SURFACE-LOCALIZED TRANSMISSION EIGENSTATES, SUPER-RESOLUTION IMAGING AND PSEUDO SURFACE PLASMON MODES

XIANCHAO WANG, YOUZI HE, YAT TIN CHOW, AND HONGYU LIU*

Abstract. We present an intriguing discovery of certain surface/curve-localized transmission eigenstates associated with the Helmholtz equation or the Schrödinger equation under generic scenarios. The existence of such localized eigenstates is topologically very robust against large deformation or even twisting of the material interface, while the eigenstates themselves are topologically sensitive to the material interface. We consider their applications in producing a super-resolution imaging scheme and generating the so-called pseudo surface plasmon resonant (PSPR) modes with a potential sensing application.

Keywords: transmission eigenstates; surface/curve localized states; super-resolution imaging; surface plasmon resonance; sensing.

1. Introduction

We first present the mathematical description of the interior transmission eigenvalue problems. Let \( \Omega \) be an open bounded and connected set in \( \mathbb{R}^N \), \( N = 2, 3 \), with a Lipschitz boundary \( \partial \Omega \). Let \( n(x) \in L^\infty(\Omega) \) and \( k \in \mathbb{R}_+ \). Consider the following system of partial differential equations (PDEs) for \( u \in H^1(\Omega) \) and \( v \in H^1(\Omega) \):

\[
\begin{align*}
\Delta u + k^2 n^2(x) u &= 0 \quad \text{in} \ \Omega, \\
\Delta v + k^2 v &= 0 \quad \text{in} \ \Omega, \\
u = v, \quad \frac{\partial u}{\partial \nu} &= \frac{\partial v}{\partial \nu} \quad \text{on} \ \partial \Omega,
\end{align*}
\]

where \( \nu \) is the exterior unit normal vector to \( \partial \Omega \). (1.1) is referred to as the interior transmission eigenvalue problem associated with the Helmholtz equation. Clearly, \( u \equiv v \equiv 0 \) are a trivial solution to (1.1). If there exists a non-trivial pair of solutions \((u, v)\) to (1.1), \( k \in \mathbb{R}_+ \) is called a transmission eigenvalue, and \( u, v \) are the associated transmission eigenfunctions.

In a similar manner, one can formulate the transmission eigenvalue problem associated with the time-harmonic Schrödinger equation by replacing the first equation in (1.1) to be \( \Delta u + k^2 u - V(x) u = 0 \), where \( V(x) \in L^\infty(\Omega) \) denotes a potential field and \( E := k^2 \) signifies an energy level. The latter transmission eigenvalue problem is connected to (1.1) by formally taking \( n^2 = 1 - V/k^2 \) for a given \( k \in \mathbb{R}_+ \). In what follows, we shall mainly present our study for the transmission eigenvalue problem (1.1) and the corresponding applications to the wave scattering governed the Helmholtz equation. However, through the aforementioned connection, all of the subsequent results can be readily extended to the Schrödinger transmission eigenvalue problem as well as applications to the quantum scattering governed by the Schrödinger equation.

The transmission eigenvalue problem arises in the scattering theory of time-harmonic waves and was first proposed in [20] in the context of a reconstruction scheme for inverse scattering problems. It also relates to the non-scattering phenomenon, a.k.a invisibility cloaks [4,5,7]. The spectral properties of the transmission eigenvalues have been extensively

*CORRESPONDING AUTHOR
studied in the literature. Under certain generic conditions on \( n \), particularly including the case with \( n \neq 1 \) being a positive constant, it is known that there exist an infinite and discrete set of eigenvalues \( 0 < k_1 \leq k_2 \leq \cdots \leq k_l \leq \cdots \to +\infty \). For each eigenvalue, the corresponding eigenspace is finite dimensional. We refer to [8, 10, 29] and the references therein for the related studies of the aforementioned properties. Recently, it is revealed in [4–6] that the transmission eigenfunctions possess rich geometric structures. It is proved that near a point on \( \partial \Omega \) where the magnitude of the curvature is sufficiently large, then the transmission eigenfunctions must be nearly vanishing. In particular, in the extreme case where the high-curvature part degenerates to become a corner, then under a certain mild regularity assumption on the transmission eigenfunctions, they must be vanishing.

