ON THE FIRST SIGN CHANGE IN MERTENS’ THEOREM

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Abstract. The function \( \sum_{p \leq x} \frac{1}{p} - \log \log(x) - M \) is known to change sign infinitely often, but so far all calculated values are positive. In this paper we prove that the first sign change occurs well before \( e^{495.702833165} \).

1. Introduction

Mertens’ Theorem states that

\[
\Delta_M(x) := \sum_{p \leq x} \frac{1}{p} - \log \log(x) - M = O\left(\log(x)^{-1}\right)
\]

for \( x \to \infty \), where \( M = 0.26149\ldots \) denotes the Mertens constant [Mer74]. Rosser and Schoenfeld observed that \( \Delta_M(x) \) is always positive for \( 1 \leq x \leq 10^8 \) and posed the question whether this would always be the case [RS62, p. 72f]. This has been answered by Robin who showed that \( \Delta_M(x) \) changes sign infinitely often [Rob83].

In this paper we show that the first sign change occurs before \( e^{495.702833165} = 1.909875\cdots \times 10^{215} \). More specifically, we prove

Theorem 1. There exists an \( x_0 \in [\exp(495.702833109), \exp(495.702833165)] \) such that \( \Delta_M(x) \) is negative for all \( x \in [x_0 - \exp(239.046541), x_0] \).

This problem is similar to the problem of bounding the Skewes number, the number in \( [2, \infty) \) where the first sign change of \( \Delta(x) = \pi(x) - \text{li}(x) \) occurs [Ske33]; this number is by now known to lie between \( 1.39 \times 10^{17} \) [PTar] and \( \exp(727.9513586) \) [SD10]. The functions \( \Delta(x) \) and \( \Delta_M(x) \) are closely related and the Prime Number Theorem, \( \Delta(x) = o(\text{li}(x)) \) for \( x \to \infty \), is in fact equivalent to \( \Delta_M(x) = o(\log(x)^{-1}) \) for \( x \to \infty \). But since \( \Delta(x) \) and \( \Delta_M(x) \) are biased in opposite directions there is no correlation between the sign changes of the two functions. On the Riemann Hypothesis, sign changes of \( \Delta_M(x) \) rather occur at points where \( \Delta(x) \approx -2\sqrt{x}/\log(x) \).

Theorem 1 is proven by an adaption of the Lehman method for bounding the Skewes number [Leh66], using explicit formulas and numerical approximations to part of the zeros of the Riemann Zeta Function from [FKBJ]. In doing this, the kernel function in Lehman’s method is replaced by the Logan function [Log88], which appears to be more suitable for this problem. This is done in such generality that it can easily be reapplied to the original Lehman method.
2. Notations

As usual \(\zeta(s)\) denotes the Riemann zeta function and zeros of \(\zeta(s)\) are denoted by \(\rho = \beta + i\gamma\) with \(\beta, \gamma \in \mathbb{R}\). The Euler constant is denoted by \(C_0 = 0.57721\ldots\) and the Mertens constant by

\[
M = C_0 - \sum_{p} \sum_{m=2}^{\infty} \frac{1}{m p^m} = 0.26149\ldots.
\]

We use the symbol \(\sum'\) to define normalized summatory functions, i.e. we define

\[
\sum'_{x<n<y} a_n := \frac{1}{2} \sum_{x<n<y} a_n + \frac{1}{2} \sum_{x\leq n\leq y} a_n.
\]

Moreover, we define the Mertens prime-counting functions

\[
\pi_M(x) = \sum'_{p<x} \frac{1}{p} \quad \text{and} \quad \pi^*_M(x) = \sum_{m=1}^{\infty} \frac{\pi_M(x^{1/m})}{m}.
\]

The Fourier transform of a function \(f\) is denoted by \(\hat{f}\) and is defined by

\[
\hat{f}(x) = \int_{-\infty}^{\infty} f(t) e^{-itx} \, dt.
\]

Finally, we will use Turing’s big theta notation for explicit estimates and write \(f(x) = \Theta(g(x))\) for \(|f(x)| \leq g(x)\).

3. Description of the Method

The method we use is similar to the Lehman method for finding regions where \(\pi(x) - \text{li}(x)\) is positive \([\text{Leh66}]\). We aim to calculate upper bounds for a weighted mean value

\[
\int_{\omega-\varepsilon}^{\omega+\varepsilon} K(y) e^{y/2} \left[ \pi_M(e^y) - \log(y) - M \right] \, dy,
\]

where \(K(y)\) is a non-negative kernel function. Using explicit formulas this mean value can be expressed as a sum over the non-trivial zeros of \(\zeta(s)\) which can be approximated numerically. Then, if an \(\omega\) can be found for which the value in (2) is negative, there must exist an \(x \in [\exp(\omega - \varepsilon), \exp(\omega + \varepsilon)]\), such that \(\pi_M(x) - \log \log(x) - M\) is negative.

