Fractional BPS Multi-Trace Fields of $\mathcal{N}=4$ SYM$_4$ from AdS/CFT

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Abstract

We prove inductively that every $k$-trace operator of $SO(6)_R$ irrep with Young tableau partition \{r$_1$, r$_2$, r$_3$\}, constructed out of $k$ chiral primaries in the twenty dimensional $SO(6)_R$ irrep, leads to a quasi primary field with protected conformal dimension. Our argument is based on perturbative evaluations of certain four point functions up to order $1/N^2$. 
1 Introduction

The Maldacena conjecture \cite{Maldacena1,Maldacena2,Maldacena3} has opened a way of finding nontrivial statements about certain conformal field theories. Its actually most fruitful example is the duality between $\mathcal{N}=4$ supersymmetric Yang Mills theory in four dimensions and type II B string theory in the $\text{AdS}_5 \times S^5$ background. Since in this highly supersymmetric gauge theory one has a free parameter, namely the gauge coupling constant, one can compare results from the weak coupling regime, which can be treated via perturbative techniques, with those from the strong coupling regime. The latter is accessible via the AdS/CFT duality. This will also be the philosophy of the current work, where we use the AdS/CFT correspondence to show that certain anomalous dimensions vanish at strong coupling.

In $\mathcal{N}=4$ superconformal Yang-Mills theory SYM$_4$ an infinite number of chiral primary fields

$$\mathcal{O}_m^I(x) = \text{tr}_{SU(N)}\{\Phi^{i_1}\Phi^{i_2}\cdots\Phi^{i_m}\} - \text{traces}$$

(1)

exists, which is formed from the scalar fields $\Phi$ in the gauge supermultiplet by a trace over the $SU(N)$ gauge labels. The superscripts $\{i\}$ denote the components of an $SO(6)_R$ vector and the rank $m$ symmetric tensor is made traceless by explicit subtraction. In Dynkin labels \cite{Dynkin} of $SU(4)$ the representation \cite{Dynkin} is $[0,m,0]$ and its dimension is

$$D_m = \frac{1}{12}(m+1)(m+2)^2(m+3).$$

(2)

It is well known that the chiral primary fields \cite{Dynkin} belong to short multiplets and that their conformal dimensions are protected

$$\Delta(\mathcal{O}_m^I) = m.$$  

(3)

In this work we consider composite fields which are produced from $k$ fields $\mathcal{O}_2(x)$ by a certain stepwise fusion, see section 2. In general these composite fields are spacetime tensors of rank $l$, they have approximate dimension

$$\Delta = 2k + l,$$

(4)

and are $SO(6)_R$ traceless tensors, which we want to describe by $SO(6)_R$ Young tableaus. Each such tableau has at most three rows of length $r_1 \geq r_2 \geq r_3 \geq 0$, \{\{r_1, r_2, r_3\}. We want to exclude contractions of $SO(6)_R$ vector labels throughout, such that

$$r_1 + r_2 + r_3 = 2k,$$

(5)

and \{\{r_1, r_2, r_3\} is a partition of $2k$. Our fusion procedure produces a tower of quasi-primary (or mixtures of quasi-primary) fields with increasing $l$. The

\footnote{We use brackets for $SU(4)_R$ Dynkin labels.}
minimal $l$ is $l_0$

$$l_0 = \begin{cases} 
0 & \text{if $r_1, r_2, r_3$ are all even;} \\
1 & \text{if only one $r_i$ is even.}
\end{cases} \quad (6)$$

It is important to point out that the composite fields with $l_0 = 0$ cannot be supersymmetric descendants, since the minimal dimension for operators composed of $k$ operators $\mathcal{O}_2$ is reached by the $l_0 = 0$ fields due to (4), and each supersymmetry charge raises the dimension by $1/2$. On the other hand, with the help of (4) one can show that the composite fields with $l_0 = 1$ can be obtained by the application of two (different) supersymmetry charges from a composite field with the same number of $SO(6)_R$ blocks. Thanks to supersymmetry the composite fields with $l_0 = 1$ inherit their properties from those with $l_0 = 0$.

