UNEXPECTED SURFACES SINGULAR ON LINES IN $\mathbb{P}^3$

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Abstract. We study linear systems of surfaces in $\mathbb{P}^3$ singular along general lines. Our purpose is to identify and classify special systems of such surfaces, i.e., those nonempty systems where the conditions imposed by the multiple lines are not independent. We prove the existence of four surfaces arising as (projective) linear systems with a single reduced member, which numerical experiments had suggested must exist. These are unexpected surfaces and we expect that our list is complete, i.e. it contains all special linear systems of affine dimension $1$, whose projectivisation has one, reduced and irreducible member.

As an application we find upper bounds for Waldschmidt constants along certain sets of general lines.

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1. Introduction

The study of linear systems of hypersurfaces in complex projective spaces with assigned base points of given multiplicity is a classical and central problem in algebraic geometry; see, e.g. [1], [21], [6], [15].

In the last few years this problem has been generalized to linear systems of hypersurfaces with assigned base loci consisting of linear subspaces of higher dimension [13, 10]. Conjectures such as [13, Conjecture 5.5] and [10, Conjectures A, B and C] suggest that the asymptotic behavior of such linear systems in the case of linear subspaces of higher dimension is similar to the case of points. In particular, after fixing sufficiently many sufficiently general linear subspaces, it is expected that the conditions imposed on forms by vanishing along these subspaces will be independent. (For efficiency we slightly abuse terminology by saying that $r$ homogeneous linear equations in a vector space of dimension $s$ are independent if the subspace of solutions has dimension either $s - r$ or 0.) On the other hand, it is interesting and important to understand special linear systems, i.e., nonempty systems for which the imposed conditions are dependent, or, equivalently, nonempty linear systems whose dimension is greater than expected from a naive conditions count.

In the classical setup of assigned base points in the case of generic points in $\mathbb{P}^2$, the well-known Segre-Harbourne-Gimigliano-Hirschowitz Conjecture (SHGH for short) [26, 16, 12, 20], provides a complete (yet still conjectural) explanation of all special linear systems of plane curves; see [4] for a nice survey and [5] for an account on recent progress. In the case of $\mathbb{P}^3$, there is an analogous conjecture due to Laface and Ugaá[/22, 23]. In addition, there
is a very nice partial result due to Brambilla, Dumitrescu and Postinghel [3] valid for points in projective spaces of arbitrary dimension.

Recently a new path of research has been opened in [7] and [17], where the authors introduce the notion of unexpected hypersurfaces; see also [28] for interesting connections with Lefschetz Properties and hypo-osculating varieties. With respect to the notation introduced in the next section, the hypersurfaces in $L = L_d(m_1, \ldots, m_s)$ being unexpected just means $L$ is special (defined below).

In the present note we focus on the classification of special linear systems with base loci assigned along positive dimensional linear subspaces in the simplest nontrivial situation, i.e., we consider linear systems of surfaces in $\mathbb{P}^3$ with vanishing conditions imposed along general lines. Our main result is Theorem 3.3.

2. Notation and basic properties

In our context of imposed base lines, we will use the same notation customarily used for linear systems with imposed base points. This has the advantage of working with familiar notation but with no danger of confusion since we clearly flag appearances of assigned base points.

Thus, $L = L_d(m_1, \ldots, m_s)$ denotes the linear system of surfaces of degree $d$ in $\mathbb{P}^3$ passing through $s$ general lines (hence the lines are in particular disjoint) with assigned multiplicities $m_1, \ldots, m_s$. If $d < m_i$ for some $i$, then clearly $L_d(m_1, \ldots, m_s) = \emptyset$, so we will always assume that $d \geq \max(m_1, \ldots, m_s)$. As is customary, if the multiplicities are repeated, then we abbreviate the notation in a natural way. For example $L_d^{(m \times s)}$ denotes a linear system of surfaces of degree $d$ with $s$ lines of the same multiplicity $m$.

Let $c_{m,d}$ be the number of conditions which vanishing to order $m$ along a line in $\mathbb{P}^3$ imposes on forms of degree $d \geq m$. This number is well-known and is worked out in [10, Lemma A.2(c)]:

$$c_{m,d} = \sum_{i=0}^{m-1} (d+1-i)(i+1) = \frac{1}{6}m(m+1)(3d+5-2m).$$

For the convenience of the reader we provide explicit formulas for a few initial values of $m$:

$c_{1,d} = d+1, \ c_{2,d} = 3d+1, \ c_{3,d} = 6d-2, \ c_{4,d} = 10d-10, \ c_{5,d} = 15d-25.$

The virtual dimension of $L$ is therefore

$$\dim_{\text{vir}}(L) = \binom{d+3}{3} - \frac{1}{6} \sum_{i=1}^{s} m_i(m_i+1)(3d+5-2m_i).$$

Note that we use here affine dimension; i.e., the dimension of the vector space of forms defining the surfaces in the linear system. As usual, the expected dimension of $L$ is

$$\dim_{\text{exp}}(L) = \max\{\dim_{\text{vir}}(L), 0\}.$$

If the actual dimension of $L$ is larger than the expected dimension, then we say that $L$ is special. In this note we are primarily interested in special systems.
3. Reasons for speciality

The conditions imposed by general lines with multiplicity 1 are always independent. In other words, linear systems of the type

\[ \mathcal{L}_d(1^{xs}) \]

are always non-special. While general assigned base points of multiplicity 1 trivially impose independent conditions, in the case of base lines of multiplicity 1 this is a non-trivial result due to Hartshorne and Hirschowitz [19].

Thus as with points it requires lines of higher multiplicities in order to get a special system. There are two easy ways to construct such special linear systems. We discuss them in the following two examples.

**Example 3.1 (Multiples of non-special systems).** Let \( \mathcal{L} = \mathcal{L}_2(1,1,1) \). Then, by the Hartshorne and Hirschowitz result, \( \mathcal{L} \) is non-special with \( \dim(\mathcal{L}) = 1 \), so the (projective) linear system \( \mathcal{L} \) contains a unique quadric \( Q \). On the other hand, for \( \mathcal{M} = 2\mathcal{L} = \mathcal{L}_4(2^{s3}) \) we have

\[ \dim_{\text{vir}}(\mathcal{M}) = -4, \text{ hence } \dim_{\text{exp}}(\mathcal{M}) = 0 \]

but of course \( \dim(\mathcal{M}) \geq 1 \) since \( 2Q \) is in \( \mathcal{M} \), and it is not hard to verify that \( \dim(\mathcal{M}) = 1 \) so \( 2Q \) is the only member of the projectivisation of \( \mathcal{M} \). In fact, all linear systems \( \mathcal{L}_{2m}(m^{s3}) \) with \( m \geq 2 \) are special of affine dimension 1.

Another instance of this is given by the linear system \( \mathcal{L}_3(1^{x4}) \) of cubics containing 4 general lines. By considering quadrics through 3 of the 4 lines with a plane containing the fourth we see that \( \mathcal{L}_3(1^{x4}) \) is nonempty.