Lehman’s method uses the Gaussian function as a kernel function but we prefer to use dilatations of the function

\[
K_c(y) := \begin{cases} 
\frac{c}{2\sinh(c)} I_0(c\sqrt{1-y^2}) & |y| < 1, \\
0 & \text{otherwise},
\end{cases}
\]

where \(I_0(t) := \sum_{n=0}^{\infty} (t/2)^{2n}/(n!)^2\) denotes the 0-th modified Bessel function. The Fourier transform of \(K_c\) is given by the Logan function (see \([\text{FKBJ}, \text{Proposition 4.1}]\))

\[
\hat{K}_c(x) = \ell_c(x) := \frac{c}{\sinh(c)} \frac{\sin(t^2 - c^2)}{\sqrt{t^2 - c^2}},
\]
which satisfies an optimality property well-suited for this problem [Log88], and which outperforms the Gaussian function in the similar context of calculating the prime-counting function analytically [FKBJ].

We define
\[ K_{c,\varepsilon}(y) := \frac{1}{\varepsilon} K_c(y/\varepsilon) \quad \text{and} \quad \ell_{c,\varepsilon}(x) := \tilde{K}_{c,\varepsilon}(x) = \ell_c(\varepsilon x). \]

Then our main result is

**Theorem 2.** Let \( 0 < \varepsilon < 10^{-3}, \ c \geq 3, \ \omega - \varepsilon > 200, \) and let \( H \geq c/\varepsilon \) be a number such that \( \beta = 1/2 \) holds for all zeros \( \rho = \beta + i\gamma \) of the Riemann zeta function with \( 0 < \gamma \leq H \).

Furthermore, let \( h = 0 \) if the Riemann hypothesis holds and \( h = 1 \) otherwise. Then we have

\[
\int_{\omega-\varepsilon}^{\omega+\varepsilon} K_{c,\varepsilon}(y-\omega) y e^{y/2} \left[ \pi_M(e^y) - \log(y) - M \right] dy \\
\leq \sum_{|\gamma| \leq c/\varepsilon} e^{-i\gamma \omega} \ell_{c,\varepsilon}(\gamma) \left( \frac{1}{\rho} - \frac{1}{\omega \rho^2} \right) \\
+ 1 + 5.4 \times 10^{-10} + \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3,
\]

where

\[
\mathcal{E}_1 \leq 0.33 e^{h \omega/2} \frac{e^{0.71 \sqrt{c \varepsilon}}}{\sinh c} \log(3c) \log \left( \frac{c}{\varepsilon} \right),
\]

\[
\mathcal{E}_2 \leq \frac{3.36 + 126 \varepsilon}{1000 \omega^2} + 2.8 \left( \frac{e}{2H} \right)^{\omega/2-1} \log(H),
\]

and

\[
\mathcal{E}_3 \leq \frac{e^{\omega/2}}{1.99H} \log(H) \left( \frac{c \ e^{3.12 \sqrt{c \varepsilon}}}{\omega \sinh(c)} + \left( \frac{e \varepsilon}{\omega} \right)^{\omega/2} \right).
\]

Moreover, if \( a \in (0, 1) \) satisfies \( ac/\varepsilon \geq 10^5 \) in addition to the previous conditions, then

\[
\sum_{a \varepsilon < |\gamma| \leq \varepsilon} \left| e^{-i\gamma \omega} \ell_{c,\varepsilon}(\gamma) \left( \frac{1}{\rho} - \frac{1}{\omega \rho^2} \right) \right| \leq 0.32 + 3.51 c \varepsilon \log \left( \frac{c}{\varepsilon} \right) \frac{\cosh(c \sqrt{1-a^2})}{\sinh(c)}.
\]

The proof needs some preparation.

4. **The explicit formula for \( \pi_M^*(x) \)**

The first ingredient is the explicit formula for \( \pi_M^*(x) \). We define the auxiliary function

\[
\tilde{Ei}(z) = \int_0^\infty e^{z-t} \frac{1}{z-t} dt,
\]

which coincides with the exponential integral \( \text{Ei}(z) \) in \( \mathbb{R} \setminus \{0\} \), and which occurs naturally in explicit formulas for prime-counting functions.
Lemma 1. Let $x > 1$. Then
\begin{equation}
\pi_M^*(x) = \log \log(x) + C_0 - \sum_{\rho}^* \tilde{\text{Ei}}(-\rho \log x) + \int_x^\infty \frac{dt}{t^2 \log(t)(t^2 - 1)}.
\end{equation}
where the star indicates that the sum over zeros is calculated as
\[
\lim_{T \to \infty} \sum_{|\gamma| < T} \tilde{\text{Ei}}(-\rho \log x).
\]

Proof. The proof is similar to the original proof of the Riemann explicit formula in [vM95].

