A NOTE ON THE NIELSEN REALIZATION PROBLEM FOR HK MANIFOLDS

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Abstract. We give an answer to the Nielsen realization problem for hyper-Kähler manifolds in terms of the same invariant used for K3 surfaces. We determine that, for some of the known deformation types, the representation of the mapping class group on the second cohomology admits a section on its image and produce an example where lifting of diffeomorphisms behaves differently than the case of homeomorphisms.

Introduction

The Nielsen realization problem was originally formulated by Nielsen in [Nie32, Section 4], and then affirmatively solved in [Ker83]. The question is whether any finite group $G$ of mapping classes of a complex curve can be lifted to a group of diffeomorphisms (which preserve the metric and the complex structure). Equivalently, one wonders if $G$ fixes any point in the Teichmüller space.

The answer to the same question for K3 surfaces is given in [FL21]: their result shows that not every finite subgroup $G \subset \text{Mod} = \pi_0(\text{Diff})$ can be lifted, but there is an invariant $\Gamma_G$ which determines if it is possible or not.

In [BK19], the authors proved that for a K3 surface the representation of the mapping class group on the second cohomology group has a section over its image, and show an example where an order two mapping class does not lift to diffeomorphisms but its action in cohomology lifts to homeomorphisms.

We address the same problems for hyper-Kähler manifolds: in the first section we determine that, for $X$ of type $K3^{[n]}$ type with $n-1$ a prime power or OG10 type, the map $\rho : \text{Mod}(X) \to O^+(H^2(X, \mathbb{Z}))$ admits a section. In the second section we show that a similar example of order two group of mapping classes can be produced for hyper-Kähler manifolds of $K3^{[n]}$ type. In the third section we define an invariant analogous to the invariant $\Gamma_G$ used for K3 surfaces and conclude that a similar condition gives an answer to the Nielsen realization problem.

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Setting

We will denote by $X$ the underlying smooth manifold of a hyper-Kähler manifold, i.e. of a simply connected complex compact Kähler manifold such that $H^0(X, \Omega_X^2) = \mathbb{C}\sigma$ where $\sigma$ is a non-degenerate holomorphic 2-form. A Kähler
class on $X$ determines a unique Kähler-Einstein metric and a 2-sphere of complex structures for which the metric is still Kähler. Let $\text{Mod}(X) = \pi_0(\text{Diff}(X))$ be its mapping class group and $\Lambda_X = H^2(X, \mathbb{Z})$ the second cohomology group endowed with the Beauville-Bogomolov-Fujiki quadratic form $q_X$, which by the Fujiki relations is a topological invariant. We are going to use the notation $S$ when dealing with a K3 surface.

There is a map from $\text{Mod}(X)$ to $O(\Lambda_X)$ whose image $\Gamma$ in fact sits in the index 2 subgroup $O^+(\Lambda_X)$, the kernel of the spinor norm, and it has finite index in that by [Ver20, Theorem 1.1].

Consider the following diagram

$$
\begin{array}{ccc}
\text{Diff}(X) & \longrightarrow & \text{Mod}(X) \\
\downarrow & & \downarrow \\
\text{Homeo}(X) & \longrightarrow & O(\Lambda_X)
\end{array}
$$

the two following statement hold:

**Theorem 0.1** ([BK19], Theorem 1.1). Let $S$ be a K3 surface. There is a section $s : O^+(\Lambda_S) \to \text{Mod}(S)$ of $\rho : \text{Mod}(S) \to O^+(\Lambda_S)$.

**Theorem 0.2** ([BK19], Theorem 1.2). Let $S$ be a K3 surface. There is a subgroup of $\text{Mod}(S)$ of order 2 which does not lift to a subgroup of order 2 of $\text{Diff}(S)$. The image of the subgroup in $O^+(\Lambda_S)$ is non-trivial and it lifts to an order 2 subgroup of $\text{Homeo}(S)$.

