Programming and Verifying
Subgame-Perfect Mechanisms

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Abstract

An extension of the WHILE-language is developed for programming game-theoretic mechanisms involving multiple agents. Examples of such mechanisms include auctions, voting procedures, and negotiation protocols. A structured operational semantics is provided in terms of extensive games of almost perfect information. Hoare-style partial correctness assertions are proposed to reason about the correctness of these mechanisms, where correctness is interpreted as the existence of a subgame-perfect equilibrium. Using an extensional approach to pre- and postconditions, we show that an extension of Hoare’s original calculus is sound and complete for reasoning about subgame-perfect equilibria in game-theoretic mechanisms. We use the calculus to verify some simple mechanisms like the Dutch auction.

1 Introduction

In recent years, games have become more prominent in different areas of computer science research. The reason for this seems to be the realisation that games form a natural generalisation of programs. This insight can be realised on a number of different levels (we shall only mention two): On a foundational level, games have been used to provide an alternative model of computation, the alternating Turing machine [3]. At a more abstract level, program logics like propositional dynamic logic have been extended to games [12].

From a game-theoretic perspective, much of this work is extremely narrow, since it mainly focuses on determined 2-player win/loss games of perfect information. On the other hand, game theory has developed a wealth of techniques to study more complicated situations where agents interact, involving more than two players, imperfect information, and preferences over outcomes which cannot be captured by simply distinguishing between winning and losing. Still, it has been suggested [13] that combining research in game theory and computer
science, we may be able to obtain a better understanding of social software, i.e.,
of the formal properties of the social processes we are involved in. The present
paper tries to contribute to this aim.

More concretely, we attempt to generalise techniques from formal program
verification to games or game-theoretic mechanisms such as auctions, voting pro-
cedures, etc. From a logical perspective, two approaches suggest themselves. On
the one hand, one might extend model checking approaches [4], where one uses,
for instance, temporal logic to specify properties of a system (program/game)
and proceeds to verify these properties using model checking. This approach
has been generalised to reason about coalitional power in games [14]. On the
other hand, one can try to extend approaches based on theorem proving using
a formal calculus in which one can derive certain properties of a system. This
approach will be taken here.

The axiomatic or compositional approach to program verification was intro-
duced by Hoare [7] and Dijkstra [5], and provided the foundation stone for formal
program verification [10, 1, 6]. In Hoare’s calculus, correctness assertions of the
form \( \{P\} \pi \{Q\} \) are used to express that program \( \pi \), when executed in a state
satisfying \( P \), will terminate in a state satisfying \( Q \) (provided it does terminate).
In generalising the program verification approach to games, this paper makes
two contributions: First, it defines a programming language which is a simple
extension of the WHILE-language sufficient to program game-theoretic mecha-
nisms. The syntax of this language is defined in section 2, and section 3 provides
a structured operational semantics in terms of extensive games of almost perfect
information. Second, we are going to extend Hoare’s calculus to reason about
the correctness of these mechanisms, where correctness is interpreted as the ex-
istence of a subgame-perfect equilibrium with a certain payoff. In section 4, we
define our new notion of correctness by providing a game-theoretic interpreta-
tion of \( \{P\} \pi \{Q\} \), also linking it to the game-theoretic notion of implementa-
tion and mechanism design. Section 5 presents an extensional calculus for reasoning
about mechanism correctness, and provides proofs of soundness and complete-
ness. Finally, section 6 illustrates the calculus in the verification of a few simple
mechanisms.

2 Syntax of MPL

Our mechanism programming language (MPL) is a simple extension of standard
imperative programming languages; more concretely, our point of departure is
the well-known WHILE-language (see e.g. [10]). We assume throughout that we
are given a nonempty set of agents or players \( Ags \), a set of mechanism variables
\( MV \), a set of function symbols \( Funs \) and a set of relation symbols \( Rels \). Using
these, we inductively define terms \( t \), boolean expressions \( B \) and mechanisms (or
game forms) \( \gamma \) as follows:
where \( a \in \text{Ags} \), \( f^k \in \text{Funs} \) is a \( k \)-ary function symbol (in case \( k = 0 \) we are dealing with constants), \( R^k \in \text{Rels} \) a \( k \)-ary relation symbol, \( x, x_a \in \text{MV} \), and \( A \subseteq \text{Ags} \) is finite and nonempty.

The last construct presents the only addition to the standard WHILE-language: \( \text{ch}_A \) lets agent \( a \in A \) choose any value for the variable \( x_a \). The agents in \( A \) are making their choice simultaneously, so in order to prevent conflicting assignments to variables, we require all the \( x_a \) to be distinct. One can think of the \( \text{ch}_A \) construct as a strategic game among \( n \) agents, where the strategic choice of an agent is represented by the value of his/her variable. While the set of agents may be infinite, we require each \( \text{ch}_A \) construct to involve only finitely many agents. In the special case where \( |A| = 1 \), we have a simple nondeterministic choice. More concretely, in case agent 1 can choose between 2 different strategies, executing \( \gamma_1 \) vs. executing \( \gamma_2 \), we can describe this situation as

\[
\text{ch}_{\{1\}}(\{x_1\}); \text{if } x_1 = 0 \text{ then } \gamma_1 \text{ else } \gamma_2;
\]

where we assume that the domain of computation is the set of natural numbers, for instance, and \( = \in \text{Rels} \) and \( 0 \in \text{Funs} \).

MPL is an extremely general programming language for a large variety of different kinds of mechanisms. In section 4 we shall use it for defining mechanisms for different kinds of auctions. Voting procedures are further examples of mechanisms which can be programmed using MPL. As an example, the well-known Borda-count procedure (see, e.g., [2]) can be programmed as follows:

\[
\text{ch}_{\text{Ags}}(\{x_1, x_2, \ldots, x_N\});
\]

\[
i := 1;
\]

\[
\text{while } i \leq K \text{ do } c_i := 0; i := i + 1;
\]

\[
a := 1;
\]

\[
\text{while } a \leq N \text{ do }
\]

\[
i := 1;
\]

\[
\text{while } i \leq K \text{ do }
\]

\[
c_i := c_i + x_a[i];
\]

\[
i := i + 1;
\]

\[
a := a + 1
\]

In this example, we assume that \( \text{Ags} = \{1, 2, \ldots, N\} \), and that the agents have to choose among \( K \) candidates. First, each agent \( a \) can cast a ballot of the form \( x_a = (p_1, p_2, \ldots, p_K) \), where \( p_i \) is the number of points the agent gives to candidate \( i \). Ballots have to be rankings of candidates, i.e., the most preferred candidate must obtain \( K \) points, the next preferred candidate \( K - 1 \) points, etc., so that the least-preferred candidate obtains 1 point. Hence, we assume
implicitly that the domain of computation contains these possible ballots, and
that the initial choice assigns a ballot to each $x_a$. (Note that since the domain
of computation will also contain the natural numbers, we need to make sure that
each $x_a$ is assigned to an element of the appropriate ballot type, but we shall
ignore this problem in order to keep the algorithm simple.) Once the ballots are
cast, $a$ is initialised to the first agent, and $i$ to the first candidate. The variable
c$_i$ counts the number of points accumulated by candidate $i$, and is initialised
to 0. The main part of the algorithm then simply sums up the points for each
candidate, where $x_a[i]$ refers to $p_i$, in case $x_a = (p_1, p_2, \ldots, p_K)$. The winner of
the vote will be the candidate accumulating the most points.

A further example of a well-known mechanism which can be programmed in
MPL is a version of Rubinstein’s negotiation protocol of alternating offers (see
[11, 8]).

```plaintext
agree := false;
optout := false;
i := 1;
while ¬optout ∧ ¬agree do
    if $i = 1$ then ch$_{11}$(x) else ch$_{12}$(x);
    if $i = 1$ then ch$_{21}$(y) else ch$_{22}$(y);
    if $y = 0$ then agree := true
    else if $y = 1$ then optout := true i := 3 − $i$
```

For simplicity, we have assumed that there are only two agents who try to reach
an agreement over, e.g., the price of a car which agent 1 wants to sell to agent
2, and so we can assume the domain of computation to be simply the natural
numbers. The negotiation procedure can end in an agreement concerning the
price, one of the agents can opt out of the negotiation (in which case some
predetermined event will occur), or the negotiation can go on forever. The
protocol starts by agent 1 making a price offer $x$. Agent 2 responds by choosing
$y$, where we interpret $y = 0$ as signalling agreement to the price offered, $y = 1$ as
a decision to opt out of the negotiation, and any other value for $y$ as signalling
the desire to make a counteroffer, upon which we get another iteration of the
loop with the roles reversed.

The above negotiation protocol is very general, and numerous instances of
it have been analysed game-theoretically [8]. We shall not go into this or the
voting mechanism in more detail, since our main aim at this point is only to
suggest the generality of the mechanism programming language defined. Section
6 will provide a more detailed and more formal treatment of examples such as
the ones given here. In the following section, we shall provide a formal semantics
for this language in terms of games. Furthermore, we will subsequently provide
a calculus for reasoning about the existence of game-theoretic equilibria in these
mechanisms, and about the payoffs the agents obtain in equilibrium.

Note that MPL only allows one to construct mechanisms with almost-perfect
information, i.e., agents are perfectly informed about all the choices made ex-
cept possibly for simultaneous moves. Different subclasses of MPL-mechanisms
correspond to various natural assumptions regarding the power of the mechanism designer and the agents in general. The class MPL(PRG) of programs is the class of MPL-mechanisms which do not contain any \( \text{ch}_A \) construct. Without this construct, MPL is simply the WHILE-language. The class MPL(PI) of perfect-information mechanisms will restrict the use of \( \text{ch}_A \) to cases where \( |A| = 1 \), i.e., where all choices involve only a single agent. Perfect-information mechanisms allow different agents to make choices at different times, but all choices are public, there are no simultaneous moves.

