UNIFORMLY LOCALLY UNIVALENT HARMONIC MAPPINGS ASSOCIATED WITH THE PRE-SCHWARZIAN NORM

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Abstract. In this paper, we consider the class of uniformly locally univalent harmonic mappings in the unit disk and build a relationship between its pre-Schwarzian norm and uniformly hyperbolic radius. Also, we establish eight ways of characterizing uniformly locally univalent sense-preserving harmonic mappings. We also present some sharp distortions and growth estimates and investigate their connections with Hardy spaces. Finally, we study subordination principles of norm estimates.

1. Introduction

Harmonic mappings play an important role in various branches of applied mathematics including the study of liquid crystals, both in theory and in practice. There are many classical approaches to deal with harmonic maps in various settings. For example, A. Aleman and A. Constantin [3] developed tools using complex analytic theory and the univalence of the labelling map to solve fluid flow problems in a surprisingly simple form. More recently, O. Constantin and M. J. Martín [15] proposed a new approach to obtain a complete solution to the problem of classifying all two dimensional ideal fluid flows with harmonic labelling maps. This approach is based on ideas from the theory of harmonic mappings by finding two harmonic maps with same Jacobians and illustrates the deep links between the fields of complex analysis and fluid mechanics. Investigations of this type have prompted renewed interest in the study of sense-preserving harmonic mappings. The present article is concerned with Schwarzian and pre-Schwarzian norms defined in the unit disk, and in particular, with certain important function spaces. In addition, we introduce several new ideas and tools for a number of problems in the case of harmonic mappings.

1.1. Basic notations. A complex-valued function $f$ in the unit disk $D = \{ z : |z| < 1 \}$ is called a harmonic mapping if it satisfies the Laplace equation $\Delta f = 4f_{zz} = f_{xx} + f_{yy} = 0$. It is known that $f$ has a canonical representation $f = h + \overline{g}$ with $g(0) = 0$, where $h$ and $g$ are analytic functions in $D$ and $J_f = |h'|^2 - |g'|^2$ denotes the Jacobian of $f$. As is usual, we call $h$ the analytic part of $f$ and $g$ the co-analytic part of $f$. Lewy [27] proved that $f = h + \overline{g}$ is locally univalent in $D$ if and only if $J_f(z) \neq 0$ in $D$. Without loss of generality, we consider harmonic mappings $f$ that are sense-preserving, i.e. $J_f > 0$ or equivalently $|h'| > |g'|$ in $D$. In this case, its dilatation $\omega_f = g'/h'$ has the property that $|\omega_f| < 1$ in

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Especially, if $|\omega_f| \leq k < 1$ in $\mathbb{D}$, then $f$ is called a $K-$quasiconformal mapping, where $K = (1 + k)/(1 - k)$. More details about planar harmonic mappings, may be found in the monograph of Duren [18] and in the survey article of Ponnusamy and Rasila [33].

For the convenience of the reader, we first list down the following notations and terminologies whose precise definitions will be presented at appropriate places.

- ULU (ULC) - uniformly locally univalent (uniformly locally convex)
- SAULU - stable analytic uniformly locally univalent
- SAULC - stable analytic uniformly locally convex
- SHULU - stable harmonic uniformly locally univalent
- SHULC - stable harmonic uniformly locally convex
- SHU (SHC) - stable harmonic univalent (stable harmonic convex)
- PSD (SD) - pre-Schwarzian derivative (Schwarzian derivative)
- PSN (SN) - pre-Schwarzian norm (Schwarzian norm)
- SBAPSN - stable bounded analytic pre-Schwarzian norm
- SBASN - stable bounded analytic Schwarzian norm
- SBHPSN - stable bounded harmonic pre-Schwarzian norm
- SBHSN - stable bounded harmonic Schwarzian norm
- $H = \{ f = h + g : f$ is a sense-preserving harmonic mapping in $\mathbb{D}$ satisfying
  the normalizations $h(0) = h'(0) - 1 = g(0) = 0 \}$
- $H_0 = \{ f = h + g \in H : g'(0) = 0 \}$

Sometimes we write $f \in$ ULU to convey that $f$ is a uniformly locally univalent function in $\mathbb{D}$. Similar convention will be followed for other cases.

1.2. ULU harmonic mappings. Let $z, a \in \mathbb{D}$. We denote the hyperbolic distance between $z$ and $a$ by

$$d_h(z, a) = 2 \tanh^{-1} \left( \frac{|z - a|}{1 - \overline{a}z} \right).$$

The hyperbolic disk in $\mathbb{D}$ with center $a \in \mathbb{D}$ and hyperbolic radius $\rho$, $0 < \rho \leq \infty$, is defined by

$$D_h(a, \rho) = \{ z \in \mathbb{D} : d_h(z, a) < \rho \}.$$

We say that a sense-preserving harmonic mapping $f = h + \overline{g}$ in $\mathbb{D}$ is a ULU harmonic mapping in $\mathbb{D}$ if $\rho(f) > 0$, where

$$\rho(f) = \inf_{z \in \mathbb{D}} \left\{ \sup_{\rho_z > 0} \{ \rho_z : f$ is univalent in $D_h(z, \rho_z) \} \right\}.$$ 

The number $\rho(f)$ is called the uniformly hyperbolic radius of $f$. Moreover, $f$ is univalent in $\mathbb{D}$ if and only if $\rho(f) = \infty$.

1.3. PSD and PSN of harmonic mappings. Let $f = h + \overline{g}$ be a sense-preserving harmonic mapping in $\mathbb{D}$ with $\omega := \omega_f = g'/h'$. Then the PSD and the PSN of $f$ are defined by

$$P_f = (\log J_f)_z = \frac{h''\overline{h'} - g''\overline{g'}}{|h'|^2 - |g'|^2} = \frac{h''}{h'} \frac{\overline{\omega} \omega'}{1 - |\omega|^2}$$
and

\[ ||P_f|| = \sup_{z \in \mathbb{D}} (1 - |z|^2)|P_f(z)|, \]

respectively. Clearly, in the analytic case \( \omega \) in (1.1) has taken to be identically 0 in \( \mathbb{D} \), and thus, throughout we use the same notations for the PSD and the PSN in the case of analytic functions as well.

The PSD has affine invariance property:

\[ (1.2) \quad P_f = P_{A \circ f}, \quad A(z) = az + b z + c, \quad a, \ b, \ c \in \mathbb{C} \quad \text{and} \quad |a| > |b|. \]

Note that \( A \circ f \) is still sense-preserving in \( \mathbb{D} \).

The above definitions of the PSD and the PSN for harmonic mappings were introduced by Hernández and Martín in [21] (see also [11]), which coincide with the corresponding analytic definitions (see [17, 31]). It is well known that the PSN of a locally univalent analytic function is an important quantity in the study of the global univalence. For example, if \( f \) is a univalent analytic function in \( \mathbb{D} \), then \( ||P_f|| \leq 6 \), which is sharp. Conversely, if \( ||P_f|| \leq 1 \) holds for a locally univalent analytic function \( f \) in \( \mathbb{D} \), then \( f \) is necessarily univalent in \( \mathbb{D} \) and the constant 1 is sharp (see [7, 8]). Recently, new criteria for the univalence of harmonic mappings in terms of the PSD or the PSN have been established in [6, 19, 21].

1.4. Relationship between ULU and PSN. Yamashita [40] showed that a locally univalent analytic function \( f \) in \( \mathbb{D} \) is ULU in \( \mathbb{D} \) if and only if \( ||P_f|| \) is bounded. Later, Kim and Sugawa [24] investigated the growth of various quantities for a ULU analytic function \( f \) in \( \mathbb{D} \) by means of finite the PSN. Since \( P_{\phi \circ f} = P_f \) for any linear transformation \( \phi(z) = az + b \ (a \neq 0) \), they just considered the following normalized function space

\[ \mathcal{B}_A = \{ f \in A : ||P_f|| < \infty \}, \]

where \( A \) is the set of analytic functions \( f \) in \( \mathbb{D} \) with the normalizations \( f(0) = f'(0) - 1 = 0 \). In fact, the space \( \mathcal{B}_A \) has the structure of a nonseparable complex Banach space under the Hornich operation (see [39]). To obtain some precise results, it was necessary to study the subset of \( \mathcal{B}_A \):

\[ \mathcal{B}_A(\lambda) = \{ f \in \mathcal{A} : ||P_f|| \leq 2 \lambda \}, \]

where \( \lambda \geq 0 \) and the factor 2 is due to only some technical reason.

Following the proof of [21, Theorem 7], we see that a sense-preserving harmonic mapping \( f \) in \( \mathbb{D} \) is ULU in \( \mathbb{D} \) if and only if \( ||P_f|| \) is bounded, which will be also proved in Section 3 by other method. Therefore, the primary aim of this paper is to extend some of the results from [24] to sense-preserving and ULU harmonic mappings in \( \mathbb{D} \) associated with finite the PSN. Since the PSD preserves affine invariance, in what follows, we only to consider the following set of normalized functions:

\[ \mathcal{B}_H = \{ f \in \mathcal{H} : ||P_f|| < \infty \}. \]

If we concern only on the PSN, then \( \mathcal{B}_H \) can be further restricted to be \( \mathcal{B}_H^0 := \mathcal{B}_H \cap \mathcal{H}_0 \).

In fact, if \( f = h + \bar{h} \in \mathcal{B}_H \) and \( A(z) = \frac{z - b_1}{1 - |b_1|^2} \ (b_1 = g'(0)) \), then it follows from (1.2) that \( ||P_{A \circ f}|| = ||P_f|| \) and it is also easy to see that \( A \circ f \in \mathcal{B}_H^0 \).
Let \( f = h + \overline{g} \) be a sense-preserving harmonic mapping in \( \mathbb{D} \). Then, motivated by the works of [20], in Section 2, we will build some sharp inequalities between \( ||P_{h + \varepsilon g}|| \) and \( ||P_{h + \varepsilon g}|| \), where \( \varepsilon_1, \varepsilon_2 \in \mathbb{D} \). In particular, we obtain the following important implication:

\[
\tag{1.3} f \in \mathcal{B}_H(\lambda) := \{ f \in \mathcal{H} : ||P_f|| \leq \lambda \} \Rightarrow \frac{h + \varepsilon g}{1 + \varepsilon g'(0)} \in \mathcal{B}_A \left( \frac{\lambda + 1}{2} \right) \quad \forall \varepsilon \in \mathbb{D},
\]

where \( \lambda \geq 0 \). In Section 3, for any given sense-preserving and ULU harmonic mapping in the unit disk, we give a relationship between its PSN and uniformly hyperbolic radius. Combining the above results with some works about ULU harmonic mappings, plenty of equivalent conditions for a sense-preserving and ULU harmonic mapping in the unit disk are obtained in Section 4. To present some sharp examples in Sections 6 and 7, we introduce a class of sense-preserving harmonic mappings with prescribed PSN in Section 5. These results help us to obtain sharp distortion, growth and covering theorems for \( \mathcal{B}_H(\lambda) \) or \( \mathcal{B}_H^0(\lambda) := \mathcal{B}_H(\lambda) \cap \mathcal{H}_0 \) in Section 5. Applying (1.3) and the corresponding results in [24] and [32], the growth of coefficients and the relationship with Hardy space for the class \( \mathcal{B}_H(\lambda) \) are considered in Sections 7 and 8, respectively. Finally, some subordination principles of the PSN estimates are also obtained in Section 9.

### 2. Some inequalities Concerning Pre-Schwarzian norm

We now state our key inequalities which will provide important connections between ULU analytic functions and ULU harmonic mappings in the unit disk.

**Theorem 2.1.** Let \( f = h + \overline{g} \) be a sense-preserving harmonic mapping in \( \mathbb{D} \). Then either \( ||P_{h + \varepsilon g}|| = ||P_f|| = \infty \) or both \( ||P_{h + \varepsilon g}|| \) and \( ||P_f|| \) are finite for each \( \varepsilon \in \mathbb{D} \). If \( ||P_f|| < \infty \), then the inequality

\[
||P_{h + \varepsilon g}|| - ||P_f|| \leq 1
\]

holds for each \( \varepsilon \in \mathbb{D} \). In particular,

\[
||P_{h + \varepsilon g}|| - ||P_f|| \leq 1.
\]

The constant 1 is sharp in the two estimates.

