Analytic mean-field Hall crystal solution at $\nu = 1/3$: composite fermion like sub-bands and correlation effects

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(Dated: May 5, 2001)

An analytic solution of the Hartree-Fock problem for a 2DEG at filling $1/3$ and half an electron per unit cell is presented. The Coulomb interaction dynamically breaks the first Landau level into three narrow sub-bands, one of which is fully occupied and the other empty, as in the composite fermion model. The localized orbitals associated to the Bloch like single electron wavefunctions are nearly static, resembling the angular momentum eigenstates within a Landau level for non-interacting fermions. Strong correlations are expected owing to the large charge density overlap between neighboring plaquettes. A numerical evaluation brings the cohesive energy close to that of the best present day models. It is also found that correlations are long range, requiring over 50 particles spread over a finite sample to approach convergence. Since presently allowed exact calculations are far from this number, the question of how relevant the considered wave-function is for the description of the ground state of the 2DEG system remains open.

PACS numbers: 73.43.Cd, 73.43.-f

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I. INTRODUCTION

It is well accepted today that the fractional quantum Hall effect results from the formation of a strongly correlated spin polarized electron liquid in the lowest Landau level, that crystallizes below filling $1/7$. It is intriguing, however, that a mean field charge density wave solution exists at all fillings, called a Hall crystal (HC), which yields in a natural way the odd-denominator rule that characterizes the effect. The state is characterized by a unit cell that contains a fractional charge of even denominator, such as $e/2$. The detailed investigation of this class of mean field states was stopped largely due to their higher energy than that of the liquid, and the very small correlation correction obtained by Yoshioka and Lee (YL) for the Wigner solid (WS), which accommodates one whole electron per plaquette. However, while in the latter case there is little overlap of electrons from neighboring cells, in the former the charge density forms continuous ridges between cells opening the way for an appreciable increase of the cohesive correlation energy. This idea was first explored in the simplest case of $1/2$ filling. The results confirmed the effect yielding a second order correlation correction an order of magnitude larger than for the best WS state. The same line of thinking was also recently considered by Mikhailov.

In this work we report results for the correlation energy of the HC at $1/3$ filling. The work rests on an analytic solution of the mean field problem constructed using symmetry considerations. We show that the electron-electron interaction breaks the single particle degeneracy of the lowest Landau level (LLL), splitting it into three remarkably flat bands. All bands contain the same number of states so that only one is filled while the other two are empty, separated by a large gap. While the HC unit cell traps $3/2$ flux quanta of the original field, it is only pierced by one flux quanta of the reduced field $B^* = B_{1/3} - B_{1/2}$, where $B_{1/3}$ and $B_{1/2}$ are the fields at $1/3$ and $1/2$ filling factor, respectively. The narrowness of the occupied band and this latter fact suggest that the magnetic Wannier states are approximate solutions of the problem and behave as nearly free quasiparticles filling the LLL of the effective field $B^*$, much as the composite fermion model predicts.

We compute the correlation energy using the YL method, that is, evaluating the second order perturbation correction to the Hartree-Fock (HF) solution. Two methods are used, one employing a Monte Carlo approach for the space
integrals over a large sample, and the other by computing in a discrete momentum space using a set defined by periodic boundary conditions applied to a comparatively small sample. As for the half filling case, both methods yield an energy correction of about 10%, an order of magnitude larger than that for the WS state, confirming the earlier conjecture that at all fillings the high electron overlap of the HC enhances significantly the cohesive energy corrections, thus re-opening the case for this state as the proper precursor in a first principles perturbative approach to the true ground state of the system in the thermodynamic limit.

One further outcome of our calculations is that over 50 particles would be required in order to approach convergence, way above current permissible sizes used in first principles numerical evaluations normally performed with up to 12 particles, number that has been extended to treat at most 27 particles. This may be understood again looking at the large inter cell overlap through hexagonal ridges, which seems to point out the importance of cooperative ring exchange effects in the system. To be operative, these need large sample sizes and would be missed by such numerical diagonalizations of the hamiltonian. If the true ground state retains the essential features of its HF approximation, then one would expect the Hall crystal to be the proper mean field precursor due to its higher correlation energy than other mean field solutions.

In Sec. 2 the single particle Hartree-Fock hamiltonian is diagonalized analytically. Section 3 is devoted to the second order correction on the mean field solution. Finally, in Section 4 we present our conclusions. Details of the calculation are given in appendices A-D. In particular, in Appendix B a formula is derived expressing the single particle Hartree-Fock Hamiltonian in terms of the magnetic translations associated to an arbitrary periodicity lattice of the HF problem. This formula, up to our knowledge, is absent in the literature.

II. BLOCH REDUCTION OF THE HF PROBLEM AT $\nu = \frac{1}{3}$

A. One particle Hamiltonian and magnetic translations

We consider a 2DEG in a strong perpendicular magnetic field. We are interested in an analytic treatment of the Hartree-Fock problem at filling 1/3. We start out by writing the Hartree-Fock hamiltonian in the compact form

$$H_{HF} = \sum_Q v(Q) \exp(-r_o^2Q^2)T_{r_o} \times Q,$$  \hspace{1cm} (1)

where the coefficients $v(Q)$ are given by

$$v(Q) = 2\pi r_o^2 \rho(Q) \exp\left(\frac{r_o^2Q^2}{4}\right) \left(\frac{1 - \delta_{Q,0}}{r_o |Q|} \exp\left(-\frac{r_o^2Q^2}{4}\right) - \sqrt{\frac{\pi}{2}} f_o \left(\frac{r_o^2Q^2}{4}\right) \right) \frac{e^2}{\varepsilon_o r_o},$$ \hspace{1cm} (2)

where $\varepsilon_o$ is the background dielectric constant. The magnetic translation operators $T$ are defined in Appendix A, while form (1) is derived in Appendix B. The particle density in real space $\rho(x)$ is assumed to be periodic under displacements covering the triangular lattice defined by the vectors

$$R = r_1 a_1 + r_2 a_2, \hspace{0.5cm} r_1, r_2 = 0, \pm 1, \pm 2, ...;$$ \hspace{1cm} (3)

$$a_1 = \frac{\sqrt{6\pi}}{\sqrt{3}} (1, 0) r_o,$$  \hspace{1cm} (4)

$$a_2 = \frac{\sqrt{6\pi}}{\sqrt{3}} (1, \sqrt{3}/2) r_o.$$  \hspace{1cm} (5)

The Fourier components of the density are defined as

$$\rho(Q) = \frac{1}{A_{cell}} \int dx \rho(x) \exp(iQ \cdot x),$$ \hspace{1cm} (4)

where $A_{cell}$ is the unit cell area

$$A_{cell} = \mathbf{n} \times a_2 = 3\pi r_o^2.$$

(5)
Here $\mathbf{n}$ is a unit vector normal to the electron gas plane. Through this cell traverse a flux $\frac{\Phi}{2}$ in units of the magnetic flux quantum $\phi_o = \hbar c/e$. Associated with the above real space lattice is the reciprocal lattice

$$Q = Q_1 s_1 + Q_2 s_2$$

$$Q_1, Q_2 = 0, \pm 1, \pm 2, \ldots$$

$$s_1 = -\frac{2}{3\sqrt{3}} \mathbf{n} \times \mathbf{a}_2,$$

$$s_2 = \frac{2}{3\sqrt{3}} \mathbf{n} \times \mathbf{a}_1,$$

$$s_i a_j = 2\pi \delta_{ij}.$$  

### B. Block diagonalization of the HF hamiltonian

We next show that it is possible to find a basis in which the matrix representation of (1) has a diagonal form composed of simple 3-dimensional blocks. In addition, the functions have such a structure that they automatically furnish the translation symmetry of the total density over the lattice (2). For this purpose we consider the basis functions $\varphi_p(x)$ defined in Appendix A, constructed over the lattice with primitive vectors

$$b_1 = a_1,$$

$$b_2 = \frac{2}{3} a_2.$$  

The magnetic translations then have the form

$$T_{\frac{1}{2}} \mathbf{n} \times \mathbf{Q} = T_{-\frac{1}{2}Q_2 b_1 + Q_1 b_2}.$$  

Since the flux piercing the unit cell of the lattice (2) is not an integral number of flux quanta, the set of translation operators obtained by varying the integers $Q_1$ and $Q_2$ in (3) do not commute and it is not possible to find common eigenfunctions to all of them. The basis we shall construct decomposes in a set of three dimensional subspaces, closing each of them under the action of translations for all values of $Q$.

A first step in finding the basis is to define a set of eigenfunctions of a translation in the vector $a_2$, which is a period of the density. For a given value of the momentum $p$ such orbitals may be written as

$$\chi^\sigma_p(x) = \frac{1}{\sqrt{2}} \left( \varphi_p(x) + \frac{\sigma}{\exp(-ia_2.p)} T_{a_2} \varphi_p(x) \right), \sigma = \pm 1.$$  

Using the formulas in Appendix A it can be readily proven that these functions satisfy the eigenvalue relation

$$T_{a_2} \chi^\sigma_p(x) = \sigma \exp(-ia_2.p) \chi^\sigma_p(x).$$  

The two values of $\sigma$ appearing in these equations will play an important role in what follows. They will allow us to impose the periodicity of the density under the shifts in $a_1$ and $a_2$, in spite of the impossibility of obtaining a basis of eigenvectors of all the magnetic translations in the lattice. It should be stressed that the range of values of $p$ defining independent functions in the new basis have been reduced in half, the two values of $\sigma$ compensating for this reduction.

The restriction comes from the singular property of the basis $\{\varphi_p\}$, that a magnetic translation is fully equivalent to a shifting of the momenta argument as implied by the relation

$$T_{R} \varphi_p(x) = \mathcal{F}_p(R) \varphi_p + \frac{\Phi}{\Phi_0} A(R) \varphi_p(x),$$  

where $\mathcal{F}_p(R)$ is a pure phase factor. See Appendix A and Ref (16) for the justification of this relation. It directly implies that the shift done in $a_2$ in constructing the new basis precisely changes the momentum $p$ of the particle in $-s_i/2$. Therefore, what has been basically done is to form linear combinations of the original functions associated with different values of the momentum. In the new basis the magnetic translation in $a_1$ has the simple effect

$$T_{a_1} \chi^\sigma_p(x) = \exp(-ia_1.p) \chi^{-\sigma}_p(x),$$  

that is, it merely changes the sign of $\sigma$ and multiplies it by a phase factor.

As the next step let us employ the fact that, although the functions are not eigenvectors of translations in $a_1$, they are eigenfunctions of the double sized translations in $2a_1$. This is because its effect, when considered as two
consecutive shifts in \( \mathbf{a}_1 \), have the simple result of making two consecutive changes of sign of \( \sigma \) that reproduce the original function. Therefore, if for \( \mathbf{p} \) and \( \sigma \) fixed we construct the triplet of states formed by the function \( \chi_{\mathbf{p}}^{(r, \sigma)}(x) \) and the other two obtained by a pair of successive translations in the vector \(-2\mathbf{a}_1/3\), the operation of performing a translation in an arbitrary multiple of these vectors leaves the triplets invariant.

