Investigating entanglement entropy at small-$x$ in DIS off protons and nuclei

G.S. Ramos and M.V.T. Machado

High Energy Physics Phenomenology Group, GFPAE IF-UFRGS
Caixa Postal 15051, CEP 91501-970, Porto Alegre, RS, Brazil

In this work we analyse the entanglement entropy in deep inelastic scattering off protons and nuclei. It is computed based on the formalism where the partonic state at small-$x$ is maximally entangled with proton being constituted by large number of microstates occurring with equal probabilities. We consider analytical expressions for the number of gluons, $N_{\text{gluon}}$, obtained from gluon saturation models for the dipole-target amplitudes within the QCD color dipole picture. In particular, the nuclear entanglement entropy per nucleon is studied. We also study the underlying uncertainties on these calculations and compare the results to similar investigations in literature.

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I. INTRODUCTION

Recently, high energy physics community make strong efforts to use statistical physics concepts to describe the outcome of particle collisions [1, 2]. As an example, the central distribution of multiplicities of particle produced in such scatterings at high energies regime is related to the entropy produced by the collisions. In this context, one subject of study in recent years is the entanglement entropy [3], $S_{EE}$. It measures how far the particle system is from a pure quantum state. Specifically, the $S_{EE}$ quantifies the level of entanglement between different subsets of degrees of freedom in a quantum state. In an entangled system its quantum state can not be factored as a product of states of its local constituents. The confinement of quarks inside hadrons is a typical example of quantum entanglement as they are both correlated and not isolated objects. One way to probe the short distance structure inside hadrons is to consider hard scattering of deeply virtual photons off nucleons or nuclei. For large momentum transfer, small transverse distances of order $1/Q$ are probed by photons having virtualities $Q^2$. One place where this occurs is in deep inelastic scattering (DIS) of leptons off hadrons. The partons, i.e. quarks and gluons, constituting those hadrons are experimentally investigated for a long time and the kinematical range available by now for DIS off protons reaches $x \gtrsim 10^{-5}$ and $0.065 \lesssim Q^2 \lesssim 10^5$ GeV$^2$ [4]. The Bjorken-$x$ variable is the longitudinal momentum fraction carried by these partons. Then, one question that arises is the tension between a non-zero entropy resulting from different configurations of quasi-free incoherent partons and the zero von Neumann entropy for the probed hadron which is a pure state in its rest frame. One answer to this issue seems to be the quantum entanglement of partons [5].

The use of different theoretical techniques in quantum chromodynamics (QCD) in order to describe the entropy production and entanglement entropy of partons has been employed nowadays. For instance, by using the dominance of gluon fusion reaction in $t\bar{t}$ production at high energy colliders in Ref. [6] it was proposed the direct detection of entanglement by measuring the angular separation of their decay products (signature of spin-entanglement). Here, we summarize some key works in literature related to these issues. In Ref. [7] the definition of dynamical entropy for dense QCD states of matter is proposed, which is written as an overlap functional between the gluon distribution at different total rapidities and saturation radius, $R_s(x) = 1/Q^2_s(x)$. The typical momentum scale in the saturated limit is the saturation scale, $Q_s$. The formalism also has been extended to the initial preequilibrium state of a heavy ion collision. The entanglement entropy between the two outgoing particles in an elastic scattering is presented in Ref. [8] by using an $S$-matrix formalism taking into account partial wave expansion of the two-body states. The identification of the physical origin of the divergence in the entropy expression appearing in [8] and the its further regularization is done in Ref. [9]. The obtained finite $S_{EE}$ is then applied to proton-proton collisions at collider energies. On the other hand, the entropy of a jet is determined in Ref. [10] by using the entropy of the hard reduced density matrix obtained from tracing over infrared states. The thermodynamical entropy associated with production of gluons is shown in Ref. [11] taking into account unintegrated gluon distribution (UGD) based on saturation approach. One important conclusion is that the thermodynamical entropy behaves like multiplicity of produced gluons and there should exist an upper bound on entropy of gluons coming from the saturated sector of gluon UGD. In Ref. [12] the authors consider the entropy of quarks and gluons by using the Wehrl entropy, $S_W$, in QCD which is the semiclassical counterpart of von Neumann entropy. They use the parton phase space QCD Wigner and Husimi distributions, which are obtained from models that include gluon saturation effects. The obtained Wehrl entropy is expressed in terms of the gauge invariant matrix element of the quark and gluon field operators. In asymptotic regime, $Y = \ln(1/x) \to \infty$ they found $S_W \propto Q_s^2(Y) \sim e^{\alpha Y}$, which agree in the same limit with the different definitions of entanglement entropy referred above [7, 11].