**Proof.** Suppose that \( f = h + \overline{g} \) is a sense-preserving harmonic mapping in \( \mathbb{D} \). Then \( h + \varepsilon g \) is a locally univalent analytic function in \( \mathbb{D} \) for each \( \varepsilon \in \mathbb{D} \). By (1.1), a direct computation shows that

\[
P_{h + \varepsilon g} = \frac{h'' + \varepsilon g''}{h' + \varepsilon g'} = P_h + \frac{\varepsilon \omega'}{1 + \varepsilon \omega},
\]

and thus,

\[
P_{h + \varepsilon g} - P_f = \frac{\varepsilon \omega'}{1 + \varepsilon \omega} + \frac{\overline{\omega}\omega'}{1 - |\omega|^2} = \frac{\varepsilon + \overline{\omega}}{1 + \varepsilon \omega} \cdot \frac{\omega'}{1 - |w|^2},
\]

where \( \omega = g'/h' \). Therefore, by the Schwarz-Pick lemma, we have

\[
(1 - |z|^2)||P_{h + \varepsilon g}(z)| - |P_f(z)|| \leq (1 - |z|^2)||P_{h + \varepsilon g}(z) - P_f(z)|| \leq \sup_{z \in \mathbb{D}} \left| \frac{\varepsilon + \omega(z)}{1 + \varepsilon \omega(z)} \right| \leq 1
\]

for every \( z \in \mathbb{D} \) and the assertion easily follows.
To show the sharpness, it suffices to consider the harmonic Koebe function \( K \) (see [13]) defined by
\[
K(z) = h(z) + g(z) = \frac{z - \frac{1}{2}z^2 + \frac{1}{6}z^3}{(1 - z)^3} + \frac{\frac{1}{2}z^2 + \frac{1}{6}z^3}{(1 - z)^3}, \quad z \in \mathbb{D}.
\]

By a direct computation, we find that
\[
P_{h-g}(z) = \frac{4 + 2z}{1 - z^2}, \quad P_K(z) = \frac{5 + 3z}{1 - z^2} - \frac{z}{1 - |z|^2},
\]
\[
P_h(z) = \frac{5 + 3z}{1 - z^2} \quad \text{and} \quad P_{h+g}(z) = \frac{6 + 2z}{1 - z^2}.
\]
It is easy to see that \( ||P_{h-g}|| = 6 \) and \( ||P_h|| = ||P_{h+g}|| = 8 \). Choosing \( \varepsilon = \pm 1 \) in (2.1), it follows that \( ||P_K|| = 7 \). In summary, we get that
\[
||P_{h-g}|| + 1 = ||P_K|| = 7 = ||P_h|| - 1 = ||P_{h+g}|| - 1.
\]

In addition, the sharpness can be seen from the harmonic half-plane mapping \( L \) (see [13]) defined by
\[
L(z) = h(z) + g(z) = \frac{2z - z^2}{2(1 - z)^2} + \frac{-z^2}{2(1 - z)^2}, \quad z \in \mathbb{D}.
\]
Elementary computations yield that
\[
P_{h+g}(z) = \frac{2}{1 - z}, \quad P_L(z) = \frac{3}{1 - z} - \frac{\bar{z}}{1 - |z|^2},
\]
\[
P_h(z) = \frac{3}{1 - z} \quad \text{and} \quad P_{h-g}(z) = \frac{4 + 2z}{1 - z^2}.
\]
As in the harmonic Koebe function, we obtain that
\[
||P_{h+g}|| + 1 = ||P_L|| = 5 = ||P_h|| - 1 = ||P_{h-g}|| - 1
\]
and the proof is complete. \( \square \)

Obviously, the assertion (1.3) is true by Theorem 2.1. Next we consider more general inequalities.

**Corollary 2.1.** Let \( f = h + \bar{g} \) be a sense-preserving harmonic mapping in \( \mathbb{D} \). If \( ||P_h|| < \infty \), then for any \( \varepsilon_1, \varepsilon_2 \in \mathbb{D} \), we have the following inequalities.

1. The sharp inequality \( |||P_{h+\varepsilon_1g}|| - ||P_{h+\varepsilon_2g}||| \leq 2 \) holds.
2. If \( |\varepsilon_1| = |\varepsilon_2| \), then \( ||P_{h+\varepsilon_1g}|| = ||P_{h+\varepsilon_2g}|| \). If \( |\varepsilon_1| \neq |\varepsilon_2| \), then
   \[
   |||P_{h+\varepsilon_1g}|| - ||P_{h+\varepsilon_2g}||| \leq |\varepsilon_1| + |\varepsilon_2| < 2.
   \]
3. If \( |\varepsilon_1| \leq |\varepsilon_2| \), then we have the sharp inequality \( |||P_{h+\varepsilon_1g}|| - ||P_{h+\varepsilon_2g}||| \leq 1 \). If \( |\varepsilon_1| > |\varepsilon_2| \), then
   \[
   |||P_{h+\varepsilon_1g}|| - ||P_{h+\varepsilon_2g}||| \leq 1 + |\varepsilon_1| + |\varepsilon_2| < 3.
   \]
Proof. Since \( ||P_h|| < \infty \), it follows from Theorem 2.1 that both \( ||P_{h+\varepsilon g}|| \) and \( ||P_{h+\varepsilon g'}|| \) are finite for each \( \varepsilon \in \mathbb{D} \).

(1) The inequality can be easily deduced from (2.1) by applying the triangle inequality once. The sharpness can be seen from the harmonic Koebe function and the harmonic half-plane mapping.

(2) Note that \( f_\varepsilon = h + \varepsilon g \) is still a sense-preserving harmonic mapping with dilatation \( \overline{\omega_f} \) for any given \( \varepsilon \in \mathbb{D} \). It follows from (1.1) that

\[
P_{h+\varepsilon g} - P_{h+\varepsilon g'} = \left( \frac{|\varepsilon_2|^2}{1-|\varepsilon_2 \omega_f|^2} - \frac{|\varepsilon_1|^2}{1-|\varepsilon_1 \omega_f|^2} \right) \overline{\omega_f} \omega_f'.
\]

Then the former part is trivial. The later part can be easily deduced from the Schwarz-Pick lemma and the triangle inequality.

(3) The former part is a direct consequence of (2.1). For the later part, using (2), the former part and the triangle inequality, we have

\[
||P_{h+\varepsilon g}|| - ||P_{h+\varepsilon g'}|| \leq ||P_{h+\varepsilon g}|| - ||P_{h+\varepsilon g'}|| + ||P_{h+\varepsilon g'}|| - ||P_{h+\varepsilon g'}||
\]

\[
\leq 1 + |\varepsilon_1| + |\varepsilon_2| < 3.
\]

The proof is complete. \( \square \)

Associated with Bieberbach’s criterion and Yamashita’s result about convex analytic functions (see [11, Theorem 1]), we get the following result.

Corollary 2.2. Let \( f = h + g \) be a sense-preserving harmonic mapping in \( \mathbb{D} \). If \( h + \varepsilon g \) is univalent (resp. convex) in \( \mathbb{D} \) for some \( \varepsilon_1 \in \mathbb{D} \), then \( ||P_{h+\varepsilon g}|| < 9 \) (resp. 7) and \( ||P_{h+\varepsilon g}|| \leq 8 \) (resp. 6) for each \( \varepsilon \in \mathbb{D} \). Furthermore, the constants 8 and 6 are sharp. Conversely, if either \( ||P_{h+\varepsilon g}|| \geq 9 \) (resp. 7) or \( ||P_{h+\varepsilon g}|| > 8 \) (resp. 6) for some \( \varepsilon_1 \), \( \varepsilon_2 \in \mathbb{D} \), then \( h + \varepsilon g \) is not univalent (resp. convex) in \( \mathbb{D} \) for any \( \varepsilon \in \mathbb{D} \).

The harmonic Koebe function \( K = h_K + g_K \) and the harmonic half-plane mapping \( L = h_L + g_L \) still show its sharpness in the corresponding cases because \( h_K(z) - g_K(z) = \frac{z}{(1-z)^2} \) is univalent in \( \mathbb{D} \) and \( h_L(z) + g_L(z) = \frac{z}{(1-z)^2} \) is univalent and convex in \( \mathbb{D} \), respectively.

3. PRE-SCHWARZIAN NORM AND UNIFORMLY HYPERBOLIC RADIUS

It is natural to ask whether there exists a generalization of Bieberbach’s criterion for univalent harmonic mappings. Let

\[
S_H = \left\{ f \in \mathcal{H} : f(z) = h(z) + g(z) = \sum_{n=1}^\infty a_n z^n + \sum_{n=1}^\infty b_n z^n \text{ is univalent in } \mathbb{D} \right\}
\]

and \( S_H^0 = S_H \cap \mathcal{H}_0 \). Set

\[
\alpha = \sup_{f \in S_H} |a_2| \quad \text{and} \quad \alpha_0 = \sup_{f \in S_H^0} |a_2|.
\]

Clunie and Sheil-Small [13] showed that if \( f = h + g \in S_H \), then \( ||P_h|| \leq 2(\alpha + 1) \), \( \alpha_0 < 12172 \) and \( \alpha_0 \leq \alpha \leq \alpha_0 + 1/2 \). They conjectured that \( \alpha_0 \leq 5/2 \), which has a special significance in many extremal problems for harmonic mappings. The estimate of \( \alpha_0 \) was improved (see [13] p. 96 and [30, Theorem 10]). Now the best known upper bound for \( \alpha_0 \) is in [2].
However, for certain geometric subfamilies of $S_H$, we have some precise coefficient estimates. For example, for the families $K_H$ and $C_H$ of convex and close-to-convex harmonic mappings in $\mathbb{D}$, respectively. We note that $K_H \subseteq C_H \subseteq S_H$. Set $K^0_H = K_H \cap \mathcal{H}_0$ and $C^0_H = C_H \cap \mathcal{H}_0$. For these special families, we know (see [13] and [38]):

$$\sup_{f \in K^0_H} |a_2| = \frac{3}{2}, \quad \sup_{f \in K_H} |a_2| = 2, \quad \sup_{f \in C^0_H} |a_2| = \frac{5}{2} \quad \text{and} \quad \sup_{f \in C_H} |a_2| = 3.$$ 

Therefore, the sharp estimate $||P_f|| \leq 5$ is obtained for all $f \in K_H$ (see [21, Theorem 4]). On the other hand, based on further research on affine and linear invariant locally univalent harmonic mappings, Graf in [19, Theorem 1] obtained that $||P_f|| \leq 7$ for $f \in C_H$ and $||P_f|| \leq 2(\alpha_0 + 1)$ for $f \in S_H$.

In this section, we will first re-certify the above partial results concerning the PSN as a direct consequence of our present study on ULU harmonic mappings. For the convenience of the reader, we include the proof here since it follows by a direct computation. Note that the PSN is in general not linear invariant.

**Theorem 3.1.** Let $f = h + \overline{g}$ be a sense-preserving and ULU harmonic mapping in $\mathbb{D}$. Then we have

$$(3.1) \quad (1 - |z|^2)|P_h(z)| \leq 2(\alpha/t + |z|) \quad \text{and} \quad (1 - |z|^2)|P_f(z)| \leq 2(\alpha_0/t + |z|)$$

for every $z \in \mathbb{D}$, where

$$t = \begin{cases} 
\frac{e^{\rho(f)} - 1}{e^{\rho(f)} + 1} & \text{if } \rho(f) < \infty, \\
1 & \text{if } \rho(f) = \infty.
\end{cases}$$

In particular, if $f$ is univalent in $\mathbb{D}$, then

$$||P_h|| \leq 2(\alpha + 1) \quad \text{and} \quad ||P_f|| \leq 2(\alpha_0 + 1).$$

**Proof.** Suppose that $f = h + \overline{g} \in ULU$. Then $f$ is univalent in each hyperbolic disk $d_k(z, \rho(f))$ for every $z \in \mathbb{D}$. Fix $z \in \mathbb{D}$ and let $\phi(\zeta) = \frac{\zeta + \sqrt{1 + \zeta^2}}{1 + \zeta^2}$ ($\zeta \in \mathbb{D}$), where $t$ is defined as above. Using the Koebe transformation, we get that

$$F_1(\zeta) = \frac{(f \circ \phi)(\zeta) - (f \circ \phi)(0)}{(f \circ \phi)(0)} = \frac{h(\phi(\zeta)) - h(z)}{th'(z)(1 - |z|^2)} + \frac{g(\phi(\zeta)) - g(z)}{th'(z)(1 - |z|^2)}$$

$$= H_1(\zeta) + G_1(\zeta)$$

and $F_1 \in S_H$. A simple computation yields that

$$|H''_1(0)| = t \left| (1 - |z|^2) \frac{h''(z)}{h'(z)} - 2\pi \right| \leq 2\alpha,$$
which implies the first inequality in (3.1). Using the affine change, we have that
\[
F_2(\zeta) = \frac{F_1(\zeta) - b_1 F_1(\bar{\zeta})}{1 - |b_1|^2} = \frac{H_1(\zeta) - b_1 G_1(\zeta)}{1 - |b_1|^2} + \frac{G_1(\zeta) - b_1 H_1(\zeta)}{1 - |b_1|^2} = H_2(\zeta) + G_2(\zeta)
\]
and \(F_2 \in S^V_n\), where \(b_1 = G'_1(0) = g'(z)/h'(z)\). Again, a straightforward computation shows that
\[
|H_2''(0)| = \left| \frac{H''_1(0) - b_1 G''_1(0)}{1 - |b_1|^2} \right| = t \left| (1 - |z|^2) \frac{h''(z)h'(z) - g''(z)g'(z)}{|h'(z)|^2 - |g'(z)|^2} - 2\tau \right| = t |(1 - |z|^2) P_f(z) - 2\tau| \leq 2\alpha_0,
\]
which implies the second inequality in (3.1). □

Next we consider stable harmonic univalent (resp. convex) mappings. A sense-preserving harmonic mapping \(f = h + \bar{g}\) in \(D\) is called SHU (resp. SHC) if \(h + \lambda \bar{g}\) is univalent (resp. convex) in \(D\) for every \(|\lambda| = 1\). The following result has some similarities with the classical estimate of the SD for SHU and SHC mappings in [12, Theorem 2], but the method of proof is different and so can also be adapted to prove [12, Theorem 2].

