A Noncommutative Mikusiński Calculus

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Abstract We set up a left ring of fractions over a certain ring of boundary problems for linear ordinary differential equations. The fraction ring acts naturally on a new module of generalized functions. The latter includes an isomorphic copy of the differential algebra underlying the given ring of boundary problems. Our methodology employs noncommutative localization in the theory of integro-differential algebras and operators. The resulting structure allows to build a symbolic calculus in the style of Heaviside and Mikusiński, but with the added benefit of incorporating boundary conditions where the traditional calculi allow only initial conditions.

Keywords Linear boundary problems · Differential algebra · Mikusinski calculus · Integro-differential operators · Localization

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1 Introduction

General Context. Linear boundary problems are a crucial concern of applied mathematics [15,1,2,50]. One might thus expect a rich algebraic theory with symbolic algorithms, so as to support the exact solution and manipulation of a suitable class of boundary problems. Alas, this is not the case yet.

There may be two reasons for this. One is that the classical algebraic theory of (nonlinear) differential equations, the differential algebra built up by Ritt [37] and Kolchin [23], does not lend itself easily to boundary conditions: The elements of a...
differential ring or field are not functions but abstract objects that cannot be evaluated at a boundary point.

The other reason is of a pragmatic nature. If a differential equation can be solved at all, one tends to delegate boundary conditions to adhoc postprocessing steps that would adapt the integration constants/functions of the “general solution”. While this may be viable for linear ordinary differential equations (LODEs), the notion of general solution is much less useful in the case of partial differential equations (LPDEs): For example, there is not much point in solving the $u_{xx} + u_{yy} = 0$ per se, but there are useful representations of those solutions that satisfy Dirichlet boundary conditions on the unit disc (Poisson kernel). So even though we will be dealing only with LODEs in this paper, the larger context of LPDEs should be kept in mind.

In the admittedly modest case of boundary problems for LODEs, an algebraic theory was set up in [39]. The decisive step was to expand the well-known structure of a differential ring, complementing its derivation by a compatible Baxter operator (integral operator). This solves at once two problems: It provides the algebraic structure needed for expressing Green’s operators (solution operators for boundary problems), and it yields an evaluation (a multiplicative functional). The notion of evaluation is the key for imposing boundary conditions on the otherwise abstract objects of a differential ring; one supplies as many evaluation as needed (in addition to the one coming from the Baxter operator).

In [40], the basic theory of [39] was refined and generalized. Moreover, the authors have introduced a multiplicative structure on boundary problems that will also be crucial for our buildup of the Heaviside calculus. The multiplication of boundary problems corresponds to the composition of their Green’s operators in reverse order; see Equation (9).

The framework of boundary problems of [39,40] was implemented several times: The first implementation was coded in MATHEMATICA/THORÆM as an external package for boundary problems with constant coefficients. This version was superseded by a new implementation as an internal THORÆM functor for generic integro-differential algebras in [42,52,43]. Now in the third generation, the most recent implementation is the Maple package IntDiffOp [26,27,28], which provides additional support for singular boundary problems (all previous packages being restricted to regular boundary problems).

**Heaviside Calculus.** Our main concern in this paper is to build a bridge from the framework of symbolic boundary problems to the classical Heaviside calculus [21, 22, 31]. Since O. Heaviside’s main idea, subsequently made rigorous by J. Mikusiński, was to treat the differential operator as a symbolic multiplier, it is perhaps natural to reflect on possible connections to contemporary symbolic methods for differential equations. But why to symbolic boundary problems?

The answer lies in Heaviside’s so-called fundamental formula, the algebraic analog of a well-known relation for the Laplace transform. If $s$ is the “symbolic multiplier” representing the differential operator and $f$ is a suitable function, then one has $sf = f' + f(0)\delta_0$. The point is that $s$, which is supposed to be invertible in this calculus, must somehow “remember” the integration constant that is lost in differentiation. We will come back to this point in more detail (Section 5). At this point
it suffices to say that an appropriate action of $s$ on functions involves an evaluation operator. It is used in the Heaviside-Mikusinski calculus for incorporating the initial values associated to a differential equation, thus yielding at once the solution of the whole initial value problem. It is thus natural—staying in the frame of LODEs—to ask if one can build up a more general calculus that would allow to incorporate multiple boundary values, given by several evaluation operators.

The basic ideas of Heaviside and Mikusiński have been vastly extended, specifically by L. Berg [5] and I. Dimovski [12]. The latter has also broached a question closely related to the one raised above, namely the setup of nonlocal convolutions [13, 49] and custom-tailored convolutions for boundary problems [14]. As far as we know, however, there are no direct generalizations of the fundamental formula from one to more evaluation operators acting on univariate functions.

**Basic setup.** Our own approach is different in many respects. It does not qualify as an operational calculus in Dimovski’s understanding [14], for whom its crucial feature is that “operators” and “operands” are merged in a single data structure, whereas algebraic analysis keeps the module of operand functions separate from the ring of operators. In this sense, we follow the line of algebraic analysis since we distinguish the ring of methorious operators (Section 4) from the module of methorious functions (Section 5). Moreover, our approach is genuinely algebraic while the hallmark of the Heaviside-Mikusiński tradition is an ingenious mix of algebra and analysis. Nevertheless, we believe that our setup is close in spirit to the original Heaviside-Mikusiński setup, enjoying the following attractive features:

- Its basic philosophy follows closely in Heaviside’s footsteps, making the differential operator invertible by “remembering” suitable boundary data.
- It provides an algebraic structure that accommodates boundary problems for an arbitrary number of evaluation points as well as nonlocal conditions.
- All boundary problems are covered uniformly, so one need not set up custom-tailored multiplications (convolutions) for each type of boundary conditions one wants to consider.
- It illuminates the passage from one to several evaluations algebraically: Localization takes place in a noncommutative ring. In contrast, convolution algebras are commutative by definition [12].
- The construction is generic in the coefficient algebra (it works for the class of umbral integro-differential algebras).

From the viewpoint of analysis, however, the class of umbral coefficient algebras is rather limited compared to Mikusiński’s setup (continuous or even $L^2$ functions on the positive half-axis). While this might be relaxed by suitable limit considerations, this is not in the interest of the present paper, where we want to focus on the algebraic aspects.

Our goal in this paper is to build a first bridge between the algebraic theory of boundary problems [39,40] and Heaviside’s tactic of using “symbolic fractions” (localization in a suitable convolution ring) for integrating differential equations with initial/boundary data. In fact, it gives a new justification (see after Proposition 10) for a notational device initiated in [40]: Since then we have written $B^{-1}$ for the Green’s
operator associated to a boundary problem $\mathcal{B}$, originally chosen in view of the anti-isomorphism (9) mentioned above.

The work in this paper may also be seen as an answer to an (implicit) question originating from [35]. The setting there was restricted to $K[x]$ coefficients and treated via Ore algebras. The object of central interest was the integro-differential Weyl algebra, which has shown to have two important quotient algebras: The ring of localized differential operators $K[\partial, \partial^{-1}] [x]$ and the usual ring of integro-differential operators $K[x][\partial, \int]$. The former has $\partial$ as a two-sided inverse, but no action on $K[x]$; the latter has the action, but $\int$ is only a right inverse of $\partial$. Now the question arises: Can one build up a structure, with an action on $K[x]$ or other coefficient algebras, such that the derivation has a two-sided inverse? We show in this paper that the answer is affirmative, provided the derivation is enhanced in a way similar to Heaviside’s symbolic multiplier $s$.

As always in algebra, localizing a ring sheds new light on its structure, especially in the noncommutative setting (here the theory of fraction rings is somewhat more delicate due to the Ore condition, see Section 4). We hope this will also be the case for our present construction. But one must bear in mind that the subject has only been touched and various issues remain in a preliminary and unsatisfactory state (see the Conclusion). Nevertheless, a wealth of new relations has been uncovered, and we are confident that they will allow interesting generalizations and refinements.

Structure of the paper. We start out by summarizing the basic theory of symbolic boundary problems, giving special emphasis to their monoid structure (Section 2). Our construction is based on a certain kind of boundary conditions, which we have called “umbral” because of certain relations to the umbral calculus (Section 3). After this preparation, we tackle the task of localization for an integro-differential algebra with an umbral character set (Section 4). For the resulting ring of fractions, we construct a module of functions on which it acts naturally (Section 5). We conclude with some remarks about possible extensions and generalizations (Section 6).

Notation. All rings are with unit but not necessarily commutative. A domain is a (commutative or noncommutative) ring without zero divisors. The zero vector space of any dimension will be denoted by $O$. We use the notation $\mathcal{F}_1 \leq \mathcal{F}_2$ for indicating that a vector space $\mathcal{F}_1$ is a subspace of a vector space $\mathcal{F}_2$.

2 The Monoid of Boundary Problems

Our basic setting is that of [41]. We review the main results here for making our present treatment more self-contained and for introducing various pieces of notation and terminology in their proper places. We start from the notion of integro-differential algebra, a natural generalization of differential algebras that permits the algebraic formulation of linear boundary problems.
Definition 1  We call \((\mathcal{F}, \partial, \int)\) an integro-differential algebra over \(K\) if \(\mathcal{F}\) is a commutative \(K\)-algebra with \(K\)-linear operations \(\partial\) and \(\int\) such that the three axioms

\[
\begin{align*}
(f')' &= f, \\
(fg)' &= f'g + fg', \\
(fg)(g') + \int(fg)' &= (f')g + f(g')
\end{align*}
\]

are satisfied, where \(\ldots'\) is the usual shorthand notation for \(\partial\).

This definition differs from Definition 4 in [41] as it uses (3) as the differential Baxter axiom rather than the variant

\[
\int fg = f\int g - \int f'\int g
\]

used earlier. In fact, one checks immediately that for commutative \(\mathcal{F}\) both axioms are equivalent (assuming the other axioms). The advantage of (3) is that it is more symmetric and that it also sufficient for the noncommutative case, where (4) must be supplemented by the corresponding dual axiom. Nevertheless, we will from now on tacitly assume that all integro-differential algebras are commutative, as in [41].

Incidentally we note the following convention on precedence: The scope of an integral sign \(\int\) covers all factors to its right, unless otherwise specified. Therefore the above term \(\int f'\int g\) is to be parsed as \(\int(f'\int g)\). This helps avoiding the proliferation of parentheses in nested integrals.

The differential Baxter axiom is in general stronger than the following pure Baxter axiom

\[
(\int f)(\int g) = \int fg + g\int f.
\]

which is the defining axiom for the so-called Rota-Baxter algebras [20,4,44]. (In fact, one may relax the conditions on \(\mathcal{F}\) to include noncommutative algebras over arbitrary commutative rings with one. Moreover, one may add a so-called weight term for incorporating the discrete setting where \(\int\) is, for example, the operator of partial summation. In that case, one must of course also adapt (2) to account for the weight. Remarkably, the differential Baxter axiom (3) is the same with or without weight.)

Every integro-differential algebra comes with a multiplicative projector, namely the evaluation \(e = 1 - \int \partial\). In fact, the strong Baxter axiom is equivalent to the weak one combined with the multiplicativity of \(e\). The presence of the character \(e\) leads to the direct sum decomposition

\[
\mathcal{F} = \mathcal{C} + \mathcal{I}
\]

where \(\mathcal{C} = \text{Im}(e) = \text{Ker}(\partial)\) is the usual ring of constants while \(\mathcal{I} = \text{Ker}(e) = \text{Im}(\int)\) is called the ideal of initialized functions (one checks immediately that the multiplicativity of \(e\) is equivalent to \(\mathcal{I}\) being an ideal). So \(e\) is the projector onto \(\mathcal{C}\) along \(\mathcal{I}\). Moreover, both \(\partial\) and \(\int\) are \(\mathcal{C}\)-linear rather than just \(K\)-linear (for \(\partial\) this is of course trivial, but for \(\int\) this is again equivalent to the multiplicativity of \(e\)).

In this paper, we want to restrict ourselves to boundary problems for LODEs. For reflecting this property in the (integro) differential structure, in [41] we have called a
differential algebra \((\mathcal{F}, \partial)\) ordinary if \(\text{Ker}(\partial) = K\). We will henceforth assume that all integro-differential algebras are ordinary in this sense.

Restricting ourselves to ordinary integro-differential algebras has a number of pleasant implications. First of all, the evaluation is now a multiplicative linear operator \(e : \mathcal{F} \to \mathcal{F}\), meaning a character (in the sense “multiplicative linear functional on an algebra”). Another consequence, which will be of some importance later, is that the polynomials behave as usual. For any integro-differential algebra \(\mathcal{F}\), we set \(x = \int 1 \in \mathcal{F}\). Now the pure Baxter axiom \((5)\) ensures that \(x^2 = 2 \int 1 \in \mathcal{F}\), and so on. It turns out that the elements of \(K[x]\) are “really” polynomials, and they satisfy the usual differential equations (see before Eq. \((6)\) and also Eq. \((8)\) in \([41]\) for more details).

