Supersymmetric BCS:
Effects of an external magnetic field and spatial fluctuations of the gap

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Abstract

Recently an \( \mathcal{N} = 1 \) supersymmetric model of BCS superconductivity was proposed realizing spontaneous symmetry breaking of a \( U(1)_R \) symmetry. Due to scalar contributions the superconducting phase transition turned out to be first order rather than second order as in standard BCS theory. Here we consider the effects of an external magnetic field and spatial fluctuations of the gap in that model. This allows us to compute the magnetic penetration length and the coherence length, and also to distinguish between type I and type II superconductors. We compare the supersymmetric and standard relativistic BCS results, where the main differences come from the different orders of the phase transition.
1 Introduction

Certain condensed matter systems have scalar and fermionic excitations showing a quasi-supersymmetric dynamics, for example BCS theory, where Nambu [1] found a certain quantum mechanical quasi-supersymmetry, in terms of which one can describe the Interating Boson Model [2]. In the past few years, some attempts to study the supersymmetric generalization of relativistic BCS theory have appeared in the literature [3, 4, 5]. However, these models include explicit supersymmetry breaking terms to stabilize the scalar sector. In this paper we proceed with the work done in [6], where a model of supersymmetric BCS superconductivity was proposed without the need of introducing explicit supersymmetry breaking mass terms to avoid Bose-Einstein condensation (BEC) of the scalars.

Apart from being an interesting problem illustrating the difficulties that the introduction of a chemical potential in a supersymmetric theory brings up, implementing the BCS mechanism within the context of a supersymmetric field theory might be interesting because some exact non-perturbative results can be computed for supersymmetric theories (e.g. [7]). Therefore, this may help to the understanding of strong coupling physics in superconductivity theories. Moreover, supersymmetric field theories are more stable and less sensitive to radiative corrections, as a result, the theory is less sensitive to the UV cutoff, which in BCS theory must be put by hand by introducing a Debye energy as a phenomenological input of the model. Another motivation is the fact that having scalar superpartners for fermions in the supersymmetric version of BCS theory might be of relevance for real condensed matter systems where scalar and fermionic excitations arise in a quasi-supersymmetric way, like in BCS theory itself, as noticed by Nambu.

The implementation of BCS superconductivity requires the presence of a Fermi surface and the introduction of a chemical potential for fermions, but these two conditions may lead to problems when scalars are included in the supersymmetric version. If the chemical potential is introduced through a baryonic $U(1)_B$ symmetry, supersymmetry demands that both fermions and scalars have the same coupling to the chemical potential. If the chemical potential is greater than the mass, a feature which turns out to be necessary for the existence of a Fermi surface, scalars undergo Bose-Einstein condensation which prevents the BCS mechanism to be at work. Fortunately, there is a $U(1)_R$ symmetry which allows us to introduce the chemical potential for fermions while keeping the scalars neutral, avoiding in this way BEC. The main difference between this supersymmetric model
and standard BCS, is that, as we will explain, IR physics of scalars makes the superconducting phase transition first order.

Our aim here is to study the supersymmetric model in the presence of an external magnetic field, which will allow us to compute the magnetic penetration length and describe the Meissner effect. We also consider the effect of spatial fluctuations of the field $\Delta$ around its vacuum expectation value, necessary to compute the coherence length. After computing these two lengths, we can compute the critical magnetic fields, $H_{c1}$ and $H_{c2}$, that we would have for a type-II superconductor, so that compared with the critical magnetic field of a type I superconductor, $H_c$, allows us to distinguish between the two types. All this is done putting emphasis in the comparison between the supersymmetric and non-supersymmetric results.

The paper is organized in the following way. In section 2, we review the relativistic BCS theory and comment on how to obtain the aforementioned characteristic quantities for a superconductor as functions of the temperature. In section 3, we briefly explain the supersymmetric BCS model. In section 4, we go on with the comparison of the quantities computed within the two theories, and comment on whether the superconductors are type I or II. Finally, we end up in section 5 with some conclusions. Details of the computations are given in the appendices. We will follow the notation of in writing the Lagrangians.

2 Relativistic BCS theory

In this section we will review the relativistic BCS theory following the line of reasoning in . As shown in , the relativistic BCS theory can be described in terms of a four fermion interaction by the Lagrangian

$$L = i \frac{1}{2}(\bar{\psi}\gamma^\mu \partial_\mu \psi - \partial_\mu \bar{\psi}\gamma^\mu \psi) - m\bar{\psi}\psi + \mu\bar{\psi}\psi + g^2(\bar{\psi}\gamma_5 \psi)(\bar{\psi}\gamma_5 \psi).$$

This Lagrangian enjoys a global $U(1)$ symmetry, this allows us to introduce the chemical potential associated with the corresponding conserved charge. BCS is a theory describing a spontaneous symmetry breaking of this $U(1)$ symmetry driven by the temperature. For this reason, to study the theory at finite temperature we have to consider the Lagrangian in Euclidean space. At this point, we have a Lagrangian which is not quadratic in the fields, to alleviate this, one has to perform a Hubbard-Stratonovich transformation by introducing an auxiliary field, $\Delta$. At the end of the day we arrive at the Lagrangian

$$L_E = \frac{1}{2}(\bar{\psi}\gamma^\mu \partial_\mu \psi - \partial_\mu \bar{\psi}\gamma^\mu \psi) - \frac{i}{2}(\bar{\psi}\gamma^\mu \partial_\mu \psi - \partial_\mu \bar{\psi}\gamma^\mu \psi) + m\bar{\psi}\psi - \mu\bar{\psi}\psi + g^2|\Delta|^2 - g^2[\Delta(\bar{\psi}\gamma_5 \psi) + \Delta(\bar{\psi}\gamma_5 \psi)]^\dagger.$$  

In this way, if we eliminate the auxiliary field through its equations of motion, we recover the original Lagrangian (1). The equations of motion set $\Delta = (\bar{\psi}\gamma_5 \psi)$, where we see that $\Delta$ is a measure of the density number of Cooper pairs.

Once we have a Lagrangian which is quadratic in fermions, our purpose is to integrate them out to obtain a one-loop effective potential for the auxiliary field $\Delta$. After performing the Matsubara

1Strictly speaking, if the symmetry is global, we are dealing with superfluidity, but transport properties are similar to those of superconductivity and we can consider this model as that of a superconductor in the limit in which the $U(1)$ is “weakly gauged”.
thermal sums, the effective potential is given by

\[ V_{\text{eff}} = g^2 |\Delta|^2 \]

\[ - \int \frac{d^3p}{(2\pi)^3} (\omega_-(p) + \omega_+(p)) \]

\[ - \frac{2}{\beta} \int \frac{d^3p}{(2\pi)^3} \left( \log(1 + e^{-\omega_-(p)}) + \log(1 + e^{-\omega_+(p)}) \right), \]

(3)

where we have written in separate lines, the classical potential, the Coleman-Weinberg potential and the thermal potential, from top to bottom respectively. The \( \omega_{\pm} \) appearing in the effective potential are the energy eigenvalues coming from the Lagrangian (2), and they are given by

\[ \omega_{\pm} = \sqrt{(\omega_0 \pm \mu)^2 + 4g^4|\Delta|^2}, \]

(4)

where \( \omega_0 \equiv \sqrt{p^2 + m^2} \). From the expression of the energy eigenvalues it is manifest why \( \Delta \) is called the gap.

