Froth-like Minimizers of a Non-Local Free Energy Functional with Competing Interactions

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Received: 18 June 2012 / Accepted: 3 December 2012
Published online: 12 June 2013 – © Springer-Verlag Berlin Heidelberg 2013

Abstract: We investigate the ground and low energy states of a one dimensional non-local free energy functional describing at a mean field level a spin system with both ferromagnetic and antiferromagnetic interactions. In particular, the antiferromagnetic interaction is assumed to have a range much larger than the ferromagnetic one. The competition between these two effects is expected to lead to the spontaneous emergence of a regular alternation of long intervals on which the spin profile is magnetized either up or down, with an oscillation scale intermediate between the range of the ferromagnetic and that of the antiferromagnetic interaction. In this sense, the optimal or quasi-optimal profiles are “froth-like”: if seen on the scale of the antiferromagnetic potential they look neutral, but if seen at the microscope they actually consist of big bubbles of two different phases alternating among each other. In this paper we prove the validity of this picture, we compute the oscillation scale of the quasi-optimal profiles and we quantify their distance in norm from a reference periodic profile. The proof consists of two main steps: we first coarse grain the system on a scale intermediate between the range of the ferromagnetic potential and the expected optimal oscillation scale; in this way we reduce the original functional to an effective “sharp interface” one. Next, we study the latter by reflection positivity methods, which require as a key ingredient the exact locality of the short range term. Our proof has the conceptual interest of combining coarse grain- ing with reflection positivity methods, an idea that is presumably useful in much more general contexts than the one studied here.

1. Introduction and Description of the Model

The competition between short-range attractive forces and long-range dipolar forces can give rise to the spontaneous formation of periodic patterns, such as stripes or bubbles, as observed in several quasi two-dimensional (2D) systems, e.g., micromagnets and magnetic films, ferrofluids, quasi-2D electron gases and high temperature superconductors,
liquid crystals, system of suspended lipidic molecules on the water surface, assemblies of diblock copolymers, martensitic phase transitions; see, e.g., [4,5,14,21,25,35,38–40,42,43,47]. From a mathematical point of view, these systems are modelled by a microscopic or mesoscopic non-convex energy functional, whose low energy states are expected to display the same pattern formation phenomenon. There are a number of rigorous indications of the emergence of regular structures, ranging from equipartition to rigorous upper and lower bounds on the minimizing energy [1,6,9,15–19,22,30,32,33]. In a few cases, the existence of periodic ground states can be rigorously proved [2,8,13,26–29,31,36,41,44,46]. Among these, a one dimensional (1D) Ising model with nearest neighbor ferromagnetic (FM) exchange and long range power law antiferromagnetic (AF) interaction, where periodicity of the ground states was proved by means of a generalized reflection positivity (RP) method [26]. Later, such proof of periodicity was extended to other systems, both in one and two dimensions, in the discrete or continuum setting [27–31]; in particular, we mention two continuous versions of the 1D spin model studied in [26], where the discrete spin Hamiltonian is replaced by an effective free energy functional and the configuration of discrete Ising spins is replaced by a magnetization profile $\sigma(x)$ with $x \in \mathbb{R}$, either assuming all possible values between $-1$ and $1$ (the “soft spin” case, see [28]), or assuming only values $\pm 1$ (the “Ising spin” case, see [29]). In both cases, a crucial technical assumption for the method of the proof to work is that the short range FM term appearing in the free energy functional is exactly local, i.e., it is modeled by a gradient term or by a local surface tension term, depending on whether one considers the soft or Ising spin case. Under this assumption, the minimizers are exactly periodic and consist of intervals of constant length $h^*$ (the optimal modulation length) in which the magnetization has constant sign, the sign oscillating from plus to minus or viceversa when one moves from a given interval to the following one. Moreover, the magnetization profiles with free energy close to the minimal one are very close to the periodic minimizers.

From a physical point of view, the locality of the surface tension term is a phenomenological (often unjustified) assumption and it should be essentially irrelevant as far as the results are concerned. In other words, if we replace a local surface tension term by a short but finite ranged one, with range much smaller than the range of the AF interaction, the magnetization profiles minimizing the free energy functional, or with free energy sufficiently close to the minimum, should still consist of a regular alternation of intervals where the magnetization is positive or negative. The exact periodicity of the minimizers may be a special feature of models with a local surface tension, but approximate periodicity should be a robust property. Therefore, it is important to understand whether the results of [28,29] can be extended to cases where the generalized RP method breaks down, due to the non-locality of the short range interaction. The extension has, on the one hand, a specific interest for the class of 1D magnetic models we are considering: in fact, we are not aware of examples of free energy functionals with strictly local penalization of gradients which can be directly derived as continuum limit of microscopic particle models. On the other hand, it has a more general conceptual importance: it is of great interest to develop methods allowing one to extend the validity of results based on RP to cases where RP does not hold exactly, possibly by combining it with coarse graining or averaging methods.

In this paper we attempt a first extension in this direction, by focusing on a free energy functional that arises naturally in 1D Ising models with competing long range interactions at positive temperature in a specific mean field limit, known as Kac limit. To be more precise, let us consider the 1D spin system described by the Hamiltonian