LONG-TIME ASYMPTOTICS OF PERTURBED FINITE-GAP KORTEWEG–DE VRIES SOLUTIONS

By

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Abstract. We apply the method of nonlinear steepest descent to compute the long-time asymptotics of solutions of the Korteweg–de Vries equation which are decaying perturbations of a quasi-periodic finite-gap background solution. We compute a nonlinear dispersion relation and show that the $x/t$ plane splits into $g+1$ soliton regions which are interlaced by $g+1$ oscillatory regions, where $g+1$ is the number of spectral gaps.

In the soliton regions, the solution is asymptotically given by a number of solitons travelling on top of finite-gap solutions which are in the same isospectral class as the background solution. In the oscillatory region, the solution can be described by a modulated finite-gap solution plus a decaying dispersive tail. The modulation is given by a phase transition on the isospectral torus and is, together with the dispersive tail, explicitly characterized in terms of Abelian integrals on the underlying hyperelliptic curve.

1 Introduction

Consider the Korteweg–de Vries (KdV) equation

\begin{equation}
V_t(x, t) = 6V(x, t)V_x(x, t) - V_{xxx}(x, t), \quad (x, t) \in \mathbb{R} \times \mathbb{R},
\end{equation}

where the subscripts denote differentiation with respect to the corresponding variables.

Following the seminal work of Gardner, Green, Kruskal, and Miura [17], one can use the inverse scattering transform to establish existence and uniqueness of (real-valued) classical solutions for the corresponding initial value problem with rapidly decaying initial conditions. We refer to, for instance, the monograph by Marchenko [28]. Our concern here is the long-time asymptotic behavior of such solutions. A classical result is that an arbitrary short-range solution of the above type eventually splits into a number of solitons travelling to the right plus a decaying radiation part travelling to the left. The first numerical evidence of such

*Research supported by the Austrian Science Fund (FWF) under Grant No. Y330.
a behavior was found by Zabusky and Kruskal [36]. The first mathematical re-
sults were given by Ablowitz and Newell [1], Manakov [27], and Šabat [31]. The
first rigorous results for the KdV equation were proved by Šabat [31] and Tanaka
[33]. Precise asymptotic behavior for the radiation part were first formally derived
by Zakharov and Manakov [37], Ablowitz and Segur [2], [32], Buslaev [6] (see
also [5]), and later rigorously justified and extended to all orders by Buslaev and
Sukhanov [7]. A detailed rigorous proof (not requiring any a priori information
on the asymptotic form of the solution) was given by Deift and Zhou [11] based
on earlier work of Manakov [27] and Its [18], and is now known as the nonlinear
steepest descent method for oscillatory Riemann–Hilbert problems. For an expos-
itory introduction to this method applied to the KdV equation we refer to [20]. For
further information on the history of this problem we refer to the survey by Deift,
Its, and Zhou [8].

In this paper, we look at solutions that are asymptotically close to a quasi-
periodic algebro-geometric finite-gap solution of the KdV equation. The under-
lying inverse scattering transform used to solve the initial value problem for this
class of solutions was developed only recently by Grunert, Egorova, and Teschl
[13], [14], [12]. However, nothing is known about their long-time asymptotic be-
havior even though attempts to describe it date back to over 35 years ago; see
Kuznetsov and Mikhaîlov [23]. The aim of this paper is to fill this gap. In case
of the discrete analog, the Toda lattice (see e.g. [35]), Kamvissis and Teschl [21],
[22] (with further extensions by Krüger and Teschl [26]) have recently extended
the nonlinear steepest descent method for Riemann–Hilbert problem deformations
to Riemann surfaces and used this extension to prove the following result for the
Toda lattice.

Let $g$ be the genus of the hyperelliptic curve associated with the unperturbed
solution. Then, apart from the phenomenon of the solitons travelling on the quasi-
periodic background, the $(n, t)$-plane contains $g + 2$ regions where the perturbed
solution is close to a finite-gap solution from the same isospectral torus. In be-
tween exist $g + 1$ regions where the perturbed solution is asymptotically close to a
modulated lattice which undergoes a continuous phase transition (in the Jacobian
variety) and which interpolates between these isospectral solutions. In the special
case of the free lattice ($g = 0$), the isospectral torus consists of a single point, and
known results are recovered. Solutions in the isospectral torus and phase transition
have been explicitly characterized in terms of Abelian integrals on the underlying
hyperelliptic curve.

Here, we use this extension for Riemann–Hilbert problems on Riemann sur-
faces to prove an analogous result, to be formulated in the next section, for the
KdV equation.

2 Main results

To set the stage, we choose a quasi-periodic algebro-geometric finite-gap background solution $V_q(x, t)$ of the KdV equation (cf. Section 3) plus another solution $V(x, t)$ of the KdV equation such that

$$\int_{-\infty}^{+\infty} (1 + |x|)^7(|V(x, t) - V_q(x, t)|)dx < \infty$$

for all $t \in \mathbb{R}$. We remark that such solutions exist. This can be shown by solving the associated Cauchy problem using the inverse scattering transform [13].

To fix our background solution $V_q$, let us consider a hyperelliptic Riemann surface $K_g$ of genus $g \in \mathbb{N}_0$ with real moduli $E_0, E_1, \ldots, E_{2g}$. We then choose a Dirichlet divisor $D_{\hat{\mu}(x, t)}$ and introduce

$$z(p, x, t) = \hat{\omega}_{E_0}(p) - \tilde{\omega}_{E_0}(D_{\hat{\mu}(x, t)}) \in \mathbb{C}^g,$$

$$\hat{\omega}_{E_0}(D_{\hat{\mu}(x, t)}) = \hat{\omega}_{E_0}(D_{\hat{\mu}}) + \frac{x}{2\pi} U_0 + 12 \frac{t}{2\pi} U_2,$$

where $\hat{\omega}_{E_0}(\tilde{\omega}_{E_0})$ is Abel’s map (for divisors), and $\hat{\omega}_{E_0}, U_0,$ and $U_2$ are some constants defined in detail in Section 3 below. Our background solution is then given in terms of Riemann theta functions (cf. (3.13)) by

$$V_q(x, t) = E_0 + \sum_{j=1}^{2g}(E_{2j-1} + E_{2j} - 2\mu_j(x, t))$$

$$= E_0 + \sum_{j=1}^{g}(E_{2j-1} + E_{2j} - 2\lambda_j) - 2\partial_x^2 \ln \theta(z(p, x, t)),$$

where $\lambda_j \in (E_{2j-1}, E_{2j}),$ $j = 1, \ldots, g.$

In order to state our main result, we begin by recalling that the perturbed KdV solution $V(x, t), x \in \mathbb{R},$ fixed $t \in \mathbb{R}$, is uniquely determined by its scattering data, that is, by the right reflection coefficient $R_+(\lambda, t), \lambda \in \sigma(H_q),$ and the eigenvalues $\rho_k \in \mathbb{R} \setminus \sigma(H_q), k = 1, \ldots, N,$ together with the corresponding right norming constants $\gamma_{+,k}(t) > 0, k = 1, \ldots, N$. Here

$$\sigma(H_q) = \bigcup_{j=0}^{g-1}[E_{2j}, E_{2j+1}] \cup [E_{2g}, \infty)$$

denotes the finite-band spectrum of the underlying background Lax operator

$$H_q(t) = -\partial_x^2 + V_q(x, t).$$
The relation between the energy $\lambda$ of the underlying Lax operator $H_q$ and the propagation speed at which the corresponding parts of the solutions of the KdV equation travel is given by

$$v(\lambda) = \frac{x}{t},$$

where

$$v(\lambda) = \lim_{\varepsilon \to 0} -\frac{12 \text{Re}(i \int_{E_0}^{(\lambda+\varepsilon,+)\omega_{p,0}})}{\text{Re}(i \int_{E_0}^{(\lambda+\varepsilon,+)\omega_{p,0}})},$$

and can be regarded as a nonlinear analog of the classical dispersion relation. Here $\omega_{p,0}$ and $\omega_{p,2}$ are Abelian differentials of the second kind on the underlying Riemann surface defined in (3.15) and (3.16). We show in Section 5 that $v$ is a decreasing homeomorphism of $\mathbb{R}$; we denote its inverse by $\zeta(v)$.

Furthermore, we define the limiting KdV solution $V_{l,v}(x,t)$ via the relation

$$\int_{x}^{\infty} (V_{l,v} - V_q)(y,t)dy = -\sum_{\rho_j < \zeta(v)} 4i \int_{E(\rho_j)}^{\rho_j} \omega_{p,0} + \frac{1}{\pi} \int_{C(v)} \log(1 - |R|^2)\omega_{p,0}$$

$$+ 2\delta(\varepsilon) \ln \left( \frac{\theta(z(p,\infty, x, t) + \delta(v))}{\theta(z(p,\infty, x, t))} \right),$$

with

$$\delta(\varepsilon) = -2 \sum_{\rho_j < \zeta(v)} A_{E(\rho_j),\varepsilon}(\rho_j) + \frac{1}{2\pi i} \int_{C(v)} \log(1 - |R|^2)\zeta, \varepsilon,$$

where $R = R_+(\lambda, t)$ is the associated reflection coefficient and $\zeta$ is a canonical basis of holomorphic differentials. Here, $C(v)$ is a contour on the Riemann surface obtained by taking the part of the spectrum $\sigma(H_q)$ which is to the left of $\zeta(v)$ and lifting it to the Riemann surface (oriented so that the upper sheet lies to its left). Here we have also identified $\rho_j$ with its lift to the upper sheet, and $E(\rho_j)$ denotes the branch point closest to $\rho_j$. If $v = x/t$, we set $V_l(x,t) = V_{l,x/t}(x,t)$.

Our main result concerning the long-time asymptotics in the soliton region is given by the following.

**Theorem 2.1.** Assume $V(x,t)$ is a classical solution of the KdV equation (1.1) and satisfies

$$\int_{-\infty}^{+\infty} (1 + |x|^{1+\eta})(|V(x,t) - V_q(x,t)|)dx < \infty,$$

for some integer $n \geq 1$. Denote by $c_k = v(\rho_k)$ the velocity of the $k$-th soliton. Let $\varepsilon > 0$ be so small that the intervals $[c_k - \varepsilon, c_k + \varepsilon], 1 \leq k \leq N$, are disjoint and lie
inside \( \nu(\mathbb{R} \setminus \sigma(H_q)) \). Then the asymptotic behavior of the solution \( \{ (x, t) | \zeta(x/t) \in \mathbb{R} \setminus \sigma(H_q) \} \), in the soliton region, can be described as follows.

If \( |x/t - c_k| < \varepsilon \) for some \( k \), then the solution is asymptotically given by a one-soliton solution on top of the limiting solution

\[
\int_{x}^{\infty} (V - V_{l,c_k})(y, t) dy = -2 \frac{\partial}{\partial x} \log (c_{l,k}(x, t)) + O(t^{-n})
\]

as well as

\[
(V - V_{l,c_k})(x, t) = 2 \frac{\partial^2}{\partial x^2} \log (c_{l,k}(x, t)) + O(t^{-n}),
\]

where

\[
c_{l,k}(x, t) = 1 + \tilde{\gamma}_k \int_{x}^{\infty} \psi_{l,c_k}(\rho_k, y, t)^2 dy
\]

and

\[
\tilde{\gamma}_k = \gamma_k \left( \frac{\theta(z(\rho_k, 0, 0) + \delta(c_k))}{\theta(z(\rho_k, 0, 0))} \right)^2 \left( \prod_{\rho_j < \zeta(c_k)} \exp \left( 2 \int_{E_0}^{\rho_k} \omega_{\rho_j, \rho_j} \right) \right)
\]

\[
\cdot \exp \left( \frac{-1}{\pi i} \int_{C(c_k)} \log(1 - |R|^2) \omega_{p_k, p_q} \right).
\]

Here \( \psi_{l,o}(p, x, t) \) denotes the Baker–Akhiezer function corresponding to the limiting KdV solution \( V_{l,o}(x, t) \), and \( \omega_{p,q} \) denotes the Abelian differential of the third kind with poles at \( p \) and \( q \).

If \( |x/t - c_k| \geq \varepsilon \) for all \( k \), then the solution is asymptotically close to the limiting solution

\[
\int_{x}^{\infty} (V - V_l)(y, t) dy = O(t^{-n})
\]

as well as

\[
V(x, t) = V_l(x, t) + O(t^{-n}).
\]

In particular, the solution splits into a sum of independent solitons in which the presence of the other solitons and the radiation part corresponding to the continuous spectrum manifest themselves in phase shifts given by (2.13). Moreover, in the periodic case considered here, there may be a stationary soliton (see the discussion in Section 5).

The proof of Theorem 2.1 is given at the end of Section 5.
Theorem 2.2. Assume \( V(x, t) \) is a classical solution of (1.1) and satisfies (2.1). Let \( D_j \) be the sector \( D_j = \{ (x, t) : \zeta(x/t) \in [E_{2j} + \varepsilon, E_{2j+1} - \varepsilon] \} \) for some \( \varepsilon > 0 \). Then

\[
\int_{x}^{+\infty} (V - V_t)(y, t)dy = 4\sqrt{\frac{i}{\phi''(z_j)t}} \text{Re}(\beta(x, t))\Lambda^\nu_1(z_j) + O(t^{-\alpha})
\]

as well as

\[
(V - V_t)(x, t) = 4\sqrt{\frac{i}{\phi''(z_j)t}} \left[ \text{Im}(\beta(x, t)) - i\text{Re}(\beta(x, t)) \sum_{k=1}^{g} \sum_{\ell=1}^{g} c_{k\ell}(\tilde{\nu})\zeta_k(z_j) \right] + O(t^{-\alpha})
\]

for any \( 1/2 < \alpha < 1 \) uniformly in \( D_j \) as \( t \to \infty \). Here

\[
\phi(p) = -24i \int_{p_0}^{p} \omega_{p_{\infty}, 2} - 2i \frac{x}{t} \int_{p_0}^{p} \omega_{p_{\infty}, 0}
\]

is the phase function,

\[
\frac{\phi''(z_j)}{i} = -\frac{12 \prod_{k=0, k \neq j}^{g} (z_j - z_k)}{iR_{2g+1}^{1/2}(z_j)} > 0,
\]

where \( R_{2g+1}^{1/2}(z) \) is the square root of the underlying Riemann surface \( \mathcal{K}_g \) and we identify \( z_j \) with its lift to the upper sheet,

\[
\Lambda^\nu_1(z_j) = \omega_{p_{\infty}, 0}(z_j) - \sum_{k=1}^{g} \sum_{\ell=1}^{g} c_{k\ell}(\tilde{\nu})\alpha_{g-1}(\tilde{\nu}_{\ell})\zeta_k(z_j),
\]

\( \omega_{p_{\infty}, 0} \) is an Abelian differential of the second kind with a second order pole at \( p_{\infty} \) (cf. (3.15)), \( \omega(p) \) denotes the value of a differential evaluated at \( p \) in the chart given by the canonical projection, and \( c_{k\ell}(\tilde{\nu}), \alpha_{g-1}(\tilde{\nu}_{\ell}) \) are constants defined in (6.24), (6.34), respectively. Moreover,

\[
\beta(x, t) = \sqrt{\nu} \exp \left( i\pi/4 - \arg(R(z_j)) + \arg(\Gamma(i\nu)) + 2\alpha(z_j) \right) \left( \frac{\phi''(z_j)}{i} \right)^{-iv} e^{-\phi(z_j)}t^{-iv} \cdot \frac{\theta(z, z_j, 0, 0)}{\theta(z, \zeta(x/t) + \delta(x/t), 0, 0)} \frac{\theta(z^*, x, t + \delta(x/t))}{\theta(z^*, z_j^*, 0, 0)} \exp \left( - \sum_{p \leq \zeta(x/t)} \int_{E(p_k)} \omega_{\zeta z_j} + \frac{1}{2\pi i} \int_{C(x/t)} \log \left( \frac{1 - |R|^2}{1 - |z_j|^2} \right) \omega_{z^* z_j^*} \right),
\]
where $\Gamma(z)$ is the gamma function, $\omega_{z_jz_j}$ an Abelian differential of the third kind defined in (3.21).

(2.22) \quad \nu = -\frac{1}{2\pi} \log (1 - |R(z_j)|^2) > 0,

and $a(z_j)$ is a constant defined in (6.9).

The proof of Theorem 2.2 is given in Section 6.

Finally, note that if $q(x, t)$ solves the KdV equation, then so does $q(-x, -t)$. Therefore, it suffices to investigate the case $t \to \infty$.

3 Algebro-geometric quasi-periodic finite-gap solutions

Since we want to choose our background solution $V_q$ from the class of algebro-geometric quasi-periodic finite-gap solutions, i.e., the class of stationary solutions of the KdV hierarchy, we present some well-known facts on this class. We use the same notation as in [16]; we also refer to this reference for proofs. As a reference for Riemann surfaces in this context, we recommend [15].

To set the stage, let $K_g$ be the Riemann surface associated with the function

$$R_{2g+1}^{1/2}(z) = \ii \prod_{j=0}^{2g} \sqrt{z - E_j}, \quad E_0 < E_1 < \cdots < E_{2g},$$

where $g \in \mathbb{N}_0$ and $(E_j)_{j=0}^{2g} \subset \mathbb{R}$. Here $\sqrt{.}$ denotes the standard root with branch cut along $(0, \infty)$. We extend $R_{2g+1}^{1/2}(z)$ to the branch cuts by setting $R_{2g+1}^{1/2}(z + i\varepsilon) = \lim_{\varepsilon \downarrow 0} R_{2g+1}^{1/2}(z + i\varepsilon)$ for $z \in \mathbb{C} \setminus \Pi$. Hence,

(3.2) \quad R_{2g+1}^{1/2}(z) = |R_{2g+1}^{1/2}(z)| \begin{cases} (-1)^{g+1} & \text{for } z \in (-\infty, E_0), \\ (-1)^{g+j}i & \text{for } z \in (E_j, E_{j+1}), \quad j = 0, \ldots, g - 1, \\ (-1)^{g+j} & \text{for } z \in (E_{2j+1}, E_{2j+2}), \quad j = 0, \ldots, g - 1, \\ i & \text{for } z \in (E_{2g}, \infty). \end{cases}

$K_g$ is a compact, hyperelliptic Riemann surface of genus $g$.

