Lower Bound for Randomized First Order Convex Optimization

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Abstract

We provide an explicit construction and direct proof for the lower bound on the number of first order oracle accesses required for a randomized algorithm to minimize a convex Lipschitz function.

1 Introduction

We prove lower bounds for the complexity of first-order optimization using a randomized algorithm for the following problem:

$$\min_{x \in \mathbb{R}^d \mid \|x\| \leq B} f(x)$$

(1)

where $f$ is convex and $L$-Lipschitz continuous with respect to the Euclidean norm. We consider a standard oracle access model: at each iteration the algorithm selects, possibly at random, a vector $x \in \mathbb{R}^d, \|x\| \leq B$ based on the oracle’s responses to previous queries. The oracle then returns the function value $f(x)$ and some subgradient $g \in \partial f(x)$ chosen by the oracle. We bound the expected number of iterations $T$, as a function of $L, B$ and $\epsilon$, needed to ensure that for any convex $L$-Lipschitz function, any valid first order oracle, and any dimension, $f(x_T) \leq \min_{x \in \mathbb{R}^d \mid \|x\| \leq B} f(x) + \epsilon$. We are interested in lower bounding dimension-independent performance (i.e. what an algorithm can guarantee in an arbitrary dimension) and in our constructions we allow the dimension to grow as $\epsilon \to 0$.

Nemirovsky and Yudin \[2\] carefully study both randomized and deterministic first order optimization algorithms and give matching upper and lower bounds for both, establishing a tight worst-case complexity for (1), whether using randomized or deterministic algorithms, of $\Theta(L^2B^2/\epsilon^2)$ oracle queries.

The lower bound for deterministic algorithms is fairly direct, well-known and has been reproduced in many forms in books, tutorials, and lecture notes in the ensuing four decades, as are the lower bounds for algorithms (whether randomized or deterministic) where the iterates $x_t$ are constrained to be in the span of previous oracle responses. When the iterates are constrained to be in this span, one can ensure the first $t + 1$ iterates are spanned by the first $t$ standard basis vectors $e_1, \ldots, e_t$, and that no point in this span can be $O(1/\sqrt{t})$-suboptimal. For deterministic algorithms, even if the iterates escape this span, one can adversarially rotate the objective function so that the algorithm only escapes in useless directions, obtaining the exact same lower bound using a very similar construction. Either way, a dimensionality of $d = \Theta(T) = \Theta(L^2B^2/\epsilon^2)$ is sufficient to construct a function requiring $\Theta(L^2B^2/\epsilon^2)$ queries to optimize.

Analyzing randomized algorithms which are allowed to leave the span of oracle responses is trickier: the algorithm may guess directions, and since even if we know the algorithm, we do not know in advance which directions it will guess, we cannot rotate the function so as to avoid these directions. Nemirovsky and Yudin \[2\] do provide a detailed and careful analysis for such randomized algorithms, using a recursive reduction argument and without a direct construction. To the best of our knowledge, this lower bound has not since been simplified, and so lower bounds for randomized algorithms are rarely ever covered in books, tutorials and courses. In this note, we provide an explicit construction establishing the following lower bound:
Theorem 1. For any $L, B, \epsilon \in (0, \frac{L}{B})$, dimension $d \geq \frac{2 L^2 B^2}{\epsilon^2} \log \frac{L B^4}{\epsilon}$, and any randomized optimization algorithm, there exists a convex $L$-Lipschitz function $f : \{x \in \mathbb{R}^d : \|x\| \leq B\} \rightarrow \mathbb{R}$ and an appropriate first order oracle such that the algorithm must make $\Omega(L^2 B^2/\epsilon^2)$ queries to the oracle in expectation in order to find an $\epsilon$-suboptimal point.

Our construction and proof directly captures the following intuition: if the dimension is large enough, blindly guessing a direction becomes increasingly difficult, and the algorithm should not gain much by such random guessing. In the standard construction used for the deterministic lower bound, guessing a direction actually does provide information on all useful directions. However, by slightly perturbing the standard construction, we are able to avoid such information leakage. To do so, we use a technique we recently developed in order to analyze finite sum structured optimization problems [3].