Since \( \mathcal{L}_{3m}(m^{x3}) \) has negative virtual dimension if (and only if) \( m \geq 8 \), yet is a multiple of the nonempty system \( \mathcal{L}_3(1^{x4}) \), it is nonempty and hence special for \( m \geq 8 \). Note for 4 general lines that there are two lines which meet each of the 4 transversally. Thus both of these transversal lines are in the base locus of \( \mathcal{L}_3(1^{x4}) \), and by considering quadrics through 3 of the 4 lines with a plane containing the fourth, it is not hard to see that the base locus is precisely the two transversals (see [9, Example 3.4.3]). Since the base locus of \( \mathcal{L}_3(1^{x4}) \) has no divisorial components, the general member of \( \mathcal{L}_{3m}(m^{x3}) \) is reduced. In particular, these give special linear systems whose general members are reduced. This is in contrast to \( \mathcal{L}_{2m}(m^{x3}) \), and to the assertion of the SHGH Conjecture for special linear systems of curves in the plane (which says that every special system has a multiple curve in its base locus).

Instead of taking multiples of a fixed system, one may add some distinct systems.

**Example 3.2 (Unions of non-special systems).** Let \( \mathcal{L} = \mathcal{L}_8(3^{x4}) \). Then

\[ \dim_{\text{vir}}(\mathcal{L}) = -19, \text{ hence } \dim_{\text{exp}}(\mathcal{L}) = 0 \]

but this system is non-empty, as it contains the element consisting of the union of four quadrics (hence it is reduced), each of which vanishes along three of the four given general lines. In fact, it can be easily checked that this is the only element of the system.
In contrast to the previous examples, it is possible for a special system to be reduced, irreducible and of affine dimension one. An extensive computer search using Singular [8] exhibited four such special linear systems, listed in Theorem 3.3. Since in each case the projectivisation of the system contains a unique, irreducible member, the speciality cannot be explained along the lines of Examples 3.1 and 3.2. We will see in section 6 how the existence of systems B) and C) is related to some important asymptotic invariants of homogeneous ideals.

**Theorem 3.3.** The following systems are special of (affine) dimension 1:

- A) $L_{10}(3^4,1^5)$;
- B) $L_{12}(4,3^5)$;
- C) $L_{12}(3^6,2)$;
- D) $L_{20}(6^5,1)$.

Thus there is a single surface of the given degree vanishing to given order along the given number of general lines.

The systems above are special as long as they are effective. Note that the Singular computation which led to the systems in Theorem 3.3 works with random rather than general lines, and so it gives strong evidence, but not proof, of the effectivity of the given systems. This Theorem is proved in subsection 4.4 and in section 5.

**Remark 3.4.** In cases A), B) and D) of Theorem 3.3 the unexpected surfaces are rational. However in case C) the unexpected surface is of general type by Corollary 5.4. This is quite surprising in view of the special effect varieties program proposed by Bocci [2].

4. Cremona transformations of $\mathbb{P}^3$

Cremona transformations have long played a prominent role in the study of special linear systems in the plane. It turns out that this is also a useful approach in our situation. We recall here two birational transformations of $\mathbb{P}^3$ which best suit our needs. We provide some background for non-experts.

4.1. A cubo-cubic transformation. First, there is the cubo-cubic transformation $C : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ induced by the linear system of cubics vanishing to order 1 along 4 general lines, for which both the image and inverse image of a general plane is a cubic surface (hence the name cubo-cubic). This compares to the more familiar plane quadratic Cremona transformation induced by the linear system of conics vanishing to order 1 at 3 general points (which by analogy is called a quadro-quadric transformation).

![Figure 1. The cubo-cubic transformation.](image-url)
As mentioned in Example 3.1, the base locus of the linear system of cubic surfaces vanishing on 4 general lines consists of 6 lines: the 4 general lines, and two additional lines which are transversal to the first 4; i.e., both transversal lines intersect each of the 4 general lines (see [9, Example 3.4.3]).

Let \( \pi : X \to \mathbb{P}^3 \) be the morphism obtained by first blowing up \( \mathbb{P}^3 \) along each of four general lines \( l_1, \ldots, l_4 \) (whose exceptional divisors we denote by \( E_1, \ldots, E_4 \)) and then blowing up the two transversal lines \( t_1, t_2 \) (whose exceptional divisors we denote by \( T_1 \) and \( T_2 \)). One can show that \( \pi \) resolves the indeterminacy of \( C \), giving a morphism \( C_0 \), as shown in Figure 1.

Denote by \( H \) the pullback via \( \pi \) to \( X \) of a plane in \( \mathbb{P}^3 \). By \( \mathbb{E} \) we denote \( E_1 + \cdots + E_4 \) and by \( T \) we denote \( T_1 + T_2 \). Thus \( C_0 \) is induced by the linear system of sections of \( 3H - E - T \). By \( H' \) we denote the pullback of a plane via \( C_0 \), so \( H' = 3H - E - T \).

The intersection product on \( X \) is determined by

\[
H^3 = 1, \quad HE_i^2 = HT_j^2 = E_iT_j^2 = -1 \quad \text{and} \quad -E_i = T_j^3 = 2
\]

with all other monomial triple intersections being 0. As \( H^3 = 1 \) and \( H^2E_i = H^2T_i = 0 \) are clear, we briefly explain the other intersections. Note that the blowup of a line in \( \mathbb{P}^3 \) is isomorphic to \( \mathbb{P}^1 \times \mathbb{P}^1 \), thus sections of \( H - E_i \) are disjoint so \( H(H - E_i)^2 = 0 \). Expanding this shows that \( HE_i^2 = -1 \).

Similarly expanding \( (H - E_i)^3 = 0 \) we get \( E_i^3 = -2 \).

Computing \( T_j^3 \) is more subtle. Consider \( (H - T_i)^2T_1 \). We will show that this intersection equals 4, then expanding, we get that \( T_j^3 = 2 \).

Each element of \( |H - T_1| \) is the proper transform of a plane \( h \subset \mathbb{P}^3 \) containing \( t_1 \). The proper transform of \( h \) is the blowup of \( h \) in the four points \( p_1, \ldots, p_4 \) where the lines \( l_i \) meet \( h \). So if \( A, B \) are different elements of \( |H - T_1| \), then the restrictions \( A \twoheadrightarrow a, B \twoheadrightarrow b \) of \( \pi \) to \( A \) and \( B \) are the blowups of \( p_1, \ldots, p_4 \) on the planes \( a \) and \( b \). Thus, \( A \cap B = e_{p_1} \cup \cdots \cup e_{p_4} \), where \( e_{p_i} \) is the exceptional curve given by the blowup of \( p_i \). Since \( \#(e_{p_i} \cap T_1) = 1 \) and the intersection is transversal, we get \( (H - T_i)^2T_1 = 4 \).

The morphism \( C_0 : X \to \mathbb{P}^3 \) factorizes into an isomorphism of \( X \) followed by a sequence of blowups of \( \mathbb{P}^3 \), as we now explain. There are four quadrics, say \( q_1, \ldots, q_4 \), where \( q_1 \) is the unique quadric which passes through the lines \( l_2, l_3, l_4, q_2 \) is the unique quadric which passes through \( l_1, l_3, l_4 \), etc. The proper transform by \( \pi \) of \( q_i \) is \( Q_i \). The divisor class of \( Q_i \) is \( 2H - E + E_i - T \).

