Hydrodynamic Collimation of GRB Fireballs

Amir Levinson  
*School of Physics and Astronomy, Tel Aviv University, Tel Aviv 69978, Israel*

David Eichler  
*Physics Department, Ben-Gurion University, Beer-Sheva*

Analytic solutions are presented for the hydrodynamic collimation of a relativistic fireball by a surrounding baryonic wind emanating from a torus. The opening angle is shown to be the ratio of the power output of the inner fireball to that of the exterior baryonic wind. The gamma ray burst 990123 might thus be interpreted as a baryon-pure jet with an energy output of order $10^{50}$ erg or less, collimated by a baryonic wind from a torus with an energy output of order $10^{52}$ erg.

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Recent observations of GRB’s and their afterglows appear to indicate that GRB ejecta are collimated. In particular, i) the high fluences exhibited by several bursts, notably GRB 990123 for which the fluence indicated an isotropic equivalent energy in excess of $M_\odot c^2$ [1], require large focusing factors if the central engine is indeed associated with a stellar mass object as widely believed, and ii) the achromatic break seen in the light curve of afterglow emission in GRB 990510 [2], can be interpreted (e.g., refs [3,4]), as due to a transition whereby the blast wave decelerates from $\Gamma > 1/\psi$ to $\Gamma < 1/\psi$, where $\psi$ is the opening angle of the outflow. A blast wave model yielded an opening angle of $\psi = 0.08$ and a corresponding beaming factor of 300 for this source [2].

If GRB’s are indeed collimated, as they appear to be, this would not be surprising. Collimation is a generic feature of outflows in astrophysics, and appears in outflows from protostars, AGN, and accretion disks around neutron stars (e.g. SS433). There is as yet no universally accepted explanation for such collimation. One possibility is collimation by magnetic hoop stress that arises when the field is wound up by rotation. Such a field configuration, however, would be kink unstable [3], and in any case is very slowly collimated [4]. Another possibility is simply that the inertial forces in the outer part of the flow collimate material that impacts upon it from within. This hypothesis merely makes the plausible assumption that the flow is unsteady or multicomponent, so that there are glancing collisions between fluid elements on different streamlines. Even a steady, one component flow can display a collimated jet if it originates from a ring, with some of it directed towards the ring’s axis of cylindrical symmetry [5]. If that part of the fluid that converges at the axis makes a dissipative ”sticky” collision, then it remains well collimated as seen by the external observer.

In GRB’s, there is an a priori reason to suspect multicomponent flows: If the central engine is a black hole, then there needs to be an accretion disk feeding it, and the luminosity from the accretion disk would be so super-Eddington that a baryon outflow would be dragged out with the photons and electron-positron pairs [6,7]. This would obscure the gamma ray burst, so we probably have to say that the gamma ray burst is seen as such only when we look not too far out its streamlines may diverge more slowly than the BPJ streamlines, thereby giving rise to a collision of the two outflows. In general, the collision of the two fluids will lead to the formation of a contact discontinuity across which the total pressure is continuous, and two oblique shocks, one in each fluid, across which the streamlines of the colliding (unshocked) fluids are deflected. The details of this structure will depend, quite generally, on the parameters of the two outflows and on the boundary conditions.

We seek to determine the structure of the shocked layers assuming that the parameters of the unshocked fluids are given. The problem is then characterized by 11 independent variables: the number density, energy density, pressure, and velocity of each of the two shocked layers, and the cross-sectional radii of the shocks and the contact discontinuity surface. In what follows subscripts $j$ and $b$ refer to quantities in the BPJ and in the baryonic wind, respectively, and subscript $s$ denotes quantities in the shocked layers. To simplify the analysis we approximate the shocked layers as cylindrically symmetric, coaxial, one dimensional flows along the channels, and denote by $a_c$, $a_j$, and $a_b$, respectively, the cross sectional radiation of the contact discontinuity surface, the inner shock surface (in the BPJ), and the outer shock surface. The energy flux incident into the shocked BPJ layer through the shock is given by