Let
\begin{equation}
\psi(x, r) = \sum_{p^m < x} \frac{\log p}{p^mr}.
\end{equation}

Then we have
\[
\pi_M^*(x) = \int_1^\infty \psi(x, r) \, dr.
\]

From [Lan08 (39)] we get the explicit formula
\[
\psi(x, r) = \frac{x^{1-r}}{1-r} - \sum_{\rho} \frac{x^{\rho-r}}{\rho - r} - \sum_{n=1}^\infty \frac{x^{-2n-r}}{-2n - r} - \frac{\zeta'(r)}{\zeta(r)}.
\]

Since $\text{Ei}(-x) = \log(x) + C_0 + o(x)$ for $x \searrow 0$ [Olv97, p. 40], and since $\log \zeta(1 + \varepsilon) = -\log(\varepsilon) + o(1)$ for $\varepsilon \searrow 0$ (an easy consequence of [Ing33, p. 26, eq. (3)]) we have
\[
\int_1^\infty \frac{x^{1-r}}{1-r} - \frac{\zeta'(r)}{\zeta(r)} \, dr = \lim_{\varepsilon \searrow 0} \left[ \text{Ei}(\varepsilon \log x) + \log \zeta(1 + \varepsilon) \right] = \log \log(x) + C_0.
\]

The sum over zeros takes the form
\[
\int_1^\infty \sum_{\rho}^* \frac{x^{\rho-r}}{\rho - r} \, dr = \sum_{\rho}^* \tilde{\text{Ei}}((\rho - 1) \log x) = \sum_{\rho}^* \tilde{\text{Ei}}(-\rho \log x),
\]

and for the sum over the trivial zeros we find
\[
\int_1^\infty \sum_{n=1}^\infty x^{-2n-r} \, dr = \int_1^\infty \sum_{n=1}^\infty x^{-(2n+1)r} \, dr = \int_1^\infty \frac{x^{-3r}}{1 - x^{-2r}} \, dr = \int_x^\infty \frac{dt}{t^2 \log(t)(t^2 - 1)}. \quad \Box
\]

5. The difference $\pi_M^*(x) - \pi_M(x)$

By definition of the Mertens constant [1], we have
\[
\pi_M(x) = \pi_M^*(x) + M - C_0 + r_M(x),
\]

where
\[
r_M(x) = \sum_{\substack{p^m \geq x \ \text{m} \geq 2}} \frac{1}{mp^m}.
\]

The term $r_M(x)$ is responsible for the positive bias in Mertens’ Theorem and needs to be bounded from above.
Lemma 2. Let $\log(x) > 200$. Then

$$r_M(x) \leq \frac{1 + 5.3 \times 10^{-10}}{\sqrt{x} \log x}.$$ 

Proof. First we consider the contribution of the squares of prime numbers which yield the main term. Let $r(t) = \psi(t) - t$, where $\psi(t) := \psi(t, 0)$ in the sense of [9] denotes the normalized Chebyshov function, and assume $|r(t)| < \varepsilon t$ for $t \geq \sqrt{x}$. Then partial summation gives

$$\sum_{p > \sqrt{x}}^{\prime} \frac{1}{p^2} < \left[ -\frac{r(t)}{t^2 \log t} \right]^{\infty}_{\sqrt{x}} + \int_{\sqrt{x}}^{\infty} \frac{dt}{t^2 \log(t)} - \int_{\sqrt{x}}^{\infty} \frac{r(t)}{t} \left( \frac{1}{t^2 \log t} \right) dt < 2 \frac{1 + 3 \varepsilon}{\sqrt{x} \log x}.$$ 

For $3 \leq m \leq \log(x)$ we use

$$\sum_{p \geq x^{1/m}} \frac{1}{p^m} \leq \frac{1}{x} + \int_{x^{1/m}}^{\infty} \frac{dt}{t^m} = \frac{1}{x} + \frac{1}{m - 1} x^{1/m - 1},$$

which gives

$$\sum_{3 \leq m \leq \log x} \frac{1}{mp^m} \leq \frac{\log x}{x} + (\zeta(2) - 1)x^{-2/3} < \frac{10^{-12}}{\sqrt{x} \log(x)}.$$ 

For $m > \log x$ we estimate trivially:

$$\sum_{p} \frac{1}{p^m} \leq \sum_{n=3}^{\infty} n^{-m} + 2^{-m} \leq 2^{-m} + \int_{2}^{\infty} \frac{dt}{t^m} = 2^{-m} \left( 1 + \frac{2}{m - 1} \right).$$

Therefore, we get

$$\sum_{p^m \geq x} \frac{1}{mp^m} \leq \frac{1.01}{\log(x)} \sum_{m \geq \log x} 2^{-m} \leq \frac{2.02 \times 2^{-\log(x)}}{\log(x)} < \frac{10^{-16}}{\sqrt{x} \log(x)}.$$ 

