**On the absence of the Boulware-Deser ghost in novel graviton kinetic terms**

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Using novel nonlinear kinetic terms for gravitons, a large class of bi-gravity models were constructed, which are potentially free of the Boulware-Deser (BD) ghost. In this work, we derive their Hamiltonians using the ADM formalism, and verify that the BD ghost is eliminated by additional constraints. The general Hamiltonian structure is analogous to that of the other class of bi-gravity models free of the BD ghost.

I. INTRODUCTION

In a general framework for Lagrangian theories free of Ostrogradsky’s scalar ghost [1, 2], novel nonlinear kinetic terms for gravitons were proposed in the language of vielbeins

$$
\mathcal{L}_{\text{kin}} = R(E^{(1)}) \wedge E^{(2)} \wedge \cdots \wedge E^{(d-1)},
$$

where $d$ is the spacetime dimension, $R(E^{(1)})$ is the curvature two-form associated with $E^{(1)}$, and $E^{(k)}$ could be different vielbeins. The Einstein-Hilbert kinetic term corresponds to the case where $E^{(k)}$ are the same. These nonlinear kinetic terms can be supplemented by some nonlinear potential terms [3–5]

$$
\mathcal{L}_{\text{pot}} = E^{(1)} \wedge \cdots \wedge E^{(d)},
$$

where $E^{(k)}$ could be different vielbeins.

Some of these novel kinetic terms are nonlinear, multi-gravity completion of the two-derivative cubic term discovered in [6]. It was shown in [7] that this cubic term is a natural generalization of perturbative Lovelock terms [8] and dRGT terms [9], so more possible terms exist in higher dimensions. They were conjectured to have nonlinear completions [3], which were constructed in [10].

However, in the literature, there are comprehensive no-go theorems for the nonlinear completion of the cubic term mentioned above. A no-go theorem for single dynamical metric theories in 4d was established in [10]: around Minkowski vacuum, the only nonlinear two-derivative term for spin-2 fields that can avoid the 6th degree of freedom is the Einstein-Hilbert term. The 6th degree of freedom of spin-2 field is a dangerous scalar mode. It always plagues a generic nonlinear completion of the consistent theory of linear massive gravity, namely Fierz-Pauli massive gravity [13]. This unhealthy scalar mode is known as the Boulware-Deser ghost [14].

However, our proposals are not ruled out by these negative results, because we consider a more general setting where all the spin-2 fields are dynamical. In addition, we use the second-order formulation, so the torsion-free condition is satisfied automatically.

Usually, a bi-gravity model reduces to a single dynamical metric model in the decoupling limit where one of the Planck masses goes to infinity. The novel kinetic terms for bi-gravity will be ruled out if they do have the same decoupling limit. However, a large class of promising bi-gravity theories identified in [13] do not have non-trivial single dynamical metric limit. Their Lagrangians contain at most one Einstein-Hilbert term, so we have only one Planck mass which can not be sent to infinity.

Unfortunately, precisely due to the fact that only one kind of curvature tensor is allowed, the linearized kinetic terms have opposite signs after diagonalization, which means one of them is a spin-2 ghost. It has been known for a long time that higher derivative gravity has the same problem. In fact, some of them are equivalent to bi-gravity models with novel kinetic terms when the couplings to matter are not introduced [15].

According to the minisuperspace analysis in [15], there are two classes of bi-gravity models where the 6th degrees of freedom could be absent. In the first class, the kinetic terms are two standard Einstein-Hilbert terms [4] and novel kinetic terms are not allowed. In the second class, the novel graviton kinetic terms could be present. We focus on the second class of bi-gravity models in this work. In 4d, the possible kinetic terms are

$$
\mathcal{L}_{1}^{\text{kin}} = R(E) \wedge E \wedge E, \\
\mathcal{L}_{2}^{\text{kin}} = R(E) \wedge E \wedge F, \\
\mathcal{L}_{3}^{\text{kin}} = R(E) \wedge F \wedge F,
$$

where $E$ and $F$ are two different vielbeins and the normalization factors are not precise. If we impose the symmetric condition and fix the second metric to Minkowski, they reduce to the two-derivative terms proposed in [16], which can also be obtained from the dimensionally deconstruction of 5d Gauss-Bonnet term [10]. In [10, 16], the second spin-2 field was assumed to be fixed, which leads to the dynamical Boulware-Deser ghost. Let us emphasize that we will not make this assumption.

The nonlinear kinetic terms can be accompanied by some
potential terms
\[ \mathcal{L}_{1}^{\text{pot}} = E \wedge E \wedge E \wedge E, \]
\[ \mathcal{L}_{2}^{\text{pot}} = E \wedge E \wedge E \wedge F, \]
\[ \mathcal{L}_{3}^{\text{pot}} = E \wedge E \wedge F \wedge F, \]
\[ \mathcal{L}_{4}^{\text{pot}} = E \wedge F \wedge F \wedge F, \]
\[ \mathcal{L}_{5}^{\text{pot}} = F \wedge F \wedge F \wedge F, \]
which are different wedge products of vielbeins $E$ and $F$. The
cosmological constant terms $\mathcal{L}_{1}^{\text{pot}}$ and $\mathcal{L}_{5}^{\text{pot}}$ are special cases
where only one vielbein is used.

After a field redefinition, the novel kinetic terms become
\[ \mathcal{L}_{2}^{\text{kin}} = \left( -\frac{1}{4} \sqrt{-g} R(g)_{\mu \nu}^{\rho \sigma} e_{\rho}^{\mu} e_{\sigma}^{\nu} \right) d^{4}x, \]
\[ \mathcal{L}_{3}^{\text{kin}} = \frac{1}{4} \sqrt{-g} R(g)_{\mu \nu}^{\rho \sigma} e_{\rho}^{\mu} e_{\sigma}^{\nu} d^{4}x. \]
where $R(g)_{\mu \nu}^{\rho \sigma}$ is the Riemann tensor associated with the
metric $g_{\mu \nu}$
\[ g_{\mu \nu} = E^{\mu A} E^{\nu B} \eta_{AB}. \]

There are the symmetric condition is satisfied. The second sym-
tetric tensor is defined as
\[ f_{\mu \nu} = F_{\mu A} F_{\nu B} \eta_{AB}. \]

The indices of the second spin-2 field $e$ can be lowered and
raised by the metric $g_{\mu \nu}$ and its inverse $g^{\mu \nu}$.

This paper is organized as follows. In section II we intro-
duce the ADM formalism. In section III we investigate two
of the simplest but representative examples. In section IV we
discuss the general structure of bi-gravity models involving
novel kinetic terms. In section V we summarize our results.

II. ADM FORMALISM

Before deriving the Hamiltonians, let us express the La-
grangians in terms of the ADM variables with the help of
Gauss-Codazzi-Ricci equations.

In the ADM formalism \[18\], the metric is
\[ ds^2 = g_{\mu \nu} dx^{\mu} dx^{\nu} = -N^2 dt^2 + \gamma_{ij} (dx^i + N^i dt)(dx^j + N^j dt) = -(n_\mu dx^\mu)^2 + \gamma_{ij} (\gamma_{\mu i} dx^\mu)(\gamma_{\nu j} dx^\nu) \]
where $N$ is the lapse function, $N^i$ is the shift vector, $\gamma_{ij}$ is
the induced metric, $n^\mu$ is the normal vector and $\gamma_{\mu i}$ is the
projector, according to the foliation of spacetime.

It is useful to introduce a local frame
\[ ds^2 = G_{\mu \nu} H^\mu H^\nu = -(H^0)^2 + \gamma_{ij} H^i H^j, \]
which we call a local ADM frame. This frame generalizes the
concept of a "local Lorentz frame". The spatial metric
\[ G_{ij} = \gamma_{ij} \]
is not necessarily flat and coincides with the induced metric $\gamma_{ij}$. The components of the local ADM frame fields are given
by the normal vector $n_\mu$ and the projector $\gamma_{\mu i}$
\[ H^0 = n_{\mu} dx^{\mu}, \]
\[ H^i = \gamma_{\mu i} dx^{\mu}. \]
For more details about the local ADM frame, we refer to section 21.5 of \[19\].

In the local ADM frame, the components of the Riemann
curvature tensor are given by the Gauss-Codazzi-Ricci relations:

- The Gauss equations are
  \[ R_{ijkl}(G) = \gamma_{i}^{\mu} \gamma_{j}^{\nu} \gamma_{k}^{\rho} \gamma_{l}^{\sigma} R_{\mu \nu \rho \sigma}(g) \]
  \[ = R_{ijkl}(\gamma) + K_{ik} K_{jl} - K_{il} K_{jk}. \]

- The Peterson-Mainardi-Codazzi equations are
  \[ R_{ijkl}(G) = \gamma_{i}^{\mu} \gamma_{j}^{\nu} \gamma_{k}^{\rho} n^{\sigma} R_{\mu \nu \rho \sigma}(g) \]
  \[ = D_{i} K_{jk} - D_{j} K_{ik}. \]

- The Ricci equations are
  \[ R_{ijkl}(G) = \gamma_{i}^{\mu} n^{\nu} \gamma_{k}^{\rho} n^{\sigma} R_{\mu \nu \rho \sigma}(g) \]
  \[ = -L_{n} K_{ik} + K_{ij} K^{j} k + N^{-1} D_{i} \partial_{k} N. \]

The extrinsic curvature $K_{ij}$ is defined as
\[ K_{ij} = \frac{1}{2} L_{n} \gamma_{ij} = \frac{1}{2N} (\gamma_{ij} - D_{i} N_{j} - D_{j} N_{i}), \]
with
\[ n_{0} = \frac{1}{N}, \quad n^{i} = -\frac{N^{i}}{N}, \]
and $D_{i}$ is the covariant derivative compatible with the induced met-
ic $\gamma_{ij}$
\[ D_{i} \gamma_{jk} = 0. \]

Let us introduce a notation
\[ N_{0} = N, \]
so the lapse function and the shift vector form a spacetime
vector
\[ N_{\mu} = (N_{0}, -N^{i}). \]
We use $R(G)$ to denote the components of the Riemann curvature tensor in a local ADM frame. We can raise the last two indices by $G^{\mu\nu}$

$$
R^0_{0j} (G) = R_{0ijk}(G) G^{0\mu} G^{kj} = \mathcal{L}_n K^j + K_i K^i - N^{-1} D_i \partial^j N, 
$$

(29)

$$
R^{ij}_{kl} (G) = R_{ijmn} (G) G^{mk} G^{nl} = R^{ij}_{kl} (\gamma) + K_i^k K_j^l - K_i^l K_j^k, 
$$

(30)

$$
R^{0i} (G) = R_{iklm} (G) G^{0\mu} G^{mi} = D_i K^k - D_k K^i, 
$$

(31)

$$
R^{ij} (G) = R_{imjn} (G) G^{mj} G^{nk} = - (D^j K^k - D^k K^j). 
$$

(32)

These equations will be used to derive the expressions of the novel kinetic terms in terms of the ADM variables.

