ERROR ANALYSIS OF AN ADI SPLITTING SCHEME FOR THE
INHOMOGENEOUS MAXWELL EQUATIONS

JOHANNES EILINGHOFF AND ROLAND SCHAUBELT*

Department of Mathematics
Karlsruhe Institute of Technology
76128 Karlsruhe, Germany

(Communicated by J. M. Sanz-Serna)

ABSTRACT. In this paper we investigate an alternating direction implicit (ADI)
time integration scheme for the linear Maxwell equations with currents, charges
and conductivity. We show its stability and efficiency. The main results es-
tablish that the scheme converges in a space similar to $H^{-1}$ with order two
to the solution of the Maxwell system. Moreover, the divergence conditions in
the system are preserved in $H^{-1}$ with order one.

1. Introduction. The Maxwell equations are the foundation of the electro mag-
netic theory and one of the basic PDEs in physics. They form a large coupled
system of six time-depending scalar equations in three space dimensions and thus
pose considerable difficulties to the numerical treatment already in the linear case.
Explicit methods like finite differences on the Yee grid [25] are efficient, but to avoid
instabilities one is restricted to small time step sizes, cf. [24]. On the other hand,
stable implicit methods for time integration can lead to very large linear systems
to be solved in every time step. Around 2000 the very efficient and uncondi-
tionally stable alternating direction implicit (ADI) scheme was introduced in [20] and
[26] for the Maxwell system on a cuboid with linear isotropic material laws. In
this scheme one splits the curl operator into the partial derivatives with a plus and
a minus sign, see (1.4), and then applies the implicit-explicit Peaceman-Rachford
method to the two subsystems, cf. (1.5). In [20] and [26] it was observed that the
resulting implicit steps essentially decouple into one-dimensional problems which
makes the algorithm very fast, see also Proposition 4.6 of [13] as well as (4.3) and
(4.4) below. There are energy-conserving variants of the ADI splitting, see e.g. [4],
[5], [11], [18], not discussed here. We refer to [13], [14] and [15] for further references
about the numerical treatment of the Maxwell system.

Despite its importance, there exists very little rigorous error analysis of the ADI
scheme in the literature, and the available results only cover systems without re-
sistancy, currents and charges. For a variant of the scheme, in [5] error estimates
have been shown for solutions in $C^6$, see also [4] and [11] for two space dimensions.

2010 Mathematics Subject Classification. Primary: 65M12; Secondary: 35Q61, 47D06, 65J10.
Key words and phrases. Maxwell equations, splitting, convergence order, preservation of diver-
gence conditions, stability.

The authors gratefully acknowledge financial support by the Deutsche Forschungsgemeinschaft
(DFG) through CRC 1173.

* Corresponding author: Roland Schnaubelt.
The paper [13] (co-authored by one of the present authors) establishes second order convergence in $L^2$ for the linear Maxwell system on a cuboid with the boundary conditions of a perfect conductor. Here the initial data belong to $H^3$ and satisfy appropriate compatibility conditions, whereas the coefficients are contained in $W^{2,3} \cap W^{1,\infty}$. We stress that the scheme is of second order classically and that the needed degree of regularity in [13] is natural for a splitting in the highest derivatives. As in our paper, the results of [13] are concerned with the time integration on the PDE level and do not treat the space discretization. Based on these and our investigations, we expect that one can develop an error analysis for the full discretization in the future, cf. [14] and [15].

In this work we study the complete Maxwell system with conductivity, currents and charges for Lipschitz coefficients and data in $H^2$. Compared to [13], we thus have to modify both the scheme and the functional analytic setting for the Maxwell equations, see (1.4), (1.5) and (2.4), which leads to substantial changes in the analysis. We establish the stability of the scheme in $L^2$ and $H^1$, and that it converges of second order in $H^{-1}$, roughly speaking, which is the natural level of regularity for our data. Moreover, the scheme preserves the divergence conditions (1.1c) and (1.2) of the Maxwell system in $H^{-1}$ up to order $\tau$. To our knowledge, such preservation results have only be shown for $C^6$–solutions in the case without charges and in two space dimensions for a related scheme in [4], see also [5] for three space dimensions.

We want to approximate the electric and magnetic fields $E(t, x) \in \mathbb{R}^3$ and $H(t, x) \in \mathbb{R}^3$ satisfying the linear Maxwell equations

\[
\begin{align*}
\partial_t E(t) &= \frac{1}{\varepsilon} \text{curl} H(t) - \frac{1}{\mu} (\sigma E(t) + J(t)) & \text{in } Q, \ t \geq 0, \\
\partial_t H(t) &= -\frac{1}{\mu} \text{curl} E(t) & \text{in } Q, \ t \geq 0, \\
\text{div}(\varepsilon E(t)) &= \rho(t), & \text{div}(\mu H(t)) &= 0 & \text{in } Q, \ t \geq 0, \\
E(t) \times \nu &= 0, & \mu H(t) \cdot \nu &= 0 & \text{on } \partial Q, \ t \geq 0, \\
E(0) &= E_0, & H(0) &= H_0 & \text{in } Q,
\end{align*}
\]

on the cuboid $Q$, where $\nu(x)$ is the outer unit normal at $x \in \partial Q$. Here the initial fields in (1.1e), the current density $J(t, x) \in \mathbb{R}^3$, the permittivity $\varepsilon(x) > 0$, the permeability $\mu(x) > 0$ and the conductivity $\sigma(x) \geq 0$ are given for $x \in Q$ and $t \geq 0$. We treat the conditions (1.1d) of a perfectly conducting boundary. As noted in Proposition 2.3, the charge density $\rho(t, x) \in \mathbb{R}$ depends on the data and (if $\sigma \neq 0$) on the solution via

\[
\rho(t) = \text{div}(\varepsilon E(t)) = \text{div}(\varepsilon E_0) - \int_0^t \text{div}(\sigma E(s) + J(s)) \, ds, \quad t \geq 0.
\]

Throughout, we assume that the material coefficients satisfy

\[
\varepsilon, \mu, \sigma \in W^{1,\infty}(Q, \mathbb{R}), \quad \varepsilon, \mu \geq \delta \quad \text{for a constant } \delta > 0, \quad \sigma \geq 0.
\]

For the initial fields and the current density we require regularity of second order and certain compatibility conditions in our theorems.

In Section 2 we present the solution theory for (1.1). In presence of conductivity, currents and charges, one has to work in the space $X_{\text{div}}$ of fields in $L^2$ satisfying the magnetic conditions $\mu H \cdot \nu = 0$ and $\text{div}(\mu H) = 0$ from (1.1) as well as the regularity $\text{div}(\varepsilon E) \in L^2(Q)$ of the charges, see (2.4). The last condition also enters in the norm of $X_{\text{div}}$. (If $\sigma$, $J$ and $\rho$ vanish, it is replaced by the equation $\text{div}(\varepsilon E) = 0$, see
The electric boundary condition is included in the domain of the Maxwell operator $M$ from (2.3) governing (1.1a) and (1.1b). It is crucial for our analysis that the domain of the part of $M$ in $X_{\text{div}}$ embeds into $H^1(Q)^6$, see Proposition 2.2.

In the ADI method one splits $\text{curl} = C_1 - C_2$ and $M = A + B$, where we put

$$A = \begin{pmatrix} \frac{-\varepsilon}{\mu} I & \frac{1}{\mu} C_1 \\ \frac{1}{\mu} C_2 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -\varepsilon I & -\frac{1}{\mu} C_2 \\ \frac{1}{\mu} C_1 & 0 \end{pmatrix},$$

$$C_1 = \begin{pmatrix} 0 & 0 & \partial_3 \\ \partial_2 & 0 & 0 \\ 0 & \partial_1 & 0 \end{pmatrix} \quad \text{and} \quad C_2 = \begin{pmatrix} 0 & \partial_3 & 0 \\ 0 & 0 & \partial_1 \\ \partial_2 & 0 & 0 \end{pmatrix}.$$ (1.4)

The domains of $A$ and $B$ are described after (3.1). Let $\tau > 0$, $T \geq 1$, and $t_0 = n\tau \leq T$ for $n \in \mathbb{N}$. The $(n + 1)$-th step of the scheme is given by

$$w_{n+1} = (I - \frac{\tau}{T} B)^{-1}(I + \frac{\tau}{T} A)[(I - \frac{\tau}{T} A)^{-1}(I + \frac{\tau}{T} B)w_n - \frac{\tau}{T^2}(J(t_n) + J(t_{n+1}), 0)].$$ (1.5)

Here we modify an approach developed in [22] for a different situation. Note that the conductivity $\sigma$ is included into the maps $A$ and $B$, whereas the current density $J$ is added to the scheme.

In Section 3 we analyze the operators $A$ and $B$ and their adjoints, showing in particular that they generate contraction semigroups (possibly up to a shift). In $L^2(Q)^6$ we can proceed as in [13], but we also have to work in the closed subspace $Y$ of $H^1(Q)^6$ equipped with the boundary conditions in (1.1d), which leads to significant new difficulties. Proposition 3.6 then yields the main estimates needed for the error analysis of (1.5). We point out that the domains of the parts $A_Y$ and $B_Y$ of $A$ and $B$ in $Y$ provide a very convenient framework for the analysis of the ADI scheme, see Sections 4 and 6. In Section 4 we explain the efficiency of the scheme. Based on the results of Section 3, its stability in $L^2$ and $H^1$ is shown in Theorem 4.2. These estimates should lead to its unconditional stability independent of the mesh size of a spatial discretization. Moreover, if $\sigma$, $J$ and $\rho$ vanish, we obtain a modified energy equality for the scheme.

Our main results are proved in the final two sections. By Theorem 5.1, the error of the scheme is bounded in $Y^*$ by $cT^2e^{\kappa T}T^2$ times certain second order norms of $E_0$, $H_0$ and $J$, where $c$ and $\kappa$ only depend on the quantities in (1.3). To show this core fact, we adopt arguments from [12] and [22] to derive the (rather lengthy) error formula (5.6). It is formulated in a weak sense with test functions in $Y$ which is needed to work in the present degree of regularity of the data. To estimate the error, one then uses the above mentioned Propositions 2.2 and 3.6. In the final Theorem 6.1 we prove a first order bound in $H^{-1}$ for the error concerning the discrete analogues of the divergence conditions $\text{div} (\mu H) = 0$ and (1.2). Besides Proposition 3.6, this proof is based on the surprisingly simple exact formula (6.3) for this error, which nicely follows from the structural properties of the scheme.

Concluding, we discuss possible extensions of our work. For data in $H^2$ one can establish variants of Theorems 5.1 and 6.1 for the $L^2$-norms of the errors. For these results one has to develop a rather intricate regularity theory in $H^2$ for the Maxwell equation and for the split operators $A$ and $B$. See our companion paper [9] and the thesis [8], where one can also find numerical experiments. The scheme loses its efficiency for non-isotropic (or nonlinear) material laws so that it does not make sense to study matrix-valued coefficients $\varepsilon$, $\mu$, or $\sigma$. On the other hand, numerically one could implement the scheme also on a union of cuboids (e.g., an L-shaped domain). Since such domains are not convex anymore, the domain of the Maxwell
operator only embeds into $H^\alpha(Q)$ for some $\alpha \in (1/2, 1)$, see e.g. [3], which should lead to reduced convergence orders. Further technical difficulties arise since some of the arguments in Section 3 heavily depend on the structure of a cuboid. These questions shall be investigated in a later paper. In [14] and [15], an error analysis was given for the full discretization of the Maxwell system (with $\sigma = 0$), using the discontinuous Galerkin method and a locally implicit time integration scheme. We expect that one can treat the full discretization for the ADI scheme combining methods in these and our papers.

2. Basic results on the Maxwell system. We first collect notation and basic results used throughout this paper. By $c$ we denote a generic constant which may depend only on $Q$ and on the constants from (1.3); i.e., on $\delta, \|\varepsilon\|_{W^{1,\infty}}$, $\|\mu\|_{W^{1,\infty}}$, or $\|\sigma\|_{W^{1,\infty}}$. We write $I$ for the identity operator and $v \cdot w$ for the Euclidean inner product in $\mathbb{R}^m$.

Let $X$ and $Y$ be real Banach spaces. On the intersection $X \cap Y$ we use the norm $\|z\|_X + \|z\|_Y$. The symbol $Y \hookrightarrow X$ means that $Y$ is continuously embedded into $X$, and $X \cong Y$ that they are isomorphic. The duality pairing between $X$ and its dual $X^*$ is denoted by $\langle x^*, x \rangle_{X^*,X}$ for $x \in X$ and $x^* \in X^*$; and the inner product by $\langle \cdot, \cdot \rangle_X$ if $X$ is a Hilbert space. In the latter case, a dense embedding $Y \hookrightarrow X$ implies that $X \hookrightarrow Y^*$, where $x \in X \cong X^*$ acts on $Y$ via $\langle x, y \rangle_{Y^*,Y} = \langle x, y \rangle_X$ for $y \in Y \hookrightarrow X$.

Let $\mathcal{B}(X,Y)$ be the space of bounded linear operators from $X$ to $Y$, and $\mathcal{B}(X) = \mathcal{B}(X,X)$. The domain $D(L)$ of a linear operator $L$ is always equipped with the graph norm $\|\cdot\|_L$ of $L$. If $Y \hookrightarrow X$, then the part $L_Y$ of $L$ in $Y$ is given by $D(L_Y) = \{y \in Y \cap D(L) \mid Ly \in Y\}$ and $L_Yy = Ly$. For two operators $L$ and $G$ in $X$, the product $LG$ is defined on $D(LG) = \{x \in D(G) \mid Gx \in D(L)\}$.

Let $\lambda$ belong to the resolvent set of a closed operator $L$ in $X$. We occasionally need the extrapolation space $X_{-1} = X^L_{-1}$ of $L$; i.e., the completion of $X$ with respect to the norm given by $\|x\|_{-1} = \|(\lambda I - L)^{-1}x\|_X$. One then has a continuous extension $L_{-1} : X \rightarrow X_{-1}$ whose resolvent operators extend those of $L$. If $L$ generates a $C_0$-semigroup $T(\cdot)$ on $X$, then $L_{-1}$ is the generator of the semigroup $T_{-1}(\cdot)$ of extensions to $X_{-1}$. This procedure can be iterated, providing $L_{-2} : X_{-1} \rightarrow X_{-2}$. If $X$ is reflexive, then $X^L_{-1}$ can be identified with the dual space of $D(L^*)$. See Section V.1.3 in [2] or Section II.5a in [10].

