Abstract. Hyperbolic polynomials are real multivariate polynomials with only real roots along a fixed pencil of lines. Testing whether a given polynomial is hyperbolic is a difficult task in general. We examine different ways of translating hyperbolicity into nonnegativity conditions, which can then be tested via sum-of-squares relaxations.

Introduction

A real form (i.e. homogeneous polynomial) $F$ in $n$ variables $x_1, \ldots, x_n$ is called hyperbolic with respect to a point $e \in \mathbb{R}^n$ if $F(e) \neq 0$ and $F(te - a)$ has only real roots in $t$ for all $a \in \mathbb{R}^n$. The simplest example is the determinant of symmetric matrices, which is hyperbolic with respect to the identity matrix, and it can be useful to think of hyperbolic polynomials as generalizations of this determinant.

Hyperbolic polynomials originate in the theory of partial differential equations (see for example [11]) but have more recently received a lot of attention in optimization (hyperbolic programming, spectrahedra) and real algebraic geometry (determinantal representations). They are also closely related to real-stable polynomials which have become important in combinatorics and theoretical computer science. Indeed, if $F$ is irreducible of degree at least 2, then stability is equivalent to hyperbolicity with respect to all unit vectors $e_1, \ldots, e_n$.

Testing whether a given polynomial is hyperbolic (with respect to a fixed point $e$) is a computationally difficult task as soon as $n \geq 3$, even though the precise complexity is only known in some special cases (see [22]). In this note, we will look at three different approaches, all of which work by translating hyperbolicity into a condition of nonnegativity. Positive (resp. nonnegative) polynomials are a staple of real algebraic geometry and, while testing nonnegativity is at least equally hard in general, there are several well established relaxation techniques, in particular based on sums of squares.

For $n = 3$, hyperbolicity is equivalent to the existence of a definite hermitian or real symmetric determinantal representation (by a celebrated result due to Helton and Vinnikov [10]). For $n \geq 4$, this is no longer true (see for example [18]). In any case, the problem of computing determinantal representations is interesting in its own right, but we do not consider it here (see [9], [14], [21], [2], [3], [4], [23], [8]).

The first method we describe is a direct translation of the classical real root counting result due to Hermite. This is the simplest approach. It is certainly well known and we keep the discussion very brief. Our second method looks at the intersection of the real and imaginary part of a polynomial, which can be viewed as parametrized curves in the plane. We use resultants to describe this intersection. This resultant factors, and we show that nonnegativity of the nontrivial factor
characterizes hyperbolicity (Thm. 3.3); one can also apply a real Nullstellensatz certificate to the real and imaginary parts directly. Our third method relies on the fact that the set of hyperbolic polynomials is known to be connected (even simply connected). One can explicitly trace a path from any given polynomial to a fixed hyperbolic polynomial and characterize hyperbolicity by evaluating a (univariate) discriminant along that path (Thm. 4.4).

We illustrate our methods by a number of examples. The different translations between hyperbolicity and nonnegativity are interesting to us in themselves. From a more practical point of view, the appeal comes mostly from the fact that sum-of-squares relaxations are already implemented in a number of software packages and therefore readily available. We have not written general code for our methods that would allow for meaningful runtime-comparisons. Rather, our results should be seen as proof-of-concept. However, the size of the semidefinite programs involved grows rapidly with each method. The examples we have suggest that while the Hermite method is the most straightforward (and possibly the best in general), the intersection method should perform better in certain cases (like curves of low degree). The discriminant method will in general lead to larger relaxations.

Finally, it should also be pointed out that we always test hyperbolicity of a polynomial with respect to a fixed point $e$. This seems to be the most important case, since the point $e$ is often in some way distinguished. However, one might also ask how to test for hyperbolicity with respect to any point. Apart from completely unspecific approaches (like quantifier elimination), it seems entirely unclear how to test this at all and it could be an interesting future problem, even in special cases.

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1. Preliminaries

Definitions 1.1. A polynomial in one variable with real coefficients is called real rooted if all its complex roots are real. Fix a point $e \in \mathbb{R}^{n+1}$. A form (i.e. a homogeneous polynomial) $F \in \mathbb{R}[x_0, x_1, \ldots, x_n]$ is called hyperbolic with respect to $e$ if $F(e) \neq 0$ and the univariate polynomial $F(te - a) \in \mathbb{R}[t]$ is real rooted for all $a \in \mathbb{R}^{n+1}$. It is called strictly hyperbolic if the roots of $F(te - a) \in \mathbb{R}[t]$ are real and distinct for all $a \in \mathbb{R}^{n+1}, a \neq 0$.

Example 1.2. A cubic form $F \in \mathbb{R}[x_0, x_1, x_2]$ in Weierstraß normal form

$$F(x_0, x_1, x_2) = x_0 x_2^2 - H(x_0, x_1)$$

with $H \in \mathbb{R}[x_0, x_1]$ homogeneous of degree 3 is hyperbolic (with respect to some point $e = (1, r, 0)$) if and only if the bivariate form $H$ factors into three real linear forms. It is strictly hyperbolic if and only if these factors are distinct. In this
case, the cubic curve defined by $F$ in the real projective plane has two connected components, while if $H$ contains an irreducible quadratic factor, it has only one connected component.

