C*-ALGEBRAS ASSOCIATED WITH HILBERT C*-QUAD MODULES OF C*-TEXTILE DYNAMICAL SYSTEMS

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Abstract. A C*-textile dynamical system \((A, \rho, \eta, \Sigma_\rho, \Sigma_\eta, \kappa)\) consists of a unital C*-algebra \(A\), two families of endomorphisms \(\{\rho_\alpha\}_{\alpha \in \Sigma_\rho}\) and \(\{\eta_a\}_{a \in \Sigma_\eta}\) of \(A\) and certain commutation relations \(\kappa\) among them. It yields a two-dimensional subshift and multi structure of Hilbert C*-bimodules, which we call a Hilbert C*-quad module. We introduce a C*-algebra from the Hilbert C*-quad module as a two-dimensional analogue of Pimsner's construction of C*-algebras from Hilbert C*-bimodules. We study the C*-algebras defined by the Hilbert C*-quad modules and prove that they have universal properties subject to certain operator relations. We also present its examples arising from commuting matrices.

1. Introduction

In [16], the author has introduced a notion of \(\lambda\)-graph system as a generalization of finite labeled graphs. The \(\lambda\)-graph systems yield C*-algebras so that its K-theory groups are related to topological conjugacy invariants of the underlying symbolic dynamical systems. He has extended the notion of \(\lambda\)-graph system to C*-symbolic dynamical system, which is a generalization of both a \(\lambda\)-graph system and an automorphism of a unital C*-algebra. It is denoted by \((A, \rho, \Sigma)\) and consists of a finite family \(\{\rho_\alpha\}_{\alpha \in \Sigma}\) of endomorphisms of a unital C*-algebra \(A\) such that \(\rho_\alpha(Z_A) \subset Z_A, \alpha \in \Sigma\) and \(\sum_{\alpha \in \Sigma} \rho_\alpha(1) \geq 1\) where \(Z_A\) denotes the center of \(A\). A \(\lambda\)-graph system \(\mathcal{L}\) yields a C*-symbolic dynamical system \((A_\mathcal{L}, \rho^\mathcal{L}, \Sigma)\) such that \(A_\mathcal{L}\) is \(C(\Omega_\mathcal{L})\) for some compact Hausdorff space \(\Omega_\mathcal{L}\) with \(\dim \Omega_\mathcal{L} = 0\). A C*-symbolic dynamical system \((A, \rho, \Sigma)\) provides a subshift \(\Delta\) over \(\Sigma\) and a Hilbert C*-bimodule \(\mathcal{H}^\rho_A\) over \(A\) which gives rise to a C*-algebra \(O_\rho\) as a Cuntz-Pimsner algebra ([19], cf. [11], [31]).

G. Robertson–T. Steger [34] have initiated a certain study of higher dimensional analogue of Cuntz–Krieger algebras from the view point of tiling systems of 2-dimensional plane. After their work, A. Kumjian–D. Pask [12] have generalized their construction to introduce the notion of higher rank graphs and its C*-algebras. Since then, there have been many studies on these C*-algebras by many authors (see for example [6], [8], [12], [32], [27], [34], etc.).

M. Nasu in [25] has introduced the notion of textile system which is useful in analyzing automorphisms and endomorphisms of topological Markov shifts. A textile system also gives rise to a two-dimensional tiling called Wang tiling. Among textile systems, LR textile systems have specific properties that consist of two commuting symbolic matrices. In [20], the author has extended the notion of textile systems to \(\lambda\)-graph systems and has defined a notion of textile systems on \(\lambda\)-graph systems, which are called textile \(\lambda\)-graph systems for short. C*-algebras associated to textile systems have been initiated by V. Deaconu ([6]).
In [23], the author has extended the notion of $C^*$-symbolic dynamical system to $C^*$-textile dynamical system which is a higher dimensional analogue of $C^*$-symbolic dynamical system. The $C^*$-textile dynamical system $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ consists of two $C^*$-symbolic dynamical systems $(\mathcal{A}, \rho, \Sigma^\rho)$ and $(\mathcal{A}, \eta, \Sigma^\eta)$ with a common unital $C^*$-algebra $\mathcal{A}$ and a commutation relation between $\rho$ and $\eta$ through a map $\kappa$ below. Set

$$\Sigma^\rho = \{(a, b) \in \Sigma^\rho \times \Sigma^\eta \mid \eta_b \circ \rho_a \neq 0\}, \quad \Sigma^{\rho\eta} = \{(a, \beta) \in \Sigma^\eta \times \Sigma^\rho \mid \rho_\beta \circ \eta_a \neq 0\}.$$ 

We require that there exists a bijection $\kappa : \Sigma^{\rho\eta} \to \Sigma^{\rho\eta}$, which we fix and call a specification. Then the required commutation relations are

$$\eta_b \circ \rho_a = \rho_\beta \circ \eta_a \quad \text{if} \quad \kappa(a, b) = (a, \beta). \quad (1.1)$$

The author has also introduced a $C^*$-algebra $\mathcal{O}_{\rho, \eta}^\kappa$ from $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ which is realized as the universal $C^*$-algebra $C^*(\mathcal{A}, S_a, T_a; x \in \mathcal{A}, \alpha \in \Sigma^\rho, a \in \Sigma^\eta)$ generated by $x \in \mathcal{A}$ and two families of partial isometries $S_a, \alpha \in \Sigma^\rho$, $T_a, a \in \Sigma^\eta$ subject to the following relations called $(\rho, \eta; \kappa)$:

$$\sum_{\beta \in \Sigma^\rho} S_{\alpha} S_{\beta}^* = 1, \quad x S_{\alpha} S_{\beta}^* = S_{\alpha} S_{\beta}^* x, \quad S_{\alpha}^* x S_{\alpha} = \rho_\alpha(x), \quad (1.2)$$

$$\sum_{b \in \Sigma^\eta} T_{b}^* T_{a} = 1, \quad x T_{b}^* T_{a} = T_{b}^* T_{a} x, \quad T_{a}^* x T_{a} = \eta_\alpha(x), \quad (1.3)$$

$$S_{\alpha} T_{b} = T_{a} S_{\beta} \quad \text{if} \quad \kappa(a, b) = (a, \beta) \quad (1.4)$$

for all $x \in \mathcal{A}$ and $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$ ([23]). The algebra is a generalization of some of higher rank graph algebras.

In the present paper, the author will introduce another kind of $C^*$-algebras associated with the $C^*$-textile dynamical systems from the view point of Hilbert $C^*$-modules. The resulting $C^*$-algebras $\mathcal{O}_{H_{\kappa}}$ are different from the above algebras $\mathcal{O}_{\rho, \eta}^\kappa$. A $C^*$-textile dynamical system provides a two-dimensional subshift and multi structure of Hilbert $C^*$-bimodules that have multi right actions and multi left actions and multi inner products. We call it a Hilbert $C^*$-quad module denoted by $H_{\kappa}$. The $C^*$-algebra $\mathcal{O}_{H_{\kappa}}$, which we will introduce in the present paper, is constructed in a concrete way from the structure of the Hilbert $C^*$-quad module $H_{\kappa}$ by a two-dimensional analogue of Pimsner’s construction from Hilbert $C^*$-bimodules. It is generated by the quotient images of creation operators on two-dimensional analogue of Fock Hilbert module by module maps of compact operators. As a result, we will show the $C^*$-algebra has a universal property subject to certain operator relations of generators.

For a $C^*$-textile dynamical system $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$, consider the set of quadruplet of symbols

$$\Sigma_\kappa = \{\omega = (a, b, a, \beta) \in \Sigma^\rho \times \Sigma^\eta \times \Sigma^\rho \times \Sigma^\eta \mid \kappa(a, b) = (a, \beta)\}. \quad (1.5)$$

Each element of $\Sigma_\kappa$ is regarded as a tile $a \downarrow \alpha \downarrow b$ of the associated two-dimensional subshift. Denote by $\mathcal{O}_{\rho}$ and by $\mathcal{O}_{\eta}$ the $C^*$-algebras associated with the $C^*$-symbolic dynamical systems $(\mathcal{A}, \rho, \Sigma^\rho)$ and $(\mathcal{A}, \eta, \Sigma^\eta)$ respectively. Let $S_{\alpha}, \alpha \in \Sigma^\rho$ and $T_{a}, a \in \Sigma^\eta$ be the generating partial isometries of $\mathcal{O}_{\rho}$ and those of $\mathcal{O}_{\eta}$,
which satisfy (1.2) and (1.3) respectively. Denote by $B$ generated by elements $S, T, \eta, \eta$. has a natural structure of a Hilbert $\mathbb{B}$ bimodule structure over $B$ and $B$. They also extend to $H$ module structure, for $\omega$ call it Hilbert $C$ module. Denote by $\hat{\rho}_\alpha$ and the vertical creation operators, on two-dimensional analog of Fock Hilbert $C$ module is generated by two kinds of creation operators, the horizontal creation operators $A$ that the algebra $\sigma$ is commutative. The main result of the paper is the following theorem, which states that the algebraic structure of the algebra $\sigma$ can be presented (Theorem 5.18).

Theorem 1.1 (Theorem 5.17). For a $C^*$-textile dynamical system $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$, the $C^*$-algebra $\mathcal{O}_{\kappa}$ associated with the Hilbert $C^*$-quad module $\mathcal{H}_\kappa$ is realized as the universal concrete $C^*$-algebra generated by the operators $z \in B_\eta, w \in B_\rho$ and partial isometries $u_\alpha, \alpha \in \Sigma^\rho, v_\alpha, \alpha \in \Sigma^\eta$ subject to the relations:

$$\sum_{\beta \in \Sigma^\rho} u_{\beta} u_\beta^* + \sum_{b \in \Sigma^\eta} v_b v_b^* = 1,$$

$$u_\alpha u_\alpha^* w = w u_\alpha u_\alpha^*, \quad v_\alpha v_\alpha^* w = w v_\alpha v_\alpha^*,$$

$$u_\alpha u_\alpha^* z = z u_\alpha u_\alpha^*, \quad v_\alpha v_\alpha^* z = z v_\alpha v_\alpha^*,$$

$$\hat{\rho}_\alpha (w) = u_\alpha w u_\alpha, \quad \hat{\rho}_\alpha (z) = v_\alpha z v_\alpha,$$

$$\hat{\eta}_\alpha (w) = u_\alpha w u_\alpha, \quad \hat{\eta}_\alpha (z) = v_\alpha z v_\alpha,$$

$$t_\eta (y) = t_\rho (y)$$

for $w \in B_\rho, z \in B_\eta, \alpha \in \Sigma^\rho, \alpha \in \Sigma^\eta, \alpha \in \mathcal{A}$. 

Thanks to the above theorem, simplicity condition of the $C^*$-algebra $\mathcal{O}_{\kappa}$ will be presented (Theorem 5.18).
Let $A, B$ be two $N \times N$ matrices with entries in nonnegative integers. They yield directed graphs $G_A = (V, E_A)$ and $G_B = (V, E_B)$ with a common vertex set $V = \{v_1, \ldots, v_N\}$ and edge sets $E_A$ and $E_B$ respectively, where the edge set $E_A$ consist of $A(i, j)$-edges from the vertex $v_i$ to the vertex $v_j$ and $E_B$ consist of $B(i, j)$-edges from the vertex $v_i$ to the vertex $v_j$. We then have two $C^*$-symbolic dynamical systems $(A_N, \rho^A, E_A)$ and $(A_N, \rho^B, E_B)$ with $A_N = \mathbb{C}^N$. Denote by $s(e), r(e)$ the source vertex and the range vertex of an edge $e$. Put

$$
\Sigma^{AB} = \{ (a, b) \in E_A \times E_B \mid r(a) = s(b) \},
$$

$$
\Sigma^{BA} = \{ (a, b) \in E_B \times E_A \mid r(a) = s(b) \}.
$$

Assume that the commutation relation

$$
AB = BA
$$

(1.10)

holds. We may take a bijection $\kappa : \Sigma^{AB} \to \Sigma^{BA}$ such that $s(\alpha) = s(a), r(b) = r(\beta)$ for $\kappa(\alpha, b) = (a, \beta)$ which we fix. This situation is called an LR-textile system introduced by Nasu ([25]). We then have a $C^*$-textile dynamical system $(A_N, \rho^A, \rho^B, E_A, E_B, \kappa)$. We set

$$
\Omega_\kappa = \{ (\alpha, a) \in E_A \times E_B \mid s(\alpha) = s(a), \kappa(\alpha, b) = (a, \beta) \text{ for some } \beta \in E_A, b \in E_B \}
$$

and define two $|\Omega_\kappa| \times |\Omega_\kappa|$-matrices $A_\kappa$ and $B_\kappa$ with entries in $\{0, 1\}$ by

$$
A_\kappa((\alpha, a), (\delta, b)) = \begin{cases} 1 & \text{if there exists } \beta \in E_A \text{ such that } \kappa(\alpha, b) = (a, \beta), \\ 0 & \text{otherwise} \end{cases}
$$

for $(\alpha, a), (\delta, b) \in \Omega_\kappa$, and

$$
B_\kappa((\alpha, a), (\beta, d)) = \begin{cases} 1 & \text{if there exists } b \in E_B \text{ such that } \kappa(\alpha, b) = (a, \beta), \\ 0 & \text{otherwise} \end{cases}
$$

for $(\alpha, a), (\beta, d) \in \Omega_\kappa$ respectively. Denote by $\mathcal{H}_\kappa^{A,B}$ the associated Hilbert $C^*$-quad module.

**Theorem 1.2 (Theorem 7.10).** The $C^*$-algebra $\mathcal{O}_{\mathcal{H}_\kappa^{A,B}}$ associated with the Hilbert $C^*$-quad module $\mathcal{H}_\kappa^{A,B}$ defined by commuting matrices $A, B$ and a specification $\kappa$ is generated by two families of partial isometries $S_{(\alpha, a)}, T_{(\alpha, a)}$ for $(\alpha, a) \in \Omega_\kappa$ satisfying the relations:

$$
\sum_{(\delta, b) \in \Omega_\kappa} S_{(\delta, b)}^* S_{(\delta, b)} + \sum_{(\beta, d) \in \Omega_\kappa} T_{(\beta, d)}^* T_{(\beta, d)} = 1,
$$

$$
S_{(\alpha, a)}^* S_{(\alpha, a)} = \sum_{(\delta, b) \in \Omega_\kappa} A_\kappa((\alpha, a), (\delta, b)) (S_{(\delta, b)}^* S_{(\delta, b)} + T_{(\delta, b)}^* T_{(\delta, b)}),
$$

$$
T_{(\alpha, a)}^* T_{(\alpha, a)} = \sum_{(\beta, d) \in \Omega_\kappa} B_\kappa((\alpha, a), (\beta, d)) (S_{(\beta, d)}^* S_{(\beta, d)} + T_{(\beta, d)}^* T_{(\beta, d)})
$$

for $(\alpha, a) \in \Omega_\kappa$. Hence the $C^*$-algebra $\mathcal{O}_{\mathcal{H}_\kappa^{A,B}}$ is $*$-isomorphic to the Cuntz-Krieger algebra $\mathcal{O}_{H_\kappa}$ for the matrix $H_\kappa = \begin{bmatrix} A_\kappa & A_\kappa^* \\ B_\kappa & B_\kappa \end{bmatrix}$.

The paper is organized as the following way: In Section 2, we will state basic facts on the $C^*$-symbolic dynamical systems and the $C^*$-textile dynamical systems. In Section 3, we will introduce Hilbert $C^*$-quad modules from $C^*$-textile dynamical...
systems. In Section 4, we will introduce Fock Hilbert $C^*$-quad modules which are two-dimensional analogue of Fock Hilbert $C^*$-bimodules, and study creation operators on the Fock Hilbert $C^*$-quad modules. In Section 5, we will prove the main result stated as Theorem 1.1. In Section 6, we will state a relationship between the $C^*$-algebras $O_{\mathcal{H}_n}$ and $O_{\rho,\eta}^n$ so that the algebra $O_{\mathcal{H}_n}$ is realized as a $C^*$-subalgebra of the tensor product $O_{\rho,\eta}^n \otimes O_2$ in a natural way. In Section 7, we will study the $C^*$-algebras arising from the Hilbert $C^*$-quad modules of the $C^*$-textile dynamical systems defined by commuting matrices and will prove Theorem 1.2.

Throughout the paper, we will denote by $\mathbb{Z}_+$ the set of nonnegative integers and by $\mathbb{N}$ the set of positive integers.

2. $C^*$-symbolic dynamical systems and $C^*$-textile dynamical systems

In this section, we will briefly state basic facts on $C^*$-symbolic dynamical systems and $C^*$-textile dynamical systems. Throughout the section, $\Sigma$ denotes a finite set with its discrete topology, that is called an alphabet. Each element of $\Sigma$ is called a symbol. Let $\Sigma^\mathbb{Z}$ be the infinite product space $\prod_{i \in \mathbb{Z}} \Sigma_i$, where $\Sigma_i = \Sigma$, endowed with the product topology. The transformation $\sigma$ on $\Sigma^\mathbb{Z}$ given by $\sigma((x_i)_{i \in \mathbb{Z}}) = (x_{i+1})_{i \in \mathbb{Z}}$ is called the full shift over $\Sigma$. Let $\Lambda$ be a shift invariant closed subset of $\Sigma^\mathbb{Z}$ i.e. $\sigma(\Lambda) = \Lambda$. The topological dynamical system $(\Lambda, \sigma|_\Lambda)$ is called a two-sided subshift, written as $\Lambda$ for brevity. Finite directed graphs present a class of subshifts called shifts of finite type. More generally, finite directed labeled graphs present a class of subshifts called sofic shifts. The author has introduced a notion of $C^*$-symbolic dynamical systems which generalize $\lambda$-graph systems and automorphisms of unital $C^*$-algebras. $C^*$-symbolic dynamical systems may be presentation of subshifts to $C^*$-algebras.

Let $\mathcal{A}$ be a unital $C^*$-algebra. In what follows, an endomorphism of $\mathcal{A}$ means a $*$-endomorphism of $\mathcal{A}$ that does not necessarily preserve the unit $1_\mathcal{A}$ of $\mathcal{A}$. The unit $1_\mathcal{A}$ is denoted by $1$ unless we specify. Denote by $Z_\mathcal{A}$ the center of $\mathcal{A}$. Let $\rho_\alpha, \alpha \in \Sigma$ be a finite family of endomorphisms of $\mathcal{A}$ indexed by symbols of a finite set $\Sigma$. We assume that $\rho_\alpha(Z_\mathcal{A}) \subset Z_\mathcal{A}, \alpha \in \Sigma$. The family $\rho_\alpha, \alpha \in \Sigma$ of endomorphisms of $\mathcal{A}$ is said to be essential if $\rho_\alpha(1) \neq 0$ for all $\alpha \in \Sigma$ and $\sum_\alpha \rho_\alpha(1) \geq 1$. It is said to be faithful if for any nonzero $x \in \mathcal{A}$ there exists a symbol $\alpha \in \Sigma$ such that $\rho_\alpha(x) \neq 0$. A $C^*$-symbolic dynamical system is a triplet $(\mathcal{A}, \rho, \Sigma)$ consisting of a unital $C^*$-algebra $\mathcal{A}$ and an essential and faithful finite family $\{\rho_\alpha\}_{\alpha \in \Sigma}$ of endomorphisms of $\mathcal{A}$. In [19, 21, 22], we have defined a $C^*$-symbolic dynamical system in a less restrictive way than the above definition. In stead of the above condition $\sum_{\alpha \in \Sigma} \rho_\alpha(1) \geq 1$ with $\rho_\alpha(Z_\mathcal{A}) \subset Z_\mathcal{A}, \alpha \in \Sigma$, we have used the condition in the papers that the closed ideal generated by $\rho_\alpha(1), \alpha \in \Sigma$ coincides with $\mathcal{A}$. All of the examples appeared in the papers [19, 21, 22] satisfy the condition $\sum_{\alpha \in \Sigma} \rho_\alpha(1) \geq 1$ with $\rho_\alpha(Z_\mathcal{A}) \subset Z_\mathcal{A}, \alpha \in \Sigma$, and all discussions in the papers well work under the new definition.

A $C^*$-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ yields a subshift $\Lambda_\rho$ over $\Sigma$ such that a word $\alpha_1 \cdots \alpha_k$ of $\Sigma$ is admissible for $\Lambda_\rho$ if and only if $(\rho_{\alpha_1} \circ \cdots \circ \rho_{\alpha_k})(1) \neq 0$ (19, Proposition 2.1). Denote by $B_k(\Lambda_\rho)$ the set of admissible words of $\Lambda_\rho$ with length $k$. Put $B_k(\Lambda_\rho) = \bigcup_{k=0}^{\infty} B_k(\Lambda_\rho)$, where $B_0(\Lambda_\rho)$ consists of the empty word. The $C^*$-algebra $O_\rho$ associated with $(\mathcal{A}, \rho, \Sigma)$ has been originally constructed in [19] from
an associated Hilbert $C^\ast$-bimodule (cf. [31], [11] etc.). It is realized as a universal
$C^\ast$-algebra $C^\ast(x, S_\alpha; x \in \mathcal{A}, \alpha \in \Sigma)$ generated by $x \in \mathcal{A}$ and partial isometries
$S_\alpha, \alpha \in \Sigma$ subject to the following relations called $(\rho)$:

$$\sum_{\beta \in \Sigma} S_\beta S_\beta^* = 1, \quad x S_\alpha S_\alpha^* = S_\alpha S_\alpha^* x, \quad S_\alpha^* x S_\alpha = \rho_\alpha(x)$$

for all $x \in \mathcal{A}$ and $\alpha \in \Sigma$. The $C^\ast$-algebra $\mathcal{O}_\rho$ is a generalization of the $C^\ast$-algebra
$\mathcal{O}_\Sigma$ associated with the $\lambda$-graph system $\Sigma$ (cf. [17]).

Let $(\mathcal{A}, \rho, \eta, \Sigma^p, \Sigma^q, \kappa)$ be a $C^\ast$-textile dynamical system. It consists of two
$C^\ast$-symbolic dynamical systems $(\mathcal{A}, \rho, \Sigma^p)$ and $(\mathcal{A}, \eta, \Sigma^q)$ with common unital $C^\ast$-
algebra $\mathcal{A}$ and commutation relations between their endomorphisms $\rho_\alpha, \alpha \in \Sigma^p$ and
$\eta_\alpha, \alpha \in \Sigma^q$ through a bijection $\kappa$ satisfying (1.1). Let $S_\alpha, \alpha \in \Sigma^p$ and $T_\alpha, \alpha \in \Sigma^q$
be the generating partial isometries of $\mathcal{O}_\rho$ and of $\mathcal{O}_\eta$, which satisfy (1.2) and (1.3)
respectively. We set two $C^\ast$-algebras

$$\mathcal{B}_\rho = C^\ast(S_\alpha x S_\alpha^* : \alpha \in \Sigma^p, x \in \mathcal{A}), \quad \mathcal{B}_\eta = C^\ast(T_\alpha T_\alpha^* : \alpha \in \Sigma^q, x \in \mathcal{A}).$$

They are realized concretely as subalgebras of $\mathcal{O}_\rho$ and of $\mathcal{O}_\eta$ respectively. Both the
algebras $\mathcal{B}_\rho$ and $\mathcal{B}_\eta$ contain the algebra $\mathcal{A}$ through the identification

$$x = \sum_{\alpha \in \Sigma^p} S_\alpha \rho_\alpha(x) S_\alpha^* = \sum_{a \in \Sigma^q} T_\alpha \eta_\alpha(x) T_\alpha^*, \quad x \in \mathcal{A}. \quad (2.1)$$

We put the projections

$$P_\alpha = \rho_\alpha(1) \text{ for } \alpha \in \Sigma^p, \quad Q_\alpha = \eta_\alpha(1) \text{ for } \alpha \in \Sigma^q.$$

Elements $w \in \mathcal{B}_\rho$ and $z \in \mathcal{B}_\eta$ are uniquely written as in the following way:

$$w = \sum_{a \in \Sigma^p} S_\alpha w_\alpha S_\alpha^* \text{ with } w_\alpha = P_\alpha w_\alpha P_\alpha, \alpha \in \Sigma^p, \quad (2.2)$$

$$z = \sum_{a \in \Sigma^q} T_\alpha z_\alpha T_\alpha^* \text{ with } z_\alpha = Q_\alpha z_\alpha Q_\alpha, \alpha \in \Sigma^q. \quad (2.3)$$

Define an alphabet set $\Sigma_\kappa$ as in (1.5). For $\omega = (\alpha, b, a, \beta) \in \Sigma_\kappa$, we set

$$\alpha = t(\omega) \in \Sigma^p, \quad b = r(\omega) \in \Sigma^q, \quad a = l(\omega) \in \Sigma^q, \quad \beta = b(\omega) \in \Sigma^p,$$

which stand for: top, right, left, bottom respectively as in the following figure.

$$\begin{array}{c}
\alpha = t(\omega) \\
\downarrow \\
\alpha = l(\omega) \\
\downarrow \\
\beta = b(\omega) \\
\end{array}$$

Define $*$-homomorphisms $\hat{\rho}_\alpha : \mathcal{B}_\rho \rightarrow \mathcal{A}$ for $\alpha \in \Sigma^p$ and $\hat{\eta}_\alpha : \mathcal{B}_\eta \rightarrow \mathcal{A}$ for $\alpha \in \Sigma^q$
by (1.6) which satisfy the equalities

$$\hat{\rho}_\alpha(w) = P_\alpha w_\alpha P_\alpha \quad \text{and} \quad \hat{\eta}_\alpha(z) = Q_\alpha z_\alpha Q_\alpha$$

for $w = \sum_{\beta \in \Sigma^p} S_\beta w_\beta S_\beta^* \in \mathcal{B}_\rho$ as in (2.2) and $z = \sum_{\beta \in \Sigma^q} T_\beta z_\beta T_\beta^* \in \mathcal{B}_\eta$ as in (2.3).

Their restrictions to $\mathcal{A}$ coincide with $\rho_\alpha$ and $\eta_\alpha$ respectively.

**Lemma 2.1.** Keep the above notations.
(i) For $\alpha \in \Sigma^\nu$ and $z = \sum_{b \in \Sigma^n} T_b z_b T_b^* \in B_\eta$ as in (2.3), put
\[
\hat{\rho}_\alpha^y(z) = \sum_{b,a,\beta} \sum_{(a,b,a,\beta) \in \Sigma_\kappa} T_b \rho_\beta(z_a) T_b^* \in B_\eta. \tag{2.4}
\]

Then $\hat{\rho}_\alpha^y : B_\eta \to B_\eta$ is a *-homomorphism such that $\hat{\rho}_\alpha^y(y) = \rho_\alpha(y)$ for $y \in A$.

(ii) For $a \in \Sigma^n$ and $w = \sum_{b \in \Sigma^\nu} S_b w_b S_b^* \in B_\rho$ as in (2.2), put
\[
\hat{\eta}_\alpha^a(w) = \sum_{\alpha,b,\beta} S_b \eta_b(w_\alpha) S_b^* \in B_\rho. \tag{2.5}
\]

Then $\hat{\eta}_\alpha^a : B_\rho \to B_\rho$ is a *-homomorphism such that $\hat{\eta}_\alpha^a(y) = \eta_\alpha(y)$ for $y \in A$.

Proof. (i) Since $z_a = \hat{\eta}_\alpha(z)$, the equality (2.4) goes to
\[
\hat{\rho}_\alpha^y(z) = \sum_{b,a,\beta} \sum_{(a,b,a,\beta) \in \Sigma_\kappa} T_b \rho_\beta(\hat{\eta}_\alpha(z)) T_b^*
\]
as in (1.7). It is easy to see that $\hat{\rho}_\alpha^y : B_\eta \to B_\eta$ yields a *-homomorphism. If in particular $z = y \in A$, we have $y = \sum_{b \in \Sigma^n} T_b \eta_b(y) T_b^*$ so that
\[
\hat{\rho}_\alpha^y(y) = \sum_{b,a,\beta} T_b \rho_\beta(\eta_\alpha(y)) T_b^* = \sum_{a,b,\beta} T_b \eta_b(\rho_\alpha(y)) T_b^* = \rho_\alpha(y).
\]

(ii) is similar to (i). \(\square\)

The commutation relations (1.1) on $A$ extend to $B_\rho$ and to $B_\eta$ as in the following lemma.

Lemma 2.2. For $\omega = (\alpha, b, a, \beta) \in \Sigma_\kappa$, we have

(i) $\eta_\beta \circ \hat{\rho}_\alpha^y(w) = \hat{\rho}_\beta \circ \hat{\rho}_\alpha^y(w)$ for $w \in B_\rho$.

(ii) $\rho_\beta \circ \hat{\eta}_\alpha^a(z) = \hat{\eta}_\beta \circ \hat{\rho}_\alpha^y(z)$ for $z \in B_\eta$.

