Orbits, forces and accretion dynamics near spinning black holes

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Abstract

We analyze the relativistic dynamical properties of Keplerian and non-Keplerian circular orbits in a general axisymmetric and stationary gravitational field, and discuss the implications for the stability of co- and counter-rotating accretion disks and tori surrounding a spinning black hole. Close to the horizon there are orbital peculiarities which can seem counterintuitive, but are elucidated by formulating the dynamics in terms of the orbital velocity actually measured by a local, zero-angular-momentum observer.

Keywords: accretion, AGN and quasar models
1 Introduction

Accretion onto supermassive black holes has emerged as the paradigmatic model for the energy source of AGNs and quasars (Lynden-Bell 1978, Rees 1984), a picture recently given compelling support by HST spectrography of the nucleus of M87 (Harms et al. 1994).

Detailed development and understanding of such models requires insight into the subtle fluid and plasma dynamics close to the edge of a spinning black hole. For near-maximal spin, accretion disks can extend almost down to the horizon (modulo the possibility of still poorly understood bar-like instabilities, see Blaes 1987). In this strong-gravity regime, general-relativistic corrections can produce non-Newtonian behaviour which, at first sight, is strange and counterintuitive (see Allen 1990). For instance, angular momentum is transported inwards rather than outwards by a viscous disk (Anderson and Lemos 1978); there is a reversal of the Rayleigh criterion for stability (which conventionally requires the specific angular momentum to increase outwards) (Abramowicz and Prasanna 1990); and spinning balls of fluid undergoing slow collapse actually become more spherical (rather than more flattened) in the latter stages of contraction toward black holes (Chandrasekhar and Miller 1974).

An insightful series of papers by Abramowicz and co-workers (Abramowicz and Lasota 1986, Abramowicz 1990, Abramowicz and Prasanna 1990) reveal how these seemingly disparate anomalies can be understood in terms of a common, non-Newtonian mechanism: an effective reversal of centrifugal force inside a region called the rotosphere, which lies within radius $r = 3m$ for a Schwarzschild black hole. (This line of thought is further pursued and expounded in Abramowicz 1992, 1993 and Abramowicz and Szukiewicz 1993.)

Viewing a thing from more than one angle often adds depth and flexibility to one’s understanding. The group of anomalies that fall under the heading “Abramowicz effect” can also be “explained” without overturning familiar Newtonian preconceptions (de Felice 1991, 1994; Page 1993). Indeed, one very simple explanation, which we shall present in this article, actually exploits the close formal resemblance between the Newtonian and general-relativistic descriptions (when suitably formulated). The basic remark is that the general relativistic formulae become simplest and almost indistinguishable from Newtonian when expressed in terms of quantities that a local observer actually measures.

The gist of the matter can be explained in a few words. The outward rocket thrust needed to hold a spacecraft in a circular orbit of radius $r$ around a central mass is proportional to $(v_K^2 - v^2)/r$ according to Newtonian theory, where $v$ and $v_K$ are the actual and Keplerian orbital velocities. This remains true in Einstein’s theory, for a spherisymmetric field, if $v$, $v_K$ are interpreted as velocities measured by a local stationary observer.

This at once makes it clear why, within $r = 3m$ (the orbital radius of a circling photon in the Schwarzschild field), the thrust needed is always outward, no matter how fast the spacecraft orbits, because here the Keplerian velocity is tachyonic: $|v_K| > c$. This is the essence of the Abramowicz effect.

Moreover, this way of looking at the phenomenon carries over trivially to the axisymmetric case when the central mass is in (steady) rotation. (By contrast, on other interpretations this extension is less straightforward (Abramowicz, Nurowski and Wex 1993).) The thrust needed to stay on an equatorial circle is now proportional to $(v_+ - v)(|v_-| + v)$, where $v_+$ and $v_-$ are the
prograde and retrograde Keplerian velocities. All velocities are understood to be measured by a local observer ("ZAMO") orbiting with zero angular momentum. Because of rotational frame-dragging, the two Keplerian velocities are not equal and opposite; they become lightlike at different radii, the retrograde orbit at the larger radius.

Such effects of frame-dragging on counter-rotating orbits have an interest which may be more than merely one of principle. Recent observations and numerical simulations suggest that counter-rotation may be a less unusual feature in old galactic nuclei than was thought just a few years ago (Binney and Tremaine 1987). The gas stream produced by a retrograde encounter between galaxies can evolve over billions of years into a counter-rotating disk of stars superimposed upon and streaming collisionlessly through the old disk. NGC4550 and NGC7217 are examples of such “two-way galaxies” (Rubin et al. 1992, Kuijken 1993).