**Proposition 1** Let \(\mathcal{F}\) be an integro-differential algebra over \(K\), and let \(K[x] = \mathcal{F}[\int ]\) be the subalgebra generated by \(x = \int 1 \in \mathcal{F}\). Then \(K[x] \leq \mathcal{F}\) is an integro-differential subalgebra.

**Example 1** The standard example of an (ordinary) integro-differential algebra is \(\mathcal{F} = C^\infty(\mathbb{R})\) with \(\partial u = u'\) in the usual sense and \(\int u = \int_0^1 u(\xi) d\xi\). This integro-differential algebra contains many important integro-differential subalgebras, for example the analytic functions \(C^\infty(\mathbb{R})\), the exponential polynomials, and of course the polynomial ring \(\mathbb{R}[x]\). An important integro-differential subalgebra of \(C^\infty(\mathbb{R})\) is formed by the holonomic power series \([8, 45]\).

Given a differential algebra \((\mathcal{F}, \partial)\), one may form the ring of differential operators \(\mathcal{F}[\partial]\). It is therefore natural to expect a similar ring of integro-differential operators \(\mathcal{F}[[\partial, \int]]\), which contains differential operators (needed for specifying differential equations) as well as integral operators (needed for specifying Green’s operators). The evaluation \(e\) can be used for writing a condition like \(2u'(0) - 3(0) = 0\) in the form \((2e\partial - 3e)u = 0\). But for boundary problems one usually needs more than one evaluation (see Example 2 below).

For algebraizing general boundary conditions, we start from a character set \(\Phi\), meaning each \(\varphi \in \Phi\) is a multiplicative linear functional just as \(e\) is. In fact, we will always assume \(e \in \Phi\). Based on the characters of \(\Phi\), we can from arbitrary Stieltjes conditions \([41]\) Def. 14]; their normal form is

\[
\sum_{\varphi \in \Phi} \left( \sum_{i \in \mathbb{N}} a_{\varphi, i} \varphi \partial^i + \varphi \int f_\varphi \right)
\]

with \(a_{\varphi, i} \in K\) and \(f_\varphi \in \mathcal{F}\) almost all zero. The summands with \(\int\) make up the so-called global part, those without the local part. The order of a boundary condition is the largest \(i\) such that \(a_i \neq 0\) in the normal form \([8]\). In the standard setting of Example 1 a typical Stieltjes condition like

\[
u''(0) - 3u(-1) + 7u(1) + \int_0^1 \xi^2 u(\xi) d\xi - \int_{-1}^1 \varphi e(\xi) d\xi = 0
\]
is encoded as \( \beta(u) = 0 \) with \( \beta = e_0 \partial^2 - 3e_{-1} + 7e_1 + e_1 \int x^2 - e_1 \int e^x + e_{-1} \int e^x \), where we have written \( e_a \) for the standard character \( f \mapsto f(a) \). Henceforth we shall always consider Example 1 with the character set \( \Phi = \{ e_a \mid a \in \mathbb{R} \} \).

For each character set \( \Phi \) one may then build up the corresponding operator ring \( \mathcal{F}_\Phi[\partial, \int] \). The details are given in [41, §3]; at this point it suffices to note that the operator ring is given by the quotient of a free algebra modulo a certain ideal of relations like the Leibniz rule, and also (3) as well as (5). The resulting operator ring decomposes nicely into a direct sum; see [41, Prop. 17] for a proof.

**Proposition 2** For any integro-differential algebra \( (\mathcal{F}, \partial, \int) \) and character set \( \Phi \), we have the decomposition \( \mathcal{F}_\Phi[\partial, \int] = \mathcal{F}[\partial] + \mathcal{F}[\int] + (\Phi) \).

Here \( (\Phi) \) is the two-sided ideal generated by \( \Phi \) in the operator ring \( R = \mathcal{F}_\Phi[\partial, \int] \). We refer to its elements as boundary operators since they may also be described in terms of Stieltjes conditions: One checks first that the collection of Stieltjes conditions can be characterized as the right ideal \( (\Phi) = \Phi R \). It turns out that \( (\Phi) \) is then the left \( \mathcal{F} \)-module generated by \( \Phi \), so every boundary operator can be written as \( f_1 \beta_1 + \cdots + f_n \beta_n \) for some functions \( f_1, \ldots, f_n \in \mathcal{F} \) and Stieljes conditions \( \beta_1, \ldots, \beta_n \in (\Phi) \). In particular, every Stieltjes condition is also a boundary operator.

The decomposition \( \mathcal{F}_\Phi[\partial, \int] = \mathcal{F}[\partial] + \mathcal{F}[\int] + (\Phi) \) reflects the basic needs for an algebraic formulation of boundary problems: We need \( \mathcal{F}[\partial] \) for the differential equation, \( (\Phi) \) for the boundary conditions, and \( \mathcal{F}[\int] \) for the solution (Green’s operators). Let us see how these ingredients look like for the simplest possible boundary problem in the standard setting of Example 1.

**Example 2** The boundary problem

\[
\begin{align*}
\frac{d^2 u}{dx^2} &= f \\
u(0) &= u(1) = 0
\end{align*}
\]

can be encoded in the standard setting \( \mathcal{F} = C^\infty(\mathbb{R}) \) by the differential operator \( T = \partial^2 \in \mathcal{F}[\partial] \) and the boundary conditions \( e_0 = e, e_1 \in (\Phi) \). Here we can choose \( \Phi = \{ e_0, e_1 \} \) or any character set containing that. The Green’s operator can be written as \( G = Ax + xB - xAx - xBx \in \mathcal{F}[\int] \), where we use the standard abbreviations \( A = \int \) and \( -B = (1 - e_1) \int \). Note that \( A \) is the integral from 0 to \( x \), and \( B \) the integral from \( x \) to 1. In the \( L^2 \) setting, \( A \) and \( B \) are adjoint operators.

The algebraic description in terms of the two functionals \( e_0 \) and \( e_1 \) contains some arbitrariness since we can clearly form any linear combination of the two functionals without changing the solution operator \( G \). In general, a boundary space is a finite-dimensional subspace of \( \mathcal{F}^* \) generated by Stieljes conditions. So the boundary space of Example 2 is \( \{ e_0, e_1 \} \). The lattice of all boundary spaces will be denoted by \( K\Phi \).

We will usually restrict the coefficient functions of the differential operators to a differential subalgebra \( \mathcal{E} \subseteq \mathcal{F} \) so as to ensure solutions in the ambient algebra \( \mathcal{F} \). This means we require that \( \mathcal{E} \) be saturated for \( \mathcal{F} \) in the sense of [41, Def. 18]. Whenever an integro-differential algebra \( (\mathcal{F}, \partial, \int) \) is specified, it is understood that a saturated coefficient algebra \( \mathcal{E} \) is set aside.
Definition 2 A boundary problem is a pair \((T, \mathcal{B})\) consisting of a monic differential operator \(T \in \mathcal{E}[\partial]\) and a boundary space \(\mathcal{B} \in K\Phi\).

So the boundary problem in Example 2 is given by \((\partial^2, [e_0, e_1])\). Conversely, we can think of a boundary problem \((T, \mathcal{B})\) as finding a solution \(u \in F\) such that

\[
\begin{align*}
Tu &= f \\
\beta(u) &= 0 & (\beta \in \mathcal{B})
\end{align*}
\]

for an arbitrary forcing function \(f \in F\). Of course it suffices that \(\beta\) ranges over any \(K\)-basis of \(\mathcal{B}\); this is how the boundary conditions are normally given in the first place. The traditional formulation in terms of \(u\) and \(f\) is more intuitive, but it conceals the fact that we are really working in the operator ring \(\mathcal{F}_{\Phi}[\partial, f]\).

Clearly every differential operator \(T\) has a certain order written as \(\text{ord} T\), which as in Example 2 usually coincides with the number of given boundary conditions (the dimension of \(\mathcal{B}\)). In fact, this a necessary—but in general not sufficient—condition for the boundary problem to be regular in the sense that there is a unique solution \(u \in F\) for every given forcing function \(f \in F\). In turns out that this is equivalent to the following formulation in terms of the relevant spaces.

Definition 3 A boundary problem \((T, \mathcal{B})\) is called regular if \(\ker(T) \cap \mathcal{B}^\perp = F\), and singular otherwise.

Here we have written \(\mathcal{B}^\perp\) for the orthogonal, meaning the space of all \(f \in F\) such that \(\beta(f) = 0\) for all \(\beta \in \mathcal{B}\). In other words, \(\mathcal{B}^\perp\) is the space of admissible functions: those that satisfy the given boundary conditions. The orthogonal, together with the analogous notion \(\mathcal{F}^\perp\) for subspaces \(\mathcal{F} \leq F\), establishes a Galois connection between \(F\) and its dual \(F^*\). This is similar to the situation in algebraic geometry with its Galois connection between affine varieties and radical ideals. See [41, §5] for more details.

The above definition of regularity is not suitable for algorithmic purposes. However, it turns out to be equivalent to the following explicit regularity test [35, Prop. 6.1], which is well-known in the special case of local boundary conditions [23, p. 184].

Lemma 1 Let \((T, \mathcal{B})\) be a boundary problem over \(F\) and choose bases \(\beta_1, \ldots, \beta_n\) for \(\mathcal{B}\) and \(u_1, \ldots, u_n\) for \(\ker(T)\). Then \((T, \mathcal{B})\) is regular iff \(\text{ord} T = \dim \mathcal{B}\) and the evaluation matrix \(\beta(u) = [\beta_i(u_j)] \in K^{n \times n}\) is regular.

Regular boundary problems are exactly those that have a Green’s operator in the classical sense (see [27] on generalized Green’s operators for singular boundary problems). In our algebraic setting, Green’s operators can be defined within the operator ring \(\mathcal{F}_{\Phi}[\partial, f]\).

Definition 4 Let \((T, \mathcal{B})\) be a regular boundary problem over an integro-differential algebra \(F\). Then \(G \in \mathcal{F}_{\Phi}[\partial, f]\) is called its Green’s operator if \(TG = 1\) and \(\text{Im}(G) = \mathcal{B}^\perp\).
The two conditions for Green’s operators are just that \( u = Gf \) satisfies the differential equation \( Tu = f \) as well as all boundary conditions \( \beta(u) = 0 \) for \( \beta \in \mathcal{B} \). Since we have assumed a coefficient algebra \( \mathcal{E} \) saturated for \( \mathcal{F} \), one can prove [41, Thm. 26] that every regular boundary problem has a Green’s operator in \( \mathcal{F}[\partial, \int] \). In a leap of faith, we write \( G = (T, \mathcal{B})^{-1} \) just as in [41, p. 533]. The relation to actual inverses will become clear in Proposition [10].

Apart from the localization, there is another good reason for the notation \( G = (T, \mathcal{B})^{-1} \). It turns out that boundary problems can be multiplied in such a way that

\[
(T_1, \mathcal{B}_1)(T_2, \mathcal{B}_2) = (T_1, \mathcal{B}_2)(T_2, \mathcal{B}_1)^{-1} (T_1, \mathcal{B}_1)^{-1} \quad (9)
\]

is satisfied. For that purpose, we have defined the multiplication of boundary problems in the fashion of a semi-direct product by

\[
(T_1, \mathcal{B}_1)(T_2, \mathcal{B}_2) = (T_1 T_2, \mathcal{B}_1 + \mathcal{B}_2). \quad (10)
\]

It is easy to check that one obtains a monoid \( \mathcal{E}[\partial] \times K\Phi \) in this way, with \((1, O)\) as the neutral element. Furthermore, it turns out that the regular boundary problems form a submonoid \( \mathcal{E}[\partial]_{\mathcal{F}} \subset \mathcal{E}[\partial] \times K\Phi \), and (9) means that \( (T, \mathcal{B}) \mapsto (T, \mathcal{B})^{-1} \) is an anti-isomorphism from \( \mathcal{E}[\partial]_{\mathcal{F}} \) to the multiplicative monoid of Green’s operators. For a proof of these facts, we refer to [41, Prop. 27].

The monoid algebra \( K\mathcal{E}[\partial]_{\mathcal{F}} \) will be called the ring of boundary problems (we suppress the qualification “regular” since we will not investigate the singular case in the frame of this paper). Its elements will supply the numerators in the localization to be constructed below (Section 4).

