A CLASS OF STOCHASTIC FREDHOLM-ALGEBRAIC EQUATIONS AND APPLICATIONS IN FINANCE

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Abstract. A class of stochastic Fredholm-algebraic equations (SFAEs) is introduced and investigated. Like backward stochastic differential equations (BSDEs), its solution includes two parts. The interesting thing is that the first part is deterministic and constrained, even though the whole system is stochastic. Our study is mainly motivated by risk indifference pricing problem. Actually, the existing risk indifference price always keeps unchangeable with respect to initial wealth, which is economically unsatisfying. Nevertheless, here a new wealth dependent risk indifference price is proposed by particular SFAEs.

1. Introduction. In this article, let \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\) be a complete filtered probability space with the filtration \(\mathbb{F} ≡ \{\mathcal{F}_t\}_{t≥0}\), on which a one-dimensional standard Wiener process \(W(\cdot)\) is defined such that \(\mathbb{F}\) is the natural filtration generated by \(W(\cdot)\).

Suppose \(a, b ∈ \mathbb{R}\) with \(a < b\), \(T ∈ \mathbb{R}^+\), \(\mathbb{R}^+\) is the set of positive constant in \(\mathbb{R}\), \(\xi : [a, b] × \Omega → \mathbb{R}\) is \(\mathcal{B}([a, b]) × \mathcal{F}_T\)-measurable, \(h : [a, b] × \Omega × [0, T] × \mathbb{R} × \mathbb{R} → \mathbb{R}\) is \(\mathcal{B}([a, b]) × \mathcal{F}_T × \mathcal{B}([0, T]) × \mathcal{B}(\mathbb{R}) × \mathcal{B}(\mathbb{R})\)-measurable such that \(s ↦ h(s, x, p, q)\) is \(\mathbb{F}\)-adapted for \((s, x, p, q) ∈ [0, T] × [a, b] × \mathbb{R} × \mathbb{R}\). We consider the following form of equation,

\[
P(x) = ξ(x) + \int_0^T h(s, x, P(s), Q(s, x))ds - \int_0^T Q(s, x)dW(s), \ x ∈ [a, b]. \tag{1}
\]

Under proper conditions on \(ξ\) and \(h\), we prove that (1) admits a unique solution \((P(\cdot), Q(\cdot, \cdot)) ∈ C([a, b]; \mathbb{R}) × C([a, b]; L^2_\mathcal{F}_T(0, T; \mathbb{R}))\). These two spaces will be defined later in this section. As to (1), we look at two special and interesting cases. Suppose \(h_1 : \Omega × [0, T] × [a, b] × \mathbb{R} → \mathbb{R}\) is a new mapping that satisfies the similar measurability as the above \(h\). Given \(x ∈ [a, b]\), we consider the parameterized backward

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stochastic differential equations ([9], [22]) as follows,
\[ Y(t, x) = \xi(x) + \int_t^T h_1(s, x, Z(s, x)) ds - \int_t^T Z(s, x) dW(s), \quad t \in [0, T]. \] (2)

By defining \( P(x) := Y(0, x) \), \( Q(s, x) = Z(s, x) \), with \( x \in [a, b] \), \( s \in [0, T] \), we see that
\[ P(x) = \xi(x) + \int_0^T h_1(s, x, Q(s, x)) ds - \int_0^T Q(s, x) dW(s), \quad x \in [a, b]. \]

It was named as stochastic Fredholm equation and played important roles in the wellposedness of backward stochastic Volterra integral equations (BSVIEs), see Section 3 of [32]. Related literature on BSVIEs can be found in e.g., [24], [25], [26], [27], [31], [32]. For the second case, we return back to (1). Actually, if \( \xi \) and \( h \) do not depend on \( \omega \), then it becomes
\[ P(x) = \xi(x) + \int_0^T h(s, x, P(x), 0) ds, \quad x \in [a, b]. \] (3)

Notice that the algebraic equations in a pointwise form appeared in [19], [28], which is distinctive from above integral form of (3). In this paper we call (1) a stochastic Fredholm-algebraic equation (SFAE). To explain the interests in (1), we give a financial motivation.

Indifference pricing is a basic method of giving suitable price for claims or positions in financial markets. Let us consider a simple financial market consisting bond and stock. Suppose \( x \) is the initial wealth of an investor, \( F \) is \( \mathcal{F}_T \)-measurable bounded financial position, \( \alpha(\cdot) \), \( \beta(\cdot) \) are the expected return rate and volatility rate of the stock, \( \pi(\cdot) \) is the investment in the stock, the following \( A \) can be seen as the terminal wealth of investor with zero initial endowment,
\[ A := \int_0^T \alpha(s) \pi(s) ds + \int_0^T \beta(s) \pi(s) dW(s). \]

Obviously, as \( \pi(\cdot) \) changes, \( A \) varies as well. We denote by \( \mathcal{A} \) is the set of \( A \) with the above form. More details can be found in Section 3. In addition, suppose the risk measure is represented via BSDE as \( \rho(x + A) = Y(0) \) where
\[ Y(t) = -x - A + \int_t^T h_1(s, Z(s)) ds - \int_t^T Z(s) dW(s), \quad t \in [0, T], \]

Here the generator \( h_1 \) is a special case of that in (2) and \( z \mapsto h_1(\cdot, z) \) is convex such that \( h_1(\cdot, 0) = 0 \). As to the connections between risk measures and BSDEs, we refer to for instance [1], [23] for more details. The translation invariance of risk measures indicates that
\[ \rho(x + A) = \rho(A) - x, \quad \rho(x - q + F + A) = \rho(F + A) - (x - q) \]
where \( x, q \in \mathbb{R} \). Suppose there exist \( \bar{A}, \tilde{A} \) such that
\[ \rho(\bar{A}) = \essinf_{A \in \mathcal{A}} \rho(A), \quad \rho(F + \tilde{A}) = \essinf_{A \in \mathcal{A}} \rho(F + A). \]

As a result, we have
\[ \rho(x + \bar{A}) = \essinf_{A \in \mathcal{A}} \rho(x + A), \quad \rho(x - q + F + \tilde{A}) = \essinf_{A \in \mathcal{A}} \rho(x - q + F + A). \]
Consequently, the risk indifference price $q$ is determined by
\[ \rho(\bar{A} - x) = \rho(x + \bar{A}) - x = \rho(x - q + F + \bar{A}) = \rho(F + \bar{A}) - x + q, \tag{4} \]
from which we see that $q = \rho(\bar{A}) - \rho(F + \bar{A})$ is independent of $x$.

In our opinion, even though above wealth-independence property of indifference pricing is widely used in the literature (e.g., [1], [12], [14], [15], [20], [30]), in some sense it seems over-perfect and economically unsatisfactory. In fact, when you only have 100 dollars initial wealth, it is easy to see that you are more reluctant to spend 50 dollars in some financial position than the scenario of 100,000 dollars initial wealth.

In the literature, there are several works on utility indifference pricing with wealth-dependent utilities, see e.g. [16], [17], and so on. Similarly, the risk indifference pricing should also be changeable with varying wealth since for instance the buyer’s or seller’s risk preference is subjective and surely depends on how wealthy he/she is ([3]). To this end, let us recall the dynamic mean-variance portfolio selection problem with constant risk aversion. With deterministic coefficients, it is shown in [2] that time consistent equilibrium investment strategy is also independent of initial wealth as above. To fill this gap, a class of state-dependent risk aversion parameter was used in [3], and the resulting form of investment strategy makes sense. We point out that the notion of state dependent risk aversion can at least date back to [13].

Let us take another example of classical entropic risk measure ([1]),
\[ \rho(\eta) := \frac{1}{\gamma} \ln \mathbb{E} \left[ \exp \left( -\gamma \eta \right) \right], \quad \eta := x + A, \]
where $\eta$ represents the terminal payoff at time $T$, $\gamma$ is the risk aversion. According to the arguments in Section 4 of [3], we can replace constant $\gamma$ by decreasing function $\gamma(\cdot)$ with respect to initial wealth, say, $\gamma(x) = \frac{x}{2}$. Notice that the entropic risk measure (or exponential utility) with state-dependent risk aversion parameter was also discussed in e.g. [8], [5]. By the BSDEs language, we then have $\rho(\eta) = Y(0)$,
\[ Y(t) = -x - A + \int_t^T \frac{\gamma(x)}{2} |Z(s)|^2 ds - \int_t^T Z(s)dW(s), \quad t \in [0, T], \]
Motivated by above points, given $x$, we consider risk measure $\rho(\eta; x) := Y(0, x)$ via
\[ Y(t, x) = -\eta + \int_t^T h(s, x, Z(s, x)) ds - \int_t^T Z(s, x)dW(s), \quad t \in [0, T]. \]
Here $h(\cdot, \cdot, 0) = 0$, $\rho$ actually depends on initial wealth $x$. As a result, (4) can be rewritten as,
\[ \text{ess inf}_{A \in \mathcal{A}} \rho(x + A, x) = \text{ess inf}_{A \in \mathcal{A}} \rho(x - q + F + A, x - q), \tag{5} \]
which implies that $q = \rho(F + \bar{A}, x - q) - \rho(\bar{A}, x)$. To see the dependence of $q$ on $x$, as well as the connection with (1), we refer to the arguments in Section 3 for more details.

The aim of this paper is to investigate the wellposedness of (1) and demonstrate detailed applications in risk indifference pricing problems. Similar as BSDEs, the solution of (1) concludes two parts ($P, Q$). As to (1), even though there is stochastic integral involved, the solution $P(\cdot)$ is still allowed to be deterministic. In addition, to guarantee the well-posedness of SFAE (1) for any fixed Lipschitz constant and time
horizon, we need to introduce somewhat structural conditions on the coefficient $h$.
We also find that our SFAE (1) have some interesting connections with BSDEs with
time delayed generators (see e.g., [6], [7]). Motivated by the risk indifference pricing
problem, we also impose some technical conditions to keep $P(\cdot)$ constrained. After
the theoretical results, we apply them into the risk indifference pricing problems.
More precisely, we introduce wealth-dependent risk indifference price that has more
economic reasonability as explained above, and describe it by particular class of
SFAEs which reduce into an algebraic equation in Markovian setting. Eventually, we
present three examples for illustration. The explicit optimal strategies and related
SFAEs are obtained, and the qualitative analysis among different parameters are
demonstrated.

