SPLINES ON THE ALFELD SPLIT OF A SIMPLEX
AND TYPE A ROOT SYSTEMS

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Abstract. In [1], Alfeld introduced a tetrahedral analog AS(∆₃) of the
Clough-Tocher split of a triangle. A formula for the dimension of the spline
space $C^r_k(AS(∆ₙ))$ is conjectured in [7]. We prove that the graded module of
$C^r_k$-splines on the cone over $AS(∆ₙ)$ is isomorphic to the module $D^{r+1}(Aₙ)$
of multiderivations on the type $Aₙ$ Coxeter arrangement. A theorem of Terao
shows that the module of multiderivations of a Coxeter arrangement is free
and gives an explicit basis. As a consequence the conjectured formula holds.

1. Introduction

The Alfeld split $AS(∆ₙ)$ of the $n$-simplex $∆ₙ$ is obtained by adding a single
central interior vertex, and then coning over the boundary of $∆ₙ$. Hence for $i > 0$,
every $i + 1$-face of $AS(∆ₙ)$ corresponds to an $i$-face of $∂(∆ₙ)$, and $AS(∆ₙ)$ has
a single interior vertex. This construction was introduced in [1] as a tetrahedral
analog of the Clough-Tocher split of a triangle, and [7] makes the following

Conjecture 1.1. [Foucart-Sorokina] The dimension of the space $C^r_k(AS(∆ₙ))$ of
splines of degree $\leq k$ in $n$-variables over the Alfeld split of $∆ₙ$ is given by

$$\dim C^r_k(AS(∆ₙ)) = \binom{k+n}{n} + \left( n^{\left(\frac{k+n-(r+1)(n+1)}{2}\right)} + \cdots \right), \quad \text{if } r \text{ is odd}$$

$$\left( n^{\left(\frac{k+n-1-(r+1)(n+1)}{2}\right)} + \cdots \right), \quad \text{if } r \text{ is even}.$$

In this brief note, we prove the conjecture. The proof hinges on a connection
between splines and the theory of hyperplane arrangements.

In [3], Billera showed that splines on a simplicial complex $Δ$ can be obtained as

the top homology module of a chain complex, and used this to solve a longstanding
conjecture of Gil Strang. Billera-Rose show in [4] that by forming the cone $\hat{Δ} \subseteq \mathbb{R}^{n+1}$

over $Δ \subseteq \mathbb{R}^n$, commutative algebra can be used to study the dimension of

$C^r_k(Δ)$: $C^r(Δ)$ is a graded module over the polynomial ring $S = \mathbb{R}[x_0, \ldots, x_n]$, and

the dimension of $C^r_k(Δ)$ is the dimension of the $k^{th}$ graded piece of $C^r(Δ)$.

A spectral sequence argument in [11] shows that $C^r(Δ)$ is a free module iff all

the lower homology modules of a certain chain complex vanish, and Corollary 4.12

of [11] shows that in this case the dimension of $C^r_k(Δ)$ is determined entirely by

local data. In [8], the algebraic geometry of fatpoints in projective space is used

to study the dimension of quotients of ideals generated by powers of linear forms.

Such modules appear in the chain complex whose top homology module is $C^r(Δ)$.

In §2 we give more details about these methods, and in §3 we show Conjecture 1.1

follows from a theorem of Terao [13]. The case $r = 1$ was recently proved in [9].

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2. Algebraic preliminaries

Throughout the paper, our references are de Boor [5] for splines and Eisenbud [6] for commutative algebra.

**Definition 2.1.** [11] For a simplicial complex \( \Delta \subseteq \mathbb{R}^n \), let \( \mathcal{R}/J(\Delta) \) be the complex of \( S = \mathbb{R}[x_0, \ldots, x_n] \) modules, with differential \( \partial_i \) the usual boundary operator in relative (modulo boundary) homology.

\[
0 \rightarrow \bigoplus_{\sigma \in \Delta_n} S \xrightarrow{\partial_{\sigma}} \bigoplus_{\tau \in \Delta_n^{0}} S/I_{\tau} \xrightarrow{\partial_{\tau}} \bigoplus_{\psi \in \Delta_n^{0-1}} S/I_{\psi} \xrightarrow{\partial_{\psi}} \cdots \xrightarrow{\partial_{\psi}} \bigoplus_{\nu \in \Delta_n^{0-2}} S/I_{\nu} \rightarrow 0,
\]

where for an interior \( i \)-face \( \gamma \in \Delta_n^0 \), we define

\[
I_{\gamma} = \langle l_{r+1}^r, \ldots, l_{n-i}^{r+1}, (x_1 + \cdots + x_{n-i})^{r+1} \rangle.
\]

This differs from the complex in [3]: the ideal \( I_{\gamma} \) is generated by \( r+1 \)st powers of (homogenizations of) linear forms whose vanishing defines the hyperplanes \( \tau \) which contain \( \gamma \). As with the complex in [3], the top homology module of \( \mathcal{R}/J(\Delta) \) computes splines of smoothness \( r \) on \( \hat{\Delta} \). Theorem 4.10 of [11] shows that if \( \Delta \) is a topological \( n \)-ball, then the module \( C^r(\hat{\Delta}) \) is free iff \( H_i(\mathcal{R}/J(\Delta)) = 0 \) for all \( i < n \).