We use the standard Sobolev spaces $W^{k,p}(\Omega)$ for $k \in \mathbb{N}_0$, $p \in [1, \infty]$ and open subsets $\Omega \subseteq \mathbb{R}^m$, where $W^0,p(\Omega) = L^p(\Omega)$. We also use their $X$-valued analogues $L^p(\Omega, X)$ and $W^1,p(J, X)$ for an open interval $J \subseteq \mathbb{R}$, see §III.1.2 in [2]. For $p \in [1, \infty)$, $s \in (0, \infty) \setminus \mathbb{N}$ and an integer $k > s$, we define the Slobodeckij spaces $W^{s,p}(\Omega) = (L^p(\Omega), W^{k,p}(\Omega))_{s/k,p}$ by real interpolation, see Section 7.32 in [1] or [19]. We set $W^{-s,p}(\Omega) = W^{s,p}_0(\Omega)^*$ for $s \geq 0$ and $p \in (1, \infty)$, where $p' = p/(p-1)$ and the subscript 0 always denotes the closure of test functions in the respective norm. We are mostly interested in the case $H^s(\Omega) := W^{s,2}(\Omega)$.

In this paper we work on the cuboid $Q = (a^-_1, a^+_1) \times (a^+_2, a^-_2) \times (a^-_3, a^+_3) \subseteq \mathbb{R}^3$ with (Lipschitz) boundary $\Gamma = \partial Q$. For $s \in [0,1]$ we use the spaces $H^s(\Gamma)$ at the boundary, see Section 2.5 of [21]. Moreover, $H^{-s}(\Gamma)$ is defined as the dual space of $H^s(\Gamma)$. We write

$$\Gamma_j^\pm = \{x \in \partial Q \mid x_j = a_j^\pm\} \quad \text{and} \quad \Gamma_j = \Gamma_j^- \cup \Gamma_j^+,$$

for $j \in \{1, 2, 3\}$ and $d_Q$ for the smallest side length of $Q$. 

5688 JOHANNES EILINGHOF AND ROLAND SCHNAUBELT
Our analysis of the Maxwell system takes place in the space \( X = L^2(Q)^6 \) with the weighted inner product
\[
((u, v) \mid (\varphi, \psi))_X = \int_Q (\varepsilon u \cdot \varphi + \mu v \cdot \psi) \, dx
\]
for \((u, v), (\varphi, \psi) \in X\). The square of the induced norm \( \| \cdot \|_X \) is twice the physical energy of the fields \((E, H)\), and because of (1.3) it is equivalent to the usual \( L^2\)−norm. We further use the Hilbert spaces

\[
H(\text{curl}, Q) = \{ u \in L^2(Q)^3 \mid \text{curl} \, u \in L^2(Q)^3 \}, \quad \| u \|_{\text{curl}}^2 = \| u \|_{L^2}^2 + \| \text{curl} \, u \|_{L^2}^2,
\]
\[
H(\text{div}, Q) = \{ u \in L^2(Q)^3 \mid \text{div} \, u \in L^2(Q) \}, \quad \| u \|_{\text{div}}^2 = \| u \|_{L^2}^2 + \| \text{div} \, u \|_{L^2}^2.
\]

Theorems 1 and 2 in Section IX.A.1.2 of [6] provide the following facts. The space of restrictions to \( \partial \) of test functions on \( \mathbb{R}^3 \) is dense in \( H(\text{curl}, Q) \) and \( H(\text{div}, Q) \). The tangential trace \( u \mapsto (u \times \nu)|_{\Gamma} \) on \( C(\overline{Q})^3 \cap H^1(Q)^3 \) has a unique continuous extension \( \text{tr}_t : H(\text{curl}, Q) \to H^{-1/2}(\Gamma)^3 \), and \( H_0(\text{curl}, Q) \) is the kernel of \( \text{tr}_t \) in \( H(\text{curl}, Q) \). We also have the integration by parts formula
\[
\int_Q \text{curl} \, u \cdot v \, dx = \int_Q u \cdot \text{curl} \, v \, dx - (\text{tr}_t u, \text{tr}_t v)_{H^{-1/2}(\Gamma)^3 \times H^{1/2}(\Gamma)^3} \tag{2.1}
\]
for all \( u \in H(\text{curl}, Q) \) and \( v \in H^1(Q)^3 \). Similarly, the normal trace \( u \mapsto (u \cdot \nu)|_{\Gamma} \) on \( C(\overline{Q})^3 \cap H^1(Q)^3 \) has a unique continuous extension \( \text{tr}_n : H(\text{div}, Q) \to H^{-1/2}(\Gamma) \). Moreover, Section 2.4 and 2.5 of [21] provide the continuous and surjective trace operator \( \text{tr} : H^1(Q) \to H^{1/2}(\Gamma) \), which is the extension of the map \( f \mapsto f|_{\Gamma} \) defined on \( C(\overline{Q}) \cap H^1(Q) \). Its kernel is the space \( H^1_0(\Gamma) \).

We also have to deal with cases of partial regularity. For instance, take a function \( f \in L^2(Q) \) with \( \partial_t f \in L^2(Q) \). We set \( Q_1 = (a^-_2, a^+_2) \times (a^-_3, a^+_3) \). A representative of \( f \) then belongs to \( H^1((a^-_1, a^+_1), L^2(Q_1)) \cong L^2(Q_1, H^1(a^-_1, a^+_1)) \), and thus possesses traces at the rectangles \( \Gamma^\pm_j = \{a^\pm_j\} \times Q_1 \) whose norms in \( L^2(\Gamma^\pm_j) \) are bounded by \( c(\|f\|_{L^2(Q)} + \|\partial_t f\|_{L^2(Q)}) \). In this way, we obtain trace operators \( \text{tr}_{\Gamma^+_j} \) and \( \text{tr}_{\Gamma^-_j} \) for \( j \in \{1, 2, 3\} \). They coincide in \( L^2(\Gamma^\pm_j) \), respectively \( L^2(\Gamma_j) \), with the respective restrictions of \( \text{tr} \, f \) if \( f \in H^1(Q) \). We usually write \( u_1 = 0 \) on \( \Gamma_2 \) instead of \( \text{tr}_{\Gamma^-_1}(u_1) = 0 \), and so on. The following lemma will often be used to check boundary conditions.

**Lemma 2.1.** For some \( j, k \in \{1, 2, 3\} \) with \( k \neq j \), let \( f \in L^2(Q) \) satisfy \( \partial_j f, \partial_k f, \partial_j \partial_k f \in L^2(Q) \) and \( f = 0 \) on \( \Gamma_j \). We then have \( \partial_k f = 0 \) on \( \Gamma_j \).

**Proof.** We only consider \( \Gamma^-_1 \), \( j = 1 \), and \( k = 2 \) since the other cases are treated analogously. By the above observations, for a.e. \( (x_2, x_3) \in Q_1 \) the map \( f(\cdot, x_2, x_3) \) is contained in \( H^1(a^-_1, a^+_1) \), and we have \( f(x_1, x_2, x_3) = f_{x_3}^{-1} \partial_1 f(t, x_2, x_3) dt \). Similarly, \( \partial_2 f(x_1, \cdot, \cdot) \) is an element of \( L^2(Q_1) \) for a.e. \( x_1 \in (a^-_1, a^+_1) \). Using the definition of weak derivatives, one checks that
\[
\partial_2 f(x) = \int_{a^-_1}^{a^+_1} \partial_2 f(t, x_2, x_3) dt \quad \text{for a.e. } x \in Q. \tag{2.2}
\]

For each integer \( n > d_Q^2 \), there is a smooth cut-off function \( \chi_n \) on \([a^-_1, a^+_1]\) such that \( \chi_n \) takes values in \([0, 1]\), vanishes on \([a^-_1, a^-_1 + 1/(2n)]\), is equal to 1 on \([a^-_1 + 1/n, a^+_1]\), and \( |\chi'_n| \) is bounded by \( cn \). We then define \( f_n(x) = \chi_n(x_1) f(x) \) for \( x \in Q \). By dominated convergence, the functions \( \partial_2 f_n = \chi_n \partial_2 f \) converge to \( \partial_2 f \) in \( L^2(Q) \) as \( n \to \infty \). In the derivative \( \partial_2 f_n = \chi'_n \partial_2 f + \chi_n \partial_1 f, \) the second summand

\[
= \int_{a^-_1}^{a^+_1} \chi'_n \partial_2 f(t, x_2, x_3) dt \quad \text{for a.e. } x \in Q.
\]
Using formula (2.2) and Hölder’s inequality, we deduce

\[
\|\chi_n \partial_2 f\|_{L^2}^2 \leq \int_{S_n} \int_{Q_1} c^2 n^2 |\partial_2 f(x_1, x_2, x_3)|^2 \, d(x_2, x_3) \, dx_1 \\
\leq c^2 n \sup_{x_1 \in S_n} \int_{Q_1} |\partial_2 f(x_1, x_2, x_3)|^2 \, d(x_2, x_3) \\
\leq c^2 \sup_{x_1 \in S_n} \int_{a_1}^{x_1} \int_{Q_1} |\partial_2 f(t, x_2, x_3)|^2 \, d(x_2, x_3) \, dt \\
eq c^2 \int_{a_1}^{a_1^{-1/n}} \int_{Q_1} |\partial_2 f(t, x_2, x_3)|^2 \, d(x_2, x_3) \, dt \to 0
\]

as \( n \to \infty \). The functions \( \partial_2 f_n \) thus tend to \( \partial_2 f \) in \( H^1((a_1^-, a_1^+), L^2(Q_1)) \) so that \( \text{tr}_{\Gamma_1^-}(\partial_2 f_n) = 0 \) converges to the trace of \( \partial_2 f \) in \( L^2(\Gamma_1^-) \).

On \( X \) we define the Maxwell operator

\[
M = \left( \begin{array}{cc} -\frac{\sigma}{\varepsilon} I & \frac{1}{\varepsilon} \text{curl} \\ \frac{1}{\mu} \text{curl} & 0 \end{array} \right), \quad D(M) = H_0(\text{curl}, Q) \times H(\text{curl}, Q). \tag{2.3}
\]

This domain includes the electric boundary condition. To encode the magnetic boundary and divergence conditions in (1.1) and the regularity of the charge density \( \rho = \text{div}(\varepsilon u) \), we introduce the subspace

\[
X_{\text{div}} := \{(u, v) \in X \mid \text{div}(\mu v) = 0, \, \text{tr}_n(\mu v) = 0, \, \text{div}(\varepsilon u) \in L^2(Q)\} \\
= \{(u, v) \in X \mid \text{div}(\mu v) = 0, \, \text{tr}_n v = 0, \, \text{div} u \in L^2(Q)\}. \tag{2.4}
\]

The above constraints are understood in \( H^{-1}(Q) \), respectively \( H^{-1/2}(\Gamma) \). The second equation in (2.4) follows from (1.3) by Remark 3.3 in [13] and because of \( \text{div}(\varepsilon u) = \nabla \varepsilon \cdot u + \varepsilon \text{div} u \). In the same way one sees that \( \text{div} v \) belongs to \( L^2(Q) \) if \( (u, v) \in X_{\text{div}} \). Equipped with the norm given by

\[
\|(u, v)\|_{X_{\text{div}}}^2 = \|(u, v)\|_X^2 + \|\text{div}(\varepsilon u)\|_{L^2}^2,
\]

\( X_{\text{div}} \) is a Hilbert space as the maps \( \text{div} : L^2(Q)^3 \to H^{-1}(Q) \) and \( \text{tr}_n : H(\text{div}, Q) \to H^{-1/2}(\Gamma) \) are continuous.

The part of \( M \) in \( X_{\text{div}} \) is denoted by \( M_{\text{div}} \). We actually have

\[
D(M_{\text{div}}^k) = D(M^k) \cap X_{\text{div}} \tag{2.5}
\]

for \( k \in \mathbb{N} \). To show this claim, let \( (u, v) \in D(M) \cap X_{\text{div}} \). As in Proposition 3.5 of [13] (for the case \( \sigma = 0 \)) one infers that \( M(u, v) \) satisfies the magnetic conditions in \( X_{\text{div}} \). Moreover, according to assumption (1.3) the function

\[
-\text{div}(\varepsilon(M(u, v)))_1 = \text{div}(\sigma u) = \nabla(\sigma \varepsilon^{-1}\varepsilon u + \sigma \varepsilon^{-1}\text{div}(\varepsilon u))
\]

belongs to \( L^2(Q) \), and thus \( M(u, v) \) to \( X_{\text{div}} \). Thereby, \( (M(u, v))_1 \) is the first component of \( M(u, v) \). Hence, \( (u, v) \) is contained in \( D(M_{\text{div}}) \), and (2.5) is shown for \( k = 1 \). By induction, (2.5) follows for all \( k \in \mathbb{N} \).

The spaces \( H(\text{curl}, Q) \) and \( H(\text{div}, Q) \) contain rather irregular \( L^2 \)-functions, e.g. from the kernels of curl and div. Nevertheless, their intersection embeds into \( H^1(Q)^3 \) if one assumes that either the tangential or the normal trace is 0. See Theorem 2.17 in [3], for instance.
Theorem 2.2. Let (1.3) hold. The domain $D(M_{\text{div}})$ is continuously embedded into $H^1(Q)^6$, with a constant only depending on the constants in (1.3). Moreover, in the sense of the traces $\text{tr}_t$, the fields $(E, H) \in D(M_{\text{div}})$ satisfy

$$
\begin{align*}
E_2 &= E_3 = 0, \\
H_1 &= 0 \quad \text{on } \Gamma_1, \\
E_1 &= E_3 = 0, \\
H_2 &= 0 \quad \text{on } \Gamma_2, \\
E_1 &= E_2 = 0, \\
H_3 &= 0 \quad \text{on } \Gamma_3.
\end{align*}
$$

Proof. Let $w = (E, H) \in D(M_{\text{div}})$. The functions curl$E = -\mu(Mw)_2$ and curl$H = \varepsilon(Mw)_1 + \sigma E$ then belong to $L^2(Q)^3$. As noted above, also div$E$ and div$H$ are contained in $L^2(Q)$. The asserted embedding then follows from Theorem 2.17 of [3]. We thus obtain tr$_tE = \nu \times \text{tr} E$ and tr$_nH = \nu \cdot \text{tr} H$, which yields the second assertion. 

We now collect the main properties of the Maxwell operators and solve (1.1). Some of these results are contained in Section XVII.B.4 in [7], for instance. The next proposition is true on any Lipschitz domain $Q$ with the same proof.

