The Josephson relation for the superfluid density in the BCS-BEC crossover

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The Josephson relation for the superfluid density is derived for a Fermi superfluid in the BCS-BEC crossover. This identity extends the original Josephson relation for Bose superfluids. It gives a simple exact relation between the superfluid density $\rho_s$ and the broken-symmetry Cooper pair order parameter $\Delta_0$ in terms of the infrared limit of the pair fluctuation propagator. The same expression holds through the entire BCS-BEC crossover, describing the superfluid density of a weak-coupling BCS superfluid as well as the superfluid density of a Bose condensate of dimer molecules.

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I. INTRODUCTION

Josephson’s relation has played an important role in developing an understanding of superfluidity since it establishes the connection between the two order parameters that are widely used to discuss Bose superfluids: the condensate density $n_c$ and the superfluid density $\rho_s$. It provides an exact relation between these quantities in a Bose superfluid in terms of the infrared behaviour of the single-particle Green’s function $D_{11}$ for bosons of mass $m_B$:

$$\rho_s = -\lim_{q \to 0} \frac{n_c m_B^2}{q^2 D_{11}(q, 0)}.$$  (1)

Here, $D_{11}(q, 0)$ is the single-particle Green’s function for bosons with momentum $q$ at zero Matsubara frequency, $\nu_m = 0$.

The simple structure of Eq. (1) has enabled a detailed analysis to be carried out of the superfluid transition in both three- and two-dimensional Bose superfluids. This includes finite-size systems where it has been used to study the Berezinskii-Kosterlitz-Thouless transition in two-dimensional superfluids. In particular, by starting from a finite two-dimensional system, Ref. 4 explicitly showed how a nonzero superfluid density can persist in the thermodynamic limit even though the condensate density vanishes.

Recently, Holzmann and Baym have extended the original phenomenological arguments of Josephson and given a microscopic proof of Eq. (1) using diagrammatic perturbation theory. This proof extends earlier discussions by Bogoliubov, Gavoret and Nozières, and Hohenberg and Martin at $T = 0$ where $\rho_s = mn$. A discussion of the Josephson relation for a Bose superfluid at finite temperatures is given by Griffin, using the dielectric diagrammatic formalism (see also Wong and Gould).

Following Holzmann and Baym, in this paper the analogous exact relation for a two-component Fermi superfluid is proven. This gives the relationship between the superfluid density and the order parameter $\Delta_0$ that represents the Bose-condensate of Cooper pairs of fermions. The Josephson relation derived in this paper is analyzed in the BCS-BEC crossover picture of Fermi superfluids, widely studied in recent years in the context of ultracold atomic gases. In the BEC limit of this crossover, where the attractive interaction between the two species of fermions is strong, the Cooper pairs reduce to dimer molecules. In this limit, the expression obtained for $\rho_s$ reduces to the usual Josephson relation for a Bose superfluid in Eq. (1).

As with the derivations for a Bose superfluid in Refs. 6 and 10, the derivation given below for a uniform system is based on exact two-fermion propagators, and is not approximate.

II. PRELIMINARIES

Consider a two-component Fermi gas (e.g., neutral Fermi atoms prepared in two different hyperfine states) with $s$-wave interactions between the two components, described by the Hamiltonian density (in this paper, $\hbar$ and also the volume $V$ are set to unity)

$$\mathcal{H} = \sum_\sigma \bar{\psi}_\sigma(x) \left( -\frac{\nabla^2}{2m_F} - \mu \right) \psi_\sigma(x) - U_0 \bar{\psi}_\uparrow(x) \psi_\downarrow(x) \psi_\downarrow(x).$$  (2)

The two components are denoted by $\sigma = \uparrow, \downarrow$ and $m_F$ is the fermion mass. The use of a momentum-independent pseudopotential interaction $U_0$ leads to ultraviolet divergencies that are regularized in the usual way by the Lippmann-Schwinger equation,

$$\frac{1}{U_0} = -\frac{m_F}{4\pi a_s} + \sum_k \frac{m_F}{k^2}.$$  (3)

Here, $a_s$ is the $s$-wave scattering length. The entire BCS-BEC crossover can be probed by “tuning” the $s$-wave scattering length from small and negative (BCS limit) through unitarity ($|a_s| = \infty$), and finally into the BEC limit where $a_s$ is small and positive.