3 Structured Operational Semantics via Games

The most detailed semantics we can provide for MPL expressions is a structured operational semantics which specifies the configurations a mechanism can be in and the possible transitions between configurations. For programs, such a semantics gives rise to an execution sequence or trace, and in case of nondeterministic programs to an execution tree. Since in the case of mechanisms we are dealing with multiple agents, we arrive at a game tree whose positions are the possible configurations of the mechanism.

As is standard in first-order logic, we will work with an interpretation \( I \) which provides us with a domain \( D_I \), and functions and relations over \( D_I \) as interpretations for the symbols in \( \text{Funs} \) and \( \text{Rels} \). Furthermore, we assume that besides the relations associated to symbols in \( \text{Rels} \), our interpretation contains an additional binary \( \geq_a^I \)-relation for every agent \( a \in \text{Ags} \). The \( \geq_a^I \) relation will be used to represent agent \( a \)'s preference over the elements of the domain. Note that mechanisms programmed in MPL cannot refer to these preferences, since \( \geq_a^I \not\in \text{Rels} \).

The only requirements on \( I \) are that the preference relations \( \geq_a^I \subseteq D_I \times D_I \) satisfy the following properties: (1) \( \geq_a^I \) must be a partial pre-order, i.e., a reflexive and transitive relation on \( D_I \), and (2) there is a uniformly worst outcome (which we denote as \( -\infty \)), i.e., there is some \( d \in D_I \) such that for all \( a \in \text{Ags} \) and \( x \in D_I \) we have \( x \geq_a^I d \). Usually, preference relations will be total orders, but our framework does not require this. The uniformly worst outcome is needed to deal with some infinite runs resulting from while-loops, it plays no substantive role in any of the examples considered.

A state \( s : MV \rightarrow D_I \) is a function assigning a domain element to each mechanism variable. Let \( S_I \) be the set of all states over \( I \). In general, whenever the intended interpretation \( I \) is clear we shall tend to omit it. The following standard logical notation will be used: \( I, s \models \varphi \) denotes that a first-order formula \( \varphi \) whose variables are all in \( MV \) is true in \( I \) at state \( s \). Similarly, we let \( \varphi^I = \{ s \in S_I | I, s \models \varphi \} \). Again, when the intended interpretation is clear, we shall often simply write \( s \models \varphi \).

Given interpretation \( I \) and an initial state \( s_0 \), we shall interpret every mechanism \( \gamma \) as a game form of almost-perfect information \( G(\gamma, s_0, I) \). Let \( Cfg \) denote the set of configurations, i.e., the set of all pairs \( (\gamma, s) \) where \( \gamma \) is a mechanism or the empty mechanism \( \Lambda \), and \( s \) is a state. We define a transition relation
\[ A \subseteq Cfg \times Cfg \text{ for } A \subseteq Ags \text{ such that } c \xrightarrow{A} c' \text{ states that the game can proceed from } c \text{ to } c' \text{ provided the agents } A \text{ make some choice/move. In case the move does not require any agent to make a choice, we will have } A = \emptyset. \text{ In the standard way (see e.g. [10]), we define the } A \xrightarrow{-} \text{ relations inductively as the smallest sets satisfying the following axioms and inference rules, the only novelty here being the definition for } \text{ch}_A: \]

\[
\begin{align*}
\langle x := t, s \rangle & \xrightarrow{0} \langle A, s^*_t \rangle \\
\langle \text{ch}_A(X), s \rangle & \xrightarrow{A} \langle A, s' \rangle \text{ where } s'(y) = s(y) \text{ for all } y \notin X \\
\langle \gamma_1, s \rangle & \xrightarrow{A} \langle A, s' \rangle \\
\langle \gamma_1; \gamma_2, s \rangle & \xrightarrow{A} \langle \gamma_2, s' \rangle \\
\langle \gamma_1; \gamma_2, s \rangle & \xrightarrow{A} \langle \gamma_1; \gamma_2, s' \rangle \\
\langle \text{if } B \text{ then } \gamma_1 \text{ else } \gamma_2, s \rangle & \xrightarrow{0} \langle \gamma_1, s \rangle \\
\langle \text{while } B \text{ do } \gamma, s \rangle & \xrightarrow{0} \langle \Lambda, s \rangle \\
\langle \text{while } B \text{ do } \gamma, s \rangle & \xrightarrow{0} \langle \Lambda, s \rangle \\
\langle \text{while } B \text{ do } \gamma, s \rangle & \xrightarrow{0} \langle \Lambda, s \rangle \\
\langle \text{while } B \text{ do } \gamma, s \rangle & \xrightarrow{0} \langle \Lambda, s \rangle
\end{align*}
\]

where \( s^*_t(y) = s(y) \) for \( y \neq x \) and \( s^*_t(x) = t^{I,s} \), the interpretation of \( t \) in \( I \) at \( s \).

Let \( Cfg^* \) be the set of all finite nonempty sequences of configurations \( c_0, c_1, \ldots, c_n \) such that \( c_i = \langle \gamma_i, s_i \rangle \) and

\[
\langle \gamma_0, s_0 \rangle \xrightarrow{A_1} \langle \gamma_1, s_1 \rangle \xrightarrow{A_2} \cdots \xrightarrow{A_n} \langle \gamma_n, s_n \rangle,
\]

and let \( Cfg_a^* \) be those sequences which end in a configuration \( c_n \) for which there is some configuration \( c_{n+1} \) and set \( A \subseteq Ags \) such that \( c_n \xrightarrow{A} c_{n+1} \) and \( a \in A \). Infinite configuration sequences as well as finite configuration sequences \( c_0, \ldots, c_n \) for which there is no \( c_{n+1} \) and \( A \) such that \( c_n \xrightarrow{A} c_{n+1} \) are called \( \text{terminal} \), and we denote the set of terminal sequences as \( Cfg^t \).

The move relations give rise to the game tree or \( \text{semi-game} \) \( G(\gamma, s_0, I) \) which starts at the initial position/configuration \( \langle \gamma, s_0 \rangle \). We interpret \( Cfg^* \) as the set of (partial) histories of the game, where each agent \( a \) gets to move at the positions which are in \( Cfg_a^* \). Note that we talk of a tree, since we can think of possible loops as infinite branches. While we shall usually refer to \( G(\gamma, s_0, I) \) as a game (omitting the “semi”), note that a semi-game lacks a link between runs/histories and preferences, for although \( I \) does contain information about the players’ preferences over outcomes, the triple \( G \) does not have any mapping
between histories of the game and outcomes. Such a mapping $\hat{o}$ will be added shortly.

A strategy for agent $a$ in semi-game $G(\gamma_0, s_0, I)$ is a function $\sigma^a : Cfg^a \rightarrow D_I$. Given a strategy profile $\sigma = (\sigma^1, \ldots, \sigma^n)$, i.e., a strategy $\sigma^a$ for every agent $a \in Ags$, we obtain a unique (possibly infinite) run which we denote as $\text{run}(\sigma)$, i.e., a maximal sequence of configurations

$$\langle \gamma_0, s_0 \rangle \xrightarrow{A_1} \langle \gamma_1, s_1 \rangle \xrightarrow{A_2} \ldots$$

where $\langle \gamma_0, s_0 \rangle$ is the initial configuration, and for all $A_{k+1} \neq \emptyset$ we have $s_{k+1}(x_i) = \sigma^i(\langle \gamma_0, s_0 \rangle, \ldots, \langle \gamma_k, s_k \rangle)$ for all $i \in A$, and $s_{k+1}(y_i) = s_k(y_i)$ otherwise. If $\text{run}(\sigma)$ is finite, we let $s_\sigma$ denote the state associated to the last configuration of $\text{run}(\sigma)$.

Preferences, Predicates, and Strategic Equilibria

Each agent has certain preferences over the various possible outcomes of the mechanism. Given interpretation $I$ and two outcomes $o, o' \in D_I$, agent $i$ prefers $o$ at least as much as $o'$ whenever $o \succeq_i o'$. Often, the elements in $D_I$ will be elements of some product space, so that, e.g., $(o_1, o_2) \in \mathbb{R} \times \mathbb{R}$ will yield outcome $o_1$ for player 1 and outcome $o_2$ for player 2, where $(o_1, o_2) \succeq_i (o_1', o_2')$ if $o_1 \geq o_1'$.

An outcome function $\hat{o} : Cfg^I \rightarrow D_I$ assigns an outcome to every terminal history, and we let $\hat{O}$ denote the the set of all outcome functions. Given a semi-game $G(\gamma, s, I)$ we then obtain a game $G(\gamma, s, I, \hat{o})$, where for each terminal sequence of configurations $\hat{\epsilon}$ the associated outcome is $\hat{o}(\hat{\epsilon})$, and agent $i$ prefers $\hat{\epsilon}_1$ to $\hat{\epsilon}_2$ iff $\hat{o}(\hat{\epsilon}_1) \succeq_i \hat{o}(\hat{\epsilon}_2)$. Given profile $\sigma$, we usually write $\hat{o}(\sigma)$ instead of $\hat{o}(\text{run}(\sigma))$, as we shall not be very careful about distinguishing $\sigma$ from $\text{run}(\sigma)$.