Proposition 2.3. Let (1.3) hold. Then the following assertions are true.

a) The operators $M$ and $M_{\text{div}}$ generate $C_0$–semigroups $(e^{tM})_{t \geq 0}$ on $X_{\text{div}}$. Moreover, $e^{tM_{\text{div}}}$ is the restriction of $e^{tM}$ to $X_{\text{div}}$, and we have $\|e^{tM}\|_X \leq 1$ and $\|e^{tM_{\text{div}}}\|_{X_{\text{div}}} \leq c(1 + t)$ for all $t \geq 0$.

b) Let $w_0 = (E_0, H_0)$ belong to $D(M_{\text{div}})$ and $(J, 0)$ to $C([0, \infty), D(M_{\text{div}})) + C^1([0, \infty), X_{\text{div}})$. There is a unique solution $w = (E, H)$ of (1.1) in $C^1([0, \infty), X_{\text{div}})$ and $C([0, \infty), D(M_{\text{div}}))$ given by

$$
\begin{align*}
(E(t), H(t)) &= e^{tM_{\text{div}}}(E_0, H_0) - \int_0^t e^{(t-s)M_{\text{div}}}(\frac{1}{2}J(s), 0) \, ds \\
\rho(t) &= \text{div}(\varepsilon E(t)) = \text{div}(\varepsilon E_0) - \int_0^t \text{div}(\sigma E(s) + J(s)) \, ds
\end{align*}
$$

for $t \geq 0$. The charge density in (1.1c) is contained in $L^2(Q)$ and satisfies

$$
\begin{align*}
\rho(t) &= \text{div}(\varepsilon E(t)) = \text{div}(\varepsilon E_0) - \int_0^t \text{div}(\sigma E(s) + J(s)) \, ds \\
&= e^{-\frac{\sigma}{\varepsilon}t} \text{div}(\varepsilon E_0) - \int_0^t e^{-\frac{\sigma}{\varepsilon}(t-s)}(\nabla(\varepsilon(t-s)) \cdot \varepsilon E(s) + \text{div} J(s)) \, ds, \quad t \geq 0.
\end{align*}
$$

c) Let $w_0 \in D(M_{\text{div}}^2)$ and $(J, 0) \in W^{2,1}([0, T], X_{\text{div}}) \cap C([0, T], D(M_{\text{div}})) =: E$ for some $T > 0$. Then $w$ belongs to $C^2([0, T], X_{\text{div}}) \cap C^1([0, T], D(M_{\text{div}})) \cap C([0, T], D(M_{\text{div}}^2))$ with norm bounded by $c(1 + T)\left(\|w_0\|_{D(M_{\text{div}})} + \|(J, 0)\|_E\right)$.

d) The adjoint of $M$ in $X$ is given by $D(M^*) = D(M)$ and

$$
M^* = \begin{pmatrix}
-\frac{\sigma}{\varepsilon} & \frac{\mu}{\varepsilon} \\
\frac{1}{\mu} & \frac{1}{\varepsilon} & \text{curl}
\end{pmatrix}.
$$

Proof. 1) If $\sigma = 0$, for instance Proposition 3.5 of [13] shows that $M$ generates a contraction semigroup on $X$. Using the dissipative perturbation theorem, see Theorem III.2.7 in [10], one can extend this result to the case $\sigma \geq 0$. In the same way one shows that the operator matrix in d) with domain $D(M)$ is a generator. Because of (2.1) it is a restriction of $M^*$, cf. Proposition 3.5 of [13]. So assertion d) has been shown.

2) We observe that the inhomogeneity $(\frac{1}{2}J, 0)$ satisfies the same assumptions as $(J, 0)$ in part b), respectively c). Let the conditions of b) and c) be true. Corollaries 4.2.5 and 4.2.6 of [23] then provide a unique solution $w = (E, H)$ in $C^1([0, \infty), X) \cap C([0, \infty), D(M))$ of (1.1a) and (1.1b) which also satisfies the electric boundary
condition and the initial conditions. It is given by Duhamel’s formula (2.6) with $M$ instead of $M_{\text{div}}$.

In view of the magnetic conditions in (1.1c) and (1.1d), we introduce the closed subspace $X_{\text{mag}} = \{(u, v) \in X \mid \text{div}(\mu v) = 0, \text{tr}_n(\mu v) = 0\}$ of $X$. As in Proposition 3.5 of [13], one sees that $M$ maps $D(M)$ into $X_{\text{mag}}$. Hence, the resolvent $(\lambda I - M)^{-1}$ for $\lambda > 0$ leaves invariant $X_{\text{mag}}$. The same is true for the operator $e^{tM}$ since it is the strong limit of $(\frac{\lambda}{\mu}(\frac{\lambda}{\mu}I - A)^{-1})^n$ in $X$ for $t > 0$, see Corollary III.5.5 of [10]. Due to Duhamel’s formula, the solution $w$ then takes values in $X_{\text{mag}}$ and thus solves (1.1).

Equation (1.1a) implies that $\partial_t \text{div}(\varepsilon E(t)) = -\text{div}(\sigma E(t) + J(t))$ in $H^{-1}(Q)$ for $t \geq 0$, whence the first part of (2.7) follows. Writing $\sigma = \frac{\varepsilon}{\varepsilon}$, we infer
\[
\partial_t \text{div}(\varepsilon E(t)) = -\varepsilon \text{div}(\varepsilon E(t)) - \nabla(\varepsilon^2)\varepsilon E(t) - \text{div} J(t),
\]
\[
\partial_t (e^{t/\varepsilon} \text{div}(\varepsilon E(t))) = e^{t/\varepsilon} (\nabla(\varepsilon^2)\varepsilon E(t) + \text{div} J(t))
\]
in $H^{-1}(Q)$. This formula leads to the second part of (2.7), and b) is established.

3) For the remaining assertions in a), we take $J = 0$. Since $e^{tM}$ is a contraction in $X$, we have $\|w(t)\|_X \leq \|w_0\|_X$. From (2.7) we then deduce the bound
\[
\|\text{div}(\varepsilon E(t))\|_{L^2} \leq \|\text{div}(\varepsilon E_0)\|_{L^2} + ct \|E_0, H_0\|_{L^2}
\]
and that $\text{div}(\varepsilon E(t))$ tends to $\text{div}(\varepsilon E_0)$ in $L^2(Q)$ as $t \to 0$. Thus, $e^{tM}$ possesses a restriction $e^{tM_{\text{div}}}$ to $X_{\text{div}}$, which forms a $C_0$-semigroup there and is bounded by $c(1 + t)$. By Paragraph II.2.3 of [10] it is generated by $M_{\text{div}}$.

4) Under the assumptions of part c), we can differentiate (2.6) in $X_{\text{div}}$ twice in $t$ (after the substitution $r = t - s$.) Hence, $w$ belongs to $C^2([0, T], X_{\text{div}})$ and
\[
w'' - w' = (M_{\text{div}} - I)w' - \frac{1}{\varepsilon}(J', 0).
\]

Inverting $M_{\text{div}} - I$, we thus obtain $w \in C^1([0, T], D(M_{\text{div}}))$. Similarly, the equation $w'' = M_{\text{div}}w - \frac{1}{\varepsilon}(J, 0)$ then implies that $w$ is contained in $C([0, T], D(M_{\text{div}}^2))$. These arguments also yield the asserted bound in c).

\[\square\]

Remark 2.4. In Proposition 2.3 we also assume that $\sigma = 0$ or $\sigma \geq \sigma_0$ for a constant $\sigma_0 > 0$. An inspection of the above proof then shows that we can omit the factors $(1 + t)$ in part a) and $(1 + T)$ in c). If $\sigma \geq \sigma_0$, then the constant $c$ also depends on $1/\sigma_0$.

3. The split operators. We now analyze the operators
\[
A = \begin{pmatrix}
\frac{-\varepsilon}{\varepsilon} I & C_1 \\
\frac{1}{\varepsilon} C_2 & 0
\end{pmatrix}
\quad \text{and} \quad
B = \begin{pmatrix}
\frac{-\varepsilon}{\varepsilon} I & -\frac{1}{\varepsilon} C_2 \\
\frac{1}{\varepsilon} C_1 & 0
\end{pmatrix}
\]
\[
C_1 = \begin{pmatrix}
0 & 0 & \partial_2 \\
0 & \partial_3 & 0 \\
0 & 0 & \partial_1
\end{pmatrix}
\quad \text{and} \quad
C_2 = \begin{pmatrix}
0 & 0 & \partial_3 \\
0 & \partial_1 & 0 \\
\partial_2 & 0 & 0
\end{pmatrix}
\]
with
\[
D(A) = \{(u, v) \in X \mid (C_1 v, C_2 u) \in X, \quad \text{tr}_\Gamma u_1 = 0, \quad \text{tr}_\Gamma, u_2 = 0, \quad \text{tr}_\Gamma, u_3 = 0\},
\]
\[
D(B) = \{(u, v) \in X \mid (C_2 v, C_1 u) \in X, \quad \text{tr}_\Gamma u_1 = 0, \quad \text{tr}_\Gamma, u_2 = 0, \quad \text{tr}_\Gamma, u_3 = 0\}.
\]
Each domain contains one half of the electric boundary conditions in \( D(M_{\text{div}}) \), see Proposition 2.2. These traces exist since they fit to the partial derivatives in \( C_{2}u \) for \( A \) and in \( C_{1}u \) for \( B \). We note that \( A \) and \( B \) map into \( X \),

\[ D(A) \cap D(B) \hookrightarrow D(M), \quad \text{and} \quad M = A + B \quad \text{on} \quad D(A) \cap D(B). \]

Neither the divergence conditions nor the magnetic boundary condition for the magnetic field are included in \( D(A) \) or \( D(B) \). We further write

\[
A_{0} = A + \begin{pmatrix} \sigma I & 0 \\ \frac{\sigma I}{\mu} & 0 \end{pmatrix} \quad \text{and} \quad B_{0} = B + \begin{pmatrix} \frac{\sigma I}{\mu} & 0 \\ 0 & 0 \end{pmatrix}, \tag{3.2}
\]

with \( D(A_{0}) = D(A) \) and \( D(B_{0}) = D(B) \) for the parts without conductivity. As in Section 4.3 in [13], one shows the following basic integration by parts formula. Let \( u, \varphi \in L^{2}(Q)^{3} \) satisfy \( C_{1}\varphi \in L^{2}(Q)^{2}, C_{2}u \in L^{2}(Q)^{3} \), and

\[
\text{tr}_{\Gamma_{3}} u \cdot \text{tr}_{\Gamma_{3}} \varphi_{1} = 0, \quad \text{tr}_{\Gamma_{1}} u_{3} \cdot \text{tr}_{\Gamma_{1}} \varphi_{2} = 0, \quad \text{tr}_{\Gamma_{2}} u_{1} \cdot \text{tr}_{\Gamma_{2}} \varphi_{3} = 0.
\]

(For instance, take \((u, \varphi) \in D(A) \) or \((\varphi, u) \in D(B) \).) We then have

\[
(C_{2}u \mid \varphi)_{L^{2}} = (u \mid -C_{1}\varphi)_{L^{2}}. \tag{3.3}
\]

In our splitting algorithm (4.1) we use resolvents and Cayley transforms of \( A \) and \( B \), and those of \( A^{*} \) and \( B^{*} \) enter in the error analysis. Their properties are stated in the next proposition. Let \( L - \kappa I \) generate a contraction semigroup for and operator \( L \) and some \( \kappa \geq 0 \). The Cayley transform

\[
\gamma_{\tau}(L) = (I + \tau L)(I - \tau L)^{-1}
\]

then exists for all \( \tau \in (0, 1/\kappa) \). Observe that \((I - \tau L)^{-1} = \tau^{-1}(\tau^{-1}I - L)^{-1}\).

**Proposition 3.1.** a) In \( X \) we have \( D(A^{*}) = D(A_{0}^{*}) = D(A) \) and \( D(B^{*}) = D(B_{0}^{*}) = D(B) \), as well as \( A_{0}^{*} = -A_{0}, B_{0}^{*} = -B_{0} \).

\[
A^{*} = \begin{pmatrix} \frac{-\sigma I}{\mu} & -\frac{1}{\mu} C_{1} \\ \frac{1}{\mu} C_{2} & 0 \end{pmatrix} \quad \text{and} \quad B^{*} = \begin{pmatrix} \frac{-\sigma I}{\mu} & \frac{1}{\mu} C_{2} \\ \frac{1}{\mu} C_{1} & 0 \end{pmatrix}.
\]

It follows \( M^{*} = A^{*} + B^{*} \) on \( D(A^{*}) \cap D(B^{*}) \hookrightarrow D(M^{*}) = D(M) \).

b) The operators \( A, B, A^{*} \) and \( B^{*} \) generate \( C_{0} \)-semigroups of contractions on \( X \). As a result, the resolvents \((I - \tau L)^{-1}\) and the Cayley transforms \( \gamma_{\tau}(L) \) are contractive for all \( L \in \{A, B, A^{*}, B^{*}\} \) and \( \tau > 0 \). Moreover, \( D(M_{\text{div}}) \hookrightarrow D(L) \).

**Proof.** Lemma 4.3 of [13] says that \( A_{0} \) and \( B_{0} \) are skew-adjoint on \( X \), and hence generate a contraction semigroup. This property is inherited by \( A \) and \( B \) due to (3.2) and the dissipative perturbation Theorem III.2.7 in [10]. In the same way one shows the generator property of the operator matrices defined in part a) on the domains \( D(A) \) and \( D(B) \), respectively. As in Lemma 4.3 of [13], equation (3.3) implies that these operator matrices are restrictions of \( A^{*} \) and \( B^{*} \). They are thus equal to these operators, respectively, since the right half plane belongs to all resolvent sets. The other assertions in b) then easily follow, using also Proposition 2.2. \( \square \)

For our error analysis we need the restrictions of the above operators to the subspace of \( H^{1} \) given by

\[
Y = \{ (u, v) \in H^{1}(Q)^{6} \mid u_{j} = 0 \text{ on } \Gamma \setminus \Gamma_{j}, \ v_{j} = 0 \text{ on } \Gamma_{j} \text{ for all } j \in \{1, 2, 3\} \}.
\]

We use on \( Y \) the weighted inner product

\[
((u, v) \mid (\varphi, \psi))_{Y} = \int_{Q} \left( \varepsilon u \cdot \varphi + \mu v \cdot \psi + \varepsilon \sum_{j=1}^{3} \partial_{j} u \cdot \partial_{j} \varphi + \mu \sum_{j=1}^{3} \partial_{j} v \cdot \partial_{j} \psi \right) dx
\]
with the induced norm $||\cdot||_Y$. Due to (1.3), this norm is equivalent to the usual one on $H^1$. The continuity of the traces implies that $Y$ is a closed subspace of $H^1(Q)^6$. We very often use that maps like $(u,v) \mapsto (\varepsilon u, \mu v)$ leave invariant $Y$ because of (1.3). Our definitions yield the embedding

$$Y \hookrightarrow D(A) \cap D(B) \cap D(A^*) \cap D(B^*) \cap D(M) \cap D(M^*).$$

We denote by $A_Y$, $B_Y$, $(A^*)_Y$, and $(B^*)_Y$ the parts of $A$, $B$, $A^*$, and $B^*$ in $Y$, respectively. Their domains are described in the next lemma.