The image of \( Q_i \) under \( C_0 \) is a line in \( \mathbb{P}^3 \), call it \( l'_i \) (as the restriction of \( 3H - E - T \) to \( Q_i \) is a \((1, 0)\) class). The images of \( T_1 \) and \( T_2 \) are lines \( l'_1, l'_2 \) transversal to \( l'_1, \ldots, l'_4 \). We note that \( \pi(T_i) \) is projection along one ruling of \( T_i = \mathbb{P}^{1} \times \mathbb{P}^{1} \) and \( C_0(T_i) \) is projection along the other ruling. In Figure 2 let \( \pi'_1 : X'_1 \to \mathbb{P}^3 \) be the blowup of the lines \( l'_i \) and let \( \pi'_2 : X'_2 \to X'_1 \) be the blowup of the proper transforms of the lines \( l'_i \).

By the universal property of blowups, \( C_0 \) lifts to \( C_1 \), and then \( C_1 \) lifts to \( C_2 \).

We claim \( C_2 \) is an isomorphism. Because \( H^2H' = 3 \), the inverse of \( C \) is defined by a four dimensional family of cubic surfaces. Since the fibers over \( l'_1, \ldots, l'_4 \) are positive dimensional, the inverse \( C' \) of \( C \) is not defined on these six lines, so the base locus of \( C' \) contains these six lines, and, as for \( C \), this is the whole base locus and a morphism \( C'_0 : X'_2 \to \mathbb{P}^3 \) resolving the indeterminacies of \( C' \) is obtained by blowing them up, \( l'_1, \ldots, l'_4 \) first, and then \( l'_1, l'_2 \). Arguing
as before, \( C'_0 \) lifts to a morphism \( C'_2 \), and since \( C \) and \( C' \) are inverse birational maps, we see that \( C_2 \) and \( C'_2 \) are inverse morphisms.

\[ X \xrightarrow{c_2} X'_2 \]

\[ \pi \]

\[ \pi' \]

\[ \mathbb{P}^3 \]

\[ \mathbb{P}^3 \]

**Figure 2.** A factorization of the morphism \( C'_2 \) as blowups of the \( t'_i \) and the \( t'_i' \).

The following transformation rules

\[ H' \mapsto 3H - E - T, \]
\[ E'_i \mapsto 2H - E + E_i - T, \]
\[ T'_i \mapsto T_i \]

(3)

define a linear transformation \( \Gamma_C : Cl(X'_2) \to Cl(X) \) on the divisor class group of \( X'_2 \) (i.e., the free \( \mathbb{Z} \)-module generated by \( H', E'_1, \ldots, E'_4, T'_1, T'_2 \)). This transformation preserves all triple products and its matrix is its own inverse. This is the map on the divisor class groups induced by pullback by the birational map \( C \) in Figure 2.

**Remark 4.1.** The divisors \( T_i \) play a role in defining \( C \) but, as explained below in Remark 4.3, we will obtain special systems by applying \( \Gamma_C \) to divisor classes in which the \( T_i \) do not occur as summands.

4.2. **Todd transformation.** There is another interesting transformation \( T : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3 \) which seems to go back to Todd [29, Introduction]. It is given by the linear system of surfaces of degree 19 vanishing to order 5 along 6 general lines \( l_1, \ldots, l_6 \) in \( \mathbb{P}^3 \). We now summarize Todd’s results.

The geometry of this Cremona map can be analyzed similarly to the cubo-cubic case. The base locus of the linear system of surfaces of degree 19 vanishing to order 5 along six general lines consists of the six lines, the 30 transversal lines which are determined by subsets of four out of the six lines, and, as explained in [31], the six twisted cubics that have the six lines as chords. Let \( \pi : X \to \mathbb{P}^3 \) be the composition of the blowing up of the source \( \mathbb{P}^3 \) in the set of base lines, followed by the blowing up in the additional 30 lines and then the 6 cubics. We denote the exceptional divisors by \( E_1, \ldots, E_6, T_1, \ldots, T_{30}, G_1, \ldots, G_6 \). We abbreviate as above \( E = E_1 + \cdots + E_6 \), \( T = T_1 + \cdots + T_{30} \), \( G = G_1 + \cdots + G_6 \). Then \( T \) induces a birational morphism \( T_0 : X \to \mathbb{P}^3 \) determined by the linear system 19\( H - 5E - T - 3G \). We have the commutative diagram 3.

The inverse map is again given by surfaces of degree 19 vanishing along a set of 6 lines, \( l'_1, \ldots, l'_{6} \). Todd shows [29, p. 59] that each base line \( l_i \) is transformed into \( D_i \), a (rational) surface of degree 12. Denote by \( t : D_i \to \)
Figure 3. The Todd transformation.

$l_i \cong \mathbb{P}^1$ the restriction of the inverse map. Todd proves that the fibers of $t$ are quintic curves which meet five of the lines $l'_1 \ldots l'_6$ in three points and the remaining line in four points. The map $t$ is in fact a morphism, and $D_i$ has multiplicity 3 along five of the lines $l'_1 \ldots l'_6$ and multiplicity 4 along the sixth. Thus, we obtain that for each $i = 1, \ldots, 6$ there is a unique surface $D_i \subset \mathbb{P}^3$ of degree 12 vanishing along 5 general lines to order 3 and along the sixth general line to order 4.

Similarly to (2), and using the same notation, the intersection product on $X$ satisfies

$H^3 = 1, HE_i^2 = -1$ and $E_i^3 = -2$

with all other monomial triple intersections of $E_i$ and $H$ being 0.

Using an argument similar to the one used above, the Todd transformation corresponds to a linear transformation $\Gamma_T$ with:

$E'_i \mapsto 12H - 3E - E_i,$

$H' \mapsto 19H - 5E,$

which preserves triple products (4) and which is self inverse.

4.3. Limits of fat disjoint lines. The Cremona transformations above will be used to show nonemptiness of the systems in Theorem 3.3. To show uniqueness of each surface, we shall rely on semicontinuity proving uniqueness in some particular position of the lines. In some cases the chosen special position involves some of the lines becoming coplanar – so they intersect; in that case the limit linear system acquires a higher multiplicity at the point of intersection of these lines:

Lemma 4.2. Let $l$ be a fixed line and let $r_t, t \in \Delta \subset \mathbb{C}$ be an analytic family of lines in $\mathbb{P}^3$ where $\Delta$ is a disk around 0, such that $l$ and $r_t$ are skew for $t \neq 0$ and the lines $l$ and $r_0$ intersect at a point $p$. Let $m, n$ be positive integers. If $F \in \mathbb{C}[[t]][X, Y, Z, W]$ is the equation of an analytic family of surfaces, such that for every $t \in \Delta \setminus 0$, $F_t$ has multiplicity at least $m$ (respectively $n$) along $l$ (respectively $r_t$), then $F_0$ has multiplicity at least $m$ (respectively $n$) along $l$ (respectively $r_0$), and multiplicity at least $m + n$ at $p$.

