THE CODISC RADIUS CAPACITY

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Abstract. We prove a generalization of Gromov’s packing inequality to symplectic embeddings of the boundaries of two balls such that the bounded components of the complements of the image spheres are disjoint. Moreover, we define a capacity which measures the size of Weinstein tubular neighborhoods of Lagrangian submanifolds. In symplectic vector spaces this leads to bounds on the codisc radius for any closed Lagrangian submanifold in terms of Viterbo’s isoperimetric inequality. Furthermore, we introduce the spherical variant of the relative Gromov radius and prove its finiteness for monotone Lagrangian tori in symplectic vector spaces.

1. Introduction

A symplectic manifold $(V,\omega)$ is a smooth $2n$-dimensional manifold $V$ together with a non-degenerate closed 2-form $\omega$. The most important examples of symplectic manifolds are $\mathbb{R}^{2n}$ with

$$\text{d}x \wedge \text{d}y := \text{d}x_1 \wedge \text{d}y_1 + \ldots + \text{d}x_n \wedge \text{d}y_n,$$

the total space of cotangent bundles $\pi : T^*Q \to Q$ of smooth manifolds $Q$ with $\text{d}\lambda_{\text{can}}$, where

$$(\lambda_{\text{can}})_u = u \circ T\pi$$

is the Liouville 1-form on $T^*Q$, and all Kähler manifolds with its Kähler form. A symplectomorphism is a diffeomorphism $\varphi$ that preserves the symplectic form $\omega$ in the sense that $\varphi^*\omega = \omega$. Symplectomorphisms can be obtained from compactly supported Hamiltonian functions $H$ on $V$ as the time-1-map of the Hamiltonian vector field $X_H$ which is defined via $i_{X_H}\omega = -\text{d}H$. The canonical lift of diffeomorphisms on a smooth manifold $Q$ to $T^*Q$ gives another class of examples.

The first observation, which is due to Liouville, is that a symplectomorphism preserves the symplectic volume $\frac{1}{n!}\omega^n$ of a symplectic manifold $(V,\omega)$. A natural question is to what extend a symplectormorphism is more special than a volume
holds. We call the restriction of a symplectic embedding of a neighborhood of the cylinder a symplectic embedding of a neighborhood of the cylinder preserving diffeomorphism. In [26] Gromov gave the following answer. Consider the cylinder
\[ Z_R = \{ x_1^2 + y_1^2 < R \} \times \mathbb{R}^{2n-2} \]
of radius \( R \). Observe that for any radius \( r \) there exists a linear volume preserving diffeomorphism of \( \mathbb{R}^{2n} \) that maps the open ball \( B_r \) of radius \( r \) into \( Z_R \). But if \( B_r \) embeds into \( Z_R \) symplectically Gromov’s non-squeezing theorem implies \( r < R \), see [26, 0.3.A]. More generally, if only a neighborhood of the sphere \( S^{2n-1}_r = \partial B_r \) in \( \mathbb{R}^{2n} \) embeds symplectically into \( Z_R \) it was shown in [24, 45, 46, 51] that \( r < R \) holds. We call the restriction of a symplectic embedding of a neighborhood of \( S^{2n-1}_r \subset \mathbb{R}^{2n} \) a \textbf{symplectic embedding} of the sphere \( S^{2n-1}_r \).

In [26, 0.3.B] Gromov proved that if two open balls \( B_{r_1} \) and \( B_{r_2} \) embed symplectically into \( B_R \) with disjoint images then the packing inequality \( r_1^2 + r_2^2 < R^2 \) holds. Ziltener asked whether a packing obstruction holds for symplectic embeddings of spheres \( S^{r_1} \) and \( S^{r_2} \) such that bounded components of the complements of the images are disjoint. If \( n = 2 \) this follows from the packing inequality for balls because the existence of embeddings of \( S^3 \) into \( \mathbb{R}^4 \) implies the existence of embeddings of \( B^4 \) into \( B^4 \), see [26, 0.3.C]. In Section 2.2 we will prove the following:

**Theorem 1.1.** Let \( n \geq 2 \). Assume that \( S^{r_1} \cup S^{r_2} \) embeds symplectically into \( B_R \subset \mathbb{R}^{2n} \) such that the bounded components of the complements of the images are disjoint. Then \( r_1^2 + r_2^2 < R^2 \).

Symplectic embeddings of balls \( B_r \subset \mathbb{R}^{2n} \) into symplectic manifolds \((V, \omega)\) exist by Darboux’s theorem. Any point has a neighborhood symplectomorphic to \( B_r \) provided \( r > 0 \) is sufficiently small, cf. [33]. A similar statement is due to Weinstein [48]. Consider a closed Lagrangian submanifold \( L \) of \((V, \omega)\), which is a submanifold such that \( \omega|_L = 0 \). A neighborhood of the zero section \( L \) in the cotangent bundle \( T^*L \) is symplectomorphic to a neighborhood of \( L \) in \((V, \omega)\). A natural question is how large such a neighborhood can be.

Consider for example a closed Lagrangian submanifold \( L \) of \( \mathbb{R}^{2n} \). We provide \( L \) with the metric \( g \) induced from \( \mathbb{R}^{2n} \). Denote by \( D^r_rL \) the \textbf{r-codisc bundle} of \( L \), which is the subset of \( T^*L \) consisting of all covectors of length at most \( r \). By Weinstein’s neighborhood theorem [48] \( D^r_rL \) embeds symplectically into \( \mathbb{R}^{2n} \) for \( r > 0 \) sufficiently small identifying the zero section of \( T^*L \) with \( L \). We estimate the largest radius \( r \) such that the \( r \)-codisc bundle embeds symplectically in terms of the volume \( \text{vol}(L) \) and \( \text{length of the shortest non-trivial closed geodesic} \) \( \text{inf}(g) \), which is bounded from below by the injectivity radius of the metric \( g \). The estimate is obtained using Viterbo’s isoperimetric inequality [47], see Section 3.2.

**Theorem 1.2.** There exists a positive constant \( \rho_n \) such that for any symplectic embedding of \( D^r_rL \) into \( \mathbb{R}^{2n} \) the codisc radius \( r \) satisfies
\[ r^n \leq \rho_n \frac{\text{vol}(L)^2}{\text{inf}(g)^n}. \]

The \textbf{Gromov radius} is defined by
\[ c_B(V, \omega) = \sup \{ r | r^2 \text{ symplectic embedding } B_r \hookrightarrow (V, \omega) \}, \]
see [26]. Due to Gromov’s non-squeezing theorem the quantity \( c = c_B \) defines a \textbf{symplectic capacity}, i.e., according to Ekeland-Hofer [19], \( c \) satisfies the following conditions: (Monotonicity) \( c(V, \omega) \leq c(V', \omega') \) provided that there exists
a symplectic embedding \((V, \omega) \hookrightarrow (V', \omega')\).  (Conformality) For any positive real number \(a\) we have \(c(V, a\omega) = ac(V, \omega)\).  (Normalization) The capacity of the open unit ball \(B\) and the open unit cylinder \(Z\) are equal to \(c(B) = \pi = c(Z)\).  Similarly, for all symplectic manifolds \((V, \omega)\) of dimension \(\geq 4\) the \textbf{spherical capacity}

\[ s(V, \omega) := \sup \{ r^2 \mid \exists \text{ symplectic embedding } S^{2n-1}_r \hookrightarrow (V, \omega) \} \]

is a symplectic capacity.  This follows from the spherical non-squeezing theorem, see [24, 45, 46, 51].

Given a symplectic manifold \((V, \omega)\) a \textbf{special capacity} on subsets \(U \subset V\) is a real number \(c(U, \omega) \in [0, \infty]\) satisfying the above conformality condition and the following:  (Non-triviality) \(c(B)\) is positive and \(c(Z)\) is finite;  (Relative monotonicity) If there exists a symplectomorphism of \((V, \omega)\) which maps \(U_1\) into \(U_2\) then \(c(U_1, \omega) \leq c(U_2, \omega)\).

