The moduli space of points in quaternionic projective space

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Abstract Let $M(n, m; \mathbb{F}P^n)$ be the configuration space of $m$-tuples of pairwise distinct points in $\mathbb{F}P^n$, that is, the quotient of the set of $m$-tuples of pairwise distinct points in $\mathbb{F}P^n$ with respect to the diagonal action of $PU(1, n; \mathbb{F})$ equipped with the quotient topology. It is an important problem in hyperbolic geometry to parameterize $M(n, m; \mathbb{F}P^n)$ and study the geometric and topological structures on the associated parameter space. In this paper, by mainly using the rotation-normalized and block-normalized algorithms, we construct the parameter spaces of both $M(n, m; \partial H^n_{\mathbb{F}})$ and $M(n, m; \mathbb{P}(V_+))$, respectively.

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1 Introduction

Let $\mathbb{F}= \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ be respectively the real numbers, the complex numbers or the quaternions, and $\langle z, w \rangle = w^* J z$ a Hermitian product in $(n + 1)$-dimensional $\mathbb{F}$-vector space $\mathbb{F}^{n,1}$ of signature $(n, 1)$, where $z = (z_1, \cdots, z_{n+1})^T$, $w = (w_1, \cdots, w_{n+1})^T$ and $^*$ denotes the conjugate transpose. The group of transformations of $\mathbb{F}^{n,1}$ that preserve this Hermitian product is the noncompact Lie group $U(1, n; \mathbb{F})$, that is,

$$U(1, n; \mathbb{F}) = \{ g \in GL(n+1, \mathbb{F}) : g^* J g = J \}.$$  

These groups are traditionally denoted by $O(n, 1) = U(1, n; \mathbb{R})$, $U(n, 1) = U(1, n; \mathbb{C})$ and $Sp(n, 1) = U(1, n; \mathbb{H})$. Denote by $\mathbb{P}$ the natural right projection from $\mathbb{F}^{n,1} - \{0\}$ to projective space $\mathbb{F}P^n$. Let $V_-, V_0, V_+$ be the subsets of $\mathbb{F}^{n,1} - \{0\}$ consisting of vectors where $\langle z, z \rangle$ is negative, zero, or positive, respectively. Their projections to $\mathbb{F}P^n$ are called isotropic, negative, and positive points, respectively. Conventionally, we denote $H^n_\mathbb{F} = \mathbb{P}(V_-), \partial H^n_\mathbb{F} = \mathbb{P}(V_0)$ and $\overline{H^n_\mathbb{F}} = H^n_\mathbb{F} \cup \partial H^n_\mathbb{F}$. The Bergman metric on $H^n_\mathbb{F}$ is given by the distance formula

$$\cosh^2 \frac{\rho(z, w)}{2} = \frac{\langle z, w \rangle \langle w, z \rangle}{\langle z, z \rangle \langle w, w \rangle}, \text{ where } z \in \mathbb{P}^{-1}(z), w \in \mathbb{P}^{-1}(w). \tag{1}$$

The center $Z(1, n; \mathbb{F})$ in $U(1, n; \mathbb{F})$ is $\{ \pm I_{n+1} \}$ if $\mathbb{F} = \mathbb{R}, \mathbb{H}$, and is the circle group $\{ e^{i \theta} I_{n+1} \}$ if $\mathbb{F} = \mathbb{C}$. We mention that $g \in U(1, n; \mathbb{F})$ acts on $\mathbb{F}P^n$ as $g(z) = \mathbb{P} g \mathbb{P}^{-1}(z)$. Therefore the holomorphic isometry group $\text{Isom}(H^n_\mathbb{F})$ of $H^n_\mathbb{F}$ is actually the quotient $PU(1, n; \mathbb{F}) = U(1, n; \mathbb{F})/Z(1, n; \mathbb{F})$. We refer to [11, 15, 18, 23] for further details.

Let $M(n, m; \mathbb{F}P^n)$ be the configuration space of $m$-tuples of pairwise distinct points in $\mathbb{F}P^n$, or equivalently, the quotient of the set of $m$-tuples of pairwise distinct points in $\mathbb{F}P^n$ with respect to the diagonal
action of $\text{PU}(1, n; \mathbb{F})$ equipped with the quotient topology. It is an important problem in hyperbolic geometry to parameterize the space $\mathcal{M}(n, m; \mathbb{F}P^n)$ and study the geometric and topological structures on the associated parameter space. We refer to such a problem the moduli problem on $\mathbb{F}P^n$.

The moduli problems of the cases $m = 1, 2$ on $\partial H^2_{\mathbb{C}}$ are trivial because $\text{U}(1, n; \mathbb{F})$ acts doubly transitively on $\partial H^2_{\mathbb{C}}$ when $\mathbb{F} = \mathbb{C}$ or $\mathbb{H}$. It is well-known that $O(n, 1)$ acts triply transitively on the boundary. To handle the cases of $m \geq 3$, one need to develop some geometric invariants or geometric tools, such as distance formula, Cartan’s angular invariant [10, 18], and cross-ratio [24] etc.

The moduli problem of $\mathcal{M}(2, 4; \partial H^2_{\mathbb{C}})$ was considered by Falbel, Parker and Platis [15, 16, 25, 26]. The main tool is the complex cross-ratio variety determined by three complex cross-ratios.

The moduli problem of $\mathcal{M}(n, m; H^2_{\mathbb{C}})$ was solved by Brehm and Et-Taoui [3, 4]. Using Bruhat decomposition, Hakim and Sandler [20] could construct many important geometric invariants in complex hyperbolic geometry. This tool helped them to arrange $n$ points in certain standard position on $\mathbb{R}P^{n-1}$ [21], and as well, to deal with the moduli problem on $\mathbb{H}^n_{\mathbb{C}}$ [22].

We need to introduce the concept of Gram matrices of $m$-tuples in $\mathbb{F}P^n$ for further discussion.

**Definition 1.1.** *Given an $m$-tuple $p = (p_1, \cdots, p_m)$ of pairwise distinct points in $\mathbb{F}P^n$ with lift $p = (p_1, \cdots, p_m)$. The following Hermitian matrix*

$$G(p) = (g_{ij}) = (p_i^* J p_j) = (\langle p_j, p_i \rangle)$$

*is called the Gram matrix associated to $p$.***

For the sake of simplicity, by a little abuse of notation, we also say that $p$ is an $m$-tuple of pairwise distinct points in $\mathbb{F}P^n$ and regard $p$ as an element in $\mathbb{F}_{n+1,m}$, the set of $(n + 1) \times m$ matrices over $\mathbb{F}$. The action of $f \in \text{U}(1, n; \mathbb{F})$ on $\mathbb{F}_{n+1,m}$ is the usual matrix multiplication, that is,

$$fp = (fp_1, \cdots, fp_m).$$

Noting that $f^* J f = J$, we have the following proposition.

**Proposition 1.1.**

$$G(p) = p^* J p = p^* f^* J f p = G(fp), \quad \forall f \in \text{U}(1, n; \mathbb{F}). \quad (2)$$

Given two $m$-tuples $p = (p_1, \cdots, p_m)$ and $q = (q_1, \cdots, q_m)$ in $\mathbb{F}P^n$ with arbitrary lifts $p = (p_1, \cdots, p_m)$ and $q = (q_1, \cdots, q_m)$. We say that $p$ and $q$ are $\text{PU}(1, n; \mathbb{F})$-congruent if there exists an $f \in \text{U}(1, n; \mathbb{F})$ such that

$$f(p_i) = q_i, \lambda_i \neq 0, i = 1, \cdots, m,$$

in language of matrix algebra, that is,

$$fp = qD, \quad D = \text{diag} (\lambda_1, \cdots, \lambda_m), \lambda_i \in \mathbb{F} - \{0\}.$$ 

Therefore

$$G(p) = p^* J p = p^* f^* J f p = D^* q^* J q D = D^* G(q) D. \quad (3)$$

Observe that an arbitrary lift of $p$ can be represented by $(p_1 \lambda_1, \cdots, p_m \lambda_m) = pD$ and

$$G(pd) = D^* p^* J p D = D^* G(p) D. \quad (4)$$

The formulae (3) and (4) imply that Gram matrices contain the information of the diagonal action of $\text{U}(1, n; \mathbb{F})$ on $p$. Moreover, a Gram matrix contains the entries $\langle p_i, p_j \rangle$, which are base material to construct the corresponding Hermitian geometric invariants. Hence Gram matrix is the priority tool in handling the moduli problem.
The moduli problem on $\partial \mathbb{H}^n_3$ was solved by Cunha and Gusevskii \cite{12,13} mainly by Gram matrix. The key idea is that one need find a suitable matrix $D$ in \cite{1} to construct corresponding normalized Gram matrix and then seek a bijection between the independent entries of normalized Gram matrix and those geometric invariants of the parameter space presenting $\mathcal{M}(n, m; \partial \mathbb{H}^n_3)$. We mention that the normalized processes in \cite{12,13} and the applications of Bruhat decomposition in \cite{20} share the same spirit in eliminating the indeterminacy of $D$ in \cite{14}.

Let $i(G(p)) = (n_+, n_-, n_0)$ be the signature of Hermitian matrix $i(G(p))$ and $V = \text{span}\{p_1, \cdots, p_m\}$ be of dimension $k + 1$. There are two different cases of the moduli problem on $\mathbb{P}(V_+)$ according to $n_+ + n_- = k + 1$ or $n_+ + n_- = k$ (see Theorem 2.2). $V$ is called parabolic in the latter case in \cite{11}. The two cases are termed by regular and non regular cases in complex hyperbolic plane \cite{13}. We still use this terminology in quaternionic setting. In non regular case, the Gram matrices are unable to distinguish different congruence classes. In regular case, the orthogonality of positive points always prevents one from taking similar normalized process in \cite{13} and makes it extremely difficult to find the bi-directional recover process between the geometric invariants and its corresponding Gram matrix. Cunha et al surmounted these difficulties with exquisite techniques on complex hyperbolic plane \cite{14}.

It is interesting to consider the moduli problems in quaternionic hyperbolic geometry. However, one may encounter the difficulty caused by the noncommutativity of quaternions. Due to this noncommutativity, it is always a huge challenge to do computations in quaternionic setting \cite{2,9,23}. Also, though in the literature there have been counterparts of terminologies such as rank, determinant and trace which are extensively used in commutative field, the properties of these concepts may be much different in quaternionic setting. One should be cautious to use them in noncommutative environment. Furthermore, another essential difference between complex and quaternionic hyperbolic geometry is due to the existence of elliptic elements of forms $\mu I_{n+1}$ in $\text{Sp}(n, 1)$, where $\mu \in \text{Sp}(1)$. This fact can make it even more difficult to define geometric invariants and determine the representative Gram matrix in its equivalent class.

By mainly using quaternionic Cartan’s angular invariant and quaternionic cross-ratio in $\mathbb{H}^n_3$, the author \cite{5} solved moduli problems of $\mathcal{M}(n, 3; \mathbb{H}^n_3)$ and $\mathcal{M}(n, 4; \partial \mathbb{H}^n_3)$, respectively.

We will continue the research in this direction. In this paper we concentrate on the moduli problems of $\mathcal{M}(n, m; \partial \mathbb{H}^n_3)$ and $\mathcal{M}(n, m; \mathbb{P}(V_+))$. As stated in \cite{13,14}, the motivation of our concerns comes from the research topic of deformation spaces of pure loxodromic subgroup, as well as the current hot research topic concerning subgroup generated by reflections in submanifolds of dimension $n - 1$ in $\mathbb{H}^n_3$.

We need several notations to illustrate our strategies for overcoming the difficulties mentioned above. At first, we figure out the relationship between the Gram matrix $G(p)$ and that of its permutation $\sigma(p)$. Using this relationship, we are free to rearrange the ordered $m$-tuple in question.

The elementary matrix obtained by swapping row $i$ and row $j$ of the identity matrix $I_m$ is denoted by $T_{ij}$. Let $\sigma$ be an element of symmetric group $S_m$. It is well-known that $\sigma$ can be expressed as the product of transpositions $\sigma = \sigma_1 \sigma_2 \cdots \sigma_l$. We denote $T_{\sigma_l} = T_{ij}$ if $\sigma_l$ is a transposition of $i \rightarrow j \rightarrow i$ and define

$$T_\sigma = T_{\sigma_1} \cdots T_{\sigma_l}.$$  

We can easily verify the following proposition.

**Proposition 1.2.** Let $p = (p_1, \cdots, p_m)$ be an $m$-tuple of points in $\mathbb{P}^{n,1} - \{0\}$. Let $\sigma$ be an element of symmetric group $S_m$. Let $\sigma(p) = (p_{\sigma(1)}, \cdots, p_{\sigma(m)})$. Then

$$G(p) = T_\sigma G(\sigma(p))T_\sigma^*.$$  

Let $v = (v_1, \cdots, v_l)$ be a row vector in $\mathbb{H}^n$ and

$$O_v = \{\mu v \mu^{-1} = (\mu v_1 \mu^{-1}, \cdots, \mu v_l \mu^{-1}) : \forall \mu \in \text{Sp}(1)\}.$$
The set $O_v$ can be thought of as the orbit of $v$ under the action of $Sp(1)/\pm 1$. The procedure of giving a coordinate to the orbit $O_v$ is termed by rotation-normalized algorithm in this paper. We mention that rotation-normalized algorithm stems both from the noncommutativity of quaternions and the existence of isometries of the form $\mu I_{n+1}$ in $Sp(n, 1)$. Such an algorithm is indigenous in quaternionic hyperbolic geometry, while obviously vacuous in complex hyperbolic geometry. We mention that rotation-normalized algorithm is involved in each moduli problem of quaternionic hyperbolic geometry.

When $V$ is parabolic, the Gram matrix $G(p)$ loses the information of configuration and only carries the information of strati-form structure (see Example 5.1 and Proposition 5.3). This strati-form structure will help us to break down the space $V = \text{span}\{p_1, \ldots, p_n\}$ into finite 2-dimensional subspaces. We mention that there exist at most $n - 1$ such 2-dimensional subspaces in $\mathbb{F}^{n, 1}$. These 2-dimensional subspaces share a common basis which is a fibre in $V_0$. In each subspace containing more than three points of the $m$-tuple, we need to introduce new invariants (the cross-ratios in $\mathbb{H} \cup \infty$) to parameterize their congruence classes. Of particular interest will be the harmonious coexistence of these 2-dimensional subspaces (see Proposition 6.4).

When $V$ is not parabolic, the Gram matrix $G(p)$ contains the full information of the congruence class of $p$. The moduli problem on $\mathbb{P}(V_0)$ is tractable for each entry in Gram matrix $G(p)$ being nonzero. On handling the moduli problem on $\mathbb{P}(V_\pm)$, the pivotal point is to find a partition of $S(m) = \{1, \ldots, m\}$ to perform rotation-normalized algorithm in each block independently. This will help us to tackle the difficulty caused by orthogonality. Such a method is termed by block-normalized algorithm.

