Analyzing Training Using Phase Transitions in Entropy—Part I: General Theory

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Abstract—We analyze phase transitions in the conditional entropy of a sequence caused by a change in the conditional variables or input distribution. Such transitions happen, for example, when training to learn the parameters of a system, since the switch from training input to data input causes a discontinuous jump in the conditional entropy of the measured system response. We show that the size of the discontinuity is of particular interest in the computation of mutual information during data transmission, and that this mutual information can be calculated as the difference between two derivatives of a single function. For large-scale systems we present a method of computing the mutual information for a system model with one-shot learning that does not require Gaussianity or linearity in the model, and does not require worst-case noise approximations or explicit estimation of any unknown parameters. The model applies to a broad range of algorithms and methods in communication, signal processing, and machine learning that employ training as part of their operation.

Index Terms—phase transition, mutual information, training, learning

I. INTRODUCTION

We begin with a brief synopsis of the results contained herein. Consider a system model that has input $x$ and output $y$, where some parameters within the model are unknown. One possible way to use the system includes supplying known inputs during a “training phase” to learn the parameters, after which the system is used during its “data phase”. An analysis of the effects of training is captured by the mutual information between the input and output at the beginning of the data phase, conditioned on the training signals:

$$I(x_{T+1}; y_{T+1}|x_T, y_T) = H(y_{T+1}|x_T, y_T) - H(y_{T+1}|x_{T+1}, y_T),$$

(1)

where $T$ is the number of training symbols, and $x_t = [x_1, \cdots, x_t]^T$. This quantity measures the amount of information that can be transferred through a single input-output pair in the data phase conditioned on the training. Generally, this mutual information increases monotonically in $T$ since a greater amount of training allows for better parameter estimates. We define

$$I'(X; Y) = \lim_{T \to \infty} I(x_{T+1}; y_{T+1}|x_T, y_T),$$

(2)

a quantity we are interested in computing.

The system model has an input process $x_t \in \mathcal{X}$, and output process $y_t \in \mathcal{Y}$ connected through a joint distribution that has random parameters that are unknown except for what is learned during the training process. Let

$$H'(Y_x|X_\delta) = \lim_{T \to \infty} H(y_{(1+\varepsilon)T+1}|x_{(1+\delta)T}, y_{(1+\varepsilon)T}),$$

(3)

where $\varepsilon \geq -1$ and $\delta \geq -1$. We also define

$$H'(Y_x|X_{\delta+}) = \lim_{T \to \infty} H(y_{(1+\varepsilon)T+1}|x_{(1+\delta)T+1}, y_{(1+\varepsilon)T}),$$

(4)

where we use $\delta^+$ to denote conditioning on the single extra input $x_{(1+\delta)T+1}$ versus $\delta$. Then (1) and (2) yield

$$I'(X; Y) = H'(Y_x|X) - H'(Y_x|X_\delta),$$

(5)

where $H'(Y_x|X)$ is defined as $H'(Y_x|X_\delta)|_{\varepsilon=0, \delta=0}$ and $H'(Y_x|X_{\delta+})$ is defined as $H'(Y_x|X_{\delta+})|_{\varepsilon=0, \delta=0}$.

We assume that:

A1: $H'(Y_x|X_{\delta+}) = \lim_{\varepsilon \to 0^+} H'(Y_x|X_{\delta+})$, 

(6)

A2: $H'(Y_x|X) = \lim_{\varepsilon \to 0^+} H'(Y_x|X)$,

(7)

and then (5) yields

$$I'(X; Y) = \lim_{\varepsilon \to 0^+} H'(Y_x|X) - \lim_{\varepsilon \to 0^+} H'(Y_x|X_{\delta+}).$$

(8)

We establish (Theorems 1 & 2 and Corollary 1) that under certain conditions

$$H'(Y_x|X_\delta) = \frac{\partial H(Y_x|X_\delta)}{\partial \varepsilon},$$

(9)

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conditioal entropy limits

Fig. 1. Sketch of $H'(Y_e|X)$, $H'(Y_e|X_e^+)$, $H(Y_e|X)$ and $H(Y_e|X_e)$ as functions of $\varepsilon$ in Phase I ($\varepsilon \leq 0$) and Phase II ($\varepsilon > 0$) where $x_i$ is iid throughout both phases. $H'(Y_e|X)$ is discontinuous at $\varepsilon = 0$ (phase change), while $H'(Y_e|X_e^+)$ is continuous at 0. The black arrow shows $I'(X;Y)$ as the difference between $H'(Y_e|X)$ and $H'(Y_e|X_e^+)$. Although $H(Y_e|X)$ is not differentiable at 0, $H(Y_e|X_e)$ is.

where $H(Y_e|X_e)$ is defined as

$$H(Y_e|X_e) = \lim_{T \to \infty} \frac{1}{T} H(y^{(1+\varepsilon)T} | x^{(1+\delta)T}).$$ (10)

When $\delta > \varepsilon$, we show that

$$H(Y_e|X_e) = H(Y_e|X_e),\ H'(Y_e|X_e) = H'(Y_e|X_e^+),$$

and (9) becomes

$$H'(Y_e|X_e^+) = \frac{\partial H(Y_e|X_e)}{\partial \varepsilon}. \quad (11)$$

We may therefore obtain $I'(X;Y)$ by computing $H'(Y_e|X)$ and $H'(Y_e|X_e^+)$ as derivatives of $H(Y_e|X_e)$ (Theorem 3). Finally, we give examples of how to use $I'(X;Y)$ and provide a bound that can be used to analyze training (Theorem 4).

The value of this formulation relies on our ability to find $H(Y_e|X_e)$ in a straightforward manner, and we provide guidance on this in Theorem 3. We make no assumptions of linearity of the system or Gaussianity in any of the processes; nor is it required to form an explicit estimate of the parameters $\theta$ during the training phase.

Qualitative sketches of $H'(Y_e|X)$, $H'(Y_e|X_e^+)$, $H(Y_e|X)$ and $H(Y_e|X_e)$ are shown in Fig. 1 during phase I (training phase, $\varepsilon \in [-1,0]$) and phase II (data phase, $\varepsilon > 0$). In this example, $x_i$ is independent and identically distributed (iid) throughout both phases; equivalently, the training and data sequences have the same distribution. The quantity $H'(Y_e|X)$ is discontinuous at $\varepsilon = 0$ since the input is no longer part of the conditioning for $\varepsilon > 0$. Both $H'(Y_e|X)$ and $H'(Y_e|X_e^+)$ decrease as $\varepsilon$ increases throughout phases I and II. Also shown are $H(Y_e|X)$ and $H(Y_e|X_e)$, which are the integrals of $H'(Y_e|X)$ and $H'(Y_e|X_e^+)$.

Although $H(Y_e|X_e)$ is not differentiable at $\varepsilon = 0$, it is differentiable for $\varepsilon \neq 0$, while $H(Y_e|X_e)$ is differentiable for all $\varepsilon > -1$.

