ON ORBITS OF THE AUTOMORPHISM GROUP ON
A COMPLETE TORIC VARIETY

IVAN BAZHOV

ABSTRACT. Let $X$ be a complete toric variety and $\text{Aut}(X)$ be the automorphism group. We give an explicit description of $\text{Aut}(X)$-orbits on $X$. In particular, we show that $\text{Aut}(X)$ acts on $X$ transitively if and only if $X$ is a product of projective spaces.

1. Introduction

Let us recall that a toric variety is a normal algebraic variety $X$ with a generically transitive and effective action of an algebraic torus $\mathbb{T}$. The complement $X \setminus O_0$ of the open $\mathbb{T}$-orbit $O_0$ is a union of prime $\mathbb{T}$-invariant divisors $D_1, \ldots, D_r$. Every $\mathbb{T}$-orbit on $X$ is uniquely defined by the divisors $D_i$ which contain the orbit. In particular, the number of $\mathbb{T}$-orbits is finite.

It is known that the group of algebraic automorphisms $\text{Aut}(X)$ of a projective or, more generally, complete toric variety $X$ is an affine algebraic group. The group $\text{Aut}(X)$ contains $\mathbb{T}$ as a maximal torus. Hence, the connected component of identity $\text{Aut}^0(X)$ is generated by the torus $\mathbb{T}$ and root subgroups, i.e., one dimensional unipotent subgroups normalized by $\mathbb{T}$. For smooth $X$ a combinatorial description of root subgroups was found by M. Demazure [8]. A generalization of this description to complete simplicial toric varieties in terms of total coordinates was given in [9]. The case of an arbitrary complete toric variety was studied in [5], see also [10]. The paper [4] worked out another method to describe the automorphism group of a projective toric variety.

It is clear that any $\text{Aut}^0(X)$- as well as any $\text{Aut}(X)$-orbit on $X$ is a union of $\mathbb{T}$-orbits. Our goal is to describe $\mathbb{T}$-orbits which lie in the same $\text{Aut}^0(X)$- or $\text{Aut}(X)$-orbit.

Let us associate with a $\mathbb{T}$-orbit $O$ the set of all prime $\mathbb{T}$-invariant divisors $D(O) := \{D_{i_1}, \ldots, D_{i_t}\}$ which do not contain $O$. Let $\Gamma(O)$ be the monoid in the divisor class group $\text{Cl}(X)$ generated by classes of the divisors in $D(O)$.

2010 Mathematics Subject Classification. 14M25, 14L30.

Key words and phrases. toric variety, automorphism, invariant divisor.
Theorem 1. Let $X$ be a complete toric variety. Then two $T$-orbits $O$ and $O'$ on $X$ lie in the same $\text{Aut}^0(X)$-orbit if and only if 
$$\Gamma(O) = \Gamma(O').$$

The proof uses techniques of paper [2]. It is shown there that for an affine toric variety any non-trivial orbit of a root subgroup meets exactly two $T$-orbits and a combinatorial description of such pairs of orbits is given. We adapt these results to a complete toric variety.

Paper [3] gives a systematic treatment of toric varieties in term of bunches of cones. These cones are images of $\Gamma(O)$ in $\text{Cl}(X) \otimes \mathbb{Q}$ and they are Gale dual to cones in $\Delta$.

Theorem 3.6 brings a description of the closures of $\text{Aut}^0(X)$-orbits on $X$. Theorem 3.7 gives a description of $\text{Aut}(X)$-orbits. As a corollary we show that the action of $\text{Aut}(X)$ on $X$ is transitive if and only if $X$ is a product of projective spaces. A related result obtained in [1, Proposition 4.1] states that every complete toric variety which admits a transitive action of a semisimple algebraic group is a product of projective spaces.

The base field $k$ is assumed to be algebraically closed and of characteristic zero.

The author is grateful to his supervisor I. V. Arzhantsev for stating the problem, the idea to apply Gale duality and useful references, and to the referee for his/her careful reading of the manuscript and many helpful comments.