Subgames of games will play a special role in the equilibrium notion to be defined subsequently. A game $G'(\gamma', s', I, \hat{o}|G')$ is a subgame of a game $G(\gamma, s, I, \hat{o})$ iff there is a finite sequence of configurations $\langle \gamma_0, s_0 \rangle \xrightarrow{A_1} \langle \gamma_1, s_1 \rangle \xrightarrow{A_2} \ldots \xrightarrow{A_n} \langle \gamma_n, s_n \rangle$ for some $n \geq 0$ such that $\langle \gamma_0, s_0 \rangle = \langle \gamma, s \rangle$ and $\langle \gamma_n, s_n \rangle = \langle \gamma', s' \rangle$. The outcome function $\hat{o}|G'$ is the restriction of $\hat{o}$ to $G'$, i.e., $\hat{o}|G'(\langle \gamma_n, s_n \rangle, \ldots, \langle \gamma_{n+k}, s_{n+k} \rangle) = \hat{o}(\langle \gamma_0, s_0 \rangle, \ldots, \langle \gamma_n, s_n \rangle, \ldots, \langle \gamma_{n+k}, s_{n+k} \rangle)$. Similarly for a strategy profile $\sigma$ for $G$, we let $\sigma|G'$ denote its restriction to $G'$, where $\sigma^a|G'(\langle \gamma_n, s_n \rangle, \ldots, \langle \gamma_{n+k}, s_{n+k} \rangle) = \sigma^a(\langle \gamma_0, s_0 \rangle, \ldots, \langle \gamma_n, s_n \rangle, \ldots, \langle \gamma_{n+k}, s_{n+k} \rangle)$.

Now that we have defined how executions of mechanisms give rise to game trees, we can apply two well-known equilibrium notions from game theory (see, e.g., [11] for a discussion of these notions). Given a strategy profile $\sigma = (\sigma^1, \ldots, \sigma^n)$ and a strategy $\tau^i$ for player $i$, let $(\tau^i, \sigma^{-i})$ denote the modified strategy profile $(\sigma^1, \ldots, \sigma^{i-1}, \tau^i, \sigma^{i+1}, \ldots, \sigma^n)$. Furthermore, let $\sigma \sim_i \tau$ denote that the strategy profiles $\sigma$ and $\tau$ differ at most regarding the strategy prescribed for player $i$. Considering any game $G(\gamma, s, I, \hat{o})$, we call a strategy profile $\sigma$ a Nash equilibrium (NE) in $G$ iff for all agents $i$ and strategies $\tau^i$ we have $\hat{o}(\sigma) \succeq_i \hat{o}(\tau^i, \sigma^{-i}))$. Furthermore, $\sigma$ is a subgame-perfect equilibrium (SPE) iff for every subgame $G'$ of $G$, $\sigma|G'$ is a Nash equilibrium in $G'$. 

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We shall usually obtain an outcome function \( \hat{o} \) from an extended predicate, to be explained now. Given a state \( s : MV \rightarrow D_I \) and an outcome \( o \in D_I \), we call \( (s, o) \) an extended state, or e-state for short. A predicate on \( I \) is simply a set of states \( P \subseteq S_I \), and hence every FOL formula \( \varphi \) containing only variables of \( MV \) gives rise to a predicate \( \varphi^I \). Similarly, an extended predicate, or e-predicate for short, is a set of e-states \( P \subseteq S_I \times D_I \), and every FOL formula which contains variables of \( MV \) plus a new outcome variable \( x_o \notin MV \) gives rise to an e-predicate. We say that e-predicate \( P \) is functional iff for every \( s \in S_I \) there exists a unique \( o \in D_I \) such that \( (s, o) \in P \). Given two predicates (or alternatively, two e-predicates), intersection, complementation, etc. can be defined simply set-theoretically. Given a predicate \( P_1 \) and an e-predicate \( P_2 \), however, we define \( P_1 \cap P_2 = \{(s, o) \in P_2 | s \in P_1 \} \).

Games can be obtained from extended predicates as follows: Given semi-game \( G(\gamma, s, I) \) and e-predicate \( Q \), let \( \hat{O}_Q \) contain all the outcome functions \( \hat{o} \) which assign an outcome satisfying \( Q \) to every finite history, i.e.,

\[
\hat{O}_Q = \{ \hat{o} \in \hat{O} \mid \forall \text{run}(\sigma) \in \text{Cfg}^f : \text{if run}(\sigma) \text{ is finite then } (s_\sigma, \hat{o}(\sigma)) \in Q \}.
\]

Note that in general, \( \hat{O}_Q \) may be empty or contain multiple outcome assignments. But given e-predicate \( Q \) and some \( \hat{o}_Q \in \hat{O}_Q \), we are able to turn the semi-game \( G(\gamma, s, I) \) into a game \( G(\gamma, s, I, \hat{o}_Q) \).

4 Mechanism Correctness

4.1 Hoare Logic: From Programs to Games

Hoare in [7] introduced correctness assertions of the form \( \{P\}\gamma\{Q\} \), where \( \gamma \) is a program and \( P \) and \( Q \) are predicates. The intended interpretation of this assertion is that in every state which satisfies \( P \), any terminating execution of program \( \gamma \) ends in a state which satisfies \( Q \). In this paper, we shall extend this approach to reason about the correctness of game-theoretic mechanisms under subgame-perfect equilibria.

In lifting standard Hoare triples to games we generalise them in two ways. We can view the postcondition \( Q \) as specifying the winning condition for the game, i.e., all plays of the game ending in a state which satisfies \( Q \) are a win, all others a loss. Note that under the partial correctness reading, infinite runs are in fact also treated as wins. Our first generalisation consists of moving from simple win/loss situations, represented by predicates, to general preference structures. This is achieved by moving from predicates to e-predicates which also specify the outcome or payoff at a state. Second and more importantly, we move from simple claims about the existence of a strategy profile satisfying the postcondition to more refined claims about the existence of a strategy profile which has an equilibrium property. This equilibrium property is generally quite complex, and it is the complexity of this equilibrium property which can present a challenge to compositionality, in particular to the Hoare inference rule for composing two programs/games (see lemma 1 below).
Before defining our mechanism correctness assertions \( \{P\} \gamma \{Q\} \), it is important to point out that we are following an *extensional* rather than an *intensional* approach (see also [10]). We assume that pre- and postconditions are predicates, i.e., semantic objects rather than formulas of some logical language. Naturally, this means that the calculus we present later is not fully syntactic. In the intensional approach, however, one runs into the problem of *expressiveness*, since it may happen that under a given interpretation the logical language is not rich enough to express all the preconditions needed. This complicates completeness proofs considerably, due to the need for an arithmetisation of syntax (Gödelisation), etc. Furthermore, we feel that this extra work yields more insights about the logic used for the assertion language (usually first-order logic) than about the game theoretic mechanisms and their equilibria, which is what we are interested in here.

Due to its fully syntactic nature, it does seem likely that the automated verification of mechanisms would benefit from using the intensional approach, and we do intend to investigate this approach in the future (see also comments in the last section). However, note that in contrast to most computer programs whose domain of computation contains at least the natural numbers, mechanisms like voting procedures often use a finite domain of computation, e.g., because there is only a small number of possible candidates running for president. In such cases, it may in fact be easier to do automatic verification using the extensions of the predicates directly. Second, even if this is not the case, the best logic to choose for automated verification may very much depend on the class of mechanisms under consideration, the theorem prover to be used, etc. Hence, for our present purposes, we decide to postpone these issues since they are more relevant for implementation, and the extensional approach conveniently allows us to do so.

### 4.2 Mechanism Correctness and Implementation

Assume that we are given some interpretation \( \mathcal{I} \), a mechanism \( \gamma \), and e-predicates \( P \) and \( Q \). Then we say that \( \{P\} \gamma \{Q\} \) is *valid* in \( \mathcal{I} \), denoted as \( \mathcal{I} \models \{P\} \gamma \{Q\} \), iff

for every \((s,o) \in P\), there is an outcome function \( \hat{o} \in \mathcal{O}_Q \) and a strategy profile \( \sigma \) such that \( \sigma \) is an SPE in \( G(\gamma, s, \mathcal{I}, \hat{o}) \) and \( \hat{o}(\sigma) = o \).

The notion defined indeed generalises the standard partial correctness assertions of Hoare in the following way: Given an arbitrary element \( d \in D_\mathcal{I} \) and a predicate \( P \subseteq S_\mathcal{I} \), let \( P^* = \{(s,d)|s \in P\} \). Then given any program \( \gamma \in MPL(PRG) \) and predicates \( P \) and \( Q \), the partial correctness assertion \( \{P\} \gamma \{Q\} \) holds in interpretation \( \mathcal{I} \) iff \( \mathcal{I} \models \{P^*\} \gamma \{Q^*\} \).

In order to link our mechanism correctness assertion to the game-theoretic literature on mechanism design and implementation theory [11, 9, 15], we shall define our version of the mechanism design problem more formally. Given a set of possible outcomes \( D_\mathcal{I} \) of the mechanism and the set of preference profiles over \( D_\mathcal{I} \), a *social choice correspondence* \( f \) maps a preference profile \( (\geq_i)_{i \in \text{Ags}} \) to a set of outcomes \( X \subseteq D_\mathcal{I} \). The idea is that at preference profile \( (\geq_i)_{i \in \text{Ags}} \), society

...
or the mechanism designer wants one of the outcomes in \( f(\geq_i) \) to be implemented or achieved. In case \( f(\geq_i) \) is empty, society is indifferent to the outcome actually realised. The mechanism design problem is to find a mechanism which implements the social choice correspondence in a non-centralised manner, i.e., no matter what the preferences of the agents are, self-interested agents will have an incentive to play so that the outcome intended by the designer will obtain. We shall now see how this problem can be translated into our mechanism correctness assertions.

For a preference profile \( (\geq_i) \) where each \( \geq_i \subseteq D_I \times D_I \), let \( \gamma \) denote the model which is obtained from \( I \) by replacing the interpretation of the preference relations by the \( \geq_i \). Furthermore, for a given social choice correspondence \( f \), let \( f^*(x) = \{ (s, o) \in S_I \times D_I | o \in f(x) \} \), and let \( Q \) be any functional e-predicate. Then we say that the pair \( (\gamma, Q) \) SPE-implements a social choice correspondence \( f \) iff for all preference profiles \( (\geq_i) \) we have

\[
\mathcal{I}(\geq_i) = \{ f^*(\geq_i) \} \gamma \{ Q \}.
\]

To see what this statement actually expresses, let us unpack the definition: \( (\gamma, Q) \) SPE-implements social choice correspondence \( f \) iff

for all preference profiles \( (\geq_i) \), for all states \( s \in S_I \), and for all \( o \in f(\geq_i) \), there is some \( \tilde{o} \in \tilde{O}_Q \) and some strategy profile \( \sigma \) such that \( \sigma \) is an SPE for \( G(\gamma, s, \mathcal{I}(\geq_i), \tilde{o}) \) and \( \tilde{o}(\sigma) = o \).

