DYNAMICS AND LAX-PHILLIPS SCATTERING FOR GENERALIZED LAMB MODELS

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Abstract. This paper treats the dynamics and scattering of a model of coupled oscillating systems, a finite dimensional one and a wave field on the half line. The coupling is realized producing the family of selfadjoint extensions of the suitably restricted self-adjoint operator describing the uncoupled dynamics. The spectral theory of the family is studied and the associated quadratic forms constructed. The dynamics turns out to be Hamiltonian and the Hamiltonian is described, including the case in which the finite dimensional systems comprises nonlinear oscillators; in this case the dynamics is shown to exist as well. In the linear case the system is equivalent, on a dense subspace, to a wave equation on the half line with higher order boundary conditions, described by a differential polynomial $p(\partial_x)$ explicitly related to the model parameters. In terms of such structure the Lax-Phillips scattering of the system is studied. In particular we determine the scattering operator, which turns out to be unitarily equivalent to the multiplication operator given by the rational function $-p(i\kappa)^*/p(i\kappa)$, the incoming and outgoing translation representations and the Lax-Phillips semigroup, which describes the evolution of the states which are neither incoming in the past nor outgoing in the future.

1. Introduction

In this paper we investigate the spectral theory, dynamics and Lax-Phillips scattering for an abstract system which models the interaction between a finite dimensional linear subsystem and an infinite dimensional wave field on a halfline. We will call such systems generalized Lamb models in that they extend the standard Lamb model (see [10]) that will be introduced shortly. Although our main concern is with linear oscillators, we will describe some properties of the models in the anharmonic case also.

To introduce our models, let us consider a $n$-dimensional lagrangian system linearized around a certain equilibrium point. Its equations of motion are given by

$$G_{ij} = Hy$$
where $y \in \mathbb{R}^n$ is the displacement from the given equilibrium point (for reference, $y = 0$), and the matrices $G$ and $H$ represent the quadratic approximation of kinetic and potential energy around the equilibrium point. $G$ is positive definite and both matrices are symmetric with respect to the standard inner product in $\mathbb{R}^n$. For technical and theoretical reasons it is more useful to endow $\mathbb{R}^n$ with the inner product given by $G$. With respect to this inner product the matrix $L = G^{-1}H$ is symmetric and the lagrangian equation take the form

$$\ddot{y} = Ly$$

with $L$ symmetric with respect to the $G$ inner product. The case of a chain of harmonic oscillators is well known and yields to a Jacobi matrix for the operator $L$.

Analogously, let us consider the wave equation on the halfline. To be definite let us consider Neumann boundary condition at the origin. Denoting with $\Delta_N$ the 1-dimensional laplacian with homogeneous Neumann boundary conditions at the origin, the wave field (we have posed equal to one the wave velocity) evolves according to the wave equation

$$\ddot{\phi} = \Delta_N \phi.$$ 

So we have two decoupled second order equations for two different oscillating systems, the finite dimensional one with generator $L$ and the infinite dimensional one with generator $\Delta_N$.

Thus on the direct sum $L^2(\mathbb{R}_+) \oplus \mathbb{R}^n$ we have the self-adjoint operator

$$A_0 = \Delta_N \oplus L$$

and the corresponding abstract wave equation

$$\ddot{\Psi} = A_0 \Psi.$$ 

In a heuristic way a coupling between the two oscillating systems could be given by posing a constraint between boundary values of the wave field at the origin and the displacement of the finite dimensional system. The prototype of this coupling is furnished by the well known Lamb model where a semi-infinite string is coupled to a single particle oscillating in the transverse direction (see section 4.1 for the general case of a chain of oscillators); the particle, with mass $M$, is subjected to the tension $T$ of the string at the origin and to a restoring harmonic force with spring constant $K$, so that the formal equations are given by the system

$$\begin{align*}
\ddot{\phi}(t, x) &= \phi''(t, x), \quad x > 0, \\
M \ddot{y}(t) &= -K y(t) + T \phi'(t, 0_+).
\end{align*}$$
plus the constraint
\[ \phi(t, 0_+) = y(t). \]

This model was proposed by Horace Lamb in 1900 as an example of dissipation in (subsystems of) conservative systems. In fact it is possible to decouple field and particle dynamics, and the particle component satisfies a reduced equation which turns out to be, for \( t > 0 \),
\[ M\ddot{y}(t) + 2T\dot{y}(t) + Ky(t) = T(\phi'_0(t, 0_+) + \dot{\phi}_0(t, 0_+)) \]

The forcing term on the right hand side depends on the evaluation at the origin of the free evolution for the wavefield of the initial data \( \phi_0, \dot{\phi}_0 \). Thus for initial data of compact support the forcing term is a pure transient definitively vanishing, and the reduced dynamics for the particle coincides for large times with that of a damped harmonic oscillator, so that the effect of interaction between particle and field reduces to damping only. This means exponentially fast return to equilibrium of the finite dimensional subsystem and correspondingly a neat transfer of energy to the field. This relaxation property towards the equilibrium position of the finite dimensional component is always true when the corresponding self-adjoint operator has empty point spectrum (see Remark 6.1) as it is the case in the Lamb model. The result has various generalizations to anharmonic oscillators (see [8]).

Some tridimensional models reduce themselves to generalized Lamb models due to symmetry. The case of an elastic spherical shell coupled to the acoustic field when radial oscillations only are allowed is treated in section 4.3, and it yields to a nontrivial generalized Lamb model. Another issue of interest of these coupled systems are given by the fact that some linear models of classical and quantum field theory reduce themselves in the ultraviolet limit, and after due renormalizations, to generalized Lamb models. For example, the Schwabl-Thirring (see [16]) model when restricted to its monopole sector (the only one where it is not trivial) and after a spring constant renormalization turns out to be equivalent to a Lamb model (see [12]). A similar phenomenon occurs for the Pauli-Fierz model describing the interaction of a charged oscillator with the electromagnetic field in dipole approximation and after mass renormalization (see [3]). In this case, however, reduction of the dynamics on its non trivial part, yields a boundary condition different from that of the Lamb model (see Example 4.2).

This discussion of motivating examples, and relevant studies existing in the literature, shows however that the usual formulation is partly formal in that it is not clear what it should be the functional setting of the Lamb system in the first place, and secondarily its Hamiltonian formulation, if any exists.
A guide to set rigorously these questions in this and in more general situations is suggested by an analysis of the coupling between field and particle. The idea is to restrict the uncoupled vector operator $A_0$ to the linear variety defined by the constraint existing between field and subsystem; the uncoupled operator on this linear variety is no more self-adjoint but it is symmetric with defect indices $(1, 1)$. All possible self-adjoint extensions different from $A_0$ itself correspond to a well defined coupling or interaction between two subsystems. The case of the Lamb model, for example, corresponds to the closed linear variety $\phi_0(0_+) = y$. The most general boundary conditions still producing a self-adjoint operator, as we will see, is of the kind $\theta \phi'(0_+) + \phi(0_+) = w \cdot y$, where $\theta \in \mathbb{R}$ and $w$ is a given vector in $\mathbb{R}^n$. The case of a chain of harmonic oscillators one of which coincides with the boundary point of the string, which is the more obvious generalization of the Lamb model, corresponds to a vector $w$ with just a single entry nonvanishing and to $\theta = 0$. A generic $w$ corresponds to nonlocal coupling between string and more than one oscillator, i.e. the interaction is not "nearest neighbour". Our first concern, is to give a rigorous account of this construction and to explicitly describe the interacting system so obtained (see Theorem 2.1) as well as its spectral properties (see Theorem 2.2). The interacting operator so constructed is a singular perturbation of the selfadjoint operator $A_0$, related to the class of one dimensional point interactions, or better point interaction with inner structure previously studied in different context and with a different formalism by many authors (see e.g. [13], [9], [1] and references therein). In passing, we note that the coupled operator we study corresponds to a boundary value problem for the wavefield only, but with an eigenparameter dependent boundary condition (see Remark 2.5): this sort of parameter dependent boundary value problems are well known in the literature, both physical and mathematical. However we do not follow this road to the study of spectral and scattering properties of the coupled operator.

As a byproduct of the construction we obtain in Section 3 the Hamiltonian structure of the system (see Theorem 3.2), which we generalize to the case of anharmonic oscillators, giving conditions for the existence of global flow (see Remark 3.3). As far as we know, a completely rigorous description of the Hamiltonian structure of such type of systems has been lacking up to now, whereas interesting, but formal treatments, are scattered in the literature (see for example [12], [7]).

In the case the symmetric operator $L$ has no degenerate eigenvalues we show that the dynamics of the system is equivalent, for a dense set of smooth initial data, to a reduced dynamics of a wave equation
on the half line which incorporates the interaction with the finite dimensional systems through a higher order boundary condition of the kind \( p(\partial_x)\phi(t,0+) = 0 \), where the polynomial \( p \) is explicitly related to the parameters entering into the definition of the model (see Theorem 5.1). This is a technical result, useful for the analysis of the Lax-Phillips scattering for the system, which is the main topic of the remaining part of the paper.

