Cliffordization, Spin and Fermionic Star Products

Allen C. Hirshfeld*, Peter Henselder† and Thomas Spernat‡
Fachbereich Physik, Universität Dortmund
44221 Dortmund

November 14, 2018

Abstract

Deformation quantization is a powerful tool for quantizing theories with bosonic and fermionic degrees of freedom. The star products involved generate the mathematical structures which have recently been used in attempts to analyze the algebraic properties of quantum field theory. In the context of quantum mechanics they provide a canonical quantization procedure for systems with either bosonic or fermionic degrees of freedom. We illustrate this procedure for a number of physical examples, including bosonic, fermionic and supersymmetric oscillators. We show how non-relativistic and relativistic particles with spin can be naturally described in this framework.

1 Introduction

The deformation quantization formulation of quantum physics was initiated by Bayen et. al. in Ref. [1], for recent reviews see e.g. [2, 3]. Physical applications of this formalism have mainly been restricted to systems involving bosonic degrees of freedom, e.g. [4]. For some time it remained unclear how spin and relativistic quantum mechanics could be described in this formalism. J. Varilly et. al. addressed this problem using Moyal products [5], in [6] they combined their methods with group theoretic arguments in order to treat particles with spin and the covariance of the Dirac equation. In contrast to such approaches we advocated in [7] using for systems involving fermionic degrees of freedom a method based on the work of Berezin and Marinov [8] in which one starts from a pseudoclassical system described in terms of Grassmann variables, and achieves quantization by use of a fermionic star product, which appeared in another context in [9]. We showed how the familiar Clifford algebra structures characteristic of particles with spin arise in this framework.

A more general program for analyzing relativistic quantum field theories using Clifford algebra structures has been pursued in recent years by Fauser [10]-[15]. In this approach these structures are derived from an underlying Grassmann algebra by the procedure of Chevalley Cliffordization [16]. In [15] Fauser discussed the Dirac equation in this context.

Another program for understanding the algebraic structures which arise in perturbative quantum field theory has been advocated by Brouder and Oekl [17, 18]; the fundamental algebraic structure they use is the circle product, introduced by Brouder in [19]. In [20] we have discussed the relation of this product to the star products used in deformation quantization.

In the present paper we attempt to clarify the relations between these different approaches to quantum physics. We believe the deformation quantization approach to be the most fundamental, relying as it does...
on Gerstenhaber’s seminal analysis of associative algebras [21]. As soon as the appropriate star product has been determined the analysis of a quantum mechanical system proceeds in a canonical fashion: the states are characterized by the relevant Wigner functions and the eigenvalues of the Hamiltonian follow from the *-genvalue equation or from the star exponential.

The paper is structured as follows. In Sec. 2 we briefly review the Chevalley Cliffordization procedure for constructing a Clifford algebra from an underlying Grassmann algebra. In Sec. 3 we elucidate the relationship between Brouder’s circle product and the Clifford algebra structure of fermionic systems. Sec. 4 clarifies the relation between Fauser’s concept of Wick isomorphism and the c-equivalence of deformation quantization. In Sec. 5 we lay out the general scheme for quantizing a given physical system, both for bosonic and for fermionic degrees of freedom. We illustrate the method for the bosonic and fermionic oscillators. We show that the fermionic angular momentum corresponds to the usual spin concept of non-relativistic quantum mechanics, and verify the basic properties of the spin vector. Sec. 6 treats a real physical system: a charged particle in a constant magnetic field. For a spinless particle the well-known Landau energy levels are recovered, as well as the eigenfunctions of the orbital angular momentum. For a particle with spin one-half the system corresponds to a supersymmetric oscillator, with its characteristic degenerate energy levels. Sec. 7 shows how the determination of exact and broken supersymmetry in terms of the Fredholm or Witten index follows in this formalism. Sec. 8 discusses different representations for the Dirac operators in this context. We determine the relevant Wigner functions and the star exponential. We also exhibit the star product analogues of the Dirac spin projectors. In Sec. 9 we follow the Foldy-Wouthuysen procedure in order to study the non-relativistic approximation to the Dirac equation and show how the conventional operator expressions may be recovered by use of the Weyl transform. Sec. 10 contains our conclusions, and an outlook for further research.

2 Chevalley Cliffordization

In this section we briefly review the construction of a Clifford algebra from a Grassmann algebra. This subject was developed by Cartan, Weyl and Chevalley [16]. We follow the notation of Fauser [10].

The starting point is a Grassmann algebra $Gr$. This is the $\mathbb{Z}$-graded algebra generated by a set of Grassmann variables $\{\theta_1, \ldots, \theta_n\}$, which satisfy the relations

$$\theta_i \theta_j = -\theta_j \theta_i \quad \forall i, j = 1 \ldots n.$$  

We also take as given a bilinear form

$$B(\theta_i, \theta_j) = g(\theta_i, \theta_j) + A(\theta_i, \theta_j),$$  

where $g$ and $A$ are the symmetric and antisymmetric parts, respectively.

We define an antiderivation on $Gr$ as a map which acts on generators and monomials according to the following rules:

$$\theta_i \theta_j = B(\theta_i, \theta_j),$$  

$$\theta_i \theta_j (uv) = (\theta_i \theta_j) v + (-1)^{\pi(u)} u (\theta_i \theta_j) v,$$  

$$uv \theta_j w = u \theta_j (v \theta_j w).$$

Here $u$ and $v$ are homogeneous monomials, and $\pi(u)$ is the grade of $u$. The mapping is then linearly extended to arbitrary elements of $Gr$. From Eq. (2.3b) with $u = v = 1$ it is clear that $\theta_i \theta_j = 0$. From Eq. (2.3c) with $v = 1$ it follows that $1 \theta_j = 0$. For homogeneous $u$ and $v$ we have

$$\pi(u \theta_j v) = \pi(v) - \pi(u).$$  

We now define the linear mapping
\[
\gamma^{\theta_i}_{\theta_j} : \text{Gr} \rightarrow \text{Gr} \quad u \mapsto \theta_i u + \theta_i \theta_j u. \tag{2.5}
\]
We easily calculate
\[
\gamma^{\theta_i}_{\theta_j} \gamma^{\theta_j}_{\theta_l} u = B(\theta_i, \theta_j) u + (\theta_i \theta_j)_{\theta_l} u + [\theta_i (\theta_j \theta_l u) - \theta_j (\theta_i \theta_l u)]. \tag{2.6}
\]
From this we see that the $\gamma_i$ are the generators of a Clifford algebra $\text{Cl}(B)$, since
\[
\\{\gamma^{\theta_i}_{\theta_j} \gamma^{\theta_j}_{\theta_l} \} := \gamma^{\theta_i}_{\theta_j} \gamma^{\theta_j}_{\theta_l} + \gamma^{\theta_j}_{\theta_l} \gamma^{\theta_l}_{\theta_i} = 2g(\theta_i, \theta_j). \tag{2.7}
\]

3 Circle Products for Grassmann Variables

By now there are a number of associative products in the literature which are used to discuss the algebraic structure of quantum mechanics and quantum field theory. In Ref. [20] we discussed the relation between Brouder’s circle product [19], which is a special case of Drinfeld’s twisted product [22], and the star product of deformation quantization. In the present section we discuss the relation of the circle product to the product encountered in the Chevalley Cliffordization procedure.

The fermionic version of the circle product [7] is
\[
u \circ_B v = u \exp \left( \sum_{i,j} B(\theta_i, \theta_j) \partial_{\theta_i} \partial_{\theta_j} \right) v. \tag{3.1}
\]
In the above formula the arrows indicate on which function the differential operators are acting. The differential operators which act to the right are left derivatives, those which act to the left are right derivatives with respect to the Grassmann variables. The following discussion is only valid for monomials, but the generalization to arbitrary elements of $\text{Gr}$ is straightforward. Since the $n$-th term in the expansion of $u \circ_B v$ is of grade $\pi(u) + \pi(v) - 2n$ one can compare the $\pi(u)$-th term with $u \circ_B v$, which is of the same grade: $\pi(v) - \pi(u)$. In fact, both turn out to be identical, i.e.
\[
u \circ_B v = u \exp \left( \sum_{i,j} B(\theta_i, \theta_j) \partial_{\theta_i} \partial_{\theta_j} \right) \pi(u) \nu \circ_B v = u \left( \sum_{\pi(u)} \prod_{i,j=1}^n \left( B(\theta_i, \theta_j) \partial_{\theta_i} \partial_{\theta_j} \right)^{k_{ij}} \right) v. \tag{3.2}
\]
where the $k_{ij}$ are either 1 or 0. To prove this equality we have to show that the three axioms of $\circ_B$ are fulfilled. The first axiom is trivial, the second one follows from the Leibniz rule
\[
\theta_i \circ_B (uv) = \sum_j B(\theta_i, \theta_j) \partial_{\theta_j} (uv)
= \sum_j B(\theta_i, \theta_j) \left[ (\partial_{\theta_j} u) v + (-1)^{\pi(u)} u (\partial_{\theta_j} v) \right]
= (\theta_i \circ_B u) v + (-1)^{\pi(u)} u (\theta_i \circ_B v) \tag{3.3}
\]
and a proof of [23] can be found in the appendix. Therefore $u \circ_B v$ is equal to the term of the expansion of $u \circ_B v$ in which all basis elements of $\theta_i$ in $u$ are cancelled by corresponding derivatives $\partial_{\theta_i}$. Such a term
will only exist if $\pi(u) \leq \pi(v)$ and if the necessary derivatives appear, i.e. the corresponding $B(\theta_i, \theta_j)$ have to be non-zero.