Focusing particularly on entanglement entropy, it has been investigated for soft gluons in the wave function of
a fast hadron in Ref. [13]. There, the entropy production in high energy collisions is also obtained in the context of color glass condensate (CGC) formalism for the hadron wavefunction. Along similar lines, in Ref. [14] the authors define the CGC density matrix and present the evolution equations for this matrix (afterward, the effective density matrix was also analyzed in detail in [13]). These equations turn out to be similar to the Lindblad evolution. At large rapidities (high energies) the obtained $S_{EE}$ grows linearly with rapidity both in the dilute and saturated regime driven by different rates. In Ref. [15], an entropy of ignorance, $S_I$, is introduced associated with the partial set of measurements on a quantum state. It is demonstrated that in the parton model the $S_I$ is equal to a Boltzmann entropy of a classical system of partons. Moreover, it was shown that the ignorance and entanglement entropies are similar at high momenta and distinct at the low ones [16]. The main point raised there is that the lack of coherence and large entropy of partons must be due to the ability to measure only a restrict number of observables rather than to the entanglement of the observed partons with the degrees of freedom which are not observed as advocated in Ref. [5].

Here, we focus on the work in Ref. [5], where the von Neumann entropy of the parton system probed in DIS is derived within the nonlinear QCD evolution formalism. Then, this entropy is interpreted as the entanglement entropy between the spatial region resolved by $c\rho$ DIS and the rest of the proton. The authors shown that there is a simple connection between the gluon distribution, $xG(x/Q^2)$, and the $S_{EE}$ with all partonic microstates being equiprobable. In particular, at small-$x$, $S_{EE}(Y) = \ln[xG(Y/Q^2)]$, where in the limit of large $Y$ the entanglement entropy is maximal. In other words, the equipartitioning of microstates that maximizes $S_{EE}$ corresponds to the parton saturation. At asymptotic regime the entropy takes the form $S_{EE} \approx \frac{2N_c}{\pi} \ln[r^2Q^2_0(Y)] Y$, with $r \sim 1/Q$ being the characteristic dipole size in DIS. In Ref. [17] an experimental test of these ideas was devised where the entropy reconstructed from the final state hadrons is compared to the entanglement entropy of the initial state partons. It is demonstrated that $S_h$ and $S_{EE}$ are in agreement at small-$x$ by using measured hadron multiplicity distributions at the Large Hadron Collider (LHC)

Motivated by those studies in this work we compute the entanglement entropy of partons within the nucleons and nuclei at high energies using analytical parametrizations for the gluon distribution function (PDF) based on parton saturation approach. In particular, the usual integrated gluon PDF, $xG(x, Q^2)$, is obtained from the unintegrated gluon distribution on the proton and nucleus using the correspondence between the color dipole picture and the $k_L$-factorization formalism in leading logarithmic approximation. We compare the results with the recent extractions of $S_{EE}$ from hadron multiplicities in DIS and proton-proton collisions at the LHC [12]. In addition, we will cover kinematical ranges relevant for future lepton-hadron colliders like LHeC/FCC-eh [18] and eRHIC [21]. We also determine the nuclear entanglement entropy per nucleon, $S_A$. The paper is organized as follows. In next section, we start by briefly reviewing the calculation of the entropy of partonic density matrix which describes DIS within the parton model. Given the proton wavefunction this matrix is obtained by reducing it with respect to the unobserved degrees of freedom and the $S_{EE}$ is identified with the von Neumann entropy. In Sec. III we present our main results and discuss the uncertainties and limitations of the approach. In last section we summarize the main conclusions and perspectives.

