Critical gaps of first-order phase transition in infinitely long Ising cylinders with antiperiodically joined circumference

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Based on the analytic expression of free energy for infinitely long Ising strip with finite width joined antiperiodically on a variety of planar lattices, we show the existence of first-order phase transition at the critical point of Ising transition. The critical gaps of the transition are also calculated analytically by measuring the discontinuities in the internal energy and the specific heat.

I. INTRODUCTION

In the study of critical phenomena, the investigations about finite-size effect have been very active[1]. These introduce more stringent test on the universality by including universal critical amplitudes and amplitude relations[2]. The finite-size effect is also important in the comparison between theory and experiment. For example, it relies on the knowledge of finite-size effect to extract the bulk properties from the experimental results. In the study of finite-size effect, a semi-infinite system for which the effect can be singled out is always of fundamental importance. Thus, the system of an infinitely long Ising cylinder with finite circumference has been studied for some times[3, 4, 5, 6, 7]. To form an infinitely long Ising cylinder, we can join the finite width periodically or antiperiodically. For periodically joined circumference (PJC, hereafter), the condition imposed on the coupling of the boundary spin variable $\sigma_{m,L}$ is $\sigma_{m,L+1} = \sigma_{m,1}$ with $-\infty < m < \infty$. For antiperiodically joined circumference (AJC, hereafter), the condition becomes $\sigma_{m,L+1} = -\sigma_{m,1}$. For the system with PJC, the finite-size scaling behaviors of the shift of critical temperature $\delta (L)$, defined as $\delta (L) = |T_{\text{max}}(L) - T_c| / T_c$ with the specific-heat peak $T_{\text{max}}(L)$ and the bulk critical temperature $T_c$, was known. The exact scaling form was shown to be $\delta (L) = b L / (L)^{2}$, where $b$ is a function of coupling ratios and independent of the size[8]. Recently, we studied the finite-size scaling of the distribution of partition function zeros of such a system[9]. The leading finite-size scaling behavior of the imaginary part of a zero labelled by $j$ is given as $\text{Im} \ z_j (L) \sim L^{-1/\nu}$ with the correlation length exponent $\nu$[9]; and the leading scaling behavior of the real part of the lowest zero ($j = 1$) can be written as $|\text{Re} \ z_1 (L) - z_c| \sim L^{-2 \lambda_{\text{zero}}}$, here $\lambda_{\text{zero}}$ is another critical exponent which is closely related to the shift exponent $\lambda$ characterizing the scaling behavior of $\delta (L)$[10, 11, 12, 13]. Our results showed that the system with PJC has $\nu = 1$ and $\lambda_{\text{zero}} = 2$.

For the system with AJC, the influences of the finite-size effect on thermodynamic quantities are quite different from those for the system with PJC. It was shown that the critical point of two-dimensional Ising phase transition is one of the partition function zeros for the system with AJC, and the singularity associated with the zero at the critical point is a first-order phase transition for finite circumference[8]. As the length of circumference goes infinite, the effect of antiperiodic boundary condition becomes negligible, and the singularity changes to be a second-order phase transition. This is a very intriguing feature. It is known that antiperiodic boundary condition may enhance the appearance of the Block walls in spin configurations. In general, the existence of Block walls will increase the free energy but suppress the energy fluctuations of the system. However, for an infinitely long Ising cylinder, based on the study of the cumulative distribution of partition function zeros we found that the existence of the zero mode of spin wave is solely responsible for the occurence of the non-analyticity at the critical point. We notice that the appearence of first-order phase transition does not contradict with the Mermin-Wigner theorem[14, 15]. Because the zero mode of spin wave always appear in the spin configurations, the system can not be viewed as an effective one-dimensional system with short-range Hamiltonian. First-order transitions are characterized by the first derivative of the free energy at the transition point. In this paper, we further analyze the nature of the first-order transition quantitatively by directly measuring the discontinuities in the internal energy and the specific heat for the systems with AJC on planar square, triangular, and hexagonal lattices with anisotropic couplings.