We introduce a quantity that measures the size of symplectic neighborhoods of Lagrangian submanifolds.  For subsets \(U\) in \(V\) the \textbf{codisc radius capacity} is defined by

\[ c_D(U) := \sup \{ r \inf(g) \mid (L, g) \subset U \text{ and } r > 0 \}, \]

where the supremum is taken over all closed Lagrangian submanifolds \(L \subset U\) of \((V, \omega)\), over all Riemannian metrics \(g\) on \(L\), and over all \(r > 0\) such that \(D_t^*(g)L\) embeds symplectically into \((V, \omega)\) mapping the zero section onto \(L\).  By Weinstein’s neighborhood theorem [48] the codisc radius capacity \(c_D(U)\) is positive for all open subsets \(U\) and all closed Lagrangian submanifolds \(U = L\) of \((V, \omega)\).  In Section 3.4 we will prove the following:

**Theorem 1.3.** The codisc radius capacity \(c_D\) is a special capacity such that

\[ c_D(Z) = \pi, \quad c_D(B) \geq \frac{\pi}{n}, \quad \text{and} \quad c_D(P) = \pi, \]

where \(Z = Z_1\) is the open unit cylinder, \(B = B_1\) is the open unit ball, and

\[ P = \{ x_1^2 + y_1^2 < 1, \ldots, x_n^2 + y_n^2 < 1 \} \]

is the open unit polydisc.

A more sensible method to measure the size of a symplectic neighborhood of a closed Lagrangian submanifold \(L\) of \(\mathbb{R}^{2n}\) was introduced by Barraud, Biran and Cornea [5, 6, 9, 10].  A symplectic embedding of the open ball \(B_r \subset \mathbb{R}^{2n}\) of radius \(r\) into \(\mathbb{R}^{2n}\) is called to be \textbf{relative} to \(L\) if the real part

\[ D_r = B_r \cap \mathbb{R}^n \]

of the ball is mapped to \(L\) and if the complement \(B_r \setminus D_r\) is mapped to \(\mathbb{R}^{2n} \setminus L\).  The \textbf{relative Gromov radius} is defined by

\[ c_B(L) = \sup \{ r^2 \mid \exists \text{ relative symplectic embedding } (B_r, D_r) \hookrightarrow (\mathbb{R}^{2n}, L) \}, \]

see [5, 6, 9, 10].  A Lagrangian submanifold \(L\) of \(\mathbb{R}^{2n}\) is called \textbf{monotone} if the Liouville class of \(L\) and the Maslov class of \(L\) are positively proportional.  Finiteness of \(c_B(L)\) follows for monotone Lagrangian tori \(L\) with [10, Theorem 1.2.2], cf. [13, 18].  As pointed out by McDuff [37, p. 125] it is not known whether the relative Gromov radius is finite in general.

Similar to the spherical capacity [51] we consider symplectic embeddings of neighborhoods \(U \subset \mathbb{R}^{2n}\) of \(S_r^{2n-1} = \partial B_r\) into \(\mathbb{R}^{2n}\) such that the induced neighborhood
$U \cap \mathbb{R}^n$ of the equatorial sphere $S^{n-1}$ is mapped to $L$. We call those embeddings to be relative to $L$ if no point from $U \setminus \mathbb{R}^n$ is mapped to $L$. The relative spherical Gromov radius is defined by

$$s(L) = \sup \{ \pi r^2 \mid \exists \text{ relative symplectic embedding } (S^{2n-1}, S^n) \hookrightarrow (\mathbb{R}^{2n}, L) \}.$$ 

Notice that $c_B(L) \leq s(L)$. We show finiteness of $s(L)$ for monotone Lagrangian tori. The proof is given in Section 4.1.

**Theorem 1.4.** Let $2n \geq 4$. The relative spherical Gromov radius is finite for any monotone Lagrangian torus in $\mathbb{R}^{2n}$.

2. Packing with empty balls

The aim of this section is to prove Theorem 1.1.

2.1. Stretching the neck. We equip $\mathbb{R}^{2n}$ with the symplectic form $dx \wedge dy$. A closed hypersurface $M$ is of restricted contact type in $\mathbb{R}^{2n}$ provided there exists a primitive 1-form of $dx \wedge dy$ that restricts to a contact form $\alpha$ on $M$. In particular $(M, \alpha)$ is a contact manifold. We denote by $\inf(\alpha)$ the minimal period of a closed Reeb orbit on $(M, \alpha)$. Because $(M, \alpha)$ appears as a hypersurface of restricted contact type in the present context $\inf(\alpha)$ is the minimal positive action of a closed characteristic on $M$.

**Theorem 2.1.** Let $n \geq 2$. Let $(M, \alpha) = (M_1, \alpha_1) \sqcup (M_2, \alpha_2)$ be a closed hypersurface in $\mathbb{R}^{2n}$ with two connected components $M_1$ and $M_2$ which are of restricted contact type and bound disjoint compact domains $W = W_1 \sqcup W_2$ in $\mathbb{R}^{2n}$, resp. If $M$ is contained in the open ball $B_R$ of radius $R$ then

$$\inf(\alpha_1) + \inf(\alpha_2) < \pi R^2.$$ 

**Proof.** The proof is an application of the compactness result in [31]. We assume the contact form $\alpha$ to be generic in the sense that 1 is not an eigenvalue of the linearized Poincaré return map for all closed Reeb orbits on $(M, \alpha)$. If $\alpha$ is not generic we replace $(M, \alpha)$ by the graph of a positive function on $M$ inside a symplectic tubular neighborhood $((-\varepsilon, \varepsilon) \times M, d(e^s \alpha))$. In view of [30, Proposition 6.1] there is a dense set of positive functions on $M$ such that the contact form obtained by restriction of $e^s \alpha$ to its graphs is generic. An application of the Arzelà-Ascoli theorem, see [33], and the Liouville flow induced by $\alpha$ allow us to undo the perturbation. For notational convenience we assume $R = 1$ so that $(M, \alpha)$ is a hypersurface of restricted contact type in the open unit ball $B$. Invoking an argument used in [25, Corollary 3.7] we find a primitive 1-form $\lambda$ of $dx \wedge dy$ which is equal to $\frac{1}{2} (x \, dy - y \, dx)$ on a neighborhood of $\mathbb{R}^{2n} \setminus B$ such that $\lambda|_{TM} = \alpha$. Collapsing the boundary sphere $S^{2n-1}$ to the hyperplane $\mathbb{C}P^{n-1}$ at infinity yields a symplectic embedding $B \subset \mathbb{C}P^n$, where $\mathbb{C}P^n$ is provided with the Fubini-Study symplectic form $\omega$. Recall that