**Lemma 3.2.** a) We have

$$D(A_Y) = D((A^*)_Y) = \{(u,v) \in Y \mid (C_1v, C_2u) \in Y\}$$

$$= \{(u,v) \in H^1(Q)^6 \mid u_j = 0 \text{ on } \Gamma \setminus \Gamma_j, \ v_j = 0 \text{ on } \Gamma_j \text{ for } j \in \{1, 2, 3\},$$

$$\partial_2 u_1, \partial_3 u_2, \partial_1 u_3, \partial_3 v_1, \partial_1 v_2, \partial_2 v_3 \in H^1(Q),$$

$$\partial_3 v_1 = 0 \text{ on } \Gamma_3, \ \partial_1 v_2 = 0 \text{ on } \Gamma_1, \ \partial_2 v_3 = 0 \text{ on } \Gamma_2,$$

$$D(B_Y) = D((B^*)_Y) = \{(u,v) \in Y \mid (C_2v, C_1u) \in Y\}$$

$$= \{(u,v) \in H^1(Q)^6 \mid u_j = 0 \text{ on } \Gamma \setminus \Gamma_j, \ v_j = 0 \text{ on } \Gamma_j \text{ for } j \in \{1, 2, 3\},$$

$$\partial_3 u_1, \partial_1 u_2, \partial_2 v_3, \partial_2 v_1, \partial_3 v_2, \partial_1 v_3 \in H^1(Q),$$

$$\partial_2 v_1 = 0 \text{ on } \Gamma_2, \ \partial_3 v_2 = 0 \text{ on } \Gamma_3, \ \partial_1 v_3 = 0 \text{ on } \Gamma_1}\}.$$

b) Let $(u,v), (\tilde{u}, \tilde{v}) \in Y$ and $C_2u, C_1v, C_1\tilde{u}, C_2\tilde{v} \in H^1(Q)^3$. Then

$$\partial_3 u_2 = \partial_2 u_3 = \partial_3 u_3 = \partial_3 v_1 = 0 \quad \text{on } \Gamma_1,$$

$$\partial_1 u_3 = \partial_3 u_1 = \partial_1 u_1 = \partial_1 v_2 = 0 \quad \text{on } \Gamma_2,$$

$$\partial_2 u_1 = \partial_1 u_2 = \partial_2 u_2 = \partial_2 v_3 = 0 \quad \text{on } \Gamma_3,$$

$$\partial_2 \tilde{u}_3 = \partial_2 \tilde{v}_2 = \partial_3 \tilde{u}_2 = \partial_3 \tilde{v}_1 = 0 \quad \text{on } \Gamma_1,$$

$$\partial_3 \tilde{u}_1 = \partial_3 \tilde{v}_3 = \partial_1 \tilde{u}_3 = \partial_3 \tilde{v}_2 = 0 \quad \text{on } \Gamma_2,$$

$$\partial_1 \tilde{u}_2 = \partial_2 \tilde{u}_1 = \partial_2 \tilde{v}_1 = \partial_1 \tilde{v}_3 = 0 \quad \text{on } \Gamma_3.$$

Here the respective first line in a) follows from Proposition 3.1 and (1.3). Part b) is a consequence of Lemma 2.1 and it yields the rest of assertion a). In a series of further lemmas we collect the basic properties of the above operators.

**Lemma 3.3.** The operators $A_Y$, $B_Y$, $(A^*)_Y$, and $(B^*)_Y$ are closed and densely defined in $Y$.

**Proof.** 1) The operators are closed as the parts of closed operators. For the density, we only treat $A_Y$ since the other cases can be handled in the same way, using Proposition 3.1 for $(A^*)_Y$ and $(B^*)_Y$.

Let $(u,v) \in Y$. We approximate $u_1 =: f$ and $v_1 =: g$ in $Y$ by maps $f_n$ and $g_n$ which are the first and fourth components of vectors $(u_n, v_n) \in D(A_Y)$, respectively. We use smooth cut-off functions $\chi_n^{(j)} : [a_j^-, a_j^+] \to [0, 1]$ with $|(|\chi_n^{(j)}|^') \leq cn$ that vanish on $[a_j^-, a_j^- + \frac{1}{4n}] \cup [a_j^+, a_j^+ - \frac{1}{4n}]$ and are equal to one on $[a_j^- + \frac{1}{4n}, a_j^+ - \frac{1}{4n}]$ for $n > (2d_Q)^{-1} =: \ell$ and $j \in \{1, 2, 3\}$. Also, $\rho_n^{(j)}$ is a standard $C^\infty$-mollifier with support in $[-\frac{1}{4n}, \frac{1}{4n}]$ that acts on $x_j$.

2) We are looking for functions $f_n \in H^1(Q)$ with $\partial_2 f_n \in H^1(Q)$ that converge to $f$ in $H^1(Q)$ and have zero traces on $\Gamma_2$ and $\Gamma_3$. We first set $\varphi_n = \chi_n^{(2)} f$ for
These maps belong to $H^1(Q)$, vanish near $\Gamma_2$ and have trace 0 on $\Gamma_3$. Due to dominated convergence, the functions $\varphi_n$ tend to $f$ and $\chi^{(2)}_n \partial_j f$ to $\partial_j f$ in $L^2(Q)$ as $n \to \infty$ and for $j \in \{1, 2, 3\}$. As in the proof of Lemma 2.1, one shows that $((\chi^{(2)}_n f))$ converges to 0 in $L^2(Q)$ since $f$ vanishes on $\Gamma_2$. Summing up, $(\varphi_n)$ has the limit $f$ in $H^1(Q)$. We then extend $\varphi_n$ by 0 to $\mathbb{R}^3$ and define
\[
\varphi_n = (\rho_n^{(2)} * \varphi_n)|_Q
\]
for all $n > \ell$. This function and $\partial_2 f_n$ belong to $H^1(Q)$, and $f_n$ vanishes near $\Gamma_2$. For a map $h \in H^1(Q) \cap C(\overline{Q})$ with $h = 0$ on $\Gamma_3$ it is clear that $\rho_n^{(2)} * h$ is also equal to 0 on $\Gamma_3$. By approximation, we thus obtain $\text{tr}_3 f_n = 0$.

3) We next have to construct functions $g_n \in H^1(Q)$ with $\partial_3 g_n \in H^1(Q)$ such that $\text{tr}_\Gamma g_n = 0$, $\text{tr}_{\Gamma_1} \partial_3 g_n = 0$ and $(g_n)$ has the limit $g$ in $H^1(Q)$. Let $\Phi$ be the linear and bounded Stein extension operator that maps functions in $H^k(Q)$ to functions in $H^k(\mathbb{R}^3)$, see Theorem 5.24 in [1]. Extending $g$ by 0 to $\mathbb{R}^3$, we set
\[
\psi_{n,m} = \left[(\rho_n^{(2)} * \rho_m^{(3)} * \Phi(\rho_m^{(1)} * (\chi_m^{(1)} g)))|_Q\right]
\]
for all $n, m > \ell$. These smooth maps and their derivatives vanish near $\Gamma_1$. Let $\eta > 0$. Arguing as in step 2), we fix an index $\bar{m} = \bar{m}(\eta) > \ell$ such that
\[
\left\|\rho_m^{(1)} * (\chi_m^{(1)} g) - g\right\|_{H^1(Q)} \leq \eta.
\]
By the properties of mollifiers, there also exists a number $\bar{n} = \bar{n}(\eta) > \ell$ with
\[
\left\|\psi_{n,\bar{m}} - \Phi(\rho_{\bar{m}}^{(1)} * (\chi_{\bar{m}}^{(1)} g))|_Q\right\|_{H^1(Q)} \leq \eta.
\]
Setting $\bar{g} = \psi_{\bar{n},\bar{m}}$, we obtain the inequality
\[
\left\|\bar{g} - g\right\|_{H^1} \leq (1 + \|\Phi\|_{B(H^1(Q), H^1(\mathbb{R}^3))})\eta.
\]
In view of the needed boundary condition of $\partial_3 g_n$, we define
\[
g_n(x) = \bar{g}(x) + \int_{\partial_3}^{x_3} (\chi_n^{(3)}(t) - 1) \partial_3 \bar{g}(x', t) \, dt =: \tilde{g}(x) + r_n(x)
\]
for $x = (x', x_3) \in Q$ and $n > \ell$. The functions $g_n$ and $\partial_3 g_n = \chi_n^{(3)} \partial_3 \tilde{g}$ are contained in $H^1(Q)$. Moreover, the traces of $g_n$ on $\Gamma_1$ and of $\partial_3 g_n$ on $\Gamma_3$ are zero by construction. The integrands of $r_n$ and their derivatives with respect to $x_3$ and $x_2$ are uniformly bounded by a constant, and these maps tend to 0 pointwise a.e. as $n \to \infty$. By dominated convergence, the functions $r_n$, $\partial_1 r_n$ and $\partial_2 r_n$ thus converge to 0 pointwise a.e. and then in $L^2(Q)$ as $n \to \infty$. The same is true for $\partial_3 r_n = (\chi_n^{(3)} - 1) \partial_3 \tilde{g}$. As a result, $(g_n)$ has the limit $g$ in $H^1(Q)$.

The other components of $(u, v)$ are treated in the same way.

We set
\[
\kappa_Y = \frac{3\|\nabla \sigma\|_{L^\infty}}{4\delta} + \frac{3\|\sigma\|_{L^\infty}\|\nabla \varepsilon\|_{L^\infty}}{4\delta^2} + \frac{\|\nabla \varepsilon\|_{L^\infty} + \|\nabla \mu\|_{L^\infty}}{2\delta^2}.
\]

(3.6)

**Lemma 3.4.** The operators $A_Y - \kappa_Y I$, $B_Y - \kappa_Y I$, $(A^*)_Y - \kappa_Y I$, and $(B^*)_Y - \kappa_Y I$ are dissipative on $Y$.

**Proof.** Again we only consider $A_Y$. Let $(u, v) \in D(A_Y)$. In view of the boundary conditions in Lemma 3.2, integration by parts yields
\[
\int_Q (\partial_j C_1 v \cdot \partial_j u + \partial_j C_2 u \cdot \partial_j v) \, dx = \int_Q (\partial_{j_2} v_3 \partial_j u_1 + \partial_{j_3} v_1 \partial_j u_2 + \partial_{j_1} v_2 \partial_j u_3)
\]
for \( j \in \{1, 2, 3\} \). The above equation, (3.3) and Hölder’s inequality imply

\[
(A(u, v) | (u, v))_Y = \int_Q \left( -\frac{\sigma \varepsilon}{2\varepsilon} |u|^2 + \frac{\varepsilon}{\varepsilon} C_1 v \cdot u + \frac{\mu}{\mu} C_2 u \cdot v - \varepsilon \sum_{j=1}^3 \partial_j \left( \frac{1}{\varepsilon} C_1 v \right) \cdot \partial_j u + \varepsilon \sum_{j=1}^3 \partial_j \left( \frac{1}{\varepsilon} C_2 u \right) \cdot \partial_j u \right) dx
\]

\[
+ \varepsilon \sum_{j=1}^3 \partial_j \left( \frac{1}{\varepsilon} C_1 v \right) \cdot \partial_j u + \mu \sum_{j=1}^3 \partial_j \left( \frac{1}{\mu} C_2 u \right) \cdot \partial_j v \right) dx
\]

\[
= -\int_Q \frac{\sigma}{2} (|u|^2 + |\partial u|^2) \, dx - \sum_{j=1}^3 \int_Q (\frac{\partial \sigma}{2} - \frac{\sigma \partial \varepsilon}{2\varepsilon}) u \cdot \partial_j u \, dx
\]

\[
- \sum_{j=1}^3 \int_Q \frac{\partial \varepsilon}{\varepsilon} C_1 v \cdot \partial_j u \, dx - \sum_{j=1}^3 \int_Q \frac{\partial \mu}{\mu} C_2 u \cdot \partial_j v \, dx
\]

\[
\leq \left( \frac{\|\nabla \sigma\|_{L^\infty}}{4\delta} + \frac{\|\sigma\|_{L^\infty} \|\nabla \varepsilon\|_{L^\infty}}{4\delta^2} \right) \int_Q (3\varepsilon |u|^2 + \varepsilon |\partial u|^2) \, dx
\]

\[
+ \frac{\|\nabla \varepsilon\|_{L^\infty} + \|\nabla \mu\|_{L^\infty}}{2\delta^2} \int_Q (\varepsilon |\partial u|^2 + \mu |\partial v|^2) \, dx
\]

\[
\leq \kappa_Y \|(u, v)\|_Y^2,
\]

where \(|\partial u|\) and \(|\partial v|\) denote the Frobenius norm of the Jacobian matrices. \(\square\)

**Lemma 3.5.** The operators \((1 + \kappa_Y) I - A_Y\), \((1 + \kappa_Y) I - B_Y\), \((1 + \kappa_Y) I - (A^*)_Y\), and \((1 + \kappa_Y) I - (B^*)_Y\) have dense range in \(Y\).