In Sec. 2 we introduce the basic concepts and techniques in the simple context of a spherically symmetric gravitational field. This introductory discussion clears the way for a concise presentation of the corresponding results for equatorial circular orbits in a general stationary axisymmetric field in Sec. 3. When the mass of the accretion disk or torus is small compared to the central black hole, the gravitational field is well approximated by the Kerr metric. In this case, explicit results are available, and are summarized in Sec. 4. Further developments and details are assembled in two Appendices.

2 Spherisymmetric fields

We begin by considering orbits in a general static, spheri symmetric geometry, described by the line-element

$$ds^2 = \eta^2(r) \, dr^2 + r^2 \, d\Omega^2 - V^2(r) \, dt^2.$$  

Without essential loss of generality we may assume the orbit confined to the equatorial plane $\theta = \pi/2$. We write $\omega = \dot{\varphi} \equiv d\varphi/dr$ for the angular velocity as measured by a comoving observer, using proper time $\tau$.

Keplerian orbits are timelike geodesics, i.e., extremals of the action $\int L \, d\tau$, where

$$L = V^2 \dot{t}^2 - r^2 \omega^2 - \eta^2 \dot{r}^2$$

and $L = 1$ on the extremal curve. For a circular Keplerian orbit of radius $r$, the angular velocity $\omega_K$ is thus found from $\partial L/\partial r = 0$ to be given by

$$\frac{r \omega_K^2}{1 + r^2 \omega_K^2} = \frac{V'(r)}{V}.$$  \hspace{1cm} (1)

With respect to a local stationary observer, a unit test mass on a circular orbit has 3-momentum (i.e., spatial projection of 4-momentum $p^\mu = dx^\mu/d\tau$)

$$p = (g_{\varphi\varphi})^{\frac{1}{2}} \, p^\varphi = r \omega$$

and 3-velocity

$$v = p/(1 + p^2)^{\frac{1}{2}} = \frac{r \omega}{(1 + r^2 \omega^2)^{\frac{1}{2}}}.$$  \hspace{1cm} (2)
in relativistic units \((c = 1)\).

Hence (1) yields

\[ \frac{v_K^2}{r} = \frac{V'(r)}{V(r)} \]  

for the Keplerian 3-velocity \(v_K\). (In the Newtonian limit \(V \approx 1 + V_{\text{Newt}}/c^2\), (1) and (3) reduce to the expected familiar results.)

The orbital radius \(r = r_{ph}\) of a freely circulating photon \(|v_K| = 1\) is determined by

\[ \frac{1}{r_{ph}} = \frac{V'(r_{ph})}{V(r_{ph})}. \]

In the case of the Schwarzschild geometry, \(V^2 = 1 - 2m/r\) and \(r_{ph} = 3m\). Inside this radius, Keplerian velocities are tachyonic \((v_K^2 > 1)\), corresponding to spacelike geodesic trajectories, \(d\tau_K^2 < 0\).

We now turn to general (non-Keplerian) circular orbits. The mechanical force or rocket thrust needed to hold a unit test mass stationary in such an orbit is given by \(F^\mu = \delta p^\mu/\delta \tau\) (the absolute derivative of 4-momentum), a purely spatial and purely radial vector. Its radial component is

\[ F_r = \frac{\partial^2 r}{\partial \tau^2} = -\frac{1}{2}(g_{rr})^{-1}\left\{ \frac{d}{d\tau}\left( \frac{\partial L}{\partial \dot{r}} \right) - \frac{\partial L}{\partial r} \right\} = \frac{1}{2\eta^{-2}} \frac{\partial L}{\partial r}. \]

This yields for the (signed) magnitude of (outward) force

\[ F = (g_{rr})^{1/2} F_r = \frac{V'}{\eta V} \left\{ 1 - \omega^2 r \left( \frac{V}{V'} - r \right) \right\}. \]  

(4)

Recalling (1), this can be re-expressed as

\[ F = \frac{V'}{\eta V} \left( 1 - \frac{\omega^2}{\omega_K^2} \right). \]  

(5)

Setting \(\omega = 0\), we obtain the force \(F_{\text{stat}} = V'/\eta V\) needed to hold up a stationary unit mass. At the photon radius \(r = r_{ph}\), we have \(\omega_K^2 = (d\varphi/d\tau)^2_K = \infty\), hence \(F = F_{\text{stat}}\) is finite for every timelike circular orbit and velocity-independent (Abramowicz and Lasota 1986).

The subtracted term in (5), proportional to \(\omega^2\), might be thought of as “centrifugal force.” Were we to adopt this terminology, we would at once make contact with Abramowicz 1990. In the rotosphere, Keplerian circular orbits are spacelike, hence \(\omega_K^2 < 0\) in (5) and “centrifugal force is reversed.”