One can now formulate the Clifford map with the help of a circle product as
\[
\gamma^B_{\theta_i} u = \left(\theta_i + \frac{\theta_i \cdot B}{B}\right) u = \theta_i \circ_B u.
\]
(3.4)

We generalize our previous notation and write for general homogeneous $u$ and $v$
\[
\gamma v u = v \circ_B u,
\]
(3.5)

which implies
\[
\gamma_u \gamma_v = \gamma_{u \circ v}.
\]
(3.6)

With this notation Eq. (2.6) reads
\[
\theta_i \circ_B \theta_j \circ_B u = \theta_i \circ_B \theta_j + \theta_j \circ_B \theta_i = 2g(\theta_i, \theta_j) u.
\]
(3.7)

and the anticommutator (2.7) can be written as
\[
\{\theta_i, \theta_j\} \circ_B u = \{\theta_i, \theta_j\} = \theta_i \circ_B \theta_j + \theta_j \circ_B \theta_i = 2g(\theta_i, \theta_j) u.
\]
(3.8)

4 The Wick Isomorphism and c-Equivalence

In quantum mechanics and quantum field theory the Clifford algebras $\mathcal{C}l(g)$ and $\mathcal{C}l(B)$ are related. Fauser [10] uses the concept of Wick isomorphism to express this relationship in terms of the grade-2 form $F = F^{ij} \theta_i \theta_j$, which is related to the antisymmetric part of $B$:
\[
\sum_{r,s} F^{rs} g(\theta_i, \theta_s) g(\theta_j, \theta_r) = \frac{1}{2} A(\theta_i, \theta_j).
\]
(4.1)

The Wick isomorphism maps a monomial $u$ into $e^{-F} u e^F$. In this section we shall discuss this relationship in terms of circle products.

We start by calculating
\[
\theta_i \circ_B F = \sum_j B(\theta_i, \theta_j) \partial_{\theta_j} F^{k l} \theta_k \theta_l = \sum_j 2B(\theta_i, \theta_j) F^{jk} \theta_k.
\]
(4.2)

This leads to
\[
\theta_i \circ_B F^n = n(\theta_i \circ_B F) F^{n-1},
\]
(4.3)

which implies
\[
\theta_i \circ_B e^F = (\theta_i \circ_B F) e^F,
\]
(4.4)

so that we find
\[
e^{-F} \left[\theta_i \circ_B (e^F u)\right] = \theta_i \circ_B u + (\theta_i \circ_B F) u.
\]
(4.5)

With this we calculate
\[
e^{-F} \gamma^g_{\theta_i} e^F u = \theta_i u + (\theta_i \circ_B F) u.
\]
(4.6)
Similarly we find

\[ e^{-F} \gamma^i_{\theta_i} \gamma^j_{\theta_j} e^F u = \theta_i \theta_j u + g(\theta_i, \theta_j)u + \theta_i(\theta_j F)u - \theta_j(\theta_i F)u \]
\[ + \theta_i(\theta_j F)u - \theta_j(\theta_i F)u + (\theta_i g)(\theta_j F)u + (\theta_j g)(\theta_i F)u \]
\[ + (\theta_i g)(\theta_j F)u. \]  \hspace{1cm} (4.7)

In this expression there are two terms that multiply \( u \) by a scalar, namely \( g(\theta_i, \theta_j)u \) and \( (\theta_i g)(\theta_j F)u \). This last term is

\[ \theta_i(\theta_j F)u - \theta_j(\theta_i F)u = \theta_i \theta_j(u - u) - \theta_j \theta_i(u - u) = \theta_i \theta_j u - \theta_j \theta_i u. \]

Hence the Wick isomorphism has induced an antisymmetric term \( A(\theta_i, \theta_j) \) that combines with the symmetric term \( g(\theta_i, \theta_j) \) to \( B(\theta_i, \theta_j) \). By symmetrizing Eq. (4.7) in \( i \) and \( j \) one sees that the anticommutator is invariant with respect to the Wick isomorphism:

\[ e^{-F} \{ \gamma^i_{\theta_i}, \gamma^j_{\theta_j} \} e^F = 2g(\theta_i, \theta_j). \]  \hspace{1cm} (4.9)

The concept of Wick isomorphism is similar to the concept of \( c \)-equivalence in the context of star products. Two star products are \( c \)-equivalent if they are related by a \( T \)-transformation:

\[ u \circ' v = T^{-1}(Tu \circ Tv) \]  \hspace{1cm} (4.10)

with \( T = \exp(T^{ij} \partial_{\theta_i} \partial_{\theta_j}) \). Transforming the Clifford maps into the circle product notation as discussed in the last section we see that the Wick isomorphism does not transform \( \circ g \) into \( \circ B \), as a \( T \)-transformation would. This can be seen from the simple fact that for \( u = 1 \) Eq. (4.7) leads to

\[ e^{-F} \left( \sigma^g_{\theta_i, \gamma^j}, e^F \right) = e^{-F} \left( \theta_i \circ \theta_j \circ e^F \right) \]
\[ = \theta_i \theta_j + g(\theta_i, \theta_j) + (\theta_i g)(\theta_j F) + (\theta_j g)(\theta_i F) + (\theta_i g)(\theta_j F), \]  \hspace{1cm} (4.11)

where a number of terms of order two appear, while in \( \theta_i \circ_B \theta_j \) the only term of order two is \( \theta_i \theta_j \). So the Wick isomorphism does not lead to a \( T \)-transformation of the corresponding circle product. But as it does induce an antisymmetric scalar part, and this scalar part is just the scalar part of the \( T \)-transformed circle product. The following result holds true:

\[ \varepsilon [\theta_{i_1} \circ_B \ldots \circ_B \theta_{i_n}] = \varepsilon \left[ e^{-F} \left( \theta_{i_1} \circ_g \theta_{i_2} \ldots \theta_{i_n} \circ_g e^F \right) \right], \]  \hspace{1cm} (4.12)

where \( \varepsilon \) projects onto the scalar part of the expression. In terms of Clifford algebras this means that although \( e^{-F} C\ell(g, V) e^{+F} \) is not equal to \( C\ell(B, V) \), the equation

\[ \varepsilon \left[ e^{-F} C\ell(g, V) e^{+F} \right] = \varepsilon \left[ C\ell(B, V) \right] \]  \hspace{1cm} (4.13)

is valid.
For $n$ odd the relation (4.12) is empty; both sides of the equation vanish. For even $n$ the left hand side yields

$$
\varepsilon \left[ \theta_{i_1} \circ_B \cdots \circ_B \theta_{i_{2m}} \right] = \sum_{\sigma \in S_{2m}} (-1)^\sigma B(\theta_{\sigma(i_1)}, \theta_{\sigma(i_2)}) \cdots B(\theta_{\sigma(i_{2m-1})}, \theta_{\sigma(i_{2m})})
$$

(4.14)

where we have used the Wick theorem as in Ref. [4]. The right hand side of Eq. (4.12) involves the term

$$
\theta_{i_1} \circ_g \cdots \circ_g \theta_{i_{2m}} = \theta_{i_1} \cdots \theta_{i_{2m}} + \sum_{\sigma \in S_{2m}} (-1)^\sigma \left[ g(\theta_{\sigma(i_1)}, \theta_{\sigma(i_2)}) \theta_{\sigma(i_3)} \cdots \theta_{\sigma(i_{2m})}
+ g(\theta_{\sigma(i_1)}, \theta_{\sigma(i_2)}) g(\theta_{\sigma(i_3)}, \theta_{\sigma(i_4)}) \theta_{\sigma(i_5)} \cdots \theta_{\sigma(i_{2m})}
+ \cdots + g(\theta_{\sigma(i_1)}, \theta_{\sigma(i_2)}) \cdots g(\theta_{\sigma(i_{2m-1})}, \theta_{\sigma(i_{2m})}) \right],
$$

(4.15)

where we have again used the Wick theorem. For the first term in this expression we find

$$
\varepsilon \left[ \left( \theta_{i_1} \cdots \theta_{i_{2m}} \right) \circ_g e^F \right] = \sum_{\sigma \in S_{2m}} (-1)^\sigma A(\theta_{\sigma(i_1)}, \theta_{\sigma(i_2)}) \cdots A(\theta_{\sigma(i_{2m-1})}, \theta_{\sigma(i_{2m})}),
$$

(4.16)

where we have used the definition of $F$ in terms of $A$, Eq. (4.11). Continuing in this way we find

$$
\varepsilon \left[ e^{-F} \left( \theta_{i_1} \circ_g \cdots \circ_g \theta_{i_{2m}} \circ_g e^F \right) \right] = \sum_{\sigma \in S_{2m}} (-1)^\sigma \sum_{X = g, A} X(\theta_{\sigma(i_1)}, \theta_{\sigma(i_2)}) \cdots X(\theta_{\sigma(i_{2m-1})}, \theta_{\sigma(i_{2m})}),
$$

(4.17)

which finishes the proof.

The result we have established in this section implies that although the result of a Wick isomorphism and a $T$-transformation on a circle product are not identical, their scalar parts are identical. The scalar parts correspond in quantum field theory to the vacuum expectation values $[20]$. This is sufficient to establish the equivalence of the two procedures in perturbative quantum field theory, where the relevant quantities are vacuum expectation values of products of field operators. For these vacuum expectation values we see that the choice of the antisymmetric part of the bilinear form $B$ is of no physical consequence.

## 5 The Quantization of Bosonic and Fermionic Systems

In this section we want to show how different specializations of the circle product (3.1) can be used for physical applications. We first consider a dynamical system involving bosonic degrees of freedom. The relevant product is then the Moyal product:

$$
f \star_M g = f \exp \left[ \frac{i\hbar}{2} \left( \partial_q \tilde{\partial}_p - \tilde{\partial}_p \partial_q \right) \right] g.
$$

(5.1)

This is obviously a special case of the circle product for the case where the symmetric part of $B$ vanishes.

