QCD at finite isospin density: chiral perturbation theory confronts lattice data

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In this paper, we calculate the equation of state of finite isospin chiral perturbation theory at next-to-leading order in the pion-condensed phase at zero temperature. We show that the transition from the vacuum phase to a Bose-condensed phase is of second order. While the tree-level result has been known for some time, surprisingly quantum effects have not yet been incorporated into the equation of state. We find that the one-loop corrections to the quantities we compute, namely the isospin density, pressure, and equation of state, increase with increasing isospin chemical potential and is quite significant particularly for large chemical potentials. It turns out that the uncertainty in the low-energy constants of chiral perturbation theory has a negligible effect on these quantities. Finally, the comparison of our results with recent lattice data is the first precision test of chiral perturbation theory at finite isospin density beyond leading order. The agreement with the lattice results is very good and improves significantly as we go from leading order to next-to-leading order in $\chi$PT.

I. INTRODUCTION

Quantum chromodynamics (QCD), the fundamental theory of strong interactions, has a rich phase structure, particularly at finite baryon densities relevant for a number of physical systems including neutron stars, neutron matter and heavy-ion collisions among others [1–3]. However, finite baryon densities are not accessible directly through QCD since the physics is non-perturbative and lattice calculations are hindered by the fermion sign problem. Though it is worth noting that some progress has been made in circumventing the sign problem through the fermion bag and Lefschetz thimble approaches [4]. There is also the additional possibility of solving QCD at finite baryon density with quantum computers since the sign problem is absent in quantum algorithms [5].

While finite baryon density is inaccessible through lattice QCD, finite isospin systems in real QCD can be studied using lattice-based methods, see Ref. [6, 7] for some early results. The most thorough of these studies were performed only recently [8–10] even though finite isospin QCD was first studied over a decade ago using chiral perturbation theory ($\chi$PT) in a seminal paper by Son and Stephanov [11]. Chiral perturbation theory [12–15] is a low-energy effective field theory of QCD that describes the dynamics of the pseudo-Goldstone bosons that are the result of the spontaneous symmetry breaking of global symmetries in the QCD vacuum. Being based only on symmetries and degrees of freedom, the predictions of $\chi$PT are model independent.

It is agreed through both lattice QCD and chiral perturbation theory studies that at an isospin chemical potential equal to the physical pion mass there is a second-order phase transition at zero temperature from the vacuum phase to a pion-condensed phase. With increasing chemical potentials there is a crossover transition to a BCS phase with a parity breaking order parameter, $\langle \bar{u}\gamma_5 d \rangle \neq 0$ or $\langle \bar{d}\gamma_5 u \rangle \neq 0$, that have the same quantum numbers as a charged pion condensate, which is nonzero above the critical chemical potential. At asymptotically large isospin densities, due to asymptotic freedom, an effective anisotropic pure glue theory can be written down and a first order deconfinement transition occurs for large enough temperatures, which has been verified through lattice QCD. Furthermore, for large temperatures of approximately 170 MeV, the pion condensates are destroyed due to thermal fluctuations. Various aspects of $\chi$PT at finite isospin density can be found in Refs. [11, 16–22]. Finite isospin systems have also been
studied in the context of QCD models including the nonrenormalizable Nambu-Jona-Lasinio model \cite{23,35}, and the renormalizable quark-meson model \cite{36,39}, with the results found there being largely in agreement with lattice QCD.

In addition to the study of pions at finite isospin chemical potential there has also been recent interest in the study of pions in the presence of external magnetic fields, which are relevant in the context of neutron stars with large fields (magnetars) and possibly in RHIC collisions, which generate magnetic fields due to accelerated charged beams of lead and gold nuclei. In neutron star cores, an isospin asymmetry is present since protons are converted into neutrons and neutrinos through electron capture. However, in the presence of a magnetic field, finite isospin systems are difficult to study due to the fermion sign problem on the lattice QCD that arises as a consequence of flavor asymmetry between up and anti-down quarks for electromagnetic interactions. The complex action problem is tackled by studying finite isospin densities for small magnetic fields, where the sign problem is mild. The lattice observes a diamagnetic phase \cite{40}, while studies in chiral perturbation theory valid for magnetic fields \(eB \ll (4\pi f_\pi)^2\) suggests that pions behave as a type-II superconductor \cite{41}.

More recently, due to the accessibility of the equation of state (EoS) of pion degrees of freedom through lattice QCD, there has been a lot of interest in the possibility of pion stars, a type of boson star that does not require the hypothesized axion, which was initially proposed as a solution to the strong CP problem in QCD. Pion stars, on the other hand, only require input from QCD and it is conjectured that pion condensation takes place in a gas of dense neutrons \cite{12}. Recent work shows that pion stars are typically much larger in size than neutron stars due to a softer equation of state and that the isospin chemical potentials at the center of such stars can be as high as 250 MeV for purely pionic stars and smaller for pion stars electromagnetically neutralized by leptons \cite{13,44}.

