Subconvexity for twisted $L$-functions on $GL_3$ over the Gaussian number field

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Abstract. Let $q \in \mathbb{Z}[i]$ be prime and $\chi$ be the primitive quadratic Hecke character modulo $q$. Let $\pi$ be a self-dual Hecke automorphic cusp form for $SL_3(\mathbb{Z}[i])$ and $f$ be a Hecke cusp form for $\Gamma_0(q) \subset SL_2(\mathbb{Z})$. Consider the twisted $L$-functions $L(s, \pi \otimes f \otimes \chi)$ and $L(s, \pi \otimes \chi)$ on $GL_3 \times GL_2$ and $GL_3$. We prove the subconvexity bounds

$$L\left(\frac{1}{2}, \pi \otimes f \otimes \chi\right) \ll_{s, r, f} N(q)^{3/4 + \epsilon}, \quad L\left(\frac{1}{2} + it, \pi \otimes \chi\right) \ll_{s, r, f} N(q)^{5/8 + \epsilon},$$

for any $\epsilon > 0$.

1. Introduction

Subconvexity for $L$-functions is one of the central problems in analytic number theory. The principal aim is to get bounds for a given $L$-function that are better than what the functional equation together with the Phragmén-Lindelöf convexity principle would imply.

The subconvexity problem for $GL_1$ and $GL_2$ over arbitrary number fields was completely solved in the seminal work of Michel and Venkatesh [MV]. More recent work on the subconvexity for $GL_2$ over number fields may be found in [BH], [Mag1], [Mag2] and [Wu].

Xiaoqing Li [Li2] and Blomer [Blo] made the first progress on the subconvexity for $GL_2$ over number fields may be found in [BH], [Mag2], [Mag3] and [Wu].

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In this paper, we shall obtain for the first time subconvexity results for GL\(_3\) over a number field other than \(\mathbb{Q}\). More precisely, we shall extend the work of Blomer [Blö] to the Gaussian number field \(\mathbb{Q}(i)\). Our main tools are the Kuznetsov formula for Hecke congruence subgroups of SL\(_2(\mathbb{Z}[i])\) and the Voronoï summation formula for SL\(_2(\mathbb{Z}[i])\); the former is indeed established in [LG] for imaginary quadratic fields and the latter in [IT] for arbitrary number fields. As alluded to above, the advances along the moment method approach rely heavily on the innovations in the analytic aspect. Likewise, our main focus here will be on the analysis of the GL\(_2(\mathbb{C})\)-Bessel kernel and Bessel integral arising in Kuznetsov and the GL\(_3(\mathbb{C})\)-Hankel transform in Voronoï. For this we must resort to the analytic theory of high-rank Bessel functions in [Qi3], especially the asymptotic formula for the GL\(_3(\mathbb{C})\)-Bessel kernel.

1.1. Main results. Let \(\mathbb{F} = \mathbb{Q}(i)\) be the Gaussian number field and \(\mathcal{O} = \mathbb{Z}[i]\) be the ring of Gaussian integers. Let \(N = N_\mathbb{F}/\mathbb{Q} = |\cdot|^2\) denote the norm on \(\mathbb{F}\).

Let \(q \in \mathcal{O}\) be a square-free Gaussian integer such that \(N(q) \equiv 1\) (mod 8) and \(\chi = \chi_q\) be the primitive quadratic Hecke character of conductor \(q\) and frequency 0.

For \(q'\mid q\) let \(H^*(q')\) be the set of the \(L^2\)-normalized Hecke newforms on \(\Gamma_0(q')\) in \(\mathbb{H}^3\) in the \(L^2\)-discrete spectrum of the Laplace-Beltrami operator. Here \(\Gamma_0(q') \subset \text{SL}_2(\mathcal{O})\) is the Hecke congruence group of level \(q'\) as defined in (3.1). Put \(B^*(q) = \bigcup_{q'\mid q} H^*(q')\) and let \(B^*(q) = \{f_j\}_{j \geq 1}\). Let the Laplacian eigenvalue of \(f_j\) be \(1 + 4t_j^2\) and denote by \(\lambda_j(n), n \in \mathcal{O} \setminus \{0\}\), the Hecke eigenvalues of \(f_j\).

Let \(\pi\) be a fixed self-dual Hecke-Maass cusp form for \(\text{SL}_3(\mathcal{O})\). Let \(A(n_1, n_2), n_1, n_2 \in \mathcal{O} \setminus \{0\}\), denote the Fourier coefficients of \(\pi\), Hecke-normalized so that \(A(1, 1) = 1\).

We consider the twisted \(L\)-function

\[
L(s, \pi \otimes \chi) = \sum_{(n) \neq 0} \frac{A(1, n)^2 \chi(n)}{N(n)^s}.
\]

and for \(f_j\) even the Rankin-Selberg \(L\)-function

\[
L(s, \pi \otimes f_j \otimes \chi) = \sum_{(n_1, n_2) \neq 0} \frac{A(n_1, n_2)^2 \lambda_j(n_2) \chi(n_2)}{N(n_1^2 n_2)^s}.
\]

(\(\lambda_j(n_2)\) is independent on the representative of the ideal \((n_2)\) only when \(f_j\) is even).

**Theorem 1.1.** Let notation be as above. Assume that \(q\) is prime. Let \(T \gg 1\). For \(\varepsilon > 0\) and \(A = A(\varepsilon)\) sufficiently large, we have

\[
\sum_{|t| \leq T} L\left(\frac{1}{2}, \pi \otimes f_j \otimes \chi\right) + \int_{-T}^{T} \left|L\left(\frac{1}{2} + it, \pi \otimes \chi\right)\right|^2 \frac{dt}{t^2 + 1} \ll T^{A} N(q)^{5/4 + \varepsilon},
\]

where \(\sum\) restricts to the even Hecke cusp forms in \(B^*(q)\). The implied constant depends only on \(\varepsilon\) and \(\pi\).

By the nonnegativity theorem of Lapid in [Lap], we have

\[
L\left(\frac{1}{2}, \pi \otimes f_j \otimes \chi\right) \geq 0.
\]

As a consequence of (1.4), we derive from (1.3) the following bound for the individual \(L\)-values.

**Corollary 1.2.** Let notation be as above. Assume that \(q\) is prime. We have

\[
L\left(\frac{1}{2}, \pi \otimes f_j \otimes \chi\right) \ll N(q)^{5/4 + \varepsilon},
\]

for any \(\varepsilon > 0\), the implied constant depending only on \(\varepsilon\), \(\pi\) and \(t_j\).
Moreover, ignoring the contribution of the cuspidal spectrum in (1.3) by nonnegativity (1.4), we have
\[
\int_{-T}^{T} |L \left( \frac{1}{2} + it, \pi \otimes \chi \right)|^2 \frac{dt}{t^2 + 1} \leq N(q)^{5/4 + \varepsilon},
\]
By the arguments in [CI §1] (see also [Blo §4]), we have the following corollary.

Corollary 1.3. Let notation be as above. Assume that \( q \) is prime. We have
\[
L \left( \frac{1}{2} + it, \pi \otimes \chi \right) \ll N(q)^{5/8 + \varepsilon},
\]
for \( \varepsilon > 0 \) and \( t \) real, the implied constant depending only on \( \varepsilon, \pi \) and \( t \).

Since the corresponding convexity bounds for \( L \left( \frac{1}{2}, \pi \otimes f_j \otimes \chi \right) \) and \( L \left( \frac{1}{2} + it, \pi \otimes \chi \right) \) are \( N(q)^{3/2 + \varepsilon} \) and \( N(q)^{3/4 + \varepsilon} \), respectively, the above bounds in Corollary 1.2 and 1.3 break their convexity bounds.

1.2. Remarks. We regard the primary novelty of this work as not in the arithmetic but rather in the analysis of \( \text{GL}_2 \)- and \( \text{GL}_3 \)-Bessel kernels and the \( \text{GL}_3 \)-Hankel transform over \( \mathbb{C} \). For example, the computations of character sums in [Blo] may be applied here without any change (thanks to the fact that \( \mathbb{Q}(i) \) is of class number one). As for the analysis, although there are obvious similarities compared to [Blo], Bessel kernels over \( \mathbb{C} \) are not only different from but more difficult to analyze than those over \( \mathbb{R} \). A remarkable reflection of this difference is that the method in [Li2] surprisingly fails to work over \( \mathbb{Q}(i) \) in the aspect of analytic conductor (the \( t \)-aspect). Roughly speaking, the analysis in [Li2] breaks down on \( \mathbb{C} \) because there is not sufficient oscillation in the weights.

This paper may be viewed as the second application of the Voronoï summation formula over number fields in Ichino-Templier [IT] and the asymptotic theory of high-rank Bessel functions in the author’s work [Qi3]. The first is the Wilton and Miller bounds for additively twisted sums of \( \text{GL}_2 \) and \( \text{GL}_3 \) Fourier coefficients over arbitrary number fields in [Qi2].

The results here over the Gaussian number field may be extended straightforwardly to the other eight imaginary quadratic number fields of class number one. Furthermore, it is very likely that the subconvexity problem under consideration may be solved over an arbitrary number field in the same manner. First, the Voronoï summation formula over number fields is well established in [IT]. Second, the spherical Kuznetsov trace formula over number fields may be found in [BM1] or [Ven] (see also [Mag1]), although the class of weight functions therein is not quite as general as in [Kuz] and [BM2, LG]. Moreover, the works in [Blo] and this paper have completed the analysis of Bessel kernels, the Bessel integral and the Hankel transform over an Archimedean local field. Yet, many details in the non-Archimedean aspect, which could be quite complicated, remain to be worked out if the class number is not one.

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1This has already been alluded to in some previous works of the author. For example, Stirling’s asymptotic formula for the gamma function, used by Xiaoqing Li and Blomer in [Li1] [Blo] to establish the asymptotic formula for Bessel kernels over \( \mathbb{R} \), becomes fairly useless in the asymptotic theory of Bessel kernels over \( \mathbb{C} \) in [Qi3].
2. Preliminaries

2.1. General notation. Throughout this article, we set $e(z) = e^{2\pi i z}$.

Let $\mathbb{F} = \mathbb{Q}(i)$ and $\mathfrak{o} = 2\mathbb{F} = \mathbb{Z}[i]$. Then $\mathfrak{o}^\times$ is the dual lattice of $\mathfrak{o}$ with respect to the trace $\text{Tr} = \text{Tr}_{\mathbb{F}/\mathbb{Q}} = 2\text{Re}$. Let $\mathfrak{o}^\times = \{1, -1, i, -i\}$ be the group of units. As convention, let $(p)$ always stand for prime ideal of $\mathfrak{o}$.

The standard notation for certain arithmetic functions on $\mathbb{Z}$ will also be used for $\mathbb{Z}[i]$, like the Möbius function $\mu$ and the Euler function $\varphi$. Namely, $\mu(n) = (-1)^k$ if $n$ is a square-free Gaussian integer with $k$ many prime factors, $\mu(n) = 0$ if $n$ is not square-free, and $\varphi(n) = |n|^2 \prod_{p \mid n} (1 - 1/|p|^2)$.

Let $dz$ be twice the Lebesgue measure on $\mathbb{C}$. In the polar coordinates, we have $dz = 2\pi x dx \phi$ for $z = xe^{i\theta}$.

2.2. Kloosterman sums. For $n_1, n_2 \in \mathfrak{o}$ and $c \in \mathfrak{o} \setminus \{0\}$ define the Kloosterman sum

\[
S(n_1, n_2; c) = \sum_{a \equiv (c)\mathfrak{o}}^* e \left( \text{Re} \left( \frac{n_1 a + n_2 \bar{a}}{c} \right) \right),
\]

where $\sum^*$ means that $a$ runs over representatives of $(\mathfrak{o}/c\mathfrak{o})^\times$ and $a \bar{a} \equiv 1 \pmod{c}$.

2.3. Hecke characters. Define $\eta(z) = z/|z|, z \in \mathbb{C} \setminus \{0\}$. For $q \in \mathfrak{o} \setminus \{0\}$ let $I_q$ denote the group of fractional ideals that are relatively prime with $q$, that is, $I_q = \{(n_1)(n_2)^{-1} : (n_1, q) = (n_2, q) = \mathfrak{o}\}$. Let $\omega$ be a character of $(\mathfrak{o}/q\mathfrak{o})^\times$ and $k$ be an integer satisfying the units consistency condition: $\omega(e) e^k = 1$ for all $e \in \mathfrak{o}^\times$. We may then form a Hecke character (Größencharakter) $\chi$ on $I_q$ such that

$\chi((n)) = \omega(n) \eta^k(n)$,

for every $n \in \mathfrak{o}$, $(n, q) = \mathfrak{o}$. In addition, we assume $\chi((n)) = 0$ if $(n, q) \neq \mathfrak{o}$. The integer $k$ is called the frequency of $\chi$. The Gauss sum $\tau(\chi)$ associated with $\chi$ is defined by

$\tau(\chi) = \eta^k(q) \sum_{a \equiv (q)\mathfrak{o}}^* \omega(a)e \left( \text{Re} \left( \frac{a}{q} \right) \right)$.

The root number $e(\chi) = i^{-k} \tau(\chi) / \sqrt{N(q)}$. See [IK] §3.8 for more details.

Now assume that $q$ is odd and square-free. Let $\omega$ be the quadratic symbol $\left( \frac{\cdot}{q} \right)$. Note that $\left( \frac{\cdot}{2} \right) = (-1)^{(N(q)-1)/4}$, so one needs $2k \equiv N(q) - 1 \pmod{8}$ for the units consistency condition. In this article, we assume that $\chi = \chi_q$ is quadratic (real), then the frequency $k = 0$ (in other words, $\chi$ is trivial at the Archimedean place) and hence $N(q) \equiv 1 \pmod{8}$.

In this case, we claim that $\tau(\chi) = \sqrt{N(q)}$ and hence $e(\chi) = 1$. To see this, let us first assume that $q$ is prime. When $N(q) = q^2$, the character $\chi_q$ is equal to the Legendre symbol $\xi_N(q) = \left( \frac{\cdot}{N(q)} \right)$ under the isomorphism $\mathfrak{o}/q\mathfrak{o} \cong \mathbb{Z}/N(q)\mathbb{Z}$, so $\tau(\chi_q) = \tau(\xi_N(q)) = \sqrt{N(q)}$ as $N(q) \equiv 1 \pmod{4}$. When $q \in \mathbb{Z}$ and $q \equiv -1 \pmod{4}$ so that $q$ is inert, the character $\chi_q$ is induced from the Legendre symbol $\xi_q = \left( \frac{\cdot}{q} \right)$ on $\mathbb{Z}/q\mathbb{Z}$ via the norm map $N : \mathfrak{o}/q\mathfrak{o} \rightarrow \mathbb{Z}/q\mathbb{Z}$, namely $\chi_q = \xi_q \cdot N$, and by [IK] §3.8, Example 5 we have $\tau(\chi) = \xi_q(-1) \cdot \tau(\xi_q)^2 = -(i \sqrt{q})^2 = -\sqrt{N(q)}$. The quadratic reciprocity law for $\mathfrak{o}$ may be applied to prove $\tau(\chi) = \sqrt{N(q)}$ for any $q$ square-free.

2.4. Stationary phase (the Van der Corput lemma).

**Lemma 2.1** (Van der Corput). Let $K \subset \mathbb{R}^d$ be a compact set that contains 0, $U$ be an open neighborhood of $K$. Let $S > 0$ and $\sqrt{S} \geq X \geq 1$. Let $u(x) \in C_0^\infty(K)$ and
Suppose that \( (\partial/\partial x)^{\alpha} u(x) \leq_{\alpha} SX^{\alpha} \) and that \( f(x) \) is real-valued, \( f(0) = 0 \), \( f'(0) = 0 \), \( \text{det} f''(0) \neq 0 \) and \( f'(x) \neq 0 \) in \( K \setminus \{0\} \). Then for any given multi-index \( \gamma \),
\[
I_\gamma(\lambda) = \int_K e(\lambda f(x)) u(x) x^\gamma dx \leq S / \lambda^{(|\gamma| + d)/2},
\]
with the implied constant depending only on \( \gamma \) when \( f \) stays in a bounded set in \( C^\infty(K) \) and \(|x|/|f'(x)| \) has a uniform bound.

