A Deterministic Convergence Framework for Exact Non-Convex Phase Retrieval

Bariscan Yonel*, †, and Birsen Yazici*, ‡, Senior Member, IEEE

Abstract—In this work, we analyze the non-convex framework of Wirtinger Flow (WF) [1] for phase retrieval and identify a novel sufficient condition for universal exact recovery through the lens of low rank matrix recovery theory. Via a perspective in the lifted domain, we establish that the convergence of WF to a true solution is geometrically implied under a condition on the lifted forward model which relates to the concentration of the spectral matrix around its expectation given that the bound is sufficiently tight. As a result, a deterministic relationship between accuracy of spectral initialization and the validity of the regularity condition is derived, and a convergence rate that solely depends on the concentration bound is obtained. Notably, the developed framework addresses a theoretical gap in non-convex optimization literature on solving quadratic systems of equations with the convergence arguments that are deterministic. Finally, we quantify a lower bound on the signal-to-noise ratio such that theoretical guarantees are valid using the spectral initialization even in the absence of pre-processing or sample truncation.

Index Terms—Wirtinger Flow, non-convex optimization, spectral method, low rank matrix recovery, phase retrieval

I. INTRODUCTION

A. Phase Retrieval

Generalized phase retrieval (GPR) is a ubiquitous problem in science and engineering. The problem consists of the recovery of an object of interest \( x \in \mathbb{C}^N \) given intensity only measurements of the form:

\[
y_m = |(a_m, x)|^2, \quad m = 1, 2, \ldots, M,
\]

where \( a_m \in \mathbb{C}^N \) denotes the \( m \)-th sampling vector. In literature, the collection \( \{a_m\}_{m=1}^M \) most prominently corresponds to models such as Gaussian sampling [1], coded diffraction patterns [2], or the rows of a known linear transformation, such as the short time Fourier transform [3], or a particular imaging operator. These models arise in problems such as X-Ray crystallography, coded diffraction imaging, optical astronomy, quantum state tomography, wave-based imaging or blind channel estimation.

GPR is commonly addressed by the following perturbed problem of the quadratic equality constraints given in (1):

\[
\text{minimize } f(z) := \frac{1}{2M} \sum_{m=1}^{M} (y_m - |(a_m, z)|^2)^2.
\]

The objective function \( f \) is non-holomorphic, and non-convex due to its invariance to global phase factors on the complex valued variable \( z \). Conventional methods from optical imaging reformulate (2) as a bilinear inverse problem by inserting the missing phase component as a variable, which is then solved by alternating minimization [4], [5], or non-convex analogs of feasibility problems [6]. However, these methods are not equipped with practical recovery guarantees, and carry the risk of getting stuck in local minima due to the non-convexity of the problem.

Despite the ill-posed nature of the problem, there has been a significant progress in the development of provably good GPR algorithms in the last decade. Such methods are characterized by either one or both of the following two principles: convexification of the equality constraints and the solution set, which include lifting based approaches [2], [7], [8], or a provably accurate initialization, followed by an algorithmic map that refines the initial estimate on the original signal domain [1], [9], [10]. Notably, lifting-based approaches reformulate inversion from the quadratic equations of the form (1) into a convex semi-definite program while squaring the dimension of the inverse problem. As a result, these solvers have demanding implementation costs due to computational complexity and memory requirements, which limit their applicability for large scale sensing problems. Essentially, methods that operate on the original signal domain evade such practical bottlenecks arising from the increased dimensionality of the inverse problem.

B. Wirtinger Flow

The latter two-step approach for exact phase retrieval on the original signal domain was most prominently popularized by the seminal Wirtinger Flow (WF) framework [1]. In contrast to other state-of-the-art exact phase retrieval methods, i.e., lifting or linear programming based approaches [10]–[12], WF solves the original non-convex problem in (2) directly.

Given an initial estimate \( z_0 \), WF performs steepest descent iterations, by means of Wirtinger derivatives of \( J \) as follows:

\[
z_{k+1} = z_k - \frac{\mu_{k+1}}{\|z_0\|^2} \nabla f(z_k),
\]

where \( \nabla f \) is defined as the complex gradient operator, and \( \mu_{k+1} \) is the step size. The premise of WF is that if \( z_0 \) is sufficiently accurate, the iterates formed by (3) provably converge to an element in the global solution set at a geometric rate.

**Definition 1.1. Global Solution Set.** We say that the points

\[
P := \{e^{j\phi}x : \phi \in [0, 2\pi]\},
\]

arXiv:2001.02855v1 [cs.IT] 9 Jan 2020
form the global solution set of \( \mathbb{C}^N \).

In general for any \( \mathbf{z} \in \mathbb{C}^N \), the non-convex set of the form \( \{ e^{j\phi} \mathbf{z} : \Phi \in [0, 2\pi) \} \) represents an equivalence under the mapping of intensity only measurements. The convergence of algorithm iterates is governed by the following distance metric, which avoids the ambiguity caused by global phase factors on the domain of \( f \).

**Definition 1.2.** Let \( \mathbf{x} \in \mathbb{C}^N \) be an element of the solution set \( P \). The distance of an element \( \mathbf{z} \in \mathbb{C}^N \) to \( \mathbf{x} \) is defined as [1]:

\[
\text{dist}(\mathbf{z}, \mathbf{x}) = \arg \min_{\phi \in [0, 2\pi]} \| \mathbf{z} - \mathbf{x} e^{j\phi} \|. \quad (5)
\]

The angle \( \hat{\phi} \) where the minimum is achieved for a given \( \mathbf{z} \in \mathbb{C}^N \) is denoted as \( \Phi(\mathbf{z}) \).

In literal terms, (5) quantifies the distance of an estimate to the closest point in \( P \). Hence, the exact phase retrieval refers to the iterates converging to any of the elements in the global solution set. Having to solve a non-convex problem, exact recovery guarantees of WF framework depend on the accuracy of the initial estimate \( \mathbf{z}_0 \) which is computed by the spectral method [9] as follows:

\[
\mathbf{Y} = \frac{1}{M} \sum_{m=1}^{M} \mathbf{y}_m \mathbf{a}_m \mathbf{a}_m^H. \quad (6)
\]

The leading eigenvector of \( \mathbf{Y} \), denoted as \( \mathbf{v}_0 \), is scaled by the square root of the normalized \( \ell_1 \)-norm of the data, i.e., \( \lambda_0 = M^{-1} \| \mathbf{y} \|_1 \) to yield the initial estimate \( \mathbf{z}_0 = \sqrt{\lambda_0} \mathbf{v}_0 \). Setting \( \text{dist}(\mathbf{z}_0, \mathbf{x}) = \epsilon \), the initial estimate determines an \( \epsilon \)-neighborhood of \( P \).

**Definition 1.3.** \( \epsilon \)-Neighborhood of \( P \). We denote the \( \epsilon \)-neighborhood of the global solution set in \( \mathbb{C}^N \) by \( E(\epsilon) \) and define it as

\[
E(\epsilon) = \{ \mathbf{z} \in \mathbb{C}^N : \text{dist}(\mathbf{z}, P) \leq \epsilon \}. \quad (7)
\]

Main result of WF framework is that for Gaussian sampling and coded diffraction patterns, the initial estimate computed by the spectral method yields a small enough \( \epsilon \), such that the following regularity condition holds with high probability for \( M = \mathcal{O}(N \log N) \).

**Condition 1.1.** Regularity Condition. The objective function \( f \) in (2) satisfies the regularity condition if, for all \( \mathbf{z} \in E(\epsilon) \) the following holds

\[
\text{Re} \left( \left( \nabla f(\mathbf{z}), (\mathbf{z} - \mathbf{x} e^{j\phi(\mathbf{z})}) \right) \right) \geq \frac{1}{\alpha} \text{dist}^2(\mathbf{z}, \mathbf{x}) + \frac{1}{\beta} \| \nabla f(\mathbf{z}) \|^2
\]

for fixed \( \alpha > 0 \) and \( \beta > 0 \) such that \( \alpha \beta > 4 \). \( (8) \)

Lemma 7.10 in [1] establishes that if the regularity condition is satisfied, the WF iterations are contractions with respect to the distance metric in (5) and all the algorithm iterates remain in \( E(\epsilon) \). Essentially the validity of the expression in (8) ensures that there exists no first order optimal point \( \mathbf{z} \in E(\epsilon) \) other than elements of \( P \).

**C. Related Work and Our Contributions**

WF inspired several variants [13]–[16], which focus on increasing the robustness of the spectral method and the gradient estimates by sample truncation. Another class of variants “reshape” the objective function, by setting \( \Phi(\mathbf{z}) \) as the measurement mismatch in magnitude instead of the squared magnitude [16], [17]. Both reportedly result in superior sample complexity of \( \mathcal{O}(N) \), and faster convergence. Spectral initialization schemes were subject to further studies such as those involving the design of optimal pre-processing functions [18]–[22], generalizations [23], or alternative formulations including orthogonality promoting methods [16].

