Local freedom in the gravitational field

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Abstract.

In a cosmological context, the electric and magnetic parts of the Weyl tensor, $E_{ab}$ and $H_{ab}$, represent the locally free curvature – i.e. they are not pointwise determined by the matter fields. By performing a complete covariant decomposition of $\nabla_c E_{ab}$ and $\nabla_c H_{ab}$, we show that the parts of the derivative of the curvature which are locally free (i.e. not pointwise determined by the matter via the Bianchi identities) are exactly the symmetrised trace–free spatial derivatives of $E_{ab}$ and $H_{ab}$ together with their spatial curls. These parts of the derivatives are shown to be crucial for the existence of gravitational waves.

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1. The problem

In a cosmological context, where there is a preferred four–velocity field $u^a$, the electric and magnetic parts of the Weyl tensor, $E_{ab}$ and $H_{ab}$, represent covariantly the locally free gravitational field, in the sense that they are not pointwise determined by the matter fields (see e.g. [1] – [5]). They do not arise explicitly in the Einstein equations. Although they are not really independent of the matter fields, being constrained by integrability conditions in the form of the Bianchi identities, it is nevertheless useful to think of them as being locally free.

The Bianchi identities involve the time derivatives and the spatial divergences and curls† of $E_{ab}$ and $H_{ab}$ (see the Appendix). The locally free parts of the curvature are not pointwise determined by the matter fields via Einstein’s equations. Similarly, the locally free parts of the derivative of the curvature are those parts not pointwise determined via the Bianchi identities by the matter fields and their derivatives and by $E_{ab}$, $H_{ab}$. The aim of this paper is to determine these locally free parts and their relation to gravitational waves in cosmology.

We find via a complete covariant decomposition that the parts of $\nabla_c E_{ab}$ and $\nabla_c H_{ab}$ which are missing from the Bianchi identities are exactly the projected totally symmetric and trace–free parts $\nabla_{(c} E_{ab)}$ and $\nabla_{(c} H_{ab)}$.‡ We call these parts the distortions of $E_{ab}$ and $H_{ab}$. The distortions represent the part of the derivative of the curvature which is uncoupled from the Bianchi identities and does not locally interact with the matter fields. The divergences are determined pointwise by the matter terms and the locally free field. The curls (together with the matter terms and the locally free field) determine the time derivatives of $E_{ab}$ and $H_{ab}$ pointwise. Therefore the distortions and the curls provide a covariant characterisation of the locally free part of the space–time gradient of the gravitational field.

We show that in the case of gravitational waves (covariantly characterised), the Laplacian is determined by the distortion. Hence not only the curls, but also the distortions of $E_{ab}$ and $H_{ab}$ must be non–zero if there are gravitational waves. This result represents a refinement of the covariant characterisation and understanding of gravitational waves in cosmology (see e.g. [6] – [11]).

In Section 2 we give the complete covariant decompositions of the derivatives of the Weyl tensor and fluid kinematic quantities. The derivations of these decompositions, involving a combination of covariant tensor methods and Young diagram methods, are presented in Section 3. In Section 4 we show how the distortions and curls determine the

† The terms divergence and curl as applied to rank–2 tensors are defined by equation (1).
‡ Angle brackets $\langle \cdots \rangle$ enclosing indices represent their totally symmetric trace-free and spatially projected part. Round and square brackets denote, respectively, symmetrisation and antisymmetrisation.
covariant gravitational wave equations. Further implications of our results are discussed in Section 5. The Appendix gives the Ricci and Bianchi identities in 1 + 3 covariant form, together with the required identities for second order derivatives.

2. The basic results

The locally free field $E_{ab}$, $H_{ab}$ and its characterisation are covariant if a covariantly defined four–velocity exists. For a single perfect fluid, there is a unique covariant four–velocity, whereas for imperfect fluids and multi–fluids there is in general more than one covariantly defined four–velocity. For simplicity, we restrict ourselves to the perfect fluid case. The results are readily extended to the imperfect fluid or multi–fluid case, given any covariant choice of four–velocity.

