BLACK HOLES, NEWTONIAN SCATTERING AND
CONFORMAL FIELD THEORY

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We discuss non-relativistic scattering by a Newtonian potential. We show that
the gray-body factors associated with scattering by a black hole exhibit the same func-
tional dependence as scattering amplitudes in the Newtonian limit, which should
be the weak-field limit of any quantum theory of gravity. This behavior arises in-
dependently of the presence of supersymmetry. The connection to two-dimensional
conformal field theory is also discussed.

1 Introduction

Although the thermodynamic properties of black holes have been understood
for some time
1,2,3,4,5,6,7,8,9,10,11,12,13,14,15
for some time, their microscopic origin was only recently illuminated. This was achieved with the aid of (super)string theory, where one should be
able to count the fundamental degrees of freedom and arrive at an expression
for the entropy
5,11. The semi-classical result (Bekenstein-Hawking entropy)
was thus confirmed. The degrees of freedom turned out to be solitonic states
(D-branes
10) and not fundamental strings at all. One could then construct
an effective theory by expanding around these solitons
5,11. The result was
a (super)conformal field theory whose validity extended in the domain of M-
theory.

It was later discovered that this effective conformal field theory was more
robust than originally expected. Indeed, not only did it give an accurate pre-
diction for the entropy, but also for the so-called gray-body factors
12,13,14,15. This was rather surprising, because there is no apparent connection between
the semi-classical derivation of these factors and the corresponding analysis in
conformal field theory. The two theories (Einstein-Maxwell gravity and super-
string theory) appear to share common properties, pointing to the existence
of a yet-to-be-discovered underlying principle on which to build a quantum
theory of gravity (possibly unified with all the other forces). In our quest for

\(^\text{a}\) Invited presentation at the Arnowitt Fest, Texas A&M University, April 1998; to appear in the Festschrift Volume "Relativity, Particle Physics, and Cosmology," published by World Scientific.
such a theory, it is important that we derive expressions for physical quantities (entropy, scattering amplitudes, etc) under as broad assumptions as possible. What is of interest is the qualitative behavior of physical quantities, since the fundamental theory is not known yet.

In this spirit, we consider the weak-field limit of Einstein gravity, which is, of course, Newtonian mechanics. We show that the scattering amplitudes in this non-relativistic limit exhibit the same behavior as the gray-body factors one obtains in the black-hole background. We conclude that this behavior is more generic than black-hole backgrounds that can be obtained from D-branes. This sheds some light on the origin of the gray-body factors, but offers no explanation on their similarity to factors obtained from (super)conformal field theory. It would be interesting to see if there is a more direct connection between non-relativistic scattering and conformal field theory.

Our discussion is organized as follows. In Section 2, we derive the gray-body factors in a black-hole background. We perform the calculation in four dimensions (Schwarzschild metric) and five dimensions (Kaluza-Klein black holes). In Section 3, we calculate scattering amplitudes for a Newtonian potential and show the similarities of the results with the gray-body factors. Finally, in Section 4, we discuss our results and their connection to conformal field theory.

2 Black holes

In this Section, we calculate the gray-body factors associated with Schwarzschild and Kaluza-Klein black holes. We follow the discussion of Maldacena and Strominger[13].

2.1 Schwarzschild black holes

The metric for a Schwarzschild black hole is (we set $c = h = 1$, but keep $G$)

$$ds^2 = -\left(1 - \frac{r_0}{r}\right)dt^2 + \left(1 - \frac{r_0}{r}\right)^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2$$

(1)

where $r_0 = 2GM$ is the radius of the horizon. We wish to study the wave equation for a massless scalar $\Psi$,

$$\Box\Psi = 0$$

(2)

Using separation of variables, we write $\Psi = R(r)\Theta(\theta)\Phi(\phi)e^{-i\omega t}$. The angular part is the same as in the case of the non-relativistic Schrödinger Equation.
with a central potential. The radial equation is

$$-\frac{1}{r^2} \left(1 - \frac{r_0}{r}\right) \frac{d}{dr} \left(r(r - r_0) \frac{dR_\ell}{dr}\right) + \frac{\ell(\ell + 1)}{r^2} \left(1 - \frac{r_0}{r}\right) R_\ell = k^2 R_\ell$$  \(3\)

where \(k = \omega\) is the wavenumber of the massless scalars. A massive scalar is described by the same equation, but in that case, \(k = \sqrt{\omega^2 - m^2}\).