A point on $K_g$ is denoted by $p = (z, \pm R_{2g+1}^{1/2}(z)) = (z, \pm)$, $z \in \mathbb{C}$, or $p_{\infty} = (\infty, \infty)$, and the projection onto $\mathbb{C} \cup \{ \infty \}$ by $\pi(p) = z$. Elements of the set $\{(E_j, 0), 0 \leq j \leq 2g\} \cup \{(\infty, \infty)\} \subseteq K_g$ are called branch points, and the sets

(3.3) \quad \Pi_{\pm} = \{(z, \pm R_{2g+1}^{1/2}(z)) \mid z \in \mathbb{C} \setminus \bigcup_{j=0}^{g-1} [E_{2j}, E_{2j+1}] \cup [E_{2g}, \infty)\} \subseteq K_g

are called the upper and lower sheet, respectively.
Next we introduce representatives \{a_j, b_j\}_{j=1}^g of a canonical homology basis for \(K_g\). For \(a_j\), we start near \(E_{2j-1}\) on \(\Pi_+\), surround \(E_{2j}\) thereby changing to \(\Pi_-\), and return to our starting point encircling \(E_{2j-1}\) again changing sheets. For \(b_j\), we choose a cycle surrounding \(E_0, E_{2j-1}\) counterclockwise (once) on \(\Pi_+\). The cycles are chosen so that their intersection matrix is

\[
a_i \circ a_j = b_i \circ b_j = 0, \quad a_i \circ b_j = \delta_{i,j}, \quad 1 \leq i, j \leq g.
\]

The corresponding canonical basis \{\zeta_j\}_{j=1}^g for the space of holomorphic differentials can be constructed by

\[
\zeta_j = \sum_{k=1}^g c_j(k) \frac{\pi^{k-1} d\pi}{R_{2g+1}^{1/2}},
\]

where the constants \(c_j(k), j, k = 1, \ldots, g\) are given by

\[
c_j(k) = C^{-1}_{jk}, \quad C_{jk} = \int_{a_k} \frac{\pi^{j-1} d\pi}{R_{2g+1}^{1/2}} = 2 \int_{E_{2k}} \frac{z^{j-1} dz}{R_{2g+1}^{1/2}} \in \mathbb{R}.
\]

The differentials satisfy

\[
\int_{a_k} \zeta_j = \delta_{j,k}, \quad \int_{b_k} \zeta_j = \tau_{k,j}, \quad \tau_{k,j} = \tau_{j,k}, \quad j, k = 1, \ldots, g.
\]

We now pick \(g\) numbers (the Dirichlet eigenvalues)

\[
(\hat{\mu}_j)_{j=1}^g = (\mu_j, \sigma_j)_{j=1}^g
\]

whose projections lie in the spectral gaps, that is, \(\mu_j \in [E_{2j-1}, E_{2j}], j = 1, \ldots, g\). Associated with these numbers is the divisor

\[
D_{\hat{\mu}}(p) = \begin{cases} 1 & p = \hat{\mu}_j, \quad j = 1, \ldots, g, \\ 0 & \text{otherwise}, \end{cases}
\]

and we can define \(g\) numbers \((\hat{\mu}_j(x, t))_{j=1}^g = (\mu_j(x, t), \sigma_j(x, t))_{j=1}^g\) via Jacobi’s Inversion Theorem by setting

\[
\hat{\alpha}_{E_0}(D_{\hat{\mu}(x, t)}) = \hat{\alpha}_{E_0}(D_{\hat{\mu}}) + \frac{x}{2\pi} U_0 + 12 \frac{t}{2\pi} U_2
\]

so that \(\hat{\mu}_j(0, 0) = \hat{\mu}_j\). Here \(U_0\) and \(U_2\) denote the \(b\)-periods of the Abelian differentials \(\omega_{p.c.0}\) and \(\omega_{p.c.2}\), respectively, defined below, and \(\hat{\alpha}_{E_0}(a_{E_0})\) is Abel’s map (for divisors). The hat (\(\hat{\cdot}\)) indicates that we regard a map as a (single-valued) map from \(\hat{K}_g\) (the fundamental polygon associated with \(K_g\) by cutting along the \(a\) and \(b\) cycles) to \(\mathbb{C}^g\).
Next we introduce
\begin{equation}
\hat{z}(p, x, t) = \hat{\xi}_{E_0} - \hat{\phi}_{E_0}(p) + \hat{a}_{E_0}(D_{\hat{\phi}(x, t)}) \in \mathbb{C}^g,
\end{equation}
where \(\hat{\xi}_{E_0}\) is the vector of Riemann constants
\begin{equation}
\hat{\xi}_{E_0,j} = \frac{j + \sum_{k=1}^{g} \tau_{j,k}}{2}, \quad j = 1, \ldots, g.
\end{equation}

The function \(\theta\) denotes the Riemann theta function associated with \(\mathcal{K}_g\) defined by
\begin{equation}
\theta(z) = \sum_{m \in \mathbb{Z}^g} \exp 2\pi i \left( \langle m, z \rangle + \frac{\langle m, \tau m \rangle}{2} \right), \quad z \in \mathbb{C}^g.
\end{equation}
Note that \(\theta(z(p, x, t))\) has precisely \(g\) zeros \(\hat{\mu}_j(x, t)\). This follows from Riemann’s Vanishing Theorem (cf. [35, Theorem A.13]).

Introduce the time-dependent Baker–Akhiezer function
\begin{equation}
\psi_q(p, x, t) = \frac{\theta(z(p, x, t)) \theta(z(p_\infty, 0, 0))}{\theta(z(p_\infty, x, t)) \theta(z(p, 0, 0))} \exp \left( -ix \int_{E_0}^{p} \omega_{p_\infty,0} - 12it \int_{E_0}^{p} \omega_{p_\infty,2} \right).
\end{equation}
Here \(\omega_{p_\infty,0}\) and \(\omega_{p_\infty,2}\) are normalized Abelian differentials of the second kind with a single pole at \(p_\infty\) and principal part \(w^{-2}dw\) and \(w^{-4}dw\) in the chart \(w(p) = \pm i z^{-1/2}\) for \(p = (z, \pm)\), respectively. The Abelian differentials are normalized so as to have vanishing \(a_j\) periods and the following expressions:
\begin{equation}
\omega_{p_\infty,0} = \frac{1}{2i} \frac{\prod_{j=1}^{g} (\pi - \lambda_j)}{R_{2g+1}^{1/2}} d\pi,
\end{equation}
with \(\lambda_j \in (E_{2j-1}, E_{2j})\), \(j = 1, \ldots, g\), and
\begin{equation}
\omega_{p_\infty,2} = \frac{1}{2i} \frac{\prod_{j=0}^{g} (\pi - \tilde{\lambda}_j)}{R_{2g+1}^{1/2}} d\pi,
\end{equation}
where \(\tilde{\lambda}_j\), \(j = 0, \ldots, g\), must be chosen to satisfy \(\sum_{j=0}^{g} \tilde{\lambda}_j = (\sum_{j=0}^{2g} E_j)/2\). We also remark that
\begin{equation}
\psi_q(p, x, t)\psi_q(p^*, x, t) = \prod_{j=1}^{g} \frac{z - \mu_j(x, t)}{z - \mu_j}, \quad p = (z, \pm).
\end{equation}
Our background KdV solution is then given by
\begin{equation}
V_q(x, t) = E_0 + \sum_{j=1}^{g}(E_{2j-1} + E_{2j} - 2\lambda_j) - 2\epsilon_x^2 \ln \theta(z(p_\infty, x, t)).
\end{equation}

The Abelian differentials of the third kind \(\omega_{q_1, q_2}\) with simple poles at \(q_1\) and \(q_2\), corresponding residues \(+1\) and \(-1\), vanishing \(a\)-periods, and holomorphic on \(\mathcal{K}_g \setminus \{q_1, q_2\}\), are explicitly given by ([16, Appendix B])
\begin{align}
\omega_{p_1, p_2} &= \left(\frac{R_{2g+1}^{1/2} + R_{2g+1}^{1/2}(p_1)}{2(\pi - \pi(p_1))} - \frac{R_{2g+1}^{1/2} + R_{2g+1}^{1/2}(p_2)}{2(\pi - \pi(p_2))} + P_{p_1, p_2}(z)\right) \frac{d\pi}{R_{2g+1}^{1/2}}, \\
\omega_{p_1, p_\infty} &= \left(\frac{R_{2g+1}^{1/2} + R_{2g+1}^{1/2}(p_1)}{2(\pi - \pi(p_1))} + P_{p_1, p_\infty}(z)\right) \frac{d\pi}{R_{2g+1}^{1/2}},
\end{align}
where \(p_1, p_2 \in \mathcal{K}_g \setminus \{p_\infty\}\) and \(P_{p_1, p}(z)\) and \(P_{p_1, p_\infty}(z)\) are polynomials of degree \(g - 1\) which are determined by the normalization \(\int_{\alpha_+} \omega_{p_1, p_2} = 0\) and \(\int_{\alpha_+} \omega_{p_1, p_\infty} = 0\), respectively. In particular,
\begin{equation}
\omega_{pp^*} = \left(\frac{R_{2g+1}^{1/2}(p)}{\pi - \pi(p)} + P_{pp^*}(\pi)\right) \frac{d\pi}{R_{2g+1}^{1/2}}.
\end{equation}

We also need the Blaschke factor
\begin{equation}
B(p, \rho) = \exp \left(\int_{E_0}^{p} \omega_{p, \rho^*} \right) = \exp \left(\int_{E_0}^{p} \omega_{p, \rho^*} \right), \quad \pi(\rho) \in \mathbb{R},
\end{equation}
where \(E(\rho) = E_0\) if \(\rho < E_0\), either \(E_{2j-1}\) or \(E_{2j}\) if \(\rho \in (E_{2j-1}, E_{2j})\), \(1 \leq j \leq g\). The Blaschke factor is a multivalued function with a simple zero at \(\rho\) and a simple pole at \(\rho^*\) and satisfies \(|B(p, \rho)| = 1, p \in \partial \Pi_+\). It is real-valued for \(\pi(p) \in (-\infty, E_0)\) and satisfies
\begin{equation}
B(E_0, \rho) = 1 \quad \text{and} \quad B(p^*, \rho) = B(p, \rho^*) = B(p, \rho)^{-1}
\end{equation}
(see, e.g., [34]).

The Baker–Akhiezer function \(\psi_q\) is a meromorphic function on \(\mathcal{K}_g \setminus \{p_\infty\}\) with an essential singularity at \(p_\infty\). The two branches are denoted by
\begin{equation}
\psi_{q, \pm}(z, x, t) = \psi_q(p, x, t), \quad p = (z, \pm).
\end{equation}
It satisfies
\begin{equation}
H_q(t)\psi_q(p, x, t) = \pi(p)\psi_q(p, x, t),
\begin{equation}
\frac{d}{dt}\psi_q(p, x, t) = P_{q, 2}(t)\psi_q(p, x, t).
\end{equation}
Here

\[ H_q(t) = \partial_x^2 + V_q(., t), \]
\[ P_{q,2}(t) = -4\partial_x^3 + 6V_q(., t)\partial_x + 3V_{q,x}(., t), \]

are the operators from the Lax pair for the KdV equation, that is,

\[ \frac{d}{dt}H_q(t) = H_q(t)P_{q,2}(t) - P_{q,2}(t)H_q(t). \]

It is well known that the spectrum of \( H_q(t) \) is time independent and consists of \( g + 1 \) bands

\[ \sigma(H_q(t)) = \bigcup_{j=0}^{g-1} [E_{2j}, E_{2j+1}] \cup [E_{2g}, \infty). \]

For further information and proofs, we refer to [16].

### 4 The inverse scattering transform and the Riemann–Hilbert problem

In this section, we recall some basic facts from the inverse scattering transform for our setting. For further background and proofs, we refer to [4], [12], and [13] (see also [29]).

Let \( \psi_{q,\pm}(z, x, t) \) be the branches of the Baker–Akhiezer function defined in the previous section. Let \( \psi_{\pm}(z, x, t) \) be the Jost functions for the perturbed problem

\[ (-\partial_x^2 + V(x, t)) \psi_{\pm}(z, x, t) = z\psi_{\pm}(z, x, t), \]

defined by the asymptotic normalization

\[ \lim_{x \to \pm\infty} e^{\mp ik(z)}(\psi_{\pm}(z, x, t) - \psi_{q,\pm}(z, x, t)) = 0, \]

where \( k(z) \) denotes the quasimomentum map

\[ k(z) = -\int_{E_0}^p \omega_{p,z}, \quad p = (z, +). \]

The asymptotics of the two projections of the Jost function are (cf. [29, Theorem 2.3])

\[ \psi_{\pm}(z, x, t) = \psi_{q,\pm}(z, x, t) \left( 1 \mp \int_{x}^{\pm\infty} (V - V_q)(y, t)dy \frac{1}{2i\sqrt{z}} + o(1/\sqrt{z}) \right), \]
as $z \to \infty$. Without loss of generality (otherwise just shift the base point $(x_0, t_0) = (0, 0)$, we assume that the poles $\mu_k$ of the Baker–Akhiezer function are all different from the eigenvalues $\rho_j$.

One has the scattering relations

$$T(z)\psi_\pm(z, x, t) = \overline{\psi_\pm(z, x, t)} + R_\pm(z)\psi_\pm(z, x, t), \quad z \in \sigma(H_q),$$

where $T(z), R_\pm(z)$ are the transmission, respectively, reflection coefficients. Here $\psi_\pm(z, x, t)$ is defined so that $\psi_\pm(z, x, t) = \lim_{\epsilon \downarrow 0} \psi_\pm(z + i\epsilon, x, t), \quad z \in \sigma(H_q)$. If we take the limit from the other side, we have $\psi_\pm(z, x, t) = \lim_{\epsilon \downarrow 0} \psi_\pm(z - i\epsilon, x, t)$.

The transmission and reflection coefficients have the well-known properties stated in the following.

**Lemma 4.1.** The transmission coefficient $T(z)$ has a meromorphic extension to $\mathbb{C}\setminus \sigma(H_q)$ with simple poles at the eigenvalues $\rho_j$. The residues of $T(z)$ are given by

$$\text{Res}_{\rho_j} T(z) = \frac{2R^{1/2}_{2g+1}(\rho_j)}{\prod_{k=1}^g (\rho_j - \mu_k)} \gamma_{\pm, j} c_j, \quad (4.6)$$

where

$$\gamma_{\pm, j}^{-1} = \int_{-\infty}^{\infty} |\psi_\pm(\rho_j, y, t)|^2 dy, \quad (4.7)$$

and $\psi_-(\rho_j, x, t) = c_j \psi_+(\rho_j, x, t)$. The numbers $\gamma_{\pm, j}^{-1}$ are referred to as norming constants.

Moreover,

$$T(z)\overline{R_+(z)} + \overline{T(z)}R_-(z) = 0, \quad |T(z)|^2 + |R_\pm(z)|^2 = 1. \quad (4.8)$$

In particular, one of the reflection coefficients, say $R(z) = R_+(z)$, and one set of norming constants, say $\gamma_j = \gamma_{+, j}$, are sufficient for us.

We define a sectionally meromorphic vector on the Riemann surface $\mathcal{K}_g$ as follows:

$$m(p, x, t) = \left\{ \begin{array}{ll}
T(z) \frac{\psi_-(z, x, t)}{\psi_{q_+}(z, x, t)} & , \quad p = (z, +) \\
\frac{1}{\psi_{q_+}(z, x, t)} T(z) \frac{\psi_+(z, x, t)}{\psi_{q_-}(z, x, t)} & , \quad p = (z, -)
\end{array} \right. \quad (4.9)$$

We are interested in the jump condition of $m(p, x, t)$ on $\Sigma$, the boundary of $\Pi_\pm$ (oriented counterclockwise when viewed from top sheet $\Pi_+$). It consists of two copies $\Sigma_\pm$ of $\sigma(H_q)$, which correspond to nontangential limits from $p = (z, +)$ with
\( \pm \text{Im}(z) > 0 \), respectively, to nontangential limits from \( p = (z, -) \) with \( \mp \text{Im}(z) > 0 \).

To formulate our jump condition, we use the following convention. When representing functions on \( \Sigma \), the lower subscript denotes the nontangential limit from \( \Pi_+ \) or \( \Pi_- \), respectively,

\[
(4.10) \quad m_\pm(p_0) = \lim_{\Pi_\mp \ni p \to p_0} m(p), \quad p_0 \in \Sigma.
\]

Use of this notation implicitly assumes that these limits exist in the sense that \( m(p) \) extends to a continuous function on the boundary away from the band edges.

Moreover, we also use symmetries with respect to the sheet exchange map

\[
(4.11) \quad p^* = \begin{cases} 
(z, \mp) & \text{for } p = (z, \pm), \\
p_\infty & \text{for } p = p_\infty,
\end{cases}
\]

and complex conjugation

\[
(4.12) \quad \overline{p} = \begin{cases} 
(\overline{z}, \pm) & \text{for } p = (z, \pm) \notin \Sigma, \\
(z, \mp) & \text{for } p = (z, \pm) \in \Sigma, \\
p_\infty & \text{for } p = p_\infty.
\end{cases}
\]

In particular, we have \( \overline{p} = p^* \) for \( p \in \Sigma \).

Note that \( \overline{\hat{m}}_\pm(p) = \hat{m}_\pm(p^*) \) for \( \hat{m}(p) = m(p^*) \) (since \( * \) reverses the orientation of \( \Sigma \)) and \( \overline{\hat{m}}_\pm(p) = \overline{\hat{m}_\pm(p^*)} \) for \( \hat{m}(p) = \overline{m(p^*)} \). Note also the following asymptotic behavior for \( m(p, x, t) \) near \( p_\infty \):

\[
(4.13) \quad m(p) = \begin{pmatrix} 1 & 1 \\ \frac{1}{2i\sqrt{2}} \int_x^\infty (V - V_q)(y, t)dy \end{pmatrix} \begin{pmatrix} -1 & 1 \\ \frac{1}{\sqrt{2}} \end{pmatrix},
\]

for \( p \) near \( p_\infty \). Here we have made use of (4.4) and

\[
(4.14) \quad T(z) = 1 + \frac{1}{2i\sqrt{2}} \int_{-\infty}^\infty (V - V_q)(y, t)dy + o\left(\frac{1}{\sqrt{2}}\right)
\]

(cf. [29, Corollary 3.7]).

We are now ready to derive the main vector Riemann–Hilbert problem.

**Theorem 4.2** (Vector Riemann–Hilbert problem). Let

\[ S_+(H(0)) = \{ R(\lambda), \lambda \in \sigma(H_q); (\rho_j, \gamma_j), 1 \leq j \leq N \} \]

be the right scattering data of the operator \( H(0) \). Then \( m(p) \) is meromorphic away from \( \Sigma \) and satisfies
(i) the jump condition

\[(4.15) \quad m_+(p) = m_-(p)J(p), \]

\[J(p) = \begin{pmatrix} 1 - |R(p)|^2 & -R(p)\Theta(p, x, t)e^{-i\phi(p)} \\ R(p)\Theta(p, x, t)e^{i\phi(p)} & 1 \end{pmatrix}, \]

for \( p \in \Sigma, \)

(ii) the divisor conditions

\[(4.16) \quad (m_1) \geq -\hat{D}(x, t) - \hat{D}_\rho, \quad (m_2) \geq -\hat{D}(x, t) - \hat{D}_\rho^*, \]

and pole conditions

\[\left( m_1(p) + \frac{-2R_{2g+1}(\rho_j)}{\prod_{k=1}^g(\rho_j - \mu_k) \pi(p) - \rho_j} \psi_q(p, x, t) \right) \geq -\hat{D}_\rho, \quad \text{near } \rho_j, \]

\[\left( \frac{-2R_{2g+1}(\rho_j)}{\prod_{k=1}^g(\rho_j - \mu_k) \pi(p) - \rho_j} \psi_q(p^*, x, t) \right) m_2(p) \geq -\hat{D}_\rho^*, \quad \text{near } \rho^*_j, \]

(iii) the symmetry condition

\[(4.18) \quad m(p^*) = m(p) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \]

(iv) the normalization

\[(4.19) \quad m(p_\infty) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}. \]

Here \((f)\) denotes the divisor of \(f\), and

\[(4.20) \quad \hat{D}_\rho = \sum_j \hat{D}_{\rho_j}, \quad \hat{D}_\rho^* = \sum_j \hat{D}_{\rho^*_j}. \]

denotes the divisor corresponding to the points \( \rho_j \equiv (\rho_j, +) \in \mathscr{K}_g \). The phase \( \phi \) is given by

\[(4.21) \quad \phi(p, \frac{x}{t}) = -24i \int_{p_0}^p \omega_{p_\infty, 2} - 2i\frac{x}{t} \int_{p_0}^p \omega_{p_\infty, 0} + i\mathbb{R} \quad \text{for } p \in \Sigma. \]

Moreover, we have set

\[(4.22) \quad \Theta(p, x, t) = \frac{\theta(z(p, x, t)) \theta(z(p^*, 0, 0))}{\theta(z(p, 0, 0)) \theta(z(p^*, x, t))} \]

so that

\[\frac{\psi_q(p, x, t)}{\psi_q(p^*, x, t)} = \Theta(p, x, t)e^{i\phi(p)}. \]
Here we have extended our definition of $R$ to $\Sigma$ so that it is equal to $R(z)$ on $\Sigma_+$ and equal to $\overline{R(z)}$ on $\Sigma_-$. In particular, the condition on $\Sigma_+$ is just the complex conjugate of the one on $\Sigma_-$ since we have $R(p^*) = \overline{R(p)}$ and $m_\pm(p^*, x, t) = \overline{m_\pm(p, x, t)}$ for $p \in \Sigma$.