In our perspective, the parameter of $PSp(n, 1)$-congruence class of $p$ is independent entries of a unique representative Gram matrix when $V$ is not parabolic. For example, the $PSp(n, 1)$-congruence class of three points in $\partial H_n^2$ is its quaternionic Cartan’s angular invariant [11][5]. We mainly rely on the rotation-normalized and block-normalized algorithms to construct such a moduli space in this paper. Our approaches sound natural and elementary.

Of course, one can construct other geometric invariants based on the independent entries of the unique Gram matrix, and search a bijective map between them. These geometric meanings of these invariants may help us to understand the configuration of points in $p$. These efforts may be involved in using Hermitian product in more positions to detour the pitfalls caused by orthogonality among positive points. We will not concentrate on that aspect in the present paper.

As should be apparent, our ideas and exposition owe a great deal to the works of the references cited above, especially to those of [13][14].

The paper is organized as follows. Section 2 contains properties of quaternions, the some basic facts in quaternionic hyperbolic geometry and the inertia of Gram matrices. These properties provide us with the tool to execute rotation-normalized algorithm and initiate the idea of block-normalized algorithm. Section 3 describes the moduli problem on $\mathbb{P}(V_0)$ for $m > 4$. This may be regarded as a generalization of that of [5], or the counterpart in quaternionic geometry of that of [13]. The application of rotation-normalized algorithm is fully described. This method will be mimicked in the more complicated cases in succeeding sections. Section 4 is devoted to describing the duality of submanifolds of dimension $n - 1$ and the polar vectors. The parameter space of $\mathcal{M}(n, 2, \mathbb{P}(V_+))$ is also constructed. In Section 5, we mainly refine the structure of Gram matrices. These refined structures are crucial in introducing new invariants in non regular case and the block-normalized algorithm in regular case. In Section 6, we construct invariants which describe the $PSp(n, 1)$-congruence classes of $V$ when $V$ is parabolic. In Section 7, we describe the moduli space of configurations of quaternionic $(n - 1)$-dimensional submanifolds when $V$ is not parabolic in conceptual style. Section 8 contains a parameter space of quaternionic hyperbolic triangles. The content of this section may be regarded as an application of somewhat conceptual results in previous sections in hyperbolic triangle groups, a current hot research topic in hyperbolic geometry.

Shortly after we completed this paper, Gou informed us that He has also considered similar problem in the boundary of quaternionic hyperbolic space [17].
2 The inertia of Gram matrices

In this section, we will recall some properties of quaternions and obtain some properties of the inertia of Gram matrices.

2.1 Properties of quaternions

Recall that a quaternion is of the form \( a = a_0 + a_1 i + a_2 j + a_3 k \in \mathbb{H} \) where \( a_i \in \mathbb{R} \) and \( i^2 = j^2 = k^2 = ijk = -1 \). Let \( \overline{a} = a_0 - a_1 i - a_2 j - a_3 k \) and \( |a| = \sqrt{\overline{a} a} = \sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2} \) be the conjugate and modulus of \( a \), respectively. We define \( \Re(a) = (a + \overline{a})/2 \) and \( \Im(a) = (a - \overline{a})/2 \). Two quaternions \( a \) and \( b \) are similar if there exists nonzero \( \lambda \in \mathbb{H} \) such that \( b = \lambda a \lambda^{-1} \).

It is useful to view \( \mathbb{H} \) as \( \mathbb{H} = \mathbb{C} \oplus \mathbb{C}j \). In this way, each quaternion \( a = a_0 + a_1 i + a_2 j + a_3 k \) can be uniquely expressed as

\[
a = (a_0 + a_1 i) + (a_2 + a_3 i)j = c_1 + c_2 j = c_1 + j c_2.
\]

It is well-known that the action of \( \text{Sp}(1)/\pm 1 \) on \( \mathbb{H} \) coincides with the action of \( \text{SO}(3) \) on \( \mathbb{R}^3 \). We recall it as the following proposition.

**Proposition 2.1.** Denote \( \vec{v} = (x, y, z)^T \) for \( v = xi + yj + zk \in \mathbb{H} \), where \( A^T \) is the transpose of matrix \( A \). For a unit quaternion \( \mu = u_0 + u_1 i + u_2 j + u_3 k \), we define

\[
M_\mu = \begin{pmatrix}
u_1^2 + u_0^2 - u_3^2 - u_2^2 & 2u_1 u_2 + 2u_0 u_3 & 2u_1 u_3 - 2u_0 u_2 \\
2u_1 u_2 - 2u_0 u_3 & u_2^2 - u_3^2 + u_1^2 & 2u_2 u_3 + 2u_0 u_1 \\
2u_1 u_3 + 2u_0 u_2 & 2u_2 u_3 - 2u_0 u_1 & u_2^2 - u_1^2 + u_3^2
\end{pmatrix}.
\]

Then \( M_\mu \in \text{SO}(3) \) and

\[
\overline{\mu} v \mu = M_\mu \vec{v}.
\]

In particular

\[
M_{\mu, 1} = \begin{pmatrix}1 & 0 & 0 \\
0 & \cos 2\beta & \sin 2\beta \\
0 & -\sin 2\beta & \cos 2\beta
\end{pmatrix}.
\]

**Lemma 2.1.** Let \( v_1 = x_1 i + y_1 j + z_1 k \) and \( v_2 = x_2 i + y_2 j + z_2 k \) such that \( \vec{v_1} \) and \( \vec{v_2} \) are linear independent. Let \( v_1 \cdot v_2 = \vec{v_2}^T \vec{v_1} \). Then there exists a unique element \( \mu \in \text{Sp}(1)/\pm 1 \) such that

\[
\begin{align*}
\overline{\mu} v_1 \mu = |v_1| i, \quad \overline{\mu} v_2 \mu = \frac{v_1 \cdot v_2}{|v_1|} i + \frac{\sqrt{(|v_1| |v_2|)^2 - (v_1 \cdot v_2)^2}}{|v_1|} j.
\end{align*}
\]

**Proof.** Let \( v_1 = x_1 i + y_1 j + z_1 k \), \( v_2 = x_2 i + y_2 j + z_2 k \) and \( \theta \) the angle between \( \vec{v}_1 \) and \( \vec{v}_2 \). Identify \( \Im(\mathbb{H}) \) with the 3-dimensional real space \( \mathbf{xyz} \). Geometrically, by rotating the plane spanned by \( v_1 \) and \( v_2 \) to \( \mathbf{xy} \) plane and then rotating around the \( z \)-axis or \( x \)-axis if necessary, we can obtain a \( \mu \) such that formulae (1) hold. It is helpful to regard this formulae as

\[
\overline{\mu} v_1 \mu = |v_1| i, \quad \overline{\mu} v_2 \mu = |v_2| \cos \theta i + |v_2| \sin \theta j.
\]

Suppose that there exists another unit quaternion \( \nu \) satisfying the above equalities. Then we have \( \nu^{-1} |v_1| i \nu \nu^{-1} = |v_1| i \) and therefore \( \nu^{-1} \mu \) is a unit complex number. Similarly we get \( \nu^{-1} \mu j \nu \nu^{-1} = j \) which implies that \( \nu^{-1} \mu = \pm 1 \). Therefore \( \nu = \mu \) or \( \nu = -\mu \). \( \square \)
Lemma 2.4 is the foundation of rotation-normalized algorithm. We give an explicit formula of such a unique $\mu$ by the following process. Note that

$$-(|v_1|i + v_1)v_1(v_1|i + v_1) = |v_1|i + v_1|^2v_1|i.$$ 

Let

$$\nu = \nu(v_1) = \left\{ \begin{array}{ll} j, & \text{provided } x_1 < 0, y_1^2 + z_1^2 = 0; \\ \frac{|v_2|i + v_2}{\sqrt{2|v_1|(|v_1| + x_1)}}, & \text{otherwise.} \end{array} \right.$$ \hfill (6)

Then

$$|\nu| = 1, \bar{\nu}v_1\nu = |v_1|i.$$ 

Let $\bar{\nu}v_2\nu = c_1 + c_2j$, where $c_1, c_2$ are complex numbers. Since $c_2 \neq 0$, we have $e^{-2\alpha}c_2 = |c_2|$ with $e^{i\alpha} = \sqrt{\frac{c_2}{|c_2|}}$. Therefore $\mu = \pm\nu e^{i\alpha}$ is the desired unit quaternion. By finding the corresponding $c_2$ and (6), we obtain the following formula:

$$\mu = \mu(v_1, v_2) = \left\{ \begin{array}{ll} \pm \sqrt{y_2^2 + z_2^2}j, & \text{provided } x_1 < 0, y_1^2 + z_1^2 = 0; \\ \pm \frac{|v_1|i + v_1}{\sqrt{2|v_1|(|v_1| + x_1)}} \sqrt{F}, & \text{otherwise,} \end{array} \right.$$ \hfill (7)

where

$$F = 2x_2(|v_1| + x_1)(y_1 + z_1i) - (|v_1| + x_1)^2(y_2 + z_2i) + (y_2 - z_2i)(y_1 + z_1i)^2.$$ 

### 2.2 The inertia of Gram matrices

In this paper, the $J$ in quaternionic Hermitian product $\langle z, w \rangle = w^*Jz$ given in Section I will be taken one of the following forms:

$$J_b = \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix} \quad \text{or} \quad J_s = \begin{pmatrix} 0 & 0 & 1 \\ 0 & I_{n-1} & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$ 

The corresponding quaternionic hyperbolic spaces are usually termed by ball model and Siegel domain model, respectively. Let $C$ be the Cayley transformation mapping the ball to the Siegel domain. Then the relation of the two models can be mainly expressed by the following two equations:

$$w^*J_bz = (Cw)^*J_s(Cz), \quad g^*J_bg = J_b = C^{-1}J_sC = C^{-1}(CgC^{-1})^*J_s(CgC^{-1})C.$$ 

Each model has its own advantage in some situations. Basically we work on Siegel domain model only in Sections 3.

Note that $g^*J_bg = J_b$ with $g = (g_1, \cdots, g_{n+1})$, that is,

$$\langle g_i, g_j \rangle = 0, \ i \neq j, \langle g_i, g_i \rangle = 1, \ i = 1, \cdots, n, \langle g_{n+1}, g_{n+1} \rangle = -1.$$ \hfill (8)

In terms of Gram matrix given by Definition 1.1, we have

$$G(g) = J_b, \forall g \in \Sp(n, 1).$$

Based on this observation, we have the following proposition.

**Proposition 2.2.** Let $p = (p_1, \cdots, p_m)$ and $q = (q_1, \cdots, q_m)$ such that $\langle p_i, p_i \rangle = \langle q_i, q_i \rangle = 1$ and $\langle p_i, p_j \rangle = \langle q_i, q_j \rangle = 0, \ i \neq j$. Then there is a $g \in \Sp(n, 1)$ such that

$$gp_i = q_i, \ i = 1, \cdots, m.$$
Proof. By the signature restriction, we have \( m \leq n \). We can extend \( p \) and \( q \) to \( f = (p, p_{m+1}, \ldots, p_{n+1}) \) and \( h = (q, q_{m+1}, \ldots, q_{n+1}) \) such that \( f, h \in \text{Sp}(n, 1) \). Then \( g = hf^{-1} \) is the desired isometry. \( \square \)

Proposition 2.2 implies the following simple result.

**Theorem 2.1.** \( \text{PSp}(n, 1) \) acts transitively on \( \mathbb{P}(V_+) \).

Let \( \mathbb{H}^{n,1} : \langle z, w \rangle = 0 \) be the orthogonal complement of the fibre \( z \mathbb{H} \) in \( \mathbb{H}^{n,1} \) and \( \dim_q(V) \) the quaternionic dimension of subspace \( V \) of \( \mathbb{H}^{n,1} \).

**Proposition 2.3.** We have the following statements concerning the orthogonal complements on \( \mathbb{H}^{n,1} \).

(i) If \( z \in V_+ \) then \( z^\perp \subset V_+ \). There exists an orthogonal basis \( \{p_2, \ldots, p_{n+1}\} \) in \( z^\perp \), \( \dim_q(z^\perp) = n \) and \( \{z, p_2, \ldots, p_{n+1}\} \) is a basis of \( \mathbb{H}^{n,1} \).

(ii) If \( z \in V_0 \) then \( z^\perp \subset V_+ \cup V_0 \) and \( z^\perp \cap V_0 = z \mathbb{H} \). There exist mutually orthogonal vectors \( \{p_2, \ldots, p_n\} \) in \( V_+ \) and \( z^\perp = \text{span}\{z, p_2, \ldots, p_n\} \).

(iii) If \( z \in V_+ \) then \( z^\perp \cap V_+ \neq \emptyset \), \( z^\perp \cap V_0 \neq \emptyset \), \( z^\perp \cap V_- \neq \emptyset \). There exist mutually orthogonal vectors \( \{p_2, \ldots, p_n,p_{n+1}\} \) such that \( \text{span}\{z, p_2, \ldots, p_n\} \subset V_+ \) and \( \{z, p_2, \ldots, p_{n+1}\} \) is a basis of \( \mathbb{H}^{n,1} \).

Proof. Let \( z \in V_- \). Then \( z^\perp \subset V_+ \). By (3), there exists an orthogonal basis \( \{p_2, \ldots, p_{n+1}\} \) in \( z^\perp \). Hence \( \dim_q(z^\perp) = n \) and \( \{z, p_2, \ldots, p_{n+1}\} \) is a basis of \( \mathbb{H}^{n,1} \). Therefore case (i) holds. Case (iii) follows similarly.

Let \( z \in V_0 \). We may assume that \( z = (1,0,\ldots,0,1)^T \). It is obvious that \( w \in z^\perp \) is of the form \( w = (q_1, q_2, \ldots, q_n, q_1)^T \). Let \( e_i \) be the standard basis of \( \mathbb{H}^{n,1} \). Then \( e_i, i = 2, \ldots, n \) belong to \( z^\perp \) and \( z^\perp = \text{span}\{z, e_2, \ldots, e_n\} \).

Recall that \( A \in \mathbb{H}_{n,n} \) is called Hermitian if and only if \( A = A^* \). Let \( H_n(\mathbb{H}) \) be the collection of \( n \times n \) Hermitian matrices. It is well-known that the right eigenvalues of \( A \in H_n(\mathbb{H}) \) are real and there exists an invertible matrix \( B \in \mathbb{H}_{n,n} \) such that \( B^*AB \) is a diagonal matrix which has only entries \( +1, -1, 0 \) along the diagonal. The numbers of \( +1 \)s, \( -1 \)s and \( 0 \)s are denoted by \( n_+, n_- \) and \( n_0 \), respectively. We denote the signature of \( A \) by

\[
i(A) = (n_+, n_-, n_0).
\]

**Proposition 2.4.** (Proposition 1.1) If \( z, w \in \mathbb{H}^{n,1} - \{0\} \) with \( \langle z, z \rangle \leq 0 \) and \( \langle w, w \rangle \leq 0 \) then either \( w = z \lambda \) for some \( \lambda \in \mathbb{H} \) or \( \langle z, w \rangle \neq 0 \).

