THE THIRD LOGARITHMIC COEFFICIENT FOR THE SUBCLASSES OF CLOSE-TO-CONVEX FUNCTIONS

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Abstract. Let $A$ denote the set of all analytic functions $f$ in the unit disk $D := \{z \in \mathbb{C} : |z| < 1\}$ normalized by $f(0) = 0$ and $f'(0) = 1$. The logarithmic coefficients $\gamma_n$ of $f \in A$ are defined by $\log f(z)/z = 2 \sum_{n=1}^{\infty} \gamma_n z^n$. In the present paper, the upper bound of the third logarithmic coefficient in general case of $f''(0)$ was computed when $f$ belongs to some familiar subclasses of close-to-convex functions.

1. Introduction and Preliminaries

Let $D := \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disk in the complex plane $\mathbb{C}$. Let $A$ denote the set of all analytic functions $f$ in the open unit disk of the form

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \quad with \quad a_1 = 1,$$

and $S$ be its subclass consisting functions that are univalent in $D$. Given a function $f \in S$, the coefficients $\gamma_n$ defined by

$$\log \frac{f(z)}{z} = 2 \sum_{n=1}^{\infty} \gamma_n z^n, \quad z \in D \setminus \{0\}, \quad \log 1 := 0,$$

are called the logarithmic coefficients of $f$. As it is commonly known, the logarithmic coefficients take a leading role in Milin conjecture (e.g., \cite[p.155]{8}), that is $f \in S$,

$$\sum_{m=1}^{n} \sum_{k=1}^{m} \left( k|\gamma_k|^2 - \frac{1}{k} \right) \le 0.$$

Milin conjecture was confirmed to be the famous Bieberbach conjecture (e.g., \cite[p.37]{6}) by De Branges \cite{2}. Sharp estimates for the class $S$ are known only for the first two coefficients,

$$|\gamma_1| \le 1, \quad |\gamma_2| \le \frac{1}{2} + \frac{1}{e} = 0.635 \cdots.$$

However, Obradović and Tuneski \cite{9} obtained an upper bound of $|\gamma_3|$ for the class $S$.

The problem of estimating the modulus of the first three logarithmic coefficients is significantly studied for the subclasses of $S$ and in some cases sharp bounds are obtained. For instance, sharp estimates for the class of starlike functions $S^*$ are given by the inequality $|\gamma_n| \le 1/n$ holds for $n \in \mathbb{N}$ \cite[p.42]{13}. Furthermore, for $f \in SS^*$, the class of strongly starlike function of order $\beta$, $(0 \le \beta \le 1)$, it holds that $|\gamma_n| \le \beta/n$ ($n \in \mathbb{N}$) \cite{11}. The bounds of $\gamma_n$ for functions in subclasses of $S$ has been widely studied in recent years, sharp estimates for the class of strongly starlike functions of certain order and gamma-starlike functions and different subclasses of close-to-convex functions are given in \cite{11}, \cite[p.116]{13}.

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and \([12]\) respectively, while non-sharp estimates for the class of Bazilevic, close-to-convex
are given in \([1,5,10]\) respectively. Let \(F_1, F_2, F_3\) denote the subclasses of \(S\) satisfying respectively the next condition
\[
\Re \{(1 - z)f'(z)\} > 0, \quad z \in \mathbb{D},
\]
\[
\Re \{(1 - z^2)f'(z)\} > 0, \quad z \in \mathbb{D},
\]
\[
\Re \{(1 - z + z^2)f'(z)\} > 0, \quad z \in \mathbb{D}.
\]

Account that each class defined above is the subclass of the well known class of close-to-
convex functions, consequently families \(F_i, i = 1,2,3\), contain only univalent functions \([6,
Vol.II,p. 2]\). The sharp bounds of \(\gamma_1, \gamma_2\) and partial results for \(\gamma_3\) of the subclasses
\(F_1, F_2, F_3\) of \(S\) were determined by Kumar and Ali \([7]\). Moreover, Cho et al. \([4]\) computed the sharp
upper bounds for the third logarithmic coefficient \(\gamma_3\) of \(f\) when \(a_2\) is real number.