In Section 6, in the case of empty point spectrum, we determine the incoming \((R^-)\) and outgoing \((R^+)\) translation representations which make the dynamics unitarily equivalent to the translation on \(L^2(\mathbb{R})\) defined by \( T^t f(x) := f(x-t) \). This provides the scattering operator \( S_p^* \) for the system by the relation \( S_p^* = R^+(R^-)^{-1} \). Moreover \( S_p^* \) turns out to be unitarily equivalent to the multiplication operator given by the rational function \(-p(i\kappa)^*/p(i\kappa)\). In Section 7 the Lax-Phillips semi-group \( Z^t, t \geq 0 \), which describes the evolution of the states which are neither incoming in the past nor outgoing in the future is completely characterized. It acts on a finite dimensional vector space, whose dimension coincides with the degree of the polynomial \( p \), by \( Z^t = e^{-tB} \), where the spectrum of the generator \( B \) is made of the resonances of the system. Such resonances correspond to the roots of the polynomial \( p \).

### 2. Singular perturbations of the free dynamics

Let us begin with some definitions. We denote by \( L^2(\mathbb{R}_+) \) the Hilbert space of square-integrable functions on the half-line \((0, +\infty)\) and by \( H^1(\mathbb{R}_+) \) and \( H^2(\mathbb{R}_+) \) the Sobolev spaces

\[
H^1(\mathbb{R}_+) := \left\{ \phi \in L^2(\mathbb{R}_+) : \phi' \in L^2(\mathbb{R}_+) \right\}, \\
H^2(\mathbb{R}_+) := \left\{ \phi \in L^2(\mathbb{R}_+) : \phi', \phi'' \in L^2(\mathbb{R}_+) \right\}.
\]

Here the prime \( \phi' \) denotes a spatial derivative. With a dot, \( \dot{\phi} \), we will denote a time derivative. We then define \( H^2_N(\mathbb{R}_+) \) as the subspace of \( H^2(\mathbb{R}_+) \) of functions which satisfy homogeneous Neumann boundary conditions at zero, i.e.

\[
H^2_N(\mathbb{R}_+) := \left\{ \phi \in H^2(\mathbb{R}_+) : \phi'(0+) = 0 \right\}.
\]

We denote by \( \langle \cdot, \cdot \rangle_2 \) and by \( \| \cdot \|_2 \) the usual scalar product and the corresponding norm on \( L^2(\mathbb{R}_+) \).

Given the \( n \)-dimensional Hilbert space \( \mathfrak{h} \) with scalar product \( \langle \cdot, \cdot \rangle \) and corresponding norm \( \| \cdot \| \), and given the symmetric operator \( L : \mathfrak{h} \to \mathfrak{h} \), we consider the self-adjoint operator

\[
A_0 : H^2_N(\mathbb{R}_+) \oplus \mathfrak{h} \subset L^2(\mathbb{R}_+) \oplus \mathfrak{h} \to L^2(\mathbb{R}_+) \oplus \mathfrak{h}, \quad A_0(\phi, y) := (\phi''', Ly) .
\]
Regarding the spectrum of $A_0$ one has

$$\sigma_{\text{ess}}(A_0) = \sigma_{\text{ac}}(A_0) = (-\infty, 0], \quad \sigma_{pp}(A_0) = \sigma(L).$$

In order to couple the two dynamical systems described by the equations $\ddot{\phi} = \phi''$ and $\ddot{y} = Ly$ we define the continuous and surjective linear operator

$$\tau : H^1(\mathbb{R}^+) \oplus \mathfrak{h} \to \mathbb{C}, \quad \tau(\phi, y) := \phi(0^+) - \langle w, y \rangle, \quad w \in \mathfrak{h},$$

and then we consider the closed symmetric operator $\dot{A}_0$ obtained by restricting $A_0$ to the kernel of $\tau$. $\dot{A}_0$ has deficiency indices $(1, 1)$ and we are interested in its self-adjoint extensions different from $A_0$ itself, which we parametrize by the real extension parameter $\theta$. Thus to each quadruple $(\mathfrak{h}, L, w, \theta)$ corresponds a different generalized Lamb model.

The next theorem completely characterizes such models.

**Theorem 2.1.** For any $\theta \in \mathbb{R}$ the linear operator

$$A : D(A) \subset L^2(\mathbb{R}^+) \oplus \mathfrak{h} \to L^2(\mathbb{R}^+) \oplus \mathfrak{h}$$

defined by

$$D(A) := \{ (\phi, y) \in H^2(\mathbb{R}^+) \oplus \mathfrak{h} : \theta \phi'(0^+) + \phi(0^+) = \langle w, y \rangle \},$$

$$A(\phi, y) := (\phi'', Ly + w \phi'(0^+))$$

is a self-adjoint extension of $\dot{A}_0$ and its resolvent is given by

$$(-A + z)^{-1} = (-A_0 + z)^{-1} + (\theta + \Gamma(z))^{-1}G_z \otimes G_{z^*},$$

where

$$\Gamma(z) := -\left( \pm \frac{1}{\sqrt{z}} + \langle w, (L + z)^{-1}w \rangle \right), \quad \pm \text{Re}\sqrt{z} > 0$$

and

$$G_z = \begin{pmatrix} \pm e^{\pm \sqrt{z}x} \sqrt{z} & (L + z)^{-1}w \end{pmatrix}, \quad \pm \text{Re}\sqrt{z} > 0. $$

**Proof.** We will make use of the mathematical procedure developed in [14] (see also [4], Theorem 2.2, for a similar proof in the case of a one-dimensional model in acoustics).

For any $z \in \rho(A_0)$, let us consider the two linear continuous operators

$$\tilde{G}(z) : L^2(\mathbb{R}^+) \oplus \mathfrak{h} \to \mathbb{C}, \quad \tilde{G}(z) := \tau(-A_0 + z)^{-1},$$

$$G(z) : \mathbb{C} \to L^2(\mathbb{R}^+) \oplus \mathfrak{h}, \quad G(z) := \tilde{G}(z^*)^*.$$

Since

$$\left( -\frac{d^2}{dx^2} + z \right)^{-1} : L^2(\mathbb{R}^+) \to H^2_N(\mathbb{R}^+)$$
has kernel
\[ \mathcal{G}_N(z; x_1, x_2) = \pm \frac{e^{\pm \sqrt{z}|x_1-x_2|} + e^{\pm \sqrt{z}(x_1+x_2)}}{2\sqrt{z}}, \quad \pm \text{Re}\sqrt{z} > 0, \]
the operators \( \tilde{G}(z) \) and \( G(z) \) are represented by the vectors \( G_z^* \) and \( G_z \) respectively, where
\[ G_z = \left( \pm \frac{e^{\pm \sqrt{z}x}}{\sqrt{z}}, -(L + z)^{-1}w \right), \quad \pm \text{Re}\sqrt{z} > 0. \]
Note that
\[ (2.1) \quad \text{Ran}(G(z)) \cap \text{D}(A_0) = \{0\}. \]
Now we define, for any \( z \in \rho(A_0) \), the map
\[ \Gamma(z) : \rho(A_0) \to \mathbb{C}, \quad \Gamma(z) := -\tau G(z), \]
i.e.
\[ \Gamma(z) := -\left( \pm \frac{1}{\sqrt{z}} + \langle w, (-L + z)^{-1}w \rangle \right), \quad \pm \text{Re}\sqrt{z} > 0. \]
At first let us note that the the function \( \Gamma \) satisfies the relation
\[ (2.2) \quad \Gamma(z) - \Gamma(w) = (z - w) \tilde{G}(w)G(z). \]
Indeed
\[ \Gamma(z) - \Gamma(w) = \tau(G(w) - G(z)) \]
and, by first resolvent identity and by the definition of \( G(z) \),
\[ (z - w)(-A_0 + z)^{-1}G(z) = G(w) - G(z). \]
Relation (2.2) implies that
\[ R(z) := (-A_0 + z)^{-1} + (\theta + \Gamma(z))^{-1}G_z \otimes G_z^* \]
satisfies the first resolvent equation
\[ (2.3) \quad R(w) - R(z) = (z - w)R(w)R(z). \]
By the definitions of \( \tilde{G}(z) \) and \( G(z) \), and since \( \Gamma(z)^* = \Gamma(z^*) \), one obtains
\[ (2.4) \quad R(z)^* = R(z^*). \]
Moreover, by (2.1), the linear operator \( R(z) \) is injective. Thus
\[ A := -R(z)^{-1} + z \]
is well defined on the domain
\[ \text{D}(A) := \text{Range}(R(z)). \]
By (2.3) such a definition is $z$-independent. By (2.4) $A$ is symmetric and is self-adjoint since

$$\text{Range}(-A \pm i) = L^2(\mathbb{R}_+) \oplus \mathfrak{h}$$

by construction. We have thus defined the self-adjoint operator

$$D(A) := \{(\phi_z, y_z) + (\theta + \Gamma(z))^{-1}(\phi_z(0_+) - \langle w, y_z \rangle) G_z, \phi_z \in H^2_N(\mathbb{R}_+)\},$$

$$(-A + z)(\phi, y) := (-A_0 + z)(\phi_z, y_z).$$

This implies

$$\phi'(0_+) = - (\theta + \Gamma(z))^{-1}(\phi_z(0_+) - \langle w, y_z \rangle)$$

and

$$\phi(0_+) = \phi_z(0_+) \mp \frac{1}{\sqrt{z}} \phi'(0_+).$$

Therefore

$$\theta \phi'(0_+) = (\theta + \Gamma(z)) \phi'(0_+) - \Gamma(z) \phi'(0_+)$$

$$= - \phi_z(0_+) + \langle w, y_z \rangle + \left(\frac{\pm 1}{\sqrt{z}} + \langle w, (-L + z)^{-1}w \rangle\right) \phi'(0_+)$$

$$= - \phi'(0_+) + \langle w, (y_z + (-L + z)^{-1}w) \phi'(0_+) \rangle$$

$$= - \phi(0_+) + \langle w, y \rangle.$$

Posing

$$A(\phi, y) \equiv (A_1(\phi, y), A_2(\phi, y))$$

one obtains

$$A_1(\phi, y)(x) = \phi''_z(x) \mp z \phi'(0_+) \frac{e^{\mp \sqrt{z} x}}{\sqrt{z}}$$

$$= \left(\phi_z(x) \mp \phi'(0_+) \frac{e^{\mp \sqrt{z} x}}{\sqrt{z}}\right)'' = \phi''(x)$$

and

$$A_2(\phi, y) = Ly_z + z \phi'(0_+) (-L + z)^{-1}w$$

$$= Ly + (-L(-L + z)^{-1} + z(-L + z)^{-1}) w \phi'(0_+)$$

$$= Ly + w \phi'(0_+).$$

\[\square\]