But there is an alternative way to write (5). Using (2) to eliminate \(\omega\) in (4) in favour of \(v\), and recalling (3), gives

\[ F = \frac{1}{\eta r} \frac{v_K^2 - v^2}{1 - v^2}. \]  

(6)

In this form, “centrifugal force” (now most naturally interpreted as the subtracted term \(v^2/\eta r(1 - v^2)\) in (6)) never changes direction, since \((1 - v^2)\) is positive for every physical (timelike) orbit. Nevertheless, (6) shows as before that to orbit in the rotosphere (where \(v_K^2 > 1\)) always requires an outward thrust \(F\), no matter how fast one moves. Indeed, speeding up is self-defeating: the thrust needed actually increases with speed in the rotosphere and becomes infinite at the speed of light.
Within the rotosphere, it takes less outward thrust to hold a body stationary than to maintain it in orbit. One might say that the special-relativistic increase of “mass” with velocity (given by the denominator of (6)) literally outweighs the effect of centrifugal force in the rotosphere (Page 1993, de Felice 1991).

It is this property that underlies the reversal of the Rayleigh criterion for stability in the rotosphere. Consider a toroidal distribution of material (e.g. incompressible fluid) held in steady, differential rotation around a gravitating mass by a prescribed non-gravitational force field. Suppose the specific angular momentum of the distribution increases outwards from the axis. An orbiting ring of material, if displaced inwards axisymmetrically, will preserve its angular momentum and will therefore orbit faster than its new surroundings. In the rotosphere, the outward force that would be called for to support it in this new orbit is therefore necessarily larger than that which actually supports the surrounding material, and this is as much as the local force field provides. The displaced ring is forced to sink further, and this triggers an instability.

The alternative “explanations” of Abramowicz and Page, however helpful and suggestive, are, of course, at bottom merely different forms of words to clothe the formulae. One may adopt either (or neither), depending on circumstances and personal taste. One advantage of Page’s form, as we shall now see, is that it is easy to adapt (6) (but less easily (5)) to the case where the central mass is rotating.

3 Stationary axisymmetric fields

We pass now to orbits in a stationary axisymmetric asymptotically flat geometry. Let

\[ \xi^\alpha_{(t)} = \partial x^\alpha / \partial t, \quad \xi^\alpha_{(\varphi)} = \partial x^\alpha / \partial \varphi \]

be the timelike and axial Killing vectors (assumed to commute). An observer with 4-velocity \( u^\alpha = dx^\alpha / d\tau \) has angular momentum

\[ \ell = p_\varphi = u \cdot \xi_{(\varphi)} = g^{}_{\varphi \alpha} u^\alpha. \]

For a free-falling (geodesic) observer, this is conserved.

Any observer orbiting in an azimuthal circle has a 4-velocity of the form

\[ u^\alpha = U^{-1}(\xi^\alpha_{(t)} + \Omega \xi^\alpha_{(\varphi)}), \tag{7} \]

where \( \Omega = d\varphi / dt \) is his angular velocity as measured by a stationary observer at infinity. The orbiting observer is a zero angular momentum (ZAM) observer (Bardeen 1970a, 1973, Thorne, Price & MacDonald 1986) if \( u \cdot \xi_{(\varphi)} = 0 \), i.e., if he orbits with the Bardeen angular velocity

\[ \Omega_B = -(\xi_{(\varphi)} \cdot \xi_{(t)}) / (\xi_{(\varphi)} \cdot \xi_{(\varphi)}) = -g_{\varphi t} / g_{\varphi \varphi}. \]

The normalizing factor in (7) is then

\[ U_{ZAM} = V = (-g^\mu)^{-\frac{1}{2}}. \]

4
The line-element is now conveniently expressible as
\[
ds^2 = g_{AB} dx^A dx^B + \rho^2 \sin^2 \theta (d\phi - \Omega_B dt)^2 - V^2 dt^2
\]  
where \(x^A \equiv (r, \theta)\) and all metric coefficients depend on \(r\) and \(\theta\) only.