Differentiating (1.2) and comparing the coefficients with (1.1), we get
\[
\gamma_1 = \frac{1}{2}a_1,
\]
\[
\gamma_2 = \frac{1}{2}a_1 + a_2 \quad \text{and} \quad \gamma_3 \quad \text{(1.3)}
\]
\[
\gamma_3 = \frac{1}{48}(3 + 2c_1 + 4c_2 + 12c_3 + 8c_1c_2 + 4c_1^3),
\]

The main aim of this paper is to determine upper bound of the third logarithmic coefficient
in general case of \(a_2\).

The following lemma are needed to prove our main results

**Lemma 1.1.** \([3]\) Let \(w(z) = c_1z + c_2z^2 + \cdots\) be a Schwarz function. Then
\[
|c_1| \leq 1, \quad |c_2| \leq 1 - |c_1|^2 \quad \text{and} \quad |c_3| \leq 1 - |c_1|^2 - \frac{|c_2|^2}{1 + |c_1|}.
\]

2. **Main Results**

**Theorem 2.1.** Let \(f \in F_1\). Then
\[
|\gamma_3| \leq \frac{15.75}{48} = 0.328125.
\]

**Proof.** Since \(f \in F_1\), an analytic self-map \(w\) of \(\mathbb{D}\) with \(w(0) = 0\) exists and
\[
(1 - z)f'(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + 2w(z) + 2w^2(z) + \cdots. \quad (2.1)
\]

Writing
\[
w(z) = c_1z + c_2z^2 + \cdots, \quad (2.2)
\]
then by using \((2.1)\) along with \((2.2)\) lead to
\[
a_2 = \frac{1}{2}(1 + 2c_1),
\]
\[
a_3 = \frac{1}{3}(1 + 2c_1 + 2c_2^2 + 2c_2),
\]
\[
a_4 = \frac{1}{4}(1 + 2c_1 + 2c_2 + 2c_3 + 2c_1^2 + 4c_1c_2 + 2c_1^3). \quad (2.3)
\]

From \((1.3)\) and \((2.3)\) after some calculations, the following was obtained
\[
\gamma_3 = \frac{1}{48}(3 + 2c_1 + 4c_2 + 12c_3 + 8c_1c_2 + 4c_1^3),
\]
and from here by using Lemma 1.1
\[ 48|\gamma_3| \leq 3 + 2|c_1| + 4|c_2| + 12|c_3| + 8|c_1||c_2| + 4|c_1|^3 \]
(2.4) \[ \leq 3 + 2|c_1| + 4|c_2| + 12 \left( 1 - |c_1|^2 - \frac{|c_2|^2}{1+|c_1|} \right) + 8|c_1||c_2| + 4|c_1|^3 =: f_1(|c_1|, |c_2|), \]
where
\[ f_1(x, y) = 3 + 2x + 4y + 12 \left( 1 - x^2 - \frac{y^2}{1+x} \right) + 8xy + 4x^3, \]
\[(x, y) \in E : 0 \leq x \leq 1, \quad 0 \leq y \leq 1 - x^2. \]
The system
\[ \frac{\partial f_1(x, y)}{\partial x} = 2 - 24x + 12 \left( \frac{y}{1+x} \right)^2 + 8y + 12x^2 = 0, \]
\[ \frac{\partial f_1(x, y)}{\partial y} = 4 - 24y + 8x = 0 \]
has unique solution \((x_1, y_1) = (1/4, 5/16) \in E \setminus \partial E \) with
(2.5) \[ f_1(x_1, y_1) = 15.75. \]
The max \( f_1(x_1, y_1) \), needs to be found when \((x, y)\) belongs to the boundary of \( E \). In that sense, we have
\[ f_1(x, 0) = 15 + 2x - 12x^2 + 4x^3 \leq 9 + \frac{10\sqrt{30}}{9} = 15.08580 \cdots \quad \text{for} \quad 0 \leq x \leq 1, \]
\[ f_1(0, y) = 15 + 4y - 12y^2 \leq \frac{46}{3} = 15.33 \cdots \quad \text{for} \quad 0 \leq y \leq 1, \]
(2.6) \[ f_1(x, 1-x^2) = 7 + 22x - 4x^2 - 16x^3 \leq 15.304035 \cdots. \]
Using (2.4), (2.5) and (2.6), concludes that
\[ 48|\gamma_3| \leq 15.75, \quad \text{i.e.} \quad |\gamma_3| \leq 0.328125. \]
This completes the proof. □