The representation of $SO(6)_R$ characterized by the partition $\{r_1, r_2, r_3\}$ is reducible if $r_3 > 0$. It decomposes into a conjugate pair of a self-dual and an antiself-dual representation with Dynkin labels

$$[r_2 - r_3, r_1 - r_2, r_2 + r_3] \oplus [r_2 + r_3, r_1 - r_2, r_2 - r_3]. \quad (7)$$

We will show that in the case $l = l_0$ these fields have protected dimensions $2n + l_0$. The scalars ($l_0 = 0$) probably belong to $\frac{1}{4}$-BPS multiplets, whereas the vectors as supersymmetric descendants might be $\frac{1}{4}$-BPS or $\frac{1}{2}$-BPS fields as well. If $r_3 = 0$ then the Dynkin labels reduce to

$$[r_2, r_1 - r_2, r_2]. \quad (8)$$

We find that the corresponding field with $l = l_0$ has a protected dimension and the scalar supposedly belongs to a $\frac{1}{4}$-BPS multiplet. Again, the BPS-type of the vector can only be determined in a case by case consideration. Finally for $r_2 = r_3 = 0$ we are led to

$$[0, r_1, 0] \quad (r_1 \text{ even}); \quad (9)$$

which is the case dealt with by Skiba [4] if $l = 0$ and belongs to $\frac{1}{2}$-BPS multiplets presumably. Our proof of protectedness is based on AdS/CFT correspondence and a perturbative expansion in $1/N^2$ and takes account only of the first order $O(1/N^2)$.

In our construction the number of $SU(N)$-traces is maximal. Other authors [1, 2] have investigated similar problems with fields containing two and three traces only, and then resolved the mixtures of the quasi-primary fields algebraically or numerically. Instead we assume that for $l = l_0$ we obtain pure quasiprimary fields. This assumption is based on the fact that in free field theory there is no freedom to place one ($l_0 = 1$) or no ($l_0 = 0$) derivatives into a normal product. However, for $l > l_0$ there is always such freedom and mixtures must appear.

Further results on vanishing anomalous dimensions have been found in [8, 9], see also [10], by an analysis of three-point functions of two $\frac{1}{2}$-BPS operators with
any other possible operator in a purely group theoretic fashion. This kinematical approach nicely complements our results in two ways: On the one hand our approach by considering four point functions is based on a $1/N$-expansion of certain four point functions and is thus dynamical. On the other hand, in the references given above, protected dimensions for operators with $SU(4)_R$ irreps contained in the reduction of $[0, p, 0] \otimes [0, q, 0]$ for any $p, q$ are obtained, whereas we only have results for even $p, q$. Nevertheless, we find vanishing anomalous dimensions for fields with $SU(4)_R$ irreps which are not contained in the reduction of any $[0, p, 0] \otimes [0, q, 0]$, but which are contained in a chain of reductions. E.g. we get a protected dimension for a field with $SU(4)_R$ irrep $[0, 2, 4]$, which cannot be obtained by a single reduction from the above factors, but which is contained in the “triple” reduction

$$
\left(\left(\left[0, 2, 0\right] \otimes \left[0, 2, 0\right]\right) \oplus \left[0, 2, 0\right]\right) \oplus \left[0, 2, 0\right] \\
= \left(\left[2, 0, 2\right] \oplus \cdots \right) \otimes \left[0, 2, 0\right] \oplus \left[0, 2, 0\right] \\
= \left[2, 2, 2\right] \oplus \cdots \otimes \left[0, 2, 0\right] = \left[0, 2, 4\right] \oplus \cdots .
$$

(10)

2 The fusion procedure

The general idea of our method to produce multitrace operators with protected dimensions is to consider a stepwise fusion process, which may be sketched by fig. 1.

![Figure 1: The fusion procedure](image)

In this picture each vertex denotes a fusion of the two incoming fields with protected dimensions to the outgoing one. By fusion we mean the formation of the normal product according to some rules given below and, since the normal product of two fields is in general reducible under $SO(6)_R$, we have to project onto an $SO(6)_R$ irrep.