This paper is organized as follows. In Sec. II, the analytic expressions of free energy density for an infinitely long Ising cylinder with PJC and AJC on a variety of planar lattices are given and discussed. In Sec. III, we perform explicit calculations for the internal energy and the specific heat. Our results show that there exists discontinuities locating at the critical point of two-dimensional Ising transition for AJC. Then, in Sec. IV we measure the discontinuities appearing in the internal energy and the specific heat explicitly. Finally, we summarize the results in Sec. V.
II. FREE ENERGY

Consider an infinitely long strip with finite width \( L \) on plane triangular or hexagonal lattices. The two infinitely extended sides may be joined periodically or antiperiodically. Then, Ising model with ferromagnetic couplings between nearest neighbors is defined on such infinitely long cylinder. The corresponding form for the free energies per site per \( k_B T \) was shown to be

\[
f = -\ln R - \frac{1}{2sL} \sum_{p=0}^{L-1} \int_0^{2\pi} d\phi \frac{2\pi}{2\pi} \ln \left \{ \frac{2\pi(p + \Delta)}{L} \right \} - A_2 \cos \phi - A_3 \cos \left \{ \frac{2\pi(p + \Delta)}{L} - \phi \right \},
\]

(1)

with \( s = 1 \) for triangular lattice and \( s = 2 \) for hexagonal lattice. This result can be obtained either by direct derivation\[6\] or by taking one side to infinite in the results of finite lattices with appropriate boundary conditions\[7, 16\].

In the above expression, \( R \) is some lattice-dependent function of temperature and irrelevant to the physical singularity of the theory, and \( A_\mu \)'s \((\mu = 0, 1, 2, \text{and } 3)\) can be expressed in terms of particular functionals as

\[
A_0 = \alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2,
\]

(2)

\[
A_1 = 2(\alpha_0\alpha_1 - \alpha_2\alpha_3),
\]

(3)

\[
A_2 = 2(\alpha_0\alpha_2 - \alpha_3\alpha_1),
\]

(4)

\[
A_3 = 2(\alpha_0\alpha_3 - \alpha_1\alpha_2),
\]

(5)

with the functionals \( \alpha_\mu \)'s given as

\[
\alpha_0 = 1 + t_1 t_2 t_3,
\]

(6)

\[
\alpha_1 = t_1 + t_2 t_3,
\]

(7)

\[
\alpha_2 = t_2 + t_3 t_1,
\]

(8)

\[
\alpha_3 = t_3 + t_1 t_2,
\]

(9)

for the triangular lattice, and

\[
\alpha_0 = 1
\]

(10)

\[
\alpha_1 = t_1 t_2,
\]

(11)

\[
\alpha_2 = t_2 t_3,
\]

(12)

\[
\alpha_3 = t_3 t_1,
\]

(13)

for the hexagonal lattice. Here the \( t_i \)'s are given in terms of the hyperbolic tangential functions as

\[
t_i = \tanh \eta_i
\]

(14)

with \( \eta_i = J_i/k_B T \), and \( J_i \) is the coupling constant given in the \( i \)'th direction as specified in Fig. 1. Note that the expression of triangular lattice with \( t_3 = 0 \) gives the result of rectangular lattice.

The \( \Delta \) value of Eq. (1) is one-half and zero for PJC and AJC respectively. Thus, the exact forms of the free energies subject to the two types of boundary conditions differ only in the possible \((p + \Delta)\)-values. To simplify the notation, we define \( p = p + \Delta \) and change the sum of Eq. (1) to be over the \( p \)-values. Then, \( p \) is integer ranging from 0 to \( L - 1 \) for AJC, and it is half integer from 1/2 to \( L - 1/2 \) for PJC. When the width of Ising strip \( L \) is extended to infinite, the sum of Eq. (1) becomes a simple integration and the two expressions of different boundary conditions reduce to the same result. In this limit, the critical temperature of the Ising transition is determined by the condition\[17\],

\[
A_0 - A_1 - A_2 - A_3 = 0.
\]

(15)

The form of the free energy density of Eq. (1) can be rewritten as

\[
f = -\ln R - \frac{1}{2sL} \sum_p \left \{ \ln \left \{ A_0 - A_1 \cos \left ( \frac{2\pi p}{L} \right ) \right \} + I(p, \eta) \right \},
\]

(16)
where
\[ I(p, \eta) = \int_0^{2\pi} \frac{d\phi}{2\pi} \ln \left[ 1 + G(p, \eta) \cos(\phi - \psi(p, \eta)) \right], \]  
(17)

\[ G(p, \eta) = \frac{[A_2^2 + A_3^2 + 2A_2A_3 \cos(2\pi p/L)]^{1/2}}{A_0 - A_1 \cos(2\pi p/L)}, \]  
(18)

and the angle \( \psi(p, \eta) \) is defined by
\[ \tan \psi(p, \eta) = \frac{A_3 \sin(2\pi p/L)}{A_2 + A_3 \cos(2\pi p/L)}. \]  
(19)

We notice that the integration of Eq. (17) may be completed to simplify the expression. However, to calculate the derivatives of free energy we have to be cautious of the continuity of the integrand to avoid ill-defined results. Particularly, if any singularity appears in the integrand, the integration becomes improper. Therefore, for concreteness, in calculating the derivatives of free energy we first proceed with the derivatives and then perform the integration in Eq. (17).

