HIGH MULTIPLICITY AND COMPLEXITY OF THE
BIFURCATION DIAGRAMS OF LARGE SOLUTIONS FOR A
CLASS OF SUPERLINEAR INDEFINITE PROBLEMS

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Abstract. This paper analyzes the existence and structure of the positive
solutions of a very simple superlinear indefinite semilinear elliptic prototype
model under non-homogeneous boundary conditions, measured by $M \leq \infty$.
Rather strikingly, there are ranges of values of the parameters involved in its
setting for which the model admits an arbitrarily large number of positive
solutions, as a result of their fast oscillatory behavior, for sufficiently large $M$.
Further, using the amplitude of the superlinear term as the main bifurcation
parameter, we can ascertain the global bifurcation diagram of the positive
solutions. This seems to be the first work where these multiplicity results have
been documented.

1. Introduction. This paper analyzes the problem of the existence and the multi-
plicity of positive solutions for the following one dimensional boundary value prob-
lem

$$\begin{cases}
-u'' = \lambda u + a(t)u^p & \text{in } (0, 1) \\
u(0) = u(1) = M
\end{cases}$$

(1)

where $M \in (0, \infty]$ and $p > 1$. The coefficient $\lambda \in \mathbb{R}$ is regarded as a real parameter,
and $a(t)$ is the symmetric piecewise constant function defined by

$$a(t) = \begin{cases}
-c & \text{if } t \in [0, \alpha) \cup (1 - \alpha, 1] \\
b & \text{if } t \in [\alpha, 1 - \alpha]
\end{cases}$$

with $\alpha \in (0, 0.5)$, $b \geq 0$ and $c > 0$. Hence, all solutions throughout this paper are
positive ones. Thanks to the maximum principle, any solution $u \geq 0$, $u \neq 0$, of (1)
must satisfy $u(t) > 0$ for all $t \in [0, 1]$.

When $M = \infty$, the solutions of (1) are referred to as large solutions, or explosive
solutions, of

$$-u'' = \lambda u + a(t)u^p$$

(2)

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in \((0, 1)\). In such case, the boundary condition should be understood as
\[
\lim_{t \downarrow 0} u(t) = \infty, \quad \lim_{t \uparrow 1} u(t) = \infty,
\]
or, shortly, as
\[
u(0) = \nu(1) = \infty.
\]
Naturally, the limits as \(M \uparrow \infty\) of the positive solutions of (1) should provide us with large solutions of (2) in \((0, 1)\). Consequently, both problems, the classical one when \(M < \infty\) and the singular boundary value problem when \(M = \infty\), are closely related.

In the very special case when \(b = 0\), (1) is a simple sublinear boundary value problem, while if \(b > 0\), then the nonlinearity of (1) changes sign in \((0, 1)\) and (1) becomes a superlinear indefinite boundary value problem, which is singular, if, in addition, \(M = \infty\). Although there is a huge amount of literature related to the sublinear problem, the problem of analyzing (1) in the general case when \(b > 0\) and \(0 < M \leq \infty\) is of a certain novelty, in spite of its simple formulation. Undoubtedly, its mathematical analysis is imperative for ascertaining the behavior of wider classes of indefinite singular boundary value problems.

More precisely, as a very special case of J. M. Fraile et al. [8], if \(b = M = 0\), then (1) possesses a unique solution (necessarily, symmetric around 0.5) if, and only if,
\[
\pi^2 < \lambda < \lambda_\alpha := \left(\frac{\pi}{1 - 2\alpha}\right)^2.
\]
Moreover, if we denote it by \(u_\lambda\), it turns out that \(u_\lambda\) bifurcates from 0 at \(\lambda = \pi^2\), and that it bifurcates from infinity at \(\lambda_\alpha\). Actually,
\[
\lim_{\lambda \uparrow \lambda_\alpha} u_\lambda(t) = \begin{cases} \ell_\alpha(t), & t \in [0, \alpha), \\ \infty, & t \in [\alpha, 0.5], \end{cases}
\]
where \(\ell_\alpha\) stands for the unique solution of the singular problem
\[
\begin{cases}
-u'' = \lambda_\alpha u - cu^p \quad \text{in} \ (0, \alpha) \\
u(0) = 0, \quad u(\alpha) = \infty
\end{cases}
\]
(see J. García-Melián et al. [10], J. López-Gómez and J. C. Sabina [20] and J. López-Gómez [16, 18], if necessary). The same result holds for all \(M > 0\) if \(b = 0\), but, in this case, \(u_\lambda\) is defined not only for \(\lambda > \pi^2\) but for all \(\lambda < \lambda_\alpha\) (see [18]).

The previous behavior changes drastically when \(b > 0\), even in the simplest case when \(M = 0\). Indeed, in such a case, it is well known there exists \(b^* > 0\) such that for every \(0 < b < b^*\) there is a unique
\[
\lambda_t = \lambda_t(b) \in (\pi^2, \lambda_\alpha)
\]
such that: a) (1) does not admit a solution if \(\lambda > \lambda_t(b)\); b) (1) admits, at least, one solution if \(\lambda \leq \lambda_t(b)\); and c) (1) admits at least two solutions for each \(\lambda \in (\pi^2, \lambda_t(b))\). Moreover,
\[
\lim_{b \downarrow 0} \lambda_t(b) = \lambda_\alpha \quad \text{and} \quad \lim_{b \uparrow b^*} \lambda_t(b) = \pi^2.
\]
On the other hand, if \(b \geq b^*\), then (1) admits a positive solution if, and only if, \(\lambda \leq \pi^2\). These results are direct consequences of the general theory developed in J. López-Gómez [14], H. Amann and J. López-Gómez [2], and R. Gómez-Reñasco and J. López-Gómez [11, 12], where some pioneering findings by H. Berestycki et al. [3, 4], and S. Alama and G. Tarantello [1] were substantially sharpened.
Rather strikingly, although (1) looks so simple, in the general case when $b > 0$ and $0 < M \leq \infty$ there are very few results concerning the global structure of the solution set of (1). Among them, J. Mawhin, D. Papini and F. Zanolin [21] found some multiplicity results of sign-changing solutions, J. López-Gómez [17] established the existence and global attractive character of the minimal solution of (1) when $M = \infty$ for sufficiently small $b > 0$, and, more recently, J. García-Melián established the general shape of the bifurcation diagram of a general multidimensional prototype of (1) for $M = \infty$ and $\lambda = 0$ using $b$ as the main bifurcation parameter. The device of using $b$ as a parameter in the context of indefinite superlinear problems goes back, at least, to J. López-Gómez [15].

The main goal of this paper is ascertaining the global bifurcation diagrams of (1) for $-\lambda > 0$ sufficiently large, using $b$ as the main bifurcation parameter, and obtaining quasi-optimal multiplicity results in the appropriate ranges of values of the secondary parameter $\lambda < 0$.

As a result of the non-homogeneous boundary conditions, measured by $M \in (0, \infty]$, it turns out that there exists $m^* > 0$ such that (1) may exhibit an arbitrarily large number of solutions for $M > m^*$ and sufficiently large $-\lambda > 0$ provided

$$b = -\lambda/(u(\alpha))^{p-1},$$

where $u(t)$ stands for the unique positive solution of

$$\begin{aligned}
-\gamma'' &= \lambda u - cw^p & \text{in } (0, \alpha) \\
u(0) &= \infty, \ u'(\alpha) = 0.
\end{aligned}$$

Up to the best of our knowledge, this multiplicity result has been observed in this paper for the first time. Based on it, we further perform a global continuation in the parameter $b$ to construct all admissible bifurcation diagrams of (1), using $b$ as the main parameter, at the values of $\lambda < 0$ where the multiplicity result holds.

The main tools which allow us to produce very precise bifurcation diagrams for equation (1) are based on a careful and detailed analysis of the time-maps associated to the corresponding phase plane systems. Analogous approaches for different kind of problems have been previously considered by Harris [13] and Dambrosio [7] in connection with non-homogeneous boundary value problems of the form

$$\begin{aligned}
-\gamma'' &= f(u) + h & \text{in } (0, 1) \\
u(0) &= A, \ u(1) = B.
\end{aligned}$$

In [13] the author studies the case of jumping nonlinearities $f(u)/u \to C, D$ as $u \to \pm \infty$, while in [7] the superlinear case $f(u)/u \to +\infty$ as $u \to \pm \infty$, is discussed. The main difference of our study with respect to previous ones like [7, 13] concerns the fact that in our situation we deal with an indefinite superlinear problem, due to the change of sign in the weight $a(t)$ and, furthermore, we also show the crucial role played by the parameter $\lambda$ when $\lambda < 0$. Such facts produce multiplicity results of positive solutions which are completely different and not comparable to the previous ones. Precisely, for problem (3) in the superlinear case one obtains a large number of sign changing oscillatory solutions (see [7]) and, as proved in [21], such kind of strong oscillatory behavior persists also for blowing-up solutions of $-\gamma'' = a(t)f(u)$, when $a(t)$ (like in our case) is negative in a neighborhood of $t = 0, 1$ and positive elsewhere, but these solutions were not, in general, positive, as it occurs in this paper, where we find for problem (1) a new and broad complementary class of multiple large solutions which are positive and oscillate around a positive level. Such multiplicity result relies on the fast oscillation of the solutions of (1) for sufficiently negative
The topological structure of the bifurcation curves associated to such positive oscillatory solutions, in dependence of the parameter $b$, is astonishingly rich and somehow surprising. Moreover, differently from [7, 13], we can treat in the same framework both the cases $M \in \mathbb{R}$ and $M = +\infty$. This gives us the advantage to obtain some new multiplicity results for positive blow-up solutions as well.

As we are describing a new phenomenon in the context of semilinear elliptic problems, which might have crucial implications in a number of fields, in this paper we focus our attention in the simplest prototype model (1), because we are looking for much completeness and depth as possible.

Except for the last section, where we adapt most of the previous results to the special case when $M = \infty$, this paper focuses into the case $M < \infty$, and it is distributed as follows.

Section 2 analyzes the restriction of (1) to the intervals $(0, \alpha)$ and $(1-\alpha, 1)$, where (1) is a sublinear problem. As far as concerns to $(0, \alpha)$, it characterizes the fine structure of the set of solutions of

$$\begin{cases} -u'' = \lambda u - cu^p & \text{in } (0, \alpha), \\ u(0) = M. \end{cases}$$

More precisely, if we denote by $\Sigma_0$ the set of solutions of (4), Section 2 shows that

$$\Gamma_{0,M} := \{(u(\alpha), u'(\alpha)) : u \in \Sigma_0 \}$$

is a differentiable strictly increasing curve. Similarly, if we denote by $\Sigma_1$ the set of solutions of

$$\begin{cases} -u'' = \lambda u - cu^p & \text{in } (1-\alpha, 1), \\ u(1) = M, \end{cases}$$

then,

$$\Gamma_{1,M} := \{(u(\alpha), u'(\alpha)) : u \in \Sigma_1 \}$$

is a differentiable decreasing curve. Actually, by the symmetries of (1), $\Gamma_{0,M}$ must be the reflection around the $u$-axis of $\Gamma_{1,M}$. The interest of these curves relies on the fact that the solutions of (1) restricted to the central interval $(\alpha, 1-\alpha)$ must be solutions of

$$-u'' = \lambda u + bu^p$$

linking $\Gamma_{0,M}$ with $\Gamma_{1,M}$ in a time $1-2\alpha$. Conversely, any solution of (5) satisfying this property provides us with a solution of (1). This technical device goes back to J. Mawhin, D. Papini and F. Zanolin [21].

Section 3 gives the main multiplicity result of this paper through a systematic use of phase portrait techniques. Section 4 introduces a series of Poincaré maps which will provide us with the local and global bifurcation diagrams in $b$ of the positive solutions of (1). Section 5 gives a series of global properties of these diagrams which are going to be pivotal in constructing all the global bifurcation diagrams of Section 6. Finally, in Section 7 we adapt most of the previous results to cover the singular case when $M = \infty$.

2. The problem (1) in the interval $[0, \alpha]$. In the interval $[0, \alpha]$ the equation (2) reduces to

$$-u'' = \lambda u - cu^p,$$
which is autonomous, and, hence, phase portrait techniques can be applied through the equivalent first order system
\[
\begin{cases}
  u' = v \\
  v' = -\lambda u + cu^p
\end{cases}
\] (7)

which admits the first integral
\[
\phi(u, v) = v^2 + \lambda u^2 - \frac{2c}{p+1} u^{p+1}.
\]

The main result of this section reveals the structure of the positive solutions of the Cauchy problem
\[
\begin{cases}
  -u'' = \lambda u - cu^p \\
  u(0) = M , \\ u'(0) = v \\
\end{cases}
\] (8)
in the interval \([0, \alpha]\), for sufficiently large \(M > 0\). Obviously, these solutions include the restrictions to \([0, \alpha]\) of the positive solutions of (1). It can be stated as follows.

**Theorem 2.1.** There exists a unique value of \(m\), denoted by \(m^*\), for which the unique solution of the Cauchy problem
\[
\begin{cases}
  -u'' = \lambda u - cu^p, \\
  u(0) = m , \\ u'(0) = 0 \\
\end{cases}
\] (9)
satisfies
\[
u(t) > 0 \quad \text{for all} \quad t \in [0, \alpha) \quad \text{and} \quad \lim_{t \uparrow \alpha} u(t) = \infty.\]

For any \(M > m^*\), there exist \(v_* < v^* < 0\) satisfying the following properties:

i) For every \(v \in (v_*, v^*)\) the unique solution of (8) satisfies \(u(t) > 0\) for all \(t \in [0, \alpha]\).

ii) Let \(u_*\) denote the unique solution of (8) with \(v = v_*\). Then, \(u_*(t) > 0\) for all \(t \in [0, \alpha]\) and \(u_*(\alpha) = 0\).

iii) Let \(u^*\) denote the unique solution of (8) with \(v = v^*\). Then, \(u^*(t) > 0\) for all \(t \in [0, \alpha]\) and \(\lim_{t \uparrow \alpha} u^*(t) = \infty\).

iv) For every \(v > v^*\), there exists \(T < \alpha\) such that the solution of (8) satisfies \(u(t) > 0\) for all \(t \in [0, T]\) and
\[
\lim_{t \uparrow T} u(t) = \infty.
\]

v) For every \(v < v_*\), there exists \(T_0 < \alpha\) such that the solution of (8) satisfies
\[
u(t) > 0 \quad \text{for all} \quad t \in [0, T_0) \quad \text{and} \quad u(T_0) = 0.
\]

Consequently, the candidates to provide us with (positive) solutions of (1) are those of (8) with \(v_* < v < v^*\).

**Proof.** The proof will be divided into several steps.

**Step 1:** Essentially, this step shows the existence and uniqueness of \(m^*\). First, we focus our attention in case \(\lambda > 0\). So, suppose \(\lambda > 0\). Then, (7) has two non-negative equilibria. Namely, \((0, 0)\) and \((\omega, 0)\), where
\[
\omega := \left(\frac{\lambda c}{2} \right)^{\frac{1}{p+1}}.
\]

As the generalized potential energy
\[
\varphi(u) := \lambda u^2 - \frac{2c}{p+1} u^{p+1}
\]
has a quadratic minimum at 0 and a quadratic maximum at \( \omega \), \((0,0)\) must be a nonlinear center, while \((\omega,0)\) is a saddle point of (7). Consequently, the phase portrait of the non-negative solutions of (6) looks like shown in Figure 1.

**Figure 1.** Phase diagram of (6) for \( \lambda > 0 \)

By simply looking at Figure 1, it becomes apparent that the solution of the problem (9) cannot blow up at time \( t = \alpha \) if \( m \leq \omega \). So, suppose \( m > \omega \) and denote by

\[
t_{\text{max}} = t_{\text{max}}(m)
\]

the global existence time of the solution of (9). Integrating the differential equation we find that, for every \( t \in [0, t_{\text{max}}) \),

\[
t = \int_{1}^{u(t)/m} \frac{d\theta}{\sqrt{\frac{2c_p}{p+1} m^{p-1} (\theta^{p+1} - 1) - \lambda (\theta^2 - 1)}}
\]

(10)

and, hence,

\[
t \leq \int_{1}^{\infty} \frac{d\theta}{\sqrt{\frac{2c_p}{p+1} m^{p-1} (\theta^{p+1} - 1) - \lambda (\theta^2 - 1)}} < \infty.
\]

Thus, \( t_{\text{max}} < \infty \) and, therefore, \( u \) blows up at a finite time. Moreover, letting \( t \uparrow t_{\text{max}} \) in (10), it becomes apparent that

\[
t_{\text{max}}(m) = \int_{1}^{\infty} \frac{d\theta}{\sqrt{\frac{2c_p}{p+1} m^{p-1} (\theta^{p+1} - 1) - \lambda (\theta^2 - 1)}}.
\]

(11)

According to (11), \( t_{\text{max}}(m) \) is decreasing with respect to \( m \). Moreover, by continuous dependence,

\[
\lim_{m \downarrow \omega} t_{\text{max}}(m) = \infty,
\]

because \((\omega,0)\) is an equilibrium, and letting \( m \uparrow \infty \) in (11) leads to

\[
\lim_{m \uparrow \infty} t_{\text{max}}(m) = 0.
\]
Therefore, there exists a unique \( m^* (> \omega) \) such that

\[ t_{\text{max}}(m^*) = \alpha. \]

By construction,

\[
 t_{\text{max}}(m) \left\{ \begin{array}{ll}
 < t_{\text{max}}(m^*) = \alpha & \text{if } m > m^*, \\
 > t_{\text{max}}(m^*) = \alpha & \text{if } m < m^*. 
\end{array} \right.
\] (12)

This shows the existence and the uniqueness of \( m^* \) when \( \lambda > 0 \). The previous proof can be easily adapted to cover the general case when \( \lambda \leq 0 \). In such case, the phase diagram looks like shown in Figure 2 and the proof of the case \( \lambda > 0 \) can be adapted almost \textit{mutatis mutandis} to cover this case. So, the technical details of the proof in this special case are omitted here in.

\textbf{Figure 2.} Phase diagram of (6) in case \( \lambda \leq 0 \)

According to (12), the unique solution of (9) blows up in a time \( t_{\text{max}}(m) \leq \alpha \) if \( m \geq m^* \), while it is globally defined in the time interval \([0, \alpha]\) if \( m < m^* \).

\textbf{Step 2:} Let \( M > m^* \) be and denote by \( v_u > 0 \) the unique value of the derivative \( v = u' \) for which \((M, v_u)\) lies on the unstable manifold of \((\omega, 0)\) (resp. \((0, 0)\)) if \( \lambda > 0 \) (resp. \( \lambda \leq 0 \)). By symmetry, \((M, -v_u)\) is the unique point on the stable manifold of that equilibrium with \( u = M \). Subsequently, for every \( v \in (-v_u, 0) \), we consider the Cauchy problem (8). According to the phase portrait, it is apparent that the solution of (8) needs a time, say \( t_{\text{min}} := t_{\text{min}}(v) \), to reach its minimum, denoted by \( m := m(v) \). So, by symmetry,

\[
m = u(t_{\text{min}}) \in (0, M), \quad u'(t_{\text{min}}) = 0, \quad u(2t_{\text{min}}) = M.
\]

Actually, \( m > \omega \) if \( \lambda > 0 \). Step 2 shows that there exists a unique \( v_0 \in (-v_u, 0) \) such that

\[
 t_{\text{min}}(v_0) = \alpha \quad \text{and} \quad t_{\text{min}}(v) \left\{ \begin{array}{ll}
 > \alpha & \text{if } v < v_0, \\
 < \alpha & \text{if } v > v_0. 
\end{array} \right.
\] (13)

As a byproduct, the boundary value problem

\[
\begin{cases}
-u'' = \lambda u - cu^p, \\
u(0) = M, \ u'(\alpha) = 0,
\end{cases}
\]

possesses a unique solution. Namely, the solution of (8) with \( v = v_0 \).
Indeed, by simply looking at the phase portrait of (6), it becomes apparent that the map \( m : (-v, 0) \to \mathbb{R}^+ \) is increasing, since two different trajectories cannot meet. Moreover,

\[
\lim_{v \uparrow 0} m(v) = M \quad \text{and} \quad \lim_{v \downarrow -v} m(v) = \begin{cases} \omega & \text{if } \lambda > 0, \\ 0 & \text{if } \lambda \leq 0. \end{cases}
\]

On the other hand, by integrating the differential equation, we are easily driven to the identity

\[
t_{\min}(v) = \int_1^{M(m(v))} \frac{d\theta}{\sqrt{2 c p + 1 m^p - 1 m^{p+1} - \lambda (\theta^2 - 1)}}.
\]

Consequently, as \( m(v) \) increases with \( v \), \( t_{\min}(v) \) decreases. Moreover, \( t_{\min}(0) = 0 \), by construction, and

\[
\lim_{v \downarrow -v} t_{\min}(v) = \infty,
\]

by continuous dependence. Therefore, (13) holds for a unique \( v_0 \in (-v, 0) \). This ends the proof of Step 2.

Setting

\[
m_0 := m(v_0),
\]

it follows from (12) that

\[
m_0 < m^*,
\]

since \( t_{\max}(m_0) > \alpha \).

**Step 3.** This step constructs \( v_* \) and \( v^* \) and completes the proof. It should be remembered that, according to Step 2, the solutions of (8) need a time larger than \( \alpha \) to reach the \( u \)-axis if \( v \in (-v, v_0) \), while they do it before time \( \alpha \) if \( v \in (v_0, 0) \).

Next, we suppose \( M > m^* \) and consider the map

\[
T : (-v, 0) \to \mathbb{R}^+
\]

defined by

\[
T(v) := t_{\min}(v) + t_{\max}(m(v)), \quad v \in (-v, 0),
\]

which measures the blow-up time of the solution of (8). According to Steps 1 and 2, \( T \) is continuous and decreasing. Moreover, by continuous dependence,

\[
\lim_{v \downarrow -v} T(v) = \infty,
\]

and, due to (12),

\[
\lim_{v \uparrow 0} T(v) = t_{\max}(M) < \alpha,
\]

because \( M > m^* \). Thus, there exists a unique

\[
v^* := v^*(\lambda, c, p, M) \in (-v, 0)
\]

such that

\[
T(v^*) = \alpha \quad \text{and} \quad T(v) \begin{cases} > \alpha & \text{if } v \in (-v, v^*) \\ < \alpha & \text{if } v \in (v^*, 0). \end{cases}
\]

Consequently, the solutions of (8) blow up before time \( \alpha \) if \( v > v^* \), while they are globally defined in \([0, \alpha]\) if \( v \in (-v, v_0) \). Moreover, \( v = v^* \) is the unique shooting speed for which the solution of (8) blows up at time \( \alpha \).

Now, we will study the behavior of the solutions of (8) with \( v \leq -v \). By simply looking at the phase portrait of (6) (cf. Figures 1 and 2), it becomes apparent that such solutions vanish for some positive time if \( v < -v \), while they stabilize to \( \omega \).
(resp. 0) if \( \lambda > 0 \) (resp. \( \lambda \leq 0 \)) and \( v = -v_u \). Integrating the differential equation, we find that the necessary time to reach the \( v \)-axis when \( v < -v_u \), denoted by \( T_0(v) \), is given through the formula

\[
T_0(v) = \int_0^M \frac{du}{\sqrt{v^2 + \frac{2c_p}{p+1}(u^{p+1} - M^{p+1}) - \lambda(u^2 - M^2)}}.
\]

Clearly, \( T_0 \) is a continuous and increasing function of \( v \), since \( v^2 \) decreases if \( v < 0 \) grows. Moreover, by continuous dependence,

\[
\lim_{v \uparrow -v_u} T_0(v) = \infty,
\]

because \( -v_u \) is the critical shooting speed of the stable manifold of the equilibrium. Also, letting \( v \downarrow -\infty \), shows that

\[
\lim_{v \downarrow -\infty} T_0(v) = 0.
\]

Therefore, there exists a unique \( v_* < -v_u \) such that

\[
T_0(v_*) = \alpha.
\]

Necessarily, by the definition of \( v_0 \) (cf. (13)),

\[
v_* < v_0 < v^*,
\]

because the solution of (8) for \( v = v_0 \) is positive in \([0, \alpha]\). From these features, the proof of the theorem can be easily completed.

Figure 3 illustrates all the possible behaviors of the solutions of problem (8) according to the different ranges of values of the shooting speed \( v \) as discussed in Theorem 2.1. The profiles on the first row correspond to two solution plots of (8) for \( v < v_* \) and \( v = v_* \), respectively. By Theorem 2.1, \( v_* \) is the unique value of \( v \) for which the solution \( u \) of (8) satisfies \( u(\alpha) = 0 \). If \( v < v_* \), then \( u \) vanishes at some \( t_0 < \alpha \).

The profiles on the second row are two solution plots for \( v \in (v_*, v_0) \) and \( v = v_0 \), respectively. According to (13), \( v_0 \) is the unique value of \( v \) for which the solution \( u \) of (8) satisfies \( u'(\alpha) = 0 \), and, thanks to (14),

\[
m_0 = u(\alpha).
\]

By (13), \( t_{\min}(v) > \alpha \) if \( v < v_0 \), and, hence, \( u'(\alpha) < 0 \) if \( v < v_0 \), as illustrated by the first plot of the second row of Figure 3.