The operators $\mathcal{O}_2$ are the single trace operators

$$
\mathcal{O}_2^I(x) = tr_{SU(N)}\left\{\Phi^i(\Phi^j - \frac{1}{6}\Phi^2\delta^{ij})\right\},
$$

(11)

We thank E. Sokatchev for bringing our attention to this point.
where $i, j$ are $SO(6)_R$ vector labels and $I$ denotes a basis of the twenty-dimensional representation space of the symmetric traceless irrep of $SO(6)_R$ under which $O^I_2$ transforms. It is well known that $O^I_2$ as chiral primary operator has protected dimension 2 and we normalize its two-point function as

$$
\langle O^I_1(x_1)O^I_2(x_2) \rangle = \frac{\delta^{I_1I_2}}{(x_{12}^2)^2}.
$$

(12)

The operators $\Psi_{D_i}$ are multitrace operators, which appear in the normal product of the two incoming operators $O_2$ and $\Psi_{D_{i-1}}$. These fields transform in the irrep $D_i$ of $SO(6)_R$.

The rules for the fusion procedure are:

1. Discard $SO(6)_R$ representations with contractions and thereby all singular terms in operator product expansions (“block number conservation”).

2. Admit only scalar intermediate quasi-primary fields.

We will show that any irrep appearing in this sequence of fusions contains at least one field with protected dimension and that thus the dimension of this field equals the sum of the dimensions of its factors.

We do not obtain every $SO(6)_R$ irrep if we start only with scalar factors. The set of fields, which are out of range of our fusion procedure with only scalar fields in the input, consists of fields with the following $SO(6)_R$ irreps:

$$
\{2r + 1, 2r + 1, 2s\} \text{ with } r \geq s,
$$

$$
\{2s, 2r + 1, 2r + 1\} \text{ with } r < s.
$$

(13)

But since these fields are spacetime vectors according to (6), they are supersymmetric descendants of fields with $l_0 = 0$. Since our proof holds for the $l_0 = 0$ fields, we can conclude that the fields with $SO(6)_R$ irrep (13) also have protected conformal dimension.

### 2.1 The first and simplest example

Besides being illustrative, the first step of the fusion procedure is the start of our inductive proof, so we repeat the arguments contained in [14, 8, 12, 14], which are useful for us. The $SO(6)_R$ representation of the operator product $O^{I_1}_2(x_1)O^{I_2}_2(x_2)$ decomposes into irreps

$$
\{2, 0, 0\} \otimes \{2, 0, 0\} = \{4, 0, 0\} \oplus \{3, 1, 0\} \oplus \{2, 2, 0\} \oplus \{2, 0, 0\} \oplus \{1, 1, 0\} \oplus \{0, 0, 0\}.
$$

(14)

The irreps in the second line contain at least one contraction and are discarded according to rule 1. Then the four-point function $\langle O^{I_1}_2(x_1)O^{I_2}_2(x_2)O^{I_3}_2(x_3)O^{I_4}_2(x_4) \rangle$...
can be analyzed in terms of conformal partial waves, where each partial wave transforms under \(SO(6)_R\) in one of the irreps of the right hand side of (14). This four-point function is calculated in [11] up to order \(O(1/N^2)\) at strong coupling by an AdS/CFT calculation. The result has the following form

\[
\langle O_2^I_1(x_1)O_2^J_2(x_2)O_2^I_3(x_3)O_2^J_2(x_4) \rangle = \left( x_{12}^2 x_{34}^2 - 2 u - 2 \right) \left( \sum_{r=0}^{2} \left[ 1 + (-1)^r (1 - Y)^2 \right] P_{I_1 I_2 J_1 J_2}^{I_3 I_4 (r)} \right) + \frac{1}{N^2} \sum_{r=0}^{2} \left[ \lambda_{(4-r,0)}(u,Y) + \mu_{(4-r,0)}(u,Y) \right] P_{I_1 I_2 J_1 J_2}^{I_3 I_4 (r)} + O(1/N^4) + \text{discarded irreps}, \tag{15}
\]

where we used the biharmonic conformal invariants

\[
u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}, \quad Y = 1 - \frac{1}{v}, \tag{16}
\]

which are suited for an operator product expansion of (13), since in this case we have

\[
x_{12} \to 0, \quad x_{34} \to 0 \quad \iff \quad u \to 0, \quad v \to 1, \quad Y \to 0. \tag{17}
\]