To solve this equation, we consider the two asymptotic limits away from the horizon \((r \to \infty)\) and near the horizon \((r \to r_0)\), respectively. We solve the wave equation in these two limits and then match the solutions.

Away from the horizon \((r \to \infty)\), Eq. \((3)\) becomes

$$-\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dR_\ell}{dr}\right) + \frac{\ell(\ell + 1)}{r^2} R_\ell = k^2 R_\ell$$  \(4\)

whose solution is given in terms of a Bessel function,

$$R_\ell = \frac{A_\ell}{\sqrt{r}} J_{\ell+\frac{1}{2}}(kr)$$  \(5\)

where we discarded the solution which is singular in the limit \(r \to 0\). At infinity, this behaves as \(R_\ell \sim \frac{A_\ell}{\sqrt{r}} \sqrt{\frac{2}{kr}} \sin(kr - \ell \pi/2)\). The incoming wave is \(\Psi_{\text{in}} = \frac{\ell+1}{2} A_\ell \sqrt{\frac{2}{kr}} e^{-ikr}\), so the incoming flux is

$$J_{\text{in}} = -2\pi i \left(\Psi_{\text{in}}^* \frac{d\Psi_{\text{in}}}{dr} - \text{c.c.}\right) = 2|A_\ell|^2$$  \(6\)

Normalizing to unity, we obtain \(A_\ell = 1/\sqrt{2}\). Therefore, Eq. \((5)\) becomes

$$R_\ell = \frac{1}{\sqrt{2r}} J_{\ell+\frac{1}{2}}(kr)$$  \(7\)

In the limit \(r \to 0\), we have

$$R_\ell \sim \frac{1}{\Gamma(\ell + \frac{1}{2})\sqrt{2r}} \left(\frac{kr}{2}\right)^{\ell+\frac{1}{2}}$$  \(8\)

Near the horizon, define a new variable \(\xi = 1 - \frac{r_0}{r}\). The horizon is at \(\xi = 0\). In terms of \(\xi\) and in the small \(\xi\) limit, Eq. \((3)\) becomes

$$-\xi \frac{d}{d\xi} \left(\xi \frac{dR_\ell}{d\xi}\right) + \frac{\ell(\ell + 1) \xi}{(1-\xi)^2} R_\ell = (k r_0)^2 R_\ell$$  \(9\)
To solve this equation, define $R_\ell = e^{ikr_0 \ln \xi (1 - \xi)^{\ell + 1}} f_\ell(\xi)$. Then we obtain for $f_\ell$:

$$\xi (1 - \xi) f_\ell'' + (1 + 2ikr_0 - (2\ell + 3 + 2ikr_0)\xi) f_\ell' - (\ell + 1)(\ell + 1 + 2ikr_0) f_\ell = 0 \quad (10)$$

whose solution is given in terms of a hypergeometric function,

$$f_\ell(\xi) = B_\ell \, 2F_1(\ell + 1 + 2ikr_0, \ell + 1; 1 + 2ikr_0; \xi) \quad (11)$$

where

$$2F_1(a, b; c; z) = 1 + \frac{ab}{c} z + \frac{a(a + 1)b(b + 1)}{c(c + 1)} \frac{z^2}{2!} \cdots \quad (12)$$

and we have discarded the solution which is not regular at $\xi = 0$. Near the horizon ($\xi \to 0$), we have

$$R_\ell \sim B_\ell e^{ikr_0 \ln \xi},$$

so the flux at the horizon is

$$f_h = -2\pi i r (r - r_0) \left( R_\ell^* \frac{dR_\ell}{dr} - \text{c.c.} \right)$$

$$= -2\pi i r_0 \left( R_\ell^* \frac{dR_\ell}{d\xi} - \text{c.c.} \right) = 4\pi r_0^2 k |B_\ell|^2 \quad (13)$$