Although only the case of $s$-wave interactions between fermions is considered in this paper, the analysis given...
below can be extended to deal with a more general pairing interaction, as well as Hubbard-type Hamiltonians that describe fermions in a lattice.

The derivation of a Josephson relation for Fermi superfluids given in this paper makes use of the structure of the grand canonical thermodynamic potential $\Omega[v_s]$ of a current-carrying Fermi superfluid to identify the change in the free energy of a superfluid when a velocity is imposed on the condensate order parameter. The superfluid density for a superfluid with velocity $v_s$ is then obtained from

$$\rho_s = \frac{\partial^2 \Omega[v_s]}{\partial v_s^2} \bigg|_{v_s=0}. \quad (4)$$

It can be shown that this is equivalent to the standard definition $\rho_s = \rho - \rho_n$, where $\rho = m \tilde{n}$ is the total mass density and the normal fluid density $\rho_n$ is given in terms of the transverse current correlation function. Explicitly, for superfluid flow along the $z$-axis,

$$\rho_s = \frac{\partial^2 F[v_s]}{\partial v_s^2} \bigg|_{v_s=0} \quad (5)$$

given in terms of the free energy $F = \Omega + \mu n$. One can also prove (see Appendix A in Ref. 19) that Eq. (4) is equivalent to $\rho_s = \rho - \rho_n$, where $\rho = m \tilde{n}$ is the total mass density and the normal fluid density $\rho_n$ is given in terms of the transverse current correlation function. Explicitly, for superfluid flow along the $z$-axis,

$$\rho_s = \left. \frac{\partial^2 \Omega[v_s]}{\partial v_s^2} \right|_{v_s=0} \rho - m \langle \hat{J}_z \rangle_z, \quad (6)$$

where $\hat{J}_z$ is the component of the total current operator in the $z$-direction.

Microscopically, the thermodynamic potential is given by the partition function $Z$,

$$\Omega = -\beta^{-1} \ln Z, \quad (7)$$

where $\beta = (k_B T)^{-1}$. Functional integral techniques allow us to express the partition function as a functional integral over fermionic Grassmann fields $\psi$ and $\bar{\psi}$ as $\Omega$.

$$Z = \int D[\psi, \bar{\psi}] e^{-S[\psi, \bar{\psi}]} \quad (8)$$

The imaginary-time action $S[\psi, \bar{\psi}]$ in Eq. (8) for a two-component Fermi superfluid is given by

$$S[\psi, \bar{\psi}] = \int d^4x \left( \sum_{\sigma} \bar{\psi}_\sigma(x) \partial_\tau \psi_\sigma(x) + \mathcal{H} \right), \quad (9)$$

where the Hamiltonian density $\mathcal{H}$ is given by Eq. (2).

Here, $x = (r, \tau)$ is used to denote the spatial coordinate $r$ and the imaginary time $\tau = it$, and $\int d^4x \equiv \int d^3r \int d\tau$.

A central aspect of the analysis in this paper (and that of Josephson) is the existence of a broken-symmetry order parameter in the superfluid phase. For Fermi superfluids, this order parameter $\Delta_0$ is given by the anomalous average

$$\Delta_0 \equiv \frac{U_0}{\beta} \sum_k (c_{1, -k} c_{\uparrow, k}). \quad (10)$$

Here, $c_{\sigma, k}$ is the Fourier-transform of the Fermi Grassmann field $\psi_\sigma(x)$,

$$\psi_\sigma(x) = \frac{1}{\sqrt{\beta}} \sum_k c_{\sigma, k} e^{ik \cdot x}, \quad (11)$$

where $k \equiv (k, \omega_n)$ is a 4-vector for the momentum $k$ and Fermi Matsubara frequency $\omega_n = \pi(2n + 1)/\beta$, $n = 0, \pm 1, \pm 2, ...$, and $k \cdot x \equiv k \cdot r - \omega_n \tau$.