We assume that this geometry also has equatorial symmetry, and we shall be interested in circular orbits in the equatorial plane \(\theta = \pi/2\). As in the previous section, their properties are derivable from the Lagrangian
\[
L = V^2 \dot{t}^2 - \rho^2 \omega^2 - \eta^2 \dot{r}^2
\]
in which the dot denotes differentiation with respect to proper time \(\tau\),
\[
\omega = \dot{\phi} - \Omega_B \dot{t}
\]
is the locally measured angular velocity relative to the local ZAM observer and all coefficients are functions of \(r\) only. The 3-velocity \(v\) of a circularly orbiting particle as measured by the local ZAM observer is
\[
v = \rho \omega /
(1 + \rho^2 \omega^2)^{1/2}.
\]

Keplerian circular orbits satisfy \(\partial L / \partial r = 0\), \(L = 1\), \(\dot{r} = 0\). This yields a quadratic equation with roots
\[
v_+ = \gamma / (\delta + \beta), \quad v_- = -\gamma / (\delta - \beta) \quad (9)
\]
giving the 3-velocities of prograde and retrograde Keplerian orbits. We have defined
\[
\alpha = \partial_r \ln \rho, \quad \beta = -\frac{1}{2} (\rho/V) \partial_r \Omega_B, \quad \gamma = \partial_r \ln V, \\
\delta = \left(\frac{\alpha \gamma + \beta^2}{2}\right)^{1/2}.
\]  
(10)

The retrograde Keplerian orbit becomes lightlike when
\[
v_- = -1 \implies \gamma - \alpha = -2\beta, \quad \delta = \frac{1}{2}(\alpha + \gamma). \quad (11)
\]

For the prograde orbit,
\[
v_+ = +1 \implies \gamma - \alpha = 2\beta, \quad \delta = \frac{1}{2}(\alpha + \gamma). \quad (12)
\]

The thrust required to hold unit mass in a non-Keplerian circular orbit is given by the absolute derivative \(F^\mu = \delta^2 x^\mu / \delta \tau^2\). The only nonvanishing component is calculable from
\[
F^r = \delta^2 r / \delta \tau^2 = \frac{1}{2} \eta^{-2} \partial L / \partial r.
\]
This gives for the (signed) magnitude of outward thrust \(F = (g_{rr})^{1/2} F^r\):
\[
F = -\frac{\alpha}{\eta} \frac{(v - v_+)(v - v_-)}{1 - v^2}.
\]  
(13)
Equations (9) and (13) are invariant, as they should be, under reparametrization of the arbitrary radial co-ordinate $r$. A more obviously geometrical form emerges if we introduce the element of proper radial distance

$$ds_r = (g_{rr})^{1/2} dr.$$  \hspace{1cm} (14)

Then

$$F = -\frac{1}{\rho} \frac{(v - v_+)(v - v_-)}{1 - v^2} \frac{d\rho}{ds_r}.$$  \hspace{1cm} (15)

The conical deficit factor $d\rho/ds_r$ (slope of circumferential radius $\rho$ versus proper radial distance) is generally not far from unity except very close to a black hole horizon, where it tends to zero.

Setting $v = 0$ in (13), and using (9) and (14), one obtains the outward force needed to hold unit mass in a ZAM orbit:

$$F_{ZAM} = d\ln V/ds_r.$$  

This generalizes to non-equatorial ZAM orbits: its covariant form is

$$(F_\alpha)_{ZAM} = u_\alpha|\beta u^\beta = \partial_\alpha \ln V.$$  \hspace{1cm} (16)

(A simple derivation is sketched in Appendix A.)

ZAM photons emitted outward from deep in the field reach infinity red shifted by a factor $V^{-1}$. At a black hole horizon, $V$ becomes zero, the ZAM orbits become the horizon’s lightlike generators and the “redshifted force” $VF_{ZAM}$ becomes the surface gravity $\kappa$. Equation (16) and these useful properties define the sense in which $V \equiv (-g^{tt})^{-1/2}$ functions as the scalar potential appropriate to ZAM observers (see, e.g., Israel 1983).

For definiteness, we take the angular momentum of the hole (or other central mass) to be positive. Then ZAM observers are dragged in the positive-\(\varphi\) direction ($\Omega_B > 0$), at least near the source. The inequalities

$$\alpha > 0, \quad \beta > 0, \quad \gamma > 0$$  \hspace{1cm} (17)

must hold, at least if the source is isolated. According to (10), these just express the conditions (respectively) that, as one moves out from the source, circumferences of equatorial circles get larger, effects of frame-dragging (as measured by $\Omega_B$) get progressively smaller, and that ZAM observers always experience inward gravitational pull. The inequalities (17) should remain valid for a larger class of (non-isolated) sources, in particular those for which a surrounding accretion disk or torus has a mass appreciably less than the central mass.

Assuming the validity of (16), we infer at once from (9) that

$$|v_-| > v_+,$$

showing that, at a radius $r_{-ph}$ where retrograde Keplerian orbits become lightlike, prograde Keplerian orbits must still exist. Thus, quite generally there is an “outer rotosphere,” characterized by $v_- < -1, v_+ < 1$, in which retrograde Keplerian orbits (only) have become superluminal, and in consequence the Rayleigh criterion is reversed for counter-rotating disks and tori.

