Learning Zero-Sum Simultaneous-Move Markov Games Using Function Approximation and Correlated Equilibrium

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Abstract

We develop provably efficient reinforcement learning algorithms for two-player zero-sum Markov games in which the two players simultaneously take actions. To incorporate function approximation, we consider a family of Markov games where the reward function and transition kernel possess a linear structure. Both the offline and online settings of the problems are considered. In the offline setting, we control both players and the goal is to find the Nash Equilibrium efficiently by minimizing the worst-case duality gap. In the online setting, we control a single player to play against an arbitrary opponent and the goal is to minimize the regret. For both settings, we propose an optimistic variant of the least-squares minimax value iteration algorithm. We show that our algorithm is computationally efficient and provably achieves a $\tilde{O}(\sqrt{d^3H^3T})$ upper bound on the duality gap and regret, without requiring additional assumptions on the sampling model.

We highlight that our setting requires overcoming several new challenges that are absent in Markov decision processes or turn-based Markov games. In particular, to achieve optimism in simultaneous-move Markov games, we construct both upper and lower confidence bounds of the value function, and then compute the optimistic policy by solving a general-sum matrix game with these bounds as the payoff matrices. As finding the Nash Equilibrium of such a general-sum game is computationally hard, our algorithm instead solves for a Coarse Correlated Equilibrium (CCE), which can be obtained efficiently via linear programming. To our best knowledge, such a CCE-based scheme for implementing optimism has not appeared in the literature and might be of interest in its own right.

1 Introduction

Reinforcement learning (Sutton and Barto, 2018) is typically modeled as a Markov Decision Process (MDP) (Puterman, 2014), where an agent aims to learn the optimal decision-making rule via interaction with the environment. In Multi-agent reinforcement learning (MARL), several agents interact with each other and with the underlying environment, and their goal is to optimize their individual returns. This problem is often formulated under the framework of Markov games (Shapley, 1953), which is a generalization of the MDP model. Powered by function approximation techniques such as deep neural networks (LeCun et al., 2015; Goodfellow et al., 2016), MARL has

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recently enjoyed tremendous empirical success across a variety of real-world applications. A partial list of such applications includes the game of Go (Silver et al., 2016, 2017), real-time strategy games (OpenAI, 2018; Vinyals et al., 2019), Texas Hold’em poker (Moravčík et al., 2017; Brown and Sandholm, 2018, 2019), autonomous driving (Shalev-Shwartz et al., 2016), and learning communication and emergent behaviors (Foerster et al., 2016; Lowe et al., 2017; Bansal et al., 2017; Jaques et al., 2018; Baker et al., 2019); see the surveys in Busoniu et al. (2008); Zhang et al. (2019).

In contrast to the vibrant empirical study, theoretical understanding of MARL is relatively inadequate. Most existing work on Markov games assumes access to either a sampling oracle or a well-explored behavioral policy, which fails to capture the exploration-exploitation tradeoff that is fundamental in real-world applications of reinforcement learning. Moreover, these results mostly focus on the relatively simple turn-based setting. An exception is the work in Wei et al. (2017), which extends the UCRL2 algorithm (Jaksch et al., 2010) for MDP to zero-sum simultaneous-move Markov games. However, their approach explicitly estimates the transition model and thus only works in the tabular setting. Problems with complicated state spaces and transitions necessitate the use of function approximation architectures. In this regard, a fundamental question is left open:

**Can we design a provably efficient reinforcement learning algorithm for Markov games under the function approximation setting?**

In this paper, we provide an affirmative answer to this question for two-player zero-sum Markov games with simultaneous moves. In particular, we study an episodic setting, where each episode consists of $H$ timesteps and the players act simultaneously at each timestep. Upon reaching the $H$-th timestep, the episode terminates and players replay the game again by starting a new episode. Here, the players have no knowledge of the system model (i.e., the transition kernel) nor access to a sampling oracle that returns the next state for an arbitrary state-action pair. Therefore, the players have to learn the system from data by playing the game sequentially through each episode and repeatedly for multiple episodes. More specifically, we study episodic Markov games under both the offline and online settings. In the offline setting, both players are controlled by a central learner, and the goal is to find an approximate Nash Equilibrium of the game, with the approximation error measured by a notion of duality gap. In the online setting, we control one of the players and play against an opponent who implements an arbitrary policy. Our goal is to minimize the total regret, defined as the difference between the cumulative return of the controlled player and its optimal achievable return when the opponent plays the best response policy. Both settings are generalizations of the regret minimization problem for MDPs.

Furthermore, to incorporate function approximation, we consider Markov games with a linear structure, motivated by the linear MDP model recently studied in Jin et al. (2019). In particular, we assume that both the transition kernel and the reward admit a $d$-dimensional linear representation with respect to a known feature mapping, which can be potentially nonlinear in its inputs. For both the online and offline settings, we propose the first provably efficient reinforcement learning algorithm without additional assumptions on the sampling model. Our algorithm is an Optimistic version of Minimax Value Iteration (OMNI-VI) with least squares estimation—a model-free approach—which constructs upper confidence bounds of the optimal action-value function to promote exploration. We show that the OMNI-VI algorithm is computationally efficient, and it provably achieves an $\tilde{O}(\sqrt{d^3 H^3 T})$ regret in the online setting and a similar duality gap guarantee in the offline setting, where $\tilde{O}$ omits logarithmic terms. Note that the bounds do not depend on the cardinalities of the state and action spaces, which can be very large or even infinite. When specialized to MDPs, our results recover the regret bounds established in Jin et al. (2019) and are thus near-optimal.
We emphasize that the Markov game model poses several new and fundamental challenges that are absent in MDPs and arise due to subtle game-theoretic considerations. Addressing these challenges require several new ideas, which we summarize as follows.

1. **Optimism via General-Sum Games.** In the offline simultaneous-move setting, implementing the optimism principle for both players amounts to constructing both upper and lower confidence bounds (UCB and LCB) for the optimal value function of the game. Doing so requires one to find, as an algorithmic subroutine, the solution of a general-sum (matrix) game where the two players’ payoff functions correspond to the upper and lower bounds for the action-value (or Q) functions of the original Markov game, even though the latter is zero-sum to begin with. This stands in sharp contrast of turn-based games (Hansen et al., 2013; Jia et al., 2019; Sidford et al., 2019), in which each turn only involves constructing an UCB for one player.

2. **Using Correlated Equilibrium.** Finding the Nash equilibrium (NE) of a general-sum matrix game, however, is computationally hard in general (Daskalakis et al., 2009; Chen et al., 2009). Our second critical observation is that it suffices to find a Coarse Correlated Equilibrium (CCE) (Moulin and Vial, 1978; Aumann, 1987) of the game. Originally developed in algorithmic game theory, CCE is a tractable notion of equilibrium that strictly generalizes NE. In contrast to NE, a CCE can be found efficiently in polynomial time even for general-sum games (Papadimitriou and Roughgarden, 2008; Blum et al., 2008). Moreover, our analysis shows that using any CCE of the matrix general-sum game are sufficient for ensuring optimism for the original Markov game. Thus, by using CCE instead of NE, we achieve efficient exploration-exploitation balance while preserving computational tractability.

3. **Concentration and Game Stability.** The last challenge is more technical, arising in the analysis of the algorithm where we need to establish certain uniform concentration bounds for the CCEs. As we elaborate later, the CCEs of a general-sum game are unstable (i.e., not Lipschitz) with respect to the payoff matrices. Therefore, standard approaches for proving uniform concentration, such as those based on covering/ε-net arguments, fail fundamentally. We overcome this issue by carefully stabilizing the algorithm, for which we make use of an ε-net in the algorithm. Moreover, we show that this can be done in a computationally efficient way.

We shall discuss the above challenges and ideas in greater details when we formally describe our algorithms. We note that our regret and duality gap bounds also imply polynomial sample complexity (or PAC) guarantees for learning the NEs of simultaneous-move Markov games. Moreover, as turn-based games can be viewed as a special case of simultaneous games, where at each state the reward and transition kernel only depend on the action of one of the players, our algorithms and guarantees readily apply to the turn-based setting. To our best knowledge, our algorithm is the first provably efficient method for two-player zero-sum Markov games with simultaneous moves under the function approximation setting.

### 1.1 Related Work

There is a large body of literature on applying reinforcement learning methods to stochastic games. In particular, under the tabular setting, the work in Littman (1994, 2001a,b); Greenwald et al. (2003); Hu and Wellman (2003); Grau-Moya et al. (2018) extends the Q-learning algorithm (Watkins and Dayan, 1992) to zero-sum and general-sum Markov games, and that in Perolat et al. (2018); Srinivasan et al. (2018) extends the actor-critic algorithm (Konda and Tsitsiklis, 2000). Most of
their convergence guarantees are asymptotic and rely on access to a sampling oracle. Particularly related to us is the work in Sidford et al. (2019), which proposes a variance-reduced variant of the minimax Q-learning algorithm with near-optimal sample complexity. We note that the theoretical results therein also require a sampling oracle, and they focus on turn-based games, a special case of simultaneous-move games. The work in Lagoudakis and Parr (2012); Perolat et al. (2015); Pérolat et al. (2016b,a,c); Yang et al. (2019) considers function approximation techniques applied to variants of value-iteration methods and establishes finite-time convergence to the NEs of two-player zero-sum Markov games. Their results are based on the framework of fitted value-iteration (Munos and Szepesvári, 2008) and the availability of a well-explored behavioral policy. In a recent work, Jia et al. (2019) studies turn-based zero-sum Markov games, where the transition model is assumed to be embedded in some $d$-dimensional feature space, extending the MDP model proposed by Yang and Wang (2019b). Assuming a sampling oracle, they propose a variant of Q-learning algorithm that is guaranteed to find an $\varepsilon$-optimal strategy using $\tilde{O}(d\varepsilon^{-2}(1 - \gamma)^{-4})$ samples, where $\gamma$ is a discount factor. In summary, all of the work above either assume a sampling oracle or a well-explored behavioral policy for drawing transitions, therefore effectively bypassing the exploration issue.

Our work builds on a line of research on provably efficient methods for MDPs without additional assumptions on the sampling model. Most of the existing work focus on the tabular setting; see e.g., Strehl et al. (2006); Jaksch et al. (2010); Osband et al. (2014); Osband and Van Roy (2016); Azar et al. (2017); Dann et al. (2017); Agrawal and Jia (2017); Jin et al. (2018); Russo (2019); Rosenberg and Mansour (2019a,b); Jin and Luo (2019); Zanette and Brunskill (2019); Simchowitz and Jamieson (2019); Dong et al. (2019b) and the references therein. Under the function approximation setting, sample-efficient algorithms have been proposed using linear function approximators (Abbasi-Yadkori et al., 2019a,b; Jin et al., 2019; Yang and Wang, 2019a; Zanette et al., 2019; Du et al., 2019b; Cai et al., 2019; Wang et al., 2019), as well as nonlinear ones (Wen and Van Roy, 2017; Jiang et al., 2017; Dann et al., 2018; Du et al., 2019b; Dong et al., 2019a; Du et al., 2019a). Among these results, our work is most related to Jin et al. (2019); Zanette et al. (2019); Cai et al. (2019), which consider linear MDP models and propose optimistic and randomized variants of least-squares value iteration (LSVI) (Bradtke and Barto, 1996; Osband et al., 2014) as well as optimistic variants of proximal policy optimization (Schulman et al., 2017). Our linear Markov game model generalizes the MDP model considered in these papers, and our OMNI-VI algorithm can be viewed as a generalization of the optimistic LSVI method proposed in (Jin et al., 2019). As mentioned before, the game structures in our problem pose fundamental challenges that are absent in MDPs, and thus their algorithms cannot be trivially extended to our game setting.

Finally, we remark that work on provably sample efficient RL methods for Markov games is quite scarce. The only comparable work we are aware of is Wei et al. (2017), which proposes a model-based algorithm that extends the UCRL2 algorithm (Jaksch et al., 2010) for tabular MDPs to the game setting. Similarly to their work, we also consider both the online and offline settings and provide guarantees in terms of duality gap and regret. On the other hand, they only consider tabular setting, which is a special case of our linear model. Moreover, their model-based algorithm explicitly estimates the Markov transition kernel and relies on the complicated technique of Extended Value Iteration, whose computational cost is quite high as it requires augmenting the state/action spaces. In comparison, our algorithm is model-free in the sense that it directly estimates the value functions; moreover, the computational cost of our algorithm only depends on the dimension $d$ of the feature and not the cardinality of the state space.
2 Background and Preliminaries

In this section, we formally describe the setup for episodic two-player zero-sum Markov games with simultaneous moves, and introduce relevant definitions and notations. We then describe the setting for turn-based games, which can be viewed as a special case of simultaneous-moves games.

2.1 Simultaneous-Move Markov Games

Denote the two players as P1 and P2. A two-player, zero-sum, simultaneous-moves, episodic Markov game is defined by the tuple

\[(S, A_1, A_2, r, P, H),\]

where \(S\) is the state space, \(A_i\) is a finite set of actions player \(i \in \{1, 2\}\) can take, \(r\) is reward function, \(P\) is transition kernel and \(H\) is the number of steps in each episode. At each step \(h \in [H]\), upon observing the state \(x\), P1 and P2 take actions \(a \in A_1\) and \(b \in A_2\), respectively, and then both receive the reward \(r_h(x, a, b)\). The system then transitions to a new state \(x' \sim P_h(\cdot|x, a, b)\) according to the transition kernel. Throughout this paper, we assume that \(A_1 = A_2 =: \mathcal{A}\) for simplicity—generalization to the setting with \(A_1 \neq A_2\) is straightforward. We also assume that the rewards \(r_h(x, a, b)\) are deterministic functions taking value in \([-1, 1]\).

Denote by \(\Delta \equiv \Delta(\mathcal{A})\) the probability simplex over the action space \(\mathcal{A}\). A stochastic policy of P1 is a sequence of \(H\) functions denoted by \(\pi := (\pi_h : S \rightarrow \Delta)_{h \in [H]}\). At each step \(h \in [H]\) and state \(x \in S\), P1 takes an action sampled from the distribution \(\pi_h(x)\) over \(\mathcal{A}\). Similarly, a stochastic policy of P2 is given by the sequence \(\nu := (\nu_h : S \rightarrow \Delta)_{h \in [H]}\).

2.1.1 Value Functions

For a fixed pair of policies \((\pi, \nu)\) for both players, the value and Q functions for the above game can be defined in a manner analogous to the episodic Markov decision process (MDP) setting:

\[
V^\pi,\nu_h(x) := \mathbb{E} \left[ \sum_{t=0}^{H} r_t(x_t, a_t, b_t) | x_h = x \right], \quad Q^\pi,\nu_h(x, a, b) := \mathbb{E} \left[ \sum_{t=0}^{H} r_t(x_t, a_t, b_t) | x_h = x, a_h = a, b_h = b \right],
\]

where the expectation is over \(a_t \sim \pi_t(x_t), b_t \sim \nu_t(x_t)\) and \(x_{t+1} \sim P_1(\cdot|x_t, a_t, b_t)\). It is convenient to set \(V_{H+1}^\pi(x) \equiv Q_{H+1}^\pi(x) \equiv 0\) for the terminal reward. Under the boundedness assumption on the reward, it is easy see that all value functions are bounded:

\[
|V^\pi,\nu_h(x)| \leq H \quad \text{and} \quad |Q^\pi,\nu_h(x, a)| \leq H, \quad \forall x, a, b, h, \pi, \nu.
\]

In the zero-sum setting, for a given initial state \(x_1\), P1 aims to maximize \(V^\pi,\nu_1(x_1)\) whereas P2 aims to minimize it. Accordingly, we introduce the value and Q (a.k.a. action-value) functions when P1 plays the best response to a fixed policy \(\nu\) of P2:

\[
V^\pi,\nu_h(x) = \max_{\pi} V^\pi,\nu_h(x) \quad \text{and} \quad Q^\pi,\nu_h(x, a, b) = \max_{\pi} Q^\pi,\nu_h(x, a, b).
\]

Analogously, when P2 plays the best response to P1’s policy \(\pi\), we define

\[
V^\pi,\nu_h(x) = \min_{\nu} V^\pi,\nu_h(x) \quad \text{and} \quad Q^\pi,\nu_h(x, a, b) = \min_{\nu} Q^\pi,\nu_h(x, a, b).
\]
A Nash Equilibrium (NE) of the game is a pair of stochastic policies \((\pi^*, \nu^*)\) that are the best response to each other; that is,

\[
V_1^{\pi^*, \nu^*}(x_1) = V_1^{\pi^*, \nu^*}(x_1) = V_1^{\pi^*, \nu^*}(x_1), \quad x_1 \in \mathcal{S}.
\]  

We assume that the game satisfies appropriate regularity conditions so that a NE exists and their values are unique.\(^1\) Correspondingly, let \(V_h^\pi(x) := V_h^{\pi^*, \nu^*}(x)\) and \(Q_h^\pi(x, a, b) := Q_h^{\pi^*, \nu^*}(x, a, b)\) denote the values of the NE at step \(h\).