In order to introduce the bosonic order parameter into the partition function $Z$, the following identity is used:

$$\exp \left\{- \int d^4x \left[ \frac{\Delta^2}{U_0} - (\Delta^* \psi_\uparrow \psi_\uparrow + \Delta \bar{\psi}_\uparrow \bar{\psi}_\uparrow) \right] \right\} \quad (12)$$

Substituting this into Eq. (11), one obtains the result

$$Z = \int D[\psi, \bar{\psi}] D[\Delta, \Delta^*] \times \exp \left\{- \int d^4x \left[ \sum_\sigma \bar{\psi}_\sigma(x) \left( \partial_\tau - \frac{\nabla^2}{2m} - \mu \right) \psi_\sigma(x) 
- \Delta^* \psi_\uparrow \psi_\uparrow + \Delta \bar{\psi}_\uparrow \bar{\psi}_\uparrow \right] \right\} \quad \equiv \int D[\psi, \bar{\psi}] D[\Delta, \Delta^*] e^{-S_{eff}}. \quad (13)$$

This integral identity (a Hubbard-Stratonovich transformation) only introduces an auxiliary Bose field $\Delta(x)$, and no approximation has been made. It is straightforward to show that the static, uniform component of this field gives the order parameter defined in Eq. (10). That is, using the partition function in Eq. (13), one can show that

$$\langle \Delta(x) \rangle = \frac{U_0}{\beta} \sum_k \langle c_{1, -k} c_{\uparrow, k} \rangle \equiv \Delta_0, \quad (14)$$

where the equilibrium average $\langle \Delta(x) \rangle$ is defined by

$$\langle \Delta(x) \rangle \equiv \frac{1}{Z} \int D[\psi, \bar{\psi}] D[\Delta, \Delta^*] \Delta(x) e^{-S_{eff}}. \quad (15)$$

It is important to emphasize that while the BCS order parameter $\Delta_0$ is usually calculated in the mean-field BCS approximation, it is not an inherently mean-field quantity. The “0” subscript on $\Delta_0$ only denotes the fact that the order parameter is related to the average occupation (macroscopic in the superfluid phase) of a pair state with zero total momentum. One obtains the mean-field approximation for $\Delta_0$ if the expectation value $\langle \cdot \cdot \rangle$ in Eq. (14) is evaluated using a mean-field expression for the partition function. Here the full partition function is used, so $\Delta_0$ is the exact value of the order parameter.

Having established the relation between the auxiliary Bose field $\Delta(x)$ and the order parameter in Eq. (14), $\Delta(x)$ can be separated as

$$\Delta(x) = \Delta_0 + \Lambda(x), \quad (16)$$
where $\Lambda(x)$ represents the fluctuations out of the static Bose-condensed pair state.

The partition function given by Eq. (13) and the identity in Eq. (10) will be used below to analyze the superfluid density in a current-carrying Fermi superfluid. First we examine the pair fluctuation propagator that describes the dynamics of the Cooper pairs. It is distinct from the two-particle Green’s function $L_{BB}$.

Using Eqs. (17) and (18), we can see from Eq. (21). Nevertheless, as discussed in Ref. 25, evaluating the pair fluctuation propagator at the Matsubara frequency gives

$$L^{-1}(q, \nu_m) = \frac{1}{A (i \nu_m - \omega_q (i \nu_m + \omega_q))},$$

$$\left( i \nu_m + \frac{F}{A} + \frac{C}{A} q^2 - D - F q^2 \right) + O(q^4, \nu_m^2),$$

where we have defined the poles of $L$ as

$$\omega_q = \frac{1}{A} \sqrt{(B + C q^2)^2 - (D + F q^2)^2}.$$  

IV. THE CURRENT-CARRYING SUPERFLUID

We now consider the properties of a current-carrying superfluid. To introduce a finite superfluid velocity $v_s$, a “phase twist” is applied to the Bose Green’s function $K$ of the Bose field $\Delta(x)$. Further discussion of the relation between the pair fluctuation propagator and the two-particle Green’s function $L$ can be found in Ref. 22.

Motivated by the similarity to the Bose Green’s function $D$, we expand the Fourier transform $L^{-1}(q, \nu_m)$ of the inverse pair fluctuation propagator in powers of $q$ as

$$L^{-1}_{11}(q, \nu_m) = (L^{-1})_{22}(q, -\nu_m) = A i \nu_m - B - C q^2 + O(q^4, \nu_m^2),$$

$$L^{-1}_{12}(q, \nu_m) = (L^{-1})_{21}(q, \nu_m) = -D - F q^2 + O(q^4, \nu_m^2),$$

where $\nu_m = 2 \pi m / \beta$, $m = 0, \pm 1, \pm 2, \ldots$ denotes the Bose Matsubara frequencies. Note that the off-diagonal element $(L^{-1})_{12}$ has the symmetry $(L^{-1})_{12}(q, -\nu_m) = (L^{-1})_{12}(q, \nu_m)$ and hence the absence of a linear term in the Matsubara frequency in its expansion.

