ARITHMETIC QUANTUM UNIQUE ERGODICITY FOR PRODUCTS OF HYPERBOLIC 2- AND 3-SPACES

ZVI SHEM-TOV AND LIOR SILBERMAN

ABSTRACT. We prove the arithmetic quantum unique ergodicity (AQUE) conjecture for sequences of Hecke–Maass forms on quotients $\Gamma \backslash (\mathbb{H}^{(2)})^r \times (\mathbb{H}^{(3)})^s$. An argument by induction on dimension of the orbit allows us to rule out the limit measure giving positive mass to closed orbits of proper subgroups despite many returns of the Hecke correspondence to neighborhoods of the orbit.

CONTENTS

1. Introduction 1
1.1. Statement of result 1
1.2. Context: QUE Conjectures for locally symmetric spaces 3
1.3. Discussion 3
1.4. Sketch of the proof of Theorem 4
1.5. Organization of the paper 5

2. Notations and background 7
2.1. Unit groups of quaternion algebras 7
2.2. Factoring $G_{\infty}$ over a number field; complex conjugation 8
2.3. The real group 11
2.4. $p$-adic and adelic groups 11
2.5. Hecke operators 12

3. Homogeneity 12
3.1. Positive Entropy 13
3.2. Measure rigidity 16

4. Submanifolds with small stabilizers 18

5. Constructing an amplifier 20

6. Non-concentration on homogenous submanifolds 23

References 26

1. INTRODUCTION

1.1. Statement of result. We prove the Arithmetic Quantum Unique Ergodicity Conjecture for hyperbolic 3-manifolds. More generally fix integers $r, s \geq 0$ so that $r + s \geq 1$ and consider the symmetric space

$$S = (\mathbb{H}^{(2)})^r \times (\mathbb{H}^{(3)})^s,$$

2010 Mathematics Subject Classification. 11F41; 37A45.
LS was supported by an NSERC Discovery Grant.
where $\mathbb{H}^{(n)}$ denotes hyperbolic $n$-space. Let $\Gamma$ be a lattice in the isometry group

\begin{equation}
G = \text{SL}_2(\mathbb{R})^r \times \text{SL}_2(\mathbb{C})^s.
\end{equation}

Then the finite-volume manifold $Y = \Gamma \backslash S$ is equipped with a family of $r + s$ commuting differential operators coming from the Laplace–Beltrami operator in each factor (the sum of which is the Laplace–Beltrami operator of $Y$). Assume that $\Gamma$ is, in addition, a congruence lattice (e.g. $\text{SL}_2(\mathbb{Z}[i]) \subset \text{SL}_2(\mathbb{C})$) so that the manifold $Y$ is also equipped with a ring of discrete averaging operators, the Hecke operators, which further commute with the differential operators noted above (see Section 2.4 for a discussion of congruence lattices and Section 2.5 for the construction of the Hecke algebra). Write $\text{dvol}$ for the Riemannian volume element normalized to have total volume 1.

**Theorem 1.** Let $\psi_j \in L^2(Y)$ be a sequence of normalized joint eigenfunctions of both the ring of invariant differential operators and of the Hecke operators. Assume that the Laplace-eigenvalues $\lambda_j \to \infty$. Then the probability measures $|\psi_j|^2 \text{dvol}(y)$ converge in the weak* topology to $\text{dvol}$. (Recall that $\mu_i$ converge weak* to $\mu$ if for every continuous function $f$ with compact support,

$$\int fd\mu_i \to \int fd\mu$$

as $i \to \infty$.)

The case $r = 1, s = 0$ (i.e. hyperbolic surfaces) is due to Lindenstrauss [8]. When $s = 0$ and $r > 1$ (more generally, when the eigenvalues of $\psi_j$ with respect to the Laplace operator corresponding to one of the $\mathbb{H}^{(2)}$ factors tend to infinity) the claim follows from existing techniques [1]. When $s > 0$ those existing techniques place significant constraints on the possible ergodic components of the limit measures under consideration but fall short of proving that the limit measure is uniform, and we introduce a new method for eliminating components other than the uniform measure. This is the main innovation of our paper; to state its realization here let $G_i$ be one of the factors isomorphic to $\text{SL}_2(\mathbb{C})$ in the product (1.1) defining $G$. Identifying $G_i$ with $\text{SL}_2(\mathbb{C})$ for the moment let $H_i = \text{SL}_2(\mathbb{R})$, and let $M_i = \{ \text{diag}(e^{i\theta}, e^{-i\theta}) \}$ be the group of diagonal matrices with entries of modulus 1. Finally set $H = H_i \times \prod_{j \neq i} G_j$ (i.e. multiply $H_i$ at the $i$th factor with the full isometry groups $G_j$ at the other factors).

**Theorem 2.** Let $\mu$ be a measure on $X = \Gamma \backslash G$ which is the weak-* limit of measures

$$\mu_j = |\phi_j|^2 \text{dvol}_X$$

where the $\phi_j \in L^2(X)$ are normalized Hecke eigenfunctions. If $g = (g_j)_j \in G$ is such that the entries of $g_i$ are algebraic numbers then

$$\mu(\Gamma gHM_i) = 0.$$
1.2. **Context: QUE Conjectures for locally symmetric spaces.** More generally, let $G$ be a semisimple Lie group, $K \subset G$ a maximal compact subgroup, $\Gamma < G$ a lattice, and consider the locally symmetric space

$$Y = \Gamma \backslash G/K,$$

with its uniform probability measure $dy$. The ring of $G$-invariant differential operators on $G/K$ is commutative and commutes with the action of $\Gamma$, hence acts on functions on $Y$. Let $\{\psi_j\}_{j \geq 1} \subset L^2(Y)$ be an orthonormal sequence of joint eigenfunctions of this ring. We remark that the Laplace–Beltrami operator of $Y$ belongs to the ring and that writing $\lambda_j$ for the corresponding eigenvalue for $\psi_j$ we necessarily have $|\lambda_j| \to \infty$. The *Quantum Unique Ergodicity (QUE) Conjecture* for locally symmetric spaces asserts the sequence $\{\psi_j\}$ becomes equidistributed on $Y$, in the sense that

$$\int_Y |\psi_j(y)|^2 dy \to \int_Y f(y) dy,$$

for all test functions $f \in \mathcal{C}_c(Y)$. In other words, the sequence $|\psi_j|^2 dy$ of probability measures converges in the weak-* topology to the uniform measure $dy$. Next, when $\Gamma$ is a congruence lattice the space $Y$ also affords a commutative family of discrete averaging operators, the *Hecke operators*, which commute with the invariant differential operators. The *Arithmetic* QUE Conjecture (AQUE) is the restricted form of the QUE Conjecture for sequences of Hecke eigenfunctions, that is for sequences where each $f_j$ is simultaneously an eigenfunction of the ring of invariant differential operators and of the ring of Hecke operators.

Investigation of Arithmetic QUE goes back to the work \cite{rudnick} of Rudnick–Sarnak, who formulated the QUE conjecture for hyperbolic surfaces and, more generally, for compact manifolds of negative sectional curvature. A major breakthrough on this problem was due to Lindenstrauss, who in \cite{lindenstrauss} established AQUE for congruence hyperbolic surfaces, that is congruence quotients $Y = \Gamma \backslash \mathbb{H}^2$ of the hyperbolic plane. More precisely, Lindenstrauss proved that weak-* limits as above were proportional to the uniform measure. This fully resolved the Conjecture in the case of uniform lattices, i.e. when the quotient is compact. For non-uniform lattices, however, the possibility remained that the constant of proportionality was strictly less than one ("escape of mass"), an alternative ruled out later by Soundararajan \cite{soundararajan}. Silberman–Venkatesh \cite{silberman1, silberman2} obtains generalizations of some of this work to the case of general semisimple groups $G$, formulating the QUE and AQUE conjectures in the context of locally symmetric spaces and obtaining further cases of AQUE.

1.3. **Discussion.** As stated above, we establish AQUE for congruence quotients of products of hyperbolic 2- and 3-spaces. The case $Y = \text{SL}_2(\mathbb{Z}[i]) \backslash \mathbb{H}^3$ is already new and contains most of the the novel ideas of this paper; reading the paper with this assumption in mind will give the reader most of the insight but avoid the technical legerdemain needed to handle number fields with multiple infinite places. For the rest of the introduction we concentrate on this case.

Thus let $G = \text{SL}_2(\mathbb{C})$ acts transitively by isometries on the space $S = \mathbb{H}^3$ with point stabilizer the maximal compact subgroup $K = \text{SU}(2)$. Let $\Gamma = \text{SL}_2(\mathbb{Z}[i])$, the group of unimodular matrices with Gaussian integer entries, which is indeed a lattice in $G$. Then $Y = \Gamma \backslash S$ is a finite-volume hyperbolic 3-manifold (or more precisely, an orbifold) and $X = Y \backslash G$ is its frame bundle. The action of $A = \{\text{diag}(a,a^{-1}) \mid a > 0\}$ on $X$ from the right corresponds to the geodesic flow on the frame bundle, with the commuting subgroup $M = \{\text{diag}(e^{i\theta}, e^{-i\theta})\}$ acting by rotating the frame around the tangent vector.
Let $H = \mathrm{SL}_2(\mathbb{R}) \subset G$. Then $Y_H = \mathrm{SL}_2(\mathbb{Z}) \backslash H / \mathrm{SO}(2) \subset Y$ is a finite-volume hyperbolic surface embedded in $Y$; its unit tangent bundle is the finite-volume $H$-orbit $X_H = \mathrm{SL}_2(\mathbb{Z}) \backslash H \subset X$. Very relevant to us will be the subset $\mathrm{SL}_2(\mathbb{Z}) \backslash HM$, corresponding to the pullback of the frame bundle of the hyperbolic 3-fold to the surface. Geometrically the unit tangent bundle of $Y_H$ embeds in the frame bundle of $Y$ in multiple ways invariant by the geodesic flow, parametrized by choosing a normal vector to the tangent vector at one point. The set of choices is thus parametrized by the group $M$ rotating the frame at the chosen point around the tangent vector.

The Casimir element in the universal enveloping algebra of the Lie algebra $G$ acts on functions on $X$ and on $Y$, where it is proportional to the Laplace–Beltrami operator. In addition there is a family of discrete averaging operators acting on functions on $X$ and commuting with the right $G$ action, hence also on functions on $Y$ (thought of as $K$-invariant functions on $X$). These Hecke operators can be constructed as follows: for each $g \in \mathrm{SL}_2(\mathbb{Q}(i))$ the set $[g] = \Gamma \backslash \Gamma g \Gamma \subset \Gamma \backslash \mathrm{SL}_2(\mathbb{Q}(i))$ is finite, so for a function $f$ on $X$ we can set

$$
(T_g f)(x) = \sum_{x \in [g]} f(sx).
$$

Since these operators are $G$-equivariant they also commute with the Casimir element, and hence with the Laplace–Beltrami operator on $Y$. They are bounded (the $L^2$ operator norm of $T_g$ is at most the cardinality of $[g]$) and it is not hard to check that the adjoint of $T_g$ is $T_{g^{-1}}$. It is a non-trivial fact that the $T_g$ commute with each other, and hence are a commuting family of bounded normal operators. For the sequel we will take a different point of view that gives better control of these operators; see the construction in Section $2.5$.

The proof of Theorem $1$ is given in Section $3$. Our strategy is the one laid down by Lindenstrauss and followed by later work on AQUE.

1. The microlocal lift of $[13]$ (see also $[7]$ for the case $s = 0$; the original such constructions in much greater generality are due to Shnirelman, Zelditch and Colin de Verdière $[15, 18, 4]$) shows that any weak-* limit on $Y$ as in Theorem $1$ is the projection from $X$ of a limit $\mu$ as in Theorem $2$ which, in addition, is an $AM$-invariant measure.

2. We show that almost every $A$-ergodic component of $\mu$ has positive entropy under the $A$-action. These arguments are standard (see $[14]$ generalizing $[11]$) but we need to adjust them to handle the fact that $M$ isn’t finite and that the group is essentially defined over a number field. What is shown is that the measure of an $\epsilon$-neighbourhood of a (compact piece) of an orbit $xAM$ in $X$ decays at least as fast as $e^{\delta}$ for some $h > 0$.

3. The classification of $A$-invariant measures in $[5]$ implies that all the ergodic components of $\mu$ other than the $G$-invariant measure are contained in sets of the form $xHM$ where $xH$ is a finite-volume $H$-orbit in $X$.

4. In a closed $H$-orbit $\Gamma g H \subset X$ the entries of $g$ are algebraic numbers, so by Theorem $2$ the exceptional possibilities are contained in a countable family of sets each of which has measure zero. It follows that $\mu$ is $G$-invariant.

5. It was shown in Zaman’s M.Sc. Thesis $[17]$ that the measure $\mu$ is a probability measure (a generalization of Soundararajan’s proof $[16]$ for the case of $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathrm{SL}_2(\mathbb{R})$).

The bulk of the paper is then devoted to realizing step $4$. Before discussing how that is achieved, let us remark on its necessity. The group $G = \mathrm{SL}_2(\mathbb{R})$ has no proper semisimple subgroups, so the issue of ruling out components supported on orbits of such subgroups did
not arise in the original work of Lindenstrauss. In followup generalizations to higher-rank groups the difficulty was addressed by choosing the lattice \( \Gamma \) so that the subgroups \( H \) that could arise did not have finite-volume orbits on \( X \) and \( G \) itself was chosen \( \mathbb{R} \)-split so that \( M \) was finite and could be ignored.

We now go further. The group \( G = \text{SL}_2(\mathbb{C}) \) has \( H = \text{SL}_2(\mathbb{R}) \) as a relevant subgroup, forcing us to confront the problem of ruling out components supported on finite-volume \( H \)-orbits head-on. It is also not \( \mathbb{R} \)-split, forcing us to contend with the infinitude of \( M \). Indeed, equip a finite-volume \( H \)-orbit \( xH \) with the \( H \)-invariant probability measure \( \nu \). Translating \( \nu \) by \( m \in M \) gives further \( A \)-invariant probability measures supported on the subsets \( xHm \), and averaging these measures to gives an \( A \)-invariant probability measure supported on \( xHM \).