The third row of Figure 3 shows the plots of two (different) solutions of (8) for two values of \( v \in (v_0, v^*) \). Finally, in the fourth row we have represented the plots of the solutions of (8) for \( v = v^* \) and some \( v > v^* \), respectively. In the first case, the solution blows up at time \( t = \alpha \), whereas the solution blows up at some \( T < \alpha \) if \( v > v^* \).

Throughout the rest of this paper, for every \( \lambda \in \mathbb{R} \) and \( M > m^* \) we denote by \( \Sigma_0 \) the set of solutions of (8) with \( v \in [v_*, v^*] \) and consider the set of points of \( \mathbb{R}^2 \) reached by the solutions of (8) at time \( \alpha \) as the shooting speed \( v \) ranges in between \( v_* \) and \( v^* \),

\[
\Gamma_0 := \{ (u(\alpha), u'(\alpha)), \quad u \in \Sigma_0 \}.
\]

The next result collects the main features of \( \Gamma_0 \).

**Theorem 2.2.** For every \( \lambda \in \mathbb{R} \) and \( M > m^* \), \( \Gamma_0 \) is a \( C^1 \)-curve in \( \mathbb{R}^+ \times \mathbb{R} \) such that:
Figure 3. Plots of the solutions of (8) according to the value of $v$.

i) $\pi_u(\Gamma_0) = \mathbb{R}^+$, where $\pi_u$ stands for the projection of $\mathbb{R}^2$ on the first component.

ii) The value of $m_0$ defined by (14) satisfies the following properties:

$(u, v) \in \Gamma_0$, $u < m_0 \implies v < 0$,

$(u, v) \in \Gamma_0$, $u > m_0 \implies v > 0$.

Note that, thanks to the proof of Theorem 2.1, $(m_0, 0) \in \Gamma_0$.

Proof. Consider the map

$S : [v_*, v^*] \rightarrow \Sigma_0,$

where, for every $v \in [v_*, v^*)$, $S(v)$ is the unique solution of (8), and the Poincaré map

$\mathcal{P} : [v_*, v^*) \rightarrow \mathbb{R}^+ \times \mathbb{R}$

defined through

$\mathcal{P}(v) := (u(\alpha), u'(\alpha)),$

where $u := S(v) \in \Sigma_0$.

Thanks to the proof of Theorem 2.1, $\mathcal{P}$ is well defined and

$\Gamma_0 = \mathcal{P}([v_*, v^*)) = \text{Im} \mathcal{P}$.
According to the theorem of differentiation of G. Peano, $\mathcal{P}$ provides us with a diffeomorphism of $[v_*, v^*]$ onto $\text{Im} \mathcal{P}$. Therefore, $\mathcal{P}$ establishes a diffeomorphism between $[v_*, v^*]$ and $\Gamma_0$. Consequently, $\Gamma_0$ is a curve of class $C^1$ in $\mathbb{R}^+ \times \mathbb{R}$.

As $\pi_u$ is a continuous map, $\pi_u(\Gamma_0)$ must be an interval and, since

$$\lim_{v \downarrow v_*} u(\alpha) = 0, \quad \lim_{v \uparrow v_*} u(\alpha) = \infty,$$

we can infer that $\pi_u(\Gamma_0) = \mathbb{R}^+$, which ends the proof of Part i). Part ii) is an easy consequence of (13) and (14). This ends the proof.

By performing the change of temporal scale

$s = 1 - t, \quad t \in [1 - \alpha, 1].$

one can easily infer the next counterpart of Theorem 2.1.

**Corollary 1.** Let $m^*$ be the value constructed by Theorem 2.1 and suppose $M > m^*$. Then, the unique solution of

$$\begin{cases}
-u'' = \lambda u - cu^p, \\
u(1) = m^*, \quad u'(1) = 0,
\end{cases}$$

satisfies

$$u(t) > 0 \quad \text{for all} \quad t \in (1 - \alpha, 1) \quad \text{and} \quad \lim_{t \downarrow 1 - \alpha} u(t) = \infty.$$

Moreover:

i) For every $v \in (-v^*, -v_*)$ the unique solution of

$$\begin{cases}
-u'' = \lambda u - cu^p, \\
u(1) = M, \quad u'(1) = v,
\end{cases}$$

satisfies $u(t) > 0$ for all $t \in [1 - \alpha, 1]$.

ii) Let $u_*$ denote the unique solution of (15) with $v = -v_*$. Then, $u_*(t) > 0$ for all $t \in (1 - \alpha, 1]$ and $u_*(1 - \alpha) = 0$.

iii) Let $u^*$ denote the unique solution of (15) with $v = -v^*$. Then, $u^*(t) > 0$ for all $t \in (1 - \alpha, 1]$ and $\lim_{t \downarrow 1 - \alpha} u^*(t) = \infty$.

iv) For every $v < -v^*$, there exists $t_{\max} < \alpha$ such that the solution of (15) satisfies $u(t) > 0$ for all $t \in (1 - t_{\max}, 1]$ and

$$\lim_{t \downarrow 1 - t_{\max}} u(t) = \infty.$$

v) For every $v > -v_*$, there exists $t_0 < \alpha$ such that the solution of (15) satisfies $u(t) > 0$ for all $t \in (1 - t_0, 1]$ and $u(1 - t_0) = 0$.

Consequently, the candidates to provide us with positive solutions of (1) are those of (15) with $-v^* < v < -v_*$.

Throughout the rest of this paper, for every $\lambda \in \mathbb{R}$ and $M > m^*$ we denote by $\Sigma_1$ the set of solutions of (15) with $v \in (-v^*, -v_*)$ and consider the set of points of $\mathbb{R}^2$ reached by the solutions of (15) at time $1 - \alpha$ as $v$ ranges in between $-v^*$ and $-v_*$,

$$\Gamma_1 := \{ (u(1 - \alpha), u'(1 - \alpha)), \quad u \in \Sigma_1 \}.$$
Corollary 2. For every $\lambda \in \mathbb{R}$ and $M > m^*$, $\Gamma_1$ is a $C^1$-curve of $\mathbb{R}^+ \times \mathbb{R}$ such that
$$\Gamma_1 = \{(u, -v) : (u,v) \in \Gamma_0\}.$$  
In particular, $\pi_u(\Gamma_1) = \mathbb{R}^+$ and
$$(u,v) \in \Gamma_1, \quad u < m_0 \implies v > 0,$$
$$(u,v) \in \Gamma_1, \quad u > m_0 \implies v < 0.$$  

The next result establishes that $\Gamma_0$ is an increasing arc of curve with respect to $u$.

Proposition 1. For every $\lambda \in \mathbb{R}$, $M > m^*$ and $t \in (0, \alpha]$, the following holds
$$v_* \leq v_1 < v_2 < v_*^* \implies u_1(t) < u_2(t) \quad \text{and} \quad u_1'(t) < u_2'(t),$$
where $u_i$ stands for the unique solution of
$$\begin{cases} -u'' = \lambda u - cu^p \\ u(0) = M, \quad u'(0) = v_i \in \mathbb{R}, \end{cases} \quad i \in \{1, 2\}.$$  

Proof. First, we will show the monotonicity of $u$. Suppose $u_1(\tilde{t}) > u_2(\tilde{t})$ for some $0 < \tilde{t} \leq \alpha$. Then, as $u_1(t) < u_2(t)$ for sufficiently small $t > 0$, there exists $\theta \in (0, \tilde{t}]$ such that
$$u_1(\theta) = u_2(\theta)$$
and, hence, setting $N := u_1(\theta)$, $u_1$ and $u_2$ are solutions of the boundary value problem
$$\begin{cases} -u'' = \lambda u - cu^p \\ u(0) = M, \quad u(\theta) = N. \end{cases} \quad (16)$$  

According to the main theorem of S. Cano-Casanova [6], (16) has a unique positive solution in $[0, \theta]$. Therefore, $u_1 = u_2$ in $[0, \theta]$, which implies $v_1 = v_2$. As this is a contradiction, necessarily $u_1(t) < u_2(t)$ for all $t \in (0, \alpha]$.  

Next, suppose that $u_1'(\tilde{t}) > u_2'(\tilde{t})$ for some $\tilde{t} \in (0, \alpha]$. Then, as
$$u_1'(0) = v_1 < u_2'(0) = v_2,$$
there exists $\theta \in (0, \tilde{t}]$ such that
$$u_1'(\theta) = u_2'(\theta).$$
Setting $N := u_1'(\theta)$, it is apparent that $u_1$ and $u_2$ solve
$$\begin{cases} -u'' = \lambda u - cu^p \\ u(0) = M, \quad u'(\theta) = N, \end{cases}$$
and, owing to S. Cano-Casanova [6], we find that $u_1 = u_2$ and, therefore, $v_1 = v_2$, which is impossible. This completes the proof. \hfill $\square$  

In Figure 4 we have represented two admissible curves $\Gamma_0$ and $\Gamma_1$ according to Theorem 2.2, Corollary 2 and Proposition 1. By Corollary 2, $\Gamma_1$ is the reflection of $\Gamma_0$ around the $u$-axis.  

Our interest in the curves $\Gamma_0$ and $\Gamma_1$ comes from the fact that the solutions of the nonlinear boundary value problem (1) are given through the solutions of the autonomous equation
$$-u'' = \lambda u + bu^p$$
connecting the curves $\Gamma_0$ and $\Gamma_1$ in the phase portrait of $(u, u')$ in time $1 - 2\alpha$. Note that $1 - 2\alpha$ is the length of the subinterval $(\alpha, 1 - \alpha)$ of $(0, 1)$ where (2) becomes
Figure 4. The curves $\Gamma_0$ and $\Gamma_1$.

More precisely, let $(u_i, v_i) \in \Gamma_i$, $i = 0, 1$, for which there exists a solution of (17), say $u_c$, such that

$$(u_c(\alpha), u'_c(\alpha)) = (u_0, v_0), \quad (u_c(1 - \alpha), u'_c(1 - \alpha)) = (u_1, v_1),$$

and let $u_\ell$ and $u_r$ be the solutions of (6) in $[0, \alpha]$ and $[1 - \alpha, 1]$ such that

$$u_\ell(0) = M, \quad (u_\ell(\alpha), u'_\ell(\alpha)) = (u_0, v_0),$$

and

$$(u_r(1 - \alpha), u'_r(1 - \alpha)) = (u_1, v_1), \quad u_r(1) = M,$$

respectively. Then, the function

$$u(t) := \begin{cases} u_\ell(t), & \text{if } t \in [0, \alpha], \\ u_c(t), & \text{if } t \in (\alpha, 1 - \alpha), \\ u_r(t), & \text{if } t \in [1 - \alpha, 1], \end{cases}$$

provides us with a solution of the superlinear indefinite problem (1).

This section concludes with a further fundamental property of $\Gamma_0$ which will be used throughout the rest of this paper.

**Theorem 2.3.** Suppose $\lambda \leq 0$ and $M > m^*$. Then, there exists a unique function of class $C^1$, $y : [0, \infty) \to \mathbb{R}$, such that

$$\Gamma_0 := \{ (x, y(x)) : x \geq 0 \}.$$

Moreover,

$$y'(x) > 0 \quad \text{for all } x \geq 0.$$

Actually, if $(x, y(x)) = \mathcal{P}(v)$, $v \in [v_*, v^*]$, i.e., $x = \pi_u(\mathcal{P}(v))$, then,

$$y'(x) = \frac{\xi'(\alpha)}{\xi(\alpha)}.$$
where \( \xi \) stands for the unique solution of
\[
\begin{cases}
-\xi'' = (\lambda - cpv^{p-1}) \xi \\
\xi(0) = 0, \quad \xi'(0) = 1,
\end{cases}
\]
u being the solution of
\[
\begin{cases}
-u'' = \lambda u - cu^p \\
u(0) = M, \quad u'(0) = v.
\end{cases}
\]

Proof. Subsequently, the notations introduced in the proof of Theorem 2.2 will be maintained. By definition,
\[
P(v) = \left( S(v)(\alpha), \frac{d}{dt}S(v)(\alpha) \right)
\]
for all \( v \in [v_*, v^*] \), where \( S(v) \) is the unique solution of
\[
\begin{cases}
-\frac{d^2}{dt^2}S(v) = \lambda S(v) - c(S(v))^p \\
S(v)(0) = M, \quad \frac{d}{dt}S(v)(0) = v.
\end{cases}
\]
Thus, according to the theorem of differentiation of G. Peano, we have that
\[
DP(v) = \left( DS(v)(\alpha), \frac{d}{dt}DS(v)(\alpha) \right)
\]
for all \( v \in [v_*, v^*] \), where \( DS(v) \) is the unique solution of
\[
\begin{cases}
-\frac{d^2}{dt^2}DS(v) = (\lambda - cp(S(v))^{p-1}) DS(v) \\
DS(v)(0) = 0, \quad \frac{d}{dt}DS(v)(0) = 1.
\end{cases}
\]
As we are imposing \( \lambda \leq 0 \) and, due to Proposition 1,
\[ DS(v)(t) \geq 0 \]
for all \( t \in [0, \alpha] \) and \( v \in [v_*, v^*] \), we obtain that
\[
-\frac{d^2}{dt^2}DS(v) \leq 0 \quad \text{in } [0, \alpha]
\]
for all \( v \in [v_*, v^*] \). Consequently,
\[ DS(v)(\alpha) > 0 \quad \text{and} \quad \frac{d}{dt}DS(v)(\alpha) > 0. \]

Therefore, the two components of \( DP(v) \) are positive real numbers for all \( v \in [v_*, v^*] \). As \( DP(v) \) is the tangent vector to the curve \( \Gamma_0 \) at \( P(v) \) for all \( v \in [v_*, v^*] \), the rest of the proof follows easily from well known features on differential geometry of curves.

3. Multiplicity results for the superlinear indefinite problem. Throughout this section we impose that \( \lambda < 0 \) and \( b = b^* \), where
\[ b^* := -\lambda/m_0^{p-1}. \]
Under this condition, the following result holds.
Theorem 3.1. Suppose
\[ b = -\lambda/m_0^{p-1}. \] (18)
Then, the number of solutions of (1) grows to infinity as \( \lambda \downarrow -\infty \). More precisely, if
\[ \lambda < -\frac{4\lambda_\alpha}{p-1}n^2 \] for some integer \( n \geq 1 \), (19)
where we are denoting
\[ \lambda_\alpha := \left( \frac{\pi}{1-2\alpha} \right)^2, \]
then, (1) possesses, at least, \( 4n \) solutions; among them, \( 2n \) are symmetric and the remaining \( 2n \) are asymmetric.

Proof. As \( \lambda < 0 \), (17) has two equilibria. Namely, 0 and the positive steady-state
\[ \Omega := \left( \frac{-\lambda}{b} \right)^{\frac{p-1}{p}}. \]
According to (18), it turns out that
\[ \Omega = m_0. \]
The first order system associated to (17) admits the first integral
\[ \psi(u,v) := v^2 + \lambda u^2 + \frac{2b}{p+1}u^{p+1} \]
whose generalized potential energy is
\[ \varphi(u) := \lambda u^2 + \frac{2b}{p+1}u^{p+1}. \]
As \( \varphi \) has a quadratic maximum at 0 and a quadratic minimum at \( \Omega \), \((0,0)\) is a saddle point and \((\Omega,0)\) is a center. Moreover, as \( \varphi(u) \) is a potential well, there is an homoclinic connection of \((0,0)\) surrounding \((\Omega,0)\) and any periodic orbit around \((\Omega,0)\). Figure 5 sketches the phase portrait of (17) in case \( \lambda < 0 \), as well as the curves \( \Gamma_0 \) and \( \Gamma_1 \).

Naturally, by the definition of \( m_0 \) (cf. (14)), if \( u_\ell(t) \) stands for the unique solution of
\[ \begin{cases} 
-\lambda u'' = \lambda u - cw^p, \\
u(0) = M, \quad u(\alpha) = m_0,
\end{cases} \]
then, the function
\[ u_0(t) := \begin{cases} 
u_\ell(t), & t \in [0,\alpha], \\
m_0, & t \in (\alpha, 1-\alpha), \\
u_\ell(1-t), & t \in [1-\alpha,1],
\end{cases} \]
provides us with a symmetric solution of problem (1). Throughout the rest of the paper we will call this function the trivial solution.

To construct more solutions, one should note that the limiting period of the small amplitude periodic oscillations around \((\Omega,0)\), as the amplitude goes down to zero, equals the period of the solutions of the linearized equation
\[ -\lambda u'' = \lambda u + pb\Omega^{p-1}u = \lambda(1-p)u, \]
which is given through
\[ \tau_\Omega := \frac{2\pi}{\sqrt{\lambda(1-p)}}. \]
Having a glance at Figure 5 it becomes apparent from Proposition 1 that $\Gamma_0$ meets the homoclinic of $(0,0)$ twice. Let $(x_0^-, y_0^-)$ denote the second crossing point, as time passes by, between them. By Theorem 2.3, $y_0^- = y(x_0^-)$. Note that $x_0^- < \Omega$. For every $(x, y) \in \Gamma_0$ with $x \in (x_0^-, \Omega)$ (necessarily $y = y(x)$), consider the initial value problem

$$\begin{cases}
-u'' = \lambda u + bu^p, \\
u(0) = x, \ u'(0) = y(x).
\end{cases}$$

(20)

Let $\tau_1(x)$ denote the time taken by the trajectory of the solution of (20) to reach $\Gamma_1$ for the first time, and $\tau(x)$ the period of the solution of (20). By definition, $\tau_1(x) < \tau(x)$ for all $x \in (x_0^-, \Omega)$. Thus, by continuous dependence and taking into account that

$$\lim_{x \uparrow \Omega} \tau(x) = \tau_\Omega,$$

it is easily seen that

$$\lim_{x \downarrow x_0^-} \tau_1(x) = \infty \quad \text{and} \quad \limsup_{x \uparrow \Omega} \tau_1(x) \leq \tau_\Omega.$$  

(21)

Thus, if we assume that

$$\tau_\Omega = \frac{2\pi}{\sqrt{\lambda(1-p)}} < 1 - 2\alpha \quad \iff \quad \lambda < -\frac{4\lambda_0}{p - 1},$$

(22)

then, $\tau_1(x) < 1 - 2\alpha$ for some $x < \Omega$, $x \sim \Omega$, and, hence, by the continuity of $\tau_1$ in $(x_0^-, x)$ along $\Gamma_0$, there exists $(x_1, y_1) \in \Gamma_0$, $y_1 = y(x_1)$, with $x_1 \in (x_0^-, x)$ such that

$$\tau_1(x_1) = 1 - 2\alpha;$$

$\tau_1$ is continuous by the transversality of the trajectories with the curves $\Gamma_0$ and $\Gamma_1$. Consequently, reasoning as at the end of Section 2, the unique solution of (17) such
that

\[(u(\alpha), u'(\alpha)) = (x_1, y_1)\]

provides us with another symmetric solution of (1) having a unique critical point (a local minimum) in the interval \((\alpha, 1 - \alpha)\).

Now, for every \((x, y) \in \Gamma_0\) with \(x \in (x^{-\infty}, \Omega)\), let \(\tau_2(x)\) denote the necessary time to reach \(\Gamma_1\) exactly twice. As for (21), (22) implies that

\[\lim_{x \downarrow x^{-\infty}} \tau_2(x) = \infty \quad \text{and} \quad \limsup_{x \uparrow \Omega} \tau_2(x) \leq \tau_\Omega < 1 - 2\alpha\]

and, therefore, there exists \((x_2, y_2) \in \Gamma_0\) \((y_2 = y(x_2))\), with \(x_2 \in (x_0, \Omega)\), such that

\[\tau_2(x_2) = 1 - 2\alpha.\]

Consequently, the unique solution of (17) such that

\[(u(\alpha), u'(\alpha)) = (x_2, y_2)\]

provides us with an asymmetric solution of (1) under condition (22). This solution has two critical points in \((\alpha, 1 - \alpha)\): a local minimum and a local maximum. Naturally, if we denote it by \(u(t)\), then, the reflected function

\[\tilde{u}(t) := u(1 - t), \quad t \in [0, 1],\]

provides us with another asymmetric solution of (1). The associated orbit \((\tilde{u}, \tilde{u}')\) leaves \(\Gamma_0\) at \(\tilde{u}(\alpha) > m_0\) and meets \(\Gamma_1\) twice ending on it. This solution also has a local minimum and a local maximum.

The four solutions that we have just constructed have been represented in the first row of Figure 6. It should be noted that they do exist provided (22) holds, which has been emphasized in the first column of Figure 6, where the requested conditions for the existence of the solutions of the corresponding row are given. Except for the solutions of the first column, the remaining solutions of Figure 6 are constructed by adding \(n \geq 1\) laps around \((\Omega, 0)\) to the solution at the top of the corresponding column.

Figure 6 consists of 12 pictures and three conditions. Each of the pictures exhibits two crossing curves. As we move from the left to the right, one of them increases and the other decreases. The increasing one represents \(\Gamma_0\), whereas the decreasing one stands for \(\Gamma_1\). The arcs of curve connecting them stand for trajectories of solutions of (17). The number of arrows counts how many times the corresponding piece of trajectory between \(\Gamma_0\) and \(\Gamma_1\) is run. One should always start at \(\Gamma_0\) and end at \(\Gamma_1\).

Next, we will construct the first solution of the second column. Let \((x_0^+, y_0^+)\) be the first crossing point, as time passes by, between \(\Gamma_0\) and the homoclinic of \((0, 0)\). Obviously, \(\Omega < x_0^+\) and, thanks to Theorem 2.3, \(y_0^+ = y(x_0^+)\). For every \((x, y) \in \Gamma_0\) with \(x \in (\Omega, x_0^+)\), let \(\tau_3(x)\) denote the time needed by the solution of (20) to reach \(\Gamma_1\) for the first time after a complete lap. By definition, \(\tau_3 < 2\tau_\Omega\) and, arguing as above, it becomes apparent that

\[\lim_{x \uparrow x_0^+} \tau_3(x) = \infty \quad \text{and} \quad \limsup_{x \downarrow \Omega} \tau_3(x) \leq 2\tau_\Omega.\]

Consequently, under condition

\[2 \cdot \frac{2\pi}{\sqrt{\lambda(1 - p)}} < 1 - 2\alpha \quad \iff \quad \lambda < \frac{-4\lambda_\alpha}{p - 1^2},\]
Condition | Symmetric solutions | Asymmetric solutions
--- | --- | ---
$\tau_\Omega < 1 - 2\alpha$ | ![Symmetric solutions](image1) | ![Asymmetric solutions](image2)
$2\tau_\Omega < 1 - 2\alpha$ | ![Symmetric solutions](image3) | ![Asymmetric solutions](image4)
$3\tau_\Omega < 1 - 2\alpha$ | ![Symmetric solutions](image5) | ![Asymmetric solutions](image6)

Figure 6. The restriction of the solutions of (1) to $[\alpha, 1 - \alpha]$ when $b = b^*$

there exists $(x_3, y_3) \in \Gamma_0$ with $x_3 > \Omega$ such that $\tau_3(x_3) = 1 - 2\alpha$. As in all previous cases, the unique solution of (17) satisfying

$$(u(\alpha), u'(\alpha)) = (x_3, y_3)$$

provides us with a symmetric solution of (1) which has three critical points in $(\alpha, 1 - \alpha)$: two local maxima and one local minimum. The remaining solutions of the first column of Figure 6 are constructed from this one by adding an additional lap every time we pass from one row to the next one, as it happens with the remaining three columns. Consequently, under condition

$$n\tau_\Omega < 1 - 2\alpha \iff \lambda < -\frac{4\lambda_\alpha}{p - 1}n^2,$$

(1) possesses, at least, $4n$ solutions, $2n$ among them being symmetric (those on the first two columns), and the remaining $2n$ solutions (those on the last two columns) asymmetric. The proof is complete.

By Theorem 3.1, it is natural to introduce the following concept.

**Definition 3.2.** A solution $u$ of (1) is said to be of type $n \geq 0$, which can be shortly expressed by writing $u \in T_n$, if it has $n \geq 0$ strict critical points in the central interval $(\alpha, 1 - \alpha)$. According to this terminology, the trivial solution $u_0$ is of type 0, i.e., $u_0 \in T_0$.

The proof of Theorem 3.1 reveals that, under conditions (18) and (19), the problem (1) has, at least, one solution of type 1 and two solutions of type $j$, for every $j \in \{1, \ldots, n\}$, besides the trivial solution (cf. Figure 6).
3.1. **A sharp pivotal property of the time-map** \( \tau_1 \). The time map \( \tau_1 \) defined in the proof of Theorem 3.1 can be extended for every \( x > \Omega \) in a natural way. Indeed, for each \( x > \Omega \), \( \tau_1(x) \) is the minimal necessary time to reach \( \Gamma_1 \) by the solution of the problem (20), where \( y(x) \) is the function introduced by Theorem 2.3. The extended function \( \tau_1 \) is defined in \( (x^0, \infty) \setminus \{ \Omega \} \). The next result shows that the singularity \( \Omega \) can be overcome.

