Exact four-spinon dynamical correlation function
in isotropic Heisenberg model

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Abstract

We discuss some properties of the exact four-spinon dynamical correlation function \( S_4 \) in the antiferromagnetic spin 1/2 \( XXX \)-model, the expression of which we derived recently. We show that the region in which it is not identically zero is different from and larger than the spin-wave continuum. We describe its behavior as a function of the neutron momentum transfer \( k \) for fixed values of the neutron energy \( \omega \) and compare it to the one corresponding to the exact two-spinon dynamical correlation function \( S_2 \). We show that the overall shapes are quite similar, even though the expression of \( S_4 \) is much more involved than that of \( S_2 \). We finish with concluding remarks.

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I. INTRODUCTION

Quantum spin chains have been the subject of intensive study during the past seven decades [1]. Experimentally, their properties are investigated via inelastic neutron scattering on (anti)ferromagnetic quasi one-dimensional compounds [2]. One important quantity that holds much of the information related to such compounds is the dynamical correlation function (DCF) $S$ of two local spin operators. Indeed, the neutron scattering intensity is directly proportional to it, see for example [2].

An important feature of these chains is that some of them, like the Heisenberg model, are amenable to exact theoretical treatment while still describing nontrivial interactions [1], see also [3]. This is because they incorporate in them a rich mathematical structure: the quantum affine algebra. The early work on such models consisted in determining exactly their static thermodynamic properties, see [1], whereas, more recently, a fuller exploitation of the quantum symmetry using bosonization techniques has allowed for a more systematic description of their dynamical properties [4]. A systematic account of this work is given in [5].

However, one aspect of these new techniques is that the exact correlation functions are usually obtained in the form of quite complicated contour integrals and this renders their manipulation somewhat cumbersome. But on the other hand, before this exact treatment, the approach to these dynamical quantities was only approximate. Indeed, if one considers for instance the DCF in the antiferromagnetic Heisenberg model, the focus has for a long time been only on what we now know to be its two-spinon contribution $S_2$. First there has been the Anderson (semi-classical) spin-wave theory [6], an approach based on an expansion in powers of $1/s$, where $s$ is the spin of the system and hence, is exact only in the classical limit $s = \infty$. It can describe with some satisfaction compounds with higher spins [7], but fails in the quantum limit $s = \frac{1}{2}$. Then there has been the so-called Müller ansatz [8], which gives an approximate expression for $S_2$ that can account for many aspects of the phenomenology for $s = \frac{1}{2}$ compounds. Only very recently could we get for this system a final exact expression for $S_2$ [9], and that gave a better account of the data [10].

Now for the spin-$\frac{1}{2}$ antiferromagnetic Heisenberg model, there is a need to go beyond the two-spinon contribution. The need is two-fold. First, it is important to see if one is able to get useful information from these complicated and compact expressions we alluded to above. Second and more important perhaps is the fact that, though the exact two-spinon contribution accounts for much of the phenomenology as we said, about 70%, in a sense that will become clear later in section 3, it still doesn’t account for all of it. This point is demonstrated in particular in [11].

The natural step forward is to look into the exact four-spinon contribution. To the best
of our knowledge, refs [12] and [13] constitute the first direct attempt in this direction. A general expression for the $n$-spinon contribution to the DCF in the anisotropic Heisenberg model is given in [13] and (a still compact one) for the isotropic limit in [12]. In ref [13], we specialize to the four-spinon case and give a discussion of some of its properties in the isotropic and Ising limits. In particular, we show that in the isotropic limit, the one of interest here, the exact four-spinon contribution is safe of any potential divergences.

In this work, we further the description of the four-spinon contribution $S_4$. We determine the region in which $S_4$ is not identically zero and show that it is larger than that of $S_2$. Also, we show that the behavior of $S_4$ as a function of the neutron momentum transfer $k$ is similar in its overall shape to that of the corresponding $S_2$. This result is to be contrasted with the fact that the expression of $S_4$ is a lot more involved than that of $S_2$, see (23) and (13) respectively.