Let us define the (eventually empty) set

$$\sigma_w(L) := \{\lambda \in \sigma(L) : w \in \mathfrak{h}_\lambda^+\},$$

where $\mathfrak{h}_\lambda$ denoted the spectral subspace relative to $\lambda$. 
Theorem 2.2.

\[ \sigma_{ess}(A) = \sigma_{ac}(A) = (-\infty, 0], \]

\[ \sigma_{pp}(A) = \sigma_{w}(L) \cup \{ \lambda \in \rho(L) \cap (0, +\infty) : \theta + \Gamma(\lambda) = 0 \} . \]

Proof. The properties regarding the essential and continuous spectrum are more or less standard and can be obtained proceeding as in [4], Theorem 2.3. Let us now come to the point spectrum.

1. Let \( \lambda \in \sigma(L) \). Then \((0, y_{\lambda})\) is an eigenvector if \( y_{\lambda} \) solves the equations

\[ Ly_{\lambda} = \lambda y_{\lambda}, \quad \langle w, y_{\lambda} \rangle = 0, \]

thus \( \lambda \in \sigma_{w}(L) \).

2. Let \( \lambda > 0 \). Then \( \phi_{\lambda}(x) := e^{-\sqrt{\lambda}x} \) solves \( \phi''_{\lambda} = \lambda \phi_{\lambda} \). Thus \((\phi_{\lambda}, y_{\lambda})\) is an eigenvector if \( \lambda \) and \( y_{\lambda} \) solve the equations

\[(2.5) \quad Ly_{\lambda} - \sqrt{\lambda} w = \lambda y_{\lambda},
\]

\[ -\theta \sqrt{\lambda} + 1 - \langle w, y_{\lambda} \rangle = 0. \]

If \( \lambda \in \rho(L) \) then

\[ y_{\lambda} = -\sqrt{\lambda} (-L + \lambda)^{-1} w \]

and \( \lambda \) must solve the equation

\[ -\theta \sqrt{\lambda} + 1 + \sqrt{\lambda} \langle w, (-L + \lambda)^{-1} w \rangle = 0. \]

If otherwise \( \lambda \in \sigma(L) \) then \((2.5)\) can be solved only if \( w \in h_{\lambda}^{\perp} \) by \( y_{\lambda} = y_{\lambda}^{\parallel} + y_{\lambda}^{\perp} \), where \( y_{\lambda}^{\parallel} \in h_{\lambda} \) and \( y_{\lambda}^{\perp} \in h_{\lambda}^{\perp} \) is defined by

\[ y_{\lambda}^{\perp} := -\sqrt{\lambda} (-L_{\lambda} + \lambda)^{-1} w, \quad L_{\lambda} := (1 - P_{\lambda})L(1 - P_{\lambda}) : h_{\lambda}^{\perp} \rightarrow h_{\lambda}^{\perp}. \]

Thus \( \lambda \in \sigma_{w}(L) \) and moreover it has to solve the equation

\[ -\theta \sqrt{\lambda} + 1 + \sqrt{\lambda} \langle w, (-L_{\lambda} + \lambda)^{-1} w \rangle = 0. \]

\[ \square \]

Remark 2.3. When \( \sigma_{w}(L) \) is empty, i.e. in the generic situation, the point spectrum of the interacting operator \( A \) is quite different from the point spectrum of the decoupled one, \( A_{0} \). In particular, the free eigenvalues of the finite dimensional subsystem disappear, and in their place could possibly appear the real solutions of the equation \( \Gamma(\lambda) + \theta = 0 \). In fact, as we shall see in Section 6, the disappeared eigenvalues, which for the uninteracting operator \( A_{0} \) are immersed in the continuum spectrum, become resonances of the interacting operator.
Remark 2.4. In the case $\sigma_w(L) \neq \emptyset$ we can suppose, without loss of generality, $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$, $L = L_1 \oplus L_2$ and $w = w_1 \oplus 0$. Then for the self-adjoint extensions given in Theorem 2.1 we have $A = A_1 \oplus L_2$, where

$$A_1 : D(A_1) \subset L^2(\mathbb{R}_+) \oplus \mathfrak{h}_1 \to L^2(\mathbb{R}_+) \oplus \mathfrak{h}_1$$

is defined by

$$D(A_1) := \left\{ (\phi, y_1) \in H^2(\mathbb{R}_+) \oplus \mathfrak{h}_1 : \theta \phi'(0_+) + \phi(0_+) = \langle w_1, y_1 \rangle_{\mathfrak{h}_1} \right\},$$

$$A_1(\phi, y_1) := (\phi'', L_1 y_1 + w_1 \phi'(0_+)),$$

i.e. the dynamics on $\mathfrak{h}_2$ is trivial, in the sense that it is decoupled from the field one. Thus the hypothesis $\sigma_w(L) \neq \emptyset$ is equivalent to the existence of a subspace on which the particles and the field are uncoupled. In other words the points of the pure point spectrum belonging to $\sigma_w(L)$ correspond to "radiationless motions", in which the interaction between oscillators and field is decoupled. Similar exceptional solutions in which the subsystem oscillates at a normal frequency of the decoupled system are known, for example, also in the classical electrodynamics of an extended charge where they are called Bohm-Weinstein modes. In that case the coupling between field and particle is defined by the charge density $\rho(x)$ of the particle, and the condition to have radiationless modes of frequency $\omega$ is that the Fourier transform of the form factor satisfies $\hat{\rho}(\omega) = 0$.

Remark 2.5. The operator $A$ can be interpreted in a formal way as a differential operator with an eigenvalue dependent boundary condition. Let us consider the secular equation for the operator $A$ and in particular its finite dimensional component, and couple it with the boundary condition for elements of the domain of the operator. We get

$$Ly + w \phi'(0_+) = \lambda y$$

$$\theta \phi'(0_+) + \phi(0_+) = \langle w, y \rangle$$

From the first equation it follows $y = -\phi'(0_+) (L - \lambda)^{-1} w$ and substituting in the second equation one gets

$$(\theta + \langle w, (L - \lambda)^{-1} w \rangle) \phi'(0_+) + \phi(0_+) = 0$$

which is, formally, a Robin boundary condition for the field at the origin. The condition contains the eigenvalue $\lambda$ and it is is known in the physical and mathematical literature as an energy dependent boundary condition. From this point of view, the boundary value problem for the coupled operator can be reduced to a boundary value problem for the field only, but eigenvalue dependent.
By the way note that, as it should be, the above eigenvalue dependent boundary condition is equivalent to the eigenvalue equation in Theorem 2.3, \( \Gamma(\lambda) + \theta = 0 \), as it is immediately seen by the position \( \sqrt{z} = \lambda \).

3. The Hamiltonian structure

In this section we are interested in describing the Hamiltonian structure of the dynamical system related to the abstract wave equation

\[
\ddot{\Psi} = A\Psi, \quad \Psi \equiv (\phi, y).
\]

The solution of the corresponding Cauchy problem is then given by the symplectic flow generated by the linear operator \( \begin{pmatrix} 0 & 1 \\ A & 0 \end{pmatrix} \).