At this point we should also note that a regular boundary problem \((T, \mathcal{B})\) need not be well-posed in the sense that its Green’s operator \( G \) is continuous in the standard setting of Example [1]. For example, consider the regular boundary problem \((\partial - 1, e\xi_0\partial^2)\) or

\[
\begin{align*}
\quad u' - u &= f \\
\quad u''(0) = 0
\end{align*}
\]

in traditional notation. One checks immediately that \( G = e\xi \int e^{-\xi} - e\xi e\xi_0 - e\xi e\xi_0\partial \in \mathcal{F}[\partial, \int] \) is its Green’s operator, meaning the general solution is given by

\[
u(x) = \int_0^1 e^{\xi - \xi_0} f(\xi) d\xi - (f(0) + f'(0)) e^\xi.
\]

Clearly, this Green’s operator \( G \) is not continuous, at least not if one endows \( C^\infty(\mathbb{R}) \) with the usual topology induced by \( C(\mathbb{R}) \). Following Hadamard, a well-posed boundary problem [16, p. 86] must be regular (meaning the solution \( u \) exists and is unique for each given \( f \)) as well as stable (meaning the solution \( u \) depends continuously on \( f \)). In the example above, the source of the instability is clear—we have imposed a second-order condition for a first-order differential equation. Going back to the algebraic setting then, we call a boundary problem \((T, \mathcal{B})\) well-posed if it is regular and \( \mathcal{B} \) can be generated by Stieltjes conditions whose order is smaller than that of \( T \); otherwise \((T, \mathcal{B})\) is called ill-posed. As one easily checks, the well-posed boundary problems form a submonoid of \( \mathcal{E}[\partial]_{\mathcal{F}} \).
The Green’s operator $G$ of a regular boundary problem $(T, B)$ of order $n$ factors naturally as

$$G = (1 - P) T^\circ,$$  \hspace{1cm} (11)

where $T^\circ$ is the so-called fundamental right inverse of $T$, defined as the Green’s operator of the initial value problem $(T, [e, e\partial, \ldots, e\partial^{n-1}])$, and $P$ is the projector onto $\ker(T)$ along $B^{\perp}$. The fundamental right inverse $T^\circ$ exists since initial value problems are always solvable \cite{41} Prop. 22. We may thus think of \eqref{11} as a two step process: Using $T^\circ$ one solves first the initial value problem (which is usually much easier), then one incorporates the boundary conditions via the projector $P$.

The submonoid relation $\mathcal{E}[\partial]|\phi \subset \mathcal{E}[\partial] \times K\Phi$ means that multiplying regular problems leads to regular problems again. It is interesting to note that the converse is also true, provided the order condition of Lemma 1 holds.

\textbf{Lemma 2} Let $(T_1, B_1), (T_2, B_2)$ be boundary problems over an integro-differential algebra $\mathcal{F}$ with $\text{ord} T_1 = \dim B_1$ and $\text{ord} T_2 = \dim B_2$. Then $(T_1, B_1)$ and $(T_2, B_2)$ are regular whenever $(T_1, B_1) \cdot (T_2, B_2)$ is.

\textbf{Proof} Let us write $m = \text{ord} T_1 = \dim B_1$ and $n = \text{ord} T_2 = \dim B_2$. Choose fundamental systems $f_1, \ldots, f_m \in \mathcal{F}$ for $T_1$ and $g_1, \ldots, g_n \in \mathcal{F}$ for $T_2$. Take $K$-bases $\beta_1, \ldots, \beta_m$ of $\mathcal{B}_1 \subseteq \mathcal{F}^*$ and $\gamma_1, \ldots, \gamma_n$ of $\mathcal{B}_2 \subseteq \mathcal{F}^*$. Then $T_2 f_1, \ldots, T_2 f_m, g_1, \ldots, g_n$ is a fundamental system for $T_1 T_2$, and a $K$-basis of $\mathcal{B}_1 T_2 + \mathcal{B}_2$ is given by $\beta_1 T_2, \ldots, \beta_m T_2, \gamma_1, \ldots, \gamma_n$. The latter fact follows since the sum $\mathcal{B}_1 T_2 + \mathcal{B}_2$ in \eqref{10} is always direct \cite{35} Prop. 3.2. By Lemma 1 the regularity of the boundary problem $(T_1, B_1) T_2 + \mathcal{B}_2$ means that its evaluation matrix

$$\begin{pmatrix}
\beta(T_2) (T_2^\circ f) & (\beta T_2) (g) \\
\gamma(T_2^\circ f) & \gamma(g)
\end{pmatrix}$$

is regular. But this is only possible if both diagonal blocks $\beta(f)$ and $\gamma(g)$ are regular, and these are just the evaluation matrices of $(T_1, B_1)$ and $(T_2, B_2)$.

For practical applications, one is not so much interested in multiplying boundary problems—thus increasing their order—as to factor them into lower-order problems. Interestingly, factorization will also be instrumental in the localization process (Lemma 10). We repeat here as the main result the so-called \textit{Factorization Theorem} \cite{41} Thm. 32.

\textbf{Theorem 1} Given a regular boundary problem $(T, B) \in \mathcal{E}[\partial]|\phi$, every factorization $T = T_1 T_2$ of the differential operator lifts to a factorization $(T, B) = (T_1, B_1) \cdot (T_2, B_2)$ with $(T_1, B_1), (T_2, B_2)$ regular and $B_2 \subseteq B$.

We note that the right factor may be chosen to be an arbitrary subspace of $B$ as long as regularity holds for $(T_2, B_2)$; in particular one can always choose initial conditions for $B_2$ to ensure regularity. In contrast, $B_1$ is determined by the choice of $T_1$ and $T_2$ alone. This is the content of the following uniqueness result \cite{41} Prop. 31), called the \textit{Division Lemma} for boundary problems (see below for the notion of subproblem).
Lemma 3  Given a regular boundary problem \((T, \mathcal{B}) \in E[\partial] \Phi\) and any factorization \(T = T_1 T_2\) of the differential operator, there is a unique boundary problem \((T_1, \mathcal{B}_1)\) such that for any regular subproblem \((T_2, \mathcal{B}_2) \leq (T, \mathcal{B})\) we have the lifted factorization \((T_1, \mathcal{B}_1) \cdot (T_2, \mathcal{B}_2) = (T, \mathcal{B})\).

The notion of subproblem was defined (under the name “right factor”, which is avoided here due to the ambiguity in cases like Example 7) for regular boundary problems in \([41, \text{Def. 29}]\), but it can be extended naturally to all boundary problems. Hence a subproblem of a boundary problem \((T, \mathcal{B}) \in E[\partial] \iota K \Phi\) is any \((T_2, \mathcal{B}_2) \in E[\partial] \iota K \Phi\) with \(T_2\) a right divisor of \(T\) and \(\mathcal{B}_2\) a subspace of \(\mathcal{B}\). We denote this relation by \((T_2, \mathcal{B}_2) \leq (T, \mathcal{B})\).

We conclude this section with a few remarks on algorithmic issues. While at the present stage of our work we do not aim at a computational realization of the localization, we see this nevertheless as a mid-term goal. In fact, most of the results in this paper are algorithmic, with two exceptions: (1) The Ore condition is needed in \(E[\partial]\) for monic operators; this is not supported by customary packages but appears to be adaptable (see the remarks after Proposition 6). (2) One would need an algorithm for determining the monomials required in Definition 5. Once these two gaps are filled, one should be able to compute in the localization since all the constructions reviewed in this section are themselves algorithmic \([41]\):

- The operator ring \(F[\partial, \{\}]\) is defined as a quotient by a certain ideal of relations. This ideal is described by a noncommutative Gröbner basis for an ideal with infinitely many generators (on another view: a noetherian and confluent rewrite system).
- If the ground algebra \(F\) and the character set \(\Phi\) are computable, the normal forms of \(F[\partial, \{\}]\) are computable as well. Typically, \(F\) is an algorithmic fragment of \(C^\infty(\mathbb{R})\) like the exponential polynomials \([39, \text{p. 176}]\), while \(\Phi\) consists of finitely many point evaluations \(f \mapsto f(a)\).
- The Green’s operator of a regular boundary problem (and hence the solutions for arbitrary forcing functions up to quadratures) can be computed as long as one has a fundamental system for the underlying homogenous differential equation. For the latter, one can in principle rely on the vast body of results from differential Galois theory \([34]\).
- The multiplication and factorization of boundary problems can be computed as long as the same is true for the constituent differential operators. In this way one can often solve higher-order boundary problems by decomposing them into smaller factors such as in a recent application to actuarial mathematics \([3]\). For the algorithmic theory of factoring linear differential operators, we refer to \([18, 46, 54]\).

As detailed in the Introduction, the algorithms addressed above are implemented in \textsc{Mathematica} and \textsc{Maple} packages.

3 Umbral Boundary Conditions

We have seen that for a given integro-differential algebra \(F\) and character set \(\Phi\), the natural choice of boundary conditions is the Stieltjes conditions, defined in \([41]\)
as the right ideal $|\Phi|$ of $\mathcal{F}_\Phi[\partial, \int]$ and characterized by their normal forms (8). The choice of $|\Phi|$ is vindicated by Theorem [1] which asserts that Stieltjes conditions are sufficient for describing arbitrary factor problems (and they are clearly also sufficient for multiplying boundary problems). In fact, Stieltjes conditions with nonzero global part appear in the left factor problem even if the original boundary problem has only local conditions; see e.g. Example 33 in [41].

For the purpose of localization we must isolate a subclass of Stieltjes conditions. First of all, we must distinguish carefully between the zero condition $0 \in |\Phi|$ and degenerate boundary conditions that only act as zero: We call a Stieltjes condition $\beta$ degenerate if $\varphi(f) = 0$ for all $f \in \mathcal{F}$. In the standard setting of Example [1] there are plenty of degenerate boundary conditions. If $f$ is a bump function supported, say, on an interval disjoint from $[0, 1]$, the Stieltjes condition $\beta_1 \int f = \int_0^1 f$ is clearly degenerate. In contrast, the integro-differential subalgebra of analytic functions $C^\infty(\mathbb{R})$ does not have degenerate global conditions of this form as we shall see below (Example 4).

The subclass of Stieltjes conditions to be chosen must be so as to ensure a sufficient supply of regular boundary problems. In particular, we shall find ourselves in the situation of embedding a singular boundary problem $(T, \mathcal{B})$ into a surrounding regular problem (Lemma [10]). Hence we must enlarge $T \in \mathcal{E}[\partial]$ by a suitable differential operator $\tilde{T} \in \mathcal{E}[\partial]$ and the boundary space $\mathcal{B}$ by new boundary conditions $\tilde{\beta}_1, \tilde{\beta}_2, \ldots$ in such a manner that the resulting evaluation matrix is regular (Lemma [1]).

The key for solving this problem is found in the fortunate fact every integro-differential algebra $\mathcal{F}$ contains the polynomial ring (Proposition [1]), so the initial value problem

$$(\partial^n, [\mathbb{E}, \mathbb{E}\partial, \ldots, \mathbb{E}\partial^{n-1}])$$

is a natural choice for extending the given boundary problem $(T, \mathcal{B})$. Then the monomials $1, x/1!, \ldots, x^{n-1}/(n-1)!$ are a fundamental system by Proposition [1] and the corresponding evaluation matrix is $I_n$. But this is not enough for embedding singular problems since the evaluation matrix of the surrounding problem involves combining the given boundary conditions $\beta \in \mathcal{B}$ with the monomials of the fundamental system (7). Indeed, the crucial step in the proof of Lemma [10] needs that $\beta(x^n) \neq 0$ for some monomial $x^n \in K[x]$. This is the motivation for the following definition.

**Definition 5** A Stieltjes condition $\beta \in \mathcal{F}_\Phi[\partial, \int]$ is called umbral if $\beta(x^n) \neq 0$ for some monomial $x^n \in K[x]$. Furthermore, we call $\Phi$ an umbral character set if every nondegenerate Stieltjes condition is umbral.

The reason for our terminology is that there is an interesting link between Stieltjes conditions $\beta \in \mathcal{F}_\Phi[\partial, \int]$ with $\beta(x^n) \neq 0$ and the umbral calculus. Indeed, every umbral condition defines a nontrivial shift-invariant operator (Proposition [5]). This is clear for the local parts $\varphi\partial^n$ but needs some justification for global terms $\varphi \int f$ with arbitrary $f \in \mathcal{F} \supset K[x]$. The point is that one may apply the following special case of the well-known “antiderivative Leibniz rule”. (In the general setting of analytic functions one writes $\int fg$ as an infinite series of iterated integrals of $f$ times iterated derivatives of $g$. Here we have $g = x^n$ so that only finitely many derivatives are nonzero and the series terminates.)
Lemma 4 In any integro-differential algebra \((\mathcal{F}, \partial, \int)\), we have the formula
\[
\int f x^n = \sum_{k=0}^{n} (-1)^k n^k x^{n-k} f^{(-k-1)}
\]
(12)
for all \(f \in \mathcal{F}\). Here \(f^{(-k)} (k \geq 0)\) is defined by \(f^{(0)} = f\) and \(f^{(-k-1)} = \int f^{(-k)}\).

Proof This is essentially the special case \(m=1\) of Equation (14) in [36], but the proof is short enough to present here. We use induction over \(n\). The base case \(n=0\) is clear, so assume (12) for \(n \geq 0\); we prove it for \(n+1\). Since \(x^{n+1} = (n+1) \int x^n\) we have
\[
\int f x^{n+1} = (n+1) \int f \int x^n = f^{(-1)} x^{n+1} - (n+1) \int f^{(-1)} x^n
\]
by the pure Baxter axiom (5). Using the induction hypothesis, the right summand becomes
\[
\sum_{k=1}^{n+1} (-1)^k (n+1)^k x^{n+1-k} f^{(-k-1)}
\]
after an index transformation. Adding the extra term \(f^{(-1)} x^{n+1}\) means extending the summation to \(k = 0, \ldots, n+1\), which yields (12) for \(n+1\).