**Proof.** The jump condition follows by using (4.5) and (4.8). By Riemann’s Vanishing Theorem (cf. [35, Theorem A.13]), the Baker–Akhiezer function $\psi_q$ has simple zeros at $\hat{\mu}_j(x, t)$ and simple poles at $\hat{\mu}_j, j = 1, \ldots, g$. Moreover, the transmission coefficient $T(z)$ has simple poles at the eigenvalues $\rho_j, j = 1, \ldots, N$. Thus the divisor conditions (4.16) are indeed satisfied. The pole conditions follow from the fact that the transmission coefficient $T(z)$ is meromorphic in $\mathbb{C} \setminus \sigma(H_q)$ with simple poles at $\rho_j$ and its residues are given by (4.6). The symmetry condition (4.18) obviously holds by the definition of the function $m(p)$. The normalization (4.19) is immediately clear from (4.13). $\square$

We note that the symmetry condition is in fact crucial for guaranteeing that the solution of this vector Riemann–Hilbert problem is unique.

**Theorem 4.3.** The vector $m(p)$ defined in (4.9) is the unique solution of the vector Riemann–Hilbert problem (4.15)–(4.19).

**Proof.** The argument is similar to that used in [22, Thm. B.1]. It suffices to show that the corresponding vanishing Riemann–Hilbert problem, where the normalization condition (4.19) is replaced with $m(p_\infty) = (0 \ 0)$, has only the trivial solution.

Let $\tilde{m}$ be some solution of the vanishing Riemann–Hilbert problem. We want to apply Cauchy’s Integral Theorem to $\tilde{m}(p)^*\tilde{m}(p)$. To handle the poles of $\tilde{m}$, we multiply $\tilde{m}(p)\tilde{m}(p)^*$ by a meromorphic differential $d\Omega$ which has zeros at $\mu$ and $\mu^*$ and a simple pole at $p_\infty$ such that $\tilde{m}(p)\tilde{m}(p)^*d\Omega(p)$ is holomorphic away from the contour. Here $\tilde{m}^*$ denotes the adjoint (transpose and complex conjugate) vector of $\tilde{m}$.

More precisely, let

$$
(4.23) \quad d\Omega = \frac{\prod_{j=1}^g (\pi - \mu_j)}{-R_{2g+1}^{1/2}} d\pi
$$

and note that $-(\prod_j (z - \mu_j))R_{2g+1}^{1/2}(z)$ is a Herglotz function; that is, it has positive imaginary part in the upper half-plane (and it is purely imaginary on $\sigma(H_q)$). Hence $\tilde{m}(p)\tilde{m}(p)^*d\Omega(p) > 0$ on $\Sigma$.

Next, consider the integral

$$
(4.24) \quad 0 = \int_D \tilde{m}(p)\tilde{m}(p)^*d\Omega(p),
$$
where $D$ is a $\varpi$-invariant contour consisting of two loops, on the upper and on the lower sheet, encircling none of the poles $\rho_j$, $\rho_j^*$. We first deform $D$ to a $\varpi$-invariant contour consisting of several parts: two pieces $D_{\pm}$ wrapping around the $\pm$ side of $\Sigma$ plus a number of small circles $D_{\pm,j}$ around the poles $\rho_j$, $\rho_j^*$, respectively. The contribution from $\Sigma$ is then given by

$$
\int_{\Sigma} \tilde{m}(p) \tilde{m}^\dagger(\varpi^*) d\Omega(p) = \int_{\Sigma} (\tilde{m}_+(p) \tilde{m}^\dagger_-(\varpi^*) + \tilde{m}_-(p) \tilde{m}^\dagger_+(\varpi^*)) d\Omega(p)
$$

(4.25)

$$
= \int_{\Sigma} \tilde{m}_-(p)(J(p) + J^\dagger(\varpi^*))\tilde{m}^\dagger_-(\varpi^*)d\Omega(p) \geq 0,
$$

and the contribution from the poles is given by

$$
\int_{D_{\pm}} \tilde{m}(p) \tilde{m}^\dagger(\varpi^*) d\Omega(p)
$$

$$
= \sum_{j=1}^N \left( \text{Res}_{\rho_j} \tilde{m}(p) \tilde{m}^\dagger(\varpi^*) d\Omega(p) + \text{Res}_{\rho_j^*} \tilde{m}(p) \tilde{m}^\dagger(\varpi^*) d\Omega(p) \right)
$$

(4.26)

$$
= 2 \sum_{j=1}^N \text{Res}_{\rho_j} \tilde{m}(p) \tilde{m}^\dagger(\varpi^*) d\Omega(p).
$$

To compute the residues, we use the pole conditions (4.17), which imply (using (3.17))

$$
\text{Res}_{\rho_j} \tilde{m}(p) \tilde{m}^\dagger(\varpi^*) d\Omega(p) = \frac{2\gamma_j}{\prod_{k=1}^x (\rho_j - \mu_k)} \psi_q(\rho_j)^2 m_2(\rho_j)^2 \geq 0.
$$

In particular, both contributions to the integral (4.24) are nonnegative and thus vanish. It follows that $\tilde{m} = 0$ vanishes along $\Sigma$, and consequently $\tilde{m}(p) = 0$ as desired.

We also need another asymptotic relation, namely

$$
m_1 \cdot m_2 = 1 + (V - V_q)(x, t) \frac{1}{2z} + o(z^{-1}),
$$

(4.27)

which is immediate from the following well-known result.

**Lemma 4.4.**

$$
T(z) \frac{\psi_-(z, x, t)}{\psi_{q,-}(z, x, t)} \frac{\psi_+(z, x, t)}{\psi_{q,+}(z, x, t)} = 1 + \frac{1}{2} (V - V_q)(x, t) \frac{1}{z} + o(z^{-1}).
$$

(4.28)

**Proof.** We use the following representation of the Jost solutions:

$$
\psi_{\pm}(z, x, t) = \psi_{q,\pm}(z, x, t) \exp \left( \mp \int_x^{\pm \infty} (m_{\pm}(z, y, t) - m_{q,\pm}(z, y, t)) dy \right),
$$

(4.29)
where
\[ m_\pm(z, x, t) = \frac{\psi'_\pm(z, x, t)}{\psi_{z, \pm}(z, x, t)}, \quad m_{q, \pm}(z, x, t) = \frac{\psi'_{q, \pm}(z, x, t)}{\psi_{q, \pm}(z, x, t)} \]
are the Weyl–Titchmarsh functions. Here the prime (’) denotes differentiation with respect to \( x \). Using the expansion of the Weyl \( m \)-functions (cf. [29, Lemma 6.1]) and the one for \( \log T(z) \) (cf. [29, Theorem 6.2]) for large \( z \) proves the claim. \( \square \)

For our further analysis, following the idea of Deift, Kamvissis, Kriecherbauer, and Zhou [9], it is convenient to rewrite the pole conditions as jump conditions. For that purpose, we choose \( \varepsilon \) so small that the discs \( |\pi(p) - \rho_j| < \varepsilon \) are inside the upper sheet \( \Pi_+ \) and do not intersect with the spectral bands. We then redefine \( m(p) \) in a neighborhood of \( \rho_j \), respectively, \( \rho_j^* \) as follows:

\[
\begin{align*}
\begin{cases}
 m(p) & \left(1 \quad 0\right), & |\pi(p) - \rho_j| < \varepsilon, \quad p \in \Pi_+, \\
 \left(0 \quad 1\right), & |\pi(p) - \rho_j| < \varepsilon, \quad p \in \Pi_-, \\
 m(p), & \text{otherwise,}
\end{cases}
\end{align*}
\]

where \( \gamma_j(p, x, t) \) is a function which is analytic in \( 0 < |\pi(p) - \rho_j| < \varepsilon, p \in \Pi_+ \) and satisfies

\[
\lim_{p \to \rho_j} \gamma_j(p, x, t) \frac{\psi_{q}(p^*, x, t)}{\psi_{q}(p, x, t)} = \frac{2R^{1/2}_{2g+1}(\rho_j)}{\prod_{k=1}^g (\rho_j - \mu_k)} \gamma_j.
\]

For example, we can choose

\[
\gamma_j(p, x, t) = \frac{-2R^{1/2}_{2g+1}(\rho_j)}{\prod_{k=1}^g (\rho_j - \mu_k)} \frac{\psi_{q}(p, x, t)}{\psi_{q}(p^*, x, t)} \gamma_j
\]

or

\[
\gamma_j(p, x, t) = \frac{-2R^{1/2}_{2g+1}(\rho_j)}{\prod_{k=1}^g (\pi(p) - \mu_k)} \frac{\psi_{q}(p, x, t)}{\psi_{q}(p^*, x, t)} \gamma_j.
\]

**Lemma 4.5.** Suppose \( m(p) \) is redefined as in (4.30). Then \( m(p) \) is meromorphic away from \( \Sigma \) and satisfies (4.15), (4.18), and (4.19). Moreover, the divisor conditions change according to

\[
(4.31) \quad (m_1) \geq -D_{\hat{\mu}(x, t)}' \quad \text{and} \quad (m_2) \geq -D_{\hat{\mu}(x, t)},
\]
and the pole conditions are replaced with the jump conditions

\[ m_+(p) = m_-(p) \begin{pmatrix} 1 & 0 \\ \gamma(p, x, t) & \pi(p) \end{pmatrix}, \quad p \in \Sigma_v(\rho), \]

\[ m_+(p) = m_-(p) \begin{pmatrix} 1 & 0 \\ -\gamma(p^*, x, t) & \pi(p) \end{pmatrix}, \quad p \in \Sigma_v(\rho^*), \]

where

\[ \Sigma_v(p) = \{ q \in \Pi_\pm : |\pi(q) - z| = \varepsilon \}, \quad p = (z, \pm), \]

is a small circle around \( p \) on the same sheet as \( p \). It is oriented counterclockwise on the upper sheet and clockwise on the lower sheet.

**Proof.** Everything except for the pole conditions follows as in the proof of Theorem 4.2. That the pole conditions (4.17) are indeed replaced with the jump conditions (4.32) when \( m(p) \) is redefined as in (4.30) can be shown by a straightforward calculation. \( \square \)

The next thing we do is deduce the one-soliton solution of our Riemann–Hilbert problem, i.e., the solution in the case where only one eigenvalue \( \rho \) corresponding to one bound state is present and the reflection coefficient \( R(p) \) vanishes identically on \( K_g \).

**Lemma 4.6** (One-soliton solution). Suppose there is only one eigenvalue and a vanishing reflection coefficient, that is, \( S_+(H(t)) = \{ R(p) \equiv 0, \ p \in \Sigma; (\rho, \gamma) \} \).

Let

\[ c_{q,\gamma}(\rho, x, t) = 1 + \gamma W(x, t)(\psi_q(\rho, x, t), \psi_q(\rho, x, t)) = 1 + \gamma \int_x^\infty \psi_q(\rho, y, t)^2 \ dy \]

and

\[ \psi_{q,\gamma}(p, x, t) = \psi_q(p, x, t) + \frac{\gamma}{z - \rho} \frac{\psi_q(p, x, t)W(x, t)(\psi_q(\rho, x, t), \psi_q(\rho, x, t))}{c_{q,\gamma}(\rho, x, t)}. \]

Here the dot (\( ' \)) denotes a derivative with respect to \( \rho \) and

\[ W(x, t)(f, g) = (f(x)g'(x) - f'(x)g(x)) \]

is the usual Wronski determinant, where the prime (\( ' \)) denotes the derivative with respect to \( x \). Then the Riemann–Hilbert problem (4.15)–(4.19) has a unique solution, which is given by

\[ m_0(p) = \begin{pmatrix} f(p^*, x, t) & f(p, x, t) \end{pmatrix}, \quad f(p, x, t) = \frac{\psi_{q,\gamma}(p, x, t)}{\psi_q(p, x, t)}. \]
In particular,

\begin{equation}
\int_x^\infty (V - V_q)(y, t) dy = -2 \frac{\partial}{\partial x} \log (c_{q,\gamma}(\rho, x, t)),
\end{equation}

or

\begin{equation}
(V - V_q)(x, t) = 2 \frac{\partial^2}{\partial x^2} \log (c_{q,\gamma}(\rho, x, t)).
\end{equation}

**Proof.** Since we assume the reflection coefficient vanishes, the jump along \( \Sigma \) disappears. Moreover, since the symmetry condition (4.18) must be satisfied, it follows that the solution of the Riemann–Hilbert problem (4.15)–(4.19) must be of the form \( m_0(p) = (f(p^*, x, t), f(p, x, t)) \). The divisor conditions (4.16) follow from the fact that the Baker–Akhiezer function \( \psi_q \) has simple zeros at \( \hat{\mu}_j(x, t) \) and simple poles at \( \hat{\mu}_j, j = 1, \ldots, g \) and by construction of \( \psi_{q,\gamma} \). It is obvious that the normalization condition (4.19) holds. Thus it is only left to check the pole conditions (4.17). For this purpose, we compute

\[
\lim_{\rho \to \rho} (z - \rho)f(p^*) = \frac{\gamma(p, x, t)}{c_{q,\gamma}(\rho, x, t)} W(x, t)(\psi_q(\rho, x, t), \psi_q(p^*, x, t))
\]

\[
= - \frac{\gamma(p, x, t)}{c_{q,\gamma}(\rho, x, t)} \frac{2R_{2g+1}(\rho)}{\prod_{k=1}^{\infty} (\rho - \mu_k)},
\]

where we have defined

\[
\gamma(p, x, t) = \frac{\psi_q(p, x, t)}{\psi_q(p^*, x, t)} = \gamma(x, t)e^{i\phi(p)},
\]

and used (cf. [16, Equ. (1.87)])

\[
W(\psi_{q,\pm}(z), \psi_{q,\pm}(z)) = \pm \frac{2R_{2g+1}(z)}{\prod_{k=1}^{\infty} (z - \mu_k)}.
\]

Moreover,

\[
\lim_{\rho \to \rho} f(p) = 1 + \frac{\gamma}{c_{q,\gamma}(\rho, x, t)} \lim_{\rho \to \rho} W(x, t)(\psi_q(\rho, x, t), \psi_q(p, x, t)) \frac{(z - \rho)}{z - \rho}
\]

\[
= 1 + \frac{\gamma}{c_{q,\gamma}(\rho, x, t)} \left[ \psi_q(\rho, x, t) \lim_{\rho \to \rho} \left( \frac{\psi_q'(p, x, t) - \psi_q'(\rho, x, t)}{z - \rho} \right)
\]

\[
- \psi_q'(\rho, x, t) \lim_{\rho \to \rho} \left( \frac{\psi_q(p, x, t) - \psi_q(\rho, x, t)}{z - \rho} \right) \right]
\]

\[
= 1 + \frac{\gamma}{c_{q,\gamma}(\rho, x, t)} W(x, t)(\psi_q(\rho, x, t), \psi_q(p, x, t)) = \frac{1}{c_{q,\gamma}(\rho, x, t)}.
\]

Hence the pole conditions (4.17) are satisfied.
The formula (4.36) follows after expanding around \( p = p_\infty \), that is,

\[
f(p, x, t) = 1 + \frac{\gamma}{(z - \rho) c_{q,\gamma}(p, x, t)} \psi_q(p, x, t) \left( \psi_q(p, x, t) m_q(p, x, t) - \psi'_q(p, x, t) \right) = 1 \mp \frac{\gamma}{c_{q,\gamma}(p, x, t)} \psi'_q(p, x, t) + O(z^{-1}), \quad p = (z, \pm),
\]

where we have used the fact that the Weyl–Titchmarsh \( m \)-function has the asymptotic expansion for \( p \) near \( p_\infty \) (cf. [29, Lemma 6.1]).

\[
m_{q, \pm}(z, x, t) = \frac{\psi'_{q, \pm}(z, x, t)}{\psi_{q, \pm}(z, x, t)} = \pm i \sqrt{z} + \frac{V_q(x, t)}{2i \sqrt{z}} + O(z^{-1}), \quad p = (z, \pm).
\]

Thus, comparing with (4.13) proves the equation (4.36).

To see uniqueness, let \( \tilde{m}_0(p) \) be a second solution which, by the symmetry condition, must be of the form \( \tilde{m}_0(p) = (\tilde{f}(p^*) \tilde{f}(p)) \). Since the divisor \( \mathcal{D}_{\tilde{\mu}(x, t)} \) is nonspecial, the Riemann–Roch theorem implies that \( \tilde{f}(p) = \alpha f(p) + \beta \) for some \( \alpha, \beta \in \mathbb{C} \). But the pole condition implies \( \beta = 0 \) and the normalization condition implies \( \alpha = 1 \).

Since up to quasi-periodic factors, \( \psi_q(\rho, x, t) \) is a function of \( x - \nu(\rho)t \), where

\[
\nu(\rho) = \frac{-12 \text{Re} \int_{E_0}^{(p^+) \omega_{p, \infty, 2}}}{\text{Re} \int_{E_0}^{(p^+) \omega_{p, \infty, 0}}},
\]

we call \( \nu(\rho) \) the velocity of the corresponding soliton.

5 The stationary phase points and the nonlinear dispersion relation

In this section, we examine the relation between the energy \( \lambda \) of the underlying Lax operator \( H_q \) and the propagation speed at which the corresponding parts of KdV solutions travel, that is, the analog of the classical dispersion relation. We set

\[
\nu(\lambda) = \lim_{\varepsilon \to 0} \frac{-12 \text{Re}(i \int_{E_0}^{(i+\varepsilon, +) \omega_{p, \infty, 2}})}{\text{Re}(i \int_{E_0}^{(i+\varepsilon, +) \omega_{p, \infty, 0}})} = \lim_{\varepsilon \to 0} \frac{-12 \text{Im}(i \int_{E_0}^{(i+\varepsilon, +) \omega_{p, \infty, 2}})}{\text{Im}(i \int_{E_0}^{(i+\varepsilon, +) \omega_{p, \infty, 0}})}.
\]

Our first aim is to show that the nonlinear dispersion relation is given by

\[
\nu(\lambda) = \frac{x}{t}.
\]

Recall that the Abelian differentials are given by (3.15) and (3.16).
For $\rho \in \mathbb{R} \setminus \sigma(H_q)$, the denominator of (5.1) is nonzero and (5.1) agrees with the soliton velocity defined in (4.39). In particular, recalling the definition of our phase $\phi$ from (4.21), this implies that

\[(5.3) \quad v(\rho) = \frac{x}{t} \quad \text{if and only if} \quad \text{Re} \phi \left( \rho, \frac{x}{t} \right) = 0\]

in this case. In particular, the definition of velocity given in (5.1) reduces precisely to the definition of the velocity of a soliton corresponding to the eigenvalue $\rho$ (cf. the discussion after Lemma 4.6).

For $\lambda \in \sigma(H_q)$, both the numerator and denominator vanish on $\sigma(H_q)$, by (3.2). Hence l’Hôpital’s rule gives

\[(5.4) \quad v(\lambda) = -\frac{12 \prod_{j=0}^{g} (\lambda - \tilde{\lambda}_j)}{\prod_{j=1}^{g} (\lambda - \lambda_j)},\]

that is,

\[(5.5) \quad v(\lambda) = \frac{x}{t} \quad \text{if and only if} \quad \phi'(\lambda, \frac{x}{t}) = 0.\]

In other words, $v(\lambda)$ coincides with a stationary phase point in this case.

Thus we discuss the stationary phase points, that is, the solutions of $\phi'(\lambda, x/t) = 0$. The solutions are given by the zeros of the polynomial

\[(5.6) \quad 12 \prod_{j=0}^{g} (z - \tilde{\lambda}_j) + \frac{x}{t} \prod_{j=1}^{g} (z - \lambda_j).\]

Since our Abelian differentials are all normalized to have vanishing $a_j$-periods, the numbers $\lambda_j$, $0 \leq j \leq g$, are real and different. Moreover, precisely one lies in each spectral gap. We denote by $\lambda_j$ the one that lies in the $j$-th gap. Similarly, each $\tilde{\lambda}_j$, $0 \leq j \leq g$, is real, the $\tilde{\lambda}_j$ are all different, and $\tilde{\lambda}_j$, $1 \leq j \leq g$, sits in the $j$-th gap. However, $\tilde{\lambda}_0$ can be anywhere (see [35, Sect. 13.5]).

The following lemma clarifies the dependence of the stationary phase points on $x/t$.