II. THEORETICAL FORMALISM

Here we follow the formalism presented in Ref. [5], where the entanglement entropy is obtained in the framework of high energy QCD using both a simplified (1+1) dimensional model of nonlinear QCD evolution and full calculations in (3+1) dimensional case described by the Balitsky-Kovchegov (BK) evolution equation. The main point is that the von Neumann (Shannon) entropy resulting from entanglement between the two regions probed in DIS can be interpreted as the $S_{EE}$. The entropy is given by the Gibbs formula, $S_{EE} = -\sum p_n \ln(p_n)$, where $p_n$ is the probability or a state with $n$ partons. Using a dipole representation, where a set of partons is represented by a set of color dipoles, the probability of microstates $p_n$ is identified to the probabilities to find $n$ color dipoles inside the proton. In the toy model (1+1) dimensional, it is found that the entanglement entropy depends on rapidity $Y$ on the following way:

$$S_{EE}(Y) = e^{\alpha_h Y} (\alpha_h Y) + (1 - e^{\alpha_h Y}) \ln (e^{\alpha_h Y} - 1),$$

where $\alpha_h$ is the BFKL intercept, $\alpha_h = 4 \ln 2 \bar{a}_s (\bar{a}_s = \alpha_s N_c / \pi)$, and with a limit $S_{EE}(\alpha_h Y \gg 1) \sim \alpha_h Y$. By defining the gluon distribution, $xG(x, Q^2)$, as the average number of partons probed with resolution $Q^2$ at a given value of $x$, one obtains $xG(x) = \langle n \rangle = e^{\alpha_h Y} = x^{-\alpha_h}$. Comparing the average number of gluon $\langle n \rangle$ and the entropy expression in Eq. (1) the following relation is obtained at the limit $\alpha_h Y \gg 1$,

$$S_{EE} = \ln \left[ xG(x, Q^2) \right],$$

which is a key result presented in [5]. The limit $\alpha_h Y \gg 1$ is satisfied by values of Bjorken-$x$ less than $\sim 10^{-3}$.

The calculation for a full (3+1) dimensional QCD is more involved. The starting point is writing down the parton cascade equation whose solution gives the probability to have $n$-dipoles, $P_n(Y; r_i)$, at rapidity $Y - y$ and transverse size $r_i$. The cascade equation conducts to the BK evolution equation for dipole amplitude. Explicitly, the entanglement entropy in (3+1) QCD is evaluated as
\[
S_{EE} = -\sum_{n=1}^{\infty} \prod_{i=1}^{n} \int d^{2}r_{i} P_{n}(Y; r_{i}) \ln [P_{n}(Y; r_{i})],
\]
\[
= \Delta_{s}Y - e^{-\Delta_{s}Y} \int_{0}^{\Delta_{s}Y} t_{n} \Phi(t_{n}) e^{\Xi(t_{n})} dt_{n}, (3)
\]
where one defines \( \Delta_{s} = \tilde{\alpha}_{s}(\ln(r^{2}Q_{s}^{2})) \) with \( r \) being the typical dipole size in DIS and \( Q_{s}(x) \) is the saturation scale. The auxiliary functions appearing in the integrand for the second term in Eq. (3) take the form, \( \Phi(t) = (1 - e^{t})/t \) and,
\[
\Xi(t_{n}) = \int_{0}^{t_{n}} \Phi(t) dt = \gamma_{E} + \Gamma(0, t_{n}) + \ln(t_{n}), \quad (4)
\]
with \( \gamma_{E} \) being the Euler-Mascheroni constant and \( \Gamma(0, t) \) is the incomplete gamma function. That term is subleading for any rapidity and in the limit of large \( Y \) (very small-\( x \)) the entropy has the form \( S_{EE} \approx \Delta_{s}Y \).