The representation dependent functions \(\lambda_{(4-r,0)}(u,Y)\) are given by

\[
\lambda_{(4,0)}(u,Y) = u^2(1 - Y),
\lambda_{(3,1)}(u,Y) = uY(1 - Y),
\lambda_{(2,2)}(u,Y) = \frac{3}{2} u(1 - Y)^2 - \frac{1}{2} u^2(1 - Y)^2,
\tag{18}
\]

as well as

\[
\mu_{(4,0)}(u,Y) = 4u^2(1 - Y)
\mu_{(3,1)}(u,Y) = 0
\mu_{(2,2)}(u,Y) = -2u^2(1 - Y). \tag{19}
\]

Moreover we need a generalized hypergeometric function [17, 13]

\[
\phi(u,v) = -4 \left( \frac{1}{1 - Y} \frac{\partial}{\partial \varepsilon} \right) \sum_{n \geq 0} \frac{u^{n+2+\varepsilon}}{\Gamma(n+1+\varepsilon)} \frac{\Gamma(n+3+\varepsilon)^2 \Gamma(n+4+\varepsilon)}{\Gamma(2n+6+2\varepsilon)} F \left[ \begin{array}{c} n + 3 + \varepsilon, n + 2 + \varepsilon \\ 2n + 6 + 2\varepsilon \end{array} ; 1 - v \right] \tag{20}
\]
To obtain the contribution of an exchanged field, which transforms in an irrep \( \{ 4 - r, r, 0 \} \), \( r = 0, 1, 2 \), of the first line of the right hand side of (14), we have to project onto this irrep by the respective projection operator \( P_{\{ 4 - r, r, 0 \}} \). Since these projectors are mutually orthogonal, application of \( P_{\{ 4 - r, r, 0 \}} \) to (15) gives only terms proportional to \( P_{\{ 4 - r, r, 0 \}} \) on the right hand side.

Note that the logarithmic terms, which appear at \( 1/N^2 \), have the form
\[
\left. P_{\{ 4 - r, r, 0 \}} \langle O O O O \rangle \right|_{\sim \log} \sim \log u \left( u^p + \text{higher powers in } u \right),
\]
with \( p = 2 \) if \( r = 0, 2 \) and \( p = 1 \) if \( r = 1 \).

The general form of the exchange of a partial wave with dimension \( \delta \) and spacetime tensor rank \( l \) produced from two scalar fields of dimensions \( \Delta_1 \) and \( \Delta_2 \) can be found with the help of the “master formula” of [15]
\[
(x^2_{13} - \Delta_1 (x^2_{24} - \Delta_2 C_{l, \Delta_1, \Delta_2}) \sum_{M=0}^{l} u^M (\delta - \Delta_1 - \Delta_2 - M) f_M (u, 1 - v),
\]
where \( f_M \) is an analytic function in \( (u, 1 - v) \) with \( f_M (0, 1 - v) \neq 0 \), and \( C_{l, \Delta_1, \Delta_2} \) is a constant containing the normalization of the exchanged field and the coupling of the external fields to the exchanged one. The dimension \( \delta \) of the exchanged composite operator built out of the components with dimensions \( \Delta_1 \) and \( \Delta_2 \) is given by
\[
\delta = \Delta_1 + \Delta_2 + l + 2t + \eta_{l,t}.
\]
Plugging this formula into (22) and expanding \( u^{\eta_{l,t}} = 1 + \eta_{l,t} \log u + \cdots \) we see that the leading term proportional to \( \log u \) begins with the constant \( \eta_{0,0} \). To achieve agreement with the strong coupling result (21), \( \eta_{0,0} \) must be zero at strong coupling and we conclude that \( \eta_{0,0} \) vanishes at all.

