Excluded minors in cubic graphs

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Abstract

Let $G$ be a cubic graph, with girth at least five, such that for every partition $X,Y$ of its vertex set with $|X|, |Y| \geq 7$ there are at least six edges between $X$ and $Y$. We prove that if there is no homeomorphic embedding of the Petersen graph in $G$, and $G$ is not one particular 20-vertex graph, then either

- $G \setminus v$ is planar for some vertex $v$, or
- $G$ can be drawn with crossings in the plane, but with only two crossings, both on the infinite region.

We also prove several other theorems of the same kind.
1 Introduction

All graphs in this paper are simple and finite. Circuits have no repeated vertices or edges; the girth of a graph is the length of the shortest circuit. If $G$ is a graph and $X \subseteq V(G)$, $\delta_G(X)$ or $\delta(X)$ denotes the set of edges with one end in $X$ and the other in $V(G) \setminus X$. We say a cubic graph $G$ is cyclically $k$-connected, for $k \geq 1$ an integer, if $G$ has girth $\geq k$, and $|\delta_G(X)| \geq k$ for every $X \subseteq V(G)$ such that both $X$ and $V(G) \setminus X$ include the vertex set of a circuit of $G$.

A homeomorphic embedding of a graph $G$ in a graph $H$ is a function $\eta$ such that

- for each $v \in V(G)$, $\eta(v)$ is a vertex of $H$, and $\eta(v_1) \neq \eta(v_2)$ for all distinct $v_1, v_2 \in V(G)$
- for each $e \in E(G)$, $\eta(e)$ is a path of $H$ with ends $\eta(v_1)$ and $\eta(v_2)$, where $e$ has ends $v_1, v_2$ in $G$; and no edge or internal vertex of $\eta(e_1)$ belongs to $\eta(e_2)$, for all distinct $e_1, e_2 \in E(G)$
- for all $v \in V(G)$ and $e \in E(G)$, $\eta(v)$ belongs to $\eta(e)$ if and only if $v$ is an end of $e$ in $G$.

We denote by $\eta(G)$ the subgraph of $H$ consisting of all the vertices $\eta(v)$ ($v \in V(G)$) and all the paths $\eta(e)$ ($e \in E(G)$). We say that $H$ contains $G$ if there is a homeomorphic embedding of $G$ in $H$.

Figure 1: Starfish

Let us say that $G$ is theta-connected if $G$ is cubic and cyclically five-connected, and $|\delta_G(X)| \geq 6$ for all $X \subseteq V(G)$ with $|X|, |V(G) \setminus X| \geq 7$. We say $G$ is apex if $G \setminus v$ is planar for some vertex $v$ (we use $\setminus$ to denote deletion); and $G$ is doublecross if it can be drawn in the plane with only two crossings, both on the infinite region. Our goal in this paper is to give a construction for all theta-connected graphs not containing Petersen (we define Petersen to be the Petersen graph.) This is motivated by a result of a previous paper [4], where we showed that to prove Tutte’s conjecture [7] that every two-edge-connected cubic graph not containing Petersen is three-edge-colourable, it is enough to prove the same for theta-connected graphs not containing Petersen, and for apex graphs.

The graph Starfish is shown in Figure 1. Our main result is the following.
1.1 Let \( G \) be theta-connected. Then \( G \) does not contain Petersen if and only if either \( G \) is apex, or \( G \) is doublecross, or \( G \) is isomorphic to Starfish.

The “if” part of 1.1 is easy and we omit it. (It is enough to check that Petersen itself is not apex or doublecross, and is not contained in Starfish.) The “only if” part is an immediate consequence of the following three theorems. The graph \( Jaws \) is defined in Figure 2.

![Figure 2: Jaws](image)

1.2 Let \( G \) be theta-connected, and not contain Petersen. If \( G \) contains Starfish then \( G \) is isomorphic to Starfish.

1.3 Let \( G \) be theta-connected, and not contain Petersen. If \( G \) contains Jaws then \( G \) is doublecross.

1.4 Let \( G \) be theta-connected, and not contain Petersen. If \( G \) contains neither Jaws nor Starfish, then \( G \) is apex.

1.2, proved in section 17, is an easy consequence of a theorem of a previous paper [3], and 1.3 is proved in section 18. The main part of the paper is devoted to proving 1.4. Our approach is as follows.

A graph \( H \) is minimal with property \( P \) if there is no graph \( G \) with property \( P \) such that \( H \) contains \( G \) and \( H \) is not isomorphic to \( G \). In Figure 3 we define four more graphs, namely \( Triplex, \ Box, \ Ruby \) and \( Dodecahedron \). A theorem of McCuaig [1] asserts

1.5 Petersen, Triplex, Box, Ruby and Dodecahedron are the only graphs minimal with the property of being cubic and cyclically five-connected.

We shall prove the following three theorems.

1.6 Petersen, Triplex, Box and Ruby are the only graphs minimal with the property of being cyclically five-connected and non-planar.
A graph $G$ is *dodecahedrally-connected* if it is cubic and cyclically five-connected, and for every $X \subseteq V(G)$ with $|X|, |V(G) \setminus X| \geq 7$ and $|\delta_G(X)| = 5$, $G \setminus X$ cannot be drawn in a disc $\Delta$ such that the five vertices in $X$ with neighbours in $V(G) \setminus X$ are drawn in $bd(\Delta)$.

1.7 *Petersen, Triplex and Box are the only graphs minimal with the property of being dodecahedrally-connected and having crossing number at least two.*

We say $G$ is *arched* if $G \setminus e$ is planar for some edge $e$.

1.8 *Petersen and Triplex are the only graphs minimal with the property of being dodecahedrally-connected and not arched.*

Then we use 1.8 to find all the graphs minimal with the property of being dodecahedrally-connected and non-apex (there are six). Let us say $G$ is *die-connected* if it is dodecahedrally-connected and $|\delta_G(X)| \geq 6$ for every $X \subseteq V(G)$ with $|X|, |V(G) \setminus X| \geq 9$. We use the last result to find all graphs minimal with the property of being die-connected and non-apex (there are nine); and then use that to find the minimal graphs with the property of being theta-connected and non-apex. There are three, namely Petersen, Starfish, and Jaws, and from this 1.3 follows.
2 Extensions

It will be convenient to denote by \(ab\) or \(ba\) an edge with ends \(a\) and \(b\) (since we do not permit parallel edges, this is unambiguous). Let \(ab\) and \(cd\) be distinct edges of a graph \(G\). They are diverse if \(a, b, c, d\) are all distinct and \(a, b\) are not adjacent to \(c\) or \(d\). We denote by \(G + (ab, cd)\) the graph obtained from \(G\) as follows: delete \(ab\) and \(cd\), and add two new vertices \(x\) and \(y\) and five new edges \(xa, xb, yc, yd, xy\).

We call \(x, y\) (in this order) the new vertices of \(G + (ab, cd)\). Multiple applications of this operation are denoted in the natural way; for instance, if \(e, f \in E(G)\) are distinct, and \(G' = G + (e, f)\), and \(g, h \in E(G')\) are distinct, we write \(G + (e, f) + (g, h)\) for \(G' + (g, h)\).

Similarly, let \(ab, cd, ef\) be distinct edges of \(G\), where \(a, b, c, d, e, f\) are all distinct. We denote by \(G + (ab, cd, ef)\) the graph obtained by deleting \(ab, cd\) and \(ef\), and adding four new vertices \(x, y, z, w\), and nine new edges \(xa, xb, ye, yd, ze, zf, wx, wy, wz\); and call \(x, y, z, w\) (in this order) the new vertices of \(G + (ab, cd, ef)\).

A path has no “repeated” vertices or edges. Its first and last vertices are its ends, and its first and last edges are its end-edges. Its other vertices and edges are called internal vertices and edges. A path with ends \(s\) and \(t\) is called an \((s, t)\)-path. If \(P\) is a path and \(s, t \in V(P)\), the subpath of \(P\) with ends \(s\) and \(t\) is denoted by \(P[s, t]\). Let \(\eta\) be a homeomorphic embedding of \(G\) in \(H\). An \(\eta\)-path in \(H\) is a path \(P\) with distinct ends both in \(V(\eta(G))\), but with no other vertex or edge in \(\eta(G)\). Let \(G, H\) both be cubic, and let \(\eta\) and \(P\) be as above, where \(P\) has ends \(s\) and \(t\), with \(s \in V(\eta(e))\) and \(t \in V(\eta(f))\). We can sometimes use \(P\) to obtain a new homeomorphic embedding \(\eta'\) of \(G\) in \(H\), equal to \(\eta\) except as follows:

- If \(e = f\), let \(e = uv\), where \(\eta(u), s, t, \eta(v)\) lie in \(\eta(e)\) in order. Define
  \[
  \eta'(e) = \eta(e)[\eta(u), s] \cup P \cup \eta(e)[t, \eta(v)].
  \]

- If \(e \neq f\) but they have a common end, let \(e = uv\) and \(f = vw\) say, and let \(g\) be the third edge of \(G\) incident with \(v\). Define \(\eta'\) by:
  \[
  \begin{align*}
  \eta'(v) &= t, \\
  \eta'(e) &= \eta(e)[\eta(u), s] \cup P, \\
  \eta'(f) &= \eta(f)[t, \eta(w)], \\
  \eta'(g) &= \eta(g) \cup \eta(f)[\eta(v), t].
  \end{align*}
  \]

- If \(e, f\) have no common end, but one end of \(e\) is adjacent to one end of \(f\), let \(e = uv\), \(f = wx\) and \(g = vw\) say. Let \(h, i\) be the third edges at \(v, w\) respectively. Define \(\eta'\) by:
  \[
  \begin{align*}
  \eta'(v) &= s, \\
  \eta'(w) &= t, \\
  \eta'(e) &= \eta(e)[\eta(u), s], \\
  \eta'(f) &= \eta(f)[t, \eta(x)], \\
  \eta'(g) &= P, \\
  \eta'(h) &= \eta(h) \cup \eta(e)[s, \eta(v)], \\
  \eta'(i) &= \eta(i) \cup \eta(f)[\eta(w), t].
  \end{align*}
  \]
In the first two cases we say that \( \eta' \) is obtained from \( \eta \) by rerouting \( e \) along \( P \), and in the third case by rerouting \( g \) along \( P \). If \( \eta \) is a homeomorphic embedding of \( G \) in \( H \), an \( \eta \)-bridge is a connected subgraph \( B \) of \( H \) with \( E(B \cap \eta(G)) = \emptyset \), such that either

- \(|E(B)| = 1, E(B) = \{e\} \) say, and both ends of \( e \) are in \( V(\eta(G)) \), or
- for some component \( C \) of \( H \setminus V(\eta(G)) \), \( E(B) \) consists of all edges of \( H \) with at least one end in \( V(C) \).

It follows that every edge of \( H \) not in \( \eta(G) \) belongs to a unique \( \eta \)-bridge. We say that an edge \( e \) of \( G \) is an \( \eta \)-attachment of an \( \eta \)-bridge \( B \) if \( \eta(e) \cap B \) is non-null.

3 Frameworks

We shall often have a cubic graph \( G \), such that \( G \) (or sometimes, most of \( G \)) is drawn in a surface, possibly with crossings, and also a homeomorphic embedding \( \eta \) of \( G \) in another cubic graph \( H \); and we wish to show that the drawing of \( G \) can be extended to a drawing of \( H \) without introducing any more crossings. For this to be true, one necessary condition is that for each \( \eta \)-bridge \( B \), all its attachments belong to the same “region” of \( G \). Each region of the drawing is bounded either by a circuit (if no crossings involve any edge incident with the region) or by one or more paths, whose first and last edges cross others and no internal edges cross others. For instance, in Figure 2, one region is bounded by the path 6-1-2-3-8; and another by two paths 6-1-13-18 and 15-20-8-3. If we list all these circuits and paths we obtain some set of subgraphs of \( G \), and it is convenient to work with this set rather than explicitly with regions of a drawing of \( G \).

Sometimes, the drawing is just of a subgraph \( G' \) of \( G \) rather than of all of \( G \), and therefore all the circuits and paths in the set are subgraphs of \( G' \). In this case we shall always be able to arrange that \( \eta(e) \) has only one edge, for every edge \( e \) of \( G \) not in \( G' \). This motivates the following definition.

We say \((G, F, C)\) is a framework if \( G \) is cubic, \( F \) is a subgraph of \( G \), and \( C \) is a set of subgraphs of \( G \setminus E(F) \), satisfying \((F1)\)--\((F7)\) below. We say distinct edges \( e, f \) are twinned if there exist distinct \( C_1, C_2 \in C \) with \( e, f \in E(C_1 \cap C_2) \).

\((F1)\) Each member of \( C \) is an induced subgraph of \( G \setminus E(F) \), with at least three edges, and is either a path or a circuit.

\((F2)\) Every edge of \( G \setminus E(F) \) belongs to some member of \( C \), and for every two edges \( e, f \) of \( G \) with a common end not in \( V(F) \), there exists \( C \in C \) with \( e, f \in E(C) \).

\((F3)\) If \( C_1, C_2 \in C \) are distinct and \( v \in V(C_1 \cap C_2) \), then either \( V(C_1 \cap C_2) = \{v\} \), or \( v \) is incident with an edge in \( C_1 \cap C_2 \), or \( v \in V(F) \).

\((F4)\) If \( C_1 \in C \) is a path, then every member of \( C \) containing an end-edge of \( C_1 \) is a path. Moreover, if also \( C_2 \in C \setminus \{C_1\} \) is a path, then every component of \( C_1 \cap C_2 \) contains an end of \( C_1 \), and every edge of \( C_1 \cap C_2 \) is an end-edge of \( C_1 \).

\((F5)\) If \( C \in C \) is a circuit then \( |V(C \cap F)| \leq 1 \), and every vertex in \( C \cap F \) has degree 1 in \( F \); and if \( C \in C \) is a path then every vertex in \( C \cap F \) is an end of \( C \) and has degree 0 or 2 in \( F \).

\((F6)\) If \( e, f \) are twinned and \( C \in C \) with \( e \in E(C) \), then \( |V(C)| \leq 6 \), and either
- If \( f \in E(C) \), and \( C \) is a circuit, and \( e, f \) have a common end in \( V(F) \), and no path in \( C \) contains any vertex of \( e \) or \( f \), or
- If \( f \in E(C) \), and \( C \) is a path with end-edges \( e, f \), and \( C \cap F \) is null, or
- If \( f \notin E(C) \), and \( C \) is a path with \( |E(C)| = 3 \), and \( e \) is an end-edge of \( C \), and no end of \( e \) belongs to \( V(F) \).

(F7) Let \( C \in \mathcal{C} \) be a path of length five, with twinned end-edges \( e, f \). Then \( |E(C')| \leq 4 \) for every path \( C' \in \mathcal{C} \setminus \{C\} \) containing \( e \). Moreover, let \( C \) have vertices \( v_0-v_1-\cdots-v_5 \) in order; then there exists \( C' \in \mathcal{C} \) with end-edges \( e \) and \( f \) and with ends \( v_0 \) and \( v_4 \).

We will prove a theorem that says, roughly, that if we have a framework \((G, F, \mathcal{C})\), and a homeomorphic embedding of \( G \) in \( H \), where \( H \) is appropriately cyclically connected, then either the drawing of \( G \) extends to an drawing of the whole of \( H \), or there is some bounded enlargement of \( \eta(G) \) in \( H \) to which the drawing does not extend, and this enlargement still has high cyclic connectivity.

These seven axioms are a little hard to digest, and before we go on it may help to see how they will be used. In all our applications of (F1)–(F7) we have some particular graph \( G \) in mind and a drawing of it that defines the framework. We could replace (F1)–(F7) just by the hypothesis that \((G, F, \mathcal{C})\) arises from one of these particular cases, but there are nine of these cases, and it seemed clearer to try to abstract the properties that we really use. Here are three examples that might help.

- The simplest application is to prove 1.6; we take \( G \) to be Dodecahedron, and \( F \) null, and \( \mathcal{C} \) to be the set of region-bounding circuits in the drawing of \( G \) in Figure 3. Suppose now some \( H \) contains \( G \); our result will tell us that either the embedding of \( G \) extends to an embedding of \( H \) (and hence \( H \) is planar), or \( H \) contains a non-planar subgraph, a bounded enlargement of \( \eta(G) \) with high cyclic connectivity. We enumerate all the possibilities for this enlargement, and check they all contain one of Petersen, Ruby, Box, Triplex. From this, 1.6 will follow.
- When we come to try to understand the graphs that contain Jaws and not Petersen, we take \( G \) to be Jaws, and \((G, F, \mathcal{C})\) to be defined by the drawing in Figure 2. Thus, \( F \) is null; \( \mathcal{C} \) will contain the seven circuits in Figure 2 that bound regions and do not include any of the four edges that cross, together with eight paths (four like 6-1-2-3-8; two like 1-6-5-4-3-8; and two like 6-1-13-18.)
- A last example, one with \( F \) non-null; when we prove 1.8, we take \( G \) to be Box, and \((G, F, \mathcal{C})\) to be defined by the drawing in Figure 3, and \( E(F) = \{f\} \) where \( f \) is the edge 13-14. In this case, take the drawing of Box given in Figure 3, and delete the edge \( f \), and we get a drawing of \( G\setminus f \) without crossings; let \( \mathcal{C} \) be the set of circuits that bound regions in this drawing. The only twinned edges are 2-13 with 5-13, and 8-14 with 11-14.

(F1)–(F7) have a number of easy consequences, for instance, the following four results.

3.1 Let \((G, F, \mathcal{C})\) be a framework.

- \( F \) is an induced subgraph of \( G \).
- Let \( e \in E(G) \setminus E(F) \). Then \( e \) belongs to at least two members of \( \mathcal{C} \), and to more than two if and only if \( e \) is an end-edge of a path in \( \mathcal{C} \) and neither end of \( e \) is in \( V(F) \); and in this case \( e \) belongs to exactly four members of \( \mathcal{C} \), all paths, and it is an end-edge of each of them.
• For every two edges $e, f$ of $G$ with a common end with degree three in $G \setminus E(F)$, there is at most one $C \in \mathcal{C}$ with $e, f \in E(C)$.

Proof. Let $e = uv$ be an edge of $E(G) \setminus E(F)$. We claim that $|\{u, v\} \cap V(F)| \leq 1$. For by (F2) there exists $C \in \mathcal{C}$ with $e \in E(C)$. If $C$ is a circuit the claim follows from (F5), and if $C$ is a path then one of $u, v$ is internal to $C$, and again it follows from (F5). Thus the first claim holds.

For the second claim, again let $e = uv$ be an edge of $E(G) \setminus E(F)$. We may assume that $u \notin V(F)$. Let $u$ be incident with $e, e_1, e_2$. By (F2) there exist $C_1, C_2 \in \mathcal{C}$ with $e, e_i \in E(C_i)$ ($i = 1, 2$). Hence $C_1 \neq C_2$, so $e$ belongs to at least two members of $\mathcal{C}$.

No other member of $\mathcal{C}$ contains $e$ and either $e_1$ or $e_2$, by (F6), since $u \notin V(F)$. Hence every other $C \in \mathcal{C}$ containing $e$ is a path with one end $u$. If $e$ is not an end-edge of any path in $\mathcal{C}$ the second claim is therefore true, so we assume it is. Hence by (F4), $C_1$ and $C_2$ are both paths with end-edge $e$, and both have one end $v$. If $v \in V(F)$, there is no path in $\mathcal{C}$ containing $e$ with one end $u$, by (F5), so we may assume that $v \notin V(F)$. Let $v$ be incident with $e, e_3, e_4$; then by (F2) there exist $C_3, C_4 \in \mathcal{C}$ with $e, e_i \in E(C_i)$ ($i = 3, 4$); and $C_3, C_4$ both have one end $u$. Hence $C_1, \ldots, C_4$ are all distinct, and no other member of $\mathcal{C}$ contains $e$. This proves the second claim.

For the third claim, let $v \in V(G)$ be incident with edges $e, f, g \in E(G) \setminus E(F)$. Suppose there exist distinct $C, C' \in \mathcal{C}$ both containing $e, f$. Thus $e, f$ are twinned. If $C$ is a circuit, then by (F6) $v \in V(F)$, and by (F5) $v$ has degree one in $F$, a contradiction. Thus $C$ is a path. By (F6) both $e, f$ are end-edges of $C$, and hence $C$ has length two, a contradiction. This proves the third claim, and hence proves 3.1.

3.2 Let $C_1, C_2 \in \mathcal{C}$ be distinct. Then $|E(C_1 \cap C_2)| \leq 2$, and if equality holds, then either

• $C_1, C_2$ are both circuits, and $C_1 \cap C_2$ is a 2-edge path with middle vertex $v$ in $V(F)$, and $v$ has degree one in $F$, or

• $C_1, C_2$ are both paths with the same end-edges $e, f$ say, and $C_1 \cap C_2$ consists of the disjoint edges $e, f$ and their ends, and $C_1, C_2$ are disjoint from $F$.

Proof. Let $e, f \in E(C_1 \cap C_2)$ be distinct. If $C_1$ is a path then by (F6) and (F4), so is $C_2$, and both $C_1$ and $C_2$ have end-edges $e, f$, and no end of $e$ or $f$ is in $V(F)$, and by (F5) $C_1, C_2$ are disjoint from $F$. But then by (F6) $|E(C_1 \cap C_2)| = 2$ (for any third edge in $E(C_1 \cap C_2)$ would also have to be an end-edge of $C_1$, which is impossible); and if $v \in V(C_1 \cap C_2)$ is not incident with $e$ or $f$, then $v$ is internal to both paths and hence is incident with an edge of $C_1 \cap C_2$, a contradiction. Thus in this case the theorem holds. We may assume then that $C_1$ and $C_2$ are both circuits. By (F6), $e, f$ have a common end, $v$ say, in $V(F)$. By (F5) no other vertex of $C_1$ or $C_2$ is in $V(F)$, and $v$ has degree one in $F$. By (F6), $E(C_1 \cap C_2) = \{e, f\}$, and hence the theorem holds. This proves 3.2.

3.3 Let $C_1, C_2 \in \mathcal{C}$ be distinct with $|E(C_1 \cap C_2)| \geq 2$. Then $|E(C_1)| \geq 4$.

Proof. Suppose that $C_1$ is a circuit. If $|E(C_1)| = 3$, then since $C_2$ is an induced subgraph of $G \setminus E(F)$ and $|E(C_1 \cap C_2)| \geq 2$ it follows that $C_1$ is a subgraph of $C_2$ which is impossible. Hence the result holds if $C_1$ is a circuit. Now let $C_1$ be a path. Let $e, f \in E(C_1 \cap C_2)$ be distinct; then by (F6), $e$ and $f$ are end-edges of $C_1$, and by (F4) $C_2$ is a path with end-edges $e, f$. Hence again $C_1$ is not a subgraph of $C_2$, and so since $C_2$ is an induced subgraph of $G \setminus E(F)$ it follows that $|E(C_1)| \geq 4$. This proves 3.3.
3.4 Let \((G, F, \mathcal{C})\) be a framework, and let \(e, f_1, f_2 \in E(G)\) be distinct. If \(e, f_1\) are twinned then \(e, f_2\) are not twinned.

**Proof.** Let \(C_1, C_1' \in \mathcal{C}\) be distinct with \(e, f_1 \in E(C_1 \cap C_1')\), and suppose that there exist \(C_2, C_2' \in \mathcal{C}\), distinct, with \(e, f_2 \in E(C_2 \cap C_2')\). At least three of \(C_1, C_1', C_2, C_2'\) are distinct, and they all contain \(e\), and so by 3.1 all of \(C_1, C_1', C_2, C_2'\) are paths and \(e\) is an end-edge of each of them. By (F6) \(C_1\) has end-edges \(e\) and \(f_1\), and \(f_2 \not\in E(C_1)\). Since \(e, f_1 \in E(C_1)\), by 3.3 \(|E(C_1)| \geq 4\); but since \(f_2 \not\in E(C_1)\), by (F6) \(|E(C_1)| \leq 3\), a contradiction. This proves 3.4.

Let \(F, G, H\) be graphs, where \(F\) is a subgraph of \(G\), and let \(\zeta, \eta\) be homeomorphic embeddings of \(F, G\) into \(H\) respectively. We say that \(\eta\) extends \(\zeta\) if \(\eta(e) = \zeta(e)\) for all \(e \in E(F)\) and \(\eta(v) = \zeta(v)\) for all \(v \in V(F)\).

Let \((G, F, \mathcal{C})\) be a framework, let \(\eta_F\) be a homeomorphic embedding of \(F\) into \(H\), and let \(J\) be the subgraph of \(F\) obtained by deleting all vertices with degree one in \(F\). Let \(G'\) be a cubic graph with \(J\) a subgraph of \(G'\). A homeomorphic embedding \(\eta\) of \(G'\) in \(H\) is said to respect \(\eta_F\) if \(\eta\) extends the restriction of \(\eta_F\) to \(J\).

Again, let \((G, F, \mathcal{C})\) be a framework, and let \(\eta_F\) be a homeomorphic embedding of \(F\) into \(H\). We list a number of conditions on the framework, \(H\) and \(\eta_F\) that we shall prove have the following property. Suppose that these conditions are satisfied, and there is a homeomorphic embedding of \(G\) in \(H\) extending \(\eta_F\); then the natural drawing of \(G \setminus E(F)\) (where the members of \(\mathcal{C}\) define the region-boundaries) can be extended to one of \(H \setminus E(\eta_F(F))\). They are the following seven conditions (E1)–(E7).

**E1** \(H\) is cubic and cyclically four-connected, and if \((G, F, \mathcal{C})\) has any twinned edges, then \(H\) is cyclically five-connected. Also, \(\eta_F(e)\) has only one edge for every \(e \in E(F)\).

**E2** Let \(e, f \in E(G) \setminus E(F)\) be distinct. If there is a homeomorphic embedding of \(G + (e, f)\) in \(H\) respecting \(\eta_F\), then there exists \(C \in \mathcal{C}\) with \(e, f \in E(C)\).

If \(e, f, g\) are distinct edges of \(E(G)\) such that no member of \(\mathcal{C}\) contains all of \(e, f, g\), but one contains \(e, f\), one contains \(e, g\) and one contains \(f, g\), we call \(\{e, f, g\}\) a trinity. A trinity is diverse if every two edges in it are diverse in \(G \setminus E(F)\).