Proof. (i) For $w = \sum_{\alpha' \in \Sigma^\nu} S_{\alpha'} w_{\alpha'} S_{\alpha'}^*$ as in (2.2), we have $S_\beta \hat{\rho}_\alpha^y(w) S_\beta = S_\beta \rho_\beta(\eta_\alpha(w)) S_\beta^*$ so that by (1.1)
\[
\hat{\rho}_\beta \circ \hat{\rho}_\alpha^y(w) = P_\beta \eta_\beta(w_\alpha) P_\beta
\]
\[
= \rho_\beta(1) \eta_\beta(\rho_\alpha(1)) \eta_b(w_\alpha) \eta_b(\rho_\alpha(1)) \rho_\beta(1)
\]
\[
= \rho_\beta(\eta_\alpha(1)) \eta_b(w_\alpha) \rho_\beta(\eta_\alpha(1))
\]
\[
= \eta_b(\rho_\alpha(1) w_\alpha \rho_\alpha(1)) = \eta_b(\hat{\rho}_\alpha(w)).
\]

(ii) is similar to (i). \(\square\)

3. Hilbert $C^*$-quad modules from $C^*$-textile dynamical systems

We fix a $C^*$-textile dynamical system $(A, \rho, \eta, \Sigma^\nu, \Sigma^\eta, \Sigma_\kappa)$. For $\omega = (\alpha, b, a, \beta) \in \Sigma_\kappa$, we put a projection
\[
E_\omega = \eta_\beta(\rho_\alpha(1)) = \rho_\beta(\eta_\alpha(1)) \in Z_A. \tag{3.1}
\]
Let $e_\omega, \omega \in \Sigma_k$ denote the orthogonal basis of the vector space $C|\Sigma_k|$ where $|\Sigma_k|$ means the cardinal number of the finite set $\Sigma_k$. Define the vector space
\[ \mathcal{H}_k = \sum_{\omega \in \Sigma_k} C e_\omega \otimes E_\omega A \] (3.2)
which is naturally isomorphic to the vector space $\sum_{\omega \in \Sigma_k} \otimes E_\omega A$.

We first endow $\mathcal{H}_k$ with right $A$-module structure and $A$-valued inner product as follows: For $\xi = \sum_{\omega \in \Sigma_k} e_\omega \otimes E_\omega x_\omega, \xi' = \sum_{\omega' \in \Sigma_k} e_{\omega'} \otimes E_{\omega'} x'_{\omega'}$ with $x_\omega, x'_{\omega'} \in A$ and $y \in A$, set
\[ \langle \xi | \xi' \rangle_A(y) := \sum_{\omega \in \Sigma_k} e_\omega \otimes E_\omega x_\omega y \in \mathcal{H}_k, \]
\[ (\xi | \xi')_A := \sum_{\omega \in \Sigma_k} x^*_\omega E_\omega x'_{\omega} \in A \]
which satisfy the relations:
\[ \langle \xi | \xi' \rangle_A(y) = \langle \xi | \xi' \rangle_A \cdot y, \quad (\xi | \xi')_A = (\xi' | \xi)_A. \]

We will further endow $\mathcal{H}_k$ with two other Hilbert $C^*$-bimodule structure. Such a system will be called a Hilbert $C^*$-quad module. Let $\xi = \sum_{\omega \in \Sigma_k} e_\omega \otimes E_\omega x_\omega, \xi' = \sum_{\omega \in \Sigma_k} e_{\omega'} \otimes E_{\omega'} x'_{\omega'} \in \mathcal{H}_k$, $w = \sum_{\omega \in \Sigma_k} \omega_{\omega'}w_{\omega'}S_{\omega}\in B_\rho$, and $w = \sum_{\omega \in \Sigma_k} T_{\omega'}z_{\omega'}T'_{\omega'\rho} \in B_{\eta}$ as in (2.2), $z = \sum_{\omega \in \Sigma_k} \sum T'_{\omega}z_{\omega}T_{\omega'} \in B_{\eta}$ as in (2.3). We define

1. The right $B_\rho$-action $\phi_\rho$ and the right $B_\eta$-action $\phi_\eta$:
\[ \phi_\rho(w)\xi := \sum_{\omega \in \Sigma_k} e_\omega \otimes E_\omega \eta_{T(\omega)}(w_{T(\omega)}) x_\omega, \quad \phi_\eta(z)\xi := \sum_{\omega \in \Sigma_k} e_\omega \otimes E_\omega \rho_{T(\omega)}(z_{T(\omega)}) x_\omega. \]

2. The left $B_\rho$-action $\varphi_\rho$ and the left $B_\eta$-action $\varphi_\eta$:
\[ \varphi_\rho(w)\xi := \sum_{\omega \in \Sigma_k} e_\omega \otimes E_\omega \eta_{T(\omega)}(w_{T(\omega)}) x_\omega, \quad \varphi_\eta(z)\xi := \sum_{\omega \in \Sigma_k} e_\omega \otimes E_\omega \rho_{T(\omega)}(z_{T(\omega)}) x_\omega. \]

3. The right $B_\rho$-valued inner product $\langle \cdot | \cdot \rangle_\rho$ and the right $B_\eta$-valued inner product $\langle \cdot | \cdot \rangle_\eta$:
\[ \langle \xi | \xi' \rangle_\rho := \sum_{\omega \in \Sigma_k} S_{\rho}(\omega) x^*_\omega E_\omega x'_{\omega} S^*_{\rho}(\omega), \quad (\xi | \xi')_\eta := \sum_{\omega \in \Sigma_k} T_{\eta}(\omega) x^*_\omega E_\omega x'_{\omega} T^*_{\eta}(\omega). \]

The following lemma is straightforward.

**Lemma 3.1.** For $\xi \in \mathcal{H}_k$ and $w, w' \in B_\rho, z, z' \in B_\eta$, we have
\[ \langle \xi \varphi_\rho(w) \varphi_\rho(w') \rangle_\rho = \langle \xi \varphi_\rho(ww') \rangle_\rho, \quad (\xi \varphi_\eta(z)) \varphi_\eta(z') = \xi \varphi_\eta(zz'), \]
\[ \phi_\rho(w)(\phi_\rho(w') \xi) = \phi_\rho(ww') \xi, \quad \varphi_\eta(z)(\varphi_\eta(z') \xi) = \varphi_\eta(zz') \xi, \]
\[ \phi_\rho(w)(\xi \varphi_\rho(w')) = (\phi_\rho(w) \xi) \varphi_\rho(w'), \quad \varphi_\eta(z)(\xi \varphi_\eta(z')) = (\phi_\eta(z) \xi) \varphi_\eta(z'). \]

**Lemma 3.2.** For $\xi, \xi' \in \mathcal{H}_k$ and $w \in B_\rho, z \in B_\eta$, we have
\[ \langle \xi | \xi' \varphi_\rho(w) \rangle_\rho = \langle \xi | \xi' \rangle_\rho \cdot w, \quad \langle \xi | \xi' \varphi_\eta(z) \rangle_\eta = \langle \xi | \xi' \rangle_\eta \cdot z, \]
\[ \langle \xi \varphi_\rho(w) | \xi' \rangle_\rho = w^* \langle \xi | \xi' \rangle_\rho, \quad \langle \xi \varphi_\eta(z) | \xi' \rangle_\eta = z^* \langle \xi | \xi' \rangle_\eta, \]
\[ \langle \phi_\rho(w) \xi | \xi' \rangle_\rho = \langle \xi | \phi_\rho(w^*) \xi' \rangle_\rho, \quad \langle \phi_\eta(z) \xi | \xi' \rangle_\eta = \langle \xi | \phi_\eta(z^*) \xi' \rangle_\eta. \]
Proof. We will show the equalities
\[\langle \xi \mid \xi' \varphi_p(w) \rangle_\rho = \langle \xi \mid \xi' \rangle_\rho \cdot w, \quad \langle \varphi_p(w) \xi \mid \xi' \rangle_\rho = \langle \xi \mid \varphi_p(w^* \xi') \rangle_\rho.\]

For \(\xi = \sum_{\omega \in \Sigma_n} e_{\omega} \otimes E_{\omega} x_{\omega}, \xi' = \sum_{\omega' \in \Sigma_n} e_{\omega'} \otimes E_{\omega'} x'_{\omega'} \in \mathcal{H}_\kappa\) with \(x_{\omega}, x'_{\omega'} \in A\), and \(w = \sum_{\gamma \in \Sigma^\rho} S_{\gamma} w_{\gamma} S^*_\gamma \in B_\rho\) as in (2.2), we have
\[
\langle \xi \mid \xi' \varphi_p(w) \rangle_\rho = \sum_{\omega, \omega' \in \Sigma_n} \langle e_{\omega} \otimes E_{\omega} x_{\omega} \mid e_{\omega'} \otimes E_{\omega'} x'_{\omega'} w_{b(\omega')} \rangle_\rho \\
= \sum_{\omega \in \Sigma_n} S_{b(\omega)} x^*_\omega E_{\omega} x'_{\omega} w_{b(\omega)} S^*_b(\omega) \\
= \left( \sum_{\omega \in \Sigma_n} S_{b(\omega)} x^*_\omega E_{\omega} x'_{\omega} S^*_b(\omega) \right) \cdot \left( \sum_{\gamma \in \Sigma^\rho} S_{\gamma} w_{\gamma} S^*_\gamma \right) \\
= \langle \xi \mid \xi' \rangle_\rho \cdot w.
\]

We also have
\[
\langle \varphi_p(w) \xi \mid \xi' \rangle_\rho = \sum_{\omega, \omega' \in \Sigma_n} \sum_{\gamma \in \Sigma^\rho} \langle \varphi_p(S_{\gamma} w_{\gamma} S^*_\gamma) (e_{\omega} \otimes E_{\omega} x_{\omega}) \mid e_{\omega'} \otimes E_{\omega'} x'_{\omega'} \rangle_\rho \\
= \sum_{\omega, \omega' \in \Sigma_n} \langle e_{\omega} \otimes E_{\omega} \eta_{\tau(\omega)}(w^*_t(\omega)) x_{\omega} \mid e_{\omega'} \otimes E_{\omega'} x'_{\omega'} \rangle_\rho \\
= \sum_{\omega \in \Sigma_n} S_{b(\omega)} x^*_\omega E_{\omega} \eta_{\tau(\omega)}(w^*_t(\omega)) x'_{\omega} S^*_b(\omega) \\
= \sum_{\omega, \omega' \in \Sigma_n} \sum_{\gamma \in \Sigma^\rho} \langle e_{\omega} \otimes E_{\omega} x_{\omega} \mid \varphi_p(S_{\gamma} w_{\gamma} S^*_\gamma) (e_{\omega'} \otimes E_{\omega'} x'_{\omega'}) \rangle_\rho \\
= \langle \xi \mid \varphi_p(w^* \xi') \rangle_\rho.
\]

The three equalities of the right hand side are similarly shown to the above equalities. \(\square\)

Hence we have
\[
\varphi_p(w^*) = \varphi_p(w)^* : \text{the adjoint with respect to the inner product } \langle \cdot \mid \cdot \rangle_\rho,
\]
\[
\varphi_\eta(z^*) = \varphi_\eta(z)^* : \text{the adjoint with respect to the inner product } \langle \cdot \mid \cdot \rangle_\eta.
\]

The following lemma is direct and shows that the two module structure are compatible to each other.

**Lemma 3.3.** For \(w \in B_\rho, z \in B_\eta\) and \(\xi \in \mathcal{H}_\kappa\), we have
\[
\text{(i)} \quad \langle \varphi_p(w) \xi \mid \varphi_\eta(z) \rangle = \varphi_p(w) \langle \xi \varphi_\eta(z) \rangle.
\]
\[
\text{(ii)} \quad \langle \varphi_\eta(z) \xi \mid \varphi_p(w) \rangle = \varphi_\eta(z) \langle \xi \varphi_p(w) \rangle.
\]

Then we have the following proposition

**Proposition 3.4.** Keep the above notations.
\[
\text{(i)} \quad (\mathcal{H}_\kappa, \varphi_p) \text{ is a right } B_\rho \text{-module with right } B_\rho \text{-valued inner product } \langle \cdot \mid \cdot \rangle_\rho \text{ and left } B_\rho \text{-action by } \varphi_p. \text{ Hence } \mathcal{H}_\kappa \text{ is a Hilbert } C^* \text{-bimodule over } B_\rho.
\]
\[
\text{(ii)} \quad (\mathcal{H}_\kappa, \varphi_\eta) \text{ is a right } B_\eta \text{-module with right } B_\eta \text{-valued inner product } \langle \cdot \mid \cdot \rangle_\eta \text{ and left } B_\eta \text{-action by } \varphi_\eta. \text{ Hence } \mathcal{H}_\kappa \text{ is a Hilbert } C^* \text{-bimodule over } B_\eta.
\]
Therefore $\mathcal{H}_k$ has multi structure of Hilbert $C^*$-bimodules, which are compatible to each other.

$\mathcal{H}_k$ is originally a Hilbert $C^*$-right module over $\mathcal{A}$, which is also compatible to the two left actions $\phi_p$ of $\mathcal{B}_p$ and $\phi_\eta$ of $\mathcal{B}_\eta$ as in the following lemma. Its proof is straightforward.

**Lemma 3.5.** For $\xi \in \mathcal{H}_k$ and $y \in \mathcal{A}$, we have

(i) $\phi_p(w)(\xi \varphi_\mathcal{A}(y)) = (\phi_p(w)\xi)\varphi_\mathcal{A}(y)$ for $w \in \mathcal{B}_p$.

(ii) $\phi_\eta(z)(\xi \varphi_\mathcal{A}(y)) = (\phi_\eta(z)\xi)\varphi_\mathcal{A}(y)$ for $z \in \mathcal{B}_\eta$.

Hence both $\phi_\eta(z)$ and $\phi_p(w)$ are right $\mathcal{A}$-module maps.

Define positive maps $\psi_p : \mathcal{A} \to \mathcal{B}_p$ and $\psi_\eta : \mathcal{A} \to \mathcal{B}_\eta$ by

$$
\psi_p(y) = \sum_{\alpha \in \Sigma^\alpha} S_\alpha y S_\alpha^* \in \mathcal{B}_p, \quad \psi_\eta(y) = \sum_{\alpha \in \Sigma^\alpha} T_\alpha y T_\alpha^* \in \mathcal{B}_\eta
$$

for $y \in \mathcal{A}$. Then we have

**Lemma 3.6.** For $\xi \in \mathcal{H}_k$ and $y \in \mathcal{A}$, we have

(i) $\xi \varphi_\mathcal{A}(\psi_p(y)) = (\xi \varphi_\mathcal{A}(\psi_\eta(y))$ for $w \in \mathcal{B}_p$.

(ii) $\xi \varphi_\mathcal{A}(\psi_\eta(y)) = (\xi \varphi_\mathcal{A}(\psi_p(y))$ for $z \in \mathcal{B}_\eta$.

Hence we have

$$
\xi \varphi_\mathcal{A}((\psi_p(y)) = \xi \varphi_\mathcal{A}(\psi_\eta(y)) = \xi \varphi_\mathcal{A}(y).
$$

**Proof.** (i) For $\xi = \sum_{\alpha \in \Sigma^\alpha} e_\alpha \otimes E_\alpha x_\alpha \in \mathcal{H}_k$ with $x_\alpha \in \mathcal{A}$, and $w = \sum_{\alpha' \in \Sigma^\alpha} S_\alpha w_\alpha S_\alpha^* \in \mathcal{B}_p$ as in (2.2), we have

$$
w \psi_p(y) = \sum_{\alpha' \in \Sigma^\alpha} S_\alpha w_\alpha S_\alpha^* \sum_{\beta \in \Sigma^\beta} S_\beta y S_\beta^* = \sum_{\alpha' \in \Sigma^\alpha} S_\alpha w_\alpha y S_\alpha^*
$$

so that

$$
\xi \varphi_\mathcal{A}(w \psi_p(y)) = \sum_{\omega \in \Sigma_n} e_\omega \otimes E_\omega x_\omega \psi_p(y)
$$

$$
= \sum_{\omega \in \Sigma_n} [(e_\omega \otimes E_\omega x_\omega) \varphi_\mathcal{A}(w)] \varphi_\mathcal{A}(y) = [\xi \varphi_\mathcal{A}(w)] \varphi_\mathcal{A}(y).
$$

(ii) is similar to (i). \(\square\)

**Lemma 3.7.** For $y \in \mathcal{A}$ and $\xi \in \mathcal{H}_k$, we have $\phi_p(y) = \phi_\eta(y) = \phi(y)\xi$.

**Proof.** Since the identities $y = \sum_{\alpha \in \Sigma^\alpha} S_\alpha (y) S_\alpha^* = \sum_{\alpha \in \Sigma^\alpha} T_\alpha (y) T_\alpha^*$ hold, we have for $\xi = \sum_{\omega \in \Sigma_n} e_\omega \otimes E_\omega x_\omega$ with $x_\omega \in \mathcal{A},$

$$
\phi_p(y) \xi = \sum_{\omega \in \Sigma_n} e_\omega \otimes E_\omega \eta(\rho(\omega))(\xi) x_\omega.
$$

On the other hand, we have

$$
\phi_\eta(y) \xi = \sum_{\omega \in \Sigma_n} e_\omega \otimes E_\omega \rho(\omega)(\xi) x_\omega.
$$

As $\eta(\rho(\omega))(\xi) = \rho(\omega)(\eta(\omega))(\xi)$, we obtain the desired equality. \(\square\)
By the above lemma, we may define the left action \( \phi \) of \( \mathcal{A} \) on \( \mathcal{H}_\kappa \) by
\[
\phi(y)\xi := \phi_p(y)\xi = \phi(y)\xi, \quad y \in \mathcal{A}, \quad \xi \in \mathcal{H}_\kappa
\]
so that \( \mathcal{H}_\kappa \) has a structure of a Hilbert \( C^* \)-bimodule over \( \mathcal{A} \). We note the following lemma.

**Lemma 3.8.** If the algebra \( \mathcal{A} \) is commutative, we have
\[
\phi_p(w)\phi_q(z) = \phi_q(z)\phi_p(w), \quad w \in \mathcal{B}_\rho, \quad z \in \mathcal{B}_\eta.
\]

**Proof.** For \( w = \sum_{a \in \Sigma^p} S_a \alpha S_a^* \) as in (2.2), \( z = \sum_{a' \in \Sigma^q} T_{a'} \beta T_{a'}^* \) as in (2.3) and \( \xi = \sum_{\omega \in \Sigma_n} e_\omega \otimes E_\omega x_\omega \) with \( x_\omega \in \mathcal{A} \), we have
\[
\phi_p(w)\phi_q(z)\xi = \sum_{\omega \in \Sigma_n} e_\omega \otimes E_\omega \eta_{r(\omega)}(w_{t(\omega)})\rho_{b(\omega)}(z_{t(\omega)})x_\omega
\]
\[
= \sum_{\omega \in \Sigma_n} e_\omega \otimes E_\omega \rho_{b(\omega)}(z_{t(\omega)})\eta_{r(\omega)}(w_{t(\omega)})x_\omega
\]
\[
= \phi_q(z)\phi_p(w)\xi.
\]

Put for \( \alpha \in \Sigma^p \) and \( \alpha \in \Sigma^q \)
\[
u_\alpha = \sum_{\omega \in \Sigma_n, \alpha = t(\omega)} e_\omega \otimes E_\omega \in \mathcal{H}_\kappa, \quad v_\alpha = \sum_{\omega \in \Sigma_n, \alpha = t(\omega)} e_\omega \otimes E_\omega \in \mathcal{H}_\kappa.
\]

**Lemma 3.9.** Keep the above notations.

(i) \( \{u_\alpha\}_{\alpha \in \Sigma^p} \) forms an essential orthogonal finite basis of \( \mathcal{H}_\kappa \) with respect to the \( \mathcal{B}_\rho \)-valued inner product \( \langle \cdot | \cdot \rangle_\eta \) as right \( \mathcal{B}_\rho \)-module through \( \phi_\eta \).

(ii) \( \{v_\alpha\}_{\alpha \in \Sigma^q} \) forms an essential orthogonal finite basis of \( \mathcal{H}_\kappa \) with respect to the \( \mathcal{B}_\rho \)-valued inner product \( \langle \cdot | \cdot \rangle_\rho \) as right \( \mathcal{B}_\rho \)-module through \( \phi_\rho \).

**Proof.** (i) For \( \alpha, \beta \in \Sigma^p \), we have
\[
\langle u_\alpha | u_\beta \rangle_\eta = \langle \sum_{\omega \in \Sigma_n, \alpha = t(\omega)} e_\omega \otimes E_\omega | \sum_{\omega' \in \Sigma_n, \beta = t(\omega')} e_{\omega'} \otimes E_{\omega'} \rangle_\eta
\]
\[
= \begin{cases} 
\sum_{\omega \in \Sigma_n, \alpha = t(\omega)} \langle e_\omega \otimes E_\omega | e_\omega \otimes E_\omega \rangle_\eta & \text{if } \alpha = \beta, \\
0 & \text{if } \alpha \neq \beta.
\end{cases}
\]

Since
\[
\sum_{\omega \in \Sigma_n, \alpha = t(\omega)} \langle e_\omega \otimes E_\omega | e_\omega \otimes E_\omega \rangle_\eta = \sum_{\omega \in \Sigma_n, \alpha = t(\omega)} T_{r(\omega)} E_{\omega} T_{r(\omega)}^*
\]
\[
= \sum_{\omega \in \Sigma_n, \alpha = t(\omega)} T_{r(\omega)} T_{r(\omega)}^* \rho_{\alpha}(1) T_{r(\omega)} T_{r(\omega)}^* = P_\alpha,
\]
we see
\[
\langle u_\alpha | u_\beta \rangle_\eta = \begin{cases} 
P_\alpha & \text{if } \alpha = \beta, \\
0 & \text{if } \alpha \neq \beta.
\end{cases}
\]

Hence we have
\[
\sum_{\alpha \in \Sigma^p} \langle u_\alpha | u_\alpha \rangle_\eta = \sum_{\alpha \in \Sigma^p} P_\alpha \geq 1.
\]
For \( \xi = \sum_{\omega \in \Sigma_{\omega}} e_{\omega} \otimes E_{\omega} x_{\omega} \in \mathcal{H}_{\omega} \) with \( x_{\omega} \in \mathcal{A} \), we have

\[
(u_{\alpha} \mid \xi)_{\eta} = \sum_{\omega \in \Sigma_{\omega}, \alpha = t(\omega)} T_{r(\omega)} E_{\omega} x_{\omega} T_{r(\omega)}^*.
\]

It then follows that

\[
u_{\alpha} \varphi_{\eta}(\langle u_{\alpha} \mid \xi \rangle_{\eta}) = \left( \sum_{\omega \in \Sigma_{\omega}, \alpha = t(\omega)} e_{\omega} \otimes E_{\omega} \right) \varphi_{\eta}\left( \sum_{\omega' \in \Sigma_{\omega}, \alpha = t(\omega')} T_{r(\omega')} E_{\omega'} x_{\omega'} T_{r(\omega')}^* \right)
= \sum_{\omega, \omega' \in \Sigma_{\omega}, \alpha = t(\omega')} (e_{\omega} \otimes E_{\omega}) \varphi_{\eta}(T_{r(\omega')} E_{\omega'} x_{\omega'} T_{r(\omega')}^*)
= \sum_{\omega \in \Sigma_{\omega}, \alpha = t(\omega)} e_{\omega} \otimes E_{\omega} x_{\omega}
\]

so that

\[
\sum_{\alpha \in \Sigma_{\alpha}} u_{\alpha} \varphi_{\eta}(\langle u_{\alpha} \mid \xi \rangle_{\eta}) = \sum_{\alpha \in \Sigma_{\alpha}, \omega \in \Sigma_{\omega}, \alpha = t(\omega)} e_{\omega} \otimes E_{\omega} x_{\omega} = \sum_{\omega \in \mathcal{H}_{\omega}} e_{\omega} \otimes E_{\omega} x_{\omega} = \xi.
\]

(ii) is similar to (i). \( \square \)

As \( \langle u_{\alpha} \mid u_{\alpha} \rangle_{\eta} = P_{\alpha} \) and \( \langle v_{\alpha} \mid v_{\alpha} \rangle_{\rho} = Q_{\alpha} \), we note that the equality

\[
\eta_{\beta}(\langle u_{\alpha} \mid u_{\alpha} \rangle_{\eta}) = \rho_{\beta}(\langle v_{\alpha} \mid v_{\alpha} \rangle_{\rho}) = E_{\omega}
\]

holds for \( \omega = (\alpha, b, a, \beta) \in \Sigma_{\kappa} \).

**Lemma 3.10.** For \( \alpha \in \Sigma_{\rho}, a \in \Sigma_{\eta} \) and \( y \in \mathcal{A} \), we have

(i) \( \phi(y) u_{\alpha} = u_{\alpha} \varphi_{\eta}(\rho_{\alpha}(y)) \) and hence \( \rho_{\alpha}(y) = \langle u_{\alpha} \mid \phi(y) u_{\alpha} \rangle_{\eta} \).

(ii) \( \phi(y) v_{\alpha} = v_{\alpha} \varphi_{\rho}(\eta_{\alpha}(y)) \) and hence \( \eta_{\alpha}(y) = \langle v_{\alpha} \mid \phi(y) v_{\alpha} \rangle_{\rho} \).

Therefore the commutation relations (1.1) are rephrased as the equality

\[
\langle v_{b} \mid \phi(\langle u_{\alpha} \mid \phi(y) u_{\alpha} \rangle_{\eta})v_{b} \rangle_{\rho} = \langle u_{\beta} \mid \phi(\langle v_{a} \mid \phi(y) v_{a} \rangle_{\rho})u_{\beta} \rangle_{\eta}
\]  

for \( \omega = (\alpha, b, a, \beta) \in \Sigma_{\kappa} \) and \( y \in \mathcal{A} \).

**Proof.** (i) It follows that

\[
\phi(y) u_{\alpha} = \phi_{\eta}(y) u_{\alpha} = \phi_{\eta}(\sum_{\alpha' \in \Sigma_{\eta}} T_{\alpha'} \eta_{\alpha}(y) T_{\alpha'}^*)\left( \sum_{\omega \in \Sigma_{\omega}, \alpha = t(\omega)} e_{\omega} \otimes E_{\omega} \right)
= \sum_{\omega \in \Sigma_{\omega}, \alpha = t(\omega)} e_{\omega} \otimes E_{\omega} \rho_{\beta}(\eta_{\alpha}(y))
= \sum_{\omega \in \Sigma_{\omega}, \alpha = t(\omega)} e_{\omega} \otimes E_{\omega} \eta_{\beta}(\rho_{\alpha}(y))
= \sum_{\omega \in \Sigma_{\omega}, \alpha = t(\omega)} [e_{\omega} \otimes E_{\omega}] \varphi_{\eta}(\sum_{b' \in \Sigma_{\eta}} T_{b'} \eta_{b'}(\rho_{\alpha}(y)) T_{b'}^*)
= u_{\alpha} \varphi_{\eta}(\rho_{\alpha}(y)).
\]

It then follows that

\[
\langle u_{\alpha} \mid \phi(y) u_{\alpha} \rangle_{\eta} = \langle u_{\alpha} \mid u_{\alpha} \varphi_{\eta}(\rho_{\alpha}(y)) \rangle_{\eta} = \langle u_{\alpha} \mid u_{\alpha} \rangle_{\eta} \cdot \rho_{\alpha}(y) = \rho_{\alpha}(y).
\]

(ii) is similar to (i). \( \square \)

More generally we have

**Lemma 3.11.** For \( \alpha \in \Sigma_{\rho}, a \in \Sigma_{\eta} \) and \( w \in \mathcal{B}_{\rho}, z \in \mathcal{B}_{\eta} \), we have
(i) \( \phi_\rho(w)u_\alpha = u_\alpha \varphi_\eta(\langle u_\alpha \mid \phi_\rho(w)u_\alpha \rangle) \), \( \phi_\rho(w)v_\alpha = v_\alpha \varphi_\rho(\langle v_\alpha \mid \phi_\rho(w)v_\alpha \rangle) \).
(ii) \( \phi_\eta(z)u_\alpha = u_\alpha \varphi_\eta(\langle u_\alpha \mid \varphi_\eta(z)u_\alpha \rangle) \), \( \phi_\eta(z)v_\alpha = v_\alpha \varphi_\eta(\langle v_\alpha \mid \varphi_\eta(z)v_\alpha \rangle) \).

**Proof.** (i) For \( \xi \in \mathcal{H}_\alpha \), we have \( \xi = \sum_{\alpha' \in \Sigma^\alpha} u_{\alpha'} \varphi_\eta(\langle u_{\alpha'} \mid \xi \rangle) \). For \( \alpha \neq \alpha' \), we have 
\[
\langle u_\alpha \mid \phi_\rho(w)u_{\alpha'} \rangle = 0 \text{ so that } \phi_\rho(w)u_\alpha = \sum_{\alpha' \in \Sigma^\alpha} u_{\alpha'} \varphi_\eta(\langle u_{\alpha'} \mid \phi_\rho(w)u_\alpha \rangle) = u_\alpha \varphi_\eta(\langle u_\alpha \mid \phi_\rho(w)u_\alpha \rangle).
\]
Similarly for \( \alpha \neq \alpha' \), we have 
\[
\langle v_\alpha \mid \phi_\rho(w)v_{\alpha'} \rangle = 0 \text{ so that } \phi_\rho(w)v_\alpha = \sum_{\alpha' \in \Sigma^\alpha} v_{\alpha'} \varphi_\rho(\langle v_{\alpha'} \mid \phi_\rho(w)v_\alpha \rangle) = v_\alpha \varphi_\rho(\langle v_\alpha \mid \phi_\rho(w)v_\alpha \rangle).
\]
(ii) is similar to (i). \( \Box \)

The following lemma states that the *-homomorphisms \( \tilde{\rho}_\alpha, \tilde{\eta}_\alpha \) on \( \mathcal{B}_\rho \) and \( \check{\eta}_\alpha, \check{\rho}_\alpha \) on \( \mathcal{B}_\eta \) are given by inner products.