A specific linear combination within each triplet which is also an eigenfunction of the translation in \(-2\mathbf{a}_1/3\) can be obtained by constructing the new basis

\[
\chi_{\mathbf{p}}^{(r, \sigma)}(x) = \sum_{s=-1,0,1} c_s^r(\mathbf{p}) T_{-\frac{2}{3}s \mathbf{b}_2} \chi_{\mathbf{p}}^\sigma(x), \quad r = -1, 0, 1. \tag{13}
\]

After solving the linear equations for the constants \(c_s^r\) obtained by imposing the condition that these functions be solutions of the eigenvalue problem

\[
T_{-\frac{2}{3}s \mathbf{b}_2} \chi_{\mathbf{p}}^{(r, \sigma)}(x) = \lambda \chi_{\mathbf{p}}^{(r, \sigma)}(x)
\]

one finds

\[
\lambda^{(r)}(\mathbf{p}) = \exp\left(\frac{2}{3}i \mathbf{p} \cdot \mathbf{a}_1 + \frac{2\pi r}{3}\right), \quad (14)
\]

\[
c_s^r(\mathbf{p}) = \frac{1}{\sqrt{3}} \exp\left(-\frac{2}{3}i \mathbf{p} \cdot \mathbf{a}_1 s - \frac{2\pi i s}{3}\right), \quad r, s = -1, 0, 1. \tag{15}
\]

Substituting in (13) yields

\[
\chi_{\mathbf{p}}^{(r, \sigma)}(x) = \frac{1}{\sqrt{3}} \sum_{s=-1,0,1} \exp(-\frac{2}{3}i \mathbf{p} \cdot \mathbf{a}_1 s - \frac{2\pi i s}{3}) T_{-\frac{2}{3}s \mathbf{b}_2} \chi_{\mathbf{p}}^\sigma(x), \tag{16}
\]

The last relation expresses that, modulo a phase factor, the states of the new basis are equivalent upon a shift of their momenta \(\mathbf{p}\) in any linear combination with integer coefficients of half the unit cell vectors of the reciprocal lattice corresponding to the periodicity of the density. The periodicity of the states under the shifts in \(\mathbf{s}_1/2\) was discussed above, and the one related with \(\mathbf{s}_2/2\) similarly follows from the relation \(\chi_{2\ell,0}^{(x)}\) in Appendix A, expressing the equivalence of a magnetic translation with a shift in momentum. The functions just defined have an alternative and more compact form given by

\[
\chi_{\mathbf{p}}^{(r, \sigma)}(x) = \frac{1}{\sqrt{6N_{\mathbf{p}}^{(3,2)}}} \sum_m \exp(i \mathbf{P}(\mathbf{p}, r, \sigma) \cdot \mathbf{m} + \frac{5\pi i}{6} m_1 m_2) T_{m \phi}(x), \tag{17}
\]

\[
N_{\mathbf{p}}^{(3,2)} = \sqrt{N_{\mathbf{p}}^{(3,2)}} \sum_{\ell} (-1)^{\ell_1 \ell_2} \exp(i \mathbf{k} \cdot \ell - \frac{\ell^2}{4\mathbf{r}^2_0}) \tag{18}
\]

\[
\ell = \ell_1(3\mathbf{c}_1) + \ell_2(2\mathbf{c}_2), \tag{19}
\]

where the effective momenta and the new elementary lattice of vectors \(\mathbf{m}\) over which the sum is performed are given by

\[
\mathbf{P}(\mathbf{p}, r, \sigma) = \mathbf{p} - r \mathbf{s}_1 + \frac{\sigma - 1}{2} \mathbf{s}_2, \tag{20}
\]

\[
\mathbf{m} = m_1 \mathbf{c}_1 + m_2 \mathbf{c}_2, \quad \mathbf{c}_1 = \frac{\mathbf{a}_1}{3}, \quad \mathbf{c}_2 = \frac{\mathbf{a}_2}{3}. \tag{21}
\]

The double sum \(\chi_{\mathbf{p}}^{(r, \sigma)}(x)\) can be evaluated to obtain an explicit formula in terms of the Elliptic Theta functions as follows

\[
\chi_{\mathbf{p}}^{(r, \sigma)}(x) = \frac{\exp\left(-\frac{x^2}{4\pi^2}\right)}{\sqrt{6\sqrt{2\pi r_0^{(3,2)} N_{\mathbf{p}}^{(3,2)}}}} \times
\]
\[
\sum_{\beta=0,...,5\alpha=0,1} \sum_{\alpha} \exp \left( 27\pi i \alpha^2 + \pi i \beta^2 + a_2 \beta + 2\pi i a_1 \frac{3a_2 + 3a_1}{2\pi i} - \frac{15\alpha}{2} + (\frac{9\beta}{2} - \frac{5}{4})\beta \right) \times \\
\Theta_3 \left( \frac{6\alpha}{\pi i} - \frac{3a_1}{\pi i} - (15 - 54\tau_1)\alpha + 2(\frac{9\tau_1}{2} - \frac{5}{4})\beta \mid 108 \tau_1 \right) \times \Theta_3 \left( \frac{a_1}{2\pi i} + \frac{5\alpha}{2} + (\frac{\tau_1}{2} + \frac{5}{12})\beta \mid \tau_1 \right),
\]

where

\[
a_1 = i \mathbf{P}(p, \mathbf{r}, \sigma) \cdot \mathbf{c}_1 + \frac{1}{2\pi^2} (\mathbf{c}_{2-i} \times \mathbf{n} \times \mathbf{c}_1) \cdot \mathbf{x}
\]
\[
a_2 = i \mathbf{P}(p, \mathbf{r}, \sigma) \cdot \mathbf{c}_2 + \frac{1}{2\pi^2} (\mathbf{c}_{2-i} \times \mathbf{n} \times \mathbf{c}_2) \cdot \mathbf{x}
\]
\[
\tau_1 = \frac{i}{6\sqrt{6}}.
\]

Let us inspect now the action of a magnetic translation by \( \frac{2}{3} \mathbf{a}_2 \) on the new functions. If such a transformation leaves the triplets invariant, then the matrix reduction of the Hartree-Fock Hamiltonian will follow. One has,

\[
T_{\frac{2}{3}\mathbf{a}_2} \chi_{\mathbf{p}}^{(r,\sigma)}(\mathbf{x}) = \frac{1}{\sqrt{3}} \sum_{s=-1,0,1} \exp \left( -\frac{2}{3} i \mathbf{p}.a_1 s - \frac{2\pi i r s}{3} \right) T_{\frac{2}{3}\mathbf{a}_2} T_{-\frac{2}{3}\mathbf{a}_1} \chi_{\mathbf{p}}^{(r-s,\sigma)}(\mathbf{x}).
\]

But after using (20) for changing the order of the two operators within the sum, it follows that

\[
T_{\frac{2}{3}\mathbf{a}_2} \chi_{\mathbf{p}}^{(r,\sigma)}(\mathbf{x}) = \chi_{\mathbf{p}}^{(r,\sigma)}(\mathbf{x}),
\]

where the square bracket defines the number among the set \{-1,0,1\} that is equivalent, modulo 3, to the integer in the argument.

For fixed \( \mathbf{p} \) and \( \sigma \) the matrix elements of the Hamiltonian \( \mathbf{H} \) in the new basis can readily be found to have the form

\[
h_{(r',r)}^{(p,\sigma)} = \sum_{\mathbf{Q}} v(\mathbf{Q}) \exp \left( -\frac{Q^2 r'^2}{4} \right) \exp \left( i \mathbf{p} \cdot \mathbf{n} \times \mathbf{Q} \frac{r'^2}{2} + \frac{2\pi i r}{3} Q_2 (r + Q_1) \right) \delta_{r',|r-Q|}.
\]

The problem has thus been reduced to the self-consistent diagonalization of a three dimensional matrix for each value of momenta \( \mathbf{p} \) and parameter \( \sigma \). The basis can be checked to have the following set of transformations properties,

\[
T_{2e_1} \chi_{\mathbf{p}}^{(r,\sigma)}(\mathbf{x}) = \exp(2i \mathbf{p} \cdot \mathbf{c}_1 - \frac{2\pi i r}{3}) \chi_{\mathbf{p}}^{(r,\sigma)}(\mathbf{x}),
\]
\[
T_{2e_2} \chi_{\mathbf{p}}^{(r,\sigma)}(\mathbf{x}) = \exp(2i \mathbf{p} \cdot \mathbf{c}_2) \chi_{\mathbf{p}}^{(r+1,\sigma)}(\mathbf{x}),
\]
\[
T_{3e_1} \chi_{\mathbf{p}}^{(r,\sigma)}(\mathbf{x}) = \exp(-3i \mathbf{p} \cdot \mathbf{c}_1) \chi_{\mathbf{p}}^{(-r,\sigma)}(\mathbf{x}),
\]
\[
T_{3e_2} \chi_{\mathbf{p}}^{(r,\sigma)}(\mathbf{x}) = \sigma \chi_{\mathbf{p}}^{(-r,\sigma)}(\mathbf{x}),
\]
\[
I \chi_{\mathbf{p}}^{(r,\sigma)}(\mathbf{x}) = \chi_{\mathbf{p}}^{(-r,\sigma)}(\mathbf{x}),
\]

where the parity transformation \( I \) is defined as usual by \( I \chi_{\mathbf{p}}^{(r,\sigma)}(\mathbf{x}) = \chi_{\mathbf{p}}^{(r,\sigma)}(-\mathbf{x}) \). As shown in Appendix A, from these symmetry properties it follows that the density associated with any Slater determinant constructed by selecting one orbital within each triplet has exact periodicity under shifts in all vectors \( \mathbf{R} \).

In order to find the solution of the mean field problem by an iterative technique it is sufficient to make an ansatz for the density in the first step, and then diagonalize numerically the matrices for a sufficiently high partition of the reduced Brillouin cell momenta \( \mathbf{p} \) defined by (16). By selecting the normalized lowest energy state within each three dimensional quantum mechanical problem, the Fourier components of the density corresponding to the new step should be constructed. Following its definition, it can be done by means of the following expression,

\[
\rho(\mathbf{Q}) = \frac{1}{A_{cell}} \sum_{\mathbf{p},\sigma = \pm 1} \int d\mathbf{x} \left| \sum_r g^0_r(\mathbf{p}) \chi_{\mathbf{p}}^{(r,\sigma)}(\mathbf{x}) \right|^2 \exp(i \mathbf{Q} \cdot \mathbf{x})
\]
\[
= \frac{\exp(-Q^2 r^2)}{A} \sum_{\mathbf{p}} \sum_{\sigma = \pm 1} \exp(i \mathbf{p} \cdot \mathbf{n} \times \mathbf{Q} \frac{r^2}{2} + \frac{2\pi i}{3} Q_2 Q_1) \times \sum_{r,r'=-1,0,1} g^0_{r'}(\mathbf{p})^* g^0_r(\mathbf{p}) \exp(-\frac{2\pi i}{3} r Q_2) \delta_{r',|r+Q|}.
\]
This formula can be obtained by evaluating the Gaussian integrals appearing after substituting the expansions defining the functions $\chi_{p}^{(r,\sigma)}$. The coefficients $g_{r}^{0}$, $g_{r}^{1}$, and $g_{r}^{2}$, define the components of the eigenvectors of the single particle HF Hamiltonian in the basis of states $\chi$. They fix the wave-functions of the filled band and the empty bands as

$$
\Phi_{p}^{(b,\sigma)}(x) = \sum_{r=-1,0,1} g_{r}^{b}(p) \chi_{p}^{(r,\sigma)}(x) \tag{27}
$$

Here $b = 0$ labels the filled band in each triplet and $b = 1, 2$ label the two empty bands.