Now we take a closer look at the four point function leading to the exchange of the field with \( SO(6)_R \) irrep \( \{ 3, 1, 0 \} \) and observe that it has a factor \( Y \). The Taylor expansion of \( Y \) in terms of \( x^\mu_{12}, x^\nu_{34} \) starts with
\[
\frac{2}{x^2_{24}} t_{\mu\nu} (x_{24}) x^\mu_{12} x^\nu_{34}.
\]
On the other hand, the numerator in the two-point function of a conformal vector field contains the tensor
\[
t_{\mu\nu} (x) = \delta_{\mu\nu} - 2 \frac{x_{\mu} x_{\nu}}{x^2}.
\]
This indicates that the exchanged field with \( SO(6)_R \) irrep \( \{ 3, 1, 0 \} \) is indeed a vector field, in agreement with (6).
2.2 The second example

We now consider the fusion $\Psi_{\{2k-2,0,0\}}^{I_1I_2...I_{k-1}}(x_1)O_2^{J_2}(x_2)$, which was also done in [17]. According to Skiba’s theorem [5], the scalar field $\Psi_{\{2k-2,0,0\}}$ has protected conformal dimension and this dimension therefore equals $2k - 2$. We want to show that the fusion products also have protected dimension. To this end we again consider the four-point function

$$\left\langle \Psi_{\{2k-2,0,0\}}^{I_1I_2...I_{k-1}}(x_1)O_2^{J_2}(x_2)\Psi_{\{2k-2,0,0\}}^{I_1I_2...I_{k-1}}(x_3)O_2^{J_2}(x_4) \right\rangle$$  \hspace{1cm} (26)

This four-point function is calculated up to order $O(1/N^2)$ by inserting the four-point function (15) into the $2k$ point function

$$\left\langle \prod_{r=1}^{k-1} O_2^{r}(x_1^{(r)})O_2^{J_2}(x_2)\prod_{r=1}^{k-1} O_2^{r}(x_3^{(r)})O_2^{J_2}(x_4) \right\rangle$$  \hspace{1cm} (27)

in all possible ways. In principle it is possible that there are also contributions of two three point correlators, which are of order $O(1/N)$ each, to the $2k$ point function for $k \geq 3$, see fig. 2. However, it is pointed out in [17] that discarding of $SO(6)_R$ contractions eliminates these contributions in calculations up to order $O(1/N^2)$.

The next step in obtaining the four-point function (26) is the projection of $2k - 2$ $SO(6)_R$ indices of the $2k$ point function (27) onto the $\{2k-2,0,0\}$ irrep. Then we perform the “local limit” $x_i^{(r)} \rightarrow x_i$ to obtain the four-point function (26). Finally we project onto the irreps which occur in the normal product $\Psi_{\{2k-2,0,0\}}^{I_1I_2...I_{k-1}}(x_1)O_2^{J_2}(x_2)$, i.e.

$$\{2k - 2, 0, 0\} \otimes \{2, 0, 0\} = \{2k, 0, 0\} \oplus \{2k - 1, 1, 0\} \oplus \{2k - 2, 2, 0\}$$

+ irreps with contractions.  \hspace{1cm} (28)
The result of this computation found in [17] yields for any irrep \( \{2k - r, r, 0\} \), \( r \in \{0, 1, 2\} \),

\[
\left\langle \Psi^{I_1, I_2, \ldots, I_{k-1}}_{\{2k-2,0,0\}}(x_1)O^I_2(x_2)\Psi^{J_1, J_2, \ldots, J_{k-1}}_{\{2k-2,0,0\}}(x_3)O^J_2(x_4) \right\rangle = (x_{13}^2)^{-2k+2}(x_{24}^2)^{-2}(k - 1)!
\]

\[
\left\{ a(k, r) + b(k, r)(1 - Y)^2 \right. \\
+ \frac{1}{N^2} \left[ c(k, r) + d(k, r)(1 - Y)^2 \\
+ e(k, r)(1 - Y)^2 \left[ \lambda_{\{4-r,r,0\}}(u, Y)\phi(u, v) + \mu_{\{4-r,r,0\}}(u, Y) \right] \right] \\
+ O(1/N^4) \right\} P^{I_1, I_2, \ldots, I_k, J_1, \ldots, J_k}_{(2k-r,r,0)} + \text{irreps with contractions}.
\]

(29)

The functions \( \mu, \lambda, \phi \) are the same as in (18), (19), (20) and the coefficient functions \( a, b, c, d, e \) are given in table 1.