Lemma 2.1 is a generalization of [Sog Lemma 1.1.6], but here \( X \) can be as large as \( \sqrt{\lambda} \), while \( X = 1 \) in [Sog], which means that the amplitudes are allowed to have moderate oscillation.

In the settings of [Sog], one works with amplitudes that involve the parameter \( \lambda \), namely, \( u(x) = u(\lambda, x) \), and, using the Van der Corput lemma, one may deduce the following stationary estimate as in [Sog Theorem 1.1.4],
\[
(\partial/\partial \lambda)^2 i_0(\lambda) \leq SP X^\beta / \lambda^{d/2 + \beta}
\]
from
\[
(\partial/\partial x)^{\alpha}(\partial/\partial \lambda)^{\beta} u(\lambda, x) \leq_{\alpha, \beta} SPX^{\alpha}/\lambda^\beta.
\]
In our settings however we have to also differentiate with respect to an additional angular parameter \( \theta \) involved in the phase \( f(x) \), so [Sog Theorem 1.1.4] is not sufficient; nevertheless the Van der Corput lemma would still do the job for us.

Proof. Following [Sog], Lemma 2.1 in higher dimensions may be deduced from the one-dimensional case by an induction and the Morse lemma. When \( d = 1 \), Lemma 2.1 can be proven by the arguments of Van der Corput as in the proof of [Sog Lemma 1.1.2]. Indeed, at the end we would get
\[
I_\gamma(\lambda) \leq_{\gamma, N} S r^{\gamma + 1} \left( 1 + \left( \frac{X + 1/r}{\lambda r} \right)^N \right),
\]
for any \( r > 0 \) and \( N \geq \gamma + 2 \). The right side is smallest when \( r = (X + \sqrt{X^2 + 4\lambda})/2\lambda \), which yields
\[
I_\gamma(\lambda) \leq S \left( \frac{X + \sqrt{X^2 + 4\lambda}}{2\lambda} \right)^{\gamma + 1} \leq S \left( \frac{X + \sqrt{\lambda}}{\lambda} \right)^{\gamma + 1}.
\]
Hence, we have the desired stationary phase bound \( S / \lambda^{(\gamma + 1)/2} \) when \( X < \sqrt{\lambda} \). Q.E.D.

We shall use the following variant of the two-dimensional Van der Corput lemma in the polar coordinates.

**Lemma 2.2.** Let \( S > 0 \) and \( \sqrt{\lambda} \geq X \geq 1 \). In the polar coordinates, let \( u(x, \phi) \) be a smooth function with support in the annulus \( A[b, c] = \{(x, \phi) : x \in [b, c]\} \) and derivatives satisfying \( \partial_x \partial_\phi u(x, \phi) \leq_{\alpha, \beta} SX^{\alpha + \beta} \) for all \( \alpha, \beta \). Let \( f(x, \phi) \) be a smooth real-valued function such that \( f(a, \theta) = 0 \), \( f'(a, \theta) = 0 \), \( \text{det} f''(a, \theta) \neq 0 \) and \( f''(a, \theta) \neq 0 \) in \( A[b, c] \setminus \{(a, \theta)\} \). Then for any given \( \alpha \) and \( \beta \),
\[
L_{\alpha\beta}(\lambda) = \int_0^{2\pi} \int_0^{\infty} e(\lambda f(x, \phi)) u(x, \phi) (x - a)^\alpha \sin^\beta((\phi - \theta)/2) dx d\phi \leq S / \lambda^{(\alpha + \beta + 2)/2},
\]
with the implied constant depending only on \( \alpha, \beta \), when \( f(x, \phi) \) stays in a bounded set in \( C^\infty(A[b, c]) \) and \( ((x - a)^2 + \sin^2(\phi - \theta)/2)/((\partial_x f(x, \phi))^2 + (\partial_\phi f(x, \phi))^2) \) has a uniform bound.
3. A review of automorphic forms and L-functions

3.1. Automorphic forms on $\mathbb{H}^3$.

3.1.1. The three-dimensional hyperbolic space. We let

$$\mathbb{H}^3 = \{w = z + jr = x + iy + jr : x, y, r \text{ real, } r > 0\}$$

denote the three-dimensional hyperbolic space, with the action of $\text{GL}_2(\mathbb{C})$ or $\text{PGL}_2(\mathbb{C}) (= \text{PSL}_2(\mathbb{C}))$ given by

$$z(g \cdot w) = \frac{(az + b)(cw + d)}{|cz + d|^2 + |c|^2r^2}, \quad r(g \cdot w) = \frac{|\det g|}{|cz + d|^2 + |c|^2r^2},$$

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbb{C}),$$

while the action of $\text{GL}_2(\mathbb{C})$ on the boundary $\partial \mathbb{H}^3 = \mathbb{C} \cup \{\infty\}$ is by the Möbius transform. $\mathbb{H}^3$ is equipped with the $\text{GL}_2(\mathbb{C})$-invariant hyperbolic metric $(dx^2 + dy^2 + dr^2)/r^2$ and hyperbolic measure $dx\,dy\,dr/r^3$. The associated hyperbolic Laplace-Beltrami operator is given by $\Delta = r^2 (\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial r^2) - r\partial/\partial r$.

3.1.2. Hecke congruence groups. For $q \in \mathcal{O} \setminus \{0\}$ define the Hecke congruence group

$$(3.1) \quad \Gamma_0(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathcal{O}) : c \equiv 0 \text{ (mod } q) \right\}.$$

$\Gamma = \Gamma_0(q)$ is a discrete subgroup of $\text{SL}_2(\mathbb{C})$ which is cofinite but not cocompact. Subsequently, we shall always assume that $q$ is square-free.

3.1.3. Maass cusp forms. The $L^2$-discrete spectrum of the Laplace-Beltrami operator $\Delta$ on $\Gamma \backslash \mathbb{H}^3$ comprises the constant function $f_0 = 1/\sqrt{\text{Vol}(\Gamma \backslash \mathbb{H}^3)}$ and an orthonormal basis of Maass cusp forms $\{f_j\}_{j \geq 1}$ which are eigenfunctions of $\Delta$. For $f_j$ with Laplacian eigenvalue $1 + 4t_j^2$, we have the Fourier expansion

$$f_j(z, r) = \sum_{n \in \mathcal{O} \cap \{0\}} \rho_j(2n) r K_{2it_j}(4\pi|n|r)e(\text{Tr}(nz)).$$

We recall the Kim-Sarnak bound in [BB1] over the field $\mathbb{F}$,

$$(3.2) \quad |\text{Im} \, t_j| \leq \frac{1}{24},$$

so $t_j$ is either real or imaginary with $|it_j| \leq \frac{1}{24}$.

Since $i \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in \Gamma_0(q)$, we infer that $f_j$ is invariant under the action of $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and hence $\rho_j(-n) = \rho_j(n)$. A Maass cusp form for $\Gamma_0(q)$ is said to be even or odd if it is an eigenfunction of the action of $i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ with eigenvalue $1$ or $-1$, respectively. We may require that each $f_j$ is either even or odd. Note that $f_j$ is even if and only if $\rho_j(en) = \rho_j(n)$ for all $e \in \mathcal{O}^\times$ and $n \in \mathcal{O} \setminus \{0\}$.

Remark 3.1. It would be of some interest to introduce the congruence group $\Gamma_0'(q) \subset \text{GL}_2(\mathcal{O})$ defined similarly as in [SJ].

$$(3.3) \quad \Gamma_0'(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathcal{O}) : c \equiv 0 \text{ (mod } q) \right\}.$$

It is clear that an even Maass cusp form for $\Gamma_0(q)$ is indeed a Maass cusp form for $\Gamma_0'(q)$.

For $n \in \mathcal{O} \setminus \{0\}$, we define the Hecke operator $T_n$ by

$$T_n f(w) = \frac{1}{4|n|} \sum_{ad=n} \sum_{b \text{ (mod } d)} f \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} w \right).$$
Hecke operators commute with each other as well as the Laplacian operator. We may further assume that every \( f_j \) is an eigenfunction of all the Hecke operators \( T_n \) with \( (n, q) = \emptyset \). Let \( \lambda_j(n) \) denote the the Hecke eigenvalue of \( T_n \) for \( f_j \), then
\[
\rho_j(n) = \rho_j(1) \lambda_j(n),
\]
if \( (n, q) = \emptyset \). The Hecke eigenvalues \( \lambda_j(n) \) are real and satisfy the Hecke relation
\[
\lambda_j(n_1)\lambda_j(n_2) = \frac{1}{4} \sum_{d|n_1n_2} \lambda_j(n_1n_2/d^2),
\]
if \( (n_1n_2, q) = \emptyset \).

Finally, let \( H^*(q) \) be the set of the \( L^2 \)-normalized newforms for \( \Gamma_0(q) \) which are eigenfunctions of all the \( T_n \). Later in §3.2, the orthonormal basis \( \{ f_j \}_{j \geq 1} \) will be constructed from all the newforms for \( \Gamma_0(q') \) with \( q' | q \).

### 3.1.4. Eisenstein series

For each cusp \( a \) of \( \Gamma = \Gamma_0(q) \), we form the Eisenstein series
\[
E_a(w; s) = \sum_{\gamma \in \Gamma \setminus \Gamma} r(\sigma_a^{-1} \gamma \cdot w)^2s,
\]
if \( \text{Re } s > 1 \) and by analytic continuation for all \( s \) in the complex plane. Here \( \Gamma_a \) denote the stability group of \( a \) in \( \text{SL}_2(\mathbb{O}) \) and \( \sigma_a \in \text{SL}_2(\mathbb{C}) \) is such that
\[
\sigma_a \infty = a \quad \text{and} \quad \sigma_a^{-1} \Gamma_0 \sigma_a = \Gamma_0.
\]
The Fourier expansion of \( E(z, r; s) \) is similar to that of a cusp form. Precisely
\[
E_a(z, r; s) = \varphi_a r^{2s} + \varphi_a(s)r^{-2-2s} + \sum_{n \in \mathbb{O} \setminus \{0\}} \varphi_a(n, s)rK_{2s-1}(2\pi|n|r)e(\text{Re}(nz)),
\]
with \( \varphi_a = 1 \) if \( a \sim \infty \) or \( \varphi_a = 0 \) if otherwise.

The continuous \( L^2 \)-spectrum of the Laplacian comprises all the \( E_a(w, \frac{1}{2} + it) \).

Following [CI §3], we compute the Fourier coefficients of \( E_a(w; s) \) by using the Eisenstein series for \( \text{SL}_2(\mathbb{O}) = \Gamma_0(1) \),
\[
E(z, r; s) = \frac{1}{4} r^{2s} \sum_{(c,d)=1} \left( |cz+d|^2 + |c|^2 r^2 \right)^{-2s},
\]
which is known to have an explicit Fourier expansion as in (3.7) with
\[
\varphi(s) = \frac{\pi \zeta_E(2s-1)}{(2s-1)\zeta_E(2s)}, \quad \varphi(n, s) = \frac{2\pi^{2s} \eta(n, s)}{\zeta_E(2s)},
\]
in which \( \zeta_E \) is the Dedekind zeta function associated with \( \mathbb{E} \),
\[
\zeta_E(s) = \sum_{(a) \neq (0)} \frac{1}{a^{2s}} = \frac{1}{4} \sum_{n \neq 0} \frac{1}{n^{2s}},
\]
and for \( n \in \mathbb{O} \setminus \{0\} \)
\[
\eta(n, s) = \sum_{(a) \supseteq (n)} \frac{n^{2s-1}}{a^2} = \frac{1}{4} \sum_{a|n} \frac{n^{2s-1}}{a^2},
\]
which satisfies the same Hecke relation as \( \lambda_j(n) \), namely,
\[
\eta(n_1, s)\eta(n_2, s) = \frac{1}{4} \sum_{d|n_1d|n_2} \eta(n_1n_2/d^2, s),
\]
for \( (n_1n_2, q) = \emptyset \). For more details, see for example [EGM §3.4, 8.2].

Since \( q \) is square-free, every cusp of \( \Gamma = \Gamma_0(q) \) is equivalent to one of \( 1/v \) with \( v|q \), and \( 1/v \sim 1/v' \) if and only if \( (v) = (v') \). These may be verified by the arguments in [Shi].
Hence the Eisenstein series for the cusp \( a \) in the sense that the expression on the right of (3.16) is independent on the choice of the \( z \).

It is understood that in the nongeneric case when \( 4 \), \( H \)

connection formulae between \( O \).

For latter applications, we have the following lemma. See [\[\text{

By the calculations in [\[\text{

\( z \) be replaced by its limit. Moreover,

\( \) is well-defined

It should be warned that the product in (3.19) is not well-defined as function on \( \mathbb{C} \setminus \{0\} \).

For latter applications, we have the following lemma. See [\[\text{

\[\text{§1.6]. For the cusp } a = 1/v \text{ let } w = q/v \text{ be the complementary divisor and define the scaling matrix

\[ \sigma_a = \left( \frac{\sqrt{w}}{\sqrt{q/v}} \right)^2 \right) \in \text{SL}_2(\mathbb{Z}), c + aw \equiv 0(\text{mod } q) \right\}. \]

Hence the Eisenstein series for the cusp \( a = 1/v \) is given by

\[ E_a(z, r; s) = \sum_{\tau \in \Gamma_a \setminus \sigma_a^{-1} \Gamma} r(\tau(z, r))^{2s} = \frac{1}{4} \left( \frac{r}{|w|} \right)^{2s} \sum_{(c, d) = 0 \atop |c|} \sum_{e \mid c} (|cz + d|^2 + |e|^2r^2)^{-2s}. \]

By the calculations in [\[\text{

\[ (3.13) \quad E_a(z, r; s) = \frac{\mu(v)\zeta_{\mathbb{E}, q}(2s)}{16|qv|^2} \sum_{\beta \gamma \in \mathbb{E}, q} \mu(\beta \gamma)|\beta \gamma|^{2s}E(\beta \gamma z, |\beta \gamma| r; s), \]

where \( \zeta_{\mathbb{E}, q}(s) \) is the local zeta-function

\[ (3.14) \quad \zeta_{\mathbb{E}, q}(s) = \prod_{(p) = 0(q)} \left( 1 - |p|^{-2s} \right)^{-1}. \]

In conclusion, by (3.9)-(3.14) we deduce that for \( n \in \mathbb{O} \setminus \{0\}, (n, q) = 0 \),

\[ (3.15) \quad \varphi_a(n, s) = \frac{2\mu(v)^2\zeta_{\mathbb{E}, q}(2s)\eta(n, s)}{|qv|^2\Gamma(2s)\zeta(2s)}. \]

3.2. The spectral Kuznetsov formula for \( \Gamma_0(q)/\mathbb{H}^3 \).

3.2.1. Bessel functions for \( \text{GL}_2(\mathbb{C}) \). Let \( \mu \in \mathbb{O} \) and \( m \in \mathbb{Z} \). We define

\[ (3.16) \quad J_{\mu, m}(z) = J_{-2\mu-\frac{1}{2}m}(z) J_{-2\mu+\frac{1}{2}m}(z), \]

with \( J_{\nu}(z) \) the classical \( J \)-Bessel function of order \( \nu \). The function \( J_{\mu, m}(z) \) is well-defined in the sense that the expression on the right of (3.16) is independent on the choice of the argument of \( z \) modulo \( 2\pi \). Next, we define

\[ (3.17) \quad J_{\mu, m}(z) = \begin{cases} \frac{2\pi^2}{\sin(2\pi\mu)}(J_{\mu, m}(4\pi z) - J_{-\mu, -m}(4\pi z)), & \text{if } m \text{ is even}, \\ \frac{2\pi^2 i}{\cos(2\pi\mu)}(J_{\mu, m}(4\pi z) + J_{-\mu, -m}(4\pi z)), & \text{if } m \text{ is odd}. \end{cases} \]

It is understood that in the nongeneric case when \( 4\mu \in 2\mathbb{Z} + m \) the right-hand side should be replaced by its limit. Moreover, \( J_{\mu, m}(z) \) is an even function when \( m \) is even. See [Qi3 §15.3].