**Lemma 5.1.** Denote by $z_j(\nu)$, $0 \leq j \leq g$, the stationary phase points, where $\nu = x/t$. Set $\lambda_0 = -\infty$ and $\lambda_{g+1} = \infty$. Then

\[(5.7) \quad \lambda_j < z_j(\nu) < \lambda_{j+1}\]

and there is always at least one stationary phase point in each spectral gap. Moreover, $z_j(\nu)$ is strictly decreasing with

\[(5.8) \quad \lim_{\nu \to -\infty} z_j(\nu) = \lambda_{j+1} \quad \text{and} \quad \lim_{\nu \to \infty} z_j(\nu) = \lambda_j.\]
Proof. Since the Abelian differential $\omega_{p,\infty} + v\omega_{p,\infty,0}$ has vanishing $a$ periods, the polynomial (5.6) must change sign in each gap except the lowest. Consequently there is at least one stationary phase point in each gap except the lowest, and the stationary phase points are all different. Furthermore, by the Implicit Function Theorem,

$$z_j' = -\frac{q(z_j)}{\tilde{q}'(z_j) + vq'(z_j)} = -\frac{\prod_{k=1}^{g}(z_j - \lambda_k)}{12\prod_{k=0,k\neq j}^{g}(z_j - z_k)},$$

where

$$\tilde{q}(z) = 12\prod_{k=0}^{g}(z - \tilde{\lambda}_k), \quad q(z) = \prod_{k=1}^{g}(z - \lambda_k).$$

Since the $\lambda_k$ are fixed points of this ordinary first order differential equation (note that since the $z_j$’s are all different, the denominator cannot vanish), the $z_j$ cannot cross these points. Combining the behavior as $v \to \pm \infty$ with the fact that there must always be at least one $\lambda_k$ in each gap, we conclude that $z_j$ must stay between $\lambda_j$ and $\lambda_{j+1}$. This also shows $z_j' < 0$ and thus $z_j(v)$ is strictly decreasing. $\square$

In other words, Lemma 5.1 tells us that as $v = x/t$ runs from $-\infty$ to $\infty$, $z_g(v)$ starts from $\infty$ and tends to $E_{2g}$, while the other stationary phase points $z_j$, $j = 0, \ldots, g-1$, stay in their spectral gaps until $z_g(v)$ has passed $E_{2g}$ and therefore left the first spectral band $[E_{2g}, \infty)$. After this has happened, the next stationary phase point $z_{g-1}(v)$ can leave its gap $(E_{2g-1}, E_{2g})$ while $z_g(v)$ remains there, traverses the next spectral band $[E_{2g-2}, E_{2g-1}]$ and so on. Finally $z_0(v)$ traverses the last spectral band $[E_0, E_1]$ and moves to $-\infty$. So, depending on the value of $x/t$, there is at most one single stationary phase point belonging to the union of the bands $\sigma(H_q)$; this is the one that solves (5.5).

Lemma 5.2. The function $v(\lambda)$ defined in (5.1) is a continuous and strictly decreasing bijection from $\mathbb{R}$ to $\mathbb{R}$.

Proof. That $v(\lambda)$ is continuous except possibly at the band edges $\lambda = E_j$ is obvious. However, at these edges, (5.1) becomes (5.4), by l’Hôpital’s rule and $v(\lambda)$ defined in (5.4) is obviously continuous at the band edges $E_j$, since $\lambda_j$ lies in the $j$-th gap and thus does not hit the band edges.

Furthermore, for large $\lambda$,

$$(5.9) \quad \lim_{\lambda \to -\infty} \frac{v(\lambda)}{-4\lambda} = 1, \quad \lim_{\lambda \to +\infty} \frac{v(\lambda)}{-12\lambda} = 1,$$

which shows $\lim_{\lambda \to \pm \infty} v(\lambda) = \mp \infty$. 


We know that \( z_j(v) \) is the inverse of \( v(\lambda) \) in the regions where there is one stationary phase point \( z_j(v) \in \sigma(H_q) \), and monotonicity follows from Lemma 5.1. For the other regions, we compute

\[
(5.10) \quad v'(z) = \frac{\prod_{j=0}^{g}(z - z_j(v(z)))}{-2i R_{2g+1}(z) \int_{E_0}^{(z,+)} \omega_{p,0}}.
\]

Order the stationary phase points so that \( z_j(v(z)) = z \) for \( z \in (E_{2j-1} - \varepsilon, E_{2j-1}] \) and \( z_j(v(z)) = z \) for \( z \in [E_{2j}, E_{2j} + \varepsilon) \). We claim that \( z_j(v(z)) < z < z_{j-1}(v(z)) \) for \( z \in (E_{2j-1}, E_{2j}) \) (set \( z_{-1} = \infty \)). In fact, (5.10) implies that \( v(z) \) can cross the curve \( z_j(v(z)) \) only from below and hence must stay above this curve since it starts on this curve at \( z = E_{2j-1} \). Similarly, it can cross the curve \( z_{j-1}(v(z)) \) only from below and then remains above this curve afterwards. This can only happen at \( z = E_{2j} \). □

In summary, we can define a function \( \zeta(x/t) \) via

\[
(5.11) \quad v(\zeta) = \frac{x}{t}.
\]

In particular, different solitons travel at different speeds and don’t collide with each other or with the parts corresponding to the continuous spectrum. Moreover, there is some \( \zeta_0 \) for which \( v(\zeta_0) = 0 \), and hence there can be stationary solitons provided \( \zeta_0 \notin \sigma(H_q) \).

**Lemma 5.3** (Stationary solitons). There exists a unique \( \zeta_0 \) such that \( v(\zeta_0) = 0 \). Moreover, if \( \zeta_0 \in \sigma(H_q) \) or \( \tilde{\lambda}_0 \in \sigma(H_q) \), then \( \zeta_0 = \tilde{\lambda}_0 \). In particular, \( \zeta_0 \in \sigma(H_q) \) if and only if \( \tilde{\lambda}_0 \in \sigma(H_q) \).

**Proof.** Existence and uniqueness of \( \zeta_0 \) follows since \( v \) is a bijection. It is left to show that \( \zeta_0 = \tilde{\lambda}_0 \) if \( \zeta \in \sigma(H_q) \) or \( \tilde{\lambda}_0 \in \sigma(H_q) \). Assume \( \zeta_0 \in \sigma(H_q) \). Then using \( v(\zeta_0) = 0 \) and (5.4), we get

\[
\prod_{j=0}^{g}(\zeta_0 - \tilde{\lambda}_j) = (\zeta_0 - \tilde{\lambda}_0) \prod_{j=1}^{g}(\zeta_0 - \tilde{\lambda}_j) = 0.
\]

Since \( \tilde{\lambda}_j \in (E_{2j-1}, E_{2j}) \), \( j = 1, \ldots, g \), it follows that \( \zeta_0 = \tilde{\lambda}_0 \). Now suppose \( \tilde{\lambda}_0 \in \sigma(H_q) \), and again use (5.4) to obtain

\[
v(\tilde{\lambda}_0) \prod_{j=1}^{g}(\tilde{\lambda}_0 - \lambda_j) = 0.
\]

Since \( \lambda_j \in (E_{2j-1}, E_{2j}) \), \( j = 1, \ldots, g \), we obtain \( v(\tilde{\lambda}_0) = 0 \) and thus \( \zeta_0 = \tilde{\lambda}_0 \). □
Concerning the other bands \( [E_{2j}, E_{2j+1}] \), the second case (ii), the soliton region (case (ii)), and the transitional region (case (iii)).

Case (i): The oscillatory region. Note that in this case, we have

\[
\frac{\phi''(z_j)}{1} = -\frac{12 \prod_{k=0, k \neq j}^{g} (z_j - z_k)}{iR_{2g+1}^{1/2}(z_j)} > 0.
\]

Suppose \( \zeta(v) = z_j(v) \), belongs to the interior of the band \( [E_{2j}, E_{2j+1}] \) (with \( E_{2g+1} = \infty \)). We introduce the “lens” contour near that band as shown in Figure 1.

The oriented paths \( C_j = C_{j1} \cup C_{j2}, C'_j = C'_{j1} \cup C'_{j2} \) are meant to be close to the band \( [E_{2j}, E_{2j+1}] \).

Concerning the other bands \( [E_{2k}, E_{2k+1}], k \neq j, k = 0, \ldots, g \) (setting \( E_{2g+1} = \infty \)), one simply constructs “lens” contours near each of the other bands \( [E_{2k}, E_{2k+1}] \).
and \([E_{2k}^*, E_{2k+1}^*]\) as shown in Figure 2.

The oriented paths \(C_k, C_k^*\) are meant to be close to the band \([E_{2k}, E_{2k+1}]\). In particular, these loops must not contain any of the eigenvalues \(\rho_j\).

An investigation of the sign of \(\text{Re}(\phi)\) shows that

\[
\text{Re}(\phi(p)) > 0 \text{ if } p \in D_{j1} \cup D_k, \quad \pi(p) < \zeta(x/t),
\]

\[
\text{Re}(\phi(p)) < 0 \text{ if } p \in D_{j2} \cup D_k, \quad \pi(p) > \zeta(x/t),
\]

with \(k = 1, \ldots, g, k \neq j\).

Observe that our original jump matrix (4.15) has the important factorization

\[
(5.13) \quad J(p) = b_-(p)^{-1}b_+(p),
\]

where

\[
(5.14) \quad b_-(p) = \begin{pmatrix} 1 & R(p^*)\Theta(p^*)e^{-i\phi(p)} \\ 0 & 1 \end{pmatrix}, \quad b_+(p) = \begin{pmatrix} 1 & 0 \\ R(p)\Theta(p)e^{i\phi(p)} & 1 \end{pmatrix},
\]

which is the right factorization for \(p \in \Sigma \setminus C(x/t) = \Sigma \cap \pi^{-1}((\zeta(x/t), \infty)), \) i.e., \(\pi(p) > \zeta(x/t)\). Similarly, we have

\[
(5.15) \quad J(p) = B_-(p)^{-1} \begin{pmatrix} 1 - |R(p)|^2 & 0 \\ 0 & \frac{1}{1 - |R(p)|^2} \end{pmatrix} B_+(p),
\]

where

\[
(5.16) \quad B_-(p) = \begin{pmatrix} 1 & 0 \\ -\frac{R(p)\Theta(p)e^{i\phi(p)}}{1 - |R(p)|^2} & 1 \end{pmatrix}, \quad B_+(p) = \begin{pmatrix} 1 & -\frac{R(p^*)\Theta(p^*)e^{-i\phi(p)}}{1 - |R(p)|^2} \\ 0 & 1 \end{pmatrix},
\]
This constitutes the right factorization for \( p \in C(x/t) = \Sigma \cap \pi^{-1}(-\infty, \zeta(x/t)) \), i.e., \( \pi(p) < \zeta(x/t) \). Here we have used \( \overline{R(p)} = R(p^*) \), for \( p \in \Sigma \). To get rid of the diagonal part in the factorization corresponding to \( \pi(p) < \zeta(x/t) \) and to conjugate the jumps near the eigenvalues, we need to find the solution of the corresponding scalar Riemann–Hilbert problem, the so-called \textit{partial transmission coefficient}. Again we seek a meromorphic solution. This means that the poles of the scalar Riemann–Hilbert problem are added to the resulting Riemann–Hilbert problem. On the other hand, a pole structure similar to the one of \( m \) is crucial for uniqueness. We address this problem by choosing the poles of the scalar problem in such a way that its zeros cancel the poles of \( m \). The right choice turns out to be \( D_{\hat{z}} \) (that is, the Dirichlet divisor corresponding to the limiting lattice defined in (2.8)).

Define a divisor \( D_{\hat{z}(x,t)} \) of degree \( g \) via

\[
\alpha_{E_0}(D_{\hat{z}(x,t)}) = \alpha_{E_0}(D_{\hat{\mu}(x,t)}) + \hat{\theta}(x/t),
\]

where

\[
\delta_\ell(x/t) = -2 \sum_{\rho_k < \zeta(x/t)} A_{E(\rho_k),\ell}(\rho_k) + \frac{1}{2\pi i} \int_{C(x/t)} \log(1 - |R|^2)\zeta_\ell
\]

with \( C(x/t) = \Sigma \cap \pi^{-1}((-\infty, \zeta(x/t)) \) and \( \zeta(x/t) \) as defined in (5.11).

The divisor \( D_{\hat{z}(x,t)} \) is nonspecial, and \( \pi(\hat{\nu}_j(x, t)) = \nu_j(x, t) \in \mathbb{R} \) with precisely one \( \nu_j \) in each spectral gap (see [22]).

We define the \textit{partial transmission coefficient} \( T \) by

\[
T(p, x, t) = \frac{\theta(z(p_\infty, x, t) + \hat{\theta}(x/t))}{\theta(z(p, x, t))} \frac{\theta(z(p, x, t))}{\theta(z(p_\infty, x, t) + \hat{\theta}(x/t))} \cdot \left( \prod_{\rho_k < \zeta(x/t)} \exp\left(-\int_{E_0} \omega_{\rho_k \rho_j} \right) \right) \exp\left(\frac{1}{2\pi i} \int_{C(x/t)} \log(1 - |R|^2)\omega_{p p_\infty} \right),
\]

where \( \hat{\theta}(x, t) \) is defined in (5.18) and \( \omega_{p_1 p_2} \) is the Abelian differential of the third kind with poles at \( p_1 \) and \( p_2 \).

The function \( T(p, x, t) \) is meromorphic in \( \mathcal{K}_g \setminus C(x/t) \), with first order poles at \( \rho_k < \zeta(x/t), \hat{\nu}_j(x, t) \) and first order zeros at \( \hat{\mu}_j(x, t) \).

\textbf{Lemma 5.4.} \( T(p, x, t) \) satisfies the scalar meromorphic Riemann–Hilbert problem

\[
T_+(p, x, t) = T_-(p, x, t)(1 - |R(p)|^2), \quad p \in C(x/t),
\]

\[
(T(p, x, t)) = \sum_{\rho_k < \zeta(x/t)} D_{\rho_k} - \sum_{\rho_k < \zeta(x/t)} D_{\rho_k} + D_{\hat{\mu}(x,t)} - D_{\hat{z}(x,t)},
\]

\[
T(p_\infty, x, t) = 1.
\]
Moreover,

(i) \[ T(p^*, x, t) T(p, x, t) = \prod_{j=1}^{g} \frac{z - \mu_j(x, t)}{z - \nu_j(x, t)}, \quad z = \pi(p); \]

(ii) \[ \overline{T(p, x, t)} = T(\overline{p}, x, t) \] and, in particular, \( T(p, x, t) \) is real-valued for \( \pi(p) \in \mathbb{R} \setminus \sigma(H_q) \).

**Proof.** The argument is similar to that used in [22, Thm. 4.3]. The solution of a Riemann–Hilbert problem on the Riemann sphere is given by the Plemelj-Sokhotsky formula. Since our problem is now set on the Riemann surface \( \mathcal{K}_g \), the Cauchy kernel is given by the Abelian differential of the third kind \( \omega_{x/t} \) (cf. [34]).

In particular, \( T(p, x, t) \) satisfies the jump condition from (5.20) along \( C(x/t) \). Next, we have to check that \( T(p, x, t) \) extends to a single-valued function on \( \mathcal{K}_g \). For that purpose, note that the only possible contribution which causes multi-valuedness may come from the \( b \)-cycles, since all Abelian differentials are normalized to have vanishing \( a \)-periods. So for the \( b_\ell \)-periods \( \ell = 1, \ldots, g \), we compute, for \( p \in C(x/t) \),

\[ \lim_{\varepsilon \to 0} \frac{T(p + i\varepsilon, x, t)}{T(p - i\varepsilon, x, t)} = \exp \left( 2\pi i \delta_{\ell} - \int_{C(x/t)} \log(1 - |R|^2) \zeta_{\ell} + \sum_{\rho_k < \zeta(x/t)} 4\pi i A_{E(p_k), \ell}(\rho_k) \right), \]

which is indeed equals 1 by the choice of \( \delta_{\ell} \) in (5.18).

Concerning the poles and zeros of \( T(p, x, t) \), we see that by Riemann’s Vanishing Theorem (cf. [35, Theorem A.13]) and the choice of the divisor \( D_{\hat{\mu}(x,t)} \) defined by (5.17), the ratio of theta functions is meromorphic with simple zeros at \( \hat{\mu}_j \) and simple poles at \( \hat{\nu}_j \). Moreover, from the product of the Blaschke factors, we see that \( T \) has simple poles at \( \rho_k \) and simple zeros at \( \rho_k^* \) for which \( \rho_k < \zeta(x/t) \).

To prove uniqueness, let \( \hat{T} \) be a second solution and consider \( \hat{T}/T \). Then \( \hat{T}/T \) has no jump and the Schwarz reflection principle implies that it extends to a meromorphic function on \( \mathcal{K}_g \). Since the poles of \( T \) cancel the poles of \( \hat{T} \), its divisor satisfies \( (\hat{T}/T) \geq -D_{\hat{\mu}(x,t)} \). Since \( D_{\hat{\mu}(x,t)} \) is nonspecial, \( \hat{T}/T \) has to be a constant, by the Riemann–Roch Theorem (cf. [35, Theorem A.2]). Setting \( p = p_{\infty} \), we see that this constant is 1, that is, \( \hat{T} = T \) as claimed.

Finally, that \( \overline{T(p, x, t)} = T(\overline{p}, x, t) \) follows from uniqueness, since both functions solve (5.20).

We also need the expansion of \( T \) around \( p_{\infty} \), which is given in the following.
Lemma 5.5. The asymptotic expansion of the partial transmission coefficient for \( p \) near \( p_\infty \) is given by

\[
T(p, x, t) = 1 \pm \frac{T_1(x, t)}{\sqrt{z}} + O\left(\frac{1}{z}\right), \quad p = (z, \pm),
\]

where

\[
T_1(x, t) = -\sum_{\rho_k < \zeta(x/t)} 2 \int_{E(\rho_k)} \omega_{p_\infty, 0} + \frac{1}{2\pi i} \int_{C(x/t)} \log(1 - |R|^2)\omega_{p_\infty, 0}
\]

\[
- i\partial_y \ln \left( \frac{\theta(z(p_\infty, y, t) + \tilde{\theta}(x/t))}{\theta(z(p_\infty, y, t))} \right) \bigg|_{y=x},
\]

and \( \omega_{p_\infty, 0} \) is the Abelian differential of the second kind defined in (3.15).

Proof. This can be verified similarly to the case of the full transmission coefficient (cf. [29, Theorems 6.2 and 6.3]) and by expanding the ratio of theta functions near \( p_\infty \).

Now that we have solved the scalar Riemann–Hilbert problem for \( T(p, x, t) \), we can conjugate our original Riemann–Hilbert problem. Since to each discrete eigenvalue there corresponds a soliton, it follows that solitons are represented in our Riemann–Hilbert problem by the pole conditions (4.32). For this reason, in this section, we study how poles can be dealt with. We follow closely the presentation of [24, Section 4].

There are two cases to distinguish for the removal of poles. If \( \rho_j > \zeta(x/t) \), then the jump at \( \rho_j \) is exponentially close to the identity, and there is nothing to do.

Otherwise, in the case \( \rho_j < \zeta(x/t) \), we follow [9], using conjugation to convert the jumps at these poles into exponentially decaying ones. It turns out that we have to handle the poles at \( \rho_j \) and \( \rho_j^* \) in one step in order to preserve symmetry and avoid introducing new poles elsewhere. Conjugation of the Riemann–Hilbert problem also serves another purpose. Namely, it allows us to separate the jump matrix into two matrices, one containing an off-diagonal term with \( \exp(-t\phi) \) and the other with \( \exp(t\phi) \). Without conjugation, the jump on \( C(x/t) = \Sigma \cap \pi^{-1}((-\infty, \zeta(x/t))] \) cannot be separated, since a diagonal matrix also appears when the jump matrix is separated.

For easy reference, we note the following result, which can be verified by a straightforward calculation.
Lemma 5.6 (Conjugation). Assume that \( \tilde{\Sigma} \subseteq \Sigma \). Let \( D \) be a matrix of the form

\[
D(p) = \begin{pmatrix}
d(p^*) & 0 \\
0 & d(p)
\end{pmatrix},
\]

where \( d : \mathcal{K}_g \setminus \tilde{\Sigma} \to \mathbb{C} \) is a sectionally analytic function. Set

\[
\tilde{m}(p) = m(p)D(p).
\]

Then the jump matrix transforms according to

\[
\tilde{J}(p) = D_-(p)^{-1}J(p)D_+(p).
\]

Moreover, \( \tilde{m}(p) \) satisfies the symmetry condition (4.18) if and only if \( m(p) \) does, and \( \tilde{m}(p) \) satisfies the normalization condition (4.19) if \( m(p) \) satisfies (4.19) and \( d(p_\infty) = 1 \).