Computing the dimension of the modules which appear in the complex \( \mathcal{R}/J(\Delta) \) is nontrivial as soon as \( n \geq 3 \). Indeed, for ideals generated by powers of linear forms in three variables, no answer is known, even if the forms are in general position. This is due to a classical connection from algebraic geometry known as Macaulay inverse systems, which translates questions about powers of linear forms into questions about fatpoints, which are points, with multiplicity, in projective space. For more details on inverse systems, see [8]. In fact, for powers of linear forms in three variables, inverse systems translate to questions about fatpoints in \( \mathbb{P}^2 \), which is the topic of the famous (and open) Segre-Harbourne-Gimigliano-Hirschowitz conjecture [10].

In the case of the Alfeld split of \( \Delta_n \), it is easy to see that at every interior \( i \)-face there are exactly \( n-i+1 \) forms incident to the face. This means that after a change of variables, we can assume that the ideals which appear are of the form

\[
I_{\gamma} = \langle l_{r+1}^i, \ldots, l_{n-i}^{r+1}, (x_1 + \cdots + x_{n-i})^{r+1} \rangle
\]

For these ideals, the corresponding fatpoint scheme consists of the coordinate points and the point \((1:1: \cdots :1)\), and the dimension of \( I_{\gamma} \) in any degree is as expected, in the sense of [10]. As a consequence, we have

**Lemma 2.2.** Conjecture 1.1 holds if \( C^r(\hat{\mathcal{A}S(\Delta_n)}) \) is a free \( S \)-module.

**Proof.** By Theorem 4.10 of [11], \( C^r(\hat{\mathcal{A}S(\Delta_n)}) \) is free iff \( H_i(\mathcal{R}/J) = 0 \) for all \( i < n \), and in this case, Corollary 4.12 of [11] shows that the dimension of \( C^r_k(\hat{\mathcal{A}S(\Delta_n)}) \) is the alternating sum of the dimensions of the degree \( k \) graded components of the modules appearing in the chain complex \( \mathcal{R}/J \). These are the dimensions of the ideals in Equation (1), which can be worked out using inverse systems. \( \square \)

Conjecture 1.1 could hold even if \( C^r(\hat{\mathcal{A}S(\Delta_n)}) \) is not free: contributions from nonzero \( H_i(\mathcal{R}/J) \) could cancel out. However, in the next section, we will see that Conjecture 1.1 is a consequence of a stronger fact: \( C^r(\hat{\mathcal{A}S(\Delta_n)}) \) is a free module.
3. Conjecture 1.1 via Arrangement Theory

In [4], Billera-Rose note that \(C^r(\hat{\Delta})\) may be computed as the kernel of the matrix

\[
\phi = \begin{bmatrix}
\partial_n & l_{r1}^n & \cdots & l_{rn}^n
\end{bmatrix},
\]

where \(\partial_n\) is the \(n^{th}\) simplicial boundary map in relative homology; each row encodes the smoothness condition across a codimension one face \(\tau_i\). The key observation needed for the proof of Conjecture 1.1 relates the matrix above to a matrix which appears in hyperplane arrangement theory, and is motivated by a result of [12], where it appears in a different guise. First, we need some definitions. A hyperplane arrangement is a collection of hyperplanes

\[\mathcal{A} = \bigcup_{i=1}^{d} H_i \subseteq \mathbb{K}^n,\]

where \(\mathbb{K}\) is typically \(\mathbb{R}\) or \(\mathbb{C}\). An arrangement is central if all hyperplanes \(H_i\) pass thru the origin, which is true iff for every \(H_i = V(l_i)\), the linear form \(l_i\) is homogeneous. Given a simplicial complex \(\Delta \subseteq \mathbb{R}^n\), by construction the codimension one faces of the cone \(\hat{\Delta}\) yield a central arrangement.