Consider the family of cubic forms

$$F_c(x_0, x_1, x_2) = x_0x_2^2 - \left(x_1 - \frac{1}{c}x_0\right)\left(x_2^2 - cx_0^2\right)$$

in one parameter $c \in \mathbb{R} \setminus \{0\}$. It is hyperbolic with respect to $(1, 0, 0)$ for $c > 0$ and not hyperbolic (with respect to any point) for $c < 0$. For $c = 1$, it is hyperbolic but not strictly hyperbolic.

In the definition of hyperbolicity, it is equivalent to ask that $F(e - ta)$ should be real rooted for all $a \in \mathbb{R}^{n+1}$, since $F$ is homogeneous. Moreover, it is sufficient to test real-rootedness for vectors $a \in \mathbb{R}^{n+1}$ orthogonal to $e$. In particular, for $e = (1, 0, \ldots, 0)$, a form $F$ with $F(e) \neq 0$ is hyperbolic with respect to $e$ if and only if the dehomogenization $f = F(1, x_1, \ldots, x_n)$ has the property that the univariate polynomial $f(ta) \in \mathbb{R}[t]$ is real rooted for all $a \in \mathbb{R}^n$. Such polynomials are called real zero polynomials. This inhomogeneous setup is preferred in several applications.

An important variant of the definition of hyperbolicity is the following: A form $F \in \mathbb{R}[x_0, \ldots, x_n]$ is called real stable if it is hyperbolic with respect to every point in the positive orthant $\mathbb{R}_{+}^{n+1}$.

2. The Hermite Method

Methods to determine the number of real roots of real univariate polynomials go back to Sturm and Hermite in the nineteenth century. Given a monic polynomial $f \in \mathbb{R}[t]$ of degree $d$, the Hermite matrix of $f$ is the real symmetric $d \times d$-matrix

$$H(f) = \begin{pmatrix}
N_0(f) & N_1(f) & \cdots & N_{d-1}(f) \\
N_1(f) & \ddots & \ddots & \\
\vdots & \ddots & \ddots & \\
N_{d-1}(f) & N_d(f) & \cdots & N_{2d-2}(f)
\end{pmatrix}$$

where $N_j(f)$ denotes the $j$-th power-sum of the complex zeros of $f$, which can be expressed in the coefficients of $f$ via the classical Newton identities. The number of distinct roots is given by the rank of $H(f)$ and the number of real roots by the
signature. In particular, \( f \) is real-rooted if and only if \( H(f) \) is a positive semidefinite matrix (see [12] for all of this and an excellent survey).

Fix \( e = (1, 0, \ldots, 0) \in \mathbb{R}^{n+1} \). Given a form \( F[x_0, \ldots, x_n] \) of degree \( d \) with \( F(e) = 1 \), we can write down the Hermite matrix \( H_{x_0}(F) \) with respect to the variable \( x_0 \), whose entries are polynomials in \( x_1, \ldots, x_n \). Then \( F \) is hyperbolic with respect to \( e \) if and only if \( H_{x_0}(F) \) is positive semidefinite for all \( a \in \mathbb{R}^n \). Equivalently, the Hermite form

\[
H(F) = u^T H_{x_0}(F) u, \quad u^T = (u_1, \ldots, u_d)
\]

in variables \( x_1, \ldots, x_n, u_1, \ldots, u_d \), which is quadratic in \( u \), is a nonnegative polynomial on \( \mathbb{R}^{n+d} \) if and only if \( F \) is hyperbolic with respect to \( e \).

Nonnegativity of \( H(F) \) can be relaxed to a sum-of-squares certificate. The Hermite form \( H(F) \) is a sum of squares in \( \mathbb{R}[x,u] \) if and only if the matrix \( H_{x_0}(F) \) can be factored into

\[
H_{x_0}(F) = V^T V
\]

where \( V \) is a matrix with entries in \( \mathbb{R}[x_1, \ldots, x_n] \) of some format \( d \times r \). For \( n \leq 2 \), this relaxation is exact, but not for \( n \geq 3 \) (see [3] or [17]).