III. DERIVATIVES OF FREE ENERGY

The dimensionless internal energy density \( \epsilon \) and specific heat \( c \) are defined as follows,
\[ \epsilon_i = \frac{\partial f}{\partial \eta_i}, \]  
(20)
\[ c = -\eta^2 \frac{\partial \epsilon}{\partial \eta} = -\eta^2 \frac{\partial^2 f}{\partial \eta^2}. \]  
(21)

Note that the measurement of Eq. (20) depends on the reference scale which is one of the coupling constants indexed by \( i \). However, the expression of the specific heat is invariant under the change of the reference scale since \( \eta_i^2 (\partial/\eta_i)^2 = \beta^2 (\partial/\beta)^2 \) for any \( i \).

The calculation of Eq. (20) contains the derivative of Eq. (17) which gives
\[ \frac{\partial I}{\partial \eta} = \int_{-\psi(p, \eta)}^{2\pi - \psi(p, \eta)} \frac{d\Phi}{2\pi} \left[ \frac{\partial G(p, \eta)}{\partial \eta} \cos \Phi \right], \]  
(22)

where we neglect the label of reference scale, and the validity for interchanging the derivative with the integration is, however, only ensured at the points where the integrand is continuous for any \( \Phi \). Using the technique of contour integration, we obtain the result of Eq. (22) as
\[ \frac{\partial I}{\partial \eta} = (1 - \frac{1}{\sqrt{1 - G^2(p, \eta)}}) \frac{\partial}{\partial \eta} \ln G(p, \eta) \ \text{for} \ 1 - G^2(p, \eta) > 0. \]  
(23)

As indicated, the analytic form of \( \partial I/\partial \eta \) given by Eq. (23) is valid only for positive function \( 1 - G^2(p, \eta) \). To inspect the domain of the functional values, we plot \( 1 - G^2(p, \eta) \) versus \( \eta \) for some \( p \)-modes in Fig. 2 which indicate that \( 1 - G^2(p, \eta) \) is a positive and concave function of \( \eta \) for a given \( p \)-mode and the absolute minimum locates at the point \( p = 0 \) and \( \eta = \eta_c \) with \( 1 - G^2(0, \eta_c) = 0 \). Here \( \eta_c \) denotes the \( \eta \) value at the critical point of ising transition. These features are summarized in the following lemma and corollary:

**Lemma 1** For the ferromagnetic couplings, \( 1 - G^2(p, \eta) \) as a function of \( \eta \) is bounded from below for a given \( p \)-mode.

**Proof.** The function of interest can be rewritten as
\[ 1 - G^2(p, \eta) = \frac{N(p, \eta)}{[A_0 - A_1 \cos(2\pi p/L)]^2}, \]  
(24)

where the numerator \( N(p, \eta) \) is given as
\[ N(p, \eta) = A_1^2 [\cos(2\pi p/L) - B(\eta)]^2 + C(\eta), \]  
(25)
with
\[ B(\eta) = \frac{(A_2A_3 + A_0A_1)}{A_1^2}, \]
\[ C(\eta) = (A_0^2 - A_2^2 - A_3^2) - \frac{(A_2A_3 + A_0A_1)^2}{A_1^2}. \]

Since the denominator of Eq. (24) is strictly positive, the sign of the function is completely determined by the numerator \( N(p, \eta) \). For the ferromagnetic couplings, we have \( B(\eta) \geq 1 \), which implies \( N(p, \eta) \geq N(0, \eta) \) for any given temperature \( \eta \). Furthermore, according to the definitions of the \( A_\mu \)'s given by Eqs. (2)-(5) we have
\[ N(0, \eta) = \left[ (\alpha_0 - \alpha_1 - \alpha_2 - \alpha_3) (\alpha_0 - \alpha_1 + \alpha_2 + \alpha_3) \right]^2 \geq 0. \] (28)

Therefore, \( N(p, \eta) \) is bound from below so is the value of \( 1 - G^2(p, \eta) \).