**E3** For every diverse trinity \(\{e, f, g\}\) there is no homeomorphic embedding of \(G + (e, f, g)\) in \(H\) extending \(\eta_F\).

**E4** Let \(v\) have degree one in \(F\), incident with \(g \in E(F)\). Let \(C_1, C_2\) be the two members of \(\mathcal{C}\) containing \(v\). For all \(e_1 \in E(C_1) \setminus E(C_2)\) and \(e_2 \in E(C_2) \setminus E(C_1)\) such that \(e_1\) and \(e_2\) have no common end, there is no homeomorphic embedding of \(G + (e_1, g) + (e_2, vy)\) in \(H\) respecting \(\eta_F\), where \(G + (e_1, g)\) has new vertices \(x, y\).

**E5** Let \(v\) have degree one in \(F\), incident with \(g \in E(F)\). Let \(u\) be a neighbour of \(v\) in \(G \setminus E(F)\), and let \(C_0\) be the (unique, by 3.1) member of \(\mathcal{C}\) that contains \(u\) and not \(v\). Let \(u\) have neighbours \(v, w_1, w_2\). Let \(G' = G + (uw_1, g)\) with new vertices \(x_1, y_1\); and let \(G'' = G' + (uw_2, vy_1)\) with new vertices \(x_2, y_2\). Let \(i = 1\) or \(2\), and let \(e = ux_i\). Let \(f\) be an edge of \(C_0\) not incident with \(w_i\) or \(w_2\), and with no end adjacent to \(w_i\). (This is vacuous unless \(|E(C_0)| \geq 6\).) There is no homeomorphic embedding of \(G'' + (e, f)\) in \(H\) respecting \(\eta_F\).
Two edges of \( G \setminus E(F) \) are distant if they are diverse in \( G \) and not twinned. Let \( C \in \mathcal{C} \). We shall speak of a sequence of vertices and/or edges of \( C \) as being in order in \( C \), with the natural meaning (that is, if \( C \) is a path, in order as \( C \) is traversed from one end, and if \( C \) is a circuit, in order as \( C \) is traversed from some starting point).

- If \( e, f, g, h \) are distinct edges of \( C \), in order, and \( e, g \) are distant and so are \( f, h \), we call \( G + (e, g) + (f, h) \) a cross extension (of \( G \), over \( C \)) of the first kind.

- If \( e, uv, f \) are distinct edges of \( C \), and either \( e, u, v, f \) are in order, or \( f, e, u, v \) are in order, and \( e, uv \) are distant and so are \( uv, f \), we call \( G + (e, uv) + (uv, f) \) a cross extension of the second kind, where \( G + (e, uv) \) has new vertices \( x, y \).

- If \( u_1v_1 \) and \( u_2v_2 \) are distant edges of \( C \) and \( u_1, v_1, u_2, v_2 \) are in order, we call \( G + (u_1v_1, u_2v_2) + (xv_1, yv_2) \) a cross extension of the third kind, where \( G + (u_1v_1, u_2v_2) \) has new vertices \( x, y \).

(E6) For each \( C \in \mathcal{C} \) and every cross extension \( G' \) of \( G \) over \( C \) of the first, second or third kinds, there is no homeomorphic embedding of \( G' \) in \( H \) extending \( \eta_F \).

(E7) Let \( C \in \mathcal{C} \) be a path with \( |E(C)| = 5 \), with vertices \( v_0 \cdots v_5 \) in order, and let \( v_0v_1 \) and \( v_4v_5 \) be twinned. Let \( G_1 = G + (v_0v_1, v_1v_5) \) with new vertices \( x_1, y_1 \); let \( G_2 = G_1 + (v_1v_2, y_1v_5) \) with new vertices \( x_2, y_2 \); and let \( G_3 = G_2 + (x_1v_0, y_2v_5) \). There is no homeomorphic embedding of \( G_3 \) in \( H \) extending \( \eta_F \).

In the proofs to come, when we need to apply (E1)–(E7), it is often cumbersome to indicate the full homeomorphic embedding involved, and we use some shortcuts. For instance, when we apply (E2), with \( e, f, \eta \) as in (E2), let \( g \) be the new edge of \( G + (e, f) \), and let \( H' \) be the graph obtained from \( \eta(G + (e, f)) \) by deleting the interior of the path \( \eta(g) \); we normally say “by (E2) applied to \( H' \) with edges \( e, f \)”, and leave the reader to figure out the appropriate homeomorphic embedding and the path \( \eta(g) \).

Whenever we wish to apply our main theorem, we have to verify directly that (E1)–(E7) hold, and this can be a lot of case-checking. We have therefore tried to design (E1)–(E7) to be as easily checked as possible consistent with implying the main result. Nevertheless, there is still a great deal of case-checking, and we have omitted almost all the details. We are making available in [5] both the case-checking and all the graphs of the paper in computer-readable form.

4 Degenerate trinities

Now (E3) was a statement about diverse trinities; our first objective is to prove the same statement about non-diverse trinities.

A trinity is a \( Y \)-trinity if some two edges in it (say \( e \) and \( f \)) have a common end \( u \), the third edge in it (\( g \) say) is not incident with \( u \), and if \( h \) denotes the third edge incident with \( u \) then there exist \( C_1, C_2 \in \mathcal{C} \) with \( e, g, h \in E(C_1) \) and \( f, g, h \in E(C_2) \). (Consequently \( g, h \) are twinned.) It is circuit-type or path-type depending whether \( g \) and \( h \) have a common end or not.

4.1 Let \((G, F, C)\) be a framework and let \( H, \eta_F \) satisfy (E1)–(E7). For every path-type \( Y \)-trinity \( \{e, f, g\} \) there is no homeomorphic embedding of \( G + (e, f, g) \) in \( H \) extending \( \eta_F \).

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Proof. Let \( u, h, C_1, C_2 \) be as above. Since the twinned edges \( g, h \) have no common end, it follows from (F6) that \( C_1 \) and \( C_2 \) are both paths with end-edges \( g, h \), and both are vertex-disjoint from \( F \). Let \( e = uw_1, f = uw_2 \). Suppose that \( \eta \) is a homeomorphic embedding of \( G \) into \( H \) extending \( \eta_F \), and \( e, f, g \) are all \( \eta \)-attachments of some \( \eta \)-bridge \( B \).

By 3.3, \( |E(C_1)| \geq 4 \), and so \( g \) is not incident with \( w_1 \), and similarly not with \( w_2 \). By (F7), at least one of \( C_1, C_2 \) has length at most four, and so we may assume that the edges of \( C_1 \) in order are \( h, e, g, f, a, g \) say. Let \( \eta' \) be obtained from \( \eta \) by rerouting \( g \) along a \( \eta \)-path in \( B \) from \( \eta(g) \) to \( \eta(e) \). Then \( \eta' \) extends \( \eta_F \), and \( g, h, f \) are both \( \eta' \)-attachments of an \( \eta' \)-bridge. By (E2) applied to \( \eta'(G) \) with edges \( g, h, f \), there exists \( C \subset \eta \) with \( g, h, f \in E(C) \), and hence with \( e \in E(C) \) since \( C \) is an induced subgraph of \( G \setminus E(F) \). But then \( e, g_1 \in E(C \setminus C_1) \), and \( C_1 \neq C \), so \( e, g_1 \) are twinned edges, and yet their common end \( w_1 \) is not in \( V(F) \), contrary to (F6). There is therefore no such \( \eta \). This proves 4.1.

Let \( \{e, f, g\} \) be a circuit-type \( Y \)-trinity, where \( e = xw_1, f = xw_2 \) and \( g = vw_3 \), where \( v, w_3 \neq x \) and \( v, x \) are adjacent in \( G \). Let \( h = vx \), and let \( w_4 \) be the third neighbour of \( v \). Since \( g, h \) are twinned and share an end, 3.1 implies that \( vw_4 \in E(F) \). Hence \( w_4 \neq w_1, w_2 \), since no member of \( C \) contains both \( v, w_4 \). (See Figure 4.) We wish to consider three rather similar graphs \( G_1, G_2, G_3 \) called expansions of the \( Y \)-trinity \( \{e, f, g\} \). Let \( G' \) be obtained from \( G \) by deleting \( x \) and the edge \( vw_3 \), and adding five new vertices \( x_1, x_2, x_3, y_1, y_2 \) and nine new edges \( x_1w_1, x_2w_2, x_3w_3 \) and \( x_iy_j \) for \( 1 \leq i \leq 3 \) and \( 1 \leq j \leq 2 \). Let \( G_1, G_2, G_3 \) be obtained from \( G' \) by deleting the edge \( y_2a \) (where \( a \) is \( x_1, x_2 \) and \( x_3 \) respectively), and adding two new edges \( vy_2, va \). Let \( x_4 = v \). (The reason we did not just replace \( v \) by a new vertex \( w_4 \), is that the edge \( vw_4 \) belongs to \( F \) and we want to preserve it.) Thus \( F \) is a subgraph of \( G_1, G_2 \) and \( G_3 \). (See Figure 4.)

4.2 Let \( (G, F, C) \) be a framework, and let \( H, \eta_F \) satisfy (E1)–(E7). Let \( \{e, f, g\} \) be a circuit-type \( Y \)-trinity, and \( G_1, G_2, G_3 \) its three expansions. Then there is no homeomorphic embedding of \( G_1, G_2 \) or \( G_3 \) in \( H \) extending \( \eta_F \). In particular, there is no homeomorphic embedding of \( G + \{e, f, g\} \) in \( H \) extending \( \eta_F \).

Proof. Let \( v, x, w_1, \ldots, w_4 \) be as in Figure 4 and let \( G_1, G_2, G_3 \) be labelled as in Figure 4, where \( e = xw_1, f = xw_2 \), and \( g = vw_3 \).

Suppose that there is a homeomorphic embedding \( \eta \) of some \( G_k \) in \( H \) extending \( \eta_F \). Let \( A \) be the subgraph of \( G_k \) induced on \( \{x_1, x_2, x_3, x_4, y_1, y_2\} \), and \( B \) the subgraph of \( G_k \) induced on the complementary set of vertices. It follows that there is a homeomorphic embedding \( \zeta \) of \( G_k \) in \( H \) such that:

- \( \zeta \) extends the restriction of \( \eta \) to \( B \) (and in particular, \( \zeta(z) = \eta(z) \) for every vertex or edge \( z \) of \( F \), different from \( x_4, w_4x_4 \))
- \( \zeta(w_4x_4) \) is a path with one end \( \eta(w_4) \) containing the one-edge path \( \eta(w_4x_4) \).

(To see this, take \( \zeta = \eta \).) Let \( Z_i = \zeta(x_iw_i) \) for \( i = 1, \ldots, 4 \). Let us choose \( k \) and \( \zeta \) such that

1. \( Z_1 \cup Z_2 \cup Z_3 \) is minimal, and subject to that \( Z_4 \) is minimal.

Since \( H \) is cyclically five-connected by (E1) since there are twinned edges in \( G \), there are five disjoint paths \( P_1, \ldots, P_5 \) of \( H \) from \( \zeta(A) \) to \( \zeta(B) = \eta(B) \). Choose \( P_1, \ldots, P_5 \) to minimize the number
of edges of $P_1 \cup \cdots \cup P_5$ that do not belong to $Z_1 \cup \cdots \cup Z_4$. It follows that each $P_i$ has only its first vertex $a_i$ say in $V(\zeta(A))$, and only its last vertex $b_i$ say in $V(\eta(B))$. Now one of $a_1, \ldots, a_5$ is different from $\zeta(x_1), \zeta(x_2), \zeta(x_3), \zeta(x_4)$, say $a_5$. Let $a_5 \in V(\zeta(h_1))$, where $h_1 \in E(A)$. From (1) (or the theory of augmenting paths for network flows) it follows easily that $$\{a_1, \ldots, a_4\} = \{\zeta(x_1), \ldots, \zeta(x_4)\},$$ and we may assume that $a_i = \zeta(x_i) \ (1 \leq i \leq 4)$.

Let $p$ be the first vertex (that is, closest to $a_5$) in $P_5$ that belongs to $\eta(B) \cup Z_1 \cup Z_2 \cup Z_3 \cup Z_4$ (this exists since $b_5 \in V(\eta(B))$), and let $P = P_5[a_5,p]$.

(2) $p \in V(\eta(B))$.

**Subproof.** Suppose not; then $p \in V(Z_i)$ for some $i$. If $i = 4$, then by replacing $Z_4[\zeta(x_4),p]$ by $P$ we obtain a homeomorphic embedding of some $G_{k'}$ (where possibly $k' \neq k$), contradicting (1), since $Z_4$ is replaced by a proper subpath and $Z_1, Z_2, Z_3$ remain unchanged. So $1 \leq i \leq 3$.

If $h_1$ is incident with $x_i$, then by rerouting $h_1$ along $P$ we obtain a contradiction to (1). Now suppose that $h_1 = ab$ where $a$ is adjacent to $x_i$. By rerouting $ax_i$ along $P$, we again obtain a contradiction to (1).

Thus, neither end of $h_1$ is adjacent to $x_i$. Consequently, $h_1 \neq y_2x_4$, and $y_1$ is not incident with $h_1$, since $1 \leq i \leq 3$. The only remaining possibility is that there is a four-vertex path of $G_k$ with vertices $x_i, a, b, x_j$ in order, for some $j \neq i$, where $\{a, b\} = \{y_2, x_4\}$, and $h_1 = bx_j$. But then there is a homeomorphic embedding of some $G_{k'}$ in $H$ mapping $G_{k'}$ to the graph obtained from $\zeta(G) \cup P$ by

Figure 4: A circuit-type $Y$-trinity, and its three expansions.
deleting the interior of \( \zeta(x, a) \), contradicting (1). This proves (2).

Hence \( p \in V(\eta(h_2)) \) for some \( h_2 \in E(B) \). Now we examine the possibilities for \( h_1 \) and \( h_2 \). Since \( \eta(h_2) \) has an interior vertex, it follows from the choice of \( \zeta \) that \( h_2 \notin E(F) \). We recall that \( v \in V(G) \cap V(F) \). Let \( C_1, C_2 \in \mathcal{C} \) be the two members of \( \mathcal{C} \) that contain \( v \), and let \( C_0 \in \mathcal{C} \) contain \( e \) and \( f \). Thus \( C_0, C_1, C_2 \) are circuits by (F6), and \( v \) is the only vertex of \( F \) in \( V(C_1 \cup C_2) \).

\[ \text{(3) } h_2 \text{ belongs to at most one of } C_0, C_1, C_2. \]

Subproof. By 3.2, \( E(C_1 \cap C_2) \) contains at most two edges, and since it contains both \( g, vx \), it follows that \( h_2 \notin E(C_1 \cap C_2) \). Since \( C_1 \) is a circuit and \( v \in V(F) \), (F5) implies that \( x, w_1 \notin V(F) \), and so neither end of \( xw_1 \) is in \( V(F) \). Since \( xw_1 \in E(C_0 \cup C_1) \), 3.2 implies that \( |E(C_0 \cap C_1)| = 1 \) and so \( h_2 \notin E(C_0 \cap C_1) \); and similarly \( h_2 \notin E(C_0 \cap C_2) \). This proves (3).

\[ \text{(4) } k = 1 \text{ or } 2. \]

Subproof. Suppose that \( k = 3 \). First, suppose that \( h_1 \) is incident with \( y_1 \). By restricting \( \zeta \) to \( G_3 \setminus y_1 \), we obtain a homeomorphic embedding \( \eta' \) of \( G \) in \( H \) respecting \( \eta_\mathcal{F} \), such that \( e, f, g \) and \( h_2 \) are all \( \eta' \)-attachments in \( E(G) \setminus E(F) \) of some \( \eta' \)-bridge. Since \( C_1, C_2 \) are the only members of \( \mathcal{C} \) containing \( g \), it follows from (E2), applied to \( \eta'(G) \) with the edges \( g, h_2 \), that \( h_2 \in E(C_1 \cup C_2) \). Since \( C_1 \) and \( C_0 \) are the only members of \( \mathcal{C} \) containing \( e \) it follows from (E2) (with the edges \( e, h_2 \)) that \( h_2 \in E(C_0 \cup C_1) \), and similarly \( h_2 \in E(C_0 \cup C_2) \). Thus \( h_2 \) belongs to two of \( C_0, C_1, C_2 \), contrary to (3). This proves that \( h_1 \) is not incident with \( y_1 \).

Suppose next that \( h_1 \) is incident with \( y_2 \). By restricting \( \eta \) to \( G_3 \setminus y_2 \) we obtain a homeomorphic embedding \( \eta' \) of \( G \) in \( H \) respecting \( \eta_\mathcal{F} \) such that \( e, f \) and \( h_2 \) are all \( \eta' \)-attachments of some \( \eta' \)-bridge. So \( h_2 \in E(C_0 \cup C_1) \), by (E2) applied to \( \eta'(G) \) with edges \( e, h_2 \), and similarly \( h_2 \in E(C_0 \cup C_2) \). By (3) it follows that \( h_2 \in E(C_0) \), and \( h_2 \notin E(C_1 \cup C_2) \). Let \( H' \) be the graph obtained from \( \zeta(G_3) \) by deleting the interiors of \( \zeta(x_1y_2) \) and \( \zeta(x_3y_1) \). There is a homeomorphic embedding of \( G \) in \( H \) respecting \( \eta_\mathcal{F} \), mapping \( G \) onto \( H' \); and from (E2) applied to \( H' \) with edges \( f, h_2 \), we deduce that \( h_2 \in E(C_1 \cup C_2) \), a contradiction. This proves that \( h_1 \) is not incident with \( y_2 \).

Thus, \( h_1 = x_3x_4 \). From (E2) applied to the restriction of \( \zeta \) to \( G_3 \setminus y_1 \) and the edges \( g, h_2 \), it follows that \( h_2 \in E(C_1 \cup C_2) \); and from the symmetry between \( C_1, C_2 \), we may assume that \( h_2 \in E(C_2) \) without loss of generality. By 3.2, \( w_1 \notin V(C_2) \), and it follows that \( h_2, e \) are disjoint edges of \( G \). From (E4) applied to the restriction of \( \zeta \) to \( G_3 \setminus y_2 \), we obtain from the paths \( \zeta(x_1y_2) \cup \zeta(x_4y_2) \) and \( P \) that \( h_2 \notin E(C_2) \), a contradiction. This proves (4).

From (4) and the symmetry between \( w_1 \) and \( w_2 \) (exchanging \( G_1 \) and \( G_2 \)) we may therefore assume that \( k = 1 \). There are three homeomorphic embeddings of \( G \) in \( H \) respecting \( F \) that we need:

- let \( H_1 \) be the graph obtained from \( \zeta(G_1) \) by deleting the interiors of \( \zeta(x_1x_4) \) and \( \zeta(x_3y_1) \)
- let \( H_2 \) be obtained from \( \zeta(G_1) \) by deleting the interiors of \( \zeta(x_1x_4) \) and \( \zeta(x_2y_2) \)
- let \( H_3 \) be obtained from \( \zeta(G_1) \) by deleting the interiors of \( \zeta(x_3y_1) \) and \( \zeta(x_2y_2) \).

For \( i = 1, 2, 3 \) there is a homeomorphic embedding \( \eta_i \) of \( G \) in \( H_i \) respecting \( F \), with \( \eta_i(z) = \eta(z) \) for each vertex and edge \( z \) of \( B \).
(5) \( h_2 \in E(C_0 \cup C_1) \).

**Subproof.** Suppose not. By (E2) applied to \( H_1 \) and the edges \( e, h_2 \), it follows that

\[
h_1 \neq x_1y_1, x_3y_1, x_1x_4, x_2y_1,
\]

and so \( h_1 \) is incident with \( y_2 \). By (E2) applied to \( H_2 \) and the edges \( g, h_2 \), we deduce that \( h_2 \in E(C_2) \). Consequently \( e, h_2 \) are disjoint, since \( w_1 \notin V(C_2) \); but then this contradicts (E4) applied to \( H_2 \) and the paths \( \zeta(x_1x_4) \) and \( P \) (extended by a subpath of \( \zeta(x_2y_2) \) if necessary).

(6) \( h_2 \in E(C_0 \cup C_2) \).

**Subproof.** Suppose not. By (E2) applied to \( H_3 \) and the edges \( f, h_2 \), it follows that

\[
h_1 \neq x_1y_1, x_2y_1, x_3y_1, x_2y_2,
\]

and so \( h_1 \) is one of \( x_1x_4, x_1y_2, x_3y_2 \). By (5), \( h_2 \in E(C_1) \), and so \( f, h_2 \) are disjoint, since \( w_2 \notin V(C_1) \). But this contradicts (E4) applied to \( H_2 \) and the paths \( \zeta(x_2, y_2) \) and \( P \) (extended by a subpath of \( \zeta(x_1x_4) \) if necessary).

From (3) and (6), it follows that \( h_2 \in E(C_0) \), and \( h_2 \notin E(C_1 \cup C_2) \). By (E2) applied to \( H_3 \) and the edges \( g, h_2 \), we deduce that \( h_1 \neq x_3y_1, x_3y_2, x_2y_2, x_4y_2 \); and by (E2) applied to \( H_3 \) and the edges \( vx, h_2 \), we deduce that \( h_1 \neq x_1x_4 \). Thus \( h_1 \) is one of \( x_1y_1, x_2y_1 \).

We recall that \( \eta_2 \) is a homeomorphic embedding of \( G \) in \( H_2 \). Suppose that \( h_2 \) is incident with \( w_1 \). Let \( \eta' \) be obtained from \( \eta_2 \) by rerouting \( e \) along \( P \); then the paths \( \zeta(x_2y_2) \) and \( \zeta(x_1w_1) \cup \zeta(x_1x_4) \) violate (E4). Similarly, if \( h_2 \) is incident with \( w_2 \), let \( \eta' \) be obtained from \( \eta_2 \) by rerouting \( f \) along \( P \); then the paths \( \zeta(x_1x_4) \) and \( \zeta(x_2y_2) \cup \zeta(x_2w_2) \) violate (E4).

Thus \( w_1, w_2 \) are not incident with \( h_2 \). Next suppose that \( h_1 = x_1y_1 \) and one end \( a \) of \( h_2 \) is adjacent to \( w_1 \). Let \( \eta' \) be obtained from \( \eta_2 \) by rerouting \( aw_1 \) along \( P \); then the paths \( \zeta(x_1x_4), \zeta(x_2y_2) \) violate (E4). Next suppose that \( h_1 = x_2y_1 \) and one end \( a \) of \( h_2 \) is adjacent to \( w_2 \). Let \( \eta' \) be obtained from \( \eta_2 \) by rerouting \( aw_2 \) along \( P \); then the paths \( \zeta(x_1x_4), \zeta(x_2y_2) \) violate (E4). In summary, then, we have shown that \( h_2 \notin E(C_0) \), incident with neither of \( w_1, w_2 \), and for \( i = 1, 2 \), if \( h_1 = x_iy_1 \) then no end of \( h_2 \) is adjacent to \( w_i \). But this contradicts (E5).

There is therefore no such \( \eta \), and the first statement of the theorem holds. The second statement of the theorem follows from the first, since \( G + (e, f, g) \) is isomorphic to \( G_3 \) (and the isomorphism fixes \( F \)). This proves 4.2. ■

**4.3** Let \( (G, F, C) \) be a framework, and let \( H, \eta_F \) satisfy (E1)-(E7). Let \( \{e_1, e_2, e_3\} \) be a trinity such that no vertex is incident with all of \( e_1, e_2, e_3 \). Then there is no homeomorphic embedding of \( G + (e_1, e_2, e_3) \) in \( H \) extending \( \eta_F \).

**Proof.** For \( i = 1, 2, 3 \) there exists \( C_i \subseteq C \) with \( \{e_1, e_2, e_3\} \setminus \{e_i\} \subseteq E(C_i) \) and \( e_i \notin E(C_i) \), since \( \{e_1, e_2, e_3\} \) is a trinity. Suppose first that \( e_1, e_2 \) have a common end \( v \) say; and let \( h \) be the third edge incident with \( v \). By hypothesis \( h \neq e_3 \). If \( v \notin V(F) \) then since \( v \) has degree two in \( C_3, C_3 \) is a circuit, and hence by (F4), \( e_1 \) is not an end-edge of \( C_2 \); and if \( v \notin V(F) \) then by (F3) either \( e_1 \) is not
an end-edge of $C_2$, or $e_2$ is not an end-edge of $C_1$, and we may assume the first. Hence in either case $e_1$ is not an end-edge of $C_2$. Since $e_1 \in E(C_2)$ and $e_2 \notin E(C_2)$, it follows that $h \in E(C_2)$. By (F3), since $e_3 \in E(C_1 \cap C_2)$, it follows that $h \in E(C_1)$, since $v$ not in $V(F)$ by (F5); and so $\{e_1, e_2, e_3\}$ is a $Y$-trinity, contrary to 4.1 and 4.2.

Thus, no two of $e_1, e_2, e_3$ have a common end. Suppose that there is a homeomorphic embedding of $G + (e_1, e_2, e_3)$ in $H$ extending $\eta_F$. Then there is a homeomorphic embedding of $G$ in $H$ extending $\eta_F$, such that $e_1, e_2, e_3$ are all $\eta$-attachments of the same $\eta$-bridge $B$ say. By (E3), $\{e_1, e_2, e_3\}$ is not diverse in $G \setminus E(F)$, so we may assume that $e_1 = a_1b_1$ and $e_2 = a_2b_2$, where $a_1, a_2$ are adjacent in $G \setminus E(F)$. Let $a_1a_2 = e_0$.

Since $e_1, e_2 \in E(C_3)$ and $C_3$ is an induced subgraph of $G \setminus E(F)$, it follows that $e_0 \in E(C_3)$. Let $a_1$ have neighbours $b_1, a_2, c_1$ and $a_2$ have neighbours $a_1, b_2, c_2$ in $G$.