Table 1: Table of coefficients \( a-e \)

|       | \( a \) | \( b \) | \( c \) | \( d \) | \( e \) |
|-------|--------|--------|--------|--------|--------|
| \{2k, 0, 0\} | 1      | \( k - 1 \) | \( (k - 1)(k - 2) \) | \( (k - 1)(k - 2)(k + 1) \) | \( k - 1 \) |
| \{2k - 1, 1, 0\} | 1      | \(-1\)  | \( (k - 1)(k - 2) \) | \(- (k - 1)(k - 2) \) | \( k - 1 \) |
| \{2k - 2, 2, 0\} | 1      | \( k - 1 \) | \( (k - 1)(k - 2) \) | \( \frac{2}{3}(k - 1)(k - 2) \) | \( k - 1 \) |

By the general rule (6), in the case \( r = 1 \) we expect that the exchanged partial wave of lowest spacetime tensor rank is a vector. Therefore we must have proportionality with \( Y \). This is in fact true since from the second line of table 1 we read off that

\[
a(k, 1) + b(k, 1) = 0 \\
c(k, 1) + d(k, 1) = 0,
\]

(30)

and \( \lambda_{\{3,1,0\}}(u, Y) \) and \( \mu_{\{3,1,0\}}(u, Y) \) allow to factor \( Y \), see (18), (19).

Since the terms proportional to \( \log u \) all stem from the \( O(1/N^2) \) contributions of the four point function, they are at least of order \( u \), and comparison with (22) reveals again that the anomalous dimension of the composite field with minimal dimension vanishes. More precisely, in the case \( \{2k, 0, 0\} \) the dimensions \( \delta = 2k, 2k + 2 \) do not acquire an anomalous term, while for the other two cases only the fields with minimal conformal dimension \( \delta = 2k, \delta = 2k + 1 \) for the scalar and the vector, respectively, have vanishing anomalous dimensions.
We mention finally that the effective perturbation expansion parameter in (29) is obviously $\frac{k}{N}$ and not $\frac{1}{N}$, which must be much smaller than one to enable such an expansion. From the critical nonlinear $O(N)$ sigma model a similar behavior is known, where the effective expansion parameter is $\frac{k}{N^{1/2}}$ [10], where $k$ denotes the number of fusions of fundamental fields.

3 The general case

For the general case we consider the fusion of a scalar normal product $\Psi_{\{r_1, r_2, r_3\}}$, where $r_i$ are all even, of protected conformal dimension $\Delta = 2k - 2 = r_1 + r_2 + r_3$ with $\mathcal{O}_2$. We claim that to any irrep of $SO(6)_R$ appearing in the reduction of $\{r_1, r_2, r_3\} \otimes \{2, 0, 0\}$ respecting the fusion rules belongs at least one field with vanishing anomalous dimension. The proof goes along similar lines as before.

First we calculate the four-point function

$$\left\langle \Psi_{\{r_1, r_2, r_3\}}^{l_1 \ldots l_{k-1}}(x_1) \mathcal{O}_2^{J_k}(x_2) \Psi_{\{r_1, r_2, r_3\}}^{l_1 \ldots l_{k-1}}(x_3) \mathcal{O}_2^{J_k}(x_4) \right\rangle$$

(31)

by evaluating the $2k$-point function

$$\left\langle \prod_{r=1}^{k-1} \mathcal{O}_2^{J_r}(x_1^{(r)}) \mathcal{O}_2^{J_k}(x_2) \prod_{r=1}^{k-1} \mathcal{O}_2^{J_r}(x_3^{(r)}) \mathcal{O}_2^{J_k}(x_4) \right\rangle$$

(32)

up to order $O(1/N^2)$. In this calculation we are confronted with the following graphs:

- At order $O(1)$ the $2k$ point function decomposes into a product of two-point functions (see fig. 3).