We give a short review of the deformation quantization procedure for the harmonic oscillator with Hamilton function

$$
H(q, p) = \frac{p^2}{2m} + \frac{m\omega^2}{2} q^2.
$$

(5.2)
For more details see the review in [3]. The Wigner functions \( \pi_n^{(M)} \) and the energy levels \( E_n \) of the harmonic oscillator can be calculated with the help of the star exponential

\[
\text{Exp}_M(Ht) = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{-it}{\hbar} \right)^n H^{\ast n}M = \sum_{n=0}^{\infty} \pi_n^{(M)} e^{-iE_n t/\hbar},
\]

(5.3)

where \( H^{\ast n}M = H \ast M \ast \cdots \ast M H \) is the \( n \)-fold star product of \( H \). For the harmonic oscillator one obtains

\[
E_n = \hbar \omega \left( n + \frac{1}{2} \right)
\]

and

\[
\pi_n^{(M)} = 2(-1)^n e^{-2H/\hbar \omega} L_n \left( \frac{4H}{\hbar \omega} \right),
\]

(5.4)

where \( L_n \) are the Laguerre polynomials. The energy levels and the Wigner functions fulfill the \( \ast \)-genvalue equation

\[
H \ast \pi_n^{(M)} = E_n \pi_n^{(M)}.
\]

(5.5)

The Wigner functions \( \pi_n^{(M)} \) are normalized according to

\[
\frac{1}{2\pi \hbar} \int \pi_n^{(M)} dq dp = 1.
\]

(5.6)

The expectation value of a phase space function \( f \) can be calculated as

\[
\langle f \rangle = \frac{1}{2\pi \hbar} \int f \ast \pi_n^{(M)} dq dp.
\]

(5.7)

We now consider dynamical systems involving fermionic degrees of freedom. These degrees of freedom are described by Grassmann variables, so for one-dimensional systems no quadratic kinetic or potential terms exist, because of the nilpotency of these variables. The simplest non-trivial system in Grassmannian mechanics is therefore a two dimensional system with Lagrange function [8]

\[
L = \frac{i}{2} \left( \theta_1 \dot{\theta}_1 + \theta_2 \dot{\theta}_2 \right) + i\omega \theta_1 \theta_2,
\]

(5.8)

where \( \theta_1, \theta_2 \) are Grassmann variables. The canonical momenta are

\[
\rho_\alpha = -\frac{i}{2} \theta_\alpha, \quad \alpha = 1, 2,
\]

(5.9)

and the Hamilton function is given by

\[
H = \dot{\theta}_1 \rho_2 - \dot{\theta}_2 \rho_1 = -i\omega \theta_1 \theta_2.
\]

(5.10)

Eq. (5.9) implies that this Hamiltonian may be seen as describing rotation. Indeed, the fermionic angular momentum, which corresponds to the spin, is

\[
S_3 = \theta_1 \rho_2 - \theta_2 \rho_1 = -i\theta_1 \theta_2,
\]

(5.11)

so that the Hamiltonian in (5.10) can also be written as \( H = \omega S_3 \). As a vector the angular momentum points out of the \( \theta_1-\theta_2 \) plane. Therefore it is natural to consider the two dimensional fermionic oscillator as embedded into a three dimensional fermionic space with coordinates \( \theta_1, \theta_2 \) and \( \theta_3 \). We choose units such that both the fermionic coordinates and momenta have dimension \( \sqrt{\hbar} \).

The appropriate star product for the quantization of fermionic degrees of freedom is given by specifying

\[
B(\theta_i, \theta_j) = \frac{\hbar}{2} \delta_{ij} \text{ in the circle product}^{[34]}, \quad \text{i.e.}
\]

\[
F \ast_p G = F \exp \left[ \frac{\hbar}{2} \sum_{i=1}^{d} \tilde{\theta}_i \tilde{\theta}_i \right] G.
\]

(5.12)
We call this product the Pauli star product, it is first mentioned in [1]. It was shown in [7] that it can be obtained by deformation quantization of a Grassmann algebra. The Pauli star product (5.12) leads to a Cliffordization of the Grassmann algebra of the \( \theta_i \), because the star-anticommutator is given by

\[
\{ \theta_i, \theta_j \}_\ast = \theta_i \ast \theta_j + \theta_j \ast \theta_i = \hbar \delta_{ij}.
\]  \tag{5.13}

The even Grassmann functions

\[
\sigma^i = \frac{1}{i\hbar} \epsilon^{ijk} \theta_j \theta_k,
\]

\( i = 1, 2, 3 \),

(5.14)

fulfill the relations

\[
[\sigma^i, \sigma^j]_\ast = 2i\epsilon^{ijk} \sigma^k
\]

and

\[
\{ \sigma^i, \sigma^j \}_\ast = 2\delta^{ij}
\]

with \( [\sigma^i, \sigma^j]_\ast = \sigma^i \ast \sigma^j - \sigma^j \ast \sigma^i \), they therefore correspond to the Pauli matrices. Note that \( \{1, \sigma^1, \sigma^2, \sigma^3\} \) is a basis of the even subalgebra of the Grassmann algebra, and that this space is closed under \( \ast \) multiplication. From Eqs. (5.11) and (5.14) we see that \( S_3 = \frac{\hbar}{2} \sigma^3 \) and \( H = \omega S_3 = \frac{\hbar \omega}{2} \sigma^3 \).

The *involution* operation in the space of Grassmann variables [8] is a mapping \( F \mapsto \bar{F} \) satisfying the conditions

\[
\bar{F} = F, \quad \bar{F_1 F_2} = \bar{F_2 F_1} \quad \text{and} \quad \bar{c F} = \bar{c} \bar{F},
\]

(5.16)

where \( c \) is a complex number and \( \bar{c} \) its complex conjugate. For the generators \( \theta_i \) of the Grassmann algebra we assume \( \overline{\theta_i} = \theta_i \), so that for \( \sigma_i \) defined in (5.14) the relation \( \overline{\sigma_i} = \sigma_i \) holds true. This corresponds to the fact that the \( 2 \times 2 \) Pauli matrices \( \sigma^i \) are hermitian.

The *Hodge dual* maps a Grassmann monomial of grade \( r \) into a monomial of grade \( d - r \), where \( d \) is the number of Grassmann basis elements:

\[
\ast (\theta_{i_1} \theta_{i_2} \cdots \theta_{i_r}) = \frac{1}{(d-r)!} \epsilon_{i_1 \cdots i_{r+1} \cdots i_r} \theta_{i_{r+1}} \cdots \theta_{i_r}.
\]  \tag{5.17}

With the help of the Hodge dual we can define the trace for \( d = 3 \) as

\[
\text{Tr}(F) = \frac{2}{\hbar^3} \int d\theta_3 d\theta_2 d\theta_1 \ast F.
\]  \tag{5.18}

The integration is given by the Berezin integral, for which we have \( \int d\theta \theta = \hbar \delta_{i,j} \), where \( \hbar \) on the right hand side is due to the fact that the variables \( \theta_i \) have units of \( \sqrt{\hbar} \). The only monomial with a non-zero trace is 1, so that by the linearity of the integral we obtain the trace rules

\[
\text{Tr}(\sigma^i) = 0 \quad \text{and} \quad \text{Tr}(\sigma^i \ast \sigma^j) = 2\delta^{ij}.
\]  \tag{5.19}

With the fermionic star product (5.12) one can—as in the bosonic case—calculate the energy levels and the *eigenfunctions* of the fermionic oscillator [7]. This can be done by using the fermionic star exponential

\[
\text{Exp}_\ast(Ht) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(- \frac{it}{\hbar}\right)^n H^{n \ast} = \pi_1^{(P)} e^{-i\omega t/2} + \pi_{-1}^{(P)} e^{i\omega t/2},
\]  \tag{5.20}

where the Wigner functions are given by

\[
\pi^{(P)}_{\pm 1/2} = \frac{1}{2} \pm \frac{i}{\hbar} \theta_1 \theta_2 = \frac{1}{2} \left(1 \pm \sigma^3\right). \tag{5.21}
\]

The \( \pi^{(P)}_{\pm 1/2} \) fulfill the *eigenvalue equation*

\[
H \ast \pi^{(P)}_{\pm 1/2} = E_{\pm 1/2} \pi^{(P)}_{\pm 1/2},
\]  \tag{5.22}

8
for the energy levels \(E_{\pm 1/2} = \pm \frac{\hbar \omega}{2}\). The Wigner functions \(\pi^{(P)}_{\pm 1/2}\) are complete, idempotent and normalized with respect to the trace, i.e. they fulfill the equations

\[
\pi^{(P)}_{+1/2} + \pi^{(P)}_{-1/2} = 1, \quad \pi^{(P)}_{\alpha} \ast_{P} \pi^{(P)}_{\beta} = \delta_{\alpha \beta} \pi^{(P)}_{\alpha} \quad \text{and} \quad \text{Tr}(\pi^{(P)}_{\pm 1/2}) = 1, \tag{5.23}
\]

respectively. Furthermore they correspond to spin up and spin down states since the Wigner functions \(\pi^{(P)}_{\pm 1/2}\) correspond to the spin projectors, and the expectation values of the spin components \(S = \frac{\hbar}{2} \sigma\) are

\[
\langle S_i \rangle = \text{Tr} \left( \pi^{(M)}_{\pm 1/2} \ast_{P} \frac{\hbar}{2} \sigma^i \right) = 0 \quad (i = 1, 2), \tag{5.24a}
\]

\[
\langle S_3 \rangle = \text{Tr} \left( \pi^{(M)}_{\pm 1/2} \ast_{P} \frac{\hbar}{2} \sigma^3 \right) = \pm \frac{\hbar}{2}, \tag{5.24b}
\]

\[
\langle S^{2*} \rangle = \text{Tr} \left( \pi^{(M)}_{\pm 1/2} \ast_{P} \frac{\hbar^2}{4} \sigma^{2*} \right) = \frac{3}{4} \hbar^2. \tag{5.24c}
\]

The star exponential \(\text{Exp}(\cdot)\) allows us to calculate the time development of the \(\sigma^i\) as

\[
\begin{align*}
\sigma^1(t) &= \text{Exp}_P(-Ht) \ast_P \sigma^1 \ast_P \text{Exp}_P(Ht) = \sigma^1 \cos(\omega t) - \sigma^2 \sin(\omega t), \tag{5.25a} \\
\sigma^2(t) &= \text{Exp}_P(-Ht) \ast_P \sigma^2 \ast_P \text{Exp}_P(Ht) = \sigma^1 \sin(\omega t) + \sigma^2 \cos(\omega t), \tag{5.25b} \\
\sigma^3(t) &= \text{Exp}_P(-Ht) \ast_P \sigma^3 \ast_P \text{Exp}_P(Ht) = \sigma^3. \tag{5.25c}
\end{align*}
\]