The whole one-loop effective potential is identified with the free energy density in the grand canonical ensemble. It is UV divergent due to the Coleman-Weinberg contribution, for which we must introduce a cut-off, \( \Lambda \), appearing here like a “Debye energy”. Considering the thermal potential at low temperatures, one can see that the dominant contribution comes from the minimum of the energy eigenvalues. Supposing \( \mu > 0 \), this minimum is reached for \( \omega_- \) (which is identified with the particle contribution) around \( \omega_0 = \mu \), and contributions from \( \omega_+ \), i.e. the anti-particle, can be neglected. The momentum space location of this minimum defines a Fermi surface, \( p^2_F = \mu^2 - m^2 \), from which it is clear that the condition \( \mu > m \) is required for the Fermi surface to exist.

The minimum of the effective potential as a function of the gap for each temperature defines a curve \( \Delta(T) \), describing a second order phase transition at a certain critical temperature \( T_c \), below which the fermion condensate appears. This curve can be found by solving the gap equation, \( \partial_{\Delta} V_{\text{eff}} = 0 \), where we have introduced the notation \( \varepsilon = |\Delta|^2 \). Explicitly, the gap equation is

\[ 1 = \frac{g^2}{\pi^2} \int_0^\Lambda dp \ p^2 \left( \frac{\tanh \left( \frac{1}{2} \beta \omega_-(p, \Delta) \right)}{\omega_-(p, \Delta)} + \frac{\tanh \left( \frac{1}{2} \beta \omega_+(p, \Delta) \right)}{\omega_+(p, \Delta)} \right), \]

(5)

Solving this equation at \( \Delta = 0 \) one can obtain the value of the critical temperature at which the phase transition takes place.

If one is interested in studying electromagnetic properties, such as the Meissner effect, one must include in (2) a U(1) gauge field, \( A_\mu \). This gauge field is going to be treated as an external field.

One can also consider fluctuations of the gap, \( \Delta(\vec{x}) = \Delta_0 + \tilde{\Delta}(\vec{x}) \), around its equilibrium position, \( \Delta_0 \), determined by eq. (5). This will allow us to compute the coherence length, which is a measure of the rigidity of the condensate. For simplicity, we will consider static fluctuations and suppose \( \Delta \) to be real.

We are going to treat the inclusion of the gauge field and the fluctuations of the gap as static perturbations. To this purpose, we write the Lagrangian (2) as \( \mathcal{L} = \bar{\Psi} O_F \Psi \), where \( \Psi^\dagger = (\psi^\dagger, \psi^\dagger_c) \), and we split the matrix \( O_F \) as \( O_F(\Delta, A) = O_{F0}(\Delta_0) + \delta O_F(\Delta, A) \). In the path integral formalism this amounts to consider the saddle point approximation, which is the approach that Gor’kov [10] followed to derive the Ginzburg-Landau effective action from the BCS theory. In this way, the free energy can be expanded as

\[ \Omega = \int d^3x \ V_{\text{cl}}(\Delta_0 + \tilde{\Delta}(\vec{x})) - \frac{1}{2\beta} \log \det O_F = \Omega_0 + \Omega_1 + \Omega_2 + \ldots \]

(6)
The complete free energy is then
\[
\Omega_0 = \int d^3x \, g^2 \Delta_0^2 - \frac{1}{2\beta} \log \det O_F ,
\]
\[
\Omega_1 = \int d^3x \, 2^2 \Delta_0 \Delta(\bar{x}) - \frac{1}{2\beta} \text{Tr}[O_F^{-1} \delta O_F] ,
\]
\[
\Omega_2 = \int d^3x \, g^2 \bar{A}^2(\bar{x}) + \frac{1}{4\beta} \text{Tr}[(O_F^{-1} \delta O_F)^2] ,
\]
where
\[
\text{Tr}[O_0^{-1} \delta O] \equiv \int d^4x \int d^4x_1 \text{tr}[O_0^{-1}(x, x_1) \delta O(x_1, x)] ,
\]
\[
\text{Tr}[(O_0^{-1} \delta O)^2] \equiv \int d^4x \int d^4x_1 \int d^4x_2 \int d^4x_3 \text{tr}[O_0^{-1}(x, x_1) \delta O(x_1, x_2)O_0^{-1}(x_2, x_3) \delta O(x_3, x)] ,
\]
\[
\delta O(x, y) = \delta^{(4)}(x - y) \delta O(x) \quad \text{and} \quad \text{tr}[.] \quad \text{is the usual trace over matrix elements.}
\]
The first term of the expansion (6), \( \Omega_0 \), is just the free energy corresponding to eq. (3), which fixes the value of the gap. The second term, \( \Omega_1 \), has two contributions, one corresponding to \( \bar{\Delta} \) (which vanishes, as it is proportional to the gap equation) and another one due to the gauge field \( A \). In momentum space we have
\[
\text{Tr}[O_F^{-1} \delta O_F] = \int d^4x \frac{1}{\beta} \sum_n \int \frac{d^3p}{(2\pi)^3} \text{tr}[O_F^{-1}(\omega_n, \vec{p}) \delta O_F(x)] ,
\]
which gives rise to terms of the form
\[
\int d^4x \frac{1}{\beta} \sum_n \int \frac{d^3p}{(2\pi)^3} h(\omega_n, p^2) \vec{p} \cdot \vec{A}(x) ,
\]
for some function \( h \) depending on \( \omega_n \) and \( p^2 \). This term does not survive the momentum integration, so this just leaves contributions involving the temporal component of the gauge field, which are interpreted as fluctuations or space inhomogeneities of the chemical potential.

Once the effective potential is computed we have to add the kinetic term for the gauge field, the complete free energy is then
\[
\Omega_{\text{tot}} = \int d^3x \left[ \frac{1}{2} E^2 + \frac{1}{2} B^2 \right] + \Omega(\Delta, A)
\]
\[
= \int d^3x \left[ f_1 \left( \vec{\nabla} - ie\vec{A} \right) \Delta \right]^2 + m^{-2} |\Delta|^2 + f_2 \bar{E}^2 + f_3 \bar{B}^2 + \Omega_0 + \ldots ,
\]
where the coefficients \( f_1, f_2, f_3 \) and \( m^{-2} \) have been introduced to account for the contributions coming from the \( \Omega_1 \) and \( \Omega_2 \) terms of the expansion of the free energy. Near \( T_c \), we can expand \( \Omega_0 \) in eq. (11) as
\[
\Omega_0(\Delta_0, T) = \int d^3x \left[ a(T - T_c) \Delta_0^2 + b \Delta_0^4 + \ldots \right] ,
\]
where
\[
a = \frac{g^4}{2\pi^2} \beta_c^2 \int_0^\Lambda dp \, p^2 \left( \text{sech}^2 \left( \frac{1}{2} \beta_c (\omega_0(p) + \mu) \right) + (\mu \rightarrow -\mu) \right) ,
\]
\[
b = \frac{g^8}{2\pi^2} \int_0^\Lambda dp \, p^2 \left( -\beta_c \text{sech}^2 \left( \frac{1}{2} \beta_c (\omega_0(p) + \mu) \right) \frac{\text{sech}^2 \left( \frac{1}{2} \beta_c (\omega_0(p) + \mu) \right)}{(\omega_0(p) + \mu)^2} + 2 \frac{\text{tanh} \left( \frac{1}{2} \beta_c (\omega_0(p) + \mu) \right)}{(\omega_0(p) + \mu)^3} + (\mu \rightarrow -\mu) \right) .
\]
Eq. (12) is just the Ginzburg-Landau (GL) free energy, which is valid near the critical temperature, so (11) must be considered as a generalization of the GL free energy, valid for all temperatures well below the cut-off scale.