Related to the theory of WF as a non-convex optimization framework, it was observed in [13] and [24], that the regularity condition can be enforced by the restricted strong convexity condition due to the local Lipschitz differentiability of the objective function. The restricted strong convexity of the objective function around the solution manifold was further studied in [25] in which, given \( \mathcal{O}(N \log(N))^2 \) measurements for the Gaussian sampling model, spectral initialization was proven to fall within the strongly convex region with high probability. In [3], the exact recovery framework for phase retrieval via non-convex optimization was extended to the deterministic model of short-time Fourier transform (STFT) samples, leveraging the redundancy resulting from overlapping STFT processing windows. In [23], the authors extended the exact recovery framework of WF to interferometric inversion with a new sufficient condition defined on arbitrary lifted forward models, characterized as a restricted isometry property (RIP) over rank-1, positive semi-definite matrices. Notably, the convergence theory in [23] is based on purely deterministic arguments. In [26], a non-convex approach based on a local RIP over rank-2 matrices is considered for solving the blind deconvolution problem. Non-convex optimization via first order methods offer exact recovery guarantees from quadratic or bilinear equations of rank-\( r \) matrices for the Gaussian measurement map [27], or if the measurement map satisfies RIP over rank-6\( r \) with a RIC less than or equal to 1/10 [28].

For further discussion on advances in non-convex low rank matrix recovery (LRMR), we refer the reader to [29], [30].

The aforementioned non-convex approaches offer exact recovery guarantees for phase retrieval based on a wide range of theoretical arguments which are predominantly probabilistic in nature, derived through the properties of statistical models assumed for the underlying measurement maps. Our contribution in this paper is the unification of these various probabilistic arguments under a single condition, from which key properties utilized in the literature such as the local RIP in [26], or the restricted strong convexity of the objective function are implied geometrically in a deterministic manner. Specifically, we show that one arrives at the restricted strong convexity property of the objective function around a global solution directly through a concentration bound of the spectral matrix due to the special structure of the set of rank-1, positive semi-definite (PSD) matrices, by interpreting WF in the lifted domain. As a result, our framework establishes that the two steps of the non-convex optimization framework blueprinted by the
semeial work of [1], i.e., the accuracy of spectral initialization, and the regularity sufficient condition, are geometric outcomes of a less restrictive sufficient condition on the following concentration bound.

\[ \|Y - (xx^H + \|x\|^2 I)\| \leq \delta \|x\|^2, \quad \text{for any } x \in \mathbb{C}^N. \quad (9) \]

(9) is by no means a novel condition, and is known to hold true with high probability for Gaussian sampling and coded diffraction patterns when \( M = O(N \log N) \) through the concentration of the Hessian of \( f \) around its expectation when evaluated at a global solution. Typically, it is used within the probabilistic analysis conducted for statistical models in relating the distance of the spectral initialization to the ground truth. In our work, we show that this is a much stronger condition than it is given credit for. Namely, we prove that if the concentration bound in (9) is sufficiently tight with \( \delta \leq 0.184 \), then the Condition (1) is redundant for the exact recovery guarantees of WF starting from the spectral initialization. In other words, there surely exists positive \( \alpha, \beta \) with \( \alpha \beta > 4 \) such that (9) is satisfied deterministically via the restricted strong convexity of the objective function in (2). Notably, the upper bound we identify on \( \delta \) supersedes the values commonly tested in literature for (9). Thereby, we establish an exact non-convex phase retrieval framework that is consistent with the original work of [1], and its various variants [3], [14]–[17], [31]–[33]. Furthermore, our result improves upon the required sample complexity identified by [25] to achieve restricted strong convexity for (2) in a neighborhood around the solution manifold for the rank-1, PSD matrix recovery problem.

D. Notation and Organization of the Paper

The rest of the paper is organized as follows. In Section II we provide a preliminary discussion on the interpretation of WF in the lifted domain. Section III contains our main results, and remarks. Section IV evaluates the robustness of WF in the presence of additive noise. In Section V we present the proofs of our results. Section VI concludes the paper.

We denote the elements of finite dimensional vector spaces with lower case bold letters. Corresponding lifted elements are denoted in bold with a tilde sign. Upper case bold and italic letters are allocated for operators that act on the vector space of \( \mathbb{C}^N \). Caligraphic letters are allocated for matrices and sets, respectively. Corresponding lifted elements of our results. Section VI concludes the paper.

II. WF IN THE LIFTED DOMAIN

We start by interpreting WF as a solver in the lifted domain, and adopt the concepts of the seminal work of PhaseLift in [2]. Lifting based approaches provide a profound perspective to the phase retrieval problem. In principle, these methods target the core issue of non-injectivity of phaseless measurement maps, which is a key step in formulating methods that guarantee exact recovery in phase retrieval literature [10], [34]. Notably, one can consider the measurement model in (1) as a mapping from a rank-1, positive semi-definite matrix \( xx^H \in \mathbb{C}^{N \times N} \) instead of a quadratic map from the signal domain in \( \mathbb{C}^N \). Lifting conceptualizes this observation:

**Definition II.1. Lifting.** Each measurement in (1) can be expressed in the form of an inner product of two rank-1 operators, \( \hat{x} = xx^H \) and \( A_m = a_m a_m^H \) such that

\[ y_m = \langle A_m, \hat{x} \rangle_F = m = 1, ..., M \quad (10) \]

where \( \langle \cdot, \cdot \rangle_F \) is the Frobenius (or in infinite dimensions the Hilbert-Schmidt) inner product. The process of transforming the signal recovery over \( \mathbb{C}^N \) to the recovery of the rank-1 unknown \( \hat{x} \in \mathbb{C}^{N \times N} \) is known as lifting.

Lifting technique introduces a new, linear measurement map \( A : \mathbb{C}^{N \times N} \rightarrow \mathbb{C}^M \), which we refer to as the lifted forward model. Specifically, for the phaseless measurement model in (1), the domain of \( A \) is constrained on the set of rank-1, positive semi-definite (PSD) matrices \( X = \{ zz^H : z \in \mathbb{C}^N \} \), with its range is in \( \mathbb{R}^M \) as follows:

\[ y = A(xx^H) \quad (11) \]

where \( y = [y_1, y_2, \ldots, y_M] \in \mathbb{R}^M \). As a result, each non-convex set of equivalent points under the mapping from the signal domain in \( \mathbb{C}^N \) to the phaseless measurements, i.e., \( \{ ze^{i\Phi} : \Phi \in [0, 2\pi) \} \) for \( z \in \mathbb{C}^N \), is compressed into a single element in the set \( X, zz^H \). Thereby, quadratic equality constraints over the signal domain are transformed to affine equality constraints in the lifted domain in \( \mathbb{C}^{N \times N} \), which define a convex manifold.

In typical inference problems, \( A \) has a non-trivial null space as the system of linear equations in (10) is severely underdetermined due to \( M \ll N^2 \). Various studies approach the phase retrieval problem over the lifted domain, leveraging the low rank structure of the unknown \( \hat{x} = xx^H \) and the subsequent LRMR theory from compressed sensing and matrix completion literature [35], [36]. The sufficient conditions on \( A \) for exact recovery of \( \hat{x} \) are primarily characterized by its null space [37]–[39] or restricted isometry properties on low rank [36], [40], [41] or PSD [2] matrices.

Knowing that (2) corresponds to the objective of the perturbed problem, WF exclusively iterates on the set of rank-1, PSD matrices by solving the following:

\[ \minimize_{x} \quad \frac{1}{2M} \|A(X) - y\|^2 \quad \text{s.t.} \quad X = zz^H, \quad (12) \]

where \( X \) denotes the optimization variable in the lifted domain. The functional constraint on \( X \) as rank-1, PSD matrix casts this minimization equivalent to minimizing over the signal domain variable \( z \), resulting with dimensionality reduction of the search space. This is practically enforced by a spectral projection within the gradient term \( \nabla f \), which can be expressed as

\[ \nabla f(z) = \frac{1}{M} A^H A(\hat{z} - \hat{x})z. \quad (13) \]
Beyond the immediate gains in practicality, the formulation in [12] reveals a theoretical advantage offered by the non-convex framework of WF, using which deterministic arguments for exact recovery as those of lifting-based methods can also be attained. Moving from the convex relaxations of rank-minimization, WF corresponds to solving a perturbed non-convex feasibility problem, reminiscent of the optimizationless PhaseLift method in [42], and Uzawa’s iterations in [35]. This yields an iterative scheme for the unrelaxed, non-convex convex feasibility problem, reminiscent of the optimizationless convex relaxations of rank-moments for exact recovery as those of lifting-based methods can also be attained. Moving from the convex relaxations of rank-

Remark. The spectral matrix Y in (6) is the backprojection estimate of the lifted signal, \( \hat{x} = xx^H \), i.e.,

\[
Y = \frac{1}{M} A^H (y),
\]

which, in the noise free case, corresponds to \( Y = \frac{1}{M} A^H A(\hat{x}) \).