With these expressions it is easy to see that the \(*\)-Heisenberg equation

\[
i \hbar \frac{df(t)}{dt} = [f(t), H(t)]_{\ast_P} \tag{5.26}
\]

for the spin is given by

\[
\frac{dS_1(t)}{dt} = -\omega S_2(t), \quad \frac{dS_2(t)}{dt} = \omega S_1(t) \quad \text{and} \quad \frac{dS_3(t)}{dt} = 0. \tag{5.27}
\]

For \(\omega = \left(\frac{\hbar}{mc}\right) B_3\), where \(B_3\) is the third component of the magnetic field \(B = (0, 0, B_3)\), this leads to the equation of motion for the spin in a magnetic field:

\[
\frac{dS}{dt} = \frac{e}{mc} B \times S. \tag{5.28}
\]

In the fermionic \(\theta\)-space the spin \(S = \frac{\hbar}{2} \sigma\) is the generator of rotations, which are described by the star exponential

\[
\text{Exp}_P(\varphi \cdot S) = \cos \frac{\varphi}{2} - i(\sigma \cdot n) \sin \frac{\varphi}{2}, \tag{5.29}
\]

where \(\varphi = \varphi n\) with the angle of rotation \(\varphi\) and rotation axis \(n\). The vector \(\theta = (\theta_1, \theta_2, \theta_3)^T\) transforms according to

\[
\text{Exp}_P(\varphi \cdot S) \ast_P \theta \ast_P \text{Exp}_P(-\varphi \cdot S) = \text{Exp}_P(\varphi \cdot S) \ast_P \theta \ast_P \text{Exp}_P(-\varphi \cdot S) = R(\varphi)\theta \tag{5.30}
\]

where \(R(\varphi)\) is the rotation matrix which satisfies

\[
R(\varphi)\theta = n(n \cdot \theta) + \cos \varphi(\theta - n(n \cdot \theta)) - \sin \varphi(n \times \theta). \quad \tag{5.31}
\]

The axial vector \(\sigma\) transforms in the same way under rotations.
6 Charged Particle with Spin in a Constant Magnetic Field

The bosonic and the fermionic oscillators can be combined to treat a physical system consisting of a charged particle with spin in a constant magnetic field in the star product formalism. We first consider the bosonic part of this problem: a charged spinless particle in a constant magnetic field. The magnetic field points in the direction of \( q_1 \) and can be described with the gauge potential \( A = \frac{e}{c}(q_2, q_1, 0) \). By minimal substitution \( p \rightarrow p - \frac{e}{c}A \) one obtains the Landau Hamiltonian

\[
H_L = \frac{1}{2m} (\tilde{p}_1^2 + \tilde{p}_2^2),
\]

where we have defined

\[
\tilde{p}_1 = p_1 - \frac{e}{c}A_1 = p_1 + \frac{m\omega}{2} q_2 \quad \text{and} \quad \tilde{p}_2 = p_2 - \frac{e}{c}A_2 = p_2 - \frac{m\omega}{2} q_1.
\]

with \( \omega = \frac{eB}{mc} \). In order to quantize this two dimensional system we transform the Moyal product (6.1) into the \((q_1, \tilde{p}_1)\)-coordinates; the resulting expression is

\[
f \ast_M g = f \exp \left[ \frac{i\hbar}{2} \left( \tilde{p}_1 \tilde{q}_1 - \tilde{q}_1 \tilde{p}_1 + \tilde{p}_2 \tilde{q}_2 - \tilde{q}_2 \tilde{p}_2 \right) + \frac{i\hbar m\omega}{2} \left( \tilde{p}_1 \tilde{q}_1 - \tilde{q}_1 \tilde{p}_1 + \tilde{p}_2 \tilde{q}_2 - \tilde{q}_2 \tilde{p}_2 \right) \right] g.
\]  

The \( \ast \)-genvalue equation

\[
\frac{1}{2m} (\tilde{p}_1^2 + \tilde{p}_2^2) \ast_M \pi_n^{(M)} = E_n \pi_n^{(M)}
\]

can easily be solved by comparison with the bosonic oscillator. As we have seen above the \( \ast \)-eigenfunctions of the bosonic oscillator depend only on the Hamiltonian. Therefore we also expect \( \pi_n^{(M)} \) to depend on \( \tilde{p}_1 \) and \( \tilde{p}_2 \) only. Taking this as an ansatz, only the second part of the star product (6.2), which can be written as

\[
\exp \left[ \frac{i\hbar}{2} \left( \tilde{p}_1 \tilde{q}_1 - \tilde{q}_1 \tilde{p}_1 + \tilde{p}_2 \tilde{q}_2 - \tilde{q}_2 \tilde{p}_2 \right) \right],
\]

has to be taken into account for the \( \ast \)-genvalue equation.

Setting \( q = \frac{\tilde{q}_1}{\omega} \) and \( p = \tilde{p}_2 \) the Landau Hamiltonian \( H_L \) reduces to the Hamiltonian of the bosonic harmonic oscillator (6.5) and Eq. (6.6) becomes the Moyal product in canonical variables. Then it is clear that the \( \ast \)-eigenfunctions of the Landau Hamiltonian are in analogy to (6.4) given by

\[
\pi_n^{(M)}(\tilde{p}_1, \tilde{p}_2) = \pi_n^{(M)}(H_L) = 2(-1)^n \exp \left( -\frac{2H_L}{\hbar \omega} \right) L_n \left( \frac{4H_L}{\hbar \omega} \right).
\]

The energy levels are the Landau levels \( E_n = \hbar \omega \left( n + \frac{1}{2} \right) \).

Since the system considered here is described in a four dimensional phase space we expect that another observable which commutes with the Hamiltonian is needed to characterize all the energy \( \ast \)-genfunctions. To find such an observable it is useful to write the star product (6.2) in the two forms

\[
f \ast_M g = f \exp \left[ \frac{i\hbar}{2} \left( \tilde{p}_1 \tilde{q}_1 - \tilde{q}_1 \tilde{p}_1 + \tilde{p}_2 \tilde{q}_2 - \tilde{q}_2 \tilde{p}_2 \right) + \frac{i\hbar m\omega}{2} \left( \tilde{p}_1 \tilde{q}_1 - \tilde{q}_1 \tilde{p}_1 + \tilde{p}_2 \tilde{q}_2 - \tilde{q}_2 \tilde{p}_2 \right) \right] g
\]  

\[
= f \exp \left[ \frac{i\hbar}{2} \left( \tilde{p}_1 \tilde{q}_1 - \tilde{q}_1 \tilde{p}_1 + \tilde{p}_2 \tilde{q}_2 - \tilde{q}_2 \tilde{p}_2 \right) \right] g
\]

by simply rearranging the terms in the argument of the exponential function. By observing that the functions \( \tilde{q}_1 = q_1 + \frac{1}{m\omega} \tilde{p}_2 \) and \( \tilde{q}_2 = q_2 - \frac{1}{m\omega} \tilde{p}_1 \) fulfill the equations

\[
(\partial_{\tilde{q}_1} - m\omega \partial_{\tilde{p}_2})\tilde{q}_1 = 0 \quad \text{and} \quad (\partial_{\tilde{q}_2} + m\omega \partial_{\tilde{p}_1})\tilde{q}_2 = 0
\]

(6.8)
it is obvious from Eqs. (6.7) that every function of the \( \tilde{q}_i \) commutes with every function of the \( \tilde{p}_i \), e.g.

\[
H_L \ast_M f(\tilde{q}_1, \tilde{q}_2) = H_L f(\tilde{q}_1, \tilde{q}_2) = f(\tilde{q}_1, \tilde{q}_2) H_L = f(\tilde{q}_1, \tilde{q}_2) \ast_M H_L.
\]  

(6.9)

This implies that \( \tilde{q}_1 \) and \( \tilde{q}_2 \) are conserved phase space functions, and also that all functions of the form \( f(\tilde{q}_1, \tilde{q}_2)\pi_n^{(M)}(\tilde{p}_1, \tilde{p}_2) \) are \( \ast \)-genfunctions of the Hamiltonian. Obviously such a function becomes a \( \ast \)-genfunction of the angular momentum

\[
J = q_1 p_2 - q_2 p_1 = -\frac{i}{\omega} H_L + \frac{m \omega}{2} (\tilde{q}_1^2 + \tilde{q}_2^2)
\]  

(6.10)

if \( f(\tilde{q}_1, \tilde{q}_2) \) is chosen to be a \( \ast \)-genfunction of \( \frac{m \omega}{2} (\tilde{q}_1^2 + \tilde{q}_2^2) \).

Using Eq. (6.8), only two terms in the argument of the exponential function contribute to the star product (6.7b) in the \( \ast \)-genvalue Eq. (6.9), so that

\[
\frac{m \omega}{2} (\tilde{q}_1^2 + \tilde{q}_2^2) \ast_M f(\tilde{q}_1, \tilde{q}_2) = \frac{m \omega}{2} (\tilde{q}_1^2 + \tilde{q}_2^2) \exp \left[ \frac{i \hbar}{2} \left( -\partial_{\tilde{q}_1} \partial_{\tilde{q}_1} - \partial_{\tilde{q}_2} \partial_{\tilde{q}_2} \right) \right] = \frac{m \omega}{2} (\tilde{q}_1^2 + \tilde{q}_2^2) \exp \left[ \frac{i \hbar}{2m \omega} \left( \partial_{\tilde{q}_1} \partial_{\tilde{q}_1} - \partial_{\tilde{q}_2} \partial_{\tilde{q}_2} \right) \right] f(\tilde{q}_1, \tilde{q}_2),
\]  

(6.11)

where we used the definition of \( \tilde{q}_i \) in the last step. Setting \( q = \tilde{p}_2 \) and \( p = m \omega \tilde{q}_1 \), the problem reduces to the one dimensional harmonic oscillator, so that \( f(\tilde{q}_1, \tilde{q}_2) \) becomes

\[
\pi_i^{(M)}(\tilde{q}_1, \tilde{q}_2) = 2(-1)^j \exp \left(-\frac{m \omega}{\hbar} (\tilde{q}_1^2 + \tilde{q}_2^2) \right) L_j \left( \frac{2m \omega}{\hbar} (\tilde{q}_1^2 + \tilde{q}_2^2) \right)
\]  

(6.12)

and the \( \ast \)-eigenvalues of \( \frac{m \omega}{2} (\tilde{q}_1^2 + \tilde{q}_2^2) \) are \( \hbar (l + \frac{1}{2}) \).