In this note we only consider Lipschitz (non-smooth) functions without an assumption of strong convexity. A reduction or simple modification to the construction can be used to establish a lower bound for smooth functions too. All of these lower bounds would match those for deterministic optimization, with a polynomial increase in the dimension required. This polynomial increase can likely be reduced to a smaller polynomial through more careful analysis.

Theorem 1 (which we reiterate also follows from the more detailed analysis of Nemirovsky and Yudin) shows that randomization cannot help first order optimization. It is important to emphasize that this should not be taken for granted, and that in other situations randomization could be beneficial. For example, when optimizing finite sum structured objectives, randomization provably reduces the oracle complexity [3]. It is thus important to specifically and carefully consider randomized algorithms when proving oracle lower bounds, and we hope this note will aid in such analysis.

We would like to thank the authors of [1] Yair Carmon, John Duchi, Oliver Hinder, and Aaron Sidford for pointing out a mistake with our original proof of Lemma 3 which has since been corrected.

## 2 Proof of Theorem 1

Without loss of generality assume $L = B = 1$. Consider a family of functions $F$ of the form

$$f(x) = \max_{1 \leq j \leq k} (\langle x, v_j \rangle - jc)$$

where $k = \frac{1}{\sqrt{d}}, c = \frac{\epsilon}{2},$ and the vectors $v_j$ are an orthonormal set in $\mathbb{R}^d$. Each of these functions is the maximum of linear functions thus convex and 1-Lipschitz. Drawing the orthonormal set of vectors $v_j$ uniformly at random specifies a distribution over the family of functions $F$. Our approach will be to show that any deterministic optimization algorithm must make at least $\Omega(1/\epsilon^2)$ oracle queries in expectation over the randomness in the choice of $f$. This implies through Yao’s minimax principle a lower bound on the expected number of queries needed by a randomized algorithm on the worst-case function in $F$. Therefore, for the remainder of the proof we need only consider deterministic optimization algorithms and functions drawn from this distribution over $F$.

First, we show that minimizing a given function $f$ amounts to finding a vector $x$ which has significant negative correlation with all of the vectors $v_j$. Consider the unit vector \( \hat{x} = -\frac{1}{\sqrt{k}} \sum_{j=1}^{k} v_j \)

$$f(\hat{x}) = \max_{1 \leq j \leq k} (\langle \hat{x}, v_j \rangle - jc) = -\frac{1}{k} - c = -2\epsilon - c \geq f(x^*)$$

Therefore, for any $x$ such that $\langle x, v_j \rangle > -\frac{c}{2}$ for some $j$, $f(x) \geq \langle x, v_j \rangle - jc \geq \frac{c}{2} - kc = \frac{c}{2} - \epsilon > f(\hat{x}) + \epsilon \geq f(x^*) + \epsilon$
Consequently, any such $x$ cannot be $\epsilon$-suboptimal. Therefore, in order to show that the expected number of oracle queries is $\Omega(k) = \Omega(1/\varepsilon^2)$, it suffices to show that the following event occurs with constant probability:

$$E = \left\{ \forall t \leq k \forall j \geq t \mid \langle x^{(t)}, v_j \rangle < \frac{\epsilon}{2} \right\}$$

(5)

Let $S_t = \text{span} \{x^{(1)}, ..., x^{(t)}, v_1, ..., v_t\}$ and let $S_t^\perp$ be its orthogonal complement. Let $P_t$ and $P_t^\perp$ be (orthogonal) projection operators onto $S_t$ and $S_t^\perp$ respectively. Consider the events

$$G_t = \left\{ \forall j \geq t \mid \left| \langle P_{t-1}^\perp x^{(t)}, v_j \rangle \right| < \frac{\epsilon}{2(\sqrt{2} + \sqrt{k-1})} \right\}$$

(6)

These events are useful because:

**Lemma 1.** $\bigcap_{t=1}^{k} G_t \implies E$

*Proof.* Let $G_{<t} = \bigcap_{t'=1}^{t-1} G_{t'}$. It suffices to show that for each $t \leq k$, $G_{<t} \implies \forall j \geq t \mid \langle x^{(t)}, v_j \rangle \mid < \frac{\epsilon}{2}$. For each $t \leq k$ and $j \geq t$

$$\left| \langle x^{(t)}, v_j \rangle \right| \leq \left\| x^{(t)} \right\| \left\| \left\langle \frac{x^{(t)}}{\|x^{(t)}\|}, P_{t-1} v_j \right\rangle \right\| + \left\| x^{(t)} \right\| \left\| \left\langle \frac{x^{(t)}}{\|x^{(t)}\|}, P_{t-1}^\perp v_j \right\rangle \right\|$$

$$\leq \|P_{t-1} v_j\| + \left\| \frac{P_{t-1} x^{(t)}}{\|x^{(t)}\|}, v_j \right\|$$

$$\leq \|P_{t-1} v_j\| + \left\| \frac{P_{t-1}^\perp x^{(t)}}{\|P_{t-1}^\perp x^{(t)}\|}, v_j \right\|$$

$$\leq \|P_{t-1} v_j\| + \frac{c}{2(\sqrt{2} + \sqrt{k-1})}$$

(7)

First, we decomposed $v_j$ into its $S_{t-1}$ and $S_{t-1}^\perp$ components and applied the triangle inequality. Next, we used that $\left\| x^{(t)} \right\| \leq 1$ and that the orthogonal projection operator $P_{t-1}^\perp$ is self-adjoint. Finally, we used that the projection operator is non-expansive and then applied the definition of $G_t$.

Next, we will prove by induction on $t$ that for all $t \leq k$ and all $j \geq t$, $G_{<t} \implies \|P_{t-1} v_j\|^2 \leq \frac{c^2(t-1)}{2(\sqrt{2} + \sqrt{k-1})^2}$. The case $t = 1$ is trivial since the left hand side is the projection of $v_j$ onto the empty set.

For the inductive step, fix any $t \leq k$ and $j \geq t$. Let $P_t$ project onto $\text{span} \{x^{(1)}, ..., x^{(t+1)}, v_1, ..., v_t\}$ (this includes $x^{(t+1)}$ in contrast with $P_1$) and let $P_t^\perp$ be the projection onto the orthogonal subspace. Since $\{x^{(1)}, ..., x^{(t-1)}, v_1, ..., v_{t-1}\}$ spans $S_{t-1}$, the Gram-Schmidt vectors

$$\begin{align*}
P_0^\perp x^{(1)} & \quad \hat{P}_0^\perp v_1 \\
P_0^\perp x^{(2)} & \quad \hat{P}_1^\perp v_2 \\
P_t^\perp x^{(t-1)} & \quad \hat{P}_{t-1}^\perp v_{t-1}
\end{align*}$$

(8)

are an orthonormal basis for $S_{t-1}$ (after ignoring any zero vectors that arise from projection).

We now write $\|P_{t-1} v_j\|$ in terms of this orthonormal basis:

$$\|P_{t-1} v_j\|^2 = \sum_{i=1}^{t-1} \left\langle \frac{P_{t-1}^\perp x^{(i)}}{\left\| P_{t-1}^\perp x^{(i)} \right\|}, v_j \right\rangle^2 + \sum_{i=1}^{t-1} \left\langle \frac{P_{t-1}^\perp v_i}{\left\| P_{t-1}^\perp v_i \right\|}, v_j \right\rangle^2$$

$$\leq \frac{c^2(t-1)}{4(\sqrt{2} + \sqrt{k-1})^2} + \sum_{i=1}^{t-1} \frac{1}{\left\| P_{t-1}^\perp v_i \right\|^2} \left\langle \hat{P}_{t-1}^\perp v_i, v_j \right\rangle^2$$