The paper is organized as follows: In Section 2, we investigate the wellposedness
of SFAEs in a systematical way. In Section 3, we apply the conclusions to risk
indifference pricing problems.

To conclude this section, we give some useful notations. For $a, b \in \mathbb{R}$ with $a < b$,
let

$C([0, T]; \mathbb{R}) := \{X : [0, T] \rightarrow \mathbb{R}, t \mapsto X(t) \text{ is continuous}\}$,

$L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}) := \{\xi : \Omega \rightarrow \mathbb{R} \mid \xi \text{ is } \mathcal{F}_T\text{-measurable, } \mathbb{E}[\xi^2] < \infty\}$,

$L^2([0, T]; \mathbb{R}) := \{X : [0, T] \times \Omega \rightarrow \mathbb{R} \mid X(\cdot) \text{ is } \mathcal{B}([0, T]) \otimes \mathcal{F}\text{-measurable}
\quad \text{and } \mathbb{F}\text{-adapted, } \|X\|_{L^2([0, T]; \mathbb{R})} := \left[\mathbb{E}\int_0^T |X(r)|^2 \, dr\right]^{\frac{1}{2}} < \infty\}$,

$L^2(\Omega; C([0, T]; \mathbb{R})) := \{X : [0, T] \times \Omega \rightarrow \mathbb{R} \mid X(\cdot) \text{ is } \mathcal{B}([0, T]) \otimes \mathcal{F}\text{-measurable,}
\quad \mathbb{F}\text{-adapted, and has continuous paths such that}
\quad \mathbb{E}\left(\sup_{r \in [0, T]} |X(r)|^2\right) < \infty\}$,

$C([a, b]; L^2_{\mathcal{F}_T}(0, T; \mathbb{R})) := \{X : [a, b] \times [0, T] \times \Omega \rightarrow \mathbb{R} \mid X \text{ is } \mathcal{B}([a, b]) \otimes \mathcal{B}([0, T]) \otimes \mathcal{F}\text{-measurable, and for any } r_0 \in [a, b], X(r_0, \cdot) \in L^2_{\mathcal{F}_T}(0, T; \mathbb{R}),
\quad \lim_{r \to r_0} \|X(r, \cdot) - X(r_0, \cdot)\|_{L^2([0, T]; \mathbb{R})} = 0\}$,

$C([a, b]; L^2_{\mathcal{F}_T}(\Omega; \mathbb{R})) := \{X : [a, b] \times \Omega \rightarrow \mathbb{R} \mid X(\cdot) \text{ is } \mathcal{B}([a, b]) \otimes \mathcal{F}\text{-measurable,}
\quad \text{and for any } r_0 \in [a, b], X(r_0, \cdot) \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}),
\quad \lim_{r \to r_0} \mathbb{E}\|X(r) - X(r_0)\|^2 = 0\}$,

$C([a, b]; L^2(\Omega; C([0, T]; \mathbb{R}))) := \{X : [a, b] \times [0, T] \times \Omega \rightarrow \mathbb{R} \mid X \text{ is } \mathcal{B}([a, b]) \otimes \mathcal{B}
\quad ([0, T]) \otimes \mathcal{F}\text{-measurable for any } r_0 \in [a, b],
\quad X(r_0, \cdot) \in L^2_{\mathcal{F}_T}(\Omega; C([0, T]; \mathbb{R})),
\quad \lim_{r \to r_0} \|X(r, \cdot) - X(r_0, \cdot)\|_{L^2(\Omega; C([0, T]; \mathbb{R}))} = 0\}$.

2. Wellposedness of stochastic Fredholm-algebraic equations. Suppose $a, b \in \mathbb{R}$, $a < b$, and for $(s, x, p, q) \in [0, T] \times [a, b] \times \mathbb{R} \times \mathbb{R}$, mapping $(x, \omega) \mapsto \xi(x, \omega)$ is
$\mathcal{B}([a, b]) \times \mathcal{F}\text{-measurable, } (s, \omega, x, p, q) \mapsto h(s, \omega, x, p, q) \text{ is } \mathcal{B}([0, T]) \times \mathcal{F}_T \times \mathcal{B}([a, b]) \times
\mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R})\text{-measurable such that } s \mapsto h(s, x, p, q) \text{ is } \mathbb{F}\text{-adapted. In this section we}
study the wellposedness of SFAE as follows,
\[ P(x) = \xi(x) + \int_0^T h(s, x, P(x), Q(s, x))ds - \int_0^T Q(s, x)dW(s), \quad x \in [a, b]. \quad (6) \]

**Definition 2.1.** A pair of \((P(\cdot), Q(\cdot, \cdot)) \in C([a, b]; \mathbb{R}) \times C([a, b]; L^2_F(0, T; \mathbb{R}))\) is called a solution of (6) if it holds true almost surely.

2.1. **A special case.** We first consider a special form of SFAE (6),

\[ P(\tau) = \xi(\tau) + \int_0^T h(s, \tau, Q(s, \tau))ds - \int_0^T Q(s, \tau)dW(s), \quad \tau \in [a, b]. \quad (7) \]

(H1) Let

\[ \xi : [a, b] \times \Omega \rightarrow \mathbb{R}, \quad h : [a, b] \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R} \]

be measurable such that \(s \mapsto h(s, \tau, q)\) is \(\mathcal{F}\)-adapted, \(q \mapsto h(s, \tau, q)\) is Lipschitz with constant \(L_q\), \(\tau \mapsto h(s, \tau, q)\) is continuous, and

\[ \xi(\cdot) \in C([a, b]; L^2_{\mathcal{F}}(\Omega, \mathbb{R})), \quad \sup_{\tau \in [a, b]} |h(\cdot, \tau, 0)| \in L^2_{\mathcal{F}}(0, T; \mathbb{R}). \]

(H2) There exists some function \(K : [a, b] \rightarrow \mathbb{R}^+\) such that

\[
\begin{align*}
\xi(\tau) + \int_0^T h(s, \tau, 0)ds &\geq 0, \quad \forall \tau \in [a, b], \\
\xi(\tau) + \int_0^T h(s, \tau, 0)ds &\leq K(\tau), \quad \forall \tau \in [a, b].
\end{align*}
\]

For later usefulness, with \(L_q\) in (H1), we define

\[ C_{1,q} := e^{(2L_q^2 + \frac{1}{2})T}[4L_q^2 + 3], \quad C_{2,q} := 2e^{(2L_q^2 + \frac{1}{2})T}[2L_q^2 + 2]. \quad (9) \]

**Lemma 2.2.** Let (H1) holds true. Then SFAE (7) admits a unique pair of \((P(\cdot), Q(\cdot, \cdot)) \in C([a, b]; \mathbb{R}) \times C([a, b]; L^2_F(0, T; \mathbb{R}))\) such that

\[
\sup_{\tau \in [a, b]} |P(\tau)|^2 + \sup_{\tau \in [a, b]} \mathbb{E} \int_0^T |Q(s, \tau)|^2ds 
\leq 2C_{1,q} \sup_{\tau \in [a, b]} \mathbb{E} |\xi(\tau)|^2 + 2C_{2,q} \sup_{\tau \in [a, b]} \mathbb{E} \int_0^T |h(s, \tau, 0)|^2ds. \quad (10)
\]

For \(i = 1, 2\), if \((P_i(\cdot), Q_i(\cdot, \cdot))\) satisfies (7) associated with \((\xi_i, h_i)\). Then for \(\tau \in [a, b]\),

\[
|P_1(\tau) - P_2(\tau)|^2 + \mathbb{E} \int_0^T |Q_1(s, \tau) - Q_2(s, \tau)|^2ds 
\leq C_{1,q} \mathbb{E} |\xi_1(\tau) - \xi_2(\tau)|^2 + C_{2,q} \mathbb{E} \int_0^T |h_1(s, \tau, Q_1(s, \tau)) - h_2(s, \tau, Q_1(s, \tau))|^2ds. \quad (11)
\]

Moreover, if (H2) also holds true, then

\[ P(x) \geq 0, \quad x \in [a, b], \quad P(x) \leq K(x), \quad x \in [a, b]. \quad (12) \]
Proof. We separate the proof into three parts.

**Step 1:** Given assumption (H1), let us consider the following BSDE with parameter \( \tau \in [a, b] \),
\[
Y(t, \tau) = \xi(\tau) + \int_t^T h(s, \tau, Z(s, \tau)) ds - \int_t^T Z(s, \tau) dW(s), \quad t \in [0, T].
\]
We prove that (13) admits a unique pair of \((Y, Z)\) such that
\[
(Y, Z) \in \mathcal{C}([a, b]; L^2_{\mathbb{F}}(\Omega; \mathbb{R})) \times \mathcal{C}([a, b]; L^2_{\mathbb{F}}(0, T; \mathbb{R}))
\]
and the following inequality holds,
\[
\sup_{t \in [0, T]} \mathbb{E}|Y(t, \tau)|^2 + \mathbb{E} \int_0^T |Z(s, \tau)|^2 ds \leq C_1 \mathbb{E}|\xi(\tau)|^2 + C_2 \mathbb{E} \int_0^T |h(s, \tau, 0)|^2 ds.
\]
In addition, if \((Y_i, Z_i), i = 1, 2\), is the solution corresponding to \((\xi_i, h_i)\), then for \( \hat{Y} := Y_1 - Y_2, \hat{Z} := Z_1 - Z_2 \), one has
\[
\sup_{t \in [0, T]} \mathbb{E} |\hat{Y}(t, \tau)|^2 + \mathbb{E} \int_0^T |\hat{Z}(s, \tau)|^2 ds \\
\leq C_1 \mathbb{E}|\xi_1(\tau) - \xi_2(\tau)|^2 + C_2 \mathbb{E} \int_0^T [|h_1(s, \tau, Z_1(\tau)) - h_2(s, \tau, Z_1(\tau))|^2] ds.
\]

For any fixed \( \tau \), the well-posedness of \((Y(\cdot, \tau), Z(\cdot, \tau))\) is followed by standard BSDEs theory, see e.g. Subsection 2.4 in [9]. We emphasize that the key point is the joint measurability of \((Y, Z)\) w.r.t. \( \mathcal{B}([0, T]) \times \mathcal{B}([a, b]) \times \mathcal{F}_T \).