Lemma 5.7 ([26], Lem. 7.2). Introduce

\[
\tilde{B}(p, \rho) = C_\rho(x, t)\frac{\theta(z(p, x, t))}{\theta(z(p, x, t) + 2A_{E_0}(\rho))}B(p, \rho).
\]

Then \( \tilde{B}(., \rho) \) is a well-defined meromorphic function with divisor

\[
(\tilde{B}(., \rho)) = -D_\tilde{\Sigma} + D_{\tilde{\mu}} - D_{\rho^*} + D_{\rho},
\]

where \( \nu \) is defined via

\[
\alpha_{E_0}(D_{\tilde{\Sigma}}) = \alpha_{E_0}(D_{\tilde{\mu}}) + 2A_{E_0}(\rho).
\]

Furthermore,

\[
\tilde{B}(p_\infty, \rho) = 1
\]

if

\[
C_\rho(x, t) = \frac{\theta(z(p_\infty, x, t) + 2A_{E_0}(\rho))}{\theta(z(p_\infty, x, t))}.
\]

Following [26], we can now show how to conjugate the jump corresponding to one eigenvalue.
Lemma 5.8. Assume that the Riemann–Hilbert problem for $m$ has jump conditions near $\rho$ and $\rho^*$ given by

$$m_+(p) = m_-(p) \begin{pmatrix} 1 & \frac{\gamma(p)}{\pi(p) - \rho} \\ \pi(p) - \rho & 1 \end{pmatrix}, \quad p \in \Sigma_\varepsilon(\rho),$$

and satisfies a divisor condition

$$(5.32) \quad (m_1) \geq -\mathcal{D}_{\hat{\nu}}, \quad (m_2) \geq -\mathcal{D}_{\tilde{\nu}},$$

Then this Riemann–Hilbert problem is equivalent to a Riemann–Hilbert problem for $\tilde{m}$ which has jump conditions near $\rho$ and $\rho^*$ given by

$$\tilde{m}_+(p) = \tilde{m}_-(p) \begin{pmatrix} 1 & \frac{\tilde{B}(p, \rho^*) \gamma(p) - \rho}{\pi(p) - \rho} \\ \pi(p) - \rho & 1 \end{pmatrix}, \quad p \in \Sigma_\varepsilon(\rho),$$

$$\tilde{m}_+(p) = \tilde{m}_-(p) \begin{pmatrix} 1 & \frac{\tilde{B}(p^*, \rho^*) \gamma(p) - \rho}{\pi(p) - \rho} \\ \pi(p) - \rho & 1 \end{pmatrix}, \quad p \in \Sigma_\varepsilon(\rho^*),$$

divisor condition

$$(5.34) \quad (\tilde{m}_1) \geq -\mathcal{D}_{\hat{\nu}}, \quad (\tilde{m}_2) \geq -\mathcal{D}_{\tilde{\nu}},$$

where $\mathcal{D}_{\hat{\nu}}$ is defined via

$$(5.35) \quad \alpha_{E_0}(\mathcal{D}_{\hat{\nu}}) = \alpha_{E_0}(\mathcal{D}_{\tilde{\nu}}) + 2\Delta_{E_0}(\rho),$$

and all remaining data conjugated (as in Lemma 5.6) by

$$(5.36) \quad D(p) = \begin{pmatrix} \tilde{B}(p^*, \rho^*) & 0 \\ 0 & \tilde{B}(p, \rho^*) \end{pmatrix}.$$

Proof. Denote by $U$ the interior of $\Sigma_\varepsilon(\rho)$. To convert $\gamma$ to $\gamma^{-1}$, introduce $D$ by

$$D(p) = \begin{cases} 
\begin{pmatrix}
1 & \frac{\pi(p) - \rho}{\gamma(p)} \\
\gamma(p) & \pi(p) - \rho
\end{pmatrix} \begin{pmatrix}
\tilde{B}(p^*, \rho^*) & 0 \\
0 & \tilde{B}(p, \rho^*)
\end{pmatrix}, & p \in U, \\
\begin{pmatrix}
0 & \frac{\pi(p) - \rho}{\gamma(p)} \\
\gamma(p) & \pi(p) - \rho
\end{pmatrix} \begin{pmatrix}
\tilde{B}(p^*, \rho^*) & 0 \\
0 & \tilde{B}(p, \rho^*)
\end{pmatrix}, & p^* \in U, \\
\tilde{B}(p^*, \rho^*) & 0 \\
0 & \tilde{B}(p, \rho^*)
\end{cases},$$

otherwise,
and note that \( D(p) \) is meromorphic away from the two circles. Now set \( m(p) = m(p)D(p) \). The claim about the divisors follows from observing where the poles of \( \tilde{B}(p, \rho) \) are located. \( \square \)

Note that Lemma 5.8 can be applied iteratively to conjugate the eigenvalues \( \rho_j < \zeta(x/t) \): start with the poles \( \mu = \mu_0 \) and apply the lemma, setting \( \rho = \rho_1 \). This gives new poles \( \mu_1 = \nu \). Then repeat this with \( \mu = \mu_1, \rho = \rho_2, \) and so on.

We now make the following conjugation step. Let

\[
\gamma_k(p, x, t) = \frac{-2R_{2g+1}(\rho_k)}{\prod_{l=1}^{g} (\rho_k - \mu_l)} \psi_q(p, x, t) \psi_q(p^*, x, t) \gamma_k
\]

and introduce

\[
(5.37) \quad D(p) = \begin{cases}
1 & \frac{\pi(p) - \rho_k}{\gamma_k(p, x, t)} D_0(p), \quad |\pi(p) - \rho_k| < \epsilon, \quad \rho_k < \zeta(x/t), \\
-\frac{\gamma_k(p, x, t)}{\pi(p) - \rho_k} & 0, \quad p \in \Pi_+, \\
0 & \frac{\gamma_k(p^*, x, t)}{\pi(p) - \rho_k}, \quad |\pi(p) - \rho_k| < \epsilon, \quad \rho_k < \zeta(x/t), \\
\frac{\pi(p) - \rho_k}{\gamma_k(p^*, x, t)} & D_0(p), \quad p \in \Pi_-, \\
D_0(p), & \text{otherwise},
\end{cases}
\]

where

\[
D_0(p) = \begin{pmatrix} T(p^*, x, t) & 0 \\ 0 & T(p, x, t) \end{pmatrix}.
\]

Note that \( D(p) \) is meromorphic in \( \mathcal{K}_g \setminus C(x/t) \) and that

\[
(5.38) \quad D(p^*) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} D(p) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

We now conjugate our problem using \( D(p) \).

**Theorem 5.9 (Conjugation).** The function \( m^2(p) = m(p)D(p) \), where \( D(p) \) is defined in (5.37), is meromorphic away from \( C(x/t) \) and satisfies the following.

(i) (Jump condition)

\[
(5.39) \quad m^2(p) = m^2(p)J^2(p), \quad p \in \Sigma,
\]

where the jump matrix is given by

\[
(5.40) \quad J^2(p) = D_{0-}(p)^{-1}J(p)D_{0+}(p).
\]

(ii) (Divisor conditions)

\[
(5.41) \quad (m_1^2) \geq -D_{\hat{\nu}(x,t)}, \quad (m_2^2) \geq -D_{\hat{\nu}(x,t)}.
\]
All jumps corresponding to poles, except for possibly one if \( \rho_k = \zeta(x/t) \), are exponentially decreasing. In that case, we keep the pole condition which is now of the form

\[
\left( m_1^2(p) + \frac{\gamma_k(p, x, t)}{\pi(p) - \rho_k} \frac{T(p^*, x, t)}{T(p, x, t)} m_2^2(p) \right) \geq -\bar{D}_{\Xi(x,t)}^*, \text{ near } \rho_k,
\]

\[
\left( \frac{\gamma_k(p^*, x, t)}{\pi(p) - \rho_k} \frac{T(p, x, t)}{T(p^*, x, t)} m_1^2(p) + m_2^2(p) \right) \geq -\bar{D}_{\Xi(x,t)}^*, \text{ near } \rho_k^*.
\]

(iii) **(Symmetry condition)**

\[ m^2(p*) = m^2(p) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \]

(iv) **(Normalization)**

\[ m^2(p_\infty) = \begin{pmatrix} 1 & 1 \end{pmatrix}. \]

**Proof.** Invoking Lemma 5.6 and (4.15), we see that the jump matrix \( J^2(p) \) is indeed given by (5.40). The divisor conditions follow from the one for \( T(p, x, t) \) and \( m(p) \). Moreover, using Lemma 5.8, one sees easily that the jump corresponding to \( \rho_k < \zeta(x/t) \) (if any) is given by

\[
J^2(p) = \begin{pmatrix} 1 & \frac{T(p, x, t)(\pi(p) - \rho_k)}{\gamma_k(p, x, t) \pi(p) - \rho_k} \\ 0 & 1 \end{pmatrix}, \quad p \in \Sigma_c(\rho_k),
\]

\[
J^2(p) = \begin{pmatrix} 1 & 0 \\ -\frac{T(p^*, x, t)(\pi(p) - \rho_k)}{\gamma_k(p^*, x, t) \pi(p) - \rho_k} & 1 \end{pmatrix}, \quad p \in \Sigma_c(\rho_k^*),
\]

and by Lemma 5.6 the jump corresponding to \( \rho_k > \zeta(x/t) \) (if any) reads

\[
J^2(p) = \begin{pmatrix} 1 & \frac{\gamma_k(p^*, x, t)(\pi(p) - \rho_k)}{T(p^*, x, t)(\pi(p) - \rho_k)} \\ 0 & 1 \end{pmatrix}, \quad p \in \Sigma_c(\rho_k),
\]

\[
J^2(p) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{\gamma_k(p^*, x, t)(\pi(p) - \rho_k)}{T(p^*, x, t)(\pi(p) - \rho_k)} \end{pmatrix}, \quad p \in \Sigma_c(\rho_k^*).
\]

That is, all jumps corresponding to the poles \( \rho_k \neq \zeta(x/t) \) are exponentially decreasing. That the pole conditions are of the form (5.42) in the case \( \rho_k = \zeta(x/t) \) can be checked directly: just use the pole conditions of the original Riemann–Hilbert problem (4.17) and the divisor condition (5.20) for \( T(p, x, t) \). Furthermore, by (4.18) and (5.38), one checks that the symmetry condition for \( m^2 \) is satisfied. From \( T(p_\infty, x, t) = 1 \), we finally deduce

\[
m^2(p_\infty) = m(p_\infty) = \begin{pmatrix} 1 & 1 \end{pmatrix},
\]

which finishes the proof.
For \( p \in \Sigma \setminus C(x/t) = \Sigma \cap \pi^{-1}(\omega(x/t), \infty) \), the jump matrix \( J^2 \) can be factorized as \( J^2 = (\tilde{b}_-)^{-1}\tilde{b}_+ \), where \( \tilde{b}_\pm = D_\pm^{-1}b_\pm D_0 \), that is,

\[
\tilde{b}_- = \begin{pmatrix}
1 & T(p, x, t)T(p^*, x, t)R(p^*)\Theta(p)e^{-i\phi(p)} \\
0 & 1
\end{pmatrix},
\]

(5.46)

\[
\tilde{b}_+ = \begin{pmatrix}
1 & 0 \\
T(p^*, x, t)R(p)\Theta(p)e^{i\phi(p)} & 1
\end{pmatrix}.
\]

For \( p \in C(x/t) = \Sigma \cap \pi^{-1}((-\infty, \zeta(x/t))) \), we can factorize \( J^2 \) as \( J^2 = (\tilde{B}_-)^{-1}\tilde{B}_+ \), where \( \tilde{B}_\pm = D_\pm^{-1}B_\pm D_\pm \), that is,

\[
\tilde{B}_- = \begin{pmatrix}
1 & T_-(p^*, x, t)R(p)\Theta(p)e^{-i\phi(p)} \\
0 & 1
\end{pmatrix},
\]

(5.47)

\[
\tilde{B}_+ = \begin{pmatrix}
1 & 0 \\
T_+(p^*, x, t)R(p^*)\Theta(p^*)e^{i\phi(p)} & 1
\end{pmatrix}.
\]

Note that by \( T(p, x, t) = T(\bar{p}, x, t) \), we have

\[
\frac{T_-(p^*, x, t)}{T_+(p, x, t)} = \frac{T_-(p^*, x, t)}{T_-(p, x, t)} \frac{1}{1 - |R(p)|^2} = \frac{T_+(p, x, t)}{T_+(p, x, t)} \frac{1}{1 - |R(p)|^2}, \quad p \in C(x/t),
\]

(5.48)

\[
\frac{T_+(p, x, t)}{T_-(p^*, x, t)} = \frac{T_+(p, x, t)}{T_+(p, x, t)} \frac{1}{1 - |R(p)|^2} = \frac{T_-(p, x, t)}{T_-(p, x, t)}, \quad p \in C(x/t).
\]

(5.49)

We are now able to redefine the Riemann–Hilbert problem for \( m^2(p) \) in such a way that the jumps of the new Riemann–Hilbert problem lie in the regions where they are exponentially close to the identity for large times. The following theorem can be proved by straightforward calculations:

**Theorem 5.10** (Deformation). Define

\[
m^3(p) = \begin{cases}
m^2(p)\tilde{B}_+(p)^{-1}, & p \in D_k \cup D_{j1}, \ k < j, \\
m^2(p)\tilde{B}_-(p)^{-1}, & p \in D_k^* \cup D_{j1}^*, \ k < j, \\
m^2(p)\tilde{b}_+(p)^{-1}, & p \in D_k \cup D_{j2}, \ k > j, \\
m^2(p)\tilde{b}_-(p)^{-1}, & p \in D_k^* \cup D_{j2}^*, \ k > j, \\
m^2(p), & \text{otherwise},
\end{cases}
\]

(5.50)

where the matrices \( \tilde{b}_\pm \) and \( \tilde{B}_\pm \) are defined in (5.46) and (5.47), respectively. Here we assume that the deformed contour is sufficiently close to the original one. Then the function \( m^3(p) \) satisfies the following.
(i) (Jump condition)

\[(5.51) \quad m_3^3(p) = m_3^3(p)J^3(p), \text{ for } p \in \Sigma, \]

where the jump matrix \(J^3\) is given by

\[
(5.52) \quad J^3(p) = \begin{cases} 
\tilde{B}_+(p), & p \in C_k \cup C_{j1}, k < j, \\
\tilde{B}_-(p)^{-1}, & p \in C^*_k \cup C^*_{j1}, k < j, \\
\tilde{b}_+(p), & p \in C_k \cup C_{j2}, k > j, \\
\tilde{b}_-(p)^{-1}, & p \in C^*_k \cup C^*_{j2}, k > j, \\
J^2(p), & \text{otherwise.}
\end{cases}
\]

(ii) (Divisor conditions)

\[
(5.53) \quad (m_3^3)^1 \geq -D_{\hat{\nu}(x,t)}, \quad (m_3^3)^2 \geq -D_{\hat{\nu}(x,t)}.
\]

The jumps on the small circles around the eigenvalues remain unchanged.

(iii) (Symmetry condition)

\[
(5.54) \quad m^3(p^*) = m^3(p) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

(iv) (Normalization)

\[
(5.55) \quad m^3(p_\infty) = \begin{pmatrix} 1 & 1 \end{pmatrix}.
\]

Here we have assumed that the reflection coefficient \(R(p)\) appearing in the jump matrices admits an analytic extension to the corresponding regions. Of course, this is not true in general; but we can always evade this obstacle by approximating \(R(p)\) by analytic functions. We relegate the details to Section 7.

The crucial observation now is that the jumps \(J^3\) on the oriented paths \(C_k, C^*_k\) are of the form \(I + \text{exponentially small}\) asymptotically as \(t \to \infty\), at least away from the stationary phase points \(z_j, z^*_j\). We thus hope to be able simply to replace these jumps by the identity matrix (asymptotically as \(t \to \infty\)), implying that the solution should asymptotically be given by the constant vector \(\begin{pmatrix} 1 & 1 \end{pmatrix}\). That this can in fact be done is shown in the next section by explicitly computing the contribution of the stationary phase points, thereby showing that they are of the order \(O(t^{-1/2})\); that is, \(m^3(p) = \begin{pmatrix} 1 & 1 \end{pmatrix} + O(t^{-1/2})\) uniformly for \(p\) away from the jump contour. Hence all that remains to obtain the leading term \(V_l\) in Theorem 2.2 is to trace back the definitions of \(m^3\) and \(m^2\) and compare with (4.13). First of all, since \(m^3\) and \(m^2\) coincide near \(p_\infty\) we have \(m^2(p) = \begin{pmatrix} 1 & 1 \end{pmatrix} + O(t^{-1/2})\).
uniformly for $p$ in a neighborhood of $p_\infty$. Consequently, by the definition of $m^2$ (see Theorem 5.9), we have

$$m(p) = (T(p^*, x, t)^{-1} T(p, x, t)^{-1}) + O(t^{-1/2}),$$

again uniformly for $p$ in a neighborhood of $p_\infty$. Finally, using the expansion of $T(p, x, t)$ near $p_\infty$ (see Lemma 5.5) and then comparing the last identity with (4.13), we obtain

$$(5.56) \quad \int_x^\infty (V - V_q)y \, dy = 2iT_1(x, t) + O(t^{-1/2}),$$

where $T_1$ is defined via (5.22), that is,

$$T_1(x, t) = - \sum_{\rho_k < \zeta(x/t)} 2 \int_{E(p_k)} \omega_{p_\infty, 0} + \frac{1}{2\pi i} \int_{C(x/t)} \log(1 - |R|^2) \omega_{p_\infty, 0}$$

$$- i\hat{c}_k \ln \left( \frac{\theta(z(p_\infty, x, t) + \delta(x/t))}{\theta(z(p_\infty, x, t))} \right).$$

Similarly, using (4.27) instead of (4.13) gives

$$(5.57) \quad (V - V_q)(x, t) = O(t^{-1/2}).$$

Hence we have computed the leading term in Theorem 2.2. The next term is computed in Section 6.

**Case (ii): The soliton region.** The case where no stationary phase points lie in the spectrum is similar to Case (i). In fact, it is much simpler since there is no contribution from the stationary phase points; there is a gap (the $j$-th gap, say) in which two stationary phase points exist. As in Case (i), an investigation of the sign of $\text{Re}(\phi)$ gives

$$\text{Re}(\phi(p)) > 0 \quad \text{if} \quad p \in D_k, \quad k < j,$$

$$\text{Re}(\phi(p)) < 0 \quad \text{if} \quad p \in D_k, \quad k > j.$$

Now we construct “lens-type” contours $C_k$ (as shown in Figure 2) around every single band lying to the left of the $j$-th gap and make use of the factorization $J^2 = (\tilde{b}_-)^{-1}\tilde{b}_+$, where the matrices $\tilde{b}_-$ and $\tilde{b}_+$ are defined in (5.46). We also construct such “lens-type” contours $C_k$ around each band lying to the right of the $j$-th gap and make use of the factorization $J^2 = (\tilde{B}_-)^{-1}\tilde{B}_+$ with the matrices $\tilde{B}_-$
and \( \tilde{B}_+ \) given by (5.47). Indeed, in place of (5.50), we set

\[
m^3(p) = \begin{cases} 
    m^2(p)\tilde{B}_+^{-1}(p), & p \in D_k, \ k < j, \\
    m^2(p)\tilde{B}_-^{-1}(p), & p \in D_k^*, \ k < j, \\
    m^2(p)\tilde{b}_+^{-1}(p), & p \in D_k, \ k > j, \\
    m^2(p)\tilde{b}_-^{-1}(p), & p \in D_k^*, \ k > j, \\
    m^2(p), & \text{otherwise.}
\end{cases}
\]

(5.58)

Now we are ready to prove Theorem 2.1 by applying Theorem A.8. If \( |\zeta(x/t) - \rho_k| > \varepsilon \) for all \( k \), we can choose \( \gamma_0' = 0 \) and \( w_0' \) by removing all jumps corresponding to poles from \( w' \). The difference between the solutions of \( w' \) and \( w_0' \) is exponentially small in the sense of Theorem A.8, that is, \( \|w' - w_0'\|_\infty \leq O(t^{-l}) \) for any \( l \geq 1 \). We have the one soliton solution (cf. Lemma 4.6) \( \hat{m}_0(p) = (\hat{f}(p^*, x, t) - \hat{f}(p, x, t)) \), where \( \hat{f}(p) = 1 \) for sufficiently large \( p \). Using Lemma 5.5, we compute

\[
m(p) = \hat{m}_0(p) \begin{pmatrix} T(p^*, x, t)^{-1} & 0 \\ 0 & T(p, x, t)^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ T_1(x, t) & \sqrt{z} \end{pmatrix} + O(z^{-1}) + O(z^{-1}) \begin{pmatrix} 1 & T_1(x, t) \sqrt{z} \\ 0 & 1 \end{pmatrix}.
\]

Comparing this expression with (4.13) yields

\[
\int_x^\infty (V - V_q)(y, t) dy = 2iT_1(x, t) + O(t^{-n}),
\]

and thus, by our definition of the limiting solution, we have, finally,

\[
\int_x^\infty (V - V_l)(y, t) dy = \int_x^\infty (V - V_q)(y, t) - (V_l - V_q)(y, t) dy = O(t^{-n}),
\]

for any \( n \geq 1 \) if \( R(p) \) has an analytic extension. This proves the second part of the theorem.