**Theorem 3.3.** Suppose \( \lambda < 0 \) and (18). Then,

\[
\lim_{x \uparrow \Omega} \tau_1(x) = \frac{2}{\sqrt{\lambda(1-p)}} \arctan \frac{y'(\Omega)}{\sqrt{\lambda(1-p)}} = \lim_{x \downarrow \Omega} \tau_1(x).
\]  
(23)

Consequently, \( \tau_1 \in C(u_0, \infty) \) if we extend \( \tau_1 \) by setting

\[
\tau_1(\Omega) := \frac{2}{\sqrt{\lambda(1-p)}} \arctan \frac{y'(\Omega)}{\sqrt{\lambda(1-p)}} > 0.
\]

Moreover,

\[
\lim_{x \downarrow x^0} \tau_1(x) = \infty, \quad \lim_{x \uparrow \infty} \tau_1(x) = 0.
\]

The value \( y'(\Omega) \) had already been calculated in Theorem 2.3.

**Proof.** Suppose \( x < \Omega = m_0 \), \( x \sim \Omega \), and let \( x_L < x < \Omega \) be the unique \( x_L = x_L(x) \) for which the trajectory of the solution of (20) reaches \( (x^0, 0) \) at a (minimal) time \( \tau_1(x)/2 \). As the nonlinearity of (20) is analytic at \( \Omega \), the differential equation of (20) can be equivalently written as follows

\[
-(u-\Omega)' = \lambda(1-p)(u-\Omega) + \sum_{j \geq 2} h_j(u-\Omega)^j, \quad u \sim \Omega,
\]  
(24)

for some coefficients \( h_j, j \geq 2 \), whose knowledge is not relevant in this proof. The series is absolutely convergent in some interval around \( \Omega \). Multiplying (24) by \( (u-\Omega)' = u' \) and rearranging terms, we are driven to

\[
\frac{d}{dt} \left\{ \frac{|u-\Omega|^2}{2} + \frac{\lambda(1-p)}{2}(u-\Omega)^2 + \sum_{j \geq 2} \frac{h_j}{j+1} (u-\Omega)^{j+1} \right\} = 0.
\]

Therefore, the orbit of the periodic solution of (20) is given through

\[
v^2 + \lambda(1-p)(u-\Omega)^2 + \sum_{j \geq 3} c_j(u-\Omega)^j = E,
\]

where

\[
v := u' = (u-\Omega)', \quad c_{j+1} := 2h_j/(j+1), \quad j \geq 2,
\]

and

\[
E = y^2(x) + \lambda(1-p)(x-\Omega)^2 + \sum_{j \geq 3} c_j(x-\Omega)^j.
\]

Consequently, shortening the notations by setting

\[
\Sigma_\xi(x) := \sum_{j \geq 3} c_j(x-\xi)^j, \quad x \sim \xi,
\]

it is easily seen that

\[
\tau_1(x) = 2 \int_{x_L}^x \frac{du}{\sqrt{y^2(x) + \lambda(1-p)(x-\Omega)^2 + \Sigma_\Omega(x) - \lambda(1-p)(u-\Omega)^2 - \Sigma_\Omega(u)}}.
\]
Equivalently, by performing the change of variable $\theta = u - \Omega$, $u \sim \Omega$,
\[
\tau_1(x) = 2 \int_{x_L - \Omega}^{x - \Omega} \frac{d\theta}{\sqrt{y^2(x) + \lambda(1 - p)(x - \Omega)^2 + \sum \theta(x) - \lambda(1 - p)\theta^2 - \sum \theta_0(\theta)}}. \tag{25}
\]
On the other hand, since
\[
x_L - \Omega < \theta = u - \Omega < x - \Omega < 0
\]
implies
\[
|\theta| \leq |x_L - \Omega|,
\]
we find the estimate
\[
|\sum \theta(\theta)| \leq \sum \nabla_j |\theta|^j \leq \sum \nabla_j |x_L - \Omega|^j. \tag{26}
\]
Consequently, shortening the notations by naming
\[
h_\pm(x, x_L) := y^2(x) + \lambda(1 - p)(x - \Omega)^2 + \sum \theta(x) \pm \sum \nabla_j |x_L - \Omega|^j
\]
and
\[
I_\pm(x, x_L) := 2 \int_{x_L - \Omega}^{x - \Omega} \frac{d\theta}{h_\pm(x, x_L) - \lambda(1 - p)\theta^2} \tag{28}
\]
it follows from (25) and (26) that
\[
I_+(x, x_L) \leq \tau_1(x) \leq I_-(x, x_L)
\]
and, therefore, to prove the first identity of (23) it suffices to show that
\[
\lim_{x \uparrow \Omega} I_\pm(x, x_L(x)) = \frac{2}{\sqrt{\lambda(1 - p)}} \arctan \left( \frac{y'(\Omega)}{\sqrt{\lambda(1 - p)}} \right). \tag{29}
\]
It should be noted that, in order to do this, we must make sure that the functions
\[
g(\theta) := h_\pm(x, x_L) - \lambda(1 - p)\theta^2, \quad x_L - \Omega \leq \theta \leq x - \Omega,
\]
are non-negative, at least for $x$ in a neighborhood of $\Omega$.

Through some elementary manipulations, it is easily seen that (28) implies
\[
I_\pm(x, x_L(x)) = \frac{2}{\sqrt{\lambda(1 - p)}} \sqrt{\frac{\lambda(1 - p)(x - \Omega) / \sqrt{h_\pm(x, x_L)}}{\sqrt{h_\pm(x, x_L)}}} \frac{d\theta}{\sqrt{1 - \theta^2}} \tag{31}
\]
\[
= \frac{2}{\sqrt{\lambda(1 - p)}} \left[ \arcsin \left( \frac{\sqrt{\lambda(1 - p)(x - \Omega)}}{\sqrt{h_\pm(x, x_L)}} \right) - \arcsin \left( \frac{\sqrt{\lambda(1 - p)(x_L - \Omega)}}{\sqrt{h_\pm(x, x_L)}} \right) \right].
\]
Now, we need to ascertain the asymptotic expansion of $x_L(x)$ in terms of $x - \Omega$, as $x \uparrow \Omega$. As the solution of (20) satisfies
\[
\frac{d}{dt} \left( \frac{v^2}{2} + \frac{\lambda}{2} u^2 + \frac{b}{p + 1} u^{p+1} \right) = 0, \tag{30}
\]
necessarily
\[
y^2(x) + \lambda x^2 + \frac{2b}{p + 1} x^{p+1} = \lambda x_L^2(x) + \frac{2b}{p + 1} x_L^{p+1}(x) \tag{31}
\]
for all $x < \Omega$, $x \sim \Omega$. By Theorem 2.3, we already know that
\[
y(x) = y'(\Omega)(x - \Omega) + o(x - \Omega) \quad \text{as } x \to \Omega. \tag{32}
\]
Moreover, a direct calculation shows that

\[ \lambda x^2 + \frac{2b}{p+1} x^{p+1} = \lambda \Omega^2 + \frac{2b}{p+1} \Omega^{p+1} + \lambda (1-p)(x-\Omega)^2 + O((x-\Omega)^3) \quad \text{as} \quad x \to \Omega, \]

because \( \Omega^{p-1} = -\lambda/b \). Thus, substituting these expansions into (31) and rearranging terms, it follows that

\[ \lambda x_L^2(x) + \frac{2b}{p+1} x_L^{p+1}(x) = \lambda \Omega^2 + \frac{2b}{p+1} \Omega^{p+1} + [(y'(\Omega))^2 + \lambda (1-p)](x-\Omega)^2 + o((x-\Omega)^2) \]

as \( x \uparrow \Omega \). By expanding the left hand side of this identity in powers of \( x - \Omega \), or, alternatively, differentiating twice with respect to \( x \) and particularizing at \( x = \Omega \), it becomes apparent that

\[ x_L(x) = \Omega + \sqrt{\frac{(y'(\Omega))^2}{\lambda(1-p)}} + 1 (x - \Omega) + o(x - \Omega) \quad (33) \]

as \( x \uparrow \Omega \). Actually, \( x_L \) is a function of class \( C^\infty \) in a neighborhood of \( \Omega \) by the theorem of differentiation of G. Peano, because \( y(x) \), and so \( \Gamma_0 \), is of class \( C^\infty \) outside the origin. Consequently, substituting (32) and (33) into (27), we are led to

\[ h_\pm(x, x_L(x)) = [(y'(\Omega))^2 + \lambda (1-p)](x-\Omega)^2 + o((x-\Omega)^2). \]

Using these asymptotic expansions, it is straightforward to check that the function \( g(\theta) \) defined above is indeed non-negative and that

\[ \lim_{x \uparrow \Omega} \sqrt{\frac{\lambda(1-p)(x-\Omega)}{h_\pm(x, x_L)}} = \frac{-\sqrt{\lambda(1-p)}}{\sqrt{(y'(\Omega))^2 + \lambda(1-p)}} = \frac{-1}{\sqrt{\frac{(y'(\Omega))^2}{\lambda(1-p)} + 1}}. \]

As a byproduct, by (33), we also have that

\[ \lim_{x \uparrow \Omega} \sqrt{\frac{\lambda(1-p)(x_L-\Omega)}{h_\pm(x, x_L)}} = -1. \]

Consequently, letting \( x \uparrow \Omega \), it becomes apparent that

\[ \lim_{x \uparrow \Omega} I_\pm(x, x_L(x)) = \frac{2}{\sqrt{\lambda(1-p)}} \left[ \frac{\pi}{2} - \arcsin \left( \frac{1}{\sqrt{(y'(\Omega))^2 + \lambda(1-p)}} + 1 \right) \right] = \frac{2}{\sqrt{\lambda(1-p)}} \arctan \frac{y'(\Omega)}{\sqrt{\lambda(1-p)}}. \]

This proves (29) and ends the proof of the first identity of (23). The previous argument can be easily adapted, almost “mutatis mutandis”, to show the validity of the second identity of (23). So, we omit the technical details here. The fact that \( \tau_1(x) \to \infty \) as \( x \downarrow x_0 \) was already shown in the proof of Theorem 3.1. To conclude the proof it remains to show that

\[ \lim_{x \uparrow \infty} \tau_1(x) = 0. \quad (34) \]

Subsequently, for every \( x > \Omega \), we denote by \( x_M = x_M(x) > x \) the first crossing point of the trajectory of the solution of (20) with the \( u \)-axis in its phase portrait. Going back to (30), it is apparent that the solution of (20) satisfies

\[ v^2 + \lambda u^2 + \frac{2b}{p+1} u^{p+1} = \lambda x_M^2 + \frac{2b}{p+1} x_M^{p+1} \]
and, hence, for every \( x > x^+_0 \),
\[
\tau_1(x) = 2 \int_x^{x_M} \frac{du}{\sqrt{\lambda(x^2_M - u^2) + \frac{2b}{p+1}(x^{p+1}_M - u^{p+1})}} \\
= 2 \int_{x/x_M}^1 \frac{d\theta}{\sqrt{\lambda(1 - \theta^2) + \frac{2b}{p+1}\theta^{p-1}(1 - \theta^{p+1})}} < 2 \int_0^1 \frac{d\theta}{\sqrt{\lambda(1 - \theta^2) + \frac{2b}{p+1}\theta^{p-1}(1 - \theta^{p+1})}}.
\]

As \( \lim_{x \to \infty} x_M(x) = \infty \), letting \( x \uparrow \infty \) in the previous estimate shows (34) and ends the proof.

As expected, the value of \( \tau_1(\Omega) \) for the nonlinear problem (20) given through Theorem 3.3 coincides with the value of \( \tau_1(\Omega) \) for its linearized problem at the steady-state \( \Omega \)
\[
\begin{cases}
-(u - \Omega)'' = \lambda(1-p)(u - \Omega),
\end{cases}
\]
\( u(0) = x, \ u'(0) = y(x). \) (35)

Indeed, the general solution of the differential equation of (35) can be expressed as
\[
u(t) = \Omega + A\sin(\sqrt{\lambda(1-p)} t) + B\cos(\sqrt{\lambda(1-p)} t),
\]
where \( A, B \in \mathbb{R} \) are arbitrary integration constants, and, hence,
\[
u'(t) = A\sqrt{\lambda(1-p)}\cos(\sqrt{\lambda(1-p)} t) - B\sqrt{\lambda(1-p)}\sin(\sqrt{\lambda(1-p)} t).
\]

As
\[
x = u(0) = \Omega + B, \quad y(x) = u'(0) = A\sqrt{\lambda(1-p)},
\]
necessarily
\[
B = x - \Omega, \quad A = \frac{y(x)}{\sqrt{\lambda(1-p)}},
\]
and, therefore, the unique solution of (35) is given through
\[
u(t) = \Omega + \frac{y(x)}{\sqrt{\lambda(1-p)}}\sin(\sqrt{\lambda(1-p)} t) + (x - \Omega)\cos(\sqrt{\lambda(1-p)} t).
\]

As \( \tau_1(x)/2 \) is the first time where \( u' \) vanishes, it becomes apparent that
\[
y(x)\cos\left(\sqrt{\lambda(1-p)} \frac{\tau_1(x)}{2}\right) = (x - \Omega)\sqrt{\lambda(1-p)}\sin\left(\sqrt{\lambda(1-p)} \frac{\tau_1(x)}{2}\right)
\]
and, consequently,
\[
\tau_1(x) = \frac{2}{\sqrt{\lambda(1-p)}} \arctan\frac{y(x)}{(x - \Omega)\sqrt{\lambda(1-p)}}. \quad (36)
\]

Letting \( x \to \Omega \) in (36), indeed gives the value of \( \tau_1(\Omega) \) calculated through Theorem 3.3.
3.2. The existence of solutions of type $T_1$. The following result establishes an 
almost optimal condition for the existence of solutions of type $T_1$, in the sense that 
it would be a characterization theorem if $\tau_1(x)$ would be decreasing. But, we have 
not been able to prove this monotonicity property yet.

**Theorem 3.4.** Suppose $\lambda < 0$ and (18). Then, the following assertions are true:

(a) The problem (1) has a solution of type $T_1$ with a local minimum in $(\alpha, 1 - \alpha)$ if 
$$\tau_1(\Omega) < 1 - 2\alpha. \quad (37)$$

(b) The problem (1) has a solution of type $T_1$ with a local maximum in $(\alpha, 1 - \alpha)$ if 
$$\tau_1(\Omega) > 1 - 2\alpha. \quad (38)$$

Note that, besides these solutions of type $T_1$, the problem admits the trivial solution $u_0$.

**Proof.** The notations introduced in the proof of Theorem 3.1 will be kept through 
this proof. Under condition (37), there exists $x \in (x_0^-, \Omega)$ such that $\tau_1(x) < 1 - 2\alpha$. 
Thus, since $\lim_{x \to x_0^-} \tau_1(x) = \infty$, there exists $\tilde{x} \in (x_0^-, x)$ such that $\tau_1(\tilde{x}) = 1 - 2\alpha$, 
which ends the proof of Part (a).

Now, suppose (38), instead of (37). Then, there exists $x > \Omega$ such that $\tau_1(x) > 1 - 2\alpha$. Therefore, owing to (34), it becomes apparent that there exists $\tilde{x} > x$ such 
that $\tau_1(\tilde{x}) = 1 - 2\alpha$. This completes the proof. \hfill \Box

Figure 7 represents the two symmetric solutions of (1) when conditions $b = b^*$ and 
$\tau_1(\Omega) > 1 - 2\alpha$ are satisfied. The symmetric solutions in case $\tau_1(\Omega) < 1 - 2\alpha$ 
had been already represented in Figure 6.

![Figure 7. The symmetric solutions of (1) in case (38).](image)

As far as concerns to solutions of type $T_1$, it should be noted that Theorem 3.4 
is substantially sharper than Theorem 3.1. Not only the solutions given by Part 
(b) were left aside of Theorem 3.1, but the condition $\tau_1(\Omega) < 1 - 2\alpha$ is much stronger 
than (37), since, actually, we do have 
$$2\tau_1(\Omega) < \tau_1. \quad (39)$$

The next result complements Theorem 3.4 ensuring that there are open ranges of 
values of the parameters involved in the setting of (1) for which any of the conditions 
(37) and (38) can be satisfied.

**Proposition 2.** Suppose $\lambda < 0$ and (18). Then, there exist $\lambda_1^- \leq \lambda_1^+ < 0$ such 
that:
(a) $\tau_1(\Omega) < 1 - 2\alpha$ if $\lambda < \lambda_1^-$.  
(b) $\tau_1(\Omega) > 1 - 2\alpha$ if $\lambda \in (\lambda_1^+, 0)$.

Proof. According to (39),

$$\tau_1(\Omega) < \frac{\tau_\Omega}{2} = \frac{\pi}{\sqrt{\lambda(1 - p)}} < 1 - 2\alpha$$

provided

$$\lambda < -\frac{1}{p - 1} \left( \frac{\pi}{1 - 2\alpha} \right)^2,$$

which concludes the proof of Part (a).

By the theorem of differentiation of G. Peano, it follows from Theorem 2.3 that the positive real number $y'(\Omega)$ can be regarded as a continuous function of $\lambda \leq 0$, say $y'(\Omega, \lambda)$. Therefore,

$$\lim_{\lambda \uparrow 0} \arctan \frac{y'(\Omega, \lambda)}{\sqrt{\lambda(1 - p)}} = \pi/2$$

and, consequently, Theorem 3.3 implies that

$$\lim_{\lambda \uparrow 0} \tau_1(\Omega) = \lim_{\lambda \uparrow 0} \left( \frac{2}{\sqrt{\lambda(1 - p)}} \arctan \frac{y'(\Omega, \lambda)}{\sqrt{\lambda(1 - p)}} \right) = \infty,$$

which completes the proof of Part (b).

3.3. The time-maps $\tau_j$, $j \geq 2$. Sharpening Theorem 3.1. Throughout this section, we suppose $\lambda < 0$ and (18). As for $\tau_1$, the time map $\tau_2$ constructed in the proof of Theorem 3.1 can be extended, in a rather natural way, to be defined for all $x \in J^* := J \setminus \{\Omega\}$, $J := (x_0^-, x_0^+)$, where, following the notations introduced in the proof of Theorem 3.1, $(x_0^+, y(x_0^+))$ and $(x_0^-, y(x_0^-))$ are the first and the second crossing points, respectively, as time increases, between $\Gamma_0$ and the homoclinic orbit through $(0, 0)$; $x_0^- < \Omega = m_0 < x_0^+$. Similarly, the map $\tau_3$ can be extended to be defined in $J^*$ too. Also, by construction (see Figure 5, if necessary), we have that

$$\tau_3(x) = \tau_1(x) + \tau(x) \quad \text{for all } x \in J^*,$$

where $\tau(x)$ stands for the period of the solution of (20).

More generally, throughout the rest of this section, for every integer number $n \geq 2$, we consider the time-maps $\tau_{2n}$ and $\tau_{2n+1}$ defined in $J^*$ through

$$\tau_{2n+1} = \tau_1 + (n-1)\tau, \quad \tau_{2n} = \tau_2 + n\tau.$$

By (40), (41) does actually make sense for every $n \geq 1$.

According to Theorem 3.3, one can easily infer, by the symmetry of the problem, that

$$\lim_{x \to \Omega} \tau_2(x) = 2\tau_1(\Omega) + \frac{\tau_\Omega - 2\tau_1(\Omega)}{2} = \tau_1(\Omega) + \frac{\tau_\Omega}{2},$$

Consequently,

$$\lim_{x \to \Omega} \tau_{2n+1}(x) = \tau_1(\Omega) + n\tau_\Omega, \quad \lim_{x \to \Omega} \tau_{2n}(x) = \tau_1(\Omega) + \left( n - \frac{1}{2} \right) \tau_\Omega.$$
for all $n \geq 1$, and, therefore, all these time-maps can be extended to $J$ so that $	au_j \in \mathcal{C}(J)$, $j \geq 1$, by simply setting

$$
\tau_{2n+1}(\Omega) := \tau_1(\Omega) + n\tau_\Omega, \quad \tau_{2n}(\Omega) := \tau_1(\Omega) + \left( n - \frac{1}{2} \right) \tau_\Omega, \quad n \geq 1. \quad (42)
$$

By continuous dependence, all these time-maps satisfy

$$
\lim_{x \downarrow x_0} \tau_n(x) = \infty, \quad \lim_{x \uparrow x_0} \tau_{n+1}(x) = \infty, \quad n \geq 1.
$$

Moreover, by construction,

$$
\tau_n(x) < \tau_{n+1}(x), \quad n \geq 1, \quad x \in J.
$$

Actually, $\tau_1$ is globally defined in $(x_0^-, \infty)$ and, due to Theorem 3.3, $\lim_{x \uparrow \infty} \tau_1(x) = 0$. Figure 8 represents the plots of $\tau_n$, $1 \leq n \leq 7$, that we have computed using Mathematica for an appropriate choice of the several parameters involved in the formulation of (1). The numerics gave the nice monotonicity properties shown in Figure 8, though we were not able to prove them analytically.

Figure 8. The graphs of the curves $\tau_n$, $1 \leq n \leq 7$.

By Corollary 2 and the symmetry of the equation (17), the solutions of type $T_{2n}$, $n \geq 2$, must appear by pairs $(u, \tilde{u})$ with

$$
\tilde{u}(t) = u(1 - t), \quad t \in [0, 1],
$$

similarly to the case $n = 1$ already analyzed in the proof of Theorem 3.1.

It should be noted that, according to (42),

$$
\tau_{n+1}(\Omega) = \tau_n(\Omega) + \frac{\tau_\Omega}{2}, \quad n \geq 1.
$$

By simply counting the number of roots of $\tau_n = 1 - 2\alpha$, for every $n \geq 1$, one can get the next substantial improvement of Theorem 3.1.
Theorem 3.5. Suppose $\lambda < 0$, (18) and
\[ \tau_n(\Omega) < 1 - 2\alpha < \tau_{n+1}(\Omega) \]
for some $n \geq 2$. Then, (1) admits, at least, one solution of type $T_1$ and two solutions of type $T_j$ for all $2 \leq j \leq n$, besides the trivial solution $u_0 \in T_0$. When $n = 1$, the problem possesses at least one solution of type $T_1$ plus the trivial solution, as already established by Theorem 3.4(a).

4. Local perturbation from $b = b^*$. The main goal of this section is to analyze the behavior of the solutions already constructed in Section 3 as the parameter $b$ perturbs from $b^*$. In Section 5 we will ascertain the global behavior of these solution as $b$ separates away from $b^*$, obtaining in this way the corresponding global bifurcation diagrams. One of the main differences between the cases $b = b^*$ and $b \neq b^*$ is that in case $b = b^*$ all solutions of odd type must be symmetric, whereas in case $b \neq b^*$ the problem (1) can admit asymmetric solutions of odd type, which play the same role as solutions of even type.

4.1. The case $b > b^*$. Now, $\Omega < m_0$. Clearly, for $b > b^*$, $b \sim b^*$, the phase portrait of (17) looks like shows Figure 9, where we also have superimposed $\Gamma_0$ and $\Gamma_1$. Besides the homoclinic through the origin, we have represented three trajectories. One exterior to the homoclinic and two interior orbits. The interior ones are rather special, as the inner one is the unique orbit which is tangent to $\Gamma_0$. The exterior one is the unique orbit passing through the crossing point between $\Gamma_0$ and $\Gamma_1$, $(m_0,0)$. The points $x_0^-$ and $x_0^+$ are defined as in the proof of Theorem 3.1. We have denoted by $x_m$ the unique value of $x \neq m_0$ for which $(x,y(x)) \in \Gamma_0$ lies on the orbit through $(m_0,0)$. All these points are going to play a very important role in the subsequent analysis. Naturally, except $m_0$, all of them are regular functions of $b$,
\[
x_0^- = x_0^-(b) < x_m = x_m(b) < x_t = x_t(b) < m_0 < x_0^+ = x_0^+(b).
\]
Although Figure 9 illustrates the special case when
\[
\Omega = \left(\frac{-\lambda}{b}\right)^{\frac{1}{p-1}} < x_m(b) < x_t = x_t(b),
\]
in general, this condition does not need to be satisfied.

Besides $(0,0)$, the homoclinic connection meets the $u$-axis at $(u_M,0)$, where
\[
u_M = u_M(b) := \left(\frac{-\lambda(p+1)}{2b}\right)^{\frac{1}{p-1}}.
\]
Thus, setting
\[
b_{m_0} := -\frac{\lambda(p+1)}{2m_0^{p-1}},
\]
we have that
\[
u_M(b) \begin{cases} 
> m_0 & \text{if } b < b_{m_0}, \\
= m_0 & \text{if } b = b_{m_0}, \\
< m_0 & \text{if } b > b_{m_0}.
\end{cases}
\]

Figure 9 represents the phase portrait of (17) in the range of values $b \in (b^*, b_{m_0})$, where we recall that $b^* = -\lambda/m_0^{p-1}$. This is the range of values of $b$ where we are going to focus our attention in this section. Figure 10 describes the most relevant general features of the phase portrait of (17) as $b$ increases from $b^*$. Precisely, Figure 10A shows the phase portrait for $b^* < b < b_{m_0}$. As $b$ reaches the critical value $b_{m_0}$
Figure 9. The phase portrait of (17) for $b > b^*$, $b \sim b^*$. (Figure 10B) and crosses it, $m_0$ moves outside the homoclinic until $b$ attains a further critical value, say $b_t > b_{m_0}$, where the homoclinic is tangent to $\Gamma_0$ and $\Gamma_1$ (Figure 10C). As $b > b_t$, the homoclinic connection cannot meet $\Gamma_0 \cup \Gamma_1$, and, actually, it shrinks to $(0,0)$ as $b \uparrow \infty$. These features are straightforward consequences of the fact that the family of homoclinic connections shrinks monotonically to $(0,0)$ as $b$ increases.