Retrograde orbits can still extend into the outer rotosphere if they are supported, for instance by the pressure gradient in a thick disk. Counter-rotation in the outer parts of such a disk must, however, give way to co-rotation in the inner parts, since no outward radial force is available to hold up the equatorial inner edge.
Two-way structures of this kind may arise astrophysically when a retrograde inflow first impinges on the outlying parts of a pre-existing co-rotating accretion disk, and may survive on the order of an accretion time. (At least in principle, one might conceive of steady-state configurations in which magnetic coupling of the disk to the forward-spinning hole supplies the torque needed to drive the specific angular momentum $\ell$ from negative to positive as the gas slowly spirals in. By itself, the Rayleigh criterion would forbid such a transition, since it requires $|\ell|$ to increase inwards for retrograde motion in the outer rotosphere. However, Archimedean buoyancy due to the denser inner parts of the disk can easily offset this. For compressible fluids, Høiland’s criterion is the one that is relevant (see Blaes 1987).)

The inner parts of disks and tori are constrained by stability and Bernoulli’s law, and must stop well short of the corresponding (co- or counter-rotating) photon orbits.

In thin (pressureless) disks, all orbits are Keplerian, and the innermost orbit cannot lie within the last stable Kepler orbit: $r = r_s$. Stability requires that the Keplerian angular momentum $|\ell_K|$ should increase outward. From

$$\ell = \rho^2 \omega = \rho v / (1 - v^2)^{1/2}$$

one sees that $|\ell_K|$ becomes infinite at both ends of the range $r_{ph} < r < \infty$, and hence must attain a minimum at a radius $r_s > r_{ph}$, determined from

$$\partial(\ell_K^2) / \partial r \big|_{r=r_s} = 0.$$  

For the special case of the Kerr geometry, simple explicit expressions are obtainable for equatorial circular orbits in both pro- and retrograde cases, and are summarized in the next section and Appendix B.

In the case of “thick disks” (tori), internally supported by fluid pressure, material at the inner equatorial edge follows a nearly Keplerian orbit subject to the condition that its binding energy, $(1 - E_K)$, be positive (Kozlowski, Jaroszyński & Abramowicz 1978; see also Appendix A). Here, the specific energy $E$ (energy per unit proper mass) is given by

$$E = -u_\alpha \xi^\alpha = \frac{1}{2} \partial L / \partial \dot{t} = (1 - v^2)^{-1/2} (V + \rho \Omega_B v).$$

For pro- and retrograde Keplerian orbits around non-extremal holes, $E_{+K}$ and $E_{-K}$ drop from infinity as one moves outward from the corresponding photon orbits at $r_{+ph}$ and $r_{-ph}$, attain minima at radii $r_{+s}$ and $r_{-s}$ respectively and finally tend to unity from below as $r \to \infty$. It follows that there exist radii $r_{eb}$ ($\epsilon = \pm$) such that $r_{eph} < r_{eb} < r_{es}$ and $E_{\epsilon K}(r_{eb}) = 1$. Inner edges of pro- and retrograde thick disks cannot fall within the corresponding radii of zero binding energy $r_{+b}$ and $r_{-b}$. This condition is less restrictive than that for thin disks.

### 4 Circular orbits in Kerr geometry

The formulae and results derived in the previous section for orbits on general stationary and axisymmetric fields take a rather simple explicit form in the case of the Kerr geometry. The explicit formulae should be a satisfactory approximation when the mass of the accreting disk or torus is
small. We collect the most interesting results here; more detail can be found in Appendix B (also Bardeen 1973, Lynden-Bell 1978, de Felice & Usseglio-Tomasset 1991, de Felice 1994).

We work in standard (Boyer-Lindquist) co-ordinates for a Kerr geometry of mass $m$ and angular momentum $ma$. It is convenient to introduce dimensionless quantities $\xi$, $a_*$ defined by

$$r = m \xi^2, \quad a = ma_*.$$

(21)

For equatorial circular orbits, the pro- and retrograde Keplerian 3-velocities $v_+$ and $v_-$ relative to a ZAM observer are given by

$$v_\epsilon = \epsilon \left( \frac{m}{r^2 - 2mr + a^2} \right)^{\frac{1}{2}} \frac{r^2 + a^2 - 2\epsilon a(mr)^{\frac{1}{2}}}{r^{\frac{3}{2}} + \epsilon am^{\frac{3}{2}}}$$

with $\epsilon = \pm 1$. In terms of the dimensionless quantities (21),

$$v_\epsilon = \epsilon \frac{\xi^4 + a_*^2 - 2\epsilon a_*\xi}{(\xi^4 - 2\xi^2 + a_*^2)^{\frac{1}{2}}(\xi^3 + 6a_*)}.$$