If \( |\zeta(x/t) - \rho_k| < \varepsilon \) for some \( k \), we choose \( \gamma_0' = \tilde{\gamma}_k \) and \( w_0' \equiv 0 \). Again we conclude that the difference between the solutions of \( w' \) and \( w_0' \) is exponentially small, that is, \( \|w' - w_0'\|_\infty \leq O(t^{-l}) \), for any \( l \geq 1 \). By Lemma 4.6, we have the one soliton solution \( \hat{m}_0(p) = (\hat{f}(p^*, x, t) - \hat{f}(p, x, t)) \) with

\[
\hat{f}(p, x, t) = 1 + \frac{\tilde{\gamma}_k}{z - \rho_k} \frac{\psi_{l,c}\lambda(\rho_k, x, t)W(x, t)(\psi_{l,c}(\rho_k, x, t), \psi_{l,c}(p, x, t))}{\psi_{l,c}(p, x, t)\lambda(x, t)},
\]

for \( p \) sufficiently large, where \( \tilde{\gamma}_k \) is defined as in (2.13). We again use

\[
m(p) = \hat{m}_0(p) \begin{pmatrix} T(p^*, x, t)^{-1} & 0 \\ 0 & T(p, x, t)^{-1} \end{pmatrix} = \begin{pmatrix} \hat{f}(p^*, x, t) & \hat{f}(p, x, t) \\ T(p^*, x, t) & T(p, x, t) \end{pmatrix}.
\]
and now expand $\hat{f}(p)$ as in the proof of Lemma 4.6. Finally, a comparison with (4.13) yields
\[
\int_\infty^x (V - V_q)(y, t)\,dy = 2iT_1(x, t) + \frac{2\tilde{y}_k\varphi_{l,c_k}(p_k, x, t)^2}{c_{l,k}(x, t)} + O(t^{-n});
\]
hence, by our definition of the limiting solution (2.8), we obtain (2.10) for any $n \geq 1$ if $R(p)$ has an analytic extension. Similarly, we obtain (2.11) by using (4.27) instead of (4.13).

Case (iii): The transitional region. In the special case where the two stationary phase points coincide (so that $z_j = z_j^* = E_k$ for some $k$), the Riemann–Hilbert problem arising above is of a different nature. We expect a similar behavior as in the constant background case [10] (cf. also the discussion in [22]). However, we do not treat this case in this paper.

6 The “local” Riemann–Hilbert problems on the small crosses

In the previous section, we reduced everything to the solution of the Riemann–Hilbert problem
\[
m_3^s(p) = m_3^s(p)J^3(p),
\]
\[
(m_1^3) \geq -D_{\Sigma(z,0)}, \quad (m_2^3) \geq -D_{\Sigma(x,t)},
\]
\[
m^3(p^*) = m^3(p)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]
\[
m^3(p_\infty) = \begin{pmatrix} 1 & 1 \end{pmatrix},
\]
where the jump matrix $J^3$ is given by (5.52). We have performed a deformation in such a way that the jumps $J^3$ on the oriented paths $C_k, C_k^*$ for $k \neq j$ are of the form “$Il +$ exponentially small” asymptotically as $t \to \infty$. The same is true for the oriented paths $C_{j1}, C_{j2}, C_{j1}^*, C_{j2}^*$, at least away from the stationary phase points $z_j, z_j^*$. The purpose of this section is to derive the actual asymptotic rate at which $m^3(p)$ tends to $\begin{pmatrix} 1 & 1 \end{pmatrix}$, again following [22]. The jump contour near the stationary phase points (cf. Figure 3) is denoted by $\Sigma^C(z_j)$ and $\Sigma^C(z_j^*)$. On these crosses, the
Figure 3. The small cross containing the stationary phase point $z_j$ and its flipping image containing $z^*_j$. Views are of the top and bottom sheets. Dotted curves lie in the bottom sheet.

Jumps read

$$
J^3 = \tilde{B}_+ = \begin{pmatrix}
1 & \frac{T^* R^* \Theta^*}{T^* (1 - R^* R)} e^{-t \phi} \\
0 & 1
\end{pmatrix}, \quad p \in C_{j1},
$$

$$
J^3 = \tilde{B}^{-1}_+ = \begin{pmatrix}
1 & 0 \\
\frac{R \Theta}{1 - R^* R} e^{t \phi} & 1
\end{pmatrix}, \quad p \in C^*_{j1},
$$

(6.1)

$$
J^3 = \tilde{b}_+ = \begin{pmatrix}
1 & 0 \\
\frac{T R \Theta}{T} e^{t \phi} & 1
\end{pmatrix}, \quad p \in C_{j2},
$$

$$
J^3 = \tilde{b}^{-1}_+ = \begin{pmatrix}
1 & -\frac{T^* R \Theta^* e^{-t \phi}}{1 - R R^*} \\
0 & 1
\end{pmatrix}, \quad p \in C^*_{j2}.
$$

To reduce our Riemann–Hilbert problem to the one corresponding to the two crosses, we proceed as follows. Take a small disc $D$ around $z_j(x/t)$ and project it onto the complex plane using the canonical projection $\pi$. Now consider the (holomorphic) Riemann–Hilbert problem in the complex plane with the very jump obtained by projection, and normalize it to be $I$ near $\infty$.

The corresponding Riemann–Hilbert problem is solved in [25, Appendix A]. To apply [25, Theorem A.1] we need to know the behavior of the jump matrix $J^3$, that is, the behavior of $T(p, x, t)$ near the stationary phase points $z_j$ and $z^*_j$.

The following lemma gives more information on the singularities of $T(p, x, t)$ near the stationary phase points $z_j$, $j = 0, \ldots, g$ and the band edges $E_j$, $j = 0, \ldots, 2g + 1$ (setting $E_{2g+1} = \infty$).
Lemma 6.1. For $p$ near a stationary phase point $z_j$ or $z_j^*$ (not equal to a band edge),

\[(6.2) \quad T(p, x, t) = (z - z_j)^{\pm i\nu} e^{\pm i\alpha(z)}, \quad p = (z, \pm),\]

where $e^{\pm}(z)$ has continuous limits near $z_j$ and

\[(6.3) \quad \nu = -\frac{1}{2\pi} \log(1 - |R(z_j)|^2) > 0.\]

Here $(z - z_j)^{\pm i\nu} = \exp(\pm i\nu \log(z - z_j))$, where the branch cut of the logarithm is along the negative real axis.

For $p$ near a band edge $E_k \in C(x/t)$,

\[(6.4) \quad T(p, x, t) = T^{\pm 1}(z) \tilde{e}^{\pm}(z), \quad p = (z, \pm),\]

where $\tilde{e}^{\pm}(z)$ is holomorphic near $E_k$ if none of the $\nu_j$ is equal to $E_k$ and $\tilde{e}^{\pm}(z)$ has a first order pole at $E_k = \nu_j$ otherwise.

Proof. Rewrite (5.19) by factorizing the jump as

\[1 - |R(p)|^2 = (1 - |R(z_j)|^2) \frac{1 - |R(p)|^2}{1 - |R(z_j)|^2}.\]

Then consider the Abelian differential $\omega_{pp^*}$ for $p \in \mathcal{K}_g \setminus \{p_\infty\}$ which is given explicitly by (3.21). This gives

\[(6.5) \quad \frac{1}{2} \int_{C(x/t)} \omega_{pp^*} = \pm \log(z - z_j) \pm \alpha(z_j) + O(z - z_j), \quad p = (z, \pm),\]

and thus

\[(6.6) \quad \int_{C(x/t)} \omega_{pp_\infty} = \pm \log(z - z_j) \pm \alpha(z_j) + O(z - z_j), \quad p = (z, \pm),\]

since $\int_{C(x/t)} f \omega_{pp_\infty} = (\int_{C(x/t)} f \omega_{pp^*})/2$ for any symmetric function $f(q) = f(q^*)$. From this, the first claim follows. For the second claim, note that

\[t(p) = \begin{cases} T(z), & p = (z, +) \in \Pi_+, \\ T(z)^{-1}, & p = (z, -) \in \Pi_- \end{cases},\]

satisfies the (holomorphic) Riemann–Hilbert problem

\[t_+(p) = t_-(p)(1 - |R(p)|^2), \quad p \in \Sigma,\]

\[t(p_\infty) = 1.\]

Thus $T(p, x, t)/t(p)$ has no jump along $C(x/t)$ and is therefore holomorphic near $C(x/t)$ away from band edges $E_k = \nu_j$ (where there is a simple pole), by the Schwarz reflection principle.
Moreover, we have the following.

**Lemma 6.2.**

(6.7) \[ e^\pm(z) = \overline{e^\mp(z)}, \quad p = (z, \pm) \in \Sigma \setminus C(x/t) \]

and

(6.8) \[ e^+(z_j) = \exp \left( \text{i} \alpha(z_j) \right) \frac{\theta(z(p_\infty, x, t) + \delta(x/t))}{\theta(z(p_\infty, x, t))} \frac{\theta(z(z_j, x, t))}{\theta(z(z_j, x, t) + \delta(x/t))} \]

\[ \cdot \exp \left( - \sum_{\rho_k \sim \zeta(x/t)} \int_{E(\rho_k)} \omega_{z_j, z_j'} + \frac{1}{4\pi i} \int_{C(x/t)} \log \left( \frac{1 - |R|^2}{1 - |R(z_j)|^2} \right) \omega_{z_j, z_j'} \right), \]

where

(6.9) \[ \alpha(z_j) = \lim_{p \to z_j} \left( \frac{1}{2} \int_{C(x/t)} \omega_{p, p^*} - \log \left( \pi(p) - z_j \right) \right). \]

Here \( \alpha(z_j) \in \mathbb{R} \) and \( \omega_{p, p^*} \) is real on \( C(x/t) \).

**Proof.** The first claim follows from the fact that

\[ T(p^*, x, t) = T(\overline{p}, x, t) = \overline{T(p, x, t)} \quad \text{for} \quad p \in \Sigma \setminus C(x/t). \]

The second claim follows by the argument used in the proof of the Lemma 6.1. \( \square \)

From Lemma 6.1, one deduces that near the stationary phase points the jumps are given by

(6.10) \[
\begin{align*}
\hat{B}_+ &= \begin{pmatrix}
1 & -\left( \frac{\phi''(z_j)}{i}(z - z_j) \right) & 2iv\frac{r}{1 - |r|^2}e^{-t\phi} \\
0 & 1 & 1
\end{pmatrix}, \quad p \in L_{j1}, \\
\hat{B}^{-1}_+ &= \begin{pmatrix}
1 & 0 \\
\frac{\phi''(z_j)}{2i}(z - z_j) & -2iv\frac{r}{1 - |r|^2}e^{t\phi}
\end{pmatrix}, \quad p \in L_{j1}^*, \\
\hat{b}_+ &= \begin{pmatrix}
1 & 0 \\
\frac{\phi''(z_j)}{2i}(z - z_j) & re^{t\phi}
\end{pmatrix}, \quad p \in L_{j2}, \\
\hat{b}^{-1}_+ &= \begin{pmatrix}
1 & 2iv \frac{r}{1 - |r|^2}e^{-t\phi} \\
0 & 1
\end{pmatrix}, \quad p \in L_{j2}^*.
\end{align*}
\]
where (cf. (6.2))

\[(6.11) \quad r = R(z_j)\Theta(z_j, x, t) \frac{e^+(z_j)}{e^+(z_j)} \left( \frac{\phi''(z_j)}{i} \right)^{iv}.\]

The error terms satisfy appropriate Hölder estimates

\[(6.12) \quad \|\hat{B}_+(p) - \hat{B}_+(p)\| \leq C|z - z_j|^\alpha, \quad p = (z, +) \in C_{j1},\]

for any \( \alpha < 1 \), and similarly for the other matrices. Thus the assumptions of [25, Theorem A.1] are satisfied, and we can conclude that the solution on \( \pi \left( \Sigma^C(z_j) \right) \) is of the form

\[(6.13) \quad M(z) = I + \frac{M_0}{z - z_j} \frac{1}{t^{1/2}} + O(t^{-\alpha}),\]

where

\[(6.14) \quad M_0 = \text{i} \sqrt{\frac{1}{\phi''(z_j)}} \begin{pmatrix} 0 & -\beta(t) \\ \beta(t) & 0 \end{pmatrix},\]

\[(6.15) \quad \beta(t) = \sqrt{\text{i} \epsilon} \frac{\pi}{4 - \arg(r) + \arg(\Gamma(\text{i}v))} e^{-\phi(z_j)t^{-iv}},\]

and \( 1/2 < \alpha < 1 \). We now lift this solution in the complex plane back to the small disc \( D \) on the Riemann surface \( \mathcal{K}_8 \) by setting

\[(6.16) \quad M(p) = \begin{cases} 
M(z), & p \in D, \\
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} M(z) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & p \in \mathcal{D}^*. 
\end{cases}\]

Thus we conclude that the solution on \( \Sigma^C(z_j) \) is given by

\[(6.17) \quad M^C(p) = I + \frac{1}{t^{1/2}} \frac{M_0}{z - z_j} + O(t^{-\alpha}), \quad p = (z, +),\]

and the one on \( \Sigma^C(z_j^*) \) by

\[(6.18) \quad \tilde{M}^C(p) = I + \frac{1}{t^{1/2}} \frac{\tilde{M}_0}{z - z_j} + O(t^{-\alpha}), \quad p = (z, -).\]

Then

\[(6.19) \quad m^4(p) = \begin{cases} 
m^3(p)M^C(p)^{-1}, & p \in D, \\
m^3(p)\tilde{M}^C(p)^{-1}, & p \in \mathcal{D}^*, \\
m^3(p), & \text{otherwise}, 
\end{cases}\]
has no jump inside $D \cup D^*$ but jumps on the boundary $\partial D \cup \partial D^*$. All jumps outside $D \cup D^*$ are of the form $\mathbb{I}+\mathbb{I}$ exponentially small, and the jump on $\partial D \cup \partial D^*$ is of the form $\mathbb{I} + O(t^{-1/2})$. In order to identify the leading behavior, it remains to rewrite the Riemann–Hilbert problem for $m^4$ as a singular integral equation, following Section A. Let the operator $C_{w^4}: L^2(\Sigma^4) \to L^2(\Sigma^4)$ be defined by

$$C_{w^4}f = C_-(fw^4)$$

for a vector valued $f$, where $w^4 = J^4 - I$ and

$$(C_{\pm}f)(q) = \lim_{p \to q \in \Sigma^4} \frac{1}{2\pi i} \int_{\Sigma^4} f \frac{\Omega^0_p}{\Omega^0_\nu_p}, \quad \Omega^0_p = \begin{pmatrix} \Omega^0_{p,q} & 0 \\ 0 & \Omega^0_{p,p} \end{pmatrix},$$

are the Cauchy operators for our Riemann surface. In particular, $\Omega^0_{p,q}$ is the Cauchy kernel given by

$$\Omega^0_{p,q} = \omega_{p,q} + \sum_{j=1}^g I^q_j(p)\zeta_j,$$

where

$$(C_{\pm}f)(q) = \lim_{p \to q \in \Sigma^4} \frac{1}{2\pi i} \int_{\Sigma^4} f \frac{\Omega^0_p}{\Omega^0_\nu_p}, \quad \Omega^0_p = \begin{pmatrix} \Omega^0_{p,q} & 0 \\ 0 & \Omega^0_{p,p} \end{pmatrix},$$

are the Cauchy operators for our Riemann surface. In particular, $\Omega^0_{p,q}$ is the Cauchy kernel given by

$$\Omega^0_{p,q} = \omega_{p,q} + \sum_{j=1}^g I^q_j(p)\zeta_j,$$

where

$$I^q_j(p) = \sum_{\ell=1}^g c_{j\ell}(\tilde{\nu}) \int_{\Sigma^4} \frac{\omega_{0,\ell,0}}{\omega_{\nu,0}},$$

Here $\omega_{q,0}$ is the (normalized) Abelian differential of the second kind with a second order pole at $q$ (cf. Remark 6.4 below), and $\omega_{p,q}$ denotes the Abelian differential of the third kind with simple poles at $p$ and $q$. Note that $I^q_j(p)$ has first order poles at the points $\tilde{\nu}$.

The constants $c_{j\ell}(\tilde{\nu})$ are chosen such that $\Omega^0_{p,q}$ is single valued, that is,

$$c_{\ell k}(\tilde{\nu})_{1 \leq \ell, k \leq g} = \left( \sum_{j=1}^g c_{j\ell}(\tilde{\nu}) \mu_{\ell}^{j-1} \frac{d\mu_{\ell}}{R_{2g+2}(\hat{\mu}_{\ell})} \right)^{-1},$$

where $c_{\ell}(j)$ are defined in (3.6) (cf. Lemma A.3).

Consider the solution $\mu^4$ of the singular integral equation

$$\mu = \begin{pmatrix} 1 & 1 \\ C_{w^4} & \mu \end{pmatrix} in L^2(\Sigma^4).$$

The solution of our Riemann–Hilbert problem is given by

$$m^4(p) = \begin{pmatrix} 1 & 1 \\ \frac{1}{2\pi i} \int_{\Sigma^4} \mu^4 w^4 \Omega^0_p \end{pmatrix}.$$
By $\|w^4\|_\infty = O(t^{-1/2})$, Neumann’s formula implies

$$\mu^4(q) = (I - C_{w^4})^{-1} \begin{pmatrix} 1 & 1 \end{pmatrix} + O(t^{-1/2}).$$

Moreover,

$$w^4(p) = \begin{cases} \frac{M_0}{z-z_j} + O(t^{-\alpha}), & p \in \partial D, \\ \frac{-M_0}{z-z_j} + O(t^{-\alpha}), & p \in \partial D^*. \end{cases}$$

Hence we obtain

$$m^4(p) = \begin{pmatrix} 1 & 1 \end{pmatrix} - \frac{1}{2\lambda_1} \int_{\partial D} \frac{1}{\pi - z_j} \frac{\Omega_\nu^{\hat{\nu}}}{\nu}$$

$$= \begin{pmatrix} 1 & 1 \end{pmatrix} - \frac{1}{2\lambda_1} \int_{\partial D^*} \frac{1}{\pi - z_j} \frac{\Omega_\nu^{\hat{\nu}}}{\nu} + O(t^{-\alpha})$$

$$= \begin{pmatrix} 1 & 1 \end{pmatrix} - \frac{1}{\sqrt{\phi''(z_j)t}} \times$$

$$\times \left( i\beta \Omega_p^{\hat{\nu},p_\infty}(z_j) - i\beta \Omega_p^{\hat{\nu},p_\infty}(z_j^0) - i\beta \Omega_p^{\hat{\nu},p_\infty}(z_j) + i\beta \Omega_p^{\hat{\nu},p_\infty}(z_j^0) \right)$$

$$+ O(t^{-\alpha}).$$

Since we need the asymptotic expansions around $p_\infty$, we note the following.