Since $e_0$ is not an end-edge of $C_3$, it is not an end-edge of $C_1$ or $C_2$, by (F4). Since $e_0$ and $e_3$ are disjoint, and $e_3 \in E(C_1 \cap C_2)$, it follows from (F6) that $e_0 \notin E(C_1 \cap C_2)$; we assume that $e_0 \notin E(C_1)$ without loss of generality. Suppose that $e_0 \in E(C_2)$. Since $e_0, e_1 \in E(C_2 \cap C_3)$, it follows from (F6) that $C_2, C_3$ are both circuits, $a_1 \in V(F)$ and $a_1c_1 \in E(F)$. Hence $a_2c_2 \in E(C_2)$ (since $e_2 \notin E(C_2)$). Moreover by (F3), $a_2$ is incident with an edge in $C_1 \cap C_2$, since $E(C_1 \cap C_2) \neq \emptyset$ and $a_2 \in V(C_1 \cap C_2)$. Since $e_0 \notin E(C_1)$ and $e_2 \notin E(C_2)$ it follows that $a_2c_2 \in E(C_1)$. Since $E(C_1 \cap C_2)$ contains $e_3$ and $a_2c_2$ and $C_2$ is a circuit, it follows from (F6) that $c_2 \in V(F)$, and so $a_1, c_2 \in V(C_2 \cap C)$. This proves that $e_0 \notin E(C_2)$.

If $a_1 \in V(F)$ then $a_1c_1 \notin E(C_2)$ by (F5), and so $e_1$ is an end-edge of $C_2$. By (F4), $C_3$ is a path, and $a_1$ is an internal vertex of it, contrary to (F5). Hence $a_1 \notin V(F)$, and similarly $a_2 \notin V(F)$.

Now $e_1, e_2, e_3$ are all $\eta$-attachments of $B$. Let $P$ be an $\eta$-path in $B$ with ends in $\eta(e_1)$ and $\eta(e_2)$, and let $\eta'$ be obtained by rerouting $e_0$ along $P$. Then $\eta'$ is a homeomorphic embedding of $G$ in $H$ extending $\eta_F$. Since $e_3$ is an $\eta$-attachment of $B$, it follows that $e_0$ and $e_3$ are $\eta'$-attachments of some $\eta'$-bridge. By (E2), applied to $\eta'(G)$ with edges $e_0, e_3$, there exists $C_4 \in C$ with $e_0, e_3 \in E(C_4)$. Since $a_1 \notin V(F)$ it follows from (F6) that $e_1 \notin E(C_4)$. But from (F4) applied to $C_3$ and $C_4$, $e_0$ is not an end-edge of $C_4$. By (F3) applied to $C_2$ and $C_4$, $a_1c_1 \in E(C_2 \cap C_4)$. Since $E(C_2 \cap C_4)$ contains both $a_1c_1$ and $e_3$, it follows that $e_3, a_1c_1$ are twinned, and similarly so are $e_3, a_2c_2$, contrary to 3.4. Thus there is no such $\eta$. This proves 4.3.

Next we need the following lemma.

**4.4** Let $\eta$ be a homeomorphic embedding of a cubic graph $G$ in a cyclically four-connected cubic graph $H$. Let $v \in V(G)$, incident with edges $e_1, e_2, e_3$, and suppose that $e_1, e_2, e_3$ are $\eta$-attachments of some $\eta$-bridge. Then there is a homeomorphic embedding $\eta'$ of $G$ in $H$, such that $\eta'(u) = \eta(u)$ for all $u \in V(G) \setminus \{v\}$, and $\eta'(e) = \eta(e)$ for all $e \in E(G) \setminus \{e_1, e_2, e_3\}$, and such that for some edge $e_4 \neq e_1, e_2, e_3$ of $G, e_1, e_2, e_3, e_4$ are $\eta'$-attachments of some $\eta'$-bridge.

**Proof.** For $1 \leq i \leq 3$, let $e_i$ have ends $v$ and $v_i$. Let $G' = G + (e_1, e_2, e_3)$, with new vertices $x_1, x_2, x_3, w$. By hypothesis, there is a homeomorphic embedding $\eta'$ of $G'$ in $H$ such that $\eta'(u) = \eta(u)$ for all $u \in V(G) \setminus \{v\}$, and $\eta'(e) = \eta(e)$ for all $e \in E(G) \setminus \{e_1, e_2, e_3\}$. Choose $\eta'$ such that

$$\eta'(v_1x_1) \cup \eta'(v_2x_2) \cup \eta'(v_3x_3)$$

is minimal. Since $H$ is cyclically four-connected, there is an $\eta'$-path with one end in

$$\bigcup_{1 \leq i \leq 3}(V(\eta'(vx_i)) \cup V(\eta'(wx_i)))$$
and the other end, \( t \), in
\[
V(\eta(G \setminus v) \cup \eta'(v_1 x_1) \cup \eta'(v_2 x_2) \cup \eta'(v_3 x_3)).
\]
From the choice of \( \eta' \) it follows that \( t \) belongs to none of \( \eta'(v_1 x_1), \eta'(v_2 x_2), \eta'(v_3 x_3) \), and so it belongs to \( \eta'(e_4) = \eta(e_4) \) for some \( e_4 \in E(G \setminus v) \). This proves 4.4.

4.5 Let \((G, F, \mathcal{C})\) be a framework, and let \( H, \eta_F \) satisfy \((E1)-(E7)\). Let \( \{e_1, e_2, e_3\} \) be a trinity. There is no homeomorphic embedding of \( G + (e_1, e_2, e_3) \) in \( H \) extending \( \eta_F \).

**Proof.** By 4.3 we may assume that \( v \in V(G) \) is incident with \( e_1, e_2 \) and \( e_3 \). Suppose \( \eta \) is a homeomorphic embedding of \( G + (e_1, e_2, e_3) \) in \( H \) extending \( \eta_F \). By 4.4 there is an edge \( e_4 \neq e_1, e_2, e_3 \) of \( G \) such that there are homeomorphic embeddings of each of \( G + (e_2, e_3, e_4), G + (e_1, e_3, e_4), G + (e_1, e_2, e_4) \) in \( H \) extending \( \eta_F \). It follows that \( e_4 \notin E(F) \). Since no vertex is incident with all of \( e_2, e_3, e_4 \), it follows from 4.3 that \( \{e_2, e_3, e_4\} \) is not a trinity; and yet \((E2)\), applied to \( \eta(G) \) with edges each pair of \( e_2, e_3, e_4 \), implies that every two of \( e_2, e_3, e_4 \) are contained in a member of \( \mathcal{C} \). Consequently there exists \( C_1 \in \mathcal{C} \) with \( e_2, e_3, e_4 \in E(C_1) \). Similarly there exist \( C_2, C_3 \in \mathcal{C} \) with \( e_1, e_2, e_4 \in E(C_2) \) and \( e_1, e_2, e_4 \in E(C_3) \). Since \( \{e_1, e_2, e_3\} \) is a trinity, \( e_i \notin E(C_i)(1 \leq i \leq 3) \), and so \( C_1, C_2, C_3 \) are all distinct.

Now if \( e_4 \) is not the end-edge of any path in \( \mathcal{C} \), then since \( C_2 \cap C_3 \) contains \( e_1 \) and \( e_4 \) it follows from \((F6)\) that \( e_1 \) and \( e_4 \) have a common end, and similarly do \( e_i \) and \( e_4 \) for \( i = 1, 2, 3 \), which is impossible. Hence \( e_4 \) is an end-edge of some path in \( \mathcal{C} \). By \((F4)\) \( C_1, C_2 \) and \( C_3 \) are all paths. Since \( e_3, e_4 \in E(C_1 \cap C_2), C_1 \) has end-edges \( e_3 \) and \( e_4 \); and since \( e_2, e_4 \in E(C_1 \cap C_3), C_1 \) has end-edges \( e_2 \) and \( e_4 \), a contradiction. This proves 4.5.

4.6 Let \((G, F, \mathcal{C})\) be a framework, and let \( H, \eta_F \) satisfy \((E1)-(E7)\). Let \( \eta \) be a homeomorphic embedding of \( G \) in \( H \) extending \( \eta_F \). For every \( \eta \)-bridge \( B \) there exists \( C \in \mathcal{C} \) such that \( e \in E(C) \) for every \( \eta \)-attachment \( e \) of \( B \).

**Proof.** Since \( \eta \) extends \( \eta_F \), it follows that \( Z \subseteq E(G) \setminus E(F) \), where \( Z \) is the set of all \( \eta \)-attachments of \( B \). Suppose, for a contradiction, that there is no \( C \in \mathcal{C} \) with \( Z \subseteq E(C) \), and choose \( X \subseteq Z \) minimal such that there is no \( C \in \mathcal{C} \) with \( X \subseteq E(C) \). By \((F2)\), \( |X| \geq 2 \); by \((E2)\), applied to \( \eta(G) \) with edges the members of \( X \), \( |X| \neq 2 \); and by 4.5, \( |X| \neq 3 \). Hence \( |X| \geq 4 \). Let \( X = \{e_1, \ldots, e_k\} \) say, where \( k \geq 4 \). For each \( i \in \{1, \ldots, k\} \), there exists \( C \in \mathcal{C} \) including \( X \setminus \{e_i\} \), from the minimality of \( X \). All these members of \( \mathcal{C} \) are different, and so every two members of \( X \) are twinned, contrary to 3.4. This proves 4.6.

5 Crossings on a region

Let \( \eta \) extend \( \eta_F \), and let \( B \) be an \( \eta \)-bridge. Since \( \eta \) extends \( \eta_F \), it follows that no \( \eta \)-attachment of \( B \) is in \( E(F) \), and so by 4.6, there exists \( C \in \mathcal{C} \) such that every \( \eta \)-attachment of \( B \) belongs to \( C \). If \( C \) is unique, we say that \( B \) sits on \( C \).

Our objective in this section is to show that if \( \eta \) extends \( \eta_F \), then for every \( C \in \mathcal{C} \) all the bridges that sit on \( C \) can be simultaneously drawn within the “region” that \( C \) bounds. There may be some bridges that sit on no member of \( \mathcal{C} \), but we worry about them later.
Let \( C \) be a path or circuit in a graph \( J \). We say paths \( P, Q \) of \( J \) cross with respect to \( C \), if \( P, Q \) are disjoint, and \( P \) has distinct ends \( p_1, p_2 \in V(C) \), and \( Q \) has distinct ends \( q_1, q_2 \in V(C) \), and no other vertex of \( P \) or \( Q \) belongs to \( C \), and these ends can be numbered such that either \( p_1, q_1, p_2, q_2 \) are in order in \( C \), or \( q_1, p_1, q_2, p_2 \) are in order in \( C \). We say that \( J \) is \( C \)-planar if \( J \) can be drawn in a closed disc \( \Delta \) such that every vertex and edge of \( C \) is drawn in the boundary of \( \Delta \). We shall prove:

5.1 Let \((G, F, C)\) be a framework, and let \( H, \eta_F \) satisfy (E1)--(E7). Let \( \eta \) be a homeomorphic embedding of \( G \) in \( H \) that extends \( \eta_F \), let \( C \subseteq C \), and let \( A \) be a set of \( \eta \)-bridges that sit on \( C \). Let \( J = \eta(C) \cup \bigcup \{ B : B \in A \} \). Then \( J \) is \( \eta(C) \)-planar.

5.1 is a consequence of the following.

5.2 Let \((G, F, C)\) be a framework, and let \( H, \eta_F \) satisfy (E1)--(E7). Let \( \eta \) be a homeomorphic embedding of \( G \) in \( H \) that extends \( \eta_F \), and let \( C \subseteq C \). Let \( P, Q \) be \( \eta \)-paths that cross with respect to \( \eta(C) \). Then for one of \( P, Q \), the \( \eta \)-bridge that contains it does not sit on \( C \).

Proof of 5.1, assuming 5.2.

Suppose that \( X,Y \subseteq V(J) \) with \( X \cup Y = V(J) \) and \( V(C) \subseteq Y \), such that \(|X \setminus Y| \geq 2\) and no edge of \( J \) has one end in \( X \setminus Y \) and the other in \( Y \setminus X \). We claim that \(|X \cap Y| \geq 4\). For let \( Y' = Y \cup (V(H) \setminus X) \); then no edge of \( H \) has one end in \( X \setminus Y' \) and the other in \( Y' \setminus X \), and \( X \cup Y' = V(H) \), and \(|X \setminus Y'| \geq 2\), and so \( X \) and \( Y' \) both includes the vertex set of a circuit of \( H \). Since \( H \) is cyclically four-connected, it follows that \(|X \cap Y'| \geq 4\), and so \(|X \cap Y| \geq 4\) as claimed.

From this and theorems 2.3 and 2.4 of [2], it follows, assuming for a contradiction that \( J \) is not \( \eta(C) \)-planar, that there are \( \eta \)-paths \( P, Q \) in \( J \) that cross with respect to \( \eta(C) \). By 5.2 the \( \eta \)-bridge containing one of \( P, Q \) does not sit on \( C \) and hence does not belong to \( A \), a contradiction. This proves 5.1.

Proof of 5.2.

We remark, first, that

(1) If \( B \) is an \( \eta \)-bridge that sits on \( C \), and \( e \in E(C) \) is an \( \eta \)-attachment of \( B \), then there is an \( \eta \)-attachment \( g \in E(C) \) of \( B \) such that \( g \neq e \) and \( g \) is not twinned with \( e \).

Subproof. By 3.1 it follows that \( B \) has at least two \( \eta \)-attachments. Suppose that every \( \eta \)-attachment different from \( e \) is twinned with \( e \); then by 3.4 there is only one other, say \( f \), and \( e, f \) are twinned, and therefore there exists \( C' \neq C \) in \( C \) containing all \( \eta \)-attachments of \( B \), contradicting that \( B \) sits on \( C \). This proves (1).

For \( e, f \in E(C) \), let

\[
\epsilon(e, f) = \begin{cases} 
3 & \text{if } e = f, \\
2 & \text{if } e \neq f, \text{ and } e, f \text{ are twinned} \\
0 & \text{if } e \neq f, \text{ and } e, f \text{ are not twinned}.
\end{cases}
\]

Let \( P \) have ends \( p_1, p_2 \), and let \( Q \) have ends \( q_1, q_2 \); and let \( B_1, B_2 \) be the \( \eta \)-bridges containing \( P, Q \) respectively. Let \( p_i \in V(\eta(e_i)) \) and \( q_i \in V(\eta(f_i)) \) for \( i = 1, 2 \), and let \( N = \epsilon(e_1, e_2) + \epsilon(f_1, f_2) \). We
prove by induction on \( N \) that one of \( B_1, B_2 \) does not sit on \( C \). We assume they both sit on \( C \), for a contradiction.

(2) Either \( e_1, e_2 \) are different and not twinned, or \( f_1, f_2 \) are different and not twinned.

Subproof. Suppose that \( e_1 \) and \( e_2 \) are equal or twinned, and so are \( f_1, f_2 \). We claim that

\[
|\{e_1, e_2, f_1, f_2\}| \leq 2,
\]

and if this set has two members then they are twinned. For suppose that \( e_1 = e_2 \). Since \( P, Q \) cross, it follows that one of \( f_1, f_2 \) equals \( e_1 \), say \( f_1 = e_1 \); and since either \( f_2 = f_1 \) or \( f_2 \) is twinned with \( f_1 \), the claim follows. So we may assume that \( e_1, e_2 \) are twinned, and similarly so are \( f_1, f_2 \). But by (F5) and (F6), only one pair of edges of \( C \) is twinned, and so again the claim holds.

Since \( B_1 \) sits on \( C \), by (1) it has an \( \eta \)-attachment \( g \neq e_1 \) that is not twinned with \( e_1 \); and so \( g \neq e_1, e_2, f_1, f_2 \). Take a minimal path \( R \) in \( B_1 \) between \( V(P \cup Q) \) and \( V(G) \), and let its end \( r \) in \( P \cup Q \) be a vertex of \( S \), say, where \( \{S, T\} = \{P, Q\} \). Let \( S' \) be a path consisting of the union of \( R \) and a subpath of \( S \) from \( r \) to an appropriate end of \( S \), chosen such that \( S', T \) cross. This contradicts the inductive hypothesis on \( N \), and so proves (2).

(3) \( e_1 \neq e_2 \) and \( f_1 \neq f_2 \).

Subproof. Suppose that \( e_1 = e_2 \), say. Since \( P, Q \) cross, one of \( f_1, f_2 \) equals \( e_1 \), say \( f_1 = e_1 = e_2 \); and by (2), \( f_2 \neq f_1 \), and \( f_1, f_2 \) are not twinned. By (1), \( B_1 \) has an \( \eta \)-attachment \( g \in E(C) \) not twinned with \( e_1 \). Hence there is a minimal path \( R \) of \( B_1 \) from \( V(P) \) to \( V(Q) \cup \eta(g) \). If it meets \( \eta(g) \), we contradict the inductive hypothesis as before, so we assume \( R \) has one end in \( V(P) \) and the other in \( V(Q) \).

Let \( f_1 = uv \), and let \( G' = G + (f_1,f_2) \) with new vertices \( x, y \). By adding \( Q \) to \( \eta(G) \) we see that there is a homeomorphic embedding \( \eta'' \) of \( G' \) in \( H \) extending \( \eta_F \) such that \( ux, vx \) and \( xy \) are all \( \eta'' \)-attachments of some \( \eta'' \)-bridge (including \( P \cup R \)). From 4.4, we may choose \( \eta'' \) extending \( \eta_F \) such that \( ux, vx, xy \) and some fourth edge \( g \) are all \( \eta'' \)-attachments of some \( \eta'' \)-bridge. In other words, we may choose a homeomorphic embedding \( \eta' \) of \( G \) in \( H \) extending \( \eta_F \) such that there exist

- an \( \eta' \)-path \( P' \) with ends \( p'_1, p'_2 \) in \( V(\eta'(f_1)) \),
- an \( \eta' \)-path \( Q' \) with ends \( q'_1, q'_2 \) disjoint from \( P' \), where \( q'_1 \) lies in \( \eta'(f_1) \) between \( p'_1 \) and \( p'_2 \), and \( q'_2 \in V(\eta'(f_2)) \),
- a path \( R' \) with one end in \( P' \), the other end in \( Q' \), and with no other vertex or edge in \( \eta'(G) \cup P' \cup Q' \), and
- a path \( S' \) with one end in \( P' \cup R' \), the other end in \( \eta'(g) \) where \( g \neq f_1 \), and with no other vertex or edge in \( \eta'(G) \cup P' \cup Q' \cup R' \).

Let \( B' \) be the \( \eta' \)-bridge containing \( P' \cup Q' \cup R' \cup S' \). By 4.6, there exists \( C' \in \mathcal{C} \) such that all \( \eta' \)-attachments of \( B' \) are in \( E(C') \). Now \( f_1 \neq f_2 \) and they are not twinned, so \( C' = C \), and hence \( B' \) sits on \( C \). Let \( T \) be an \( \eta' \)-path in \( P' \cup R' \cup S' \) with one end in \( \eta'(f_1) \) and the other in \( \eta'(g) \), chosen such that \( Q', T \) cross with respect to \( \eta'(C) \). Then both \( Q', T \) are contained in \( B' \), and yet \( B' \) sits on
$C$, and $\epsilon(f_1, g) \prec \epsilon(f_1, f_1)$, contrary to the inductive hypothesis. This proves (3).

(4) $e_1, e_2$ are not twinned, and $f_1, f_2$ are not twinned.

**Subproof.** Suppose that $f_1, f_2$ are twinned, say. Let $f_1 = v_1x_1$ and $f_2 = v_2x_2$ where either $C$ is a circuit and $v_1 = v_2 \in V(F)$, or $C$ is a path with ends $v_1, v_2$. By (1), there is an $\eta$-attachment of $B_2$ different from $f_1, f_2$; and so there is a minimal $\eta$-path $R$ in $B_2$ from $V(Q)$ to $V(P) \cup |\eta(C \setminus \{f_1, f_2\})|$. From the inductive hypothesis, $R$ does not meet $\eta(C \setminus \{f_1, f_2\})$, and so it meets $P$. Let $R$ have ends $r_1 \in V(P)$ and $r_2 \in V(Q)$, and for $i = 1, 2$, let $P_i = P[p_i, r_i]$ and $Q_i = Q[q_i, r_i]$

Now for $i = 1, 2$, $x_i \notin V(F)$ by (F5) (since if $C$ is a circuit then $v_1 \in V(F)$ by (F6)). For $i = 1, 2$, let $g_i$ be the edge of $G$ not in $C_i$ incident with $x_i$, and let $h_i$ be the edge of $C$ different from $f_i$ that is incident with $x_i$.

Now since either $C$ is a path and $f_1, f_2$ are end-edges of $C$, or $C$ is a circuit and $f_1, f_2$ have a common end, and since $P, Q$ cross, we may assume that $e_1 = f_1$, and $p_1$ lies in $\eta(f_1)$ between $q_1$ and $\eta(v_1)$. It follows that $e_2 \neq f_1, f_2$ by (2).

Suppose first that either $e_2 = h_1$ or $x_1$ is adjacent to an end of $e_2$. By rerouting $h_1$ along $P$, we obtain a homeomorphic embedding $\eta'$ of $G$ in $H$ extending $\eta_F$, such that $g_1, h_1$ and $f_2$ are all $\eta'$-attachments of some $\eta'$-bridge (containing $Q \cup R$). Since no member of $C$ contains all of $g_1, h_1$ and $f_2$, this contradicts 4.6. Hence $e_2 \neq h_1$ and $x_1$ is not adjacent to any end of $e_2$.

By (F6), $|V(C)| \leq 6$, so either $e_2 = h_2$, or $C$ is a circuit and $x_2$ is adjacent to an end of $e_2$. Suppose first that $C$ is a path; so $e_2 = h_2$. By rerouting $h_2$ along $P_2 \cup R \cup Q_2$ and adding $P_1$ and $Q_1$, we obtain a homeomorphic embedding (in $H$, extending $\eta_F$) of a cross extension of $G$ over $C$ of the third kind, contrary to (E6). Thus $C$ is a circuit, and so $v_1 \in V(F)$, and therefore $\{f_1, g_2, h_2\}$ is a circuit-type $Y$-trinity. But then by rerouting $h_2$ along $P_2 \cup R \cup Q_2$ and adding $P_1$ and $Q_1$ we obtain a homeomorphic embedding (in $H$, extending $\eta_F$) of an expansion of this $Y$-trinity of the first or second type, contrary to 4.2. This proves (4).

(5) $e_1, e_2$ have no common end, and $f_1, f_2$ have no common end.

**Subproof.** Suppose that $e_1, e_2$ have a common end, say. Since $e_1, e_2$ are not twinned by (4), it follows from 3.1 that $v$ has degree three in $G \setminus E(F)$; and so by (F5), $v \notin V(F)$. Since $P, Q$ cross, we may assume that $f_1 = e_1$ and $p_1, q_1, \eta(v)$ are in order in $\eta(e_1)$. Let $f, e_1, e_2$ be the three edges of $G$ incident with $v$. Let $\eta'$ be obtained from $\eta$ by rerouting $e_1$ along $P$. Then $\eta'$ is a homeomorphic embedding of $G$ in $H$ extending $\eta_F$, and $f, f_1$ and $f_2$ are $\eta'$-attachments in $E(G) \setminus E(F)$ of the $\eta'$-bridge containing $Q$. From 4.6, there exists $C' \subseteq C$ with $f, f_1, f_2 \in E(C')$. Since $f \notin E(C)$ it follows that $C' \neq C$, and so $f_1, f_2$ are twinined, contrary to (4). This proves (5).

Thus $e_1, e_2$ have no common end, and nor do $f_1, f_2$. By (E6), we may assume that one end of $e_1$ is adjacent to one end of $e_2$. Since $P, Q$ cross, we may therefore assume that for some edge $g = uv$ of $C$, $u$ is an end of $e_1$, $v$ is an end of $e_2$, $f_1 \in \{e_1, g\}$, and if $f_1 = e_1$ then $p_1, q_1, \eta(u)$ are in order in $\eta(e_1)$. Let $u$ be incident with $g, e_1, g_1$ and $v$ with $g, e_2, g_2$.

Suppose that $u \notin V(F)$. Let $\eta'$ be the homeomorphic embedding obtained from $\eta$ by rerouting $g$ along $P$. By (E2), applied to $\eta'(G)$ with edges $f_2, g_1$, it follows that there exists $C_1 \subseteq C$ with $f_2, g_1 \in E(C_1)$. By (F3), $C_1$ contains one of $e_1, g$, say $h$. Hence $h, f_2$ are twinined; and since $f_1, f_2$ are not twinined, it follows that $\{e_1, g\} = \{f_1, h\}$. If $h = g$, then $f_1 = e_1$; and since $g$ is not an end-edge
of $C$, (F6) implies that $f_2 = e_2$ and $g_2 \in E(F)$. But then $\{e_1, g_1, e_2\}$ is a Y-type trinity, and adding $P$ and $Q$ provides a homeomorphic embedding (in $H$, extending $\eta_F$) of an expansion of this trinity, contrary to 4.2. Thus $h \neq g$, and so $h = e_1$ and $f_1 = g$, and $f_2 \neq e_1, g, e_2$. If $C$ is a circuit, (F6) implies that the end of $e_1$ different from $u$ belongs to $V(F)$; but then by (F5), $v \notin V(F)$, and the symmetry between $u$ and $v$ implies that the end of $e_2$ different from $v$ belongs to $V(F)$, contrary to (F5). If $C$ is a path, then (F6) implies that $e_1$ is an end-edge of $C$, and $v \notin V(F)$; but then the symmetry between $u, v$ implies that $e_2$ is also an end-edge of $C$, a contradiction.

This proves that $u \in V(F)$. Consequently $v \notin V(F)$, and it follows (by exchanging $P,Q$, and exchanging $e_1, e_2$) that $f_2 \neq e_2$. Since $C$ contains $e_1, g$, it follows that $u$ is not an end of $C$, and so by (F5), $g_1 \in E(F)$. By (F2) there exists $C_2 \subset C$ containing $g, g_2$, since $v \notin V(F)$. Since $g_1 \in E(F)$, we deduce that $e_1 \in E(C_2)$. Since $f_1, f_2$ are not twinned, it follows that $f_2 \notin E(C_2)$. Thus $g_2 \in E(C_2) \setminus E(C)$, and $f_2 \in E(C) \setminus E(C_2)$, and $f_2, g_2$ have no common end, since $f_2 \neq e_2$. But rerouting $g$ along $P$ gives a homeomorphic embedding of $G$ in $H$ extending $\eta_F$, and adding $\eta(g)$ and $Q$ to it contradicts (E4). This proves 5.2.