**Theorem 3.2.** Let \(f = h + \bar{g}\) be a sense-preserving harmonic mapping in \(D\). If \(f\) is SHU (resp. SHC), then we have
\[
||P_{h+\varepsilon g}|| \leq 6 \text{ (resp. 4) } \text{ and } ||P_{h+\varepsilon \bar{g}}|| \leq 6 \text{ (resp. 4)}
\]
for each \(\varepsilon \in \overline{D}\). All estimates are sharp.

**Proof.** If \(f = h + \bar{g}\) is SHU (resp. SHC) in \(D\), then both \(h + \varepsilon g\) and \(h + \varepsilon \bar{g}\) are univalent (resp. convex) in \(D\) for each \(\varepsilon \in \overline{D}\) (see [20]). It follows from Bieberbach’s criterion (resp. [11, Theorem 1]) that \(||P_{h+\varepsilon g}|| \leq 6\) (resp. 4) for each \(\varepsilon \in \overline{D}\).

Fix \(\varepsilon \in \overline{D}\) and let \(f_\varepsilon = h + \varepsilon \bar{g}\). For all \(z_0 \in D\), it follows from [21, Lemma 1] that \(P_{f_\varepsilon}(z_0) = P_{h-\varepsilon \omega(z_0)g}(z_0)\) and thus,
\[
(1 - |z_0|^2)|P_{f_\varepsilon}(z_0)| = (1 - |z_0|^2)|P_{h-\varepsilon \omega(z_0)g}(z_0)|,
\]
where \(\omega = g'/h'\). This implies that \(||P_{f_\varepsilon}|| \leq \sup_{\lambda \in \overline{D}} ||P_{h+\lambda \bar{g}}||\) and the assertion follows.

To show that all estimates are sharp, it is enough to consider the analytic functions
\[
k(z) = \frac{z}{(1 - z)^2} \text{ and } l(z) = \frac{1 + z}{1 - z}
\]
that belong to the families of SHU and SHC mappings with \(||P_k|| = 6\) and \(||P_l|| = 4\), respectively. □

Combining Corollary 2.1 and Theorem 3.1 (resp. Theorem 3.2), we can obtain a few similar results as that of Corollary 2.2 for univalent harmonic mappings (resp. SHU and
Theorem 3.3. Let \( f = h + \overline{g} \) be a sense-preserving harmonic mapping in \( \mathbb{D} \). If \( ||Pf|| \leq M \), then \( f \) is univalent in the hyperbolic disk \( D_h(z, t) \) for each \( z \in \mathbb{D} \). Consequently, \( f \) is ULU in \( \mathbb{D} \) and its uniformly hyperbolic radius \( \rho(f) \) is no less than \( t \). Here \( t = 2 \tanh^{-1}(1/(8(M + 1))) \).

Proof. Fix \( \varepsilon \in \mathbb{D} \). Let \( f_\varepsilon = f + \varepsilon \overline{g} = \phi_\varepsilon + \overline{\psi}_\varepsilon \), where \( \phi_\varepsilon = h + \varepsilon g \) and \( \psi_\varepsilon = g + \overline{\varepsilon}h \). By the hypothesis, (1.2) and (2.1), we get that

\[ ||P\phi_\varepsilon|| \leq ||Pf_\varepsilon|| + 1 = ||Pf|| + 1 \leq M + 1. \]

It follows from [37, Theorem 2] that \( \phi_\varepsilon \) is univalent in \( D_h(z, t) \) for each \( z \in \mathbb{D} \), where \( t \) as the above. By Hurwitz’s theorem, we know that for each \( z \in \mathbb{D} \), \( h + \lambda g \) is univalent in \( D_h(z, t) \) for every \( |\lambda| = 1 \). Therefore, it follows from [20, Corollary 2.2] that \( f \) is univalent in \( D_h(z, t) \) for each \( z \in \mathbb{D} \). This ends the proof. \( \square \)

4. Stable geometric properties of ULU analytic and harmonic mappings

In this section, we will show a great number of equivalent conditions for sense-preserving and ULU harmonic mappings in \( \mathbb{D} \). First we will introduce some notations. Let \( f = h + \overline{g} \) be a sense-preserving harmonic mapping in \( \mathbb{D} \). Set

\[ \rho^*(f) = \inf_{z \in \mathbb{D}} \left\{ \sup_{\rho_z > 0} \{ \rho_z : f \text{ is convex in } D_h(z, \rho_z) \} \right\}. \]

If \( \rho^*(f) > 0 \), then we say that \( f \in \text{ULC} \). The SD and the SN of \( f \) were investigated in details by Hernández and Martín [21] (see also [11]) and they were defined by

\[ S_f = S_h + \frac{\overline{\omega}}{1 - |\omega|^2} \left( \frac{h''}{h'} \omega' - \omega'' \right) - \frac{3}{2} \left( \frac{\overline{\omega}\omega'}{1 - |\omega|^2} \right)^2 \]

and

\[ ||S_f|| = \sup_{z \in \mathbb{D}} (1 - |z|^2)^2 |S_f(z)|, \]

respectively, where \( S_h \) is the classical Schwarzian derivative of a locally univalent function \( h \) defined by

\[ S_h = \frac{h''}{h'} - \frac{3}{2} \left( \frac{h''}{h'} \right)^2. \]

If \( g \) is a constant, then it is clear that \( S_f = S_h \) and \( ||S_f|| = ||S_h|| \). Analogous to some features of the PSN, if \( f \) is a univalent analytic function in \( \mathbb{D} \), we have the sharp inequality \( ||S_f|| \leq 6 \). Conversely, if \( ||S_f|| \leq 2 \) for a locally univalent analytic function \( f \) in \( \mathbb{D} \), then, according to Krauss-Nehari’s criterion, \( f \) is univalent in \( \mathbb{D} \) and the constant 2 is sharp (see [26, 30]). There are some criteria for the univalence of harmonic mappings in terms of the SN (see [19, 21, 22]), but these results are not sharp.

Next, we present equivalent conditions for sense-preserving and ULU harmonic mappings in \( \mathbb{D} \) based on the following result.

Lemma 4.1. ([17, p. 44] and [40, Theorem 2]) Let \( f \) be a locally univalent analytic function in \( \mathbb{D} \). Then the following are equivalent.
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(1) \( f \in \text{ULU}; \)
(2) \( f \in \text{ULC}; \)
(3) \( ||P_f|| < \infty; \)
(4) \( ||S_f|| < \infty; \)
(5) There exists a constant \( m > 0 \), and a univalent analytic function \( F \) in \( \mathbb{D} \) such that \( f' = (F')^m \).

To describe our results, we introduce the following abbreviations analogous to the paper [20]. Below, let \( f = h + g \) be a sense-preserving harmonic mapping in \( \mathbb{D} \) and \( \varepsilon_1, \varepsilon_2 \in \mathbb{D} \) with \( \varepsilon_1 \neq \varepsilon_2 \). If \( h + \varepsilon_1 g \) is ULU (resp. ULC) in \( \mathbb{D} \) if and only if \( h + \varepsilon_2 g \) is ULU (resp. ULC) in \( \mathbb{D} \), then we say that \( f \) is SHULU (resp. SHULC). Similarly, if \( h + \varepsilon_1 g \) is ULU (resp. ULC) in \( \mathbb{D} \) if and only if \( h + \varepsilon_2 g \) is ULU (resp. ULC) in \( \mathbb{D} \), then we say that \( h + g \) is SAULU (resp. SAULC). If \( ||P_{h+\varepsilon_1 g}|| \) (resp. \( ||S_{h+\varepsilon_1 g}|| \) ) is bounded if and only if \( ||P_{h+\varepsilon_2 g}|| \) (resp. \( ||S_{h+\varepsilon_2 g}|| \) ) is bounded, then we say that \( f \) has SBHPSN (resp. SBHSN). Similarly, if \( ||P_{h+\varepsilon_1 g}|| \) (resp. \( ||S_{h+\varepsilon_1 g}|| \) ) is bounded if and only if \( ||P_{h+\varepsilon_2 g}|| \) (resp. \( ||S_{h+\varepsilon_2 g}|| \) ) is bounded, then we say that \( h + g \) has SBAPSN (resp. SBASN).

Theorem 4.1. (Equivalent conditions) Let \( f = h + g \) be a sense-preserving harmonic mapping in \( \mathbb{D} \). Then the following conditions are equivalent.

(1) \( h + g \) is SAULU;
(2) \( h + g \) is SAULC;
(3) \( h + g \) has SBAPSN;
(4) \( h + g \) has SBASN;
(5) For any two points \( \varepsilon_1, \varepsilon_2 \in \overline{\mathbb{D}} \) with \( \varepsilon_1 \neq \varepsilon_2 \), there exists a constant \( m_1 > 0 \), and a univalent analytic function \( F_1 \) such that \( (h + \varepsilon_1 g)' = (F_1')^{m_1} \) if and only if there exists a constant \( m_2 > 0 \), and a univalent analytic function \( F_2 \) such that \( (h + \varepsilon_2 g)' = (F_2')^{m_2} \);

(6) \( f \) is SHULU;
(7) \( f \) is SHULC;
(8) \( f \) has SBHPSN;
(9) \( f \) has SBHSN.

Proof. To simplify the proof, we use the equivalent diagram below. If we apply Lemma 4.1 to \( h + \varepsilon_1 g \) and \( h + \varepsilon_2 g \), we see that (Ai) and (Bi) \((i = 1, 2, 3, 4)\) hold. On the other hand, (A5), (AB) and (B5) are the direct consequences of Theorem 2.1 and Corollary 2.1. Clearly, the following implications are easy to obtain

\[
(1) \iff (2) \iff (3) \iff (4) \iff (5) \iff (8).
\]
Theorems 3.1 and 3.3 to $\infty$ and thus, sup

To complete the proof, we need to show that (6) $\iff$ (7) $\iff$ (8) $\iff$ (9). If we apply Theorems 3.1 and 3.3 to $h + \varepsilon \overline{g}$ and $h + \varepsilon g$, then we obtain the inclusions (A6) and (B6).

From [21, Theorem 7], (A7) and (B7) follow.

To prove (A8) and (B8), it suffices to show that each $f = h + \overline{g}$ in ULU also belongs to ULC. To do this, let us assume that $f = h + \overline{g}$ is ULU in $\mathbb{D}$. Then $M = ||P_f|| < \infty$ and thus, sup $\varepsilon \in \mathbb{C} ||P_{f + \varepsilon g}|| \leq M + 1$ by (2.1). Following the proof and notations of Theorem 3.3, we see that for each $z \in \mathbb{D}$, $h + \lambda g$ is convex in $D_h(z,(2 - \sqrt{3})t)$ for every $|\lambda| = 1$ by the classical result on the radius of convexity (see [17, p. 44]), where $t = 2 \tanh^{-1}(1/(8(M + 2)))$. It follows from [20, Theorem 3.1] that $f$ is convex in the hyperbolic disk $D_h(z,(2 - \sqrt{3})t)$ for each $z \in \mathbb{D}$, which means that $f$ is ULC in $\mathbb{D}$.

Again, by the bridge (AB), we prove that (6) $\iff$ (7) $\iff$ (8) $\iff$ (9). This completes the proof.

Remarks. In the remarks below, let $f = h + \overline{g}$ be sense-preserving in $\mathbb{D}$.

