Multilevel Particle Filters for the Non-Linear Filtering Problem in Continuous Time

BY AJAY JASRA\textsuperscript{1}, FANGYUAN YU\textsuperscript{2} & JEREMY HENG\textsuperscript{3}

\textsuperscript{1}Computer, Electrical and Mathematical Sciences and Engineering Division, King Abdullah University of Science and Technology, Thuwal, 23955, KSA. E-Mail: \texttt{ajay.jasra@kaust.edu.sa}

\textsuperscript{2}Department of Statistics & Applied Probability, National University of Singapore, Singapore, 117546, SG. E-Mail: \texttt{stafy@nus.edu.sg}

\textsuperscript{3}ESSEC Business School, Singapore, 139408, SG. E-Mail: \texttt{heng@essec.edu}

Abstract

In the following article we consider the numerical approximation of the non-linear filter in continuous-time, where the observations and signal follow diffusion processes. Given access to high-frequency, but discrete-time observations, we resort to a first order time discretization of the non-linear filter, followed by an Euler discretization of the signal dynamics. In order to approximate the associated discretized non-linear filter, one can use a particle filter (PF). Under assumptions, this can achieve a mean square error of $O(\epsilon^2)$, for $\epsilon > 0$ arbitrary, such that the associated cost is $O(\epsilon^{-4})$. We prove, under assumptions, that the multilevel particle filter (MLPF) of [15] can achieve a mean square error of $O(\epsilon^2)$, for cost $O(\epsilon^{-3})$. This is supported by numerical simulations in several examples.

**Key words:** Multilevel Monte Carlo, Particle Filters, Non-Linear Filtering.

1 Introduction

The non-linear filtering problem in continuous-time is found in many applications in finance, economics and engineering; see e.g. [1]. We consider the case where one seeks to filter an unobserved diffusion process (the signal) with access to an observation trajectory that is, in theory, continuous in time and following a diffusion process itself. The non-linear filter is the solution to the Kallianpur-Striebel formula (e.g. [1]) and typically has no analytical solution. This has lead to a substantial literature on the numerical solution of the filtering problem; see for instance [1, 7].

In practice, one has access to very high-frequency observations, but not an entire trajectory and this often means one has to time discretize the functionals associated to the path of the observation and signal. This latter task can be achieved by using the approach in [18], which is the one used in this article, but improvements exist; see for instance [5, 6]. Even under such a time-discretization, such a filter is not available analytically, for most problems of interest. From here one must often discretize the dynamics of the signal (such as Euler), which in essense leads to a high-frequency discrete-time non-linear filter. This latter object can be approximated using particle filters in discrete time, as in, for instance, [1]; this is the approach followed in this article. Alternatives exist, such as unbiased methods [9] and integration-by-parts, change of variables along with Feynman-Kac particle methods [7], but, each of these schemes has its advantages and pitfalls versus the one followed in this paper. We refer to e.g. [6] for some discussion.

Particle filters generate $N$ samples (or particles) in parallel and sequentially approximate non-linear filters using sampling and resampling. The algorithms are very well understood mathematically; see for instance [7] and the references therein. Given the particle filter approximation of the discretized filter, using an Euler method for the signal, one can expect that to obtain a mean square error (MSE), relative to the true filter, of $O(\epsilon^2)$, for $\epsilon > 0$ arbitrary, such that the associated cost is $O(\epsilon^{-4})$. This follows from standard results on discretizations and particle filters. In a related context of regular, discrete time observations and dynamics, with the signal following a diffusion, [15] (see also [14]), show that when the MSE for a particle filter is $O(\epsilon^2)$, the cost is $O(\epsilon^{-3})$ and one can improve particle filters using the multilevel Monte Carlo (MLMC) method [11, 12], as we now explain.

MLMC is an approach which can help to approximate expectations w.r.t. probability measures that are induced by discretizations, such as an Euler method. The idea is to create a collapsing sum representation of an expectation w.r.t. an accurate discretization and interpolate with differences of expectations of increasingly coarse (in terms of the discretization) probability measures. Then, if one can sample from appropriate couplings of the pairs of probability measures in the differences of the expectations, one can reduce the computational effort to achieve a given MSE. In the case of [15], one can achieve a MSE $O(\epsilon^2)$, for cost $O(\epsilon^{-2.5})$ for a class of processes.

In this paper we apply the methodology of [15], which combines particle filters with the MLMC methodology (termed the multilevel particle filter), to the non-linear filtering problem in continuous-time. The main issue is that in-order to mathematically understand the application of this methodology to this new context several new results are required. The main difference to the case of [15], other than the processes involved, is the fact that one averages
over the data in the analysis of filters in continuous-time. This requires one to analyze the properties of several
time-discretized Feynman-Kac semigroups, in order to verify the mathematical improvements of the approach (see
also [10]). Under assumptions, we prove that to achieve a MSE $O(\epsilon^2)$ one requires a cost $O(\epsilon^{-3})$. This is verified
in several numerical examples. We remark that the mathematical results are of interest beyond the context of this
article, for instance, unbiased estimation; see [2] for example.

This article is structured as follows. In Section 2 we formalize the problem of interest. Our approach is detailed
in Section 3. The theoretical results are presented in Section 4. In Section 5 our numerical results are given. The
proofs of our theoretical results are housed in the appendix.

## 2 Problem

### 2.1 Notations

Let $(X, \mathcal{X})$ be a measurable space. For $\varphi : X \to \mathbb{R}$ we write $B_b(X)$ as the collection of bounded measurable functions.
Let $\varphi : \mathbb{R}^d \to \mathbb{R}$, $\text{Lip}_p(\mathbb{R}^d)$ denotes the collection of real-valued functions that are Lipschitz w.r.t. $\|\cdot\|_2$ $(\|\cdot\|_p$ denotes the $\ell_p$-norm of a vector $x \in \mathbb{R}^d)$. That is, $\varphi \in \text{Lip}_p(\mathbb{R}^d)$ if there exists a $C < +\infty$ such that for any $(x, y) \in \mathbb{R}^{2d}$
$$|\varphi(x) - \varphi(y)| \leq C\|x - y\|_2.$$

We write $\|\varphi\|_{\text{Lip}}$ as the Lipschitz constant of a function $\varphi \in \text{Lip}_2(\mathbb{R}^d)$. For $\varphi \in B_b(X)$, we write the supremum norm $\|\varphi\| = \sup_{x \in X} |\varphi(x)|$. $\mathcal{P}(X)$ denotes the collection of probability measures on $(X, \mathcal{X})$. For a measure $\mu$ on $(X, \mathcal{X})$ and a $\varphi \in B_b(X)$, the notation $\mu(\varphi) = \int_X \varphi(x) \mu(dx)$ is used. $B(\mathbb{R}^d)$ denote the Borel sets on $\mathbb{R}^d$. $dx$ is used to denote the Lebesgue measure. For $(X \times Y, \mathcal{X} \otimes \mathcal{Y})$ a measurable space and $\mu$ a non-negative measure on this space, we use the tensor-product of function notations for $(\varphi, \psi) \in B_b(X) \times B_b(Y)$, $\mu(\varphi \otimes \psi) = \int_X \varphi(x) \psi(y) \mu(dx, dy)$.

Let $K : X \times X \to [0, \infty)$ be a non-negative operator and $\mu$ be a measure then we use the notations $\mu K(dy) = \int_X \mu(dx)K(x, dy)$ and for $\varphi \in B_b(X)$, $K(\varphi)(x) = \int_X \varphi(y)K(x, dy)$. For $A \in \mathcal{X}$ the indicator is written $1_A(x)$. $\mathcal{N}_s(\mu, \Sigma)$ (resp. $\psi_s(x; \mu, \Sigma)$) denotes an $s-$dimensional Gaussian distribution (density evaluated at $x \in \mathbb{R}^s$) of mean $\mu$ and covariance $\Sigma$. If $s = 1$ we omit the subscript $s$. For a vector/matrix $X, X^*$ is used to denote the transpose of $X$. For $A \in \mathcal{X}$, $\delta_A(d\mu)$ denotes the Dirac measure of $A$, and if $A = \{x\}$ with $x \in X$, we write $\delta_x(d\mu)$. For a vector-valued function in $d-$dimensions (resp. $d-$dimensional vector), $\varphi(x)$ (resp. $\varphi$) say, we write the $i^{th}$-component ($i \in \{1, \ldots, d\}$) as $\varphi^{(i)}(x)$ (resp. $x^{(i)}$). For a $d \times q$ matrix $X$ we write the $(i, j)^{th}$ entry as $x^{(ij)}$. For $\mu \in \mathcal{P}(X)$ and $X$ a random variable on $X$ with distribution associated to $\mu$ we use the notation $X \sim \mu(\cdot)$. For a finite set $A \in \mathcal{X}$, we write $\text{Card}(A)$ as the cardinality of $A$.

### 2.2 Model

Let $(\Omega, \mathcal{F})$ be a measurable space. On $(\Omega, \mathcal{F})$ consider the probability measure $\mathbb{P}$ and a pair of stochastic processes $\{Y_t\}_{t \geq 0}, \{X_t\}_{t \geq 0}$, with $Y_t \in \mathbb{R}^{d_y}, X_t \in \mathbb{R}^{d_x}$ $(d_y, d_x) \in \mathbb{N}^2$, $d_x, d_y < +\infty$, with $X_0 = x_0 \in \mathbb{R}^{d_x}$ given:

$$dY_t = h(X_t)dt + dB_t$$
$$dX_t = b(X_t)dt + \sigma(X_t)dW_t$$

where $h : \mathbb{R}^{d_x} \to \mathbb{R}^{d_y}, b : \mathbb{R}^{d_x} \to \mathbb{R}^{d_x}, \sigma : \mathbb{R}^{d_x} \to \mathbb{R}^{d_x \times d_x}$ with $\sigma$ non-constant and of full rank and $\{B_t\}_{t \geq 0}, \{W_t\}_{t \geq 0}$ are independent standard Brownian motions of dimension $d_y$ and $d_x$ respectively. To minimize certain technical difficulties, the following assumption is made throughout the paper:

**D1** We have:

1. $\sigma^{(ij)}$ is bounded with $\sigma^{(ij)} \in \text{Lip}_2(\mathbb{R}^{d_x})$, $(i, j) \in \{1, \ldots, d_x\}^2$ and $a(x) := \sigma(x)\sigma(x)^*$ is uniformly elliptic.
2. $(h^{(i)}, b^{(j)})$ are bounded and $(h^{(i)}, b^{(j)}) \in \text{Lip}_2(\mathbb{R}^{d_x}) \times \text{Lip}_2(\mathbb{R}^{d_x}), (i, j) \in \{1, \ldots, d_y\} \times \{1, \ldots, d_x\}$.

Now, we introduce the probability measure $\overline{\mathbb{P}}$ which is equivalent to $\mathbb{P}$ defined by the Radon-Nikodym derivative

$$Z_T := \frac{d\overline{\mathbb{P}}}{d\mathbb{P}} = \exp \left\{ \int_0^T h(X_s)^*dY_s - \frac{1}{2} \int_0^T h(X_s)^*h(X_s)ds \right\}$$

2
with, under \( \mathbb{P} \), \( \{X_t\}_{t \geq 0} \) following the dynamics (2) and independently \( \{Y_t\}_{t \geq 0} \) is a standard Brownian motion. We have the solution to the Zakai equation for \( \varphi \in \mathcal{B}_b(\mathbb{R}^{d_x}) \)

\[
\gamma_t(\varphi) := \mathbb{E}\left[ \varphi(X_t) \exp \left\{ \int_0^t h(X_s)^* dY_s - \frac{1}{2} \int_0^t h(X_s)^* h(X_s) \, ds \right\} | \mathcal{Y}_t \right]
\]

where \( \mathcal{Y}_t \) is the filtration generated by the process \( \{Y_s\}_{0 \leq s \leq t} \). Our objective is to, recursively in time, estimate the filter, for \( \varphi \in \mathcal{B}_b(\mathbb{R}^{d_x}) \)

\[
\eta_t(\varphi) := \frac{\gamma_t(\varphi)}{\gamma_t(1)}.
\]

### 2.3 Discretized Model

In practice, we will have to work with a discretization of the model in (1)-(2), for several reasons:

1. One only has access to a finite, but possibly very high frequency data.
2. \( Z_T \) is typically unavailable analytically.
3. There may not be a non-negative and unbiased estimator of the transition densities induced by the model (1)-(2).

We will assume access to path of data \( \{Y_t\}_{0 \leq t \leq T} \) which is observed at a high frequency, as mentioned above.

Let \( l \in \{0, 1, \ldots, \} \) be given and consider an Euler discretization of step-size \( \Delta_l = 2^{-l}, k \in \{1, 2, \ldots, 2^l T\} \), \( \tilde{X}_0 = x_* \):

\[
\tilde{X}_{k\Delta_l} = \tilde{X}_{(k-1)\Delta_l} + b(\tilde{X}_{(k-1)\Delta_l}) \Delta_l + \sigma(\tilde{X}_{(k-1)\Delta_l}) [W_{k\Delta_l} - W_{(k-1)\Delta_l}].
\]

(3)

It should be noted that the Brownian motion in (3) is the same as in (2) under both \( \mathbb{P} \) and \( \mathbb{P} \). Then, for \( k \in \{0, 1, \ldots, \} \) define:

\[
G^l_k(x_{k\Delta_l}) := \exp \left\{ h(x_{k\Delta_l})^*(y_{(k+1)\Delta_l} - y_{k\Delta_l}) - \frac{\Delta_l}{2} h(x_{k\Delta_l})^* h(x_{k\Delta_l}) \right\}
\]

and note that for any \( T \in \mathbb{N} \)

\[
Z^l_T(x_0, x_{\Delta_l}, \ldots, x_{T-\Delta_l}) := \prod_{k=0}^{2^l T-1} G^l_k(x_{k\Delta_l}) = \exp \left\{ \sum_{k=0}^{2^l T-1} \left[ h(x_{k\Delta_l})^*(y_{(k+1)\Delta_l} - y_{k\Delta_l}) - \frac{\Delta_l}{2} h(x_{k\Delta_l})^* h(x_{k\Delta_l}) \right] \right\}
\]

is simply a discretization of \( Z_T \) (of the type of [18]). Then set for \( (t, \varphi) \in \mathbb{N} \times \mathcal{B}_b(\mathbb{R}^{d_x}) \)

\[
\gamma^l_t(\varphi) := \mathbb{E}[\varphi(X_t) Z^l_t(\tilde{X}_0, \tilde{X}_{\Delta_l}, \ldots, \tilde{X}_{T-\Delta_l}) | \mathcal{Y}_t]
\]

\[
\eta^l_t(\varphi) := \frac{\gamma^l_t(\varphi)}{\gamma^l_t(1)}
\]

For notational convenience \( \eta^l_0(dx) = \delta_{x_*}(dx) \). For \( (l, p, t, \varphi) \in \mathbb{N} \times \{0, 1, \ldots, \} \times \{\Delta_l, 2\Delta_l, \ldots, 1 - \Delta_l\} \times \mathcal{B}_b(\mathbb{R}^{d_x}) \) one can also set

\[
\gamma^l_{p+t}(\varphi) := \mathbb{E}\left[ \varphi(X_{p+t}) Z^l_p(\tilde{X}_0, \tilde{X}_{\Delta_l}, \ldots, \tilde{X}_{p-\Delta_l}) \left( \prod_{k=0}^{t\Delta_l-1} G^l_{p\Delta_l^{-1}+k}(\tilde{X}_{p+k\Delta_l^{-1}}) \right) | \mathcal{Y}_{p+t} \right]
\]

\[
\eta^l_{p+t}(\varphi) := \frac{\gamma^l_{p+t}(\varphi)}{\gamma^l_{p+t}(1)}
\]

where we define \( Z^l_0(x_{-\Delta_l}) = 1 \).