Note that this notion of implementation is a weak notion which does not ask every but only some equilibrium profile to yield the desired outcome, hence strictly speaking we are dealing with mechanism design rather than implementation theory. In the remainder of this section, we shall look at a few concrete examples of mechanism design.

### 4.3 Auctions

Over the domain of natural numbers, the mechanism

\[
ch_{\{1,2\}}(\{x_1, x_2\})
\]

can represent a sealed-bid auction where the two players simultaneously choose their bids, e.g., in euros, in order to obtain some desirable object, say a piano. Since this game is atomic, the notions of SPE and NE coincide, and hence we can phrase the existence of Nash equilibria using the correctness notion defined earlier.

Consider the case of a second-price auction where the player who makes the highest bid has to pay the price of the loser’s bid. We assume that our model \( \mathcal{I} \) has the natural numbers as its domain, and contains two constants \( v_1 \) and \( v_2 \) whose values denote the private valuations of the players. Instead of representing outcomes as pairs \( o = (o_1, o_2) \) we shall assume that there are two outcome variables \( o_1 \) and \( o_2 \) which determine the payoffs of player 1 and 2, respectively.
A player’s payoff is 0 if he fails to obtain the piano, and his valuation minus the other player’s bid if he does obtain the piano. The preference ordering over elements of the domain is the obvious one: \( d_1 \geq d_2 \) iff \( d_1 \geq d_2 \). Note that a player’s preference relation is completely determined by his valuation.

The postcondition of the second-price auction is the e-predicate expressed by the following formula \( \psi \):

\[
(x_1 \geq x_2 \rightarrow (o_1 = v_1 - x_2 \land o_2 = 0)) \land (x_1 < x_2 \rightarrow (o_1 = 0 \land o_2 = v_2 - x_1))
\]

It is easy to see that this postcondition expresses the payoffs of the players in the second-price auction. Note also that the postcondition formalises the tie-breaking rule which assigns the object to player 1 in case the bids are equal. Now consider the e-predicate expressed by the following formula \( \varphi \):

\[
(v_1 \geq v_2 \rightarrow (o_1 = v_1 - v_2 \land o_2 = 0)) \land (v_1 < v_2 \rightarrow (o_1 = 0 \land o_2 = v_2 - v_1))
\]

We claim that \( I \models \{ \varphi \} \text{ch}_{\{1,2\}}(\{x_1, x_2\})\{\psi \} \): If player 1’s valuation is at least as high as player 2’s valuation, then the auction has a Nash-equilibrium in which player 2’s payoff is 0 and player 1’s payoff is the difference between the valuations. Similarly in case player 2’s valuation is higher.

To see why this is so, note that it is a well-known result in game theory (see, e.g., [11]) that in a second-price sealed-bid auction, bidding your valuation results in a Nash equilibrium (in fact, it is even a dominant strategy). Hence, if each player bids \( x_i = v_i \), the outcomes are the ones specified by \( \varphi \), and the strategies are in equilibrium.

In fact, from the validity of \( \{ \varphi \} \text{ch}_{\{1,2\}}(\{x_1, x_2\})\{\psi \} \) we can derive some information about the nature of the winning strategies. For suppose w.l.o.g. that \( v_1 \geq v_2 \). Using precondition \( \varphi \), we know that \( o_1 = v_1 - v_2 \) and \( o_2 = 0 \). Now we can distinguish two cases: In the first case, we have a Nash equilibrium (and hence also a SPE) where player 1 bids less than player 2, i.e., \( x_1 < x_2 \). Now using the postcondition \( \psi \) and the fact that the outcome variables \( o_1 \) and \( o_2 \) are never changed by any mechanism, we know that \( o_1 = 0 \) and \( o_2 = v_2 - x_1 \). Hence \( x_1 = v_2 = v_1 \) and \( x_2 > v_2 = v_1 \), i.e., the players’ valuations must be the same and player 2 must bid higher than his valuation. It is easy to check that these bids indeed constitute a Nash equilibrium. In the second case, we have a Nash equilibrium with \( x_1 \geq x_2 \). Again using the postcondition, \( o_2 = 0 \) and \( o_1 = v_1 - x_2 \). Hence, \( x_2 = v_2 \) and \( x_1 \geq v_2 \). Thus, player 2 bids his valuation and player 1 bids at least player 2’s valuation. Again, these bid combinations all constitute Nash equilibria, and our intended equilibrium, where each player bids his own valuation, is included in this second case.

In a private-value environment, a sealed-bid second-price auction is essentially outcome equivalent with an English auction, where bidders keep increasing the price over a number of bidding rounds until there is no more bidder who wants to obtain the object for a higher price. In an English auction, bidding slightly more than the second-highest valuation will suffice to obtain the object. Analogously, we can consider a sealed-bid first-price auction where the winner has to pay his own bid rather than the second-highest bid. The first-price
auction is essentially outcome equivalent to the Dutch (or descending) auction, where the auctioneer continues to lower the price of the object until a player decides to take the object for the current price. If the players’ valuations are not public, the safe strategy is to stop the auction just below one’s valuation, the result being that the player with the highest valuation will obtain the object for the price of almost his valuation.

Contrary to these results, we shall show in section 6 that from the perspective of SPEs, the Dutch auction is also similar to a sealed-bid second-price auction. In order to apply SPEs as a solution concept, we need to assume that players’ preferences are public. In an auction, this means that players know each other’s valuations. In this case, however, if \( v_1 > v_2 \), player 1 can wait longer before calling out to stop the Dutch auction, he can wait until the prices reach \( v_2 \) or just above. Hence, when preferences are public, it would seem that Dutch auction and second-price auction share a SPE. We will verify this claim in section 6, thereby also obtaining the precise conditions for this equivalence.

Finally, a further remark relating auction preconditions to the notion of SPE-implementation. In a second-price auction, we want to SPE-implement the social choice correspondence \( f \) which assigns to a preference profile \((v_1, v_2)\) the outcome \((o_1, o_2)\) with \( o_1 = v_1 - v_2 \) and \( o_2 = 0 \) in case \( v_1 \geq v_2 \) and \( o_2 = v_2 - v_1 \) and \( o_1 = 0 \) in case \( v_1 < v_2 \). While the precondition \( \varphi \) given above does capture this social choice correspondence in an intuitive sense, note that it is not the precondition used in our definition of SPE-implementation. This is because SPE-implementation, as we defined it, requires a correctness claim for each preference profile separately. In contrast, our precondition \( \varphi \) covers all preference profiles in one precondition, since it conditions the assigned outcomes on the relationship between the valuation constants. This formulation leads to a much more general result and hence is usually preferable. In the next section, we shall present an example using the notion of SPE-implementation literally.

4.4 Solomon’s Dilemma

The biblical dilemma of Solomon (1 Kings 3:16-28) has often been used to illustrate the basic idea of implementation theory [11, 9]. In the same spirit, we shall use it here to illustrate our notion of SPE-implementation. The game-theorist will get the additional benefit of seeing a well-known example of implementation theory translated into our framework. Solomon’s dilemma is that two women have come before him with a small child, both claiming to be the mother of the child.

He sent for a sword, and when it was brought, he said, “Cut the living child in two and give each woman half of it.” The real mother, her heart full of love for her son, said to the king, “Please, Your Majesty, don’t kill the child! Give it to her!” But the other woman said, “Don’t give it to either of us; go on and cut it in two.” Then Solomon said, “Don’t kill the child! Give it to the first woman, she is its real mother.”
The story exemplifies the need for a mechanism very well: Since Solomon does not know who the real mother is (i.e., he does not know the women’s preferences), he cannot impose the outcome of his choice function directly. Rather, he needs to devise a mechanism which will provide an incentive to the women to reveal this information to him.

To mathematically model Solomon’s situation, we consider three outcomes: $a$ (baby is given to Anne, player 1), $b$ (baby is given to Bess, player 2), and $c$ (baby is cut in two). Solomon has to consider two possible situations: In case Anne is the real mother, the preference profile is given by $\theta_1$, in case Bess is the real mother, the preference profile is $\theta_2$.

\[
\begin{align*}
\theta_1 : & \ a >_1 b >_1 c \quad \text{and} \quad b >_2 c >_2 a \\
\theta_2 : & \ a >_1 c >_1 b \quad \text{and} \quad b >_2 a >_2 c
\end{align*}
\]

Solomon’s problem is to find a mechanism which implements the social choice correspondence $f$ for which $f(\theta_1) = \{a\}$ and $f(\theta_2) = \{b\}$. In spite of Solomon’s apparent cleverness, it turns out that $f$ is not Nash-implementable (see [9] for a proof). However, by slightly modifying the problem, one can obtain an implementation nonetheless.

Let us consider the situation where instead of quarreling about a child, Anne and Bess argue about who is the owner of a painting. Furthermore, we allow Solomon to impose fines on the two women, i.e., we allow for monetary side payments. We can then think of the possible outcomes as triples $(x, m_1, m_2)$, where $x \in \{0, 1, 2\}$ denotes who obtains the painting (0 denoting that it is cut in two), and $m_i$ denotes the fine player $i$ has to pay to Solomon. Now suppose that the legitimate owner of the painting has valuation $v_H$ and the other woman has valuation $v_L$, where $v_H > v_L > 0$. Then if player $i$ does not get the painting, her payoff is $-m_i$. If she does get the painting, her payoff will be $v_H - m_i$ in case she is the legitimate owner, and $v_L - m_i$ otherwise. If player $i$ is the legitimate owner, these payoffs will then induce a preference profile $\theta_i$ in the obvious way.