(9)
The inequality follows from the definition of $G_{<t}$. We must now bound the second term of (9). Focusing on the inner product one individual term in the sum

$$
\left| \left\langle \hat{P}_{t-1}^\perp v_i, v_j \right\rangle \right| = \left| \langle v_i, v_j \rangle - \left\langle \hat{P}_{t-1}^\perp v_i, v_j \right\rangle \right|
$$

$$
= \left| \langle P_{t-1}^\perp v_i, v_j \rangle + \left\langle \frac{P_{t-1}^\perp x^{(i)}}{\|P_{t-1}^\perp x^{(i)}\|} v_i \right\rangle \left\langle \frac{P_{t-1}^\perp x^{(i)}}{\|P_{t-1}^\perp x^{(i)}\|} v_j \right\rangle \right|
$$

$$
\leq \left| \langle P_{t-1}^\perp v_i, P_{t-1}^\perp v_j \rangle \right| + \left\langle \frac{P_{t-1}^\perp x^{(i)}}{\|P_{t-1}^\perp x^{(i)}\|} v_i \right\rangle \left\langle \frac{P_{t-1}^\perp x^{(i)}}{\|P_{t-1}^\perp x^{(i)}\|} v_j \right\rangle
$$

By the Cauchy-Schwarz inequality and the inductive hypothesis, the first term is bounded by $\| P_{t-1}^\perp v_i \| \| P_{t-1}^\perp v_j \| \leq \frac{c^2(i-1)}{2(\sqrt{2}+\sqrt{k-1})^2}$. By the definition of $G_{<t}$, the second term is bounded by $\frac{c^2}{4(\sqrt{2}+\sqrt{k-1})^2}$. Furthermore, by our choice of $c = \frac{1}{2}$ and $k = \frac{1}{2\sqrt{k}}$, we conclude that

$$
\left| \left\langle \hat{P}_{t-1}^\perp v_i, v_j \right\rangle \right| \leq \frac{c^2(2i-1)}{4(\sqrt{2}+\sqrt{k-1})^2}
$$

$$
= \frac{c}{4(\sqrt{2}+\sqrt{k-1})} \frac{2i-1}{2\sqrt{k}(\sqrt{2}+\sqrt{k-1})}
$$

$$
\leq \frac{c}{4(\sqrt{2}+\sqrt{k-1})} \frac{2i-1}{2k}
$$

$$
\leq \frac{c}{4(\sqrt{2}+\sqrt{k-1})}
$$

We have now upper bounded the inner products in (9), it remains to lower bound the norm in the denominator. Rewriting the projection $\hat{P}_{t-1}^\perp$ as in (10):

$$
\left\| \hat{P}_{t-1}^\perp v_i \right\|^2 = \left\| \hat{P}_{t-1}^\perp v_i, v_i \right\|
$$

$$
= 1 - \| P_{t-1}^\perp v_i \|^2 - \left\langle \frac{P_{t-1}^\perp x^{(i)}}{\|P_{t-1}^\perp x^{(i)}\|}, v_i \right\rangle^2
$$

$$
\geq 1 - \frac{c^2(i-1)}{2(\sqrt{2}+\sqrt{k-1})^2} - \frac{c^2}{4(\sqrt{2}+\sqrt{k-1})^2}
$$

This quantity is at least $\frac{1}{2}$ because $c = \frac{1}{2\sqrt{k}} < 1$ so

$$
\frac{c^2(i-1)}{2(\sqrt{2}+\sqrt{k-1})^2} + \frac{c^2}{4(\sqrt{2}+\sqrt{k-1})^2} \leq \frac{2i-1}{(\sqrt{2}+\sqrt{k-1})^2}
$$