In practical terms, (9) becomes a condition on the lifted Gram operator, \( A^H A \). Thereby, the main result of this paper is that, if the concentration bound

\[
\| \frac{1}{M} A^H A(xx^H) - (xx^H + ||x||^2 I) \| \leq \delta ||x||^2,
\]

holds for any \( x \in \mathbb{C}^N \) with a fixed \( \delta \) that is sufficiently small, then the spectral structure of the set of rank-1, PSD matrices, the iterations in (3) are guaranteed to converge to a solution in \( P \) via the restricted strong convexity of \( \mathcal{F} \) in \( E(\epsilon) \). As \( \delta \) gets smaller, the spectral initialization yields more accurate estimates due to favorable properties of the lifted Gram operator over the set of rank-1, PSD matrices. Due to the fact that shrinkage on the value of \( \delta \) is correlated with increasing the number of measurements \( M \), the existence suboptimal minima accordingly vanishes. Hence, there exists a phase transition below which the tightness of the concentration bound can deterministically guarantee exact recovery from (11) using WF.

III. MAIN RESULTS

In this section, we establish (15) as a sufficient condition for exact phase retrieval under an arbitrary measurement model. Thereby, we use the form in (14) as a crucial element of our approach. Note that the lifted forward model in (11) may be a realization from a statistical model, or a deterministic measurement map. Accordingly, (15) is a condition satisfied with some probability at least \( p \) in general, with \( p = 1 \) corresponding to the deterministic case. The following lemma characterizes the Gram operator of the lifted forward model over the set of rank-1, positive semi-definite matrices.

Lemma III.1. Assume that (15) holds with probability at least \( p \) for any \( x \in \mathbb{C}^N \). Then, the Gram operator of the lifted forward model can be expressed as follows over the set of rank-1, PSD matrices:

\[
\frac{1}{M} A^H A = I + \mathcal{R} + \Delta,
\]

where for \( \hat{x} \in \mathcal{X} \), \( \mathcal{R}(\hat{x}) = ||x||^2 I \) which satisfies

\[
\mathcal{R}(\hat{x} - \hat{x}) = (||x||^2 - ||x||^2) I,
\]

and \( \Delta : \mathcal{X} \rightarrow \mathbb{C}^{N \times N} \) is a perturbation operator satisfying

\[
\max_{v \in \mathbb{C}^N \setminus \{0\}} \| \mathcal{R}(vv^H) \| \leq \delta.
\]

Proof. See Section V-A

Specifically in the case of the Gaussian model, the operator \( \mathcal{R} : \mathbb{C}^{N \times N} \rightarrow \mathbb{C}^{N \times N} \) characterizes the effect of the 4th moments of the sampling vectors. This term captures the diagonal bias of the spectral matrix in estimating the lifted signal, using the fact that the expectation of the Gram operator is linear on \( \mathbb{C}^{N \times N} \).

We begin by considering the spectral initialization scheme. Namely, through Lemma III.1, the concentration bound in (15) indicates a proper scaling factor for the unit-norm eigenvector of \( Y \). This is derived in the following corollary which implies that the lifted forward model \( A \) is a tight frame.

Corollary III.1. Assume that the assumptions of Lemma III.1 hold. Then, \( A \) satisfies the following identity for any \( x \in \mathbb{C}^N \) with probability \( p \):

\[
(2 - \delta)||x||^2 \leq \frac{1}{M} \|y\|^2 \leq (2 + \delta)||x||^2,
\]

where \( y = A(\hat{x}) \in \mathbb{R}^M \) are the phaseless measurements in (11).

Proof. See Section V-A

As a result, \( \lambda_0 = (2M)^{-1/2} ||y|| \) is an estimator for the energy of the signal \( ||x||^2 \). Using the norm estimate obtained by Corollary III.1 for \( \lambda_0 \), the distance of the spectral initialization yields the following \( \epsilon \)-neighborhood as a function of the concentration parameter \( \delta \).

Lemma III.2. Assume that (15) holds with at least probability \( p \) for any \( x \in \mathbb{C}^N \). Let \( z_0 = \sqrt{\lambda_0} v_0 \), where \( v_0 \) is the eigenvector corresponding to the leading eigenvalue of the spectral matrix \( Y \) in (14), and \( \lambda_0 \) is the signal energy estimate obtained as

\[
\lambda_0 = \frac{1}{\sqrt{2M}} ||y||.
\]
Then the initial estimate $z_0$ satisfies $\text{dist}(z_0, x) \leq c^2 \|x\|^2$, where
\begin{equation}
  c^2 \leq 1 + \sqrt{1 + \frac{\delta}{2}} - 2\sqrt{(1 - 2\delta) \left(1 + \frac{\delta}{2}\right)^{1/2}}.
\end{equation}

**Proof.** See Section V-B.

Observe that a valid estimate via spectral initialization requires $\delta < 0.5$ by default. Next, we introduce the following lemma to characterize the relation between the distance metric introduced in (5), and the distance in the lifted domain.

**Lemma III.3.** Assume that (15) holds with at least probability $p$ for any $x \in \mathbb{C}^N$. Let $x$ be the ground truth signal. Then, for any $z \in E(\epsilon)$, we have
\begin{equation}
  h_1(\delta)\text{dist}(z, x)\|x\| \leq \|\hat{x} - x\|_F \leq h_2(\delta)\text{dist}(z, x)\|x\|,
\end{equation}
where $h_1 = \sqrt{(1 - \epsilon)(2 - \epsilon)}$, $h_2 = (2 + \epsilon)$, with $\epsilon$ satisfying (21).

**Proof.** See Section V-C.

Result of Lemma III.3 establishes that the distance between the lifted signals $\hat{x}$, $x$ is of the rate of the distance of the signals in $\mathbb{C}^N$, when $z \in E(\epsilon)$. Essentially, the distance metric in (5) locally tracks the error on the constraint set of rank-1, PSD matrices in the lifted domain. The outcome of this result, and Lemma III.1 is the following local restricted isometry-type property.

**Lemma III.4.** Under the assumptions of Lemmas III.3 and III.4 for any $z \in E(\epsilon)$, the lifted forward model satisfies
\begin{equation}
  (1 - \delta)\|\hat{x} - x\|^2_p \leq \frac{1}{M}\|A(\hat{x} - x)\|_2^2 \leq (2 + \delta)\|\hat{x} - x\|^2_p,
\end{equation}
where $\delta = \frac{\sqrt{2(2+\epsilon)}}{\sqrt{(1-\epsilon)(2-\epsilon)}} \cdot \delta$.

**Proof.** See Section V-D.

**Remark.** We refer to $\delta$ as the local restricted isometry constant (RIC) of the lifted forward model over rank-2 matrices. However, note that we abuse the definition of the restricted isometry property here, as the upper bound in (23) is up to a scale $(2 + \delta)$ instead of $(1 + \delta)$. This is a simple consequence of the diagonal bias term of $\|x\|^2 I$ in the spectral matrix, resulting from $R$ in Lemma III.1.

The key impact of Lemma III.4 is the fact that (23) is derived as a deterministic consequence of the concentration bound of the spectral matrix. The Lemmas III.3 and III.4 culminate to yield the Lipschitz differentiability of the objective function.

**Lemma III.5.** In the setup of Lemmas III.3 and III.4, for any $z \in E(\epsilon)$, the objective function $f$ in (2) is local Lipschitz differentiable at $x \in P$ with
\begin{equation}
  \|\nabla f(z)\| \leq c(\delta) \cdot \text{dist}(z, x)\|x\|^2
\end{equation}
where $c(\delta) = (1 + \epsilon)(2 + \epsilon)(2 + \delta)$ is the local Lipschitz constant.