**Lemma 3.12.** For \( \alpha \in \Sigma^\rho, a \in \Sigma^\eta \) and \( w \in \mathcal{B}_\rho, z \in \mathcal{B}_\eta \), we have

(i) \( \tilde{\rho}_\alpha(w) = \langle u_\alpha \mid \phi_\rho(w)u_\alpha \rangle \), \( \tilde{\eta}_\alpha(z) = \langle v_\alpha \mid \phi_\eta(z)v_\alpha \rangle \).
(ii) \( \check{\rho}_\alpha^\rho(z) = \langle u_\alpha \mid \phi_\eta(z)u_\alpha \rangle \), \( \check{\eta}_\alpha^\rho(w) = \langle v_\alpha \mid \phi_\rho(w)v_\alpha \rangle \).

**Proof.** (i) For \( w = \sum_{\alpha' \in \Sigma^\alpha} S_{\alpha'} w_{\alpha'} S_{\alpha'}^* \in \mathcal{B}_\rho \) as in (2.2), we have
\[
\phi_\rho(w)u_\alpha = \sum_{\omega \in \Sigma_\alpha} e_\omega \otimes E_{\omega T_\rho(\omega)}(w_\alpha)
\]
so that
\[
\langle u_\alpha \mid \phi_\rho(w)u_\alpha \rangle = \langle \sum_{\omega \in \Sigma_\alpha} e_\omega \otimes E_{\omega T_\rho(\omega)} | \sum_{\omega \in \Sigma_\alpha} e_\omega \otimes E_{\omega T_\eta(\omega)}(w_\alpha) \rangle
\]
\[
= \sum_{\omega \in \Sigma_\alpha} T_{\alpha T_\rho(\omega)} E_{\omega T_\eta(\omega)}(w_\alpha) T_{\alpha T_\rho(\omega)}^*
\]
\[
= \sum_{(\alpha, b) \in \Sigma_\eta} T_{b T_\rho(\omega)} S_{\alpha} w_{\alpha} S_{\alpha}^* T_{b T_\rho(\omega)}^* = P_{\alpha} w_{\alpha} = \tilde{\rho}_\alpha(w).
\]
The other equality for \( \tilde{\eta}_\alpha(z) \) is similarly shown to the above equalities.

(ii) For \( z = \sum_{\alpha \in \Sigma^\eta} T_{\alpha} z_{\alpha} T_{\alpha}^* \in \mathcal{B}_\eta \) as in (2.3), we have
\[
\phi_\eta(z)u_\alpha = \sum_{\omega \in \Sigma_\alpha} e_\omega \otimes E_{\omega \rho_\eta(\omega)}(z_\omega)
\]
so that
\[
\langle u_\alpha \mid \phi_\eta(z)u_\alpha \rangle = \langle \sum_{\omega \in \Sigma_\alpha} e_\omega \otimes E_{\omega \rho_\eta(\omega)} | \sum_{\omega \in \Sigma_\alpha} T_{\omega \rho_\eta(\omega)}(z_\omega) T_{\omega \rho_\eta(\omega)}^* \rangle = \check{\rho}_\alpha^\rho(z).
\]
The other equality for \( \check{\eta}_\alpha^\rho(w) \) is similarly shown to the above equalities. \( \Box \)

We will next study the norms on \( \mathcal{H}_\alpha \) induced by the two inner products \( \langle \cdot \mid \cdot \rangle_\eta \) and \( \langle \cdot \mid \cdot \rangle_\rho \)

**Lemma 3.13.** For \( \xi = \sum_{\omega \in \Sigma_\alpha} e_\omega \otimes E_{\omega x_\omega} \in \mathcal{H}_\alpha \) with \( x_\omega \in \mathcal{A} \), we have

(i) \( \| \xi \|_\rho = \max_{\beta \in \Sigma^\beta} \| \sum_{\alpha \in \Sigma^\eta} x_{\alpha \beta}^* x_{\alpha \beta} \| \).
(ii) \( \|\langle \xi | \xi \rangle\| = \max_{\eta \in \Sigma^n} \| \sum_{\alpha \in \Sigma^p} x^*_{\alpha,\beta} x_{\alpha,\beta} \| \),
where \( E_\omega x_\omega = x_{\alpha,\beta} = x_{a,\beta} \) for \( \omega = (\alpha, b, a, \beta) \in \Sigma_k \).

Proof. (i) We have
\[
\|\langle \xi | \xi \rangle\| = \| \sum_{\omega \in \Sigma_n} S_{b(\omega)} x^*_\omega E_\omega x_\omega S_{b(\omega)}^* \|
\]
\[
= \| \sum_{(\alpha, \beta) \in \Sigma^p} S_\beta x^*_{\alpha,\beta} x_{\alpha,\beta} S_\beta^* \|
\]
\[
= \| \sum_{\beta \in \Sigma^p} S_\beta (\sum_{a \in \Sigma^q} x^*_{a,\beta} x_{a,\beta}) S_\beta^* \|
\]
\[
= \max_{\beta \in \Sigma^p} \| S_\beta (\sum_{a \in \Sigma^q} x^*_{a,\beta} x_{a,\beta}) S_\beta^* \|.
\]
Since \( x_{a,\beta} = x_{a,\beta} P_\beta \), we have for \( \beta \in \Sigma^p \)
\[
\| S_\beta (\sum_{a \in \Sigma^q} x^*_{a,\beta} x_{a,\beta}) S_\beta^* \| \leq \| \sum_{a \in \Sigma^q} x^*_{a,\beta} x_{a,\beta} \| \leq \| \sum_{a \in \Sigma^q} P_\beta x^*_{a,\beta} x_{a,\beta} P_\beta \|
\]
one has
\[
\| S_\beta (\sum_{a \in \Sigma^q} x^*_{a,\beta} x_{a,\beta}) S_\beta^* \| = \| \sum_{a \in \Sigma^q} x^*_{a,\beta} x_{a,\beta} \|.
\]
Therefore we have
\[
\|\langle \xi | \xi \rangle\| = \max_{\beta \in \Sigma^p} \| \sum_{a \in \Sigma^q} x^*_{a,\beta} x_{a,\beta} \|.
\]

(ii) is similar to (i). \( \square \)

Define positive maps \( \lambda_\rho : B_\rho \to A \) and \( \lambda_\eta : B_\eta \to A \) by
\[
\lambda_\rho(w) = \sum_{\alpha \in \Sigma^p} \rho_\alpha(w), \quad w \in B_\rho, \quad \lambda_\eta(z) = \sum_{\alpha \in \Sigma^q} \eta_\alpha(z), \quad z \in B_\eta.
\]
(3.5)

Then we have for \( \xi, \xi' \in \mathcal{H}_\kappa \)
\[
\langle \xi | \xi' \rangle_A = \lambda_\rho(\langle \xi | \xi' \rangle_\rho) = \lambda_\eta(\langle \xi | \xi' \rangle_\eta).
\]
(3.6)

Put \( C_\rho = \| \lambda_\rho(1) \|, C_\eta = \| \lambda_\eta(1) \| \). As \( \lambda_\rho(1) = \sum_{\alpha \in \Sigma^p} \rho_\alpha(1) \geq 1 \), one sees \( C_\rho \geq 1 \) and similarly \( C_\eta \geq 1 \). Define the three norms for \( \xi \in \mathcal{H}_\kappa \)
\[
\| \xi \|_A = \| \langle \xi | \xi \rangle_A \|^{\frac{1}{2}}, \quad \| \xi \|_\rho = \| \langle \xi | \xi \rangle_\rho \|^{\frac{1}{2}}, \quad \| \xi \|_\eta = \| \langle \xi | \xi \rangle_\eta \|^{\frac{1}{2}}.
\]
(3.7)

Lemma 3.14. The following inequalities hold for \( \xi \in \mathcal{H}_\kappa \):
\[
\| \xi \|_\rho \leq \| \xi \|_A \leq C_\rho^{\frac{1}{2}} \| \xi \|_\rho \quad \text{and} \quad \| \xi \|_\eta \leq \| \xi \|_A \leq C_\eta^{\frac{1}{2}} \| \xi \|_\eta.
\]
Hence the three norms \( \| \xi \|_A, \| \xi \|_\rho, \| \xi \|_\eta \) are equivalent to each other.

Proof. For \( \xi = \sum_{\omega \in \Sigma_n} c_\omega \otimes E_\omega x_\omega \in \mathcal{H}_\kappa \) with \( x_\omega \in A \), where \( E_\omega x_\omega = x_{\alpha,\beta} = x_{a,\beta} \) for \( \omega = (\alpha, b, a, \beta) \in \Sigma_k \), we have
\[
\|\langle \xi | \xi \rangle_\rho\| = \max_{\beta \in \Sigma^p} \| \sum_{a \in \Sigma^q} x^*_{a,\beta} x_{a,\beta} \|.
\]
We then have
\[
\| \xi \|_A = \| \langle \xi | \xi \rangle_A \|^{\frac{1}{2}} = \| \sum_{\omega \in \Sigma_n} x^*_\omega x_\omega \|^{\frac{1}{2}} = \| \sum_{\beta \in \Sigma^p} \sum_{a \in \Sigma^q} x^*_{a,\beta} x_{a,\beta} \|^{\frac{1}{2}}.
\]

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Since
\[ \| \sum_{a \in \Sigma} x_{a,\beta}x_{a,\beta} \| \leq \| \sum_{\beta \in \Sigma'} \sum_{a \in \Sigma} x_{a,\beta}x_{a,\beta} \|, \]
we have
\[ \| \langle \xi | \xi \rangle \rho \| \leq \| \xi \|_A. \]
On the other hand, by the equality \( \langle \xi | \xi \rangle_A = \lambda_P(\langle \xi | \xi \rangle_B) \), we have
\[ \| \langle \xi | \xi \rangle_A \| \leq \| \lambda_P \| \| \langle \xi | \xi \rangle_B \| = \| \lambda_P(1) \| \| \langle \xi | \xi \rangle_B \|. \]
Therefore we have
\[ \| \xi \|_B \| \xi \|_A \leq \| \xi \|_A \leq C \| \xi \|_B \]
and similarly \( \| \xi \|_C \| \xi \|_D \leq \| \xi \|_D \leq C \| \xi \|_C \).

4. **FOCK HILBERT C*-QUAD MODULES AND CREATION OPERATORS**

In this section, we will consider relative tensor products of Hilbert \( C^* \)-quad modules and introduce Fock space of Hilbert \( C^* \)-quad modules which is two-dimensional analogue of Fock space of Hilbert \( C^* \)-bimodules. The Hilbert \( C^* \)-module \( H_\kappa \) is originally a Hilbert \( C^* \)-right module \(( H_\kappa, \phi_A )\)
over \( A \) with \( A \)-valued inner product \( \langle \cdot | \cdot \rangle_A \). It has two other multi structure of Hilbert \( C^* \)-bimodules. The Hilbert \( C^* \)-bimodule \(( \phi_\rho, H_\kappa, \phi_\rho )\) over \( B_\rho \) and the Hilbert \( C^* \)-bimodule \(( \phi_\eta, H_\kappa, \phi_\eta )\) over \( B_\eta \). This situation is written as in the figure:

\[ \begin{array}{c}
B_\rho \\
\phi_\rho \downarrow \\
B_\eta \xrightarrow{\phi_\eta} H_\kappa \\
\phi_\rho \uparrow \\
B_\eta \\
\end{array} \]

There exist faithful completely positive maps \( \lambda_\rho : B_\rho \rightarrow A \) and \( \lambda_\eta : B_\eta \rightarrow A \) satisfying (3.6) so that the three norms induced by their respect inner pronuts \( \langle \cdot | \cdot \rangle_A \), \( \langle \cdot | \cdot \rangle_\rho \), \( \langle \cdot | \cdot \rangle_\eta \) are equivalent to each other. The Hilbert \( C^* \)-right module \(( H_\kappa, \phi_A )\) over \( A \) with multi structure of Hilbert \( C^* \)-bimodules \(( \phi_\rho, H_\kappa, \phi_\rho )\) over \( B_\rho \) and \(( \phi_\eta, H_\kappa, \phi_\eta )\) over \( B_\eta \) is called a Hilbert \( C^* \)-quad module over \(( A; B_\rho, B_\eta )\). We will define two kinds of relative tensor products
\[ H_\kappa \otimes_\eta H_\kappa, \quad H_\kappa \otimes_\rho H_\kappa \]
as Hilbert \( C^* \)-quad modules over \(( A; B_\rho, B_\eta )\). The latter one should be written vertically as
\[ H_\kappa \\
\otimes_\rho \\
H_\kappa \]
rather than horizontally \( H_\kappa \otimes_\rho H_\kappa \). The first relative tensor product is defined as
\[ H_\kappa \otimes_\eta H_\kappa := H_\kappa \otimes_{B_\eta} H_\kappa \]
the relative tensor product as Hilbert $C^*$-modules over $B_\eta$, the left $\mathcal{H}_\kappa$ is a right $B_\eta$-module through $\varphi_\eta$ and the right $\mathcal{H}_\kappa$ is a left $B_\eta$-module through $\phi_\eta$. It has a right $B_\rho$-valued inner product and a right $B_\eta$-valued inner product defined by
\[
\langle \xi \otimes \eta \mid \xi' \otimes \eta \rangle_{\rho} := \langle \xi \mid \phi_\eta((\xi \mid \xi')_{\rho}) \rangle \\
\langle \xi \otimes \eta \mid \xi' \otimes \eta \rangle_{\eta} := \langle \xi \mid \phi_\eta((\xi \mid \xi')_{\eta}) \rangle
\]
respectively. It has two right actions, $\text{id} \otimes \varphi_\rho$ from $B_\rho$ and $\text{id} \otimes \varphi_\eta$ from $B_\eta$. It also has two left actions, $\phi_\rho \otimes \text{id}$ from $B_\rho$ and $\phi_\eta \otimes \text{id}$ from $B_\eta$. By these operations $\mathcal{H}_\kappa \otimes \mathcal{H}_\kappa$ is a Hilbert $C^*$-bimodule over $B_\rho$ and also is a Hilbert $C^*$-bimodule over $B_\eta$. It also has a right $\mathcal{A}$-valued inner product defined by
\[
\langle \xi \otimes \eta \mid \xi' \otimes \eta \rangle_{\mathcal{A}} := \lambda_\eta(\langle \xi \otimes \eta \mid \xi' \otimes \eta \rangle_{\eta}) = \lambda_\rho(\langle \xi \otimes \eta \mid \xi' \otimes \eta \rangle_{\rho})
\]
and a right $\mathcal{A}$-action $\text{id} \otimes \varphi_{\mathcal{A}}$ and a left $\mathcal{A}$-action $\phi \otimes \text{id}$. By these structure $\mathcal{H}_\kappa \otimes \mathcal{H}_\kappa$ is a Hilbert $C^*$-quad module over $(\mathcal{A}; B_\rho, B_\eta)$.

We denote the above operations $\phi_\rho \otimes \text{id}$, $\phi_\eta \otimes \text{id}$, $\text{id} \otimes \varphi_\rho$, $\text{id} \otimes \varphi_\eta$ still by $\phi_\rho$, $\phi_\eta$, $\varphi_\rho$, $\varphi_\eta$ respectively. Similarly we consider the other relative tensor product defined by
$$
\mathcal{H}_\kappa \otimes \rho \mathcal{H}_\kappa := \mathcal{H}_\kappa \otimes_{B_\rho} \mathcal{H}_\kappa
$$
the relative tensor product as Hilbert $C^*$-modules over $B_\rho$, the left $\mathcal{H}_\kappa$ is a right $B_\rho$-module through $\varphi_\rho$ and the right $\mathcal{H}_\kappa$ is a left $B_\rho$-module through $\phi_\rho$. By symmetrically to the above, $\mathcal{H}_\kappa \otimes \rho \mathcal{H}_\kappa$ is a Hilbert $C^*$-quad module over $(\mathcal{A}; B_\rho, B_\eta)$.

The following lemma is routine.

**Lemma 4.1.** Let $\mathcal{H}_i = \mathcal{H}_\kappa$, $i = 1, 2, 3$. The correspondences
\[
(\xi_1 \otimes \eta \xi_2) \otimes \rho \xi_3 \in (\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3) \quad \rightarrow \quad \xi_1 \otimes \eta (\xi_2 \otimes_{\rho} \xi_3) \in \mathcal{H}_1 \otimes \eta (\mathcal{H}_2 \otimes_{\rho} \mathcal{H}_3),
\]
\[
(\xi_1 \otimes \rho \xi_2) \otimes \eta \xi_3 \in (\mathcal{H}_1 \otimes \rho \mathcal{H}_2) \otimes \eta \mathcal{H}_3 \quad \rightarrow \quad \xi_1 \otimes_{\rho} (\xi_2 \otimes_{\eta} \xi_3) \in \mathcal{H}_1 \otimes_{\rho} (\mathcal{H}_2 \otimes_{\eta} \mathcal{H}_3)
\]
yield isomorphisms of Hilbert $C^*$-quad modules respectively.

We write the isomorphism class of the former Hilbert $C^*$-quad modules as $\mathcal{H}_1 \otimes \eta \mathcal{H}_2 \otimes_{\rho} \mathcal{H}_3$ and that of the latter ones as $\mathcal{H}_1 \otimes_{\rho} \mathcal{H}_2 \otimes_{\eta} \mathcal{H}_3$ respectively.

We note that the direct sum $B_\eta \oplus B_\rho$ has a structure of a Hilbert $C^*$-quad module by the following operations: For $b_1 \oplus b_2, b'_1 \oplus b'_2 \in B_\eta \oplus B_\rho$ and $y \in \mathcal{A}$, set
\[
(b_1 \oplus b_2)\varphi_\mathcal{A}(y) := b_1 \psi_\eta(y) \oplus b_2 \psi_\rho(y) \in B_\eta \oplus B_\rho,
\]
\[
(b_1 \oplus b_2 \mid b'_1 \oplus b'_2)_{\mathcal{A}} := \lambda_\eta(b_1 b'_1') + \lambda_\rho(b_2 b'_2') \in \mathcal{A}
\]
It is direct to see
\[
\langle b_1 \oplus b_2 \mid (b'_1 \oplus b'_2)\varphi_\mathcal{A}(y) \rangle_{\mathcal{A}} = (b_1 \oplus b_2 \mid b'_1 \oplus b'_2)_{\mathcal{A}} \cdot y
\]
so that $B_\eta \oplus B_\rho$ is a Hilbert $C^*$-right module over $\mathcal{A}$. Its Hilbert $C^*$-bimodule structure over $B_\rho$ and over $B_\eta$ are defined as follows: For $w \in B_\rho, z \in B_\eta$, set
Let us define the Fock Hilbert $C$-module which is the completion of the algebraic direct sum $\bigoplus C$ the Fock space of Hilbert $quad$ modules under the norm hence the algebraic direct sum as Hilbert $C$ product $\langle \cdot | \cdot \rangle$. Hence $y$ $Hence$ $\| \|_n$. The left $B$-action $\eta$ and the right $B$-module map $\phi$ of the Hilbert $C$-module $\phi$, we have $\phi(b_1 \oplus b_2) = \phi(b_1) \phi(b_2)$. The right $B$-module map $\phi$ from $A$ to $B$ is a right $\phi$-valued inner products. As in Lemma 3.14, both of the two norms $\| \|_\rho$ and $\| \|_\eta$ induced by the inner products are equivalent to the norm $\| \|_A$. For $\xi \in \mathcal{H}_\eta$ we define operators $s_\xi$ and $t_\xi$ from $F_0(\kappa)$ to $F_1(\kappa)$ by

$$ s_\xi(b_1 \oplus b_2) = \xi \varphi_\eta(b_1), \quad t_\xi(b_1 \oplus b_2) = \xi \varphi_\rho(b_2) $$

for $b_1 \oplus b_2 \in B_\eta \oplus B_\rho$.

**Lemma 4.2.**

(i) $s_\xi$ is a right $B_\eta$-module map from $F_0(\kappa)$ to $F_1(\kappa)$.  
(ii) $t_\xi$ is a right $B_\rho$-module map from $F_0(\kappa)$ to $F_1(\kappa)$.  
(iii) Both the maps $s_\xi, t_\xi : F_0(\kappa) \rightarrow F_1(\kappa)$ are right $A$-module maps.

**Proof.** For $b_1 \oplus b_2 \in B_\eta \oplus B_\rho$ and $z \in B_\eta$, we have

$$ s_\xi((b_1 \oplus b_2) \varphi_\eta(z)) = s_\xi(b_1 z) = \xi \varphi_\eta(b_1 z) = (s_\xi(b_1 \oplus b_2)) \varphi_\eta(z). $$

Hence $s_\xi$ is a right $B_\eta$-module map and similarly $t_\xi$ is a right $B_\rho$-module map. For $y \in A$, by Lemma 3.6, we have

$$ s_\xi((b_1 \oplus b_2) \varphi_\rho(y)) = \xi \varphi_\rho(b_1 \psi_\rho(y)) = (\xi \varphi_\rho(b_1)) \varphi_\rho(y) = (s_\xi(b_1 \oplus b_2)) \varphi_\rho(y). $$

Hence $s_\xi$ and similarly $t_\xi$ are right $A$-module maps. \(\square\)
Lemma 4.3. For $\xi, \xi' \in \mathcal{H}_\kappa$, we have

(i) $s^*_\xi \xi' = \langle \xi | \xi' \rangle_\eta \oplus 0$ in $\mathcal{B}_\eta \oplus \mathcal{B}_\rho$.
(ii) $t^*_\xi \xi' = 0 \oplus \langle \xi | \xi' \rangle_\rho$ in $\mathcal{B}_\eta \oplus \mathcal{B}_\rho$.

Proof. For $b_1 \oplus b_2 \in \mathcal{B}_\eta \oplus \mathcal{B}_\rho$, we have

$$
\langle b_1 + b_2 | s^*_\xi \xi' \rangle_A = \langle \xi \varphi_\eta (b_1) \mid \xi' \rangle_A = \lambda_\eta (b_1^* \xi'_\eta) = \lambda_\eta (\langle b_1 + b_2 | \langle \xi \mid \xi' \rangle_\eta \oplus 0 \rangle) = \langle b_1 + b_2 \mid \langle \xi \mid \xi' \rangle_\eta \oplus 0 \rangle_A
$$

so that $s^*_\xi \xi' = \langle \xi | \xi' \rangle_\eta \oplus 0$.

(ii) is similar to (i).

For $\xi \in \mathcal{H}_\kappa$ and $\xi_1 \otimes \pi_1 \cdots \otimes \pi_{n-1} \xi_n \in F_n(\kappa)$ with $(\pi_1, \ldots, \pi_{n-1}) \in \Gamma_n$, set

$$
s_\xi (\xi_1 \otimes \pi_1 \cdots \otimes \pi_{n-1} \xi_n) = \xi_0 \xi_1 \otimes \pi_1 \cdots \otimes \pi_{n-1} \xi_n, \quad (4.1)
$$

$$
t_\xi (\xi_1 \otimes \pi_1 \cdots \otimes \pi_{n-1} \xi_n) = \xi_\rho \xi_1 \otimes \pi_1 \cdots \otimes \pi_{n-1} \xi_n. \quad (4.2)
$$

The following lemma is direct.

Lemma 4.4. For $\xi \in \mathcal{H}_\kappa$ and $n = 1, 2, \ldots$, we have

(i) $s_\xi$ is a right $\mathcal{B}_\eta$-module map from $F_n(\kappa)$ to $F_{n+1}(\kappa)$.
(ii) $t_\xi$ is a right $\mathcal{B}_\rho$-module map from $F_n(\kappa)$ to $F_{n+1}(\kappa)$.
(iii) Both the maps $s_\xi, t_\xi : F_n(\kappa) \to F_{n+1}(\kappa)$ are right $\mathcal{A}$-module maps.

Denote by $s^*_\xi, t^*_\xi : F_{n+1}(\kappa) \to F_n(\kappa)$ the adjoints of $s_\xi, t_\xi : F_n(\kappa) \to F_{n+1}(\kappa)$ with respect to the right $\mathcal{A}$-valued inner products.

Lemma 4.5. For $\xi \in \mathcal{H}_\kappa$ and $\xi_1 \otimes \pi_1 \xi_2 \otimes \pi_2 \cdots \otimes \pi_n \xi_{n+1} \in F_{n+1}(\kappa)$, we have

(i) $s^*_\xi (\xi_1 \otimes \pi_1 \xi_2 \otimes \pi_2 \cdots \otimes \pi_n \xi_{n+1}) = \begin{cases} \phi_\eta (\langle \xi_1 \mid \xi_1 \rangle_\eta) \xi_2 \otimes \pi_2 \cdots \otimes \pi_n \xi_{n+1} & \text{if } \pi_1 = \eta, \\ \phi_\rho (\langle \xi_1 \mid \xi_1 \rangle_\rho) \xi_2 \otimes \pi_2 \cdots \otimes \pi_n \xi_{n+1} & \text{if } \pi_1 = \rho, \\ 0 & \text{if } \pi_1 = \eta, \end{cases}$
(ii) $t^*_\xi (\xi_1 \otimes \pi_1 \xi_2 \otimes \pi_2 \cdots \otimes \pi_n \xi_{n+1}) = \begin{cases} \phi_\eta (\langle \xi_1 \mid \xi_1 \rangle_\eta) \xi_2 \otimes \pi_2 \cdots \otimes \pi_n \xi_{n+1} & \text{if } \pi_1 = \eta, \\ \phi_\rho (\langle \xi_1 \mid \xi_1 \rangle_\rho) \xi_2 \otimes \pi_2 \cdots \otimes \pi_n \xi_{n+1} & \text{if } \pi_1 = \rho, \\ 0 & \text{if } \pi_1 = \eta, \end{cases}$

Proof. (i) Let $\gamma = \eta$ or $\rho$. For $\xi_1 \otimes \theta_1 \xi_2 \otimes \theta_2 \cdots \otimes \theta_{n-1} \xi_n \in F_n(\kappa)$, we have

$$
\langle \xi_1 \otimes \theta_1 \xi_2 \otimes \theta_2 \cdots \otimes \theta_{n-1} \xi_n \mid s^*_\xi (\xi_1 \otimes \pi_1 \xi_2 \otimes \pi_2 \cdots \otimes \pi_n \xi_{n+1}) \rangle_\gamma = \langle s_\xi (\xi_1 \otimes \pi_1 \xi_2 \otimes \pi_2 \cdots \otimes \pi_{n+1} \xi_{n+1}) \rangle_\gamma = \delta (\langle \xi_1 \mid \xi_1 \rangle_\eta) \phi_\eta (\langle \xi_1 \mid \xi_1 \rangle_\eta) \xi_2 \otimes \pi_2 \cdots \otimes \pi_n \xi_{n+1} + \delta (\langle \xi_1 \mid \xi_1 \rangle_\rho) \phi_\rho (\langle \xi_1 \mid \xi_1 \rangle_\rho) \xi_2 \otimes \pi_2 \cdots \otimes \pi_n \xi_{n+1} + 0
$$

Hence the desired formulae hold with respect to the inner products $\langle \cdot \mid \cdot \rangle_\eta, \langle \cdot \mid \cdot \rangle_\rho$ and hence to the $\mathcal{A}$-valued inner product $\langle \cdot \mid \cdot \rangle_\mathcal{A}$ because of the equality

$$
\langle \cdot \mid \cdot \rangle_\mathcal{A} = \lambda_\eta (\langle \cdot \mid \cdot \rangle_\eta) = \lambda_\rho (\langle \cdot \mid \cdot \rangle_\rho).
$$

\[\square\]
We have shown in the proof of the above lemma that the adjoints of $s_\xi, t_\xi : F_n(\kappa) \rightarrow F_{n+1}(\kappa)$ with respect to the other two inner products $\langle \cdot | \cdot \rangle_\eta, \langle \cdot | \cdot \rangle_\rho$ have the same form as above. We denote by $\tilde{\phi}_\rho, \tilde{\phi}_\eta, \check{\phi}_\rho, \check{\phi}_\eta$ the right $B_\rho$-action, the right $B_\eta$-action, the left $B_\rho$-action, the left $B_\eta$-action on $F_n(\kappa)$ and hence on $F_\kappa$ respectively. The left actions $\check{\phi}_\rho$ of $B_\rho$ and $\check{\phi}_\eta$ of $B_\eta$ satisfy the following equalities

$$
\check{\phi}_\rho(w)(b_1 \oplus b_2) = wb_2,
\check{\phi}_\eta(z)(b_1 \oplus b_2) = zb_1,
$$

$$
\check{\phi}_\rho(w)(\xi_1 \otimes_{n_1} \xi_2 \otimes_{n_2} \cdots \otimes_{n_{n-1}} \xi_n) = (\phi_p(w)\xi_1) \otimes_{n_1} \xi_2 \otimes_{n_2} \cdots \otimes_{n_{n-1}} \xi_n),
\check{\phi}_\eta(z)(\xi_1 \otimes_{n_1} \xi_2 \otimes_{n_2} \cdots \otimes_{n_{n-1}} \xi_n) = (\phi_\eta(z)\xi_1) \otimes_{n_1} \xi_2 \otimes_{n_2} \cdots \otimes_{n_{n-1}} \xi_n)
$$

for $w \in B_\rho, z \in B_\eta, b_1 \oplus b_2 \in B_\eta \oplus B_\rho$ and $\xi_1 \otimes_{n_1} \xi_2 \otimes_{n_2} \cdots \otimes_{n_{n-1}} \xi_n \in F_n(\kappa)$. The following lemma is direct.