Here, we will take into account an analytical expression for the gluon PDF \cite{22}, which is valid for \( Q^{2} \leq 50 \) GeV\(^2\) and allow us to obtain the number of gluons down to very small virtualities, \( Q^{2} \ll 1 \) GeV\(^2\). This is an advantage compared to the usual PDFs extracted from fitting initial conditions at \( Q^{2} = Q_{s}^{2} = 2 \) GeV\(^2\) and further DGLAP evolution. Another advantage is that it is an explicit function of the saturation scale, \( Q_{s}(x) \). Starting from the GBW saturation model \cite{22} which nicely describes all data on \( F_{2} \), \( F_{L} \), exclusive vector meson production and diffractive structure function in the small-\( x \) regime one obtains the unintegrated gluon distribution (UGD), \( \alpha_{s}F(x, k_{\perp}) = N_{0}(k_{\perp}/Q_{s}^{2}) \exp(-k_{\perp}/Q_{s}^{2}) \), with \( N_{0} = 3\sigma_{0}/4\pi^{2} \). The usual integrated gluon PDF can be calculated from the UGD,
\[
xG(x, Q^{2}) = \int_{0}^{Q^{2}} dk_{\perp}^{2} F(x, k_{\perp}),
\]
\[
= \frac{3\sigma_{0}}{4\pi^{2}} \alpha_{s}Q_{s}^{2} \left[ 1 - \left( 1 + \frac{Q^{2}}{Q_{s}^{2}} \right) e^{-\frac{Q^{2}}{Q_{s}^{2}}} \right] (5)
\]
where \( Q_{s}(x) = (x_{0}/x)^{\lambda/2} \) gives the transition between the dilute and saturated gluon system. In numerical calculations in next section, we use the updated values for the model parameters (fit result including charm): \( \sigma_{0} = 27.32 \) mb, \( \lambda = 0.248 \) and \( x_{0} = \frac{4.2 \times 10^{-3}}{23} \). The presence of the nucleon saturation scale will be useful when investigating the entropy for DIS off nuclei as discussed in next section.

### III. RESULTS AND DISCUSSIONS

Here, we will focus on the numerical calculation of the entanglement entropy in the small-\( x \) limit both for electron-proton and electron-ion collisions. In Fig. 1 one presents \( S_{EE} \) for DIS off proton as a function of \( x \) (\( 10^{-5} \leq x \leq 10^{-2} \)) for representative photon virtualities. We start with a very low scale, \( Q^{2} = 0.65 \) GeV\(^2\), typical of a soft regime which in general can not be addressed by DGLAP evolution starting in an initial hard scale \( Q_{0}^{2} \sim 2 \) GeV\(^2\). Notice that the gluon distribution we are using is obtained from the color dipole cross section including parton saturation, which describes successfully the proton structure function, \( F_{2}(x, Q^{2}) \) at very low-\( x \) \cite{22}. The results for virtualities \( Q^{2} = 2 \) and \( Q^{2} = 10 \) GeV\(^2\) are also presented. It is very clear the transition from soft to hard scales. Using the parametrization for the saturation scale, \( Q_{s}^{2}(x) = (x_{0}/x)^{\lambda} \) (with \( \lambda = 0.248 \)), one verifies that \( Q_{s}^{2} \) is of order \( Q_{s}^{2} = 0.63 \) GeV\(^2\) at \( x \leq 10^{-3} \). The advantage of using an analytical expression for \( xG \) is to trace back the behavior in terms of scaling variable \( \tau = Q^{2}/Q_{s}^{2} \). At \( \tau \ll 1 \), the series expansion gives \( xG \propto \xi^{4}Q_{s}^{2} \) and \( S_{EE} \propto -\lambda \log(Q^{2}) \).