$$\int_{\mathbb{C}P^n} \omega = \pi,$$ 

that $\mathbb{C}P^n$ is a monotone symplectic manifold, and that through any two distinct points in $\mathbb{C}P^n$ it passes a unique complex line. With [26, 0.2.B] we have that for any compatible almost complex structure on $\mathbb{C}P^n$ and any pair of distinct points $p_1 \in W_1$ and $p_2 \in W_2$ there exists a possibly non-unique holomorphic sphere through $p_1$ and $p_2$, which is homologous to $\mathbb{C}P^1$. For each $N \in \mathbb{N}$ we choose an almost
complex structure $J_N$ which is equal to the complex structure of $\mathbb{C}P^n$ restricted to $\mathbb{C}P^{n-1}$. Moreover, in a neighborhood of $M$ the almost complex structure $J_N$ is subject to the process of stretching the neck: A neighborhood of $M \subset B$ is symplectomorphic to \(([-\varepsilon, \varepsilon] \times M, d(e^s\alpha))\) for $\varepsilon > 0$. We assume that the points $p_1$ and $p_2$ inside $W$ are contained in the complement of this neighborhood. Denote by $V$ the concave filling cut out of $\mathbb{C}P^n$ by $(M, \alpha)$. We form a symplectic manifold
\[ W \cup \([N, N] \times M\) \cup V \]
by identifying the $\varepsilon$-collar neighborhoods of $M$ in $W$ and $V$ with the corresponding $\varepsilon$-collars on the neck
\[ [-N, \varepsilon, N + \varepsilon] \times M, \]
see [22, p. 273–276]. The symplectic form on $W \cup V$ is $\omega$ and on the neck $d(\tau \alpha)$, where $\tau$ is a smooth strictly increasing function on $[-N - \varepsilon, N + \varepsilon]$ that equals $e^{+N}$ on $[-N - \varepsilon, -N - \varepsilon/2]$ and $e^{-N}$ on $[N + \varepsilon/2, N + \varepsilon]$. The resulting manifold is symplectomorphic to $\mathbb{C}P^n$. A symplectomorphism is obtained by following the Liouville flow parallel to the $R$-direction, see [31, p. 158]. To finish the construction of $J_N$ it suffices to define $J_N$ on $[-N - \varepsilon, N + \varepsilon] \times M$. Choose a complex structure $j$ on the contact structure $\ker \alpha$ that is compatible with $d\alpha$. By definition $J_N$ is the unique translation invariant almost complex structure which sends $\partial_s$ to the Reeb vector field of $\alpha$ and coincides with $j$ on $\ker \alpha$. Under the identifying symplectomorphism this defines $J_N$ near $M$. On $\mathbb{C}P^n \setminus ((-\varepsilon, \varepsilon) \times M)$ the almost complex structure $J_N$ does not depend on $N$. By the above discussion we find for each $N \in \mathbb{N}$ a $J_N$-holomorphic map
\[ w_N: \mathbb{C}P^1 \longrightarrow \mathbb{C}P^n, \quad C_N := w_N(\mathbb{C}P^1), \]
such that $p_1, p_2 \in C_N, C_N$ intersects $\mathbb{C}P^{n-1}$ positively in exactly one point, and $C_N$ has energy
\[ \int_{C_N} \omega = \pi. \]
We quote the compactness result on [31, p. 192–193], which applies to the present situation because $M$ is of restricted contact type in $B$. Therefore, formulated in the language of [11], a subsequence of $w_N$ converges to a holomorphic building. The lowest level of the building, which corresponds to components in $W \cup ([0, \infty) \times M)$, consists of finite energy planes only, cf. [31, p. 193, Fig. 14]. At least two of them, say $u_1$ and $u_2$, pass through the points $p_1$ and $p_2$, resp. Moreover, the total Hofer-energy
\[ E(u) = \sup_{\tau} \int_{\mathbb{C}} u^* \omega_\tau \]
satisfies
\[ E(u_1) + E(u_2) < \pi, \]
see [31]. Here the supremum is taken over all smooth strictly increasing functions $\tau$ on $[-\varepsilon, \infty)$ that agree with $e^s$ on $[-\varepsilon, -\varepsilon/2]$ and tend to 1 as $s \to \infty$. The symplectic form $\omega_\tau$ equals $d\lambda$ on $W \setminus ([\varepsilon, \varepsilon] \times M)$ and $d(\tau \alpha)$ on $([-\varepsilon, 0] \times M)$. Because the primitive $\lambda$ extends to $\tau \alpha$ on the cylindrical end the finite energy planes $u_1$ and $u_2$ are asymptotic to closed Reeb orbits of period less or equal to its Hofer-energy, see [28] and cf. [25, Lemma 6.3]. In other words, we have found closed Reeb orbits in each component of $(M_1, \alpha_1) \cup (M_2, \alpha_2)$ having period $T_1$ and $T_2$, resp., such that $T_1 + T_2 < \pi$. \[ \Box \]
2.2. **Proof of Theorem 1.1.** The image $S_r$ of a symplectic embedding of $S^{2n-1}_r$ is a hypersurface of restricted contact type. This follows with the Mayer-Vietoris sequence for the de Rham cohomology, cf. the proof of [25, Corollary 3.7]. Moreover, the minimal positive action equals $\pi r^2$. Theorem 2.1 implies that the sum of the smallest actions of $S_{r_1}$ and $S_{r_2}$ is bounded by the action of $S^{2n-1}_R$. Therefore, Theorem 1.1 follows.

2.3. **Superadditivity.** Smooth boundaries of bounded convex domains $K$ in $\mathbb{R}^{2n}$ are of restricted contact type. Moreover, the action-capacity representation theorem for the Hofer-Zehnder capacity $c_{HZ}$ implies that $c_{HZ}(K)$ is the minimal positive action of a closed characteristic on $\partial K$, see [32]. Because $c_{HZ}$ has inner regularity Theorem 2.1 yields that

$$c_{HZ}(K_1) + c_{HZ}(K_2) \leq c_{HZ}(B_R)$$

for disjoint convex subsets $K_1$ and $K_2$ of $B_R$. Artstein-Avidan and Ostrover [4] proved that

$$c_{HZ}(K_1)^{1/2} + c_{HZ}(K_2)^{1/2} \leq c_{HZ}(K_1 + K_2)^{1/2}$$

for bounded convex sets without assuming that $K_1$ and $K_2$ are disjoint. Furthermore, [34, Corollary 1.3] says that

$$c_{HZ}(U_1) + c_{HZ}(U_2) \leq c_{HZ}(B_R)$$

for open disjoint subsets $U_1$ and $U_2$ of $B_R$. In particular, $c_{HZ}$ tends to zero on $B_1 \setminus B_{1-\varepsilon}$ as $\varepsilon \to 0$ while the spherical capacity [51] as well as the orbit capacity [25, 24] are equal to $\pi$ for all $\varepsilon \in (0,1)$. Hence, neither the spherical capacity nor the orbit capacity equal the Hofer-Zehnder capacity.

2.4. **Codisc radii.** In the proof of Theorem 2.1 the existence of finite energy surfaces contained in the symplectic filling $W$ follows without making use of the restricted contact type property of $M \subset B$. In order to obtain the estimates on the periods it suffices that the filling $W$ is exact. Hence, Theorem 2.1 continues to hold e.g. for images of codisc bundles.

**Corollary 2.2.** Let $n \geq 2$. Let $M = M_1 \cup M_2$ be a closed hypersurface in $\mathbb{R}^{2n}$ with two connected components that bound disjoint compact domains $W = W_1 \cup W_2$. Let $\lambda$ be a primitive 1-form of $dx \wedge dy$ on $W$ such that $\alpha_1 = \lambda|_{T M_1}$ and $\alpha_2 = \lambda|_{T M_2}$ are contact forms. If $M \subset B_R$ then

$$\inf(\alpha_1) + \inf(\alpha_2) < \pi R^2.$$  

**Proof.** We assume the splitting situation of the proof of Theorem 2.1 and consider again the sequence $w_N$ of $J_N$-holomorphic spheres. The energy

$$\int_{C_N} \omega = \int w_N^* \omega + \int w_N^* d(\tau \alpha)$$

decomposes into the sum of two integrals. The first one is taken over

$$w_N^{-1}(CP^n \setminus ([-\varepsilon, \varepsilon] \times M))$$

and the second over

$$w_N^{-1}([-N - \varepsilon, N + \varepsilon] \times M)).$$

Here $\tau$ is a smooth increasing function on $[-N - \varepsilon, N + \varepsilon]$ that equals $e^{s+N}$ on $[-N - \varepsilon, -N - \varepsilon/2]$ and $e^{s-N}$ on $[N + \varepsilon/2, N + \varepsilon]$. Notice that the integral is independent of the choice of $\tau$, cf. [31, p. 159]. Smoothing the function that is
defined to be $e^{s+N}$ on $[-N - \varepsilon, -N]$, 1 on $[-N, N]$, and $e^{s-N}$ on $[N, N + \varepsilon]$, by functions $\tau$ as described we get

$$\int_{C_N} \omega = \int w_N^N \omega + \int w_N^N d\alpha,$$

where the first integral is taken over

$$w_N^{-1}(CP^n \setminus M)$$

and the second over

$$w_N^{-1}([-N, N] \times M).$$

By [11, Lemma 9.2 and Theorem 10.3] there exists a subsequence of $w_N$ which converges to a holomorphic building. Its lowest level contains energy surfaces $u_1$ and $u_2$ such that $p_1 \in \text{im}(u_1)$ and $p_2 \in \text{im}(u_2)$. [11, Proposition 5.6] implies that these energy surfaces are asymptotic to finitely many periodic Reeb orbits on $(M, \alpha)$ with total period $T$. With [11, Lemma 9.1] their $\omega$-energy

$$E_{\omega}(u_1) + E_{\omega}(u_2)$$

is less than $\pi$, where

$$E_{\omega}(u) = \int_{w^{-1}(W)} u^* d\lambda + \int_{w^{-1}([0, \infty) \times M)} u^* d\alpha.$$

Therefore, by smoothing out the integrand and employing an approximation argument as above, we see that the total period $T$ is less than $\pi$. □

**Remark 2.3.** Assuming the situation of Corollary 2.2 let $Y$ be the Liouville vector field on $(W, dx \wedge dy)$ defined by $\lambda$. Because $W$ is compact the flow of $Y$ exists on $(-\infty, 0]$. Therefore, $W$ decomposes into the Lagrangian skeleton $Y^{-1}(0)$ and the negative half-symplectization $((-\infty, 0] \times M, d(e^s \alpha))$. If $Y^{-1}(0)$ represents a cycle of dimension at most $2n - 3$ then the finite energy surfaces $u_1$ and $u_2$ obtained in the proof of Corollary 2.2 can be homotoped with fixed (asymptotic) boundary conditions into $M$. This can be used to estimate the minimal total period $\inf(\alpha)$ of a null-homologous Reeb link in $(M, \alpha)$ introduced in [24]. Theorem 2.1 and Corollary 2.2 generalize accordingly.