This paper is organized as follows. In the next section, we briefly discuss the antiferromagnetic spin $1/2$ Heisenberg model. We describe the spinon Hilbert-space structure and define the dynamical correlation function. In section 3, we succinctly review the properties of the two-spinon contribution. We give the exact expression of $S_2$ and that of the Müller ansatz, and briefly compare their main features. In section 4, we give the exact expression of $S_4$ and discuss in detail its properties we mentioned above. The last section comprises concluding remarks.

II. THE EXACT DCF IN THE XXX-MODEL

The antiferromagnetic $s = \frac{1}{2}$ XXX-Heisenberg model is defined as the isotropic limit of the $XXZ$-anisotropic Heisenberg Hamiltonian:

$$H_{XXZ} = -\frac{1}{2} \sum_{n=-\infty}^{\infty} \left( \sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \Delta \sigma_n^z \sigma_{n+1}^z \right),$$  \hspace{1cm} (1)

where $\Delta = (q + q^{-1})/2$ is the anisotropy parameter. The isotropic antiferromagnetic limit is obtained via the limit $q \rightarrow -1^-$ or equivalently $\Delta \rightarrow -1^-$. Here $\sigma_n^{x,y,z}$ are the usual Pauli matrices acting at the site $n$. The exact diagonalization directly in the thermodynamic limit of the Hamiltonian in (1) is possible using the $U_q(\mathfrak{sl}(2))$ quantum group symmetry present in the model [3]. The resulting Hilbert space $\mathcal{F}$ consists of $n$-spinon energy eigenstates $|\xi_1, ..., \xi_n; \epsilon_1, ..., \epsilon_n; i \rangle$ built on the two vacuum states $|0\rangle_i$, $i = 0, 1$ such that:

$$H_{XXZ}|\xi_1, ..., \xi_n; \epsilon_1, ..., \epsilon_n; i \rangle = \sum_{j=1}^{n} e(\xi_j)|\xi_1, ..., \xi_n; \epsilon_1, ..., \epsilon_n; i \rangle,$$  \hspace{1cm} (2)

where $e(\xi_j)$ is the energy of spinon $j$ and $\xi_j$ is a spectral parameter living on the unit circle. In the above equation, $\epsilon_j = \pm 1$ and the index $i$ refers to the boundary condition on the spin
The translation operator $T$ which shifts the spin chain by one site acts on the spinon eigenstates in the following manner:

$$T|\xi_1, \ldots, \xi_n\rangle_{\epsilon_1, \ldots, \epsilon_n; i} = \prod_{i=1}^{n} \tau(\xi_i)|\xi_1, \ldots, \xi_n\rangle_{\epsilon_1, \ldots, \epsilon_n; 1-i},$$

(3)

where $\tau(\xi_j) = e^{-ip_j}$ and $p_j$ is the lattice momentum of spinon $j$. The expressions of the spinon energy and lattice momentum in terms of the spectral parameter are quite cumbersome in the anisotropic case, see [5,13], but simplify considerably in the isotropic limit, see eq (12) below. The completeness relation in $\mathcal{F}$ reads:

$$I = \sum_{i=0,1} \sum_{n \geq 0} \sum_{\{\epsilon_j = \pm 1\}_{j=1}^n} \frac{1}{n!} \oint \prod_{j=1}^{n} d\xi_j \frac{1}{2\pi i \xi_j} |\xi_1, \ldots, \xi_n\rangle_{\epsilon_1, \ldots, \epsilon_n; i} \langle \epsilon_1, \ldots, \epsilon_n | \xi_1, \ldots, \xi_n|.$$

(4)

The two-point DCF is the Fourier transform of the zero-temperature vacuum-to-vacuum two-point function, i.e., it is defined by:

$$S^{i,-}(\omega, k) = \int_{-\infty}^{\infty} dt \sum_{m \in \mathbb{Z}} e^{i(\omega t + km)} \langle 0|\sigma^+_m(t)\sigma^-_0(0)|0\rangle_i,$$