By [Büttel Table 1] (10) holds with $\varepsilon = 1.752 \times 10^{-10}$ and so the assertion follows. \qed

6. Evaluating the sum over zeros

The next problem is to approximate the following integral over the sum over zeros

$$\int_{-\varepsilon}^{\varepsilon} K_{c, \varepsilon}(y - \omega) y e^{\eta/2} \sum_{\rho}^\ast \widetilde{E}_\varepsilon(-\rho y) dy.$$ 

Here, integral and sum may be interchanged, since the sum converges locally in $L^1$. Therefore, we may treat each summand individually.
6.1. Asymptotic Expansion of the Summands. Since the Logan kernel should also be of interest for the question on finding regions where \( \pi(x) - \text{li}(x) \) is positive, the following Lemma is presented in a more general version, which also covers the classical case.

**Lemma 3.** Let \( 0 < \varepsilon < \omega \), and let \( K \in L^1([-\varepsilon, \varepsilon]) \) satisfy \( \|K\|_{L^1} = 1 \). Let \( a \in [0, 1] \), let \( \rho = \beta + i\gamma \), where \( 0 \leq \beta \leq 1 \) and \( \gamma \in \mathbb{R} \setminus \{0\} \), and let

\[
\Phi_{\omega, \rho, a} = \int_{\omega - \varepsilon}^{\omega + \varepsilon} K(y - \omega) y e^{(\frac{1}{2} - a) y} \tilde{E}_i((a - \rho)y) \, dy.
\]

Then we have

\[
\Phi_{\omega, \rho, a} = \sum_{j=1}^{k} (j - 1)! F^{(-j)}_{\omega, \rho}(0) \left( \frac{(\frac{1}{2} - \beta) \omega}{(\omega - \varepsilon)^k |\gamma|^{k+1}} \right) + \Theta \left( \frac{\varepsilon^{j-2 - \frac{1}{2} + i\beta} (\frac{1}{2} - \beta) \omega (\varepsilon \omega)^m}{\omega^{j-1}} \right),
\]

where \( F^{(-1)}_{\omega, \rho}(0) = -e^{(\frac{1}{2} - \rho) \omega} \tilde{K}(\frac{1}{2} - \frac{1}{2\pi}) \) and

\[
F^{(-j)}_{\omega, \rho}(0) = (-1)^j e^{(\frac{1}{2} - \rho) \omega} \sum_{n=0}^{m} \left( \frac{(n + j - 2)!}{(n)!} \right) \left( \frac{(-i)^n \tilde{K}(n) (\frac{1}{2} - \frac{1}{2\pi})}{\omega^{n+j-1}} \right)
\]

\[
+ \Theta \left( \frac{\varepsilon^{j-2 + i(\frac{1}{2} - \beta) \omega} (\varepsilon \omega)^m}{\omega^{j-1}} \right).
\]

**Proof.** By definition of \( \tilde{E}_i \) we have

\[
\Phi_{\omega, \rho, a} = \int_{\omega - \varepsilon}^{\omega + \varepsilon} K(y - \omega) y e^{(\frac{1}{2} - a) y} \int_{0}^{\infty} e^{(a - \rho - r)y} \frac{a - \rho - r}{a - \rho - r} \, dr \, dy
\]

\[
= \int_{0}^{\infty} \frac{1}{a - \rho - r} \int_{\omega - \varepsilon}^{\omega + \varepsilon} K(y - \omega) y e^{(\frac{1}{2} - \rho - r)y} \, dy \, dr.
\]

Now let

\[
F^{(-j)}_{\omega, \rho}(r) := (-1)^j \int_{\omega - \varepsilon}^{\omega + \varepsilon} y^{1-j} K(y - \omega) e^{(\frac{1}{2} - \rho - r)y} \, dy,
\]

which is well defined since \( \omega > \varepsilon \) and satisfies \( \frac{d}{dr} F^{(-j)}_{\omega, \rho} = F^{(-1-j)}_{\omega, \rho} \). Then partial summation gives

\[
\Phi_{\omega, \rho, a} = - \int_{0}^{\infty} \frac{F_{\omega, \rho}^{(0)}(r)}{r + \rho - a} \, dr = \sum_{j=1}^{k} (j - 1)! F_{\omega, \rho}^{(-j)}(0) (r + \rho - a)^{-j} - k! \int_{0}^{\infty} \frac{F_{\omega, \rho}^{(-k)}(r)}{(r + \rho - a)^{k+1}} \, dr.
\]