- At order $O(1/N^2)$ we have to distribute respectively two legs of the $O(1/N^2)$ part of the four point function (33) to the external points $x_1^{(r)}$, $x_2$ and $x_3^{(r)}$, $x_4$ in all possible ways. Then we must connect the residing external points via two point functions. This leads to four classes of graphs shown in figures 4, 5, 6.

Then we sum up the $k!$ graphs from $O(1)$ and the $\binom{k}{2}^2 (k - 2)!$ graphs from $O(1/N^2)$. To be able to perform the local limit $x_i^{(r)} \to x_i$ we must first project onto the incoming irreps. In this limit we then obtain the four-point function (31).

In the following step we project onto the block-conserving irreps $\{r'_1, r'_2, r'_3\}$, which occur in the reduction of $\{r_1, r_2, r_3\} \otimes \{2, 0, 0\}$. Thus we need the following projections:
At order $O(1)$

$$P_{J_1...J_k}^{I_1...I_k} \prod_{\{r'_1, r'_2, r'_3\}} P^{K_1...K_k} \prod_{\{r_1, r_2, r_3\}} K_{I_1...I_{k-1}} P_{I_{k-1}}^{l_1...l_m} \prod_{m=1}^{k} P_{\{2,0,0\}}^{l_m} J_m,$$

and at order $O(1/N^2)$, say for class (4), see fig.

$$P_{J_1...J_k}^{I_1...I_k} \prod_{\{r'_1, r'_2, r'_3\}} P^{K_1...K_k} \prod_{\{r_1, r_2, r_3\}} K_{I_1...I_{k-1}} P_{I_{k-1}}^{l_1...l_m} \prod_{m=1}^{k} P_{\{2,0,0\}}^{l_m} J_m \prod_{\{2,0,0\}}^{l_k} J_{k-1}. \quad (34)$$

To avoid the evaluation of these projectors we make an ansatz for the four-point function

$$P_{J_1...J_k}^{I_1...I_k} \prod_{\{r'_1, r'_2, r'_3\}} P^{K_1...K_k} \prod_{\{r_1, r_2, r_3\}} K_{I_1...I_{k-1}} P_{I_{k-1}}^{l_1...l_m} \prod_{m=1}^{k} P_{\{2,0,0\}}^{l_m} J_m \prod_{\{2,0,0\}}^{l_k} J_{k-1}. \quad (35)$$

The functions $A, B, C, D, E$ are polynomials resulting from the combinatorics of the graphs contributing to this four point function. They depend on the representations $\{r'_1, r'_2, r'_3\}$ and on $\{r_1, r_2, r_3\}$. In the local limit two (four) external legs of the four point function eventually coincide, as one can see from the graphs of class (1), (2) and (3). This means that we must evaluate the bracket containing $\lambda \phi + \mu$ at $u = Y = 0$, which gives rise to the $C$ and $D$ function. The factor $(1 - Y)^2$ arises from crossing, namely a propagator $(x_{23}^2)^{-2}$ or $(x_{14}^2)^{-2}$. In the case of $l_0 = 1$ (two $r'_i$ are odd and the third is even) we have the constraints

$$A + B = 0, \quad C + D = 0. \quad (36)$$

Finally the same argument for the protectedness of the conformal dimension of quasiprimary fields with $SO(6)_R$ representation and spacetime tensor rank $l = l_0 \in \{0, 1\}$ applies. We compare the terms proportional to $\log u$ in the
operator product expansion from (35) with those from the generic exchange (22) and observe that the former starts with (at least) one power of $u$, while the latter begins with a constant. Therefore we conclude that the fields with minimal canonical conformal dimension have vanishing anomalous dimension, i.e. the canonical conformal dimension is exact.

This completes the proof.

4 Summary

In this note we described a simple method of finding quasiprimary fields with protected conformal dimensions in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in four dimensions in a recursive way. We find vanishing anomalous dimensions for spacetime scalars as well as spacetime vectors. We noted that the vectors are supersymmetry descendants of spacetime scalars with the same number of $O_2$ constituents, thus inheriting the protectedness of the conformal dimension from the scalars.

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References

[1] J. M. Maldacena, “The large $N$ limit of superconformal field theories and supergravity,” Adv. Theor. Math. Phys. 2 (1998) 231 [Int. J. Theor. Phys. 38 (1999) 1113] [arXiv:hep-th/9711200].