**Remark 2.4.** We consider a Riemannian manifold $(L, g)$. Using the metric $g$ we identify the tangent bundle of $L$ with $T^* L$. The canonical Liouville 1-form of $T^* L$ induces a contact form

$$\alpha = \lambda_{\text{can}}|_{T^*_r(g)L}$$

on the cosphere bundle $S^*_r(g)L$ of radius $r$. According to [22] non-trivial closed geodesics on $(L, g)$ and closed Reeb orbits on $S^*_r(g)L$ are in one-to-one correspondence. The speed curve $\tilde{c}$ of a closed geodesic $c$, which is parametrized proportional to arc length, with speed $|\tilde{c}| = r$ is contained in $S^*_r(g)L$. It defines a closed Reeb orbit $\gamma$ by reparametrizing $\tilde{c}$ by $1/r^2$. The action of $\gamma$ and the length of $c$ are related via

$$\int_{\gamma} \alpha = \int_{\tilde{c}} \lambda_{\text{can}} = r \text{length}(c).$$

The length of the shortest non-trivial closed geodesic on $(L, g)$ is denoted by $\inf(g)$, which is bounded by the injectivity radius from below. We have

$$r \inf(g) = \inf(\alpha).$$
Assume in the following that $L$ decomposes into closed submanifolds $L_1 \sqcup L_2$. The metric $g$ defines Riemannian manifolds $(L_1, g_1)$ and $(L_2, g_2)$. If the closure of the codisc bundles $D^*_r(g_1)L_1 \sqcup D^*_r(g_2)L_2$ embed symplectically into $B_R$ such that the images are disjoint Corollary 2.2 implies

$$r_1 \inf(g_1) + r_2 \inf(g_2) < \pi R^2.$$ 

If in addition $(L, g)$ has no contractible closed geodesics we obtain in view of Remark 2.3 that $2r \inf(g) \leq \inf(\alpha)$ using the bundle projection. This implies

$$2r_1 \inf(g_1) + 2r_2 \inf(g_2) < \pi R^2.$$ 

In Section 3.4 we continue the discussion on the size of symplectically embedded codisc bundles.

2.5. **More than two components.** We consider a closed hypersurface

$$M = M_1 \sqcup \ldots \sqcup M_k$$

of $B \subset \mathbb{R}^{2n}$ with $k$ connected components. We assume that the bounded components $W_1, \ldots, W_k$ of the complements of $M_1, \ldots, M_k$ are pairwise disjoint and that $dx \wedge dy$ admits a primitive 1-form on the closure of $W = W_1 \sqcup \ldots \sqcup W_k$ that restricts to contact forms $\alpha_1, \ldots, \alpha_k$ on $M_1, \ldots, M_k$, resp. In Corollary 2.2 we considered the case $k = 2$. The proof of Theorem 2.1 and Corollary 2.2 generalizes to hypersurfaces $M$ with $k \geq 3$ connected components provided that there exists a holomorphic curve $C$ through $k$ generic points for any (generic) compatible almost complex structure. Therefore,

$$\inf(\alpha_1) + \ldots + \inf(\alpha_k) < \int_C \omega.$$ 

Observe that the compactness result in [11] which we used in the above proofs applies to holomorphic curves of higher genus.

**Definition 2.5.** The smallest positive action of a closed characteristic on $M$ divided by $\pi$ is denoted by $a_k$.

It follows from Corollary 2.2 that $a_k < 1/2$ for all $k$.

**Example 2.6.** We consider $B \subset \mathbb{C}P^2$. Through $k = 3d - 1$ generic points there exists a holomorphic sphere of degree $d$, which has area $\int_C \omega = d\pi$, see [40, Proposition 7.4.8]. Therefore,

$$a_{3d-1} < \frac{d}{3d-1}.$$ 

Taking genus $\frac{1}{2}(d-1)(d-2)$ curves which pass through $k = \frac{1}{2}d(d+3)$ points in general position and whose symplectic area equals $d\pi$ into account we get

$$a_{d(d+3)/2} < \frac{2}{d+3},$$ 

see [26, 0.2.B] and [29].

### 3. THE SIZE OF A WEINSTEIN NEIGHBORHOOD

The aim of this section is to prove Theorem 1.2 and Theorem 1.3.
3.1. The action-area inequality. Let \((X, \omega)\) be a symplectic manifold which is symplectically aspherical, i.e., the symplectic area \(\int_{S^2} f^* \omega\) vanishes for all smooth maps \(f: S^2 \to X\). We assume that \((X, \omega)\) is either closed, of bounded geometry in the sense of Gromov [26], or compact with convex contact type boundary. In the latter case we replace \(X\) by its completion so that \((X, \omega)\) has positive cylindrical ends as introduced in [11].

Let \(L \subset X\) be a closed Lagrangian submanifold. The Gromov width of \(L\) is defined by

\[ \sigma(L) = \sup_J \sigma(L, J), \]

where \(\sigma(L, J)\) is the minimal symplectic area \(\int_D w^* \omega\) of a non-constant \(J\)-holomorphic disc \(w: D \to X\) with boundary on \(L\), see [26]. The supremum is taken over all almost complex structures \(J\) that are tamed by \(\omega\) and have adapted boundary or asymptotic conditions, resp. Notice that \(\sigma(L, J) = \infty\) if no such disc exists and that \(\sigma(L, J) > 0\) by Gromov’s compactness theorem, cf. [21].

Denote by \(D^* L\) the unit codisc bundle of \(L\) w.r.t. a Riemannian metric. On the unit cotangent bundle \(S^* L\) the canonical Liouville 1-form \(\lambda_{\text{can}}\) defines a contact form \(\alpha = \lambda_{\text{can}}|_{T^* L}\).

The aim is to compare the Gromov width \(\sigma(L)\) with the minimal period of a closed Reeb orbit \(\inf(\alpha)\).

**Theorem 3.1.** If the closure of \(D^* L\) embeds symplectically into \((X, \omega)\) then

\[ \inf(\alpha) < \sigma(L). \]

**Proof.** The proof is based on a stretching the neck argument along the lines of Theorem 2.1. Denote by \(M\) the image of \(S^* L\) and assume that the contact form \(\alpha\) on \(M\) is generic. Identify \(D^* L\) with its image \(W\) in \(X\) and denote the Liouville primitive of \(\omega|_W\) by \(\lambda\). Set \(V = X \setminus W\) so that \(X\) decomposes as \(W \cup M \cup V\). For each \(N \in \mathbb{N}\) we define a compatible almost complex structure \(J_N\) on \((X, \omega)\) as in the proof of Theorem 2.1 such that the sequence \(J_N\) only depends on \(N\) in the distinguished neighbourhood of \(M\). We choose \(J_N\) to be cylindrical, resp., to ensure uniform \(C^0\)-bounds on all holomorphic discs. This requires a modification of \(J_N\) in a neighborhood of \(\partial V \setminus M\), resp., near the ends of \(V\). We assume that \(\sigma(L)\) is finite. Hence, there is a sequence of \(J_N\)-holomorphic discs

\[ w_N: (D, \partial D) \to (X, L) \]

with energy

\[ 0 < \int_D w_N^* \omega \leq \sigma(L). \]

Moreover, we assume that \(w_N(0)\) is contained in \(V \setminus ([0, \varepsilon] \times M)\). As in [31, p. 163] we choose a Riemannian metric on \(X\) of bounded geometry which is independent of \(N\) on \(X \setminus ([\varepsilon, \varepsilon] \times M)\) and is equal to a product metric on the neck \([-N, N] \times M\). An application of the mean value theorem to the path \(w_N(t), t \in [0, 1]\), shows that there are no uniform gradient bounds on \(w_N\). In other words, after passing to a subsequence \(w_{\nu}\), there exists a sequence \(z_{\nu} \to z_0\) in \(D\) such that

\[ R_{\nu} = |\nabla w_{\nu}(z_\nu)| \to \infty. \]

We call \(z_0\) a **bubbling off point**. We claim that there are only finitely many bubbling off points. In view of [31, Lemma 3.2] it is enough to show that there
exists $c > 0$ such that for any (subsequence of a) bubbling off sequence $z_\nu \to z_0$ and for any $\varrho > 0$

$$\liminf_{\nu \to \infty} \int_{D_\nu(z_\nu)} w_\nu^* \omega > c.$$  

If a bubbling off point is contained in the interior of $D$ the bubbling off argument on [31, p. 163–167] shows that there exists a finite energy plane $v$ with Hofer-energy $E(v) \leq \sigma(L)$ in $W \cup ([0, \infty) \times M)$, $\mathbb{R} \times M$, or $((-\infty, 0) \times M) \cup V$. In the first two cases we get $\inf(\alpha) \leq E(v)$; in the third, invoking the compactness theorem

[11, Theorem 10.5], $E(v)$ is bounded from below by a uniform positive constant.