For a gravitational field with perfect fluid source, the basic covariant variables are: the fluid scalars $\Theta$ (expansion), $\rho$ (energy density), $p$ (pressure); the fluid spatial vectors $\dot{u}_a$ (4–acceleration), $\omega_a$ (vorticity); the spatial trace–free symmetric tensors $\sigma_{ab}$ (fluid shear), $E_{ab}$, $H_{ab}$; and the projection tensor $h_{ab} = g_{ab} + u_a u_b$, which projects orthogonal to the fluid 4–velocity vector $u^a$. The covariant spatial derivative is defined by

$$D_c S^{ab \cdots q} = h_c^r h^a_p \cdots h^q_b \cdots \nabla_r S^{pq \cdots}.$$ 

It follows that $D_c h_{ab} = 0$. The covariant projected permutation tensor is $\varepsilon_{abc} = \eta_{abcd} u^d$, satisfying $D_a \varepsilon_{bcd} = 0$ and $\varepsilon_{abc} \varepsilon^{def} = 3! h_{[a}^d h_{b}^e h_{c]}^f$. The covariant spatial divergence and curl of rank–2 tensors are defined by

$$\text{(div } S\text{)}_a = D^b S_{ab}, \quad \text{(curl } S\text{)}_{ab} = \varepsilon_{cd(a} D^c S_{b)}^d,$$

(1)

generalising the expressions for vectors:

$$\text{div } V = D^b V_b, \quad \text{(curl } V\text{)}_a = \varepsilon_{abc} D^b V^c.$$ 

For brevity, we shall henceforth omit the brackets around ‘curl’ unless this leads to ambiguity.

The aim now is, through the covariant 1 + 3 splittings of vectors and rank–2 and rank–3 tensors, to find the complete covariant decomposition of the derivatives of scalars, vectors, and rank–2 symmetric tensors. Any space–time vector $S_a$ may be covariantly split as:

$$S_a = -S_b u_b^a u_a + h_a^c S_c.$$ 

As a special case, the derivatives of the fluid scalars have the decomposition

$$\nabla_a \Theta = -\dot{\Theta} u_a + D_a \Theta,$$

and similarly for $\rho$ and $p$. 
Any spatial tensor \( S_{ab} = h^c_a h^d_b S_{cd} \) may be covariantly decomposed as:

\[
S_{ab} = S_{(ab)} + \frac{1}{3} h_{cd} S^{cd} h_{ab} + \varepsilon_{abc} S^c,
\]

where \( S_{(ab)} \equiv h_{(a} c h_{b)} d S_{cd} - \frac{1}{3} h_{cd} \varepsilon_{abc} S_{ab} \) is the projected symmetric trace–free part, and the spatial vector \( S^a \) is equivalent to the skew part:

\[
S_{(ab)} = \varepsilon_{abc} S^c \quad \Leftrightarrow \quad S_a = \frac{1}{2} \varepsilon_{abc} S^{bc}.
\]

If \( S_{ab} = S_{(ab)} \), then \( (\text{curl} \, S)_{ab} = (\text{curl} \, S)_{(ab)} \). If \( S_{ab} = S_{[ab]} \), then we find the identities

\[
\text{curl} \, S_{ab} = D_{(a} S_{b)} - \frac{2}{3} D^c S_{cb} h_{ab} + D^b S_{ab} = \text{curl} \, S_a.
\]

The decomposition of the derivative of the fluid 4–velocity \( u^a \) is in our notation

\[
\nabla_b u_a = -\tilde{u}_a u_b + D_b u_a, \quad (5)
\]

\[
D_b u_a = \sigma_{ab} + \frac{1}{3} \Theta h_{ab} + \omega_c \varepsilon_{abc}; \quad (6)
\]

\[
\sigma_{ab} = D_{(a} u_{b)} - \frac{1}{3} D^c u_c h_{ab}, \quad \omega_a = -\frac{1}{2} \text{curl} \, u_a, \quad (7)
\]

where \( (6) \) is a particular case of \( (2) \). For the vorticity, using \( (2) - (4) \) we find

\[
\nabla_b \omega_a = -\omega_c \tilde{u}^c u_a u_b - \tilde{\omega}(a) u_b - u_a \left\{ \sigma_{bc} \omega^c + \frac{2}{3} \Theta \omega_b \right\} + D_b \omega_a, \quad (8)
\]

\[
D_b \omega_a = D_{(a} \omega_{b)} + \frac{1}{3} D^c \omega_c h_{ab} - \frac{1}{2} \text{curl} \, \omega_c \varepsilon_{abc}, \quad (9)
\]

where \( \tilde{\omega}_{(a)} \equiv h^b_a \tilde{\omega}_b, \) and similarly for the 4–acceleration \( \dot{u}_a \).