(5)

where $\omega$ and $k$ are the neutron energy and momentum transfer respectively and $\sigma^{\pm}$ denotes $(\sigma^x \pm i\sigma^y)/2$. The DCF is such that:

$$S^{i,-}(\omega, k) = S^{i,-}(\omega, -k) = S^{i,-}(\omega, k + 2\pi),$$

(6)

the two relations that express the reflection and periodicity symmetries on the linear chain. Inserting the completeness relation and using the Heisenberg relation:

$$\sigma^{x,y,z}_m(t) = \exp(iH_{XXZ}t)T^{-m}\sigma^{x,y,z}_0(0)T^m \exp(-iH_{XXZ}t),$$

(7)

we can write the transverse DCF as the sum of $n$-spinon contributions:

$$S^{i,-}(\omega, k) = \sum_{n \text{ even}} S_n^{i,-}(\omega, k),$$

(8)

where the $n$-spinon DCF $S_n$ is given by:

$$S_n^{i,-}(\omega, k) = \frac{2\pi}{n!} \sum_{m \in \mathbb{Z}} \sum_{\epsilon_1, \ldots, \epsilon_n} \oint \prod_{j=1}^{n} \frac{d\xi_j}{2\pi i \xi_j} e^{im(k+\sum_{j=1}^{n} p_j)} \delta\left(\omega - \sum_{j=1}^{n} \epsilon_j\right)$$

$$\times X_{\epsilon_1, \ldots, \epsilon_n}^{i,m}(\xi, \ldots, \xi_1) X_{\epsilon_1, \ldots, \epsilon_n}^{i,-m}(\xi, \ldots, \xi_n),$$

(9)

relation in which $X^i$ denotes the form factor:

$$X^i_{\epsilon_1, \ldots, \epsilon_n}(\xi_1, \ldots, \xi_n) \equiv \langle 0|\sigma^+_0(0)|\xi_1, \ldots, \xi_n\rangle_{\epsilon_1, \ldots, \epsilon_n}.$$

(10)
Note that each $S_n$ must satisfy relations (8).

The exact expression of this form factor has been determined in [5]. Using this form factor, we can give an exact expression for the $n$-spinon DCF in the anisotropic case, see [13], and determine exactly its isotropic limit, see [12]. This limit is obtained via the replacement [5,13]:

$$\xi = i e^{-2i\epsilon \rho} ; \quad q = -e^{-\epsilon} , \quad \epsilon \to 0^+ ,$$

where $\rho$ is the spectral parameter suited for this limit. The expressions of the energy $e$ and momentum $p$ in terms of $\rho$ then read:

$$e(\rho) = \frac{\pi}{\cosh(2\pi \rho)} = -\pi \sin p ; \quad \cot p = \sinh(2\pi \rho) ; \quad -\pi \leq p \leq 0 . \quad (12)$$

The transverse two-spinon DCF $S_2$ does not involve a contour integration and has been given in [9]. The four-spinon one $S_4$ involves only one contour integration and its expression is given in [13]. In the next section, we review the properties of $S_2$ and in the one that follows it, we discuss those of $S_4$.

### III. THE EXACT TWO-SPINON DCF

The exact expression of the two-spinon DCF of the spin $\frac{1}{2}$ XXX-model is given in [3] and reads:

$$S_2^+(\omega, k - \pi) = \frac{1}{4} e^{-I(\rho)} \sqrt{\omega^2 - \omega^2_{2l}} \Theta(\omega - \omega_{2l}) \Theta(\omega_{2u} - \omega) , \quad (13)$$

where $\Theta$ is the Heaviside step function and the function $I(\rho)$ is given by:

$$I(\rho) = \int_0^{+\infty} \frac{dt}{t} \frac{\cosh(2t) \cos(4\rho t) - 1}{\sinh(2t) \cosh(t)} e^t . \quad (14)$$

$\omega_{2u(l)}$ is the upper (lower) bound of the two-spinon excitation energies called the des Cloizeaux and Pearson [8,9] upper (lower) bound or limit. They read:

$$\omega_{2u} = 2\pi \sin \frac{k}{2} ; \quad \omega_{2l} = \pi | \sin k | . \quad (15)$$

The quantity $\rho = \rho_1 - \rho_2$ and is related to $\omega$ and $k$ by the relation:

$$\cosh \pi \rho = \sqrt{\frac{\omega_{2u}^2 - \omega_{2l}^2}{\omega^2 - \omega_{2l}^2}} , \quad (16)$$

a relation obtained using eq [12] and the energy-momentum conservation laws:
\[ \omega = e_1 + e_2; \quad k = -p_1 - p_2. \]  

(17)

Note that the explicit expression of \( S_2 \) given in eq (13) satisfies the reflection and periodicity symmetries expressed in (14).