Using the expansion in Eq. (22), the pair fluctuation propagator becomes

$$L(q, \nu_m) = \frac{1}{A (i \nu_m - \omega_q (i \nu_m + \omega_q))} \times$$

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This shows that in order for the poles of the pair fluctuation propagator to be gapless, we must have $B = D$, giving a Bogoliubov-Anderson mode with velocity

$$v = \frac{1}{A} \sqrt{2 B (C - F)}. $$

The condition for a gapless mode to exist can also be written as

$$(L^{-1})_{11}(0, 0) + (L^{-1})_{22}(0, 0) - (L^{-1})_{12}(0, 0) - (L^{-1})_{21}(0, 0) = 0,$$

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a result that will be made use of in deriving the Josephson relation in Sec. IV.
static value of the order parameter $\Delta_0$ as in Eq. (16) and applying the phase twist, one finds
\[
\Delta(x) \rightarrow \Delta_0 e^{i m \mathbf{v} \cdot \mathbf{r}} + \Lambda(x) = \Delta(x) + \Delta_0 \left( e^{i m \mathbf{v} \cdot \mathbf{r}} - 1 \right). \tag{28}
\]
Writing the dependence on the superfluid velocity $\mathbf{v}_s$ in this way emphasizes that the proceeding analysis is based on the full Bose pairing field $\Delta(x)$ that includes all fluctuations about the static order parameter $\Delta_0$.

Using Eq. (28) in Eq. (13), it is seen that the effect of imposing a phase twist on the order parameter is to generate a new term in the effective action:
\[
S_{\text{eff}}[\mathbf{v}_s] = S_{\text{eff}}[0] + \delta S[\mathbf{v}_s], \tag{29}
\]
where
\[
\delta S = -\Delta_0 \int d^4x \left[ (e^{i m \mathbf{v} \cdot \mathbf{r}} - 1)\Phi(x) + (e^{-i m \mathbf{v} \cdot \mathbf{r}} - 1)\Phi(x) \right], \tag{30}
\]
and $\Phi, \bar{\Phi}$ are defined in Eq. (21). Note that the term in the effective action $S_{\text{eff}}$ in Eq. (13) involving $|\Delta(x)|^2/U_0$ is unchanged by the phase twist to the order parameter since the imaginary-time integral over any term linear in $\Lambda(x)$ vanishes.

Using Eq. (29) in the partition function defined in Eq. (13), the thermodynamic potential of the current-carrying superfluid is found to be
\[
\Omega[\mathbf{v}_s] = -\frac{1}{\beta} \ln \int D[\psi, \bar{\psi}] D[\Delta, \Delta^*] e^{-S_{\text{eff}}[0] - \delta S[\mathbf{v}_s]} \tag{31}
\]
Applying the definition of the superfluid density given by Eq. (1) to Eq. (31), one finds
\[
\rho_s = \frac{1}{\beta} \left\langle \frac{\partial^2 \delta S}{\partial \mathbf{v}_s^2} \right\rangle - \frac{1}{\beta} \left\langle \left( \frac{\partial \delta S}{\partial \mathbf{v}_s} \right)^2 \right\rangle. \tag{32}
\]
Here, $\langle \cdots \rangle$ denotes the equilibrium average in the current-free state, given by
\[
\langle \cdots \rangle = \frac{1}{Z[0]} \int D[\psi, \bar{\psi}] D[\Delta, \Delta^*] \langle \cdots \rangle e^{-S_{\text{eff}}[0]} \tag{33}
\]
Note that $\langle (\partial \delta S/\partial \mathbf{v}_s) \rangle|_{\mathbf{v}_s = 0} = 0$, by symmetry.