The electric and magnetic parts of the Weyl tensor, \( E_{ab} = C_{acbd} u^c u^d \) and \( H_{ab} = \frac{1}{2} \varepsilon_{abcd} \tilde{C}^{cd} b^a u^c \), satisfy \( E_{ab} = E_{(ab)} \) and \( H_{ab} = H_{(ab)} \). Their derivatives have the following covariant decomposition:

\[
\nabla_c E_{ab} = -\tilde{E}_{(ab)} u_c - 2 u_{(a} E_{b)} d \dot{u}^d u_c + 2 u_{(a} \left\{ E_{b)} d \left( \sigma_{dc} + \omega_{dc} \right) + \frac{2}{3} \Theta E_{b)} c \right\} + D_c E_{ab}, \quad (10)
\]

\[
D_c E_{ab} = \tilde{E}_{cab} + \frac{2}{3} D^d E_{d(a} h_{b)} c - \frac{2}{3} \text{curl} \, E_{d(a} \varepsilon_{b) c}, \quad (11)
\]

where \( \tilde{\omega}_{(ab)} \equiv h_{(a} c h_{b)} d \tilde{\omega}_{cd} - \frac{1}{3} h_{cd} \tilde{\omega}_{ab} \), and where \( \tilde{E}_{cab} \) is completely symmetric and trace–free:

\[
\tilde{E}_{cab} = \tilde{E}_{(cab)} = D_{(c} E_{ab)}. \quad (12)
\]

Note that the projected time derivative in \( (10) \) satisfies \( h^d_a h^e_b \tilde{E}_{de} = h^d_{[a} h^e_{b]} \tilde{E}_{de} \). This decomposition is derived in the next section.

A similar decomposition applies to \( \nabla_c H_{ab} \). The generalisation \( (11) \) of \( (6) \) suggests that we call \( \tilde{E}_{cab} \) and \( \tilde{H}_{cab} \) the distortions of \( E_{ab} \) and \( H_{ab} \), by analogy with the shear \( \sigma_{ab} \). These represent the divergence–free and curl–free spatial variation of the Weyl tensor.†

† Note that the term distortion in the case of a rank–2 tensor is not intended to imply the geometrical significance that it has for a vector \( \mathbf{f} \).
The decomposition (10) – (12) applies to the derivative of any spatial, symmetric and trace-free rank–2 tensor; thus we have for example
\[
D_c\sigma_{ab} = \tilde{\sigma}_{cab} + \frac{3}{5}D^d\sigma_{d(a}h_{b)c} - \frac{2}{5}\text{curl}\sigma_{d(a}\varepsilon_{b)c}^d, \tag{13}
\]
in the case of the shear \(\sigma_{ab}\).

Using the notation of [4], the covariant splitting of the Ricci and Bianchi identities given in [1] is considerably simplified. For convenience, these equations are given in the appendix, in the case of a perfect fluid source for the gravitational field. All the covariant time and spatial derivatives of the scalar and vector variables – \(\Theta, \rho, p, u_a, \omega_a\) and \(\dot{u}_a\) – appear in the covariant Ricci and Bianchi equations. This includes the distortions \(D_{(b}u_{a)} \equiv \sigma_{ab}, D_{(b}\omega_{a)}D_{(a}\dot{u}_{a)}\). But for the tensor variables \(E_{ab}, H_{ab}\) and \(\sigma_{ab}\), the distortions do not appear in any of these equations. In the case of the shear, the divergence and curl terms occur in the Ricci constraint equations (A4) and (A6) respectively, but the distortion \(\tilde{\sigma}_{cab}\) does not occur in any of the Ricci or Bianchi identities. However, the evolution of \(\tilde{\sigma}_{cab}\) is governed by the distortion of \(E_{ab}\), as may be seen by taking the symmetric and trace–free part of the spatial derivative of the shear propagation equation (A3).