As in Section 3, the solutions of (17) connecting $\Gamma_0$ with $\Gamma_1$ in a time $1 - 2\alpha$ provide us with solutions of (1), as it was described in Section 2. But, since the phase portrait changes when $b$ increases from $b^*$, the time-maps $\tau_n$, $n \geq 1$, also change. Most of our efforts to understand what is going on when $b$ perturbs from $b^*$ rely on the appropriate definitions of these time-maps, which will be subsequently denoted by $\tau_n(x,b)$, in order to emphasize their dependence on $b$.

For every $b \in (b^*, b_{m_0})$, we denote by

$$
\tau_1(\cdot,b) : D_1 = D_1(b) := (x^+_0, x_t] \cup (m_0, \infty) \rightarrow [0, \infty)
$$

the Poincaré map defined, for every $x \in D_1$, as the minimal time needed by the solution of (20) to reach $\Gamma_1$. Figure 11 shows the corresponding orbits of two of these solutions for some $x < x_t$ (A) and $x > m_0$ (B). By continuous dependence,

$$\lim_{x \downarrow x_0} \tau_1(x,b) = \infty.
$$

Moreover,

$$\lim_{x \downarrow m_0} \tau_1(x,b) = 0,
$$

(43)
because $m_0$ is not an equilibrium of (17). Also, the proof of (34) (for $b = b^*$) adapts \textit{mutatis mutandis} to show that

$$
\lim_{x \uparrow \infty} \tau_1(x, b) = 0
$$

for all $b > b^*$. Note that $\tau_1(x_t, b)$ is the minimal time needed by the solution of (20), with $x = x_t$, to connect $(x_t, y(x_t)) \in \Gamma_0$ with $\Gamma_1$.

Subsequently, for every $b \in (b^*, b_{m_0})$, we denote by

$$
\tau_{1,s}(\cdot, b) : D_{1,s} = D_{1,s}(b) := [x_t, m_0] \to [0, \infty)
$$
Figure 11. The time-map \((x, y(x)) \mapsto \tau_1(x, b)\) for \(b^* < b < b_{m_0}\)

the Poincaré-map defined, for every \(x \in (x_t, m_0)\), as the minimal time needed by the solution of (20) to reach \(\Gamma_1\) exactly twice, while

\[
\begin{align*}
\tau_{1,s}(x_t, b) &:= \lim_{x \downarrow x_t} \tau_{1,s}(x, b) = \tau_1(x_t, b) \\
\tau_{1,s}(m_0, b) &:= \lim_{x \uparrow m_0} \tau_{1,s}(x, b) = \tau(m_0, b), \quad (44)
\end{align*}
\]

where, for every \(x \in (x_0^-, x_0^+)\), \(\tau(x, b)\) stands for the period of the orbit through \((x, y(x))\). Figure 12 shows the corresponding orbits of two of these solutions for some \(x \sim x_t\) (A) and \(x \sim m_0\) (B).

Figure 12. The time-map \((x, y(x)) \mapsto \tau_{1,s}(x, b)\) for \(b^* < b < b_{m_0}\)
Similarly, we may introduce the Poincaré map

\[ \tau_{1,a}(\cdot, b) : D_{1,a} := [x_{m_0}, m_0] \rightarrow [0, \infty) \]

defined, for every \( x \in [x_{m_0}, x_t] \), as the minimal time needed by the solution of (20) to reach \( \Gamma_1 \) twice (see Figure 13A), and, for every \( x \in [x_t, m_0] \), as the minimal time needed to reach \( \Gamma_1 \) for the first time (see Figure 13B).

![Figure 13. The time-map \((x, y(x)) \mapsto \tau_{1,a}(x, b)\) for \(b^* < b < b_{m_0}\)](image)

Note that

\[ \lim_{x \to x_t} \tau_{1,a}(x, b) = \tau_{1,a}(x_t, b) = \tau_1(x_t, b) \]

and, hence, by (44), we obtain that

\[ \tau_{1,a}(x_t, b) = \tau_{1,s}(x_t, b) = \tau_1(x_t, b), \]

though, by definition,

\[ \tau_1(x, b) < \tau_{1,a}(x, b), \quad x_{m_0} \leq x < x_t, \]
\[ \tau_{1,a}(x, b) < \tau_{1,s}(x, b), \quad x_t < x \leq m_0. \]

Incidentally, the solutions of \( \tau_1 = 1 - 2\alpha \) and \( \tau_{1,s} = 1 - 2\alpha \) provide us with symmetric solutions of (1), whereas the solutions constructed from \( \tau_{1,a} = 1 - 2\alpha \) are asymmetric, except at \( x = x_t \), and, by definition of the time maps, all these solutions provide us with solutions of (1) with a single critical point. Conversely, any solution of problem (1) with a single critical point in \((\alpha, 1 - \alpha)\) must be of some of these forms. Actually, this is why we have introduced all these time-maps.

Naturally, in order to look for solutions of type \( T_2 \) of (1), we must introduce the Poincaré map

\[ \tau_2(\cdot, b) : D_2 = D_2(b) := (x_0^-, x_{m_0}] \cup [m_0, x_0^+) \rightarrow [0, \infty) \]
defined, for every \( x \in D_2 \setminus \{x_{m_0}, m_0\} \), as the minimal time needed by the solution of (20) to reach \( \Gamma_1 \) exactly twice, and
\[
\tau_2(x_{m_0}, b) := \lim_{x \uparrow x_{m_0}} \tau_2(x, b) = \tau_{1,a}(x_{m_0}, b),
\]
\[
\tau_2(m_0, b) := \lim_{x \downarrow m_0} \tau_2(x, b) = \tau_{1,a}(m_0, b).
\] (47)

By continuous dependence and the symmetry properties of our problem, these identities are consistent. Figure 14 shows the corresponding orbits of two of these solutions for some \( x \in (x_0^-, x_{m_0}) \) (A) and \( x \in (m_0, x_0^+) \) (B). It is easy to see that (1)

\[\text{Figure 14. The time-map } (x, y(x)) \mapsto \tau_2(x, b) \text{ for } b^* < b < b_{m_0}\]

cannot admit a solution of type \( T_2 \) passing through \( (x, y(x)) \) if \( x_{m_0} < x < m_0 \). Incidentally, by symmetry, each of the solutions on Figure 14(A) must be a reflection around \( t = 1/2 \) of some solution in Figure 14(B).

More generally, we may introduce the time maps
\[
\tau_{2n+1}(x, b) := \tau_1(x, b) + n\tau(x, b), \quad x \in D_1(b),
\]
\[
\tau_{2n+1,s}(x, b) := \tau_{1,s}(x, b) + n\tau(x, b), \quad x \in D_{1,s}(b),
\]
\[
\tau_{2n+1,a}(x, b) := \tau_{1,a}(x, b) + n\tau(x, b), \quad x \in D_{1,a}(b),
\]
\[
\tau_{2n}(x, b) := \tau_2(x, b) + (n - 1)\tau(x, b), \quad x \in D_2(b),
\] (48)

for all \( n \geq 1 \).

According to (43), it follows from (44) and (48) that
\[
\lim_{x \downarrow m_0} \tau_3(x, b) = \tau(m_0, b) = \tau_{1,s}(m_0, b).
\]

Thus, the graph of \( \tau_{1,s} \) in the interval \([x_t, m_0]\) does connect the graph of \( \tau_1 \) in \([x_0^-, x_t]\) (left branch of \( \tau_1 \)) with the graph of \( \tau_3 \) in \([m_0, x_0^+]\) (right branch of \( \tau_3 \)), as illustrated by Figure 15, where those graphs have been plotted together. Consequently, by (48), the graph of \( \tau_{2n+1,s} \) in \([x_t, m_0]\) must connect the graph of \( \tau_{2n+1} \) in the interval \([x_0^-, x_t]\) with the graph of \( \tau_{2(n+1)+1} \) in \([m_0, x_0^+]\), for all \( n \geq 1 \). Similarly, by (47), the graph of \( \tau_{1,a} \) in \([x_{m_0}, m_0]\) does connect the graph of \( \tau_2 \) in \([x_0^-, x_{m_0}]\) with the graph
of $\tau_2$ in $[m_0, x_0^+]$. As, according to (45), the graphs of $\tau_1$, $\tau_{1,s}$ and $\tau_{1,a}$ cross at $x_t$; owing to (46) and (48), it becomes apparent that the graphs of all these functions look like shown by Figure 15, where we have collected the plots of the first seven time maps for an appropriate special case with $b = b^* + 0.01$. Note that

$$\lim_{x \uparrow x_0^+} \tau(x, b) = \infty,$$

and

$$\lim_{x \downarrow x_0^-} \tau_n(x, b) = \infty = \lim_{x \uparrow x_0^+} \tau_{n+1}(x, b), \quad n \geq 1.$$

![Figure 15](image_url)

**Figure 15.** The graphs of the curves (48) and (49) for $b^* < b < b_{m_0}$

More precisely, Figure 15 represents the graphs of the functions $\theta_j$, $j \geq 1$, defined by

$$\theta_{2n+1}(x, b) := \begin{cases} 
\tau_{2n+1}(x, b) & \text{if } x \in (x_0^-, x_t], \\
\tau_{2n+1, s}(x, b) & \text{if } x \in (x_t, m_0], \\
\tau_{2(n+1)+1}(x, b) & \text{if } x \in (m_0, x^+_0),
\end{cases} \quad (50)$$

for all $n \geq 0$, and

$$\theta_{2n}(x, b) := \begin{cases} 
\tau_{2n}(x, b) & \text{if } x \in (x_0^-, x_{m_0}], \\
\tau_{2(n-1)+1, a}(x, b) & \text{if } x \in (x_{m_0}, m_0), \\
\tau_{2n}(x, b) & \text{if } x \in [m_0, x^+_0),
\end{cases} \quad (51)$$

for all $n \geq 1$. According to (48) and (49), it is apparent that

$$\theta_{2n+1} := \theta_1 + n\tau, \quad \theta_{2n} := \theta_2 + (n-1)\tau, \quad n \geq 1.$$

The next result collects some global monotonicity properties of these auxiliary functions.
Lemma 4.1. For every integer \( n \geq 0 \) and \( b \in (b^*, b_{m_0}) \), we have that

\[
\theta_{2n+1}(x, b) = \begin{cases} 
< \theta_{2(n+1)}(x, b) & \text{if } x \in (x_0^-, x_1), \\
\tau_1(x, b) + n\tau(x, b) < \tau_2(x, b) + n\tau(x, b) & \text{if } x = x_t, \\
\tau_2(n+1)(x, b) & \text{if } x \in (x_1, x_0^+). 
\end{cases}
\] (52)

Moreover, for every \( x \in (x_0^-, x_0^+) \),

\[
\max \{ \theta_{2n+1}(x, b), \theta_{2(n+1)}(x, b) \} < \min \{ \theta_{2(n+1)+1}(x, b), \theta_{2(n+2)}(x, b) \}. \] (53)

Note that Figure 15 is in complete agreement with Lemma 4.1.

Proof. Suppose \( x \in (x_0^-, x_{m_0}] \). Then, by (50), (51), (48) and (49), taking into account that \( \tau_1(x, b) < \tau_2(x, b) \), we find that

\[
\theta_{2n+1}(x, b) = \tau_{2n+1}(x, b) = \tau_1(x, b) + n\tau(x, b) < \tau_2(x, b) + n\tau(x, b) = \tau_2(2n+1)(x, b). 
\]

Suppose \( x \in (x_{m_0}, x_t) \). Then, by (50), (48), (49), (46) and (51),

\[
\theta_{2n+1}(x, b) = \tau_{2n+1}(x, b) = \tau_1(x, b) + n\tau(x, b) < \tau_{1,a}(x, b) + n\tau(x, b) = \tau_{2n+1,a}(x, b) = \tau_{2(n+1)}(x, b). 
\]

Finally, for every \( x \in (m_0, x_0^+) \), we find that

\[
\theta_{2n+1}(x, b) = \tau_{2n+1}(x, b) = \tau_1(x, b) + n\tau(x, b) > \tau_2(x, b) + n\tau(x, b) = \tau_{2(n+1)}(x, b), 
\]

because \( \tau_3 = \tau_1 + \tau > \tau_2 \), which completes the proof of (52).

Naturally, (53) relies on (52). For every \( x \in (x_0^-, x_{m_0}] \), using (52), (51), (48), (49) and (50), we find that

\[
\max \{ \theta_{2n+1}(x, b), \theta_{2(n+1)}(x, b) \} = \theta_{2(n+1)}(x, b) = \tau_{2(n+1)}(x, b) = \tau_2(x, b) + n\tau(x, b) < \tau_1(x, b) + (n+1)\tau(x, b) = \tau_{2(n+1)+1}(x, b) = \min \{ \theta_{2(n+1)+1}(x, b), \theta_{2(n+2)}(x, b) \}. 
\]

Similarly, for every \( x \in (x_{m_0}, x_t) \), we obtain that

\[
\max \{ \theta_{2n+1}(x, b), \theta_{2(n+1)}(x, b) \} = \theta_{2(n+1)}(x, b) = \tau_{2n+1,a}(x, b) = \tau_{1,a}(x, b) + n\tau(x, b) < \tau_1(x, b) + (n+1)\tau(x, b) = \tau_{2(n+1)+1}(x, b) = \min \{ \theta_{2(n+1)+1}(x, b), \theta_{2(n+2)}(x, b) \}. 
\]
because $\tau_{1,a} < \tau < \tau_1 + \tau$. Analogously, for every $x \in [x_1, m_0]$, we have that

$$\max \{\theta_{2n+1}(x, b), \theta_{2(n+1)}(x, b)\} = \theta_{2n+1}(x, b) = \tau_{2n+1,a}(x, b) = \tau_{1,a}(x, b) + n\tau(x, b)$$

$$< \tau_{1,a}(x, b) + (n + 1)\tau(x, b)$$

$$= \tau_{2(n+1)+1,a}(x, b) = \theta_{2(n+2)}(x, b)$$

$$= \min \{\theta_{2(n+1)+1}(x, b), \theta_{2(n+2)}(x, b)\},$$

because $\tau_{1,s} < \tau < \tau_{1,a} + \tau$. Finally, for any $x \in (m_0, x_0^+)$, we have that

$$\max \{\theta_{2n+1}(x, b), \theta_{2(n+1)}(x, b)\} = \theta_{2n+1}(x, b) = \tau_{2(n+1)+1}(x, b)$$

$$= \tau_1(x, b) + (n + 1)\tau(x, b)$$

$$< \tau_2(x, b) + (n + 1)\tau(x, b)$$

$$= \tau_{2(n+2)}(x, b) = \theta_{2(n+2)}(x, b)$$

$$= \min \{\theta_{2(n+1)+1}(x, b), \theta_{2(n+2)}(x, b)\},$$

which concludes the proof. 

Since

$$\lim_{b \downarrow b^*} x_{m_0}(b) = \lim_{b \downarrow b^*} x_t(b) = \lim_{b \downarrow b^*} \Omega(b) = m_0,$$

we have that

$$\lim_{b \downarrow b^*} D_{1,a}(b) = \lim_{b \downarrow b^*} D_{1,s}(b) = \{m_0\},$$

and the graphs of all these $\theta_j$'s in the intervals $[x_{m_0}(b), m_0]$ shrink either to a single point, or a certain segment, as $b \downarrow b^*$. Moreover, the crossing points between the homoclinic and $\Gamma_0$ satisfy

$$\lim_{b \downarrow b^*} x_{m_0}^{+}(b) = x_0^{+},$$

where $x_0^+ = x_0^+(b^*)$ are those introduced in the proof of Theorem 3.1. Naturally, by well known results on continuous dependence, the next result holds.

**Proposition 3.** For every $n \geq 1$ and $\varepsilon > 0$ sufficiently small,

$$\lim_{b \downarrow b^*} \tau_n(x, b) = \tau_n(x)$$

uniformly in

$$x \in [x_0^-(b^*) + \varepsilon, m_0 - \varepsilon] \cup [m_0 + \varepsilon, x_0^+(b^*) - \varepsilon].$$

Proposition 3 guarantees that, for any $n \geq 1$, the graphs of the time map $\tau_n(\cdot, b)$ constructed in this section converges, as $b \downarrow b^*$, to the graph of the corresponding time map $\tau_n$ constructed in Section 3; the convergence being uniform in the complement in $(x_0^-(b^*), x_0^+(b^*))$ of any neighborhood of $m_0$. This has been illustrated by Figure 15, where the graphs of the perturbed Poincaré maps, $\tau_n(\cdot, b)$, for $b > b^*$, plotted with continuous lines, are very close to the graphs of the corresponding unperturbed time maps $\tau_n$, plotted with dashed lines. Not surprisingly, in between $x_{m_0}$ and $m_0$ (very close if $b \sim b^*$), the nature of these graphs changes drastically. Indeed, in the interval $[x_{m_0}(b), m_0]$ each $\tau_{2n+1}$ breaks down and its perturbed left branch can be connected with the perturbed right branch of $\tau_{2(n+1)+1}$ through the graph of $\tau_{2n+1,a}$, while the graph of $\tau_{2n}$ breaks down and its perturbed left branch can be connected with its perturbed right branch through the graph of $\tau_{2n-1,a}$. In other words, the unperturbed $\tau_n$'s of Section 3 perturb into the $\theta_n$'s constructed through (50) and (51). The next theorem provides us with the precise behavior of these time-maps as $b \downarrow b^*$. 

Theorem 4.2. The following assertions are true:

(a) \( \lim_{b \downarrow b^*} \tau_{1,a}(x, b) = \tau_2(\Omega) \) for all \( x \in [x_{m_0}(b), m_0] \), as well as for the linearized problem.

(b) \( \lim_{b \downarrow b^*} \tau_{1,s}(m_0, b) = \tau(\Omega) \).

(c) \( \lim_{b \downarrow b^*} \lim_{x \downarrow m_0} \tau_3(x, b) = \tau(\Omega) \).

Thus, owing to Parts (a), (b) and (45), the function \( \tau_{1,s} \) must jump from a value very close to \( \tau_2(\Omega) \) to a value very close to \( \tau(\Omega) \) in the interval \( [x_t(b), m_0] \) for sufficiently close \( b > b^* \), whereas, due to Parts (b), (c) and Proposition 3, the graph of \( \tau_3(., b) \) approximates the graph of \( \tau_3 \) in \( (m_0, x_0^t) \) as \( b \downarrow b^* \) plus a vertical segment linking \( \tau(\Omega) \) with \( \tau_3(\Omega) = \tau_1(\Omega) + \tau(\Omega) \) at \( x = m_0 \).

Naturally, thanks to (48) and (49), we also have that

\[
\lim_{b \downarrow b^*} \tau_{2n-1,a}(x, b) = \tau_{2n}(\Omega) \quad \text{for all} \quad x \in [x_{m_0}(b), m_0],
\]

\[
\lim_{b \downarrow b^*} \tau_{2n-1,s}(m_0, b) = n \tau(\Omega),
\]

and that

\[
\lim_{b \downarrow b^*} \lim_{x \downarrow m_0} \tau_{2n+1}(x, b) = n \tau(\Omega)
\]

for all \( n \geq 1 \).

Proof. Part (b) is an immediate consequence of (44), as

\[
\lim_{b \downarrow b^*} \tau_{1,s}(m_0, b) = \lim_{b \downarrow b^*} \tau(m_0, b) = \tau(\Omega).
\]

Part (c) follows directly from (48) and (43), as

\[
\lim_{b \downarrow b^*} \lim_{x \downarrow m_0} \tau_3(x, b) = \lim_{b \downarrow b^*} \lim_{x \downarrow m_0} \tau(m_0, b) = \tau(\Omega).
\]

Therefore, we only have to prove Part (a).

We begin with the study of the linearized problem

\[ -u'' = \lambda u + pb\Omega^{p-1} u = \lambda(1 - p)u, \]

whose integral curve through a generic point \((x, y)\) in the phase plane \((u, v)\) is the ellipse with equation

\[ v^2 + \lambda(1 - p)(u - \Omega)^2 = y^2 + \lambda(1 - p)(x - \Omega)^2. \]  

Figure 16 represents three of these integral curves. Precisely, Figure 16 shows the tangent straight lines

\[ v = \pm y'(m_0)(u - m_0) \]  

(55)

to the curves \( \Gamma_0 \) and \( \Gamma_1 \) at \((m_0, 0)\) (subsequently referred to as \( \tilde{\Gamma}_0 \) and \( \tilde{\Gamma}_1 \), respectively), the integral curves through \((m_0, 0), (x_t, y_t)\), and a generic point \((x, y)\) with \( x_{m_0} < x < x_t \), where \((x_{m_0}, y_{m_0})\) stands for the other crossing point between the integral curve through \((m_0, 0)\) and the line \( \tilde{\Gamma}_0 \), and \((x_t, y_t)\) denotes the crossing point with \( \tilde{\Gamma}_0 \) of the unique integral curve tangent to it.

For every \( x \in [x_{m_0}, x_t) \), we define \( \tilde{\tau}_{1,a}(x) \) as the time needed to cross \( \tilde{\Gamma}_1 \) twice, starting at the point \((x, y) \in \tilde{\Gamma}_0 \) and moving along the orbit up to reach \((x_2, y_2)\), while \( \tilde{\tau}_{1,a}(x_t) \) stands for the necessary time to reach \( \tilde{\Gamma}_1 \) for the first time starting at \((x_t, y_t)\). By continuous dependence,

\[ \tilde{\tau}_{1,a}(x_t) := \lim_{x \uparrow x_t} \tilde{\tau}_{1,a}(x). \]
We want to prove that $	ilde{\tau}_{1,a}(x) = \tau_2(\Omega)$ for all $x \in [x_{m_0}, x_t]$.

To this end, we will show that $\tilde{\tau}_{1,a}(x)$ is actually constant as a function of $x$, and that $\tilde{\tau}_{1,a}(x_t) = \tau_2(\Omega)$.

Substituting (55) in (54) and solving yields

$$x_2 = \frac{(y'(m_0))^2(2m_0 - x) + \lambda(1 - p)(2\Omega - x)}{(y'(m_0))^2 + \lambda(1 - p)}.$$

Similarly, substituting (55) in

$$v^2 + \lambda(1 - p)(u - \Omega)^2 = y_t^2 + \lambda(1 - p)(x_t - \Omega)^2$$

and imposing that $x_t$ is a double solution, shows that

$$x_t = \frac{(y'(m_0))^2m_0 + \lambda(1 - p)\Omega}{(y'(m_0))^2 + \lambda(1 - p)}.$$

Consequently,

$$x_t - \Omega = \frac{(y'(m_0))^2}{(y'(m_0))^2 + \lambda(1 - p)}(m_0 - \Omega)$$

and, therefore, since $m_0 > \Omega$, $x_t > \Omega$. As it is apparent from Figure 16, $x_2 \geq x_t > \Omega$, since $x \in [x_{m_0}, x_t]$.

Subsequently, we denote by $x_L = x_L(x)$ and $x_M = x_M(x)$ the minimum and maximum values of the $u$-coordinate along the orbit $E_x$ passing through $(x, y)$, and, for every pair of points, $(x_a, y_a), (x_b, y_b) \in E_x$, $\tilde{\tau}_{x_a, x_b}$ stands for the necessary time to reach $(x_b, y_b)$ from $(x_a, y_a)$ for the first time along $E_x$. By definition,

$$\tilde{\tau}_{1,a}(x) = \tilde{\tau}_{x,x_L} + \tilde{\tau}_{x_L, x_M} - \tilde{\tau}_{x_M, x}$$

(see Figure 16).