Using Eq. (31) to evaluate Eq. (32) gives
\[
\rho_s = \frac{\Delta_0 m_B^2}{\beta} \int d^4x \left( \mathbf{v}_s \cdot \mathbf{r} \right)^2 \langle \Phi(x) + \Phi(x) \rangle - \frac{\Delta_0^2 m_B^2}{\beta} \int d^4x \int d^4x' \langle \mathbf{v}_s \cdot \mathbf{r} \rangle \langle \mathbf{v}_s \cdot \mathbf{r}' \rangle |K_{11} + K_{22} - K_{12} - K_{21}|(x, x') \tag{34}
\]
where $\mathbf{v}_s \equiv \mathbf{v}_s/\mathbf{v}_s$ and $K_{ij}(x, x')$ denote the elements of the matrix two-particle Green’s function defined in Eq. (19). Making use of the identity
\[
\langle \tilde{\Phi}(x) \rangle = \langle \Phi(x) \rangle = \frac{\Delta_0}{U_0}, \tag{35}
\]
Eq. (34) is naturally given in terms of the inverse pair fluctuation propagator,
\[
\rho_s = -\frac{\Delta_0^2 m_B^2}{\beta} \int d^4x d^4x' \langle \mathbf{v}_s \cdot \mathbf{r} \rangle \langle \mathbf{v}_s \cdot \mathbf{r}' \rangle |(L^{-1})_{11}| + (L^{-1})_{22} - (L^{-1})_{12} - (L^{-1})_{21}|(x, x'). \tag{36}
\]
Fourier transforming this expression, one finds (taking $\mathbf{v}_s = \mathbf{z}$ to lie along the $z$-axis)
\[
\rho_s = -\frac{\Delta_0^2 m_B^2}{2} \lim_{q \to 0} \frac{\partial^2}{\partial q_z^2} \left| (L^{-1})_{11} + (L^{-1})_{22} - (L^{-1})_{12} - (L^{-1})_{21} \right|(q, 0), \tag{37}
\]
with static matrix elements $(L^{-1})_{ij}(q, \nu_m = 0)$. In arriving at this result, a gapless Bose excitation spectrum has been assumed, using the result in Eq. (26).

Using the expansion in Eq. (22) to evaluate the second-derivative in Eq. (37), we see that
\[
\lim_{q \to 0} \frac{\partial^2}{\partial q_z^2} \left| (L^{-1})_{11} + (L^{-1})_{22} - (L^{-1})_{12} - (L^{-1})_{21} \right|(q, 0) = 4F - 4C. \tag{38}
\]
This allows us to more compactly write Eq. (37) as
\[
\rho_s = 2\Delta_0^2 m_B^2(C - F). \tag{39}
\]
Now, from the static (1,1) matrix element $L_{11}(q, 0)$ in Eq. (23), one also finds
\[
\lim_{q \to 0} \frac{1}{q^2 L_{11}(q, 0)} = \lim_{q \to 0} \frac{|D^2 - B^2 + 2(DF - BC)q^2|}{Bq^2}. \tag{40}
\]
Assuming that $L$ has a gapless excitation spectrum (such that $B = D$), this reduces to
\[
\lim_{q \to 0} \frac{1}{q^2 L_{11}(q, 0)} = 2F - 2C. \tag{41}
\]
Comparing Eqs. (39) and (11), one finally obtains
\[
\rho_s = -\lim_{q \to 0} \frac{\Delta_0^2 m_B^2}{q^2 L_{11}(q, 0)}. \tag{42}
\]
This expression gives the precise analogue for Fermi superfluids of the Josephson relation for Bose superfluids in Eq. (11). We see that the single-particle Green’s function $D$ for bosons has been replaced by the pair fluctuation propagator $L$, and the square of the BCS order parameter $\Delta_0^2$ plays the role of the square of the order parameter $\Phi_0^2 \equiv |\langle \psi \rangle|^2 = n_c. \tag{43}$

of a Bose superfluid.

For a Bose superfluid, Eq. (43) gives a simple relation between the order parameter $\Phi_0$ and the condensate density $n_c$, and these two quantities can be interchanged in
the Josephson relation in Eq. (11). This is not the case in a Fermi superfluid, however, where the condensate density is not a simple function of the order parameter $\Delta_0$. This can be seen from the mean-field expression for the condensate density in a BCS superfluid, given by\(^{28}\)

$$n_c = \frac{1}{\beta^2} \sum_{k,\omega_n} G_{0,21}(k,\omega_n) G_{0,12}(k,\omega_n'),$$

(44)

where $G_0$ is the mean-field $2 \times 2$ matrix BCS Green’s function. Equation (44) emphasizes the direct role of the order parameter in Josephson’s relation, in contrast to the indirect role played by the condensate density.

Equation (44) gives an exact relation between the superfluid density $\rho_s$ and the order parameter $\Delta_0$ in terms of the static pair fluctuation propagator $L^0(q,0)$. It can immediately be used to study superfluidity in Fermi superfluids. In Sec. V, the Josephson relation is studied within the BCS approximation for the pair fluctuation propagator. We see how the resulting expression for the superfluid density reduces to the Landau formula for BCS quasiparticle excitations.