The components of the new tensor $e_{\mu}^\nu$ in the local ADM frame are

$$
e_0^0 = (-) e_{\nu}^\mu n_\mu, \\
e_i^0 = e_\nu^\mu \nu^\nu i, \\
e_0^0 = e_\mu^\nu \nu^\mu i, \\
e_{ij} = e_\nu^\mu \nu^\mu j, \\
e_{i0} = e_\mu^\nu \nu^\mu ij, \\
e_{ij0} = e_\mu^\nu \nu^\mu i, .
$$

(33)

The left hand sides $e_0^0, e_i^0, e_{ij}$, $e_{i0}$ are the fundamental fields in the local ADM frame. Let us emphasize that they are not the components of $e_\mu^\nu$ and the indices 0, i, j, k, are those of a local ADM frame.

## III. EXAMPLES

In this section, we carry out the Hamiltonian analyses of two concrete examples of novel kinetic terms. The discussion of the general structure of a linear combination of the nonlinear kinetic terms is postponed to the next section.

### A. $\mathcal{L}_{EH}$

In this subsection, we briefly review the Hamiltonian structure of the Einstein-Hilbert kinetic term. The general features shared by other kinetic terms are emphasized in this well-understood example.

Using (29) and (30), we have

$$
\mathcal{L}_{EH} = \sqrt{-g} R(g) \\
= N \sqrt{\gamma} [2 \mathcal{L}_n K + K^2 + K_i K^i + R(\gamma) - 2N^{-1} D_i \partial^j N] \\
= N \sqrt{\gamma} [R(\gamma) + K_i K^i - K^2] \\
+ \partial_\mu (N \sqrt{\gamma} 2 n^{\mu} K) - \partial_\mu (\sqrt{\gamma} 2 \partial^\mu N). 
$$

(34)

The Ricci scalar contains a second order time derivative term $\mathcal{L}_n K$. It can be eliminated by supplementing the action by the York-Gibbons-Hawking term on the space-like boundaries

$$
S_{YGH} = \int d^3 x [\sqrt{\gamma} (-2K)]^t_f, 
$$

(35)

which cancel the time component of the first total derivative term in the last line of (34). The second order space derivative terms can be cancelled by terms on the time-like boundaries.

Introducing the matrix

$$
(M_0)^{ijkl} = \gamma^{ij} \gamma^{kl} - \gamma^{ik} \gamma^{jl}, 
$$

(36)

the Lagrangian becomes

$$
\mathcal{L}_{EH} = N \sqrt{\gamma} [ - K_{ij} (M_0)^{ijkl} K_{kl} + R(\gamma)]. 
$$

(37)

Note that $M_0$ is the Kulkarni-Nomizu product of two inverse induced metrics

$$
(M_0)^{ijkl} = \frac{1}{2} (\gamma \otimes \gamma)^{ijkl}, 
$$

(38)

which is related to the Wheeler-DeWitt metric. $(M_0)^{ijkl}$ appears in the novel kinetic terms as well.

The conjugate momenta are

$$
\Pi_N = \frac{\partial \mathcal{L}_{EH}}{\partial N} = 0, \\
\Pi_{N^i} = \frac{\partial \mathcal{L}_{EH}}{\partial N^i} = 0, 
$$

(39)

$$
\Pi^i = \frac{1}{2N} \frac{\partial \mathcal{L}_{EH}}{\partial K_{ij}} = \left( \frac{1}{2} \right) \sqrt{\gamma} (M_0)^{ijkl} \mathcal{L}_n \gamma_{kl}, 
$$

(40)

where the first line gives primary constraints.

Inverting the relation between $\Pi^i$ and $\mathcal{L}_n \gamma_{kl}$, we have

$$
\mathcal{L}_n \gamma_{kl} = (-2) \gamma^{-1/2} \Pi^i (M_0^{-1})^i_{jkl}, 
$$

(41)

where the inverse matrix of $(M_0)^{ijkl}$ is introduced

$$
(M_0^{-1})_{ijkl} = \frac{1}{2} \gamma^{-1/2} \gamma_{ij} \gamma_{kl} - \gamma_{ik} \gamma_{jl}, 
$$

(42)

with

$$
(M_0)^{ijkl} (M_0^{-1})_{klmn} = \delta_m^i \delta_n^j. 
$$

(43)

The Hamiltonian is

$$
\mathcal{H}_{EH} = \gamma_{ij} \Pi^i - \mathcal{L}_{EH} \\
= (N \mathcal{L}_n \gamma_{ij} + D_i N_j + D_j N_i) \Pi^j - \mathcal{L}_{EH} \\
= -N \gamma^{-1/2} \Pi^i (M_0^{-1})^i_{jkl} K_{kl} + \sqrt{\gamma} R \\
- N^i D_j 2 \Pi^j. 
$$

(44)

In the second equality, we reconstruct $\mathcal{L}_n \gamma_{ij}$ from $\gamma_{ij}$, which generates the $D_i N_j, \Pi^j$ terms. In the third equality, we express $\mathcal{L}_n \gamma_{ij}$ in terms of $\Pi^j$ using the inverse of $M_0$. These are two common steps in expressing a Hamiltonian in terms of the canonical variables.
The momenta are tensor densities, so their covariant derivatives should be
\[ D_i \Pi^j \rightarrow \sqrt{\gamma} D_i (\gamma^{-1/2} \Pi^j). \] (45)

The total Hamiltonian that contains the information of primary constraints is
\[ H_{EH}^T = H_{EH} + \lambda^\mu \Pi_{N^\mu}, \] (46)
where \( \lambda^\mu \) are Lagrange multipliers associated with the primary constraints.

To preserve the primary constraints \( \Pi_{N^\mu} \approx 0 \) in time, we require the time derivative of \( \Pi_{N^\mu} \) vanish on the constraint surface
\[ \dot{\Pi}_{N^\mu} = \left\{ \Pi_{N^\mu}, \int d^3x \mathcal{H}_{EH} \right\} = (-) \frac{\partial H_{EH}}{\partial N^\mu} = (-) \tilde{C}_\mu \approx 0, \] (47)
where \( A \approx 0 \) means \( A \) vanishes on the constraint surface. They are secondary constraints. Since \( N^\mu \) are Lagrange multipliers, \( \tilde{C}_\mu \) do not contain \( \Pi^\nu \)
\[ \tilde{C}_0 = (-) \left[ \gamma^{-1/2} \Pi^{ij} (M_0^{-1})_{ij,kl} \Pi^{kl} + \sqrt{\gamma} R \right] \approx 0, \] (48)
\[ \tilde{C}_1 = (-2) \sqrt{\gamma} D^j (\gamma^{-1/2} \Pi_{ij}) \approx 0, \] (49)
where \( \tilde{C}_0 \approx 0 \) is known as the Hamiltonian constraint and \( \tilde{C}_1 \approx 0 \) the momentum constraint or the diffeomorphism constraint. There are no more independent constraints from the time derivatives of secondary constraints.

Note that \( \tilde{C}_0 \) and \( \tilde{C}_1 \) are first-class constraints, as their Poisson brackets vanish on the constraint surface. To see this, we compute the Poisson brackets of smeared constraints
\[ C_1[\alpha] = \int d^3x \alpha^i(x) \tilde{C}_i, \quad C_2[\alpha] = \int d^3x \alpha(x) \tilde{C}_0, \] (50)
where \( \alpha(x) \) and \( \alpha^i(x) \) are test functions. The result is Dirac’s hypersurface deformation algebra
\[ \{ C_1[\alpha], C_2[\beta] \} = C_1[\alpha \beta], \] (51)
\[ \{ C_1[\alpha], C_2[\beta] \} = C_2[\alpha \beta], \] (52)
\[ \{ C_2[\alpha], C_2[\beta] \} = C_1[\alpha \beta], \] (53)
where boundary terms are neglected, the Poisson bracket is defined as
\[ \{ A, B \} = \int d^3x \left[ \frac{\delta A}{\delta \gamma_{ij}(x)} \frac{\delta B}{\delta \Pi^{ij}(x)} - \frac{\delta B}{\delta \gamma_{ij}(x)} \frac{\delta A}{\delta \Pi^{ij}(x)} \right] \] (54)
and the vector in the last bracket \( \{ 53 \} \) is defined as
\[ f^i(\alpha, \beta, \gamma_{ij}) = \gamma_{ij} (\alpha \partial_j \beta - \beta \partial_j \alpha). \] (55)
\( f^i \) is also known as structure functions\(^3 \) due to the dependence on the phase space variables \( \gamma_{ij} \).