Define the following shorthand for conditional expectation for the step-\(h\) transition:

\[
\mathbb{P}_h[V](x, a, b) := \mathbb{E}_x^{x' \sim \mathbb{P}_h(\cdot | x, a, b)}[V(x')] = \int V(x') \mathbb{d}_h(x'|x, a, b).
\]

While not explicitly needed in our analysis, we note that the value/Q functions for the NE satisfy the Bellman equation

\[
\begin{align*}
Q_h^\pi(x, a, b) &= r_h(x, a, b) + (\mathbb{P}_h V_h^{\pi^*})(x, a, b), \quad \text{(2a)} \\
\text{and} \quad V_h^\pi(x) &= \max_{A \in \Delta} \min_{B \in \Delta} \mathbb{E}_{a \sim \Delta, b \sim B} Q_h^\pi(x, a, b) = \min_{B \in \Delta} \max_{a \sim \Delta} \mathbb{E}_{a \sim \Delta, b \sim B} Q_h^\pi(x, a, b). \quad \text{(2b)}
\end{align*}
\]

The fixed-policy and best-response value/Q functions, \(V_h^{\pi^*, \nu^*}, V_h^{\pi^*, \nu^*}, Q_h^{\pi^*, \nu^*}, Q_h^{\pi^*, \nu^*}\) and \(Q_h^{\pi^*, \nu^*}\), satisfy a similar set of Bellman equations; we omit the details.

The following weak duality result, which follows immediately from definition, relates the above value and Q functions.

**Proposition 1** (Weak duality). For each policy pair \((\pi, \nu)\) and each \(h \in [H]\), \((x, a, b) \in \mathcal{S} \times \mathcal{A} \times \mathcal{A}\), we have

\[
\begin{align*}
Q_h^{\pi^*, \nu^*}(x, a, b) &\leq Q_h^\pi(x, a, b) \leq Q_h^{\pi^*, \nu^*}(x, a, b), \\
V_h^{\pi^*, \nu^*}(x) &\leq V_h^\pi(x) \leq V_h^{\pi^*, \nu^*}(x).
\end{align*}
\]

### 2.1.2 Linear Structures

We assume that both the reward function and transition kernel have a linear structure.

**Assumption 1** (Linearity and Boundedness). For each \((x, a, b) \in \mathcal{S} \times \mathcal{A} \times \mathcal{A}\) and \(h \in [H]\), we have

\[
\begin{align*}
&\mathbb{r}_h(x, a, b) = \phi(x, a, b) \pi_h \quad \text{and} \quad \mathbb{P}_h(\cdot | x, a, b) = \phi(x, a, b) \mu_h(\cdot),
\end{align*}
\]

where \(\phi : \mathcal{S} \times \mathcal{A} \times \mathcal{A} \to \mathbb{R}^d\) is a known feature map, \(\theta_h \in \mathbb{R}^d\) is an unknown vector and \(\{\mu_h(i)\}_{i \in [d]}\) are \(d\) unknown (signed) measures on \(\mathcal{S}\). We assume that \(\|\phi(\cdot, \cdot, \cdot)\| \leq 1, \|\theta_h\| \leq \sqrt{d}\) and \(\|\theta_h\| \leq \sqrt{d}\) for all \(h \in [H]\).

The above assumption implies that the Q functions are linear.

**Lemma 1** (Linearity of value function). Under Assumption 1, for any policy pair \((\pi, \nu)\) and any \(h \in [H]\), there exists a vector \(w_h^{\pi, \nu} \in \mathbb{R}^d\) such that

\[
Q_h^{\pi, \nu}(x, a, b) = \langle \phi(x, a, b), w_h^{\pi, \nu} \rangle, \quad \forall (x, a, b) \in \mathcal{S} \times \mathcal{A} \times \mathcal{A}.
\]

\(^1\)This holds, e.g., when the state space is compact (Maitra and Parthasarathy, 1970, 1971).
Proof. By Bellman equation and linearity of $r_h$ and $\mathbb{P}_h$, we have

$$Q^π_h(x,a,b) = r_h(x,a,b) + \mathbb{P}_h V^π_{h+1}(x,a,b) = \phi(x,a,b)^\top \theta_h + \int V^π_{h+1}(x')\phi(x,a,b)^\top d\mu_h(x').$$

Letting $w^π_h := \theta_h + \int V^π_{h+1}(x')d\mu_h(x')$ proves the lemma.

Remark 1. Since $Q^π_h(x,a,b) = Q^π_h(br(\pi))(x,a,b)$, where $br(\pi) \in \arg\min Q^π_h(x,a,b)$ is the best response policy to $\pi$, it follows immediately from Lemma 1 that $Q^π_h(x,a,b) = \langle \phi(x,a,b), w^π_h \rangle$ for some $w^π_h \in \mathbb{R}^d$. Similarly, we have $Q^π_h(x,a,b) = \langle \phi(x,a,b), w^π_h \rangle$ for some $w^π_h \in \mathbb{R}^d$.

The linear setting above covers the tabular setting as a special case, where $d = |S| \cdot |A|^2$ and $\phi(x,a,b)$ is the indicator vector for the tuple $(x,a,b)$. It is also clear that MDPs are a special case of Markov games when P2 plays a fixed and known policy. In particular, our setting covers both tabular MDPs as well as the linear MDP setting considered in Jin et al. (2019). Finally, as we elaborate below, turn-based Markov Games can also be viewed as a special case of our setting.

2.2 Turn-Based Markov games

In turn-based games, at each state only one player takes an action. Without loss of generality, we may partition the state space as $S = S_1 \cup S_2$, where $S_i$ are the states at which it is player $i$’s turn to play. For each state $x \in S$, let $I(x) \in \{1,2\}$ indicate the current player to play so that we have $x \in S_{I(x)}$. At each step $h \in [H]$, player $I(x)$ observes the current state $x$ and takes an action $a_i$; then the two players receive the reward $r_h(x,a)$, and the system transitions to a new state $x' \sim \mathbb{P}_h(\cdot|x,a)$.

The value/Q functions $V^π_h(x), Q^π_h(x,a)$ etc., as well as the corresponding NE of the game, can be defined in a completely analogous way as in the simultaneous-move setting. Similarly to Assumption 1, we also assume that the game has a linear structure.

Assumption 2 (Linearity and Boundedness, Turn-Based). For each $(x,a) \in S \times A$ and $h \in [H]$, we have

$$r_h(x,a) = \phi(x,a)^\top \theta_h \quad \text{and} \quad \mathbb{P}_h(\cdot|x,a) = \phi(x,a)^\top \mu_h(\cdot),$$

where $\phi : S \times A \to \mathbb{R}^d$ is a known feature map, $\theta_h \in \mathbb{R}^d$ is an unknown vector and $\{\mu^{(i)}_h\}_{i \in [d]}$ are $d$ unknown (signed) measures on $S$. We assume that $\|\phi(\cdot,\cdot)\| \leq 1$, $\|\mu_h(S)\| \leq \sqrt{d}$ and $\|\theta_h\| \leq \sqrt{d}$ for all $h \in [H]$.

One can view a turn-based Markov game as a special case of a simultaneous-move Markov game, where at each state only one of the players is “active” and the other player’s action has no influence on the reward or the transition. Formally, for each $x \in S$, the values of the functions $r_h(x,a,b)$, $\mathbb{P}_h(\cdot|x,a,b)$ and $\phi(x,a,b)$ are independent of $b$; for each $x \in S_2$, they are independent of $a$.

2.3 Notation

If $x \geq Cy$ holds for a universal absolute constant $C > 0$, we write $x \gtrless y, x = \Omega(y)$ and $y = O(x)$. For each real number $u$, define the clipping operation $\Pi_H(u) = \max\{\min\{u,H\},-H\}$. We use $\|\cdot\|$.  

\footnote{The assumption $S_1 \cap S_2 = \emptyset$ is satisfied if one incorporates the “turn” of the player as part of the state.}
Another way to interpret the above objective is as follows. Define the \( \text{Duality gap guarantees:} \quad \text{Gap}(K) := \sum_{k=1}^{K} \left[ V_{1}^{\pi^{k}, \nu^{k}}(x_{1}^{k}) - V_{1}^{\pi^{k}, \nu^{k}}(x_{1}^{k}) \right]. \) (4)

Another way to interpret the above objective is as follows. Define the \textit{exploitability} (Davis et al., 2014) of P1 and P2, respectively, as

\[
\text{Exploit}_{1}(\pi^{k}, \nu^{k}) := V_{1}^{\pi^{k}, \nu^{k}}(x_{1}^{k}) - V_{1}^{\pi^{k}, \nu^{k}}(x_{1}^{k}), \quad \text{and} \quad \text{Exploit}_{2}(\pi^{k}, \nu^{k}) := V_{1}^{\pi^{k}, \nu^{k}}(x_{1}^{k}) - V_{1}^{\pi^{k}, \nu^{k}}(x_{1}^{k}),
\]

both of which are nonnegative by Proposition 1. Here \( \text{Exploit}_{i}(\pi^{k}, \nu^{k}) \) measures the potential loss of player \( i \in \{1, 2\} \) in the \( k \)-th episode if the other player unilaterally switched to the best response.
policy. Then the total duality gap can be written equivalently as

$$\text{Gap}(K) = \sum_{k=1}^{K} \left[ \text{Exploit}_{1}(\pi^k, \nu^k) + \text{Exploit}_{2}(\pi^k, \nu^k) \right],$$

which is the sum of the exploitability of both players accumulated over $K$ episodes. Also note that in special cases of MDPs, $\text{Gap}(K)$ reduces to the usual notion of expected total regret.

**Sample complexity (PAC) guarantees:** Another performance metric is the sample complexity for finding an approximate NE. In particular, suppose that for all episodes the initial states $x_1$ are sampled from the same fixed distribution. We are interested in the number of episodes $K$ (or equivalently the number of samples $T = KH$) needed to find a policy pair $(\pi, \nu)$ satisfying

$$V^*_{1,\nu}(x_1) - V^{\pi,\nu}_{1}(x_1) \leq \epsilon$$

with probability at least $1 - \delta$.

Note that in light of Proposition 1, the above inequality implies that $(\pi, \nu)$ is an $\epsilon$-approximate NE in the sense that

$$V^*_{1,\nu}(x_1) - \epsilon \leq V^{\pi,\nu}_{1}(x_1) \leq V^{\pi,\nu}_{1}(x_1) + \epsilon;$$

that is, $(\pi, \nu)$ satisfies the definition (1) of NE up to an $\epsilon$ error. As we will discuss in details after presenting our main theorem, a bound on the total duality gap implies a bound on the sample complexity.

### 3.2 Algorithm

We now present our algorithm, Optimistic Minimax Value Iteration (OMNI-VI) with least squares estimation, which is given as Algorithm 1. In each episode $k$, the algorithm involves first constructing the policies for both players (lines 3–12), and then executing the policy to play the game (lines 13–17). The construction of the policy is done through backward induction in the timestep $h$. In each time step, we first compute upper/lower estimates $\overline{w}_h, \underline{w}_h \in \mathbb{R}^d$ of the linear coefficients of the Q-function. This is done by approximately solving the Bellman equation (2) using (regularized) least-squares estimation, for which we use empirical data from the previous $k - 1$ episodes to estimate the unknown transition kernel $P_h$ (lines 4–6). Then, to encourage exploration, we construct UCB/LCB for the Q function by adding/subtracting an appropriate bonus term (lines 7–8). The bonus takes the form $\beta \sqrt{\phi^\top (\Lambda_h^k)^{-1}\phi}$, which is common in the literature of linear bandits (Lattimore and Szepesvári, 2018). The next and crucial step, which we elaborate on below, is to convert these bounds into UCB/LCB for the value function (lines 9–11).
Algorithm 1 Optimistic Minimax Value Iteration (Simultaneous Move, Offline)

1: for episode \( k = 1, 2, \ldots, K \) do
2:     Receive initial state \( x_0^k \)
3:     for step \( h = H, H - 1, \ldots, 2, 1 \) do \( \triangleright \) update policy
4:         \( \Lambda_h^k \leftarrow -k-1 \sum_{t=1}^{k-1} \phi(x_t^h, a_t^h, b_t^h)\phi(x_t^h, a_t^h, b_t^h)^	op + I. \)
5:         \( \overline{w}_h^k \leftarrow (\Lambda_h^k)^{-1} \sum_{t=1}^{k-1} \phi(x_t^h, a_t^h, b_t^h) [r_h(x_t^h, a_t^h, b_t^h) + \nabla_h^k(x_t^h, a_t^h, b_t^h)]. \)
6:         \( \underline{w}_h^k \leftarrow (\Lambda_h^k)^{-1} \sum_{t=1}^{k-1} \phi(x_t^h, a_t^h, b_t^h) [r_h(x_t^h, a_t^h, b_t^h) + \nabla_h^k(x_t^h, a_t^h, b_t^h)]. \)
7:         \( \overline{Q}_h^k(\cdot, \cdot, \cdot) \leftarrow \Pi_H \{ (\overline{w}_h^k)\top \phi(\cdot, \cdot, \cdot) + \beta \sqrt{\phi(\cdot, \cdot, \cdot)\top (\Lambda_h^k)^{-1} \phi(\cdot, \cdot, \cdot)} \}. \)
8:         \( \underline{Q}_h^k(\cdot, \cdot, \cdot) \leftarrow \Pi_H \{ (\underline{w}_h^k)\top \phi(\cdot, \cdot, \cdot) - \beta \sqrt{\phi(\cdot, \cdot, \cdot)\top (\Lambda_h^k)^{-1} \phi(\cdot, \cdot, \cdot)} \}. \)
9:     For each \( x \), let \( \sigma_h^k(x) \leftarrow \text{FIND_CCE} (\overline{Q}_h^k, \underline{Q}_h^k, x). \)
10: \( \overline{V}_h^k(x) \leftarrow \mathbb{E}_{(a,b) \sim \sigma_h^k(x)} \overline{Q}_h^k(x, a, b) \) for each \( x \).
11: \( \underline{V}_h^k(x) \leftarrow \mathbb{E}_{(a,b) \sim \sigma_h^k(x)} \underline{Q}_h^k(x, a, b) \) for each \( x \).
12: end for
13: for step \( h = 1, 2, \ldots, H \) do \( \triangleright \) execute policy
14:     Sample \( (a_h^k, b_h^k) \sim \sigma_h^k(x_h^k). \)
15:     P1 takes action \( a_h^k \); P2 takes action \( b_h^k \).
16:     Observe next state \( x_{h+1}^k \).
17: end for
18: end for

Note that the UCB/LCB \( \overline{V}_h(x) \) and \( \underline{V}_h(x) \) for the value functions must correspond to the actions \((a', b')\) that would be actually played at state \( x \), i.e., \( \overline{V}_h(x) = \overline{Q}_h(x, a', b') \) (in expectation w.r.t. randomness of the stochastic policy; similarly for \( \underline{V}_h(x) \)), so that the upper/lower bounds can be tightened up using empirical observations from these actions. To construct these bounds, one may be tempted to let each player independently compute the maximin or minimax values and actions. That is, one may let P1 play the action \( a' = \arg \max_a \min_b \overline{Q}_h(x, a, b) \) and P2 play \( b' = \arg \min_b \max_a \overline{Q}_h(x, a, b) \), and then set \( \overline{V}_h(x) \leftarrow \overline{Q}_h(x, a', b') \) and \( \underline{V}_h(x) \leftarrow \underline{Q}_h(x, a', b') \). Unfortunately, such a \( \overline{V}_h(x) \) is not a valid upper bound for the true value, since \( \overline{Q}_h \neq \overline{Q}_h \) in general and hence \( \overline{Q}_h(x, a', b') \neq \max_a \min_b \overline{Q}_h(x, a, b) \).