6 The bridges between twins

To apply these results about frameworks, we have to choose a homeomorphic embedding $\eta$ of $G$ in $H$, and there is some freedom in how we do so. If we choose it carefully we can make several problems disappear simultaneously. The most important consideration is to ensure that each $\eta$-bridge has at least two $\eta$-attachments, but that is rather easy. With more care, we can also discourage $\eta$-bridges from having $\eta$-attachments in certain difficult places. To do so, we proceed as follows.

Let $(G, F, C)$ be a framework, and let $\eta$ be a homeomorphic embedding of $G$ in $H$ extending $\eta_F$, as usual. An edge $e$ of $G$ is a twin if there exists $f$ such that $e, f$ are twinned. (Thus, stating that “$e, f$ are twins” does not imply that they are twinned with each other.) An edge $e \in E(G) \setminus E(F)$ is

- **central** if it does not belong to any path in $C$ and is not a twin;
- **peripheral** if $e$ is an internal edge of some path in $C$
- **critical** if either $e$ is a twin or $e$ is an end-edge of some path in $C$.

By (F4) and (F6), no edge is both peripheral and critical, so every edge of $E(G) \setminus E(F)$ is of exactly one of these three kinds.

An edge $f \in E(H)$ is said to $\eta$-attach to $e \in E(G)$ if there is a path $P$ of $H$ with no internal vertex in $V(\eta(G))$ with $f \in E(P)$ and with one end a vertex of $\eta(e)$. (Thus $f$ $\eta$-attaches to $e$ if and only if either $f \in E(\eta(e))$ or $f$ belongs to an $\eta$-bridge for which $e$ is an $\eta$-attachment.) Let

- $L_1(\eta)$ be the set of edges in $E(H)$ that $\eta$-attach to some central edge of $G$;
- $L_2(\eta)$ be the set of edges in $E(H)$ that $\eta$-attach to an edge of $G$ which is either peripheral or central;
- $L_3(\eta)$ be the set of edges in $E(H)$ that attach to two edges of $G$ that are not twinned; and
- $L_4(\eta)$ be the set of edges in $E(H)$ that attach to two edges of $G$.  

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We say that \( \eta \) is \textit{optimal} if it is chosen (among all homeometric embeddings of \( G \) in \( H \) extending \( \eta_F \)) with the four-tuple of cardinalities of these sets lexicographically maximum; that is, for every homeomorphic embedding \( \eta' \) extending \( \eta_F \), there exists \( j \in \{1, \ldots, 5\} \) such that \( |L_i(\eta)| = |L_i(\eta')| \) for \( 1 \leq i < j \), and \( |L_j(\eta)| > |L_j(\eta')| \) if \( j \leq 4 \). In this section we study the properties of optimal embeddings.

6.1 Let \( \eta \) be an optimal homeomorphic embedding of \( G \) in \( H \) extending \( \eta_F \). Then every \( \eta \)-bridge has at least two \( \eta \)-attachments.

\textbf{Proof.} Let \( e \in E(G) \setminus E(F) \). Let us say an \( \eta \)-bridge is \textit{singular} if \( e \) is its only \( \eta \)-attachment, and \textit{nonsingular} otherwise. Suppose that there is a singular \( \eta \)-bridge. Let \( e = uv \), let \( p_1, \ldots, p_r \) be the set of vertices of \( \eta(e) \) that belong to nonsingular \( \eta \)-bridges, and let \( p_0 = \eta(u) \) and \( p_{r+1} = \eta(v) \), numbered such that \( p_0, p_1, \ldots, p_{r+1} \) are in order in \( \eta(e) \). For \( 0 \leq i \leq r \) let \( P_i = \eta(e)[p_i, p_{i+1}] \). Choose \( j \) with \( 0 \leq j \leq t \) such that some singular \( \eta \)-bridge contains a vertex of \( P_j \). Since \( H \) is three-connected, there is an \( \eta \)-bridge \( B \) containing a vertex \( b \) of the interior of \( P_j \) and containing a vertex \( a \) of \( \eta(G) \) not in \( P_j \). From the definition of \( p_1, \ldots, p_r \), it follows that \( B \) is singular. Hence there exists \( i \neq j \) with \( 0 \leq i \leq r \) such that \( a \) belongs to \( P_i \), and from the symmetry we may assume that \( i < j \). Let \( P \) be an \( \eta \)-path in \( B \) between \( a, b \). Let \( \eta' \) be obtained from \( \eta \) by rerouting \( e \) along \( P \). For every edge \( f \) of \( E(H) \), every \( \eta \)-attachment of \( f \) is also an \( \eta' \)-attachment. Consequently \( L_i(\eta) \subseteq L_i(\eta') \) for \( 1 \leq i \leq 4 \). But the edge of \( P_j \) incident with \( p_j \) belongs to \( L_4(\eta') \setminus L_4(\eta) \), contrary to the optimality of \( \eta \). This proves 6.1.

6.2 Let \( \eta \) be an optimal homeomorphic embedding of \( G \) in \( H \) extending \( \eta_F \). Let \( C \in \mathcal{C} \) be a path, and suppose that \( B \) is an \( \eta \)-bridge and all its \( \eta \)-attachments are edges of \( C \). Then its \( \eta \)-attachments are pairwise diverse in \( C \).

\textbf{Proof.} We claim first

(1) If \( e, f \) are edges of \( C \) with a common end \( v \), and \( g \) is the third edge of \( G \) incident with \( v \), then \( v \notin V(F) \), and either \( g \) is central, or \( g \) is peripheral and one of \( e, f \) is an end-edge of \( C \).

\textit{Subproof.} Certainly \( v \notin V(F) \) by (F5), since \( C \) is a path. If \( g \) does not belong to any path of \( C \) then it is not a twin by (F6), and so it is central. Thus we may assume that there is a path \( C' \in \mathcal{C} \) containing \( g \). By (F4), \( C' \) contains one of \( e, f \), say \( e \), and \( e \) is an end-edge of both \( C, C' \). Now (F1) implies that \( g \) is not an end-edge of \( C' \), and so by (F6), \( g \) is not a twin, and by (F4) \( g \) is not an end-edge of any path in \( C \), that is, \( g \) is peripheral. This proves (1).

(2) No two \( \eta \)-attachments of \( B \) in \( C \) have a common end.

\textit{Subproof.} Suppose that \( e, f \) are \( \eta \)-attachments of \( B \), and they have a common end \( v \). Let \( g \) be the third edge of \( G \) incident with \( v \). Choose a path \( P \) in \( B \) from a vertex \( a \) of \( \eta(e) \) to a vertex \( b \) of \( \eta(f) \). Let \( \eta' \) be obtained from \( \eta \) by rerouting \( f \) along \( P \). Then \( \eta' \) is a homeomorphic embedding of \( G \) in \( H \) extending \( \eta_F \) (note that \( g \notin E(F) \) since \( v \notin V(F) \) by (1)). Moreover, since no \( \eta \)-attachment of \( B \) is central, it follows that \( L_1(\eta) \subseteq L_1(\eta') \), and therefore equality holds. In particular, the edge of \( \eta(e) \) incident with \( \eta(v) \) therefore does not belong to \( L_1(\eta') \), and so \( g \) is not central. We deduce from
(1) that \(g\) is peripheral and one of \(e, f\) is an end-edge of \(C\), and from the symmetry we may assume that \(e\) is an end-edge of \(C\). Thus \(f\) is peripheral, and it follows that \(L_2(\eta) \subseteq L_2(\eta')\), and therefore equality holds. But the edge of \(\eta(e)\) incident with \(\eta(v)\) belongs to \(L_2(\eta')\), and does not belong to \(L_2(\eta)\) since \(e\) is an end-edge of \(C\), a contradiction. This proves \(6.2\).

To complete the proof, suppose that some two \(\eta\)-attachments \(e, f\) of \(B\) in \(C\) are not diverse in \(C\). Then by \((2)\), there are consecutive vertices \(u, v, w, x\) of \(C\), such that \(e = uv\) and \(f = wx\). Let the third edge of \(G\) at \(v\) be \(g\) and at \(w\) be \(h\). Choose a path \(P\) in \(B\) from a vertex \(a\) of \(\eta(e)\) to a vertex \(b\) of \(\eta(f)\). Let \(\eta'\) be obtained from \(\eta\) by rerouting \(vw\) along \(P\). Then \(\eta'\) is a homeomorphic embedding of \(G\) in \(H\) extending \(\eta_F\). Since no \(\eta\)-attachment of \(B\) is central, it follows that \(L_1(\eta) \subseteq L_1(\eta')\), and therefore equality holds. In particular, the edge of \(\eta(e)\) incident with \(\eta(v)\) does not belong to \(L_1(\eta')\), and so \(g\) is not central. From \((1)\), it follows that \(g\) is peripheral and \(e\) is an end-edge of \(C\). Similarly \(h\) is peripheral and \(f\) is an end-edge of \(C\). Hence \(L_2(\eta) \subseteq L_2(\eta')\), and therefore equality holds. But the edge of \(\eta(e)\) incident with \(\eta(v)\) belongs to \(L_2(\eta')\) and not to \(L_2(\eta)\) since \(e\) is an end-edge of \(C\), a contradiction. This proves \(6.2\).

If \(C \in \mathcal{C}\), we denote by \(\mathcal{A}(C)\) the set of all \(\eta\)-bridges that sit on \(C\). If \(e, f\) are twinned edges of \(G\), we denote by \(\mathcal{A}(e, f)\) the set of all \(\eta\)-bridges that have no attachments different from \(e, f\). Thus, if \(\eta\) is optimal, then by \((1)\) every bridge belongs to \(\mathcal{A}(C)\) for some \(C\) or to \(\mathcal{A}(e, f)\) for some \(e, f\), and to only one such set (except that \(\mathcal{A}(e, f) = \mathcal{A}(f, e)\)). The next four theorems are all about a pair of twinned edges \(e, f\), and it is convenient first to set up some notation. Thus, let \(e, f\) be twinned edges of \(G\). Let there be \(r\) vertices \(p_1, \ldots, p_r\) of \(\eta(e)\) that belong to an \(\eta\)-bridge with an \(\eta\)-attachment different from \(e\) and \(f\), and let \(\eta(e)\) have ends \(p_0\) and \(p_{r+1}\), numbered such that \(p_0, \ldots, p_{r+1}\) are in order in \(\eta(e)\). For \(0 \leq i \leq r\), let \(P_i = \eta(e)[p_i, p_{i+1}]\). Let \(q_0, \ldots, q_{s+1} \in V(\eta(f))\) and \(Q_0, \ldots, Q_s\) be defined similarly.

6.3 Let \(\eta\) be an optimal homeomorphic embedding of \(G\) in \(H\) extending \(\eta_F\), and let \(e, f\) be twinned edges of \(G\). With notation as above, for every \(B \in \mathcal{A}(e, f)\) there exist \(i\) and \(j\) with \(0 \leq i \leq r\) and \(0 \leq j \leq s\) such that \(B \cap \eta(e) \subseteq P_i\) and \(B \cap \eta(f) \subseteq Q_j\).

**Proof.** Suppose that some member \(B\) of \(\mathcal{A}(e, f)\) meets both \(P_i\) and \(P_j\), where \(0 \leq i < j \leq r\). Let \(P\) be an \(\eta\)-path in \(B\) between some \(a \in V(P_i)\) and some \(b \in V(P_j)\). Let \(\eta'\) be obtained from \(\eta\) by rerouting \(e\) along \(P\). Since no \(\eta\)-attachment of \(B\) is central or peripheral, and no edge of \(B\) is in \(L_3(\eta)\), it follows that \(L_i(\eta) \subseteq L_i(\eta')\) for \(i = 1, 2, 3\), and so equality holds in all three. Let \(B'\) be an \(\eta\)-bridge containing \(p_i\); then \(B'\) has an \(\eta\)-attachment different from \(e, f\), say \(g\). Consequently \(e, g\) are not twinned, and in particular, the edge of \(P_j\) incident with \(p_j\) is in \(L_3(\eta')\), a contradiction. This proves \(6.3\).

6.4 Let \(\eta\) be an optimal homeomorphic embedding of \(G\) in \(H\) extending \(\eta_F\), and let \(e, f\) be twinned edges of \(G\). Suppose that \(e, f\) have a common end \(v\), and let \(e = uv\) and \(f = vw\). Then \(\mathcal{A}(e, f)\) can be numbered as \(\{B_1, \ldots, B_k\}\), such that

- \(B_i\) has only one edge \(c_id_i\) for \(1 \leq i \leq k\);
- \(\eta(u), c_1, \ldots, c_k, \eta(v)\) are in order in \(\eta(e)\), and \(\eta(w), d_1, \ldots, d_k, \eta(v)\) are in order in \(\eta(f)\); and
• for 1 ≤ i < k, one of \( \eta(e)[c_i,c_{i+1}] \) \( \eta(f)[d_i,d_{i+1}] \) contains a vertex of some \( \eta \)-bridge not in \( A(e,f) \).

**Proof.** Using the notation established earlier, we may assume that \( \eta(v) = p_0 = q_0 \).

(1) Suppose that \( M,N \) are disjoint \( \eta \)-paths, from \( m \) to \( m' \) and from \( n \) to \( n' \) respectively, such that

- \( \eta(u),m,n,\eta(v),m',n',\eta(w) \) are in order in the path \( \eta(e) \cup \eta(f) \); and

- no edge of \( M \cup N \) belongs to \( L_2(\eta) \).

Then there exist \( i,j \) with 0 ≤ \( i \leq r \) and 0 ≤ \( j \leq s \) such that \( m,n \) belong to \( P_i \) and \( m',n' \) belong to \( P_j \).

**Subproof.** Let \( m \) be in \( P_i \) and \( n \) be in \( P_h \) where 0 ≤ \( h \leq i \leq r \). Let

\[
\eta'(e) = \eta(e)[\eta(u),m] \cup M \cup \eta(f)[m',\eta(v)].
\]

and

\[
\eta'(f) = \eta(e)[\eta(v),n] \cup N \cup \eta(f)[n',\eta(w)].
\]

Then \( \eta' \) is a homeomorphic embedding of \( G \) in \( H \) extending \( \eta_F \). Since no edge of the \( \eta \)-bridges containing \( M \) or \( N \) belongs to \( L_1(\eta) \) or to \( L_2(\eta) \), and \( e,f \) are critical, it follows that \( L_i(\eta) \subseteq L_i(\eta') \) for \( i = 1,2 \), and so equality holds in both. Let \( B \) be the \( \eta \)-bridge containing \( p_i \). Then there is an \( \eta \)-attachment \( g \neq e,f \) of \( B \). Choose \( C \subseteq C \) containing \( e,g \) (this is possible by \( E_2 \) applied to \( \eta(G) \) with edges \( e,g \)). From \( F_6 \), \( C \) is a circuit, and so \( g \) is not critical from \( F_5 \). Hence \( g \) is either central or peripheral, and so the edges of \( \eta(e) \) incident with \( p_i \) belongs to \( L_2(\eta') \), a contradiction. This proves (1).

To complete the proof, for 0 ≤ \( i \leq r \) and 0 ≤ \( j \leq s \) let \( A_{ij} \) be the set of all \( B \in A(e,f) \) with \( B \cap \eta(e) \subseteq P_i \) and \( B \cap \eta(f) \subseteq Q_j \). From (1), \( A(e,f) = \bigcup A_{ij} \). For each \( i,j \) let \( J_{ij} \) be the union of all members of \( A_{ij} \). Suppose that some \( |E(J_{ij})| \geq 2 \). Since \( H \) is cyclicly five-connected by \( E_1 \), we may assume (by exchanging \( e \) and \( f \) if necessary) that there are \( b_1,b',b_2 \) in \( P_i \), in order, such that \( b_1 \) and \( b_2 \) both belong to \( J_{ij} \), and \( b' \) belongs to some \( \eta \)-bridge \( B' \notin A_{ij} \). Since \( b' \neq p_1, \ldots, p_r \) it follows that \( B' \in A(e,f) \), and so \( B' \in A_{ij'} \), for some \( j' \neq j \). In particular, \( J_{ij} \) and \( J_{ij'} \) are disjoint. By 6.1 it follows that there is a path \( M \) in \( J_{ij} \) and a path \( N \) in \( J_{ij'} \) violating (1) (possibly with \( M,N \) exchanged). This proves that each \( J_{ij} \) has at most one edge, and in particular from 6.3, each \( \eta \)-bridge in \( A(e,f) \) has only one edge. The result follows from (2) applied to the paths of length one formed by these \( \eta \)-bridges. This proves 6.4. □

6.5 Let \( \eta \) be an optimal homeomorphic embedding of \( G \) in \( H \) extending \( \eta_F \), and let \( e,f \) be twinned edges of \( G \). Suppose that \( e,f \) are disjoint, and there is no path \( C \subseteq C \) of length five with end-edges \( e,f \). Then

- there is at most one \( \eta \)-bridge in \( A(e,f) \), and any such \( \eta \)-bridge has only one edge;

- no other \( \eta \)-bridge contains any vertex of \( \eta(e) \cup \eta(f) \); and

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• \( \mathcal{A}(C) = \emptyset \) for every member of \( C \) containing \( e \) or \( f \).

**Proof.** Now there is a path in \( C \) with end-edges \( e, f \), and so every member \( C \) of \( C \) containing \( e \) or \( f \) is a path, by (F4). Moreover, if \( e, f \in E(C) \) then \( C \) has length at most four by hypothesis and (F6), and \( C \) has end-edges \( e, f \), and therefore every member of \( \mathcal{A}(C) \) has an \( \eta \)-attachment some edge of \( C \) different from \( e, f \). By 6.2, this implies that \( \mathcal{A}(C) = \emptyset \). On the other hand, if \( C \subseteq C \) contains just one of \( e, f \) then \( C \) has length three by (F6), and again \( \mathcal{A}(C) = \emptyset \) by 6.2. This proves the third assertion. Consequently, \( r = s = 0 \) (in our previous notation). Since \( H \) is cyclically five-connected by (E1), it follows that the union of all \( \eta \)-bridges in \( \mathcal{A}(e, f) \) and the paths \( \eta(e), \eta(f) \) contains no circuit; and so there is at most one \( \eta \)-bridge in \( \mathcal{A}(e, f) \) and any such \( \eta \)-bridge has only one edge. This proves 6.5.

6.6 Let \( \eta \) be an optimal homeomorphic embedding of \( G \) in \( H \) extending \( \eta_F \), and let \( e, f \) be twinned edges of \( G \). Suppose that \( e, f \) are disjoint, and there exists \( C \subseteq C \) of length five with end-edges \( e, f \). Then:

• \( \mathcal{A}(C') \) is empty for every \( C' \neq C \) in \( C \) containing \( e \) or \( f \);

• the vertices of \( C \) can be numbered in order as \( v_0 v_1 \cdots v_5 \), such that for each \( B \in \mathcal{A}(C) \), its only \( \eta \)-attachments are \( v_1 v_2 \) and \( v_4 v_5 \) (and we may assume that \( e = v_0 v_1 \) and \( f = v_4 v_5 \), possibly after exchanging \( e, f \));

• \( \mathcal{A}(e, f) \) can be numbered as \( \{B_1, \ldots, B_k\} \) such that \( B_i \) has exactly one edge \( c_i d_i \) for \( 1 \leq i \leq k \), where \( c_i \in V(\eta(e)) \) and \( d_i \in V(\eta(f)) \); and

• \( \eta(v_0), c_1, \ldots, c_k, \eta(v_1) \) are in order in \( \eta(e) \), and \( \eta(v_4), d_1, \ldots, d_k, \eta(v_5) \) are in order in \( \eta(f) \).

**Proof.** Let \( C \subseteq C \) of length five with end-edges \( e, f \).

(1) **The first assertion of the theorem is true.**

**Subproof.** By (F7), every other path in \( C \) containing \( e \) or \( f \) has length at most four. If \( C' \subseteq C \) contains both \( e, f \), then \( \mathcal{A}(C') = \emptyset \) by 6.2, since each member of \( \mathcal{A}(C') \) has an \( \eta \)-attachment in \( C \) different from \( e, f \); and if \( C' \subseteq C \) contains just one of \( e, f \), then it has length three by (F6), and again \( \mathcal{A}(C') = \emptyset \) by 6.2. This proves (1).

(2) **The second assertion is true.**

**Subproof.** Let \( C \) have vertices \( v_0 v_1 \cdots v_5 \) in order, where \( e = v_0 v_1 \) and \( f = v_4 v_5 \). Let \( B \in \mathcal{A}(C) \). By 6.2, one of \( e, f \) is an \( \eta \)-attachment of \( B \), say \( f \); and since \( B \) has two \( \eta \)-attachments in \( C \) and they are diverse in \( C \) by 6.2, and \( e, f \) are twinned, it follows that the only other \( \eta \)-attachment of \( B \) is \( v_1 v_2 \). Let \( B' \in \mathcal{A}(C) \) with \( B' \neq B \); we claim that \( v_1 v_2 \) and \( v_4 v_5 \) are the \( \eta \)-attachments of \( B' \). For if not, then by the previous argument \( v_0 v_1 \) and \( v_3 v_4 \) are \( \eta \)-attachments of \( B' \), contrary to (E6). This proves (2).

In our earlier notation, we may assume that \( p_0 = \eta(v_0) \) and \( q_0 = \eta(v_1) \). Suppose that \( B \) is an \( \eta \)-bridge not in \( \mathcal{A}(e, f) \) that meets \( \eta(e) \). Then from 6.1 and 4.6, \( B \in \mathcal{A}(C') \) for some \( C' \subseteq C \).
containing e, and hence \( B \in A(C) \) from (1); but this contradicts (2). Consequently \( r = 0 \).

(3) Suppose that \( M, N \) are disjoint \( \eta \)-paths, from \( m \) to \( m' \) and from \( n \) to \( n' \) respectively, where \( \eta(v_0), m, n, \eta(v_1) \) are in order in \( \eta(e) \), and \( \eta(v_4), n', m', \eta(v_5) \) are in order in \( \eta(f) \). Then there exists \( j \) with \( 0 \leq j \leq s \) such that \( m', n' \) belong to \( Q_j \).

Subproof. Suppose not; then there exist distinct \( j, j' \) with \( m' \in V(Q_j) \) and \( n' \in V(Q_{j'}) \), and consequently \( j < j' \). Let \( B \) be the \( \eta \)-bridge containing \( q_{j'} \); then \( B \notin A(e, f) \) from the definition of \( q_1, \ldots, q_s \), and so \( B \) has an \( \eta \)-attachment \( g \neq e, f \). From 4.6, and (1) it follows that \( B \in A(C) \), and \( g = v_1v_2 \). In particular, \( B \) is disjoint from \( M, N \). Choose an \( \eta \)-path \( P \) in \( B \) from \( q_{j'} \) to \( V(\eta(v_1v_2)) \); then \( M, N, P \) contradict (E7). This proves (3).

For \( 0 \leq j \leq s \) let \( A_j \) be the set of all \( B \in A(e, f) \) with \( B \cap \eta(f) \subseteq Q_j \). From (1), \( A(e, f) = \bigcup A_j \). For each \( j \) let \( J_j \) be the union of all members of \( A_j \). Suppose that some \( |E(J_j)| \geq 2 \). Since \( H \) is cyclically five-connected by (E1), there are distinct \( b_1, b', b_2 \) in \( \eta(e) \), in order, such that \( b_1 \) and \( b_2 \) both belong to \( J_j \), and \( b' \) belongs to some \( \eta \)-bridge \( B' \notin A_j \). Since \( b' \neq p_1, \ldots, p_r \) it follows that \( B' \in A(e, f) \), and so \( B' \in A_{j'} \), for some \( j' \neq j \). In particular, \( J_j \) and \( J_{j'} \) are disjoint. By 6.1 it follows that there is a path \( M \) in \( J_j \) and a path \( N \) in \( J_{j'} \) violating (1) (possibly with \( M, N \) exchanged). This proves that \( |E(J_j)| \leq 1 \) for \( 0 \leq j \leq s \). Thus every \( \eta \)-bridge in \( C(e, f) \) has only one edge, and no two of them have ends in the same \( Q_j \). The result follows from (3) applied to the paths of length one formed by these \( \eta \)-bridges. This proves 6.6.

7 Flattenable graphs

Let \( (G, F, C) \) be a framework and let \( H, \eta_F \) satisfy (E1). We say that \( H \) is flattenable onto \( (G, F, C) \) via \( \eta_F \) if there is

- a homeomorphic embedding \( \eta \) of \( G \) in \( H \) extending \( \eta_F \)
- a set of \( \eta \)-bridges \( B(C) \), for each \( C \in \mathcal{C} \), and
- an edge \( N(e) \) of \( \eta(e) \), for each edge \( e \) of \( G \setminus E(F) \) such that for some edge \( f \neq e \), \( e \) and \( f \) are twinned and have no common end

with the following properties. For each \( C \in \mathcal{C} \), if \( C \) is a circuit let \( P(C) \) be \( \eta(C) \), and if \( C \) is a path let \( P(C) \) be the maximal subpath of \( \eta(C) \) that contains \( \eta(g) \) for every \( g \in E(C) \) that is not an end-edge of \( C \), and does not contain any edge \( N(e) \). Then we require:

- every \( \eta \)-bridge belongs to exactly one set \( B(C) \)
- if \( B \in B(C) \) then \( B \cap \eta(G) \subseteq P(C) \)
- for \( C \in \mathcal{C} \), \( P(C) \cup \bigcup \{ B : B \in B(C) \} \) is \( P(C) \)-planar.

The main result, that everything so far has been directed towards, and of which all the other results in the paper will be a consequence, is the following.
7.1 Let \((G, F, \mathcal{C})\) be a framework, and let \(H, \eta_F\) satisfy (E1)–(E7). Suppose that there is a homeomorphic embedding of \(G\) in \(H\) extending \(\eta_F\). Then \(H\) is flattenable onto \((G, F, \mathcal{C})\) via \(\eta_F\).