From the equations of motion for $\vec{A}$ (in the London limit, i.e. $\Delta = 0$, and in the Coulomb gauge, $\vec{\nabla} \cdot \vec{A} = 0$) one sees that the magnetic field is exponentially suppressed inside the superconductor, which is the footprint of the Meissner effect. The corresponding magnetic penetration length, $\lambda$, is given by

$$\nabla^2 \vec{A} = \frac{e^2 f_1}{f_3} |\Delta_0|^2 \vec{A} \Rightarrow \frac{1}{\lambda^2} = \frac{e^2 f_1}{f_3} |\Delta_0|^2.$$  \hspace{1cm} (13)

Another characteristic length is the coherence length, $\xi$, which characterizes the distance over which the superconducting electron concentration cannot drastically change. This length is obtained from the equations of motion for $\Delta(x)$ obtained from (11). In absence of gauge field, we get

$$\nabla^2 \Delta = \frac{1}{m^2 f_1} \Delta \Rightarrow \frac{1}{\xi^2} = \frac{1}{m^2 f_1}.$$ \hspace{1cm} (14)

Hence to determine these lengths we must be able to identify the coefficients $f_1$, $f_3$ and $m^{-2}$ in (11). The identification of these coefficients is explained in appendix B.

We also want to compute the specific heat and the critical magnetic field above which superconductivity is destroyed. The former was already computed in [6] at constant $\mu$, instead of constant charge density. This way is more convenient because, when neutral scalars are considered in the SUSY case, they do not contribute to the charge density constraint $\rho = d\Omega/d\mu$. The specific heat is computed through the formula

$$S = - \left( \frac{\partial V_{\text{eff}}}{\partial T} \right)_\mu, \hspace{0.5cm} c = T \left( \frac{dS}{dT} \right)_\mu = -T \left( \frac{\partial^2 V_{\text{eff}}}{\partial T^2} + \frac{\partial^2 V_{\text{eff}}}{\partial T \partial \varepsilon} \frac{\partial \varepsilon}{\partial T} \right), \hspace{0.5cm} \text{where} \hspace{0.5cm} \frac{\partial \varepsilon}{\partial T} = - \frac{\partial T}{\partial \varepsilon} \frac{\partial V_{\text{eff}}}{\partial \varepsilon}.$$ \hspace{1cm} (15)

The critical magnetic field is obtained by equating the work done for holding the magnetic field out of the superconductor with the condensation energy,

$$\frac{H_c^2(T)}{8\pi} = V_n(T) - V_s(T),$$ \hspace{1cm} (16)

where $V_n$ and $V_s$ are the free energies per unit volume in the normal and superconducting phase at zero field. This is the critical magnetic field above which superconductivity would be destroyed in a type I superconductor, where the GL parameter is $\kappa \equiv \lambda/\xi \ll 1$. When $\kappa \gg 1$, we are dealing with type II superconductors. Instead of a discontinuous breakdown of superconductivity in a first order phase transition at $H_c$, like in type I superconductors, type II superconductors exhibit an intermediate Abrikosov vortex state in between two critical magnetic fields,

$$H_{c1} \approx \frac{\phi_0}{2\pi \lambda^2}, \hspace{1cm} H_{c2} \approx \frac{\phi_0}{2\pi \xi^2}.$$ \hspace{1cm} (17)

$H_{c1}$ is the value of the magnetic field for which a single vortex with flux quantum $\phi_0 = \pi/\mu$ would appear, whereas near $H_{c2}$ vortices are as closely packaged as the coherence length allows. In this vortex state the magnetic field penetrates in regular arrays of flux tubes of non-superconducting material surrounded by a superconducting current. Superconductivity disappears completely for magnetic fields above $H_{c2}$. 

6
3 Supersymmetric BCS theory

The supersymmetric version of BCS theory [6], is based on an $\mathcal{N} = 1$ SUSY theory described by the following Kähler potential,

$$K = \Phi^\dagger \Phi + g^2 (\Phi^\dagger \Phi)^2,$$  \hspace{1cm} (18)

and no superpotential. In [6] it was necessary to include up to four chiral superfields. Two superfields with the mentioned Kähler potential for each one, in order to have Dirac fermions, just as in standard BCS theory; and two additional free superfields with canonical Kähler potential to ensure that the $U(1)_R$ symmetry is non-anomalous, so that we can introduce a chemical potential for this symmetry in a consistent way.

The chemical potential is introduced in this theory for the $U(1)_R$ symmetry in such a way that scalars belonging to chiral superfields with Kähler potential (18) are neutral and, as a consequence, their fermionic superpartners have $R$-charge $-1$. This is done in this way in order to prevent these scalars from coupling to the chemical potential, thus avoiding Bose-Einstein condensation (BEC) of the scalars. However, the additional superfields, added to ensure that the $U(1)_R$ symmetry is non-anomalous, must contain fermions with $R$-charge $+1$ and therefore scalars with $R$-charge $+2$, which couple to the chemical potential and condense. Nevertheless, the four chiral superfields are decoupled form each other, then the sector with scalars suffering from BEC does not participate in the thermodynamics of the sector with Kähler potential (18), in which we are interested. For this reason, it is sufficient to study the one superfield model (18).

In this minimal setup, with one chiral superfield, this particular choice of $R$-charges prevents the introduction of a superpotential, which may lead to mass terms, in this way we are dealing with a massless theory.

This theory also has a baryonic $U(1)_B$ symmetry. If the chemical potential is introduced for this $U(1)_B$ symmetry instead for the $U(1)_R$ symmetry, this would inevitably lead to BEC for the scalar in the presence of a Fermi surface. In this case, there is no reason why we cannot introduce a superpotential that generates mass terms, for example $W = m \Phi \Phi$. With this superpotential we obtain the following scalar energy eigenvalues,

$$\omega_{S\pm} = \sqrt{p^2 + m^2 + 4g^4 \Delta^2} \pm \mu.$$  \hspace{1cm} (19)

As we have seen, the existence of a Fermi surface requires $\mu > m$, for which $\omega_{S-}$ would become negative at low momentum near the non-superconducting phase ($\Delta = 0$), making the thermal contribution to the effective potential ill-defined, resulting in the appearance of BEC.

Once the Kähler potential (18) is expanded in terms of component fields, we have to perform, as in the previous section, the Hubbard-Stratonovich transformation introducing the auxiliary field $\Delta$, which now is related with the fermion condensate through

$$\Delta = \frac{\psi \bar{\psi}}{1 + 4g^2 v^2}.$$  \hspace{1cm} (20)

After expanding the scalar field to quadratic order around its vacuum expectation value, $v$, which turns out to be zero, going to Euclidean space and introducing the chemical potential for the $R$-symmetry we arrive at the following Lagrangian

$$\mathcal{L}_E = (\partial^a \phi^a \partial^a \phi^* + \bar{\psi} \sigma^0 (\partial^\tau + \mu) \psi + i \bar{\psi} \bar{\sigma}^i \partial^i \psi) + 4g^4 |\Delta|^2 |\phi|^2 - g^2 \Delta (\psi \bar{\psi}) - g^2 \Delta^* (\bar{\psi} \psi),$$  \hspace{1cm} (21)

with classical potential $V_\Delta = g^2 |\Delta|^2$.