![Figure 1: Entanglement entropy as a function of x for virtualities Q^2 = 0.63, 2, 10 GeV^2 in DIS off protons. For the low scale Q^2 = 0.63 GeV^2 the maximum entropy at small-x is shown (long-dashed line, with Q^2 = Q_s^2(x)).](image-url)
At low \(Q^2\) and very small-\(x\) it would be interesting to compare our calculation to the analytical expression of \(xG_A(x, \mu^2)\) at next-to-leading-order (NLO) level by Jones-Martin-Ryskin-Teubner (JMRT) [26]. In this case, the parameters of the NLO gluon fit are determined by a global analysis taking into account DESY-HERA data and the LHCb measurements of exclusive \(J/\psi\) production in proton-proton collisions (the probed Bjorken-\(x\) reaches \(x \sim 10^{-6}\), with \(\mu^2 \simeq m_c^2\), in charmonium photoproduction extracted from ultra-peripheral pp collisions). For sake of comparison, in Fig. 1 we present the result for the \(S_{EE}\) using at low \(Q^2\) scales the following limit \(Q^2 \simeq Q_s^2(x)\). This is represented by the long-dashed curve at \(Q^2 = 0.65\) GeV\(^2\). In what follows we consider only the kinematical ranges on \(Q^2\) where \(S_{EE}\) is equal or smaller than its maximum.

The determination of \(S_{EE}\) from data was recently done in Ref. [17]. For DIS off proton at small-\(x\) in DESY-HERA energy range, \(\sqrt{s_{ep}} \simeq 225\) GeV, the authors considered Monte Carlo simulations (PYTHIA 6) for the multiplicity distribution in order to obtain the entropy of the final state hadrons, \(S_{hadron}\), and compared it to the entanglement entropy determined by the gluon distribution. The main point is that the \(S_h\) and the entropy of initial state \(S_{EE}\) obey an inequality, \(S_h \geq S_{EE}(Y)\), if the second law of thermodynamics applies to entanglement entropy. For instance, they used the leading order Parton Distribution Function (PDF) set MSTW [27] and demonstrated that the entropy reconstructed from the final state hadrons is not correlated to \(S_{EE}\) at \(Q^2 = 2\) and \(Q^2 = 10\) GeV\(^2\). In both virtualities, one has a flat behavior \(S_{hadron} \approx 1.5\) for any \(x\) against a powerlike behavior for \(S_{EE}\). Our results using a saturated gluon distribution for \(Q^2 \geq 2\) GeV\(^2\) is somewhat similar to those from MSTW PDF presented in Ref. [17], as expected for a kinematic range where \(Q^2 \geq Q_s^2(x)\). It is argued that DESY-HERA experiment did not cover the kinematic regime where the expression of \(S_{EE}\) in terms of gluon distribution applies and the Monte Carlo models do not encode quantum entanglement. It is expected that the available range for \(x\) will be amplified in the proposed ep(A) colliders like the Large Electron-Hadron Collider (LHeC). For LHeC with energy \(\sqrt{s_{ep}} \geq 1\) TeV, DIS kinematics cover \(2.010^{-6} \leq x \leq 0.8\) and \(2 \leq Q^2 \leq 10^5\) GeV\(^2\).

In order to investigate the entanglement entropy in the case of DIS off nuclei, for simplicity we will consider the geometric scaling property of the parton saturation approaches. That is, the DIS cross section in eA collisions at small-\(x\) is directly related to the cross section for a proton target. The nuclear effects are absorbed in the nuclear saturation scale, \(Q_s^2(x, A) = \Delta A R^2_A/\pi R^2_p\) where \(Q_s(x) \sim A^{1/3} Q_s(x)\), with \(\Delta \simeq 1.27\) [28] and the normalization of cross section is rescaled relative to ep by the change \(\sigma_A \rightarrow \sigma_A (\pi R^2_A/\pi R^2_p)\sigma_0 \sim A^{2/3} \sigma_0\). Here, \(R_A \simeq 1.12 A^{1/3}\) is the nuclear radius. Therefore, the simplest extension of the gluon distribution in nuclei is given by:

\[
xG_A(x, Q^2) = \frac{3R^2_A}{4\pi Q_s^2} \left[ 1 - \left(1 + \frac{Q^2}{Q_s^2}\right) e^{-\frac{Q^2}{Q_s^2}} \right].
\]