The goal of this paper is to revisit the equation of state for finite isospin QCD in the regime of validity of \(\chi PT\), where we expect \(\mu_I \ll 4\pi f_\pi\). The equation of state (at tree level) was originally calculated in Ref. \cite{11} of QCD. In this paper, we calculate the equation of state within \(\chi PT\) and incorporate leading order quantum corrections.

We begin in Sec. II with a brief overview of chiral perturbation theory and how to parametrize the fluctuations around the ground state. We derive the Lagrangian that is needed for all next-to-leading order (NLO) calculations within \(\chi PT\) at finite isospin chemical potential allowing for a charged pion condensate. In Sec. III, we use this NLO Lagrangian to calculate the renormalized one-loop free energy at finite \(\mu_I\). In Sec. IV, we calculate the isospin density, the pressure, and the equation of state in the pion-condensed phase. Our results are compared to those of recent lattice simulations.

### II. \(\chi PT\) LAGRANGIAN AT \(O(p^4)\)

We begin with the chiral perturbation theory Lagrangian (in the isospin limit) at \(O(p^2)\)

\[
\mathcal{L}_2 = \frac{f^2}{4} \text{Tr} \left[ \nabla^\mu \Sigma^\dagger \nabla_\mu \Sigma \right] + \frac{f^2 m^2}{4} \text{Tr} \left[ \Sigma + \Sigma^\dagger \right],
\]

where \(f\) is the (bare) pion decay constant and \(m\) the (bare) pion mass. The covariant derivatives at finite isospin are defined as follows

\[
\nabla_\mu \Sigma = \partial_\mu \Sigma - i [v_\mu, \Sigma],
\]

\[
\nabla_\mu \Sigma^\dagger = \partial_\mu \Sigma^\dagger - i [v_\mu, \Sigma^\dagger],
\]

where \(v_\mu = \delta_\mu_0 \mu_I \frac{\hat{\Sigma}}{\Sigma}\) with \(\mu_I\) denoting the isospin chemical potential and \(\gamma_3\) the third Pauli matrix.

It is well known that chiral perturbation theory encodes the interactions among the Goldstone bosons (pions) that arise due to the spontaneous breaking of chiral symmetry by the QCD vacuum, i.e.

\[
\Sigma_{ji} \equiv \langle \bar{\psi}_i R \psi_j L \rangle = 0
\]

Under chiral rotations, i.e. \(SU(2)_L \times SU(2)_R\),

\[
\psi_L \rightarrow L \psi_L,
\]

\[
\psi_R \rightarrow R \psi_R.
\]

As such \(\Sigma\) transforms as

\[
\Sigma \rightarrow L \Sigma R^\dagger.
\]

We note that the Lagrangian above (and chiral perturbation theory at all orders) is symmetric under \(SU(2)_L \times SU(2)_R\) only in the chiral limit and zero isospin chemical potential. Both quark masses (or equivalently the mass term above) and isospin chemical potentials separately break this symmetry explicitly down to the vector subgroup \(SU(2)_L + SU(2)_R \equiv SU(2)_V\). Furthermore, for \(\mu_I > m_\pi\), where \(m_\pi\) is the physical pion mass equal to \(m\) at tree level, pion condensation occurs and the ground state explicitly breaks this symmetry further down to a \(U(1)\) corresponding to a phase rotation of the charged pions.

#### A. Ground State

We briefly review the ground state of \(\chi PT\) at finite isospin using the \(O(p^2)\) Lagrangian above and the parametrization,

\[
\Sigma = \cos \alpha \mathbf{1} + i \hat{\phi}_1 \gamma_1 \sin \alpha,
\]

where \(\hat{\phi}_1\) is a unit vector, i.e. \(\hat{\phi}_1 \cdot \hat{\phi}_1 = 1\). This parametrization guarantees that \(\Sigma^\dagger \Sigma = 1\). The static Hamiltonian at \(O(p^2)\) then becomes

\[
\mathcal{H}_2 = -\mathcal{L}_2 = -f^2 m^2 \left[ \cos \alpha + \frac{\mu_I^2}{2m^2} \sin^2 \alpha (\hat{\phi}_1^2 + \hat{\phi}_2^2) \right].
\]
By minimizing the above expression with respect to $\alpha$, we get the well-known result that pion condensation occurs for $\mu_I \geq m$ with $\cos \alpha = \frac{m^2}{\mu_I^2}$. We also note that since the Hamiltonian does not explicitly depend on $\phi_3$ and $\phi_I \phi_i = 1$, neutral pions do not condense in the vacuum.