$$
= \left( \frac{\sqrt{i-\frac{1}{2}}}{\sqrt{2}+\sqrt{k-1}} \right)^2
$$

$$
\leq \frac{1}{2} \left( \frac{\sqrt{i-\frac{1}{2}}}{k} \right)^2
$$

$$
\leq \frac{1}{2}
$$
Combining this with (11) and returning to (9) we have that
\[
\|P_{t-1}v_j\|^2 \leq \frac{c^2(t-1)}{4(\sqrt{2} + \sqrt{k-1})^2} + \sum_{i=1}^{t-1} \left( \frac{\hat{P}_{t-1}v_i}{\|\hat{P}_{t-1}v_i\|^2} \right)^2
\]
\[
\leq \frac{c^2(t-1)}{4(\sqrt{2} + \sqrt{k-1})^2} + \sum_{i=1}^{t-1} \frac{4c}{4(\sqrt{2} + \sqrt{k-1})}
\]
\[
= \frac{c^2(t-1)}{4(\sqrt{2} + \sqrt{k-1})^2} + \frac{4c^2(t-1)}{4(\sqrt{2} + \sqrt{k-1})^2} = \frac{c^2(t-1)}{2(\sqrt{2} + \sqrt{k-1})^2}
\]
which completes the inductive step. Finally, we return to (7) and conclude that
\[
\left| \langle x(t), v_j \rangle \right| \leq \|P_{t-1}v_j\| + \frac{c}{2(\sqrt{2} + \sqrt{k-1})}
\]
\[
\leq \sqrt{\frac{c^2(t-1)}{2(\sqrt{2} + \sqrt{k-1})^2} + \frac{c}{2(\sqrt{2} + \sqrt{k-1})}}
\]
\[
= \frac{c(\sqrt{2} + \sqrt{t-1})}{2(\sqrt{2} + \sqrt{k-1})} \leq \frac{c}{2}
\]
This completes the proof.

In our model of computation, the oracle can provide the algorithm with any subgradient at the query point. We can therefore design a resisting oracle which returns subgradients that are as uninformative as possible. At a given point \(x\), the subdifferential of \(f\) is
\[
\partial f(x) = \text{Conv} \left\{ v_\ell : \ell \in \arg \max_{1 \leq j \leq k} (\langle x, v_j \rangle - jc) \right\}
\]
and our resisting oracle will return as a subgradient
\[
v_\ell \quad \text{where} \quad \ell = \min \left\{ \arg \max_{1 \leq j \leq k} (\langle x, v_j \rangle - jc) \right\}
\]
That is, the returned subgradient will always be a single vector \(v_\ell\) for the smallest value of \(\ell\) that corresponds to a valid subgradient.

**Lemma 2.** For each \(t \leq k\), let \(g(t) \in \partial f(x(t))\) be the subgradient returned by the oracle. Then \(G_{<t} \implies g(t) \in \{v_1, ..., v_t\}\).

**Proof.** This follows from the structure of the objective function \(f\) and our choice of subgradient oracle. In the proof of Lemma 1 we established that \(G_{<t} \implies \forall j \geq t \left| \langle x(t), v_j \rangle \right| < \frac{c}{2} \). Thus for any \(j > t\)
\[
\langle x(t), v_j \rangle - tc > -\frac{c}{2} - tc = \frac{c}{2} - (t+1)c > \langle x(t), v_j \rangle - jc
\]
Therefore, no \(j > t\) can index a maximizing term in \(f\) so \(g(t) \subseteq \{v_1, ..., v_t\}\).

By Lemma 1 and the chain rule of probability
\[
\mathbb{P}[E] \geq \mathbb{P} \left[ \bigcap_{t=1}^{k} G_t \right] = \prod_{t=1}^{k} \mathbb{P}[G_t \mid G_{<t}]
\]
Focusing on a single term in the product:
Lemma 3. For any $t \leq k$, $\Pr \left[ G_t \mid G_{<t} \right] > 1 - (k - t + 1) \exp \left( \frac{-c^2 (d - 2t + 1)}{8(n + \sqrt{d} + k - 1)} \right)$

Proof. The key to lower bounding $\Pr \left[ G_t \mid G_{<t} \right]$ is to show that for $j \geq t$ the vector $\frac{p_{\perp}^{j-1} v_t}{\|p_{\perp}^{j-1} v_t\|}$ is uniformly distributed on the unit sphere in $S_{t-1}$ conditioned on $G_{<t}$ and $\{v_1, \ldots, v_{t-1}\}$. If we can show this, then the inner product in the definition of $G_t$ is effectively between a fixed vector and a random unit vector, and the probability that this is large decreases rapidly as the dimension grows.

Fix an arbitrary $t \leq k$ and $j \geq t$. Let $V_{<t} := \{v_1, \ldots, v_{t-1}\}$ be any set of orthonormal vectors in $\mathbb{R}^d$. We will show that the density $p_{V_{<t}} (V_{\geq t} \mid G_{<t}, V_{<t})$ is invariant under rotations which preserve $\{x^{(1)}, \ldots, x^{(t-1)}, v_1, \ldots, v_{t-1}\}$.