**Proof.** 1) As above we only consider \((1 + \kappa_Y) I - A_Y\). We know from Lemma 3.3 that \(D(A_Y)\) is dense in \(Y\). Let \((f, g) \in D(A_Y)\). We look for fields \((u, v) \in D(A_Y)\) with \(((1 + \kappa_Y) I - A)(u, v) = (f, g)\); i.e.,

\[
(1 + \kappa_Y + \frac{\sigma}{\varepsilon}) u_1 - 1 \partial_2 v_3 = f_1, \quad (1 + \kappa_Y) v_3 - 1 \partial_2 u_1 = g_3,
\]

\[
(1 + \kappa_Y + \frac{\sigma}{\varepsilon}) u_2 - 1 \partial_3 v_1 = f_2, \quad (1 + \kappa_Y) v_1 - 1 \partial_3 u_2 = g_1,
\]

\[
(1 + \kappa_Y + \frac{\sigma}{\varepsilon}) u_3 - 1 \partial_1 v_2 = f_3, \quad (1 + \kappa_Y) v_2 - 1 \partial_1 u_3 = g_2.
\]

We formally insert in each line the second equation into the first one, obtaining

\[
(\varepsilon (1 + \kappa_Y) + \frac{\sigma}{2}) u_1 - 1 \frac{1}{1+\kappa_Y} D_2 u_1 = \varepsilon f_1 + 1 \frac{1}{1+\kappa_Y} \partial_2 g_3 =: h_1,
\]

\[
(\varepsilon (1 + \kappa_Y) + \frac{\sigma}{2}) u_2 - 1 \frac{1}{1+\kappa_Y} D_3 u_2 = \varepsilon f_2 + 1 \frac{1}{1+\kappa_Y} \partial_3 g_1 =: h_2,
\]

\[
(\varepsilon (1 + \kappa_Y) + \frac{\sigma}{2}) u_3 - 1 \frac{1}{1+\kappa_Y} D_1 u_3 = \varepsilon f_3 + 1 \frac{1}{1+\kappa_Y} \partial_1 g_2 =: h_3.
\]

Here we have set \(D_j = \partial_j \frac{1}{\mu} \partial_j\) with domain

\[
D(D_j) := \{ \varphi \in L^2(Q) \mid \partial_j \varphi \in L^2(Q), \quad D_j \varphi \in L^2(Q), \quad \varphi = 0 \text{ on } \Gamma_j \}
\]

\[
= \{ \varphi \in L^2(Q) \mid \partial_j \varphi \in L^2(Q), \partial_j^2 \varphi \in L^2(Q), \quad \varphi = 0 \text{ on } \Gamma_j \}
\]

for \( j \in \{1, 2, 3\} \), where we have used (1.3). Since \((f, g) \in D(A_Y)\), the map \(h_j\) belongs to \(H^1(Q)\) and satisfies \(h_j = 0\) on \(\Gamma \setminus \Gamma_j\), see Lemma 3.2. We also define \(D(\partial_j) = \{ \varphi \in L^2(Q) \mid \partial_j \varphi \in L^2(Q), \quad \varphi = 0 \text{ on } \Gamma_j \}\).
2) Let $j = 2$. We are looking for a function $w \in D(D_2)$ solving (3.8), where we put $h := h_1$. To this aim, we abbreviate

$$Lw = ((1 + \kappa_Y)\varepsilon + \frac{\sigma}{2})w - \frac{1}{1 + \kappa_Y} \partial_2 \left( \frac{1}{\mu} \partial_2 w \right)$$

for $w \in D(D_2)$. As in the proof of Lemma 4.3 in [13], where $\sigma = 0$ and $\kappa_Y = 0$, by means of the Lax–Milgram lemma we obtain a unique map $w \in D(D_2)$ with $Lw = h$, and $w$ thus satisfies (3.8). Moreover, Theorem VI.2.7 in [16] shows that $L$ is given by the symmetric, closed, positive definite and densely defined bilinear form

$$(w, \bar{w}) \mapsto \left( \left( (1 + \kappa_Y)\varepsilon + \frac{\sigma}{2} \right)w, \bar{w} \right)_{L^2} + \frac{1}{1 + \kappa_Y} \left( \frac{1}{\mu} \partial_2 w, \partial_2 \bar{w} \right)_{L^2}$$

on $D(\partial_2)$, and that $L$ is invertible and self-adjoint in $X$.

We check that $w$ satisfies the properties of $u_1$ needed for $(u, v) \in D(A_Y)$. Let $k \in \{1, 2, 3\}$ and $\varphi \in H^2_0(Q)$. Since $\partial_2 h = \partial_2 \partial_2 w$ in $H^{-2}(Q)$ and $\partial_2 w$ belongs to $L^2(Q)$, the derivative $\partial_2 \partial_2 w$ is contained in $H^{-1}(Q)$ and hence $D_2 \partial_2 h \in H^{-2}(Q)$ by (1.3). We can thus compute

$$\langle L\partial_2 w, \varphi \rangle_{H^{-1} \times H^2_0}$$

for $Q$.

We check that $w$ vanishes if $\varphi$ vanishes if $\varphi \in D(\partial_2)$ because $H^2_0(Q)$ is dense in $H^1(Q)$. We have shown the equation

$$L\partial_2 w = \partial_2 h - \partial_2 ((1 + \kappa_Y)\varepsilon + \frac{\sigma}{2})w + \frac{1}{1 + \kappa_Y} \partial_2 \left( \frac{1}{\mu} \partial_2 w \right) =: \psi(h) \quad (3.9)$$

in $D(\partial_2)^*$. Theorem VI.2.23 in [16] yields the isomorphism $D(\partial_2) \cong D(L^{1/2})$, so that $D(\partial_2)^* \cong D(L^{1/2})^*$. Using also Theorem V.1.4.12 in [2], we deduce that $\partial_2 w = L^{-1}\psi(h)$ belongs to $D(\partial_2) \cong D(L^{1/2})$, and hence $w$ and $\partial_2 w$ are contained in $H^1(Q)$. We already know that $w = 0$ on $\Gamma_2$.

3) To prove that $w = 0$ on $\Gamma_3$, we approximate $h$ by functions $h_n = \chi_n^3 h$ with $n > (2d_Q)^{-1}$, cf. step 2) of the proof of Lemma 3.3. As in that proof one sees that $h_n$ vanishes if $|\alpha_3 - a_3| \leq 1/(2n)$ and that $(h_n)$ tends to $h$ in $H^1(Q)$ as $n \to \infty$. The map $w_n := L^{-1}h_n \in D(D_2)$ thus converges to $w$ in $D(\partial_2)$. We take functions $\phi_n \in C_0^{\infty}(\alpha_3, a_3^+)$ which are equal to 1 on $[\alpha_3^+ + 1/(2n), a_3^+ - 1/(2n)]$; i.e., $h_n = \phi_n h_n$. Since $L$ is injective and

$$Lw_n = h_n = \phi_n h_n = \phi_n Lw_n = L(\phi_n w_n),$$

we obtain $w_n = \phi_n w_n = 0$ on $\Gamma_3$. Formula (3.9) and assumption (1.3) next yield

$$\|\partial_2 (w_n - w)\|_{L^2} = \|L^{-1}_2 \psi(h_n - h)\|_{L^2} \leq c \|\psi(h_n - h)\|_{D(\partial_2)^*}.$$
\[
\leq c \left[ \| \partial_k h_n - \partial_k h \|_{L^2} + \| \partial_k \left( (1 + \kappa_Y) \varepsilon + \frac{\sigma}{2} \right) (w_n - w) \|_{L^2} \right. \\
+ \left. \frac{1}{1 + \kappa_Y} \left\| \partial_k \left( \frac{1}{\nu} \right) \partial_2 (w_n - w) \right\|_{L^2} \right]
\]

for \( k \in \{1, 2, 3\} \). The right-hand side tends to 0 as \( n \to \infty \), so that \( w_n \) converges to \( w \) in \( H^1(Q) \). As a result, \( w \) has the trace 0 on \( \Gamma_3 \). We set \( u_1 = w \).

4) In a final step we define

\[
v_3 = \frac{1}{1 + \kappa_Y} g_3 + \frac{1}{1 + \kappa_Y} \frac{1}{\nu} \partial_2 u_1 \in H^1(Q),
\]

compare (3.7). The trace \( v_3 \) on \( \Gamma_3 \) vanishes since \((f,g) \in D(A_Y)\) and \( \partial_2 u_1 = 0 \) on \( \Gamma_3 \) by Lemma 2.1. We differentiate the above equation w.r.t. \( x_2 \) and insert the equations \( Lu_1 = h_1 \) and (3.8). It follows

\[
\partial_2 v_3 = \frac{1}{1 + \kappa_Y} \partial_2 g_3 + \left( (1 + \kappa_Y) \varepsilon + \frac{1}{2} \right) u_1 - h_1 = -\varepsilon f_1 + \left( \varepsilon (1 + \kappa_Y) + \frac{\sigma}{2} \right) u_1
\]
in \( L^2(Q) \); i.e., \( u_1 \) and \( v_3 \) satisfy (3.7). This equation also yields that \( \partial_2 v_3 \) belongs to \( H^1(Q) \) and has trace 0 on \( \Gamma_2 \), as required in \( D(A_Y) \). The other components are treated in the same way.

We now easily derive the basic properties of our split operators on \( Y \) by means of the Lumer–Phillips theorem, see e.g. Section II.3.b in [10]. Recall the definition of \( \kappa_Y \) in (3.6).

**Proposition 3.6.** Let \( L \in \{A, B, A^*, B^*\} \). The part \( L_Y \) of \( L \) in \( Y \) generates a \( C_0 \)-semigroup on \( Y \) bounded by \( e^{\kappa_Y t} \). The resolvent \( (I - \tau L_Y)^{-1} \) is the restriction of \( (I - \tau L)^{-1} \) to \( Y \) and it satisfies

\[
\| (I - \tau L_Y)^{-1} \|_{\mathcal{B}(Y)} \leq \frac{1}{1 - \tau \kappa_Y}
\]

for all \( 0 < \tau < \frac{1}{\kappa_Y} \), so that \( \| (I - \tau L_Y)^{-1} \|_{\mathcal{B}(Y)} \leq 2 \) for all \( 0 < \tau \leq \frac{1}{\kappa_Y} \). The Cayley transforms are dominated by

\[
\| \gamma_\tau (L_Y) \|_{\mathcal{B}(Y)} \leq e^{3 \kappa_Y \tau}
\]

for all \( 0 < \tau \leq \tau_0 \) and a constant \( \tau_0 \in (0, (2 \kappa_Y)^{-1}] \) only depending on \( \kappa_Y \).

**Proof.** The generation property and the resolvent bounds follow from Lemmas 3.3–3.5 and the Lumer–Phillips theorem. Let \( 0 < \tau < \frac{1}{\kappa_Y} \) and \( h \in Y \). The function \( w = (I - \tau L_Y)^{-1} h \) then satisfies \( h = (I - \tau L_Y) w = (I - \tau L) w \) and hence \( w = (I - \tau L)^{-1} h \), which is the asserted restriction property. For \( \text{Re} \ z > 0 \) we define

\[
\bar{\gamma}_\tau(z) = \frac{1 - \tau(z - \kappa_Y)}{1 + \tau(z - \kappa_Y)}.
\]

It is easily seen that

\[
\sup_{\text{Re} \ z > 0} | \bar{\gamma}(z) | = \frac{1 + \tau \kappa_Y}{1 - \tau \kappa_Y} \leq e^{3 \kappa_Y \tau}
\]

for all \( 0 < \tau \leq \bar{\tau} \) and a constant \( \bar{\tau} \in (0, 1/\kappa_Y) \) only depending on \( \kappa_Y \). Since \( L_Y - \kappa_Y I \) generates a contraction semigroup on a Hilbert space, Theorem 11.5 of [17] provides an \( H^\infty \)-functional calculus for \( \kappa_Y I - L_Y \) and the desired estimate

\[
\| \gamma_\tau(L_Y) \|_{\mathcal{B}(Y)} = \| \bar{\gamma}_\tau(\kappa_Y I - L_Y) \|_{\mathcal{B}(Y)} \leq \sup_{\text{Re} \ z > 0} | \bar{\gamma}_\tau(z) | \leq e^{3 \kappa_Y \tau}
\]
for all $\tau \in (0, \tilde{\tau})$, where we recall (3.4). (In this argument we use the complexification of $Y$ and the natural extension of $L_Y$.) We set $\tau_0 = \min\{\tilde{\tau}, (2\kappa_Y)^{-1}\}$. \hfill $\square$

4. The ADI splitting scheme. Let $\tau > 0$. We set $t_n = n\tau$ for $n \in \mathbb{N}_0$ and assume that $(\mathbf{J}(t), 0) \in D(A)$ for all $t \geq 0$. The ADI splitting scheme is given by the operators

\[
S_{\tau, n+1}^J \tilde{w} = S_{\tau}^{(2)} \left[ S_{\tau}^{(1)} \tilde{w} - \frac{\tau}{2\tau} (\mathbf{J}(t_n) + \mathbf{J}(t_{n+1}), 0) \right],
\]

\[
S_{\tau}^{(1)} = (I - \frac{\tau}{2} A)^{-1} (I + \frac{\tau}{2} B) : D(B) \to D(A),
\]

\[
S_{\tau}^{(2)} = (I - \frac{\tau}{2} B)^{-1} (I + \frac{\tau}{2} A) : D(A) \to D(B),
\]

for $\tilde{w} \in D(B)$. Note that $(\frac{1}{2}\mathbf{J}(t), 0) \in D(A)$. For $n \in \mathbb{N}_0$ and $w_0 \in D(B)$, we further write

\[
(E_{n}, H_{n}) = w_{n} = S_{\tau, n}^J \cdots S_{\tau, 1}^J w_0,
\]

\[
(E_{n+1/2}, H_{n+1/2}) = w_{n+1/2} = S_{\tau}^{(1)} S_{\tau, n}^J \cdots S_{\tau, 1}^J w_0.
\]

Remark 4.1. Let $w_0 \in D(B_H)$ and $(\frac{1}{2}\mathbf{J}(t), 0) \in D(A_H)$ for all $t \in \mathbb{R}$. Proposition 3.6 then yields $w_n \in D(B_H)$ and $w_{n+1/2} \in D(A_H)$ for all $n \in \mathbb{N}_0$.

The operators $S_{\tau}^{(k)}$ contain implicit steps. For $\sigma = 0$ and $\mathbf{J} = 0$ it was pointed out in [20] and [26] that these steps decouple into (essentially) one dimensional problems, see also [13]. We now extend this observation to our setting. To this aim, for $\lambda \in \{\epsilon, \mu\}$ we define the operators

\[
D_{\lambda}^{(1)} := C_1 \frac{1}{\lambda} C_2 = \begin{pmatrix}
\partial_2 \frac{1}{\lambda} \partial_2 & 0 & 0 \\
0 & \partial_3 \frac{1}{\lambda} \partial_3 & 0 \\
0 & 0 & \partial_1 \frac{1}{\lambda} \partial_1 
\end{pmatrix}
\]

and

\[
D_{\lambda}^{(2)} := C_2 \frac{1}{\lambda} C_1 = \begin{pmatrix}
\partial_3 \frac{1}{\lambda} \partial_3 & 0 & 0 \\
0 & \partial_1 \frac{1}{\lambda} \partial_1 & 0 \\
0 & 0 & \partial_2 \frac{1}{\lambda} \partial_2 
\end{pmatrix}
\]

on the domains $D(\partial_{22}) \times D(\partial_{33}) \times D(\partial_{11})$ and $D(\partial_{33}) \times D(\partial_{11}) \times D(\partial_{22})$, respectively, where $D(\partial_{kk})$ is the set of $f \in L^2(Q)$ with $\partial_{kk} f, k f \in L^2(Q)$ and $f = 0$ on $\Gamma_k$.