To this end, given measurable process \( z(\cdot, \cdot) \in \mathcal{C}([a, b]; L^2_{\mathbb{F}}(0, T; \mathbb{R})) \), we define
\[
\psi(\tau) := \xi(\tau) + \int_0^T h(s, \tau, z(s, \tau)) ds,
\]
and consider the following simple BSDE,
\[
M(t, \tau) = \psi(\tau) - \int_t^T Z(s, \tau) dW(s), \quad \forall t \in [0, T].
\]

The existence of \((M(\cdot, \tau), Z(\cdot, \tau))\) follows from martingale representation theorem. To see their joint measurability, let us take a careful look at \( \psi(\cdot) \). We observe that
\[
\begin{align*}
&\mathbb{E} \int_0^T \left| h(s, \tau_2, z(s, \tau_2)) - h(s, \tau_1, z(s, \tau_1)) \right|^2 ds \\
\leq &\ 2 L^2_q \mathbb{E} \int_0^T |z(s, \tau_2) - z(s, \tau_1)|^2 ds \\
+ &\ 2 \mathbb{E} \int_0^T \left| h(s, \tau_2, z(s, \tau_1)) - h(s, \tau_1, z(s, \tau_1)) \right|^2 ds \rightarrow 0, \quad \tau_2 \rightarrow \tau_1,
\end{align*}
\]
\[
\sup_{\tau \in [a, b]} \mathbb{E} \int_0^T |h(s, \tau, z(s, \tau))|^2 ds \\
\leq &\ 2 L^2_q \sup_{\tau \in [a, b]} \mathbb{E} \int_0^T |z(s, \tau)|^2 ds + 2 \sup_{\tau \in [a, b]} \mathbb{E} \int_0^T |h(s, \tau, 0)|^2 ds < \infty.
\]
Since \( \xi(\cdot) \in C([a, b]; \mathbb{L}^2_{\mathcal{F}_T}(\Omega; \mathbb{R})) \), we see that \( \psi(\cdot) \in C([a, b]; \mathbb{L}^2_{\mathcal{F}_T}(\Omega; \mathbb{R})) \) and introduce

\[
\psi^N(\tau) := \sum_{k=1}^{N-1} \psi_{k-1}I_{[r_{k-1}, r_k)}(\tau) + \psi_{N-1}I_{[r_{N-1}, r_N]}(\tau),
\]

satisfying

\[
\sup_{\tau \in [a, b]} \mathbb{E}[|\psi^N(\tau) - \psi(\tau)|^2] \to 0, \quad N \to \infty.
\]

Here \( r_k, 0 \leq k \leq N \) are the grid points in \([a, b]\) with \( a = r_0 < r_1 < r_2 \cdots < r_N = b \), \( \psi_{k-1} \) is \( \mathcal{F}_T \)-measurable random variable in \( \mathbb{L}^2_{\mathcal{F}_T}(\Omega; \mathbb{R}) \). Given \( \psi_{k-1} \), let \( \lambda_{k-1}(t) := \mathbb{E}_t[\psi_{k-1}] \). Moreover, by martingale representation theorem, there exists a unique \( \mu_{k-1}(\cdot) \in \mathbb{L}^2(0, T; \mathbb{R}) \) such that

\[
\lambda_{k-1}(t) = \psi_{k-1} - \int_t^T \mu_{k-1}(s) dW(s).
\]

As a result, we know that

\[
M^N(t, \tau) = \psi^N(\tau) - \int_t^T Z^N(s, \tau) dW(s),
\]

where

\[
M^N(t, \tau) := \sum_{k=1}^{N} \lambda_{k-1}(t)I_{[r_{k-1}, r_k)}(\tau), \quad Z^N(t, \tau) := \sum_{k=1}^{N} \mu_{k-1}(t)I_{[r_{k-1}, r_k)}(\tau).
\]

We see that both \( M^N(\cdot, \cdot) \) and \( Z^N(\cdot, \cdot) \) are \( \mathcal{B}([0, T]) \times \mathcal{B}([a, b]) \times \mathcal{F}_T \)-measurable. In addition, for any fixed \( N \geq 1 \),

\[
\left\{ \begin{array}{l}
\mathbb{E} \sup_{s \in [0, T]} |M^N(s, \tau_2) - M^N(s, \tau_1)|^2 + \mathbb{E} \int_0^T |Z^N(s, \tau_2) - Z^N(s, \tau_1)|^2 ds \\
\leq \sum_{k=1}^{N} \left[ \mathbb{E} \sup_{s \in [0, T]} |\lambda_{k-1}(s)|^2 + \mathbb{E} \int_0^T |\mu_{k-1}(s)|^2 ds \right] \sum_{k=1}^{N} \left[ I_{[r_{k-1}, r_k)}(\tau_2) - I_{[r_{k-1}, r_k)}(\tau_1) \right]^2 \\
\sup_{\tau \in [a, b]} \mathbb{E} \sup_{s \in [0, T]} |M^N(s, \tau)|^2 + \mathbb{E} \int_0^T |Z^N(s, \tau)|^2 ds \leq K \sup_{\tau \in [a, b]} \mathbb{E} |\psi^N(\tau)|^2 < \infty,
\end{array} \right.
\]

which implies that

\[
(M^N(\cdot, \cdot), Z^N(\cdot, \cdot)) \in C([a, b]; \mathbb{L}^2_{\mathcal{F}_T}(\Omega; C([0, T]; \mathbb{R}))) \times C([a, b]; \mathbb{L}^2_{\mathcal{F}_T}(0, T; \mathbb{R})).
\]

By the standard BSDEs theory, for \( N, N_1, N_2 \to \infty \),

\[
\left\{ \begin{array}{l}
\sup_{\tau \in [a, b]} \mathbb{E} \sup_{s \in [0, T]} |M^{N_1}(s, \tau) - M^{N_2}(s, \tau)|^2 \\
\quad \quad \quad + \sup_{\tau \in [a, b]} \mathbb{E} \int_0^T |Z^{N_1}(s, \tau) - Z^{N_2}(s, \tau)|^2 ds \\
\leq K \sup_{\tau \in [a, b]} \mathbb{E} |\psi^{N_1}(\tau) - \psi^{N_2}(\tau)|^2 \to 0,
\end{array} \right.
\]

\[
\left\{ \begin{array}{l}
\sup_{\tau \in [a, b]} \mathbb{E} \sup_{s \in [0, T]} |M^N(s, \tau) - M(s, \tau)|^2 + \sup_{\tau \in [a, b]} \mathbb{E} \int_0^T |Z^N(s, \tau) - Z(s, \tau)|^2 ds \\
\leq K \sup_{\tau \in [a, b]} \mathbb{E} |\psi^N(\tau) - \psi(\tau)|^2 \to 0.
\end{array} \right.
\]
Hence there exists $M_0 \in C([a, b]; L^2_F(\Omega; C([0, T]; \mathbb{R})))$ and $Z_0 \in C([a, b]; L^2_F(0, T; \mathbb{R}))$ such that
\[
\sup_{\tau \in [a, b]} \mathbb{E} \sup_{t \in [0, T]} |M_0(s, \tau) - M(s, \tau)|^2 + \sup_{\tau \in [a, b]} \mathbb{E} \int_0^T |Z_0(s, \tau) - Z(s, \tau)|^2 ds = 0.
\]
Therefore, we choose $(M, Z)$ as the measurable solution of (16). At this moment, we define
\[
Y(t, \tau) := M(t, \tau) - \int_t^\tau h(s, \tau, z(s, \tau)) ds.
\]
It is easy to see that $(Y, Z)$ satisfies BSDE of
\[
Y(t, \tau) = \xi(\tau) + \int_t^\tau h(s, \tau, z(s, \tau)) ds - \int_t^\tau Z(s, \tau) dW(s), \quad t \in [0, T].
\]
By standard contraction arguments, one obtains the existence and uniqueness of measurable solutions $(Y, Z)$ to (13).

To show the way to obtain explicit forms of $C_{1,q}, C_{2,q}$, we give a sketch of proof for (14). In fact, for $\tau \in [a, b]$ and $\beta := 2L^2_q + \frac{3}{2}$, by using Itô’s formula to $e^{\beta Y(\cdot, \tau)}$ on $[0, T]$, we have
\[
\mathbb{E} \int_0^T \beta e^{\beta s}|Y(s, \tau)|^2 ds + \mathbb{E} \int_0^T e^{\beta s} |Z(s, \tau)|^2 ds
\leq e^{\beta T}\mathbb{E}|\xi(\tau)|^2 + [2L^2_q + 1] \mathbb{E} \int_0^T e^{\beta s}|Y(s, \tau)|^2 ds
\quad + \frac{1}{2} \mathbb{E} \int_0^T e^{\beta s}|Z(s, \tau)|^2 ds + \mathbb{E} \int_0^T e^{\beta s}|h(s, \tau, 0)|^2 ds,
\]
from which
\[
\mathbb{E} \int_0^T e^{\beta s}|Y(s, \tau)|^2 ds + \mathbb{E} \int_0^T e^{\beta s} |Z(s, \tau)|^2 ds
\leq 2e^{\beta T}\mathbb{E}|\xi(\tau)|^2 + 2e^{\beta T}\mathbb{E} \int_0^T |h(s, \tau, 0)|^2 ds.
\]
In addition,
\[
\sup_{t \in [0, T]} e^{\beta t} \mathbb{E}|Y(t, \tau)|^2 \leq e^{\beta T}\mathbb{E}|\xi(\tau)|^2 + \mathbb{E} \int_0^T e^{\beta s}|Y(s, \tau)|^2 ds
\quad + 2L^2_q \mathbb{E} \int_0^T e^{\beta s}|Z(s, \tau)|^2 ds + 2\mathbb{E} \int_0^T e^{\beta s}|h(s, \tau, 0)|^2 ds.
\]
Combining (17) and (18) together, we arrive at (14).

