Quadratic Growth and Strong Metric Subregularity of the Subdifferential for a Class of Non-prox-regular Functions

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Abstract
This paper mainly studies the quadratic growth and the strong metric subregularity of the subdifferential of a function that can be represented as the sum of a function twice differentiable in the extended sense and a subdifferentially continuous, prox-regular, twice epi-differentiable function. For such a function, which is not necessarily prox-regular, it is shown that the quadratic growth, the strong metric subregularity of the subdifferential at a local minimizer, and the positive definiteness of the subgradient graphical derivative at a stationary point are equivalent. In addition, other characterizations of the quadratic growth and the strong metric subregularity of the subdifferential are also given. Besides, properties of functions twice differentiable in the extended sense are examined.

Keywords Quadratic growth · Strong metric subregularity · Second-order differentiability in the extended sense · Second-order epi-differentiability · Prox-regularity

Mathematics Subject Classification 49J53 · 90C31 · 90C46

1 Introduction

Quadratic growth is an important property of extended-real-valued functions, which plays a central role in optimization [1–8, 10–12]. It can be used for justifying the
linear convergence of various optimization algorithms \cite{3, 7, 15} as well as analyzing perturbations of optimization problems \cite{4}. Especially, for many favorable classes of functions, quadratic growth is closely related to critical point stability \cite{1, 2, 5, 6, 10–12}.

For proper lower semicontinuous convex functions, Aragón Artacho and Geoffroy \cite{1} showed that the quadratic growth and the strong metric subregularity of the subdifferential at a minimizer are equivalent, and they can be characterized by the positive definiteness of the subgradient graphical derivative at a stationary point. For proper lower semicontinuous nonconvex functions, Drusvyatskiy et al. \cite{8} proved the validity of the quadratic growth under the strong metric subregularity of the subdifferential at a local minimizer. Moreover, Drusvyatskiy and Ioffe \cite{6} established that the converse holds whenever the function under consideration is semi-algebraic, and it may fail if the function is not semi-algebraic. It is worth noting that the approach of \cite{6} is based on some facts from semi-algebraic geometry.

Using tools of second-order variational analysis, Chieu et al. \cite{5} showed that for proper lower semicontinuous functions, the positive definiteness of the subgradient graphical derivative at a stationary point guarantees that the point is a local minimizer and the subdifferential is strongly metrically subregular, which implies the quadratic growth by \cite[Corollary 3.3]{8}. Furthermore, the quadratic growth, the strong metric subregularity of the subdifferential at a local minimizer, and the positive definiteness of the subgradient graphical derivative at a stationary point are equivalent whenever the function is either subdifferentially continuous, prox-regular, and twice epi-differentiable or variationally convex.

More recent developments in this direction can be found in \cite{10–12, 15}, where the authors investigated composite models under certain assumptions on the component functions that make the composite function subdifferentially continuous, prox-regular, and twice epi-differentiable.

To the best of our knowledge, all known results on the equivalence relationship between the quadratic growth and the strong metric subregularity of the subdifferential, except for the one of Drusvyatskiy and Ioffe \cite{6}, are established only for subclasses of the class of subdifferentially continuous and prox-regular functions. This observation leads us to the question if such an equivalence relationship is valid for functions that are neither subdifferentially continuous and prox-regular nor semi-algebraic.

In the current work, we study the quadratic growth and the strong metric subregularity of the subdifferential of functions that can be represented as the sum of a function twice differentiable in the extended sense and a subdifferentially continuous, prox-regular, twice epi-differentiable function. The second-order differentiability in the extended sense was introduced by Rockafellar and Wets \cite{16}, which amounts to the differentiability of the subdifferential \cite[Theorem 13.2]{16}. This property is stronger than the strict differentiability, but weaker than the second-order differentiability. It is known from \cite[Theorem 13.51]{16} that every $C^2$-lower function on an open set is twice differentiable in the extended sense almost everywhere in that open set. In addition, there exists a maximum of two infinitely differentiable functions that is twice differentiable in the extended sense but not twice differentiable (see Example 3.1). On the other hand, the class of subdifferentially continuous, prox-regular, twice epi-differentiable functions has been recognized as a favorable class of functions in variational analysis.
and optimization [5, 10–12, 16]. Therefore, the class of functions under consideration is a big class of functions encompassing subdifferentially continuous, prox-regular, twice epi-differentiable functions, and twice differentiable functions.

For a function from the just mentioned class, which is not necessarily prox-regular, our main result (Theorem 4.1) shows that the quadratic growth, the strong metric subregularity of the subdifferential at a local minimizer, and the positive definiteness of the subgradient graphical derivative at a stationary point are equivalent. It improves [5, Theorem 3.7] since the class of functions examined in Theorem 4.1 is strictly larger than the one appearing in [5, Theorem 3.7] even in the case where the extended second-order differentiability is replaced by the classical second-order differentiability. In addition, other characterizations of the quadratic growth as well as the strong metric subregularity of the subdifferential are also given. Among other things, we prove the equivalence between the quadratic growth and the strong metric subregularity of the subdifferential at a stationary point that fulfils the second-order necessary optimality condition via the second subderivative. This partly answers the question in [6, p. 637] of characterizing the quadratic growth in terms of the subdifferential without having to assume that the point under investigation is a local minimizer. For the composition model, our result (Corollary 4.1) extends [12, Theorem 6.3] by relaxing the continuously twice differentiable assumption. Besides, properties of functions twice differentiable in the extended sense are examined.

The rest of the paper is organized as follows. Section 2 collects notions from variational analysis that are needed in the sequel. Section 3 investigates functions that are twice differentiable in the extended sense. The focuses of this section are on sum rules and chain rules for second subderivative, parabolic subderivative, and subgradient graphical derivative, which are used for proving the main results reported in Sect. 4. Section 4 is devoted to the study of quadratic growth and strong metric subregularity of the subdifferential. Here we specially pay the attention to the relationship between these two properties. Besides, we are also interested in characterizations of quadratic growth and strong metric subregularity of the subdifferential via the second subderivative. Section 5 summarizes the main results of the paper and presents some remarks on this research direction.

2 Preliminaries

This section recalls some concepts and their properties from variational analysis [13, 14, 16], which are needed for our analysis. Unless otherwise stated, \( \mathbb{R}^n \) is a Euclidean space with inner product \( \langle \cdot, \cdot \rangle \) and norm \( \| \cdot \| \), and \( \overline{\mathbb{R}} := \mathbb{R} \cup \{ \infty \} \). The closed ball with center \( \bar{x} \) and radius \( \varepsilon > 0 \) is denoted by \( B_{\varepsilon}(\bar{x}) \), that is \( B_{\varepsilon}(\bar{x}) := \{ x \in \mathbb{R}^n \mid \|x - \bar{x}\| \leq \varepsilon \} \).

**Definition 2.1** [13, 14, 16]. Let \( f : \mathbb{R}^n \to \overline{\mathbb{R}} \) and \( \text{dom } f := \{ x \in \mathbb{R}^n \mid f(x) < \infty \} \). The proximal subdifferential of \( f \) at \( \bar{x} \in \text{dom } f \) is defined by

\[
\partial_p f(\bar{x}) := \left\{ v \in \mathbb{R}^n \mid \lim_{x \to \bar{x}} \inf \frac{f(x) - f(\bar{x}) - \langle v, x - \bar{x} \rangle}{\|x - \bar{x}\|^2} > -\infty \right\}.
\]
The regular subdifferential (also called the Fréchet subdifferential) of \( f \) at \( \bar{x} \in \text{dom } f \) is given by

\[
\hat{\partial} f(\bar{x}) := \left\{ v \in \mathbb{R}^n \mid \liminf_{x \to \bar{x}} \frac{f(x) - f(\bar{x}) - \langle v, x - \bar{x} \rangle}{\|x - \bar{x}\|} \geq 0 \right\}.
\]

The limiting subdifferential (also called the Mordukhovich subdifferential) of \( f \) at \( \bar{x} \in \text{dom } f \) is defined by

\[
\partial f(\bar{x}) := \left\{ v \in \mathbb{R}^n \mid \exists x_k \to \bar{x}, v_k \to v \text{ with } v_k \in \hat{\partial} f(x_k) \right\},
\]

where \( x_k \to \bar{x} \) means \( x_k \to \bar{x} \) along with \( f(x_k) \to f(\bar{x}) \). If \( \bar{x} \notin \text{dom } f \), one puts

\[
\partial f(\bar{x}) = \hat{\partial} f(\bar{x}) = \partial_p f(\bar{x}) := \emptyset.
\]

**Definition 2.2** [16]. A function \( f : \mathbb{R}^n \to \mathbb{R} \) is said to be prox-regular at \( \bar{x} \in \text{dom } f \) for \( \bar{v} \in \partial f(\bar{x}) \) if there exist \( r, \varepsilon > 0 \) such that for all \( x, u \in B_\varepsilon(\bar{x}) \) with \( |f(u) - f(\bar{x})| < \varepsilon \) we have

\[
f(x) \geq f(u) + \langle v, x - u \rangle - \frac{r}{2}\|x - u\|^2 \quad \text{for all } v \in \partial f(u) \cap B_\varepsilon(\bar{v}). \tag{1}
\]

Moreover, \( f \) is said to be subdifferentially continuous at \( \bar{x} \) for \( \bar{v} \) if whenever \((x_k, v_k) \to (\bar{x}, \bar{v})\) with \( v_k \in \partial f(x_k) \) one has \( f(x_k) \to f(\bar{x}) \).

From (1), it follows that \( \partial f(u) \cap B_\varepsilon(\bar{v}) \subseteq \partial_p f(u) \) whenever \( \|u - \bar{x}\| < \varepsilon \) with \( |f(u) - f(\bar{x})| < \varepsilon \). Furthermore, if \( f \) is subdifferentially continuous at \( \bar{x} \) for \( \bar{v} \), then the inequality \(|f(u) - f(\bar{x})| < \varepsilon^*\) in the definition of prox-regularity above can be removed.

The following result is a direct consequence of (1), which is very useful for us to verify the prox-regularity in the sequel.

**Lemma 2.1** [16, Theorem 13.36]. If \( f : \mathbb{R}^n \to \mathbb{R} \) is prox-regular and subdifferentially continuous at \( \bar{x} \) for \( \bar{v} \) then there exist \( r, \varepsilon > 0 \) such that

\[
\langle v_2 - v_1, x_2 - x_1 \rangle \geq -r \|x_2 - x_1\|^2,
\]

for every \( x_1, x_2 \in B_\varepsilon(\bar{x}), \ v_1 \in \partial f(x_1) \cap B_\varepsilon(\bar{v}), \ v_2 \in \partial f(x_2) \cap B_\varepsilon(\bar{v}) \).

**Definition 2.3** [16]. Given a function \( f : \mathbb{R}^n \to \mathbb{R} \) with \( f(\bar{x}) \in \mathbb{R} \), the subderivative of \( f \) at \( \bar{x} \) is the function \( df(\bar{x}) : \mathbb{R}^n \to [-\infty, \infty] \) defined by

\[
df(\bar{x})(w) = \liminf_{w' \to w} \frac{f(\bar{x} + tw') - f(\bar{x})}{t} \quad \text{for all } w \in \mathbb{R}^n.
\]
The second subderivative of \( f \) at \( \bar{x} \) for \( v \in \mathbb{R}^n \) and \( w \in \mathbb{R}^n \) is given by
\[
d^2 f(\bar{x} | v)(w) = \lim_{t \downarrow 0} \Delta^2_f(\bar{x}, v)(w'),
\]
where \( \Delta^2_f(\bar{x}, v)(w') := \frac{f(\bar{x} + tw') - f(\bar{x}) - t\langle v, w' \rangle}{\frac{1}{2}t^2} \).