Fig. 2 is similar to Fig. 1 except that the distribution of $x_i$ in phase II differs from phase I, causing a discontinuity in $H'(Y_e|X_e^+)$ at $\varepsilon = 0$. As a result, both $H(Y_e|X_e)$ and $H(Y_e|X_e)$ are not differentiable at $\varepsilon = 0$, but are differentiable for $\varepsilon \in (-1,0)$ and $\varepsilon > 0$.

Phase transitions in entropy have been used in other contexts, including the “information bottleneck” [1], minimum mean-square error (MMSE) analysis [2], [3], and random Boolean networks [4]. Here, we focus on the applications to training of phase transitions caused by the change of input from training signals to data signals. The remainder of the paper is devoted to explanations and justifications of the above assumptions and statements. We begin by establishing 3.

II. DERIVATIVE RELATIONSHIP BETWEEN $H'$ AND $H$

Define

$$H'(Y_e) = \lim_{T \to \infty} H(y^{(1+\varepsilon)T+1} | y^{(1+\varepsilon)T}), \quad (12)$$

$$H(Y_e) = \lim_{T \to \infty} \frac{1}{T} H(y^{(1+\varepsilon)T}), \quad (13)$$

which can be considered as $H'(Y_e|X_0)$ and $H(Y_e|X_0)$ with $\delta = -1$. We show that, under some conditions, $H'(Y_e)$ is the derivative of $H(Y_e)$, provided that this derivative exists.

Theorem 1. If there exists a $\kappa > 0$ so that $H(y_{t+1}|y_t)$ is monotonic in $t$ when $t \in [(1+\varepsilon - \kappa)T, (1+\varepsilon + \kappa)T]$, then

$$H'(Y_e) = \frac{\partial H(Y_e)}{\partial \varepsilon}. \quad (14)$$
Proof. Without loss of generality, we assume that $H(y_{t+1}|y_t)$ is monotonically decreasing. Using the definition of $H(Y_e)$ in (13), we have

\[
\frac{1}{\kappa} (H(Y_{e+\kappa}) - H(Y_e)) = \lim_{T \to \infty} \frac{H(y_{(1+\epsilon+\kappa)T}) - H(y_{(1+\epsilon)T})}{\kappa T} = \lim_{T \to \infty} \frac{\sum_{t=(1+\epsilon)T+1}^{(1+\epsilon+\kappa)T} H(y_t|y_{t-1})}{\kappa T} \leq \lim_{T \to \infty} \frac{\kappa T \cdot H(y_{(1+\epsilon)T+1}|y_{(1+\epsilon)T})}{\kappa T} = \lim_{T \to \infty} H(y_{(1+\epsilon)T+1}|y_{(1+\epsilon)T}).
\] (15)

Similarly to (15), we also have

\[
\lim_{T \to \infty} H(y_{(1+\epsilon)T+1}|y_{(1+\epsilon)T}) \leq \frac{1}{\kappa} (H(Y_e) - H(Y_{e-\kappa})).
\] (16)

Let $\kappa \to 0^+$ for both (15) and (16); because we assume that the derivative of $H(Y_e)$ exists, these limits both equal this derivative. Then, the definition of $H'(Y_e)$ in (12) yields (14).

An integral equivalent of (14) is:

**Theorem 2.** For a process $\mathcal{Y}$ that satisfies:

1) $H(y_{t+1}|y_t)$ is bounded for all $t < T$ as $T \to \infty$;
2) there exists a $\kappa > 0$ so that $H(y_{t+1}|y_t)$ is monotonic in $t$ when $t \in [(1+\epsilon-\kappa)T,(1+\epsilon+\kappa)T]$ for all $\epsilon > -1$, except for a finite number of points as $T \to \infty$;

then we have

\[
H(Y_e) = \int_{-1}^{\epsilon} H'(Y_e) d\epsilon.
\] (17)

The proof is omitted.

Theorems 1 and 2 are consequences of the entropy chain rule and letting an infinite sum converge to an integral (standard Riemann sum approximation). Such an analysis has also been used in the context of computing mutual information; for example [3]–[7], where the mutual information between a high-dimensional input vector and a high-dimensional output vector is considered, and the chain rule is applied along the dimension of the input, thus producing a summation of mutual information between a scalar input and the vector output, conditioned on all the previous scalar inputs. In the limit as the dimension goes to infinity, the summation converges to an integral.

Two examples are given.

**Example 1:** Let $\mathcal{Y}$ be a stationary process where the joint distribution of any subset of the sequence of random variables is invariant with respect to shifts in the time index [10], and where

\[
\lim_{T \to \infty} H(y_{T+1}|y_T) = H(\mathcal{Y}) = \lim_{T \to \infty} \frac{1}{T} H(y_T),
\]

where $H(\mathcal{Y})$ is called the “entropy rate” of $\mathcal{Y}$. For all $\epsilon > -1$, we have

\[
H(Y_e) = \lim_{T \to \infty} \frac{1}{T} H(y_{(1+\epsilon)T}) = (1+\epsilon) H(\mathcal{Y}),
\]

and

\[
H'(Y_e) = \lim_{T \to \infty} H(y_{(1+\epsilon)T+1}|y_{(1+\epsilon)T}) = H(\mathcal{Y}).
\] (18)

Because of stationarity, $H(y_{t+1}|y_t)$ is monotonically decreasing in $t$, and we see that $H'(Y_e)$ is the derivative of $H(Y_e)$, as expected.

However, we are generally interested in non-stationary processes, and the next example is a simple example containing a “phase change” at $t = T$.

**Example 2:** Let

\[
y_t = \begin{cases} 
 b_t, & t = 1, \ldots, T; \\
 b_{(t-1 \mod T) + 1}, & t = T + 1, T + 2, \ldots,
\end{cases}
\] (19)

where $b_1, b_2, \ldots$ are iid with entropy 1. There are two phases in $\mathcal{Y}$: the first phase contains iid elements, while the second phase contains repetitions of the first. Clearly

\[
H'(Y_e) = \begin{cases} 
 1, & \epsilon \in [-1, 0); \\
 0, & \epsilon \geq 0,
\end{cases}
\] (20)

and $H(y_{t+1}|y_t)$ is bounded and monotonic for all $t$.

Theorem 2 (or inspection) yields

\[
H(Y_e) = \int_{-1}^{\epsilon} H'(Y_u) du = \begin{cases} 
 1 + \epsilon, & \epsilon < 0; \\
 1, & \epsilon \geq 0.
\end{cases}
\]

Note that $H(Y_e)$ is differentiable everywhere but $\epsilon = 0$. This point will reappear later.

**A. Processes where $H'(Y_e)$ is not the derivative of $H(Y_e)$**

We examine, through two examples, what can go wrong when the conditions of Theorems 1 and 2 are not met.