The first two brackets \( \{ 51, 52 \} \) can be derived in a simple way due to the fact that \( C_1 \) generates an operator in the phase space
\[ \{ C_1[\alpha], \gamma_{ij} \} = -L_\alpha \gamma_{ij}, \] (56)
\[ \{ C_1[\alpha], \Pi^{ij} \} = -L_\alpha \Pi^{ij}, \] (57)
which are the Lie derivatives of the canonical variables along \( \alpha \). So \( \tilde{C}_1 \approx 0 \) is also known as the diffeomorphism constraint. Using \( \{ 56 \} \) and \( \{ 57 \} \), the Lie derivative \( L_\alpha \) acts on the test functions \( (\beta, \beta) \) after integrating by parts, so the first two brackets \( \{ 51, 52 \} \) share a similar form.

**B. \( L_{2^{\text{kin}}} \)**

Let us investigate the simplest example of novel kinetic terms
\[ L_{2^{\text{kin}}} = (-) \frac{1}{4} \sqrt{-g} R (g)_{\mu\nu} e_{\rho\sigma} d^4x. \] (58)

Since the antisymmetric part of \( e_{\mu\nu} \) in \( L_{2^{\text{kin}}} \) is projected out, we will simply assume \( e_{\mu\nu} \) is symmetric. In the local ADM frame, we have
\[ e_0^i = e^i_0, \quad e_{ij} = e_{ji}. \] (59)

Using \( \{ 29, 32 \} \), we can derive the explicit expression of \( L_{2^{\text{kin}}} \) in the local ADM frame
\[ \gamma^{-1/2} L_2 = N (L_n \gamma_{ij}) (L_n \gamma_{kl}) \left[ (M_1)_{ijkl} - \frac{1}{8} e_0^0 (M_0)_{ijkl} \right] + N (L_n \gamma_{ij}) (L_n e_{kl}) \frac{1}{2} (M_0)_{ijkl} \]
\[ + L_n (L_n g) (L_n e_{ij}) (L_n e_{kl}) \frac{1}{2} (M_0)_{ijkl} \]
\[ + e_n (M_0)_{ijkl} e_{kl}, \]
(60)
where \( M_0 \) and \( M_0^{-1} \) are defined in \( \{ 36, 42 \} \) and
\[ (M_1)_{ijkl} = -\frac{1}{2} (\gamma_{ij} e_{kl} - \gamma_{ik} e_{jl}) + \frac{1}{8} e_m^m (M_0)_{ijkl}. \] (61)

From the definition in \( \{ 33 \} \), we know \( e_{ij} \) is tangent to a constant time slice, so its Lie derivative is
\[ L_n e_{ij} = \frac{1}{N} (e_{ij} - N K D_k e_{ij} - e_{kj} D_i e_k - e_{ik} D_j e_k). \] (62)

We supplement the action by a boundary term analogous to the York-Gibbons-Hawking term
\[ \int d^3x \sqrt{-\gamma} K_{ij}(M_0)_{ijkl} e_{kl} \big|_{t_i}^{t_f}, \] (63)
on the space-like boundaries to eliminate the second order time derivative term in the Lagrangian. These boundary terms generate time derivative terms \( \partial_t e_{ij} \) in \( \{ 58 \} \), so \( e_{ij} \) is also a

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\(^3\) Therefore, Dirac’s hypersurface deformation algebra is not a Lie algebra.
dynamical tensor field. In contrast, there is no time derivative acting on $e_0^i$, $e_0^0$, which are the counterparts of the lapse function $N$ and the shift vector $N^i$ in the ADM metric.

The conjugate momenta are

$$
\Pi^{ij} = \frac{\partial L_2}{\partial \dot{\gamma}_{ij}} = 2\sqrt{\gamma} \left( \mathcal{L}_n \gamma_{kl} \right) \left[ (M_1)^{ijkl} - \frac{1}{8} e_0^0 (M_0)^{ijkl} \right] + \sqrt{\gamma} \frac{1}{2} \left( \mathcal{L}_n e_{kl} \right) (M_0)^{ijkl} + \sqrt{\gamma} N^{-1} D_k \left[ N e_0^j (-)(M_0)^{ijk} \right],
$$

(64)

$$
\pi^{ij} = \frac{\partial L_2}{\partial \dot{e}_{ij}} = \sqrt{\gamma} \frac{1}{2} \left( \mathcal{L}_n \gamma_{kl} \right) (M_0)^{ijkl},
$$

$$
\Pi_{\mu} = \frac{\partial L_2}{\partial \dot{N}_\mu} = 0, \quad \pi_{e_\mu} = \frac{\partial L_2}{\partial \dot{e}_\mu} = 0,
$$

(66)

where the last line contains 8 primary constraints.

Then we obtain the Hamiltonian by the Legendre transform

$$
\mathcal{H}_2 = \gamma_{ij} \Pi^{ij} + \dot{e}_{ij} \pi^{ij} - L_2
$$

$$
= \frac{1}{2} e_0^0 N \left[ \gamma^{-1/2} \pi^{ij} \pi_{kl} (M_0^{-1})_{ijkl} + \gamma^{1/2} R \right]
$$

$$
+ \pi^{ij} D_i (2N e_0^0 \dot{\gamma}_{ij})
$$

$$
+ \gamma^{-1/2} \pi^{ij} \pi_{kl} \left[ - \frac{1}{2} \epsilon_{mn} (M_0^{-1})_{ijkl} + (\epsilon_{ij} \gamma_{kl} - 2 \gamma_{ik} \epsilon_{jl}) \right]
$$

$$
+ 2N \gamma^{-1/2} \Pi^{ij} \pi_{kl} (M_0^{-1})_{ijkl}
$$

$$
+ \gamma^{1/2} R \pi_{kl} (M_0^{-1})_{ijkl} - N \gamma^{1/2} D_i D_j (M_0 e)^{ij}
$$

$$
+ \Pi^{ij} (D_i N_j + D_j N_i)
$$

$$
+ \pi^{ij} (N^k D_k e_{ij} + e_{kj} D_i N^k + e_{ik} D_j N^k).
$$

(67)

The derivation is similar to the case of $\mathcal{H}_{	ext{EH}}$. We first reconstruct $\mathcal{L}_n \gamma_{ij}$, $\mathcal{L}_n e_{ij}$ from $\dot{\gamma}_{ij}$, $\dot{e}_{ij}$. Then, by inverting [64, 65], we express $\mathcal{L}_n e_{ij}$, $\mathcal{L}_n e_\mu$ in terms of $\Pi^{ij}$, $\pi^{ij}$.

The corresponding total Hamiltonian is

$$
\mathcal{H}_2 = \mathcal{H}_2 + \lambda_1^\mu \Pi_{\mu}^\nu + \lambda_2^\mu \Pi_{e_\mu},
$$

(68)

where $\lambda_1^\mu$, $\lambda_2^\mu$ are Lagrange multipliers associated with the primary constraints.

Note that $N$, $N^i$, $e_0^0$, $e_0^i$ are Lagrange multipliers in $\mathcal{H}_2$. By computing time derivatives of the primary constraints, we obtain 8 secondary constraints which do not contain $N$, $N^i$, $e_0^0$, $e_0^i$. All the secondary constraints are first-class constraints. To show this, we introduce the smeared constraints

$$
\mathcal{C}_1[\dot{\alpha}] = \int d^3 x \left( \Pi^{ij} \mathcal{L}_\alpha \gamma_{ij} + \pi^{ij} \mathcal{L}_\alpha e_{ij} \right),
$$

(69)

$$
C_2[\alpha] = \int d^3 x \alpha \left[ \gamma^{-1/2} \pi^{ij} \pi_{kl} \left[ - \frac{1}{2} \epsilon_{mn} (M_0^{-1})_{ijkl} + (\epsilon_{ij} \gamma_{kl} - 2 \gamma_{ik} \epsilon_{jl}) \right] + 2\gamma^{-1/2} \Pi^{ij} \pi_{kl} (M_0^{-1})_{ijkl} + \gamma^{1/2} R_i e_{kl} (M_0 e)^{ij} - \gamma^{1/2} D_i D_j (M_0)^{ijkl} e_{kl} \right],
$$

(70)

$$
C_3[\alpha] = \int d^3 x \left( \pi_{ij} \mathcal{L}_{\bar{\alpha}} \gamma_{ij} \right),
$$

(71)

$$
C_4[\alpha] = \int d^3 x \alpha \left[ \gamma^{-1/2} \pi^{ij} \pi_{kl} (M_0^{-1})_{ijkl} + \gamma^{1/2} R \right],
$$

(72)

where $\alpha(x)$ and $\alpha^i(x)$ are test functions.

After a straightforward computation, the Poisson brackets of the smeared constraints are

$$
\{C_1[\alpha], C_1[\beta]\} = C_1[\mathcal{L}_\alpha \beta],
$$

(73)

$$
\{C_1[\alpha], C_2[\beta]\} = C_2[\mathcal{L}_\alpha \beta],
$$

(74)

$$
\{C_1[\alpha], C_3[\beta]\} = C_3[\mathcal{L}_\alpha \beta],
$$

(75)

$$
\{C_1[\alpha], C_4[\beta]\} = C_4[\mathcal{L}_\alpha \beta],
$$

(76)

$$
\{C_2[\alpha], C_2[\beta]\} = C_2[\mathcal{L}_\alpha \beta],
$$

(77)

$$
\{C_3[\alpha], C_2[\beta]\} = C_4[\mathcal{L}_\alpha \beta],
$$

(78)

$$
\{C_4[\alpha], C_2[\beta]\} = C_3[\mathcal{L}_\alpha \beta],
$$

(79)

$$
\{C_4[\alpha], C_3[\beta]\} = \{C_3[\alpha], C_4[\beta]\} = \{C_4[\alpha], C_4[\beta]\} = 0,
$$

(80)

where the structure functions $\mathcal{L}_{\bar{\alpha}}$ are defined as

$$
f^i(\alpha, \beta, k^j) = k^{ij} (\alpha \partial_j \beta - \beta \partial_j \alpha),
$$

(81)

with

$$
k^{ij} = \gamma^{ij} \quad \text{or} \quad k^{ij} = e^{ij} = \gamma^{ik} \gamma^{j\ell} e_{k\ell}.
$$