Lemma 5 Let \(\beta = \varphi / \mathcal{F}\) be a global condition in \(\mathcal{F}[\partial, \int]\). Then we have \(\beta = \varphi \tilde{\beta}\) as a functional \(K[x] \rightarrow K\), where
\[
\tilde{\beta} = \sum_{k=0}^{\infty} b_k \partial^k : K[x] \rightarrow K[x]
\]
(13)
is a shift-invariant operator with expansion coefficients \(b_k = (-1)^k \varphi(f^{(-k-1)})\).

Proof By a well-known result of the umbral calculus [11, Thm. 2.1.7], incidentally also presented in [38, Prop. 92], the operator \(\tilde{\beta}\) is shift-invariant. For seeing that \(\beta = \varphi \tilde{\beta}\), we apply Lemma 4 to obtain \(\beta(x^n) = \sum_k b_k \varphi(n^k x^{n-k}) = \sum_k b_k \varphi \partial^k(x^n)\).

The operator expansion (13) allows us to associate a shift invariant operator with a Stieltjes condition of the special form \(\beta = \varphi / f\). But this association generalizes immediately to arbitrary Stieltjes conditions since the local conditions are unproblematic.

Proposition 3 Let \(\beta\) be a Stieltjes condition in \(\mathcal{F}[\partial, \int]\). Then there is an associated shift-invariant operator
\[
\tilde{\beta} = \sum_{k=0}^{\infty} b_k \partial^k : K[x] \rightarrow K[x]
\]
(14)
with coefficients \(b_k = \beta(x^k / k!)\) such that \(\beta = e \tilde{\beta}\). Clearly, the associated operator \(\tilde{\beta}\) is nonzero iff \(\beta\) is an umbral condition.
Proof Let \( S_\varphi : K[x] \to K[x] \) be the shift operator \( f(x) \mapsto f(x + \varphi) \) with \( \varphi = \varphi(x) \in K \). Using the normal form (8) we can write the given boundary condition as

\[
\hat{\beta} = \sum_{\varphi \in \Phi} \varphi(T_\varphi + \hat{\beta}_\varphi) = e \sum_{\varphi \in \Phi} S_\varphi(T_\varphi + \hat{\beta}_\varphi),
\]

where \( T_\varphi = \sum_i a_{\varphi,i} \partial^i \) is clearly shift-invariant while \( \hat{\beta}_\varphi \) is the shift-invariant operator corresponding to \( \beta_\varphi = \varphi \int f_\varphi \) according to Lemma 5. Then the terms \( S_\varphi(T_\varphi + \hat{\beta}_\varphi) \) in the second sum are shift-invariant, hence we have \( \beta = e \hat{\beta} \) with the shift-invariant operator

\[
\hat{\beta} = \sum_{\varphi \in \Phi} S_\varphi(T_\varphi + \hat{\beta}_\varphi),
\]

and the formula \( b_\beta = \beta(x^k/k!) = e \hat{\beta}(x^k/k!) \) for its expansion coefficients follows from the general result referred to in the proof of Lemma 5.

Umbral boundary conditions appear to be abundant, at least in the cases most crucial to us, especially in the \( C^\infty \) setting (containing many important integro-differential subalgebras—notably the analytic and holonomic functions as well as the exponential polynomials).

Proposition 4 In the standard setting of Example 7 the point evaluations form an umbral character set.

Proof Consider an arbitrary nondegenerate Stieltjes condition

\[
\beta : \mathcal{F} \to \mathbb{R}, \quad \beta(u) = \sum_{\varphi \in \Phi} \sum_{i=0}^k a_{\varphi,i} u^{(i)}(\varphi) + \sum_{\varphi \in \Phi} \int_0^\varphi f_\varphi(\xi) u(\xi) d\xi
\]

so that there is a function \( u \in \mathcal{F} \) with \( \beta(u) \neq 0 \). Let \( R \) be the maximum of \( |\varphi| \) for all \( \varphi \) with \( a_{\varphi,i} \neq 0 \) or \( f_\varphi \neq 0 \). We consider now the Banach space \( C^k(K) \) on the compact interval \( K = [-R,R] \). Its norm \( \| \cdot \|_k \) is given by

\[
\| f \|_k = \sum_{i=0}^k \| u^{(i)} \|_\infty
\]

for all \( f \in C^k(K) \). There is a little known generalization of the Weierstrass approximation theorem due to Nachbin 32, which asserts (in a simple special case) that \( \mathbb{R}[x] \) is dense in \( C^k(K) \). Hence we may choose a polynomial sequence \( p_n \) that converges to \( u \in \mathcal{F} \subseteq C^k(K) \) in the \( C^k \) topology.

One checks immediately that \( \beta : C^k(K) \to \mathbb{R} \) is a continuous functional with respect to the \( C^k \) norm. In detail, one has \( |\beta(u)| \leq C \| u \|_k \) with operator-norm bound

\[
C = \sum_{\varphi \in \Phi} \sum_{i=0}^k a_{\varphi,i} + \sum_{\varphi \in \Phi} \| f_\varphi \|_\infty.
\]

Therefore we have \( \beta(p_n) \to \beta(u) \neq 0 \). This is impossible if \( \beta \) vanishes on all of \( \mathbb{R}[x] \). Hence there is some \( p \in \mathbb{R}[x] \) with \( \beta(p) \neq 0 \). Clearly there exists a smallest monomial \( x^m \) in \( p \) with \( \beta(x^m) \neq 0 \).
Not every character set is umbral, though. It is natural to introduce the following **necessary conditions** for a character set $\Phi$ to be umbral (we write $\overline{\varphi}$ for the canonical value $\varphi(x) \in K$ of a character $\varphi \in \Phi$):

1. The character set $\Phi$ must clearly be **separative** in the sense that $\overline{\varphi} = \overline{\chi}$ implies $\varphi = \chi$. Otherwise $\beta = \varphi - \chi$ is a nonzero Stieltjes condition with $\beta(K[x]) = 0$.

2. Every character $\varphi \in \Phi$ must be **complete** in the sense that every global condition of the form $\beta = \varphi \int f$ is umbral whenever it is nondegenerate. This may also be expressed as $f \perp \varphi K[x] \Rightarrow f = 0$. Here orthogonality refers to the nondegenerate bilinear form $\langle f|g \rangle = \varphi \int f g$. If this bilinear form is positive definite with $K = \mathbb{R}$ or $K = \mathbb{C}$, we have a pre-Hilbert space $(\mathcal{F}, \langle \cdot|\cdot \rangle_\varphi)$. Following the terminology of [17], completeness of $\varphi$ then means that any $\varphi$-orthonormal basis of $K[x]$ is complete in $(\mathcal{F}, \langle \cdot|\cdot \rangle_\varphi)$.

These conditions are most likely not sufficient. At the moment we cannot give a counterexample for corroborating this claim. But it is intuitively clear that the completeness properties associated with each $\varphi$ separately cannot prevent linear dependencies between the actions of distinct global conditions $\varphi \int f$. It remains an interesting task to formulate stronger conditions on $\Phi$ that ensure an umbral character set in a natural way.

**Example 3** As an example of a nonseparative character set, consider the exponential polynomials $K[x, e^x]$ with the nonstandard character $\varphi$ defined by $\varphi(e^x) = 1$ and $\varphi(x^n) = 1$ for all $n$, effectively mixing evaluation at $1 \in K$ for monomials with evaluation at $0 \in K$ for the exponential. Choosing $K = \mathbb{R}$ for simplicity, let $e$ be the honest evaluation $f \mapsto f(1)$. Then clearly $\varphi$ and $e$ coincide on $K[x]$ while they are in fact distinct characters on $K[x, e^x]$ since $\varphi(e^x) = 1 \neq e = e(e^x)$.

This shows that separativity is necessary but not sufficient. Nevertheless, it ensures that all local boundary conditions are indeed umbral.

**Proposition 5** Let $\Phi$ be a separative character set for an integro-differential algebra $(\mathcal{F}, \partial, \int)$. Then every local boundary condition is umbral.

**Proof** Using (8), every local boundary condition can be written in the form

$$\beta = \sum_{i=1}^r \sum_{j=1}^s a_{ij} \varphi_i \partial_j^{i-1}.$$ 

Assuming $\beta(K[x]) = 0$, we show that all coefficients $a_{ij}$ are zero. Collecting them in the vector

$$a = (a_{1,1}, a_{1,2}, \ldots, a_{1,s}; \ldots; a_{r,1}, a_{r,2}, \ldots, a_{r,s})^T \in K^n,$$

we obtain $n = rs$ linear homogeneous equations for the unknowns $a_{ij}$ by applying $\beta$ to the monomials $1, x, x^2/2!, \ldots, x^{n-1}/(n-1)!$. The matrix of the corresponding sys-
tem $Ma = 0$ is given by $M = (M_{m_1}(\phi_1) \cdots M_{m_r}(\phi_r)) \in K^{n \times n}$ with blocks

$$M_{m_i}(x) \equiv \begin{pmatrix}
1 & x & \cdots & x^{s-1} \\
\frac{1}{2} & \frac{x^2}{2} & \cdots & \frac{x^{s-1}}{2} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{(s-1)!} & \frac{x^2}{(s-2)!} & \cdots & \frac{x^{s-1}}{(s-2)!} \\
\frac{1}{(s-2)!} & \frac{x^2}{(s-2)!} & \cdots & \frac{x^{s-2}}{(s-2)!} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{(n-1)!} & \frac{x^2}{(n-2)!} & \cdots & \frac{x^{n-s}}{(n-2)!} \\
\frac{1}{(n-2)!} & \frac{x^2}{(n-2)!} & \cdots & \frac{x^{n-s}}{(n-2)!}
\end{pmatrix} \in K[x]^{n \times n}. \tag{15}$$

Since $\Phi$ is separative, the values $\phi_1, \ldots, \phi_r$ are mutually distinct. Then we may apply the subsequent Lemma 6 to obtain $\det M \neq 0$, hence $Ma = 0$ has the unique solution $a = 0$.

**Lemma 6** The determinant of $M(x) = (M_{m_1}(x_1), \ldots, M_{m_r}(x_r)) \in K[x_1, \ldots, x_r]^{n \times n}$, with $n = rs$ and blocks (15), is given by

$$\det M(x) = V(r)^s \sf(s - 1)^r/\sf(n - 1), \tag{16}$$

where $V(r) = \prod_{1 \leq i < j \leq r} (x_j - x_i) / V(k)$ is the $r \times r$ Vandermonde determinant and $\sf(i) = 1!2! \cdots i!$ denotes the superfactorial.

**Proof** The determinant is a special case of [17, Thm. 1.1] or [29, Thm. 20], apart from the linear factor $V(k)$ in $k$-th row of $M(x)$. Thus we have

$$\det M(x) = V(r)^s \prod_{i=1}^r \prod_{j=1}^{s-1} j! \left( \prod_{k=0}^{n-1} \frac{1}{k!} \right) = V(r)^s \sf(s - 1)^r / \sf(n - 1),$$

which is (16).

Let us now turn to completeness. In the analysis setting, every $\varphi$ is complete since we know already the stronger result that every Stieltjes condition is umbral (Proposition 4). It is nevertheless instructive to have a closer look at this case.

**Example 4** Consider the point evaluation $\varphi = e_a$ in the standard setting of Example 1. Here we get an inner product

$$\langle f | g \rangle_a = \beta(g) = \int_a^b f(\xi) g(\xi) \, d\xi,$$

which can be extended to all continuous functions $f, g$ on $[0, a]$ and so gives rise to the pre-Hilbert Hausdorff space $(C[0, a], \langle \cdot | \cdot \rangle_a)$. It is well-known that $(x^m)_{m \in \mathbb{N}}$ is a complete sequence in this space [7, V.24]. We may assume $a = 1$ by a scale transformation. By the usual Gram-Schmidt process, the monomials $x^m$ can be transformed
to an orthonormal basis \((e_n)\) of \(C[0,1]\), which consists in this case of the Legendre polynomials

\[
e_n = \frac{\sqrt{n+1/2}}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n.
\]

In every pre-Hilbert Hausdorff space \(E\), an orthonormal basis \((e_n)\) has the following well-known property [7 Prop. V.2.5]: If \(f \in E\) is nonzero, we have \(\langle e_m, f \rangle \neq 0\) for some \(m \in \mathbb{N}\). Applying this to \(E = C[0,1] \supset C^\infty[0,1]\) and fixing any \(f \in C^\infty[0,1] \subset C^\infty(\mathbb{R})\), we see that \(\phi\) is complete in the sense defined above. For if \(\langle x^m, f \rangle \neq 0\) for all \(m \in \mathbb{N}\), then \(f\) vanishes on \([0,1]\) and hence \(\phi \int f\) is degenerate.