C. Functions that vanish at the origin are exact solutions

In order to proceed within an analytical context let us consider the observation from former numerical studies, that the particle density when the number of flux quanta piercing the unit cell is a half integer essentially vanishes at all lattice points. Then, let us first assume that the density rigorously vanishes at this set of points. If so is the case, the wave-function of any of the filled states should then also vanish at those points. This requirement follows from the fact that the Hartree-Fock particle density is a sum over the individual densities of all occupied orbitals,

$$
\sum_{p,\sigma=\pm 1} |\sum_{r} g_{r}^{0}(p) \chi_{p}^{(r,\sigma)}(x)|^{2} = \rho(x). \tag{28}
$$

We can then use this property to fix the coefficients of the wave functions within each triplet.

After imposing the vanishing conditions at the origin, the coefficients defining the functions $g_{r}^{0} \tag{27}$ are fully determined and take the form

$$
\begin{align*}
g_{0}^{0}(p) &= \frac{1}{N_{p}^{0}}, \\
g_{-1}^{0}(p) &= -\frac{1}{N_{p}^{0}} \chi_{p}^{(0,+)}(0) \chi_{p}^{(+1,-1)}(0) - \chi_{p}^{(0,-1)}(0) \chi_{p}^{(-1,-1)}(0) \\
g_{+1}^{0}(p) &= -\frac{1}{N_{p}^{0}} \chi_{p}^{(-1,+1)}(0) \chi_{p}^{(-1,+1)}(0) - \chi_{p}^{(0,+1)}(0) \chi_{p}^{(-1,+1)}(0) \\
1 &= |g_{0}^{0}(p)|^{2} + |g_{-1}^{0}(p)|^{2} + |g_{+1}^{0}(p)|^{2}.
\end{align*} \tag{29}
$$

Note that the coefficients $g$ are all independent of $\sigma$. This completes the definition of our functions. That they are true solutions of the Hartree-Fock problem is proven in Appendix C.

The particle density may now be computed replacing these functions in Eq. $28$. The real space particle density thus obtained is shown in Fig. 4.

A main property to be noticed in this figure is the formation of sharp hexagonal channels surrounding the low density regions at the center of which the vanishing density occurs. These structures mark the difference with the Wigner solid whose unit cell encloses one flux quanta. The charge density in this latter instance is made up essentially of well localized gaussians centered at each lattice point. In our case there is strong overlap, suggesting that cooperative ring exchange involving many unit cells is a large contribution to the correlation energy.

Further, the insertion of the calculated density in the eigenvalue equation associated with the matrix representation of the Hamiltonian in each triplet, Eqs. $2$ and $24$, allows for the evaluation of the one particle spectrum of the system. As it was expected, three energy bands appear, each associated with a value of the index $r$ and covering the full range of $p$ within each triplet. We also note that states associated with $\sigma = \pm 1$ turn out to be degenerate. The bands dispersion relations are illustrated in Fig. 2.

Note the narrowness of the bands as compared with the gaps separating them. This fact leads to the idea that in this mean field approximation the Coulomb interaction, although breaking the degeneracy of the first Landau level, it reorganizes the states in three equally populated separate sets that again are approximately degenerate, as if they were Landau levels of a renormalized problem. Since at $1/3$ filling one band is full and the other two empty, one expects the electrons to behave dynamically similarly to filling one, except for a different effective mass, as the composite fermion model suggests. The magnetic Wannier states are expected to be approximate solutions, playing the role of the angular momentum states in the non-interacting problem at filling one.
FIG. 1: The density of particles associated with the HC state. Its structure indicates the high overlapping of the localized Wannier-like states in terms of which the HF determinant always can be expressed.

\[
\begin{align*}
\rho_1 & \approx 0.02 \\
\rho_2 & \approx 0.04 \\
\rho_3 & \approx 0.06
\end{align*}
\]

FIG. 2: Triplet of narrow bands in which the Landau level is split by the action of the Coulomb interaction.

These properties seem to corroborate the possibility of tracing a link between weakly interacting composite fermions and the Bloch or Wannier orbitals in the mean field solutions considered earlier by one of us.\(^4,5\) To finish this section we note that the mean field energy per particle obtained from the above formalism for the solution we have constructed confirms the value reported in an earlier numerical computation: \(^6\)

\[
\epsilon = -0.362 \frac{e^2}{\varepsilon_0 r_0}.
\]
III. CORRELATION ENERGY IN SECOND ORDER

In order to obtain the energy correction to second order of perturbation theory we proceed similarly as in Ref \(8\), starting with the expression

\[
E^{(2)} = \sum_{i} \langle \Psi_{HF} | (H - H_{HF}) | \Psi_{i} \rangle \frac{1}{E_{HF} - E_{i}} \langle \Psi_{i} | (H - H_{HF}) | \Psi_{HF} \rangle \tag{31}
\]

Here \(\Psi_{HF}, E_{HF}\) are the mean field Slater determinant and total Hartree-Fock energy associated with the ground state, respectively, and \(H\) is the projection of the exact many particle Hamiltonian onto the first Landau level. The many particle excited states \(\Psi_{i}\) are Slater determinants constructed with the basis \(27\), mixing states in the filled band with states in the empty bands. It follows that \(\langle \Psi_{HF} | \Psi_{i} \rangle = 0\), a property that allowed to write the last equality in \(31\). In the second quantized representation the Hamiltonian \(H\) will have non-vanishing matrix elements linking the HF state and excited states of the form \(| \Phi_{i} \rangle = a_{\eta} a_{\eta'} a_{\xi}^{+} a_{\xi'}^{+} | \Phi_{HF} \rangle\), where \(a_{\xi}^{+}\) creates an electron of quantum numbers \(\xi\), etc. The index \(i\) is a shorthand notation for the set of two pairs of filled \((\eta, \eta' \in F)\) and empty \((\xi, \xi' \in T - F)\) electron states, where \(T\) and \(F\) are the set of all states in the Landau level and the filled ones, respectively. The indices \(\eta = (0, \mathbf{p}, \sigma)\) and \(\xi = (b, \mathbf{p}, \sigma)\) for \(b = 1, 2\) denote the quantum numbers of filled and empty states, respectively. The total energy of an excited state is \(E_{i} = E^{(HF)} + \epsilon (\xi') - \epsilon(\eta) - \epsilon(\eta')\). Then, the second order correction can be rewritten in the form

\[
E^{(2)} = \sum_{(\eta, \eta')} \sum_{(\xi, \xi')} | \langle \Phi_{HF} | H a_{\eta} a_{\eta'} a_{\xi}^{+} a_{\xi'}^{+} | \Phi_{HF} \rangle |^{2} \frac{1}{\epsilon(\eta) + \epsilon(\eta') - \epsilon(\xi) - \epsilon(\xi')},
\]

where the total projected Hamiltonian is

\[
H = \frac{\epsilon_{0}^{2}}{2} \int \int dx dx' \frac{\Psi^{\ast}(x) \Psi^{\ast}(x')}{|x - x'|} \frac{1}{|x - x'|} \Psi(x') \Psi(x) \tag{33}
\]

The matrix elements of the Coulomb interaction are given by

\[
M(\alpha, \alpha' | \beta', \beta) = \int \int dx dx' \Phi_{\alpha}^{\ast}(x) \Phi_{\alpha'}^{\ast}(x') \frac{1}{|x - x'|} \Phi_{\beta}(x') \Phi_{\beta}(x), \tag{34}
\]

where we have used the shorthand notation \(\Phi_{\alpha} = \Phi_{\mathbf{p}}^{(b, \sigma)}\). By using the anti-commutation relations \([a_{\alpha}, a_{\alpha'}^{+}] = \delta_{\alpha, \alpha'}\), formula \(31\) can be expressed as

\[
E^{(2)} = \frac{\epsilon_{0}^{4}}{2} \sum_{(\eta, \eta')} \sum_{(\xi, \xi')} | \int \int dx dx' \frac{\Phi_{\eta, \eta'}^{\ast}(x, x')}{|x - x'|} \Phi_{\xi, \xi'}(x, x') |^{2} \frac{1}{\epsilon(\eta) + \epsilon(\eta') - \epsilon(\xi) - \epsilon(\xi')} \tag{35}
\]

where the two particle states \(\Phi_{\eta, \eta'}\) are defined by

\[
\Phi_{\eta, \eta'}(x, x') = \frac{\Phi_{\eta}(x) \Phi_{\eta'}^{\ast}(x') - \Phi_{\eta'}(x) \Phi_{\eta}^{\ast}(x')}{\sqrt{2}}. \tag{36}
\]

The pairs \((\eta, \eta')\) and \((\xi, \xi')\) are considered as unordered.

A. Correlation energy: first evaluation

As pointed out above the single particle bands are remarkably flat. We can use this property to simplify the calculation of the energy correction. First, we approximate the filled band energies appearing in the denominator of
by their mean value,
\[ \epsilon(\eta) = \epsilon_0. \]

In addition, and slightly more crudely, we substitute the energies in the excited bands by a common energy equal to half the sum of the mean energies of the two bands,
\[ \epsilon(\xi) = \frac{\epsilon_1 + \epsilon_2}{2}. \]

This last approximation is taken in view of the small relative gap separating these two bands.