### 3 Approach

For notational convenience, throughout this Section, we omit the \( \tilde{} \) notation from \( X \) for the Euler discretization.
3.1 Particle Filter

Let \( l \in \{0, 1, \ldots \} \) be given, we consider approximating \( \eta^l_t(\varphi) \) using a particle filter. For \( p \in \{0, 1, \ldots \} \) set

\[
u^l_p := (x_p, x_{p+\Delta_l}, \ldots, x_{p+1}) \in (\mathbb{R}^{d_s})^{\Delta_l+1} := E_l.
\]

For \( \varphi \in \mathcal{B}_b(\mathbb{R}^{d_s}) \), we define, for any \( l \geq 0 \), \( \varphi^l : E_l \to \mathbb{R} \)

\[
\varphi^l(x_0, x_{\Delta_l}, \ldots, x_1) := \varphi(x_1).
\]

Set, with \( p \in \{0, 1, \ldots \} \)

\[
\mathbf{G}^l_p(\nu^l_p) := \prod_{k=0}^{\Delta_l-1} \mathbf{G}^l_{p\Delta_l+1+k}(x_{p+k\Delta_l}).
\]

Denote by \( M^l : \mathbb{R}^{d_x} \to \mathcal{P}(E_l) \) the joint Markov transition of \((x_0, x_{\Delta_l}, \ldots, x_1)\) defined via the Euler discretization (3) and a Dirac on a point \( x \in \mathbb{R}^{d_x} \): for \((x, \varphi) \in \mathbb{R}^{d_x} \times \mathcal{B}_b(E_l)\),

\[
M^l(\varphi)(x) := \int_{E_l} \varphi(x_0, x_{\Delta_l}, \ldots, x_1) \delta_x(dx_0) \left[ \prod_{k=1}^{\Delta_l-1} \psi_{\Delta_l k}(x_{k\Delta_l}, x_{(k-1)\Delta_l} + b(x_{(k-1)\Delta_l})\Delta_l, a(x_{(k-1)\Delta_l})\Delta_l) \right] dx_{\Delta_l}, \ldots, x_1).
\]

For \( p \in \mathbb{N} \), define the operator \( \Phi^l_p : \mathcal{P}(E_l) \to \mathcal{P}(E_l) \) with \((\mu, \varphi) \in \mathcal{P}(E_l) \times \mathcal{B}_b(E_l)\) as:

\[
\Phi^l_p(\mu)(\varphi) := \frac{\mu(\mathbf{G}^l_{p-1}M^l(\varphi))}{\mu(\mathbf{G}^l_{p-1})}
\]

(4)

where, to clarify, \( \mu(\mathbf{G}^l_{p-1}M^l(\varphi)) = \int_{E_l} \mu(dx_{p-1}, x_{p-1+\Delta_l}, \ldots, x_p))\mathbf{G}^l_{p-1}(x_{p-1}, x_{p-1+\Delta_l}, \ldots, x_{p-\Delta_l})M^l(\varphi)(x_p) \).

Now, define, for \((p, \varphi) \in \{0, 1, \ldots \} \times \mathcal{B}_b(E_l)\),

\[
\pi^l_p(\varphi) := \int_{\mathbb{R}^{d_x}} \eta^l_p(dx) \int_{E_l} M^l(x, du) \varphi(u).
\]

Then one can establish that for \( p \in \mathbb{N} \)

\[
\pi^l_p(\varphi) = \Phi^l_p(\pi^l_{p-1})(\varphi).
\]

Moreover, for \((t, \varphi) \in \mathbb{N} \times \mathcal{B}_b(\mathbb{R}^{d_x})\)

\[
\eta^l_t(\varphi) = \frac{\pi^l_t(\mathbf{G}^l_{t-1}\varphi)}{\pi^l_t(\mathbf{G}^l_{t-1})}
\]

(5)

The objective of the PF is to provide an approximation of the formulae (4) and (5).

Let \( N \in \mathbb{N} \) be given, then the particle filter generates a system of random variables on \((E_l^N)^{n+1}\) at a time \( n \in \{0, 1, \ldots \} \) according to the probability measure

\[
\mathbb{Q}(du_{0}^{1:N}, \ldots, u_{n}^{1:N}) = \left( \prod_{i=1}^{N} M^l(x_{i-1}, du_{0}^{i}) \right) \prod_{i=1}^{n} \prod_{p=1}^{N} \Phi^l_{t}(\pi^l_{p-1})(du_{p}^{i})
\]

where for \( \varphi \in \mathcal{B}_b(E_l)\)

\[
\pi^l_{p-1}(\varphi) := \frac{1}{N} \sum_{i=1}^{N} \varphi(u_{p-1}^{i}).
\]

The particle filter is summarized in Algorithm 1. For \((t, \varphi) \in \mathbb{N} \times \mathcal{B}_b(\mathbb{R}^{d_x})\) one can approximate \( \eta^l_t(\varphi) \) via

\[
\eta^{l,N}_t(\varphi) := \frac{\pi^{l,N}_t(\mathbf{G}^l_{t-1}\varphi)}{\pi^{l,N}_t(\mathbf{G}^l_{t-1})}.
\]

(6)

For \((l, p, t, \varphi) \in \mathbb{N} \times \{0, 1, \ldots \} \times \{\Delta_l, 2\Delta_l, \ldots, 1 - \Delta_l\} \times \mathcal{B}_b(\mathbb{R}^{d_x})\) one can also estimate the filter at time \( p + t \), \( \eta^l_{p+t}(\varphi) \), as

\[
\eta^{l,N}_{p+t}(\varphi) := \frac{\sum_{i=1}^{N} \left( \prod_{k=0}^{\Delta_l-1} \mathbf{G}^l_{p\Delta_l+1+k}(x_{p+k\Delta_l}) \varphi(x_{p+1}) \right)}{\sum_{i=1}^{N} \prod_{k=0}^{\Delta_l-1} \mathbf{G}^l_{p\Delta_l+1+k}(x_{p+k\Delta_l})}.
\]

(6)
Algorithm 1 Particle Filter.

1. Initialize: For \( i \in \{1, \ldots, N\} \), generate \( u_{i}^{l} \) from \( M^{l}(x_{*}, \cdot) \). Set \( p = 1 \).

2. Update: For \( i \in \{1, \ldots, N\} \), generate \( u_{i}^{l} \) from \( \Phi_{l}^{l}((\eta_{l}^{N})^{l})(\cdot) \). Set \( p = p + 1 \) and return to the start of 2.

3.2 Coupled Particle Filter

Let \( L \in \mathbb{N} \) be given, in multilevel estimation, the basic idea is to approximate, for \( \varphi \in B_{b}(\mathbb{R}^{d_{x}}) \)
\[
\eta_{l}^{L}(\varphi) = \eta_{l}^{0}(\varphi) + \sum_{i=1}^{L} \left[ \eta_{l}^{i} - \eta_{l}^{i-1}(\varphi) \right].
\]

Normally \( L \) is chosen to target a specific bias and this is the strategy considered here. There is a complication as \( L \) also determines the level frequency of the data that are used - this is discussed below. We focus on the term \( [\eta_{l}^{i} - \eta_{l}^{i-1}(\varphi), l \in \mathbb{N} \), as one can use the PF above for approximating the term \( \eta_{l}^{0}(\varphi) \) (see (6)).

For \( (l, \varphi) \in \mathbb{N} \times B_{b}(\mathbb{R}^{d_{x}} \times \mathbb{R}^{d_{z}}) \) (resp. \( (l, \varphi) \in \mathbb{N} \times B_{b}(\mathbb{R}^{d_{x}}) \)), we define \( \varphi^{l}: E_{l} \times E_{l-1} \to \mathbb{R} \) (resp. \( \varphi^{l}: (E_{l} \times E_{l-1})^{2} \to \mathbb{R} \))
\[
\varphi^{l}((x_{0}, x_{\Delta_{1}}, \ldots, x_{1}), (x'_{0}, x'_{\Delta_{1}}, \ldots, x'_{1})) := \varphi(x_{1}, x'_{1})
\]
(reg. \( \varphi^{l}((x_{0}, x_{\Delta_{1}}, \ldots, x_{1}), (x'_{0}, x'_{\Delta_{1}}, \ldots, x'_{1}), (v_{0}, v_{\Delta_{1}}, \ldots, v_{1}), (v'_{0}, v'_{\Delta_{1}}, \ldots, v'_{1})) := \varphi(x_{1}, x'_{1}, v_{1}, v'_{1}) \)). The following exposition closely follows [13], with modifications to the context here. Let \( \bar{P}^{l}: \mathbb{R}^{d_{x}} \times \mathbb{R}^{d_{x}} \to \mathcal{P}(\mathbb{R}^{d_{x}})^{\Delta_{l-1}} \times \mathbb{R}^{d_{x}} \Delta_{l-1}) \) be a Markov kernel, for paths \( (x_{\Delta_{1}}, \ldots, x_{1}) \) and \( (x'_{\Delta_{1}}, \ldots, x'_{1}) \) contracted by using the same Brownian increments in the discretization (3) (see e.g. [11] or [16, Section 3.3]). Let \( \bar{M}^{l}: \mathbb{R}^{d_{x}} \times \mathbb{R}^{d_{x}} \to \mathcal{P}(E_{l} \times E_{l-1}) \) be a Markov kernel defined for \( (u, v, \varphi) \in \mathbb{R}^{d_{x}} \times \mathbb{R}^{d_{x}} \times B_{b}(E_{l} \times E_{l-1}) \)
\[
\bar{M}^{l}(\varphi)((u, v)) := \int_{E_{l} \times E_{l-1}} \varphi(u, u') \delta_{u}(dx_{0}) \delta_{v}(dx_{0}^{l-1}) \bar{P}^{l}\left(\left(x_{0}, x_{0}^{l-1}\right), d((x_{\Delta_{1}}, \ldots, x_{1}), (x'_{\Delta_{1}}, \ldots, x_{1}^{l-1}))\right).
\]

Note that for any \( (u, v) \in \mathbb{R}^{d_{x}} \times \mathbb{R}^{d_{x}} \), \( (A, B) \in B(E_{l}) \lor B(E_{l-1}) \)
\[
\bar{M}^{l}(u, v, A \times E_{l-1}) = M^{l}(u, A) \quad \text{and} \quad \bar{M}^{l}(u, v', E_{l} \times B) = M^{l-1}(v, B).
\]

Let \( (p, \nu, \varphi) \in \mathbb{N} \times B_{b}(E_{l} \times E_{l-1}) \times \mathcal{P}(E_{l} \times E_{l-1}) \) and define the probability measure:
\[
\Phi_{l}^{l}(\mu) := \mu\left(\left\{ F_{p-1, \mu, l} \land F_{p-1, \mu, l-1}\right\}\bar{M}^{l}(\varphi) + \left(1 - \mu\left\{ F_{p-1, \mu, l} \land F_{p-1, \mu, l-1}\right\}\right)\times \right)
\]
\[
\left(\nu \otimes \mu\right)\left(\left\{ F_{p-1, \mu, l} \otimes F_{p-1, \mu, l-1}\right\}\bar{M}^{l}(\varphi)\right)\]  \hspace{1cm} (7)

where for \( (u, v) \in E_{l} \times E_{l-1} \)
\[
\bar{F}_{p-1, \mu, l}(u, v) = \frac{F_{p-1, \mu, l}(u, v) - \left(\left\{ F_{p-1, \mu, l}(u, v) \land F_{p-1, \mu, l-1}(u, v)\right\}\right)}{\mu\left(F_{p-1, \mu, l} - \left\{ F_{p-1, \mu, l} \land F_{p-1, \mu, l-1}\right\}\right)}
\]
\[
\bar{F}_{p-1, \mu, l-1}(u, v) = \frac{F_{p-1, \mu, l-1}(u, v) - \left(\left\{ F_{p-1, \mu, l-1}(u, v) \land F_{p-1, \mu, l-1}(u, v)\right\}\right)}{\mu\left(F_{p-1, \mu, l-1} - \left\{ F_{p-1, \mu, l-1} \land F_{p-1, \mu, l-1}\right\}\right)}
\]
\[
F_{p-1, \mu, l}(u, v) = \frac{\hat{G}_{p-1, \mu, l}(u) \otimes 1}{\mu\left(\hat{G}_{p-1, \mu, l}\right)}
\]
\[
F_{p-1, \mu, l-1}(u, v) = \frac{1 \otimes \hat{G}_{p-1, \mu, l-1}(v)}{\mu\left(\hat{G}_{p-1, \mu, l-1}\right)}
\]
\[
\hat{G}_{p-1, \mu, l}(u) = \frac{\hat{G}_{p-1, \mu, l}(u)}{\mu\left(\hat{G}_{p-1, \mu, l}\right)}
\]
\[
\hat{G}_{p-1, \mu, l-1}(v) = \frac{\hat{G}_{p-1, \mu, l-1}(v)}{\mu\left(\hat{G}_{p-1, \mu, l-1}\right)}
\]
and for \( ((u, v), (u', v')) \in (E_{l} \times E_{l-1}) \times (E_{l} \times E_{l-1}) \) and \( \varphi \in B_{b}(E_{l} \times E_{l-1}) \)
\[
\bar{M}^{l}(\varphi)((u, v), (u', v')) = \bar{M}^{l}(\varphi)(u, v').
\]
Algorithm 2 Sampling from $\tilde{\Phi}_p^I(\tilde{\pi}_{p-1}^I)(\cdot)$.

1. With probability $\tilde{\pi}_{p-1}^I\left(\{ F_{p-1}, \tilde{x}_{p-1,1}^I \land F_{p-1}, \tilde{x}_{p-1,1}^I \} \right)$ generate $w_p \in (E_t \times E_{t-1})$ according to

$$\frac{\sum_{i=1}^N F_{p-1}, \tilde{x}_{p-1,i}^I \land F_{p-1}, \tilde{x}_{p-1,i}^I (w_{p-1}^{l,i}) M^I((x_{p,i}^I, x_{p-1,i}^I), \cdot)}{\sum_{i=1}^N F_{p-1}, \tilde{x}_{p-1,i}^I (w_{p-1}^{l,i}) \land F_{p-1}, \tilde{x}_{p-1,i}^I (w_{p-1}^{l,i})}.$$

2. Otherwise, generate $w_p \in (E_t \times E_{t-1})$ according to

$$\frac{1}{N^2} \sum_{i=1}^N \sum_{j=1}^N \left\{ F_{p-1}, \tilde{x}_{p-1,i}^I (w_{p-1}^{l,i}) \land F_{p-1}, \tilde{x}_{p-1,i}^I (w_{p-1}^{l,i}) \right\} M^I((x_{p,i}^I, x_{p-1,i}^I), \cdot).$$

Define for $\varphi \in B_0(E_t \times E_{t-1})$

$$\tilde{\pi}_0^I(\varphi) := M^I(\varphi)(x_*, x_*)$$

and for $(p, \varphi) \in \mathbb{N} \times B_0(E_t \times E_{t-1})$

$$\tilde{\pi}_p^I(\varphi) = \tilde{\Phi}_p^I(\tilde{\pi}_{p-1}^I)(\varphi).$$

Now it can be shown that (see [15]) that for $(t, \varphi) \in \mathbb{N} \times B_0(\mathbb{R}^{d_t})$

$$\eta^I_t(\varphi) = \frac{\tilde{\pi}_{t-1}^I((G_{t-1}^I \varphi) \otimes 1)}{\tilde{\pi}_{t-1}^I(G_{t-1}^I \otimes 1)} \quad \text{and} \quad \eta_t^{-1}(\varphi) = \frac{\tilde{\pi}_{t-1}^I(1 \otimes (G_{t-1}^I \varphi^{-1}))}{\tilde{\pi}_{t-1}^I(1 \otimes G_{t-1}^I \varphi^{-1})}.$$}

The objective of the coupled particle filter (CPF) is to provide an approximation of the formulae (7) and (9).