**Example 3:** $H(y_{t+1}|y_t)$ oscillates as $t$ increases

Consider the process

\[
\mathcal{Y} = (b_1, b_1, b_2, b_2, \ldots, b_k, b_k, \ldots),
\] (21)
where $b_1, b_2, \ldots$ are iid unit-entropy random variables. Then, for all $t \geq T$,
\[
H(y_{t+1}|y_t) = \begin{cases} 
1, & t \text{ even} \\
0, & t \text{ odd} 
\end{cases}
\]
and $\mathcal{H}'(\mathcal{Y}_\epsilon)$ does not exist for any $\epsilon$. However, $\mathcal{H}(\mathcal{Y}_\epsilon)$ exists for all $\epsilon > -1$ with
\[
\mathcal{H}(\mathcal{Y}_\epsilon|\mathcal{X}) = \lim_{T \to \infty} \frac{1}{T} \frac{(1 + \epsilon)T}{2} = 1 + \frac{\epsilon}{2},
\]
which is differentiable for all $\epsilon > 0$. The conditions for Theorem 1 are not met and the derivative relationship (14) does not hold.

**Example 4:** $H(y_{t+1}|y_t)$ is unbounded

Consider a process $\mathcal{Y}$ with independent elements whose entropies are
\[
H(y_t) = \begin{cases} 
T, & t = \frac{1}{2}T - 3 \\
1, & \text{otherwise}
\end{cases}
\]
It is clear that $H(y_t|y_{t-1})$ is unbounded at $t = \frac{1}{2}T - 3$. Both $\mathcal{H}'(\mathcal{Y}_\epsilon)$ and $\mathcal{H}(\mathcal{Y}_\epsilon)$ exist with
\[
\mathcal{H}'(\mathcal{Y}_\epsilon) = 1, \quad \epsilon \geq -1,
\]
\[
\mathcal{H}(\mathcal{Y}_\epsilon) = \begin{cases} 
1 + \epsilon, & \epsilon \in (-1, -\frac{1}{2}) \\
2 + \epsilon, & \epsilon \geq -\frac{1}{2},
\end{cases}
\]
but the conditions for Theorem 2 are not met and the integral relationship (17) does not hold everywhere.

Nonetheless, these examples can still be accommodated by expanding the definition of $\mathcal{H}'(\mathcal{Y}_\epsilon)$. In Example 3, $\mathcal{H}'(\mathcal{Y}_\epsilon)$ is not a good representative of $H(y_{t+1}|y_t)$ when $t = (1 + \epsilon)T$ because of its oscillatory behavior. A better representative of $H(y_{t+1}|y_t)$ when $t = (1 + \epsilon)T$ can be found by averaging:
\[
\tilde{\mathcal{H}}'(\mathcal{Y}_\epsilon) = \lim_{\kappa \to 0^+} \lim_{T \to \infty} \frac{1}{\kappa T} \frac{\sum_{t=(1+\epsilon-\frac{1}{2})T}^{(1+\epsilon-\frac{1}{2})T-1} H(y_{t+1}|y_t)}{T}.
\]
With the new definition, for Example 3,
\[
\tilde{\mathcal{H}}'(\mathcal{Y}_\epsilon) = \lim_{\kappa \to 0^+} \lim_{T \to \infty} \frac{\kappa T}{2} = \frac{1}{2},
\]
which is the derivative of $\mathcal{H}(\mathcal{Y}_\epsilon|\mathcal{X})$ shown in (22). Thus, averaging smooths out the oscillation and expands the class of processes for which Theorem 1 holds.

In Example 4, $\tilde{\mathcal{H}}'(\mathcal{Y}_\epsilon)$ is unbounded at $\epsilon = -\frac{1}{2}$. By allowing an impulse function in $\tilde{\mathcal{H}}'(\mathcal{Y}_\epsilon)$ at $\frac{1}{2}$, we may then consider $\tilde{\mathcal{H}}(\mathcal{Y}_\epsilon)$ as the integral of $\tilde{\mathcal{H}}'(\mathcal{Y}_\epsilon)$, thereby expanding the class of processes for which Theorem 2 holds. We do not pursue these issues any further.

### III. Input Process and Mutual Information

#### A. The input process and the parameters

In the remainder, we consider a triple $(\mathcal{X}, \mathcal{Y}, \mathcal{G})$, where $x_t \in \mathcal{X}$ and $y_t \in \mathcal{Y}$ are the input and output processes of a system, and $\mathcal{G}$ is the parameter set. We assume $t = 1, \ldots, B$, where $B$ is the blocklength, defined as the period of time for which the parameters are considered constant. Let $T = \tau B$ be the training time, with $\tau \in (0, 1]$, where $\tau$ is the fraction of the total blocklength devoted to learning the unknown parameters through training. The input and output are connected through a conditional distribution parameterized by $g_B \in \mathcal{G}$, whose value is unknown. We note that $g_B$ is indexed by $B$, indicating that the parameter set is allowed to grow in size as $B \to \infty$ (and $T = \tau B \to \infty$). We generally drop the use of $B$, and substitute $\frac{T}{\tau}$ in its place, for a fixed $\tau$.

We make the following assumption:

A3: $p(y_{\frac{T}{\tau}}|x_{\frac{T}{\tau}}; g_{\frac{T}{\tau}}) = \prod_{t=1}^{\frac{T}{\tau}} p(y_t|x_t; g_{\frac{T}{\tau}})$, $\prod_{t=T+1}^{T+1} p(x_t)$, $\prod_{t=T+1}^{T+1} p(x_{t+1})$,

where $p(y_t|x_t; g_{\frac{T}{\tau}})$ is a fixed conditional distribution for all $t = 1, 2, \ldots, \frac{T}{\tau}$ and $p(x_t)$ is a fixed distribution for all $t = T + 1, T + 2, \ldots, \frac{T}{\tau}$. Equation (28) says that the system is memoryless and time invariant (given the input and parameters) and (29) says that the input $x_t$ is iid and independent of $x_{\tau}$ for all $t > T$. The distributions of $x_t$ during training and afterward can therefore differ.

We use the common convention of writing $p(x_T)$ and $p(x_t)$ when we mean $p_{x|T}(\cdot)$ and $p_{x|t}(\cdot)$, even though these functions can differ. Under A3, the distributions of $(\mathcal{X}, \mathcal{Y}, \mathcal{G})$ are described by the set of known distributions

$$P(T, \tau) = \{p(y|x; g_{\frac{T}{\tau}}), p(g_{\frac{T}{\tau}}), p(x_T), p(x_{T+1})\},$$

which depends on $\tau$. These distributions are used to calculate all of the entropies and mutual informations throughout, and hence, these quantities may depend on $\tau$ and can be thought of as “ergodic” in the sense that they average over realizations of $g_{\frac{T}{\tau}}$.

Both Theorems 1 and 2 can be generalized to include conditioning on $\mathcal{X}$, thus leading to the following corollary.