If a bubbling off point is contained on the boundary $\partial D$ we distinguish following

[28, 23] two cases: We view $w_\nu$ as a $J_\nu$-holomorphic map on the upper half plane $\mathbb{H}$ such that the bubbling off point equals $0$. Using Hofer’s Lemma [33, Lemma 6.4.5] we modify $z_\nu = x_\nu + iy_\nu$ such that

$$R_\nu y_\nu \to r$$

for some $r \in [0, \infty]$, and that there exists a sequence $\varepsilon_\nu \searrow 0$ with $\varepsilon_\nu R_\nu \to \infty$ and

$$|\nabla w_\nu(z)| \leq 2R_\nu$$

for all $z \in \mathbb{H}$ with $|z - z_\nu| \leq \varepsilon_\nu$. The first case is $r = \infty$. With the rescaling argument on [23, p. 560] we obtain a finite energy plane $v$ in $W \cup ([0, \infty) \times M)$, $\mathbb{R} \times M$, or $((-\infty, 0) \times M) \cup V$, which has Hofer-energy $E(v)$ uniformly bounded from below as in the above argument. It remains to consider the case $r < \infty$. Replace the sequence $w_\nu$ by the rescaled sequence

$$u_\nu(z) = w_\nu(x_\nu + z/R_\nu).$$

Set $\zeta_\nu = iR_\nu y_\nu$, and observe that $\zeta_\nu \to ir$ and $|\nabla u_\nu(\zeta_\nu)| = 1$. Hence we get

$$|\nabla u_\nu(z)| \leq 2$$

for all $z \in \mathbb{H}$ with $|z - \zeta_\nu| \leq \varepsilon_\nu R_\nu$. Identifying $[-N, N] \times M$ with $[0, 2N] \times M$ we see

$$2\nu \leq \text{dist}\left(u_\nu(0), \{2\nu\} \times M\right).$$

With the mean value theorem this implies

$$\nu \leq \text{dist}\left(0, u_\nu^{-1}\left(\{2\nu\} \times M\right) \cap D_{\varepsilon_\nu R_\nu}(\zeta_\nu)\right)$$

for all sufficiently large $\nu$. In other words, each $u_\nu$ maps the half-disc $D^+_R$ into $W \cup ([0, \infty) \times M)$ provided $R \ll \nu$. Hence, a subsequence of $u_\nu$ converges in $C^\infty_{\text{loc}}$ to a non-constant holomorphic map

$$u: (\mathbb{H}, \mathbb{R}) \to \left(W \cup ([0, \infty) \times M), L\right),$$

see [23, Proposition 6.1 Case 1.1] and [31, p. 168]. With the mean value inequality, see [2, Remark 3.54], and the argument before [23, Lemma 6.2] a neighborhood of $\infty \in \mathbb{H}$ is mapped by $u$ into a compact neighborhood of $L$. In view of the finiteness of the Hofer energy of $u$ the boundary removable of singularities theorem

[40, Theorem 4.1.2] applies. That means $u$ extends to a non-constant holomorphic disc map with boundary on $L$. Because $W \cup ([0, \infty) \times M)$ provided with $\omega_\nu$ is symplectomorphic to $D^+ L$, see [31, Lemma 2.10], this is a contradiction, i.e., the case $r < \infty$ can not occur. Consequently, there are only finitely many bubbling off points. Denote the finite set of bubbling off points by $\Gamma \subset \bar{D}$. Recall that $\Gamma \neq \emptyset$. In the complement of any neighborhood of $\Gamma$ the sequence $w_\nu$ admits
uniform gradient bounds. Applying the mean value theorem we get $C^0$-bounds such that a subsequence $w_{\nu}$ converges in $C^\infty_{\text{loc}}(\bar{D} \setminus \Gamma)$ to a punctured holomorphic disc $w$ in $W \cup ([0, \infty) \times M)$ with boundary in $L$. The Hofer-energy $E(w)$ is strictly bounded from above by $\sigma(L)$. We claim that $w$ is not constant. Observe that for $\varrho > 0$ sufficiently small and $z \in \Gamma$ we have a uniform bound
\[
\liminf_{\nu \to \infty} \int_{D_\varrho(z)} w_{\nu}^* \omega > c.
\]
Arguing by contradiction we see that all the circles, resp., chords $w_{\nu}(\partial D_\varrho(z))$ converge in $C^\infty_{\text{loc}}$ to a point in $L$. In both cases as on [40, p. 85–86] we can extent $w_{\nu}(D_\varrho(z))$ smoothly to sphere maps into $X$. If $\nu \gg 1$ we can assume that the symplectic areas are positive. This contradicts our assumption that $(X, \omega)$ is symplectically aspherical. Therefore, $w$ is a non-constant punctured holomorphic disc. All its boundary singular points can be removed by the above argument. We assume that all its removable interior punctures are removed as well. With [31, Proposition 2.11] $w$ is a finite energy disc in $W \cup ([0, \infty) \times M)$ with boundary on $L$ and positive punctures. Taking the primitive $\lambda$ into account, which vanishes along $L$, an application of Stokes’s theorem yields
\[
\inf(\alpha) \leq E(w) < \sigma(L).
\]
This proves the Theorem 3.1.

3.2. Proof of Theorem 1.2. In the two-dimensional case a closed connected Lagrangian submanifold $L$ is an embedded curve in the plane. The isoperimetric inequality implies that the enclosed bounded domain $D$ has area less that or equal to $\frac{1}{4\pi} \text{length}(L)^2$. Notice that $L$ divides $D^*_\varrho L$ into two components of equal area. Precisely one component is mapped into $D$. Since a symplectomorphism preserves the area the area of $D^*_\varrho L$ is $\leq \frac{1}{2\pi} \text{length}(L)^2$. Because the metric on $L$ is a positive multiple of the metric induced by $\mathbb{R}/2\pi \mathbb{Z}$ there exists $\varepsilon > 0$ such that the area of $D^*_\varrho L$ equals $4\pi \varepsilon r$ and $\inf(g) = 2\pi \varepsilon$. It follows that
\[
\varrho \leq \frac{1}{4\pi} \frac{\text{length}(L)^2}{\inf(g)}.
\]
Let $2n \geq 4$. Consider a symplectic embedding of the closure of $D^*_\varrho L$ into $\mathbb{R}^{2n}$. Notice that the Lagrangian submanifold $L$ is displaceable. A theorem of Chekanov [14] implies that the Gromov width $\sigma(L)$ is bounded by the displacement energy $d(L)$ of $L$. In [47] Viterbo proved an isoperimetric inequality $d(L)^n \leq \rho_n \text{vol}(L)^2$ for a positive constant $\rho_n$. As explained in Remark 2.4 we have $r \inf(g) = \inf(\alpha)$ for the contact form $\alpha = \lambda_{\text{can}}|_{T^*\mathbb{R}^n L}$. Theorem 3.1 yields
\[
(r \inf(g))^n \leq \rho_n \text{vol}(L)^2.
\]
This proves Theorem 1.2.