The pro- and retrograde photon orbits $v_\epsilon = \epsilon$ have radii $r_{\epsilon ph} = m \xi_{\epsilon ph}^2$, where

$$\xi_{\epsilon ph}(3 - \xi_{\epsilon ph}^2) = 2\epsilon a_*.$$

(Another (prograde) solution is $\xi_{+ ph} = 0$, corresponding to the singular equatorial ring $r = 0, \theta = \pi/2$ inside the horizon. Although of no astrophysical relevance, this has interesting implications for the source structure of the analytically extended Kerr geometry (Israel 1977).) For the Schwarzschild case ($a_* = 0$), $\xi_{\epsilon ph}^2 = 3$, and we recover the familiar result $r_{ph} = 3m$ for both pro- and retrograde photon orbits. For maximally rotating Kerr ($a_* = 1$) we obtain $\xi_{- ph} = 2, \xi_{+ ph} = 1$: the retrograde photon circles out at radius $4m$, the prograde one on the horizon itself, at radius $m$.

The innermost stable Kepler orbits (pro- and retrograde) are at radii $r_{\epsilon s} = m \xi_{\epsilon s}^2$, where

$$\xi_{\epsilon s}[4 - (3\xi_{\epsilon s}^2 - 2)^{\frac{1}{2}}] = 3\epsilon a_*.$$

This gives $\xi_{\epsilon s}^2 = 6$, i.e., $r_s = 6m$ for Schwarzschild. For maximal Kerr, $\xi_{- s} = 3, \xi_{+ s} = 1$. Counter-rotating thin disks have inner edges beyond $r_{- s} = 9m$, far outside the outer rotosphere, which terminates at $r_{- ph} = 4m$. On the other hand, co-rotating thin disks can extend within the ergosphere, almost to the horizon.

Keplerian orbits have energy per unit proper mass

$$e_{\epsilon K} = \frac{\xi^3 - 2\xi + \epsilon a_*}{\xi^2(\xi^3 - 3\xi + 2\epsilon a_*)^{\frac{1}{2}}}.$$

(22)

Orbits of zero binding energy, $1 - e_{\epsilon K} = 0$, which are believed to fence off the inner edges of thick disks (see Kozlowski et al. 1978 and Appendix A) are at radii $r_{\epsilon b} = m \xi_{\epsilon b}^2$, where

$$\xi_{\epsilon b}(2 - \xi_{\epsilon b}) = \epsilon a_*.$$
For Schwarzschild, $r_b = 4m$. For retrograde orbits in extremal Kerr, $r_{-b} = (3 + 2\sqrt{2})m$. Prograde orbits never reach zero binding in extremal Kerr; evaluation of (22) gives

$$1 - \mathcal{E}_{+K} = 1 - 3^{-\frac{1}{2}} = 0.423 \quad \text{for } a_* = 1, \ r = m,$$

a well-known result (Bardeen 1970b), which gives the maximum energy extractible from accretion onto a Kerr black hole.

At the radius $r_{-ph}$ of the retrograde photon orbit, the specific energy of the prograde Kepler orbit is

$$\mathcal{E}_{+K}(r_{-ph}) = \frac{1}{2\sqrt{2}} \frac{3\xi^2 - 7}{\xi(\xi^2 - 3)^{\frac{1}{2}}}, \quad \xi(\xi^2 - 3) = 2a_*.$$

This gives a positive binding energy for $(a/m) > 8\sqrt{2} - 11 = 0.3137$. If the hole is spinning faster than this (very moderate) value, a thick disk whose outer parts are retrograde can extend into the outer rotosphere provided its inner edge is co-rotating. That such hybrid objects can play more than a transitory role in the fuelling of quasar activity by retrograde accretion is a priori perhaps unlikely, but really deserves further study.

### 5 Concluding remarks

We hope that the simple treatment we have presented will contribute to the understanding of the general-relativistic dynamics of the inner parts of accretion disks near spinning black holes, in particular, the effects of frame-dragging, which, as we have seen, is very different for co-rotating and counter-rotating disks. Recent observational evidence suggests that retrograde accretion may occur sometimes or even fairly often in galactic nuclei.