**Corollary 1.** Under Assumption A3, for $\epsilon > 0$,
\[
\mathcal{H}'(\mathcal{Y}_\epsilon|\mathcal{X}) = \frac{\partial \mathcal{H}(\mathcal{Y}_\epsilon|\mathcal{X})}{\partial \epsilon},
\]

\[ \mathcal{H}'(Y_t | X_t) = \frac{\partial \mathcal{H}(Y_t | X_t)}{\partial \varepsilon}. \] (29)

If \( x_t \) are iid for all \( t \), then we have
\[ \mathcal{H}'(Y_t | X_t) = \frac{\partial \mathcal{H}(Y_t | X_t)}{\partial \varepsilon} \] (30)
for all \( \varepsilon, \delta > 0 \) and \( \varepsilon \neq \delta \). Also,
\[ \mathcal{H}'(Y_t | X_t) = \left. \frac{\partial \mathcal{H}(Y_t | X_t)}{\partial \varepsilon} \right|_{\varepsilon=0}. \] (31)

**Proof.** Under Assumption A3, for all \( \delta \geq 0 \), we have
\[ H(y_{t+1} | x_{(1+\delta)T} \lor T, y_t) \leq H(y_{t+1} | x_{(1+\delta)T} \lor T, y_t) \]
\[ = H(y_t | x_{(1+\delta)T} \lor T, y_t), \] (32)
when \( T+1 \leq t \leq (1+\delta)T-1 \) or \( t \geq (1+\delta)T+1 \). Here, we use that the input is iid and the system is memoryless and time invariant; the inequality follows from the fact that conditioning reduces entropy. Therefore, \( \forall \kappa \in (0, \varepsilon) \), \( \mathcal{H}(y_{t+1} | x_{(1+\delta)T} \lor T, y_t) \) is monotonic decreasing in \( t \) for \( t \in [1+\varepsilon-\kappa, 1+\varepsilon+\kappa)T \). Then, Theorem 1 yields (28).

Also, \( \forall \kappa \in (0, \varepsilon), \delta > 2\varepsilon \), \( \mathcal{H}(y_{t+1} | x_{(1+\delta)T} \lor T, y_t) \) is monotonic decreasing in \( t \) for \( t \in [1+\varepsilon-\kappa, 1+\varepsilon+\kappa)T \). Then, Theorem 1 yields
\[ \mathcal{H}'(Y_t | X_t) = \frac{\partial \mathcal{H}(Y_t | X_t)}{\partial \varepsilon}. \] (33)
Assumption A3 yields
\[ \mathcal{H}'(Y_t | X_t) = \mathcal{H}'(Y_t | X_{t+1}), \] (34)
where \( \mathcal{H}'(Y_t | X_{t+1}) \) is defined in (4), and
\[ \mathcal{H}'(Y_t | X_t) = \mathcal{H}'(Y_t | X_{t+1}). \] (35)
Therefore, (33) becomes (29).

If \( x_t \) are iid for all \( t \), then (32) is valid for all \( t \leq (1+\delta)T-1 \) or \( t \geq (1+\delta)T+1 \). Therefore, Theorem 1 yields (30). By taking \( \varepsilon = 0 \) and \( \delta > 0 \), (30), (34), and (35) then yield (31).

**B. Computation of \( \mathcal{I}'(X_t \lor T; Y_t) \) and significance of assumptions**

We defer our applications of \( \mathcal{I}'(X_t \lor T; Y_t) \) until Section II-C but now show it may readily be computed with the help of Corollary 1. It is shown in (5) that \( \mathcal{I}'(X_t \lor T; Y_t) \) can be computed as the difference between \( \mathcal{H}'(Y_t | X_t) \) and \( \mathcal{H}'(Y_t | X_{t+1}) \). These quantities can be computed as derivatives of \( \mathcal{H}(Y_t | X_t) \) and \( \mathcal{H}(Y_t | X_{t+1}) \).

**Theorem 3.** Under Assumptions A1–A3, we have
\[ \mathcal{I}'(X_t \lor T; Y_t) = \lim_{\varepsilon \to 0} \frac{\partial \mathcal{H}(Y_t | X_t)}{\partial \varepsilon} - \lim_{\varepsilon \to 0} \frac{\partial \mathcal{H}(Y_t | X_{t+1})}{\partial \varepsilon}. \] (36)

**Proof.** Corollary 1 and (8) yield (36).
training. We show in the following theorem that both $\mathcal{I}(\mathcal{X}_1'; \mathcal{Y}_c')$ and $\mathcal{I}(\mathcal{X}_2'; \mathcal{Y}_c)$ have close relationship with $\mathcal{I}(\mathcal{X}; \mathcal{Y})$:

**Theorem 4.** Under $A3$, for all $\varepsilon_1, \varepsilon_2 \geq 0$, $\varepsilon_1 > \varepsilon_2$, $\tau \in (0, 1)$,
\[
\mathcal{I}(\mathcal{X}_1'; \mathcal{Y}_c') \geq \mathcal{I}(\mathcal{X}_2'; \mathcal{Y}_c),
\]
and,
\[
\mathcal{I}(\mathcal{X}_{-1}; \mathcal{Y}_{-1}) \geq \mathcal{I}(\mathcal{X}; \mathcal{Y})
\]
where $\hat{g}_T$ is any estimate of $g_T$ that is a function of $(x_T, y_T)$.

**Proof.**
Assumption $A3$ implies that $x_t$ is independent of $(x_k, y_k)$ when $k \neq t$ and $t \geq T + 1$. Therefore, for all $t \geq T + 1$,
\[
I(x_t+1; y_{t+1} | x_t, y_t) - I(x_t; y_t | x_{t-1}, y_{t-1})
\]
\[(a) = (H(x_{t+1}) - H(x_{t+1} | x_t, y_{t+1})) \]
\[\quad - (H(x_t) - H(x_t | x_{t-1}, y_{t-1}))
\]
\[\quad = H(x_t | x_{t-1}, y_{t-1}) - H(x_{t+1} | x_t, y_{t+1})
\]
\[(b) \geq (T x_{t+1} | x_{t-1}, y_{t-1}, y_{t+1}) - H(x_{t+1} | x_t, y_{t+1})
\]
\[(c) \geq 0.
\]
Here, (a) uses the independence between $x_t$ and $(x_{t-1}, y_{t-1})$, (b) uses Assumption $A3$, (c) uses conditioning to reduce entropy. Thus, $I(x_t; y_t | x_{t-1}, y_{t-1})$ is monotonically increasing with $t$ for all $t > T$. In the limit when $T \to \infty$, (46) yields (47). A sufficient condition for achieving equality in (c) is when $g_T$ can be estimated perfectly from $(x_T, y_T)$, and both entropies on the right hand side of (b) equal $H(x_{t+1} | g_T, y_{t+1})$.

Moreover,
\[
I(x^{T+1}; y_T | x_T)
\]
\[= I(x^{T+1}; y_T | x_T) + I(x^{T+1}; y_T | x_T, y_T)
\]
\[= I(x^{T+1}; y_T | x_T, y_T),
\]
where the first equality uses the chain rule and the second uses that $x^{T+1}$ is independent of $(x_T, y_T)$. Therefore,
\[
I(x^{T+1}; y_T | x_T, y_T) = H(y_T | x_T) - H(y_T | x_T)
\]
\[\leq H(y_T) - H(y_T | x_T) = I(x_T; y_T),
\]
which, with (40), yields (42).

