Generalized action-angle coordinates in toric contact spaces

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Abstract

In this paper we are concerned with completely integrable Hamiltonian systems in the setting of contact geometry. Unlike the symplectic case, contact structures are automatically Hamiltonian. Using the Jacobi brackets defined on contact manifolds, we discuss the commutativity of the first integrals for contact Hamiltonian systems and introduce the generalized contact action-angle variables. We exemplify the general scheme in the case of the five-dimensional toric Sasaki-Einstein spaces $T^{1,1}$ and $Y^{p,q}$.

1 Introduction

There has been considerable interest recently in contact geometry in connection with some modern developments in mathematics and theoretical physics [1, 2]. The theory of contact structures is linked to many geometric backgrounds as symplectic geometry, Riemannian and complex geometry, analysis and dynamics [3, 4]. Contact spaces have shown their usefulness in gauge theories of gravity, black holes in higher dimensions, branes. Sasaki-Einstein manifolds whose metric cones are Calabi-Yau manifolds find applications in string theory in connection with AdS/CFT correspondence which relates quantum gravity in a certain background to ordinary quantum field theory without gravity.

The isolated systems are conservative and their standard description is given in terms of Hamilton’s equations of motion in the phase space which has a natural symplectic structure. Over the last decades, there have been many attempts to extend the symplectic Hamiltonian mechanics. For example, contact Hamiltonian mechanics is a natural candidate for a geometric description of

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non-dissipative and dissipative systems [5]. Contact geometries are adequate in describing mechanical systems where the Hamilton function explicitly depends on time. Contact Hamiltonian dynamics has been used in thermodynamics [6] and in description of dissipative systems at the mesoscopic level [7].

The aim of this paper is to investigate contact Hamiltonian dynamics for a class of toric contact structures. In the case of toric contact spaces, the system is completely integrable if the toric action is effective and preserve the contact structures. By analogy to standard symplectic dynamics, we introduce the action-angle variables and indicate the possibility to evaluate the frequencies of the flow of toric action.

The paper is organized as follows: In Section 2 we review fundamentals on contact geometry and contact Hamiltonian dynamics. The contact dynamics is presented first in a coordinate-free manner and then in special local coordinates. In Section 3 we illustrate the contact Hamiltonian dynamics through two examples related to toric Sasaki-Einstein spaces in five-dimensions. In Section 4 we provide some closing remarks.

2 Preliminaries

2.1 Contact geometry

A contact manifold \((M, \eta)\) is a \((2n + 1)\)-dimensional manifold \(M\) endowed with a contact 1-form \(\eta\) such that [1]

\[ \eta \wedge (d\eta)^n \neq 0. \]  

(1)

Associated with a contact form \(\eta\) there exists a unique vector field \(R_\eta\) called the Reeb vector field defined by the contractions (interior products):

\[ i(R_\eta)\eta = 1, \]  

(2a)

\[ i(R_\eta)d\eta = 0. \]  

(2b)

The tangent bundle \(TM\) may be decomposed into [8]

\[ TM = \mathbb{R}R_\eta \oplus \mathcal{H}, \]  

(3)

and by duality we have for the cotangent bundle \(T^*M\)

\[ T^*M = \mathbb{R}\eta \oplus \mathcal{K}. \]  

(4)

The subbundle \(\mathcal{H}\) (or the horizontal distribution), of rank \(2n\), is the kernel of \(\eta\). \(\mathcal{K}\) is the annihilator of \(\ker d\eta = \mathbb{R}R_\eta\) and its sections are called semi-basic forms satisfying the relation

\[ i(R_\eta)\varphi = 0. \]  

(5)

According to [8] every vector field \(X\) on \(M\) may be decomposed as

\[ X = (i(X)\eta)R_\eta + \hat{X}. \]  

(6)
where $\hat{X}$ is the horizontal part of $X$. Every 1-form $\psi$ may be in turn decomposed as

$$\psi = (i(R_\eta)\psi)\eta + \hat{\psi},$$

where $\hat{\psi}$ is the semi-basic component of $\psi$.

The mapping

$$\eta^\sharp : X \mapsto -i(X)d\eta,$$

(8)
carries any vector field $X$ on $M$ into a semi-basic form. We will denote the inverse isomorphism of $\eta^\sharp$ by $\eta^\#$.