Let $G$ be a finite group of $\text{Mod}(X)$, one can ask the following:

**Problem** (Nielsen realization). Does there exists an Einstein metric $g$ on $X$ such that $G$ is realizable as a subgroup of $\text{Isom}(X, g)$? Can the metric $g$ chosen to be also Kähler?

We want to generalize to the situation where $X$ is a higher dimensional hyper-Kähler manifold. In the first section we give a generalization of the analogue of Theorem 0.1 for hyper-Kähler manifolds for some deformation types, in the second section we conclude an analogue of Theorem 0.2 for hyper-Kähler of K3[η]-type. In the third section we address the Nielsen realization problem, giving a generalization of [FL21, Theorem 1.2].

1. **Sections of the representation map**

As in [Loo21], we consider the following Teichmüller spaces with the respective period maps:

$$
\begin{array}{ccc}
\mathcal{T}_{HK} & \longrightarrow & \mathcal{T}_{Ein} \\
\mathcal{P}_{HK} & \longrightarrow & \mathcal{P}_{Ein} \\
\mathcal{D}_{HK} & \longrightarrow & \text{Gr}^+(3, \Lambda_X \otimes \mathbb{R})
\end{array}
$$

where $\mathcal{T}_{HK}$ is the Teichmüller space of hyper-Kähler metrics with unitary volume and $\mathcal{T}_{Ein}$ is the Teichmüller space of Einstein metrics with unitary volume, the sets of such metrics up to isotopy. The space $\mathcal{D}_{HK} = \{[v] \in \mathbb{P}(\Lambda_X \otimes \mathbb{C})| q_X(v) = 0, q_X(v, \bar{v}) > 0\}$ is the period domain of $\mathcal{T}_{HK}$ and $\text{Gr}^+(3, \Lambda_X \otimes \mathbb{R})$ is the one of
$\mathcal{T}_{\text{Ein}}$, the vertical maps are the period maps associating to a metric the class $[\sigma_X]$ in the first case and the positively-oriented 3-space $P$ determined by the real and imaginary parts of $\sigma_X$ in the second case. The period maps give to the Teichmüller spaces the structure of respectively a complex and a differentiable space. The map $\mathcal{T}_{\text{HK}} \to \mathcal{T}_{\text{Ein}}$ simply forgets the choice of the complex structure and hence is a locally trivial 2-sphere bundle (inducing a bijection between the sets of connected components) and each connected component $C$ of $\mathcal{T}_{\text{Ein}}$ is mapped diffeomorphically onto a simply connected subset of $\text{Gr}^+ (3, \mathbb{A}_X \otimes \mathbb{R})$, the Grassmannian of oriented positive definite 3-spaces. In particular, $C$ is simply connected.

Denote by $\text{Mod}(X)_C$ the $\text{Mod}(X)$-stabilizer of the connected component $C$ and consider the Torelli group $T(X) = \ker(\rho)$. The monodromy group $\text{Mon}^2(X) \subseteq \Gamma \subseteq \text{O}^+ (\mathbb{A}_X)$ is the image of $\text{Mod}(X)_C$ and can be identified with the group of isometries coming from parallel transport operators [Ver13, Theorem 7.2].

In the case of K3 surfaces, $\text{Mod}(S)_C$ maps isomorphically onto $\text{O}^+ (\mathbb{A}_S) = \text{Mon}^2(S)$ via $\rho$ giving the isomorphism

$$\text{Mod}(S) \cong T(S) \rtimes \text{O}^+ (\mathbb{A}_S)$$

which implies Theorem 0.1. In this particular case, $T(S)$ permutes transitively the connected components of $\mathcal{T}_{\text{HK}}$ implying that the moduli space of marked K3 surfaces $\mathcal{M}_{\mathbb{A}_S} = \mathcal{T}_{\text{HK}} / T(S)$ is connected.