**Proposition 2.5.** Let \( p = (p_1, \ldots, p_m) \) be an \( m \)-tuple of pairwise distinct points in \( \partial \mathbb{H}^m_\mathbb{H} \) with lift \( p = (p_1, \ldots, p_m) \) and \( m \geq 2 \). Then \( G(p) \) has a negative eigenvalue.

Proof. Let \( q = p_1 + p_2 \mu \) with \( \mu = -\langle p_1, p_2 \rangle \). By Proposition 2.4

\[
\langle q, q \rangle = -2\langle p_1, p_2 \rangle < 0.
\]

Suppose that the eigenvalues of \( G(p) \) are all non-negative. Then there exists an invertible matrix \( S \in \mathbb{H}_{m,m} \) such that

\[
S^*G(p)S = \text{diag}(1, \ldots, 1, 0, \ldots, 0).
\]

Then \( x^*S^*p^*JpSx \geq 0, \forall x \in \mathbb{H}^m \). This contradicts (9) with \( x = S^{-1}l \) and \( l = (1, \mu, 0, \ldots, 0)^T \in \mathbb{H}^m \). \( \square \)
The following proposition is obvious.

**Proposition 2.6.** Let $S$ be an invertible matrix. Then $i(A) = i(S^*AS)$. Furthermore assume that $S^*AS = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$. Then

$$i(A) = i(A_1) + i(A_2).$$

Let $p = (p_1, \cdots, p_t)$ and $q = (q_1, \cdots, q_t)$ such that $\langle p_i, q_j \rangle = 0$ for all $i, j$. Then

$$(p, q)^{*}J(p, q) = \begin{pmatrix} G(p) & 0 \\ 0 & G(q) \end{pmatrix}. \quad (10)$$

We can now prove the following crucial result.

**Theorem 2.2.** Let $p = (p_1, \cdots, p_m) \in \mathbb{H}_{n+1,m}$, $V = \text{span}\{p_1, \cdots, p_m\}$ and

$$\dim_q V = k + 1, \ i(G(p)) = i(p^*Jp) = (n_+, n_-, n_0).$$

Then

$$k \leq n_+ + n_- \leq k + 1, n_+ \leq n, n_- \leq 1, n_+ + n_- + n_0 = m.$$  

In particular, we have the following statements.

1. If $p_i \in V_0, i = 1, \cdots, m$ then $n_+ = k, n_- = 1$.
2. If $p_i \in V_+, i = 1, \cdots, m$ then there are three cases:
   1. $n_+ = k, n_- = 1$, in this case $V$ is hyperbolic;
   2. $n_+ = k + 1, n_- = 0$, in this case $V$ is elliptic;
   3. $n_+ = k, n_- = 0$, in this case $V$ is parabolic.

**Proof.** Let $t = k + 1$. Without loss of generality, we assume that $p_1, \cdots, p_t$ are linearly independent and

$$p_j = p_1\lambda_{1j} + \cdots + p_t\lambda_{tj}, j = t + 1, \cdots, m.$$  

Let $q = (p_1, \cdots, p_t) \in \mathbb{H}_{n+1,t}$. Then $p = q(I_t, \Lambda)$, where $\Lambda = (\lambda_{ij}), i = 1, \cdots, t, j = t + 1, \cdots, m$. Let

$$S = \begin{pmatrix} I_t & -\Lambda \\ 0 & I_{m-t} \end{pmatrix}. \text{ Direction computation shows that} \ S^*G(p)S = S^*p^*JpS = \begin{pmatrix} q^*Jq & 0 \\ 0 & 0 \end{pmatrix}. \ \text{Therefore, by Proposition 2.6 we have that} \ i(p^*Jp) = i(q^*Jq).$$

This implies that $n_+ + n_- \leq k + 1$.

If $V \cap V_\perp \neq \emptyset$ then there exists a $z \in V_\perp$ such that $V = z\mathbb{H} \oplus (z^\perp \cap V)$. In the space $z^\perp \cap V$ there exist $k$ mutually orthogonal positive lines $q_1, \cdots, q_k$ such that $V = \text{span}\{z, q_1, \cdots, q_k\}$. By (10) we have $n_+ = k, n_- = 0$ and $V$ is hyperbolic in this case.

By Proposition 2.5, a space with two different null lines must contain negative lines. If $V \cap V_\perp = \emptyset$ and $V \cap V_0 \neq \emptyset$ then there exists a unique $z\mathbb{H} \in V_0$. The space $z^\perp \cap V$ contains only $k$ mutually orthogonal positive lines $q_1, \cdots, q_k$. In this case $n_+ = k, n_- = 0$ and $V$ is parabolic.

If $V \subset V_+$, then $V$ contains $k + 1$ mutually orthogonal positive lines $q_1, \cdots, q_{k+1}$. In this case $n_+ = k + 1, n_- = 0$ and $V$ is elliptic.

It follows from Proposition 2.3 and 2.5 that the statements of (1) and (2) hold. \qed
Remark 2.1. Since any \( m \)-tuple \( p = (p_1, \ldots, p_m) \) in \( \mathbb{F}^{n,1} \) span a space \( V = \text{span}\{p_1, \ldots, p_m\} \) which is definitely contained in a copy of \( \mathbb{F}^{m,1} \). In other words, there exists a \( g \in \text{Sp}(n,1) \) such that \( g(V) \subset \mathbb{F}^{m,1} \). So if one consider the moduli problem of points in \( \mathbb{F}^n \), it is enough to assume that \( n \leq m \).

Furthermore, for moduli problem of points in \( \mathbb{H}^m \), one can further assume that \( n \leq m - 1 \).

3 Moduli problem on \( \mathbb{P}(V_0) \)

In this section, we will consider the moduli problem on \( \mathbb{P}(V_0) \) for \( m > 4 \). The application of rotation-normalized algorithm is fully described. This method will be mimicked conceptually to more complicated cases in Sections 6 and 7.

3.1 Semi-normalized Gram matrix

We recall the following definition in [1, 5].

Definition 3.1. The quaternionic Cartan’s angular invariant of a triple \( p = (p_1, p_2, p_3) \) of pairwise distinct points in \( \mathbb{H}_H^m \) is the angular invariant \( A_H(p) \), \( 0 \leq A_H(p) \leq \frac{\pi}{2} \), given by

\[
A_H(p) = A_H(p_1, p_2, p_3) := \arccos \frac{\mathcal{R}(-\langle p_1, p_2, p_3 \rangle)}{\langle p_1, p_2, p_3 \rangle},
\]

where \( p_1, p_2, p_3 \) are lifts of \( p_1, p_2, p_3 \), respectively.

Proposition 3.1. Let \( p = (p_1, \ldots, p_m) \) be an \( m \)-tuple of pairwise distinct points in \( \partial \mathbb{H}_H^n \). Then the equivalence class of Gram matrices associated to \( p \) contains a matrix \( G = (g_{ij}) \) with

\[
g_{ii} = 0, \ i = 1, \ldots, m, \ g_{i-1,i} = 1, i = 2, \ldots, m, \ g_{13} = -e^{-i\lambda},
\]

where \( \lambda = A_H((p_1, p_2, p_3)). \)

Proof. Let \( p = (p_1, \ldots, p_m) \) be an arbitrary lift of \( p \). We want to obtain a diagonal matrix \( D \) such that \( G(D) \) is the desired Gram matrix.

Note that \( \langle p_i, p_j \rangle \neq 0 \) for \( i \neq j \). Firstly we obtain the solutions \( \lambda_i, i = 2, \ldots, m \) of the equations below:

\[
\langle p_1, p_2 \lambda_2 \rangle = 1, \ \langle p_2 \lambda_2, p_3 \lambda_3 \rangle = 1, \ldots, \langle p_{m-1} \lambda_{m-1}, p_m \lambda_m \rangle = 1.
\]

Next, by (6) we let

\[
\lambda_1 = \frac{\nu(\langle p_1, p_3 \lambda_3 \rangle)}{\sqrt{|\langle p_1, p_3 \lambda_3 \rangle|}} = \frac{\nu(\langle p_2, p_1 \rangle \langle p_2, p_3 \rangle^{-1} \langle p_1, p_3 \rangle)}{\sqrt{|\langle p_2, p_1 \rangle \langle p_2, p_3 \rangle^{-1} \langle p_1, p_3 \rangle|}}
\]

By the property of quaternionic Cartan’s angular invariant, \( \langle p_1 \lambda_1, p_3 \lambda_3 \lambda_1 \rangle \) is a unit complex with negative real part and therefore

\[
\langle p_1 \lambda_1, p_3 \lambda_3 \lambda_1 \rangle = -e^{-i\lambda}.
\]

Let \( \mu_1 = \lambda_1 \); for \( i \geq 2 \), \( \mu_i = \lambda_i \lambda_1 \) when \( i \) is odd, and \( \mu_i = \lambda_i \lambda_1^{-1} \) when \( i \) is even. Then \( G(D) \) is the desired Gram matrix with

\[
D = \text{diag}(\mu_1, \ldots, \mu_m).
\]

\[\Box\]
Definition 3.2. The Gram matrix $G$ as in Proposition 3.1 of the form

$$G(n) = (g_{ij}) = \begin{pmatrix}
0 & 1 & g_{13} & g_{14} & \cdots & g_{1m} \\
1 & 0 & 1 & g_{24} & \cdots & g_{2m} \\
g_{13} & 1 & 0 & 1 & \cdots & g_{3m} \\
g_{14} & g_{24} & 1 & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & 0 & 1 \\
g_{1m} & g_{2m} & g_{3m} & \cdots & 1 & 0
\end{pmatrix} \quad (14)$$

is called the semi-normalized Gram matrix.

Proposition 3.2. ([13] Theorems 2.1, 2.2) Let $G = (g_{ij})$ be a Hermitian $m \times m$-matrix, $m > 2$ with

$$g_{ii} = 0, \ i = 1, \ldots, m, \ g_{i-1,i} = 1, \ i = 2, \ldots, m, \ g_{13} = -e^{-i\Delta},$$

where $\Delta \in [0, \pi/2]$. Let $i(G) = (n_+, n_-, n_0)$. Then $G$ is a semi-normalized Gram matrix associated with some ordered $m$-tuple $p = (p_1, \cdots, p_m)$ of pairwise distinct isotropic points in $\partial H^0_H$ if and only if

$$n_+ \leq n, \ n_- = 1, \ n_+ + n_- + n_0 = m. \quad (15)$$

Proof. Suppose that $G$ is a semi-normalized Gram matrix associated with some ordered $m$-tuple $p = (p_1, \cdots, p_m)$ of pairwise distinct isotropic points in $\partial H^0_H$. It follows from Theorem 2.2 that $n_+ \leq n, \ n_- = 1, \ n_+ + n_- + n_0 = m$.

Conversely, suppose that $G = (g_{ij})$ is of the form (14) with

$$i(G) = (n_+, 1, m - n_+ - 1).$$

There exists an invertible matrix $S$ such that $S^*GS = B$, where $B$ is the diagonal $m \times m$ matrix with

$$b_{ii} = 1 \text{ for } 1 \leq i \leq n_+, \ b_{ii} = -1 \text{ for } i = n_+ + 1, \text{ and } b_{ij} = 0 \text{ for all other indices.}$$

Now let $A = (a_{ij})$ be the $(n + 1) \times m$-matrix such that $a_{ii} = 1$ for $1 \leq i \leq n_+$, $a_{ii} = -1$ for $i = n_+ + 1$, and $a_{ij} = 0$ for all other indices. Then $A^*JA = B = S^*GS$, which implies that

$$(S^*)^{-1}A^*JAS^{-1} = G.$$ 

Then $p = AS^{-1}$ is the desired lift of $p = (p_1, \cdots, p_m)$ to get the semi-normalized Gram matrix $G$. \qed

3.2 The parameter space of moduli space

The following lemma shows that a semi-normalized Gram matrix is just an equivalent class, and also indicates the necessity of performing rotation-normalized algorithm.

Lemma 3.1. Suppose that the Gram matrix $G(p)$ is a semi-normalized Gram matrix for $p = (p_1, \cdots, p_m)$. Then $G(pD)$ is still a semi-normalized Gram matrix with $D = \text{diag}(\mu_1, \cdots, \mu_m)$ if only if

$$D = \mu I_m = \text{diag}(\mu, \cdots, \mu), \mu e^{-i\Delta} = e^{-i\Delta} \mu, \mu \in \text{Sp}(1).$$

Proof. It follows from

$$(p_{i-1} \mu_{i-1}, p_i \mu_i) = 1, \ i = 2, \cdots, m$$

that all those $\mu_i$ with $i$ odd are equal, and so do for all those $\mu_i$ with $i$ even. The fact $(p_1 \mu_1, p_3 \mu_3) = -e^{-i\Delta}$ implies $\mu_1 = \mu_3$. Hence $\mu_1 = \mu_2 = \cdots = \mu_m := \mu$ and $\mu e^{-i\Delta} = e^{-i\Delta} \mu$. \qed
Set \( t = \frac{(m-1)(m-2)}{2} \). We can represent a semi-normalized Gram matrix by a \( t \)-vector:

\[
v_G = (g_{13}, g_{14}, g_{24}, \ldots, g_{1m}, \ldots, g_{m-2,m}). \tag{16}\]

Also we represent

\[
G = G(v_G). \tag{17}
\]

Recall that two Hermitian matrices \( H \) and \( \tilde{H} \) are equivalent if there exists a diagonal matrix \( D \) such that \( \tilde{H} = D^*HD \) (see [5, 13]). By Lemma 3.1 we obtain the following result.

**Lemma 3.2.** Let \( G \) and \( \tilde{G} \) be two semi-normalized Gram matrices represented by \( V(G) \) and \( V(\tilde{G}) \). Then \( \tilde{G} \) and \( G \) are equivalent if and only if

\[
O_{v_G} \simeq O_{v_{\tilde{G}}}. \tag{18}
\]

From this, Proposition 3.2 can be reformulated as follows.

**Proposition 3.3.** Let \( v = (v_1, \cdots, v_t) \) with \( v_1 = -e^{-i\alpha}, \alpha \in [0, \pi/2] \). Let \( i(G(v)) = (n_+, n_-, n_0) \). Then \( G(v) \) is a semi-normalized Gram matrix associated with some ordered \( m \)-tuple \( p = (p_1, \cdots, p_m) \) of distinct isotropic points in \( \partial H^n_H \) if and only if

\[
n_+ \leq n, \ n_- = 1, \ n_+ + n_- + n_0 = m. \tag{19}\]

**Definition 3.3.**

\[ V(n, m) = \{ v = (v_1, \cdots, v_t) : i(G(v)) = (n_+, n_-, n_0) \text{ with } n_+ \leq n, \ n_- = 1 \}. \]

By Lemma 3.2 there is an equivalent relation in \( V(n, m) \) defined by \( \simeq \). Therefore the configuration space \( \mathcal{M}(n, m; \partial H^n_C) \) can be thought of as the quotient of \( V(n, m) \) under this equivalent relation. That is

\[
\mathcal{M}(n, m; \partial H^n_C) = V(n, m)/\simeq.
\]

Based on this observation, we are ready to construct the parameter space \( \mathbb{M}(n, m) \) for \( V(n, m)/\simeq \) with rotation-normalized algorithm. We mainly rely on Lemma 2.1 to execute rotation-normalized algorithm.