Let $w_0 \in D(B_H)$ and $(\frac{1}{2}\mathbf{J}(t), 0) \in D(A_H)$ for all $t \in \mathbb{R}$. Remark 4.1 then yields $(E_n, H_n) \in D(B_H)$ for each $n \in \mathbb{N}$. The above definitions lead to

\[
(1 + \frac{\tau^2}{4\tau}) E_{n+1/2} = (1 - \frac{\tau^2}{4\tau}) E_n - \frac{\tau^2}{2\tau} C_2 H_n + \frac{\tau}{2\tau} C_1 H_{n+1/2},
\]

\[
H_{n+1/2} = H_n - \frac{\tau^2}{2\tau} C_1 E_n + \frac{\tau}{2\tau} C_2 E_{n+1/2},
\]

with $(E_{n+1/2}, H_{n+1/2}) \in D(A_H)$. Because of this regularity, we can insert the second equation into the first one and infer

\[
((1 + \frac{\tau^2}{4\tau}) I - \frac{\tau^2}{4\tau} D_{\mu}^{(1)} ) E_{n+1/2} = (1 - \frac{\tau^2}{4\tau}) E_n + \frac{\tau^2}{2\tau} \text{curl} H_n - \frac{\tau^2}{4\tau} C_1 \frac{\mu}{\mu} C_1 E_n,
\]

\[
H_{n+1/2} = H_n - \frac{\tau^2}{2\tau} C_1 E_n + \frac{\tau}{2\tau} C_2 E_{n+1/2}
\]

in $L^2(Q)^3$, using curl $= C_1 - C_2$. Observe that $E_{n+1/2}$ belongs to the domain of $D_{\mu}^{(1)}$. Due to (4.2), the implicit part of these equations splits into (essentially) one dimensional problems; one only has to solve parameter-dependent elliptic equations in one space variable. Similarly, the second half step is rewritten as

\[
(1 + \frac{\tau^2}{4\tau}) E_{n+1} = (1 - \frac{\tau^2}{4\tau}) E_{n+1/2} + \frac{\tau}{2\tau} C_1 H_{n+1/2} - \frac{\tau}{2\tau} C_2 H_{n+1}
\]

\[- (1 - \frac{\tau^2}{4\tau}) \frac{\mu}{4\tau}(\mathbf{J}(t_n) + \mathbf{J}(t_{n+1})),
\]
\[
H_{n+1} = H_{n+\frac{1}{2}} + \frac{\tau}{2\mu} C_2 E_{n+\frac{1}{2}} - \frac{\tau}{2\mu} C_1 E_{n+1} - \frac{\tau}{2\mu} C_2 E_{n+\frac{1}{2}} (J(t_n) + J(t_{n+1})).
\]
As above we then obtain the essentially one-dimensional problem
\[
((1 + \frac{\tau^2}{4\mu}) I - \frac{\tau^2}{4\mu} D^{(2)}) E_{n+1} = \left(1 - \frac{\tau^2}{4\mu}\right) E_{n+\frac{1}{2}} + \frac{\tau}{2\mu} \text{curl} H_{n+\frac{1}{2}}
\]
\[
- \frac{\tau}{4\mu} C_1^2 E_{n+\frac{1}{2}} - \left(1 - \frac{\tau^2}{4\mu}\right) \frac{\tau}{2\mu} (J(t_n) + J(t_{n+1}))
\]
\[
+ \frac{\tau}{4\mu} C_1^2 C_2 E_{n+\frac{1}{2}} (J(t_n) + J(t_{n+1})),
\]
\[
H_{n+1} = H_{n+\frac{1}{2}} + \frac{\tau}{2\mu} C_2 E_{n+\frac{1}{2}} - \frac{\tau}{2\mu} C_1 E_{n+1}
\]
\[
- \frac{\tau}{2\mu} C_2 E_{n+\frac{1}{2}} (J(t_n) + J(t_{n+1})).
\]
(4.4)

Using the Cayley transforms \(\gamma_r(L)\) from (3.4) and induction, we further deduce from (4.1) the closed expression of the scheme
\[
w_n = (I - \frac{\tau}{2\mu} B)^{-1} \gamma_r(A) \gamma_r(B) \gamma_r(A)^{n-1} (I + \frac{\tau}{2\mu} B) w_0
\]
\[- (I - \frac{\tau}{2\mu} B)^{-1} \sum_{k=1}^n \gamma_r(A) \gamma_r(B)^n \gamma_r(A)^{n-k} (I + \frac{\tau}{2\mu} A) (\frac{\tau}{2\mu} (J(t_k-1) + J(t_k)), 0).
\]

Propositions 3.1 and 3.6 then easily yield the unconditional stability of the scheme in \(X\) and \(Y\). Recall the definition of \(\kappa_1 \geq 0\) in (3.6) and that of \(\tau_0 > 0\) in Proposition 3.6. Both depend only on the constants in (1.3).

**Theorem 4.2.** Let (1.3) hold, \(n \in \mathbb{N}\), \(\tau \in (0, 1]\) and \(T \geq n\tau\). Take \(w_0 \in D(B)\) and \((J, 0) \in C([0, T], D(A))\). We then have
\[
\|w_n\|_{L^2} \leq c \|w_0\|_{B} + c T \max_{t \in [0, T]} \| (J, 0) \|_{A},
\]
\[
\|(I - \frac{\tau}{2\mu} B) w_n\|_X \leq \|(I + \frac{\tau}{2\mu} B) w_0\|_{X} + T \max_{t \in [0, T]} \| (I + \frac{\tau}{2\mu} A) (\frac{1}{\varepsilon} J(t), 0) \|_{X}.
\]
If also \(0 < \tau \leq \tau_0\), \(w_0 \in D(B_Y)\) and \((\frac{1}{\varepsilon} J, 0) \in C([0, T], D(A_Y))\), we obtain
\[
\|w_n\|_{H^1} \leq c e^{\delta_Y^* T} \left( \|w_0\|_{B_Y} + T \max_{t \in [0, T]} \| (\frac{1}{\varepsilon} J(t), 0) \|_{A_Y} \right),
\]
\[
\|(I - \frac{\tau}{2\mu} B) w_n\|_Y \leq e^{\delta_Y^* T} \left( \| (I + \frac{\tau}{2\mu} B_Y) w_0\|_Y + T \max_{t \in [0, T]} \| (I + \frac{\tau}{2\mu} A_Y) (\frac{1}{\varepsilon} J(t), 0) \|_Y \right).
\]
The constants \(c > 0\) only depend on the constants from (1.3).

**Remark 4.3.** a) In the above result, we can drop the factor \(\frac{1}{\varepsilon}\) in the assumptions if \(\varepsilon\) also belongs to \(W^{2, \delta}(Q)\) since \(H^1(Q) \hookrightarrow L^6(Q)\).

b) In Theorem 4.2, for \(\sigma = 0\) and \(J = 0\) the inequality in \(X\) is actually an equality since then the operators \(A\) and \(\hat{B}\) are skew-adjoint in \(X\) by Lemma 4.3 of [13], and hence their Cayley transforms are unitary in \(X\). This can be viewed as a modified energy preservation of the scheme in the conservative case.

5. **Convergence of the ADI scheme.** For the semigroups from Proposition 2.3, \(\tau \in (0, 1]\) and \(j \in \mathbb{N}_0\), we define
\[
\Lambda_{j+1}(\tau) = \frac{1}{j! \tau^{j+1}} \int_0^T s^j e^{(\tau-s)M} ds \quad \text{and} \quad \Lambda_{j+1}^{\text{div}}(\tau) = \frac{1}{j! \tau^{j+1}} \int_0^T s^j e^{(\tau-s)M_{\text{div}}} ds,
\]
as well as \(\Lambda_0(\tau) = e^{\tau M}\) and \(\Lambda_{j+1}^{\text{div}}(\tau) = e^{\tau M_{\text{div}}}\). By Proposition 2.3, these operators are uniformly bounded on \(X\), respectively \(X_{\text{div}}\), and \(\Lambda_j(\tau)\) is the restriction of \(\Lambda_j(\tau)\) to \(X_{\text{div}}\). Standard semigroup theory shows that these operators leave invariant \(D(M^k)\), respectively \(D(M_{\text{div}}^k)\), and commute with \(M^k\), respectively \(M_{\text{div}}^k\). For
$j \geq k$ they actually map into $D(M^k)$, respectively $D(M^k_{\text{div}})$. One can further check that

$$
\Lambda_j(\tau) = \frac{1}{j!} I + \tau M \Lambda_{j+1}(\tau) \quad \text{and} \quad \Lambda^\text{div}_j(\tau) = \frac{1}{j!} I + \tau M_{\text{div}} \Lambda^\text{div}_{j+1}(\tau).
$$

(5.1)

Our first main result establishes the second order convergence of the ADI scheme in $Y^\ast$. According to (3.6) the number $\kappa_Y \geq 0$ only depends on the constants in (1.3), and we have $\kappa_Y = 0$ in the case of constant coefficients. We use the number $\tau_0 > 0$ from Proposition 3.6, which only depends on $\kappa_Y$.

**Theorem 5.1.** Let (1.3) hold, $T \geq 1$, $0 < \tau \leq \min\{1, \tau_0\}$, $w_0 = (E_0, H_0) \in D(M^2_{\text{div}})$ and $(J, 0)$ belong to $E := C([0, T], D(M^2_{\text{div}})) \cap W^{2, 1}([0, T], X_{\text{div}})$. Let $w = (E, H)$ be the solution of (1.1) and $w_n = S^J_{\tau, n} \cdots S^J_{\tau, 1} w_0$ be its approximation from (4.1). For all $n \tau \leq T$ and $(\varphi, \psi) \in Y$, we then have

$$
| (w_n - w(n\tau)) (\varphi, \psi) | \leq c \varepsilon^2 T^2 e^{\delta \kappa_Y T} \left( \|w_0\|_{D(M^2_{\text{div}})} + \|(J, 0)\|_E \right) \|(\varphi, \psi)\|_Y.
$$

The constant $c > 0$ only depends on the constants from (1.3).

Let $\sigma = 0$ or $\sigma \geq \sigma_0$ for a constant $\sigma_0 > 0$, in addition. Then we can omit the factor $(1 + T)$ in the above inequality, where $c$ also depends on $1/\sigma_0$ if $\sigma \geq \sigma_0$.

**Proof.** Proposition 2.3 yields a solution $w \in C([0, T], D(M^2_{\text{div}}))$ of (1.1). Recall from Proposition 3.1 that $D(M_{\text{div}}) \hookrightarrow D(A) \cap D(B)$. So the scheme (4.1) is well-defined. The properties of $A$, $B$ and $M$ contained in (3.5) and Propositions 2.2, 2.3, 3.1 and 3.6 are freely used below. We start from the Taylor expansion

$$
\left( \frac{1}{\varepsilon} J(n \tau + s), 0 \right) = \left( \frac{1}{\varepsilon} J(n \tau) + \frac{\varepsilon}{n} J'(n \tau) + \int_{n \tau}^{n \tau + s} (n \tau + s - r) \frac{1}{\varepsilon} J''(r) \, dr, 0 \right)
$$

in $X_{\text{div}}$ for $n \tau + s \in [0, T]$. This equation, Duhamel’s formula (2.6) and the above definitions yield the expression

$$
w((n + 1) \tau) = \Lambda_0(\tau) w(n \tau) + \tau \Lambda_1(\tau) \left( -\frac{1}{\varepsilon} J(n \tau), 0 \right) + \tau^2 \Lambda_2(\tau) \left( -\frac{1}{\varepsilon} J'(n \tau), 0 \right) + R_n(\tau)
$$

for the solution, where

$$
R_n(\tau) = \int_0^\tau e^{(\tau - s) M} \left( \int_{n \tau}^{n \tau + s} (n \tau + s - r) \left( -\frac{1}{\varepsilon} J''(r), 0 \right) \, dr \right) ds.
$$

We insert (5.2) for $s = \tau$ into the definition of $S^J_{\tau, n+1}$ from (4.1) with $\tilde{w} = S^J_{\tau, n} \cdots S^J_{\tau, 1} w_0 \in D(B)$ and obtain

$$
S^J_{\tau, n+1} \cdots S^J_{\tau, 1} w_0 = (I - \frac{\tau}{2} B)^{-1} (I + \frac{\tau}{2} A) \left( (I - \frac{\tau}{2} A)^{-1} (I + \frac{\tau}{2} B) S^J_{\tau, n} \cdots S^J_{\tau, 1} w_0 \right.
$$

$$
+ \tau \left( -\frac{1}{\varepsilon} J(n \tau), 0 \right) + \frac{\tau^2}{2} \left( -\frac{1}{\varepsilon} J'(n \tau), 0 \right) + r_n(\tau) \left. \right)
$$

with the remainder

$$
r_n(\tau) = \frac{\tau}{2} \int_{n \tau}^{(n + 1) \tau} ((n + 1) \tau - r) \left( -\frac{1}{\varepsilon} J''(r), 0 \right) \, dr.
$$

In the next step we take the inner product in $X$ of the fields $y = (\varphi, \psi) \in Y$ with the difference of (5.4) and (5.3). In the following we write $A^*_Y$ and $B^*_Y$ instead of $(A^*)_Y$ and $(B^*)_Y$, respectively. Putting several operators as adjoints on the side of $y$, we arrive at the formula

$$
(S^J_{\tau, n+1} \cdots S^J_{\tau, 1} w_0 - w((n + 1) \tau)) \, | \,(\varphi, \psi)\rangle_X
$$

$$
= (S^J_{\tau, n} \cdots S^J_{\tau, 1} w_0 - w(n \tau)) \, | \,(I + \frac{\tau}{2} B^*) (I + \frac{\tau}{2} A^*_Y) (I - \frac{\tau}{2} A^*_Y)^{-1} (I - \frac{\tau}{2} B^*_Y)^{-1} y\rangle_X
$$

(5.5)
where we used that $I + \frac{\tau}{2} A^*$ and $(I - \frac{\tau}{2} A^*)^{-1}$ commute on $Y \hookrightarrow D(A^*) \cap D(B^*)$. We abbreviate

$$
\chi(\tau) = (I - \frac{\tau}{2} A^*)^{-1}(I - \frac{\tau}{2} B^*)^{-1}(\varphi, \psi) \in D(A^*_\gamma).
$$

