Reentrant stability of BEC standing wave patterns

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We describe standing wave patterns induced by an attractive finite-ranged external potential inside a large Bose-Einstein Condensate (BEC). As the potential depth increases, the time independent Gross-Pitaevskii equation develops pairs of solutions that have nodes in their wavefunction. We elucidate the nature of these states and study their dynamical stability. Although we study the problem in a two-dimensional BEC subject to a cylindrically symmetric square well potential of a radius that is comparable to the coherence length of the BEC, our analysis reveals general trends, valid in two and three dimensions, independent of the symmetry of the localized potential well, and suggestive of the behavior in general, short- and large-range potentials. One set of nodal BEC wavefunctions resembles the single particle $n$ node bound state wavefunction of the potential well, the other wavefunctions resemble the $n-1$ node bound-state wavefunction with a kink state pinned by the potential. The second state, though corresponding to the lower free energy value of the pair of $n$ node BEC states, is always unstable, whereas the first can be dynamically stable in intervals of the potential well depth, implying that the standing wave BEC can evolve from a dynamically unstable to stable, and back to unstable status as the potential well is adiabatically deepened, a phenomenon that we refer to as “reentrant dynamical stability”.

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I. INTRODUCTION

As a superfluid, the dilute gas Bose-Einstein condensate (BEC) is a coherent quantum system and behaves in many respects like a single-particle quantum wave. Stationary single particle bound states have wave functions with nodes, giving standing wave patterns. In a BEC, nodal structures, such as the one pictured in Fig. 1, could be imaged and manipulated directly since the experimental BEC wave functions extend over tens of microns. Can BECs support such patterns as long-lived structures? A fundamental difference between single particle and BEC wave dynamics is the nonlinearity of the BEC evolution that stems from the inter-particle interactions. This nonlinearity leads to stationary standing wave solutions such as kink states or dark solitons that have been created in BEC experiments \textsuperscript{1,2}. However, the same nonlinearity also makes the kink states unstable in two and three dimensions \textsuperscript{3,4,5}, as has been observed in cold atom experiments \textsuperscript{6}. The work of this paper was motivated by the basic question: when and how can a BEC support a long-lived standing wave induced by a local potential well in dimensions higher than one? The standing wave gives a stationary interference pattern in the full single-particle density that vanishes \textsuperscript{7} on the surfaces at which the stationary BEC wavefunction changes sign. The existence of such structures may, as we discuss below, impact fundamental science phenomena and cold atom applications.

We analyze the nodal BEC wavefunctions in a two-dimensional (pancake shaped) BEC; although the trends uncovered by our analysis, suitably generalized, are valid in higher dimensions. The 2D trap geometry, realized in cold atom experiments \textsuperscript{8,9}, offers unprecedented prospects for “wavefunction engineering”: focused laser-beams can, in principle, access and image each point of the BEC wavefunction. We consider a large, dilute 2D BEC in a homogeneous potential that is subjected to a superimposed attractive potential well of cylindrical symmetry with a radius comparable to the coherence length of the surrounding BEC. We study the existence of cylindrically symmetric standing wave solutions of the time-independent Gross-Pitaevskii equation, elucidating their nature and investigating their dynamical stability.

Our analysis reveals general trends: as the depth of the finite-ranged potential well increases near the resonance value (though still above) at which a single-particle bound state of $n$ nodes forms, a pair of $n$ node standing
wave solutions to the time independent Gross-Pitaevskii equation appears, as found in Ref. [10]. Upon further deepening of the well, the last node of one of the two new BEC wavefunctions moves inside the well and the corresponding wavefunction resembles that of the single particle bound state. We refer to this solution as the “atomic nodal BEC” state. At the same well depth, the other $n$ node BEC-wavefunction resembles the wavefunction of the $n-1$ node atomic bound state with an extra soliton pinned near the potential edge and we refer to it as the “nonlinear nodal BEC” solution. Of the two $n$ node BEC wavefunctions, the nonlinear nodal BEC state has the lower free energy, but we find that it is always dynamically unstable. If the atomic nodal BEC wavefunction varies on a length scale comparable to or greater than the healing length of the surrounding BEC, then the atomic BEC solution is also unstable when it first forms. Upon further deepening of the well, the stationary atomic nodal BEC wavefunction can become stable, then go unstable again as the well depth is further increased—a phenomenon that we refer to as “reentrant dynamical stability”. We show that the stability analysis can be reduced to the simpler study of the spectrum of two distinct single-particle-Hamiltonian-like operators.

The size of the stability islands, in terms of the range of well depths, for the atomic nodal BEC standing wave depends sensitively on the ratio of the potential range to the BEC coherence length: the more the potential range exceeds the coherence length, the broader the regions of instability become. If the potential well is confined to a spatial region much smaller than the BEC coherence length, the regions of instability are confined to narrow intervals of the well depth and the time scale at which the instability sets in increases markedly.

The existence of stable, localized BEC standing waves could impact many areas of cold atom physics. In cold atom traps standing waves could play a role in the physics of localized objects moving through a BEC [14]. If the object interacts strongly and attractively with BEC bosons, the dynamics of object acceleration may bring the BEC into an unstable nodal state, disturbing the system and possibly providing another mechanism of invalidating the picture of superfluid, dissipationless motion [12]. A class of BEC objects of particular, fundamental interest consists of a BEC with self-localized polaron-like impurity atoms [10, 17]. For sufficiently strong impurity-boson attractions, a standing wave BEC configuration combined with a localized, nodal impurity wavefunction may give more complex, localized excited state polaron objects. Another class of BEC objects that was proposed [18] consists of mesoscopic ultra-cold molecules. Localized potentials of ions embedded in the BEC may capture cold atoms in micron-sized orbits. If the BEC can be engineered to be in an atomic nodal BEC state, a gentle lowering of the boson-boson interactions can bring the system into a true molecular state, assuming that no instabilities are encountered in the adiabatic dynamics. Finally, we remark that long-lived, localized standing wave BEC patterns may resolve the challenge of observing the BEC response to external perturbations. For instance, a ring BEC can sense rotation: as the trap rotates in the plane of the ring, the BEC cannot follow as such motion would imply vorticity. Hence, in the rotating trap frame, the BEC flows. How can we observe that flow? If a local dip in the ring potential supports a standing wave BEC with fringes (across which the BEC cannot flow), the rotation would shift the position of the fringes, perhaps destroying the standing wave.

For a more whimsical application of standing wave nodal BEC states, one could look to the hypothesized dark matter BECs. If, as has been speculated, the dark matter haloes that surround most galaxies are scalar BECs of ultra-light boson particles [11, 12, 13], steep gravitational potentials may support standing wave dark matter BEC density patterns.

The relevance of any of the prospects for standing wave BECs hinge on their stability. In this manuscript we study the standing wave patterns and their stability in a linear stability analysis (dynamical stability). The paper is organized as follows: Section II describes the multiple, stationary solutions to the Gross-Pitaevskii equation in the presence of an attractive potential well of finite radius. We describe the linear stability analysis in Section III and we analyze the dynamical stability of the standing wave solutions in Section IV. In Section V, we describe a general analysis based on the study of the spectrum of effective single-particle Hamiltonians. Section VI concludes.