This procedure can be described conceptually as follows:

In case \( \alpha = 0 \), or equivalently, \( -e^{-i\alpha} = -1 \), we basically need to find two entries \( v_i \) and \( v_j \) in \( v \in V(n, m) \) with \( \Im(v_i) \) and \( \Im(v_j) \) being linearly independent to specific the parameters for its representing equivalent class, whilst only a quaternion in \( \mathbb{H} - \mathbb{C} \) in the case of \( \alpha \neq 0 \).

The above conceptual description is a motivation of the definition of the following sets.

Let

\[
\mathbb{R}^{2+} = \{ v \in \mathbb{H} : v = x_0 + x_1i + x_2j, x_2 > 0 \}, \mathbb{R}^{1+} = \{ v \in \mathbb{C} : v = x_0 + x_1i, x_1 > 0 \}.
\]

**Definition 3.4.** We define the following sets.

\[
P(\mathbb{C}) = \{ v \in V(n, m) : v_i \notin \mathbb{R}, v_i \in \mathbb{C}, \text{ for } i = 2, \cdots, t \};
\]

\[
P(j) = \{ v \in V(n, m) : v_i \notin \mathbb{R}, v_i \in \mathbb{C}, \text{ for } i < j, v_j \in \mathbb{R}^{2+} \}, j = 2, \cdots, t;
\]

\[
Z(\mathbb{R}) = \{ v \in V(n, m) : v_i \in \mathbb{R} \text{ for } i = 1, \cdots, t \};
\]

\[
Z(\mathbb{C}, i) = \{ v \in V(n, m) : v_t \in \mathbb{R}, \text{ for } t < i, v_i \in \mathbb{R}^{1+} \}, i = 2, \cdots, t;
\]

\[
Z(i, j) = \{ v \in V(n, m) : v_t \in \mathbb{R}, t < i, v_i \in \mathbb{R}^{1+}; v_t \in \mathbb{C}, t < j, v_j \in \mathbb{R}^{2+} \}, j = 2, \cdots, t, 2 \leq i < j.
\]
We remark that the sets defined above is roughly divided by two cases: $A \neq 0$ and $A = 0$. Each case is refined according to the positions in which Lemma 2.1 acts. Roughly speaking, such a $Z(i, j)$ looks like

$$Z(i, j) = (\mathbb{R}_{i-1}^*, \mathbb{R}_{j-i-1}^*, \mathbb{C}_{i-1}^*, \mathbb{C}_{j-i-1}^*, \mathbb{H}_{i-1}^*, \mathbb{H}_{j-i-1}^*).$$

Let

$$P(n, m) = P(\mathbb{C}) \cup P(j), Z(n, m) = Z(\mathbb{R}) \cup Z(\mathbb{C}, j) \cup Z(i, j)$$

and

$$\mathcal{M}(n, m) = P(n, m) \cup Z(n, m).$$

**Theorem 3.1.** $\mathcal{M}(n, m)$ is a parameter space of $V(n, m)/ \sim$.

**Proof.** Let $v = (v_1, \ldots, v_l) \in V(n, m)$, where $v_i = -e^{-i\theta}$. We define a map

$$\psi: O_v \in V(n, m)/ \sim \rightarrow \mathcal{M}(n, m) \quad (20)$$

by the following steps:

The equivalent class $O_v$ with $A \neq 0$ will be mapped to an element in $P(n, m)$. It is obvious that $\bar{\mu}\nu\mu \in V(n, m)$ if and only if $\mu \in U(1)$. If all entries of $v$ are complex numbers, then $O_v$ is represented by $v$ itself. Equivalently, the parameter of $O_v$ assigned by $\psi$ in $\mathcal{M}(n, m)$ is $v$ which belongs to $P(\mathbb{C})$. Otherwise, let $j$ be the smallest index among entries of $v$ such that $v_j \in \mathbb{H} - \mathbb{C}$. Let $\mu = \mu(\Im(v_1), \Im(v_j))$ given by (16). Therefore $O_v$ is assigned to the parameter $\bar{\mu}\nu\mu$, which belongs to $P(j)$.

The equivalent class $O_v$ with $A = 0$ belongs to $Z(n, m)$. More precisely, if all entries of $v$ are reals, then $O_v$ is represented by $v$ itself belonging to $Z(\mathbb{R})$. We divide the remainder into two cases. If all entries of $v$ are complex numbers with $i$ being the smallest index such that $v_i \in \mathbb{C} - \mathbb{R}$, then we assign $O_v$ to $\bar{\mu}\nu\mu$, which belongs to $Z(\mathbb{C}, j)$. For the latter case, let $i$ be the smallest index such that $v_i \in \mathbb{C} - \mathbb{R}$ and $j$ the smallest index such that $v_j \in \mathbb{H} - \mathbb{C}$. Let $\mu = \mu(\Im(v_i), \Im(v_j))$. Then we assign $O_v$ to $\bar{\mu}\nu\mu$, which belongs to $Z(i, j)$.

By Lemma 2.1 and the construction of $P(n, m)$ and $Z(n, m)$ above, the map $\psi$ is bijection. Therefore $\mathcal{M}(n, m)$ is a parameter space of $V(n, m)/ \sim$.

**Theorem 3.2.** The configuration space $\mathcal{M}(n, m; \partial \mathbb{H}_R^m)$ is homeomorphic to $\mathcal{M}(n, m)$

**Proof.** Let $m(p) \in \mathcal{M}(n, m; \partial \mathbb{H}_R^m)$ be the point represented by $p = (p_1, \ldots, p_m)$. We can get a semi-normalized Gram matrix $G$ with arbitrary lift of $p$. Proposition 3.3 and Theorem 3.1 imply that we can define a map

$$\tau: m(p) \in \mathcal{M}(n, m; \partial \mathbb{H}_R^m) \rightarrow \psi(v_G) \in \mathcal{M}(n, m).$$

This map is a bijection. Such a map is a homeomorphism because $\mathcal{M}(n, m)$ has the topology structure induced from $\mathbb{H}^m$.

We conclude this section by some remarks. Firstly, if we allow $m = 3$ in our process then we get the parameter of quaternionic Cartan’s angular invariant $A$ (in fact a complex number $-e^{-i\theta}$); while the case of $m = 4$ is exactly the result in [5]. Secondly it seems that the parameters of $m$-tuples in $Z(\mathbb{R})$, $Z(\mathbb{R}) \cup Z(\mathbb{C}, i) \cup P(\mathbb{C})$ can be thought of as $m$-tuples living in a copy of $\partial \mathbb{H}_R^m$ and $\partial \mathbb{H}_R^m$, respectively.

4 Moduli space on $\mathbb{P}(V_{+})$ of case $m = 2$

In this section we will describe the configuration of two submanifolds of dimension $n - 1$. The author believe that this fact is well-known in quaternionic hyperbolic geometry. However we did not find any proof of it in the literature. The parameter space of $\mathcal{M}(n, 2; \mathbb{P}(V_{+}))$ is also constructed.
4.1  The duality of submanifold of dimension $n - 1$ and polar vector

It follows from Proposition 2.3 that $p^\perp$ is an $n$-dimensional subspace of $\mathbb{H}^{n,1}$ for any vector $p \in V_+$. 

Definition 4.1. We define

$$l_p = \mathbb{P}(p^\perp \cap (V_0 \cup V_-)) = \mathbb{P}(p^\perp) \cap H_{\mathbb{H}}^n,$$  \hspace{1cm} (21)

for $p \in V_+$. $l_p$ is a totally geodesic submanifold with boundary in $H_{\mathbb{H}}^n$, which is equivalent to $H_{\mathbb{H}}^{n-1}$. 

We call $p \in V_+$ a polar vector of $l_p$. Sometimes we drop off $V_0$ in (21), and call $l_p$ an $(n - 1)$-submanifold in $H_{\mathbb{H}}^n$. Also for each $(n - 1)$-submanifold $M$, we can find a vector $p \in V_+$ such that $pH$ is the unique fibre with the property $M \subset \mathbb{P}(p^\perp)$. Due to this duality, the configuration of $m$-tuples of distinct $(n - 1)$-submanifolds is equivalent to the configuration of $m$-tuples of pairwise distinct positive points.

As in [27], we define the angle $\theta \in [0, \pi/2]$ between any pair of intersecting $(n - 1)$-submanifolds $l_{p_1}$ and $l_{p_2}$ by

$$\cos^2(\theta) = \frac{\langle p_1, p_2 \rangle \langle p_2, p_1 \rangle}{\langle p_1, p_1 \rangle \langle p_2, p_2 \rangle}.$$ 

This is clearly invariant under quaternionic hyperbolic isometries.

We need a formula to calculate the distance between a negative point and an $(n - 1)$-submanifold.

Lemma 4.1. (27 Corollary 7.7) Let $z$ be any point of $H_{\mathbb{H}}^n$ with lift $z$. Then

$$\cosh^2 \left( \frac{\rho(l_p, z)}{2} \right) = 1 - \frac{\langle z, p \rangle \langle p, z \rangle}{\langle z, z \rangle \langle p, p \rangle} \geq 1. \hspace{1cm} (22)$$

Proof. Let $\Pi_{l_p}$ be the orthogonal projection from $\mathbb{H}^{n,1}$ to $p^\perp$. Then we can express a lift of $z$ as $z = \Pi_{l_p}(z) + p\mu$. Since $\langle \Pi_{l_p}(z), p \rangle = 0$, we have $\langle z, p \rangle = \langle p, p \rangle \mu$. Then $|\langle z, \Pi_{l_p}(z) \rangle|^2 = |\mu|^2 |\langle \Pi_{l_p}(z), \Pi_{l_p}(z) \rangle|^2$ and

$$\langle z, z \rangle = |\lambda|^2 |\langle \Pi_{l_p}(z), \Pi_{l_p}(z) \rangle|^2 + |\mu|^2 |\langle p, p \rangle|^2.$$ 

Hence

$$\cosh^2 \left( \frac{\rho(l_p, z)}{2} \right) = \cosh^2 \left( \frac{\rho(\mathbb{P}(\Pi_{l_p}(z)), z)}{2} \right) = \frac{|\lambda|^2 |\langle \Pi_{l_p}(z), \Pi_{l_p}(z) \rangle|^2}{\langle z, z \rangle |\Pi_{l_p}(z), \Pi_{l_p}(z) \rangle} = 1 - \frac{\langle z, p \rangle \langle p, z \rangle}{\langle z, z \rangle \langle p, p \rangle}.$$ 

The configuration of two positive lines in $V_+$ can be described as follows.

Theorem 4.1. (27 Proposition 7.8) Let $p_1, p_2$ be two points in $V_+$ with distinct projections in $\mathbb{P}(V_+)$, $V = \text{span}\{p_1, p_2\}$ and

$$t = \frac{|\langle p_1, p_2 \rangle|}{\sqrt{\langle p_1, p_1 \rangle \langle p_2, p_2 \rangle}}.$$ 

Then we have the following statements.

(i) $|\langle p_1, p_2 \rangle|^2 < |\langle p_1, p_1 \rangle \langle p_2, p_2 \rangle|$ if and only if $V \subset V_+$. In this case, $p_1^\perp \cap p_2^\perp \cap V_- = \emptyset$

and the angle between $l_{p_1}$ and $l_{p_2}$ is $\arccos t$. 

(ii) \(|\langle p_1, p_2 \rangle| = \langle p_1, p_1 \rangle\langle p_2, p_2 \rangle\) if and only if there exists a unique fibre \(z \mathbb{H} \in V_0\) such that 

\[ V \cap V_0 = z \mathbb{H}, V \cap V_- = \emptyset. \]

In this case

\[ p_1^\perp \cap p_2^\perp \subset V_+ \cup V_0, p_1^\perp \cap p_2^\perp \cap V_0 = (p_1 - p_2) \mathbb{H}, \]

which implies that \(l_{p_1}\) and \(l_{p_2}\) intersect in a unique point in \(\partial H_\mathbb{H}^n\).

(iii) \(|\langle p_1, p_2 \rangle|^2 > \langle p_1, p_1 \rangle\langle p_2, p_2 \rangle\) if and only if \(V \cap V_0 \neq \emptyset, V \cap V_- \neq \emptyset\).

In this case

\[ (p_1^\perp \cap V_-) \cap (p_2^\perp \cap V_-) = \emptyset \]

and

\[ \cosh \left( \frac{\rho(l_{p_1}, l_{p_2})}{2} \right) = \cosh \left( \frac{\rho(P(z), P(w))}{2} \right) = t, \]

where \(V \cap p_1^\perp = z \mathbb{H}\) and \(V \cap p_2^\perp = \mathbb{w} \mathbb{H}\).

**Proof.** By normalization and the transitivity of \(\text{PSp}(n, 1)\) on \(\mathbb{P}(V_+)\), we may assume that

\[ \langle p_1, p_1 \rangle = \langle p_2, p_2 \rangle = 1, t = \langle p_1, p_2 \rangle \geq 0, \]

where \(p_1 = (0, 1, 0, \cdots, 0)^T\) and \(p_2 = (x_1, \cdots, x_{n+1})^T\). With the above assumption we have

\[ t = x_2 \geq 0, \sum_{i=1}^n |x_i|^2 - |x_{n+1}|^2 = 1. \]

Let \(u = p_1 \lambda_1 + p_2 \lambda_2 \in V\). Then

\[ \langle u, u \rangle = |\lambda_1|^2 + |\lambda_2|^2 + 2 \Re(\lambda_2 \lambda_1) t. \]

(24)

We need to consider the following three cases \(t < 1, = 1, > 1\), respectively.

Note that \(t < 1\) if and only if \(\langle u, u \rangle > 0\). This implies that \(V \subset V_+\) for \(t < 1\). In this case, there exists a \(z = (z_1, 0, z_3, \cdots, z_n, 1)^T \in p_1^\perp \cap p_2^\perp \cap V_-\) satisfying the following equation

\[ \bar{x}_1 z_1 + \bar{x}_z z_3 + \cdots + \bar{x}_n z_n - \bar{x}_{n+1} = 0. \]

In fact \(\mathbb{P}(p_1^\perp \cap p_2^\perp \cap V_-)\) is equivalent to \(H_\mathbb{H}^{n-2}\) and the angle between \(l_{p_1}\) and \(l_{p_2}\) is \(\arccos t\).