Here, the trivial bound

\[
\left| F_{\omega, \rho}^{(-k)}(r) \right| \leq \int_{\omega - \varepsilon}^{\omega + \varepsilon} \frac{|K(y)|}{(\omega + y)^{k+1}} e^{(\frac{1}{2} - \beta - r)(y+\omega)} \, dy \leq \frac{e^{\frac{\varepsilon}{\omega}}}{(\omega - \varepsilon)^{k-1}} e^{(\frac{1}{2} - \beta)\omega} e^{r(\varepsilon - \omega)},
\]

yields

\[
\int_{0}^{\infty} \left| F_{\omega, \rho}^{(-k)}(r) \right| (r + \rho - a)^{-k+1} \, dr \leq \frac{e^{\frac{\varepsilon}{\omega}}}{(\omega - \varepsilon)^{k}} e^{(\frac{1}{2} - \beta)\omega} e^{r(\varepsilon - \omega)}.
\]
Thus, we have Lemma 5 (Butb, Lemma 4.5) which gives

\[ F_{\omega, \rho}(0) = -e^{(\frac{j}{4} - \frac{1}{2}) \omega} \int_{-\varepsilon}^{\varepsilon} K(y)e^{-i(\frac{j}{4} - \frac{1}{2})y} dy = -e^{(\frac{j}{4} - \frac{1}{2}) \omega} \tilde{K}\left(\frac{\rho}{i} - \frac{1}{2i}\right). \]

For larger values of \( j \) we use the Taylor series expansion

\[ \frac{1}{(\omega + y)^n} = \sum_{n=0}^{\infty} \binom{u + n - 1}{n} \frac{(-y)^n}{\omega^{n+n}} \]

and

\[ \int_{-\varepsilon}^{\varepsilon} K(y)y^n e^{-i(\frac{j}{4} - \frac{1}{2})y} dy = i^n \tilde{K}^{(n)}\left(\frac{\rho}{i} - \frac{1}{2i}\right), \]

which gives

\[ F_{\omega, \rho}^{(-j)}(0) = (-1)^j e^{(\frac{j}{4} - \frac{1}{2}) \omega} \sum_{n=0}^{\infty} \binom{j + n - 2}{n} \frac{(-i)^n \tilde{K}^{(n)}\left(\frac{\rho}{i} - \frac{1}{2i}\right)}{\omega^{n+j-1}}. \]

From (14) we get

\[ \left| \tilde{K}^{(n)}\left(\frac{\rho}{i} - \frac{1}{2i}\right) \right| \leq e^{\frac{j}{n} \omega} \varepsilon^n \]

and the inequality \( \binom{a}{b} \leq \left(\frac{ea}{b}\right)^b \), which follows from Stirling’s lower bound for \( b! \), implies

\[ \binom{j + n - 2}{n} \leq e^n \left(1 + \frac{j - 2}{n}\right)^n \leq e^{n + j - 2}. \]

Thus, we have

\[ \sum_{n=m+1}^{\infty} \binom{j + n - 2}{n} \frac{\left| \tilde{K}^{(n)}\left(\frac{\rho}{i} - \frac{1}{2i}\right) \right|}{\omega^{n+j-1}} \leq e^{j - 2 + \frac{\varepsilon}{2}} \sum_{n=m+1}^{\infty} \frac{\varepsilon^n}{\omega^n} = e^{j - 2 + \frac{\varepsilon}{2}} \frac{(\varepsilon/\omega)^{m+1}}{\omega^{m+1} - 1 - \varepsilon/\omega}, \]

which confirms the bound in (12).

6.2. **Bounds for the Kernel Function.** We need some bounds to estimate the tails of the sum over zeros. These are provided by the following two Lemmas from Butb and Buta:

**Lemma 4** (Buta, Lemma 2). Let \( 0 < \varepsilon < 10^{-3} \) and \( c \geq 3 \). Then we have

\[ \sum_{|\gamma| > \frac{c}{\varepsilon}}^* \frac{|\ell_{c, \varepsilon}(\frac{\rho}{i} - \frac{1}{2i})|}{|\gamma|} \leq 0.32 \frac{e^{0.71\sqrt{\varepsilon \varepsilon}}}{\sinh(c)} \log(3c) \log\left(\frac{c}{\varepsilon}\right). \]

**Lemma 5** (Butb, Lemma 4.5). Let \( 0 < \varepsilon < 10^{-3} \) and \( c \geq 3 \), and let \( a \in (0, 1) \) satisfy \( ac/\varepsilon > 10^3 \). Then we have

\[ \sum_{\frac{ae}{\varepsilon} < |\gamma| \leq \frac{c}{\varepsilon}}^* \frac{|\ell_{c, \varepsilon}(\frac{\rho}{i} - \frac{1}{2i})|}{|\gamma|} \leq \frac{1 + 11c \varepsilon}{\pi ca^2 \sinh(c)} \cosh(c\sqrt{1 - a^2}) \log\left(\frac{c}{\varepsilon}\right) \frac{c}{\varepsilon}. \]