The construction in step 1 is such that if \( \nu \) occurs in the ergodic decomposition of \( \mu \) then so do its \( M \)-translates, so we need to rule out the averaged measure on \( xHM \) as occurring in \( \mu \) (fortunately there are only countable many such subsets). Now previous technology as mentioned in step 2 could show that \( (\varepsilon \text{-neighbourhoods of pieces of}) \) subgroup orbits such as \( xH \) have small measure but \( xHM \) is not of this form. Instead the subgroup generated by \( HM \) is all of \( G \), so the only orbit of a closed subgroup containing \( xHM \) is all of \( X \).

1.4. Sketch of the proof of Theorem 2 Let \( U \subset xHM \) be a bounded open set. Then showing that \( \mu(xHM) = 0 \) amounts to bounding the mass Hecke eigenfunctions can give to \( \varepsilon \)-neighbourhoods \( U_e \). In other words, our goal is to prove

\[
\int_{U_e} |\phi(x)|^2 dx = o_e(1),
\]

for any Hecke-eigenfunction \( \phi \), \textit{uniformly in} \( \phi \).

We find a Hecke operator (“amplifier”) \( \tau \) which acts on \( \phi \) with a large eigenvalue \( \Lambda \) yet geometrically “smears” \( U_e \) around the space. The operator \( \tau \) is a linear combination of operators \( T_h \) so there is a subset \( \text{supp}(\tau) \subset \text{SL}_2(\mathbb{Z}[i]) \backslash \text{SL}_2(\mathbb{Q}(i)) \) so that

\[
\Lambda \phi(x) = (\tau \ast \phi)(x) = \sum_{s \in \text{supp}(\tau)} \tau(s) \phi(sx).
\]

Thus if \( |\tau(s)| \leq 1 \) for each \( s \in \text{supp}(\tau) \) then by Cauchy–Schwartz

\[
\Lambda^2 \int_{U_e} |\phi(x)|^2 dx \leq \#\text{supp}(\tau) \int_{U_e} \sum_{s \in \text{supp}(\tau)} |\phi(sx)|^2 dx,
\]

so that

\[
\mu_{\phi}(U_e) \leq \frac{\#\text{supp}(\tau)}{\Lambda^2} \sum_{s \in \text{supp}(\tau)} \mu_{\phi}(sU_e)
\]

\[
\leq \frac{\#\text{supp}(\tau)}{\Lambda^2} \max_{s \in \text{supp}(\tau)} \#\{s' \in \text{supp}(\tau) \mid sU_e \cap s'U_e \neq \emptyset\} \mu_{\phi}(\bigcup_{s \in \text{supp}(\tau)} sU_e)
\]

\[
\leq \frac{\#\text{supp}(\tau)}{\Lambda^2} \max_{s \in \text{supp}(\tau)} \#\{s' \in \text{supp}(\tau) \mid sU_e \cap s'U_e \neq \emptyset\}.
\]

The best kind of smearing is thus if there were very few non-empty intersections between the translates \( sU_e \), where a bound would follow as long as we could arrange for \( \Lambda^2 \) to be large compared with the size of the support of \( \tau \). That is too much to hope for, but with better geometric and spectral arguments one requires less stringent control over the intersections. For various implementations of this strategy see [11, 1, 14, 2, 12], all in
settings where $U$ is a piece of an orbit $xL$ for a subgroup $L$. Then an intersection between $sU_ε$ and $U_ε$ (say) implies that $s$ is $O(ε)$-close to the bounded neighbourhood $UU^{-1}$ in $xLx^{-1}$ and if $τ$ is such that the elements $s$ in its support have sufficiently small denominators relative to $ε$ then all the $s$ causing the intersections are jointly contained in a single conjugate of $L$. For certain kinds of groups $L$ (e.g. tori) this implies that there will be very few intersections. There are also exceptional cases ("division algebras of prime degree", see [13, Prop. 4.10]) where the lattice $Γ$ is such that even for larger groups $L$ the set of $s ∈ Γ$ which create an intersection jointly generate a torus and again there are few intersections.

In the case we consider, namely $U ⊂ xHM$, the set $UU^{-1}$ (let alone $U_εU_ε^{-1}$) is an open subset of $G$ (a reflection of the fact that $HM$ generates $G$), so the number of intersections may be large: $s ∈ UU^{-1}$ holds generically (rational points of large height are equidistributed in $G$), and the argument fails. Instead we bound the measure of the pieces through an induction argument on their dimension. For this lift the picture to $G$, replacing $x ∈ X$ with a representative $g ∈ G$ and $L = gHM$ which is, in fact, an irreducible real algebraic subvariety of $G$. An intersection $sL ∩ L$ with $s ∈ SL_2(ℚ(i))$ is then also a subvariety, and hence one of two possibilities must hold: either the intersection has strictly smaller dimension, in which case we call the intersection (and, by abuse of language, the element $s$) transverse, or it is not, in which case we call $sL$ (and $s$) parallel to $L$ and must actually have $sL = L$ by the irreducibility of $L$. By induction we may assume that the measures of all transverse intersections $sU_ε ∩ U_ε$ are small (one needs to show that $sU_ε ∩ U_ε$ is contained in a decreasing neighbourhood of $sU ∩ U$). On the other hand the parallel elements stabilize the subvariety $L$ and this forces them to lie in a proper subgroup $S ⊂ SL_2(ℚ(i))$ (here we gained over the naive attempt which considered the much larger subgroup generated by $SL_2(ℚ(i)) ∩ LL^{-1}$).

For example, the stabilizer of $HM$ under the left action of $G$ on itself is exactly $H$, so parallel intersections can only arise from $s ∈ SL_2(ℚ)$. We now introduce a final idea, allowing us to deal with a situation where the proper subgroup $S$ noted above is not a torus. Let $p ≡ 1 (4)$ be a prime number so that $p = (a + bi)(a - bi)$ for integers $a, b$ and set

$$g_p = \text{diag}(a + bi, \frac{1}{a + bi}).$$

It turns out that for any Hecke eigenfunction $ψ$, the eigenvalue of $λ_p$ of one of the two operators $T_{g_p}, T_{g_p^{-1}}$ is at least comparable to the square root of the size of its support. In addition, the supports $|g_p|, |g_p^2|$ of these operators do not intersect $SL_2(ℚ)$: complex conjugation exchanges the Gaussian prime $a + bi$ with the distinct Gaussian prime $a - bi$, so the ratio of matrices from the two double cosets must contain both primes and cannot be real), so these operators themselves do not cause intersections. Making use of these as building blocks and combining the contribution from many primes we then construct an amplifier which has good spectral properties and at the same time avoids intersections caused by $SL_2(ℚ)$.

Each closed orbit $xHM$ is the projection of $L = gHM$ where the entries of $g$ are algebraic but need not be rational, so greater care must be taken to define the complex conjugation which limits the intersection and correspondingly the arithmetic progression from which we select the primes needs to be smaller. Also, we need to consider the stabilizers of the subvarieties of $L$ that will appear in the recursive argument. We show each of these stabilizers is either contained in a conjugate of $H$, and then similarly to the case of $H$ itself we can avoid it entirely, or are tori (which are already known to cause few intersections).
1.5. **Organization of the paper.** In Section 2 we fix our notation and examine the algebraic structure of forms of SL_{2} over number fields, constructing the complex conjugations which control returns of Hecke translates to real submanifolds. We take the adelic point of view of the Hecke operators, which is more convenient than the one used for the introduction. In Section 3 we then reduce Theorem 1 to Theorem 2. Before giving the proof of Theorem 2 in Section 4 we devote Section 4 to classifying the stabilizers of the subvarieties that can occur in the recursion and Section 5 to constructing the amplifier.

**Acknowledgements.** We would like to thank Elon Lindenstrauss and Manfred Einsiedler for very useful suggestions and fruitful discussions. This work forms part of the PhD thesis of the first author at the Hebrew University of Jerusalem. The first author was supported by the ERC grant HomDyn no. 833423.

2. **Notations and background**

For a comprehensive reference on the theory of algebraic groups over number fields see [10]. The theory of automorphic forms over such groups is developed, for example, in the textbook [3]. The particular case of forms of SL_{2} over number fields is articulated in detail in [6].

2.1. **Unit groups of quaternion algebras.** Varieties (including algebraic groups) will be named in blackboard bold font.

Fix a number field \( F \), and let \( V = |F| \) be the set of places of \( F \), \( V_{\infty} \subseteq V \) the set of archimedian places, divided into real and complex places as \( V_{\infty} = V_{\mathbb{R}} \sqcup V_{\mathbb{C}} \). For a place \( v \in V \) write \( F_{v} \) for the completion of \( F \) at \( v \). If the place is finite write \( \mathcal{O}_{v} \subseteq F_{v} \) for the maximal compact subring \( \{ x \in F_{v} \mid |x|_{v} \leq 1 \} \), and let \( \mathfrak{o}_{v} \in \mathcal{O}_{v} \) be a uniformizer. We normalize the associated absolute value so that \( q_{v} = |\mathfrak{o}_{v}|_{v}^{-1} \) is the cardinality of the residue field \( \mathcal{O}_{v}/\mathfrak{o}_{v} \mathcal{O}_{v} \). For a rational prime \( p \) let \( F_{p} = \prod_{v \mid p} F_{v} \) and similarly set \( F_{\infty} = \prod_{v \in \infty} F_{v} \). We write \( \mathfrak{A}_{F} \) for the ring of finite adeles \( \prod_{v \in \infty} (F_{v} : \mathcal{O}_{v}) \) and \( \mathfrak{a}_{F} \) for the ring of adeles \( F_{\infty} \times \mathfrak{A}_{F} \).

If \( G \) is an \( F \)-variety and \( E \) is an \( F \)-algebra write \( G(E) \) for the \( E \)-points of \( G \), equipped with the analytic topology if \( E \) is a local field extending \( F \). For a place \( v \) of \( F \) we also write \( G_{v} = G(F_{v}) \). We further write \( G = G_{\infty} \) for the manifold (Lie group) \( \prod_{v \in \infty} G_{v} \).

Let \( D \) be a quaternion algebra over \( F \), that is either the matrix algebra \( M_{2}(F) \) or a division algebra of dimension 4 over \( F \) (for a first reading one can consider the special case \( F = \mathbb{Q}(i), D = M_{2}(F) \)). Let \( \mathbb{D} \) be the variety such that \( \mathbb{D}(E) = D \otimes_{F} E \), and let \( \det: \mathbb{D} \to \mathbb{A}^{1} \) be the reduced norm. Let \( \mathbb{G} = \mathbb{D}^{1} \) be the algebraic group of elements of reduced norm 1 in \( \mathbb{D} \):

\[
\mathbb{G}(E) = \{ g \in D \otimes_{F} E \mid \det(g) = 1 \}.
\]

For each place \( v \in V \) we have the algebra \( D_{v} = D \otimes_{F} F_{v} \). When \( v \) is complex necessarily \( D_{v} \simeq M_{2}(\mathbb{C}) \) and thus \( G_{v} \simeq SL_{2}(\mathbb{C}) \). When \( v \) is real \( D_{v} \) is either the split algebra \( M_{2}(\mathbb{R}) \) or Hamilton’s quaternions \( \mathbb{H} \), and corresponding \( G_{v} \) is one of the groups \( SL_{2}(\mathbb{R}) \) and \( \mathbb{H}^{1} \simeq SU(2) \). We suppose there are \( s \) complex places and \( r + t \) real places divided into \( r \) of the first form and \( t \) of the second so that

\[
G = G_{\infty} \simeq SL_{2}(\mathbb{R})^{r} \times SL_{2}(\mathbb{C})^{s} \times SU(2)^{t}.
\]
2.2. Factoring $G_\infty$ over a number field; complex conjugation. We have the factorization $G_\infty = \prod_{v|\infty} G_v$. For a complex place $v \in \mathcal{V}_C$, the usual extension of scalars realizes $G_v$ as the group of complex points of the $\mathbb{C}$-group $G \times \mathbb{C}$. However, when thought of as a Lie group this group has closed subgroups which are not complex – the key example for us being the subgroup $\text{SL}_2(\mathbb{R})$ of $\text{SL}_2(\mathbb{C})$. We thus would like to think of the group $G_v$ as the group of real points of an algebraic group defined over an extension of $F$.

Let $N$ be any finite Galois extension of $\mathbb{Q}$ containing $F$. Given a complex place $v$ we construct a number field $E \subset N$ of index 2 equipped with a real place $\tilde{w}$ and an algebraic group $\tilde{G}$ defined over $\tilde{E}$ factoring as an $\tilde{E}$-group into a product $\tilde{G} = \prod_{i=1}^{r+p+3} \tilde{G}_i$ such that:

1. The group $G(F)$ embeds in $\tilde{G}(E)$.
2. We have an isomorphism

$$\tilde{G}(E_w) = \prod_{i=1}^{r+p+3} \tilde{G}_i(E_w) \to G_\infty$$

where the factors correspond.

3. For a rational prime $p$ splitting completely in $N$ we have an embedding $E \hookrightarrow \mathbb{Q}_p$ such that for $1 \leq i \leq s$ we have $\tilde{G}_i(\mathbb{Q}_p) \simeq (\text{SL}_2(\mathbb{Q}_p))^2$, for $s < i \leq r + t + s$ we have $\tilde{G}_i(\mathbb{Q}_p) \simeq \text{SL}_2(\mathbb{Q}_p)$ and the resulting factorization of $\tilde{G}(\mathbb{Q}_p)$ is isomorphic to the factorization of $G_p$ into $2s + r + t$ copies of $\text{SL}_2(\mathbb{Q}_p)$ coming from the $2s + r + t$ places over $F$ over $\mathbb{Q}_p$. We fix an embedding of $E$ in $\mathbb{Q}_p$ as above and use it to view $E$ as a subfield of $\mathbb{Q}_p$.