Observe that \(\langle u, u \rangle \geq 0\) for \(t = 1\). Note that \(\langle u, u \rangle = 0\) if and only if \(\lambda_2 = -\lambda_1\). Therefore \((p_1 - p_2) \mathbb{H}\) is the unique fibre in \(V \cap V_0\). It is obvious that \(\langle p_1, (p_1 - p_2) \rangle = \langle p_2, (p_1 - p_2) \rangle = 0\). Each \(z\) in \(p_1^\perp \cap p_2^\perp\) is of the form \((z_1, 0, z_3, \cdots, z_n+1)^T\) satisfying the following equation

\[ \bar{x}_1 z_1 + \bar{x}_z z_3 + \cdots + \bar{x}_n z_n - \bar{x}_{n+1} z_{n+1} = 0. \]

Noting that \(p_2 = (x_1, 1, x_3, \cdots, x_{n+1})^T\) and \(|x_1|^2 + \sum_{i=3}^{n} |x_i|^2 = |x_{n+1}|^2\), we have

\[ |\bar{x}_{n+1} z_{n+1}|^2 = |\bar{x}_1 z_1 + \bar{x}_z z_3 + \cdots + \bar{x}_n z_n|^2 \]

\[ \leq (|x|^2 + \sum_{i=3}^{n} |x_i|^2)(|z_1|^2 + \sum_{i=3}^{n} |z_i|^2). \]

This implies that \(p_1^\perp \cap p_2^\perp \subset V_+ \cup V_0\) and \(p_1^\perp \cap p_2^\perp \cap V_0 = (p_1 - p_2) \mathbb{H}\).
We consider the case \( t > 1 \). Noting that \( x_2 = t > 1 \), we have \( |x_{n+1}|^2 - (|x_1|^2 + \sum_{i=3}^{n} |x_i|^2) = |x_2|^2 - 1 > 0 \). Similarly each \( z \in \mathbb{P}\mathbb{P}_2 \) of the form \( (z_1, 0, z_3, \cdots, z_{n+1})^T \) satisfying the following equation
\[
\bar{x}_1 z_1 + \bar{x}_3 z_3 + \cdots + \bar{x}_n z_n - \bar{x}_{n+1} z_{n+1} = 0.
\]

Direct computation shows that \( V \cap \mathbb{P}_2 = \mathbb{P}\mathbb{H}, \) where \( z = (x_1, 0, x_3, \cdots, x_{n+1})^T \mathbb{H} \in V_-, \) and \( V \cap \mathbb{P}_2 = \mathbb{W}\mathbb{H}, \) where \( w = (x_1, (|x_2|^2 - 1)\bar{x}_2^{-1}, x_3, \cdots, x_{n+1})^T \mathbb{W} \in V_-. \) Hence
\[
(\mathbb{P}_2^+ \cap V_-) \cap (\mathbb{P}_2^+ \cap V_-) = \emptyset.
\]

We mention that \( l_{\mathbb{P}_i} = \mathbb{P}(\mathbb{P}_2^\perp \cap V_-), i = 1, 2 \) are two totally geodesic submanifolds which are equivalent to \( \mathbb{H}^{n-1}_2. \) It follows from (I) that
\[
|\langle \mathbb{P}_1, \mathbb{P}_2 \rangle| = \cosh\left( \frac{\rho(\mathbb{P}(\mathbb{P}_2), \mathbb{P}(\mathbb{P}_2))}{2} \right) = |x_2|.
\]

Let \( z = (z_1, \cdots, z_{n+1})^T \in \mathbb{P}_2^\perp \cap V_- \) and, for simplicity, denote by
\[
X = \sqrt{\sum_{i \neq 2, n+1} |x_i|^2}, \quad Z = \sqrt{\sum_{i \neq 2, n+1} |z_i|^2}.
\]

Then \( 1 - |x_2|^2 = X^2 - |x_{n+1}|^2 \) and
\[
|x_2 z_2|^2 = |\bar{x}_n z_{n+1} - (\bar{x}_1 z_1 + \bar{x}_3 z_3 + \cdots + \bar{x}_n z_n)|^2 \geq (|\bar{x}_n | z_{n+1}^2 | - X Z)^2.
\]

Let
\[
K = \cosh^2 \left( \frac{\rho(\mathbb{P}(\mathbb{P}_1^\perp \cap V_-), \mathbb{P}(\mathbb{P}_2)))}{2} - |\langle \mathbb{P}_1, \mathbb{P}_2 \rangle|^2.
\]

By Lemma 4.1 and (25), we obtain
\[
K = 1 + \frac{|z_2|^2}{|z_{n+1}|^2 - |z_1|^2 - |z_3|^2 - \cdots - |z_n|^2 - |x_2|^2} = \frac{|z_2|^2 + (X^2 - |x_{n+1}|^2)(|z_{n+1}|^2 - Z^2 - |z_2|^2)}{|z_{n+1}|^2 - Z^2 - |z_2|^2} \geq 0.
\]

This inequality implies that the real geodesic connecting \( z \) and \( w \) is the shortest curve form \( l_{\mathbb{P}_1} \) to \( l_{\mathbb{P}_2} \). \( \square \)

### 4.2 Moduli space on \( \mathbb{P}(V_+) \) of case \( m = 2 \)

We need the following fact, which is easy to verified. We refer to [2] [8] for more details of \( \text{Sp}(1, 1). \)

**Lemma 4.2.** Let \( g \in \text{Sp}(2, 1) \) and \( e_2 = (0, 1, 0)^T \in \mathbb{H}^{2,1} \) such that \( ge_2 = e_2 \mu. \) Then \( g \) is of the form
\[
g = \begin{pmatrix}
a & 0 & b \\
0 & \mu & 0 \\
c & 0 & d
\end{pmatrix},
\]
where
\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix} \in \text{Sp}(1, 1) \quad \text{and} \quad \mu \in \text{Sp}(1).
\]
**Theorem 4.2.** The configuration space $\mathcal{M}(n, 2)$ is homeomorphic to $\mathbb{R}^\geq = \{t \in \mathbb{R} : t \geq 0\}$.

**Proof.** By Remark 2.1.1 we can work in $\mathbb{H}^{2, 1}$ in this situation. Noting the normalization (23), we only need to show that there exists a $g \in \text{Sp}(2, 1)$ such that $g p_1 = q_1\lambda_1$ and $g p_2 = q_2\lambda_2$ when $G((p_1, p_2)) = G((q_1, q_2)) = G((q_1, q_2)) = \left(\begin{array}{cc} 1 & t \\ t & 1 \end{array}\right)$. Noting Proposition 2.2, we only need to consider the case $t \neq 0$. Observe that $t \neq 0$ implies $\lambda_1 = \lambda_2$. Since $\text{Sp}(2, 1)$ acts transitively on $\mathbb{P}(V_+)$, we may further assume that

$$p_1 = q_1 = (0, 1, 0)^T, \quad p_2 = (x_1, t, x_3), \quad q_2 = (y_1, t, y_3)^T,$$

where $|x_3|^2 - |x_1|^2 = |y_3|^2 - |y_1|^2 = t^2 - 1$. By Lemma 4.2 we need to find an element $f = \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in \text{Sp}(1, 1)$ mapping $(x_1, x_3)^T$ to $(y_1, y_3)^T \mu$. The fact that $\text{Sp}(1, 1)$ acts doubly transitively on $\partial \mathbb{H}^1_{\mathbb{H}}$, transitively on $\mathbb{H}^1_{\mathbb{H}}$, and $\mathbb{P}(V_+)$ respectively, completes the proof. \qed

## 5 The structure of Gram matrices of points on $\mathbb{P}(V_+)$

In this section, we provide a 1-normalized Gram matrix for an $m$-tuple on $\mathbb{P}(V_+)$. The main purpose of this section is to refine the structures of Gram matrices. These refined structures are crucial in introducing new invariants in non regular case and the block-normalized algorithm in regular case.

### 5.1 1-normalized Gram matrix

**Proposition 5.1.** Let $p = (p_1, \cdots, p_m)$ be an $m$-tuple of pairwise distinct points in $\mathbb{P}(V_+)$. Then the equivalence class of Gram matrices associated to $p$ contains a matrix $G = (g_{ij})$ with

$$g_{ii} = 1, \quad i = 1, \cdots, m, \quad g_{ij} \geq 0, \quad j = 2, \cdots, m.$$

**Proof.** Let $p = (p_1, \cdots, p_m)$ be an arbitrary lift of $p$. We want to obtain a diagonal matrix $D_1$ such that $G(pD_1)$ is the desired Gram matrix.

We may assume that $\langle p_i, p_i \rangle = 1$ by noticing that

$$\langle p_i \lambda_i, p_i \lambda_i \rangle = 1, \quad \text{for } \lambda_i = \sqrt{\frac{1}{\langle p_i, p_i \rangle}}.$$

For $i = 2, \cdots, m$, let

$$\lambda_i = \begin{cases} \frac{\langle p_1, p_i \rangle \langle p_1, p_i \rangle}{\langle p_i, p_i \rangle}, & \text{provided } \langle p_1, p_i \rangle \neq 0; \\ 1, & \text{otherwise}. \end{cases}$$

(26)

Then there exists a $\lambda_1 \in \text{Sp}(1)$ such that $\lambda_1 \lambda_2 \lambda_3 (p_2, p_3) \lambda_2 \lambda_1$ is a complex number with no-negative imaginary part if $\langle p_2, p_3 \rangle \neq 0$. Then $G(p_1 \lambda_1, p_2 \lambda_2 \lambda_1, \cdots, p_m \lambda_m \lambda_1)$ is the desired Gram matrix. In other words, $G(pD_1)$ is the desired Gram matrix with

$$D_1 = \text{diag}\left(\sqrt{\frac{1}{\langle p_1, p_1 \rangle}} \lambda_1, \sqrt{\frac{1}{\langle p_2, p_2 \rangle}} \lambda_2 \lambda_1, \cdots, \sqrt{\frac{1}{\langle p_m, p_m \rangle}} \lambda_m \lambda_1\right).$$

(27) \qed
Definition 5.1. The Gram matrix \( G \) as in Proposition 5.1 of the form

\[
G = (g_{ij}) = \begin{pmatrix}
1 & g_{12} & g_{13} & \cdots & g_{1m} \\
 g_{12} & 1 & g_{23} & \cdots & g_{2m} \\
 g_{13} & g_{23} & 1 & \cdots & g_{3m} \\
 g_{14} & g_{24} & g_{34} & \cdots & g_{4m} \\
 \vdots & \vdots & \vdots & \ddots & \vdots \\
 g_{1m} & g_{2m} & g_{3m} & \cdots & 1
\end{pmatrix}
\tag{28}
\]

is called the 1-normalized Gram matrix.

The following result can be shown similarly as Proposition 3.2.

Theorem 5.1. ([14, Proposition 3.2]) Let \( G = (g_{ij}) \) be a Hermitian \( m \times m \)-matrix, \( m > 2 \) with

\[
g_{ii} = 1, \quad i = 1, \ldots, m, \quad g_{1j} \geq 0, \quad j = 2, \ldots, m.
\]

Let \( i(G) = (n_+, n_-, n_0) \). Then \( G \) is a 1-normalized Gram matrix associated with an \( m \)-tuple of pairwise distinct points in \( \mathbb{P}(V_+) \) if and only if

\[
1 \leq n_+ + n_- \leq n + 1, \quad n_+ \leq n, \quad n_- \leq 1, \quad n_+ + n_- + n_0 = m.
\]

Remark 5.1. The number 1 of 1-normalized Gram matrix is equivoke with meaning that we normalize the Gram matrix in the view point standing in our ordered position 1, as well as with the meaning that we normalize the points in \( \mathbb{P}(V_+) \) with the properties \( \langle p_i, p_i \rangle = 1 \). It specifies the entries in row 1 (together column 1) and leaves entries in other rows ambiguity (even in the complex case). This phenomenon motivates the development of block-normalized algorithm. By the content in Section 2.2, we can state similar Theorem 5.1 for other normalized form of Gram matrix because of the invariability of (29). So we can focus on constructing of the parameter space in the sequence.

5.2 The structure of Gram matrices of points on \( \mathbb{P}(V_+) \)

In what follows, we assume that \( G(p) \) is already a 1-normalized Gram matrix. The following proposition may be regarded as a generalization of Theorem 4.1 (ii)

Proposition 5.2. Let \( p = (p_1, \cdots, p_t) \) be a \( t \)-tuple of pairwise distinct points in \( \mathbb{P}(V_+) \) satisfying

\[
\langle p_i, p_j \rangle = 1, \quad i, j = 1, \cdots t
\]

and \( V = \text{span}\{p_1, \cdots, p_t\} \). Then there exists a unique fibre \( z \mathbb{H} \in V_0 \) such that

\[
V \subset \mathbb{H}^+, \quad V \cap V_0 = z \mathbb{H}, \quad V \cap V_- = \emptyset.
\]

In fact

\[
z = p_2 - p_1, \quad V = \text{span}\{p_1, p_2\} = \text{span}\{z, p_1\}.
\]

Proof. Let \( u = p_1 \lambda_1 + p_2 \lambda_2 \in V \). Then

\[
\langle u, u \rangle = |\lambda_1|^2 + |\lambda_2|^2 + 2\Re(\lambda_2 \lambda_1) \geq 0.
\]

Note that \( \langle u, u \rangle = 0 \) if and only if \( \lambda_1 = -\lambda_2 \). Hence \( (p_2 - p_1)\mathbb{H} \) is the unique fibre of the intersection \( \text{span}\{p_1, p_2\} \cap V_0 \) and \( \text{span}\{p_1, p_2\} \cap V_- = \emptyset \). Noting that \( \langle p_i - p_j, p_i - p_j \rangle = 0 \) and \( \langle p_i - p_j, p_2 - p_1 \rangle = 0 \).
for \(i \neq j\), by Proposition 2.4 we have \((p_2 - p_1)H = (p_1 - p_2)H\). Since \(\langle p_i, (p_1 - p_2) \rangle = 0, i = 1, \ldots, t\), we have \(V \subset (p_2 - p_1)\perp\). It follows from \(p_i - p_1 \in (p_2 - p_1)H\) that there exist \(\lambda_i\) such that
\[
p_i = p_1 + (p_2 - p_1)\lambda_i = p_2\lambda_i + p_1(1 - \lambda_i), i = 1, \ldots, t.
\]
This implies that
\[
V = \text{span}\{p_1, p_2\} = \text{span}\{z, p_1\}
\]
and therefore \(V \cap V_0 = zH, V \cap V_\perp = \emptyset\).

The information of \(\lambda_i\) disappears in the sub Gram matrix \(G((p_1, \ldots, p_t))\). Moreover, such information can not be rebuilt through the relationships with other points in some situations. This implies that the Gram matrix loses the configuration information of such a \(t\)-tuple. We provide the following explicit example in ball model to illustrate this phenomenon. We remind that Cunha et al. provided a proof of similar example involving the fixed point theory of complex hyperbolic isometries in [14, Section 5].

**Example 5.1.** Let \(z = (1, 0, 1)^T \in V_0\) and \(p_1 = (0, 1, 0) \in V^+\). Let \(p_i = p_1 + iz, i = 2, 3\). Then
\[
G((p_1, p_2, p_3)) = G((p_3, p_2, p_1)) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.
\]
We claim that \((P(p_1), P(p_2), P(p_3))\) and \((P(p_3), P(p_2), P(p_1))\) are not \(\text{PSp}(2, 1)\)-congruent.