Proof. Without loss of generality we may assume that $l$ is the line $X = Y = 0$ and $r_t$ is $Y - tW = Z = 0$, so $p = [0 : 0 : 0 : 1]$. The only claim that needs proof is the multiplicity at $p$, which is a local issue, so we will work in the regular local ring $R = \mathbb{C}[[t]][x, y, z]_p$, where $x = X/W, y = Y/W, z = Z/W$ and the lines are given by the ideals $I_l = (x, y)$ and $I_{r_t} = (y - t, z)$. Then
it is easy to see that \( R/I^n_m \) and \( R/I^n_{ri} \) are Cohen-Macaulay modules, and 
\[
\dim R/I^n_m + \dim R/I^n_{ri} = 4 = \dim R.
\]
Hence by \cite{27} V.B.6, Corollary after Theorem 4\( l \) one has \( \text{Tor}^{R}_1(R/I^n_m, R/I^n_{ri}) = 0 \). Therefore, tensoring the exact sequence \( 0 \to I^n_m \to R \to R/I^n_m \to 0 \) with \( R/I^n_{ri} \) we obtain the exact sequence
\[
0 \to I^n_m/(I^n_m \cdot I^n_{ri}) \to R/I^n_{ri} \to R/(I^n_m + I^n_{ri}) \to 0.
\]
Then injectivity of the first map gives
\[
I^n_m \cap I^n_{ri} = I^n_m \cdot I^n_{ri}
\]
in \( R \). Any \( F \) as in the claim of course belongs to \( I^n_m \cap I^n_{ri} \), and it follows that it belongs to \( I^n_m \cdot I^n_{ri} \). In particular, \( F_0 \in I^n_m \cdot I^n_{r0} \) has multiplicity at least \( m + n \) at \( p \).

4.4. First part of the proof of main theorem. We are now in position to prove part of Theorem 3.3:

Proof of Theorem 3.3. We see immediately from \cite{5} that the duodecic vanishing to order 3 along 5 general lines and to order 4 along the sixth line is the image under \( T_0 \) of the exceptional divisor \( E_i \) of \( \pi \). In particular, it is a unique and irreducible member of the (projective) linear system in Theorem 3.3]. Moreover surfaces of this kind are rational. We will show in section \cite{6} that these surfaces allow us to extend the list of known Waldschmidt constants of configurations of general lines in \( \mathbb{P}^3 \) obtained in \cite{1]} Proposition B.2.1].

The Todd transformation explains also the linear system in Theorem 3.3. Indeed, it is easy to check with the rules stated in \cite{5} that the system \( 20H - 6E + 5E_6 \) is the image under \( \Gamma_T \) of the system \( 8H - E - 5E_6 \), which is nonempty by \cite{11}. To check that \( 8H - E - 5E_6 \) has (projectively) a single element let \( \Pi \subset X \) be the strict transform of a general plane through \( l_6 \), and consider the residual exact sequence
\[
0 \to \mathcal{O}_X(7H - E - 4E_6) \to \mathcal{O}_X(8H - E - 5E_6) \to \mathcal{O}_\Pi((8H - E - 5E_6)|_\Pi) \to 0.
\]
Since the system restricted to the plane \( \Pi \) is the planar system of curves of degree two with five general base points, which consists of the unique conic through the five points, it will be enough to show that \( 7H - E - 4E_6 \) is empty. Specialize the lines so that for \( i = 1, \ldots, 4, l_i \) and \( l_{i+1} \) intersect at a point \( P_i \). By Lemma \cite{12} the limit of \( 7H - E - 4E_6 \) consists of surfaces which have multiplicity 2 at the points \( P_i \). Call \( \Pi_i \) the strict transform of the plane \( P_i \cup l_6 \) and \( \Pi_i' \) the strict transform of the plane \( l_i \cup l_{i+1} \) (where for projective subspaces \( A, B \subset \mathbb{P}^n \), \( A \cup B \) denotes the span of \( A \) and \( B \)).

The restriction of the specialized system to \( \Pi_i \) consists of septic containing 5\( l_6 \), double at \( P_i \) and with three further points, which are not aligned if the lines are general with the given restrictions; so this system is empty, which means that the four planes \( \Pi_i \) are fixed components of the specialized system. The residual consists of cubic surfaces passing through the six (special) lines. It is not hard to see that the restriction of this residual to the planes \( \Pi_i' \) and \( \Pi_i'' \) is nonempty, so these are also fixed components.

What remains is the system \( \mathcal{L}_1(1^{\times 2}) \), obviously empty. By semicontinuity, \( 7H - E - 4E_6 \) is also empty for lines in general position.
Finally, the linear system in Theorem 3.3 A) is explained by the means of the cubo-cubic transformation \( \tilde{C} \) based at the first 4 fat lines. To this end we show that already the linear system \( 10H - 3E \) is special. Indeed, we have \( \dim_{\exp}(10H - 3E) = 54 \) but the actual dimension is 56 since this system is the image of the system \( 6H - E \), which has the expected dimension 56 and is non special by the aforementioned result of Hartshorne and Hirschowitz. Then, vanishing along an additional line imposes at most 11 conditions, so that the system \( 10H - 3E - E_5 - \cdots - E_9 \) has dimension at least 1. To show that it is exactly 1, we exhibit again a specialization for which the dimension is 1. Denote, \( Q_i \) for each \( \{1, 2, 3, 4\} \) the unique quadric containing all lines \( l_j \) with \( j \in \{1, 2, 3, 4\}, j \neq i \). Let \( \{P^1_i, P^2_i\} = Q_i \cap l_5, \{P^1_2, P^2_2\} = Q_2 \cap l_5 \), and \( \{P^3_1, P^3_2\} \) two points in \( Q_3 \). Specialize the four last lines so that

\[
\begin{align*}
l_6 &= P^1_1 \cup P^3_1, & l_7 &= P^1_2 \cup P^3_1, \\
l_8 &= P^1_1 \cup P^3_2, & l_9 &= P^2_2 \cup P^3_2.
\end{align*}
\]

By Lemma 4.2 the limit system has multiplicity 2 at each point \( P^i \). If the points \( P^1 \) are general, the restriction of the limit system to \( Q_i \equiv \mathbb{P}^1 \times \mathbb{P}^1 \) consists of curves of bidegree \((10, 10)\) containing three triple sections \( l^1_i, i = 2, 3, 4 \) of type \((3, 0)\) with two triple points at \( l_1 \cap Q_i \), which by Bézout forces the sections of type \((0, 1)\) through these two points to split twice each; the residual are curves of bidegree \((1, 6)\), with two double points \( P^1_1, P^2_2 \) and passing through 8 additional general simple points; this is known to be empty, see [24]. The same analysis applied to \( Q_2, Q_3 \) shows that the three quadrics \( Q_1, Q_2, Q_3 \) are fixed components of the system. The residual system consists of surfaces of degree 4 passing with multiplicity 1 through all lines except \( l_4 \); it is not hard to see that the only such surface splits as \( Q_4 + (P^1_1 \cup P^3_1 \cup P^3_2) + (P^2_2 \cup P^2_2 \cup P^3_2) \).

The remaining case C) is the most interesting and it is dealt with in the next section.

**Remark 4.3.** Note that in the proof above we obtained special systems by applying cubo-cubic or Todd transformations to particular divisors. Similarly, by applying them to \( aH - (E_1 + \cdots + E_4) \) or to \( aH - (E_1 + \cdots + E_5) \) respectively, we obtain systems \((3a - 8)H - (a - 3)(E_1 + \cdots + E_4) \) and \((19a - 72)H - (5a - 19)(E_1 + \cdots + E_6) \). For \( a \) big enough the resulting systems have smaller expected dimensions than their actual dimensions, thus producing many examples of unexpected surfaces.