The geometric structures of the transmission eigenfunctions discussed above are of a local feature. In this paper, we present an intriguing discovery of certain global properties of the transmission eigenfunctions. Indeed, we find that under a generic scenario, either the transmission eigenfunction \( u \) or \( v \) must be localized on the boundary surface in \( \mathbb{R}^3 \) or the boundary curve in \( \mathbb{R}^2 \). That is, the energy of \( u \) (resp. \( v \)) is localized around \( \partial \Omega \) and barely enters into the bulk \( \Omega \). For terminological convenience, those particular eigenmodes are referred to as the surface-localized eigenstates (SLEs). The existence of SLEs are shown to be topologically very robust against large deformation or even twisting of the boundary surface/curve \( \partial \Omega \). However, the SLEs themselves are topologically sensitive to the change of the boundary. Our study unveils a significant physical phenomenon that is completely unknown before. Moreover, it provides a new perspective on wave localization, which is a central topic to many practical applications including surface plasmon resonances [3, 12, 21, 28, 33], topologically robust states in quantum Hall effect [11, 16, 31], directional optical waveguide [14, 15], photonic transport [19, 30, 32], and cloaking due to anomalous localized resonance [2, 22, 27]. We consider applications of our discovery in producing a super-resolution imaging scheme and generating the so-called pseudo surface plasmon resonant (PSPR) modes with a potential sensing application.

2. Surface-localized transmission eigenstates

In order to focus on the physical insights and simplify the exposition, unless specified otherwise, we shall mainly consider the case that \( n \) is a positive constant and \( n \neq 1 \). Nevertheless, it is emphasized that all the subsequent results hold for the case that \( n \) is a variable function satisfying some generic requirements. Physically, \( n \) signifies the refractive index of an inhomogeneous medium supported in \( \Omega \) and the transmission eigenvalue \( k \) signifies a wavenumber. Note that if \((u, v)\) is a pair of eigenfunctions to (1.1), so is \( \alpha \cdot (u, v) \) for any \( \alpha \in \mathbb{C} \setminus \{0\} \). Hence, throughout we shall normalize \( v \) (or \( u \) in some occasions). Moreover, we also note the following scaling property of \( k \) in (1.1) with respect to the size of \( \Omega \): for \( \rho \in \mathbb{R}_+ \), \( \rho \cdot k \) is an eigenvalue to (1.1) associated with \( \Omega_\rho := \frac{1}{\rho} \Omega \). Hence, we always assume that \( \text{diam}(\Omega) \sim 1 \) in order to calibrate our study.

Our localizing discovery states that under the following condition:

\[
k \cdot n \gg 1,
\]

if \( n > 1 \) in \( \Omega \), then the transmission eigenfunction \( v \) in (1.1) is an SLE, whereas if \( 0 < n < 1 \) in \( \Omega \), then the transmission eigenfunction \( u \) in (1.1) is an SLE. Next, we present some typical numerical results to verify and demonstrate the SLEs in different scenarios. We mainly discuss the case \( n > 1 \) and briefly remark the case \( 0 < n < 1 \). The rigorous justification of the above localizing result is highly technical and lengthy, and shall be presented in a forthcoming theoretical paper [9].
2.1. SLEs for high-contrast mediums. First, we consider the scenario that \( n \) is sufficiently large, which corresponds to the case that a high-contrast medium is located inside \( \Omega \) (the medium outside \( \Omega \) possesses an \( n \equiv 1 \)). In Fig. 1, we calculate a transmission eigenvalue \( k = 1.0080 \) for \( \Omega \) being a unit disk and plot the corresponding eigenfunctions \( u \) and \( v \). It is clearly seen that \( v \) is an SLE. However, it is pointed out that the eigenfunction \( u \) is not a SLE. In fact, this phenomenon also holds for the other cases in our subsequent discussion. That is, \( u \) and \( v \) cannot be SLEs simultaneously. Moreover, in Fig. 1, we note that \( \text{diam}(\Omega) \), being 1, is much smaller than the underlying wavelength, being \( 2\pi/k \approx 2\pi \). Such an observation is critical for our subsequent development of the super-resolution imaging scheme. Fig. 1, (c) and (d), present two more SLEs, and in particular, in (d), we note a significant localization phenomenon at the concave part of \( \partial \Omega \), which is also a critical ingredient for our subsequent development of the super-resolution wave imaging.