The above argument shows in fact that every global condition \(\phi \int f\) over \(C^\infty(\mathbb{R})\) is umbral or degenerate. Working over \(C^\infty(\mathbb{R})\), we can exclude the degenerate case. Indeed, the identity theorem of complex analysis ensures that \(f = 0\) whenever \(f\) vanishes on \([0,1]\). This is in stark contrast to the \(C^\infty\) case, which has plenty of degenerate global conditions \(\phi \int f\) as we have observed at the beginning of Section 3.

It becomes clear from the above example that completeness is really an analytic property of some sort. It is therefore not surprising that one can construct “purely algebraic” examples lacking this property.

**Example 5** To give an example of an incomplete character on \(\mathcal{F} = K[x, e^x]\) with \(K = \mathbb{R}\), we proceed similar to Example 3. Define \(\phi(x^n) = 0\) for \(n > 0\) and \(\phi(e^x) = e\), \(\phi(1) = 1\). Then clearly \(\beta = \phi \int 1\) is nondegenerate since \(\beta(e^x) = \phi \int e^x = \phi(e^x - 1) = e - 1 \neq 0\). But we have \(\beta(x^n) = \phi \int x^n = \phi(e^{nx}/n!) = 0\) for all \(m \in \mathbb{N}\).

As noted above, separativity and completeness are most likely not strong enough to ensure an umbral character set. Note that umbrality of \(\Phi\) implies that two Stieltjes conditions in \(\mathcal{F}_\phi[\partial, \int]\) are linearly independent on \(K[x]\) whenever they are linearly independent on \(\mathcal{F}\). If one of the two conditions is local and the other global, it is reasonable to expect this property to follow from completeness. This expectations is fulfilled.

**Lemma 7** Let \((\mathcal{F}, \partial, \int)\) be an integro-differential algebra with a complete character \(\phi\). Then a nondegenerate global condition \(\phi \int f\) never coincides on \(K[x]\) with any local condition based on \(\phi\).

**Proof** Assume \(\beta = \phi \int f\) coincides on \(K[x]\) with \(a_0 \phi + a_1 \phi \partial + \cdots + a_s \phi \partial^s\), where \(s\) is chosen minimal. Then the umbral expansion (13) breaks off at \(k = s\) and we have \(\phi(f^{(-k-1)}) = 0\) for \(k > s\). For \(f = f^{(-s-1)}\), we get \(\phi(f^{(-k-1)}) = 0\) for all \(k \in \mathbb{N}\), so the condition \(\phi \int f\) is degenerate since \(\phi\) is complete. For showing that this cannot happen, it suffices to prove that all \(\phi \int f^{(-k)}\) are nondegenerate whenever \(\phi \int f\) is. We use induction on \(k \in \mathbb{N}\). The base case \(k = 0\) is trivial, so assume the claim for fixed \(k\). By the induction hypothesis we can choose \(g \in \mathcal{F}\) with \(\phi \int f^{(-k)} g = 1\). Using (4) with \(g\) in place of \(f\) and \(f^{(-k)}\) in place of \(g\), we have

\[
\int f^{(-k)} g = f^{(-k-1)} g - \int f^{(-k-1)} g'
\]
so our choice of $g$ entails $1 = \varphi(f(-k-1)g) - \varphi(f(-k-1)g')$. If the first summand on the right-hand side is different from 1, then $g'$ witnesses to $\varphi(f(-k-1))$ being nondegenerate, and the induction is complete. Otherwise $\varphi(f(-k-1))$ must be nonzero, and we may use the special case $(\int h)^2 = 2 \int \int h$ of the pure Baxter axiom (5) to derive

$$(\varphi(f(-k-1))(f(-k)) = \varphi(f(-k))f(-k) = \frac{1}{2} \varphi(\int f(-k))^2 = \frac{1}{2} \varphi(f(-k-1))^2 \neq 0,$$

and again the induction is complete.

4 The Ring of Methorious Operators

Let us start by reviewing the general setting for localization in a noncommutative unital ring $R$, following [30 § 4.10]. As in the commutative case, the denominator set $S$ (the elements that should become invertible) must clearly form a multiplicative set $S$, meaning a submonoid of $(R^\times, \cdot)$. Clearly we must stipulate $0 \notin S$, otherwise the localization is the zero ring. If $R$ is a domain, one might want to take $S = R^\times$. While this is always possible in the commutative setting, one needs an additional condition if $R$ is not commutative.

Indeed, let us strive for a localization $S^{-1}R$ on the left, meaning all elements have the form $s^{-1}r$ with $s \in S$ and $r \in R$. Since $s^{-1}, r \in S^{-1}R$ this must also be possible for the reverse product so that $rs^{-1} = s^{-1}r$ for some $s \in S$ and $r \in R$. Multiplying out, we get the necessary condition

$$Sr \cap Rs \neq \emptyset \quad \text{for all } r \in R \text{ and } s \in S, \quad (17)$$

known as the left Ore condition; the set $S$ is then called left permutable.

If $R$ is a domain, this condition is actually sufficient for guaranteeing the existence of a unique localization $S^{-1}R \supseteq R$, which can be constructed essentially as in the commutative case. However, if $R$ has zero divisors, in general one does not get an embedding $R \subseteq S^{-1}R$. In this case, the extension $\varepsilon : R \to S^{-1}R$ is a ring homomorphism that is not injective (and of course not surjective). Its kernel contains at least those $r \in R$ that yield $sr = 0$ for some $s \in S$ since this implies $\varepsilon(s)\varepsilon(r) = 0$ and hence $\varepsilon(r) = 0$, due to $\varepsilon(s)$ being invertible in $S^{-1}R$. In the classical localization $S^{-1}R$, the kernel should be optimal in the sense that it contains no other elements than these necessary ones.

**Definition 6** Let $R$ be an arbitrary ring with $S \subseteq R$. Then $\varepsilon : R \to S^{-1}R$ is called a left ring of fractions if

(a) all elements $\varepsilon(s)$ with $s \in S$ are invertible in $S^{-1}R$,
(b) every element of $S^{-1}R$ has the form $\varepsilon(s)^{-1}\varepsilon(r)$ for some $s \in S$, $r \in R$,
(c) and the kernel of $\varepsilon$ is given by $\{ r \in R \mid sr = 0 \text{ for some } s \in S \}$.

The ring homomorphism $\varepsilon$ is called the extension.

The missing condition is now easy to establish. Assume $s \in S$ is a right zero divisor so that $rs = 0$ for some $r \in R$. Then also $\varepsilon(r)\varepsilon(s) = 0$ and hence $\varepsilon(r) = 0$.
since \( \varepsilon(s) \) is invertible in \( S^{-1}R \). But this implies \( \tilde{s}r = 0 \) for some \( \tilde{s} \in S \) by item (c) of Definition 6. Accordingly, one calls a set \( S \) with the property
\[
(\forall r \in R) \ (0 \in rS \Rightarrow 0 \in \tilde{s}r)
\]
left reversible. Together with the left Ore condition \((17)\), this turns out to be sufficient for the existence of a left ring of fractions.

**Theorem 2** Let \( R \) be an arbitrary ring. Then for any \( S \subseteq R \), the left ring of fractions \( S^{-1}R \) exists iff \( S \) is multiplicative, left permutable and left reversible.

**Proof** See [30, Thm. 10.6].

The setting would become much nicer when the extension \( \varepsilon: R \to S^{-1}R \) is injective so that we can regard \( R \subseteq S^{-1}R \) as an embedding. Unfortunately, the localization of \( \mathcal{F}_\Phi[\partial, \int] \) that we will work out in the sequel is not of this type. In fact, one can easily show that the injectivity of \( \varepsilon \) is equivalent to having only regular elements in \( S \).

Following [9, §5.1], we call an element \( s \in S \) regular if it is both left and right regular, where left regular means \( rs = 0 \) implies \( r = 0 \) for all \( r \in R \) while right regular means \( sr = 0 \) implies \( r = 0 \) for all \( r \in R \). As \( \mathcal{F}_\Phi[\partial, \int] \) contains plenty of zero divisors, it is hard to achieve a regular denominator set \( S \).

Of course there are analogous definitions for the right ring of fractions \( SR^{-1} \), but in general the existence of \( S^{-1}R \) does not imply the existence of \( SR^{-1} \) or vice versa, and even when both exist they need not be isomorphic. In fact, we shall be dealing with a case that is left permutable (Lemma 11) but not right permutable (Proposition 7).

The localization that we shall construct is based on the monoid \( \mathcal{E}[\partial, \Phi] \) of regular boundary problems over an integro-differential algebra \( (\mathcal{F}, \partial, \int) \) with umbral character set \( \Phi \). The ring \( R \) to be localized is the ring of boundary problems \( K\mathcal{E}[\partial, \Phi] \), which is a monoid algebra. For such cases, the above setting can be somewhat simplified.

A monoid \( S \) is called left permutable if it satisfies the Ore condition \((17)\), with respect to itself, meaning \( Ss \cap \tilde{S}S \neq \emptyset \) for all \( s, \tilde{s} \in S \). In the absence of addition, the analog of condition \((15)\) says that for all \( s, s_1, s_2 \in S \) with \( s_1s = s_2s \) there must exist \( \tilde{s} \in S \) with \( \tilde{s}s_1 = \tilde{s}s_2 \); in this case we call \( S \) left reversible (as a monoid). We call \( S \) a left Ore monoid if \( S \) is both left permutable and left reversible; cf. also [33, Def. 1.3].

The following lemma is a special case of [48, Lem. 6.6] but we include its proof since it dispenses with the technical machinery used there. It tells us that we can transfer left permutability from \( S \) to the monoid algebra \( KS \).

**Lemma 8** If \( S \) is a left permutable monoid, then \( S \) is a left permutable subset of \( KS \). Similarly, if \( S \) is a left reversible monoid, then \( S \) is a left reversible subset of \( KS \).

**Proof** Given \( s_1 \in S, \lambda_i \in K \) and \( s \in S \), we have to find \( \tilde{s}_1 \in S, \lambda_i \in K \) and \( \tilde{s} \in S \) such that
\[
\tilde{s}(\lambda_1s_1 + \cdots + \lambda_ns_n) = (\lambda_1\tilde{s}_1 + \cdots + \lambda_n\tilde{s}_n)s.
\]
Using the Ore condition in $S$, we can successively find $\tilde{l}_n, \ldots, \tilde{l}_1 \in S$ and $\tilde{r}_1, \ldots, \tilde{r}_n \in S$ such that

$$\tilde{l}_ns_1 = \tilde{r}_1s$$
$$\tilde{l}_{n-1}(\tilde{l}_ns_2) = \tilde{r}_2s$$
$$\ldots = \ldots$$
$$\tilde{l}_1(\tilde{l}_2 \cdots \tilde{l}_{n}s_n) = \tilde{r}_ns$$

is fulfilled. Multiplying these equations by $\lambda_1 \tilde{l}_1 \cdots \tilde{l}_{n-1}$, $\lambda_2 \tilde{l}_1 \cdots \tilde{l}_{n-2}$, $\ldots$, $\lambda_n$ on the left yields the system

$$\tilde{s}(\lambda_1s_1) = (\tilde{\lambda}_1s_1)s$$
$$\tilde{s}(\lambda_2s_2) = (\tilde{\lambda}_2s_2)s$$
$$\ldots = \ldots$$
$$\tilde{s}(\lambda_ns_n) = (\tilde{\lambda}_ns_n)s$$

if we set $\tilde{s} = \tilde{l}_1 \cdots \tilde{l}_n$ and $\tilde{s}_i = \tilde{l}_1 \cdots \tilde{l}_{n-i} \tilde{r}_i$ with coefficients $\tilde{\lambda}_i = \lambda_i$. Summing these equations gives the desired Ore condition (19). The proof of the second statement follows immediately by induction on the number of terms.

**Corollary 1** If $S$ is a left Ore monoid, the left ring of fractions $S^{-1}(KS)$ exists.

**Proof** Immediate from Lemma 8 and Theorem 2.

We shall now prove that $E[\partial]$ is a left Ore monoid. The multiplication of boundary problems is realized by the semi-direct product (10) of the multiplicative monoid $E[\partial]$ acting on the additive monoid of boundary spaces. Projecting onto the first factor, it is then clear that the monic differential operators of $E[\partial]$ must form a left Ore monoid. A coefficient algebra with this property shall be called left extensible.

**Example 6** Left permutability requires some common left multiple for given differential operators $T_1, T_2 \in E[\partial]$. Hence it may seem tempting to restrict ourselves to those coefficient algebras $(E, \partial)$ that even have a least common left multiple $\text{lclm}(T_1, T_2)$. But this setting is not suitable for our purposes: As far as we know, there are only two natural examples: The rational functions $E = K(x)$ allow an adaption of the Euclidean algorithm [47, p. 23], and of course one may always take $F = K$. The first case yields a differential field, which excludes the existence of an integro-differential structure [41, p. 518]. The second case restricts us to differential operators with constant coefficients, which is excessively restritive.

