We study monopoles and corresponding 't Hooft tensor in a generic gauge theory. This issue is relevant to the understanding of color confinement.

I. MOTIVATION

The stringent experimental upper limits on the observation of free quarks in Nature indicate that quarks are absolutely confined due to some symmetry. The deconfining transition is a change of symmetry, i.e. an order disorder transition. Color is an exact symmetry. What can then be the extra symmetry responsible for confinement? The answer is provided by DUALITY. Infrared modes exist with topologically non trivial spatial boundary conditions (Homotopy) which can be labelled by the eigenvalues of conserved topological charges, i.e. in terms of a dual symmetry. Many examples of dual symmetries are known in statistical mechanics [1] and in field theory [2]. In (2+1) dimensions the homotopy group is $\Pi_1$ and the excitations are vortices, in (3+1) dimensions the homotopy group is $\Pi_2$ and the excitations are monopoles. This is equivalent to extend the formulation of the theory to a spacetime with an arbitrary but finite number of line-like singularities in each configuration (monopoles) [3].

II. MONOPOLES

A prototype is the 't Hooft-Polyakov monopole [4][5] in the $SU(2)$ gauge theory interacting with a Higgs scalar $\phi(r) = \hat{\phi}(r)\sigma_i$ in the adjoint representation. It is a static soliton solution made stable by its non trivial homotopy. In the "hedgehog" gauge at large $r$

$$\phi^i \simeq \frac{r^i}{|F|}$$

(1)

a mapping of the sphere $S_2$ at spatial infinity on the group, with non trivial homotopy. In the unitary gauge, where $\hat{\phi}^i \equiv \frac{\phi^i}{|\phi|} = \delta^i_3 \sigma_3$ is diagonal, a line singularity appears starting from the location of the monopole.

A gauge invariant field strength $F_{\mu\nu}$ can be defined ('t Hooft tensor), which coincides with the abelian field strength of the residual symmetry $F^3_{\mu\nu} = \partial_\mu A^3_\nu - \partial_\nu A^3_\mu$ in the unitary gauge [4]

$$F_{\mu\nu} = Tr(\hat{\phi} G_{\mu\nu}) - \frac{i}{g} Tr \left( \hat{\phi} [D_\mu \hat{\phi}, D_\nu \hat{\phi}] \right).$$

(2)

The monopole configuration has zero electric field ($F_{0i} = 0$). The magnetic field $H_i = \frac{1}{2} \epsilon_{ijk} F_{jk}$ is the field of a Dirac monopole of charge 2

$$\vec{H} = \frac{1}{g} \frac{\vec{r}}{4\pi r^3} + \text{Dirac String}$$

(3)
In a compact formulation, like is lattice, the Dirac string is invisible and a violation of Bianchi identity occurs. A magnetic current can be defined as

\[ j_\mu = \partial_\nu \tilde{F}_{\mu\nu} \]  

(4)

with \( \tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F_{\rho\sigma} \). A non zero value of it signals the violation of Bianchi identities. The current defined by eq.(4) is identically conserved due to the antisymmetry of \( \tilde{F}_{\mu\nu} \):

\[ \partial^\mu j_\mu = 0. \]  

(5)

This conservation law is the dual symmetry.

The dual symmetry is well defined also in absence of Higgs breaking. In a theory with no Higgs any operator \( \phi \) in the adjoint representation can be used and monopoles will be located at the zeroes of \( \phi \). The unitary gauges corresponding to different choices of \( \phi \) will differ by a gauge transformation which is regular everywhere except for a finite number of points. Creating a monopole means adding a singularity, independent on the choice of \( \Phi \) [9].

Recently some special groups like \( G_2 \) and \( F_4 \) became of interest, since they have no center and seem to confine [6], in contrast with the idea that center vortices could be the configurations responsible for confinement. However, for the group \( G_2 \) and \( F_4 \) (unlike \( SU(N) \) groups) it proves impossible to identify the abelian field strength in the unitary gauge with a ’t Hooft tensor of the form of eq.(2), because no solution exists for \( \phi \) such that this equation is true. Still there are monopoles and it is possible to define magnetic conserved currents. However, the approach above has to be modified for a more general construction of a ’t Hooft like tensor.