With this simplifying substitution (35) can be expressed in the simpler form
\[
E^{(2)} = \frac{\epsilon^4}{2(2\epsilon_0 - \epsilon_1 - \epsilon_2)} \int \int dx' dx \int \int dy' dy \frac{1}{|x - x'|} \frac{1}{|y - y'|} \times \\
(\pi_f(x', y')\pi_f(x, y) - \pi_f(x', y')\pi_f(x, y)) \pi_e(y, x)\pi_e(y', x'),
\]
where \( \pi_f \) is the projection operator on the subspace of states of the filled band and \( \pi_e \) the projector associated to the subspace of states formed with the union of the empty bands. The projectors have the following expression in terms of the corresponding densities
\[
\pi_f(x, x') = \sum_{p, \sigma} \Phi_p^{(0, \sigma)}(x)(\Phi_p^{(0, \sigma)}(x'))^*, \\
\pi_e(x, x') = \sum_{p, \sigma} \Phi_p^{(b, \sigma)}(x)(\Phi_p^{(b, \sigma)}(x'))^*, \\
\rho_e(Q) = \frac{1}{2\pi r_o^2} \delta_{Q, 0} - \rho_b(Q),
\]
where \( \delta_{Q, 0} \) is the ordinary Kronecker delta and \( P(x, x') \) is the projection operator onto the first Landau level, defined in Appendix A. These expressions can be obtained from formula (31) in Appendix B. After evaluating a few spatial integrals, the following formula for the correlation energy per particle \( \epsilon^{(2)} = E^{(2)}/N \) is obtained
\[
\epsilon^{(2)} = \frac{\epsilon^4}{2(2\epsilon_0 - \epsilon_1 - \epsilon_2)} \sum_{Q_1, Q_2, Q_3} \sum_{l=0}^{\nu} \rho_f(Q_1)\rho_f(Q_2)\rho_e(Q_3)\rho_e(-Q_1 - Q_2 - Q_3) \\
\exp\left(\frac{(Q_1 + Q_2 + Q_3)^2 r_o^2}{4}\right) \int dz \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{4r_o^2}\right) \exp\left(-\frac{z^2}{2} - (Q_1 + Q_3) \frac{n \times z}{2} - i(Q_2 + Q_3) \frac{z}{2}\right) \\
I_0\left(\frac{1}{8} (\frac{z}{r_o} - n \times (Q_1 + Q_3)r_o - i(Q_2 + Q_3)r_o)^2\right) \times \\
\exp\left(\frac{1}{8} (\frac{z}{r_o} - n \times (Q_1 + Q_3)r_o - i(Q_2 + Q_3)r_o)^2 - \exp\left(-\frac{1}{8} (\frac{z}{r_o} - n \times (Q_1 + Q_3)r_o - i(Q_2 + Q_3)r_o)^2\right)\right).
\]

Further progress in the evaluation of the energy correction is aided by noting the presence of an exponential factor in the squared Fourier wave-vectors, allowing to take just a few of them only, for sufficient convergence. We thus assume the 36 Fourier components associated with the shortest values of \( Q \) are nonzero, only. Of these, those associated with the longest \( Q \) are slightly altered to assure that the sum rule
\[
\sum_Q |\rho_f(Q)|^2 \exp\left(\frac{Q^2 r_o^2}{2}\right) = \frac{\nu}{(2\pi r_o^2)^2}
\]
is satisfied. This condition is used to cancel various fictitious divergences in the formula defining the correlation energy. Then, we obtain
FIG. 3: Second order correction to the energy per particle as a function of inverse number of sample points in the Monte-Carlo integration.

\[
\epsilon^{(2)} = \frac{(e^2/\varepsilon_o r_0)^2}{4 \nu \sqrt{\pi (2e_0 - e_1 - e_2)}} \sum_{Q_1} \sum_{Q_2} \sum_{Q_3} \Delta_f(Q_1) \Delta_f(Q_2) \Delta_e(Q_3) \Delta_e(Q_1 + Q_2 + Q_3) \times \\
\exp(-i \frac{r_o}{2} (Q_2 + Q_3) \cdot n \times (Q_1 + Q_3)r_o + \frac{(Q_2 + Q_3)^2 r_o^2}{2}) \times \\
\int du \exp\left(-\frac{(u - i(Q_2 + Q_3)r_o)^2}{4} - i r_o(Q_2 + Q_3), \frac{(u - i(Q_2 + Q_3)r_o)}{2}\right) \times \\
I_0\left(\frac{1}{8}(u - i(Q_2 + Q_3)r_o)^2\right) \sinh\left(\frac{1}{8}(u - i(Q_2 + Q_3)r_o)^2\right) \times \\
\left(\frac{1}{\sqrt{(u + n \times (Q_1 + Q_3)r_o)^2}} - \frac{\delta_{Q_2 + Q_3, 0}}{\sqrt{u^2}}\right),
\]

where the order parameters of the filled and empty bands have been defined as usual,

\[
\Delta_f(Q) = 2\pi r_o^2 \rho_f(Q) \exp\left(\frac{Q^2 r_o^2}{4}\right),
\]

\[
\Delta_e(Q) = 2\pi r_o^2 \rho_e(Q) \exp\left(\frac{Q^2 r_o^2}{4}\right).
\]

The integrals in Eq. (41) were calculated using the Monte-Carlo method with a variable number of sample points in the range \( N = 10000 - 40000 \). The region of integration was taken as a square of size \( L = 50 \) in units of the magnetic length \( r_0 \), centered at the origin. At 1/3 filling this sample contains 132 particles and traps 396 flux quanta. The results for the correlation energy as a function of \( 1/N \) are shown in Fig. 3. In units \( e^2/\varepsilon_o r_0 \) the estimated value for the correlation energy is

\[
\epsilon^{(2)} \simeq -0.04 \pm 0.01.
\]

Adding this correction to the Hartree-Fock energy one obtains the value \( \epsilon \simeq -0.40 \pm 0.01 \), to be compared with the result of Yoshioka and Lee with its own correction \( \epsilon \simeq -0.394 \), and that of Laughlin \( \epsilon \simeq -0.416 \), all in the same units.

B. Correlation energy: dependence on the sample size

The lack of a definite convergence pattern provided by the method just described prompted us to try a different procedure. Using the formulas derived in Appendix D it is possible to recast Eq. (11) to give it the form,
where the function $V$ is defined by

$$V(\alpha, \alpha', \beta, \beta') = \sum \frac{2\pi}{a} r_0 |P_{\beta', r'\beta} - P_{\alpha', r'\alpha} + Q^*|^2 \exp(-r_0^2(P_{\beta', r'\beta} - P_{\alpha', r'\alpha} + Q^*)^2) \times \exp\left(-i r_0^2 n \times Q^* \cdot (P_{\alpha, r_\alpha} + P_{\beta, r_\beta} - P_{\beta', r'\beta} - P_{\alpha', r'\alpha})\right),$$

The form of the pure phase factors $F^*$ and $F^*_p$, the momenta $P_{\alpha, r}$ and the special reciprocal lattice vectors $Q^*$ are all specified in Appendix D. As before, the functions $g_{\alpha}(p)$ are the coefficients determining the single particle HF excitations. We remark that once the coefficients for the filled band are known the other set of coefficients for the empty bands can be evaluated analytically as the two eigenvectors of the $(3x3)$ matrix representation of the single particle HF hamiltonian, orthogonal to the vector $q_\theta(p)$.

In order to evaluate numerically expression (43) we restrict the Hilbert space of the single particle HF problem by defining a cell of sides $2N_1a_1$, $2N_2a_2$ with $N_1$, $N_2$ integers, and imposing periodic boundary conditions over its borders. Using the symmetry properties (25)

$$T_{N_1\omega c_1} \chi_p^{(r, \sigma)}(x) = \exp(-i p \cdot c_1 N_1) \chi_p^{(r, \sigma)}(x),$$
$$T_{N_2\omega c_2} \chi_p^{(r, \sigma)}(x) = \exp(-i p \cdot c_2 N_2) \chi_p^{(r, \sigma)}(x).$$

these constraints can be translated into the relations

$$-6p \cdot c_1 = 2\pi n_{1, 1} \frac{N_1}{N_1},$$
$$-6p \cdot c_2 = 2\pi n_{2, 2} \frac{N_2}{N_2}.$$

They restrict the values of the quasimomentum $p$ to a discrete set, as expressed in compact form by the condition

$$p = \frac{n_1 t_1}{N_1} + \frac{n_2 t_2}{N_2},$$

$$0 \leq n_1 < N_1, 0 \leq n_2 < N_2,$$

where $t_1$ and $t_2$ are defined in Appendix A. The cell chosen traps $6N_1N_2$ flux quanta and contains $2N_1N_2$ particles. The integrals over the continuum in the Brillouin zone of Eq. (43) are now restricted to summations over a finite number of values of $p$. Formula (43) was first evaluated by for an elongated sample with $N_1 = 1$, and $N_2 = N$ varying from 1 to 13. In order to keep the computing time within our allowed practical limits, the values of $n_1$ and $n_2$ in the expression defining the summation argument

$$Q^* = n_1 \frac{3}{2} t_1 + n_2 \frac{2}{3} t_2,$$
FIG. 4: Correlation energy as a function of the parameter N. The circles indicate the calculated values for elongated samples of the form \((N_1, N_2) = (1, N)\). The black dots signal the values corresponding more appropriate samples with form defined by \((N_1, N_2) = (N, N)\).

were restricted to the regions \(-4 \leq n_1 \leq 4\) and \(-4 \leq n_2 \leq 4\). The results of this calculation are shown as empty circles in Fig. (4). Note that the correlation energy per particle grows with N, approaching rather slowly the thermodynamic limit whose value \(\epsilon^{(2)} \sim -0.06 e^2/e_o r_o\) exceeds the result \((42)\) of the previous method. The energy correction was also calculated for samples with an aspect ratio close to one, taking \((N_1, N_2) = (N, N)\), up to \(N = 5\), an upper bound set by computing limitations. The results are shown in Fig. (4) by filled circles. As can be observed, the energy values for those samples grow similarly with \(N\) than those associated with elongated samples. This property indicates that these more appropriate samples also need values of \(N\) greater than five in order to approach convergence. However, already at \(N = 5\) the number of particles involved is \(2N^2 = 50\), considerably higher than the numbers handled in exact diagonalization.

The most relevant finding of this last method of evaluation is the need for very high sample sizes in order to approach the thermodynamical limit. This outcome strongly supports the relevance of long distance correlations in the attainment of the true character of the ground state through numerical computations.

IV. SUMMARY

An analytic solution of the Hartree-Fock problem for a 2DEG at filling \(1/3\) and \(1/2\) electron per unit cell is found by employing symmetry considerations and a special complete set of common eigenfunctions of the magnetic translations. A triplet of bands arises that turn out surprisingly flat as functions of the two dimensional quasimomentum. The Coulomb interaction breaks the first Landau level in three narrow sub-bands resembling effective Landau levels of composite fermions, one of which is fully occupied and the other two, empty. The energy per particle reproduces an earlier numerical result for this quantity \(5\). We find that the charge density forms hexagonal rings throughout the lattice, suggesting strong long range correlations owing to cooperative ring exchange. An evaluation of the correlation energy using second order perturbation theory yields a correction an order of magnitude larger than that for the localized single-particle features of the usual Wigner Crystal state \(6\) and lowers the energy to make it comparable with the lowest values obtained using other models. Our results also suggest that the thermodynamic limit in a cell with aspect ratio near one is not reached if less than 50 particles are included.