Remark 1. If $X$ is hyper-Kähler of dimension bigger than 2 then $\text{Mod}(X)$ could be just an extension of $T(X)$ and $\Gamma$, similarly $\text{Mod}(X)_C$ could be an extension of $\text{Mon}^2(X)$ and $T(X) \cap \text{Mod}(X)_C$, but by [Ver20, Remark 2.5] the intersection $T(X) \cap \text{Mod}(X)_C$ is always finite. Moreover, $T(X)$ acts on $\pi_0(\mathcal{T}_{\text{HK}})$ with finitely many orbits, each connected component has finite stabilizer and an element of $T(X)$ which fixes an element $g \in \mathcal{T}_{\text{HK}}$ fixes the entire connected component of $g$ ([Ver20, Theorem 3.1]). In general, the moduli space of marked hyper-Kähler manifolds $\mathcal{M}_{\mathbb{A}_X} = \mathcal{T}_{\text{HK}} / T(X)$ could have more connected components, but each one is simply connected.

Rephrasing what we said before, $\text{Mon}^2(X) \cong \text{Mod}(X)_C$ precisely when $T(X) \cap \text{Mod}(X)_C$ is trivial and $\text{Mod}(X) \cong T(X) \rtimes \Gamma$ exactly when $\rho$ admits a section on its image. There could be a section of $\rho$ over its image even if $\text{Mon}^2(X)$ is a proper subgroup of $\text{O}^+ (\mathbb{A}_X)$ and on the other hand a priori there is still the possibility that $\text{Mon}^2(X) = \Gamma = \text{O}^+ (\mathbb{A}_X)$ but $\text{Mod}(X)$ is not the semidirect product of $T(X)$ and the stabilizer of a component.

Here we generalize the proof of [BK19, Theorem 1.1] in some cases. Note that in this setting most of the groups we consider are discrete.

Lemma 1.1. If $X$ is such that $\text{Aut}(X) \to \text{O}^+ (\mathbb{A}_X)$ is injective, then $T(X)$ acts freely on $\mathcal{T}_{\text{Ein}}$. In particular, the projection $\mathcal{T}_{\text{Ein}} \to \mathcal{T}_{\text{Ein}} / T(X) = \mathcal{M}_{\mathbb{A}_X}$ is a principal $T(X)$-bundle.

Proof. Suppose that there are $[\varphi] \in T(X)$ and $[g] \in \mathcal{T}_{\text{Ein}}$ such that $\varphi(g) = g'$ is isotopic to $g$, then the path connecting $g'$ to $g$ connects $\varphi$ to a diffeomorphism $\varphi'$ which fixes $g$. The diffeomorphism $\varphi'$ acts by isometries on the 2-sphere of
complex structures associated to $g$, so it must preserve a complex structure and hence $\varphi' \in \text{Aut}(X)$ is an automorphism acting trivially in cohomology, in conclusion $\varphi' = \text{id}$ and this implies $[\varphi] = [\text{id}]$. □

**Proposition 1.2.** Let $X$ be hyper-Kähler such that $\text{Aut}(X) \to O^+(A_X)$ is injective, then $\rho : \text{Mod}(X) \to O^+(A_X)$ has a section over its image $\Gamma$.

**Proof.** We set

$$M = \mathcal{T}_{\text{Ein}} \times_{\text{Mod}(X)} \text{E Mod}(X).$$

Since $\mathcal{T}_{\text{Ein}} \to \mathcal{M}_{A_X}$ is a principal $\mathcal{T}(X)$-bundle, arguing as in [Gia09, Corollary 4.6], we find that

$$M \cong \mathcal{M}_{A_X} \times \Gamma \mathcal{E} \Gamma.$$  

The long exact sequence of homotopy groups associated to the fibration $\mathcal{M}_{A_X} \to M \to B \Gamma$ implies that

$$1 = \pi_1(\mathcal{M}_{A_X}) \to \pi_1(M) \to \pi_1(B \Gamma) \to \pi_0(\mathcal{M}_{A_X})$$

is exact. The natural projection map $M \to B \Gamma$ induces an injection $\pi_1(M) \hookrightarrow \pi_1(B \Gamma) = \Gamma$. From the description of $M$, the map $M \to B \Gamma$ must factor as

$$M \to \text{B Mod}(X) \to B \Gamma.$$  

Hence, the induced map $s : \pi_1(M) \to \pi_1(\text{B Mod}(X)) = \pi_0(\text{Mod}(X)) = \text{Mod}(X)$ is a splitting of $\rho : \text{Mod}(X) \to O^+(A_X)$. □

**Corollary 1.3.** If $X$ is $K^2[n]$ type with $n - 1$ a power of a prime, or $X$ is of OG10-type, then $\rho : \text{Mod}(X) \to O^+(A_X)$ has a section.