The distortion arises in the covariant Laplacian, which is central to the wave equation (see Section 4). Consider the distortion of say \(H_{ab}\):
\[
\tilde{H}_{cab} = D_{(c}H_{ab)} - \frac{2}{5}h_{(ab}D^dH_{c)d}. \tag{14}
\]

Taking the divergence, we get
\[
D^c\tilde{H}_{cab} = \frac{1}{3}D^2H_{ab} - \frac{4}{15}D_{(c}D^cH_{b)c} + \frac{4}{3}D^cD_{(c}H_{b)c},
\]
where \(D^2 = D^aD_a\) is the covariant Laplacian. The last term on the right may be converted into a divergence term plus curvature correction terms, using the commutation identity (A13):
\[
D^2H_{ab} = 3D^c\tilde{H}_{cab} - 2\left(\rho - \frac{1}{3}\Theta^2\right)H_{ab} - \frac{6}{5}D_{(a}D^cH_{b)c} + \mathcal{N}[H]_{ab}, \tag{15}
\]
where
\[
\mathcal{N}[H]_{ab} = 4\omega^c\omega_cH_{ab} + 2\Theta\sigma_{(a}H_{b)c} - 6\omega^c\omega_{(a}H_{b)c} + 2\Theta\omega^c\varepsilon_{cd(a}H_{b)}^d
- 2\sigma^c(\sigma_{(a}H_{b)c}^d - 6E^c(\sigma_{(a}H_{b)c})^d + 2\omega^c\varepsilon_{d(e}\sigma_{(a}H_{b)d}^{c)}
+ 2\omega^c\varepsilon_{cd(a}\sigma_{b)e}H^{de} + 2\omega_c\varepsilon_{cd(a}H_{b)}^{d} + 2\omega_c\varepsilon_{cd(a}H_{(bd)}^{d} + 2\omega_c\varepsilon_{cd(a}H_{(bd)}^{d}.
\]
then forces $E_{ab}$, $H_{ab}$ and the spatial gradients of $\rho$, $p$ and $\Theta$ to vanish. In an almost FRW universe (which is considered in Section 4), these covariant quantities are small \[3\], and the term $N[H]_{ab}$ is non–linear and may be neglected.

3. The decompositions

The derivations of the covariant decompositions (5) – (9) for the vectors $u_a$ and $\omega_a$ (and the corresponding expressions for $\dot{u}_a$) involve a straightforward application of the $1 + 3$ splitting of rank–2 tensors together with (2) – (4). For the tensor derivatives, we need the $1 + 3$ splitting of rank–3 tensors, the generalisation to rank–3 tensors of the decomposition (2), and the properties of the covariant tensor curl and divergence.

A: The space–time splitting. Any rank–3 space–time tensor has the covariant $1 + 3$ decomposition

$$S_{abc} = \alpha u_a u_b u_c + V_a u_b u_c + W_b u_c u_a + Z_c u_a u_b + A_{ab} u_c + B_{bc} u_a + C_{ca} u_b + F_{abc},$$

where $V_a, \ldots, F_{abc}$ are spatial tensors. Since $E_{ab} = E_{(ab)}$ and $E_{ab} u^b = 0$, we have $u^a u^b E_{ab;c} = 0$ and so

$$E_{ab;c} = 2 V_{(a} u_{b)} u_c + A_{ab} u_c + 2 u_{(a} B_{b)c} + F_{abc},$$

where $A_{ab} = A_{(ab)}$ and $F_{abc} = D_c E_{ab}$. Contracting with $u^c$ and $u^b$ and using (6) gives

$$A_{ab} + V_{a} u_{b} + V_{b} u_{a} = - \dot{E}_{ab},$$

$$B_{ac} + V_{a} u_{c} = \left( \frac{1}{2} \Theta h_{ac} + \sigma_{ac} + \omega_{ac} - \dot{u}_{ac} \right) E_{ab},$$

and (10) follows.

B: The spatial splitting. Any rank–3 tensor in $n$ dimensions may be decomposed as

$$M_{abc} = M_{(abc)} + S_{abc} + T_{abc} + M_{[abc]},$$

where

$$S_{abc} = \frac{2}{3} M_{[abc]} + \frac{2}{3} M_{[a|bc]} = - S_{bac},$$

$$T_{abc} = \frac{2}{3} M_{a[bc]} + \frac{2}{3} M_{b[a|c]} = - T_{acb}.$$  \[17\]

In terms of Young diagrams \[12\], this is represented by the irreducible decomposition

$$a \otimes b \otimes c = a b c \oplus b a c \oplus c b a \oplus b c a,$$

so the 4 terms on the right of (16) are linearly independent. For $n = 3$, they have $10 + 8 + 8 + 1 = 27$ independent components. (In general, they have

$$\frac{1}{3!} n(n+1)(n+2) + \frac{1}{3} n(n+1)(n-1) + \frac{1}{3} n(n+1)(n-1) + \frac{1}{3} n(n-1)(n-2) = n^3$$
free components.) Since the terms in (14) are linearly independent, we can split off their trace–free parts independently. Because of the symmetry, each has (at most) one trace term. Using the appropriate metric \( h_{ab} \):

\[
M_{(abc)} = M_{(abc)} + \frac{3}{n + 2} M_{(a h_{bc})},
\]