![Figure 1](image-url)

**Figure 1.** (a)&(b): eigenfunctions \( u \) and \( v \) to (1.1) associated with \( n = 30 \), where \( k = 1.0080 \); (c): eigenfunction \( v \) to (1.1) associated with \( n = 30 \); (d): eigenfunction \( v \) to (1.1) associated with \( n = 10 \).

2.2. SLEs for high-wavenumber modes. Next, we consider the case that \( n \) is regular, namely \( n \sim 1 \), and according to (2.1), \( k \) should be sufficiently large in order to have the SLEs. Fig. 2 presents two examples in both 2D and 3D.

2.3. Topological robustness of the existence of SLEs. The existence of the SLEs is topologically very robust against large deformation or even twisting of the material interface \( \partial \Omega \); see Fig. 3.
2.4. SLEs for variable refractive inhomogeneities and coated objects. As remarked earlier, the SLEs also exist for variable refractive inhomogeneities. In Fig. 4 (a), the eigenfunction $v$ is associated with $n = 30$ in the outside thin layer and $n = 4$ in the inside triangle. It is emphasized that the outside layer is not required to be very thin in order to exhibit the SLEs. We choose this specific example for our subsequent use. Fig. 4, (b), corresponds to a coated object, where the inside kite-domain is an insulator. That is, in (1.1), $u$ does not exist in the inside kite-domain and we impose a zero Dirichlet condition of $u$ on the boundary of the inside kite-domain.

Finally, we briefly remark the case $0 < n < 1$ in (1.1). All of the previous localization results still hold, but with $v$ replaced to be $u$. To elucidate this phenomenon, we consider a simple case by assuming that $n$ is a constant. Set $\tilde{k} = k/n$. It is directly verified that the first two equations in (1.1) become, $(\Delta + \tilde{k}^2)u = 0$ and $(\Delta + \tilde{k}^2 n^{-2})v = 0$, while the transmission conditions on $\partial \Omega$ remain unchanged. Noting that $n^{-1} > 1$, one therefore has by using the previous localization results that $u$ should be an SLE, provided $\tilde{k}n^{-1}$ sufficiently large. If $n$ is variable, such a scaling argument does not work. However, it is remarked that the localization results still hold for $u$, whose behaviour is similar to those presented in the previous examples for $v$ associated with $n > 1$. 
Figure 4. Transmission eigenfunction $v$. (a): $n = 30$ in the outside layer and $n = 4$ in the inside triangle; (b): $n = 30$ in the outside layer and the inside kite is an insulator.

### 3. Applications of SLEs

In this section, we consider two interesting and practically important applications of the SLEs presented in the previous section.

#### 3.1. Super-resolution wave imaging.

We briefly introduce the inverse scattering problem of recovering an inhomogeneous medium by its associated scattering measurements. Let $u^i(x) = e^{ikx \cdot \hat{\theta}}$ be an incident plane wave, which is an entire solution to $\Delta u^i + k^2 u^i = 0$ in $\mathbb{R}^N$. Here, $\hat{\theta} \in S^{N-1} := \{ x \in \mathbb{R}^N; |x| = 1 \}$ is a unit vector, signifying the incident direction. The scattering due to the impinging of plane waves on the refractive inhomogeneity $n$ supported in $\Omega$ is governed by the following Helmholtz system:

$$
\begin{align*}
\Delta u + k^2 n^2 u &= 0 & \text{in } \mathbb{R}^N, \\
u &= u^i + u^s & \text{in } \mathbb{R}^N, \\
\lim_{|x| \to \infty} |x|^{N-1} \left( \frac{\partial u^s}{\partial |x|} - iku^s \right) &= 0,
\end{align*}
$$