$$T_{jk}^{0k} n_k = (p_j + \rho_j c^2) \Gamma_j U_j \sin \delta_j,$$

and the momentum flux in the axial direction incident through the shock is

$$T_{jz}^{zj} n_t = p_j + \rho_j c^2 \Gamma_j U_j \cos \theta_j \sin \delta_j.$$

Here $n_k$ are the components of a unit vector normal to the shock surface, $\theta_j$ is the angle between the velocity of the fluid just upstream the shock and the BPJ axis, $\Gamma_j$ and $U_j$ are the Lorentz factor and 4-velocity of the unshocked BPJ, and $\delta_j$ is the deflection angle (i.e., the angle between the velocity of upstream fluid and the shock surface), which is related to $\theta_j$ through: $\sin \delta_j = (\sin \theta_j + \cos \theta_j da_j/dz) / (\sqrt{\Gamma + (da_j/dz)^2})$. With the above results the energy and momentum equations of the shocked fluid can be written as

$$\frac{d}{dz} [T_{kj}(a_e^2 - a_b^2)] = T_{jk}^{0k} n_k dS/dz = (p_j + \rho_j c^2) \Gamma_j U_j \pi a_j (\sin \theta_j + \cos \theta_j da_j/dz).$$
and

\[ \frac{d}{dz} \left[ T_{s_j}^{zz} \pi(a_c^2 - a_j^2) \right] = T_{s_j}^{zl} n_l dS/dz = p_j + (p_j + \rho_j c^2)U_j^2 \cos \theta_j \pi a_j (\sin \theta_j + \cos \theta_j da_j/dz), \]

(4)

where \( T_{s_j}^{zz} \), \( T_{s_j}^{zl} \) are the energy and momentum fluxes in the axial (z) direction of the shocked BPJ fluid, and \( dS = \pi a_j \sqrt{1 + (da_j/dz)^2} dz \) is an element of the shock surface between \( z \) and \( z + dz \). The projection of the momentum equation on the direction normal to the shock surface yields

\[ p_{sj} = T_j^{kl} n_k = p_j + (p_j + \rho_j c^2)U_j^2 \sin^2 \delta_j. \]

(5)

The last equation holds provided the transverse momentum flux incident through the shock exceeds the free expansion pressure of the shocked fluid. The above equations must be supplemented by an equation of state, \( p_{sj} = p_{sj}(\rho_{sj}) \), and a continuity equation that governs the density profile of the shocked fluid. Likewise, the equations describing the structure of the shocked baryonic wind can be written in the same form with the subscript \( j \) replaced by \( b \). Finally, the requirement that the net momentum flux across the contact discontinuity surface vanishes implies that \( p_{sb} = p_{sj} \).

A complete treatment requires numerical integration of the coupled set of 11 equations derived above, subject to appropriate boundary conditions, and is left for future investigation. Below, we consider for illustration a simple situation where the BPJ is assumed to be pressure dominated, and the assumption is then justified a posteriori. This corresponds to the limit \( a_j = \sin \delta_j = 0 \). We further suppose that the shocked baryonic layer is very thin, so that to a good approximation we can set \( a_b = a_c \) in the above equations. Adopting a relativistic equation of state for the BPJ, \( p = \rho c^2/3 \), mass, energy and momentum conservation imply:

\[ p_{s_j}^{3/4} \Gamma_{s_j} \pi a_c^2 = C, \]

(6)

\[ (p_{sj} + \rho_{sj} c^2) \Gamma_{s_j} \pi a_c^2 c \simeq 4p_{sj} \Gamma_{s_j}^2 \pi a_c^2 c = L_j, \]

(7)

\[ p_{sj} = p_b + (p_b + \rho_b c^2)U_b^2 \sin^2 \delta_b, \]

(8)

where \( C \) and \( L_j \), the corresponding BPJ power, are constants. Equations (6), (8) can be combined to yield a relation between the Lorentz factor, the pressure, and the cross sectional radius:

\[ (\Gamma_{s_j}/\Gamma_{s_jo}) = (a_c/a_{co}) = (p_{sj}/p_{sjo})^{-1/4}, \]

(9)

where \( \Gamma_{sjo} \), \( a_{co} \), and \( p_{sjo} \) are the corresponding values at \( z = z_o \). On substituting eq. (9) into eq. (8), one obtains a differential equation for \( a_c(z) \):

\[ (a_c/a_{co})^{-4} = p_b/p_{sjo} + \frac{(p_b + \rho_b c^2)U_b^2 (\sin \theta_b + \cos \theta_b da_c/dz)^2}{p_{sjo} \left[ 1 + (da_c/dz)^2 \right]^2}. \]

(10)