(82)

The constraints in $C_1[\alpha]$ is the bi-gravity version of the diffeomorphism constraint. The first term in this constraint is the momentum constraint in general relativity, which generates the orbit of $\gamma_{ij}$, $\Pi_{ij}$ in the phase space. The second term in $C_1[\alpha]$ generates the displacement of the second set of canonical variables

$$
\{C_1[\alpha], e_{ij}\} = -\mathcal{L}_\alpha e_{ij},
$$

(83)

$$
\{C_1[\alpha], \pi^{ij}\} = -\mathcal{L}_\alpha \pi^{ij}.
$$

(84)

Following the same argument as the diffeomorphism constraint in general relativity, the Poisson brackets related to $C_1$ have the same forms.
Since the Hamiltonian is a linear combination of the secondary constraints
\[
\oint d^3x \mathcal{H}_2 = C_1[N^i] + C_2[N] + C_3[N e_0^i] + C_4[N e_0^0/2],
\]
the time derives of the secondary constraints do not lead to new constraints. The primary constraints are of first class as well, so we have 16 first class constraints in total. We can count the number of dynamical variables
\[
(10 + 10) \times 2 - 16 \times 2 = (2 + 2) \times 2,
\]
where a symmetric rank-2 tensor has 10 independent components, in the phase space each degree of freedom corresponds to two canonical variables and first class constraints eliminate two copies of independent variables. Therefore, \( \mathcal{L}_{3 \text{kin}} \) contains \((2 + 2)\) dynamical degrees of freedom, corresponding to two interacting massless gravitons\(^4\).

C. \( \mathcal{L}_{3 \text{kin}} \)

The metric \( g_{\mu \nu} \) has only two dynamical degrees of freedom because the bi-gravity kinetic terms are covariant. In \( \mathcal{L}_{2 \text{kin}} \), the 6th degrees of freedom of \( e_{\mu \nu} \) are eliminated by the additional gauge symmetries. In this subsection, we show that even though the additional gauge invariances are broken in
\[
\mathcal{L}_{3 \text{kin}} = \frac{1}{4} \sqrt{-g} R(g)_{\mu \nu [\gamma e_{\rho e_{\sigma}]} d^4x}
\]
due to higher order interaction terms, there exist two constraints which can eliminate the 6th degree of freedom of the second spin-2 field. To minimalize the number of degrees of freedom, we impose the symmetric condition \([17]\)
\[
e_{\mu \nu} = e_{\nu \mu},
\]
as part of the definition of the model. In terms of the ADM variables, the explicit expression of \( \mathcal{L}_{3 \text{kin}} \) is
\[
\gamma^{-1/2} \mathcal{L}_{3 \text{kin}} = N (\, L_1 \gamma_{ij}) (L_2 \gamma_{kl}) \left[ (M_1^{ijkl} + e_0^0 (M_2)^{ijkl} + e_0^m e_0^n (M_3)^{ijkl mn} \right] + N (L_1 \gamma_{ij} (L_2 e_{0 i}) (M_4)^{ijkl} + (L_1 \gamma_{ij}) D_k [N e_0^0 (M_5)^{ijkl}] + N R_{0} [ (M_6)^{ij} + e_0^0 (M_7)^{ij} + e_0^k e_0^l (M_8)^{ijkl}] + N D_i D_j (M_9)^{ij}, \]
where \( M_i \) are functions of the dynamical fields \( (\gamma_{ij}, e_{ij}) \)
\[
(M_1)^{ijkl} = -\frac{1}{4} \gamma^{ij}_{kl} (m_1)^{ijkl} - \frac{1}{2} (m_{1a})^{ijkl}, \quad (M_2)^{ijkl} = \frac{1}{4} \gamma^{ij}_{kl} (m_{3a})^{ijkl}, \quad (M_3)^{ijkl mn} = \frac{1}{4} \gamma^{ij}_{kl} (m_{3b})^{ijkl mn},
\]
\[
(M_4)^{ijkl} = (-\frac{1}{2}) (m_{1b})^{ijkl}, \quad (M_5)^{ijkl} = \frac{1}{2} (m_{2a})^{ijkl}, \quad (M_6)^{ij} = 0, \quad (M_7)^{ij} = (m_{3a})^{ij}, \quad (M_8)^{ijk l} = (m_{3b})^{ijk l}, \quad (M_9)^{ij} = -(m_1)^{ij},
\]
and
\[
(m_1)^{ijkl} = \gamma^{ij}_{kl} e_k [e_i l] - 2 e^{ij}_{kl} e_{k l}, \quad (m_{1a})^{ijkl} = - \frac{1}{2} [ e_m^m (\gamma_{ij} e^{ij}_{kl} - \gamma^{ik} e^{jl}) + e^{ik} e^{jl} + \gamma^{ik} e^{jm} e_{ij} + \gamma^{ij} e^{km} e_m l \right], \quad (m_{1b})^{ijkl} = 2 [ e_m^m (M_0)^{ijkl} + (\gamma^{ik} e^{jl} + e^{ik} \gamma^{jl}) - (\gamma^{ij} e^{kl} + e^{ij} \gamma^{kl})], \quad (m_{2a})^{ijkl} = (\gamma^{ik} e^{jl} - \gamma^{jl} e^{ik}) e_m m + e^{ij} \gamma^{ij}, \quad (m_{3a})^{ij} = \gamma^{ij} e_i l - 2 e^{ij}, \quad (m_{3b})^{ijkl} = 2 (M_0^{-1})^{ijkl},
\]
\[
(\partial \mu [N \gamma^{1/2} \lambda^\mu K_{ij} (m_1)^{ij}])
\]
to cancel the second order time derivative terms.

We can see the explicit expression of \( \mathcal{L}_{3 \text{kin}} \) is considerably more complicated than that of \( \mathcal{L}_{2 \text{kin}} \). Nevertheless, we can derive the corresponding Hamiltonian
\[
\mathcal{H}_3 = \dot{\gamma}_{ij} \Pi^{ij} + \dot{e}_{ij} \pi^{ij} - \mathcal{L}_3 = \mathcal{H}_2 + \frac{1}{2} \gamma^{-1/2} (\pi M_4^{-1})_{ij} (\pi M_4^{-1})_{kl} (M_2)^{ijkl} + \gamma^{1/2} R_{ij} (M_7)^{ij} + (\pi M_4^{-1})_{ij} D_k [N e_0^0 (M_5)^{ijkl}] + N e_0^m e_0^n [\gamma^{-1/2} (\pi M_4^{-1})_{ij} (\pi M_4^{-1})_{kl} (M_3)^{ijkl mn} + \gamma^{1/2} R_{ij} (M_8)^{ijkl mn} - N \gamma^{-1/2} (\pi M_4^{-1})_{ij} (\pi M_4^{-1})_{kl} (M_1)^{ijkl} + N \gamma^{-1/2} (\pi M_4^{-1})_{ij} - N \gamma^{-1/2} (D_i N_j + D_j N_i) + \Pi^{ij} (D_i N_j + D_j N_i) + \pi^{ij} (N_k D_k e_{ij} + e_{kj} D_i N_k + e_{ik} D_j n_k),
\]
\[
(102)
\]
\[^4\text{The no-go theorem [33] is evaded because one of the spin-2 kinetic terms has a wrong sign. Another way to evade this no-go theorem is to introduce additional fields [32].}\]
where the conjugate momenta are
\[ \Pi^{ij} = \frac{\partial L_3}{\partial \dot{e}^{ij}}, \quad \pi^{ij} = \frac{\partial L_3}{\partial \dot{c}^{ij}} = \sqrt{\gamma} \left( E_n \gamma_{kl} \right) (M_4)^{ijkl}, \] (103)
a shorthand notation is used
\[ (\pi M_4^{-1})_{mn} = \pi^{ij} (M_4^{-1})_{ijmn}, \] (104)
and \( M_4^{-1} \) is the inverse of \( M_4 \)
\[ (M_4)^{ijkl} (M_4^{-1})_{klmn} = \delta^i_m \delta^j_n. \] (105)

The main difference between the Hamiltonians of \( \mathcal{L}_2 \) and \( \mathcal{L}_3 \) is that \( e_0^i \) are not Lagrange multipliers in the latter case. This is due to the fact that \( \mathcal{L}_3 \) has less gauge symmetries. We also expect some secondary constraints of \( \mathcal{L}_3 \) are of second class.

The primary constraints come from the variables that do not have time-derivative terms
\[ \Pi_N = \Pi_N = \pi_{e_0 \gamma} = \pi_{e_0 \dot{c}} = 0. \] (106)
The corresponding total Hamiltonian is
\[ \mathcal{H}_3 = \mathcal{H}_3 + \lambda^i_1 \Pi N^i + \lambda^i_2 \Pi e_0^i, \] (107)
where \( \lambda^i_1, \lambda^i_2 \) are Lagrange multipliers associated with the primary constraints.

Secondary constraints are obtained from the requirement that primary constraints are preserved in time. In the smeared form, the secondary constraints are
\[ \mathcal{C}_1[\alpha] = \int d^3 x \left( \Pi^{ij} L_\alpha \gamma_{ij} + \pi^{ij} L_\alpha \dot{e}_{ij} \right), \] (108)
\[ \mathcal{C}_2[\alpha] = \int d^3 x \left\{ \frac{1}{2} (\pi M_4^{-1})_{ij} D_k \left[ \alpha e_0^i (M_5)^{ijkl} \right] + \alpha \gamma^{-1/2} (\pi M_4^{-1})_{ij} (\pi M_4^{-1})_{kl} (M_4)^{ijkl} - \alpha \gamma^{-1/2} \Pi^{ij} (\pi M_4^{-1})_{ij} + \alpha \gamma^{1/2} D_i D_j (M_9)^{ij} \right\}, \] (109)
\[ \mathcal{C}_3[\alpha] = \int d^3 x \left\{ (\pi M_4^{-1})_{ij} D_k \left[ \alpha^l (M_5)^{ijkl} \right] +2 \alpha^m e_0^m \left[ \gamma^{-1/2} (\pi M_4^{-1})_{ij} (\pi M_4^{-1})_{kl} (M_4)^{ijkl} \right] \right\}, \] (110)
\[ \mathcal{C}_4[\alpha] = \int d^3 x \left( \gamma^{-1/2} (\pi M_4^{-1})_{ij} (\pi M_4^{-1})_{kl} (M_2)^{ijkl} + \gamma^{1/2} R_{ij} (M_7)^{ij} \right), \] (111)

where \( \alpha(x) \) and \( \alpha^i(x) \) are test functions.