**Example 2.1.** For the cubic

\[
F = x_0^3 - \frac{x_0^2 x_1}{2} - x_0 x_1^2 - \frac{x_0 x_2^2}{2} + \frac{x_1^3}{2}
\]

the Hermite form is given by

\[
H(F) = 3u_1^2 + u_1 u_2 x_1 + \frac{9}{4} u_2^2 x_1 + \frac{9}{2} u_1 u_3 x_1^2 + \frac{1}{4} u_2 u_3 x_1^3 + \frac{33}{16} u_3 x_1^4 + u_2^2 x_2^2 + 2 u_1 u_3 x_2^2 \\
+ \frac{3}{2} u_2 u_3 x_1 x_2 + \frac{5}{2} u_2^2 x_1^2 x_2 + \frac{1}{2} u_2^2 x_2^2
\]

Since \( F \) is hyperbolic, this should be a sum of squares in \( x_1, x_2, u_1, u_2, u_3 \). Indeed, we computed

\[
H(F) = 3 \left( \frac{3}{4} x_1^2 u_3 + \frac{1}{3} x_2^2 u_3 + \frac{1}{6} x_1 u_2 + u_1 \right)^2 + \frac{13}{6} \left( -\frac{3}{26} x_1^2 u_3 + \frac{1}{26} x_2^2 u_3 + x_1 u_2 \right)^2 \\
+ \left( x_1 x_2 u_3 + \frac{1}{2} x_2 u_2 \right)^2 + \frac{3}{4} x_2^2 u_2^2 + \frac{9}{26} \left( x_1^2 u_3 + \frac{1}{36} x_2^2 u_3 \right)^2 + \frac{47}{288} x_2^4 u_3^2.
\]

3. The Intersection Method

In this section we characterize hyperbolicity of multivariate polynomials by separating into real and imaginary part. For the remainder of this section, we will use the following notation: We fix \( e = (1, 0, \ldots, 0) \in \mathbb{R}^{n+1} \). Given a form \( F \in \mathbb{R}[x_0, \ldots, x_n] \), we put

\[
f_x(t) = F(te - x) = F(t, x) \text{ where } x = (x_1, \ldots, x_n)
\]

Recall that \( F \) is hyperbolic with respect to \( e \) if and only if \( f_a(t) \) is real rooted for all \( a \in \mathbb{R}^n \). For any \( a \in \mathbb{R}^n \), the real and imaginary parts of \( f_a(t) \) are polynomials in \( a \), i.e. we can write

\[
f_x(t) = f_{\text{Re}}(t_1, t_2, x) + if_{\text{Im}}(t_1, t_2, x)
\]

where \( t = t_1 + it_2 \).
Lemma 3.1. A form $F \in \mathbb{R}[x_0, \ldots, x_n]$ with $F(e) \neq 0$ is hyperbolic with respect to $e$ if and only if the two polynomials $f_{\text{Re}}, f_{\text{Im}} \in \mathbb{R}[t_1, t_2, x]$ have no common real zero $(s_1, s_2, a) \in \mathbb{R}^{n+2}$ with $s_2 \neq 0$.

Proof. This is simply restating that $f_a(t)$ must be real rooted for all $a \in \mathbb{R}^n$. □

We can express the condition in the Lemma using resultants: Recall that two non-zero polynomials $g, h \in \mathbb{R}[t]$ have a common factor in $\mathbb{R}[t]$ if and only if the resultant $\text{Res}(g, h) = 0$ in $\mathbb{R}$. The resultant is a polynomial in the coefficients of $g$ and $h$. Now if we let $f = \sum_{j=0}^d c_j t^j$ with variable coefficients and write $t = t_1 + it_2$, $f(t_1, t_2) = f_{\text{Re}}(t_1, t_2) + if_{\text{Im}}(t_1, t_2)$, then $f_{\text{Re}}$ has degree $d$ in $t_1$ while $f_{\text{Im}}$ has degree $d - 1$. If $f_{\text{Re}} = \sum_{j=0}^d a_j t_1^j$, and $f_{\text{Im}} = \sum_{j=0}^{d-1} b_j t_1^j$ with $a_j, b_j \in \mathbb{R}[t_2]$, the resultant is given by the determinant of the $(2d - 1) \times (2d - 1)$-Sylvester matrix

$$
\text{Res}(f_{\text{Re}}, f_{\text{Im}}) = \det \begin{pmatrix}
 a_0 & b_0 & & \\
 \vdots & \ddots & \ddots & \\
 & \ddots & \ddots & \ddots \\
 a_d & b_{d-1} & & \\
 & \ddots & \ddots & \ddots \\
 & & a_d & b_{d-1}
\end{pmatrix}
$$

Since $a_d = c_d$ and $b_{d-1} = dc_dt_2$, and since $t_2$ divides the last $d$ columns of the Sylvester matrix, we have

$$
t_2^d | \text{Res}(f_{\text{Re}}, f_{\text{Im}}) \quad \text{and} \quad c_d | \text{Res}(f_{\text{Re}}, f_{\text{Im}}).
$$

Remark 3.2. In fact, no higher power of $t_2$ and $c_d$ divides $\text{Res}(f_{\text{Re}}, f_{\text{Im}})$, but it seems rather complicated to give a direct proof of this fact. (For example, for $f = c_dt^d + t^{d-1}$, one can compute $\text{Res}(f_{\text{Re}}, f_{\text{Im}}) = r \cdot c_d \cdot t_2^{d^2 - 2d + 2} (1 + 4t_2^2 c_d^2)^{d-1}$, where $r$ is a (large) integer, which shows that $\text{Res}(f_{\text{Re}}, f_{\text{Im}})$ cannot in general be divisible by $c_d^2$. Similary, one can examine for example $f = t^d + 1$ to show that $t_2$ does not in general occur to a higher power than $d$.