**Pressure confinement** - We consider first the possibility that the BPJ is collimated by the pressure of the baryonic wind. Such collimation will occur naturally in the region located within the acceleration zone of the BPJ and above the acceleration zone of the baryonic wind (roughly above its critical point), even if the two outflows have the same equation of state. The reason is that in this region the density profile of the BPJ declines with radius as \( r^{-3} \) (for a conical BPJ) while that of the wind declines as \( r^{-2} \). Taking \( \delta_b = 0 \) in equation (8) and using eq. (9) one finds,

\[ a_c(z) = a_{co}[p_b(z)/p_{bo}]^{-1/4}. \]

(11)

Adopting an equation of state of the form \( p_b(n_b) \propto n_b^\alpha \) for the baryonic fluid yields, \( a_c(z) = a_{co}[n_b(z)/n_{bo}]^{-\alpha/4} \). In case of a conical (or spherical) wind, \( n_b(z) \propto z^{-2} \), and the cross sectional radius scales as \( a_c(z) \propto z^{\alpha/2} \). Convergence occurs for \( \alpha < 2 \), which is the case for both relativistic gas (\( \alpha = 4/3 \)) and non-relativistic gas (\( \alpha = 5/3 \)).

The above analysis does not take into account effects associated with the formation of a rarefaction wave in the baryonic outflow due to the divergence of its stream lines near the interface separating the two fluids. This will result in a steeper decline of the pressure supporting the BPJ at radii where the wind is highly supersonic, and the consequent alteration of the profile of the contact discontinuity surface. We naively expect collimation to take place predominantly over a range of radii at which the Mach number of the baryonic wind is mild, unless some fraction of the wind power is dissipated, e.g., due to formation of shocks, far enough out. This suggests that only a modest beaming factor may be attainable in this case. The determination of the level of collimation in this scenario is left for future work.
**BPJ confinement by a wind emanating from a torus** - As a second example we consider a baryonic wind emanating from a thin torus of radius $R$ located at $z = 0$, where $z$ is the axial coordinate, and centered around the central engine ejecting the BPJ (see Fig. 1). We suppose that on scales of interest the baryonic wind is highly supersonic, so that momentum transfer from the wind into the BPJ is dominated by ram pressure. We can therefore neglect the kinetic pressure, $p_k$, in eq. (10). Now the ram pressure of the baryonic outflow just upstream the oblique shock at some position $z$ is related to the total wind power, $L_b$, through:

\[
(p_k + p_r c^2) U_b^2 = \beta_b L_b / (4\pi^2 c a_r r), \tag{12}
\]

where $r = [z^2 + (R - a_c)^2]^{1/2}$ is the distance between a point on the torus and the nearest point to it at the shock (see figure 1), and $\beta_b$ is the (terminal) wind velocity. The angle between the wind velocity and the BPJ axis is given by $\tan \theta_b = (R - a_c)/z$. Substituting eq. (12) into eq. (10), and using eq. (7) yields,

\[
\frac{(R - a_c + da_c/dz \ln z)^2}{1 + (da_c/dz)^2} = a_{co}^2 \alpha \frac{[z^2 + (R - a_c)^2]^{3/2}}{a_c^3} \tag{13}
\]

with $\chi = \pi^{-1}\beta_b \Gamma_{z_{in}}^2 (L_b/L_j)$ being approximately the ratio of baryonic wind and BPJ powers. At large enough axial distances from the plane of the torus, $z >> R$, an approximate solution to eq. (13) can be obtained by expanding the various terms in powers of $R/z$, and becomes more exact at large $z$. To second order the BPJ in this regime is conical, viz., $a_c(z) = b + a_2 z$, where $\alpha$, the opening angle, and $b$ are constants, and satisfy the relation:

\[
\chi (R/a_{co} - b/a_{co})^2 = \frac{(1 + \alpha^2)^{5/2}}{\alpha^3}. \tag{14}
\]

The parameter $b/a_{co}$ depends on the properties of the solution near the central engine (at $z < R$) and must be determined numerically. The requirement that the BPJ is confined by the ram pressure of the baryonic outflow at $z = 0$ yields,

\[
(R/a_{co} - 1) = \chi, \tag{15}
\]

where it has been assumed that $da_c(z = 0)/dz = 0$. Combining the last two equations, one finds

\[
\frac{(1 + \alpha^2)^{5/2}}{\alpha^3} = \chi (\chi + 1 - b/a_{co})^2. \tag{16}
\]

For $\chi >> 1$ the latter equation implies that

\[
\alpha \simeq \chi^{-1} + O(\chi^{-2}). \tag{17}
\]

Numerical integration of eq. (13) confirmed that the BPJ is indeed conical with an opening angle given by eq. (16), except for a small range ($z < R$) near the central engine. BPJ profiles computed numerically for different values of the parameter $\chi$ are exhibited in Fig. 2, and the corresponding dependence of $\alpha$ on $\chi$ is displayed in Fig. 3.