**Proof.** See Section V-E.

Invoking the result of Lemma III.5 to establish the regularity condition for $f$, it is sufficient to show that for any $z \in E(\epsilon)$
\begin{equation}
  \text{Re} \left(\langle \nabla f(z) \rangle \right) \geq \left(\frac{1}{\alpha} + \frac{c^2(\delta)\|x\|^4}{\beta}\right) \|A(z, x)\|^2
\end{equation}
which is equivalent to the restricted strong convexity of the objective function in $E(\epsilon)$. By the definition of strong convexity around the closest solution $\hat{x}$ to an estimate $x \in E(\epsilon)$, this condition, is implied if the objective function satisfies
\begin{equation}
  f (z) \geq f (\hat{x}) + \text{Re} \left(\langle \nabla f (\hat{x}) \rangle \right) + \frac{L}{2}\|z - \hat{x}\|^2,
\end{equation}
where $L$ equals to the multiplier of the distance term in (25). Having $f (\hat{x})$ and $\nabla f (\hat{x})$ equal 0 by definition, the restricted strong convexity in $E(\epsilon)$ is simply reduced to the following condition:
\begin{equation}
  f (z) \geq \frac{1}{\alpha} \left(\frac{1}{2} + \frac{c^2(\delta)\|x\|^4}{\beta}\right) \|A(z, x)\|^2.
\end{equation}

Thus, the regularity condition is satisfied by setting $\alpha$ and $\beta$ such that $\alpha \beta > 4$, and
\begin{equation}
  \frac{1}{\alpha \|x\|^2} + \frac{c^2(\delta)\|x\|^2}{\beta} \leq (1 - \delta)h_1^2(\delta) := h(\delta).
\end{equation}

The final form we derive in (29) results in a number of notable outcomes regarding the non-convex optimization theory of the WF framework:

1) We show that there exists a regime in which the regularity condition holds by default. This regime is characterized by the concentration bound of the spectral matrix in (15), as $\delta$ is solely a function of $\epsilon$. Observe that the validity of this regime depends on attaining $\delta < 1$ which constrains the tightness of the concentration property in (15). This numerically yields an upper bound of $\delta \leq 0.184$ as shown in Figure 1.

2) Theorem 7.10 provides an interpretation of the algorithm parameters consistent with the original work of [1]: Figure 2 demonstrates the range of values the constants $c$ and $h$ can get in the valid region of $\delta$. Notably, the values of these $O(1)$ constants characterize the convergence rate of the algorithm, as $\alpha$ and $\beta$ are required to be sufficiently large constants for (29) to hold. Observe that (29) implies setting $\alpha = O(1/\|x\|^2)$, and $\beta = O(\|x\|^2)$, hence $\alpha \beta = O(1)$. Since we clearly have $h < 2$, and $c > 4$, the condition in (29) holds with $\alpha \beta > 4$ by definition. Hence, the regularity condition is satisfied, and a step size $\mu^s \leq 2/\beta$ can be chosen to yield a convergence rate of $2\mu^s/\alpha$ via [1]. This step size $\mu^s$ is then $O(1/\|x\|^2)$. Hence, the definition of the updates in (3) requires an approximate normalization term of $\|z_0\|^2$ on a scalar entity $\mu_{k+1} = O(1)$.
achievable geometric convergence rate as a function of the geometric convergence of the algorithm via \cite{1}, we obtain the best achievable rate of convergence of the algorithm under which exact recovery of any \(x \in \mathbb{C}^N\) is guaranteed. Simply fixing \(\kappa = \alpha\beta\) and re-organizing \eqref{29}, we obtain an equation of the form

\[
\frac{1}{|x|^2} \left( 1 + \frac{c^2(\delta)}{\kappa} \|x\|^4 - h(\delta) \|x\|^2 \alpha \right) \leq 0. \tag{30}
\]

Since we have \(\alpha > 0\) by definition and \(h(\delta) > 0\) by constraining \(\delta\), it suffices to consider the non-negativity of the discriminant of the quadratic equation with respect to \(\alpha\) in \eqref{30} for the overall condition to hold true, which yields

\[
h^2(\delta) \|x\|^4 - \frac{4}{\kappa} c^2(\delta) \|x\|^4 \geq 0. \tag{31}
\]

As a result, knowing that \(4/\kappa\) is an upper bound on the rate of convergence of the algorithm via \cite{1}, we obtain the best achievable geometric convergence rate as a function of the concentration bound parameter as

\[
\frac{4}{\alpha \beta} \leq \frac{h^2(\delta)}{c^2(\delta)} := r(\delta) = \left( \frac{(1 - \hat{\delta})(1 - \epsilon)(2 - \epsilon)}{(2 + \hat{\delta})(1 + \epsilon)(2 + \epsilon)} \right)^2. \tag{32}
\]

Thereby, beyond the \(\delta < 0.184\) bound derived for the existence of \((\alpha, \beta)\) to satisfy the regularity condition, the evolution of \(r(\delta)\) characterizes the practicality of the algorithm with respect to the concentration bound parameter, which is provided in Figure \ref{3}.

Organizing the arguments developed in this section, we state the following for exact phase retrieval via Wirtinger Flow:

**Theorem III.1.** Assume that given the sample complexity \(M = O(h(N))\), we have

\[
\| \frac{1}{M} A^H A (x x^H) - (x x^H + \|x\|^2 I) \| \leq \delta \|x\|^2.
\]

with some probability \(p(\delta)\), for any \(x \in \mathbb{C}^N\). Then the initial estimate \(z_0\) obtained from the spectral matrix in \eqref{6} using the scaling factor in \eqref{20} satisfies

\[
\text{dist}^2(z_k, x) \leq \epsilon^2 \|x\|^2,
\]

where \(\epsilon^2 \leq 1 + \sqrt{1 + \frac{\delta}{\kappa} - 2 \sqrt{(1 - 2\delta)(1 + \frac{\delta}{2})^{1/2}}}\), and for the iterates generated by \eqref{3} with a fixed step size \(\frac{\mu}{\|x\|^2} = \mu \leq \frac{2}{\beta}\), we have

\[
\text{dist}^2(z_k, x) \leq \epsilon^2 (1 - \frac{2\mu}{\alpha})^k \|x\|^2,
\]

if \(\delta \leq 0.184\), with an achievable rate of convergence of

\[
\frac{2\mu}{\alpha} \leq \left( \frac{(1 - \hat{\delta})(1 - \epsilon)(2 - \epsilon)}{(2 + \hat{\delta})(1 + \epsilon)(2 + \epsilon)} \right)^2,
\]

where \(\hat{\delta} = \frac{\sqrt{2(1 + \epsilon)}}{\sqrt{(1 - \epsilon)(2 - \epsilon)}}\).

In establishing Theorem III.1 for exact phase retrieval it is necessary to use the specific structure of the diagonal bias term in the expectation of the spectral matrix. This is in contrast to our work in \cite{23}, in which the spectral matrix is an unbiased
estimator of the lifted signal, and there exists a direct relation of the concentration bound of the Hessian and a restricted isometry property of the lifted forward model. Nonetheless, Corollary III.1 highlights a key advantage of the non-convex framework of WF. Via the removal of convex relaxations and solving the perturbed problem in the lifted domain over the set of rank-1, PSD matrices, the RIP-type properties required by semi-definite programming and lifting-based approaches are relaxed to smaller, more specific domain of matrices. Under the lens of LRMR theory, WF not only offers computational advantages, but also less stringent theoretical means to achieve exact recovery guarantees. It can also be observed that the advantages, but also less stringent theoretical means to achieve the lens of LRMR theory, WF not only offers computational relaxed to smaller, more specific domain of matrices. Under semi-definite programming and lifting-based approaches are solved the perturbed problem in the lifted domain over the set of rank-1, PSD matrices, the RIP-type properties required by the spectral matrix in the phase retrieval problem requires \( \text{RIP} \) and shown in Figure 4.

The concentration bound that we study is known to hold for \( M = O(N \log N) \) for statistical models such as Gaussian sampling and coded diffraction patterns. Via the established deterministic convergence framework given the concentration bound, our result proves that the restricted strong convexity property of the objective function requires \( O(N \log N) \) samples. This is a log \( N \) factor less than the sample complexity reported in [25].

Our sufficient condition has to hold only over the rank-1, PSD matrices, which is less stringent than those studied in LRMR literature [27], [28]. Additionally, a universal upper bound on \( \epsilon \) via Figure 1 is attained for the exact recovery regime to hold. Hence, the concept of sufficient accuracy of the spectral initialization is captured by a quantitative measure.

Notably, the upper bound on the concentration property of the spectral matrix in the phase retrieval problem requires a smaller constant than the one in the interferometric inversion problem we studied in [23] (0.184 as opposed to 0.214), in which the relative phase information of a pair of measurements is retained. This is indeed an intuitive outcome, as more information is lost when measurements are phaseless, compared to the interferometric case. A similar outcome is observed in the upper bound obtained for the geometric convergence rate of the algorithm, which approaches to 0 for the case of interferometric inversion as \( \delta \to 0 \). As a result, the impact of the additional loss of phase information is directly captured in the sufficient conditions and the performance of the algorithm in solving the different types of quadratic systems of equations.