### B. Parametrizing Fluctuations

Since the goal of this paper is to study the equation of state of the pion condensed phase including quantum corrections, it is natural to expand the chiral perturbation theory Lagrangian around the pion condensed ground state. The Goldstone manifold as a consequence of chiral symmetry breaking is $SU(2)_L \times SU(2)_R/SU(2)_V$. As such, we proceed by first parametrizing the condensed vacuum as follows

$$\Sigma_\alpha = A_\alpha \Sigma_0 A_\alpha^\dagger$$

(9)

$$A_\alpha = e^{i \frac{1}{2} \left( \phi_1 \tau_1 + \phi_2 \tau_2 \right)} = \cos \frac{\alpha}{2} + i (\hat{\phi}_1 \tau_1 + \hat{\phi}_2 \tau_2) \sin \frac{\alpha}{2}$$

(10)

where we, for the purposes of this paper, choose $\hat{\phi}_1 = 1$ and $\hat{\phi}_2 = 0$ without any loss of generality. Note that $\alpha = 0$ reproduces the normal vacuum with $\Sigma_0 = 1$ as required. Then the fluctuations (which are axial) around this condensed vacuum are parametrized as

$$\Sigma = L_\alpha \Sigma_0 R_\alpha$$

(11)

with

$$L_\alpha = A_\alpha U A_\alpha^\dagger$$

(12)

$$R_\alpha = A_\alpha^\dagger U A_\alpha^\dagger$$

(13)

We emphasize that the fluctuations parameterized by $L_\alpha$ and $R_\alpha$ around the ground state depend on $\alpha$ since the broken generators (of QCD) need to be rotated appropriately as the condensed vacuum rotates with the angle $\alpha$. $U$ is an $SU(2)$ matrix that parameterizes the fluctuations around the ground state:

$$U = \exp \left( i \frac{\phi_0 \tau_0}{2 f} \right)$$

(14)

With the parameterizations stated above, we get

$$\Sigma = A_\alpha (U \Sigma_0 U) A_\alpha$$

(15)

$$= A_\alpha (U \Sigma_0 U) A_\alpha$$

As we show later in this paper, this parameterization not only produces the correct linear terms that vanish at $O(p^2)$ (see following subsection), the divergences of one-loop diagrams also cancel using counterterms from the $O(p^4)$ Lagrangian. Furthermore, the parametrization produces a Lagrangian that is canonical in the fluctuations and has the correct limit when $\alpha = 0$, whereby

$$\Sigma = U \Sigma_0 U = U^2 = \exp \left( i \frac{\phi_0 \tau_0}{f} \right)$$

(16)

as expected.

We would like to emphasize the importance of using $L_\alpha$ and $R_\alpha$ instead of $L = U$ and $R = U^\dagger$. If the latter set is used, Eq. (11) is replaced by

$$\Sigma_{\text{wrong}} = U \Sigma_0 U = U A_\alpha \Sigma_0 A_\alpha U$$

(17)

and one finds that the kinetic term of the Lagrangian is not properly normalized. This is in itself not problematic since the canonical normalization can be achieved by a field redefinition. This field redefinition changes the mass and interaction terms of the Lagrangian and only at the minimum of the LO effective potential do the masses coincide with the correct expressions, Eqs. (24)–(27) below. Moreover, if one computes the one-loop effective potential, it turns out that the counterterms cancel the divergences only at the classical minimum. Thus one cannot renormalize the NLO effective potential away from the LO minimum and therefore not find the NLO minimum, which shows that the $\Sigma_{\text{wrong}}$ in Eq. (17) cannot be correct.

### C. Leading-order Lagrangian

Using the parameterization of Eq. (15) discussed above, we can write down the Lagrangian in terms of the fields $\phi_\alpha$, which parametrizes the Goldstone manifold

$$\mathcal{L}_2 = \mathcal{L}_2^{\text{linear}} + \mathcal{L}_2^{\text{static}} + \mathcal{L}_2^{\text{quadratic}} + \cdots$$

(18)

where

$$\mathcal{L}_2^{\text{static}} = f^2 \left( m^2 \cos \alpha + \frac{\mu_I^2}{2} \sin^2 \alpha \right)$$

(19)

$$\mathcal{L}_2^{\text{linear}} = f \left( -m^2 \sin \alpha + \mu_I^2 \cos \alpha \sin \alpha \right) \phi_1 + f \mu_1 \sin \alpha \phi_2$$

(20)

$$\mathcal{L}_2^{\text{quadratic}} = \frac{1}{2} \left( \partial_\mu \phi_\alpha \right) \left( \partial^\mu \phi_\alpha \right) + \mu_1 \cos \alpha \left( \phi_1 \partial_0 \phi_2 - \phi_2 \partial_0 \phi_1 \right)$$

(21)

The inverse propagator in the $\phi_\alpha$ basis is

$$D^{-1}_Q = \begin{pmatrix} D_{Q1}^{-1} & 0 \\ 0 & m_1^2 \end{pmatrix}$$

(22)

$$D_{Q1}^{-1} = \begin{pmatrix} p^2 - m_1^2 & ip_0 m_3 \\ -ip_0 m_3 & p^2 - m_1^2 \end{pmatrix}$$

(23)