Now we want to study the magnetic response of the superconductor, therefore we have to turn on an external $U(1)$ gauge field. We have two possibilities: the gauge field can be turned on for the
baryonic $U(1)_B$ symmetry or the $U(1)_R$ symmetry\textsuperscript{2}. In either case supersymmetry is broken by the background. In principle, the magnetic response could depend on this choice, but as explained at the end of appendix B, this is not the case, at least for a small enough gauge coupling\textsuperscript{3}. For definiteness we introduce the gauge field through the baryonic $U(1)_B$ symmetry, therefore the Lagrangian (21) is modified to

\[
\mathcal{L}_E = \left( \partial^a \phi \partial^a \phi^* + \bar{\psi} \sigma^0 (\partial^a - ieA^a) \psi + i \bar{\psi} \sigma^i (\partial^i - ieA^i) \psi \right)
- ieA^a \phi \partial^a \phi^* + ieA^a \phi^* \partial^a \phi + e^2 A^a A^a |\phi|^2 + 4g^4 |\Delta|^2 |\phi|^2 - g^2 \Delta (\psi \psi) - g^2 \Delta^* (\bar{\psi} \bar{\psi}) .
\]

Given this Lagrangian, we can construct the bosonic and fermionic matrices (whose explicit form is written in appendix A) and split them as explained in the previous section, so that we have to add to eq. (6), the scalar contribution

\[
+ \frac{1}{2\beta} \log \det O_{S0} + \frac{1}{2\beta} \text{Tr}[O_{S0}^{-1} \delta O_S] - \frac{1}{4\beta} \text{Tr}[(O_{S0}^{-1} \delta O_S)^2] + \ldots
\]

The determination of the quantities explained in the case of relativistic BCS theory is now completely analogous for the supersymmetric case. For example, the effective potential analogous to (3), becomes now

\[
V_{\text{eff}} = g^2 \Delta_0^2
- \frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} (\omega_+ + \omega_- - 2\omega_S)
- \frac{1}{\beta} \int \frac{d^3 p}{(2\pi)^3} \left( \log(1 + e^{-\beta \omega_+}) + \log(1 + e^{-\beta \omega_-}) - 2 \log(1 - e^{-\beta \omega_S}) \right) ,
\]

where the energy eigenvalues are

\[
\omega_{F\pm} = \sqrt{(p \pm \mu)^2 + 4g^4 \Delta_0^2} , \quad \omega_{S1,2} = \sqrt{p^2 + 4g^4 \Delta_0^2} .
\]

4 Comparison between the SUSY model and relativistic BCS

4.1 Gap

According to the previous section, we have to solve the gap equation $\partial_r V_{\text{eff}} = 0$. Just as shown in [6], the explicit form of the gap equation is

\[
1 = \frac{g^2}{2\pi^2} \int_0^\Lambda dp \ p^2 \left( \frac{\tanh \left( \frac{1}{2} \beta \sqrt{4g^4 \Delta_0^2 + (p - \mu)^2} \right)}{\sqrt{4g^4 \Delta_0^2 + (p - \mu)^2}} + \frac{\tanh \left( \frac{1}{2} \beta \sqrt{4g^4 \Delta_0^2 + (p + \mu)^2} \right)}{\sqrt{4g^4 \Delta_0^2 + (p + \mu)^2}} \right)
- \frac{2 \coth \left( \frac{1}{2} \beta \sqrt{4g^4 \Delta_0^2 + p^2} \right)}{\sqrt{4g^4 \Delta_0^2 + p^2}} ,
\]

where the second line is the new contribution due to the scalar. For the standard relativistic BCS

\textsuperscript{2}R-symmetry can only be gauged within the context of supergravity. The present model can be easily embedded in $\mathcal{N} = 1$ supergravity, in such a way that in the usual laboratory setup, where energy configurations are much lower than the Planck scale, the supergravity multiplet can be ignored.

\textsuperscript{3}However, things are different if we turn on the gauge fields associated with the $U(1)_B$ and $U(1)_R$ symmetries at the same time. In this case, there is a linear combination of the previously introduced gauge fields which defines a massless rotated gauge field. Therefore, the magnetic field associated to this rotated gauge field can penetrate the superconductor without being subject to the Meissner effect. This case is similar to that of [11, 12], so it would be interesting to study how the phenomenology presented there translates into the present supersymmetric model.
As explained in [6], the IR physics of massless scalars makes the phase transition first order rather than second order. First order phase transitions in usual superconductivity have already been explained in [13], where a gauge field takes a non zero vacuum expectation value appearing as a $\langle A^2|\Delta_0|\rangle^3$ term in the free energy, which inevitably leads to a first order phase transition. This gauge field plays the same role as the scalar in the supersymmetric case. Near $\Delta_0 = 0$ and at low momentum, we can approximate the contribution of the scalar as

$$
\partial_\Delta V_{\text{eff}} \approx \frac{2g^4T}{\pi^2} \int_0^\infty dp \frac{1}{p^2 + 4g^4\Delta_0^2} \approx \frac{-2g^6T}{\pi} \Delta_0 ,
$$

Figure 1: (a) Effective potential as function of the gap for different temperatures ($T = 2, 1.3, 1, 0.75, 0$ from top to bottom) in the relativistic BCS case. (b) Corresponding gap as function of the temperature ($\mu = 0.0065, g = 0.54, \Lambda = 8.4$). (c) ($T = 1.1, 1, 0.92, 0.84, 0.7, 0.47, 0$) and (d) analogous figures for the SUSY case ($\mu = 0.65, g = 3.9, \Lambda = 52$). The thicker red lines in (d) correspond to metastable solutions and the curved dashed line represents the maximum that separates both minima in the effective potential. From now on, all figures and data are given in units of $T_c$.

case the effective potential develops a non-trivial minimum at $\Delta_0 \neq 0$, which will approach the origin as we increase the temperature reaching the zero gap value at the critical temperature $T_c$, (fig. 1 (a) and (b)). The phase transition is second order. For the SUSY case, at low temperatures, $T < T_{c1}$, the effective potential has a unique non-trivial minimum (zone 1 of fig. 1 (c) and (d)), above $T_{c1}$ (zone 2) a zero gap metastable minimum appears becoming the dominant one for $T > T_{c2}$ (zone 3), so that the minimum with non-vanishing gap becomes metastable until it disappears at $T_{c3}$, jumping to the non-superconducting zero gap solution for $T > T_{c3}$ (zone 4). Therefore, the phase transition is first order in the supersymmetric case. For short, we will be calling $T_c \equiv T_{c3}$ and $\Delta_c \equiv \Delta(T_{c3})$ from now on.
which leads to the analogous $O(|\Delta_0|^3)$ term in the free energy.

At low temperatures, we can neglect in (26) the antiparticle and the scalar contributions for momenta near the Fermi surface, where the main contribution to the integral comes from. By doing so, we are taking the non-relativistic limit and connecting with the standard BCS result. Specifically, one performs the ad hoc approximation by which one substitutes $dp^3$ by $4\pi p^2 dp$ and the integral is done in the interval $|p - p_F| < \Lambda$ around the Fermi surface. After these approximations and at large cut-off ($\Lambda/T_c \gg 1$), one can show that the behaviour of the gap near the critical temperature $T_c$ follows a universal behaviour

$$\frac{\Delta_0(T)}{\Delta_0(0)} \approx \eta \sqrt{1 - \frac{T}{T_c}},$$

(28)

where $\eta = 1.74$ can be computed numerically from eq. (12). In the supersymmetric case we obtain numerically the following behaviour near the critical temperature:

$$\frac{\Delta_0(T) - \Delta_c}{\Delta_0(0)} \approx \eta \left(1 - \frac{T}{T_c}\right)^\alpha,$$

(29)

where $\alpha \approx 0.5$ and now $\eta$ depends on the parameters $g$, $\mu$ and $\Lambda$.