V. RELATION TO LANDAU’S FORMULA FOR A BCS SUPERFLUID

An important check of the Josephson relation for Fermi superfluids is that it reproduces Landau’s well-known formula for the superfluid density when the normal fluid is comprised of BCS quasiparticle excitations\(^{28}\)

$$\rho_s = \rho + 2 \sum_k \frac{\partial f}{\partial E_k} k_z^2.$$  

(45)

Here, $f = [\exp(\beta E_k) + 1]^{-1}$ is the Fermi thermal distribution for BCS quasiparticles of energy $E_k = \sqrt{\xi_k^2 + \Delta_0^2}$ and $k_z$ is the z-component of $k$. The total mass density $\rho$ is given by

$$\rho = m_F \sum_k \left[ 1 - \frac{\xi_k}{E_k} (1 - 2f) \right].$$

(46)

The result given by Eqs. (45) and (46) for the superfluid density is mean-field insofar as it ignores the contribution to the normal fluid arising from bosonic collective modes, as discussed in Refs. 19,29. Consequently, we should be able to reproduce Eq. (45) by evaluating the Josephson relation in Eq. (42) using a mean-field BCS approximation.

Evaluating the pair fluctuation propagator $L^{-1}(q,0)$ within the BCS mean-field approximation amounts to evaluating the two-particle Green’s function defined in Eq. (19) as a loop of two single-particle mean-field BCS Green’s functions: $K = \sum G_k G_0$ (schematically). Explicitly, Eqs. (17) and (18) become

$$(L^{-1})_{12}(q,0) = (L^{-1})_{22}(q,0) = -\frac{1}{U_0} - \frac{1}{\beta} \sum_{k,\omega_n} G_{0,11}(k,\omega_n) G_{0,22}(k - q,\omega_n)$$

(47)

and

$$(L^{-1})_{11}(q,0) = (L^{-1})_{21}(q,0) = -\frac{1}{\beta} \sum_{k,\omega_n} G_{0,11}(k,\omega_n) G_{0,12}(k - q,\omega_n).$$

(48)

In this approximation, $L^{-1}$ is equivalent to the negative of the inverse pair fluctuation propagator $M$ defined in Refs. 19,29. Reference 29 showed explicitly that the poles of $L$ describe the gapless (i.e., $B = D$) Bogoliubov-Anderson spectrum at small $q$ throughout the entire BCS-BEC crossover. Furthermore, the combination of inverse matrix elements

$$(L^{-1})_{11}(q,0) - (L^{-1})_{12}(q,0) = -(C - F) q^2 + \cdots$$

(49)

that enters the expression for the superfluid density in Eqs. (37) and (39) is proportional to the static inverse pair fluctuation propagator $G_{12}(q,0)$ for BCS quasiparticles of energy $\xi_q = \sqrt{\xi_q^2 + \Delta_0^2}$.

The result given by Eqs. (45) and (46) for the superfluid density is mean-field insofar as it ignores the contribution to the normal fluid arising from bosonic collective modes, as discussed in Refs. 19,29. Consequently, we should be able to reproduce Eq. (45) by evaluating the Josephson relation in Eq. (42) using a mean-field BCS approximation.

Evaluating the pair fluctuation propagator $L^{-1}(q,0)$ within the BCS mean-field approximation amounts to evaluating the two-particle Green’s function defined in Eq. (19) as a loop of two single-particle mean-field BCS Green’s functions: $K = \sum G_k G_0$ (schematically). Explicitly, Eqs. (17) and (18) become

$$(L^{-1})_{12}(q,0) = (L^{-1})_{22}(q,0) = -\frac{1}{U_0} - \frac{1}{\beta} \sum_{k,\omega_n} G_{0,11}(k,\omega_n) G_{0,22}(k - q,\omega_n)$$

(47)

and

$$(L^{-1})_{11}(q,0) = (L^{-1})_{21}(q,0) = -\frac{1}{\beta} \sum_{k,\omega_n} G_{0,11}(k,\omega_n) G_{0,12}(k - q,\omega_n).$$

(48)