Instead, we must coordinate both players for their choices of actions, which is done by solving the general-sum matrix game with payoff matrices \( \overline{Q}_h(x, \cdot, \cdot) \) and \( \underline{Q}_h(x, \cdot, \cdot) \). As computing the NE for general-sum games is intractable, we find an (approximate) CCE of the matrix game instead. For technical reasons discussed in the Introduction (and further elaborated in the next paragraph), the subroutine \text{FIND_CCE} for finding the CCE is implemented in a specific way as follows. Let \( Q \) be the class of functions \( Q : S \times A \times A \rightarrow \mathbb{R} \) with the parametric form

\[
Q(x, a, b) = \Pi_H \{ \langle w, \phi(x, a, b) \rangle + \rho \beta \sqrt{\phi(x, a, b)\top A\phi(x, a, b)} \}.
\]

(5)

where the parameters \((w, A, \rho) \in \mathbb{R}^d \times \mathbb{R}^{d \times d} \times \mathbb{R} \) satisfy \( \| w \| \leq 2H\sqrt{dK}, \| A \|_F \leq \beta^2\sqrt{d} \) and \( \rho \in \{ \pm 1 \} \). Let \( \mathcal{Q}_e \) be a fixed \( e \)-covering of \( Q \) with respect to the \( \ell_\infty \)-norm \( \| Q - Q' \|_\infty := \sup_{x, a, b} | Q(x, a, b) - Q'(x, a, b) | \). With these notations, we present the subroutine \text{FIND_CCE} in Algorithm 2. The algorithm finds, as
an surrogate of the CCE of the game \((Q^k_h(x, \cdot, \cdot), Q^k_l(x, \cdot, \cdot))\) of interest, the CCE of a nearby game in the finite \(\epsilon\)-cover \(Q_e \times Q_e\).

Algorithm 2 FIND_CCE

1: Input: \(Q^k_h, Q^k_l, x\).
2: Pick a pair \((\bar{Q}, \bar{Q}_e)\) in \(Q_e \times Q_e\) satisfying \(\|\bar{Q} - \bar{Q}_h\|_\infty \leq \epsilon\) and \(\|\bar{Q} - \bar{Q}_l\|_\infty \leq \epsilon\).
3: For the input \(x\), let \(\bar{\sigma}(x)\) be the CCE (cf. equation (3)) of the matrix game with payoff matrices \(\bar{Q}(x, \cdot, \cdot)\) for P1 and \(\bar{Q}_e(x, \cdot, \cdot)\) for P2.
4: Output: \(\bar{\sigma}(x)\).

The implementation of FIND_CCE is motivated by the following technical considerations. Note that both \(Q^k_h\) and \(Q^k_l\) belong to a relatively simple class of (quadratic) functions of the feature vectors \(\phi\) parametrized by the low-dimensional tuple \((\pi^k_h, \omega^k_h, \Lambda^k_h)\). One may consider computing \((V^k_h, V^k_l)\) by directly using the CCE of the game with payoff matrices \((Q^k_h, Q^k_l)\). As we show in Appendix F, the CCE and its value of a general-sum game are not Lipschitz in the game’s payoff matrices. Therefore, even when \((Q^k_h, Q^k_l)\) is simple, \((V^k_h, V^k_l)\) constructed in this way could potentially be a complicated and ill-behaved function of \(\phi\). It is difficult to establish concentration bounds for \((V^k_h, V^k_l)\) uniformly over such a complex function class.\(^3\) Instead, FIND_CCE only makes use of a finite set of payoff matrices in the \(\epsilon\)-cover \(Q_e \times Q_e\). Doing so ensures that the CCE \(\sigma^k_h\) output by FIND_CCE only takes a finite number of values. Consequently, the pair \((V^k_h, V^k_l)\) constructed using \(\sigma^k_h\) also takes values in a finite set, hence concentration can be established by a union-bound-type argument over this set. The small price we pay is that \(\sigma^k_h\) computed by FIND_CCE is only an approximate CCE of the game \((Q^k_h, Q^k_l)\); nevertheless, we can make the approximation error sufficiently small by choosing a small enough \(\epsilon\).

3.3 Theoretical Guarantees

In each episode \(k\), Algorithm 2 computes a joint (correlated) policy \(\sigma^k_h\). As NE requires the policies to be in product form, we marginalize \(\sigma^k_h\) into a pair of independent policies \(\pi^k_h(x) := P_1\sigma^k_h(x)\) and \(\nu^k_h(x) := P_2\sigma^k_h(x)\) for each player. Our main theoretical result is the following bound on the total duality gap (4) of these policy pairs. Recall that \(T = KH\) is the total number of timesteps.

Theorem 1 (Offline, Simultaneous Moves). Under Assumption 1, there exists a constant \(c > 0\) such that the following holds for each fixed \(p \in (0, 1)\). Set \(\beta = c dH \sqrt{i} \log(2dT/p)\) in Algorithm 1, and set \(\epsilon = \frac{1}{KH}\) in Algorithm 2. Then with probability at least \(1 - p\), Algorithm 1 satisfies bound

\[
\text{Gap}(K) \lesssim \sqrt{d^3 H^3 T i^2}.
\]

The proof is given in Section 5. Below we provide discussion and remarks on this theorem.

\(^3\)It is worth noticing that this issue does not exist in the tabular setting, as each \((V^k_h, V^k_l)\) is just a pair of finite-dimensional vectors and one can directly build an \(\epsilon\)-cover of the relevant set of vectors.
Optimality of the bound: The theorem provides an (instance-independent) bound scaling with $\sqrt{T}$. As the total duality gap reduces to the usual regret in the special case of MDPs, our bound is optimal in $T$ in view of known minimax lower bounds for MDPs (Lattimore and Szepesvári, 2018). Also note that our bound is independent of cardinality $|S| \cdot |A|$ of the state/action spaces, but rather depends only on dimension $d$ of the feature space, thanks to the use of function approximation. To investigate the tightness of the dependence of our bound on $d$ and $H$, we recall that our setting covers the standard tabular MDPs and linear bandits as special cases. A direct reduction from the known lower bounds on tabular MDPs gives a lower bound $\Omega(\sqrt{dH^2T})$ for the case of nonstationary transitions (Jin et al., 2018; Azar et al., 2017). Our bound is off by a factor of $\sqrt{H}$, which may be improved by using a “Bernstein-type” bonus term (Azar et al., 2017; Jin et al., 2018). Results from linear bandits give the lower bound $\Omega(d\sqrt{T})$. The additional $\sqrt{d}$ factor in our bound is due to a covering argument applied to the $d$-dimensional feature space for establishing uniform concentration bounds.

Computational complexity: Our algorithm can be implemented efficiently, with a computational complexity polynomial in $H, K, d$ and $|A|$. In particular, note that a CCE of a general-sum game can be found in polynomial time (Papadimitriou and Roughgarden, 2008). Moreover, in Algorithm 1 we do not need to compute $\overline{Q}(x, \cdot, \cdot), \overline{V}(x)$ and $\overline{\sigma}(x)$ etc. for all $x \in S$; rather, we only need to do so for the states $\{x^k_H\}$ actually encountered in the algorithm. Similarly, we do not need to explicitly maintain the (exponentially large) $\epsilon$-net $Q_\epsilon$ in FIND_CCE (Algorithm 2). It suffices if we can find an element in $Q_\epsilon$ that is $\epsilon$-close to a given function in $Q$, which can be done efficiently on the fly. Indeed, each function in $Q$ has a succinct representation using $(w, A) \in \mathbb{R}^d \times \mathbb{R}^{d \times d}$, so we only need to (implicitly) maintain a covering of the space of $(w, A)$ and find a nearby element from this covering when needed, see Appendix E for details.

Sample complexity guarantees: It is a standard fact that the above regret bound can be converted into a (PAC) bound on the sample complexity. For simplicity we assume that the initial state $x_1$ is fixed. After $K$ episodes, we may let $(\pi, \nu)$ be a random policy pair chosen with probabilities $P((\pi, \nu) = (\pi^k, \nu^k)) = \frac{1}{K}, k \in [K]$. Dividing the regret bound in Theorem 1 by $K = T/H$ gives

$$\frac{1}{K} \sum_{k=1}^K \left[ V_1^{x, \pi^k} (x_1^k) - V_1^{x, \pi^k} (x_1^k) \right] \lesssim \sqrt{\frac{d^3 H^5 \delta^2}{T}}.$$  

It then follows from Markov’s inequality that with probability at least $1 - \delta$:

$$V_1^{x, \nu} (x_1) - V_1^{x, \nu} (x_1) \lesssim \sqrt{\frac{d^3 H^5 \delta^2}{T \delta^2}}.$$  

Therefore, we can find an $\epsilon$-approximate NE (meaning that the last RHS is bounded by $\epsilon$) with a sample complexity of $T = O\left(\frac{d^3 H^5 \delta^2}{\epsilon^2 \delta^2}\right)$.

---

4This can be done by linear programming—as the inequalities in the definition (3) of CCE are linear in $\sigma$—or by no-regret learning with self-play (Blum et al., 2008).

5For the general case where $x_1$ is sampled from a fixed distribution, we can simply add an additional time step at the beginning of each episode.
3.4 Turn-Based Games

In this section, we consider turn-based Markov games, which is a special case of simultaneous-move Markov games. Algorithm 1 can be specialized to this setting. For completeness, we provide the resulting algorithm in Algorithm 4 in Appendix A. Note that for turn-based games, the FIND_CCE routine is simplified to the subroutines FIND_MAX and FIND_MIN given in Algorithm 5, because each state is controlled by a single player and hence finding a CCE reduces to computing a maximizer or minimizer.

As a corollary of Theorem 1, we have the following bound on the total duality gap, which is defined in the same way as in (4). We prove this bound in Appendix D.

Corollary 1 (Offline, Turn-Based). Under Assumption 2, there exists a constant c > 0 such that, for each fixed p ∈ (0, 1), by setting β = cdH √t with t := log(2dT/p) in Algorithm 4, then with probability at least 1 − p, Algorithm 4 satisfies bound

\[ \text{Gap}(K) \lesssim \sqrt{d^3H^3T^2}. \]

4 Main Results for the Online Setting

In this section, we consider the online setting, where we control P1 and play against an arbitrary (and potentially adversarial) P2. Our goal is to maximize the reward of P1. Below we describe the performance metrics, followed by our algorithms and theoretical guarantees.

4.1 Setup and Performance Metrics

We consider the episodic setting as described in Section 3.1. Let π = (πk) and ν = (νk) be the policy sequences for P1 and P2, respectively, where ν is arbitrary. We do not know P2’s choice of ν nor the Markov model of the game a priori, and would like learn a good policy π online so as to optimize the reward \( \sum_{k} V_{π_k, ν_k}^*(x_k^1) \) received by P1 over K episodes. To this end, we are interested in bounding, for each ν, the total (expected) regret

\[ \text{Regret}_π(K) := \sum_{k=1}^{K} \left[ V_1^*(x_k^1) - V_{π_k, ν_k}^*(x_k^1) \right], \] (6)

where \( x_k^1 \) is the (arbitrary) initial state in the k-th episode. If we can obtain a bound on \( \text{Regret}_π(K) \) that scales sublinearly with K for all ν, then we are guaranteed that regardless of ν, the reward collected by P1 is no worse (in the long run) than its optimal worst-case reward, that is, the NE value \( V_1^* \).

We note that a special case of the above setting is when P2 is omniscient and always plays the best response to P1’s policy, i.e.,

\[ ν_k = \text{br}(π_k) \in \arg \min_{ν' ∈ Δ} V_{1, ν'}^k(x_k^1), \quad ∀ k ∈ [K]. \]

Note that in this case, we have \( V_{1, π_k}^k(x_k^1) = V_{1, π_k}^k(x_k^1) \) by definition.
4.2 Algorithm

We adapt the Optimistic Minimax Value Iteration algorithm to the online setting, as given in Algorithm 3. This algorithm can be viewed as a one-sided version of Algorithm 1: we compute least-squares estimate for the linear coefficients and then construct UCBs for the value functions—we do not need to construct LCBs as $P2$ is not controlled by us. Constructing the UCBs is done by finding the NE of the zero-sum matrix game with the payoff matrix $Q_h^k(x, \cdot)$. Recalling the definition of NE, we see that the pair $(\pi_h^k(x), B_0) \in \Delta \times \Delta$ computed in the algorithm satisfies

$$E_{a \sim \pi_h^k(x), b \sim B_0} [Q_h^k(x, a, b)] = \max_{A \in \Delta} E_{a \sim A, b \sim \pi_h^k(x)} [Q_h^k(x, a, b)] = \min_{B \in \Delta} E_{a \sim \pi_h^k(x), b \sim B} [Q_h^k(x, a, b)],$$

for each $x \in S$.

**Algorithm 3** Optimistic Minimax Value Iteration (Simultaneous Move, Online)

1: for episode $k = 1, 2, \ldots, K$ do
2:     Receive initial state $x_1^k$.
3:     for step $h = H, H - 1, \ldots, 2, 1$ do \hspace{1cm} \triangleright update policy
4:         $\Lambda_h^k \leftarrow \sum_{\tau=1}^{k-1} \phi(x_h^\tau, a_h^\tau, b_h^\tau)\phi(x_h^\tau, a_h^\tau, b_h^\tau)\top + I$.
5:         $w_h^k \leftarrow (\Lambda_h^k)^{-1}\sum_{\tau=1}^{k-1} \phi(x_h^\tau, a_h^\tau, b_h^\tau) [r_h(x_h^\tau, a_h^\tau, b_h^\tau) + V_{h+1}^k(x_{h+1})]$.
6:         $Q_h^k(\cdot, \cdot, \cdot) \leftarrow \Pi_H \{ (w_h^k)\top \phi(\cdot, \cdot, \cdot) + \beta \sqrt{\phi(\cdot, \cdot, \cdot)\top (\Lambda_h^k)^{-1} \phi(\cdot, \cdot, \cdot)} \}.$
7:         For each $x$, let $(\pi_h^k(x), B_0)$ be the NE of the matrix game with payoff matrix $Q_h^k(x, \cdot, \cdot)$.
8:         $V_h^k(\cdot) \leftarrow E_{a \sim \pi_h^k(\cdot), b \sim B_0} [Q_h^k(\cdot, a, b)]$.
9:     end for
10: for step $h = 1, 2, \ldots, H$ do \hspace{1cm} \triangleright execute policy
11:     $P1$ take action $a_h^k \sim \pi_h^k(x_h^k)$.
12:     Let $P2$ play; denote its action by $b_h^k$.
13:     Observe next state $x_{h+1}^k$.
14: end for
15: end for

Due to the one-sided nature of the online setting, some of the difficulties in the offline setting—pertaining to general-sum games and CCE—no longer exist here. In particular, Algorithm 3 no longer requires the FIND_CCE subroutine that makes use of an $\epsilon$-net. Technically, this is due to the fact that zero-sum games are more well-behaved than general-sum games. In particular, a zero-sum game is Lipschitz in the payoff matrix, hence uniform concentration can be established in a more straightforward manner (cf. the discussion in Section 3.2).

4.3 Regret Bound Guarantees

We establish the following bound on the total regret (6) achieved by Algorithm 3.