**Corollary 2** For \( p = 0 \) the minimum of \( 1 - G^2(p, \eta) \) as a function of \( \eta \) locates at the bulk critical point \( \eta = \eta_c \) with zero value, and for \( p \neq 0 \) the minimum is larger than zero.

**Proof.** The value of \( N(0, \eta) \) given by Eq. (28) vanishes only at the critical point. This is due to the facts as follows: The critical condition of Eq. (15) is equivalent to
\[ \alpha_0 - \alpha_1 - \alpha_2 - \alpha_3 = 0. \] (29)

Moreover, for the ferromagnetic couplings we have \( \alpha_0 > \alpha_k \geq 0 \) for \( k = 1, 2, \) and 3. This leads the expression, \( \alpha_0 - \alpha_1 + \alpha_2 + \alpha_3 \), in Eq. (28) to be strictly positive. Hence, among the minima of \( N(p, \eta) \) for various \( p \)-modes there exist an absolute minimum \( N(0, \eta_c) \) whose value is zero. In other words for \( p \neq 0 \) we have
\[ N(p, \eta) > N(0, \eta) \geq N(0, \eta_c) = 0. \] (30)

Hence, the domain of the functional value of \( 1 - G^2(p, \eta) \) is specified by the conditions as
\[ 1 - G^2(p, \eta) > 0 \quad \text{for} \quad p \neq 0 \] (31)

and
\[ 1 - G^2(0, \eta) \geq 1 - G^2(0, \eta_c) = 0, \] (32)

and the proof is completed.

According to the corollary, the case of \( 1 - G^2(p, \eta) < 0 \) never happens for the case of ferromagnetic couplings. However, the situation, \( 1 - G^2(p, \eta) = 0 \), does occur at one point for which the expression of Eq. (23) is invalid. Thus, the result of Eq. (23) fails only at the critical point \( \eta = \eta_c \) for AJC, and it is valid at any temperature for the system with PJC. With the exclusion of the exceptional point from the result of Eq. (23), we can obtain the internal energy \( \epsilon \) of Eq. (20) as
\[ \epsilon = \epsilon_s + \frac{1}{2sL} \sum_p D_\epsilon(p, \eta), \] (33)

with
\[ \epsilon_s = -\frac{\partial \ln R}{\partial \eta} - \frac{1}{2sL} \sum_p \frac{\partial}{\partial \eta} \ln \left[ A_2^2 + A_3^2 + 2A_2A_3 \cos \left( \frac{2\pi p}{L} \right) \right]^{1/2}, \] (34)
\[ D_\epsilon(p, \eta) = \frac{1}{\sqrt{1 - G^2(p, \eta)}} \frac{\partial}{\partial \eta} \ln G(p, \eta). \] (35)

Likewise, using the definition given by Eq. (21) we can perform the derivative of \( \epsilon \) to obtain the specific heat \( c \) as
\[ c = c_s + \frac{\eta^2}{2sL} \sum_p D_c(p, \eta), \] (36)
with
\[ c_s = \eta^2 \frac{\partial^2 \ln R}{\partial \eta^2} + \frac{\eta^2}{2sL} \sum_p \frac{\partial^2}{\partial \eta^2} \ln \left[ A_2^2 + A_3^2 + 2A_2A_3 \cos \frac{2\pi p}{L} \right]^{1/2}, \] (37)

\[ D_c(p, \eta) = -\frac{\partial}{\partial \eta} D_c(p, \eta). \] (38)

For the case of periodic boundary condition, our numerical results from the measurement of specific heat indicate that the shift of critical temperature, \( \delta (L) \), does obey the scaling form \[ \delta (L) = b \ln L / (L)^2 \] with \( b = 0.135 \) for the rectangular lattice, 0.225 for the triangular lattice, and 0.166 for the hexagonal lattice when the case of isotropic couplings is considered. For the case of AJC, the situation, \( 1 - G^2(p, \eta) = 0 \), as shown in the corollary, occurs at one point, \( p = 0 \) and \( \eta = \eta_c \). Then, the left and right derivatives of the free energy are discontinuous at this location, and this leads to the finite jumps at the critical point. Thus, the first-order phase transition occurs at \( \eta_c \) for the system with AJC.