2. Preliminaries

Toric varieties and fans. Let $N \cong \mathbb{Z}^d$ be the lattice of one-parameter subgroups of a torus $T$, $M$ be the dual lattice of characters, and $\langle \cdot, \cdot \rangle : M \times N \to \mathbb{Z}$ be the natural pairing. Also we put $N_\mathbb{Q} = N \otimes_{\mathbb{Z}} \mathbb{Q}$, $M_\mathbb{Q} = M \otimes_{\mathbb{Z}} \mathbb{Q}$.

With any toric variety $X$ one associates a fan $\Delta = \Delta_X$ of rational polyhedral cones in $N_\mathbb{Q}$, e.g., see [9, 7, 11]. Let $\Delta(1) = \{\tau_1, \ldots, \tau_r\}$ be the set of one-dimensional cones in $\Delta$ and $\sigma(1)$ be the set of one-dimensional faces of a cone $\sigma$. The primitive lattice generator of a ray $\tau$ is denoted as $v_\tau$ and $v_i = v_{\tau_i}$.

There exists a one-to-one correspondence $\sigma \leftrightarrow O_\sigma$ between cones $\sigma$ in $\Delta$ and $T$-orbits $O_\sigma$ on $X$ such that $\dim O_\sigma = \dim \Delta - \dim \sigma$. A toric variety $X$ is complete if and only if the fan $\Delta$ is complete.

Every prime $T$-invariant divisor $D_i$ is a closure of an orbit $O_{\tau_i}$ for some $\tau_i \in \Delta(1)$. A $T$-orbit $O_\sigma$ is contained in the intersection of divisors $D_i$ corresponding to rays $\tau_i \in \sigma(1)$.

There is a natural map from the free abelian group $\text{Div}_T(X)$ generated by all prime $T$-invariant divisors to the divisor class group $\text{Cl}(X)$: $D \mapsto [D]$. If $X$ is complete then the following sequence is exact [7].
Theorem 4.1.3:
\[(1) \quad 0 \to M \to \text{Div}_T(X) \to \text{Cl}(X) \to 0,\]
where \(m \in M\) goes to \(\sum_{i=1}^{r} \langle m, v_i \rangle D_i\).

Example 2.1. Let us give some examples of complete toric surfaces.

(i) The projective plane \(\mathbb{P}^2\), the fan \(\Delta_{\mathbb{P}^2}\) is shown in Figure 1(i), \(\Delta_{\mathbb{P}^2}(1)\) is generated by \((1, 0), (0, -1), (-1, 1)\), \(\text{Cl}(\mathbb{P}^2) \cong \mathbb{Z}\).

(ii) The Hirtzebruch surface \(\mathcal{H}_s\), its fan is shown in Figure 1(ii), \(\Delta_{\mathcal{H}_s}(1), s \geq 1,\) is generated by \((1, 0), (0, -1), (-1, s), (0, 1)\), \(\text{Cl}(\mathcal{H}_s) \cong \mathbb{Z}^2\).

(iii) The weighted projective plane \(\mathbb{P}(1, 1, s), s \geq 2,\) is defined by the fan in Figure 1(iii), \(\Delta_{\mathbb{P}(1, 1, s)}(1)\) is generated by \((1, 0), (0, -1), (-1, s), (0, 1)\), \(\text{Cl}(\mathbb{P}(1, 1, s)) \cong \mathbb{Z}\).

(iv) The variety \(\mathcal{B}_s, s \geq 1,\) is defined by the fan in Figure 1(iv), \(\Delta_{\mathcal{B}_s}(1)\) is generated by \((s, 1), (s, -1), (-s, -1), (-s, 1)\), \(\text{Cl}(\mathcal{B}_s) \cong \mathbb{Z}^2 \oplus \mathbb{Z}_{2s}\).

Figure 1. Fans of \(\mathbb{P}^2, \mathcal{H}_s, \mathbb{P}(1, 1, s),\) and \(\mathcal{B}_s\).

Root subgroups and the automorphism group. Let \(X\) be a complete toric variety and \(\Delta\) be the corresponding fan.