**Theorem 2** (Online, Simultaneous Move). Under Assumption 1, there exists a constant $c > 0$ such that the following holds for each fixed $p \in (0, 1)$ and any policy sequence $v$ for $P2$. Set $\beta = cdH\sqrt{T}$ with $t := \log(2dT/p)$. Then with probability at least $1 - p$, Algorithm 3 achieves the regret bound

$$\text{Regret}_v(K) \lesssim \sqrt{d^3H^3T^2}.$$
The proof is given in Appendix C. Note that the regret bound holds for any policy \( \nu \) of \( \mathcal{P}_2 \) and any initial states \( \{x^k_1\} \). Moreover, the bound is sublinear in \( T \)—scaling with \( \sqrt{T} \) in particular—and depends polynomially on \( d \) and \( H \). As our regret reduces to the standard regret notion in the special cases of MDPs and linear bandits, the discussion in Section 3.2 on the optimality of bounds, also applies here.

4.4 Turn-Based Games

The algorithm above can be specialized to the special case of online turn-based games. For completeness we provide resulting algorithm in Appendix A as Algorithm 6. Note that in the turn-based setting, we only need to solve a unilateral maximization or minimization problem, rather than solving zero-sum games as is needed in the simultaneous-move setting.

As an immediate corollary of Theorem 2, we have the following regret bound for turn-based games in the online setting. We prove this bound in Appendix D.

**Corollary 2 (Online, Turn-based).** Under Assumption 2, there exists a constant \( c > 0 \) such that the following holds for each fixed \( p \in (0, 1) \) and any policy sequence \( \nu \) for \( \mathcal{P}_2 \). Set \( \beta = cdH\sqrt{T} \) with \( \iota := \log(2dT/p) \) in Algorithm 6. Then with probability at least \( 1 - p \), Algorithm 6 achieves the regret bound

\[
\text{Regret}_\nu(K) \lesssim \sqrt{d^3H^3T\iota^2}.
\]

5 Proof of Theorem 1

In this section, we prove Theorem 1 for the online setting of simultaneous games. We shall make use of the technical lemmas given in Appendix B. For ease of exposition, we denote by \( \phi^k_h := \phi(x^k_h, a^k_h, b^k_h) \) the feature vector encountered in the \( h \)-th step of the \( k \)-th episode. Our proof consists of five steps:

1. **Uniform concentration:** We begin by showing that an empirical estimate of the transition kernel \( \mathbb{P}_h \), when acting on the value functions maintained by the algorithm, concentrates around its expectation. See Section 5.1.

2. **Least-squares estimation error:** Using the above concentration result, we derive high probability bounds on the errors of our least-squares estimates of the true Q functions \( Q_{\pi,\nu}^h \), recursively in the timestep \( h \). See Section 5.2.

3. **UCB and LCB:** We next show that the UCBs and LCBs constructed in the algorithms are indeed valid bounds on the true value functions \( V_{\pi,\nu}^h \) and \( V_{\pi,\nu}^h \). See Section 5.3.

4. **Recursive decomposition of duality gap:** We derive a recursive formula for the difference between the UCB and LCB in terms of the timestep \( h \). This difference in turn bounds the duality gap of interest. See Section 5.4.

5. **Establishing final bound:** Bounding each term in the above recursive decomposition in terms of the least-squares estimation errors, we establish the desired bound on the total duality gap and thereby completing the proof of the theorem. See Section 5.5.

Below we provide the details of each step.
5.1 Uniform Concentration

The quantity $\sum_{\tau \in [k-1]} \phi_h^T \nabla_{h+1}^k (x_{h+1}^T)$ can be viewed as an empirical estimate of the unknown population quantity $\sum_{\tau \in [k-1]} \phi_h^T (P_h \nabla_{h+1}^k) (x_{h+1}^T, a_{h+1}^T, b_{h+1}^T)$. To control the least-squares estimation error, we need to show that the empirical estimate concentrates around its population counterpart. The main challenge in doing so is that $\nabla_{h+1}^k$ is constructed using data from previous episodes and hence depends on $\phi_h^T$ for all $\tau \in [k-1]$. We overcome this issue by noting that $\nabla_{h+1}^k$ is computed using the CCE of a finite class of games with payoff matrices in the $\epsilon$-net $Q_\epsilon \times Q_\epsilon$, as is done in FIND_CCE. Therefore, we can prove a concentration bound valid uniformly over this class of games and thereby establish following concentration result. Here we recall that $\|v\|_A := \sqrt{v^T A v}$ denotes the weighted $\ell_2$ norm of a vector $v$.

**Lemma 2** (Concentration). Under the setting of Theorem 1, for each $p \in (0, 1)$, the following event $\mathcal{E}$ holds with probability at least $1 - p/2$:

$$
\left\| \sum_{\tau \in [k-1]} \phi_h^T \left[ \nabla_{h+1}^k (x_{h+1}^T) - \left( P_h \nabla_{h+1}^k \right) (x_{h+1}^T, a_{h+1}^T, b_{h+1}^T) \right] \right\| \lesssim dH \sqrt{\log(dT/p)}, \quad \forall (k, h) \in [K] \times [H],
$$

$$
\left\| \sum_{\tau \in [k-1]} \phi_h^T \left[ \nabla_{h+1}^k (x_{h+1}^T) - \left( P_h \nabla_{h+1}^k \right) (x_{h+1}^T, a_{h+1}^T, b_{h+1}^T) \right] \right\| \lesssim dH \sqrt{\log(dT/p)}, \quad \forall (k, h) \in [K] \times [H].
$$

**Proof.** Let $\mathcal{F}_{\tau-1} := \mathcal{F}(x^1, a^1, \ldots, x_t^{t-1}, a_t^{t-1})$. Note that $\phi_h^T, a^T \in \mathcal{F}_{\tau-1}$.

Fix a pair $(\tilde{Q}, Q)$ in the $\epsilon$-net $Q_\epsilon \times Q_\epsilon$. For each $x \in \mathcal{S}$, let $\tilde{\sigma}(x)$ be the CCE of $(\tilde{Q}(x, \cdot, \cdot), Q(x, \cdot, \cdot))$ in the sense of equation 3, and set $V(x) := \mathbb{E}_{(a, b) \sim \tilde{\sigma}(x)} \left[ \tilde{Q}(x, a, b) \right]$. The random variable $V(x_{h+1}^T) = P_h \tilde{V}(x_{h+1}^T)$ in $\mathcal{F}_{\tau-1}$ is zero-mean and $H$-bounded. Applying Lemma 12 gives

$$
\left\| \sum_{\tau \in [k-1]} \phi_h^T \left[ \tilde{V}(x_{h+1}^T) - \left( P_h \tilde{V} \right) (x_{h+1}^T, a_{h+1}^T, b_{h+1}^T) \right] \right\| \lesssim dH \sqrt{\log(dT/p)}
$$

with probability at least $2^{-O(d^2 \log(dT/p))}$. Now note that $|Q_\epsilon \times Q_\epsilon| = (N_\epsilon)^2 \leq 4 \left( 1 + \frac{8H\sqrt{dK}}{\epsilon} \right)^2 \left( 1 + \frac{8\sqrt{dK}}{\epsilon} \right)^2 d^2$ by Lemma 11. By a union bound, the above inequality holds for all $(\tilde{Q}, Q) \in Q_\epsilon \times Q_\epsilon$ with probability at least $1 - p/2$.

Now, for any $(\tilde{Q}_{h+1}^k, Q_{h+1}^k) \in Q \times Q$ (Lemma 8), let $(\tilde{Q}, Q) \in Q_\epsilon \times Q_\epsilon$ be the pair in the net chosen in FIND_CCE. Recall that this pair satisfies $\|\tilde{Q} - Q_h^k\|_\infty \leq \epsilon$ and $\|Q - Q_h^k\|_\infty \leq \epsilon$. By construction, $\nabla_{h+1}^k(x) = \mathbb{E}_{(a, b) \sim \tilde{\sigma}(x)} \left[ Q_{h+1}^k(x, a, b) \right]$. Therefore, the difference $\Delta(x) := \nabla_{h+1}^k(x) - V(x)$ satisfies

$$
|\Delta(x)| = \left| \mathbb{E}_{(a, b) \sim \tilde{\sigma}(x)} \left[ Q_{h+1}^k(x, a, b) - \tilde{Q}(x, a, b) \right] \right| \leq \mathbb{E}_{(a, b) \sim \tilde{\sigma}(x)} \left| Q_{h+1}^k(x, a, b) - \tilde{Q}(x, a, b) \right| \leq \epsilon, \quad \forall x \in \mathcal{S}.
$$
It follows that
\[
\sum_{\tau \in [k-1]} \phi_h^T \left[ \nabla_{h+1}^k (x_{h+1}^\tau) - \left( \mathbf{P}_h \nabla_h^k (x_{h+1}^\tau, a_h^\tau, b_h^\tau) \right) \right] \left( \Lambda_h^k \right)^{-1} \\
\leq \sum_{\tau \in [k-1]} \phi_h^T \left[ \nabla (x_{h+1}^\tau) - \left( \mathbf{P}_h \nabla_h (x_{h+1}^\tau, a_h^\tau, b_h^\tau) \right) \right] + \sum_{\tau \in [k-1]} \phi_h^T \left[ \Delta (x_{h+1}^\tau) - (\mathbf{P}_h \Delta) (x_{h+1}^\tau, a_h^\tau, b_h^\tau) \right] \left( \Lambda_h^k \right)^{-1} \\
\leq dH \sqrt{\log (dT/p)} + \epsilon \sum_{\tau \in [k-1]} \| \phi_h^T \| (\Lambda_h^k)^{-1} \\
\leq dH \sqrt{\log (dT/p)} + \epsilon k,
\]
where the last step follows from \( \Lambda_h^k \succeq I \) and \( \| \phi_h^T \| \leq 1 \). Recalling our choice \( \epsilon = \frac{1}{\sqrt{dT}} \) proves the first inequality in the lemma. The second inequality can be proved in a similar fashion. 

\[\square\]

### 5.2 Least-squares Estimation Error

Here we bound the difference between the algorithm’s action-value functions (without bonus) and the true action-value functions of any policy pair \((\pi, \nu)\), recursively in terms of the step \( h \).

**Lemma 3** (Least-squares error bound). The quantities \{\( w_h^k, w_h^k, \nabla_h^k, \nabla_h^k \)\} in Algorithm 1 satisfy the following. If \( \beta = dH \sqrt{\tau} \), where \( \tau = \log(2dT/p) \), then on the event \( \mathcal{E} \) in Lemma 2, we have for all \((x, a, b, h, k) \in \mathcal{S} \times \mathcal{A} \times \mathcal{A} \times [H] \times [K]\) and any policy pair \((\pi, \nu)\):

\[
\begin{align*}
\left| \left( \phi(x, a, b), w_h^k \right) - Q^\pi_{h+1}(x, a, b) - \mathbf{P}_h (\nabla_{h+1}^k - V_{h+1}^{\pi, \nu})(x, a, b) \right| & \leq \rho_h^k(x, a, b), \quad (8a) \\
\left| \left( \phi(x, a, b), w_h^k \right) - Q^\pi_{h+1}(x, a, b) - \mathbf{P}_h (\nabla_{h+1}^k - V_{h+1}^{\pi, \nu})(x, a, b) \right| & \leq \rho_h^k(x, a, b), \quad (8b)
\end{align*}
\]

where \( \rho_h^k(x, a, b) := \beta \| \phi(x, a, b) \| (\Lambda_h^k)^{-1} \).

**Proof.** We only prove the first inequality \((8a)\). The second inequality can be proved in a similar fashion.

By Lemma 1 and Bellman equation we have the equality
\[
(\phi_h^T)^T w_{h+1}^{\pi, \nu} = Q^\pi_{h+1}(x_{h+1}^\tau, a_{h+1}^\tau, b_{h+1}^\tau) + \left( \mathbf{P}_h V_{h+1}^{\pi, \nu} \right)(x_{h+1}^\tau, a_{h+1}^\tau, b_{h+1}^\tau)
\]
for all \( \tau \in [k-1] \). Multiplying the above equality by \((\Lambda_h^k)^{-1} \phi_h^T\) and summing over \( \tau \), we obtain that
\[
w_{h+1}^{\pi, \nu} - \left( \Lambda_h^k \right)^{-1} w_{h+1}^{\pi, \nu} = \left( \Lambda_h^k \right)^{-1} \left( \sum_{\tau \in [k-1]} \phi_h^T (\phi_h^T)^T \right) w_{h+1}^{\pi, \nu} \\
= \left( \Lambda_h^k \right)^{-1} \sum_{\tau \in [k-1]} \phi_h^T \cdot \left[ r_h(x_{h+1}^\tau, a_{h+1}^\tau, b_{h+1}^\tau) + \left( \mathbf{P}_h V_{h+1}^{\pi, \nu} \right)(x_{h+1}^\tau, a_{h+1}^\tau, b_{h+1}^\tau) \right],
\]
where the first equality above holds because \( \sum_{\tau \in [k-1]} \phi_h^T (\phi_h^T)^T = \Lambda_h^k - I \). On the other hand, recall that by algorithm specification we have \( w_h^k = \left( \Lambda_h^k \right)^{-1} \sum_{\tau \in [k-1]} \phi_h^T \cdot \left[ r_h(x_{h+1}^\tau, a_{h+1}^\tau, b_{h+1}^\tau) + \left( \mathbf{P}_h V_{h+1}^{\pi, \nu} \right)(x_{h+1}^\tau, a_{h+1}^\tau, b_{h+1}^\tau) \right] \). It
follows that
\[ \overline{w}_h^k - w_{h_1}^{\pi,\nu} = - (\Lambda_h^k)^{-1} \overline{w}_h^{\pi,\nu} + (\Lambda_h^k)^{-1} \sum_{\tau \in [k-1]} \phi_h^\tau \cdot \left[ \nabla_{h+1}^k (x_{h+1}^\tau) - (\mathbb{P}_h V_{h+1}^{\pi,\nu})(x_h^\tau, a_h^\tau, b_h^\tau) \right] \]
\[ = - (\Lambda_h^k)^{-1} \overline{w}_h^{\pi,\nu} + (\Lambda_h^k)^{-1} \sum_{\tau \in [k-1]} \phi_h^\tau \cdot \left[ \nabla_{h+1}^k (x_{h+1}^\tau) - (\mathbb{P}_h V_{h+1}^{\pi,\nu})(x_h^\tau, a_h^\tau, b_h^\tau) \right] \]
\[ + (\Lambda_h^k)^{-1} \sum_{\tau \in [k-1]} \phi_h^\tau \cdot \left[ \mathbb{P}_h (\nabla_{h+1}^k - V_{h+1}^{\pi,\nu})(x_h^\tau, a_h^\tau, b_h^\tau) \right]. \]

whence for each \((x, a, b)\):
\[ \langle \phi(x, a, b), \overline{w}_h^k \rangle - Q_{h,\nu}^\pi(x, a, b) = \langle \phi(x, a, b), q_1 + q_2 + q_3 \rangle. \]

We apply Cauchy-Schwarz to bound each RHS term:

1. First term: we have
\[ |\langle \phi(x, a, b), q_1 \rangle| \leq \|w_{h_1}^{\pi,\nu}\|_{(\Lambda_h^k)^{-1}} \cdot \|\phi(x, a, b)\|_{(\Lambda_h^k)^{-1}} \]
\[ \leq \|w_{h_1}^{\pi,\nu}\| \cdot \|\phi(x, a, b)\|_{(\Lambda_h^k)^{-1}} \leq H \sqrt{d} \cdot \|\phi(x, a, b)\|_{(\Lambda_h^k)^{-1}}, \]
where the last two steps follow from \(\Lambda_h^k \geq I\) and \(\|w_{h_1}^{\pi,\nu}\| \lesssim H \sqrt{d}\) (Lemma 7).

2. Second term: we have
\[ |\langle \phi(x, a, b), q_2 \rangle| \leq dH \sqrt{\log(dT/p)} \cdot \|\phi(x, a, b)\|_{(\Lambda_h^k)^{-1}} \]
by Lemma 2.