Once the BPJ Lorentz factor exceeds $\alpha^{-1} \sim \chi$, it will remain conical with the same opening angle regardless of the external conditions. This occurs at a distance $z = R\chi^2/(1 + \chi)$ from the central engine. Thus the baryonic wind must extend to at least this scale.

The apparent luminosity, $L_{app}$, exhibited by a beamed source having a power $L_j$ is roughly $L_{app} \sim f^{-1} L_j$, where $f = \Delta \Omega/2\pi$ is the corresponding beaming factor. For the above model we find

\[
f = \pi \alpha^2 / 2\pi = 1/2\chi^2 = (\pi^2 / 2)(L_j/L_b)^2, \tag{18}
\]

and

\[
L_{app} \sim (2/\pi^2)(L_b/L_j)L_b. \tag{19}
\]

In conclusion, we have shown [equation (19)] that an inner relativistic jet collimated by an outer modestly relativistic jet can appear brighter to the observer within the beam by a factor that is inversely proportional to the intrinsic power of the inner jet. When all else is the same, a low luminosity inner jet actually appears brighter than a high luminosity one to the observer within its beam. The cost, of course, is that, fewer observers are within the beam. Ironically, most of the energy of the GRB is the unseen outer jet.
Now consider the remarkable GRB 990123, which had an isotropic equivalent energy output of about $3 \times 10^{54}$ ergs.

Applying equations (18) through (19), we propose that the actual energy in the gamma ray fireball was only about $\eta^2 10^{50}$ ergs, (it may have even been smaller than average) and an outer baryonic wind output of order $\eta 10^{52.5}$ erg. The corresponding beaming factor is $f = 10^{-5} \pi^2 \eta^2 / 2$. Here $\eta$ is assumed to be of order unity or less, otherwise the energetics strain theoretical models.

A rough observational lower limit on $\eta$ can in principle be derived from the consideration that GRB 990123 could not have a collimation angle that is implausibly smaller than for average GRB's. Although it was an unusual burst, the a priori chance of seeing it is unlikely to have been less than $10^{-3}$, and, considering that it is a BeppoSax-located GRB and that there has been at least one other burst of comparable (if somewhat smaller) intrinsic magnitude, probably considerably more than $10^{-3}$. The largest uncertainty is the beaming factor for the average GRB. If we take that as being of order $10^{-3}$, corresponding to an opening angle of about 0.06 radians, this suggests a value for $\eta$ of at least of order 0.3. This means that the minimum energy requirement for GRB 990123 is still of order $10^{52}$ ergs or more. On the other hand, the fact that it can come out as a baryonic wind relaxes the demands on theoretical models. For example, a magnetocentrifugal wind could plausibly release several percent of a solar rest energy over 30 seconds, whereas it would be hard to get this much from $\nu - \bar{\nu}$ annihilation.

Another example is GRB990510, for which the achromatic steepening of the optical and radio light curves observed after about 1 day, can be modeled as due to the evolution of a jet with an opening angle of ~ 0.08 rad, and a corresponding beaming factor of 300. The prompt GRB emission should be at least as beamed as the afterglow emission, but could in fact be more beamed. The isotropic gamma-ray emission inferred for this source is $3 \times 10^{53}$ ergs. Employing equations (18) and (19) yields a total energy of $< 10^{51}$ ergs for the BPJ and $3 \times 10^{52}$ ergs for the confining wind.

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FIG. 1. Schematic illustration of BPJ confinement by the ram pressure of a baryon rich wind (represented by the dashed lines) emanating from a torus of radius $R$. The torus is centered around the central engine ejecting the BPJ. Fluid elements of the baryonic outflow collide with the BPJ at an angle $\delta(z)$, and are deflected when passing through an oblique shock. The shocked wind layer is assumed to be very thin. At distances larger than roughly $R$ the BPJ becomes conical with a semi-opening angle $\alpha$. 
FIG. 2. Cross sectional radius of the BPJ versus axial distance from the central engine, computed numerically for different values of $\chi$. 
FIG. 3. A plot of $\alpha^{-1}$ against $\chi$, obtained from the numerical integration of eq. (13).