Let \( H^{(1)}_\nu(z) \) and \( H^{(2)}_\nu(z) \) be the Hankel functions of order \( \nu \). It follows from the connection formulae between \( H^{(1)}_\nu(z) \), \( H^{(2)}_\nu(z) \) and \( J_{\nu}(z) \), \( J_{-\nu}(z) \) in [\[\text{

\[ (3.18) \quad J_{\mu, m}(z) = \pi^2 i(e^{2\pi i}H^{(1)}_{\mu, m}(4\pi z) + (-)^{m+1}e^{-2\pi i}H^{(2)}_{\mu, m}(4\pi z)). \]

with

\[ (3.19) \quad H^{(1, 2)}_{\mu, m}(z) = H^{(1, 2)}_{2\mu+\frac{1}{2}m}(z) H^{(1, 2)}_{2\mu-\frac{1}{2}m}(z). \]

It should be warned that the product in (3.19) is not well-defined as function on \( \mathbb{C} \setminus \{0\} \).
Lemma 3.2. Let $K$ be a nonnegative integer. We have

\begin{align}
H^{(1)}_{\nu}(z) &= \left( \frac{2}{\pi z} \right)^{1/2} \exp\left(i\frac{\pi - \nu - \frac{1}{2}}{2\pi} \right) \left( \sum_{k=0}^{K-1} \frac{(-)^k \nu, k}{(2\pi)^k} + E^{(1)}_K(z) \right), \\
H^{(2)}_{\nu}(z) &= \left( \frac{2}{\pi z} \right)^{1/2} \exp\left(-i\frac{\pi - \nu - \frac{1}{2}}{2\pi} \right) \left( \sum_{k=0}^{K-1} \frac{\nu, k}{(2\pi)^k} + E^{(2)}_K(z) \right),
\end{align}

with $(\nu, k) = \Gamma(\nu + k + \frac{1}{2})/k!\Gamma(\nu - k + \frac{1}{2})$, of which (3.20) is valid when $z$ is such that $-\pi + \frac{\delta}{2} \leq \arg z \leq 2\pi - \frac{\delta}{2}$, and (3.21) when $-2\pi + \frac{\delta}{2} \leq \arg z \leq \pi - \frac{\delta}{2}$, $\delta$ being an acute angle, and

\begin{equation}
\zeta^{a}(d/dz) E^{(1,2)}_K(z) \ll_{\delta, \alpha, K} (|z|^2 + 1)/|z|^K,
\end{equation}

for $|z| \gg |z|^2 + 1$ and $\arg z$ in the range indicated as above.

Moreover, by [Q13 Corollary 6.17], we have the following integral representation

\begin{equation}
J_{\mu, m}(x e^{i\phi}) = 4\pi m \int_0^{\infty} Y^m y^{\mu - 1} E(y e^{i\phi})^{-m} J_m(4\pi x Y(y e^{i\phi})) dy,
\end{equation}

with

\begin{equation*}
Y(z) = |z + z^{-1}|, \quad E(z) = (z + z^{-1}) / |z + z^{-1}|. \quad \tag{3.23}
\end{equation*}

The integral on the right of (3.23) is absolutely convergent if $|\Re \mu| < \frac{1}{2}$.

In this article, we are mainly concerned with the spherical Bessel function

\begin{equation}
J_{\mu}(z) = J_{\mu, 0}(z) = \frac{2\pi^2}{\sin(2\pi \mu)} \left( J_{-2\mu} (4\pi z) J_{-2z} (4\pi z) - J_{2\mu} (4\pi z) J_{2\mu} (4\pi z) \right), \quad \tag{3.24}
\end{equation}

which is associated with the spherical principal series representation of $\text{PSL}_2(\mathbb{C})$ induced from the character $\chi_{\mu}(\frac{z}{z-1}) = |z|^{2\mu}$. Non-spherical Bessel functions $J_{\mu, \pm 1}(z)$ and $J_{\mu, \pm 2}(z)$ however will arise in the derivatives of $J_{\mu}(z)$.

3.2. The spectral Kuznetsov formula for $\Gamma_0(q) \backslash \mathbb{H}^3$. Let $h(t)$ be an even function satisfying the following two conditions,

- $h(t)$ is holomorphic on a neighborhood of the strip $|\Im t| \leq \sigma$,
- $h(t) \ll (|t| + 1)^{-\theta}$,

for some $\sigma > 1/2$ and $\theta > 1$. In view of [LG Theorem 11.3.3, along with (3.4, 3.15), we have the following spectral Kuznetsov trace formula, specialized to the spherical case. For $(n_1, n_2, q) = \emptyset$,

\begin{equation}
\sum_{j} \omega_j h(t_j) h_n(n_1, n_2) + \frac{1}{2\pi} \sum_{q \neq 1} \int_{-\infty}^{\infty} \omega_q(t) h(t) \eta(n_1, \frac{1}{2} + it) \eta(n_2, \frac{1}{2} - it) dt
= \frac{1}{2\pi^2} \sum_{q \equiv 1} \delta_{n_1, n_2} H + \frac{1}{8\pi^2} \sum_{q \equiv 1} \sum_{c \equiv 0} S(n_1, n_2; c) H \left( \frac{\sqrt{n_1 n_2}}{2c} \right),
\end{equation}

where $\sum'$ restricts to the even Hecke cusp forms for $\Gamma_0(q)$,

\begin{equation}
\omega_j = \frac{|p_j(1)|^2}{\sinh(2\pi t_j)}, \quad \omega_q(t) = \frac{|\phi_q(1, \frac{1}{2} + it)|^2}{\sinh(2\pi t)}, \quad \tag{3.25}
\end{equation}

\begin{equation}
H = \int_{-\infty}^{\infty} h(t) t^2 dt, \quad H(z) = \int_{-\infty}^{\infty} h(t) J_{\mu}(z) t^2 dt,
\end{equation}

\begin{equation}
\omega_j, \omega_q, H, H(z)
\end{equation}
in which \( J_n(z) \) is the Bessel function as in (3.24), \( \delta_{n_1 n_2} \) is the Kronecker \( \delta \)-symbol, \( \delta^{\times 2} = \{ e^2 : e \in \mathcal{O}^{\times} \} = \{ 1, -1 \} \), and \( S(n_1, e n_2, c) \) is the Kloosterman sum defined by (2.1). It follows from (3.15) that

\[
\omega_a(t) = \frac{2\pi |\zeta_{p,q}(1 + 2it)|^2}{|q|^2 |\zeta_{p}(1 + 2it)|^2},
\]

if \( a = 1/\nu \).

**Remark 3.3.** The spectral weight function \( h(2it, p) \) in [LG] Theorem 11.3.3 is chosen to be supported on \( \{(2it, p) : p = 0 \} \) and here \( h(t) = h(2it, 0) \). When \( p = 0 \), the corresponding representations of \( SL_2(\mathbb{R}) \) are spherical and the cusp forms \( f_j \) are their \( SU_2 \)-fixed vectors. Moreover, for \( n \in \mathcal{O}' \setminus \{ 0 \} \), we have \( \rho(2n) = \pi C(n; 2it, 0)/\Gamma(1 + 2it) \) and similarly \( \varphi_n(2n, \frac{1}{2} + it) = \pi B_n(n, 2it, 0)/\Gamma(1 + 2it) \) if \( C(n; 2it, 0) \) and \( B_n(n, 2it, 0) \) are the Fourier coefficients in [LG]. When \( it \) is imaginary so that the infinite component of \( f_j \) is a unitary principal series, it follows from Euler’s reflection formula that

\[
\omega_j = \frac{\rho_j(1) \Gamma(1 + 2it_j)}{2\pi} = \frac{\rho_j(1)^2 \Gamma(1 + 2it_j) \Gamma(1 - 2it_j)}{\sin(2\pi t_j)}.
\]

When \( s_j = i t_j \) is real, say \( (0, \frac{1}{2}) \), and we are in the complementary series case (although Selberg’s conjecture asserts that this case does not exist), the formula in [LG] Theorem 11.3.3 is not so accurate. As suggested in [Qi4] Theorem 2.1, there should be a correction factor \( G_{s_j,0} = \Gamma(1 + 2s_j)/\Gamma(1 - 2s_j) \) in the denominator, and hence

\[
\omega_j = \frac{\rho_j(1) \Gamma(1 + 2s_j)^2 \Gamma(1 - 2s_j)}{2\pi} = \frac{\rho_j(1)^2 \Gamma(1 + 2s_j) \Gamma(1 - 2s_j)}{2\pi} = \frac{\rho_j(1)^2 s_j}{\sin(2\pi s_j)}.
\]

The author however overlooked the simple fact that complementary series are spherical \( (d = 0) \) in the notation of [Qi4] and would like to take the chance here to correct and simplify the formula of \( G_{s,0} \) in [Qi4] Theorem 2.1 as follows,

\[
G_{s,d} = \begin{cases} 
1, & \text{if } s \text{ is imaginary}, \\
\frac{\pi i}{\sinh(\pi t)} (J_{2t}(4\pi x) - J_{-2t}(4\pi x)), & \text{if } s \in (0, \frac{1}{2}) \text{ and } d = 0.
\end{cases}
\]

**Remark 3.4.** In the real case, the Bessel kernel is

\[
\left\{ \begin{array}{ll}
\frac{\pi t}{\sinh(\pi t)} (J_{2t}(4\pi x) - J_{-2t}(4\pi x)), & \text{if } \epsilon = 1, \\
4 \cosh(\pi t) K_{2t}(4\pi x), & \text{if } \epsilon = -1.
\end{array} \right.
\]

The Bessel functions \( J_{2it} \) and \( K_{2it} \) have quite different asymptotics on \( \mathbb{R}_+ \) and must be treated separately. However, unlike the real case, the unit \( x \) here does not play an essential role.

For \( q = q'q'' \) and Hecke newform \( f \in H^*(q') \), let \( S(q''; f) \) denote the linear space spanned by the forms \( f_{|d}(z, r) = f(dz, dr), \) with \( d|q'' \). The space of cusp forms for \( \Gamma_0(q) \) decomposes into the orthogonal sum of \( S(q'', f) \). By the calculations in [ILS] §2, one may construct an orthonormal basis of \( S(q''; f) \) in terms of \( f_{|d} \). Recall that \( (n_1, n_2, 2q) = \mathcal{O} \).

Using this collection of bases as our \( \{ f_j \} \), the sum in (3.35) can be arranged into a sum over the even newforms in \( B^*(q) = \bigcup_q H^*(q) \). With ambiguity, we denote by \( f_j \) the newforms in \( B^*(q) \) (instead of the cusp forms in an orthonormal basis for \( \Gamma_0(q) \)), let \( t_j, \lambda_j(n) \) be as before, and denote by \( \omega_j^* \) the new weights. In addition, let

\[
\omega^*(t) = \sum_a \omega_a(t).
\]
We then obtain for \((n_1, n_2, q) = \emptyset\),
\[
\sum_j \omega_j^* h(t_j) \lambda_j(n_1) \overline{\lambda_j(n_2)} + \frac{1}{2\pi} \int_{-\infty}^{\infty} \omega^*(t)h(t) \eta \left( n_1, \frac{t}{2} + it \right) \eta \left( n_2, \frac{t}{2} - it \right) dt
\]
\[(3.29)\]
\[
= \frac{1}{2\pi i} \sum_{e \in \mathfrak{O}_S} \delta_{n_1, n_2} H + \frac{1}{8\pi^2} \sum_{e \in \mathfrak{O}_S/\mathfrak{O}^2} \sum_{q|e} S(n_1, en_2; e) \frac{1}{|e|^2} H \left( \frac{\sqrt{en_1n_2}}{2\pi} \right).
\]

Following [ILS] §2, we may derive the formula
\[
\omega_j^* = \frac{\pi Z_q(1, f_j)}{\text{Vol}(\text{SL}_2(\mathfrak{O})/\mathfrak{H}^3)|q|^2 Z(1, f_j)},
\]
with
\[
Z(s, f_j) = \frac{1}{4} \sum_{n \in \mathfrak{O}^\times} \frac{\lambda_j(n^2)}{|n|^{2s}}, \quad Z_q(s, f_j) = \frac{1}{4} \sum_{n \in \mathfrak{O}_S} \frac{\lambda_j(n^2)}{|n|^{2s}}.
\]

According to [EGM] §7.1, Theorem 1.1, we have \(\text{Vol}(\text{SL}_2(\mathfrak{O})/\mathfrak{H}^3) = 2\zeta(2)/\pi^2\). We also note that \(Z_q(s, f_j)\) is a finite Euler product and \(Z_q(1, f_j)\) is usually dispensable. Moreover, for \(f_j \in H^*(q')\), we have
\[
Z(s, f_j) = \zeta_{\mathfrak{O}, q'}(2s)\zeta_{\mathfrak{O}}(2s)^{-1} L(s, \text{Sym}^2 f_j).
\]

For our purpose, we only need the lower bound
\[
\omega_j^* \gg |q|^{-2-\varepsilon}(|i_j| + 1)^{-\varepsilon}.
\]

Moreover, in view of (3.28), we have
\[
\omega^*(t) = \frac{2\pi v(q)\zeta_{\mathfrak{O}}(1 + 2it)^2}{|q|^2|\zeta_{\mathfrak{O}}(1 + 2it)|^2},
\]
with the standard definition \(v(q) = |q|^2 \prod_{p \equiv 1(q)} (1 + 1/|p|^2)\). We also have the lower bound
\[
\omega^*(t) \gg |q|^{-2-\varepsilon} \min \{ |\varepsilon|^{-\varepsilon}, |\varepsilon|^2 \}.
\]

These two lower bounds are consequences of the estimate in [Mol] Theorem 1 applied to \(L(s, \text{Sym}^2 f_j)\) and \(\zeta_{\mathfrak{O}}(s + 2it)\), with the latter viewed as the \(L\)-function for the Hecke character \(\chi_{-2\mathfrak{O}}(n) = |n|^{-4it}\). Suffice it to say, [Mol] Theorem 1 is a very broad generalization of [Iwa] Theorem 2 to a large class of \(L\)-functions which satisfy an assumption much weaker than the Ramanujan hypothesis.