4. The latter two identifications are compatible with the first embedding, in such a way that if an element $\gamma \in G(F)$ embeds to a real-valued matrix in $\tilde{G}_i(D_w)$ (say) then its image in $\tilde{G}_1(\mathbb{Q}_p)$ lies in the diagonal subgroup of $\tilde{G}_1(\mathbb{Q}_p) \simeq (\text{SL}_2(\mathbb{Q}_p))^2$.

This will allow us to choose Hecke operators which "avoid" a real subgroup in a particular complex place.

Thus let $v$ be a complex place of $F$ and let $N$ be any finite Galois extension of $\mathbb{Q}$ containing $F$ (in Section 3 $N$ will be the Galois closure of a splitting field of $D$; in Section 5 we make a different choice depending on the subvariety we are trying to avoid). Let $w$ be a place of $N$ extending $v$, let $c_w \in \text{Gal}(N/\mathbb{Q})$ the element acting as complex conjugation in the completion of $N$ at $v$, let $E = N^{c_w}$ be the fixed field of $c_w$, and also write $w$ for the restriction of $w$ to $E$ – a real place of that field. The situation is simpler when $F$ contains $E$ (e.g. our running example of $F = N = \mathbb{Q}(i)$ where $E = \mathbb{Q}$) but we will not assume this is the case.

Writing $F = \mathbb{Q}[x]/(f)$ for an irreducible polynomial $f \in \mathbb{Q}[x]$ and factoring $f$ in the extensions $N$ and $E$ of $\mathbb{Q}$ we see that the $E$-algebra $A = F \otimes \mathbb{Q} E$ is étale, we have $A \simeq \prod_i E_i$ which we can interpret both as an isomorphism of $F$-algebras and as an isomorphism of $E$-algebras. From the first point viewing each embedding $F \hookrightarrow E_i$ with the embedding $w_i : E_i \to \mathbb{C}$ we obtain an enumeration of the archimedean places of $F$, with $E_i \simeq N$ for complex places (say those are indexed by $1 \leq i \leq s$) and $E_i \simeq E$ for real places (say for $s < i \leq s + r + t$). Without loss of generality we assume the inclusion $F \hookrightarrow E_i \hookrightarrow \mathbb{C}$ is the place $v$ fixed above.

Now thinking of $A$ as a $E$-algebra let $\tilde{G} = G \times F A$ thought of as an algebraic group over $E$. Equivalently $\tilde{G} = (\text{Res}_E^F G) \times_F E$ (Weil restriction of scalars). The factorization of $A$ then gives a factorization $\tilde{G} = \prod_i \tilde{G}_i$ as groups over $E$, where $\tilde{G}_i(E_w) \simeq G(F_i)$ if $v$ is the $i$th archimedean place of $F$ in terms of our enumeration above. Having done this we also write $G_i$ for $G_v$, especially in situations where we would like $v$ to vary over finite places.

The embedding $F \hookrightarrow A$ gives an inclusion $G(F) \to \tilde{G}(E)$. 
Finally let $p$ be a rational prime which splits completely in $N$ (hence also in $F \subset N$). Choosing any place $w_p : N \to \mathbb{Q}_p$ lying over $p$ all embeddings of $F$ into $\mathbb{Q}_p$ factor through $w_p$ (as above, with different embeddings of $F$ into $N$). Restricting $w_p$ to $E$ we then have

$$E_i \otimes_F \mathbb{Q}_p \simeq \begin{cases} \mathbb{Q}_p & E_i = E \\ \mathbb{Q}_p \times \mathbb{Q}_p & E_i = N \end{cases}$$

since when $E_i = N$ there are two places of $N$ lying over the place $w_p$ of $E$. Furthermore, $\text{Gal}(N/E) = \{c_w, \text{id}\}$ acts transitively on these places so that $c_w$ swaps them.

Finally let $\gamma \in G(F)$ and suppose that its image in $\hat{G}_1(E_w)$ lies in the subgroup $g_1 \hat{G}_1(E_w)^c \tilde{g}_1^{-1}$ for some $g_1 \in \hat{G}_1(E)$ (we think of $\hat{G}_1(E_w)^c$ as the “standard” copy of $\text{SL}_2(\mathbb{R})$ in $\hat{G}_1(E_w) \simeq \text{SL}_2(\mathbb{C})$). Since the image of $g_1^{-1}g_1 \in \hat{G}_1(E)$ is fixed by $c_w$, this persists when we embed our group in $\mathbb{Q}_p$, using the place $w_p$, and it follows that the image of $\gamma$ in $\hat{G}_1(E_{w_p})$ lies in $g_1 \hat{G}_1(E_{w_p})^c \tilde{g}_1^{-1}$, in other words in a particular conjugate of the diagonal subgroup $\hat{G}_1(E_{w_p})^c$ (so-called because when we identified $\hat{G}_1(E_{w_p})^c \simeq (\text{SL}_2(\mathbb{Q}_p))^2$ the Galois automorphism $c_w$ acts by exchanging the factors).

Before going any further, it may be helpful to have two concrete examples of the setup so far.

**Example 3.** Let $F = \mathbb{Q}(\xi^3)$ where $\xi^3 = 2$ and let $D = M_2(F)$ be the matrix algebra over $F$. The field $F$ has one real place $v'$ (coming from the unique real cube root of 2) and one complex place $v$ (coming from the two non-real cube roots), so the corresponding Lie group is $\text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{R})$ with the symmetric space $\mathbb{H}^3 \times \mathbb{H}^2$. Let us now examine how this fits in the framework above.

For this let $\omega$ be a cube root of unity so that $N = F(\omega) = F(\sqrt{-3})$ is the normal closure of $F$ in which the Galois conjugates of $\xi$ are $\xi, \xi^2, \xi \omega^2$.

We choose the complex embedding $v$ so that $w(\xi) = \sqrt[3]{2}(1/3)$ and extend this to $N$ by $w(\omega) = e(1/3)$. The Galois group $\text{Gal}(N/\mathbb{Q})$ is the full permutation group on the cube roots of 2 in $N$; among its elements let $c_w$ be the one fixing $\xi \omega^2$ and exchanging $\xi, \xi \omega$. Note that $w(\xi \omega^2) = \sqrt[3]{2} \in \mathbb{C}$ so $c_w$ is complex conjugation at $w$, and one can also check that $c_w(\omega) = \omega^2$. We thus obtain the subfield $E = N^{c_w} = \mathbb{Q}(\xi, \omega^2) \subset N$, a field which is abstractly isomorphic to $F$ but disjoint from it as a subfield of $N$.

Next, we have $F \otimes_F E = E[x]/(x^2 - 2) \simeq E[x]/(x^2 + \xi \omega^2 x + \xi^2 \omega)(x - \xi^{-1} \omega^2) \simeq E_1 \oplus E_2$ where $E_1 \simeq N = E(\omega)$ and $E_2 \simeq E$ (and also $E_2 \simeq F$). Extending scalars in $D$ we obtain the algebra

$$\tilde{D} = D \otimes_F (E_1 \oplus E_2) \simeq (D \otimes_F E_1) \oplus (D \otimes_F E_2) \simeq (D \otimes_F N) \oplus (D \otimes_F E)$$

It follows that $\tilde{D} \simeq M_2(N) \oplus M_2(E)$. Restricting to elements of determinant one produces from the algebras $D$, $\tilde{D}$ the $F$-group $G(K) = \{g \in M_2(K) \mid \det(g) = 1\}$ and the $E$-group $\hat{G}(K) = \{(g_1, g_2) \in M_2(K[\omega]) \times M_2(K) \mid \det(g_1) = \det(g_2) = 1\}$ for any field extension $K$ of $F$, $E$ respectively. It may seem odd to distinguish between $E$ and $F$ (which are after all isomorphic fields in this example), so let us put this distinction use. We have $G_m = \text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{R})$, but also $\hat{G}(E_w) \simeq \text{SL}_2(N_w) \times \text{SL}_2(E_w) \simeq \text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{R})$. Furthermore the isomorphism of the two groups $G_m \simeq \hat{G}(E_w)$ preserves the diagonal inclusion of $\text{SL}_2(F)$ in both of them—exactly because the complex place $w$ of $N$ restricts to the complex place of $F$ but the real place of $E$. Furthermore complex conjugation in the first factor is now algebraic: it is the obvious automorphism of the algebra $K[\omega] = K[x]/(x^2 + x + 1)$.

---

2We use the number theory convention $e(z) = \exp(\pi i z)$
Thus let $p$ be a rational prime which splits completely in $N$. Let $v_1, v_2, v_3$ be the three places of $F$ lying over $p$, giving us the group $G_p = \text{SL}_2(F_{v_1}) \times \text{SL}_2(F_{v_2}) \times \text{SL}_2(F_{v_3})$ together with the diagonal embedding of $\mathbb{G}(F) = \text{SL}_2(F)$ in $G_w \times G_p$. In the proof of Theorem 2 we will obtain elements $\gamma \in \mathbb{G}(F)$ whose image in the factor $G_v \simeq \text{SL}_2(\mathbb{C})$ lie in $\text{SL}_2(\mathbb{R})$, that is are fixed by the complex conjugation, and we will want to choose elements of $G_p$ which "avoid" those $\gamma$ in some sense. For this let $v_1$ be a place of $E$ lying over $p$. Observe that since $p$ splits in $N$, $\mathbb{Q}_p$ contains the cube roots of unity, $E_{v_1}[\omega] \simeq \mathbb{Q}_p \oplus \mathbb{Q}_p$ and this isomorphism gives us the other two places of $F$ in

$$\tilde{G}(E_{v_1}) \simeq \text{SL}_2(\mathbb{Q}_p \oplus \mathbb{Q}_p) \times \text{SL}_2(\mathbb{Q}_p) \simeq G_p.$$ 

However, all this is compatible with the action of $c_w$ in the first factor, mapping $\omega$ to $\omega^2 = -1 - \bar{\omega}$, which amounts to exchanging the two copies of $\mathbb{Q}_p$. Accordingly let $\gamma \in \mathbb{G}(F)$ be such that $v(\gamma)$ is real. Then the image of $\gamma$ in $\tilde{G}(E_{v_1})$ is fixed by $c_w$, so the same must hold for the image in $\tilde{G}(E)$ and hence the image in $\tilde{G}(E_{v_1})$. In short, we have chosen the factors in the isomorphism $G_p \simeq (\text{SL}_2(\mathbb{Q}_p))^3$ so that those $\gamma$ which are real in the first complex place are diagonal (have the same image) in the first two $p$-adic places. We could then "avoid" (in the precise sense we need) such $\gamma$ by choosing Hecke operators (see below) living at exactly one of the two places exchanged by $c_w$.

**Example 4.** Continuing with the same field $F$, let $D = \left( -\xi, -\bar{\xi} \atop F \right)$ be the quaternion algebra with $F$-basis $1, i, j, k$ such that $i^2 = j^2 = -\xi$ and $ij = -ji = k$. The reduced norm of this algebra is then the quadratic form

$$(2.1) \quad \det(a + bi + c j + dk) = a^2 + \xi b^2 + \xi c^2 + \bar{\xi} d^2$$

which is positive definite in the real embedding $\nu'$ for which $\nu'(\xi) = \frac{1}{\sqrt{2}}$. Then $D_{\nu'}$ is a non-commutative real division algebra, in other words is isomorphic to Hamilton quaternions – whereas we still have $D_{\nu} \simeq M_2(\mathbb{C})$ since the only complex division algebra is $\mathbb{C}$.

Viewing the quaternions as a 2-dimensional vector space over $\mathbb{C}$ and letting the invertible quaternions act on themselves by multiplication shows that the group of norm-$1$ quaternions isomorphic to $\text{SU}(2)$, and hence for the group $\mathbb{G}$ of norm-$1$ elements in $D$ we have

$$G_w \simeq \text{SL}_2(\mathbb{C}) \times \text{SU}(2)$$

With the corresponding symmetric space $\mathbb{H}^3$. The rest of the discussion in the previous example continues unchanged: for any prime $p$ we can still realize $G_p \simeq \tilde{G}(E_{v_1})$ so that complex conjugation in the first factor of $\tilde{G}$ corresponds to both complex conjugation in $G_v$ and to swapping the first two factors in $G_p \simeq (\text{SL}_2(\mathbb{Q}_p))^3$.

This elucidates the extra difficulty of proving our results in the case of lattices such as $\text{SL}_2(\mathbb{Z}[i])$ which do not act cocompactly on $\mathbb{H}^3$ but act on it in a more complicated fashion than a Bianchi group such as $\text{SL}_2(\mathbb{Z}[i])$. Consider an element of $\text{SL}_2(F)$ which is real in the complex embedding. When $F = \mathbb{Z}[i]$ (say) complex conjugation is a Galois automorphism of $F$ and it is clear how it acts on $\text{SL}_2(F_{v_1}) \times \text{SL}_2(F_{v_2})$ for the two places of $F$ lying over a rational prime $p$ splitting in $F$; elements of $\text{SL}_2(\mathbb{Q}(i))$ which are real in the complex embedding lie in $\text{SL}_2(\mathbb{R})$ and clearly embed diagonally. On the other hand in the present example $F$ has no Galois automorphism and so showing that elements of $D^1$ that are real at the complex place have identical images at two of the three $p$-adic places most naturally involves going beyond $F$. 

2.3. **The real group.** We return to the isomorphism $G_v \simeq (SL_2(\mathbb{C}))^r (SL_2(\mathbb{R}))^r (SU(2))^r$ and fix some subgroups of this group.