II. STANDING WAVE BEC WAVEFUNCTIONS

We consider a stationary, dilute two-dimensional BEC of $N$ bosons, each of mass $m_b$. Such systems can be realized with cold atom harmonic oscillator traps by increasing one of the the trap frequencies $\omega_z$ to ensure that the corresponding energy significantly exceeds the chemical potential $\mu$ of the BEC system, $\hbar \omega_z \gg \mu$. The bosons interact mutually via a short-range interaction described by a contact interaction potential, $v(r-r') = g \delta(r-r')$, where $r$ is a 2D vector and $\delta(r)$ the 2D delta function. In the dilute regime, the interaction strength $g$ relates to the scattering length $a$ that characterizes low energy scattering in three dimensions and to the ground state length $l_z$ of the $\omega_z$ frequency, $l_z = \sqrt{\hbar/m_b\omega_z}$, by the relation $g = (\sqrt{8\pi\hbar^2/m_b}) (a/l_z) [14, 20]$. Here we assume that the $N$ bosons are contained by a 2D-box potential of macroscopic area $\Omega$, corresponding to an average density $n_0 = N/\Omega$. The BEC is described by a wavefunction (or order parameter) $\psi_0(r)$ that is normalized by requiring $\int_\Omega d^2r |\psi_0(r)|^2 = N$. In addition, the BEC experiences an attractive external potential $V_{\text{ext}}(r)$ of finite range. The wavefunction satisfies the time-independent Gross-Pitaevskii equation

$$\left(-\frac{\hbar^2}{2m_b} \nabla^2 + V_{\text{ext}}(r) + g|\psi_0|^2\right)\psi_0 = \mu \psi_0. \quad (1)$$
Far from the center of the external potential, the wavefunction tends to a constant value, $\psi_o \to \sqrt{\rho_o}$, so that the chemical potential is related to the density by $\mu = gn_o$. The natural length scale of the problem is the coherence length of the BEC, $\xi$, defined by $\hbar^2/m\xi^2 = \mu$. Scaling the energy by $\mu$, the length by $\xi$, $x = r/\xi$, and the density by $n_o$, so that $\phi_o = \psi_o/\sqrt{n_o}$, we obtain the dimensionless form of the Gross-Pitaevskii equation,

$$\left( -\frac{\nabla^2}{2} + U_o(x) + |\phi_o|^2 \right) \phi_o = \phi_o, \quad (2)$$

where $U_o(x)$ denotes the scaled external potential $U_o(x) = V_{ext}(r = x\xi)/\mu$, and the large distance boundary condition becomes $\lim_{x \to \infty} \phi_o(x) = 1$.

We perform the calculations for an attractive external potential that has a square well shape of radius $b$ and depth $\pi^2\hbar^2\eta^2/8m_b^2$. In three dimensions the resonances for noninteracting particles in this potential would occur when $\eta$ is an odd integer; while in 2D the noninteracting resonances occur at $\eta \approx 2.45, 4.48, \ldots$, relating to solutions of a transcendental equation involving Bessel functions. Eq. (2) then reads

$$\left( -\frac{\nabla^2}{2} - \frac{\pi^2\eta^2}{8(b/\xi)^2} \Theta(b/\xi - x) + |\phi_o|^2 \right) \phi_o = \phi_o, \quad (3)$$

where $\Theta$ is the unit step function. For a given well radius $b/\xi$, we vary the dimensionless well depth $\eta$ and solve Eq. (3).

To integrate Eq. (3) subject to the proper boundary conditions, we use a scheme introduced in Ref. [10] to construct the time-independent BEC wavefunctions in the ion-neutral atom polarization potential. The boundary conditions introduce parameters $A_1$ and $A_2$: at the origin, $\phi'(x = 0) = 0$ and $\phi_o(x = 0) = A_1$, whereas far from the localized potential, $\phi_o(x) \approx 1 + A_2 \exp(-2x)$. We pick the radius of the potential well, $x_m = b/\xi$, as the matching point for the inward and outward integration procedure. Integrating inward from large $x$-values for different choices of $A_2$, we find the wavefunction and its derivative at the matching point, $(\phi_o(x_m), \phi'_o(x_m))$, tracing out a parametric curve that is independent of the potential (since the equation is integrated in the region where $U_o(x) = 0$ for $x > x_m$). Similarly, the outward integration from $x = 0$ for different $A_1$ values yields a second $(\phi_o(x_m), \phi'_o(x_m))$ curve, this one dependent on the potential. As the values of the wavefunction and its derivative must match at $x_m$, the intersections of both curves (and the corresponding $A_1$ and $A_2$ parameter values) determine the stationary solutions. Fig. 2 shows the curves for well radius $b = 1.0\xi$ and well depth $\eta = 2.9$. These parameters result in five crossings and thus five solutions. As mentioned, the “outside” curve doesn’t depend on well depth $\eta$. When $\eta$ is small, there is only a single crossing of the outside curve with the “inside” curve, corresponding to a single solution—the familiar nodeless BEC. As the well depth $\eta$ increases, the inside curve starts to swirl and pinches through the outside curve, generating a pair of new solutions. When the well depth has reached the $\eta = 2.9$ value shown in Fig. 2 the inside curve pinches once more through the outside curve near $\phi_o(b) \approx -0.8$, generating two additional time-independent Gross-Pitaevskii solutions.

![FIG. 2: Solving Eq. 3 graphically, matching inner (solid) and outer (dashed) integrations at the potential edge, $x_n = b/\xi$, by locating the crossing points in the $(\phi_o(b/\xi), \phi'_o(b/\xi))$ plane. The example shown, for a well radius of $b = 1.0\xi$ and well depth $\eta = 2.9$, gives five crossing points and thus five solutions to the GP equation, Eq. 3.](https://example.com/fig2.png)

The new solutions differ qualitatively from the nodeless BEC solution. As $\eta$ increases, each of one of the new pair of BEC wavefunctions has one more radial node than the number of nodes of the previous pair. At the well depth where the new pair of solutions first exists, the two new solutions are degenerate and they have the last node located slightly beyond the well radius. For a well radius of $b = 1.0\xi$, we find the critical well depth for the one node solutions is $\eta \approx 1.2444$. In Fig. 2(a) for $\eta = 1.25$, the two new solutions have visibly split. As the well depth increases, the two nodal solutions continue to differentiate: the node of one moves inside the potential well as the well depth increases, while the node of the other solution moves outside the potential well as depicted in Fig. 2(b) for well depth $\eta = 2.5$. As the well depth increases further, two new solutions, each with two nodes, appear near $\eta \approx 2.86$: and we show the five solutions for $\eta = 3.5$ in Fig. 2(c).

We clarify the nature of the different nodal solutions. In accordance with Levinson’s theorem, the deepening of the potential well produces bound states in the single particle wavefunction description. Each time the zero energy scattering phase goes through a resonance (increasing by $\pi$), a new bound state forms with one more node than the previously formed bound state. We note that one class of the time-independent Gross-Pitaevskii solutions—the solution in which the last node moves inward as the potential depth increases—starts to resemble the noninteracting (NI) zero energy wavefunction. In Fig. 2(c), the solid line curves show three BEC wavefunc-
tions with zero, one, and two nodes (the dashed curve wavefunctions will be discussed below). Compare these wavefunctions to the NI wavefunction plotted in Fig. 4 for the same well depth. The NI bound states with zero and one node exhibit a qualitative similarity to the BEC wavefunctions with zero and one nodes, except that the former vanish at large distance whereas the latter tend to a constant value (Figs. 4 and 3(c)). Moreover, the “nonlinear nodal BEC” solutions, when they exist, are well approximated as
\[
\psi_{n,i}(r) \approx \psi_{a,i-1}(r)\chi(r-R),
\]
where \(\chi(r) = \tanh(r/\xi)\) and \(R\) is the position of the outermost node. This hyperbolic tangent is the familiar nonlinear kink or soliton solution: the \(\psi_{n,i}\) solution has a soliton that is trapped or pinned outside the potential well. We illustrate this point graphically in Fig. 5 by plotting \(\psi_{n,1}(r)/\psi_{a,0}(r)\) for different well depths \(\eta\). Note that the quotient follows the hyperbolic tangent shape. The red dashed (1 node) wavefunction in Fig. 3 can be interpreted as the red solid (nodeless) wavefunction with a kink superimposed on it. Similarly, the blue dashed (2 node) wavefunction can be viewed as the blue solid wavefunction with a kink. For deeper wells, this pattern repeats, e.g., for deeper wells the green (2 node) wavefunction develops a kinked partner with 3 nodes.

Which solution has the highest energy? By fixing the density far away from the localized potential we fix the chemical potential so that we have to determine the free energy, \(F = H - \mu N\), where \(H\) is the Hamiltonian and
has the lowest free energy. The introduction of excitations in Fig. 3.

Energies in Fig. 6 are the same as those of the wavefunction \( \psi \), as given by Eq. 5, for a well radius of \( N = \int d^2 r \left| \psi_o(r) \right|^2 \). In fact, we calculate the difference between the free-energy of the \( \psi_o \) solution and that of the homogeneous system of the same density (without external potential),

\[
\Delta F = \int d^2 r \left[ \frac{|\nabla \psi_o|^2}{2} + U_o|\psi_o|^2 + \frac{1}{2}(|\psi_o|^4 + 1) - \phi_o^2 \right].
\]

In Fig. 6 we plot \( \Delta F \) vs. the well depth \( \eta \) for the nodeless and the two nodal solutions for a well radius of \( b = 1.0 \xi \). For ease of comparison, the colors/dashings of the free energies in Fig. 6 are the same as those of the wavefunctions in Fig. 3.