(3.1)

where $u^s$ is the scattered field and the last limit in (3.1) is the Sommerfeld radiation condition that characterizes the outgoing nature of the scattered wave. The total field $u \in H^1_{loc}(\mathbb{R}^N)$ to (3.1) possesses the following asymptotic expansion as $|x| \to \infty$:

$$
u(x, e^{ikx \cdot \hat{\theta}}) = u^i(x) + \frac{e^{ik|x|}}{|x|(N+1)/2} u_\infty(\hat{x}, e^{ikx \cdot \hat{\theta}}) + O\left( \frac{1}{|x|(N+1)/2} \right),$$

which holds uniformly for all directions $\hat{x} := x/|x| \in S^{N-1}$. $u_\infty(\hat{x}, \hat{\theta}, k) = u_\infty(\hat{x}, e^{ikx \cdot \hat{\theta}})$ is known as the scattering amplitude which encodes the information of the refractive index $n$.

We are concerned with the inverse problem of imaging the support of the inhomogeneity, namely $\Omega$, by knowledge of the scattering amplitude for $k$ from an open interval, say $I = (\kappa_0, \kappa_1)$. It can be recast as the following nonlinear operator equation

$$
\mathcal{F}(\Omega, n) = u_\infty(\hat{x}, \hat{\theta}, k), \quad \hat{x} \in S^{N-1}, \quad \hat{\theta} \in S^{N-1}, \quad k \in I.
$$

(3.2)

where $\mathcal{F}$ is defined by the Helmholtz system (3.1). Such an inverse problem is a prototypical model for many industrial and engineering applications including medical imaging and nondestructive testing. There is the well-known Abbe diffraction limit for imaging the fine details of $\partial \Omega$ [23]. In fact, one has a minimum resolvable distance of $d = \lambda/(2N)$, where $\lambda$ and $N$ stand for the wavelength and numerical aperture respectively. In modern optics, the Abbe resolution limit is roughly about half of the wavelength. Next, based on the use
of the SLEs, we develop an imaging scheme that can break the Abbe resolution limit in recovering the fine details of $\partial \Omega$ for (3.2), independent of $n$, in certain scenarios of practical interest.

The proposed imaging scheme consists of three phases. In Phase I, with the far-field data in (3.2), we apply the so-called linear sampling method to determine all the transmission eigenvalues located in the interval $I$. The theoretical foundation as well as the numerical implementation for Phase I have been established in several existing works and can be conveniently found in [8,25]. In Phase II, we let $k \in I$ be a computed transmission eigenvalue from Phase I and consider the following optimization problem,

$$
\min_{g \in L^2(S^{N-1})} \left\| \int_{S^{N-1}} u_{\infty}(\hat{x}, \hat{\theta}, k) g(\hat{\theta}) \, ds(\hat{\theta}) \right\|_{L^2(S^{N-1})} \quad \text{subject to} \quad \| g \|_{H^1(B_R)} \sim 1, \quad (3.3)
$$

where $B_R$ is an a-priori ball containing $\Omega$ and $v^k_g$ is known as the Herglotz wave. The solvability of the optimization problem (3.3) is guaranteed by the fact that $k$ is an eigenvalue (cf. [17]). Let $K_L = \{ k_1, k_2, \cdots, k_L \}$, $L \in \mathbb{N}$, and $v^g_{k_j}, j = 1, \ldots, L$, be the computed transmission eigenvalues and Herglotz waves in Phases I&II. According to [17], $v^g_{k_j}$ approximates the underlying transmission eigenfunction $v$ associated with the eigenvalue $k_j$ (cf. (1.1)), and hence is an SLE if (2.1) is fulfilled. In Phase II, we introduce the following imaging functional:

$$
I_{K_L}(z) = -\ln \sum_{k_j \in K_L} |v_{k_j}^g(z)|, \quad z \in B_R. \quad (3.5)
$$

Note that an SLE possesses the following distinctive features related to the global profile of $\partial \Omega$: 1. it nearly vanishes immediately after leaving $\partial \Omega$; 2. it globally localizes on $\partial \Omega$; 3. it locally vanishes at the cusp/highly-curved places on $\partial \Omega$. Hence, it can be easily inferred that $I_{K_L}(z)$ possesses the following indicating behaviours: it is relatively large if $z$ is located inside $\Omega$ or at the corner/highly-curved places of $\partial \Omega$. Thus, via such indicating behaviours, one can recover $\partial \Omega$. In what follows, we present two examples to demonstrate the super-resolution imaging effect.