Definition 2.2. An element \(m \in M\) is called a Demazure root of \(\Delta\) if the following conditions hold:
\[\exists \tau \in \Delta(1) : \langle m, v_{\tau} \rangle = -1 \text{ and } \forall \tau' \in \Delta(1) \setminus \{\tau\} : \langle m, v_{\tau'} \rangle \geq 0.\]
Let us denote the set of all roots of the fan $\Delta$ as $R$. Denote by $\eta_m$, $m \in R$, the unique ray generator $v_\tau$ such that $\langle m, v_\tau \rangle = -1$. A root $m \in R \cap (-R)$ is called semisimple.

**Example 2.3.** The sets of roots for the varieties of Example 2.1 are easy to picture.

(i) Roots for $P^2$ are pictured in Figure 2(i).
(ii) Roots for $H_s$ are pictured in Figure 2(ii).
(iii) The set of roots for $P(1,1,s)$ and for $H_s$ coincide.
(iv) The set of roots for $B_s$ is empty.

A root $m \in R$ defines a root subgroup $H_m \subset \text{Aut}(X)$, see [2, 6, 11]. The root subgroups $H_m$ are unipotent one-parameter subgroups normalized by $T$.

**Lemma 2.4.** For every point $Q \in X \setminus X^{H_m}$ the orbit $H_m \cdot Q$ meets exactly two $T$-orbits $O_1$ and $O_2$ on $X$ and $\dim O_{\sigma_1} = 1 + \dim O_{\sigma_2}$.

**Proof.** Let $U_i$, $i \in I$, be affine charts corresponding to cones in $\Delta$ which contain $\eta_m$. Due to the proof of [11] Proposition 3.14], the charts $U_i$ are preserved by $H_m$ and the complement $V = X \setminus \bigcup_{i \in I} U_i$ is fixed by $H_m$ pointwise. Applying [2] Proposition 2.1] to affine charts $U_i$ completes the proof. \qed

**Definition 2.5.** A pair of $T$-orbits $(O_{\sigma_1}, O_{\sigma_2})$ from Lemma 2.4 is called $H_m$-connected.

**Lemma 2.6.** ([2] Lemma 2.2]

A pair $(O_{\sigma_1}, O_{\sigma_2})$ is $H_m$-connected if and only if

\[(2) \quad m|_{\sigma_2} \leq 0 \text{ and } \sigma_1 = \sigma_2 \cap m^\perp \text{ is a facet of cone } \sigma_2.\]

\qed

Every automorphism $\psi \in \text{Aut}(N, \Delta)$ of the lattice $N$ preserving the fan $\Delta$ defines an automorphism of $X$ denoted by $P_\psi$. 

![Figure 2. Demazure roots for $P^2$, $H_s$, and $P(1,1,s)$.](image-url)
Theorem 2.7. ([6] Corollary 4.7, [8], [11] page 140) Let $X$ be a complete toric variety. Then $\text{Aut}(X)$ is a linear algebraic group generated by $\mathbb{T}$, $H_m$ for $m \in \mathbb{R}$, and $P_\psi$ for $\psi \in \text{Aut}(N, \Delta)$. Moreover, the group $\text{Aut}^0(X)$ is generated by $\mathbb{T}$ and $H_m$ for $m \in \mathbb{R}$. □

3. Main results

Let $X$ be a complete toric variety defined by the fan $\Delta$ with $\Delta(1) = \{\tau_1, \ldots, \tau_r\}$, $v_i = v_{\tau_i}$, $D_i = D_{\tau_i}$. Let us consider the collection $\tilde{\Delta}$ of all cones generated by some rays in $\Delta(1)$. For a cone $\sigma \in \tilde{\Delta}$ let us define a monoid

$$\Gamma(\sigma) = \sum_{\tau_i \in \Delta(1) \setminus \sigma(1)} \mathbb{Z}_{\geq 0}[D_i].$$

For a $\mathbb{T}$-orbit $\mathcal{O}_\sigma$ the monoids $\Gamma(\mathcal{O}_\sigma)$ (see Introduction) and $\Gamma(\sigma)$ coincide. Let us put

$$\Upsilon(\Delta) = \{\Gamma(\sigma) : \sigma \in \Delta\}.$$

Example 3.1. Let us find $\Upsilon(\Delta)$ for every fan of Example 2.1.