Remark 3.2. In [47] Viterbo proved for the volume $\text{vol}(L)$ of a closed Lagrangian submanifold $L$ in $\mathbb{R}^{2n}$ w.r.t. the induced metric $g$ that
\[
d(L)^n \leq \rho_n \text{vol}(L)^2
\]
with a positive constant
\[
\rho_n \leq \sqrt{2n(n-3)} \ n^n.
\]
With the inequality \( r \inf(g) \leq d(L) \) we get for the radius of a symplectically embedded codisc bundle taken w.r.t. the induced metric \( \frac{\text{vol}(L)^2}{\inf(g)^n} \geq \frac{r^n}{\rho_n} \).

This inequality remains valid for all Riemannian metrics \( g \) induced by any Hamiltonian deformation of \( L \). As Álvarez Paiva explained to the author a computation of the greatest value of \( r \) in the above inequality is related to questions in systolic geometry.

3.3. Non-embeddability of the cotangent bundles. Let \((X,\omega)\) be a symplectically aspherical symplectic manifold as described in Section 3.1. In [14] Chekanov proved for displaceable Lagrangian submanifolds \( L \) the inequality

\[
0 < \sigma(L) \leq d(L) < \infty
\]

for the displacement energy \( d(L) \) of \( L \).

**Corollary 3.3.** Let \( L \subset (X,\omega) \) be a closed displaceable Lagrangian submanifold. Then there is no symplectic embedding of \( T^*L \) into \( (X,\omega) \) relative \( L \).

**Proof.** Let \( g \) be a metric on \( L \). Arguing by contradiction we find for all positive \( r \) a symplectic embedding of the \( r \)-codisc bundle of \( L \). With Theorem 3.1 and Chekanov’s result [14] we find

\[
r \inf(g) = \inf(\lambda_{TS^*_r(g)L}) \leq d(L).
\]

Letting \( r \) tend to infinity yields a contradiction. \( \square \)

**Remark 3.4.** In the particular case the symplectic form \( \omega = d\lambda \) on \( X \) is exact the restriction of \( \lambda \) to \( TL \) is a closed 1-form on \( L \). Its cohomology class \( \lambda_L \), the so-called Liouville class, cf. [43], is independent of the choice of the primitive \( \lambda \) provided \( X \) is simply connected. Recall that a norm on the space of cohomology 1-classes \( m \) can be defined via \( \|m\| = \inf\{\sup_L |\mu| \mid \mu \in m\} \), cf. [7]. If a neighborhood of the closure of the \( ||\lambda_L|| \)-codisc bundle of \( L \) embeds symplectically relative \( L \) the image \( L_\lambda \) of the section into \( T^*L \) representing \( -\lambda_L \) is an exact Lagrangian submanifold of \( (X,d\lambda) \), see [3, Section 7]. This was pointed out to the author by Polterovich. With Chekanov’s result [14] \( L_\lambda \) is not displaceable. In particular, no subcritical Stein manifold contains a symplectically embedded cotangent bundle of a closed manifold, cf. [15]. Corollary 3.3 serves as a generalization to the symplectically aspherical case.

**Remark 3.5.** To give an example of non-embeddability of the cotangent bundles in the presence of holomorphic spheres we make the following remark. Barraud, Biran and Cornea [5, 6, 9, 10] defined the relative Gromov radius \( c_B(L) \) of a closed Lagrangian submanifold \( L \) in a symplectic manifold \((V,\omega)\) to be the supremum over all \( \pi r^2 \) such that there exists a symplectic embedding \( \varphi: B_r \to V \) with \( \varphi^{-1}(L) \) equal to \( B_r \cap \mathbb{R}^n \subset \mathbb{C}^n \). The relative Gromov radius of the zero section of \( T^*L \) is not finite. Hence, if a cotangent bundle embeds symplectically into \((V,\omega)\) the relative Gromov radius of the image of the zero section must be infinite in \((V,\omega)\). Consider a closed Lagrangian submanifold \( L \) in \( T^*Q \times \mathbb{C}P^1 \), where \( Q \) is any closed manifold. Then \( c_B(L) \) is bounded by the absolute Gromov radius \( c_B(T^*Q \times \mathbb{C}P^1) = \pi \). Hence, no cotangent bundle does embed symplectically into \( T^*Q \times \mathbb{C}P^1 \). Notice that it is not known in general whether the relative Gromov radius of a closed Lagrangian
submanifold \( L \) in \( \mathbb{R}^{2n} \) is finite. Examples in the monotone case can be found in [10, 13, 18]. Its spherical variant will be discussed in Section 4.

### 3.4. Proof of Theorem 1.3

We consider the torus \( \mathbb{R}^n/2\pi\mathbb{Z}^n \) with the metric induced from \( \mathbb{R}^n \) so that the shortest non-trivial closed geodesics have length \( 2\pi \).

The \( n \)-fold product of the maps

\[
(g, p) \mapsto \sqrt{1 + 2p \varepsilon^q}, \quad \text{resp.}, \quad \sqrt{1/n + 2p \varepsilon^q}
\]

embed the 1/2-codisc, resp., the 1/2n-codisc bundle symplectically into \( \mathbb{R}^{2n} \). The images of the zero section are the Clifford tori \( T_1 \) and \( T_{1/\sqrt{n}} \) which equal the product of \( n \) circles in \( \mathbb{R}^2 \times \cdots \times \mathbb{R}^2 \) of radius 1 and \( 1/\sqrt{n} \), resp. The action on the corresponding cosphere bundles induced by the shortest non-trivial closed geodesics equal \( \pi \) and \( \pi/n \), resp. This shows that \( c_D(P) \geq \pi \) and \( c_D(B) \geq \frac{\pi}{n} \).

Consider a metric \( g \) on \( S^1 \). After a reparametrization there exist a positive constant \( \varepsilon \) such that \( g = \varepsilon^2 g_0 \), where \( g_0 \) is the metric induced by \( \mathbb{R}/2\pi\mathbb{Z} \). Therefore, \( \inf(g) = 2\pi \varepsilon \). Assume that \( D^\ast_r(g)S^1 \) embeds into \( \mathbb{R}^2 \) preserving the area such that \( S^1 \) is mapped into the open unit disc. This implies that \( 1/2 \) times the area of \( D^\ast_r(g)S^1 \) is \( < \pi \). Because the area of \( D^\ast_r(g)S^1 \) is equal to \( 4\pi r \varepsilon \) we get \( \varepsilon r < 1/2 \).

Hence, \( r \inf(g) < \pi \). This proves that \( c_D \) is a capacity in dimension 2.

If \( 2n \geq 4 \) we argue as follows. Non-trivial closed geodesics \( c \) on Riemannian manifolds \( (L, g) \) are in one-to-one correspondence with closed Reeb orbits \( \gamma \) on \( S^\ast_r(g)L \) for all positive \( r \), see [22]. The correspondence assigns to a geodesic \( c \) which is assumed to have constant speed \( |\dot{c}| = r \) the reparametrized speed curve \( \gamma = c \circ 1/r^2 \). The action \( \int_\gamma \alpha \), where \( \alpha = \lambda_{can}|_{TS^\ast_r(g)L} \), equals \( \int_\varepsilon \lambda_{can} = r \text{length}(c) \).

Therefore,

\[
r \inf(g) = \inf(\alpha).
\]

On the other hand the action-area inequality in Theorem 3.1 and Chekanov’s result [14] yield

\[
\inf(\alpha) < \sigma(L) \leq d(L).
\]

The displacement energy \( d \), which is known to be a special capacity, see [33], takes the value \( \pi \) on the open unit ball and the open unit symplectic cylinder. Hence, \( c_D(Z) \leq \pi \). This proves Theorem 1.3.

**Remark 3.6.** Notice that \( c_D(L) \leq \sigma(L) \) and that for subsets \( U \) of \( \mathbb{R}^{2n} \)

\[
c_D(U) \leq \sigma(U) := \sup \{ \sigma(L) \mid L \subset U \}.
\]

With Hermann’s work [27] one obtains upper bounds for the codisc radius in terms of the Floer-Hofer resp. the Viterbo capacity, as well as with [25, 24] in terms of the orbit capacity in dimensions \( \geq 4 \).