Fortunately it turns out that we do not need least common left multiples since left extensible coefficient algebras are easy to come by, as the next lemma shows. In particular, any integro-differential algebra $F$ allows $E = K[x]$ so that $E[\partial] = A_1(K)$ is the Weyl algebra. As another example, consider the standard setting of Example 1, where for $E$ one may take the larger ring of analytic functions $C^\infty(\mathbb{R})$.

**Proposition 6** Any left Noetherian differential domain $(E, \partial)$ is left extensible.
Proof If \( \mathcal{E} \) is a left Noetherian domain, it satisfies left permutability by [9, Thm. 5.4] and is therefore a left Ore domain. But \( \mathcal{E}[[\partial]] \) is a special case of a skew-polynomial algebra and therefore inherits the property of being a left Ore domain by [9, Prop. 5.9]. But this is not enough since we need the monic differential operators of \( \mathcal{E}[[\partial]] \) to form a left Ore monoid.

Hence assume \( T_1T = T_2T \) for some monic differential operators \( T_1, T_2, T \in \mathcal{E}[[\partial]] \). Since \( \mathcal{E}[[\partial]] \) is a domain, \( (T_1 - T_2)T = 0 \) implies \( T_1 = T_2 \) because \( T = 0 \) is not possible. Hence the monic differential operators from \( \mathcal{E}[[\partial]] \) form a left reversible monoid. But this monoid is indeed a left Ore monoid: According to [10, Lem. 1.5.1], the set of all monic polynomials in the skew polynomial ring \( \mathcal{E}[[\partial]] \) is left permutable whenever \( \mathcal{E} \) is a left Noetherian domain.

Let us remark that the generalization from least common left multiples to common left multiples is crucial even for the basic case \( \mathcal{E} = K[x] \). For example when \( T_1 = \partial + x \) and \( T_2 = \partial^2 + x\partial + x + 1 \), the least common left multiple of \( T_1 \) and \( T_2 \) in \( K(x)[\partial] \) is

\[
\partial^3 + \frac{x^2 - 1}{x} \partial^2 + (x^2 + 1 + x) \partial + \frac{x^2 - 1 + x^3}{x},
\]

and so gives \( T = x\partial^3 + (2x^2 - 1) \partial^2 + (x^3 + x^2 + x) \partial + (x^3 + x^2 - 1) \) when taken in \( \mathcal{E}[[\partial]] \). This shows that the least common left multiple of two monic operators in \( \mathcal{E}[[\partial]] \) need not be monic. The conventional computation tools for the Weyl algebra are therefore not directly applicable for computing monic common left multiples in \( \mathcal{E}[[\partial]] \), but some recent methods [6, 53] can be adapted to this purpose [A. Bostan, private communication].

As explained at the beginning of Section 3, the main tool for ensuring the left Ore condition in \( \mathcal{E}[[\partial]]_{\Phi} \) is the embedding of singular boundary problems into regular ones. This will be achieved in the Regularization Lemma 10 by successively embedding the Stieltjes conditions generating a given boundary space. Hence the crucial step is to embed a single Stieltjes condition, which we require to be umbral so that we can build up a polynomial fundamental system.

Lemma 9 Let \( \beta \in \mathcal{F}_\Phi[[\partial], [\cdot]] \) be an umbral Stieltjes condition over a given integro-differential algebra \( \mathcal{F} \). Then there is a \( k \in \mathbb{N} \) such that \( (\partial^{k+1}, [\varepsilon, \ldots, \varepsilon\partial^{k-1}, \beta]) \) is a regular boundary problem.

Proof Since \( \beta \) is umbral, there is a minimal monomial \( x^k \) with \( \beta(x^k) \neq 0 \). Clearly, \( u = (1, x/1!, \ldots, x^k/k!) \) is a fundamental system for \( \partial^{k+1} \). Using the boundary conditions \( \gamma = (\varepsilon, \ldots, \varepsilon\partial^{k-1}, \beta) \), we obtain the \((n+1) \times (n+1)\) evaluation matrix

\[
\gamma(u) = \begin{pmatrix}
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & \vdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & 0 \\
0 & \cdots & 0 & 0 & \beta(x^k/k!)
\end{pmatrix},
\]

where the off-diagonal bottom entries vanish by the assumption on \( x^k \).
Note that $k$ in Lemma 9 can also be zero, for example if $\beta = \mathbf{e}$. In that case, it is not necessary to add any initial conditions since the first-order problem $(\partial, [\mathbf{e}])$ is already regular.

**Example 7** As a more or less typical case consider $\beta = \mathbf{e}_1 - \mathbf{e}_0$. Here the minimal monomial is $x$, which leads back to our standard Example 2 the regular boundary problem $(\partial^2, [\mathbf{e}_0, \mathbf{e}_1 - \mathbf{e}_0]) = (\partial^2, [\mathbf{e}_0, \mathbf{e}_1])$. As mentioned earlier, we cannot split off the first-order subproblem $(\partial, [\mathbf{e}_1 - \mathbf{e}_0])$ since it is singular (this is an example of a subproblem that is not a right factor): If it were not, we would know from the Division Lemma 8 that the corresponding unique left factor is $(\partial, [\mathbf{f}^1_0])$, as shown in [41] Ex. 28] and also below (Example 8). But of course multiplying out just gives the degenerate problem

$$(\partial, [\mathbf{f}^1_0]) \cdot (\partial, [\mathbf{e}_1 - \mathbf{e}_0]) = (\partial^2, [\mathbf{e}_0, \mathbf{e}_1])$$

and not the desired regularized problem $(\partial^2, [\mathbf{e}_0, \mathbf{e}_1])$.

Based on the case of a single boundary condition, we can now embed an arbitrary boundary problem into a regular one—provided we work with an umbral character set. This is the subject of the following **Regularization Lemma**.

**Lemma 10** Let $\Phi$ be an umbral character set for an integro-differential algebra $\mathcal{F}$. Then for an arbitrary boundary problem $(T, \mathcal{B}) \in \mathcal{E}[\partial] \times K\Phi$ there is a regular boundary problem $(S, \mathcal{A}) \in \mathcal{E}[\partial]_{\Phi}$ that has $(T, \mathcal{B})$ as a subproblem.

**Proof** Let $(T, \mathcal{B}) \in \mathcal{E}[\partial] \times K\Phi$ be an arbitrary but fixed boundary problem with $T$ a differential operator of order $n > 0$ and $\mathcal{B} = [\beta_1, \ldots, \beta_m] \leq \mathcal{F}^*$ of dimension $m$. We write $\mathcal{F}_{\beta}$ for the space of initial conditions $[\mathbf{e}, \ldots, \mathbf{e}\partial^{n-1}]$ and $\mathcal{B}_k$ for the partial boundary space $[\beta_1, \ldots, \beta_k]$ with the convention that $\mathcal{B}_0 = \emptyset$. We will now prove that for each $k = 0, \ldots, m$ there is a regular boundary problem $(S, \mathcal{A})$ that has $(T, \mathcal{B}_k)$ as a subproblem. Taking $k = m$, the theorem follows.

We use induction on $k$. The base case follows by setting $(S, \mathcal{A}) = (T, \mathcal{F}_{\beta_1})$. For the induction step, assume $(\tilde{S}, \mathcal{A})$ is a regular boundary problem that has $(T, \mathcal{B}_{k-1})$ as a subproblem. We have to construct a regular problem $(\tilde{S}, \mathcal{A})$ that has $(T, \mathcal{B}_k)$ as a subproblem. Letting $\tilde{G}$ be the Green’s operator of $(\tilde{S}, \mathcal{A})$, assume first that $\beta_k \tilde{G}$ is degenerate. Since $\beta_k \tilde{G}$ vanishes on $\mathcal{F}$, we obtain

$$\text{Im}(\tilde{G}) = \mathcal{A} \gamma \leq [\beta_k]^{-1}$$

and hence $[\beta_k] \leq \mathcal{A}$.

so we may set $(S, \mathcal{A}) = (\tilde{S}, \mathcal{A})$ in that case. Now assume $\beta_k \tilde{G}$ is nondegenerate and hence umbral. Lemma 8 yields a regular problem $(\tilde{T}, \mathcal{B}) = (\partial^{r+1}, [\mathbf{e}, \ldots, \mathbf{e}\partial^{r-1}, \beta_k \tilde{G}])$. We define the boundary problem

$$(S, \mathcal{A}) = (\tilde{T}, \mathcal{B}) \cdot (\tilde{S}, \mathcal{A}) = (\tilde{T} \tilde{S}, [\mathbf{e}\tilde{S}, \ldots, \mathbf{e}\partial^{r-1}\tilde{S}, \beta_k \tilde{G} \tilde{S} + \mathcal{A}])$$

$$= (\tilde{T} \tilde{S}, [\mathbf{e}\tilde{S}, \ldots, \mathbf{e}\partial^{r-1}\tilde{S}, \beta_k] + \mathcal{A}),$$

where the last equality follows since the conditions $\beta_k u = 0$ and $\beta_k \tilde{G} \tilde{S} u = 0$ are equivalent for $u \in \mathcal{A} \gamma$. The boundary problem $(S, \mathcal{A})$ is clearly regular since it is the product of two regular problems, and it has $(T, \mathcal{B}_k)$ as a subproblem because $\mathcal{B}_{k-1} \leq \mathcal{A}$ by the induction hypothesis.
We are now ready to prove left permutability for the monoid $\mathcal{E}[\partial]\Phi$ by merging the given factors into a singular problem that is subsequently embedded into a regular problem by virtue of Lemma 10.

**Lemma 11** Let $\Phi$ be an umbral character set for an integro-differential algebra $\mathcal{F}$ with left extensible coefficient algebra $\mathcal{E}$. Then $\mathcal{E}[\partial]\Phi$ is a left permutable monoid.

**Proof** Given $(T_1, \mathcal{A}_1), (T_2, \mathcal{A}_2) \in \mathcal{E}[\partial]\Phi$, we must find $(\tilde{T}_1, \tilde{\mathcal{A}}_1), (\tilde{T}_2, \tilde{\mathcal{A}}_2) \in \mathcal{E}[\partial]\Phi$ such that $(\tilde{T}_1, \tilde{\mathcal{A}}_1) \cdot (T_1, \mathcal{A}_1) = (\tilde{T}_2, \tilde{\mathcal{A}}_2) \cdot (T_2, \mathcal{A}_2)$. Since $\mathcal{E}$ is left extensible, Proposition 6 yields a common left multiple $T$ and cofactors $\tilde{T}_1$ and $\tilde{T}_2$ such that $T = \tilde{T}_1 T_1 = \tilde{T}_2 T_2$. Now set $\mathcal{B} = \mathcal{A}_1 + \mathcal{A}_2$. By Lemma 11 there is a regular boundary problem $(S, \mathcal{A})$ that has $(T, \mathcal{B})$ as a subproblem. But then the boundary problem $(S, \mathcal{A})$ has $(T_1, \mathcal{A}_1)$ and $(T_2, \mathcal{A}_2)$ as regular subproblems, and Lemma 3 yields regular boundary problems $(\tilde{T}_1, \tilde{\mathcal{A}}_1)$ and $(\tilde{T}_2, \tilde{\mathcal{A}}_2)$ such that

$$(S, \mathcal{A}) = (\tilde{T}_1, \tilde{\mathcal{A}}_1) \cdot (T_1, \mathcal{A}_1) = (\tilde{T}_2, \tilde{\mathcal{A}}_2) \cdot (T_2, \mathcal{A}_2)$$

as claimed.

**Remark 1** Lemma 11 is also true if one replaces $\mathcal{E}[\partial]\Phi$ by the monoid of well-posed boundary problems defined before Eq. (11). This can be seen readily by inspecting the above proof (and the proofs of Lemma 3 and 10).

Let us give a simple example of a nontrivial Ore quadruple $(T_1, \mathcal{A}_1), (T_2, \mathcal{A}_2), (\tilde{T}_1, \tilde{\mathcal{A}}_1), (\tilde{T}_2, \tilde{\mathcal{A}}_2)$ that will also serve a good purpose later on.