We can also determine the expressions for the gap at zero temperature. The non-relativistic and relativistic BCS expressions are:

$$|\Delta_0(0)_{\text{BCS}}| \approx \frac{\Lambda}{g^2} e^{-\frac{2\pi^2}{g^2 \mu^2}} \approx 12 \pi e^{-\gamma T_c/g^2},$$

$$|\Delta_0(0)_{\text{rel. BCS}}| \approx \frac{\Lambda^3}{6\pi^2} \approx T_c/g^2,$$

(30)

where in the non-relativistic limit we have considered the aforementioned approximation and $\gamma$ is the Euler-Mascheroni constant. Both the gap at zero temperature and the critical temperature depend on the cut-off (the Debye energy in the standard BCS case), in such a way that the cut-off dependence disappears from the formula for the gap once it is expressed in terms of the critical temperature. We have also shown here the dependence on the cut-off because we want to compare with the supersymmetric case, since one of the virtues of supersymmetry is the softening of divergences due to cancellations between fermionic and bosonic contributions. As the thermal integrals in (3) and (24) are convergent, it is sufficient to study the dependence of the gap with the cut-off at zero temperature. Comparing the relativistic expression with the supersymmetric one,

$$|\Delta_0(0)_{\text{SUSY BCS}}| \approx \frac{\Lambda}{g^2} e^{-\frac{2\pi^2}{g^2 \mu^2} - \frac{2}{3}},$$

(31)

we observe that this softening in the cut-off dependence appears in the present system. However, this reduction in the power of the cut-off does not appear if we compare the supersymmetric expression with the non-relativistic one, where the dependence is linear rather than cubic as in the relativistic case. This is because the power of the cut-off has already been reduced after performing the substitution $dp^3 \rightarrow 4\pi p^2 dp$ in the non-relativistic case. Finally, just as in the non-supersymmetric cases (30), one can check that the supersymmetric case (31) is also linear with the critical temperature with no dependence on the cut-off.

4.2 Specific heat

The comparison between the SUSY and the relativistic BCS case for the specific heat, computed with eq. (15), is shown in fig. 2. When only fermions are considered the jump in the specific heat at the critical temperature is finite, as it is characteristic for second order phase transitions, whereas it is infinite when the scalar is included.
The behaviour of the specific heat in the different temperature regimes is explained in [6], whose expressions we show here. At high temperatures ($\Delta_0 = 0$) the different contributions to the specific heat corresponding to the scalar, the fermion and its antiparticle, are the following ones

\[
c_s|_{\Delta_0=0} = \frac{4\pi^2 T^3}{15}, \quad c_{F \pm}|_{\Delta_0=0} = \frac{7\pi^2 T^3}{60}, \quad \text{for } T \gg \mu. \tag{32}
\]

At low temperatures the antiparticle and the scalar can be neglected and the main contribution comes from the region near the Fermi surface $p \sim \mu$, so that the behaviour of the specific heat in the normal phase and in the superconducting phase is given by

\[
c_{F-}|_{\Delta_0=0} \sim \frac{\mu^2 T}{6}, \quad c_{F-}|_{\Delta_0 \neq 0} \sim e^{-\frac{2\mu^2 \Delta_0(T=0)}{\Lambda}}. \tag{33}
\]

The last expression shows a way to compute the value of the gap at zero temperature. Indeed, by performing a fit of the plots in fig. 2 one can obtain the value of the gap at zero temperature shown in fig. 1.

### 4.3 Magnetic penetration length and coherence length

The magnetic penetration length, obtained from eq. (13), is plotted in fig. 3. In both the relativistic and the SUSY BCS theory, the magnetic penetration length is a monotonically increasing function.

For the fermionic contribution alone, the magnetic penetration length diverges near the critical temperature. Whereas in the supersymmetric case, it reaches a finite value at $T_c$ and it jumps to infinity for $T > T_c$, which is the expected behaviour for a first order phase transition. This behaviour is explained basically from the gap dependence. Expanding the $f_1$ coefficient as a power series of the temperature and the gap, one obtains

\[
\begin{align*}
f_{1,\text{rel BCS}} &= a_0 + a_1 \Delta_0^2 + a_2 (T - T_c) + \ldots, \\
f_{1,\text{SUSY BCS}} &= a_0 + a_1 (\Delta_0^2 - \Delta_c^2) + \ldots, \tag{34}
\end{align*}
\]

There is a $1/2$ factor between the results presented here from those of [6], because here we are considering just one complex scalar field and a Weyl fermion.

For high enough values of the chemical potential and the cut-off a counterintuitive non-trivial minimum can appear before reaching the critical temperature, which would mean that there is a range of temperatures where the Meissner effect is enhanced with increasing temperature. This odd behaviour can be avoided by restricting the parameter range of validity to not very high values of the chemical potential and the cut-off. However, this restriction is relaxed in the SUSY case where the cut-off dependence is softened.

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Figure 2: Specific heat a function of the temperature. (a) Relativistic BCS ($\mu = 0.0065$, $g = 0.54$, $\Lambda = 8.4$). (b) SUSY BCS ($\mu = 0.65$, $g = 3.9$, $\Lambda = 52$).

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\[^1\]There is a $1/2$ factor between the results presented here from those of [6], because here we are considering just one complex scalar field and a Weyl fermion.

\[^2\]For high enough values of the chemical potential and the cut-off a counterintuitive non-trivial minimum can appear before reaching the critical temperature, which would mean that there is a range of temperatures where the Meissner effect is enhanced with increasing temperature. This odd behaviour can be avoided by restricting the parameter range of validity to not very high values of the chemical potential and the cut-off. However, this restriction is relaxed in the SUSY case where the cut-off dependence is softened.
where one can check that the coefficients shown do not vanish. The ellipsis stands for higher powers of the temperature, taking into account the temperature dependence of the gap, (28) or (29). Substituting this expansion in (13), and using (28) or (29), one finds the behaviour

\[ \lambda_{\text{rel, BCS}} \sim \left(1 - \frac{T}{T_c}\right)^{-1/2}, \quad (\lambda - \lambda_c)_{\text{SUSY BCS}} \sim \left(1 - \frac{T}{T_c}\right)^{\alpha}. \]

Figure 3: Magnetic penetration length as a function of the temperature. (a) Relativistic BCS (\(\mu = 0.0065, g = 0.54, \Lambda = 8.4\)). (b) SUSY BCS (\(\mu = 0.65, g = 3.9, \Lambda = 52\)).

The behaviour of the magnetic penetration length at zero temperature can be computed analytically. According to the dependence of the coefficients \(f_1\) and \(m^{-2}\) with the cut-off, given in appendix B, and that of the gap, we find the following expressions for the magnetic penetration length,

\[ \lambda_{\text{rel, BCS}} \approx \frac{4g^3}{3\sqrt{3\pi^2e}}\Lambda^2, \quad \lambda_{\text{SUSY BCS}} \approx \frac{2\pi}{ec}(1 + 4c^2)^{3/4}\Lambda^{-1}, \]  \(\text{where } c = \exp\left[-\frac{\pi^2}{g^2\mu^2} - \frac{3}{2}\right].\)

The coherence length, \(\xi\), obtained from eq. (14), is plotted in fig. 4. It is a monotonically increasing function of the temperature for both the relativistic and the SUSY BCS theory. The behaviour near the critical temperature is the same as for the magnetic penetration length. To see this, expand the \(m^{-2}\) coefficient as we did with the \(f_1\) coefficient. Using the gap equation, one can see that the \(m^{-2}\) coefficient has a global \(\Delta^2_0\) factor so that the expansions are

\[ m_{\text{rel, BCS}}^{-2} = \Delta_0^2(\beta_0 + \beta_1(\Delta_0^2 - \Delta_c^2) + \ldots), \quad m_{\text{SUSY BCS}}^{-2} = \Delta_0^2(\beta_0 + \beta_1(\Delta_0^2 - \Delta_c^2) + \ldots). \]