**Remark 2.1.** If \( f \in F_1 \), where \( f''(0) \) is real number, then \[ |\gamma_3| \leq \frac{1}{288} \left( 11 + 5\sqrt{30} \right) = 0.323466 \cdots. \]

**Theorem 2.2.** Let \( f \in F_2 \). Then
\[ |\gamma_3| \leq 0.258765 \cdots. \]

**Proof.** Since \( f \in F_2 \), there exists an analytic self-map \( w \) of \( \mathbb{D} \) with \( w(0) = 0 \) and
(2.7) \[ (1 - z^2)f'(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + 2w(z) + 2w^2(z) + \cdots. \]
The coefficients can be found by comparing the notations given in (2.2) and (2.7)
\[ a_2 = c_1, \]
\[ a_3 = \frac{1}{3}(1 + 2c_2 + 2c_1^2), \]
(2.8) \[ a_4 = \frac{1}{2}(c_1 + c_3 + 2c_1c_2 + c_1^3). \]
From (1.3) and (2.8) after some calculations, the following was obtained
\[ \gamma_3 = \frac{1}{12} (c_1 + 3c_3 + 2c_1c_2 + c_1^3), \]
and from here by using Lemma 1.1
\[ 12|\gamma_3| \leq |c_1| + 3|c_3| + 2|c_1||c_2| + |c_1|^3 \]
(2.9)
\[ \leq |c_1| + 3 \left( 1 - |c_1|^2 - \frac{|c_2|^2}{1 + |c_1|} \right) + 2|c_1||c_2| + |c_1|^3 =: f_2(|c_1|, |c_2|), \]
where
\[ f_2(x, y) = x + 3 \left( 1 - x^2 - \frac{y^2}{1 + x} \right) + 2xy + x^3, \]
\((x, y) \in E : 0 \leq x \leq 1, \ 0 \leq y \leq 1 - x^2.\)

From the system
\[ \frac{\partial f_2(x, y)}{\partial x} = 1 - 6x + 3 \left( \frac{y}{1 + x} \right)^2 + 2y + 3x^2 = 0, \]
\[ \frac{\partial f_2(x, y)}{\partial y} = -\frac{6y}{1 + x} + 2x = 0, \]
only one solution \((x_2, y_2)\) lies in the interior of \(E\), where
\[ x_2 = \frac{4 - \sqrt{7}}{6} = 0.22570 \ldots, \]
\[ y_2 = \frac{47 - 14\sqrt{7}}{108} = 0.092217 \ldots, \]
and
(2.10) \[ f_2(x_2, y_2) = 3.10518 \ldots. \]

On the boundary of \(E\), we have the next property
\[ f_2(x, 0) = 3(1 - x^2) + x + x^3 \leq 2 + \frac{4}{9}\sqrt{6} = 3.08866 \quad \text{for} \quad 0 \leq x \leq 1, \]
\[ f_2(0, y) = 3(1 - y^2) \leq 3 \quad \text{for} \quad 0 \leq y \leq 1, \]
(2.11) \[ f_2(x, 1 - x^2) = 6x - 4x^3 \leq 2\sqrt{2} = 2.82842 \ldots. \]
Consequently (2.9), (2.10) and (2.11) yield
\[ 12|\gamma_3| \leq 3.10518 \ldots, \quad \text{i.e.,} \quad |\gamma_3| \leq 0.258765 \ldots. \]

\[ \square \]