For $p \in \{0, 1, \ldots\}$ set $w_p^i = (u_{p,i}^l, \tilde{w}_p^i) \in E_t \times E_{t-1}$. A CPF for sequentially approximating $[\eta^I_t - \eta_t^{-1}](\varphi)$ is then generated on $((E_t \times E_{t-1})^N)^{n+1}$ at a time $n \in \{0, 1, \ldots\}$ according to the probability measure

$$\tilde{Q}(d(w_0^{l,1:N}, \ldots, w_n^{l,1:N})) = \left\{ \prod_{i=1}^N M^I((x_*, x_*), du_0^{l,i}) \right\} \left( \prod_{i=1}^n \prod_{p=1}^N \Phi^I_p(\tilde{\pi}_{p-1}^I)(dw_p^{l,i}) \right)$$

where for $\varphi \in B_0(E_t \times E_{t-1})$

$$\tilde{\pi}_{p-1}^I(\varphi) := \frac{1}{N} \sum_{i=1}^N \varphi(u_{p-1}^{l,i}).$$

To run a CPF, one must understand how to sample from $\tilde{\Phi}_p^I(\tilde{\pi}_{p-1}^I)(\cdot)$ which is detailed in Algorithm 2. The CPF is then described in Algorithm 3. Then one can approximate $[\eta^I_t - \eta_t^{-1}](\varphi), \varphi \in B_0(\mathbb{R}^{d_t})$ via (9)

$$[\eta^I_t - \eta_t^{-1}]^N(\varphi) := \frac{\tilde{\pi}^{l,N}_{t-1}(G_{t-1}^I \varphi) \otimes 1}{\tilde{\pi}^{l,N}_{t-1}(G_{t-1}^I \otimes 1)} - \frac{\tilde{\pi}^{l,N}_{t-1}(1 \otimes (G_{t-1}^I \varphi^{-1}))}{\tilde{\pi}^{l,N}_{t-1}(1 \otimes G_{t-1}^I \varphi^{-1})}. $$

For $(l, p, t, \varphi) \in \{2, 3, \ldots\} \times \{0, 1, \ldots\} \times \{\Delta_{t-1}, 2\Delta_{t-1}, \ldots, 1 - \Delta_{t-1}\} \times B_0(\mathbb{R}^{d_t})$ one can also estimate the differences of the filter at time $p + t$, $[\eta^I_{p+t} - \eta_{p+t}^{-1}](\varphi)$, as

$$[\eta^I_{p+t} - \eta_{p+t}^{-1}]^N(\varphi) := \frac{\sum_{i=1}^N \left( \prod_{k=0}^{\Delta_{t-1}-1} G_{p+t+k,I}^I (x_{p+t+k}^{l,i}) \varphi(x_{p+t+k}^{l,i}) \right)}{\sum_{i=1}^N \prod_{k=0}^{\Delta_{t-1}-1} G_{p+t+k,I}^I (x_{p+t+k}^{l,i})} - \frac{\sum_{i=1}^N \left( \prod_{k=0}^{\Delta_{t-1}-1} G_{p+t+k,I}^{-1} (x_{p+t+k}^{l,i}) \varphi(x_{p+t+k}^{l,i}) \right)}{\sum_{i=1}^N \prod_{k=0}^{\Delta_{t-1}-1} G_{p+t+k,I}^{-1} (x_{p+t+k}^{l,i})}. $$

3.2.1 **Comment on the Operator $\tilde{\Phi}_p^I$**

As noted above, the recursion (8) provides a type of coupling of $(\eta^I_t, \eta_t^{-1})$ given in (9). However, this representation is by no means unique, nor, as discussed in [13] for the purposes of multilevel estimation optimal in any sense.
The approach corresponds to a so-called maximal coupling of the resampling indices, which ensures that one can consistently estimate differences such as \([\eta_t^l - \eta_t^{l-1}](\varphi)\). When \(d_x = 1\), an alternative scheme which uses Wasserstein coupling is used in [14] (see also [13]). This latter procedure is considered in Section 5, but is not mathematically analyzed.

### 3.3 Multilevel Estimation

The multilevel estimate is then constructed as follows.

1. **Level 0**: Run a PF as in Algorithm 1 with \(N_0\) samples, independently of all other levels.
2. **Level \(l \in \{1, \ldots, L\}**: Run a CPF as in Algorithm 3 with \(N_l\) samples, independently of all other levels.

The estimate of \(\eta_t^l(\varphi)\) is then

\[
\eta_t^{L,ML}(\varphi) := \eta_t^{0,N_0}(\varphi) + \sum_{l=1}^{L} [\eta_t^l - \eta_t^{l-1}]^{N_l}(\varphi) \tag{11}
\]

where \(\eta_t^{0,N_0}(\varphi)\) is as (6) (with \(l = 0\), \(N = N_0\)) and \([\eta_t^l - \eta_t^{l-1}]^{N_l}(\varphi)\) is as (10) (with \(N = N_l\)).

For \((l,p,t,\varphi) \in \{1, \ldots, L-1\} \times \{0,1, \ldots, N\} \times \{\Delta_1, 2\Delta_1, \ldots, 1 - \Delta_1\} \times \mathcal{B}_d(\mathbb{R}^{d_x})\) one can also obtain estimates of \(\eta_t^{L,l}(\varphi)\) as a by-product of the above procedure. This estimate is

\[
\eta_t^{L,l,ML}(\varphi) := \sum_{m=l+1}^{L} \left\{ \frac{\sum_{i=1}^{N_m} \left( \prod_{k=0}^{\Delta m_{i-1}^{-1}} G_{p\Delta m_{i-1}^{-1} + k} (x^{l,i}_{p+k\Delta m_{i-1}}) \right) \varphi(x^{m,i}_{p+l})}{\sum_{i=1}^{N_m} \left( \prod_{k=0}^{\Delta m_{i-1}^{-1}} G_{p\Delta m_{i-1}^{-1} + k} (x^{m-1,i}_{p+k\Delta m_{i-1}}) \right) \varphi(x^{m-1,i}_{p+l})} + \frac{\sum_{i=1}^{N_l} \left( \prod_{k=0}^{\Delta l_i^{-1}} G_{p\Delta l_i^{-1} + k} (x^{l,i}_{p+k\Delta l_i}) \right) \varphi(x^{l,i}_{p+l})}{\sum_{i=1}^{N_l} \prod_{k=0}^{\Delta l_i^{-1}} G_{p\Delta l_i^{-1} + k} (\bar{x}^{l,i}_{p+k\Delta l_i})} \right\} \tag{12}
\]

where the \(m^{th}\)—term of the summand on the R.H.S. has been obtained by a CPF at level \(m\) and the second term in the sum on the R.H.S. is the level \(l\) particles from the coupled particle filter run targeting \((\eta_t^l, \eta_t^{l-1})\).

### 4 Theoretical Results

We consider the estimate (11) in our analysis. The estimate (12) can also be analyzed with the same approach and only notational complications. \(\mathbb{E}\) is used to denote expectations w.r.t. the simulated process, which averages over the dynamics of the data, under the probability measure \(\mathcal{F}\). The proofs needed for the following result are given in the appendix. Below, \(A := \{(l,q) \in \{1, \ldots, L\}^2 : l \neq q\}\). The following theorem gives a bound on the MSE.

**Theorem 4.1**. Assume (D1). Then for any \(t \in \{0,1, \ldots\}\), there exists a \(C < +\infty\) such that for any \(L \in \{1,2, \ldots\}\), \((N_0, \ldots, N_L, \varphi) \in \mathbb{N}^{L+1} \times \mathcal{B}_d(\mathbb{R}^{d_x}) \cap \text{Lip}\lVert \varphi \rVert_\text{Lip}(\mathbb{R}^{d_x})\)

\[
\mathbb{E}[\varphi^2] \leq C(\lVert \varphi \rVert + \lVert \varphi \rVert_{\text{Lip}})^2 \left( \sum_{l=0}^{L} \frac{\Delta_l}{N_l} + \sum_{l=1}^{L} \sum_{q=1}^{L} \Pi_\text{A}(l,q) \frac{\Delta_l^{1/2} \Delta_q^{1/2}}{N_l N_q} + \Delta_L \right). \tag{13}
\]
Proof. One has
\[

\mathbb{E}[(\eta_t^{L, ML}(\varphi) - \eta_t(\varphi))^2] \leq 2\mathbb{E}[(\eta_t^{L, ML} - \eta_t^L)(\varphi)^2] + 2\mathbb{E}[(\eta_t^L - \eta_t)(\varphi)^2].
\]

The left-most term on the R.H.S. can be dealt with by the \(C_2\)-inequality and Propositions A.1, A.2 and A.3 in the appendix. For the right-most term on the R.H.S. one can use Remark A.2.

\[\square\]

Remark 4.1. We note that the constant \(C\) in Theorem 4.1 depends upon \(t\). As seen in [13], the task of bounding the asymptotic variance uniformly in \(t\) (for models as in [15]) is particularly difficult and one expects even more arduous calculations for the finite-sample variance. All of our below discussion does not consider \(t\), although this is of course a very important issue.

We note that if one considers (6), then the MSE associated to this estimate can be upper-bounded (using the \(C_2\)-inequality, Remark A.4 along with Lemma A.2 and Remark A.2) by
\[
C(\|\varphi\| + \|\varphi\|_{\text{Lip}})^2 \left(\frac{1}{N} + \Delta L\right).
\]

(13)

Note that the bias term is \(O(\Delta L)\) and not \(O(\Delta L^2)\) (as in e.g. [15]) as our results averages over the uncertainty in the data, as is often done in the literature in the analysis of continuous-time particle filters (e.g. [1]). The order of the bias can be improved by using higher-order discretization methods.

Let \(\epsilon > 0\) be arbitrary. To obtain a bound on the MSE, in Theorem 4.1, of \(O(\epsilon^2)\) one can choose \(L\) so that \(\Delta L = O(\epsilon^2)\). Then setting \(N_l = O(\epsilon^{-2}\Delta L^{-1/4}\Delta L^{3/4})\), the upper-bound in Theorem 4.1 is \(O(\epsilon^2)\). The associated cost to achieve this MSE is \(O(\sum_{l=0}^L \Delta L^{-1} N_l) = O(\epsilon^{-3})\). If one considers (13), then setting \(\Delta L = O(\epsilon^2)\) and \(N = O(\epsilon^{-2})\), gives a cost of \(O(\Delta L^{-1} N) = O(\epsilon^{-4})\). We note that if \(\sigma\) in (2) were constant, it is straightforward to deduce from the arguments in the appendix that the MSE associated to (11) is upper-bounded by a term of \(O(\sum_{l=0}^L \frac{1}{N_l} + \Delta L)\). Then, choosing \(N_l = O(\epsilon^{-2}\Delta L)\), one can show that the cost to achieve a MSE of \(O(\epsilon^2)\) is \(O(\epsilon^{-2}\log(\epsilon)^2)\) for the MLPF method. For the PF, again the cost is \(O(\Delta L^{-1} N) = O(\epsilon^{-4})\). One issue here is that by increasing \(L\), one must have access to data that are recorded at a frequency of \(2^{-L}\). This either creates a bottleneck of the multilevel procedure (one cannot exceed the frequency at which the data are observed), or one may linearly interpolate the data. In the latter case, one may want to use the robust filter (see [3], also [1, Chapter 5]).

5 Numerical Results

5.1 Models

We set \(d_x = d_y = 1\) and \(X_0 = x_0 = 0\). We will consider four different models for the signal, but \(h(x) = x\) in all cases. Data are generated from the process under the probability measure \(\mathbb{P}\).

Ornstein-Uhlenbeck Process (OU). Consider the following OU process:
\[
\text{d}X_t = \theta(\mu - X_t)\text{d}t + \sigma \text{d}W_t.
\]

The constants in the example are \(\theta = 1\), \(\mu = 0\) and \(\sigma = 0.5\).

Langevin Stochastic Differential Equation. Here the SDE is given by
\[
\text{d}X_t = \frac{1}{2} \nabla \log \pi(X_t) \text{d}t + \text{d}W_t
\]

where \(\pi(x)\) denotes a probability density function. The density \(\pi(x)\) is chosen as the \(t\)-distribution with 10 degrees of freedom (zero location, unit scale).

Geometric Brownian Motion (GBM). Next consider the GBM process:
\[
\text{d}X_t = \mu X_t \text{d}t + \sigma X_t \text{d}W_t
\]

with constants \(\sigma = 0.2\) and \(\mu = 0.02\).

An SDE with a Non-Linear Diffusion Term. Finally
\[
\text{d}X_t = \theta(\mu - X_t)\text{d}t + \frac{1}{\sqrt{1 + X_t^2}} \text{d}W_t
\]

with constants \(\theta = 0\), \(\mu = 0\).
5.2 Simulation Settings

We will compare the MLPF (as described in Section 3.3) and PF. We will also implement the ML method based upon the approach of [14] mentioned in Section 3.2.1. For the case of a constant diffusion coefficient (resp. non-constant) of the signal, we expect that one can set \( \Delta_L = \mathcal{O}(\epsilon^2) \) with \( N_l = \mathcal{O}(\epsilon^{-2} \Delta_l^{3/2}) \) (resp. \( N_l = \mathcal{O}(\epsilon^{-2} \Delta_l L) \)) to achieve a MSE of \( \mathcal{O}(\epsilon^2) \) and cost \( \mathcal{O}(\epsilon^{-2}) \) (resp. \( \mathcal{O}(\epsilon^{-2} \log(\epsilon)^2) \)). We note that, to our knowledge, there is no proof of this result in our context and it is a topic to be considered in future work.

For each example, the multilevel estimators are considered at levels \( L \in \{4, \ldots, 9\} \) (which correspond to a particular \( \epsilon \)). For the OU, the ground truth is computed through a Kalman filter. For the other examples, results from particle filters at level \( L = 10 \) with \( 100 \times 2^{10} \) particles are used as approximations to the ground truth. For each level of the PF algorithm, \( N_l = 100 \times \Delta_l \) particles are used. All results are averaged over multiple runs. For each level of the MLPF algorithm, \( N_l \) is set as described in Section 4 (or just as in the above paragraph, for the method in [14]). We adopt adaptive resampling for all approaches.

5.3 Results

We will present the cost against MSE plot, where both cost and MSE is in \( \log_{10} \)-scale. 100 time units are considered and we compute the expected value of the state at the terminal time-point. The results are presented in Figure 1. The plot displays the expected behaviour that is given in the theory for the MLPF and PF. In each case the gradient for the PF is around -2 (so that the cost is \( \mathcal{O}(\epsilon^{-4}) \) for a MSE of \( \mathcal{O}(\epsilon^2) \)). For the MLPF on the first row of Figure 1, one sees a gradient of about -1 which corresponds roughly to the cost \( \mathcal{O}(\epsilon^{-2} \log(\epsilon)^2) \) for a MSE of \( \mathcal{O}(\epsilon^2) \). Similarly on the second row of Figure 1 one sees a gradient of about -1.5 which corresponds to the cost \( \mathcal{O}(\epsilon^{-3}) \) for a MSE of \( \mathcal{O}(\epsilon^2) \). We can also observe that the conjectured improvements of the method of [14] seem to be confirmed in these examples.

![Figure 1: Cost against MSE plots. The PF is the red line, MLPF is the black and the method of [14] the blue.](image-url)
Acknowledgements
AJ was supported by KAUST baseline funding. FY was supported by AcRF tier 1 grant R-155-000-182-112.

A Proofs
Some operators are now defined. Let \((l, p, n) \in \{0, 1, \ldots \} \times \{0, 1, \ldots \}^2\), \(n > p\), \((x, \varphi) \in E_l \times B_0(E_l)\)

\[
Q_{p,n}^l(\varphi)(u_p) := \int \left( \prod_{q=p}^{n-1} \varphi(u_n)G_q^l(u_q) \right) \prod_{q=p+1}^n M_l(u_{q-1}, du_q),
\]

where we use the convention \(Q_{p,p}^l(\varphi)(u_p) = \varphi(u_p)\). In addition, for \((l, p, n) \in \{0, 1, \ldots \} \times \{0, 1, \ldots \}^2\), \(n > p\), \((x, \varphi) \in E_l \times B_0(E_l)\):

\[
D_{p,n}^l(\varphi)(u_p) := \frac{Q_{p,n}^l(\varphi - \pi_n^l(\varphi))(u_p)}{\pi_n^l(Q_{p,n}^l(1))}
\]

where \(D_{p,p}^l(\varphi)(u_p) = \varphi(u_p) - \pi_p^l(\varphi)\).

Throughout our arguments, \(C\) is a finite constant whose value may change from line to line, but does not depend upon \(l\) nor \(N\). The particular dependencies of a given constant will be clear from the statement of a given result.