1. The pre-Schwarzian norm $||P_g||$ and the Schwarzian norm $||S_g||$ can be unbounded even if $f$ and $g$ are univalent and locally univalent in $\mathbb{D}$, respectively. For example, let

$$f_n(z) = h_n(z) + \overline{g_n(z)} = z - 1 + \frac{1}{\lambda(z-1)^n} \quad (n \geq 2 \text{ and } 0 < |\lambda| < 1/(n^2-1)).$$

It is easy to see that $f_n$ is sense-preserving and univalent in $\mathbb{D}$ and $g_n$ is locally univalent in $\mathbb{D}$ for any $n \geq 2$. However, we have that

$$||P_{g_n}|| = \sup_{z \in \mathbb{D}} (1 - |z|^2)^{n-1} \to \infty$$

and

$$||S_{g_n}|| = \sup_{z \in \mathbb{D}} (1 - |z|^2)^{n-2} \to \infty$$

as $n \to \infty$.

2. On one hand, the dilatation of $f$ can be expressed as square of certain analytic function if both $h$ and $g$ are ULU in $\mathbb{D}$. It follows from Lemma 4.1 that $||P_h||$ and $||P_g||$ are bounded. Let $k = \max\{||P_h||, ||P_g||\} + 1$ and set

$$h_1(z) = \int_0^z (h'(\zeta))^k d\zeta \quad \text{and} \quad g_1(z) = \int_0^z (g'(\zeta))^k d\zeta$$
in the proof of [10] Theorem 2. Note that \( h_1 \) and \( g_1 \) are analytic and univalent in \( \mathbb{D} \) such that \( h' = (h_1')^{2k} \) and \( g' = (g_1')^{2k} \). Thus, we have that \( \omega_f = g'/h' = (g_1/h_1')^{2k} \). Furthermore, if \( f \) is univalent in \( \mathbb{D} \), then \( f \) can be lifted to a regular minimal surface given by conformal (or isothermal) parameters in \( \mathbb{D} \).

(3) On the other hand, the function \( f \), with the dilatation \( \omega_f = q^2 \), for some analytic function \( q \) may not belong to ULU. For instance, let

\[
f(z) = h(z) + g(z) = e^{-2z^2} + \frac{z - 5}{z - 1}, \quad z \in \mathbb{D}.
\]

A simple computation infers that

\[
\omega_f(z) = \frac{g'(z)}{h'(z)} = \left( e^{\frac{z+1}{z-1}} \right)^2
\]

so that \(|\omega_f(z)| < 1 \) in \( \mathbb{D} \) and thus, \( f \) is sense-preserving in \( \mathbb{D} \). However, \(|\omega_f(z)| \geq 1 \) in \( \mathbb{D} \), which implies that \( f \) is not ULU in \( \mathbb{D} \) by Theorem 4.1.

(4) If the analytic part \( h \) is univalent in \( \mathbb{D} \), then \( f \) is certainly ULU in \( \mathbb{D} \) by Corollary 2.2 and Theorem 4.1. However, the above example shows that even if the co-analytic part \( g \) is univalent in \( \mathbb{D} \), \( f \) may not belong to ULU.

5. Some Precise Examples

In this section, we consider a family of harmonic mappings and compute their PSNs and then discuss the univalency of the corresponding mapping. We next introduce

\[
F_{a,b,\theta}(z) = H_{a,b}(z) + G_{a,b,\theta}(z), \quad e^{-i\theta}G_{a,b,\theta}(z) := G_{a,b}(z) = H_{a+1,b}(z) - H_{a,b}(z),
\]

where \( a, b, \theta \in \mathbb{R} \) and

\[
H_{a,b}(z) = \int_0^z \frac{(1 + t)^a}{(1 - t)^b} dt.
\]

If \( a = b \), we denote \( H_{a,a} \) by \( H_a \). Clearly, \( H_{a,b} \in \mathcal{A} \) and \( H_{a,b}(z) = -H_{-b,-a}(-z) \). Therefore, it is easy to see that \( F_{a,b,\theta} \in \mathcal{H}_0 \) with dilatation \( \omega(z) = e^{i\theta}z \) and

\[
F_{a,b,\theta}(z) = -F_{-b,-a,\theta+\pi}(-z), \quad z \in \mathbb{D}.
\]

In general, computing the PSN and verifying the univalency of a given harmonic mapping are not so easy. Below, we also try to give partial answers to this issue. Moreover, as a byproduct of our investigation, we present some sharp inequalities in Section 4 and give certain properties of the family \( \mathcal{B}_H(\lambda) \) \( (\lambda \geq 1) \). In the following results, we use the following well-known facts: If \( h \) is a normalized (i.e. \( h(0) = h'(0) = 1 = 0 \) analytic function in \( \mathbb{D} \) satisfying the condition

\[
\text{Re} \left( 1 + \frac{zh''(z)}{h'(z)} \right) > -\frac{1}{2}
\]

for \(|z| < 1\), then \( h \) is convex in some direction and hence it is close-to-convex (univalent) in the unit disk. For details and its importance see [34].
Proposition 5.1. For the functions $H_{a,b}$ and $H_a$ defined by (5.2), we have the following properties:

1. $\|P_{H_{a,b}}\| = 2\max\{|a|, |b|\}$. Thus, if $\max\{|a|, |b|\} \leq 1/2$, then the functions $H_{a,b}$ are univalent in $\mathbb{D}$. If $\max\{|a|, |b|\} > 3$, then the functions $H_{a,b}$ are not univalent in $\mathbb{D}$.

2. If $\min\{|a|, |b|\} + |a - b| \leq 1$, then the functions $H_{a,b}$ are close-to-convex and univalent in $\mathbb{D}$.

3. If $a \leq 0 \leq b \leq a + 3$, then the functions $H_{a,b}$ are convex in one direction and univalent in $\mathbb{D}$. Furthermore, if $a \leq 0 \leq b \leq a + 2$, then the functions $H_{a,b}$ are convex in $\mathbb{D}$.

4. The function $H_a$ is univalent in $\mathbb{D}$ if and only if $|a| \leq 1$.

Proof. (1) By computation, for all $z \in \mathbb{D}$, we have that

$$(1 - |z|^2)|P_{H_{a,b}}(z)| = (1 - |z|^2) \left| \frac{a + b + (b - a)z}{1 - z^2} \right| \leq |a + b| + |b - a| = 2\max\{|a|, |b|\}.$$ 

Note that $\lim_{r \to 1^-} (1 - r^2)|P_{H_{a,b}}(r)| = 2|b|$ and $\lim_{r \to 1^+} (1 - r^2)|P_{H_{a,b}}(r)| = 2|a|$. Therefore, we get that $\|P_{H_{a,b}}\| = 2\max\{|a|, |b|\}$ and the result follows by Becker’s univalence criterion.

Note that if $\max\{|a|, |b|\} > 3$, then $\|P_{H_{a,b}}\| > 6$ and thus, the functions $H_{a,b}$ can not be univalent in $\mathbb{D}$.

(2) We observe that

$$H'_{a,b}(z) = \left(1 + \frac{z}{1 - z}\right)^a (1 - z)^{a-b} = \left(\frac{1 + z}{1 - z}\right)^b (1 + z)^{a-b},$$

and thus,

$$\arg(H'_{a,b}(z)) < \frac{\pi}{2} \min\{|a| + |a - b|, |b| + |a - b|\}, \quad z \in \mathbb{D}.$$ 

If $\min\{|a|, |b|\} + |a - b| \leq 1$, then we have $|\arg(H'_{a,b}(z))| < \frac{\pi}{2}$ in $\mathbb{D}$ and thus, by Noshiro-Warschawski’s theorem (see [17]), the functions $H_{a,b}$ are close-to-convex and univalent in $\mathbb{D}$.

(3) For $a \leq 0 \leq b \leq a + 3$, we see that

$$\Re \left(1 + \frac{zH''_{a,b}(z)}{H'_{a,b}(z)}\right) = 1 + \Re \frac{az}{1 + z} + \Re \frac{bz}{1 - z} > 1 + \frac{a}{2} - \frac{b}{2} = \frac{2 + a - b}{2} \geq -\frac{1}{2}$$

for all $z \in \mathbb{D}$ and thus, the functions $H_{a,b}$ are convex in one direction and univalent in $\mathbb{D}$. Also, it is clearly that if $a \leq 0 \leq b \leq a + 2$, then

$$\Re \left(1 + \frac{zH''_{a,b}(z)}{H'_{a,b}(z)}\right) > 0, \quad z \in \mathbb{D},$$

and thus, the functions $H_{a,b}$ are convex in $\mathbb{D}$.

(4) It is a direct consequence of [24] Lemma 2.1 because of $H_a(z) = -H_{-a}(-z)$. \hfill $\Box$

Proposition 5.2. For all $\theta \in \mathbb{R}$, the family of harmonic mappings $F_{a,b,\theta}$ defined by (5.1) has the following properties:
Note that the dilatation of $\theta$ yields that 

$|P_{F_{a,a,\theta}}| = 2|a| + 1 = |P_{H_a}| + 1$ for all $a \in \mathbb{R}$.

If $a \in (-\infty, -1) \cup [0, \infty)$, then $|P_{F_{a,a+1,\theta}}| = |2a + 1| = |P_{H_{a,a+1}}| - 1$. However, $|P_{F_{a,a+1,\theta}}| = 1$ for each $a \in (-1, 0)$.

If $a \leq 0 \leq b \leq a + 3$, then the functions $F_{a,b,\theta}$ are close-to-convex and univalent in $\mathbb{D}$.

The functions $F_{a,a+1,\theta}$ are univalent in $\mathbb{D}$ if and only if $a \in [-1, 0]$.

**Proof.** By a straightforward computation, we have that

$$P_{F_{a,b,\theta}}(z) = \frac{a + b + (b - a)z}{1 - z^2} - \frac{\overline{z}}{1 - |z|^2}, \quad z \in \mathbb{D}.$$  

(1) It follows from (1.2) and (5.3) that $|P_{F_{a,a,\theta}}| = |P_{F_{a,a,\theta_a}}|$. So we only need to consider the case $a \geq 0$. The conclusion can be easily got by (2.1), Proposition 5.1 and the fact that

$$|P_{F_{a,a,\theta}}| \geq \lim_{r \to 1^-} (1 - r^2)|P_{F_{a,a,\theta}}(r)| = 2a + 1 = |P_{H_a}| + 1, \quad a \geq 0.$$  

(2) We first consider the case $a \geq 0$. Note that

$$h_a(z) := H_{a,a+1}(z) + e^{i(\pi - \theta)}G_{a,a+1,\theta}(z) = H_a(z).$$  

It follows from (2.1) and Proposition 5.1 that

$$2a + 1 = |P_{H_{a,a+1}}| - 1 \leq |P_{F_{a,a,\theta}}| \leq |P_{H_a}| + 1 = 2a + 1, \quad a \geq 0.$$  

Obviously, $|P_{F_{a,a+1,\theta}}| = 2a + 1 = |P_{H_{a,a+1}}| - 1$ for each $a \geq 0$ and all $\theta \in \mathbb{R}$. For the case $a \leq -1$, the conclusion follows, since $|P_{F_{a,a+1,\theta}}| = |P_{F_{-(a+1),-(a+1),\theta}}|$. Next we will certify that $|P_{F_{a,a+1,\theta}}| = 1$ for each $a \in (-1, 0)$. A basic computation states that

$$|1 - z^2|^2 - (2a + 1)^2(1 - |z|^2)^2 - (z - \overline{z})^2) = -4a(a + 1)(1 - |z|^2)^2 > 0, \quad z \in \mathbb{D},$$  

which means that

$$(1 - |z|^2)|P_{F_{a,a+1,\theta}}(z)| = \frac{(2a + 1)(1 - |z|^2) + z - \overline{z}}{1 - z^2} \leq 1, \quad z \in \mathbb{D}.$$  

It yields that $|P_{F_{a,a+1,\theta}}| \leq 1$. Note that $\lim_{r \to 1^-} (1 - r^2)|P_{F_{a,a+1,\theta}}(ir)| = 1$ and thus, we obtain $|P_{F_{a,a+1,\theta}}| = 1$.

(3) If $a \leq 0 \leq b \leq a + 3$, then from the proof of Proposition 5.1 we find that

$$\text{Re} \left(1 + \frac{zH'_{a,b}(z)}{H_{a,b}(z)}\right) > -\frac{1}{2}, \quad z \in \mathbb{D}.$$  

Note that the dilatation of $F_{a,b,\theta}$ is $e^{i\theta}z$ for all $a, b \in \mathbb{R}$ and each $\theta \in \mathbb{R}$. As a consequence, it follows from [9] Theorem 1 that the functions $F_{a,b,\theta}$ are close-to-convex and univalent for all $\theta \in \mathbb{R}$ if $a \leq 0 \leq b \leq a + 3$.