---

1 Consider e.g. a theory with an $SO(3)$ symmetric Lagrangian with the ground state picking up a vev say in the $z$-direction. If the vev is rotated to the $y$-direction, then the (un)broken generators must be rotated accordingly.
where the masses are
\[
\begin{align*}
m_1 &= \sqrt{m^2 \cos \alpha - \mu_1^2 \cos 2\alpha} , \\
m_2 &= \sqrt{m^2 \cos \alpha - \mu_1^2 \cos^2 \alpha} , \\
m_3 &= 2\mu_1 \cos \alpha , \\
m_4 &= \sqrt{m^2 \cos \alpha + \mu_1^2 \sin^2 \alpha} ,
\end{align*}
\]
and with \(D_Q^{-1}\) representing the inverse propagator for the charged pions. The dispersion relation can be found using the zeros of the inverse propagator \(D^{-1}\). We find that the energies associated with the three pion modes are as follows
\[
\begin{align*}
E^2_\pm &= p^2 + \frac{1}{2} \left( m_1^2 + m_2^2 + m_3^2 \right) \\
&\pm \frac{1}{2} \sqrt{4p^2m_3^2 + (m_1^2 + m_2^2 + m_3^2)^2 - 4m_1^2m_2^2} \\
E_4^2 &= p^2 + m_4^2 ,
\end{align*}
\]
where \(p\) represents the spatial three-momenta. The full propagator can then be written in terms of the dispersion relations as follows
\[
\begin{align*}
D &= \begin{pmatrix} D_Q & 0 \\
0 & (p^2 - m_4^2)^{-1} \end{pmatrix} , \\
D_Q &= \frac{1}{(p^0 - E_+^2)(p^0 - E_-^2)} \left( p^2 - m_2^2 - ip_0m_3 \quad p^2 - m_1^2 \right)
\end{align*}
\]
Expanding the Lagrangian \(L_2\) beyond the quadratic terms, we get for terms with three and four fields
\[
\begin{align*}
L_2^{\text{cubic}} &= \frac{(m^2 - 4\mu_1^2 \cos \alpha ) \sin \alpha}{6f} \phi_1(\phi_a \phi_a) - \frac{\mu_1 \sin \alpha}{f} \left[ \phi_2^0 \phi_2 + \phi_3^0 \phi_2 \right] , \\
L_2^{\text{quartic}} &= \frac{1}{24f^2} \left[ (m^2 \cos \alpha - 4\mu_1^2 \cos 2\alpha) \phi_1^0 + \right. \\
&\left. \left( m^2 \cos \alpha - 4\mu_1^2 \cos^2 \alpha \right) \phi_2^0 + \right. \\
&\left. \left( m^2 \cos \alpha + 4\mu_1^2 \sin^2 \alpha \right) \phi_3^0 \right] , \\
&\left. - \frac{\mu_1 \cos \alpha}{3f^2} \left( \phi_1 \phi_0 \phi_2 - \phi_2 \phi_0 \phi_1 \right) + \right. \\
&\left. \frac{1}{6f^2} \left[ \phi_a \phi_b \partial^\mu \phi_a \partial^\mu \phi_b - \phi_a \phi_a \partial_\mu \phi_b \partial^\mu \phi_b \right] \right] .
\end{align*}
\]

D. Next-to-leading order Lagrangian

In order to perform calculations to NLO, we need the terms in the Lagrangian that contribute at \(O(p^4)\). They are
\[
\begin{align*}
L_4 &= \frac{1}{4} l_1 \left( \text{Tr} \left[ D_\mu \Sigma^\dagger D^\nu \Sigma \right] \right)^2 + \frac{1}{4} l_2 \text{Tr} \left[ D_\mu \Sigma^\dagger D_\nu \Sigma \right] \text{Tr} \left[ D^\mu \Sigma^\dagger D^\nu \Sigma \right] + \frac{1}{16} (l_3 + l_4) m^4 (\text{Tr} [\Sigma + \Sigma^\dagger])^2 \\
&+ \frac{1}{8} l_4 m^2 \text{Tr} \left[ D_\mu \Sigma^\dagger D^\nu \Sigma \right] \text{Tr} [\Sigma + \Sigma^\dagger] ,
\end{align*}
\]
where \(l_i\) are the so-called low-energy constants [13]. The unrenormalized parameters are defined as
\[
l_i = -\frac{\gamma_i}{2(4\pi)^2} \left[ \frac{1}{\epsilon} + 1 + \ln \left( \frac{\Lambda^2}{m^2} \right) - \bar{l}_i \right] ,
\]
where \(\bar{l}_i\) are the renormalized low-energy constants and
\[
\begin{align*}
\gamma_1 &= \frac{1}{3} , \quad \gamma_2 = \frac{2}{3} , \quad \gamma_3 = -\frac{1}{2} , \quad \gamma_4 = 2 .
\end{align*}
\]
In writing the NLO Lagrangian above, we have ignored contributions at finite isospin through the Wess-Zumino-Witten (WZW) Lagrangian, which is of the form
\[
\mathcal{L}_{\text{WZW}} \sim \epsilon_{\mu
u\rho\sigma} \mu_1 \text{Tr} \left[ \gamma_3 (\Sigma \partial_\mu \Sigma^\dagger) (\Sigma \partial_\nu \Sigma^\dagger) (\Sigma \partial_\rho \Sigma^\dagger) \right] (37)
\]
with the leading contribution at \(O(p^4)\). There is also a separate contribution at zero external field at the same order but neither of these contributions through the WZW action is necessary for the renormalization of the divergences at 1-loop as we will see in the following section.