Inserting the expansions (34) and (37) in the expression for the coherence length, (14), we find

\[ \xi_{\text{rel, BCS}} \sim \left(1 - \frac{T}{T_c}\right)^{-1/2}, \quad (\xi - \xi_c)_{\text{SUSY BCS}} \sim \left(1 - \frac{T}{T_c}\right)^{\alpha}. \]

Thus, the coherence length exhibits the same behaviour as the magnetic penetration length. However, the behaviour of the coherence length at zero temperature is different from that of the magnetic penetration length. Now, the dependence on the cut-off is

\[ \xi_{\text{rel, BCS}} \approx \frac{9\sqrt{6\pi^4}}{4g^4}\Lambda^{-5}, \]

\[ \xi_{\text{SUSY BCS}} \approx \frac{g}{2(1 + 4c^2)^{3/4}}\left(2\pi^2 + g^2\mu^2\left(\frac{32c^4 + 16c^2 + 5}{(1 + 4c^2)^{5/2}} - 2\log\frac{1 + \sqrt{1 + 4c^2}}{2c}\right)\right)^{^{-1/2}} + O(\Lambda^{-2}). \]
If we take the quotient between these two characteristic lengths,

$$\kappa = \frac{\lambda}{\xi} = \frac{1}{\sqrt{2e_m f_1 \Delta_0}},$$  \hspace{1cm} (41)

we get the Ginzburg-Landau parameter, shown in fig. 5. As the coherence length behaves in the same way as the magnetic penetration length near the phase transition, at leading order, the GL parameter will take a finite constant value. Depending on the value of the GL parameter one has a type I ($\kappa \ll 1$) or a type II superconductor ($\kappa \gg 1$). In the GL theory $\kappa$ is defined near the critical temperature and the critical value differentiating between the two types of superconductor is $\kappa = 1/\sqrt{2}$. As shown in fig. 5, $\kappa \gg 1$, since we are considering small values of gauge coupling $e$, then the superconductors are type II in both the relativistic BCS case and the SUSY case.

Figure 5: Ginzburg-Landau parameter as a function of temperature. (a) Relativistic BCS ($\mu = 0.0065$, $g = 0.54$, $\Lambda = 8.4$). (b) SUSY BCS ($\mu = 0.65$, $g = 3.9$, $\Lambda = 52$).

**4.4 Critical magnetic fields**

For a type I superconductor, the critical magnetic field is obtained by equating the energy per unit volume, associated with holding the field out against the magnetic pressure, with the condensation
energy. That is eq. (16),
\[
\frac{H_c^2(T)}{8\pi} = V_n(T) - V_s(T).
\]

The behaviour near the critical temperature is found by preforming expansions similar to those for the \(f_1\) and \(m^{-2}\) coefficients. In the standard relativistic BCS theory the gap is expanded around \(\Delta_0 = 0\). On the other hand, when the scalar is considered, \(H_c\) does not make sense above \(T_{c2}\), since the superconducting minimum in the effective potential becomes metastable, but we can perform the expansion around \(\Delta_{c2} \equiv \Delta_0(T_{c2})\). According to the dependence of the gap with the temperature (28) in the relativistic BCS theory and due to the fact that \(\Delta_0\) is linear with the temperature near \(T_{c2}\) in the SUSY BCS theory, we have
\[
\begin{align*}
\frac{H_{c,\text{rel, BCS}}^2}{8\pi} &= \partial_T \partial_T (V_n(T) - V_s(T)) \bigg|_{\Delta_0 = 0}^{T = T_c (T - T_c) \Delta_0^2} \\
&+ \frac{1}{2} \partial_T^2 (V_n(T) - V_s(T)) \bigg|_{\Delta_0 = 0}^{T = T_c} \Delta_0^4 + \ldots \quad (42) \\
\frac{H_{c,\text{SUSY BCS}}^2}{8\pi} &= \partial_T (V_n(T) - V_s(T)) \bigg|_{\Delta_0 = \Delta_{c2}}^{T = T_{c2}} (T - T_{c2}) + \ldots \quad (43)
\end{align*}
\]
from which we find a linear and square root behaviour near the critical temperatures,
\[
H_{c,\text{rel, BCS}} \sim \left(1 - \frac{T}{T_c}\right), \quad H_{c,\text{SUSY BCS}} \sim \sqrt{\frac{T}{T_{c2}}}, \quad (44)
\]
as shown in fig. 6 (a) and (b), respectively. The expressions for the critical magnetic field at zero temperature are
\[
H_{c,\text{rel, BCS}} \approx \frac{\sqrt{2\pi g}}{3\pi^2} \Lambda^3, \quad H_{c,\text{SUSY BCS}} \approx \sqrt{8\pi} \left[-1 - \frac{g^2}{\pi^2} \left(\frac{1}{\sqrt{1 + \frac{4c^2}{2c}}} \log \frac{1 + \sqrt{1 + 4c^2}}{2c} \right) \frac{c}{g}\right] \Lambda. \quad (45)
\]
As expected the dependence on the cut-off is milder in the supersymmetric case.

As explained at the end of sec. 2, for type II superconductors, which are characterized by the appearance of Abrikosov vortices in a mixed superconductor-normal state, there are two critical magnetic fields, \(H_{c1}\) and \(H_{c2}\),
\[
H_{c1} \approx \frac{\phi_0}{2\pi \lambda^2}, \quad H_{c2} \approx \frac{\phi_0}{2\pi \xi^2}.
\]
They are plotted in fig. 6 together with \(H_c\). It is easy to obtain the behaviour of these two critical magnetic fields in the different regimes using the expressions for the magnetic penetration length and the coherence length.

If \(H_c \ll H_{c2}\), we will be able to see the intermediate vortex state as we decrease the applied magnetic field, i.e. the superconductor is type II. On the contrary if \(H_c \gg H_{c2}\), we reach the pure superconducting state without the formation of any vortex, and we have type I superconductivity. According to fig. 6, we have type II superconductivity in both the relativistic and the SUSY BCS theory, in agreement with the prediction obtained in the previous section by computing the GL parameter.

Given that \(H_c\) ends at \(T_{c2}\) instead of \(T_{c3}\), as \(H_{c1}\) and \(H_{c2}\), and that there is a range of parameters where \(H_{c2} < H_c\) at zero temperature, one can find a crossing between the two magnetic fields and a crossover between type I and type II behaviour as we increase the temperature. However, this crossing effect disappears if the gauge coupling is sufficiently small.
5 Conclusions

In this paper we have computed some relevant quantities in superconductivity as a function of the temperature for the supersymmetric BCS model proposed in [6] and for standard relativistic BCS theory, comparing both results with the aim of clarify the role of supersymmetry. These quantities are the gap, the specific heat, the magnetic penetration length, which describes the Meissner effect, the coherence length and the different critical magnetic fields. To compute these quantities we have considered spatial fluctuations of the gap and introduced an external gauge field for the baryonic $U(1)_B$ symmetry. Both the gap fluctuations and the gauge field are introduced as perturbations, so once the effective potential is expanded and the coefficients in the generalized Ginzburg-Landau equation (11) are identified, the computation of the aforementioned quantities is straightforward.