3. Third term: recalling that \(\sum_{\tau \in [k-1]} \phi_h^\tau (\phi_h^\tau)^\top = \Lambda_h^k - I\) and \(\mathbb{P}_h (\cdot|x_h^\tau, a_h^\tau, b_h^\tau) = (\phi_h^\tau)^\top \mu_h(\cdot)\), we have
\[ \langle \phi(x, a, b), q_3 \rangle = \left\langle \phi(x, a, b), (\Lambda_h^k)^{-1} \sum_{\tau \in [k-1]} \phi_h^\tau (\phi_h^\tau)^\top \int (\nabla_{h+1}^k - V_{h+1}^{\pi,\nu})(x')d\mu_h(x') \right\rangle \]
\[ = \left\langle \phi(x, a, b), \int (\nabla_{h+1}^k - V_{h+1}^{\pi,\nu})(x')d\mu_h(x') \right\rangle - \left\langle \phi(x, a, b), (\Lambda_h^k)^{-1} \int (\nabla_{h+1}^k - V_{h+1}^{\pi,\nu})(x')d\mu_h(x') \right\rangle \]
\[ = \mathbb{P}_h (\nabla_{h+1}^k - V_{h+1}^{\pi,\nu})(x, a, b) + \left\langle \phi(x, a, b), (\Lambda_h^k)^{-1} \int (\nabla_{h+1}^k - V_{h+1}^{\pi,\nu})(x')d\mu_h(x') \right\rangle. \]

But
\[ |p_2| \lesssim \|\phi(x, a, b)\|_{(\Lambda_h^k)^{-1}} \cdot H \sqrt{d}, \]
where we use the facts that \(\Lambda_h^k \geq I, \|\mu_h(S)\| \leq \sqrt{d}\) and \(\nabla_{h+1}^k(\cdot) \leq H, |V_{h+1}^{\pi,\nu}(\cdot)| \leq H.\)
Combining, we obtain
\[
| \langle \phi(x, a, b), \overline{w}_h^k \rangle - Q_h^{\pi_h}(x, a, b) - \mathbb{P}_h(\overline{V}_h^{\pi_h} - V_{h+1}^{\pi_h})(x, a, b) | \leq dH \| \phi(x, a, b) \| (A_h)_{\pi_h} - 1 \leq \beta \| \phi(x, a, b) \| (A_h)_{\pi_h} - 1
\]
der under our choice of $\beta \sim dH \sqrt{t}$. This completes the proof of the inequality (8a) in the lemma. \qed

The above lemma can be specialized to the value functions of the best response (cf. Remark 1); e.g.,
\[
| \langle \phi(x, a, b), \overline{w}_h^k \rangle - Q_h^{\pi_h}(x, a, b) - \mathbb{P}_h(\overline{V}_h^{\pi_h} - V_{h+1}^{\pi_h})(x, a, b) | \leq \epsilon_h^k(x, a, b).
\]
We will make use of this bound and its variants in the proofs of our main theorems.

### 5.3 Upper and Lower Confidence Bounds

With the above bounds on the estimation errors, we can show that $\overline{V}_h^k$ and $V_h^k$ constructed in the algorithm are indeed lower and upper bounds for the true value function. To this end, we state a simple lemma first.

**Lemma 4 (Algorithm 2 finds 2ε-CCE).** For each $(k, h, x)$, $\sigma_h^k(x)$ is an $2\epsilon$-CCE of $(\overline{Q}_h^k(x, \cdot, \cdot), Q_h^k(x, \cdot, \cdot))$ in the sense that
\[
\mathbb{E}_{(a, b) \sim \tilde{\pi}(x)} \left[ \overline{Q}_h^k(x, a, b) \right] \geq \mathbb{E}_{b \sim \mathcal{P}_2(\tilde{\pi}(x))} \left[ \overline{Q}_h^k(x, a', b) \right] - 2\epsilon, \quad \forall a' \in A,
\]
\[
\mathbb{E}_{(a, b) \sim \tilde{\pi}(x)} \left[ Q_h^k(x, a, b) \right] \leq \mathbb{E}_{a \sim \mathcal{P}_1(\tilde{\pi}(x))} \left[ Q_h^k(x, a, b') \right] + 2\epsilon, \quad \forall b' \in A.
\]

**Proof.** Let $(\tilde{Q}, \overline{Q})$ be the elements in the $\epsilon$-net that are closest to $(\overline{Q}_h^k, Q_h^k)$, as specified in Algorithm 2. This means that $| \overline{Q}_h^k(x, a, b) - \tilde{Q}(x, a, b) | \leq \epsilon$ and $| Q_h^k(x, a, b) - \overline{Q}(x, a, b) | \leq \epsilon$ for all $(x, a, b)$. Fix an arbitrary $x \in S$. Because $\sigma_h^k(x) = \tilde{\pi}(x)$ is an CCE of $(\tilde{Q}(x, \cdot, \cdot), \overline{Q}(x, \cdot, \cdot))$, we have for all $a' \in A$:
\[
\mathbb{E}_{(a, b) \sim \tilde{\pi}(x)} \left[ \overline{Q}_h^k(x, a, b) \right] = \mathbb{E}_{(a, b) \sim \tilde{\pi}(x)} \left[ Q_h^k(x, a, b) \right] + \mathbb{E}_{(a, b) \sim \tilde{\pi}(x)} \left[ \overline{Q}_h^k(x, a, b) - Q_h^k(x, a, b) \right]
\geq \mathbb{E}_{b \sim \mathcal{P}_2(\tilde{\pi}(x))} \left[ \overline{Q}_h^k(x, a', b) \right] - \epsilon
\]
\[
= \mathbb{E}_{b \sim \mathcal{P}_2(\tilde{\pi}(x))} \left[ \overline{Q}_h^k(x, a', b) \right] + \mathbb{E}_{b \sim \mathcal{P}_2(\tilde{\pi}(x))} \left[ \overline{Q}_h^k(x, a', b) - \overline{Q}_h^k(x, a', b) \right] - \epsilon
\geq \mathbb{E}_{b \sim \mathcal{P}_2(\tilde{\pi}(x))} \left[ \overline{Q}_h^k(x, a', b) \right] - 2\epsilon.
\]
This proves the first inequality in the lemma. The second inequality can be proved in a similar fashion. \qed

We can now establish the UCB and LCB properties.

**Lemma 5 (UCB and LCB).** Under the setting of Theorem 1, on the event $\mathcal{E}$ in Lemma 2, we have for each $(x, a, b, k, h)$:
\[
\overline{Q}_h^k(x, a, b) - 2(H - h + 1)\epsilon \leq Q_h^{\pi_h}(x, a, b) \leq Q_h^{\pi_h}(x, a, b) \leq \overline{Q}_h^k(x, a, b) + 2(H - h + 1)\epsilon
\]
and
\[
\overline{V}_h^k(x) - 2(H - h + 2)\epsilon \leq V_h^{\pi_h}(x) \leq V_h^{\pi_h}(x) \leq \overline{V}_h^k(x) + 2(H - h + 2)\epsilon.
\]

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Proof. The inequalities (b) and (ii) follow from Proposition 1. Below we only prove the upper bounds (c) and (iii). The lower bounds (a) and (i) can be proved in a similar fashion.

We fix $k$ and perform induction on $h$. The base case $h = H + 1$ holds since the terminal cost is zero. Now assume that the bounds (c) and (iii) hold for step $h + 1$; that is, $\overline{Q}_h^k(x,a,b) \geq Q_{h+1}^k(x,a,b) - 2(H-h)e$ and $\overline{V}_h^k(x) \geq V_{h+1}^k(x) - 2(H-h+1)e$ for all $(x,a,b)$. By inequality (8a) in Lemma 3 applied to $(\tilde{\pi}, v^k)$ with $\tilde{\pi}$ being the best response to $v^k$, we have for each $(x,a,b)$:

$$\left| \langle \phi(x,a,b), \overline{w}_h^k \rangle - Q_h^{*,v^k}(x,a,b) - \mathbb{P}_h \left( \overline{V}_h^k - V_{h+1}^{*,v^k} \right)(x,a,b) \right| \leq \rho_h^k(x,a,b),$$

whence

$$\langle \phi(x,a,b), \overline{w}_h^k \rangle + \rho_h^k(x,a,b) \geq Q_h^{*,v^k}(x,a,b) + \mathbb{P}_h \left( \overline{V}_h^k - V_{h+1}^{*,v^k} \right)(x,a,b),$$

where we recall that $\rho_h^k(x,a,b) := \beta \|\phi(x,a,b)\|_{(A_h^k)}^{-1}$. Under the induction hypothesis, we obtain

$$\langle \phi(x,a,b), \overline{w}_h^k \rangle + \rho_h^k(x,a,b) \geq Q_h^{*,v^k}(x,a,b) - 2(H-h+1)e \geq 0.$$

We can now lower-bound $\overline{Q}_h^k(x,a,b)$:

$$\overline{Q}_h^k(x,a,b) = \Pi_H \left\{ \langle \phi(x,a,b), \overline{w}_h^k \rangle + \rho_h^k(x,a,b) \right\} \quad \text{by construction}$$

$$\geq \Pi_H \left\{ Q_h^{*,v^k}(x,a,b) - 2(H-h+1)e \right\} \quad u \geq v \implies \max \{ \min \{ u, H \}, -H \} \geq \max \{ \min \{ v, H \}, -H \}$$

$$\geq \Pi_H \left\{ Q_h^{*,v^k}(x,a,b) - 2(H-h+1)e \right\} \quad \Pi_H \text{ is non-expansive}$$

$$\geq Q_h^{*,v^k}(x,a,b) - 2(H-h+1)e. \quad Q_h^{*,v^k}(x,a,b) \in [-H,H]$$

This proves the inequality (c) for step $h$.

Finally, recall that $v_h^k(x) := \mathcal{P}_2 c_h^k(x)$, and let $\text{br}(v_h^k(x))$ denote the best response to $v_h^k(x)$ with respect to $Q_h^{*,v^k}(x,\cdot,\cdot)$; i.e.,

$$\text{br}(v_h^k(x)) := \arg \max_{a \in A} \mathbb{E}_{a \sim A, b \sim v_h^k(x)} \left[ Q_h^{*,v^k}(x,a,b) \right].$$

We then have for all $x$:

$$\overline{V}_h^k(x) := \mathbb{E}_{(a,b) \sim c_h^k(x)} \left[ \overline{Q}_h^k(x,a,b) \right] \quad \text{by construction}$$

$$\geq \mathbb{E}_{a' \sim \text{br}(v_h^k(x)), b \sim \mathcal{P}_2 c_h^k(x)} \left[ \overline{Q}_h^k(x,a',b) \right] - 2e \quad \text{$c_h^k(x)$ is $2e$-CCE by Lemma 4}$$

$$\geq \mathbb{E}_{a' \sim \text{br}(v_h^k(x)), b \sim \mathcal{P}_2 c_h^k(x)} \left[ Q_h^{*,v^k}(x,a',b) \right] - 2(H-h+1)e - 2e \quad \text{inequality (c) we just proved}$$

$$= \mathbb{E}_{a \sim \text{br}(v_h^k(x)), b \sim v_h^k(x)} \left[ Q_h^{*,v^k}(x,a,b) \right] - 2(H-h+2)e \quad \text{definition of } \pi^k_h(x) \text{ and } v_h^k(x)$$

$$= V_h^{*,v^k}(x) - 2(H-h+2)e.$$

This proves inequality (iii) for step $h$. \qed
5.4 Recursive Decomposition of Duality Gap

Thanks to Lemma 5 established above, the difference of the UCB and LCB, namely \( \delta_h^k := \nabla_h^k (x_h^k) - \nabla_h^k (x_h^k) \), is an (approximate) upper bound on the duality gap \( V_h^{*,k,s} (x_h^k) - V_h^{*,k,s} (x_h^k) \). Setting the stage for bounding the duality gap, we show below that \( \delta^k_h \) can be decomposed recursively into the sum of \( \delta_{h+1}^k \) and some error terms.

**Lemma 6** (Recursive decomposition). Let

\[
\delta_h^k := \nabla_h^k (x_h^k) - \nabla_h^k (x_h^k),
\]

\[
\zeta_h^k := \mathbb{E} \left[ \delta_{h+1}^k | x_h^k, a_h^k, b_h^k \right] - \delta_{h+1}^k,
\]

\[
\tau_h^k := \mathbb{E}_{b \sim \pi_h(x_h^k)} \left[ Q_h^k (x_h^k, a_h^k, b) - Q_h^k (x_h^k, a_h^k, b_h^k) \right],
\]

\[
\xi_h^k := \mathbb{E}_{a \sim \pi_h^k (x_h^k)} \left[ Q_h^k (x_h^k, a, b_h^k) - Q_h^k (x_h^k, a_h^k, b_h^k) \right].
\]

Then on the event \( \mathcal{E} \) in Lemma 2, we have for all \((k, h)\),

\[
\delta_h^k \leq \delta_{h+1}^k + \zeta_h^k + \tau_h^k \leq 4 \beta \sqrt{\phi_h^k} (\Lambda_h^k)^{-1} \phi_h^k.
\]

**Proof.** For each \((x, a, b, k, h)\), by construction we have

\[
\overline{Q}_h^k (x, a, b) - Q_h^k (x, a, b) = \left[ \left( w_h^k \right)^\top \phi (x, a, b) + \beta \| \phi (x, a, b) \| (\Lambda_h^k)^{-1} \right] - \left[ \left( w_h^k \right)^\top \phi (x, a, b) - \beta \| \phi (x, a, b) \| (\Lambda_h^k)^{-1} \right] = \left( w_h^k - w_h^k \right)^\top \phi (x, a, b) + 2 \beta \| \phi (x, a, b) \| (\Lambda_h^k)^{-1}.
\]

The inequalities (8a) and (8b) in Lemma 3 ensure that

\[
\left( w_h^k - w_h^k \right)^\top \phi (x, a, b) \leq \mathbb{P}_h \left( \nabla_h^k - \nabla_h^k \right) (x, a, b) + 2 \beta \| \phi (x, a, b) \| (\Lambda_h^k)^{-1},
\]

hence by plugging back we obtain the bound

\[
\overline{Q}_h^k (x, a, b) - Q_h^k (x, a, b) \leq \mathbb{P}_h \left( \nabla_h^k - \nabla_h^k \right) (x, a, b) + 4 \beta \| \phi (x, a, b) \| (\Lambda_h^k)^{-1}. \tag{9}
\]

On the other hand, observe that by definition,

\[
\delta_h^k := \nabla_h^k (x_h^k) - \nabla_h^k (x_h^k)
\]

\[
= \mathbb{E}_{(a, b) \sim \pi_h^k (x_h^k)} \left[ \overline{Q}_h^k (x_h^k, a, b) \right] - \mathbb{E}_{(a, b) \sim \pi_h^k (x_h^k)} \left[ Q_h^k (x_h^k, a, b) \right]
\]

\[
= \overline{Q}_h^k (x_h^k, a_h^k, b_h^k) - Q_h^k (x_h^k, a_h^k, b_h^k)
\]

\[
+ \mathbb{E}_{(a, b) \sim \pi_h^k (x_h^k)} \left[ \overline{Q}_h^k (x_h^k, a, b) \right] - \mathbb{E}_{(a, b) \sim \pi_h^k (x_h^k)} \left[ Q_h^k (x_h^k, a, b) \right] = \overline{Q}_h^k (x_h^k, a_h^k, b_h^k) - Q_h^k (x_h^k, a_h^k, b_h^k) + \tau_h^k - \zeta_h^k.
\]