We also need bounds for the derivatives \( \ell^{(n)}_{c, \varepsilon}(\frac{\rho}{i} - \frac{1}{2i}) \) occurring in (12), for calculations not assuming the Riemann hypothesis.
Lemma 6. Let $0 < \varepsilon \leq \delta < c/100$, and let $z \in \mathbb{C}$ satisfy $|\Re(z)| \geq c/\varepsilon$ and $|\Im(z)| \leq \frac{1}{2}$. Then
\[
|\ell_{c,\varepsilon}^{(n)}(z)| \leq n! \frac{c e^{1.56\sqrt{\delta}c}}{\sinh(c)} \left(\frac{2\varepsilon}{\delta}\right)^n.
\]

Proof. The bound follows from the Cauchy formula
\[
\ell_{c,\varepsilon}^{(n)}(z) = \frac{n!}{2\pi i} \oint_{|z-\xi| = \frac{\delta}{2\varepsilon}} \ell_{c,\varepsilon}(\xi) (z-\xi)^{n+1} d\xi
\]
if we show that
\[
|\ell_{c,\varepsilon}(\xi)| \leq \frac{c e^{1.56\sqrt{\delta}c}}{\sinh(c)}
\]
holds in the range of integration. By basic properties of $\ell_{c,\varepsilon}$ it suffices to prove this bound for $\varepsilon = 1$ under the conditions $\Re(\xi) \geq c - \delta$ and $0 \leq \Im(\xi) \leq \delta$. To this end we use
\[
|\sin(z)| \leq e^{\Im(z)}
\]
and bound $|\Im(\sqrt{\xi^2 - c^2})|$ in this region.

Let $a + ib$ be a square root of $\xi^2 - c^2$ with $b \geq 0$. Take $\xi = x + iy$, then $b$ is real analytic as a function of $x$ resp. $y$ in $x, y > 0$ and satisfies
\[
\frac{x^2y^2}{b^2} - b^2 = x^2 - y^2 - c^2.
\]
In particular, $b$, as a function of $x$ (resp. $y$), has non-vanishing derivative as can be seen by differentiating \eqref{eq:derivative} with respect to $x$ (resp. $y$). Since $\lim_{x \to 0} b(x) > \lim_{x \to \infty} b(x)$, $b(x)$ is monotonously decreasing for $x \in (0, \infty)$, and since $b(y) \to \infty$ for $y \to \infty$, $b(y)$ is monotonously increasing for $y \in [0, \infty)$. Therefore, we have
\[
|\Im(\sqrt{\xi^2 - c^2})| \leq |\Im(\sqrt{(c - \delta + i\delta)^2 - c^2})|
\]
\[
\leq \sqrt{2}|1 + i|\delta c\sin\left(\frac{\pi}{4} + \frac{1}{2}\arctan\left(\frac{\delta c - \delta^2}{\delta c}\right)\right)
\]
\[
\leq 2^{3/4} \sin(1.81) \sqrt{\delta c} \leq 1.56 \sqrt{\delta} c,
\]
where the condition $\delta < c/100$ was used. Inserting this into \eqref{eq:bound} gives the desired bound \eqref{eq:bound}.

7. Proof of Theorem 1

By Lemma 1 and Lemma 2 we have
\[
\pi_M(e^y) - \log(y) - M = \pi_M^*(e^y) - \log(y) - C_0 + r_M(e^y)
\]
\[
\leq - \sum_{\rho} \tilde{Ei}(-\rho y) + \frac{1 + 5.4 \times 10^{-10}}{y} e^{-y/2}
\]
for \( y > 200 \), where we estimated the integral in (8) trivially by \( e^{-3y} \). Therefore

\[
\int_{\omega - \varepsilon}^{\omega + \varepsilon} K_{c, \varepsilon}(y - \omega) y e^{y/2} \frac{\pi M(e^{y}) - \log(y) - M}{y} dy \leq - \sum_{\rho} \Phi_{\omega, \rho, 0} + 1 + 5.4 \times 10^{-10},
\]

with \( \Phi_{\omega, \rho, 0} \) as defined in Lemma 3 with \( K = K_{c, \varepsilon} \) and \( \tilde{K} = \ell_{c, \varepsilon} \). We subdivide the sum over zeros into two parts. For \( 0 < \gamma \leq H \) we choose \( k = 2 \) and \( m = 0 \) in Lemma 3 which gives

\[
(20) \quad - \sum_{|\gamma| \leq H} \Phi_{\omega, \rho, 0} \leq \sum_{|\gamma| \leq \varepsilon/\varepsilon} e^{-i\gamma \omega} \ell_{c, \varepsilon}(\gamma) \left( \frac{1}{\rho} - \frac{1}{\omega^{p^2}} \right) + \sum_{\frac{\varepsilon}{\varepsilon} < |\gamma| \leq H} \frac{\ell_{c, \varepsilon}(\gamma)}{\gamma} \left( 1 + \frac{\varepsilon}{c\omega} \right) + \frac{1}{\omega^2} \sum_{|\gamma| < H} \left( \frac{2.72 \varepsilon}{\gamma^2} + \frac{2.01}{|\gamma|^3} \right),
\]