In the limit $\xi \to 1$ (large $r$), we obtain

$$R_\ell \sim B_\ell \, \frac{\Gamma(2\ell + 1)\Gamma(1 + 2ikr_0)\Gamma(\ell + 1 + 2ikr_0)}{\Gamma(\ell + 1)\Gamma(\ell + 1 + 2ikr_0)} \Gamma(1 - \xi)^\ell$$

$$= B_\ell \, \frac{\Gamma(2\ell + 1)\Gamma(1 + 2ikr_0)\Gamma(\ell + 1 + 2ikr_0)}{\Gamma(\ell + 1)\Gamma(\ell + 1 + 2ikr_0)} \left( \frac{r}{r_0} \right)^\ell \quad (14)$$

Matching the two asymptotic forms (Eqs. (8) and (14)), we obtain

$$B_\ell = \frac{\Gamma(\ell + 1)\Gamma(\ell + 1 + 2ikr_0)\Gamma(2\ell + 1)\Gamma(1 + 2ikr_0)}{\Gamma(\ell + 1)\Gamma(\ell + 1 + 2ikr_0)\Gamma(\ell + 1 + 2ikr_0)\Gamma(\ell + 1 + 2ikr_0)} \frac{\sqrt{k}}{2} \left( \frac{kr_0}{2} \right)^\ell \quad (15)$$

The gray-body factors (decay rates at the horizon) are given by

$$\Gamma_\ell = \frac{\pi f_h^2}{k^2 e^{4\pi kr_0} - 1}$$

$$= \frac{\pi(\Gamma(\ell + 1))^2}{2^{2\ell + 2}(\Gamma(\ell + \frac{1}{2}))^2(\Gamma(2\ell + 1))^2} k^{2\ell - 1} r_0^{2\ell + 1} e^{-2\pi k r_0} |\Gamma(\ell + 1 + 2ikr_0)|^2 \quad (16)$$

They may also be written in terms of the Hawking temperature, $T_H = \frac{1}{4\pi r_0^2}$, and horizon area $A = 4\pi r_0^2$:

$$\Gamma_\ell = \frac{\pi(\Gamma(\ell + 1))^2}{2^{2\ell + 2}(\Gamma(\ell + \frac{1}{2}))^2(\Gamma(2\ell + 1))^2} \times k^{2\ell - 1} (T_H A)^{2\ell + 1 + 2k/(2\pi T_H)} |\Gamma(\ell + 1 + ik/(2\pi T_H))|^2 \quad (17)$$
2.2 Kaluza-Klein black holes

Consider five-dimensional space-time with an internal periodic fifth dimension $x_5$. The metric in five dimensions can be split into a four-dimensional metric $g_{\mu\nu}$, gauge field $A_\mu$ and scalar $\chi$, with dynamics governed by the action:

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left( R - 2\partial_{\mu}\chi \partial^{\mu}\chi - e^{-2\sqrt{3}\chi} F_{\mu\nu} F^{\mu\nu} \right)$$  \hspace{1cm} (18)