Since \( \mathcal{C}_3 \) involves the non-dynamical variables \( e_0^i \), one can in principle solve this equation and express \( e_0^i \) in terms of the dynamical variables \( \gamma_{ij}, \epsilon_{ij}, \pi^{ij} \). We will not proceed in this way to avoid making \( \mathcal{H}_3 \) highly nonlinear in \( \pi \) and simply think of \( \mathcal{C}_3 \) as one of the secondary constraints arising from the stability of primary constraints.

The Hamiltonian is a linear combination of the secondary constraints
\[ \int d^3 x \mathcal{H}_3 = \mathcal{C}_1[N^i] - \mathcal{C}_2[N] - \mathcal{C}_3[N e_0^i / 2] - \mathcal{C}_4[N e_0^0], \] (112)

To examine the existence of tertiary constraints, we should compute time derivatives of secondary constraints. They are the Poisson brackets of the total Hamiltonian and secondary constraints. If they do exist, we should compute the time derivatives of the tertiary constraints and repeat the same step until no independent constraints are found.

Since the diagonal diffeomorphism invariance is not broken in \( \mathcal{L}_3 \), there should be 8 first class constraints related to 4 gauge transformations. They eliminate most of the dynamical variables in \( g_{\mu \nu} \) and only 2 degrees of freedom are propagating dynamically.

For the components of the second symmetric spin-2 field \( e_{\mu \nu} \), we know 4 of them are not dynamical, so \( e_{\mu \nu} \) contains at most 6 dynamical degrees of freedom. Let us remind the reader that at the linearized level \( \mathcal{L}_3 \) reduces to two linearized Einstein-Hilbert terms. But there are more dynamical degrees of freedom in \( e_{\mu \nu} \) at the nonlinear level because the second copy of gauge symmetries are broken by high order interactions. This is analogous to massive gravity, where the gauge symmetries of the linearized kinetic term are broken by mass terms. The difference is that the additional gauge symmetries in \( \mathcal{L}_3 \) are broken by high order interaction terms, rather than mass terms.

In massive gravity, the 6th degree of freedom is known to be ghost-like, which is called the Boulware-Deser ghost. One may suspect that in \( \mathcal{L}_3 \) the 6th degree of freedom of \( e_{\mu \nu} \) is also dangerous. In fact, \( \mathcal{L}_3 \) should be supplemented by some mass terms, otherwise the helicity-1 modes in \( e_{\mu \nu} \) will become strongly coupled due to the lack of kinetic terms. Therefore, \( \mathcal{L}_3 \) is a natural kinetic term for a massive spin-2 field, together with a massless one. The 6th degree of freedom in \( e_{\mu \nu} \) is closely related to the Boulware-Deser ghost.

The ghost-like 6th degree of freedom in \( \mathcal{L}_3 \) should not be propagating. What are the constraints that can eliminate this dangerous degree of freedom? The first one can be easily identified with the secondary constraint \( \mathcal{C}_4[\alpha] \), which is generated by the time derivative of the primary constraint \( \pi_{e_0^0} = 0 \).

To eliminate the 6th degree of freedom, we need one more constraint if \( \mathcal{C}_4[\alpha] \) is related to a second class constraint. In dRGT massive gravity, the BD ghost is eliminated by a

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5 Therefore, \( e_0^i \) are not gauge parameters. As \( \mathcal{C}_3 \) does not generate gauge symmetry, we expect \( \mathcal{C}_3 \) is related to second class constraint.

6 To count the number of degrees of freedom, we need to derive all the independent constraints from the stability of primary constraints. Then we should compute the Poisson brackets among the constraints and diagonalize them to determine the numbers of first class and second class constraints.
We expect that the tertiary constraint generated by the time derivative of $C_4[\alpha]$ is the additional constraint we are looking for. So our goal is to show that the smeared tertiary constraint

$$C_5[\alpha] = \{C_4[\alpha], \int d^3x \mathcal{H}_3(x)\} \approx 0,$$

(113)
is an independent constraint and does not fix $(N, N^i, e_0^0)$. In this way, it eliminates at least one more dynamical variable in the phase space.

It is straightforward to derive $C_5[\alpha]$. The result is length because $M_i$ have complicated dependence on $\gamma_{ij}$ and $e_{ij}$. But the real obstacle is that, from the explicit expression of $C_5[\alpha]$, we cannot immediately figure out whether the tertiary constraint is an equation for the dynamical canonical variables, which should not determine $N, N^i$ or $e_0^0$.

This obstacle stems from the fact that the matrix $M_4^{-1}$ in the Hamiltonian $\mathcal{H}_3$ does not have a closed form expression. In particular, the canonical momentum $\pi_{ij}$ is always contracted with $M_4^{-1}$ because when we invert the relation between velocity $\dot{\gamma}_{ij}$ and momentum $\pi_{ij}$, $M_4^{-1}$ is generated, making the obstacle more formidable.

To derive the result of Poisson brackets, we also need to compute the variation of $M_4^{-1}$ with respect to $\gamma_{ij}$ and $e_{ij}$. We make use of the identity below

$$\delta(M_4^{-1})_{ijkl} = (-)(M_4^{-1})_{ijab} \delta(M_4)_{abcd}(M_4^{-1})_{cdkl}. \quad (114)$$

which can be derived from the definition \[105\] of $M_4^{-1}$ and a symmetric property of $M_4$

$$(M_4)_{ijkl} = (M_4)_{klij}. \quad (115)$$

Therefore, $C_5[\alpha]$ contains numerous terms involving $M_4^{-1}$ and with more complicated index contraction than the terms in the secondary constraints. It is not clear how to avoid the unwanted terms

$$\{\alpha \partial N\}(\ldots), (N \partial \alpha)(\ldots), \quad (116)$$

$$\{\alpha D \partial N\}(\ldots), (N D \partial \alpha)(\ldots), \quad (117)$$
in the result. For example, some typical unwanted terms are

$$\{\alpha \partial N\} \left[ \gamma^{1/2} e^{0i} e_{ij} (M_4^{-1})_{kilm} (M_4^{-1}) \pi^{kl} (M_4^{-1}) \pi^f \right], \quad (118)$$

We will not carry out the full procedure, but only show that there are at least two additional constraint equations for the $(10 - 4) \times 2$ dynamical variables of $e_{ij\nu}$. The number of independent constraints remains the same after the diagonalization. The two additional constraints may be related to first-class or second-class constraints, but at least 1 degree of freedom in the 6 dynamical degrees of freedom of $e_{ij\nu}$ is eliminated.

$N, N^i$ are gauge parameters, so they should be arbitrary. If this constraint fixes $e_0^0$, then this is an equation for $e_0^0$ and does not eliminate the second variable of the 6th degree of freedom.

The unwanted terms do not involve $N^i$ and $e_0^0$ due to the following brackets

$$\{C_4[\alpha], C_i[\beta]\} = C_4[-\mathcal{D}_i\alpha], \quad (122)$$

$$\{C_4[\alpha], C_i[\beta]\} = 0. \quad (123)$$

Using these two brackets, the tertiary constraint becomes

$$C_5[\alpha] = C_4[-\mathcal{D}_i\alpha] - \{C_4[\alpha], C_2[N] + C_3[N e_0^0 / 2]\}, \quad (124)$$

where the second term generates the unwanted derivative terms of $\alpha$ and $N$ mentioned above.

To make one step further, we notice that in the single metric limit\[8\]

$$e_{ij} \rightarrow \gamma_{ij}, \quad (125)$$

$M_4^{-1}$ has a closed form expression

$$(M_4^{-1})_{ijkl} \rightarrow -(M_0^{-1})_{ijkl}, \quad (M_4^{-1})_{ijkl} \rightarrow -(M_0^{-1})_{ijkl}. \quad (126)$$

In this limit, the second term of [124] is simplified

$$\{C_4[\alpha], C_2[N] + C_3[N e_0^0 / 2]\} \bigg|_{e_{ij} \rightarrow \gamma_{ij}} = \int d^3x \left\{ (\alpha N)(\pi^{ij} + 2\Pi^{ij})(R_{ij} - \frac{1}{4} \gamma_{ij} R) + (\alpha N) \left( \frac{1}{2} e_0^0 e_0^0 \pi R_{ij} - \frac{1}{2} e_0^0 e_0^0 \pi R + e_0^0 e_0^0 \pi^{ijkl} R_{ijkl} - 2e_0^0 e_0^0 \pi^k R_{jk} + \frac{3}{4} e_0^0 e_0^0 \pi^k R_{ijkl} \right) + 2(\alpha N) \gamma^{1/2} (R_{ij} - \frac{1}{2} R \gamma_{ij}) D^i e^{0j} + \gamma^{-1/2} \delta_K(\alpha N) \left[ \left( \frac{1}{8} \pi^2 - \frac{1}{2} \pi \pi^{ij} \right) e_0^0 k - \frac{1}{2} (\pi^k - \frac{1}{2} \pi \pi^k) e_0^0 j \right] + \gamma^{-1} (\alpha N) \left[ - \frac{1}{32} \pi^3 + \frac{3}{16} \pi \pi \pi \pi^{ij} - \frac{1}{4} \pi \pi^k \pi^k \pi^k + e_0^0 e_0^0 \left( - \frac{1}{16} \pi^3 + \frac{1}{4} \pi \pi \pi \pi^{ij} - \frac{1}{4} \pi \pi^k \pi^k \pi^k \right) + e_0^0 e_0^0 \left( \frac{1}{16} \pi^3 + \frac{1}{4} \pi \pi \pi \pi^{ij} - \frac{1}{4} \pi \pi^k \pi^k \pi^k \right) - \frac{3}{8} \pi^k \pi_{jk} \pi k + \frac{5}{32} \pi^k \pi^k \right] \right. \left. - \Pi \left( \frac{1}{16} \pi^2 - \frac{1}{8} \pi \pi \pi \pi^{ij} \right) + \Pi^j \left( \frac{1}{2} \pi \pi \pi \pi^{ij} - \frac{1}{2} \pi \pi \pi \pi^{ij} \right) \right\} \bigg|_{e_{ij} \rightarrow \gamma_{ij}}, \quad (127)$$

This is a trivial single metric limit without a fixed fiducial metric.
where the unwanted terms simply organize into
\[ \frac{1}{2} C_3 \hat{f} (\alpha, N, \gamma) \bigg|_{\epsilon ij \rightarrow \gamma ij} . \tag{128} \]

We want to emphasize that the single metric limit is taken after the Poisson bracket is computed.