At each infinite $v$ where $D_v$ splits, let $A_v \subset G_v$ be the group corresponding to the group of diagonal matrices with positive real entries under the isomorphism above. Also let $K_v$ be a compatible maximal compact subgroup (corresponding to the subgroup $SU(2)$ at a complex place, $SO(2)$ at a real place). From these let $M_v = Z_{K_v}(A_v)$ be the centralizer of $A_v$ in $K_v$, so that $M_v$ is the group $\{ \pm I \} \subset SL_2(\mathbb{R})$ at a real place or the group $U(1) \subset SL_2(\mathbb{C})$ of diagonal matrices with inverse entries both of modulus 1. Observe that in either case the group $A_v M_v$ consists of the $F_v$-points of a maximal $F_v$-split algebraic torus of $G \times F_v$.

At the real places $v$ where $D_v$ remains a division algebra $K_v = G_v \simeq SU(2)$ is a maximal subgroup. With this choice $K = K_m = \prod_{v | \infty} K_v$ is a maximal compact subgroup of $G$ and $G/K \simeq \left( \mathbb{H}^3(\mathbb{R}) \right)^r / \left( \mathbb{H}^2(\mathbb{R}) \right)^r$.

We fix (arbitrarily) once and for all a left-invariant Riemannian metric on $G$ (equivalently, a positive definite quadratic form on Lie $G$). This induces a left-invariant metric on $X = \Gamma \backslash G$ where the quotient map is non-expansive. For a subset $U \subset G$ we write $U_\varepsilon$ for its $\varepsilon$-neighborhood with respect to this metric. This metric is used in Sections 5 and 6 but in both cases we can first restrict our attention to compact subsets of $G$. The choice of metric thus affects some overall constants but not the bottom line. For example in Section 5 we establish bounds of the form $\mu(U_\varepsilon) \leq Ce^{3\varepsilon}$; with a different metric $U_\varepsilon$ would be contained in $U_{\varepsilon'}$ with respect to the new one and the bound would be identical except for the value of the constant $C$.

2.4. **$p$-adic and adelic groups.** Let $R \subset D$ be an order, that is a subring which is an $\mathcal{O}_F$-lattice in $D$. Then for every finite place $v$ of $F$, $R_v = R \otimes_{\mathcal{O}_F} \mathcal{O}_v$ is an order in $D_v$, that is a compact open $\mathcal{O}_v$-subalgebra of $D_v$. Its group of units $R_v^\times$ is then a compact subgroup of $G_v$ which is a maximal compact subgroup $K_v$ of almost all places.

For all but finitely many places, $D_v \simeq M_2(F_v)$ and then $G_v \simeq SL_2(F_v)$ and $K_v \simeq SL_2(\mathcal{O}_v)$ (we don’t choose $K_v$ at the finitely many places where $D_v$ is a division algebra or where $R_v$ is not a maximal order).

Let $G(\mathbb{A}) = \prod_{v < \infty} (G_v : K_v)$ be the restricted direct product of the $G_v$ and let $G(\mathbb{A}) = G_\infty \times G(\mathbb{A}_F)$. Then the diagonal embedding $G(F) \hookrightarrow G(\mathbb{A})$ realizes $G(F)$ as a lattice there. Fix an open compact subgroup $K_f \subset G(\mathbb{A}_F)$. Then there exists a finite set $S$ of finite places (including all places where $K_v$ was left undefined above) such that $K_f = H \times \prod_{S \not \subset \infty} K_v$ for some open compact subgroup $H < \prod_{v \not \in S} G_v$. We will generally ignore all places in $S$.

Let $\Gamma = G(F) \cap K_f$. Its image in $G_\infty$ is a lattice, and we have the identification

$$X = \Gamma \backslash G \simeq G(F) \backslash G(\mathbb{A}) / K_f$$

given by mapping the coset $\Gamma g_\infty$ to the double coset $[g_\infty, 1] = G(F)(g_\infty, 1)K_f$ (here 1 is the unit element of $\mathbb{A}_F$).

A congruence subgroup of $G$ is one that contains a lattice $\Gamma$ as above as a finite-index subgroup. Since equidistribution modulo $\Gamma$ implies equidistribution modulo any lattice containing it (assuming the eigenfunctions are properly invariant) we assume the lattice has this form without loss of generality.

We further set $Y = X / K = \Gamma \backslash S$ where $S = G / K$ is the symmetric space of $G$. Equipping $X$ with the $G$-invariant probability measure we study functions (“automorphic forms”) in $L^2(X)$, and identify $L^2(Y)$ with the subset of $K$-invariant functions. For each noncompact

---

3 We use here that $G$ is a form of the simply connected algebraic group $SL_{2,\mathbb{Q}}$; in general the adelic quotient would correspond to a disjoint union of quotients $\Gamma \backslash G$. 
factor $G_i$ of $G$ let $\omega_i$ be the Casimir element in the universal enveloping algebra $U(\mathfrak{g}_i \otimes \mathbb{R} \mathbb{C})$. Then $\omega_i$ acts on the right on smooth functions on $G$ and $X$, equivariantly with respect to the right $G$-actions. It thus descends to a differential operator on functions on $S$ where it coincides with the Laplace–Beltrami operator on the irreducible symmetric space $G_i/K_i$. A Maass form on $Y$ is a function $\psi \in L^2(Y)$ which is a joint eigenfunction of the $\omega_i$.

2.5. Hecke operators. For every finite place $v$, the convolution algebra of locally constant compactly supported functions on the totally disconnected group $G_v$ acts on the right by convolution on the space of smooth functions on $G(F) \backslash G(A)$. At a place $v$ where $K_v$ is defined and contained in $K_i$, the subalgebra of bi-$K_v$-invariant functions $H_v = \mathcal{H}(G_v : K_v) = C_c^\infty(K_v \backslash G_v/K_v)$ preserves the subspace of right-$K_i$ invariant functions, and hence acts on the space of functions on $G(F) \backslash G(A)/K_i = \Gamma \backslash G$. We note that the actions of the different $H_v$ commute with each other and with the right $G$-action on our space, and call the algebra of operators on functions generated by all of them the Hecke algebra.

Let $\mathcal{P}$ be the set of rational primes $p$ which split completely in $N$, and such that every place $v$ of $F$ above $p$ has $K_v$ as a factor of $K_i$, a set of positive natural density in the primes. Those are the primes whose Artin symbol in $\text{Gal}(f) = \text{Gal}(N/\mathbb{Q})$ is trivial and the claim follows immediately from the effective Chebotarev Density Theorem [10, Theorem 1.2]. For such $p$ let $\mathcal{H}_p = \otimes_{v|p} \mathcal{H}_v$ be the “Hecke algebra at $p$”. We will only consider Hecke operators in the restricted Hecke algebra $\mathcal{H} = \otimes_{p \in \mathcal{P}} \mathcal{H}_p$ generated by the $\mathcal{H}_p$.

Since the right actions of $G_v$ and $G_i$ on $G(F) \backslash G(A_F)$ commute, the Hecke operators commute with the differential operators of the previous section. A Hecke–Maass form is a Maass form $\psi \in L^2(Y)$ which is also a joint eigenfunction of the Hecke algebra. Since the group actions commute it also follows that if $\phi \in L^2(X)$ is any other element of the irreducible representation generated by $\psi$ then $\phi$ is also an eigenfunction of the Hecke algebra with the same eigenvalues as $\psi$.

3. Homogeneity

In this section we invoke the necessary machinery to deduce Theorem [11] from Theorem [2]. The main new ingredient (which was already known to experts) is the statement of a so-called “diophantine lemma” (Proposition [10] below) for groups defined over a number field.

As described in the introduction let $\{\psi_j\}_{j \geq 1} \subset L^2(Y)$ be a normalized sequence of Hecke–Maass forms with Laplace eigenvalues tending to infinity. Let $\tilde{\mu}_j$ be the corresponding probability measures on $Y$ (recall that those are the measures with density $|\psi_j(x)|^2$ with respect to the Riemannian measure); our ultimate goal is to show that the $\tilde{\mu}_j$ converge to the normalized Riemannian volume on $Y$, or equivalently that this is the only subsequential limit. Accordingly (passing to a subsequence) we assume the $\tilde{\mu}_j$ converge weak-* to a measure $\tilde{\mu}$ on $Y$. The main result of [17] is that any such limiting measure $\tilde{\mu}$ on $Y$ is a probability measure, even if $Y$ non-compact.

Again passing to a subsequence there is an infinite place $v$ such $G_v$ is non-compact and such that the Laplace–Beltrami eigenvalues of $\psi_j$ with respect to the Laplace operator at $v$ tend to infinity; without loss of generality we may assume it is the first place in our enumeration. Then by the microlocal lift of [13] there are Hecke eigenfunctions $\phi_j \in L^2(X)$ such that any subsequential weak-* limit $\mu$ of the associated measures $\mu_j$ has the following properties:

1. $\mu$ projects to $\tilde{\mu}$ under the map $X \to Y$ (in particular, $\mu$ is a probability measure).
2. $\mu$ is $A_1$-invariant.
Passing to a subsequence yet again we may assume that the $\mu_j$ themselves converge, and would like to show that the limit $\mu$ is the $G$-invariant probability measure on $X$. At this point the differential operators exit the stage: in the sequel we only use the fact that $\phi_j$ are normalized Hecke eigenfunctions on $X$ and that $\mu$ has the two properties above.

The rest of the section is divided as follows: in Section 3.1 we show that every ergodic component of $\mu$ has positive entropy with respect to the action of $A_1$. In Section 3.2 we then invoke a measure rigidity theorem of Einsiedler–Lindenstrauss and interpret its results, classifying the possible ergodic components of $\mu$.

3.1. Positive Entropy. Let $a \in A_1$ be non-trivial. Under the isomorphism $G_1 \simeq \text{SL}_2(F_1)$, $a$ is a diagonal matrix with distinct real positive entries, so $T_1 = \text{Z}_{G_1}(a)$ is the group $A_1M_1$ of all diagonal matrices in $G_1$. Setting $G_{\infty, \geq 2} = \prod_{i \geq 2} G_i$, the centralizer of $a$ in $G$ is then $T = T_1 \cdot \prod_{i \geq 2} G_i$.

For a compact neighborhood of the identity $U \subset T$ recall our notation $U_\varepsilon$ for an $\varepsilon$-neighborhood of $U$ in $G$. We will establish the following result:

**Proposition 5.** There is a constant $h > 0$ such that for any compact subset $\Omega \subset G$ (expected to be large) and any $U$, we have for any $\varepsilon$ small enough (depending on $\Omega, U$) and any Hecke eigenfunction $\phi_j \in L^2(X)$ that for all $g \in \Omega$,

$$\mu_j(\Gamma g U_\varepsilon) \ll_{\Omega, U} \varepsilon^h.$$ 

The key point is that the implied constant is independent of $\phi$, so that the limiting measure $\mu$ satisfies the same inequality.

This result is essentially contained in [14] (on some level already in [11]) except that [14] assumes that $G$ is $\mathbb{R}$-split, which is not the case for $\text{SL}_2(\mathbb{C})$. In fact all that is needed there is that the centralizer $T_1$ at some infinite place $G_v$ is a torus.

3.1.1. Diophantine Lemma. We begin by reviewing a notion of "denominator" for elements of $F$ and $\mathbb{G}(F)$. The construction is the natural generalization to number fields of the notion used in [14] for groups over the rationals. For further discussion see Remark 11.

**Definition 6.** The denominator of $x_v \in F_v$ is the natural number $\text{denom}_v(x_v) = \max\{|x_v|_v, 1\}$. Equivalently if $x_v = \sigma_v^k y$ with $y \in \mathcal{O}_v^\times$ then

$$\text{denom}_v(x_v) = \begin{cases} q_v^{-k} & k \leq 0 \\ 1 & k \geq 0 \end{cases}.$$ 

We now extend this definition. First, for $x \in \mathbb{A}_F$ or $x \in \mathbb{A}_F$ we set

$$\text{denom}_x = \prod_{v \nmid \infty} \text{denom}_v(x_v),$$

where all but finitely many of the factors are 1 since $x_v \in \mathcal{O}_v$ for all but finitely many $v$. Second for $x \in F$ let $\text{denom}(x)$ be the denominator of its image in $\mathbb{A}_F$.

We further extend the definition to matrix algebras over the above rings. Specifically for $x_v \in M_N(F_v)$ let $\text{denom}_v(x_v)$ be the largest of the denominators of the matrix entries (equivalently this is the denominator of the fractional ideal they generate), and again extend this to $M_N(\mathbb{A}_F)$, $M_N(\mathbb{A}_F)$ and $M_N(F)$ by multiplying over the places and restriction, respectively.

Finally, the product formula $\prod_v |x|_v = 1$ for $x \in F^\times$ implies

$$(\prod_v |x|_v) \cdot \text{denom}_x \geq 1,$$
for all nonzero $x \in F$, and hence also for all $x \in GL_n(F)$.

**Remark 7.** We could have defined the denominator of $x \in M_N(\mathbb{A}_F)$ by taking the largest of the denominators of its entries (call that denom$'$ for the nonce); the two notions are equivalent in that denom$'$(x) ≤ denom(x) ≤ denom$'$(x)$^2$ for all x. Our choice agrees with defining for $x \in M_N(F)$ the denominator as the denominator of the fractional ideal generated by the matrix entries. In the sequel the precise choice of denominator affects the exponents in Proposition [10] and thus the precise entropy $h$ we obtain in Proposition [12] but does not change the positivity of $h$, which suffices for our purposes (and ultimately by determining the limit exactly we prove the measure has maximal entropy anyway).

The following is an immediate calculation and we omit the proof.

**Lemma 8.** Let $x_v, y_v \in M_N(F_v)$. Then

\[ \text{denom}_v(x_v + y_v), \text{denom}_v(x_v y_v) \leq \text{denom}_v(x_v) \text{denom}_v(y_v), \]

and in particular if $y_v \in GL_N(O_v)$ then \( \text{denom}_v(x_v y_v) = \text{denom}_v(x_v) \). Furthermore there is a constant $C$ depending only on $n$ such that

\[ \text{denom}_v(x_v^{-1}) \leq \text{denom}_v(x_v)^C \]

for $x \in SL_N(F_v)$.

**Corollary 9.** Multiplying place-by-place the same inequalities hold for the denominators in $M_N(\mathbb{A}_F)$ and $M_N(F)$.