**Lemma 4.6.** For $w \in B_\rho, z \in B_\eta$, we have for $n = 0, 1, \ldots$

(i) $\tilde{\phi}_\rho(w)$ is a right $B_\rho$-module map from $F_n(\kappa)$ to $F_n(\kappa)$.

(ii) $\check{\phi}_\eta(z)$ is a right $B_\eta$-module map from $F_n(\kappa)$ to $F_n(\kappa)$.

(iii) Both the maps $\check{\phi}_\rho(w), \check{\phi}_\eta(z)$ are right $A$-module maps on $F_\kappa$.

**Lemma 4.7.** For $\xi \in H_\kappa, w \in B_\rho, z \in B_\eta$, we have

(i) $t_\xi \check{\phi}_\rho(w)$ and hence $t_\xi = \sum_{\alpha \in \Sigma^n} t_{\nu_\alpha} \check{\phi}_\rho((\nu_\alpha | \xi_\rho))$.

(ii) $s_\xi \check{\phi}_\eta(z)$ and hence $s_\xi = \sum_{\alpha \in \Sigma^n} s_{\nu_\alpha} \check{\phi}_\eta((\nu_\alpha | \xi_\eta))$.

**Proof.** (i) We have

$$
\check{\phi}_\rho(w)(\xi_1 \otimes_{n_1} \xi_2 \otimes_{n_2} \cdots \otimes_{n_{n-1}} \xi_n) = \xi \phi_\rho(w) \xi_1 \otimes_{n_1} \xi_2 \otimes_{n_2} \cdots \otimes_{n_{n-1}} \xi_n
$$

$$
= \xi \otimes_{\rho} (\phi_\rho(w)\xi_1) \otimes_{n_1} \xi_2 \otimes_{n_2} \cdots \otimes_{n_{n-1}} \xi_n
$$

$$
= (t_\xi \check{\phi}_\rho(w))\xi_1 \otimes_{n_1} \xi_2 \otimes_{n_2} \cdots \otimes_{n_{n-1}} \xi_n).
$$

(ii) is similar to (i). \qed


By Lemma 3.6, we have $\phi_\eta(\psi_\rho(y)) = \phi_\rho(\psi_\rho(y)) = \phi_A(y)$ for $y \in A$, the above lemma implies the equalities:

$$
t_\xi \phi_A(y) = t_\xi \check{\phi}_\rho(\psi_\rho(y)) \quad \text{and} \quad s_\xi \phi_A(y) = s_\xi \check{\phi}_\eta(\psi_\eta(y)) \quad \text{for} \quad y \in A.
$$

(4.3)

The following lemma is immediate.

**Lemma 4.8.** For $\xi \in H_\kappa, w \in B_\rho, z \in B_\eta$, we have

(i) $\check{\phi}_\rho(w)s_\xi = s_{\phi_\rho(w)\xi}$ and $\check{\phi}_\rho(w)t_\xi = t_{\phi_\rho(w)\xi}$.

(ii) $\check{\phi}_\eta(z)s_\xi = s_{\phi_\eta(z)\xi}$ and $\check{\phi}_\eta(z)t_\xi = t_{\phi_\eta(z)\xi}$.

We set

$$
s_\alpha = s_{\nu_\alpha} \quad \text{for} \quad \alpha \in \Sigma^n \quad \text{and} \quad t_\alpha = t_{\nu_\alpha} \quad \text{for} \quad \alpha \in \Sigma^n.
$$

(4.4)

By Lemma 4.7, we have for $\xi \in H_\kappa$

$$
s_\xi = \sum_{\alpha \in \Sigma^n} s_\alpha \check{\phi}_\eta((\nu_\alpha | \xi_\eta)), \quad t_\xi = \sum_{\alpha \in \Sigma^n} t_\alpha \check{\phi}_\rho((\nu_\alpha | \xi_\rho)).
$$

(4.5)
Lemma 4.9. Keep the above notations.

\[ \sum_{a \in \Sigma^\rho} s_\alpha s_\alpha^* = P_1 + P_\rho \quad \text{and} \quad \sum_{a \in \Sigma^\eta} t_a t_a^* = P_1 + P_\eta. \]

Hence

\[ \sum_{a \in \Sigma^\rho} s_\alpha s_\alpha^* + \sum_{a \in \Sigma^\eta} t_a t_a^* + P_0 = 1_{F_n} + P_1. \quad (4.6) \]

Proof. For \( \xi_1 \otimes \pi_1 \xi_2 \otimes \pi_2 \cdots \otimes \pi_{n-1} \xi_n \in F_n(\kappa) \) with \( 2 \leq n \in \mathbb{N} \), we have

\[
s_\alpha s_\alpha^*(\xi_1 \otimes \pi_1 \xi_2 \otimes \pi_2 \cdots \otimes \pi_{n-1} \xi_n) = \begin{cases} u_\alpha \otimes_\eta \varphi_\eta((u_\alpha \mid \xi_1)\xi_2 \otimes \pi_2 \cdots \otimes \pi_{n-1} \xi_n) & \text{if } \pi_1 = \eta, \\ 0 & \text{if } \pi_1 = \rho. \end{cases}
\]

As \( u_\alpha \otimes_\eta \varphi_\eta((u_\alpha \mid \xi_1)\xi_2 = u_\alpha \varphi_\eta((u_\alpha \mid \xi_1) \otimes_\eta \xi_2 \) and \( \sum_{a \in \Sigma^\rho} u_\alpha \varphi_\eta((u_\alpha \mid \xi_1) = \xi_1 \), we have

\[ \sum_{a \in \Sigma^\rho} s_\alpha s_\alpha^*(\xi_1 \otimes \pi_1 \xi_2 \otimes \pi_2 \cdots \otimes \pi_{n-1} \xi_n) = \begin{cases} \xi_1 \otimes \pi_1 \xi_2 \otimes \pi_2 \cdots \otimes \pi_{n-1} \xi_n & \text{if } \pi_1 = \eta, \\ 0 & \text{if } \pi_1 = \rho. \end{cases} \]

Hence we have

\[ \sum_{a \in \Sigma^\rho} s_\alpha s_\alpha^*|_{\oplus_{n=2}^{\infty} F_n(\kappa)} = P_\rho|_{\oplus_{n=2}^{\infty} F_n(\kappa)}. \]

For \( \xi \in F_1(\kappa) = \mathcal{H}_\kappa \), we have \( s_\alpha s_\alpha^* \xi = s_\alpha((u_\alpha \mid \xi) \oplus 0) = u_\alpha \varphi_\eta((u_\alpha \mid \xi) = \xi \) so that

\[ \sum_{a \in \Sigma^\rho} s_\alpha s_\alpha^* \xi = \sum_{a \in \Sigma^\rho} u_\alpha \varphi_\eta((u_\alpha \mid \xi) = \xi. \]

Hence we have

\[ \sum_{a \in \Sigma^\rho} s_\alpha s_\alpha^*|_{F_1(\kappa)} = 1_{F_1(\kappa)}. \]

As \( s_\alpha s_\alpha^*(b_1 \oplus b_2) = 0 \) for \( b_1 \oplus b_2 \in B_\eta \oplus B_\rho \), we have

\[ \sum_{a \in \Sigma^\rho} s_\alpha s_\alpha^*|_{F_0(\kappa)} = 0. \]

Therefore we conclude that

\[ \sum_{a \in \Sigma^\rho} s_\alpha s_\alpha^* = P_\rho + P_1 \quad \text{and similarly} \quad \sum_{a \in \Sigma^\eta} t_a t_a^* = P_\eta + P_1. \]

As \( P_\eta + P_\rho + P_0 + P_1 = 1_{F_n} \), one has

\[ \sum_{a \in \Sigma^\rho} s_\alpha s_\alpha^* + \sum_{a \in \Sigma^\eta} t_a t_a^* + P_0 = 1_{F_n} + P_1. \]

\[ \square \]
Lemma 4.10. $s^*_\alpha s_\xi = \tilde{\varphi}_\eta(\langle \xi \mid \zeta \rangle_\eta)$ and $t^*_\alpha t_\xi = \tilde{\varphi}_\rho(\langle \xi \mid \zeta \rangle_\rho)$ for $\zeta, \xi \in \mathcal{H}_\kappa$.

Proof. The equalities for $\xi_1 \otimes \pi_1 \otimes \pi_2 \cdots \otimes \pi_{n-1} \otimes \xi_n \in F_n(\kappa)$

$$s^*_\alpha s_\xi(\xi_1 \otimes \pi_1 \otimes \pi_2 \cdots \otimes \pi_{n-1} \otimes \xi_n) = s^*_\alpha(\xi \otimes \eta \xi_1 \otimes \pi_1 \otimes \pi_2 \cdots \otimes \pi_{n-1} \otimes \xi_n)$$

$$= \phi_\eta(\langle \xi \mid \eta \rangle_\eta) \xi_1 \otimes \pi_1 \otimes \pi_2 \cdots \otimes \pi_{n-1} \otimes \xi_n$$

hold so that $s^*_\alpha s_\xi = \tilde{\varphi}_\eta(\langle \xi \mid \zeta \rangle_\eta)$ on $\otimes_{n=1}^\infty F_n(\kappa)$. As

$$s^*_\alpha s_\xi(b_1 \otimes b_2) = s^*_\alpha(\xi \varphi_\eta(b_1)) = \langle \xi \mid \xi \varphi_\eta(b_1) \rangle_\eta \otimes 0 = \langle \xi \mid \xi \rangle_\eta b_1 \otimes 0 = \phi_\eta(\langle \xi \mid \xi \rangle_\eta)(b_1 \otimes b_2),$$

we have $s^*_\alpha s_\xi = \tilde{\varphi}_\eta(\langle \xi \mid \zeta \rangle_\eta)$ on $F_0(\kappa)$. Hence $s^*_\alpha s_\xi = \tilde{\varphi}_\eta(\langle \xi \mid \zeta \rangle_\eta)$ on $F_\kappa$ and similarly $t^*_\alpha t_\xi = \tilde{\varphi}_\rho(\langle \xi \mid \zeta \rangle_\rho)$. □

As $\tilde{\varphi}_\rho(y) = \tilde{\varphi}_\eta(y)$ on $F_n(\kappa), n = 1, 2, 3, \ldots$ for $y \in \mathcal{A}$, we write $\tilde{\varphi}_\rho(y)(= \tilde{\varphi}_\eta(y))$ as $\tilde{\varphi}(y)$ on $F_n(\kappa), n = 1, 2, 3, \ldots$ for $y \in \mathcal{A}$.

Lemma 4.11. For $\alpha \in \Sigma^\rho, \sigma \in \Sigma^\eta$ and $y \in \mathcal{A}$, we have

$$s^*_\alpha \tilde{\varphi}(y)s_\sigma = \tilde{\varphi}_\eta(\rho_\alpha(y)), \quad t^*_\alpha \tilde{\varphi}(y)t_\sigma = \tilde{\varphi}_\rho(\eta_\alpha(y)),$$

$$s_\alpha s^*_\alpha \tilde{\varphi}_\eta(y) = \tilde{\varphi}(y)s_\alpha s^*_\alpha, \quad t_\alpha t^*_\alpha \tilde{\varphi}_\rho(y) = \tilde{\varphi}(y)t_\alpha t^*_\alpha.$$

Proof. The equalities on $F_n(\kappa), n = 1, 2, \ldots$

$$s^*_\alpha \tilde{\varphi}(y)s_\sigma(\xi_1 \otimes \pi_1 \cdots \otimes \pi_{n-1} \otimes \xi_n) = s^*_\alpha(\phi(y)u_\alpha \otimes \eta \xi_1 \otimes \pi_1 \cdots \otimes \pi_{n-1} \otimes \xi_n)$$

$$= \phi_\eta(\langle u_\alpha \mid \phi(y)u_\alpha \rangle_\eta) \xi_1 \otimes \pi_1 \cdots \otimes \pi_{n-1} \otimes \xi_n$$

$$= \phi_\eta(\langle \rho_\alpha(y) \rangle_\eta)(\xi_1 \otimes \pi_1 \cdots \otimes \pi_{n-1} \otimes \xi_n)$$

imply $s^*_\alpha \tilde{\varphi}(y)s_\sigma = \tilde{\varphi}_\eta(\rho_\alpha(y))$ on $F_n(\kappa), n = 1, 2, \ldots$. We have on $F_0(\kappa)$

$$s^*_\alpha \tilde{\varphi}(y)s_\sigma(b_1 \otimes b_2) = \langle u_\alpha \mid \phi(y)u_\alpha \varphi_\eta(b_1) \rangle_\eta \otimes 0 = \langle u_\alpha \mid \varphi_\eta(\rho_\alpha(y)b_1) \rangle_\eta \otimes 0 = \langle u_\alpha \mid u_\alpha \rangle_\eta \rho_\alpha(y)b_1 \otimes 0 = \rho_\alpha(y)b_1 \otimes 0 = \phi_\eta(\rho_\alpha(y))(b_1 \otimes b_2)$$

so that $s^*_\alpha \tilde{\varphi}(y)s_\sigma = \tilde{\varphi}_\eta(\rho_\alpha(y))$ on $F_0(\kappa)$. Thus we have $s^*_\alpha \tilde{\varphi}(y)s_\sigma = \tilde{\varphi}_\eta(\rho_\alpha(y))$ on $F_\kappa$. We similarly have $t^*_\alpha \tilde{\varphi}(y)t_\sigma = \tilde{\varphi}_\rho(\eta_\alpha(y))$.

We also have

$$s_\alpha s^*_\alpha \tilde{\varphi}(y)(\xi_1 \otimes \pi_1 \otimes \pi_2 \cdots \otimes \pi_{n-1} \otimes \xi_n)$$

$$= \begin{cases} u_\alpha \varphi_\eta(\langle u_\alpha \mid \phi(y)\xi_1 \rangle_\eta) \otimes \eta \xi_2 \otimes \pi_2 \cdots \otimes \pi_{n-1} \otimes \xi_n & \text{if } \pi_1 = \eta, \\
0 & \text{if } \pi_1 \neq \eta. \end{cases}$$

Since we have

$$u_\alpha \varphi_\eta(\langle u_\alpha \mid \phi(y)\xi_1 \rangle_\eta) = u_\alpha \varphi_\eta(\rho_\alpha(y)u_\alpha \mid \xi_1 \rangle_\eta) = \phi(y)u_\alpha \varphi_\eta(\langle u_\alpha \mid \xi_1 \rangle_\eta),$$
the equality \( s_\alpha s_\alpha^* \bar{\phi}(y) = \bar{\phi}(y) s_\alpha s_\alpha^* \) on \( F_n(\kappa), n = 1, 2, \ldots \) holds. For \( b_1 \oplus b_2 \in F_0(\kappa), \) the equality \( s_\alpha s_\alpha^* \phi_\eta(y)(b_1 \oplus b_2) = \bar{\phi}(y)s_\alpha s_\alpha^*(b_1 \oplus b_2) = 0 \) holds so that we conclude \( s_\alpha s_\alpha^* \phi_\eta(y) = \bar{\phi}(y)s_\alpha s_\alpha^* \) on \( F_n \) and similarly \( ta_t^* \phi_\rho(y) = \bar{\phi}_\rho(y)ta_t^* \).

**Lemma 4.12.** For \( \omega = (\alpha, b, a, \beta) \in \Sigma_\kappa \) and \( y \in A, \) we have

\[
t_a s_\beta t_b^* s_\alpha^* \bar{\phi}(y) = \bar{\phi}(y) t_a s_\beta t_b^* s_\alpha^*.
\]

**Proof.** By the preceding lemma, we have

\[
t_a s_\beta t_b^* s_\alpha^* \bar{\phi}(y) = t_a s_\beta t_b^* t_a s_\beta t_b^* \bar{\phi}(y)s_\alpha s_\alpha^* = t_a s_\beta \bar{\phi}(\eta_\beta(\rho_\alpha(y)))t_b^* s_\alpha^*,
\]

\[
\bar{\phi}(y)t_a s_\beta t_b^* s_\alpha^* = t_a t_b^* \bar{\phi}(y)t_a s_\beta s_\beta t_b^* s_\alpha^* = t_a s_\beta \bar{\phi}(\rho_\beta(\eta_\alpha(y)))t_b^* s_\alpha^*.
\]

The desired equality holds by (1.1). \( \square \)

**Lemma 4.13.** For \( \alpha \in \Sigma^\rho, a \in \Sigma^\eta \) and \( y \in A, \) we have

\[
\bar{\phi}_\rho(S_\alpha y S_\alpha^*) P_\rho = s_\alpha \bar{\phi}_\rho(y) s_\alpha^*, \quad \bar{\phi}_\eta(T_\alpha y T_\alpha^*) P_\eta = t_\alpha \bar{\phi}_\eta(y) t_\alpha^*.
\]

**Proof.** For \( \xi_1 \otimes \eta_1 \otimes \eta_2 \otimes \cdots \otimes \eta_{n-1}, \xi_n \in F_n(\kappa), n = 2, 3, \ldots, \) we have

\[
\bar{\phi}_\rho(S_\alpha y S_\alpha^*) P_\rho(\xi_1 \otimes \eta_1 \otimes \eta_2 \otimes \cdots \otimes \eta_{n-1}, \xi_n) = \left\{
\begin{array}{cl}
\phi_\rho(S_\alpha y S_\alpha^*) \eta_1 \otimes \eta_2 \otimes \cdots \otimes \eta_{n-1}, \xi_n & \text{if } \eta_1 = \eta,
0 & \text{if } \eta_1 = \rho.
\end{array}\right.
\]

For \( \xi_1 = \sum_{\omega \in \Sigma_\kappa} e_\omega \otimes E_\omega \eta_\rho(x_\omega) \) with \( x_\omega \in A, \) we have

\[
\phi_\rho(S_\alpha y S_\alpha^*) \xi_1 = \sum_{\omega \in \Sigma_\kappa, t(\omega) = \alpha} e_\omega \otimes E_\omega \eta_\rho(x_\omega).
\]

On the other hand, we have

\[
s_\alpha \bar{\phi}_\rho(y) s_\alpha^*(\xi_1 \otimes \eta_1 \otimes \eta_2 \otimes \cdots \otimes \eta_{n-1}, \xi_n) = \left\{
\begin{array}{cl}
s_\alpha \phi(y) \phi_\eta(u_\alpha | \xi_1, \eta_1) \otimes \eta_2 \otimes \cdots \otimes \eta_{n-1}, \xi_n & \text{if } \eta_1 = \eta,
0 & \text{if } \eta_1 = \rho.
\end{array}\right.
\]

As \( u_\alpha | \xi_1 = \sum_{\omega \in \Sigma_\kappa, a=t(\omega)} T_\omega E_\omega x_\omega T_\omega^* \), we have

\[
s_\alpha \phi(y) \phi_\eta(u_\alpha | \xi_1, \eta_1) \xi_2 = s_\alpha \phi_\eta(y(u_\alpha | \xi_1, \eta_1) \xi_2 = \sum_{\omega \in \Sigma_\kappa, t(\omega) = \alpha} u_\alpha \otimes \eta_\rho(T_\omega(y)(u_\alpha | \xi_1, \eta_1) \xi_2 = \sum_{\omega \in \Sigma_\kappa, t(\omega) = \alpha} u_\alpha \phi_\eta(T_\omega(y)(x_\omega T_\omega^*)) \otimes \eta_2 = \sum_{\omega \in \Sigma_\kappa, t(\omega) = \alpha} (e_\omega \otimes E_\omega \eta_\rho(x_\omega) \otimes \eta_2.
\]

Hence we have \( \bar{\phi}_\rho(S_\alpha y S_\alpha^*) P_\rho = s_\alpha \bar{\phi}_\rho(y) s_\alpha^* \) on \( F_n(\kappa), n = 2, 3, \ldots, \)

For \( \xi \in H_\kappa, \) we have \( \phi_\rho(y) s_\alpha^* \xi = \phi_\rho(y)(\langle u_\alpha | \xi_1, \eta_1 \rangle \eta_1 \eta_1 \oplus 0) = 0 \) so that

\[
s_\alpha \bar{\phi}_\rho(y) s_\alpha^* \xi = \bar{\phi}_\rho(S_\alpha y S_\alpha^*) P_\rho \xi = 0.
\]

Therefore we have \( \bar{\phi}_\rho(S_\alpha y S_\alpha^*) P_\rho = s_\alpha \bar{\phi}_\rho(y) s_\alpha^* \) on \( F_\kappa. \)

The other equality \( \bar{\phi}_\eta(T_\alpha y T_\alpha^*) P_\eta = t_\alpha \bar{\phi}_\eta(y) t_\alpha^* \) is similarly shown. \( \square \)
**Lemma 4.14.** For \( w \in B_\rho, z \in B_\eta \) and \( \alpha \in \Sigma^p, \beta \in \Sigma^q \), we have
\[
s_\alpha s_\alpha^* \bar{\phi}_p(w) = \bar{\phi}_p(w)s_\alpha s_\alpha^*, \qquad \bar{t}_\beta s_\alpha^* \bar{\phi}_p(w) = \bar{\phi}_p(w)t_\beta s_\alpha^*,
\]
and hence
\[
\bar{P}_\rho \bar{\phi}_p(w) = \bar{\phi}_p(w)P_\rho, \quad \bar{P}_\eta \bar{\phi}_p(w) = \bar{\phi}_p(w)P_\eta,
\]
\[
\bar{P}_\rho \bar{\phi}_\eta(z) = \bar{\phi}_\eta(z)P_\rho, \quad \bar{P}_\eta \bar{\phi}_\eta(z) = \bar{\phi}_\eta(z)P_\eta.
\]

**Proof.** For \( \xi_1 \otimes_{\pi_1} \xi_2 \otimes_{\pi_2} \cdots \otimes_{\pi_{n-1}} \xi_n \in F_n(\kappa) \), we have
\[
\bar{\phi}_p(w) s_\alpha s_\alpha^*(\xi_1 \otimes_{\pi_1} \xi_2 \otimes_{\pi_2} \cdots \otimes_{\pi_{n-1}} \xi_n)
= \begin{cases} 
\phi_p(w) u_\alpha \varphi_\eta((u_\alpha | \xi_1)_\eta) \otimes_{\eta} \xi_2 \otimes_{\pi_2} \cdots \otimes_{\pi_{n-1}} \xi_n & \text{if } \pi_1 = \eta, \\
0 & \text{if } \pi_1 = \rho
\end{cases}
\]
and
\[
s_\alpha s_\alpha^* \bar{\phi}_p(w) (\xi_1 \otimes_{\pi_1} \xi_2 \otimes_{\pi_2} \cdots \otimes_{\pi_{n-1}} \xi_n)
= \begin{cases} 
u_\alpha \varphi_\eta((u_\alpha | \phi_p(w) \xi_1)_\eta) \otimes_{\eta} \xi_2 \otimes_{\pi_2} \cdots \otimes_{\pi_{n-1}} \xi_n & \text{if } \pi_1 = \eta, \\
0 & \text{if } \pi_1 = \rho
\end{cases}
\]
Since \( \phi_p(w) u_\alpha = u_\alpha \varphi_\eta((u_\alpha | \phi_p(w) u_\alpha)_\eta) \), we have
\[
\phi_p(w) u_\alpha \varphi_\eta((u_\alpha | \xi_1)_\eta) = u_\alpha \varphi_\eta((u_\alpha | \phi_p(w) u_\alpha \varphi_\eta((u_\alpha | \xi_1)_\eta))_\eta)
= u_\alpha \varphi_\eta((u_\alpha | \phi_p(w) \sum_{\beta \in \Sigma^q} u_\beta \varphi_\eta((u_\beta | \xi_1)_\eta))_\eta)
= u_\alpha \varphi_\eta((u_\alpha | \phi_p(w) \xi_1)_\eta).
\]
Hence \( s_\alpha s_\alpha^* \bar{\phi}_p(w) = \bar{\phi}_p(w) s_\alpha s_\alpha^* \) holds. Similarly by \( \phi_\eta(z) u_\alpha = u_\alpha \varphi_\eta((u_\alpha | \phi_\eta(z) u_\alpha)_\eta) \),
the equality \( \phi_\eta(z) s_\alpha s_\alpha^* = s_\alpha s_\alpha^* \phi_\eta(z) \) holds. As \( P_\rho = \sum_{\alpha \in \Sigma^p} s_\alpha s_\alpha^* \) on \( \otimes_{n=2}^\infty F_n(\kappa) \),
we see that \( P_\rho \) commutes with \( \bar{\phi}_p(w) \) and \( \bar{\phi}_\eta(z) \).

The other four equalities in the right hand side are similarly shown. \( \square \)

Let us denote by \( \mathcal{L}_A(H_\kappa) \) and \( \mathcal{L}_A(F_\kappa) \) the \( C^* \)-algebras of all bounded adjointable right \( A \)-module maps on \( H_\kappa \) and on \( F_\kappa \) with respect to the right \( A \)-valued inner products respectively. For \( L \in \mathcal{L}_A(H_\kappa) \), define \( \bar{L} \in \mathcal{L}_A(F_\kappa) \) by
\[
\bar{L}(\xi_1 \otimes_{\pi_1} \xi_2 \otimes_{\pi_2} \cdots \otimes_{\pi_{n-1}} \xi_n) = (L \xi_1) \otimes_{\pi_1} \xi_2 \otimes_{\pi_2} \cdots \otimes_{\pi_{n-1}} \xi_n
\]
for \( \xi_1 \otimes_{\pi_1} \xi_2 \otimes_{\pi_2} \cdots \otimes_{\pi_{n-1}} \xi_n \in F_n \).

**Lemma 4.15.** For \( L \in \mathcal{L}_A(H_\kappa) \) and \( \alpha \in \Sigma^p, \beta \in \Sigma^q \), we have
\[
s_\alpha^* \bar{L} s_\alpha = \bar{\phi}_\eta((u_\alpha | L u_\alpha)_\eta), \quad t_\beta^* \bar{L} t_\beta = \bar{\phi}_p((u_\alpha | L u_\alpha)_\rho).
\]

**Proof.** For \( \xi_1 \otimes_{\pi_1} \cdots \otimes_{\pi_{n-1}} \xi_n \in F_n(\kappa), n = 1, 2, \ldots \), we have
\[
s_\alpha^* \bar{L} s_\alpha (\xi_1 \otimes_{\pi_1} \cdots \otimes_{\pi_{n-1}} \xi_n) = s_\alpha^* (L u_\alpha \otimes_{\eta} \xi_1 \otimes_{\pi_1} \cdots \otimes_{\pi_{n-1}} \xi_n)
= \bar{\phi}_\eta((u_\alpha | L u_\alpha)_\eta) \xi_1 \otimes_{\pi_1} \cdots \otimes_{\pi_{n-1}} \xi_n.
\]
For \( b_1 \oplus b_2 \in B_\eta \oplus B_\rho \), we have
\[
s_\alpha^* \bar{L} s_\alpha (b_1 \oplus b_2) = s_\alpha^* (L u_\alpha \varphi_\eta(b_1)) = (u_\alpha | L u_\alpha \varphi_\eta(b_1))_\eta \oplus 0 = (u_\alpha | L u_\alpha)_\eta b_1 \oplus 0.
\]
Since \( (u_\alpha | L u_\alpha)_\eta b_1 \oplus 0 = \bar{\phi}_\eta((u_\alpha | L u_\alpha)_\eta)(b_1 \oplus b_2) \) we have
\[
s_\alpha^* \bar{L} s_\alpha = \bar{\phi}_\eta((u_\alpha | L u_\alpha)_\eta) \quad \text{on } F_\kappa.
\]
The other equality for $t_a^*L_t$ is similarly shown. \hfill \Box

**Lemma 4.16.** For $w \in B_\rho, z \in B_\eta$ and $\alpha \in \Sigma^\rho, \eta \in \Sigma^\eta$, we have

$$s_\alpha^* \bar{\varphi}_\rho(w)s_\alpha = \bar{\varphi}_\rho(\tilde{\rho}_\alpha(w)) \quad t_a^* \bar{\varphi}_\eta(z)t_a = \bar{\varphi}_\eta(\tilde{\eta}_a(z)).$$

Proof. By Lemma 3.12, we have

$$\tilde{\rho}_\alpha(w) = \langle u_\alpha | \varphi_\rho(w)u_\alpha \rangle_\eta, \quad \bar{\varphi}_\eta(z) = \langle u_\alpha | \varphi_\eta(z)u_\alpha \rangle_\eta.$$

Hence the preceding lemma implies

$$s_\alpha^* \bar{\varphi}_\rho(w)s_\alpha = \bar{\varphi}_\rho(\langle u_\alpha | \varphi_\rho(w)u_\alpha \rangle_\eta) = \bar{\varphi}_\eta(\tilde{\rho}_\alpha(w)),$$

$$s_\alpha^* \bar{\varphi}_\eta(z)s_\alpha = \bar{\varphi}_\eta(\langle u_\alpha | \varphi_\eta(z)u_\alpha \rangle_\eta) = \bar{\varphi}_\eta(\tilde{\eta}_a(z)).$$

The other two equalities in the right hand side are similarly shown. \hfill \Box

**Corollary 4.17.** For $w = \sum_{\alpha \in \Sigma^\rho} S_\alpha w_\alpha S_\alpha^*$ as in (2.2) and $z = \sum_{\alpha \in \Sigma^\eta} T_\alpha z_\alpha T_\alpha^*$ as in (2.3), we have

$$\bar{\varphi}_\rho(w) = \sum_{\alpha \in \Sigma^\rho} s_\alpha^* \bar{\varphi}_\rho(w)s_\alpha = P_\rho \bar{\varphi}_\rho(w)P_\rho + P_0 \bar{\varphi}_\rho(w)P_0,$$  \hspace{1cm} (4.7)

$$\bar{\varphi}_\eta(z) = \sum_{\alpha \in \Sigma^\eta} t_a^* \bar{\varphi}_\eta(z)t_a = P_\rho \bar{\varphi}_\eta(z)P_\rho + P_0 \bar{\varphi}_\eta(z)P_0.$$  \hspace{1cm} (4.8)

Proof. As $P_\rho + P_\eta + P_1 + P_0 = 1$ on $F_\kappa$ and $\sum_{\alpha \in \Sigma^\rho} s_\alpha^* s_\alpha = P_\rho + P_1$, one has

$$\bar{\varphi}_\rho(w) = \sum_{\alpha \in \Sigma^\rho} s_\alpha^* \bar{\varphi}_\rho(w)s_\alpha = \bar{\varphi}_\rho(\tilde{\rho}_\alpha(w)) = \bar{\varphi}_\rho(w_\alpha),$$

since $s_\alpha^* \bar{\varphi}_\rho(w)s_\alpha = \bar{\varphi}_\rho(\tilde{\rho}_\alpha(w)) = \bar{\varphi}_\rho(w_\alpha)$, we have the equality (4.7). The equality (4.8) is similarly shown. \hfill \Box

**Lemma 4.18.**

(i) $\bar{\varphi}_\rho : B_\rho \rightarrow \mathcal{L}_A(F_\kappa)$ is a faithful $*$-homomorphism.