(18)

\[
S_{abc} = \hat{S}_{abc} + \frac{2}{n - 1} S_{[a h_{b]c]},
\]

(19)

\[
T_{abc} = \hat{T}_{abc} + \frac{2}{n - 1} T_{[b h_{c]a]},
\]

(20)

where \( M_a = M_{(abc)} h^{bc} \), \( S_a = S_{abc} h^{bc} \), \( T_a = T_{abc} h^{ac} \), and \( \hat{S}_{abc}, \hat{T}_{abc} \) are trace–free. If \( M_{abc} = M_{(abc)} \), then symmetrising (16) on \( ab \) gives

\[
M_{abc} = M_{(abc)} + T_{(ab)c},
\]

(21)

and (17) becomes

\[
T_{abc} = \frac{4}{3} M_{[abc]}.\]

(22)

Notice that, when \( M_{abc} = M_{[abc]} \), anti–symmetrising (21) on \( bc \) gives, using (24), \( \frac{3}{2} T_{abc} = T_{(ab)c} - T_{(ac)b} \), so \( T_{(ab)c} \) is equivalent to \( T_{abc} \). As before, we can split off the trace–free parts of (21). If \( M_{abc} = M_{[abc]} \), then \( T_{[abc]} = 0 \) and (16) becomes

\[
M_{abc} = S_{abc} + M_{[abc]}.
\]

(23)

Specialising now to the case of spatial rank–3 tensors, so that \( n = 3 \), simplifications result from (3):

\[
M_{[abc]} = M_\varepsilon_{abc}, \quad \hat{S}_{abc} = \hat{S}_{d c \varepsilon_{ab}}^d, \quad \hat{T}_{abc} = \hat{T}_{d a \varepsilon_{bc}}^d,
\]

(24)

where \( \hat{S}_{cd} = \frac{1}{2} S_{abc} \varepsilon^{ab}_d = \hat{S}_{(cd)} \) and similarly for \( \hat{T}_{cd} \). If \( M_{abc} = M_{(abc)} \), then by (18), (20) – (22) and (24), we have

\[
M_{abc} \equiv M_{(abc)} = M_{(abc)} + \frac{2}{5} M_{(a h_{bc})} + \frac{1}{2} \hat{T}_{d (a \varepsilon_{bc})}^d + \frac{1}{2} T_{[b h_{c]a]} + T_{[a h_{c]b]},
\]

(25)

where \( M_a = \frac{2}{5} M_d^a - \frac{1}{3} M_d^a \) and \( T_a = \frac{2}{3} M_d^a - \frac{2}{3} M_d^a \). If in addition \( M_a^a = 0 \), then \( M_a = T_a \) and (25) becomes

\[
M_{abc} \equiv M_{(abc)} = M_{(abc)} + \frac{9}{10} M_{(a h_{bc})} + \hat{T}_{d (a \varepsilon_{bc})}^d.
\]

(26)

If \( M_{abc} = M_{[abc]} \), then (23) – (24) give

\[
M_{abc} \equiv M_{[abc]} = S_{[a h_{b]c]} + \hat{S}_{d c \varepsilon_{ab}}^d + M_\varepsilon_{abc},
\]

(27)

where \( S_a = M_{ab}^b \).

Equations (23) – (27) give the complete covariant decomposition of any rank–3 spatial tensor. An example of (27) is provided by the decomposition of the commutators \( \gamma_{c}^{ab} \) of an orthonormal triad [13]. The decomposition (20) applies to \( D_c E_{ab}, D_c H_{ab} \) and \( D_c \sigma_{ab} \). In the case of \( D_c E_{ab} \), contracting (26) with \( h^{bc} \) gives \( D^b E_{ab} = \frac{3}{2} M_a \), while contraction with \( \varepsilon_{cb}^e \) and symmetrisation on \( ea \) gives \( \text{curl} E_{ea} = -\frac{3}{2} \hat{T}_{ea} \). Thus (11) is established. The corresponding relations for \( H_{ab} \) and \( \sigma_{ab} \) follow immediately.
4. Gravitational waves

The identification of the distortions as the parts of the Weyl derivative that are uncoupled from the Bianchi identities, suggests that they play an important role in the existence of gravitational waves. We show here that this is indeed the case.