Function \( f \) is said to be \textit{twice epi-differentiable} at \( \bar{x} \in \mathbb{R}^n \) for \( v \in \mathbb{R}^n \) if for every \( w \in \mathbb{R}^n \) and choice of \( t_k \downarrow 0 \) there exists \( w_k \to w \) such that
\[
\Delta^2_{f_k}(\bar{x}, v)(w_k) \to d^2 f(\bar{x} | v)(w).
\]

It is well known that fully amenable functions [16], including the maximum of finitely many \( C^2 \)-functions, are subdifferentially continuous prox-regular and twice epi-differentiable lower semicontinuous proper functions [16,Corollary 13.15 and Proposition 13.32].

**Definition 2.4** [16]. Let \( \Omega \) be a nonempty subset of \( \mathbb{R}^n \) and \( \bar{x} \in \mathbb{R}^n \).

(i) The (Bouligand-Severi) tangent cone to \( \Omega \) at \( \bar{x} \in \Omega \) is given by
\[
T_{\Omega}(\bar{x}) := \{ v \in \mathbb{R}^n | \exists t_k \downarrow 0, u_k \to v \text{ with } \bar{x} + t_k u_k \in \Omega \forall k \in \mathbb{N}^* \}.
\]

If \( \bar{x} \notin \Omega \) then one puts \( T_{\Omega}(\bar{x}) := \emptyset \). Here, \( \mathbb{N}^* \) denotes the set of nonzero natural numbers.

(ii) The second-order tangent set to \( \Omega \) at \( \bar{x} \) for \( w \in \mathbb{R}^n \) is defined by
\[
T^2_{\Omega}(\bar{x}, w) = \left\{ u \in \mathbb{R}^n | \exists t_k \downarrow 0, u_k \to u \text{ with } \bar{x} + t_k u + \frac{1}{2}t_k^2 u_k \in \Omega \forall k \in \mathbb{N}^* \right\}.
\]

\( \Omega \) is called \textit{parabolically derivable} at \( \bar{x} \) for \( w \in \mathbb{R}^n \) if \( T^2_{\Omega}(\bar{x}, w) \neq \emptyset \), and for each \( u \in T^2_{\Omega}(\bar{x}, w) \) there exist \( \varepsilon > 0 \) and a mapping \( \xi : [0, \varepsilon] \to \Omega \) such that \( \xi(0) = \bar{x} \), \( \xi'_+(0) = w \) and \( \xi''_+(0) = u \), where
\[
\xi'_+(0) := \lim_{t \downarrow 0} \frac{\xi(t) - \xi(0)}{t} \quad \text{and} \quad \xi''_+(0) := \lim_{t \downarrow 0} \frac{\xi(t) - \xi(0) - t\xi'_+(0)}{\frac{1}{2}t^2}.
\]

**Definition 2.5** [16]. The subgradient graphical derivative of \( f : \mathbb{R}^n \to \mathbb{R} \) at \( \bar{x} \) for \( \bar{v} \in \partial f(\bar{x}) \) is the set-valued mapping \( D(\partial f)(\bar{x} | \bar{v}) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n \) defined by
\[
D(\partial f)(\bar{x} | \bar{v})(w) := \{ z | (w, z) \in T_{\text{gph } \partial f}(\bar{x}, \bar{v}) \} \quad \text{for all} \quad w \in \mathbb{R}^n.
\]

If \( f \) is twice epi-differentiable, prox-regular, subdifferentially continuous at \( \bar{x} \) for \( \bar{v} \), then it is known from [16,Theorem 13.40] that
\[
D(\partial f)(\bar{x} | \bar{v}) = \partial h \quad \text{with} \quad h = \frac{1}{2}d^2 f(\bar{x} | \bar{v}).
\]
Definition 2.6 [12]. A function \( f : \mathbb{R}^n \to \overline{\mathbb{R}} \) is said to be parabolically regular at \( \bar{x} \) for \( \bar{v} \in \mathbb{R}^n \) if \( f(\bar{x}) \in \mathbb{R} \) and for all \( w \) with \( d^2 f(\bar{x}, \bar{v})(w) < \infty \) there exist \( t_k \downarrow 0 \) and \( w_k \to w \) such that

\[
\lim_{k \to \infty} \Lambda_{t_k}^2 f(\bar{x}, \bar{v})(w_k) = d^2 f(\bar{x}, \bar{v})(w) \quad \text{and} \quad \limsup_{k \to \infty} \frac{\|w_k - w\|}{t_k} < \infty.
\]

A nonempty set \( \Omega \subset \mathbb{R}^n \) is called parabolically regular at \( \bar{x} \) for \( \bar{v} \) if its indicator function \( \delta_\Omega \) is parabolically regular at \( \bar{x} \) for \( \bar{v} \).

Definition 2.7 [16]. Let \( f : \mathbb{R}^n \to \overline{\mathbb{R}}, \bar{x} \in \text{dom } f \), and \( w \in \mathbb{R}^n \) with \( df(\bar{x})(w) \in \mathbb{R} \).

(i) The parabolic subderivative of \( f \) at \( \bar{x} \) for \( w \) with respect to \( z \) is

\[
d^2 f(\bar{x})(w|z) := \liminf_{n \to \infty} \frac{f(\bar{x} + t_n w + \frac{1}{2} t_n^2 z) - f(\bar{x}) - t_n df(\bar{x})(w)}{\frac{1}{2} t_n^2}.
\]

(ii) \( f \) is said to be parabolically epi-differentiable at \( \bar{x} \) for \( w \) if

\[\text{dom } d^2 f(\bar{x})(w|\cdot) = \{z \in \mathbb{R}^n \mid d^2 f(\bar{x})(w|z) < \infty \} \neq \emptyset,\]

and for every \( z \in \mathbb{R}^n \) and every \( t_k \downarrow 0 \) there exists \( z_k \to z \) such that

\[
d^2 f(\bar{x})(w|z) = \lim_{k \to \infty} \frac{f(\bar{x} + t_k w + \frac{1}{2} t_k^2 z_k) - f(\bar{x}) - t_k df(\bar{x})(w)}{\frac{1}{2} t_k^2}.
\]

As shown by Mohammadi and Sarabi [12,Proposition 3.6], a function \( f : \mathbb{R}^n \to \overline{\mathbb{R}} \) with \( \bar{v} \in \partial_p f(\bar{x}) \) is parabolically regular at \( \bar{x} \) for \( \bar{v} \) if and only if

\[
d^2 f(\bar{x}, \bar{v})(w) = \inf_{z \in \mathbb{R}^n} \left\{ d^2 f(\bar{x})(w|z) - \langle z, \bar{v} \rangle \right\} \quad \text{for all } w \in K_f(\bar{x}|\bar{v}),
\]

where \( K_f(\bar{x}|\bar{v}) := \{ w \in \mathbb{R}^n \mid df(\bar{x})(w) = \langle \bar{v}, w \rangle \} \) is called the critical cone of \( f \) at \( (\bar{x}, \bar{v}) \). Furthermore, if \( f \) is parabolically regular at \( \bar{x} \) for \( \bar{v} \) and \( w \in \text{dom } d^2 f(\bar{x}, \bar{v}) \) then there exists \( z \in \text{dom } d^2 f(\bar{x})(w|\cdot) \) such that

\[
d^2 f(\bar{x}, \bar{v})(w) = d^2 f(\bar{x})(w|z) - \langle z, \bar{v} \rangle.
\]

3 Twice Differentiability in the Extended Sense

The concept of twice differentiability of functions in the extended sense was introduced by Rockafellar and Wets [16,Definition 13.1], which came from the desire to develop second-order differentiability at \( \bar{x} \) without having to assume the existence of the first derivatives at every point in some neighborhood of \( \bar{x} \).
This section investigates properties of functions that are twice differentiable in the extended sense with special attention paying to sum rules and chain rules of equality form for second subderivative, parabolic subderivative, and subgradient graphical derivative.

**Definition 3.1** [16]. Let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) be finite at \( \bar{x} \). We say that

(i) \( f \) is *differentiable* (resp., *strictly differentiable*) at \( \bar{x} \) if there exists an \((1 \times n)\)-matrix \( \nabla f(\bar{x}) \), called the Jacobian (matrix) of \( f \) at \( \bar{x} \), such that

\[
\lim_{x \to \bar{x}} \frac{f(x) - f(\bar{x}) - \nabla f(\bar{x})(x - \bar{x})}{\|x - \bar{x}\|} = 0
\]

(resp., \( \lim_{x, u \to \bar{x}} \frac{f(x) - f(u) - \nabla f(\bar{x})(x - u)}{\|x - u\|} = 0 \));

(ii) \( f \) is *twice differentiable* at \( \bar{x} \) (in the classical sense) if it is differentiable on a neighborhood \( U \) of \( \bar{x} \) and there exists an \( n \times n \) matrix \( \nabla^2 f(\bar{x}) \), called the Hessian (matrix) of \( f \) at \( \bar{x} \), such that

\[
\lim_{x \in U \to \bar{x}} \frac{\nabla f(x) - \nabla f(\bar{x}) - \nabla^2 f(\bar{x})(x - \bar{x})}{\|x - \bar{x}\|} = 0;
\]

(iii) \( f \) is *twice differentiable at \( \bar{x} \) in the extended sense* if it is differentiable at \( \bar{x} \), and there exist an \( n \times n \) matrix \( A \), a neighborhood \( U \) of \( \bar{x} \) and a subset \( D \) of \( U \) with \( \mu(U \setminus D) = 0 \) such that \( f \) is Lipschitz on \( U \), differentiable at every point in \( D \), and

\[
\lim_{x \in D \to \bar{x}} \frac{\nabla f(x) - \nabla f(\bar{x}) - A(x - \bar{x})}{\|x - \bar{x}\|} = 0,
\]

where \( \mu \) denotes the Lebesgue measure on \( \mathbb{R}^n \). This matrix \( A \), necessarily unique, is then called the Hessian (matrix) of \( f \) at \( \bar{x} \) in the extended sense and is likewise denoted by \( \nabla^2 f(\bar{x}) \).

It is easy to see that if \( f \) is twice differentiable at \( \bar{x} \) then it is twice differentiable at \( \bar{x} \) in the extended sense, and the Hessian and the extended Hessian coincide.

The following example provides a maximum of two infinitely differentiable functions that is twice differentiable in the extended sense but not twice differentiable.