**Proof.** In these cases $\text{Mon}^2(X) = \Gamma = O^+(A_X)$ and $\text{Aut}(X) \to O^+(A_X)$ is injective. □

**Question 1.** What can be be said about the other known deformation types?

The group $\text{Mon}^2(X)$ is available for all the known deformation types. If it is maximal, then $\Gamma = O^+(X)$, but if it is a proper subgroup of $O^+(A_X)$, then $\Gamma$ is not known by the author.

Moreover, in the case $\text{Aut}(X) \to O^+(A_X)$ is not injective the argument given does not work, but it is a priori not clear if the same result might hold or not.

2. **Lift of an Order 2 Subgroup**

We now consider the Hilbert scheme of points $S^{[n]}$ of a K3 surface $S$ and its symmetric product $S^{(n)}$. Notice that for $f \in \text{Diff}(S)$ the induced map $f^{(n)}$ fixes the singular locus $\Delta = \{(x_1, \ldots, x_n) \in S^{(n)} \mid \exists i \neq j \text{ such that } x_i = x_j\} \subset S^{(n)}$ and hence it lifts via the resolution

$$S^{[n]} \to S^{(n)}$$

to an element $f^{[n]} \in \text{Diff}(S^{[n]})$ which fixes the exceptional locus, by the proof of [Boi12 Proposition 1.2]. This gives the inclusion

$$\Psi : \text{Diff}(S) \hookrightarrow \text{Diff}(S^{[n]})$$

since two elements $f, g \in \text{Diff}(S)$ such that $f^{[n]} = g^{[n]}$ must coincide: by contracting the exceptional divisor $f^{(n)} = g^{(n)}$ and then restricting to the small diagonal $S \cong$
We recall the following construction from [BK19, Section 3]:

**Theorem 2.1.** Let \( \{x_1, \ldots, x_n\} \in S^{(n)} \) \( x_1 = \cdots = x_n \) \( \subset \Delta \subset S^{(n)} \) one gets \( f = g \). With a similar argument, there is an injection
\[
O(\mathcal{A}_S) \hookrightarrow O(\mathcal{A}_{S^{[n]}})
\]
with a retraction again given by contraction and restriction.

For the same reason, if two elements \( f^{[n]} \) and \( g^{[n]} \) lie in the same path connected component of \( \text{Diff}(S^{[n]}) \), then since the action in cohomology stays the same for all elements in the image of a path from \( f^{[n]} \) to \( g^{[n]} \), it must fix the exceptional divisor, and so restricting to \( S \subset \Delta \) we get a path in \( \text{Diff}(S) \) which connects the preimages \( f \) and \( g \) via \( \Psi \). Hence there is also an injection
\[
\text{Mod}(S) \hookrightarrow \text{Mod}(S^{[n]})
\]
and a commutative diagram
\[
\begin{array}{ccc}
\text{Diff}(S) & \longrightarrow & \text{Mod}(S) & \longrightarrow & O(\mathcal{A}_S) \\
\downarrow & & \downarrow & & \downarrow \\
\text{Diff}(S^{[n]}) & \longrightarrow & \text{Mod}(S^{[n]}) & \longrightarrow & O(\mathcal{A}_{S^{[n]}})
\end{array}
\]

We recall the following construction from [BK19, Section 3]: \( S \) is topologically homeomorphic to \( 3(M \times M) \# 2(N) \), where \( N \) denotes the compact and simply-connected topological 4-manifold with intersection form the negative \( E_8 \)-lattice and \( M \) is the 2-sphere. Let \( f_0 : M \times M \to M \times M \) be given by \( f(x, y) = (y, x) \). Consider the equivariant connected sum \( 3(M \times M) \), the sum of three copies of \( (M \times M, f_0) \), remembering that \( f_0 \) has fixed points. Attaching two copies of \( N \), we get a continuous involution \( f : S \to S \).