(ii) $\bar{\varphi}_\eta : B_\eta \rightarrow \mathcal{L}_A(F_\kappa)$ is a faithful $*$-homomorphism.

Proof. (i) It is enough to show that $\varphi_\rho : B_\rho \rightarrow \mathcal{L}_A(H_\kappa)$ is injective. For $w = \sum_{\alpha \in \Sigma^\rho} S_\alpha w_\alpha S_\alpha^*$ as in (2.2), suppose that $\varphi_\rho(w) = 0$ on $H_\kappa$. By Lemma 3.12, we have $\tilde{\rho}_\alpha(w) = 0$ for all $\alpha \in \Sigma^\rho$ so that $w_\alpha = 0$ for all $\alpha \in \Sigma^\rho$, which shows $w = 0$. (ii) is similar to (i). \hfill \Box

5. THE $C^*$-ALGEBRAS ASSOCIATED TO THE HILBERT $C^*$-QUAD MODULES

In this section, we will study the $C^*$-algebras generated by the operators $s_\xi, t_\xi$ for $\xi \in H_\kappa$. For $\xi, \zeta \in F_\kappa$, denote by $\theta_{\xi, \zeta}$ the rank one operator on $F_\kappa$ defined by

$$\theta_{\xi, \zeta}(\gamma) = \xi \varphi_A(\langle \zeta | \gamma \rangle_A) \quad \text{for} \quad \gamma \in F_\kappa.$$

It is immediate to see that the operators $\theta_{\xi, \zeta}$ for $\xi, \zeta \in F_\kappa$ are $A$-module maps through $\varphi_A$. Let us denote by $K_\kappa(F_\kappa)$ the $C^*$-subalgebra of $\mathcal{L}_A(F_\kappa)$ generated by the rank one operators $\theta_{\xi, \zeta}$ for $\xi, \zeta \in F_\kappa$. Put the projections for $\alpha \in \Sigma^\rho, \eta \in \Sigma^\eta$

$$p_\alpha = S_\alpha S_\alpha^* \in B_\rho \subset F_\kappa(\kappa), \quad q_\alpha = T_\alpha T_\alpha^* \in B_\eta \subset F_\kappa(\kappa).$$

They are regarded as vectors in $F_\kappa$.

**Lemma 5.1.** $\sum_{\alpha \in \Sigma^\rho} \theta_{p_\alpha, p_\alpha} + \sum_{\alpha \in \Sigma^\eta} \theta_{q_\alpha, q_\alpha} = P_0 :$ the projection on $F_\kappa$ onto $F_0(\kappa).$
Proof. For $\alpha \in \Sigma^p$ and $b_2 \in \mathcal{B}^p$, we have
\[
\theta_{p\alpha \cdot p\alpha}(b_2) = p\alpha \varphi_A(\langle p\alpha \mid b_2 \rangle_A) \\
= p\alpha \psi_{\eta}(\lambda_\eta(p\alpha b_2)) \\
= p\alpha \sum_{\alpha', \beta' \in \Sigma^p} S_{\alpha'} S_{\beta'}^* p\alpha b_2 S_{\beta'} S_{\alpha'}^* = p\alpha b_2
\]
so that $\sum_{\alpha \in \Sigma^p} \theta_{p\alpha \cdot p\alpha}(b_2) = b_2$ for $b_2 \in \mathcal{B}^p$. Similarly we have $\theta_{q\alpha \cdot q\alpha}(b_1) = q\alpha b_1$ for $q\alpha \in \mathcal{B}_\eta$ so that $\sum_{\alpha \in \Sigma^p} \theta_{q\alpha \cdot q\alpha}(b_1) = b_1$. As $\theta_{q\alpha \cdot q\alpha}(b_2) = 0$ for $b_2 \in \mathcal{B}^p$, $\theta_{p\alpha \cdot p\alpha}(b_1) = 0$ for $b_1 \in \mathcal{B}_\eta$ and
\[
\theta_{p\alpha \cdot p\alpha}(\xi) = \theta_{q\alpha \cdot q\alpha}(\xi) = 0 \quad \text{for} \quad \xi \in F_n(\kappa), n = 1, 2, \ldots,
\]
the operator $\sum_{\alpha \in \Sigma^p} \theta_{p\alpha \cdot p\alpha} + \sum_{\alpha \in \Sigma^p} \theta_{q\alpha \cdot q\alpha}$ is the projection on $F_\kappa$ onto $F_0(\kappa)$. □

Put $\epsilon_\omega := \epsilon_\omega \otimes E_\omega \in \mathcal{H}_\kappa$ for $\omega \in \Sigma^\kappa$. Then we see
\[
\langle \epsilon_\omega \mid \epsilon_{\omega'} \rangle_A = \begin{cases} E_\omega & \text{if } \omega = \omega', \\
0 & \text{if } \omega \neq \omega'.
\end{cases}
\]

Lemma 5.2. \{ $\epsilon_\omega$ \}_{\omega \in \Sigma^\kappa} \text{ forms an orthogonal basis of } \mathcal{H}_\kappa \text{ with respect to the } \mathcal{A}-\text{valued inner product } \langle \cdot \mid \cdot \rangle_A \text{ as a right } \mathcal{A}\text{-module through } \varphi_A.

Proof. For $\xi = \sum_{\omega \in \Sigma^\kappa} \epsilon_\omega \otimes E_\omega x_\omega$ with $x_\omega \in \mathcal{A}$, one has
\[
\langle \epsilon_\omega \mid \xi \rangle_A = \sum_{\omega' \in \Sigma^\kappa} \langle \epsilon_\omega \mid \epsilon_{\omega'} \rangle_A x_\omega = E_\omega x_\omega
\]
so that
\[
\xi = \sum_{\omega \in \Sigma^\kappa} \epsilon_\omega \varphi_A(E_\omega x_\omega) = \sum_{\omega \in \Sigma^\kappa} \epsilon_\omega \varphi_A(\langle \epsilon_\omega \mid \xi \rangle_A). \tag{5.1}
\]
□

Lemma 5.3. $\sum_{\omega \in \Sigma^\kappa} \theta_{\omega \cdot \omega} \epsilon_\omega = P_1$ the projection on $F_\kappa$ onto $F_1(\kappa)$.

Proof. By (5.1), we have $\xi = \sum_{\omega \in \Sigma^\kappa} \theta_{\omega \cdot \omega} \epsilon_\omega (\xi)$ for $\xi \in \mathcal{H}_\kappa$. Since $\theta_{\omega \cdot \omega} (\xi') = 0$ for $\xi' \in F_n(\kappa)$ with $n \neq 1$, we have $\sum_{\omega \in \Sigma^\kappa} \theta_{\omega \cdot \omega} \epsilon_\omega = P_1$. □

By the preceding lemmas, we have

Corollary 5.4. $P_0, P_1 \in K_A(F_\kappa)$.

The $C^*$-subalgebra of $L_A(F_\kappa)$ generated by the operators $s_\xi, t_\xi$ for $\xi \in \mathcal{H}_\kappa$ is denoted by $T_{\mathcal{H}_\kappa}$ and is called the Toeplitz quad module algebra.

Definition. The $C^*$-algebra $\mathcal{O}_{\mathcal{H}_\kappa}$ associated with the Hilbert $C^*$-quad module $\mathcal{H}_\kappa$ is defined as the quotient $C^*$-algebra of $T_{\mathcal{H}_\kappa}$ by the ideal $T_{\mathcal{H}_\kappa} \cap K_A(F_\kappa)$.

We set the quotients of the operators in $\mathcal{O}_{\mathcal{H}_\kappa}$ for $\alpha \in \Sigma^\kappa, a \in \Sigma^\eta$:
\[
U_\alpha := [s_\alpha] \in \mathcal{O}_{\mathcal{H}_\kappa} \quad V_a := [t_a] \in \mathcal{O}_{\mathcal{H}_\kappa}.
\]

Since
\[
\sum_{\alpha \in \Sigma^\kappa} s_\alpha s_\alpha^* + \sum_{\alpha \in \Sigma^\eta} t_a t_a^* + P_0 = 1 + P_1
\]
and $P_0, P_1 \in K_A(F_\kappa)$ we have
\[
\sum_{\alpha \in \Sigma^\kappa} U_\alpha U_\alpha^* + \sum_{a \in \Sigma^\eta} V_a V_a^* = 1. \tag{5.2}
\]
We also have for $w \in \mathcal{B}_\rho$

$$\tilde{\phi}_\rho(w) = \sum_{\alpha \in \Sigma^\rho} \tilde{\phi}_\rho(w)s_\alpha s_\alpha^* + \sum_{\alpha \in \Sigma^\eta} \tilde{\phi}_\rho(w)t_\alpha^* t_\alpha + \tilde{\phi}_\rho(w)P_0 - \tilde{\phi}_\rho(w)P_1.$$ 

As $\tilde{\phi}_\rho(w)s_\alpha = s_{\phi_\rho(w)_\alpha}$ and $\tilde{\phi}_\rho(w)t_\alpha = t_{\phi_\rho(w)_\alpha}$ by Lemma 4.8, the operators $\tilde{\phi}_\rho(w)$ for $w \in \mathcal{B}_\rho$ and similarly $\tilde{\phi}_\rho(z)$ for $z \in \mathcal{B}_\eta$ belong to $T_{\mathcal{H}_\kappa} \cap \mathcal{K}_\mathcal{A}(F_\kappa)$. Let $\Phi_\rho(w), \Phi_\eta(z) \in \mathcal{O}_{\mathcal{H}_\kappa}$ denote the quotient images of $\tilde{\phi}_\rho(w), \tilde{\phi}_\rho(z)$ for $w \in \mathcal{B}_\rho, z \in \mathcal{B}_\eta$ in the quotient $\mathcal{O}_{\mathcal{H}_\kappa} = T_{\mathcal{H}_\kappa} / \mathcal{K}_\mathcal{A}(F_\kappa)$. The following lemma is clear by (4.5).

**Lemma 5.5.** The $C^*$-algebra $\mathcal{O}_{\mathcal{H}_\kappa}$ is generated by the partial isometries $U_\alpha, V_\alpha$ for $\alpha \in \Sigma^\rho, \alpha \in \Sigma^\eta$ and the elements $\Phi_\rho(w), \Phi_\eta(z)$ for $w \in \mathcal{B}_\rho, z \in \mathcal{B}_\eta$.

We will show that the $C^*$-algebra $\mathcal{O}_{\mathcal{H}_\kappa}$ has a universal property subject to the operator relations inherited from Lemma 4.14 and 4.16. The following proposition is direct.

**Proposition 5.6.** The operators $\Phi_\rho(w), \Phi_\eta(z), U_\alpha, V_\alpha$ for $w \in \mathcal{B}_\rho, z \in \mathcal{B}_\eta, \alpha \in \Sigma^\rho, \alpha \in \Sigma^\eta$ satisfy the relations:

$$\sum_{\alpha \in \Sigma^\rho} U_\alpha U_\alpha^* + \sum_{\alpha \in \Sigma^\eta} V_\alpha V_\alpha^* = 1,$$

$$U_\alpha U_\alpha^* \Phi_\rho(w) = \Phi_\rho(w)U_\alpha U_\alpha^*, \quad V_\alpha V_\alpha^* \Phi_\rho(w) = \Phi_\rho(w)V_\alpha V_\alpha^*,$$

$$U_\alpha U_\alpha^* \Phi_\eta(z) = \Phi_\eta(z)U_\alpha U_\alpha^*, \quad V_\alpha V_\alpha^* \Phi_\eta(z) = \Phi_\eta(z)V_\alpha V_\alpha^*,$$

$$\Phi_\rho(\tilde{\rho}_\alpha(w)) = U_\alpha^* \Phi_\rho(w)U_\alpha, \quad \Phi_\eta(\tilde{\eta}_\alpha(z)) = U_\alpha^* \Phi_\eta(z)U_\alpha,$$

$$\Phi_\rho(y) = \Phi_\eta(y)$$

for $w \in \mathcal{B}_\rho, z \in \mathcal{B}_\eta, \alpha \in \Sigma^\rho, \alpha \in \Sigma^\eta, y \in \mathcal{A}$.

The ten relations above are called the relations ($\mathcal{H}_\kappa$). We will henceforth prove that the $C^*$-algebra $\mathcal{O}_{\mathcal{H}_\kappa}$ has the universal property subject to the relations ($\mathcal{H}_\kappa$). Let $\mathcal{B}_\kappa$ be the $C^*$-subalgebra of $\mathcal{O}_{\mathcal{H}_\kappa}$ generated by the operators $\Phi_\rho(w), \Phi_\eta(z)$ for $w \in \mathcal{B}_\rho, z \in \mathcal{B}_\eta$.

**Lemma 5.7.** Assume that the algebra $\mathcal{A}$ is commutative. Then $\mathcal{B}_\kappa$ is commutative by the relations ($\mathcal{H}_\kappa$).

**Proof.** As the algebra $\mathcal{A}$ is commutative, the algebras $\mathcal{B}_\rho$ and $\mathcal{B}_\eta$ are both commutative by Lemma 3.8. Hence it is enough to prove that $\Phi_\rho(w)$ commutes with $\Phi_\eta(z)$ for $w \in \mathcal{B}_\rho, z \in \mathcal{B}_\eta$. For $\alpha \in \Sigma^\rho$, it follows that

$$\Phi_\eta(z)\Phi_\rho(w)U_\alpha U_\alpha^* = U_\alpha \Phi_\eta(\tilde{\rho}_\alpha^*(z))\Phi_\rho(\tilde{\rho}_\alpha(w))U_\alpha^*.$$ 

As $\tilde{\rho}_\alpha(w) \in \mathcal{A}$, we have $\Phi_\rho(\tilde{\rho}_\alpha^*(w)) = \Phi_\eta(\tilde{\rho}_\alpha^*(w))$ so that

$$\Phi_\eta(\tilde{\rho}_\alpha^*(z))\Phi_\rho(w) = \Phi_\eta(\tilde{\rho}_\alpha^*(z))\Phi_\rho(w)$$

$$\Phi_\rho(\tilde{\rho}_\alpha(w))\Phi_\eta(z) = \Phi_\eta(\tilde{\rho}_\alpha(w))\Phi_\eta(z).$$ 

Both the elements $\tilde{\rho}_\alpha^*(z), \tilde{\rho}_\alpha(w)$ belong to the commutative algebra $\mathcal{B}_\eta$, so that

$$\Phi_\eta(z)\Phi_\rho(w)U_\alpha U_\alpha^* = U_\alpha \Phi_\rho(\tilde{\rho}_\alpha(w))\Phi_\eta(\tilde{\rho}_\alpha^*(z))U_\alpha^* = \Phi_\rho(w)\Phi_\eta(z)U_\alpha U_\alpha^*.$$ 

Similarly we have

$$\Phi_\eta(z)\Phi_\rho(w)V_\alpha V_\alpha^* = \Phi_\rho(w)\Phi_\eta(z)V_\alpha V_\alpha^*.$$
for \( a \in \Sigma^\eta \). As \( \sum_{\alpha \in \Sigma^\rho} U_\alpha U_\alpha^* + \sum_{a \in \Sigma^\eta} V_a V_a^* = 1 \), one concludes that
\[
\Phi_\eta(z) \Phi_\rho(w) = \Phi_\rho(w) \Phi_\eta(z).
\]

Put \( \Sigma^{\rho,\eta} = \Sigma^\rho \cup \Sigma^\eta \). We set for \( \gamma \in \Sigma^{\rho,\eta} \) and \( X \in \mathcal{B}_\kappa \)
\[
\rho^\rho_\gamma(X) = W^\gamma_* X W^\gamma \quad \text{where} \quad W^\gamma = \begin{cases} U_\alpha & \text{if } \gamma = \alpha \in \Sigma^\rho, \\ V_a & \text{if } \gamma = a \in \Sigma^\eta. \end{cases}
\]
Since \( U^*_\alpha \mathcal{B}_\kappa U_\alpha \subset \mathcal{B}_\kappa \) and \( V^*_a \mathcal{B}_\kappa V_a \subset \mathcal{B}_\kappa \), we have
\[
\rho^\rho_\gamma(\mathcal{B}_\kappa) \subset \mathcal{B}_\kappa,
\]
so that we have a family of endomorphisms \( \rho^\rho_\gamma, \gamma \in \Sigma^{\rho,\eta} \) on \( \mathcal{B}_\kappa \). In what follows, we assume that the algebra \( \mathcal{A} \) is commutative, so that the algebras \( \mathcal{B}_\rho, \mathcal{B}_\eta \) and \( \mathcal{B}_\kappa \) are all commutative.

**Lemma 5.8.** The triplet \((\mathcal{B}_\kappa, \rho^\rho, \Sigma^{\rho,\eta})\) is a \( C^* \)-symbolic dynamical system.

**Proof.** Since \( \sum_{\gamma \in \Sigma^{\rho,\eta}} W^\gamma_* W^\gamma = \sum_{\alpha \in \Sigma^\rho} U_\alpha U_\alpha^* + \sum_{a \in \Sigma^\eta} V_a V_a^* = 1 \) and \( W^\gamma_* W^\gamma \) commutes with \( \mathcal{B}_\kappa \), the family \( \rho^\rho_\gamma, \gamma \in \Sigma^{\rho,\eta} \) yields an endomorphisms on \( \mathcal{B}_\kappa \). We have
\[
\sum_{\gamma \in \Sigma^{\rho,\eta}} \rho^\rho_\gamma(1) = \sum_{\alpha \in \Sigma^\rho} U^*_\alpha U_\alpha + \sum_{a \in \Sigma^\eta} V^*_a V_a = \sum_{\alpha \in \Sigma^\rho} \Phi_\rho(\rho_\alpha(1)) + \sum_{a \in \Sigma^\eta} \Phi_\eta(\eta_\alpha(1)) = \sum_{\alpha \in \Sigma^\rho} \Phi_\rho(1) + \sum_{a \in \Sigma^\eta} \Phi_\eta(1) \geq 2.
\]

Hence \( (\mathcal{B}_\kappa, \rho^\rho, \Sigma^{\rho,\eta}) \) is a \( C^* \)-symbolic dynamical system. \( \square \)

For \( \mu = \mu_1 \cdots \mu_n \in \mathcal{B}_n(\Sigma^{\rho,\eta}) \) where \( \mu_1, \ldots, \mu_n \in \Sigma^{\rho,\eta} \), denote by
\[
|\mu|_\rho = \text{the number of symbols of } \Sigma^\rho \text{ appearing in the word } \mu_1 \cdots \mu_n,
\]
\[
|\mu|_\eta = \text{the number of symbols of } \Sigma^\eta \text{ appearing in the word } \mu_1 \cdots \mu_n.
\]
Hence \( |\mu|_\rho + |\mu|_\eta = n \). For \( n \in \mathbb{Z}_+ \), denote by \( \mathcal{F}_n \) the \( C^* \)-subalgebra of \( \mathcal{O}_{\mathcal{H}_\kappa} \) generated by the operators \( W_{\gamma_1 \cdots \gamma_n} b W^*_{\gamma'_1 \cdots \gamma'_n} \) for \( b \in \mathcal{B}_\kappa \) and \( \gamma_1 \cdots \gamma_n, \gamma'_1 \cdots \gamma'_n \in \mathcal{B}_n(\Sigma^{\rho,\eta}) \) such that \( |\gamma_1 \cdots \gamma_n|_\rho = |\gamma'_1 \cdots \gamma'_n|_\rho \) and \( |\gamma_1 \cdots \gamma_n|_\eta = |\gamma'_1 \cdots \gamma'_n|_\eta \). Since \( \sum_{\gamma \in \Sigma^{\rho,\eta}} W^\gamma_* W^\gamma = 1 \) and \( W^\gamma_* W^\gamma \) commutes with \( \mathcal{B}_\kappa \), the equality
\[
W_{\gamma_1 \cdots \gamma_n} b W^*_{\gamma'_1 \cdots \gamma'_n} = \sum_{\gamma_{n+1} \in \Sigma^{\rho,\eta}} W_{\gamma_1 \cdots \gamma_n} W_{\gamma_{n+1}} \rho^\rho_{\gamma_{n+1}}(b) W^*_{\gamma_{n+1}} W^*_{\gamma'_1 \cdots \gamma'_n}
\]
gives rise to an embedding \( \mathcal{F}_n \hookrightarrow \mathcal{F}_{n+1}, n \in \mathbb{Z}_+ \). Let \( \mathcal{F}_{\mathcal{H}_\kappa} \) be the \( C^* \)-subalgebra of \( \mathcal{O}_{\mathcal{H}_\kappa} \) generated by \( \cup_{n=0}^{\infty} \mathcal{F}_n \).

We define 2-parameter unitary groups on \( \mathcal{H}_\kappa \) by setting
\[
\theta_\alpha^\rho(z) := u_\alpha \varphi(z) \in \mathcal{H}_\kappa, \quad \alpha \in \Sigma^\rho, z \in \mathbb{B}_\eta
\]
\[
\theta_\alpha^\eta(w) := v_\alpha \varphi(w) \in \mathcal{H}_\kappa, \quad \alpha \in \Sigma^\eta, w \in \mathbb{B}_\rho
\]
for \( r_1, r_2 \in \mathbb{R}/\mathbb{Z} = T \). They extend on \( \mathcal{H}_\kappa \otimes \pi_1 \cdots \otimes \pi_{n-1} \mathcal{H}_\kappa \) for \((\pi_1, \ldots, \pi_{n-1}) \in \Gamma_{n-1}\) by
\[
\theta_{r_1}^\otimes \mathcal{H}_\kappa = \theta_{r_1} \otimes \pi_1 \cdots \otimes \pi_{n-1} \theta_{r_1}^\rho, \quad \theta_{r_2}^\otimes \mathcal{H}_\kappa = \theta_{r_2} \otimes \pi_1 \cdots \otimes \pi_{n-1} \theta_{r_2}^\eta.
\]
which naturally extend on $F_n(\kappa)$. We put
\[ u^p_{r_1} = \sum_{n=0}^\infty \theta^p_{r_1}\otimes n, \quad u^n_{r_2} = \sum_{n=0}^\infty \theta^n_{r_2}\otimes n \quad \text{on } F_\kappa = \oplus_{n=0}^\infty F_n(\kappa) \]
for $r_1, r_2 \in \mathbb{R}/\mathbb{Z} = \mathbb{T}$, where $\theta^p_{r_1}\otimes n, \theta^n_{r_2}\otimes n$ for $n = 0$ are defined by $\theta^p_{r_1}\otimes (b_1 \oplus b_2) = e^{2\pi i r_1} b_1 \oplus b_2, \theta^n_{r_2}\otimes (b_1 \oplus b_2) = b_1 \oplus e^{2\pi i r_2} b_2$. We note that $u^p_{r_1} u^n_{r_2} = u^n_{r_2} u^p_{r_1}$. Define
\[ g(r_1, r_2) = \text{Ad}(u^p_{r_1} u^n_{r_2}) \quad \text{on } F_\kappa \text{ for } (r_1, r_2) \in \mathbb{T}^2. \]

The following lemma is straightforward.

**Lemma 5.9.** For $w \in B_\rho, z \in B_\eta, \alpha \in \Sigma^\rho, \kappa \in \Sigma^\eta$ and $(r_1, r_2) \in \mathbb{T}^2$, we have
\[ g(r_1, r_2)(\tilde{\phi}_\rho(w)) = \tilde{\phi}_\rho(w), \quad g(r_1, r_2)(\tilde{\phi}_\eta(z)) = \tilde{\phi}_\eta(z), \]
\[ g(r_1, r_2)(s_\alpha) = e^{2\pi i r_1} s_\alpha, \quad g(r_1, r_2)(t_\alpha) = e^{2\pi i r_2} t_\alpha. \]

It is easy to see that $g(r_1, r_2)(\mathcal{K}_A(F_\kappa)) = \mathcal{K}_A(F_\kappa)$ so that $g(r_1, r_2)$ defines an automorphism on $\mathcal{O}_H_\kappa$ for $(r_1, r_2) \in \mathbb{T}^2$, which is still denoted by $g(r_1, r_2)$. The automorphisms $g(r_1, r_2), (r_1, r_2) \in \mathbb{T}^2$ define an action
\[ g : (r_1, r_2) \in \mathbb{T}^2 \rightarrow g(r_1, r_2) \in \text{Aut}(\mathcal{O}_H_\kappa) \]
of $\mathbb{T}^2$, called the gauge action on $\mathcal{O}_H_\kappa$. Define a faithful conditional expectation $\mathcal{E}_H_\kappa$ from $\mathcal{O}_H_\kappa$ onto the fixed point algebra $(\mathcal{O}_H_\kappa)^g$ by setting
\[ \mathcal{E}_H_\kappa(X) = \int_{(r_1, r_2) \in \mathbb{T}^2} g(r_1, r_2)(X) \, dr_1 \, dr_2, \quad X \in \mathcal{O}_H_\kappa. \]
Then the following lemma holds. Its proof is routine.

**Lemma 5.10.** The fixed point algebra $(\mathcal{O}_H_\kappa)^g$ of $\mathcal{O}_H_\kappa$ under the action $g$ of $\mathbb{T}^2$ coincides with $\mathcal{F}_H_\kappa$.

We will prove that the algebra $\mathcal{O}_H_\kappa$ has a universal property subject to the relations $(H_\kappa)$. Let us denote by $\mathcal{O}_H_\kappa^{unii}$ the universal $C^*$-algebra generated by the operators $w \in B_\rho, z \in B_\eta$ and partial isometries $u_\alpha, \alpha \in \Sigma^\rho, v_\alpha, \kappa \in \Sigma^\eta$ satisfying the following operator relations:
\[ \sum_{\beta \in \Sigma^\rho} u_\beta u^*_\beta + \sum_{b \in \Sigma^\eta} v_b v^*_b = 1, \]
\[ u_\alpha u^*_\alpha w = w u_\alpha u^*_\alpha, \quad v_\alpha v^*_\alpha w = w v_\alpha v^*_\alpha, \]
\[ u_\alpha u^*_\alpha z = z u_\alpha u^*_\alpha, \quad v_\alpha v^*_\alpha z = z v_\alpha v^*_\alpha, \]
\[ \tilde{\rho}_\alpha(w) = u_\alpha w u_\alpha, \quad \tilde{\rho}_\alpha^*(z) = v_\alpha z v_\alpha, \]
\[ \rho_\gamma^\eta(y) = \iota_\rho(y), \quad \iota_\eta(y) = \iota_\eta(y), \]
\[ \iota_\eta(y) = \iota_\eta(y), \quad \iota_\eta(y) = \iota_\eta(y), \]
\[ \rho_\gamma^\eta(x) = u_\gamma x w_\gamma, \quad \text{where } \ i = \begin{cases} u_\alpha & \text{if } \gamma = \alpha \in \Sigma^\rho, \\ v_\alpha & \text{if } \gamma = \kappa \in \Sigma^\eta. \end{cases} \]

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Since $u^*_n B^\text{uni}_\kappa u_n \subset B^\text{uni}_\kappa$ and $v^*_a B^\text{uni}_\kappa v_a \subset B^\text{uni}_\kappa$, we have

$$\rho^*_\gamma (B^\text{uni}_\kappa) \subset B^\text{uni}_\kappa,$$

so that we have a family of endomorphisms $\rho^*_\gamma, \gamma \in \Sigma^{\rho,\eta}$ on $B^\text{uni}_\kappa$. Similarly as in the preceding lemma, we have

**Lemma 5.11.** The triplet $(B^\text{uni}_\kappa, \rho^\text{uni}, \Sigma^{\rho,\eta})$ is a C*-symbolic dynamical system.