[2] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “Gauge theory correlators from non-critical string theory,” Phys. Lett. B 428, 105 (1998) [arXiv:hep-th/9802109].

[3] E. Witten, “Anti-de Sitter space and holography,” Adv. Theor. Math. Phys. 2, 253 (1998) [arXiv:hep-th/9802150].

[4] F. A. Dolan and H. Osborn, “On short and semi-short representations for four dimensional superconformal symmetry,” arXiv:hep-th/0209050.

[5] W. Skiba, “Correlators of short multi-trace operators in $N = 4$ supersymmetric Yang-Mills,” Phys. Rev. D 60, 105038 (1999) [arXiv:hep-th/9907088].

[6] A. V. Ryzhov, “Quarter BPS operators in $N = 4$ SYM,” JHEP 0111 (2001) 046 [arXiv:hep-th/0109064].
[7] E. D’Hoker and A. V. Ryzhov, “Three-point functions of quarter BPS operators in N = 4 SYM,” JHEP 0202 (2002) 047 [arXiv:hep-th/0109063].

[8] B. Eden and E. Sokatchev, “On the OPE of 1/2 BPS short operators in N = 4 SCFT(4),” Nucl. Phys. B 618 (2001) 259 [arXiv:hep-th/0106249].

[9] S. Ferrara and E. Sokatchev, “Universal properties of superconformal OPEs for 1/2 BPS operators in 3 ≤ D ≤ 6,” New J. Phys. 4 (2002) 2 [arXiv:hep-th/0110174].

[10] P. J. Heslop and P. S. Howe, “OPEs and 3-point correlators of protected operators in N = 4 SYM,” Nucl. Phys. B 626 (2002) 265 [arXiv:hep-th/0107212].

[11] G. Arutyunov, S. Frolov and A. C. Petkou, “Operator product expansion of the lowest weight CPOs in N = 4 SYM(4) at strong coupling,” Nucl. Phys. B 586 (2000) 547 [Erratum-ibid. B 609 (2001) 539] [arXiv:hep-th/0005182].

[12] F. A. Dolan and H. Osborn, “Superconformal symmetry, correlation functions and the operator product expansion,” Nucl. Phys. B 629 (2002) 3 [arXiv:hep-th/0112251].

[13] B. Eden, A. C. Petkou, C. Schubert and E. Sokatchev, “Partial non-renormalisation of the stress-tensor four-point function in N = 4 SYM and AdS/CFT,” Nucl. Phys. B 607, 191 (2001) [arXiv:hep-th/0009106].

[14] G. Arutyunov, B. Eden, A. C. Petkou and E. Sokatchev, “Exceptional non-renormalization properties and OPE analysis of chiral four-point functions in N = 4 SYM(4),” Nucl. Phys. B 620, 380 (2002) [arXiv:hep-th/0103230].

[15] K. Lang and W. Rühl, “The Critical O(N) sigma model at dimensions 2 < d < 4: Fusion coefficients and anomalous dimensions,” Nucl. Phys. B 400 (1993) 597.

[16] K. Lang and W. Rühl, “Critical nonlinear O(N) sigma models at 2 < d < 4: The Degeneracy of quasiprimary fields and its resolution,” Z. Phys. C 61 (1994) 495.

[17] L. Hoffmann, L. Mesref, A. Meziane and W. Rühl, “Multi-trace quasi-primary fields of N = 4 SYM(4) from AdS n-point functions,” Nucl. Phys. B 641 (2002) 188 [arXiv:hep-th/0112191].
Appendix: Graphs contributing to $2k$ point functions

Figure 3: $O(1)$ contribution: The $2k$ point function decomposes into a product of two-point functions

Figure 4: Class (1) graphs where both pairs of legs coincide.
Figure 5: Class (2) graphs where the left pair of legs of the four-point function insertion coincide. The class (3) graphs are analogous to the class (2) graphs, with the right pair of legs of the inserted four point function coinciding.

Figure 6: Class (4) graphs: no legs of the inserted four point function coincide