Finally if $G$ is a linear algebraic $F$-group fixing an $F$-embedding $\rho: G \to SL_N$ allows us to define the denominators of elements of $G(F)$, $G(\mathbb{A}_F)$, $G(F)$. The same reasoning as above would show that changing the embedding gives an equivalent definition in the sense above (i.e. up to multiplying by constants and raising to powers). In our case letting $D$ act on itself by multiplication gives an embedding $G \to SL_4(F)$ and using the order $R$ to define the integral structure we further have $\rho(K_v) \subset SL_4(O_v)$ whenever $K_v$ is defined. It follows that our local denominator is a bi-$K_v$-invariant function on $G_v$, so every basic Hecke operator (the characteristic function of a double coset $K_v g, K_v$) has a well-defined denominator. Furthermore (identifying $G_v \simeq SL_2(F_v)$) this double coset has a representative $a_v = \begin{pmatrix} \sigma_v^m & 0 \\ 0 & \sigma_v^{-m} \end{pmatrix}$ for some $m \geq 0$ in which case we call $2m$ the radius of the Hecke operator; its denominator is then $q_v^m$.

Given constants $c_1, c_2$ the set of **potential Hecke operators** is the set of $g \in G(\mathbb{A}_F)$ of denominators at most $c_1 e^{-c_2}$. We will be using Hecke operators of uniformly bounded radius so the main effect here is to bound the set of places $v$ under consideration. Say that a potential Hecke operator causes an intersection at $g_v \in \Omega$ if there is $\gamma \in G(F)$ such that

\[ \gamma g_v U_{\epsilon} K_{\kappa} \cap g_v U_{\epsilon} g \kappa K_{\kappa} \neq \emptyset, \]

in which case we say $\gamma$ is involved in the intersection.

**Proposition 10.** One can choose $c_1, c_2, \epsilon_0 > 0$ such that if $\epsilon < \epsilon_0$ then for all $g_v \in \Omega$ there is an algebraic $F$-torus $S \subset G$ such that all $\gamma \in G(F)$ involved in intersections at $g_v$ lie in $S(F)$.

**Proof.** The first step of the proof is to show that there is a choice of $c_1, c_2, \epsilon_0$ so that for any $g_v \in \Omega$, all $\gamma \in G(F)$ involved in intersections at $g_v$ commute with each other, so that the $F$-group generated by them is commutative. In fact we simply take $c_1 = 1$ and show
that there is a choice of $c_2, \varepsilon$ as above. For this, suppose we have $b_1, b_2 \in U_\varepsilon$ and $k_\ell \in K_\ell$ such that $\gamma g_{\varepsilon} b_1 = g_\varepsilon b_2 g_{k_\ell}$, or equivalently
\[
\gamma = g_\varepsilon b_2 b_1^{-1} g_{k_\ell}^{-1} g_\varepsilon b_1.
\]
By hypothesis $\text{denom}(\gamma) = \text{denom}(g_{k_\ell}) = \text{denom}(g_\varepsilon) \leq \varepsilon^{-c_2}$. In addition since $\Omega$ is compact we have that $g_\varepsilon b_2 b_1^{-1} g_\varepsilon^{-1}$ is $O(\varepsilon)$-close to an element of $UU^{-1}$.

The matrix commutator $[\gamma_1, \gamma_2] = \gamma_1 \gamma_2 - \gamma_2 \gamma_1$ (interpreted via the fixed embedding in $SL_2$) is a polynomial function $\mathbb{G}^2 \to SL_2$ with coefficients in $F$. Thus we have a constant $c$ so that if $\gamma_1, \gamma_2 \in \mathbb{G}(F)$ then $\text{denom}([\gamma_1, \gamma_2]) \leq c \text{denom}(\gamma_1)^i \text{denom}(\gamma_2)^i$. Now suppose that $\gamma_1, \gamma_2$ are both involved in intersections at $g_\varepsilon$. Then $\gamma_1$ and $\gamma_2$ are $C_\Omega \varepsilon$-close to elements of $g_\varepsilon UU^{-1} g_\varepsilon^{-1}$. Since the commutator is a smooth function on $G \times G$ the element $[\gamma_1, \gamma_2] \in G$ is $O(\Omega U \varepsilon)$-close to the commutator of two elements drawn from $g_\varepsilon UU^{-1} g_\varepsilon^{-1}$. Since $T_1$ is commutative (recall that we defined $T_1 = Z_{G_1}(a) = A_1M_1$, that is the group of diagonal matrices in $G_1$) we conclude that $\| [\gamma_1, \gamma_2] \|_1 = O(\Omega U \varepsilon)$, and that for $i \geq 2$ we have $\| [\gamma_1, \gamma_2] \|_i = O(\Omega U \varepsilon)$ if we also assume $\varepsilon < 1$.

Suppose $\gamma_1, \gamma_2$ do not commute. By the product formula we then have
\[
\text{denom}( [\gamma_1, \gamma_2] ) \cdot \prod_i \| [\gamma_1, \gamma_2] \|_i \geq 1, \tag{3.1}
\]
that is
\[
ces_{-c_2} O(\Omega U \varepsilon) \cdot O(\Omega U \varepsilon)^{i} \cdot O(\Omega U \varepsilon)^{i-1} \geq 1.
\]

If $c_2 < \frac{1}{\varepsilon}$ then (3.1) is impossible for any $\varepsilon$ small enough. We conclude that there are $c_2, \varepsilon_0 > 0$ such that the $\gamma$ that are involved in intersections commute, so the $F$-subgroup of $\mathbb{G}$ generated by those $\gamma$ is commutative. If this subgroup is finite its elements are semisimple (and contained in a torus), and otherwise the only commutative connected subgroups of $SL_2$ (even over its algebraic closure) are either tori or unipotent. Note that tori are self-centralizing whereas the component group of the centralizer of a unipotent subgroup is represented by the center of $SL_2$.

This concludes the argument when $D$ is a division algebra, since in that case $\mathbb{G}(F)$ consists entirely of semisimple elements and the maximal commutative $F$-subgroups are all tori. When $D = M_2(F)$ we need to rule out the possibility that some $\gamma$ causing an intersection is unipotent. The basic idea for this is that a unipotent element that is close to a torus is close to the identity element. Thus if $\gamma$ is unipotent then the bound for the denominator of $\gamma$ forces it to be the identity element. To prove it, choose $U$ and $\varepsilon_0$ small enough so that every $g \in U_\varepsilon U_0^{-1}$ is close enough to the identity to have $|\text{Tr}(g) - 2|_i < 1$ at each infinite place. Now suppose that $\gamma \in \mathbb{G}(F)$ is involved in an intersection at $g_\varepsilon$, and that $\gamma$ is unipotent. Then $|\text{Tr}(\gamma) - 2|_i < 1$ so $\text{Tr}(\gamma) = 2$ (the alternative was that $\text{Tr}(\gamma) = -2$).

Writing $\gamma = g_\varepsilon b_2 b_1^{-1} g_\varepsilon^{-1}$ we get that $|\text{Tr}(b_2 b_1^{-1})| = 2$. At the factor $G_1$ each $b_2 b_1^{-1}$ is $\varepsilon^\eta$-close to an element of $T_1$, for some absolute $\eta > 0$, so this element has trace $2 + O(\varepsilon^\eta)$ (as measured by $|.|_1$); since $T_1$ is a (fixed) torus this element is $O(\varepsilon^\eta)$-close to the identity element. Finally, since $\Omega$ is compact, conjugation by $g_\varepsilon$ does not change this fact: the image of $\gamma$ itself in $G_1$ is $O(\varepsilon^\eta)$-close to the identity element. Thus $\gamma = 1d$ if $c_2 < \eta$ and $\varepsilon$ is small enough.

\[\square\]

Remark 11. The concrete argument using the commutator already appears in [Lem. 3.3] and two more general versions appears in [Sec. 4] for group over $\mathbb{Q}$. The new observation here is that one only needs information about the $\gamma$ at a single real place (assuming
boundedness at the other real places). In particular we cannot just restrict scalars to $\mathbb{Q}$ and apply the earlier result.

3.1.2. **Bounds on the mass of tubes.** The arguments of [14, §5] (in fact already of [11] for the group at hand) now establish the following, using the diophantine lemma we proved above instead of the analogous lemmas in those papers, noting that our notion of denominator is equivalent to the notion we’d obtain over $\mathbb{Q}$ for the group $\text{Res}^{F}_{Q} G$.

**Proposition 12.** Fix compact subsets $\Omega \subset G, C \subset T_1 \times G_{m,\geq 2}$. Then for any normalized Hecke eigenfunction $\phi \in L^2(X)$ (pulled back to a function on $G$) and any $g_{\infty} \in \Omega$ we have

$$\int_{g_{\infty} - U} |\phi(x)|^2 \text{dvol}(x) = O(e^h)$$

for a universal constant $h > 0$ where the implied constant depends on $\Omega, C$ but not on $\phi$.

**Corollary 13.** $A_1$ acts on almost every ergodic component of $\mu$ with positive entropy.

The fact that Proposition 12 implies Corollary 13 is standard. Roughly speaking, it follows by the (relative) Shannon–McMillan–Brieman Theorem and the fact that the shape of a generic atom in a refinement of an appropriate partition of $X$ by the action of $A_1$ is approximated by sets $U_\varepsilon$ as in Proposition 12. For the basic definitions of entropy and precise statements from which the fact that positive entropy of a.e. ergodic component follows from bounds as in Proposition 12 see the appendix and Section 4.1 in [14].

3.2. **Measure rigidity.** In this section we interpret $\mu$ as a measure on $G(F) \backslash G(\mathbb{A})$. Let $\nu$ be any finite place at which $G_{\ell}$ is non-compact. Then [8 Thm. 8.1] shows that $\mu$ is $G_{\nu}$-recurrent: if $B \subset G(F) \backslash G(\mathbb{A})$ is any set of positive measure then for almost every $x \in B$ the set of returns $\{ g_{\nu} \in G_{\nu} \mid xg_{\nu} \in B \}$ is unbounded.

**Lemma 14.** For $\mu$-almost every $x \in G(F) \backslash G(\mathbb{A})$ the group

$$S(x) = \{ h \in M_1 \times G_{m,\geq 2} \times G_{\nu} \mid xh = h \}$$

is finite.

**Proof.** Let $x = G(F)g$ with $g = (g_{\infty}, g_{\ell}) \in G(\mathbb{A})$ where $g_{\ell} \in K_{\ell}$, and $h \in G(\mathbb{A})$ with $h_1 \in M_1$, and suppose that $G(F)g \cdot h = G(F)g$. Then there exists $\gamma \in G(F)$ such that $g \cdot h = \gamma g$ or equivalently

$$\gamma = ghg^{-1}.$$ 

Thus conjugation by $g$ (and the inclusion $G(F) \hookrightarrow G_1$) embeds $S(x)$ in $G(F) \cap g_1(A_1 M_1)g_1^{-1}$. As in Section 3.1.1 above this intersection consists of the $F_1$-points of the diagonal $F_1$-torus of $G_1$, which is one-dimensional, so if the group in the statement was infinite it would be Zariski-dense in the conjugate torus; equivalently the Zariski closure of this group would be an $F$-torus $T \subset G$ with $g_1^{-1}Tg_1$ diagonal.

For each torus $T$ the set of $g_1 \in G_1$ that diagonalize it lies in two $A_1 M_1$-cosets (due to the effect of the Weyl group). Since there are countably many such tori the set of $x$ for which $S(x)$ is infinite is contained in the countable union of images in $G(F) \backslash G(\mathbb{A})$ of the sets of the form

$$(g_1 A_1 M_k) \times G_{m,\geq 2} \times K_{\ell},$$

and we need to show this is a null set. Since our measure is right-$K_{\ell}$-invariant we may instead consider the image of the set in $X = \Gamma \backslash G_{\infty}$, and it remains to recall that the positive entropy argument showed that the images of the sets $(g_1 A_1 M_k) \times G_{m,\geq 2}$, have $\mu$-measure zero in a strong quantitative form (the argument gave a uniform bound for the mass of $\varepsilon$-neighborhoods of compact parts of such images). \qed
At this point we have verified the hypotheses of the following measure rigidity result.

**Definition 15.** A measure $\nu$ on $X$ is **homogeneous** if it is the unique $H$-invariant probability measure on a closed $H$-orbit for a closed subgroup $H < G$.

**Theorem 16** (Einsiedler–Lindenstrauss [5, Thm. 1.5]). Let $\mu$ be an $A_1$-invariant probability measure on $X$ such that

1. $\mu$ has positive entropy on a.e. ergodic component with respect to the action of $A_1$.
2. $\mu$ is $G$-recurrent.
3. For $\mu$-a.e. $z \in \Gamma \setminus \mathbb{G} \times \mathbb{G}$, the group
   \[ \{h \in M_1 \times G_{\alpha,\geq 2} \times G_v \mid zh = z\} \]
   is finite.

Then $\mu$ is a convex combination of homogeneous measures. Furthermore, for each such component $\nu$ the associated group $H$ contains a semisimple algebraic subgroup of $G_1$ of real rank 1 which further contains $A_1$ and conversely $H$ is a finite-index subgroup of an algebraic subgroup of $G$ (here we think of $G_1$ and $G$ as real algebraic groups).

By strong approximation the lattice $\Gamma$ is irreducible in $G$, so that any $G_1$-invariant measure is in fact $G$-invariant. Thus every component $\nu$ for which $H$ contains $G_1$ is the $G$-invariant measure. In particular this happens when $G_1 \cong \text{SL}_2(\mathbb{R})$ (the corresponding place of $F$ is real) since $\text{SL}_2(\mathbb{R})$ has no proper semisimple subgroups, at which point Theorem 1 follows directly. For the remainder of the paper we will thus assume that $G_1 \cong \text{SL}_2(\mathbb{C})$ (the place is complex) and that $H$ contains a conjugate of $\text{SL}_2(\mathbb{R})$ there (these being the only proper semisimple subgroups).