The momentum along $x_5$ gives rise to a charge. Thus, we obtain charged black hole solutions with four-dimensional metric,

$$ds^2 = -\frac{1}{\sqrt{\Delta}} \left( 1 - \frac{r_0}{r} \right) dt^2 + \sqrt{\Delta} \left( \frac{dr^2}{1 - r_0/r} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right)$$  \hspace{1cm} (19)

where $\Delta = e^{-4\chi/\sqrt{3}} = 1 + \frac{r_0}{r} \sinh^2 \gamma$, and gauge field $A_0 = -\frac{r_0}{4r\Delta} \sinh(2\gamma)$. The ADM mass, charge, entropy and Hawking temperature of the black hole, respectively, are

$$M = \frac{r_0}{8G} \left( 3 + \cosh(2\gamma) \right) , \quad Q = \frac{r_0}{4G} \sinh(2\gamma)$$

$$S = \frac{\pi r_0^2}{G} \cosh \gamma , \quad T_H = \frac{1}{4\pi r_0 \cosh \gamma}$$  \hspace{1cm} (20)

The scattering of a neutral massless scalar is described by the wave equation

$$\Delta \frac{\partial^2 \Psi}{\partial t^2} - \frac{1}{r^2} \left( 1 - \frac{r_0}{r} \right) \frac{d}{dr} \left( r(r - r_0) \frac{d\Psi}{dr} \right) + \frac{1}{r^2} \left( 1 - \frac{r_0}{r} \right) L^2 \Psi = 0$$  \hspace{1cm} (21)

The eigenvalues of $L^2$ are $\ell(\ell + 1)$, so the radial part of the wavefunction for scalars of energy $\omega = k$ satisfies

$$-\frac{1}{r^2} \left( 1 - \frac{r_0}{r} \right) \frac{d}{dr} \left( r(r - r_0) \frac{dR_\ell}{dr} \right) + \frac{\ell(\ell + 1)}{r^2} \left( 1 - \frac{r_0}{r} \right) R_\ell = k^2 \Delta R_\ell$$  \hspace{1cm} (22)

Working as before, we solve this equation in the two asymptotic limits, away from the horizon ($r \to \infty$) and near the horizon ($r \to r_0$), respectively. We then match the two solutions.

Away from the horizon, we obtain the same form (4) as before. Near the horizon, we obtain (after the change of variables $\xi = 1 - \frac{r_0}{r}$),

$$-\xi \frac{d}{d\xi} \left( \xi \frac{dR_\ell}{d\xi} \right) + \frac{\ell(\ell + 1)}{(1 - \xi)^2} R_\ell = (kr_0)^2 \cosh^2 \gamma R_\ell$$  \hspace{1cm} (23)
which only differs from Eq. (9) by the substitution $r_0 \to r_0 \cosh \gamma$. Therefore, the gray-body factors are (cf. Eq. (16))

$$
\Gamma_\ell = \frac{\pi (\Gamma(\ell + 1))^2}{2^{2\ell+2}(\Gamma(\ell + \frac{3}{2}))^2(\Gamma(2\ell + 1))^2} \times k^{2\ell-1}\frac{2^{\ell+1}(\cosh \gamma)^{2\ell+1}e^{-2\pi kr_0 \cosh \gamma} \Gamma(\ell + 1 + 2ikr_0 \cosh \gamma)}{\Gamma(\ell + 1 + 2ikr_0 \cosh \gamma)}
$$

(24)

It terms of the Hawking temperature (Eq. (20)), we obtain

$$
\Gamma_\ell \sim k^{2\ell-1}e^{-k/(2T_H)}\Gamma(\ell + 1 + ik/(2\pi T_H))^2
$$

(25)

which is of the same form as Eq. (17).