A vector field $X$ on $(M, \eta)$ is an infinitesimal contact automorphism if and only if there exists a differentiable function $\rho$ such that

$$\mathcal{L}(X)\eta = \rho\eta.$$  

(9)

In what follows we shall write eq. (6) in the form

$$X_f = f R_\eta + \hat{X}_f,$$

(10)

where $f R_\eta$ and $\hat{X}_f$ are, respectively, the vertical and horizontal components with

$$f = i(X_f)\eta.$$  

(11)

With the help of Cartan’s formula connecting the Lie derivative with the interior product, $\mathcal{L}(X) = d \circ i(X) + i(X) \circ d$, eq. (9) may be written

$$df + i(X_f)d\eta = \rho\eta.$$  

(12)

Using the properties (2) of the contact form $\eta$ we have

$$\rho = i(R_\eta)df.$$  

(13)

The condition $\rho = 0$ expresses the fact that $f$ is a first integral of the vector field $R_\eta$ being a constant along the flow of the vector field $R_\eta$.

A chosen contact form $\eta$ on $M$ defines an isomorphism $\Phi$ from the vector space of infinitesimal contact automorphisms onto the set $C^\infty(M)$ of smooth functions on $M$:

$$\Phi(X_f) = f = i(X_f)\eta,$$

(14)

with the inverse

$$\Phi^{-1}(f) = f R_\eta + \eta^\sharp(df - (i(R_\eta)df)\eta).$$  

(15)

Let us remark that the Reeb vector field $R_\eta = \Phi^{-1}(1)$ is an infinitesimal automorphism of the contact form $\eta$ with $\rho = 0$. 

3
2.2 Contact Hamiltonian systems

On a symplectic manifold \((M, \Omega)\), \(\Omega\) being the symplectic form, the system of the 1-st order differential equations

\[
\dot{x} = X_H,
\]

for some smooth function \(H\) on \(M\) is called a Hamiltonian system.

The development of the theory of completely integrable systems in contact geometry has been more recent starting with the influential work of Banyaga and Molino \cite{9}.

In the frame of contact geometry, the vector field \(X_f = \Phi^{-1}(f)\) \cite{14} is called the contact Hamiltonian vector field and the analog of \(\dot{x} = X_H\)

\[
\dot{x} = X_f,
\]

represents the contact Hamiltonian equation corresponding to \(f\). Taking into account \(\ref{13}\) we get that \(X_f\) is an infinitesimal automorphism of \(\eta\) if and only if \(df\) is semi-basic \cite{10}.

It is often convenient to consider the Reeb vector field \(R_\eta\) as the Hamiltonian vector field with \(1 = \eta(R_\eta)\) as the Hamiltonian. In this case the Hamiltonian contact structure is said to be of Reeb type and the Hamiltonian is understood to be the constant function 1.

In connection with the isomorphism \(\Phi\) \cite{13}, the Lie algebra structure of \(C^\infty(M)\) is given by the Jacobi bracket \cite{11} \cite{12}

\[
[f, g]_\eta = \Phi[X_f, X_g] = -i(X_g)df + fi(R_\eta)dg
\]

\[
= -i(X_f)di(X_g)d\eta + fi(R_\eta)dg - gi(R_\eta)df.
\]

Assuming that \(f\) and \(g\) are first integrals of the vector field \(R_\eta\) we have

\[
[f, g]_\eta = d\eta(X_f, X_g).
\]

A Hamiltonian contact structure of Reeb type is said to be completely integrable if there exists \((n + 1)\) first integrals \(f_0 = 1, f_1, \ldots, f_n\) that are independent and in involution. In addition a completely integrable contact Hamiltonian system is said to be of toric type if the corresponding vector fields \(X_{f_0} = R_\eta, X_{f_1}, \ldots, X_{f_n}\) form the Lie algebra of a torus \(T^{n+1}\). The action of a torus \(T^{n+1}\) on a contact \((2n + 1)\)-dimensional manifold \((M, \eta)\) is completely integrable if it is effective and preserve the contact structure \(\eta\) \cite{13}.