To see the above (15), we observe that
\[
\hat{Y}(t, \tau) = \xi_1(\tau) - \xi_2(\tau) + \int_t^\tau \left[ h_2(s, \tau, Z_1(s, \tau)) - h_2(s, \tau, Z_2(s, \tau))
\quad + h_1(s, \tau, Z_1(s, \tau)) - h_2(s, \tau, Z_1(s, \tau)) \right] ds - \int_t^\tau \tilde{Z}(s, \tau) dW(s).
\]
The similar as (14), we can obtain (15) as well.

**Step 2:** In this step, we prove the results on $(P, Q)$. We define two maps $P : [a, b] \to \mathbb{R}$, $Q : [a, b] \to L^2_F(0, T; \mathbb{R})$, i.e.,
\[
P(\tau) = Y(0, \tau), \quad Q(\cdot, \tau) = Z(\cdot, \tau), \quad \tau \in [a, b].
Obviously, they are well-defined. For the uniqueness, suppose there are \((P_i(\cdot), Q_i(\cdot, \cdot))\) satisfying

\[
P_i(\tau) = \xi(\tau) + \int_0^T h(s, \tau, Q_i(s, \tau))ds - \int_0^T Q_i(s, \tau)dW(s), \quad \tau \in [a, b].
\]

Let

\[
Y_i(t, \tau) := P_i(\tau) - \int_0^t h(s, \tau, Q_i(s, \tau))ds + \int_0^t Q_i(s, \tau)dW(s), \quad \tau \in [a, b],
\]

\[
Z_i(\cdot, \tau) := Q_i(\cdot, \tau). \quad \text{Here both} \ (Y_1, Z_1) \ (Y_2, Z_2) \ \text{satisfy} \ (13). \ \text{By the uniqueness of BSDEs}
\]

\[
|P_1(\tau) - P_2(\tau)|^2 + \mathbb{E} \int_0^T |Q_1(s, \tau) - Q_2(s, \tau)|^2ds = 0, \quad \tau \in [a, b].
\]

Estimates (10)-(11) can be deduced easily from (14) and (15). The remaining part is to prove (12) under (H2). To this end, we rewrite (13) as,

\[
\begin{cases}
Y'(t, \tau) = \xi'(\tau) + \int_0^t [h(s, \tau, Z(s, \tau)) - h(s, \tau, 0)] ds - \int_t^T Z(s, \tau)dW(s), \\
\xi'(\tau) := \xi(\tau) + \int_0^\tau h(s, \tau, 0)ds, \quad Y'(t, \tau) := Y(t, \tau) + \int_0^t h(s, \tau, 0)ds.
\end{cases}
\]

Therefore our conclusion follows from (8) and comparison theorems of BSDEs.  \( \square \)

2.2. The well-posedness of SFAE. We are ready to give the well-posedness of (6). To this end, we introduce

(H3) Let

\[
\xi : [a, b] \times \Omega \to \mathbb{R}, \quad h : [0, T] \times [a, b] \times \mathbb{R} \times \mathbb{R} \times \Omega \to \mathbb{R}
\]

be measurable such that \( s \mapsto h(s, x, p, q) \) is \( \mathbb{F} \)-adapted, \( x \mapsto h(s, x, p, q) \) is continuous, and

\[
\xi(\cdot) \in C([a, b]; L^2_{\mathbb{F}}(\Omega; \mathbb{R})), \quad \sup_{x \in [a, b]} |h(\cdot, x, 0, 0)| \in L^2_{\mathbb{F}}(0, T; \mathbb{R}),
\]

\[
|h(s, x, p_1, q_1) - h(s, x, p_2, q_2)| \leq \tilde{L}_p|p_1 - p_2| + \tilde{L}_q|q_1 - q_2|,
\]

\[
\begin{cases}
[h(s, x, p_1, q) - h(s, x, p_2, q)] [p_1 - p_2] \leq 0, \\
\tilde{L}_q, \tilde{L}_q > 0, \quad s \in [0, T], \quad x \in [a, b], \quad p_i \in [0, \infty), \quad q_i, q \in \mathbb{R}.
\end{cases}
\]

Moreover, there exists Lipschitz function \( H : [a, b] \to \mathbb{R}^+ \) such that

\[
\xi(x) + \int_0^T h(s, x, H(x), 0)ds \geq 0, \quad \xi(x) + \int_0^T h(s, x, 0, 0)ds \leq H(x). \tag{20}
\]

Given \( \tilde{L}_p, \tilde{L}_q \), we define \( \tilde{C}_{1,q}, \tilde{C}_{2,q} \) in a similar way as (9).

**Theorem 2.3.** Let (H3) hold true. Then (6) admits a unique solution

\[
(P(\cdot), Q(\cdot, \cdot)) \in C([a, b]; \mathbb{R}) \times C([a, b]; L^2_{\mathbb{F}}(0, T; \mathbb{R})). \tag{21}
\]
such that $0 \leq P(x) \leq H(x)$, $x \in [a, b]$. Moreover, given $(\xi, h, \mu)$ satisfying (H3), $i = 1, 2$, if $(P_i(\cdot), Q_i(\cdot, \cdot))$ is the corresponding solution, then

$$
|P_1(x) - P_2(x)|^2 + \mathbb{E} \int_0^T |Q_1(s, x) - Q_2(s, x)|^2 ds
\leq K_2 \mathbb{E} \int_0^T \left| h_1(s, x, P_2(x), Q_2(s, x)) - h_2(s, x, P_2(x), Q_2(s, x)) \right|^2 ds
+ K_1 \mathbb{E} |\xi_1(x) - \xi_2(x)|^2,
$$

(22)

where $K_1$, $K_2$ are defined as follows,

$$
K_1 := \bar{C}_{2,q} + 4 \bar{C}_{2,q} T e^{\bar{L}_q^2 T} |\bar{L}_p|^2, \quad K_2 := 2 \bar{C}_{2,q} + 4 \bar{C}_{2,q} T^2 e^{\bar{L}_q^2 T} |\bar{L}_p|^2.
$$

Proof. We split the proof of the wellposedness into two steps.

**Step 1:** Given $\tilde{C}_{2,q}$ in (9) associated with $\tilde{L}_p$, $\tilde{L}_q$, let $\mu_0$ be a positive constant defined as $\mu_0 := \frac{1}{3\bar{C}_{2,q}} \mathbb{E} |\mu_0|^2 \wedge 1$. For this $\mu_0$, we prove that the following equation admits a unique solution $(P(\cdot), Q(\cdot, \cdot)) \in C([a, b]; \mathbb{R}) \times C([a, b]; L^2_p(0, T; \mathbb{R}))$

$$
P(x) = \xi(x) + \int_0^T h(s, x, \mu_0 P(x), Q(s, x)) ds - \int_0^T Q(s, x) dW(s).
$$

(23)

To do so, we first claim that the following equation admits a unique pair of solution $(P(\cdot), Q(\cdot, \cdot))$

$$
P(x) = \xi(x) + \int_0^T h(s, x, \mu_0 [0 \vee P(x)] \wedge H(x)], Q(s, x)) ds - \int_0^T Q(s, x) dW(s).
$$

(24)

Here e.g., $0 \vee P(x) := \max\{P(x), 0\}$. In fact, given $i = 1, 2$, $p_i(\cdot) \in C([a, b]; \mathbb{R})$, let us look at

$$
P_i(x) = \xi(x) + \int_0^T h(s, x, \mu_0 [(0 \vee p_i(x)] \wedge H(x)], Q_i(s, x)) ds - \int_0^T Q_i(s, x) dW(s).
$$

From (H3) we see that

$$
q \mapsto h(s, x, \mu_0 [(0 \vee p_i(x)] \wedge H(x)], q) \text{ is Lipschitz,}
$$

$$
x \mapsto h(s, x, \mu_0 [(0 \vee p_i(x)] \wedge H(x)], 0)
$$

is continuous,

$$
|h(s, x, \mu_0 [(0 \vee p_i(x)] \wedge H(x)], 0)| \leq |h(s, x, 0, 0)| + \bar{L}_p |\mu_0 [(0 \vee p_i(x)] \wedge H(x)]|.
$$

It then follows from Lemma 2.2 that there exists a unique pair $(P_i(\cdot), Q_i(\cdot, \cdot))$ such that

$$
|P_1(x) - P_2(x)|^2 + \mathbb{E} \int_0^T |Q_1(s, x) - Q_2(s, x)|^2 ds
\leq \bar{C}_{2,q} \mathbb{E} \int_0^T h(s, x, \mu_0 [(0 \vee p_2(x)] \wedge H(x)], Q_1(s, x))
- h(s, x, \mu_0 [(0 \vee p_2(x)] \wedge H(x)], Q_1(s, x))|^2 ds
\leq \bar{C}_{2,q} \mu_0^2 |\bar{L}_p|^2 T |p_1(x) - p_2(x)|^2.
$$

In the above we use the fact that

$$
|x_1 \vee K - x_2 \vee K| \leq |x_1 - x_2|, \quad x_i \in \mathbb{R}, \quad K \in \mathbb{R}^+.
$$
If \( \frac{1}{ \| Z \|_{L^2}^2} \geq 1 \), then \( \mu_0 = 1 \), and we can obtain the existence and uniqueness of solution to (6) by contraction. If \( \frac{1}{ \| Z \|_{L^2}^2} < 1 \), then we have \( \mu_0^2 := \frac{1}{ \| Z \|_{L^2}^2} \), and

\[
\sup_{x \in [a,b]} |P_1(x) - P_2(x)|^2 + \sup_{x \in [a,b]} \mathbb{E} \int_0^T |Q_1(s, x) - Q_2(s, x)|^2 ds \\
\leq \frac{1}{3} \sup_{x \in [a,b]} |p_1(x) - p_2(x)|^2.
\]