In our setup, given a form $F$ with $F(e) \neq 0$, we may assume $F(e) = 1$ so that $f_k(t)$ is monic in $t$. We denote by $\text{Res}_{t_1}(f_{\text{Re}}, f_{\text{Im}})$ the resultant of $f_{\text{Re}}, f_{\text{Im}} \in \mathbb{R}[x, t_1, t_2]$ with respect to the variable $t_1$, which is quasi-homogeneous in the coefficients of $F$ (see also [5, Ch. 12]). The relation to hyperbolicity is easy to guess, but care has to be taken to account for possible exceptional cases. We show the following.

Theorem 3.3. Let $e = (1, 0, \ldots, 0)$. Given a form $F \in \mathbb{R}[x_0, \ldots, x_n]$ of degree $d$ with $F(e) = 1$, the resultant of $f_{\text{Re}}$ and $f_{\text{Im}}$ with respect to $t_1$ is a polynomial in $t_2$ and factors into

$$
\text{Res}_{t_1}(f_{\text{Re}}, f_{\text{Im}}) = t_2^p \cdot R_F
$$

for some $p \geq d$, and $R_F$ is a polynomial in $t_2, x_1, \ldots, x_n$ not divisible by $t_2$.

1. If $F$ is hyperbolic with respect to $e$, then $R_F$ has constant sign, i.e. it is everywhere non-negative or everywhere non-positive.
2. Conversely, if $R_F(t_2, x_1, \ldots, x_n)$ does not vanish in any point $(s, a) \in \mathbb{R} \times \mathbb{R}^n$ with $s \neq 0$, then $F$ is hyperbolic with respect to $e$. 

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Proof. We have already observed that the resultant is always divisible by $t_2^3$.

(1) Let $F$ be hyperbolic of degree $d$ and write $S = \{(s,a) \in \mathbb{R} \times \mathbb{R}^n \mid s \neq 0\}$. Suppose for contradiction that $\mathcal{R}_F$ is indefinite, i.e. there is a point $(s,a)$ such that the sign of $\mathcal{R}_F$ is not constant in any neighborhood of $(s,a)$. This implies that $\mathcal{R}_F$ has an irreducible factor $Q$ that changes sign in $(s,a)$. We distinguish two cases:

Assume first that $(s,a) \in S$. Note that $Q$ cannot be a polynomial in $x_1, \ldots, x_n$ alone (independent of $t_2$), since this would imply that $f_a(t)$ vanishes identically. Therefore, $Q$ must have a zero $(s',a') \in S$ (in fact in any neighborhood of $(s,a)$) such that $Q(t_2, a')$ changes sign at $t_2 = s'$. Thus $\text{Res}_{t_1} (f_{\text{Re}}, f_{\text{Im}})(t_2, a')$ has a real root at $t_2 = s' \neq 0$ of odd multiplicity. It follows that $f_{\text{Re}}$ and $f_{\text{Im}}$ have an odd number of intersection points (counted with multiplicity) with second coordinate $s'$. Thus there is a real such point, i.e. a real number $r'$ such that $r' + is'$ is a non-real root of $f_a(t)$, contradicting hyperbolicity.

If $(s,a) \notin S$, then $Q$ changes sign along the hyperplane $\mathbb{R}^{n+1} \setminus S$, which would imply $Q = t_2 \mathcal{R}_F$.

(2) Suppose that $F$ is not hyperbolic. Then there is a point $a \in \mathbb{R}^n \setminus \{0\}$ for which $f_a(t)$ has a non-real zero. Then $f_{\text{Re}}(t_1, t_2, a)$ and $f_{\text{Im}}(t_1, t_2, a)$ have a real intersection point $(t_1, t_2) = (r, s)$ with $s \neq 0$, so that $(s,a)$ is a point in $S$ with $\text{Res}_{t_1}(f_{\text{Re}}, f_{\text{Im}})(s,a) = 0$ and hence $\mathcal{R}_F(s,a) = 0$. \hfill \Box

Remark 3.4. It is natural to ask whether the stronger assumption in (2) is really needed or whether the criterion in (1) is in fact necessary and sufficient for hyperbolicity. We do not see how to show this without some further information about the factor $\mathcal{R}_F$ in the resultant. For instance, the converse in (1) would hold if $\mathcal{R}_F$ were generically irreducible or at least square-free. This should be expected and is verified in the examples below, but does not seem so easy to prove. Thus we make the following conjecture. If true, it would allow for a neater version of Thm. 3.3.