To calculate $\tilde{\tau}_{x_L, x_M}$, we consider the general solution of the linearized differential equation, which is given by

$$u(t) = \Omega + A \sin \left( \sqrt{\lambda(1 - p)} t \right) + B \cos \left( \sqrt{\lambda(1 - p)} t \right).$$

Figure 16. Some significant integral curves in the linearized case
subjected to the initial conditions \((u(0), u'(0)) = (x_L, 0)\). Then, \(\hat{t}_{xL,xM}\) is the first (positive) time where \(u'\) vanishes. As
\[
A = 0, \quad B = x_L - \Omega,
\]
we have that
\[
u(t) = \Omega + (x_L - \Omega) \cos \left(\frac{\sqrt{\Lambda(1-p)}}{\pi} t\right)
\]
and, hence, the first (positive) time where \(u' = 0\) is
\[
\hat{t}_{xL,xM} = \frac{\pi}{\sqrt{\Lambda(1-p)}}.
\]
Similarly, the unique solution of (57) such that \((u(0), u'(0)) = (x, y)\) satisfies
\[
A = \frac{y}{\sqrt{\Lambda(1-p)}} = \frac{y'(m_0)(x - m_0)}{\sqrt{\Lambda(1-p)}} < 0, \quad B = x - \Omega,
\]
and, therefore, the first (positive) time where \(u' = 0\) is
\[
\hat{t}_{xL} = \begin{cases} 
\frac{1}{\sqrt{\Lambda(1-p)}} \arctan \left( \frac{y'(m_0)}{\sqrt{\Lambda(1-p)}} \frac{x - m_0}{x - \Omega} \right) & \text{if } x < \Omega, \\
\frac{1}{\sqrt{\Lambda(1-p)}} \arctan \left( \frac{y'(m_0)}{\sqrt{\Lambda(1-p)}} \frac{x - m_0}{x - \Omega} \right) & \text{if } x = \Omega, \\
\frac{1}{\sqrt{\Lambda(1-p)}} \arctan \left( \frac{y'(m_0)}{\sqrt{\Lambda(1-p)}} \frac{x - m_0}{x - \Omega} \right) & \text{if } x > \Omega,
\end{cases}
\]
where it should be noted that
\[
\hat{t}_{xM,xL} = \hat{t}_{xL,xM} = \frac{\pi}{\sqrt{\Lambda(1-p)}}.
\]
Analogously, the unique solution such that \((u(0), u'(0)) = (x_2, y_2)\) is given by (57) with
\[
A = \frac{y_2}{\sqrt{\Lambda(1-p)}} = \frac{y'(m_0)(x_2 - m_0)}{\sqrt{\Lambda(1-p)}} > 0, \quad B = x_2 - \Omega > 0,
\]
where \(x_2\) is given by (56). Thus, since \(\hat{t}_{x_2,x_M}\) is the first positive time where \(u' = 0\), we obtain that
\[
\hat{t}_{x_2,x_M} = \frac{-1}{\sqrt{\Lambda(1-p)}} \arctan \left( \frac{y'(m_0)}{\sqrt{\Lambda(1-p)}} \frac{(y'(m_0))^2(m_0 - x) + \lambda(1-p)(2\Omega - m_0 - x)}{(y'(m_0))^2(2m_0 - \Omega - x) + \lambda(1-p)(\Omega - x)} \right).
\]
Consequently, setting
\[
\beta := (y'(m_0))^2(m_0 - x) + \lambda(1-p)(2\Omega - m_0 - x),
\]
\[
\gamma := (y'(m_0))^2(2m_0 - \Omega - x) + \lambda(1-p)(\Omega - x),
\]
differentiating with respect to \(x\), and using (56), we find that
\[
\frac{\hat{t}_{x_2,x_M}'}{\hat{t}_{x_2,x_M}} = \tilde{\tau}_{x_2,x_M} - \frac{\hat{t}_{x_2,x_M}}{\hat{t}_{x_2,x_M}} = \frac{y'(m_0)(m_0 - \Omega)}{y^2 + \lambda(1-p)(x - \Omega)^2} - \frac{y'(m_0)((y'(m_0))^2 + \lambda(1-p))^2(m_0 - \Omega)}{(y'(m_0))^2\beta^2 + \lambda(1-p)\gamma^2} = \frac{y'(m_0)(m_0 - \Omega)}{y^2 + \lambda(1-p)(x_2 - \Omega)^2} - \frac{y'(m_0)(m_0 - \Omega)}{y^2 + \lambda(1-p)(x_2 - \Omega)^2} = 0,
\]
for all \( x \neq \Omega \), because \((x, y), (x_2, y_2) \in E_x\). As, thanks to the theorem of differentiation of G. Peano, the function \( \tilde{\tau}_{1, a}(x) \) is differentiable in \( x \), necessarily it must be constant for all \( x \). Therefore,
\[
\tilde{\tau}_{1, a}(x) = \tilde{\tau}_{1, a}(x_t)
\]
for all \( x \in [x_{m_0}, x_t] \).

Adapting our previous calculations, it is easily seen that
\[
\tilde{\tau}_1(x_t) = \frac{2}{\sqrt{\lambda(1-p)}} \left( \pi - \arctan \frac{\sqrt{\lambda(1-p)}}{y'(m_0)} \right),
\]
which equals
\[
\tau_2(\Omega) = \frac{\tau(\Omega)}{2} = \frac{1}{\sqrt{\lambda(1-p)}} \left( \pi + 2 \arctan \frac{y'(m_0)}{\sqrt{\lambda(1-p)}} \right),
\]
by Theorem 3.3, since \( \arctan \theta + \arctan 1/\theta = \pi/2 \) for all positive \( \theta \). This completes the proof of Part (a) for the linearization.

Now, we will prove Part (a) for the nonlinear problem. The basic idea will be adapting the technical device introduced in the proof of Theorem 3.3 by expanding \( x_{m_0}(b) \) and \( x_t(b) \) as power series centered at \( b = b^* \) and then letting \( b \to b^* \) in the formulas for the time maps. By construction, \( x_t(b) \) is uniquely determined by
\[
\begin{align*}
\left\{ \begin{array}{l}
v^2 + \lambda u^2 + \frac{2b}{p+1} u^{p+1} = y^2(x_t(b)) + \lambda x_t^2(b) + \frac{2b}{p+1} x_t^{p+1}(b), \\
v'(x_t(b)) = y'(x_t(b)).
\end{array} \right.
\end{align*}
\]
Thus, since \( y(x_t) < 0 \),
\[
v(u) = -\sqrt{y^2(x_t(b)) + \lambda x_t^2(b) + \frac{2b}{p+1} x_t^{p+1}(b) - \lambda u^2 - \frac{2b}{p+1} u^{p+1}}
\]
and, hence, \( x_t(b) \) may be characterized as
\[
y(x_t(b))y'(x_t(b)) + \lambda x_t(b) + bx_p^p(b) = 0 \quad (58)
\]
for all \( b > b^*, \ b \sim b^* \). Differentiating (58) with respect to \( b \) yields
\[
\left[ y'(x_t(b))^2 + y(x_t(b))y''(x_t(b)) + \lambda + bpx_t^{p-1}(b) \right] x'_t(b) + x''_t(b) = 0
\]
and, so, particularizing at \( b^* \) yields
\[
x'_t(b^*) = -\frac{m_0^p}{(y'(m_0))^2 + \lambda(1-p)}.
\]
Note that, thanks to the implicit function theorem and the theorem of differentiation of G. Peano, as a consequence of
\[
(y'(m_0))^2 + \lambda(1-p) > 0
\]
it follows the existence and the uniqueness of a real analytic function \( x_t(b), \ b \sim b^* \), such that
\[
x_t(b) = m_0 - \frac{m_0^p}{(y'(m_0))^2 + \lambda(1-p)} (b - b^*) + O((b - b^*)^2), \quad b \sim b^*. \quad (59)
\]
Similarly, \( x_{m_0}(b) \) is given through
\[
y^2(x_{m_0}(b)) + \lambda x_{m_0}^2(b) + \frac{2b}{p+1} x_{m_0}^{p+1}(b) = \lambda m_0^2 + \frac{2b}{p+1} m_0^{p+1}. \quad (60)
\]
To show its analyticity in $b$ one can argue as follows. Setting 

$$H(x, b) := y^2(x) + \lambda x^2 + \frac{2b}{p + 1}x^{p+1} - \lambda m_0^2 - \frac{2b}{p + 1}m_0^{p+1},$$

we consider the map

$$\hat{H}(x, b) := \begin{cases} 
(x - m_0)^{-1}H(x, b), & \text{if } x \neq m_0, \ x \sim m_0, \\
D_x H(m_0, b), & \text{if } x = m_0,
\end{cases}$$

which is real analytic around $(x, b) = (m_0, b^*)$. It should be noted that $H(m_0, b) = 0$ for all $b \sim b^*$. Moreover,

$$\hat{H}(m_0, b^*) = 2(y(m_0)y'(m_0) + \lambda m_0 + b^* m_0^p) = 0$$

and

$$D_x \hat{H}(m_0, b^*) = 2\{[y'(m_0)]^2 + \lambda(1 - p)\} > 0.$$ 

Consequently, the analyticity of $x_{m_0}(b)$, $b \sim b^*$, with $x_{m_0}(b^*) = m_0$, follows from the implicit function theorem applied to $\hat{H}$ at $(m_0, b^*)$. Now, differentiating twice in (60) with respect to $b$, or, alternatively, differentiating with respect to $b$ in

$$\hat{H}(x_{m_0}(b), b) = 0,$$

yields

$$x_{m_0}(b) = m_0 - \frac{2m_0^p}{(y'(m_0))^2 + \lambda(1 - p)}(b - b^*) + O((b - b^*)^2), \quad b \sim b^*. \quad (61)$$

Subsequently, we consider a generic point $x \in [x_{m_0}(b), x_1(b)]$, which can be parameterized as

$$x(s, b) := (1 - s)x_{m_0}(b) + sx_1(b), \quad s \in [0, 1].$$

According to (59) and (61),

$$x(s, b) = m_0 + \frac{(s - 2)m_0^p}{(y'(m_0))^2 + \lambda(1 - p)}(b - b^*) + O((b - b^*)^2) \quad (62)$$

for all $b \sim b^*$, uniformly in $s \in [0, 1]$.

Let $x_L(s, b)$ denote the minimum $u$-coordinate of the orbit through the point $(x(s, b), y(x(s, b)))$. It satisfies

$$\lambda x_L^2(s, b) + \frac{2b}{p + 1}x_L^{p+1}(s, b) = y^2(x(s, b)) + \lambda x^2(s, b) + \frac{2b}{p + 1}x^{p+1}(s, b). \quad (63)$$

The fact that it is real analytic can be shown by brute force by computing all the Taylor series and checking that the coefficients can be locally estimated by those of a convergent series. Differentiating twice with respect to $b$ in (63) we find that

$$x_L(s, b) = m_0 + \frac{\partial x_L}{\partial b}(s, b^*) (b - b^*) + O((b - b^*)^2) \quad (64)$$

as $b \downarrow b^*$, where

$$\frac{\partial x_L}{\partial b}(s, b^*) = -m_0^p + \sqrt{(m_0^p + \lambda(1 - p)\partial_b x(s, b^*))^2 + \lambda(1 - p)(y'(m_0)\partial_b x(s, b^*))^2}.$$

For every $s \in [0, 1]$ and $b \sim b^*$, let $(x_2(s, b), y_2(s, b))$ be the second crossing point of the orbit through $(x(s, b), y(x(s, b)))$ with the curve $\Gamma_1$, starting at $(x(s, b), y(x(s, b)))$. It is given by

$$y^2(x_2(s, b)) + \lambda x_2^2(s, b) + \frac{2b}{p + 1}x_2^{p+1}(s, b) = y^2(x(s, b)) + \lambda x^2(s, b) + \frac{2b}{p + 1}x^{p+1}(s, b).$$
Being the existence already known, to prove its analyticity in \( b \), we introduce the mappings
\[
J(\xi, s, b) := y^2(\xi) + \lambda \xi^2 + \frac{2b}{p+1} \xi^{p+1} - y^2(x(s, b)) - \lambda x^2(s, b) - \frac{2b}{p+1} x^{p+1}(s, b),
\]
and
\[
\tilde{J}(\xi, s, b) := \begin{cases} 
(\xi - x(s, b))^{-1} J(\xi, s, b), & \text{if } \xi \neq x(s, b), \\
D_\xi J(m_0, s, b), & \text{if } \xi = x(s, b),
\end{cases}
\]
for \( \xi \sim m_0, s \in [0, 1], \) and \( b \sim b^* \), which are real analytic at \( (m_0, s, b^*) \) for all \( s \in [0, 1] \). It should be noted that \( J(x(s, b), s, b) = 0 \) and that \( \tilde{J}(x(s, b^*), s, b^*) = 0 \) for all \( s \in [0, 1] \). Moreover,
\[
D_\xi \tilde{J}(x(s, b^*), s, b^*) = 2[(y'(m_0))^2 + \lambda(1 - p)] > 0.
\]
Consequently, the analyticity of \( x_2(s, b) \) is an easy consequence of the implicit function theorem. Moreover, by implicit differentiation, we obtain that
\[
x_2(s, b) = m_0 - \left( \frac{2m_0^p}{(y'(m_0))^2 + \lambda(1 - p)} + \partial_3 x(s, b^*) \right) (b - b^*) + O((b - b^*)^2) \tag{65}
\]
as \( b \to b^* \).

The time \( \tau_{1, a}(x(s, b), b) \) is given through
\[
\tau_{1, a}(x(s, b), b) = I_1(s, b) + I_2(s, b),
\]
where
\[
I_1(s, b) := \int_{x_L(s, b)}^{x_2(s, b)} \frac{du}{\sqrt{f(x(s, b), b, u)}}, \quad I_2(s, b) := \int_{x_L(s, b)}^{x_2(s, b)} \frac{du}{\sqrt{f(x(s, b), b, u)}},
\]
with
\[
f(x, b, u) := y^2(x) + \lambda(x^2 - u^2) + \frac{2b}{p+1} (x^{p+1} - u^{p+1}).
\]
By developing \( f \) around \( (x, b, u) = (m_0, b^*, m_0) \), we have that
\[
f(x(s, b), b, u) = \left( [(y'(m_0))^2 + \lambda(1 - p)] (x(s, b) - m_0)^2 + 2m_0^p (x(s, b) - m_0)(b - b^*) \right.
\]
\[
- 2m_0^p (b - b^*)(u - m_0) - \lambda(1 - p)(u - m_0)^2 + R(x(s, b), b, u),
\]
where
\[
R(x(s, b), b, u) := \sum_{i+j+k \geq 3} c_{ijk}(x(s, b) - m_0)^i(b - b^*)^j(u - m_0)^k
\]
for some constants \( c_{ijk} \) whose explicit knowledge is not important in this proof. According to (62), (64) and (65), and taking into account that \( u \in [x_L(s, b), x(s, b)] \) in the first integral, while \( u \in [x_L(s, b), x_2(s, b)] \) in the second one, we find that
\[
|u - m_0| \leq |\partial_3 x_L(s, b^*)| \cdot |b - b^*| + O(|b - b^*|^2),
\]
\[
|x(s, b) - m_0| \leq |\partial_3 x(s, b^*)| \cdot |b - b^*| + O(|b - b^*|^2),
\]
as \( b \to b^* \). Thus,
\[
|R(x(s, b), b, u)| \leq \sum_{i+j+k \geq 3} |c_{ijk}| |x(s, b) - m_0|^i|b - b^*|^j|u - m_0|^k \leq \sum_{h \geq 3} c_h(s)|b - b^*|^h,
\]
for some positive \( c_h(s), h \geq 3 \). Subsequently, we set
\[
g(s, \xi) := \sum_{h \geq 3} c_h(s) \xi^h, \quad s \in [0, 1], \quad \xi \geq 0, \quad \xi \sim 0.
\]
Note that
\[
\lim_{b \downarrow b^*} \frac{g(s, |b - b^*|)}{(b - b^*)^2} = 0
\]
for all \( s \in [0, 1] \). Setting
\[
h_{\pm}(s, b) := [(y'(m_0))^2 + \lambda(1 - p)](x(s, b) - m_0)^2 + 2m_0^p(x(s, b) - m_0)(b - b^*) \pm g(s, |b - b^*|)
\]
\[
f_{\pm}(s, b, u) := h_{\pm}(s, b) - 2m_0^p(b - b^*)(u - m_0) - \lambda(1 - p)(u - m_0)^2,
\]
and
\[
I_{1,\pm}(s, b) := \int_{x_L(s, b)}^{x_{\pm}(s, b)} \frac{du}{\sqrt{f_{\pm}(s, b, u)}}, \\
I_{2,\pm}(s, b) := \int_{x_L(s, b)}^{x_{\pm}(s, b)} \frac{du}{\sqrt{f_{\pm}(s, b, u)}},
\]
we find that
\[
I_{j,+}(s, b) \leq I_j(s, b) \leq I_{j,-}(s, b), \quad j \in \{1, 2\},
\]
for all \( s \in [0, 1] \) and \( b > b^* \) sufficiently close to \( b^* \).

Now, performing the change of variable \( \theta = u - m_0 \) and completing squares in the radicand of all the integrals, one can proceed as in the proof of Theorem 3.3 to obtain that
\[
I_{1,\pm}(s, b) = \frac{1}{\sqrt{\lambda(1 - p)}} (\arcsin \theta_{\pm}(s, b) - \arcsin \theta_{L,\pm}(s, b)),
\]
\[
I_{2,\pm}(s, b) = \frac{1}{\sqrt{\lambda(1 - p)}} (\arcsin \theta_{2,\pm}(s, b) - \arcsin \theta_{L,\pm}(s, b)),
\]
where
\[
\theta_{\pm}(s, b) := \frac{\lambda(1 - p)(x(s, b) - m_0) + m_0^p(b - b^*)}{\sqrt{m_0^{2p}(b - b^*)^2 + \lambda(1 - p)h_{\pm}(s, b)}},
\]
\[
\theta_{L,\pm}(s, b) := \frac{\lambda(1 - p)(x_L(s, b) - m_0) + m_0^p(b - b^*)}{\sqrt{m_0^{2p}(b - b^*)^2 + \lambda(1 - p)h_{\pm}(s, b)}},
\]
\[
\theta_{2,\pm}(s, b) := \frac{\lambda(1 - p)(x_2(s, b) - m_0) + m_0^p(b - b^*)}{\sqrt{m_0^{2p}(b - b^*)^2 + \lambda(1 - p)h_{\pm}(s, b)}}.
\]

Now, setting
\[
\theta(s) := \lim_{b \downarrow b^*} \theta_{\pm}(s, b), \quad \theta_L(s) := \lim_{b \downarrow b^*} \theta_{L,\pm}(s, b), \quad \theta_2(s) := \lim_{b \downarrow b^*} \theta_{2,\pm}(s, b),
\]
we find from (62), (64), (65), (67), and (66), letting \( b \downarrow b^* \), that

\[
\theta(s) = \frac{\lambda(1 - p)\partial b x(s, b^*) + m_0^p}{\sqrt{m_0^{2p} + \lambda(1 - p) \left\{ [(y'(m_0))^2 + \lambda(1 - p)] \left( \frac{\partial x}{\partial b} (s, b^*) \right)^2 + 2m_0^p \frac{\partial x}{\partial b} (s, b^*) \right\}}},
\]

\[
\theta_L(s) = \frac{\lambda(1 - p)\partial b x_L(s, b^*) + m_0^p}{\sqrt{m_0^{2p} + \lambda(1 - p) \left\{ [(y'(m_0))^2 + \lambda(1 - p)] \left( \frac{\partial x}{\partial b} (s, b^*) \right)^2 + 2m_0^p \frac{\partial x}{\partial b} (s, b^*) \right\}}},
\]

\[
\theta_2(s) = \frac{\lambda(1 - p)\partial b x_2(s, b^*) + m_0^p}{\sqrt{m_0^{2p} + \lambda(1 - p) \left\{ [(y'(m_0))^2 + \lambda(1 - p)] \left( \frac{\partial x}{\partial b} (s, b^*) \right)^2 + 2m_0^p \frac{\partial x}{\partial b} (s, b^*) \right\}}},
\]

Thus,

\[
\lim_{b \downarrow b^*} I_{1,\pm}(s, b) = \frac{1}{\sqrt{\lambda(1 - p)}} (\arcsin \theta(s) - \arcsin \theta_L(s)),
\]

\[
\lim_{b \downarrow b^*} I_{2,\pm}(s, b) = \frac{1}{\sqrt{\lambda(1 - p)}} (\arcsin \theta_2(s) - \arcsin \theta_L(s)),
\]

and, consequently, thanks to (68), we find that

\[
I_1(s) := \lim_{b \downarrow b^*} I_1(s, b) = \frac{1}{\sqrt{\lambda(1 - p)}} (\arcsin \theta(s) - \arcsin \theta_L(s)),
\]

\[
I_2(s) := \lim_{b \downarrow b^*} I_2(s, b) = \frac{1}{\sqrt{\lambda(1 - p)}} (\arcsin \theta_2(s) - \arcsin \theta_L(s)),
\]

for all \( s \in [0, 1] \). Finally, invoking again (62), (64), and (65), and differentiating with respect to \( s \), it is easily seen, after some tedious, but straightforward, manipulations, that

\[
\frac{dI_1}{ds}(s) = -\frac{dI_2}{ds}(s) = \frac{y'(m_0)}{(y'(m_0))^2 + \lambda(1 - p)(1 - s)^2} \quad \text{for all } s \in [0, 1],
\]

and, therefore,

\[
I_1(s) + I_2(s) = I_1(1) + I_2(1) = \tau_{1,\alpha}(x_t(b^*), b^*)
\]

for all \( s \in [0, 1] \). As

\[
\tau_{1,\alpha}(x_t(b^*), b^*) = \frac{1}{\sqrt{\lambda(1 - p)}} (\arcsin \theta(1) + \arcsin \theta_2(1) - 2 \arcsin \theta_L(1))
\]

\[
= \frac{1}{\sqrt{\lambda(1 - p)}} \left( 2 \arcsin \left( \frac{(y'(m_0))^2}{(y'(m_0))^2 + \lambda(1 - p)} + \pi \right) \right)
\]

\[
= \frac{1}{\sqrt{\lambda(1 - p)}} \left( 2 \arctan \left( \frac{y'(m_0)}{\sqrt{\lambda(1 - p)}} + \pi \right) \right) = \tau_2(\Omega),
\]

and all these times equal the corresponding times when \( x \) runs over the interval \([x_t(b), m_0]\), instead of \([x_{m_0}(b), x_t(b)]\), the proof is complete. \( \square \)
The results established by Theorem 4.2 are extremely sharp. Indeed, since \(x_t(b) \in (x_{m_0}(b), m_0)\), Part (a), in particular, entails that

\[
\lim_{b \downarrow b^*} \tau_1(x_t(b), b) = \tau_2(\Omega),
\]

which is far from being obvious, as using directly some (not valid) continuous dependence argument, one might have been tempted to argue that \(\tau_1(x_t(b), b)\) approximates \(\tau_1(\Omega)\) as \(b \downarrow b^*\), which is not true. Moreover, by Proposition 3,

\[
\lim_{x \downarrow m_0} \lim_{b \downarrow b^*} \tau_3(x, b) = \lim_{x \downarrow m_0} \tau_3(x, b^*) = \lim_{x \downarrow m_0} \tau_3(x) = \tau_3(\Omega) > \tau(\Omega),
\]

which, according to Theorem 4.2(c), shows that one cannot commute the double limit. This is utterly attributable to the lack of uniformity in the limit as \(b \downarrow b^*\).

It should be noted that if the equation

\[
\tau_{2n+1,s}(x, b) = 1 - 2\alpha
\]

has some solution \(x \in (x_{m_0}(b), m_0)\) for a \(b\) sufficiently close to \(b^*\), then the graph of the orbit through \((x, y(x))\) \(\in \Gamma_0\) must approach \((m_0, 0)\) as \(b \downarrow b^*\). Therefore, these solutions must perturb from the trivial solution \(u_0\). Similarly, if for some of these \(b\)'s

\[
\tau_{2n-1,a}(x, b) = 1 - 2\alpha,
\]

then the graph of the orbit through \((x, y(x))\) must approximate \((m_0, 0)\) as \(b \downarrow b^*\), and, therefore, these asymmetric solutions must perturb from \(u_0\) too.

Throughout the rest of this paper, we will use the following terminology, extending Definition 3.2.

**Definition 4.3.** Let \(u\) be a solution of (1) and an integer \(n \geq 1\). Then:

- \(u\) is said to be of type \(T_n\) if \(\tau_n(u(\alpha), b) = 1 - 2\alpha\).
- \(u\) is said to be of type \(T_{2n-1,a}\) if \(\tau_{2n-1,a}(u(\alpha), b) = 1 - 2\alpha\).
- \(u\) is said to be of type \(T_{2n-1,s}\) if \(\tau_{2n-1,s}(u(\alpha), b) = 1 - 2\alpha\).

It should be noted that the solutions of type \(T_n\) possess \(n\) local extrema in \((\alpha, 1 - \alpha)\) and that, similarly, the solutions of type \(T_{2n-1,j}\), \(j \in \{s, a\}\), exhibit \(2n - 1\) local extrema there in. Also, note that the solutions of type \(T_{2n}\) and \(T_{2n-1,a}\) are asymmetric, whereas the solutions of type \(T_{2n-1}\) and \(T_{2n-1,s}\) are symmetric.

As an immediate consequence of the previous analysis, the following multiplicity result, sharpening Theorems 3.4 and 3.5, holds. Subsequently, given two arbitrary functions \(u, v \in C[0, 1]\), we will denote

\[
\|u - v\|_\infty := \max_{t \in [0, 1]} |u(t) - v(t)|.
\]

**Theorem 4.4.** The following assertions are true:

(a) Suppose

\[
\tau_1(\Omega) > 1 - 2\alpha.
\]

Then, there exists \(\varepsilon > 0\) such that (1) has, at least, two solutions of type \(T_1\) for every \(b \in (b^*, b^* + \varepsilon)\). Moreover, the trivial solution \(u_0\) perturbs into a solution of type \(T_1\) as \(b\) separates away from \(b^*\), in the sense that there exists a family of solutions of type \(T_1\) of (1), say \(u^i\), \(\varepsilon > 0\), \(\varepsilon \sim 0\), with \(b = b^* + \varepsilon\), such that

\[
\lim_{\varepsilon \rightarrow 0} \|u^i - u_0\|_\infty = 0.
\]
(b) Suppose
\[ \tau_1(\Omega) < 1 - 2\alpha < \tau_2(\Omega). \]
Then, there exists \( \varepsilon > 0 \) such that (1) has, at least, two solutions of type \( T_1 \) for every \( b \in (b^*, b^* + \varepsilon) \). Moreover, the trivial solution \( u_0 \) perturbs into one of these solutions as \( b \) separates away from \( b^* \). Precisely, there exist a family of solutions of type \( T_1 \), say \( u^1_\varepsilon, \varepsilon > 0, \varepsilon \sim 0, \) such that
\[ \lim_{\varepsilon \downarrow 0} \| u^1_\varepsilon - u_0 \|_\infty = 0. \]

(c) Suppose
\[ \tau_{2n+1}(\Omega) < 1 - 2\alpha < \tau_{2n+2}(\Omega) \]
for some integer \( n \geq 1 \). Then, there exists \( \varepsilon > 0 \) such that (1) possesses, at least, \( 2(2n+1) \) solutions for each \( b \in (b^*, b^* + \varepsilon) \). More precisely, (1) possesses, at least, two solutions of type \( T_j \), for each \( 2 \leq j \leq 2n+1 \), one solution of type \( T_1 \) and an additional solution of type \( T_{2n+1} \). Moreover, the trivial solution \( u_0 \) perturbs, at least, into one solution of type \( T_{2n+1} \).