### 3.3. Hecke-Maass cusp forms for \(\text{SL}_3(\mathfrak{O})\)

Let \(\pi\) be a Hecke-Maass cusp form (or a cuspidal representation) for \(\text{SL}_3(\mathfrak{O})\). Suppose that the Archimedean Langlands parameter of \(\pi\) is the triple \((\mu_1, \mu_2, \mu_3)\), satisfying \(\mu_1 + \mu_2 + \mu_3 = 0\). Let \(A(n_1, n_2)\), with \(n_1, n_2 \in \mathfrak{O} \setminus \{0\}\), denote the diagonal coefficients of \(\pi\). Since all the \(\begin{pmatrix} e_1 e_2 \\ e_1 \\ 1 \end{pmatrix}\) may be generated from the diagonal elements in \(\text{SL}_3(\mathfrak{O})\) and the central elements \(e_3\), we infer that \(A(e_1 n_1, e_2 n_2) = A(n_1, n_2)\) for all \(e_1, e_2 \in \mathfrak{O}^\times\). Further, we assume that \(\pi\) is Hecke normalized in the sense that \(A(1, 1) = 1\). We have the multiplicative relation
\[
A(n_1 m_1, n_2 m_2) = A(n_1, n_2) A(m_1, m_2), \quad (n_1 n_2, m_1 m_2) = \emptyset,
\]
and the Hecke relation
\[
A(n_1, n_2) = \frac{1}{4} \sum_{d|n_1, d|n_2} \mu(d) A(n_1/d, 1) A(1, n_2/d).
\]
In this article, we assume that \( \pi \) is self-dual so that \((\mu_1, \mu_2, \mu_3) = (\mu, 0, -\mu)\) and \(A(n_1, n_2) = A(n_2, n_1) = A(n_1, n_3) = A(n_3, n_1)\). It is known by [GI] that \(\pi\) comes from the symmetric square lift of a Hecke-Maass form on \(GL_2(\mathbb{A})\). It follows from the Kim-Sarnak bound for \(GL_2(\mathbb{A})\) in [BB1] that \(\mu\) is either imaginary or real with
\[
|\text{Re} \mu| \leq \frac{7}{12},
\]
and, together the Hecke relation (3.37), that
\[
A(n_1, n_2) \leq |n_1 n_2|^{7/16 + \varepsilon}.
\]

Rankin-Selberg theory (see [JS]) implies the bound
\[
\sum_{|n| \leq X} |A(n, 1)|^2 \ll X^{\frac{3}{2}}, \quad X \geq 1.
\]
From this and the Kim-Sarnak bound (3.39) we deduce that for \(a_1, a_2 \in \mathbb{A}\) and \(X \geq 1\) that
\[
\sum_{|n| \leq X} |A(a_1 n, a_2)|^2 \ll |a_1 a_2|^{7/16 + \varepsilon} X^2.
\]
Indeed, in view of (3.36), the left-hand side is bounded by
\[
\sum_{a_1(a_1 a_2)^{\infty}} \sum_{|n| \leq X/|a_1|} |A(a a_1 n, a_2)|^2 \ll \sum_{a_1(a_1 a_2)^{\infty}} |A(a a_1, a_2)|^2 \sum_{|n| \leq X/|a_1|} |A(n, 1)|^2,
\]
and (3.41) then follows from (3.39) and (3.40). By Cauchy-Schwarz, we have
\[
\sum_{|n| \leq X} |A(a_1 n, a_2)| \ll |a_1 a_2|^{7/16 + \varepsilon} X^2.
\]

### 3.4. The Voronoï summation formula for \(SL_3(\mathbb{A})\).

The \(GL_3\) Voronoï summation formula over \(\mathbb{Q}\) was first discovered by Miller and Schmid [MS2]. In the adelic setting, the extension of the formula over an arbitrary number field may be found in the work of Ichino and Templier [IT].

#### 3.4.1. Hankel transforms and Bessel kernels for \(GL_3(\mathbb{C})\).

For a smooth compactly supported function \(w\) on \(\mathbb{C} \setminus \{0\}\), we associate a function \(W\) on \(\mathbb{C} \setminus \{0\}\) such that the following sequence of identities are satisfied, (see [IT] (1.1) or [QL] Theorem 3.15)
\[
M_m W(2s) = G_m(s, \pi)M_m w(2(1 - s)), \quad m \in \mathbb{Z},
\]
where \(M_m\) is the Mellin transform of order \(m\),
\[
M_m w(s) = \int_{\mathbb{C} \setminus \{0\}} w(z) (z/|z|)^m |z|^{1-2s} \, dz = 2 \int_0^{2\pi} \int_0^{\infty} w(xe^{i\theta}) e^{im\theta} \, d\theta \cdot x^{1-s} \, dx,
\]
in which \(G_m(s, \pi)\) is the gamma factor for \(\pi\) of order \(m\) given by
\[
G_m(s, \pi) = \prod_{\ell=1,2,3} \Gamma \left( s - \frac{\mu_\ell}{2} + \frac{1}{2} |m| \right) \Gamma \left( 1 - s + \frac{\mu_\ell}{2} + \frac{1}{2} |m| \right),
\]
\(W\) is called the Hankel transform of \(w\) (of index \((\mu_1, \mu_2, \mu_3)\)). It is known that the Hankel transform admits an integral kernel \(J(\mu_1, \mu_2, \mu_3)\) (see [QL] §3.4), namely,
\[
W(u) = \int_{\mathbb{C} \setminus \{0\}} w(z) J(\mu_1, \mu_2, \mu_3)(uz) \, dz.
\]
3.4.2. *The Voronoi summation formula for $SL_3(\mathbb{O})$.* Translating the adèlic form of the Voronoi summation formula in [11, Theorem 1] into the classical language, we have the following formula in terms of classical quantities. See [Q12, Proposition 3.4 and Remark 3.5].

**Proposition 3.5.** Let $w$ be a smooth compactly supported function on $\mathbb{C} \setminus \{0\}$. For $n_1, n_2 \in \mathbb{O} \setminus \{0\}$, let $A(n_1, n_2)$ be the $(n_1, n_2)$-th Fourier coefficient of a Hecke-Maass form $\pi$ for $SL_3(\mathbb{O})$. Let $a, b, c \in \mathbb{O}$ be such that $c \neq 0$, $(a, c) = \mathbb{O}$ and $a \bar{a} \equiv 1 (\text{mod } c)$. Then we have

$$
\sum_{n_2 \in \mathbb{O} \setminus \{0\}} A(n_1, n_2) e \left( \frac{an_2}{c} \right) w \left( \frac{n_2^2}{2} \right) = \frac{1}{4|c^2n_1|^2} \sum_{n_{3|n_1}} |n_3|^2 \sum_{a \in \mathbb{O} \setminus \{0\}} A(n_4, n_3) S(\bar{a}n_1, cn_2/n_3) W \left( \frac{n_2^2n_4}{4c^3n_1} \right),
$$

where $S(\bar{a}n_1, cn_2/n_3)$ is a Kloosterman sum as defined in (2.11) and $W$ is the Hankel transform of $w$ given by (3.43) or (3.44).

**Remark 3.6.** It should be warned that our normalization of Hankel transforms in [Q13] is slightly different from that in [MS1, MS2] in order to be consistent with the Fourier transform and the classical Hankel transform when the rank is one and two; see [Q12, Remark 3.5] for the difference. As such, the look of the Voronoi formula here is also different from those in the literature.

3.5. *$L$-functions and their approximate functional equations.* Let $q$ be squarefree such that $N(q) > 1$ and $N(q) \equiv 1 (\text{mod } 8)$. Let $\chi$ be the primitive quadratic Hecke character of conductor $q$ and frequency 0. Let $f_j$ be an even Hecke-Maass newform for $\Gamma_0(q')$ with $q' | q$. Let $E(w, s)$ be the Eisenstein series for $SL_2(\mathbb{O})$. Let $\pi$ be a fixed self-dual Hecke-Maass form for $SL_3(\mathbb{O})$.

**3.5.1. $L$-functions $L(s, \pi \otimes \chi), L(s, \pi \otimes f_j \otimes \chi)$ and $L(s, \pi \otimes E(\cdot, \frac{1}{2} + it) \otimes \chi)$.* The $L$-function attached to the twist $\pi \otimes \chi$ is defined by

$$
L(s, \pi \otimes \chi) = \sum_{(n) \neq 0} \frac{A(1, n)\chi(n)}{N(n)^s}.
$$

The Rankin-Selberg $L$-function $L(s, \pi \otimes f_j \otimes \chi)$ is defined by

$$
L(s, \pi \otimes f_j \otimes \chi) = \sum_{(n_1), (n_2) \neq 0} \frac{A(n_1, n_2)\lambda_j(n_2)^s \chi(n_2)}{N(n_1^2n_2)^s}.
$$

This definition requires only the Hecke eigenvalues $\lambda_j(n)$ with $(n, q) = \mathbb{O}$.

Recall that the root number $e(\chi) = 1$ (see [2.3] and that $\pi$ is a symmetric lift of a $GL_2(\mathbb{O})$-automorphic form ([GJ]). By the results on local $\varepsilon$-factors in [Sch §1.2, 3] and [Kna §§4], we infer that the conductors of $\pi \otimes \chi$ and $\pi \otimes f_j \otimes \chi$ are $q^3$ and $q^6$, respectively, and that both of the root numbers $e(\pi \otimes \chi) = e(\pi \otimes f_j \otimes \chi) = 1$.

The completed $L$-function for $\pi \otimes \chi$ is $\Lambda(s, \pi \otimes \chi) = N(q)^{3/2} \gamma(s, \pi \otimes \chi) L(s, \pi \otimes \chi)$, where $\gamma(s, \pi \otimes \chi) = \gamma(s, \pi) = \gamma(s)$, with

$$
\gamma(s) = \gamma(s, \mu) = 2^3(2\pi)^{-3/2} \Gamma(s + \mu) \Gamma(s - \mu).
$$

$\Lambda(s, \pi \otimes \chi)$ is entire and has the following functional equation

$$
\Lambda(s, \pi \otimes \chi) = \Lambda(1 - s, \pi \otimes \chi).
$$
Let \( \Lambda(s, \pi \otimes f_j \otimes \chi) = N(q)^{3v} \gamma(s, \pi \otimes f_j \otimes \chi)L(s, \pi \otimes f_j \otimes \chi) \), with the gamma factor 
\[
\gamma(s, \pi \otimes f_j \otimes \chi) = \gamma(s, \tau_j),
\]
defined by 
(3.48) \[
\gamma(s, t) = \gamma(s - it) \gamma(s + it).
\]
\( \Lambda(s, \pi \otimes f_j \otimes \chi) \) is also entire and satisfies the functional equation 
(3.53) \[
\Lambda(s, \pi \otimes f_j \otimes \chi) = \Lambda(1 - s, \pi \otimes f_j \otimes \chi).
\]

Similar as (3.46), we define 
(3.49) \[
L \left( s, \pi \otimes E \left( \cdot, \frac{1}{2} + it \right) \otimes \chi \right) = \sum_{(n_1), (n_2) \neq 0} \frac{A(n_1, n_2) \eta(n_2) \chi(n_2)}{n(n_1)^s}.
\]
We have 
(3.50) \[
L \left( s, \pi \otimes E \left( \cdot, \frac{1}{2} + it \right) \otimes \chi \right) = L \left( s + it, \pi \otimes \chi \right) L \left( s - it, \pi \otimes \chi \right),
\]
and hence 
(3.51) \[
L \left( \frac{1}{2}, \pi \otimes E \left( \cdot, \frac{1}{2} + it \right) \otimes \chi \right) = \left| L \left( \frac{1}{2} + it, \pi \otimes \chi \right) \right|^2.
\]

### 3.5.2. Approximate functional equations for \( L(s, \pi \otimes f_j \otimes \chi) \) and \( L(s, \pi \otimes \chi) \)

Following Blomer [Blo], for positive integer \( A' \), we introduce 
(3.52) \[
p(s, t) = p(s, t, \mu) = \prod_{k=0}^{A'-1} \prod_{\pm} (s \pm it + \mu + k) (s \pm it + k) (s \pm it - \mu + k)
\]
so that \( p(s, t) \) kills the right most \( A' \) many poles of each of the gamma factors in \( \gamma(s, t) \) defined by (3.47), (3.48).

We have the following approximate functional equation for \( L(s, \pi \otimes f_j \otimes \chi) \), (see [IK, Theorem 5.3])
(3.53) \[
L \left( \frac{1}{2}, \pi \otimes f_j \otimes \chi \right) = 2 \sum_{(n_1), (n_2) \neq 0} \frac{A(n_1, n_2) \eta(n_2) \chi(n_2)}{|n_1 n_2|^s} V \left( |n_1 n_2|/|q|^3, \tau_j \right),
\]
with 
(3.54) \[
V(y, t) = \frac{1}{2\pi i} \int_{(3)} G(v, t) y^{-2v} \frac{dv}{v}, \quad y > 0,
\]
in which 
(3.55) \[
G(v, t) = \frac{\gamma \left( \frac{1}{2} + v, t \right)}{\gamma \left( \frac{1}{2}, t \right)} \frac{p \left( \frac{1}{2} + v, t \right) p \left( \frac{1}{2} - v, t \right)}{p \left( \frac{1}{2}, t \right)^2}.
\]
Note that the second quotient in (3.55) is even in \( v \) and is equal to 1 when \( v = 0 \). Similarly, the approximate functional equation for \( L(s, \pi \otimes E \left( \cdot, \frac{1}{2} + it \right) \otimes \chi) \), along with (3.51), yields 
(3.56) \[
|L \left( \frac{1}{2} + it, \pi \otimes \chi \right)|^2 = 2 \sum_{(n_1), (n_2) \neq 0} \frac{A(n_1, n_2) \eta(n_2) \chi(n_2)}{|n_1 n_2|^s} V \left( |n_1 n_2|/|q|^3, t \right).
\]

**Lemma 3.7.** Let \( U > 1 \) and \( \varepsilon > 0 \). Let \( A \) be a positive integer.

(1). We have 
(3.57) \[
p \left( \frac{1}{2}, t \right)^2 V(y, t) \ll_{A, A'} \left( |t| + 1 \right)^{2A'} \left( 1 + \frac{y}{(|t| + 1)\varepsilon} \right)^{-A}.
\]
and 
(3.58) \[
V(y, t) = \frac{1}{2\pi i} \int_{|v|=U} G(v, t) y^{-2v} \frac{dv}{v} + O_{\varepsilon} \left( \left( |t| + 1 \right)^{6A} \right).
\]
(2). When \( \Re v > 0 \), the function \( G(v, t) (\frac{1}{2}, t)^2 \) is even in \( t \) and holomorphic in the region \( |\Im t| < A' + \frac{9}{32} = A' + \frac{1}{2} - \frac{\varepsilon}{32} \), and satisfies in this region the uniform bound

\[
(3.59) \quad G(v, t) p (\frac{1}{2}, t)^2 \ll A', \Re \varepsilon (|t| + 1)^{12A' + 6\Re \varepsilon}.
\]

Proof. (3.57) may be found in [IK] Proposition 5.4. The expression of \( V(y, t) \) in (3.58) is due to Blomer, [Blo] Lemma 1. The estimate in (3.59) is a consequence of Stirling’s approximation.

It follows from (3.57) that \( V(|m^2_n|^3, t) \) decays rapidly for \( |m^2_n| > (|y|(|t|^2 + 1))^{1+\varepsilon} \), so we can effectively take the sums in (3.53) and (3.56) above to be finite.

4. Properties of the rank-two Bessel kernel and the Bessel integral

In this section, we shall analyze the Bessel kernel \( J_\nu(z) \) and the Bessel integral \( H(z) \) that occur in the GL_2 Kuznetsov trace formula. It should be noted that the estimates for \( J_\nu(z) \) obtained here are very crude in the \( t \) aspect.