where we used \( \varepsilon \leq 10^{-3} \). For \( \gamma > H \) we have

\[
(21) \quad \sum_{|\gamma| \leq H} |\Phi_{\omega, \rho, 0}| \leq e^{h\omega/2} \sum_{|\gamma| > H} \left| \ell_{c, \varepsilon}(\frac{\varepsilon}{\gamma} - \frac{1}{2\gamma}) \right| \frac{1}{|\gamma|^3} \left( \sum_{j=1}^{k} \frac{(j-1)!}{\omega^{j-1}} H^{1-j} \right) + e^{h\omega/2} \sum_{|\gamma| > H} \frac{k! \varepsilon^2}{|\gamma|^{k+1}}
\]

for arbitrary \( k \geq 2 \) and \( m \geq 1 \), where \( h = 0 \) if the Riemann hypothesis holds and \( h = 1 \) otherwise. So the inequality in (3) holds with

\[
(22) \quad \mathcal{E}_1 = \sum_{\frac{\varepsilon}{\varepsilon} < |\gamma| \leq H} \frac{\ell_{c, \varepsilon}(\gamma)}{\gamma} \left( 1 + \frac{\varepsilon}{c\omega} \right) + e^{h\omega/2} \sum_{|\gamma| > H} \frac{\ell_{c, \varepsilon}(\frac{\varepsilon}{\gamma} - \frac{1}{2\gamma})}{\gamma} \left( \sum_{j=1}^{k} \frac{(j-1)!}{\omega^{j-1}} H^{1-j} \right),
\]

\[
(23) \quad \mathcal{E}_2 = \frac{1}{\omega^2} \sum_{\rho} \left( \frac{2.72 \varepsilon}{\gamma^2} + \frac{2.01}{|\gamma|^3} \right) + e^{h\omega/2} \sum_{|\gamma| > H} \frac{k! \varepsilon^2}{(\omega - \varepsilon)^k |\gamma|^{k+1}},
\]

and

\[
(24) \quad \mathcal{E}_3 = e^{h\omega/2} \sum_{|\gamma| > H} \sum_{j=2}^{k} \frac{(j-1)!}{|\gamma|^j} \left( \sum_{n=1}^{m} \frac{n+j-2}{n} \right) \frac{\ell_{c, \varepsilon}(\frac{\varepsilon}{\gamma} - \frac{1}{2\gamma})}{\omega^{n+j-1}} + e^{j-2+\varepsilon/2(\varepsilon^2+m+1)} \frac{(\omega - \varepsilon)^k |\gamma|^{k+1}}{(\omega - \varepsilon)^k |\gamma|^{k+1}}.
\]

We proceed by bounding \( \mathcal{E}_k \). To this end we choose \( k = m = \lfloor \omega/2 \rfloor \). In (22) we take \( H = \frac{\varepsilon}{\varepsilon} \), which gives

\[
(25) \quad \mathcal{E}_1 \leq e^{h\omega/2} \sum_{\frac{\varepsilon}{\varepsilon} < |\gamma|} \frac{\ell_{c, \varepsilon}(\gamma)}{\gamma} \sum_{j=0}^{k-1} \frac{j! \varepsilon^j}{\omega^j (\varepsilon^j/j)!},
\]
where the inner sum is bounded by
\[ \sum_{j=0}^{\infty} \left( \frac{\varepsilon}{2c} \right)^j \leq \left( 1 - \frac{1}{6000} \right)^{-1} \leq 1.0002, \]
since \( c \geq 3 \). Using this and (15) in (25) gives (14).

In (23) we use the bounds \( \sum \gamma \gamma^{-2} < 0.0463 \) and \( \sum \gamma |\gamma|^{-3} < 0.00167 \) from [Ros41, Lemma 17], the bound
\[ \sum_{|\gamma|>T} |\gamma|^{-k} \leq T^{1-k} \log(T) \]
for \( T \geq 2\pi c \) and \( k \geq 2 \) from [Leh66, Lemma 2], the bound
\[ \sum_{|\gamma| \leq T} \rho \leq \frac{1}{2} - \frac{1}{200 \times 1000} \sum_{|\gamma| \leq T} \frac{\ell_{c,2}(\gamma)}{\gamma} \]
and the inequality \((\omega - \varepsilon)^k \geq e^{-\varepsilon} \omega^k\), which follows from \( k \leq \omega/2 \), and get
\[ E_2 \leq 0.00336 + 0.126 \varepsilon + e^{\omega/2} \frac{e^{2\pi k}!}{(\omega H)^k} \log(H) \leq \frac{3.36 + 126 \varepsilon}{1000 \omega^2} + 2.8 \left( \frac{e}{2H} \right)^{\omega/2-1} \log(H). \]