### 3.5. Monotone Lagrangian submanifolds

By Damian’s proof of the Audin conjecture in the monotone case [18] the minimal Maslov number of an orientable monotone Lagrangian submanifold \( L \) which admits a metric of non-positive sectional curvature equals 2. The minimizing class can be represented by a holomorphic disc with boundary on \( L \) for any compatible almost complex structure. Therefore, the minimal symplectic area \( \inf(L) \) and the Gromov width \( \sigma(L) \) are equal. Hence, with [50, Corollary 2.6] we get

\[
\sigma(L) = \inf(L) \leq \frac{\pi}{n}
\]
provided $L \subset B$. With the arguments from Section 3.4 this yields a special capacity $c_{D,m}$ such that

$$c_{D,m}(Z) = \pi \quad \text{and} \quad c_{D,m}(B) = \frac{\pi}{n}$$

by restricting the definition of $c_D$ to Lagrangian submanifolds that are orientable, monotone, and admit a metric of non-positive sectional curvature. We remark that still the supremum is taken over all Riemannian metrics on $L$. Similar to [50, Corollary 3.3] an application of the capacity $c_{D,m}$ implies:

**Corollary 3.7.** Let $L \subset B_R$ be an orientable closed monotone Lagrangian submanifold which admits a metric of non-positive sectional curvature. Then for all Riemannian metrics $g$ on $L$ and all radii $r$ such that the corresponding $r$-codisc bundle embeds into $\mathbb{R}^{2n}$ symplectically we have

$$r \inf(g) \leq \frac{\pi}{n} R^2.$$ 

### 3.6. Relation to the link capacity.

In [50] the link capacity is defined. A variant of it can be obtained as follows. For open subsets $U \subset \mathbb{R}^{2n}$ and closed oriented monotone Lagrangian submanifolds $L \subset \mathbb{R}^{2n}$ which admit a Riemannian metric $g$ of non-positive sectional curvature consider symplectic embeddings of $D^*_r(g)L$ into $U$ relative $L$. Then

$$\ell_m^+(U) := \sup \{2r \inf(g) \mid D^*_r(g)L \hookrightarrow U, g, \text{ and } r > 0\},$$

where the supremum is taken over all metrics $g$ of non-positive sectional curvature. For the class of Lagrangian submanifolds $L$ under consideration we define

$$a^+_m(U) := \sup \{\inf(L) \mid L \subset U\},$$

cf. [50, Theorem 2.5]. The action-area inequality from Theorem 3.1 implies:

**Corollary 3.8.** For all open subsets $U$ of $\mathbb{R}^{2n}$ we have

$$\ell_m^+(U) \leq 2a^+_m(U).$$

With [50, Theorem 3.1] we obtain

$$\ell_m^+(Z) = \pi, \quad \ell_m^+(B) \in \left[\frac{\pi}{n} \frac{2\pi}{n}\right].$$

Motivated by the work of Cieliebak and Mohnke on the Lagrangian capacity [17, 16] we *conjecture* that the link capacity on the unit ball equals $\pi/n$.

### 4. The relative spherical Gromov radius

The aim of this section is to prove Theorem 1.4.

#### 4.1. Proof of Theorem 1.4.

Recall that a Lagrangian submanifold $L$ in $\mathbb{R}^{2n}$ is monotone if there exists a positive real number $\eta$, the so-called *monotonicity constant* of $L$, such that the Liouville class $\lambda_L$ and the Maslov class $\mu_L$ of $L$ satisfy $\lambda_L = \eta \mu_L$. The proof of Theorem 1.4 below will show that monotone Lagrangian tori satisfy

$$s(L) \leq 4\eta.$$ 

Notice that the minimal positive symplectic area of a smooth disc with boundary on $L$ is equal to $2\eta$. Moreover, it is attained by a holomorphic disc with Maslov number 2, see [13, 18]. A theorem of Chekanov [14] implies that

$$s(L) \leq 2d(L),$$
where $d(L)$ denotes the displacement energy of $L$, see [33]. If in addition there is a metric of non-positive sectional curvature on $L$ we get with [50, Theorem 2.5]

$$s(L) \leq \frac{2}{k} e^k_{EH}(L)$$

for all $k \in \mathbb{N}$ and the $k$-th Ekeland-Hofer capacity [19].

**Proof of Theorem 1.4.** The proof of the theorem is an application of the relative neck stretching argument due to Abbas [2]. We consider a relative symplectic embedding $\varphi$ of $(S^{2n-1}, S^{n-1})$ into $(\mathbb{R}^{2n}, L)$. For $\varepsilon > 0$ small enough we can assume that the neighborhood on which $\varphi$ is defined contains the spherical shell $U_\varepsilon = B_r + \varepsilon \setminus B_r - \varepsilon$.

We consider the symplectic ellipsoid

$$E = \left\{ \frac{x_1^2 + y_1^2}{r_1^2} + \ldots + \frac{x_n^2 + y_n^2}{r_n^2} < 1 \right\}$$

for real numbers $r - \varepsilon < r_1 < r_2 < \ldots < r_n < r + \varepsilon$ and denote by $M = \varphi(\partial E)$ the image of the boundary. Because $M$ is simply connected we find a global primitive 1-form $\lambda$ of $dx \wedge dy$ such that

$$\lambda = \varphi_* \left( \frac{1}{2} (x \, dy - y \, dx) \right)$$

on $\varphi(U_\varepsilon)$. Notice that $\lambda$ vanishes on the tangent spaces of $L \cap \varphi(U_\varepsilon)$. Denote by $\alpha$ the restriction of $\lambda$ to $TM$. Let $W$ and $V$ be the closures of the components of the complement of $M$ such that

$$\mathbb{R}^{2n} = W \cup_M V$$

is a decomposition into a Liouville filling $(W, \lambda)$ of the contact type hypersurface $(M, \alpha)$ and a symplectic manifold $(V, d\lambda)$ with concave boundary $(M, \alpha)$. Observe that the $(n - 1)$-sphere $K = L \cap M$ bounds a domain inside the torus $L$ such that one component $L \cap W$ or $L \cap V$ of the complement is simply connected. This follows with the Jordan-Schoenflies theorem [12, 36, 42] applied to the universal cover $\mathbb{R}^n$.

We choose the radii $r_1, \ldots, r_n$ such that in addition there squares are rationally independent. Then all closed Reeb orbits on $(M, \alpha)$ correspond to intersections of $\partial E$ with the complex coordinate axes and each Reeb chord of the Legendrian submanifold $K$ is contained in a closed Reeb orbit. Moreover, the contact form $\alpha$ and the pair $(\alpha, K)$ are generic in the sense of [2, Chapter 3.2], i.e., the linearized Poincaré return map restricted to the contact structure $\ker(\alpha)$ at any periodic point of the Reeb flow has no eigenvalue 1, and whenever the isotopic image $K'$ of $K$ under the Reeb flow intersects $K$ itself, the contact structure $\ker(\alpha)$ is spanned by the tangent spaces to $K$ and $K'$ at the intersection points. Therefore, the genericity assumptions of the compactness theorem in [2] are satisfied. In order to define a sequence of almost complex structures on $\mathbb{R}^{2n}$ we describe a symplectic neighborhood $U \subset \varphi(U_\varepsilon)$ of $M$. Let $Y$ be the Liouville vector field dual to $\lambda$. Following its flow near $M$ in forward and backward time we obtain a symplectomorphic model