Let $R$ be any rotation $R^T = I_d \otimes I_d$ such that $\forall w \in \text{span} \{x^{(1)}, \ldots, x^{(t-1)}, v_1, \ldots, v_{t-1}\} R w = R^T w = w$. We will show that $p_{V_{<t}} (V_{\geq t} \mid G_{<t}, V_{<t}) = p_{V_{<t}} (RV_{\geq t} \mid G_{<t}, V_{<t})$. To begin

\[
p_{V_{<t}} (V_{\geq t} \mid G_{<t}, V_{<t}) = \frac{\Pr \left[ G_{<t} \mid V \right] p_{V} (V)}{\Pr \left[ G_{<t} \mid V_{<t} \right] p_{V_{<t}} (V_{<t})}
\]

and

\[
p_{V_{<t}} (RV_{\geq t} \mid G_{<t}, V_{<t}) = \frac{\Pr \left[ G_{<t} \mid V \right] p_{V} (RV)}{\Pr \left[ G_{<t} \mid V_{<t} \right] p_{V_{<t}} (V_{<t})}
\]

Since $V = \{v_1, \ldots, v_k\}$ is marginally distributed uniformly, $p_V (V) = p_V (RV)$, so it only remains to show that $\Pr \left[ G_{<t} \mid V \right] = \Pr \left[ G_{<t} \mid RV \right]$. Recall that at this time we are considering an arbitrary deterministic algorithm minimizing a randomly selected $f$. Thus for any particular $V$, which fixes $f$, either $G_{<t}$ holds or it does not—so the probabilities are either 0 or 1.

We will show by induction that for every $i < t$, if $\Pr \left[ G_{<t} \mid V \right] = 1$ then $\Pr \left[ G_{<t} \mid RV \right] = 1$ too. The case $i = 1$ is trivial since $G_{<1}$ is independent of $V$. Consider now some $1 < i < t$, and suppose that $\Pr \left[ G_{<i} \mid V \right] = 1$. Since $G_{<i} \implies G_{<s}$ for $s < i$, $\Pr \left[ G_{<s} \mid V \right] = 1$ for all $s \leq i$ and by the inductive hypothesis $\Pr \left[ G_{<s} \mid RV \right] = 1$ for all $s < i$. Thus, it just remains to show that $\Pr \left[ G_{i-1} \mid G_{<i-1}, RV \right] = 1$. Let $P_i$ be the projection operator onto $\{x^{(1)}, \ldots, x^{(i)}, v_1, \ldots, v_i\}$ where the $x^i$ are the oracle queries made by the algorithm when $f$ is determined by $RV$. For any $\ell \geq i - 1$, consider $\left\langle \frac{p_{\perp}^{\ell-1} x^{(i-1)}}{\|p_{\perp}^{\ell-1} x^{(i-1)}\|}, R v_i \right\rangle$.

Since $G_{<i-1}$ holds when $f$ is determined by $V$, by Lemma 2 the queries $\{x^{(1)}, \ldots, x^{(i-1)}\}$ are determined by $\{v_1, \ldots, v_{i-1}\}$. Since $G_{<i-1}$ holds when $f$ is determined by $RV$ and $\{v_1, \ldots, v_{i-2}\} = \{v_1, \ldots, v_{i-2}\}$, $x^{(i-1)} = x^{(i-1)}$. Furthermore, since $R$ preserves $\{x^{(1)}, \ldots, x^{(i-2)}, v_1, \ldots, v_{i-2}\}$, it is also the case that $p_{\perp}^{\ell-2} = p_{\perp}^{\ell-2}$. Finally, since $P_{\perp}^{\ell-2} x^{(i-1)} = x^{(i-1)} - P_{\perp}^{\ell-2} x^{(i-1)} \in \text{span} \{x^{(1)}, \ldots, x^{(i-1)}, v_1, \ldots, v_{i-2}\}$, it is unchanged by $R^T$, therefore