(i) For $\mathbb{P}^2$ there is one monoid $\mathbb{Z}_{\geq 0}$, see Figure 3(i).

(ii) For $\mathcal{H}_s$ there are two monoids of rank 2, see Figure 3(ii).

(iii) For $\mathbb{P}(1, 1, s)$ there are monoids $\mathbb{Z}_{\geq 0}$ and $s\mathbb{Z}_{\geq 0}$, see Figure 3(iii).

(iv) For $\mathbb{B}_s$ there are four monoids generated by 2 elements, four monoids generated by 3 elements and a monoid generated by all 4 elements $[D_1]$, $[D_2]$, $[D_3]$, $[D_4]$. We picture monoids generated by 2 elements in 3-dimensional space remembering that $2s[D_3 - D_1] = 0$, see Figure 3(iv).

Lemma 3.2. Let $\sigma_1$, $\sigma_2 \in \tilde{\Delta}$. If there exists a root $m \in \mathbb{R}$ such that

$$\Delta(1) = \sigma_1(1) \cup \{\tau_k\}$$

holds, then

$$\sigma_1 \in \Delta \iff \sigma_2 \in \Delta.$$

Proof. Let us assume that $\sigma_1 \in \Delta$. There exists a $\mathbb{T}$-orbit $\mathcal{O}_{\sigma_1}$ on $X$. By Lemmas 2.4 and 2.6 there exists an orbit $\mathcal{O}_2$ which meets $H_m \cdot \mathcal{O}_{\sigma_1}$. By conditions (2), the cone $\sigma_1$, and the root $m$ define $\sigma_2$ uniquely and the cone corresponding to $\mathcal{O}_2$ is $\sigma_2$. Hence, $\sigma_2 \in \Delta$.

Conversely, by (2), $\sigma_1$ is a face of $\sigma_2$ and $\sigma_1 \in \Delta$. □

Lemma 3.3. Let $\sigma_1$, $\sigma_2 \in \tilde{\Delta}$ with $\sigma_2(1) = \sigma_1(1) \cup \{\tau_k\}$ for some $1 \leq k \leq r$. Then conditions (2) hold for some root $m \in \mathbb{R}$ if and only if

$$\Gamma(\sigma_1) = \Gamma(\sigma_2).$$

Proof. We have

$$\Gamma(\sigma_1) = \sum_{\tau_i \notin \sigma_1(1)} \mathbb{Z}_{\geq 0}[D_i] = \mathbb{Z}_{\geq 0}[D_k] + \sum_{\tau_i \notin \sigma_2(1)} \mathbb{Z}_{\geq 0}[D_i],$$

$$\Gamma(\sigma_2) = \sum_{\tau_i \notin \sigma_2(1)} \mathbb{Z}_{\geq 0}[D_i],$$

where $D_k$ is the corresponding divisor.
Figure 3. Collections $\Upsilon(\Delta)$ for $P^2$, $H_s$, $\mathbb{P}(1,1,s)$, and $B_s$.

and the condition $\Gamma(\sigma_1) = \Gamma(\sigma_2)$ can be written as

$$[D_k] = \sum_{\tau_i \not\in \sigma_2(1)} a_i [D_i]$$

for some $a_i \in \mathbb{Z}_{\geq 0}$, $\tau_i \not\in \sigma_2(1)$. Let us put $a_k = -1$ and $a_i = 0$ for $\tau_i \in \sigma_1(1)$. Due to exact sequence (1), condition (3) is equivalent to existence of $m \in M$ such that $\langle m, v_i \rangle = a_i$ for all $1 \leq i \leq r$. It means that $m$ is a root and (2) is satisfied for $\sigma_1$, $\sigma_2$. □

The following lemma shows how we can reconstruct the fan $\Delta$ from $\Upsilon(\Delta)$ and $\Delta(1)$.

Lemma 3.4. Let $\sigma$ be in $\tilde{\Delta}$. Then $\Gamma(\sigma) \in \Upsilon(\Delta)$ if and only if $\sigma \in \Delta$.

Proof. The implication $\sigma \in \Delta \Rightarrow \Gamma(\sigma) \in \Upsilon(\Delta)$ is trivial.