2.3 Formulae in local coordinates

In view of the concrete examples which will be studied in the next Sections, it is convenient to write some of the above formulae in local coordinates.
Let us consider in a neighborhood $U$ of a point $x$ of $M$ an adapted system of local coordinates $(x^0, x^1, \ldots, x^n, y^1, \ldots, y^n)$. According to Darboux’s theorem, in the case of contact geometry, the contact form can be written as

$$
\eta = dx^0 - \sum_{k=1}^{n} y^k dx^k, \tag{20}
$$

and the Reeb vector field defined by $\eta$ is

$$
R_\eta = \frac{\partial}{\partial x^0}. \tag{21}
$$

In the above adapted system of local coordinates, a vector field can be written as

$$
X = a_0 \frac{\partial}{\partial x^0} + \sum_{k=1}^{n} a_k \frac{\partial}{\partial x^k} + \sum_{k=1}^{n} b_k \frac{\partial}{\partial y^k}. \tag{22}
$$

According to eq. (11), a vector field $X_f$ is associated with a function

$$
f = a_0 - \sum_{k=1}^{n} a_k y^k, \tag{23}
$$

and from eq. (13) we get that

$$
\rho = \frac{\partial f}{\partial x^0}. \tag{24}
$$

On the other hand, from eq. (12) we get

$$
a_k = -\frac{\partial f}{\partial y^k}; \\
b_k = \frac{\partial f}{\partial x^k} + \frac{\partial f}{\partial x^0} y^k. \tag{25}
$$

Therefore a vector field $X_f = \Phi^{-1}(f)$ has in an local system of coordinates the form

$$
X_f = \left(f - y^k \frac{\partial f}{\partial y^k}\right) \frac{\partial}{\partial x^0} - \frac{\partial f}{\partial y^k} \frac{\partial}{\partial x^k} + \left(\frac{\partial f}{\partial x^k} + y^k \frac{\partial f}{\partial x^0}\right) \frac{\partial}{\partial y^k}
\right. \\
= f R_\eta + \eta^\sharp \left[ \left( \frac{\partial f}{\partial x^k} + \frac{\partial f}{\partial x^0} y^k \right) dx^k + \frac{\partial f}{\partial y^k} dy^k \right]. \tag{26}
$$

Here and in the sequel, we use the convention that repeated indices are summed over.

Finally, the Jacobi bracket of two functions $f$ and $g$ may be expressed as

$$
[f, g]_\eta = \left(f - y^k \frac{\partial f}{\partial y^k}\right) \frac{\partial g}{\partial x^0} - \left(g - y^k \frac{\partial g}{\partial y^k}\right) \frac{\partial f}{\partial x^0} + \left(\frac{\partial f}{\partial x^k} + y^k \frac{\partial f}{\partial x^0}\right) \frac{\partial g}{\partial y^k} - \left(\frac{\partial g}{\partial x^k} + y^k \frac{\partial g}{\partial x^0}\right) \frac{\partial f}{\partial y^k}. \tag{27}
$$
3 5-dimensional Sasaki-Einstein spaces

A contact Riemannian manifold $M$ equipped with a metric $g$ is Sasakian if its metric cone

$$(C(M), \tilde{g}) = (\mathbb{R}_+ \times M, d\rho^2 + r^2 g),$$

is Kähler. Here $r \in (0, \infty)$ may be considered as a coordinate on the positive real line $\mathbb{R}_+$. Moreover if the Sasaki manifold is Einstein

$$\text{Ric}_g = 2ng,$$

then the Kähler metric cone is Ricci flat ($\text{Ric}_{\tilde{g}} = 0$), i.e. a Calabi-Yau manifold.

3.1 Sasaki-Einstein space $T^{1,1}$

One of the most familiar examples of homogeneous toric Sasaki-Einstein 5-dimensional manifold is the space $T^{1,1}$. The metric on $T^{1,1}$ may be written down explicitly by utilizing the fact that it is a $U(1)$ bundle over $S^2 \times S^2$. We choose the coordinates $(\theta_i, \phi_i)$, $i = 1, 2$ to parametrize the two spheres $S^2$ in the standard way, while the angle $\psi \in [0, 4\pi)$ parametrizes the $U(1)$ fiber. Using these parametrizations the metric on $T^{1,1}$ may be written as \cite{14, 15}

$$ds^2(T^{1,1}) = \frac{1}{6} (d\theta_1^2 + \sin^2 \theta_1 d\phi_1^2 + d\theta_2^2 + \sin^2 \theta_2 d\phi_2^2)
+ \frac{1}{9} (d\psi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2)^2.$$  \hspace{1cm} (28)

In what follows we introduce $\nu = \frac{1}{4} \psi$ so that $\nu$ has canonical period $2\pi$.