**Example 3.1** (Extended second-order differentiability of a maximum function). Let \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) be the function given by \( f(x, y) = \max\{x^4, y^4\} \) for \((x, y) \in \mathbb{R}^2\). Obviously, \( f \) is locally Lipschitz and
\[ \nabla f(x, y) = \begin{cases} (4x^3, 0) & \text{if } |x| > |y|, \\ (0, 4y^3) & \text{if } |x| < |y|, \\ (0, 0) & \text{if } x = y = 0. \end{cases} \]

For any \((x_0, y_0) \in \mathbb{R}^2\) with \(|x_0| = |y_0| \neq 0\), we see that

\[
\begin{align*}
\lim_{x \to x_0^+} & \frac{f(x, y) - f(x_0, y_0)}{x - x_0} = \lim_{x \to x_0^+} \frac{\max[x^4 - x_0^4, 0]}{x - x_0} = \max[4x_0^3, 0], \\
\lim_{x \to x_0^-} & \frac{f(x, y) - f(x_0, y_0)}{x - x_0} = \lim_{x \to x_0^-} \frac{\max[x^4 - x_0^4, 0]}{x - x_0} = \min[4x_0^3, 0] \neq \max[4x_0^3, 0].
\end{align*}
\]

The latter shows that \(f\) is not differentiable at \((x_0, y_0) \in \mathbb{R}^2\) with \(|x_0| = |y_0| \neq 0\). Therefore, \(f\) is not twice differentiable at \((0, 0)\). Furthermore, for \(D := \{(x, y) \in \mathbb{R}^2\} \neq |x| \neq |y|\), we have \(\mu(\mathbb{R}^2 \setminus D) = 0\) and \(\lim_{(x, y) \to (0, 0)} \frac{\nabla f(x, y) - \nabla f(0, 0)}{\|(x, y) - (0, 0)\|} = 0\), where \(\mu\) is the Lebesgue measure on \(\mathbb{R}^2\). This proves that \(f\) is twice differentiable at \((0, 0)\) in the extended sense, and \(\nabla^2 f(0, 0) = 0 \in \mathbb{R}^{2 \times 2}\).

The following example shows that there exists a function that is twice differentiable in the extended sense, but neither twice differentiable nor prox-regular.

**Example 3.2** (Extended second-order differentiability does not imply either second-order differentiability or prox-regularity). Consider the function \(g : \mathbb{R} \to \mathbb{R}\) given by

\[
g(x) = \begin{cases} x^{10/3} \cos \frac{1}{x} + x^4 & \text{if } x \geq 1, \\
x^{10/3} \cos \frac{1}{x} + \frac{(2n+1)(2n^2+2n+1)}{n^3(n+1)^3}x + \frac{1}{(n+1)^3} - \frac{1}{n^3} & \text{if } x \in \left[\frac{1}{n+1}, \frac{1}{n}\right], \; n = 1, 2, \ldots \\
0 & \text{if } x = 0, \\
g(-x) & \text{if } x < 0. \end{cases}
\]

**Claim 1**: \(g\) is twice differentiable at \(\bar{x} = 0\) in the extended sense, but it is not twice differentiable at \(\bar{x}\) in the classical sense. Indeed, we see that \(g\) is differentiable at \(\bar{x}\), and

\[
\nabla g(x) = \begin{cases} \frac{10}{3}x^{7/3} \cos \frac{1}{x} + x^{4/3} \sin \frac{1}{x} + 4x^3 & \text{if } x > 1, \\
\frac{10}{3}x^{7/3} \cos \frac{1}{x} + x^{4/3} \sin \frac{1}{x} + \frac{(2n+1)(2n^2+2n+1)}{n^3(n+1)^3} & \text{if } x \in \left(\frac{1}{n+1}, \frac{1}{n}\right), \; n = 1, 2, \ldots \\
0 & \text{if } x = 0, \\
-\nabla g(-x) & \text{if } x \in (-\infty, 0) \setminus \left\{ -\frac{1}{n} \mid n \in \mathbb{N}^* \right\}. \end{cases}
\]

Put \(U = (-1, 1), \; D = (-1, 1) \setminus \left\{ \frac{1}{n} \mid n \in \mathbb{N}^* \right\}, \; \text{and } A = 0\). Then \(\mu(U \setminus D) = 0\), \(g\) is Lipschitz on \(U\) with constant \(\kappa = 1\), and differentiable at every point in \(D\), where \(\mu\) is the Lebesgue measure on \(\mathbb{R}\). Furthermore, for each \(x \in \left(\frac{1}{n+1}, \frac{1}{n}\right)\) with \(n \in \mathbb{N}^*\) we have
\[
\left| \frac{\nabla g(x) - \nabla g(\bar{x}) - A(x - \bar{x})}{|x - \bar{x}|} \right| \leq \left| \frac{10}{3} x^{4/3} \cos \frac{1}{x} + x^{1/3} \sin \frac{1}{x} \right| + \frac{(2n+1)(2n^2+2n+1)}{n^3(n+1)^3|x|} \\
\leq \frac{10}{3} x^{4/3} + x^{1/3} + \frac{(2n+1)(2n^2+2n+1)}{n^3(n+1)^2} \to 0 \quad \text{as } n \to \infty.
\]

Combining this with \( \nabla g(x) = -\nabla g(-x) \) for all \( x \in (-\infty, 0) \setminus \{ -\frac{1}{n} | n \in \mathbb{N}^* \} \), we get

\[
\lim_{x \to \bar{x}} \frac{\nabla g(x) - \nabla g(\bar{x}) - A(x - \bar{x})}{|x - \bar{x}|} = 0.
\]

Hence, \( g \) is twice differentiable at \( \bar{x} \) in the extended sense. On the other hand, since \( g \) is not differentiable at each point \( \frac{1}{n} \) with \( n \in \mathbb{N}^* \), \( g \) is not twice differentiable at \( \bar{x} \) in the classical sense. Therefore, the extended second-order differentiability does not imply the classical second-order differentiability.

**Claim 2:** \( g \) is not prox-regular at \( \bar{x} \) for \( \bar{v} = 0 \). Fix \( r > 0 \) and put \( u_k = \frac{1}{2k\pi} \), \( x_k = \frac{1}{\frac{7}{2} + 2k\pi} \) for every \( k \in \mathbb{N}^* \). Then for each \( k \in \mathbb{N}^* \) there exist \( m_k, n_k \in \mathbb{N}^* \) such that \( u_k \in \left( \frac{1}{m_k+1}, \frac{1}{m_k} \right) \) and \( x_k \in \left( \frac{1}{n_k+1}, \frac{1}{n_k} \right) \). This implies that \( 2k\pi < m_k + 1 \) for all \( k \). So we have

\[
\langle \nabla g(u_k) - \nabla g(x_k), u_k - x_k \rangle + r |u_k - x_k|^2
\]

\[
= \left( \frac{10}{3} \frac{1}{(2k\pi)^{4/3}} + \frac{(2m_k+1)(2m_k^2+2m_k+1)}{m_k^3(m_k+1)^3} \right) - \frac{1}{\left( \frac{7}{2} + 2k\pi \right)^{4/3}} \\
- \frac{(2n_k+1)(2n_k^2+2n_k+1)}{n_k^3(n_k+1)^3} \left( \frac{1}{2k\pi} - \frac{1}{\frac{7}{2} + 2k\pi} \right) + r \left( \frac{1}{2k\pi} - \frac{1}{\frac{7}{2} + 2k\pi} \right)^2
\]

\[
\leq \frac{\pi}{4k\pi} \left( \frac{10}{3} \frac{1}{(2k\pi)^{4/3}} + \frac{5}{(m_k+1)^3} - \frac{1}{\left( \frac{7}{2} + 2k\pi \right)^{4/3}} + r \frac{\pi}{4k\pi} \left( \frac{7}{2} + 2k\pi \right) \right)
\]

\[
\leq \frac{\pi}{4k\pi} \left( \frac{10}{3} \frac{1}{(2k\pi)^{4/3}} + \frac{5}{(2k\pi)^3} - \frac{1}{\left( \frac{7}{2} + 2k\pi \right)^{4/3}} + r \frac{\pi}{4k\pi} \left( \frac{7}{2} + 2k\pi \right) \right) < 0,
\]

for all \( k \) large enough. Note that \( \lim_{k \to \infty} \left( u_k, \nabla g(u_k) \right) = \lim_{k \to \infty} \left( x_k, \nabla g(x_k) \right) = (0, 0) \). Therefore, by Lemma 2.1, \( f \) is not prox-regular at \( \bar{x} = 0 \) for \( \bar{v} = 0 \).

**Example 3.3** (Second-order differentiability does not imply prox-regularity). Consider the function \( f: \mathbb{R} \to \mathbb{R} \) defined by

\[
f(x) := \int_0^x g(t)dt \quad \text{where} \quad g(x) = \begin{cases} x^2 \sin \frac{1}{x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}
\]

We see that \( \nabla f(x) = g(x) \) for all \( x \in \mathbb{R} \), and \( f \) is twice differentiable at every point in \( \mathbb{R} \) with

\[
\nabla^2 f(x) = \nabla^2 g(x) = \begin{cases} 2x \sin \frac{1}{x^2} - \frac{2}{x} \cos \frac{1}{x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}
\]
We next prove that $f$ is not prox-regular at $\bar{x} := 0$ for $\nabla f(\bar{x}) = 0$. Arguing by contradiction, suppose that $f$ is prox-regular at 0 for 0. Then, by Lemma 2.1, there exist $r, \epsilon > 0$ such that

$$
\langle g(u) - g(x), u - x \rangle = \langle \nabla f(u) - \nabla f(x), u - x \rangle \geq -r |u - x|^2,
$$

for every $u, x \in \mathbb{B}_\epsilon(\bar{x})$. Thus, for each $k \in \mathbb{N}^*$ sufficiently large, choosing $u_k = \frac{1}{\sqrt{2k}}$ and $x_k = \frac{1}{\sqrt{2 + 2k}}$, we have

$$
\langle g(u_k) - g(x_k), u_k - x_k \rangle + r |u_k - x_k|^2
= -\frac{1}{2 + 2k^2} \left( \frac{1}{\sqrt{2k}} - \frac{1}{\sqrt{2 + 2k}} \right) + r \left( \frac{1}{\sqrt{2k}} - \frac{1}{\sqrt{2 + 2k}} \right)^2
= \left( \frac{1}{\sqrt{2k}} - \frac{1}{\sqrt{2 + 2k}} \right) \left( -\frac{1}{2 + 2k^2} + r \left( \frac{1}{\sqrt{2k}} - \frac{1}{\sqrt{2 + 2k}} \right) \right)
= \left( \frac{1}{\sqrt{2k}} - \frac{1}{\sqrt{2 + 2k}} \right) \left( -\frac{1}{2 + 2k^2} + r \frac{\sqrt{2 + 2k}}{\sqrt{2k}} \right) < 0.
$$

This contradicts (4) since $\lim_{k \to \infty} u_k = \lim_{k \to \infty} u_k = 0 = \bar{x}$. Therefore, $f$ is not prox-regular at $\bar{x} = 0$ for $\bar{v} = 0$.

The following lemma collects some properties of extended twice differentiable functions that will be used in the sequel.

**Lemma 3.1** [16, Theorem 13.2]. Let $f : \mathbb{R}^n \to \mathbb{R}$ be twice differentiable at a point $\bar{x}$ in the extended sense. Then, $\partial f(\bar{x}) = \{ \nabla f(\bar{x}) \}$ and there exists a neighborhood $U$ of $\bar{x}$ such that

$$
\emptyset \neq \partial f(x) \subset \nabla f(\bar{x}) + \nabla^2 f(\bar{x})(x - \bar{x}) + o(\|x - \bar{x}\|)\mathbb{B},
$$

for every $x \in U$. Furthermore, $f$ is strictly differentiable at $\bar{x}$, and

$$
f(x) = f(\bar{x}) + \langle \nabla f(\bar{x}), x - \bar{x} \rangle + \frac{1}{2} \langle x - \bar{x}, \nabla^2 f(\bar{x})(x - \bar{x}) \rangle + o(\|x - \bar{x}\|^2).
$$

Here, $o(t)$ stands for some function of $t$ with $\lim_{t \to 0} \frac{o(t)}{t} = 0$.