**Conjecture 3.5.** For a generic form $F$, the factor $\mathcal{R}_F$ of the resultant in Theorem 3.3 is irreducible.

For quadratic and cubic forms, the condition in Thm. 3.3 can be made quite a bit more explicit, since hyperbolicity can be decided by looking only at the discriminant of $f_x(t)$ with respect to $t$. We will take a closer look at this case and see how it compares to our general analysis above.

**Quadratic Forms.** Let $F = x_0^2 + f_1(x)x_0 + f_2(x) \in \mathbb{R}[x_0, x]$ be a quadratic form. It is hyperbolic with respect to $e = (1,0,\ldots,0)$ if and only if the polynomial $f_x(t) = t^2 + f_1(x)t + f_2(x)$ is real-rooted for all $x \in \mathbb{R}^n$. This will be the case if and only if $f_1^2 - 4f_2$ is nonnegative in $x$ (compare also [18] and [2]). This is a quadratic form in $x$, hence it is nonnegative if and only if it is a sum of squares in $\mathbb{R}[x]$. Let us see how this translates into real and imaginary parts, which are given by

$$f_{\text{Re}}(t_1, t_2) = (t_1^2 - t_2^2) + t_1 f_1 + f_2$$
$$f_{\text{Im}}(t_1, t_2) = 2t_1 t_2 + t_2 f_1$$
Thus the resultant of these two bivariate polynomials \( f_{\text{Re}}, f_{\text{Im}} \) with respect to \( t_1 \) is given by
\[
\text{Res}_{t_1}(f_{\text{Re}}, f_{\text{Im}}) = \begin{vmatrix} 1 & 2t_2 & 0 \\ f_1 & f_1t_2 & 2t_2 \\ f_2 - t_2^2 & 0 & f_1t_2 \end{vmatrix} = t_2^2(4f_2 - f_1^2 - 4t_2^2)
\]
Thus \( \mathcal{R}_F \) in Theorem 3.3 is
\[
\mathcal{R}_F = 4f_2 - f_1^2 - 4t_2^2.
\]
Indeed, this polynomial is nonpositive if and only if \( f_1^2 - 4f_2 \) is nonnegative.

**Example 3.6.** Let \( F = x_0^2 - x_1^2 - x_2^2 - \cdots - x_n^2 \). Then
\[
\text{Res}_{t_1}(f_{\text{Re}}, f_{\text{Im}}) = 4t_2^3(x_1^2 + x_2^2 + \cdots + x_n^2)(1 + t_2^2(x_1^2 + x_2^2 + \cdots + x_n^2))
\]
which does not vanish for \( t_2 \in \mathbb{R} \setminus \{0\} \) and any \( x \in \mathbb{R}^n \). Hence \( F \) is hyperbolic. Also note that \( f_1^2 - 4f_2 = 4(x_1^2 + \cdots + x_n^2) \) is a sum of squares.

**Cubic Forms.** Let
\[
F(x) = x_0^3 + f_1(x)x_0^2 + f_2(x)x_0 + f_3(x)
\]
be a cubic form. Again, hyperbolicity of \( F \) with respect to \( e = (1, 0, \ldots, 0) \) is equivalent to \( t^3 + f_1(x)t^2 + f_2(x)t + f_3(x) \) being real rooted in \( t \) for all \( x = a \in \mathbb{R}^n \). This is the case if and only if the cubic discriminant \( \Delta \) of \( f_x(t) \) is nonnegative for all \( x \). It is given by
\[
\Delta = 18f_1f_2f_3 - 4f_2^3 + f_1^2f_2^2 - 4f_1^3f_3 - 27f_3^2
\]
Again, we compare this to our resultant. The real and imaginary parts are given by
\[
\begin{align*}
\text{Res}_{t_1}(f_{\text{Re}}, f_{\text{Im}}) &= t_3^2 + t_2^3f_1 + t_1(f_2 - 3t_2^2) - t_2^2f_1 + f_3 \\
\text{Res}_{t_1}(f_{\text{Im}}, f_{\text{Re}}) &= 3t_2^3 + 2t_1t_2f_1 + t_2f_2 - t_3^2.
\end{align*}
\]
Thus the resultant of \( f_{\text{Re}} \) and \( f_{\text{Im}} \) with respect to \( t_1 \) comes out as
\[
\text{Res}_{t_1}(f_{\text{Re}}, f_{\text{Im}}) = \begin{vmatrix} 1 & 0 & 3t_2 & 0 & 0 \\ f_1 & 1 & 2t_2f_1 & 3t_2 & 0 \\ f_2 - 3t_2^2 & f_1 & t_2f_2 - t_3^2 & 2t_2f_1 & 3t_2 \\ f_3 - t_2^2f_1 & f_2 - 3t_2^2 & 0 & t_2f_2 - t_3^2 & 2f_1t_2 \\ 0 & f_3 - t_2^2f_1 & 0 & t_2f_2 - t_3^2 \end{vmatrix} = -t_2^2[\Delta + t_2^2g_2 + t_2^4g_3 + t_2^6g_4].
\]
where
\[
g_2 = 4(f_1^2 - 3f_2)^2, \quad g_3 = 32(f_1^2 - 3f_2), \quad g_4 = 64.
\]
Thus \( \mathcal{R}_F \) in Thm. 3.3 is the polynomial
\[
\mathcal{R}_F = \Delta + t_2^2g_2 + t_2^4g_3 + t_2^6g_4.
\]
In this case, we find indeed that nonnegativity of \( \Delta \) is equivalent to nonnegativity of \( \mathcal{R}_F \). To see this, note that \( \mathcal{R}_F \) is nonnegative if and only if the cubic equation
\[
\Delta + t_2^2g_2 + t_2^4g_3 + t_2^6g_4 = 0
\]
in $t_2^2$ has no positive real root. Assume that $\Delta$ is nonnegative. This means that $4(f_1^2 - 3f_2)^3 - (2f_1^3 - 9f_1f_2 + 27f_3)^2$ is nonnegative. Therefore, $f_1^2 - 3f_2$ and hence $g_3$ must be nonnegative. This shows that (1) has only nonnegative coefficients and hence no positive solution in $t_2^2$. Thus $\mathcal{R}_F$ is nonnegative.