### IV. Robustness

In this section, we assess the robustness of the WF algorithm in the presence of additive noise in the measurements. We show that for the general problem setting of

\[
y = \mathcal{A}(\tilde{x}) + \eta,\tag{33}
\]

the results presented in Theorem III.1 for the noise-free case in Section III are attained up to a bounded perturbation for \( \mathbb{E}[\eta] = 0 \). It should be noted that, the \( \ell_2 \) mismatch function minimized in the problem formulation fits the data to i.i.d. additive white Gaussian noise model for \( \{\eta_m\}_{m=1}^M \) in the maximum likelihood sense. Despite this, the results presented in this section have no specification of the distribution of the noise term \( \eta \), similar to those of [13], which were derived for the Poisson loss function.

Our main focus is establishing the validity of the spectral initialization for our exact recovery guarantees with respect to the SNR of measurements in (33) by utilizing the arguments developed over the lifted domain. Namely, for our convergence theory to hold, we have derived numerical constraints on both the concentration bound (i.e. \( \delta \)), and the distance of the initial estimate obtained from the spectral method (i.e. \( \epsilon \)). These constraints characterize the amount of perturbation the algorithm can tolerate, which is stated in the following lemma and shown in Figure 4.

**Lemma IV.1.** Consider the spectral matrix formed by (6), using the noisy measurements in (33). Moreover, let the concentration bound in (15) hold with some probability \( p \) as stated in the setup of Theorem III.1. Then, the spectral matrix \( \mathbf{Y} \) satisfies,

\[
\mathbb{E}_\eta [\| \mathbf{Y} - (\mathbf{x}\mathbf{x}^H + \|\mathbf{x}\|^2\mathbf{I}) \|] \leq \left( \delta + \frac{(2 + \delta)}{\sqrt{\text{SNR}}} \|\mathbf{x}\|^2 \right), \tag{34}
\]

where SNR stands for signal to noise ratio, defined as \( \text{SNR} = \|\mathcal{A}(\tilde{x})\|^2 / \mathbb{E}[\|\eta\|^2] \).

**Proof.** See Section VI-F.

Analogous to the one-to-one relationship of the \( \epsilon \)-distance of the spectral initialization and the concentration bound \( \delta \) in the noise-free phase retrieval problem, the noisy scenario has the additional dependence on the SNR of the measurements through \( \delta := \delta + (2 + \delta)/\sqrt{\text{SNR}} \). With the presence of the SNR term, there exists a level of noise as a function of \( \delta \).
beyond which the concentration bound in (35) is insufficient to guarantee an effective spectral initialization. This restriction is directly determined by two constraints: \( \epsilon < 1 \) and \( \delta < 1 \), in order to retain a valid regime where convergence arguments from Theorem II.1 hold true for the noise-free component of the gradients. Thereby, we obtain a region over the \((\delta, \text{SNR})\) domain such that the spectral method produces a valid estimator under \( \delta \).

A direct manner to determine this region is by enforcing \( \hat{\delta} \leq 0.184 \), through which accuracy of spectral initialization and subsequent arguments within the \( E(\epsilon) \) are preserved, yielding an SNR lower bound of

\[
\text{SNR(dB)} \geq 20 \log \frac{2 + \delta}{0.184 - \delta}.
\]  

(35)

Figure 4 depicts that the lowest SNR value of 20.7dB is attained at \( \delta = 0 \). Although it is derived in a straight-forward manner, (35) is the best lower bound that can be characterized by our framework. This is rather surprising at first glance, since the \( \delta \) term is only affected by the perturbation resulting from noise through the \( \epsilon \) parameter, as \( \delta \) and \( \epsilon \) are properties of the underlying lifted forward model \( A \), which is independent of noise. However, these still prove to be consequential for the stability of the algorithm under additive noise because of constraints that arise from the convergence arguments, beyond those related to the validity of the spectral initialization.

In particular, under additive noise and the assumptions of Lemma IV.1 within the valid region for the spectral initialization defined by (35), the convergence guarantees of WF are perturbed by a constant factor that is a function of SNR.

**Theorem IV.1.** Assume that the assumptions of Lemma IV.1 hold. Then, for the identical procedure and values of constants \( \alpha, \beta \) in the setup of Theorem III.1 we have

\[ E_{\eta} [\text{dist}(z_k, x)] \lesssim \epsilon(1 - \frac{2\mu}{\alpha})^\frac{1}{2} \|x\| + \alpha' \frac{(2 + \delta)}{\sqrt{\text{SNR}}} \|x\|. \]  

(36)

where \( \mu = \mu_k/\|z_0\|^2 \leq 2/\beta \), \( \alpha' = O(1) \) such that \( \alpha = \alpha'/\|z_0\|^2 \), and \( \epsilon^2 \leq 1 + \sqrt{1 + \frac{1}{2} - 2\sqrt{(1 - 2\hat{\delta})(1 + \frac{1}{2})^{1/2}}} \), with \( \hat{\delta} = \delta + \frac{(2 + \alpha)}{\sqrt{\text{SNR}}} \).

Proof. See Section V-G

As a result of Theorem IV.1 we observe a crucial element for determining the trade-off between the \( \alpha \) and \( \beta \) parameters. The stability guarantees directly incentivize allocating a small value for the parameter \( \alpha \), which is inversely related to the magnitude of the \( \beta \) parameter. Since \( \mu \leq 2/\beta \) by definition, a tighter stability bound requires a trade-off from the step size of the algorithm. This outcome is indeed expected, as lower SNR in the measurements means more variance for the gradient estimates at the iterative refinement stage, hence one should take less confident steps to counter inaccurate update terms. Our framework perfectly captures this phenomenon, and requires small step sizes for improved stability in the algorithm performance while optimizing the noisy landscape of the objective function over the signal domain.

Furthermore, to guarantee that the iterates formed using the noisy measurements remain in the \( E(\epsilon) \), there is an effective upper bound on the \( \alpha \) parameter such that

\[ \alpha' \leq \frac{\epsilon \sqrt{\text{SNR}}}{(1 + \epsilon)(2 + \delta)}. \]  

(37)

Equivalently, this is a lower bound on the value \( (1/\alpha \|x\|^2) \) component of (29) which requires the following

\[ \frac{(1 + \epsilon)(2 + \delta)}{\epsilon \sqrt{\text{SNR}}} \leq \frac{1}{\alpha \|x\|^2} < (1 - \hat{\delta})(1 - \epsilon)(2 - \epsilon), \]  

(38)

for a finite \( \beta \) to exist to attain a practical step-size for the algorithm. Thereby, on expectation, iterative updates that are contractions with respect to the distance metric can be achieved, if SNR is sufficiently high to satisfy

\[ \frac{(1 + \epsilon)(2 + \delta)}{\epsilon \sqrt{\text{SNR}}} < (1 - \hat{\delta})(1 - \epsilon)(2 - \epsilon), \]  

(39)

where both \( \epsilon \), and \( \hat{\delta} \) have SNR dependence. Since the left and right-hand-sides of the inequality monotonically decrease and increase, respectively, with increasing SNR, there exists a transition point at any fixed \( \delta < 0.184 \) value beyond which the inequality holds true. This numerical characterization of the SNR requirements for the convergence of WF precisely corresponds to the lower bound in (35).

As depicted in Figure 3 alternative techniques that improve the accuracy of the spectral estimation must be pursued in order to sustain the convergence guarantees given our sufficient condition below the provided SNR lower bound. We call this region on the \((\delta, \text{SNR})\) domain as the truncation region, referring to the techniques developed in the phase retrieval literature that deploy processing schemes to improve the accuracy of spectral estimators [19], [22].

V. PROOFS

In this section we present the proofs of our arguments used in establishing Theorem III.1 and its corollaries. Notably, our results are derived in a deterministic framework based on geometric arguments unlike the probabilistic theory of original WF theory [1]. The probabilistic nature of the convergence analysis prominent in phase retrieval literature is thereby compressed into a single condition on the lifted forward model.

A. PROOFS OF LEMMA II.1 AND COROLLARY II.7

1) **Lemma II.7.** Reprising (15), from the definition of the lifted forward model \( A \), and spectral matrix \( Y \) we have

\[ \| \frac{1}{M} A^H A x - (x + \|x\|^2 I) \| \leq \delta \|x\|^2. \]  

(40)

Over the set of rank-1 matrices matrices, i.e., \( R_1 = \{uv^H : u, v \in \mathbb{C}^N\} \), we define the operator \( R : R_1 \to \text{span}(I) \subset \mathbb{C}^{N \times N} \), such that \( R(uv^H) = (uv^H)I \). Then, we define \( A^H A - R(\hat{x}) = \Delta(\hat{x}) \), and by (40) we have that \( \| \Delta(\hat{x}) \| \leq \delta \|x\|^2 \).