IV. MEASUREMENT OF DISCONTINUITY

To speculate how the finite jumps appear at the critical point, we may consider, for instance, Eq. (35) for the internal energy. By direct calculation, one can show that the quantity, \( \partial [\ln G(0, \eta)] / \partial \eta \), vanishes at the critical point \( \eta_c \). In this manner, although the denominator, \( \sqrt{1 - G^2(0, \eta)} \), also vanishes, the value of \( D_c(0, \eta) \) evaluated in the limit of approaching \( \eta_c \) can be finite. The existence of the jumps is then related to the sign change of the functions, \( \partial [\ln G(0, \eta)] / \partial \eta \) and \( \sqrt{1 - G^2(0, \eta)} \), across the critical point. While the function \( \partial [\ln G(0, \eta)] / \partial \eta \) appears to be continuous function with nonzero slope intersecting the real line at the critical point, its value essentially reverses sign across the critical point. On the other hand, the function \( \sqrt{1 - G^2(0, \eta)} \) must have its minimum at the critical point, and hence it does not change sign across the critical point. Thus, as approaching the critical point the left and right limits of Eq. (35) do not coincide for the zeroth mode and form the jump. Because the zero mode is absent from PJC, the discontinuity only appears in the case of AJC.

To measure the discontinuity appearing in the case of AJC, we introduce the critical gaps, say \( \Delta_s \) and \( \Delta_c \) for the internal energy and the specific heat respectively, defined as

\[ \Delta_s = \lim_{T \to T_c^+} \epsilon(T) - \lim_{T \to T_c^-} \epsilon(T), \] (39)
\[ \Delta_c = \lim_{T \to T_c^+} c(T) - \lim_{T \to T_c^-} c(T). \] (40)

Thus, \( \Delta_s \) is the amount of latent heat per site released from the first-order phase transition. Since only the zero mode contributes to the discontinuity, according to Eqs. (34) and (36) we have

\[ \Delta_s = \frac{1}{2sL} \lim_{\epsilon \to 0} \left[ D_c(0, \eta_c - \epsilon) - D_c(0, \eta_c + \epsilon) \right], \] (41)
\[ \Delta_c = \frac{\eta_c^2}{2sL} \lim_{\epsilon \to 0} \left[ D_c(0, \eta_c - \epsilon) - D_c(0, \eta_c + \epsilon) \right]. \] (42)

Then, we may extract the size-dependence from the critical gaps by introducing the gap amplitudes, \( a_s \) and \( a_c \), as

\[ \Delta_s = \frac{a_s}{sL} \quad \text{and} \quad \Delta_c = \frac{a_c}{sL}. \] (43)

Note that in the above equation we have isolated the the proportionality factor \( s \) between total number of lattice sites \( N \) and the length \( L \) along the finite edge. Owing to the inversive proportionality to \( L \), the critical gaps in fact vanish in the thermodynamic limit for which the non-analiticity changes to be the second-order phase transition.

The explicit forms for the gap amplitudes can be expressed as functions of the coupling ratios \( r_{ij} = J_i / J_j \), and the results read

\[ a_s = 2 + r_{31} \left( \frac{1 - t_3^3}{t_3 + t_2} \right)^2 + r_{21} \left( \frac{1 + t_3^3}{t_3 + t_2} \right)^2, \] (44)
\[ \frac{a_c}{(\eta_c^2)} = (r_{31} - r_{21})^2 \left[ \frac{1 - t_3^2}{t_3 + t_2} \right] + 4r_{31} r_{21} \left( \frac{1 - t_3^2}{t_3 + t_2} \right)^2. \] (45)
for triangular lattice, and

\[
a_c = r_{12} \left[ \frac{(1 + t_3^2)(1 - t_1^2)}{(t_3 + t_1)(1 - t_1 t_3)} \right]_c + r_{32} \left[ \frac{(1 - t_3^2)(1 + t_1^2)}{(t_3 + t_1)(1 - t_1 t_3)} \right]_c + \left[ \frac{1 - t_3^2}{t_2} \right]_c \tag{46}
\]

\[
\frac{a_c}{(\eta_2^2)_c} = (r_{32} - r_{12})^2 \left[ \frac{(1 - t_3^4)(1 - t_1^4)}{(t_3 + t_1)^2(1 - t_1 t_3)^2} \right]_c + 4r_{32}r_{12} \left[ \frac{(1 - t_3^2)(1 - t_1^2)}{(t_3 + t_1)^2} \right]_c + \left[ \frac{1 - t_3^4}{t_2^2} \right]_c \tag{47}
\]

for hexagonal lattice. Here the subscript \( c \) indicates that the functions of temperature contained in the parentheses are evaluated at the critical point of the Ising transition determined by Eq. (15). Note that the reference scales for Eqs. (44) and (46) are employed as \( J_1 \) and \( J_2 \) respectively, according to our convention shown in Fig. 1, and the permutation of the other coupling strengths leave the results invariant in each case. Again, the results for square lattice can be obtained from Eqs. (44) and (45) by putting \( t_3 \) and \( r_{31} \) to zero.