The properties of \( S_2 \) have been discussed in [10]. There, a comparison with the Müller ansatz [8] is given. This latter was derived from the properties of some solutions to the Bethe-ansatz equations, from numerical calculations on finite spin chains and from an analysis of phenomenological results. It reads:

\[
S_2^{(a)}(\omega, k - \pi) = \frac{A}{2\pi} \frac{\Theta(\omega - \omega_{2l}) \Theta(\omega_{2u} - \omega)}{\sqrt{\omega^2 - \omega_{2l}^2}},
\]

(18)

where \( A \) is a constant determined in such a way to fit best the phenomenology [8]. There are two main differences between the exact expression (13) and the approximate one (18), [10]. First, the two-spinon threshold at \( \omega_{2l} \) is more singular in (13) than in (18). Second, at the upper two-spinon boundary \( \omega_{2u} \), \( S_2 \) vanishes smoothly whereas \( S_2^{(a)} \) has a sharp cut-off.

But if one defines the frequency moments of the DCF:

\[
K_n(k) = \int_{-\infty}^{+\infty} d\omega \, \omega^n S(\omega, k),
\]

(19)

one shows that as \( k \to 0 \), the moment of \( S_2 \) vanishes as:

\[
K_n^{(2)}(k) \sim \omega_{2u}^{n+1}(k),
\]

(20)

and the same holds for the Müller ansatz \( S_2^{(a)} \).

Actually, the frequency moments (19) are particular cases of a set of sum rules the DCF is known to satisfy exactly. For example, we know that the first moment is exactly equal to [14]:

\[
K_1(k) = \frac{4 \ln 2 - 1}{6} (1 - \cos k).
\]

(21)

It turns out that the same frequency moment for \( S_2 \) is such that [10]:

\[
\frac{K_1^{(2)}(k)}{K_1(k)} \simeq 70\%.
\]

(22)

This means that, according to this sum rule, \( S_2 \) is way off the total DCF \( S \) by roughly 30%. In fact, other exact sum rules confirm this trend, see [10]. Of course, those remaining 30% are “filled”, so to speak, by the \( n > 2 \)-spinon contributions. The natural question that comes to mind is: how much \( S_4 \) takes up from these 30%? Obviously, one has first to study the behavior of this contribution before trying to answer this question, and this we do in the remainder of this paper.
IV. THE EXACT FOUR-SPINON DCF

The analytic expression of the four-spinon DCF has been worked out in [13] and for $0 \leq k \leq \pi$ it reads:

$$S_4^-(\omega, k - \pi) = C_4 \int_{-\pi}^{0} dp_3 \int_{-\pi}^{0} dp_4 F(\rho_1, ..., \rho_4) ,$$

where $C_4$ is a numerical constant and the integrand $F$ is given by:

$$F(\rho_1, ..., \rho_4) = \sum_{(p_1, p_2)} f(\rho_1, ..., \rho_4) \frac{\sum_{\ell=1}^{4} |g_{\ell}(\rho_1, ..., \rho_4)|^2}{\sqrt{W_u^2 - W^2}}.$$  \hspace{1cm} (23)

The different quantities involved in this expression are:

$$W = \omega + \pi (\sin p_3 + \sin p_4) ;$$

$$W_u = 2\pi \left| \sin \frac{K}{2} \right| ;$$

$$K = k + p_3 + p_4 ;$$

$$\cot p_j = \sinh(2\pi \rho_j) , \quad -\pi \leq p_j \leq 0 .$$  \hspace{1cm} (24)