Naturally, we say a mapping $F : \mathbb{R}^n \to \mathbb{R}^m$, $x \mapsto (F_1(x), F_2(x), \ldots, F_m(x))$ is twice differentiable at $\bar{x}$ in the extended sense if $F_k$ is twice differentiable at $\bar{x}$ in the extended sense for every $k = 1, 2, \ldots, m$. In the sequel, for such a mapping $F$, the symbol $\nabla^2 F(\bar{x})(w, v)$ stands for $(\langle \nabla^2 F_1(\bar{x})w, v \rangle, \langle \nabla^2 F_2(\bar{x})w, v \rangle, \ldots, \langle \nabla^2 F_m(\bar{x})w, v \rangle)$ for all $v, w \in \mathbb{R}^n$.

The following theorem provides sum rules of equality form for gradient graphical derivative, second subderivative, and parabolic subderivative.
Theorem 3.1 Suppose that \( \varphi : \mathbb{R}^n \to \mathbb{R} \) is twice differentiable at \( \bar{x} \) in the extended sense, \( \psi : \mathbb{R}^n \to \mathbb{R} \) is proper lower semicontinuous around \( \bar{x} \), and \( \bar{v} \in \partial (\varphi + \psi)(\bar{x}) \). Then, one has

\[
D\partial (\varphi + \psi)(\bar{x}|\bar{v})(w) = \nabla^2 \varphi(\bar{x})(w) + D\partial \psi(\bar{x}|\bar{v} - \nabla \varphi(\bar{x}))(w),
\]

(7)

and

\[
d^2(\varphi + \psi)(\bar{x})(w|z) = \langle w, \nabla^2 \varphi(\bar{x})w \rangle + d^2 \psi(\bar{x}|\bar{v} - \nabla \varphi(\bar{x}))(w|z),
\]

(8)

for every \( w \in \mathbb{R}^n \) and \( z \in \mathbb{R}^n \).

Proof We first prove (7). To this end, take any \( w \in \mathbb{R}^n \) and \( z \in D\partial (\varphi + \psi)(\bar{x}|\bar{v})(w) \). Then, there exist sequences \( t_k \downarrow 0 \) and \( (w_k, z_k) \to (w, z) \) such that

\[
\bar{v} + t_k z_k \in \partial (\varphi + \psi)(\bar{x} + t_k w_k) \quad \text{for all } k \in \mathbb{N}^*.
\]

Since \( \varphi \) is twice differentiable at \( \bar{x} \) in the extended sense, it is Lipschitz continuous around \( \bar{x} \), and by the sum rule of subdifferential [14, Theorem 2.19] and (5), we get

\[
\partial (\varphi + \psi)(\bar{x} + t_k w_k) \subset \partial \varphi(\bar{x} + t_k w_k) + \partial \psi(\bar{x} + t_k w_k)
\subset \nabla \varphi(\bar{x}) + t_k \nabla^2 \varphi(\bar{x})(w_k) + o(\|t_k w_k\|)B + \partial \psi(\bar{x} + t_k w_k),
\]

for all \( k \in \mathbb{N}^* \) sufficiently large. Thus, for such numbers \( k \), it holds that

\[
\bar{v} - \nabla \varphi(\bar{x}) + t_k \left( z_k - \nabla^2 \varphi(\bar{x})(w_k) + \frac{o(\|t_k w_k\|)}{t_k} \right) \in \partial \psi(\bar{x} + t_k w_k),
\]

or equivalently,

\[
\left( \bar{x}, \bar{v} - \nabla \varphi(\bar{x}) \right) + t_k \left( w_k, z_k - \nabla^2 \varphi(\bar{x})(w_k) + \frac{o(\|t_k w_k\|)}{t_k} \right) \in \text{gph} \partial \psi.
\]

On the other hand,

\[
\left( w_k, z_k - \nabla^2 \varphi(\bar{x})(w_k) + \frac{o(\|t_k w_k\|)}{t_k} \right) \to (w, z - \nabla^2 \varphi(\bar{x})(w)) \quad \text{as } k \to \infty.
\]

Therefore,

\[
(w, z - \nabla^2 \varphi(\bar{x})(w)) \in T_{\text{gph} \partial \psi} \left( \bar{x}, \bar{v} - \nabla \varphi(\bar{x}) \right).
\]

In other words,

\[
z - \nabla^2 \varphi(\bar{x})(w) \in D\partial \psi(\bar{x}|\bar{v} - \nabla \varphi(\bar{x})(w)).
\]
This shows that
\[
D\partial (\varphi + \psi)(\bar{x}|\bar{v})(w) \subset \nabla^2 \varphi(\bar{x})(w) + D\partial \psi(\bar{x}|\bar{v} - \nabla \varphi)(\bar{x})(w). \tag{10}
\]

Conversely, by using (10) and noting that \( -\varphi \) is also twice differentiable at \( \bar{x} \) in the extended sense with \( \nabla^2 (-\varphi)(\bar{x}) = -\nabla^2 \varphi(\bar{x}) \), we have
\[
D\partial \psi(\bar{x}|\bar{v} - \nabla \varphi)(\bar{x})(w) = D\partial (\varphi + \psi + (-\varphi))(\bar{x}|\bar{v} - \nabla \varphi)(\bar{x})(w)
\subset D\partial (\varphi + \psi)(\bar{x}|\bar{v})(w) + \nabla^2 (-\varphi)(\bar{x})(w)
= D\partial (\varphi + \psi)(\bar{x}|\bar{v})(w) - \nabla^2 \varphi(\bar{x})(w).
\]

This infers that
\[
\nabla^2 \varphi(\bar{x})(w) + D\partial \psi(\bar{x}|\bar{v} - \nabla \varphi)(\bar{x})(w) \subset D\partial (\varphi + \psi)(\bar{x}|\bar{v})(w). \tag{11}
\]

From (10) and (11), it follows that
\[
D\partial (\varphi + \psi)(\bar{x}|\bar{v})(w) = \nabla^2 \varphi(\bar{x})(w) + D\partial \psi(\bar{x}|\bar{v} - \nabla \varphi)(\bar{x})(w) \text{ for every } w \in \mathbb{R}^n.
\]

We next justify the validity of (8). Take any \( w \in \mathbb{R}^n \). Since \( \varphi \) is twice differentiable at \( \bar{x} \) in the extended sense, by (6), we see that
\[
\{ w, \nabla^2 \varphi(\bar{x})w \} = \lim_{w' \to w} \Delta^2 \varphi(\bar{x}|\nabla \varphi(\bar{x}))(w').
\]

Therefore,
\[
d^2 (\varphi + \psi)(\bar{x}|\bar{v})(w) = \liminf_{w' \to w} \Delta^2 (\varphi + \psi)(\bar{x}|\bar{v})(w')
= \liminf_{w' \to w} \left[ \Delta^2 (\bar{x}|\nabla \varphi(\bar{x}))(w') + \Delta^2 \psi(\bar{x}|\bar{v} - \nabla \varphi(\bar{x}))(w') \right]
= \{ w, \nabla^2 \varphi(\bar{x})w \} + \liminf_{w' \to w} \Delta^2 \psi(\bar{x}|\bar{v} - \nabla \varphi(\bar{x}))(w')
= \{ w, \nabla^2 \varphi(\bar{x})w \} + d^2 \psi(\bar{x}|\bar{v} - \nabla \varphi(\bar{x}))(w).
\]

Finally, we show that (9) holds. The differentiability of \( \varphi \) at \( \bar{x} \) gives us that
\[
d (\varphi + \psi)(\bar{x})(w) = \liminf_{w' \to w} \frac{(\varphi + \psi)(\bar{x} + tw') - (\varphi + \psi)(\bar{x})}{t}
= \liminf_{w' \to w} \left[ \frac{\varphi(\bar{x} + tw') - \varphi(\bar{x})}{t} + \frac{\psi(\bar{x} + tw') - \psi(\bar{x})}{t} \right]
= \nabla \varphi(\bar{x})w + \liminf_{w' \to w} \frac{\psi(\bar{x} + tw') - \psi(\bar{x})}{t}
= \nabla \varphi(\bar{x})w + d \psi(\bar{x})(w) \quad \forall w \in \mathbb{R}^n.
\]
Since \( \varphi \) is twice differentiable at \( \bar{x} \) in the extended sense, by (6), we get

\[
\lim_{z' \to z} \varphi(\bar{x} + tw + \frac{1}{2}t^2 z') - \varphi(\bar{x}) - t \nabla \varphi(\bar{x}) w = \langle w, \nabla^2 \varphi(\bar{x}) w \rangle + \nabla \varphi(\bar{x})z \quad \forall w \in \mathbb{R}^n, z \in \mathbb{R}^n.
\]

Therefore,

\[
d^2(\varphi + \psi)(\bar{x})(w|z) = \liminf_{z' \to z} \frac{(\varphi + \psi)(\bar{x} + tw + \frac{1}{2}t^2 z') - (\varphi + \psi)(\bar{x}) - td(\varphi + \psi)(\bar{x})(w)}{\frac{1}{2}t^2}
\]

\[
= \liminf_{z' \to z} \left[ \frac{\varphi(\bar{x} + tw + \frac{1}{2}t^2 z') - \varphi(\bar{x}) - t \nabla \varphi(\bar{x}) w}{\frac{1}{2}t^2} + \frac{\psi(\bar{x} + tw + \frac{1}{2}t^2 z') - \psi(\bar{x}) - td\psi(\bar{x})(w)}{\frac{1}{2}t^2} \right]
\]

\[
= \langle w, \nabla^2 \varphi(\bar{x}) w \rangle + \nabla \varphi(\bar{x})z + \liminf_{z' \to z} \frac{\psi(\bar{x} + tw + \frac{1}{2}t^2 z') - \psi(\bar{x}) - td\psi(\bar{x})(w)}{\frac{1}{2}t^2}
\]

\[
= \langle w, \nabla^2 \varphi(\bar{x}) w \rangle + \nabla \varphi(\bar{x})z + d^2(\psi)(\bar{x})(w|z) \quad \forall w \in \mathbb{R}^n, z \in \mathbb{R}^n.
\]

This finishes the proof. \( \square \)

Let \( \psi : \mathbb{R}^n \to \overline{\mathbb{R}} \) be finite at \( \bar{x} \in \mathbb{R}^n \). Assume that there exists a neighborhood \( O \) of \( \bar{x} \) on which \( \psi \) can be represented as

\[
\psi(x) = g \circ F(x) \quad \text{for all } x \in O,
\]

where \( F : \mathbb{R}^n \to \mathbb{R}^m \) is twice differentiable at \( \bar{x} \) in the extended sense, and \( g : \mathbb{R}^m \to \overline{\mathbb{R}} \) is proper lower semicontinuous, convex, and Lipschitz continuous around \( F(\bar{x}) \) relative to its domain with constant \( \ell \in \mathbb{R}_+^* \), that is, there exists a neighborhood \( V \) of \( F(\bar{x}) \) such that \( |g(y_1) - g(y_2)| \leq \ell \| y_1 - y_2 \| \) for all \( y_1, y_2 \in \text{dom } g \cap V \).