Since $M^* = A^* + B^*$ on $Y$, we obtain

$$
\Sigma_1(\tau) = \left( w(\eta n) \left[ [I - \Lambda_0(\tau)^*] + \frac{\tau}{2} (I + \Lambda_0(\tau)^*) M^* + \frac{\tau^2}{4} (I - \Lambda_0(\tau)^*) B^* A^* \right] \chi(\tau) \right)_X.
$$

By means of (5.1) we expand

$$
I - \Lambda_0(\tau)^* = -\tau M^* - \frac{1}{2} \tau^2 (M^*)^2 - \tau^3 \Lambda_3(\tau)^* (M^*)^3 \quad \text{on } D((M^*)^3),
$$

$$
I + \Lambda_0(\tau)^* = 2I + \tau M^* + \tau^2 \Lambda_2(\tau)^* (M^*)^2 \quad \text{on } D((M^*)^2),
$$

$$
I - \Lambda_0(\tau)^* = -\tau \Lambda_1(\tau)^* M^* \quad \text{on } D(M^*).
$$

Because of $Y \hookrightarrow D(B^*)$, $w(\eta n) \in D(M_{\text{div}}^2) \hookrightarrow D(M^2)$ and the crucial embedding $D(M_{\text{div}}) \hookrightarrow D(A) \cap D(B)$ from Proposition 3.1, the above expansions yield

$$
\Sigma_1(\tau) = \left( w(\eta n) \left( [I - \Lambda_3(\tau)]_{-2} M_{-2} M_{-1} M^* + \frac{\tau^3}{4} \Lambda_2(\tau)_{-2} M_{-2} M_{-1} M^*

- \frac{\tau^3}{4} \Lambda_1(\tau)_{-1} M_{-1} B^* A^*_\gamma \right] \chi(\tau) \right)_{D(M^2) \times X_{M^*_2}}
$$

$$
= \tau^3 \left( [I - M^* \Lambda_3(\tau)] + \frac{1}{2} M^2 \Lambda_2(\tau) w(\eta n) \right) M^* \chi(\tau) \right)_X
$$

where we employ the extrapolation space $X_{M^*_2} \cong D(M^2)^*$, see [2] or [10]. We rewrite the term $\Sigma_2(\tau)$ as

$$
\tau \left( \left( -\frac{1}{2} J(\eta n), 0 \right) \left[ [I + \frac{\tau}{2} A^*] (I - \frac{\tau}{2} A^*_\gamma) - \Lambda_1(\tau)^* (I - \frac{\tau}{2} B^*) (I - \frac{\tau}{2} A^*_\gamma) \right] \chi(\tau) \right)_X
$$

$$
= \tau \left( \left( -\frac{1}{2} J(\eta n), 0 \right) \left[ [I + \frac{\tau}{2} A^* A^*_\gamma - \Lambda_2(\tau)^* M^* + \frac{\tau}{2} M^*

+ \frac{\tau}{4} \Lambda_2(\tau)_{-1} M_{-1} M^* - \frac{\tau^2}{4} \Lambda_1(\tau)^* B^* A^*_\gamma \right] \chi(\tau) \right)_{D(M) \times X_{M^*_1}}
$$

$$
= \tau \left( \left( -\frac{1}{2} J(\eta n), 0 \right) \left[ [I + \frac{\tau}{2} A^* A^*_\gamma - \tau^2 \Lambda_3(\tau)_{-1} M_{-1} M^* + \frac{\tau^2}{4} \Lambda_2(\tau)_{-1} M_{-1} M^* \right. \right.
$$

$$
\left. + \frac{\tau}{2} \Lambda_2(\tau)_{-1} M_{-1} M^* - \frac{\tau^2}{4} \Lambda_1(\tau)^* B^* A^*_\gamma \right] \chi(\tau) \right)_{D(M) \times X_{M^*_1}}
$$
\[-\frac{\tau^2}{2} \Lambda_1(\tau)^* B^* A_Y^* \chi(\tau) \bigg|_{D(M) \times X} = \tau^3 \left( \frac{1}{2} A \left( \frac{1}{2} J(n \tau), 0 \right) \bigg| A^* \chi(\tau) \right)_X + \tau^3 \left( M A_3(\tau) \left( \frac{1}{2} J(n \tau), 0 \right) \bigg| M^* \chi(\tau) \right)_X
\]

Similarly, (5.1) implies
\[
\Sigma_3(\tau) = \tau^2 \left( -\frac{1}{2} J'(n \tau), 0 \right) \left[ \frac{1}{2} I + \frac{1}{2} A^* - (\frac{1}{2} I + \tau A_3(\tau)^* M^*) + \frac{1}{2} \Lambda_2(\tau)^* B^* \right] \cdot (I - \frac{1}{2} B^*)^{-1} y)_X
\]

We recursively insert the above expressions in (5.5) and obtain (omitting the subscript \( Y \) several times)
\[
\left( S_{\tau, n} \cdots S_{\tau, 1} w(0) - w(n \tau) \right) \left( \varphi, \psi \right)_X
\]

(5.6)
The terms \( r_k(\tau) \) and \( R_k(\tau) \) are bounded in \( X \) by \( cT^2 \int_{k\tau}^{(k+1)\tau} \| (J''(s), 0) \|_X \) ds. Propositions 2.2, 2.3, 3.1 and 3.6 then imply

\[
\left| \left( S'_{r,n} \cdots S'_{r,1} \right) u_0 - u(n\tau) \right| ||(\varphi, \psi)\|_{L^2}
\leq cT^3 e^{6\kappa_Y n\tau} \sum_{k=0}^{n-1} \left( \| w(k\tau) \|_{D(M^2_{div})} + \| (J(k\tau), 0) \|_{D(M_{div})} + \| (J'(k\tau), 0) \|_X \right)
+ \frac{1}{\tau} \int_{k\tau}^{(k+1)\tau} \| (J''(s), 0) \|_X \| (\varphi, \psi) \|_{H^1}
\leq cT^2 T^2 e^{6\kappa_Y T} \left( \| u_0 \|_{D(M^2_{div})} + \| (J, 0) \|_E \right) \| (\varphi, \psi) \|_{H^1},
\]

where we use \( \tau \leq 1 \) and \( c \) only depends on the constants from (1.3). These arguments also yield the addendum.

\[
6. \text{Almost preservation of the divergence conditions in } H^{-1}. \text{ The solution } (E, H) \text{ of (1.1) fulfills the Gaussian laws (2.7) and } \text{div}(\mu H(t)) = 0. \text{ We now show that the scheme (4.1) satisfies a discrete analogue of these divergence conditions up to an error of order } \tau \text{ in } H^{-1}(Q). \text{ We recall that the numbers } \kappa_Y \geq 0 \text{ and } \tau_0 > 0 \text{ from (3.6) and Proposition 3.6 only depend on the constants in (1.3), and that } \kappa_Y = 0 \text{ if the coefficients are constant.}
\]

**Theorem 6.1.** Let (3.3) hold, \( T > 0, \tau \in (0, \min\{1, \tau_0\}) \), \( n \in \mathbb{N}_0 \), and \( n\tau \leq T \). Take \( u_0 = (E_0, H_0) \) in \( D(B_Y) \) and \( (\frac{1}{2}J, 0) \) in \( C([0, T], D(A_Y)) \cap C^1([0, T], X) \). Let \( w_n = (E_n, H_n) \) be given by (4.1). We then have

\[
\left\| \left( \text{div}(\varepsilon E_n), \text{div}(\mu H_n) \right) - \left( \text{div}(\varepsilon E_0), 0 \right) \right\|
\leq cT e^{6\kappa_Y T} \left[ \left\| u_0 \right\|_{H^1} + \tau \left\| B_Y u_0 \right\|_{H^1} + T \max_{t \in [0, T]} \left\| (J(t), 0) \right\|_{H^1} + \tau \left\| A_Y \left( \frac{1}{2} J(t), 0 \right) \right\|_{H^1} \right]
\]

\[
+ cT \int_0^T \left\| (J'(s), 0) \right\|_{L^2} \text{ ds}
\]

for a constant \( c \geq 0 \) only depending on the constants in (1.3).

In the proof, see (6.4), we will see that the integral on the left-hand side of the above inequality can be replaced by the sum

\[
\sum_{k=0}^{n-1} \frac{\tau}{2} \left( \text{div}(J(t_k) + J(t_{k+1})) \right), 0).
\]

Moreover, as in Remark 4.3 one can drop the factor \( \frac{1}{2} \) in the assumption if \( \varepsilon \) also belongs to \( W^{2,3}(Q) \).

**Proof.** We first derive a recursion formula for the divergence of \( w_n \) which is then estimated by means of Propositions 3.1 and 3.6. Remark 4.1 says that \( w_n \) belongs to \( D(B_Y) \) and \( w_{n+1/2} \) to \( D(A_Y) \). We often use this regularity, these propositions and that \( \tau \leq 1 \) without further notice. Take \( k \in \mathbb{N}_0 \) with \( k + 1 \leq T/\tau \).

1) As before (4.3), the definition (4.1) yields

\[
\left( \frac{1}{2} E_{k+1/2} \right) - \frac{\tau}{2} \left( \frac{1}{2} C_2 H_{k+1/2} \right) = \left( 1 - \frac{\tau}{2} \right) \left( \frac{1}{2} C_2 \right) \left( E_k \right) - \frac{\tau}{2} \left( \frac{1}{2} C_2 H_k \right)
\]

\[
+ \frac{\tau}{2} \left( \frac{1}{2} C_2 H_{k+1/2} \right) \]
in $Y$. To obtain a convenient recursion formula, we insert the second line into the first one and the first line into the second one. It follows

$$
\begin{align*}
(1 + \frac{\tau^2}{4\tau^\varepsilon})H_{k+1/2}^{1/2} & = \frac{\tau}{2} \left( \frac{1}{\varepsilon} C_1 \left[ \frac{\tau}{2\mu} C_2 E_{k+1/2} + H_k - \frac{\tau}{2\mu} C_1 E_k \right] \\
+ \frac{\tau}{2} \left( \frac{1}{\mu} C_2 \left[ \frac{\tau}{2\varepsilon} C_1 H_{k+1/2} + (1 - \frac{\tau^2}{4\tau}) E_k - \frac{\tau}{2\varepsilon} C_2 H_k \right] \right) \\
- \frac{\tau}{2} \left( \frac{1}{\mu} C_2 \sigma\tau \epsilon E_{k+1/2} \right) \right) - \frac{\tau}{2} \left( \frac{1}{\mu} C_1 E_k \right)
\end{align*}
$$

in $L^2(\Omega)$\textsuperscript{6}. Using $\text{curl} = C_1 - C_2$ and the definition (4.2), we reorder the above identity and infer the first half of the recursion step

$$
\begin{align*}
\left( \frac{\varepsilon E_{k+\frac{1}{2}}}{2} - \frac{\tau^2}{4} D^2_{\mu}(k+\frac{1}{2}) \right) & = \left( \frac{\varepsilon E_k}{2} - \frac{\tau^2}{4} C_1 \frac{1}{\mu} C_1 E_k \right) - \frac{\tau}{2} \left( \frac{1}{\mu} C_2 \sigma\tau \epsilon E_{k+\frac{1}{2}} + E_k \right) \\
& - \frac{\tau}{2} \left( \frac{1}{\mu} C_1 H_{k+\frac{1}{2}} + E_k \right) + \frac{\tau}{2} \left( \text{curl} E_k \right)
\end{align*}
$$

in $L^2(\Omega)$\textsuperscript{6}. Similarly, (4.1) leads to the expression

$$
\begin{align*}
\left( (1 + \frac{\tau^2}{4\tau^\varepsilon})E_{k+1} \right) + \frac{\tau}{2} \left( \frac{1}{\mu} C_2 H_{k+1} \right) \\
= \frac{\tau}{2} \left( \frac{1}{\mu} C_1 E_{k+1} \right)
\end{align*}
$$

in $Y$. Proceeding as in the first half step, we conclude

$$
\begin{align*}
(1 + \frac{\tau^2}{4\tau^\varepsilon})E_{k+1} & = -\frac{\tau}{2} \left( \frac{1}{\varepsilon} C_2 \left[ \frac{\tau}{2\mu} C_1 E_{k+1} + H_{k+1/2} + \frac{\tau}{2\mu} C_2 E_{k+1/2} \right] \\
& + \left( \frac{\tau}{2\mu} C_1 \sigma\tau \epsilon E_{k+1} \right) + \frac{\tau}{2} \left( \frac{1}{\varepsilon} C_2 \left[ \frac{\tau}{2\mu} C_1 H_{k+1} + (1 - \frac{\tau^2}{4\tau}) E_k + \frac{\tau}{2\varepsilon} C_2 H_k \right] \right) \\
& + \left( \frac{\tau}{2} \frac{1}{\mu} C_2 E_{k+1/2} \right) \right) \\
& - \frac{\tau}{2} \left( \frac{1}{\mu} C_1 H_{k+1/2} + E_k \right) + \frac{\tau}{2} \left( \frac{1}{\mu} C_2 E_{k+1/2} \right) \\
& - \frac{\tau}{2} \left( \frac{1}{\mu} C_1 H_{k+1/2} + E_k \right) + \frac{\tau}{2} \left( \frac{1}{\mu} C_2 E_{k+1/2} \right)
\end{align*}
$$

in $L^2(\Omega)$\textsuperscript{6}. Again with (4.2) and $\text{curl} = C_1 - C_2$, this equation implies the second step of the recursion

$$
\begin{align*}
\left( \frac{\varepsilon E_{k+1}}{2} - \frac{\tau^2}{4} D^2_{\mu}(k+1) \right) & = \frac{\tau}{2} \left( \frac{1}{\mu} C_2 \sigma\tau \epsilon E_{k+1} \right) \\
+ \frac{\tau}{2} \left( \frac{1}{\mu} C_1 H_{k+1/2} + E_k \right) + \frac{\tau}{2} \left( \frac{1}{\mu} C_2 E_{k+1/2} \right) \\
+ \frac{\tau}{2} \left( \frac{1}{\mu} C_2 \sigma\tau \epsilon E_{k+1} \right) + \frac{\tau}{2} \left( \frac{1}{\mu} C_2 \sigma\tau \epsilon E_{k+1} \right)
\end{align*}
$$

in $L^2(\Omega)$\textsuperscript{6}. Again with (4.2) and $\text{curl} = C_1 - C_2$, this equation implies the second step of the recursion