A.1 Results for the Coupled Particle Filter
Set, for \(l \in \{1, \ldots \}, (n, p, \varphi) \in \{0, 1, \ldots \}^2 \times B_0(\mathbb{R}^d_{x})\), \(n > p\)

\[
T_{p,n}^l(\varphi) := E[(D_{p,n}^l(G_n^l\varphi))(U_{p,1}^{l,1}) - D_{p,n}^{l-1}(G_n^{l-1}\varphi^{l-1})(U_{p,1}^{l-1,1}))^2] + E[(G_p^l(U_{p,1}^{l,1}) - G_p^{l-1}(U_{p,1}^{l-1,1}))^2] + E[(\pi_p^l(G_p^l) - \pi_p^{l-1}(G_p^{l-1}))^2]
\]

and if \(n = p\)

\[
T_{p,n}^l(\varphi) := E[(D_{p,n}^l(G_n^l\varphi))(U_{p,1}^{l,1}) - D_{p,n}^{l-1}(G_n^{l-1}\varphi^{l-1})(U_{p,1}^{l-1,1}))].
\]

Lemma A.1. Assume (D1). Then for any \(n \in \{0, 1, \ldots \}\), there exists a \(C < +\infty\) such that for any \((l, N, \varphi) \in \mathbb{N} \times \mathbb{N} \times B_0(\mathbb{R}^d_{x})\)

\[
E[(\hat{\pi}_n^{l,N} - \hat{\pi}_n^l)((G_n^l\varphi^l) \otimes 1 - 1 \otimes (G_n^{l-1}\varphi^{l-1}))^2] \leq \frac{C}{N} \sum_{p=0}^n T_{p,n}^l(\varphi).
\]

Proof. We have the following standard Martingale plus remainder decomposition [8]

\[
(\hat{\pi}_n^{l,N} - \hat{\pi}_n^l)((G_n^l\varphi^l) \otimes 1 - 1 \otimes (G_n^{l-1}\varphi^{l-1})) = \sum_{p=0}^n (\hat{\pi}_n^{l,N} - \hat{\pi}_n^{l,N}(\hat{\pi}_{p-1}^l))((D_{p,n}^l(G_n^l\varphi^l) \otimes 1 - 1 \otimes D_{p,n}^{l-1}(G_n^{l-1}\varphi^{l-1})) + \sum_{p=0}^{n-1} \left\{ \frac{\hat{\pi}_n^{l,N}(D_{p,n}^l(G_n^l\varphi^l) \otimes 1)}{\hat{\pi}_n^{l,N}(G_p^l \otimes 1)}[\hat{\pi}_n^l - \hat{\pi}_n^{l,N}][G_p^l \otimes 1] - \frac{\hat{\pi}_n^{l,N}(1 \otimes D_{p,n}^{l-1}(G_n^{l-1}\varphi^{l-1}))}{\hat{\pi}_n^{l,N}(1 \otimes G_p^{l-1})}[\hat{\pi}_n^l - \hat{\pi}_n^{l,N}](1 \otimes G_p^{l-1}) \right\}.
\]

Using the \(C_2\)-inequality multiple times:

\[
E[(\hat{\pi}_n^{l,N} - \hat{\pi}_n^l)((G_n^l\varphi^l) \otimes 1 - 1 \otimes (G_n^{l-1}\varphi^{l-1}))^2] \leq C \left( \sum_{p=0}^n E[T_1(p)^2] + \sum_{p=0}^{n-1} E[T_2(p)^2] \right)
\]

where

\[
T_1(p) := (\hat{\pi}_n^{l,N} - \hat{\pi}_n^{l,N}(\hat{\pi}_{p-1}^l))(D_{p,n}^l(G_n^l\varphi^l) \otimes 1 - 1 \otimes D_{p,n}^{l-1}(G_n^{l-1}\varphi^{l-1}))
\]

\[
T_2(p) := \frac{\hat{\pi}_n^{l,N}(D_{p,n}^l(G_n^l\varphi^l) \otimes 1)}{\hat{\pi}_n^{l,N}(G_p^l \otimes 1)}[\hat{\pi}_n^l - \hat{\pi}_n^{l,N}][G_p^l \otimes 1] - \frac{\hat{\pi}_n^{l,N}(1 \otimes D_{p,n}^{l-1}(G_n^{l-1}\varphi^{l-1}))}{\hat{\pi}_n^{l,N}(1 \otimes G_p^{l-1})}[\hat{\pi}_n^l - \hat{\pi}_n^{l,N}](1 \otimes G_p^{l-1}).
\]
It thus suffices to control the terms \(T_1(p), p \in \{0, 1, \ldots, n\}\) and \(T_2(p), p \in \{0, 1, \ldots, n - 1\}\) in an appropriate way. Now, by the conditional Marcinkiewicz-Zygmund inequality
\[
\mathbb{E}[T_1(p)^2] \leq \frac{C}{N} \mathbb{E}[(D_{p,n}^l(G_n^l\varphi))(U_p^{l,1}) - D_{p,n}^{l-1}(G_n^{l-1}\varphi^{l-1})(U_p^{l-1,1}))^2].
\]
(15)
For \(T_2(p)\) we have
\[
T_2(p) = T_3(p) + T_4(p) + T_5(p)
\]
where
\[
T_3(p) := [\hat{\pi}_p^l - \hat{\pi}_p^{l,N}](G_p^l \otimes 1) \frac{(D_{p,n}^l(G_n^l\varphi)(G_p^l \otimes 1))}{\hat{\pi}_p^{l,N}(G_p^l \otimes 1)(1 \otimes G_p^{l-1})}\left\{\hat{\pi}_p^l(1 \otimes G_p^{l-1}) - \hat{\pi}_p^{l,N}(G_p^l \otimes 1)\right\}
\]
(16)
\[
T_4(p) := [\hat{\pi}_p^l - \hat{\pi}_p^{l,N}](G_p^l \otimes 1) \frac{1}{\hat{\pi}_p^{l,N}(1 \otimes G_p^{l-1})}\left\{\hat{\pi}_p^{l,N}(D_{p,n}^l(G_n^{l-1}\varphi^{l-1})) - \hat{\pi}_p^l(1 \otimes D_{p,n}^{l-1}(G_n^{l-1}\varphi^{l-1}))\right\}
\]
(17)
\[
T_5(p) := \frac{\hat{\pi}_p^{l,N}(1 \otimes D_{p,n}^{l-1}(G_n^{l-1}\varphi^{l-1})))}{\hat{\pi}_p^l(1 \otimes G_p^{l-1})}[\hat{\pi}_p^l - \hat{\pi}_p^{l,N}](G_p^l \otimes 1 - 1 \otimes G_p^{l-1}).
\]
(18)
By using the \(C_2\)-inequality three times \(\mathbb{E}[T_2(p)^2] \leq C \sum_{j=3}^5 \mathbb{E}[T_j(p)^2]\), so we consider bounding the R.H.S. of this inequality. For \(T_3(p)\), using Cauchy-Schwarz twice gives
\[
\mathbb{E}[T_3(p)^2] \leq \mathbb{E}[(\hat{\pi}_p^l - \hat{\pi}_p^{l,N})(G_p^l \otimes 1)^{8/4}]^{1/4}[\left\{\frac{\hat{\pi}_p^{l,N}(D_{p,n}^l(G_n^l\varphi)(G_p^l \otimes 1))}{\hat{\pi}_p^{l,N}(G_p^l \otimes 1)(1 \otimes G_p^{l-1})}\right\}]^{8/4}\left\{\mathbb{E}[\hat{\pi}_p^l(1 \otimes G_p^{l-1}) - \hat{\pi}_p^{l,N}(G_p^l \otimes 1)]\right\}^{4/2}.
\]
For the left-most term on the R.H.S. one can apply Proposition A.3. For the middle-term on the R.H.S. one can apply Hölder, Lemma A.2 and Corollary A.3. For the right-most term on the R.H.S. one has \(\mathbb{E}[(\hat{\pi}_p^l(1 \otimes G_p^{l-1}) - \hat{\pi}_p^{l,N}(G_p^l \otimes 1))^4]^{1/2} \leq \mathbb{E}[(G_p^l(U_p^{l,1}) - G_p^{l-1}(U_p^{l-1,1}))^4]^{1/2}\). Hence, we have that
\[
\mathbb{E}[T_3(p)^2] \leq \frac{C}{N^2} \mathbb{E}[(G_p^l(U_p^{l,1}) - G_p^{l-1}(U_p^{l-1,1}))^4]^{1/2}.
\]
(19)
For \(T_4(p)\), using almost the same strategy yields
\[
\mathbb{E}[T_4(p)^2] \leq \frac{C}{N} \mathbb{E}[(D_{p,n}^l(G_n^l\varphi)(U_p^{l,1}) - D_{p,n}^{l-1}(G_n^{l-1}\varphi^{l-1})(U_p^{l-1,1}))^4]^{1/2}.
\]
(20)
For \(T_5(p)\), we have
\[
\mathbb{E}[T_5(p)^2] \leq \mathbb{E} \left[ \left(\frac{\hat{\pi}_p^{l,N}(1 \otimes D_{p,n}^{l-1}(G_n^{l-1}\varphi^{l-1}))}{\hat{\pi}_p^{l,N}(1 \otimes G_p^{l-1})}\right)^2 \right] + \mathbb{E} \left[ \left(\frac{\hat{\pi}_p^{l,N}(1 \otimes G_p^{l-1})(G_n^{l-1}\varphi^{l-1}))}{\hat{\pi}_p^{l,N}(1 \otimes G_p^{l-1})}\right)^2 \right] \right).
\]
Using Cauchy-Schwarz four times, Lemma A.2 and Corollary A.3 gives
\[
\mathbb{E}[T_5(p)^2] \leq \frac{C}{N} \left( \mathbb{E}[(\hat{\pi}_p^l(G_p^l) - \hat{\pi}_p^{l,N}(G_p^l)^{1/2} + \mathbb{E}[(G_p^l(U_p^{l,1}) - G_p^{l-1}(U_p^{l-1,1}))^4]^{1/2}\right).
\]
(21)
Combining (19)-(21) gives
\[
\mathbb{E}[T_2(p)^2] \leq \frac{C}{N} T_{p,n}(\varphi).
\]
(22)
The proof is easily completed by noting the bounds in (14), (15) and (22).

**Lemma A.2.** Assume (D1). Then for any \((p, n, r) \in \{0, 1, \ldots\}^2 \times \mathbb{N}, n \geq p,\) there exists a \(C < +\infty\) such that for any \(l \in \{0, 1, \ldots\}, (i, \varphi) \in \{1, \ldots, N\} \times B_i(E_i)\)

\[
\max\{\mathbb{E}[\pi_p^l(Q_p,n(\varphi)^{-1}), \mathbb{E}[\pi_p^{l,N}(Q_p,n(\varphi)^{-1}), \mathbb{E}[Q_p,n(\varphi)(U_p^{l,i})^r]^{1/r}] \leq C\|\varphi\|.
\]

Proof. We start by considering $\mathbb{E}[Q_{p,n}(\varphi)(U^{l,i}_p)^r]^{1/r}$. Applying Jensen’s inequality, we have the upper-bound:

$$
\mathbb{E}[Q_{p,n}^l(\varphi)(U^{l,i}_p)^r] \leq \|\varphi\|^r \mathbb{E}[A_{p,n}(\varphi)_p]^r \prod_{q=p+1}^{n-1} C_{q}^l(U_q)^r
$$

where $U_{p+1}, \ldots, U_n$ is a Markov chain of initial distribution $M^l(U^{l,i}_p, \cdot)$ and transition $M^l$. As $h$ is bounded, we have the upper-bound

$$
\mathbb{E}[Q_{p,n}^l(\varphi)(U^{l,i}_p)^r] \leq C\|\varphi\|^r \mathbb{E}\left[\exp\left\{\frac{\Delta^{-1}}{2} \sum_{s_1=0}^{p} \sum_{s_2=1}^{d_p} h^{(s_2)}(X^{l,i}_{p+s_1}) (Y^{r(s_2)}_{p+s_1,\Delta_i} - Y^{r(s_2)}_{p+s_1,\Delta_i})\right\} \times \exp\left\{\frac{n-1}{2} \sum_{q=p+1}^{d_p} \sum_{s_1=0}^{p} \sum_{s_2=1}^{d_p} h^{(s_2)}(X^{l,i}_{q+s_1}) (Y^{r(s_2)}_{q+s_1,\Delta_i} - Y^{r(s_2)}_{q+s_1,\Delta_i})\right\}\right].
$$

Taking expectations w.r.t. the process $\{Y_t\}$ we have

$$
\mathbb{E}[Q_{p,n}^l(\varphi)(U^{l,i}_p)^r] \leq C\|\varphi\|^r \mathbb{E}\left[\exp\left\{\frac{\Delta^{-1}}{2} \sum_{s_1=0}^{p} \sum_{s_2=1}^{d_p} h^{(s_2)}(X^{l,i}_{p+s_1})^2 \Delta_i\right\} \times \exp\left\{\frac{n-1}{2} \sum_{q=p+1}^{d_p} \sum_{s_1=0}^{p} \sum_{s_2=1}^{d_p} h^{(s_2)}(X^{l,i}_{q+s_1})^2 \Delta_i\right\}\right].
$$

Then using the fact that $h$ is bounded, it clearly follows

$$
\mathbb{E}[Q_{p,n}^l(\varphi)(U^{l,i}_p)^r]^{1/r} \leq C\|\varphi\|.
$$

For the terms $\mathbb{E}[\pi^l_p(Q_{p,n}^l(\varphi))^{-r}]$ and $\mathbb{E}[\pi^l_p(N(Q_{p,n}^l(\varphi)))^{-r}]$ one can apply (the conditional) Jensen’s inequality and essentially the same argument as above and hence the proof is omitted. $\square$

A.1.1 Additional Technical Results for Coupled Particle Filters

The following Section is essentially an adaptation of [15, Lemmata D.3-D.4]. Many of the arguments are very similar to that article, with a modification to the context here. The entire proofs are included for completeness of this paper.

For $(i,l,n) \in \{1, \ldots, N\} \times N \times \{0,1, \ldots\}$:

- $\widehat{U}_n^{l,i}, \bar{U}_n^{l,i}$ denote the particles immediately after resampling
- $(I_n^{l,i}, \bar{I}_n^{l-1,i}) \in \{1, \ldots, N\}$ represent the resampled indices of $(u_n^{l,i}, \bar{u}_n^{l-1,i})$ and let $I_n^{l}(i) := I_n^{l,i}$ and $\bar{I}_n^{l-1}(i) := \bar{I}_n^{l-1,i}$.