4.1. Analysis of the Bessel kernel.

Lemma 4.1. Let \( t \) be real. Let \( K \) be a nonnegative integer. We have

\[
(4.1) \quad J_\nu(z) = e(4 \Re z) W(z) + e(-4 \Re z) W(-z) + E(z),
\]

where \( W(z) \) and \( E(z) \) are real analytic functions satisfying

\[
(4.2) \quad z^a z^b (\partial/\partial z)^a (\partial/\partial z)^b W(z) \ll_{a, b, K} 1/|z|, \quad |z| \gg (|t| + 1)^{2},
\]

\[
(4.3) \quad (\partial/\partial z)^a (\partial/\partial z)^b E(z) \ll_{a, b, K} (|t| + 1)^{2K}/|z|^{1+K}, \quad |z| \gg (|t| + 1)^{2}.
\]

We have the uniform bound

\[
(4.4) \quad |t J_\nu(z)| \ll (|t| + 1)^3 \min \{1, 1/|z|\}.
\]

Proof. The identity (4.1) comes from the expression of \( J_\nu(z) \) as in (3.18, 3.19), with \( m = 0 \), by truncating the asymptotic expansions of \( H_2^{(1)} \) and \( H_2^{(2)} \) as in (3.20) and (3.21) in Lemma 3.2. To be precise,

\[
W(z) = 2\pi \sum_{k \neq 0} \sum_{K=0}^{K-1} \frac{(2it, k)(2it, k')}{-8\pi i} \frac{1}{z^{K+1/2} e^{1/2}}.
\]

Thus (4.2) is transparent and (4.3) follows from (3.22) Moreover, applying Lemma 3.2 with \( K = 0 \), we derive the estimate \( J_\nu(z) \ll 1/|z| \) when \( |z| \gg (|t| + 1)^2 \), which is actually stronger than (4.4). Furthermore, estimating the integral in (3.23), with \( m = 0 \), by

\[
J_0(\chi x) \ll \min \{1, 1/\sqrt{x}\}, \quad x > 0,
\]

we get

\[
J_\nu (xe^{\theta}) \ll \int_{1/2}^2 dy \frac{1}{y} + \int_{1/2}^{1/2} dy \frac{1}{\sqrt{y - 1/|y|}} \ll 1,
\]

for all \( x > 1 \). This is also stronger than (4.4) when \( 1 \ll |z| < (|t| + 1)^2 \). Finally, suppose \( |z| \ll 1 \). We have

\[
|J_\nu(z)| \ll_{\Re \nu} \frac{|z|^\nu \exp(|\Im z|)}{\Gamma(\nu + \frac{1}{2})}, \quad \Re \nu > -\frac{1}{2},
\]

as a consequence of Poisson’s integral representation (see [Wat] 3.3 (6)),

\[
J_\nu(z) = \frac{(\frac{1}{2})^{\nu}}{\Gamma(\nu + \frac{1}{2})} \int_0^{\pi} e^{i\nu \cos \theta} \sin 2\nu \theta d\theta.
\]
The reader is referred to \[ \text{Wat} \ 3.31 (1) \] for \( \nu \) real, where the bound is slightly different and cleaner, but one should note that the integral of \( \sin^{2\nu} \theta \) therein needs to be replaced by that of \( |\sin^{2\nu} \theta| = (\sin \theta)^{2\Re \nu} \) here for complex values of \( \nu \). Applying (4.6) with \( \nu = \pm 2it \) to the two products of Bessel functions in the definition of \( J_\nu(z) \) as in (3.24), along with \( |\Gamma \left( \frac{1}{2} + 2it \right)|^2 = \pi / \cosh (2\pi t) \) (due to Euler’s reflection formula), we infer that
\[
|J_\nu(z)| \leq |t| \coth (2\pi t) \leq |t| + 1,
\]
if \( |z| \leq 1 \).

Q.E.D.

For later applications, we would like to further generalize the second part of Lemma 4.1 to the derivatives of \( J_\nu(z) \). For this we need the following lemma.

**Lemma 4.2.** Let \( t \) be real. Let \( J_{\alpha,\beta}(z) \) be the Bessel function defined by (3.16) and (3.17). Then there are polynomials \( P^0_\alpha(Y,Z) \) and \( P^1_\alpha(Y,Z) \) of degree \( |\alpha|/2 \) and \( (|\alpha| - 1)/2 \) respectively such that
\[
2z^\alpha z^\beta (\partial / \partial z)^\alpha (\partial / \partial z)^\beta J_\alpha(z) = P^0_\alpha P^0_\alpha J_\alpha(z)
\]
\[
+ z^\alpha z^\beta P^0_\alpha P^0_\alpha J_{\alpha+1/2}(z) + J_{\alpha-1/2}(z) + J_{\alpha+1/2}(z) + J_{\alpha-1/2}(z)
\]
\[
+ izP^1_\alpha P^0_\alpha J_{\alpha+1/2}(z) + J_{\alpha-1/2}(z) + izP^1_\alpha P^0_\alpha J_{\alpha+1/2}(z) + J_{\alpha-1/2}(z),
\]
in which \( P^0_\alpha \ldots \) are the shorthand notation for \( P^a(-4t^2, z^2) \ldots \).

**Proof.** First, by the Bessel differential equation for \( J_\nu(z) \) (\[ \text{Wat} \ 3.1 (1) \]),
\[
z^\alpha z^\beta (\partial / \partial z)^\alpha (\partial / \partial z)^\beta J_\alpha(z) = 0,
\]
it is straightforward to prove that we may write
\[
z^\alpha (\partial / \partial z)^\alpha J_\nu(z) = z P^0_\nu (v^2, z^2) J'_\nu(z) + P^0_\nu (v^2, z^2) J_\nu(z),
\]
for certain (integer-coefficient) polynomials \( P^0_\nu(Y,Z) \) and \( P^1_\nu(Y,Z) \) of degree \( |\alpha|/2 \) and \( (|\alpha| - 1)/2 \) respectively. Second, we have the recurrence relation (\[ \text{Wat} \ 3.2 (2) \])
\[
2J'_\nu(z) = J_{\nu-1}(z) - J_{\nu+1}(z).
\]
From these and the definitions in (3.16) and (3.17) follows (4.7) by direct calculations.

Q.E.D.

**Lemma 4.3.** We have
\[
t(4t^2 + 1)z^\alpha z^\beta z^\alpha \partial / \partial z^\beta \partial / \partial z^\beta J_\nu(z) \ll_{\alpha,\beta} (|t| + 1)^5 \min \{1, 1/|z| \} (|z| + |t| + 1)^{\nu + \beta}.
\]

**Proof.** The estimates may be proven by the arguments in the proof of Lemma 4.1 applied to the Bessel functions in (4.7) in Lemma 4.2. The conscientious reader however might have already noticed that there are two technical issues: (4.6) is only valid for \( \Re \nu > -\frac{1}{2} \) and (3.23) for \( |\Re \mu| < \frac{1}{2} \). These issues however may be addressed as follows.

By using the recurrence formula (\[ \text{Wat} \ 3.2 (1) \])
\[
J_{\nu-1}(z) = \frac{2\nu}{z} J_\nu(z) - J_{\nu+1}(z),
\]
we deduce from (4.6) that (see \[ \text{Wat} \ 3.31 (2) \] for \( \nu \) real)
\[
J_\nu(z) \ll_{\Re \nu} \frac{\exp(\text{Im } z)}{|\Gamma (\nu + \frac{1}{2})|} \left( |\nu + 1| + \frac{|z|^2}{|\nu + \frac{1}{2}|} \right), \quad -\frac{3}{2} < \Re \nu \leq -\frac{1}{2}.
\]
Thus we are now able to estimate all the Bessel functions in (4.7), like \( J_{\nu-1/2}(z) \ldots \), by applying either (4.6) or (4.8).

Moreover, we have the following lemma.
Lemma 4.4. Suppose that $|\text{Re} \mu| < (2k + 1)/8$. For any $|z| > 1$, we have
\begin{equation}
J_{\mu,m}(z) \ll_{\mu,m,1} 1 + (|\mu| + 1)^k/|z|^{k + 1/2}.
\end{equation}

Proof of Lemma 4.4 First, for the integral in (3.23), we introduce a smooth partition of unity $(1 - \nu(y)) + \nu(y) \equiv 1$ for the domain of integration $(0, \infty) = [1/3, 3] \cup \{0, 1/2\} \cup [2, \infty)$. The integral over $[1/3, 3]$ is absolutely convergent (for all $\mu$) and bounded. Next, we consider the integral over $[2, \infty)$, while that over $(0, 1/2)$ may be treated in the same way after changing the variable $y$ to $1/y$. We apply to the $J_m(4\pi x |\nu e^{i\phi} + y^{-1} e^{-i\phi}|)$ in (3.23) the following asymptotic expansion (see [Wat 7.21 (1)]),
\begin{equation}
J_m(x) = \left(\frac{2}{\pi x}\right)^{1/2} \left(\cos \left(x - \frac{i}{2} m \pi - \frac{1}{4} \pi\right) \sum_{l=0}^{(k-1)/2} \frac{(-1)^l \cdot (m, 2l)}{(2x)^{2l}} \right)
\end{equation}
\begin{equation}
- \sin \left(x - \frac{i}{2} m \pi - \frac{1}{4} \pi\right) \sum_{l=0}^{(k-2)/2} \frac{(-1)^l \cdot (m, 2l + 1)}{(2x)^{2l+1}} + O \left(\frac{1}{x^k}\right),
\end{equation}
in which $(m, l)$ is the coefficient defined as in Lemma 3.2. The error term contributes an absolutely convergent integral with the bound $1/x^{k+1/2}$. Thus we have to consider integrals of the form
\begin{equation}
\int_2^{\infty} w_{\mu,m}(y, \phi) e(\pm 2xy) dy,
\end{equation}
with
\begin{equation}
w_{\mu,m}(y, \phi) = \frac{y^{\mu - 1} (\nu e^{i\phi} + y^{-1} e^{-i\phi})^m}{(\nu e^{i\phi} + y^{-1} e^{-i\phi})^{m+1} + 1} e \left(\pm 2 (|\nu e^{i\phi} + y^{-1} e^{-i\phi}| - y)\right).
\end{equation}
Note that $|\nu e^{i\phi} + y^{-1} e^{-i\phi}| - y = O(1)$ when $y \geq 2$. So by repeating partial integration $k - l$ many times we get absolute convergence when $|\text{Re} \mu| < (2k + 1)/8$ and the bound $(|\mu| + 1)^k/|z|^{k + 1/2}$. Now (4.9) follows after collecting the bounds that we have established.

Q.E.D.

Using (4.6, 4.8), (4.9) along with (3.18, 3.19), we may deduce the estimates in this lemma from (4.7) in Lemma 4.2

Q.E.D.

4.2. Bounds for the Bessel integral. Let $A'$ be a positive integer as in (3.5.2). Subsequently, we shall fix the choice of spectral weight
\begin{equation}
h(t) = k(t) G(u, t),
\end{equation}
with
\begin{equation}
k(t) = e^{-\gamma / T^2} p \left(\frac{t}{2}, t\right)^2 p(t)/g(t),
\end{equation}
p and $k(t)$ defined as in (3.5.2), $G(u, t)$ as in (3.5.5), and
\begin{equation}
p(t) = \prod_{k=0}^{2k-1} \left(4t^2 + (k + 1)^2\right), \quad g(t) = \left(t + (A' + 1)^3\right)^{kA'}.
\end{equation}
Note that by the Kim-Sarnak bound for $it_j$ and $\mu$ in (3.2, 3.38), we have
\begin{equation}
k(t) > 0,
\end{equation}
and
\begin{equation}
k(t) \rightarrow A' e^{-\gamma / T^2}, \quad t \rightarrow \infty,
\end{equation}
where
if \( t \) is the spectral parameter \( t_j \) of a Maass form \( f_j \) or the Eisenstein series. In view of (3.59) in Lemma 3.7(2), we have
\[
(4.15) \quad h(t) \ll A', \mu \left( |t| + 1 \right)^{6} e^{-\left(\text{Re}(t) \right)^{2}/T^{2}}, \quad |\text{Im} t| < A' + \frac{\pi}{\mu}.
\]

**Lemma 4.5.** Let \( H(z) \) be defined as in (3.27) with the choice of \( h(t) \) given by (4.10)-(4.12) and also (3.47), (3.48), (3.52), (3.55). We have
\[
(4.16) \quad H(z) \ll T^{3+\epsilon} \min \left\{ |z|^{A'}, 1/|z| \right\}.
\]

**Proof.** First let \( |z| \leq 1 \). By the definitions in (3.24) and (3.27), we have
\[
H(z) = 4\pi^{2}i \int_{-\infty}^{\infty} h(t) J_{2\mu} \left( 4\pi |z| \right) J_{2\mu} \left( 4\pi |z| \right) \frac{t^{2}}{\sinh (2\pi t)} |t|^{2} dt.
\]
Since we have included the polynomial \( p(t) \) in the definition of \( h(t) \) as in (4.10)-(4.12), \( h(t)/\sinh (2\pi t) \) is holomorphic when \( |\text{Im} t| < A' + \frac{\pi}{\mu} \). We now shift the line of integration to \( \text{Re}(it) = A' \) and estimate the resulting integral by (4.15) and (4.6) along with Stirling’s formula. Then follows (4.16) for \( |z| \leq 1 \). The case \( |z| > 1 \) follows from (4.3) in Lemma 4.1.

5. Properties of the rank-three Bessel kernel

In order to apply the Voronoi formula for \( SL_{3}(\mathbb{C}) \), one must understand the asymptotic behaviour of the Hankel transform or its Bessel kernel for \( GL_{3}(\mathbb{C}) \). This is done in the author’s work (Qi3), combining the high-dimensional stationary phase method, due to Hörmander, for certain formal integrals and the asymptotic theory for Bessel differential equations. Some discussions may be found in the remarks after (Qi2) Proposition 5.3.

Let the Bessel kernel \( J_{(\mu_{1},\mu_{2},\mu_{3})}(z) \) be defined by (3.43) and (3.44) in §3.4.1. The following asymptotic expansion formula for \( J_{(\mu_{1},\mu_{2},\mu_{3})}(z) \) will be the foundation of our analysis in this article. See (Qi3) Theorem 16.6.

**Lemma 5.1.** Let the Bessel kernel \( J_{(\mu_{1},\mu_{2},\mu_{3})}(z) \) be given as in §3.4.1. Let \( K \) be a nonnegative integer. For \( |z| \geq 1 \), we have the asymptotic expansion
\[
J_{(\mu_{1},\mu_{2},\mu_{3})}(z) = \sum_{\ell=1} e \left( \frac{3}{2} \Re z^{3/2} + \Re \sigma z \right) \left( \sum_{k+l=K} B_{k+l} e^{-k+i\ell} \right) + O_{K,\mu_{1},\mu_{2},\mu_{3}} \left( |z|^{-K+3/2} \right),
\]
where \( B_{k+l} = B_{k+l}(\mu_{1},\mu_{2},\mu_{3}) \) is a symmetric polynomial in \( \mu_{1},\mu_{2},\mu_{3} \) of degree \( 2k \). Moreover, similar asymptotic expansions are valid for all the partial derivatives of \( J_{(\mu_{1},\mu_{2},\mu_{3})}(z) \).

Moreover, we have the following bounds for \( J_{(\mu_{1},\mu_{2},\mu_{3})}(z) \) when \( |z| \leq 1 \). See (Qi2) Lemma 5.1 and its proof.

**Lemma 5.2.** Suppose that \( \text{Re} \mu_{1}, \text{Re} \mu_{2}, \text{Re} \mu_{3} < \sigma \). For \( |z| \leq 1 \), we have
\[
\zeta \sum_{n=1} \left( \frac{3}{2} \Re z^{3/2} + \Re \sigma z \right)^{n/2} J_{(\mu_{1},\mu_{2},\mu_{3})}(z) \ll a_{\beta,\mu_{1},\mu_{2},\mu_{3}} 1/|z|^{2\sigma}.
\]

Both (Qi3) Theorem 16.6 and (Qi2) Lemma 5.1 concern only \( J_{(\mu_{1},\mu_{2},\mu_{3})}(z) \) but not its derivatives. However, the former may be generalized to the derivatives of \( J_{(\mu_{1},\mu_{2},\mu_{3})}(z) \) by using the full (Qi3) Theorem 11.24, while the generalization of the latter is very straightforward.

6. Analysis of the Hankel transform

This section is devoted to the analysis of the \( GL_{3}-\text{Hankel transform of the GL}_{2}-\text{Bessel integral.} \)
6.1. Oscillatory double integrals. Consider

\[ I(\lambda, \theta) = \int_0^{2\pi} \int_0^\infty e(\lambda f(x, \phi; \theta)) w(x, \phi; \lambda, \theta) \, dx \, d\phi, \]

with

\[ f(x, \phi; \theta) = 3x^2 \cos(2\phi + \theta) - 2x^3 \cos 3\theta - \cos 3\phi. \]

Suppose that \( w(x, \phi; \lambda, \theta) \) is supported in \( \{(x, \phi) : x \in [\rho, 21/6]\} \) and its derivatives satisfy

\[ x^{\alpha} \partial_x^\alpha \partial_\phi^\beta \partial_\theta^\gamma c_\phi^\delta w(x, \phi; \lambda, \theta) \ll a, \beta, \gamma, \delta \cdot S \cdot X^{|a + \beta + \gamma + \delta|}, \]

with \( S > 0, X \geq 1 \). We have

\[ f'(x, \phi; \theta) = (6x (\cos (2\phi + \theta) - x \cos 3\phi), -6x^2 (\sin (2\phi + \theta) - x \sin 3\phi), \]

and hence \( f(x, \phi; \theta) \) has a unique stationary point at \((1, \theta)\).