For $n \in \mathbb{Z}_+$, denote by $\mathcal{F}^\text{uni}_n$ the C*-subalgebra of $\mathcal{O}^\text{uni}_{\mathcal{H}_\kappa}$ generated by the operators $w_{\gamma_1 \cdots \gamma_n} u^*_{\gamma_1} \cdots u^*_{\gamma_n}$ for $b \in B^\text{uni}_\kappa$ and $\gamma_1 \cdots \gamma_n \in B_n(\Sigma^{\rho,\eta})$ such that $|\gamma_1 \cdots \gamma_n|_\rho = |\gamma_1 \cdots \gamma_n|_\eta$ and $|\gamma_1 \cdots \gamma_n|_{\rho} = |\gamma_1 \cdots \gamma_n|_{\eta}$. Since $\sum_{\gamma \in \Sigma^{\rho,\eta}} w_{\gamma} u^*_{\gamma} = 1$, and $w_{\gamma} u^*_{\gamma}$ commutes with $B^\text{uni}_\kappa$, the equality

$$w_{\gamma_1 \cdots \gamma_n} b u^*_{\gamma_1} \cdots u^*_{\gamma_n} = \sum_{n+1 \in \Sigma^{\rho,\eta}} w_{\gamma_1 \cdots \gamma_n} w_{\gamma_{n+1}} \rho_{\gamma_{n+1}} (b) u^*_{\gamma_{n+1}} \cdots u^*_{\gamma_n}$$

gives rise to an embedding $\mathcal{F}^\text{uni}_n \hookrightarrow \mathcal{F}^\text{uni}_{n+1}, n \in \mathbb{Z}_+$. Let $\mathcal{F}^\text{uni}_{\mathcal{H}_\kappa}$ be the C*-subalgebra of $\mathcal{O}^\text{uni}_{\mathcal{H}_\kappa}$ generated by $\cup_{n=0}^{\infty} \mathcal{F}^\text{uni}_n$. By the universality of the algebra $\mathcal{O}^\text{uni}_{\mathcal{H}_\kappa}$ subject to the relations $(\mathcal{H}_\kappa)$, the correspondences for each $(r_1, r_2) \in \mathbb{T}^2$

$$w \in \mathcal{O}^\text{uni}_{\mathcal{H}_\kappa} \mapsto w \in \mathcal{O}^\text{uni}_{\mathcal{H}_\kappa}, \quad z \in \mathcal{O}^\text{uni}_{\mathcal{H}_\kappa} \mapsto z \in \mathcal{O}^\text{uni}_{\mathcal{H}_\kappa},$$

$$u_\alpha \in \mathcal{O}^\text{uni}_{\mathcal{H}_\kappa} \mapsto e^{2\pi i r_1} u_\alpha \in \mathcal{O}^\text{uni}_{\mathcal{H}_\kappa}, \quad v_\alpha \in \mathcal{O}^\text{uni}_{\mathcal{H}_\kappa} \mapsto e^{2\pi i r_2} v_\alpha \in \mathcal{O}^\text{uni}_{\mathcal{H}_\kappa},$$

give rise to an automorphism of $\mathcal{O}^\text{uni}_{\mathcal{H}_\kappa}$, which we denote by $g^\text{uni}_{(r_1, r_2)}$. Similarly to the preceding discussions, $g^\text{uni}$ yields an action of $\mathbb{T}^2$ on $\mathcal{O}^\text{uni}_{\mathcal{H}_\kappa}$, called the gauge action on $\mathcal{O}^\text{uni}_{\mathcal{H}_\kappa}$. Define similarly to the preceding discussions a faithful conditional expectation $E_{\mathcal{H}_\kappa}$ from $\mathcal{O}^\text{uni}_{\mathcal{H}_\kappa}$ onto the fixed point algebra $(\mathcal{O}^\text{uni}_{\mathcal{H}_\kappa})^\text{uni}$ by setting

$$E^\text{uni}_{\mathcal{H}_\kappa}(X) = \int_{(r_1, r_2) \in \mathbb{T}^2} g_{(r_1, r_2)}(X) \, dr_1 \, dr_2, \quad X \in \mathcal{O}^\text{uni}_{\mathcal{H}_\kappa}.$$

Similarly to the previous discussions, we have

**Lemma 5.12.** The fixed point algebra $(\mathcal{O}^\text{uni}_{\mathcal{H}_\kappa})^\text{uni}$ of $\mathcal{O}^\text{uni}_{\mathcal{H}_\kappa}$ under the gauge action $g^\text{uni}$ of $\mathbb{T}^2$ coincides with $\mathcal{F}^\text{uni}_{\mathcal{H}_\kappa}$.

By the universality of the algebra $\mathcal{O}^\text{uni}_{\mathcal{H}_\kappa}$ subject to the relations $(\mathcal{H}_\kappa)$, there exists a surjective *-homomorphism $\Psi : \mathcal{O}^\text{uni}_{\mathcal{H}_\kappa} \longrightarrow \mathcal{O}_{\mathcal{H}_\kappa}$ satisfying

$$\Psi(w) = \Phi_\rho(w), \quad \Psi(z) = \Phi_\eta(z),$$

$$\Psi(u_\alpha) = U_\alpha, \quad \Psi(v_\alpha) = V_\alpha$$

for $w \in B_\rho, z \in B_\eta, \alpha \in \Sigma^{\rho,\eta}$. We will prove that there exists a *-homomorphism $\pi_{\kappa} : B_\kappa \longrightarrow B^\text{uni}_\kappa$ such that $\pi_{\kappa}(\Phi_\rho(w)) = w, \pi_{\kappa}(\Phi_\eta(z)) = z$.

**Lemma 5.13.** For $\alpha \in \Sigma^{\rho,\eta}$, we have

(i) The correspondence $U_\alpha \Phi_\eta(z) U^*_\alpha \in U_\alpha \Phi_\eta(B_\eta) U^*_\alpha \mapsto u_\alpha z u_\alpha^* \in u_\alpha B_\eta u_\alpha^*$ for $z \in B_\eta$ yields a *-homomorphism.

(ii) The correspondence $V_\alpha \Phi_\rho(w) V^*_\alpha \in V_\alpha \Phi_\rho(B_\rho) V^*_\alpha \mapsto v_\alpha w v_\alpha^* \in v_\alpha B_\rho v_\alpha^*$ for $w \in B_\rho$ yields a *-homomorphism.

**Proof.**

(i) As $U^*_\alpha U_\alpha = \Phi_\eta(\rho_\alpha(1)) = \Phi_\eta(P_\alpha)$ and $u_\alpha P_\alpha = u_\alpha \rho_\alpha(1) = u_\alpha$, the maps

$$Ad(U^*_\alpha) : U_\alpha \Phi_\eta(z) U^*_\alpha \in U_\alpha \Phi_\eta(B_\eta) U^*_\alpha \longrightarrow U^*_\alpha U_\alpha \Phi_\eta(z) U^*_\alpha U_\alpha \in \Phi_\eta(P_\alpha B_\eta P_\alpha),$$

$$Ad(u_\alpha) : P_\alpha z P_\alpha \in P_\alpha B_\eta P_\alpha (\subset B_\eta) \longrightarrow u_\alpha z u_\alpha^* \in u_\alpha B_\eta u_\alpha^*$$

for $z \in B_\eta$...
are \(\ast\)-homomorphisms. As \(\Phi_\eta : B_\eta \rightarrow \Phi_\eta(B_\eta)\) is a \(\ast\)-isomorphism, the desired map

\[
U_\alpha\Phi_\eta(z)U_\alpha^* \in U_\alpha\Phi_\eta(B_\eta)U_\alpha^* \rightarrow u_\alpha z u_\alpha^* \in u_\alpha B_\eta u_\alpha^*
\]

for \(z \in B_\eta\) yields a \(\ast\)-homomorphism.

(ii) is similar to (i). \(\square\)

**Lemma 5.14.**

(i) For \(\alpha \in \Sigma^\rho\), the correspondence:

\[
\Phi_{\gamma_1}(x_{j_1})\Phi_{\gamma_2}(x_{j_2}) \cdots \Phi_{\gamma_n}(x_{j_n})U_\alpha U_\alpha^* \in B_\kappa U_\alpha U_\alpha^* \rightarrow x_{j_1}x_{j_2} \cdots x_{j_n}u_\alpha u_\alpha^* \in B_\kappa u_\alpha u_\alpha^*
\]

for \(x_{j_k} \in B_\rho(\gamma_k = \rho)\) and \(x_{j_k} \in B_\eta(\gamma_k = \eta)\) gives rise to a \(\ast\)-homomorphism from \(B_\kappa U_\alpha U_\alpha^*\) to \(B_\kappa u_\alpha u_\alpha^*\).

(ii) For \(a \in \Sigma^\kappa\), the correspondence:

\[
\Phi_{\gamma_1}(x_{j_1})\Phi_{\gamma_2}(x_{j_2}) \cdots \Phi_{\gamma_n}(x_{j_n})V_a V_a^* \in B_\kappa V_a V_a^* \rightarrow x_{j_1}x_{j_2} \cdots x_{j_n}v_a v_a^* \in B_\kappa v_a v_a^*
\]

for \(x_{j_k} \in B_\rho(\gamma_k = \rho)\) and \(x_{j_k} \in B_\eta(\gamma_k = \eta)\) gives rise to a \(\ast\)-homomorphism from \(B_\kappa V_a V_a^*\) to \(B_\kappa v_a v_a^*\).

**Proof.** (i) Since \(\hat{\rho}_\alpha(w) \in A \subset B_\eta\) for \(w \in B_\rho\), we see \(\Phi_\rho(\hat{\rho}_\alpha(w)) = \Phi_\eta(\hat{\rho}_\alpha(w))\) so that

\[
U_\alpha^* \Phi_\rho(w)U_\alpha = \Phi_\rho(\hat{\rho}_\alpha(w)) = \Phi_\eta(\hat{\rho}_\alpha(w)) = U_\alpha^* \Phi_\eta(z)U_\alpha = \Phi_\eta(\hat{\rho}_\eta(z))
\]

for \(w \in B_\rho, z \in B_\eta\). For \(x_{j_k} \in B_\eta\) or \(B_\rho\), put

\[
\hat{x}_{j_k} := \begin{cases} 
\hat{\rho}_\alpha(x_{j_k}) & \text{if } x_{j_k} \in B_\rho, \\
\hat{\rho}_\eta(x_{j_k}) & \text{if } x_{j_k} \in B_\eta.
\end{cases}
\]

We then have \(\hat{x}_{j_k} \in B_\eta\) so that

\[
\Phi_{\gamma_1}(x_{j_1})\Phi_{\gamma_2}(x_{j_2}) \cdots \Phi_{\gamma_n}(x_{j_n})U_\alpha U_\alpha^* = U_\alpha U_\alpha^* \Phi_{\gamma_1}(x_{j_1})U_\alpha U_\alpha^* \Phi_{\gamma_2}(x_{j_2})U_\alpha U_\alpha^* \cdots U_\alpha U_\alpha^* \Phi_{\gamma_n}(x_{j_n})U_\alpha U_\alpha^*
\]

\[
= U_\alpha \Phi_\eta(\hat{x}_{j_1}) \Phi_\eta(\hat{x}_{j_2}) \cdots \Phi_\eta(\hat{x}_{j_n})U_\alpha^*
\]

\[
= U_\alpha \Phi_\eta(\hat{x}_{j_1} \hat{x}_{j_2} \cdots \hat{x}_{j_n})U_\alpha^*.
\]

By the preceding lemma, the correspondence

\[
U_\alpha \Phi_\eta(\hat{x}_{j_1} \hat{x}_{j_2} \cdots \hat{x}_{j_n})U_\alpha^* \in U_\alpha \Phi_\eta(B_\eta)U_\alpha^* \rightarrow u_\alpha x_{j_1}x_{j_2} \cdots x_{j_n}u_\alpha^* \in u_\alpha B_\eta u_\alpha^*
\]

gives rise to a \(\ast\)-homomorphism from \(U_\alpha \Phi_\eta(B_\eta)U_\alpha^*\) to \(u_\alpha B_\eta u_\alpha^*\). Since we have

\[
u_\alpha x_{j_1}x_{j_2} \cdots x_{j_n} u_\alpha^* = u_\alpha u_\alpha^* x_{j_1} u_\alpha u_\alpha^* x_{j_2} u_\alpha u_\alpha^* \cdots u_\alpha u_\alpha^* x_{j_n} u_\alpha u_\alpha^*
\]

\[
= x_{j_1}x_{j_2} \cdots x_{j_n} u_\alpha u_\alpha^*,
\]

we have a desired \(\ast\)-homomorphism from \(B_\alpha U_\alpha U_\alpha^*\) to \(B_\kappa u_\alpha u_\alpha^*\).

(ii) is similar to (i). \(\square\)

The above \(\ast\)-homomorphisms of (i) and of (ii) are denoted by

\[
\pi_\alpha : \Phi_{\gamma_1}(x_{j_1}) \Phi_{\gamma_2}(x_{j_2}) \cdots \Phi_{\gamma_n}(x_{j_n}) U_\alpha U_\alpha^* \in B_\kappa B_\kappa U_\alpha U_\alpha^* \rightarrow x_{j_1}x_{j_2} \cdots x_{j_n} u_\alpha u_\alpha^* \in B_\kappa u_\alpha u_\alpha^*
\]

\[
\pi_\alpha : \Phi_{\gamma_1}(x_{j_1}) \Phi_{\gamma_2}(x_{j_2}) \cdots \Phi_{\gamma_n}(x_{j_n}) V_a V_a^* \in B_\kappa B_\kappa V_a V_a^* \rightarrow x_{j_1}x_{j_2} \cdots x_{j_n} v_a v_a^* \in B_\kappa v_a v_a^*.
\]

**Lemma 5.15.** There exists a \(\ast\)-homomorphism

\[
\pi_\kappa : B_\kappa \rightarrow B_\kappa^{\text{uni}}
\]

such that \(\pi_\kappa(\Phi_\rho(w)) = w\) for \(w \in B_\rho\) and \(\pi_\kappa(\Phi_\eta(z)) = z\) for \(z \in B_\eta\).
Proof. Since \( \sum_{\alpha \in \Sigma^p} U_\alpha U_\alpha^* + \sum_{a \in \Sigma^n} V_a V_a^* = 1 \) and \( \sum_{\alpha \in \Sigma^p} u_\alpha u_\alpha^* + \sum_{a \in \Sigma^n} v_a v_a^* = 1 \) by putting
\[
\pi_\kappa(X) := \sum_{\alpha \in \Sigma^p} \pi_\alpha(XU_\alpha U_\alpha^*)u_\alpha u_\alpha^* + \sum_{a \in \Sigma^n} \pi_a(XV_a V_a^*)v_a v_a^*
\]
for \( X \in \mathcal{B}_\kappa \), we have a desired \(*\)-homomorphism from \( \mathcal{B}_\kappa \) to \( \mathcal{B}_\kappa^{uni} \).

We will consider the \(*\)-homomorphism \( \Psi : \mathcal{O}_{\mathcal{H}_\kappa}^{uni} \rightarrow \mathcal{O}_{\mathcal{H}_\kappa} \) again. By the above discussions we have

**Lemma 5.16.** The restriction \( \Psi|_{\mathcal{B}_\kappa^{uni}} : \mathcal{B}_\kappa^{uni} \rightarrow \mathcal{B}_\kappa \) of \( \Psi \) to the subalgebra \( \mathcal{B}_\kappa^{uni} \) is the inverse of \( \pi_\kappa : \mathcal{B}_\kappa \rightarrow \mathcal{B}_\kappa^{uni} \). Hence \( \Psi|_{\mathcal{B}_\kappa^{uni}} : \mathcal{B}_\kappa^{uni} \rightarrow \mathcal{B}_\kappa \) is a \(*\)-isomorphism.

Therefore we reach the main result of the paper:

**Theorem 5.17.** The \( C^* \)-algebra \( \mathcal{O}_{\mathcal{H}_\kappa} \) associated with the Hilbert \( C^* \)-quad module \( \mathcal{H}_\kappa \) is canonically \(*\)-isomorphic to the universal \( C^* \)-algebra \( \mathcal{O}_{\mathcal{H}_\kappa}^{uni} \) generated by the operators \( w \in \mathcal{B}_\rho, z \in \mathcal{B}_\eta \) and partial isometries \( u_\alpha, \alpha \in \Sigma^p, v_a, a \in \Sigma^n \) satisfying the operator relations:

\[
\sum_{\beta \in \Sigma^p} u_\beta u_\beta^* + \sum_{b \in \Sigma^n} v_b v_b^* = 1, \quad (5.3)
\]
\[
u_\alpha u_\alpha^* w = w u_\alpha u_\alpha^*, \quad v_a v_a^* w = w v_a v_a^*, \quad (5.4)
\]
\[
u_\alpha u_\alpha^* z = z v_a v_a^*, \quad v_a v_a^* z = z v_a v_a^*, \quad (5.5)
\]
\[
\tilde{\rho}_\alpha(w) = u_\alpha w u_\alpha, \quad \tilde{\eta}_\alpha(z) = v_a z v_a, \quad (5.6)
\]
\[
\tilde{\rho}_\alpha(z) = u_\alpha^* z u_\alpha, \quad \tilde{\eta}_\alpha(w) = v_a^* w v_a, \quad (5.7)
\]
\[
\iota_\rho(y) = \iota_\rho(y) \quad (5.8)
\]

for \( w \in \mathcal{B}_\rho, z \in \mathcal{B}_\eta, \alpha \in \Sigma^p, a \in \Sigma^n, y \in \mathcal{A} \) where \( \iota_\rho : \mathcal{A} \hookrightarrow \mathcal{B}_\rho \) and \( \iota_\eta : \mathcal{A} \hookrightarrow \mathcal{B}_\eta \) are natural embeddings.

Proof. The triplets \((\mathcal{B}_\kappa^{uni}, \rho^{uni}, \Sigma^{(uni)})\) and \((\mathcal{B}_\kappa, \rho^*, \Sigma^{(\eta)})\) are both the \( C^* \)-symbolic dynamical systems. As in the discussions of the proof of [22, Lemma 3.2], the above lemma implies that the restriction \( \Psi|_{\mathcal{F}_{\mathcal{H}_\kappa}^{uni}} : \mathcal{F}_{\mathcal{H}_\kappa}^{uni} \rightarrow \mathcal{F}_{\mathcal{H}_\kappa} \) of \( \Psi \) to the subalgebra \( \mathcal{F}_{\mathcal{H}_\kappa}^{uni} \) is a \(*\)-isomorphism. The diagram:

\[
\begin{array}{ccc}
\mathcal{O}_{\mathcal{H}_\kappa}^{uni} & \xrightarrow{\Psi} & \mathcal{O}_{\mathcal{H}_\kappa} \\
\mathcal{E}_{\mathcal{H}_\kappa}^{uni} \downarrow & & \downarrow \mathcal{E}_{\mathcal{H}_\kappa} \\
\mathcal{F}_{\mathcal{H}_\kappa}^{uni} & \xrightarrow{\Psi|_{\mathcal{F}_{\mathcal{H}_\kappa}^{uni}}} & \mathcal{F}_{\mathcal{H}_\kappa}
\end{array}
\]

is commutative. Since the conditional expectation \( \mathcal{E}_{\mathcal{H}_\kappa}^{uni} \) is faithful and the restriction \( \Psi|_{\mathcal{F}_{\mathcal{H}_\kappa}^{uni}} \) is \(*\)-isomorphic, one concludes that \( \Psi : \mathcal{O}_{\mathcal{H}_\kappa}^{uni} \rightarrow \mathcal{O}_{\mathcal{H}_\kappa} \) is a \(*\)-isomorphism by a routine argument as in [5, 2.9 Proposition].

The above theorem implies the following: Suppose that there exist two families of partial isometries \( \hat{u}_\alpha, \alpha \in \Sigma^p, \hat{v}_a, a \in \Sigma^n \) in a unital \( C^* \)-algebra \( \mathcal{D} \) and two
Lemma 5.19. For \( \omega = (a, b, \alpha, \beta) \in \Sigma_{\kappa} \), we have

\[ P_{\omega} = C_{\omega} C_{\omega}^*, \quad \omega \in \Sigma_{\kappa}, \]

\[ P_{\omega} = C_{\omega} C_{\omega}^*, \quad \omega \in \Sigma_{\kappa}. \]
(iii) $C_\omega \notin B_\kappa$.

**Proof.** (i) By (5.10) and (5.11), we have for $x \in A$,

$$C_\omega \Phi(x) = V_\alpha U_\beta V_\alpha^* \Phi(\rho_\alpha(x)) U_\alpha^* = V_\alpha U_\beta \Phi(\eta_\beta(\rho_\alpha(x))) U_\alpha^*$$

and similarly $C_\omega V_\alpha U_\beta = U_\alpha V_\beta$.

(ii) The equations (5.16) are straightforward.

(iii) Suppose that $C_\omega \in B_\kappa$. Since $U_\alpha U_\alpha^*$ commutes with $\Phi_\rho(w)$, $\Phi_\eta(z)$ for $w \in B_\rho$, $z \in B_\eta$, it commutes with $B_\kappa$ and hence with $C_\omega$. Hence we have

$$V_\alpha U_\beta = C_\omega U_\alpha V_\beta = U_\alpha U_\alpha^* C_\omega U_\beta.$$  

As $V_\alpha U_\alpha^* U_\beta v = 0$, one has $V_\alpha U_\beta = 0$. Since $\Phi(\rho_\beta(\eta_\alpha(y))) = U_\beta^* V_\alpha^* y V_\alpha U_\beta = 0$, one has $\rho_\beta(\eta_\alpha(y)) = 0$ for all $y \in A$, a contradiction. \qed

For the two $C^*$-symbolic dynamical systems $(A, \rho, \Sigma^\rho)$ and $(A, \eta, \Sigma^\eta)$ in the $C^*$-textile dynamical system $(A, \rho \cup \eta, \Sigma^{\rho, \eta}, \kappa)$, we define their union $(A, \rho \cup \eta, \Sigma^{\rho, \eta})$ as the following way, where $\Sigma^{\rho, \eta} = \Sigma^\rho \cup \Sigma^\eta$. For $\gamma \in \Sigma^{\rho, \eta}$, define an endomorphism $(\rho \cup \eta)_\gamma$ on $A$ by setting

$$(\rho \cup \eta)_\gamma = \begin{cases} 
\rho_\gamma & \text{if } \gamma \in \Sigma^\rho, \\
\eta_\gamma & \text{if } \gamma \in \Sigma^\eta.
\end{cases}$$

It is easy to see that the triplet $(A, \rho \cup \eta, \Sigma^{\rho, \eta})$ is a $C^*$-symbolic dynamical system. Hence we have a $C^*$-algebra $O_{\rho \cup \eta}$ from $(A, \rho \cup \eta, \Sigma^{\rho, \eta})$. Denote by $S_\alpha$, $\alpha \in \Sigma^\rho$ and $S_\alpha$, $\alpha \in \Sigma^\eta$ its generating partial isometries satisfying the relations:

$$\sum_{\beta \in \Sigma^\rho} S_\beta S_\beta^* + \sum_{b \in \Sigma^\eta} S_b S_b^* = 1, \quad S_\alpha S_\alpha^* x = x S_\alpha^* S_\alpha, \quad S_\alpha S_\alpha^* = \rho_\alpha(x), \quad S_\alpha^* S_\alpha = \eta_\alpha(x) \quad (5.17)$$

for $x \in A, \alpha \in \Sigma^\rho, \alpha \in \Sigma^\eta$. Then we have

**Proposition 5.20.**

(i) The correspondence:

$$S_\alpha \mapsto U_\alpha, \quad S_\alpha \mapsto V_\alpha, \quad x \mapsto \Phi(x)$$

for $\alpha \in \Sigma^\rho, \alpha \in \Sigma^\eta, x \in A$ yield a *-isomorphism between the $C^*$-algebras $O_{\rho \cup \eta}$ and $O_{A, \kappa}$. Therefore we have

$$O_{A, \kappa} \cong O_{\rho \cup \eta}. \quad (5.20)$$

(ii) Hence the $C^*$-algebra $O_{A, \kappa}$ is realized as the universal $C^*$-algebra generated by partial isometries $U_\alpha, V_\alpha$ for $\alpha \in \Sigma^\rho, \alpha \in \Sigma^\eta$ and elements $x \in A$ subject to the relations:

$$\sum_{\beta \in \Sigma^\rho} U_\beta U_\beta^* + \sum_{b \in \Sigma^\eta} V_b V_b^* = 1,$$

$$U_\alpha^* \Phi(x) = \Phi(x) U_\alpha U_\alpha^*, \quad V_\alpha^* \Phi(x) = \Phi(x) V_\alpha V_\alpha^*,$$

$$U_\alpha^* \Phi(x) U_\alpha = \Phi(\rho_\alpha(x)), \quad V_\alpha^* \Phi(x) V_\alpha = \Phi(\eta_\alpha(x)).$$
Proof. By the universality of the algebra $\mathcal{O}_{\rho,\eta}$ subject to the relation (5.17), (5.18), (5.19), the correspondences
\[
S_\alpha \rightarrow U_\alpha \quad S_a \rightarrow V_a, \quad x \rightarrow \Phi(x)
\]
for $\alpha \in \Sigma^\rho, a \in \Sigma^\eta, x \in \mathcal{A}$ yield a $*$-homomorphism from $\mathcal{O}_{\rho,\eta}$ to $\mathcal{O}_{\mathcal{A},\kappa}$ which we denote by $\Psi$. Since the action $g(r_1, r_2), (r_1, r_2) \in \mathbb{T}^2$ preserves $\mathcal{O}_{\mathcal{A},\kappa}$, the automorphisms $g_t$ for $t \in \mathbb{T}$ give rise to an action on $\mathcal{O}_{\mathcal{A},\kappa}$ which we denote by $g^A_t$. It satisfies
\[
g^A_t(\Phi(x)) = \Phi(x), \quad g^A_t(U_\alpha) = e^{2\pi i t}U_\alpha, \quad g^A_t(V_a) = e^{2\pi i t}V_a
\]
for $\alpha \in \Sigma^\rho, a \in \Sigma^\eta, x \in \mathcal{A}$ and $t \in \mathbb{T}$. Let $\hat{g}$ be the gauge action on $\mathcal{O}_{\rho,\eta}$ which satisfies
\[
\hat{g}_t(x) = x, \quad \hat{g}_t(S_\alpha) = e^{2\pi i t}S_\alpha, \quad \hat{g}_t(S_a) = e^{2\pi i t}S_a
\]
for $\alpha \in \Sigma^\rho, a \in \Sigma^\eta, x \in \mathcal{A}$ and $t \in \mathbb{T}$. Hence we have
\[
\Psi \circ \hat{g}_t = g^A_t \circ \Psi \quad \text{for} \quad t \in \mathbb{T}.
\]
Let $(\mathcal{O}_{\mathcal{A},\kappa})^{g^A}$ be the fixed point algebra of $\mathcal{O}_{\mathcal{A},\kappa}$ under the action $g^A$. Denote by $\mathcal{E}^A : \mathcal{O}_{\mathcal{A},\kappa} \rightarrow (\mathcal{O}_{\mathcal{A},\kappa})^{g^A}$ the conditional expectation defined by the formula:
\[
\mathcal{E}^A(X) = \int_{\mathbb{T}} g^A_t(X) \, dt \quad \text{for} \quad X \in \mathcal{O}_{\mathcal{A},\kappa}.
\]
Denote by $\mathcal{E}^{\rho,\eta}_\kappa : \mathcal{O}_{\rho,\eta} \rightarrow (\mathcal{O}_{\rho,\eta})^{\hat{g}}$ the conditional expectation similarly defined to the above by the gauge action $\hat{g}$. Since we have
\[
\Psi \circ \mathcal{E}^{\rho,\eta}_\kappa = \mathcal{E}^A \circ \Psi
\]
and $\mathcal{E}^{\rho,\eta}_\kappa$ is faithful, by a routine argument as in [5, 2.9 Proposition], one concludes that $\Psi$ is injective and hence $*$-isomorphic. \hfill $\square$

6. Relation between $\mathcal{O}_{\mathcal{H},\kappa}$ and $\mathcal{O}_{\rho,\eta}^\kappa$

For a $C^*$-textile dynamical system $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$, the author has introduced a $C^*$-algebra $\mathcal{O}_{\rho,\eta}^\kappa$ in [23]. It is realized as the universal $C^*$-algebra $C^*(x, S_\alpha, T_a; x \in \mathcal{A}, \alpha \in \Sigma^\rho, a \in \Sigma^\eta)$ generated by $x \in \mathcal{A}$ and two families of partial isometries $S_\alpha, \alpha \in \Sigma^\rho, T_a, a \in \Sigma^\eta$ subject to the relations (1.2), (1.3) and (1.4) that are called the relations $(\rho, \eta; \kappa)$. In this section we will describe a relationship between the two algebras $\mathcal{O}_{\mathcal{H},\kappa}$ and $\mathcal{O}_{\rho,\eta}^\kappa$. As both of the $C^*$-algebras $\mathcal{O}_{\rho}$ and $\mathcal{O}_{\eta}$ are naturally regarded as $C^*$-subalgebras of $\mathcal{O}_{\rho,\eta}^\kappa$, the algebras $B_\rho$ and $B_\eta$ may be realized as the $C^*$-subalgebra of $\mathcal{O}_{\rho,\eta}^\kappa$ generated by $S_\alpha x S_\alpha^*$ for $x \in \mathcal{A}, \alpha \in \Sigma^\rho$ and that of $\mathcal{O}_{\rho,\eta}^\kappa$ generated by $T_a x T_a^*$ for $x \in \mathcal{A}, a \in \Sigma^\eta$ respectively.