First, we consider a convex domain and let $\Omega$ be a square with $n = 10$. The synthetic far-field data are computed at 100 observation directions, 100 incident directions and 3000 wavenumbers within the interval $I = [0.6,0.9]$, all equally distributed. By Phase I, we can determine four transmission eigenvalues to be $k_1 = 0.6219$, $k_2 = 0.6896$, $k_3 = 0.7858$ and $k_4 = 0.8370$. By Phase II, we can also determine 4 Herglotz waves, approximating the underlying eigenfunctions. Since $n$ is large, according to our study in Section 2, those computed eigenfunctions turn out to be SLEs. We present the reconstruction results in Fig. 5, (a)–(c), by using 1, 2 and 4 SLEs, respectively. One readily sees that the square is already finely reconstructed with 4 SLEs. For comparisons, we also present the reconstruction results by using a sampling type method developed in several works [13,18,24,26] by using the multiple frequency scattering data in (3.2). The reconstruction results are presented in Fig. 5, (d)–(e). It can be seen that the reconstructions basically yield a spot without any resolution of the square-shape object. In fact, one can also implement the other popular imaging schemes including the linear sampling method or the factorization method [10], the reconstruction effects shall remain almost the same. This is clearly justifiable due to the Abbe resolution limit by noting that the imaging frequencies range from 0.6 to 0.9, while the object is a square of side length 2.
With the reconstruction in Fig. 5, (c) in hand, one can further refine the recovery as follows. We first select a threshold value, with which one cuts a domain $\tilde{\Omega}$ from Fig. 5, (c), serving as an approximation to $\Omega$. In order to refine the approximation, one can further implement the following imaging functional

$$J_{K,L}(z) = \left| \sum_{k_j \in K} (v_{k_j} g_j \chi_{\tilde{\Omega}})(z) \right|^p, \quad z \in \tilde{\Omega},$$

(3.6)

where $p \in \mathbb{R}_+$ is sufficiently large. According to the localizing property of $v_{k_j} g_j$, one infers that $J_{K,L}(z)$ should be of a large amplitude when $z$ is located around $\partial \Omega$. Hence, (3.6) is naturally expected to further enhance the imaging reconstruction in Fig. 5, (c), provided the SLEs $v_{k_j} g_j$, $j = 1, \ldots, 4$, are finely reconstructed. However, a finer recovery of $v_{k_j} g_j$ is computationally very costly and due to the limited computing resources available to us, we choose to explore along this direction in our future study. It is noted that a finer recovery of $v_{k_j} g_j$ would also lead to an improved imaging resolution for (3.5). However, as is seen from Fig. 5, even with a relatively rough recovery of the SLEs, $v_{k_j} g_j$, $j = 1, \ldots, 4$, the proposed imaging scheme can produce super-resolution effect compared to the existing schemes.

Two remarks are in order. First, it is arguable that the super-resolution effect comes from the high-contrast medium parameter $n$ in the specific example (cf. [1]). Indeed, according to (2.1), a high-contrast $n$ leads to a relatively small $k$ that can induce the desired SLE for the reconstruction, which is a matter of fact. However, in practice, for a regular refractive inhomogeneity, one may first coat the object via indirect means with a thin layer of high-contrast medium (cf. Fig. 4), then apply the same reconstruction procedure as above. According to the results in Fig. 4, one would have the same super-resolution reconstructions.
as in Fig. 5, (a)–(c). Second, it is observed through this example that the super-resolution is achieved at the cost of a large amount of computations and a restrictive requirement on the high-precision of the measurement data. This is unobjectionable due to the increasing capabilities of physical apparatus nowadays.

Finally, we present another example in Fig. 6 where the target object is kite-shaped with \( n = 100 \). The imaging frequencies are chosen within \([1, 3.1]\), and the computed transmission eigenvalues in Phase I are \( k_1 = 1.2387, k_2 = 1.4771, k_3 = 2.0513, k_4 = 3.0013 \). The reconstructions by our proposed method are given in (a)–(c), whereas the reconstructions by the same sampling-type method as before are given in (d)–(f). (g)–(i) present the combined results of the above two reconstructions. Clearly, Fig. 6, (i) yields a very nice reconstruction of the kite-object.