As with noninteracting systems, the nodeless solution (red line) has the lowest free energy. The introduction of an attractive well smoothly lowers \( \Delta F \) from 0 to negative values as the well is turned on. At the well depths where the nodal solutions first appear, their \( \Delta F \) is positive for a range of well depths before becoming negative [22]. The nonlinear pinned soliton solutions of \( n \) nodes (depicted by the red and blue dashed lines in Fig. 6) have a \( \Delta F \) value that differs from that of the atomic \( n - 1 \) node solutions by a nearly constant value—the energy cost of creating the kink.

The solutions with nodes are obviously not the ground state of the system, but are they metastable? In the next sections, we undertake a stability analysis to investigate whether they might be long-lived.

### III. LINEAR STABILITY ANALYSIS: FORMALISM

In this section and the following we test whether or not the \( \phi_o \) function is stable with respect to small perturbations. We use linear stability analysis, which investigates the response of the system to a weak perturbation caused, for instance, by a modulation of the potential around the time-independent external potential \( U_o \), \( U(x,t) = U_o(x) + \delta U(x,t) \). The response of the system is described by the time-dependent Gross-Pitaevskii equation,

\[
i \frac{\partial \phi(x,t)}{\partial t} = \left[ -\nabla^2 + |\phi(x,t)|^2 + U(x,t) \right] \phi(x,t).
\]

The time-independent solution to the equation with \( U = U_o \) evolves as \( \phi(x,t) = \exp(-it)\phi_o(x) \). The system’s small amplitude response

\[
\phi(x,t) \approx \exp(-it) \left[ \phi_o(x) + \delta \phi(x,t) \right],
\]

to the potential perturbation \( U(x,t) = U_o(x) + \delta U(x,t) \) evolves according to the linearized Gross-Pitaevskii equation for the \( \delta \phi \) evolution,

\[
i \frac{\partial \delta \phi}{\partial t} = \hat{h}^- \delta \phi + \phi_o \left( \phi_o^* \delta \phi + \phi_o \delta \phi^* \right) + \phi_o \delta U,
\]

where the \( \hat{h}^- \) operator,

\[
\hat{h}^- = -\nabla^2 \left( \frac{|\phi_o|^2 - 1}{2} \right),
\]

represents the Hartree-Hamiltonian of the \( \phi_o \) system.

Note that the linearized Gross-Pitaevskii equation, Eq. 6, couples \( \delta \phi \) and \( \delta \phi^* \), mixing wavefunction fluctuations and their complex conjugates. To describe the coupling, we separate out the positive from the negative frequency components in the Fourier transform,

\[
\delta \phi(x, \omega) = \int_{-\infty}^{\infty} dte^{i\omega t} \delta \phi(x,t); \\
\delta \phi(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{-i\omega t} \delta \phi(x, \omega).
\]
We define the positive, $\delta \phi_p$, and negative, $\delta \phi_n$, frequency amplitude functions to have non-zero values for $\omega > 0$,

$$
\delta \phi_p(x, \omega) = \delta \phi(x, \omega) \quad \text{if} \quad \omega > 0;
$$

$$
= 0 \quad \text{if} \quad \omega < 0;
$$

$$
\delta \phi_n(x, \omega) = \delta \phi^*(x, -\omega) \quad \text{if} \quad \omega > 0;
$$

$$
= 0 \quad \text{if} \quad \omega < 0;
$$

so that

$$
\delta \phi(x, \omega) = \delta \phi_p(x, \omega) + \delta \phi_n^*(x, -\omega). \tag{12}
$$

If the time-dependent function $\delta U(x, t)$ is real-valued, then the negative and positive frequency amplitudes are identical, $\delta U_n(x, \omega) = \delta U_p(x, \omega)$, where $\delta U_n$ and $\delta U_p$ are defined as in Eq. (11). In the time-domain,

$$
\delta \phi(x, t) = \int_0^{\infty} d\omega \, e^{-i\omega t} \delta \phi_p(x, \omega)
$$

$$
+ \int_{-\infty}^{0} d\omega \, e^{-i\omega t} \delta \phi_n^*(x, \omega)
$$

$$
= \delta \phi_p(x, t) + \delta \phi_n^*(x, t), \tag{13}
$$

where, in accordance with the Fourier transformation,

$$
\delta \phi_{p(n)}(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \, e^{-i\omega t} \delta \phi_{p(n)}(x, \omega), \tag{14}
$$

and we remember that $\delta \phi_{p(n)}(x, \omega) = 0$ if $\omega < 0$.

Inserting Eq. (13) into the linearized Gross-Pitaevskii equation of Eq. (8), we identify the $e^{-i\omega t}$ and the $e^{i\omega t}$ components. Taking the negative of the complex conjugate of the $e^{i\omega t}$ components, we obtain

$$
\omega \delta \phi_p = \left[ \hat{h} + |\phi_0|^2 \right] \delta \phi_p + \phi_0^2 \delta \phi_n + \phi_o \delta U_p;
$$

$$
\omega \delta \phi_n = -\phi_0^2 \delta \phi_p - \left[ \hat{h} + |\phi_0|^2 \right] \delta \phi_n - \phi_o^2 \delta U_n. \tag{15}
$$

Multiplying by $e^{-i\omega t}$ and integrating over $(2\pi)^{-1} \int d\omega$, we obtain the time-domain version,

$$
\frac{i}{\partial t} \begin{pmatrix} \delta \phi_p \\ \delta \phi_n \end{pmatrix} = \hat{\mathcal{L}} \begin{pmatrix} \delta \phi_p \\ \delta \phi_n \end{pmatrix} + \begin{pmatrix} \phi_o \delta U_p \\ -\phi_o^2 \delta U_n \end{pmatrix}, \tag{16}
$$

where we have introduced the $\hat{\mathcal{L}}$ operator used in Ref. 24,

$$
\hat{\mathcal{L}} = \begin{pmatrix} \hat{h} + |\phi_0|^2 & \phi_0^2 \\ -\phi_0^2 & -\left[ \hat{h} + |\phi_0|^2 \right] \end{pmatrix}. \tag{17}
$$

The equations that diagonalize the $\hat{\mathcal{L}}$ operator,

$$
\hat{\mathcal{L}} \begin{pmatrix} u_j(x) \\ v_j(x) \end{pmatrix} = \omega_j \begin{pmatrix} u_j(x) \\ v_j(x) \end{pmatrix}, \tag{18}
$$

are the Bogoliubov-de Gennes (BdG) equations 25.

If $\omega_j$ is an eigenvalue of $\hat{\mathcal{L}}$ then $\omega_j^*$ is also an eigenvalue. If $\phi_o$ is real-valued, this statement can be verified by taking the complex conjugate of the BdG equations, but this property is true for complex-valued $\phi_o$ wavefunctions as well. Note that if $\begin{pmatrix} u_j(x) \\ v_j(x) \end{pmatrix}$ is a right-eigenvector of $\hat{\mathcal{L}}$, as shown in Eq. (15), then $\begin{pmatrix} u_j^*(x) \\ -v_j^*(x) \end{pmatrix}$ is a left-eigenvector of eigenvalue $\omega_j^*$. By integrating

$$
\int d^3x \ \begin{pmatrix} u_j^*(x) \\ -v_j^*(x) \end{pmatrix} \hat{\mathcal{L}} \begin{pmatrix} u_j(x) \\ v_j(x) \end{pmatrix}
$$

$$
= \omega_j^* \left( \langle u_j|v_j^* \rangle - \langle v_j^*|v_j \rangle \right)
$$

$$
= \omega_j \left( \langle u_j|u_j \rangle - \langle v_j^*|v_j \rangle \right) = 0, \tag{19}
$$

where $\langle u_j|v_j \rangle = \int d^3x \ u_j^*(x) v_j(x)$, we find that if $\omega_j^* \neq \omega_j$, $\langle u_j^*|u_j \rangle - \langle v_j^*|v_j \rangle = 0$.