We begin by determining the quadratic form corresponding to \(-A\):

**Theorem 3.1.** Let us denote by \(Q\) the quadratic form of \(-A\).

1. If \(\theta = 0\) then

\[
D(Q) = \{ (\phi, y) \in H^1(\mathbb{R}_+) \oplus h : \phi(0_+) = \langle w, y \rangle \},
\]

\[
Q : D(Q) \to \mathbb{R}, \quad Q(\phi, y) = \|\phi\|^2_2 - \langle Ly, y \rangle.
\]

2. If \(\theta \neq 0\) then \(D(Q) = H^1(\mathbb{R}_+) \oplus h\) and

\[
Q : H^1(\mathbb{R}_+) \oplus h \to \mathbb{R}, \quad Q(\phi, y) = \|\phi\|^2_2 - \langle Ly, y \rangle - \frac{1}{\theta} |\phi(0_+) - \langle w, y \rangle|^2.
\]

**Proof.** For any \((\phi, y) \in D(A)\) one has

\[
Q(\phi, y) = \|\phi\|^2_2 - \langle Ly, y \rangle + (\phi^*)(0_+)(\phi(0) - \langle w, y \rangle)
\]

\[
= \|\phi\|^2_2 - \langle Ly, y \rangle - \theta |\phi'(0_+)|^2.
\]

Thus the proof is done if \(Q\) is bounded from below and closed. This follows from

\[
|\phi(0_+)|^2 \leq a \|\phi\|^2_2 + b \|\phi\|^2_2, \quad a > 0, \ 0 < b < 1.
\]

\[\square\]

Let us make \(D(Q) \subset L^2(\mathbb{R}^3) \oplus h\) a Banach space with norm

\[
\| (\phi, y) \|^2_Q := Q(\phi, y) + (\sup \sigma(A) + 1)(\|\phi\|^2 + \|y\|^2)
\]

and define

\[\mathcal{H}_\phi := D(Q) \oplus L^2(\mathbb{R}_+) \oplus h.\]

Then one has the following
**Theorem 3.2.** The linear operator
\[
\begin{pmatrix}
0 & 1 \\
A & 0
\end{pmatrix} : D(A) \oplus D(Q) \subset \mathcal{H}_0 \to \mathcal{H}_0,
\]
is the generator of a strongly continuous group of evolution
\[U^t : \mathcal{H}_0 \to \mathcal{H}_0\]
which preserves the energy
\[E((\phi, y), (\dot{\phi}, \dot{y})) := \frac{1}{2} \left( Q(\phi, y) + \|\dot{\phi}\|_2^2 + \|\dot{y}\|_2^2 \right).\]

Such an operator is the Hamiltonian linear vector field corresponding to the quadratic Hamiltonian \(E\) via the canonical symplectic form on \(L^2(\mathbb{R}_+) \oplus h \oplus L^2(\mathbb{R}_+) \oplus h\) given by
\[
\Omega((\phi_1, y_1, \dot{\phi}_1, \dot{y}_1), (\phi_2, y_2, \dot{\phi}_2, \dot{y}_2)) := \langle \phi_1, \dot{\phi}_2 \rangle_2 - \langle \dot{\phi}_1, \dot{\phi}_2 \rangle_2 + \langle y_1, \dot{y}_2 \rangle - \langle \dot{y}_1, y_2 \rangle.
\]

**Proof.** The operator \(A\) is self-adjoint and bounded from above. Thus the result concerning evolution generation follows from the theory of abstract wave equations (see e.g. [6], chapter 2, section 7). The results about the Hamiltonian structure follows from the theory of linear Hamiltonian systems in infinite dimensions (see e.g. [5], chapter 2). \(\square\)

**Remark 3.3.** The above results can be immediately extended to a nonlinear situation. Indeed, given the potential function \(V\) let us consider the Hamiltonian
\[H : \mathcal{H}_0 \to \mathbb{R},\]
where
\[H((\phi, y), (\dot{\phi}, \dot{y})) = \frac{1}{2} \left( \|\phi'\|_2^2 + \|\dot{\phi}\|_2^2 + \|\dot{y}\|_2^2 \right) + V(y)\]
when \(\theta = 0\) and
\[H((\phi, y), (\dot{\phi}, \dot{y})) = \frac{1}{2} \left( \|\phi'\|_2^2 + \|\dot{\phi}\|_2^2 + \|\dot{y}\|^2 - \frac{1}{\theta} |\phi(0_+) - \langle w, y \rangle|^2 \right) + V(y)\]
when \(\theta \neq 0\).

The non linear Hamiltonian vector field corresponding, via the canonical symplectic form on \(L^2(\mathbb{R}_+) \oplus h \oplus L^2(\mathbb{R}_+) \oplus h\), to \(H\) is given by
\[X_H : D(A) \oplus D(Q) \subset \mathcal{H}_0 \to \mathcal{H}_0,\]
\[X_H((\phi, y), (\dot{\phi}, \dot{y})) := (\dot{\phi}, \dot{y}, \phi'', -\nabla V(y) + w \phi'(0_+)).\]

Obviously \(X_H = X_E + B\), where \(X_E\) is the linear Hamiltonian vector field corresponding to the quadratic Hamiltonian \(E\) and \(B\) is vector field \(B((\phi, y), (\dot{\phi}, \dot{y})) := (0, 0, -(Ly + \nabla V(y)), 0)\). Thus if \(V\) is twice
differentiable then by Segal’s existence theorem (see [17]) \( X_H \) generates a local continuous non linear symplectic flow on \( \mathcal{H}_\circ \). Since \( Q \) is bounded from below, if \( V(y) \geq c_1\|y\|^2 - c_2, \quad c_1 > 0, \quad c_2 \geq 0, \) then such a flow is global.

4. Examples

Example 4.1. The dynamics of the Lamb model (see [10]) we already described in the introduction, given by the equations

\[
\ddot{\phi}(t, x) = \phi''(t, x) \\
M\ddot{y}(t) = -Ky(t) + T\phi'(t, 0+) \\
y(t) = \phi(t, 0+)
\]

is described by the self-adjoint extension \( A \) corresponding to

\[
\dim \mathfrak{h} = 1, \quad \langle x, y \rangle = \frac{M}{T} x^* y, \quad Ly = -\frac{K}{M} y, \quad w = \frac{T}{M}, \quad \theta = 0.
\]

The similar model with \( n \) point masses

\[
\ddot{\phi}(t, x) = \phi''(t, x) \\
M_1\ddot{y}_1(t) = -K_1(y_1(t) - y_2(t)) + T\phi'(t, 0+) \\
M_2\ddot{y}_2(t) = -K_2(y_2(t) - y_3(t)) + K_1(y_1(t) - y_2(t)) \\
\vdots \\
M_{n-1}\ddot{y}_{n-1}(t) = -K_{n-1}(y_{n-1}(t) - y_n(t)) + K_{n-2}(y_{n-2}(t) - y_{n-1}(t)) \\
M_n\ddot{y}_n(t) = -K_n y_n(t) + K_{n-1}(y_{n-1}(t) - y_n(t)) \\
y_1(t) = \phi(t, 0+)
\]

is described by the self-adjoint extension \( A \) corresponding to

\[
\dim \mathfrak{h} = n, \quad \langle x, y \rangle = \frac{1}{T} \sum_{j=1}^n M_j x^*_j y_j,
\]

\[
L = \begin{pmatrix}
-\frac{K_1}{M_1} & \frac{K_1}{M_1} - \frac{K_2}{M_2} & 0 & 0 & \ldots & 0 \\
\frac{K_1}{M_2} - \frac{K_3}{M_3} & -\frac{K_2}{M_2} & 0 & \ldots & 0 \\
0 & \frac{K_2}{M_3} - \frac{K_3}{M_3} & -\frac{K_3}{M_3} & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \ldots & \frac{K_{n-1}}{M_{n-2}} & -\frac{K_{n-2}}{M_{n-2}} - \frac{K_{n-1}}{M_{n-1}} & 0 \\
0 & \ldots & 0 & \frac{K_{n-2}}{M_{n-1}} - \frac{K_{n-1}}{M_{n-1}} & 0 \\
0 & \ldots & 0 & 0 & -\frac{K_{n-1}}{M_n} + \frac{K_n}{M_n}
\end{pmatrix}
\]
\[ w = \left( \frac{T}{M_1}, 0, \ldots, 0 \right), \quad \theta = 0. \]

The corresponding Hamiltonian is given by
\[
H : \{ (\phi, y) \in H^1(\mathbb{R}_+) \oplus \mathbb{C}^n : \phi(0^+) = y_1 \} \oplus L^2(\mathbb{R}_+) \oplus \mathbb{C}^n \to \mathbb{R}
\]
\[
H((\phi, y), (\dot{\phi}, \dot{y})) := \frac{1}{2} \left( \|\dot{\phi}\|_2^2 + \|\phi'\|_2^2 + \frac{1}{T} \sum_{k=1}^n M_k |y_k|^2 + \frac{1}{T} \Lambda y \cdot y \right),
\]
where the matrix \( \Lambda \) is given by
\[
\Lambda = \begin{pmatrix}
K_1 & -K_1 & 0 & \ldots & 0 \\
-K_1 & K_1 + K_2 & -K_2 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \ldots & -K_{n-2} & K_{n-2} + K_{n-1} & -K_{n-1} \\
0 & \ldots & 0 & -K_{n-1} & K_{n-1} + K_n
\end{pmatrix}
\]
and \( \cdot \) denotes the standard inner product in \( \mathbb{C}^n \).

In the following examples we describe models, from classical electrodynamics and theoretical acoustic respectively, which are not interpretable as standard Lamb models in that \( \theta \neq 0 \).