The modified integral \( e(2\pi \cos 3\theta) I(\lambda, \theta) \) may be considered as a \( GL_3 \times GL_2 \)-type Fourier-Hankel convolution and results in an oscillatory function of \( GL_1 \)-type. Similar integrals of type \( GL_3 \times GL_1, GL_2 \times GL_1 \) and \( GL_1 \times GL_2 \) have been studied in the author's previous work. See [Hör] §5.3 and [Q1] §5.1.3.

First, we would like to apply Hörmander’s elaborate partial integration (see the proof of [Hör] Theorem 7.7.1) in the polar coordinates. For this, we define

\[ g(x, \phi; \theta) = \left( \partial_x f(x, \phi; \theta) \right)^2 / x^3 + \left( \partial_\phi f(x, \phi; \theta) \right)^2 / x^5 \]

\[ = 36\left( \left( \sqrt{x} - 1 / \sqrt{x} \right)^2 + 2(1 - \cos(\phi - \theta)) \right). \]

An important observation is that

\[ g(x, \phi; \theta) > \begin{cases} 4x, & \text{if } x > 3/2, \\ 4x, & \text{if } x < 2/3. \end{cases} \]

We then introduce the differential operator

\[ D = \frac{\partial_x f(x, \phi; \theta)}{x^3 g(x, \phi; \theta)} \frac{\partial_x f(x, \phi; \theta)}{x^3 g(x, \phi; \theta)} + \frac{\partial_\phi f(x, \phi; \theta)}{x^3 g(x, \phi; \theta)} \frac{\partial_\phi f(x, \phi; \theta)}{x^3 g(x, \phi; \theta)} \]

so that \( D \left( e(\lambda f(x, \phi; \theta)) \right) = 2\pi i \hat{\chi} \cdot e(\lambda f(x, \phi; \theta)) \). Consequently,

\[ D^* = -\partial_x \frac{\partial_x f(x, \phi; \theta)}{x^3 g(x, \phi; \theta)} - \partial_\phi \frac{\partial_\phi f(x, \phi; \theta)}{x^3 g(x, \phi; \theta)} \]

is the adjoint and

\[ I(\lambda, \theta) = \int_0^{2\pi} \int_0^\infty e(\lambda f(x, \phi; \theta)) D^{*A} w(x, \phi; \lambda) \, dx \, d\phi \]

for any integer \( A \geq 0 \). By a straightforward inductive argument, it may be shown that \( D^{*A} w \) is a linear combination of all the terms occurring in the expansions of

\[ \partial_x^{\alpha} \partial_\phi^{\beta} \left( (\partial_x f(x, \phi; \theta) / x^3)^{\alpha} (\partial_\phi f(x, \phi; \theta) / x^5)^{\beta} g^A w \right) / g^{2A}, \quad \alpha + \beta = A. \]

Moreover, we have

\[ x^{a+2} \partial_x^{\alpha} \partial_\phi^{\beta} \left( \partial_x f(x, \phi; \theta) / x^3 \right), \quad x^{a+3} \partial_x^{\alpha} \partial_\phi^{\beta} \left( \partial_\phi f(x, \phi; \theta) / x^5 \right) \ll x + 1, \]

\[ x^2 \partial_\phi g(x, \phi; \theta) \ll (x + 1)^2, \quad x^{a+1} \partial_x^{\alpha} \partial_\phi^{\beta} g(x, \phi; \theta) \ll 1, \quad \partial_\phi^{\beta} g(x, \phi; \theta) \ll 1, \quad (\alpha \geq 2, \beta \geq 1), \]

\[ \partial_x \partial_\phi g(x, \phi; \theta) \equiv 0. \]

Let \( \alpha, \beta, \alpha \leq \alpha \) and \( \beta, \beta' \leq \beta \). From the estimates above, it is straightforward to prove that

\[ x^{a+2} \partial_x^{\alpha} \partial_\phi^{\beta} \left( \partial_x f(x, \phi; \theta) / x^3 \right)^{\alpha} (\partial_\phi f(x, \phi; \theta) / x^5)^{\beta} \ll \frac{(x + 1)^{a+\beta}}{x^{2a+3\beta}}. \]
\[
\frac{x^\gamma \partial^\alpha \partial^\beta g (x, \phi; \theta)^A}{g (x, \phi; \theta)^{2\lambda}} \leq \sum_{\alpha_1 + \alpha_2 = \lambda} \sum_{\beta_1 \leq \beta} \sum_{a_1 + 2a_2 = \alpha_1} (x + 1)^{2\alpha_1} x^{\alpha_1 + \alpha_2} g (x, \phi; \theta) |x^\alpha \partial^\gamma \partial^\delta w| |x^\gamma \partial^\alpha \partial^\beta w| \int_0^1 \cdots \int_0^1 \text{d}x d\phi.
\]

Combining these, we deduce the following estimate,

\[
I (\lambda, \theta) \leq \sum_{\alpha_1 + 2a_2 + \alpha + \beta \leq A} \frac{1}{\lambda^A} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \frac{X^\alpha \partial^\gamma \partial^\delta w}{X^\alpha \partial^\gamma \partial^\delta w} \int_0^{2\pi} \cdots \int_0^{2\pi} \text{d}x d\phi.
\]

**Lemma 6.1.** For either \( \rho > 2 \) or \( \rho < 1/2 \), we have

\[
\lambda^2 \frac{\partial^\gamma \partial^\delta f (\lambda, \theta)}{\partial^\gamma \partial^\delta} \leq X \frac{X}{\lambda^2} \left( \frac{X}{\lambda^2} \right)^A.
\]

Proof. When \( \rho > 2 \), it follows from (6.3), (6.4) and (6.5) that \( I (\lambda, \theta) \) is bounded by

\[
\sum_{\alpha_1 + \beta_1 + 2a_2 + \alpha + \beta \leq A} \frac{1}{\lambda^A} \int_0^{2\pi} \frac{X^\alpha \partial^\gamma \partial^\delta w}{X^\alpha \partial^\gamma \partial^\delta w} \int_0^{2\pi} \cdots \int_0^{2\pi} \text{d}x d\phi.
\]

Similarly, when \( \rho < 1/2 \), \( I (\lambda, \theta) \) is bounded by

\[
\sum_{\alpha_1 + \beta_1 + 2a_2 + \alpha + \beta \leq A} \frac{1}{\lambda^A} \int_0^{2\pi} \frac{X^\alpha \partial^\gamma \partial^\delta w}{X^\alpha \partial^\gamma \partial^\delta w} \int_0^{2\pi} \cdots \int_0^{2\pi} \text{d}x d\phi.
\]

In general, one applies the two estimates above to the integrals obtained after differentiation.

Q.E.D.

Next, we consider the case when \( 1/2 \leq \rho \leq 2 \) and aim to get the stationary phase estimate for the derivatives of \( I (\lambda, \theta) \). For this, the following lemma is very useful, which can be viewed as a substitute for the Morse lemma. It may be proven by straightforward algebraic and trigonometric computations.

**Lemma 6.2.** Let \( f (x, \phi; \theta) \) be defined as in (6.2). We may write

\[
f = p \cdot (x - 1)^2 + 2q \cdot (x - 1) \sin (\phi + \theta) + r \cdot \sin^2 (\phi - \theta) + \frac{1}{2}.
\]

with

\[
p(x, \phi, \theta) = -(2x + 1) \cos (2\phi + \theta),
\]

\[
q(x, \phi, \theta) = 2 (x^2 + x + 1) \sin (5\phi + \theta) / 2,
\]

\[
r(x, \phi, \theta) = \cos (2(\phi - \theta) / 2) (2 \cos 2(2\phi + \theta) + \cos 2(\phi + 2\theta)).
\]

When \( 1/2 \leq x \leq 2/\rho \), it is clear that

\[
(6.6) \quad \partial^\alpha \partial^\beta f (x, \phi, \theta) \leq 1.
\]

We now compute the derivatives of \( I (\lambda, \theta) \) explicitly. On changing the variable to \( \phi + \theta \) and then differentiating under the integral, with the help of Lemma 6.2, some calculations show that \( \lambda^\gamma \partial^\lambda \partial^\beta f (\lambda, \theta) \) may be expressed as a linear combination of integrals of the form

\[
(6.7) \quad \chi^{(a + \beta + \tau) / 2} \int_0^{2\pi} \cdots \int_0^{2\pi} \text{w} (x, \phi, \theta) \partial^\beta f (x, \phi, \theta) \frac{1}{\chi^{(a + \beta + \tau)}} \text{d}x d\phi.
\]

with \( a + \beta + \gamma + \chi \leq 2 \gamma \) and \( \kappa + \gamma + 2\delta \leq 2 \delta \) such that \( \kappa + \tau \) is even and that \( \delta' = \delta \) if \( \kappa = \tau = 0 \). Moreover, \( v_{\alpha \beta} \) are defined by

\[
(p \cdot Y^2 + 2q \cdot YZ + r \cdot Z^2) \gamma - \gamma = \sum_{\alpha \beta \gamma} v_{\alpha \beta} \cdot Y^\alpha Z^\beta.
\]
and \( w_{x\tau} \) are defined by \( w_{00} \equiv 1 \) and
\[
\overline{c}_\sigma^{-\phi} e \left( \lambda \left( p \cdot Y^2 + 2q \cdot YZ + r \cdot Z^2 \right) \right) e \left( \lambda \left( p \cdot Y^2 + 2q \cdot YZ + r \cdot Z^2 \right) \right) = \sum \lambda^{(x_\tau + r)} w_{x\tau} \cdot Y^x \tau;
\]
in the second identity, the arguments of \( p, q \) and \( r \) on the left are \((x, \phi + \theta, \theta)\) while those of \( w_{x\tau} \) on the right are \((x, \phi, \theta)\).

The integral in (6.7) is of the form \( I_{a+x, \beta+y} (\lambda) \) as defined in (2.2) in Lemma 2.2 whose amplitude \( u(x, \phi) \) satisfies
\[
\overline{c}_\phi^x \overline{c}_\phi^y u(x, \phi) \leq_{\nu, \mu} X^{X+\delta-(\alpha+\beta+x+y)/2} \cdot X^{\nu+\mu},
\]
which follows easily from (6.3) and (6.5). In conclusion, Lemma 2.2 yields the following stationary phase estimates.

**Lemma 6.3.** Assume that \( 1/2 \leq \rho \leq 2 \) and that \( 1 \leq X \leq \sqrt{\kappa} \). We have
\[
\lambda^{1} \frac{\partial^{\gamma+\delta}}{\partial \kappa^{\gamma} \partial \theta^{\delta}} I(\lambda, \theta) \leq_{\gamma, \delta} \frac{S X^{X+\delta}}{\lambda}.
\]

**6.2. Analysis of the Hankel transform.** To start with, the following lemma is a direct consequence of Lemma 5.1 and 5.2. Note that in Lemma 5.2 we may choose \( \sigma = \frac{1}{2} \) (> \( \frac{1}{2} \geq |\text{Re} \mu| \)), say.

**Lemma 6.4.** Suppose that \( w(z) \) is a smooth function with support in \( \{ z : |z| \in [1, 2] \} \). Let \( W(u) \) be the Hankel transform of \( w(z) \) defined as in (3.44). Then
\[
w^\alpha u^\beta (\overline{c}/\overline{c}u)^\alpha (\overline{c}/\overline{c}u)^\beta W(u) \leq_{\alpha, \beta} |w|_{L^\infty} \cdot (|u|^{1/3} + 1)^{\alpha+\beta}/|u|^{2/3},
\]
where \( |w|_{L^\infty} \) is the sup-norm of \( w \).

Another simple consequence of Lemma 5.1 is the following lemma.

**Lemma 6.5.** Let \( S > 0 \) and \( X \geq 1 \). Let \( \gamma, \delta \) and \( A \) be nonnegative integers. Suppose that \( w(z) \) is a smooth function with support in \( \{ z : |z| \in [1, 2] \} \) and derivatives satisfying \( (\overline{c}/\overline{c}z)^\alpha (\overline{c}/\overline{c}z)^\beta w(z) \leq_{\alpha, \beta} X^{X+\beta} \) for all \( \alpha, \beta \). Let \( W(u) \) be the Hankel transform of \( w(z) \) defined as in (3.44). Then for \( |u| \geq 1 \) we have
\[
w^{\alpha} u^{\beta} (\overline{c}/\overline{c}u)^\alpha (\overline{c}/\overline{c}u)^\beta W(u) \leq_{\gamma, \delta, A} S X^{2A}/|u|^{(2A+2-\gamma-\delta)/3}.
\]

**Proof.** By Lemma 5.1 with \( K \) large, say \( K = 2A \), we have to bound integrals of the following form
\[
\sum_{\xi=1} e^{(3\xi(uz)^{1/3} + 3\xi(\overline{u}z)^{1/3})} a(\xi z^{1/3}) dz,
\]
with the amplitude \( a(z) \) satisfying
\[
(\overline{c}/\overline{c}z)^\alpha (\overline{c}/\overline{c}z)^\beta a(z) \leq S X^{\alpha + \beta} |u|^{(\gamma + \delta - 2)/3}.
\]
Changing the variable \( z \) to \( z^3 \), the integral turns into
\[
9 \int e^{(3u^{1/3}z + 3\overline{u}^{1/3}z)} a(z)|z|^4 dz.
\]
Now define the differential operator \( D = (\overline{c}/\overline{c}z)(\overline{c}/\overline{c}z) \) so that \( D(e^{(3u^{1/3}z + 3\overline{u}^{1/3}z)}) = -36\pi^2 |u|^{2/3} e^{(3u^{1/3}z + 3\overline{u}^{1/3}z)} \). By repeating partial integration with respect to \( D \) we get the desired estimate. Q.E.D.

Next, we consider the Hankel transform of functions of the form
\[
w(z, \Lambda) = e(-4Re(\Lambda \sqrt{z}))v(\sqrt{z}) + e(4Re(\Lambda \sqrt{z}))v(-\sqrt{z}).
\]
Let \( y \) and satisfies 
\[
|B(x)|^{\alpha} \leq \langle x \rangle \delta \langle x \rangle \beta \langle x \rangle^\gamma 
\]
for \( S > 0 \) and \( X \geq 1 \). Note that these estimates are equivalent to 
\[
|\partial_t \rangle \langle \partial_t| \leq \langle x \rangle \delta \langle x \rangle \beta \langle x \rangle^\gamma 
\]
in the polar coordinates.

**Lemma 6.6.** Let \( \gamma, \delta, A, K \) be nonnegative integers. Let \( w(z, \Lambda) \) be defined as above and \( W(u, \Lambda) \) be the Hankel transform of \( w(z, \Lambda) \). Define
\[
\hat{W}(u, \Lambda) = e(-2 \text{Re}(u/\Lambda^2)) W(u, \Lambda).
\]

Let \( y \gg 1 \). When either \( |\Lambda| > 2^{1/3} \) or \( |\Lambda| < 2^{1/3} \), we have
\[
y^z \frac{\partial^\gamma \partial^\beta}{\partial y^\gamma \partial \theta^\beta} \hat{W}(ye^{\theta}, \Lambda) \leq \frac{S}{y^{2/3}} \left( \frac{|\Lambda| + y/|\Lambda|^2 + X}{|\Lambda| y/|\Lambda|^2} \min \left\{ \frac{X}{|\Lambda| y/|\Lambda|^2}, \frac{X}{y^{1/3}} + |\Lambda|^2 X^{\delta \gamma \beta} \right\}^{A} + \frac{y^{1/3} + |\Lambda|^2}{y^{2/3}} \right).
\]

When \( 2^{1/3} \leq |\Lambda| \leq 2^{1/3} \), under the assumption \( X \leq \sqrt{\pi}/|\Lambda| \), we have
\[
y^z \frac{\partial^\gamma \partial^\beta}{\partial y^\gamma \partial \theta^\beta} \hat{W}(ye^{\theta}, \Lambda) \leq \frac{S}{y^{2/3}} \left( \frac{|\Lambda|^2 X^{\delta \gamma \beta}}{y^{1/3}} + \frac{y^{1/3}}{y^{2/3}} \right).
\]
All the implied constants depend only on \( \gamma, \delta, A, K \).