### 5. The linear system \( \mathcal{L}_{12}(3 \times 6, 2) \)

This section is devoted to the system C) in Theorem 5.3.

For the system \( \mathcal{L} = \mathcal{L}_{12}(3 \times 6, 2) \) we have

\[ \dim_{\text{vir}}(\mathcal{L}) = -2, \text{ hence } \dim_{\exp}(\mathcal{L}) = 0. \]

We will now show that nevertheless the projectivisation of this system is non-empty and contains a single irreducible element (and no other elements). We do not see how to show the speciality of this system using birational transformations of the ambient space, so we take a different approach.
Proof of Theorem 3.3, part II. Let \( l_1, \ldots, l_7 \) be general lines in \( \mathbb{P}^3 \) and let \( f : X \to \mathbb{P}^3 \) be the blowup of \( \mathbb{P}^3 \) along the first six lines with exceptional divisors \( E_1, \ldots, E_6 \), respectively. As usual we write \( E \) for the union of the exceptional divisors \( E_1, \ldots, E_6 \) and denote by \( H \) the pullback of the hyperplane bundle to \( X \). Then \( K_X = -4H + E \). We study the morphism defined by the anti-canonical system

\[ M = -K_X = 4H - E. \]

The divisors in this system correspond to quartics in \( \mathbb{P}^3 \) vanishing along the first six lines. By the aforementioned result of Hartshorne and Hirschowitz we have

\[ h^0(X, M) = 5, \]

hence \( M \) defines a rational map \( \varphi_M : X \to \mathbb{P}^4 \). The map is a morphism; one sees this by looking at reducible quartics containing the six lines (namely, products of two quadrics, each containing three of the six lines), and concluding that a base point on the blowup of one of the six lines implies that there would be a transversal to all six general lines. Note that \( \varphi_M \) contracts lines transversal to any 4 of the 6 given lines (there are \( \binom{6}{4} = 30 \) such contracted lines). By [30] (or by computer calculations) the image \( Y \) of \( \varphi_M \) is a quartic hypersurface, and \( \varphi_M \) is generically 1 : 1.

5.1. Existence. Let now \( C \subset \mathbb{P}^4 \) be the image of the seventh line under \( \varphi_M \). Then \( C \) is a rational normal curve of degree 4. It is well known that its secant (chordal) variety is a determinantal threefold \( T \) of degree 3 in \( \mathbb{P}^4 \), singular along \( C \), see for example [18], Proposition 9.7.

Note that \( T \cap Y \) is an element in \( \mathcal{O}_{\mathbb{P}^4}(3)|_Y \), so it pulls back to a surface \( D \) of degree 12 in \( \mathbb{P}^3 \), which vanishes to order 3 along the first six lines.

Moreover, since \( T \) is singular along \( C \) and \( C \subset T \cap Y \) we conclude that \( D \) is singular along \( L_7 \). Hence \( D \) has vanishing orders along the lines \( l_i \) with \( i = 1, \ldots, 7 \) corresponding exactly to those required for part C) of Theorem 3.3 Thus the existence of duodecics \( D \) vanishing to order 3 along six general lines and singular along the seventh line is established.

5.2. Uniqueness. To show that the surface is unique we apply semicontinuity, proving it in the particular case when the lines \( l_1, \ldots, l_5 \) are chosen to be disjoint lines in a general cubic surface, and \( \ell_6, \ell_7 \) are general.

A general cubic \( \Sigma \) in \( \mathbb{P}^3 \) can be understood as the blowup \( \sigma : \Sigma \to \mathbb{P}^2 \) of \( \mathbb{P}^2 \) at 6 general points, \( A_1, \ldots, A_6 \), embedded by the system \( \mathcal{O}_{\mathbb{P}^2}(3) \otimes I_{A_1} \cdots \otimes I_{A_6} \). The pullback of \( \mathcal{O}_{\mathbb{P}^2}(1) \) on the cubic surface is denoted by \( h \). The 27 lines on \( \Sigma \) are identified as the 6 exceptional divisors \( a_1, \ldots, a_6 \), the \( \binom{6}{2} = 15 \) lines through pairs of points, and the 6 conics through 5 out of 6 points. We choose \( l_1, \ldots, l_5 \) to be five of these latter lines.

Denote by \( \pi : W \to \mathbb{P}^3 \) the blowup of the 7 lines, \( l_1, \ldots, l_7 \), (in this particular position) with exceptional divisors \( E_1, \ldots, E_7 \). Let \( S \subset W \) (resp. \( N \subset W \)) be the proper transform of the cubic (resp. of the duodecic \( D \)). In \( S \) the divisor \( E_i|_S \) has class \( 2h - a + a_i \) for \( i = 1, \ldots, 5 \) (where as usual \( a = a_1 + \cdots + a_6 \). \( S \) is the blowup \( \pi : S \to \Sigma \) of the cubic at 6 additional points, \( F_1, F_2, F_3, G_1, G_2, G_3 \), namely its intersection points with \( \ell_6 \) and \( \ell_7 \). Denote the exceptional curves of this blowup by \( f_1, f_2, f_3, g_1, g_2, g_3 \) respectively, and
set \( f = f_1 + f_2 + f_3, \ g = g_1 + g_2 + g_3 \). The fact that these points \( S \cap (l_6 \cup l_7) \) lie on two lines translates to

\[
(6) \quad h^0(S, \mathcal{O}_S(3h - a - f)) = h^0(S, \mathcal{O}_S(3h - a - g)) = 2.
\]

**Remark 5.1.** We claim that for a general choice of the lines \( l_6, l_7 \), no three of the image points of \( F_1, F_2, F_3, G_1, G_2, G_3 \) in \( \mathbb{P}^2 \) are collinear. To prove this, first observe that \( F_1, F_2 \) (resp. \( G_1, G_2 \)) can be chosen arbitrarily, and then \( F_3 \) (resp. \( G_3 \)) is determined as the third intersection of \( l_6 = F_1 \cap F_2 \) (resp. \( l_7 = G_1 \cap G_2 \)) with the cubic surface \( \Sigma \). So we can assume that the image points of \( F_1, F_2 \) (resp. \( G_1, G_2 \)) in \( \mathbb{P}^2 \) are not aligned with any \( A_i \). This already implies that the images of \( F_1, F_2, F_3 \) (or equivalently \( G_1, G_2, G_3 \)) are not collinear, because if they were, by \( [6] \) there would be at least one section of \( \mathcal{O}_S(3h - a - f) \) vanishing on the line containing them, and then the six points \( A_1, \ldots, A_6 \) would belong to a conic, which contradicts their being general points. Now fix a choice of \( F_1, F_2, F_3 \) and let \( t \subset \Sigma \) be the pullback of the triangle through their images in \( \mathbb{P}^2 \). Consider the rational map

\[
\tau : \Sigma \times \Sigma \dashrightarrow \Sigma
\]

\[
(G_1, G_2) \mapsto G_3
\]

which is clearly dominant. If \( U \) is the open subset of \( \Sigma \times \Sigma \) where \( \tau \) is defined, then choosing \( (G_1, G_2) \in U \setminus ((t \times \Sigma) \cup (\Sigma \times t) \cup \tau^{-1}(t)) \) guarantees that the images of \( F_1, F_2, G_k \) in \( \mathbb{P}^2 \) are not aligned. By symmetry, a general choice of \( l_6 \) and \( l_7 \) will give that the images of \( G_i, G_j, F_k \) are not aligned either.