**Theorem 2.1.** Let \( X \) be a hyper-Kähler of \( K3^{[n]} \) type. There is a subgroup of \( \text{Mod}(X) \) of order 2 which does not lift to an order 2 subgroup of \( \text{Diff}(X) \). The image of this group in \( O(\mathcal{A}_X) \), which is not trivial, lifts to a subgroup of order 2 in \( \text{Homeo}(X) \).

**Proof.** Let \( f \in \text{Homeo}(S) \) be the topological involution described above and consider the induced action \( \bar{\phi} \in O^+(\mathcal{A}_S) \subset O^+(\mathcal{A}_{S^{[n]}}) \). Clearly we can put \( \bar{\phi} = s(\bar{\phi}) \in \text{Mod}(S^{[n]}) \), where \( s : O^+(\mathcal{A}_S) \to \text{Mod}(S^{[n]}) \) is obtained by composing the section of \( \text{Mod}(S) \to O^+(\mathcal{A}_S) \), which exists by Theorem 0.1, with the inclusion in diagram (1).

Let \( h^{[n]} \in \text{Diff}(S^{[n]}) \) be a lift of \( \bar{\phi} \), then \( h^{[n]} \) cannot be an involution. Otherwise, since the exceptional divisor is preserved, it would determine an involution \( h \in \text{Diff}(S) \) which acts in cohomology as \( \bar{\phi} \) and this is a contradiction to [Theorem 3.1].

The statement for general \( X \) follows by Ehresmann’s Lemma.

This provides an example of order two subgroup of \( \text{Mod}(X) \) which does not admit a lift to \( \text{Diff}(X) \), but whose representation in second cohomology lifts to \( \text{Homeo}(X) \).

### 3. Nielsen realization

Let \( G \) be a finite subgroup of \( \text{Mod}(X) \), by abuse of notation its image in \( O^+(\mathcal{A}_X) \) will be sometimes denoted again by \( G \). We want to give an answer to the Nielsen
realization problem in terms of a lattice which is invariant, as done in [FL21] Theorem 2.1 for K3 surfaces.

Since $\text{Gr}^+(3, \Lambda_X \otimes \mathbb{R})$ is the symmetric space of $O^+(\Lambda_X \otimes \mathbb{R})$, it is non-positively curved and then $G$ must fix a point $P$. This means that $P$ is a $G$-invariant positive 3-space, and hence there is a linear representation $G \to \text{SO}(P)$ of $P$.

Let $I_G$ be the sum of all the irreducible $G$-subrepresentations of $\Lambda_X \otimes \mathbb{R}$ which are isomorphic to any of the ones appearing in $P$.

**Definition 3.1.** Let $\Gamma_G = I_G^+ \cap \Lambda_X$.

Notice that in [FL21] this is denoted by $L_G$ but this might lead to confusion because $\Lambda_X$ is sometimes denoted by $\mathbf{L}$ and $L_G$ denotes the coinvariant lattice $(L^G)^\perp$ where $L^G = \{ v \in L | g(v) = v \forall g \in G \}$, but the coinvariant lattice and $\Gamma_G$ in fact differ in general. For example, $\Gamma_G$ is always negative definite but if $G$ comes from the action of non-symplectic automorphisms then the coinvariant lattice $L_G$ is not definite.

We say that a linear form $\delta \in H^2(X, \mathbb{Z})^\vee$ is negative if its kernel has signature $(3, b_2(X) - 4)$. If $C$ is a connected component of the Teichmüller space, let $\Delta_C \subset H^2(X, \mathbb{Z})^\vee$ be the set of indivisible negative forms which are represented by an irreducible rational curve for a hyper-Kähler metric belonging to $C$.