The globally defined contact 1-form $\eta$ is:

$$\eta = \frac{1}{3} (2d\nu + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2),$$  \hspace{1cm} (29)

and the Reeb vector field $R_\eta$ has the form

$$R_\eta = \frac{3}{2} \frac{\partial}{\partial \nu}.$$ \hspace{1cm} (30)

We employ the basis \cite{15} for an effectively acting $T^3$ action

$$e_1 = \frac{\partial}{\partial \phi_1} + \frac{1}{2} \frac{\partial}{\partial \nu},$$

$$e_2 = \frac{\partial}{\partial \phi_2} + \frac{1}{2} \frac{\partial}{\partial \nu},$$  \hspace{1cm} (31)

$$e_3 = \frac{\partial}{\partial \nu},$$

which preserves the contact structure $\eta$.

As it was explained in Section 2.2 the effective action of the torus $T^3$ on the Sasaki-Einstein space $T^{1,1}$ is completely integrable. Let $\mathcal{F} = (f_0, f_1, f_2)$...
be the set of independent first integrals in involution and \( \mathcal{X} = (R_\eta, X_{f_1}, X_{f_2}) \) the corresponding set of infinitesimal automorphisms of \( \eta \). Let \( T \) be a compact connected component of the level set \( \{ f_1 = c_1, f_2 = c_2 \} \) and \( df_1 \wedge df_2 \neq 0 \) on \( T \). Then \( T \) is diffeomorphic to a \( T^3 \) torus. There exist a neighborhood \( U \) of \( T \) and a diffeomorphism \( \phi : U \to T^3 \times D \)

\[
\phi(x) = (\vartheta_0, \vartheta_1, \vartheta_2, y_1, y_2),
\]

where \( D \in \mathbb{R}^2 \), such that the contact form has the following canonical expression

\[
\eta_0 = (\phi^{-1})^* \eta = y_0 d\vartheta_0 + y_1 d\vartheta_1 + y_2 d\vartheta_2. \tag{33}
\]

We refer to the local coordinates \( (y_i, \vartheta_i) \) as \textit{generalized contact action-angle coordinates} \cite{9, 17}. Note that \( \eta_0(\frac{\partial}{\partial \vartheta_i}) = y_i \) are the contact Hamiltonians of the independent set of vector fields \( \mathcal{X} \). Let us remark that the action of the torus \( T^3 \) is given by translations of the angles \( \vartheta_i \).

Taking into account the 1-form \( \eta \) \cite{29} it is convenient to choose

\[
\vartheta_0 = \frac{2}{3} \nu, \quad \vartheta_1 = \phi_1, \quad \vartheta_2 = \phi_2, \tag{34}
\]

and accordingly we have

\[
y_0 = 1, \quad y_1 = \frac{1}{3} \cos \theta_1, \quad y_2 = \frac{1}{3} \cos \theta_2. \tag{35}
\]

These functions are first integrals of the Hamiltonian contact structure

\[
f_0 = y_0 \equiv 1, \quad f_i = y_i = \frac{1}{3} \cos \theta_i, \quad i = 1, 2, \tag{36}
\]

which are independent and in involution

\[
[f_i, f_j]_\eta = [f_i, f_j]_\eta = 0, \quad i, j = 1, 2, \tag{37}
\]

as can be seen through a direct evaluation of the respective Jacobi brackets \cite{27}.

The flows of the set \( \mathcal{X} \) on invariant tori is quasi-periodic

\[
(\vartheta_0, \vartheta_1, \vartheta_2) \to (\vartheta_0 + t\omega_0, \vartheta_1 + t\omega_1, \vartheta_2 + t\omega_2), \tag{38}
\]

where the frequencies \( \omega_i \) depend only on \( y_0 \).