V. ACKNOWLEDGMENTS

This work was supported in part by Fondecyt Grants 1020829 and 7020829, the ICTP Associateship Program, and the Third World Academy of Sciences. Helpful discussions with A. Gonzalez, G. Baskaran, N. H March and K. Esfarjani are also acknowledged.
VI. APPENDIX A

A. Eigenfunctions of magnetic translations

Consider a 2DEG constrained to move in a plane of area \( A \) under a perpendicular magnetic field \( B \). A useful basis set of single particle Bloch-like states in the lowest Landau level can be defined in terms of linear combinations of the normalized zero angular momentum eigenfunction

\[
\phi(x) = \frac{1}{\sqrt{2\pi r_o}} \exp(-\frac{x^2}{4r_o^2})
\]  

in the compact form\(^\text{7,15,16}\),

\[
\varphi_k(x) = \frac{1}{N_k} \sum_\ell (-1)^{\ell_1 \ell_2} \exp(i \ell \cdot k) T_\ell \phi(x),
\]  

\[
N_k = \sqrt{N_{\phi_0}} \sqrt{\sum_\ell (-1)^{\ell_1 \ell_2} \exp(-i \ell \cdot \frac{\ell^2}{4r_o^2})}.
\]

Due to its role in the above definition the function \( \phi \) is called the "seed" function. The sum runs over all integers \( \ell_1, \ell_2 \) defining a planar lattice \( L \), through \( \ell = \ell_1 b_1 + \ell_2 b_2 \), where the unit cell intercepts one flux quantum, so that \( \mathbf{n} \cdot b_1 \times b_2 = 2\pi r_o^2 \). The magnetic translation operators \( T_\mathbf{R} \) acting on any function \( f \) are defined by

\[
T_\mathbf{R} f(x) = \exp(\frac{2ie}{\hbar c} A(\mathbf{R}) \cdot x) f(x - \mathbf{R}),
\]

where the vector potential is assumed in the axial gauge \( A(x) = B(-x_2, x_1, 0)/2 \) and the electron charge \( e \) is taken with its negative sign. This basis was employed before to obtain exact mean field solutions of the related problem at filling 1/2\(^\text{8}\). For arbitrary vectors \( \mathbf{R}_1 \) and \( \mathbf{R}_2 \) the translation operators satisfy the commutation relation

\[
T_{\mathbf{R}_1} T_{\mathbf{R}_2} = \exp(\frac{ie}{\hbar c} A(\mathbf{R}_1) \cdot \mathbf{R}_2) T_{\mathbf{R}_2} T_{\mathbf{R}_1}.
\]

As it may be easily verified, the functions \( \varphi_k \) satisfy the eigenvalue equation

\[
T_\ell \varphi_k(x) = \lambda_k(\ell) \varphi_k(x),
\]

\[
\lambda_k(\ell) = (-1)^{\ell_1 \ell_2} \exp(-i \ell \cdot k).
\]

Arranged in an arbitrary Slater determinant these functions are exact solutions of the Hartree-Fock problem\(^\text{16,17}\). This strong property happens because the HF single particle Hamiltonian commutes with all translations leaving \( L \) invariant\(^\text{19}\). The functions \( \varphi_k \) are common eigenfunctions of the commuting magnetic translations. Moreover, the set of eigenvalues \( \lambda_k(\ell) \) uniquely determines them. Therefore, the HF hamiltonian associated with the Slater determinant can not change those eigenvalues and the \( \varphi_k \) should be eigenfunctions.

Finally, let us show that the effect of an arbitrary translation on the basis functions is equivalent to a shift in the momentum label, modulo a phase factor\(^\text{24}\). Operating twice with the translation operator involving an arbitrary vector \( \mathbf{a} \) and a vector in the lattice \( \ell \) and using Eqs. \( \text{19} \) and \( \text{30} \) one readily gets,

\[
T_\ell T_\mathbf{a} \varphi_p(x) = \lambda_p(\ell) T_\mathbf{a} \varphi_p(x)
\]

\[
= \exp(2\frac{i e}{\hbar c} A(\mathbf{a}) \cdot \ell) T_\ell T_\mathbf{a} \varphi_p(x).
\]

Again using \( \text{30} \) we have

\[
T_\ell T_\mathbf{a} \varphi_p(x) = \lambda_{p+2\frac{i e}{\hbar c} A(\mathbf{a})}(\ell) T_\mathbf{a} \varphi_p(x).
\]
Then, taking into account that the set of eigenvalues defines uniquely the wave-functions modulo a phase, it follows that

\[ T_a \varphi_p(x) = F_p(a) \varphi_{p+2\pi\hbar A(a)}(x), \]  

(53)

from which also follows,

\[ F_p(a) = \frac{\varphi_p(0)}{\varphi_{p+2\pi\hbar A(a)}(a)}, \]  

(54)

That is, a magnetic translation is equivalent to a shift in the quasi-momentum.

B. A \( \varphi_{\mu} \)-transform

Any function \( f \) in the first Landau level, and its inverse, can be represented as

\[ f(x) = \sum_{p \in \tilde{B}} c(p) \varphi_p(x), \]

\[ = \int_{p \in \tilde{B}} \frac{dp}{(2\pi)^2} \hat{c}(p) \tilde{\varphi}_p(x), \]

\[ \hat{c}(p) = \int d^2x \tilde{\varphi}_p^*(x) f(x), \]

where \( \tilde{\varphi}_p(x) = \sqrt{S} \varphi_p(x), \hat{c}(p) = \sqrt{S} c(p) \), with \( S = 2\pi \phi_0^2 N_\phi \), with \( N_\phi \) the number of flux quanta in the system area, \( \tilde{B} \) is the Brillouin zone defined by the unit cell vectors

\[ t_1 = -\frac{1}{\tilde{r}_0} n \times b_2, \]

\[ t_2 = \frac{1}{\tilde{r}_0} n \times b_1. \]

The orthogonality and completeness relations in the first Landau level of the modified functions take the forms

\[ \int_A dx \tilde{\varphi}_p^*(x) \tilde{\varphi}_{p'}(x) = (2\pi)^2 \delta(p-p'), \]

\[ P(x,x') = \sum_{p \in \tilde{B}} \varphi_p(x) \varphi_p^*(x') = \int_{p \in \tilde{B}} \frac{dp}{(2\pi)^2} \tilde{\varphi}_p(x) \tilde{\varphi}_p^*(x') \]  

(55)

C. The orbitals \( \chi_{\mu}^{(r,\sigma)} \) as special cases of the basis functions \( \varphi_p(x) \)

Let us now verify that the functions \( \chi_{\mu}^{(r,\sigma)} \) constructed to diagonalize in blocks the HF Hamiltonian in Sec. 2 are simply constant phase factors multiplied by a particular kind of basis functions \( \varphi_p \). These differ from the orbitals considered in Sec. 2 and defined by the unit cell vectors \( \{7\} \) in that their basis vectors are given by

\[ b_1 = \frac{2}{3} a_1, \]

\[ b_2 = a_2. \]  

(56)

To start out it will be useful to consider the \( \varphi_p \)-transform of a translated "seed" function \( \phi(x) \). For this purpose we notice that it is possible to fix the second argument of the projector operator to be equal to the translation vector \( a \).
It follows that the projector operator in the first Landau level can be rewritten as

\[
P(x, a) = \frac{1}{2\pi r_0^2} \exp\left(-\frac{(x - a)^2}{4r_0^2}\right) \exp\frac{ie}{\hbar c} A(a, x) \tag{57}
\]

\[
= \frac{1}{\sqrt{2\pi r_0^2}} T_a \phi(x) = \int_{p \in B} \frac{dp}{(2\pi)^2} \bar{\varphi}_p(x) \varphi_p(a),
\]

where the last equality is merely the definition of the projector operator in coordinate space. Therefore, these relations determine the following \( \varphi_p \)-transform of the translated "seed" function

\[
T_a \phi(x) = \int_{p \in B} \frac{dp}{(2\pi)^2} \left( \sqrt{2\pi r_0^2} \varphi_p(a) \right) \bar{\varphi}_p(x).
\tag{58}
\]

Next we will consider the definition

\[
\chi_p^{(r, \sigma)}(x) = \frac{1}{\sqrt{6N_p^{(3,2)}}} \sum_m \exp(i P^{(p, r, \sigma)} m + \frac{5\pi i}{6} m_1 m_2) T_m \phi(x),
\]

and represent the vectors \( m \) in terms of alternative vectors \( \ell \) and the indices \( \alpha \) and \( \beta \) defined through the expressions

\[
\begin{align*}
m &= \ell + \delta, \\
\ell &= \ell_1 2 \mathbf{c}_1 + \ell_2 3 \mathbf{c}_2, \\
\delta &= u \mathbf{c}_1 + v \mathbf{c}_2.
\end{align*}
\]

The symbols \( \alpha, \beta, \ell_1 \), and \( \ell_2 \) are specified by the relations

\[
\begin{align*}
u &= [m_1] = \begin{cases} 0 & \text{if } m_1 = 0 \Modulo(2), \\ 1 & \text{if } m_1 = 1 \Modulo(2) \end{cases}, \\
u &= [m_1] = \begin{cases} -1 & \text{if } m_2 = -1 \Modulo(3), \\ 1 & \text{if } m_2 = 1 \Modulo(3) \end{cases}, \\
\ell_1 &= \frac{m_1 - \langle m_1 \rangle}{2}, \\
\ell_2 &= \frac{m_2 - \langle m_2 \rangle}{3}.
\end{align*}
\]

The use of these alternative definitions allows to write (59) in the form

\[
\chi_p^{(r, \sigma)}(x) = \frac{1}{\sqrt{6N_p^{(3,2)}}} \sum_\ell (-1)^{\ell_1 \ell_2} \exp(i P^{(p, r, \sigma)} \ell) T_\ell \times \\
\left\{ \sum_{u=0,1} \sum_{v=-1,0,1} \exp\left(\frac{5\pi i}{6} uv + i P^{(p, r, \sigma)} (uc_1 + vc_2)\right) T_{uc_1 + vc_2} \phi(x) \right\}.
\]

Now, it can be noticed that the above expression differs form the definition (47) only in that the "seed" function is changed by a superposition of itself translated to the points \( uc_1 + vc_2 \). However, it was shown in Ref. (16) that the basis functions are completely independent of an arbitrary change in the "seed" for any other function in the first Landau level.