This can be generalized to a four-dimensional black hole obtained from string theory. To do this, we start with ten-dimensional spacetime and compactify the six dimensions on a torus. The four-dimensional metric is given by Eq. (19), where

$$
\Delta = f(\gamma_1)f(\gamma_2)f(\gamma_3)f(\gamma_4), \quad f(\gamma_i) = 1 + \frac{r_0 \sinh^2 \gamma_i}{r}
$$

(26)

The parameters $\gamma_i$ ($i = 1, \ldots, 4$) are related to the charges of the black hole. The ADM mass, entropy and Hawking temperature of the black hole, respectively, are (cf. Eq. (20))

$$
M = \frac{r_0}{8G} \sum_{i=1}^{4} \cosh(2\gamma_i), \quad S = \frac{\pi r_0^2}{G} \prod_{i=1}^{4} \cosh \gamma_i, \quad T_H = \frac{1}{4\pi r_0} \prod_{i=1}^{4} \frac{1}{\cosh \gamma_i}
$$

(27)

Working as before, we arrive at the same results provided we substitute $r_0 \to r_0 \cosh \gamma_1 \cosh \gamma_2 \cosh \gamma_3 \cosh \gamma_4$. Once again, the gray-body factor is of the same form (25), where $T_H$ is given by Eq. (27).

Next, we consider the non-relativistic limit of Newtonian scattering. Even though there is no horizon present, the scattering amplitudes exhibit the same behavior near the center of gravity.

### 3 Newtonian scattering

Consider a heavy body of mass $M$ (e.g. the black hole of the previous Section) and an incident beam of light particles of reduced mass $m$. The particles have (non-relativistic) relative speed $v = k/m$ in the $z$-direction. Their scattering by the heavy body is described by the non-relativistic Schrödinger Equation

$$
- \frac{1}{2m} \nabla^2 \Psi + \frac{GMm}{r} \Psi = E\Psi
$$

(28)
where \( E = \frac{k^2}{2m} \). This Equation can be solved exactly for the given boundary conditions, by using parabolic coordinates. Normalizing the incident flux to unity, we obtain

\[
\Psi = \frac{1}{\sqrt{v}} \Gamma(1 + i\eta) e^{-\pi\eta/2} e^{ikz} F(-i\eta, 1; 2ikr \sin^2 \frac{1}{2}\theta) \tag{29}
\]

where \( \eta = \frac{GMm^2}{k} \) and \( F \) is the hypergeometric function

\[
F(a, b; z) = 1 + \frac{a}{b} z + \frac{a(a+1)}{b(b+1)} \frac{z^2}{2!} + \frac{a(a+1)(a+2)}{b(b+1)(b+2)} \frac{z^3}{3!} + \cdots \tag{30}
\]

This solution can be expanded in partial waves,

\[
\Psi = \sum_{\ell=0}^{\infty} R_{\ell}(r) P_{\ell}(\cos \theta) \tag{31}
\]

where the \( R_{\ell} \) satisfy the Radial Equation

\[
-\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dR_{\ell}}{dr} \right) + \left( \frac{2\eta}{r} + \frac{\ell(\ell+1)}{r^2} \right) R_{\ell} = k^2 R_{\ell} \tag{32}
\]

We obtain

\[
R_{\ell}(r) = \frac{1}{\sqrt{v}} \frac{\Gamma(\ell + 1 + i\eta)}{\Gamma(2\ell + 1)} e^{-\pi\eta/2} (2ikr)^{\ell} e^{ikr} F(\ell + 1 + i\eta, 2\ell + 2; -2ikr) \tag{33}
\]

Close to the center of gravity \((r \to 0)\), we obtain

\[
|R_{\ell}|^2 \sim \frac{1}{v} \frac{\Gamma(\ell + 1 + i\eta)^2}{(\Gamma(2\ell + 1))^2} (2\eta)^{2\ell} e^{-\pi\eta} \frac{2\eta}{e^{2\pi\eta} - 1} \tag{34}
\]

In particular, the particle density at the center of gravity is found from Eq. (29), if we set \( r = 0 \),

\[
|\Psi(0)|^2 = \frac{1}{\sqrt{v}} \frac{\Gamma(1 + i\eta)^2}{\Gamma(2\ell + 1)^2} e^{-\pi\eta} = \frac{1}{v} \frac{2\pi\eta}{e^{2\pi\eta} - 1} \tag{35}
\]