We now fix a particular representative for this conjugacy class. Let $\tilde{H}_1 \subset \tilde{G}_1$ be the fixed points of the automorphism $c_w$. Indeed the isomorphism $G_1 \cong \text{SL}_2(\mathbb{C})$ has $c_w$ act via complex conjugation so $H_1 = \tilde{H}_1(E_w)$ is exactly $\text{SL}_2(\mathbb{R})$, the subgroup of matrices fixed by complex conjugation. Recall that our choice of $A_1$, the positive diagonal subgroup makes it a subgroup of this $H_1$, in fact a real Cartan subgroup there.

**Proposition 17.** Let $\beta$ be a component of $\mu$ as in the Theorem. Then the support of $\beta$ is contained in the image in $X$ of a submanifold of $G$ of the form

\[ L = g_1H_1M_1G_{\alpha,\geq 2} \]

where $g_1 \in \tilde{G}_1(\tilde{\mathbb{Q}} \cap E_w)$.

**Proof.** Let $\beta$ be a component of $\mu$ as in the theorem, supported on an $H$-orbit $xH \subset X$ where $H$ contains a conjugate of the fixed group $H_1$. Let $\tilde{H} \subset G = \tilde{G}(E_w)$ be the semisimple $E_w \cong \mathbb{R}$-algebraic subgroup of which $H$ is a finite-index subgroup. Note that $\tilde{H} \cap G_1 = H \cap G_1$ because $\text{SL}_2(\mathbb{R})$ is a maximal proper closed subgroup of $\text{SL}_2(\mathbb{C})$, and for the same reason $H_1$ is the projection to $G_1$ of any closed subgroup of $G$ that contains $H_1$.

Writing $x = \Gamma g$ for some $g \in G$ the translate $xHg^{-1}$ is the $H'$-orbit of the identity coset and supports an $H'$-invariant measure; in other words $\Gamma \cap H'$ is a lattice in $H'$. By the Borel Density Theorem this lattice is Zariski-dense in $\tilde{H}'$ making $\tilde{H}_1$ defined over $E$ in view of $\Gamma \subset G(F) \subset \tilde{G}(E)$.

The intersection $G_1 \cap H' = g_1(G_1 \cap H)g_1^{-1}$ is also defined over $E$, where we write $g_1 \in \tilde{G}_1(E_w)$ for the first coordinate of $g$. The group $g_1(G_1 \cap H)g_1^{-1}$ and the fixed group $H_1$ are then conjugate subgroups of $G_1$ defined over $E$, so by the Lemma below they are conjugate by an element of $\tilde{G}_1(E_w \cap \tilde{\mathbb{Q}})$; let $g_1'$ be such an element. We then have
Conjugating $A_1 < H$ (an avatar of the $A_1$-invariance of $\mu$), gives a Cartan subgroup $(g_1^{-1}g'_1)^{-1}A_1(g_1^{-1}g'_1) \subset H_1$, and since all Cartan subgroups of a semisimple Lie group are conjugate we obtain $h_1 \in H_1$ such that

$$h_1A_1h_1^{-1} = (g_1^{-1}g'_1)^{-1}A_1(g_1^{-1}g'_1)^{-1}.$$ 

so that $h_1^{-1}(g_1^{-1}g'_1)^{-1} \in N_{G_1}(A_1) = N_{H_1}(A_1)M_1$ and thus that $(g_1^{-1}g'_1)^{-1} \in H_1M_1$.

Finally in terms of all those elements,

$$\Gamma gH \subset \Gamma g_1(g_1^{-1}g)(G_1 \cap H)G_{\infty \geq 2} = \Gamma g_1(g_1^{-1}g'_1)H_1(g_1^{-1}g'_1)^{-1}G_{\infty \geq 2} \subset \Gamma g'_1H_1H_1MG_{\infty \geq 2} \subset \Gamma g'_1H_1MG_{\infty \geq 2}.$$ 

□

Now there are countably many submanifolds of the form above, so to prove our main theorem and show that $\mu$ is the $G$-invariant measure it suffices to rule out each submanifold separately.

**Lemma 18.** Let $G$ be an algebraic group over $\mathbb{R} \cap \overline{\mathbb{Q}}$. Suppose that $\mathbb{H}_1, \mathbb{H}_2$ are two subgroups of $G$ and $H_1(\mathbb{R})$ is contained in $g\mathbb{H}_2(\mathbb{R})g^{-1}$ for some $g \in G(\mathbb{R})$. Then $H_1(\mathbb{R})$ is contained in $g'\mathbb{H}_2(\mathbb{R})g'^{-1}$ for some $g' \in G(\mathbb{R} \cap \overline{\mathbb{Q}})$.

**Proof.** The statement “there exists $g \in G$ such that $g\mathbb{H}_1g^{-1} \subset \mathbb{H}_2$” can be expressed in first-order logic in the language of fields. Thus the result follows from the fact that the first order theory of real closed fields is complete. □

## 4. Submanifolds with small stabilizers

In the previous section we showed that components of the limit measure $\mu$ other than the uniform measure are supported in images in $X = \Gamma \backslash G_{\infty}$ of submanifolds of $G_{\infty}$ of the form $L = g_1H_1M_1G_{\infty \geq 2}$. In this section we will use elementary arguments to show that such manifolds $L$ and their submanifolds cannot be left-invariant by subgroup of $G_1$ which are "too large" in a technical sense. In the next section we will then construct Hecke operators which avoid such subgroups, allowing us to prove Theorem 12 in the last section.

Since the calculations will take place in the group $G_1$ rather than the entire group $G_{\infty}$, for this section only we write $G$ for this group (recall that it is defined over $E$) and $G$ for the group $G_1(E_w) \simeq SL_2(\mathbb{C})$ that we usually denote $G_1$. Similarly we will drop the subscript 1 for algebraic and Lie subgroups of $G$, especially $H \simeq SL_2(\mathbb{R})$ and $M \simeq U(1)$. We write $\mathbb{R}$ for $E_w$ and $\mathbb{C} = N_w$.

**Definition 19.** Let $R < G$ be an (abstract) subgroup. We say that $R$ is small (in $G$) if there exist a finite extension $K \subset \mathbb{C}$ of $E$ and a finite-index subgroup $R' < R$ such that $R'$ is contained in the $K$-points $B(K)$ of a $K$-subgroup $B$ of $G$ of one of the following forms:

1. Multiplicative type: $B$ is diagonalizable over $\mathbb{C}$.
2. $SL_2$-type: $B$ is $G(\mathbb{K})$-conjugate to a subgroup of $\mathbb{H} \times_E K$. 


Remark 20. Observe that $\mathbb{G}(\mathbb{C}) \simeq \text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{C})$ in such a way that the image of $\mathbb{H}(\mathbb{C})$ is the subgroup $\{(g, g) \mid g \in \text{SL}_2(\mathbb{C})\}$ (with $c_w$ acting on the complexified group by exchanging the two factors). This allows us to check that the subgroup $\text{SU}_2 \Gamma_{\text{Shem-Tov–Silberman}: \text{AQUE on } \Gamma \backslash (\mathbb{H}^2)^* \times (\mathbb{H}^3)^s}$ is also of $\text{SL}_2$-type: its complexification is the subgroup $\{(g, g^{-1}) \mid g \in \text{SL}_2(\mathbb{C})\}$ which is conjugate to $\mathbb{H}$ by the element $\left( \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \right)$.

We will be interested in the stabilizers of algebraic varieties. On the one hand they will arise for us as abstract subgroups stabilizing sets of points. On the other hand, we will also be analyzing them as algebraic groups. To connect the two points of view we rely on the following:

Theorem 21 ([9 Cor. 1.81]). Let $\mathbb{G}$ be an algebraic group acting on an algebraic variety $\mathcal{X}$. Let $\mathbb{P} < \mathcal{X}$ be a closed subvariety. Then the stabilizer $\mathcal{S} = \{g \in \mathbb{G} \mid g\mathbb{P} = \mathbb{P}\}$ is a closed subgroup of $\mathbb{G}$. Furthermore if $\mathcal{K}$ is an extension of the field of definition such that $\mathbb{P}(\mathcal{K})$ is Zariski-dense in $\mathbb{P}$ then $\mathcal{S}(\mathcal{K}) = \{g \in \mathbb{G}(\mathcal{K}) \mid g\mathbb{P}(\mathcal{K}) = \mathbb{P}(\mathcal{K})\}$.

Definition 22. We say that a submanifold $L \subset G$ has small stabilizers if its Zariski closure $\bar{L} \subset \bar{G}$ over $\bar{\mathbb{Q}}$ is defined over a finite extension $\mathbb{F}^*/\mathbb{E}$ contained in $\mathbb{R}$ and we have

- For every $F'$-subvariety $\mathbb{P} \subset L$ such that $\mathbb{P}(\mathbb{R})$ is Zariski-dense in $\mathbb{P}$, the stabilizer $S_\mathbb{P} = \{s \in \mathbb{G}(F') \mid s\mathbb{P}(\mathbb{R}) = \mathbb{P}(\mathbb{R})\}$,

is a small subgroup of $G$.

We begin our analysis of stabilizers of subvarieties of the submanifolds $gHM$ with the observation that $HM$ is, in fact, an $E$-subvariety, as the reader can verify by writing explicit algebraic equations in the real and complex parts of the matrix entries. Each translate $gHM$ is then also a real subvariety which, in the case of interest where $g$ has algebraic entries, is defined over a number field.

As a preliminary step we rule out one possible kind of stabilizer:

Lemma 23. Let $P \subset L = gHM$ be a real subvariety, and let $\mathcal{S}$ be its stabilizer and $\mathcal{S} = \mathcal{S}(\mathbb{R})^o$ (identity component in the analytic topology). Then $\mathcal{S}$ is one of:

1. the real points of an algebraic torus; or
2. compact (as a Lie group); or
3. contained in a conjugate of $H = \text{SL}_2(\mathbb{R})$.

Proof. We have $\dim \mathcal{S} \leq \dim P \leq \dim L = 4$, and since $L$ itself is connected but not the coset of a subgroup, the dimension of $\mathcal{S}$ is actually at most 3.

Other than possibilities 1-3 above the only other closed connected subgroups of $\text{SL}_2(\mathbb{C})$ of dimension at most 3 are solvable with unipotent radical of dimension 2, so we suppose $\mathcal{S}^o$ is of that form. Conjugating $\mathcal{S}$ (and translating $L$ and $P$ appropriately) we may assume that $\mathcal{S}$ contains all the matrices of the form $u(z) = \left( \begin{smallmatrix} 1 & z \\ 0 & 1 \end{smallmatrix} \right)$ for $z \in \mathbb{C}$, which we can now rule out by by showing that no orbit of this group is contained in any submanifold $L$ as above, let alone in a submanifold $P$.

Thus fix $l = ghm$ for some $g \in G$, $h \in H$, $m \in M$. Setting $y = gh$ that $u(z)ghm \in gHM$ is equivalent to

$$y^{-1}u(z)y \in HM,$$
and for $y = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the left hand side is
\[ y^{-1}u(z)y = I_2 + \begin{pmatrix} cd & d^2 \\ -c^2 & -cd \end{pmatrix} z. \]

Now the only complex subspace of the tangent space at the identity of $HM$ is the Lie algebra of $AM$, that is the diagonal matrices, whereas the image of the derivative of the map $z \mapsto y^{-1}u(z)y$ is never of this form since $c, d$ can’t both vanish. \qed

We can now prove the following assertion:

**Proposition 24.** Let $F' \subset \mathbb{R}$ be a finite extension of $E$, and let $g \in \mathbb{G}(F')$. Then the submanifold $L = gHM$ of $G$ has small stabilizers.

**Proof.** We observed earlier $L$ is Zariski-closed and defined over $F'$; let $L$ be the underlying subvariety, if $F'$ an $F'$-subvariety with dense real points, and $S$ its stabilizer. We need to show that $S(F')$ is small.

If we are in the first possibility in Lemma \[23\] then $S$ is diagonal, hence of multiplicative type. In the second possibility $S$ is contained in a maximal compact subgroup, i.e., a conjugate of SU(2), so either $S$ is conjugate to SU(2) (over a number field by Lemma \[18\]) and hence of SL₂-type as observed in Remark \[20\], or $S$ is contained in a torus and hence is of multiplicative type. The only remaining possibility is that $S$ is conjugate to a subgroup of SL₂(\mathbb{R}), which follows from Lemma \[18\] as well. \qed

5. Constructing an amplifier

Given a Hecke eigenfunction $\phi \in L^2(X)$ and a small subgroup $S$ of $G_1$, in section this we construct a Hecke operator that acts on $\phi$ with large eigenvalue (making it an “amplifier” for $\phi$) while at the same time avoiding the orbits of $S$ in products of trees $\mathcal{G}_1(\mathbb{Q}_p)/\mathcal{G}_1(\mathbb{Z}_p)$.

We begin with the observation that having constructed the factorization $\mathcal{G} = \prod_i \mathcal{G}_i$ over $E$, we can now extend the field $N$ to contain the field $K$ evincing the smallness of $S$ as in Definition \[19\] in such a way that $E$ contains the field of definition $F'$ of $S$. Thus for the rest of the paper the fields $E$ and $N$ and the factorizations of $G$ will depend on $S$ (and ultimately on the subvariety of $G$ we are trying to avoid) rather than just on $\mathcal{G}$ as in the first run through Section 2. Since we are ruling out the subvarieties one-by-one this is not a problem.