The parametrization based on the color dipole picture and parton saturation formalism is quite reliable and it describes correctly inclusive \(\gamma p\) and \(\gamma A\) interactions at small-\(x\). In particular, the geometric scaling property described above reproduces without further fitting procedure the experimental data on the energy and centrality dependence of multiplicity of charged particles at RHIC and LHC [28]. The main features of the measured ratios of central and semi-central to peripheral pA and dA collisions, \(R_{cp}\), are also roughly described. More recently, the same approach was demonstrated to describe all exclusive processes in ep and eA collisions at small-\(x\) like Deeply Virtual Compton Scattering (DVCS) and exclusive vector mesons production. Predictions for exclusive Z\(^0\) photoproduction, Timelike DVCS and exclusive dilepton production are presented for instance in Refs. [29–31].

In Fig. 2 we calculate the corresponding nuclear entanglement entropy from the analytical parametrization for the nuclear gluon PDF. We consider the virtualities \(Q^2 = 5, 10\) and 50 GeV\(^2\) and the following nuclei: lead (Pb), gold (Au), calcium (Ca) and silicon (Si). Nuclei Pb and Au are reference for future electron-ion colliders like LHeC and eRHIC. The case \(Q^2 = 2\) is interesting as the nuclear saturation scale (squared) is enhanced by a factor \(A^{1/9}\) compared to saturation scale for proton target. This is factor 10 for lead (\(A = 208\)) and 5 for calcium (\(A = 40\)). Therefore, in the model we are using here the scale \(Q_s^2\) is of order 2 GeV\(^2\) already at \(x \simeq 10^{-2}\) for Pb and \(x \simeq 10^{-3}\) for Ca, whereas in the proton case it
occurs at \( x \approx 10^{-5} \) (see Fig. 1). This means that the \( S_{EE} \) will reach to its maximum value for larger value of \( x \) compared to DIS off nucleons due to the faster gluon saturation in the nuclear case. We see that entropy plateau already appears for lead and gold at a sufficiently hard scale \( Q^2 = 5 \text{ GeV}^2 \).

The topic of entanglement entropy and its connection to nuclear shadowing was addressed recently in Ref. [32]. The authors claim that the gluon shadowing is due to a reduction of the entanglement between the observed and unobserved degrees of freedom for gluons in a nucleus compared to those in free nucleon. Specifically, the nuclear entanglement entropy is given by \( S_A = A \ln[ xG_A(x,Q^2)/A ] \), and \( S_A/A \) is the entanglement entropy per nucleon \( (xG_N/A = xG_A/A \) is the nuclear gluon density per nucleon). Then, in [32] nuclear shadowing is a direct measure of the variation of the entanglement entropy by nucleon. For two nuclei having atomic number \( A \) and \( B \), respectively, the nuclear ratio takes the form,

\[
R_{A/B}^Q(x,Q^2) = \left( \frac{B}{A} \right) \left[ \frac{xG_A(x,Q^2)}{xG_B(x,Q^2)} \right] = \exp \left( \frac{S_A}{A} - \frac{S_B}{B} \right).
\]

Accordingly, the number of degrees of freedom per nucleon investigated in DIS in a nucleus of atomic number \( A \), \( m_A \), is smaller that those for a free nucleon, \( m_D \) (gluons in deuterium are considered as those in a free nucleon). In [32] one estimates the nuclear entanglement using the Page approach [33, 34] for the average entanglement entropy of a subsystem applied to DIS in a nucleus target. In such approach, one considers the Hilbert space with dimension \( N = mn \) of a quantum bipartite system having dimensions \( m \) and \( n \), respectively. The Page conjecture provides analytical expressions for the entanglement entropy in both cases \( m \leq n \) and \( m \geq n \) (see Ref. [32] for details). Moreover, it is proposed that antishadowing is connected to the conservation of total entropy, that is \( \int_0^1 [S_A(x) - S_D(x)] \, dx = 0 \).