**Lemma 6.3.**

$$\Omega_p^{\hat{\nu},p_\infty}(z_j) = \Lambda_1^{\hat{\nu}}(z_j) \zeta + \Lambda_2^{\hat{\nu}}(z_j) \zeta^2 + O(\zeta^3)$$

for $\zeta = z^{-1/2}$ being the local chart near $p_\infty$ and

$$\Lambda_1^{\hat{\nu}}(z_j) = \omega_{p_\infty,0}(z_j) - \sum_{k=1}^{g} \sum_{\ell=1}^{r} c_{k\ell}(\hat{\nu})\alpha_{g-1}(\hat{\nu}_\ell)\zeta_k(z_j),$$

$$\Lambda_2^{\hat{\nu}}(z_j) = \omega_{p_\infty,1}(z_j) - \frac{1}{2} \sum_{k=1}^{g} \sum_{\ell=1}^{r} c_{k\ell}(\hat{\nu})\zeta_k(z_j),$$

where $\omega_{q,k}$, $k = 0, 1, \ldots$, is an Abelian differential of the second kind with a single pole of order $k + 2$ at $q$ and $\alpha_{g-1}(\hat{\nu}_\ell)$ denotes a constant defined in Remark 6.4 below.
**Proof.** Use the local coordinate \( \zeta = z^{-1/2} \) near \( p_\infty = (\infty, \infty) \) and expand the differential \( \omega_{pp_\infty} \) as is done in [34, Theorem 4.1] and \( \int_{p_\infty}^p \omega_{0,0} \) using the expression (6.33). For \( g \geq 1 \), one obtains \( \omega_{0,0}(\zeta) = -\alpha_{g-1}(\hat{\nu}) - \zeta + O(\zeta^2) \), and the claimed formulas for \( \Lambda_{1}(z_j) \) and \( \Lambda_{2}(z_j) \) follow. \( \square \)

**Remark 6.4.** The Abelian differential appearing in Lemma 6.3 is given explicitly by

(6.33) \[ \omega_{0,0} = \frac{R_{2g+1}^{1/2} + R_{2g+1}^{1/2}(\hat{\nu}) + \frac{R_{2g+1}^{1/2}(\hat{\nu})}{2R_{2g+1}^{1/2}(\hat{\nu})} (\pi - \nu) + P_{0,0} \cdot (\pi - \nu)^2}{2(\pi - \nu)^2 R_{2g+1}^{1/2}} d\pi, \]

with \( P_{0,0} \) a polynomial of degree \( g - 1 \) which has to be determined from the normalization. We let

(6.34) \[ P_{0,0}(z) = \sum_{j=0}^{g-1} \alpha_j(\hat{\nu}) z^j. \]

Concerning the Abelian differential \( \omega_{p_\infty,0} \), we refer to (3.15). The differential \( \omega_{p_\infty,1} \) is given by

(6.35) \[ \omega_{p_\infty,1} = \left( - \frac{R_{2g+1}^{1/2}}{2} + P_{p_\infty,1} \right) \frac{d\pi}{R_{2g+1}^{1/2}}, \]

where \( P_{p_\infty,1} \) is a polynomial of degree \( g - 1 \) which has to be determined as usual by the vanishing \( a_j \)-periods.

Note the relations

(6.36) \[ \omega_{p_\infty,0}(z_j^*) = -\omega_{p_\infty,0}(z_j), \]
\[ \omega_{p_\infty,1}(z_j^*) + \omega_{p_\infty,1}(z_j) = -1, \]

and

(6.37) \[ c_{kl}(\hat{\nu}^*) = -c_{kl}(\hat{\nu}), \quad \zeta_{k}(z_j^*) = -\zeta_{k}(z_j). \]

Moreover, the coefficients \( a_j(\hat{\nu}), j = 0, \ldots, g - 1 \) of the polynomial \( P_{\hat{\nu},0} \) satisfy

(6.38) \[ a_j(\hat{\nu}^*) = -a_j(\hat{\nu}), \quad j = 0, \ldots, g - 1. \]

We come to the proof of our main result.

**Proof of Theorem 2.2.** As in the previous section, the asymptotic behavior can be read off by using

(6.39) \[ m(p) = m^4(p) \begin{pmatrix} 1 & 0 \\ 1 & 1/T(p^*,x,t) \end{pmatrix}, \]
for $p$ near $p_\infty$ and comparing with (4.13). From that one deduces

$$m_2(p) = m_2^4(p) T(p)^{-1}$$

$$= 1 + \left( \sqrt{\frac{i}{\phi''(z_j)t}} (i\beta \Lambda^\phi_T(z_j) - i\beta \Lambda^\phi_T(z^*_j)) - T_1(x, t) + O(t^{-\alpha}) \right) \frac{1}{\sqrt{z}} + O(z^{-1}),$$

where we have used (6.29), (6.30) and (5.21). Comparing this asymptotic expansion with (4.13) yields

$$\int_x^{+\infty} (V - V_q)(y)dy = 2 \sqrt{\frac{i}{\phi''(z_j)\xi}} (\beta \Lambda^\phi_T(z_j) - \beta \Lambda^\phi_T(z^*_j)) + 2i T_1(x, t) + O(t^{-\alpha}).$$

Invoking (6.36), (6.37) and (6.38) gives

$$\Lambda^\phi_T(z_j) = -\Lambda^\phi_T(z_j),$$

which therefore

$$(6.40) \int_x^{+\infty} (V - V_q)(y, t)dy = 4 \sqrt{\frac{i}{\phi''(z_j)t}} \text{Re} \{\beta(x, t)\} \Lambda^\phi_T(x, t) + 2i T_1(x, t) + O(t^{-\alpha}).$$

Finally, using the definition of the limiting solution (2.8) proves the claim. Note that one obtains the same result by comparing the expressions for the component $m_1$. 

Similarly one obtains (2.17) by using (4.27) instead of (4.13). 

7 Analytic approximation

In this section, we show how to get rid of the analyticity assumption on the reflection coefficient $R(p)$. To this end, following the ideas of [11] (see also [25, Sect. 6]), we split $R(p)$ into an analytic part $R_{a,t}$ and a small residual $R_{r,t}$. The analytic part is moved to regions of the Riemann surface, while $R_{r,t}$ remains on $\Sigma = \pi^{-1}(\sigma(H_q))$. This needs to be done in such a way that the rest is of $O(t^{-1})$ and the growth of the analytic part can be controlled by the decay of the phase.

In order to avoid problems when one of the poles $\nu_j$ hits $\Sigma$, we have to make the approximation in such a way that the nonanalytic $R_{r,t}$ vanishes at the band edges. That is, we split $R$ as

$$R(p) = R(E_{2j}) \frac{(z - E_{2j})(E_{2j+1} - i)}{(E_{2j+1} - E_{2j})(z - i)} + R(E_{2j+1}) \frac{(z - E_{2j+1})(E_{2j} - i)}{(E_{2j} - E_{2j+1})(z - i)} + Q_j(p) \tilde{R}(p), \quad p = (z, \pm),$$

(7.1)
where $Q_j(p)$ is a rational function with first order zeros at $E_{2j}, E_{2j+1}$ and with all other zeros and poles away from $\Sigma$, and approximate $\hat{R}$. Note that if $R \in C^I(\Sigma)$, then $\hat{R} \in C^{I-1}(\Sigma)$. We use different splittings for different bands depending on whether the band contains our stationary phase point $z_j(x/t)$.

We begin with some preparatory lemmas. For the bands containing no stationary phase points, we split $R$ based on the Fourier transform associated with the background operator $H_q$. Given $R \in C^I(\Sigma)$, we can write

\begin{equation}
R(p) = \int_\mathbb{R} \hat{R}(x) \psi_q(p, x, 0) dx,
\end{equation}

where $\psi_q(p, x, t)$ denotes the time-dependent Baker–Akhiezer function and (cf. [12], [13])

\begin{equation}
\hat{R}(x) = \frac{1}{2\pi i} \oint_{\Sigma} R(p) \psi_q(p, x, 0) \frac{i \prod_{j=1}^s (\pi(p) - \mu_j)}{2R_{2j+1}^{1/2}(p)} d\pi(p).
\end{equation}

If we make use of (3.14), the above expression for $R(p)$ takes the form

\begin{equation}
R(p) = \int_\mathbb{R} \hat{R}(x) \theta_q(p, x, 0) \exp(ik(x/p)) dx,
\end{equation}

where $k(p) = -i \int_{E_0}^{p} \omega_{p,\infty,0}$ and $\theta_q(p, n, t)$ collects the remaining parts in (3.14). Using $k(p)$ as a new coordinate and performing $l$ integration by parts yields (cf. [13])

\begin{equation}
|\hat{R}(x)| \leq \frac{\text{const}}{1 + |x|^l},
\end{equation}

provided that $R \in C^I(\Sigma)$.

**Lemma 7.1.** Suppose $\hat{R} \in L^1(\mathbb{R})$, $x^l \hat{R}(x) \in L^1(\mathbb{R})$ and let $\beta > 0$. Then $R(p) = R_{a,l}(p) + R_{r,l}(p)$, where $R_{a,l}(p)$ is analytic in the region $0 < \text{Im}(k(p)) < \varepsilon$, and

\begin{equation}
|R_{a,l}(p)e^{-\beta t}| = O(t^{-l}), \quad 0 < \text{Im}(k(p)) < \varepsilon,
\end{equation}

\begin{equation}
|R_{r,l}(p)| = O(t^{-l}), \quad p \in \Sigma.
\end{equation}

**Proof.** Let

\begin{equation*}
R_{a,l}(p) = \int_{x=-K(t)}^{\infty} \hat{R}(x) \theta_q(p, x, 0) \exp(ik(x/p)) dx
\end{equation*}

with $K(t) = \beta_0 t/\varepsilon$ for some positive $\beta_0 < \beta$. Then for $0 < \text{Im}(k(p)) < \varepsilon$,

\begin{equation*}
|R_{a,l}(k)e^{-\beta t}| \leq Ce^{-\beta t} \int_{x=-K(t)}^{\infty} |\hat{R}(x)|e^{-\text{Im}(k(p))x} dx
\end{equation*}

\begin{equation*}
\leq Ce^{-\beta t} e^{K(t)e}\|F\|_1 = \|\hat{R}\|_1 e^{-(\beta-\beta_0)t},
\end{equation*}

where $\|F\|_1$ is the $l^1$ norm of $F$. 

Note that if $R \in C^I(\Sigma)$, then $\hat{R} \in C^{I-1}(\Sigma)$. We use different splittings for different bands depending on whether the band contains our stationary phase point $z_j(x/t)$.
which proves the first claim. Similarly, for \( p \in \Sigma, \)
\[
|R_{r,t}(k)| \leq C \int_{x=K(t)}^{\infty} \frac{x' |\hat{R}(-x)|}{x^l} dx \leq C \frac{\|x' \hat{R}(-x)\|_1}{K(t)^l} \leq \frac{\tilde{C}}{t^l}.
\]

For the band that contains \( z_j(x/t), \) we need to take the small vicinities of the stationary phase points into account. Since the phase is cubic near these points, we cannot use it to dominate the exponential growth of the analytic part away from \( \Sigma. \) Hence we take the phase as a new variable and use the Fourier transform with respect to this new variable. Since this change of coordinates is singular near the stationary phase points, there is a price to be paid, namely, additional smoothness requirements for \( R(p). \)

Without loss of generality, we choose the path of integration in our phase \( \phi(p), \) defined in (4.21), such that \( \phi(p) \) is continuous (and thus analytic) in \( D_{j,1} \) with continuous limits on the boundary (cf. Figure 1).

**Lemma 7.2.** Suppose \( R(p) \in C^5(\Sigma). \) Then
\[
R(p) = R_0(p) + \frac{z - z_j}{z - i} H(p), \quad p = (z, \pm) \in \Sigma \cap D_{j,1},
\]
where \( R_0(p) \) is a real rational function on \( \mathbb{M} \) such that \( H(p) \) vanishes at \( z_j, z_j^* \) to order three and has a Fourier transform
\[
H(p) = \int_{\mathbb{R}} \hat{H}(x)e^{x\phi(p)} dx,
\]
with \( x\hat{H}(x) \) integrable. Here \( \phi \) denotes the phase defined in (4.21).

**Proof.** We begin by choosing a rational function \( R_0(p) = a(z) + b(z)R_{2g+1}(z) \)
with \( p = (z, \pm) \) such that \( a(z), b(z) \) are real-valued rational functions which are chosen such that \( a(z) \) matches the values of \( \text{Re}(R(p)) \) and its first four derivatives at \( z_j \) and \( i^{-1}b(z)R_{2g+1}(z) \) matches the values of \( \text{Im}(R(p)) \) and its first four derivatives at \( z_j, \) respectively. Moreover, all poles are chosen away from \( \Sigma. \) Since \( R(p) \) is \( C^5, \) we infer that \( H(p) \in C^4(\Sigma) \) and vanishes together with its first three derivatives at \( z_j, z_j^*. \)

Note that the phase \( \phi(p)/i \) has a maximum at \( z_j^* \) and a minimum at \( z_j. \) Thus \( \phi(p)/i \) restricted to \( \Sigma \cap D_{j,1} \) gives a one-to-one coordinate transformation \( \Sigma \cap D_{j,1} \rightarrow [\phi(z_j^*)/i, \phi(z_j)/i]; \) hence we express \( H(p) \) in this new coordinate (setting it equal to zero outside this interval). The coordinate transformation looks locally like a cube root near \( z_j \) and \( z_j^*. \) However, by our assumption that \( H \) vanishes there, \( H \) is still \( C^2 \) in this new coordinate, and the Fourier transform with respect to this new coordinates exists and has the required properties. \( \square \)
Moreover, as in Lemma 7.1 we obtain the following.

**Lemma 7.3.** Let \( H(p) \) be as in Lemma 7.2. Then \( H(p) = H_{a,l}(p) + H_{r,l}(p) \), where \( H_{a,l}(p) \) is analytic in the region \( \text{Re}(\phi(p)) < 0 \) and

\[
|H_{a,l}(p)e^{\phi(p)t/2}| = O(1), \quad p \in \overline{D}, \quad |H_{r,l}(p)| = O(t^{-1}), \quad p \in \Sigma.
\]

**Proof.** Choose \( H_{a,l}(p) = \int_{x=-K(t)}^{\infty} \hat{H}(x)e^{t\phi(p)}dx \) with \( K(t) = t/2 \). Then, proceeding as in Lemma 7.1, we obtain

\[
|H_{a,l}(p)e^{\phi(p)t/2}| \leq \|\hat{H}\|_1 e^{-K(t)(p)x/2} \leq \|\hat{H}\|_1
\]

and

\[
|H_{r,l}(p)| \leq \frac{1}{K(t)} \int_{x=K(t)}^{\infty} x|\hat{H}(x)|dx \leq \frac{C}{t}.
\]

Clearly an analogous splitting exists for \( p \in \Sigma \cap D_j \).

Now we are ready for our analytic approximation step. First recall that our jump is given in terms of \( \tilde{b}_\pm \) and \( \tilde{B}_\pm \) defined in (5.14) and (5.16), respectively. While \( \tilde{b}_\pm \) are already in the correct form for our purpose, \( \tilde{B}_\pm \) are not, since they contain the nonanalytic expression \( |T(p)|^2 \). To remedy this, we rewrite \( \tilde{B}_\pm \) in terms of the left rather than the right scattering data. For this purpose, let us write \( R_r(p) \equiv R_+(p) \) and \( R_l(p) \equiv R_-(p) \) for the right and left reflection coefficients, respectively. Moreover, let \( T_r(p, x, t) \equiv T(p, x, t) \) and \( T_l(p, x, t) \equiv T(p)/T_r(p, x, t) \) be the right and left partial transmission coefficients, respectively.

With this notation,

\[
J^2(p) = \begin{cases} 
\tilde{b}_-(p)^{-1}\tilde{b}_+(p), & \pi(p) > \zeta(x/t), \\
\tilde{B}_-(p)^{-1}\tilde{B}_+(p), & \pi(p) < \zeta(x/t),
\end{cases}
\]

where

\[
\tilde{b}_- = \begin{pmatrix} T_r(p, x, t) & 1 \\
0 & 1
\end{pmatrix} R_r(p) \Theta(p^*) e^{-t\phi(p)}
\]

\[
\tilde{b}_+ = \begin{pmatrix} T_r(p^*, x, t) & 1 \\
0 & 1
\end{pmatrix} R_r(p) \Theta(p) e^{-t\phi(p)}
\]

and

\[
\tilde{B}_- = \begin{pmatrix} 1 & 0 \\
-T_r(p, x, t) R_r(p) \Theta(p^*) e^{-t\phi(p)} & 1
\end{pmatrix}
\]

\[
\tilde{B}_+ = \begin{pmatrix} 1 & 0 \\
-T_r(p^*, x, t) R_r(p^*) \Theta(p) e^{-t\phi(p)} & 1
\end{pmatrix}
\]
Using (4.8), we can write

\[
\tilde{B}_- = \begin{pmatrix}
\frac{1}{\tilde{T}(p^{*}, x)} R(p) \Theta(p) e^{-i\phi(p)} & 0 \\
0 & 1
\end{pmatrix},
\]

\[
\tilde{B}_+ = \begin{pmatrix}
1 & \frac{1}{\tilde{T}(p^{*}, x)} R(p^*) \Theta(p^*) e^{-i\phi(p)} \\
0 & 1
\end{pmatrix}.
\]

We now write \( R_r(p) = R_{a,r}(p) + R_{r,t}(p) \) by splitting \( \tilde{R}_r(p) \) defined via (7.1) according to Lemma 7.1 for \( \pi(p) \in [E_{2k}, E_{2k+1}] \) with \( k < j \) (i.e., not containing \( \zeta(x/t) \)) and according to Lemma 7.3 for \( \pi(p) \in [E_{2j}, \zeta(x/t)] \). In the same way, we write \( R_l(p) = R_{a,l}(p) + R_{l,t}(p) \) for \( \pi(p) \in [\zeta(x/t), E_{2j+1}] \) and \( \pi(p) \in [E_{2k}, E_{2k+1}] \) with \( k > j \). For \( \beta \) in Lemma 7.1, we choose

\[
(7.12) \quad \beta = \begin{cases}
\min_{p \in C_k} -\text{Re}(\phi(p)) > 0, & \pi(p) > \zeta(x/t), \\
\min_{p \in C_1} \text{Re}(\phi(p)) > 0, & \pi(p) < \zeta(x/t).
\end{cases}
\]

In this way, we obtain

\[
\tilde{b}_\pm(p) = \tilde{b}_{a,t,\pm}(p) \tilde{b}_{r,t,\pm}(p) = \tilde{b}_{r,t,\pm}(p) \tilde{b}_{a,t,\pm}(p),
\]

\[
\tilde{B}_\pm(p) = \tilde{B}_{a,t,\pm}(p) \tilde{B}_{r,t,\pm}(p) = \tilde{B}_{r,t,\pm}(p) \tilde{B}_{a,t,\pm}(p).
\]

Here \( \tilde{b}_{a,t,\pm}(p), \tilde{b}_{r,t,\pm}(p) \) denote the matrices obtained from \( \tilde{b}_\pm(p) \) by replacing \( R_r(p) \) with \( R_{a,r}(p), R_{r,t}(p) \), respectively. Similarly for \( \tilde{B}_{a,t,\pm}(p), \tilde{B}_{r,t,\pm}(p) \). We can now move the analytic parts into regions of the Riemann surface as in Section 5, while leaving the rest on \( \Sigma \). Hence, rather than satisfying (5.52), the jump now reads

\[
(7.13) \quad J^3(p) = \begin{cases}
\tilde{b}_{a,t,\pm}(p), & p \in C_k, \quad \pi(p) > \zeta(x/t), \\
\tilde{b}_{a,t,-}(p)^{-1}, & p \in C_k^*, \quad \pi(p) > \zeta(x/t), \\
\tilde{b}_{r,t,-}(p)^{-1}\tilde{b}_{r,t,+}(p), & p \in \pi^{-1}(\zeta(x/t), +\infty)), \\
\tilde{B}_{a,t,\pm}(p), & p \in C_k, \quad \pi(p) < \zeta(x/t), \\
\tilde{B}_{a,t,-}(p)^{-1}, & p \in C_k^*, \quad \pi(p) < \zeta(x/t), \\
\tilde{B}_{r,t,-}(p)^{-1}\tilde{B}_{r,t,+}(p), & p \in \pi^{-1}(\zeta(x/t), +\infty)).
\end{cases}
\]

By construction, \( R_{a,t}(p) = R_0(p) + (\pi(p) - \pi(z_j))H_{a,t}(p) \) satisfies the required Lipschitz estimate in a vicinity of the stationary phase points (uniformly in \( t \)), and the jump is \( J^3(p) = I + O(t^{-1}) \). The remaining parts of \( \Sigma \) can be handled analogously, and hence we can proceed as in Section 6.