We see that

\[
(\text{dom } \psi) \cap O = \{ x \in O \mid F(x) \in \text{dom } g \}. \quad (13)
\]

Following [10, Definition 3.2], the composition \( \psi = g \circ F \) is said to satisfy the metric subregularity qualification condition (MSQC) at \( \bar{x} \in \text{dom } \psi \) if there exist a constant \( \kappa \in \mathbb{R}_+ \) and a neighborhood \( U \) of \( \bar{x} \) such that

\[
d(x, \text{dom } \psi) \leq \kappa d(F(x), \text{dom } g) \quad \text{for all } x \in U. \quad (14)
\]

**Proposition 3.1** Let \( \psi : \mathbb{R}^n \to \overline{\mathbb{R}} \) be a function that is represented as (13) with the composition \( g \circ F \) satisfying MSQC at \( \bar{x} \). Then, we have

\[
d(\psi)(\bar{x})(w) = d(g(F(\bar{x}))(\nabla F(\bar{x}) w) \quad \text{for all } w \in \mathbb{R}^n, \quad \partial_p \psi(\bar{x}) = \partial \psi(\bar{x}) = \nabla F(\bar{x})^* \partial g(F(\bar{x})),
\]
and $T_{\text{dom } \psi}(\bar{x}) = \{ w \in \mathbb{R}^n \mid \nabla F(\bar{x}) w \in T_{\text{dom } g}(F(\bar{x})) \}$.

If we assume further that $w \in T_{\text{dom } \psi}(\bar{x})$ and $g$ is parabolically epi-differentiable at $F(\bar{x})$ for $\nabla F(\bar{x}) w$, then the following assertions hold:

(i) $z \in T^2_{\text{dom } \psi}(\bar{x}, w) \Leftrightarrow \nabla F(\bar{x}) z + \nabla^2 F(\bar{x})(w, w) \in T^2_{\text{dom } g}(F(\bar{x}), \nabla F(\bar{x}) w)$, and

(ii) $\nabla \psi(\bar{x})$ is parabolically epi-differentiable at $\bar{x}$ for $\nabla F(\bar{x}) w$;

(iii) $\text{dom } d^2 \psi(\bar{x})(w) = T^2_{\text{dom } \psi}(\bar{x}, w)$;

(iv) $\psi$ is parabolically epi-differentiable at $\bar{x}$ for $\nabla F(\bar{x}) w$.

**Proof** Since $F: \mathbb{R}^n \to \mathbb{R}^m$ is twice differentiable at $\bar{x}$ in the extended sense, by Lemma 3.1, we get

$$F(x) = F(\bar{x}) + \langle \nabla F(\bar{x}), x - \bar{x} \rangle + \frac{1}{2} \nabla^2 F(\bar{x})(x - \bar{x}, x - \bar{x}) + o(\|x - \bar{x}\|^2) \quad (15)$$

and $F$ is strictly differentiable at $\bar{x}$. The latter along with the composition $g \circ F$ satisfying MSQC at $\bar{x}$ implies by [10, Theorem 3.4] that

$$d \psi(\bar{x})(w) = d g(F(\bar{x}))(\nabla F(\bar{x}) w) \text{ for all } w \in \mathbb{R}^n, \quad (16)$$

and by [10, Theorem 3.6] that

$$\partial_p \psi(\bar{x}) \subset \partial \psi(\bar{x}) = \nabla F(\bar{x})^* \partial g(F(\bar{x})). \quad (17)$$

We next prove that $\nabla F(\bar{x})^* \partial g(F(\bar{x})) \subset \partial_p \psi(\bar{x})$. To this end, take any $y \in \partial g(F(\bar{x}))$. Since $g$ is convex, we have $y \in \partial_p g(F(\bar{x}))$. Hence,

$$\liminf_{x \to \bar{x}} \frac{\psi(x) - \psi(\bar{x}) - \nabla F(\bar{x})^* y, x - \bar{x}}{\|x - \bar{x}\|^2} = \liminf_{x \to \bar{x}} \frac{g(F(\bar{x}) + \nabla F(\bar{x})(x - \bar{x}) + \frac{1}{2} \nabla^2 F(\bar{x})(x - \bar{x}, x - \bar{x}) + o(\|x - \bar{x}\|^2)) - g(F(\bar{x})) - \nabla y, F(\bar{x})(x - \bar{x})}{\|x - \bar{x}\|^2}$$

$$= \liminf_{x \to \bar{x}} \frac{g(F(\bar{x}) + \Delta(x)) - g(F(\bar{x})) - \nabla y, F(\bar{x})(x - \bar{x}, x - \bar{x})}{\|x - \bar{x}\|^2} \geq \liminf_{x \to \bar{x}} \frac{g(F(\bar{x}) + \Delta(x)) - g(F(\bar{x})) - \nabla y, F(\bar{x})(x - \bar{x}, x - \bar{x})}{\|x - \bar{x}\|^2} - \frac{1}{2} \|y\| \cdot \|\nabla^2 F(\bar{x})\| \to -\infty,$$

where $\Delta(x) := \nabla F(\bar{x})(x - \bar{x}) + \frac{1}{2} \nabla^2 F(\bar{x})(x - \bar{x}, x - \bar{x}) + o(\|x - \bar{x}\|^2) \to 0$ as $x \to \bar{x}$. This shows that $\nabla F(\bar{x})^* y \in \partial_p \psi(\bar{x})$, and thus

$$\nabla F(\bar{x})^* \partial g(F(\bar{x})) \subset \partial_p \psi(\bar{x}). \quad (18)$$

From (17) and (18), it follows that

$$\partial_p \psi(\bar{x}) = \partial \psi(\bar{x}) = \nabla F(\bar{x})^* \partial g(F(\bar{x})).$$
Furthermore, by (16) and [12,Proposition 2.2], we get

$$T_{\text{dom} \psi}(\bar{x}) = \text{dom } d\psi(\bar{x}) = \big\{ w \in \mathbb{R}^n \mid \nabla F(\bar{x})w \in \text{dom } dg(F(\bar{x})) \big\} = \big\{ w \in \mathbb{R}^n \mid \nabla F(\bar{x})w \in T_{\text{dom } g}(F(\bar{x})) \big\}. $$

Let us now suppose further that $w \in T_{\text{dom} \psi}(\bar{x})$ and $g$ is parabolically epi-differentiable at $F(\bar{x})$ for $\nabla F(\bar{x})w$. Since $g$ is Lipschitz continuous around $F(\bar{x})$ relative to its domain, and $\nabla F(\bar{x})w \in T_{\text{dom } g}(F(\bar{x}))$, by [12,Proposition 4.1], $\text{dom } g$ is parabolically derivable at $F(\bar{x})$ for $\nabla F(\bar{x})w$. Hence, the proofs of (i) and (ii)–(iv) can be, respectively, done as the ones of [11,Theorem 4.5] and [12,Theorem 4.4], where $F$ was assumed to be twice differentiable at $\bar{x}$, but they actually needed the quadratic expansion of (15) and the strict differentiability of $F$ at $\bar{x}$, which are valid under the second-order differentiability in the extended sense. □

In order to prove the next proposition, we need the following lemma whose proof is the one of [12,Proposition 4.6]. For completeness, we provide the proof with more details.

**Lemma 3.2** Suppose $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is a proper lower semicontinuous convex function with $f(\bar{x}) \in \mathbb{R}$, $\bar{v} \in \partial f(\bar{x})$, $w \in K_f(\bar{x}, \bar{v})$, and $f$ is parabolically epi-differentiable at $\bar{x}$ for $w$. Then, $d^2 f(\bar{x})(w\cdot)$ is proper lower semicontinuous and convex. Furthermore, $d^2 f(\bar{x})(w\cdot)^*(v) = \infty$ whenever $v \in \mathbb{R}^n \setminus A(\bar{x}, w)$, and $d^2 f(\bar{x})(w\cdot)^*(v) = -d^2 f(\bar{x}, v)(w)$ if $v \in A(\bar{x}, w)$ and $f$ is parabolically regular at $\bar{x}$ for $v$, where $A(\bar{x}, w) := \{ v \in \partial f(\bar{x}) \mid d f(\bar{x})(w) = \langle v, w \rangle \}$ and $d^2 f(\bar{x})(w\cdot)^*$ is the Fenchel conjugate of $d^2 f(\bar{x})(w\cdot)$.

**Proof** Since $f$ is a lower semicontinuous function, $f(\bar{x}) \in \mathbb{R}$ and $d f(\bar{x})(w) = \langle \bar{v}, w \rangle \in \mathbb{R}$, by [16,Proposition 13.64], $d^2 f(\bar{x})(w\cdot)$ is lower semicontinuous and

$$d^2 f(\bar{x})(w\cdot) - \langle \bar{v}, z \rangle \geq d^2 f(\bar{x}, \bar{v})(w) \quad \forall z \in \mathbb{R}^n.$$

Noting that $f$ is convex and $\bar{v} \in \partial f(\bar{x})$, we have

$$d^2 f(\bar{x}, \bar{v})(w) = \liminf_{w' \to w} \frac{f(\bar{x} + tw') - f(\bar{x}) - t \langle \bar{v}, w' \rangle}{\frac{1}{2} t^2} \geq 0. \quad (19)$$

Thus, $d^2 f(\bar{x})(w\cdot)^* > -\infty$ for all $z \in \mathbb{R}^n$. Combining this with $\text{dom } d^2 f(\bar{x})(w\cdot) \neq \emptyset$ (due to the parabolic epi-differentiability of $f$ at $\bar{x}$ for $w$), we see that $d^2 f(\bar{x})(w\cdot)$ is a proper function. By [16,Example 13.62],

$$\text{epi } d^2 f(\bar{x})(w\cdot) = T_{\text{epi } f}(\bar{x}, f(\bar{x}))(w, df(\bar{x})(w)),$$

and since $f$ is parabolically epi-differentiable at $\bar{x}$ for $w$, $\text{epi } f$ is parabolically derivable at $(\bar{x}, f(\bar{x}))$ for $(w, df(\bar{x})(w))$. This implies that $d^2 f(\bar{x})(w\cdot)$ is convex since $f$ is convex.
Take any \( v \in \mathbb{R}^n \). Let us consider the following two cases.

**Case 1** \( v \in \mathcal{A}(\bar{x}, w) \). Then \( w \in K_f(\bar{x}, v) \) and by [12, Proposition 3.6], we have

\[
-d^2 f(\bar{x}, v)(w) = -\inf_{z \in \mathbb{R}^n} \{ d^2 f(\bar{x})(w|z) - \langle v, z \rangle \} = d^2 f(\bar{x})(w|\cdot)^*(v),
\]

due to the parabolic regularity of \( f \) at \( \bar{x} \) for \( v \).