**Theorem 3.2.** Let $G$ be a finite subgroup of $\text{Mod}(X)$.

- $G$ lifts to a group of isometries of an Einstein metric if and only if $G$ fixes a connected component $C$ of $T_{\text{Ein}}$, and $\Gamma_G$ does not contain any element of $\Delta_C$.
- $G$ lifts to a group of isometries for a hyper-Kähler metric if and only if $G$ fixes a connected component $C$ of $T_{\text{Ein}}$, $\Gamma_G$ does not contain any element of $\Delta_C$ and $\Gamma_G^C$ contains the trivial representation (in this case the metric can be chosen so that $X$ is projective and $G$ acts by algebraic automorphisms).

Similarly, a finite subgroup of $O^+(\Lambda_X)$ lifts under the same conditions when it is contained in $\text{Mon}^2(X)$.

**Proof.** From the description in [Loo21] Section 5], each connected component $C$ of the Teichmüller space is mapped diffeomorphically onto

$$\text{Gr}^+(3, \Lambda_X \otimes \mathbb{R})_{\Delta_C} = \text{Gr}^+(3, \Lambda_X \otimes \mathbb{R}) \setminus \bigcup_{\delta \in \Delta_C} \text{Gr}^+(3, \delta^\perp \otimes \mathbb{R})$$

which is connected (and simply connected if $b_2(X) > 5$). This in particular means that if $G$ comes from a group of isometries for an Einstein metric, then the image $P$ via the period map is $G$-invariant and not orthogonal to any $\delta$, hence $\Gamma_G$ does not contain any $\delta$. If $G$ preserves a metric which is also Kähler, then the positive cone must be preserved by $G$ and we can find a $G$-invariant Kähler class which spans the trivial representation in $\Gamma_G^C$. For a detailed description of the period map in the differential setting, we refer to [Loo21] Section 2.1].

Suppose now that $G$ is a subgroup of $\text{Mod}(X)$ which preserves a connected component of the Teichmüller space and for which $\Gamma_G$ does not contain any element in $\Delta_C$. We argue as in the proof of [FL21] Theorem 1.2]: among the $G$-invariant 3-spaces $P \subset \Gamma_G^C \otimes \mathbb{R}$, the ones such that $P^\perp \cap \Lambda_X = \Gamma_G$ are dense, so we can find
a positive-definite $P \subset \Gamma \otimes \mathbb{R}$ such that $P^\perp \cap \mathcal{A}_X = \Gamma \cap \mathcal{X}$. Now, since $P$ does not lie in any $\delta \cdot \mathcal{A}_X$, the surjectivity of the period map ensures that $G$ lifts to a group of isometries for a metric $g$ provided that it preserves a component $\mathcal{C}$ of the Teichmüller space. The last hypothesis is needed: if $G$ acts transitively on the components of $T_{Ein}$, then no metrics can be preserved. However, if $G$ fixes a component of the Teichmüller space, then it singles out counterimages of elements in $G$ (notice that those could be not unique). Lastly, having the trivial representation in $\Gamma$ means that $G$ fixes a positive class $0 = k \in \mathcal{P}$ and hence the orientation determines a complex structure on $X$ that makes $g$ a Kähler metric.

The trivial representation is spanned by a positive integral $(1,1)$-class, so we can conclude using Huybrechts’ projectivity criterion.

The situation could be much more complicated than for K3 surfaces: as already noticed in [Mar11] Question 10.5] the stabilizer $\text{Mod}(X)_C$ could depend on the component $\mathcal{C}$ and it could intersect nontrivially the Torelli group, so it could happen that not every subgroup of $\text{O}^+(\mathcal{A}_X)$ is the image of some stabilizer of a component and even those which are could have elements acting trivially on $\mathcal{A}_X$. In case $G \subseteq \text{Mon}^2(X)$ is the image of a group which intersects non-trivially $\text{T}(X)$, then a lift could be found but it would be an extension of $G$.

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