**Proof.** In view of Lemma 5.1 we have to bound the following integral and its derivatives
\[
I(u^{1/3}, \Lambda) = \sum \sum \int \int e(\text{Re}(6\xi u^{1/3} - 4\xi \Lambda \sqrt{z} - 2u^2)) v(\xi \sqrt{z}) W(\xi u^{1/3}) dz,
\]
in which
\[
W(z) = \sum \sum \sum \frac{B_k B_l}{z^{1/2} z^{1/2} z^{1/2}},
\]
with the coefficients \( B_k = B_k(\mu, 0, -\mu) \) as in Lemma 5.1. Substituting the variables \( z \) and \( u \) by \( \xi^6 \) and \( u^3 \), we have
\[
I(u, \Lambda) = \int \int e(\text{Re}(6u^{1/2} - 4u^{1/2} - 2u^3)) \ a(z) dz/|z|,
\]
where the weight function \( a(z) = 36|z|^{1/2} v(z^{1/2}) W(u^{1/2}) \) is supported in \( \{z : |z| \in [1, 2^{1/6}]\} \) and satisfies \( \frac{\partial}{\partial z} |z|^6 \partial |z|^6 a(z) \leq a, \beta, \gamma, \Lambda^2 X^{\delta \gamma \beta} \) (by our assumption that \( |\Lambda| \gg 1 \)).

Without loss of generality, we assume that \( \Lambda > 0 \). Letting \( z = xe^{i\theta} \) and \( u = ye^{i\theta} \) and then replacing \( x \) by \( xy/\Lambda \), we have
\[
I(ye^{i\theta}, \Lambda) = \frac{2y}{\Lambda} \int \int \int e\left( \frac{2y^3}{\Lambda^3} f(x, \phi; \theta) \right) a \left( \frac{xy e^{i\theta}}{\Lambda} \right) dx d\phi.
\]
According to the notation in 5.1 the phase function \( f(x, \phi; \theta) \) is given by (6.2) and this integral is of the form \( I(\lambda, \theta) \) as in (6.1) if one let \( \lambda = 2y^3/\Lambda^3, \rho = \Lambda/y \). Consequently, the estimates in this lemma follow directly from Lemma 6.1 and Lemma 6.5. Recall that the variable \( u \) was changed into \( u^3 \) at an early stage of the proof.

**Remark 6.7.** Note that we have modified the Hankel-transform integral \( W \) by a phase factor. It is the same phenomenon as in (C1) and (B10). this phase factor comes up naturally in both the analytic and the arithmetic part.
Finally, we conclude this section with the following lemma as the complex analogue of Lemma 10 in Blomer [Blo]. It should be stressed that our normalization of the Hankel transform is different from that of Blomer, as indicated in Remark 3.6.

**Lemma 6.8.** Fix $\varepsilon > 0$, $T \geq 1$, nonnegative integers $\gamma, \delta$, and integer $A$ sufficiently large. Then there are integers $A'$ and $A''$ sufficiently large (in terms of $\varepsilon, \gamma, \delta$, and $A$) with the following property. Let $X > 1$, and let $v(z)$ be a smooth function with support in \( \{ z : |z| \in [1, 2] \} \) satisfying \((\partial/\partial x)^\alpha (\partial/\partial \phi)^\beta v(xe^i) \ll_{\alpha, \beta} X^{\alpha+\beta} \) for all $\alpha, \beta$. Define $H(z)$ as in (3.27) with the choice of $h(t)$ in (4.1) (see also (3.3)), with $\Re v = \varepsilon$. For $A \neq 0$, define

\[
W(z) = w(z, \Lambda) = H(\Lambda \sqrt{z})v(z),
\]
and its Hankel transform $W(u) = W(u, \Lambda)$ as in (3.44). Let

\[
\tilde{W}(u, \Lambda) = e\left(-2\Re \left(u/\Lambda^2\right)\right)W(u, \Lambda).
\]

Then for any $Q > (TX)^{A''}(y + 1/y)(|\Lambda| + 1/|\Lambda|)$ we have the following bounds

\[
y^\gamma \frac{\partial^{\gamma+\delta}}{\partial y^{\gamma}\partial \theta^{\delta}} \tilde{W}(ye^{i\theta}, \Lambda) \ll_{\varepsilon, \gamma, \delta, A} T^{5+\varepsilon} Q^\varepsilon.
\]

\[
\begin{cases}
Q^{-A}, & \text{if } |\Lambda| \leq Q^{-\varepsilon/12}, \\
Q^\varepsilon/|\Lambda|^{2/3}, & \text{if } y \ll Q^\varepsilon, |\Lambda| > Q^{-\varepsilon/12}, \\
Q^{-A}, & \text{if } y > Q^\varepsilon, |\Lambda| > Q^{-\varepsilon/12}, |\Lambda| < y^{1/3}/2 \text{ or } |\Lambda| > 2y^{1/3}, \\
Q^\varepsilon |\Lambda|/y^{5/3}, & \text{if } y > Q^\varepsilon, y^{1/3}/2 \leq |\Lambda| \leq 2y^{1/3}.
\end{cases}
\]

In particular, the following uniform bound holds:

\[
y^\gamma \frac{\partial^{\gamma+\delta}}{\partial y^{\gamma}\partial \theta^{\delta}} \tilde{W}(ye^{i\theta}, \Lambda) \ll_{\varepsilon, \gamma, \delta} T^{5+\varepsilon} Q^\varepsilon/|\Lambda|^y.
\]

Proof. The first and second bound follow directly from Lemma 4.5 and 6.4. The other two bounds will follow mainly from Lemma 6.6 but require some work. We partition the integral (3.27) defining $H(z)$ into three parts according to the subdivision of the real line into three ranges:

\[
|t| \geq TQ^{\varepsilon/12}, \quad |t| \leq \sqrt{|\Lambda|} - 1, \quad |t| < TQ^{\varepsilon/12}.
\]

According to our decomposition we write $H = H^0 + H^1 + H^2$, $w = w^0 + w^1 + w^2$, and $\tilde{W} = \tilde{W}^0 + \tilde{W}^1 + \tilde{W}^2$.

In the first case, we use the exponential decay of $h(t)$ to see that

\[
\|w^0\|_{L^\infty} \ll T^{5+\varepsilon} Q^{\varepsilon/2} \exp(-Q^{\varepsilon/6}).
\]

Hence Lemma 4.4 implies that $\tilde{W}^0$ and its derivatives are negligible.

In the second case, we insert the expression (4.1) for the Bessel function $J_\nu(z)$ from Lemma 1.1 into (3.27). For the main terms, apply Lemma 6.6 with

\[
S = \|h(t)\|^2/|\Lambda| \ll (|t| + 1)^{2+\varepsilon} \exp(-t^2/T^2)/|\Lambda|
\]

and $X$ as in the present lemma, getting the desired bounds for $\tilde{W}^1$. For the error term containing $E_k$, Lemma 4.3 and 3.3 in Lemma 4.1 show that it is negligibly small. Note that in this case we have necessarily $|\Lambda| \geq 1$.

In the third case, as $J_\nu(z)$ is even, we may artificially write $2J_\nu(A\sqrt{z})$ in the form of (6.8) with the $v$ function being $v_0(z, \Lambda) = e(4\Re(\Lambda z))J_\nu(\Lambda z)$. It follows from Lemma 4.3 that

\[
(\partial/\partial z)^\alpha (\partial/\partial \bar{z})^\beta (h(t)^2\tilde{v}_0(z, \Lambda)) \ll_{\alpha, \beta} (|t| + 1)^{4+\varepsilon} \exp(-t^2/T^2) (|\Lambda| + (|t| + 1)/|\Lambda|)^{\alpha+\beta}.
\]
Note that \( \frac{|q|^{-1/2}}{3^{1/10}} < |\lambda| < |(l + 1)^2 < T^2 \frac{|q|^{1/6}}{3^{1/10}} \) and hence
\[
X + |\lambda| + (k + 1)/|\lambda| < X + 2T^2 \frac{|q|^{1/6}}{3^{1/10}} < \frac{Q^2}{3^{1/10}}.
\]
By the first estimate in Lemma 6.6 with the X there being \( X + |\lambda| + (k + 1)/|\lambda| \), we infer that \( W \) and its derivatives are negligibly small.

Q.E.D.

7. Setup

We are now ready to start the proof of Theorem 1.1. We shall follow Blomer [Blo] very closely.

We start with introducing the spectral mean of \( L \)-values,
\[
\sum_j \omega_j^* k(t_j) \left( \frac{1}{2} \pi \otimes f_j \otimes \chi \right) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \omega^*(t) k(t) \left| L \left( \frac{1}{2} + it, \pi \otimes \chi \right) \right|^2 dt,
\]
in which the spectral weight \( k(t) \) is defined as in (4.11). In view of (3.33), (3.34), (4.13), (4.14), along with the positivity of the \( L \)-values, we need to prove that this is bounded by \( |q|^{1/2+\epsilon} \). Applying the approximate functional equations (3.53), (3.56), the spectral mean may be written as
\[
\frac{1}{8} \sum_{n_1, n_2} A(n_1, n_2) \chi(n_2) \left( \sum_j \omega_j^* k(t_j) \lambda_j(n_2) V \left( \frac{|n_1^2 n_2|}{|q|^3}; t_j \right) \right.
\]
\[
+ \frac{1}{2\pi} \int_{-\infty}^{\infty} \omega^*(t) k(t) \eta \left( n_2, \frac{1}{2} + it \right) V \left( \frac{|n_1^2 n_2|}{|q|^3}; t \right) dt \right).
\]

From now on, we shall not pay attention to the dependence of implied constants on \( T \). It should be noted that the dependency on \( T \) in our estimates will be polynomial at each step.

By (3.57) in Lemma 3.7 (1), we may truncate the sum over \( n_1, n_2 \) at \( |n_1^2 n_2| \leq |q|^{3+\epsilon} \) with the cost of a negligible error. We then apply (3.58) in Lemma 3.7 (1), in which we choose \( U = \log |q|^{1/2} \). The error term is again negligible, and we need to prove
\[
\sum_{|n_1^2 n_2| \leq |q|^{1+\epsilon}} A(n_1, n_2) \gamma(n_2) \left( \sum_j \omega_j^* h(t_j) \lambda_j(n_2) \right.
\]
\[
+ \frac{1}{2\pi} \int_{-\infty}^{\infty} \omega^*(t) h(t) \eta \left( n_2, \frac{1}{2} + it \right) dt \right) \leq |q|^{1/2+\epsilon},
\]
uniformly in \( v \in \left[ e - i \log |q|^{\epsilon}, e + i \log |q|^{\epsilon} \right] \).

We now apply the Kuznetsov formula (3.29) with \( n_1 = 1 \) therein, obtaining a diagonal term
\[
\frac{1}{2\pi} \sum_{|n_1| \leq |q|^{1/2+\epsilon}} A(n_1, 1) \frac{|n_1^2|^{1/2+\epsilon}}{|n_1|^{2+\epsilon}},
\]
and an off-diagonal term
\[
\frac{1}{8\pi} \sum_{|n_1^2 n_2| \leq |q|^{1+\epsilon}} A(n_1, n_2) \gamma(n_2) \sum_{q \mid c} S(n_2, c; c) \frac{H}{|c|^2} \left( \frac{\sqrt{n_2}}{2c} \right).
\]

It follows from (3.40), along with Cauchy-Schwarz, that the diagonal term is bounded by \( |q|^{\epsilon} \). As for the off-diagonal term, we reformulate it by the Hecke relation (3.37) for
\( A(n_1, n_2) \), getting
\[
\sum_{|\delta| n_1^2 \leq \epsilon^4} \frac{\mu(\delta) \chi(\delta) A(n_1, 1)}{\delta^3 n_1^2 \left(1 + 2\epsilon\right)} \sum_{|n_2| \leq |q|^{1/\epsilon} \sqrt{\delta}} \frac{A(1, n_2) \chi(n_2)}{|n_2|^{1+\epsilon}} \sum_{|c| \leq N^{1/2}} \frac{S(\delta n_2, c; \epsilon)}{|c|^2} H \left( \frac{\sqrt{\epsilon} \delta n_2}{2c} \right).
\]
Note that in this sum, we may assume that \( \delta \) is square-free and that \( (\delta, q) = 1 \). Finally, by a smooth partition of unity, it is reduced to proving the following proposition.

**Proposition 7.1.** Let \( q \in \mathcal{O} \) be prime. Let \( \delta \in \mathcal{O} \) be square-free such that \( (\delta, q) = 1 \). Let \( \epsilon = 1 \) or \( i \). Assume
\[(7.1) \quad N \leq \|q\|^{3+\epsilon}/|\delta|^3.\]
Put \( X = (\log |q|)^2 \). Let \( v \) be a smooth function supported on \([1, 2]\) satisfying \( v(x) \leq |x|^2 \) for all \( \alpha \). Suppose that \( H(z) \) is the Bessel transform of \( h(t) \) given by (3.27), with \( h(t) \) defined as in (4.10), (4.12).
Define
\[(7.2) \quad S^0_\delta(q, N) = \sum_{q|c} \frac{1}{|c|^2} \sum_{n} A(1, n) \chi(n) S(\delta n, c; \epsilon) w \left( \frac{n}{N^\epsilon}; \frac{\sqrt{\epsilon} \delta N}{2c} \right),\]
with
\[(7.3) \quad w(z; A) = v(|z|) H \left( A \sqrt{z} \right).\]
Then
\[(7.4) \quad S^0_\delta(q, N) \ll \epsilon \|q\|^{7/2+\epsilon}|\delta|N + |q|^{1/2+\epsilon}N,\]
for any \( \epsilon > 0 \).

### 8. Application of the Voronoï summation

We now consider the sum \( S^\epsilon_\delta(q, N) \) defined in (7.2). In order to prepare for the application of Voronoï summation, we open the Kloosterman sum \( S(\delta n, c; \epsilon) \) and split the \( n \)-sum into residue classes modulo \( c \), and \( S^\epsilon_\delta(q, N) \) becomes
\[
\sum_{q|c} \frac{1}{|c|^2} \sum_{b \pmod{c}} \left( \Re \left( \frac{eb}{c} \right) \right) \sum_{a \pmod{c}} \chi(a) e \left( \Re \left( \frac{\delta ab}{c} \right) \right) \sum_{n=a \pmod{c}} A(1, n) w \left( \frac{n}{N^\epsilon}; \frac{\sqrt{\epsilon} \delta N}{2c} \right).
\]
We then detect the summation condition \( n \equiv a \pmod{c} \) by primitive additive characters modulo \( c_1 \),
\[
S^\epsilon_\delta(q, N) = \sum_{q|c} \frac{1}{|c|^4} \sum_{c_1|c} \sum_{b \pmod{c_1}} \left( \Re \left( \frac{eb}{c} \right) \right) \sum_{a \pmod{c_1}} \chi(a) e \left( \Re \left( \frac{\delta ab}{c} - \frac{ab_1}{c_1} \right) \right) \sum_{n=a \pmod{c_1}} A(1, n) e \left( \frac{nb_1}{c_1} \right) w \left( \frac{n}{N^\epsilon}; \frac{\sqrt{\epsilon} \delta N}{2c} \right).
\]
For the innermost sum we can now apply the Voronoï formula in Proposition 3.5 getting
\[(8.1) \quad T_{\delta n_1, n_2} \left( c_1; c_1, q \right) W \left( \frac{n_1^2 n_2 N}{8c_1^2}; \frac{\sqrt{\epsilon} \delta N}{2c} \right),\]
where
\[ T_{\delta,m,n}^*(c,c_1;q) \]
with the function \( W(u;\Lambda) \) is the Hankel transform of \( w(z;\Lambda) \), which has been studied in §6.2 (in particular, see Lemma 6.8).