In the single metric limit, it is clear that
- \( C_3 [\alpha] \) is an independent constraint which involves cubic momentum terms \( \pi \pi \pi, \pi \pi \pi \Pi \),
- and \( C_5 [\alpha] \) does not fix \( N \) in terms of the dynamical variables because \( N \) is always multiplied by the test function \( \alpha \).

We show that in the subspace of the phase space where \( \epsilon ij \) and \( \gamma ij \) coincide, the tertiary constraint \( C_5 [\alpha] \) is an equation for the dynamical variable. Together with the secondary \( C_4 [\alpha] \), the tertiary constraint \( C_5 [\alpha] \) eliminates the 6th degree of freedom.

The spirit of the single metric limit is similar to the minispacer approximation: we consider a special subspace of the phase space where the nonlinear structure is considerably simplified and tests at the nonlinear level are possible. The validities of certain statements in these subspaces are only necessary conditions, but non-trivial and beyond the linearized level.

Interestingly, all the unwanted terms are absorbed into the secondary constraint \( C_3 \), which might be true beyond the single metric limit. To verify this, we make use of the explicit definition of \( M_4^{-1} \)
\[ \left( [-\epsilon \mu]^m (M_0)^{ijkl} + (\gamma^{ik} e^{jl} + e^{ik} \gamma^{jl})
- (\gamma^{ij} e^{kl} + e^{ij} \gamma^{kl}) \right) (M_4^{-1})_{klmn} = \delta_m^i \delta_n^j \tag{129} \]
which can reduce the number of \( M_4^{-1} \) in \( C_5 [\alpha] \). After a long computation, the unwanted terms (\( \sim 200 \) terms) do reduce to
\[ -\frac{1}{2} C_3 \hat{f} (\alpha, N, \gamma) \bigg|_{\epsilon ij \rightarrow \gamma ij} . \tag{130} \]
without taking the single metric limit. Therefore, the independent constraint in \( C_5 [\alpha] \) reads
\[ C_5 [\alpha] = \left\{ C_4 [\alpha / N], \int d^3 x \mathcal{H}^T \right\} + C_4 [\mathcal{L} \bar{N} (\alpha / N)] - \frac{1}{2} C_3 \hat{f} (\alpha / N, N, \gamma) \approx 0. \tag{131} \]
The 6th degree of freedom of \( \epsilon_{\mu\nu} \) is eliminated by
\[ C_4 [\alpha] \approx 0, \quad C_5 [\alpha] \approx 0. \tag{132} \]

IV. GENERAL SITUATION

Now we are ready to discuss the general situation, where the Lagrangian is a linear combination of three kinetic terms and five potential terms
\[ \mathcal{L} = a_1 \mathcal{L} \text{EH} + a_2 \mathcal{L} \text{kin} + a_3 \mathcal{L} \text{kin} + c_1 \mathcal{L} \text{pot} + c_2 \mathcal{L} \text{pot} + c_3 \mathcal{L} \text{pot} + c_4 \mathcal{L} \text{pot} + c_5 \mathcal{L} \text{pot} , \tag{133} \]
and at least one of the coefficients of the novel kinetic term is not zero
\[ a_2 \neq 0 \quad \text{or} \quad a_3 \neq 0. \tag{134} \]

As we discuss below, there are 2 types of bi-gravity models, which extend the main features of the two examples \( \mathcal{L} = \mathcal{L} \text{kin} + \mathcal{L} \text{kin} \).

The definitions of the kinetic terms are the same as those in the examples
\[ \mathcal{L} \text{pot} = \frac{\epsilon^{\mu\nu}}{\sqrt{\gamma}} \mathcal{L} \text{pot}, \tag{138} \]
\[ \mathcal{L} \text{pot} = \frac{\epsilon^{\mu\nu}}{\sqrt{\gamma}} \mathcal{L} \text{pot}, \tag{139} \]
\[ \mathcal{L} \text{pot} = \frac{\epsilon^{\mu\nu}}{\sqrt{\gamma}} \mathcal{L} \text{pot}, \tag{140} \]
\[ \mathcal{L} \text{pot} = \frac{\epsilon^{\mu\nu}}{\sqrt{\gamma}} \mathcal{L} \text{pot}, \tag{141} \]
\[ \mathcal{L} \text{pot} = \frac{\epsilon^{\mu\nu}}{\sqrt{\gamma}} \mathcal{L} \text{pot}. \tag{142} \]

In the local ADM frame, the explicit expression of the general Lagrangian is
\[ \gamma^{-1/2} \mathcal{L} = N (\mathcal{L}_n \gamma_{ij})(\mathcal{L}_n \gamma_{kl}) \left[ (M_1)^{ijkl} + e_0^0 (M_2)^{ijkl} 
+ e_0^m e_0^n (M_3)^{ijkl}_{mn} \right] \]
\[ + N (\mathcal{L}_n \gamma_{ij})(\mathcal{L}_n \gamma_{kl}) (M_4)^{ijkl} 
+ (\mathcal{L}_n \gamma_{ij}) D_k [N e_0^l (M_5)^{ijkl}_{j} ] \]
\[ + N R_{ij} (M_6)^{ijkl} + e_0^0 (M_7)^{ijkl} + e_0^k e_0^l (M_8)^{ijkl}_{kl} \tag{143} \]
\[ + N D_i D_j (M_9)^{ijkl} \]
\[ + N \left[ e_0^0 M_{10} + e_0^i e_0^j (M_{11})_{ij} + M_{12} \right] , \tag{144} \]
which is explicitly linear in \( N, N^i, e_0^0 \).

The matrices \( M_i \) are functions of the dynamical fields \( \gamma_{ij}, e_{ij} \)
\[ (M_1)^{ijkl} = -\frac{1}{4} \gamma^{ij} (M_1)^{kl} - \frac{1}{2} (M_1)^{ijkl} + \frac{1}{4} \gamma^{ijkl} (M_3)^{ijkl}, \tag{145} \]
\[ (M_2)^{ijkl} = \frac{1}{4} \gamma^{ijkl} (M_3)^{ijkl}, \tag{146} \]
\[ (M_3)^{ijkl}_{mn} = \frac{1}{4} \gamma^{ijkl} (M_3)^{ijkl}_{mn}, \tag{147} \]
\[ (M_4)^{ijkl} = -\frac{1}{2} (M_1)^{ijkl}, \quad (M_5)^{ijkl} = \frac{1}{2} (M_2)^{ijkl}, \tag{148} \]
\[(M_0)^{ij} = (m_{3a})^{ij}, \quad (M_2)^{ij} = (m_{3a})^{ij}, \quad (M_0)^{ij}_{kl} = (m_{3b})^{ij}_{kl}, \quad (M_0)^{ij} = -(m_1)^{ij}, \]
\[M_{10} = \sum_{n=2}^{5} c_n (n-1) e_i [i_1 \ldots e_{i_{n-2}}], \]
\[(M_{11})_{kl} = \sum_{n=3}^{5} c_n (n-1)(n-2) e_i [i_1 \ldots e_{i_{n-3}}], \]
\[M_{12} = \sum_{n=1}^{4} c_n e_i [i_1 \ldots e_{i_{n-1}}], \]
and
\[(m_1)^{ij} = 2a_1 \gamma^{ij} - a_2 (M_0)^{ijkl} e_{kl}
+ a_3 (\gamma^{ij} e_k e_i - 2e_i^{kl} e^k), \quad (m_{1a})^{ijkl} = a_2 (\gamma^{ij} e^k e^l - \gamma^{ik} e^j), \]
\[2a_3 \left[ e_{m}^{ij} (\gamma^{ij} e^k e_l - \gamma^{ik} e^j) + e^{ik} e^{jl} + e^{jm} e^m - e^{ij} e^k e^l + \gamma^{ij} e^{km} e^m \right], \]
\[(m_{1b})^{ijkl} = (-a_2) (M_0)^{ijkl}, \]
\[-2a_3 \left[ e_{m}^{ij} (\gamma^{ij} e^k e_l - \gamma^{ik} e^j) + e^{ik} e^{jl} + e^{jm} e^m - e^{ij} e^k e^l + \gamma^{ij} e^{km} e^m \right], \]
\[(m_{2a})^{ijkl} = (-a_2) (M_0)^{ijkl}, \]
\[4a_3 (\gamma^{ik} e^j - \gamma^{k} e^{ij} e_{m}^{m} + e^{kl} e^{jl}), \]
\[(m_{3a})^{ij} = -\frac{1}{2} a_2 \gamma^{ij} + 2a_3 (M_0^{-1})^{ijkl} e_{kl}, \]
\[(m_{3b})^{ijkl} = 2a_3 (M_0^{-1})^{ijkl}, \]
\[(m_{3c})^{ij} = a_1 \gamma^{ij} - a_2 (M_0^{-1})^{ijkl} e_{kl}, \]