**A Appendix: Singular integral equations**

The solution of a Riemann–Hilbert problem in the complex plane can be reduced to the solution of a singular integral equation (see [3]). In our case, the underlying
space is a Riemann surface $\mathbb{M}$. The purpose of this appendix is to generalize this reduction to a meromorphic vector Riemann–Hilbert problem with simple poles at $\rho, \rho^*$ of the type

$$m_+(p) = m_-(p)J(p), \quad p \in \Sigma,$$

$$(m_1) \geq -D_{\hat{\mu}} - D_\rho, \quad (m_2) \geq -D_{\hat{\mu}} - D_{\rho^*},$$

$$\left( m_1(p) - \frac{R_{2^2+2}(\rho)}{\prod_{k=1}^g \gamma_j \psi_q(p)} \frac{\gamma_j}{\psi_q(p)} m_2(p) \right) \geq -D_{\hat{\mu}}^* \text{ near } \rho,$$

$$\left( -\frac{R_{2^2+2}(\rho)}{\prod_{k=1}^g \gamma_j \psi_q(p)} \frac{\gamma_j}{\psi_q(p)} m_1(p) + m_2(p) \right) \geq -D_{\hat{\mu}}^* \text{ near } \rho^*,$$

$$m(p^*) = m(p) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$m(p_\infty) = \begin{pmatrix} 1 & 1 \end{pmatrix},$$

Since we require the symmetry condition (4.18) for our Riemann–Hilbert problems, we need to adapt the usual Cauchy kernel to preserve this symmetry. Moreover, we keep the single soliton as an inhomogeneous term which plays the role of the leading asymptotics in our applications.

Concerning the jump contour $\Sigma$ and the jump matrix $J$, we make assumptions contained in the following.

**Hypothesis A.1.** The contour $\Sigma$ consists of a finite number of smooth oriented finite curves in $\mathbb{M}$ that intersect at most finitely many times, each intersection being transversal. $\Sigma$ does not contain any of the points $\hat{\mu}$ and is invariant under $p \mapsto p^*$. It is oriented so that under the mapping $p \mapsto p^*$, sequences converging from the positive side to $\Sigma$ are mapped to sequences converging to the negative side. The divisor $D_{\hat{\mu}}$ is nonspecial.

The jump matrix $J$ is nonsingular and can be factorized as $J = b_+^{-1}b_+ = (I - w_-)^{-1}(I + w_+)$, where $w_\pm = \pm(b_\pm - I)$ are Hölder continuous and satisfy

$$w_\pm(p^*) = -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} w_\pm(p) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad p \in \Sigma.$$

Moreover,

$$\|w\|_\infty = \|w_+\|_{L^\infty(\Sigma)} + \|w_-\|_{L^\infty(\Sigma)} < \infty.$$

**Remark A.2.** The assumption that that none of the poles $\hat{\mu}$ lies on $\Sigma$ can be made without loss of generality if the jump is analytic, since we can perturb $\Sigma$.
slightly without changing the value at \( p_\infty \) (which is the only value we are eventually interested in). Alternatively, the case where one (or more) of the poles \( \hat{\mu}_j \) lies on \( \Sigma \) can be included if one assumes that \( w_\pm \) has a first order zero at \( \hat{\mu}_j \). In fact, in this case, one can replace \( \mu(s) \) with \( \bar{\mu}(s) = (\pi(s) - \mu_j)^{-1} w_\pm(s) \).

Otherwise one could also assume that the matrices \( w_\pm \) are Hölder continuous and vanish at such points. Then one can work with the weighted measure \( -iK^{1/2}_{2g+1}(p)d\pi \) on \( \Sigma \). In fact, one can show that the Cauchy operators are still bounded in this weighted Hilbert space (cf. [19, Thm. 4.1]).

Our first step is to replace the classical Cauchy kernel with a "generalized" Cauchy kernel appropriate to our Riemann surface. In order to get a single valued kernel, we again need to admit \( g \) poles. We follow the construction from [30, Sec. 4].

**Lemma A.3 ([22, 25]).** Let \( \mathcal{D}_{\hat{\mu}} \) be nonspecial and introduce the differential

\[
\Omega^\hat{\mu}_\rho = \omega_{p,\rho} + \sum_{j=1}^{g} I_j^\hat{\mu}_\rho(p)\zeta_j,
\]

where

\[
I_j^\hat{\mu}_\rho(p) = \sum_{\ell=1}^{g} c_{j\ell}(\hat{\mu}) \int_{\rho}^{p} \omega_{\hat{\mu}_\ell,0}.
\]

Here \( \omega_{q,0} \) is the (normalized) Abelian differential of the second kind with a second order pole at \( q \) (cf. Remark 6.4) and the matrix \( c_{j\ell} \) is defined as the inverse matrix of \( \eta_\ell(\hat{\mu}_j) \), where \( \zeta_\ell = \eta_\ell(z)dz \) is the chart expression in a local chart near \( \hat{\mu}_j \) (the same chart used to define \( \omega_{\hat{\mu}_\ell,0} \)).

Then \( \Omega^\hat{\mu}_\rho \) is single valued as a function of \( p \) with first order poles at the points \( \hat{\mu} \).

Next we show that the Cauchy kernel introduced in (A.4) indeed has the correct properties. We write \( L^p(\Sigma) = L^p(\Sigma, \mathbb{C}^2) \).

**Theorem A.4 ([22, 25]).** Set

\[
\Omega^{\hat{\mu}_\rho}_p = \begin{pmatrix} \Omega^\hat{\mu}_\rho & 0 \\ 0 & \Omega^\hat{\mu}_\rho \end{pmatrix}
\]

and define the matrix operators as follows. Given a \( 2 \times 2 \) matrix \( f \) defined on \( \Sigma \) with Hölder continuous entries, let

\[
(Cf)(p) = \frac{1}{2\pi i} \int_{\Sigma} f(s)\Omega^{\hat{\mu}_\rho}_p, \quad \text{for} \quad p \notin \Sigma,
\]
and

\[(C \pm f)(q) = \lim_{p \to q \in \Sigma} (Cf)(p)\]

from the left and right of \(\Sigma\), respectively (with respect to its orientation). Then

(i) The operators \(C \pm\) are given by the Plemelj formulas

\[
(C_+ f)(q) - (C_- f)(q) = f(q),
\]

\[
(C_+ f)(q) + (C_- f)(q) = \frac{1}{\pi i} \int_{\Sigma} f \, \Omega_{\tilde{\mu}, \tilde{\rho}}^{-, \rho},
\]

and extend to bounded operators on \(L^2(\Sigma)\). Here \(\tilde{f}\) denotes the principal value integral, as usual, and the bound can be chosen independently of the divisor as long as it stays some finite distance away from \(\Sigma\).

(ii) \(Cf\) is a meromorphic function off \(\Sigma\) with divisor given by \(((Cf)_{j1}) \geq -D_{\tilde{\mu}}\), and \(((Cf)_{j2}) \geq -D_{\tilde{\rho}}\).

(iii) \(((Cf)(\rho^*)) = (0 \ast), \quad (Cf)(\rho) = (\ast 0)\).

Furthermore, \(C\) restricts to \(L^2_s(\Sigma)\), that is,

\[(A.9) \quad (Cf)(p^*) = (Cf)(p) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad p \in \mathbb{M} \setminus \Sigma\]

for \(f \in L^2_s(\Sigma)\) and if \(w_{\pm}\) satisfy (A.2), we also have

\[(A.10) \quad C_{\pm}(fw_{\pm})(p^*) = C_{\pm}(fw_{\pm})(p) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad p \in \Sigma.\]

Now let the operator \(C_w : L^2_s(\Sigma) \to L^2_s(\Sigma)\) be defined by

\[(A.11) \quad C_w f = C_+(fw_-) + C_-(fw_+), \quad f \in L^2_s(\Sigma)\]

for a 2 \times 2 matrix valued \(f\), where \(w_+ = b_+ - \mathbb{I}\) and \(w_- = \mathbb{I} - b_-\). Recall from Lemma 4.6 that the unique solution corresponding to \(J \equiv \mathbb{I}\) is given by \(m_0(p) = (f(p^*) \quad f(p))\) for some given \(f(p)\) with \((f) \geq -D_{\tilde{\mu}} - D_{\tilde{\rho}}\). Since we assumed \(D_{\tilde{\mu}}\) to be away from \(\Sigma\), we clearly have \(m_0 \in L^2_s(\Sigma)\).

**Theorem A.5** ([22, 25]). Assume Hypothesis A.1, and let \(m_0 \in C^2\).

Suppose \(m\) solves the Riemann–Hilbert problem (A.1). Then

\[(A.12) \quad m(p) = (1 - c_0)m_0(p) + \frac{1}{2\pi i} \int_{\Sigma} \mu(s)(w_+(s) + w_-(s))\Omega_{\tilde{\mu}, \tilde{\rho}}^{\tilde{\rho}, \rho},\]

where

\[
\mu = m_+b_+^{-1} = m_-b_-^{-1} \quad \text{and} \quad c_0 = \left(\frac{1}{2\pi i} \int_{\Sigma} \mu(s)(w_+(s) + w_-(s))\Omega_{\tilde{\mu}, \tilde{\rho}}^{\tilde{\rho}, \rho}\right).\]
Here \((m)_{j}\) denotes the \(j\)'th component of a vector. Furthermore, \(\mu\) solves

\[
(\mathbb{I} - C_w)\mu = (1 - c_0)m_0(p).
\]

Conversely, suppose \(\tilde{\mu}\) solves

\[
(\mathbb{I} - C_w)\tilde{\mu} = m_0,
\]

and that

\[
\tilde{c}_0 = \left( \frac{1}{2\pi i} \int \tilde{\mu}(s)(w_+(s) + w_-(s))\Omega_{\rho_\infty}^{\tilde{\mu}} \right) \neq 0.
\]

Then \(m\) defined via (A.12) with \((1 - c_0) = (1 - \tilde{c}_0)^{-1}\) and \(\mu = (1 - \tilde{c}_0)^{-1}\tilde{\mu}\) solves the Riemann–Hilbert problem (A.1) and \(\mu = m_{\pm}b_{\pm}^{-1}\).

Hence we have a formula for the solution of our Riemann–Hilbert problem \(m(z)\) in terms of \((\mathbb{I} - C_w)^{-1}m_0\). This clearly raises the question of bounded invertibility of \(\mathbb{I} - C_w\) which is answered by Fredholm theory (cf., e.g., [38]):

**Lemma A.6** ([22, 25]). Assume Hypothesis A.1. Then the operator \(\mathbb{I} - C_w\) is Fredholm of index zero,

\[
\text{ind}(\mathbb{I} - C_w) = 0.
\]

By the Fredholm alternative, it follows that showing bounded invertibility of \(\mathbb{I} - C_w\) requires only showing that \(\ker(\mathbb{I} - C_w) = 0\), the latter being equivalent to unique solvability of the corresponding vanishing Riemann–Hilbert problem.

**Corollary A.7.** Under Hypothesis A.1, a unique solution of the Riemann–Hilbert problem (A.1) exists if and only if the corresponding vanishing Riemann–Hilbert problem, where the normalization condition is given by \(m(p_\infty) = \begin{pmatrix} 0 & 0 \end{pmatrix}\), has at most one solution.

We are interested in comparing two Riemann–Hilbert problems associated with respective jumps \(w_0\) and \(w\) with \(\|w - w_0\|_\infty\) small, where

\[
\|w\|_\infty = \|w_+\|_{L^\infty(\Sigma)} + \|w_-\|_{L^\infty(\Sigma)}.
\]

For such a situation we have the following result.

**Theorem A.8** ([24]). Assume that for some data \(w'_0\), the operator

\[
\mathbb{I} - C_{w'_0} : L^2(\Sigma) \to L^2(\Sigma)
\]

has a bounded inverse, where the bound is independent of \(t\).
Furthermore, assume $w^t$ satisfies
\begin{equation}
\| w^t - w^t_0 \|_\infty \leq \alpha(t)
\end{equation}
for some function $\alpha(t)$ which satisfies $\alpha(t) \to 0$ as $t \to \infty$. Then $(\mathbb{I} - C^\omega)^{-1} : L^2(\Sigma) \to L^2(\Sigma)$ also exists for sufficiently large $t$, and the associated solutions of the Riemann–Hilbert problems (A.1) only differ by $O(\alpha(t))$.

Acknowledgements. We thank Ira Egorova for most helpful discussions on this topic and for pointing out several misprints in the original version of this manuscript.

References

[1] M. J. Ablowitz and A. C. Newell, *The decay of the continuous spectrum for solutions of the Korteweg–de Vries equation*, J. Math. Phys. 14 (1973), 1277–1284.

[2] M. J. Ablowitz and H. Segur, *Asymptotic solutions of the Korteweg–de Vries equation*, Stud. Appl. Math. 57 (1977), 13–44.

[3] R. Beals and R. Coifman, *Scattering and inverse scattering for first order systems*, Comm. Pure Appl. Math. 37 (1984), 39–90.

[4] A. Boutet de Monvel, I. Egorova, and G. Teschl, *Inverse scattering theory for one-dimensional Schrödinger operators with steplike finite-gap potentials*, J. Amer. Math. Soc. 106 (2008), 271–316.

[5] A. M. Budylin and V. S. Buslaev, *Quasiclassical integral equations and the asymptotic behavior of solutions of the Korteweg–de Vries equation for large time values*, Dokl. Akad. Nauk 348 (1996), 455–458.

[6] V. S. Buslaev, *Use of the determinant representation of solutions of the Korteweg–de Vries equation for the investigation of their asymptotic behavior for large times*, Uspekhi Mat. Nauk 36:4 (1981), 217–218.

[7] V. S. Buslaev and V. V. Sukhanov, *Asymptotic behavior of solutions of the Korteweg–de Vries equation*, J. Soviet Math. 34 (1986), 1905–1920.

[8] P. A. Deift, A. R. Its, and X. Zhou, *Long-time asymptotics for integrable nonlinear wave equations*, in *Important Developments in Soliton Theory*, Springer, Berlin, 1993, pp. 181–204.

[9] P. Deift, S. Kamvissis, T. Kriecherbauer, and X. Zhou, *The Toda rarefaction problem*, Comm. Pure Appl. Math. 49 (1996), 35–83.

[10] P. Deift, S. Venakides, and X. Zhou, *The collisionless shock region for the long time behavior of solutions of the KdV equation*, Comm. Pure Appl. Math. 47 (1994), 199–206.

[11] P. Deift, X. Zhou, *A steepest descent method for oscillatory Riemann–Hilbert problems*, Ann. of Math. (2) 137 (1983), 295–368.

[12] I. Egorova, K. Grunert, and G. Teschl, *On the Cauchy problem for the Korteweg–de Vries equation with steplike finite-gap initial data I. Schwartz-type perturbations*, Nonlinearity 22 (2009), 1431–1457.

[13] I. Egorova and G. Teschl, *On the Cauchy problem for the Korteweg–de Vries equation with steplike finite-gap initial data II. Perturbations with finite moments*, J. Anal. Math. 115 (2011), 71–102.
[14] I. Egorova and G. Teschl, A Paley-Wiener theorem for periodic scattering with applications to the Korteweg-de Vries equation, Zh. Mat. Fiz. Anal. Geom. 6 (2010), 21–33.

[15] H. Farkas and I. Kra, Riemann Surfaces, 2nd edition, Springer, New York, 1992.

[16] F. Gesztesy and H. Holden, Soliton Equations and their Algebro-Geometric Solutions, Volume I. (1+1)-Dimensional Continuous Models, Cambridge University Press, Cambridge, 2003.

[17] C. S. Gardner, J. M. Green, M. D. Kruskal, and R. M. Miura, A method for solving the Korteweg–de Vries equation, Phys. Rev. Lett. 19 (1967), 1095–1097.

[18] A. R. Its, Asymptotic behavior of the solutions to the nonlinear Schrödinger equation, and isomonodromic deformations of systems of linear differential equations, Soviet Math. Dokl. 24 (1981), 452–456.

[19] I. Gohberg and N. Krupnik, One-Dimensional Linear Singular Integral Equations, Birkhäuser, Basel, 1992.

[20] K. Grunert and G. Teschl, Long-time asymptotics for the Korteweg–de Vries Equation via nonlinear steepest descent, Math. Phys. Anal. Geom. 12 (2009), 287–324.

[21] S. Kamvissis and G. Teschl, Stability of periodic soliton equations under short range perturbations, Phys. Lett. A. 364 (2007), 480–483.

[22] S. Kamvissis and G. Teschl, Stability of the periodic Toda lattice under short range perturbations, arXiv:0705.0346v5.

[23] E. A. Kuznetsov and A. V. Mikhailov, Stability of stationary waves in nonlinear weakly dispersive media, Soviet Phys. JETP 40 (1975), 855–859.

[24] H. Krüger and G. Teschl, Long-time asymptotics for the Toda lattice in the soliton region, Math. Z. 262 (2009), 585–602.

[25] H. Krüger and G. Teschl, Long-time asymptotics of the Toda lattice for decaying initial data revisited, Rev. Math. Phys. 21 (2009), 61–109.

[26] H. Krüger and G. Teschl, Stability of the periodic Toda lattice in the soliton region, Int. Math. Res. Not. IMRN 2009, 3996–4031.

[27] S. V. Manakov, Nonlinear Fraenhofer diffraction, Soviet Phys. JETP 38 (1974), 693–696.

[28] V. A. Marchenko, Sturm–Liouville Operators and Applications, Birkhäuser, Basel, 1986.

[29] A. Mikikits-Leitner and G. Teschl, Trace formulas for Schrödinger operators in connection with scattering theory for finite-gap backgrounds, in Spectral Theory and Analysis, Birkhäuser, Basel, 2011, pp. 107–124.

[30] Yu. Rodin, The Riemann Boundary Problem on Riemann Surfaces, D. Reidel Publishing Co., Dordrecht, 1988.

[31] A. B. Šabat, On the Korteweg–de Vries equation, Soviet Math. Dokl. 14 (1973), 1266–1270.

[32] H. Segur and M. J. Ablowitz, Asymptotic solutions of nonlinear evolution equations and a Painlevé transcendent, Phys. D 3 (1981), 165–184.

[33] S. Tanaka, Korteweg–de Vries equation; Asymptotic behavior of solutions, Publ. Res. Inst. Math. Sci. 10 (1975), 367–379.

[34] G. Teschl, Algebro-geometric constraints on solitons with respect to quasi-periodic backgrounds, Bull. London Math. Soc. 39 (2007), 677–684.

[35] G. Teschl, Jacobi Operators and Completely Integrable Nonlinear Lattices, Amer. Math. Soc., Providence, RI, 2000.

[36] N. J. Zabusky and M. D. Kruskal, Interaction of solitons in a collisionless plasma and the recurrence of initial states, Phys. Rev. Lett. 15 (1965), 240–243.
[37] V. E. Zakharov and S. V. Manakov, *Asymptotic behavior of nonlinear wave systems integrated by the inverse method*, Soviet Phys. JETP **44** (1976), 106–112.

[38] X. Zhou, *The Riemann–Hilbert problem and inverse scattering*, SIAM J. Math. Anal. **20** (1989), 966–986.

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(Received August 22, 2010)