In this approximation, $L^{-1}$ is equivalent to the negative of the inverse pair fluctuation propagator $M$ defined in Refs. 19,29. Reference 29 showed explicitly that the poles of $L$ describe the gapless (i.e., $B = D$) Bogoliubov-Anderson spectrum at small $q$ throughout the entire BCS-BEC crossover. Furthermore, the combination of inverse matrix elements

$$(L^{-1})_{11}(q,0) - (L^{-1})_{12}(q,0) = -(C - F) q^2 + \cdots$$

(49)

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Applying Eq. (53) to the last line and integrating by parts again, one finds
\[
\rho_s = 2 \sum_k \frac{\Delta_0^2}{E_k^2} \left[ \frac{1 - 2f}{2E_k} + \frac{\partial f}{\partial E_k} \right] k^2_z + 2m_F \sum_k \frac{\Delta_0^2}{2E_k^3} \left[ k + \frac{k^2}{m_F} \left( \frac{3\Delta_0^2}{E_k^2} - 2 \right) \right]. \tag{55}
\]
The integral in the second line vanishes exactly and our expression for the superfluid density reduces to
\[
\rho_s = 2 \sum_k \frac{\Delta_0^2}{E_k^2} \left[ \frac{1 - 2f}{2E_k} + \frac{\partial f}{\partial E_k} \right] k^2_z. \tag{56}
\]
Rearranging the mean-field expression for the mass density \(\rho\) in Eq. (46) using integration by parts and Eq. (53), one can write as
\[
\rho = -m_F \sum_k k^2 \frac{\partial f}{\partial E_k} \left[ 1 - \frac{\xi_k}{E_k} (1 - 2f) \right] = 2 \sum_k \frac{\Delta_0^2}{E_k^2} \frac{1 - 2f}{2E_k} k^2_z - 2 \sum_k \frac{\xi_k}{E_k^2} \frac{\partial f}{\partial E_k} k^2_z. \tag{57}
\]
Combining this result with Eq. (56), we see that it reduces to Eq. (45). Thus, evaluating Josephson’s relation provided by Eqs. (42) using the mean-field approximation given by Eqs. (17) and (18) gives us Landau’s formula for the superfluid density in a BCS superfluid.

In Sec. VII, we employ the same mean-field approximation used in this section to show that the Josephson relation for a Fermi superfluid reduces to the analogous expression given by Eq. (11) for a Bose superfluid, in the BEC limit of the BCS-BEC crossover.

VI. JOSEPHSON’S RELATION IN THE BEC LIMIT

An obvious feature of the Josephson relation for a Fermi superfluid is that it must reduce to Eq. (11) in the BEC limit of the BCS-BEC crossover, where the Cooper pairs are tightly-bound dimer molecules. In this limit, where the s-wave scattering length \(a_s\) is small and positive, the chemical potential becomes large and negative, roughly equal to half the dimer binding energy \(\mu = -1/2m_F a_s^2\). In this case, \(\mu \gg \Delta_0, k_B T; f \to 0\), and Eq. (14) can be solved analytically to give the condensate density of dimers.\(^{25,32}\)

\[
n_c(T) \simeq \sum_k \frac{\Delta_0^2(T)}{4k^2} \simeq \left( \frac{m_F a_s}{8\pi} \right) \Delta_0^2(T). \tag{58}
\]

Within the same mean-field approximation [given by Eqs. (17) and (18)], one can show in the BEC limit that the inverse pair fluctuation propagator \(L^{-1}\) reduces to\(^{25,32}\)

\[
L^{-1}(q, \nu_m) = \left( \frac{m_F a_s}{8\pi} \right) D^{-1}(q, \nu_m), \tag{59}
\]

where
\[
D^{-1}(q, \nu_m) \equiv \left( i\nu_m - c_q - n_c U_{\text{mol}} - n_c U_{\text{mol}} \right) - n_c U_{\text{mol}} \tag{60}
\]

is the inverse single-particle Green’s function for the Bose-condensed dimer molecules, analogous to the Green’s function that enters Eq. (1). Here, \(c_q = q^2/2m_B\) while \(U_{\text{mol}} = 4\pi(2a_s)/m_B\) is the mean-field interaction between dimers, which predicts a dimer scattering length of \(a_{\text{mol}} = 2a_s^{25}\) instead of the exact result \(a_{\text{mol}} = 0.6a_s^{31}\). Substituting Eqs. (58) and (59) into Eq. (42), one immediately obtains the Josephson relation in Eq. (11) for a condensate of dimer molecules.