Expanding the Lagrangian [34] to second order in the fields, we obtain
\( \mathcal{L}_4^{\text{linear}} = (l_1 + l_2) \frac{4 \mu_4^2}{f} \cos \alpha \sin^3 \phi_1 + \frac{m_2^2 \mu_1^2}{f} (2 \sin \alpha - 3 \sin^2 \alpha) \phi_1 - (l_3 + l_4) \frac{2m_4}{f} \sin \alpha \cos \phi_1 \)

\( \mathcal{L}_4^{\text{quadratic}} = (l_1 + l_2) \frac{2 \mu_4^2 \sin^2 \alpha}{f^2} \left[ (1 + 2 \cos 2\alpha) \phi_1^2 + \cos^2 \alpha \phi_2^2 - \sin^2 \alpha \phi_3^2 \right] + \frac{l_4 m_2^2 \cos \alpha}{f^2} \left[ (-5 + 9 \cos 2\alpha) \phi_1^2 + (1 + 3 \cos 2\alpha) \phi_2^2 - 6 \sin^2 \alpha \phi_3^2 \right] - (l_3 + l_4) \frac{m_2^4}{f^2} \left[ (\cos 2\alpha) \phi_1^2 + \cos^2 \alpha (\phi_2^2 + \phi_3^2) \right] - (l_1 + l_2) \frac{4 \mu_1^2 \sin \alpha \sin 2\alpha}{f^2} (\phi_2 \partial_0 \phi_1 - \phi_1 \partial_0 \phi_2) + l_4 \frac{2m_2^2 \mu_1}{f^2} \left[ - \cos^2 \alpha \phi_2 \partial_0 \phi_1 + \cos 2\alpha \phi_1 \partial_0 \phi_2 \right] + l_1 \frac{2 \mu_1^2}{f^2} \left[ 2 \sin^2 \alpha (\partial_\mu \phi_3) (\partial^\mu \phi_3) + l_2 \frac{2 \mu_1^2}{f^2} \sin^2 \alpha (\partial_\mu \phi_3) (\partial^\mu \phi_3) \right] + (l_1 + l_2) \frac{4 \mu_1^2 \sin^2 \alpha}{f^2} (\phi_2 \partial_0 \phi_1) + 2(\partial_0 \phi_2) + (\partial_0 \phi_3^2) ) + l_4 \frac{m_2^2 \cos \alpha}{f^2} (\partial_\mu \phi_3) (\partial^\mu \phi_3) \right) (40) \)

Eqs. (39)–(41) from \( \mathcal{L}_2 \), the Wess-Zumino-Witten term (37), and Eqs. (38)–(40) from \( \mathcal{L}_4 \) provide us with all the terms needed for a NLO calculation within \( \chi \)PT.

### III. ONE-LOOP EFFECTIVE POTENTIAL

Using the dispersion relations for the pions, we can write down the one-loop contribution to the effective potential as follows

\[ V_{\text{eff}} = V_{\text{eff},Q} + V_{\text{eff},0} = \frac{1}{2} \int_p (E_+ + E_-) + \frac{1}{2} \int_p E_4 (41) \]

where \( V_{\text{eff},Q} \) is the contribution to the one-loop effective potential due to the charged pions and \( V_{\text{eff},0} \) due to the neutral pion. The integral is defined as

\[ \int_p = \left( \frac{e^{\gamma_E} \Lambda^2}{4\pi} \right)^{-\epsilon} \int \frac{d^d p}{(2\pi)^d} , \]

where \( \Lambda \) is the renormalization scale in the modified minimal subtraction (\( \overline{\text{MS}} \)) scheme and \( d \) is the number of spatial dimensions. Using Eq. (A1), we find

\[ V_{\text{eff},0} = \frac{1}{2} \int_p \sqrt{p^2 + m_4^2} \]

\[ = - \frac{m_4^2}{4(4\pi)^2} \left[ \frac{1}{\epsilon} + \frac{3}{2} + \log \left( \frac{\Lambda^2}{m_4^2} \right) \right] . \]

The calculation of \( V_{\text{eff},Q} \) requires isolating the ultraviolet divergences, which can be done by expanding \( E_\pm \) in powers of \( \frac{1}{p} \), which gives

\[ E_+ + E_- = 2p + \frac{2(m_1^2 + m_2^2) + m_3^2}{4p} - \frac{8(m_1^4 + m_2^4) + 4(m_1^2 + m_2^2)m_3^2 + m_3^4}{64p^3} + ... \]