An alternative proof of this curious property is possible. To see it let us represent the "seed" functions in (60) by their \( \varphi_p \)-transform (58), yieldng

\[
\chi_p^{(r, \sigma)}(x) = \frac{1}{\sqrt{6N_p^{(3,2)}}} \sum_{\alpha=0,1} \sum_{\beta=-1,0,1} \exp\left(\frac{5\pi i}{6} uv + i P^{(p, r, \sigma)} (uc_1 + vc_2)\right) \times \\
\int \frac{dq}{(2\pi)^2} \sqrt{2\pi r_0^2} \varphi_q (uc_1 + vc_2) \bar{\varphi}_q (x) \sum_\ell \exp(i P^{(p, r, \sigma)} - q) \ell,
\]
after employing the character of eigenfunctions of the operators $T_{l}$ that the functions $\tilde{\varphi}_{q}$ have. But now the following identity can be employed

$$\sum_{\ell \in A} \exp(i((q - q').\ell)) = \frac{(2\pi)^2}{2\pi r_0^2} \delta(q, q'),$$

where the $\delta$ function is non-vanishing for equal arguments modulo a vector of the reciprocal lattice $Q = \frac{3}{2}Q_{1}t_{1} + \frac{3}{2}Q_{2}t_{2}$. This relation allows to show the desired connection between the basis functions $\chi_{p}^{(r, \sigma)}$ and $\tilde{\varphi}_{p}(x)$,

$$\chi_{p}^{(r, \sigma)}(x) = F(p, r, \sigma) \tilde{\varphi}_{p(r, \sigma)}(x),$$

where the phase factor is given by

$$F(p, r, \sigma) = \frac{1}{\sqrt{6\sqrt{2\pi^2}A_{p}^{(3,2)}}} \sum_{u=0,1 \; v=-1,0,1} \exp\left(\frac{5\pi i}{6} uv + i p \cdot (u c_{1} + v c_{2})\right) \tilde{\varphi}_{p(r, \sigma)}^{u}(u c_{1} + v c_{2}).$$

### D. Periodicity of the Slater determinants of triplet orbitals

Here we show that the single particle density associated with a Slater determinant formed with a set of functions, each one corresponding to an arbitrary linear combinations of the triplet of orbitals characterized by the indices $(p, \sigma)$, is periodic over the lattice $R$. To consider this question, let us use the basic property that the density of a many electron Slater determinant constructed with orthogonal orbitals is the sum of the individual densities of each orbital. Writing the selected linear combinations as

$$\Psi_{p}^{(\sigma)}(x) = \sum_{r} C_{p}^{(r, \sigma)} \chi_{p}^{(r, \sigma)}(x),$$

where the coefficients $C_{p}^{(r, \sigma)}$ are arbitrary, and considering the symmetry properties, it follows that

$$\rho(x) = \sum_{p} \sum_{\sigma = \pm 1} \left|\Psi_{p}^{(\sigma)}(x)\right|^2$$

$$= \sum_{p} \sum_{\sigma = \pm 1} \left|\Psi_{p}^{(-\sigma)}(x)\right|^2$$

$$= \sum_{p} \sum_{\sigma = \pm 1} \sum_{r} C_{p}^{(r, \sigma)} C_{p}^{(r', \sigma)} \left(\exp(ip \cdot a_{1})T_{a_{1}} \chi_{p}^{(r', \sigma)}(x)\right)^{*} \exp(ip \cdot a_{1})T_{a_{1}} \chi_{p}^{(r, \sigma)}(x)$$

$$= \sum_{p} \sum_{\sigma = \pm 1} \sum_{r} C_{p}^{(r', \sigma)} C_{p}^{(r, \sigma)} \left(\exp\left(\frac{ie}{\hbar c} A(a_{1}) \cdot x\right) \chi_{p}^{(r', \sigma)}(x - a_{1})\right)^{*} \exp\left(\frac{ie}{\hbar c} A(a_{1}) \cdot x\right) \chi_{p}^{(r, \sigma)}(x - a_{1})$$

$$= \sum_{p} \sum_{\sigma = \pm 1} \left|\Psi_{p}^{(\sigma)}(x - a_{1})\right|^2$$

$$= \rho(x - a_{1}).$$

One can show in a similar way that $\rho(x) = \rho(x - a_{2}).$

### VII. APPENDIX B

Here we shall derive the general formula expressing the single particle Hartree-Fock hamiltonian as a sum of magnetic translation operations. The arguments of these operators are spatial vectors determined by the reciprocal lattice associated with the Fourier components of the density. It is worth remarking here that the lattice considered in this Appendix is not restricted in any way. Therefore, the representation of the single particle HF hamiltonian is valid for a general though periodic mean field problem.
Consider the action of the kernels of the direct and exchange interactions defining the HF hamiltonian over any of the elements $\varphi_p$ of the complete basis functions as follows

$$\sim^{(d)}(x) = H_{HF}^{(d)} \varphi_p(y') = \frac{e^2}{\varepsilon_o} \int dy \ dy' \ P(x, y') \frac{\rho(y) - n_0/e^2}{|y - y'|} \varphi_p(y'), \quad (62)$$

$$\sim^{(e)}(x) = H_{HF}^{(e)} \varphi_p(y') = -\frac{e^2}{\varepsilon_o} \int dy \ dy' \ P(x, y') \frac{\rho(y, y')}{|y - y'|} \varphi_p(y'), \quad (63)$$

where $n_0$ is the jellium background charge density making the overall system neutral. The representation we shall derive is valid for both direct and exchange kernels and will be discussed below for each case separately.

### A. Direct Coulomb interaction

The potential term representing the direct Coulomb interaction can be written in its Fourier transform representation as follows

$$v^{(d)}(x) = \frac{e^2}{\varepsilon_o} \int dx' \ \rho(x') - \frac{n_0/e^2}{|x - x'|}$$

$$= \frac{e^2}{\varepsilon_o} \sum_{Q \neq 0} \rho(Q) \int dx' \ \exp(iQ \cdot x)$$

$$= \sum_Q \frac{2\pi e^2 \rho(Q)(1 - \delta_{Q, 0})}{|Q|} \exp(iQ \cdot x)$$

$$= \sum_Q v^{(d)}(Q) \exp(iQ \cdot x). \quad (64)$$

Then, using the following representation of the kernel for the projection operator in the first Landau level

$$P(x, x') = \frac{1}{2\pi r_0^2} \exp\left(-\frac{(x - x')^2}{4r_0^2}\right) \exp\left[\frac{ie}{\hbar c} A(x') \cdot x\right]$$

we obtain,

$$\sim^{(d)} \varphi_p(x) = \int dy' \ P(x, y') \sum_Q v^{(d)}(Q) \exp(iQ \cdot y') \varphi_p(y'). \quad (65)$$

Recalling definition (47) we obtain for each translation term

$$L^{(d)}_\ell(x) = \int dy' \ P(x, y') \sum_Q v^{(d)}(Q) \exp(iQ \cdot y') T_\ell \phi(y'),$$

$$= \frac{1}{\sqrt{2\pi r_0}} \sum_Q v^{(d)}(Q) \exp(-\frac{1}{4}r_0^2Q^2) \exp\left(i\frac{Q \cdot \ell}{2}\right) \exp\left(-\frac{1}{4\pi r_0^2}(x - (\ell + r_0^2Q \times n))^2\right)$$

$$- \frac{i}{2r_0} x \cdot (\ell + r_0^2n \times Q),$$

$$= \sum_Q v^{(d)}(Q) \exp(-\frac{1}{4}r_0^2Q^2) \exp\left(i\frac{Q \cdot 1}{2}\right) T_{\ell + r_0^2n \times Q} \phi(x). \quad (66)$$

This last relation can be further simplified by employing relation (52) and the similar property

$$T_{R_1} T_{R_2} = \exp\left[i\frac{e}{\hbar c} A(R_1) \cdot R_2\right] T_{R_1 + R_2},$$

leading to the formula
\[
\int dy' P(x, y') \sum_Q v^{(d)}(Q) \exp(iQy')T_0 \phi(x) = \sum_Q v^{(d)}(Q) \exp(-\frac{1}{4} r_0^2 Q^2) T_{r_0^2 n \times Q} T_0 \phi(x).
\]

After adding corresponding the integrals over all translations \( \ell \) one obtains

\[
H_{HF}^{(d)} \varphi_p(x) = \sum_Q v^{(d)}(Q) \exp(-\frac{1}{4} r_0^2 Q^2) T_{r_0^2 n \times Q} \varphi_p(x).
\]

Since the equality is valid for any element of the complete basis \( \varphi_p(x) \) it then follows,

\[
H_{HF}^{(d)} = \sum_Q v^{(d)}(Q) \exp(-\frac{1}{4} r_0^2 Q^2) T_{r_0^2 n \times Q}.
\]

### B. Exchange interaction

The derivation of the analogous representation for the exchange interaction kernel is more involved and needs for some special properties of the one particle density matrix. Then, let us initially consider these properties.

#### 1. One particle density matrix transformations

The definition of the one-particle density matrix in terms of the Slater determinant \( \Phi(x_1, x_2, x_3, x_4, \ldots, x_N) \) can be transformed by performing a simultaneous magnetic translation operation in a vector \( R \) of the periodic lattice over all the particle coordinates. Since this is assumed to be a symmetry transformation of the system, this map should leave the many particle state invariant. Thus by assumption

\[
\rho(x_1, x_2) = \int \ldots \int d{x_3} d{x_4} \ldots d{x_N} |\Phi(x_1, x_2, x_3, x_4, \ldots, x_N)|^2
\]

\[
= \int \ldots \int d{x_3} d{x_4} \ldots d{x_N} |\prod_{i=3}^{N} T_{R}(x_i) \Phi(x_1, x_2, x_3, x_4, \ldots, x_N)|^2
\]

\[
= \sum_{x} \Psi_x(x_1) \Psi_x^*(x_2),
\]

\[
= \sum_{x} T_{R}(x_1) \Psi_x(x_1) (T_{R}(x_2) \Psi_x(x_2))^*,
\]

\[
= \sum_{x} \exp\left(\frac{i e A(R)}{\hbar c} (x_1 - x_2)\right) \exp(\Psi_x(x_1 - R) (\Psi_x(x_2 - R))^* \rho(x_1, x_2),
\]

\[
= \exp\left(\frac{i e A(R)}{\hbar c} (x_1 - x_2)\right) \rho(x_1 - R, x_2 - R),
\]

an expression that furnishes the transformation law of the one particle density matrix under spatial shifts in the vectors of the periodic lattice.