For $(l,n) \in \{0,1, \ldots\}$, let $S_n^l$ be the particle indices that choose the same ancestor at each resampling stage:

$$
S_n^l = \{i \in \{1, \ldots, N\} : I_n^{l}(i) = \bar{I}_n^{l-1}(i), I_n^{l-1} \circ I_n^{l}(i) = \bar{I}_n^{l-1} \circ \bar{I}_n^{l-1}(i), \ldots, I_0 \circ \ldots \circ I_n^{l}(i) = \bar{I}_0 \circ \ldots \circ \bar{I}_n^{l-1}(i)\}.
$$

For $n = -1$, set $S_0^l = \{1, \ldots, N\}$. Let, for $(l,n) \in \{0,1, \ldots\}$

\[
\mathcal{G}_n^l = \sigma \left( \left\{ U_p^{l,i}, \bar{U}_p^{l-1,i}, \widehat{U}_p^{l,i}, \bar{U}_p^{l-1,i}, I_p^{l,i}, \bar{I}_p^{l-1,i} : 0 \leq p < n, 1 \leq i \leq N \right\} \cup \left\{ U_n^{l,i}, \bar{U}_n^{l-1,i}, 1 \leq i \leq N \right\} \right) \lor \mathcal{Y}_{n+1}^l,
\]
\[
\mathcal{G}_n^l = \sigma \left( \left\{ U_p^{l,i}, \bar{U}_p^{l-1,i}, \widehat{U}_p^{l,i}, \bar{U}_p^{l-1,i}, I_p^{l,i}, \bar{I}_p^{l-1,i} : 0 \leq p < n, 1 \leq i \leq N \right\} \cup \left\{ U_n^{l,i}, \bar{U}_n^{l-1,i}, \bar{U}_n^{l,i}, \bar{I}_n^{l-1,i}, 1 \leq i \leq N \right\} \right) \lor \mathcal{Y}_{n+1}^l.
\]

To avoid ambiguity in the subsequent notations, we set for $(i,l,n) \in \{1, \ldots, N\} \times N \times \{0,1, \ldots\}$

\[
\begin{align*}
\overline{u}_{n}^{l,i} & = (x_{n,n}^{l,i}, x_{n,n+\Delta_1}^{l,i}, \ldots, x_{n,n+\Delta_{l-1}}^{l,i}) \in E_l \\
\widehat{u}_{n}^{l,i} & = (x_{n,n}^{l-1,i}, \overline{u}_{n}^{l-1,i}, \overline{u}_{n}^{l-1,i}, x_{n,n+\Delta_1}^{l-1,i}, \ldots, x_{n,n+\Delta_{l-1}}^{l-1,i}) \in E_{l-1}.
\end{align*}
\]
Lemma A.3. Assume (D1). Then for any \((n, r) \in \{0, 1, \ldots \} \times \mathbb{N}\), there exists a \(C < +\infty\) such that for any \((l, N) \in \mathbb{N} \times \mathbb{N}\)

\[
\mathbb{E} \left[ \frac{1}{N} \sum_{i \in \mathcal{S}_{n-1}^l} \| X_{n,n+1}^{l,i} - \bar{X}_{n,n+1}^{l-1,i} \|_2^r \right]^{1/r} \leq C \Delta_l^{1/2}.
\]

Proof. The proof is by induction on \(n\). The case \(n = 0\) follows immediately, for instance by [15, Proposition D.1]. For a general \(n\); following the first four lines of the proof of [15, Lemma D.3], one has

\[
\mathbb{E} \left[ \frac{1}{N} \sum_{i \in \mathcal{S}_{n-1}^l} \| X_{n,n+1}^{l,i} - \bar{X}_{n,n+1}^{l-1,i} \|_2^r \right]^{1/r} \leq C \mathbb{E} \left[ \frac{1}{N} \sum_{i \in \mathcal{S}_{n-1}^l} \| X_{n-1,n}^{l,i} - \bar{X}_{n-1,n}^{l-1,i} \|_2^r \right]^{1/r} + C \Delta_l^{1/2}.
\]

Now, \(I_{n-1} = I_{n-1}^{l-1,i} \) for \(i \in \mathcal{S}_{n-1}^l\). The conditional distribution of \((X_{n-1,n}^{l,i}, X_{n-1,n}^{l-1,i})\) \((i \in \mathcal{S}_{n-1}^l)\) given \(\mathcal{S}_{n-1}^l\) and \(\mathcal{G}_{n-1}^l\) is

\[
\frac{\sum_{i \in \mathcal{S}_{n-2}} \mathbb{G}_{n-1}^l(I_{n-1}^i) \mathcal{G}_{n-1}^l(I_{n-1}^{l-1,i}) \delta(X_{n-1,n}^{l,i}, X_{n-1,n}^{l-1,i})}{\sum_{i \in \mathcal{S}_{n-2}} \mathbb{G}_{n-1}^l(I_{n-1}^i) \mathcal{G}_{n-1}^l(I_{n-1}^{l-1,i}) \delta(X_{n-1,n}^{l,i}, X_{n-1,n}^{l-1,i})}.
\]

Almost surely, it follows:

\[
\mathbb{E} \left[ \frac{\text{Card}(\mathcal{S}_{n-1}^l)}{N} \right] \leq \frac{\text{Card}(\mathcal{S}_{n-2}^l)}{N}.
\]

As \(h\) is bounded, there exists \(-\infty < C < \bar{C} < +\infty\), such that for any \((l, n) \in \mathbb{N} \times \{0, 1, \ldots \}\) and \(u_n \in E_{l1}\), almost surely

\[
\mathbb{G}_n^l(u_n) \leq \mathcal{G}_n^l \quad \mathbb{G}_n^l(u_n) \geq \mathcal{G}_n^l
\]

where

\[
\mathcal{G}_n^l = \mathcal{G}_n \exp \left\{ \sum_{i=0}^{\Delta_l^{-1} - 1} \sum_{k=1}^{d_y} \mathcal{G}_{n+\Delta_l}^i \left( (Y(k)_{n+(i+1)\Delta_l} - Y(k)_{n+i\Delta_l}) (Y(k)_{n+(i+1)\Delta_l} - Y(k)_{n+i\Delta_l}) \right) \right\}, \quad \mathcal{G}_n^l = \mathcal{G}_n \exp \left\{ \sum_{i=0}^{\Delta_l^{-1} - 1} \sum_{k=1}^{d_y} \mathcal{G}_{n+\Delta_l}^i \left( (Y(k)_{n+(i+1)\Delta_l} - Y(k)_{n+i\Delta_l}) (Y(k)_{n+(i+1)\Delta_l} - Y(k)_{n+i\Delta_l}) \right) \right\}. \]

Moreover, for any \(r \in \mathbb{N}\), it is straightforward to verify that these upper and lower bounds have finite \(L_r\) and \(L_{-r}\) moments that do not depend upon \(l\).
Therefore
\[
\mathbb{E} \left[ \frac{1}{N} \sum_{i \in S_{n-1}} \| X_{n-1,n}^{i,i} - \bar{X}_{n-1,n}^{i,i} \|_2^2 \right] = \mathbb{E} \left[ \frac{1}{N} \sum_{i \in S_{n-1}} \mathbb{E} \left[ \| X_{n-1,n}^{i,i} - \bar{X}_{n-1,n}^{i-1,i} \|_2^2 \right] \mathcal{S}_{n-1} \right] = \mathbb{E} \left[ \frac{\text{Card}(S_{n-1})}{N} \left( \sum_{i \in S_{n-2}} \sum_{k=1}^{N} G_{n-1}^{i,(U_{n,k}^{i,i})} \wedge \sum_{k=1}^{N} G_{n-1}^{i-1,(U_{n,k}^{i-1,i})} \| X_{n-1,n}^{i,i} - \bar{X}_{n-1,n}^{i,i} \|_2^2 \right) \right] = \mathbb{E} \left[ \frac{\text{Card}(S_{n-1})}{N} \left( \sum_{i \in S_{n-2}} \sum_{k=1}^{N} G_{n-1}^{i,(U_{n,k}^{i,i})} \wedge \sum_{k=1}^{N} G_{n-1}^{i-1,(U_{n,k}^{i-1,i})} \| X_{n-1,n}^{i,i} - \bar{X}_{n-1,n}^{i,i} \|_2^2 \right) \right] \leq \mathbb{E} \left[ \frac{\text{Card}(S_{n-1})}{N} \left( \sum_{i \in S_{n-2}} \sum_{k=1}^{N} G_{n-1}^{i,(U_{n,k}^{i,i})} \wedge \sum_{k=1}^{N} G_{n-1}^{i-1,(U_{n,k}^{i-1,i})} \| X_{n-1,n}^{i,i} - \bar{X}_{n-1,n}^{i,i} \|_2^2 \right) \right].
\]

Taking expectations w.r.t. the data on the time interval \([n-1,n]\) yields:
\[
\mathbb{E} \left[ \frac{1}{N} \sum_{i \in S_{n-1}} \| X_{n-1,n}^{i,i} - \bar{X}_{n-1,n}^{i-1,i} \|_2^2 \right] \leq C \mathbb{E} \left[ \frac{1}{N} \sum_{i \in S_{n-2}} \| X_{n-1,n}^{i,i} - \bar{X}_{n-1,n}^{i,i} \|_2^2 \right].
\]
The result follows by induction. \(\square\)

**Corollary A.1.** Assume (D1). Then for any \((n,r) \in \mathbb{N} \times \mathbb{N}\), there exists a \(C < +\infty\) such that for any \((l,N) \in \mathbb{N} \times \mathbb{N}\)
\[
\mathbb{E} \left[ \frac{1}{N} \sum_{i \in S_{n-1}} \| \bar{X}_{n-1,n}^{i,i} - \bar{X}_{n-1,n}^{l-1,i} \|_2^2 \right]^{1/r} \leq C \mathbb{E} \left[ \frac{1}{N} \sum_{i \in S_{n-2}} \| X_{n-1,n}^{i,i} - \bar{X}_{n-1,n}^{i,i} \|_2^2 \right].
\]

**Proof.** Easily follows from the proof of Lemma A.3. \(\square\)

**Lemma A.4.** Assume (D1). Then for any \(n \in \{-1,0,1,\ldots\}\), there exists a \(C < +\infty\) such that for any \((l,N) \in \mathbb{N} \times \mathbb{N}\)
\[
1 - \mathbb{E} \left[ \text{Card}(S_{n}^{l}) \right] \leq C \mathbb{E}^{1/2}.
\]

**Proof.** The proof is by induction, with the initialization clear. We have
\[
1 - \mathbb{E} \left[ \text{Card}(S_{n}^{l}) \right] \leq \left( 1 - \mathbb{E} \left[ \sum_{i=1}^{N} G_{n-1}^{l,(U_{n,k}^{i,i})} \wedge \sum_{k=1}^{N} G_{n-1}^{i-1,(U_{n,k}^{i-1,i})} \right] \right) \leq \frac{1}{2} \mathbb{E} \left[ \sum_{i \in S_{n-1}^{l}} \sum_{k=1}^{N} G_{n-1}^{i,(U_{n,k}^{i,i})} - \sum_{k=1}^{N} G_{n-1}^{i-1,(U_{n,k}^{i-1,i})} \right] + \frac{1}{2} \sum_{i \in S_{n-1}^{l}} \sum_{k=1}^{N} G_{n-1}^{i,(U_{n,k}^{i-1,i})} - \sum_{k=1}^{N} G_{n-1}^{i-1,(U_{n,k}^{i-1,i})} \right] \right) \leq \mathbb{E} \left[ \frac{\text{Card}(S_{n-1})^{(l)}}{N} \left( \frac{G_{n-1}^{l}}{G_{n}^{l}} + \frac{G_{n-1}^{i-1}}{G_{n}^{i-1}} \right) \right] \leq C \left( 1 - \mathbb{E} \left[ \text{Card}(S_{n-1}^{l}) \right] \right).
\]
To conclude the result, we must appropriately deal with the left-most term on the R.H.S. of (25). We have

\[
\mathbb{E}\left[ \sum_{i \in S_{n-1}} \left| \frac{G_n^l(U_{n}^{l,i})}{\sum_{k=1}^{N} G_n^l(U_{n}^{l,k})} - \frac{G_n^{l-1}(U_{n}^{l-1,i})}{\sum_{k=1}^{N} G_n^{l-1}(U_{n}^{l-1,k})} \right| \right] \leq T_1 + T_2
\]

where

\[
T_1 := \mathbb{E}\left[ \frac{1}{\sum_{k=1}^{N} G_n^l(U_{n}^{l,k})} \sum_{i \in S_{n-1}} \left| G_n^l(U_{n}^{l,i}) - G_n^{l-1}(U_{n}^{l-1,i}) \right| \right]
\]

\[
T_2 := \mathbb{E}\left[ \sum_{i \in S_{n-1}} G_n^l(U_{n}^{l-1,i}) \left( \frac{\sum_{k=1}^{N} G_n^{l-1}(U_{n}^{l-1,k}) - \sum_{k=1}^{N} G_n^l(U_{n}^{l,k})}{\sum_{k=1}^{N} G_n^{l-1}(U_{n}^{l-1,k}) \sum_{k=1}^{N} G_n^l(U_{n}^{l,k})} \right) \right]
\]

For \( T_1 \), applying Cauchy-Schwarz and recalling the bounds from (23) and (24),

\[
T_1 \leq \mathbb{E}\left[ \frac{1}{(G_n^l)^2} \right]^{1/2} \mathbb{E}\left[ \left( \frac{1}{N} \sum_{i \in S_{n-1}} \left| G_n^l(U_{n}^{l,i}) - G_n^{l-1}(U_{n}^{l-1,i}) \right| \right)^2 \right]^{1/2}
\]

Then applying Jensen and noting that the left-most term on the R.H.S. is upper-bounded by a constant that does not depend upon \( l \) nor \( N \) we have

\[
T_1 \leq C \mathbb{E}\left[ \text{Card}(S_{n-1}) \frac{1}{N} \sum_{i \in S_{n-1}} \left| G_n^l(U_{n}^{l,i}) - G_n^{l-1}(U_{n}^{l-1,i}) \right|^2 \right]^{1/2}.
\]

Conditioning upon \( \tilde{G}_{n-1}^l \) and applying Lemma A.9 followed by Corollary A.1 yields the upper-bound

\[
T_1 \leq C \Delta_l^{1/2}.
\]

For \( T_2 \), using the bounds (23) and (24), one has

\[
T_2 \leq \mathbb{E}\left[ \text{Card}(S_{n-1}) \frac{1}{N} \sum_{i \in S_{n-1}} \left| G_n^l(U_{n}^{l,i}) - G_n^{l-1}(U_{n}^{l-1,i}) \right| \right].
\]

Then it easily follows

\[
T_2 \leq \mathbb{E}\left[ \text{Card}(S_{n-1}) \left( \frac{G_n^l}{G_n^l \tilde{G}_n^l} \right) \left( \left| \frac{1}{N} \sum_{i \in S_{n-1}} \left( G_n^l(U_{n}^{l,i}) - G_n^{l-1}(U_{n}^{l-1,i}) \right) \right| + \frac{\text{Card}(S_{n-1})}{N} \left( \frac{G_n^l}{G_n^l \tilde{G}_n^l} \right) \right) \right]
\]

Now we set

\[
T_3 := \mathbb{E}\left[ \text{Card}(S_{n-1}) \left( \frac{G_n^l}{G_n^l \tilde{G}_n^l} \right) \left| \frac{1}{N} \sum_{i \in S_{n-1}} \left( G_n^l(U_{n}^{l,i}) - G_n^{l-1}(U_{n}^{l-1,i}) \right) \right| \right],
\]

\[
T_4 := \mathbb{E}\left[ \text{Card}(S_{n-1}) \left( \frac{G_n^l}{G_n^l \tilde{G}_n^l} \right) \frac{\text{Card}(S_{n-1})}{N} \left( \frac{G_n^l}{G_n^l \tilde{G}_n^l} \right) \right].
\]

For \( T_3 \), applying Cauchy Schwarz and Jensen

\[
T_3 \leq \mathbb{E}\left[ \left( \frac{\text{Card}(S_{n-1})}{N} \left( \frac{G_n^l}{G_n^l \tilde{G}_n^l} \right) \right)^2 \right]^{1/2} \mathbb{E}\left[ \left( \frac{1}{N} \sum_{i \in S_{n-1}} \left( G_n^l(U_{n}^{l,i}) - G_n^{l-1}(U_{n}^{l-1,i}) \right) \right)^2 \right]^{1/2}.
\]
Again, noting that the left-most term on the R.H.S. is upper-bounded by a constant that does not depend upon \( l \) nor \( N \)

\[
T_3 \leq C \mathbb{E} \left[ \frac{1}{N} \sum_{i \in S_{n-1}^l} \left( G_n^l(U_n^{i,l}) - G_n^{l-1}(U_n^{i,l-1}) \right)^2 \right]^{1/2}.
\]

Then, applying the above arguments, we have

\[
T_3 \leq C \Delta_l^{1/2}.
\]

For \( T_4 \), taking expectations w.r.t. the data on the time interval \([n, n + 1]\) yields:

\[
T_4 \leq C \mathbb{E} \left[ \text{Card}(\{S_{n-1}^l\}) \right].
\]

Thus, we have

\[
\mathbb{E} \left[ \sum_{i \in S_{n-1}^l} \left| \frac{G_n^l(U_n^{i,l})}{\sum_{k=1}^N G_n^l(U_n^{i,k})} - \frac{G_n^{l-1}(U_n^{i,l-1})}{\sum_{k=1}^N G_n^{l-1}(U_n^{i,k-1})} \right| \right] \leq C \left( \Delta_l^{1/2} + 1 - \mathbb{E} \left[ \frac{\text{Card}(\{S_{n-1}^l\})}{N} \right] \right). \tag{27}
\]

Combining (25), (26), (27), one can conclude the result via induction. \( \square \)

### A.1.2 Rate Proofs for the Coupled Particle Filter

**Lemma A.5.** Assume (D1). Then for any \( n \in \{0, 1, \ldots \} \), there exists a \( C < +\infty \) such that for any \((l, N, \varphi) \in \mathbb{N} \times \mathbb{N} \times \mathbb{B}_0(\mathbb{R}^d) \cap \text{Lip}_2(\mathbb{R}^d)\)

\[
\sum_{p=0}^n T_{p,n}(\varphi) \leq C(\|\varphi\| + \|\varphi\|_{\text{Lip}})^2 \Delta_l.
\]

**Proof.** We consider the three expectations in the definition of \( T_{p,n}(\varphi) \) (for \( p < n \), the case \( p = n \) also follows easily from our arguments). For the terms of the type \( \mathbb{E}[(\pi_p^l(G_p^l) - \pi_p^{l-1}(G_p^{l-1}))^{41/2}] \), one can apply Corollary A.2, so we need only consider the other two expectations. As the term \( \mathbb{E}[(D_{p,n}^l(G_n^l, \varphi^l))(U_p^{l,1}) - D_{p,n}^{l-1}(G_n^{l-1, \varphi^{l-1}})(U_p^{l-1,1}))^{41/2} \) is more difficult to deal with and the approach can be used for the other expectation, we consider only this term.