The numerical results of the gap amplitudes for various ratios of coupling constants are summarized in Tables. I and II. Since the expressions of Eqs. (44)-(47) all tend to be positive definite, by means of Eqs. (39) and (40), the left limit for the value of the internal energy or the specific heat at the critical point is always greater than the right limit. For both the triangular and hexagonal lattices, the gaps of the internal energy increase when the coupling constants are strengthened, however for the specific heat the behaviors appear to be rather complicated. Here, we record solely some of the typical values subject to the cases of isotropic couplings: For the triangular lattice \( a_c \) and \( a_c \) are 6.0 and 0.905212 respectively, while reducing to the rectangular lattices the values are 4.0 and 1.098589. On the other hand, the gap amplitudes for the hexagonal lattice are subsequently 3.464102 and 1.734378 for the internal energy density and specific heat.

V. SUMMARY

Based on the exact study of infinitely long Ising cylinders with the length of circumference \( L \), we show analytically that there exists first-order phase transition for the case of antiperiodically joined circumference. The transition point locates at the critical point of the Ising transition. This peculiar phenomena arises because of the existence of the zero mode of spin wave. The soft mode of spin waves causes the free energy to behave less smoothly around the critical temperature, and this leads to the finite jumps in the derivative quantities, such as the internal energy and specific heat of the system. The finite jumps, referred as the critical gaps, are found to be inversely proportional to the width of the system. We then extract the size-dependent factors from the critical gaps by introducing the gap amplitudes. The magnitudes of the gap amplitudes for the internal energy and specific heat are also given explicitly in terms of the ratios of the coupling strengths. In general, the gaps of the internal energy increase when the coupling constants are strengthened, however for the specific heat the behaviors appear to be rather complicated. In the limit of \( L \to \infty \), the nature of the non-analiticity becomes the conventional two-dimensional Ising transition.

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[1] V. Privman (ed.), *Finite Size Scaling and Numerical Simulation of Statistical Systems* (World Scientific, Singapore 1990).
[2] M. N. Barber, in *Phase Transitions and Critical Phenomena*, vol. 8, ed. C. Domb and J. L. Lebowitz, (London: Academic, 1983).
[3] L. Onsager, Phys. Rev. 65, 117 (1944).
[4] M. E. Fisher, in *Proc. 1970 E. Fermi Int. School of Physics*, M. S. Green ed. (Academic, NY, 1971) Vol. 51, p. 1.
[5] A. F. Ferdinand and M. E. Fisher, Phys. Rev. 185, 832 (1969).
[6] T. M. Liaw, M. C. Huang, S. C. Lin, and M. C. Wu, Phys. Rev. B 60, 12994 (1999).
[7] M. C. Wu, M. C. Huang, Y. P. Luo, and T. M. Liaw, J. Phys. A32, 4887 (1999).
[8] M. C. Huang, T. M. Liaw, Y. P. Luo, and S. C. Lin, *First-Order Phase Transition in Infinitely Long Ising Cylinders*, e-print: cond-mat/0407731.
[9] C. Itzykson, R.B. Pearson, and J.B. Zuber, Nucl. Phys. B 220, 415 (1983).
[10] W. Janke and R. Kenna, Phys. Rev. B65, 064110 (2002).
[11] C.N. Yang and T.D. Lee, Phys. Rev. 87, 404 (1952).
[12] T.D. Lee and C.N. Yang, Phys. Rev. 87, 410 (1952).
[13] M. E. Fisher, *Lecture Note in Theoretical Physics*, edited by E. Brittin (University of Colorado Press, Boulder, 1965), Vol. 7c, pp. 1–159.
[14] N.D. Mermin and H. Wagner, Phys. Rev. Lett. 17, 1133 (1966).
[15] P.W. Anderson and Y. Yuval, J. Phys. C4, 607 (1971).
[16] M.C. Wu and C.K. Hu, J. Phys. A35, 5189 (2002).
[17] V. N. Plechko, Theor. Math. Phys. 64, 748 (1985); Physica A152, 51 (1988); Phys. Lett. A157, 335 (1991).
Figure 1. The planar (a) triangular and (b) hexagonal lattices with directional coupling constants, $J_1$, $J_2$, and $J_3$.