Gravitational waves in cosmology are gauge–invariant transverse tensor perturbations of the homogeneous and isotropic FRW spacetime. The covariant characterisation of gravitational waves in terms of the Weyl tensor was introduced by Hawking [6] and developed in the covariant and gauge–invariant perturbation theory of Ellis and Bruni [3]. In this approach, the fundamental equations are the covariant propagation and constraint equations (A1) – (A12) given in the Appendix, with the right hand sides set to zero (i.e. linearised). Gravitational waves are described by \( E_{ab} \) and \( H_{ab} \), governed by the linearised form of the Bianchi identities (A9) – (A12). In the absence of scalar and vector modes, the gradients of \( \rho \), \( p \) and \( \Theta \) are zero, together with \( \dot{u}_a \) and \( \omega_a \). It follows from the linearised version of equations (A4), (A11) and (A12) that \( \sigma_{ab}, E_{ab} \) and \( H_{ab} \) are divergence–free. These tensors satisfy wave equations (28), provided their curls are non–zero (strictly speaking, the curl of the curl must be non–zero). For example, taking the time derivative of the linearisation of equation (A10), using the linearisation of equations (A1), (A6), (A9) and the identities (A14) and (A15), we get

\[
\Box^2 H_{ab} \equiv - \ddot{H}_{ab} + D^2 H_{ab} = \frac{7}{3} \Theta \dot{H}_{ab} + \left( \frac{2}{3} \Theta^2 - 2p \right) H_{ab},
\]

in agreement with [7] and [11].

The derivation of (28) reflects the known result [7] that a necessary covariant condition for gravitational waves in cosmology is

\[
D^b E_{ab} = 0 \neq \text{curl} \ E_{ab}, \quad D^b H_{ab} = 0 \neq \text{curl} \ H_{ab}.
\]

Now the identity (14) derived in Section 2 for the covariant Laplacian allows us to refine this result. In the linearised case, the term \( \mathcal{N}[H]_{ab} \) given by (15) vanishes, and we have

\[
D^2 H_{ab} = 3D^c \tilde{H}_{cab} - 2 \left( \rho - \frac{1}{3} \Theta^2 \right) H_{ab},
\]

since the divergence is zero for gravitational waves. It follows that if the distortion of \( H_{ab} \) vanishes (strictly, if the divergence of the distortion vanishes), then \( H_{ab} \) satisfies a covariant Helmholtz equation

\[
D^2 H_{ab} = - \left( \frac{6 \mathcal{K}}{a^2} \right) H_{ab},
\]

where we have used the background Friedmann equation to evaluate the coefficient in (30), with \( a \) the scale factor and \( \mathcal{K} = 0, \pm 1 \) the spatial curvature of the background.
For $K = 0$, the Laplacian of $H_{ab}$ vanishes and there can be no wave propagation. For $K \neq 0$, $H_{ab}$ is proportional to a tensor eigenfunction of the covariant Laplacian. These covariant eigenfunctions satisfy [6]

$$D^2 Q^{(k)}_{ab} = -\frac{k^2}{a^2} Q^{(k)}_{ab}, \quad Q^{(k)}_{ab} = Q^{(k)}_{(ab)}, \quad D^b Q^{(k)}_{ab} = 0 = G^{(k)}_{ab},$$

where $k$ determines the comoving wave number. For $K = -1$, (31) shows that there is no real value of $k$ (and thus no smooth eigenfunction), which forces $H_{ab} = 0$, so that there are no gravitational waves. For $K = +1$, (31) implies $k = \sqrt{6}$, so that there is at most one wavelength admitted. However, this rules out general gravitational waves. A similar result follows if we start with zero distortion of $E_{ab}$. Thus we have the extension of (29):

**a necessary covariant condition for the existence of gravitational waves in cosmology is**

$$D^b H_{ab} = 0, \quad \text{curl} \ H_{ab} \neq 0 \neq D_{(c} H_{ab)} ,$$

$$D^b E_{ab} = 0, \quad \text{curl} \ E_{ab} \neq 0 \neq D_{(c} E_{ab)} .$$

(32)