To prove (43), we first show following inequality:
\[
I(x^{T+1}; y_T | x_T, y_T)
\]
\[= H(x^{T+1} | x_T, y_T) - H(x^{T+1} | x_T, y_T)
\]
\[\geq \sum_{t=T+1} \left( H(x_t | x_{t-1}, y_T) - H(x_t | x_{t-1}, y_T) \right)
\]
\[\geq \sum_{t=T+1} \left( H(x_t | x_{t-1}, y_{t-1}) - H(x_t | x_{t-1}, y_{t-1}) \right)
\]
\[= \sum_{t=T+1} I(x_t; y_t | x_{t-1}, y_{t-1}).
\]
Here, (a) uses the chain rule, (b) uses conditioning to reduce entropy. Equality in (b) can be achieved when $g_T$ is estimated perfectly from $(x_T, y_T)$. Then, by normalizing (47) by $\tau$ and letting $T \to \infty$, we see that the sum converges to the integral, which proves (43). Then, (41) yields (44).

Also,
\[
I(x^{T+1}; y_T | x_T, y_T)
\]
\[= H(x^{T+1}) - H(x^{T+1} | x_T, y_T, y_{T+1})
\]
\[\geq H(x^{T+1}) - H(x^{T+1} | x_T, y_T, g_T, y_{T+1})
\]
\[\geq H(x^{T+1}) - H(x^{T+1} | g_T, y_{T+1})
\]
\[= I(x^{T+1}; y_T | g_T, y_{T+1}),
\]
where (a) uses that $g_T$ is a function of $(x_T, y_T)$, and (b) uses conditioning to reduce entropy. By taking the limit $T \to \infty$, we have (45).

**D. Discussion of lower bounds**

The quantities in (42)–(45) have a long history of being studied in various contexts. For example, $\mathcal{I}(\mathcal{X}_{-1}; \mathcal{Y}_{-1})$ describes the mutual information between the input and output in a whole block without any training. This quantity is studied in [11]–[13] in the context of a linear system with additive Gaussian noise and vector inputs and outputs; analytical expressions are obtained in special cases when the blocklength is larger than the dimension of the input, or the noise power is small.

The right-hand side of (42) $\mathcal{I}(\mathcal{X}; \mathcal{Y})$ describes the mutual information between the input and output in the data phase conditioned on $T$ training, and the right-hand side of (43) integrates the mutual information between each input and output pair in the data phase where all the output (including data phase) is used to refine the estimate of $g_T$. In [14], [15], linear systems with
additive Gaussian noise are considered, starting with \( I(X; Y) \), which is then lower bounded by an integral similar to the right-hand side of (43). This integral is further lower-bounded by obtaining a linear minimum mean-squared error (LMMSE) estimate of \( g_Z \). This lower bound is maximized when \( \tau = 0 \) since any amount of training at the expense of data is harmful to the total throughput for the block; the unknown parameters can, in theory, be jointly estimated with the data. In general, when the input is iid during the whole block, (42) and (43) are maximized when \( \tau = 0 \).

In contrast, the right-hand sides of (44) and (45) assume \( g_Z \) is learned only through the training phase (and not through the data phase)—so-called “one-shot” learning, which is our primary interest. In particular, (44) does not form an explicit estimate of \( g_Z \), and is an upper bound on (45), which forms an explicit estimate derived only from the training. Special cases of one-shot learning with explicit estimates of system parameters are analyzed in [16]–[18]. A linear system with additive Gaussian noise is analyzed in [16], where the system parameters are estimated in the training phase, and the worst-case noise analysis then produces a lower bound. The worst-case noise analysis is developed also in [16] where the estimation error is treated as additive Gaussian noise. In [17], [18], systems with additive Gaussian noise and one-bit quantization at the output are considered, where the Bussgang decomposition is used to reformulate the nonlinear quantizer as a linear function with additional noise, followed by a worst-case noise analysis. Training times that maximize these lower bounds are generally nonzero since only the training phase is used to learn the unknown parameters, and further refinement of these parameters during the data phase is not performed.

We are interested in optimizing the amount of training in a one-shot system, and this is provided by analyzing \((1 - \tau)I'(X; Y)\) shown in (44). We compute

\[
\tau_{\text{opt}} = \arg\max_{\tau} (1 - \tau)I'(X; Y) \tag{48}
\]

and the corresponding rate per receive symbol is

\[
R_{\text{opt}} = (1 - \tau_{\text{opt}})I'(X; Y)|_{\tau=\tau_{\text{opt}}} \tag{49}
\]

As shown in (43), this is an upper bound on the rate achievable by any method that estimates the unknown parameters from the training data.

Crucially, according to Theorem 3, derivatives of \( H(Y_e | X_0) \), which can be obtained from the single entropy function \( H(Y_e | X) \), are needed to compute this. Because of the large blocksize limit used in the analysis, tractable answers can sometimes be obtained for models that do not necessarily have Gaussianity or linearity. This is the subject of the next section.

IV. COMPUTATION OF \( H(Y_e | X_0) \) AND AN EXAMPLE

Closed-form expressions for \( H(Y_e | X_0) \) are not necessarily always available, but we may derive this quantity from \( H(Y_e | X) \), when this is available. For example, in some cases \( H(Y_e | X) \) can be obtained through methods employed in statistical mechanics by treating the conditional entropy as free energy in a large-scale system. Free energy is a fundamental quantity [19]–[21] that has been analyzed through the powerful “replica method”, and this, in turn, has been applied to entropy calculations in machine learning [22]–[25] and wireless communications [26]–[28], in both linear and nonlinear systems.

The entropy \( H(Y_e | X) \) (equivalent to \( \epsilon = 0 \)) is considered in [22]–[25], where the input is multiplied by an unknown vector as an inner product and the passes through a nonlinearity to generate a scalar output. In [22], [23], [25], the input are iid, while orthogonal inputs are considered in [24]. In [22]–[24], the nonlinearity outputs the sign of the input, while a sigmoid nonlinearity with a continuous output is considered in [25].

The entropy \( H(Y_e | X) \) for MIMO systems is considered in [26]–[28], where the inputs are iid in the training phase and are iid in the data phase, but the distributions in two phases can differ. In [26], a linear system is considered where the output is the result of the input multiplied by an unknown matrix, plus additive noise, while in [27], [28] uniform quantization is added at the output.

As we now show, these results can be leveraged to compute \( H(Y_e | X_0) \) for \( \epsilon, \delta > -1 \).