Applying the inequality (9), we obtain

\[
\delta_h^k \leq \mathbb{P}_h \left( \nabla_h^k - \nabla_h^k \right) (x_h^k, a_h^k, b_h^k) + 4 \beta \| \phi (x_h^k, a_h^k) \| (\Lambda_h^k)^{-1} + \tau_h^k - \zeta_h^k
\]

\[
= \mathbb{E} \left[ \delta_{h+1}^k | x_h^k, a_h^k, b_h^k \right] + 4 \beta \| \phi_h^k \| (\Lambda_h^k)^{-1} + \tau_h^k - \zeta_h^k
\]

\[
= \delta_{h+1}^k + \zeta_h^k + 4 \beta \| \phi_h^k \| (\Lambda_h^k)^{-1} + \tau_h^k - \zeta_h^k
\]

as desired. \( \square \)
5.5 Establishing Duality Gap Bound

We are now ready to prove Theorem 1. First observe that on the event \( E \) in Lemma 2 (which holds with probability at least \( 1 - p/2 \)), we have

\[
\text{Gap}(K) := \sum_{k=1}^{K} \left[ V_{1}^{x,k}(x_1) - V_{1}^{\pi^{k},*}(x_1) \right] \quad \text{definition}
\]

\[
\leq \sum_{k=1}^{K} \left[ V_{1}^{k}(x_1) - V_{1}^{\pi^{k},*}(x_1) \right] + 8KH\epsilon \quad \text{Lemma 5}
\]

\[
= \sum_{k=1}^{K} \delta_{1}^{k} + 8KH\epsilon \quad \text{definition}
\]

\[
\leq \sum_{k=1}^{K} \sum_{h=1}^{H} (\zeta_{h}^{k} + \gamma_{h}^{k} - \gamma_{h}^{k}) + 4\beta \sum_{k=1}^{K} \sum_{h=1}^{H} \sqrt{(\phi_{h}^{k})^\top (\Lambda_{h}^{k})^{-1} \phi_{h}^{k}} + 8KH\epsilon. \quad \text{Lemma 6}
\]

- For the first term, we know that \( (\zeta_{h}^{k} + \gamma_{h}^{k} - \gamma_{h}^{k}) \) is a martingale difference sequence (with respect to both \( h \) and \( k \)), and \( |\zeta_{h}^{k} + \gamma_{h}^{k} - \gamma_{h}^{k}| \leq 6H \). Hence by Azuma-Hoeffding, we have with probability at least \( 1 - p/2 \),

\[
\sum_{k=1}^{K} \sum_{h=1}^{H} (\zeta_{h}^{k} + \gamma_{h}^{k} - \gamma_{h}^{k}) \lesssim H \cdot \sqrt{KH}.
\]

- For the second term, we apply the elliptical potential Lemma 10 to obtain

\[
\sum_{h=1}^{H} \sum_{k=1}^{K} \sqrt{(\phi_{h}^{k})^\top (\Lambda_{h}^{k})^{-1} \phi_{h}^{k}} \leq \sum_{h=1}^{H} \sqrt{\sum_{k=1}^{K} (\phi_{h}^{k})^\top (\Lambda_{h}^{k})^{-1} \phi_{h}^{k}} \quad \text{Jensen’s or Cauchy-Schwarz}
\]

\[
\leq \sum_{h=1}^{H} \sqrt{K} \cdot \sqrt{2 \log \left( \frac{\det \Lambda_{h}^{K}}{\det \Lambda_{h}^{0}} \right)} \quad \text{Lemma 10}
\]

\[
\leq \sum_{h=1}^{H} \sqrt{K} \cdot \sqrt{2 \log \left( \frac{(\lambda + K \max_{h} \| \phi_{h}^{k} \|^2)^d}{\lambda^d} \right)} \quad \text{by construction of } \Lambda_{h}^{k}
\]

\[
\leq \sum_{h=1}^{H} \sqrt{K} \cdot \sqrt{2d \log \left( \frac{\lambda + K}{\lambda} \right)} \quad \| \phi_{h}^{k} \| \leq 1, \forall h, k \text{ by assumption}
\]

\[
\leq H \sqrt{2Kd}.
\]

- For the third term, we have \( 8KH\epsilon \leq 8 \) by the choice \( \epsilon = \frac{1}{KH} \).

Combining the above inequalities, we obtain that with probability at least \( 1 - p \),

\[
\text{Gap}(K) \lesssim H \sqrt{HK} + 4\beta \cdot H \sqrt{2Kd} + 8 \lesssim \sqrt{d^3 H^3 T^2},
\]

by our choice of \( \beta \asymp dH \sqrt{t} \) and the fact that \( T = KH \). This completes the proof of Theorem 1.
6 Conclusion

In this paper, we develop provably efficient reinforcement learning methods for zero-sum Markov Games with simultaneous moves and a linear structure. To ensure efficient exploration, our algorithms construct appropriate UCB/LCB for both players and make crucial use of the concept of Coarse Correlated Equilibrium. We provide regret bounds under both the offline and online settings. Corollaries of these bounds apply to turn-based games and the tabular settings. Our results build on and generalize work on learning MDPs with linear structures, and at the same time highlight the crucial differences and new challenges in the game setting.

A number of directions are of interest for future research. An immediate step is to investigate whether the dependence on the dimension $d$ and horizon $H$ in our bounds can be improved and what are the optimal scaling. It would also be interesting to improve our online regret bounds so that we can compete with the best response to the opponent (not just competing with the NE). Generalizations to general-sum Markov games, as well as to games with more complicated, nonlinear structures, are also of great interest.

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Appendices

A Algorithms for Turn-based Games

In this section, we present our algorithms for turn-based games for both the offline and online settings. Note that these algorithms are derived by specializing the corresponding simultaneous-move algorithms, Algorithms 1 and 3, to turn-based games.

A.1 Offline Setting

In the offline setting, the algorithm for turn-based games is given in Algorithm 4.
Algorithm 4 Optimistic Minimax Value Iteration (Turn-Based, Offline)

1: for episode $k = 1, 2, \ldots, K$ do
2:  receive initial state $x^k_1$.
3:  for step $h = H, H - 1, \ldots, 2, 1$ do $\triangleright$ update policy
4:    $\Lambda^k_h \leftarrow \sum_{t=1}^{H} \phi(x^t_h, a^t_h)\phi(x^t_h, a^t_h) + I$.
5:    $\varpi^k_h \leftarrow (\Lambda^k_h)^{-1} \sum_{t=1}^{H} \phi(x^t_h, a^t_h) \left[ r_h(x^t_h, a^t_h) + V^k_{h+1}(x^t_h) \right]$.
6:    $\nu^k_h \leftarrow (\Lambda^k_h)^{-1} \sum_{t=1}^{H} \phi(x^t_h, a^t_h) \left[ r_h(x^t_h, a^t_h) + V^k_{h+1}(x^t_h) \right]$.
7:    $Q^k_h(\cdot, \cdot) \leftarrow \Pi_H \left\{ \left( \varpi^k_h \right)^\top \phi(\cdot, \cdot) + \beta \sqrt{\phi(\cdot, \cdot) \top (\Lambda^k_h)^{-1} \phi(\cdot, \cdot)} \right\}$.
8:    $Q^k_h(\cdot, \cdot) \leftarrow \Pi_H \left\{ \left( \nu^k_h \right)^\top \phi(\cdot, \cdot) - \beta \sqrt{\phi(\cdot, \cdot) \top (\Lambda^k_h)^{-1} \phi(\cdot, \cdot)} \right\}$.
9:  end for
10: for step $h = 1, 2, \ldots, H$ do $\triangleright$ execute policy
11:    if $I(x^k_h) = 1$, P1 takes action $a^k_h = \pi^k_h(x^k_h)$,
12:      else if $I(x^k_h) = 2$, P2 takes action $a^k_h = v^k_h(x^k_h)$.
13:    observe next state $x^k_{h+1}$.
14: end for
15: end for

The algorithm involves the subroutines FIND_MAX and FIND_MIN, which are derived by specializing the FIND_MAX routine in Algorithm 2 to the turn-based setting. For completeness we provide below a description of these two subroutines. Let $Q$ be the class of functions $Q : \mathcal{S} \times \mathcal{A} \to \mathbb{R}$ with the parametric form

$$Q(x, a) = \langle w, \phi(x, a) \rangle + \rho \beta \sqrt{\phi(x, a) \top A \phi(x, a)},$$

where the parameters $(w, A, \rho)$ satisfy $\|w\| \leq 2H\sqrt{dk}, \|A\|_F \leq \beta^2 \sqrt{d}$ and $\rho \in \{\pm 1\}$. Let $Q_\epsilon$ be a fixed $\epsilon$-covering of $Q$ with respect to the $\ell_\infty$ norm. With these notations, the subroutine FIND_MAX is given in Algorithm 5, and the subroutine FIND_MIN is given by $\text{FIND_MIN}(Q, x) = \text{FIND_MAX}(-Q, x)$.

Algorithm 5 FIND_MAX

1: Input: $Q, x$.
2: Pick $\bar{Q} \in Q_\epsilon$ satisfying $\| \bar{Q} - Q \|_\infty \leq \epsilon$.
3: For the input $x$, let $\bar{a} = \arg\max_a \bar{Q}(x, a)$.
4: Output: $\bar{a}$.

Informally, one may simply think of FIND_MAX($Q, x$) as $\arg\max_a Q(x, a)$ and FIND_MIN($Q, x$) as $\arg\min_a Q(x, a)$. As in the simultaneous game setting, these subroutines are introduced for technical reasons in the analysis.
A.2 Online Setting

In the online setting, the algorithm for turn-based games is given in Algorithm 6.

Algorithm 6 Optimistic Minimax Value Iteration (Turn-Based, Online)

1: for episode $k = 1, 2, \ldots, K$ do
2:  Receive initial state $x^k_1$. 
3:  for step $h = H, H - 1, \ldots, 2, 1$ do \hfill $\triangleright$ update policy
4:    \hspace{1em} $\Lambda^k_h \leftarrow \sum_{\tau=1}^{k-1} \phi(x^\tau_h, a^\tau_h)\phi(x^\tau_h, a^\tau_h)^\top + I.$ 
5:    \hspace{1em} $w^k_h \leftarrow (\Lambda^k_h)^{-1} \sum_{\tau=1}^{k-1} \phi(x^\tau_h, a^\tau_h) \left[ r_h(x^\tau_h, a^\tau_h) + V^k_{h+1}(x^\tau_{h+1}) \right].$ 
6:    \hspace{1em} $Q^k_h(\cdot, \cdot) \leftarrow \Pi_H \left\{ (w^k_h)^\top \phi(\cdot, \cdot) + \beta \sqrt{\phi(\cdot, \cdot)^\top (\Lambda^k_h)^{-1} \phi(\cdot, \cdot)} \right\}.$ 
7:    \hspace{1em} $V^k_h(\cdot) \leftarrow \begin{cases} \max_a Q^k_{h+1}(\cdot, a) & \text{if } I(\cdot) = 1, \\ \min_a Q^k_{h+1}(\cdot, a) & \text{if } I(\cdot) = 2. \end{cases}$ 
8:  end for
9:  for step $h = 1, 2, \ldots, H$ do \hfill $\triangleright$ execute policy
10:     if $I(x^k_h) = 1$, take action $a^k_h = \arg \max_a Q^k_h(x^k_h, a)$, 
11:     else do nothing and let P2 play. 
12:     Observe next state $x^k_{h+1}$. 
13:  end for 
14: end for

B Technical Lemmas

The proofs of our main Theorems 1 and 2 involve several common steps. We summarize these steps as several lemmas, which are either proved below or are standard in the literature.

B.1 Boundedness of Linear Coefficients

We begin with two simple lemmas about boundedness of the linear coefficients of $Q$ functions.

Lemma 7 (True coefficients are bounded). Under Assumption 1, for each policy pair $(\pi, \nu)$ of P1 and P2, the linear coefficient of their action-value function $Q^\pi_\nu(x, a, b) = \langle \phi(x, a, b), w^\pi_\nu \rangle$ satisfies

$$||w^\pi_\nu|| \leq 2H\sqrt{d}, \quad \forall h \in [H].$$

Proof. From the Bellman equation, we have

$$\phi(x, a, b)^\top w^\pi_\nu = Q^\pi_\nu(x, a, b) = r_h(x, a, b) + (\mathbb{P}^\pi_\nu V^\pi_\nu)(x, a, b)$$

$$= \phi(x, a, b)^\top \theta_h + \int V^\pi_\nu_{h+1}(x')\phi(x, a, b)^\top d\mu_h(x'), \quad \forall x, a, b, h.$$

Assuming that $\{\phi(x, a, b)\}$ spans $\mathbb{R}^d$ and solving the linear equation, we obtain

$$w^\pi_\nu = \theta_h + \int V^\pi_\nu_{h+1}(x')d\mu_h(x').$$
Under the normalization Assumption 1, we have \( \| \theta_h \| \leq \sqrt{d}, \| \mu_h(S) \| \leq \sqrt{d} \) and \( |V_{h+1}^{\pi,v}(x')| \leq H \). It follows that
\[
\| w_h^{\pi,v} \| \leq \sqrt{d} + H\sqrt{d} \leq 2H\sqrt{d}
\]
as desired.

An immediate consequence of the above lemma is that \( \| w_h^{\pi,v} \| \leq 2H\sqrt{d} \) and \( \| w_h^{v,\nu} \| \leq 2H\sqrt{d} \); cf. Remark 1.

**Lemma 8 (Algorithm coefficients are bounded).** The coefficients \{\( w_h^k \), \( w_h^k \)\} in Algorithm 1 and the coefficients \{\( w_h^k \)\} in Algorithm 3 satisfy
\[
\| w_h^k \| \leq 2H\sqrt{d}, \quad \| w_h^k \| \leq 2H\sqrt{d}, \quad \text{and} \quad \| w_h^k \| \leq 2H\sqrt{d}, \quad \forall (k,h) \in [K] \times [H].
\]

**Proof.** We only prove the last inequality. The other two inequalities can be established in exactly the same way. For each \( k \) and \( h \), we have
\[
\| w_h^k \| = \left\| \left( \Lambda_h^k \right)^{-1} \sum_{t=1}^{k-1} \phi(x_h^t, a_h^t, b_h^t) \left[ r_h(x_h^t, a_h^t, b_h^t) + V_{h+1}^k(x_{h+1}^t) \right] \right\|
\leq \sum_{t=1}^{k-1} \left\| \left( \Lambda_h^k \right)^{-1} \phi(x_h^t, a_h^t, b_h^t) \right\| \cdot 2H \quad \text{\( r_h \) \leq H, \( V_{h+1}^k \) \leq H}
\leq \sum_{t=1}^{k-1} \left\| \left( \Lambda_h^k \right)^{-1/2} \left\| \phi(x_h^t, a_h^t, b_h^t) \right\|_{\Lambda_h^k}^{-1} \cdot 2H \quad \Lambda_h^k \succeq I \text{ and Jensen’s}
\leq \sqrt{k} \sum_{t=1}^{k-1} \left\| \phi(x_h^t, a_h^t, b_h^t) \right\|_{\Lambda_h^k}^{-1} \cdot 2H \quad \text{Lemma 9}
\leq \sqrt{kd} \cdot 2H,
\]
thereby proving the last inequality in the lemma.

**B.2 Inequalities for Summations**

We next state two lemmas for summations. The first lemma is from Jin et al. (2019, Lemma D.1).

**Lemma 9 (Simple upper bound).** If \( \Lambda_t = \lambda I + \sum_{i \in [t]} \phi_i \phi_i^T \), where \( \phi_i \in \mathbb{R}^d \) and \( \lambda > 0 \), then
\[
\sum_{i \in [t]} \phi_i^T \Lambda_t^{-1} \phi_i \leq d.
\]

The second lemma is from Abbasi-Yadkori et al. (2011, Lemma 11), Jin et al. (2019, Lemma D.2).