The Hamiltonian is derived by the Legendre transform
\[\mathcal{H} = \dot{\gamma}^{ij} \Pi^{ij} + \dot{e}_{ij} \pi^{ij} - \mathcal{L} = (-) N e_0^2 \left\{ \gamma^{-1/2} (\pi M_4^{-1})_{ij} (\pi M_4^{-1})_{kl} (M_3)^{ijkl} + \gamma^{1/2} R_{ij} (M_7)^{ij} + M_{10} \right\}
- (\pi M_4^{-1})_{ij} D_k [N e_0^2 (M_5)^{ijk}], \]
\[-N e_0^m e_0^n \left\{ \gamma^{-1/2} (\pi M_4^{-1})_{ij} (\pi M_4^{-1})_{kl} (M_5)^{ijkl} m n + \right\}
\[+ \gamma^{1/2} R_{ij} (M_5)^{ij} m n + \gamma^{1/2} (M_1)^{mn} + N \gamma^{-1/2} (\pi M_4^{-1})_{ij} (\pi M_4^{-1})_{kl} (M_4)^{ijkl} \]
\[+ N \gamma^{-1/2} \Pi^{ij} (\pi M_4^{-1})_{ij} - N \gamma^{1/2} R_{ij} (M_4)^{ij} \]
\[+ N \gamma^{1/2} D_i D_j (M_5)^{ij} - N \gamma^{1/2} M_{12} \]
\[+ \Pi^{ij} (D_i N_j + D_j N_i)
+ \pi^{ij} (N k D_k e_{ij} + e_{kj} D_i N_k + e_{ik} D_j N_k), \]

where a shorthand notation is used
\[(\pi M_4^{-1})_{mn} = \pi^{ij} (M_4^{-1})_{ijmn}, \]

and \(M_4^{-1}\) is the inverse of \(M_4\)

\[(M_4)^{ijkl} (M_4)^{klmn} = \delta_{m}^{i} \delta_{n}^{j}. \]

The conjugate momenta are defined as
\[\Pi^{ij} = \frac{\partial \mathcal{L}}{\partial \dot{\gamma}^{ij}}, \quad \pi^{ij} = \frac{\partial \mathcal{L}}{\partial \dot{e}_{ij}} = \sqrt{\gamma} (\mathcal{L} n \gamma_{kl}) (M_4)^{ijkl}, \]
\[\Pi_{N^\mu} = \frac{\partial \mathcal{L}}{\partial \dot{N}^\mu} = 0, \quad \pi_{e_0^\mu} = \frac{\partial \mathcal{L}}{\partial \dot{e}_{0^\mu}} = 0, \]

where the last line contains 8 primary constraints.

To encode the information of primary constraints, we introduce the total Hamiltonian
\[\mathcal{H}^T = \mathcal{H} + \lambda_1^a \Pi_{N^\mu} + \lambda_2^a \Pi_{e_0^\mu}, \]

where \(\lambda_1^a, \lambda_2^a\) are Lagrange multipliers associated with 8 primary constraints.

The total Hamiltonian is a linear combination of constraints, so it vanishes on the constraint surface
\[\mathcal{H}^T \approx 0. \]

According to their Hamiltonian structures, the 4d bi-gravity models involving novel kinetic terms are classified into two types
\[9\]

9 This classification can be easily generalized to higher dimensions, where more derivative terms are allowed. In Type A models, there is at most one \(F\) vielbein in the wedge products.
are of first-class. There are no more independent constraints. Let us define the smeared secondary constraint as

\[
C_1[\bar{\alpha}] = \frac{1}{a_2} \int d^3 x \alpha^i(x) \frac{\delta}{\delta N^i(x)} \int d^3 y \mathcal{H}(y),
\]

\[
C_2[\alpha] = \frac{1}{a_2} \int d^3 x \frac{\alpha^i(x) \delta}{\delta N(x)} \int d^3 y \mathcal{H}(y)
- C_3[N_0^i] - C_4[N_0^0 / 2],
\]

\[
C_3[\bar{\alpha}] = \frac{1}{a_2} \int d^3 x \frac{\alpha^i(x) \delta}{\delta N(x) \delta e_0^i(x)} \int d^3 y \mathcal{H}(y),
\]

\[
C_4[\alpha] = \frac{1}{a_2} \int d^3 x \frac{2 \alpha(x) \delta}{\delta e_0^0} \int d^3 y \mathcal{H}(y)
\]

Their Poisson brackets are the same as those in the case of $\mathcal{L}_2^\text{kin}$

\[
\{C_1[\bar{\alpha}], C_1[\bar{\beta}]\} = C_1[\mathcal{L}_{\bar{\alpha}} \bar{\beta}],
\]

\[
\{C_1[\bar{\alpha}], C_2[\beta]\} = C_2[\mathcal{L}_{\bar{\alpha}} \beta],
\]

\[
\{C_1[\bar{\alpha}], C_3[\bar{\beta}]\} = C_3[\mathcal{L}_{\bar{\alpha}} \bar{\beta}],
\]

\[
\{C_1[\bar{\alpha}], C_4[\beta]\} = C_4[\mathcal{L}_{\bar{\alpha}} \beta],
\]

\[
\{C_2[\alpha], C_2[\beta]\} = C_1[\mathcal{f}(\alpha, \beta, \gamma) - C_3[\mathcal{f}(\alpha, \beta, e)],
\]

\[
\{C_3[\bar{\alpha}], C_3[\bar{\beta}]\} = C_4[\mathcal{L}_{\bar{\alpha}} \bar{\beta}],
\]

\[
\{C_4[\alpha], C_2[\beta]\} = C_3[\mathcal{f}(\alpha, \beta, \gamma)],
\]

\[
\{C_3[\bar{\alpha}], C_3[\bar{\beta}]\} = \{C_3[\bar{\alpha}], C_4[\beta]\} = \{C_4[\alpha], C_4[\beta]\} = 0,
\]

where the structure functions are given by

\[
f^i(\alpha, \beta, k^{ij}) = k^{ij} (\alpha \partial_j \beta - \beta \partial_j \alpha).
\]

The Poisson brackets involving primary constraints vanish. Therefore, both the primary and the secondary constraints are first class constraints.

Type A models have $(2 + 2)$ dynamical degrees of freedom corresponding to two massless gravitons. The gauge symmetries are

- diagonal diffeomorphism invariance
  \[
  \delta g_{\mu \nu} = L \xi g_{\mu \nu}, \quad \delta e_{\mu \nu} = L \xi e_{\mu \nu},
  \]

- additional “diffeomorphism invariance”
  \[
  \delta e_{\mu \nu} = L \xi g_{\mu \nu},
  \]

- diagonal local Lorentz invariance
  \[
  \delta F^A_{\mu} = \omega^A_B F^B_{\mu}, \quad \delta F^A_{\mu} = \omega^A_B F^B_{\mu},
  \]

- additional “local Lorentz invariance”
  \[
  \delta F^A_{\mu} = \omega^A_B F^B_{\mu},
  \]

where $\xi^\mu$, $\xi'^\mu$ are four vectors and $\omega^A_B$, $\omega'^A_B$ are antisymmetric. At the linear level, the two diffeomorphism invariances reduce to two sets of linearized symmetries of the Lagrangians, which consists of two decoupled linearized Einstein-Hilbert kinetic terms.

In the minisuperspace approximation, the right hand side of

\[
\{C_4[\alpha], C_2[\beta]\} = C_3[\mathcal{f}(\alpha, \beta, \gamma)]
\]

vanishes. It was speculated in [13] that the two commuting Hamiltonian-like constraints are of first-class. Here we can see this statement is indeed true and we have two sets of first class constraints.

The nonlinear models in Type A have the same amount of dynamical degrees of freedom as the linearized theories, which contain two free massless spin-2 fields whose kinetic terms have opposite signs.

**B. Type B**

In Type B models the additional gauge symmetries are broken by the derivative interaction of $\mathcal{L}_3^\text{kin}$ or the potential interactions $\mathcal{L}_3^\text{pot}$, $\mathcal{L}_4^\text{pot}$, $\mathcal{L}_5^\text{pot}$. A direct consequence is that $\epsilon_0^i$ are not Lagrange multipliers in the Hamiltonian.

The smeared secondary constraints of Type B models are

\[
C_1[\bar{\alpha}] = \int d^3 x \left( \Pi^{ij} \mathcal{L}_{\bar{\alpha}} \gamma_{ij} + \pi^{ij} \mathcal{L}_{\bar{\alpha}} e_{ij} \right),
\]
We conclude that the constraint.

According to the definition of \( D_3 \), we determine the smeared tertiary constraint

\[ \mathcal{C}_3[\alpha] = \int d^3 x \left\{ \left( \pi M_4^{-1} \right)_{ij} D_k \left[ \alpha \varepsilon_{0} (M_5)_{ij}^{jk} \right] + \alpha \gamma^{-1/2} (\pi M_4^{-1})_{ij} (\pi M_4^{-1})_{kl} (M_1)_{ijkl} - \alpha \gamma^{-1/2} \Pi^{ij} (\pi M_4^{-1})_{ij} + \alpha \gamma^{1/2} R_{i j} (M_6)_{ij} \right. \]

\[ + \left. \alpha^{1/2} D_i N D_j (M_9)_{ij} + \alpha^{1/2} M_{12} \right\}, \quad (192) \]

As before, the changes are the Lie derivatives of the canonical variables

\[ C \equiv \left( \gamma^{-1/2} (\pi M_4^{-1})_{ij} (\pi M_4^{-1})_{kl} (M_1)_{ijkl} \right)_{mn} \]

\[ + \gamma^{1/2} R_{ij} (M_8)_{ij} + \gamma^{1/2} (M_11)_{mn} \}, \quad (193) \]

Now let us examine the smeared tertiary constraint

\[ C_3[\alpha] = \{ C_3[\alpha], \int d^3 x \mathcal{H}(x) \} \approx 0. \quad (195) \]

We expect that \( C_3[\alpha] \) is an independent constraint for the dynamical variables.