Now consider the residual exact sequence:

\[
0 \rightarrow \mathcal{O}_W(N - S) \rightarrow \mathcal{O}_W(N) \rightarrow \mathcal{O}_S(N|_S) \rightarrow 0
\]

We need to see that the global sections of the sheaf in the middle have dimension (at most) 1; we do so by proving that the global sections of the two other sheaves have dimensions 0 and 1 respectively.

The restriction of the class of the duodecic to \( S \) is:

\[
(7) \quad N|_S = 12H|_S - 3E|_S + E_T|_S = 12(3h - a) - 3 \sum_{i=1}^{5} (2h - a + a_i) - 3f - 2g = 6h + 3a_6 - 3f - 2g.
\]

Thus \( a_6 \) (which as a line in \( \mathbb{P}^3 \) is the unique common transversal to \( l_1, \ldots, l_5 \)) is a fixed component of the restricted system \( N|_S \). After subtraction of this fixed part the residual corresponds to the planar system \( \mathcal{O}_{\mathbb{P}^2}(6) \otimes I_{l_1}^1 \otimes I_{l_2}^1 \otimes I_{l_3}^1 \otimes I_{l_4}^1 \otimes I_{l_5}^1 \otimes I_{l_6}^1 \otimes I_{l_7}^1 \) (by a slight abuse of notation we identify the points \( F_1, F_2, F_3, G_1, G_2, G_3 \) with their images under \( \sigma \) in \( \mathbb{P}^2 \)). Any sextic in this system splits by the Bézout theorem into the sum of three conics: \( C_i \) through \( F_1, F_2, F_3, G_j, G_k \) for \( i = 1, 2, 3 \) and \( j, k \) such that \( \{i, j, k\} = \{1, 2, 3\} \). Note that these conics are irreducible by Remark 5.1. Thus, the dimension of this system is 1 and consequently \( h^0(S, \mathcal{O}_S(N|_S)) = 1 \). It remains to see that \( h^0(W, \mathcal{O}_W(N - S)) = 0 \), which we prove in Proposition 5.2 below. \( \square \)
Proposition 5.2. Let \(l_1, \ldots, l_5 \subset \mathbb{P}^3\) be disjoint lines in a general cubic surface, and \(l_6, l_7\) general lines in \(\mathbb{P}^3\). Let \(W \rightarrow \mathbb{P}^3\) be the blowup of the seven lines, and denote \(E_i\) the exceptional divisors, \(E = E_1 + \cdots + E_7\) as above. Then \(H^0(\mathcal{O}_W(9H - 2E - E_6)) = 0\).

Proof. By projection formula we have
\[
H^0(W, \mathcal{O}_W(9H - 2E - E_6)) = H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(9) \otimes I_{l_6}^2 \cdots \otimes I_{l_6}^2 \otimes I_{l_7}^2 \otimes I_{l_7}^2).
\]

We specialize further \(l_7\) to \(l_7'\), which is the line on \(\Sigma\), the proper transform of the line \(A_1 \vee A_2 \subset \mathbb{P}^2\). This line intersects \(l_1\) and \(l_2\) and no other line \(l_i\). Let \(P_1 = l_1 \cap l_7'\) and \(P_2 = l_2 \cap l_7'\).

By semicontinuity and Lemma \(\text{[1,2]}\) it will be enough to show that there is no surface \(T\) of degree 9 in \(\mathbb{P}^3\) which is singular along all 7 lines, has multiplicity 3 along \(l_6\), and multiplicity 4 at \(P_1\) and \(P_2\). The restriction of such a surface to \(\Sigma\) would be, in the notations as above,
\[
T|_\Sigma = 9(3h - a) - 2 \sum_{i=1}^5 (2h - a + a_i) - 2(h - a_1 - a_2) - 3f = 5h + a_1 + a_2 - (a_3 + a_4 + a_5) + a_6 - 3f.
\]

After taking out the fixed components \(a_1 + a_2 + a_6\) we are left with the planar system \(\mathcal{O}_{\mathbb{P}^2}(5) \otimes I_{P_1}^2 \otimes I_{P_2}^2 \otimes I_{l_6}^2 \otimes I_{A_1} \otimes I_{A_2} \otimes I_{A_3}\), which by Remark \(\text{[3]}\) is non-effective. Therefore \(\Sigma\) must be a component of \(T\).

Let \(U = T - \Sigma\), which would be a surface of degree 6 passing through all 7 lines, with multiplicity 3 along \(l_6\), and multiplicity 3 at \(P_1\) and \(P_2\). Let \(\Pi_1, \Pi_2\) be the two planes \(\Pi_i = l_6 \vee P_i\), \(i = 1, 2\). The restriction \(U|_{\Pi_i}\) is a plane curve of degree 6 containing \(l_6\) as a triple component, and vanishing at \(P_i\) to order 3 and vanishing in 4 additional points, which are intersection points \(\Pi_i \cap l_j\) for \(j \neq 6\) and such that \(P_i \notin l_j\). Since such a curve does not exist, \(\Pi_1\) and \(\Pi_2\) must be components of \(U\).

Let \(V = U - \Pi_1 - \Pi_2\) be a surface of degree 4 passing through all 7 lines, and singular at \(P_1\) and \(P_2\). A similar computation as above shows that \(V|_{\Sigma}\) is a system of divisors \(h + a_1 + a_2 + a_6\) passing through \(F_1, F_2, F_3\), which is non-effective, so \(\Sigma\) is again a component of \(V\). But \(V - \Sigma\) would then be a plane containing \(l_6, P_1, P_2\), which is clearly impossible, and we are done.

Lemma 5.3. Let \(l_1, \ldots, l_7 \subset \mathbb{P}^3\) be general lines in \(\mathbb{P}^3\), let \(W \rightarrow \mathbb{P}^3\) be the blowup of the seven lines, and denote \(N \subset W\) the pullback of the unique surface in \(\mathcal{L}_{12}(3 \times 6, 2)\). Then \(N\) is smooth.

Proof. By semicontinuity of multiplicities it is enough to check smoothness for a particular choice of lines. We do this computationally, see the Appendix.

Corollary 5.4. Let \(D\) be the unique surface in \(\mathcal{L}_{12}(3 \times 6, 2)\), and \(N \subset W\) its smooth model. Then \(D\) is a surface of general type, with \(p_g = 6, q = 0,\) and \(K^2_N = 8\).