We have touched in passing on an issue which has recently had considerable exposure: Are the paradoxical effects inside circular photon orbits best “explained” as a reversal of centrifugal force, or is it preferable to suppose that kinetic energy has weight? Heuristic explanations are a subjective matter in which each individual is free to prescribe for himself. Simplicity (a purely subjective criterion) will be the deciding factor in each case. An explanation so subtle that it needs explaining loses its raison d’être. We cede the last word to Richard Feynman:

“What do I mean by understanding? Nothing deep or accurate—just to be able to see some of the qualitative consequences of the equations without solving them in detail.” (Feynman 1947)

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A Circular orbits in stationary axisymmetric geometry: covariant approach

For any steady motion in a circle (not necessarily equatorial) centred on the symmetry axis, the normalized 4-velocity is

\[ u^\alpha = U^{-1} U^\alpha, \quad U^\alpha = \xi^\alpha_{(t)} + \Omega \xi^\alpha_{(\phi)}, \quad (A1) \]

with

\[ -U^2 = U_\alpha U^\alpha = g_{\phi\phi} \Omega^2 + 2 \Omega g_{\phi t} + g_{tt}. \quad (A2) \]

Axial symmetry and stationarity imply

\[ u^\beta \partial_\beta U = u^\beta \partial_\beta \Omega = 0. \quad (A3) \]

The force on a unit mass in this orbit is

\[ F_\alpha = u_\alpha \xi^{\alpha}_{(\phi)} = U^{-1} U_\alpha U^\beta. \]

Substituting from (A1) for \( U_\alpha \), and using (A3) and Killing’s equations \( \xi_{\alpha|\beta} = -\xi_{\beta|\alpha} \) for both Killing vectors, we easily reduce this to

\[ F_\alpha = -U^{-2} (U_{\beta|\alpha} - \xi_{(\phi)\beta} \partial_\alpha \Omega) U^\beta, \]

or, by (A2),

\[ F_\alpha = U^{-1} (\partial_\alpha U + \ell \partial_\alpha \Omega), \quad (A4) \]

where \( \ell = u_\alpha \xi^\alpha_{(\phi)} \) is the specific angular momentum.

A ZAM orbit is defined by \( \ell = 0 \). Then \( U_{\text{ZAM}} = V \) and (A4) reduces to (16) in the text.

The energy per unit rest-mass is

\[ \mathcal{E} = -u^\alpha \xi^{\alpha}_{(t)} = -u^\alpha (U u_\alpha - \Omega \xi^{\alpha}_{(\phi)}) = U + \Omega \ell, \quad (A5) \]

where we have used (A1). For a ZAM orbit,

\[ \mathcal{E}_{\text{ZAM}} = -u_{\text{ZAM}}^\alpha \xi^{\alpha}_{(t)} = U_{\text{ZAM}} = V. \quad (A6) \]

From (A1) and its analogue for \( U_{\text{ZAM}} \) we find

\[ U \ell = U^\alpha \xi_{(\phi)\alpha} = (U^\alpha - U_{\text{ZAM}}^\alpha) \xi_{(\phi)\alpha} = (\Omega - \Omega_B) g_{\phi\phi}. \quad (A7) \]

Similarly, subtracting its ZAMO analogue from (A2) and recalling \( \Omega_B = -g_{\phi t}/g_{\phi\phi} \),

\[ V^2 - U^2 = g_{\phi\phi} (\Omega - \Omega_B)^2. \]
The relative 3-velocity $v$ of the particle and a ZAM observer in the same spatial orbit is given by

$$(1 - v^2)^{-\frac{1}{2}} = -u_\alpha u^\alpha_{ZAM} = -U^{-1}\xi_{(t)}\xi^\alpha = V/U,$$  \hspace{1cm} (A8)

where we have used (A1) and (A6). From (A8) and (A7),

$$v = V^{-1}(g_{\varphi\varphi})^{\frac{1}{2}}(\Omega - \Omega_B).$$  \hspace{1cm} (A9)

(A5) allows us to write (A4) in the alternative form

$$F_\alpha = U^{-1}(\partial_\alpha E - \Omega \partial_\alpha \ell).$$  \hspace{1cm} (A10)

All these formulae of course extend at once to a continuous medium in steady rotation, with energy per unit rest-mass

$$E = -u_\alpha \xi^\alpha_{(t)} = U + \Omega \ell = V(1 - v^2)^{-\frac{1}{2}} + \Omega_B \ell$$

and angular momentum per unit rest-mass

$$\ell = u_\alpha \xi^\alpha_{(\varphi)} = (g_{\varphi\varphi})^{\frac{1}{2}} v/(1 - v^2)^{\frac{1}{2}}.$$

These formulae agree with (18) and (20) in the case of equatorial orbits.