Next, we represent the rank-1, PSD matrix \( \tilde{z} \) as a linear combination of rank-1 elements in \( R_1 \). Letting \( e = z - x \), we have

\[ \tilde{e} = z - \bar{x} - ex^H - xe^H. \]  

(41)
B. Proof of Lemma III.2
Thus, the condition in Corollary III.1 is implied by (40), and
\[ x \in \mathbb{R}^n \]
As shown in [1] we know that (15) implies \( |\hat{e}| + 2|\hat{e}|^2 \geq (1 - 2\delta)|x|^2 \), where \( |\hat{e}| = 1 \) is the leading eigenvector of the spectral matrix \( Y \). Using Corollary III.1, we know that \( \lambda_0 \) is necessarily linear over \( \mathbb{C} \).

\[ \hat{z} = \hat{e} + \hat{e}^H \hat{x} + \hat{x}^H \hat{e} + \hat{x}^H \hat{x}^H \hat{e}^H \hat{x} \]

Using the definition of the distance metric, and the lower bound from [1], for \( z_0 = \sqrt{\lambda_0} v_0 \), we have
\[ \text{dist}^2(z_0, x) \leq \left( \frac{\lambda_0}{|x|^2} + 1 - 2 \sqrt{\frac{\lambda_0}{|x|^2}} \| x \| \sqrt{1 - 2\delta} \right) \| x \|^2. \]

Since the right-hand-side is a convex quadratic function of \( \sqrt{\lambda_0} \), its maximum value is achieved at the boundary values of \( \sqrt{\lambda_0} \). Setting \( \lambda_0 = (\sqrt{1 - \delta/2})^2 \), the upper bound is monotonically greater than at \( \lambda_0 = (\sqrt{1 - \delta/2})^2 \) for all valid values for \( \delta \), hence we conclude that
\[ \text{dist}^2(z_0, x) \leq \left( \sqrt{1 + \frac{\delta}{2}} + 1 - 2 \sqrt{(1 - 2\delta)(1 + \frac{\delta}{2})} \| x \|^2. \]

C. Proof of Lemma III.3
1) Proof of the Upper Bound: Let \( \hat{x} \) be the closest solution in \( P \) to an arbitrary \( z \in E(e) \). From reverse triangle inequality we have \( (1 - \epsilon)||z|| \leq ||z|| \leq (1 + \epsilon)||z|| \). Setting \( e = z - x \), by (41) we have
\[ \hat{z} - \hat{x} = \hat{e} + \hat{e}^H \hat{x} + \hat{x}^H \hat{e}. \]

Then, for the Frobenius norm of the error in the domain, we have
\[ \| \hat{z} - \hat{x} \|_F \leq \| \hat{e} \|_F + \| \hat{e} \|_F \| \hat{x} \|_F + \| \hat{x} \|_F \hat{e}. \]

Since all the elements on the right-hand-side are rank-1, and \( z \in E(e) \), by definition, we have \( \| \cdot \| = \| \cdot \|_F \), and
\[ \| \hat{z} - \hat{x} \|_F \leq \| e \|_F + 2\| x \| \| e \| \leq (2 + \epsilon)\| e \| \| x \|, \]

which yields the upper bound as \( \| e \| = \text{dist}(z, x) \).
2) Proof of the Lower Bound in Lemma III.3: Expanding \( \| \hat{z} - \hat{x} \|_F \) by the definition of the Frobenius inner product, we have
\[ \| \hat{z} - \hat{x} \|_F^2 = \| \hat{z} \|_F^2 + \| \hat{x} \|_F^2 - 2\text{Re}(\hat{z}, \hat{x})_F. \]

Due to rank-1 property, the Frobenius inner product reduces to \( 2\text{Re}(\hat{z}, \hat{x})_F = 2(\| z, x \|_2^2, \) and \( \| \hat{z} \|_F^2 = \| x \|_4^4, \| \hat{x} \|_F^2 = \| x \|_4^2. \) Hence, \( (55) \) is equal to \( \| x \|_4^4 + \| x \|_4^4 - 2\| z, x \|_2^2, \) and
\[ \| x \|_4^4 - (\| z, x \|_2^2) + (\| x \|_4^4 - (\| z, x \|_2^2) = \| x \|_4^4 + (\| z, x \|_2^2) \| |x|^2 - (\| z, x \|_2^2) |x|^2 + (\| x \|_4^4 - (\| z, x \|_2^2) \| x \|_4^2 - |x, z|). \]

Since \( \text{dist}^2(z, x) = \| x \|^2 + \| x \|^2 - 2\| z, x \|_2^2 = \| x \|^2 \geq 2, \) we can lower bound \( (54) \) using \( (55) \) as
\[ \| \hat{z} - \hat{x} \|_F^2 \geq \min \left( (\| x \|_4^2 + (\| z, x \|_2^2), (\| x \|_4^2 + (\| z, x \|_2^2) \right) \times (\| x \|^2 + |x, z|^2 - 2|z, x|). \]

Knowing that \( \text{dist}^2(z, x) \leq \epsilon^2 \| x \|^2 \) and the result from the reverse triangle inequality on \( |z| \), the terms within the minimization are further lowered on using
\[ 2(\| z, x \|_2^2) \geq \| x \|^2 + \| x \|^2 - \epsilon^2 \| x \|^2 \]
\[ (\| z, x \|_2^2) \geq 2(1 - \epsilon)\| x \|^2. \]
We then get the bound on the scalar multiplying $\text{dist}^2(z, x)$ as
\[
\min \left( (\|z\|^2 + |(z, x)|, (\|x\|^2 + |(z, x)|) \right)
\geq ((1 - \epsilon)^2 + (1 - \epsilon))\|x\|^2,
\]
which yields the lower bound of Lemma [III.3]
\[
\|\tilde{z} - \tilde{x}\|_F \geq \sqrt{(1 - \epsilon)(2 - \epsilon)} \text{ dist}(z, x) \|x\|,
\]
and the proof is complete.

D. Proof of Lemma [III.4]

From Lemma [III.1] we have
\[
\frac{1}{M} \|A(\tilde{z} - \tilde{x})\|_F^2 = \langle A^H(\tilde{z} - \tilde{x}) \rangle F = (\tilde{z} - \tilde{x} + (\|z\|^2 - |x|^2)I + \Delta(\tilde{z} - \tilde{x}), \tilde{z} - \tilde{x})_F.
\]
From the linearity of the inner product, (60) becomes $\|\tilde{z} - \tilde{x}\|^2_F + (\|z\|^2 - |x|^2)^2 (I, \tilde{z} - \tilde{x})_F + (\Delta(\tilde{z} - \tilde{x}), \tilde{z} - \tilde{x})_F$. Since the Frobenius inner product of a matrix with the identity matrix $I$ is simply the sum of its diagonal terms, and the lifted signals have the auto-correlation of their entries at diagonals, the second term reduces to $(\|z\|^2 - |x|^2)^2 = \|z\|^2 + |x|^2 - 2\|z\|^2 |x|^2$. From Cauchy-Schwarz, $\|z\|^2 - |x|^2)^2 \leq 4\|z\|^2 + |x|^2$. Hence, we obtain
\[
\frac{1}{M} \|A(\tilde{z} - \tilde{x})\|_F^2 \leq 2\|\tilde{z} - \tilde{x}\|^2_F + (\Delta(\tilde{z} - \tilde{x}), \tilde{z} - \tilde{x})_F,
\]
and in the other direction, since $(\|z\|^2 - |x|^2)^2$ is the square of a real valued quantity, it is lower bounded by 0, which yields
\[
\frac{1}{M} \|A(\tilde{z} - \tilde{x})\|_F^2 \geq \|\tilde{z} - \tilde{x}\|^2_F + (\Delta(\tilde{z} - \tilde{x}), \tilde{z} - \tilde{x})_F.
\]

It remains to upper bound the quantity $\|\Delta(\tilde{z} - \tilde{x})\|_F$. Using the definition in (51), and the linearity of $\Delta$, from Cauchy Schwartz inequality, we have, for any $z \in E(\epsilon)$,
\[
\|\Delta(\tilde{z} - \tilde{x})\|_F \leq \|\tilde{z} - \tilde{x}\|_F (\|\Delta(\tilde{e})\| + \|\Delta(\tilde{x})\|) + \|\Delta(\tilde{x})\|_F
\leq \sqrt{2}\|\tilde{z} - \tilde{x}\|_F \|\tilde{e}\|_F + \|\tilde{x}\|_F + \|\tilde{x}\|_F
\leq \sqrt{2}\|\tilde{z} - \tilde{x}\|_F \delta (\|\tilde{e}\|^2 + 2\|\tilde{x}\|_F)\leq \sqrt{2}(2 + \delta) \|\tilde{z} - \tilde{x}\|_F \text{ dist}(z, x) \|x\|.
\]

Finally, using the lower bound from Lemma [III.3] where $\epsilon < 1$, we obtain
\[
\sqrt{2}(2 + \delta) \|\tilde{z} - \tilde{x}\|_F \text{ dist}(z, x) \|x\| \leq \frac{\sqrt{2}(2 + \delta)}{(1 - \epsilon)(2 - \epsilon)} \|\tilde{z} - \tilde{x}\|^2_F.
\]

Thereby, setting $\hat{\delta} = \frac{\sqrt{2}(2 + \epsilon)}{(1 - \epsilon)(2 - \epsilon)} \delta$, and combining the bounds (61), (62) and (64), the proof is complete.