The function $f$ is given by:

$$f(\rho_1, ..., \rho_4) = \exp \left[ - \sum_{1 \leq j < j' \leq 4} I(\rho_{jj'}) \right] ,$$

where $\rho_{jj'} = \rho_j - \rho_{j'}$ and the function $g_{\ell}$ reads:

$$g_{\ell} = (-1)^{\ell+1}(2\pi)^4 \sum_{j=1}^{4} \cosh(2\pi \rho_j)$$

$$\times \sum_{m=\Theta(j-\ell)}^{\infty} \frac{\prod_{i \neq j} \Gamma(m - \frac{1}{2} + i\rho_{ji}) \prod_{i,j} \Gamma(m + 1 + i\rho_{ji})}{\prod_{i \neq j} \Gamma(m + \frac{1}{2} + i\rho_{ji}) \prod_{i,j} \Gamma(m - \frac{1}{2} + i\rho_{ji})} ,$$

where $\Theta$ is the Heaviside step function. In (24), the sum $\sum_{(p_1, p_2)}$ is over the two pairs $(p_1, p_2)$ and $(p_2, p_1)$ solutions of the energy-momentum conservation laws:

$$W = -\pi (\sin p_1 + \sin p_2) ; \quad K = -p_1 - p_2 .$$  \hspace{1cm} (25)

They read:

$$(p_1, p_2) = \left( -\frac{K}{2} + \arccos \left( \frac{W}{2\pi \sin \frac{K}{2}} \right) , -\frac{K}{2} - \arccos \left( \frac{W}{2\pi \sin \frac{K}{2}} \right) \right) .$$

(26)
Note that the solution in (29) is allowed as long as \( W_l \leq W \leq W_u \) where \( W_u \) is given in (25) and:

\[
W_l = \pi |\sin K| .
\]  

(30)

We henceforth put ourselves in the interval \( 0 \leq k \leq \pi \). To get the behavior of \( S_4 \) for the values of \( k \) outside this interval, one uses the symmetry relations (6) given in section 2. In the work [13], we have discussed the behavior of the function \( F \) given in (24). We have shown that the series \( g_\ell \) is convergent. We have also shown that in the region where two \( \rho_i \)'s or more get equal, the function \( g_\ell \) is finite. The function \( f \) going to zero in these same regions [10], this means the integrand \( F \) of \( S_4 \) has a nice regular behavior there. Furthermore, we have shown that \( F \) is exponentially convergent when one of the \( \rho_i \)'s goes to infinity, which means the two integrals over \( p_3 \) and \( p_4 \) in (23) do not yield infinities. All these analytic results pave the way to “safe” numerical manipulations.

The first thing we wish to discuss in this work is the extent of the region in the \((k, \omega)\)-plane in which \( S_4 \) is not identically zero, and compare it to that of \( S_2 \). Remember that from (13), \( S_2 \) is zero identically outside the spin-wave continuum \( \omega_{2l}(k) \leq \omega \leq \omega_{2u}(k) \), where \( \omega_{2l,u}(k) \) are given in (15). From the condition \( W_l \leq W \leq W_u \) discussed after eq (29), one infers that in order for \( S_4 \) to be nonzero identically, one has to have \( \omega_{4l} \leq \omega \leq \omega_{4u} \), where:

\[
\begin{align*}
\omega_{4l}(k) &= 3\pi \sin(k/3) \quad \text{for} \quad 0 \leq k \leq \pi/2 ; \\
\omega_{4l}(k) &= 3\pi \sin(k/3 + 2\pi/3) \quad \text{for} \quad \pi/2 \leq k \leq \pi ; \\
\omega_{4u}(k) &= 4\pi \cos(k/4) \quad \text{for} \quad 0 \leq k \leq \pi .
\end{align*}
\]  

(31)

All these branches are plotted in fig. 1.

The first thing we immediately see is that the \( S_4 \)-region, i.e., the region in which \( S_4 \) is not identically zero, is not confined to the spin-wave continuum delimited by the dCP branches \( \omega_{2l,u} \) given in (15). This means that, a fortiori, the full \( S \) is also not confined to the \( S_2 \)-region. This fact is confirmed by early finite chain numerical calculations [8] and the phenomenology [2]. Furthermore, fig. 1 shows that for \( 0 \leq k/\pi \leq 1/2 \), the \( S_4 \)-region is entirely beyond the \( S_2 \)-region. This means that for this interval, we may expect \( S_2 \) to be dominant in \( S \) within the spin-wave continuum. However, for \( 1/2 \leq k/\pi \leq 1 \), there is overlap between the two regions such that the \( S_2 \)-region is more or less within the \( S_4 \)-region. We may therefore expect here the contribution of \( S_4 \) to play a rôle, and hence, we expect \( S_2 \) to be a little less dominant within the spin-wave continuum.