Considering the quotient between the magnetic penetration length and the coherence length, which defines the GL parameter, shows that the studied superconductors are type II. This is confirmed when studying the relation between the critical magnetic fields $H_c$ and $H_{c2}$.

The main differences between the standard relativistic BCS results and those from the supersymmetric model lie in the different orders of the phase transition. For example, the infinite jumps found in the supersymmetric model for the specific heat, magnetic penetration length and coherence length are those characteristic of a first order phase transition. Another important difference is the softening in the cut-off dependence in all quantities thanks to supersymmetry.

It would be very interesting to see if there are realistic condensed matter systems with an effective quasi supersymmetric dynamics with thermodynamic and response properties similar to the present system, summarized in the different figures. It would also be interesting to construct extensions of the present supersymmetric model, in particular, to describe supersymmetric color superconductivity and compare the resulting dynamics with the standard phenomenology of QCD color superconductivity [14, 15, 16].

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A Scalar and Fermionic matrices

Writing the quadratic part of the Lagrangian in the presence of a baryonic gauge field, (22), as

$$\mathcal{L} = \Phi^4 O_S \Phi + \Psi^4 O_F \Psi,$$

with $\Phi = \begin{pmatrix} \phi(p) \\ \phi^*(p) \end{pmatrix}$ and $\Psi = \begin{pmatrix} \psi_1(p) \\ \psi_2(p) \\ \psi_1^*(-p) \\ \psi_2^*(-p) \end{pmatrix}$,

the scalar and fermionic matrices after the splitting $O(\Delta, A) = O_0(\Delta_0) + \delta O(\Delta, A)$, are given in momentum space by

$$O_{S0} = \begin{pmatrix} \frac{1}{2} (\omega^2 + p^2) + 2g^4 \Delta_0^2 & 0 \\ 0 & \frac{1}{2} (\omega^2 + p^2) + 2g^4 \Delta_0^2 \end{pmatrix},$$

$$\delta O_S = \begin{pmatrix} e(A_+ \omega + \bar{A} \cdot \vec{p}) + \frac{i}{2} e^2 (A_+^2 + \bar{A}^2) + 2g^4 (2\Delta_0 + \Delta_0^2) & 0 \\ 0 & -e(A_+ \omega + \bar{A} \cdot \vec{p}) + \frac{i}{2} e^2 (A_+^2 + \bar{A}^2) + 2g^4 (2\Delta_0 + \Delta_0^2) \end{pmatrix},$$

$$O_{F0} = \begin{pmatrix} \frac{1}{2} \omega + \frac{1}{2} \Omega & 0 & 0 & -g^2 \Delta_0 \\ 0 & \frac{1}{2} \omega - \frac{1}{2} \Omega & 0 & 0 \\ 0 & 0 & \frac{1}{2} \omega + \frac{1}{2} \mu & -g^2 \Delta_0 \\ -g^2 \Delta_0 & 0 & 0 & \frac{1}{2} \omega - \frac{1}{2} \Omega \end{pmatrix},$$

$$\delta O_F = \begin{pmatrix} -\frac{1}{2} e A_+ - \frac{1}{2} e A & 0 & 0 & -g^2 \Delta \\ 0 & -\frac{1}{2} e A_+ + \frac{1}{2} e A & 0 & 0 \\ 0 & 0 & \frac{1}{2} e A_+ + \frac{1}{2} e A & 0 \\ -g^2 \Delta & 0 & 0 & \frac{1}{2} e A_+ - \frac{1}{2} e A \end{pmatrix}. \quad (46)$$

If instead of introducing a baryonic $U(1)_B$ gauge field, we had introduced $U(1)_R$ gauge field, the scalar and fermionic matrices would have been the following ones:

$$\delta O_S = \begin{pmatrix} 2g^4 (2\Delta_0 + \Delta_0^2) & 0 \\ 0 & 2g^4 (2\Delta_0 + \Delta_0^2) \end{pmatrix},$$

$$\delta O_F = \begin{pmatrix} \frac{1}{2} e A_+ + \frac{1}{2} e A & 0 & 0 & -g^2 \Delta \\ 0 & \frac{1}{2} e A_+ - \frac{1}{2} e A & 0 & 0 \\ 0 & 0 & \frac{1}{2} e A_+ - \frac{1}{2} e A & 0 \\ -g^2 \Delta & 0 & 0 & \frac{1}{2} e A_+ + \frac{1}{2} e A \end{pmatrix}. \quad (47)$$

The energy eigenvalues computed for the $O_{S0}$ and $O_{F0}$ matrices are

$$\omega_{F+} = \sqrt{(p + \mu)^2 + 4g^4 \Delta_0^2}, \quad \omega_{S, 1, 2} = \sqrt{p^2 + 4g^4 \Delta_0^2}. \quad (48)$$

B $m^{-2}$, $f_1$ and $f_3$ coefficients

In momentum space, the $m^{-2}$ and $f_1$ terms in (11) are given by

$$\int d^3x m^{-2} \Delta \Delta^* = \int \frac{d^3k}{(2\pi)^3} m^{-2} \Delta^*(\vec{k}) \Delta(\vec{k}), \quad (49)$$

$$\int d^3x f_1 \partial^2 \Delta \partial^2 \Delta^* = \int \frac{d^3k}{(2\pi)^3} f_1 \bar{k}^2 \Delta^*(\vec{k}) \Delta(\vec{k}), \quad (50)$$

16
where we have considered time independent perturbations. Thus, we have to find in (6) a term quadratic in $\Delta$ and expand its coefficient up to quadratic order in momentum. The zero order term will correspond to $m^{-2}$ and the coefficient of the quadratic term in momentum will be identified with $f_1$. Terms quadratic in $\Delta$ are found in $\Omega_2$ and, if scalars are considered, in $\Omega_1$. The $\Omega_1$ term only contributes to the $m^{-2}$ coefficient, which in momentum space becomes

$$\frac{1}{2\beta} \text{Tr} \left[ O_{S0}^{-1} \delta O_S \right] = \frac{1}{2\beta^2} \int d^4x \sum_n \int \frac{d^3K}{(2\pi)^3} \text{Tr} \left[ O_{S0}^{-1}(\omega_n, \vec{K}) \delta O_S(\vec{x}) \right],$$

(51)

where we have to sum over Matsubara frequencies, $\omega_n = 2n\pi/\beta$ for bosonic frequencies and $\omega_n = (2n+1)\pi/\beta$ for fermionic ones. Once the Matsubara sums are done, we have to consider the piece quadratic in $\Delta$ (supposing $\Delta$ to be real)

$$\frac{1}{2\beta} \text{Tr} \left[ O_{S0}^{-1} \delta O_S \right]_{\Delta\Delta} = \frac{1}{\beta} \int d^4x \int \frac{d^3K}{(2\pi)^3} B_{\Delta\Delta}(\vec{K}) \Delta(\vec{K}) \Delta(\vec{x})$$

$$= \frac{1}{\beta} \int d^3K (2\pi)^3 \Delta(\vec{K}) \int \frac{d^3k}{(2\pi)^3} \Delta^*(\vec{k}) \Delta(\vec{k}).$$

(52)

Hence, we identify the first contribution to $m^{-2}$ as

$$m^{-2} = \frac{1}{\beta} \int \frac{d^3K}{(2\pi)^3} B_{\Delta\Delta}(\vec{K}) + \ldots$$

(53)

Let us elaborate now on the $\Omega_2$ contribution,

$$\frac{1}{4\beta} \text{Tr}[(O_{F0}^{-1} \delta O_F)^2] = \frac{1}{4\beta} \int d^4x_1 \int d^4x_2 \text{Tr}[\delta O_F(\vec{x}_1)O_{F0}^{-1}(x_1, x_2)\delta O_F(\vec{x}_2)O_{F0}^{-1}(x_2, x_1)],$$