The solvability of (24) follows from contraction method. To show \( (P(\cdot, Q(\cdot, \cdot)) \) satisfies (23), we prove that \( 0 \leq P(x) \leq H(x) \), \( x \in [a, b] \). To this end, for any \( \alpha \in [0, H(x)] \), we have

\[
\begin{align*}
\left\{ \begin{array}{ll}
h(\cdot, x, \alpha, 0) - h(\cdot, x, 0, 0) & \leq 0, \text{ a.s.} \\
[h(\cdot, x, \alpha, 0) - h(\cdot, x, H(x), 0)](\alpha - H(x)) & \leq 0, \text{ a.s.}
\end{array} \right.
\end{align*}
\]

which is implied by (19). Consequently, if \( H(x) > 0 \),

\[
h(s, x, H(x), 0) \leq h(s, x, \alpha, 0) \leq h(s, x, 0, 0), \quad \text{a.s., } \alpha \in [0, H(x)].
\]

The conclusion is obvious if \( H(x) = 0 \). By taking \( \alpha := \mu_0 \left[ (0 \lor P(x)) \land H(x) \right] \), it follows from Lemma 2.2 and (20) that \( 0 \leq P(x) \leq H(x) \). Hence we obtain the well-posedness of (23).

**Step 2:** For the above \( \mu_0 < 1 \), we continue to prove that there exists constant \( 0 < \mu_1 \leq 1 - \mu_0 \) such that the existence and uniqueness of \( (P(\cdot, Q(\cdot, \cdot)) \) for the following equation holds,

\[
P(x) = \xi(x) + \int_0^T h(s, x, [\mu_0 + \mu_1] P(x), Q(s, x)) ds - \int_0^T Q(s, x) dW(s).
\]

To this end, given \( l(\cdot) \in C([a, b]; \mathbb{R}) \), and

\[
\hat{h}(s, x, p, q) := h(s, x, p + \mu_1 \left[ (0 \lor l(x)) \land H(x) \right], q), \quad p, q \in \mathbb{R},
\]

we claim that the following equation admits a unique solution

\[
P(x) = \xi(x) + \int_0^T \hat{h}(s, x, \mu_0 P(x), Q(s, x)) ds - \int_0^T Q(s, x) dW(s).
\]

In fact, since \( \hat{h} \) satisfies (19) and \( \mu_0^2 := \frac{1}{ \| Z \|_{L^2}^2} \), by following the similar arguments ranging from (24) to (25) of Step 1, there exists a unique \( P, Q \) satisfying

\[
P(x) = \xi(x) + \int_0^T \hat{h}(s, x, \mu_0 \left[ (0 \lor P(x)) \land H(x) \right], Q(s, x)) ds - \int_0^T Q(s, x) dW(s).
\]

On the other hand, for

\[
\begin{align*}
\alpha & := \mu_0 \left[ (0 \lor P(x)) \land H(x) \right] \in [0, (1 - \mu_1) H(x)], \\
0 & \leq \alpha_1 := \alpha + \mu_1 \left[ (0 \lor l(x)) \land H(x) \right] \leq H(x),
\end{align*}
\]

one obtains that

\[
\begin{align*}
\left\{ \begin{array}{l}
[h(\cdot, x, \alpha, 0) - h(\cdot, x, 0, 0)] \cdot \alpha_1 & = [h(\cdot, x, \alpha_1, 0) - h(\cdot, x, 0, 0)] \cdot \alpha_1 \leq 0, \\
[h(\cdot, x, \alpha, 0) - h(\cdot, x, H(x), 0)] \cdot (\alpha_1 - H(x)) & = [h(\cdot, x, \alpha_1, 0) - h(\cdot, x, H(x), 0)] \cdot (\alpha_1 - H(x)) \leq 0, \quad \text{a.s.}
\end{array} \right.
\end{align*}
\]
Multiplying we then obtain that which is also implied by (19). Similar as (26), we see that
\[ h(s, x, H(x), 0) \leq \hat{h}(s, x, \alpha, 0) \leq h(s, x, 0, 0), \quad \text{a.s.} \quad \alpha \in [0, H(x)]. \quad (28) \]
It then follows from Lemma 2.2 and (20) that \( 0 \leq P(x) \leq H(x) \). Consequently, we finish the above declaration. For \( p_i(\cdot) \in C([a, b]; \mathbb{R}) \) and
\[ \hat{h}_i(s, x, p, q) := h(s, x, p + \mu_i \lfloor 0 \lor p_i(x) \rfloor \land H(x), q), \quad p, q \in \mathbb{R}, \]
there exists a unique pair \((P_i(\cdot), Q_i(\cdot, \cdot))\) such that \( 0 \leq P_i(x) \leq H(x), \quad x \in [a, b], \) and
\[ P_1(x) = \xi(x) + \int_0^T \hat{h}_i(s, x, \mu_0 P_i(x), Q_i(s, x))ds - \int_0^T Q_i(s, x)dW(s). \quad (29) \]
As a result, the mapping \( \Psi : C([a, b]; \mathbb{R}) \to C([a, b]; \mathbb{R}), \quad P_i(\cdot) := (\Psi(p_i)) (\cdot) \) is well-defined. Moreover,
\[ P_1(x) - P_2(x) = \int_0^T [H_1(s, x) + H_2(s, x) + A(s, x)(Q_1(s, x) - Q_2(s, x))] ds \]
\[ - \int_0^T (Q_1(s, x) - Q_2(s, x))dW(s), \quad x \in [a, b], \]
where
\[ A(s, x) := \frac{H_3(s, x)}{(Q_1(s, x) - Q_2(s, x)) \land 0}, \]
\[ H_1(s, x) := \hat{h}_1(s, x, \mu_0 P_1(x), Q_1(s, x)) - \hat{h}_1(s, x, \mu_0 P_2(x), Q_1(s, x)), \]
\[ H_2(s, x) := \hat{h}_2(s, x, \mu_0 P_2(x), Q_1(s, x)) - \hat{h}_2(s, x, \mu_0 P_2(x), Q_1(s, x)), \]
\[ H_3(s, x) := \hat{h}_2(s, x, \mu_0 P_2(x), Q_1(s, x)) - \hat{h}_2(s, x, \mu_0 P_2(x), Q_2(s, x)). \]
For any \( x \in [a, b] \), by introducing a new probability \( \mathbb{P}_{A,x} \) as,
\[ \frac{d\mathbb{P}_{A,x}}{d\mathbb{P}} = \exp \left\{ \int_0^T A(s, x)dW(s) - \frac{1}{2} \int_0^T |A(s, x)|^2ds \right\}, \]
we then obtain that
\[ P_1(x) - P_2(x) = \mathbb{E}_{\mathbb{P}_{A,x}} \int_0^T [H_1(s, x) + H_2(s, x)] ds, \quad x \in [a, b]. \]
Multiplying \( P_1(x) - P_2(x) \) on both sides, and making use of condition (19),
\[ \left\{ \begin{array}{l}
(P_1(x) - P_2(x))\mathbb{E}_{\mathbb{P}_{A,x}} \int_0^T H_1(s, x)ds \leq 0,
(P_1(x) - P_2(x))\mathbb{E}_{\mathbb{P}_{A,x}} \int_0^T H_2(s, x)ds
\end{array} \right. \quad (30) \]
\[ \leq \frac{1}{2} \mu_1 T L P |p_1(x) - p_2(x)|^2 + \frac{1}{2} \mu_1 T L P |p_1(x) - p_2(x)|^2. \]
From (30) we conclude that
\[ [1 - \frac{1}{2} \mu_1 L P T] |P_1(x) - P_2(x)|^2 \leq \frac{1}{2} \mu_1 L P T |p_1(x) - p_2(x)|^2. \quad (31) \]
We choose \( \mu_1 = \min \{ \frac{1}{2} L P T, 1 - \mu_0 \} \), and obtain that
\[ [1 - \frac{1}{2} \mu_1 L P T] \geq \frac{1}{2} \mu_1 L P T + \frac{1}{2}. \]
Therefore $\Psi$ is a contraction map. Thanks to fixed point theorem, there exists a unique $P(\cdot)$ such that $\Psi(P) = P$. Plugging this $P(\cdot)$ into

$$P(x) = \xi(x) + \int_0^T h(s, x, \mu_0 P(x) + \mu_1 \left[ (0 \vee P(x)) \land H(x) \right], Q(s, x)) ds - \int_0^T Q(s, x) dW(s),$$

one can also obtain $Q(\cdot, \cdot) \in C([a, b]; L^2_T(0, T; \mathbb{R}))$ as well. Using the same arguments as (28), one has $0 \leq P(x) \leq H(x)$ and the solvability of (27).

**Step 3:** Given $\mu_1$ in Step 2, if $\frac{1}{2L_x T} \geq 1 - \mu_0$, then $\mu_0 + \mu_1 = 1$ and the proof is finished. Otherwise $\mu_1 = \frac{1}{2L_x T}$ and there exists $\mu_2 > 0$ such that $\mu_0 + \mu_1 + \mu_2 \leq 1$. Repeating the above procedures, and choosing $\mu_2 := \min\{\frac{1}{2L_x T}, 1 - \mu_0 - \mu_1\}$, one obtains the well-posedness of

$$P(x) = \xi(x) + \int_0^T h(s, x, [\mu_0 + \mu_1 + \mu_2] P(x), Q(s, x)) ds - \int_0^T Q(s, x) dW(s).$$

By induction we get the solvability of (6) such that $0 \leq P(x) \leq H(x)$, $x \in [a, b]$.