(d) Suppose
\[ \tau_{2n}(\Omega) < 1 - 2\alpha < n\tau(\Omega) \]
for some integer \( n \geq 1 \). Then, there exists \( \varepsilon > 0 \) such that (1) possesses, at least, \( 4n \) solutions for every \( b \in (b^*, b^* + \varepsilon) \). More precisely, (1) possesses, at least, two solutions of type \( T_j \) for every \( j \in \{2, \ldots, 2n\} \), one solution of type \( T_1 \), and a solution of type \( T_{2n-1,s} \). Moreover, the trivial solution \( u_0 \) perturbs, at least, into one solution of type \( T_{2n-1,s} \).

(e) Suppose
\[ n\tau(\Omega) < 1 - 2\alpha < \tau_{2n+1}(\Omega) \]
for some integer \( n \geq 1 \). Then, there exists \( \varepsilon > 0 \) such that (1) possesses, at least, \( 4n \) solutions for every \( b \in (b^*, b^* + \varepsilon) \). More precisely, (1) possesses, at least, two solutions of type \( T_j \) for every \( j \in \{2, \ldots, 2n\} \), one solution of type \( T_1 \), and a solution of type \( T_{2n+1} \). Moreover, the trivial solution \( u_0 \) perturbs, at least, into one solution of type \( T_{2n+1} \).

Proof. For Part (b), one should take into account that, thanks to Theorem 4.2,
\[ \tau(\Omega) \geq \lim_{b \downarrow b^*} \tau(x, b) \geq \lim_{b \downarrow b^*} \tau_{1,s}(x, b) \geq \lim_{b \downarrow b^*} \tau_{1,a}(x, b) = \tau_2(\Omega) \]
for all \( x \in [x_t(b), m_0] \), and, consequently, \( u_0 \) must have a perturbed solution of type \( T_1 \). Actually, the proof of Theorem 4.2 can be adapted to show that \( \lim_{b \downarrow b^*} \tau_{1,s}(x, b) \) is well defined for all \( x \in [x_t(b), m_0] \) and
\[ \left\{ \lim_{b \downarrow b^*} \tau_{1,s}(x, b) : x \in [x_t(b), m_0] \right\} = [\tau_2(\Omega), \tau(\Omega)]; \]
the end-points reached at \( x_t(b) \) and \( m_0 \), respectively. This information is needed in the proof of Part (d).

4.2. The case \( b < b^* \). Now, \( \Omega > m_0 \) and the phase portrait of (17) looks like shows Figure 17 for all \( 0 < b < b^* \). Naturally, Figure 17 is reminiscent of Figure 9 and, consequently, we will not paraphrase its construction again. The main difference we can observe between the cases \( b > b^* \) and \( b < b^* \) is that the relative positions of the points \( x_{m_0}, x_t \) and \( m_0 \) have been interchanged, so that, in case \( b < b^* \),
\[ x_{m_0}(b) < m_0 < x_t(b) < x_{m_0}(b) < x_{m_0}^1(b). \]
Note that \( \Omega(b) \uparrow \infty \) if \( b \downarrow 0 \).
We already know that the solutions of (17) connecting $\Gamma_0$ with $\Gamma_1$ in a time $1 - 2\alpha$ provide us with the solutions of (1), as described in Section 2. But, as the phase portrait changes when $b$ separates away from $b^*$, the time-maps $\tau_n = \tau_n(\cdot, b)$, $n \geq 1$, also change with $b$. In order to understand what is going on when $0 < b < b^*$ we should first introduce an exhaustive list of Poincaré maps, much like in the previous section, which will provide us with all the solutions of (1).

For every $b \in (0, b^*)$, we denote by

$$
\tau_1(\cdot, b) : D_1(b) := (x_0^-, m_0) \cup [x_t, \infty) \to [0, \infty)
$$

the Poincaré map defined, for every $x \in D_1$, as the minimal time needed by the solution of (20) to reach $\Gamma_1$. Figure 18 shows the corresponding orbits of two of these solutions for some $x < m_0$ (A) and $x > x_t$ (B). Note that $\tau_1(x_t, b)$ is the minimal time needed by the orbit through $(x_t, y(x_t))$ to reach $\Gamma_1$.

By continuous dependence, it is apparent that, for every $b \in (0, b^*)$,

$$
\lim_{x \downarrow x_0^-} \tau_1(x, b) = \infty, \quad \tau_1(m_0, b) := \lim_{x \uparrow m_0} \tau_1(x, b) = 0,
$$

since $m_0$ is not an equilibrium. In particular, $\tau_1(\cdot, b)$ admits a continuous extension to $m_0$ by setting $\tau_1(m_0, b) = 0$.

Next, we denote by

$$
\tau_{1,s}(\cdot, b) : D_{1,s}(b) := [m_0, x_t] \to [0, \infty)
$$

the Poincaré-map defined, for every $x \in (m_0, x_t)$, as the minimal time needed by the solution of (20) to reach $\Gamma_1$ exactly twice, whereas

$$
\tau_{1,s}(x_t, b) := \lim_{x \uparrow x_t} \tau_{1,s}(x, b) = \tau_1(x_t, b) \quad \tau_{1,s}(m_0, b) := \lim_{x \downarrow m_0} \tau_{1,s}(x, b) = \tau(m_0, b).
$$

(72)
Figure 18. The time-map \((x, y(x)) \mapsto \tau_1(x, b)\) for \(0 < b < b^*\)

Figure 19. The time-map \((x, y(x)) \mapsto \tau_{1,s}(x, b)\) for \(0 < b < b^*\)

Figure 19 shows the orbit of one of these solutions with \(x \in (m_0, x_t)\).

Now, for every \(0 < b < b^*\), we consider the map
\[
\tau_{1,a}(:b) : D_{1,a}(b) := [m_0, x_{m_0}] \to [0, \infty)
\]
defined, for every \(x \in [m_0, x_t]\), as the minimal time needed by the solution of (20) to reach \(\Gamma_1\) for the first time (see Figure 20A), and, for every \(x \in (x_t, x_{m_0}]\), as the minimal time needed by the solution of (20) to cross \(\Gamma_1\) twice (see Figure 20B).

Since
\[
\lim_{x \uparrow x_t} \tau_{1,a}(x, b) = \tau_{1,a}(x_t, b) = \tau_1(x_t, b)
\]
it follows from (72) that
\[
\tau_{1,a}(x_t, b) = \tau_{1,s}(x_t, b) = \tau_1(x_t, b).
\]

Moreover,
\[
\begin{align*}
\tau_{1,a}(x, b) &< \tau_{1,s}(x, b), \quad m_0 \leq x < x_t, \\
\tau_1(x, b) &< \tau_{1,a}(x, b), \quad x_t < x \leq x_{m_0}.
\end{align*}
\]
Figure 20. The time-map \((x, y(x)) \mapsto \tau_{1,a}(x, b)\) for \(0 < b < b^*\)

By construction, the solutions of \(\tau_{1,s} = 1 - 2\alpha\) and \(\tau_1 = 1 - 2\alpha\) provide us with symmetric solutions of (1), whereas the solutions given by \(\tau_{1,a} = 1 - 2\alpha\) are asymmetric, except at \(x = x_t\). All these solutions provide us with solutions of (1) with a single critical point in \((\alpha, 1 - \alpha)\). Conversely, any solution of (1) with a single critical point in \((\alpha, 1 - \alpha)\) must be of one of these forms.

As in the previous sections, we also introduce the Poincaré map

\[
\tau_2(\cdot, b) : D_2(b) := (x_0^-, m_0) \cup [x_{m_0}, x_0^+) \to [0, \infty)
\]

defined, for every \(x \in D_2(b) \setminus \{m_0\}\), as the minimal time needed by the solution of (20) to reach \(\Gamma_1\) exactly twice, while \(\tau_2(m_0, b)\) is the minimal time needed by the solution of (20) to reach \(\Gamma_1\). Figure 21 shows the corresponding orbits of two of these solutions for some \(x \in (x_0^-, m_0)\) (A) and \(x \in (x_{m_0}, x_0^+)\) (B). It should be observed that (1) cannot admit a solution of type \(T_2\) passing through \((x, y(x))\) if \(m_0 \leq x \leq x_{m_0}\).

Figure 21. The time-map \((x, y(x)) \mapsto \tau_2(x, b)\) for \(0 < b < b^*\)
By continuous dependence and symmetry, it is easily seen that
\[ \tau_{1,a}(x_{m_0}, b) = \tau_2(x_{m_0}, b) = \tau_2(m_0, b) = \tau_{1,a}(m_0, b), \]
and, by symmetry, each of the solutions on Figure 21(A) must be a reflection of some solution in Figure 21(B).

Once defined the time maps \( \tau_1(\cdot, b) \), \( \tau_{1,a}(\cdot, b) \), \( \tau_{1,s}(\cdot, b) \) and \( \tau_2(\cdot, b) \) in the whole interval \( 0 < b < b^* \), one can introduce *mutatis mutandis*, as in Section 4.1, the Poincaré maps (48) and (49). Similarly, one can extend the \( \theta_n \)'s introduced in (50) and (51) to the case \( 0 < b < b^* \) by setting
\[
\begin{align*}
\theta_{2n+1}(x, b) &= \begin{cases} 
\tau_{2(n+1)+1}(x, b) & \text{if } x \in (x_0^-, m_0], \\
\tau_{2n+1,s}(x, b) & \text{if } x \in (m_0, x_t], \\
\tau_{2n+1}(x, b) & \text{if } x \in (x_t, x_0^+),
\end{cases}
\end{align*}
\]
for all \( n \geq 0 \), and
\[
\begin{align*}
\theta_{2n}(x, b) &= \begin{cases} 
\tau_{2n}(x, b) & \text{if } x \in (x_0^-, m_0], \\
\tau_{2(n-1)+1,a}(x, b) & \text{if } x \in (m_0, x_{m_0}], \\
\tau_{2n}(x, b) & \text{if } x \in [x_{m_0}, x_0^+],
\end{cases}
\end{align*}
\]
for all \( n \geq 1 \). These functions satisfy the following counterpart of Lemma 4.1, whose repetitive proof will be omitted here.

**Lemma 4.5.** For every integer \( n \geq 0 \) and \( b \in (0, b^*) \), we have that
\[
\theta_{2n+1}(x, b) \begin{cases} > \theta_{2(n+1)+1}(x, b) & \text{if } x \in (x_0^-, x_t], \\
= \theta_{2(n+1)+1}(x_t, b) & \text{if } x = x_t, \\
< \theta_{2(n+1)+1}(x, b) & \text{if } x \in (x_t, x_0^+).
\end{cases}
\]
Moreover,
\[
\max \{ \theta_{2n+1}(x, b), \theta_{2(n+1)+1}(x, b) \} < \min \{ \theta_{2(n+1)+1}(x, b), \theta_{2(n+2)+1}(x, b) \}
\]
for all \( x \in (x_0^-, x_0^+) \).

Naturally, as \( x_{m_0}(b) \downarrow m_0 \) if \( b \uparrow b^* \), the following counterpart of Proposition 3 holds.

**Proposition 4.** For every \( n \geq 1 \) and \( \varepsilon > 0 \) sufficiently small,
\[
\lim_{b \uparrow b^*} \tau_n(x, b = \tau_n(x)
\]
uniformly in
\[
x \in [x_0^-(b^*) + \varepsilon, m_0 - \varepsilon] \cup [m_0 + \varepsilon, x_0^+(b^*) - \varepsilon].
\]

Similarly, the following counterpart of Theorem 4.2 holds.

**Theorem 4.6.** The following assertions are true:
(a) \( \lim_{b \uparrow b^*} \tau_{1,a}(x, b) = \tau_2(\Omega) \) for all \( x \in [m_0, x_{m_0}(b)] \).
(b) \( \lim_{b \uparrow b^*} \tau_{1,s}(m_0, b) = \tau(\Omega) \).
(c) \( \lim_{b \uparrow b^*} \lim_{x \uparrow m_0} \tau_3(x, b) = \tau(\Omega) \).

Naturally, thanks to (48) and (49), we also have that
\[
\lim_{b \uparrow b^*} \tau_{2n-1,a}(x, b) = \tau_{2n}(\Omega) \quad \text{for all } x \in [m_0, x_{m_0}(b)],
\]
\[
\lim_{b \uparrow b^*} \tau_{2n-1,s}(m_0, b) = n\tau(\Omega),
\]
and that
\[
\lim_{b \uparrow b^*} \lim_{x \uparrow m_0} \tau_{2n+1}(x, b) = n\tau(\Omega)
\]
for all \( n \geq 1 \).

According to these properties, the graphs of these functions, for \( 0 < b < b^* \) should be of the type illustrated in Figure 22. Through the remaining of this paper we will use the concepts introduced in Definition 4.3 extended to \( 0 < b < b^* \). By simply looking at Figure 22 one can easily infer the next counterpart of Theorem 4.4 for \( b \in (0, b^*) \). Figure 22 collects the plots of the first seven Poincaré maps for an appropriate choice of the parameters with \( b = b^* - 0.01 \).

![Figure 22. The graphs of the curves (48) and (49) for 0 < b < b^*](image)

**Theorem 4.7.** The following assertions are true:

(a) Suppose \( \tau_1(\Omega) > 1 - 2\alpha \). Then, there exists \( \varepsilon > 0 \) such that (1) has, at least, two solutions of type \( T_1 \) for each \( b \in (b^* - \varepsilon, b^*) \). Moreover, the trivial solution \( u_0 \) perturbs into a solution of type \( T_1 \) as \( b \) separates away from \( b^* \).

(b) Suppose \( \tau_1(\Omega) < 1 - 2\alpha < \tau_2(\Omega) \). Then, there exists \( \varepsilon > 0 \) such that (1) has, at least, two solutions of type \( T_1 \) for every \( b \in (b^* - \varepsilon, b^*) \). Moreover, the trivial solution \( u_0 \) perturbs, at least, into one solution of type \( T_1 \) as \( b \) separates away from \( b^* \).

(c) Suppose (69) for some integer \( n \geq 1 \). Then, there exists \( \varepsilon > 0 \) such that (1) possesses, at least, \( 2(2n+1) \) solutions for each \( b \in (b^* - \varepsilon, b^*) \). More precisely, (1) possesses, at least, two solutions of type \( T_j \), for each \( 2 \leq j \leq 2n + 1 \), one solution of type \( T_1 \) and an additional solution of type \( T_{2n+1} \). Moreover, the trivial solution \( u_0 \) perturbs, at least, into a solution of type \( T_{2n+1} \).

(d) Suppose (70) for some integer \( n \geq 1 \). Then, there exists \( \varepsilon > 0 \) such that (1) possesses, at least, \( 4n \) solutions for every \( b \in (b^* - \varepsilon, b^*) \). More precisely, (1) possesses, at least, two solutions of type \( T_j \) for every \( j \in \{2, \ldots, 2n\} \), one solution of type \( T_1 \), and a solution of type \( T_{2n-1} \). Moreover, the trivial solution \( u_0 \) perturbs into one solution of type \( T_{2n-1} \).

(e) Suppose (71) for some integer \( n \geq 1 \). Then, there exists \( \varepsilon > 0 \) such that (1) possesses, at least, \( 4n \) solutions for every \( b \in (b^* - \varepsilon, b^*) \). More precisely,
Lemma 5.2. The following assertions are true:

(1) possesses, at least, two solutions of type $T_j$ for every $j \in \{2, \ldots, 2n\}$, one solution of type $T_1$, and a solution of type $T_{2n+1}$. Moreover, the trivial solution $u_0$ perturbs into one solution of type $T_{2n+1}$.

5. Some general existence and non-existence results.

Lemma 5.1. Suppose that $b \in (0, b_{m_0})$ and (1) possesses a solution of type $T_n$ for some $n \geq 2$. Then,

(a) (1) has, at least, two solutions of type $T_j$ for all $2 \leq j \leq n$, if $n$ is even.
(b) (1) has, at least, two solutions of type $T_j$ for all $2 \leq j \leq n - 1$, if $n$ is odd.

In both cases, (1) exhibits, at least, a solution of type $T_1$.

Proof. Under the assumptions of the lemma, there exist $x \in (x^-_0(b), x^+_0(b))$ such that

$$\tau_n(x, b) = 1 - 2\alpha.$$  \hfill (73)

Consequently, thanks to Lemmas 4.1 and 4.5, the equation

$$\tau_j(x, b) = 1 - 2\alpha$$

has, at least, two solutions for all $2 \leq j \leq n - 1$ (see Figures 8, 15 and 22) and one solution for $j = 1$. If, in addition, $n$ is even, then, by reflection and symmetry, the solutions of (73) must appear by pairs. This ends the proof. \hfill $\square$

Lemma 5.2. The following assertions are true:

(a) Suppose $b_{m_0} \leq b < b_1$. Then, the problem (1) cannot admit any solution of type $T_{2n}$ with $n \geq 1$.
(b) Suppose $b \geq b_1$. Then, (1) admits, at most, solutions of type $T_1$. Moreover, any of these solutions $u$ has a unique local maximum in $(\alpha, 1 - \alpha)$ and satisfies

$$u_L := \min_{[\alpha, 1-\alpha]} u = u(\alpha) > m_0.$$  \hfill (73)

Proof. Remember that $b_{m_0}$ is the unique value of $b > b^*$ for which $(m_0, 0)$ belongs to the homoclinic trajectory through $(0, 0)$ (cf. Figure 10B if necessary), and that $b_1 > b_{m_0}$ is the unique value of $b$ for which this homoclinic is tangent to $\Gamma_0$, and, hence, to $\Gamma_1$ (see Figure 10C). Suppose

$$b_{m_0} < b < b_1.$$  \hfill (73)

By simply looking at the phase portraits of (17) it is easily seen that (1) cannot admit any solution of type $T_2$.

Indeed, the exterior trajectories of the solutions of (20) look like shown in Figure 23A and, consequently, they might provide us, at most, with solutions of type $T_1$ with a single local maximum in $(\alpha, 1 - \alpha)$. The interior trajectories are orbits of periodic solutions around $\Omega$. Let $u$ be denote the solution of (20) with $P = (x, y(x)) \in \Gamma_0$. Its first critical point occurs at some time, say $t_1$ such that $u(t_1) = x_L$. The second one at some further time $t_2 > t_1$ such that $u(t_2) = x_M$. But $(u(t), u'(t))$ cannot reach the curve $\Gamma_1$ without crossing again $(x_L, 0)$. Therefore, (1) cannot admit a solution of type $T_2$. Consequently, it cannot admit any solution of even order, however it might admit solutions of type $T_{2n+1}$, $T_{2n+1,s}$, and $T_{2n+1,a}$, for some integer $n \geq 0$. The proof of the lemma in this case can be adapted mutatis mutandis to cover the case $b = b_{m_0}$.

Now, suppose $b \geq b_1$. Then, the proof of the lemma is straightforward, as a glance to the phase portrait reveals that (1) might only admit solutions of type
Figure 23. The phase portrait in case $b_{m_0} < b < b_t$.

Figure 24. The phase portrait in case $b > b_t$.

Weighting Lemma 5.1 against Lemma 5.2 might cause a little bit of confusion, as their results seem paradoxical. On one hand, Lemma 5.1 establishes that if (1) admits a solution of type $T_n$, $n \geq 1$, then the problem must have solutions of type $T_j$ for all $1 \leq j \leq n$ provided $0 < b < b_{m_0}$. On the other, Lemma 5.2 establishes that, in case $b_{m_0} \leq b < b_t$, (1) might have a solution of type $T_{2n+1}$ but not solutions...
of type $T_{2n}$. As a byproduct, the structure of the diagram of Poincaré maps given by Figure 22, if $b^* < b < b_{m_0}$, should change as $b$ crosses $b_{m_0}$. Actually, the graphs of the $\tau_{2n}$’s in Figure 22 should disappear when $b$ crosses $b_{m_0}$. In order to realize what is going on, we will study all the possible solutions of (1) in case

$$b_{m_0} \leq b < b_t.$$ 

As in Section 4, we introduce the points $x_0^+ = x_0^+(b)$ and $x_t = x_t(b)$ similarly, and define the time map

$$\tau_1 : D_1 := (x_0^+, x_t] \cup (m_0, \infty) \to [0, \infty),$$

for every $x \in D_1(b)$, as the minimal time needed by the solution of (20) to reach $\Gamma_1$. Figure 25 shows two trajectories corresponding to $x \in (x_0^-, x_t)$ and $x > m_0$.

![Figure 25. The Poincaré map $\tau_1$ in case $b_{m_0} < b < b_t$](image)

By continuous dependence,

$$\lim_{x \downarrow x_0^-} \tau_1(x, b) = \infty, \quad \lim_{x \downarrow m_0} \tau_1(x, b) = 0.$$ 

Similarly, we introduce the map

$$\tau_{1,s} : D_{1,s} := (x_t, x_0^+) \to [0, \infty),$$

defined, for every $x \in D_{1,s}(b)$, as the time needed by the solution of (20) to reach $\Gamma_1$ exactly twice (see Figure 26).

By continuous dependence, we have that

$$\lim_{x \downarrow x_t} \tau_{1,s}(x, b) = \tau_1(x_t, b), \quad \lim_{x \uparrow x_0^-} \tau_{1,s}(x, b) = \infty.$$ 

Thus, $\tau_{1,s}$ may be regarded as a sort of continuous prolongation of $\tau_1$ from $(x_0^-, x_t)$ to the wider interval $(x_0^-, x_0^+)$. In addition, we introduce the map

$$\tau_{1,a} : D_{1,a} := (x_0^-, x_0^+) \to [0, \infty).$$
as follows. For every $x \in (x_0^-, x_t)$, $\tau_{1,a}(x,b)$ is the time needed by the solution of (20) to reach $\Gamma_1$ twice. For every $x \in (x_t, x_0^+)$, $\tau_{1,a}(x,b)$ is the minimal time needed by the solution of (20) to reach $\Gamma_1$. Finally,

$$\tau_{1,a}(x_t, b) := \lim_{x \to x_t} \tau_{1,a}(x,b) = \tau_{1}(x_t, b) = \tau_{1,s}(x_t, b).$$

Figure 27 shows the trajectories of these solutions in each of these situations.
Note that, by construction,
\[ \tau_1(x, b) < \tau_{1,a}(x, b), \quad x \in (x_0^-, x_t), \]
\[ \tau_{1,a}(x, b) < \tau_{1,s}(x, b), \quad x \in (x_t, x_0^+). \]
Consequently, the relative positions of the graphs of these curves are in complete agreement with Figure 28.

Finally, for every integer \( n \geq 1 \), we introduce the maps
\[ \tau_{2n+1}(x, b) := \tau_1(x, b) + n \tau(x, b), \quad x \in D_1(b), \]
\[ \tau_{2n+1,s}(x, b) := \tau_{1,s}(x, b) + n \tau(x, b), \quad x \in D_{1,s}(b), \]
\[ \tau_{2n+1,a}(x, b) := \tau_{1,a}(x, b) + n \tau(x, b), \quad x \in D_{1,a}(b), \]
for all \( b \in [b_{m_0}, b_t) \), where \( \tau(x, b) \) is the period of the solution of (20).

According to Lemma 5.2, if the solution of (20) provides us with a solution of (1), necessarily
\[ x \in \Sigma := \bigcup_{n \geq 0} \tau_{2n+1}^{-1}(1 - 2\alpha) \cup \tau_{2n+1,s}^{-1}(1 - 2\alpha) \cup \tau_{2n+1,a}^{-1}(1 - 2\alpha). \]
Conversely, if \( x \in \Sigma \), then, the solution of (20) provides us with a solution of (1).

It should be noted that, by construction, we have that
\[ \tau_{2n+1,a}(x, b) < \tau_{2(n+1)+1}(x, b), \quad x \in (x_0^-, x_t), \]
\[ \tau_{2n+1,s}(x, b) < \tau_{2(n+1)+1,a}(x, b), \quad x \in (x_t, x_0^+), \]
for all \( n \geq 0 \). Consequently, the set of graphs of all these Poincaré maps looks like shown in Figure 28, which shows the plots of some of these time maps.