(4) Obviously, it follows from (3) that the functions $F_{a,a+1,\theta}$ are univalent in $\mathbb{D}$ for all $\theta \in \mathbb{R}$ if $a \in [-1, 0]$. For the remaining part, we use the method of contradiction. Assume that there exists some $a > 0$ such that the functions $F_{a,a+1,\theta}$ are univalent in $\mathbb{D}$ for all $\theta \in \mathbb{R}$. Therefore, the function $F_{a,a+1,\theta}$ is stable univalent for each $\theta \in \mathbb{R}$ and thus $H_{a,a+1} + \lambda G_{a,a+1,\theta}$ is univalent in $\mathbb{D}$ for each $\lambda \in \mathbb{D}$ (see [20]), especially for $\lambda = e^{-i\theta}$. 


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However, \( H_{a,a+1} + e^{-i\theta}G_{a,a+1,\theta} = H_{a+1,a+1} \) is not univalent in \( \mathbb{D} \) by Proposition 5.1 when \( a > 0 \). This is a contradiction. Using (5.3), the similar contradiction can be obtained for the case \( a < -1 \). This completes the proof. \( \square \)

From the proof of Proposition 5.2, the two families of harmonic mappings \( F_{a,a,\theta} \) and \( F_{a,a+1,\theta} \) provide sharp results for several of the inequalities in Section 2. For simplicity, let

\[
h_{a,b,\theta,\varphi}(z) = H_{a,b}(z) + e^{i\varphi}G_{a,b,\theta}(z) = H_{a,b}(z) + e^{i(\theta+\varphi)}(H_{a+1,b}(z) - H_{a,b}(z)).
\]

Clearly, \( h_{a,b,\theta,-\theta} = H_{a+1,b} \) and \( h_{a,b,\theta,\pi-\theta} = 2H_{a,b} - H_{a+1,b} = H_{a,b-1} \). We have the following results from Propositions 5.1 and 5.2:

- \(|P_{h_{a,a,\theta,\pi-\theta}}| + 1 = |P_{H_{a}}| + 1 = |P_{F_{a,a,\theta}}| = 2a + 1 = |P_{h_{a,a,\theta,-\theta}}| - 1, a \geq 1/2.\)
- \(|P_{H_{a}}| + 1 = |P_{F_{a,a,\theta}}| = 2a + 1 = |P_{h_{a,a,\theta,-\theta}}| - 1, 0 \leq a \leq 1/2.\)
- \(|P_{F_{a,a,\theta}}| = 2a + 1 = |P_{h_{a,a,\theta,-\theta}}| + \varepsilon, a = (1 + \varepsilon)/4 \in [1/4, 1/2].\)
- \(|P_{F_{a,a,\theta}}| = 2a + 1 = |P_{h_{a,a,\theta,-\theta}}| - \varepsilon, a = (1 - \varepsilon)/4 \in [0, 1/4].\)
- \(|P_{h_{a,a+,1,\theta,-\theta}}| + 1 = |P_{F_{a,a+1,\theta}}| = 2a + 1 = |P_{H_{a,a+1}}| - 1 = |P_{h_{a,a+,1,\theta,-\theta}}| - 1, a \geq 0.\)

Similar results may be stated for \( F_{a,a,\theta} \) and \( F_{a,a+1,\theta} \) when \( a < 0 \). For these functions, we know that \(|P_{F_{a,b,\theta}}| \geq 1 \) and \( \omega_{F_{a,b,\theta}} = e^{i\theta}z \) for all \( a, b = a \) or \( a + 1, \theta \in \mathbb{R} \). These things do not happen accidentally. Our next result, which is a parallel result to [12, Theorem 3], demonstrates the reason behind these.

We denote by \( \mathcal{A}(\lambda) \) (resp. \( \mathcal{A}_0(\lambda) \)) the set of all admissible dilatations of \( f \in \mathcal{B}_H(\lambda) \) (resp. \( \mathcal{B}_0^0(\lambda) \)); i.e., \( \omega \in \mathcal{A}(\lambda) \) (or \( \mathcal{A}_0(\lambda) \)) if there exists a harmonic mapping \( f = h + \overline{g} \in \mathcal{B}_H(\lambda) \) (or \( \mathcal{B}_H(\lambda) \)) with dilatation \( \omega \).

**Theorem 5.1.** The following conditions are equivalent.

1. \( \lambda \geq 1; \)
2. There exists \( \omega \in \mathcal{A}_0(\lambda) \) with \( |\omega'(0)| = 1; \)
3. The set \( \{ \mu \cdot I : |\mu| = 1 \} \) is contained in \( \mathcal{A}_0(\lambda); \)
4. Every automorphism \( \sigma \) of the unit disk is an admissible dilatation in \( \mathcal{B}_H(\lambda) \).

**Proof.** The scheme of the proof is to show that (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3) \( \Rightarrow \) (1) and (3) \( \Leftrightarrow \) (4). We only show (1) \( \Rightarrow \) (2) and (3) \( \Rightarrow \) (1). The remaining implications are similar to corresponding proofs of [12, Theorem 3] since the PSD preserves affine invariance.

We now show that (1) \( \Rightarrow \) (2): For any given \( \lambda \geq 1 \), we choose \( |a| = \frac{\lambda-1}{2} \) so that \( P_{F_{a,a,\theta}} \in \mathcal{B}_H^0(\lambda) \) with the dilatation \( e^{i\theta}z \) by Proposition 5.2. Then (2) follows.

Next, we prove that (3) \( \Rightarrow \) (1): If (3) is satisfied, then for any given \( \mu \) with \( |\mu| = 1 \), there is a harmonic function \( f_\mu = h_\mu + \overline{g_\mu} \in \mathcal{B}_H^0(\lambda) \) with dilatation \( \mu z \). Since \( g_\mu'(0) = 0 \), by (1.3), \( h_\mu + \varepsilon g_\mu' \in \mathcal{B}(\lambda + 1/2) \) for each \( \varepsilon \in \mathbb{D} \). It follows from [24, Theorem 2.3] that for any \( \varepsilon \in \mathbb{D}, \)

\[
|h_\mu'(z) + \varepsilon g_\mu'(z)| = |h_\mu'(z)| \cdot |1 + \varepsilon \mu z| \geq \left( \frac{1 - |z|}{1 + |z|} \right)^{\lambda+1}, \quad z \in \mathbb{D}.
\]
Since \( \varepsilon \in \mathbb{D} \) and \(|\mu| = 1\), for any given \( z \neq 0 \) in the unit disk, we can get

\[
|h'_\mu(z)| \geq \frac{(1 - |z|)^{\lambda+1}}{(1 + |z|)^{\lambda+1}},
\]

by choosing \( \varepsilon \cdot \mu = -\overline{z}/|z| \) in (5.4). Clearly, the above inequality holds for \( z = 0 \). Note that \( h_\mu \) is locally univalent in \( \mathbb{D} \) and \( h'_\mu(0) = 1 \) for each \( \mu \in \partial \mathbb{D} \). We obtain that the analytic function \( \frac{1}{h'_\mu} \) satisfies

\[
\frac{1}{|h'_\mu(z)|} \leq \frac{(1 + |z|)^{\lambda+1}}{(1 - |z|)^{\lambda+1}}, \quad z \in \mathbb{D},
\]

which implies that \( \lambda \geq 1 \) by the maximum modulus principle. Otherwise we would get \( 1/|h'_\mu(0)| < 1 \), which contradicts \( h'_\mu(0) = 1 \). This completes the proof.

Compared to [12, Theorem 3], since the PSD is in general not linear invariant, we are not sure whether the conditions in Theorem 5.1 are equivalent to that there exists \( \omega \in \mathcal{A}(\lambda) \) (or \( \mathcal{A}_0(\lambda) \)) with \(|||\omega^*||| = 1\). Here \(|||\omega^*|||\) is the hyperbolic norm of the dilatation \( \omega \) of a sense-preserving harmonic mapping in \( \mathbb{D} \), i.e.,

\[
||\omega^*|| = \sup_{z \in \mathbb{D}} \frac{|\omega'(z)|(1 - |z|^2)}{1 - |\omega(z)|^2}.
\]

The hyperbolic norm plays a distinguished role in the analysis of the order of affine and linear invariant families of harmonic mappings with bounded SD (see [12]).

6. Growth estimate for the class \( \mathcal{B}_H(\lambda) \)

To study the growth estimate for the class \( \mathcal{B}_H(\lambda) \), we need the following result which characterizes harmonic mappings in \( \mathcal{B}_H(\lambda) \).

**Proposition 6.1.** A harmonic mapping \( f \in \mathcal{H} \) belongs to \( \mathcal{B}_H(\lambda) \) if and only if for each pair of points \( z, z_0 \) in \( \mathbb{D} \), the inequality

\[
|A(z) - A(z_0)| \leq \lambda d_h(z, z_0)
\]

holds, where \( A(z) = \log J_f(z) \).

**Proof.** Assume that \( f = h + \overline{\eta} \in \mathcal{B}_H(\lambda) \). Then \( |P_f(z)| \leq \lambda/(1 - |z|^2) \) holds in \( \mathbb{D} \). We observe that

\[
A_\eta = (\log J_f)_\eta = (\log J_f)_{\overline{z}} = \overline{P_f} = \overline{A_z}.
\]

Therefore, for two points \( z, z_0 \) in \( \mathbb{D} \), we have that

\[
|A(z) - A(z_0)| \leq \left| \int_{\Gamma} A_\zeta(\zeta)d\zeta + A_\overline{\zeta}(\zeta)d\overline{\zeta} \right|
\]

\[
\leq \int_{\Gamma} (|A_\zeta(\zeta)| + |A_{\overline{\zeta}}(\zeta)|)|d\zeta|
\]

\[
\leq \int_{\Gamma} \frac{2\lambda}{1 - |\zeta|^2}|d\zeta| = \lambda d_h(z, z_0),
\]

where \( \Gamma \) is the hyperbolic geodesic joining \( z \) and \( z_0 \).
Conversely, we assume that the inequality $|A(z) - A(z_0)| \leq \lambda d_{\mathbb{H}}(z_1, z_2)$ holds for each pair of points $z, z_0$ in $\mathbb{D}$. It suffices to prove that $(1 - |z_0|^2)|P_f(z_0)| \leq \lambda$ for each $z_0 \in \mathbb{D}$. Fix $z_0 \in \mathbb{D}$. If $A_z(z_0) = 0$, then $P_f(z_0) = 0$ and $(1 - |z_0|^2)|P_f(z_0)| \leq \lambda$. Otherwise, choose a curve
\[
\gamma = \{z = z_0 + re^{-i\theta} : r \in (0, 1 - |z_0|), \ \theta = \arg(A_z(z_0))\}.
\]
Clearly, $A(z)$ is infinitely differentiable in $\mathbb{D}$ owing to $J_f(z) > 0$. Thus, we have the representation
\[
A(z) - A(z_0) = A_z(z_0)(z - z_0) + A_{zz}(z_0)(\bar{z} - \bar{z_0}) + \sum_{i+j>1} C_{ij}(z - z_0)^i(\bar{z} - \bar{z_0})^j
\]
for some complex constants $C_{ij}$, which implies that
\[
\lim_{\gamma \ni z \to z_0} \frac{A(z) - A(z_0)}{z - z_0} = A_z(z_0) + \overline{A_z(z_0)}e^{2i\theta} = 2A_z(z_0) = 2P_f(z_0).
\]
The desired inequality $(1 - |z_0|^2)|P_f(z_0)| \leq \lambda$ follows from the equality
\[
\lim_{\gamma \ni z \to z_0} \frac{|A(z) - A(z_0)|}{d_{\mathbb{H}}(z, z_0)} = (1 - |z_0|^2)|P_f(z_0)|.
\]
The proof is complete. $\square$

**Theorem 6.1.** (Distortion theorem) Let $f = h + \overline{g} \in \mathcal{B}_H(\lambda)$ for some $\lambda \geq 0$ with $b_1 = g'(0)$, and let $H_{a,b}$ and $H_a$ be defined by (5.2). Then for each $z \in \mathbb{D}$, we have
\[
(1) \ (1 - |b_1|^2)H_a^*(|z|) \leq J_f(z) \leq (1 - |b_1|^2)H_a^*\left((|z|)\right);
(2) \ \sqrt{1 - |b_1|^2}H_a^*\left(|z|\right) \leq |h'(z)| \leq (1 + |b_1|)H_{a,b}^*\left(|z|\right);
(3) \ |g'(z)| \leq \left(|z| + |b_1|\right)H_{a,b}^*\left(|z|\right);
(4) \ -\sqrt{1 - |b_1|^2}H_a^*\left(|z|\right) \leq |h(z)| \leq (1 - |b_1|)H_{a,b}^*\left(|z|\right) + |b_1|H_{a,b}^*\left(|z|\right);
(5) \ |g(z)| \leq H_{a,b}^*\left(|z|\right) - (1 - |b_1|)H_{a,b}^*\left(|z|\right);
(6) \ |f(z)| \leq (1 + |b_1|)H_{a,b}^*\left(|z|\right).
\]

The estimates in (1) are sharp for all $\lambda \geq 0$. The right sides of (2)-(6) are sharp for all $\lambda \geq 1$ and the left side of (6) is sharp for $\lambda = 1$. Moreover, if $f \in \mathcal{B}_H^0(\lambda)$, then the left sides of (2) and (4) are sharp for all $\lambda \geq 0$.

