (Au + v) = \|A\|^2 Au - 2nAu + \|A\|^2 v - 2nAu$$

and this proves the Lemma. \(\blacksquare\)

3. Proof of Theorem 1

As the contact vector field \(v\) is a conformal vector field by Lemma 4, we have \(JA = AJ\), that is, \(\Psi = 0\). Then Lemma 3 implies

$$\int_M (1 - f^2) \|A\|^2 = \int_M (2n - 2n(2n + 1)f^2),$$

that is,

$$\int_M (1 - f^2) \|A\|^2 = \int_M (2n(1 - f^2) - 4nf^2). \quad (18)$$

Therefore, we get the inequality

$$\int_M (1 - f^2) \|A\|^2 \leq \int_M 2n(1 - f^2).$$

Moreover, if the equality holds, then by Equation (18), we get \(f = 0\), which in view of (vi), (vii) of Lemma 1, we conclude that \(Au = v\) and that the contact vector field \(v\) is a unit vector field. As \(v\) is a conformal vector field, combining \(Au = v\) with Lemma 5, we get \(\|A\|^2 v = 2nv\), that is, \(\|A\|^2 = 2n\). Therefore, \(M\) is a Clifford hypersurface (cf. [5]).

The converse is trivial.

4. Proof of Theorem 2

As the Reeb function \(f\) is a constant along the integral curves of the contact vector field \(v\), we have \(v(f) = 0\). Note that \(\text{div}(fv) = v(f) + f\text{div}v = -2nf^2\), which on integration gives \(f = 0\), and consequently, the contact vector field \(v\) is a unit vector field. Then Lemma 3, implies

$$\int_M \|A\|^2 = \int_M \left(2n + \frac{1}{2} \|\Psi\|^2\right), \quad (19)$$

which proves the inequality

$$\int_M \|A\|^2 \geq 2n\text{Vol}(M).$$

If the equality holds, then by Equation (4.1), we get that \(\Psi = 0\), that is, \(JA = AJ\). Thus, by Lemma 4, the contact vector field \(v\) is a conformal vector field. Using Lemma 5, we get \(\|A\|^2 = 2n\). Therefore, \(M\) is a Clifford hypersurface (cf. [5]).

The converse is trivial.

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