**Example 4.2.** The renormalized Pauli-Fierz model.

A three dimensional charged oscillator characterized by frequency \( \omega \), mass \( m \) and electric charge \( e \) interacting with the electromagnetic field in dipole approximation has a dynamics described, in the point limit and after mass renormalization, by a well defined self-adjoint operator which couples particle momentum and vector electromagnetic potential. Dynamics and its main properties, classical and quantum, are constructed and studied in [3]. This is the point limit of the Pauli-Fierz model for the case of a quadratic potential energy. Due to the dipole approximation, the action of this operator is non trivial (i.e. different from the free uncoupled dynamics) only on the radial component of the field, and by standard decomposition using vector spherical harmonics it turns out that on this monopole subspace, the restricted dynamics for every couple \((\phi, p)\) constituted by a component of the vector potential on the non trivial subspace, and a corresponding component of the particle momentum, is given by the coupled system

\[
\ddot{\phi}(t, r) = \phi''(t, r)
\]
\[
\ddot{p}(t) = -\frac{3m}{2e} \omega^2 \phi(t, 0^+) + \frac{3m}{2e^2} \phi(t, 0^+) = \frac{1}{e} \dot{p}(t).
\]
Writing the field $\phi$ which appears in the evolution equation for $p$ in terms of its derivative $\phi'$ and $p$ by means of the boundary condition one obtains

$$
\ddot{\phi}(t, r) = \phi''(t, r) \\
\ddot{p}(t) = -\omega^2 p(t) + e \omega^2 \phi'(t, 0_+) \\
\frac{2e^2}{3m} \phi'(t, 0_+) + \phi(t, 0_+) = \frac{2e}{3m} p(t).
$$

Let us remark here that by Newton’s law $\dot{p} = -m \omega^2 q$, where $q$ is a component of the particle position. Thus the Cauchy initial datum for $\dot{p}$ is obtained from the initial position.

In conclusion the dynamics of the renormalized Pauli-Fierz model in dipole approximation and with quadratic external potential is described by the self-adjoint operator $A$ corresponding to

$$\dim \mathfrak{h} = 1, \quad \langle x, y \rangle = \frac{2x^* y}{3m \omega^2}, \quad Ly = -\omega^2 y, \quad w = e \omega^2, \quad \theta = \frac{2e^2}{3m}.$$

The corresponding Hamiltonian is given by

$$H : H^1(\mathbb{R}_+) \oplus \mathbb{C} \oplus L^2(\mathbb{R}_+) \oplus \mathbb{C} \to \mathbb{R}$$

$$H((\phi, p), (\dot{\phi}, \dot{p})) := \frac{1}{2} \left( \| \dot{\phi} \|^2 + \| \phi' \|^2 \right) + \frac{1}{3m} \left( \frac{|\dot{p}|^2}{\omega^2} + |p|^2 - \frac{3m}{2e} |\phi(0_+)|^2 - |p|^2 \right).$$

As recalled in the introduction, a field-particle interaction which reduces to the standard ($\theta = 0$) Lamb model in the point limit and after spring constant renormalization is the Schwabl-Thirring model, in which a scalar field interacts with a scalar oscillator (for details, in a different framework, see [12]).

**Example 4.3.** A spherical elastic shell in the acoustic field.

Let us consider the exterior problem for a spherical shell of mass $M$, radius $R_0$ and constant surface density $\rho = \frac{M}{4\pi R_0^2}$ undergoing radial motion only, and interacting with an irrotational acoustic field in the linear approximation. The shell is elastic, i.e. on every surface element acts a restoring force proportional to the radius, and of Young modulus $K$. If small radial oscillations around $R_0$ are considered, introducing the variables $\psi$, related to the acoustic potential $\phi$ by $\phi(R_0 + r) = \frac{\psi(r)}{r}$, $r > 0$, and the radius $R_0 + R(t)$ and taking into account the continuity
of velocity $\phi'$ of the acoustic field at the boundary one obtains the
equations of motion (assuming propagation velocity equal to one)
\[
\ddot{\psi}(t, r) = \psi''(t, r)
\]
\[
M\ddot{R}(t) = -KR(t) + 4\pi R_0 \rho \psi(t, 0+)
\]
\[
\frac{\psi'(t, 0+)}{R_0} - \frac{\psi(t, 0+)}{R_0^2} = \dot{R}(t).
\]
This system can be converted to a generalized Lamb system by intro-
ducing the new variable (in fact a sort of total momentum)
\[
P := MR - 4\pi R_0 \rho \psi(0+).
\]
Re-writing the above equations in terms of the new variable $P$, ex-
pressing the field $\psi$ which appears in the evolution equation for $P$ in
terms of its derivative $\psi'$ and $P$ by means of the boundary condition,
one obtains
\[
\ddot{\psi}(t, r) = \psi''(t, r)
\]
\[
\ddot{P}(t) = -\frac{K}{M + 4\pi R_0^3 \rho} P(t) - \frac{4\pi K R_0^2 \rho}{M + 4\pi R_0^3 \rho} \psi'(t, 0+) - \frac{M R_0}{M + 4\pi R_0^3 \rho} \psi'(t, 0+) + \psi(t, 0+) = -\frac{R_0^2}{M + 4\pi R_0^3 \rho} P(t).
\]
As regards the initial datum for $\dot{P}$ it can be recovered from the one for
$R$ as in the previous example.

By defining $\omega^2 := \frac{K}{M}$ the above system is described by the self-adjoint
operator $A$ corresponding to
\[
\dim \mathfrak{h} = 1, \quad \langle x, y \rangle = \frac{x^* y}{4\pi K}, \quad Ly = -\frac{\omega^2}{1 + R_0} y,
\]
\[
w = -\frac{4\pi \omega^2 R_0^2}{1 + R_0}, \quad \theta = -\frac{R_0}{1 + R_0}.
\]
The corresponding Hamiltonian is given by
\[
H : H^1(\mathbb{R}_+) \oplus \mathbb{C} \oplus L^2(\mathbb{R}_+) \oplus \mathbb{C} \to \mathbb{R}
\]
\[
H((\psi, P), (\dot{\psi}, \dot{P})) := \frac{1}{2} \left( \|\dot{\psi}\|^2 + \|\psi'\|^2 \right) + \frac{1}{8\pi M} \left( \frac{\|\dot{P}\|^2}{\omega^2} + \frac{\|P\|^2}{1 + R_0} \right) + \frac{1 + R_0}{2R_0} \left| \psi(0+) + \frac{R_0^2}{1 + R_0} \frac{P}{M} \right|^2.
\]
An analysis of interaction of elastic surfaces with acoustic fields from
the point of view of Lax-Phillips scattering theory is given in [2]. For
a study of one dimensional models in acoustics in the framework of self-adjoint extensions we refer to [4].

5. Wave equations with high-order boundary conditions

From now on we will consider self-adjoint operators $A$ which are self-adjoint extensions corresponding (according to Theorem 2.1) to $L$'s and $w$'s such that

$$\{L^k w\}_{0}^{n-1}$$

is a basis in $\mathcal{h}$.

Note that the examples given in Section 4 satisfy such hypothesis.

With respect to the orthonormal base obtained from $\{L^k w\}_{0}^{n-1}$ by the Schmidt orthogonalization procedure the linear operator $L$ is represented by a Jacobi matrix. However we prefer to consider here the unitary isomorphism $\mathcal{h} \simeq \mathbb{C}^n$ induced by the orthonormal system $\{\hat{e}_i\}_{1}^{n}$ made of the eigenvectors of $L$. For any vector $y \in \mathcal{h}$ and for any linear operator $M : \mathcal{h} \to \mathcal{h}$ we pose

$$y \equiv (y_1, \ldots, y_n), \quad M \equiv \begin{pmatrix} M_{11} & \ldots & M_{1n} \\ \vdots & \ddots & \vdots \\ M_{n1} & \ldots & M_{nn} \end{pmatrix}.$$ 

With these notations

$$w \equiv (w_1, \ldots, w_n), \quad L \equiv \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & \lambda_n \end{pmatrix},$$

where $\sigma(L) = \{\lambda_1, \ldots, \lambda_n\}$. We introduce the diagonal matrix

$$W \equiv \begin{pmatrix} w_1 & 0 & \ldots & 0 \\ 0 & w_2 & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & w_n \end{pmatrix},$$

the Vandermonde matrix

$$V \equiv \begin{pmatrix} 1 & 1 & \ldots & 1 \\ \lambda_1 & \lambda_2 & \ldots & \lambda_n \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \ldots & \lambda_n^{n-1} \end{pmatrix},$$

and then we define

$$M := VW.$$
Since
\[ \det M = \prod_{1 \leq i < j \leq n} (\lambda_j - \lambda_i) \prod_{1 \leq i \leq n} w_i \neq 0 , \]
our hypothesis (5.1) is equivalent to
\[ \lambda_i \neq \lambda_j \quad \text{and} \quad \sigma_w(L) = \emptyset . \]
Thus under our hypothesis the spectrum of \( L \) is simple and, by Theorem 2.2, \( A \) has no eigenvalue immersed in the continuous spectrum.

Let us denote by \( S(\mathbb{R}_+) \) the space of rapidly decreasing smooth functions on \([0, +\infty)\). We define the dense subspace \( D \subset H^\circ \) by
\[ D := \{ (\phi, y, \dot{\phi}, \dot{y}) \in H^\circ : \phi \in S(\mathbb{R}_+), \dot{\phi} \in S(\mathbb{R}_+), \]
\[ y = M^{-1}v(\phi), \dot{y} = M^{-1}v(\dot{\phi}) \} , \]
where
\[ v(\phi) = \sum_{k=1}^{n} p_k(\partial_x)\phi(0_+) \hat{e}_k , \]
and \( p_k(\partial_x) \) is the differential operator with constant coefficients associated to the polynomial recursively defined by
\[ p_1(z) = \theta z + 1 , \quad p_k(z) = z^2 p_{k-1}(z) - \langle w, L^{b-2}w \rangle z , \quad k \geq 2 . \]

The next theorem is the main technical point as regards the successive study of the Lax-Phillips scattering of generalized Lamb models. It says that \( D \) is invariant under the flow \( U^t_0 \) and that on this dense subspace a generalized Lamb model is equivalent to a wave equation with a high order boundary condition at zero.