**Proof of the Claim.** Suppose that the two triples above are \(\text{PSp}(2, 1)\)-congruent. Then there exist a \(g \in \text{Sp}(2, 1)\) such that
\[
gp_1 = p_3\lambda_1, gp_2 = p_2\lambda_2, gp_3 = p_1\lambda_3.
\]
It follows from
\[
\langle gp_i, gp_i \rangle = \langle gp_i, gp_j \rangle = \langle p_i, p_i \rangle = 1
\]
that \(\lambda_i \in \text{Sp}(1)\) and \(\tilde{\lambda}_j \lambda_i = 1\), and therefore \(\lambda_1 = \lambda_2 = \lambda_3 := \lambda\). Hence
\[
g^2z = g(p_2 - p_1) = (p_2 - p_3)\lambda = -z\lambda,
\]
which contradicts
\[
gz = g(p_3 - p_2) = (p_1 - p_2)\lambda = -z2\lambda.
\]

If \(V\) is parabolic, by Proposition 5.2 we can refine Theorem 2.2 as follows.

**Proposition 5.3.** Let \(p = (p_1, \ldots, p_m) \in \mathbb{H}_{n+1,m}\), \(V = \text{span}\{p_1, \ldots, p_m\}\) and
\[
\dim_q V = k + 1, \ i(G(p)) = i(p^*Jp) = (k, m - k).
\]
Then \(S(m) = \{1, \ldots, m\}\) has a partition:
\[
S_i = \{s_{i1}, \ldots, s_{it_i}\}, s_{i1} < \cdots < s_{it_i}, i = 1, \ldots, k
\]
with the properties
\[
S(m) = \bigcup_{i=1}^k S_i; \langle p_{s_{it}}, p_{s_{jd}} \rangle = 1, 1 \leq l, d \leq t_i; \langle p_{s_{it}}, p_{s_{jd}} \rangle = 0, i \neq j
\]
and in each
\[ p_{S_i} := (p_{s_{i1}}, \ldots, p_{s_{it}}) \]
we can not partition likewise as in (33).

There exists a common \( z_0 \in V_0 \) such that \( p \in z_0^\perp \) and
\[ p_{s_{it}} = p_{s_{i1}} + z_0 \lambda_{it}, 1 < l \leq \text{Card}(S_i), i = 1, \ldots, k, \]  
(33)
where \( \text{Card}(S_i) \) is the cardinality of \( S_i \). We define
\[ V_i = \text{span}\{p_{s_{i1}}, \ldots, p_{s_{iti}}\} = \text{span}\{p_{s_{i1}}, z_0\}, i = 1, \ldots, k. \]  
(34)

If \( V \) is not parabolic, we can refine Theorem 2.2 as follows.

**Proposition 5.4.** Let \( p = (p_1, \ldots, p_m) \in \mathbb{H}_{n+1,m}, V = \text{span}\{p_1, \ldots, p_m\} \) and
\[ \dim_q V = k + 1, \ i(G(p)) = i(p^*Jp) = (k,1,m-k-1) \text{ or } (k+1,0,m-k-1). \]

Then \( S(m) = \{1, \ldots, m\} \) has a partition:
\[ S_i = \{s_{i1}, \ldots, s_{iti}\}, s_{i1} < \cdots < s_{iti}, i = 1, \ldots, s \]  
(35)
with the properties
\[ S(m) = \bigcup_{i=1}^s S_i; \langle p_{s_{id}}, p_{s_{jd}} \rangle = 0, i \neq j \]  
(36)
and in each \( p_{S_i} := (p_{s_{i1}}, \ldots, p_{s_{iti}}) \) we can not partition likewise as above.

It is helpful to keep in mind that there are no relationships among the blocked-entries corresponding to each components \( p_{S_i} \) in the diagonal matrix \( D \) in (4). This is the motivation of refinement of Theorem 2.2. Furthermore, when \( V \) is not parabolic, we still need to partition the components \( S_i \) in some situations.

## 6 Moduli space on \( \mathbb{P}(V_+) \) of case \( m \geq 3 \): non regular cases

We will work on the Siegel domain in this section. We will construct invariants which describe the \( \text{PSp}(n,1) \)-congruence classes when \( V \) is parabolic.

We first recall the following fact of isometries in \( \text{Sp}(n,1) \) fixing \( \infty \).

**Lemma 6.1.** (c.f. [11, Lemma 3.3.1]) Let \( z_\infty = (1,0,\cdots,0,0)^T, \mathbb{P}(z_\infty) = \infty \) and
\[ G_\infty = \{g \in \text{Sp}(n,1) : g(\infty) = \infty\}. \]

Then \( g \in G_\infty \) is of the form
\[ g = \begin{pmatrix} \lambda & \gamma^* & s \\ 0 & U & \beta \\ 0 & 0 & \mu \end{pmatrix}, \]
(37)
where \( \lambda, \mu, s \in \mathbb{H}, \beta, \gamma \in \mathbb{H}^{n-1}, U \in \text{Sp}(n-1), |\mu \bar{\gamma}| = 1, \ \Re(\bar{s} \mu) = -\frac{1}{2} |\beta|^2, \beta = -U \gamma \mu. \)
Let \( p = (p_1, \cdots, p_m) \) and \( q = (q_1, \cdots, q_m) \) be two ordered \( m \)-tuples of pairwise distinct points in \( \mathbb{P}(V) \) such that \( V(p) \) and \( V(q) \) are parabolic. Observe that if \( p \) and \( q \) are \( \text{PSp}(n, 1) \)-congruent then they have the same structure given by Proposition 5.3. Since \( \text{Sp}(n, 1) \) acts doubly transitively on \( \partial \mathbb{H}^n \), we can further assume that \( p, q \in \mathbb{Z}_\infty^+ \). As showed by Example 5.1 besides the information of structure, other conditions are needed for \( p, q \) being \( \text{PSp}(n, 1) \)-congruent.

In what follows, we assume that \( m \geq 3 \), \( V(p) = \text{span}\{p_1, \cdots, p_m\} \) is parabolic and \( V(p) \subset \mathbb{Z}_\infty^+ \). It is obvious that 
\[
\mathbb{Z}_\infty^+ = (z_1, \cdots, z_n, 0)^T := (z_1, \alpha^T, 0)^T.
\]
Therefore the action of \( g \in G_\infty \) on \( \mathbb{Z}_\infty^+ \) can be expressed by 
\[
g : \begin{pmatrix} z_1 \\ \alpha \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} \lambda z_1 + \gamma^* \alpha \\ U \alpha \\ 0 \end{pmatrix}.
\]
The restriction of the Hermitian form \( \langle , \rangle \) on \( \mathbb{Z}_\infty^+ \) is the usual inner product on \( \mathbb{H}^{n-1} \), i.e., 
\[
\langle (k_1, \alpha_1, 0)^T, (k_2, \alpha_2, 0)^T \rangle = \alpha_2^* \alpha_1.
\]
For \( g \) of the form (37), we define the map 
\[
\Pi : g \in G_\infty \rightarrow \tilde{g} = \begin{pmatrix} \lambda & \gamma^* \\ 0 & U \end{pmatrix} \in \tilde{G}_\infty.
\]
Then \( \Pi \) is a homomorphism with 
\[
\ker(\Pi) = \left\{ \begin{pmatrix} 1 & 0 & s \\ 0 & I_{n-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ with } \Re(s) = 0 \right\}
\]
and its homomorphic image \( \tilde{G}_\infty = \Pi(G_\infty) \) is a subgroup of \( \text{GL}(n, \mathbb{H}) \). The action of \( G_\infty \) on \( \mathbb{Z}_\infty^+ \) can be expressed by the projection action of \( \tilde{G}_\infty \) on \( \mathbb{H}^{n-1} = (z_1, \alpha^T)^T \).

Noting Proposition 5.3 and \( G(p) \) being a 1-normalized Gram matrix, we have 
\[
\tilde{p}_{s_{il}} = (k_{il}, \alpha_i^T)^T, \text{ for } s_{il} \in S_i
\]
and 
\[
\alpha_i^* \alpha_i = 1, 1 \leq i \leq k.
\]
Therefore there exists a \( U \in \text{Sp}(n-1) \) such that \( g = \text{diag}(1, U, 1) \in \text{Sp}(n, 1) \) satisfying 
\[
U(\alpha_1, \cdots, \alpha_k) = (e_1, \cdots, e_k),
\]
where \( e_i, 1 \leq i \leq k \) are \( k \) vectors in the standard basis of \( \mathbb{H}^{n-1} \). Therefore we may further reformulate (39) as 
\[
\tilde{p}_{s_{il}} = (k_{il}, e_i^T)^T, \text{ for } s_{il} \in S_i.
\]
In order to parameterize the moduli space, we introduce the following map \( \phi \) to give the corresponding coordinates in \( \mathbb{H} \cup \infty \) for vectors in \( V_l = \text{span}\{p_{s_{il}}, \mathbb{Z}_\infty\} \):
\[
\phi(z_\infty) = \infty, \phi(\tilde{p}_{s_{il}}) = k_{il}, 1 \leq l \leq t_i; 1 \leq i \leq k.
\]
Let \( h = \begin{pmatrix} \lambda & \gamma^* \\ 0 & U \end{pmatrix} \) with \( U(e_1, \ldots, e_k) = (e_1, \ldots, e_k) \) and \( \gamma = (c_1, \ldots, c_{n-1})^T \). Note that
\[
\begin{pmatrix} \lambda & \gamma^* \\ 0 & U \end{pmatrix} \begin{pmatrix} k_{il} \\ e_i \end{pmatrix} = \begin{pmatrix} \lambda k_{il} + \gamma^* e_i \\ e_i \end{pmatrix} = \begin{pmatrix} \lambda k_{i1} + c_i \\ e_i \end{pmatrix}.
\]
This means the restriction of \( h \) in \( V_i \) is
\[
h_i : k_{il} \to \lambda k_{i1} + c_i, 1 \leq i \leq k.
\]

The above treatment can be thought of as introducing the inhomogeneous coordinates in each \( V_i \). Form this point of view, the restriction of an element \( \tilde{g} \) of form \((38)\) to \( V_i \) is a quaternionic Möbius transformation in \( \Gamma_\infty \), the isotropy group at \( \infty \) in \( \text{PS}_\Delta L(2, \mathbb{H}) \) \([7]\).

Summarizing the above descriptions, we have so far defined a map
\[
\Pi_i : g \in G_\infty \to h_i = \begin{pmatrix} \lambda & c_i \\ 0 & 1 \end{pmatrix} \in \Gamma_\infty
\]
and the action of \( g \) on \( \mathbb{H}_\infty^+ \) is inherited by the actions of \( h_i \) on \( V_i \), which is identified with \( \mathbb{H} \).

Observe that the coordinates defined by \((42)\) contain the information of \( \lambda_{il} \) in \((33)\). To distinguish between \( \text{PSp}(n, 1) \)-congruence classes of \( m \)-tuples in degenerate case is the same as distinguishing the \( h_i \)-congruence classes in \( V_i \) for all \( i \). For this purpose, we need to introduce new geometric invariants which are invariant under the action of \( h_i \).

**Definition 6.1.** ([2, Definition 4.2]) The quaternionic cross-ratio of four points \( z_1, z_2, z_3, z_4 \in \mathbb{H} \cup \infty \) is defined as
\[
[z_1, z_2, z_3, z_4] = (z_1 - z_3)(z_1 - z_4)^{-1}(z_2 - z_4)(z_2 - z_3)^{-1}.
\]

**Lemma 6.2.** ([2, Proposition 4.1]) Given three distinct \( z_1, z_2, z_3 \in \mathbb{H} \), the element \( f \in \text{PS}_\Delta L(2, \mathbb{H}) \) defined by
\[
f(z) = (z_3 - z_2)(z_3 - z_1)^{-1}(z - z_1)(z - z_2)^{-1}
\]
maps \( z_1 \) to \( 0 \), \( z_2 \) to \( \infty \) and \( z_3 \) to \( 1 \). Moreover, all elements \( f \in \text{PS}_\Delta L(2, \mathbb{H}) \) with the same property are of the form:
\[
\lambda I_2 \circ f(z) = \lambda f(z)\lambda^{-1}
\]
with \( \lambda \in \mathbb{H} \setminus \{0\} \).

It follows from \([7]\) that an element \( f \in \text{PS}_\Delta L(2, \mathbb{H}) \) fixing \( 0, 1, \infty \) is of the form \( f = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \lambda I_2 \).

Based on this observation and \([2, Proposition 4.4]\), the cross-ratios enjoy the following properties.

**Lemma 6.3.** (1) For any \( z \in \mathbb{H} \) such that \( z \neq 0 \) and \( z \neq 1, [z, 1, 0, \infty] = z \);

(2) Given distinct points \( z_1, z_2, z_3, z \in \overline{\mathbb{H}}, \)
\[
[f(z), f(z_3), f(z_2), f(z_1)] = \lambda_f[z, z_3, z_2, z_1] \lambda_f^{-1},
\]
where \( \lambda_f \) is a quaternion solely depending on \( f \in \text{PS}_\Delta L(2, \mathbb{H}) \). In particular, for \( h_i \) given by \((44)\), we have
\[
[h_i(z), h_i(z_3), h_i(z_2), h_i(z_1)] = \lambda[z, z_3, z_2, z_1] \lambda^{-1}.
\]
Definition 6.2. We introduce the following geometric invariants in component $p_{s_i} = (p_{s_{i1}}, \cdots, p_{s_{it_i}})$ when $\text{Card}(S_i) \geq 3$:

$$
\chi(\tilde{p}_{s_{i1}}, \tilde{p}_{s_{i2}}, \tilde{p}_{s_{i3}}) = (k_{it} - k_{it})(k_{ij} - k_{it})^{-1} = [k_{it}, k_{ij}, k_{it}, \infty].
$$

We mention that since the points $\tilde{p}_{s_{i1}}, \tilde{p}_{s_{i2}}, \tilde{p}_{s_{i3}}$ are all distinct, $\chi$ is finite and $\chi \neq 0, 1$. Therefore

$$
\chi \in \mathbb{H} - \{0, 1\}.
$$

To sort out the conditions for $p$ and $q$ being $\text{PSp}(n, 1)$-congruent, w.o.l.g., we may assume that $p, q \in \mathbb{Z}_{\infty}^3$ have the same structure given by Proposition 5.3. We denote the corresponding coordinates of $\tilde{q}_{s_i}$ by

$$
\tilde{q}_{s_{il}} = (w_{il}, e_{il}^T), \text{ for } s_{il} \in S_i
$$

and compute the corresponding invariants of $q$ in the same manner as these of $p$.

We first obtain the necessary and sufficient condition of two triples being $\text{PSp}(n, 1)$-congruent directly.