(54)

plus the analogous scalar case, and the extra term $\int d^3x g^2\Delta^2$ for $m^{-2}$. The previous expression in momentum space is

$$\frac{1}{4\beta} \text{Tr}[(O_{F0}^{-1} \delta O_F)^2] = \frac{1}{4\beta} \int d^4x_1 d^4x_2 \sum_{m,n} \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \frac{d^3q_1}{(2\pi)^3} \frac{d^3q_2}{(2\pi)^3}$$

$$\times e^{-i(\omega_m - \omega_n)(\tau_1 - \tau_2)} e^{-i(\vec{k}_1 - \vec{k}_2 + \vec{q}_1)\vec{x}_1} e^{-i(\vec{k}_1 + \vec{k}_2 + \vec{q}_2)\vec{x}_2}$$

$$\times \text{Tr}[\delta O_F(\vec{q}_1)O_{F0}^{-1}(\omega_m, \vec{k}_1)\delta O_F(\vec{q}_2)O_{F0}^{-1}(\omega_n, \vec{k}_2)]$$

$$= \frac{1}{4\beta} \sum_n \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \text{Tr}[\delta O_F(\vec{k}_2 - \vec{k}_1)O_{F0}^{-1}(\omega_n, \vec{k}_1)\delta O_F(\vec{k}_1 - \vec{k}_2)O_{F0}^{-1}(\omega_n, \vec{k}_2)]$$

$$= \frac{1}{\beta} \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} F(\vec{k}_1, \vec{k}_2).$$

(55)

Taking the piece quadratic in $\Delta$ in (55), we have

$$\Omega_2|_{\Delta\Delta} = \frac{1}{\beta} \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \Delta^*(\vec{k}_2 - \vec{k}_1) \Delta(\vec{k}_2 - \vec{k}_1) F_{\Delta\Delta}(\vec{k}_1, \vec{k}_2).$$

(56)

Assuming that second order corrections are located close to each other in momentum space, we can expand the momenta around their average value, $\vec{k}_1 = \vec{K} - \vec{k}/2$, $\vec{k}_2 = \vec{K} + \vec{k}/2$, so that the corresponding free energy term is

$$\Omega_2|_{\Delta\Delta} = \frac{1}{\beta} \int \frac{d^3K}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} \Delta^*(\vec{k}) \Delta(\vec{k}) F_{\Delta\Delta}(\vec{K}, \vec{k}),$$

(57)
Expanding \( \mathcal{F}_{\Delta \Delta} \) up to quadratic order in \( k \), we identify the \( m^{-2} \) and \( f_1 \) coefficients with

\[
m^{-2} = g^2 + \frac{1}{\beta} \int \frac{d^3 K}{(2\pi)^3} \left( \mathcal{F}_{\Delta \Delta}(\vec{K},0) - B_{\Delta \Delta}(\vec{K},0) + \mathfrak{B}_{\Delta \Delta}(\vec{K}) \right),
\]

\[
f_1 = \frac{1}{2\beta} \int \frac{d^3 K}{(2\pi)^3} \left( \partial^2_k \mathcal{F}_{\Delta \Delta}(\vec{K},0) - \partial^2_k B_{\Delta \Delta}(\vec{K},0) \right),
\]

once the analogous bosonic contribution is included.

The \( f_3 \)-term in the Ginzburg-Landau free energy \((11)\) will have a contribution coming from the gauge field kinetic term plus contributions coming from the part of \( \Omega_2 \) quadratic in the gauge field, which will be proportional to the square of the gauge coupling, \( e^2 \),

\[
f_3 = \frac{1}{2} + \mathcal{O}(e^2),
\]

As the gauge coupling is assumed to be small we can simply take \( f_3 = \frac{1}{2} \). From this identification for the \( f_3 \) coefficient we see that there is no significant difference between turning on a baryonic gauge field \((46)\) or an \( R \)-symmetry \((47)\) gauge field, since differences would appear to order \( \mathcal{O}(e^2) \).

Once these coefficients are computed, one can study their cut-off dependence in both the relativistic and supersymmetric case. We will restrict ourselves to the zero temperature regime, where integrals can be performed analytically. In the zero temperature limit, the explicit forms of the coefficients \( f_1 \) and \( m^{-2} \) are

\[
f_1 \big|_{T=0} = \frac{128g^4 \Delta^4_0 \mu + 80g^8 \Delta^4_0 \mu(\Lambda + \mu)^2 + \mu^2(\Lambda + \mu)^5 + 2g^4 \Delta^2_0(\Lambda + \mu)^3 (3\Lambda^2 + 6\Lambda \mu + 8\mu^2)}{96\pi^2 \Delta_0 \left( 4g^4 \Delta^2_0 + (\Lambda + \mu)^2 \right)^{5/2}}
+ \left( \mu \to -\mu \right)
+ \frac{g^8 \Delta^2_0 \Lambda^3}{2\pi^2 \left( 4g^4 \Delta^2_0 + \Lambda^2 \right)^{5/2}},
\]

\[
m^{-2} \big|_{T=0} = g^2 + \frac{g^4}{4\pi^2} \left( \frac{(\Lambda + \mu)(5\mu^2 - 2\Lambda \mu - \Lambda^2)}{\sqrt{4g^4 \Delta^2_0 + (\Lambda + \mu)^2}} + 2 \left( \mu^2 - 6g^4 \Delta^2_0 \right) \log \left( \frac{\mu + \sqrt{4g^4 \Delta^2_0 + \mu^2}}{\Lambda + \mu + \sqrt{4g^4 \Delta^2_0 + (\Lambda + \mu)^2}} \right) + (\mu \to -\mu) \right)
+ \frac{g^4}{2\pi^2} \left( \frac{12g^4 \Delta^2_0 \Lambda + \Lambda^3}{\sqrt{4g^4 \Delta^2_0 + \Lambda^2}} - 12g^4 \Delta^2_0 \text{csch}^{-1} \left( \frac{2g^2 \Delta_0}{\Lambda} \right) \right),
\]

where the last line in \((61)\) or \((62)\) corresponds to the scalar contribution. Taking into account the cut-off dependence of the gap at zero temperature, \((30)\), we find the following expressions for the \( f_1 \) and \( m^{-2} \) coefficients at leading order in \( \Lambda \):

\[
f_{1 \text{ rel.BCS}} = \frac{35\pi^8}{8g^6\Lambda^{-10}}, \quad f_{1 \text{ SUSY BCS}} = \frac{g^4}{8\pi^2} (1 + 4c^2)^{-3/2} + \mathcal{O}(\Lambda^{-2}),
\]

\[
m_{-2 \text{ rel.BCS}} = g^2 + \mathcal{O}(\Lambda^{-4}), \quad m_{-2 \text{ SUSY BCS}} = g^2 + \frac{g^4 \mu^2}{2\pi^2} \left( \frac{32c^4 + 16c^2 + 5}{(1 + 4c^2)^{5/2}} - 2 \log \frac{1 + \sqrt{1 + 4c^2}}{2c} \right) + \mathcal{O}(\Lambda^{-2}),
\]

where \( c = \exp \left[ -\frac{x^2}{g^2 \mu^2} - \frac{3}{2} \right] \). We must stress that the coefficient \( f_1 \) in the relativistic BCS theory does not vanish at zero temperature, because \( \Lambda \) is a physical cut-off acting like a “Debye energy”, which takes a finite value.
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