**Proof.** Since there is a homeomorphic embedding of \(G\) in \(H\) extending \(\eta_F\), there is an optimal one, say \(\eta\). We will prove that \(\eta\) provides the required flattening. We begin with

1. If \(e, f \in E(G)\) are twinned and have a common end, there exists \(C \in \mathcal{C}\) containing \(e, f\) such that
   \[
   \eta(C) \cup \bigcup \{B : B \in \mathcal{A}(C) \cup \mathcal{A}(e, f)\}
   \]
   is \(\eta(C)\)-planar.

   **Subproof.** Let the two members of \(\mathcal{C}\) that contain \(v\) be \(C_1, C_2\), where \(v\) is the common end of \(e\) and \(f\). Let \(e = uv\) and \(f = vw\), and let \(c_1d_1, \ldots, c_kd_k\) be the edges of \(H\) with one end in \(\eta(e)\) and the other in \(\eta(f)\) (these are the edges of the bridges in \(\mathcal{A}(e, f)\) numbered as in 6.4). By 5.1 we may assume that \(k \geq 1\) Now
   \[
   \eta(C_i) \cup \bigcup \{B : B \in \mathcal{A}(e, f)\}
   \]
   is \(\eta(C_i)\)-planar for \(i = 1, 2\). We claim that for either \(i = 1\) or \(i = 2\), no member of \(\mathcal{A}(C_i)\) meets \(\eta(e) \cup \eta(f)\) between \(c_1\) and \(d_1\). If not, there are disjoint \(\eta\)-paths \(R_1, R_2\) such that for \(i = 1, 2\), \(R_i\) has one end \(r_i\) in \(\eta(e) \cup \eta(f)\) between \(c_1\) and \(d_1\), and its other end \(s_i\) is in \(\eta(C_i)\) and not in \(\eta(e) \cup \eta(f)\). Let \(s_i \in V(\eta(g_i)) (i = 1, 2)\). If \(g_1, g_2\) have no common end, this contradicts (E4), and if they have a common end, this contradicts 4.2. (To see this, in each case delete an appropriate end-edge of the subpath of \(\eta(e) \cup \eta(f)\) between \(c_1, d_1\).) We may therefore assume that no member of \(\mathcal{A}(C_1)\) meets \(\eta(e) \cup \eta(f)\) between \(c_1\) and \(d_1\). But then by 5.1, the claim holds. This proves (1).

   For edges \(e, f\) as in (1), let \(D(e, f)\) be some \(C \in \mathcal{C}\) satisfying (1).

2. Let \(e, f\) be twinned, with no common end. Then there are edges \(N(e)\) of \(\eta(e)\) and \(N(f)\) of \(\eta(f)\), and distinct paths \(C_1, C_2 \in \mathcal{C}\), both with end-edges \(e, f\) and with the following property, where for \(i = 1, 2\), \(P(C_i)\) denotes the component of \(\eta(C_i) \setminus \{N(e), N(f)\}\) containing \(\eta(g)\) for each internal edge \(g\) of \(C\).

   - \(\mathcal{A}(C) = \emptyset\) for all \(C \in \mathcal{C}\) containing either \(e\) or \(f\) and different from \(C_1\)
   - \(B \cap \eta(G) \subseteq P(C_1)\) for all \(B \in \mathcal{A}(C_1)\)
   - \(B \cap \eta(G) \subseteq P(C_2)\) for all \(B \in \mathcal{A}(e, f)\), and
   \[
   P(C_2) \cup \bigcup \{B : B \in \mathcal{A}(e, f)\}
   \]
   is \(P(C_2)\)-planar.

   **Subproof.** By 3.2 there are at least two paths in \(\mathcal{C}\) with end-edges \(e, f\), and by (F6) every such path has length at most five. If there is no path in \(\mathcal{C}\) with end-edges \(e, f\) and with length exactly five, the claim follows from 6.5, so we assume that some such path has length five, say \(C_1\). By 6.6, \(\mathcal{A}(C)\) is empty for every \(C \neq C_1\) in \(\mathcal{C}\) containing \(e\) or \(f\), so the first assertion of the claim holds. Moreover, also by 6.6,
the vertices of $C_1$ can be numbered in order as $v_0, v_1, \ldots, v_5$, such that for each $B \in A(C)$, its only $\eta$-attachments are $v_1 v_2$ and $v_4 v_5$ (and we may assume that $e = v_0 v_1$ and $f = v_4 v_5$, possibly after exchanging $e, f$);

- $A(e, f)$ can be numbered as $\{B_1, \ldots, B_k\}$ such that $B_i$ has exactly one edge $c_i d_i$ for $1 \leq i \leq k$, where $c_i \in V(\eta(e))$ and $d_i \in V(\eta(f))$; and

- $\eta(v_0), c_1, \ldots, c_k, \eta(v_1)$ are in order in $\eta(e)$, and $\eta(v_4), d_1, \ldots, d_k, \eta(v_5)$ are in order in $\eta(f)$.

Let $N(e)$ be the edge of $\eta(e)$ incident with $\eta(v_1)$, and $N(f)$ be the edge of $\eta(f)$ incident with $\eta(v_5)$. Then $B \cap \eta(G) \subseteq P(C)$ for all $B \in A(C)$, so the second assertion holds.

By (F7) there exists $C_2 \in C$ with end-edges $e$ and $f$ and with ends $v_1$ and $v_5$. It follows that $N(e)$ and $N(f)$ are the end-edges of $C_2$, and so $B \cap \eta(G) \subseteq P(C_2)$ for all $B \in A(e, f)$. From the second and third bullets above,

$$P(C_2) \cup \bigcup (B : B \in A(e, f))$$

is $P(C_2)$-planar. So the third assertion holds. This proves (2).

For $e, f$ as in (2), choose $C_1, C_2$ as in (2), and define $D(e, f) = C_2$. For each edge $e$ that is twinned with an edge disjoint from $e$, choose $N(e)$ as in (2). Since no edge of $e$ is twinned with more than one other edge, by 3.4, this is well-defined. For each $C \in C$, if $C$ is a circuit let $P(C) = C$, and if $C$ is a path let $P(C)$ be the maximal subpath of $\eta(C)$ that contains $\eta(g)$ for every $g \in E(C)$ that is not an end-edge of $C$, and does not contain any edge $N(e)$.

(3) For every path $C \in C$, $B \cap \eta(G) \subseteq P(C)$ for each $B \in A(C)$.

For let $C \in C$ be a path. If $P(C) = C$ the claim is true, so we may assume that some edge $e$ of $C$ is twinned with some other edge $f$ disjoint from $e$, and so $N(e)$ is defined. Choose $C_1, C_2$ satisfying (2), where $C_2 = D(e, f)$. If $C \neq C_1$ then $A(C) = \emptyset$ and the claim is trivial, by the first assertion of (2); while if $C = C_1$ then the claim holds by the second assertion of (2). This proves (3).

For each $C \in C$, let $B(C)$ be the following set of $\eta$-bridges:

- if $C = D(e, f)$ for some pair $e, f$ of twinned edges with a common end, let $B(C) = A(C) \cup A(e, f)$
- if $C = D(e, f)$ for some pair $e, f$ of twinned edges with no common end, let $B(C) = A(e, f)$
- otherwise, let $B(C) = A(C)$.

Now let $B$ be an $\eta$-bridge. We claim that $B$ belongs to exactly one set $B(C)$. For if $B$ sits on some $C' \in C$, then for $C \in C$, $B \in C$ if and only if $C = C'$; and otherwise, $B$ belongs to $A(e, f)$ for a unique pair $e, f$ of twinned edges, and then for $C \in C$, $B \in B(C)$ if and only if $C = D(e, f)$.

Also, we claim that if $B \in B(C)$ then $B \cap \eta(G) \subseteq P(C)$; for this is trivial if $C$ is a circuit, so we assume that $C$ is a path. By (3) the claim holds if $B \in A(C)$, so we may assume that $B = D(e, f)$ for some pair $e, f$ of disjoint twinned edges, and $B \in A(e, f)$. But then the claim holds by the third assertion of (2).
Finally, we claim that \( P(C) \cup \bigcup (B : B \in \mathcal{B}(C)) \) is \( P(C) \)-planar for each \( C \in \mathcal{C} \). If \( C = D(e, f) \) for some pair \( e, f \) with a common end, the claim follows from (1) and the definition of \( D(e, f) \). If \( C = D(e, f) \) for some pair of disjoint twinned edges, the claim follows from the third assertion of (2) and the definition of \( D(e, f) \), since \( \mathcal{A}(C) = \emptyset \) from the first assertion of (2). And otherwise, the claim follows from 5.1. This proves that \( \eta \) provides a flattening satisfying the theorem, and so proves 7.1.

8 Augmenting paths

We need three more techniques for the second half of the paper, all developed in [3], and in this section we describe the first. If \( F \) is a subgraph of \( G \) and of \( H \), and \( \eta \) is a homeomorphic embedding of \( G \) in \( H \), we say it fixes \( F \) if \( \eta(e) = e \) for all \( e \in E(F) \) and \( \eta(v) = v \) for all \( v \in V(F) \).

Let \( G \) be cubic, and let \( F \) be a subgraph of \( G \) with minimum degree \( \geq 2 \) (possibly null). Let \( X \subseteq V(G) \), such that \( \delta_G(X) \cap E(F) = \emptyset \). Let \( n \geq 1 \), let \( G_0 = G \), and inductively for \( 1 \leq i \leq n \) let \( G_i = G_{i-1} + (e_i, f_i) \) with new vertices \( v_i, v_i' \), where \( e_i, f_i \) are edges of \( G_{i-1} \) not in \( E(F) \). Let \( \eta_F \) be the identity homeomorphic embedding of \( G_0 \) to itself; and for \( 1 \leq i \leq n \), let \( \eta_i \) be obtained from \( \eta_{i-1} \) by replacing \( e_i \) and \( f_i \) by the corresponding two-edge paths of \( G_i \). Thus \( \eta_n \) is a homeomorphic embedding of \( G \) in \( G_i \); it fixes \( F \), and \( \eta(v) = v \) for all \( v \in V(G) \), and \( \eta(e) = e \) for all \( e \in E(G) \) except edges of \( G \) in \( \{e_1, f_1, \ldots, e_i, f_i\} \).

Let \( \delta_G(X) = \{x_1 y_1, \ldots, x_k y_k\} \), where \( x_1, \ldots, x_k \in X \) are all distinct, and \( y_1, \ldots, y_k \in V(G) \setminus X \) are all distinct. Suppose in addition:

- \( e_1 \in E(G) \) has both ends in \( X \), and \( f_n \in E(G) \) with both ends in \( V(G) \setminus X \)
- for \( 1 \leq i < n \) there exists \( j \in \{1, \ldots, k\} \) such that \( f_i \) is the edge of \( \eta_{i-1}(x_j y_j) \) incident with \( y_j \), and \( e_{i+1} \) is the edge of \( \eta_i(x_j y_j) \) incident with \( v_i \) and not with \( y_j \)
- if \( f_1 \in E(\eta(x_j y_j)) \) (that is, \( f_1 = x_j y_j \)) where \( 1 \leq j \leq k \), then \( e_1 \) is not incident with \( x_j \) in \( G \), and no end of \( e_1 \) is adjacent in \( G \setminus E(F) \) to \( x_j \); similarly, if \( e_n \in E(\eta(x_j y_j)) \) then \( e_n \) is not incident with \( y_j \) in \( G \), and no end of \( e_n \) is adjacent in \( G \setminus E(F) \) to \( y_j \)
- for \( 2 \leq i \leq n - 1 \), let \( e_i \in E(\eta_{i-1}(x_j y_j)) \) and \( f_i \in E(\eta_{i-1}(x_j' y_j')) \); then \( j' \neq j \), and \( x_j \) is not adjacent to \( x_j' \) in \( G \setminus E(F) \), and \( y_j \) is not adjacent to \( y_j' \) in \( G \setminus E(F) \).

(See Figure 5.)

In these circumstances we call \( G_n \) an \( X \)-augmentation of \( G \) (modulo \( F \)), and \( (e_1, f_1), \ldots, (e_n, f_n) \) an \( X \)-augmenting sequence of \( G \) (modulo \( F \)). Note that we permit \( n = 1 \). The following is proved in lemma 3.4 of [3], applied to \( F, H \setminus E(F) \) and \( X \).

8.1 Let \( G \) be cubic and let \( F \) be a subgraph of \( G \) with minimum degree at least two. Let \( X \subseteq V(G) \) with \( \delta_G(X) \cap E(F) = \emptyset \), such that the edges in \( \delta_G(X) \) pairwise have no common end. Let \( H \) be cubic such that \( F \) is a subgraph of \( H \), and let \( \eta \) be a homeomorphic embedding of \( G \) in \( H \) fixing \( F \). Then either

- there exists \( X' \subseteq V(H) \) with \( |\delta_H(X')| = |\delta_G(X)| \), such that for \( v \in V(G) \), \( v \in X \) if and only if \( \eta(v) \in X' \), or
9 Jumps on a dodecahedron

Now we begin the second part of the paper. First we prove the following variant of 1.5 (equivalent to 1.6).

9.1 Let $H$ be cyclically five-connected and cubic. Then $H$ is non-planar if and only if $H$ contains one of Petersen, Triplex, Box and Ruby.

Proof. “If” is clear. For “only if”, let $H$ be cyclically five-connected and cubic, and contain none of the four graphs. By 1.5 it follows that $H$ contains Dodecahedron. Let $G = $ Dodecahedron, let $F$ and $\eta F$ be null, and let $C$ be the set of circuits of $G$ that bound regions in the drawing in Figure 3; then $(G, F, C)$ is a framework. We claim that (E1)--(E7) are satisfied. Most are trivial, because $F$ is null, and there are no twinned edges, and no paths in $C$. Also, (E6) is vacuously true because no member of $C$ has length $\geq 6$; so the only axiom that needs work is (E2).

Let $e, f \in E(G)$ such that no member of $C$ contains both $e$ and $f$; we claim that $G + (e, f)$ contains one of Petersen, Triplex, Box, Ruby. Up to isomorphism of $G$ there are five possibilities for $e, f$, namely (setting $e = ab$ and $f = cd$) $(a, b, c, d) = (1, 2, 6, 15), (1, 2, 10, 15), (1, 2, 15, 20), (1, 2, 18, 19), (1, 2, 19, 20)$. In the first three cases $G + (e, f)$ contains Ruby, and in the last two it contains Box.

Thus, (E2) holds; and so $H$ is planar, by 7.1. This proves 9.1.

Next, a small repair job. The definition of “dodecahedrally-connected” in [3] differs from the one in this paper, and our objective of the remainder of this section is to prove them equivalent. To do so, we essentially have to repeat the proof of 9.1 with slightly different hypotheses.

In this section we fix a graph $F$, and we need to look at several graphs such that $F$ is a subgraph of all of them. If $G, H$ are cubic, and $F$ is a subgraph of them both, and there is a homeomorphic embedding of $G$ in $H$ fixing $F$, we say that $H F$-contains $G$.

Let $G$ be cubic, and let $F$ be a subgraph of $G$, such that every vertex in $F$ has degree $\geq 2$ in $F$. Let $C$ be a circuit of $G$ of length four, with vertices $a_1, a_2, a_3, a_4$ in order, none of them in
Let \( a_i \) be adjacent to \( b_i \not\in V(C) \) for \( 1 \leq i \leq 4 \), where \( b_1, \ldots, b_4 \) are all distinct, and not in \( V(F) \), and are pairwise non-adjacent. A \( C \)-leap of \( G \) means a graph \( G + (e, f) \), where \( e \in E(G) \) and \( f \in E(G) \setminus E(F) \), with no vertex in \( V(C) \).

### 9.2
Let \( G \) be cubic and cyclically four-connected, with \( |V(G)| \geq 8 \). Let \( F \) be a subgraph of \( G \) such that every vertex in \( F \) has degree \( \geq 2 \) in \( F \). Let \( C \) be a circuit of \( G \) of length 4, disjoint from \( F \). Let \( L \) be a set of cubic graphs such that \( F \) is a subgraph of each of them. Suppose that every \( C \)-leap of \( G \) \( F \)-contains a member of \( L \). Let \( H \) be a cyclically five-connected cubic graph containing \( F \) as a subgraph, that does not \( F \)-contain any member of \( L \). Then \( H \) does not \( F \)-contain \( G \).

**Proof.** Let \( X = V(C) \). Then \( \delta_G(X) \cap E(F) = \emptyset \) since \( X \cap V(F) = \emptyset \). Since \( G \) is cyclically four-connected and \( |V(G)| \geq 8 \) it follows that no two members of \( \delta_G(X) \) have a common end.

Suppose that \( H \) \( F \)-contains \( G \). Let us apply 8.1. Since \( H \) is cyclically five-connected, \( 8.1(i) \) does not hold, and so \( 8.1(ii) \) holds. Let \( (e_1, f_1), \ldots, (e_n, f_n) \) be an \( X \)-augmenting sequence of \( G \), such that there is a homeomorphic embedding of the corresponding \( X \)-augmentation \( G' \) in \( H \) fixing \( F \). From condition (iii) in the definition of “\( X \)-augmenting sequence”, it follows that \( n = 1 \), and so \( G' = G + (e_1, f_1) \). Thus \( G' \) is a \( C \)-leap of \( G \), and therefore \( F \)-contains a member of \( L \). But \( H \) \( F \)-contains \( G' \), and so \( H \) \( F \)-contains any member of \( L \), a contradiction. This proves 9.2.

It is convenient from now on to make the following convention. When we speak of a graph \( G + (e, f) \) and the vertices of \( G \) are numbered \( 1, \ldots, n \), the new vertices of \( G + (e, f) \) will be assumed to be numbered \( n + 1 \) and \( n + 2 \) (in order), unless we specify otherwise.

Let \( G \) be Dodecahedron, and let \( F \) be a circuit of \( G \) of length five. If \( e, f \in E(G) \setminus E(F) \), and at most one of \( e, f \) has an end in \( V(F) \), and \( e, f \) are not incident with the same region of \( G \), we call \( G + (e, f) \) a hop extension of \( (G, F) \); and if in addition \( e, f \) are diverse, we call \( G + (e, f) \) a jump extension of \( (G, F) \). We begin with the following lemma.

### 9.3
Let \( G \) be Dodecahedron, and let \( F \) be a circuit of \( G \) of length five. Let \( H \) be a cyclically five-connected cubic graph, such that \( F \) is a subgraph of \( H \). Suppose that

- \( H \) \( F \)-contains no jump extension of \( (G, F) \)
- for every \( X \subseteq V(H) \setminus V(F) \) with \( |\delta_H(X)| = 5 \) and \( X \neq V(H) \setminus V(F) \), there is no homeomorphic embedding \( \eta \) of \( G \) in \( H \) fixing \( F \) such that \( \eta(v) \in X \) for all \( v \in V(G) \setminus V(F) \).

Let \( e, f \) be diverse edges of \( G \) not in \( E(F) \); then \( H \) does not \( F \)-contain \( G + (e, f) \).

**Proof.** Suppose it does. Hence \( G + (e, f) \) is not a jump extension of \( (G, F) \), and so both \( e, f \) have ends in \( V(F) \). Let us number the vertices of Dodecahedron as in Figure 3, and from the symmetry we may assume that \( e \) is 2-7 and \( f \) is 5-10. Let \( G' = G + (e, f) \) with new vertices 21, 22 say. Let \( X = \{6, 7, \ldots, 20\} \). From the second bullet and 8.1, there is an \( X \)-augmenting sequence of \( G' \) modulo \( F \), say \( (e_1, f_1), \ldots, (e_n, f_n) \), and a homeomorphic embedding \( \eta'' \) of the corresponding \( X \)-augmentation \( G'' \) in \( H \) fixing \( F \). Now \( e_1 (= a_1 b_1 \text{ say}) \) has both ends in \( X \), but \( f_1 \) does not, so \( f_1 \) is one of 1-6, 2-21, 7-21, 3-8, 4-9, 5-22, 10-22, 21-22; and from the symmetry we may assume that \( f_1 \) is one of 1-6, 2-21, 7-21, 3-8, 21-22.

Suppose that \( f_1 \) is one of 1-6, 3-8. Then \( e_1, f_1 \) are diverse, from the third condition in the definition of \( X \)-augmenting sequence; but then \( G + (e_1, f_1) \) is a jump extension of \( (G, F) \) \( F \)-contained
in $G' + (e_1, f_1)$ and hence in $H$, a contradiction. Similarly if $f_1$ is 7-21 then $G + (e_1, 2-7)$ is a jump extension $F$-contained in $H$. Thus $f_1$ is one of 21-22, 2-21, and in particular $n = 1$. Assume $f_1$ is 21-22. Then we may assume that $e_1, 2-7$ are not diverse in $G$ (for otherwise $G + (e_1, 2-7)$ is a jump extension $F$-contained in $H$), and similarly $e_1, 5-10$ are not diverse in $G$. But this is impossible. Finally, assume that $f_1$ is 2-21. we may assume that $e_1, 2-7$ are not diverse in $G$, and so $e_1$ is one of 7-11, 7-12, 6-11, 11-16, 8-12, 12-17.

If $e_1$ is one of 7-12, 8-12, 12-17, rerouting 7-12 along 21-22 gives a jump extension of $(G, F)$ $F$-contained in $H$; and if $e_1$ is one of 7-11, 6-11, 11-16, rerouting 7-11 along 21-22 gives a jump extension of $(G, F)$ $F$-contained in $H$, again a contradiction. This proves 9.3.

9.4 Let $G, F, H$ be as in 9.3. Then $H$ $F$-contains no hop extension of $(G, F)$.

**Proof.** Let $L$ be the set of all graphs $G + (e, f)$ where $e, f$ are diverse edges of $G$ not in $E(F)$. By 9.3, $H$ $F$-contains no member of $L$. Let $G$ be labelled as in Figure 3. (We do not specify the circuit $F$ at this stage; it is better to preserve the symmetry.) Let $G_1 = G + (a, b)$ be a hop extension of $G$, and suppose that $H$ $F$-contains $G_1$. Thus $G_1 \notin L$. From the symmetry of $G$, we may therefore assume that $a$ is 15-20 and $b$ is 16-17. Thus the edges 16-17 and 15-20 are not in $E(F)$. Since $F$ is a circuit of length five, it follows that 16-20 is not in $E(F)$, and hence 16,20 are not in $V(F)$. Let $C$ be the circuit 16-20-21-22-16 of $G_1$. Then no vertex of $C$ is in $V(F)$, and $H$ is cyclically five-connected, so we can apply 9.2. We deduce that $H$ $F$-contains some $C$-leap $G_2 = G_1 + (e, f)$.

Now $e$ is one of 16-20, 20-21, 21-22, 16-22. Since $F$ is not yet specified, there is a symmetry of $G_1$ exchanging the edges 16-20 and 21-22; and one exchanging 20-11 and 16-22. Thus we may assume that $e$ is one of 21-22, 20-21.

Now $f$ is an edge of $G$ not incident with either of 16,20. Since $e$ is one of 21-22, 20-21, and $f \notin E(F)$, $H$ $F$-contains $G + (15-20, f)$ in $G_2$, and so $G + (15-20, f) \notin L$. Consequently $f$, 15-20 are not diverse, so $f$ is one of

$$6-15, 10-15, 1-6, 6-11, 5-10, 10-14, 14-19, 18-19.$$ 

Suppose first that $e$ is 21-22. Then by the same argument, $f$ and 16-17 are not diverse in $G$, and so $f$ is one of 6-11, 18-19. If $f$ is 6-11, rerouting 6-15 along 24-23-21 gives a member of $L$ $F$-contained in $H$ (in future we just say “works”) and if $f$ is 18-19, rerouting 17-18 along 22-23-24 works. Thus the claim holds if $e$ is 21-22.

Now we assume that $e$ is 20-21. If $f$ is one of 1-6,6-11,6-15 then rerouting 6-15 along 23-24 works; if $f$ is one of 10-15,5-10,10-14, rerouting 10-15 along 23-24 works; and if $f$ is 14-19 or 18-19 then rerouting 19-20 along 23-24 works. Thus in each case we have a contradiction. This proves 9.4.

Next we need another similar lemma. Let $G$ be Dodecahedron, labelled as in Figure 3, and let $G_1 = G + (1-6, 2-7)$. Let $G_2 = G_1 + (6-21, 2-22)$. Thus the edge 1-6 of $G$ has been subdivided to become a path 1-21-23-6 of $G_2$, and 2-7 has become 2-24-22-7.

9.5 Let $G, F, H$ be as in opposite. Then $H$ does not $F$-contain $G_2$. 

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Proof. Let \( X = \{6, 7, \ldots, 20\} \). By the second bullet hypothesis about \( H \), and 8.1, there is an \( X \)-augmenting sequence of \( G_2 \) modulo \( F \), say \((e_1, f_1), \ldots, (e_n, f_n)\), and a homeomorphic embedding \( \eta' \) of the corresponding \( X \)-augmentation \( G' \) in \( H \) fixing \( F \). Now \( e_1 (= a_1b_1 \text{ say}) \) has both ends in \( X \), but \( f_1 \) does not, so \( f_1 \) is one of

\[
1-21, 21-23, 6-23, 2-24, 22-24, 7-22, 3-8, 4-9, 5-10, 21-22, 23-24,
\]

and from the symmetry we may assume that \( f_1 \) is one of

\[
1-21, 21-23, 6-23, 5-10, 4-9, 21-22.
\]

If \( f_1 \) is one of 5-10, 4-9 then by the third condition in the definition of \( X \)-augmenting sequence, it follows that \( e_1, f_1 \) are diverse in \( G \), and \( H \) contains the jump extension \( G + (e_1, f_1) \), a contradiction. Similarly if \( f_1 \) is 6-23 then \( e_1, 1-6 \) are diverse in \( G \), again a contradiction. Thus \( f_1 \) is one of 1-21, 21-23, 21-22. Hence \( H \) \( F \)-contains \( G + (1-6, e_1) \), and so by 9.4, \( G + (1-6, e_1) \) is not a hop extension of \((G, F)\). Consequently \( f_1 \) is one of 10-15, 6-15, 6-11, 7-11. If \( f_1 \) is one of 6-11, 7-11, then rerouting 1-6 along 25-26 gives a jump extension of \((G, F)\) \( F \)-contained in \( H \); while if \( f_1 \) is one of 6-15, 10-15, rerouting 6-15 along 25-26, and then rerouting 7-11 along 23-24, give the desired jump extension. This proves 9.5.