Let us assume that $\Gamma(\sigma) \in \Upsilon(\Delta)$. According to the definition of $\Upsilon(\Delta)$ there exists a cone $\sigma' \in \Delta$ such that $\Gamma(\sigma') = \Gamma(\sigma)$. Without loss of generality we may assume $\sigma(1) = \{\tau_1, \ldots, \tau_p\}$, $\sigma'(1) = \{\tau_s, \tau_{s+1}, \ldots, \tau_{p+q}\}$. If $\sigma(1) \cap \sigma'(1) \neq \emptyset$, then $\sigma(1) \cap \sigma'(1) = \{\tau_s, \tau_{s+1}, \ldots, \tau_p\}$, otherwise we may assume $s = p + 1$. Consider a sequence of cones in $\tilde{\Delta}$

$$\sigma = \varsigma_1, \varsigma_2, \ldots, \varsigma_{s+q} = \sigma',$$

$$\varsigma_i = \text{cone}_Q(\tau_i, \ldots, \tau_p), \ 1 \leq i \leq s - 1,$$

$$\varsigma_s = \text{cone}_Q(\tau_s, \ldots, \tau_p) \text{ if } s \leq p, \ \varsigma_s = \text{cone}_Q(0) \text{ otherwise},$$
Theorem 3.6. \( \Gamma(\sigma) = \Gamma(\sigma') = \Gamma(\sigma) = \Gamma(\sigma') \).

Moreover,
\[
\Gamma(\sigma) \subset \Gamma(\sigma_i) \subset \Gamma(s_i), \text{ hence, } \Gamma(s_i) = \Gamma(\sigma) \text{ for } 1 \leq i \leq s,
\]
\[
\Gamma(\sigma') \subset \Gamma(s_{i+1}) \subset \Gamma(s_i), \text{ hence, } \Gamma(s_{i+1}) = \Gamma(\sigma') \text{ for } 1 \leq i \leq q.
\]

By Lemma 3.3 for \( s_i, s_{i+1} \) and some roots \( m_i \in M \) conditions (2) are satisfied for all \( 1 \leq i \leq s + q - 1 \). By Lemma 3.2 \( s_i \in \Delta \) for \( s + q - 1 \geq i \geq 1 \). Hence, \( \sigma \in \Delta \).

Proof. Since \( \text{Aut}^0(X) \) is generated by \( T \) and \( H_m, m \in \mathbb{R} \), orbits \( \mathcal{O}_\sigma \) and \( \mathcal{O}_{\sigma'} \) lie in the same \( \text{Aut}^0(X) \)-orbit if and only if there exists a sequence
\[
(\sigma) = s_1, s_2, \ldots, s_l = \sigma',
\]
of cones from \( \Delta \) such that \( (\mathcal{O}_{s_i}, \mathcal{O}_{s_{i+1}}) \) is \( H_m \)-connected for some \( m_i \in \mathbb{R} \), \( 1 \leq i \leq l - 1 \).

If \( \Gamma(\sigma) = \Gamma(\sigma') \) we can take sequence (4) as the required one.

Conversely, let us assume there is sequence (5). By Lemma 3.3 we have \( \Gamma(\sigma) = \Gamma(s_1) = \ldots = \Gamma(s_l) = \Gamma(\sigma') \) and \( \Gamma(\sigma) = \Gamma(\sigma') \).

Example 3.5. For every toric variety \( X \) an \( \text{Aut}^0(X) \)-orbit on \( X \) consists of \( T \)-orbits and can be pictured as a set of cones. We describe these orbits for the varieties from previous examples.

(i) The action of \( \text{Aut}^0(\mathbb{P}^2) \) on \( \mathbb{P}^2 \) is transitive, see Figure 4(i).

(ii) There are two orbits on \( \mathbb{H}_8 \), see Figure 4(ii).

(iii) There are two orbits on \( \mathbb{P}(1, 1, s) \): the regular locus and the singular point, see Figure 4(iii).

(iv) Any orbit of \( \text{Aut}^0(\mathbb{B}_s) \) on \( \mathbb{B}_s \) is a \( T \)-orbit.