**Step 4:** We prove estimate (22). For notational simplicity, let

$$(\tilde{P}(\cdot), \tilde{Q}(\cdot, \cdot), \tilde{\xi}(\cdot)) := (P_1(\cdot) - P_2(\cdot), Q_1(\cdot, \cdot) - Q_2(\cdot, \cdot), \xi_1(\cdot) - \xi_2(\cdot)),$$

and $\tilde{L}_{p,1}$, $\tilde{L}_{q,1}$ be the Lipschitz constant of $h_1(s, x, p, q)$. For $P_i(\cdot)$,

$$P_1(x) - P_2(x) = \xi_1(x) - \xi_2(x) - \int_0^T (Q_1(s, x) - Q_2(s, x)) dW(s)$$

$$+ \int_0^T \left[ H_4(s, x) + H_6(s, x) + B(s, x) (Q_1(s, x) - Q_2(s, x)) \right] ds,$$

where $x \in [a, b]$,

$$\left\{
\begin{array}{l}
B(s, x) := \frac{H_5(s, x)}{(Q_1(s, x) - Q_2(s, x)) I_{[Q_1(s, x) - Q_2(s, x)] \neq 0}}, \\
H_4(s, x) := h_1(s, x, P_1(x), Q_1(s, x)) - h_1(s, x, P_2(x), Q_1(s, x)), \\
H_5(s, x) := h_1(s, x, P_2(x), Q_1(s, x)) - h_1(s, x, P_2(x), Q_2(s, x)), \\
H_6(s, x) := h_1(s, x, P_2(x), Q_2(s, x)) - h_2(s, x, P_2(x), Q_2(s, x)).
\end{array}
\right.$$ 

For any $x \in [a, b]$, by introducing a new probability $\mathbb{P}_{B, x}$ as,

$$\frac{d\mathbb{P}_{B, x}}{d\mathbb{P}} = \exp \left\{ \int_0^T B(s, x) dW(s) - \frac{1}{2} \int_0^T |B(s, x)|^2 ds \right\} := \exp \{ \mathfrak{B}(T) \},$$

one can rewrite (32) as,

$$P_1(x) - P_2(x) = \mathbb{E}_{\mathbb{P}_{B, x}} [\xi_1(x) - \xi_2(x)] + \mathbb{E}_{\mathbb{P}_{B, x}} \int_0^T [H_4(s, x) + H_6(s, x)] ds.$$
Multiplying $\bar{P}(x) := P_1(x) - P_2(x)$ on both sides, and making use of condition (19),

$$|\bar{P}(x)|^2 = \left[ E_{P_{B,x}} \xi(x) + E_{P_{B,x}} \int_0^T H_4(s, x)ds + E_{P_{B,x}} \int_0^T H_6(s, x)ds \right] \bar{P}(x)$$

$$\leq \frac{1}{2} |\bar{P}(x)|^2 + \left[ E_{P_{B,x}} \xi(x) \right]^2 + \left[ E_{P_{B,x}} \int_0^T H_6(s, x)ds \right]^2,$$

where for example,

$$\left[ E_{P_{B,x}} \xi(x) \right] \cdot \bar{P}(x) = 2 \cdot \left[ E_{P_{B,x}} \xi(x) \right] \cdot \frac{1}{2} \bar{P}(x) \leq \left[ E_{P_{B,x}} \xi(x) \right]^2 + \frac{1}{4} |\bar{P}(x)|^2.$$

A direct calculation shows that,

$$\left\{ \begin{array}{l}
\left[ E_{P_{B,x}} \xi(x) \right]^2 = \left[ E e^{S_x(T)} \xi(x) \right]^2 \leq E e^{2S_x(T)} \cdot E |\xi(x)|^2 \\
\left[ E_{P_{B,x}} \int_0^T H_6(s, x)ds \right]^2 \leq e^{L_{q,1}^2T} \cdot \left[ E \int_0^T |H_6(s, x)|ds \right]^2.
\end{array} \right.$$

To sum up, we derive the following estimate

$$|\bar{P}(x)|^2 \leq 2e^{L_{q,1}^2T} \cdot \left[ E |\xi(x)|^2 + \left( E \int_0^T |H_6(s, x)|ds \right)^2 \right]. \quad (33)$$

Let $\tilde{h}_i(s, x, Q(s, x)) := h_i(s, x, P_i(x), Q(s, x))$, from Lemma 2.2 the following equation admits a unique pair of solution,

$$\bar{P}_i(x) = \xi_i(x) + \int_0^T \tilde{h}_i(s, x, \bar{Q}_i(s, x))ds - \int_0^T \tilde{Q}_i(s, x)dW(s), \quad (34)$$

such that,

$$|\bar{P}_1(x) - \bar{P}_2(x)|^2 + E \int_0^T |\bar{Q}_1(s, x) - \bar{Q}_2(s, x)|^2ds$$

$$\leq \tilde{C}_{1,q} E |\xi_1(x) - \xi_2(x)|^2 + \tilde{C}_{2,q} E \int_0^T |\tilde{h}_1(s, x, \bar{Q}_1(s, x)) - \tilde{h}_2(s, x, \bar{Q}_2(s, x))|^2ds.$$

By the uniqueness of (34) we know that $\bar{P}_i(\cdot) = P_i(\cdot)$ and $\bar{Q}_i(\cdot, \cdot) = Q_i(\cdot, \cdot)$. Hence

$$|\bar{P}_1(x) - \bar{P}_2(x)|^2 + E \int_0^T |\bar{Q}_1(s, x) - \bar{Q}_2(s, x)|^2ds$$

$$\leq \tilde{C}_{2,q} E \int_0^T |h_1(s, x, P_1(x), Q_2(s, x)) - h_2(s, x, P_2(x), Q_2(s, x))|^2ds$$

$$+ \tilde{C}_{1,q} E |\xi_1(x) - \xi_2(x)|^2.$$

Recall $H_6(\cdot, x)$ above, from (33) we arrive at

$$E \int_0^T |h_1(s, x, P_1(x), Q_2(s, x)) - h_2(s, x, P_2(x), Q_2(s, x))|^2ds$$

$$\leq 2e^{L_{q,1}^2T} |\bar{P}(x)|^2 + 2E \int_0^T |H_6(s, x)|^2ds$$

$$\leq [4Te^{L_{q,1}^2T} |\bar{P}(x)|^2 + 2 + 4T^2e^{L_{q,1}^2T} |\bar{P}(x)|^2] \left( E \int_0^T |H_6(s, x)|^2ds. \right)$$

Then (22) is easy to see. \hfill \Box
Remark 1. Given $\bar{x} \in [a, b]$, $(P, Q)$ satisfying (6) and (21), we look at the following BSDE,

$$Y(t, \bar{x}) = \xi(\bar{x}) + \int_t^T h(s, \bar{x}, P(\bar{x}), Q(s, \bar{x}))ds - \int_t^T Z(s, \bar{x})dW(s), \ t \in [0, T].$$

Under condition (H3), the existence and uniqueness of $(Y, Z)$ is easy to see. Let $t = 0$, one has

$$Y(0, \bar{x}) = \xi(\bar{x}) + \int_0^T h(s, \bar{x}, P(\bar{x}), Q(s, \bar{x}))ds - \int_0^T Z(s, \bar{x})dW(s).$$

Recall that $(P, Q)$ is the solution to (6), one has

$$Y(0, \bar{x}) - P(\bar{x}) = \int_0^T [Q(s, \bar{x}) - Z(s, \bar{x})]dW(s).$$

Taking expectation on both sides, one has $Y(0, \bar{x}) = P(\bar{x})$. In addition, $Q = Z$. As a result,

$$Y(t, \bar{x}) = \xi(\bar{x}) + \int_t^T h(s, \bar{x}, Y(0, \bar{x}), Z(s, \bar{x}))ds - \int_t^T Z(s, \bar{x})dW(s).$$

(35)

For fixed $\bar{x}$, it can be seen as a particular case of BSDEs with time delayed generators, see e.g. [6], [7]. For the general time-delayed BSDEs, it was proved in [6] that the corresponding existence and uniqueness of solution holds with either small time horizon or small Lipschitz constant. However, if we only consider the above special class (35), we can drop the aforementioned requirement of small time horizon or small Lipschitz constant by the above Theorem 2.3. As a tradeoff, we need some kind of monotonicity conditions, see (19) in (H3).

Remark 2. Let us give some careful observations on the non-increasing condition of $p \mapsto h(s, x, p, q)$ in (19). On the one hand, if we drop this condition, by the proof of Step 1 of Theorem 2.3, we can obtain the existence and uniqueness of $(P, Q)$ for (6) when either time horizon $T$ or Lipschitz constant $L_p$ is small enough. Similar conclusion also holds for BSDEs with time delayed generators, see [6]. From this viewpoint, such a non-increasing condition is somewhat structural conditions to ensure the well-posedness of SFAE (6) for any fixed $L_p$ and $T$. On the other hand, the following example in some sense shows the necessity of such non-increasing condition. For $x \in [a, b]$, let $\xi(x) = x - b - 1$, $T = 1$, and $h(s, x, p, q) = p$. Here $p \mapsto h(s, x, p, q)$ is increasing, and $\xi(x) \leq -1$. It is obvious to see that (6) is ill-posed.

2.3. An extension. We discuss one extension by relaxing the continuity of $x \mapsto (P(x), Q(\cdot, x))$. To this end, we introduce two more spaces as follows,

$$L^2(a, b; \mathbb{R}) := \left\{ X : [a, b] \to \mathbb{R} \mid X(\cdot) \text{ is measurable,} \right\},$$

$$\|X\|_{L^2(a, b; \mathbb{R})} := \left( \int_a^b |X(r)|^2dr \right)^{\frac{1}{2}} < \infty,$$

$$L^2(a, b; L^2_p(0, T; \mathbb{R})) := \left\{ X : [0, T] \times [a, b] \times \Omega \to \mathbb{R} \mid X(\cdot, \cdot) \text{ is measurable,} \right\},$$

and for almost $x \in [a, b]$, $X(\cdot, x)$ is $\mathbb{F}$-adapted,

$$\|X\|_{L^2(a, b; L^2_p(0, T; \mathbb{R}))} := \left\{ \mathbb{E} \int_a^b \int_0^T |X(r, x)|^2drdx \right\}^{\frac{1}{2}} < \infty.$$
Definition 2.4. A pair of \((P(\cdot), Q(\cdot, \cdot))\) \(\in L^2(a, b; \mathbb{R}) \times L^2(a, b; L^2_p(0, T; \mathbb{R}))\) is called a solution of (6) if it holds true almost surely.