From these lemmas we deduce a kind of variant of 9.1:

9.6 Let \( G \) be Dodecahedron, and let \( F \) be a circuit of \( G \) of length five. Let \( H \) be a cyclically five-connected cubic graph, such that \( F \) is a subgraph of \( H \). Suppose that

- \( H \) \( F \)-contains no jump extension of \((G, F)\)
- for every \( X \subseteq V(H) \setminus V(F) \) with \(|\delta_H(X)| = 5 \) and \( X \neq V(H) \setminus V(F) \), there is no homeomorphic embedding \( \eta \) of \( G \) in \( H \) fixing \( F \) such that \( \eta(v) \in X \) for all \( v \in V(G) \setminus V(F) \).

Then \( H \) is planar, and can be drawn in the plane such that \( F \) bounds the infinite region.

Proof. Let \( C \) be the set of the following eleven subgraphs of \( G = \) Dodecahedron: the six circuits that bound regions (in the drawing in Figure 3) that contain no edge incident with the infinite region, and for each \( e \in E(F) \), the path \( C \setminus e \) where \( C \neq F \) is the boundary of a region incident with \( e \). Let \( \eta_F \) be the identity homeomorphic embedding on \( F \). By hypothesis there is a homeomorphic embedding of \( G \) in \( H \) extending \( \eta_F \). We apply 7.1 to \((G, F, C)\) and \( H, \eta_F \). There are no twinned edges and all members of \( C \) have at most five edges; so we have to check only (E2) and (E6). (Note that in this case, the paths in \( C \) are not induced subgraphs of \( G \); this is the only one of our applications when this is so.) But the truth of (E2) and (E6) follows from the three lemmas above 9.3, 9.4, 9.5; and so by 7.1, the result follows. This proves 9.6.

As we said earlier, we need this to prove the equivalence of the definitions of dodecahedrally-connected given in this paper and in [3], and now we turn to that. Let \( G \) be Dodecahedron, and let \( F \) be a circuit of \( G \) of length five. Let \( H \) be a cubic graph, and let \( X \subseteq V(H) \). We say that \( H \) is \textit{placid} on \( X \) if

- \(|V(H) \setminus X| \geq 7\), and \( \delta_H(X) \) is a matching of cardinality five
\begin{itemize}
\item \( \{x_i y_i : 1 \leq i \leq 5\} \) is an enumeration of \( \delta_H(X) \), with \( x_i \in X \) (1 \( \leq i \leq 5 \))
\item there is a homeomorphic embedding of \( G \) in \( H' \) mapping \( F \) to the circuit \( y_1 \cdot y_2 \cdot y_3 \cdot y_4 \cdot y_5 \cdot y_1 \), and
\item there is no homeomorphic embedding of any jump extension of \((G, F)\) in \( H' \) mapping \( F \) to \( y_1 \cdot y_2 \cdot y_3 \cdot y_4 \cdot y_5 \cdot y_1 \),
\end{itemize}

where \( H' \) is obtained from \( H|\{X \cup \{y_1, y_2, y_3, y_4, y_5\}\} \) by deleting all edges with both ends in \( \{y_1, y_2, y_3, y_4, y_5\} \), and adding new edges \( y_1 y_2, y_2 y_3, y_3 y_4, y_4 y_5, y_1 y_5 \).

We say that a graph \( H \) is \textit{strangely connected} if \( H \) is cubic and cyclically five-connected, and there is no \( X \subseteq V(H) \) such that \( H \) is placid on \( X \). (This is the definition of “dodecahedrally-connected” in [3].)

9.7 \textit{A graph \( H \) is dodecahedrally-connected if and only if it is strangely connected.}

\textbf{Proof.} We may assume that \( H \) is cubic and cyclically five-connected. Suppose first that it is not dodecahedrally-connected. Let \( X \subseteq V(H) \) with \( |X|, |V(H) \setminus X| \geq 7 \) and \( |\delta_H(X)| = 5 \), \( \delta_H(X) = \{x_1 y_1, \ldots, x_5 y_5\} \) say where \( x_1, \ldots, x_5 \in V(H) \), such that \( H|X \) can be drawn in a disc with \( x_1, \ldots, x_5 \) on the boundary in order. Let us choose such \( X \) with \( |X| \) minimum. Since \( H \) is cyclically five-connected it follows that \( x_1, \ldots, x_5 \) are all distinct and so are \( y_1, \ldots, y_5 \). Also, from the planarity of \( H|X \) it follows that \( |X| \geq 9 \), and so from the minimality of \( X \), no two of \( x_1, \ldots, x_5 \) are adjacent. Let \( H' \) be obtained from \( H \) as in the definition of “placid”, and let \( F' \) be the circuit made by the five new edges. It follows easily that \( H' \) is cyclically five-connected, and hence from 1.6 contains \( G = \text{Dodecahedron} \). Take a planar drawing of \( H' \), and choose a homeomorphic embedding \( \eta \) of \( G \) in \( H' \) such that the region of \( \eta(G) \) including \( r \) is minimal, where \( r \) is the region of \( H' \) bounded by \( F' \). It follows easily that \( F' \subseteq \eta(G) \), and so from the symmetry of \( G \) we may choose \( \eta \) mapping \( F \) to \( F' \). Hence \( H \) is placid on \( X \) (the final condition in the definition of “placid” holds because of the planarity of \( H' \)) and so \( H \) is not strangely connected, as required.

For the converse, suppose that \( H \) is not strangely connected, and let \( X, x_i y_i \) (1 \( \leq i \leq 5 \)), \( F \) and \( H' \) be as in the definition of “strangely connected”, such that \( H \) is placid on \( X \) via \( x_1 y_1, \ldots, x_5 y_5 \). Choose \( X \) minimal. By 9.6, \( H|X \) can be drawn in a disc with \( x_1, \ldots, x_5 \) on the boundary in order; and so \( H \) is not dodecahedrally-connected. This proves 9.7. \( \Box \)

10 Adding jumps to repair connectivity

Now that we have reconciled the two definitions of “dodecahedrally-connected”, we can apply results of [3] about this kind of connectivity.

The idea behind 9.2 is that cyclic five-connectivity is better than cyclic four-connectivity, and we begin with a graph \( G \) that is cyclically five-connected, except for the circuit \( C \). We use the cyclic five-connectivity of \( H \) to prove that if \( H \) contains \( G \) then \( H \) also contains a slightly larger graph where the circuit \( C \) has been expanded to a circuit of length five by adding an edge to \( G \). This can be useful, as we saw in the previous section. However, it has the defect that the edge we add to \( G \) to expand the circuit \( C \) might create a new circuit of length four, with its own problems. We can apply 9.2 again to this new circuit, but the process can go on forever. In fact, there is a stronger theorem; one can expand the circuit \( C \) to a longer circuit, without adding any new circuits of length four, just
by adding a bounded number of edges. That is essentially the content of the next result, proved in [3]. (We also weaken the hypothesis on $G$, allowing it to have more than one circuit of length four.) But first we need some definitions.

Let $\mathcal{L}$ be a set of cubic graphs. We say that a graph $H$ is killed by $\mathcal{L}$ if there is a homeomorphic embedding of some $G' \in \mathcal{L}$ in $H$. Let $G$ be cubic, and let $C$ be a circuit of $G$ of length four, with vertices $a_1, a_2, a_3, a_4$ in order. Let $a_i$ be adjacent to $b_i \notin V(C)$ for $1 \leq i \leq 4$, where $b_1, \ldots, b_4$ are all distinct and pairwise non-adjacent. We denote by $\mathcal{P}(C, \mathcal{L})$ the set of all pairs $(e, f)$ such that $f \in E(G)$ is incident with one of $b_1, \ldots, b_4$, say $b_i$, $f \neq a_ib_i, e \in E(C)$ is incident with $a_i$, and $G + (e, f)$ is not killed by $\mathcal{L}$.

Let $e = uv$ and $f = wx$ be edges of a cubic graph $G$. If $u, v \neq w, x$, and $u$ is adjacent to $w$, and no other edge has one end in $\{u, v\}$ and the other in $\{w, x\}$, we denote by $(e, f)^* \in \mathcal{L}$ the pair of edges $(e', f')$, where $e' (\neq e, uw)$ is incident with $u$ and $f' (\neq f, uw)$ is incident with $w$.

We shall frequently have to list the members of some set $\mathcal{P}(C, \mathcal{L})$ explicitly, and we can save some writing as follows. Clearly $(e, f) \in \mathcal{P}(C, \mathcal{L})$ if and only if $(e, f)^* \in \mathcal{P}(C, \mathcal{L})$, and so we really need only to list half the members of $\mathcal{P}(C, \mathcal{L})$. If $X$ is a set of pairs of edges for which $(e, f)^*$ is defined for each $(e, f) \in X$, we denote by $X^*$ the set $X \cup \{(e, f)^* : (e, f) \in X\}$.

If $e \in E(C)$ and $e, f$ are diverse in $G$, we call $G + (e, f)$ an $A$-extension of $G$. Now let $e \in E(C)$ and $f \notin E(G) \setminus E(C)$ such that $e, f$ are not diverse in $G$ but have no common end. Let $G' = G + (e, f)$ with new vertices $x_1, y_1$. Label the vertices of $C$ as $a_1, \ldots, a_4$ in order, and their neighbours not in $V(C)$ as $b_1, \ldots, b_4$ respectively, as before, such that $e = a_1a_2$ and $f$ is incident with $b_1$, $f = b_1c_1$ say. If $g \in E(G)$, not incident in $G$ with $a_1, b_1, c_1, d_1$ (where $b_1$ is adjacent in $G$ to $a_1, c_1, d_1$) we call $G' + (b_1y_1, g)$ a $B$-extension (of $G$) via $(e, f)$. If $g \in E(G)$ incident with $b_2$ and not with $c_1$ or $a_2$, we call $G' + (x_1y_1, g)$ a $C$-extension (of $G$) via $(e, f)$ onto $g$. We call $G' + (a_1x_1, a_3b_3)$ a $D$-extension via $(e, f)$. Finally, we say $(e, f)$ and $(e', f')$ are $C$-opposite if $e, e' \in E(C)$ and the labelling can be chosen as before with $e = a_1a_2$, $f = b_1c_1$, $e' = a_3a_4$, and $f' = b_3c_3$. Let $(e, f), (e', f')$ be $C$-opposite, with labels as above. Let $G'' = G' + (e', f')$ with new vertices $x_2, y_2$; then we call $G'' + (a_1x_1, a_3x_2)$ an $E$-extension via $(e, f), (e', f')$.

We say a graph $G$ is quad-connected if

- $G$ is cubic and cyclically four-connected
- $|V(G)| \geq 10$, and if $G$ has more than one circuit of length four then $|V(G)| \geq 12$
- for all $X \subseteq V(G)$ with $|\delta_G(X)| \leq 4$, one of $|X|, |V(G) \setminus X| \leq 4$

The following is a restatement of 9.2 in this language (with $F$ removed, because we no longer need it.)

**10.1** Let $G$ be cubic and cyclically four-connected, with $|V(G)| \geq 8$. Let $C$ be a circuit of $G$ of length 4, and let $\mathcal{L}$ be a set of cubic graphs. Suppose that every $A$-extension of $G$ is killed by $\mathcal{L}$, and $\mathcal{P}(C, \mathcal{L}) = \emptyset$. Let $H$ be a cyclically five-connected cubic graph that is not killed by $\mathcal{L}$. Then there is no homeomorphic embedding of $G$ in $H$.

Here is the strengthening, proved in [3].

**10.2** Let $G$ be quad-connected, and let $C$ be a circuit of $G$ of length four. Let $\mathcal{L}$ be a set of cubic graphs, such that
• every $A$-extension of $G$ is killed by $L$

• for every $(e, f) \in \mathcal{P}(C, L)$, every $B$-extension via $(e, f)$ is killed by $L$, and so is the $D$-extension via $(e, f)$

• for all $(e, f_1), (e, f_2) \in \mathcal{P}(C, L)$ such that $f_1, f_2$ have no common end, the $C$-extension via $(e, f_1)$ onto $f_2$ is killed by $L$, and

• for all $C$-opposite $(e_1, f_1), (e_2, f_2) \in \mathcal{P}(C, L)$, the $E$-extension via $(e_1, f_1), (e_2, f_2)$ is killed by $L$.

Let $H$ be a dodecahedrally-connected cubic graph such that $H$ is not killed by $L$. Then there is no homeomorphic embedding of $G$ in $H$.

The other result of [3] that we need is the following. Let $n \geq 5$ be an integer, with $n \geq 10$ if $n$ is even. The $n$-biladder is the graph with vertex set $\{a_1, \ldots, a_n, b_1, \ldots, b_n\}$, where for $1 \leq i \leq n, a_i$ is adjacent to $a_{i+1}$ and to $b_i$, and $b_i$ is adjacent to $b_{i+2}$ (where $a_{n+1}, b_{n+1}, b_{n+2}$ mean $a_1, b_1, b_2$). Thus, Petersen is isomorphic to the 5-biladder, and Dodecahedron to the 10-biladder. The following follows from theorem 1.4 of [3].

10.3 Let $G$ be cubic and cyclically five-connected. Let there be a homeomorphic embedding of $G$ in $H$, where $H$ is dodecahedrally-connected. Then either

• there exist $e, f \in E(G)$, diverse in $G$, such that there is a homeomorphic embedding of $G+(e, f)$ in $H$, or

• $G$ is isomorphic to an $n$-biladder for some $n$, and there is a homeomorphic embedding of the $(n+2)$-biladder in $H$, or

• $G$ is isomorphic to $H$.

11 Graphs with crossing number at least two

At the end of the proof of 9.1, there were five statements left to the reader to verify, that five particular graphs contain either Ruby or Box. In the remainder of the paper there will be many more similar statements left to the reader; unfortunately, we see no way of avoiding this, since there are simply too many of them to include full details of each. But perhaps 95% of them are of the form that “Graph $G$ contains Petersen”, where $G$ is cubic and cyclically five-connected; and here is a quick method for checking such a statement. Choose a circuit $C$ of $G$ with $|E(C)| = 5$, arbitrarily (there always is one, in this paper). Let $C$ have vertices $v_1, \ldots, v_5$ in order. Let $u_1, \ldots, u_5$ be vertices of a 5-circuit of Petersen, in order. Check if there is a homeomorphic embedding $\eta$ of Petersen in $G$ with $\eta(u_i) = v_i$ ($1 \leq i \leq 5$). (This is easy to do by hand.) It is proved in [6] that such a homeomorphic embedding exists if and only if $G$ contains Petersen.

This makes checking for containment of Petersen much easier. But even so, there are too many cases to reasonably do them all by hand, and we found it very helpful to write a simple computer programme to check containment for us. We suggest that the reader who wants to check these cases should do the same thing. There is a computer file available online with all the details of the case-checking [5].

In this section, we prove 1.7, which we restate as:
11.1 Let $H$ be dodecahedrally-connected. Then $H$ has crossing number $\geq 2$ if and only if it contains one of Petersen, Triplex or Box.

Dodecahedral connectivity cannot be replaced by cyclic 5-connectivity, because the graph of Figure 6 is a counterexample. The graphs Window, Antibox, and Drape are defined in Figure 7. We prove 11.1 in three steps, as follows.

11.2 Let $H$ be a dodecahedrally-connected graph containing Antibox; then $H$ contains Petersen, Triplex or Box.

11.3 Let $H$ be a cyclically five-connected cubic graph containing Drape; then $H$ contains Petersen, Triplex, Box or Antibox.

11.4 Let $H$ be a cyclically five-connected cubic graph containing Window, but not Petersen, Triplex, Box or Drape. Then $H$ has crossing number $\leq 1$.

Proof of 11.1, assuming 11.2, 11.3, 11.4. “If” is clear and we omit it. For “only if”, let $H$ be dodecahedrally-connected, and contain none of Petersen, Triplex or Box. By 11.2 it does not contain Antibox, and by 11.3 it does not contain Drape. We may assume from 9.1 that it contains Ruby (in fact it must, for no dodecahedrally-connected graph is planar), and hence Window, since Ruby contains Window. From 11.4, this proves 11.1.

Proof of 11.2. We shall apply 10.2, with $G =$ Antibox, $C$ the quadrangle of $G$, and $L =$ {Petersen, Triplex, Box}. Thus, $V(C) = \{1, 2, 3, 4\}$. We find that every $A$-expansion is killed by $L$. In detail, let $G'$ be $G + (ab, cd)$, where $(a, b, c, d)$ is as follows; in each case $G'$ contains the specified member of $L$.

Petersen: $(1, 2, 7, 10), (1, 2, 8, 11), (1, 2, 8, 12), (1, 2, 9, 11), (1, 2, 11, 14), (1, 2, 13, 14), (1, 4, 6, 10), (1, 4, 6, 13), (1, 4, 7, 10), (1, 4, 7, 14), (1, 4, 9, 13), (1, 4, 11, 14), (1, 4, 13,
Figure 7: Window, Drape and Antibox.

14).

Triplex: (1, 2, 5, 12), (1, 2, 6, 10), (1, 2, 10, 12), (1, 4, 5, 9), (1, 4, 8, 11), (1, 4, 9, 11).

Box: (1, 2, 9, 13), (1, 4, 10, 12).

In future we shall omit this kind of detail (because in the future it will get worse). The full details are in [5].

We find that \( P(C, L) = \{(1\text{-}2, 5\text{-}9), (1\text{-}2, 6\text{-}13), (3\text{-}4, 8\text{-}11), (3\text{-}4, 7\text{-}14)\}^* \). Then we verify the hypotheses (ii)-(iv) of 10.2. This proves 11.2.

Proof of 11.3. We apply 10.1, with \( G = \text{Drape}, C \) the quadrangle of \( G \) with vertex set \( \{5, 12, 13, 14\} \), and \( L = \{\text{Petersen, Triplex, Box, Antibox}\} \). We find that every \( A \)-extension of \( G \) is killed by \( L \), and \( P(C, L) = \emptyset \), so from 10.1, this proves 11.3.

Proof of 11.4. Let \( G \) be Window, let \( F \) and \( \eta_F \) be null, and let \( C \) be the subgraphs of \( G \) induced
on the following nine sets:

1, 2, 3, 4;
1, 2, 5, 6, 9;
2, 3, 6, 7, 10;
3, 4, 7, 8, 11;
1, 4, 5, 8, 12;
5, 9, 10, 11, 12;
6, 9, 10, 11, 12;
7, 9, 10, 11, 12;
8, 9, 10, 11, 12.

Then \((G, F, C)\) is a framework. We claim that (E1)–(E7) hold. The only twinned edges are 9-11 and 10-12, and again the only axiom that needs work is (E2). But if \(e, f \in E(G)\) are not both in some member of \(C\), then \(G + (e, f)\) contains one of Petersen, Triplex, Box, Antibox, Drape, and so (E2) holds. From 7.1, this proves 11.4.

12 Non-projective-planar graphs

Now we digress, to prove a result that we shall not need; but it is pretty, and follows easily from the machinery we have already set up. The graph Twinplex is defined in Figure 8. We shall show the following.

12.1 Let \(H\) be dodecahedrally-connected. Then \(H\) cannot be drawn in the projective plane if and only if \(H\) contains one of Triplex, Twinplex, Box.

Proof. “If” is easy and we omit it. For “only if”, suppose that \(H\) contains none of Triplex, Twinplex, Box; we shall show that it can be drawn in the projective plane. If \(H\) has crossing number \(\leq 1\) this is true, so by 11.1 we may assume that \(H\) contains Petersen.
Let $G_0 = \text{Petersen}$. We may assume that $H$ is not isomorphic to $G_0$, so by 10.3 either there are edges $ab, cd$ of $G_0$ diverse in $G_0$ and a homeomorphic embedding of $G_0 + (ab, cd)$ in $H$, or $H$ contains the 7-biladder. The former is impossible, because from the symmetry of $G_0$ we may assume that $(a, b, c, d) = (4, 5, 6, 8)$, and then $G_0 + (ab, cd)$ is isomorphic to Twinplex, a contradiction. Hence there is a homeomorphic embedding of $G$ in $H$, where $G$ is the 7-biladder. Let $V(G) = \{a_1, \ldots, a_7, b_1, \ldots, b_7\}$, as in the definition of “biladder”. Let $C$ be the subgraphs of $G$ induced on the following vertex sets:

\[
\begin{align*}
&b_1, b_2, \ldots, b_7; \\
&a_1, a_2, a_3, b_3, b_1; \\
&a_2, a_3, a_4, b_4, b_2; \\
&a_3, a_4, a_5, b_5, b_3; \\
&a_4, a_5, a_6, b_6, b_4; \\
&a_5, a_6, a_7, b_7, b_5; \\
&a_6, a_7, a_1, b_1, b_6; \\
&a_7, a_1, a_2, b_2, b_7. \\
\end{align*}
\]

(These are the face-boundaries of an embedding of $G$ in the projective plane.) Let $F$ and $\eta_F$ be null; then $(G, F, C)$ is a framework, and we claim that (E1)–(E7) hold. All except (E2), (E3) and (E6) are obvious. To check (E2), let $G' = G + (ab, cd)$ where $ab, cd \in E(G)$ are not both in any member of $C$. There are twelve possibilities for $(a, b, c, d)$ up to isomorphism of $G$; in one case $G'$ contain Box, in three others it contains Twinplex, and in the other eight it contains Triplex. (As usual, we omit the details; they are also not in the appendix [5], because we don’t really need the result.) Thus, (E2) holds. For (E3), the only diverse trinity (up to isomorphism of $G$) is $\{a_1a_2, b_1b_3, b_2b_7\}$, and $G + (a_1a_2, b_1b_3, b_2b_7)$ contains Twinplex. Hence (E3) holds. For (E6), we need only check cross extensions over the circuit with vertex set $\{b_1, \ldots, b_7\}$, since all other members of $C$ have only five edges. There are four possibilities (up to isomorphism of $G$). Let $G' = G + (b_1b_3, b_2b_4)$ with new vertices $x, y$; then the possibilities are $G' + (ab, cd)$ where $(a, b, c, d)$ is $(b_1, x, b_2, y)$, $(b_1, x, b_2, b_7)$, $(b_1, b_6, b_2, b_7)$, $(b_1, b_6, b_5, b_7)$. The first contains Box, and the other three contain Triplex. Hence (E6) holds, and from 7.1, this proves 12.1.

13 Arched graphs

We say a graph $H$ is \textit{arched} if $H \setminus e$ is planar for some edge $e$. In this section we prove 1.8, which we restate as:

13.1 \textbf{Let $H$ be dodecahedrally-connected. Then $H$ is arched if and only if it does not contain Petersen or Triplex.}

We start with the following lemma.

13.2 \textbf{Let $G$ be Box, let $G'$ be obtained by deleting the edge 13-14, and let $C$ be the set of circuits of $G'$ that bound regions in the drawing in Figure 3. Let $e, f \in E(G)$, with no common end, and not both in any member of $C$. Then either $G + (e, f)$ has a Petersen or Triplex minor, or (up to exchanging $e$ and $f$, and automorphisms of $G$) $e$ is 13-14 and $f$ is 1-2 or 1-4.}
We leave the proof to the reader (the details are in the Appendix [5]).

13.3 Let $G$ be Box, and let $H$ be cyclically five-connected, and not contain Petersen or Triplex. Let $\eta$ be a homeomorphic embedding of $G$ in $H$ such that $\eta(13-14)$ has only one edge, $g$ say. Then $H \setminus g$ is planar, and so $H$ is arched.

**Proof.** We apply 7.1, taking $F$ to be the subgraph of $G$ consisting of 13-14 and its ends, and $\eta_F$ the restriction of $\eta$ to $F$. Let $C$ be as in 13.2. Then $(G,F,C)$ is a framework, and we claim that (E1)–(E7) hold. (E2) follows from 13.2, and (E5) and (E6) are vacuously true, because all members of $C$ have five edges. Also, (E3) and (E7) are vacuously true. For (E4), it suffices from symmetry to check

\[
\begin{align*}
G + (1-2,13-14) + (3-6,13-16) \\
G + (1-2,13-14) + (3-6,14-16) \\
G + (1-2,13-14) + (5-6,13-16) \\
G + (1-4,13-14) + (3-6,13-16),
\end{align*}
\]

but all four contain Triplex. Hence (E4) holds, so from 7.1, this proves 13.3.

The graph Superbox is defined in Figure 9. (It is isomorphic to Box + (1-4,13-14).)

![Figure 9: Superbox.](image)

13.4 Let $G$ be Superbox, let $G'$ be obtained by deleting the edge 15-16, and let $C$ be the set of circuits of $G'$ that bound regions in the drawing in Figure 9. Let $e,f \in E(G)$ with no common end, and not both in any member of $C$. Then either $G + (e,f)$ has a Petersen or Triplex minor, or (up to exchanging $e,f$ and automorphisms of $G$) $e$ is 15-16 and $f$ is 1-2 or 1-11.

We leave the proof to the reader. (Actually, it follows quite easily from 13.2.)

13.5 Let $G$ be Superbox, and let $H$ be cyclically five-connected, and not contain Petersen or Triplex. Let $\eta$ be a homeomorphic embedding of $G$ in $H$ such that $\eta(15-16)$ has only one edge, $g$ say. Then $H \setminus g$ is planar, and so $H$ is arched.
Proof. We apply 7.1 to \((G, F, C)\), where \(F\) consists of 15-16 and its ends, and \(\eta_F\) is the restriction of \(\eta\) to \(F\), and \(C\) is as in 13.4. Because of 13.4, it remains to verify (E4), (E5) and (E6), because (E3), (E7) are vacuous. Checking (E4) is exactly like in 13.3 (indeed, by deleting 14-16 from \(G\) we obtain Box, so actually we could deduce that (E4) holds now from the fact that it held in the proof of 13.3). For (E5), we must check

\[
G + (1-11, 15-16) + (6-11, 15-18) + (ab, cd)
\]

where \((ab, cd)\) is either \((11-17, 10-14)\) or \((11-19, 5-14)\); and both contain Triplex. Thus (E5) holds.