The above orthogonalization relation suggests the $\langle |u_j \rangle - \langle v_j \rangle \rangle$ form as a scalar product and we could try to normalize the $\langle u_j, v_j \rangle$ solutions by requiring $\langle u_j|u_j \rangle - \langle v_j|v_j \rangle = 1$. This, however is not possible: if $\begin{pmatrix} u_{j,+}(x) \\ v_{j,+}(x) \end{pmatrix}$ is a right eigenvector with eigenvalue $\omega_j$, then direct substitution shows that $\begin{pmatrix} u_{j,-}(x) \\ v_{j,-}(x) \end{pmatrix} = \begin{pmatrix} v_{j,+}^*(x) \\ u_{j,+}^*(x) \end{pmatrix}$ is a right eigenvector of eigenvalue $-\omega_j$. If the $j$-th eigenvector is normalized by requiring $\langle u_{j,+}|u_{j,+} \rangle - \langle v_{j,+}|v_{j,+} \rangle = 1$ then $\langle u_{j,-}|u_{j,-} \rangle - \langle v_{j,-}|v_{j,-} \rangle = 1$. It is then convenient to distinguish between a “+” family, $(u_{j,+}, v_{j,+})$, and a “−” family with $(u_{j,-}, v_{j,-}) = -\delta_{j,j'}$.

We expand the positive and negative frequency components of the wavefunction fluctuations as

$$
\begin{pmatrix} \delta \phi_p(x, t) \\ \delta \phi_n(x, t) \end{pmatrix} = \sum_{j,s} \begin{pmatrix} u_{j,s}(x) \\ v_{j,s}(x) \end{pmatrix}, \tag{20}
$$

where the $s$ subscript indicates the family, $s = +1, -1$, and the $\langle u_{j,s}, v_{j,s} \rangle$ denotes the right eigenvector of $\hat{\mathcal{L}}$ with eigenvalue $\omega_{j,s}$. Substitution of Eq. (20) into the linearized Gross-Pitaevskii equation, Eq. (10), yields

$$
\sum_{k,s'} \frac{\partial \langle u_{k,s'}(x) | u_{k,s'}^*(x) \rangle}{\partial t} = \omega_{k,s'} \langle u_{k,s'}(x) | u_{k,s'}(x) \rangle
$$

$$
- \sum_{k,s'} \omega_{k,s'} c_{k,s}(t) \begin{pmatrix} u_{k,s}(x) \\ v_{k,s}(x) \end{pmatrix}
$$

$$
+ \left( \phi_o(x) \delta U_p(x, t) - \phi_o^2(x) \delta U_n(x, t) \right). \tag{21}
$$

Multiplying from the left by $\langle u_{j,s}^*, v_{j,s} \rangle$ and integrating over the position coordinate $x$ we obtain

$$
\frac{\partial c_{j,s}}{\partial t} = \omega_{j,s} c_{j,s}
$$

$$
+ s \int d^3x \ u_{j,s}^*(x) \phi_o(x) \delta U_p(x, t)
$$

$$
+ \int d^3x \ v_{j,s}^*(x) \phi_o^2(x) \delta U_n(x, t). \tag{22}
$$

Assuming that the perturbation was turned on at a time $t_0$, the solution that satisfies the corresponding boundary
condition, \( c_{j,s}(t') = 0 \) when \( t' < t_0 \), is
\[
c_{j,s}(t) = s \int_{t_0}^{t} d\tau \exp(-i\omega_{j,s} [t - \tau]) \times \int d^3x' \left[ u_{j,s}(x') \phi_0(x') \delta U_p(x', \tau) + v_{j,s}(x') \phi_0^*(x') \delta U_m(x', \tau) \right].
\]
(23)

If \( \omega_{j,s} \) has a finite imaginary part, one of the \( c \) amplitudes grows exponentially. Even if the imaginary part of the \((j, s)\) frequency corresponds to a damped mode, \( \omega_{j,s} = \omega - i\Gamma \), another mode \((n, s)\) exists for which \( \omega_{n,s} = \omega_{j,s}^* = \omega + i\Gamma \), so that \( c_{n,s}(t) \) grows exponentially. In that case, even the smallest of perturbations gives rise to an exponentially growing response. In physical and simulated BECs, the effects of perturbations are always present, caused either by thermal or quantum fluctuations in physical systems or by round-off errors in numerical simulations. From these considerations the criterion for dynamical stability in linear stability analysis follows: all eigenvalues of the BdG equations need to be real-valued for the system to be dynamically stable.

For studying the stability of the standing wave Gross-Pitaevskii solutions of Section II (in which case the \( \phi_0 \) wavefunction, and hence \( L \), is real-valued, and \( \phi_0^2 = |\phi_0|^2 \)), we find it convenient to calculate the sum and difference vectors instead of the \( u \) and \( v \) functions,
\[
f_j^\pm = u_j \pm v_j,
\]
(24)
where, from now on, the subscripts \( j \) fully characterize the mode (including the “family”). By adding and subtracting Eq. 18 the BdG equations take the form of a diagonal set of coupled eigenvalue-type equations,
\[
\hat{h}_j^m + f_j^m = \omega_j f_j^m;
\hat{h}_j^m - f_j^m = \omega_j f_j^m;
\]
(25)
where the \( \hat{h}_j^\pm \) operators,
\[
\hat{h}_j = -\frac{\nabla^2}{2} + U_0 + \left[ 2\phi_0^2 - 1 \right] \pm \phi_0^2,
\]
are single-particle-Hamiltonian-like operators. For a cylindrically symmetric potential and standing wave pattern, the modes will be eigenvectors of the angular momentum operator. We perform the separation of variables into rational and angular functions,
\[
f_j^\pm(x) = f_{m,\nu}(\rho)e^{im\theta},
\]
(26)
where \( \rho \) is the dimensionless radial coordinate measured in units of the healing length \( \xi \), \( \theta \) is the azimuthal angle, and \(|m| = 0, 1, 2, \ldots \) is the azimuthal quantum number. Henceforth we will take the radial quantum number \( \nu \) as implicit. Using Eqs. (21) and (26), the BdG equations take on the form
\[
h_m^+ f_m^- = \omega_m f_m^+;
\]
\[
h_m^- f_m^+ = \omega_m f_m^-;
\]
(27)
where
\[
h_m^\pm = -\frac{1}{2\rho} \frac{\partial}{\partial \rho} \frac{\partial}{\partial \rho} + \frac{m^2}{2\rho^2} + U_o(\rho) + 3\phi_o^2 - 1;
\]
\[
h_m^- = -\frac{1}{2\rho} \frac{\partial}{\partial \rho} \frac{\partial}{\partial \rho} + \frac{m^2}{2\rho^2} + U_o(\rho) + \phi_o^2 - 1;
\]
(28)
denote the radial \( \hat{h}_j^\pm \) operators for fixed azimuthal quantum number.

What is the physical meaning of the \( f_m^+ \) and \( f_m^- \) fluctuation functions? At finite oscillation frequency, \( f_m^+ \) is proportional to the density fluctuation and \( f_m^- \) to the phase fluctuation. This can be seen by writing the wavefunction as \( \phi = \sqrt{n_0}e^{i\alpha} \), defining the equilibrium density and phase values as \( n_0, \alpha_0 \), and introducing their fluctuations \( n = n_0 + \delta n, \alpha = \alpha_0 + \delta \alpha \), so that
\[
\delta \phi = \frac{\delta n}{2\sqrt{n_0}} + i\sqrt{n_0}\delta \alpha.
\]
(29)
We see that we can isolate the density and phase fluctuations by taking
\[
\delta \phi + \delta \phi^* = \frac{\delta n}{\sqrt{n_0}};
\]
\[
\delta \phi - \delta \phi^* = 2i\sqrt{n_0}\delta \alpha.
\]
(30)
But from Eq. 13, \( \delta n = \delta n_0 + \delta n_o \), and putting \( \delta n_o = v_j \) and \( \delta \phi_o = u_j \), we find, with Eqs. (24) and (26), that
\[
\delta \phi + \delta \phi^* = f_m^+(\rho)e^{im\theta}e^{-i\omega_mt} + c.c.,
\]
\[
\delta \phi - \delta \phi^* = f_m^-(\rho)e^{im\theta}e^{-i\omega_mt} - c.c.
\]
(31)
Comparing Eqs. (30) and (31), we obtain
\[
\delta n = 2\sqrt{n_0}\text{Re}[f_m^+(\rho)e^{im\theta}e^{-i\omega_mt}];
\]
\[
\delta \alpha = \frac{1}{\sqrt{n_0}}\text{Im}[f_m^-(\rho)e^{im\theta}e^{-i\omega_mt}];
\]
(32)
so that \( f_m^+ \) and \( f_m^- \) are indeed proportional to the density and phase fluctuations, respectively.