TABLE I: The entanglement entropy, \( S_{EE} \), in proton-proton collisions at the LHC predicted by gluon saturation PDF, Eq. (5), using procedure from Ref. [17].

| \(|\sqrt{s}_{pp} \, (\text{TeV})\) | \(|y| < 0.5\) | \(|y| < 1.0\) | \(|y| < 1.5\) | \(|y| < 2.0\) | \(|y| < 2.4\) |
|---|---|---|---|---|---|
| 7.00 | 1.56 | 1.38 | 1.21 | 1.05 | 0.92 |
| 2.36 | 1.23 | 1.11 | 0.94 | 0.77 | 0.65 |
| 0.90 | 1.03 | 0.87 | 0.70 | 0.53 | 0.41 |

In order to compare our calculations with those in Ref. [32], in Fig. 3 the ratio \( S_A/S_D \) is presented as a function of Bjorken-\( x \) for a fixed value of virtuality, \( Q^2 = 1.7 \text{ GeV}^2 \). We consider the nuclides Pb (solid line), Xe (dashed line), Ca (long dashed line) and C (dotted line). Notice that the ratios obtained in [32] are not dependent on \( Q^2 \) and the degrees on freedom \( m_A \) are obtained by fitting the EPPS16 [35] output for the gluon shadowing at \( Q^2 = 1.7 \text{ GeV}^2 \). Our results for lead and carbon are in agreement to those in [32], obtaining \( S_{Pb}/S_D \approx 0.5 \) and \( S_{C}/S_D \approx 0.85 \) at \( x = 10^{-4} \) (the same ratios there give 0.3 and 0.7 for equal values of \( x \), respectively). The nuclear gluon density we are taking into account describes correctly the nuclear shadowing at small-\( x \) for a large variety of nuclei (see Ref. [36] for the corresponding phenomenology).

Finally, we discuss the case when proton-proton collisions are considered. In Ref. [17] the authors modify the multiplicity distribution, \( P(N) \), doing an extrapolation in order to reflect a single proton as in \( ep \) collisions. The procedure is based on the assumption that final state hadrons are produced coherently by the proton-proton collisions. Moreover, they consider the typical scale in an average \( pp \) reaction as being the saturation scale, \( Q^2 \approx \langle p_T^2 \rangle \approx Q_s^2(x) \). Here, we do not argue about the reliability of hypothesis considered in the extraction of \( S_{hadron} \) in \( pp \) case. In Table II we present the entanglement entropy given by Eq. (2) using the scale \( Q^2 = Q_s^2(x) \) and the degrees of freedom distribution concerning the hadron pseudorapidity, \( y \). Our results for lead are in agreement with those in [32] and following the same procedure proposed in [17] to compare it to final state \( S_{hadron} \). Specifically, a selection on hadron rapidity, \( y \), is taken into account based on the different experimental cuts for multiplicity distribution concerning the hadron pseudorapidity. Thus, \( S_{hadron} \) is extracted from experimental data from CMS collaboration [37], which are consistent with similar measurements done by ATLAS and ALICE collaborations. On the other hand, \( S_{EE} = \ln(N_{gluon}) \) is obtained computing the number of gluons \( N_{gluon} \) by units of rapidity after integration of the gluon PDF over the given rapidity range at a fixed \( Q^2 \).