**Lemma 10 (Elliptical potential lemma).** Suppose that \( \{ \phi_i \} \geq 0 \) is a sequence in \( \mathbb{R}^d \) satisfying \( \| \phi_i \| \leq 1, \forall t \). Let \( \Lambda_0 \in \mathbb{R}^{d \times d} \) be a positive definite matrix, and \( \Lambda_t = \Lambda_0 + \sum_{i \in [t]} \phi_i \phi_i^T \). If the smallest eigenvalue of \( \Lambda_0 \) satisfies \( \lambda_{\min} (\Lambda_0) \geq 1 \), then
\[
\log \left( \frac{\det \Lambda_t}{\det \Lambda_0} \right) \leq \sum_{j \in [t]} \phi_j^T \Lambda_{j-1}^{-1} \phi_j \leq 2 \log \left( \frac{\det \Lambda_t}{\det \Lambda_0} \right), \forall t.
\]
B.3 Covering and Concentration Inequalities for Self-normalized Processes

The first lemma below is useful for establishing uniform concentration. Recall the function class \( Q \) defined in the text around equation (5).

**Lemma 11 (Covering).** The \( \epsilon \)-covering number of \( Q \) with respect to the \( \ell_\infty \) norm satisfies

\[
N_\epsilon \leq 2 \left( 1 + \frac{8H \sqrt{dk}}{\epsilon} \right)^d \left( 1 + \frac{8\beta^2 \sqrt{d}}{\epsilon^2} \right)^{d^2}
\]

**Proof.** For any two functions \( Q, Q' \in Q \) with parameters \((w, A, \rho)\) and \((w', A', \rho)\), we have

\[
\|Q - Q'\|_\infty = \sup_{x, a, b} |\Pi_H \left\{ \langle w, \phi(x, a, b) \rangle + \rho \beta \sqrt{\phi(x, a, b)^\top A \phi(x, a, b)} \right\} - \Pi_H \left\{ \langle w', \phi(x, a, b) \rangle - \rho \beta \sqrt{\phi(x, a, b)^\top A' \phi(x, a, b)} \right\}|.
\]

\[
\leq \sup_{\phi : \|\phi\| \leq 1} \left| \langle w - w', \phi \rangle + \rho \beta \sqrt{\phi \top A \phi - \rho \beta \sqrt{\phi \top A' \phi}} \right|
\]

\[
\leq \sup_{\phi : \|\phi\| \leq 1} |\langle w - w', \phi \rangle| + \sup_{\phi : \|\phi\| \leq 1} \sqrt{\langle \phi \top (A - A') \phi \rangle}
\]

\[
\leq \|w - w'\| + \sqrt{\|A - A'\|_F},
\]

where the second last inequality follows due to the fact that \( |\sqrt{x} - \sqrt{y}| \leq \sqrt{|x - y|} \) holds for any \( x, y \geq 0 \).

Therefore, a \( 0 \)-cover \( C_0 \) of \{ \pm 1 \}, an \( \epsilon \)/2-cover \( C_w \) of \( \{ w \in \mathbb{R}^d : \|w\| \leq 2H \sqrt{dk} \} \) and an \( \epsilon^2 \)/4-cover \( C_A \) of \( \{ A \in \mathbb{R}^{d \times d} : \|A\|_F \leq \beta^2 \sqrt{d} \} \) implies an \( \epsilon \)-cover of \( Q \). It follows that

\[
N_\epsilon \leq |C_0| |C_w| |C_A| \leq 2 \left( 1 + \frac{8H \sqrt{dk}}{\epsilon} \right)^d \left( 1 + \frac{8\beta^2 \sqrt{d}}{\epsilon^2} \right)^{d^2},
\]

where the last step follows from standard bounds on the covering number of Euclidean Balls, e.g., Vershynin (2012, Lemma 5.2).

The next lemma, originally from Abbasi-Yadkori et al. (2011, Theorem 1), is now standard in the bandit literature.

**Lemma 12 (Concentration for self-normalized processes).** Suppose \( \{\epsilon_t\}_{t \geq 1} \) is a scalar stochastic process generating the filtration \( \{\mathcal{F}_t\}_{t \geq 0} \), and \( \epsilon_t | \mathcal{F}_{t-1} \) is zero mean and \( \sigma \)-subGaussian. Let \( \{\phi_t\}_{t \geq 1} \) be an \( \mathbb{R}^d \)-valued stochastic process with \( \phi_t \in \mathcal{F}_{t-1} \). Suppose \( \Lambda_0 \in \mathbb{R}^{d \times d} \) is positive definite, and \( \Lambda_t = \Lambda_0 + \sum_{s=1}^t \phi_s \phi_s^\top \). Then for each \( \delta \in (0, 1) \), with probability at least \( 1 - \delta \), we have

\[
\left\| \sum_{s=1}^t \phi_s \epsilon_s \right\|_{\Lambda_t^{-1}}^2 \leq 2\sigma^2 \log \left[ \frac{\det(\Lambda_t)^{1/2} \det(\Lambda_0)^{-1/2}}{\delta} \right], \quad \forall t \geq 0.
\]
C Proof of Theorem 2

In this section, we prove Theorem 2 for the online setting of simultaneous games. We shall make use of the technical lemmas given in Appendix B. Recall the shorthand \( \phi^k_h := \phi(x_h^k, a_h^k, b_h^k) \). The proof follows a similar strategy as that for the proof of Theorem 1 in Section 5. In particular, our proof consists of five steps as presented in the subsections to follow.

C.1 Uniform Concentration

In the online setting, the value function estimate \( V_{k+1}^h(x) \) is computed using the NE of the zero-sum game defined by a single payoff matrix \( Q_{k+1}^h(x, \cdot, \cdot) \). It is easier to establish uniform concentration in this setting. To see why, we recall the function class \( V \) defined in the text around equation (5), and introduce the related function class

\[
V := \left\{ V : \mathcal{S} \to \mathbb{R}, V(x) = \max_{A \in \Delta} \min_{B \in \Delta} E_{a \in A, b \in B} Q(x, a, b), Q \in \mathcal{Q} \right\}.
\]

In words, \( V \) contains the values of the NEs of the zero-sum games in \( \mathcal{Q} \). As we show in the lemma below, an \( \epsilon \)-cover of the set \( \mathcal{Q} \) immediately induces an \( \epsilon \)-cover of the set \( V \), thanks to the non-expansiveness of the maximin operator for zero-sum games. (Note that general-sum games and their CCEs do not have such a non-expansiveness property in general; see Appendix F for details.)

Lemma 13 (Covering). The \( \epsilon \)-covering number of \( V \) with respect to the \( \ell_\infty \) norm is upper bounded by

\[
N_\epsilon \leq 2 \left( 1 + \frac{8H \sqrt{d}k}{\epsilon} \right)^d \left( 1 + \frac{8\beta^2 \sqrt{d}}{\epsilon^2} \right)^{d^2}.
\]

Proof. For any two functions \( V, V' \in V \), let them take the form \( V(\cdot) = \max_{A \in \Delta} \min_{B \in \Delta} E_{a \in A, b \in B} Q(\cdot, a, b) \) and \( V'(\cdot) = \max_{A \in \Delta} \min_{B \in \Delta} E_{a \in A, b \in B} Q'(\cdot, a, b) \) with \( Q, Q' \in \mathcal{Q} \). Since the maximin operator is non-expansive, we have

\[
\| V - V' \|_\infty = \sup_x \left| \max_{A \in \Delta} \min_{B \in \Delta} E_{a \in A, b \in B} Q(\cdot, a, b) - \max_{A \in \Delta} \min_{B \in \Delta} E_{a \in A, b \in B} Q'(\cdot, a, b) \right|
\]

\[
\leq \sup_{x, a, b} |Q(x, a, b) - Q'(x, a, b)|
\]

\[
= \| Q - Q' \|_\infty.
\]

Therefore, an \( \epsilon \)-cover of \( \mathcal{Q} \) induces an \( \epsilon \)-cover of \( V \), and hence the \( \epsilon \)-covering number of \( V \) is upper bounded by the \( \epsilon \)-covering number of \( \mathcal{Q} \). Recalling that the latter number is bounded in Lemma 11, we complete the proof of the desired bound.

Lemma 14 (Concentration). Under the setting of Theorem 2, for each \( p \in (0, 1) \), the following event \( \mathcal{E} \) holds with probability at least \( 1 - p/2 \):

\[
\left\| \sum_{\tau \in [k-1]} \phi^k_h \left[ V_{h+1}^k(x_{h+1}) - \left( P_{h} V_{h}^k \right) (x_h^k, a_h^k, b_h^k) \right] \right\|_{(A_h^k)^{-1}} \lesssim dH \sqrt{\log(dT/p)}, \quad \forall (k, h) \in [K] \times [H].
\]
Proof. Let \( \mathcal{F}_{\tau - 1} := \mathcal{F}(x^1, a^1, \ldots, x^{\tau - 1}, a^{\tau - 1}) \). Note that \( \phi^r, a^r \in \mathcal{F}_{\tau - 1} \).

Set \( \epsilon = \frac{1}{K} \) and let \( \mathcal{V}_\epsilon \) be a minimal \( \epsilon \)-net of \( \mathcal{V} \). Fix a function \( \tilde{V} \in \mathcal{V}_\epsilon \). The random variable \( \tilde{V}(x^\tau_{\tau + 1}) - \mathbb{P}_h \tilde{V}(x^\tau_{\tau + 1}) \mid \mathcal{F}_{\tau - 1} \) is zero-mean and \( 2H \)-bounded. Applying Lemma 12 gives

\[
\left\| \sum_{\tau \in [k-1]} \phi^r_h \left( \tilde{V}(x^\tau_{\tau + 1}) - \mathbb{P}_h \tilde{V}(x^\tau_{\tau + 1}, a^r_h, b^r_h) \right) \right\|_{(A^h)^{-1}} \lesssim dH \sqrt{\log(dT/p)}
\]

with probability at least \( 2^{-\Omega(d^2 \log(dT/p))} \). Now note that \( |\mathcal{V}_\epsilon| = \mathcal{N}_\epsilon \leq 2 \left( 1 + \frac{8H \sqrt{dT}}{\epsilon} \right)^d \left( 1 + \frac{\rho^2 \sqrt{dT}}{\epsilon} \right)^d \) by Lemma 13. By a union bound, the above inequality holds for all \( \tilde{V} \in \mathcal{V}_\epsilon \) with probability at least \( 1 - p/2 \).

Now, for each \( V^k_h+1 \in \mathcal{V} \) (the inclusion follows from Lemma 8), let \( \tilde{V} \in \mathcal{V}_\epsilon \) be the closest point in the net. The difference \( \Delta = V^k_{h+1} - \tilde{V} \) satisfies \( \|\Delta\|_\infty \leq \epsilon \). It follows that

\[
\left\| \sum_{\tau \in [k-1]} \phi^r_h \left[ V^k_{h+1}(x^\tau_{h+1}) - \left( \mathbb{P}_h V^k_{h+1} \right)(x^\tau_{h+1}, a^r_h, b^r_h) \right] \right\|_{(A^h)^{-1}} \\
\leq \left\| \sum_{\tau \in [k-1]} \phi^r_h \left[ \tilde{V}(x^\tau_{h+1}) - \left( \mathbb{P}_h \tilde{V} \right)(x^\tau_{h+1}, a^r_h, b^r_h) \right] \right\|_{(A^h)^{-1}} + \left\| \sum_{\tau \in [k-1]} \phi^r_h \left[ \Delta(x^\tau_{h+1}) - \left( \mathbb{P}_h \Delta \right)(x^\tau_{h+1}, a^r_h, b^r_h) \right] \right\|_{(A^h)^{-1}} \\
\leq dH \sqrt{\log(dT/p)} + \epsilon \sum_{\tau \in [k-1]} \|\phi^r_h\|_{(A^h)^{-1}} \\
\leq dH \sqrt{\log(dT/p)} + \frac{1}{K} \cdot k,
\]

where the last step follows from \( \epsilon = \frac{1}{K}, \Lambda^h \geq I \) and \( \|\phi^r_h\| \leq 1 \). This completes the proof of the lemma. \( \square \)

C.2 Least-squares Estimation Error

Here we bound the difference between the algorithm’s value function (without bonus) and the true value function of any policy \( \pi \), recursively in terms of the step \( h \).

Lemma 15 (Least-squares error bound). The quantities \( \{w^k_h, V^k_h\} \) in Algorithm 3 satisfy the following. If \( \beta = dH \sqrt{\tau} \), then on the event \( \mathcal{E} \) in Lemma 14, we have for all \( (x, a, b, h, k) \) and any policy pair \( (\pi, v) \):

\[
\left| \left\langle \phi(x, a, b), w^k_h \right\rangle - Q^\pi_{\tau, h}(x, a, b) - \mathbb{P}_h (V^k_{h+1} - V^{\pi, h}_{h+1})(x, a, b) \right| \leq \rho^\beta_h(x, a, b),
\]

(10)

where \( \rho^\beta_h(x, a, b) := \beta \sqrt{\phi(x, a, b) \top (A^h)^{-1} \phi(x, a, b)} \).

Proof. The proof is essentially identical to that of Lemma 3, except that we use the concentration result in Lemma 14 instead of Lemma 2. \( \square \)

C.3 Upper Confidence Bounds

Here we establish the desired UCB property.

\[ \text{Page 29} \]
**Lemma 16 (UCB).** On the event $\mathcal{E}$ in Lemma 2, we have for all $(x, a, b, k, h)$:

$$Q_h^k(x, a, b) \geq Q_h^k(x, a, b), \quad V_h^k(x) \geq V_h^*(x).$$

**Proof.** We fix $k$ and perform induction on $h$. The base case $h = H$ holds since the terminal cost is zero. Now assume that the bounds hold for step $h + 1$; that is, $Q_{h+1}^k(x, a, b) \geq Q_{h+1}^k(x, a, b)$ and $V_{h+1}^k(x) \geq V_{h+1}^*(x), \forall (x, a, b)$. By construction we have

$$Q_h^k(x, a, b) = \Pi_H \left\{ \left( \phi(x, a, b), \omega_h^k \right) + \beta \|\phi(x, a, b)\|_{(\Lambda_h^k)^{-1}} \right\}.$$  

On the other hand, note that $Q_h^* = Q_{h, k}^{\pi^*, v^*}$ and $V_h^* = V_{h, k}^{\pi^*, v^*}$, hence by inequality (10) in Lemma 3 applied to $(\pi, v) = (\pi^*, v^*)$, we have

$$\left| \left( \phi(x, a, b), \omega_h^k \right) - Q_h^k(x, a, b) - \Pi_h(V_{h+1}^k - V_{h+1}^*)(x, a, b) \right| \leq \beta \|\phi(x, a, b)\|_{(\Lambda_h^k)^{-1}}.$$  

Plugging back we obtain

$$Q_h^k(x, a, b) \geq \Pi_H \left\{ Q_h^k(x, a, b) + \Pi_h(V_{h+1}^k - V_{h+1}^*)(x, a, b) \right\}.$$  

Under the induction hypothesis, we have $V_{h+1}^k(x) - V_{h+1}^*(x) \geq 0$ for each $x \in \mathcal{S}$, whence

$$Q_h^k(x, a, b) \geq \Pi_H \left\{ Q_h^k(x, a, b) \right\} = Q_h^k(x, a, b).$$  

Consequently, we have

$$V_h^k(x) = \max_{A \in \Delta} \min_{B \in \Delta} \mathbb{E}_{a \sim A, b \sim B} \left[ Q_h^k(x, a, b) \right] \quad \text{algorithm specification}$$

$$\geq \max_{A \in \Delta} \min_{B \in \Delta} \mathbb{E}_{a \sim A, b \sim B} \left[ Q_h^k(x, a, b) \right]$$

$$= V_h^*(x). \quad \text{definition}$$  

We conclude that the bounds hold for step $h$. \hfill \Box

### C.4 Recursive Regret Decomposition

Thanks to Lemma 16, the difference $V_1^k(x_1^k) - V_1^{\pi^*, v^*}(x_1^k)$ between the empirical value (with bonus) and true value of the agent’s policy $\pi^*$, is an upper bound on the regret $V_1^k(x_1^k) - V_1^{\pi^*, v^*}(x_1^k)$ of interest. We next derive a recursive (in $h$) formula for this difference.