Theorem 3.6. Let \( \mathcal{O} \) and \( \mathcal{O}' \) be \( T \)-orbits. Then
\[
\text{Aut}^0(X) \cdot \mathcal{O} \subset \text{Aut}^0(X) \cdot \mathcal{O}' \iff \Gamma(\mathcal{O}) \subset \Gamma(\mathcal{O}').
\]

Proof. Let us put \( \sigma_{\max} = \text{cone}_{\mathbb{Q}}(v_i : [D_i] \notin \Gamma(\mathcal{O})) \). If \( \mathcal{O} \) corresponds to a cone \( \sigma \in \Delta \), then \( \sigma_{\max} \subset \sigma \). Hence, \( \sigma_{\max}(1) = \sigma_{\max} \cap \Delta(1) \) and \( \Gamma(\sigma_{\max}) = \Gamma(\mathcal{O}) \).

By Lemma 3.4 there exists a \( T \)-orbit \( \mathcal{O}_{\max} \) corresponding to \( \sigma_{\max} \). It is easy to see that \( \sigma_{\max} \subset \sigma \) and \( \mathcal{O} \subset \mathcal{O}_{\max} \).

Since \( \mathcal{O} \) is an arbitrary \( T \)-orbit with \( \Gamma(\mathcal{O}) = \Gamma(\mathcal{O}_{\max}) \), we have
\[
\mathcal{O}_{\max} \subset \text{Aut}^0(X) \cdot \mathcal{O} \quad \text{and} \quad \mathcal{O}_{\max} = \overline{\text{Aut}^0(X) \cdot \mathcal{O}}.
\]

Without loss of generality we may assume \( \overline{\mathcal{O}} = \overline{\text{Aut}^0(X) \cdot \mathcal{O}} \) and \( \overline{\mathcal{O}}' = \overline{\text{Aut}^0(X) \cdot \mathcal{O}'} \). It means that
\[
\sigma(1) = \{v_i : [D_i] \notin \Gamma(\mathcal{O})\} \quad \text{and} \quad \sigma'(1) = \{v_i : [D_i] \notin \Gamma(\mathcal{O}')\}.
\]
We have
\[ \Gamma(O) \subseteq \Gamma(O') \iff \Delta(1) \setminus \sigma(1) \subseteq \Delta(1) \setminus \sigma'(1) \]
\[ \iff \sigma' \prec \sigma \]
\[ \iff \sigma \prec \sigma' \]
\[ \iff \overline{O} \subseteq \overline{O'} \]
\[ \iff \text{Aut}^0(X) \cdot O \subseteq \text{Aut}^0(X) \cdot O'. \]

\[ \square \]

**Theorem 3.7.** Torus orbits \( O_\sigma \) and \( O_{\sigma'} \) on \( X \) lie in the same \( \text{Aut}(X) \)-orbit if and only if there exists an automorphism \( \phi : \text{Cl}(X) \rightarrow \text{Cl}(X) \) with the following properties

- \( \phi(\Gamma(O_\sigma)) = \Gamma(O_{\sigma'}) \),
- \( \phi(\Upsilon(\Delta)) = \Upsilon(\Delta) \),
- there exists a permutation \( f \) of elements in \( \{1, \ldots, r\} \) such that \( \phi([D_i]) = [D_{f(i)}] \).

**Proof.** Let us assume \( O_\sigma \) and \( O_{\sigma'} \) lie in the same \( \text{Aut}(X) \)-orbit. By Theorem 2.7 there exists \( P_\psi \in \text{Aut}(X) \) defined by some \( \psi \in \text{Aut}(N, \Delta) \) that maps \( \text{Aut}^0(X) \cdot O_\sigma \) to \( \text{Aut}^0(X) \cdot O_{\sigma'} \). The automorphism \( P_\psi \) defines an automorphism \( \phi : \text{Cl}(X) \rightarrow \text{Cl}(X) \) with the following properties:

- \( \phi([D_i]) = [P_\psi(D_i)] = [D_{\psi(i)}] \);
- \( \phi(\Gamma(\sigma_1)) = \Gamma(\sigma_2) \);
- \( \phi(\Upsilon(\Delta)) = \Upsilon(\Delta) \).
Conversely, for every $\phi$ let us find $P_\psi \in \text{Aut}(X)$ mapping $\text{Aut}^0(X)$-orbit corresponding to $\Gamma(\sigma)$ to $\text{Aut}^0(X)$-orbit corresponding to $\Gamma(\sigma') = \phi(\Gamma(\sigma))$. There is a commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & \text{Div}_T(X) & \longrightarrow & \text{Cl}(X) & \longrightarrow & 0 \\
& & \downarrow \psi^* & \downarrow \phi & & \downarrow \phi & & \downarrow \phi & \\
0 & \longrightarrow & M & \longrightarrow & \text{Div}_T(X) & \longrightarrow & \text{Cl}(X) & \longrightarrow & 0
\end{array}
$$

where $\tilde{f}$ is defined by $f$. There exists an automorphism $\psi^* : M \rightarrow M$ making the diagram commutative. Let us define $\psi : N \rightarrow N$ as dual to $\psi^*$. One can check that $\psi$ preserves $\Delta(1)$. By Lemma 3.4, the cone $\psi(\varsigma), \varsigma \in \Delta,$ belongs to $\Delta$, since $\Gamma(\psi(\varsigma)) = \phi(\Gamma(\varsigma)) \in \Upsilon(\Delta)$. The automorphism $\psi$ defines the desired $P_\psi$.

Example 3.8. It is easy to see that $\text{Aut}^0$- and $\text{Aut}$-orbits on $\mathbb{P}^2$, $\mathcal{H}_s$, and $\mathbb{P}(1,1,s)$ coincide.

Let us describe the $\text{Aut}(\mathcal{B}_s)$-orbits on $\mathcal{B}_s$. For $s > 1$ the group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ acts on $\Upsilon(\Delta)$ and defines orbits on $\mathcal{B}_s$ in the following way: the open $\text{Aut}(\mathcal{B}_s)$-orbit coincides with the open $T$-orbit, one dimensional torus orbits belong to the same $\text{Aut}(\mathcal{B}_s)$-orbit, and four $T$-stable points form two $\text{Aut}(\mathcal{B}_s)$-orbits, see Figure 4(iv). For $s = 1$ all $T$-fixed points on $\mathcal{B}_1$ are in the same $\text{Aut}(\mathcal{B}_1)$-orbit.

Theorem 3.9. Let $X$ be a complete toric variety. The group $\text{Aut}(X)$ acts on $X$ transitively if and only if

$$X \cong \prod_{i=1}^{s} \mathbb{P}^{k_i}$$

for some $k_i \in \mathbb{N}$.

Proof. Let us assume that the action of $\text{Aut}(X)$ is transitive. Then $X$ is smooth, in particular, all cones of $\Delta$ are simplicial. It is easy to see that open $\text{Aut}^0(X)$- and open $\text{Aut}(X)$-orbits coincide. Hence, $\Upsilon(\Delta)$ consists of one monoid.

Let $\sigma \in \Delta$ be a cone of maximal dimension $d$. We may assume that $\sigma(1) = \{ \tau_1, \ldots, \tau_d \}$. There is a cone $\sigma' \in \Delta$ of dimension $d$ which meets $\sigma$ along the facet without $\tau_1$: $\sigma'(1) = \{ \tau_2, \ldots, \tau_d, \tau_{d+1} \}$.

Since $\Upsilon(\Delta)$ contains only one monoid, the following monoids coincide:

$$\Gamma(\sigma') = \mathbb{Z}_{\geq 0}[D_1] + \sum_{i>d+1} \mathbb{Z}_{\geq 0}[D_i],$$

$$\Gamma(\sigma) = \mathbb{Z}_{\geq 0}[D_{d+1}] + \sum_{i>d+1} \mathbb{Z}_{\geq 0}[D_i].$$
We have
\[ [D_1] = a_{d+1}[D_{d+1}] + \sum_{i > d+1} a_i[D_i], \]
\[ [D_{d+1}] = b_1[D_1] + \sum_{i > d+1} b_i[D_i], \]
where \( a_{d+1}, b_1 \in \mathbb{Z}_{\geq 0} \) and \( a_i, b_i \in \mathbb{Z}_{\geq 0} \) for \( d + 2 \leq i \leq r \). Let us put \( a_1 = -1, b_{d+1} = -1 \) and \( a_i = 0, b_i = 0 \) for \( 2 \leq i \leq d \).