In this new setup, Solomon wishes to implement the social choice rule $f$ for which $f(\theta_i) = \{(i, 0, 0)\}$, i.e., the painting is given to the legitimate owner and nobody has to pay any fines (we assume here that Solomon does not engage in dispute resolution to make money). More precisely, Solomon is looking for a pair $(\gamma, Q)$ which SPE-implements $f$, i.e., for which

\[
\mathcal{I}[\theta_1] = \{o = (1, 0, 0)\} \gamma Q \quad \text{and} \quad \mathcal{I}[\theta_2] = \{o = (2, 0, 0)\} \gamma Q.
\]

The following mechanism $\gamma$ achieves this goal: First, Anne is asked whether the painting is hers or not. If she says no, the painting is given to Bess and no fines are imposed. Otherwise, Bess is asked the same question. If Bess answers the painting is not hers, it is given to Anne, again without imposing any fines. Finally, in case both players have claimed to be the owner of the painting, Anne is fined a small amount $\varepsilon > 0$ and Bess gets the painting but has to pay a large amount $M$ for which $v_L < M < v_H$. The mechanism $\gamma$ can be programmed as follows, where we take the real numbers as our domain:
\[ \text{ch}_{(1)}(\{x_1\}); \]
\[ \text{if } x_1 > 0 \text{ then } \text{owner} := 2 \]
\[ \text{else } \text{ch}_{(2)}(\{x_2\}); \]
\[ \text{if } x_2 > 0 \text{ then } \text{owner} := 1 \text{ else } \text{owner} := 0 \]

As for the payoff specification, let \( Q \) be the e-predicate corresponding to the following formula:
\[
(\text{owner} = 1 \rightarrow o = (1, 0, 0)) \\
\land (\text{owner} = 2 \rightarrow o = (2, 0, 0)) \\
\land (\text{owner} = 0 \rightarrow o = (2, \varepsilon, M))
\]

Game theoretically, it is easy to verify that for preference profile \( \theta_i \), the following game form has a subgame-perfect equilibrium yielding outcome \((i, 0, 0)\). We will return to this example in section 6 and give a formal verification of this mechanism.

\[
\begin{array}{ccc}
1 & \rightarrow & 2 \\
\downarrow & & \downarrow \\
(2,0,0) & & (2,\varepsilon,M) \\
\end{array}
\]

\[
(1,0,0)
\]

5 Axiomatic Mechanism Verification

5.1 A Hoare-style Calculus

Below we present a calculus for deriving the correctness assertions we introduced above. Note that the calculus is a natural generalisation of the standard Hoare calculus, where the only addition is an axiom for the new construct \( \text{ch}_A \). Given e-predicate \( P \), we let \( P[x/t] = \{(s, o) \in S_I \times D_I | (s, t, o) \in P\} \).

\[
\begin{array}{c}
\{Q[x/t]\} x := t(Q) \hspace{1cm} \text{(ass.)} \\
\{\text{wpre}(\text{ch}_A(X), Q, I)\} \text{ch}_A(X)\{Q\} \hspace{1cm} \text{(choice)} \\
\{P\} \gamma_1 \{R\} \quad \{R\} \gamma_2 \{Q\} \hspace{1cm} \text{(comp.)} \\
\{P \cap B^2\} \gamma_1 \{Q\} \quad \{P \cap \overline{B^2}\} \gamma_2 \{Q\} \hspace{1cm} \text{(if)} \\
\{P\text{ if } B \text{ then } \gamma_1 \text{ else } \gamma_2\} \{Q\} \\
\{P\} \text{ while } B \text{ do } \gamma \{P \cap \overline{B^2}\} \hspace{1cm} \text{(while)} \\
P \subseteq P', \ \{P'\} \gamma \{Q'\}, \ Q' \subseteq Q \hspace{1cm} \{P\} \gamma \{Q\} \hspace{1cm} \text{(l.c.)}
\end{array}
\]
In the choice axiom, $\text{wpre}(\gamma, Q, I)$ refers to the weakest precondition of $Q$ under $\gamma$. Given interpretation $I$, mechanism $\gamma$, and e-predicate $Q$, we define $\text{wpre}$ as follows:

$$\text{wpre}(\gamma, Q, I) = \{(s, o) \in S_I \times D_I \mid \exists \hat{o} \in \hat{O}_Q \exists \sigma : \sigma \text{ is an SPE in } G(\gamma, s, I, \hat{o}) \text{ and } \hat{o}(\sigma) = o\}$$

Note that by definition, $I \models \{\text{wpre}(\gamma, Q, I)\} \gamma \{Q\}$, and for every e-predicate $P$ such that $I \models \{P\} \gamma \{Q\}$, we have $P \subseteq \text{wpre}(\gamma, Q, I)$. Weakest preconditions will play an important role in the completeness proof of section 5.3.

Let $\Delta_I$ be the smallest set of correctness assertions $\{P\} \gamma \{Q\}$ over $I$ which includes the axioms and is closed under the inference rules above. We shall usually write $\{P\} \gamma \{Q\} \in \Delta_I$ as $I \vdash \{P\} \gamma \{Q\}$. In order to gain some intuitions regarding this calculus, the reader may wish to consult section 6 before proceeding with the subsequent soundness and completeness results.

Before establishing soundness and completeness of the calculus presented, some further comments regarding the choice axiom are in order. As mentioned, the calculus is extensional in the sense that preconditions and postconditions are semantic rather than syntactic objects, predicates rather than formulas of, say, first-order logic. As a consequence, we do not get a syntactic proof system, but rather what one might call a compositional proof methodology. Hence, while the precondition of the choice axiom may seem tautological, it still suffices to reduce reasoning about subgame-perfect equilibria in complex games to reasoning about Nash equilibria in simple games. Hence, while we are still in need of a semantic argument to establish the Nash equilibrium, it is a simpler semantic argument which applies only to the simplest game, the atomic choice game. As the examples in section 6 will illustrate, this decomposition is achieved by moving the complexity from the mechanism into the mechanism’s postcondition or payoff assignment, and it is this which the calculus allows one to do. In other words, the complexity is moved from the dynamic to the static part, from the mechanism to the predicates describing pre- and postconditions.

In verification practice, it turns out that the precondition of the choice axiom is often rather analogous to the precondition of the assignment axiom, where Nash equilibrium strategies are substituted for the choice variables in the precondition. Slightly more formally, suppose that the postcondition $Q$ is a functional e-predicate which simply assigns outcomes based on the choice variables, and that $Q$ only contains these choice variables and no other variables. An example of such a postcondition is the postcondition $\psi$ of the second-price auction discussed in section 4.3. Since this postcondition depends on the state only in terms of the choice variables, we can say that the weakest precondition of the choice construct is simply $Q$ where each choice variable $x_i$ is replaced by the Nash equilibrium strategy of player $i$ in the choice game played in any state with payoffs given by $Q$. In fact, this is precisely what happened with the precondition $\varphi$ of the second-price auction where $x_i$ is replaced by $v_i$. In general, however, things are not quite so simple, as the analysis of the Dutch auction in section 6 will illustrate.
5.2 Soundness

The following lemma presents the first of the two most difficult cases of the subsequent soundness result. It guarantees that equilibria of subgames can be composed into equilibria of the supergame.

Lemma 1 (Composition) If we have both \( \mathcal{I} \models \{P\}\gamma_1\{R\} \) and \( \mathcal{I} \models \{R\}\gamma_2\{Q\} \) then \( \mathcal{I} \models \{P\}\gamma_1;\gamma_2\{Q\} \).

Proof. Let \((s,o) \in P\), and consider \(G(\gamma_1;\gamma_2,s,I)\). By our first assumption, there is an outcome function \(\widehat{o}_1 \in \widehat{O}_R\) and a strategy profile \(\sigma_1\) such that \(\sigma_1\) is an SPE in \(G_1(\gamma_1,s,I,\widehat{o}_1)\) and \(\widehat{o}_1(\sigma_1) = o\).

Now for every finite run \(\tau_1\) of \(G_1\) ending in some terminal state \(t\) with \(\widehat{o}_1(\tau_1) = o_t\), since \((t,o_t) \in R\), we know by our second assumption that there is some outcome function \(\widehat{o}_i \in \widehat{O}_Q\) and some strategy profile \(\sigma_i\) such that \(\sigma_i\) is an SPE in \(G_i(\gamma_2,t,I,\widehat{o}_i)\) and \(\widehat{o}_i(\sigma_i) = o_t\). Taken together, \(\sigma_1\) and the \(\sigma_i\) induce a strategy profile \(\sigma\) for \(G\), and similarly \(\widehat{o}_1\) (for the infinite runs of \(G_1\)) and the \(\widehat{o}_i\) induce an outcome function \(\widehat{o} \in \widehat{O}_Q\) for \(G\). Hence, it remains to show that \(\sigma\) is an SPE and that \(\widehat{o}(\sigma) = o\).

First, it is easily seen that \(\widehat{o}(\sigma) = o\), for \(\widehat{o}_1(\sigma_1) = o\), and so in case \(\sigma_1\) is finite, \((s_{\sigma_1}, o) \in R\), from which by definition it follows that \(\widehat{o}(\sigma) = o\). Second, we need to show that \(\sigma\) is an SPE in \(G(\gamma_1;\gamma_2,s,I,\widehat{o})\). So consider any subgame \(G'(\pi,t,I)\) of \(G\). In the easy case, \(G'\) will be a subgame of some \(G_{\nu'}\), where \(t'\) is a terminal state in \(G_1\), for in this case, our second assumption immediately guarantees the equilibrium property. In the more complicated case, \(G'\) lies partly in \(G_1\). For simplicity, we shall for the rest of this argument assume that \(\sigma = \sigma_1;\sigma_2\) refers to its restriction to \(G'\). So consider any strategy profile \(\tau_1;\tau_2\) for \(G'\) such that \(\sigma = \sigma_1;\sigma_2 \sim \tau_1;\tau_2 = \tau\), where \(\sigma_1\) and \(\tau_1\) both yield finite runs. Suppose further that \(\widehat{o}(\sigma) = o_0\) and \(\widehat{o}(\tau) = o_2\), as depicted below.