In order to construct effectively the flow of \( X_f \) and find the frequencies \( \omega_i \) we define the family of 1-forms

\[
\eta_t = \eta_0 + tdf, \tag{39}
\]

where \( f \) is one of the first integrals of the Reeb vector field \( R_\eta \). We observe that \( \eta_t \) is a contact form also having the Reeb vector field \( R_\eta \). Following \cite{9} we consider the vector field \( X = -f R_\eta \) and let \( \phi_t \) the flow of this vector field.
Because $f$ is a first integral of the $T^3$ action, $\phi_t$ commutes with this action. Using the Moser’s deformation \[9, 18\] we have

$$\mathcal{L}(X)\eta_t = -df = -\frac{\partial\eta_t}{\partial t},$$

which imply

$$\frac{d}{dt}(\phi_t^*\eta_t) = \phi_t^*(\mathcal{L}(X)\eta_t + \frac{\partial\eta_t}{\partial t}) = 0.$$  (40)

Therefore $\phi_t^*\eta_t = \eta_0$ and we can obtain the coordinates in which the 1-form (39) has the canonical expression. In our case choosing the first integrals $f_i = y_i$ as in ec.(36), a simple calculation permits us to extract the frequencies:

$$\omega_i = \ln \cos \theta_i, \quad i = 1, 2.$$  (41)

### 3.2 Sasaki-Einstein space $Y^{p,q}$

The metric of the Sasaki-Einstein space $Y^{p,q}$ is given by the line element \[15\]

$$ds^2 = \frac{1-y}{6} (d\theta^2 + \sin^2 \theta \, d\phi^2) + \frac{1}{w(y)q(y)}dy^2 + \frac{q(y)}{9} (d\psi - \cos \theta \, d\phi)^2$$

$$+ w(y) \left[ d\alpha + \frac{a - 2y + y^2}{6(a - y^2)}(d\psi - \cos \theta \, d\phi) \right]^2,$$

where

$$w(y) = \frac{2(a - y^2)}{1 - y},$$

$$q(y) = \frac{a - 3y^2 + 2y^3}{a - y^2}.$$  (43)

A detailed analysis of the metric $Y^{p,q}$ \[19\] showed that it is globally well-defined and there are a countable infinite number of Sasaki-Einstein manifolds characterized by two relatively prime positive integers $p, q$ with $p < q$. For $0 < a < 1$ one can take the range of the angular coordinates $(\theta, \phi, \psi)$ to be $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$, while $y$ lies between the negative and the smallest positive zeros of $q(y)$. Finally, the period of $\alpha$ is chosen so as to describe a principal $S^1$ bundle over $B_4 = S^2 \times S^2$. For any $p$ and $q$ coprime, the space $Y^{p,q}$ is topologically $S^2 \times S^3$ and one may take \[15, 19\]

$$0 \leq \alpha \leq 2\pi \ell,$$  (45)

where

$$\ell = \frac{q}{3q^2 - 2p^2 + p(4p^2 - 3q^2)^{1/2}}.$$  (46)

The contact 1-form $\eta$ is \[15, 20\]

$$\eta = -2yda + \frac{1 - y}{3}(d\psi - \cos \theta d\phi),$$  (47)
The orbits of the Reeb vector field \( R_\eta \) may or may not close. The geometries \( Y^{p,q} \) with \( 4p^2 - 3q^2 \) a square are examples of quasi-regular manifolds for which the orbits of the Reeb vector field close corresponding to a locally free \( U(1) \) action on \( Y^{p,q} \). On the other hand, for \( 4p^2 - 3q^2 \) not a square the orbits of the Reeb vector field do not close generating an action \( \mathbb{R} \) on \( Y^{p,q} \), with the orbits densely filling the orbits of a torus and in this case the Sasaki-Einstein manifold is said to be irregular.