9. Transformation of the character sum

9.1. The computations in [Blo] §6 may be applied almost identically for the character sum \( T_{\delta,m,n}^*(c,c_1;q) \), so we shall only give a summary of the final results in the following lemma (see [Blo] Lemma 12]). Note that \( \chi(-1) = 1 \).

**Lemma 9.1.** Let \( q, \delta \in \mathcal{O} \) be square-free. Let \( \epsilon = 1 \) or \( i \). Assume that
\[ (\delta, q) = (\delta', c_2) = \emptyset, \quad \delta_0|c_2, \quad n_1|c_1, \]
with the following notation,
\[ c = qr = c_1c_2, (\delta_0) = (\delta, r) = (\delta, c), \quad c' = c/\delta_0, c'_1 = c_2/\delta_0, \delta' = \delta/\delta_0, r' = r/\delta_0. \]
Then
\[ T_{\delta,m,n}^*(c,c_1;q) = e \left( -\text{Re} \left( \frac{\ell_0 c_1 c_2 n_1 n_2}{c'} \right) \right) \varphi(c_1) \varphi(c_1/n_1) \frac{\mu(\delta_0)\chi(\delta)}{|\delta_0|^2} |\epsilon|^2 \]
\[ \cdot V_{\delta,m,n}^*(c,c_1;q) \sum_{b_2,b_1(\text{mod } q)} \sum_{b_2,b_1(\text{mod } q)} \mu(f_2) \frac{f_1}{|f_1|^2} \left( \text{Re} \left( \frac{\ell_0 c_1 c_2 n_1 n_2}{f_1} \right) \right), \]
where
\[ V_{\delta,m,n}^*(c,c_1;q) = \sum_{b_2,b_1(\text{mod } q)} \chi(b_2b_3) \chi(b_2r' + c_2b_3c_1) \]
the summation is subject to the following conditions,
\[ f_1f_2d_2 = r', \quad (d_2', f_1n_1n_2) = (f_1, f_2) = (f_1, f_2, q) = \emptyset, \quad \mu^2(f_1) = 1 = \mu(f_1)_n, \]
and
\[ n'_1 = n_1/f_2. \]
We have \( T_{\delta,m,n}^*(c,c_1;q) = 0 \) if one of the conditions \( (r', c_2) = \emptyset, \delta_0|c_2 \) is not satisfied.

**Remark 9.2.** In the course of computations, one needs the Selberg-Kuznetsov identity for Kloosterman sums over the Gaussian field \( \mathcal{F} \),
\[ S(n_1, n_2; c) = \frac{1}{4} \sum_{d|n_1, d|n_2, d|c} |d|^2 S \left( \frac{n_1n_2}{d^2}, 1; \frac{c}{d} \right), \]
or, after the Möbius inversion,
\[ S(n_1n_2, 1; c) = \frac{1}{4} \sum_{d|n_1, d|n_2, d|c} |d|^2 \mu(d) S \left( \frac{n_1}{d}, \frac{n_2}{d}; \frac{c}{d} \right), \]
which should be thought of as a kind of the Hecke multiplicativity relation. This identity may be proven using either the Kloosterman-weighted Kuznetsov formula for \( \text{SL}_2(\mathcal{O}) \) in [LG] or the elementary arguments in [Mat].
9.2. As observed in [10], the character sum \( V_{\lambda^{-1}}^\epsilon(c, c_1; q) \) defined in (9.4) has been studied in the work of Conrey and Iwaniec [1]. Again, the calculations in [1] §10 are valid over the Gaussian field in an obvious way.

As in in [1] (10.2), we define \( H_r(m, m_1, m_2; q) \) as follows,

\[
H_r(m, m_1, m_2; q) = \sum_{u,v \equiv (\mod q)} \chi(v(u + vm_1)(vr - m)(ur + nm_1))e \left( \Re \left( \frac{um_2^r}{q} \right) \right).
\]

Moreover, define as in [1] (10.7)

\[
H(w; q) = \sum_{u,v \equiv (\mod q)} \chi(uv(u + 1)(v + 1))e \left( \frac{(uv - 1)w}{q} \right).
\]

Put

\[
\begin{align*}
(h) &= (r, q), \quad (k) = (nm_1m_2, q/h), \quad \ell = q/hk.
\end{align*}
\]

According to [1] (10.12, 10.13), we have

\[
H_r(m, m_1, m_2; q) = \left[ \frac{|h|^2}{\varphi(k)} R(m; k) R(m_1; k) R(m_2; k) H(hkmn_1m_2; \ell), \right.
\]

if \((h, mm_1m_2) = \sigma\), or else \(H_r(m, m_1, m_2; q) = 0\), where \(R(m; k) = S(m, 0; k)\) is the Ramanujan sum.

By comparing (9.4) and (9.7), we observe that

\[
V_{\lambda^{-1}, n_1, n_2}^\epsilon(c, c_1; q) = H_r(\overline{c}c_2^n n_1, -c_2, \overline{c}c_2^n n_1; q).
\]

Consequently, we have the the following lemma.

**Lemma 9.3.** Let notation be as in Lemma 9.1. When the assumptions in (9.2) and (9.5) hold, we have

\[
V_{\lambda^{-1}, n_1, n_2}^\epsilon(c, c_1; q) = \left[ \frac{|h|^2}{\varphi(k)} R(n_1 n_2 c_2^n; k) R(c_2^n; k) R(n_1 c_2^n; k) H(-\overline{c}c_2^n n_2, n_2^{\ell}hkr; \ell) \right] \cdot
\]

At this point, we assume that \(q\) is prime. Following [1] §7, we substitute (9.3) and (9.11) into (8.1) and simplify the resulting sum. Indeed, using (6.11) in Lemma 6.8 (7.1), along with (3.32), it may be shown that the contributions from the terms with \(c_2^n = q\), \(h = q\) or \(k = q\) are small (negligibly small or at most \(|q|^{7/16} + eN\)). Thus we may assume that \(c_2^n = h = k = 1\) and in particular that \(c_1 = \ell = q, (d'_2 n'_2, q) = \sigma\) and \(n'_2 | f_1\). Let
\[ g = f_1/n_1'. \] Now the simplified sum that we need to consider is as follows,

\[
\frac{N^2}{|q|^2} \sum_{d, \delta = \delta} \mu(d) \varphi(d) \prod_{n=1}^{\infty} \frac{A(n)}{n} e \left( \frac{\tau_0 n_2 n_2' qrd_2^2}{\delta^2 g} \right) W \left( \frac{n_2 N}{8f_2 n_1'(qgd_2^2)^3}; \sqrt{\epsilon \delta N} \right),
\]

subject to the conditions

\[
(f_2 gn_1'(d_2', \delta) = (f_2 gn_1', g_1') = (f_2, g_1') = \emptyset, \\
(d_2', n_1'n_2) = (d_2'n_1n_2, q) = \emptyset, \\
\mu'(g_1') = 1.
\]

Furthermore, let

\[
(s) = (n_2, \delta g), \quad n_2' = n_2/s,
\]

pull the factor \( s \) out of the numerator and denominator of the exponential, and then introduce the new variable \( r \) to relax the coprimality condition \((d_2', n_2') = \emptyset\) by Möbius inversion and separate these two variables. It reduces to proving that the following sum has the same bound as in (7.4).

\[
\frac{N^2}{|q|^2} \sum_{d, \delta = \delta} \mu(d) \varphi(d) \prod_{n=1}^{\infty} \frac{A(n)}{n} e \left( \frac{\tau_0 n_2 n_2' qrd_2^2}{\delta^2 g/s} \right) W \left( \frac{n_2 N}{8f_2 n_1'(qgd_2^2)^3}; \sqrt{\epsilon \delta N} \right),
\]

with

\[
\sum_{d, \delta = \delta} \frac{A(rsn_2', n_1)}{|d_2'|^2} e \left( \frac{\tau_0 n_2 n_2' qrd_2^2}{\delta^2 g/s} \right) W \left( \frac{sn_2 N}{8n_1 r^2(qgd_2^2)^3}; \sqrt{\epsilon \delta N} \right).
\]

Note that the fraction in the exponential is in lowest terms.

It is left to estimate the sum \( S_{g,n_1,r,s}(q, N) \). For this, we shall prove the following lemma in the last section.

**Lemma 10.1.** Let notation be as above. Assume that

\[ 1 \leq D_2 \leq |q|^{1/2} N^{1/2}, \quad 1 \leq N_2 \leq |q|^{1/2} N^{1/2}. \]

Define

\[
S_{g,n_1,r,s}(q, N, D_2, N_2) = \sum_{d_2 \leq D_2} \sum_{n_1 \leq N_2} \frac{A(rsn_2', n_1)}{|d_2'|^2} e \left( \frac{\tau_0 n_2 n_2' qrd_2^2}{\delta^2 g/s} \right) W \left( \frac{sn_2 N}{8n_1 r^2(qgd_2^2)^3}; \sqrt{\epsilon \delta N} \right).
\]

Then

\[
S_{g,n_1,r,s}(q, N, D_2, N_2) \ll |q|^{1/2} |\delta|^{1/2} N^{3/2} D_2 \left( \frac{|q|}{s} \right)^{1/2} + N_2.
\]
Granted that Lemma 10.1 holds, we can now finish the proof of Proposition 7.1. Using (6.11) in Lemma 6.8 to determine the range of summation and then applying a dyadic partition to (10.2), it follows from Lemma 10.1 that the sum in (10.1) is bound by

\[ |q|^c N^{1/2} \sum_{d_0 N^{1/2}} \frac{1}{|d_0|^1/2} \sum_{f_2 g n_1'} \sum_{s|d_0} \int_{|f_2 g n_1'| \leq |q|^{1/2+\epsilon}} |f_2 g n_1'|^{7/16} \left| \frac{|f_2 g n_1'|}{s} \right| D_2 \left( \frac{|q|^{1/2}}{s} + N_2 \right), \]

for some \( D_2, N_2 \) in the range (10.3), with \( n_1 = f_2 n_1' \). Note here that (10.3) and (7.1) imply \( |f_2 g n_1'| \leq |q|^{1/2+\epsilon} \). Inserting (10.3), the sum above is further bounded by the sum of

\[ |q|^c N^{3/2} \sum_{d_0 N^{1/2}} \frac{|d_0|^3}{|d_0|^2} \sum_{f_2 g n_1'} \sum_{s|d_0} \int_{|f_2 g n_1'| \leq |q|^{1/2+\epsilon}} |f_2 g n_1'|^{7/16} \left| \frac{|f_2 g n_1'|}{s} \right|^2 \leq |q|^c \frac{|d_0|^{3/2} N^{3/2}}{|d_0|} \leq |q|^{1/2+\epsilon} N, \]

as desired.

11. Completion of the proof

We start the proof of Lemma 10.1 with separating variables in the first argument of \( \tilde{W} \) in (10.4) by a standard Mellin inversion technique. Precisely, in the polar coordinates,

\[ \tilde{W}(ye^{i\theta}, A) = \sum_{m=-\infty}^{\infty} e^{im\theta} \int_{-\infty}^{\infty} T_Y(t, m; A)e^{i\theta} dt, \quad Y/4 \leq y \leq 4Y, \]

with

\[ T_Y(t, m; A) = \frac{1}{4\pi^2} \int_{0}^{2\pi} \tilde{W}(ye^{i\theta}, A)e^{-im\theta} y^{-it} \frac{d\theta}{y}, \]

where \( w(x) \) is a fixed smooth function such that \( w(x) \equiv 1 \) on \([1/4, 4]\) and \( w(x) \equiv 0 \) on \((0, 1/8) \cup (8, \infty)\), say, and \( Y = \frac{|s|N_2^2}{16n_1^2(q \delta)^3}|D_2| \). It should be pointed out that one has to repeat partial integration for both \( y \) and \( \theta \) (at least twice) to secure the convergence of the \( m \)-sum and the \( \theta \)-integral. For this, we apply the estimates for the derivatives of \( \tilde{W}(ye^{i\theta}, A) \) in Lemma 6.8. It is crucial that the right-hand side of (6.11) in Lemma 6.8 is up to the implied constant independent of the differential orders \( \gamma \) and \( \delta \). From the uniform bound (6.12) in Lemma 6.8 we infer that \( S_{\gamma, \delta}^{e, \delta, n_1}(q, N, D_2, N_2) \) is bounded by

\[ |q|^c \left| \frac{|g|^4 \delta_0^{n_1^2} n_1^2 y^{3/2}}{|\delta|^{1/2} |s|^3/2 N_2} \right| \sum_{D_2 \leq |d'_2| N_1^2} \sum_{N_2 \leq |n'_2| < 2N_2} \sum_{(d'_2, q) = \theta} \alpha(d'_2) \beta(n'_2) A(rsn'_2, n_1), \]

(11.1)

for some complex coefficients \( \alpha(d'_2), \beta(n'_2) \) of absolute value at most 1.

The main ingredient for estimating (11.1) is the following analogue of [Blo 13], which in turn is a small variation of Lemma 11.1 in Conrey-Iwaniec [CI].
LEMMA 11.1. Let $q \in \mathcal{O}$ be square-free, $c \in \mathcal{O} \setminus \{0\}$, $a, b \in \mathcal{O}$, with $(a, c) = (b, q) = \mathcal{O}$. Let $\alpha(d), \beta(n)$ be complex numbers for $d, n \in \mathcal{O} \setminus \{0\}$ with $1 \leq |d| \leq D$, $1 \leq |n| \leq N$. Assume that $|\alpha(d)| \leq 1$. Then

\begin{equation}
\sum_{\substack{d, n \in \mathcal{O} \setminus \{0\} \atop (d, n|q)c}} \alpha(d)\beta(n)\epsilon(adn/c)H(bdn, q) \ll (\beta[|qc/\epsilon|B_N + N|D(|qc| + D),
\end{equation}

for any $\epsilon > 0$, the implied constant depending only on $\epsilon$. As usual, $|\beta| = (\sum_n |\beta(n)|^2)^{1/2}$.

The proof of [Bl] Lemma 13 may be directly applied here. In our settings, the core would be the following bound of Conrey-Iwaniec

\begin{equation}
g(x, \psi) \ll |q|^{-1/2},
\end{equation}

for the character sum

\begin{equation}
\sum_{uv, (uv, \mod q)} \chi(uv(u+1)(v+1))\psi(uv-1),
\end{equation}

where $\chi(\mod q)$ is the non-principal quadratic character and $\psi(\mod q)$ is any character; see [CI] (11.10, 11.12). For proving (11.3), we may assume that $q$ is prime as $g(x, \psi)$ is multiplicative (the $\epsilon$ in (11.3) may be removed for $q$ prime). The proof in [CI] §13, 14 employs the Riemann hypothesis for varieties over finite fields if $\psi \neq \chi$ and some elementary arguments if $\psi = \chi$. Their proof may be extended to any finite field and in particular to $\mathcal{O}/q\mathcal{O}$.

We apply Lemma [11.1] with $c = \delta'g/s$ for the double sum in (11.1). In this way, we see that (11.4) is bound by

\begin{equation}
|q|^{-1/4} \delta_0 \delta g^2 n^2 r^2 |D_2| \left( \sum_{N \leq |n'| < 2N} |A(rsn', n)|^2 \right)^{1/2} \left( \frac{|q^\delta g|}{s} + N_2 \right),
\end{equation}

where $D_2, N_2$ are restricted by (10.3), which, along with (7.1), implies in particular $D_2 \leq |q^\delta g/s|$. From (3.42) we infer

\begin{equation}
\left( \sum_{N \leq |n'| < 2N} |A(rsn', n)|^2 \right)^{1/2} \ll |q|^{-1} |n|s^{7/16} N_2,
\end{equation}

so that (11.5) is bound by

\begin{equation}
|q|^{-1/4} \delta_0 \delta g^2 n^2 r^2 |n|s^{7/16} |D_2| \left( \frac{|q^\delta g|}{s} + N_2 \right).
\end{equation}

This completes the proof of Lemma [11.1].

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