**Remark 2.2.** If \( f \in F_2 \), where \( f''(0) \) is real number, then [4]
\[ |\gamma_3| \leq \frac{1}{972} \left( 95 + 23\sqrt{46} \right) = 0.258223 \ldots. \]

**Theorem 2.3.** Let \( f \in F_3 \). Then
\[ |\gamma_3| \leq \frac{17.75}{48} = 0.36979 \ldots. \]
Proof. Proceeding similarly as in the previous proofs, there exists an analytic self-map \( w \) of \( D \) with \( w(0) = 0 \) and

\[
(1 - z + z^2) f'(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + 2w(z) + 2w^2(z) + \cdots.
\]

(2.12)

Substituting (2.2) into (2.12) and after comparing coefficients leads to

\[
a_2 = \frac{1}{2}(1 + 2c_1),
\]

\[
a_3 = \frac{2}{3}(c_1 + c_2 + c_1^2),
\]

(2.13)

\[
a_4 = \frac{1}{4}(2c_2 + 2c_3 + 2c_1^2 + 2c_1^3 + 4c_1c_2 - 1).
\]

By using (1.3) along with (2.13), upon simplification

\[
\gamma_3 = \frac{1}{48}(-5 - 2c_1 + 4c_2 + 12c_3 + 8c_1c_2 + 4c_1^3),
\]

(2.14)

and from here by using Lemma 1.1

\[
48|\gamma_3| \leq 5 + 2|c_1| + 4|c_2| + 12|c_3| + 8|c_1||c_2| + 4|c_1|^3
\]

(2.15)

\[
5 + 2|c_1| + 4|c_2| + 12 \left(1 - |c_1|^2 - \frac{|c_2|^2}{1 + |c_1|}\right) + 8|c_1||c_2| + 4|c_1|^3 =: f_3(|c_1|, |c_2|),
\]

where

\[
f_3(x, y) = 5 + 2x + 4y + 12 \left(1 - x^2 - \frac{y^2}{1 + x}\right) + 8xy + 4x^3,
\]

\((x, y) \in E: 0 \leq x \leq 1, 0 \leq y \leq 1 - x^2\).

The system

\[
\begin{align*}
\frac{\partial f_3(x, y)}{\partial x} &= 2 - 24x + 12 \left(\frac{y}{1 + x}\right)^2 + 8y + 12x^2 = 0, \\
\frac{\partial f_3(x, y)}{\partial y} &= 4 - \frac{24y}{1 + x} + 8x = 0
\end{align*}
\]

has unique solution \((x_3, y_3) = (1/4, 5/16)\) belongs to the interior of \(E\) and

(2.16)

\[
f_1(x_3, y_3) = 17.75.
\]

On the boundary of \(E\), the following cases are given

\[
f_3(x, 0) = 17 + 2x - 12x^2 + 4x^3 \leq 11 + \frac{10\sqrt{30}}{9} = 17.08580 \cdots \text{ for } 0 \leq x \leq 1,
\]

\[
f_3(0, y) = 17 + 4y - 12y^2 \leq 17.33 \cdots \text{ for } 0 \leq y \leq 1,
\]

(2.17)

\[
f_3(x, 1 - x^2) = 9 + 22x - 4x^2 - 20x^3 \leq 16.56455 \cdots.
\]

Equations (2.14), (2.15) and (2.16) show that

\[
|\gamma_3| \leq \frac{17.75}{48} = 0.36979 \cdots.
\]

Remark 2.3. Let \( f \in \mathcal{F}_3 \), where \( f''(0) \) is real number. Then \( |\gamma_3| \leq \frac{1}{7776} \left(743 + 131\sqrt{262}\right) = 0.368238 \cdots. \)
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