**Proof.** (1) The conclusion can be easily obtained by choosing $z_0 = 0$ in Proposition 6.1.
(2) Since $f \in \mathcal{B}_H(\lambda)$, by Lindelöf’s inequality, we get that
\[
|\omega_f(z)| \leq \frac{|z| + |b_1|}{1 + |b_1||z|}.
\]
and thus,
\[ |h'(z)| = \left( \frac{J_f(z)}{1 - |w_f(z)|^2} \right)^{\frac{1}{2}} \]
\[ \leq (1 - |b_1|^2)^{\frac{1}{2}} \left( 1 - \left( \frac{|z| + |b_1|}{1 + |b_1|} \right)^2 \right)^{-\frac{1}{2}} \left( \frac{1 + |z|}{1 - |z|} \right)^{\frac{1}{2}} \]
\[ = (1 + |b_1 z|)H_{\frac{1}{2}, \frac{1}{2}}(1) \]
and
\[ |h'(z)| \geq (J_f(z))^{\frac{1}{2}} \geq 1 - |b_1|^2 \left( \frac{1 - |z|}{1 + |z|} \right)^{\frac{1}{2}} = \sqrt{1 - |b_1|^2} H_{\frac{1}{2}}(1). \]

(3) It follows from Lindelöf's inequality and the proof of (2) that
\[ |g'(z)| = |w_f(z)h'(z)| \leq (|z| + |b_1|)H_{\frac{1}{2}, \frac{1}{2}}(1). \]

(4) Integrating inequalities in (2) yields (4).

(5) Integrating inequality in (3) yields (5).

(6) Applying the triangle inequality and the results in (4) and (5), we obtain
\[ |f(z)| \leq |h(z)| + |g(z)| \leq (1 + |b_1|)H_{\frac{1}{2}, \frac{1}{2}}(1). \]

Let \( f_\varepsilon = \frac{h + \varepsilon g}{1 + \varepsilon b_1} \) (\( \varepsilon \in \mathbb{D} \)). Then \( f_\varepsilon \) belongs to \( B_\Lambda(\frac{1}{2}) \). By [24, Theorem 2.3], we have
\[ |(h' + \varepsilon g')(z)| \geq |1 + \varepsilon b_1| \left( \frac{1 - |z|}{1 + |z|} \right)^{\frac{1}{2}} \geq (1 - |b_1|)H_{\frac{3}{2}}(1). \]

Especially, since \( \varepsilon \) is arbitrary, we get that
\[ |h'(z)| - |g'(z)| \geq (1 - |b_1|)H_{\frac{3}{2}}(1). \]

For \( 0 < r < 1 \) we choose \( z_0 \) such that \( |f(z_0)| \) is the minimum of \( |f(z)| \) on \( |z| = r \). If \( f \) is univalent in \( \mathbb{D} \) and \( \Lambda \) is the preimage of the segment \([0, f(z_0)]\), then for \( |z| = r \), we have that
\[ |f(z)| \geq |f(z_0)| = \int_0^{|z|} |df(z)| \geq \int_0^{|z|} (|f_\varepsilon(z)| - |f_\varepsilon(z)|)dz \geq -(1 - |b_1|)H_{\frac{3}{2}}(1). \]

Next we consider the sharpness part. The equality occurs in (1) if we take
\[ f(z) = H_{\frac{1}{2}}(z) + b_1 H_{\frac{1}{2}}(z) \quad \text{for each } \lambda \geq 0. \]

For each \( \lambda \geq 1 \), the equalities in the right sides of (2)-(6) are attained for
\[ f(z) = f_\Lambda(z) = F_{a,b,0}^{-1, \frac{1}{2}, \frac{1}{2}}(z) + b_1 F_{a,b,0}^{-1, \frac{1}{2}, \frac{1}{2}}(z) \]
at \( z = r \in [0, 1) \), where \( F_{a,b,0} \) is defined by (5.1). Note that \( f_\Lambda \in B_H(\lambda) \) by (1.2) and Proposition 5.2. Similarly, for each \( \lambda \geq 0 \), the function \( H_{\frac{1}{2}} \) provides the sharpness for the left sides of (2) and (4) at \( z = -r \in (-1, 0] \) when \( f \in B_H(\lambda) \). It follows from Proposition 5.2 that \( F_{a,b,0}^{-1, \frac{1}{2}, \frac{1}{2}} \) is univalent in \( \mathbb{D} \) for \( \lambda = 1 \). The equality in the left side of (6) occurs for \( f = F_{0,1,0} - b_1 F_{0,1,0} \in B_H(1) \) and \( z = -r \in (-1, 0] \). We complete the proof. \( \square \)
The following result can be directly deduced from Theorem 6.1.

**Corollary 6.1.** (Growth and covering theorem) Let \( f = h + \bar{g} \in B_H(\lambda) \) with \( b_1 = g'(0) \), and let \( H_{a,b} \) and \( H_a \) be defined by (6.2). If \( \lambda > 1 \), then \( f, h \) and \( g \) satisfy the same growth condition

\[
|f(z)| (h(z), \ g(z)) = O(1 - |z|)^{\frac{1}{\lambda - 1}} \quad \text{as } |z| \to 1.
\]

If \( \lambda < 1 \), then \( f \) (resp. \( h, g \)) is bounded by

\[
(1 + |b_1|) H_{\lambda + 1}(1) \quad (\text{resp. } (1 - |b_1|) H_{\lambda + 1}(1) + |b_1| H_{\lambda + 1}(1) \quad H_{\lambda + 1}(1) - (1 - |b_1|) H_{\lambda + 1}(1)).
\]

For all \( \lambda > 0 \), the image \( h(\mathbb{D}) \) contains the disk \( \{ |z| < -\sqrt{1 - |b_1|^2} H_{\frac{\lambda}{2}}(-1) \} \). If \( f \in B_H(\lambda) \cap S_H \), then the image \( f(\mathbb{D}) \) contains the disk \( \{ |z| < -(1 - |b_1|) H_{\lambda + 1}(-1) \} \).

If \( f \in B_H(\lambda) \cap S^0_H \) for some \( \lambda \in [0, 1] \), then

\[
-H_{\lambda + 1}(-1) \geq -H_1(-1) = 2 \log 2 - 1 = 0.38629 \cdots.
\]

This result is an improvement over the non-sharp known result that \( f(\mathbb{D}) \supseteq \{ w : |w| < 1/16 \} \) if \( f \in S^0_H \).

In Corollary 6.1 the case \( \lambda = 1 \) is critical. By Theorem 6.1, we have that, for \( f \in B_H(1) \),

\[
|f(z)| \leq (1 + |b_1|) H_1(|z|) = (1 + |b_1|) \left( -2 \log(1 - |z|) - |z| \right), \quad z \in \mathbb{D},
\]

which shows that functions in \( B_H(1) \) need not be bounded. The following result gives a sufficient condition for the boundedness of mappings in \( B_H(1) \).

**Proposition 6.2.** Let \( f = h + \bar{g} \) be a sense-preserving harmonic mapping in \( \mathbb{D} \). If \( f \) satisfies the condition

\[
\beta(f) := \lim_{|z| \to 1^-} \left( (1 - |z|^2) |P_f(z)| - 1 \right) \log \frac{1}{1 - |z|^2} < -2,
\]

then \( f, h \) and \( g \) are bounded in \( \mathbb{D} \).

**Proof.** Without loss of generality, we can assume that \( g(0) = 0 \). It follows from (1.2) that \( \beta(A \circ f) = \beta(f) \) for any affine harmonic mapping \( A \) defined in (1.2). Let \( A \circ f = H + \bar{G} \).

It is easy to check that both \( h \) and \( g \) are bounded in \( \mathbb{D} \) if and only if both \( H \) and \( G \) are bounded in \( \mathbb{D} \). Note that \( A \circ f \) is also sense-preserving in \( \mathbb{D} \). Thus, it is enough to consider the case \( f = h + \bar{g} \in H_0 \) and prove that both \( h \) and \( g \) are bounded in \( \mathbb{D} \).

By assumption, there exist \( \beta < -2 \) and \( r_0 \in (1 - 1/(2e), 1) \) such that

\[
|P_f(z)| \leq \frac{1}{1 - |z|^2} + \frac{\beta}{(1 - |z|^2) \log(1/1 - |z|^2)}
\]

for \( z \in D_{r_0} = \{ z : r_0 < |z| < 1 \} \). Fix \( z \in D_{r_0} \) and let \( \Gamma \) be a line segment from \( z \) to \( z_0 := r_0 e^{i \arg z} \) in the proof of Proposition 6.1. Then we have

\[
|\log J_f(z)| \leq 2 \int_{r_0}^{1-} |P_f(\zeta)||d\zeta| + C_1,
\]
where \( C_1 = \max_{\theta \in [0, 2\pi]} |J_f(r_0 e^{i\theta})| < \infty \). By (6.1) and (6.2), we see that

\[
|\log J_f(z)| \leq \log \frac{1 + |z|}{1 - |z|} + \int_{r_0}^{|z|} \frac{2\beta dt}{(1 - t^2) \log(1/(1 - t^2))} + C_1
\]

\[
\leq \log \frac{1 + |z|}{1 - |z|} + \int_{r_0}^{|z|} \frac{\beta dt}{(1 - t) \log(1/(2(1 - t)))} + C_1
\]

\[
= \log \frac{1 + |z|}{1 - |z|} + \beta \log \log \frac{1}{2(1 - |z|)} + C_2,
\]

where \( C_2 = C_1 - \beta \log \log \frac{1}{2(1 - r_0)} \). Exponentiating the last inequality shows that

\[
|J_f(z)| = |h'(z)|^2 (1 - |\omega_f(z)|^2) \leq e^{C_2} \frac{1 + |z|}{1 - |z|} \left( \log \frac{1}{2(1 - |z|)} \right)^\beta.
\]

Using that \( f \in \mathcal{B}_H^0(1) \), we have \( |\omega_f(z)| \leq |z| \) in \( \mathbb{D} \) and thus, we find that

\[
|g'(z)| \leq |h'(z)| = \left( \frac{J_f(z)}{1 - |\omega_f(z)|^2} \right)^{1/2} \leq e^{C_2/2} \frac{1 + |z|}{1 - |z|} \left( \log \frac{1}{2(1 - |z|)} \right)^{\beta/2}.
\]

Since \( \beta/2 < -1 \), the function \( \left( \frac{1}{\log(1/(2(1 - t)))} \right)^{\beta/2} / (1 - t) \) is integrable on the interval \( [r_0, 1) \). It follows that both \( h \) and \( g \) are bounded in \( \mathbb{D} \) so that \( f \) is also bounded in \( \mathbb{D} \). \( \square \)

Remark. Let \( f = h + \overline{f} \) be a sense-preserving harmonic mapping in \( \mathbb{D} \). If \( h \) and \( g \) are unbounded in \( \mathbb{D} \), then the boundedness of \( f \) is uncertain. For instance, let’s recall the function \( F_{0,1,\theta} \) defined by (5.1):

\[
F_{0,1,\theta}(z) = H_{0,1}(z) + G_{0,1,\theta}(z) = -\log(1 - z) + e^{i\theta}(-z - \log(1 - z)),
\]

we see that \( H_{0,1} \) and \( G_{0,1,\theta} \) are unbounded in \( \mathbb{D} \). However, \( F_{0,1,0}(z) = \overline{z} - 2|z| \log |1 - z| \) and \( F_{0,1,\pi}(z) = \overline{z} - 2 \arg(1 - z) \) are unbounded and bounded in \( \mathbb{D} \), respectively. Furthermore, it follows from Proposition 5.2 that \( ||P_{F_{0,1,\theta}}|| = 1 \) for any \( \theta \in \mathbb{R} \).

By Theorem 6.1 and [16] Theorem 5.1, we conclude the Hölder continuity of mappings in \( \mathcal{B}_H(\lambda) \).