If the medium is a fluid, with local energy density $\mu$ and pressure $P$, the force is provided by the pressure gradient:

$$F_\alpha = \partial_\alpha P/(\mu + P).$$

In a “barytropic” fluid, $\mu$ depends only on $P$. Then $F_\alpha$ is a gradient, and we have the usual stringent consequences exemplified by von Zeipel’s theorem (Eddington 1926). In particular, the free surface ($P = 0$) of a barytropic fluid torus is also a surface of constant specific energy: $E = E_0$. If the surface is closed, fluid at the surface must have positive binding energy ($E_0 < 1$), since the fluid orbit at the outer equatorial edge is Keplerian or sub-Keplerian. (On the verge of accretion, the inner equatorial edge is expected to be cusped and slightly super-Keplerian, see Kozlowski et al. 1978.)

To conclude, let us briefly link up the present approach with the force formula (15) for equatorial circular orbits.

Axi-stationarity and (A4) imply generally that $F_\varphi = F_t = 0$. For an equatorial orbit, the poloidal component is zero also, and the force is purely radial. For a specific orbit we can take $\Omega$ to be a given (position-independent) number. (The apparent dependence on $\nabla \Omega$ in (A4) is actually illusory, because it is cancelled by a part of $\nabla U$.) Then (A4) reduces to the scalar relation

$$F = \frac{1}{2U^2} \frac{d}{ds_r} (U^2) = \frac{1}{2V^2(1 - v^2)} \frac{d}{ds_r} (V^2)$$

where $ds_r$ is the element of proper radial distance, and we have used (A8).

According to (A2), $U^2$ depends quadratically on $\Omega - \Omega_B$, which in turn is simply proportional to $v$ according to (A9); all coefficients are purely geometrical, i.e., velocity-independent. Putting all this together, we arrive at

$$F = -\frac{1}{2V^2(1 - v^2)} \frac{dg_{\varphi\varphi}}{ds_r} \frac{V^2}{g_{\varphi\varphi}} (v - v_+)(v - v_-)$$

$$= -\frac{1}{2g_{\varphi\varphi}} \frac{dg_{\varphi\varphi}}{ds_r} \frac{(v - v_+)(v - v_-)}{(1 - v^2)},$$  \hspace{1cm} (A11)
in agreement with (15). (A11) itself identifies the quadratic roots \( v_+ \), \( v_- \) as Keplerian orbital velocities, for which \( F = 0 \).

**B More on equatorial Kerr orbits**

In Boyer-Lindquist co-ordinates, the Kerr equation \( \theta = \pi/2 \) has a metric of the form (8), i.e.,

\[
ds^2 = \eta^2 dr^2 + \rho^2(d\varphi - \Omega_B dt)^2 - V^2 dt^2,
\]

with

\[
\rho^2 = q(r)/r, \quad \rho^2 V^2 = \Delta(r), \quad \Omega_B = 2ma/q(r)
\]

and

\[
qu(r) = r^3 + a^2r + 2ma^2, \quad \Delta(r) = r^2 - 2mr + a^2.
\]

The quantities defined in (10) take the explicit form

\[
\alpha = (r^3 - ma^2)/qr, \quad \beta = ma(3r^2 + a^2)/qr\Delta^{1/2}
\]

\[
\gamma = m[(r^2 + a^2)^2 - 4ma^2r]/qr
\]

To compute \( \delta \), it helps to note the factorization

\[
[(r^2 + a^2)^2 - 4ma^2r](r^3 - ma^2) + ma^2(3r^2 + a^2)^2 = rq^2.
\]

Then it easily follows that

\[
\delta = (m/r\Delta)^{1/2}.
\]

Substituting these expressions into (9) produces the formulae given in Sec. 4 for the Keplerian velocities \( v_\pm \). The Keplerian energies \( \mathcal{E}_{\pm K} \) are then found from (20). It helps to note the factorization

\[
(1 - v_\epsilon^2)^{1/2} = \frac{\rho\xi^3}{m(\xi^3 + \epsilon a_*)} \left( \frac{\xi^3 - 3\xi + 2\epsilon a_*}{\xi^4 - 2\xi^2 + a_*^2} \right)^{1/2}
\]

in the notation of Sec. 4.

The Keplerian angular momenta, prograde and retrograde, are

\[
\ell_\epsilon = \rho v_\epsilon (1 - v_\epsilon^2)^{-1/2} = me \frac{\xi^4 + a_*^2 - 2\epsilon a_* \xi}{\xi^2(\xi^3 - 3\xi + 2\epsilon a_*)^{1/2}}.
\]

The condition for marginal Keplerian stability, \( \partial(\ell^2)/\partial r = 0 \), factorizes as

\[
(\xi^3 + \epsilon a_*)(\xi^4 - 6\xi^2 + 8\epsilon a_* \xi - 3a_*^2) = 0.
\]

The vanishing of the second factor is most easily treated as a quadratic equation for \( a_* \) in terms of \( \xi \), and leads to the results stated in Sec. 4.