The next feature we discuss in this work is the behavior of \( S_4 \) as a function of \( k \) for fixed \( \omega \). Figs. 2a, 3a and 4a show the behavior of \( S_4 \) for \( \omega/\pi = 0.45, 0.5 \) and 0.75 respectively. Note that we have scaled \( S_4 \) to appropriate units. Figs. 2b, 3b and 4b show the behavior of \( S_2 \) as a function of \( k \) for the same values of \( \omega \). Let us for example discuss the case \( \omega/\pi = 1/2 \).
We see from fig. 3b that the function $S_2$ vanishes for (roughly) $k/\pi \leq 0.8$. Looking back into fig. 1, this corresponds indeed to the region outside the spin-wave continuum for $\omega/\pi = 1/2$, i.e., $k/\pi \leq 5/6$. The function $S_2$ starts at $k/\pi = 5/6$ with a large value and goes to a minimum at $k/\pi = 1$.

Fig. 3a shows that the function $S_4$ has a somewhat similar behavior. From the figure, we read that $S_4$ too is not vanishing for (roughly) $0.8 \leq k/\pi \leq 1$. From fig. 1, that corresponds within the $S_4$-region to $0.84 \leq k/\pi \leq 1$. But fig. 1 shows also that for $\omega/\pi = 0.5$, $S_4$ may be non-vanishing in the interval $0 \leq k/\pi \leq 0.16$. That contribution doesn’t appear on fig. 3a, presumably because $S_4$ there is negligible. This is confirmed in the case $\omega/\pi = 0.45$ which is close to the case $\omega/\pi = 0.5$: there we see $S_4$ having a very small contribution in the corresponding interval. From fig. 3a, we see that $S_4$, like $S_2$, starts from a large value at its lower boundary $k/\pi = 0.84$ and decreases while moving to $k/\pi = 1$.

The case $\omega/\pi = 0.75$ is practically the same. Fig. 4b shows $S_2$ starting from zero at roughly $k/\pi = 0.25$, getting to a maximum and sharply dropping to zero a little further. It then starts sharply again from a large value a little after $k/\pi = 0.7$ and decreases to a minimum at $k/\pi = 1$. This is also consistent with fig. 1: the function $S_2$ starts to be nonvanishing for $\omega/\pi = 0.75$ at $k/\pi = 0.2447$. It stays nonvanishing until $k/\pi$ reaches the value 0.2699. It remains identically zero until $k/\pi$ reaches the value 0.7301 at which we enter back into the $S_2$-region.

As we said, $S_4$ has in this case too the same overall behavior as that of $S_2$. In fig. 4a, we see that $S_4$ starts to increase from the value zero at $k = 0$. It goes quickly to a maximum before dropping sharply to zero a little before $k/\pi = 0.3$. It stays at zero till a little after $k/\pi = 0.7$ and increases sharply. Then it decreases while wiggling to $k/\pi = 1$. This overall behavior is also consistent with fig. 1. Indeed, for $\omega/\pi = 0.75$, the $S_4$-region starts at $k = 0$ and extends first to $k/\pi = 0.2413$. Then we get outside this region from this value of $k$ till $k/\pi = 0.7587$. Then $S_4$ is no more identically zero beyond this point until we reach the point $k = \pi$. As we said, this is quite consistent with fig. 4a.