**Theorem 6.2.** Let \( f = h + \overline{f} \in \mathcal{B}_H(\lambda) \) for some \( \lambda \in [0, 1) \). Then \( h + \varepsilon g \) is Hölder continuous of exponent \( \frac{1 - \lambda}{2} \) in \( \mathbb{D} \) for each \( \varepsilon \in \mathbb{R} \). Moreover, \( f \) is Hölder continuous of exponent \( \frac{1 - \lambda}{2} \) in \( \mathbb{D} \).

### 7. Coefficient estimates for the class \( \mathcal{B}_H(\lambda) \)

Throughout the section we consider \( f = h + \overline{f} \in \mathcal{B}_H \), where

\[
h(z) = \sum_{n=1}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n
\]

with \( a_1 = 1 \) and \( \mathcal{B}_H \) is defined in Section 1.4. For \( \varepsilon \in \mathbb{D} \), we now introduce \( f_\varepsilon \) by

\[
f_\varepsilon(z) := \frac{h(z) + \varepsilon g(z)}{1 + \varepsilon b_1} = \frac{1}{1 + \varepsilon b_1} \sum_{n=1}^{\infty} (a_n + \varepsilon b_n) z^n.
\]
We first determine the estimate for $a_2$.

**Theorem 7.1.** If $f \in B_H(\lambda)$, then we have

$$
|a_2| \leq \frac{1}{2} \min \left\{ (1 - |b_1|^2)\lambda + 2|b_1b_2|, \min\{1 + \varepsilon b_1|(|\lambda + 1)| + 2|\varepsilon b_2|\} \right\}.
$$

If $f \in B_H^0(\lambda)$, then $|a_2| \leq \lambda/2$ and the estimate is sharp for all $\lambda > 0$.

**Proof.** Let $\varepsilon \in \overline{D}$ and $f \in B_H(\lambda)$ for some $\lambda > 0$. Then $f_\varepsilon$ defined above belongs to $B_A((\lambda + 1)/2)$, and thus, we have $|P_{f_\varepsilon}(0)| \leq ||P_{f_\varepsilon}|| \leq \lambda + 1$ so that

$$
\left| \frac{h''(0) + \varepsilon g''(0)}{h'(0) + \varepsilon g'(0)} \right| = \left| \frac{2a_2 + 2\varepsilon b_2}{1 + \varepsilon b_1} \right| \leq \lambda + 1,
$$

which implies that $|a_2| \leq \frac{1}{2}((1 + \varepsilon b_1)(|\lambda + 1| + 2|\varepsilon b_2|)).$

On the other hand, for $f \in B_H(\lambda)$, it follows from (1.1) that

$$
|P_f(0)| = \left| \frac{h''(0)h'(0) - g''(0)g'(0)}{|h'(0)|^2 - |g'(0)|^2} \right| = \frac{|2a_2 - 2b_1b_2|}{1 - |b_1|^2} \leq ||P_f|| = \lambda,
$$

and thus, $|a_2| \leq \frac{1}{2}(1 - |b_1|^2)\lambda + 2|b_1b_2|).$ Inequality (7.1) follows if we estimate the last two estimates for $|a_2|$. If $f \in B_H^0(\lambda)$, then $b_1 = 0$ and thus, (7.1) reduces to $|a_2| \leq \lambda/2$. The function $H_{\lambda/2} = H_{\lambda/2,\lambda/2}$ defined by (5.2) provides the sharpness for each $\lambda > 0$. □

In order to indicate estimates for the coefficients of $f \in B_H(\lambda)$, we consider the integral mean $I_p(r, f)$ of $f$ defined by

$$
I_p(r, f) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta,
$$

where $p$ is a positive real number. Set $M_p(r, f) = (I_p(r, f))^{1/p}$, $0 < r < 1$.

**Definition 7.1.** For $0 < p < \infty$, the Hardy space $H^p$ is the set of all functions $f$ analytic in $\mathbb{D}$ for which $\|f\|_p := \sup\{M_p(r, f) : 0 < r < 1\} < +\infty$, where $M_p(r, f)$ is defined as above.

Let $h^p$ denote the analogous space of harmonic mappings $f$ in $\mathbb{D}$ with $\|f\|_p$ defined similarly (see [18]).

In [9], Aleman and Martín constructed convex harmonic mappings that do not belong to $h^{1/2}$ which settles the question raised by Duren [18]. It is worth pointing out that the space $h^p$ is well-behaved for $p \geq 1$ whereas $H^p$ is comparatively well-behaved for all $p > 0$, such is not the case for $h^p$, $0 < p < 1$.

Since $f \in B_H(\lambda)$ implies that $f_\varepsilon \in B_A((\lambda + 1)/2)$ for each $\varepsilon \in \overline{D}$, it follows from the result of [24, p. 190] that

$$
|a_n + \varepsilon b_n| = O(n^{(\lambda+1)/2-1})
$$

uniformly for $\varepsilon \in \overline{D}$ as $n \to \infty$ and thus, we obtain that $|a_n| + |b_n| = O(n^{(\lambda-1)/2})$ as $n \to \infty$. Especially, $|a_n| = O(n^{(\lambda-1)/2})$ and $|b_n| = O(n^{(\lambda-1)/2})$ as $n \to \infty$. Moreover, if $\lambda < 1$ and $f$ is univalent in $\mathbb{D}$, then, by Corollary 6.1, $f$ is bounded and thus,

$$
\text{Area}(f(\mathbb{D})) = \pi \left( \sum_{n=1}^{\infty} n(|a_n|^2 - |b_n|^2) \right) < \infty,
$$
which implies that \( \sqrt{|a_n|^2 - |b_n|^2} = o(n^{-1/2}) \) as \( n \to \infty \).

Combining the results from \cite{24} Section 3 and the implication \((1.3)\), we can get a series of results. We omit detailed proofs, but it might be appropriate to include some necessary explanations. In fact, we only need to modify the conditions by replacing the parameter \( \lambda \) in the theorems of \cite{24} Section 3 by \((\lambda + 1)/2\) at appropriate places.

**Theorem 7.2.** Let \( f = h + \overline{g} \in \mathcal{B}_H(\lambda) \). Then, for any \( a > 0 \) and a real number \( p \), we have
\[
(7.2) \quad I_p(r, h' + \varepsilon g') = O \left( (1 - r)^{-\alpha(p(\lambda + 1)/2) - a} \right),
\]
for each \( \varepsilon \in \overline{\mathbb{D}} \) and thus, in particular,
\[
|a_n| + |b_n| = O \left( n^{\alpha((\lambda + 1)/2) - 1 + a} \right).
\]

For \( p > 0 \), we get that
\[
(7.3) \quad I_p(r, f) = O \left( (1 - r)^{p - \alpha(p(\lambda + 1)/2) - a} \right).
\]

Here \( \alpha(\lambda) = \frac{\sqrt{1 + \lambda^2} - 1}{2} \).

**Proof.** The former part can be deduced from \cite{24} Theorem 3.1. For the later part, it follows from \((7.2)\) and \cite{16} Theorem 5.5 that
\[
M_p(r, h') = O \left( (1 - r)^{-\alpha(p(\lambda + 1)/2) + a}/p \right)
\]
and
\[
M_p(r, h) = O \left( (1 - r)^{1 - \alpha(p(\lambda + 1)/2) + a}/p \right),
\]
respectively. Similar conclusions hold for \( g \). Because \( M_p(r, f) \leq 2^p(M_p(r, h) + M_p(r, g)) \), we finally obtain that
\[
M_p(r, f) = O \left( (1 - r)^{1 - \alpha(p(\lambda + 1)/2) + a}/p \right),
\]
which implies \((7.3)\). \( \square \)

**Theorem 7.3.** Let \( f = h + \overline{g} \in \mathcal{B}_H(\lambda) \) for some \( \lambda \) with \( 1.982 < \lambda \leq 5 \). If there exists a constant \( \varepsilon \in \overline{\mathbb{D}} \) such that \( h + \varepsilon g \) is univalent in \( \mathbb{D} \), then \( |a_n + \varepsilon b_n| = O \left( n^{(\lambda - 3)/2} \right) \) as \( n \to \infty \). In particular, if \( h \) is univalent in \( \mathbb{D} \), then \( |a_n| = O \left( n^{(\lambda - 3)/2} \right) \) as \( n \to \infty \). Moreover, if \( h + \varepsilon g \) is univalent in \( \mathbb{D} \) for every \( |\varepsilon| = 1 \), then \( |a_n| + |b_n| = O \left( n^{(\lambda - 3)/2} \right) \) as \( n \to \infty \). The three estimates are sharp.

**Proof.** If \( f \in \mathcal{B}_H(\lambda) \) for some \( \lambda \) with \( 1.982 < \lambda \leq 5 \), then as before we have \( f_{\varepsilon} \in \mathcal{B}_A(\frac{\lambda + 1}{2}) \) \((1.491 < \frac{\lambda + 1}{2} \leq 3) \) for each \( \varepsilon \in \overline{\mathbb{D}} \). The results follow from \cite{24} Theorem 3.2.

To show the sharpness, we construct a family of functions
\[
T_{\lambda, \theta}(z) = t_{\lambda}(z) + e^{i\theta}z t_{\lambda}(z) = \sum_{n=1}^{\infty} a_n z^n + \sum_{n=2}^{\infty} b_n z^n, \quad z \in \mathbb{D},
\]
where \( \lambda \in (1.982, 5] \), \( \theta \in \mathbb{R} \) and
\[
t_{\lambda}(z) = \frac{1 - (1 - z)^{(1-\lambda)/2}}{(1 - \lambda)/2}.
\]
First, we show that \( T_{\lambda, \theta} \in \mathcal{B}_H(\lambda) \) for all \( \lambda > 1 \) and \( \theta \in \mathbb{R} \). It suffices to prove that \( \|P_{T_{\lambda, \theta}}\| = \lambda \) due to \( T_{\lambda, \theta} \in \mathcal{H} \). By computation, we find that

\[
P_{T_{\lambda, \theta}}(z) = \frac{1 + \lambda}{2} \cdot \frac{1}{1 - z} - \frac{\pi}{1 - |z|^2}.
\]

Also, we note that \( \|P_{T_{\lambda, \theta}}\| = 1 + \lambda \). If we get \( \|P_{T_{\lambda, \theta}}\| \leq \lambda \), then it follows from (2.1) that \( \|P_{T_{\lambda, \theta}}\| = \lambda \). Indeed we may let \( z = x + iy \in \mathbb{D} \). By computation, we obtain

\[
4\lambda^2(1 - z)^2 - ((1 + \lambda)(1 - |z|^2) - 2\pi(1 - z))^2 = (\lambda - 1)[(1 - x)^3(1 - x + \lambda(3 + x)) + 2(3 + 3\lambda - 2x + (1 - \lambda)x^2)y^2 - (\lambda - 1)y^4] \\
\geq (\lambda - 1)(2(\lambda + 1) - 2x + (1 - \lambda)x^2)y^2 - 2(\lambda - 1)y^4] \\
\geq 2(\lambda - 1)(3 + 3\lambda - 2 + 1 - \lambda - \lambda + 1)y^2 \geq 0,
\]

which clearly implies that

\[
(1 - |z|^2)|P_{T_{\lambda, \theta}}(z)| \leq \lambda
\]

and thus, \( \|P_{T_{\lambda, \theta}}\| \leq \lambda \). Next, we show that the functions \( T_{\lambda, \theta} \) are univalent in \( \mathbb{D} \) for each \( 1 < \lambda \leq 5 \) and all \( \theta \in \mathbb{R} \). A simple computation shows that

\[
\Re \left( 1 + z \frac{t'_\lambda(z)}{t'_{\lambda}(z)} \right) = \Re \left( 1 + \frac{1 + \lambda}{2} \cdot \frac{z}{1 - z} \right) > 1 - \frac{1 + \lambda}{4} \geq -\frac{1}{2}, \quad z \in \mathbb{D},
\]

for \( 1 < \lambda \leq 5 \). According to a well-known result, the function \( t_\lambda \) is univalent and convex in one direction (and hence, close-to-convex) in \( \mathbb{D} \). Note that the dilatation of \( T_{\lambda, \theta} \) is \( e^{i\theta}z \). It follows from [9, Theorem 1] that the functions \( T_{\lambda, \theta} \) are univalent in \( \mathbb{D} \) for each \( 1 < \lambda \leq 5 \) and all \( \theta \in \mathbb{R} \). Therefore, \( T_{\lambda, \theta} \) is SHU and thus, \( t_\lambda + \varepsilon z t'_\lambda \) is univalent in \( \mathbb{D} \) for each \( \varepsilon \in \mathbb{D} \). Finally, by Stirling’s formula, we have

\[
|a_n| = \frac{2\Gamma((\lambda + 2n - 1)/2)}{(\lambda - 1)n\Gamma((\lambda - 1)/2)} \sim \frac{2}{\lambda - 1} n^{(\lambda - 3)/2} \text{ as } n \to \infty.
\]

Note that \( b_n = e^{i\theta}a_{n-1} \) for each \( n > 1 \). Hence, \( |a_n| + |b_n| = |a_n| + |a_{n-1}| = O\left(n^{(\lambda - 3)/2}\right) \) as \( n \to \infty \).