Now supposing that \(\widehat{o}(\tau_1;\sigma_2) = o_1\), we know by definition of \(\sigma\) that \(o_1 \geq o_2\), and that \(\widehat{o}_1(\tau_1) = o_1\). Furthermore, since \(\sigma_1\) was an SPE in \(G_1\), we know also that \(\widehat{o}_1(\sigma_1) \geq o_1\). Since \(o_0 = \widehat{o}(\sigma) = \widehat{o}_1(\sigma_1)\), we can conclude by transitivity that \(o_0 \geq o_2\).
Finally, note that the case where either \( \sigma_1 \) or \( \tau_1 \) or both are infinite can be treated by a simplification of the above argument. \( \square \)

The following lemma isolates the arguments needed to prove the soundness of the inference rule for iteration. Our assumption that our model \( \mathcal{I} \) contains a uniformly worst element is needed here.

**Lemma 2** If \( \mathcal{I} \models \{ P \cap B \} \gamma \{ P \} \) then \( \mathcal{I} \models \{ P \} \text{while} B \gamma \{ P \cap B \} \).

**Proof.** Roughly speaking, the proof is an iterated application of the preceding composition lemma, but a few subtleties have to be dealt with, in particular the possibility of newly arising infinite runs.

Suppose that \( (s, o) \in P \). In order to define a strategy \( \sigma \) and outcome function \( \hat{o} \) for \( G(\text{while} B \gamma \gamma, s, \mathcal{I}) \), we shall inductively define strategy profile \( \sigma_n \) and outcome function \( \hat{o}_n \) for game \( G_n \) which consists of the first \( n \) iterations of game \( G \). Game \( G_0 \) simply consists of configuration \((\Lambda, s)\), strategy profile \( \sigma_0 \) consists of doing nothing, and as an outcome function we take \( \hat{o}_0((\Lambda, s)) = o \). Note that \( \hat{o}_0 \in \hat{O}_P \).

For the inductive step, define \( G_{n+1} \) as \( G_n \) where for every terminal state \((t, o_t) \in P \cap B^2 \) in \( G_n \), we concatenate \( G_t(\gamma, t, \mathcal{I}) \) to \( t \). By our assumption, for each such terminal state, we have an outcome function \( \hat{o}_t \) and a SPE strategy profile \( \sigma_t \), and we define \( \sigma_{n+1} \) and \( \hat{o}_{n+1} \) in the natural way, by extending \( \sigma_n \) and \( \hat{o}_n \) to \( G_{n+1} \) using the \( \hat{o}_t \) and \( \sigma_t \).

Now with slight abuse of notation, we can define strategy profile \( \sigma \) and outcome function \( \hat{o} \) for \( G \) as follows: We take \( \sigma = \bigcup \sigma_i \), i.e., we simply take the profile generated by the \( \sigma_i \). Similarly, we define \( \hat{o} = \bigcup \hat{o}_i \), i.e., every run \( \tau \) of \( G \) which is part of some \( G_i \) is evaluated according to \( \hat{o}_i \). Furthermore, there may be new infinite runs in \( G \) which are not part of any \( G_i \), but are instead generated by an infinite number of plays of \( \gamma \) itself. Given such an infinite run \( \tau \), we define \( \hat{o}(\tau) = o_c \) in case there is some \( j \) such that for all \( k \geq j \) we have \( \hat{o}_k(G_k) = o_c \); otherwise, we let \( \hat{o}(\tau) = -\infty \). Thus, for infinite runs which converge on a certain outcome \( o_c \), we assign \( o_c \) to the run, and otherwise simply take the uniformly worst outcome. Note that \( \hat{o} \in \hat{O}_P^\gamma \).

Observe first that \( \hat{o}(\sigma) = o \). For we have \( \hat{o}_1(\sigma_1) = o, \hat{o}_2(\sigma_2) = \hat{o}_1(\sigma_1) = o, \) etc., and so in case \( \sigma \) is finite, there is some maximal \( k \) such that \( \hat{o}(\sigma) = \hat{o}_k(\sigma_k) = o \). In case \( \sigma \) is infinite, we have a constant and hence converging sequence of outcomes consisting of \( o \) only.

Hence, all we need to show is that \( \sigma \) is an SPE in \( G(\text{while} B \gamma \gamma, s, \mathcal{I}, \hat{o}) \). So consider any subgame \( G' \) of \( G \) and a strategy \( \tau \sim_i \sigma \) such that \( \hat{o}(\sigma) = o_0 \) and \( \hat{o}(\tau) = o_2 \). Now the reasoning can proceed along the lines of the composition lemma and the figure given there: In case \( \tau \) yields a run which lies in \( G_k \), we can show by induction on \( k \) that \( o_0 \geq o_2 \), each step involving the reasoning carried out in the composition lemma. On the other hand, in case \( \tau \) is an infinite run generated by infinitely many \( \gamma \)-repetitions, we need to distinguish two cases: In the easy case where \( \hat{o}(\tau) = -\infty \), the result is obvious. In the more complicated case, \( \hat{o}(\tau) = o_c \) due to a sequence of outcomes which converges on \( o_c \). Suppose
$k$ is the smallest number for which $\hat{o}_{k}(\tau|G_{k}) = o_{\gamma}$. Then again we can apply the reasoning of the composition lemma $k$ times to show that $o_{0} \geq o_{\gamma}$. \hfill \Box

**Theorem 3 (Soundness)** If $I \vdash \{P\} \gamma \{Q\}$ then $I \models \{P\} \gamma \{Q\}$.

*Proof.* The proof is by induction on the length of the derivation, so we start with showing the validity of the axioms. The soundness of the $\text{ch}_{A}$ axiom follows by definition.

For $I \vdash \{Q[x/t]\}x := t\{Q\}$, suppose that $(s,o) \in Q[x/t]$. We know that all runs in $G(x := t,s,I)$ are finite. Since no choices need to be made in $G(x := t,s,I)$, the one existing strategy profile $\sigma$ is trivially an equilibrium in $G(x := t,s,I,\hat{o})$ for any outcome function $\hat{o} \in \hat{O}_{Q}$, and in particular for the outcome function $\hat{o}$ which assigns $o$ to $\sigma$. Note that since $(s,o) \in Q[x/t]$, $(s^{x},o) \in Q$, and hence $\hat{o} \in \hat{O}_{Q}$.

Turning to the inference rules, note that the case of composition is treated in lemma 1, and the logical consequence rule is an easy consequence of the semantic definition of $I \models \{P\} \gamma \{Q\}$. For conditional branching, the conclusion follows directly from the two premises, given that $G(\text{if } B \text{ then } \gamma_{1} \text{ else } \gamma_{2}, s,I)$ is either $G_{1}(\gamma_{1},s,I)$ or $G_{2}(\gamma_{2},s,I)$. Finally, lemma 2 takes care of iteration. \hfill \Box

Note that the soundness result also holds for Nash equilibria: If in the definition of $I \models \{P\} \gamma \{Q\}$ we replace SPE by NE, the above soundness result can still be proved. This is as it should be, since every subgame-perfect equilibrium is also a Nash equilibrium.

### 5.3 Completeness

Like in the completeness proof for the standard Hoare calculus, the notion of a weakest precondition plays an important role for our calculus as well. The following lemma contains the essential argument for the completeness result.

**Lemma 4 (Decomposition)** If $I \models \{P\} \gamma_{1}; \gamma_{2}\{Q\}$, then for some $R$ we have $I \models \{P\} \gamma_{1}\{R\}$ and $I \models \{R\} \gamma_{2}\{Q\}$.

*Proof.* Our assumption is $I \models \{P\} \gamma_{1}; \gamma_{2}\{Q\}$. Let $R = \text{wpre}(\gamma_{2},Q,I)$, then all we need to show is that $I \models \{P\} \gamma_{1}\{R\}$. So supposing that $(s,o) \in P$, we need to provide an outcome function $\hat{o}_{1} \in \hat{O}_{R}$ and a strategy profile $\sigma_{1}$ such that $\sigma_{1}$ is an SPE in $G_{1}(\gamma_{1},s,I,\hat{o}_{1})$ and $\hat{o}_{1}(\sigma_{1}) = o$.

Consider the outcome function $\hat{o} \in \hat{O}_{Q}$ and the strategy profile $\sigma$ for $G(\gamma_{1}; \gamma_{2}, s, I)$ provided by our assumption. We let $\sigma_{1} = \sigma|_{G_{1}}$. As for the definition of $\hat{o}_{1}$, for every infinite run $\tau$ of $G_{1}$ we let $\hat{o}_{1}(\tau) = \hat{o}(\tau)$. If on the other hand $\tau$ is finite, we define $\hat{o}_{1}(\tau) = \hat{o}(\tau \cdot \sigma_{r})$, where $\sigma_{r} = \sigma|_{G_{r}}$. By our assumption, we have $\hat{o}_{1}(\sigma_{1}) = \hat{o}(\sigma) = o$. Furthermore, since $(s_{\tau},\hat{o}(\tau \cdot \sigma_{r})) \in R$, $\hat{o}_{1} \in \hat{O}_{R}$.

Hence, all we need to show is that $\sigma_{1}$ is an SPE in $G_{1}(\gamma_{1},s,I,\hat{o}_{1})$. So consider any subgame $G' = (\pi,t,I,\hat{o}_{1})$ of $G_{1}$, and a strategy profile $\tau_{1} \sim_{i} \sigma_{1},$
where we take $\hat{o}_1(\sigma_1) = o_0$ and $\hat{o}_1(\tau_1) = o_1$. Assume first that both $\sigma_1$ and $\tau_1$ are finite. Considering $G' = (\pi; \gamma_2, t, I, \hat{o})$, we know that there is a profile $\sigma_2$ (derived from $\sigma$) such that $\sigma_1 \cdot \sigma_2$ is an SPE in $G'$ and $\sigma_1 \cdot \sigma_2 \sim_i \tau_1 \cdot \sigma_2$. The situation is depicted below.