**Remark 3.1.** Embedding \(\Delta \subseteq \mathbb{R}^n\) in the hyperplane \(x_{n+1} = 1 \subseteq \mathbb{R}^{n+1}\) and coning with the origin yields the complex \(\hat{\Delta}\). The set of splines on \(\hat{\Delta}\) has the additional structure of being a graded \(\mathbb{R}[x_0, \ldots, x_n]\) module. Homogenizing splines in \(C_k^r(\Delta)\) using the variable \(x_{n+1}\) yields the formula \(\dim C_k^r(\Delta) = \dim C_k^r(\hat{\Delta})\). The point is that coning is necessary in order to apply the tools of commutative algebra.

One of the main algebraic objects associated to a central arrangement is the graded \(S\)–module \(D(\mathcal{A})\) of vector fields tangent to \(\mathcal{A}\). Recall \(\text{Der}_\mathbb{R}(S)\) is the free \(S\)–module with basis \(\partial/\partial(x_i)\).

**Definition 3.2.** The module of derivations of \(\mathcal{A}\) is

\[D(\mathcal{A}) = \{\theta \mid \theta(l_i) \in \langle l_i \rangle \text{ for all } V(l_i) \in \mathcal{A}\} \subseteq \text{Der}_\mathbb{R}(S),\]

and the module of \(m\)–multiderivations of \(\mathcal{A}\) is

\[D^m(\mathcal{A}) = \{\theta \mid \theta(l_i) \in \langle l_i^m \rangle \text{ for all } V(l_i) \in \mathcal{A}\} \subseteq \text{Der}_\mathbb{R}(S).\]

An arrangement \(\mathcal{A}\) is \(m\)–multifree if \(D^m(\mathcal{A}) \simeq \oplus S(-d_i)\). In this case the \(d_i\) are called the \(m\)–multieponents of \(\mathcal{A}\).

It follows from the definition that the \(m\)–multiderivations can be computed as the kernel of a matrix similar to \(\phi\) above: \(D^m(\mathcal{A})\) is the kernel of

\[
\psi = \begin{bmatrix}
\delta_n & l_1^m & \cdots & l_d^m
\end{bmatrix},
\]

where \(\delta_n\) is a \(d \times n\) matrix, such that if \(l_j = \sum a_j^i x_i\), then the \(j^{th}\) row of \(\delta_n\) is \([a_1^j, \ldots, a_d^j]\). To relate \(AS(\Delta_n)\) to arrangements, we need the following:

**Definition 3.3.** The reflection arrangement of type \(A_n\) is defined by the reflecting hyperplanes of \(\text{SL}(n)\), which are given by \(\bigcup_{1 \leq i < j \leq n+1} V(x_j - x_i) = \mathcal{A}_n \subseteq \mathbb{R}^{n+1}\).
Theorem 3.4 (Terao, [13]). $D^m(A_n)$ is $m$-multifree for all $m$, with exponents
\[
\begin{aligned}
\left\{ \frac{m(n+1)}{2}, \ldots, \frac{m(n+1)}{2} \right\} & \text{ if } m \text{ is even} \\
\left\{ \frac{(m-1)(n+1)}{2} + 1, \ldots, \frac{(m-1)(n+1)}{2} + n \right\} & \text{ if } m \text{ is odd.}
\end{aligned}
\]

Terao proves the theorem for all reflection groups, not just those of type $A$. For the free $S = \mathbb{R}[x_0, \ldots, x_n]$ module $S(-a)$ generated in degree $a$,
\[
\dim_S S(-a)_k = \binom{n - a + k}{n}.
\]
Terao’s formula does not include the constant derivation, which corresponds to the constant splines contributing the term $\binom{k+n}{n}$ in Conjecture 1.1. So it follows from Terao’s theorem that the dimension of $D^{r+1}(A_n)_k$ is equal to the formula of Conjecture 1.1. Before we prove the theorem, we give a simple motivating example.

Example 3.5. Let $\Delta_2$ have vertices $\{e_0 = (0,0), e_1 = (1,2), e_2 = (2,1)\}$, and place a central vertex at $(1,1)$. Order the triangles of $\Delta_2$ as $[e_0,e_1], [e_0,e_2], [e_1,e_2]$ and the rays as $e_0, e_1, e_2$. Note that the linear forms vanishing on the rays are $x_1 - x_2, x_1 - 1, x_2 - 1$. Homogenizing with respect to $x_3$, we obtain
\[
\phi = \begin{bmatrix}
-1 & 1 & 0 & (x_1 - x_2)^{r+1} & 0 & 0 \\
1 & 0 & -1 & 0 & (x_1 - x_3)^{r+1} & 0 \\
0 & -1 & 1 & 0 & 0 & (x_2 - x_3)^{r+1}
\end{bmatrix}
\]
On the other hand, the matrix computing $D^{r+1}(A_2)$ is
\[
\psi = \begin{bmatrix}
1 & -1 & 0 & (x_1 - x_2)^{r+1} & 0 & 0 \\
1 & 0 & -1 & 0 & (x_1 - x_3)^{r+1} & 0 \\
0 & 1 & -1 & 0 & 0 & (x_2 - x_3)^{r+1}
\end{bmatrix}
\]
The signs in first and third rows of the matrices differ, but this can be fixed by reversing the orientation of $e_0$ and $e_2$. So for the choice of vertices above
\[
C^r(AS(\Delta_2)) \simeq D^{r+1}(A_2).
\]

Theorem 3.6. $C^r(AS(\Delta_n)) \simeq D^{r+1}(A_n)$.