Given a harmonic mapping \( f \in \mathcal{H} \), let \( \gamma(f) \) denote the infimum of exponents \( \gamma \) such that \( |a_n| + |b_n| = O\left(n^{\gamma - 1}\right) \) as \( n \to \infty \), that is,

\[
\gamma(f) = \lim_{n \to \infty} \frac{\log n(|a_n| + |b_n|)}{\log n}.
\]

For the subset \( X \) of \( \mathcal{H} \), we let \( \gamma(X) = \sup_{f \in X} \gamma(f) \). There are some investigations about \( \gamma(f) \) (resp. \( \gamma(X) \)) if \( f \) (resp. \( X \)) is restricted to be analytic or special families of analytic functions. The reader can refer to [10, 24, 28] and [31, Chapter 10] for some details on this problem.

For each \( \lambda \in (0, \infty) \) and \( \varepsilon \in \mathbb{D} \), we introduce

\[
\mathcal{A}_H(\lambda, \varepsilon) = \left\{ f_\varepsilon : f_\varepsilon(z) = \frac{h(z) + \varepsilon g(z)}{1 + \varepsilon g'(0)} \text{ and } f = h + g \in \mathcal{B}_H(\lambda) \right\}
\]

and obtain the following theorem.
Theorem 7.4. For each $\lambda \in (0, \infty)$ and $\varepsilon \in \overline{D}$, we have
\[
\max\{(\lambda - 1)/2, 0\} \leq \gamma(B_H(\lambda)) (\gamma(A_H(\lambda, \varepsilon))) \leq \alpha((\lambda + 1)/2),
\]
where $\alpha(\lambda) = \sqrt{1 + 4\lambda^2 - 1/2}$. In particular, $\gamma(B_H(\lambda)) = O((\lambda + 1)^2)$ and $\gamma(A_H(\lambda, \varepsilon)) = O((\lambda + 1)^2)$ as $\lambda \to 0$.

We continue the discussion by mentioning a connection with integral means for univalent analytic functions. For a univalent harmonic mapping $f = h + \overline{g} \in S_H$, a complex number $\varepsilon \in \overline{D}$ and a real number $p$, we let
\[
\beta_f(\varepsilon)(p) = \lim_{r \to 1^-} \frac{\log I_p(r, f')}{\log \frac{1}{1 - r}}.
\]
Clearly, for a univalent analytic function $f \in A \cap S_H$,
\[
\beta_f(\varepsilon)(p) = \lim_{r \to 1^-} \frac{\log I_p(r, f')}{\log \frac{1}{1 - r}}.
\]
Brennan conjectured that $\beta_f(-2) \leq 1$ for univalent analytic functions $f$ (see [31, Chapter 8]).

As a corollary to Theorem 7.2, we have

Theorem 7.5. For $f \in B_H(\lambda)$ and a real number $p$,
\[
\beta_f(\varepsilon)(p) \leq \alpha(|p|)(\lambda + 1)/2 = \frac{\sqrt{1 + p^2(1 + \lambda)^2} - 1}{2}
\]
holds for each $\varepsilon \in \overline{D}$. In particular, the Brennan conjecture is true for every univalent harmonic mapping $f$ with $\|P_f\| \leq \sqrt{2} - 1$.

8. THE SPACE $B_H(\lambda)$ AND THE HARDY SPACE

For a harmonic mapping $f = h + \overline{g}$ in $\mathbb{D}$, the Bloch seminorm is given by (see [14])
\[
\|f\|_{\mathcal{B}_H} = \sup_{z \in \mathbb{D}} (1 - |z|^2)(|h'(z)| + |g'(z)|),
\]
and $f$ is called a (harmonic) Bloch mapping when $\|f\|_{\mathcal{B}_H} < \infty$. Let BMOA (resp. BMOH) denote the class of analytic functions (resp. harmonic mappings) that have bounded mean oscillation on the unit disk $\mathbb{D}$ (see [11]). Kim [24] showed some relationships among $\mathcal{B}_A(\lambda)$, $H^p$ and BMOA (see also [25]). Combined with the study on Bloch, BMO and univalent harmonic mappings (see [11]), a generalization of Kim’s result is given in [32]. Basic properties about analytic Bloch functions may be obtained from [5, 31].

Our results are based on the following observation. It follows from Theorem 6.1 (6) that the inequality
\[
|f(z)| \leq (1 + |b_1|) \int_0^{|z|} \left( \frac{1 + t}{1 - t} \right)^{\lambda + 1} dt, \quad z \in \mathbb{D},
\]
holds for every $f \in B_H(\lambda)$, which implies that
- $f$ is bounded when $\lambda < 1$,
- $f(z) = O(- \log(1 - |z|))$ ($|z| \to 1$) when $\lambda = 1$, and
- $f(z) = O((1 - |z|)^{1 - \frac{\lambda + 1}{2}})$ ($|z| \to 1$) when $\lambda > 1$. 

On the other hand, the proofs of our results are similar to that of results of \[32\] Section 4. Let

\[ T_H(\lambda) = \{ f = h + \overline{g} \in \mathcal{H} : \| T_f \| \leq 2\lambda \} \]

with

\[ \| T_f \| := \sup_{z \in \mathbb{D}, \theta \in [0, 2\pi]} (1 - |z|^2) \left| \frac{h''(z) + e^{i\theta} g''(z)}{h'(z) + e^{i\theta} g'(z)} \right|. \]

For the one parameter family \( T_H(\lambda) \), the authors showed its relationship with Hardy spaces in \[32\] Section 4. Note that \( \| T_f \| = \sup_{\theta \in [0, 2\pi]} \| P_{h + e^{i\theta} g} \| \). If \( f \in \mathcal{B}_H(\lambda) \), then it is easy to see that \( f \in T_H(\frac{\lambda+1}{2}) \) from (2.1). Therefore, applying the above observation and replacing \( h + e^{i\theta} g \) (\( \theta \in [0, 2\pi] \)) (resp. \( \lambda \)) to \( h + \varepsilon g \) (\( \varepsilon \in \mathbb{D} \)) (resp. \( (\lambda + 1)/2 \)) in corresponding proofs of \[32\] Section 4, then we can easily obtain the following results. So we omit their proofs.

**Theorem 8.1.**

1. If \( \lambda < 1 \), then \( \mathcal{B}_H(\lambda) \cap S_H \subset h^\infty \).
2. If \( \lambda = 1 \), then \( \mathcal{B}_H(\lambda) \cap S_H \subset \text{BMHOH} \).
3. If \( \lambda > 1 \), then \( \mathcal{B}_H(\lambda) \cap S_{HK} \subset h^p \) for every \( 0 < p < 2/(\lambda - 1) \), where \( K \geq 1 \) and \( S_{HK} = \{ f = h + \overline{g} \in S_H : f \text{ is } K\text{-quasiconformal} \} \).

**Theorem 8.2.** Let \( \lambda \geq 1 \). Then \( \mathcal{B}_H(\lambda) \subset h^p \) with \( 0 < p < p_0(\lambda) = \frac{4}{(\lambda+5)(\lambda-1)} \), where \( p_0(\lambda) = \infty \) if \( \lambda = 1 \).

**Remark.** Theorem 8.2 can be directly obtained by choosing \( p - \alpha(p(\lambda + 1)/2) > 0 \) in Theorem 7.2

**Corollary 8.1.** A uniformly locally univalent harmonic mapping \( f \) in \( \mathbb{D} \) is contained in the Hardy space \( h^p \) for some \( p = p(f) > 0 \).

9. **Subordination principles for the estimate of PSN**

In this section, \( \mathcal{A}_D \) denotes the class of analytic functions \( \phi \) from \( \mathbb{D} \) into itself and \( \mathcal{A}_D^0 \) denotes the subclass of \( \mathcal{A}_D \) with the normalization \( \phi(0) = 0 \). If \( f \) and \( F \) are restricted to be analytic, then we say that \( f \) is said to be subordinate (resp. weakly subordinate) to \( F \) (written \( f \prec F \) (resp. \( f \preceq F \))) if there exists a function \( \phi \in \mathcal{A}_D^0 \) (resp. \( \phi \in \mathcal{A}_D \)) such that \( f(z) = F(\phi(z)) \) in \( \mathbb{D} \).

In 2000, Schaubroeck \[35\] generalized the notion of subordination to harmonic mappings. A harmonic mapping \( f \) is subordinate to a harmonic mapping \( F \), still denoted by \( f \prec F \), if there is a function \( \phi \in \mathcal{A}_D^0 \) such that \( f = F \circ \phi \).

Note that if the analytic function \( F \) is univalent in \( \mathbb{D} \), then \( f \prec F \) if and only if that \( f(0) = F(0) \) and \( f(\mathbb{D}) \subseteq F(\mathbb{D}) \). However, this property is not true for harmonic mappings. As in \[24\], a harmonic mapping \( f \) is said to be weakly subordinate to the harmonic mapping \( F \) if \( f(\mathbb{D}) \subseteq F(\mathbb{D}) \).

In this article, \( f = h + \overline{g} \preceq F = H + \overline{G} \) means that there exists a function \( \phi \in \mathcal{A}_D \) such that \( h = H \circ \phi \) and \( g = G \circ \phi \). Clearly, if \( f \preceq F \), then \( f \) is weakly subordinate to \( F \) in the sense of Muir. The following result is a generalization of \[24\] Theorem 4.1].
Theorem 9.1. (Subordination principle I) Let \( f = h + \overline{g} \) be a harmonic mapping in \( \mathbb{D} \) and \( F = H + \overline{G} \in \mathcal{B}_H \). If \( h' + \overline{g'} \preceq H' + \overline{G'} \), then we have \( ||P_f|| \leq ||P_F|| \). In this case, \( f \) is ULU in \( \mathbb{D} \).

Proof. By assumption, there exists a function \( \phi \in \mathcal{A}_D \) such that \( h' = H' \circ \phi \) and \( g' = G' \circ \phi \). Therefore, \( f \) is sense-preserving in \( \mathbb{D} \) since \( F \) is sense-preserving in \( \mathbb{D} \). Moreover, we have \( P_h = (P_H \circ \phi)\phi' \) and
\[
\omega_f(z) = \frac{g'(z)}{h'(z)} = \frac{g'(w)}{H'(w)} = \omega_F(w), \quad w = \phi(z).
\]
Consequently,
\[
\frac{\omega_f(z)\omega'_f(z)}{1 - |\omega_f(z)|^2} = \frac{\omega_F(w)\omega'_F(w)}{1 - |\omega_F(w)|^2} \phi'(z).
\]
It follows from (1.1) that \( P_f = (P_F \circ \phi)\phi' \). By Schwarz-Pick’s lemma,
\[
|\phi'(z)| \leq \frac{1 - |\phi(z)|^2}{1 - |z|^2}
\]
and using this, we find that
\[
(1 - |z|^2)|P_f(z)| = (1 - |z|^2)|\phi'(z)P_F(\phi(z))| \leq (1 - |\phi(z)|^2)|P_F(\phi(z))| \leq ||P_F||.
\]
The desired conclusion follows. \( \square \)

Often, the property of a sense-preserving harmonic mapping is mainly decided by its analytic part. As another example of it, we have

Theorem 9.2. (Subordination principle II) Let \( f = h + \overline{g} \) be a sense-preserving harmonic mapping in \( \mathbb{D} \) and \( F = H + \overline{G} \in \mathcal{B}_H \) such that \( h' \preceq H' \). Then we have \( ||P_f|| \leq ||P_F|| + 2 \). Thus, \( f \) is ULU in \( \mathbb{D} \).

Proof. Since \( F \in \mathcal{B}_H \), we know that \( H \in \mathcal{B}_A \) by Theorem 4.1. Clearly, \( \mathcal{B}_A \subseteq \mathcal{B}_H \). It follows from the assumption and Theorem 9.1 that \( ||P_h|| \leq ||P_H|| \). Using the inequality (2.1) twice, we obtain that
\[
||P_f|| \leq ||P_h|| + 1 \leq ||P_H|| + 1 \leq ||P_F|| + 2
\]
and the proof is complete. \( \square \)

Similar to Theorem 9.2, few other results on subordination of analytic functions in [24, Section 4] can be transplanted to the case of sense-preserving harmonic mappings by considering its analytic parts.

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