**Case 2** \( v \in \mathbb{R}^n \setminus \mathcal{A}(\bar{x}, w) \). Then either \( v \notin \partial f(\bar{x}) \) or \( d f(\bar{x})(w) \neq \langle v, w \rangle \). Put

\[
v_t(z) := \frac{f(\bar{x}+tw + \frac{1}{2}t^2 z) - f(\bar{x}) - t d f(\bar{x})(w)}{\frac{1}{2} t^2} \quad \forall z \in \mathbb{R}^n, t > 0.
\]

We have

\[
v^*_t(v) = \frac{f(\bar{x}) + f^*(v) - \langle v, \bar{x} \rangle}{\frac{1}{2} t^2} + \frac{d f(\bar{x})(w) - \langle v, w \rangle}{\frac{1}{2} t} \quad \forall v \in \mathbb{R}^n, t > 0.
\]

Since \( f \) is parabolically epi-differentiable at \( \bar{x} \) for \( w \), by [16, Example 13.59], epi \( v_t \) converges to epi \( d^2 f(\bar{x})(w|\cdot) \) as \( t \downarrow 0 \). Noting that \( v_t(\cdot) \) and \( d^2 f(\bar{x})(w|\cdot) \) are proper lower semicontinuous and convex functions, by [16, Theorem 11.34], the latter implies that epi \( v^*_t \) converges to epi \( d^2 f(\bar{x})(w|\cdot)^* \) as \( t \downarrow 0 \). So, for any sequence \( t_k \downarrow 0 \), by [16, Proposition 7.2], there exists a sequence \( v_k \to v \) such that

\[
d^2 f(\bar{x})(w|\cdot)^*(v) = \lim_{k \to \infty} v^*_t(v_k).
\]

If \( v \notin \partial f(\bar{x}) \) then \( f(\bar{x}) + f^*(v) - \langle v, \bar{x} \rangle > 0 \). Thus, by lower semicontinuity of \( f^* \), we see that

\[
d^2 f(\bar{x})(w|\cdot)^*(v) = \lim_{k \to \infty} v^*_t(v_k)
= \lim_{k \to \infty} \frac{1}{t_k} \left( \frac{f(\bar{x}) + f^*(v_k) - \langle v_k, \bar{x} \rangle}{t_k} + d f(\bar{x})(w) - \langle v_k, w \rangle \right)
= \infty.
\]

If \( d f(\bar{x})(w) \neq \langle v, w \rangle \) then, by (19) and [16, Proposition 13.5], \( \langle v, w \rangle < d f(\bar{x})(w) \).

On the other hand, we have

\[
f(\bar{x}) + f^*(v_k) - \langle v_k, \bar{x} \rangle = f(\bar{x}) + \sup_{x \in \mathbb{R}^n} \{ \langle v_k, x \rangle - f(x) \} - \langle v_k, \bar{x} \rangle \geq 0 \quad \forall k.
\]

Therefore,

\[
d^2 f(\bar{x})(w|\cdot)^*(v) = \lim_{k \to \infty} v^*_t(v_k)
= \lim_{k \to \infty} \left( \frac{f(\bar{x}) + f^*(v_k) - \langle v_k, \bar{x} \rangle}{\frac{1}{2} t_k^2} + \frac{d f(\bar{x})(w) - \langle v_k, w \rangle}{\frac{1}{2} t_k} \right)
\geq \lim_{k \to \infty} \frac{d f(\bar{x})(w) - \langle v_k, w \rangle}{\frac{1}{2} t_k} = \infty.
\]
So, we arrive at the desired conclusion. □

Following Mohammadi and Sarabi [12], we say that function $\psi(x) := g \circ F$ with $(\bar{x}, \bar{v}) \in \text{gph } \partial \psi$ satisfies the basic assumptions at $(\bar{x}, \bar{v})$ if the following conditions hold:

(H1) the metric subregularity qualification condition (14) is valid at $\bar{x}$;

(H2) for each $y \in L(\bar{x}, \bar{v})$, $g$ is parabolically epi-differentiable at $F(\bar{x})$ for every $u \in K_g(F(\bar{x}), y)$;

(H3) $g$ is parabolically regular at $F(\bar{x})$ for every $y \in L(\bar{x}, \bar{v})$.

Here

$$L(\bar{x}, \bar{v}) := \{ y \in \partial g(F(\bar{x})) \mid \nabla F(\bar{x})^* y = \bar{v} \}$$

and

$$K_g(F(\bar{x}), y) := \{ w \in \mathbb{R}^m \mid dg(F(\bar{x}))(w) = (\bar{v}, w) \}$$

are the set of Lagrangian multipliers associated with $(\bar{x}, \bar{v})$, and the critical cone of $g$ at $(F(\bar{x}), y)$, respectively.

Let us consider the following optimization problem:

$$\min_{x \in \mathbb{R}^n} -\langle z, \bar{v} \rangle + d^2 g(F(\bar{x}))(\nabla F(\bar{x})w | \nabla F(\bar{x})z + \nabla^2 F(\bar{x})(w, w)). \quad (20)$$

**Proposition 3.2** Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a function that is represented as (12) with the composition $\psi = g \circ F$ satisfying the basic assumptions (H1)-(H3) at $(\bar{x}, \bar{v})$. Then, the following assertions hold:

(i) For each $w \in K_\psi(\bar{x}, \bar{v})$, the dual problem of (20) is

$$\max_{y \in L(\bar{x}, \bar{v})} \left\{ y, \nabla^2 F(\bar{x})(w, w) \right\} + d^2 g(F(\bar{x}), y)(\nabla F(\bar{x}))(w) ; \quad (21)$$

the optimal values of the primal and dual optimization problems (20) and (21) are equal and finite. Furthermore, $L(\bar{x}, \bar{v}, w) \cap \tau B \neq \emptyset$, where $L(\bar{x}, \bar{v}, w)$ is the optimal solution set of (21) and

$$\tau := \kappa \ell \| \nabla F(\bar{x}) \| + \kappa \| \bar{v} \| + \ell \quad (22)$$

with $\ell$ and $\kappa$ given in (12) and (14), respectively.

(ii) $\psi$ is parabolically regular at $\bar{x}$ for $\bar{v}$, and

$$d^2 \psi(\bar{x}, \bar{v})(w) = \max_{y \in L(\bar{x}, \bar{v})} \left\{ \left\{ y, \nabla^2 F(\bar{x})(w, w) \right\} + d^2 g(F(\bar{x}), y)(\nabla F(\bar{x})w) \right\}$$

$$= \max_{y \in L(\bar{x}, \bar{v}) \cap \tau B} \left\{ \left\{ y, \nabla^2 F(\bar{x})(w, w) \right\} + d^2 g(F(\bar{x}), y)(\nabla F(\bar{x})w) \right\},$$

for every $w \in \mathbb{R}^n$, where $\tau$ is given by (22).

(iii) $\psi$ is twice epi-differentiable at $\bar{x}$ for $\bar{v}$. 
Under the assumption of Proposition 3.2, Remark 3.1 only for all $F$ of case where $F$ regularity of Lemma 3.2, we see that $[12,Theorem 3.8]$, follows that the Fenchel dual problem of $(20)$ is a proper lower semicontinuous convex function. Hence, from $[16,Example 11.41]$ it follows that the Fenchel dual problem of $(20)$ is

$$
\max_{\nabla F(\bar{x})^*y = \bar{v}} \left\{ y, \nabla^2 F(\bar{x})(w, w) \right\} - d^2 g(F(\bar{x}))(\nabla F(\bar{x})w)^*(y). \tag{23}
$$

Pick any $y \in \mathbb{R}^m$ with $\nabla F(\bar{x})^*y = \bar{v}$. If $y \notin \partial g(F(\bar{x}))$ then, by Lemma 3.2,

$$
d^2 g(F(\bar{x}))(\nabla F(\bar{x})w)^*(y) = \infty. \tag{24}
$$

Otherwise, we get $y \in \Lambda(\bar{x}, \bar{v})$. Then, by $(H2)$, $g$ is parabolically regular at $F(\bar{x})$ for $y$. Note that $y \in \partial g(F(\bar{x}))$ and $d g(F(\bar{x}))(\nabla F(\bar{x})w) = \langle y, \nabla F(\bar{x})w \rangle$. So, by Lemma 3.2, we see that

$$
d^2 g(F(\bar{x}))(\nabla F(\bar{x})w)^*(y) = -d^2 g(F(\bar{x}), y)(w). \tag{25}
$$

From (24) and (25), it follows that problem (23) can be written as problem (21). The rest of the proof (i) runs as the one of $[12,Theorem 5.2]$, and the proof of (ii) is similar to the proof of $[12,Theorem 5.4]$. So, they are omitted. Finally, we see that, by (ii), $\psi$ is parabolically regular at $\bar{x}$ for $\bar{v} \in \partial \psi(\bar{x}) = \partial_p \psi(\bar{x})$, and, by $[12,Theorem 4.4]$, $\psi$ is parabolically epi-differentiable at $\bar{x}$ for every $w \in K_{\psi}(\bar{x}, \bar{v})$. Therefore, by $[12,Theorem 3.8]$, $\psi$ is twice epi-differentiable at $\bar{x}$ for $\bar{v}$. $\square$

**Remark 3.1** Under the assumption of Proposition 3.2, $g$ is parabolically regular at $(\bar{x})$ only for all $y \in \partial g(F(\bar{x}))$ with $\nabla F(\bar{x})^*y = \bar{v}$. Thus, we cannot apply $[12,Proposition 4.6]$ to transforming (23) into (21), since $[12,Proposition 4.6]$ requires the parabolic regularity of $g$ at $F(\bar{x})$ for every $y \in \partial g(F(\bar{x}))$. That is why Lemma 3.2 is utilized. We note that the results in Propositions 3.1 and 3.2 were established in $[12]$ for the case where $F$ is twice differentiable in the classical sense.

### 4 Quadratic Growth and Strong Metric Subregularity of the Subdifferential

Let $f : \mathbb{R}^n \to \mathbb{R}$ and $\bar{x} \in \text{dom } f$. Recall that $\bar{x}$ is said to be a *strong local minimizer* of $f$ with modulus $\kappa > 0$ if for some $\gamma > 0$ the following *quadratic growth condition* (QGC) holds:

$$
f(x) - f(\bar{x}) \geq \frac{\kappa}{2} \|x - \bar{x}\|^2 \quad \text{for all} \quad x \in B_\gamma(\bar{x}). \tag{26}
$$
The exact modulus for QGC of $f$ at $\bar{x}$ is given by
\[
\text{QG}(f; \bar{x}) := \sup \{ \kappa > 0 \mid \bar{x} \text{ is a strong local minimizer of } f \text{ with modulus } \kappa \}. \tag{iii}
\]

One says the subdifferential mapping $\partial f$ is strongly metrically subregular at $\bar{x}$ for $0 \in \partial f(\bar{x})$ with modulus $\kappa > 0$ if there exists $\varepsilon > 0$ such that
\[
\kappa d(0; \partial f(x)) \geq \|x - \bar{x}\| \quad \text{for all } x \in B_\varepsilon(\bar{x}),
\]
where $d(0; \partial f(x))$ is the distance from 0 to the set of subgradients $\partial f(x)$. If there exists $\kappa > 0$ such that $\partial f$ is strongly metrically subregular at $\bar{x}$ for $0 \in \partial f(\bar{x})$ with modulus $\kappa$, then $\partial f$ is said to be strongly metrically subregular at $\bar{x}$ for 0.

**Lemma 4.1** [8, Corollary 3.3]. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a proper lower semicontinuous function and let $\bar{x} \in \text{dom } f$ with $0 \in \partial f(\bar{x})$. Suppose that $\partial f$ is strongly metrically subregular at $\bar{x}$ for 0 with modulus $\kappa > 0$ and there are real numbers $r \in (0, \kappa^{-1})$ and $\delta > 0$ such that
\[
f(x) \geq f(\bar{x}) - r \frac{1}{2} \|x - \bar{x}\|^2 \quad \text{for all } x \in B_\delta(\bar{x}).
\]
Then for any $\alpha \in (0, \kappa^{-1})$, there exists a real number $\eta > 0$ such that
\[
f(x) \geq f(\bar{x}) + \alpha \frac{1}{2} \|x - \bar{x}\|^2 \quad \text{for all } x \in B_\eta(\bar{x}).
\]

**Lemma 4.2** [5, Lemma 3.6]. Let $h : \mathbb{R}^n \to \mathbb{R}$ be a proper function. Suppose that $h$ is positively homogeneous of degree 2 in the sense that $h(\lambda w) = \lambda^2 h(w)$ for all $\lambda > 0$ and $w \in \text{dom } h$. Then for any $w \in \text{dom } h$ and $z \in \partial h(w)$, we have $\langle z, w \rangle = 2h(w)$.