**Figure 6.** Reconstructions of a kite-object by multiple SLEs (top row) and multiple-frequency direct sampling method (middle row), respectively, where \( n = 10 \) for all cases. Bottom row gives the corresponding reconstructions of combining the above two reconstruction means.
3.2. Pseudo surface plasmon resonances and potential applications. Surface plasmon resonance (SPR) is the resonant oscillation of conducting electrons at the interface between negative and positive permittivity materials stimulated by incident light. It is a non-radiative electromagnetic surface wave that propagates in a direction parallel to the negative permittivity/dielectric material interface \([3, 12, 21, 28, 33]\). Clearly, the SPR wave is a surface-localized mode. It is in this sense that the SLE can be viewed as a certain SPR. Indeed, viewed from the inside of \(\Omega\) (this is unobjectionable since \(v\) is only supported in \(\Omega\)), the behaviour of a SLE is very much like a SPR. However, SPR usually occurs in the quasi-static regime (subwavelength scale), whereas SLE can occur in both the quasi-static regime and the high-frequency regime. Moreover, the SPR is usually generated from direct light incidence, whereas the generation of SLEs is rather indirect according to our earlier study.

As is known that the SPR can have many industrial and engineering applications including color-based biosensors, different lab-on-a-chip sensors and diatom photosynthesis \([21]\). In what follows, we show that the SLEs can also be generated through direct wave incidences. This will pave the way for the proposal of an interesting SLE sensing that is similar to the SPR sensing.

First, we recall that assuming \(\mathbb{R}^N \setminus \overline{\Omega}\) connected, the Herglotz waves of the form (3.4) are dense in the space \(\{v \in H^1(\Omega); (\Delta + k^2)v = 0 \text{ in } \Omega\}\). Hence, for any transmission eigenfunction \(v\) to (1.1), there exists \(g \in L^2(S^{N-1})\) such that \(v_k^g \approx v \text{ in } H^1(\Omega)\). Next, for a refractive inhomogeneity \(n, 0 < n < 1\), supported in \(\Omega\) with \(\mathbb{R}^N \setminus \overline{\Omega}\) connected, we let \(k_0\) be an eigenvalue to (1.1) with the eigenfunctions denoted as \((u_\Omega, v_\Omega)\) such that \(u_\Omega\) is an SLE. Let \(u_\Omega^k\) be a Herglotz wave function of the form (3.4) such that \(u_\Omega^k = u_\Omega \text{ in } H^1(\Omega)\). Now, we consider the scattering problem (3.1) with the incident field \(u^i = v_k^g\). It is straightforward to show that if \(u_\infty(x, v_k^g) \equiv 0\) (equivalent to \(u^s(x, v_k^g) = 0\) in \(\mathbb{R}^N \setminus \overline{\Omega}\) by Rellich’s Theorem \([10]\)), one then has the transmission eigenvalue problem (1.1) with \(k = k_0\), \(u_\Omega = u^i_\Omega\) and \(v = u^i_\Omega\), where \(u\) is the total field to (3.1). Conversely, noting that \(u^i \approx u_\Omega\) from our earlier construction, one can show (cf. \([4]\)) that \(u_\infty \approx 0\), and more importantly \(u^i_\Omega \approx u_\Omega\). Since \(u_\Omega\) is an SLE, we see that \(u^i_\Omega\) is also a SLE (at least approximately). Set

\[
  w = \begin{cases} 
    u - u^i & \text{in } \mathbb{R}^N \setminus \overline{\Omega}, \\
    u & \text{in } \Omega.
  \end{cases} \tag{3.7}
\]

Clearly, \(w\) is generated from a direct incidence on the inhomogeneity \(n\). \(w \approx 0\) in \(\mathbb{R}^N \setminus \overline{\Omega}\) and \(w \approx u_\Omega\). That is, \(w\) is localized around \(\partial \Omega\), which exhibits a similar behaviour to the SPR oscillation. In what follows, we refer to \(w\) as a pseudo plasmon resonant (PSPR) mode. In Fig. 7 (a)–(c), we present a numerical illustration of the generation of a PSPR mode.