Proof. Denote \(\pi_6 : X \rightarrow \mathbb{P}^3\) the blowup of the first 6 lines, and \(\tau : W \rightarrow X\) the blowup of \(l_7\). By Lemma \(\text{[5,3]}\) \(N\) is the strict transform of \(D\) in \(W\), and by adjunction the canonical class of \(N\) is given as \(K_N = (K_W + N)|_N\). Since
\( h^0(\mathcal{O}_W(K_W)) = h^1(\mathcal{O}_W(K_W)) = 0, p_g(N) = h^0(\mathcal{O}_W(K_W + N)). \) This can be computed, with the help of the map \( \phi_M \) introduced in the proof of the existence of \( D \). Indeed, \( K_W + N = -4H + E + 12H - 3E + E_7 = -2K_W + E_7 = \pi^*(2M) - E_7 \), so
\[
p_g(N) = h^0(W, \mathcal{O}_W(\pi^*(2M) - E_7)) = h^0(X, \mathcal{O}_X(2M) \otimes I_{L_7}) = h^0(Y, \mathcal{O}_Y(2) \otimes I_C),
\]
where as before \( Y = \phi_M(X) \) is a quartic threefold in \( \mathbb{P}^4 \), and \( I_C \) is the ideal sheaf of the quartic curve \( C = \phi_M(L_7) \). Now the residual exact sequence
\[
0 \to \mathcal{O}_{\mathbb{P}^4}(-2) \to \mathcal{O}_{\mathbb{P}^4}(2) \to \mathcal{O}_Y(2) \to 0
\]
gives
\[
p_g(N) = h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2) \otimes I_C) = \dim_C(I_C)_2,
\]
where \( I_C \) is the homogeneous ideal of \( C \) in the coordinate ring of \( \mathbb{P}^4 \), and \((I_C)_2 \) is the degree 2 piece. The ideal of a rational normal quartic curve is well known: it is generated by 6 independent quadrics [18 Examples 1.14 and 9.3]. So \( p_g(N) = 6 \). The selfintersection of the canonical divisor is computed as the intersection number of three divisor classes on \( W \), which by [11] is
\[
\chi(W, \mathcal{O}_W(2K_W + N)) = \chi(W, \mathcal{O}_W(2K_W + 2N)) - \chi(W, \mathcal{O}_W(2K_W + N)),
\]
and these two Euler characteristics can be computed as virtual dimensions with [11]:
\[
\chi(W, \mathcal{O}_W(2K_W + N)) = \chi(W, \mathcal{O}_W(16H - 4E + E_7)) = \dim_{\vir}(L_{16}(4^{X^6}, 2)) = 20
\]
\[
\chi(W, \mathcal{O}_W(2K_W + N)) = \chi(W, \mathcal{O}_W(4H - E + E_7)) = \dim_{\vir}(L_4(4^{X^6})) = 5
\]
Therefore, by Riemann-Roch we have
\[
15 = \chi(N, \mathcal{O}_N(2K_N)) = \frac{1}{2}((2K_N)^2 - 2K_N \cdot K_N) + 1 - q + p_g = 8 + 1 - q + 6,
\]
so the irregularity vanishes, and \( \chi(\mathcal{O}_N) = 7 \).

6. WALDSCHMIDT CONSTANTS

The rest of this note is devoted to asymptotic invariants of the homogeneous ideal \( I = I(Z_s) \) of \( s \) reduced general lines in \( \mathbb{P}^3 \). Recall that \( I = \oplus(I)_d \) and that the initial degree of \( I \) is defined as the number
\[
\alpha(I) = \min \{ d : (I)_d \neq 0 \}.
\]
The \( m \)-th symbolic power in this situation is
\[
I^{(m)} = \bigcap I(L_i)^m.
\]
The asymptotic counterpart of \( \alpha(I) \) is the Waldschmidt constant of \( I \), defined as
\[
\tilde{\alpha}(I) = \inf \frac{\alpha(I^{(m)})}{m} = \lim_{m \to \infty} \frac{\alpha(I^{(m)})}{m}.
\]
These constants were first studied by Waldschmidt \[32\] for ideals defining finite sets of points in \(\mathbb{P}^n\). They are very hard to compute in general. For ideals of general lines in \(\mathbb{P}^3\), Waldschmidt constants were studied in \[14, 10, 11\] and computed for up to 5 lines; see \[14, \text{Corollary 1.1}\] for up to 4 lines or \[10, \text{Proposition B.2.1}\]. In this section we extend these results to 6 lines. We expect that the value of the Waldschmidt constant for 7 lines is determined by the duodecics of type C in Theorem 3.3 and that Waldschmidt constants of a greater number of lines in \(\mathbb{P}^3\) are governed by \[13, \text{Conjecture 5.5}\] (see \[10, \text{Conjecture A}\] for a generalization).

**Proposition 6.1.** Let \(Z_6\) be the union of 6 general lines in \(\mathbb{P}^3\). Then

\[
\hat{\alpha}(I) = \frac{72}{19}
\]

for \(I = I(Z_6)\).

**Proof.** Let \(I\) be the homogeneous ideal of the union of six general lines \(L_1, \ldots, L_6\). By Theorem 3.3 for each \(i = 1, \ldots, 6\), there exists a duodecic \(D_i\) vanishing along \(L_j, j \neq i\), to order 3 and along \(L_i\) to order 4. Then the symmetrized divisor \(D = \sum_{i=1}^{6} D_i\) has degree 72 and vanishing order 19 along all lines. Hence

\[
\hat{\alpha}(I) \leq \frac{72}{19}.
\]

In order to see the reverse inequality, it suffices to show that \(h^0(\mathcal{O}_{\mathbb{P}^3}(d) \otimes I^{(m)}) = 0\), whenever \(\frac{d}{m} < \frac{72}{19}\). In fact, it suffices to show that

\[
h^0(\mathcal{O}_{\mathbb{P}^3}(72m - 1) \otimes I^{(19m)}) = 0
\]

for all \(m \geq 1\). Here the Todd transformation again comes into the picture. We check easily, that the Todd transformation of the system \((72m - 1)H - 19mE\) (in the notation from section 4) is \(-19H + mE\) which is obviously non-effective.

The reader might find it convenient to have an overview of known Waldschmidt constants for \(s \leq 6\) general lines in \(\mathbb{P}^3\) as well as the expected value for \(s \geq 7\).

| \(s\) | 1 | 2 | 3 | 4 | 5 | 6 |
|-------|---|---|---|---|---|---|
| \(\hat{\alpha}(I_s)\) | 1 | 2 | 2 | 5/3 | 10/7 | 72/17 |

For \(s = 7\) we expect that \(\hat{\alpha}(I_7) = 21/5\) and for \(s \geq 8\) we conjecture that \(\hat{\alpha}(I_s)\) equals the largest real root of the polynomial

\[\tau^3 - 3s\tau + 2s\]

This is inspired by the conjectures given in \[13, 10\], but it is stronger, because it is specific about the values of \(s\) for which it is conjectured to hold.

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APPENDIX

// This is a text file, to be made available in the authors’
// pages, containing scripts to
// check some of the claims made in the paper,
// in particular that the surface N in section 5
// is smooth.
// These scripts can be run in
// Singular, https://www.singular.uni-kl.de/

ring R=32003,(x,y,z,w),dp;
LIB "primdec.lib"; option(redSB);

// This procedure shows the degrees of the
// generators of an ideal
// We use it to quickly check the dimension of the homogeneous
// component in its initial degree,
// in particular for uniqueness of the 12-ic
proc writedegrees(ideal CC) {
    "generators degrees:";
    string ss="";
    int i;
    for (i=1;(i<=size(CC))&&(i<=500);i++) {
        ss=ss+string(deg(CC[i]))+" ";
    }
}
We are interested in sets of 7 general lines, and we choose 7 "nice" lines to argue by semicontinuity. These turn out to be general enough for our purposes.