For (E6), we need only check cross extensions over the circuit bounding the infinite region, since all other members of \(C\) have length five; and from symmetry, it suffices to check

\[
G + (1-11, 10-14) + (1-17, 10-18) \\
G + (1-11, 10-14) + (6-11, 5-14) \\
G + (1-11, 10-14) + (1-5, 6-10) \\
G + (1-5, 6-10) + (1-17, 10-18).
\]

All four contain Petersen. Hence (E6) holds, and from 7.1, this proves 13.5.

Proof of 13.1. “Only if” is easy and we omit it. For “if”, let \(H\) be dodecahedrally-connected, and not contain Petersen or Triplex. Since graphs of crossing number \(\leq 1\) are arched, we may assume from 11.1 that \(G\) contains Box. Choose a homeomorphic embedding of \(G\) in \(H\), where \(G\) is either Box or Superbox, such that \(|E(S)|\) is minimum, where \(S = \eta(15-16)\) if \(G\) is Box, and \(S = \eta(17-18)\) if \(G\) is Superbox. We claim that \(|E(S)| = 1\). For suppose not. Since \(H\) is three-connected, there is an \(\eta\)-path \(P\) with one end in \(V(S)\) and the other, \(t\), in \(V(\eta(G)) \setminus V(S)\). Let \(t \in \eta(f)\) say, and let \(e = 15-16\) if \(G\) is Box, and \(e = 17-18\) if \(G\) is Superbox. If \(e, f\) have a common end in \(G\), then by rerouting \(f\) along \(P\) we contradict the minimality of \(|E(S)|\). If some edge \(g\) of \(G\) joins an end of \(e\) to an end of \(f\), then by rerouting \(g\) along \(P\) we contradict the minimality of \(|E(S)|\). Hence \(e, f\) are diverse in \(G\). By the symmetry we may therefore assume, by 13.2 and 13.4, that either \(G\) is Box and \(f = 1-4\), or \(G\) is Superbox and \(f = 1-2\). In the first case, by adding \(P\) to \(\eta(G)\) we obtain a homeomorphic embedding of Superbox contradicting the minimality of \(|E(S)|\). In the second case, by adding \(P\) to \(\eta(G \setminus \{3-8,6-7\})\) we obtain a homeomorphic embedding of Box contradicting the minimality of \(|E(S)|\).

This proves our claim that \(|E(S)| = 1\). From 13.3 and 13.5, \(H\) is arched. This proves 13.1.

14 The children of Drum

The graph Drum is defined in Figure 10.

14.1 Let \(H\) be dodecahedrally-connected, and not isomorphic to Triplex. Then \(H\) is arched if and only it contains none of Petersen, Drum.

Proof. Since Drum contains Triplex (delete 9-10) “only if” follows from 13.1. For “if”, let \(H\) be dodecahedrally-connected, not isomorphic to Triplex, and not arched, and suppose that \(H\) does
not contain Petersen. We must show that $H$ contains Drum. By 13.1, $H$ contains Triplex; and so by 10.3, since Triplex is not a biladder, it follows that $H$ contains Triplex + $(e, f)$, where $e, f$ are diverse edges of Triplex. But for all such choices of $e, f$, Triplex + $(e, f)$ either contains Petersen or is isomorphic to Drum. This proves 14.1.

In Figure 11 we define the graphs *Firstapex*, *Secondapex*, *Thirdapex*, *Fourthapex*, and *Sailboat*. They all contain Drum. We call the first four of them *Apex-selectors*.

14.2 Let $H$ be dodecahedrally-connected, and not isomorphic to Triplex or Drum. Then $H$ is arched if and only if it contains none of Petersen, an Apex-selector, or Sailboat. 

**Proof.** As in 14.1, “only if” is easy, and for “if” we may assume that $H$ contains Drum, by 14.1. By 10.3 $H$ contains Triplex + $(e, f)$ where $e, f$ are diverse edges of Drum. There are (up to isomorphism of Drum) 26 possibilities for {$e, f$}; let $e = ab, f = cd$, and $G' = \text{Drum} + (ab, cd)$. If $(a, b, c, d)$ is one of 

(1, 2, 11, 13), (1, 3, 8, 13), (3, 7, 5, 9), (3, 11, 9, 14), (7, 14, 11, 13),

$G$ is isomorphic to Firstapex, Secondapex, Thirdapex, Fourthapex and Sailboat respectively, and in all other cases $G$ contains Petersen. This proves 14.2.

Let us say $H$ is *doubly-apex* if it has two vertices $u, v$ such that the graph obtained from $H$ by identifying $u$ and $v$ is planar. Sailboat is doubly-apex (identify 15 and 16) but the Apex-selectors are not, and Petersen is not. The main result of this section is the following.

14.3 Let $H$ be dodecahedrally-connected. Then $H$ is either arched or doubly-apex if and only if it does not contain Petersen or an Apex-selector.

14.3 follows from the following.
14.4 Let $H$ be dodecahedrally-connected, and contain Sailboat but not Petersen or any Apex-selector. Then $H$ is doubly-apex.

Proof of 14.3 assuming 14.4.

"If" is easy, and we omit it. For "only if", let $H$ not contain Petersen or an Apex-selector. If $H$ is isomorphic to Triplex or Drum it is doubly-apex as required. Otherwise, by 14.2 either it is arched or it contains Sailboat; and in the latter case by 14.4 it is doubly-apex. This proves 14.3.

It remains to prove 14.4. That will require several lemmas. Let $C$ be the set of the subgraphs of Sailboat induced on the following vertex sets (which bound the regions when Sailboat is drawn in...
the plane with 15 and 16 identified):

1, 2, 3, 4, 5;
1, 2, 7, 11, 6;
2, 3, 8, 12, 7;
3, 4, 9, 13, 8;
4, 5, 10, 14, 9;
15, 6, 1, 5, 10, 16;
15, 6, 11, 16;
16, 11, 7, 12, 15;
15, 12, 8, 13, 16;
16, 13, 9, 14, 15;
15, 14, 10, 16.

Let Boat(1), . . . , Boat(7) be Sailboat + (ab, cd) where respectively (a, b, c, d) is

(2, 7, 12, 15), (7, 12, 6, 15), (1, 6, 11, 16), (2, 7, 11, 16), (6, 11, 12, 15), (9, 14, 12, 15), (6, 15, 12, 15).

14.5 Let G be Sailboat, and let ab and cd be edges of G such that no member of C contains them both. Then G + (ab, cd) contains Petersen or an Apex-selector or one of Boat(1), . . . , Boat(7).

Proof. If a = c then since no member of C contains ab and cd it follows that a = 15 or 16, and then G + (ab, cd) is isomorphic to Boat(7). We assume therefore that a, b ̸= c, d.

Up to the symmetry of Sailboat and exchanging ab with cd, there are 88 cases to be checked. Let G’ = G + (ab, cd). If (a, b, c, d) is (1, 6, 11, 16) or (6, 15, 7, 11), G’ is (isomorphic to) Boat(3). If (a, b, c, d) is (7, 12, 6, 11) or (2, 7, 11, 16), G’ is Boat(4). If (a, b, c, d) is (2, 7, 12, 15) or (7, 11, 8, 12), G’ is Boat(1). If (a, b, c, d) is (1, 6, 14, 15) or (6, 11, 12, 15), G’ is Boat(5). If (a, b, c, d) is (7, 12, 6, 15) or (8, 12, 14, 15), G’ is Boat(2). If (a, b, c, d) is (9, 14, 12, 15) or (10, 14, 6, 15), G’ is Boat(6). If (a, b, c, d) = (2, 3, 12, 15), G’ contains Firstapex; if (a, b, c, d) = (1, 6, 10, 14), (3, 8, 12, 15) or (7, 11, 8, 13) it contains Secondapex; if (a, b, c, d) is one of

(1, 5, 6, 11), (1, 5, 14, 15), (1, 2, 6, 15), (1, 2, 11, 16), (2, 7, 15, 18), (8, 12, 11, 16)

G’ contains Thirdapex; and in the remaining 66 cases, G’ contains Petersen. This proves 14.5.

14.6 Let H be dodecahedrally-connected, and not contain Petersen or an Apex-selector. Then H contains none of Boat(1), . . . , Boat(7).

Proof.

(1) H does not contain Boat(1).

Subproof. Let L1 consist of Petersen and the four Apex-Selectors, and let C be the quadrangle of Boat(1). Then every A-extension of Boat(1) is killed by L1, and P(C, L1) = ∅, so the claim
follows from 10.1. This proves (1).

(2) *H does not contain Boat(2).*

Subproof. Let $C$ be the quadrangle of Boat(2). Then every $A$-extension of Boat(2) is killed by $\mathcal{L}_1$, and

$$\mathcal{P}(C, \mathcal{L}_1) = \{(17-18, 6-11), (17-18, 7-11)\}^*.$$  

The result follows from 10.2. This proves (2).

(3) *H does not contain Boat(3) or Boat(4).*

Subproof. Let $G$ be Boat(3) or Boat(4), and $\mathcal{L}_3 = \mathcal{L}_1 \cup \{\text{Boat}(2)\}$. Let $C$ be the quadrangle of $G$. Then every $A$-extension of $G$ is killed by $\mathcal{L}_3$, and $\mathcal{P}(C, \mathcal{L}_3) = \emptyset$, so the result follows from (2) and 10.1. This proves (3).

(4) *H does not contain Boat(5) or Boat(6).*

Subproof. Let $G$ be Boat(5) or Boat(6), and let $\mathcal{L}_4 = \mathcal{L}_3 \cup \{\text{Boat}(3), \text{Boat}(4)\}$. Let $C$ be the quadrangle of $G$. Then every $A$-extension of $G$ is killed by $\mathcal{L}_4$, and $\mathcal{P}(C, \mathcal{L}_4) = \emptyset$, so the result follows from (2), (3) and 10.1. This proves (4).

(5) *H does not contain Boat(7).*

Subproof. Let $G$ be Boat(7), and let $C$ be its circuit of length 3. Let $X = V(C)$. Suppose that there is a homeomorphic embedding of $G$ in $H$; then by 8.1, there is a $X$-augmenting sequence $(e_1, f_1), \ldots, (e_n, f_n)$ of $G$ such that $H$ contains $G + (e_1, f_1) + \ldots + (e_n, f_n)$. From the definition of “$X$-augmentation” it follows that $n = 1$ since $|E(C)| = 3$; and so $H$ contains $G(e_1, f_1)$ for some $e_1 \in E(C)$ and $f_1 \in E(G \setminus X)$. But for all such $e_1, f_1, G + (e_1, f_1)$ contains a member of $\mathcal{L}_1$ or one of Boat(2), Boat(5), Boat(6), a contradiction by (2) and (4). This proves (5).

From (1)–(5), this proves 14.6.

Proof of 14.4.

Let $H$ be dodecahedrally-connected and not contain Petersen or an Apex-selector. Let $\eta$ be a homeomorphic embedding of $G$ in $H$, where $G$ is Sailboat. Let $V(F) = \{15, 16\}$ and $E(F) = \emptyset$; and let $\eta_F$ be the restriction of $\eta$ to $F$. Let $C$ be as before. Then $(G, F, C)$ is a framework, and we claim that (E1)–(E7) hold. By 14.6 $H$ contains none of Boat(1),..., Boat(7), so by 14.5 (E2) holds. All the others are clear except for (E6), and for (E6) we need only consider cross-extensions of $G$ on some of the paths in $C$, namely the ones with vertex sets

$$\{15, 6, 1, 5, 10, 16\}, \{16, 11, 7, 12, 15\}, \{15, 12, 8, 13, 16\}$$

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(and two more, that from symmetry we need not consider). We need to examine

\[ G + (6-15, 10-16) + (1-6, 16-18) \]
\[ G + (6-15, 10-16) + (6-17, 16-18) \]
\[ G + (6-15, 5-10) + (6-17, 10-18) \]
\[ G + (6-15, 5-10) + (1-6, 10-16) \]
\[ G + (11-16, 12-15) + (11-17, 15-18) \]
\[ G + (12-15, 13-16) + (12-17, 16-18); \]

they contain Thirdapex, Boat(3), Boat(3), Petersen, Boat(3) and Boat(3) respectively. Hence (E6) holds, and from 7.1, this proves 14.4.

15 Dodecahedrally connected non-apex graphs

The graphs Diamond, Bigdrum and Concertina are defined in Figure 12.

In this section we prove the following.

15.1 Let \( H \) be dodecahedrally-connected. Then \( H \) is apex if and only if it contains none of Petersen, Jaws, Starfish, Diamond, Concertina, Bigdrum.
Let $\text{Square}(1)$ be Secondapex $+$ $(14-16, 11-13)$. Let $\text{Square}(2)$, ..., $\text{Square}(5)$ be Fourthapex $+$ $(ab, cd)$ where $(a, b, c, d)$ is 

$$(1, 5, 10, 12), (1, 11, 6, 10), (6, 14, 13, 16), (12, 13, 15, 16)$$

respectively. Let $\text{Square}(6)$ and $\text{Square}(7)$ be Thirdapex $+$ $(ab, cd)$ where $(a, b, c, d)$ is $(3, 15, 14, 16)$ and $(2, 3, 8, 9)$ respectively.

**15.2** Let $H$ be dodecahedrally-connected, and not contain any of Petersen, Jaws, Starfish, Diamond, Concertina, Bigdrum. Then it contains none of $\text{Square}(1)$, ..., $\text{Square}(7)$.

**Proof.**

(1) $H$ does not contain $\text{Square}(1)$.

**Subproof.** Let $G$ be $\text{Square}(1)$, let $C$ be the quadrangle of $G$, and let

$$\mathcal{L}_1 = \{\text{Petersen}, \text{Jaws}, \text{Starfish}, \text{Diamond}, \text{Concertina}, \text{Bigdrum}\}.$$ 

Every $A$-extension of $G$ is killed by $\mathcal{L}_1$ (indeed, by $\{\text{Petersen}, \text{Jaws}, \text{Starfish}\}$), and

$$\mathcal{P}(C, \mathcal{L}_1) = \{(13-18, 5-12), (13-18, 10-12), (13-18, 1-11), (13-18, 6-11)\}^*.$$ 

(Note that $G + (13-18, 1-5)$ is isomorphic to Jaws, and $G + (16-17, 3-8)$ to Starfish.) Then we verify the hypotheses of 10.2; and find that all the various extensions listed in 10.2 contain Petersen, except for the $B$-extensions

$$G + (13-18, 12-5) + (12-20, 4-15)$$

$$G + (13-18, 12-5) + (12-20, 11-18)$$

$$G + (13-18, 12-5) + (12-20, 15-16).$$

(which contain Jaws, Diamond, and Concertina respectively) and the $C$-extension

$$G + (13-18, 12-5) + (19-20, 1-11)$$

(which contains Jaws), and isomorphic extensions. Hence, from 10.2, this proves (1).

Now let

$$\mathcal{L}_2 = \{\text{Petersen}, \text{Square}(1), \text{Diamond}, \text{Concertina}, \text{Bigdrum}\}.$$ 

(Jaws and Starfish are no longer necessary, since they both contain $\text{Square}(1)$.)

(2) $H$ does not contain $\text{Square}(2)$.

**Subproof.** We apply 10.1 to the quadrangle $C$ of $\text{Square}(2)$, with $\mathcal{L} = \mathcal{L}_2$. All $A$-extensions are killed by $\mathcal{L}_2$, and $\mathcal{P}(C, \mathcal{L}_2) = \emptyset$, so the result follows from 10.1. This proves (2).
(3) $H$ does not contain $\text{Square}(3)$.

Subproof. Let $C$ be the quadrangle of $G = \text{Square}(3)$; we apply 10.2, with $\mathcal{L} = \mathcal{L}_2$. All $A$-extensions are killed by $\mathcal{L}_2$, and
\[
\mathcal{P}(C, \mathcal{L}_2) = \{(6-11, 13-16), (6-11, 14-16)\}^*.
\]
We verify the hypotheses of 10.2. This proves (3).

(4) $H$ does not contain $\text{Square}(4)$.

Subproof. Now let $\mathcal{L}_4 = \mathcal{L}_2 \cup \{\text{Square}(2), \text{Square}(3)\}$. The result follows from 10.1, applied to the quadrangle of $\text{Square}(4)$ and $\mathcal{L}_4$, using (2) and (3). This proves (4).

(5) $H$ does not contain $\text{Square}(5)$.

Subproof. Let $\mathcal{L}_5 = \mathcal{L}_4 \cup \{\text{Square}(4)\}$, and $C$ the quadrangle of $G = \text{Square}(5)$. Then all $A$-extensions are killed by $\mathcal{L}_5$, and
\[
\mathcal{P}(C, \mathcal{L}_5) = \{(13-17, 6-11)\}^*;
\]
and we verify the hypotheses of 10.2 to prove (5).

(6) $H$ does not contain $\text{Square}(6)$.

Subproof. Let $\mathcal{L}_6 = \mathcal{L}_5 \cup \{\text{Square}(5)\}$, and $C, G$ as usual. All $A$-extensions are killed by $\mathcal{L}_6$, and
\[
\mathcal{P}(C, \mathcal{L}_6) = \{(17-18, 3-8), (17-18, 8-14)\}^*;
\]
and again the result follows from 10.2. This proves (6).

(7) $H$ does not contain $\text{Square}(7)$.

Subproof. Let $\mathcal{L}_7 = \mathcal{L}_6 \cup \{\text{Square}(6)\}$, and $C, G$ as usual. Then all $A$-extensions are killed by $\mathcal{L}_7$, and $\mathcal{P}(C, \mathcal{L}_7) = \emptyset$, so (7) follows from 10.1.

From (1)–(7), this proves 15.2.

The graph $\text{Extrapex}$ is defined in Figure 13. We say that $G$ is an $\text{Apex-forcer}$ if either it is an Apex-selector or it is $\text{Extrapex}$. By the $\text{Non-apex family}$ we mean
\[
\{\text{Petersen, Diamond, Concertina, Bigdrum, Square}(1), ..., \text{Square}(7)\}.
\]

15.3 Let $G$ be an $\text{Apex-forcer}$. Let $\mathcal{C}$ be the set of circuits that bound regions in the planar drawing of $G \setminus 16$. If $ab$ and $cd$ are edges of $G$ with $a, b \neq c, d$, and no member of $\mathcal{C}$ contains them both, then either $G + (ab, cd)$ contains a member of the $\text{Non-apex family}$, or one of $a, b, c, d$ is 16 and the other three belong to some member of $\mathcal{C}$.

We leave the proof to the reader (the details are in the Appendix [5]). If $G$ is an $\text{Apex-forcer}$, and $\eta$ is a homeomorphic embedding of $G$ in $H$, we define the spine of $\eta$ to be $\eta(13-16) \cup \eta(14-16) \cup \eta(15-16)$. 47
Let $H$ be cubic and cyclically four-connected, and contain no member of the Non-apex family. Let $H$ contain some Apex-forcer. Then there is a homeomorphic embedding $\eta$ of some Apex-forcer in $H$ such that its spine has only three edges.

**Proof.** Choose an Apex-forcer $G$ and a homeomorphic embedding $\eta$ of $G$ in $H$, such that its spine is minimal. Suppose its spine has more than three edges; then since $H$ is cyclically four-connected, there is an $\eta$-path $P$ with one end in $\eta(e)$ and the other in $\eta(f)$, where $f$ is one of 13-16, 14-16, 15-16 and $e$ is not incident with 16. If $e$ and $f$ have a common end then by rerouting $e$ along $P$ we obtain a new homeomorphic embedding with smaller spine, a contradiction. Similarly, it follows that no edge of $G \setminus 16$ joins an end of $e$ to an end of $f$. Let $C$ be as in 15.3. By 15.3 there exists $C \in \mathcal{C}$ such that $e \in E(C)$ and $f$ has an end in $V(C)$. Let $e = ab$ and let $f$ be incident with $c, 16$. Now we must examine cases.

If $G$ is Firstapex, we may assume that $(a, b, c) = (2, 8, 13)$ from the symmetry. Then $\eta(G \setminus 6-12) \cup P$ yields a homeomorphic embedding of Secondapex with smaller spine, a contradiction. (We apologize for this awkward notation; by $G \setminus 6-12$ we mean the graph obtained from $G$ by deleting the edge 6-12. We use the same notation below.)

If $G$ is Secondapex, there are three possibilities for $(a, b, c)$: $(1, 5, 13)$ (when $\eta(G \setminus 6-11) \cup P$ yields a homeomorphic embedding of Firstapex), $(1, 11, 14)$ (when $\eta(G \setminus 1-5) \cup P$ yields a homeomorphic embedding of Fourthapex), and $(3, 8, 14)$ (when $\eta(G) \cup P$ yields a homeomorphic embedding of Extrapex), in each case contradicting the minimality of the spine. If $G$ is Thirdapex, the possibilities for $(a, b, c)$ are: $(1, 5, 13)$ or $(2, 3, 14)$ (when $\eta(G \setminus 8-9) \cup P$ yields a homeomorphic embedding of Fourthapex), $(6, 10, 14)$ (when $\eta(G \setminus 1-11) \cup P$ yields a homeomorphic embedding of Thirdapex), and $(9, 10, 14)$ (when $\eta(G \setminus 2-7) \cup P$ yields a homeomorphic embedding of Firstapex), in each case a contradiction.

If $G$ is Fourthapex, the possibilities are: $(1, 5, 13)$ (when $\eta(G \setminus 4-9) \cup P$ yields a homeomorphic embedding of Thirdapex), $(6, 10, 13)$ (when $\eta(G) \cup P$ yields a homeomorphic embedding of Extrapex), $(1, 2, 14)$ (when $\eta(G \setminus 4-9) \cup P$ yields a homeomorphic embedding of Secondapex), and $(1, 11, 14)$ (when $\eta(G \setminus 10-12) \cup P$ yields a homeomorphic embedding of Thirdapex), in each case a contradiction. (We have used a symmetry of Fourthapex not evident from the drawing, exchanging 13 with 15 and 1 with 9.)

If $G$ is Extrapex, the possibilities are: $(1, 2, 13)$ (when $\eta(G \setminus \{7-13, 1-6\}) \cup P$ yields a homeomorphic
embedding of Secondapex) and (2, 7, 14) (when \( \eta(G \setminus \{2-3,10-11\}) \cup P \) yields a homeomorphic embedding of Thirdapex), in each case a contradiction.

Hence the spine has only three edges. This proves 15.4.

**Proof of 15.1.**

“Only if” is easy, and we omit it. For “if”, let \( H \) be dodecahedrally-connected, and not contain any of Petersen, Jaws, Starfish, Diamond, Concertina, Bigdrum. By 15.2 it contains none of Square(1),..., Square(7). We may assume that \( H \) is not arched or doubly-apex, for such graphs are apex; and so by 14.3 \( H \) contains an Apex-selector. By 15.4, there is a homeomorphic embedding \( \eta \) of some Apex-forcer \( G \) in \( H \) such that its spine has only three edges. Let \( F \) be the subgraph of \( G \) induced on \( \{13,14,15,16\} \), and let \( \eta_F \) be the restriction of \( \eta \) to \( F \). Let \( C \) be as in 15.3; then \((G,F,C)\) is a framework, and \( H, \eta_F \) satisfy (E1). We claim they satisfy (E2)–(E7). (E2) follows from 15.3, and (E3), (E7) are vacuously true. For (E4), (E5) and (E6) a large amount of case-checking is required, for \( G = \) Firstapex, Secondapex, Thirdapex, Fourthapex and Extrapex, separately. (In the case-checking we use that \( H \) contains none of Petersen, Jaws, Starfish, Diamond, Concertina, Bigdrum, and we could also use that it contains none of Square(1)–Square(7). In fact we find that we don’t need to use all of the latter; we just need that \( H \) does not contain Square(2).) The details are in the Appendix [5]. From 7.1, this proves 15.1.

### 16 Die-connected non-apex graphs

![Figure 14: Antilog and Log.](image-url)
Our next real objective in this paper is to modify 15.1 to find all the cubic graphs \( G \) minimal with the properties that they are non-apex and dodecahedrally-connected, and \( |\delta(X)| \geq 6 \) for all \( X \subseteq V(G) \) with \( |X|, |V(G) \setminus X| \geq 7 \). (There are only three such graphs, namely Petersen, Jaws and Starfish, as we shall see in the next section.) Diamond, Concertina and Bigdrum all have subsets \( X \) with \( |\delta(X)| = 5 \) and \( |X|, |V(G) \setminus X| \geq 9 \), so they are rather far from having the property we require; and a convenient half-way stage is afforded by "die-connectivity". We recall that a graph \( G \) is die-connected if it is dodecahedrally-connected (and hence cubic and cyclically five-connected) and \( |\delta(X)| \geq 6 \) for all \( X \subseteq V(G) \) with \( |X|, |V(G) \setminus X| \geq 9 \). In this section we find all minimal graphs that are non-apex and die-connected. The graphs Log, Antilog, and Dice(1),..., Dice(4) are defined in Figures 14, 15 and 16. We shall show the following.

16.1 Let \( H \) be die-connected. Then \( H \) is apex if and only if \( H \) contains none of Petersen, Jaws, Starfish, Log, Antilog, Dice(1), Dice(2), Dice(3), Dice(4).

We begin with the following.

16.2 Any die-connected graph that contains Diamond also contains one of Petersen, Antilog, Dice(4).

Proof. Let \( H \) be die-connected, and contain no member of \( \mathcal{L} = \{\text{Petersen, Antilog, Dice(4)}\} \). We claim first that

(1) \( H \) does not contain Diamond +\((1,2,10,11)\).
Subproof. Let $C$ be the quadrangle of $G = \text{Diamond } + (1-2, 10-11)$. Then all $A$-extensions are killed by $\mathcal{L}$, and

$$\mathcal{P}(C, \mathcal{L}) = \{(2-19, 4-5), (11-20, 10-13)\}^*.$$

We verify the hypotheses of 10.2 (the $E$-extension is isomorphic to $\text{Dice}(4)$). This proves (1).

Now let $\mathcal{L}' = \{\text{Petersen, Antilog, Diamond } + (1-2, 10-11)\}$, and $X = \{1, \ldots, 9\}$.