5. Conclusion

We have found the complete covariant decomposition of the derivatives of the electric and magnetic parts of the Weyl tensor. This extends the covariant characterisation of the locally free gravitational field to its space–time gradient. The derivatives decompose into time derivatives and spatial curls, divergences and distortions (totally symmetric trace-free parts). The covariant Ricci and Bianchi equations given in the Appendix involve all these parts of $\nabla c E_{ab}$ and $\nabla c H_{ab}$ except for the distortions $\hat{E}_{cab} \equiv D_{(c} E_{ab)}$ and $\hat{H}_{cab} \equiv D_{(c} H_{ab)}$, which do not occur in the equations. They may be thought of as the uncoupled parts of the derivatives, containing at most $2 \times 7 = 14$ independent components out of the maximal 40 independent components of $\nabla c C_{abcd}$. Together with the curls, the distortions are crucial for the existence of gravitational waves.

At any spacetime event $\mathcal{P}$, specification of $C_{abcd}$ and $\nabla_c C_{abcd}$ determines a first–order Taylor approximation to $C_{abcd}$ near $\mathcal{P}$. The divergences are determined algebraically at $\mathcal{P}$ via the Bianchi constraints (A11) and (A12). The curls algebraically determine the time derivatives via the Bianchi propagation equations (A9) and (A10), so determining the future evolution of the system. Together with the distortions at $\mathcal{P}$, they are free data at that point, in that they are not determined by the matter there (i.e. by $\rho, p, u^a$ and their first derivatives). Rather their values there are determined non–locally, their time evolutions along the fluid flow–line through $\mathcal{P}$ being determined by the free data at earlier times (through spatial derivatives of the Bianchi propagation equations (A9) and (A10) for $E_{ab}$ and $H_{ab}$), and their spatial derivatives being restricted by spatial derivatives of the Bianchi constraint equations (A11) and (A12).
It is clear from this that the distortions at $\mathcal{P}$ are the least coupled parts of the derivative of the Weyl tensor. Together with the curls, they are the free data that can be given at a point. Since the curls contain at most $2 \times 5 = 10$ independent components, there are at $\mathcal{P}$ in general $14 + 10 = 24$ unconstrained components of $\nabla_e C_{abcd}$. These tensor properties at a point are reminiscent of Penrose’s result \[14\] that the totally symmetrised derivatives of the Weyl spinor are freely specifiable at a point in the vacuum case (see \[13\] for the non–vacuum generalisation, and \[10\] for further discussion and application, of Penrose’s result).

We have shown how the distortions of $H_{ab}$ and $E_{ab}$ must be non–vanishing if there are to be gravitational waves in cosmology, thus refining the known covariant condition that the curls must be non–zero. Further investigation of the role of the distortions is clearly warranted. In particular, the covariant classification of cosmological solutions via $E_{ab}$, $H_{ab}$ and their divergences and curls (see \[1\], \[4\], \[5\], \[9\], \[10\], \[17\]), could be extended by considering also the distortions.

Finally, we note that the electromagnetic analogue of these results follows from the covariant decomposition

$$D_b E_a = D_{(b} E_{a)} + \frac{1}{3} D^c E_c h_{ab} - \frac{1}{2} \text{curl} E_c \varepsilon_{ab}^c,$$

and similarly for the magnetic field vector $H_a$. This shows that the distortions of $E_a$ and $H_a$ are ‘free’ in the sense that they are not directly constrained by Maxwell’s equations (see \[2\]). Together with the curls, these distortions, as in the tensor case, are crucial for the existence of electromagnetic waves.
Appendix A. Covariant Ricci and Bianchi equations

The Ricci identity for $u^a$ and the Bianchi identities (incorporating the field equations via the Ricci tensor), are covariantly split into propagation and constraint equations in [1]. For a perfect fluid source, these equations are in our notation the following:

**Ricci:**
\[
\dot{\Theta} + \frac{1}{3} \Theta^2 - D^a \dot{u}_a + \frac{1}{2} (\rho + 3p) = - \sigma_{ab} \sigma^{ab} + 2 \omega_a \omega^a + \dot{u}_a \dot{u}^a ,
\]
\[
\dot{\omega}(a) + \frac{2}{3} \Theta \omega_a + \frac{1}{2} \text{curl} \, \dot{u}_a = \sigma_{ab} \omega^b ,
\]
\[
\dot{\sigma}_{(ab)} + \frac{2}{3} \Theta \sigma_{ab} - D_{(a} \dot{u}_{b)} + E_{ab} = - \sigma_{c(a} \sigma_{b)}^c
\]
\[
\frac{2}{3} D^a \Theta - D^b \sigma_{ab} + \text{curl} \, \omega_a = 2 \varepsilon_{abc} \omega^b \dot{u}^c ,
\]
\[
D^a \omega_a = \ddot{u}^a \omega_a ,
\]
\[
H_{ab} - \text{curl} \, \sigma_{ab} - D_{(a} \omega_{b)} = 2 \ddot{u}_{(a} \omega_{b)} ,
\]

**Bianchi:**
\[
\dot{\rho} + (\rho + p) \Theta = 0 ,
\]
\[
(\rho + p) \dot{u}_a + D_a p = 0 ,
\]
\[
\dot{E}_{(ab)} + \Theta E_{ab} + \frac{\rho + p}{2} \sigma_{ab} - \text{curl} \, H_{ab} = 3 \sigma_{c(a} E_{b)}^c + \omega^c \varepsilon_{cd(a} E_{b)}^d
\]
\[
- 2 \ddot{u}^c \varepsilon_{cd(a} H_{b)}^d ,
\]
\[
\dot{H}_{(ab)} + \Theta H_{ab} + \text{curl} \, E_{ab} = 3 \sigma_{c(a} H_{b)}^c + \omega^c \varepsilon_{cd(a} H_{b)}^d
\]
\[
+ 2 \ddot{u}^c \varepsilon_{cd(a} E_{b)}^d ,
\]
\[
D^b E_{ab} - \frac{1}{3} D_a \rho = \varepsilon_{abc} \sigma^b_d H^{cd} - 3 H_{ab} \omega^b ,
\]
\[
D^b H_{ab} - (\rho + p) \omega_a = - \varepsilon_{abc} \sigma^b_d E^{cd} + 3 E_{ab} \omega^b .
\]

These equations generalise those given for the case of irrotational dust in [1]. In the case of small anisotropies and inhomogeneities, i.e. when the cosmology is almost FRW, the right hand sides of the above equations can be set to zero. The resulting linearised equations are the basis for covariant and gauge–invariant perturbation theory [3].

By projecting the Ricci identity for rank–2 tensors, we can generalise the identities in [1] for the commutation of covariant time and spatial derivatives, and for the curl of the curl:
Commutation of spatial derivatives (exact, non–linear):
\[
D_{[a}D_{b]}S^{cd} = \frac{2}{3} \left( \frac{1}{3} \Theta^2 - \rho \right) S_{[a}^{(c}h_{b]}^{d)} - 2S_{[a}^{(c} \left\{ E_{b]}^{d)} - \frac{1}{3} \Theta \sigma_{b]}^{d)} \right\} \\
+ 2h_{[a}^{(c} \left\{ E_{b]e} - \frac{1}{3} \Theta \sigma_{b]e} \right\} S^{d) e} - 2\sigma_{[a}^{(c} \sigma_{b]e} S^{d) e} \\
- 2S_{[a}^{(c} \left\{ \omega_{b]d)} - \omega_{c}^{e}h_{b]}^{d)} + \frac{1}{3} \Theta \epsilon_{b]}^{d) e} \omega^{e} \right\} \\
+ 2 \left\{ \sigma_{[a}^{(c} \epsilon_{b]ef} + \sigma_{[b}^{(c} \epsilon_{a]f} \right\} \omega^{f} S^{d) e} \\
+ h_{[a}^{(c} \left\{ \omega_{b]e} + \frac{1}{3} \Theta \epsilon_{b]ef} \omega^{e} \right\} S^{d) e} - \epsilon_{abc} \omega^{e} \hat{S}^{(cd)}. \right) \quad (A13)
\]

Commutation of curl and time derivative (linearised):
\[
(\text{curl } S_{ab})' = \text{curl } \dot{S}_{ab} - \frac{1}{3} \Theta \text{curl } S_{ab}. \quad (A14)
\]

Curl of curl (linearised):
\[
\text{curl curl } S_{ab} = -D^2 S_{ab} + \frac{2}{3} D_{(a}D_{b)c} + \left( \rho - \frac{1}{3} \Theta^2 \right) S_{ab}, \quad (A15)
\]
where $D^2 = D^aD_a$ is the covariant Laplacian.
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