For forms of degree at least 4, it is not enough to consider only the discriminant, as the following simple example shows.

**Example 3.7.** The quartic

$$F = x_0^4 - x_1^4 - x_2^4$$

is not hyperbolic with respect to $e = (1, 0, 0)$. However, $f_0(t) = t^4 - (x_1^4 + x_2^4)$ has distinct roots in $t$ for all $(x_1, x_2) \neq (0, 0)$, hence the discriminant of $f_0(t)$ has constant sign.

That $F$ is not hyperbolic is however reflected in the fact that

$$\mathcal{R}_F = 256(t_2^4 - x_1^4 - x_2^4)(4t_2^4 + x_1^4 + x_2^4)^2$$

is clearly neither nonnegative nor nonpositive.

**Example 3.8.** Consider our parametrized cubic

$$F_c(x_0, x_1, x_2) = x_0x_2^2 - x_1 - \frac{1}{c}x_0 \left( x_1^2 - cx_0^2 \right)$$

As noted, it is hyperbolic with respect to $(1, 0, 0)$ if and only if $c > 0$. Indeed, substituting $c = b^2$, we can represent the discriminant $\Delta$ as the sum of squares

$$\Delta = (2b^6 - 2)^2 x_0^6 + x_0^2 y_0^4 b_1^6 + 20x_0^4 y_0^2 b_1^4 + 4y_0^6 b_1^2 + 12x_0^2 y_0^4 b_1^2 + 12x_0^4 y_0^2 b_1^2.$$

### 3.1 Using the real Nullstellensatz

Since the conditions in Theorem 3.3 need to be satisfied for any $x \in \mathbb{R}^n$ and the resultants quickly become quite large, it is not clear how useful this method is in practice. Of course, it is not necessary to rely on resultants to test whether the real and imaginary part of a polynomial intersect. One can also employ the real Nullstellensatz, which will also translate into a sums-of-squares condition.

The real Nullstellensatz is the following general criterion for infeasibility; see [1], [15].

**Theorem 3.9** (Real Nullstellensatz). A system $f_1, \ldots, f_k \in \mathbb{R}[x]$ of real polynomials in variables $x = (x_1, \ldots, x_n)$ has no common zero in $\mathbb{R}^n$ if and only if there exist polynomials $q_1, \ldots, q_k$ and a sum of squares $s$ in $\mathbb{R}[x]$ such that

$$s + q_1 f_1 + \cdots + q_k f_k = -1$$

Reading the identity in the real Nullstellensatz modulo the ideal generated by $f_1, \ldots, f_k$, we obtain the following equivalent formulation: If $\mathcal{I}$ is an ideal $\mathbb{R}[x]$, then the real variety $\mathcal{V}_\mathbb{R}(\mathcal{I})$ defined by $\mathcal{I}$ in $\mathbb{R}^n$ is empty if and only if $-1$ is a sum of squares in the residue ring $\mathbb{R}[x]/\mathcal{I}$.

Testing this sum-of-squares condition can be translated into a semidefinite program, either directly or combined with a Gröbner basis computation working in $\mathbb{R}/\mathcal{I}$. We did some experiments in Macaulay2 with the SOSm2 package ([7], [20]).
Applying this to our problem, we are given a form $F \in \mathbb{R}[x_0, x]$ and wish to test for hyperbolicity with respect to $e = (1, 0, \ldots, 0)$. We form $f_x(t)$ and decompose into real and imaginary part. Then $F$ is hyperbolic if and only if $f_{\text{Re}}$ and $f_{\text{Im}}$ have no common real root in $t_1, t_2, x$ with $t_2 \neq 0$ (Lemma 3.1). This is equivalent to the system

$$f_{\text{Re}}, f_{\text{Im}}, 1 - yt_2$$

with one additional variable $y$ being infeasible. Thus we obtain the following criterion for hyperbolicity.