The following result provides some characterizations of the quadratic growth and the strong metric subregularity of the subdifferential.

**Theorem 4.1** Let $f : \mathbb{R}^n \to \mathbb{R}$ be the function defined by $f(x) = \varphi(x) + \psi(x)$ for every $x \in \mathbb{R}^n$, where $\varphi : \mathbb{R}^n \to \mathbb{R}$ is twice differentiable at $\bar{x}$ in the extended sense, $0 \in \nabla \varphi(\bar{x}) + \partial \psi(\bar{x})$, and $\psi : \mathbb{R}^n \to \mathbb{R}$ is subdifferentially continuous, prox-regular, and twice epi-differentiable at $\bar{x}$ for $-\nabla \varphi(\bar{x})$. Then, the following assertions are equivalent:

(i) The quadratic growth condition (26) is satisfied.
(ii) $\partial f$ is strongly metrically subregular at $\bar{x}$ for 0, and
\[
\langle \nabla^2 \varphi(\bar{x})w, w \rangle + d^2 \psi(\bar{x}) - \nabla \varphi(\bar{x})(w) \geq 0 \quad \text{for all } w \in \mathbb{R}^n. \tag{27}
\]
(iii) $\partial f$ is strongly metrically subregular at $\bar{x}$ for 0, and $\bar{x}$ is a local minimizer of $f$.
(iv) For all $w \in \text{dom } D\partial \psi(\bar{x}) - \nabla \varphi(\bar{x}))\setminus \{0\}$ and $z \in D\partial \psi(\bar{x}) - \nabla \varphi(\bar{x}))(w)$, we have
\[
\langle \nabla^2 \varphi(\bar{x})w, w \rangle + \langle z, w \rangle > 0.
\]
There exists a real number $c > 0$ such that
\[
\langle \nabla^2 \varphi(\bar{x}) w, w \rangle + \langle z, w \rangle \geq c \|w\|^2,
\]
for all $w \in \text{dom} D\partial \psi(\bar{x} - \nabla \varphi(\bar{x}))$ and $z \in D\partial \psi(\bar{x} - \nabla \varphi(\bar{x}))(w)$.

For every $w \in \mathbb{R}^n \setminus \{0\}$, we have
\[
(\nabla^2 \varphi(\bar{x}) w, w) + d^2 \psi(\bar{x} - \nabla \varphi(\bar{x}))(w) > 0.
\]

If one of the above assertions holds then
\[
\text{QG}(f; \bar{x}) = \inf \left\{ \frac{(\nabla^2 \varphi(\bar{x}) w, w) + \langle z, w \rangle}{\|w\|^2} \mid w \in \text{dom} D\partial \psi(\bar{x} - \nabla \varphi(\bar{x})), z \in D\partial \psi(\bar{x} - \nabla \varphi(\bar{x}))(w) \right\},
\]
with the convention that $0/0 = \infty$.

\textbf{Proof} Under our assumption, by Theorem 3.1, we have
\[
D\partial (\varphi + \psi)(\bar{x}|0)(w) = \nabla^2 \varphi(\bar{x})(w) + D\partial \psi(\bar{x} - \nabla \varphi(\bar{x}))(w),
\]
and
\[
d^2 (\varphi + \psi)(\bar{x}|0)(w) = \{w, \nabla^2 \varphi(\bar{x})w\} + d^2 \psi(\bar{x} - \nabla \varphi(\bar{x}))(w),
\]
for every $w \in \mathbb{R}^n$. By (30) and [16, Theorem 13.24], we see that (i) $\iff$ (vi) and (iii) $\Rightarrow$ (ii).

We next prove that $0 \in \partial_p f(\bar{x})$. Since $\psi$ is subdifferentially continuous and prox-regular at $\bar{x}$ for $-\nabla \varphi(\bar{x})$, we get
\[
\liminf_{x \to \bar{x}} \frac{\psi(x) - \psi(\bar{x}) + \langle \nabla \varphi(\bar{x}), x - \bar{x} \rangle}{\|x - \bar{x}\|^2} > -\infty.
\]
On the other hand, from the extended second-order differentiability of $\varphi$ at $\bar{x}$, by (6), it follows that
\[
\varphi(x) = \varphi(\bar{x}) + \langle \nabla \varphi(\bar{x}), x - \bar{x} \rangle + \frac{1}{2} \langle x - \bar{x}, \nabla^2 \varphi(\bar{x})(x - \bar{x}) \rangle + o(\|x - \bar{x}\|^2),
\]
which gives us the following estimations
\[
\liminf_{x \to \bar{x}} \frac{\varphi(x) - \varphi(\bar{x}) - \langle \nabla \varphi(\bar{x}), x - \bar{x} \rangle}{\|x - \bar{x}\|^2} = \liminf_{x \to \bar{x}} \frac{\frac{1}{2} \langle x - \bar{x}, \nabla^2 \varphi(\bar{x})(x - \bar{x}) \rangle + o(\|x - \bar{x\}|^2)}{\|x - \bar{x}\|^2} \geq -\frac{1}{2} \|\nabla^2 \varphi(\bar{x})\| > -\infty.
\]
Therefore,
\[
\liminf_{x \to \tilde{x}} \frac{f(x) - f(\tilde{x})}{\|x - \tilde{x}\|^2} = \liminf_{x \to \tilde{x}} \left[ \frac{\varphi(x) - \varphi(\tilde{x}) - \langle \nabla \varphi(\tilde{x}), x - \tilde{x} \rangle}{\|x - \tilde{x}\|^2} + \frac{\psi(x) - \psi(\tilde{x}) + \langle \nabla \varphi(\tilde{x}), x - \tilde{x} \rangle}{\|x - \tilde{x}\|^2} \right] \\
\geq \liminf_{x \to \tilde{x}} \frac{\varphi(x) - \varphi(\tilde{x}) - \langle \nabla \varphi(\tilde{x}), x - \tilde{x} \rangle}{\|x - \tilde{x}\|^2} + \liminf_{x \to \tilde{x}} \frac{\psi(x) - \psi(\tilde{x}) + \langle \nabla \varphi(\tilde{x}), x - \tilde{x} \rangle}{\|x - \tilde{x}\|^2} > -\infty.
\]

This shows that \(0 \in \partial_{p} f(\tilde{x})\).

Hence, by (29) and [5, Theorem 3.2], implication \((v) \Rightarrow (i) \Rightarrow (iii)\) holds, and
\[
QG(f; \tilde{x}) \geq \inf \left\{ \frac{\langle \nabla^2 \varphi(\tilde{x})w, w \rangle + \langle z, w \rangle}{\|w\|^2} \mid w \in \text{dom}D\partial \psi(\tilde{x}| - \nabla \varphi(\tilde{x})), \ z \in D\partial \psi(\tilde{x}| - \nabla \varphi(\tilde{x}))(w) \right\}.
\]  

(31)

We now prove \((ii) \Rightarrow (i)\). Suppose \(\partial f\) is strongly metrically subregular at \(\tilde{x}\) for 0 with modulus \(\kappa > 0\), and (27) holds. By (30), we get
\[
d^2 f(\tilde{x}|0)(w) \geq 0 \text{ for all } w \in \mathbb{R}^n.
\]  

(32)

Fix an arbitrary \(r \in (0, \kappa^{-1})\). Then, there exists a real number \(\delta > 0\) such that
\[
f(x) \geq f(\tilde{x}) - \frac{r}{2} \|x - \tilde{x}\|^2 \text{ for all } x \in \mathbb{B}_\delta(\tilde{x}).
\]  

(33)

Indeed, suppose by contrary that this claim does not hold. Then, for each \(k \in \mathbb{N}\), there exists \(x_k \in \mathbb{B}_{1/k}(\tilde{x})\) with
\[
f(x_k) < f(\tilde{x}) - \frac{r}{2} \|x_k - \tilde{x}\|^2.
\]

Put \(t_k := \|x_k - \tilde{x}\|\) and \(w_k := t_k^{-1}(x_k - \tilde{x})\) for \(k \in \mathbb{N}\). We see that \(_k \downarrow 0\) as \(k \to \infty\). Furthermore, passing to a subsequence if necessary, we may assume that \(\{w_k\}\) converges to some \(\tilde{w} \in \mathbb{R}^n\) as \(k \to \infty\). So we have
\[
d^2 f(\tilde{x}|0)(\tilde{w}) = \liminf_{\tau \downarrow 0} \frac{f(\tilde{x} + \tau w) - f(\tilde{x}) - \tau \langle 0, w \rangle}{\frac{1}{2} \tau^2} \\
\leq \liminf_{k \to \infty} \frac{f(\tilde{x} + t_k w_k) - f(\tilde{x})}{\frac{1}{2} t_k^2} \\
= \liminf_{k \to \infty} \frac{f(x_k) - f(\tilde{x})}{\frac{1}{2} \|x_k - \tilde{x}\|^2} \leq -\frac{r}{2} < 0.
\]
This contradicts (32). Therefore, there exists a real number \( \delta > 0 \) such that (33) holds. By Lemma 4.1, the quadratic growth condition (26) holds, and we have \((ii) \implies (i)\).

Finally, we prove \((i) \implies (v)\) and

\[
\text{QG}(f; \bar{x}) \leq \inf \left\{ \frac{\langle \nabla^2 \varphi(\bar{x}) w, w \rangle + \langle z, w \rangle}{\|w\|^2} \mid w \in \text{dom} D \partial \psi(\bar{x} | - \nabla \varphi(\bar{x})), \ z \in D \partial \psi(\bar{x} | - \nabla \varphi(\bar{x}))(w) \right\}. \tag{34}
\]

Suppose that \( \bar{x} \) is a strong local minimizer with modulus \( \kappa \) as in (26). We derive from (26) and (2) that

\[
d^2 f(\bar{x}|0)(w) \geq \kappa \|w\|^2 \quad \text{for all } w \in \mathbb{R}^n. \tag{35}
\]

Since \( \psi \) is subdifferentially continuous, prox-regular, and twice epi-differentiable at \( \bar{x} \) for \( - \nabla \varphi(\bar{x}) \in D \partial \psi(\bar{x}) \), it follows from (3) that

\[
D(\partial \psi)(\bar{x} | - \nabla \varphi(\bar{x})) = \partial h \quad \text{with} \quad h(\cdot) := \frac{1}{2} d^2 \psi(\bar{x} | - \nabla \varphi(\bar{x}))(\cdot) \tag{36}
\]

Note from (2) and (35) that \( h \) is proper and positively homogeneous of degree 2. By Lemma 4.2, for any \( z \in D(\partial \psi)(\bar{x} | - \nabla \varphi(\bar{x}))(w) = \partial h(w) \), we obtain from (35) and (36) that

\[
\langle z, w \rangle = 2h(w) = d^2 \psi(\bar{x} | - \nabla \varphi(\bar{x}))(w). \tag{37}
\]