We also introduce the corresponding assumptions as follows.

\(\text{(H3-1)}\) Let

\[
\xi : [a, b] \times \Omega \to \mathbb{R}, \quad h : [0, T] \times [a, b] \times \mathbb{R} \times \Omega \to \mathbb{R}
\]

be measurable such that \(s \mapsto h(s, x, p, q)\) is \(\mathbb{F}\)-adapted, and

\[
\begin{align*}
\xi(\cdot) & \in L^2(a, b; L^2_p(\Omega, \mathbb{R})), \quad h(\cdot, 0, 0) \in L^2(a, b; L^2_p(0, T; \mathbb{R})), \\
|h(s, x, p_1, q_1) - h(s, x, p_2, q_2)| & \leq \bar{L}_p|p_1 - p_2| + \bar{L}_q|q_1 - q_2|, \\
[h(s, x, p_1, q) - h(s, x, p_2, q)] [p_1 - p_2] & \leq 0, \\
\bar{L}_q, \bar{L}_q & > 0, \quad s \in [0, T], \quad x \in [a, b], \quad p_i \in [0, \infty), \quad q_i, q \in \mathbb{R}.
\end{align*}
\]

Moreover, there exists function \(H \in L^2(a, b; \mathbb{R})\) such that

\[
\xi(x) + \int_0^T h(s, x, H(x), 0) ds \geq 0, \quad \xi(x) + \int_0^T h(s, x, 0, 0) ds \leq H(x).
\]

Using almost the same proof as Theorem 2.3, we obtain the following conclusion.

Theorem 2.5. Let \(\text{(H3-1)}\) hold true. Then SFAE (6) admits a unique solution \((P(\cdot), Q(\cdot, \cdot))\) \(\in L^2(a, b; \mathbb{R}) \times L^2(a, b; L^2_p(0, T; \mathbb{R}))\) such that 0 \(\leq P(x) \leq H(x), \quad x \in [a, b], \quad \text{a.e.}\) Moreover, given \((\xi_i, h_i)\) satisfying \(\text{(H3-1)}, i = 1, 2\), if \((P_i(\cdot), Q_i(\cdot, \cdot))\) is the corresponding solution, then

\[
\begin{align*}
\int_a^b \left| P_1(x) - P_2(x) \right|^2 dx + \mathbb{E} \int_a^b \int_0^T |Q_1(s, x) - Q_2(s, x)|^2 ds dx \\
& \leq K_2 \mathbb{E} \int_a^b \int_0^T \left| h_1(s, x, P_2(x), Q_2(s, x)) - h_2(s, x, P_2(x), Q_2(s, x)) \right|^2 ds dx \\
& \quad + K_1 \mathbb{E} \int_a^b |\xi_1(x) - \xi_2(x)|^2 dx,
\end{align*}
\]

where \(K_1, \ K_2\) are defined as in Theorem 2.3.

3. Applications to state dependent risk indifference prices.

3.1. Problem formulation. For simplicity, suppose there are two financial assets in the market, i.e., one non-risky asset (e.g., bond) used as numeraire, and one risky asset (e.g., stock) described as,

\[
\begin{align*}
& \frac{dS(r)}{S(r)} = \alpha(r) S(r) dr + \beta(r) S(r) dW(r), \quad r \in [0, T], \\
& S(0) = s_0,
\end{align*}
\]

where \(\alpha(\cdot), \ \beta(\cdot)\) are bounded adapted processes, \(\epsilon \leq |\beta(\cdot)| \leq K, \ 0 < \epsilon < K\). Let \(\pi(\cdot) \in L^2(0, T; \mathbb{R})\) be dollar amount of the agent invested in the risky asset. Given initial wealth \(x\), the wealth process is,

\[
X(s) = x + \int_0^s \alpha(u) \pi(u) du + \int_0^s \beta(u) \pi(u) dW(u), \quad s \in [0, T].
\]
In the following, let $x \in [a, b]$, where $a \geq 0$. As a generalization of the example in the Introduction, for given $\pi$ to minimize risk measure $\rho(X^\pi(T); x) := y^\pi(0; x)$, where $y^\pi(\cdot; x)$ satisfies a BSDE

$$y^\pi(t; x) = -X^\pi(T) + \int_t^T h(s, x, z^\pi(s; x))ds - \int_t^T z^\pi(s; x)dW(s), \quad t \in [0, T].$$

(37)

Here $h : [0, T] \times [a, b] \times \mathbb{R} \times \Omega \to \mathbb{R}$ is measurable such that $h(\cdot, \cdot, 0) = 0$, $s \mapsto h(s, x, z)$ is $\mathbb{F}$-adapted, $z \mapsto h(s, x, z)$ is Lipschitz continuous and convex.

**Remark 3.** Notice that the generator of (37) depends on $x$. This implies that $(g(\cdot, z(\cdot), \pi(\cdot)))$ should do so as well. As is shown later, such state dependence feature enables us to make qualitative analysis between optimal strategy and initial wealth.

By taking $p(\cdot; x) := \pi(\cdot; x)\beta(\cdot), \theta(\cdot) := \alpha(\cdot)\beta^{-1}(\cdot)$, for $t \in [0, T]$, we can rewrite (36), (37) as

$$\begin{cases}
X^p(t; x) = x + \int_0^t \theta(u)p(u; x)du + \int_0^t p(u; x)dW(u), \\
y^p(t; x) = -X^p(T; x) + \int_t^T h(s, x, z^p(s; x))ds - \int_t^T z^p(s; x)dW(s).
\end{cases}$$

For the above backward equation, let $t = 0$, and it is easy to see

$$y^p(0; x) = -x + \int_0^T g(s, x, Z^p(s; x), p(s; x))ds - \int_0^T Z^p(s; x)dW(s),$$

(38)

where for $s \in [0, T], x \in [a, b],

$$\begin{cases}
Z^p(s; x) := z^p(s; x) + p(s; x), \\
g(s, x, z, p(s; x)) := h(s, x, z - p(s; x)) - p(s; x)\theta(s).
\end{cases}$$

By the convexity of $h$, for $s \in [0, T], x \in [a, b]$, we define the Fenchel-Legendre transform as,

$$G(s, x, \theta(s)) := \sup_{r \in \mathbb{R}} \tilde{h}(s, x, \theta(s), r), \quad \text{a.s.}, \quad \tilde{h}(s, x, \theta(s), r) := -\theta(s) \cdot r - h(s, x, r).$$

(39)

Given $L$ the Lipschitz constant of $h$, $G < \infty$ is lead by $|\theta(\cdot)| < L$ (see page 35 in [9]).

In the following, we assume that the supremum in (39) is reachable. More precisely, for given $\theta(\cdot)$, suppose there exists measurable mapping $\tilde{p} : [0, T] \times [a, b] \times \Omega \to \mathbb{R}$ such that for $x \in [a, b], \tilde{p}(\cdot, x) \in L^2_{\mathbb{F}}(0, T; \mathbb{R})$, (38) admits a unique pair of $(\tilde{Y}(\cdot, x), \tilde{Z}(\cdot, x))$, and for $s \in [0, T],

$$G(s, x, \theta(s)) = -\theta(s)[\tilde{Z}(s, x) - \tilde{p}(s, x)] - h(s, x, \tilde{Z}(s, x) - \tilde{p}(s, x)) \quad \text{a.s.}$$

(40)

From (40), for any measurable $p(\cdot, \cdot)$ satisfying $p(\cdot, x) \in L^2_{\mathbb{F}}(0, T; \mathbb{R})$, it is easy to see

$$-G(s, x, \theta(s)) - \theta(s)\tilde{Z}(s, x) = g(s, x, \tilde{Z}(s, x), \tilde{p}(s, x)) \leq g(s, x, \tilde{Z}(s, x), p(s, x)) \leq g(s, x, Z(s, x), p(s, x)).$$

By the comparisons theorem of BSDEs (e.g., Theorem 2.2 and Proposition 3.1 of [9]),

$$\pi(s; x) := \tilde{p}(s; x)\beta^{-1}(s), \quad s \in [0, T], \quad x \in [a, b],$$

(41)
is an optimal strategy in the sense that $y^\pi(0; x)$ is the minimized risk measure. We will provide three explicit examples to verify assumption (40).

Remark 4. Notice that our main interest is not to find general sufficiency for optimal investment strategy, but to introduce the subsequent state dependent risk indifference price. Therefore, we introduce assumption (40) and skip the relevant improvements.

We point out that (40) also enables us to directly use the technique of comparison theorems which is also quite useful in other financial problems, see e.g., Section 3 in [9], Section 2 in [21], Section 4 in [11], Theorem 7.17 in [14].

The introduced $G$ of (39) is helpful for later study on risk indifference price. It is worthy mentioning that equality (40) does not imply that $G$ is a linear function with respect to $\theta(\cdot)$ due to the reliance of $\hat{Z}, \hat{p}, \hat{Y}$ on $\theta(\cdot)$.