The coupled BdG equations, Eqs. (27), can be cast in an alternate form by noting that the repeated application of \( \omega_m \) yields decoupled eigenvalue equations,
\[
\omega_m^2 f_m^+ = h_m^- h_m^+ f_m^+;
\]
\[
\omega_m^2 f_m^- = h_m^- h_m^- f_m^-;
\]
(33)
so that \( f_m^+ \) is the eigenvector of \( h_m^- h_m^+ \), whereas \( f_m^- \) is the eigenvector of \( h_m^- h_m^- \). While this eigenvalue problem is distinct from the Hamiltonian eigenvalue problem (\( h_m^- h_m^+ \) has a fourth derivative in space, for instance), there is one important similarity: since the product operators \( h_m^- h_m^+ \) and \( h_m^- h_m^- \) are both self-adjoint, their eigenvalues \( \omega_m^2 \) are real-valued. As a consequence, \( \omega_m \) is either entirely real (if \( \omega_m^2 > 0 \)) or entirely imaginary (if \( \omega_m^2 < 0 \)). The former case, real \( \omega_m \), corresponds to oscillations about a possibly stable equilibrium solution \( \phi_o \) for mode \( m \); whereas an imaginary \( \omega_m \) indicates that the mode \( m \) drives the system away from an unstable equilibrium solution \( \phi_o \).
IV. LINEAR STABILITY ANALYSIS: 2D BEC STANDING WAVE PATTERNS

We subject the standing-wave $\phi_0$ wavefunctions of Section II to the stability analysis of Section III. We use the $f_m^\pm$ form of the BdG equations, Eq. (28), and solve Eqs. (29) for $m^\pm$ and $f_m^\pm$. In the following two subsections we describe typical examples to uncover the general stability picture. The first case deals with a shallow well that supports nodal BECs with a single node and the second case deals with deeper wells and nodal BECs with many nodes. We highlight the following key results: (i) the nonlinear nodal BECs are always unstable, and (ii) the atomic nodal BECs can transition from stable to unstable and from unstable to stable as the well depth increases.

1. Single node BEC states in a well of coherence length radius

We consider the stability of a BEC with one node induced by a potential well of radius $b = 1.0\xi$, as pictured in Fig. 3. While the azimuthal quantum number $m$ has to be an integer to ensure the single-valuedness of $f_m$ in Eq. (26), we can formally solve Eqs. (27) for any value of $m$. Varying $m$ continuously, we plot Im$[\omega_m]$ for the nonlinear (a) and atomic (b) nodal BEC solutions with one node for various well depths.

When the atomic and nonlinear nodal solutions first appear with increasing $\eta$, they are identical. For a well-depth of $\eta = 1.25$, we show the two wavefunctions with one node in Fig. 3(a). The blue line shows the atomic nodal BEC wavefunction and the red dashed line represents the nonlinear nodal BEC wavefunction. Note that in spite of their similarity, the Im$[\omega_m]$ spectra differ markedly, as seen in Fig. 7. The nonlinear nodal BEC has a finite value of Im$[\omega_m=0]$, while the atomic nodal BEC has Im$[\omega_m=0] = 0$, a general feature that distinguishes atomic from nonlinear solutions. In Fig. 7, we see that a small variation of $\eta$ from $\eta = 1.2445$ (within 0.0001 of the critical well depth for nodal solutions) to $\eta = 1.25$, drives the left-edge of the Im$[\omega_m]$ lobe in opposite directions: Im$[\omega_m=0] = 0$ for the critical well depth where the nonlinear solutions first exist; and $\omega_m=0$ becomes entirely imaginary for the nonlinear nodal BEC solutions and entirely real for the atomic nodal BEC solutions. The nonlinear nodal BECs have at least one imaginary-valued frequency mode, the radially symmetric $m = 0$ mode, so that the nonlinear nodal BECs are always unstable. The instability of the nonlinear nodal BECs has features in common with the snake instability of a kinkwise nodal line or plane in an otherwise homogeneous 2D or 3D condensate [4], but an in-depth discussion would take us further afield.

We now discuss the atomic nodal solutions. Fig. 7(b) shows Im$[\omega_m]$ for the one-node atomic nodal BEC state. As mentioned above, Im$[\omega_m=0] = 0$ and the lowest possible angular momentum of an exponentially growing mode is $m \geq 1$. For the lowest three well depth values shown in Fig. 7(b), the $m = 1$ mode has an imaginary-valued frequency and the atomic nodal BEC is unstable. As the well increases, from $\eta = 1.8$ to $\eta = 2.2$, for instance, we don’t find any imaginary $\omega_m$ values for integer $m$ and the atomic nodal BEC is stable. Between $\eta = 2.2$ and $\eta = 2.6$, the Im$[\omega_m]$ lobe shifts to larger $m$ values and passes through the $m = 2$ line, indicating the instability of the mode with azimuthal quantum number 2. In the $\eta = 2.6$ to $\eta = 3.0$ interval, the Im$[\omega_m]$ lobe has shifted to values between $m = 2$ and $m = 3$, so that this regime of well-depths is stable again. Therefore, as the potential deepens from $\eta = 1.25$ to 3.0, the single node atomic nodal BEC transitions from instability with exponentially growing $m = 1$ mode to stability, then returns to an unstable status, this time caused by the exponential
growth of the \( m = 2 \) mode, then enters another island of stability. We refer to this trend as \textit{reentrant stability} to describe the entering and exiting into and out of islands of stability.

2. \textit{Multi-node BEC states in broad, deep wells}

We now discuss nodal BECs in a wider, deeper well with radius \( b = 4.0 \xi \) and depth \( \eta = 11.5 \). This well has nodal BEC solutions with up to six nodes; Fig. 8(a) shows the five node atomic and six node atomic and nonlinear nodal BECs. Fig. 8(b) shows the corresponding \( \text{Im}[\omega_m] \) spectra. As in the previous subsection, we find that the nonlinear solution has an imaginary-valued \( m = 0 \) mode, while the two atomic solutions do not, following the general \( m = 0 \) behavior mentioned above. More striking, however, are the spectra’s multi-peaked structures which we now explain.

To understand the \( \text{Im}[\omega_m] \) structure we revisit the BdG equations (27), \( h^+_m f^+_m = \omega_m f^+_m \). The frequency \( \omega_m \) is either entirely real or entirely imaginary. Therefore, if we treat \( m \) as a continuous variable and we assume \( \omega_m \) to be a continuous function of \( m \) then \( \omega_m \) has to pass through zero to switch from real to imaginary values. When \( \omega_m \) vanishes, either \( h^+_m f^+_m = 0 \) or \( h^-_m f^-_m = 0 \), corresponding to a \( f^+_m \) or \( f^-_m \) eigenvector of the \( h^\pm_m \) operator with zero eigenvalue.

Consider the \( h^\pm_m \) operators of (28), which we write as

\[
V^\pm_m = \frac{m^2}{2\rho^2} + U_0(\rho) + \phi_0^2(2 \pm 1) - 1 ,
\]

that are effective potentials of the single-particle-Hamiltonian-like operators \( h^\pm_m \). We show examples of \( V^\pm_m \) for \( m = 0 \) in Fig. 8(c) for the two atomic nodal BECs of Fig. 8(a). The short-scale, oscillatory structure of the \( V^\pm \) potentials stems from the \( \phi_0 \) contributions, but for high-node BEC states in sufficiently deep wells (for which \( \phi_0^2 < U_0 \)) the overall \( V^\pm \) shape is dominated by the external \( U_0 \) potential. For example, Fig. 8(c) shows \( V^\pm \) for the state with five (six) nodes and the \( V^\pm \) potentials themselves support five (six) negative energy levels. As \( m \) increases, the angular momentum barrier, \( m^2/2\rho^2 \), grows and raises the energy-levels to positive values (29). At a value \( m_1 \), a \( V^\pm_{m_1} \) level crosses the zero energy dividing line and \( h^+_m f^+_m = 0 \) so that the \( h^+_m \) BdG equation becomes an eigenvalue equation with zero eigenvalue. This \( m_1 \) value denotes the left-hand side of the first peak or lobe in the \( \text{Im}[\omega_m] \) spectrum. As \( m \) increases further to \( m_2 \) at which the corresponding \( V^\pm_{m_2} \) level crosses over, \( h^-_m f^-_m = 0 \), the \( h^-_m \) BdG equation reduces to another eigenvalue equation with zero eigenvalue and the \( m_2 \) value denotes the right-hand edge of the lobe that started at \( m_1 \). The pattern repeats as many times as there are levels in the \( V^\pm_{m=0} \) potentials, generating the other peaks in the \( \text{Im}[\omega_m] \) spectrum.