**Proposition 3.10.** Let $F \in \mathbb{R}[x_0, \ldots, x_n]$ be a form of degree $d$ with $F(e) \neq 0$ and let $\mathcal{I}$ be the ideal generated by $f_{\text{Re}}, f_{\text{Im}}, 1 - yt_2$ in $A = \mathbb{R}[t_1, t_2, x, y]$. Then $F$ is hyperbolic with respect to $e$ if and only if $-1$ is a sum of squares in $A/\mathcal{I}$. \[\square\]

### 4. The discriminant method

Our final method for testing hyperbolicity is based on an observation due to Nuij in [19], also used in [14]. We will work in the following setup. Let $e = (1, 0, \ldots, 0) \in \mathbb{R}^{n+1}$, $d \geq 1$, $x = (x_1, \ldots, x_n)$ as before, and consider the sets

$$\mathcal{F} = \{ F \in \mathbb{R}[t, x] \mid F \text{ is homogeneous of degree } d \text{ and } F(e) = 1 \}$$

$$\mathcal{H} = \{ F \in \mathcal{F} \mid F \text{ is hyperbolic with respect to } e \}.$$

Note that $F$ lies in $\mathcal{H}$ if and only if $F(t, a)$ is real rooted in $t$ for all $a \in \mathbb{R}^n$ (c.f. §1).

Nuij constructed an explicit path in the space of polynomials connecting any given polynomial to a fixed polynomial in $\mathcal{H}$. We consider the following operators on polynomials $\mathcal{F} \subset \mathbb{R}[t, x] = \mathbb{R}[t, x_1, \ldots, x_n]$.

$$T^t_s: F \mapsto F + s\ell \frac{\partial F}{\partial t} \quad (\ell \in \mathbb{R}[x] \text{ a linear form})$$

$$G_s: F \mapsto F(t, sx)$$

$$H_s = (T^t_1)^d \cdots (T^t_n)^d$$

$$N_s = H_{1-s}G_s,$$

where $s \in \mathbb{R}$ is a parameter. For fixed $s$, all of these are linear operators on $\mathbb{R}[t, x]$ taking the affine-linear subspace $\mathcal{F}$ to itself. Clearly, $G_s$ preserves hyperbolicity for any $s \in \mathbb{R}$, and $G_0(f) = t^d$ for all $F \in \mathcal{F}$. The operator $H_s$ is used to smoothen the polynomials along the path $s \mapsto G_s(F)$. The exact statement is the following.

**Proposition 4.1** (Nuij [19]). For $s \geq 0$, the operators $T_s^t$ preserve hyperbolicity. Moreover, the following holds:

(1) For any $F \in \mathcal{F}$, we have $N_1(F) = F$.

(2) The polynomial $N_0(F)$ lies in $\text{int}(\mathcal{H})$ and is independent of $F$.

(3) For $F \in \mathcal{H}$, we have $N_s(F) \in \text{int}(\mathcal{H})$ for all $s \in [0, 1)$. \[\square\]

For $F \in \mathcal{F}$, we call $[0, 1] \ni s \mapsto N_s(F)$ the $N$-path of $F$.

**Corollary 4.2.** A form $F \in \mathcal{F}$ is hyperbolic if and only if the $N$-path does not cross the boundary of $\mathcal{H}$, i.e. $N_s(F) \in \mathcal{H}$ for all $s \in (0, 1)$. \[\square\]
The boundary of \( \mathcal{H} \) is a subset of the hypersurface in \( \mathcal{F} \) defined by the vanishing of the discriminant \( \Delta \in \mathbb{R}[x] \) of polynomials in \( \mathcal{F} \) with respect to the variable \( t \). The polynomial \( \Delta \) can be expressed via the Sylvester matrix and is homogeneous of degree \( 2d-2 \). We can test for the criterion in Cor. 4.2 by restricting the discriminant to the \( N \)-path, as follows: Let \( F \in \mathcal{F} \) and write

\[
\Delta_N(F) = \Delta(N_s(F)) \in \mathbb{R}[s,x].
\]

We call \( \Delta_N \) the \( N \)-path discriminant. Our preceding discussion translates to the following statement:

**Corollary 4.3.** Let \( F \in \mathcal{F} \). If \( \Delta_N(F)(s,a) \neq 0 \) for all \( s \in (0,1) \) and \( a \in \mathbb{R}^n \), then \( F \) is hyperbolic.

The converse is not quite true, but we have the following characterization of hyperbolicity, which is analogous to what we found for the intersection method.

**Theorem 4.4.** If a polynomial \( F \in \mathcal{F} \) is hyperbolic, then

\[
\Delta_N(F)(s,a) \geq 0
\]

holds for all \( s \in [0,1] \) and \( a \in \mathbb{R}^n \). Conversely, if \( \Delta_N(F)(s,a) > 0 \) holds for all \( s \in [0,1] \) and \( a \in \mathbb{R}^n \setminus \{0\} \), then \( F \) is strictly hyperbolic.