We have the upper-bound

\[
\mathbb{E}[(D_{p,n}^l(G_n^l, \varphi^l))(U_p^{l,1}) - D_{p,n}^{l-1}(G_n^{l-1, \varphi^{l-1}})(U_p^{l-1,1}))^{4}] \leq 40 \sum_{j=1}^5 T_j
\]

where

\[
T_1 := \mathbb{E} \left[ \left( \frac{1}{\pi_p^l(Q_{p,n}(1))} \{Q_{p,n}(G_n^l, \varphi^l)(U_p^{l,1}) - Q_{p,n}^{l-1}(G_n^{l-1, \varphi^{l-1}})(U_p^{l-1,1}) \} \right)^4 \right],
\]

\[
T_2 := \mathbb{E} \left[ \left( \frac{Q_{p,n}^{l-1}(G_n^{l-1, \varphi^{l-1}})(U_p^{l-1,1})}{\pi_p^l(Q_{p,n}(1))} \{Q_{p,n}^l(1) - \pi_p^l(Q_{p,n}(1)) \} \right)^4 \right],
\]

\[
T_3 := \mathbb{E} \left[ \left( \frac{Q_{p,n}^l(1)(U_p^{l-1,1})}{\pi_p^{l-1}(Q_{p,n}(1))} \{Q_{p,n}^l(1) - \pi_p^l(Q_{p,n}(1)) \} \right)^4 \right],
\]

\[
T_4 := \mathbb{E} \left[ \left( \frac{m_p^l(G_n^l, \varphi^l)}{\pi_p^l(Q_{p,n}(1))} \{Q_{p,n}^l(1)(U_p^{l,1}) - Q_{p,n}^{l-1}(U_p^{l-1,1}) \} \right)^4 \right],
\]

\[
T_5 := \mathbb{E} \left[ \left( \frac{Q_{p,n}^l(1)(U_p^{l,1})}{\pi_p^l(Q_{p,n}(1))} \{Q_{p,n}^l(1) - \pi_p^l(Q_{p,n}(1)) \} \right)^4 \right].
\]

We will give bounds on the terms \( T_1 \) and \( T_2 \) only. The proofs for appropriate bounds on \( T_3, T_4, T_5 \) are very similar and hence omitted.
For $T_1$ applying Cauchy-Schwarz and Lemma A.2

$$
T_1 \leq C\mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N}(Q_{p,n}^{i}(G_n^i\varphi')(U_p^{i,1}) - Q_{p,n}^{i-1}(G_n^{i-1}\varphi')((U^{i-1,1})_p)^{8}\right]^{1/2}
$$

$$
\leq C\left(\mathbb{E}\left[\frac{1}{N}\sum_{i\in\mathcal{S}_p_{-1}}(Q_{p,n}^{i}(G_n^i\varphi')(U_p^{i,1}) - Q_{p,n}^{i-1}(G_n^{i-1}\varphi')((U^{i-1,1})_p)^{8}\right]\right)^{1/8}
$$

$$
+ \mathbb{E}\left[\frac{1}{N}\sum_{i\in(S_{p-1})^c}(Q_{p,n}^{i}(G_n^i\varphi')(U_p^{i,1}) - Q_{p,n}^{i-1}(G_n^{i-1}\varphi')((U^{i-1,1})_p)^{8}\right]^{1/8}4
$$

For the left-most term on the R.H.S. conditioning upon $\mathcal{G}_p^{i-1}$ and applying conditional Jensen, followed by Lemma A.9 gives the upper-bound

$$
T_1 \leq C\left(\|\varphi\|_2 + \|\varphi\|_{\text{Lip}}\right)^4 \left(\Delta_1^{1/2} + 1 - \mathbb{E}\left[\frac{\text{Card}(\mathcal{S}_p^{i-1})}{N}\right]\right)^{4}.
$$

Then applying Lemma A.4 one has

$$
T_1 \leq C(\|\varphi\|_2 + \|\varphi\|_{\text{Lip}})^4 \Delta_1^2.
$$

For $T_2$ applying Cauchy-Schwarz and Corollary A.2

$$
T_2 \leq C\mathbb{E}\left[\left(\frac{Q_{p,n}^{i-1}(G_n^{i-1}\varphi')(U_p^{i-1,1})}{\pi_p^{i}(Q_{p,n}^{i,1}(G_n^{i-1}\varphi')((U^{i-1,1})_p)^{8}}\right)^{1/2}\Delta_1^2.
$$

Applying Hölder (twice) and Lemma A.2 one has

$$
T_2 \leq C(\|\varphi\|_2 + \|\varphi\|_{\text{Lip}})^4 \Delta_1^2.
$$

Hence we have shown that

$$
\mathbb{E}\left[\left(D_p^{i,1}(G_n^i\varphi')(U_p^{i,1}) - D_p^{i-1}(G_n^{i-1}\varphi')(U^{i-1,1})\right)^{4}\right] \leq C(\|\varphi\|_2 + \|\varphi\|_{\text{Lip}})^4 \Delta_1^2
$$

and the argument can be concluded from here.

**Remark A.1.** One can also prove the following result, using the arguments in Lemmata A.1 and A.5. Assume (D1). Then for any $(n,p) \in \{0,1,\ldots\}^2$, $n \geq p$ there exists a $C < +\infty$ such that for any $(l,N,\varphi) \in \mathbb{N} \times \mathbb{N} \times \mathcal{B}_b(\mathbb{R}^{d_x}) \cap \text{Lip}_{\|\_\|}(\mathbb{R}^{d_x})$

$$
\mathbb{E}\left[\left(\pi_p^{i,N}(D_p^{i,n}(G_n^i\varphi') \otimes 1) - \pi_p^{i,N}(1 \otimes D_p^{i-1,n}(G_n^{i-1}\varphi'))\right)^2\right] C(\|\varphi\|_2 + \|\varphi\|_{\text{Lip}})^2 \Delta_1^2.
$$

**Proposition A.1.** Assume (D1). Then for any $n \in \{0,1,\ldots\}$, there exists a $C < +\infty$ such that for any $(l,N,\varphi) \in \mathbb{N} \times \mathbb{N} \times \mathcal{B}_b(\mathbb{R}^{d_x}) \cap \text{Lip}_{\|\_\|}(\mathbb{R}^{d_x})$

$$
\mathbb{E}\left[\left([\eta_n^l - \eta_n^{l-1}]^N(\varphi) - [\eta_n^l - \eta_n^{l-1}](\varphi)\right)^2\right] \leq C(\|\varphi\|_2 + \|\varphi\|_{\text{Lip}})^2 \Delta_1^2.
$$
Proof. The result follows by using [15, Lemma C.5] along with Lemmata A.1, A.5, along with Corollary A.2 (see also Remark A.3).

Proposition A.2. Assume (D1). Then for any \( n \in \{0, 1, \ldots \} \), there exists a \( C < +\infty \) such that for any \( (l, N, \varphi) \in \mathbb{N} \times \mathbb{N} \times B_0(\mathbb{R}^{d_m}) \cap \text{Lip}_{\|\cdot\|_2}(\mathbb{R}^{d_m}) \)

\[
\mathbb{E} \left[ \|\varphi - \varphi_n\|^{N} \right] \leq C(\|\varphi\| + \|\varphi\|_{\text{Lip}}) \frac{\Delta_l^{1/2}}{N}.
\]

Proof. We first note, that one can prove

\[
\mathbb{E} \left[ \|\tilde{\varphi}_n - \tilde{\varphi}_{n-1}\|^{N} \right] \leq C(\|\varphi\| + \|\varphi\|_{\text{Lip}}) \frac{\Delta_l^{1/2}}{N}.
\]

by using the decomposition

\[
\mathbb{E} \left[ \|\tilde{\varphi}_n - \tilde{\varphi}_{n-1}\|^{N} \right] = \sum_{p=0}^{n-1} \mathbb{E} \sum_{j=1}^{3} T_j(p)
\]

where \( T_3(p), T_4(p), T_5(p) \) are defined in (19)-(21). Then one can use Cauchy-Schwarz arguments, Lemmata A.1, A.5 and Proposition A.3 (see also Remark A.1) to deduce (28). The result follows by using a similar argument as in the proof of Lemma A.1.

A.2 Results for the Non-Linear Filter

In this section, we consider the case of the non-linear filter, with a probability space \((\Omega, \mathcal{F})\), with \(\mathcal{F}_t\) the filtration, that includes \(\{Y_t\}_{t \geq 0}\) as standard Brownian motion independent of a diffusion process \(\{X^*_t\}_{t \geq 0}\) which obeys (2) with initial condition \(x \in \mathbb{R}^{d_m}\) and associated Euler discretization (with the same Brownian increments) at level \(l\) \((\tilde{X}^*_l, \tilde{X}^*_l, \ldots)\). We will also consider another diffusion process \(\{X^*_t\}_{t \geq 0}\) which obeys (2), initial condition \(x_* \in \mathbb{R}^{d_m}\) and the same Brownian motion as \(\{X^*_t\}_{t \geq 0}\) and associated Euler discretization (with the same Brownian increments) at level \(l\) \((\tilde{X}^{x_*}_l, \tilde{X}^{x_*}_l, \ldots)\). Expectations are written \(\mathbb{E}\). We set for \((p, n) \in \{0, 1, \ldots\}^2\), \(n+1 > p\)

\[
Z^x_{p,n+1} = \exp \left\{ \int_p^{n+1} h(X^*_{s,t}) dY_s - \frac{1}{2} \int_p^{n+1} h(X^*_{s,t})^2 ds \right\}
\]

with the convention that \(Z^x_{p,n+1} = Z^x_{n+1}\). The technical results in this appendix are critical in proving the results in Appendix A.1. Although some of the results are more-or-less known in the literature (e.g. [18]), we give the proofs for the completeness of the article.

Lemma A.6. Assume (D1). Then for any \((n,r) \in \{0, 1, \ldots\} \times \mathbb{N}\), there exists a \( C < +\infty \) such that for any \((l, \varphi, x) \in \{0, 1, \ldots\} \times B_0(\mathbb{R}^{d_m}) \cap \text{Lip}_{\|\cdot\|_2}(\mathbb{R}^{d_m}) \times \mathbb{R}^{d_m}\)

\[
\mathbb{E} \left[ \|\varphi(\tilde{X}^x_{n+1}) Z^l_{n+1}(\tilde{X}^x_{0}, \tilde{X}^x_{l}, \ldots, \tilde{X}^x_{n+1-l}) \|^{1/r} \right] \leq C(\|\varphi\| + \|\varphi\|_{\text{Lip}}) \Delta_l^{1/2}.
\]

Proof. We have

\[
\mathbb{E} \left[ \|\varphi(\tilde{X}^x_{n+1}) Z^l_{n+1}(\tilde{X}^x_{0}, \tilde{X}^x_{l}, \ldots, \tilde{X}^x_{n+1-l}) \|^{1/r} \right] \leq C(T_1 + T_2)
\]

where

\[
T_1 := \mathbb{E} \left[ \|\varphi(\tilde{X}^x_{n+1}) Z^l_{n+1}(\tilde{X}^x_{0}, \tilde{X}^x_{l}, \ldots, \tilde{X}^x_{n+1-l}) \|^{1/r} \right]
\]

\[
T_2 := \mathbb{E} \left[ \|\varphi(X^x_{n+1}) Z^l_{n+1}(X^x_{0}, X^x_{l}, \ldots, X^x_{n+1-l}) \|^{1/r} \right].
\]

The term \(T_2\) can be treated with a very similar proof to \(T_1\) along the lines of [4, Theorem 21.3], so we will give a proof for \(T_1\) only.

One has

\[
T_1 \leq C(T_3 + T_4)
\]
where
\[
T_3 := \mathbb{E} \left[ \left( \varphi(\tilde{X}_{0}^x) - \varphi(x_{n+1}) \right) Z_{n+1}^l(\tilde{X}_0^x, \tilde{X}_{\Delta}, \ldots, \tilde{X}_{n+1-\Delta}) \right]^2
\]
\[
T_4 := \mathbb{E} \left[ \left( \varphi(x_{n+1}) \left( Z_{n+1}^l(\tilde{X}_0^x, \tilde{X}_{\Delta}, \ldots, \tilde{X}_{n+1-\Delta}) - Z_{n+1}^l(0, X_1^x, \ldots, X_{n+1-\Delta}) \right) \right]^2 \right].
\]

We now need to appropriately upper-bound \( T_3 \) and \( T_4 \). For \( T_3 \), taking expectations of \( Z_{n+1}^l(\tilde{X}_0^x, \tilde{X}_{\Delta}, \ldots, \tilde{X}_{n+1-\Delta}) \) w.r.t. the process \( \{Y_t\} \) and using the fact that \( h \) is bounded (as in the proof of Lemma A.2) along with the fact that \( \varphi \in \text{Lip}_{||\cdot||_2}(\mathbb{R}^{d_x}) \) gives the upper-bound
\[
T_3 \leq C(||\varphi|| + ||\varphi||_{\text{Lip}})^r \mathbb{E}[||\tilde{X}_{n+1}^x - X_{n+1}^x||_2^2].
\]

Then using standard results on Euler discretization of diffusion processes (e.g. [17])
\[
T_3 \leq C(||\varphi|| + ||\varphi||_{\text{Lip}})^r \Delta_t^{r/2}.
\]

For \( T_4 \) as \( \varphi \in \mathcal{B}_0(\mathbb{R}^{d_x}) \), one has
\[
T_4 \leq (||\varphi|| + ||\varphi||_{\text{Lip}})^r \mathbb{E} \left[ \left| Z_{n+1}^l(\tilde{X}_0^x, \tilde{X}_{\Delta}, \ldots, \tilde{X}_{n+1-\Delta}) - Z_{n+1}^l(0, X_1^x, \ldots, X_{n+1-\Delta}) \right|^2 \right].
\]