**Theorem 5.1.** Let \( U^t_0 \) the strongly continuous group of evolution provided by Theorem 3.2. Then
\[ U^t_0 : D \to D \]
and
\[ U^t_0(\phi_0, M^{-1}v(\phi_0), \dot{\phi}, M^{-1}v(\dot{\phi}_0)) \]
\[ = (\phi(t), M^{-1}v(\phi(t)), \dot{\phi}(t), M^{-1}v(\dot{\phi}(t))) \]
where \( \phi(t, x) \) solves the equations
\[ \partial_t \phi(t, x) = \partial_{xx} \phi(t, x) , \quad x > 0 , \]
\[ (5.2) \]
\[ p(\partial_x) \phi(0_+, 0) = 0 \]
\[ \phi(0, x) = \phi_0(x) , \quad \dot{\phi}(0, x) = \dot{\phi}_0(x) . \]
Here \( p(\partial_x) \) denotes the constant coefficients differential operator of degree \( 2n + 1 \) (\( 2n \) if \( \theta = 0 \)) associated to the polynomial
\[
p(z) = p_{n+1}(z) - \sum_{i,j=1}^n \lambda_i^n (V^{-1})_{ij} p_j(z) .
\]

**Proof.** Let \((\phi(t), \dot{\phi}(t))\) be the solution of the Cauchy problem
\[
\begin{align*}
\frac{d^2}{dt^2} (\phi(t), y(t)) &= A(\phi(t), y(t)) \\
(\phi(0), y(0)) &= (\phi_0, y_0) , \\
(\dot{\phi}(0), \dot{y}(0)) &= (\dot{\phi}_0, \dot{y}_0)
\end{align*}
\]
(5.3) with \((\phi_0, y_0, \dot{\phi}_0, \dot{y}_0) \in D\) and let us suppose that \((\phi(t), \dot{\phi}(t))\) is in \( S(\mathbb{R}_+) \) for all times. Then deriving with respect to time the boundary condition \(2k\)-times, \( k = 0, \ldots, n \), using \( \dot{\phi}(t, 0) = \phi''(t, 0_+) \), one obtains the \( n + 1 \) equations
\[
\langle w, L^k y(t) \rangle = p_{k+1}(\partial_x) \phi(t, 0_+) , \quad k = 0, 1, \ldots, n .
\]
The first \( n \) of such equations can be can be rewritten as
\[
My(t) = v(\phi(t))
\]
so that
\[
y(t) = M^{-1} v(\phi(t)) , \quad \dot{y}(t) = M^{-1} v(\dot{\phi}(t))
\]
for all times. Moreover, inserting the expression for \( y(t) \) into the \( n'\)th equation one obtains \( p(\partial_x) \phi(t, 0_+) = 0 \), so \( \phi \) satisfies \((5.2)\).

Conversely let \( \phi(t) \) be the solution of \((5.2)\) and put
\[
y(t) := M^{-1} v(\phi(t)) .
\]
Then the \( n \) equations
\[
\langle w, L^k y(t) \rangle = p_{k+1}(\partial_x) \phi(t, 0_+) , \quad k = 0, 1, \ldots, n - 1 .
\]
are satisfied. The first equation says that \( (\phi, y) \) and \( (\dot{\phi}, \dot{y}) \) are in \( D(A) \). Deriving each equation two times with respect to time and using \( \ddot{\phi}(t, 0_+) = \phi''(t, 0_+) \) one obtains
\[
\langle w, L^k \ddot{y}(t) \rangle = p_{k+1}(\partial_x) \phi''(t, 0_+) = p_{k+2}(\partial_x) \phi(t, 0_+) + \langle w, L^k w \rangle \phi'(t, 0_+)^2 \\
= \langle w, L^k (Ly(t) + w\phi'(t, 0_+)) \rangle , \quad k = 0, 1, \ldots, n - 1 ,
\]
which implies \( \ddot{y}(t) = Ly(t) + w\phi'(t, 0_+) \). So \((\phi, y)\) is the solution of \((5.3)\). This also show, by unicity, that if the initial conditions of \((5.3)\) are in \( S(\mathbb{R}_+) \) then they are in \( S(\mathbb{R}_+) \) for all times. This justifies the assumption we made at the beginning of the proof. \( \square \)
The next lemma makes more explicit the polynomial $p$ appearing in the previous theorem:

**Lemma 5.2.**

$$p(z) = (\theta z + 1) \det(z^2 - L) - z \sum_{j=1}^{n} \left( \sum_{k=1}^{\bar{j}} a_{j-k} \langle w, L^{k-1}w \rangle \right) z^{2(n-j)},$$

where

$$a_0 = 1, \quad a_j := (-1)^j \sum_{i_1 < \cdots < i_j} \lambda_{i_1} \cdots \lambda_{i_j}, \quad 1 \leq j \leq n.$$ 

**Proof.** Put

$$\tilde{a}_j := -\sum_{i=1}^{n} \lambda_i^n (V^{-1})_{ij}, \quad 1 \leq j \leq n, \quad \tilde{a}_{n+1} := 1$$

and

$$b_{jk} := -\sum_{i=1}^{n} \lambda_i^{k-j}|w_i|^2, \quad 1 \leq k \leq n+1, \quad 1 \leq j \leq k-1.$$ 

By the definitions of $p(z)$, $p_k(z)$, $\tilde{a}_j$ and $b_{jk}$ one has

$$p(z) = (\theta z + 1) \sum_{j=1}^{n+1} \tilde{a}_j z^{2(j-1)} + z \left( \sum_{j=2}^{n} b_{j,n+1} z^{2(j-1)} + \sum_{j=1}^{n} \tilde{a}_j \sum_{i=1}^{j-1} b_{ij} z^{2(j-1)} \right)$$

$$= (\theta z + 1) \sum_{j=1}^{n+1} \tilde{a}_j z^{2(j-1)} + z \sum_{j=1}^{n} \left( \sum_{k=j+1}^{n+1} b_{jk} \tilde{a}_k \right) z^{2(j-1)}$$

$$= (\theta z + 1) p_a(z^2) + z p_b(z^2),$$

where

$$p_a(z) = \sum_{j=1}^{n+1} \tilde{a}_j z^{j-1} \equiv \sum_{j=0}^{n} a_j z^{n-j},$$

$$p_b(z) = \sum_{j=1}^{n} b_j z^{j-1} \equiv \sum_{j=1}^{n} b_j z^{n-j},$$

$$\tilde{b}_j = -\sum_{k=1}^{n+1} \sum_{i=j+1}^{n} \lambda_i^{k-j-1} a_i |w_k|^2 = -\sum_{i=j+1}^{n+1} \tilde{a}_i \langle w, L^{i-j-1}w \rangle.$$ 

Since

$$\sum_{j=1}^{n+1} \tilde{a}_j \lambda_i^{j-1} = \lambda_i^n - \sum_{j=1}^{n} \sum_{i=1}^{n} \lambda_i^n (V^{-1})_{ij} V_{jk} = \lambda_k^n - \lambda_k^n = 0,$$
The eigenvalues $\lambda_1, \ldots, \lambda_n$ are the roots of $p_a$. Thus
\[ \tilde{a}_j = \sum_{i_1 < \cdots < i_{n-j+1}} (-1)^{n-j+1} \lambda_{i_1} \cdots \lambda_{i_{n-j+1}}, \quad 1 \leq j \leq n, \quad \tilde{a}_{n+1} = 1. \]
\[ \square \]

The next result gives the relation between the roots of $p$ and eigenvalues and resonances of the self-adjoint extension $A$:

**Lemma 5.3.** Suppose $\det L \neq 0$ and let us define the couple $(\phi, y)$ by
\[ \phi(x) := e^{zx}, \quad y := M^{-1}v(\phi). \]
Then
\[ p(z) = 0 \iff \begin{cases} \phi'' = z^2 \phi, \\ Ly + w\phi'(0_+) = z^2 y \\ \theta\phi'(0_+) + \phi(0_+) = \langle w, y \rangle, \end{cases} \implies z \in \mathbb{C} \setminus i\mathbb{R}. \]

Hence
\[ p(z) = 0, \quad \text{Re}(z) \leq 0, \iff z = -\sqrt{\lambda}, \quad \lambda \in \sigma_{pp}(A) \cap (0, +\infty). \]

**Proof.** Since $\phi'' = z^2 \phi$ implies $z \neq 0$ and $p(0) = (-1)^n \det L \neq 0$, we can take $z \neq 0$. By the definition of $(\phi, y)$ we only need to show that
\[ p(z) = 0 \iff Ly + wz = z^2 y^2. \]
Moreover, beside $\langle w, y \rangle = p_1(z)$, one has
\[ \langle w, L^k y \rangle = p_{k+1}(z) = z^2 p_k(z) - \langle w, L^{k-1} w \rangle z, \quad k = 1, \ldots, n - 1. \]
The above equalities, together with $p(z) = 0$, gives
\[ \langle w, L^n y \rangle = p_{n+1}(z) = z^2 p_n(z) - \langle w, L^{n-1} w \rangle z \]
Thus
\[ \langle w, L^k(Ly + wz - z^2 y) \rangle = 0, \quad k = 0, \ldots, n - 1, \]
i.e. $Ly + wz = z^2 y$. Reversing the above argument one has that
\[ Ly + wz = z^2 y \] implies $p(z) = 0$.
Suppose now $p(iv) = 0$, $v \in \mathbb{R}$, so that $Ly + ivw = -v^2 y$. Since $\langle w, y \rangle = iv\theta + 1$ we have
\[ \langle (L + v^2) y, y \rangle = v^2 \theta - iv. \]
Since $L + v^2$ is symmetric we have $v = 0$. But $v \neq 0$ by $\det L \neq 0$. \[ \square \]

**Remark 5.4.** By the previous lemma we have that the polynomial $p$ has no purely imaginary roots. The ones in the left half-plane are real and correspond to eigenvalues, the ones in the right half plane give rise to non-normalizable solutions of the eigenvalue equations and correspond to resonant states.
6. LAX-PHILLIPS SCATTERING

From now on we suppose that

\[(6.1) \quad \sigma_{pp}(A) = \emptyset.\]

Here we remark that the successive results hold, with the appropriate modifications, even without this hypothesis which is just a convenient choice in order to simplify the exposition.