This may be viewed as blackbody spectrum of the wavenumber \( k \) at temperature

\[
T = \frac{k}{2\pi\eta} = \frac{v^2}{2\pi GM} \tag{36}
\]
The ensemble consists of particles of varying masses and wavenumbers, but of constant incoming speed. If we express the partial-wave rates (Eq. (34)) in terms of this temperature, we obtain

$$|R_\ell|^2 = m^2 \frac{2^{2\ell}}{(\Gamma(2\ell + 1))^2} e^{-k/2T} k^{2\ell-1} r^{2\ell-1} \Gamma(\ell + 1 + i\kappa/(2\pi T))^2$$  \hspace{1cm} (37)

This is of the same functional form as the gray-body factors $\Gamma^\ell$ (Eq. (16)) that were derived by using the exact black-hole potential. Of course, the temperature in the non-relativistic case is arbitrary, because there is no horizon effect. Still, the similarity with Eq. (17) is non-trivial. It should also be pointed out that the differential equations in the two cases are different; their respective solutions are expressed in terms of different hypergeometric functions.

4 Discussion

The microscopic calculation of the entropy of black holes in superstring theory has left little doubt that strings hold the key to the discovery of the quantum theory of gravity. On the other hand, the fact that the microscopic calculation involves the counting of solitonic states ($D$-branes) shows that a more fundamental theory is needed that will provide the missing underlying principle on which quantum gravity should be based.

The first microscopic calculation seemed to rely heavily on supersymmetry. The agreement between the microscopic and macroscopic calculations was guaranteed by supersymmetry, which ensured that the number of supersymmetric (BPS) states was invariant when the string coupling was varied. It was therefore surprising to discover that there was agreement between the two approaches that went beyond the demands of supersymmetry. Such an agreement was demonstrated at a fairly detailed level with non-extremal black holes and gray-body factors.

We have discussed the behavior of gray-body factors for non-supersymmetric black holes. The goal was to understand the origin of their behavior. We have shown that there is a striking agreement with partial-wave amplitudes in the non-relativistic limit. This agreement is rather non-trivial as the calculations in the two cases (exact and non-relativistic approximation) rely on different differential equations possessing solutions that are expressed in terms of different hypergeometric functions. Therefore, the behavior of the gray-body factors seems to be universal.

In the cases we studied, there is no corresponding superstring theory, so a microscopic calculation is not readily available. However, it should be pointed out that one may still derive the functional dependence of the gray-body factors...
from conformal field theory. Indeed, if we introduce the chiral operator \( \Theta(\sigma^+) \) of conformal dimension \( \ell + 1 \), the thermal correlators at temperature \( T \) are

\[
\langle \Theta^\dagger(0) \Theta(\sigma^+) \rangle_T \sim \frac{1}{\sinh^{2(\ell+1)} \pi T \sigma^+} \tag{38}
\]

The gray-body factors are

\[
\Gamma^\ell \sim \int d\sigma^+ e^{-ik\sigma^+} \langle \Theta^\dagger(0) \Theta(\sigma^+) \rangle_T \tag{39}
\]

where we identify \( \sigma^+ \equiv \sigma^+ + 2i/T \). We obtain

\[
\Gamma^\ell \sim e^{-k/2T} |\Gamma(\ell + 1 + ik/(2\pi T))|^2 \tag{40}
\]

in agreement with our earlier results.

In conclusion, we have shown that there is an agreement between

(a) gray-body factors calculated from black-hole dynamics;

(b) partial-wave scattering amplitudes in the non-relativistic limit;

(c) thermal correlators in chiral conformal field theory.

We found agreement at a fairly detailed level, even though the three calculations bared little resemblance to one another. It might be interesting to extend these results to higher space-time dimensions and more general classes of black holes. Such explorations should shed light on the yet-to-be-discovered underlying principle of quantum gravity and the information loss paradox.

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