We next propose a potential sensing application of the PSPR mode. Let \((\Omega, n)\) be an a-priori known inhomogeneity. Due to a certain reason, it is supposed that \(\partial \Omega\) has some fine defects, namely, the support of the inhomogeneity actually becomes \(\partial \tilde{\Omega}\). Following the spirit of SPR sensing, one can detect the boundary defects as follows. Let \(u^i\) be an incident field that can generate a PSPR \(w\) associated with \((\Omega, n)\) as above. The field impinges on \((\tilde{\Omega}, n)\), and we let \(\tilde{w}\) be the associated field according to (3.7). In Fig. 7(d), 7(e) and 7(f), we present the corresponding numerical results. It can be seen that the difference \(\tilde{w} - w\) is highly sensitive with respect to the boundary defects \(\partial \tilde{\Omega} - \partial \Omega\), and hence it can be used for the fine detection of the defects. It is a promising sensing technique that is worth of further investigation.
Figure 7. Generation of the PSPR mode, where $k = 7.6548$ and $n = 1/4$. $u$, $w$ and $(\tilde{u}, \tilde{w})$, respectively, denote the corresponding fields associated with $(\Omega, n)$ and $(\tilde{\Omega}, n)$.

Acknowledgment

The work was supported by the startup fund of City University of Hong Kong and the Hong Kong RGC General Research Funds, 12302017, 12301218 and 12302919.

References

[1] H. Ammari, Y. Chow and J. Zou, Super-resolution in imaging high contrast targets from the perspective of scattering coefficients, J. Math. Pures Appl. (9), 111 (2018), 191–226.
[2] H. Ammari, G. Ciraolo, H. Kang, H. Lee and G. W. Milton, Spectral theory of a Neumann-Poincaré-type operator and analysis of cloaking due to anomalous localized resonance, Arch. Ration. Mech. Anal., 208 (2013), 667–692.
[3] D. J. Bergman and M. I. Stockman, Surface plasmon amplification by stimulated emission of radiation: quantum generation of coherent surface plasmons in nanosystems, Phys. Rev. Lett., 90 (2003), 027402.
[4] E. Blåsten and H. Liu, On vanishing near corners of transmission eigenfunctions, J. Funct. Anal., 273 (2017), 3616–3632.
[5] E. Blåsten and H. Liu, Scattering by curvatures, radiationless sources, transmission eigenfunctions and inverse scattering problems, arXiv: 1808.01425, 2018.
[6] E. Blåsten, X. Li, H. Liu and Y. Wang, On vanishing and localizing of transmission eigenfunctions near singular points: a numerical study, Inverse Problems, 33 (2017), 105001.
[7] E. Blåsten, L. Päivärinta and J. Sylvester, Corners always scatter, Commun. Math. Phys., 331 (2014), 725–753.
[8] F. Cakoni, D. Colton, and H. Haddar, Inverse Scattering Theory and Transmission Eigenvalues, SIAM, Philadelphia, 2016.
[9] Y. T. Chow, Y. He, H. Liu and X. Wang, On surface localization of transmission eigenfunctions, in preparation, 2020.
[10] D. Colton and R. Kress, Inverse Acoustic and Electromagnetic Scattering Theory, 3rd. ed., Springer, New York, 2013.
A. Elgart, G. M. Graf and J. H. Shenker, *Equality of the bulk and the edge Hall conductances in a mobility gap*, Commun. Math. Phys., 259 (2005), 185–221.

D. R. Fredkin and I. D. Mayergoyz, *Resonant behavior of dielectric objects (electrostatic resonances)*, Phys. Rev. Lett., 91 (2003), 253902.

R. Griesmaier, *Multi-frequency orthogonality sampling for inverse obstacle scattering problems*, Inverse Problems, 27 (2011), no. 8, 085005, 23 pp.

B. I. Halperin, *Quantized Hall conductance, current-carrying edge states, and the existence of extended states in a two-dimensional disordered potential*, Phys. Rev. B, 25 (1982), 2185–2190.