Since we already supposed \(\sigma_w(L) = \emptyset\), by Theorem 2.2 the above hypothesis means

\[\{ \lambda \in \rho(L) \cap (0, +\infty) : \theta + \Gamma(\lambda) = 0 \} = \emptyset,\]

i.e. we are supposing that there are no strictly positive solutions \(x\) of the equation

\[\frac{1}{x} + \sum_{j=1}^{n} \frac{|w_j|^2}{-\lambda_j + x^2} = \theta.\]

This is true if and only if

\[\sigma(L) \subset (-\infty, 0), \quad \theta \leq 0.\]

Thus hypothesis \((6.1)\) is satisfied by both Examples 4.1 and 4.3.

By Lemma 5.3 \((6.1)\) implies that the polynomial \(p\) has no negative real root and that the complex ones (which appear in complex-conjugate pairs since \(p\) has real coefficients) are all contained in the right half-plane. Thus

\[\sigma_{pp}(A) = \emptyset \iff \text{Roots}(p) \subset \{ z \in \mathbb{C} : \Re(z) > 0 \} .\]

By functional calculus, since \(A\) is injective and negative by \((6.1)\) we have that

\[U^t_0 = \begin{pmatrix} \cos t\sqrt{-A} & \sqrt{-A^{-1}} \sin t\sqrt{-A} \\ -\sqrt{-A} \sin t\sqrt{-A} & \cos t\sqrt{-A} \end{pmatrix}.\]

Moreover \(U^t_0\) extends to a strongly continuous unitary group

\[U^t : \mathcal{H} \to \mathcal{H},\]

where \(\mathcal{H}\) is the Hilbert space given by the completion of \(\mathcal{H}_0\) with respect to the scalar product corresponding to energy norm

\[\|(\phi, y, \dot{\phi}, \dot{y})\|_E := E(\phi, y, \dot{\phi}, \dot{y})^{1/2}.\]

**Remark 6.1.** By Theorem 2.2 our hypothesis \(\sigma_{pp}(A) = \emptyset\) says that \(A\) has purely absolutely continuous spectrum. Thus

\[\lim_{t \to \pm \infty} \|y(t)\| = 0.\]
Indeed by functional calculus and Riemann-Lebesgue lemma, denoting by $P(d\lambda)$ the projection-valued measure corresponding to $A$, one has, for any $(\phi, y, \dot{\phi}, \dot{y}) \in \mathcal{H}$,

$$\lim_{t \to \pm \infty} \lambda_i y_i(t) = \lim_{t \to \pm \infty} \langle \sqrt{-A}(0, \hat{e}_i), \sqrt{-A} \cos t \sqrt{-A}(\phi, y) + \sin t \sqrt{-A}(\dot{\phi}, \dot{y}) \rangle_{L^2(\mathbb{R}^+) \otimes \mathbb{H}}$$

$$= \lim_{t \to \pm \infty} \int_0^\infty \cos t \lambda \langle \sqrt{-A}(0, \hat{e}_i), P(d\lambda) \sqrt{-A}(\phi, y) \rangle_{L^2(\mathbb{R}^+) \otimes \mathbb{H}}$$

$$+ \lim_{t \to \pm \infty} \int_0^\infty \sin t \lambda \langle \sqrt{-A}(0, \hat{e}_i), P(d\lambda)(\dot{\phi}, \dot{y}) \rangle_{L^2(\mathbb{R}^+) \otimes \mathbb{H}} = 0$$

Analogously one has

$$\lim_{t \to \pm \infty} \|\dot{y}(t)\| = 0.$$

Define

$$\mathcal{F} := \{(f_-, f_+) \in \mathcal{S}_0(\mathbb{R}) \times \mathcal{S}_0(\mathbb{R}) : p(\partial_x) f_- + p(-\partial_x) f_+ = 0\},$$

where

$$\mathcal{S}_0(\mathbb{R}) := \left\{ f \in \mathcal{S}(\mathbb{R}) : \int_\mathbb{R} f(x) \, dx = 0 \right\}$$

$$\equiv \left\{ f \in \mathcal{S}(\mathbb{R}) : f = g', \ g \in \mathcal{S}(\mathbb{R}) \right\}.$$

By considering the Fourier transform (denoted by $\hat{\cdot}$) of the differential equation

$$p(\partial_x) f_- + p(-\partial_x) f_+ = 0$$

one obtains

$$p(i\kappa) \hat{f}_-(\kappa) + p(-i\kappa) \hat{f}_+(\kappa) = 0.$$

Thus one has the following result, which permits to define what will be the scattering operator.

**Lemma 6.2.**

$$(f_-, f_+) \in \mathcal{F} \iff \hat{f}_+(\kappa) = -\frac{p(i\kappa)}{p(-i\kappa)} \hat{f}_-(\kappa).$$

Since $p$ has real coefficients, we have

$$\left| \frac{p(i\kappa)}{p(-i\kappa)} \right| = 1,$$

therefore

$$S_p : \mathcal{S}_0(\mathbb{R}) \to \mathcal{S}_0(\mathbb{R}), \quad (S_p f)(k) := -\frac{p(i\kappa)}{p(-i\kappa)} \hat{f}(k).$$
extends to an unitary map on $L^2(\mathbb{R})$.

Let $\phi(t)$ be the solution of (5.2) with initial data $(\phi, y, \dot{\phi}, \dot{y}) \in \mathcal{D}$. Then

$$\phi(t, x) = a(x + t) + b(t - x),$$

where the couple $(a, b)$ is determined on an half-line up to a constant $c$ by

(6.3) \quad a(x) = -\frac{1}{2} \int_{x}^{\infty} (\dot{\phi}(y) + \phi'(y)) \, dy + c, \quad x \geq 0,

(6.4) \quad b(-x) = \frac{1}{2} \int_{x}^{\infty} (\dot{\phi}(y) - \phi'(y)) \, dy - c, \quad x \geq 0.

The functions $a(x)$ and $b(-x)$ are then determined for the remaining values of $x < 0$ by solving the differential equation

(6.5) \quad p(\partial_x)a + p(-\partial_x)b = 0.

The following central result holds:

**Theorem 6.3.** The map $I_\circ \equiv (I^-_\circ, I^+_\circ)$ defined by

$$I_\circ : \mathcal{D} \to \mathcal{F}, \quad I_\circ(\phi, y, \dot{\phi}, \dot{y}) := (f_-, f_+), \quad f_- = a', \quad f_+ = b'.$$

is one-to-one and extends to an unitary map

$$I : \mathcal{H} \to \text{Graph}(S_p).$$

**Proof.** It is easy to check that the map $I_\circ : \mathcal{D} \to \mathcal{F}$ is injective. It is surjective too. Indeed, if $(f_+, f_-) \in \mathcal{F}$ then $I_\circ(\phi, y, \dot{\phi}, \dot{y}) = (f_+, f_-)$ where

$$\begin{align*}
(\phi, y, \dot{\phi}, \dot{y}) &= (\phi, M^{-1}v(\phi), \dot{\phi}, M^{-1}v(\dot{\phi})), \\
\phi(x) &= a(x) + b(-x), \quad \dot{\phi}(x) = a'(x) + b'(-x), \\
a(x) := \int_{-\infty}^{x} f_-(y) \, dy, \quad b(x) := \int_{-\infty}^{x} f_+(y) \, dy.
\end{align*}$$

Let us now show that

$$\| (\phi, y, \dot{\phi}, \dot{y}) \|_E^2 = \| I^+_\circ(\phi, y, \dot{\phi}, \dot{y}) \|_2^2 + \| I^-_\circ(\phi, y, \dot{\phi}, \dot{y}) \|_2^2.$$