Lemma 6.1. Let $S_\alpha, x \in \Sigma^\rho$ and $T_a, a \in \Sigma^\eta$ be partial isometries in the algebra $\mathcal{O}_{\rho,\eta}^\kappa$ satisfying the relations (1.2), (1.3) and (1.4).

(i) For $\alpha \in \Sigma^\rho$ and $z = \sum_{b \in \Sigma^\eta} T_b z_b T_b^* \in B_\eta$ as in (2.3),
\[
S_\alpha^* z S_\alpha = \sum_{(\alpha, b, a, \beta) \in \Sigma_\kappa} T_b \rho_\beta (z_a) T_b^* = \hat{\rho}_\alpha^\eta (z).
\]
Let us denote by $C$. Proof. (i) By [23, Lemma 4.2] the following formulae hold for $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$,
\[
T_a^* a T_a = \sum_{\kappa(a,b) = (a,\beta)} S_{\beta}^\eta \eta (w_a) S_{\beta}^\rho = \tilde{\rho}_a^\alpha (w).
\] (6.2)

It then follows that
\[
S_{\alpha}^* a S_{\alpha} = \sum_{a \in \Sigma^\eta} S_{\alpha}^* T_a z_a T_a^* S_{\alpha} = \sum_{a \in \Sigma^\eta} T_b \rho_{\beta}(Q_a) S_{\beta}^* a z_a \sum_{\kappa(a,b) = (a,\beta)} S_{\beta}^\rho \rho_{\beta}(Q_a) T_b^*
\]
\[
= \sum_{\kappa(a,b) = (a,\beta)} T_b \rho_{\beta}(Q_a z_a Q_a) T_b^* = \tilde{\rho}_a^\alpha (z).
\]

(ii) is similar to (i). □

Let us denote by $S_1, S_2$ isometries satisfying $S_1 S_1^* + S_2 S_2^* = 1$. The $C^*$-algebra generated by them is the Cuntz algebra $O_2$ of order 2. In the tensor product $C^*$-algebra $O_{\rho,\eta}^\kappa \otimes O_2$, we put
\[
\widehat{u}_a = S_{\alpha} \otimes S_1 \text{ for } \alpha \in \Sigma^\rho, \quad \widehat{v}_a = T_a \otimes S_2 \text{ for } a \in \Sigma^\eta,
\]
\[
\pi_{\rho}(w) = w \otimes 1 \text{ for } w \in B_{\rho}, \quad \pi_{\eta}(z) = z \otimes 1 \text{ for } z \in B_{\eta}.
\]

By the above lemma, the following lemma is straightforward.

**Lemma 6.2.** The operators $\widehat{u}_a, \widehat{v}_a$ for $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$ and $\pi_{\rho}(w), \pi_{\eta}(z)$ for $w \in B_{\rho}, z \in B_{\eta}$ satisfy the relations $(H_\kappa)$.

We thus have

**Theorem 6.3.** Assume that $(A, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ satisfies condition (I). Then the correspondences

\[
U_a \in \mathcal{O}_{H_\kappa} \rightarrow S_{\alpha} \otimes S_1 \in \mathcal{O}_{\rho,\eta}^\kappa \otimes O_2,
\]
\[
V_a \in \mathcal{O}_{H_\kappa} \rightarrow T_a \otimes S_2 \in \mathcal{O}_{\rho,\eta}^\kappa \otimes O_2,
\]
\[
w \in B_{\rho} \rightarrow w \otimes 1 \in \mathcal{O}_{\rho,\eta}^\kappa \otimes O_2,
\]
\[
z \in B_{\eta} \rightarrow z \otimes 1 \in \mathcal{O}_{\rho,\eta}^\kappa \otimes O_2
\]
for $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$ give rise to a $*$-isomorphism from $\mathcal{O}_{H_\kappa}$ to the $C^*$-subalgebra of $\mathcal{O}_{\rho,\eta}^\kappa \otimes O_2$ generated by the partial isometries $S_{\alpha} \otimes S_1, T_a \otimes S_2$ for $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$ and the elements $w \otimes 1, z \otimes 1$ for $w \in B_{\rho}, z \in B_{\eta}$. That is
\[
\mathcal{O}_{H_\kappa} \cong C^*(S_{\alpha} \otimes S_1, T_a \otimes S_2, w \otimes 1, z \otimes 1 : \alpha \in \Sigma^\rho, a \in \Sigma^\eta, w \in B_{\rho}, z \in B_{\eta}).
\]

We will present an example. Let $\alpha, \beta$ be automorphisms of a unital commutative $C^*$-algebra $A$. Put $\Sigma^\rho = \{\alpha\}, \Sigma^\eta = \{\beta\}$ and define $\rho_{\alpha} = \alpha, \eta_{\beta} = \beta$. We have two $C^*$-symbolic dynamical systems $(A, \alpha, \{\alpha\}), (A, \beta, \{\beta\})$. Assume that $\alpha \circ \beta = \beta \circ \alpha$. Put $\Sigma^{\alpha \beta} = \{(\alpha, \beta)\}, \Sigma^{\beta \alpha} = \{\beta, \alpha\}$. The specification $\kappa : \Sigma^{\alpha \beta} \rightarrow \Sigma^{\beta \alpha}$ is
unique and satisfies \( \kappa(\alpha, \beta) = (\beta, \alpha) \). We have a \( C^* \)-textile dynamical system \((\mathcal{A}, \alpha, \beta, \{\alpha\}, \{\beta\}, \kappa)\). We denote by \( \mathcal{H}_{\kappa}^{\alpha, \beta} \) the associated Hilbert \( C^* \)-quad module. Since \( \Sigma_\kappa = \{(\alpha, \beta, \beta, \beta)\} \) a singleton and \( E_\omega = \beta(\alpha(1)) = 1 \) so that \( H_{\kappa}^{\alpha, \beta} = \mathcal{A} \).

As \( \alpha \circ \beta = \beta \circ \alpha \), they induce an action of \( Z^2 \) on \( \mathcal{A} \). By the universality subject to the relations (1.2), (1.3) and (1.4), one easily sees that the algebra \( O_{\kappa}^{\alpha, \beta} \) for the \( C^* \)-textile dynamical system \((\mathcal{A}, \alpha, \beta, \{\alpha\}, \{\beta\}, \kappa)\) is *-isomorphic to the crossed product \( \mathcal{A} \times_{\alpha, \beta} Z^2 \). Take implementing unitaries \( U, V \) in \( \mathcal{A} \times_{\alpha, \beta} Z^2 \) for the action such that

\[
\alpha(x) = U^* x U, \quad \beta(x) = V^* x V \quad \text{for} \ x \in \mathcal{A}.
\]

Since both \( U, V \) are unitaries, we have \( \alpha^{-1}(x) = U x U^* \), \( \beta^{-1}(x) = V x V^* \) for \( x \in \mathcal{A} \) which belong to \( \mathcal{A} \). Hence we have

\[
\mathcal{B}_\alpha = C^*(U x U^* \mid x \in \mathcal{A}) = \mathcal{A}, \quad \mathcal{B}_\beta = C^*(V x V^* \mid x \in \mathcal{A}) = \mathcal{A}.
\]

For \( w \in \mathcal{B}_\alpha, z \in \mathcal{B}_\beta \) we have

\[
w = U \alpha(w) U^*, \quad z = V \alpha(z) V^*.
\]

We will write down the Hilbert \( C^* \)-quad module structure for \( H_{\kappa}^{\alpha, \beta} = \mathcal{A} \) defined in Section 3. For \( \xi = x, \xi' = x' \in \mathcal{H}_{\kappa}^{\alpha, \beta} = \mathcal{A}, y \in \mathcal{A}, w \in \mathcal{B}_\alpha = \mathcal{A}, z \in \mathcal{B}_\beta = \mathcal{A}, \)

0. The right \( \mathcal{A} \)-module and the right \( \mathcal{A} \)-valued inner product \( \langle \cdot | \cdot, \cdot \rangle_\mathcal{A} \):

\[
\xi \varphi \mathcal{A}(y) = xy \quad \text{for} \ y \in \mathcal{A}, \quad \langle \xi | \xi', \cdot \rangle_\mathcal{A} = x^* x'.
\]

1. The right action of \( \mathcal{B}_\alpha \) and the right action of \( \mathcal{B}_\beta \):

\[
\xi \varphi \mathcal{A}(w) = x \alpha(w), \quad \xi \varphi \mathcal{A}(z) = x \beta(z).
\]

2. The right \( \mathcal{B}_\alpha \)-valued inner product and the right \( \mathcal{B}_\beta \)-valued inner product:

\[
\langle \xi | \xi', \cdot \rangle_\mathcal{B} = U x^* x' U^* = \alpha^{-1}(x^* x'), \quad \langle \xi | \xi', \cdot \rangle_\mathcal{B} = V x^* x' V^* = \beta^{-1}(x^* x').
\]

3. The left action of \( \mathcal{B}_\alpha \) and the left action of \( \mathcal{B}_\beta \):

\[
\phi_\alpha(w) \xi = \beta(\alpha(w)) x, \quad \phi_\beta(z) \xi = \alpha(\beta(z)) x.
\]

We then have

\[
\hat{\rho}_\alpha(w) = \alpha(w), \quad \hat{\eta}_\beta(z) = \beta(z) \quad \text{for} \ w \in \mathcal{B}_\alpha = \mathcal{A}, z \in \mathcal{B}_\beta = \mathcal{A}.
\]

For the *-homomorphisms \( \hat{\rho}_\alpha : \mathcal{B}_\beta \to \mathcal{B}_\beta \) and \( \hat{\eta}_\beta : \mathcal{B}_\alpha \to \mathcal{B}_\alpha \), we have for \( z \in \mathcal{B}_\beta, \)

\[
\hat{\rho}_\alpha(z) = V \alpha(z) V^* = V \alpha(V^* z V) V^* = \beta^{-1}(\alpha(\beta(z))) = \alpha(z)
\]

and similarly \( \hat{\eta}_\beta(w) = \beta(w) \) for \( w \in \mathcal{B}_\alpha \). Hence the \( C^* \)-algebra \( O_{\hat{\mathcal{H}}^{\alpha, \beta}} \) defined by the Hilbert \( C^* \)-quad module \( \mathcal{H}_{\kappa}^{\alpha, \beta} \) is the universal \( C^* \)-algebra generated by two isometries \( u, v \) and elements \( x \in \mathcal{A} \) subject to the relations

\[
u=1,
\nu x=xu^*,
\nu^* x = xv^*,
\alpha(x) = u^* x u, \quad \beta(x) = v^* x v
\]

for \( x \in \mathcal{A} \). By the universality of the algebra \( O_{\hat{\mathcal{H}}^{\alpha, \beta}} \) the correspondence

\[
u \in O_{\hat{\mathcal{H}}^{\alpha, \beta}} \to \hat{\nu} = U \otimes S_1 \in \mathcal{A} \times_{\alpha, \beta} Z^2 \otimes O_2,
\nu \in O_{\hat{\mathcal{H}}^{\alpha, \beta}} \to \hat{\nu} = V \otimes S_2 \in \mathcal{A} \times_{\alpha, \beta} Z^2 \otimes O_2,
\]

\[
x \in \mathcal{A} \to \hat{x} = x \otimes 1 \in \mathcal{A} \times_{\alpha, \beta} Z^2 \otimes O_2
\]
gives rise to a ∗-homomorphism. If in particular, the action \((n, m) \in \mathbb{Z}^2 \rightarrow \alpha^n \circ \beta^m \in \text{Aut}(A)\) is outer, \((A, \alpha, \beta, \{\alpha\}, \{\beta\}, \kappa)\) satisfies condition (I) so that the above correspondence gives rise to a ∗-isomorphism, that is
\[
\mathcal{O}_{H_{n,m}^A} \cong C^*(U \otimes S_1, V \otimes S_2, x \mid x \in A) \subset A \times_\alpha, \beta \mathbb{Z}^2 \otimes \mathcal{O}_2.
\]

7. TEXTILE SYSTEMS OF COMMUTING MATRICES

We will study \(C^\ast\)-algebras associated with the Hilbert \(C^\ast\)-quad modules defined by textile systems of commuting matrices (cf. [25]). Let \(A\) be an \(N \times N\) matrix with entries in nonnegative integers. We may consider a directed graph \(G_A = (V, E_A)\) with vertex set \(V = \{v_1, \ldots, v_N\}\) and edge set \(E_A\) consisting of \(A(i, j)\) edges from the vertex \(v_i\) to the vertex \(v_j\). Denote by \(\Sigma^A = E_A\). Let \(\mathcal{A}\) be the \(N\)-dimensional commutative \(C^\ast\)-algebra \(\mathbb{C}^N\) with minimal projections \(E_1, \ldots, E_N\) such that
\[
\mathcal{A} = \mathbb{C}E_1 + \cdots + \mathbb{C}E_N.
\]
We set for \(\alpha \in \Sigma^A, i, j = 1, \ldots, N\)
\[
\hat{A}(i, \alpha, j) = \begin{cases} 
1 & \text{if } s(\alpha) = v_i, r(\alpha) = v_j, \\
0 & \text{otherwise}
\end{cases}
\]
where \(s(\alpha), r(\alpha)\) mean the source vertex, the range vertex of an edge \(\alpha\) respectively.

We define an endomorphism \(\rho^A_\alpha\) on \(\mathcal{A}\) for \(\alpha \in \Sigma^A:\)
\[
\rho^A_\alpha(E_i) = \sum_{j=1}^N \hat{A}(i, \alpha, j)E_j, \quad i = 1, \ldots, N.
\]
Then we have a \(C^\ast\)-symbolic dynamical system \((\mathcal{A}, \rho^A, \Sigma^A)\). Let \(B\) be another \(N \times N\) matrix with entries in nonnegative integers such that
\[
AB = BA. \tag{7.1}
\]
Consider the associated directed graph \(G_B = (V, E_B)\) and the \(C^\ast\)-symbolic dynamical system \((\mathcal{A}, \rho^B, \Sigma^B)\) for \(B\). Let \(S_\alpha, \alpha \in E_A, T_\alpha, \alpha \in E_B\) be the generating partial isometries of the associated \(C^\ast\)-algebras \(\mathcal{O}_{\rho^A}\) and \(\mathcal{O}_{\rho^B}\) respectively. They satisfy the relations:
\[
\sum_{\beta \in E_A} S_\beta S_\beta^* = 1, \quad xS_\alpha S_\alpha^* = S_\alpha S_\alpha^* x, \quad S_\alpha^* x S_\alpha = \rho^A_\alpha(x),
\]
\[
\sum_{\beta \in E_B} T_\beta T_\beta^* = 1, \quad xT_\alpha T_\alpha^* = T_\alpha T_\alpha^* x, \quad T_\alpha^* x T_\alpha = \rho^B_\alpha(x),
\]
for all \(x \in \mathcal{A}\) and \(\alpha \in E_A, \alpha \in E_B\) respectively. The \(C^\ast\)-algebras \(\mathcal{O}_{\rho^A}\) and \(\mathcal{O}_{\rho^B}\) are isomorphic to the Cuntz-Krieger algebras \(\mathcal{O}_A\) and \(\mathcal{O}_B\) respectively. Put subalgebras
\[
\mathcal{B}_A = C^*(S_\alpha T_\alpha T_\alpha^* : \alpha \in E_A, i = 1, \ldots, N) \subset \mathcal{O}_{\rho^A},
\]
\[
\mathcal{B}_B = C^*(T_\alpha E_i E_i^* : \alpha \in E_B, i = 1, \ldots, N) \subset \mathcal{O}_{\rho^B}.
\]
Since \(S_\alpha T_\alpha S_\alpha^* \neq 0\) if and only if \(S_\alpha T_\alpha S_\alpha^* = E_i\), which is equivalent to \(r(\alpha) = v_i\). Hence \(\mathcal{B}_A\) is of \(|E_A|\)-dimension, and similarly \(\mathcal{B}_B\) is of \(|E_B|\)-dimension. By the identities
\[
E_i = \sum_{\alpha \in E_A} S_\alpha \rho^A_\alpha(E_i)S_\alpha^* = \sum_{\alpha \in E_A} \sum_{j=1}^N \hat{A}(i, \alpha, j)S_\alpha E_j S_\alpha^*, \quad i = 1, \ldots, N,
\]
Lemma 7.2. We write the projection \( E_i \) belongs to \( B_A \) so that \( A \subset B_A \) and similarly \( A \subset B_B \). The equality (7.1) implies that the cardinal numbers of the sets of the pairs of directed edges

\[
\Sigma_{AB}^{(i,j)} = \{(a, b) \in E_A \times E_B \mid s(a) = v_i, r(a) = s(b), r(b) = v_j\},
\]

\[
\Sigma_{BA}^{(i,j)} = \{(a, \beta) \in E_B \times E_A \mid s(a) = v_i, r(a) = s(\beta), r(\beta) = v_j\}
\]

coincide with each other for each \( v_i \) and \( v_j \), so that one may take a bijection \( \kappa : \Sigma_{AB}^{(i,j)} = \bigcup_{i,j=1}^N \Sigma_{AB}^{(i,j)} \rightarrow \Sigma_{BA}^{(i,j)} = \bigcup_{i,j=1}^N \Sigma_{BA}^{(i,j)} \) such that

\[
s(a) = s(a), \quad r(b) = r(\beta) \quad \text{if} \quad \kappa(a, b) = (a, \beta).
\]

We then have

**Lemma 7.1.** For \( (\alpha, b) \in \Sigma_{AB}^{(i,j)}, (\alpha, \beta) \in \Sigma_{BA}^{(i,j)} \) with \( \kappa(\alpha, b) = (\alpha, \beta) \), we have

\[
\rho_B^\alpha \circ \rho_A^\alpha(E_i) = \rho_B^\alpha \circ \rho_B^\alpha(E_i), \quad i = 1, \ldots, N.
\]

**Proof.** We have for \( i, k = 1, \ldots, N \)

\[
\rho_B^\alpha \circ \rho_A^\alpha(E_i)E_k = \sum_{j=1}^N \hat{A}(i, \alpha, j)\rho_B^\alpha(E_j)E_k = \sum_{j=1}^N \hat{A}(i, \alpha, j)\hat{B}(j, b, k)E_k.
\]

Hence \( \rho_B^\alpha \circ \rho_A^\alpha(E_i)E_k = E_k \) if and only if \( v_i = s(\alpha), r(\alpha) = s(b), r(b) = v_k \). Similarly for \( (\alpha, \beta) \in \Sigma_{BA}^{(i,j)} \), we have \( \rho_B^\alpha \circ \rho_B^\alpha(E_i)E_k = E_k \) if and only if \( v_i = s(\alpha), r(\alpha) = s(\beta), r(\beta) = v_k \). The condition \( \kappa(\alpha, b) = (\alpha, \beta) \) implies \( r(\alpha) = s(b), r(\alpha) = s(\beta), s(\alpha) = s(a), r(b) = r(\beta) \). Therefore

\[
\rho_B^\alpha \circ \rho_B^\alpha(E_i)E_k = \rho_B^\alpha \circ \rho_B^\beta(E_i)E_k \quad \text{for} \quad i, k = 1, \ldots, N
\]

and hence \( \rho_B^\alpha \circ \rho_B^\alpha = \rho_B^\alpha \circ \rho_B^\alpha \) if \( \kappa(\alpha, b) = (a, \beta) \).

We thus have a \( C^* \)-textile dynamical system

\[
(A, \rho_A^\alpha, \rho_B^\alpha, E_A, E_B, \kappa).
\]

We set \( E_\kappa = \{(\alpha, b, a, \beta) \in E_A \times E_B \times E_B \mid \kappa(\alpha, b) = (a, \beta)\} \).

For \( a \in E_B \) and \( \alpha \in E_A \), we will describe the -homomorphisms

\[
\hat{\eta}_a^\alpha : \mathbb{B}_{\rho^\alpha} (= \mathbb{B}_A) \rightarrow \mathbb{B}_{\rho^\alpha} (= \mathbb{B}_A) \quad \text{and} \quad \hat{\eta}_a^\beta : \mathbb{B}_{\rho^\beta} (= \mathbb{B}_B) \rightarrow \mathbb{B}_{\rho^\beta} (= \mathbb{B}_B)
\]

which will be denoted by \( \hat{\rho}_a^\alpha \) and by \( \hat{\rho}_a^\beta \) respectively. We set

\[
E_{AB} = \Sigma_{AB}, \quad E_{BA} = \Sigma_{BA}.
\]

For \( (\alpha, \beta) \in E_{AB} \) there uniquely exists \( (a, b) \in E_{AB} \) such that \( \kappa(a, b) = (a, \beta) \). We then define a map for \( a \in E_B \)

\[
\kappa_a : \beta \in \{\beta \in E_A \mid (\alpha, b) \in E_{BA}\} \rightarrow \alpha \in \{\alpha \in E_A \mid \kappa(\alpha, b) = (a, \beta) \text{ for some } b \in E_B\}.
\]

Similarly, we then define a map for \( \alpha \in E_A \)

\[
\kappa_a : b \in \{b \in E_B \mid (\alpha, b) \in E_{AB}\} \rightarrow \alpha \in \{\alpha \in E_B \mid \kappa(a, b) = (a, \beta) \text{ for some } \beta \in E_A\}.
\]

We write the projection \( E_i \) also as \( E_{\kappa_i} \). Hence \( w, z \in \mathbb{B}_B \) are uniquely written as \( w = \sum_{\alpha \in E_A} w(\alpha)S_\alpha E_{\kappa(\alpha)} S_\alpha^* \), \( z = \sum_{a \in E_B} z(a)T_aE_{\kappa(a)} T_a^* \) for \( w(\alpha), z(a) \in \mathbb{C} \).

**Lemma 7.2.** Keep the above notations. We have

\[
\hat{\rho}_a^\alpha(w) = \sum_{b \in E_A} w(\kappa_a(b))S_\beta E_{\kappa(\beta)} S_\beta^* \quad \text{and} \quad \hat{\rho}_a^\beta(z) = \sum_{b \in E_B} z(\kappa_a(b))T_b E_{\kappa(b)} T_b^*.
\]
Proof. We have
\[
\hat{\rho}_a^A(w) = \sum_{\beta, b, \alpha \in \text{E}_n} S_\beta \rho_\alpha^B(w(\alpha)E_{r(\alpha)})S_\beta^* = \sum_{\beta, b, \alpha \in \text{E}_n} w(\alpha)S_\beta T_{r(\alpha)} E_{r(\alpha)} T_{b} S_\beta^*.
\]

Since \( T_{r(\alpha)} E_{r(\alpha)} T_{b} = E_{r(b)} \) and \( \alpha = \kappa_\alpha(\beta) \), we see
\[
\hat{\rho}_a^A(w) = \sum_{\beta \in \text{A}} w(\kappa_\alpha(\beta))S_\beta E_{r(\beta)} S_\beta^*.
\]

We similarly have
\[
\hat{\rho}_a^B(z) = \sum_{b, \alpha, \beta \in \text{E}_n} T_b \rho_\alpha^A(z(a)E_{r(\alpha)}) T_{b}^* = \sum_{b, \alpha, \beta \in \text{E}_n} z(a)T_b S_\beta E_{r(\beta)} S_\beta^* T_{b}^*.
\]

Since \( S_\beta E_{r(\alpha)} S_\beta^* = E_{r(\beta)} \) and \( a = \kappa_\alpha(b) \), we see
\[
\hat{\rho}_a^B(z) = \sum_{b \in \text{B}} w(\kappa_\alpha(b))T_b E_{r(\beta)} T_{b}^*.
\]

\( \square \)

For \( \omega = (\alpha, b, a, \beta) \in \text{E}_n \), put \( v(\omega) = r(b)(= r(\beta)) \in V \) and
\( E_\omega = E_{v(\omega)} (= \rho_\alpha^B(\rho_\alpha^A(1)) = \rho_\alpha^A(\rho_\alpha^B(1))). \)

Then the Hilbert \( C^* \)-quad module \( \mathcal{H}_\kappa \) is written as \( \mathcal{H}_\kappa^{A,B} \) and regarded as
\( \mathcal{H}_\kappa^{A,B} = \sum_{\omega \in \text{E}_n} \oplus C E_\omega. \)

For \( \xi = \sum_{\omega \in \text{E}_n} \oplus \xi(\omega)E_{v(\omega)}, \xi' = \sum_{\omega \in \text{E}_n} \oplus \xi'(\omega)E_{v(\omega)} \in \mathcal{H}_\kappa^{A,B} \) and \( y = \sum_{v \in V} y(v)E_v \in \text{A} \) with \( \xi(\omega), \xi'(\omega) \) and \( y(v) \in \text{C}, \) we see

0. The right \( \text{A} \)-module structure and the right \( \text{A} \)-valued inner product are written as follows:
\[
\xi \varphi_\alpha(y) = \sum_{\omega \in \text{E}_n} \oplus \xi(\omega)E_{v(\omega)} y(v(\omega)), \quad (\xi | \xi')_\alpha = \sum_{\omega \in \text{E}_n} \overline{\xi(\omega)}\xi'(\omega)E_{v(\omega)}.
\]

Furthermore for \( w = \sum_{\alpha \in \text{E}_n} w(\alpha)S_{\alpha} E_{r(\alpha)} S_{\alpha}^* \), \( z = \sum_{a \in \text{B}} z(a)T_{a} E_{r(\alpha)} T_{a}^* \) with \( w(\alpha), z(a) \in \text{C}, \)

1. The right \( \text{B}_A \)-action \( \varphi_\alpha \) and the right \( \text{B}_B \)-action \( \varphi_B \) are written as follows:
\[
\xi \varphi_\alpha(w) = \sum_{\omega \in \text{E}_n} \oplus \xi(\omega)E_{v(\omega)} \varphi_\alpha(w) = \sum_{\omega \in \text{E}_n} \oplus \xi(\omega)w(b(\omega))E_{v(\omega)},
\]
\[
\xi \varphi_B(z) = \sum_{\omega \in \text{E}_n} \oplus \xi(\omega)E_{r(\omega)} \varphi_B(z) = \sum_{\omega \in \text{E}_n} \oplus \xi(\omega)z(r(\omega))E_{v(\omega)}.
\]

2. The left \( \text{B}_A \)-action \( \phi_\alpha \) and the left \( \text{B}_B \)-action \( \phi_B \) are written as follows:
\[
\phi_\alpha(w)\xi = \sum_{\omega \in \text{E}_n} \oplus \xi(\omega)E_{v(\omega)} \phi_\alpha(w)
\]
\[
= \sum_{\omega \in \text{E}_n} \oplus \xi(\omega)E_{v(\omega)} \rho_\alpha^B(w(t(\omega))E_{v(\omega)}(t(\omega)))
\]
\[
= \sum_{\omega \in \text{E}_n} \oplus \xi(\omega)w(t(\omega))E_{v(\omega)} \rho_\alpha^B(E_{r(\omega)}(t(\omega))).
\]
As \( \rho^B_{(l(\omega))}(E_{r(l(\omega))}) = E_{v(\omega)} \), we have

\[
\phi_A(w)\xi = \sum_{\omega \in E_A} \xi(\omega)w(t(\omega))E_{v(\omega)}.
\]

We also have

\[
\phi_B(z)\xi = \sum_{\omega \in E_A} \xi(\omega)E_{v(\omega)}\phi_B(z) = \sum_{\omega \in E_A} \xi(\omega)E_{v(\omega)}\rho^A_{(l(\omega))}(z(l(\omega))) = \sum_{\omega \in E_A} \xi(\omega)z(l(\omega))E_{v(\omega)}\rho^A_{(l(\omega))}(E_{r(l(\omega))}).
\]

As \( \rho^A_{b(\omega)}(E_{r(l(\omega))}) = E_{v(\omega)} \), we have

\[
\phi_B(z)\xi = \sum_{\omega \in E_A} \xi(\omega)z(l(\omega))E_{v(\omega)}.
\]

3. The right \( B_A \)-valued inner product \( \langle \cdot , \cdot \rangle_A \) and the right \( B_B \)-valued inner product \( \langle \cdot , \cdot \rangle_B \) are written as follows:

\[
\langle \xi | \xi' \rangle_A = \sum_{\omega \in E_A} S_{b(\omega)}\xi(\omega)E_{v(\omega)}\xi'(\omega)S^*_{b(\omega)}, \quad \langle \xi | \xi' \rangle_B = \sum_{\omega \in E_A} T_{r(\omega)}\xi(\omega)E_{v(\omega)}\xi'(\omega)T^*_{r(\omega)}.
\]

As \( E_{v(\omega)} = S^*_{b(\omega)}S_{b(\omega)} = T^*_{r(\omega)}T_{r(\omega)} \), we have

\[
\langle \xi | \xi' \rangle_A = \sum_{\omega \in E_A} \xi(\omega)\xi'(\omega)S_{b(\omega)}S^*_{b(\omega)}, \quad \langle \xi | \xi' \rangle_B = \sum_{\omega \in E_A} \xi(\omega)\xi'(\omega)T_{r(\omega)}T^*_{r(\omega)}.
\]

4. The positive maps \( \lambda_A : B_A \rightarrow A \) and \( \lambda_B : B_B \rightarrow A \) are written as follows:

\[
\lambda_A(w) = \sum_{\alpha \in E_A} w(\alpha)E_{r(\alpha)}, \quad \lambda_B(z) = \sum_{\alpha \in E_B} z(\alpha)E_{r(\alpha)}.
\]