Therefore, for every \( w \in \text{dom} D \partial \psi(\bar{x} | - \nabla \varphi(\bar{x})) \) and \( z \in D \partial \psi(\bar{x} | - \nabla \varphi(\bar{x}))(w) \), by (29), (30), (35), and (37), we get

\[
\langle \nabla^2 \varphi(\bar{x}) w, w \rangle + \langle z, w \rangle = \langle \nabla^2 \varphi(\bar{x}) w, w \rangle + d^2 \psi(\bar{x} | - \nabla \varphi(\bar{x}))(w) = d^2 f(\bar{x}|0)(w) \geq \kappa \|w\|^2,
\]

which clearly verifies \((v)\) and

\[
\kappa \leq \inf \left\{ \frac{\langle \nabla^2 \varphi(\bar{x}) w, w \rangle + \langle z, w \rangle}{\|w\|^2} \mid w \in \text{dom} D \partial \psi(\bar{x} | - \nabla \varphi(\bar{x})), \ z \in D \partial \psi(\bar{x} | - \nabla \varphi(\bar{x}))(w) \right\}.
\]

Since \( \kappa \) is an arbitrary modulus of the strong local minimizer \( \bar{x} \), the latter implies that (34) holds. So by (31) and (34) we get (28).

\[\square\]

**Remark 4.1** By choosing \( \varphi := 0 \), we can get [5,Theorem 3.7] from Theorem 4.1. In the case where \( \varphi \) is twice continuously differentiable and \( \psi \) is twice epi-differentiable and convex, other characterizations of the quadratic growth as well as the strong metric subregularity of the subdifferential can be found in [15,Theorem 7.8]. We note that (27) is the second-order necessary condition via the second derivative for \( \bar{x} \) to be a local minimizer of \( f \) [16,Theorem 10.24]; hence, it is generally weaker than the assumption that \( \bar{x} \) is a local minimizer of \( f \).

We next consider the composite optimization problem

\[
\min_{x \in \mathbb{R}^n} f(x) := \varphi(x) + g(F(x)), \tag{38}
\]
where \( \varphi : \mathbb{R}^n \to \mathbb{R} \) is twice differentiable at \( \bar{x} \) in the extended sense, \( F : \mathbb{R}^n \to \mathbb{R}^m \) is twice differentiable, and \( g : \mathbb{R}^m \to \mathbb{R} := (-\infty, +\infty) \) is a proper lower semicontinuous convex function Lipschitz continuous around \( F(\bar{x}) \) relative to its domain with constant \( \ell \in \mathbb{R}_+ \).

The Lagrangian associated with (38) is defined by

\[
L(x, y) = \varphi(x) + \langle F(x), y \rangle - g^*(y),
\]

where \( g^*(y) := \sup_{v \in \mathbb{R}^m} \{ \langle y, v \rangle - g(v) \} \) is the Fenchel conjugate of \( g \) (see [12]).

**Corollary 4.1** Let \( 0 \in \nabla \varphi(\bar{x}) + \partial \psi(\bar{x}) \), where \( \varphi : \mathbb{R}^n \to \mathbb{R} \) is twice differentiable at \( \bar{x} \) in the extended sense, and \( \psi := g \circ F \) with \( F : \mathbb{R}^n \to \mathbb{R}^m \) being twice differentiable at \( \bar{x} \) in the extended sense and \( g : \mathbb{R}^m \to \mathbb{R} \) being a proper lower semicontinuous convex function Lipschitz continuous around \( F(\bar{x}) \) relative to its domain. Assume that the basic assumptions (H1)-(H3) hold for \( \psi \) at \( (\bar{x}, \bar{v}) \) with \( \bar{v} := -\nabla \varphi(\bar{x}) \), and \( \psi \) is prox-regular at \( \bar{x} \) for \( \bar{v} \). Then, the following assertions are equivalent:

(i) The quadratic growth condition (26) is satisfied.

(ii) \( \partial f \) is strongly metrically subregular at \( \bar{x} \) for 0, and

\[
\max_{y \in A(\bar{x}, \bar{v})} \left\{ \left\langle \nabla^2_{xx} L(\bar{x}, y)w, w \right\rangle + d^2 g(F(\bar{x}), y) \langle \nabla F(\bar{x}) w \rangle \right\} \geq 0
\]

for all \( w \in K_\psi(\bar{x}, \bar{v}) \).

(iii) \( \partial f \) is strongly metrically subregular at \( \bar{x} \) for 0, and \( \bar{x} \) is a local minimizer of \( f \).

(iv) For all \( w \in \text{dom} D\partial \psi(\bar{x} - \nabla \varphi(\bar{x})) \setminus \{0\} \) and \( z \in D\partial \psi(\bar{x} - \nabla \varphi(\bar{x}))(w) \), we have

\[
\langle \nabla^2 \varphi(\bar{x})w, w \rangle + \langle z, w \rangle > 0.
\]

(v) There exists a real number \( c > 0 \) such that

\[
\langle \nabla^2 \varphi(\bar{x})w, w \rangle + \langle z, w \rangle \geq c \|w\|^2,
\]

for all \( w \in \text{dom} D\partial \psi(\bar{x} - \nabla \varphi(\bar{x})) \) and \( z \in D\partial \psi(\bar{x} - \nabla \varphi(\bar{x}))(w) \).

(vi) For every \( w \in K_\psi(\bar{x}, \bar{v}) \setminus \{0\} \), we have

\[
\max_{y \in A(\bar{x}, \bar{v})} \left\{ \langle \nabla^2_{xx} L(\bar{x}, y)w, w \rangle + d^2 g(F(\bar{x}), y) \langle \nabla F(\bar{x}) w \rangle \right\} > 0.
\]

If one of the above assertions holds then

\[
\text{QG}(f; \bar{x}) = \inf \left\{ \frac{\langle \nabla^2 \varphi(\bar{x})w, w \rangle + \langle z, w \rangle}{\|w\|^2} \mid w \in \text{dom} D\partial \psi(\bar{x} - \nabla \varphi(\bar{x})), z \in D\partial \psi(\bar{x} - \nabla \varphi(\bar{x}))(w) \right\},
\]

with the convention that \( 0/0 = \infty \).
Proof Under the given assumption, \( \psi \) is prox-regular and subdifferentially continuous at \( \bar{x} \) for \( \bar{v} \), and by Proposition 3.2, \( \psi \) is twice epi-differentiable at \( \bar{x} \) for \( \bar{v} \). Furthermore, since \( g \) is Lipschitz continuous relative to its domain and \( F \) is Lipschitz continuous around \( \bar{x} \), the composition \( \psi = g \circ F \) is subdifferentially continuous at \( \bar{x} \) for \( \bar{v} \). On the other hand, by Proposition 3.2, we have

\[
d^2 \psi (\bar{x}, \bar{v})(w) = \max_{y \in \Lambda(\bar{x}, \bar{v})} \left\{ \langle \nabla^2 \phi(\bar{x})w, w \rangle + d^2 \psi (\bar{x}, \bar{v})(w) \right\}
\]

which gives us that

\[
\langle \nabla^2 \varphi(\bar{x})w, w \rangle + d^2 \psi (\bar{x}, \bar{v})(w) = \max_{y \in \Lambda(\bar{x}, \bar{v})} \left\{ \langle \nabla^2_{xx} L(\bar{x}, y)w, w \rangle + d^2 g(F(\bar{x}), y)(\nabla F(\bar{x})w) \right\}
\]

for every \( w \in \mathbb{R}^n \). Therefore, noting that \( d^2 \psi (\bar{x}, \bar{v}) \) is a proper lower semicontinuous function with \( \text{dom} \ d^2 \psi (\bar{x}, \bar{v}) = K \psi (\bar{x}, \bar{v}) \), we get the desired conclusion by applying Theorem 4.1 to the function \( f := \phi + \psi \) with \( \psi := g \circ F \). \( \square \)

Remark 4.2 Under \((H1)-(H3)\), Mohammadi and Sarabi [12, Theorem 6.3] showed that \((iii) \Leftrightarrow (vi)\) when \( \varphi \) and \( F \) are twice continuously differentiable around \( \bar{x} \). Since the latter implies the prox-regularity of \( \psi \), Corollary 4.1 is an extension of [12,Theorem 6.3].

Example 4.1 Consider the following optimization problem:

\[
\min_{x \in \mathbb{R}} \varphi(x) + \psi(x),
\]

where \( \varphi(x) = 2x + g(x) \) with \( g(x) \) being taken from Example 3.2, and \( \psi(x) := \delta_{\mathbb{R}^2_+} \circ F(x) \) with \( F(x) = (F_1(x), F_2(x)) \), \( F_1(x) = -x \) and \( F_2(x) = -x^3 \). By Example 3.2, \( \varphi \) is twice differentiable at \( \bar{x} = 0 \) in the extended sense and not prox-regular at \( \bar{x} = 0 \) for \( \bar{v} = 0 \). Put

\[
\Gamma = \{ x \in \mathbb{R} | F_i(x) \leq 0, \ i = 1, 2 \} = \mathbb{R}_+ \text{ and } g(y) : = \delta_{\mathbb{R}^2_+}(y).
\]

Then, \( g \) satisfies \((H2)\) and \((H3)\). Furthermore, we see that

\[
d(x, \text{dom } \psi) = d(x, \Gamma) = \begin{cases} 0 & \text{if } x \geq 0, \\ |x| & \text{if } x < 0, \end{cases}
\]

and

\[
d(F(x), \text{dom } g) = d(F(x), \mathbb{R}^2_+) = \begin{cases} 0 & \text{if } x \geq 0, \\ \sqrt{x^2 + x^6} & \text{if } x < 0, \end{cases}
\]
which infers that $d(x, \text{dom } \psi) \leq d(F(x), \text{dom } g)$. This shows that $(H1)$ holds at $\bar{x}$.

We next prove that $\bar{x}$ is a strong local minimizer. Indeed, for all $x \in \Gamma \cap [-1, 1]$ and $n \in \mathbb{N}^*$, we have

$$x + x^{10/3} \cos \frac{1}{x} \geq 0 \quad \text{and} \quad \frac{(2n + 1)(2n^2 + 2n + 1)}{n^3(n + 1)^3} x + \frac{1}{(n + 1)^3} - \frac{1}{n^3} \geq x^4 \geq 0.$$ 

Therefore, we get

$$\varphi(x) - \varphi(\bar{x}) \geq x \geq x^2 \quad \text{for all } x \in \Gamma \cap [-1, 1].$$

Thus, $\bar{x}$ is a strong local minimizer. By Corollary 4.1, the assertions $(ii)$-$(vi)$ hold.

## 5 Conclusion

We have proved some characterizations of the quadratic growth and the strong metric subregularity of the subdifferential of a function that can be represented as the sum of a function twice differentiable in the extended sense and a subdifferentially continuous, prox-regular, twice epi-differentiable function. Especially, for such a function, we have shown that the quadratic growth, the strong metric subregularity of the subdifferential at a local minimizer, and the positive definiteness of the subgradient graphical derivative at a stationary point are equivalent. Our results are new even for the case where the second-order differentiability in the extended sense is replaced by the second order differentiability. In this research direction, it seems to us that finding out to which extent the established results can be applied to the analysis of convergence of numerical algorithms is a very interesting issue [3, 9, 15], which requires further investigation. Moreover, in order to widen the range of applications of the obtained results, more researches on the class of functions that are twice differentiable in the extended sense are needed.

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