#### 2. The density determines the density matrix

Below we will show that the Fourier components of the density completely determine the whole one-particle density matrix. For this purpose, let us use the new variable \( z = x - x' \) in the one-particle density matrix, so that \( \rho(x, x') = \tilde{\rho}(x, z) \), where the tilde over \( \rho \) underlines the different functional expression of the new definition. It now transforms under spatial shifts as

\[
\tilde{\rho}(x - R, z) = \exp\left(-\frac{i e A(R)}{\hbar c} z\right) \tilde{\rho}(x, z).
\]

(69)
Therefore, the function
\[
\tilde{g}(x, z) = \exp(-ieA(x), z) \tilde{\rho}(x, z),
\]
is fully periodic in the variable $x$ under the lattice shifts. That is $\tilde{g}(x - R, z) = \tilde{g}(x, z)$. Then, Fourier expanding $\tilde{g}$ leads to the following expression for $\tilde{\rho}$
\[
\tilde{\rho}(x, z) = \sum_Q \exp(i(Q - \frac{e}{\hbar c}A(R)), x) \tilde{\rho}(Q, z),
\]
where
\[
\tilde{\rho}(Q, z) = \frac{1}{\mathcal{A}_{cell}} \int dx' \exp(-iQ, x) \exp(i\frac{e}{\hbar c}A(z), x') \tilde{\rho}(x', z).
\]
(70)

Further, let us consider the density written as the sum
\[
\sum_x \psi \psi^*(x') = \sum_p \sum_{r, \sigma} \sum_{r', \sigma'} f_{r, \sigma}(p) \lambda^{(r, \sigma)}(p)(x) \Omega_{r', \sigma'}^{(r, \sigma)}(p) \lambda^{(r', \sigma')}(x'),
\]
and the fact that each $\lambda^{(r, \sigma)}(p)$ in the lower Landau level can be expanded as a linear combination of the complete basis functions $\{T_{\ell} \phi(\mathbf{x})\}$ defined over a lattice with one flux quantum per unit cell. Then, a generic term in the integral in (70) has the form
\[
\tilde{\rho}_{\ell, \ell'}(Q, z) = \frac{1}{\mathcal{A}_{cell}} \int dx \exp(-iQ, x) \exp(i\frac{e}{\hbar c}A(z), x) T_{\ell} \phi(\mathbf{x})(T_{\ell'} \phi(\mathbf{x'}) | x'=x-z)^*.
\]
Since the function $\phi$ is a Gaussian, its magnetic translations have also this character, and thus the integrals in (70) can be calculated explicitly. We obtain,
\[
\tilde{\rho}_{\ell, \ell'}(Q, z) = \tilde{\rho}_{\ell, \ell'}(Q, 0) \exp(-z^2/4r_0^2 - 1/4r_0^2 \mathbf{Q} \cdot (\mathbf{n} \times \mathbf{z} + i \mathbf{z})).
\]
But since the phase factor here is independent of the indices $\ell, \ell'$, the summation of all the contributions in (70) leads to the relation
\[
\tilde{\rho}(Q, z) = \rho(Q) \exp(-z^2/4r_0^2 - 1/4r_0^2 \mathbf{Q} \cdot (\mathbf{n} \times \mathbf{z} + i \mathbf{z})),
\]
(71)
which expresses the interesting result that the full one-particle density matrix is completely determined by the particle density.

We now consider the exchange term (78). Using the above results the integrals in
\[
\varphi^{(e)}(x) = -e^2 \int dy' P(x, y') \int dz \sum_Q \exp(i(Q - \frac{e}{\hbar c}A(z)), y') \frac{\tilde{\rho}(Q, z)}{|z|} \varphi_p(y' - z),
\]
(72)
can be calculated explicitly. For this purpose, let us consider again the integral of a generic term $T_{\ell} \phi$ in the sum over $\ell$ defining the functions $\varphi_p$. The integral for this term is then reduced to two simple integrals, which after some algebra can be explicitly evaluated to obtain
\[
I^{(e)}(x) = \sum_Q \psi^{(e)}(Q) \exp(-1/4r_0^2 \mathbf{Q}^2 \mathbf{x}) \exp(i\frac{Q, \ell}{2}) T_{i+\mathbf{z} \times \mathbf{Q}} x \mathbf{Q}^2 \phi(x),
\]
(73)
\[
\sum_Q \psi^{(e)}(Q) \exp(-1/4r_0^2 \mathbf{Q}^2 \mathbf{x}) T_{i+z} x \mathbf{Q}^2 T_{\ell} x \phi(x),
\]
(73)
\[
\psi^{(e)}(Q) = -\frac{2\pi e^2 r_0}{\varepsilon_0} \sqrt{\frac{\pi}{2}} \frac{\tilde{\rho}(Q, 0)}{\mathbf{Q}^2} \exp(1/4r_0^2 \mathbf{Q}^2) I_0(1/4r_0^2 \mathbf{Q}^2).
\]
(74)
Henceforth, after adding all terms for different values of \( \ell \) follows

\[
H_{HF}^{(c)} \varphi_p(x) = \left( \sum_Q v^{(c)}(Q) \exp(-\frac{1}{4}r_0^2Q^2)|T_{r_0^2n \times Q}\right) \varphi_p(x).
\]

Using the independence and completeness of the basis formed by the \( \varphi_p \)'s in the first Landau level, we thus obtain

\[
H_{HF}^{(c)} = \sum_Q v^{(c)}(Q) \exp(-\frac{1}{4}r_0^2Q^2)|T_{r_0^2n \times Q}.
\]

Combining this results and the previous one \(^{67}\) the following representation for the one-particle Hartree-Fock Hamiltonian in the first Landau level follows,

\[
H_{HF} = \sum_Q (v^{(d)}(Q) + v^{(c)}(Q)) \exp(-\frac{1}{4}r_0^2Q^2)|T_{r_0^2n \times Q}.
\]

with the coefficients defined by

\[
v(Q) = 2\pi r_0^2 \rho(Q) \left( \frac{(1 - \delta_{Q,0})}{r_0 |Q|} - \sqrt{\frac{\pi}{2}} \exp\left(\frac{1}{4}r_0^2Q^2\right) I_0\left(\frac{1}{4}r_0^2Q^2\right) \right) \frac{e^2}{\varepsilon_o r_0}. \tag{76}
\]

\[\text{VIII. APPENDIX C}\]

In this section we prove that the eigenfunctions that vanish at all lattice points \( \mathbf{R} \) are solutions of the HF equations. For this purpose, let us consider the Fourier components

\[
\rho_{y}(Q) = \frac{1}{A_{cell}} \int dx \rho_{y}(x) \exp(-iQ \cdot x)
\]

of the configuration space periodically extended density operator

\[
\rho_{y}(x) = \delta(x - y) = \sum_{\mathbf{R}} \delta(x - y + \mathbf{R})
\]

\[
= \frac{1}{A_{cell}} \sum_{Q} \exp(-iQ \cdot y) \exp(iQ \cdot x).
\]

We will examine the projection of this operator in the first Landau level. For this aim it is useful to consider the set of operators

\[
P_i = \partial_i + \frac{e}{\hbar c} A_i(x)
\]

\[
= \partial_i + \frac{i}{2r_0^2} \epsilon_{ij} x_j
\]

\[
G_i = \partial_i - \frac{e}{\hbar c} A_i(x)
\]

\[
= \partial_i - \frac{i}{2r_0^2} \epsilon_{ij} x_j
\]

\[
\epsilon_{ij} \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},
\]

where the indices \( i, j \) take the values 1, 2. These quantities satisfy the commutation relations
\[ [P_i, G_j] = 0, \]
\[ [P_i, P_j] = -\frac{\epsilon_{ij}}{r_0^2}, \]
\[ [G_i, G_j] = \frac{\epsilon_{ij}}{r_0^2}. \]

With their use, the spatial Fourier modes can be identically represented as

\[
\exp(-iQ \cdot x) = \exp\left(-\frac{1}{2}(P_i - G_i) \cdot \epsilon_{ik}Q_k(2r_0^2)\right) \\
= \exp\left(-\frac{1}{2}(P - G) \cdot n \times Q(2r_0^2)\right),
\]

(77)

where the 3D-vectors \( P \) and \( G \) are defined as \((P_1, P_2, 0)\) and \((G_1, G_2, 0)\). But the two components \( P_i \) are linear combinations of the rising and lowering operators which transform functions of a Landau level in functions of the higher and lower contiguous levels. Therefore, the projection onto the first Landau level of the Fourier component can be obtained by simply omitting the operator \( P \) in the above formula. Then, the projection of the density operator will be

\[
\rho_{0y}^{(p)}(x) = \frac{1}{A_{cell}} \sum_{Q} \exp(iQ \cdot y) \exp\left(-\frac{1}{2}G \cdot n \times Q(2r_0^2)\right) \\
= \frac{1}{A_{cell}} \sum_{Q} \exp(iQ \cdot y) T_{n \times Q(2r_0^2)} \\
= \frac{1}{A_{cell}} \sum_{x^*} \exp(ix^* \cdot n \times y(2r_0^2)) T_{x^*},
\]

where a new set of spatial lattice vectors was defined in terms of the Fourier wavevectors,

\[ x^* = n \times Qr_0^2. \]

Let us consider now the identity

\[
\rho_0 \sum_r g_r^0(p) |\chi_p^{(r, \sigma)}\rangle = 0,
\]

where the three coefficients \( g_r^0(p) \), \( r = 1, 2, 3 \), are fixed by the condition that the linear combination of the functions of the two triplets of functions \(|\chi_p^{(r, \sigma)}\rangle\) for \( \sigma = \pm 1 \) vanish at the origin \( x = 0 \). Projecting the above identity onto the first Landau level by acting with \( P \) at the left it follows

\[
P \rho_0 \sum_r g_r^0(p) |\chi_p^{(r, \sigma)}\rangle = P \rho_0 P \rho_0 \sum_r g_r^0(p) |\chi_p^{(r, \sigma)}\rangle \\
= \rho_{0y}^{(p)}(x) = \frac{1}{A_{cell}} \sum_{Q} \exp(iQ \cdot y) T_{n \times Q(2r_0^2)} \\
= \frac{1}{A_{cell}} \sum_{x^*} \exp(ix^* \cdot n \times y(2r_0^2)) T_{x^*},
\]

(78)

implying that the projected periodic density has the defined functions as eigenvectors with zero eigenvalues. Let us inspect further the commutation properties of the projected density operator. Using the representation of the single particle Hamiltonian in terms of the magnetic translations found in Appendix B and the relations

\[
T_{x^*}T_{y^*} = \exp\left(\frac{ie}{\hbar c} A(x^*) \cdot y^*\right) T_{x^*+y^*}, \\
T_{y^*}T_{x^*} = \exp\left(\frac{ie}{\hbar c} A(x^*) \cdot y^*\right) T_{x^*+y^*},
\]

it follows that
\[
\left[ H_{HF}, \rho_0^{(p)} \right] = \sum_{x^*} \sum_{y^*} u(x^*) [T_{x^*}, T_{y^*}]
\]

\[(79)\]

\[
= \sum_{x^*} \sum_{y^*} u(x^*) (\exp(i e \hbar^{-1} A(x^*), z^*) - \exp(-i e \hbar^{-1} A(x^*), z^*)) T_{x^*}.
\]

Considering that \( u \) is the coefficient of the superposition of translations giving the single particle HF Hamiltonian which is a function of \( Q^2 \), and using the inversion symmetry of the solution under consideration \( u(x^*) = u(-x^*) \), one finds

\[
\left[ H_{HF}, \rho_0^{(p)} \right] = 0.
\]

(80)

Hence, acting with the Hamiltonian on \( \rho_0 \) and employing the definition of the projection operators and the periodic density operator it follows that

\[
H_{HF} \rho_0^{(p)} \sum_r g_r^0(p) \mid \chi^{(r,\sigma)}_p \rangle = \tilde{\rho}_0 H_{HF} \sum_r g_r^0(p) \mid \chi^{(r,\sigma)}_p \rangle
\]

\[
= \int dx' P(x, x') \rho_0(x') \sum_r h_r(p) \chi^{(r,\sigma)}_p (x')
\]

\[
= \int dx' P(x, x') \sum_R \delta(x' - R) \sum_r h_r(p) \chi^{(r,\sigma)}_p (x')
\]

\[
= \sum_R P(x, R) \sum_r h_r(p) \chi^{(r,\sigma)}_p (R)
\]

\[
= \sum_R \frac{1}{2\pi r_0^2} \exp(-\frac{(x - R)^2}{4r_0^2}) \exp(i e \hbar^{-1} A(R), x) \sum_r h_r(p) \chi^{(r,\sigma)}_p (R)
\]

\[
= \sum_R \frac{1}{\sqrt{2\pi r_0^2}} \sum_r h_r(p) \chi^{(r,\sigma)}_p (R) T_R \phi(x)
\]

\[
= 0.
\]

Finally, taking into account that the set of functions \( T_R \phi(x) \) for all values of the lattice \( R \) are linearly independent, it follows that

\[
\sum_r h_r(p) \chi^{(r,\sigma)}_p (R) = 0.
\]

(81)

In addition, since the coefficients of the linear combinations of the triplet functions for the two signs of \( \sigma \) are uniquely defined by the vanishing condition of the function at the origin (or any point of the lattice \( R \)), we have,

\[
h_r(p) = \varepsilon g_r^0(p),
\]

where \( \varepsilon \) is here a normalization constant. Finally, we obtain

\[
H_{HF} \sum_r g_r^0(p) \mid \chi^{(r,\sigma)}_p \rangle = \sum_r h_r(p) \mid \chi^{(r,\sigma)}_p \rangle,
\]

\[
= \varepsilon \sum_r g_r^0(p) \mid \chi^{(r,\sigma)}_p \rangle.
\]

That is, the special functions that vanish at all lattice points are eigenfunctions of the one particle HF Hamiltonian.