In relation to the angular variables \( \phi, \psi, \alpha \), the basis for an effectively acting \( T^3 \) action is [15, 21]

\[
\begin{align*}
e_1 &= \frac{\partial}{\partial \phi} + \frac{\partial}{\partial \psi}, \\
e_2 &= \frac{\partial}{\partial \phi} - \frac{(p-q)\ell}{2} \frac{\partial}{\partial \alpha}, \\
e_3 &= \ell \frac{\partial}{\partial \alpha}.
\end{align*}
\]

In order to put the contact form (47) and the Reeb vector field (48) in the canonical forms (20) and (21) we introduce the angle variables

\[
\begin{align*}
\vartheta_0 &= \frac{\psi}{3}, \quad \vartheta_1 = -6\alpha - \psi, \quad \vartheta_2 = \phi,
\end{align*}
\]

and the generalized action variables

\[
y_0 = 1, \quad y_1 = \frac{y}{3}, \quad y_2 = \frac{y - 1}{3} \cos \theta.
\]

These functions are first integrals of the Hamiltonian contact structure, independent and in involution.

The corresponding set of infinitesimal automorphisms is \( \mathcal{X} = (R_\eta, X_{y_1}, X_{y_2}) \).

\section{Concluding remarks}

An important point of interest in physics is to find the conserved quantities and investigate the integrability of the systems. Having in mind that Sasaki-Einstein manifolds have become of significant interest in many areas of physics, we investigated the integrability in the frame of contact geometry. For example, in string theory, in connection with AdS/CFT correspondence, a larger class models consists of type IIB string theory on the background \( AdS_5 \times M_5 \) with \( M_5 \) a five-dimensional Sasaki-Einstein space.
The analyses of integrability of geodesics of the five-dimensional Sasaki-Einstein spaces $T^{1,1}$ and $Y^{p,q}$ show that the geodesic motions in these spaces are completely integrable. The description of the integrability of geodesic in terms of action-angle variables gives a comprehensive understanding of dynamics. The presence of resonant frequencies gives way to chaotic behavior when the integrable Hamiltonian is perturbed by a small non-integrable piece. The action-angle approach offers strong support for the observation that certain classical string configurations in $AdS_5 \times M_5$ with $M_5$ in a large class of Einstein spaces is non-integrable.

In the present paper we move the analysis of integrability from the ten-dimensional phase space for the geodesic motions in Sasaki-Einstein spaces to the integrability in five-dimensional contact geometry. Unlike the symplectic case, the contact structures are automatically Hamiltonian. Moreover, for the $Y^{p,q}$ and $T^{1,1}$ toric manifolds, the torus action $T^3$ is effective and preserve the contact structure implying the complete integrability.

It is possible to introduce generalized action-angle variables which are similar to the ones in Hamiltonian dynamics. However the explicit construction of action-angle variables, as in the case of symplectic geometry, is more less equivalent to solving the equations. This is a quite difficult task and a general method does not exist, even the equations are integrable. In particular we used the Moser’s deformation of the contact forms and analyzed the flow of the Hamiltonian contact vector fields.

Motivated by the recent applications of contact geometry in some physical problems, the contact integrability deserves further studies. It would be interesting to extend the action-angle formulation for a better understanding of time-dependent and dissipative Hamiltonian systems.

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References

[1] C. P. Boyer, K. Galicki, Sasakian geometry, Oxford Mathematical Monographs, Oxford University Press, Oxford (2008).

[2] J. Sparks, Sasaki-Einstein manifolds, Surv. Diff. Geom. 16 (2011) 265–324; arXiv:1004.2461

[3] S. Ianus, M. Visinescu, G. E. Vilcu, Conformal Killing-Yano tensors on manifolds with mixed 3-structures SIGMA 5 (2009) 022; arXiv:0902.3968

[4] S. Ianus, M. Visinescu, G. E. Vilcu, Hidden symmetries on Euclideanised Kerr-NUT-(A)dS metrics in certain scaling limits SIGMA 8 (2012) 058; arXiv:1205.6036
[5] A. Bravetti, H. Cruz, D. Tapias, Contact Hamiltonian mechanics, Annals Phys. 376 (2017) 17–39; arXiv:1604.08266.

[6] A. Bravetti, C. S. Lopez-Monsalvo, F. Nettel, Contact symmetries and Hamiltonian thermodynamics, Annals Phys. 361 (2015) 377–400; arXiv:1409.7340.