Thus, the Wigner functions of the Landau problem are \( \pi_{nl}^{(M)}(\tilde{q}_1, \tilde{q}_2, \tilde{p}_1, \tilde{p}_2) = \pi_i^{(M)}(\tilde{q}_1, \tilde{q}_2) \pi_{nl}^{(M)}(\tilde{p}_1, \tilde{p}_2) \) and lead with the \( \ast \)-genvalue equation

\[
H_L \ast_M \pi_{nl}^{(M)} = E_n \pi_{nl}^{(M)}
\]  

(6.13)

to the Landau levels \( E_n = \hbar \omega (n + \frac{1}{2}) \), whereas the equation \( J \ast_M \pi_{nl}^{(M)} = j_{nl} \pi_{nl}^{(M)} \) gives rise to the angular momentum eigenvalues \( j_{nl} = \hbar (l - n) \). For a treatment of the problem of a charged particle in a constant magnetic field using star products and holomorphic coordinates see Ref. [23].

In order to include the interaction of the spin with the magnetic field one has to consider both the bosonic and the fermionic sectors. Therefore we first combine the bosonic and the fermionic star products to the Moyal Pauli star product

\[
F \ast_{MP} G = F \exp \left[ i \frac{\hbar}{2} \sum_{i=1}^{3} \left( \partial_{\tilde{q}_i} \partial_{\tilde{p}_i} - \partial_{\tilde{p}_i} \partial_{\tilde{q}_i} \right) \right] G.
\]  

(6.14)

The Poisson bracket corresponding to this star product was considered in [25]. The realization of the “Feynman Trick” [24] for including the influence of a magnetic field is given in the star product formalism by

\[
\left[ \left( p - \frac{e}{c} A \right) \cdot \sigma \right]^{2*_{MP}} = \left( p - \frac{e}{c} A \right)^{2*_{M}} - \frac{\hbar c}{e} \sigma \cdot B.
\]  

(6.15)

Therefore an interaction term \( H_I = -\frac{\hbar c}{2mc} \sigma \cdot B \) is induced in the Hamiltonian. With Eq. (6.14) the interaction term can be written as

\[
H_I = -\frac{e \hbar}{2mc} B \sigma^3 = -i \omega \theta_1 \theta_2,
\]  

(6.16)
which is just the fermionic oscillator \((5.10)\). Thus, the system describing a charged particle with spin in a constant magnetic field can be regarded as the sum of a two-dimensional bosonic and fermionic oscillators with the same parameter \(\omega\), i.e. a supersymmetric oscillator \([20]\). The projectors for this system are products of the projectors of the bosonic and the fermionic oscillators, i.e. \(\pi^{(M)}_n \pi^{(P)}_{\pm 1/2}\) and the energy levels are the sum of the single energy levels: \(E_{n,\pm 1/2} = \hbar \omega (n + \frac{1}{2} \pm \frac{1}{2})\).

7 The Supersymmetric Oscillator and the Witten Index

The harmonic oscillator can be factorized into \(H = \omega \bar{a} a\) by using the holomorphic coordinates

\[
a = \sqrt{\frac{m \omega}{2}} \left(q + i \frac{p}{m \omega}\right) \quad \text{and} \quad \bar{a} = \sqrt{\frac{m \omega}{2}} \left(q - i \frac{p}{m \omega}\right).
\]

(7.1)

The Moyal product for holomorphic coordinates is

\[
f \ast_M g = f \exp \left[\frac{\hbar}{2} \left(\bar{\partial}_f \bar{\partial}_a - \bar{\partial}_a \bar{\partial}_f\right)\right] g.
\]

(7.2)

Since the projectors \(\pi^{(M)}_n\) depend on \(H\) only, they have the same functional form as in Eq. \((5.4)\).

In the fermionic case one can also go over to holomorphic coordinates

\[
f = \frac{1}{\sqrt{2}}(\theta_2 + i \theta_1) \quad \text{and} \quad \bar{f} = \frac{1}{\sqrt{2}}(\theta_2 - i \theta_1).
\]

(7.3)

The fermionic oscillator in these coordinates has the form \(H = \omega \bar{f} f\) and the Pauli star product becomes

\[
F \ast_P G = F \exp \left[\frac{\hbar}{2} \left(\bar{\partial}_f \bar{\partial}_a + \bar{\partial}_a \bar{\partial}_f\right)\right] G.
\]

(7.4)

The fermionic projectors in holomorphic coordinates are \(\pi^{(P)}_{\pm 1/2} = \frac{1}{2} \pm \frac{i}{2} \bar{f} f\).

The bosonic and the fermionic oscillator Hamiltonian can be combined to the supersymmetric Hamiltonian \(H = \omega (\bar{a} a + \bar{f} f)\). The corresponding supersymmetric star product consists of the bosonic and fermionic star products \([52]\) and \([63]\):

\[
F \ast_{SU} G = F \exp \left[\frac{\hbar}{2} \left(\bar{\partial}_a \bar{\partial}_\bar{f} - \bar{\partial}_\bar{f} \bar{\partial}_a\right)\right] G.
\]

(7.5)

The supersymmetric projectors \(\pi^{(SU)}_{n_F, n_B} = \pi^{(M)}_{n_B} \pi^{(P)}_{n_F = \pm 1/2}\) are products of the bosonic and the fermionic projectors.

One can define the functions \(Q_+ = \frac{1}{\sqrt{h}} \bar{a} \bar{f}\) and \(Q_- = - \frac{1}{\sqrt{h}} \bar{a} \bar{f}\), which satisfy

\[
Q_+ \ast_{SU} \pi^{(SU)}_{n_F, n_B} \ast_{SU} Q_- = \hbar \pi^{(SU)}_{n_F+1, n_B} \quad \text{and} \quad Q_- \ast_{SU} \pi^{(SU)}_{n_F, n_B} \ast_{SU} Q_+ = \hbar \pi^{(SU)}_{n_F-1, n_B+1},
\]

(7.6)

and thus relate the otherwise distinct bosonic and fermionic sectors. The functions \(\pi^{(P)}_{\pm 1/2}\), \(Q_+\) and \(Q_-\) fulfill the relations

\[
\pi^{(P)}_{\pm 1/2} \ast_{SU} \pi^{(P)}_{\mp 1/2} = \pi^{(P)}_{\mp 1/2}, \quad Q_+ \ast_{SU} \pi^{(P)}_{\mp 1/2} = Q_\pm \quad \text{and} \quad \pi^{(P)}_{\pm 1/2} \ast_{SU} Q_\mp = Q_\pm,
\]

(7.7)

so that these functions form a Fredholm quadruple \(\Xi\), with which one can define the index \([27]\)

\[
\text{ind} \Xi = \text{tr} \left[\pi^{(P)}_{-1/2} - \frac{1}{\hbar} Q_+ \ast_{SU} Q_-\right] - \text{tr} \left[\pi^{(P)}_{1/2} - \frac{1}{\hbar} Q_- \ast_{SU} Q_+\right]
\]

\[
= \text{tr} \left[\pi^{(P)}_{-1/2} \left(\frac{1}{2} - \frac{a \bar{a}}{\hbar}\right)\right] - \text{tr} \left[\pi^{(P)}_{1/2} \left(\frac{1}{2} - \frac{a \bar{a}}{\hbar}\right)\right],
\]

(7.8)
where the trace “tr” is the sum over all states
\[
\text{tr}[F] = \sum_{n=0}^{\infty} \sum_{n_s=\pm1/2} \int d^2a \text{Tr}(\pi_n^{(M)} \pi_n^{(P)} \star U F).
\] (7.9)

The trace “Tr” is defined as in (5.18). The second terms in the round brackets of (7.8) give the sum of the numbers of the bosonic states. Since all bosonic states with \( E > 0 \) appear as pairs in the bosonic and the fermionic sector, these two terms cancel each other. The first term in the round brackets counts the number of bosonic states, so that the index is the difference of the number of bosonic states in the bosonic and the fermionic sector. Because of the pairing of states with \( E > 0 \) the index will be zero if there is a state with \( E = 0 \) in the bosonic and the fermionic sector and one if only one of the sectors has a state with \( E = 0 \). This index is the Witten index [28], which reveals whether the supersymmetry is exact or broken.

8 The Dirac Equation

With the Grassmannian representation of the Pauli matrices [5,41] it is possible to give a Grassmannian representation of the Dirac \( \gamma \)-matrices with two sets of \( \sigma^i \). Starting with the variables \( \theta_1, \ldots, \theta_6 \) one can build two triples of \( \sigma^i = \frac{2}{\hbar} \epsilon^{ijk} \theta_j \theta_k \), one for \( i, j, k \in \{1, 2, 3\} \) and one for \( i, j, k \in \{4, 5, 6\} \), by which the tensor structure of \( \hat{\alpha}^i = \hat{\sigma}^i \otimes \hat{\sigma}^4 \) and \( \hat{\beta} = \hat{\sigma}^3 \otimes \hat{1} \) in the Dirac representation as \( 4 \times 4 \) matrices is reproduced. The four functions defined as
\[
\alpha_i = \sigma^i \sigma^4 \quad (i = 1, 2, 3) \quad \text{and} \quad \beta = \sigma^6,
\] (8.1)
fulfill the equations
\[
\{\alpha_k, \alpha_l\}_{\star P} = 2 \delta_{kl}, \quad \{\alpha_k, \beta\}_{\star P} = 0 \quad \text{and} \quad \beta_{\star P} \beta = 1,
\] (8.2)
where we used the Pauli star product [5,12] for \( d = 6 \). Conceptually we have turned around Dirac’s ansatz. While Dirac tried to find (matrix) quantities \( \hat{\alpha} \) and \( \hat{\beta} \) that fulfill the Dirac algebra, we look for a product such that the relations of the Dirac algebra are fulfilled. This leads us to the Pauli star product.