By definition, we know that $\hat{o}_1(\sigma_1) = \hat{o}(\sigma_1 \cdot \sigma_2) = o_0$ and $\hat{o}_1(\tau_1) = \hat{o}(\tau_1 \cdot \sigma_2) = o_1$, and hence we must have $o_0 \geq_i o_1$.

Note that in case either $\sigma_1$ or $\tau_1$ or both are infinite, a simplified version of the above argument can be applied. 

The above lemma is what distinguishes subgame-perfect equilibria from Nash equilibria, since only the former can be decomposed in the way shown by the decomposition lemma. For Nash equilibria, the above lemma fails: when defining $\hat{o}_1$ in the above proof, we cannot be sure that $\hat{o}_1 \in \hat{O}_R$, since a subprofile of an equilibrium profile may itself not be an equilibrium profile. Consequently, also the following completeness result does not hold for Nash equilibria.

**Theorem 5 (Completeness)** If $I \models \{P\} \gamma \{Q\}$ then $I \vdash \{P\} \gamma \{Q\}$.

**Proof.** The proof proceeds by induction on the structure of $\gamma$. For $x := t$, note that for any state $s$, the game $G(x := t, s, I)$ contains only a single finite run ending in state $s_t^x$. Observe that $P \subseteq Q[x/t]$; if $(s, o) \in P$, every run terminates in state $(s_t^x, o) \in Q$, and hence $(s, o) \in Q[x/t]$. Applying the logical consequence rule to the assignment axiom, we then obtain $I \vdash \{P\}|x := t\{Q\}$.

For $\text{ch}_A$, we use the axiom and the logical consequence rule, and for $\gamma_1; \gamma_2$, we can appeal to the decomposition lemma, induction hypothesis, and the composition rule. The case of $\text{if } B \text{ then } \gamma_1 \text{ else } \gamma_2$ is straightforward, so we only need to deal with the while-loop.

For iteration, suppose that $I \models \{P\} \text{while } B \text{ do } \gamma \{Q\}$. Similarly, to the proof of the decomposition lemma, we let $R = \text{wpre}(\text{while } B \text{ do } \gamma, Q, I)$. First, we shall establish that $I \models \{R \cap B^2\} \gamma \{R\}$. By definition, we have $I \models \{R\} \text{while } B \text{ do } \gamma \{Q\}$. From this, $I \models \{R \cap B^2\} \gamma; \text{while } B \text{ do } \gamma \{Q\}$ is easily seen to follow. Now we can apply the decomposition lemma: Since the $R$ provided by the proof of the decomposition lemma is precisely the one we defined above, we can conclude that $I \models \{R \cap B^2\} \gamma \{R\}$. 

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Now using the induction hypothesis and applying the while-rule, we obtain
\[ I \vdash \{ R \}\text{while } B \text{ do } \gamma \{ R \cap B \} \]

Since \( P \subseteq R \) and \( R \cap B \subseteq Q \), we can apply the logical consequence rule to derive \( I \vdash \{ P \}\text{while } B \text{ do } \gamma \{ Q \} \). \( \square \)

6 Applying the Calculus - Some Examples

6.1 Solomon’s Dilemma

Consider again Solomon’s 2-stage mechanism given in section 4.4, where we will replace the variable \( \text{owner} \) by \( w \) to save space. We will show one of the two required correctness claims, namely that \( I[\theta_1] \vdash \{ o = (1, 0, 0) \} \)
\( \text{ch}_{1}(\{ x_1 \}) \);
\( \text{if } x_1 > 0 \text{ then } w := 2 \)
\( \text{else } \text{ch}_{2}(\{ x_2 \}) \);
\( \text{if } x_2 > 0 \text{ then } w := 1 \text{ else } w := 0 \)
\( \{ (w = 1 \rightarrow o = (1, 0, 0)) \land (w = 2 \rightarrow o = (2, 0, 0)) \land (w = 0 \rightarrow o = (2, \varepsilon, M)) \} \),
corresponding to the situation where player 1 is the real owner of the painting. Note that for ease of notation we are now simply representing (extended) predicates by formulas in first-order logic.

Denoting the postcondition by \( Q_0 \), we have \( I[\theta_1] \vdash \{ o = (2, \varepsilon, M) \} \)
\( w := 0 \{ Q_0 \} \) and \( I[\theta_1] \vdash \{ o = (1, 0, 0) \} \)
\( w := 1 \{ Q_0 \} \) using the assignment axiom. Hence, by the if-rule we have \( I[\theta_1] \vdash \)
\( \{ (x_2 > 0 \rightarrow o = (1, 0, 0)) \land (x_2 \leq 0 \rightarrow o = (2, \varepsilon, M)) \} \)
\( \text{if } x_2 > 0 \text{ then } w := 1 \text{ else } w := 0 \)
\( \{ Q_0 \} \).

Denote the new precondition by \( Q_1 \). Since in \( \theta_1 \), we have \( (1, 0, 0) >_2 (2, \varepsilon, M) \), we know that when choosing a value for \( x_2 \), player 2 will choose the outcome \( (1, 0, 0) \), and hence we have \( I[\theta_1] \vdash \{ o = (1, 0, 0) \} \)
\( \text{ch}_{2}(\{ x_2 \}) \{ Q_1 \} \). On the other hand, we know by the assignment rule that \( I[\theta_1] \vdash \{ o = (2, 0, 0) \} \)
\( w := 2 \{ Q_0 \} \).

Hence, using the if-rule and composition, we have \( I[\theta_1] \vdash \)
\( \{ (x_1 > 0 \rightarrow o = (2, 0, 0)) \land (x_1 \leq 0 \rightarrow o = (1, 0, 0)) \} \)
\( \text{if } x_1 > 0 \text{ then } w := 2 \)
\( \text{else } \text{ch}_{2}(\{ x_2 \}) \);
\( \text{if } x_2 > 0 \text{ then } w := 1 \text{ else } w := 0 \)
\( \{ Q_0 \} \).

where we denote the new precondition by \( Q_2 \). Finally, since \( (1, 0, 0) >_1 (2, 0, 0) \), player 1 will choose \( (1, 0, 0) \) in an equilibrium, and so we have \( I[\theta_1] \vdash \{ o = \}

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(1, 0, 0)}ch_{1,2}(\{x_1\})\{Q_2\}. Using the composition rule, we have thereby succeeded in verifying the original claim, that the 2-stage mechanism does indeed provide an SPE-implementation solving Solomon’s (modified) dilemma.

6.2 Auctions

Second-Price Sealed-Bid Auction

We have already presented the sealed-bid second-price auction in section 4.3. We argued that in the relevant model $\mathcal{I}$ where two players have private valuations represented by the constants $v_1$ and $v_2$, we have $\mathcal{I} \models \{(v_1 \geq v_2 \rightarrow (o_1 = v_1 - v_2 \land o_2 = 0)) \land (v_1 < v_2 \rightarrow (o_1 = 0 \land o_2 = v_2 - v_1))\}$ and $\{(x_1 \geq x_2 \rightarrow (o_1 = v_1 - x_2 \land o_2 = 0)) \land (x_1 < x_2 \rightarrow (o_1 = 0 \land o_2 = v_2 - x_1))\}$, due to the fact that we obtain a Nash equilibrium if each player bids his valuation, i.e. $x_i = v_i$. We abbreviate the given precondition with $P$ and the postcondition with $R$. Note that $P$ is not the weakest precondition of $G(\mathcal{I})$, and hence $\mathcal{I} \not\models \{P\}ch_{1,2}(\{x_1, x_2\}\{R\})$ is not an axiom. This is because there are equilibria other than the one mentioned. For example, suppose that $v_1 \geq v_2$. Then if $v_2 \leq x_1 = x_2 \leq v_1$, we also have a Nash equilibrium. Hence, for $v_2 \leq k \leq v_1$, we can also consider the following precondition $P_k$

$$(v_1 \geq v_2 \rightarrow (o_1 = v_1 - k \land o_2 = 0)) \land (v_1 < v_2 \rightarrow (o_1 = 0 \land o_2 = v_2 - k))$$

for which we also have $\mathcal{I} \models \{P_k\}ch_{1,2}(\{x_1, x_2\}\{R\})$. Consequently, $P_k \lor P$ is weaker than $P$ for $k \neq v_2$, and hence $\mathcal{I} \not\models \{P\}ch_{1,2}(\{x_1, x_2\}\{R\})$ is indeed not an axiom. Still, it can be easily obtained from the choice axiom using the logical consequence rule.

Dutch Auction

We shall now illustrate the calculus in action for verifying the more complex Dutch auction which involves a while loop. In fact, we shall illustrate that the Dutch auction is equivalent to the preceding sealed-bid second-price auction in the very weak sense that the Dutch auction has the same subgame-perfect equilibrium as the sealed-bid second-price auction, where the player with the higher valuation receives the object, paying the price of the other player’s valuation. More formally, we shall show that both implement the same social choice correspondence defined in section 4.3, under certain conditions.