**Example 8** In the standard setting of Example 1, consider the two simplest first-order problems on $[0, 1]$, namely

$$(T_1, \mathcal{A}_1) = (\partial, [\mathcal{E}_0]) \quad \text{and} \quad (T_2, \mathcal{A}_2) = (\partial, [\mathcal{E}_1]).$$

In that case, we have of course $T = \partial$, and we apply Lemma 10 to the boundary problem $(S, \mathcal{A}) = (\partial, [\mathcal{E}_0, \mathcal{E}_1])$. We end up with

$$(\partial, [\mathcal{E}_0]) \cdot (\partial, [\mathcal{E}_0]) = (\partial^2, [\mathcal{E}_0, \mathcal{E}_1]),$$

which is already regular. The corresponding cofactors are then

$$(T_1, \tilde{\mathcal{A}}_1) = (\partial, [\mathcal{E}_0]) \quad \text{and} \quad (T_2, \tilde{\mathcal{A}}_2) = (\partial, [\mathcal{E}_0])$$

since clearly $(\partial^2, [\mathcal{E}_0]) = (\partial^2, [\mathcal{E}_0, \mathcal{E}_1]) = (\partial^2, [\mathcal{E}_0, \mathcal{E}_1]).$

As mentioned before, $\mathcal{E}[\partial]\Phi$ is left permutable but not right permutable. This means there are boundary problems that do not have a common right factor. Actually, more is true: Even if we start from two distinct problems with the same differential operator, any common right multiple comes from a singular factor.
Proposition 7 Let $\Phi$ be an arbitrary character set for an integro-differential algebra $\mathcal{F}$ with coefficient algebra $\mathcal{E}$. Assume $(T, \mathcal{B}_1), (T, \mathcal{B}_2) \in \mathcal{E}[\partial]_{\Phi}$ have a common right multiple

$$(T, \mathcal{B}_1) (S, \mathcal{C}_1) = (T, \mathcal{B}_2) (S, \mathcal{C}_2)$$

(20)

for some right factors $(S, \mathcal{C}_1), (S, \mathcal{C}_2) \in \mathcal{E}[\partial] \times K \Phi$. Then both $(S, \mathcal{C}_1)$ and $(S, \mathcal{C}_2)$ are singular whenever $\mathcal{B}_1 \neq \mathcal{B}_2$.

Proof Assume $\mathcal{B}_1 \neq \mathcal{B}_2$. Projecting onto the boundary spaces, we have $\mathcal{B}_1 S + \mathcal{C}_1 = \mathcal{B}_2 S + \mathcal{C}_2$. If $\mathcal{B}_1 = O$ or $\mathcal{B}_2 = O$, we have $T = 1$, so in fact $\mathcal{B}_1 = \mathcal{B}_2$. Choosing $\beta_1 \in \mathcal{B}_1 \setminus \mathcal{B}_2$ we have $\beta_1 S \in \mathcal{B}_2 S + \mathcal{C}_2$ but $\beta_1 S \notin \mathcal{B}_2 S$. Hence we can write $\beta_1 S = \beta_2 S + \gamma_2$ for some $\beta_2 \in \mathcal{B}_2$ and nonzero $\gamma_2 \in \mathcal{C}_2$. But then $\gamma_2 = (\beta_1 - \beta_2) S \in \text{Ker}(S)^\perp \cap \mathcal{C}_2$ implies that $(S, \mathcal{C}_2)$ is singular. By symmetry, we see that $(S, \mathcal{C}_2)$ is singular as well.

Having established that $\mathcal{E}[\partial]_{\Phi}$ is left permutable, the only thing missing for the localization is left reversibility, which is very easy in our case. Hence we obtain the desired left ring of fractions.

Theorem 3 Let $\Phi$ be an umbral character set for the integro-differential algebra $(\mathcal{F}, \partial, \mu)$ with left extensible coefficient algebra $\mathcal{E}$. Then there exists a left ring of fractions $K \mathcal{E}[\partial]_{\Phi}$ of the ring of boundary problems $K \mathcal{E}[\partial]_{\Phi}$ with denominator set $\mathcal{E}[\partial]_{\Phi}$.

Proof By Corollary 1 it suffices to check that $\mathcal{E}[\partial]_{\Phi}$ is an Ore monoid. Since left permutability of $\mathcal{E}[\partial]_{\Phi}$ has been shown in Lemma 11 it remains to show that $\mathcal{E}[\partial]_{\Phi}$ is a left reversible monoid. So assume we have $(T_1, \mathcal{B}_1) (T, \mathcal{B}) = (T_2, \mathcal{B}_2) (T, \mathcal{B})$ for some regular problems $(T_1, \mathcal{B}_1), (T_2, \mathcal{B}_2), (T, \mathcal{B})$. By left extensibility of $\mathcal{E}$ we have $TT_1 = TT_2$ for some monic $T \in \mathcal{E}[\partial]$ of order $n$. Let $\mathcal{B} = [e, \ldots, e T^{n-1}]$ be the corresponding space of initial conditions so that $(T, \mathcal{B})$ is a regular problem. Then we obtain the regular product problem

$$((T, \mathcal{B})(T_1, \mathcal{B}_1)) (T, \mathcal{B}) = ((T, \mathcal{B})(T_2, \mathcal{B}_2)) (T, \mathcal{B}),$$

where the two parenthesized factors have the same differential operator by the choice of $T$. Since $(T, \mathcal{B})$ is a regular right factor of this problem, the Division Lemma implies that $(T, \mathcal{B})(T_1, \mathcal{B}_1) = (T, \mathcal{B})(T_2, \mathcal{B}_2)$. We conclude that the monic differential operators of $\mathcal{E}[\partial]$ are indeed left reversible.

The fraction ring $K \mathcal{E}[\partial]_{\Phi}$ shall be called the ring of meθorios operators (from the Greek word μεθόριος meaning “making up a boundary”). Note that it exists in particular in the smooth setting: If one chooses $\mathcal{F} = C^\infty(\mathbb{R})$, every character is umbral (Proposition 4), and any Noetherian domain can be used for the coefficient algebra (Proposition 6), in particular $\mathcal{E} = K[x]$ or $\mathcal{E} = C^\infty(\mathbb{R})$. In the sequel, we shall confine ourselves to the latter setting.

As already observed after Theorem 2 we must expect that $\varepsilon : K \mathcal{E}[\partial]_{\Phi} \to K \mathcal{E}[\partial]_{\Phi}^*$ is not an embedding. For example, we have $N = (\partial_x x_0) - (\partial_x x_1) \in \text{Ker}(\varepsilon)$ in the standard setting of $\mathcal{F} = C^\infty(\mathbb{R})$. For checking this, we use the characterization
of $\ker(\varepsilon)$ given in Definition 6. It suffices to find a regular problem that annihilates $N$ from the left. Indeed, we have $(\partial, [\int^0_1]) N = 0$ since we know from Example 8 that

$$(\partial, [\int^0_1])(\partial, [e_0]) = (\partial, [\int^0_1]) = (\partial, [\int^0_1]).$$

At the moment we do not know $\ker(\varepsilon)$ in explicit form. However, we have the following conjecture.

**Conjecture 1** Let $\Phi$ be an umbral character set for an integro-differential algebra $\mathcal{F}$ with left extensible coefficient algebra $\mathcal{E}$, and let $\varepsilon: K\mathcal{E}[\partial]_\Phi \to K\mathcal{E}[\partial]_\Phi^*$ be the extension into the ring of methorious operators. Then we have $\sum \lambda_i (T_i, B_i) \in \ker(\varepsilon)$ iff $\sum \lambda_i G_i \in (\Phi)$, where $G_i$ is the Green’s operator of $(T_i, B_i)$.

The previous example is a case in point. The intuitive reason for our conjecture is this: Generically, a linear combination of Green’s operators has a finite-dimensional cokernel (meaning its image is annihilated by just finitely many functionals—in the case of a single operator these functionals are the given boundary conditions). Such an operator is in some sense “almost invertible”. But if a linear combination degenerates into an element of $(\Phi)$, its image becomes one-dimensional, and we cannot expect to invert such an “operator”.

### 5 The Module of Methorious Functions

Extending an operator ring is much more useful if its elements may still be viewed as operating on some—presumably extended—domain of “functions”. As explained in the Introduction, the operational calculus of Mikusiński avoids this problem by merging “operators” and “operands” in the Mikusiński field. In contrast, we shall follow the algebraic analysis approach of keeping operators and operands in separate structures.

To this end we construct a suitable module of fractions that extends a given integro-differential algebra $(\mathcal{F}, \partial, \int)$. In analogy to the Mikusiński approach we would have to start from a ring $R$ of integral operators acting on $\mathcal{F}$, construct its localization $R^*$ and let it act on the corresponding localization $\mathcal{F}^*$. However, this does not work for the following reason: The natural candidate for $R$ would be the $K$-algebra of Green’s operators for regular boundary problems (since this is the only ring over which we have sufficient control). But this monoid/ring is dual to the regular boundary problems, as we know from (9). Since $\mathcal{E}[\partial]_\Phi$ is a left Ore monoid but not a right Ore monoid, $R$ is a right but not a left Ore ring. Hence we can only create a right ring of fractions $R^*$ and correspondingly only a right module of fractions that should extend $\mathcal{F}$. But the given integro-differential algebra $\mathcal{F}$ only has a natural left action of $R \subseteq \mathcal{F}[\partial, \int]$.

In fact, it is not quite true that one cannot construct a left module of fractions from a given left module $M$ over a right Ore ring $R$. If $R^*$ is the right ring of fractions, one may of course construct the usual scalar extension $M^* = R^* \otimes_R M$. One may then refer to $M^*$ as a left module of fractions [48 Prop. 7.2]. But the problem with this construction is that we do not know anything about its structure. For example, there
is no useful characterization of the kernel of $M \to M^*$ as in the Ore construction of Theorem 3 below.

We must therefore take the differential operators as our starting point, and this is why we have constructed the localization $K\mathscr{E}[\partial]_\Phi$ of the ring of boundary problems $K\mathscr{E}[\partial]_\Phi$. One might be tempted to take just the simpler ring $\mathscr{E}[\partial]$ instead of the unwieldy monoid ring $K\mathscr{E}[\partial]_\Phi$. But this would be too simplistic: In that case one gets a two-sided inverse $\partial^{-1}$ of the differential operator, no new functions are generated, and of course we cannot tackle boundary problems in such a setting. It is essential to work with the richer structure $K\mathscr{E}[\partial]_\Phi$, for which we will have to set up a suitable left action on $\mathcal{F}$.

But first let us briefly review the general setting for the localization of modules. As in Theorem 2, we go back to the setting of general rings $R$. In this case we can construct the module of fractions in pretty much the same way as the ring of fractions [51, Cor. II.3.3].

**Theorem 4** Let $M$ be a left $R$-module, and let $S \subseteq R$ be a multiplicative, right permutative and right reversible denominator set $S \subseteq R$. Then there exists a left $S^{-1}R$-module $S^{-1}M$, and the kernel of the extension $\mu: M \to S^{-1}M$ consists of those $u \in M$ for which there exists an $s \in S$ with $su = 0$.

Now let $(\mathcal{F}, \partial, \int)$ be an integro-differential algebra and $\mathcal{F}_\Phi[\partial, \int]$ its ring of operators. For setting up a suitable left action of $\mathscr{E}[\partial]_\Phi$ on $\mathcal{F}$, we recall the fundamental formula [31] of the Mikusiński calculus

$$ sf = f^\prime + f(0) \delta_0, $$

where $s$ denotes the inverse of the standard convolution operator $l$, defined by $lf = \int_l f(\xi) d\xi$, and $\delta_0$ is simply the multiplicative identity generated by the construction of the Mikusiński field. Of course one may write $\delta_0 = 1$ but we keep the explicit notation for intuitive reasons since we can interpret (21) as the distributional derivative of a function $f: \mathbb{R} \to \mathbb{R}$ that is continuous except for a jump at 0. In that case, $\delta_0$ is the Dirac distribution concentrated at 0. At any rate, the intuitive content of (21) is that $s$ is a kind of enhanced differential operator that memorizes the function value at 0 that is otherwise lost. Taking this clue, we define the action of $\mathscr{E}[\partial]_\Phi$ on $\mathcal{F}$ by

$$ (T, \mathcal{B}) \cdot f = Tf + Pf (T, \mathcal{B}), $$

where $P$ is the projector onto $\text{Ker}(T)$ along $\mathcal{B}$. In the special case of $T = \partial$, this recovers (21) if we think of $(\partial, [e_0])$ as some kind of algebraic representation of the Dirac distribution $\delta_0$.

The definition (22) has the consequence that we must extend $\mathcal{F}$ so that it contains the new elements $(T, \mathcal{B})$, prior to the extension effected by the localization. This is somewhat unsatisfying but appears to be inevitable in view of the facts pointed out at the opening of the section. Perhaps in the future one can find a more powerful localization that generates at once the extension objects needed in the fundamental formula (21). At any rate, we must now also define the action of $\mathcal{E}[\partial]_\Phi$ on the new elements, but this is clearly

$$ (\tilde{T}, \tilde{\mathcal{B}}) \cdot f (T, \mathcal{B}) = f (\tilde{T}T, \tilde{\mathcal{B}}T + \mathcal{B}), $$

(23)
where the boundary problem on the right side is the usual product \((10)\) in the problem monoid \(\mathcal{E}[\partial]\). This will ensure that the action is well-defined.

The intuitive meaning of the ideal element \(f(T, \mathcal{B})\) is that it records various integration constants that were lost under differentiation. When the element is created by \((22)\), it encodes all the boundary values \(\beta(f)\) for \(\beta \in \mathcal{B}\), and the function \(f\) is confined to the kernel of \(T\). Further action by a boundary problem \((\hat{T}, \mathcal{B})\) according to \((23)\) leads to a new differential operator \(\hat{T}T\) while leaving \(f\) intact. Hence we see that the function inside an ideal element is always in the kernel of the associated differential operator. We may therefore restrict the ideal elements \(f(T, \mathcal{B})\) to those with \(Tf = 0\). After the action of \((23)\), the new boundary space is \(\hat{T}T + \mathcal{B}\), so the old boundary conditions \(\beta \in \mathcal{B}\) are retained (the ones for which we have boundary values). The new boundary conditions \(\beta T\) for \(\beta \in \mathcal{B}\) give zero on \(f\) since actually \(Tf = 0\). In some sense, we have added redundant boundary data.