**IX. APPENDIX D**

In this Appendix we sketch the derivation of the formula employed in the evaluation of the sample size dependence of the correction to the energy per particle. As before, we use the shorthand notation

\[
\Phi_\alpha \equiv \Phi^{(b_\alpha, \sigma_\alpha)}_{\beta_\alpha} = \sum_r g_{r_\alpha}^{b_\alpha}(p_\alpha) \chi^{(r,\sigma_\alpha)}_\alpha(x)
\]
\[ \sum_r d_{p_\alpha}^{(b_\alpha, r, \sigma_\alpha)} \phi_{p_\alpha, r}(x), \]

\[ d_{p_\alpha}^{(b_\alpha, r, \sigma_\alpha)} = \mathcal{F}(p_\alpha, r, \sigma_\alpha) d_{p_\alpha}^{(0, r, \sigma_\alpha)}(p_\alpha) \]

\[ P_{\alpha, r} = P(p_{\alpha}, r, \sigma_\alpha) = p_\alpha - r s_1 + \frac{\sigma_\alpha - 1}{2} s_2. \]

The relations expressing the functions \( \chi^{(r, \sigma_\alpha)} \) in terms of \( \phi_p \) times a phase factor has also been employed in order to define the new coefficients \( d_{p_\alpha}^{(b_\alpha, r, \sigma_\alpha)} \). By representing the Coulomb potential by its Fourier transform, and substituting all the expansions of the functions \( \phi_p \) in terms of magnetic translations of the "seed" in the matrix elements \( [\mathcal{M}] \), the resulting Gaussian integrals for each term can be evaluated. After some algebra it is possible to obtain the expression

\[
M(\alpha, \alpha' | \beta', \beta) = \int \int dx dx' \Phi_\alpha^*(x) \Phi_{\alpha'}(x') \frac{1}{|x - x'|} \Phi_{\beta'}(x') \Phi_\beta(x),
\]

\[
= \sum_{\alpha} \sum_{\alpha'} \sum_{r, r', \sigma_\beta} \left( d_{p_\alpha}^{(b_\alpha, r, \sigma_\alpha)} d_{p_{\alpha'}}^{(b_{\alpha'}, r', \sigma_{\alpha'})} \right)^* d_{p_{\beta'}}^{(0, r', \sigma_{\beta'})} d_{p_{\beta}}^{(0, r, \sigma_{\beta})} \times
\int \int dx dx' \varphi_{p_\alpha, r, \sigma_\alpha}(x) \varphi_{p_{\alpha'}, r', \sigma_{\alpha'}}(x') \frac{1}{|x - x'|} \varphi_{p_{\beta'}, r', \sigma_{\beta'}}(x') \varphi_{p_{\beta}, r, \sigma_{\beta}}(x),
\]

\[
= \bar{N}_{\phi_\alpha} \sum_{\alpha} \sum_{\alpha'} \sum_{r, r', \sigma_\beta} \left( d_{p_\alpha}^{(b_\alpha, r, \sigma_\alpha)} d_{p_{\alpha'}}^{(b_{\alpha'}, r', \sigma_{\alpha'})} \right)^* d_{p_{\beta'}}^{(0, r', \sigma_{\beta'})} d_{p_{\beta}}^{(0, r, \sigma_{\beta})} \times
\delta(P_{\beta', r, \beta'} + P_{\beta, r, \beta} - P_{\alpha, r, \alpha} - P_{\alpha', r, \alpha'}) \times
\varphi_{p_\alpha}(n \times (P_{\beta', r, \beta'} - P_{\alpha', r, \alpha'})) \varphi_{p_{\alpha'}}(-n \times (P_{\beta', r, \beta'} - P_{\alpha', r, \alpha'}))^2 \times
\frac{2\pi}{N_{P_{\beta', r, \beta'}}^2} \frac{2\pi}{N_{P_{\beta, r, \beta}}^2} \exp(-r_0^2(P_{\beta', r, \beta'} - P_{\alpha', r, \alpha'})^2) \times
\exp \left[ -i r_0^2 n \times Q^* \cdot (P_{\alpha, r, \alpha} + P_{\beta, r, \beta} - P_{\beta', r, \beta'} - P_{\alpha', r, \alpha'}) \right],
\]

where the summation vectors \( Q^* \) and the norm functions are defined as follows,

\[
Q^* = n_1 \frac{3}{2} s_1 + n_2 s_2,
\]

\[
N_{P_{\beta', r, \beta'}}^2 = \sqrt{\bar{N}_{\phi_\alpha}} \sqrt{\sum_{\ell} (-1)^{\ell_1 \ell_2} \exp(i \ell \cdot \ell - \frac{\ell^2}{4r_0^2})},
\]

\[
\ell = \ell_1 (2 c_1) + \ell_2 (3 c_2),
\]

and, as before, \( \delta(P', P) \) equals one when the arguments are same modulo vectors of the class \( Q^* \) defined above, and zero otherwise. It is possible to show now that the ratio multiplying this latter function a simply a phase factor. For this purpose let us consider the definition of the \( \varphi_{p_\alpha, r, \alpha} \),

\[
\varphi_{p_\alpha, r, \alpha}(x) = \frac{1}{N_{p_\alpha, r, \alpha}} \sum_{\ell} (-1)^{\ell_1 \ell_2} \exp(i \ell_1 p_{\alpha, r, \alpha} \cdot \ell) T_{\ell} \phi(x),
\]

from which follows directly a connection between the value of this function at the origin and the norm functions appearing in (83)

\[
\varphi_{p_\alpha, r, \alpha}(0) = \sqrt{\sum_{\ell} (-1)^{\ell_1 \ell_2} \exp(i \ell_1 p_{\alpha, r, \alpha} \cdot \ell - \frac{\ell^2}{4r_0^2})} = \frac{N_{p_\alpha, r, \alpha}^{(2.3)}}{\sqrt{2 \pi r_0^2 \bar{N}_{\phi_\alpha}}}. \]

Further, let us recall the formula connecting a translated function of the basis \( \varphi_P \) and the same function for a shifted momenta (55). After considering this relation evaluated at \( x = 0 \), the following expression can be obtained
\[
\varphi_{\alpha',\alpha}(\mathbf{n} \times (\mathbf{P}_{\beta',r_{\beta'}} - \mathbf{P}_{\alpha',r_{\alpha'}}) r_{0}^2) = \varphi_{\alpha',\alpha}(\mathbf{n} \times (\mathbf{P}_{\alpha',r_{\alpha'}} - \mathbf{P}_{\beta,r_{\beta}}) r_{0}^2) \\
= \mathcal{F}_{\alpha',\alpha}(\mathbf{n} \times (\mathbf{P}_{\beta,r_{\beta}} - \mathbf{P}_{\alpha',r_{\alpha'}}) r_{0}^2) \varphi_{\alpha',\alpha} - \frac{2e}{\hbar c} A(\mathbf{n} \times (\mathbf{P}_{\alpha',r_{\alpha'}} - \mathbf{P}_{\beta',r_{\beta'}}) r_{0}^2)(0).
\]

But the new momenta argument of the function in the rhs can be simplified as

\[
-\frac{2e}{\hbar c} A(\mathbf{n} \times (\mathbf{P}_{\alpha',r_{\alpha'}} - \mathbf{P}_{\beta,r_{\beta}}) r_{0}^2) = \frac{2}{r_{0}^2} \frac{n \times (\mathbf{n} \times (\mathbf{P}_{\alpha',r_{\alpha'}} - \mathbf{P}_{\beta,r_{\beta}}) r_{0}^2)}{r_{0}^2} = \mathbf{P}_{\beta,r_{\beta}} - \mathbf{P}_{\alpha',r_{\alpha'}}.
\]

to produce

\[
\varphi_{\alpha',\alpha}(\mathbf{n} \times (\mathbf{P}_{\beta',r_{\beta'}} - \mathbf{P}_{\alpha',r_{\alpha'}}) r_{0}^2) = \mathcal{F}_{\alpha',\alpha}(\mathbf{n} \times (\mathbf{P}_{\beta,r_{\beta}} - \mathbf{P}_{\alpha',r_{\alpha'}}) r_{0}^2) \varphi_{\alpha',\alpha}(0).
\]

Similarly it also follows

\[
\varphi_{\alpha',\alpha'}(-\mathbf{n} \times (\mathbf{P}_{\beta',r_{\beta'}} - \mathbf{P}_{\alpha',r_{\alpha'}}) r_{0}^2) = \mathcal{F}_{\alpha',\alpha'}(\mathbf{n} \times (\mathbf{P}_{\beta',r_{\beta'}} - \mathbf{P}_{\alpha',r_{\alpha'}}) r_{0}^2) \varphi_{\alpha',\alpha'}(0),
\]

which finally leads to the desired results

\[
\frac{\varphi_{\alpha',\alpha}(\mathbf{n} \times (\mathbf{P}_{\beta',r_{\beta'}} - \mathbf{P}_{\alpha',r_{\alpha'}}) r_{0}^2) \varphi_{\alpha',\alpha'}(-\mathbf{n} \times (\mathbf{P}_{\beta',r_{\beta'}} - \mathbf{P}_{\alpha',r_{\alpha'}}) r_{0}^2)}{N_{\beta',r_{\beta'}}^{(2)} N_{\alpha',r_{\alpha'}}^{(2)}} = \frac{2\pi r_{0}^2 N_{\phi_{0}}^2}{(86)}.
\]

The expression for the matrix element allows to directly write formula employed in Section 3 for evaluating the correlation energy.

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