E. Proof of Lemma [III.5]

Recall the definition of the gradient in (13). By Lemma [III.4] a RIP-type property is satisfied locally for $\tilde{z} - \tilde{x}$ if $z \in E(\epsilon)$. As a result we can express $\nabla f(z) = \hat{z}z - \hat{x}x + (\|z\|^2 + |x|^2)z + \Delta(\tilde{z} - \tilde{x})z = \|z\|^2 z - (\tilde{\hat{x}}^T z)\hat{x} + (\|z\|^2 + |x|^2)z + \Delta(\tilde{z} - \tilde{x})z$, with $\tilde{x}$ again denoting the closest solution in $P$ to a given $z \in C^N$. Then, we upper bound $\nabla f(z)$ as follows:
\[
\|\nabla f(z)\| \leq \|\|z\|^2 z - (\tilde{\hat{x}}^T z)\hat{x}\| + \|\|z\|^2 - \|x\|^2\| ||z||
\leq \|\Delta(\tilde{z} - \tilde{x})\| ||z||
\]
from which, knowing that $\|z\|^2 z - (\tilde{\hat{x}}^T z)\hat{x} = (\|z\|^2 - \|\hat{x}\|^2)z + (\tilde{\hat{x}}^T z)\hat{x} - \|\hat{x}\|^2 z$, we obtain
\[
\|\nabla f(z)\| \leq \|z\| (\|\tilde{\hat{x}}^T z\| + \|\tilde{\hat{x}}\|) \|z\| + \|\tilde{x} - \tilde{x}\|_F + \|\Delta(\tilde{z} - \tilde{x})\|_F.
\]

Again considering the expression $\tilde{z} - \tilde{x} = \tilde{e} + \text{exh} + \text{exh}^T$, we have $\|\Delta(\tilde{z} - \tilde{x})\| \leq \|\Delta(\tilde{e})\| + \|\Delta(\text{exh} + \text{exh}^T)\|$. Since $\text{exh} + \text{exh}^T$ is at most rank-2 by definition, let $\text{exh} + \text{exh}^T = \sum_{i=1}^2 \lambda_i v_i v_i^H$, by which we obtain $\|\Delta(\text{exh} + \text{exh}^T)\| \leq \lambda_1 \|\Delta v_1 \|_2^2 + \lambda_2 \|\Delta v_2 \|_2^2 \|$. Thereby, using Lemma [III.1] we have $\|\Delta(\tilde{z} - \tilde{x})\| \leq \delta (\|e\|^2 + \lambda_1 + \lambda_2)$. Furthermore, $\lambda_1 + \lambda_2$ precisely corresponds to the nuclear norm of $\text{exh} + \text{exh}^T$, which is upper bounded as $\|\text{exh} + \text{exh}^T\|_n \leq 2\|e\| \|x\|$. As a result, invoking the upper bound on the error in the lifted domain via Lemma [III.3] we obtain
\[
\|\nabla f(z)\| \leq \|z\| (\|\tilde{\hat{x}}^T z\| + \|\tilde{\hat{x}}\| + (1 + \delta)(2 + \epsilon)\|x\|) \|\tilde{\hat{x}}^T z\| + (2 + \epsilon)\|x\| \|\tilde{\hat{x}}\| \|\tilde{\hat{x}}\| \|\tilde{\hat{x}}\| = 1 + \epsilon)(2 + \epsilon)\|x\| \|\tilde{\hat{x}}\|.
\]

In (67) we’ve used the fact that, for $z \in E(\epsilon)$, $\|z\| \leq 1 + \epsilon \|x\|$. Hence, the proof of local Lipschitz differentiability is complete, with a constant $\delta = (1 + \epsilon)(2 + \epsilon)$.
(2 + \delta)\|\hat{z}\|_2^2$, with $\|z\| = 1$. Reorganizing the last inequality in (70), we obtain
\[
\left\| \frac{1}{M} A^H(\eta) \right\| \leq \sqrt{2 + \delta} \left( \frac{1}{\sqrt{M}} \|A(\hat{x})\| \right) \|\eta\| \|A(\hat{x})\|. 
\] (71)

Invoking the upper bound from Corollary III.1 once again, along with the definition of SNR, and Jensen’s inequality, we obtain
\[
\mathbb{E}[\eta] \left[ \sqrt{2 + \delta} \left( \frac{1}{\sqrt{M}} \|A(\hat{x})\| \right) \|\eta\| \right] \leq \frac{2 + \delta}{\sqrt{\text{SNR}}} \|\hat{x}\|_F,
\] and plugging the bound in (72) into (69) the proof is complete.

G. Proof of Theorem IV.1

Given the lifted domain definition of the clean gradient term in (13), using the noisy measurements $y = A(\hat{x}) + \eta$ with $\eta \in \mathbb{R}^M$, we define
\[
\nabla \hat{f}(z) = \left( \frac{1}{M} A^H A(\hat{x} - \hat{x}) + \frac{1}{M} A^H(\eta) \right) z.
\] (73)

Having $\nabla \hat{f}(z) = \frac{1}{M} A^H A(\hat{x} - \hat{x}) z$ as the ideal gradient estimate and setting $\mu_{k+1} = \mu' \times \mu$, we analyze the following updates:
\[
z_{k+1} = (z_k - \mu' \|x\|^2 \nabla \hat{f}(z_k)) + \mu' \frac{1}{M} A^H(\eta) z_k
\] (74)

where $z_{k+1}$ denotes the iterate obtained from the ideal update given the current estimate $z_k$. We now approach the proof by induction. Starting from the first iteration $k = 0$, we have
\[
z_1 = \hat{z}_1 + \frac{\mu}{\|x\|^2} \frac{1}{M} A^H(\eta) z_0,
\] (75)

dist$(z_1, x) \leq \|z_1 - \hat{x}_1\| \leq \|z_1 - \hat{z}_1\| + \|\hat{z}_1 - \hat{x}\|$
\[
= \text{dist}(\hat{z}_1, x) + \|z_1 - \hat{z}_1\|. 
\] (76)

Furthermore given (37), the iterates are guaranteed to stay in the $\epsilon$-neighborhood determined by the spectral initialization, which via Theorem III.1 guarantees
\[
\text{dist}(\hat{z}_1, x) \leq \epsilon \|x\| (1 - \frac{2\mu'}{\alpha'})^{1/2}, 
\] (77)

under the validity of (35), where $\mu'/\alpha' = \mu/\alpha$. Repeating for the next iteration, under the validity of (35) and (37), we have that
\[
\text{dist}(z_2, x) \leq \|z_2 - \hat{z}_2\| + \text{dist}(\hat{z}_2, x)
\leq (1 - \frac{2\mu}{\alpha}) \epsilon \|x\| + \sum_{l=1}^2 (1 - \frac{2\mu'}{\alpha'})^{1/2} \|z_l - \hat{z}_l\|,
\] (78)

and by induction, we obtain
\[
\text{dist}(z_k, x) \leq \epsilon (1 - \frac{2\mu}{\alpha})^{k/2} \|x\| + \sum_{l=1}^k (1 - \frac{2\mu}{\alpha})^{l/2} \|z_l - \hat{z}_l\|.
\] (79)

Recalling that $\|z_l - \hat{z}_l\| = \left\| \frac{\mu'}{\|x\|^2} \frac{1}{M} A^H(\eta) z_{l-1} \right\|$, we can bound the term within the summation as follows:
\[
\|z_l - \hat{z}_l\| \leq \frac{\mu'}{\|x\|^2} \frac{1}{M} A^H(\eta) \|z_{l-1}\| \leq \frac{(2 + \delta)}{\sqrt{\text{SNR}}} \|x\|^2 := U_l. 
\] (80)

After summing both sides in (80), we approximately obtain
\[
\sum_{l=1}^k U_l \approx \frac{(2 + \delta)}{\sqrt{\text{SNR}}} \sum_{l=1}^k \mu' \left[ 1 - \left( \frac{2\mu'}{\alpha'} \right) \right]^{l-1} \|x\|
\leq \frac{(2 + \delta)}{\sqrt{\text{SNR}}} \|x\| \frac{1 + \sqrt{1 - \frac{2\mu'}{\alpha'}}}{(1 + \sqrt{1 - \frac{2\mu'}{\alpha'}})^2}
\leq \frac{(2 + \delta)}{\sqrt{\text{SNR}}} \|x\| \frac{2\mu' \alpha'}{2\mu'^2}
\] (82)

which completes the proof.