The above construction shows that the number of peaks in the \( \text{Im}[\omega_m] \) spectrum is equal to the number of negative energy levels of the \( V^\pm_{m=0} \) potentials. Moreover, the widths of the \( \text{Im}[\omega_m] \) peaks, \( \Delta m \), are related to the energy splitting of corresponding states in the \( V^\pm_{m=0} \) and in the \( V^\pm_{m=0} \) potentials. We illustrate this point with

![Fig. 8](image_url)
Fig. 8(c): as the BEC density \( \phi_0^2 \) in the potential well is smaller in the six node than in the five node BEC, the \( V_{m=0}^+ \) and \( V_{m=0}^- \) potentials resemble each other more in the six node atomic BEC state than in the five node atomic BEC state. As a consequence, the difference in energy between corresponding pairs of \( V_{m=0}^+ \) and \( V_{m=0}^- \) eigenstates is smaller in the six node than in the five node case. Therefore, the \( m \) value at which the \( V^+ \) and the \( V^- \) state reaches zero energy differ by a smaller amount \( \Delta m \) in the six node than in the five node situation. Generally, as the atomic nodal BEC with the maximal number of nodes has the smallest BEC density \( \phi_0^2 \), this time-independent BEC solution has the smallest peak widths \( \Delta m \).

The above analysis reveals that common methods for bound state physics can determine relevant instability properties of standing wave BECs. For instance, we consider the upper imaginary azimuthal quantum number \( m_c \) above which all mode eigenfrequencies \( \omega_m \) take on purely real values. We estimate \( m_c \) for the atomic nodal BEC state of highest node number contained in a potential well of sufficient depth to ensure that the BEC density satisfies \( |\phi_0|^2 \ll U_0 \) inside the well (for the nodeless case, this is obviously not true). As we ramp up the \( m \) quantum number, the last bound state of the \( V_m^- \) potential becomes unbound at \( m = m_c \). The corresponding \( V_m^- \) eigenstate (resonant) wavefunction, which we shall refer to as \( \chi_o^- \) (and we assume \( \chi_o^- \) to be normalized) is a nodeless function roughly centered on the position of the potential minimum of \( V_m^- \), i.e., on \( \rho = b \). At the zero energy crossing point,

\[
\langle \chi_o^- | -\frac{\nabla^2}{2} + U_0 + \frac{m_c^2}{2\rho^2} + [\phi_0^2 - 1] | \chi_o^- \rangle = 0
\]

Neglecting the kinetic energy term and the \( [\phi_0^2 - 1] \) contributions and estimating the angular momentum barrier contribution by \( m_o^2/2b^2 \), we find \( m_c \approx b\sqrt{2U_0} \), or with our notation for the square well potential

\[
m_c \approx \frac{\pi \eta}{2},
\]

which overestimates \( m_c \) because we neglected the kinetic energy and the \( \phi_0^2 \) terms. Nonetheless, we expect this approximation to be reasonable for deep wells. For example, in Fig. 8 the observed \( m_c \) of \( \sim 15 \) can be compared to the \( m_c \) estimate of 18.1 from Eq. (36).

All three nodal BEC-states shown in Fig. 8 are unstable. The five node atomic BEC has imaginary modes for \( m = 2, 3, 5, 8, 11 \), and 14. In Fig. 9 we show the density fluctuation pattern of the \( m = 3 \) mode, proportional to \( f_3^m(\rho) \cos(m\theta) \) in accordance with Eq. (32). If the \( m = 3 \) mode were the only unstable mode or if it were the dominant unstable mode (growing at the largest rate \( r_m^{-1} = \Im(\omega_m) \)), the density variation would take on the form pictured in Fig. 8 in the initial stage \( (t < \tau_m) \) of the break-up of the nodal BEC state. The intricate oscillatory pattern of Fig. 8 highlights the role of interference in the wave dynamics of the instability. Only when the interference fringes “match up” properly to initiate the break-up does the instability set in. In addition to the restriction of \( m \) values to integers, which is an expression of the angular boundary condition \( \delta\phi(\rho, \theta) = \delta\phi(\rho, \theta + 2\pi) \), the radial oscillations have to line up to create the conditions at which the instability can set in.

We interpret the finite \( \Delta m \) width of the \( \Im[\omega_m] \) lobes as the range of \( m \) values for which the radial interference fringes match up sufficiently well to trigger the break-up of the nodal BEC structure.

The six node atomic nodal BEC whose \( \Im[\omega_m] \) spectrum is shown in Fig. 8 has unstable modes for \( m = 6 \) and \( m = 15 \). However, the widths of its \( \Im[\omega_m] \) lobes satisfy \( \Delta m < 1 \). This means that as the depth of the potential well is varied, the imaginary-valued \( \omega_m \) intervals can shift off of all integer \( m \)'s leading to a reentrant stability behavior similar to that discussed in the previous subsection.

Can we make a general statement about the stability of stationary multi-node BEC wavefunctions? The multi-lobed structure of the \( \Im[\omega_m] \) spectrum can actually promote stability: if the unstable \( m \) intervals, \( \Delta m \), are sufficiently narrow, \( \Delta m \ll 1 \), the \( \Im[\omega_m] \) lobes may not overlap any integer \( m \) values. The progression from the five node to the six node \( \Im[\omega_m] \) spectrum shown in Fig. 8(b) indicates an interesting trend: while the increase in nodes increases the number of lobes, their \( \Delta m \) widths decrease markedly. Furthermore, the \( \Delta m \) widths decrease for wells of smaller radius, as we now show.

We can derive a rough estimate for the peak widths \( \Delta m \) along the lines of the above argument that associates the vanishing of \( \omega_m \) with the zero-energy crossing of bound state levels of \( V_m^- \). We assume that the BEC density inside the well remains sufficiently small to determine the splitting between the \( V_m^+ \) and \( V_m^- \) eigenen-
energies by perturbation theory. The necessary condition reduces to \( \phi_o^2 \ll |U_o - 1| \) in the relevant well region—the region in which the the \( V_{m}^\pm \) eigenstates have significant density. We determine the \( V_{m}^+ \) and \( V_{m}^- \) eigenstate energies as perturbations on the eigenstate of the average potential \( \overline{V}_m = (V_m^+ + V_m^-)/2 \),

\[
\overline{V}_m(\rho) = U_o(\rho) + 2\phi_o^2(\rho) - 1 + \frac{m^2}{2\rho^2}.
\]

(37)

We denote the eigenstate of the \( \overline{V}_m \) potential with \( \nu \) nodes by \( \chi_{m,\nu} \) and we denote its energy by \( E_{\nu}(m) \). In first order perturbation theory, the energy of the corresponding eigenstates of the \( V_{m}^+ \) and \( V_{m}^- \) potentials, \( E_{\nu}^+(m) \) and \( E_{\nu}^-(m) \), are then equal to

\[
E_{\nu}^\pm(m) = E_{\nu}(m) \pm \Delta E_{\nu}(m),
\]

(38)

where \( \Delta E_{\nu}(m) \) is the first-order \( \phi_o^2 \) energy splitting,

\[
\Delta E_{\nu}(m) = \langle \chi_{m,\nu}|\phi_o^2|\chi_{m,\nu}\rangle.
\]

(39)

The slope of the \( m \) variation of \( E_{\nu}(m) \), \( \partial E_{\nu}/\partial m \), can be calculated with the Hellmann-Feynman theorem (the derivative of the hamiltonian expectation value with respect to a hamiltonian parameter is the expectation value of the derivative of the hamiltonian operator):

\[
\frac{\partial E_{\nu}}{\partial m} = \langle \chi_{m,\nu}|\overline{V}_m|\chi_{m,\nu}\rangle = \langle \chi_{m,\nu}\frac{m}{\rho^2}|\chi_{m,\nu}\rangle.
\]

(40)

Then, the difference \( \Delta m_{\nu} \) in \( m \) values at which the \( E_{\nu}^+ \) and the \( E_{\nu}^- \) values cross the zero-energy line can be calculated from the matrix elements for \( m = m_{o,\nu} \) at which the \( \nu \)-level of the average \( \overline{V}_m \) potential crosses the zero energy line,

\[
\Delta m_{\nu} = \frac{2\Delta E_{\nu}(m_{o,\nu})}{\partial E_{\nu}/\partial m} = \frac{\langle \chi_{m_{o,\nu}}|\phi_o^2|\chi_{m_{o,\nu}}\rangle}{m_{o,\nu}\langle \chi_{m_{o,\nu}}|\rho^{-2}|\chi_{m_{o,\nu}}\rangle},
\]