Figure 2. The values of $1 - G^2(p, \eta)$ as a function of $\eta$ for a given $p$-mode specified in the curve. Here we take $L = 6$. 
TABLE I: A list of some values of the gap amplitudes and the critical temperatures for the triangular lattice where conventionally $J_1$ is used as the reference scale and the ratios of coupling constants are given as $r_{21} = J_2/J_1$ and $r_{31} = J_3/J_1$.

| $r_{21}$ | $r_{31}$ | $\eta_{J_1}^{-1}$ | $a_e(r_{21}, r_{31})$ | $a_e(r_{21}, r_{31})$ |
|---------|---------|-----------------|----------------------|----------------------|
| $\frac{1}{16}$ | $\frac{1}{16}$ | .9749969067 | 2.980303124 | .9945387423 |
| $\frac{1}{8}$ | $\frac{1}{8}$ | 1.120609893 | 3.127565685 | .9938210612 |
| $\frac{1}{4}$ | $\frac{1}{4}$ | 1.359594319 | 3.357757709 | 1.001236145 |
| $\frac{1}{2}$ | $\frac{1}{2}$ | 1.745136094 | 3.697719840 | 1.025135541 |
| 1 | 1 | 2.368135472 | 4.172052393 | 1.070033044 |
| 2 | 2 | 3.387294494 | 4.794576091 | 1.124851910 |
| 4 | 4 | 5.080921897 | 5.561554227 | 1.158588497 |
| 8 | 8 | 7.939611600 | 6.455308161 | 1.136952941 |
| 16 | 16 | 12.83539772 | 7.444318063 | 1.046706223 |
| $\frac{1}{16}$ | $\frac{1}{16}$ | 1.248671356 | 3.265311339 | .986744193 |
| $\frac{1}{8}$ | $\frac{1}{8}$ | 1.471030695 | 3.492162591 | .9857854706 |
| $\frac{1}{4}$ | $\frac{1}{4}$ | 1.845058443 | 3.839911133 | 1.002261462 |
| $\frac{1}{2}$ | $\frac{1}{2}$ | 2.464588209 | 4.335206961 | 1.044751239 |
| 1 | 1 | 3.490273808 | 4.990854252 | 1.104142168 |
| 2 | 2 | 5.202643928 | 5.80054669 | 1.147526636 |
| 4 | 4 | 8.096230369 | 6.742319380 | 1.136541280 |
| 8 | 8 | 13.05035679 | 7.788265258 | 1.054645265 |
| 16 | 16 | 1.674907167 | 3.724368061 | .970816527 |
| $\frac{1}{16}$ | $\frac{1}{16}$ | 2.034628498 | 4.095394305 | .9707838264 |
| $\frac{1}{8}$ | $\frac{1}{8}$ | 2.650888515 | 4.637804574 | 1.003032167 |
| $\frac{1}{4}$ | $\frac{1}{4}$ | 3.690116886 | 5.363639576 | 1.065812648 |
| $\frac{1}{2}$ | $\frac{1}{2}$ | 5.438377279 | 6.261540830 | 1.124696886 |
| 1 | 1 | 8.398210101 | 7.304442724 | 1.133261620 |
| 2 | 2 | 13.46287486 | 8.461116081 | 1.067581484 |
| 4 | 4 | 2.383122015 | 4.521379708 | .9433175594 |
| 8 | 8 | 3.001777420 | 5.165044415 | .9477520819 |
| 16 | 16 | 4.069256996 | 6.039616654 | 1.002114103 |
| $\frac{1}{16}$ | $\frac{1}{16}$ | 5.884122781 | 7.122162939 | 1.079055707 |
| $\frac{1}{8}$ | $\frac{1}{8}$ | 8.964879141 | 8.372913203 | 1.119838839 |
| $\frac{1}{4}$ | $\frac{1}{4}$ | 14.23051189 | 9.753480244 | 1.084320289 |
| $\frac{1}{2}$ | $\frac{1}{2}$ | 3.640956906 | 6.000000000 | .9052117204 |
| 1 | 1 | 4.766244030 | 7.172866022 | .9199187375 |
| 2 | 2 | 6.696628670 | 8.639380759 | .9984555121 |
| 4 | 4 | 9.890370850 | 10.32255458 | 1.079677481 |
| 8 | 8 | 15.59947050 | 12.16074113 | 1.094445451 |
| 16 | 16 | 6.003554839 | 8.866600995 | .864293479 |
| $\frac{1}{16}$ | $\frac{1}{16}$ | 8.138513992 | 11.08035144 | .8938292729 |
| $\frac{1}{8}$ | $\frac{1}{8}$ | 11.76824556 | 13.65416899 | .9930189587 |
| $\frac{1}{4}$ | $\frac{1}{4}$ | 17.92975829 | 16.43931998 | 1.071852827 |
| $\frac{1}{2}$ | $\frac{1}{2}$ | 10.60355406 | 14.54302699 | .8300549058 |
| 1 | 1 | 14.76046755 | 18.81937049 | .8743124974 |
| 2 | 2 | 21.75359011 | 23.54209421 | .9878571688 |
| 4 | 4 | 19.71670567 | 25.86585276 | .8066370472 |
| 8 | 8 | 27.92219046 | 34.25059835 | .8619605402 |
| 16 | 16 | 37.89016754 | 48.49746442 | .7926740283 |
TABLE II: A list of some values of the gap amplitudes and the critical temperatures for the hexagonal lattice where conventionally $J_2$ is used as the reference scale and the ratios of coupling constants are given as $r_{12} = J_1/J_2$ and $r_{32} = J_3/J_2$.