Proposition 6.1. $\tilde{p}_i = (\tilde{p}_{s_{i1}}, \tilde{p}_{s_{i2}}, \tilde{p}_{s_{i3}})$ and $\tilde{q}_i = (\tilde{q}_{s_{i1}}, \tilde{q}_{s_{i2}}, \tilde{q}_{s_{i3}})$ are $\text{PSp}(n, 1)$-congruent if and only if there exists a $\lambda \in \mathbb{H} - \{0\}$ such that

$$
\chi(\tilde{p}_{s_{i1}}, \tilde{p}_{s_{i2}}, \tilde{p}_{s_{i3}}) = \lambda \chi(\tilde{q}_{s_{i1}}, \tilde{q}_{s_{i2}}, \tilde{q}_{s_{i3}})\lambda^{-1}.
$$

Proof. Assume that $\tilde{p}_i$ and $\tilde{q}_i$ are $\text{PSp}(n, 1)$-congruent. Let $p_i$ and $q_i$ be the corresponding triples in $\mathbb{H}^{n,1}$. Then there exists a $g \in G_{\infty}$ such that $g(p_{s_{i1}}, p_{s_{i2}}, p_{s_{i3}}) = (q_{s_{i1}}, q_{s_{i2}}, q_{s_{i3}})$. As before, we know that $\nu_1 = \nu_2 = \nu_3 := \nu$ and $|\nu| = 1$. This implies that $g\tilde{p}_i = \tilde{q}_i\nu$, i.e.,

$$
\lambda k_{it} + \gamma^*e_i = w_{il} \nu, U e_i = e_i \nu, l = 1, 2, 3.
$$

Therefore

$$
\lambda(k_{i2} - k_{i3}) = (w_{i2} - w_{i3})\nu, \lambda(k_{i1} - k_{i3}) = (w_{i1} - w_{i3})\nu.
$$

Hence

$$
\lambda(k_{i1} - k_{i3})(k_{i2} - k_{i3})^{-1}\lambda^{-1} = (w_{i1} - w_{i3})(w_{i2} - w_{i3})^{-1}.
$$

Conversely, suppose that $\chi(\tilde{p}_{s_{i1}}, \tilde{p}_{s_{i2}}, \tilde{p}_{s_{i3}}) \sim \chi(\tilde{q}_{s_{i1}}, \tilde{q}_{s_{i2}}, \tilde{q}_{s_{i3}})$. Then there exists a $\lambda$ such that

$$
\lambda(k_{i1} - k_{i3})(k_{i2} - k_{i3})^{-1}\lambda^{-1} = (w_{i1} - w_{i3})(w_{i2} - w_{i3})^{-1}.
$$

We may further require that $|\lambda| = \frac{|k_{i1} - k_{i3}|}{|w_{i1} - w_{i3}|}$. Let $\nu = (w_{i1} - w_{i3})^{-1}\lambda(k_{i1} - k_{i3})$. Then

$$
\lambda(k_{i1} - k_{i3}) = (w_{i1} - w_{i3})\nu, \lambda(k_{i2} - k_{i3}) = (w_{i2} - w_{i3})\nu.
$$

From the above two equalities, we have $\lambda(k_{i1} - k_{i2}) = (w_{i1} - w_{i2})\nu$. We can find a $\gamma \in \mathbb{H}^{n-1}$ and a $U \in \text{Sp}(n-1)$ satisfying

$$
\lambda k_{i1} + \gamma^*e_i = w_{i1} \nu, U e_i = e_i \nu.
$$

The above equalities also imply

$$
\lambda k_{i2} + \gamma^*e_i = w_{i2} \nu, \lambda k_{i3} + \gamma^*e_i = w_{i3} \nu.
$$

With $\lambda, \gamma, U$ above, we can construct a $g \in G_{\infty}$ of the form (37) satisfying

$$
g(p_{s_{i1}}, p_{s_{i2}}, p_{s_{i3}}) = (q_{s_{i1}}, q_{s_{i2}}, q_{s_{i3}}, \nu_3^3).
$$
Hence and 1 to $f,g \in \chi$. Observe that $\chi(p_1,p_2,p_3) = 3$ and $\chi(p_3,p_2,p_1) = 3/2$. Therefore $(P(p_1),P(p_2),P(p_3))$ and $(P(p_3),P(p_2),P(p_1))$ are not PSp(2,1)-congruent. For the case of more than three points, it is convenient to use the quaternionic cross-ratios.

**Proposition 6.2.** Let $p = (z_1, \ldots, z_m)$ and $q = (w_1, \ldots, w_m)$ be two ordered $m$-tuples of pairwise distinct points in $\mathbb{H}$, $m \geq 4$. Then $p$ and $q$ are congruent with respect to the diagonal action of $PS_\Delta L(2,\mathbb{H})$ if and only if there exists a $\lambda \in \mathbb{H} - \{0\}$ such that

$$[z_j, z_3, z_2, z_1] = \lambda[w_j, w_3, w_2, w_1] \lambda^{-1}, \forall 4 \leq j \leq m.$$  

(49)

**Proof.** If there is an $f \in PS_\Delta L(2,\mathbb{H})$ such that $f(z_j) = w_j, j = 1, \ldots, m$. Then by Lemma 6.3 the conditions of (49) hold.

Conversely, assume that

$$[z_j, z_3, z_2, z_1] = \lambda[w_j, w_3, w_2, w_1] \lambda^{-1}, \forall 4 \leq j \leq m.$$  

By Lemma 6.2 we can find $f, g \in PS_\Delta L(2,\mathbb{H})$ such that $f(z_3) = 1, f(z_2) = 0, f(z_1) = \infty$, and $g(w_3) = 1, g(w_2) = 0, g(w_1) = \infty$. It follows from Lemma 6.3 that

$$f(z_j) = [f(z_j), 1, 0, \infty] = [f(z_j), f(z_3), f(z_2), f(z_1)] = \lambda_j[z, z_3, z_2, z_1] \lambda^{-1}_f$$

and

$$g(w_j) = [g(w_j), 1, 0, \infty] = [g(w_j), g(w_3), g(w_2), g(w_1)] = \lambda_g[w, w_3, w_2, w_1] \lambda^{-1}_g.$$  

Therefore our assumption implies that

$$h(z_j) = g^{-1} \circ \lambda_g(\lambda_f \lambda)^{-1} I_2 \circ f(z_i) = w_i, i = 1, \ldots, m.$$  

Hence $p$ and $q$ are $PS_\Delta L(2,\mathbb{H})$-congruent.  

**Definition 6.3.** For $S_i = \{s_{i1}, \ldots, s_{ik}\}, i = 1, \ldots, k$ with Card($S_i$) > 3, we associate with $S_i$ the following geometric invariants:

$$\chi_{i0} = \chi(p_{s_{i1}}, p_{s_{i2}}, p_{s_{i3}}), \chi_{i1} = \chi(p_{s_{i1}}, p_{s_{i2}}, p_{s_{i4}}), \ldots, \chi_{i(t_i-3)} = \chi(p_{s_{i1}}, p_{s_{i2}}, p_{s_{i4}})$$

and

$$X_i(p) = (\chi_{i0}, \ldots, \chi_{i(t_i-3)}).$$

Let $X(p)$ be the vector whose components consisting of $X_i(p)$ above.

Taking $z_1 = w_1 = \infty$ in Proposition 6.2, we get the following proposition.

**Proposition 6.3.** Let $p_{S_i}$ and $q_{S_i}$ belong to $z_{\infty}^\perp$ with the same Gram matrix whose entries are all equal to 1. Then $p_{S_i}$ and $q_{S_i}$ are PSp(n,1)-congruent if and only if there exists a $\lambda \in \mathbb{H} - \{0\}$ such that

$$X_i(p) = \lambda X_i(q) \lambda^{-1}.$$  

We still need to generalize the above result to the case of $G(p)$ and $G(q)$ having stratum structure.
Proposition 6.4. Let \( p = (p_1, \ldots, p_m) \) and \( q = (q_1, \ldots, q_m) \) be two \( m \)-tuples of pairwise distinct positive points of non regular case. We also assume that \( p \) and \( q \) have the same structure given by Proposition 5.3 with the property \( \text{Card}(S_i) \geq 3 \) for some \( i \). Then \( p \) and \( q \) are \( \text{PSp}(n, 1) \)-congruent if only if there exists a \( \lambda \in \mathbb{H} - \{0\} \) such that

\[
X(p) = \lambda X(q)\lambda^{-1}.
\]

Proof. W.o.l.g, we assume that \( p, q \in \mathbb{Z}_\infty^m \). If there is an \( f \in \text{PSp}(n, 1) \) such that \( f(p_i) = q_i, j = 1, \ldots, m \), then \( f \in G_\infty \) is of the form

\[
f = \begin{pmatrix}
\lambda & \gamma^* & * \\
0 & U & * \\
0 & 0 & *
\end{pmatrix}
\]

and \( p, q \) must have the same structure given by Proposition 5.3. Here and in what follows, \( * \) stands for an arbitrary entry satisfying constraint that the corresponding matrix \( f \) belongs to \( \text{Sp}(n, 1) \). By our normalization, we have

\[
U(e_1, \ldots, e_k) = (e_1, \ldots, e_k)
\]

and in each block of index \( S_i \), we also have

\[
\lambda k_{il} + \gamma^* e_i = w_{il}, 1 \leq l \leq \text{Card}(S_i).
\]

Therefore we have \( X(p) = \lambda X(q)\lambda^{-1} \).

Conversely, suppose that \( X(p) = \lambda X(q)\lambda^{-1} \). By Proposition 6.3 for two specific blocks \( p_{S_i} \) and \( q_{S_i} \), we can construct an element \( f_i \in \text{Sp}(n, 1) \) of the form

\[
f_i = \begin{pmatrix}
\lambda_i & \gamma_i^* & * \\
0 & U_i & * \\
0 & 0 & *
\end{pmatrix}
\]

such that

\[
U_i e_i = e_i, \lambda_i k_{il} + \gamma_i^* e_i = w_{il}, 1 \leq l \leq \text{Card}(S_i).
\]

It is a pleasant surprise that we can adjust \( f_i \) to a suitable transformation which works for \( p \) wholly as follows. First, it follows from Lemma 6.3 that \( \lambda_i = \lambda \). Let \( U \in \text{Sp}(n - 1) \) having the property \( U(e_1, \ldots, e_k) = (e_1, \ldots, e_k) \). It is obvious that

\[
h_i = \begin{pmatrix}
\lambda & \gamma_i^* & * \\
0 & U & * \\
0 & 0 & *
\end{pmatrix}
\]

also maps \( p_{S_i} \) to \( q_{S_i} \). Note that \( k \leq n - 1 \). Let

\[
\gamma = (\gamma_1^* e_1, \ldots, \gamma_k^* e_k, *, \ldots, *)^T
\]

and

\[
h = \begin{pmatrix}
\lambda & \gamma^* & * \\
0 & U & * \\
0 & 0 & *
\end{pmatrix}
\]

Then one has the equations (52), and therefore \( p \) and \( q \) are congruent up to \( h \).

By the above proof and Section 4.2, we have the following result which means that structures of Gram matrices determine their congruent classes when \( \text{Card}(S_i) \leq 2 \) for all \( i \).
Proposition 6.5. Let $p = (p_1, \ldots, p_m)$ and $q = (q_1, \ldots, q_m)$ be two $m$-tuples of pairwise distinct positive points of non regular case with the same structure given by Proposition 5.3 and $\text{Card}(S_i) \leq 2$, $i = 1, \ldots, k$. Then $p$ and $q$ are $\text{PSp}(n, 1)$-congruent.

In order to describe the parameter space, we need the following result.

Proposition 6.6. The coordinates of $O_{X(p)}$ given by rotation-normalized algorithm is well defined.

Proof. If both $h_1, h_2 \in \text{PSp}(n, 1)$ map $V$ to a subspace of $\mathbb{Z}_+^k$. Then the coordinates in (12) may be different from each other, which implies that $X(p)$ in Definition 6.3 is dependent on the map $\phi$ in (12). However, since $h_1^{-1}h_2 \in G_\infty$, Lemma 6.3 and Proposition 6.4 imply that the coordinates of $O_{X(p)}$ given by rotation-normalized algorithm is well defined.

Summarizing the previous results, we obtain the main result of this section.

Theorem 6.1. Let $p = (p_1, \ldots, p_m)$ be an $m$-tuple of pairwise distinct positive points given by Proposition 5.3. Then the $\text{PSp}(n, 1)$-congruence class of $p$ is determined uniquely by the partition structure of $S(m) = \bigcup_{i=1}^k S_i$ and the coordinates of $O_{X(p)}$ given by rotation-normalized algorithm.

Therefore the moduli space can be described as follows.

Theorem 6.2. The moduli space of $p = (p_1, \ldots, p_m)$ given by Proposition 5.3 can be identified with the set $M_1 \times \cdots \times M_k$, where $M_1 \times \cdots \times M_k$ are the coordinates of $O_{X(p)}$ given by rotation-normalized algorithm.

7 Moduli space on $\mathbb{P}(V_+)$ of case $m \geq 3$: regular cases

In this section, we describe the moduli space of configurations of quaternionic $(n-1)$-dimensional submanifolds when $V$ is not parabolic in conceptual style. The basic idea is to find a partition of $S(m) = \{1, \ldots, m\}$ to perform rotation-normalized algorithm in each block.

We begin with 1-normalized matrix of $p$. Proposition 5.4 roughly shows that we can treat the mutually orthogonal blocks $p_{S_i} = (p_{s_{i1}}, \ldots, p_{s_{it}}), i = 1, \ldots, s$ separately. Equivalently, we can perform the rotation-normalized algorithm separately. This is the structure of Gram matrix at top level. For each block $p_{S_i}$, there may still exist $0$s in $G(p_{S_i})$. We may need to partition $S_i$ into more small blocks to perform rotation-normalized algorithm. We call such a partition process, together with similar 1-normalized process in each small blocks, block-normalized algorithm. The output of block-normalized algorithm is a special kind of Gram matrix, which is still not unique and can be viewed as an equivalent class. We still need to apply rotation-normalized algorithm to get the parameters.

We describe block-normalized algorithm conceptually as follows.

Block-normalized algorithm:

Step 1: Let $O_{il}$ be the number of entries being zero in $il$th row of $G(p_{S_i})$ and record the set of columns of these entries being nonzero as $P_{il}$. Let $n_i = \min\{O_{i1}, \ldots, O_{it}\}$ and $K_i$ the set of indices $il$ such that $O_{il} = n_i$. Let $c_{i1}$ be the smallest integer in $K_i$ and denote the corresponding $P_{il}$ of $c_{i1}$ as $S_{i1}$. In other words, $c_{i1}$ is the smallest index in $S_i = \{s_{i1}, \ldots, s_{it}\}$ such that the cardinality of nonzero entries in the $c_{i1}$th row of $G(S_i)$ is the largest among those of the others; the set of columns of nonzero entries is recorded as $S_{i1}$. It is obvious that $c_{i1} \in S_{i1}$.
Step 2: Repeating the process in Step 1 for the remainder of \( S_i - S_{i1} \), we obtain \( c_{i2} \) and \( S_{i2} \). It is obvious that we can continue this process only finite steps. We denote by \( \tau_i \) the number of steps and record the corresponding numbers in each step as \( c_{ij} \) and \( S_{ij} \) for \( 1 \leq j \leq \tau_i \). Then we have

\[
S_i = \bigcup_{j=1}^{\tau_i} S_{ij}.
\]

Step 3: In each subindex set \( S_{ij} \), we perform the \( c_{ij} \)-normalized process to \( G(p_{S_{ij}}) \). We denote such result of the sub Gram matrix as \( G_b(p_{S_{ij}}) \). In other words, the entries of \( G_b(p_{S_{ij}}) \) have the following properties:

\[
g_{tt} = 1, \quad g_{c_{ij}t} \geq 0, \quad g_{tc_{ij}} \geq 0, \quad t \in S_{ij}.
\]

As in (27) of Section 5, we record the corresponding normalized sub-diagonal matrix as \( D_{ij} \). That is

\[
G_b(p_{S_{ij}}) = G(p_{S_{ij}}D_{ij}).
\]

Step 4: Let

\[
D_i = \text{diag}(D_{i1}, \cdots, D_{i\tau_i}), \quad D_b = \text{diag}(D_1, \cdots, D_s).
\]

We define

\[
G_b(p_{S_i}) = G(p_{S_i}D_i)
\]

and

\[
G_b(p) = G(pD_b).
\]

**Definition 7.1.** The Gram matrix \( G_b(p) \) obtained by the above block-normalized algorithm is called the block-normalized matrix of \( G(p) \).