Now, by the Mean Value Theorem (MVT)
\[
Z_{n+1}^l(\tilde{X}_0^x, \tilde{X}_{\Delta}, \ldots, \tilde{X}_{n+1-\Delta}) - Z_{n+1}^l(0, X_1^x, \ldots, X_{n+1-\Delta}) = \left( H_{n+1}^l(\tilde{X}_0^x, \tilde{X}_{\Delta}, \ldots, \tilde{X}_{n+1-\Delta}) - H_{n+1}^l(0, X_1^x, \ldots, X_{n+1-\Delta}) \right) \right]
\]
\[
\int_0^1 H_{n+1}^{l,s}(\tilde{X}_0^x, \tilde{X}_{\Delta}, \ldots, \tilde{X}_{n+1-\Delta}) \right. \]
where for any \((l, n, (x_0, x_{\Delta}, \ldots, x_{n+1-\Delta}), (x_0', x_{\Delta}', \ldots, x_{n+1-\Delta}'), s) \in \{0, 1, \ldots\}^2 \times (\mathbb{R}^{d_x})^{2(n+1)} \times [0, 1] \)
\[
H_{n+1}^l(x_0, x_{\Delta}, \ldots, x_{n+1-\Delta}) = \log[Z_{n+1}^l(x_0, x_{\Delta}, \ldots, x_{n+1-\Delta})]
\]
\[
\hat{H}_{n+1}^{l,s}(x_0, x_{\Delta}, \ldots, x_{n+1-\Delta}, x_0', x_{\Delta}', \ldots, x_{n+1-\Delta}') = \exp\{sH_{n+1}^l(x_0, x_{\Delta}, \ldots, x_{n+1-\Delta}) + (1-s)H_{n+1}^l(x_0', x_{\Delta}', \ldots, x_{n+1-\Delta}').\}
\]

Then, by using (33) in (32) and applying Cauchy-Schwarz
\[
T_4 \leq (||\varphi|| + ||\varphi||_{\text{Lip}})^r \mathbb{E} \left[ \left| \int_0^1 \hat{H}_{n+1}^{l,s}(\tilde{X}_0^x, \tilde{X}_{\Delta}, \ldots, \tilde{X}_{n+1-\Delta}) \right. \right]
\]
\[
\mathbb{E} \left[ \left| H_{n+1}^l(\tilde{X}_0^x, \tilde{X}_{\Delta}, \ldots, \tilde{X}_{n+1-\Delta}) - H_{n+1}^l(0, X_1^x, \ldots, X_{n+1-\Delta}) \right|^2 \right]^{1/2} \right]
\]

Now, taking expectations w.r.t. the process \( \{Y_t\} \) and using the fact that \( h \) is bounded, there exists a \( C < +\infty \) such that
\[
\sup_{l \geq 0} \sup_{s \in [0, 1]} \mathbb{E} \left[ \left| \hat{H}_{n+1}^{l,s}(\tilde{X}_0^x, \tilde{X}_{\Delta}, \ldots, \tilde{X}_{n+1-\Delta}) \right. \right]
\]
\[
so, via Jensen, we need only deal with the right-most expectation on the R.H.S of (34), call it \( T_5 \). Now
\[
H_{n+1}^l(\tilde{X}_0^x, \tilde{X}_{\Delta}, \ldots, \tilde{X}_{n+1-\Delta}) - H_{n+1}^l(0, X_1^x, \ldots, X_{n+1-\Delta}) = M_{n+1}^l - R_{n+1}^l
\]
where
\[
M_{n+1}^l := \Delta_{l}^{-1(n+1)-1} \sum_{k=0}^{(n+1)-1} \left( h(\tilde{X}_{k\Delta}) - h(X_{k\Delta}^x) \right)(Y_{(k+1)\Delta} - Y_{k\Delta})\]
\[
R_{n+1}^l := \Delta_{l}^{-1} \left( \sum_{k=0}^{(n+1)-1} \left( h(\tilde{X}_{k\Delta}) - h(X_{k\Delta}^x) \right)(h(X_{k\Delta})) \right)\]
and we set $M_0 = R_0 = 0$. Thus, applying the $C_{2r}$--inequality, one has

$$T_5^2 \leq C \left( \mathbb{E}[|M_{n+1}^{d_r}|^{2r}] + \mathbb{E}[|R_{n+1}^{d_r}|^{2r}] \right). \quad (35)$$

We first focus on the first term on the R.H.S. of (35). Applying $C_{2r}$--inequality $d_y$--times, we have

$$\mathbb{E}[|M_{n+1}^{d_r}|^{2r}] \leq C \sum_{i=1}^{d_y} \mathbb{E} \left[ \Delta_i^{-(n+1)-1} \sum_{k=0}^{\Delta_i^{-(n+1)-1}} \left[ h^{(i)}(x_{k \Delta_i}) - h^{(i)}(x_{k+1}) \right]^2 \right]. \quad (36)$$

We consider just the $i^{th}$ summand on the R.H.S., as the argument to be used is essentially exchangeable w.r.t. $i$. As $\{M_{n,t}, F_{n, \Delta_i} \}_{n \in \{0,1, \ldots \}}$ is a Martingale, applying the Burkholder-Gundy-Davis (BGD) inequality, Minkowski inequality, along with $h^{(i)} \in \text{Lip} || \cdot ||_r(\mathbb{R}^{d_x})$:

$$\mathbb{E} \left[ \sum_{k=0}^{\Delta_i^{-(n+1)-1}} \left[ h^{(i)}(x_{k \Delta_i}) - h^{(i)}(x_{k+1}) \right]^2 \right] \leq C \Delta_i^{-r} \mathbb{E} \left[ \sum_{k=0}^{\Delta_i^{-(n+1)-1}} \left| x_{k \Delta_i} - x_{k+1} \right|^{2r} \right]^{1/r}. \quad (37)$$

Then using standard results on Euler discretization of diffusion processes:

$$\mathbb{E} \left[ \sum_{k=0}^{\Delta_i^{-(n+1)-1}} \left[ h^{(i)}(x_{k \Delta_i}) - h^{(i)}(x_{k+1}) \right]^2 \right] \leq C \Delta_i^{-r}. \quad (38)$$

Thus, on returning to (36), we have shown that

$$\mathbb{E}[|M_{n+1}^{d_r}|^{2r}] \leq C \Delta_i^{-r}. \quad (39)$$

Noting that as $h^{(i)} \in \text{Lip} || \cdot ||_2(\mathbb{R}^{d_x})$ and $h^{(i)} \in \mathcal{B}_b(\mathbb{R}^{d_x})$, it follows that $(h^{(i)})^2 \in \text{Lip} || \cdot ||_2(\mathbb{R}^{d_x})$. So using very similar calculations to those for $M_{n+1}$ (except not requiring to apply the BGD inequality), one can prove that

$$\mathbb{E}[|R_{n+1}^{d_r}|^{2r}] \leq C \Delta_i^{-r}. \quad (40)$$

Thus combining (37)-(40) with (35), one has that $T_5 \leq C \Delta_i^{-r/2}$ and hence that

$$T_4 \leq C(|| \varphi || + || \varphi ||_{\text{Lip}})^r \Delta_i^{-r/2}. \quad (41)$$

Noting (30) and using the bounds (31) and (41)

$$T_1 \leq C(|| \varphi || + || \varphi ||_{\text{Lip}})^r \Delta_i^{-r/2}. \quad (42)$$

As noted above, a similar bound can be obtained for $T_2$ and noting (29), the proof is hence concluded. \hfill \Box

**Lemma A.7.** Assume (D1). Then for any $(p,n,r) \in \{0,1,\ldots\} \times \mathbb{N}$, $n > p$ there exists a $C < +\infty$ such that for any $(x,x_*) \in \mathbb{R}^{d_x} \times \mathbb{R}^{d_x}$

$$\mathbb{E} \left[ \left| Z_{p,n}^x - Z_{p,n}^{x_*} \right|^{1/r} \right] \leq C \|x - x_*\|_2. \quad (43)$$

**Proof.** This result can be proved in a similar manner to considering (33) in the proof of Lemma A.6, that is by using the MVT and a Martingale plus remainder method. The main difference is that one must use the result (which can be deduced by [19, Corollary v.11.7] and the Grönwall’s inequality)

$$\sup_{t \in [p,n]} \mathbb{E}[\|X_t^x - X_t^{x_*}\|_2^{2r}]^{1/2r} \leq C \|x - x_*\|_2. \quad (44)$$

The proof is omitted due to the similarity to the proof associated to (33). \hfill \Box

**Lemma A.8.** Assume (D1). Then for any $(n,r) \in \{0,1,\ldots\} \times \mathbb{N}$, there exists a $C < +\infty$ such that for any $(\varphi,x,x_*) \in \mathcal{B}_b(\mathbb{R}^{d_x}) \cap \text{Lip} || \cdot ||_2(\mathbb{R}^{d_x}) \times \mathbb{R}^{d_x} \times \mathbb{R}^{d_x}$

$$\mathbb{E} \left[ \left| \varphi(X_{n+1}^x)Z_{n+1}^x - \varphi(X_{n+1}^{x_*})Z_{n+1}^{x_*} \right|^{1/r} \right] \leq C(|| \varphi || + || \varphi ||_{\text{Lip}}) \|x - x_*\|_2. \quad (45)$$
Proof. We have
\[ E \left[ \left| \varphi(X_{n+1}^x)Z_{n+1}^x - \varphi(X_{n+1}^{x^*})Z_{n+1}^{x^*} \right|^{r/2} \right] \leq T_1 + T_2 \]
where
\[ T_1 := E \left[ \left| \varphi(X_{n+1}^x) - \varphi(X_{n+1}^{x^*}) \right| Z_{n+1}^x \right]^{1/r} \]
\[ T_2 := E \left[ \left| \varphi(X_{n+1}^{x^*}) \right| \left| Z_{n+1}^x - Z_{n+1}^{x^*} \right| \right]^{1/r} \].
So we proceed to control the two terms in \( T_1 \) and \( T_2 \).
For \( T_1 \), apply Cauchy-Schwarz to obtain the upper-bound
\[ T_1 \leq E[|Z_{n+1}^x|^{2r}]^{1/(2r)} E \left[ \left| \varphi(X_{n+1}^x) - \varphi(X_{n+1}^{x^*}) \right|^{2r} \right]^{1/(2r)} \]
As \( E[|Z_{n+1}^x|^{2r}]^{1/(2r)} \leq C \) and using \( \varphi \in \text{Lip}_\| \|_2(\mathbb{R}^d) \) along with (40) yields
\[ T_1 \leq C(\| \varphi \| + |\| \varphi \|_{\text{Lip}}|) \| x - x_* \|_2. \] \tag{41} \]
For \( T_2 \) using \( \varphi \in B_b(\mathbb{R}^d) \)
\[ T_2 \leq (\| \varphi \| + |\| \varphi \|_{\text{Lip}}|) E \left[ |Z_{n+1}^x - Z_{n+1}^{x^*}|^{r} \right]^{1/r} \]
Applying Lemma A.7 gives
\[ T_2 \leq C(\| \varphi \| + |\| \varphi \|_{\text{Lip}}|) \| x - x_* \|_2. \] \tag{42} \]
Noting (41) and (42) allows one to conclude. \( \square \)

Lemma A.9. Assume \( (D1) \). Then for any \( (n, r) \in \{0, 1, \ldots \} \times \mathbb{N} \), there exists a \( C < +\infty \) such that for any \( (l, \varphi, x, x_*) \in \mathbb{N} \times B_b(\mathbb{R}^d) \cap \text{Lip}_\| \|_2(\mathbb{R}^d) \times \mathbb{R}^d \times \mathbb{R}^d \)
\[ E \left[ \left| \varphi(\tilde{X}_{n+1}^x)Z_{n+1}^l(\tilde{X}_0^x, \tilde{X}_\Delta^x, \ldots, \tilde{X}_{n-\Delta}^x) - \varphi(\tilde{X}_{n+1}^{x^*})Z_{n+1}^{l-1}(\tilde{X}_0^x, \tilde{X}_{\Delta-1}^x, \ldots, \tilde{X}_{n-\Delta-1}^x) \right|^{r/2} \right] \leq C(\| \varphi \| + |\| \varphi \|_{\text{Lip}}|) \left( \sum_{j=1}^3 T_j \right). \]
Proof. The expectation in the statement of the Lemma is upper-bounded by \( \sum_{j=1}^3 T_j \) where
\[ T_1 := E \left[ \left| \varphi(\tilde{X}_{n+1}^x)Z_{n+1}^l(\tilde{X}_0^x, \tilde{X}_\Delta^x, \ldots, \tilde{X}_{n-\Delta}^x) - \varphi(\tilde{X}_{n+1}^{x^*})Z_{n+1}^{l-1} \right|^{1/r} \right] \]
\[ T_2 := E \left[ \left| \varphi(\tilde{X}_{n+1}^x)Z_{n+1}^{l-1} - \varphi(\tilde{X}_{n+1}^{x^*})Z_{n+1}^{l-1} \right|^{1/r} \right] \]
\[ T_3 := E \left[ \left| \varphi(\tilde{X}_{n+1}^x)Z_{n+1}^{l-1}(\tilde{X}_0^x, \tilde{X}_{\Delta-1}^x, \ldots, \tilde{X}_{n-\Delta-1}^x) - \varphi(\tilde{X}_{n+1}^{x^*})Z_{n+1}^{l-1} \right|^{1/r} \right] \]
The proof is completed by applying Lemma A.6 to \( T_1 \) and \( T_3 \), and Lemma A.8 to \( T_2 \). \( \square \)

Lemma A.10. Assume \( (D1) \). Then for any \( (n, p, r) \in \{0, 1, \ldots \} \times \mathbb{N}^2 \), there exists a \( C < +\infty \) such that for any \( (l, \varphi, x) \in \{0, 1, \ldots \} \times B_b(\mathbb{R}^d) \cap \text{Lip}_\| \|_2(\mathbb{R}^d) \times \mathbb{R}^d \)
\[ E \left[ \left| \frac{E[\varphi(\tilde{X}_{n+1}^x)Z_{n+1}^l(\tilde{X}_0^x, \tilde{X}_\Delta^x, \ldots, \tilde{X}_{n-\Delta}^x)|Y_n]}{E[Z_p(\tilde{X}_0^x, \tilde{X}_\Delta^x, \ldots, \tilde{X}_{p-\Delta}^x)|Y_n]} - \frac{E[\varphi(\tilde{X}_{n+1}^x)Z_n^l|Y_n]}{E[Z_p|Y_n]} \right|^{r/2} \right] \leq C(\| \varphi \| + |\| \varphi \|_{\text{Lip}}|) \left( \sum_{j=1}^3 T_j \right). \]
Proof. We have
\[ E \left[ \left| \frac{E[\varphi(\tilde{X}_{n+1}^x)Z_{n+1}^l(\tilde{X}_0^x, \tilde{X}_\Delta^x, \ldots, \tilde{X}_{n-\Delta}^x)|Y_n]}{E[Z_p(\tilde{X}_0^x, \tilde{X}_\Delta^x, \ldots, \tilde{X}_{p-\Delta}^x)|Y_n]} - \frac{E[\varphi(\tilde{X}_{n+1}^x)Z_n^l|Y_n]}{E[Z_p|Y_n]} \right|^{r/2} \right] \leq T_1 + T_2 \]
where

\[
T_1 := \mathbb{E} \left[ \frac{\mathbb{E}[\varphi(\tilde{X}_{n+1}^x)Z_n^x(\tilde{X}_0^x, \tilde{X}_x^x, \ldots, \tilde{X}_{x-n}^x)]}{\mathbb{E}[Z_p^x(\tilde{X}_0^x, \tilde{X}_x^x, \ldots, \tilde{X}_{x-p}^x)]} \right] \cdot 1^{1/r}
\]

\[
T_2 := \mathbb{E} \left[ \frac{\mathbb{E}[\varphi(\tilde{X}_{n+1}^x)Z_n^x(\tilde{X}_0^x, \tilde{X}_x^x, \ldots, \tilde{X}_{x-n}^x)]}{\mathbb{E}[Z_p^x(\tilde{X}_0^x, \tilde{X}_x^x, \ldots, \tilde{X}_{x-p}^x)]} \right]^{1/r}.
\]

\[T_3\] can be dealt with in a similar way to \(T_1\), except one uses approaches similar to [4, Theorem 21.3] (which is a similar MVT, Martingale plus remainder method that has been used in the proof of Lemma A.6), so we treat the former only.