In this approach to the Dirac theory we combined two copies of the three dimensional fermionic spaces which in Sec. 5 appeared to be suitable to describe spin. Thereby the subalgebra of the Grassmann algebra which contains only elements of even grade was used. From the algebraic point of view one can ask whether it is necessary to use a Grassmann algebra with six generators to reproduce the Dirac algebra [8,2]. Indeed, the functions
\[
\alpha_i = \sqrt{\frac{2}{\hbar}} \sigma^i \theta_5 \quad \text{and} \quad \beta = \frac{2i}{\hbar} \theta_4 \theta_5
\] (8.3)
also fulfill the Dirac algebra [8,2] by using five Grassmann variables and the star product [5,12] for \( d = 5 \). One is lead to this representation by constructing the Dirac Hamiltonian as a supercharge from supersymmetric quantum mechanics [7].

Since the Clifford algebra of the Dirac matrices is four dimensional, it should also be possible to start with a Grassmann algebra generated by \( \theta_1, \ldots, \theta_4 \) that is turned into a Clifford algebra with the Pauli star product [5,12] for \( d = 4 \). Indeed the dimensionless variables
\[
\alpha^i = \sqrt{\frac{2}{\hbar}} \theta_i \quad \text{and} \quad \beta = \sqrt{\frac{2}{\hbar}} \theta_4
\] (8.4)
obey the relations [8,2] and form another representation of the Dirac algebra. With respect to the Pauli star product the generators of the Grassmann algebra become here generators of the Clifford algebra, as in the Cliffordization procedure described in the first sections.
The four dimensional representation of the Dirac algebra can also be motivated by considerations of the symmetries of spacetime. With the definition of $\sigma^i$ in Eq. (5.14) we can reproduce the commutation relations of the corresponding Pauli matrices, and in Eq. (5.20) it was shown that $S_i = i\hbar/2 \sigma^i$ generate rotations in the Grassmann algebra. So far only the even part of the Grassmann algebra was involved, so that the question arises what transformations the $\theta_i$ are related to. The definition $K_i = i\sqrt{\hbar/2} \theta_i = i\hbar/2 \alpha^i$ leads to the commutation relations

$$[S_i, S_j]_{sp} = i\hbar\epsilon^{ijk} S^k, \quad [S_i, K_j]_{sp} = i\hbar\epsilon^{ijk} K_k \quad \text{and} \quad [K_i, K_j]_{sp} = -i\hbar\epsilon^{ijk} S_k,$$  

so that we can identify the components of $K$ as generators of the Lorentz boosts. The star exponential $\exp_p(\omega \cdot K)$ transforms $\alpha^\mu = (1, \alpha)$ with $\mu = 0, 1, 2, 3$ like a four vector:

$$\exp_p(\omega \cdot K) \ast_p \alpha^\mu \ast_p \exp_p(\omega \cdot K) = \exp_p(\omega \cdot K) \ast_p \alpha^\mu \ast_p \exp_p(\omega \cdot K) = \Lambda_{\nu}^{\mu}(\omega) \alpha^\nu,$$

where $\Lambda_{\nu}^{\mu}(\omega)$ is the matrix representation of a Lorentz boost. In contrast to Eqs. (5.30) the signs of the parameters—$\omega$ in this case—are not changed by the involution because $\overline{K} = -K$ compared to $\overline{S} = S$.

As one can see in Eq. (5.30) $\alpha = \sqrt{2/\hbar} \theta$ behaves like a vector under rotation and therefore should be mapped into $\mathcal{P}(\alpha) = -\alpha$ by the parity transformation $\mathcal{P}$, which cannot be represented without a further extension of the algebra. By introducing an additional generator $\theta_4$ to the three dimensional Grassmann algebra and extending the star exponential (5.12) to $d = 4$ a representation of the parity transformation can be given by

$$\mathcal{P}(F) = \beta \ast_p F \ast_p \beta$$

with the definition $\beta = \sqrt{2/\hbar} \theta_4$. The scalar 1 and the axial vector $\sigma$ defined in (5.11) are indeed invariant with respect to this transformation.

The three representations (8.1), (8.3) and (8.4) were built by starting with a representation of rotations in a Grassmann algebra with generators $\{\theta_1, \theta_2, \theta_3\}$, and as such the rotations are generated by

$$S_i = \frac{\hbar}{2} \sigma^i = \frac{1}{2i} \epsilon^{ijk} \theta_j \theta_k = -i \frac{\hbar}{4} \epsilon^{ijk} \alpha^j \ast_p \alpha^k$$

in all representations. Since $S_i$ can be given solely in terms of $\alpha^i$ and since all versions of the Dirac algebra $\{\alpha^i, \beta\}$ are equivalent, $\alpha$ behaves like a vector in all three representations, i.e.

$$\exp(\varphi \cdot S) \ast_p \alpha \ast_p \exp(-\varphi \cdot S) = R(\varphi)\alpha.$$  

The same argumentation is also valid for Lorentz boosts generated by $K_i = i\hbar/2 \alpha^i$ and the parity transformation with $\beta$—thus equations (8.6) and (8.7) hold true for all three definitions of $\{\alpha^i, \beta\}$.

It will now be shown that the rotations (8.1) and the Lorentz boosts (8.6) can be combined into one equation. Before doing so it is useful to introduce Grassmann functions that correspond to the Dirac matrices $\gamma^\mu$:

$$\gamma^0 = \beta \quad \text{and} \quad \gamma^i = \beta \ast_p \alpha^i \Rightarrow \{\gamma^\mu, \gamma^\nu\}_{sp} = 2g^{\mu\nu}.$$  

Eq. (8.6) is multiplied with $\beta$ from the left in order to get the different signs in the two star exponentials which occur in Eq. (8.6) for the rotations. Since $\beta$ anticommutes with $K_i \propto \alpha^i$, Eq. (8.6) becomes

$$\exp_p(-\omega \cdot K) \ast_p \gamma^\mu \ast_p \exp_p(\omega \cdot K) = \Lambda_{\nu}^{\mu}(\omega) \gamma^\nu.$$  

With the definition $\sigma^{\mu\nu} = \frac{1}{2} [\gamma^\mu, \gamma^\nu]_{sp}$ the six generators of the Lorentz transformation can be written as

$$K_i = \frac{i\hbar}{2} \alpha^i = i \frac{h}{2} \gamma^0 \ast_p \gamma^i = \frac{h}{2} \sigma^0 i,$$

$$S_i = -i \frac{h}{4} \epsilon^{ijk} \alpha^j \ast_p \alpha^k = i \frac{h}{4} \epsilon^{ijk} \gamma^j \ast_p \gamma^k = \frac{h}{4} \sum_{j<k} \epsilon^{ijk} \sigma^{jk}.$$  

(8.12a)
Therefore all Lorentz transformations are generated by \( \frac{\hbar}{2} \sigma^{\mu\nu} \) with \( \mu < \nu \). Because \( \beta \) commutes with \( S_i \propto \varepsilon \gamma^k \alpha^j \ast \alpha^k \), one can replace \( \alpha \) by \( \gamma \) in Eq. (5.18) and the resulting equation can finally be unified with (8.11) to

\[
\text{Exp}_P \left( \frac{\hbar}{4} \sigma^{\mu\nu} \omega_{\mu\nu} \right) \ast \gamma^\mu \ast \text{Exp}_P \left( \frac{\hbar}{4} \sigma^{\mu\nu} \omega_{\mu\nu} \right) = \Lambda^\nu_{\mu} (\omega_{\mu\nu}) \gamma^\nu.
\]

(8.13)

This is the usual form of Lorentz transformation known from Dirac theory.

For all representations of the Clifford algebra with \( d = 4, 5 \) or 6 generators \( \theta_i \) a trace can be defined as in Eq. (5.18):

\[
\text{Tr}(F) = \frac{4}{\hbar^2} \int d\theta_1 \ldots d\theta_{d-1} \ast F
\]

(8.14)

and with \( \text{Tr}(1) = 4 \) all the well-known trace rules for the \( \gamma \)-matrices are reproduced. The trace \( \text{Tr}(F) \) projects out the part of \( F \) that is proportional to 1 just like the map \( \varepsilon \) that was used in Eq. (4.12), which is the fermionic version of taking the vacuum expectation value. So \( \varepsilon \) can be made explicit by a Berezin integral.

With \( \alpha_i \) and \( \beta \) the Dirac Hamiltonian is given by

\[
H_D = c \alpha \cdot p + \beta mc^2
\]

(8.15)

and by using \( H_D \ast_M p H_D = c^2 p^2 + m^2 c^4 \) one can calculate the star exponential as

\[
\text{Exp}_{MP}(H_D t) = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{t}{\hbar} \right)^n H_D^{n \ast MP} = \pi^{(MP)}_{-E}(p) e^{iEt/\hbar} + \pi^{(MP)}_{+E}(p) e^{-iEt/\hbar}
\]

(8.16)

with the Wigner functions

\[
\pi_{\pm E}^{(MP)}(p) = \frac{1}{2} \left( 1 \pm \frac{H_D}{E} \right)
\]

(8.17)

and \( E = \sqrt{c^2 p^2 + m^2 c^4} \). The energy projectors \( \pi_{\pm E}^{(MP)}(p) \) are idempotent, complete and fulfill the \( \ast \)-value equations

\[
H_D \ast_M p \pi_{\pm E}^{(MP)}(p) = \pm E \pi_{\pm E}^{(MP)}(p).
\]

(8.18)

One can find projectors that are \( \ast \)-functions of the spin as well, which is defined by the equation \( S_u = \frac{1}{4} \gamma^5 \ast \gamma (\gamma \cdot u) \). The quantization axis \( u \) is a unit vector orthogonal to \( p \), so that the equations \( S_u \ast p \ S_u = (\frac{1}{2})^2 \) and \([H_D, S_u] \ast p = 0 \) hold. For \( S_u \) the star exponential is

\[
\text{Exp}_{p}(S_u \varphi) = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{\varphi}{\hbar} \right)^n S_u^{\ast n \ast p} = \pi^{(p)}_{-s}(u) e^{i\varphi/2} + \pi^{(p)}_{+s}(u) e^{-i\varphi/2}
\]