Performing the integral \( \int_p \), the terms proportional to \( p \) and \( \frac{1}{p} \) vanish in dimensional regularization since there is no scale in either of the integrals. The last term on the other hand is logarithmically divergent in both the infrared and the ultraviolet. In order to avoid the introduction of unphysical infrared divergences, we add a mass term in the denominator by making the substitution \( p^2 \to p^2 + \Lambda_\text{IR}^2 \), where we choose \( \Lambda_\text{IR}^2 = m_4^2 \). Then the divergent piece is

\[ V_{\text{eff},Q}^{\text{div}} = - \frac{8(m_1^4 + m_2^4) + 4(m_1^2 + m_2^2)m_3^2 + m_3^4}{64} \left[ \frac{1}{2} \int_p \frac{1}{(p^2 + m_4^2)^2} \right] \]

\[ = - \frac{8(m_1^4 + m_2^4) + 4(m_1^2 + m_2^2)m_3^2 + m_3^4}{64} \left[ \frac{2}{(4\pi)^2} \left( \frac{1}{\epsilon} + \log \left( \frac{\Lambda^2}{m_4^2} \right) \right) \right] , \]

where we have used Eq. (A2). The finite piece is

\[ V_{\text{eff},Q}^{\text{fin}} = \frac{1}{2} \int_p \left[ E_+ + E_- - \left( 2p + \frac{2(m_1^2 + m_2^2) + m_3^2}{4p} - \frac{8(m_1^4 + m_2^4) + 4(m_1^2 + m_2^2)m_3^2 + m_3^4}{64(p^2 + m_4^2)^2} \right) \right] . \]
The expression for the divergent pieces can be written in terms of \( \alpha \) using the explicit expressions for \( m_4 \), Eqs. (24)–(27). We find

\[
V_{\text{eff}}^{\text{div}} = V_{\text{eff},Q}^{\text{div}} + V_{\text{eff,0}}^{\text{div}} = - \frac{1}{4(4\pi)^2} \left[ \frac{1}{\epsilon} + \frac{3}{2} + \log \left( \frac{\Lambda^2}{m_4^2} \right) \right] (m^4 \cos^2 \alpha + 2m^2 \mu_I^2 \cos \alpha \sin^2 \alpha + \mu_I^4 \sin^4 \alpha)
\]

\[
- \frac{1}{4(4\pi)^2} \left[ \frac{1}{\epsilon} + \log \left( \frac{\Lambda^2}{m^2} \right) \right] (2m^4 \cos^2 \alpha + 2m^2 \mu_I^2 \cos \alpha \sin^2 \alpha + \mu_I^4 \sin^4 \alpha).
\]

The one-loop counterterm comes from the static part of the Lagrangian, \( \mathcal{L}^\text{static}_4 \), given by Eq. (38). After renormalization, the effective potential has the form

\[
V_{\text{eff}} = -f^2 m^2 \cos \alpha - \frac{1}{2} f^2 \mu_I^2 \sin^2 \alpha + \frac{1}{4(4\pi)^2} \left[ \frac{3}{2} + \bar{l}_3 - 4\bar{l}_4 + \log \left( \frac{m_4^2}{m^2} \right) \right] m^4 \cos^2 \alpha
\]

\[
+ \frac{1}{2(4\pi)^2} \left[ \frac{1}{2} - 2\bar{l}_4 + \log \left( \frac{m_4^2}{m^2} \right) \right] m^2 \mu_I^2 \cos \alpha \sin^2 \alpha + \frac{1}{4(4\pi)^2} \left[ \frac{1}{2} - \frac{2}{3} \bar{l}_1 - \frac{4}{3} \bar{l}_2 + \log \left( \frac{m_4^2}{m^2} \right) \right] \mu_I^4 \sin^4 \alpha + V_{\text{eff},Q}^{\text{fin}}.
\]

The thermodynamic functions can be derived from minimizing the effective potential \( \mathcal{V}_{\text{eff}} \) with respect to the parameter \( \alpha \). This can also be used to show that the linear term vanishes on-shell i.e. for the value of \( \alpha \) that minimizes \( V_{\text{eff}} \). At tree-level this is straightforward. The leading order potential is minus Eq. (19), and the value of \( \alpha \) that minimizes it, satisfies \( m^2 = \mu_I^2 \cos \alpha \). At next-to-leading-order, this requires a bit more work. The one-loop contributions to the one-point functions arise from the cubic terms in Eq. (32). In a consistent NLO calculation, it suffices to evaluate the one-loop contribution to the one-point function using the LO value of \( \alpha \), i.e. use that \( m^2 = \mu_I^2 \cos \alpha \) at tree level. One can then show that the one-loop correction to the one-point function is equal to the derivative of \( V_{\text{eff}} \) in of Eq. (41) with respect to \( \alpha \) evaluated at \( m^2 = \mu_I^2 \cos \alpha \). The conclusion is that the linear term vanishes at the minimum of the effective potential, as it should.