**Lemma 17 (Recursive formula).** Let

$$\delta_h^k := V_h^k(x_h^k) - V_h^{\pi^*, v^*}(x_h^k),$$

$$\xi_h^k := \mathbb{E} \left[ \delta_{h+1}^k | x_h^k, a_h^k, b_h^k \right] - \delta_{h+1}^k,$$

$$\gamma_h^k := \mathbb{E}_{a \sim \pi_h^k(x_h^k)} \left[ Q_h^k(x_h^k, a, b_h^k) \right] - Q_h^k(x_h^k, a_h^k, b_h^k),$$

$$\tilde{\gamma}_h^k := \mathbb{E}_{a \sim \pi^k(x_h^k), b \sim \nu_h^k(x_h^k)} \left[ Q_h^{\pi, v^*}(x_h^k, a, b) \right] - Q_h^{\pi, v^*}(x_h^k, a_h^k, b_h^k).$$

Then on the event $\mathcal{E}$ in Lemma 2, we have for all $(k, h)$,

$$\delta_h^k \leq \delta_{h+1}^k + \xi_h^k + \gamma_h^k - \tilde{\gamma}_h^k + 2\beta \sqrt{\phi_h^k \top (\Lambda_h^k)^{-1} \phi_h^k}.$$
Proof. By algorithm specification and the fact that \((\pi^k_h(x^k_h), B_0)\) is the Nash equilibrium of \(Q^k_h(x^k_h, \cdot, \cdot)\), we have
\[
V^k_h(x^k_h) = \min_b \mathbb{E}_{a \sim \pi^k_h(x^k_h)} \left[ Q^k_h(x^k_h, a, b) \right]
\leq \mathbb{E}_{a \sim \pi^k_h(x^k_h)} \left[ Q^k_h(x^k_h, a, b^k_h) \right] = Q^k_h(x^k_h, a^k_h, b^k_h) + \gamma^k_h,
\]
and by definition we have
\[
V^k_{\pi^k, \nu^k}(x^k_h) = \mathbb{E}_{a \sim \pi^k, \nu^k(x^k_h, b \sim \nu^k(x^k_h))} \left[ Q^k_{\pi^k, \nu^k}(x^k_h, a, b) \right] = Q^k_{\pi^k, \nu^k}(x^k_h, a^k_h, b^k_h) + \gamma^k_h.
\]
It follows that
\[
\delta^k_h \leq Q^k_h(x^k_h, a^k_h, b^k_h) - Q^k_{\pi^k, \nu^k}(x^k_h, a^k_h, b^k_h) + \gamma^k_h - \tilde{\gamma}^k_h.
\]
On the other hand, by construction of \(Q^k_h\) and Lemma 8, we have for all \((x, a, b)\),
\[
Q^k_h(x, a, b) - Q^k_{\pi^k, \nu^k}(x, a, b) \leq \mathbb{P}_h(V^k_{h+1} - V^k_{h+1}(x, a, b)) + 2\beta \sqrt{\phi(x, a, b)^\top (\Lambda^k_h)^{-1} \phi(x, a, b)}.
\]
Combining, we obtain
\[
\delta^k_h \leq \mathbb{P}_h(V^k_{h+1} - V^k_{h+1}(x^k_h, a^k_h, b^k_h)) + \gamma^k_h - \tilde{\gamma}^k_h + 2\beta \sqrt{\phi(x, a, b)\top (\Lambda^k_h)^{-1} \phi(x, a, b)}
\]
\[
= \mathbb{E} \left[ \delta^k_{h+1} \bigg| x^k_h, a^k_h, b^k_h \right] + \gamma^k_h - \tilde{\gamma}^k_h + 2\beta \sqrt{\phi(x, a, b)\top (\Lambda^k_h)^{-1} \phi(x, a, b)}
\]
\[
= \delta^k_{h+1} + \tilde{\gamma}^k_h + \gamma^k_h - \tilde{\gamma}^k_h + 2\beta \sqrt{\phi(x, a, b)\top (\Lambda^k_h)^{-1} \phi(x, a, b)}
\]
as desired. \(\square\)

### C.5 Establishing Regret Bound

We are now ready to prove Theorem 2.

First observe that
\[
\text{Regret}(K) := \sum_{k=1}^{K} \left[ V^*_1(x^k_1) - V^*_{\pi^k, \nu^k}(x^k_1) \right] \quad \text{definition}
\]
\[
\leq \sum_{k=1}^{K} \left[ V^*_1(x^k_1) - V^*_{\pi^k, \nu^k}(x^k_1) \right] \quad V^*_1(x^k_1) \geq V^*_1(x^k_1) \text{ by Lemma 16}
\]
\[
= \sum_{k=1}^{K} \delta^k_h \quad \text{definition}
\]
\[
\leq \sum_{k=1}^{K} \sum_{h=1}^{H} (\tilde{\gamma}^k_h + \gamma^k_h - \tilde{\gamma}^k_h) + 2\beta \sum_{k=1}^{K} \sum_{h=1}^{H} \sqrt{(\phi^k_h)^\top (\Lambda^k_h)^{-1} \phi^k_h} \quad \text{Lemma 17}
\]
- For the first term, we know that \((\tilde{\gamma}^k_h + \gamma^k_h - \tilde{\gamma}^k_h)\) is a martingale difference sequence (with respect to both \(h\) and \(k\)), and \(|\tilde{\gamma}^k_h + \gamma^k_h - \tilde{\gamma}^k_h| \leq 6H\). Hence by Azuma-Hoeffding, we have w.h.p.
\[
\sum_{k=1}^{K} \sum_{h=1}^{H} (\tilde{\gamma}^k_h + \gamma^k_h - \tilde{\gamma}^k_h) \lesssim H \cdot \sqrt{KHt} = H \sqrt{t}.
\]
• For the second term, we apply the elliptical potential Lemma 10 to obtain

\[
\sum_{h=1}^{H} \sum_{k=1}^{K} \sqrt{(\phi_{h}^{k})^\top (\Lambda_{h}^{k})^{-1} \phi_{h}^{k}} \leq \sum_{h=1}^{H} \sqrt{K} \sqrt{\sum_{k=1}^{K} (\phi_{h}^{k})^\top (\Lambda_{h}^{k})^{-1} \phi_{h}^{k}}
\]

Jensen’s or Cauchy-Schwarz

\[
\leq \sum_{h=1}^{H} \sqrt{K} \cdot 2 \log \left( \frac{\det \Lambda_{h}^{k}}{\det \Lambda_{h}^{0}} \right)
\]

\[
\leq \sum_{h=1}^{H} \sqrt{K} \cdot 2 \log \left( \frac{\lambda + K \max_k \| \phi_{h}^{k} \|^2}{\lambda^d} \right)
\]

by construction of \( \Lambda_{h}^{k} \)

\[
\leq \sum_{h=1}^{H} \sqrt{K} \cdot 2d \log \left( \frac{\lambda + K}{\lambda} \right)
\]

\[
\| \phi_{h}^{k} \| \leq 1, \forall h, k \text{ by assumption}
\]

\[
\leq H \sqrt{2Kd}.
\]

Combining, we obtain that

\[
\text{Regret}(K) \lesssim H \sqrt{3T} + \beta \cdot H \sqrt{2Kd} \lesssim \sqrt{d^3 H^3 T^2}
\]

by our choice of \( \beta \approx dH \sqrt{T} \). This completes the proof of Theorem 2.

### D Proof of Corollaries 1 and 2

**Proof of Corollary 1:** We prove Corollary 1 by specializing Theorem 1 to the turn-based setting. Specifically, as argued in Section 2.2, linear turn-based game is a special case of linear simultaneous games with

\[
\phi(x, a, b) \equiv \phi(x, a), \quad r_h(x, a, b) \equiv r(x, a), \quad P_h(x, a, b) \equiv P_h(x, a), \quad \text{if } x \in S_1,
\]

\[
\phi(x, a, b) \equiv \phi(x, b), \quad r_h(x, a, b) \equiv r(x, b), \quad P_h(x, a, b) \equiv P_h(x, b), \quad \text{if } x \in S_2.
\]

(11)

Moreover, Algorithm 1, when applied to the turn-based setting, degenerates to Algorithm 4. To see this, note that under the degeneration of \( \phi(x, a, b) \) in (11), the values \( \overline{Q}_h^k \) and \( \underline{Q}_h^k \) computed in Algorithm 1 only depend on the action of the active player; that is,

\[
\overline{Q}_h^k(x, a, b) \equiv \overline{Q}_h^k(x, a), \quad \text{if } x \in S_1,
\]

\[
\underline{Q}_h^k(x, a, b) \equiv \underline{Q}_h^k(x, b), \quad \text{if } x \in S_2.
\]

(12)

In this case, one can verify that finding the CCE (cf. (3)) as done in FIND_CCE degenerates to a unilateral maximization or minimization problem, namely \( \arg \max_a \overline{Q}(x, a) \) or \( \arg \min_a \underline{Q}(x, a) \). This is exactly what the subroutines FIND_MAX and FIND_MIN compute. With the above reduction, Corollary 1 follows directly from Theorem 1.

**Proof of Corollary 2:** Similarly, we prove Corollary 2 by specializing Theorem 2 to the turn-based setting. The argument is essentially the same as that in the proof of Corollary 1 above. We omit the details.
E Efficient Implementation of FIND_CCE

The main computation step in FIND_CCE is to find an element in the fixed $\epsilon$-cover $Q_\epsilon$ that is close to a given function $Q$. Here we discuss how to efficiently implement this procedure without explicitly maintaining the cover $Q_\epsilon$.

Note that each element in $Q_\epsilon$ is defined by a pair $(w, A) \in \mathbb{R}^d \times \mathbb{R}^{d \times d}$. Therefore, $Q_\epsilon$ is induced, up to scaling, by an $\epsilon$-cover $C_w$ in $\ell_2$ norm of the Euclidean ball $\{ w \in \mathbb{R}^d : ||w|| \leq 1 \}$ as well as an $\epsilon$-cover $C_A$ of the ball $\{ A \in \mathbb{R}^{d \times d} : ||A||_F \leq 1 \}$. We may augment $C_w$ into a cover $C_{w,\infty}$ in the $\ell_\infty$ norm; similarly for $C_A$. Such an $\ell_\infty$ cover implies covering in $\ell_2$ and allows for efficient computation of near neighbors. The price we pay is an additional dimension factor $d$ in the covering number, which eventually goes in to the log term.

We now provide the details focusing on $C_w$; the idea applies similarly to $C_A$. For a given approximation accuracy $\epsilon > 0$, set $\epsilon_0 := \frac{\epsilon}{\sqrt{d}}$. We discretize the interval $G := [-1, 1]$ into an $\epsilon_0$ grid

$$G_{\epsilon_0} := \left\{ k \epsilon_0 : k = -\left\lfloor \frac{1}{\epsilon_0} \right\rfloor, -\left\lceil \frac{1}{\epsilon_0} \right\rceil + 1, \ldots, -2, -1, 0, 1, 2, \ldots, \left\lfloor \frac{1}{\epsilon_0} \right\rfloor - 1, \left\lceil \frac{1}{\epsilon_0} \right\rceil \right\}.$$  

We then let $C_{w,\infty} := (G_{\epsilon_0})^d$. The log cardinality of $C_{w,\infty}$ is

$$\log |C_{w,\infty}| = \log |G_{\epsilon_0}|^d = \log \left( 1 + 2 \left\lfloor \frac{1}{\epsilon_0} \right\rfloor \right)^d \leq d \log \left( 1 + \frac{2\sqrt{d}}{\epsilon} \right).$$

Compare this bound with the log cardinality of the optimal $\epsilon$-cover in $\ell_2$ norm of $\{ w \in \mathbb{R}^d : ||w|| \leq 1 \}$: $\log |C_w| \approx d \log (1 + \frac{2}{\epsilon})$. We see that the former is only logarithmic larger than the latter.

Moreover, for each $w$ in the ball $\{ w' \in \mathbb{R}^d : ||w'|| \leq 1 \}$, we can efficiently find a $\bar{w} \in C_{w,\infty}$ that satisfies $||\bar{w} - w|| \leq \epsilon$. To do this, we simply let

$$\bar{w}_i = \left\lfloor \frac{|w_i|}{\epsilon_0} \right\rfloor \cdot \epsilon_0 \cdot \text{sign}(w_i), \quad \text{for each } i \in [d].$$

Note that for each $i \in [d]$, we have

$$||w|| \leq 1 \implies |w_i| \leq 1 \implies \left\lfloor \frac{|w_i|}{\epsilon_0} \right\rfloor \in \left\{ 0, 1, \ldots, \left\lfloor \frac{1}{\epsilon_0} \right\rceil \right\} ,$$

whence $\bar{w}_i \in G_{\epsilon_0}$ and consequently $\bar{w} \in (G_{\epsilon_0})^d =: C_{w,\infty}$. Moreover, we have

$$||\bar{w} - w||^2 = \sum_{i \in [d]} \left( \left\lfloor \frac{|w_i|}{\epsilon_0} \right\rfloor \cdot \epsilon_0 \cdot \text{sign}(w_i) - w_i \right)^2$$

$$= \epsilon_0^2 \sum_{i \in [d]} \left( \left\lfloor \frac{|w_i|}{\epsilon_0} \right\rfloor \cdot \text{sign}(w_i) - \frac{|w_i|}{\epsilon_0} \cdot \text{sign}(w_i) \right)^2$$

$$\leq \epsilon_0^2 \sum_{i \in [d]} 1 \cdot (\text{sign}(w_i))^2$$

$$= \epsilon$$

$$\epsilon_0 := \frac{\epsilon}{\sqrt{d}}$$

by choice

as desired.
F Stability of the Value of CCE

In the analysis of our algorithms (in particular, in proving uniform concentration in the proof of Theorem 1), we encounter the following question: Is the value of the CCE of a general-sum game stable under perturbation to the payoff matrices? Here we show that the answer is negative in general, by demonstrating a counter example.

Specifically, consider a two-player general-sum matrix game, and recall our convention that player 1 tries to maximize and player 2 tries to minimize (see Section 2.3). Let \( u_i : A \times A \rightarrow \mathbb{R} \) be the payoff matrix of player \( i \in \{1, 2\} \), such that player \( i \) receives the payoff \( u_i(a, b) \) when players 1 and 2 take actions \( a \) and \( b \), respectively. Let \( \sigma \in \Delta(A \times A) \) be any notion of (coarse) correlated equilibrium that is unique; e.g., the social-optimal or max-entropy CCE. Then the expected payoff of player \( i \) is

\[
V_i(u_1, u_2) := \mathbb{E}_{(a, b) \sim \sigma} [u_i(a, b)].
\]

We are interested in the question: Is \((V_1, V_2) \in \mathbb{R}^2\) a Lipschitz function of \((u_1, u_2)\) with respect to the \(\ell_\infty\) norm? That is, does it exist a constant \(C\) such that

\[
\max_{i \in \{1, 2\}} |V_i(u_1, u_2) - V_i(u'_1, u'_2)| \leq C \cdot \max_{j \in \{1, 2\}} \max_{a, b \in A} |u_j(a, b) - u'_j(a, b)|?
\]

The answer is no in general. Consider two games with payoff matrix pairs

\[
(u_1, u_2) = \begin{pmatrix} 1 + \epsilon, -1 - \epsilon \\ -1, 0 \end{pmatrix} \quad \text{and} \quad (u'_1, u'_2) = \begin{pmatrix} 1 - \epsilon, -1 + \epsilon \\ -\epsilon, 1 \end{pmatrix},
\]

where \(\epsilon > 0\) can be arbitrarily small. Note that the two pairs of payoff matrices \((u_1, u_2)\) and \((u'_1, u'_2)\) can be made arbitrarily close. Both games have a unique CCE, with values

\[
(V_1(u_1, u_2), V_2(u_1, u_2)) = (1 + \epsilon, -1 - \epsilon) \quad \text{and} \quad (V_1(u'_1, u'_2), V_2(u'_1, u'_2)) = (0, 0).
\]

This shows that \((V_1, V_2)\) are not Lipschitz in the payoff matrices.\(^6\)

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\(^6\)We learned the example from https://mathoverflow.net/questions/347366/perturbation-of-the-value-of-a-general-sum-game-at-a-equilibirum
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