(41)

Assuming that the \( \chi_{m_{o,\nu}} \) function is centered on the potential edge \( \rho = b \), we can estimate the denominator as \( m_{o,\nu}/b^2 \). Denoting \( \langle \chi_{m_{o,\nu}}|\phi_o^2|\chi_{m_{o,\nu}}\rangle \) by \( \overline{\phi_o^2_{m_{o,\nu}}} \), we find

\[
\Delta m_{\nu} \approx \frac{2b^2}{m_{o,\nu}} \overline{\phi_o^2_{m_{o,\nu}}}. 
\]

(42)

Since the \( \chi_{m_{o,\nu}} \) function is centered on the edge of the \( U_o \) potential where the \( \phi_o \) function is reaching towards its asymptotic value, we try an even cruder approximation \( \phi_o^2 \rightarrow 1 \), resulting in

\[
\Delta m_{\nu} \approx \frac{2b^2}{m_{o,\nu}}.
\]

(43)

which overestimates the \( \Delta m \) widths. Nevertheless, we believe that the general trends predicted by Eq. (43) are correct.

From Eq. (43), we expect that the peak widths \( \Delta m \) of the \( \text{Im}[\omega_{m}] \) spectrum narrows (i) as the lobe is situated at larger \( m \) values, and (ii) as the radius of the potential well decreases. The former effect can be observed in the spectrum of Fig. 3(b). The latter effect implies that narrow wells with a subcoherence-length radius, \( b \ll \xi \), have peaks of widths \( \Delta m \ll 1 \), implying a decreased likelihood of lobe-overlap with an integer \( m \) value (hence, increased likelihood of stability).

V. DYNAMICAL STABILITY FROM SINGLE-PARTICLE HAMILTONIAN-LIKE EIGENSPECTRA: RECIPE

The treatment of Section IV makes explicit use of the cylindrical symmetry by introducing the device of a continuously varying azimuthal quantum number \( m \). In this section, we propose a different, more general procedure to test the dynamical stability of localized, potential-induced standing wave BEC patterns. The recipe we describe below is based on the same concept of continuously varying frequencies \( \omega_j \) that have to vanish as they cross over from real to imaginary values. At the crossing points, the BdG equations reduce to a single-particle-Hamiltonian-like eigenproblem for zero eigenvalue. As in Section IV, the general recipe allows one to infer the stability of a standing wave pattern from the eigenspectra of single-particle-like operators alone, circumventing the need of conducting the full linear stability analysis. Unlike the technique of Section IV, the recipe is independent of the symmetry of the potential and of the standing wave density \( |\phi_o|^2 \) and can be applied in three as well as two dimensions.

Recipe: We consider the eigenspectrum of the single-particle-Hamiltonian-like operators

\[
\hat{h}^\pm = -\nabla^2/2 + U_o(x) + 2(\phi_o^2 - 1) \pm \phi_o^2.
\]

(44)

We identify pairs of “similar” eigenstates \( j \) of the \( \hat{h}^+ \) and \( \hat{h}^- \) operators with eigenvalues \( E_j^+ \) and \( E_j^- \) respectively and with the same quantum number \( j \), indicating eigenstates that would go over into each other in the limit that \( \phi_o \rightarrow 0 \) in Eq. (44). We suggest that the standing wave solution \( \phi_o \) of the time-independent Gross-Pitaevskii equation is unstable if there is one or more quantum numbers \( j \) for which \( E_j^- < 0 \) and \( E_j^+ > 0 \), and that it is stable otherwise.

Justification: The justification of the recipe is based on the structure of the BdG equations and the continuity of its eigenfrequencies \( \omega_j \) with respect to a continuously varying parameter \( \epsilon \) that appears in the Hamiltonian. Such parameter could be introduced in different ways (multiplying the chemical potential or the interaction strength, for instance). Here we choose to vary the
depth of the external potential

\[ U_\epsilon(x) = \epsilon U_0(x), \]

and we let \( \epsilon \) range from 0 to 1, assuming that a standing wave solution has appeared by \( \epsilon = \epsilon_c \), with \( \epsilon_c \leq 1 \). The standing wave function \( \phi_{o,\epsilon} \) solves the time-independent Gross-Pitaevskii equation

\[
\left(-\frac{\nabla^2}{2} + U_\epsilon(x) + [\phi_{o,\epsilon}(x)]^2 - 1\right)\phi_{o,\epsilon}(x) = 0 \tag{46}
\]

with \( \epsilon > \epsilon_c \). The BdG equations for fixed \( \epsilon \) and real-valued \( \phi_o \) function,

\[
\hat{h}_\epsilon^+ f_{j,\epsilon}^+ = \omega_j(\epsilon) f_{j,\epsilon}^+; \\
\hat{h}_\epsilon^- f_{j,\epsilon}^- = \omega_j(\epsilon) f_{j,\epsilon}^-; \tag{47}
\]

with

\[
\hat{h}_\epsilon^\pm = -\frac{\nabla^2}{2} + U_\epsilon(x) + 2\phi_{o,\epsilon}(x)^2 - 1 \pm \phi_{o,\epsilon}^2(x), \tag{48}
\]

has solutions that are pairs of functions \( (f_{j,\epsilon}^+, f_{j,\epsilon}^-) \). As an eigenvalue of the Hermitian \( \hat{h}_\epsilon^+ \hat{h}_\epsilon^- \) operator, \( \omega_j^2(\epsilon) \) is real-valued so that \( \omega_j(\epsilon) \) is either entirely real or entirely imaginary. To cross from real to imaginary values, \( \omega_j(\epsilon) \) has to pass through zero, at which point the BdG equations reduce to a Hamiltonian eigenvalue problem of vanishing eigenvalue. If \( \chi_j^- \) is the zero-eigenvalue, the j-type eigenvector of the \( \hat{h}_\epsilon^- \) operator for \( \epsilon = \epsilon_j^- \),

\[
\hat{h}_\epsilon^- \chi_j^- = 0 \tag{49}
\]

then \( (f_{j,\epsilon}^+ = 0, f_{j,\epsilon}^- = \chi_j^-) \) solves the BdG equations for that specific \( \epsilon \) value. Conversely, if \( \chi_j^+ \) is the zero-eigenvalue, j-type eigenvector of the \( \hat{h}_\epsilon^+ \) operator for \( \epsilon = \epsilon_j^+ \),

\[
\hat{h}_\epsilon^+ \chi_j^+ = 0 \tag{50}
\]

then \( (f_{j,\epsilon}^+ = \chi_j^+, f_{j,\epsilon}^- = 0) \) solves the BdG equations for \( \epsilon = \epsilon_j^+ \). In general, \( \hat{h}_\epsilon^\pm \chi_j, = E_{j,\epsilon}^\pm(\epsilon)\chi_j^\pm, \) and the zero eigen energy happens at specific \( \epsilon \)-values, \( \epsilon = \epsilon_j^\pm \) for which \( E_{j,\epsilon}^\pm(\epsilon_j^\pm) = 0 \). As \( (0, \chi_j^+) \) and \( (\chi_j^+, 0) \) solve the BdG equations for specific values of the parameter \( \epsilon \), the \( f_{j,\epsilon}^\pm \) functions must be of the same type (same number of nodes, same quantum numbers) as the \( \chi_j^\pm \) wavefunctions. As \( \omega_j(\epsilon) \) passes through zero at \( \epsilon = \epsilon_j^\pm \), it either changes sign or switches between real and imaginary values. We argue for the latter: as an eigenvalue of the \( \hat{h}_\epsilon^+ \hat{h}_\epsilon^- \) operator, \( \omega_j^2(\epsilon) \) and its derivative with respect to \( \epsilon \) are continuous. As a consequence, \( \omega_j^2(\epsilon) \) switches sign at \( \epsilon = \epsilon_j^\pm \), corresponding to \( \omega_j(\epsilon) \) switching between real and imaginary values. If \( \epsilon_j^- < 1 \) so that the \( E_{j,\epsilon}^-(\epsilon) \) eigenvalue has already been lowered through the zero-energy dividing line by \( \epsilon = 1 \) whereas \( \epsilon_j^+ > 1 \) so that \( E_{j,\epsilon}^+(\epsilon) > 0 \) and the \( \hat{h}_j^+ \) eigenvalue has not been brought below the zero-energy line yet, then \( \omega_j \) has moved through 0 only once, switching its value from real to imaginary value. If both \( E_{j,\epsilon}^- < 0 \) and \( E_{j,\epsilon}^+ < 0 \), then \( \omega_j(\epsilon) \) moved twice through 0 as \( \epsilon \) varied from \( \epsilon_c \) to 1 switching from real to imaginary and back to real. The upshot is that if both eigenvalues \( E_{j,\epsilon}^\pm \) find themselves on the same side of zero, then \( \omega_j \) is real, if they are on opposite sides of the zero energy dividing line, \( \omega_j \) is imaginary.