We consider the case when the input \( x_t \) are iid for all \( t \), and the distribution set \( P(T, \tau) \) defined in (27) can be simplified as

\[
P(T, \tau) = \{ p(y|x; g^T_Z), p(g^T_Z), p(x) \}. \tag{50}
\]

The following theorem assumes that we have \( H(Y_e | X) \) available as a function of \( (\tau, \epsilon) \) for all \( \epsilon \geq 0 \):

**Theorem 5.** Assume that A3 is met and \( x_t \) are iid for all \( t \). Define

\[
F(\tau, \epsilon) = H(Y_e | X), \tag{51}
\]

where \( \epsilon \geq 0 \) and \( H(Y_e | X) \) is defined in (10). Then

\[
H(Y_e | X_0) = (1 + u) \cdot F \left( (1 + u)\tau, \frac{\epsilon - u}{1 + \delta} \right), \tag{52}
\]

for all \( \epsilon, \delta \in (-1, \frac{1}{\tau} - 1) \), where \( u = \min(\epsilon, \delta) \).
Proof. According to (10) and (51), we have
\[
\lim_{T \to \infty} \frac{1}{T} H(y_{(1+\varepsilon)T}|x_T) = F(\tau, \varepsilon),
\]
which is computed using \( \mathcal{P}(T, \tau) \) defined in (50). When \( \delta \geq \varepsilon > -1 \), we have
\[
\mathcal{H}(\mathcal{Y}_c|\mathcal{X}_d) = \lim_{T \to \infty} \frac{1}{T} H(y_{(1+\varepsilon)T}|x_{(1+\delta)T})
\]
(53)
\[
= \lim_{T \to \infty} \frac{1}{T} H(y_{(1+\varepsilon)T}|x_{(1+\delta)T}).
\]
Let \( T = (1 + \varepsilon)T \), and we have
\[
\mathcal{H}(\mathcal{Y}_c|\mathcal{X}_d) = \lim_{T \to \infty} \frac{1 + \varepsilon}{T} H(y_{T}|x_T).
\]
Since \( \mathcal{P}(T, \tau) \) only depends on \( T \), we can rewrite \( \mathcal{P}(T, \tau) \) as
\[
\mathcal{P}(T, \tau) = \mathcal{P}(\tilde{T}, (1 + \varepsilon)\tau),
\]
and therefore
\[
\mathcal{H}(\mathcal{Y}_c|\mathcal{X}_d) = (1 + \varepsilon) \cdot F((1 + \varepsilon)\tau, 0).
\]
(54)
For \( \varepsilon > \delta > -1 \), let \( \tilde{T} = (1 + \delta)T \), and then (53) yields
\[
\mathcal{H}(\mathcal{Y}_c|\mathcal{X}_d) = \lim_{T \to \infty} \frac{1 + \delta}{T} H(y_{(1 + \delta)T}|x_T)
\]
\[
= (1 + \delta) \cdot F\left( (1 + \delta)\tau; \frac{\varepsilon - \delta}{1 + \delta} \right).
\]
(55)
By combining (54) and (55), we obtain (52). \( \square \)

The following example provides a brief demonstration of how to apply the main theorems.

Example 5: Bit flipping through random channels

Let
\[
y_t = x_t \oplus g_{k_t}, \quad t = 1, \ldots, T,
\]
where the binary input \( x_t \) is XOR’ed with a random unknown bit \( g_{k_t} \) intended to model an impairment through channel \( k_t \). The \( x_t \) are iid equally likely to be zero or one, Bernoulli(\( \frac{1}{2} \)). Let \( \alpha > 0 \) be a parameter, where \( \alpha \cdot \frac{T}{\tau} \) is the number of possible unique channels whose unknown states are stored in the vector \( \mathbf{g}_{\frac{T}{\tau}} = [g_1, g_2, \ldots, g_{\frac{T}{\tau}}] \) comprising iid Bernoulli(\( \frac{1}{2} \)) random variables that are independent of the input. The integer channels \( k_1, \ldots, k_{\frac{T}{\tau}} \) used during training and data are iid uniformly sampled from \( \{1, 2, \ldots, \alpha \cdot \frac{T}{\tau}\} \), and are known but are not under our control. We wish to send training signals through these channels to learn \( \mathbf{g}_{\frac{T}{\tau}} \); the more entries of this vector that we learn, the more channels become useful for sending data, but the less time we have to send data before the blocklength \( \frac{T}{\tau} \) runs out and \( \mathbf{g}_{\frac{T}{\tau}} \) changes.

We want to determine the optimum \( \tau \) as \( T \to \infty \) using (48). We therefore compute \( \mathcal{H}(\mathcal{Y}_c|\mathcal{X}) \) and then use Theorem 5 to obtain \( \mathcal{H}(\mathcal{Y}_c|\mathcal{X}_d) \), which is used to compute \( \mathcal{T}(\mathcal{X}, \mathcal{Y}) \) through Theorem 3.

By definition, \( \mathcal{H}(\mathcal{Y}|\mathcal{X}) = \lim_{T \to \infty} \frac{1}{T} H(y_T|x_T) \). The model (56) yields
\[
H(y_T|x_T) \overset{(a)}{=} H(\{g_{k_t}, 1 \leq t \leq T\}|x_T)
\]
\overset{(b)}{=} H(\{g_{k_t}, 1 \leq t \leq T\}) \overset{(c)}{=} \mathbb{E}_{k_T}(\{|k_t, 1 \leq t \leq T\}),
\]
where \( k_T = [k_1, k_2, \ldots, k_T] \) and where (a) uses \( g_{k_t} = y_t \oplus x_t \) from (56), (b) uses the independence between \( x_T \) and \( g_t \), and (c) uses the independence between \( g_{k_t} \) and \( g_{k_t} \) when \( t \neq k \), and the entropy of \( g_t \) is 1 (using log2(\( \cdot \))).

We use \( |\cdot| \) to denote the cardinality of a set. Define \( A_T = \{k_t, 1 \leq t \leq T\} \) and we have \( |A_T| = \sum_{i=1}^{\frac{T}{\tau}} \mathbb{1}_{(i \in A_T)} \), where \( \mathbb{1}_{(\cdot)} \) is the indicator function. Then,
\[
H(y_T|x_T) = \mathbb{E}(A_T) = \sum_{i=1}^{\frac{T}{\tau}} \mathbb{E}(\mathbb{1}_{(i \in A_T)}).
\]
\[
= \sum_{i=1}^{\frac{T}{\tau}} P(i \in A_T) = \frac{a_T}{\tau} (1 - P(1 \notin A_T))
\]
\[
= \frac{a_T}{\tau} (1 - \prod_{t=1}^{T} P(1 \neq k_t)) = \frac{a_T}{\tau} (1 - (1 - \frac{\tau}{a_T})^T).
\]
Therefore,
\[
\mathcal{H}(\mathcal{Y}|\mathcal{X}) = \lim_{T \to \infty} \frac{a_T}{\tau} (1 - (1 - \frac{\tau}{a_T})^T) = \frac{a_T}{\tau} (1 - e^{-\frac{\tau}{a_T}}).
\]
(57)