**IV. THERMODYNAMICS**

In this section, we investigate the thermodynamics of the pion-condensed phase using the effective potential \( \mathcal{V}_{\text{eff}} \). In addition to the equation of state \( \epsilon(P) \), where \( \epsilon \) is the energy density and \( P \) the pressure, we are also interested in the isospin density \( n_I \) and the pressure \( P \) as a function of the isospin chemical potential \( \mu_I \). In order to evaluate these quantities we need to know the low-energy constants \( \bar{l}_i \). They have the following values and uncertainties \( \bar{l}_i \):

\[
\bar{l}_1 = -0.4 \pm 0.6, \quad \bar{l}_2 = 4.3 \pm 0.1, \quad \bar{l}_3 = 2.9 \pm 2.4, \quad \bar{l}_4 = 4.4 \pm 0.2.
\]

In order to show that there is a second-order transition at a critical chemical potential \( \mu_I^c \), we expand the effective potential in powers of \( \alpha \) up to \( \mathcal{O}(\alpha^4) \) to obtain an effective Landau-Ginzburg energy functional,

\[
V_{\text{eff}}^{\text{LG}} = a_0 + a_2(\mu_I)\alpha^2 + a_4(\mu_I)\alpha^4.
\]

The critical isospin chemical potential is defined by \( \mu_I^c \) by the vanishing of the coefficient of the \( \alpha^2 \) term, i.e. \( a_2(\mu_I^c) = 0 \). If \( a_4(\mu_I^c) > 0 \), the transition is second order. Using the techniques of Ref. [46], we find

\[
V_{\text{eff}}^{\text{LG}} = -f^2 m^2 + \frac{1}{(4\pi)^2} \left[ \frac{3}{2} \left( \frac{1}{4} - \bar{l}_3 - \bar{l}_4 \right) \right] m^4 + \left( \frac{1}{2} f^2 (m^2 - \mu_I^2) - \frac{1}{(4\pi)^2} \left[ \left( \frac{1}{4} \bar{l}_1 - \bar{l}_4 \right) m^4 + \bar{l}_4 m^2 \mu_I^2 \right] \right) \alpha^2 + \mathcal{O}(\alpha^4).
\]

Expressing Eq. (54) in terms of of the physical pion mass \( m_\pi^2 \), Eq. (B5), and the pion decay-constant, Eq. (B6), we obtain

\[
V_{\text{eff}}^{\text{LG}} = -f^2 m_\pi^2 + \frac{3m_\pi^4}{8(4\pi)^2} + \frac{1}{2} f^2 \left( m_\pi^2 - \mu_I^2 \right) \alpha^2 + \mathcal{O}(\alpha^4).
\]
The coefficient $a_2(\mu_I)$ vanishes when $\mu^c_I = m_\pi$, which is independent of the choice of infrared cutoff, $\Lambda_{IR}$. One can also find the quartic coefficient $\mathcal{O}(\alpha^4)$ evaluated at $\mu_I = m_\pi$. One finds

$$a_4(\mu^c_I) = \frac{1}{8}\beta^2 m^2 \left\{ 1 - \frac{m^2}{2(4\pi)^2 f^2} \left[ 1 + \frac{8}{3} f_1 + \frac{16}{3} f_2 - \frac{8}{3} g_3 \right] \right\} > 0 \; ,$$

(56)

which means that the onset of pion condensation is a second-order transition.

The isospin density is defined by

$$n_I = \frac{\partial V_{\text{eff}}}{\partial \mu_I} \; .$$

(57)

In Fig. 1, we show the isospin density normalized by $m_\pi^3$ as a function of the chemical potential $\mu_I$, normalized by $m_\pi$. The red curve shows the tree-level result and the blue curve the one-loop result with the uncertainty shown by the dashed lines. The uncertainty arises due to the uncertainty in the low-energy constants. We also show the lattice points from Ref. [43].

![FIG. 1. Normalized isospin density as a function of the normalized isospin chemical potential. The red curve shows the tree-level result and the blue curve the one-loop result and uncertainty. The points are lattice data from Ref. [43].](image)

In Fig. 2, we show the pressure normalized to $m_\pi^4$ as a function of the isospin chemical potential normalized to $m_\pi$. The red curve is the tree-level result, while the blue curve is the one-loop result. The dashed blue lines show the uncertainty in the pressure due to the uncertainty in the low-energy constants. We have normalized the pressure to zero in the normal vacuum and at the critical chemical potential the pressure increases steadily with the chemical potential. The NLO pressure increases faster than the LO pressure and is in excellent agreement with the lattice results.

![FIG. 2. The normalized pressure as a function of the normalized isospin chemical potential. The tree-level result is in red and the one-loop result in blue. The dashed line is the lattice results from Ref. [43].](image)

The energy density is defined by

$$\epsilon = -P + n_I \mu_I \; ,$$

(59)

and can be used to find the EoS, $\epsilon = \epsilon(P)$. In Fig. 3 we show the normalized equation of state. The LO result is in red while the NLO result is in blue. The dashed blue lines show the uncertainty in the EoS due to the uncertainty in the low-energy constants. The black dashed line shows the lattice results from Ref. [43].