**Figure 28.** The Poincaré maps in case \( b_{m_0} < b < b_t \)

To explain the transition from the set of \( \tau_n \)'s described in Figure 15 to the scheme shown in Figure 28, as \( b \) crosses the critical value \( b_{m_0} \), one should be aware that
\[ \lim_{b \uparrow b_{m_0}} x_{m_0}(b) = x_0^-(b_{m_0}), \quad \lim_{b \downarrow b_{m_0}} x_0^+(b) = m_0 = x_0^+(b_{m_0}). \]
Actually, $x_0^+(b)$ crosses $m_0$ as $b$ crosses $b_{m_0}$. Consequently,

$$\bigcap_{b<b_{m_0}} D_2(b) \subset \{ x_0^+(b_{m_0}) \},$$

i.e., the domains of definition of the $\tau_{2n}$'s shrink towards the end points of the intervals as $b \uparrow b_{m_0}$, which explains why such time maps cannot be defined for $b \geq b_{m_0}$. Actually, by continuous dependence, it is easily seen that for any sequences $\varepsilon_n > 0$, $n \geq 1$, such that $\lim_{n \to \infty} \varepsilon_n = 0$, and $x_n \in D_2(b_{m_0} - \varepsilon_n)$, $n \geq 1$, one has that

$$\lim_{n \to \infty} \tau_2(x_n, b_{m_0} - \varepsilon_n) = \infty.$$ 

Therefore, the graphs of the $\tau_{2n}$'s in Figure 15 grow up to infinity as $b \uparrow b_{m_0}$ while their domains of definition shrink to the limiting points, $x_0^+ (b_{m_0})$. Consequently, letting $b \uparrow b_{m_0}$ in Figure 15, we exclusively keep the local structures bifurcated when $b$ separated away from $b = b^*$, i.e., the left branch of $\tau_{2n+1}$, and the whole branches of $\tau_{2n+1,s}$ and $\tau_{2n+1,a}$, for all $n \geq 0$. As in Figure 15, $\tau_{2n+1,s}$ linked the left branch of $\tau_{2n+1}$ with the right branch of $\tau_{2(n+1)+1}$, and $\tau_{2n+1,a}$ linked the left and the right branches of the $\tau_{2n}$'s, it becomes apparent that indeed there is a continuous transition between all these time maps as $b$ crosses $b_{m_0}$, though, at first glance, the natures of Figures 15 and 28 might look so different. It should be remarked that such a transition still respects Lemma 4.1 as $b$ crosses $b_{m_0}$, because the $\tau_{2n}$'s gradually shorten their supports around their edges. This allows us to overcome the apparent paradox between the results established by Lemmas 5.1 and 5.2.

In the light of this discussion, it becomes apparent how emphasizing the super-linear character of (1), by increasing the value of $b$, has the effect of magnifying dramatically the local effects emerged from $\Omega = m_0$ as $b$ separated away from $b^*$.

Later, as $b$ increases and crosses $b_t$, the point $x_0^-(b)$ increases, while $x_0^+(b)$ decreases, until they meet at $b = b_t$. Therefore, all the graphs shown in Figure 28 collapse as $b$ crosses $b_t$, except the right branch of $\tau_1$ emerging from zero at $m_0$. A careful analysis of the phase portrait of (17) as $b$ crosses $b_t$ reveals that the graphs of the Poincaré maps $\tau_{2n+1}$, $\tau_{2n+1,s}$ and $\tau_{2n+1,a}$, $n \geq 0$, do actually blow up as $b \uparrow b_t$.

The next result sharpens Lemma 5.2(b) by establishing that (1) cannot admit any solution for sufficiently large $b$.

**Lemma 5.3.** There exists $b_c \geq b_t$ such that (1) cannot admit any solution if $b \geq b_c$.

**Proof.** Let $x > m_0$, $b \geq b_t$, and denote by $u$ the unique solution of (20). Then, arguing as in the last part of the proof of Theorem 3.3 and setting

$$x_M = \max_{[a,1-a]} u > m_0,$$
it becomes apparent that
\[
\tau_1(x, b) = 2 \int_0^{x_M} \frac{\alpha^2}{\sqrt{\lambda(x_M - \xi^2) + \frac{2b}{p+1}(\alpha_M^{p+1} - \xi^{p+1})}}
\]
\[
= 2 \int_{x/x_M}^1 \frac{d\theta}{\sqrt{\lambda(1 - \theta^2) + \frac{2b}{p+1}x_M^{p-1}(1 - \theta^{p+1})}}
\]
\[
< 2 \int_0^1 \frac{d\theta}{\sqrt{\lambda(1 - \theta^2) + \frac{2b}{p+1}x_M^{p-1}(1 - \theta^{p+1})}}.
\]

Thus, letting \( b \uparrow \infty \) in this estimate, we are led to
\[
\lim_{b \uparrow \infty} \tau_1(x, b) = 0 \quad \text{uniformly in } x \in (m_0, \infty).
\]
Consequently, there exists \( b_c \geq b_t \) such that
\[
\tau_1(x, b) < 1 - 2\alpha \quad \text{for all } x > m_0 \text{ and } b \geq b_c.
\]
Therefore, (1) cannot admit any solution of type \( T_1 \). Lemma 5.2(b) completes the proof.

Similarly, the following result holds.

**Lemma 5.4.** For sufficiently small \( b > 0 \), the problem (1) cannot admit a solution of type \( T_n \), nor of type \( T_{2n-1,s} \), nor \( T_{2n-1,a} \), with \( n \geq 2 \).

**Proof.** On the contrary, suppose that (1) admits one of such solutions. Then, there are \( t_1, t_2 \in [\alpha, 1 - \alpha] \), with \( t_1 < t_2 \), such that
\[
u(t_1) = u'(t_2) = 0,
\]
with \( u'(t) > 0 \) for all \( t \in (t_1, t_2) \). Thus, setting \( v = u' \), we have that
\[
v^2(t) + \lambda u^2(t) + \frac{2b}{p+1}u^{p+1}(t) = \lambda u^2(t_1) + \frac{2b}{p+1}u^{p+1}(t_1)
\]
\[
= \lambda u^2(t_2) + \frac{2b}{p+1}u^{p+1}(t_2).
\]
By looking at the phase portrait for the range \( 0 < b < b^* \), it should be noted that \( u(t) \) must be periodic and that
\[
u_1 := u(t_1) < m_0 < \Omega(b) < u_2 := u(t_2).
\]
Thus,
\[
1 - 2\alpha \geq t_2 - t_1 = \int_{u_1}^{u_2} \frac{d\xi}{\sqrt{-\lambda(\xi^2 - u_1^2)} - \frac{2b}{p+1}(\xi^{p+1} - u_1^{p+1})}
\]
\[
= \int_1^{u_2/u_1} \frac{d\theta}{\sqrt{-\lambda(\theta^2 - 1)} - \frac{2b}{p+1}u_1^{p-1}(\theta^{p+1} - 1)},
\]
and, hence,
\[
1 - 2\alpha > \int_1^{u_2/u_1} \frac{d\theta}{\sqrt{-\lambda(\theta^2 - 1)}}.
\]
Moreover,
\[
\frac{u_2}{u_1} > \frac{\Omega(b)}{m_0}
\]
and, consequently, we find that
\[
1 - 2\alpha > \int_1^{\Omega(b)/m_0} \frac{d\theta}{\sqrt{-\lambda(\theta^2 - 1)}}.
\]
(74)
As
\[
\lim_{b \downarrow 0} \int_1^{\Omega(b)/m_0} \frac{d\theta}{\sqrt{-\lambda(\theta^2 - 1)}} = \int_1^{\infty} \frac{d\theta}{\sqrt{-\lambda(\theta^2 - 1)}} = \infty,
\]
the estimate (74) cannot be satisfied for small \( b > 0 \). This concludes the proof. \( \Box \)

The proof of Lemma 5.4 does actually entail that
\[
\lim_{b \downarrow 0} \tau_2(x, b) = \infty \quad \text{for all} \quad x \leq m_0 \quad \text{and} \quad x \geq x_{m_0}(b),
\]
(75)
and that the period of any periodic solution around \( \Omega \) blows up to \( \infty \) as \( b \downarrow 0 \). On the other hand, we already know that
\[
\tau_{1,a}(x_{m_0}, b) = \tau_2(x_{m_0}, b) = \tau_2(m_0, b) = \tau_{1,a}(m_0, b).
\]
Thus,
\[
\lim_{b \downarrow 0} \tau_{1,a}(m_0, b) = \infty = \lim_{b \downarrow 0} \tau_{1,a}(x_{m_0}, b).
\]
Consequently, it seems a reasonable conjecture that
\[
\lim_{b \downarrow 0} \tau_{1,a}(x, b) = \infty \quad \forall \quad x > m_0.
\]
(76)
It should be noted that, since
\[
\tau_{1,a}(x, b) < \tau_{1,s}(x, b), \quad m_0 \leq x < x_t,
\]
if condition (76) holds, then
\[
\tau_{1,a}^{-1}(1 - 2\alpha) = \theta = \tau_{1,s}^{-1}(1 - 2\alpha)
\]
for sufficiently small \( b > 0 \) and, therefore, in such case, (1) cannot admit any solution
\[
u \in T_{1,a} \cup T_{1,s}
\]
for this range of \( b \)'s. The next result characterizes the structure of the solution set of (1) with \( M < \infty \) for sufficiently small \( b > 0 \) under conjecture (76).

**Lemma 5.5.** There exist \( \varepsilon > 0 \) and a differentiable curve \( u : [0, \varepsilon] \to C_{-}[0,1] \) such that, for every \( b \in (0, \varepsilon) \), \( u[b] \) provides us with a solution of type \( T_1 \) of (1), and \( u[0] \) is the unique solution of (1) with \( b = 0 \). In particular, the solutions \( u[b] \) bifurcate from \( u[0] \) at \( b = 0 \). Moreover, the map \( b \to u[b] \) is pointwise increasing.

Furthermore, there exists \( \delta > 0 \) such that for every \( b \in (0, \delta) \) the problem (1) admits a solution \( U[b] \) such that
\[
\max_{[\alpha, 1 - \alpha]} u[b] < m_0 < \max_{[\alpha, 1 - \alpha]} U[b]. \quad (77)
\]
Consequently, the solutions \( U[b] \) bifurcate from infinity at \( b = 0 \).
Proof. Throughout this proof it is appropriate to denote
\[ a(t, b) = a(t), \]
to emphasize the dependence of the weight function \( a \) on the parameter \( b \geq 0 \).

According to Theorem 4.4 of [18], the problem (1) with the choice \( b = 0 \) possesses a unique positive solution. Let us denote it by \( u[0] \). As in the interval \([0, 1]\) we have that
\[
\left( -\frac{d^2}{dt^2} - \lambda - a(t, 0)(u[0])^{p-1} \right) u[0] = 0,
\]
almost everywhere, and
\[
u[0](0) = u[0](1) = M > 0,
\]
it is apparent that \( u[0] \) provides us with a positive strict supersolution of the second order operator
\[ \mathcal{L} := -\frac{d^2}{dt^2} - \lambda - a(t, 0)(u[0])^{p-1} \]
under Dirichlet boundary conditions in \([0, 1]\). Thus, owing to Theorem 2.1 of [19], the principal eigenvalue of \( \mathcal{L} \), denoted by \( \sigma[\mathcal{L}] \), is positive, i.e. \( \sigma[\mathcal{L}] > 0 \).

Subsequently, we consider the nonlinear differential operator
\[ \mathcal{F} : D(\mathcal{F}) := C^1_0[0, 1] \times [0, \infty) \rightarrow C^1_0[0, 1] \]
defined through
\[
\mathcal{F}(v, b) := v + M - \left( -\frac{d^2}{dt^2} + \omega \right)^{-1} [(\lambda + \omega)(v + M) + a(\cdot, b)(v + M)^p], \quad (v, b) \in D(\mathcal{F}),
\]
where \( \omega > -\pi^2 \) is fixed. Naturally, the change of variable
\[ u = v + M \]
establishes a bijection between the solutions of (1) and the zeros of \( \mathcal{F} \). By construction, we have that
\[ \mathcal{F}(u[0] - M, 0) = 0. \]
Moreover, differentiating with respect to \( v \) leads to
\[
D_v \mathcal{F}(u[0] - M, 0) = \text{Id} - \left( -\frac{d^2}{dt^2} + \omega \right)^{-1} [\lambda + \omega + pa(\cdot, 0)(u[0])^{p-1}]
\[
= \text{Id} - \left( -\frac{d^2}{dt^2} + \omega \right)^{-1} [\lambda + \omega - p\chi_{[0,\alpha)\cup(1-\alpha,1]}(u[0])^{p-1}],
\]
where, as usual, for any given measurable subset \( L \subset \mathbb{R} \), we have denoted by \( \chi_L \) the characteristic function of \( L \). Note that \( D_v \mathcal{F}(u[0] - M, 0) \) is a Fredholm operator of index zero and suppose
\[ D_v \mathcal{F}(u[0] - M, 0)\psi = 0 \]
for some \( \psi \in C^1_0[0, 1], \psi \neq 0 \). Then, by elliptic regularity theory,
\[ \psi \in C^{2-}[0, 1] \cap C^2([0, 1] \setminus \{\alpha, 1 - \alpha\}) \]
and
\[ -\psi'' = \lambda \psi - p\chi_{[0,\alpha)\cup(1-\alpha,1]}(u[0])^{p-1}\psi, \]
almost everywhere. Thus, zero is an eigenvalue of

\[ \mathfrak{M} := -\frac{d^2}{dt^2} - \lambda + p c\chi_{[0,\alpha)\cup(1-\alpha,1]}(u[0])^{p-1} \]

\[ = \mathcal{L} + (p - 1)c\chi_{[0,\alpha)\cup(1-\alpha,1]}(u[0])^{p-1} \]

and, therefore, as the principal eigenvalue is dominant, \( \sigma[\mathfrak{M}] \leq 0 \). But this is impossible, because, by the monotonicity of the principal eigenvalue with respect to the potential, we should have that

\[ \sigma[\mathfrak{M}] > \sigma[\mathcal{L}] > 0. \]

Consequently, \( D_0\mathfrak{J}(u[0] - M, 0) \) is a topological isomorphism. Therefore, the existence of the solution curve \( u[b], b > 0, b \sim 0 \), is a direct consequence of the implicit function theorem. Actually, as \( \mathfrak{M} \) satisfies the strong maximum principle, by differentiating with respect to \( b \) in \( \mathcal{L} \) it is apparent that \( D_0u[b] > 0 \) for sufficiently small \( b > 0 \). Consequently, the map \( b \to u[b] \) is increasing for such \( b \)'s. Furthermore, as in the interval \( [\alpha, 1-\alpha] \) we have that

\[ -u''[0] = \lambda u[0] < 0, \]

the solution \( u[b] \) must be strictly convex for sufficiently small \( b > 0 \). Thus, \( u'[b](\alpha) < 0 \) for sufficiently small \( b > 0 \), and, hence, owing to Theorem 2.1, \( u[b] < m_0 \) in \( (\alpha, 1-\alpha) \) and it possesses a unique critical point (a local minimum) for small \( b > 0 \).

As a byproduct of these properties, thanks to the fact that \( \tau_1(m_0, b) = \tau(m_0, b) \uparrow \infty \) as \( b \downarrow 0 \), it follows that, for every \( b \in [0, \delta) \), (1) possesses a further solution, denoted by \( U[b] \), such that \( U[b](\alpha) > m_0 \). Actually, according to Lemma 5.4, we have that

\[ U[b](\alpha) \in \tau^{-1}_1(1-2\alpha) \cup \tau^{-1}_{1,s}(1-2\alpha) \]

for sufficiently small \( b > 0 \), and, consequently,

\[ \max_{[\alpha, 1-\alpha]} U[b] > \Omega(b), \]

because \( U[b](\alpha) > m_0 \). This ends the proof.

6. Global bifurcation diagrams. The main goal of this section is to ascertain all the possible global bifurcation diagrams of the set of solutions of (1) regarding \( b \geq 0 \) as the main bifurcation parameter. It turns out that the structure of these global diagrams becomes more and more complex the more negative the secondary parameter \( \lambda \) is, in complete agreement with the multiplicity results found in Section 3.

In the light of the analysis already done in the previous sections, it becomes apparent that, as a general tendency, the \( \tau_n(\cdot, b) \)'s decrease with \( b \) for \( b < b^* \), whereas they are increasing functions of \( b \) when \( b > b^* \); independently on whether the conjecture (76) holds or not. Indeed, by construction, (75) implies that \( \tau_n(\cdot, b), n \geq 3, \) must grow as \( b \downarrow 0 \) and adjust to the profiles of the \( \tau_n \)'s constructed in Section 3 at the critical value of the parameter \( b = b^* \), which plays the role of a sort of organizing center in our mathematical analysis. Similarly, thanks to Lemmas 5.3 and 5.4, the \( \tau_n \)'s must grow as \( b > b^* \) separates away from \( b^* \), except for the right branch of \( \tau_1 \) which approximates zero uniformly as \( b \uparrow \infty \). Incidentally, this does not mean that all these functions will be pointwise monotonic in \( b \). As a matter of fact, our numerical computations show that they are not.

At a first instance, in order to construct the global bifurcation diagrams of (1), it might be of great help to assume that all the graphs of the Poincaré maps \( \tau_n \),
$\tau_{2n-1,a}$, $\tau_{2n-1,s}$, $n \geq 1$, are well-shaped, as those already represented in Figures 15 and 22. As those shown in these figures, and some others that we have computed but not included here, adjust to this property, there is no serious reason to doubt that, in general, they will have a similar shape. Anyway, even if in the general case this is not true, the lack of such assumption would not change substantially the topological nature of the global bifurcation diagrams, and the readers should be able to implement very easily by themselves all the necessary changes in the forthcoming discussion.

Throughout this section we assume conjecture (76). Consequently, according to Lemmas 5.4 and 5.5, for small $b > 0$ all the global bifurcation diagrams consist of two curves consisting of solutions of type $T_1$: one emanating from the unique solution of (1) for $b = 0$, and the other one bifurcating from infinity at $b = 0$.

6.1. The case $\tau_1(\Omega) < 1 - 2\alpha < \tau_2(\Omega)$. Throughout this section we assume that

$$\tau_1(\Omega) = \tau_1(\Omega, b^*) < 1 - 2\alpha < \tau_2(\Omega, b^*) = \tau_2(\Omega). \quad (78)$$

According to Theorems 3.4, 4.4(b) and 4.7(b), the local bifurcation diagram around $b^*$ looks like shown in Figure 29. The trivial solution, $u_0$, exhibits a bilateral bifurcation to one solution of type $T_1$.

![Figure 29. An admissible bifurcation diagram under condition (78)](image)

In Figure 29, as in the remaining global bifurcation diagrams of this section, we are representing the value of $u(\alpha)$ versus the parameter $b$. It should be remembered that

$$(u(\alpha), y(u(\alpha))) \in \Gamma_0.$$ 

Although in Figure 29, and in all subsequent bifurcation diagrams, we are representing a bifurcation from infinity at $b = 0$, it should be noted that (77) does not necessarily entail

$$\lim_{b \downarrow 0} u(\alpha, b) = \infty,$$

but, simply,

$$\lim_{b \downarrow 0} u(1/2, b) = \infty.$$
Actually, because of the superlinear character of the boundary value problem (1), the solution $u(t) := u(t; b)$ might exhibit a spike-like behavior around $1/2$ with $u(\alpha, b), b \sim 0$, bounded. In such cases, we are representing such a bifurcation from infinity as an abuse of notation.

Figure 29 provides us with the same bifurcation diagram recently found by J. García-Melián [9] for a general multidimensional prototype of (1) with $\lambda = 0$ and $M = \infty$, which was already suggested by the mathematical analysis of J. López-Gómez [17].

6.2. The case $\tau_2(\Omega) < 1 - 2\alpha < \tau(\Omega)$. Throughout this section we assume that

$$\tau_2(\Omega) = \tau_2(\Omega, b^*) < 1 - 2\alpha < \tau(\Omega) = \tau(\Omega, b^*).$$  \hfill (79)

A significant difference with respect to the previous case is that now the trivial solution, $u_0$, exhibits a bilateral bifurcation to a solution of type $T_{1,s}$. By simply having a look at Figures 15 and 22 it is easily realized that these solutions will be defined until we reach the first values of $b$ where

$$\min \tau_{1,s}(\cdot; b) = 1 - 2\alpha,$$

which has been marked with a thick dot on the bifurcation diagram. In the case represented, there are two such values of $b$. Beyond, the solutions become of type one.

Another significant difference is that, under condition (79), the problem (1) with $b = b^*$ possesses two additional solutions of type $T_2$, which can be globally path-followed in the parameter $b$ until

$$\tau_2(m_0, b) = \tau_2(x_{m_0}, b) = 1 - 2\alpha,$$

At these two values of $b$, the solutions of type $T_2$ become solutions of type $T_{1,a}$ until they further meet the branch of solutions of type $T_{1,s}$.

Figure 30 shows an admissible global bifurcation diagram under condition (79).

![Figure 30. An admissible bifurcation diagram under condition (79)](image)

In this case, there are two values of $b$, denoted by $b_- < b^* < b_+$, such that

$$\min \tau_{1,a}(\cdot; b) = 1 - 2\alpha.$$
In each of these values two different situations might occur. Either,
\[ \min \tau_{1,a}(\cdot, b_{\pm}) \in \tau_{1,a}^{-1}(1 - 2\alpha), \]  
(80)
or not. When condition (80) holds, the three solution curves meet at the solution
\[ u \]  
determined by the unique \( x \) such that
\[ \min \tau_{1,a}(\cdot, b_{\pm}) = \tau_{1,a}(x, b_{\pm}), \]
which has been the situation illustrated by Figure 30 at \( b_{+} \). If not, the branch of
solutions of type \( T_{1,a} \) must exhibit a turning point besides its crossing point with
the branch of solutions of type \( T_{1,s} \), which has been the situation illustrated by
Figure 30 at \( b = b_{-} \). The remaining features of the diagram can be easily inferred
in the light of the results of Section 5.

6.3. The case \( \tau(\Omega) < 1 - 2\alpha < \tau_{3}(\Omega) \). Figure 31 shows an admissible global
bifurcation diagram in case
\[ \tau(\Omega) < 1 - 2\alpha < \tau_{3}(\Omega). \]  
(81)

![Figure 31](image)

**Figure 31.** An admissible bifurcation diagram under condition (81)

The unique significant difference with respect to the diagram of Figure 30 is
that, in the considered situation, according to Theorems 4.4(e) and 4.7(e), the
trivial solution \( u_{0} \) perturbs, as \( b \) separates away from \( b^{*} \), into a solution of type \( T_{3} \),
instead of type \( T_{1,s} \), as it happened under condition (79). Further, these solutions of
type \( T_{3} \) become solutions of type \( T_{1,s} \) as \( |b - b^{*}| \) increases, and beyond the situation
evolves as in the previous case.

6.4. The case \( \tau_{3}(\Omega) < 1 - 2\alpha < \tau_{4}(\Omega) \). Now, we assume that
\[ \tau_{3}(\Omega) < 1 - 2\alpha < \tau_{4}(\Omega). \]  
(82)
As in the previous cases, a careful analysis based on Figures 15 and 22 and the
general properties of Section 5 reveals that an admissible global bifurcation diagram
is the one sketched in Figure 32.

As already predicted by Theorem 3.5, the problem (1) has, at least, six solutions
for \( b = b^{*} \). Among them, two of type \( T_{3} \), two of type \( T_{2} \), and one of type \( T_{1} \), besides
the trivial solution $u_0$. According to Theorems 4.4 and 4.7, the solutions of type $T_j$, $j = 1, 2, 3$, perturb, as $b$ separates away from $b^*$, into solutions of the same type, while the trivial solution $u_0$ perturbs into an additional solution of type $T_3$. The $b$-evolution of the perturbed solutions as $b$ moves far away from $b^*$ has been ascertained by analyzing how the Poincaré maps illustrated in the Figures 15 and 22 vary as discussed at the beginning of Section 6.

Basically, much like in Figure 31, the diagram consists of two main curves. One filled in by solutions of odd type, joining to infinity the solution of the sublinear problem associated to (1) by switching to zero the parameter $b$, plus a sort of close loop filled in by asymmetric solutions, which bifurcates from the *odd curve* at some points on the arcs of solutions of type $T_{1,s}$, and behaves globally like the loop shown in Figure 31. Along the *odd curve*, the trivial solution perturbs into solutions of type $T_3$, which further switch to solutions of type $T_{1,s}$ before reaching, eventually, the solution arcs consisting of solutions of type $T_1$. Note that, in any circumstances, $m_0 = u_0(\alpha)$ is the value of $u(\alpha)$ where solutions of type $T_3$ become of type $T_{1,s}$. So, these transition points are at the same level in Figure 32.

A careful comparison between the structure of the global bifurcation diagrams of Figures 31 and 32 reveals that they are really very similar, in the sense that we are going to explain now. Indeed, under condition (81), as $\tau_3(\Omega)$ decreases crossing the value $1 - 2\alpha$, there is some critical value of the parameters where the trivial solution $u_0$ must perturb into three solutions of type $T_3$. This entails that the structure of the bifurcation diagram in a neighborhood of $u_0$ is $S$-shaped for slightly perturbed values of the parameters. Initially, the $S$ is small, as it perturbs from $u_0$, but, as $\tau_3(\Omega)$ separates away from $1 - 2\alpha$, the size of the $S$ gradually grows until it reaches a significant size, which has been the situation illustrated in Figure 32.

The structure of the solutions on the diagram fits the theoretical analytical results of Section 5, of course. In particular, for every $b \in (0, b^*)$ the problem has a solution of type $T_2$ whenever it has a solution of type $T_3$, however in Figure 32 there are
some values of $b > b^*$ where the model has some solution of type $T_3$ and no solution of type $T_2$. According to Lemma 5.2, necessarily $b \geq b_{m_0}$.