Since

$$I^+_\circ = S_p I^-_\circ$$

and $S_p$ is unitary we need to show that

$$\| (\phi, y, \dot{\phi}, \dot{y}) \|_E^2 = 2 \| I^-_\circ(\phi, y, \dot{\phi}, \dot{y}) \|_2^2.$$
Let \((\phi(t, x), y(t), \dot{\phi}(t, x), \dot{y}(t))\) the solution of
\[
\begin{cases}
\ddot{\phi}(t, x) - \phi''(t, x) = 0 \\
\dot{y}(t) - Ly(t) - w\phi'(t, 0_+) = 0 \\
\theta\phi'(t, 0_+) + \phi(t, 0_+) - \langle w, y(t) \rangle = 0
\end{cases}
\]
with initial data \((\phi(x), y, \dot{\phi}(x), \dot{y})\). Then, by using the evolution equation for \(y\) and the boundary conditions, one has
\[
\frac{d}{dt} \left( ||\dot{y}(t)||^2 - \langle y(t), Ly(t) \rangle - \theta|\phi'(0_+)|^2 \right) = \dot{\phi}^*(t, 0_+)\phi'(t, 0_+) + (\phi')^*(t, 0_+)\dot{\phi}(t, 0_+)
\]
and so
\[
\int_0^\infty \left( \dot{\phi}^*(t, 0_+)\phi'(t, 0_+) + (\phi')^*(t, 0_+)\dot{\phi}(t, 0_+) \right) dt = - \left( ||\dot{y}||^2 - \langle y, Ly \rangle - \theta|\phi'(0_+)|^2 \right).
\]
By conservation of energy one has
\[
\begin{align*}
2E(\phi, y, \dot{\phi}, \dot{y}) &= ||\dot{\phi}||_2^2 + ||\phi'||_2^2 + ||\dot{y}||^2 - \langle y, Ly \rangle - \theta|\phi'(0_+)|^2 = \\
&= \int_0^\infty \left( \dot{\phi}^*(x)\dot{\phi}(x) + (\phi')^*(x)\phi'(x) \right) dx \\
&\quad - \int_0^\infty \left( \dot{\phi}^*(t, 0_+)\phi'(t, 0_+) + (\phi')^*(t, 0_+)\dot{\phi}(t, 0_+) \right) dt.
\end{align*}
\]
Inserting in the last equation \(\phi(x, t) = a(t + x) + b(t - x)\) and using \(b' = S_{p}a'\), we have
\[
E(\phi, y, \dot{\phi}, \dot{y}) = 2||b'||_2^2 = 2||I_{p}^\rightarrow(\phi, y, \dot{\phi}, \dot{y})\|_2^2.
\]
We define now the maps \(R_{0}^\pm : \mathcal{D} \to \mathcal{S}_0(\mathbb{R})\) by
\[
R_{0}^\pm(\phi, y, \dot{\phi}, \dot{y})(x) := I_{0}^\pm(\phi, y, \dot{\phi}, \dot{y})(-x)
\]
and the orthogonal spaces \(\mathcal{H}^\pm\) as the closure, with respect to the energy norm, of
\[
\mathcal{D}^\pm := \{ (\phi, y, \dot{\phi}, \dot{y}) \in \mathcal{D} : R_{0}^\pm(\phi, y, \dot{\phi}, \dot{y}) \in \mathcal{S}_0^\pm(\mathbb{R}) \},
\]
where
\[
\mathcal{S}_0^\pm(\mathbb{R}) := \{ f \in \mathcal{S}_0(\mathbb{R}) : f(x) = 0, \ \pm x \leq 0 \}.
\]
The next theorem show that the subspaces \(\mathcal{H}^-\) and \(\mathcal{H}^+\) are incoming and outgoing in the sense of Lax-Phillips scattering theory (see [11],
The proof is a straightforward consequence of the previous definitions.

**Theorem 6.4.** The subspace \( \mathcal{H}^- \) is incoming and the subspace \( \mathcal{H}^+ \) is outgoing, i.e.

\[
\begin{align*}
U^s \mathcal{H}^- & \subset U^t \mathcal{H}^-, \quad s < t \leq 0, \quad \bigcap_{t<0} U^t \mathcal{H}^- = \{0\}, & \bigcup_{t \in \mathbb{R}} U^t \mathcal{H}^- = \mathcal{H}, \\
U^t \mathcal{H}^+ & \subset U^s \mathcal{H}^+, \quad t > s \geq 0, \quad \bigcap_{t>0} U^t \mathcal{H}^+ = \{0\}, & \bigcup_{t \in \mathbb{R}} U^t \mathcal{H}^+ = \mathcal{H}.
\end{align*}
\]

By the previous theorem and by [11], chapter II, sections 2 and 3, there follows

**Theorem 6.5.** The unitary maps

\[
R^\pm : \mathcal{H} \to L^2(\mathbb{R}),
\]

defined as the closures of the maps \( R_\pm \), provide incoming and outgoing translation representations of \( U^t \), i.e.

\[
\begin{align*}
R^\pm U^t (R^\pm)^{-1} & = T^t, \\
R^\pm \mathcal{H}^\pm & = L^2(\mathbb{R}_\pm), \\
S_p^* & = R^+(R^-)^{-1},
\end{align*}
\]

where

\[
T^t : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad T^t f(x) := f(x-t).
\]

**Proof.** The thesis follows from the Theorem 6.3 and from simple computations. Otherwise one can use, as we said, the general theory contained in [11]. \(\square\)

### 7. THE LAX-PHILLIPS SEMIGROUP

We are now interested in the evolution of the states which are neither incoming in the past nor outgoing in the future. To this end one defines

\[
Z^t := PU^t P,
\]

where \( P \) is the orthogonal projection onto

\[
\mathcal{K} := \mathcal{H} \ominus (\mathcal{H}^- \oplus \mathcal{H}^+).
\]

Since \( \mathcal{H}^+ \) and \( \mathcal{H}^- \) are orthogonal it is known (see [11], [15] section XI.11) that \( Z^t \) is a strongly continuous semigroup of contractions on \( \mathcal{K} \) for positive times:

\[
\forall t \geq 0, \quad Z^t : \mathcal{K} \to \mathcal{K}, \quad \|Z^t\| \leq 1, \quad \lim_{t \to \infty} Z^t = 0.
\]
The next theorem completely characterizes such a semigroup (let us remark that, by (6.1), all the roots of $p$ have positive real part):

**Theorem 7.1.** The vector space $K$ is finite dimensional,
$$\dim K = \deg(p).$$
It is generated by the vectors

\[(7.1) \quad (R^+)^{-1} \phi_{kj}, \quad k = 0, 1, \ldots, \nu_j - 1, \quad j = 1, \ldots, m,
\]

where $z_1, \ldots, z_m$ are the roots of the polynomial $p$ and $\nu_1, \ldots, \nu_m$ the respective multiplicities. Moreover

\[Z^t = e^{-tB}, \quad \sigma(B) = \{z_1, \ldots, z_m\},\]

and the matrix representing $B$ with respect to the basis (7.1) is the direct sum

\[B = \bigoplus_{j=1}^m B_j\]

where $B_j$ is the $\nu_j \times \nu_j$ matrix

\[
\begin{pmatrix}
  z_j & 1 & 0 & \ldots & 0 \\
  0 & z_j & 2 & \ldots & \vdots \\
  0 & 0 & z_j & \ddots & 0 \\
  \vdots & \vdots & \ddots & \ddots & \nu_j - 1 \\
  0 & 0 & \ldots & 0 & z_j
\end{pmatrix}.
\]

**Proof.** Since $R_+ : \mathcal{H} \to L^2(\mathbb{R})$ is unitary, $R^+ \mathcal{H}^\pm = L^2(\mathbb{R}_\pm)$ and $R^+(R^-)^{-1} = S_p^*$ we have

\[R^+ K = L^2(\mathbb{R}) \oplus \left( (L^2(\mathbb{R}_-) \cap S_p^* L^2(\mathbb{R}_-)) \oplus L^2(\mathbb{R}_+) \right).\]

By our hypotheses $p(i\zeta) \neq 0$ for any $\zeta$ in the upper complex plane $\mathbb{C}_+$. Thus by Paley-Wiener theorems the analytic extension to $\mathbb{C}_+$ of the Fourier transform of $f \in S_{\mathbb{R}} \cap S_p^* S_{\mathbb{R}}$

\[\hat{f}(\zeta) = \frac{-p(-i\zeta)}{p(i\zeta)} \hat{g}(\zeta),\]

has no poles and has zeroes of order $\nu_j$ at $iz_j$, i.e.

\[
\frac{d^k}{d\zeta^k} \hat{f}(\zeta) \bigg|_{\zeta = iz_j} = \frac{(-i)^k}{\sqrt{2\pi}} \int_{-\infty}^{0} x^k e^{z_j x} f(x) dx = 0, \quad k = 0, 1, \ldots, \nu_j - 1.
\]

Thus

\[L^2(\mathbb{R}_-) \cap S_p^* L^2(\mathbb{R}_-) = \left\{ x^k e^{z_j x}, \quad k = 0, 1, \ldots, \nu_j - 1, \quad j = 1, \ldots, m \right\} \perp\]
and the finite dimensional subspace $\mathcal{K}$ is whose generated by vectors (7.1). These vectors are independent and so the dimension of $\mathcal{K}$ is $\sum_{j=1}^{m} \nu_j = \deg p$. To determine the action of $Z^t$ on $\mathcal{K}$ is enough to note that the evolution of vectors (7.1) in the outgoing representation is given by

$$\phi_{kj}(x - t) \equiv (x - t)^{k-1} e^{z_j(x-t)} \chi(-\infty,0](x - t)$$

Thus

$$\left. \frac{d}{dt} \phi_{kj}(x - t) \right|_{t=0} = - \left( k \phi_{k-1,j}(x) + z_j \phi_{kj}(x) \right)$$

and the proof is done.

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