Proof. In the matrix for $\psi$, the rows are indexed by pairs $\{i,j\}$ with $1 \leq i < j \leq n+1$, and the columns are indexed by $m \in \{1, \ldots, n+1\}$. In the row of $\delta_n$ indexed by $\{i,j\}$, there is a $+1$ in column $i$ and a $-1$ in column $j$.

Choose vertices for $AS(\Delta_n)$ as follows: for $i \in \{1, \ldots, n\}$, let $e_i$ be the $i^{th}$ standard basis vector of $\mathbb{R}^n$, and define
\[
\begin{aligned}
\mathbf{v} &= \sum_{i=1}^n e_i \\
e_0 &= 0 \\
e_i &= \mathbf{v} + \epsilon_i, \quad i \in \{1, \ldots, n\}
\end{aligned}
\]
Let $\Delta_n = conv\{e_0, \ldots, e_n\}$, and let $\mathbf{v}$ be the interior vertex of $AS(\Delta_n)$. In the matrix $\phi$, the component $\partial_n$ also has a single $+1$ and $-1$ in each row. The columns are indexed by the maximal faces of $\Delta_n$. Since $\Delta_n$ has $n+1$ vertices, each maximal face is determined by leaving out a single vertex. The rows of $\phi$ are indexed by the codimension one faces, which are determined by leaving out a pair $\{i,j\}$ of vertices.

Index the columns as $\overline{e_i}, \ldots, \overline{e_1}, \overline{e_0}$, and the rows as $\overline{e_n}, \overline{e_{n-1}}, \ldots, \overline{e_0}$. Then in the row indexed by $\overline{e_i} \overline{e_j}$ with $i < j$, there will be a $+1$ in the column indexed by $\overline{e_j}$, and a $-1$ in the column indexed by $\overline{e_i}$, up to multiplying by $-1$, which does not change the smoothness condition across faces. In particular, the $\partial_n$ block of
\( \phi \) agrees with the \( \delta_n \) block of \( \psi \). We need only check that the equations of the hyperplanes agree, which follows from our choice of vertex locations of \( \text{AS}(\Delta)_n \).

To be explicit, let

\[
M = \begin{bmatrix}
  x_1 & \vdots & \vdots \\
  x_2 & v & e_0 & e_1 & \cdots & e_n \\
  \vdots & \vdots & \vdots \\
  x_{n+1} & 1 & 1 & 1 & \cdots & 1
\end{bmatrix}
\]

If \( L_{ij} \) is the facet which misses \( e_i \) and \( e_j \) and \( M_{ij} \) is the matrix obtained from \( M \) by deleting the columns containing \( e_i \) and \( e_j \), then then \( L_{ij} = \det(M_{ij}) \). Since any two sets of \( n + 2 \) points in general position in \( \mathbb{P}^n \) are projectively equivalent, the result is in fact independent of the choice of \( v, e_0, \ldots, e_n \).

\[\square\]

**Example 3.7.** In the case of Example 3.5, the matrix \( M \) is

\[
\begin{bmatrix}
  x_1 & 1 & 0 & 1 & 2 \\
  x_2 & 1 & 0 & 2 & 1 \\
  x_3 & 1 & 1 & 1 & 1
\end{bmatrix}
\]

so for example \( L_{01} = \det \begin{bmatrix} x_1 & 1 & 2 \\ x_2 & 1 & 1 \\ x_3 & 1 & 1 \end{bmatrix} = x_2 - x_3 \),

which is the equation of the line \( L_{01} \) connecting \( v \) and \( e_2 \), as expected. Note that combining Theorem 3.6 with Terao’s theorem yields

**Corollary 3.8.** \( C^r(\text{AS}(\Delta_n)) \) is a free \( S \)-module, and Conjecture 1.1 holds.

**Closing Remarks:** Macaulay2 computations were essential to our work; we are investigating extensions of the results here to other types. The Macaulay2 package is available at [http://www.math.uiuc.edu/Macaulay2](http://www.math.uiuc.edu/Macaulay2).

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