6.5. **The case** $\tau_4(\Omega) < 1 - 2\alpha < 2\tau(\Omega)$. When

$$\tau_4(\Omega) < 1 - 2\alpha < 2\tau(\Omega),$$

(83)

a further loop of higher order asymmetric solutions emerges from the solutions of type $T_4$ of (1) at $b = b^*$. These solutions must bifurcate from the solutions of type $T_{3,s}$ perturbed from the trivial solution. An admissible bifurcation diagram following these patterns has been represented in Figure 33.

![Figure 33](image)

**Figure 33.** An admissible bifurcation diagram under condition (83)

6.6. **The case** $2\tau(\Omega) < 1 - 2\alpha < \tau_5(\Omega)$. When

$$2\tau(\Omega) < 1 - 2\alpha < 2\tau_5(\Omega),$$

(84)

the only difference with respect to the previous case is the fact that the trivial solution perturbs into solutions of type $T_5$, instead of solutions of type $T_{3,s}$, as $b$ moves away from $b^*$. These solutions of type $T_5$ further become of type $T_{3,s}$ as $b$ reaches some critical values. Figure 34 shows an admissible bifurcation diagram.

6.7. **The case** $\tau_5(\Omega) < 1 - 2\alpha < \tau_6(\Omega)$. When

$$\tau_5(\Omega) < 1 - 2\alpha < \tau_6(\Omega),$$

(85)

the main difference with respect to the previous case is that the odd curve exhibits an additional wind emerged from the trivial solution as $\tau_5(\Omega)$ crossed $1 - 2\alpha$. The other features of the bifurcation diagram can be explained reasoning as in the previous cases.

At this step, it should be rather apparent how to get all the admissible global bifurcation diagrams using $b$ as the main parameter as the value of the secondary parameter $\lambda$ becomes more and more negative.
7. Large solutions. This section is devoted to the construction of large solutions of equation (2), i.e., the solutions of (1) with $M = \infty$. As the basic idea for constructing them is letting $M \uparrow \infty$ in the analysis already done in Section 2, this section emphasizes the dependence on $M > m^*$ of the differentiable curves $\Gamma_0$ and $\Gamma_1$ constructed in Theorem 2.2 and Corollary 2 by renaming them $\Gamma_{0,M}$ and $\Gamma_{1,M}$, respectively. Essentially, the methodology adopted in this section consists in
showing that, for the appropriate range of values of $x$, the limiting curves
\[ \Gamma_{0,\infty} := \lim_{M \uparrow \infty} \Gamma_{0,M}, \quad \Gamma_{1,\infty} := \lim_{M \uparrow \infty} \Gamma_{1,M}, \]
(86)
do actually possess similar properties to those of $\Gamma_{0,M}$ and $\Gamma_{1,M}$, and play an analogous role in the construction of the solutions of (1) carried out in Sections 3, 4, 5 and 6. According to Theorem 2.3, $\Gamma_{0,M}$ is the graph of a smooth function denoted by $y(x)$. In this section, it is appropriate to rename this function by $y_M$ and set
\[ \Gamma_{0,M} = \{ (x, y_M(x)) : x \geq 0 \}. \]
Although most of the proofs in this section follow the general patterns of the proofs of Section 2, by the sake of completeness, we will detail their main parts.

We begin our analysis with the following pivotal proposition which will provide us with $\Gamma_{0,\infty}$ in a compact way, without passing to the limit as $M \uparrow \infty$. 

**Proposition 5.** Suppose $\lambda \leq 0$. Then, the following properties hold:

(a) For every $x \geq 0$, the singular boundary value problem
\[ \begin{cases} -u'' = \lambda u - cu^p, \\ u(0) = \infty, \quad u(\alpha) = x, \end{cases} \]
(87)
has a unique positive solution.

(b) There exists a unique value of $x$, denoted by $m_{0,\infty}$, for which the solution of (87) satisfies $u'(\alpha) = 0$. In other words, $m_{0,\infty} = u(\alpha)$, where $u(t)$ is the (unique) solution of the singular problem
\[ \begin{cases} -u'' = \lambda u - cu^p, \\ u(0) = \infty, \quad u'(\alpha) = 0. \end{cases} \]
(88)

(c) For every $x \geq 0$, denote by $y_\infty(x)$ the value of $u'(\alpha)$, where $u(t)$ is the unique solution of (87). Then,
\[ y_\infty(x) \begin{cases} < 0 & \text{if } 0 \leq x < m_{0,\infty} \\ = 0 & \text{if } x = m_{0,\infty} \\ > 0 & \text{if } x > m_{0,\infty} \end{cases} \]

Throughout the rest of this section, we will consider the curve
\[ \Gamma_{0,\infty} := \{ (x, y_\infty(x)) : x \geq 0 \}. \]
Although (86) holds, the proof is postponed.

**Proof.** First, we will prove the existence and the uniqueness of $m_{0,\infty}$. Let $t_{\min}^\infty(x)$ denote the time needed by a solution of (6) with $u(0) = \infty$ to reach the $u$-axis in the phase plane (see Figure 2) at the point $x$, i.e., the necessary time to attain its minimum value $x$. Then,
\[ t_{\min}^\infty(x) = \int_x^\infty \frac{d\xi}{\sqrt{-\lambda(\xi^2 - x^2) + \frac{2c}{p+1}(\xi^{p+1} - x^{p+1})}} = \int_1^\infty \frac{d\theta}{\sqrt{-\lambda(\theta^2 - 1) + \frac{2c}{p+1}x^p(\theta^{p+1} - 1)}} \]
and, hence, $t_{\min}^\infty$ is decreasing and it satisfies
\[ \lim_{x \downarrow 0} t_{\min}^\infty(x) = \infty, \quad \lim_{x \uparrow \infty} t_{\min}^\infty(x) = 0. \]
Therefore, there exists a unique value of $x > 0$ for which
\[ t_{\min}^\infty(x) = \alpha. \]
Let us denote it by $m_{0,\infty}$. Due to the monotonicity of $t_{\min}^\infty$, we have that
\[
 t_{\min}^\infty(x) = \begin{cases} > \alpha & \text{if } 0 \leq x < m_{0,\infty}, \\ = \alpha & \text{if } x = m_{0,\infty}, \\ < \alpha & \text{if } x > m_{0,\infty}. \end{cases}
\]
(89)
Note that $t_{\min}^\infty(0) = \infty$. This proves the first sentence of Part (b). To show that (88) admits a unique solution, we proceed by contradiction. Suppose it admits two solutions, $u_1 \neq u_2$. Then, by the uniqueness of the Cauchy problem at $t = \alpha$, we infer that $u_1(\alpha) \neq u_2(\alpha)$, but this contradicts the uniqueness of $m_{0,\infty}$ and completes the proof of Part (b).

Thanks to (89), we have that the solution of (87), if it exists, satisfies $u'(\alpha) < 0$ when $0 \leq x < m_{0,\infty}$, whereas $u'(\alpha) > 0$ if $x > m_{0,\infty}$.

Now, we will prove Part (a). By Part (b), we already know that (87) does indeed admit a positive solution if $x = m_{0,\infty}$. Suppose $0 \leq x < m_{0,\infty}$ and denote by $T_\infty(v)$ the backward blow-up time of the solution of
\[
 \begin{cases} -u'' = \lambda u - cu^p, \\ u(\alpha) = x, \quad u'(\alpha) = v < 0, \end{cases}
\]
(90)
which is given by
\[
 T_\infty(v) = \int_{x}^{\infty} \frac{d\xi}{\sqrt{v^2 - \lambda(\xi^2 - x^2) + \frac{2c}{p+1} (\xi^{p+1} - x^{p+1})}}.
\]
Obviously, $T_\infty(v)$ is increasing in $(-\infty, 0)$ and, due to (89), it satisfies
\[
 \lim_{v \downarrow 0} T_\infty(v) = 0 \quad \text{and} \quad \lim_{v \uparrow 0} T_\infty(v) = t_{\min}^\infty(x) > \alpha.
\]
Therefore, there exists a unique $v < 0$ such that $T_\infty(v) = \alpha$, which completes the proof of Part (a) in this case.

Suppose $x > m_{0,\infty}$ and let $v_u(x) > 0$ denote the intersection between the unstable manifold passing through $(0,0)$ in the phase plane of (6) and the straight line $u = x$. For every $v \in (0, v_u(x))$, let $(m(v),0)$ be the crossing point between the orbit through $(x,v)$ and the $u$-axis. With these notations, it is easy to realize that the backward blow-up time of the solution of (90) is given by
\[
 T_\infty(v) = t_{\min}^\infty(m(v)) + t_x(m(v)),
\]
where
\[
 t_x(m(v)) := \int_{m(v)}^{x} \frac{d\xi}{\sqrt{v^2 - \lambda(\xi^2 - x^2) + \frac{2c}{p+1} (\xi^{p+1} - x^{p+1})}}
\]
is the necessary time to reach $x = u(\alpha)$ from the minimum $m(v)$. According to (89), we find that
\[
 \lim_{v \downarrow 0} T_\infty(v) = t_{\min}^\infty(x) < \alpha.
\]
Moreover, by continuous dependence,
\[
 \lim_{v \uparrow v_u(x)} T_\infty(v) = \infty,
\]
which implies the existence of a $v(x) \in (0, v_u(x))$ such that $T_\infty(v(x)) = \alpha$. To show the uniqueness and, hence, complete the proof of Part (a), we will prove the
monotonicity of $T_\infty(v)$. As $m(v)$ is decreasing in $(0, v_u(x))$, $t_{\infty min}(m(v))$ is increasing. Moreover,

$$t_x(m(v)) = \int_1^{x/m(v)} \frac{d\theta}{\sqrt{-\lambda(\theta^2 - 1) + \frac{2c}{p+1}m^{-1}(v)\theta^{p+1} - 1}},$$

and, consequently, $t_x(m(v))$ is increasing. Therefore, we conclude that $T_\infty(v)$ is also increasing when $x > m_{0,\infty}$. This completes the proof of Part (a).

Part (c) is a straightforward consequence of the construction that we have just carried out in this proof.

The next two results are corollaries from Proposition 5 and its proof.

**Corollary 3.** Suppose $\lambda \leq 0$, $M > 0$, and $u(t)$ solves

$$\left\{ \begin{array}{l}
-u'' = \lambda u - cu^p \\
u(0) = M, \quad u(\alpha) = x.
\end{array} \right.$$ 

Then, $u'(\alpha) > y_\infty(x)$.

**Proof.** Suppose $u'(\alpha) = y_\infty(x)$. Then, by the uniqueness of the solution for the associated Cauchy problem, $M = \infty$, which is impossible. Suppose $u'(\alpha) < y_\infty(x)$. Then, since $T_\infty$ is increasing, we find that

$$T_\infty(u'(\alpha)) < T_\infty(y_\infty(x)) = \alpha,$$

which is also impossible, because $u$ is defined in $[0, \alpha]$. Therefore, $u'(\alpha) > y_\infty(x)$.

**Corollary 4.** Any solution of (87) must achieve its minimum in $(\alpha/2, \alpha]$. In particular, it is decreasing in $(0, \alpha/2)$.

**Proof.** Going back to the construction of the solutions of (87) in the proof of Proposition 5, the result is obvious. Indeed, for every $x \in [0, m_{0,\infty}]$, the solution $u$ of (87) is decreasing and, hence,

$$\inf_{[0,\alpha]} u = u(\alpha) = x,$$

whereas, for every $x > m_{0,\infty}$,

$$\inf_{[0,\alpha]} u = m(u'(\alpha)) = u(t_{\infty min}(x)),$$

where $t_{\infty min}(x) \in (0, \alpha)$, because $u'(\alpha) = y_\infty(x) > 0$. It remains to prove that $t_{\infty min}(x) > \alpha/2$. This follows very easily by contradiction. Suppose $t_{\infty min}(x) \leq \alpha/2$. Then, by reflection around $t_{\infty min}$ and uniqueness, it is apparent that $u(t)$ solves the problem

$$\left\{ \begin{array}{l}
-u'' = \lambda u - cu^p \\
u(0) = \infty, \quad u(2t_{\infty min}) = \infty,
\end{array} \right.$$ 

and, hence, $u$ cannot be a solution of (87), because $2t_{\infty min}(x) \leq \alpha$. This ends the proof.

The next result establishes the monotonicity in $M$ of the functions $y_M(x)$ constructed in Theorem 2.3 to parameterize the curves $\Gamma_{0,M}$.

**Lemma 7.1.** For each $x \geq 0$, the map $M \mapsto y_M(x)$, $M > m^*$, is decreasing; $m^*$ is the value constructed in Theorem 2.1.
Proof. Let $m^* < M_1 < M_2$ and denote by $u_i$, $i = 1, 2$, the unique solution of
\[
\begin{aligned}
-u'' &= \lambda u - cu^p \\
u(0) &= M_i, \quad u(\alpha) = x.
\end{aligned}
\]
We want to prove that, under these conditions,
\[y_{M_1}(x) = u'_1(\alpha) > u'_2(\alpha) = y_{M_2}(x)\]
on the contrary, assume that $u'_1(\alpha) \leq u'_2(\alpha)$. Suppose $u'_1(\alpha) = u'_2(\alpha)$. Then, by the
uniqueness of the Cauchy problem associated to (6) at $t = \alpha$, we find that $u_1 = u_2$
and, in particular, $M_1 = M_2$, which is a contradiction. Thus, $u'_1(\alpha) < u'_2(\alpha)$. As
$u_1(\alpha) = u_2(\alpha) = x$, this implies that there exists $\varepsilon > 0$ such that $u_2(t) < u_1(t)$ for
all $t \in (\alpha - \varepsilon, \alpha)$. Therefore, since
\[M_1 = u_1(0) < u_2(0) = M_2,
\]
there exists $\theta \in (0, \alpha)$ such that $u_1(\theta) = u_2(\theta) = \tilde{u}$. Consequently, the boundary
value problem
\[
\begin{aligned}
-u'' &= \lambda u - cu^p \\
u(\theta) &= \tilde{u}, \quad u(\alpha) = x,
\end{aligned}
\]
possesses two solutions in $(\theta, \alpha)$, which contradicts the main result of [6]. The proof
is complete. □

As an immediate consequence of these results, we find the next one.

**Theorem 7.2.** For every $x \in [0, m_{0,\infty})$,
\[y_\infty(x) = \lim_{M \uparrow \infty} y_M(x)\]
In particular, the first relation of (86) holds. Moreover, $y_\infty \in C^1[0, m_{0,\infty})$ and it
satisfies the implicit relation
\[G_\infty(x, y_\infty(x)) = \alpha, \quad 0 \leq x < m_{0,\infty},\]
where
\[G_\infty(x, y) := \int_x^\infty \frac{d\xi}{\sqrt{y^2 - \lambda(\xi^2 - x^2) + \frac{2c}{p+1}(\xi^{p+1} - x^{p+1})}}.\]
Furthermore,
\[y'_\infty(x) = -\frac{\partial G_\infty}{\partial y}(x, y_\infty(x)) \quad \frac{\partial G_\infty}{\partial y}(x, y_\infty(x)).\]
for all $x \in [0, m_{0,\infty})$.

**Proof.** Fix $x \geq 0$. According to Corollary 3, the family $\{y_M(x)\}_{M > m^*}$ is bounded
from below by $y_\infty(x)$ and, owing to Lemma 7.1, it is decreasing in $M$. Therefore, the
limit
\[\hat{y}_\infty(x) := \lim_{M \uparrow \infty} y_M(x) \geq y_\infty(x)\]
is well defined.

Subsequently, for all $(x, y) \in ([0, \infty) \times \mathbb{R}) \setminus \{(0, 0)\}$ and $M \in (m^*, \infty]$, we set
\[G_M(x, y) = \int_x^M \frac{d\xi}{\sqrt{y^2 - \lambda(\xi^2 - x^2) + \frac{2c}{p+1}(\xi^{p+1} - x^{p+1})}}\]
and denote
\[m_{0, M} := y_M^{-1}(0)\].
By Corollary 3 and Lemma 7.1,

\[ m_{0,M_1} < m_{0,M_2} < m_{0,\infty} \]

if \( m^* < M_1 < M_2 < \infty \). Thus, the limit

\[ \tilde{m}_{0,\infty} := \lim_{M \to \infty} m_{0,M} \leq m_{0,\infty} \]

is well defined. Moreover, by definition,

\[ \alpha = t_{\min,M}(m_{0,M}) = \int_{0}^{m_{0,M}} \frac{d\xi}{\sqrt{-\lambda(\xi^2 - 1) + \frac{2c}{p+1}m_{0,M}^{p-1}(\xi^{p+1} - 1)}} \]

for all \( M > m^* \). Thus, letting \( M \to \infty \), shows that \( t_{\min}(\tilde{m}_{0,\infty}) = \alpha \). As \( t_{\min} \) is monotone, necessarily \( \tilde{m}_{0,\infty} = m_{0,\infty} \).

Let \( x \in [0, m_{0,\infty}) \). Then, for sufficiently large \( M > m^* \), we have that \( 0 \leq x < m_{0,M} \) and, hence,

\[ G_M(x, y_M(x)) = \alpha, \quad (91) \]

by construction. Thus, letting \( M \to \infty \) in (91), yields

\[ G_{\infty}(x, \tilde{y}_\infty(x)) = \alpha. \]

As \( G_{\infty}(x, \cdot) \) is (strictly) increasing for \( y \leq 0 \), necessarily

\[ \tilde{y}_\infty(x) = y_\infty(x) \quad \forall \ x \in [0, m_{0,\infty}). \]

This ends the proof of the first assertion.

The regularity of \( y_\infty \) in \( [0, m_{0,\infty}) \) is a straightforward consequence of the implicit function theorem applied to the equation

\[ G_{\infty}(x, y_\infty(x)) = \alpha, \quad 0 \leq x < m_{0,\infty}, \]

based on the fact that

\[ \frac{\partial G_{\infty}}{\partial y}(x, y_\infty(x)) = -\int_{x}^{\infty} \frac{y_\infty(x)}{\sqrt{[y_\infty(x)]^2 - \lambda(\xi^2 - x^2) + \frac{2c}{p+1}(\xi^{p+1} - x^{p+1})}} d\xi > 0 \]

because \( y_\infty(x) < 0 \) for all \( x \in [0, m_{0,\infty}) \). The proof is complete.

We need a last preliminary result about the positive blow-up solutions of (6).

**Lemma 7.3.** Suppose \( \lambda \leq 0, \ v < 0 \), and let denote by \( u(t) = u(t; x) \) the unique solution of the initial value problem

\[ \begin{cases} -u'' = \lambda u - cw^p & \text{in } (-\infty, 0] \\ u(0) = x \geq 0, \ u'(0) = v. \end{cases} \]

Then, \( u \) is positive and blows up in finite time

\[ T(x) := -G_{\infty}(x, v) < 0. \]

Moreover, \( T \) is a non-decreasing function such that

\[ \lim_{x \to \infty} T(x) = 0. \]
Proof. The unique delicate point is the monotonicity of $T$. To show it we will proceed by contradiction. Suppose that there are $0 \leq x_1 < x_2$ such that $T(x_1) > T(x_2)$ and let $u_i$, $i = 1, 2$, denote the unique solution of

$$\begin{cases} -u'' = \lambda u - cu^p \\ u(0) = x_i, \ u'(0) = v. \end{cases}$$

Then, there exists $\theta \in (T(x_1), 0)$ such that $u_1(\theta) = u_2(\theta)$ and, hence, the problem

$$\begin{cases} -u'' = \lambda u - cu^p \\ u(\theta) = u_1(\theta), \ u'(0) = v. \end{cases}$$

admits two solutions in $(\theta, 0)$, which is impossible.

The last result we need in order to cover the general case when $M \leq \infty$ is the next counterpart of Theorem 2.3 for $M = \infty$.

**Theorem 7.4.** Suppose $\lambda \leq 0$. Then, $y_\infty \in C^1[0, \infty)$ and it satisfies $y'_\infty(x) > 0$ for all $x \geq 0$.

**Proof.** Fix $\varepsilon \in (0, \alpha/2)$. According to Proposition 5 (with $\alpha = \varepsilon$), for every $x \geq 0$, the problem

$$\begin{cases} -u'' = \lambda u - cu^p, \\ u(0) = \infty, \ u(\varepsilon) = x, \end{cases} \quad (92)$$

has a unique positive solution. Let $y'_\infty : [0, \infty) \to \mathbb{R}$ denote the map defined by $y'_\infty(x) := u'(\varepsilon)$, where $u$ is the unique solution of (92). By Proposition 5, $y'_\infty$ satisfies

$$y'_\infty(x) \begin{cases} < 0 & \text{if } 0 \leq x < m^\varepsilon_{0, \infty} \\ = 0 & \text{if } x = m^\varepsilon_{0, \infty} \\ > 0 & \text{if } x > m^\varepsilon_{0, \infty} \end{cases}$$

where $m^\varepsilon_{0, \infty}$ is the unique value of $x$ for which $u'(\varepsilon) = 0$.

Thanks to Theorem 7.2 (applied with $\alpha = \varepsilon$), we obtain that $y'_\infty \in C^1[0, m^\varepsilon_{0, \infty})$ and that it satisfies the implicit relation

$$G_\infty(x, y'_\infty(x)) = \varepsilon \quad \forall x \in [0, m^\varepsilon_{0, \infty}).$$

As

$$\frac{\partial G_\infty}{\partial y}(x, y'_\infty(x)) > 0,$$

because $y'_\infty(x) < 0$ for all $x \in [0, m^\varepsilon_{0, \infty})$, and, thanks to Lemma 7.3,

$$\frac{\partial G_\infty}{\partial x}(x, y'_\infty(x)) \leq 0,$$

by implicit differentiation, it becomes apparent that

$$\frac{d}{dx}y'_\infty(x) = -\frac{\partial G_\infty}{\partial y}(x, y'_\infty(x)) \geq 0$$

for every $x \in [0, m^\varepsilon_{0, \infty})$.

Let $\bar{u}$ be the unique solution of

$$\begin{cases} -u'' = \lambda u - cu^p, \\ u(0) = \infty, \ u'(\alpha/2) = 0, \end{cases}$$

and set $\bar{x}(\varepsilon) := \bar{u}(\varepsilon)$;
it is defined by Proposition 5 (b) applied at $\alpha/2$. Similarly, we denote
\[ x(\varepsilon) := u(\varepsilon), \]
where $u$ is the unique solution of (87) with $x = 0$. Then,
\[ \Gamma_{0,\infty} = \mathcal{P}_{\alpha-\varepsilon} \left( \{ (x, y_{\infty}(x)) : x \in [g(\varepsilon), \bar{x}(\varepsilon)) \} \right), \]
where $\mathcal{P}_{\alpha-\varepsilon}$ is the Poincaré map
\[ x \in [g(\varepsilon), \bar{x}(\varepsilon)) \rightarrow \mathcal{P}_{\alpha-\varepsilon}(x) := (u(\alpha), u'(\alpha)), \]
$u(t)$ being the unique solution of the Cauchy problem
\[
\begin{cases}
-u'' = \lambda u - cu^p \\
u(\varepsilon) = x, \quad u'(\varepsilon) = y_{\infty}(x).
\end{cases}
\tag{93}
\]
As $\mathcal{P}_{\alpha-\varepsilon}$ is a diffeomorphism and
\[ \{ (x, y_{\infty}(x)) : x \in [g(\varepsilon), \bar{x}(\varepsilon)) \} \]
is a curve of class $C^1$, $\Gamma_{0,\infty}$ must be a curve of class $C^1$.

The differential of the Poincaré map $\mathcal{P}_{\alpha-\varepsilon}$ is given by $(DS(\alpha), \frac{d}{dt}DS(\alpha))$, where $DS(t)$ is the solution of
\[
\begin{cases}
-\frac{d^2}{dt^2}DS = (\lambda - cp^{p-1}) DS \\
DS(\varepsilon) = 1, \quad \frac{d}{dt}DS(\varepsilon) = \frac{d}{dx}y_{\infty}(x) \geq 0,
\end{cases}
\]
and $u$ is the solution of (93). Obviously, the same device used in the proof of Theorem 2.3 shows that both components of $D\mathcal{P}_{\alpha-\varepsilon}$ are strictly positive. Consequently, $\Gamma_{0,\infty}$ is the graph of an increasing $C^1$-function. Therefore, $y_{\infty}$ must be increasing and of class $C^1$. The proof is complete. \qed

Naturally, the change of variables $\hat{t} = 1 - t$ provides us with the counterparts of the previous results in the interval $[1 - \alpha, 1]$. As in Section 2, $\Gamma_{1,\infty}$ must be the reflection around the $u-$axis of $\Gamma_{0,\infty}$.

Remark 1. (a) Combining the results of this section with the techniques of Sections 3–6, it is easily realized that all the results for $M < \infty$ found in this paper hold for $M = \infty$ too, except for the local perturbation result established by Lemma 5.5, which can be proven following the general patterns of M. Bertsch and R. Rostamian [5]. J. López-Gómez [17] and J. García-Melián [9], though the proof, being rather technical and outside the scope of this work, will appear elsewhere.

(b) Throughout all this section we have used that some integrals, like the one defining $t_{\min}^\infty(x)$ in the proof of Proposition 5, are finite and tend to 0 as $x \uparrow \infty$. This is a very special case of the so-called Keller-Osserman condition, which allows us to generalize, substantially, our results by dealing with more general classes of nonlinearities, but this is far from being our goal in this work. So, we stop our analysis here.

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