We mention that our strategy in block-normalized algorithm is from parts to entirety. We deal with the diagonal blocks separately. In this scale \( p_{S_i} \) and \( p_{S_j} \) are totally independent. In each block \( p_{S_i} \), all processes are explicitly recorded by the corresponding sub-diagonal matrices \( D_i = \text{diag}(D_{i1}, \cdots, D_{i\tau_i}) \).

In this way the entries \( \langle p_{s_{ii}}, p_{s_{id}} \rangle \) in off-diagonal blocks of \( p_{S_i} \) are all determined definitely by \( D_i \). We describe the structure of \( G_b(p) \) in the following proposition in more details.

**Proposition 7.1.** The block-normalized Gram matrix \( G_b(p) \) has the following characteristics.

1. If we view the block-normalized Gram matrix \( G_b(p) \) in its permuted position with index \( S_i \), then \( G_b(p) \) consists of blocks submatrix \( G_b(p_{S_i}) \), the entries of the corresponding off-diagonal blocks matrices are zero (see (55) in Proposition 5.4).

2. In the \( c_{i1} \)-th row (and column) of submatrix \( G_b(p_{S_i}) \), the first \( \text{Card}(S_{i1}) \) entries are nonzero real numbers, the others are zeros (see Step 2 of block-normalized algorithm).

3. In the \( c_{i2} \)-th row (and column) of submatrix \( G_b(p_{S_i}) \), the entries with index between \( \text{Card}(S_{i1}) + 1 \) and \( \text{Card}(S_{i1}) + \text{Card}(S_{i2}) \) are nonzero real numbers, the entries with index bigger than \( \text{Card}(S_{i1}) + \text{Card}(S_{i2}) \) are zeros; the entries in the \( c_{ij} \)-th row (and column) of submatrix \( G_b(p_{S_i}) \) can be described similarly when \( j = 3, \cdots, \tau_i \).

4. \( G_b(p_{S_i}) \) can not be block diagonal according to our partition in Proposition 5.4.

Similarly to Lemma 3.1, we have the following result.
Lemma 7.1. Suppose that $G(q)$ is a block-normalized Gram matrix $G_b(p)$ for $p = (p_1, \ldots, p_m)$. Then $G(qD_r)$ is still a block-normalized Gram matrix with

$$D_r = \text{diag}(\mu_1, \ldots, \mu_m)$$

if only if every $\mu_t$ with $t \in S_{ij}$ is the same quaternion of modulus 1, i.e.,

$$\mu_t = \mu_{ij}, \forall t \in S_{ij}, \text{ where } \mu_{ij} \in \text{Sp}(1).$$

Summarizing the previous treatments, we have the following procedure.

Theorem 7.1. Let $p = (p_1, \ldots, p_m)$ be an $m$-tuple of pairwise distinct positive point given by Proposition 5.4. We can assign the PSp($n,1$)-congruence class of $p$ a coordinate as follows.

1. Obtain a block-normalized matrix $G(pD_b)$ by performing the block-normalized algorithm, where $D_b$ is given by (56).

2. Perform the rotation-normalized algorithm to each block $S_{ij}$ (as the case of $m$-tuple of $P(V_0)$ in Section 3). This is equivalent to choosing a specific $\mu_{ij} \in \text{Sp}(1)$. Combine them to the corresponding whole rotation normalized diagonal matrix $D_r$.

3. The independent entries of

$$G(pD_bD_r),$$

that is, all the entries above the diagonal entries, are the desired coordinate of the PSp($n,1$)-congruent class of $p$.

We now are ready to give a conceptual description of the parameter space $M(n,m)$ in regular case. We mimic conceptually the method used in Section 3.2 as follows.

The procedure of constructing parameter space:

For a partition $S$ of $S(m) = \{1, \ldots, m\}$ as

$$S_i = \{s_{i1}, \ldots, s_{i\tau_i}\}, s_{i1} < \cdots < s_{i\tau_i}, i = 1, \ldots, s$$

with sub partitions

$$S_i = \bigcup_{j=1}^{\tau_i} S_{ij},$$

Let $\text{Card}(S_{ij}) = \sigma_{ij}$. As in Section 3.2 we construct the parameter space $M(n,\sigma_{ij})$ of $S_{ij}$. Let

$$M(n, i) = M(n, \sigma_{i1}) \times \cdots \times M(n, \sigma_{i\tau_i}) \times C_i,$$

where the set of $C_i$ is the corresponding space of the off-diagonal sub-blocks. Let

$$M(n, m, S) = M(n, i) \times \cdots \times M(n, s).$$

The Hermitian matrix constructed from the entries of the parameter space $M(n, m, S)$ should subject to analogous constraints as those of Theorem 5.1. Then the union of the parameter spaces determined by all possible partitions

$$M(n, m) = \bigcup_{S} M(n, m, S)$$

is a parameter space of the configuration space $M(n, m; \mathbb{P}(V_+))$ when $V$ is not parabolic.

Therefore, the moduli space can be described as follows.

Theorem 7.2. The moduli space of $p = (p_1, \ldots, p_m)$ given by Proposition 5.4 can be identified with the set

$$M(n, m) = \bigcup_{S} M(n, m, S).$$
8 Quaternionic hyperbolic triangles

In this section, we will give a parameter space of quaternionic hyperbolic triangles. This section may be regarded as an application of somewhat conceptual results in previous sections in triangle groups, a current hot research topic since the seminal work of Goldman and Parker [19].

We will work on ball model and begin with some notations. Let \( p_t \) be the normalized polar vector of the quaternionic line \( l_t, t = 1, 2, 3 \). That is \( l_t \) is a quaternionic 1-dimensional submanifold corresponding to \( p_t \) with \( \langle p_t, p_t \rangle = 1 \).

**Definition 8.1.** A quaternionic hyperbolic triangle is a triple \((l_1, l_2, l_3)\) of quaternionic lines in quaternionic hyperbolic space \( \mathbb{H}^2 \).

For pair of quaternionic lines \( l_{t-1} \) and \( l_{t+1} \), let \( r_t = |\langle p_{t-1}, p_{t+1} \rangle| \), where the indices are taken mod 3. By Theorem 4.1, the number \( r_t < 1, r_t = 1 \) and \( r_t > 1 \) means that the quaternionic lines \( l_{t-k} \) intersects \( l_{t+1} \) intersect at \( \mathbb{H}^2 \), with angle \( \varphi_t = \arccos r_t \), intersect at \( \partial \mathbb{H}^2 \), are ultra-parallel with the distance \( \ell_t = 2 \cosh^{-1} r_t \), respectively.

We define that the following quaternion for a triple of points \( p = (p_1, p_2, p_3) \) in \( V_+ \):

\[
\langle p_1, p_2, p_3 \rangle = \langle p_2, p_1 \rangle / \langle p_3, p_2 \rangle / \langle p_1, p_3 \rangle.
\]

**Definition 8.2.** The angular invariant of the quaternionic hyperbolic triangle \((l_1, l_2, l_3)\) is defined by

\[
\mathcal{A}(p) = \mathcal{A}(p_1, p_2, p_3) = \begin{cases} 
\arccos \frac{\Re(\langle p_1, p_2, p_3 \rangle)}{\|p_1, p_2, p_3\|}, & \text{provided } \langle p_1, p_2, p_3 \rangle \neq 0; \\
\pi/2, & \text{otherwise.}
\end{cases}
\]

When \( \mathcal{A}(p) = \pi/2 \), we may have two cases: \( \langle p_1, p_2, p_3 \rangle = 0 \) or \( \Re(\langle p_1, p_2, p_3 \rangle) = 0 \). For example, \( \langle p_1, p_2, p_3 \rangle = i \) when \( p_1 = (0, 1, 0)^T, p_2 = (1, 1, 1)^T, p_3 = (i, 1, 1)^T \).

It is obvious that

\[
\mathcal{A}(p) = \mathcal{A}(p_1, p_2, p_3) = \mathcal{A}(fp_1, fp_2, fp_3) \in [0, \pi], \forall f \in \text{Sp}(2, 1).
\]

It is easy to verify the following proposition.

**Proposition 8.1.** Let \( p_1, p_2, p_3 \in V_+ \), let \( \sigma \) be a permutation of 1, 2, 3, let \( \lambda_i \in \mathbb{H} - \{0\} \). Then

\[
\mathcal{A}(p_1, p_2, p_3) = \mathcal{A}(p_{\sigma(1)}, p_{\sigma(2)}, p_{\sigma(3)}) = \mathcal{A}(p_1 \lambda_1, p_2 \lambda_2, p_3 \lambda_3).
\]

In this specific case, as in the 1-normalized process, we have the following proposition.

**Proposition 8.2.** Let \( p = (p_1, p_2, p_3) \) be a triple of points in \( V_+ \). Then the equivalence class of Gram matrices associated to \( p \) contains a unique matrix \( G = (g_{ij}) \) with \( g_{ii} = 1, g_{12} = r_1 \geq 0, g_{13} = r_2 \geq 0 \) and \( g_{23} = r_1 (\cos \mathcal{A} + \sin \mathcal{A} i) \in \mathbb{C} \), where \( \sin \mathcal{A} \geq 0 \).

**Proof.** By appropriate rescaling we may assume that \( p_i \) are normalized vectors, i.e., \( g_{ii} = 1, i = 1, 2, 3 \). For \( i = 2, 3 \), let

\[
\lambda_i = \begin{cases} 
\frac{\langle p_i, p_i \rangle}{\|p_i, p_i\|}, & \text{provided } \langle p_i, p_i \rangle \neq 0; \\
1, & \text{otherwise.}
\end{cases}
\]

It is known that there exists a \( \lambda_1 \) of norm 1 such that \( \lambda_1 \lambda_3 \langle p_2, p_3 \rangle \lambda_2 \lambda_1 \) is a complex number with non-negative imaginary part. Then \( G(p_1 \lambda_1, p_2 \lambda_2 \lambda_1, p_3 \lambda_3 \lambda_1) \) is the desired Gram matrix. \( \square \)
The Gram matrix in Proposition 8.2 is called the normalized Gram matrix, which is of the form

\[ G = (g_{ij}) = \begin{pmatrix} 1 & r_3 & r_2 \\ r_3 & 1 & r_1 e^{i\lambda} \\ r_2 & r_1 e^{-i\lambda} & 1 \end{pmatrix}. \] (61)

We call such a quaternionic hyperbolic triangle a \((r_1, r_2, r_3; \mathbb{A})\)-triangle.

Let \(G(i, j)\) be the submatrix consisting of entries in row and column index with \(i, j\). It is easy to verify the following proposition.

**Proposition 8.3.** Let \(G\) be the normalized matrix of \((r_1, r_2, r_3; \mathbb{A})\) triangle.

1. Since \(i(G) \neq (3, 0, 0)\), \(r_1^2 + r_2^2 + r_3^2 \neq 0\).

2. \[ \det G(1, 2) = 1 - r_3^2, \quad \det G(1, 3) = 1 - r_2^2, \quad \det G(2, 3) = 1 - r_1^2 \] (62)

and

\[ \det G = 1 - (r_1^2 + r_2^2 + r_3^2) + 2r_1r_2r_3 \cos \lambda. \] (63)

Let \(V = \text{span}\{p_1, p_2, p_3\}\). We assume that \(p_1, p_2, p_3\) are pairwise distinct points in \(\mathbb{P}(V_+)\), therefore \(\dim_q(V) \geq 2\). We mention that beginning with three points in \(\mathbb{H}_R^2\) and then constructing the quaternionic lines, one may obtain that the corresponding polar vectors \(p_1, p_2, p_3\) which may be the same in the view point of \(\mathbb{P}(V_+)\). In this situation the three points lie in the closure of a common quaternionic line and \(\dim_q(V) = 1\), however this case is not so interesting [6].

**Proposition 8.4.** We enumerate the possibilities of the signatures \(i(G)\) corresponding to quaternionic hyperbolic triangle groups.

1. If \(V\) is parabolic then \(i(G) = (1, 0, 0)\), \(\dim_q(V) = 2\) and it corresponds to \((1, 1, 1; 0)\)-triangle.

2. If \(V\) is elliptic then \(i(G) = (2, 0, 0), \dim_q(V) = 2\).

3. If \(V\) is hyperbolic then \(i(G) = (1, 1, 0), \dim_q(V) = 2\) or \((2, 1, 0), \dim_q(V) = 3\).

It follows from Proposition 6.1 that the parameter space of \((1, 1, 1; 0)\)-triangle is \(\mathbb{C} - \{0, 1\}\).

By Proposition 8.4 and the properties of determinant of complex matrices, we have the following result.

**Theorem 8.1.** For any \(\mathbb{A} \in [0, \pi/2]\) there exists a quaternionic hyperbolic \((r_1, r_2, r_3; \mathbb{A})\)-triangle in \(\mathbb{H}^2_{\mathbb{H}}\) if and only if

\[ \det G = 1 - (r_1^2 + r_2^2 + r_3^2) + 2r_1r_2r_3 \cos \lambda \leq 0. \] (64)

Moreover, \(\det G = 0\) if and only if there exist \(\lambda_i, \in \mathbb{H}\) with \(\sum_{i=1}^3 |\lambda_i| = 0\) such that \(p_1\lambda_1 + p_2\lambda_2 + p_3\lambda_3 = 0\).

We need to replace \(\mathbb{A} \in [0, \pi/2]\) with \(\mathbb{A} \in [-\pi, \pi]\) in Theorem 8.1 for complex hyperbolic geometry. We refer to [28, 29] etc. for more details of the complex hyperbolic triangle groups. Cao and Huang [6] have addressed the discreteness of quaternionic ideal triangle groups, which corresponds to the quaternionic hyperbolic triangle of the types \((1, 1, 1; \mathbb{A})\). We mention that the angular invariant \(\mathbb{A}\) given by Definition 8.2 is different from that of [6]. It is of current interest to settle the problems of faithful and discrete presentations both on complex and quaternionic hyperbolic geometries.

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