![FIG. 3. The normalized equation of state at tree level is shown in red at one loop in blue. The dashed line is the lattice results from Ref. [43].](image)

In Fig. 3, we show the pressure normalized to $m_\pi^4$ as a function of the isospin chemical potential normalized to $m_\pi$. The red curve is the tree-level result, while the blue curve is the one-loop result. The dashed blue lines show the uncertainty in the pressure due to the uncertainty in the low-energy constants. We have normalized the pressure to zero in the normal vacuum and at the critical chemical potential the pressure increases steadily with the chemical potential. The NLO pressure increases faster than the LO pressure and is in excellent agreement with the lattice results.

![FIG. 3. The normalized equation of state at tree level is shown in red at one loop in blue. The dashed line is the lattice results from Ref. [43].](image)
the NLO equation of state is softer than the LO one and that the difference increases steadily with the pressure $P$. Moreover, the NLO EoS is in better agreement with the lattice results than the LO EoS and is generally very good.

In conclusion, we have derived the $\chi$PT Lagrangian which is necessary for all NLO calculations at finite isospin. We have applied this Lagrangian calculating the pressure, isospin density, as well as the equation of state. Our results are in excellent agreement with the lattice results of Ref. [43] and are the first test of $\chi$PT in the pion-condensed phase beyond leading order. The Lagrangian we have derived can be used to calculate e.g. the one-loop corrections to the quasiparticle masses in the pion-condensed phase. Here a nontrivial check would be to show that one of the branches is a massless Goldstone boson. The Lagrangian for three-flavor QCD can be derived in the same way and opens up the possibility to study quantum effects in phases that involve pion or kaon condensation. Finally, an NNLO calculation of the thermodynamic functions is within reach. Work in this direction is in progress [48].

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Appendix A: Dimensional Regularized Integrals

We specifically need two integrals in $d = 3 - 2\epsilon$ dimensions,

$$\int_p \frac{1}{(p^2 + m^2)^2} = \frac{4}{(4\pi)^2} \left( \frac{\Lambda^2}{m^2} \right)^\epsilon \left[ \frac{1}{\epsilon} + \frac{3}{2} + O(\epsilon) \right].$$  \(A1\)

$$\int_p \frac{1}{p^2 - m^2} = \frac{m^4}{2(4\pi)^2} \left( \frac{\Lambda^2}{m^2} \right)^\epsilon \left[ \frac{1}{\epsilon} + O(\epsilon) \right].$$  \(A2\)

We need a single integral in $d = 4 - 2\epsilon$ dimensions,

$$\int_p \frac{1}{p^2 - m^2} = \frac{im^2}{(4\pi)^2} \left( \frac{\Lambda^2}{m^2} \right)^\epsilon \left[ \frac{1}{\epsilon} + 1 + O(\epsilon) \right].$$  \(A3\)

Appendix B: Mass renormalization

In order to show the second-order nature of the phase transition to a Bose-condensed phase at $\mu_F^2 = m_\pi^2$, where $m_\pi$ is the physical mass in the vacuum, we need to express it in terms of the parameters $m$ and $f$ of the chiral Lagrangian. The relevant interaction terms are found by setting $\alpha = 0$ in the Lagrangian $\mathcal{L}_2$ and the relevant counterterms are found by setting $\alpha = 0$ in the Lagrangian $\mathcal{L}_4$. The inverse propagator for the pion is

$$k^2 - m^2 - i\Sigma(k^2) - i\Sigma_{ct}(k^2),$$  \(B1\)

where the one-loop self-energy and the associated counter-term in the vacuum are

$$\Sigma(k^2) = \frac{2k^2}{3f^2} \int_p \frac{1}{p^2 - m^2} - \frac{m^2}{6f^2} \int_p \frac{1}{p^2 - m^2},$$  \(B2\)

$$\Sigma_{ct}(k^2) = -2ik^2 \frac{m^2}{f^2} l_4 - \frac{2im^4}{f^2} (l_5 + l_4).$$  \(B3\)

Here the integral is in Minkowski space in $d = 4 - 2\epsilon$. The physical pion mass $m_\pi$ is defined as the pole of the propagator, or

$$m_\pi^2 - m^2 - i\Sigma(m_\pi^2) - i\Sigma_{ct}(m_\pi^2) = 0.$$  \(B4\)

Solving this equation self-consistently to NLO yields

$$m_\pi^2 = m^2 + \frac{im^2}{2f^2} \int_p \frac{1}{p^2 - m^2} + \frac{2m^4}{f^2} l_3,$$

$$= m^2 \left[ 1 - \frac{m^2}{2(4\pi)^2 f^2 l_3} \right],$$  \(B5\)

where we have used Eq. (55) with $i = 3$, and Eq. (A3). The pion-decay constant $f_\pi$ can be determined in a similar manner at NLO,

$$f_\pi^2 = f^2 \left[ 1 + \frac{2m^2}{(4\pi)^2 f^2 l_4} \right].$$  \(B6\)