(2) Every $X$-augmentation of Diamond contains a member of $\mathcal{L}'$.

Subproof. Let $(e_1, f_1), \ldots, (e_n, f_n)$ be an $X$-augmenting sequence, and suppose the corresponding $X$-augmentation contains no member of $\mathcal{L}'$. In particular, Diamond $+$ $(e_1, f_1)$ contains no member of $\mathcal{L}'$, and so (by checking all possibilities) it follows that $f_1$ is 6-10 and $e_1$ is one of 1-2, 1-7, 4-9. In particular, $n \geq 2$. Since $f_1 = 6-10$ it follows that $e_2 = 6-20$. If $e_1$ is 1-7 or 4-9 there is no possibility for $f_2$. Thus $e_1$ is 1-2, and then $f_2$ is 9-12, and $n \geq 3$, and $e_3$ is 9-22. Again by checking cases it follows that $f_3$ is 7-17, and hence $n \geq 4$ and $e_4$ is 7-24; and there is no possibility for $f_4$, a contradiction. This proves (2).

From (1), (2) and 8.1, the result follows since $H$ is die-connected. This proves 16.2.

16.3 Every die-connected graph that contains Bigdrum also contains one of Petersen, Diamond or $\text{Dice}(2)$.

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Proof. Let $H$ be die-connected, and contain no member of $\mathcal{L} = \{\text{Petersen, Diamond, Dice}(2)\}$. We claim first

(1) $H$ does not contain Bigdrum + (3-8, 10-11).

Subproof. Let $G = \text{Bigdrum} + (3-8, 10-11)$, and let $C$ be the quadrangle of $G$. Then all $A$-extensions are killed by $\mathcal{L}$, and

$$\mathcal{P}(C, \mathcal{L}) = \{(8-11, 9-13), (19-20, 10-14)\}^*.$$ The result follows from 10.2 by checking all the various extensions (in particular, $G + (8-19, 5-9) + (11-20, 10-14) + (8-21, 20-23)$ is isomorphic to Dice(2)). This proves (1).

Now let $\mathcal{L}' = \{\text{Petersen, Diamond, Bigdrum} + (3-8, 10-11)\}$ and $X = \{1, \ldots, 9\}$. We claim that

(2) Every $X$-augmentation of Bigdrum contains a member of $\mathcal{L}'$.

Subproof. Let $(e_1, f_1), \ldots, (e_n, f_n)$ be an $X$-augmenting sequence, such that the corresponding $X$-augmentation contains no member of $\mathcal{L}'$. Then by checking cases it follows that $(e_1, f_1)$ is one of $(3-8, 6-10), (4-7, 9-13)$, and by the symmetry we may assume the first. Then $n \geq 2$, and $e_2$ is 6-20; and there is no possibility for $f_2$, a contradiction. This proves (2).

From (1), (2) and 8.1, this proves 16.3.

16.4 Any die-connected graph that contains Concertina also contains one of Petersen, Log, Diamond, Bigdrum, Dice(1), Dice(3).

Proof. Let $H$ be a die-connected graph that contains no member of $\mathcal{L} = \{\text{Petersen, Log, Diamond, Bigdrum, Dice}(1), \text{Dice}(3)\}$. Let Conc(1), Conc(2), Conc(3) be Concertina + $(e, f)$ where $(e, f)$ is (4-8, 10-11), (6-7, 17-18), (8-9, 16-17); and let Conc(4) be Concertina + (2-3, 8-11) + (8-20, 16-17).

(1) $H$ does not contain Conc(1).

Subproof. Let $C$ be the quadrangle of $G = \text{Conc}(1)$. All $A$-extensions are killed by $\mathcal{L}$, and

$$\mathcal{P}(C, \mathcal{L}) = \{(8-11, 9-17), (19-20, 2-10)\}^*;$$ and the result follows by verifying the other hypotheses of 10.2. (The $E$-extension is isomorphic to Dice(1).) Thus proves (1).

Let Conc(21) be Conc(2) + (7-19, 1-5), let Conc(211) be Conc(21) + (1-2, 3-4), and let Conc(212) be Conc(21) + (1-2, 3-7).

(2) $H$ does not contain Conc(211) or Conc(212).
Subproof. Let \( G = \text{Conc}(211) \) and let \( C \) be its quadrangle. Then all \( A \)-extensions are killed by \( \mathcal{L} \), and
\[
P(C, \mathcal{L}) = \{(2-23, 1-12)\}^*,
\]
and the result for \( \text{Conc}(211) \) follows by verifying the other hypotheses of 10.2.

Now let \( G = \text{Conc}(212) \) and let \( C \) be its quadrangle. Again all \( A \)-extensions are killed by \( \mathcal{L} \), and again
\[
P(C, \mathcal{L}) = \{(2-23, 1-12)\}^*,
\]
and again the result follows from 10.2. (\( \text{Conc}(212) + (3-24, 1-22) \) is isomorphic to \( \text{Dice}(3) \).) This proves (2).

(3) \( H \) does not contain \( \text{Conc}(21) \).

Subproof. Let \( \mathcal{L}_1 = \mathcal{L} \cup \{\text{Conc}(211), \text{Conc}(212)\} \). Let \( X = \{1, 2, 10, 11, 12, 13, 14, 15, 16\} \); we claim that every \( X \)-augmentation of \( \text{Conc}(21) \) contains a member of \( \mathcal{L}_1 \). For suppose not, and let the corresponding sequence be \((e_1, f_1), \ldots, (e_n, f_n)\). By checking cases, \( e_1 \) is 12-16 and \( f_1 \) is 14-18; and so \( n \geq 2 \), and \( e_2 \) is 14-20, and there is no possibility for \( f_2 \). Hence (3) follows from 8.1 and (2).

(4) \( H \) does not contain \( \text{Conc}(2) \).

Subproof. Let \( \mathcal{L}_2 = \mathcal{L} \cup \{\text{Conc}(21)\} \), \( G = \text{Conc}(2) \), and \( C \) the quadrangle of \( G \). Then all \( A \)-extensions are killed by \( \mathcal{L}_2 \), and
\[
P(C, \mathcal{L}_2) = \{(19-20, 6-9), (19-20, 9-17)\}^*,
\]
and the result follows by verifying the hypotheses of 10.2. This proves (4).

(5) \( H \) does not contain \( \text{Conc}(3) \).

Subproof. Let \( \mathcal{L}_3 = \mathcal{L} \cup \{\text{Conc}(2)\} \), \( G = \text{Conc}(3) \), and \( C \) the quadrangle of \( G \). Then all \( A \)-extensions are killed by \( \mathcal{L}_3 \), and
\[
P(C, \mathcal{L}_3) = \{(9-19, 4-8)\}^*,
\]
and the result follows by verifying the hypotheses of 10.2. This proves (5).

(6) \( H \) does not contain \( \text{Conc}(4) \).

Subproof. Let \( \mathcal{L}_4 = \mathcal{L} \cup \{\text{Conc}(2), \text{Conc}(3)\} \), and \( X = \{3, 4, 5, 6, 7, 8, 9, 17, 18\} \). We claim that every \( X \)-augmentation of \( G = \text{Conc}(4) \) contains a member of \( \mathcal{L}_4 \). Suppose not, and let the corresponding sequence be \((e_1, f_1), \ldots, (e_n, f_n)\). By checking cases, \( e_1 \) is 3-7 and \( f_1 \) is 1-5; so \( n \geq 2 \), and \( e_2 \) is 5-24, and there is no possibility for \( f_2 \), a contradiction. Hence (6) follows from 8.1.

Let \( \mathcal{L}_5 = \mathcal{L}_4 \cup \{\text{Conc}(1), \text{Conc}(4)\} \), and \( X = \{1, \ldots, 9\} \). We claim that every \( X \)-augmentation of \( G = \text{Concertina} \) contains a member of \( \mathcal{L} \). Suppose not, and let \((e_1, f_1), \ldots, (e_n, f_n)\) be the corresponding sequence. By checking cases \((e_1, f_1)\) is one of \((2-3, 8-11), (4-8, 2-10)\); so \( n \geq 2 \), and in either case there is no possibility for \( f_2 \). Hence the result follows from (1), (4), (5), (6) and 8.1. This proves 16.4.
Proof of 16.1. “Only if” is easy, and we omit it. For “if”, let $H$ contain none of the given graphs. By 16.2, 16.3, 16.4 it contains none of Diamond, Bigdrum, Concertina; and so by 15.1 it is apex. This proves 16.1.

17 Theta-connected non-apex graphs

We recall that $G$ is theta-connected if it is cubic and cyclically five-connected, and $|\delta(X)| \geq 6$ for all $X \subseteq V(G)$ with $|X|, |V(G) \setminus X| \geq 7$ (and hence it is dodecahedrally-connected). None of the graphs of Figures 14–16 are theta-connected, and our next objective is to make a version of 16.1 for theta-connected graphs. It becomes much simpler:

17.1 Let $H$ be theta-connected. Then $H$ is apex if and only if it contains none of Petersen, Jaws and Starfish.

For the proof we use 17.2 below. A domino in a cubic graph $H$ is a subgraph $D$ with $|V(D)| = 7$, consisting of the union of three paths $P_1, P_2, P_3$ of lengths two, three and three respectively, which have common ends and otherwise are disjoint. The middle vertex of $P_1$ is called the centre of the domino, and the other four vertices of degree two are its corners; an attachment sequence is some sequence $(x_1, \ldots, x_5)$ where $x_1, \ldots, x_4$ are the corners, $x_5$ is the centre, $x_1x_2$ is an edge, and $x_2, x_3$ have a common neighbour. (See Figure 17.)

A domino $D$ in $G$ with attachment sequence $(x_1, \ldots, x_5)$ is said to be crossed if

- there are two disjoint connected subgraphs $P, Q$ of $G$, both edge-disjoint from $D$, with $V(P \cap D) = \{x_1, x_3\}$ and $V(Q \cap D) = \{x_2, x_4, x_5\}$, and
- there are two disjoint connected subgraphs $P, Q$ of $G$, both edge-disjoint from $D$, with $V(P \cap D) = \{x_1, x_3, x_5\}$ and $V(Q \cap D) = \{x_2, x_4\}$.

17.2 Let $D$ be a crossed domino with attachment sequence $x_1, \ldots, x_5$, in a cyclically five-connected cubic graph $G$ with $|V(G)| \geq 14$. Let $x_5$ be incident with $g \notin E(D)$. Let $H$ be a cubic graph, cyclically five-connected, and let $\eta$ be a homeomorphic embedding of $G$ in $H$. Then either

- there exists $X \subseteq V(H)$ with $|\delta_H(X)| = 5$, such that for all $v \in V(G)$, $\eta(v) \in X$ if and only if $v \in V(D)$, or
• $H$ contains Petersen, or

• for some $e \in E(D)$ and $f \in E(G \setminus V(D))$ there is a homeomorphic embedding $\eta'$ of $G + (e, f)$ in $H$, or

• for some $e \in \{x_1x_2, x_3x_4\}$, and for some $f \in E(G \setminus V(D))$ such that $f, g$ are diverse in $G$, there is a homeomorphic embedding $\eta'$ of

$$G + (e, g) + (yx_5, f)$$

in $H$, where $x, y$ are the new vertices of $G + (e, g)$.

**Proof.** Let $X = V(D)$. We assume that (i) and (ii) are false. Since $|V(G)| \geq 14$ and $|\delta_G(X)| = 5$, and since (i) is false, it follows from 8.1 that there is an $X$-augmentation $G'$ of $G$, and a homeomorphic embedding $\eta'$ of $G'$ in $G$. Let $(e_1, f_1), \ldots, (e_n, f_n)$ be the corresponding sequence. If $n = 1$ then (iii) is true, so we assume that $n \geq 2$. For $1 \leq i \leq 5$, let $x_i$ be adjacent in $G$ to $y_i \in V(G) \setminus V(D)$. Let the neighbours of $x_5$ in $G$ be $y_5, x_6, x_7$, where $x_6$ is adjacent to $x_1$. Let $G_1 = G + (e_1, f_1)$ with new vertices $s_1, t_1$, and let $D_1$ be the subgraph of $G_1$ induced on $V(D) \cup \{s_1, t_1\}$.

Suppose first that $f_1 = x_1y_1$. Then since $e_1$ and $f_1$ are diverse in $G$, it follows that $e_1 = a_1b_1$ say where $a_1, b_1 \in \{x_3, x_4, x_5, x_7\}$, that is, $e_1$ is one of $x_3x_4, x_3x_7, x_5x_7$. If $f_1$ is 3-4 or 3-7, let $P, Q$ be disjoint paths of $G_1$ from $x_2$ to $x_4$ and from $t_1$ to $x_5$, with no vertices or edges in $D_1$ except their ends; and let $R$ be a path of $G \setminus V(D)$ between $V(P)$ and $V(Q)$ with no internal vertex or edge in $P$ or $Q$. Then $D_1 \cup P \cup Q \cup R$ is homeomorphic to Petersen, and so $G_1$ and hence $H$ contains Petersen, and (ii) is true, a contradiction. So $e_1 = x_5x_7$. Let $P, Q$ be disjoint paths of $G_1$ from $t_1$ to $x_3$ and from $x_2$ to $x_5$, with no vertices or edges in $D_1$ except their ends, and let $R$ be as before. Then $D_1 \cup P \cup Q \cup R$ again is homeomorphic to Petersen, a contradiction.

Hence $f_1 \neq x_1y_1$, and so by symmetry $f_1 \neq x_2y_2, x_3y_3, x_4y_4$; and hence $f = x_5y_5$. Hence $e_1$ is 1-2 or 3-4, and by symmetry we may assume the first. Also, $e_2 = x_5t_1$, and there are (up to the symmetry) three possibilities for $f_2$, namely $f_2 = x_1y_1, f_2 = x_4y_4$, and $f_4 \in E(G \setminus V(D))$. In the third case the theorem is true, so we assume for a contradiction that one of the first two cases hold. Let $G_2 = G_1 + (e_2, f_2)$, with new vertices $s_2, t_2$, and let $D_2$ be the subgraph of $G_2$ induced on $V(D) \cup \{s_1, t_1, s_2, t_2\}$.

If $f_2 = x_1y_1$, let $P, Q$ be disjoint paths of $G_2$ from $t_2$ to $x_3$ and from $t_1$ to $x_4$ with no vertices or edges in $D_2$ except their ends; then $D_2 \cup P \cup Q$ is homeomorphic to Petersen, a contradiction. But if $f_2 = x_4y_4$, let $P, Q$ be disjoint paths of $G_2$ from $x_2$ to $t_2$ and $t_1$ to $x_3$, with no vertices or edges in $D_2$ except their ends; then $D_2 \cup P \cup Q$ is homeomorphic to Petersen, a contradiction. This proves 17.2.

**Proof of 17.1.** “Only if” is easy and we omit it. For “if”, let $H$ be theta-connected and not contain Petersen, Jaws or Starfish.

(1) $H$ does not contain Antilog.

Subproof. Let $G$ be Antilog, let $X = \{1, \ldots, 7\}$, and let $D = G|X$. Then $D$ is a crossed domino of $G$. But the following all contain Petersen:
(i) \(G + (e, f)\) for all \(e \in E(D)\) and \(f \in E(G \setminus X)\)

(ii) \(G + (1-6, 5-10) + (5-22, xy)\) for all \(xy \in E(G \setminus X)\) with \(x, y \neq 10, 14, 15\).

From 17.2, this proves (1).

Let \(L = \{\text{Petersen, Jaws}\}\).

(2) \(H\) does not contain \(\text{Log}\).

Subproof. Let \(\text{Log}(1)\) be \(\text{Log} + (1-2, 8-13)\), let \(C\) be its quadrangle, and let \(L_1 = L \cup \{\text{Antilog}\}\). All \(A\)-extensions are killed by \(L_1\), and

\[
\mathcal{P}(C, L_1) = \{(21-22, 2-9), (21-22, 13-9)\}^*,
\]

and it follows by verifying the hypotheses of 10.2 that \(H\) does not contain \(\text{Log}(1)\).

Let \(\text{Log}(2)\) be \(\text{Log} + (1-2, 9-13)\), let \(C\) be its quadrangle, and \(L_2 = L_1 \cup \{\text{Log}(1)\}\). All \(A\)-extensions are killed by \(L_2\), and

\[
\mathcal{P}(C, L_2) = \emptyset,
\]

and so by 10.1 \(H\) does not contain \(\text{Log}(2)\).

Now let \(G = \text{Log}, X = 1, \ldots, 7\), and \(L_3 = L_2 \cup \{\text{Log}(2)\}\). For any edge \(e\) of \(G|X\) and edge \(f\) of \(G\) not in \(G\)\(|X\) (we permit \(f\) to have one end in \(X\)), if \(e, f\) are diverse then \(G + (e, f)\) contains a member of \(L_3\); and so \(H\) does not contain \(\text{Log}\), by (1) and 8.1. This proves (2).

(3) \(H\) does not contain \(\text{Dice}(1)\).

Subproof. Let \(\text{Dice}(11) = \text{Dice}(1) + (1-2, 20-23)\), let \(C\) be its quadrangle, and \(L_4 = \{\text{Petersen, Jaws, Log, Antilog}\}\). All \(A\)-extensions are killed by \(L_4\), and \(\mathcal{P}(C, L_4) = \emptyset\), so by 10.1 \(H\) does not contain \(\text{Dice}(11)\).

Now let \(L_5 = L_4 \cup \{\text{Dice}(11)\}\), let \(G = \text{Dice}(1), X = \{1, \ldots, 7\}\) and \(D = G|X\); then \(D\) is a crossed domino in \(G\). For all \(e \in E(D)\) and \(f \in E(G \setminus X)\), \(G + (e, f)\) contains a member of \(L_4\); and for all \(xy \in E(G \setminus X)\) with \(x, y \neq 9, 10, 11\), \(G + (1-2, 5-10) + (5-28, xy)\) contains Petersen. Hence the result follows from 17.2. This proves (3).

(4) \(H\) does not contain \(\text{Dice}(2)\).

Subproof. Let \(G = \text{Dice}(2), X = \{1, \ldots, 7\}\) and \(L_6 = \{\text{Petersen, Antilog, Dice}(1)\}\). For all \(e \in E(G|X)\) and \(f \in E(G) \setminus E(G|X)\), if \(e, f\) have no common end then \(G + (e, f)\) contains a member of \(L_6\); so (4) follows from (1), (3) and 8.1.

(5) \(H\) does not contain \(\text{Dice}(3)\).

Subproof. Let \(\text{Dice}(31) = \text{Dice}(3) + (3-4, 13-14)\), let \(C\) be its quadrangle, and \(L_4\) as before. All \(A\)-extensions are killed by \(L_4\), and \(\mathcal{P}(C, L_4) = \emptyset\), so by 10.1 \(H\) does not contain \(\text{Dice}(31)\).

Let \(L_7 = L_4 \cup \{\text{Dice}(31)\}\). Let \(G = \text{Dice}(3), X = \{1, \ldots, 7\}\), and \(D = G|X\). Then \(D\) is a crossed domino in \(G\). For all \(e \in E(D)\) and \(f \in E(G \setminus X)\), \(G + (e, f)\) contains a member of \(L_7\). Moreover, for all \(xy \in E(G \setminus X)\) with \(x, y \neq 15, 16, 18\),

\[
G + (1-2, 5-15) + (5-28, xy)
\]
both contain Petersen or Log. From (1)–(3) and 17.2, this proves (5).

(6) \( H \) does not contain Dice(4).

Subproof. Let \( G = \text{Dice}(4), X = \{1, \ldots, 7\} \) and \( D = G|X \). Then \( D \) is a crossed domino in \( G \). But for all \( e \in E(D) \) and \( f \in E(G \setminus X), G + (e, f) \) contains Petersen or Log; and for all \( xy \in E(G \setminus X) \) with \( x, y \neq 16, 21, 23, \)

\[
G + (1-2, 5-21) + (5-28, xy) \\
G + (3-4, 5-21) + (5-28, xy)
\]

both contain Petersen or Log. The result follows from (2) and 17.2. This proves (6).

From (1)–(6) and 16.2, this proves 17.1.

The reader may have noticed that Starfish hardly ever is needed for anything. There is an explanation, the following (previously stated as 1.2).

17.3 Every dodecahedrally-connected graph \( H \) containing Starfish either is isomorphic to Starfish or contains Petersen.

Proof. If \( H \) “properly” contains \( G = \text{Starfish} \), then by 10.3 \( H \) contains a graph \( G' = G + (e, f) \) for some choice of diverse edges \( e, f \) of \( G \). But every such graph \( G' \) contains Petersen. This proves 17.3.

From 17.3 we obtain a slightly stronger reformulation of 17.1, previously stated as 1.3.

17.4 Let \( H \) be theta-connected, and not isomorphic to Starfish. Then \( H \) is apex if and only if it contains neither of Petersen, Jaws.

The proof is clear.

18 Excluding Petersen

In this section we prove 1.3, thereby completing the proof of 1.1. We restate it:

18.1 Let \( H \) be theta-connected, and contain Jaws but not Petersen. Then \( H \) is doublecross.

Proof. Let Jaws(1) be Jaws + (1-2, 3-4), let Jaws(11) be Jaws(1) + (3-22, 1-6), and let Jaws(12) be Jaws(1) + (21-22, 1-6).

(1) \( H \) does not contain Jaws(11) or Jaws(12).

Subproof. Let \( G \) be Jaws(11), and let \( X = V(G) \setminus \{1, 2, 3, 21, 22, 23, 24\} \). If \( ab \in E(G|X) \) and \( cd \in E(G) \setminus E(G|X) \), with \( a, b \neq c, d \) and with \( a, b \) non-adjacent to any of \( c, d \) that are in \( X \), then \( G + (ab, cd) \) contains Petersen. Hence the result follows from 8.1 when \( G \) is Jaws(11).
When \( G \) is Jaws(12), the argument is not so simple. Again we apply 8.1 to the same set \( X \). Let \((e_1, f_1), \ldots, (e_k, f_k)\) be an augmenting sequence. By checking cases, we find that \( f_1 \) is not an edge of \( G \setminus X \) (because every choice of \( e_1 \in E(G \mid X) \) and \( f_1 \in E(G \setminus X) \) gives a Petersen), and so \( k \geq 2 \); and having fixed \((e_1, f_1)\), we try all the possibilities for \((e_2, f_2)\). Again, there is no case with \( f_2 \in E(G \setminus X) \), and so for each surviving choice of \((e_2, f_2)\) we try the possibilities for \((e_3, f_3)\). We find in every case that there is no choice of \((e_3, f_3)\). (See the Appendix [5].) This proves (1).

(2) \( H \) does not contain Jaws(1).

**Subproof.** Let \( C \) be the quadrangle of \( G = \text{Jaws}(1) \), and let \( L = \{\text{Petersen, Jaws(11), Jaws(12)}\} \). Then all \( A \)-extensions are killed by \( L \), and \( \mathcal{P}(C, L) = \emptyset \), so (2) follows from 10.1.

Let Jaws(2) be Jaws + (8, 3, 5, 6) + (21, 3, 22, 6), let Jaws(21) be Jaws(2) + (6, 7, 11, 12), and let Jaws(22) be Jaws(2) + (7, 8, 19, 10).

(3) \( H \) does not contain Jaws(21).

We apply 10.2 to the quadrangle \{25, 26, 12, 7\}, taking \( L \) to be \{Petersen, Jaws1\}. Again, see the Appendix for details. (Note that Jaws(21) has two circuits of length four, but it is quad-connected; this was the reason we extended 10.2 to quad-connected graphs instead of graphs \( G \) that were cyclically five-connected except for one circuit of length four.)

(4) \( H \) does not contain Jaws(22).

This is easier; we apply 10.1 to the quadrangle \{8, 20, 26, 25\}, taking \( L \) to be \{Petersen, Jaws1, Jaws(21)\}.

(5) \( H \) does not contain Jaws(2).

Let \( X = \{6, 7, 8, 21, 22, 23, 24\} \). We apply 8.1 to \( X \), and try all possibilities for the first three terms of the augmenting sequence; and find in each case contains one of Petersen, Jaws(1), Jaws(21), Jaws(22). (See the Appendix.)

Now let \( C_1 \) be the set of the seven circuits of Jaws that bound regions in the drawing in Figure 2, not containing 1-6, 3-8, 13-18 or 15-20. Let \( C_2 \) be the set of paths of Jaws induced on the following
sets:

6, 1, 2, 3, 8;
8, 3, 4, 5, 6, 1;
1, 6, 7, 8, 3;
3, 8, 20, 15;
15, 20, 19, 18, 13;
13, 18, 17, 16, 15, 20;
20, 15, 14, 13, 18;
18, 13, 1, 6.

Let $G = \text{Jaws}$, let $F$ and $\eta_F$ be null, and let $C = C_1 \cup C_2$; then $(G, F, C)$ is a framework. By hypotheses, there is a homeomorphic embedding $\eta$ of $G$ in $H$. We claim that (E1)–(E7) hold.

Since $F$ is null, (E4), (E5) are vacuously true, and (E1), (E3) are obvious. It remains to check (E2), (E6) and (E7). For (E2) we check that if $e, f \in E(G)$, not both in some member of $C$, then $G + (e, f)$ contains either Petersen or Jaws(1); so (E2) follows from (2). For (E6) it is only necessary to check cross extensions on the circuit with vertex set $\{4, 5, 11, 17, 16, 10\}$ and the path with vertex set $\{1, 6, 5, 4, 3, 8\}$, since all the other circuits and paths are too short or are equivalent by symmetry. Hence we must check

\[
G + (4-5, 16-17) + (4-21, 17-22)
\]
\[
G + (4-5, 16-17) + (4-10, 11-17)
\]
\[
G + (4-10, 11-17) + (4-21, 17-22)
\]
\[
G + (4-10, 11-17) + (10-16, 5-11)
\]
\[
G + (3-8, 5-6) + (3-4, 1-6)
\]
\[
G + (3-8, 5-6) + (3-21, 6-22);
\]

but they all contain Petersen, except the last which contains Jaws(2). Hence (E6) holds.

For (E7) we must check

\[
G + (3-8, 5-6) + (3-21, 1-6) + (8-21, 1-24);
\]

but this contains Petersen. Hence (E7) holds. From 7.1, this proves 18.1.

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