Recall from that section that the group $\mathcal{G}_1$ is defined over $E$ so that $\mathcal{G}_1(E) = SL_2(E_1)$ where since we assume $G_1 = SL_2(\mathbb{C})$ we have $E_1 = N$, and in that case for each prime $p$ splitting completely in $N$ we can fix a place $w_p$ of $N$ lying over $p$ so that $E_1 \otimes_E N_{w_p} \simeq \mathbb{Q}_p \times \mathbb{Q}_p$ or equivalently so that $G_{p,1} = \mathcal{G}_1(E_{w_p}) \simeq SL_2(\mathbb{Q}_p) \times SL_2(\mathbb{Q}_p)$ (with $c_w$ exchanging the two factors). Then
\[ G_p = \prod_{v \mid p} G_v = \prod_i \mathcal{G}_1(E_{w_p}) = G_{p,1} \times G_{p,2}, \]

and omitting finitely many primes we may also assume that with this identification the maximal compact subgroups $K_{p,i}$ isomorphic to $SL_2(\mathbb{Z}_p)$ or $SL_2(\mathbb{Z}_p)^2$ of the factors are factors of the adelic open compact subgroup $K_1$. Let $K_p$ be their product, so under the assumptions this section, $K_p \simeq SL_2(\mathbb{Z}_p)^2$.

**Definition 25.** A Hecke operator $\tau \in \mathcal{H}_p$ one-sided if it is supported in the image of the first factor of $G_{p,1}$. 

Identifying $G_{p,1}$ with $\text{SL}_2(\mathbb{Q}_p)^2$ and $K_p$ with $\text{SL}_2(\mathbb{Z}_p)^2$ the basic Hecke operators $\tau_{p^j}$ will be the one-sided Hecke operator corresponding to the characteristic function of the double coset

$$\text{SL}_2(\mathbb{Z}_p) \begin{pmatrix} p^j & 0 \\ 0 & p^{-j} \end{pmatrix} \text{SL}_2(\mathbb{Z}_p),$$

that is summing over the sphere of radius $2j$ in the Bruhat-Tits tree of the first factor of $G_{p,1}$.

**Lemma 26.** Let $S \subset \mathbb{G}_1(E)$ be a small subgroup. Then there is a constant $C$ (depending on $S$) such that for all primes $p > C$ (splitting completely in $N$) the $S$-orbit of the identity coset in $G_p/K_p$:

1. Does not meet the support of any one-sided $\tau_{p^l}$, if $S$ is of $\text{SL}_2$-type.
2. Meets the support of any one-sided $\tau_{p^l}$ at most $C$ times, if is of multiplicative type.

**Proof.** Suppose first that $S$ is of $\text{SL}_2$-type and let $g \in G(K)$ be such that $g\mathbb{H}(K)g^{-1} \cap S$ is of finite index $C$ in $S$ with coset representatives $\{s_1, \ldots, s_C\}$. Since the prime $p$ splits completely in the field $K \subset N$, the element $h$ embeds in $\mathbb{G}_1(E_w)$; omitting finitely many primes we may also assume that $g$ and the $s_i$ all lie in $K_{p,1}$.

Now any element of the $S$-orbit has the form $s_i g(h, h) g^{-1}$ for some $h \in \text{SL}_2(\mathbb{Q}_p)$. Assuming $s_i, g \in K_{1,p}$ if this meets the support of a one-sided Hecke operator the second coordinate of this element must lie in the compact subgroup $\text{SL}_2(\mathbb{Z}_p)$ which by the KAK decomposition in $\text{SL}_2(\mathbb{Q}_p)$ forces $h \in \text{SL}_2(\mathbb{Z}_p)$ the element is in the identity coset.

Suppose now instead that $g\mathbb{T}(K)g^{-1} \cap S$ is of finite index $C$ in $S$ where $\mathbb{T}$ is the diagonal torus of $\mathbb{G}_1$. Again assume that $g \in K_p$; since our torus is $K$-split and $p$ splits completely in $K$ the orbit of $g\mathbb{T}(\mathbb{Q}_p \times \mathbb{Q}_p)g^{-1}$ is the product of two apartments (=geodesics) passing through the origin of the trees $\text{SL}_2(\mathbb{Q}_p)/\text{SL}_2(\mathbb{Z}_p)$. In particular in the first coordinate the orbit meets the translates by $s_i^{-1}$ of each sphere (the support of $\tau_{p^l}$) at most twice. 

Next we construct the standard amplifier in $\text{SL}_2(\mathbb{Q}_p)$. The fundamental observation (often attributed to Iwaniec) is that it is impossible for the eigenvalues of $\tau_p$, $\tau_{p^2}$ to be simultaneously small.

**Lemma 27.** Given $\phi$ we can choose $\tau$ to be either $\tau_p$ or $\tau_{p^2}$ so that the corresponding eigenvalue $\lambda$ satisfies $|\lambda| \gg \#\text{supp}(\tau)^{1/2}$.

**Proof.** Let $\lambda_{p^j}$ be the eigenvalue of $\tau_{p^j}$ acting on $\phi$. A direct calculation in the tree gives the convolution identity

$$(5.1) \quad \tau_{p^j}^2 = p(p+1) + (p-1)\tau_p + \tau_{p^2},$$

$$(5.2) \quad \tau_{p^2}^2 = p^3(p+1) + p^2(p-1)\tau_p + p(p-1)\tau_{p^2} + (p-1)\tau_{p^3} + \tau_{p^4}.$$ 

The first identity implies $(\lambda_{p^j})^2 = p(p+1) + (p-1)\lambda_p + \lambda_{p^2}$, so at least one of $|\lambda_{p^j}| \gg p$ or $|\lambda_{p^2}| \gg p^2$ must hold with the implied constants absolute.

Combining the spectral calculation and the control of intersections we have

**Corollary 28.** Let $S \subset \mathbb{G}(F)$ be a subgroup, and assume the image of $S$ under the projection to $\mathbb{G}_1(E)$ is small. Then there exist an absolute constant $L$, a constant $C$ (depending on $S$), and a set of positive density $P_S \subset \mathcal{P}$ such that for every prime $p \in P_S$ there exists a finite set $J_p \subset \mathcal{H}_p$ of basic Hecke operators such that:
(1) For each $h_p \in J_p$, 
\[ p^\ell \ll \#\text{supp}(h_p) \ll p^\ell \]
for some $\ell \ll L$ where the constants are absolute (we can take $L = 4$).

(2) For $x_p \in G_p/K_p$ other than the origin we have $|\langle h_p * h_p^* \rangle(x_p)| \ll p^{-\ell-1}$.

(3) For each $h_p \in J_p$, the number of intersections of the $S$-orbit in $G_p/K_p$ and the support of any of $h_p, h_p^*, h_p * h_p^*$ is bounded above by $C$.

(4) For each Hecke eigenfunction $\phi \in L^2(X)$, at least one $h_p \in J_p$ acts on $\phi$ with eigenvalue $\lambda_p$ satisfying 
\[ |\lambda_p| \geq (\#\text{supp} h_p)^{1/2}. \]

Proof. Lemma 26 and Lemma 27 together show that the claim holds for all primes which are large enough (depending on $S$) and split completely in a number field $N$ which depends on $S$ (and $\mathfrak{g}$). The Chebotarev Density Theorem shows that the set of split primes has positive density.

Our global amplifier will amplify a specified Hecke-eigenfunction $\phi$. However, we need some control of its action on its orthogonal complement as well. Accordingly for a self-adjoint Hecke operator $\tau \in \mathcal{H}$ we denote by $c = c(\tau)$ the smallest non-negative constant such that the spherical transform of $\tau$ on the unitary dual is bounded below by $-c$. In particular the spectrum of $\tau$ acting on $L^2(X)$ is contained in $[-c, \infty)$ and therefore (a fact which can be taken as a not-quite-equivalent definition) we have for all $R \in L^2(X)$ that
\[ \langle \tau, R, R \rangle \geq -c \|R\|_2^2. \]

Proposition 29 (Global construction). Continuing with the hypotheses of Corollary 28 let also $\varepsilon > 0$. Then there is $Q = Q(S, \varepsilon)$ such that for every Hecke-eigenfunction $\phi \in L^2(X)$, there exists $\tau \in \text{Span}_C (\cup_{p \leq Q} J_p)$ acting on $\phi$ with eigenvalue $\lambda$ satisfying 
\[ \text{supp}(\tau) \text{ meets the } S\text{-orbit of the identity in } \prod_p G_p/K_p \text{ in at most } \frac{\Delta}{\|\tau\|_{\infty}} \text{ points.} \]

(1) $c(\tau) \leq \varepsilon \Delta$.

Proof. For $Q \in \mathbb{Z}_{\geq 1}$ and $\ell \leq L$ let $P_\ell$ be the set of primes $p \in [Q, 2Q] \cap P$ for which there is an operator $h_p$ with eigenvalue $\lambda_p$ as in the conclusion of Corollary 28 with support of size comparable to $p^\ell$, and fix $\varepsilon$ so that $P = P_\ell$ consists of at least $1/\ell$ of the primes on in $[Q, 2Q] \cap P$. Then we have $\#P \gg Q/\log Q$ where the constant depends on $S$ through the density of $P$.

For each $p \in P$ let $\zeta_p$ be a complex number of magnitude 1 such that $\zeta_p \lambda_p$ is a positive real number, and finally set
\[ \tau_1 = (\sum_{p \in P} \zeta_p h_p) * (\sum_{p \in P} \zeta_p h_p)^* \]
and
\[ \tau = \tau_0 = \tau_1 - \tau_1(1) \delta, \]
where $\delta$ is the identity element of the (full) Hecke algebra, and $\tau_1(1)$ is the value of the function $\tau$ at the identity coset. In other words $\tau$ is obtained from $\tau_1$ by restricting away from a single point.

It is clear that $\tau$ is self-adjoint. To compute its $\ell^\infty$ norm (as a function on $K_1 \backslash \mathbb{G}(A_1)/K_1$) we start with the fact that, as functions on that space, we have the pointwise identity
\[ \tau_1 \leq \sum_{p \in P} h_p * h_p^* + \sum_{p < q \in P} (h_p * h_q^* + h_q * h_p^*). \]
Since the summands on the right have disjoint supports, it suffices to estimate each separately. First, if \( p \neq q \) then the support of the convolution is the product of the supports since \( G_p \) and \( G_q \) are disjoint commuting subgroups, so those terms are bounded by 2. At a single prime \( p \) the local construction gives the bound \( p^{f-1} \) for the values of \( h_p \cdot h_p \) away from the origin, and we conclude that \( \| \tau \|_{\infty} \ll \ell^{f-1} \).

Next, the eigenvalue of \( \tau_1 \) is clearly

\[
(\sum_{p \in \mathbb{P}} |\lambda_p|)^2 \gg \left( \sum_{p \in \mathbb{P}} \sqrt{\#\text{supp}(h_p)} \right)^2 \gg \left( \#PQ^{f/2} \right)^2 \gg \frac{Q^{f+2}}{\log^2 Q},
\]

and since \( h_p h_q^*(1) = 0 \) whenever \( p \neq q \) and \( h_q^*(1) = \#\text{supp}(h_p) \) we have

\[
\tau_1(1) = \sum_{p \in \mathbb{P}} \#\text{supp}(h_p) \ll \frac{Q^{1+f/2}}{\log Q}.
\]

Subtracting the two gives

\[
\Lambda \gg \frac{Q^{2+f}}{\log^2 Q}.
\]

For primes \( p, q \) let \( T_{p, q} \) be the set of points in \( \text{supp} h_p \cdot h_q^* \) that meet the \( S \)-orbit of the identity. When \( p = q \) these sets are of uniformly bounded size by the third item of Corollary 28 whereas when \( p \neq q \) this is the product of the corresponding subsets of \( G_p/K_p \) and \( G_q/K_q \) so again uniformly bounded. We conclude that the product of \( \| \tau_{\infty} \| \) with the total number of intersections satisfies

\[
(5.4) \quad \frac{\| \tau_{\infty} \| \sum_{p, q \in \mathbb{P}} \#T_{p, q}}{\Lambda} \ll \frac{\#P^2}{\Lambda} \ll Q^{-1}. \]

Similarly since \( \tau_1 \) is positive definite, \( c(\tau) = \tau_1(1) \) and hence

\[
(5.5) \quad \frac{c(\tau)}{\Lambda} = \frac{\tau_1(1)}{\Lambda} \ll Q^{-1-f/2} \log Q.
\]

Finally if we take \( Q \) large enough we can ensure both right-hand-sides of Eqs. 5.4 and 5.5 are less than \( \epsilon \).

6. NON-CONCENTRATION ON HOMOGENOUS SUBMANIFOLDS

We have shown that to rule out non-Haar components it suffices to rule out components supported in images in \( X = \Gamma \backslash G \) of submanifolds \( gHM \subset G \), which we finally do in this section. As in Section 2, instead of bounding \( \mu(\Gamma gHM) \) we will bound \( \mu(\Gamma U) \) where \( U \subset gHM \) is a bounded neighborhood. Unlike the positive entropy arguments we now need to treat \( U \) as a subset of \( gHM \) rather than try for additional uniformity by fixing a single \( U \subset HM \) and translating by \( g \) later.

Since will calculate in \( G \) and \( \mathbb{G}(\hat{\Lambda}) \) but ultimately make statements about subsets of \( X \) we introduce the notation \( \pi_G \): \( G \to X = \Gamma \backslash G \) and \( \pi_\Lambda \): \( \mathbb{G}(\hat{\Lambda}) \to X = \mathbb{G}(F) \backslash \mathbb{G}(\hat{\Lambda})/K \), for the quotient maps.

**Definition 30.** An algebraic piece of \( G \) will be a triple \((U, L, F')\) where \( F'/E \) is a finite extension contained in \( E_u \), \( L \subset \hat{\mathbb{G}}_1 \) is an irreducible subvariety defined over \( F' \) and irreducible over \( E_u \), and \( U \subset (L \times \hat{\mathbb{G}}_{\geq 2}')(E_u) \) is Zariski-dense, and also bounded and relatively open in the analytic topology.
This parametrization is redundant (\(U\) determines \(L\)) but it is easier to directly keep track
of \(L\) and its field of definition. We also write \(\bar{L} = L \times \hat{G}_{\geq 2}\).