Ideal \( L_1 = x, y \);
Ideal \( L_2 = z, w \);
Ideal \( L_3 = x+w, y+z \);
Ideal \( L_4 = x-y+w, 2x+z \);
Ideal \( L_5 = 2y+z+2w, x-z+w \);
Ideal \( L_6 = 2y+z, 2x-w \);
Ideal \( L_7 = x+z+2w, y+w \);

The rational map \( \phi_M \) of section 5 is given by quartics through \( L_1 \)–\( L_6 \).

We first check that there are 5 independent such quartics.

Ideal \( M_1 = \text{intersect}(L_1, L_2, L_3, L_4, L_5, L_6) \);
\( M_1 = \text{std}(M_1) \);
\( \text{writedegrees}(M_1) \);
Ideal \( M = M_1[1], M_1[2], M_1[3], M_1[4], M_1[5] \);

Next we check that the image of the map is a quartic hypersurface.

Ring \( S = \text{32003, } (x, y, z, w, a, b, c, d, e), \text{dp} \);
Ideal \( M = \text{fetch}(R, M) \);
Ideal \( \text{image} = a-M[1], b-M[2], c-M[3], d-M[4], e-M[5] \);
\( \text{image} = \text{std}(\text{image}) \);
\( \text{image} = \text{eliminate}(\text{image}, xyzw) \);
\( \text{image} \); // Output: a single polynomial of degree 4

A line meeting this quartic properly will do so in 4 points. If its preimage consists of 4 points, this will prove that the map is generically 1:1.

Setring \( R \);
Ideal \( \phi_{\text{lin}} = M[2], M[3], M[5] \); // The preimage of line \([a:0:0:b:0]\)
\( \phi_{\text{lin}} = \text{std}(\phi_{\text{lin}}) \);
\( \phi_{\text{lin}} = \text{sat}(\phi_{\text{lin}}, M)[1] \);
\( \text{hilb}(\phi_{\text{lin}}) \); // consists of four pts

Next we check uniqueness of the 12-ic for these lines.

Ideal \( I = \text{intersect}(L_1^2, L_2^3, L_3^3, L_4^3, L_5^3, L_6^3, L_7^3) \);
\( I = \text{std}(I) \);
\( \text{writedegrees}(I) \);
Poly \( D = I[1] \); // Equation of the 12-ic

And now we will check the singularities of \( N \) (lemma 5.3)
First we want to see that all singularities lie on \( N_E \)
We do it by computing the jacobian ideal of \( D \)
and quotienting by the ideals of the lines
with the corresponding multiplicity

\[
\begin{align*}
\text{ideal } m &= x, y, z, w; \\
\text{ideal } \text{singD} &= \text{jacob}(D); \\
\text{singD} &= \text{std}(\text{singD}); \\
\text{singD} &= \text{quotient}(\text{singD}, L1); \text{ //These computations take some time} \\
\text{singD} &= \text{quotient}(\text{singD}, L1); \\
\text{singD} &= \text{quotient}(\text{singD}, L2); \\
\text{singD} &= \text{quotient}(\text{singD}, L2); \\
\text{singD} &= \text{quotient}(\text{singD}, L2); \\
\text{singD} &= \text{sat}(\text{singD}, m)[1]; \\
\text{singD} &= \text{quotient}(\text{singD}, L3^3); \\
\text{singD} &= \text{quotient}(\text{singD}, L4^3); \\
\text{singD} &= \text{quotient}(\text{singD}, L5^3); \\
\text{singD} &= \text{quotient}(\text{singD}, L6^3); \\
\text{singD} &= \text{quotient}(\text{singD}, L7^3); \\
\text{singD} &= \text{sat}(\text{singD}, m)[1]; \\
\text{singD} &= \text{quotient}(\text{singD}, L3^3); \\
\text{singD} &= \text{quotient}(\text{singD}, L4^3); \\
\text{singD} &= \text{quotient}(\text{singD}, L5^3); \\
\text{singD} &= \text{quotient}(\text{singD}, L6^3); \\
\text{singD} &= \text{quotient}(\text{singD}, L7^3); \\
\text{singD} &= \text{sat}(\text{singD}, m)[1]; \\
\text{singD} &= \text{quotient}(\text{singD}, L3^3); \\
\text{singD} &= \text{quotient}(\text{singD}, L4^3); \\
\text{singD} &= \text{quotient}(\text{singD}, L5^3); \\
\text{singD} &= \text{quotient}(\text{singD}, L6^3); \\
\text{singD} &= \text{quotient}(\text{singD}, L7^3); \\
\text{singD} &= \text{sat}(\text{singD}, m)[1]; \\
\text{singD} &= \text{quotient}(\text{singD}, L3^3); \\
\text{singD} &= \text{quotient}(\text{singD}, L4^3); \\
\text{singD} &= \text{quotient}(\text{singD}, L5^3); \\
\text{singD} &= \text{quotient}(\text{singD}, L6^3); \\
\text{singD} &= \text{quotient}(\text{singD}, L7^3); \\
\text{singD} &= \text{sat}(\text{singD}, m)[1]; \\
\end{align*}
\]

Output: 1, so OK

Now let us analyze singularities on the exceptional divisors
Each \( E_{\text{i}} \) is isomorphic to \( P^1 \times P^1 \), and \( E_{\text{i}} \cap N \),
the tangent cone of \( N \) along \( l_{\text{i}} \),
is given by a bihomogeneous polynomial of bidegree \( (12-m, m) \)
where \( m \) is the multiplicity of \( D \) along \( l_{\text{i}} \).
We shall show that each of these tangent cones
is in fact smooth,
so not only \( N \rightarrow D \) is a resolution of \( D \),
it is an embedded resolution.
By genericity of the lines and symmetry,
it is enough to do it for the double line
and one of the triple ones.
By semicontinuity of multiplicity,
it is enough to do it for these particular lines.
The computation is especially simple
for the "coordinate" lines \( L1 \) and \( L2 \)

\[
\begin{align*}
\text{poly } D1 &= \text{reduce}(D, \text{std}(L1^3)); \text{ //The tangent cone to a double line} \\
\text{poly } D2 &= \text{reduce}(D, \text{std}(L2^4)); \text{ //The tangent cone to a triple line} \\
\end{align*}
\]

We change the ordering to account
for bihomogeneous coordinates

\[
\begin{align*}
\text{ring } T &= 32000, (x, y, z, w), (dp(2), dp(2)); \\
\text{poly } D1 &= \text{fetch}(R, D1); \\
\text{poly } D2 &= \text{fetch}(R, D2); \\
\end{align*}
\]

singD1 = sat(singD1, m1)[1];
singD1 = sat(singD1, m2)[1];
singD1; // Output: 1 so the surface is transverse to E1

ideal singD2 = jacob(D2);
singD2 = sat(singD2, m1)[1]; // Takes some time
singD2 = sat(singD2, m2)[1];
singD2; // Output: 1 so the surface is transverse to E2

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