Using \( Q^2 = Q_s^2 \), we obtains an analytical expression for \( S_{EE} \), which reads,

\[
S_{EE}(Q^2 = Q_s^2) = \ln \left( Q_s^2(x) \right) + S_0,
\]

where \( S_0 = \ln[3(e - 2)R^2_e/4\pi\alpha_s] \approx 2 \) for \( \alpha_s = 0.2 \) and \( S_{EE} = S_0 \) when \( Q_s^2 = 1 \text{ GeV}^2 \). From Tab. II we show an agreement between the \( S_{EE} \) predicted by the saturation model for the gluon PDF and the entropy recon-

FIG. 3: Ratio \( S_A/S_D \) as a function of \( x \) at \( Q^2 = 1.7 \text{ GeV}^2 \) for different nuclei. It is shown prediction for lead (Pb), xenon (Xe), calcium (Ca) and carbon (C).
structed from hadrom multiplicity at very small-$x$. Interestingly, the PDFs give small $S_{EE}$ compared to data when the saturation scale enters into a soft regime even after considering their uncertainties and good agreement for $Q_s \approx 1$ GeV. For instance, it was found in Ref. $^{17}$ that the extracted entropy is $S_{EE} \approx 2, 3.5, \text{and} 3.7$ for $|y| < 0.5, 1.5, \text{and} 2.4 \text{at} 7 \text{TeV}$, respectively. From Table 4 we verify a better agreement at midrapidity. Similar behavior was also verified in Ref. $^{17}$. It should be noticed that the larger the $|y|$ interval the bigger the average ($\langle x \rangle$). Quantitatively, at 7 TeV one has $\langle x \rangle = 1.41 \cdot 10^{-4}$ for $|y| < 0.5$ in contrast to $\langle x \rangle = 3.08 \cdot 10^{-4}$ for $|y| < 2.4$.

IV. SUMMARY

We have investigated the entanglement entropy in deep inelastic scattering for $ep$ and $eA$ collisions. The theoretical formalism is based on the von Neumann entropy written in terms of the gluon number as a function of Bjorken-$x$ and photon virtualities $Q^2$. Specifically, we consider an analytical expression for the gluon density in proton related to the parton saturation physics within the color dipole picture. The integrated gluon density, $xG$, is then extracted from the corresponding unintegrated one. The approach is able to describe all the important observables in DIS at small-$x$ and up to intermediate $Q^2 \approx 50$ GeV$^2$. Based on geometric scaling property, an extrapolation is done in order to obtain the nuclear gluon density, which also has been tested against nuclear ratios data in $eA$ collisions. The obtained nuclear entanglement entropy is compared to other proposals in literature. In $eA$ case, it was found that the results are similar to those in Ref. $^{17}$ with deviations only at very low scales, $Q^2 \lesssim 1$ GeV$^2$. The origin of this deviation is traced back to the behavior of gluon PDF below saturation scale, $Q_s(x)$. In $eA$ case, we analyse the relation between gluon shadowing and the descreening of the entropy per nucleon proposed in Ref. $^{32}$. The results corroborate the main results found in that reference. The direct comparison to data is done in Table 4 with $S_{EE}$ in agreement with final state hadron entropy in the region $|y| < 1.0$ extracted from CMS data. The results are quite similar to those obtained in Ref. $^{17}$ using the usual gluon PDFs like MSTW parametrization not including saturation aspects or higher twist effects. The main results is that the entanglement entropy at scale $Q^2 \approx Q^2_{s,T}$ behaves as $S_{EE} \propto \ln(Q^2_{s,T}(x))$ for a proton target, $T = p$, as well as a nuclear one, $T = A$.

In summary, our study shed light on the entanglement entropy in hard scattering processes using analytical tools which could bring a better understanding on the underlying dynamics in a quantum bipartite system. The detailed investigation on the entropy production and the entanglement entropy in these processes are crucial to understand the dynamics of multiparticle production in $pp$ and $AA$ collisions at high energies (see Ref. $^{38}$ for a review). For instance, the thermalization present in those reactions in accelerators like LHC and RHIC could be explained as due to the high degree of entanglement in the wavefunction of colliding particle $^{39,42}$.

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