Through the chain rule, we have
\[
\mathcal{H}(\mathcal{Y}_c|\mathcal{X}) = \mathcal{H}(\mathcal{Y}|\mathcal{X}) + \lim_{T \to \infty} \frac{1}{T} H(y_{T+1}, y_{T+2}, \ldots, y_{(1+\varepsilon)T}|x_T, y_T).
\]
Since
\[
\varepsilon \tau \geq H(y_{T+1}, y_{T+2}, \ldots, y_{(1+\varepsilon)T}|x_T, y_T)
\]
\[
\geq H(y_{T+1}, y_{T+2}, \ldots, y_{(1+\varepsilon)T}|x_T, y_T, \mathbf{g}_{\frac{T}{\tau}})
\]
\[
= H(x_{T+1}, x_{T+2}, \ldots, x_{(1+\varepsilon)T}|x_T, y_T, \mathbf{g}_{\frac{T}{\tau}})
\]
\[
= H(x_{T+1}, x_{T+2}, \ldots, x_{(1+\varepsilon)T}) = \varepsilon \tau,
\]
we conclude that
\[
F(\tau, \varepsilon) = \mathcal{H}(\mathcal{Y}_c|\mathcal{X}) = \frac{a_T}{\tau} (1 - e^{-\frac{\tau}{a_T}}) + \varepsilon.
\]
(58)
From (56), we have
\[ p(y_{(1:t)}|x_{(1:t)}; g_{(1:t)}) = \prod_{t=1}^{\tau} p(y_t|x_t; g_{(1:t)}) , \]
where \( p(y_t|x_t; g_{(1:t)}) = \mathbb{I}_{y_t=x_t \oplus g_{y_t}} \) for all \( t \). It is clear that Assumption A3 is met and \( x_t \) are iid independent of \( g_{(1:t)} \). Theorem 3 yields
\[ H(Y_{\tau}|X_{\tau}) = \begin{cases} \frac{2}{\tau}(1-e^{-\frac{2}{\tau}}(1+\varepsilon)), & \varepsilon \leq \delta; \\ \frac{2}{\tau}(1-e^{-\frac{2}{\tau}}(1+\delta)) + (\varepsilon - \delta), & \delta < \varepsilon, \end{cases} \]
for \( \varepsilon, \delta \in (-1, \frac{1}{\tau}-1) \). Then, Corollary 1 yields
\[ H'(Y_{\tau}|X_{\tau}) = \frac{\partial H(Y_{\tau}|X_{\tau})}{\partial \varepsilon} = e^{-\frac{2}{\tau}(1+\varepsilon)}, \]
\[ H'(Y_{\tau}|X_{\tau}) = \frac{\partial H(Y_{\tau}|X_{\tau})}{\partial \varepsilon} = \begin{cases} e^{-\frac{2}{\tau}(1+\varepsilon)}, & \varepsilon < \delta; \\ 1, & \varepsilon > \delta. \end{cases} \]
Therefore, Assumption A1 holds, and Lemma 1 allows us to conclude that A2 also holds.

From Theorems 3 and 4 we obtain
\[ T'(X; \mathcal{Y}) = 1 - e^{-\frac{2}{\tau}}, \]
\[ T(X; \mathcal{Y}) \geq (1-\tau)(1-e^{-\frac{2}{\tau}}). \]  (59)

Also, (48) yields \( \tau_{\text{opt}} = \arg\max_{\tau}(1-\tau)(1-e^{-\frac{2}{\tau}}) \), or
\[ \tau_{\text{opt}} = \begin{cases} -a \ln a, & a \to 0; \\ \frac{1}{e}, & a \to \infty; \\ \tau_{\text{opt}} - a, & a = \frac{1}{e}. \end{cases} \]  (60)

When \( a \) is small, \( \tau_{\text{opt}} \) is larger than \( a \); when \( a \) is large, \( \tau_{\text{opt}} \) saturates at \( \frac{1}{e} \); and \( a = \frac{1}{e} \) is the dividing line between \( \tau_{\text{opt}} > a \) and \( \tau_{\text{opt}} < a \). The corresponding rates per receive symbol are
\[ R_{\text{opt}} = \begin{cases} (1 + a \ln a)(1-a), & a \to 0; \\ \frac{1}{e}, & a \to \infty; \\ (1-\frac{1}{e})^2, & a = \frac{1}{e}. \end{cases} \]  (61)

The next section generalizes our results from scalar inputs and outputs to higher dimensional objects and develops a channel coding theorem to provide operational significance to \( (1-\tau)T'(X; \mathcal{Y}) \).

V. GENERALIZATION TO HIGH-DIMENSIONAL PROCESSES, AND CHANNEL CODING THEOREM

A. High-dimensional processes

We define the input and output processes as \( \mathcal{X} = (x_1, x_2, \ldots) \) and \( \mathcal{Y} = (y_1, y_2, \ldots) \), which now comprise vectors, matrices, or tensors. For simplicity, we consider \( x_t \) and \( y_t \) as vectors. Denote \( X_t = [x_1, x_2, \ldots, x_t] \) and \( X^t = [x_t, x_{t+1}, \ldots, x^t] \), and similar notation is used for \( Y_t \) and \( Y^t \). The notation here differs from previous sections; we use \( x_t \) as the \( t \)-th vector in the process, and \( X_t, X^{t+1} \) as the collection of first \( t \) vectors and the last \( \frac{\tau}{2}-t \) vectors in the process. The vectors \( y_t \) have length \( N \), which is generally a function of blocklength \( \frac{\tau}{2} \).

Similarly to (2), (3), (4), and (10), we define
\[ H'(Y_{\tau}|X_{\tau}) = \lim_{T \to \infty} \frac{1}{N} H(Y_{(1+\varepsilon)T+1}|X_{(1+\varepsilon)T}, Y_{(1+\varepsilon)T}), \]  (62)
\[ H'(Y_{\tau}|X_{\tau}) = \lim_{T \to \infty} \frac{1}{N} H(Y_{(1+\varepsilon)T+1}|X_{(1+\varepsilon)T}, Y_{(1+\varepsilon)T}), \]  (63)
\[ T'(X; \mathcal{Y}) = \lim_{T \to \infty} \frac{1}{N} I(X_{(1+\varepsilon)T+1}; Y_{(1+\varepsilon)T}, Y_{(1+\varepsilon)T}), \]  (64)
\[ H(Y_{\tau}|X_{\tau}) = \lim_{T \to \infty} \frac{1}{NT} I(Y_{(1+\varepsilon)T}|X_{(1+\varepsilon)T}). \]  (65)

Furthermore, similar to (39) and (40), we define
\[ T'(X; \mathcal{Y}) = \lim_{T \to \infty} \frac{1}{N} I(X_{(1+\varepsilon)T}; Y_{(1+\varepsilon)T+1}, Y_{(1+\varepsilon)T}), \]  (66)
\[ T(X; \mathcal{Y}) = \lim_{T \to \infty} \frac{T}{NT} I(X_{(1+\varepsilon)T+1}; Y_{(1+\varepsilon)T+1}, Y_{(1+\varepsilon)T}), \]  (67)
where \( P(T, \tau) \) is defined as in (27)
\[ P(T, \tau) = \{ p(y|x; G_{\tau}), p(G_{\tau}), p(X_T), p(X_{T+1}) \}. \]  (68)

and \( G_{\tau} \) are the unknown parameters.