**III. MONOPOLES AND ’T HOOFT TENSOR FOR GENERIC GAUGE GROUP**

Let \( G \) be the gauge group, which we shall assume to be compact and simple. To define a monopole current we have to isolate an \( SU(2) \) subgroup, and break it to its third component, say \( T_3 \), by some ”Higgs field” \( \phi \) in the adjoint representation [2].

Each of the roots of the Lie algebra identifies an \( SU(2) \) subgroup. Since any root can be transformed to a simple root by a transformation of the group one can restrict the choice to the simple roots, which are as many as the rank \( r \) of the group and are represented by the circles in the Dynkin diagram of the Lie algebra. We are using here the language of any textbook on group representations, e.g. [3]. The ”Higgs fields” \( \phi^i \) are can be easily shown to be equal in the unitary gauge to the fundamental weights \( \mu^i \) corresponding to each simple root \( \vec{\alpha}_i \), it transforms in the adjoint representation of \( G \). The little group of \( \phi^i \) is the product of the \( U(1) \) generated by \( \phi^i \) times a group \( H \) which has as Dynkin diagram a diagram (connected or not connected) obtained by erasing from the diagram of \( G \) the root \( \alpha_i \) and the links which connect it to the rest (Levi subgroup), modulo a subgroup of the center of \( H \times U(1) \). Indeed \( \phi = \mu^i \) commutes with all the roots different from \( \alpha_i \).

There are \( r \) independent dual charges \( Q^a (a = 1, \ldots, r) \), whose values are in one-to-one correspondence with the homotopy classes of \( \Pi_2 \), and which are coupled to the abelian field strengths along \( \Phi^a \) in the unitary gauge, \( F^a_{\mu\nu} \).

\[ j^a_\nu = \partial_\mu \tilde{F}^a_{\mu\nu} \]  

(6)

\[ \partial_\nu j^a_\nu = 0 \]  

(7)

\[ Q^a = \int d^3 x j^a_0 (\vec{x}, t) \]  

(8)

The field strengths \( F^a_{\mu\nu} \) can be written in a gauge invariant form (’t Hooft tensors). The result for a generic group can be given in terms of the ”Higgs” fields \( \hat{\phi}^a (x) \), which coincide with the fundamental weights \( \phi^a \) in the unitary

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gauge, and of the set of non zero values of the numbers \( \lambda^j_i = (\phi^i \vec{\alpha})^2 \), which are characteristic constants of the group \( G \). The explicit form is

\[
F_{\mu\nu}^i = \text{Tr}(\phi^i G_{\mu\nu}) - \frac{i}{g} \sum \frac{1}{\lambda^j_i} \text{Tr} \left( \phi^i [D_\mu \phi^i, D_\nu \phi^i] \right) + \\
+ \frac{i}{g} \sum_{i \neq j} \frac{1}{\lambda^j_i \lambda^j_j} \text{Tr} \left( \phi^i [[D_\mu \phi^i, \phi^j], [D_\nu \phi^i, \phi^j]] \right) + \ldots \tag{9}
\]

The sum runs on the different values of \( \lambda^j_i \) each taken once. For \( SU(N) \) groups \( [\phi^i, E_{\vec{\alpha}}] = (\vec{c}^i \cdot \vec{\alpha}) E_{\vec{\alpha}} \) and \( (\vec{c}^i \cdot \vec{\alpha}) = 0, \pm 1 \), so that the 't Hooft tensor is the usual one

\[
F_{\mu\nu}^a = \text{Tr}(\phi^a G_{\mu\nu}) - \frac{i}{g} \text{Tr}(\phi^a [D_\mu \phi^a, D_\nu \phi^a]). \tag{10}
\]

For a generic group the projector is more complicated and in principle depends on the root chosen \( [7] \).

### IV. CONCLUDING REMARKS

- Experimental limits on the existence of free quarks indicate that confinement is an absolute property due to some symmetry. The deconfining transition is a change of symmetry.
- Color is an exact symmetry: the extra symmetry responsible for confinement is a dual symmetry related to spatial homotopy. Three dimensional physical space implies that the extra symmetry is generated by magnetic charges.
- Higgs breaking of magnetic gauge symmetry produces dual superconductivity and confinement. The transition to the Coulomb phase is deconfinement.
- Magnetic conserved current for a generic gauge group can be identified, and with them the corresponding 't'Hooft tensors. They label the dual degrees of freedom.

### V. REFERENCES

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