$$([-\delta, \delta] \times M, d(e^\varepsilon \alpha))$$

of $U$ for $\delta > 0$, see [22], which we call the neck. Notice that $Y$, which is mapped to $\partial_s$, is tangent to $L \cap U$ so that the intersection of the Lagrangian submanifold
$L$ with $U$ corresponds precisely to $[-\delta, \delta] \times K$. In the same way we obtain for each $N \in \mathbb{N}$ a symplectomorphic copy of the neck
\[
([-N - \delta, N + \delta] \times M, d(\tau \alpha))
\]
with Lagrangian submanifold $[-N - \delta, N + \delta] \times K$, where $\tau$ is a smooth strictly increasing function on $[-N - \delta, N + \delta]$, which equals $e^{s/2N}$ on $[-N - \delta, -N - \delta/2]$ and $e^{-s/2N}$ on $[N + \delta/2, N + \delta]$, see [31, p. 158]. We define a translation invariant almost complex structure on the $N$-neck as follows: On ker($\alpha$) it is required to restrict to a complex structure compatible with $d\alpha$ and on its complement it is required to map $\partial_u$ to the Reeb vector field of $\alpha$. This defines an almost complex structure $J_N$ on $U$ for each $N \in \mathbb{N}$. Near the boundary of $U$ it is independent of $N$ and therefore can be extended to $\mathbb{R}^{2n}$ in a uniform way, see [26]. Moreover, $J_N$ equals the complex structure of $C^n$ outside a fixed large ball. We consider the case where $L \cap W$ is simply connected. The case of $L \cap V$ being simply connected can not occur because otherwise there would be a Reeb chord on $K$ with negative action by the analogue of the following argument: Let $p$ be a point in the interior of $L \cap W$. Because the Lagrangian torus $L$ is monotone we can apply Damian’s result [18, Theorem 1.5.(c)]. Therefore, we find for any $N$ a $J_N$-holomorphic disc $u_N : D \to \mathbb{R}^{2n}$ through $p$ with boundary on $L$ and Maslov index 2. In particular, the energy of $u_N$ is
\[
\int_D u_N^*d\lambda = 2\eta
\]
for all $N$, where $\eta$ is the monotonicity constant of $L$. Notice that the boundary curves $u_N(\partial D) \subset L$ are not entirely contained in $L \cap W$ because these are not contractible in $L$. By Abbas’s compactness theorem [2] a subsequence of $u_N$ converges to a holomorphic building of total Hofer-energy equal to $2\eta$. Its level structure corresponds to $W \cup \{(0, \infty) \times M\}$, several (or non) copies of $\mathbb{R} \times M$, and $((-\infty, 0) \times M) \cup V$. The boundary of the building lies in $(L \cap W) \cup \{(0, \infty) \times K\}$, the corresponding copies of $\mathbb{R} \times K$, and $((-\infty, 0) \times K) \cup (L \cap V)$ resp., cf. [11, 31]. Moreover, the building consists of punctured holomorphic spheres and discs with Lagrangian boundary conditions such that at least one disc is contained in each level. Over the interior punctures these are asymptotic to closed Reeb orbits of $\alpha$ and over boundary punctures to non-constant Reeb chords of $(\alpha, K)$, see [1, 28]. In particular, in $W \cup \{(0, \infty) \times M\}$ there exists a finite energy disc $u$ with at least one boundary puncture. Its Hofer-energy satisfies
\[
E(u) = \sup_\tau \int_{D \setminus \Gamma} u^*d\lambda_\tau < 2\eta,
\]
where $\Gamma \subset D$ is the set of punctures of $u$. The supremum is taken over all smooth (strictly) increasing functions $\tau$ on $[-\delta, \infty)$ which equal $e^s$ on $[-\delta, -\delta/2]$ and converge to 1 as $s \to \infty$. The 1-form $\lambda_\tau$ is given by $\lambda$ on $W \setminus ([-\delta, 0] \times M)$ and by $\tau \alpha$ on $[-\delta, \infty) \times M$. Notice that $\lambda_\tau$ vanishes on vectors tangent to $[-\delta, \infty) \times K$. The boundary curves $u(\partial D \setminus \Gamma)$ can be homotoped into $[-\delta, \infty) \times K$ relative neighborhoods of the punctures $\Gamma$ because $L \cap W$ is simply connected. An application of Stokes’s theorem (taking the asymptotic and the boundary conditions into account) yields that the sum of all periods of Reeb orbits over (the possibly empty set of) interior punctures of $u$ and of the longueurs (i.e., the actions w.r.t. $\alpha$) of all Reeb chords over (the non-empty set of) boundary punctures of $u$ is strictly less than $2\eta$. 


Therefore, we get
\[ \frac{\pi}{2} (r - \varepsilon)^2 < 2\eta, \]
because the left hand side is precisely the shortest length of a Reeb chord in \( \partial E \) starting and ending on \( \partial E \cap \mathbb{R}^n \). Letting \( \varepsilon \) tend to zero this proves the theorem. \( \square \)

Remark 4.1. Our proof requires the existence of a holomorphic discs \( D \) through any given point on a Lagrangian submanifold \( L \) with \( \partial D \subset L \) for any admissible almost complex structure such that the energy of \( D \) is uniformly bounded and \( \partial D \) is not contractible in \( L \). Theorem 1.4 generalizes accordingly. By [18, Theorem 3.3.(b)] this is the case if \( L \) is a Lagrangian submanifold of a Liouville symplectic manifold \( (X, \lambda) \) convex at infinity such that any compact set in \( (X, d\lambda) \) is displaceable. \( L \) itself is required to be closed, oriented, and monotone such that the total singular \( \mathbb{Z}_2 \)-homology of the universal cover \( \tilde{L} \) has finite dimension over \( \mathbb{Z}_2 \) and the \( \mathbb{Z}_2 \)-Euler characteristic of \( \tilde{L} \) does not vanish. The resulting discs in this situation all have Maslov index 2. Moreover, we used in the proof that any hypersurface of contact type symplectomorphic to the sphere separates \( X \) and that any smoothly embedded \( (n-1) \)-sphere in \( L \) bounds a simply connected domain in \( L \).

Notice that the 2-torus \( L \) has this property. Moreover, any manifold \( L \) such that any smoothly embedded \( (n-1) \)-sphere in \( L \) bounds a homeomorphic \( n \)-disc satisfies this too. Examples can be obtained with the Jordan-Schoenflies theorem [12, 36, 42] if the universal cover is \( \mathbb{R}^n \) with \( n \geq 3 \). Therefore, the inequality \( s(L) \leq 2d(L) \) holds in the situation described.

4.2. A remark on dimension 4. In the case of a Stein surface \( X \) and a monotone Lagrangian 2-torus \( L \) both quantities \( c_B(L) \) and \( s(L) \) coincide: Consider a symplectic embedding \( \varphi \) of \( S^3_1 \) into \( X \) relative \( L \). Then \( \varphi(S^3_1) \) cuts an exact symplectic filling of out \( X \). By a theorem of Gromov [26] \( \varphi \) extends (after restriction to a smaller neighborhood of \( S^3_1 \)) to the ball \( B^4_1 \), see [40, Theorem 9.4.2]. It follows from the proof of Theorem 1.4 that the intersection of \( \varphi(B^4_1) \) with \( L \) is a 2-disc. Therefore, \( \varphi^{-1}(L) \) is a local Lagrangian knot inside the ball \( B^4_1 \) in the sense of [20]. As Polterovich pointed out to the author with [20, Theorem 1.1.A, Proposition 5.1.A.2]) one can assume that the local Lagrangian knot is isotopic to \( \mathbb{R}^2 \) through local Lagrangian knots whose trace of the non-flat regions stay inside \( B^4_1 \). Hence, with Hamiltonian isotopy extension, cf. [43, p. 43] and [39, p. 96], \( \varphi \) can be Hamiltonianally isotoped to \( \psi \) inside \( B^4_1 \) such that \( \psi \) coincides with \( \varphi \) near \( \partial B^4_1 \) and maps \( \mathbb{R}^2 \cap B^4_1 \) to \( L \). Hence, \( \varphi \) extends to a symplectic embedding of \( B^4_1 \) relative \( L \).

4.3. A non-monotone example. We consider a closed Lagrangian torus
\[ L = L' \times S^1_\varrho \]
in \( \mathbb{R}^{2n-2} \times \mathbb{R}^2 \) such that the Lagrangian torus \( L' \subset \mathbb{R}^{2n-2} \) is rational. This means that the minimal positive symplectic area \( \inf(L') \) of a disc with boundary on \( L' \) is positive. We choose the radius \( \varrho \) of the circle \( S^1_\varrho = \partial D_\varrho \) such that
\[ \frac{\inf(L')}{\varrho} \]
is a natural number. This implies that \( L \) itself is rational with \( \inf(L) = \varrho \).

Notice that for the complex structure of \( \mathbb{C}^n \) the family
\[ \{\ast\} \times D_\varrho \subset L' \times \mathbb{R}^2 \]
of holomorphic discs defines a smooth filling. With transversality as in [40, 49] and Gromov compactness, see [21, 26, 35], we get that for all tamed almost complex structures \( J \) standard at infinity and any point \( p \) on \( L \) there exists a \( J \)-holomorphic map
\[
u: (D, \partial D, 1) \to (\mathbb{R}^{2n}, L, p)
\]
with symplectic area
\[
\int_D \nu^*(\mathrm{d}x \wedge \mathrm{d}y) = \pi \varrho^2,
\]
see [26, 2.3.D.]. In view of Remark 4.1 we get:

**Corollary 4.2.** The relative spherical Gromov radius of the rational Lagrangian torus \( L \) described above satisfies
\[
s(L) \leq 2\pi \varrho^2.
\]

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