Now, we have

\[T_1 \leq T_3 + T_4\]

where

\[
T_3 := \mathbb{E} \left[ \left( \frac{\mathbb{E}[\varphi(\tilde{X}_{n+1}^x)Z_n^x(\tilde{X}_0^x, \tilde{X}_x^x, \ldots, \tilde{X}_{x-n}^x)]}{\mathbb{E}[Z_p^x(\tilde{X}_0^x, \tilde{X}_x^x, \ldots, \tilde{X}_{x-p}^x)]} - \frac{\mathbb{E}[\varphi(\tilde{X}_{n+1}^x)Z_n^x(\tilde{X}_0^x, \tilde{X}_x^x, \ldots, \tilde{X}_{x-n}^x)]}{\mathbb{E}[Z_p^x(\tilde{X}_0^x, \tilde{X}_x^x, \ldots, \tilde{X}_{x-p}^x)]} \right)^{1/r} \right]
\]

\[
T_4 := \mathbb{E} \left[ \left( \frac{\mathbb{E}[\varphi(\tilde{X}_{n+1}^x)Z_n^x(\tilde{X}_0^x, \tilde{X}_x^x, \ldots, \tilde{X}_{x-n}^x)]}{\mathbb{E}[Z_p^x(\tilde{X}_0^x, \tilde{X}_x^x, \ldots, \tilde{X}_{x-p}^x)]} - \frac{\mathbb{E}[\varphi(\tilde{X}_{n+1}^x)Z_n^x(\tilde{X}_0^x, \tilde{X}_x^x, \ldots, \tilde{X}_{x-n}^x)]}{\mathbb{E}[Z_p^x(\tilde{X}_0^x, \tilde{X}_x^x, \ldots, \tilde{X}_{x-p}^x)]} \right)^{1/r} \right]
\]

For \(T_3\) applying Cauchy-Schwarz and conditional Jensen,

\[
T_3 \leq \mathbb{E} \left[ \left( \frac{\mathbb{E}[\varphi(\tilde{X}_{n+1}^x)Z_n^x(\tilde{X}_0^x, \tilde{X}_x^x, \ldots, \tilde{X}_{x-n}^x)]}{\mathbb{E}[Z_p^x(\tilde{X}_0^x, \tilde{X}_x^x, \ldots, \tilde{X}_{x-p}^x)]} - \frac{\mathbb{E}[\varphi(\tilde{X}_{n+1}^x)Z_n^x(\tilde{X}_0^x, \tilde{X}_x^x, \ldots, \tilde{X}_{x-n}^x)]}{\mathbb{E}[Z_p^x(\tilde{X}_0^x, \tilde{X}_x^x, \ldots, \tilde{X}_{x-p}^x)]} \right)^{1/r} \right] \times \mathbb{E} \left[ \left( \frac{\mathbb{E}[\varphi(\tilde{X}_{n+1}^x)Z_n^x(\tilde{X}_0^x, \tilde{X}_x^x, \ldots, \tilde{X}_{x-n}^x)]}{\mathbb{E}[Z_p^x(\tilde{X}_0^x, \tilde{X}_x^x, \ldots, \tilde{X}_{x-p}^x)]} - \frac{\mathbb{E}[\varphi(\tilde{X}_{n+1}^x)Z_n^x(\tilde{X}_0^x, \tilde{X}_x^x, \ldots, \tilde{X}_{x-n}^x)]}{\mathbb{E}[Z_p^x(\tilde{X}_0^x, \tilde{X}_x^x, \ldots, \tilde{X}_{x-p}^x)]} \right)^{1/r} \right] \frac{1}{2^{1/(2r)}}
\]

For the left-most expectation on the R.H.S. one can use \(\varphi \in \mathcal{B}_b(\mathbb{R}^{d_x})\), the Hölder and Jensen inequalities along with the proof approaches in the proof of Lemma A.2 to establish that the expectation is upper-bounded by a constant \(C\) that does not depend upon \(l, x\). For the right-most expectation on the R.H.S. one can use the ideas in (33) to deduce that

\[
T_3 \leq C(\|\varphi\| + \|\varphi\|_{\text{Lip}}) \Delta_l^{1/2}.
\]

The proof for \(T_4\) is similar, except using the ideas for (30) instead of (33). That is, \(T_4 \leq C(\|\varphi\| + \|\varphi\|_{\text{Lip}}) \Delta_l^{1/2}\). Hence

\[
T_1 \leq C(\|\varphi\| + \|\varphi\|_{\text{Lip}}) \Delta_l^{1/2}.
\]

This completes the argument.

\[\square\]

**Remark A.2.** One can easily deduce: Assume (D1). Then for any \((n, r) \in \{0, 1, \ldots\} \times \mathbb{N}\), there exists a \(C < +\infty\) such that for any \((l, \varphi, x) \in \{0, 1, \ldots\} \times \mathcal{B}_b(\mathbb{R}^{d_x}) \cap \text{Lip}_{\|\|_2}(\mathbb{R}^{d_x}) \times \mathbb{R}^{d_x}\)

\[
\mathbb{E} \left[ \left( \frac{\mathbb{E}[\varphi(\tilde{X}_{n+1}^x)Z_n^x(\tilde{X}_0^x, \tilde{X}_x^x, \ldots, \tilde{X}_{x-n}^x)]}{\mathbb{E}[Z_p^x(\tilde{X}_0^x, \tilde{X}_x^x, \ldots, \tilde{X}_{x-p}^x)]} - \frac{\mathbb{E}[\varphi(\tilde{X}_{n+1}^x)Z_n^x(\tilde{X}_0^x, \tilde{X}_x^x, \ldots, \tilde{X}_{x-n}^x)]}{\mathbb{E}[Z_p^x(\tilde{X}_0^x, \tilde{X}_x^x, \ldots, \tilde{X}_{x-p}^x)]} \right)^{1/r} \right] \leq C(\|\varphi\| + \|\varphi\|_{\text{Lip}}) \Delta_l^{1/2}.
\]

**Corollary A.2.** Assume (D1). Then for any \((n, p, r) \in \{0, 1, \ldots\} \times \mathbb{N}^2\), there exists a \(C < +\infty\) such that for any \((l, \varphi, x) \in \mathbb{N} \times \mathcal{B}_b(\mathbb{R}^{d_x}) \cap \text{Lip}_{\|\|_2}(\mathbb{R}^{d_x}) \times \mathbb{R}^{d_x}\)

\[
\mathbb{E} \left[ \left( \frac{\mathbb{E}[\varphi(\tilde{X}_{n+1}^x)Z_n^x(\tilde{X}_0^x, \tilde{X}_x^x, \ldots, \tilde{X}_{x-n}^x)]}{\mathbb{E}[Z_p^x(\tilde{X}_0^x, \tilde{X}_x^x, \ldots, \tilde{X}_{x-p}^x)]} - \frac{\mathbb{E}[\varphi(\tilde{X}_{n+1}^x)Z_n^{x-1}(\tilde{X}_0^x, \tilde{X}_{x-1}^x, \ldots, \tilde{X}_{x-n-1}^x)]}{\mathbb{E}[Z_p^{x-1}(\tilde{X}_0^x, \tilde{X}_{x-1}^x, \ldots, \tilde{X}_{x-p-1}^x)]} \right)^{1/r} \right] \leq C(\|\varphi\| + \|\varphi\|_{\text{Lip}}) \Delta_l^{1/2}.
\]
Proof. Can be easily proved by using Lemma A.10.

\[E[\mathbb{E}[\varphi(\bar{X}_{n+1}^x)Z_n^{l,N}(\Delta_0,\Delta_1,\ldots,\bar{X}_{n}^{x-\Delta_1})]1/r] \leq C(\|\varphi\| + \|\varphi\|_{Lip}D_{1/2}).\] (43)

For any \((n,r) \in \mathbb{N}^2\), there exists a \(C < +\infty\) such that for any \((l,\varphi,x) \in \mathbb{B}_0(\mathbb{R}^d_e) \cap \text{Lip}_{\|\|} (\mathbb{R}^d_e) \times \mathbb{R}^d_e:\)

\[E\left[\mathbb{E}[\varphi(\bar{X}_{n+1}^x)Z_n^{l,N}(\Delta_0,\Delta_1,\ldots,\bar{X}_{n}^{x-\Delta_1})]1/r\right] \leq C(\|\varphi\| + \|\varphi\|_{Lip}D_{1/2}).\]

For any \((n,r) \in \{0,1,\ldots\} \times \mathbb{N}\), there exists a \(C < +\infty\) such that for any \((l,\varphi,x) \in \mathbb{B}_0(\mathbb{R}^d_e) \cap \text{Lip}_{\|\|} (\mathbb{R}^d_e) \times \mathbb{R}^d_e:\)

\[E\left[\mathbb{E}[\varphi(\bar{X}_{n+1}^x)Z_n^{l,N}(\Delta_0,\Delta_1,\ldots,\bar{X}_{n}^{x-\Delta_1})]1/r\right] \leq C(\|\varphi\| + \|\varphi\|_{Lip}D_{1/2}).\]

A.3 Results for the Particle Filter

Proposition A.3. Assume (D1). Then for any \((p,n,r) \in \{0,1,\ldots\}^2 \times \mathbb{N}\), \(n \geq p\), there exists a \(C < +\infty\) such that for any \(l \in \{0,1,\ldots\}\), \((N,\varphi) \in \mathbb{N} \times \mathbb{B}_0(\mathbb{E})\)

\[E[\|\pi_p^{l,N} - \pi_p^l\rangle\mathcal{Q}_{p,n}(\varphi)]1/r \leq \frac{C\|\varphi\|}{\sqrt{N}}.\]

Proof. The proof is by induction on \(p\) for any fixed \(n \geq p\). If \(p = 0\) one can apply the conditional Marcinkiewicz-Zygmund inequality to yield:

\[E[\|\pi_p^{l,N} - \pi_p^l\rangle\mathcal{Q}_{p,n}(\varphi)]1/r \leq \frac{C}{\sqrt{N}}E[\mathcal{Q}_{p,n}(\varphi)(U_p^{l,1})]1/r.\]

Then one has the result by Lemma A.2.

For the induction step, we have the standard decomposition via Minkowski

\[E[\|\pi_p^{l,N} - \pi_p^l\rangle\mathcal{Q}_{p,n}(\varphi)]1/r \leq T_1 + T_2 + T_3\] (44)

where

\[T_1 = E[\|\pi_p^{l,N} - \Phi_p^{l}(\pi_p^{l,N})\rangle\mathcal{Q}_{p,n}(\varphi)]1/r\]

\[T_2 = E[\Phi_p^{l}(\pi_p^{l,N})\rangle\mathcal{Q}_{p,n}(\varphi)]1/r\]

\[T_3 = E[\pi_{p-1}^{l,N} - \pi_{p-1}^l\rangle\mathcal{Q}_{p-1,n}(\varphi)]1/r\]

By the same argument as for the initialization

\[T_1 \leq \frac{C\|\varphi\|}{\sqrt{N}}.\] (45)

For \(T_2\) applying Hölder

\[T_2 \leq E[\pi_{p-1}^l(G_{p-1}^{l})^{-3r}]1/(3r)E[\Phi_p^{l}(\pi_p^{l,N})\rangle\mathcal{Q}_{p,n}(\varphi)]3r]1/(3r)\]

\[\leq E[\|\pi_p^{l,N} - \pi_p^l\rangle\mathcal{Q}_{p,n}(\varphi)]3r]1/(3r).\]
For the left-most term on the R.H.S. one can apply Lemma A.2. For the middle term on the R.H.S. one can apply the conditional Jensen inequality and Lemma A.2. For the right-most term on the R.H.S. one can apply the induction hypothesis. Hence
\[ T_2 \leq \frac{C\|\varphi\|}{\sqrt{N}}. \] (46)

For \( T_3 \), one can use Cauchy-Schwarz, Lemma A.2 and the induction hypothesis to yield
\[ T_3 \leq \frac{C\|\varphi\|}{\sqrt{N}}. \] (47)

Combining (45)-(47) with (44) concludes the proof.

**Remark A.4.** It is straightforward to extend Proposition A.3 to the following result, under (D1): for any \((p, n, r) \in \{0, 1, \ldots\}^2 \times \mathbb{N}, n \geq p\), there exists a \(C < +\infty\) such that for any \(l \in \{0, 1, \ldots\}, (N, \varphi) \in \mathbb{N} \times \mathcal{B}_b(E_l)\)
\[ \mathbb{E}[|\pi_{p,N}^l - \pi_p^l(Q_{p,n}^l(G_n^{l}(\varphi)))|^{1/r}]^{1/r} \leq \frac{C\|\varphi\|}{\sqrt{N}}. \]

**Corollary A.3.** Assume (D1). Then for any \((p, n, r) \in \{0, 1, \ldots\}^2 \times \mathbb{N}, n \geq p\), there exists a \(C < +\infty\) such that for any \(l \in \{0, 1, \ldots\}, (N, \varphi) \in \mathbb{N} \times \mathcal{B}_b(E_l)\)
\[ \mathbb{E}[|\pi_{p,N}^l(D_{p,n}^l(G_n^{l}(\varphi)))|^{1/r}]^{1/r} \leq \frac{C\|\varphi\|}{\sqrt{N}}. \]

**Proof.** Noting that \(\pi_p^l(D_{p,n}^l(G_n^{l}(\varphi))) = 0\) a.s., the result follows immediately by Cauchy-Schwarz, Lemma A.2 and Proposition A.3 (see Remark A.4). \(\square\)

**References**

[1] Bain, A. & Crisan, D. (2009). *Fundamentals of Stochastic Filtering*. Springer: New York.

[2] Blanchet, J., Glynn, P. & Pei, Y. (2019). Unbiased Multilevel Monte Carlo. arXiv preprint.

[3] Clark, J. M. C. (1978). The design of robust approximations to the stochastic differential equations of non-linear filtering. In *Communications Systems and Random Process Theory* (Swirzynski, J. K.), 25 721–734. Sijthoff & Noordhoff.

[4] Crisan, D. (2011). Discretizing the continuous-time filtering problem: Order of Convergence. In *The Oxford Handbook on Non-Linear Filtering* (Crisan, D. & Rozovskii, B.), 572–597. Oxford: OUP.

[5] Crisan, D. & Ortiz-Latorre, S. (2013). A Kusuoka-Lyons-Victoir particle filter. *Proc. Roy. Soc. A*, 469, 2156.

[6] Crisan, D. & Ortiz-Latorre, S. (2019). A high order time discretization of the solution of the non-linear filtering problem. arXiv preprint.

[7] Del Moral, P. (2013). *Mean Field Simulation for Monte Carlo Integration*. Chapman & Hall: London.

[8] Del Moral, P., Doucet, A. & Jasra, A. (2012). On adaptive resampling procedures for sequential Monte Carlo methods. *Bernoulli*, 18, 252–272.

[9] Fearnhead, P., Papaspiliopoulos, O., Roberts, G. O. & Stuart, A. (2010). Random-weight particle filtering of continuous time processes. *J. R. Statist. Soc. Ser. B*, 72, 497-512.

[10] Ferre, G. & Stolz, G. (2019). Error estimates on ergodic properties of discretized Feynman-Kac Semigroups. arXiv preprint.

[11] Giles, M. B. (2008). Multilevel Monte Carlo path simulation. *Op. Res.*, 56, 607-617.

[12] Heinrich, S. (2001). Multilevel Monte Carlo methods. In *Large-Scale Scientific Computing*, (eds. S.Margenov, J. Wasniewski & P. Yalamov), Springer: Berlin
[13] Jasra, A., & Yu, F. (2018). Central limit theorems for coupled particle filters. arXiv:1810.04900.

[14] Jasra, A., Ballesio, M., Von Schwerin, E., & Tempone, R. (2019). A coupled particle filter for multilevel estimation. Technical Report.

[15] Jasra, A., Kamatani, K., Law K. J. H. & Zhou, Y. (2017). Multilevel particle filters. SIAM J. Numer. Anal., 55, 3068-3096.

[16] Jasra, A., Kamatani, K., Osei, P. P. & Zhou, Y. (2018). Multilevel particle filters: normalizing constant estimation. Statist. Comp., 28, 47–60.

[17] Kloeden, P. E. & Platen, E. (1992). Numerical Solution of Stochastic Differential Equations. Springer: Berlin.

[18] Picard, J. (1984). Approximations of non-linear filtering problems and order of convergence. Filtering and control of random processes. Lecture Notes in Control and Information Sciences, 61, 219–236. Springer: Berlin.

[19] Rogers, L. C. G. & Williams, D. (2000). Diffusions, Markov Processes and Martingales, Vol 2. CUP: Cambridge.