Theorems 1-5, Corollary 1, and Lemma 1 can all be generalized, and here we only show the generalization of Theorem 1 for simplicity:

**Theorem 1.** If there exists a \( \kappa > 0 \) so that \( H(Y_{t+1}|Y_t) \) is monotonic in \( t \) when \( t \in [(1+\varepsilon-\kappa)T, (1+\varepsilon+\kappa)T] \), then
\[ H'(Y_{\tau}) = \frac{\partial H(Y_{\tau})}{\partial \varepsilon}. \]  (69)

For the generalizations of Theorems 2, 3, Corollary 1, and Lemma 1 the definitions in (62)-(67) are used.

B. Channel coding theorem

We now provide an operational description of the mutual information inequality (44). We consider a communication system where the channel is constant for blocklength \( \frac{\tau}{2} \), and then changes independently and stays constant for another blocklength, and so on. The
According to the definition in (39), we have
\[ I(x_{T+1}; y_{T+1} | x_T, y_T) > I'(X; Y) - \kappa, \]
and (74) yields
\[ R_T > (1 - \tau)(I'(X; Y) - \kappa), \]
which means any rate \( R \leq (1 - \tau)(I'(X; Y) - \kappa) \) is achievable.

By taking the limit \( \kappa \downarrow 0 \), we finish the proof. \( \square \)

This theorem shows that rates below \((1 - \tau)I'(X; Y)\) are achievable when \( T \) is chosen large enough. Only an achievability statement is given here since \((1 - \tau)I'(X; Y)\) is a lower bound on \( R_T \) for large \( T \).

VI. DISCUSSION AND CONCLUSION

A. Number of unknowns and bilinear model

In general, a finite number of unknowns in the model leads to uninteresting results as \( T \to \infty \). For example, consider a system modeled as
\[ y_t = gx_t + v_t, \quad t = 1, 2, \ldots \]
where \( g \) is the unknown gain of the system, \( x_t, y_t \) are the input and corresponding output, \( v_t \) is the additive noise, \( \tau \) is the fraction of time used for training. This system is bilinear in the gain and the input. We assume that \( v_t \) is modeled as iid Gaussian \( \mathcal{N}(0, 1) \), independent of the input. The training signals are \( x_t = 1 \) for all \( t = 1, 2, \ldots, T \), and the data signals \( x_t \) are modeled as iid Gaussian \( \mathcal{N}(0, 1) \) for all \( t = T + 1, T + 2, \ldots \). An analysis similar to Example 5 produces

\[ I(X; Y) \geq \frac{1 - \tau}{2} \mathbb{E}_g \log(1 + g^2), \]

and therefore \( \tau_{\text{opt}} = 0 \) maximizes this bound. This result reflects the fact that \( g \) is learned perfectly for any \( \tau > 0 \) because there is only one unknown parameter for \( T \) training symbols as \( T \to \infty \). Hence, trivially, it is advantageous to make \( \tau \) as small as possible.

More interesting is the “large-scale” model

\[ y_t = f(Gx_t + v_t), \quad t = 1, 2, \ldots, \]
where \( x_t \) and \( y_t \) are the \( t \)th input and output vectors with dimension \( M \) and \( N \), \( G \) is an \( N \times M \) unknown random matrix that is not a function of \( t \), \( v_1, v_2, \ldots \) are iid unknown vectors with dimension \( N \) and known distribution (not necessarily Gaussian), and \( f(\cdot) \) applies a possibly nonlinear function \( f(\cdot) \) to each element of its input. The training interval \( T \) is used to learn \( G \). Let \( M \)
and $N$ increase proportionally to the blocklength $\frac{T}{\tau}$, and define the ratios
\[ \alpha = \frac{N}{M}, \quad \beta = \frac{T}{\tau M}. \] (77)

This model can be used in large-scale wireless communication, signal processing, and machine learning applications. In wireless communication and signal processing [14]–[18], [26], [28], [31], $x_t$ and $y_t$ are the transmitted signal and the received signal at time $t$ in a multiple-input–multiple-output (MIMO) system with $M$ transmitters and $N$ receivers, $G$ models the channel coefficients between the transmitters and receivers, $\frac{T}{\tau}$ is the coherence time during which the channel $G$ is constant, $v_t$ is the additive noise at time $t$, $f(\cdot)$ models receiver effects such as quantization in analog-to-digital converters (ADC’s) and nonlinearities in amplifiers. A linear receiver, $f(x) = x$, is considered in [14], [15], [26], [31]. Single-bit ADC’s with $f(x) = \text{sign}(x)$ are considered in [17], [18], and low-resolution ADC’s with $f(x)$ modeled as a uniform quantizer are considered in [27], [28]. The training and data signals can be chosen from different distributions, as in [16]–[18]. Conversely, the training and data signals can both be iid, as in [15], [26]–[28].

In machine learning, (76) is a model of a single layer neural network (perceptron) [22]–[24] and $x_t$ is the input to the perceptron with dimension $M$. $y_t$ is the scalar decision variable ($N = 1$) at time $t$, $G$ holds the unknown weights of the perceptron, and $f(\cdot)$ is the nonlinear activation function. A perceptron is often used as a classifier, where the output of the perceptron is the class label of the corresponding input. In [22], [23], iid inputs are used to learn the weights, and orthogonal inputs are used in [24]. Binary class classifiers are considered in [22], [24]. Training employs $T$ labeled input–output pairs $(x_t, y_t)$, and the trained perceptron then classifies new inputs before it is retrained on a new dataset. Generally, both the training and data are modeled as having the same distribution.

To obtain optimal training results for (76), Theorems [3] and [5] show that a starting point for computing $\mathcal{I}'(\mathcal{X};\mathcal{Y})$ is $\mathcal{H}(\mathcal{Y}_t|\mathcal{X})$ for $\varepsilon \geq 0$. Fortunately, $\mathcal{H}(\mathcal{Y}_t|\mathcal{X})$ results can sometimes be found in the existing literature; in [26]–[28], $\mathcal{H}(\mathcal{Y}_t|\mathcal{X})$ is used to calculate the mean-square error of the estimated input signal, conditioned on the training. We may employ these same $\mathcal{H}(\mathcal{Y}_t|\mathcal{X})$ results to quickly derive the training-based mutual information using the derivative analysis presented herein. Part II of this paper [32] focuses on this.

B. Models for which assumptions are superfluous

Assumptions A1 and A2 presented in the Introduction are likely superfluous for certain common system models, such as when the distributions on $x_t$ are iid through the training and data phases, and the transition probabilities can be written as a product as in Assumption A3. However, as Lemma [1] shows, we have not yet characterized for which models A1 and A2 are automatically satisfied without additional assumptions on $\mathcal{H}'$, and think that this would be an interesting research topic for further work.

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