**V. DISCUSSION AND CONCLUSION**

In this work, we have discussed the behavior of the exact four-spinon contribution $S_4$ to the dynamical correlation function $S$ of the $s = 1/2$ antiferromagnetic Heisenberg model and compared it to the one of the exact two-spinon contribution $S_2$. We first reviewed the model and the spinon structure of the corresponding Hilbert space. We then gave a brief account of the results concerning $S_2$ and its comparison to the Müller ansatz. The first thing concerning $S_4$ we discussed is the region in the $(k, \omega)$-plane in which $S_4$ is not identically zero. We found it to be different and larger than that of $S_2$, i.e., the spin-wave continuum.
Both regions are drawn in fig. 1. We then discussed the behavior of $S_4$ as a function of $k$ for fixed values of $\omega$, and compared it to the one of $S_2$. These behaviors are plotted in figs. 2, 3 and 4. We have found that the overall shape of $S_4$ is more or less the same as that of $S_2$.

The first thing to emphasize we think is the overall similarity between the shapes of $S_4$ and $S_2$. This is not at all expected from the outset, given the more complicated expression of $S_4$, see (23), as compared to that of $S_2$, see (13). Does this mean that the shape of the other $n > 2$-spinon contributions and hence of the total $S$ is already more or less “traced” by that of $S_2$? We think that at this stage, it is too early for such an inference: this is a preliminary investigation into the behavior of $S_4$ and clearly more work is needed.

In any case, it would have been interesting to measure $S_4$ for other (larger) values of $\omega$, especially in regions where $S_2$ is identically zero whereas $S_4$ is not, see fig. 1. But as $\omega$ increases, the structure of the function $F(\rho_1, ..., \rho_4)$ of eq(24) in the $(p_3, p_4)$-plane gets “richer”, which means numerically harder to handle. To illustrate this point, we have plotted for the reader in figs. 5 the function $F$ for $k/\pi = 1/2$ and $\omega = 2\pi$ (5a), $\omega = 3\pi$ (5b). One can see that as $\omega$ increases, the function $F$ gets distributed nontrivially in larger areas in the $(p_3, p_4)$-plane, with an increasing more involved structure. This is the main reason why we preferred to defer the discussion of these regions to future work. Also, for the same reason, we have deferred the systematic discussion of $S_4$ as a function of $\omega$ for fixed values $k$.

One other interesting question we haven’t touched on in this work but merely alluded to at the end of section 3 is the following: how much $S_4$ accounts for in the total $S$? In other words, is $S_2 + S_4$ better an approximation to the total $S$ than $S_2$ alone, and if yes, by how much? As we said, we know that $S_2$ accounts for about 70% of the total $S$, which means that roughly 30% are left for the $n > 2$-spinon contributions. To tackle this question as it should, one has to rely on a certain number of sum rules $S$ is known to satisfy exactly. Then one compares the contribution of $S_2 + S_4$ to the exact result and carries a discussion thereon.

The other interesting question one may ask is the physical interpretation and implications of the behavior of the four-spinon DCF we have described in this work. What would certainly be interesting is to be able to systematically measure $S$ outside the spin-wave continuum so that we are assured of having the effects of the two-spinon contribution eliminated.

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Figure captions

Fig. 1: The regions in the \((k, \omega)\)-plane inside which \(S_2\) (dashed lines) and \(S_4\) (solid lines) are not identically zero. The boundaries are indicated as defined in the text.

Figs. 2–4: \(S_4\) (a) and \(S_2\) (b) as functions of \(k\) for fixed \(\omega\). The values of \(\omega\) are indicated. Note that for the sake of illustration, \(S_4\) is plotted to appropriate units.

Figs. 5: The function \(F(\rho_1, \rho_2, \rho_3, \rho_4)\) of relation (23) plotted in the \((p_3, p_4)\)-plane for \(k = 0.5\pi\) and \(\omega = 2\pi\) (a), \(\omega = 3\pi\) (b). Note that each \(F\) is scaled to appropriate units.
Figure 1
$\omega = 0.45\pi$

$S_4$

$k_\pi$

Figure 2a
\omega = 0.45\pi

\text{figure 2b}
\( \omega = 0.5\pi \)

figure 3a
$\omega = 0.5\pi$

Figure 3b
$\omega = 0.75\pi$

Figure 4a
\[ \omega = 0.75\pi \]
\[ k = 0.5\pi \]
\[ \omega = 2\pi \]
$k = 0.5\pi$

$\omega = 3\pi$

figure 5b