(8.19)

with the Wigner functions

\[
\pi^{(p)}_{\pm s}(u) = \frac{1}{2} \pm \frac{i}{\hbar} S_u.
\]

(8.20)

These are the star product analogues of the Dirac spin projectors and they obey the \( \ast \)-value equation

\[
S_u \ast p \pi^{(p)}_{\pm s}(u) = \pm \frac{\hbar}{2} \pi^{(p)}_{\pm s}(u).
\]

(8.21)

Since we have for \( p \cdot u = 0 \):

\[
[\beta, \gamma^5 \ast \gamma (\gamma \cdot p)] \ast p = 0 \quad \text{and} \quad [p \cdot \alpha, \gamma^5 \ast \gamma (\gamma \cdot u)] \ast p = 0
\]

(8.22)

the Wigner functions \( \pi_{\pm E}^{(MP)}(p) \) and \( \pi_{\pm s}^{(p)}(u) \) and the observables \( H_D \) and \( S_u \) commute under the star product. The Wigner functions for the Dirac problem are therefore given by

\[
\pi_{\pm E, \pm s}(p, u) = \pi_{\pm E}^{(MP)}(p) \ast_M p \pi_{\pm s}^{(p)}(u)
\]

(8.23)
The Dirac Wigner functions are idempotent:
\[
\pi^{(MP)}_{\pm \pm}(p, u) = \pm E^{(MP)}_{\pm \pm}(p, u) \quad \text{and} \quad S_u \pi^{(MP)}_{\pm \pm}(p, u) = \frac{\hbar}{2} \pi^{(MP)}_{\pm \pm}(p, u).
\] (8.24)

The Dirac Wigner functions are normalized to 1.

With the relations
\[
[H_D, x_i]_{*_{MP}} = -i\hbar \alpha_i \quad \text{and} \quad \{H_D, \alpha_i\}_{*_{MP}} = 2cp_i
\] (8.25)

one can calculate the time development of the position as
\[
x_i(t) = \text{Exp}_{MP}(-H_D t) *_{MP} x_i *_{MP} \text{Exp}_{MP}(H_D t)
\]
\[
= x_i + \frac{c^2 p_i t}{*_{MP} H_D^{-1*_{MP}}} + \frac{i\hbar c}{2} (\alpha_i - cp_i *_{MP} H_D^{-1*_{MP}}) *_{MP} H_D^{-1*_{MP}} *_{MP} (\text{Exp}_{MP}(2H_D t) - 1),
\] (8.26)

where
\[
H_D^{-1*_{MP}} = \frac{H_D}{c^2 p^2 + m^2 c^4}
\] (8.27)

is the inverse under the Moyal-Pauli star product. In Eq. (8.26) the first two terms correspond to the classical motion while the last term is the well-known term that represents the Zitterbewegung [29].

It is also possible to derive the Dirac equation in the star product formalism by using the fact that in the rest frame it should coincide with the *-genvalue Eq. (8.19). By setting \( p = 0 \) this equation becomes
\[
(\gamma^0 mc \mp mc) *_p \pi^{(MP)}_{\pm E}(0) = 0 \quad \text{with} \quad \pi^{(MP)}_{\pm E}(0) = \frac{1}{2} (1 \pm \gamma^0).
\] (8.28)

The solution \( \pi^{(MP)}_{\pm E}(0) \) follows from Eq. (8.17). According to this, the equations in (8.25) can be boosted into a moving frame by \( S = \text{Exp}_{P}(\omega \cdot K) \), where the parameter \( \omega \) depends on the momentum \( p \) of the particle in the moving frame:
\[
S^{-1} *_p (\gamma^0 mc \mp mc) *_p \pi^{(MP)}_{\pm E}(0) *_p S = (S^{-1} *_p \gamma^0 *_p S mc \mp mc) *_p S^{-1} *_p \pi^{(MP)}_{\pm E}(0) *_p S = 0.
\]

Eq. (8.21) leads to
\[
S^{-1} *_p \gamma^0 *_p S = \frac{\dot{p}}{mc},
\] (8.29)

so that with the definition
\[
\pi^{(MP)}_{\pm m}(p) = S^{-1} *_p \pi^{(MP)}_{\pm E}(0) *_p S
\] (8.30)

the equation above turns into
\[
(\dot{p} \mp mc) *_{MP} \pi^{(MP)}_{\pm m}(p) = 0 \quad \text{with} \quad \pi^{(MP)}_{\pm m}(p) = \frac{\dot{p} \pm mc}{2mc},
\] (8.31)

which corresponds to the Dirac equation and the well-known energy projector respectively.

The same discussion as for the Lorentz boost of the energy *-genvalue Eq. (8.15) can be repeated for the spin *-genvalue Eq. (8.21) with its solution (8.20). By assuming that \( S_u = \frac{1}{2} \gamma_5 *_p (\gamma \cdot u) \) is a valid spin observable in the rest frame it takes on the form
\[
S_u = S^{-1} *_p S_u *_p S = -\frac{\hbar}{2} \gamma_5 *_p \psi
\] (8.32)
in the moving frame by applying a boost with \( S = \text{Exp}_p(\omega \cdot K) \). The condition \( u^2 = 1 \) and \( u \cdot p = 0 \) have to be translated into \( u^\mu u_\mu = -1 \) and \( u^\mu p_\mu = 0 \) respectively to ensure that \( S_u * p, S_u = \left( \frac{1}{2} \right)^2 \) and \([S_u, H_D] * p = 0\) hold true in every frame. Finally, the relativistic version of the spin \( * \) eigenvalue equation and its solution become

\[
S_u * p \pi^{(P)}_{\pm s}(u) = \frac{\hbar}{2} \gamma_5 * p \pi^{(P)}_{\pm s}(u) = \pm \frac{\hbar}{2} \pi^{(P)}_{\pm s}(u) \quad \text{with} \quad \pi^{(P)}_{\pm s}(u) = \frac{1}{2} \pm \frac{1}{\hbar} S_u = \frac{1}{2} \pm \frac{\gamma_5 * p \cdot \gamma}{2} \tag{8.33}
\]

by replacing \( S_u \) with \( S_u \) in both \( 8.31 \) and \( 8.33 \). One can see that the spin projectors \( \pi^{(P)}_{\pm s} \) take on the form which is known from the Dirac theory. As in Eq. \( 8.23 \), the two projectors in Eqs. \( 8.31 \) and \( 8.33 \) can be combined to

\[
\pi^{(MP)}_{\pm m, \pm s}(p, u) = \pi^{(MP)}_{\pm m} (p) *_{MP} \pi^{(P)}_{\pm s}(u) = \pi^{(P)}_{\pm s}(u) *_{MP} \pi^{(MP)}_{\pm m} (p), \tag{8.34}
\]

which is a projector corresponding to the four-spinors \( u \) and \( v \) in the Dirac theory. It fulfills both \( * \) eigenvalue equations in \( 8.31 \) and \( 8.33 \), is idempotent, and is normalized with respect to the trace \( 8.11 \).

9 The Non-Relativistic Limit of the Dirac Equation

In order to calculate the non-relativistic limit of the Dirac Hamiltonian it is straightforward to translate the Foldy-Wouthuysen transformation \( 30 \) into the star product formalism. The time development of the Wigner function is given by \( 31 \)

\[
i\hbar \frac{\partial \pi(t)}{\partial t} = [H(t), \pi(t)]_{* MP} \, . \tag{9.1}
\]

This can be translated into an equation for the unitary transformed Wigner function

\[
\pi'(t) = U(t) *_{MP} \pi(t) *_{MP} U(t)^{-1}, \tag{9.2}
\]

which leads to

\[
i\hbar \partial_t \pi'(t) = [H'(t), \pi'(t)]_{* MP}, \tag{9.3}
\]

with

\[
H'(t) = U(t) *_{MP} (H(t) - i\hbar \partial_t) *_{MP} U(t)^{-1}. \tag{9.4}
\]

The Hamiltonian can be written as

\[
\frac{H}{mc^2} = \beta + E + \mathcal{O} \tag{9.5}
\]

with

\[
\beta + E = \frac{1}{2} \left( \frac{H}{mc^2} + \beta *_{MP} \frac{H}{mc^2} \beta \right) \quad \text{and} \quad \mathcal{O} = \frac{1}{2} \left( \frac{H}{mc^2} - \beta *_{MP} \frac{H}{mc^2} \beta \right). \]

The function \( E \) has positive parity and \( \mathcal{O} \) is a function with negative parity. It is assumed that \( E \) and \( \mathcal{O} \) are of order \( (\frac{1}{c^2})^3 \) and \( (\frac{1}{c^2})^4 \) respectively.

Following the conventional Foldy-Wouthuysen procedure we choose

\[
U(t) = \text{Exp}_{MP} \left( \frac{i\beta}{2} *_{MP} \mathcal{O} \right) = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{i\beta}{2} *_{MP} \mathcal{O} \right)^n *_{MP}, \tag{9.6}
\]

so that \( 8.31 \) gives

\[
\frac{H'}{mc^2} = \beta *_{MP} \left( 1 + \frac{1}{2} \mathcal{O} \right) \mathcal{O} *_{MP} - \frac{1}{8} \mathcal{O}^4 *_{MP} + \mathcal{O} - \frac{1}{8} \left[ \mathcal{O}, \left( \mathcal{O}, \mathcal{O} \right) *_{MP} + \frac{i\hbar}{mc^2} \mathcal{O} *_{MP} \right] *_{MP}
\]

\[
+ \frac{1}{2} \beta *_{MP} \mathcal{O}, \mathcal{O} *_{MP} - \frac{1}{3} \mathcal{O}^3 *_{MP} + \frac{i\hbar}{2mc^2} \beta *_{MP} \mathcal{O} + \ldots, \tag{9.7}
\]
where the first row contains even functions only, whereas the second row consists of odd functions only. This shows that (9.7) can be written as

\[ \frac{H'}{mc^2} = \beta + \mathcal{E}' + \mathcal{O}'. \]  

(9.8)

Repeating this transformation with

\[ U(t) = \text{Exp}_{MP} \left( \frac{i\beta}{2} *_{MP} \mathcal{O}' \right) \]  

(9.9)

leads to

\[ \frac{H''}{mc^2} = \beta + \mathcal{E}', \]  

(9.10)

where all terms of the order \((\frac{1}{c^5})\) or higher are neglected.