[7] M. Grmela, Contact geometry of mesoscopic thermodynamics and dynamics, Entropy 16 (2014) 1652–1686.

[8] P. Libermann, Legendre foliation on contact manifolds, Differential Geometry and its Applications 1 (1991) 57–76.

[9] A. Banyaga, P. Molino, Géométrie des formes de contact complètement intégrables de type torique, Séminaire Gaston Darboux de Géométrie et Topologie Différentielle, 1991-1992 (Montpellier), Univ. Montpellier II, Montpellier (1993), pp. 1–25.

[10] B. Jovanović, Noncommutative integrability and action-angle variables in contact geometry, J. Symplectic Geom. 10 (2012) 535–561; arXiv:1103.3611.

[11] P. Libermann, Ch.-M. Marle, Symplectic geometry and analytical mechanics, Mathematics and its Applications, Vol. 35, D. Reidel Publishing Co., Dordrecht (1987).

[12] C. P. Boyer, Completely integrable contact Hamiltonian systems and toric contact structures on $S^2 \times S^3$, SIGMA 7 (2011) 058; arXiv:1101.5587.

[13] E. Lerman, Contact toric manifolds, J. Symplectic Geom. 1 (2003) 785–828; arXiv:math/0107201.

[14] P. Candelas, X. C. de la Ossa, Comments on conifolds, Nucl. Phys. B 342 (1990) 246–268.

[15] D. Martelli, J. Sparks, Toric geometry, Sasaki-Einstein manifolds and a new infinite class of AdS/CFT duals, Commun. Math. Phys. 262 (2006) 51–89; arXiv:hep-th/0411238.

[16] L. Banyaga, The geometry surrounding the Arnold-Liouville theorem, Progress in Mathematics, vol. 172, Advances in Geometry (J.-L. Brylinski, R. Brylinski, V. Nistor, B. Tsygan, P. Xu, eds.), Birkhäuser, Boston (1999).

[17] B. Jovanović, V. Jovanović Contact flows and integrable systems, J. Geom. Phys. 87 (2015) 217–232; arXiv:1212.2918.

[18] H. Geiges, Contact geometry, Handbook Diff. Geom., Vol. 2, (J. J. E. Dillen, L. C. A. Verstraelen, eds.), North-Holland, Amsterdam (2006), pp. 315–382; arXiv:math/0307242.
[19] J. P. Gauntlett, D. Martelli, J. Sparks, D. Waldram, \textit{Sasaki-Einstein metrics on $S^2 \times S^3$}, Adv. Theor. Math. Phys. 8 (2004) 711–734; arXiv:hep-th/0403002.

[20] M. Visinescu, \textit{Killing forms on the five-dimensional Einstein-Sasaki $Y(p,q)$ spaces}, Mod. Phys. Lett. A 27 (2012) 1250217; arXiv:1207.2581.

[21] V. Slesar, M. Visinescu, G. E. Vilcu, \textit{Toric data, Killing forms and complete integrability of geodesics in Sasaki-Einstein spaces $Y^{p,q}$}, Annals Phys. 361 (2015) 548–562; arXiv:1506.04483.

[22] V. Slesar, M. Visinescu, G. E. Vilcu, \textit{Hidden symmetries on toric Sasaki-Einstein spaces}, EPL 110 (2015) 31001.

[23] E. M. Babalic, M. Visinescu, \textit{Complete integrability of geodesic motion in Sasaki-Einstein toric $Y^{p,q}$ spaces}, Mod. Phys. Lett. A 30 (2015) 1550180; arXiv:1505.03976.

[24] M. Visinescu, \textit{Integrability of geodesics and action-angle variables in Sasaki-Einstein space $T^{1,1}$}, Eur. Phys. J. C 76 (2016) 498; arXiv:1604.03705.

[25] M. Visinescu, \textit{Action-angle variables for geodesic motions in Sasaki-Einstein spaces $Y^{p,q}$}, Prog. Theor. Exp. Phys. 2017 (2017) 013A01; arXiv:1611.01275.

[26] P. Basu, L. A. Pando Zayas, \textit{Chaos rules out integrability of strings on $AdS_5 \times T^{1,1}$}, Phys. Lett. B 700 (2011) 243–248; arXiv:1103.4107.