In (24) we use (26) again and the bound from Lemma 6, where we choose \( \delta = 4 \varepsilon \), which gives
\[ E_3 \leq e^{\omega/2} \sum_{j=2}^{k} H^{1-j} \log(H) \left( \frac{e^{3.12 \sqrt{\varepsilon}}}{\sinh(c)} \sum_{n=1}^{m} \frac{j-1}{\omega} (n+j-2)! \omega^{n+j-2} 2^{-n} \right. \]
\[ \left. + \frac{1.002e^{j-1} (j-1)!}{e^{j-1}} \left( \frac{e\varepsilon}{\omega} \right)^{m+1} \right). \]

Since \( n+j-2 \leq \omega \) we have \((n+j-2)!/\omega^{n+j-2} \leq 1/\omega \), so the inner sum is bounded by \(1/(2\omega)\).
In the second summand, we use the bound \((j-1)!/\omega^{j-1} \leq 2^{j-1} \). Since \( \sum_{j=1}^{\infty} H^{-j} \leq 1.001/H \), \( \sum_{j=1}^{\infty} (2H/e)^{-j} \leq 1.001e/(2H) \), and \( m+1 \geq \omega/2 \), we obtain bound the in (6). Finally, the estimate in (7) follows from (16) since
\[ \sum_{\frac{\pi}{2} < |\gamma| \leq \frac{\pi}{2}} \frac{\ell_{c,2}(\gamma)}{\rho} \leq \left( 1 + \frac{1}{200 \times 1000} \right) \sum_{\frac{\pi}{2} < |\gamma| \leq \frac{\pi}{2}} \frac{\ell_{c,2}(\gamma)}{\gamma} \]
\[ \square \]

8. Numerical Results

To locate potential regions where the left hand side of (3) should be small, the function
\[ \sigma_T(y) = \sum_{|\gamma| \leq T} \frac{e^{i\gamma y}}{1 - i\gamma}. \]
has been evaluated for \( T = 10^6 \) at all points in \( 10^{-7} \mathbb{Z} \cap [1, 2500] \). This has been done using the method for fast multiple evaluation of trigonometric sums from [FKBJ]. A more detailed search with \( T = 10^8 \) around 495.7028078, the first point where \( \sigma_{10^6}(y) \) turned out to be
Table 1. Values of $y \in [1, 2500]$ for which $\sigma_{10^6}(y) < -0.95$.

| $y$           | $\sigma_{10^6}(y)$ |
|---------------|--------------------|
| 495.7028078   | -0.9972...         |
| 1423.957207   | -0.9740...         |
| 1623.9204309  | -0.9807...         |
| 1859.1291846  | -1.0511...         |
| 2107.5263606  | -1.0214...         |
| 2285.3917834  | -1.0454...         |
| 2430.3039554  | -1.0172...         |
| 2447.6661764  | -1.0028...         |

promisingly small, revealed a short region of length $\approx 2.8 \times 10^{-8}$ about $495.702833137$ where $\sigma_{10^6}(y) < -1$.

Theorem 1 now follows by an application of Theorem 1 with $\omega = 495.702833137$, $c = 280$, $\varepsilon = 2.8 \times 10^{-8}$, $H = 10^{11}$ (which has been reported in [FKBJ]) and $a = 0.4$.

The sum over zeros was calculated using approximations to the zeros with imaginary part up to $4 \times 10^9$ which were given within an absolute accuracy of $2^{-64}$. The sum was evaluated using multiple precision arithmetic, which gave the bound

$$\sum_{|\gamma|\leq 4\times 10^9} e^{-i\omega \epsilon \ell(\gamma)} \left( \frac{1}{\rho} - \frac{1}{\omega \rho^2} \right) \leq -1.00015419.$$  

The sum in (27) is then bounded by $1.2 \times 10^{-11}$ and we have

$$\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 \leq 1.2 \times 10^{-12} + 1.37 \times 10^{-8} + 1.6 \times 10^{-24} \leq 1.38 \times 10^{-8}.$$  

Thus, the left hand side of (28) is bounded by

$$-1.00015419 + 1.2 \times 10^{-11} + 1 + 5.4 \times 10^{-10} + 1.38 \times 10^{-8} < -0.000154.$$  

Consequently, there exists an $x \in [\exp(w - \varepsilon), \exp(w + \varepsilon)]$ such that $\pi_M(x) - \log \log(x) - M < -0.000154/(\sqrt{x} \log x)$. Obviously, we have

$$\pi_M(x - y) - \log \log(x - y) - M \leq \pi_M(x) - \log \log(x) - M + \int_{x-y}^{x} \frac{dt}{t \log t} \leq -\frac{0.000154}{\sqrt{x} \log(x)} + \frac{y}{(x-y) \log(x - y)},$$

which is negative for $y \leq 0.00015 \sqrt{x}$. Since $0.00015 \sqrt{x} > \exp(239.046541)$ the assertion of Theorem 1 follows.  

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