We must therefore regulate redundancy in the ideal elements. This can be achieved by declaring that \(f(T, \mathcal{B})\) should be the same as \(\hat{G}f(T, \mathcal{B})(T, \mathcal{B})\) for any \((T, \mathcal{B}) \in \mathcal{E}[\partial]_{\mathcal{P}}\) with Green’s operator \(\hat{G}\). This can be understood as follows. The boundary conditions in \(\hat{G}f(T, \mathcal{B})(T, \mathcal{B})\) are \(\beta \in \mathcal{B}\) and \(\beta T\) with \(\beta \in \mathcal{B}\). The former yield zero on \(\hat{G}f\) since \(\hat{G}\) is the Green’s operator of \((T, \mathcal{B})\), the latter give back \(\beta T(\hat{G}f) = \beta(f)\) just as for the ideal element \(f(T, \mathcal{B})\). This leads to the following definition.

**Definition 7** Let \(\mathcal{I}\) be the subspace of \(\mathcal{F} \otimes_K \mathcal{E}[\partial]_{\mathcal{P}}\) generated by the ideal elements \(f \otimes (T, \mathcal{B}) \equiv f(T, \mathcal{B})\) with \(Tf = 0\). Furthermore, let \(\mathcal{I}_0\) be the subspace of \(\mathcal{I}\) generated by the elements

\[
f(T, \mathcal{B}) - \hat{G}f(T\hat{T}, \mathcal{B}\hat{T} + \mathcal{B}).
\]

Then we define the module of methorious functions to be \(\mathcal{F}_{\mathcal{P}} = \mathcal{F} \otimes \mathcal{I} / \mathcal{I}_0\).

Let us now check that \(\mathcal{F}_{\mathcal{P}}\) is indeed a module.

**Proposition 8** Let \((\mathcal{F}, \partial, \int)\) be an integro-differential algebra with character set \(\Phi\). The definitions \((22)\) and \((23)\) induce a monoid action of \(\mathcal{E}[\partial]_{\mathcal{P}}\) on \(\mathcal{F}_{\mathcal{P}}\) such that it becomes a \(K\mathcal{E}[\partial]_{\mathcal{P}}\)-module.

**Proof** Any monoid \(E\) acting on a \(K\)-algebra \(A\) extends to an action of the monoid ring \(K[E]\) that makes \(A\) into a \(K[E]\)-module. Hence it suffices to verify the statement about the monoid action.

First of all we must check that \((23)\) does not depend on the representative \(f(T, \mathcal{B}) \in \mathcal{I}\), meaning it maps the subspace \(\mathcal{I}_0\) into itself. This follows immediately from the associativity of the multiplication \((10)\) in \(\mathcal{E}[\partial]_{\mathcal{P}}\).

Clearly we have \((1, 0) \cdot f = f + 0(1, 0) = f\) and \((1, 0) \cdot (T, \mathcal{B})f = (T, \mathcal{B})f\), so the unit element is respected. Associativity is immediate in the case of \((23)\), so it remains to check that \((T, \mathcal{B}) \cdot ((T, \mathcal{B}) \cdot f)\) and \(((T, \mathcal{B})(T, \mathcal{B})) \cdot f\) are equal. The former is

\[
(T, \mathcal{B}) \cdot \left( \hat{T}f + Pf(T, \mathcal{B}) \right) = TTf + P\hat{T}f(T, \mathcal{B}) + Pf(T\hat{T}, \mathcal{B}\hat{T} + \mathcal{B}),
\]

where \(P\) is the projector onto \(\text{Ker}(T)\) along \(\mathcal{B}^\perp\) and \(\hat{P}\) is correspondingly the projector onto \(\text{Ker}(\hat{T})\) along \(\mathcal{B}^\perp\). If \(G\) and \(\hat{G}\) are the Green’s operators of \((T, \mathcal{B})\) and \((\hat{T}, \mathcal{B})\),
respectively, one can easily check that the projector associated with the composite problem
\[(T, \mathcal{B})(\tilde{T}, \mathcal{B}) = (T\tilde{T}, \mathcal{B}\tilde{T} + \mathcal{B})\]
is \(\tilde{P} + \tilde{G}P\tilde{T}\), meaning it projects onto \(\text{Ker}(T\tilde{T})\) along \((\mathcal{B}\tilde{T} + \mathcal{B})^\perp\). Hence we obtain for the other side of the prospective equality
\[
(T\tilde{T}, \mathcal{B}\tilde{T} + \mathcal{B}) \cdot f = T\tilde{T}f + \tilde{P}f + G\tilde{P}f(T\tilde{T}, \mathcal{B}\tilde{T} + \mathcal{B})
\]
where the first and the last term is identical to the corresponding terms on the left hand side while the middle terms are equal since their difference is in \(\mathcal{F}_0\).

Since \(\mathcal{F}_\Phi\) is a \(K\mathcal{E}[\partial]_\Phi\)-module, we obtain the \(K\mathcal{E}[\partial]_\Phi^*\)-module \(\mathcal{F}_\Phi^*\) of methorious hyperfunctions by localization via Theorem[4] We cannot expect the extension \(\mu: \mathcal{F}_\Phi \to \mathcal{F}_\Phi^*\) to be injective since its kernel
\[
\mathcal{K} = \{ \phi \in \mathcal{F}_\Phi \mid \exists (T, \mathcal{B}) \in \mathcal{E}[\partial]_\Phi: (T, \mathcal{B}) \cdot \phi = 0 \}
\]
contains elements like \((\partial, [\varepsilon_1]) - (\partial, [\varepsilon_0])\), which is annihilated upon multiplying with \((\partial, \{ f^1_0 \})\) from the left (see the example before Conjecture[4]). But fortunately no elements of \(\mathcal{F}\) are lost.

**Proposition 9** Let \((\mathcal{F}, \partial, \varphi)\) be an integro-differential algebra with character set \(\Phi\). Then we have an embedding \(\mathcal{F} \subset \mathcal{F}_\Phi\).

**Proof** Assume \((T, \mathcal{B}) \cdot f = 0\) for some \(f \in \mathcal{F}\) and \((T, \mathcal{B}) \in \mathcal{E}[\partial]_\Phi\). Then \(Tf + Pf(T, \mathcal{B}) = 0\) implies that \(Tf = 0\) and \(Pf = 0\). But the former means that \(f \in \text{Ker}(T)\) and the latter that \(f \in \mathcal{B}^\perp\). Since \((T, \mathcal{B})\) is a regular problem, we have a direct sum \(\text{Ker}(T) + \mathcal{B}^\perp = \mathcal{F}\) and so \(f = 0\).

In the module of methorious hyperfunctions, we can finally justify our earlier notation \((T, \mathcal{B})^{-1}\) for the Green’s operator of a regular boundary problem \((T, \mathcal{B}) \in \mathcal{E}[\partial]_\Phi\). To avoid confusion with \((T, \mathcal{B})^{-1} \in K\mathcal{E}[\partial]_\Phi^*\) we refrain from the reciprocal notation for Green’s operators in the scope of the following theorem. But the result of the theorem is of course that it anyway does not matter how we interpret \((T, \mathcal{B})^{-1} f\) since it amounts to the same.

**Proposition 10** We have \((T, \mathcal{B})^{-1} \cdot f = Gf\) for all \(f \in \mathcal{F}\). Moreover, if \(Tf = 0\) then \((T, \mathcal{B})^{-1} \cdot f (T, \mathcal{B}) = f\).

**Proof** We have \((T, \mathcal{B}) \cdot Gf = TGf + PGf(T, \mathcal{B}) = f\) since \(TG = 1\) and \(PG = 0\). Multiplying by \((T, \mathcal{B})^{-1} \in K\mathcal{E}[\partial]_\Phi^*\) yields the first result claimed. Now assume \(Tf = 0\). We obtain \((T, \mathcal{B}) \cdot f = 0 + Pf(T, \mathcal{B}) = f(T, \mathcal{B})\) since \(Pf = f\) for \(f \in \text{Ker}(T)\). Again we multiply by \((T, \mathcal{B})^{-1}\) to gain the result.

We conclude with an example that hints at possible applications of our noncommutative Mikusiński calculus. The classical Mikusiński calculus has only one fundamental formula since boundary values (or rather: initial values) are only processed at 0. In contrast, there are plenty of fundamental formulae in the noncommutative Mikusiński calculus.
Example 9 Let us write $\partial_\xi$ and $\delta_\xi$ as abbreviations for the problems $(\partial, [\xi]) \in K \mathcal{E}[[\partial]]$ and $(\partial, [\xi]) \in F \mathcal{E}[[\partial]]$, respectively. Then we have $\partial_\xi f = f' + f(\xi) \delta_\xi$ by the definition of the action (22). So we have algebraic representations for all the Dirac distributions. But there are other methorious functions that do not have any distributional counterpart. For example, let us consider $\partial_F = (\partial, [\int_0^1 f(\xi) d\xi]) \in F \mathcal{E}[[\partial]]$ and $\varepsilon = (\partial, [\int_0^1]) \in F \mathcal{E}[[\partial]]$. This yields the fundamental formula

$$\partial_F f = f' + \left( \int_0^1 f(\xi) d\xi \right) \varepsilon,$$

so $\varepsilon$ is a kind of “smeared out” distribution that keeps the mean value of a given function $f$.

Example 10 Finally let us see how one solves inhomogeneous boundary problems in the noncommutative Mikusiński calculus. We use the standard setting of $\mathcal{F} = \mathcal{C}^\infty(\mathbb{R})$ and consider the problem

$$\begin{align*}
\frac{d^2}{dx^2} u &= f \\
u(0) &= a, u(b) = b
\end{align*}$$

for given boundary values $a, b \in \mathbb{R}$. We know how to compute the Green’s operator $G$ of the homogeneous problem $(\partial^2, [\xi])$ as mentioned at the end of Section 2. In this case the projector $P$ onto $\ker(\partial^2) = [1, x]$ along $[\xi, [\xi]]$ is given by $Pu = u(0)(1-x) + u(1)x$. Hence we have

$$\begin{align*}
(\partial^2, [\xi]) u &= u'' + \left( u(0)(1-x) + u(1)x \right)(\partial^2, [\xi]) \\
n &= f + \left( a(1-x) + bx \right)(\partial^2, [\xi]), \\
u &= Gf + a(1-x) + bx
\end{align*}$$

by Proposition 10.

6 Conclusion

As already indicated at various points above, our construction has several loose ends. To begin with, it would be preferrable to localize in a subring (or even all) of $\mathcal{F}_\phi[\partial, f]$ rather than the monoid ring $K \mathcal{E}[[\partial]]$. This would have the advantage that one has a natural action on the underlying integro-differential algebra $(\mathcal{F}, \partial, [\xi])$, and the somewhat artificial action on the module of methorious functions $\mathcal{F}_\phi$ would be unnecessary. Regarding the latter, we have already remarked in Section 5 that the current two-stage process of creating the localization $\mathcal{F}_\phi^*$ is unsatisfactory: In Mikusiński’s setup, all the “ideal elements” like $s$ and $\delta_\xi$ are an immediate result of the localization while we have to supply them offhand, prior to localization. We would like to find a better formulation in the future that will avoid this kind of inadequacy.

One possible path towards such an improved localization is suggested by the prominent appearance of singular problems in the proof of Lemma 11. It may be
worthwhile to expand the ring $K\mathcal{E}[\partial]_{\Phi}$ to include (all or some) singular boundary problems. Of course this will also increase the kernel of the corresponding extension (we cannot expect to invert singular problems), but perhaps the resulting ring of fractions is more natural. In particular, it may be possible to interpret the sum of two (singular) boundary problems in some useful way, in contrast to the formal sums of $K\mathcal{E}[\partial]_{\Phi}$. The results of [27,25] will be useful for the work in this direction.

Staying with the current construction, there are some obvious open questions: First of all, the kernel of the extension $\varepsilon$ into the methorious operators should be determined, possibly by proving Conjecture [1]. Likewise, the kernel [24] of the extension $\mu$ into the methorious hyperfunctions is to be computed.

Finally, we would like to draw attention to the interesting link between integro-differential algebras and the umbral calculus (Section 3), which deserves to be studied in more depth. There is also an intriguing relation [19] between the umbral calculus and the Rota-Baxter algebras (both of these being favourite topic of G.-C. Rota), which might benefit from the new perspective afforded by boundary problems in integro-differential algebras (since the latter are special cases of differential Rota-Baxter algebras and hence of plain Rota-Baxter algebras). As a more mundane goal, it would also be important to find a better characterization of umbral character sets that strengthens the separativity and completeness conditions given before Example [3].

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