Consider now a translate of an algebraic piece \(U\) by some element \(g_f \in \mathbb{G}(A_f)\). Let
\(\gamma \in \mathbb{G}(F)\) be such that \(\gamma g_f \in K_f\). Then
\[
\pi_h(U g_f) = \pi_h(\gamma U g_f) = \pi_h(\gamma U)
\]

For a subset \(U \subset G\) write \(\bar{U}\) for its closure in the analytic topology and \(\bar{U}^1\) for its
projection to \(G_1\).

**Definition 31.** We say the translate \(U g_f\) is *transverse* to \(U\) if for each \(\gamma \in \mathbb{G}(F)\) with
\(\gamma g_f \in K_f\) the \(F'\)-Zariski closure of the intersection \(\gamma \bar{U}^1 \cap \bar{U}^1 \subset G_1\) is of smaller
dimension than \(\bar{L}\). Abusing notation we say that \(g_f\) itself is transverse to \(U\), and otherwise say that it is
parallel to \(U\).

Observe that the definition only depends on the coset of \(g_f\) in \(\mathbb{G}(A_f)/K_f\).

**Lemma 32.** Let \(\gamma \in \mathbb{G}(F)\) and \(g_f \in \mathbb{G}(A_f)\) be such that \(\gamma g_f \in K_f\). Suppose further that the
\(F'\)-Zariski closure of \(\gamma \bar{U}^1 \cap \bar{U}^1\) is of dimension \(\dim \bar{L}\). Then \(\gamma \bar{L} = \bar{L}\).

For a function \(\phi \in L^2(X)\) and a measurable subset \(N \subset G\) (resp. \(N \subset \mathbb{G}(A_F)\)) we write
\(\phi_N\) for the restriction of \(\phi\) to \(\pi_G(N)\) (resp. to \(\pi_h(N)\)).

**Proposition 33.** Let \((U, L, F')\) be an algebraic piece of \(G\) and let \(\tau\) be a self-adjoint Hecke
operator. Write \(P\) for the set of parallel elements in the support of \(\tau\). Then there exists
a finite collection \(V\) of algebraic pieces \((V, L', F'')\) of \(G\) such that the \(L'\) are irreducible
proper \(F''\)-subvarieties of \(L\) and such that for every \(\delta > 0\) there exists \(\varepsilon > 0\) such that for
any eigenfunction \(\phi \in L^2(X)\) of \(\tau\) with eigenvalue \(\Lambda > 0\), we have
\[
\|\phi_U\| \leq \frac{1}{\Lambda} \left( \#P \|\tau\|_\infty + c(\tau) + \frac{\|\tau\|_\infty}{\|\phi_U\|_2^2} \sum_{\nu \in V} \|\phi_\nu\|^2 \right).
\]

Here \(c(\tau)\) is the constant defined in (5.3).

For the proof we will use the following easy and elementary observation:

**Lemma 34.** Suppose \(X\) is a metric space and \(A, B \subset X\) bounded subsets of \(X\). Then for
each \(\delta > 0\) there exists \(\varepsilon > 0\) such that \(A_\varepsilon \cap B_\varepsilon \subset (A \cap B)_\delta\).

**Proof of Proposition 33.** We estimate the expression
\[
(\tau, \phi_U, \phi_U)
\]
in two different ways: a geometric upper bound and a spectral lower bound.

On the geometric side let \(S = P \cup T \subset \mathbb{G}(A_f)\) be a set of representatives for the support of
\(\tau\), partitioned into the subsets of transverse and parallel elements. By the triangle inequality
\[
(\tau, \phi_U, \phi_U) \leq \|\tau\|_\infty \sum_{g_t \in S} \int_X \left| \phi_U(g_t) \bar{\phi}_U(g) \right| dg.
\]

When \(g_t \in P\) we naively apply Cauchy–Schwartz and the unitarity of the right \(\mathbb{G}(A_f)\)-action to get
\[
\int_X \left| \phi_U(g_t) \bar{\phi}_U(g) \right| dg \leq \|\phi_U\|^2
\]

Now suppose \(g_t \in T\). The images of \(U_\varepsilon\) and \(U_\varepsilon g_t\) in \(X\) intersect if and only if there is
\(\gamma \in \mathbb{G}(F)\) such that \(\gamma g_t \in K_f\) and such that \(\gamma U_\varepsilon \cap U_\varepsilon \neq \emptyset\). These \(\gamma\) belong to the intersection
are finitely many of them.

By Lemma 34 for each such \( \gamma \) and any \( \delta > 0 \) the boundedness of \( U \) ensures the existence of \( \varepsilon > 0 \) (depending also on \( \gamma \)) such that
\[
U_e \cap \gamma U_e \subset (\bar{U} \cap \gamma \bar{U})_\delta.
\]

Since we assumed that \( g_1 \) is transverse the \( F' \)-Zariski closure \( \bar{M} \) of \( U^1 \cap \gamma U^1 \) in \( \bar{G}_1 \) has \( \dim \bar{M} < \dim \bar{L} \). The irreducible components of \( \bar{M} \times_{F'} \bar{E}_w \) are defined over some finite extension \( F'' \) of \( F' \), and we can then cover \( \bar{U} \cap \gamma \bar{U} \) with finitely many algebraic pieces \( (V, L', F'') \) where \( L' \) is an irreducible component of \( \bar{M} \times_{F'} \bar{F}'' \). The union of the \( \delta \)-neighborhoods of these pieces then covers the intersection \( U_e \cap \gamma U_e \).

Now for one piece \( V \) we have
\[
\int_{V_\delta} |\phi(g_1)|^2 \overline{dg} \leq \frac{1}{2} \left( \int_{V_\delta} |\phi(\gamma g)|^2 dg + \int_{V_\delta} |\phi(g)|^2 dg \right) = \frac{1}{2} \left( \int_{\bar{V}_\delta} |\phi(g)|^2 dg + \int_{V_\delta} |\phi(g)|^2 dg \right).
\]

Accordingly, let \( V \) denote the set of pieces \( V \) and \( \gamma^{-1} V \) arising as components of \( \bar{M} \) as \( g_1 \) ranges over \( T \) and \( \gamma \) ranges over the finite set causing intersections (note that \( \gamma^{-1} V_\delta = (\gamma^{-1} V)_\delta \)).

Since this set of pieces is finite we can choose \( \varepsilon \) small enough for all of them. Combining the bounds for parallel and transverse intersections then gives the geometric-side estimate
\[
(6.3) \quad \langle \tau, \phi U_e, \phi U_e \rangle \leq \# P \| \tau \|_\infty \| \phi U_e \|^2 + \| \tau \|_\infty \sum_{V \in V} \| \phi V_\delta \|^2.
\]

We next obtain a spectral lower bound. Returning to Eq. (6.1), to the extent \( \phi U_e \) "approximates" \( \phi \) we expect \( \tau \phi U_e \) to be approximately \( \Lambda \phi U_e \). This is not literally true; instead we write \( \phi U_e = a\phi + \phi^\perp \) for some \( \phi^\perp \) orthogonal to \( \phi \). Here \( a = \langle \phi U_e, \phi \rangle = \| \phi U_e \|^2 \) since \( \phi \) is normalized. Since also \( \| \phi^\perp \| \leq \| \phi U_e \| \) we have
\[
(6.4) \quad \langle \tau, \phi U_e, \phi U_e \rangle = a^2 \langle \tau, \phi, \phi \rangle + 2\Re \langle \tau, \phi, \phi^\perp \rangle + \langle \tau, \phi^\perp, \phi^\perp \rangle \geq \| \phi U_e \|^4 \Lambda \phi U_e - c(\tau) \| \phi U_e \|^2.
\]

Putting Eqs. (6.3) and (6.4) together and dividing by \( \Lambda \| \phi U_e \|^2 \) finally established the Proposition. \( \square \)

We can now prove our main result.

**Theorem 35.** Let \( \mu \) be a weak-* limit of normalized Hecke-eigenfunctions on \( X \) and let \((U, \mathbb{L}, F')\) be an algebraic piece of \( G \) so that \( \mathbb{L} \) has small stabilizers. Then \( \mu(\pi_G(U)) = 0 \).

**Proof.** We show by induction on \( \dim \mathbb{L} \) that for every \( \eta > 0 \) there is \( \varepsilon > 0 \) such that \( \| \phi U_e \|^2 \leq \eta \) for all Hecke eigenfunctions \( \phi \). It would follow that \( \mu(\pi_G(U)) \leq \eta \) for all limits.

When \( U \) and \( \mathbb{L} \) are empty ("dimension −1") there is nothing to prove, so we assume \( \dim \mathbb{L} \geq 0 \). Let \( S < G(F) \) be the stabilizer of \( \mathbb{L} \) which is small by hypothesis. With \( \alpha > 0 \) to be chosen later let \( \tau \) be the Hecke operator constructed in Proposition 29 so that \( \tau \phi = \Lambda \phi \) and \( \# P \| \tau \|_\infty, c(\tau) \leq \alpha \Lambda \) where \( P \) is the set of elements in the support of \( \tau \) which lie on the
By Lemma 32, every element in the support of \( \tau \) parallel to \( U \) lies in the \( S \)-orbit so Proposition 33 produces a finite set \( V \) of algebraic pieces contained in \( U \) (hence themselves having small stabilizers) and such that for any \( \delta > 0 \) there is \( \epsilon > 0 \) such that

\[
\| \phi_{U_{\epsilon}} \|^2 \leq \frac{1}{\bar{\chi}} \left( \# P \| \tau \|_{\infty} + c(\tau) + \sum_{\phi_{U_{\delta}} \in V} \| \phi_{U_{\delta}} \|^2 \right).
\]

By induction we can choose \( \delta \) small enough so that for each \( V \in V \) we have \( \| \phi_{U_{\delta}} \|^2 \leq \frac{1}{\pi \chi} \alpha \). With this choice (and the corresponding \( \epsilon \)) we have

\[
\| \phi_{U_{\epsilon}} \|^2 \leq \alpha \left( 2 + \frac{1}{\| \phi_{U_{\delta}} \|^2} \right),
\]

and hence \( \| \phi_{U_{\epsilon}} \|^2 \leq 3 \sqrt{\alpha} \) (if we always choose \( \alpha < 1 \)). The Theorem follows upon choosing \( \alpha \) small enough.

**Proof of Theorem** 2 Without loss of generality \( i = 1 \). Let \( g \in G = \prod_i G_i \) have its 1st coordinate in \( \tilde{G}_1(\mathbb{Q}) \). Then Proposition 24 shows that \( L = g(H_1 M_1) G_{i > 2} \) has small stabilizers. By Theorem 33 for every bounded open \( U \subset L \) we have \( \mu(\pi_G(U)) = 0 \) and covering \( L \) by countably many such \( U \) we conclude that \( \mu(\pi_G(L)) = 0 \).

**REFERENCES**

[1] Jean Bourgain and Elon Lindenstrauss. Entropy of quantum limits. *Comm. Math. Phys.*, 233(1):153–171, 2003.

[2] Shimon Brooks and Elon Lindenstrauss. Joint quasimodes, positive entropy, and quantum unique ergodicity. *Invent. Math.*, 198(1):219–259, 2014.

[3] Daniel Bump. *Automorphic forms and representations*, volume 55 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1997.

[4] Yves Colin de Verdière. Ergodicité et fonctions propres du laplacien. *Comm. Math. Phys.*, 102(3):497–502, 1985.

[5] Manfred Einsiedler and Elon Lindenstrauss. On measures invariant under diagonalizable actions: the rank-one case and the general low-entropy method. *J. Mod. Dyn.*, 2(1):83–128, 2008.

[6] Elon Lindenstrauss. Entropy of quantum unique ergodicity for \( \Gamma \backslash \mathbb{H} \times \mathbb{H} \). *Internat. Math. Res. Notices*, 2001(17):913–933, 2001.

[7] Elon Lindenstrauss. Invariant measures and arithmetic quantum unique ergodicity. *Ann. of Math. (2)*, 163(1):165–219, 2006.

[8] J. S. Milne. *Algebraic groups*, volume 170 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1997.

[9] Vladimir Platonov and Andrei Rapinchuk. *Algebraic groups and number theory*, volume 139 of *Pure and Applied Mathematics*. Academic Press Inc., Boston, MA, 1994. Translated from the 1991 Russian original by Rachel Rowen.

[10] [. Rudnick and Peter Sarnak. The behaviour of eigenstates of arithmetic hyperbolic manifolds. *Ann. of Math. (2)*, 161(1):195–213, 1999.

[11] Zvi Shem-Tov. Positive entropy using hecke operators at a single place. *Int. Math. Res. Not. IMRN*, 09 2020.

[12] Lior Silberman and Akshay Venkatesh. Quantum unique ergodicity for locally symmetric spaces. *Geom. Funct. Anal.*, 17(3):960–998, 2007. arXiv:math.RT/0407413.

[13] Lior Silberman and Akshay Venkatesh. Entropy bounds and quantum unique ergodicity for Hecke eigenfunctions on division algebras. In *Probabilistic methods in geometry, topology and spectral theory*, volume 739 of *Contemp. Math.*, pages 171–197. Amer. Math. Soc., [Providence], RI, [2010] 2019.

[14] Alexander I. Šnirel’man. Ergodic properties of eigenfunctions. *Uspekhi Mat. Nauk.*, 29(6(180)):181–182, 1974.

[15] Kannan Soundararajan. Quantum unique ergodicity for \( \mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H} \). *Ann. of Math. (2)*, 172(2):1529–1538, 2010.
[17] Asif Zaman. Escape of mass on hilbert modular varieties. Master’s thesis, The University of British Columbia, 2012. M.Sc. thesis.
[18] Steven Zelditch. Pseudodifferential analysis on hyperbolic surfaces. *J. Funct. Anal.*, 68(1):72–105, 1986.

EINSTEIN INSTITUTE OF MATHEMATICS, THE HEBREW UNIVERSITY OF JERUSALEM, ISRAEL
Email address: zvi.shem-tov@mail.huji.ac.il

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BRITISH COLUMBIA, VANCOUVER BC V6T 1Z2, CANADA
Email address: lior@math.ubc.ca