Hence we have

\[
\lambda_A(\langle \xi | \xi' \rangle_A) = \langle \xi | \xi' \rangle_A, \quad \lambda_B(\langle \xi | \xi' \rangle_B) = \langle \xi | \xi' \rangle_A.
\]

Put \( p_\alpha = S_{\alpha}E_{r(\alpha)}S^*_\alpha \) for \( \alpha \in E_A \) and \( q_\alpha = T_{\alpha}E_{r(\alpha)}T^*_\alpha \) for \( \alpha \in E_B \). Hence

\[
B_A = \sum_{\alpha \in E_A} C_{p_\alpha}, \quad B_B = \sum_{\alpha \in E_B} C_{q_\alpha}.
\]

By Lemma 7.2, we have

\[
\hat{\rho}^A_\alpha(w) = \sum_{\beta \in E_A} w(\kappa_\alpha(\beta))p_\beta \quad \text{for} \quad w = \sum_{\alpha \in E_A} w(\alpha)p_\alpha \in B_A, \quad (7.2)
\]

\[
\hat{\rho}^B_\alpha(z) = \sum_{\beta \in E_B} z(\kappa_\alpha(\beta))q_\beta \quad \text{for} \quad z = \sum_{\alpha \in E_B} z(\alpha)q_\alpha \in B_B. \quad (7.3)
\]

We define \( \kappa_B : E_A \times E_B \times E_A \rightarrow \{0, 1\} \) and \( \kappa_A : E_B \times E_A \times E_B \rightarrow \{0, 1\} \) by

\[
\kappa_B(\alpha, a, b) = \begin{cases} 1 & \text{if } \kappa_\alpha(b) = a, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \kappa_A(a, \alpha, b) = \begin{cases} 1 & \text{if } \kappa_\alpha(b) = a, \\ 0 & \text{otherwise}. \end{cases}
\]
We identify the $C^*$-algebra $\mathcal{O}_{\mathcal{H}_{\mathcal{A},\mathcal{B}}}$ with the universal $C^*$-algebra subject to the relation $(\mathcal{H}_{\mathcal{A},\mathcal{B}})$, and denote the generating partial isometries by $u_\alpha, \alpha \in E_A$ and $v_a, a \in E_B$. Since $\hat{\rho}_\alpha^A(w) = v_a^*wv_a$ and $\hat{\rho}_\beta^B(z) = u_a^*zu_a$ with (7.2) and (7.3), we have

**Lemma 7.3.**

\[ v_a^*p_a v_a = \sum_{\beta \in E_A} \kappa_B(\alpha, a, \beta) p_\beta, \quad u_\alpha^*q_a u_\alpha = \sum_{b \in E_B} \kappa_A(a, \alpha, b) q_b. \]

Define $|E_A| \times |E_A|$-matrix $A^E = [A^E(\alpha, \beta)]_{\alpha, \beta \in E_A}$ and $|E_B| \times |E_B|$-matrix $B^E = [B^E(a, b)]_{a, b \in E_B}$ by

\[ A^E(\alpha, \beta) = \begin{cases} 1 & \text{if } r(\alpha) = s(\beta), \\ 0 & \text{if } r(\alpha) \neq s(\beta), \end{cases} \quad B^E(a, b) = \begin{cases} 1 & \text{if } r(a) = s(b), \\ 0 & \text{if } r(a) \neq s(b), \end{cases} \]

respectively. We then have for $\delta \in E_A$,

\[ E_{r(\delta)} = \sum_{\beta \in E_A} S_\beta \hat{\rho}_\delta^A(E_{r(\delta)}) S_\beta^* = \sum_{\beta \in E_A} S_\beta \hat{A}(r(\delta), \beta, r(\beta)) E_{r(\beta)} S_\beta^* = \sum_{\beta \in E_A} A^E(\delta, \beta) p_\beta \]

and similarly for $d \in E_B$

\[ E_{r(d)} = \sum_{b \in E_B} B^E(d, b) q_b. \]

Since we know that

\[ \hat{\rho}_\alpha^A(p_\delta) = \begin{cases} E_{r(\delta)} & \text{if } \delta = \alpha, \\ 0 & \text{otherwise}, \end{cases} \quad \hat{\rho}_\alpha^A(q_d) = \begin{cases} E_{r(d)} & \text{if } d = \alpha, \\ 0 & \text{otherwise}, \end{cases} \]

and $\sum_{\delta \in E_A} p_\delta = \sum_{d \in E_B} q_d = 1$, we have

\[ u_\alpha^*u_\alpha = \sum_{\beta \in E_A} A^E(\alpha, \beta) p_\beta, \quad v_a^*v_a = \sum_{b \in E_B} B^E(a, b) q_b. \]

Therefore we have

**Proposition 7.4.** The $C^*$-algebra $\mathcal{O}_{\mathcal{H}_{\mathcal{A},\mathcal{B}}}$ is *-isomorphic to the universal $C^*$-algebra generated by two families of projections $\{p_\alpha\}_{\alpha \in E_A}$, $\{q_a\}_{a \in E_B}$ and two families of partial isometries $\{u_\alpha\}_{\alpha \in E_A}$, $\{v_a\}_{a \in E_B}$ subject to the relations:

\[ \sum_{\beta \in E_A} p_\beta = \sum_{b \in E_B} q_b = \sum_{\beta \in E_A} u_\beta u_\beta^* + \sum_{b \in E_B} v_b v_b^* = 1, \quad (7.4) \]

\[ u_\alpha u_\alpha^* p_\alpha = u_\alpha u_\alpha^*, \quad v_a v_a^* q_a = v_a v_a^*, \quad (7.5) \]

\[ u_\alpha u_\alpha^* q_a = q_a u_\alpha u_\alpha^*, \quad v_a v_a^* p_\alpha = p_\alpha v_a v_a^*, \quad (7.6) \]

\[ u_\alpha^* u_\alpha = \sum_{\beta \in E_A} A^E(\alpha, \beta) p_\beta, \quad v_a^* v_a = \sum_{b \in E_B} B^E(a, b) q_b, \quad (7.7) \]

\[ u_\alpha^* q_a u_\alpha = \sum_{b \in E_B} \kappa_A(a, \alpha, b) q_b, \quad v_a^* p_\alpha v_a = \sum_{\beta \in E_A} \kappa_B(a, \alpha, \beta) p_\beta \quad (7.8) \]
for \( \alpha \in E_A \) and \( a \in E_B \) where

\[
\kappa_A(\alpha, a, b) = \begin{cases} 
1 & \text{if } \kappa(a, b) = (a, \beta) \text{ for some } \beta \in E_A, \\
0 & \text{otherwise},
\end{cases}
\]

\[
\kappa_B(\alpha, a, \beta) = \begin{cases} 
1 & \text{if } \kappa(a, b) = (a, \beta) \text{ for some } b \in E_B, \\
0 & \text{otherwise}.
\end{cases}
\]

**Proof.** The relations (7.4), (7.5) and (7.7) imply

\[
u_a^* q_d v_a = \begin{cases} 
\sum_{b \in E_B} B^E(a, b) q_b & \text{if } d = a, \\
0 & \text{otherwise}.
\end{cases}
\]

The above two relations are equivalent to (5.5) and (5.6). Since the two \( C^* \)-algebras \( B \) and \( B \) are generated by the projections \( \{p_a\}_{\alpha \in E_A} \) and \( \{q_a\}_{a \in E_B} \) respectively, we see that the relations (7.4), (7.5), (7.6), (7.7) and (7.8) are equivalent to the relations \( (H_{k_i}^{A, B}) \).

We will further study the above operator relations.

**Lemma 7.5.** \( p_a \) commutes with \( q_a \) for all \( \alpha \in E_A, a \in E_B \).

**Proof.** For \( \alpha \in E_A, a \in E_B \), by (7.4), we have

\[
p_a = \sum_{\beta \in E_A} u_{\beta} u_{\beta}^* p_a + \sum_{b \in E_B} v_b v_b^* p_a.
\]

By (7.5), we have

\[
u_a^* q_d v_a = \begin{cases} 
u_a^* u_{a}^* & \text{if } d = a, \\
0 & \text{otherwise}.
\end{cases}
\]

so that \( p_a = u_{\alpha} u_{\alpha}^* + \sum_{b \in E_B} v_b v_b^* p_a \) and hence \( q_a p_a = q_a u_{\alpha} u_{\alpha}^* + v_a v_a^* p_a \). Since \( q_a \) commutes with \( u_{\alpha} u_{\alpha}^* \) and \( p_a \) commutes with \( v_a v_a^* \), we have

\[
q_a p_a = u_{\alpha} u_{\alpha}^* q_a + p_a v_a v_a^*.
\]

which is symmetrically equal to \( p_a q_a \).

Put for \( \alpha, \beta \in E_A \) and \( a, b \in E_B \)

\[
k_{AB}(\alpha, b) = \begin{cases} 
1 & \text{if } \kappa(a, b) = (a, \beta), \\
0 & \text{otherwise},
\end{cases}
\]

\[
k_{BA}(\alpha, \beta) = \begin{cases} 
1 & \text{if } \kappa(a, b) = (a, \beta), \\
0 & \text{otherwise}.
\end{cases}
\]

**Lemma 7.6.** For \( \alpha \in E_A, a \in E_B \), we have

\[
u_a^* u_{\alpha} = \sum_{b \in E_B} k_{AB}(\alpha, b) q_b,
\]

\[
u_a^* v_a = \sum_{\delta \in E_A} k_{BA}(\alpha, \delta) p_\delta.
\]

**Proof.** By (7.4) and (7.8) and the equality \( \sum_{a \in E_B} \kappa_A(a, \alpha, b) = \kappa_{AB}(\alpha, b) \), we have

\[
u_a^* u_a = \sum_{a \in E_B} u_a^* q_a u_a = \sum_{\alpha \in E_A} \sum_{b \in E_B} \kappa_A(a, \alpha, b) q_b = \sum_{b \in E_B} \kappa_{AB}(\alpha, b) q_b.
\]

The equality for \( v_a^* v_a \) is similarly shown.
Lemma 7.7. For $\alpha \in E_A$, $a \in E_B$, if $r(\alpha) = r(a)$, then $u^*_\alpha u_a = v^*_a v_a$.

Proof. The condition $r(\alpha) = r(a)$ implies $\sum_{b \in E_B} \kappa_{AB}(\alpha, b) q_b = \sum_{b \in E_B} B^E(a, b) q_b$. By the equality for $u^*_\alpha u_a$ in the preceding lemma and the equality for $v^*_a v_a$ in (7.7), we have $u^*_\alpha u_a = v^*_a v_a$. □

Put

$$\Omega_\kappa = \{(\alpha, a) \in E_A \times E_B | s(\alpha) = s(a), \kappa(\alpha, b) = (a, \beta) \text{ for some } \beta \in E_A, b \in E_B\}$$

and $e_{\alpha, a} = p_\alpha q_a$ for $(\alpha, a) \in \Omega_\kappa$.

Lemma 7.8. 

(i) $\sum_{(\alpha, a) \in \Omega_\kappa} e_{(\alpha, a)} = 1$.

(ii) The $C^*$-subalgebra $B_\kappa$ of $\mathcal{O}_{\Omega_\kappa}$ generated by the subalgebras $B_\rho$ and $B_\eta$ is $*$-isomorphic to the direct sum $\bigoplus_{(\alpha, a) \in \Omega_\kappa} \mathcal{C} e_{(\alpha, a)}$. It is the $C^*$-algebra of all complex valued continuous functions on $\Omega_\kappa$.

Proof. By the equality (7.9), one sees that $p_\alpha q_a \neq 0$ if and only if $u^*_\alpha q_u u_\alpha \neq 0$ or $v^*_a v_a \neq 0$. The latter condition is equivalent to the condition that there exist $b \in E_B$ such that $\kappa_\alpha(a, b) \neq 0$ or there exists $\beta \in E_A$ such that $\kappa_B(a, b, \beta) \neq 0$, which is also equivalent to the condition that there exist $b \in E_B$ and $\beta \in E_A$ such that $\kappa(\alpha, b) = (a, \beta)$. Hence we have $p_\alpha q_a \neq 0$ if and only if $(\alpha, a) \in \Omega_\kappa$. Therefore we have

$$\sum_{(\alpha, a) \in \Omega_\kappa} e_{(\alpha, a)} = \left( \sum_{\alpha \in E_A} p_{\alpha} \right) \cdot \left( \sum_{a \in E_B} q_{a} \right) = 1.$$

As $p_\alpha q_a \cdot p_{\alpha'} q_{a'} = 0$ if $\alpha \neq \alpha'$ or $a \neq a'$, the $C^*$-algebra $B_\kappa$ is $*$-isomorphic to $\bigoplus_{(\alpha, a) \in \Omega_\kappa} \mathcal{C} e_{(\alpha, a)}$. □

We define two $|\Omega_\kappa| \times |\Omega_\kappa|$-matrices $A_\kappa$ and $B_\kappa$ with entries in $\{0, 1\}$ by

$$A_\kappa((\alpha, a), (\delta, b)) = \begin{cases} 1 & \text{if there exists } \beta \in E_A \text{ such that } \kappa(\alpha, b) = (a, \beta), \\ 0 & \text{otherwise} \end{cases}$$

for $(\alpha, a), (\delta, b) \in \Omega_\kappa$, and

$$B_\kappa((\alpha, a), (\beta, d)) = \begin{cases} 1 & \text{if there exists } b \in E_B \text{ such that } \kappa(\alpha, b) = (a, \beta), \\ 0 & \text{otherwise} \end{cases}$$

for $(\alpha, a), (\beta, d) \in \Omega_\kappa$ respectively. They represent the concatenations of edges as in the following figures respectively:

$$\begin{array}{ccc}
\circ & \overset{\alpha}{\rightarrow} & \circ \\
\downarrow a & & \downarrow b \\
\circ & \overset{\beta}{\rightarrow} & \circ \\
\end{array} \quad \text{and} \quad \begin{array}{ccc}
\circ & \overset{\alpha}{\rightarrow} & \circ \\
\downarrow a & & \downarrow b \\
\circ & \overset{\beta}{\rightarrow} & \circ \\
\end{array} \begin{array}{ccc}
\circ & \overset{\alpha}{\rightarrow} & \circ \\
\downarrow d & & \downarrow \circ \\
\circ & \overset{\beta}{\rightarrow} & \circ \\
\end{array}$$

Proposition 7.9. The $C^*$-algebra $\mathcal{O}_{\mathcal{H}^A_{\kappa}, B}$ is $*$-isomorphic to the universal $C^*$-algebra generated by a family $\{e_{(\alpha, a)}(a, a) \}_{(\alpha, a) \in \Omega_\kappa}$ of projections and two families of
partial isometries \( \{u_\alpha\}_{\alpha \in E_A}, \{v_\alpha\}_{\alpha \in E_B} \) subject to the relations:

\[
\sum_{(\alpha, a) \in \Omega_\kappa} e_{(\alpha, a)} = \sum_{\beta \in E_A} u_\beta u_\beta^* + \sum_{b \in E_B} v_b v_b^* = 1, \tag{7.10}
\]

\[
u_\alpha u_\alpha^* = \sum_{a \in E_A} u_\alpha u_\alpha^* e_{(\alpha, a)} = \sum_{a \in E_A} e_{(\alpha, a)} u_\alpha u_\alpha^*, \tag{7.11}
\]

\[
v_a v_a^* = \sum_{a \in E_A} v_a v_a^* e_{(\alpha, a)} = \sum_{a \in E_A} e_{(\alpha, a)} v_a v_a^*, \tag{7.12}
\]

\[
u_\alpha^* e_{(\alpha, a)} u_\alpha = \sum_{(\delta, b) \in \Omega_\kappa} A_\kappa((\alpha, a), (\delta, b)) e_{(\delta, b)}, \tag{7.13}
\]

\[
v_a^* e_{(\alpha, a)} v_a = \sum_{(\beta, d) \in \Omega_\kappa} B_\kappa((\alpha, a), (\beta, d)) e_{(\beta, d)} \tag{7.14}
\]

for \( \alpha \in E_A \) and \( a \in E_B \).

**Proof.** Let \( u_\alpha, \alpha \in E_A \) and \( v_\alpha, a \in E_B \) be the partial isometries as in Proposition 7.4. The equalities (7.11) and (7.12) follow from the equalities (7.4) and (7.5), (7.6). As \( u_\alpha^* p_\alpha u_\alpha = u_\alpha u_\alpha^* \) by (7.5), we have by (7.8)

\[
u_\alpha^* e_{(\alpha, a)} u_\alpha = u_\alpha^* q_\alpha u_\alpha = \sum_{b \in E_B} \kappa_\alpha(a, \alpha, b) q_b
\]

\[
= \sum_{b \in E_B} \kappa_\alpha(a, \alpha, b) \sum_{b \in E_A} p_b q_b
\]

\[
= \sum_{(\delta, b) \in \Omega_\kappa} A_\kappa((\alpha, a), (\delta, b)) e_{(\delta, b)}.
\]

The equality (7.14) is similarly shown. Hence the equalities (7.10), \ldots, (7.14) follow from the equalities (7.4), \ldots, (7.8). Conversely, from the projections \( e_{(\alpha, a)}, (\alpha, a) \in \Omega_\kappa \) by putting

\[
p_\alpha = \sum_{a \in E_B} e_{(\alpha, a)}, \quad q_\alpha = \sum_{a \in E_A} e_{(\alpha, a)}
\]

the equalities (7.4), \ldots, (7.8) follow from the equalities (7.10), \ldots, (7.14). \( \Box \)

We then see the following theorem:

**Theorem 7.10.** The \( C^*\)-algebra \( \mathcal{O}_{\mathcal{H}_\kappa A, B} \) associated with the Hilbert \( C^*\)-quad module \( \mathcal{H}_\kappa A, B \) defined by commuting matrices \( A, B \) and a specification \( \kappa \) is generated by partial isometries \( S_{(\alpha, a)}, T_{(\alpha, a)} \) for \( (\alpha, a) \in \Omega_\kappa \) satisfying the relations:

\[
\sum_{(\delta, b) \in \Omega_\kappa} S_{(\delta, b)} S_{(\delta, b)}^* + \sum_{(\beta, d) \in \Omega_\kappa} T_{(\beta, d)} T_{(\beta, d)}^* = 1,
\]

\[
S_{(\alpha, a)}^* S_{(\alpha, a)} = \sum_{(\delta, b) \in \Omega_\kappa} A_\kappa((\alpha, a), (\delta, b)) S_{(\delta, b)} S_{(\delta, b)}^* + T_{(\delta, b)} T_{(\delta, b)}^*,
\]

\[
T_{(\alpha, a)}^* T_{(\alpha, a)} = \sum_{(\beta, d) \in \Omega_\kappa} B_\kappa((\alpha, a), (\beta, d)) S_{(\beta, d)} S_{(\beta, d)}^* + T_{(\beta, d)} T_{(\beta, d)}^*
\]

for \( (\alpha, a) \in \Omega_\kappa \).
Proof. The algebra $\mathcal{O}_{\mathcal{H}^A,a}$ is generated by $u_\alpha, \alpha \in E_A, v_\alpha, \alpha \in E_A$ and $e_{(\alpha,a)}, (\alpha,a) \in \Omega_\kappa$ as in the preceding proposition. For $(\alpha,a) \in \Omega_\kappa$, put
\[
S_{(\alpha,a)} = e_{(\alpha,a)}u_\alpha, \quad T_{(\alpha,a)} = e_{(\alpha,a)}v_\alpha.
\]
(7.15)
Denote by $C^*(S_{(\alpha,a)}, T_{(\alpha,a)} : (\alpha,a) \in \Omega_\kappa)$ the $C^*$-subalgebra of $\mathcal{O}_{\mathcal{H}^A,a}$ generated by elements $S_{(\alpha,a)}, T_{(\alpha,a)}, (\alpha,a) \in \Omega_\kappa$. We have
\[
S^*_{(\alpha,a)}S_{(\alpha,a)} = u_\alpha^* e_{(\alpha,a)}u_\alpha = \sum_{(\delta,b)\in \Omega_\kappa} A_\kappa((\alpha,a), (\delta,b))e_{(\delta,b)}.
\]
As $e_{(\alpha,a)}u_\beta = 0$ for $\beta \neq \alpha$, and $e_{(\alpha,a)}v_b = 0$ for $b \neq a$, we have
\[
e_{(\alpha,a)} = \sum_{\beta \in E_A} e_{(\alpha,a)}u_\beta u_\beta^* e_{(\alpha,a)} + \sum_{b \in E_B} e_{(\alpha,a)}v_b v_b^* e_{(\alpha,a)} = S_{(\alpha,a)}S^*_{(\alpha,a)} + T_{(\alpha,a)}T^*_{(\alpha,a)}
\]
so that $e_{(\alpha,a)}$ belongs to the algebra $C^*(S_{(\alpha,a)}, T_{(\alpha,a)} : (\alpha,a) \in \Omega_\kappa)$ and the equality
\[
S^*_{(\alpha,a)}S_{(\alpha,a)} = \sum_{(\delta,b)\in \Omega_\kappa} A_\kappa((\alpha,a), (\delta,b))(S_{(\delta,b)}S^*_{(\delta,b)} + T_{(\delta,b)}T^*_{(\delta,b)})
\]
holds. Similarly we have
\[
T^*_{(\alpha,a)}T_{(\alpha,a)} = \sum_{(\beta,d)\in \Omega_\kappa} B_\kappa((\alpha,a), (\beta,d))(S_{(\beta,d)}S^*_{(\beta,d)} + T_{(\beta,d)}T^*_{(\beta,d)}).
\]
As $\sum_{(\alpha,a)\in \Omega_\kappa} e_{(\alpha,a)} = 1$ and $e_{(\alpha,a)}u_\beta = 0$ for $\beta \neq \alpha$, we have
\[
u_\alpha = \sum_{a \in \Sigma^2} e_{(\alpha,a)}u_\alpha = \sum_{(\alpha,a)\in \Omega_\kappa} S_{(\alpha,a)}
\]
since $u_\alpha$ and similarly $v_\alpha$ belong to the algebra $C^*(S_{(\alpha,a)}, T_{(\alpha,a)} : (\alpha,a) \in \Omega_\kappa)$. Therefore the $C^*$-algebra generated by $e_{(\alpha,a)}, u_\alpha, v_\alpha$ coincides with the subalgebra $C^*(S_{(\alpha,a)}, T_{(\alpha,a)} : (\alpha,a) \in \Omega_\kappa)$. 

Put $n = |\Omega_\kappa|$. Define a $2n \times 2n$-matrix $H_\kappa$ with entries in $\{0,1\}$ by the block matrix
\[
H_\kappa = \begin{bmatrix} A_\kappa & A_\kappa \\ B_\kappa & B_\kappa \end{bmatrix}.
\]
Denote by $I_n$ and $I_{2n}$ the identity matrices of size $n$ and of size $2n$ respectively.

Lemma 7.11.
(i) $\mathbb{Z}^{2n}/(H_\kappa - I_{2n})\mathbb{Z}^{2n} \cong \mathbb{Z}^n/(A_\kappa + B_\kappa - I_n)\mathbb{Z}^n$.
(ii) $\text{Ker}(H_\kappa - I_{2n})$ in $\mathbb{Z}^{2n} \cong \text{Ker}(A_\kappa + B_\kappa - I_n)$ in $\mathbb{Z}^n$.

Proof. (i) Put a $2n \times 2n$ block matrix $\bar{H}_\kappa = \begin{bmatrix} A_\kappa & I_n \\ B_\kappa & 0 \end{bmatrix}$. Then we easily see
\[
\mathbb{Z}^{2n}/(H_\kappa - I_{2n})\mathbb{Z}^{2n} \cong \mathbb{Z}^{2n}/(\bar{H}_\kappa - I_{2n})\mathbb{Z}^{2n}.
\]

Define a map
\[
\Psi : (x_1, \ldots, x_n, y_1, \ldots, y_n) \in \mathbb{Z}^{2n} \rightarrow (x_1 + y_1, \ldots, x_n + y_n) \in \mathbb{Z}^n
\]
which is a surjective homomorphism of abelian groups from $\mathbb{Z}^{2n}$ to $\mathbb{Z}^n$. Since we know $\Psi((\tilde{H}_\kappa - I_{2n})\mathbb{Z}^{2n}) = (A_\kappa + B_\kappa - I_n)\mathbb{Z}^n$, the homomorphism $\Psi: \mathbb{Z}^{2n} \rightarrow \mathbb{Z}^n$ induces an isomorphism from $\mathbb{Z}^{2n}/(\tilde{H}_\kappa - I_{2n})\mathbb{Z}^{2n}$ to $\mathbb{Z}^n/(A_\kappa + B_\kappa - I_n)\mathbb{Z}^n$. Therefore $\mathbb{Z}^{2n}/(H_\kappa - I_{2n})\mathbb{Z}^{2n}$ is isomorphic to $\mathbb{Z}^n/(A_\kappa + B_\kappa - I_n)\mathbb{Z}^n$.

(ii) The groups Ker$(H_\kappa - I_{2n})$ in $\mathbb{Z}^{2n}$ and Ker$(A_\kappa + B_\kappa - I_n)$ in $\mathbb{Z}^n$ are the torsion free part of $\mathbb{Z}^{2n}/(H_\kappa - I_{2n})\mathbb{Z}^{2n}$ and that of $\mathbb{Z}^n/(A_\kappa + B_\kappa - I_n)\mathbb{Z}^n$ respectively, so that they are isomorphic to each other. □

Therefore we reach the following theorem.

**Theorem 7.12.** The $C^*$-algebra $\mathcal{O}_{H_\kappa^{A,B}}$ associated with the Hilbert $C^*$-quad module $\mathcal{H}_\kappa^{A,B}$ defined by commuting matrices $A, B$ and a specification $\kappa$ is isomorphic to the Cuntz-Krieger algebra $\mathcal{O}_{H_\kappa^{A,B}}$ for the matrix $H_\kappa^{A,B}$. Its $K$-groups $K_*(\mathcal{O}_{H_\kappa^{A,B}})$ are computed as

$$K_0(\mathcal{O}_{H_\kappa^{A,B}}) = \mathbb{Z}^n/(A_\kappa + B_\kappa - I_n)\mathbb{Z}^n,$$

$$K_1(\mathcal{O}_{H_\kappa^{A,B}}) = \text{Ker}(A_\kappa + B_\kappa - I_n) \text{ in } \mathbb{Z}^n,$$

where $n = |\Omega_\kappa|$.

We will finally present a concrete example. For $1 < N, M \in \mathbb{N}$, let $A$ and $B$ be the $1 \times 1$ matrices $[N]$ and $[M]$ respectively. The directed graph $G_A$ associated to the matrix $A = [N]$ is a graph consists of a vertex denoted by $v$ with $N$-self directed loops denoted by $E_A$. Similarly the directed graph $G_B$ consists of the vertex $v$ with $M$-self directed loops denoted by $E_B$. We fix a specification $\kappa : E_A \times E_B \rightarrow E_B \times E_A$ defined by exchanging $\kappa((\alpha, a), (\beta, b)) = (a, \alpha)$ for $(\alpha, a) \in E_A \times E_B$. Hence $\Omega_\kappa = E_A \times E_B$ so that $|\Omega_\kappa| = |E_A| \times |E_B| = N \times M$. We then know $\kappa_\alpha((\alpha, a), (\beta, b)) = 1$ if and only if $b = a$. And $\kappa_B((\alpha, a), (\beta, b)) = 1$ if and only if $\beta = \alpha$ as in the following figures respectively.

$$\begin{align*}
o & \rightarrow ^\alpha \rightarrow o & \delta \\
a & \downarrow _{a=b} & ^a \downarrow \\
\end{align*}$$

and

$$\begin{align*}
o & \rightarrow ^\alpha \rightarrow o & \rightarrow ^\alpha = ^\beta \rightarrow o \\
\end{align*}$$

In particular, for the case $N = 2$ and $M = 3$, we write $E_A = \{1, 2\}, E_B = \{1, 2, 3\}$ and $\Omega_\kappa$ as

$$\Omega_\kappa = E_A \times E_B = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3)\}.$$

The $6 \times 6$ matrices $A_\kappa$ and $B_\kappa$ are written along the above ordered basis in order as:

$$A_\kappa = \begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
\end{bmatrix} \quad \text{and} \quad B_\kappa = \begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 \\
\end{bmatrix}.$$
respectively so that we have

\[ A_\kappa + B_\kappa - I = \begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1
\end{bmatrix}. \]

It is easy to see that

\[ \mathbb{Z}^6/(A_\kappa + B_\kappa - I) \mathbb{Z}^6 \cong \mathbb{Z}/8\mathbb{Z}, \quad \text{Ker}(A_\kappa + B_\kappa - I) \text{ in } \mathbb{Z}^6 \cong \{0\}. \]

Therefore the \( C^* \)-algebra \( \mathcal{O}_{H_{A,B}} \) for \( A = [2], B = [3] \) and \( \kappa = \text{exchange} \) is a Cuntz-Krieger algebra stably isomorphic to the Cuntz algebra \( \mathcal{O}_9 \), whereas the \( C^* \)-algebra \( \mathcal{O}_{[2],[3]} \) considered in [23] is isomorphic \( \mathcal{O}_2 \otimes \mathcal{O}_3 \) which is isomorphic to \( \mathcal{O}_2 \). We will further study these \( C^* \)-algebras \( \mathcal{O}_{H_{A,B}} \) in a forthcoming paper.

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