**Proof.** Suppose first that \( F \) is hyperbolic. By continuity, we may assume \( F \in \text{int}(\mathcal{H}) \), which means that \( F \) is strictly hyperbolic. It follows that \( N_s(F)(t,a) \) has distinct real roots in \( t \) for all \( a \in \mathbb{R}^n \setminus \{0\} \), hence \( \Delta_N(F)(s,a) > 0 \) for all \( s \in [0,1] \). Since \( \Delta_N(F)(0,a) > 0 \) for all \( a \in \mathbb{R}^n \setminus \{0\} \), we conclude that \( \Delta_N(F)(s,a) \geq 0 \) holds for all \( s \in [0,1] \).

If \( F \) is not hyperbolic, then \( F(t,a) \) has a non-real root for some \( a \in \mathbb{R}^n \setminus \{0\} \). Since \( N_0(F)(t,a) \) has distinct real roots, it follows that \( \Delta_N(F)(s,a) \) must vanish for some \( s \in (0,1] \).

In the case of curves \( (n = 2) \), the positivity condition on the hyperbolicity discriminant can be related to a beautiful result due to Marshall:

**Theorem 4.5 ([16]).** A polynomial \( h \in \mathbb{R}[s,t] \) satisfies \( h(a,b) \geq 0 \) for all \( a \in [0,1] \) and \( b \in \mathbb{R} \) if and only if there exist sums of squares \( \sigma_1, \sigma_2 \in \mathbb{R}[s,t] \) such that

\[
h = \sigma_1 + \sigma_2 \cdot s(1-s).
\]

The proof of Marshall’s theorem is quite intricate. Unfortunately, the degree of the sums of squares \( \sigma_1 \) and \( \sigma_2 \) cannot be bounded in terms of the degree of \( h \) alone. Therefore, Thm. 4.5 does not translate into a criterion that can be checked by a single semidefinite program. Nevertheless, an SDP hierarchy of growing degrees can be employed. The analogue of Thm. 4.5 does not hold if more than one variable is unbounded. We refer to Marshall’s book [15] for a broader discussion.

**Corollary 4.6.** If a form \( F \in \mathbb{R}[t,x_1,x_2] \) is hyperbolic, then there exist sums of squares \( \sigma_1, \sigma_2 \in \mathbb{R}[s,y] \) such that

\[
\Delta_N(F)(s,y,1) = \sigma_1 + \sigma_2 s(1-s).
\]

**Proof.** The polynomial \( \Delta_N(F)(s,x_1,x_2) \) is homogeneous in \( x_1, x_2 \), hence, if it is non-negative for \( s \in [0,1] \), then so is the dehomogenization \( \Delta_N(F)(s,y,1) \).
Example 4.7. The hyperbolic cubic

\[ F = t^3 - \frac{t^2 x_1}{2} - tx_1^2 - \frac{tx_1^2}{2} + \frac{x_1^3}{2} \]

has the N-path discriminant

\[
\Delta_N(F) = \frac{294698s^6}{4} + \frac{51283s^6}{2} - 3316s^6x_1^2 + \frac{392497}{16} - 36s^6x_1^2 + \frac{121698s^6}{2}
\]

\[ - 39350s^6x_1^4 - 143390s^6x_1^4x_2^2 + 20316s^6x_1^4x_2^2 - 139200s^6x_1^4x_2^2 + 108s^6x_1^4x_2^2 - 34632s^6x_2^2
\]

\[ + 89581s^6x_1^4 + 3398905s^6x_1^4x_2^2 - 51420s^6x_1^4x_2^2 + 332832s^6x_1^4x_2^2 - 108s^6x_1^4x_2^2 + 82980s^6x_2^2
\]

\[ - 111308s^6x_1^4 - 433316s^6x_1^4x_2^2 + 68980s^6x_1^4x_2^2 - 429120s^6x_1^4x_2^2 + 36s^6x_1^4x_2^2 - 107136s^6x_2^2
\]

\[ + 79632s^6x_1^4 + 315648s^6x_1^4x_2^2 - 51840s^6x_1^4x_2^2 + 314640s^6x_1^4x_2^2 + 78024s^6x_2^2 - 31104sx_1^4
\]

\[ - 124416s^6x_1^4x_2^2 + 20736sx_1^4x_2^2 - 124416sx_1^4x_2^2 - 31104sx_2^2 + 5184sx_1^4 + 20736sx_1^4x_2^2
\]

\[ - 3456sx_1^4x_2^2 + 20736sx_1^4x_2^2 + 5184sx_2^2. \]

It can be verified numerically that \( \Delta_N(F) \) is indeed nonnegative for \( 0 \leq s \leq 1 \), but we could not find a nice rational representation of the form in Cor. 4.6.

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