Due to equations (6) there exist \( m_1, m'_1 \in M \) such that
\[ \langle m_1, v_i \rangle = a_i \text{ for } 1 \leq i \leq r, \]
\[ \langle m'_1, v_i \rangle = b_i \text{ for } 1 \leq i \leq r. \]

Since \( \langle m_1, v_i \rangle = \langle m'_1, v_i \rangle = 0, 2 \leq i \leq d \), the characters \( m_1 \) and \( m'_1 \) are proportional. Hence, \( a_i = -b_i \) for all \( i \) in particular, \( a_i = b_i = 0 \) for \( i > d + 1 \). We have \([D_1] = [D_{d+1}]\) and \( \langle m_1, v_1 \rangle = -1, \langle m_1, v_{d+1} \rangle = 1, \langle m_1, v_i \rangle = 0, i \neq 1, d + 1 \). It means that \( m_1 \in \mathbb{R} \) is a semisimple root.

Since we can take any element of \( \sigma(1) \) instead of \( \tau_1 \) there exists a set of semisimple roots \( m_1, \ldots, m_d \) such that \( \langle m_i, v_j \rangle = -\delta_{ij} \) for all \( 1 \leq i, j \leq d \). Hence, \( m_1, \ldots, m_d \) are linearly independent.

In [10, Theorem 3.18] it is shown that if \( X \) is a complete toric variety of dimension \( d \) and there are \( d \) linearly independent semisimple roots \( m_1, \ldots, m_d \), then
\[ X \cong \prod_{i=1}^{s} \mathbb{P}^{k_i} \]
for some \( k_i \in \mathbb{N} \). Applying this theorem completes the proof in one direction.

Conversely, the group
\[ \prod_{i=1}^{s} \text{PGL}(k_i + 1), \]
acts on \( \prod_{i=1}^{s} \mathbb{P}^{k_i} \) transitively.

\[ \square \]

References

[1] I. Arzhantsev, S. Gaifullin: Homogeneous toric varieties. J. Lie Theory 20 (2010), 283-293.
[2] I. Arzhantsev, K. Kuyumzhiyan, M. Zaidenberg: Flag varieties, toric varieties, and suspensions: three instances of infinite transitivity. arXiv:1003.3164v1 (2010), 25 p.
[3] F. Berchtold, J. Hausen: Bunches of cones in the divisor class group — a new combinatorial language for toric varieties. Int. Math. Res. Not. 6 (2004), 261-302.
[4] W. Bruns, J. Gubeladze: Polytopal linear groups. J. Algebra 218 (1999), 715-737.
[5] D. Bühler: *Homogener Koordinatenring und Automorphismengruppe vollständiger torischer Varietäten*. Diplomarbeit, University of Basel, 1996.

[6] D. Cox: *The homogeneous coordinate ring of a toric variety*. J. Alg. Geometry 4 (1995), 17-50.

[7] D. Cox, J. Little, H. Shenck: *Toric varieties*. Graduate Studies in Math. 124, AMS, Providence, RI, 2011.

[8] M. Demazure: *Sous-groupes algébriques de rang maximum du groupe de Cremona*. Ann. Sci. École Norm. Sup. 3 (1970) 507-588.

[9] W. Fulton: *Introduction to toric varieties*. Ann. of Math. Studies 131, Princeton University Press, Princeton, NJ, 1993.

[10] B. Nill: *Complete toric varieties with reductive automorphism group*. Math. Z. 252 (2006), 767-786.

[11] T. Oda: *Convex bodies and algebraic geometry — An introduction to the theory of toric varieties*. Springer Verlag, Berlin, 1988.

**Algebra Department, Faculty of Mechanics and Mathematics, Moscow State University, Leninskie Gory 1, Moscow, 119991, Russia**

**E-mail address:** ibazhov@gmail.com