The application of the above method to the 2D-cylindrically symmetric situation described in Sections II and IV leads to the investigation of \( E_{m}^\pm \) for integer, fixed values of \( m = 0, \pm 1, \pm 2, \ldots \), from which stability or instability can be determined. In contrast, the treatment of Section IV used the quantum number \( m \) itself as the continuously varying parameter and calculated the rate of the instability \( \text{Im}[\omega_m]^{-1} \) as a function of \( m \). A single \( \text{Im}[\omega_m]^{-1} \) curve then reveals excitations of different \( j \)-types: the different \( \text{Im}[\omega_m]^{-1} \)-lobes shown in Fig. (8) correspond to \( j \)-functions of different \( j \)-types, here nodes of different vibrational quantum numbers \( \nu \) (with different numbers of nodes). While convenient for studying the 2D cylindrically symmetric standing wave pattern, the method of a continuously varying quantum number cannot readily be generalized to describe non-symmetric \( \phi_o \)-patterns or three-dimensional structures; but the concept outlined here of varying a parameter in the effective Hamiltonian can be.

VI. CONCLUSIONS

We have studied the existence and dynamical stability of stationary BEC states with nodes in the BEC wavefunction. In particular, we investigated large 2D BEC systems in the presence of an attractive potential well with a coherence length-sized radius. The potential is cylindrically symmetric and of the square well type. From the center of the potential well, the BEC tends to a uniform density. As the depth of the attractive well increases, BEC solutions with radial nodes in the wavefunction appear in pairs, each pair exhibiting one more node than the last pair. While the calculations are carried out for this very specific case, we discuss which trends are independent of the symmetry and can be applied to three-dimensional standing wave BEC patterns.

We have classified the nodal BEC solutions to the time-independent Gross-Pitaevskii-equation to be of two types. One type we refer to as the atomic nodal BEC state since the wavefunction resembles that of single-particle-like bound states in the attractive potential well. The other type we refer to as the nonlinear BEC state since the \( n \) node BEC wavefunction resembles the \( n - 1 \) node atomic BEC wavefunction with a soliton attached outside the potential well.

Our stability analysis shows that the nonlinear nodal BEC state is always unstable and that atomic nodal
BEC states can transition from stable to unstable to stable status as the attractive well depth is adiabatically increased—a behavior which we refer to as reentrant stability. In general, the stable regimes grow as the radius of the attractive potential is decreased below the BEC healing length scale.

We describe a more general analysis of the dynamical instability that is based on the eigenspectra of two effective single-particle-Hamiltonian-like operators, which gives new insights, can be applied to low symmetry standing wave patterns patterns, and is valid in three dimensions.

Given the dynamical stability in specific parameter regimes, atomic nodal BEC states could find applications, for instance, in a BEC ring as a rotation sensor. Nearby instabilities may be exploited as a threshold measurement tool.

This work serves as an introduction to the existence and dynamical stability of nodal BEC states. Many challenges, on the theory and on the experimental front, remain. We have shown that nodal BEC states can be dynamically stable, but the manufacturing of these states may require creativity. Perhaps the phase imprinting techniques that have generated vortices and dark solitons \([27, 28]\) can create the required initial conditions. Conversely, the non-adiabatic dynamics of BECs in steep localized potential wells may exhibit signatures of the instabilities described above: if the BEC dynamics brings the system close to a nodal wavefunction, it can break up if the potential depth is such that the nodal state is unstable.

We believe that standing wave BECs with nodal surfaces separated by macroscopic scale distances suggest new prospects and we hope that this work will stimulate interest in standing wave BECs.

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[1] J. Denschlag et al., Science 287, 97 (2000).
[2] Z. Dutton, M. Budde, C. Sowe, and L.V. Hau, Science 293, 663 (2001).
[3] C. Josserand and Y. Pomeau, Europhys. Lett. 30, 43 (1995).
[4] A.E. Muryshev, H.B. van den Heuvel, G.V. Shlyapnikov, Phys. Rev. A 60, R2665 (1999).
[5] P.O. Fedichev, A.E. Muryshev, and G.V. Shlyapnikov, Phys. Rev. A, 60, 3220 (1999).
[6] B.P. Anderson et al., Phys. Rev. Lett. 86, 2926 (2001).
[7] Quantum fluctuations will give a finite density at the nodal surfaces, so that a more correct statement would refer to a “nearly vanishing density”.
[8] Z. Hadzibabic, P. Krüger, M. Cheneau, B. Battelier, and J. Dalibard, Nature 441, 1118 (2006).
[9] A trap technology that provides a particularly promising environment for wavefunction engineering was recently reported in: K. Henderson, C. Ryu, C. MacCormick, and M.G. Boshier, New J. Phys. 11, 043030 (2009).
[10] P. Massignan, C.J. Pethick, and H. Smith, Phys. Rev. A 71, 023606 (2005).
[11] S.-J. Sin, Phys. Rev. D 50, 3650 (1994).
[12] C.G. Böhrer and T. Harko, J. Cosmol. Astropart. Phys. 06, 025 (2007).
[13] P. Sikivie and Q. Yang, Phys. Rev. Lett. 103 111301 (2009).
[14] G.E. Astrakharchik and L.P. Pitaevskii, Phys. Rev. A 70, 013608 (2004).
[15] D.C. Roberts and Y. Pomeau, Phys. Rev. Lett. 95, 145303 (2005).
[16] F.M. Cucchietti and E. Timmermans, Phys. Rev. Lett. 96, 210401 (2006).
[17] R.M. Kalas and D. Blume, Phys. Rev. A 73 043608 (2006).
[18] R. Cote, V. Kharchenko, and M.D. Lukin, Phys. Rev. Lett. 89 093001 (2002).
[19] D.S. Petrov, and G.V. Shlyapnikov, Phys. Rev. A 64, 012706 (2001).
[20] I. Bloch, J. Dalibard, and W. Zwerger, Rev. Mod. Phys. 80, 885 (2008).
[21] The existence of nodal BEC solutions with \(j\) nodes at well depths shallower than that necessary for the existence of the noninteracting single particle bound state with \(j\) nodes is due to the presence of the BEC chemical potential creating an effectively deeper well.
[22] The reason that \(\Delta F > 0\) for well depths near those where the nodal solutions first exist is related to the appearance of the nodal BEC solutions at well depths shallower than that for the corresponding non-interacting bound states, cf. [21].
[23] Y. Castin and R. Dum, Phys. Rev. Lett. 79, 3553 (1997).
[24] A thorough analysis of the diagonalization of the \(\hat{L}\) operator is given by: Y. Castin, in Coherent atomic matter waves, Lecture Notes of the Les Houches Summer School, edited by R. Kaiser, C. Westbrook, and F. David, EDP Sciences and Springer-Verlag (2001); and arXiv:cond-mat/0105058.
[25] The \(\hat{L}\) operator is actually not quite diagonalizable, as discussed in [21]. This problem can be remedied by introducing a projection operator that projects out the overlap of \(\delta \phi\) with \(\phi_\circ\). However, the modified \(\hat{L}\) operator has the same spectrum as the \(\hat{L}\) operator, so the conclusions remain unaltered.
[26] \(V_{m}^{-}\) goes to 0 for large distances and \(V_{m}^{+}\) goes to 2 since \(\phi_{m}^{\circ} \to 1\); thus the \(V_{m}^{-}\) energy level passing through 0 corresponds to an unbinding transition while for \(V_{m}^{+}\) it...
[27] L. Dobrek, M. Gajda, M. Lewenstein, K. Sengstock, G. Birkl, and W. Ertmer, Phys. Rev. A 60, R3381 (1999).

[28] S. Burger et al, Phys. Rev. Lett. 83, 5198 (1999).