$$
\begin{align*}
\left( \frac{\varepsilon E_{k+1}}{2} - \frac{\tau^2}{4} D^2_{\mu}(k+1) \right) & = \frac{\tau}{2} \left( \frac{1}{\mu} C_2 \sigma\tau \epsilon E_{k+1} \right) \\
+ \frac{\tau}{2} \left( \frac{1}{\mu} C_1 H_{k+1/2} + E_k \right) + \frac{\tau}{2} \left( \frac{1}{\mu} C_2 E_{k+1/2} \right)
\end{align*}
$$
in $L^2(Q)^6$. For $\varphi \in H^1(Q)^3$ we have $0 = \text{div} \, \text{curl} \, \varphi = \text{div} \, C_1 \varphi - \text{div} \, C_2 \varphi$ in $H^{-1}(Q)$. Let $\lambda \in (\varepsilon, \mu)$. Since $D^{(1)}_\lambda = C_1 \frac{1}{\lambda} C_2 \frac{1}{\lambda}$ by (4.2), it follows $\text{div} \, C_2 \frac{1}{\lambda} C_2 v = \text{div} \, D^{(1)}_\lambda v$ in $H^{-1}(Q)$ for $v \in H^1(Q)^3$ with $C_2 v \in H^1(Q)^3$, and similarly $\text{div} \, C_1 \frac{1}{\lambda} C_1 u = \text{div} \, D^{(2)}_\lambda u$ in $H^{-1}(Q)$ for $u \in H^1(Q)^3$ with $C_1 u \in H^1(Q)^3$. From (6.2) and (6.1) we thus deduce the recursion formula

\[
\begin{align*}
&\left( \text{div}[\varepsilon \mathbf{E}_{k+1} - \frac{\tau}{4} D^{(1)}_\mu \mathbf{E}_{k+1}] \right) \\
&- \left( \text{div}[\mu \mathbf{H}_{k+1} - \frac{\tau}{4} D^{(1)}_\lambda \mathbf{H}_{k+1}] \right) \\
&= \left( \text{div}[\varepsilon \mathbf{E}_k - \frac{\tau}{4} D^{(2)}_\mu \mathbf{E}_k] \right) - \left( \text{div}[\mu \mathbf{H}_k - \frac{\tau}{4} D^{(1)}_\lambda \mathbf{H}_k] \right) \\
&+ \frac{\tau}{2} \left( \text{div}[C_1 \frac{\tau}{4} \mathbf{E}_{k+\frac{1}{2}} + \mathbf{E}_{k+1}] \right) + \frac{\tau}{2} \left( - \text{div}[C_2 \frac{\tau}{4} \mathbf{J}(t_k) + \mathbf{J}(t_{k+1})] \right) \\
&- \frac{\tau}{2} \left( \text{div}[1 - \frac{\tau}{4} \mathbf{E}(t_k) + \mathbf{E}(t_{k+1})] \right)
\end{align*}
\]

in $H^{-1}(Q)^6$. For $n \leq T/\tau$ an easy induction then yields

\[
\begin{align*}
&\left( \text{div}[\varepsilon \mathbf{E}_n - \frac{\tau}{4} D^{(2)}_\mu \mathbf{E}_n] \right) \\
&- \left( \text{div}[\mu \mathbf{H}_n - \frac{\tau}{4} D^{(1)}_\lambda \mathbf{H}_n] \right) \\
&= \left( \text{div}[\varepsilon \mathbf{E}_0 - \frac{\tau}{4} D^{(2)}_\mu \mathbf{E}_0] \right) - \sum_{k=0}^{n-1} \left( \text{div}[\frac{\tau}{4} \mathbf{E}_{k+1} + \frac{\tau}{4} \mathbf{E}_{k+\frac{1}{2}} + \frac{\tau}{4} \mathbf{E}_k] \right) \\
&- \sum_{k=0}^{n-1} \left[ \frac{\tau}{2} \left( - \text{div}[\mathbf{J}(t_k) + \mathbf{J}(t_{k+1})] \right) + \frac{\tau^2}{8} \left( \text{div}[C_1 \frac{\tau}{4} \mathbf{E}(t_k) - \mathbf{E}_k] \right) \right] \\
&+ \frac{\tau^3}{16} \left( \text{div}[D^{(1)}_\mu \frac{\tau}{4} \mathbf{J}(t_k) + \mathbf{J}(t_{k+1})] \right) + \frac{\tau^2}{8} \left( \text{div}[\frac{\tau}{4} \mathbf{J}(t_k) + \mathbf{J}(t_{k+1})] \right)
\end{align*}
\]

in $H^{-1}(Q)^6$. We reorder these terms and use $\text{div}(\mu \mathbf{H}_0) = 0$ to derive the crucial formula for the charges of the ADI scheme

\[
\begin{align*}
&\left( \text{div}[\varepsilon \mathbf{E}_n] \right) - \left( \text{div}[\varepsilon \mathbf{E}_0] \right) \\
&+ \frac{\tau}{2} \sum_{k=0}^{n-1} \left( \text{div}[\frac{\tau}{4} \mathbf{E}_{k+1} + \frac{\tau}{4} \mathbf{E}_{k+\frac{1}{2}} + \frac{\tau}{4} \mathbf{E}_k] \right) + \left( \text{div}[\mathbf{J}(t_k) + \mathbf{J}(t_{k+1})] \right)
\end{align*}
\]
Similarly, it follows
\[
\tau^2 \left( \frac{\text{div}(D_{\mu}^{(2)}E_n)}{\text{div}(D_{\xi}^{(1)}H_n)} \right) - \frac{\tau^2}{4} \left( \frac{\text{div}(D_{\mu}^{(2)}E_0)}{\text{div}(D_{\xi}^{(1)}H_0)} \right) + \frac{\tau^2}{8} \left( \text{div}[C_1 \frac{2}{\varepsilon}(E_n - E_0)] \right)
\]
\[
+ \sum_{k=0}^{n-1} \left[ \frac{\tau^3}{16} \left( \text{div}[D_{\mu}^{(1)}(\varepsilon J(t_k) + J(t_{k+1}))] \right) - \text{div}[C_1 \frac{2}{\varepsilon}(J(t_k) + J(t_{k+1}))] \right] + \frac{\tau^2}{8} \left( \text{div}[\varepsilon J(t_k) + J(t_{k+1})] \right)
\]
in $H^{-1}(Q)^6$.

2) The term for the current density $J$ on the left-hand side of (6.3) can be exchanged by the corresponding integral with the asserted error order since
\[
\left\| \sum_{k=0}^{n-1} \frac{\tau}{2} \text{div} [J(t_k) + J(t_{k+1})] - \int_0^{n\tau} \text{div} J(s) \, ds \right\|_{H^{-1}} \leq \left\| \sum_{k=0}^{n-1} \left[ \int_{t_k}^{t_k+1} (J(t_k) - J(s)) \, ds + \int_{t_k+1}^{t_{k+1}} (J(t_{k+1}) - J(s)) \, ds \right] \right\|_{L^2} \leq \sum_{k=0}^{n-1} \tau \int_{t_k}^{t_{k+1}} \|J'(r)\|_{L^2} \, dr = \tau \int_0^{T} \|J'(r)\|_{L^2} \, dr.
\]

To treat the first and main summand on the right-hand side of (6.3), we insert the closed expression (4.5) for the ADI scheme obtaining
\[
\begin{align*}
(D_{\mu}^{(2)}E_n) & = \left( \begin{array}{cc} 0 & C_2 \\ C_1 & 0 \end{array} \right) B_0 S_{r,n} \cdots S_{r,1} w_0 = K B_0^2 S_{r,n}^2 \cdots S_{r,1}^2 w_0 \\
& = \frac{2}{\tau} K (B + S) \frac{\varepsilon}{2} (B_Y + S)(I - \frac{\tau}{2} B_Y)^{-1} \left[ \gamma_S (A_Y) \gamma_S (B_Y) \gamma_S (A_Y) \right]^{n-1} (I + \frac{\tau}{2} B_Y) w_0 \\
& - \sum_{k=0}^{n-1} \left[ \gamma_S (A_Y) \gamma_S (B_Y) \right]^{k} \left( I + \frac{\tau}{2} A_Y \right) \frac{\tau}{2} \left( J(t_{n-k-1}) + J(t_{n-k}) \right)
\end{align*}
\]
in $L^2(Q)^6$, where we have put
\[
K = \left( \begin{array}{cc} \varepsilon I & 0 \\ 0 & \mu I \end{array} \right) \quad \text{and} \quad S = B_0 - B = \left( \begin{array}{cc} \frac{\tau}{2} I & 0 \\ 0 & 0 \end{array} \right).
\]

Proposition 3.6 then implies
\[
\frac{\tau^2}{4} \left\| \frac{\text{div} D_{\mu}^{(2)}E_n}{\text{div} D_{\xi}^{(1)}H_n} \right\|_{H^{-1}} \leq c r e^{6\kappa_\gamma n r} \left[ \|w_0\|_{H^1} + r \|B_Y w_0\|_{H^1} \right] + n r \max_{t \in [0,T]} \left( \|J(t), 0\|_{H^1} + r \|A_Y(\frac{\tau}{2} J(t), 0)\|_{H^1} \right).
\]

Similarly, it follows
\[
\frac{\tau^2}{4} \left\| \frac{\text{div} D_{\mu}^{(2)}E_0}{\text{div} D_{\xi}^{(1)}H_0} \right\|_{H^{-1}} \leq c r^2 \|B_0 w_0\|_{H^1} \leq c r (\|w_0\|_{H^1} + r \|B_Y w_0\|_{H^1}).
\]

In the same way, the remaining term of third order in (6.3) is bounded by
\[
\frac{\tau^3}{16} \left\| \sum_{k=0}^{n-1} \text{div} D_{\mu}^{(1)}(\varepsilon J(t_k) + J(t_{k+1})) \right\|_{H^{-1}}
\]
\[ \leq c\tau^2 n \max_{t \in [0,T]} \left( \| (\text{J}(t), 0) \|_{H^1} + \tau \| A_Y \left( \frac{1}{\tau} \text{J}(t), 0 \right) \|_{H^1} \right). \]

The other terms in (6.3) can be estimated analogously. The assertion now follows from formula (6.3), the inequality \( \tau n \leq T \), and the above estimates. \( \Box \)

REFERENCES

[1] R. A. Adams and J. F. Fournier, *Sobolev Spaces*, 2nd edition, Elsevier, Amsterdam, 2003.
[2] H. Amann, *Linear and Quasilinear Parabolic Problems. Volume I: Abstract Linear Theory*, Birkhäuser, Basel, 1995.
[3] C. Amrouche, C. Bernardi, M. Dauge and V. Girault, Vector potentials in three-dimensional non-smooth domains, *Math. Methods Appl. Sci.*, 21 (1998), 823–864.
[4] W. Chen, X. Li and D. Liang, Energy-conserved splitting FDTD methods for Maxwell’s equations, *Numer. Math.*, 108 (2008), 445–485.
[5] W. Chen, X. Li and D. Liang, Energy-conserved splitting finite-difference time-domain methods for Maxwell’s equations in three dimensions, *SIAM J. Numer. Anal.*, 48 (2010), 1530–1554.
[6] R. Dautray and J.-L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology*, Volume 3: Spectral Theory and Applications, Springer, Berlin, 1990.
[7] R. Dautray and J.-L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology*, Volume 5: Evolution Problems I, Springer, Berlin, 1992.
[8] J. Eilinghoff, Error estimates of splitting methods for wave type equations, Ph.D. thesis, Karlsruhe, 2017, see https://publikationen.bibliothek.kit.edu/1000075070.
[9] J. Eilinghoff and R. Schnaubelt, Error estimates in \( L^2 \) of an ADI splitting scheme for the inhomogeneous Maxwell, preprint, see http://www.math.kit.edu/iana3/$\sim$schnaubelt/media/adi-strong.pdf.
[10] K.-J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Springer, New York, 2000.
[11] L. Gao, B. Zhang and D. Liang, The splitting finite-difference time-domain methods for Maxwell’s equations in two dimensions, *J. Comput. Appl. Math.*, 205 (2007), 207–230.
[12] E. Hansen and A. Ostermann, Dimension splitting for evolution equations, *Numer. Math.*, 108 (2008), 557–570.
[13] M. Hochbruck, T. Jahnke and R. Schnaubelt, Convergence of an ADI splitting for Maxwell’s equations, *Numer. Math.*, 129 (2015), 535–561.
[14] M. Hochbruck and A. Sturm, Error analysis of a second-order locally implicit method for linear Maxwell’s equations, *SIAM J. Numer. Anal.*, 54 (2016), 3167–3191.
[15] M. Hochbruck and A. Sturm, Upwind discontinuous Galerkin space discretization and locally implicit time integration for linear Maxwell’s equations, preprint 2017/12 of CRC 1172, see http://www.waves.kit.edu/downloads/CRC1173_Preprint_2017-12.pdf.
[16] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin, 1995.
[17] P. C. Kunstmann and L. Weis, Maximal \( L^p \)-regularity for parabolic equations, Fourier multiplier theorems and \( H^\infty \)-functional calculus, In *Functional Analytic Methods for Evolution Equations* (eds. M. Iannelli, R. Nagel and S. Piazzera), Springer-Verlag, 1855 (2004), 65–311.
[18] J. Lee and B. Fornberg, A split step approach for the 3-D Maxwell’s equations, *J. Comput. Appl. Math.*, 158 (2003), 485–505.
[19] A. Lunardi, *Interpolation Theory*, Edizione della Normale, Pisa, 2009.
[20] T. Namiki, 3-D ADI-FDTD method-unconditionally stable time-domain algorithm for solving full vector Maxwell’s equations, *IEEE Trans. Microwave Theory Tech.*, 48 (2000), 1743–1748.
[21] J. Nečas, *Direct Methods in the Theory of Elliptic Equations*, Springer-Verlag, Heidelberg, 2012.
[22] A. Ostermann and K. Schratz, Error analysis of splitting methods for inhomogeneous evolution equations, *Appl. Numer. Math.*, 62 (2012), 1436–1446.
[23] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.
[24] A. Taflove and S. C. Hagness, *Computational Electrodynamics: The Finite-Difference Time-Domain Method*, Second edition. With 1 CD-ROM (Windows). Artech House, Inc., Boston, MA, 2000.
[25] K. S. Yee, Numerical solution of initial boundary value problems involving Maxwell’s equations in isotropic media, *IEEE Trans. Antennas Propagation*, 14 (1966), 302–307.

[26] F. Zheng, Z. Chen and J. Zhang, Toward the development of a three-dimensional unconditionally stable finite-difference time-domain method, *IEEE Trans. Microwave Theory Tech.*, 48 (2000), 1550–1558.

Received December 2017; revised June 2018.

E-mail address: johannes.eilinghoff@kit.edu

E-mail address: schnaubelt@kit.edu