\[
\left\langle \frac{p_{\perp}^{\ell-1} x^{(i-1)}}{\|p_{\perp}^{\ell-1} x^{(i-1)}\|}, R v_i \right\rangle = \left\langle R^T \frac{p_{\perp}^{\ell-1} x^{(i-1)}}{\|p_{\perp}^{\ell-1} x^{(i-1)}\|}, v_i \right\rangle = \left\langle \frac{p_{\perp}^{\ell-1} x^{(i-1)}}{\|p_{\perp}^{\ell-1} x^{(i-1)}\|}, v_i \right\rangle \leq \frac{c}{2(\sqrt{2} + \sqrt{k - 1})}
\]

since $G_{i-1}$ holds when $f$ is determined by $V$. Therefore, we conclude that $p_{V_{<t}} (V_{\geq t} \mid G_{<t}, V_{<t})$ is invariant under rotations that preserve $\{x^{(1)}, \ldots, x^{(t-1)}, v_1, \ldots, v_{t-1}\}$.

For a given $j \geq t$, the marginal density of $v_j$ conditioned on $G_{<t}, V_{<t}$ is invariant under $R$. By Lemma 2 since the optimization algorithm is deterministic, the queries $\{x^{(1)}, \ldots, x^{(t)}\}$ are completely determined given $G_{<t}, G_{<t}, V_{<t}$, thus the projection $p_{\perp}^{j-1}$ is also determined by $G_{<t}, V_{<t}$. Therefore, the random vectors $\frac{p_{\perp}^{j-1} v_j}{\|p_{\perp}^{j-1} v_j\|}$ and $\frac{p_{\perp}^{j-1} R v_j}{\|p_{\perp}^{j-1} R v_j\|}$ have the same density. The rotation $R$ preserves $S_{t-1}$ and $R$ preserves length, so $\frac{p_{\perp}^{j-1} R v_j}{\|p_{\perp}^{j-1} R v_j\|} = R \left( \frac{p_{\perp}^{j-1} v_j}{\|p_{\perp}^{j-1} v_j\|} \right)$ and we conclude that the distribution of $\frac{p_{\perp}^{j-1} v_j}{\|p_{\perp}^{j-1} v_j\|}$ conditioned on $G_{<t}, V_{<t}$ is spherically symmetric on $S_{t-1}$. 

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We can now lower bound $\mathbb{P}[G_t \mid G_{<t}] = \mathbb{E}_{V_{<t}}[\mathbb{P}[G_t \mid G_{<t}, V_{<t}] \geq \inf_{V_{<t}} \mathbb{P}[G_t \mid G_{<t}, V_{<t}]]$. For any $V_{<t}$,

$$\mathbb{P}[G_t \mid G_{<t}, V_{<t}] = \mathbb{P} \left[ \forall j \geq t \left| \frac{P_{G_1}^{\perp} \langle x(t), v_j \rangle}{\left\| P_{G_1}^{\perp} x(t) \right\|} \right| > \frac{c}{2(\sqrt{2} + \sqrt{k-1})} \mid G_{<t}, V_{<t} \right]$$

$$\geq 1 - \sum_{j=t}^{k} \mathbb{P} \left[ \left| \frac{P_{G_1}^{\perp} \langle x(t), v_j \rangle}{\left\| P_{G_1}^{\perp} x(t) \right\|} \right| > \frac{c}{2(\sqrt{2} + \sqrt{k-1})} \mid G_{<t}, V_{<t} \right]$$

$$\geq 1 - \sum_{j=t}^{k} \mathbb{P} \left[ \left| \frac{P_{G_1}^{\perp} \langle x(t), v_j \rangle}{\left\| P_{G_1}^{\perp} x(t) \right\|} \right| > \frac{c}{2(\sqrt{2} + \sqrt{k-1})} \mid G_{<t}, V_{<t} \right]$$

The first term in the inner product is fixed given $G_{<t}, V_{<t}$, and we showed above that the second term is a unit vector that is distributed spherically symmetrically on the unit sphere in $S_{G_1}^{d-1}$ given $G_{<t}, V_{<t}$. Therefore, each probability is equal to $\mathbb{P} \left( \langle u, e_1 \rangle \geq \frac{c}{2(\sqrt{2} + \sqrt{k-1})} \right)$ where $u$ is uniformly random on the unit sphere in $\mathbb{R}^d$ where $d = \dim(S_{G_1}^{d-1}) \geq k - 2t + 2$.