VI. CONCLUSION

This paper analyzes the exact recovery guarantees of the non-convex phase retrieval framework of Wirtinger Flow through a novel perspective in the lifted domain. Our approach quantifies a regime in which the concentration bound of the spectral matrix geometrically implies the validity of the regularity condition. As a result, we identify a sufficient condition under which the convergence to the true solution is guaranteed deterministically via Wirtinger Flow, starting from the estimate obtained from the spectral initialization. Notably, our results address a theoretical gap that exists in phase retrieval literature, in which convergence arguments are predominantly probabilistic in nature. Furthermore, the deterministic convergence arguments developed in this paper rely on a less stringent restricted isometry type property than those of state-of-the art low rank matrix recovery methods. Although numerical simulations on specific problem domains are beyond the scope of this paper, our results culminate into a framework that is highly relevant to applications such as wave-based imaging, in which the underlying scattering phenomenon is typically a deterministic map. Namely, processing back-scattered data without phase offers several advantages for radar imaging, such as reducing implementation costs, bypassing the autofocus problem, and increased robustness to phase errors. Especially in the context of passive imaging, via exact phase retrieval theory the reliance on transmitter related terms such as location or the transmitted waveform can be eliminated to obtain a robust framework. In future work, we will address the problem of designing a lifted forward model from randomly illuminated 2D Fourier slices to satisfy our sufficient condition, and attain an exact phaseless wave-based imaging theory via Wirtinger Flow.
REFERENCES

[1] E. J. Candès, X. Li, and M. Soltanolkotabi, “Phase retrieval via Wirtinger flow: Theory and algorithms,” IEEE Trans. Inf. Theory, vol. 61, no. 4, pp. 1985–2007, Apr. 2015.

[2] E. J. Candès and T. Strohmer, “Phasemax: Convex phase retrieval via basis pursuit,” IEEE Trans. Signal Processing, vol. 66, no. 5, pp. 1241–1274, Aug. 2013.

[3] T. Bendory, Y. C. Eldar, and N. Boumal, “Non-convex phase retrieval from stft measurements,” IEEE Transactions on Information Theory, vol. 64, no. 1, pp. 467–484, 2018.

[4] R. W. Gerchberg, “A practical algorithm for the determination of the phase from image and diffraction plane pictures,” Optik, vol. 35, pp. 237–246, 1972.

[5] J. R. Fienup, “Reconstruction of an object from the modulus of its fourier transform,” Optics letters, vol. 3, no. 1, pp. 27–29, 1978.

[6] H. H. Bauschke, P. L. Combettes, and D. R. Luke, “Hybrid projection–reflection method for phase retrieval,” JOSA A, vol. 20, no. 6, pp. 1025–1034, 2003.

[7] E. J. Candès, Y. Eldar, T. Strohmer, and V. Voroninski, “Phase retrieval via matrix completion,” SIAM J. Imag. Sci., vol. 6, no. 1, pp. 199–225, 2013.

[8] P. Hand and V. Voroninski, “An elementary proof of convex phase retrieval in the natural parameter space via the linear program phasemax,” arXiv preprint arXiv:1611.03935, 2016.

[9] S. Bahmani, J. Romberg et al., “Phase retrieval using alternating direction method of multipliers,” Mathematical Programming, vol. 149, no. 1, pp. 47–81, 2015. [Online]. Available: http://dx.doi.org/10.1007/s10107-013-0738-9

[10] P. Netrapalli, P. Jain, and S. Sanghavi, “Phase retrieval using alternating minimization,” in Advances in Neural Information Processing Systems, 2013, pp. 2796–2804.

[11] H. Zhang, Y. Zhou, Y. Liang, and Y. Chi, “Reshaped wirtinger flow for solving quadratic equations via truncated amplitude flow,” SIAM J. Imag. Sci., vol. 11, no. 2, pp. 5254–5281, 2017.

[12] J. Sun, Q. Qu, and J. Wright, “A geometric analysis of phase retrieval,” Foundations of Computational Mathematics, pp. 1–71, 2017.

[13] W. Luo, W. Alghambbi, and Y. M. Lu, “Optimal spectral initialization for signal recovery with applications to phase retrieval,” IEEE Transactions on Signal Processing, vol. 67, no. 9, pp. 2347–2356, 2019.

[14] R. Ghods, A. S. Lan, T. Goldstein, and C. Studer, “Linear spectral estimators and an application to phase retrieval,” arXiv preprint arXiv:1806.03547, 2018.

[15] B. Gao and Z. Xu, “Phaseless recovery using the gauss–newton method,” IEEE Transactions on Signal Processing, vol. 65, no. 22, pp. 5885–5896, 2017.

[16] B. Yanel and B. Yazici, “A generalization of Wirtinger flow for exact interferometric inversion,” submitted to SIAM Journal of Imaging Sciences.

[17] J. Sun, Q. Qu, and J. Wright, “A geometric analysis of phase retrieval,” Foundations of Computational Mathematics, vol. 18, no. 5, pp. 1131–1198, 2018.

[18] S. Sanghavi, R. Ward, and C. D. White, “The local convexity of solving systems of quadratic equations,” Results in Mathematics, vol. 71, no. 3-4, pp. 569–608, 2017.

[19] X. Li, S. Ling, T. Strohmer, and K. Wei, “Rapid, robust, and reliable blind deconvolution via nonconvex optimization,” Applied and Computational Harmonic Analysis, 2018.

[20] Q. Zheng and J. Lafferty, “A convergent gradient descent algorithm for rank minimization and semidefinite programming from random linear measurements,” in Advances in Neural Information Processing Systems, 2015, pp. 109–117.

[21] S. Tu, R. Boczar, M. Simchowitz, M. Soltanolkotabi, and B. Recht, “Low-rank solutions of linear matrix equations via.procrustes flow,” arXiv preprint arXiv:1507.03566, 2015.

[22] L. Wang, X. Zhang, and Q. Gu, “A unified computational and statistical framework for nonconvex low-rank matrix estimation,” arXiv preprint arXiv:1610.05275, 2016.

[23] Y. Chi, Y. M. Lu, and Y. Chen, “Nonconvex optimization meets low-rank matrix factorization: An overview,” arXiv preprint arXiv:1809.09573, 2018.

[24] Y. Chen and E. J. Candès, “Solving random quadratic systems of equations is nearly as easy as solving linear systems,” in Advances in Neural Information Processing Systems, 2015, pp. 739–747.

[25] H. Zhang, Y. Zhou, Y. Liang, and Y. Chi, “Reshaped wirtinger flow and incremental algorithms for solving quadratic systems of equations,” 2017, preprint.

[26] M. Soltanolkotabi, “Structured signal recovery from quadratic measurements: Breaking sample complexity barriers via nonconvex optimization,” IEEE Transactions on Information Theory, vol. 65, no. 4, pp. 2374–2400, 2019.

[27] A. S. Bandeira, J. Cahi, D. G. Mixon, and A. A. Nelson, “Saving phase: Injectivity and stability for phase retrieval,” Applied and Computational Harmonic Analysis, vol. 37, no. 1, pp. 106–125, 2014.

[28] J.-F. Cai, E. J. Candès, and Z. Shen, “A singular value thresholding algorithm for matrix completion,” SIAM J. Opti., vol. 20, no. 4, pp. 1956–1982, 2010.

[29] B. Recht, M. Fazel, and P. A. Parrilo, “Guaranteed minimum-rank solutions of linear matrix equations via nuclear norm minimization,” SIAM review, vol. 52, no. 3, pp. 471–501, 2010.

[30] B. Recht, W. Xu, and B. Hassibi, “Necessary and sufficient conditions for success of the nuclear norm heuristic for rank minimization,” in Decision and Control, 2008. CDC 2008. 47th IEEE Conference on. IEEE, 2008, pp. 3065–3070.

[31] B. Recht, W. Xu, and B. Hassibi, “Null space conditions and thresholds for rank minimization,” Mathematical programming, vol. 127, no. 1, pp. 175–202, 2011.

[32] S. Oymak, K. Mohan, M. Fazel, and B. Hassibi, “A simplified approach to recovery conditions for low rank matrices,” in Information Theory Proceedings (ISIT), 2011 IEEE International Symposium on. IEEE, 2011, pp. 2318–2322.

[33] T. T. Cai, “Sharp rip bound for sparse signal and low-rank matrix recovery,” Appl. Comput. Harmon. Anal, vol. 35, pp. 74–93, 2013.

[34] S. Bhojanapalli, B. Neyshabur, and N. Srebro, “Global optimality of local search for low rank matrix recovery,” in Advances in Neural Information Processing Systems, 2016, pp. 3873–3881.

[35] L. Demanet and P. Hand, “Stable optimizationless recovery from phaseless linear measurements,” Journal of Fourier Analysis and Applications, vol. 23, no. 6, pp. 175–202, 2019.