As mentioned in section 4.3, in a Dutch auction, the auctioneer continues to lower the price of an object until a player decides to take the object for the current price. Over the domain of natural numbers, the Dutch auction is captured by the following mechanism $\alpha$:

$$p := init;$$
Variable \( w \) keeps track of the winner, \( p \) keeps track of the current price, and is initialised to some value \( \text{init} \). For each offer, both players can choose a non-negative number signaling their desire to buy the object for price \( p \). As the algorithm is written down here, in case both players want to buy the object, player 1 gets it. Note that it is also subtleties like these which provide an argument for formally specifying and verifying mechanisms. The following postcondition \( Q \) naturally assigns payoffs at the end of the Dutch auction:

\[
(w = 1 \rightarrow (o_1 = v_1 - p \land o_2 = 0)) \\
\land (w = 2 \rightarrow (o_1 = 0 \land o_2 = v_2 - p)) \\
\land (w = 0 \rightarrow (o_1 = 0 \land o_2 = 0))
\]

Our goal will be to show that \( \mathcal{I} \vdash \{P\} \alpha \{Q\} \), i.e., just like the sealed-bid auction \( \text{ch}_{\{1,2\}}(\{x_1, x_2\}), R \) SPE-implements our desired social choice correspondence, so does \((\alpha, Q)\).

As in standard program verification, the art of proving the correctness of a while-loop lies in finding an invariant which remains true at the beginning of every loop execution. Consider the following invariant \( Inv \):

\[
v_1 \geq v_2 > 0 \land p \geq v_2 \land w \in \{0, 1, 2\} \\
\land (w = 1 \rightarrow (o_1 = v_1 - p \land o_2 = 0)) \\
\land (w = 2 \rightarrow (o_1 = 0 \land o_2 = v_2 - p)) \\
\land (w = 0 \rightarrow (o_1 = v_1 - v_2 \land o_2 = 0))
\]

Note that in order to simplify the exposition we have restricted ourselves to the case where \( v_1 \geq v_2 \), but this restriction is in no way essential. The invariant is similar to the desired postcondition \( Q \), the main difference lies in the situation where there is no winner. In that case, our desired outcome will be the SPE of the remaining subgame, the outcome designated by our social choice function, \( o_1 = v_1 - v_2 \) and \( o_2 = 0 \). Besides these winning conditions, we state the range of variable \( w \) as well as two conditions for \( v_2 \). First, \( v_2 \) must never be greater than the current price, for our equilibrium strategies force us to exit the loop at \( v_2 \). If, e.g., the auction started with a price below \( v_2 \), player 1 could immediately take the object and thereby receive a payoff higher than \( v_1 - v_2 \). Second, \( v_2 \) must be strictly greater than 0, for otherwise, it would be optimal for player 1 to take the object in the last round, where the price \( p = 1 \), and hence obtaining a payoff lower than \( v_1 - v_2 \). Note that the need for these additional constraints was discovered in the verification process and hence the “discovery” of these crucial side conditions should be regarded as a result of the verification effort.

We will now proceed to show that \( Inv \) is indeed an invariant, i.e., that \( \mathcal{I} \vdash \)
\{Inv \land p > 0 \land w = 0\}
ch_{1,2}(\{x_1, x_2\})
\text{if } x_1 > 0 \text{ then } w := 1
\quad \text{else if } x_2 > 0 \text{ then } w := 2
\quad \text{else } p := p - 1
\{Inv\}

Note that in fact, \( p > 0 \) is already implied by \( Inv \) which means that if \( Inv \) is indeed an invariant, the auction can never terminate due to the price having reached 0. Hence, for the purposes of verifying the desired equilibrium, the condition \( p > 0 \) is redundant in the guard condition of the while-loop.

To begin with, applying the assignment rule and the if-rule, it is easy to check that \( I \vdash \)
\begin{align*}
\{ v_1 \geq v_2 > 0 \land p \geq v_2 \land (x_1 > 0 \rightarrow (o_1 = v_1 - p \land o_2 = 0)) \\
\land ((x_1 = 0 \land x_2 > 0) \rightarrow (o_1 = 0 \land o_2 = v_2 - p)) \\
\land ((x_1 = 0 \land x_2 = 0) \rightarrow Inv[p/p - 1])\}
\text{if } x_1 > 0 \text{ then } w := 1
\quad \text{else if } x_2 > 0 \text{ then } w := 2
\quad \text{else } p := p - 1
\{Inv\},
\end{align*}

where \( Inv[p/p - 1] \) results from substituting \( p - 1 \) for \( p \) in \( Inv \). Denote the new precondition as \( Inv_2 \). Now we claim that \( I \vdash \)
\begin{align*}
\{ v_1 \geq v_2 > 0 \land p \geq v_2 \land w = 0 \land (p \leq v_2 \rightarrow (o_1 = v_1 - p \land o_2 = 0)) \\
\land (p > v_2 \rightarrow (o_1 = v_1 - v_2 \land o_2 = 0))\}
\text{ch}_{1,2}(\{x_1, x_2\})
\{Inv_2\}
\end{align*}

Assume that \( v_1 \geq v_2 > 0 \), and consider a state \( s \) where \( p \geq v_2 \) and \( w = 0 \). We distinguish two cases. First, if \( p \leq v_2 \) (i.e., \( p = v_2 \)), both players asking for the object, i.e., \( x_1 > 0 \) and \( x_2 > 0 \), constitutes a Nash equilibrium in the game with payoffs according to \( Inv_2 \), with payoffs \( o_1 = v_1 - p \) and \( o_2 = 0 \). Second, suppose that \( p > v_2 \). In this case, both players declining the object, i.e., \( x_1 = x_2 = 0 \), constitutes a Nash equilibrium. Player 2 should not ask for it since the price exceeds his valuation, and player 1 should not ask for it since the price will be lower in the next round; formally, declining the object yields \( o_1 = v_1 - v_2 \), whereas demanding the object only yields \( o_1 = v_1 - p \). Note that here it is essential that \( v_2 > 0 \), since it allows us to conclude that also \( p - 1 > 0 \), i.e., we have not reached the last auction round yet, there will be another round with a lower price.

Denote the new precondition as \( Inv_3 \). Note that \( Inv \land w = 0 \subseteq Inv_3 \). Hence, by using the composition rule and the logical consequence rule, we have established that \( Inv \) is indeed an invariant of the loop. Hence, we can apply the while rule to derive that \( I \vdash \)
\{Inv\}
while $p > 0 \land w = 0$ do
  $\text{ch}_{\{1,2\}}(\{x_1, x_2\})$:
  if $x_1 > 0$ then $w := 1$
  else if $x_2 > 0$ then $w := 2$
  else $p := p - 1$

$\{\text{Inv} \land \neg(p > 0 \land w = 0)\}$

So to conclude the verification of the Dutch auction, it suffices to note two things. First, $\text{Inv} \land \neg(p > 0 \land w = 0) \subseteq Q$, and hence we can apply the logical consequence rule to obtain the desired postcondition $Q$. Second, we have $I \vdash$

$\{v_1 \geq v_2 > 0 \land \text{init} \geq v_2 \land o_1 = v_1 - v_2 \land o_2 = 0\}$
$p := \text{init}$;
$w := 0$
$\{\text{Inv}\}$

Hence, using the composition rule, we have now shown that $I \vdash$

$\{v_1 \geq v_2 > 0 \land \text{init} \geq v_2 \land o_1 = v_1 - v_2 \land o_2 = 0\}$
$p := \text{init}$;
$w := 0$;
while $p > 0 \land w = 0$ do
  $\text{ch}_{\{1,2\}}(\{x_1, x_2\})$:
  if $x_1 > 0$ then $w := 1$
  else if $x_2 > 0$ then $w := 2$
  else $p := p - 1$

$\{(w = 1 \rightarrow (o_1 = v_1 - p \land o_2 = 0)) \land (w = 2 \rightarrow (o_1 = 0 \land o_2 = v_2 - p))\}
\land (w = 0 \rightarrow (o_1 = 0 \land o_2 = 0))\}$

Note that the verification process has revealed two crucial details which had to be added to our original precondition $P$. First, $\text{init} \geq v_2$. This means that we need to make sure that we start the auction at a price that is high enough. If the players’ valuations are not known, the choice of the initial price can indeed be a problem. On the other hand, the condition tells us exactly what “high enough” means; in particular, the initial price does not need to exceed everybody’s valuation. Second, $v_2 > 0$. Hence, it does not suffice if only a single player has a non-zero valuation of the object. The problem here lies in the fact that in order to obtain the object one has to pay at least something, and if the other player’s valuation is zero, that something is more than the other player’s valuation, and hence the payoff is in turn lower than expected. Hence, we have succeeded in verifying that $(\alpha, Q)$ does indeed implement the social choice correspondence of section 4.3 associated with the second-price auction, on condition that $\text{init} \geq v_2 > 0$.

Finally, it should be emphasised again that the weak equivalence of the Dutch auction and the sealed-bid second-price auction demonstrated here is very weak indeed, since these auctions are very different. Crucially, in the sealed-bid second-price auction, a player does not need to know the other player’s valuation. It suffices that each player submits his own valuation as a bid. In
the Dutch auction, however, obtaining the same equilibrium outcome requires the player with the higher valuation to know the valuation of the other player so that he can decide to shout out just at the right moment. Hence, the two auctions do not satisfy the same knowledge preconditions. The standard result concerning the equivalence between Dutch auction and first-price auction does take these knowledge preconditions into account.

7 Conclusions

Two main directions for future research present themselves: On the foundational side, the question arises whether the present approach can also be applied to other equilibrium notions. We have already remarked that while the calculus presented can also be used to reason about Nash equilibria, the non-compositional nature of these equilibria stands in the way of a complete calculus. Hence, alternative equilibrium notions that promise to be amenable to our approach will be refinements of subgame-perfect equilibria. Second, we mentioned already that an intensional approach to pre- and postconditions is worth developing. For this, the crucial question is whether the logic used (FOL) and the expressiveness results obtained for programs can be carried over to mechanisms.

At the most general level, we hope that this paper has shown that tools from computational logic can be extended from program verification to the verification of game-theoretic mechanisms. The examples provided should suffice to convince the reader of the variety of possible applications of such an extension. The semantics of the correctness assertions for mechanisms is more complex than for programs, but this is counterbalanced by the fact that the mechanisms we would like to verify (e.g., spectrum auctions for telecommunication markets) may turn out to be simpler than their counterparts in computer software (e.g., operating systems).

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