For the Dirac Hamiltonian \( H = \alpha \cdot (c \mathbf{p} - e\mathbf{A}) + \beta mc^2 + e\varphi \) we have

\[ \mathcal{E} = \frac{e\varphi}{mc^2} \quad \text{and} \quad \mathcal{O} = \alpha \cdot \frac{cp - e\mathbf{A}}{mc^2}. \]  

(9.11)

Up to terms of order \((\frac{1}{c^4})\) in \( \frac{H''}{mc^2} \) the transformed Hamiltonian \( H'' \) is therefore given by

\[ \begin{align*}
H'' &= mc^2 \beta \ *_{MP} \left( 1 + \frac{1}{2} \mathcal{O}'^{2*_{MP}} - \frac{1}{8} \mathcal{O}'^{4*_{MP}} \right) + mc^2 \mathcal{E} - \frac{mc^2}{8} \left[ \mathcal{O}, \left( \mathcal{O}'_{*_{MP}} + \frac{i\hbar}{mc^2} \mathcal{O} \right) \right]_{*_{MP}} \\
&= \beta \left( mc^2 + \frac{(p - e\mathbf{A})^{2*_{MP}}}{2m} - \frac{p^4}{8m^3c^2} \right) - \frac{e\hbar}{2mc} \beta \ *_{MP} \mathbf{\sigma} \cdot \mathbf{B} + e\varphi \\
&- \frac{e\hbar}{4m^2c^2} \mathbf{\sigma} \cdot (\mathbf{E} \times \mathbf{p}) - \frac{e\hbar^2}{8m^2c^2} \text{div} \mathbf{E}.
\end{align*} \]  

(9.12)

In order to compare this result with the conventional operator expression one has to apply a Weyl transformation \( \Theta_W \) [32], which transforms a product of phase space variables into the totally symmetrized product of the corresponding operators and the \( \sigma^i, \alpha_i \) and \( \beta \) into the corresponding matrices. The Hamilton operator corresponding to (9.12) is then

\[ \hat{H}'' = \beta \left( mc^2 + \frac{(\hat{p} - e\mathbf{A})^2}{2m} - \frac{\hat{p}^4}{8m^3c^2} \right) - \frac{e\hbar}{2mc} \beta \hat{\mathbf{\sigma}} \cdot \hat{\mathbf{B}} + e\hat{\varphi} \\
- \frac{e\hbar}{4m^2c^2} \hat{\mathbf{\sigma}} \cdot (\hat{\mathbf{E}} \times \hat{\mathbf{p}}) - \frac{ie\hbar^2}{8m^2c^2} \hat{\mathbf{\sigma}} \cdot \text{rot} \hat{\mathbf{E}} - \frac{e\hbar^2}{8m^2c^2} \text{div} \hat{\mathbf{E}}, \]  

(9.13)

which is the conventional result. We have used the relation

\[ \Theta_W (\mathbf{E} \times \mathbf{p}) = \frac{1}{2} (\mathbf{E} \times \hat{\mathbf{p}} - \hat{\mathbf{p}} \times \mathbf{E}) = \mathbf{E} \times \hat{\mathbf{p}} + \frac{i\hbar}{2} \text{rot} \mathbf{E}. \]  

(9.14)

10 Conclusions

Starting from an underlying Grassmann algebra a process of Chevalley Cliffordization leads to a Clifford algebra. The product in this algebra is essentially a fermionic star product which arises in the quantization of physical systems involving fermionic degrees of freedom. This product is important for analysis of the algebraic structure of quantum field theories. It also provides a canonical procedure for
quantizing physical systems with either bosonic or fermionic degrees of freedom. The concept of spin in relativistic and non-relativistic quantum mechanics can be clarified in this framework.

Clifford algebras can be taken as the starting point for a fruitful analysis of many mathematical structures which arise in theoretical physics [33], not only in quantum mechanics and field theory, but also in classical mechanics [34]. Starting from an underlying Grassmann algebra these structures in classical mechanics may be seen as arising from a Cliffordization procedure involving a fermionic star product. An additional deformation of the theory by use of a Moyal star product for the bosonic variables then leads to its quantum version. In a subsequent paper [35] we shall further elucidate this unified approach for treating classical and quantum mechanical dynamical systems.

Appendix

In this Appendix we show that the representation [62] fulfills Axiom [23c], i.e.

\[(uv)_B w = u_B(v_Bw).\]  \hspace{1cm} (A.1)

Without restriction of generality we choose \(u = \theta_1 \ldots \theta_r\), \(v = \theta_{r+1} \ldots \theta_s\) and \(w = \theta_i \ldots \theta_t\) with \(t \geq s\). Using the abbreviations \(B(\theta_i, \theta_j) = B_{i,j}\) and \(\partial_\theta = \partial_i\) we find for the left hand side:

\[
(uv)_B w = uv_B \frac{1}{s!} \left( \sum_{i,j} B_{i,j} \tilde{\partial}_i \tilde{\partial}_j \right)^s w
\]

\[
= \theta_i \ldots \theta_s \left( \sum_{\sigma \in S_{s,t}} B_{1, i_{\sigma(1)}} \ldots B_{s, i_{\sigma(s)}} (\tilde{\partial}_{i_{\sigma(1)}} \tilde{\partial}_{i_{\sigma(s)}}) \ldots (\tilde{\partial}_{i_{\sigma(s)}} \tilde{\partial}_{i_{\sigma(1)}}) \right) \theta_{t_1} \ldots \theta_{t_t}
\]

\[
= \theta_i \ldots \theta_s \left( \sum_{\sigma \in S_{s,t}} B_{1, i_{\sigma(1)}} \ldots B_{s, i_{\sigma(s)}} \tilde{\partial}_{i_{\sigma(1)}} \ldots \tilde{\partial}_{i_{\sigma(s)}} \theta_{t_1} \ldots \theta_{t_t} \right)
\]

\[
= (-1)^{(s-1)/2} \sum_{\sigma \in S_{s,t}} B_{1, i_{\sigma(1)}} \ldots B_{s, i_{\sigma(s)}} \tilde{\partial}_{i_{\sigma(1)}} \ldots \tilde{\partial}_{i_{\sigma(s)}} \theta_{t_1} \ldots \theta_{t_t}, \hspace{1cm} (A.2)
\]

where \(S_{s,t}\) is the set of all permutations of \(s\) elements out of \(t\).

For the right hand side of Eq. (A.1) we first calculate

\[
v_B w = \theta_{r+1} \ldots \theta_s \left( \sum_{\sigma \in S_{s-r,t}} B_{r+1, i_{\sigma(r+1)}} \ldots B_{s, i_{\sigma(s)}} (\tilde{\partial}_{i_{\sigma(r+1)}} \tilde{\partial}_{i_{\sigma(s)}}) \ldots (\tilde{\partial}_{i_{\sigma(s)}} \tilde{\partial}_{i_{\sigma(r+1)}}) \right) \theta_{t_1} \ldots \theta_{t_t}
\]

\[
= (-1)^{(s-r)(s-r-1)/2} \sum_{\sigma \in S_{s-r,t}} B_{r+1, i_{\sigma(r+1)}} \ldots B_{s, i_{\sigma(s)}} \tilde{\partial}_{i_{\sigma(s)}} \ldots \tilde{\partial}_{i_{\sigma(r+1)}} \theta_{t_1} \ldots \theta_{t_t}. \hspace{1cm} (A.3)
\]
This result lead to
\[
\begin{align*}
u_{ij}(v,w) &= (-1)^{(s-r)(s-r-1)/2} \sum_{\sigma \in S_{s-r,t}} B_{r+1,i\sigma(r+1)} \cdots B_{s,i\sigma(s)} \theta_1 \cdots \theta_r \\
&\times \left[ \sum_{\sigma' \in S_{r,t}} B_{1,i\sigma'(1)} \cdots B_{r,i\sigma'(r)} \tilde{\theta}_1 \cdots \tilde{\theta}_r \tilde{\theta}_{\sigma'(r)} \cdots \tilde{\theta}_{\sigma'(1)} \right] \tilde{\theta}_{\sigma(r)} \cdots \tilde{\theta}_{\sigma(r+1)} \theta_1 \cdots \theta_t \\
&= (-1)^{[s-r+1]+r(r-1)/2} \sum_{\sigma' \in S_{r,t}} B_{1,i\sigma'(1)} \cdots B_{r,i\sigma'(r)} B_{r+1,i\sigma(r+1)} \cdots B_{s,i\sigma(s)} \\
&\times \tilde{\theta}_{\sigma'(r)} \cdots \tilde{\theta}_{\sigma'(1)} \tilde{\theta}_{\sigma(r)} \cdots \tilde{\theta}_{\sigma(r+1)} \theta_1 \cdots \theta_t \\
&= (-1)^{s(s-1)/2} \sum_{\sigma' \in S_{r,t}} B_{1,i\sigma'(1)} \cdots B_{r,i\sigma'(r)} B_{r+1,i\sigma(r+1)} \cdots B_{s,i\sigma(s)} \\
&\times \tilde{\theta}_{\sigma'(r)} \cdots \tilde{\theta}_{\sigma'(1)} \tilde{\theta}_{\sigma(r)} \cdots \tilde{\theta}_{\sigma(r+1)} \theta_1 \cdots \theta_t \\
&= (-1)^{s(s-1)/2} \sum_{\sigma \in S_{s,t}} B_{1,i\sigma(1)} \cdots B_{s,i\sigma(s)} \tilde{\theta}_{\sigma(1)} \cdots \tilde{\theta}_{\sigma(r)} \theta_1 \cdots \theta_t,
\end{align*}
\] (A.4)

which is the same result as before. In the last step we used the fact that a term in the sum will be zero if \(\sigma(i) = \sigma'(j)\), because of the fermionic character of the derivatives.

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