According to the definition of \( C_3[\alpha] \), we compute the complete Poisson brackets, which is a lengthy result. When \( \alpha_3 \neq 0 \), \( C_3[\alpha] \) contains cubic momentum terms, so it cannot be written as a linear combination of the secondary constraints. When \( \alpha_3 = 0 \), no cubic momentum term appears in \( C_3[\alpha] \), but there is no apparent way to express \( C_3[\alpha] \) in terms of other constraints. We conclude that \( C_3[\alpha] \) is an independent constraint.

As before, \( C_3[\alpha] \) contains a lot of unwanted terms

\[ \left( \alpha \partial N \right)(\ldots), \left( N \partial \alpha \right)(\ldots), \quad (197) \]

\[ \left( \alpha D \partial N \right)(\ldots), \left( ND \partial \alpha \right)(\ldots), \quad (198) \]

but we expect them to be absorbed into

\[ \frac{1}{2} C_3[f(\alpha, N, \gamma)]. \quad (199) \]

It is difficult to directly check this statement in the general case due to a proliferation of terms in \( C_5[\alpha] \). However, in the single metric limit \( \varepsilon_{ij} \rightarrow \gamma_{ij} \), the unwanted terms are greatly simplified. In this limit, we are able to verify that they do organize into \( C_3 \).

Here we would like to discuss some general properties of the diffeomorphism constraint. We can see the secondary constraints in \( C_3 \) determine \( \varepsilon_0^0 \) in terms of the canonical variables \( \gamma_{ij}, \varepsilon_{ij}, \pi^{ij} \). In addition, the quaternary constraint

\[ C_4[\alpha] = \{ C_4[\alpha], \int d^3 x \mathcal{H}(x) \} \approx 0 \quad (200) \]

is usually an equation for \( \varepsilon_0^0 \) because the Poisson bracket of \( C_4 \) and \( C_5 \) does not vanish. Since \( \varepsilon_0^0, \varepsilon_0^i \) are not arbitrary functions, the diffeomorphism constraint is modified such that its Poisson brackets with other constraints only change the test functions. In the smeared form, the diffeomorphism constraint should be

\[ C_{\text{diff}}[\alpha] = C_1[\alpha] + \int d^3 x \left( \pi_{\varepsilon_0^0} \mathcal{L}_\alpha \varepsilon_0^0 + \pi_{\varepsilon_0^i} \mathcal{L}_\alpha \varepsilon_0^i \right). \quad (202) \]

Then \( C_{\text{diff}}[\alpha] \) generates an orbit in the phase space

\[ \{ C_1[\alpha], A \} = -\mathcal{L}_\alpha A, \quad (203) \]

where \( A \) can be the canonical variables

\[ A = \gamma_{ij}, \varepsilon_{ij}, \varepsilon_0^0, \varepsilon_0^i, \Pi^{ij}, \pi^{ij}, \pi_{\varepsilon_0^0}, \pi_{\varepsilon_0^i}, \]

but not the gauge parameters and their conjugate momenta. The changes are the Lie derivatives of the canonical variables along \( \alpha \). After integrating by parts, we have

\[ \{ C_{\text{diff}}[\alpha], C_1[\beta] \} = C_1[\mathcal{L}_\alpha \beta], \quad i = 1, 3 \quad (205) \]

and

\[ \{ C_{\text{diff}}[\alpha], C_1[\beta] \} = C_1[\mathcal{L}_\alpha \beta], \quad i = 2, 4. \quad (206) \]

Therefore, the linear combinations of the primary and secondary constraints in \( C_{\text{diff}}[\alpha] \) are first class constraints. This is analogous to the Hamiltonian structure of spatially covariant gravity, where one of the first class constraints is identified with a linear combination of the momentum constraint and the lapse primary constraint.

Now we continue the counting of dynamical degrees of freedom. We have two constraint equations \( C_4 \) and \( C_5 \) for

\[ \{ C_4[\alpha], A \} = -\mathcal{L}_\alpha A, \quad (203) \]

However, the Poisson bracket \( \{ C_4[\alpha], C_5[\beta] \} \) vanishes, so \( \varepsilon_0^0 \) remains arbitrary. The diffeomorphism constraint should not contain \( \pi_{\varepsilon_0^0} \mathcal{L}_\alpha \varepsilon_0^0 \). \quad (201)
(2 + 6) × 2 dynamical variables. If they are related to second class constraints, there are at most (2 + 5) degrees of freedom because a second class constraint removes 1/2 degree of freedom. If $C_5$ is related to a first class constraint, the number of dynamical degrees of freedom will be at most $(2 + 9/2)$ as a first class constraint eliminates 1 degree of freedom. Usually, a first class constraint is related to a gauge symmetry, then the arbitrary gauge parameter will not appear in the constraints. The primary constraint of the gauge parameter should be a first class constraint is related to a gauge symmetry, then the first class constraint eliminates 1 degree of freedom. Usually, the primary constraint of the gauge parameter should be a first class constraint.

A well-known example is Weyl gravity

$$\mathcal{L}_{\text{Weyl}} = R(E) \wedge E \wedge F + E \wedge F \wedge F,$$

(207)

which is conformal invariant and contains $(2 + 4)$ degrees of freedom. The independent part of the tertiary constraint

$$\bar{C}_5[\alpha] = \int d^3x \left( 2\Pi_i^i - \pi^{ij} D_i \partial_j \right) \alpha$$

(208)

is related to the generator of conformal transformations

$$\delta \gamma_{\mu \nu} = 2\alpha \gamma_{\mu \nu}, \quad \delta \epsilon_{\mu \nu} = -\nabla_{\mu} \partial_{\nu} \alpha.$$  

(209)

The non-dynamical variable $\epsilon_{0^0}$ remains arbitrary. After diagonalizing the constraint brackets, $C_5$ is supplemented by some terms involving $\pi_{e_0^i}$ in order to preserve the equation for $\epsilon_{0^i}$ under a conformal transformation. Then we have one more pair of first class constraints. Note that Weyl gravity is also a nonlinear completion of Fierz-Pauli theory at the partially massless points of the parameter space.

V. CONCLUSION

In summary, we show that the BD ghost can be removed by additional constraints in the bi-gravity models with novel kinetic terms. There are two important features of the general Hamiltonian structure:\n
- The first key point is that the Hamiltonian is always linear in $\epsilon_{0^0}$, so the time derivative of the primary constraint $\pi_{e_0^0} = 0$ generates a secondary constraint

$$C_4 = \{\pi_{e_0^0}, \int d^3x \mathcal{H}^E \} \approx 0,$$

(210)

which does not contain $\epsilon_{0^0}$. From the expression of the secondary constraint, we know it is an equation for the dynamical variables $\gamma_{ij}, e_{ij}, \pi^{ij}$.\n
- The second key point is that in $C_4$ the spatial derivative term is the Riemann curvature tensor of the induced metric $\gamma_{ij}$. This indicates the Poisson bracket of $C_4[\alpha]$ and $C_4[\beta]$ vanishes

$$\{C_4[\alpha], C_4[\beta]\} = 0,$$

(211)

because $C_4$ does not involve $\Pi^{ij}$, which is the conjugate momentum of the induced metric.

If $C_4$ is a first class constraint, it already eliminates 1 dynamical degree of freedom because of the first key point.

When $C_4$ is related to a second class constraint, an independent tertiary constraint $C_5$ is generated by the time derivative of $C_4$. This tertiary constraint will not contain $\epsilon_{0^i}$ due to the second key point. The lapse function $N$ and the shift vector $N^i$ are gauge parameters associated with the diffeomorphism invariance, so they will not appear in the constraint equations. Then $C_5$ is an equation for the dynamical variables. $C_4$ could correspond to a first or second class constraint, but in both cases at least 1 degree of freedom is removed by the two constraints $C_4$ and $C_5$.

The general Hamiltonian structure of these bi-gravity models is analogous to Hassan-Rosen bi-gravity theory. The reason is that both of them are nonlinear completions of the same linear theory, Fierz-Pauli massive gravity. As a result, the degree of freedom eliminated by $C_4$ and $C_5$ is a nonlinear Ostrogradsky’s scalar ghost, namely the Boulware-Deser ghost.

We expect that the general constraint structure of the novel two-derivative terms can be extended to any dimension

$$R(E) \wedge E \wedge \cdots \wedge E \wedge F \wedge \cdots \wedge F$$

(212)

and to the cases of novel higher-derivative terms

$$R(E) \wedge \cdots \wedge R(E) \wedge E \wedge \cdots \wedge E \wedge F \wedge \cdots \wedge F.$$  

(213)

For multi-gravity generalizations, we can introduce different $F^{(k)}$. The Boulware-Deser ghost should be absent as well. There are additional primary constraints for $\gamma_{ij}^{(k)}$ because they are functions of $\mathcal{L}_n \gamma_{ij}$. To understand the origin of these primary constraints, we can eliminate

$$e_{\mu \nu}^{(k)} = E^{(k)} \mu B^{(k)} \eta_{AB},$$

(214)

by their equations of motion, which is possible in most of the cases. Then a multi-gravity theory with novel derivative terms becomes a model of higher curvature gravity, so there are at most two dynamical spin-2 fields. An interesting question is whether the resulting higher derivative gravity model is more general than that from a bi-gravity theory.

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14 These two features originate in the antisymmetric structures in the Lagrangians and can be easily generalized to other dimensions.

15 However, from the explicit expression of $C_5$, it is not apparent that $N$ is unconstrained.

16 In Type B models, the Hamiltonian is quadratic in $\epsilon_{0^i}$, so the corresponding secondary constraint can be solved and $\epsilon_{0^i}$ are functions of the dynamical variables.
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