F. D. M. Haldane and S. Raghu, *Possible realization of directional optical waveguides in photonic crystals with broken time-reversal symmetry*, Phys. Rev. Lett., 100 (2008), 013904.

Y. Hatsugai, *The Chern number and edge states in the integer quantum hall effect*, Phys. Rev. Lett., 71 (1993), 3697–3700.

Y. He, T. Li, H. Liu, X. Wang and H. Zhang, *An interior-resonant-mode-based imaging method for inverse medium scattering problems*, in preparation, 2020.

K. Ito, B. Jin and J. Zou, *A direct sampling method for inverse electromagnetic medium scattering*, Inverse Problems, 29 (2013), no. 9, 095018, 19 pp.

A. B. Khanikaev, S. H. Mousavi, W.-K. Tse, M. Kargarian, A. H. MacDonald and G. Shvets, *Photonic topological insulators*, Nat. Mater., 12 (2013), 233–239.

A. Kirsch, *The denseness of the far field patterns for the transmission problem*, IMA J. Appl. Math., 37 (1986), 213–225.

V. V. Klimov, Nanoplasmonics, CRC Press, 2014.

H. Li and H. Liu, *On anomalous localized resonance and plasmonic cloaking beyond the quasi-static limit*, Proc. R. Soc. A, 474 (2018), 20180165.

S. G. Lipson, H. Lipson and D. S. Tannhauser, Optical Physics, Cambridge University Press, 1995.

J. Li, H. Liu, Q. Wang, *Fast imaging of electromagnetic scatterers by a two-stage multilevel sampling method*, Discrete Contin. Dyn. Syst. Ser. S, 8, (2015), no. 3, 547–561.

H. Liu, X. Liu, X. Wang and Y. Wang, *On a novel inverse scattering scheme using resonant modes with enhanced imaging resolution*, Inverse Problems, 35 (2019), 125012.

X. Liu, *A novel sampling method for multiple multiscale targets from scattering amplitudes at a fixed frequency*, Inverse Problems, 33, (2017), 085011.

G. W. Milton and N.-A. P. Nicorovici, *On the cloaking effects associated with anomalous localized resonance*, Proc. R. Soc. A, 462 (2006), 3027–3059.

F. Ouyang and M. Isaacson, *Surface plasmon excitation of objects with arbitrary shape and dielectric constant*, Philos. Mag., 60 (1989), 481–492.

P. Pääväranta and J. Sylvester, *Transmission eigenvalues*, SIAM J. Math. Anal., 40 (2008), 738–753.

M. C. Rechtsman, J. M. Zeuner, Y. Plotnik, Y. Lumer, D. Podolsky, F. Dreisow, S. Nolte, M. Segev, and A. Szameit, *Photonic Floquet topological insulators*, Nature, 496 (2013), 196.

D. J. Thouless, M. Kohmoto, M. P. Nightingale and M. Den Nijs, *Quantized hall conductance in a two dimensional periodic potential*, Phys. Rev. Lett., 49 (1982), 405.

Z. Wang, Y. D. Chong, J. D. Joannopoulos and M. Soljacic, *Reflection-free one-way edge modes in a gyromagnetic photonic crystal*, Phys. Rev. Lett., 100 (2008), 013905.

S. Zeng, D. Baillargeat, H. P. Ho and K. T. Yong, *Nanomaterials enhanced surface plasmon resonance for biological and chemical sensing applications*, Chemical Society Reviews, 43 (2014), 3426–3452.

**Department of Mathematics, University of Hong Kong, Kowloon, Hong Kong, China**

*E-mail address*: xcwang90@gmail.com

**Department of Mathematics, Hong Kong Baptist University, Kowloon, Hong Kong, China**

*E-mail address*: 18481469@life.hkbu.edu.hk

**Department of Mathematics, University of California, Riverside, USA**

*E-mail address*: yattinc@ucr.edu

**Department of Mathematics, City University of Hong Kong, Kowloon, Hong Kong, China**

*E-mail address*: hongyu.liuip@gmail.com, hongyliu@cityu.edu.hk