Of course, one expects Eq. (42) to reduce to Eq. (11) in the BEC limit at any level of approximation, and not just at the mean-field level at which Eqs. (59) and (60) have been derived. The fact that Eq. (59) is a mean-field result only means that the molecular self-energies \(\Sigma_{\text{mol}}(q, \nu_m)\) that enter the dimer Green’s function \(D^{-1}\) in Eq. (60) are mean-field. Explicitly, writing down the exact single-particle Green’s function for a dimer molecule,

\[
D^{-1}(q, \nu_m) \equiv \left( i\nu_m - c_q + n_c U_{\text{mol}} - n_c U_{\text{mol}} - n_c U_{\text{mol}} \right) - n_c U_{\text{mol}} \tag{61}
\]

Eq. (60) corresponds to the result

\[
\mu_{\text{mol}} - \Sigma_{11}(q, \nu_m) = -n_c U_{\text{mol}} \tag{62}
\]

and

\[
\Sigma_{12}(q, \nu_m) = n_c U_{\text{mol}}. \tag{63}
\]

We anticipate that, going past the mean-field approximation used to obtain the results in Eqs. (52) and (53), one still arrives at the identity given by Eq. (59), except that the self-energies will incorporate beyond-mean-field contributions. Using Eq. (61) in Eq. (11) shows that a momentum-independent self-energy will always lead to the well-known mean-field result, \(\rho_s = m_B n_c(T)\). As first shown in Ref. 24, including contributions from fluctuations, the superfluid density in the BEC limit is actually given by Landau’s formula for a normal fluid comprised of gapless Bogoliubov excitations of the BEC of dimer molecules.

VII. SUMMARY

Josephson’s relation for Bose superfluids gave the first explicit identity connecting the two key order parameters in the theory of superfluids: the broken-symmetry
order parameter and the superfluid density. It is remarkable that such a simple relation exists between two such different quantities: the superfluid density that describes the response of a system to a transverse current probe, as in Eq. (6), and the order-parameter in a Bose superfluid, associated with the macroscopic occupation of a single-particle state. Extending the recent analysis by Holzmann and Baym, the analogous identity has been derived for a two-component $s$-wave Fermi superfluid. This gives an exact relation between the superfluid density $\rho_s$ and the BCS order parameter $\Delta_0$ in terms of the infrared limit of the static pair fluctuation propagator $L(q, \nu_m = 0)$.

Using mean-field BCS theory to evaluate the pair fluctuation propagator, we have seen that the Josephson relation derived in this paper reduces to the Josephson relation for a Bose superfluid in the BEC limit and also Landau’s formula for the superfluid density of a Fermi gas with BCS quasiparticle excitations. At first glance, it might seem surprising that the Josephson relation—which expresses the superfluid density in terms of the propagator for collective phase fluctuations—manages to reproduce this Landau formula for a normal fluid of single-particle Fermi BCS excitations. However, this propagator is actually a correlation function for the gradient of the phase of the order parameter, and consequently, is directly related to the current correlation function.

In turn, it is well-known that Landau’s formula can be obtained by a direct evaluation of the longitudinal and transverse components of the current correlation function within the BCS approximation.

The simple structure of the Josephson relation derived in this paper should simplify the calculation of the superfluid density using the standard tools of diagrammatic perturbation theory developed for the BCS-BEC crossover problem to evaluate the pair fluctuation propagator $L$. It also opens the way to giving a rigorous analysis of the superfluid transition in Fermi systems, along the lines of those given for Bose superfluids.

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The pair fluctuation propagator can be transformed via unitary transformation into a $2 \times 2$ matrix propagator describing fluctuations of the phase and amplitude of the order parameter. In the long wavelength limit, $(L^{-1})_{11} - (L^{-1})_{12}$ is the inverse propagator for phase fluctuations.\cite{30}

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From the definition of the superfluid density in Eq. (4), $\rho_s$ is associated with the change $\delta F$ in the free energy due to a small “twist” of the order parameter \textit{phase} $\phi$. By $U(1)$ gauge symmetry, the free energy can only depend on the gradient of the phase. Performing a gradient expansion of the free energy, it follows that $\delta F[\Delta_0, e^{i\phi}] \simeq \delta F[\Delta_0, (\nabla \phi \cdot \nabla \phi)]$ is expressed in terms of the correlation function for the gradient $\nabla \phi$ of the phase. This is directly related to the $q^2$ term in the expansion of the inverse phase fluctuation propagator $(L^{-1})_{11} - (L^{-1})_{12}$.

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