| $r_{12}$ | $r_{32}$ | $\eta_{2c}^{-1}$ | $a_c(z)(r_{12},r_{32})$ | $a_c(r_{12},r_{32})$ |
|---------|---------|------------------|--------------------------|--------------------------|
| $\frac{1}{16}$ | $\frac{1}{16}$ | .1418239113 | .7771176184 |
| $\frac{1}{8}$ | $\frac{1}{8}$ | .2051068932 | .8582854546 |
| $\frac{1}{4}$ | $\frac{1}{4}$ | .308086098 | 1.108064395 |
| $\frac{1}{2}$ | $\frac{1}{2}$ | .4686005983 | 1.546763926 |
| 1 | 1 | .6558798475 | 1.751829109 |
| 2 | 2 | .7724015781 | 1.326421433 |
| 4 | 4 | .882560748 | 1.045650432 |
| 8 | 8 | .7883670985 | 1.037622919 |
| 16 | 16 | .7883671029 | 1.037621633 |
| $\frac{1}{16}$ | $\frac{1}{16}$ | .2932579533 | .861858018 |
| $\frac{1}{8}$ | $\frac{1}{8}$ | .4052684179 | 1.194463434 |
| $\frac{1}{4}$ | $\frac{1}{4}$ | .5808413289 | 1.731487056 |
| $\frac{1}{2}$ | $\frac{1}{2}$ | .7855971775 | 1.997314795 |
| 1 | 1 | .932011965 | 1.632312177 |
| 2 | 2 | .971670407 | 1.172806220 |
| 4 | 4 | .9723443373 | 1.136392329 |
| 8 | 8 | .9723445189 | 1.136352737 |
| 16 | 16 | .9723445189 | 1.136352737 |
| $\frac{1}{16}$ | $\frac{1}{16}$ | .5447699531 | 1.448073250 |
| $\frac{1}{8}$ | $\frac{1}{8}$ | .7347791228 | 1.804948783 |
| $\frac{1}{4}$ | $\frac{1}{4}$ | .9621164771 | 2.068392041 |
| $\frac{1}{2}$ | $\frac{1}{2}$ | 1.161682658 | 1.897602675 |
| 1 | 1 | 1.235234439 | 1.302590480 |
| 2 | 2 | 1.239071584 | 1.170089563 |
| 4 | 4 | 1.239077752 | 1.169219758 |
| 8 | 8 | 1.239077752 | 1.169219758 |
| $\frac{1}{16}$ | $\frac{1}{16}$ | .9422627892 | 1.822399379 |
| $\frac{1}{8}$ | $\frac{1}{8}$ | 1.202728348 | 1.924993360 |
| $\frac{1}{4}$ | $\frac{1}{4}$ | 1.469558246 | 1.996128066 |
| $\frac{1}{2}$ | $\frac{1}{2}$ | 1.621073672 | 1.510617828 |
| 1 | 1 | 1.640855146 | 1.158833094 |
| 2 | 2 | 1.641017920 | 1.146471498 |
| 4 | 4 | 1.641017920 | 1.146471498 |
| 8 | 8 | 1.641017920 | 1.146471498 |
| 16 | 16 | 1.641017920 | 1.146471498 |
| 16 | 16 | 1.641017920 | 1.146471498 |

\[ a_c(r_{12},r_{32}) = J_1/J_2 \]
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