Imagining a unit sphere with “up” and “down” corresponding to $\pm e_1$, $\mathbb{P} \left( \langle u, e_1 \rangle \geq \frac{c}{2(\sqrt{2} + \sqrt{k-1})} \right)$ is the surface area of the “end caps” of the sphere lying above and below circles of radius $R := \sqrt{1 - \left( \frac{c}{2(\sqrt{2} + \sqrt{k-1})} \right)^2}$, which is strictly smaller than the surface area of a full sphere of radius $R$. Therefore,

$$\mathbb{P} \left[ \frac{\left\| P_{G_1}^{\perp} x(t) \right\|}{\left\| P_{G_1}^{\perp} v_j \right\|} \right] \geq \frac{c}{2(\sqrt{2} + \sqrt{k-1})} \mid G_{<t}, V_{<t} \right]$$

$$\leq \frac{\text{SurfaceArea}_{d-2t+2}(R)}{\text{SurfaceArea}_{d-2t+2}(1)}$$

$$= \frac{\left(1 - \left( \frac{c}{2(\sqrt{2} + \sqrt{k-1})} \right)^2 \frac{d-2t+1}{2} \right)^{\frac{d-2t+1}{2}}}{\left(1 - \left( \frac{c}{2(\sqrt{2} + \sqrt{k-1})} \right)^2 \frac{d-2t+1}{2} \right)^{\frac{d-2t+1}{2}}}$$

$$\leq \exp \left( - \left( \frac{c}{2(\sqrt{2} + \sqrt{k-1})} \right)^2 \frac{d-2t+1}{2} \right)$$

With the final inequality coming from the fact that $1 - x \leq \exp(-x)$. This holds for each $j \geq t$, therefore,

$$\mathbb{P}[G_t \mid G_{<t}] \geq \inf_{V_{<t}} \mathbb{P}[G_t \mid G_{<t}, V_{<t}]$$

$$\geq 1 - (k - t + 1) \exp \left( - \left( \frac{c}{2(\sqrt{2} + \sqrt{k-1})} \right)^2 \frac{d-2t+1}{2} \right)$$

$$= 1 - (k - t + 1) \exp \left( - \frac{c^2(d-2t+1)}{8(\sqrt{2} + \sqrt{k-1})^2} \right)$$

Finally, bringing together Lemma 3 and Lemma 19.

**Lemma 4.** For any $\epsilon \in (0, \frac{1}{2})$, and dimension $d \geq \frac{2}{\epsilon} \log \frac{1}{\epsilon}$, $\mathbb{P}[E] > \frac{15}{16}$, where the probability is over the random choice of $\{v_j\}$.

**Proof.** By Lemma 3 for all $t$

$$\mathbb{P}[G_t \mid G_{<t}] > 1 - (k - t + 1) \exp \left( - \frac{c^2(d-2t+1)}{8(\sqrt{2} + \sqrt{k-1})^2} \right)$$
Combining this with Equation (19):

\[ P[E] \geq \prod_{t=1}^{k} P[G_t \mid G_{<t}] \]

\[ > \prod_{t=1}^{k} \left( 1 - (k - t + 1) \exp \left( \frac{-c^2(d - 2t + 1)}{8(\sqrt{2} + \sqrt{k - 1})^2} \right) \right)^k \]

\[ \geq 1 - k^2 \exp \left( \frac{-c^2(d - 2k + 1)}{40k} \right) \]

Thus, when \( \epsilon < \frac{1}{2} \) and \( d \geq \frac{2}{{\epsilon^2}} \log \frac{1}{\epsilon} \geq \frac{1}{\epsilon^2} \log \frac{1}{\epsilon} + 2k - 1 \) then \( P[E] > \frac{15}{16} \)

Thus \( E \) occurs with constant probability when the dimension is sufficiently large, and when \( E \) does occur, the algorithm must make at least \( k \) queries the subgradient oracle of \( f \) in order to find an \( \epsilon \)-suboptimal solution. Thus the expected number of oracle queries for any deterministic algorithm on the specified distribution over \( F \) is at least \( \frac{45k}{16} = \Omega(1/\epsilon^2) \), applying Yao’s minimax principle completes the proof.

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