Suppose the agent wants to buy one financial position (or liability) represented by bounded $\mathcal{F}_T$-measurable random variable $F$. Let the price be $q(x)$ which depends on initial wealth $x$. Hence the real initial wealth becomes $x_q := x - q(x)$. Likewise, the optimal strategy $\hat{\pi}(\cdot; x_q)$ associated with $F$ and $x_q$ satisfies

$$\rho(X^\pi(T; x_q) + F; x_q) = \inf_{\pi} \rho(X^\pi(T; x_q) + F; x_q). \quad (42)$$

Definition 3.1. A function $q(\cdot)$ is called (buyer’s) risk indifference price of $F$ on $[a, b]$, if for any initial wealth $x \in [a, b]$ and optimal strategies $\pi(\cdot; x), \hat{\pi}(\cdot; x_q)$, one has $0 \leq q(x) \leq x$,

$$\rho(X^\pi(T; x_q) + F; x_q) = \rho(X^\pi(T, x); x). \quad (43)$$

Remark 5. In contrast with existing indifference prices (e.g., [1], [10], [11], [12], [14], [15], [20], [30]), above notion allows the dependence on initial wealth in a reasonable way, the point of which is inspired by the state dependent risk aversion in [3]. We introduce and establish the well-posedness of SFAE that describes this price, and carry out some quantitative analysis between initial wealth and risk indifference price.

3.2. Risk indifference pricing via stochastic Fredholm equation. In this part, we choose initial wealth $x \in [0, b]$, and prove that risk indifference price $q(\cdot)$ satisfies the SFAE introduced in Section 2. For $G$ appearing in (39) and financial position $F$, we introduce the following hypotheses.

(H4) Let $0 < \alpha_1 < F < \alpha_2 < b$ with constants $\alpha_1, \alpha_2, G(\cdot, 0, \theta(\cdot)) \in L_2^2(0, T; \mathbb{R})$ and

$$\begin{align*}
\left\{ \begin{array}{l}
G(s, x_1, \theta(s)) - G(s, x_2, \theta(s)) \geq 0, \\
G(s, x_1, \theta(s)) - G(s, x_2, \theta(s)) \leq \bar{L}|x_1 - x_2|,
\end{array} \right. \\
x_1 \in [0, b], \quad s \in [0, T], \quad \bar{L} > 0.
\end{align*} \quad (44)$$

From (H4) and Lemma 2.2, there exists a unique pair of $(\bar{P}(\cdot), \bar{Q}(\cdot, \cdot))$ satisfying

$$\bar{P}(x) = -x - \int_0^T \left[ G(s, x, \theta(s)) + \bar{Q}(s, x)\theta(s) \right] ds - \int_0^T \bar{Q}(s, x)dW(s), \quad (45)$$
where \( x \in [0, b] \). Based on \( \hat{P}(\cdot) \), we consider the following SFAE,
\[
\hat{P}(x) = F + x + \hat{P}(x) + \int_0^T \left[ G(s, x - \hat{P}(x), \theta(s)) + \hat{Q}(s, x)\theta(s) \right] ds \\
+ \int_0^T \hat{Q}(s, x)dW(s), \quad x \in [0, b].
\] (46)

**Theorem 3.2.** Given \( G \) in (39), suppose (40) and (H4) hold, \( \alpha_2 \hat{L}T < \alpha_1 \). Then (46) is solvable, and \( \hat{P}(\cdot) \) is the risk indifference price on \([\alpha_2, b]\).

**Proof.** We first prove the solvability of (46). Since \( \alpha_2 \hat{L}T < \alpha_1 \), we arrive at
\[
\int_0^T |G(s, x, \theta(s)) - G(s, x - \alpha_2, \theta(s))| ds \leq \hat{L}_2T \leq \alpha_1.
\] (47)

For any \( x \in [0, b], p, q \in \mathbb{R}, s \in [0, T] \), let
\[
\xi(x) := F + \hat{P}(x) + x, \quad h(s, x, p, q) := G(s, x - p, \theta(s)) + q\theta(s).
\]

According to (H4), we have
\[
\begin{cases}
\xi(x) + \int_0^T h(s, x, \alpha_2, 0) ds = F + x + \hat{P}(x) + \int_0^T G(s, x - \alpha_2, \theta(s)) ds \geq 0, \\
\xi(x) + \int_0^T h(s, x, 0, 0) ds = F + \hat{P}(x) + \int_0^T G(s, x, \theta(s)) ds \leq \alpha_2.
\end{cases}
\]

Consequently, from Theorem 2.3, (46) admits a unique solution \((\hat{P}(\cdot), \hat{Q}(\cdot, \cdot))\) such that \( 0 \leq \hat{P}(x) \leq \alpha_2, \ x \in [0, b] \).

Next we verify that \( \hat{P}(\cdot) \) is indeed risk indifference price. Let \( \hat{x} := x - \hat{P}(x) \in [0, b], \ x \in [\alpha_2, b] \). From (40) we define optimal strategy \( \bar{p}(\cdot), \bar{p}(\cdot) \) associated with \( x \) and \( \hat{x} \), respectively. Due to Definition 3.1, it is suffice to prove
\[
\hat{Y}(0; x) := Y^\theta(0; x) = Y^{\bar{p}}(0; \hat{x}) := \hat{Y}(0; \hat{x}), \quad x \in [\alpha_2, b], \ \hat{x} \in [0, b].
\] (48)

Actually, we see that \( \hat{Y}(0; x) \) and \( \hat{Y}(0; \hat{x}) \) satisfy
\[
\begin{cases}
\hat{Y}(0; x) = -x - \int_0^T \left[ G(s, x, \theta(s)) + \bar{Z}(s; x)\theta(s) \right] ds - \int_0^T \bar{Z}(s; x)dW(s), \\
\hat{Y}(0; \hat{x}) = -\hat{x} - F - \int_0^T \left[ G(s, \hat{x}, \theta(s)) + \bar{Z}(s; \hat{x})\theta(s) \right] ds - \int_0^T \bar{Z}(s; \hat{x})dW(s).
\end{cases}
\] (49)

From (H4) and Lemma 2.2, the solutions of (49) are unique in the sense of
\[
(\hat{Y}(0;, \cdot), \bar{Z}(::)) \in C([\alpha_2, b]; \mathbb{R}) \times C([\alpha_2, b]; L^2_T(0, T; \mathbb{R})), \\
(\hat{Y}(0; :, \cdot), \bar{Z}(::)) \in C([0, b]; \mathbb{R}) \times C([0, b]; L^2_T(0, T; \mathbb{R})).
\]

We compare (45) with the first equation in (49). By the uniqueness in Lemma 2.2,
\[
\hat{P}(x) = \hat{Y}(0; x), \quad \hat{Q}(\cdot, x) = \bar{Z}(\cdot; x), \quad a.s. \ x \in [\alpha_2, b].
\]

Substituting \( \hat{P}(x) \) of (46) into the second equation of (49), we conclude that
\[
\hat{Y}(0; \hat{x}) = \hat{P}(x) - \int_0^T \left[ \bar{Z}(s; \hat{x}) - \hat{Q}(s, x) \right] \theta(s) ds - \int_0^T \left[ \bar{Z}(s; \hat{x}) - \hat{Q}(s, x) \right] dW(s).
\]
Hence \( \hat{Y}(0; \hat{x}) = \hat{P}(x) \) which leads to (48). 
\[\square\]
Remark 6. For later comparison, we point one simple case. As to \( f \) in (37), if it does not depend on \( x \), then (44), (47) are trivial, and \( \tilde{P}(x) \equiv q_0 := E_{\tilde{P}} F \) where

\[
\frac{dP_0}{dP} = \exp \left\{ \int_0^T \theta(s) dW(s) - \frac{1}{2} \int_0^T \theta(s)^2 ds \right\}.
\] (50)

Given \( P_0 \) in (50), we rewrite the risk indifference price \( \tilde{q}(\cdot) \equiv \tilde{P}(\cdot) \) as,

\[
\tilde{q}(x) = E_{\tilde{P}} F + E_{\tilde{P}} \int_0^T \left[ G(s, x - q(s), \theta(s)) - G(s, x, \theta(s)) \right] ds, \quad x \in [0, b].
\] (51)

Suppose \( G_x \) exists. By defining

\[
L(x) := E_{\tilde{P}} \int_0^T G_x(s, x - q(s), \theta(s)) ds,
\]

we can conclude that

\[
q_x(x) = \frac{1}{1 + L(x)} E_{\tilde{P}} \int_0^T \left[ G_x(s, x - q(s), \theta(s)) - G_x(s, x, \theta(s)) \right] ds.
\] (52)

Remark 7. From (39), we see that \( G(\cdot, x, \theta) \) is deterministic if \( \theta \) and \( f(\cdot, x, z) \) are \( \omega \)-independence. In this case, (51) then becomes

\[
\tilde{q}(x) = E_{\tilde{P}} F + \int_0^T \left[ G(s, x - q(s), \theta(s)) - G(s, x, \theta(s)) \right] ds, \quad x \in [0, b].
\]

It is a special deterministic algebraic equation.

At this very moment, we make some quantitative analysis.

\( \diamond \) If \( G \) is increasing in \( x \), by (51) we derive that \( 0 \leq q(\cdot) \leq q_0 \). In other words, if the buyer balances the risk in \( \omega \)-independence way by taking the wealth \( x \) into consideration, he/she can save some money by a lower price than the \( \text{sophisticated} \) case in Remark 6.

\( \diamond \) If \( G_x \) is decreasing (or increasing) in \( x \), from (52) it yields \( q_x(x) \geq 0 \) (or \( q_x(x) \leq 0 \)), which shows the buyer’s price has positive (or negative) correlation with initial wealth \( x \).

To verify above conclusions and analysis, we point out three examples.

Example 3.1. Given bounded function \( k : [0, T] \times [0, b] \rightarrow \mathbb{R}^+ \), \( \alpha(\cdot), \beta(\cdot) \) in (36), let us consider

\[
h(s, x, z) := k(s, x) \left[ \sqrt{1 + |z|^2} - 1 \right], \quad s \in [0, T], \quad x \in [0, b], \quad z \in \mathbb{R}.
\]

If \( k(\cdot, x) \) does not dependent on \( x \), similar generator was used in Section 6 of [24] and Section 3 of [31] for dynamic risk measures via BSVIEs. From (39), we have

\[
\tilde{h}(s, x, \theta(s), r) = -\theta(s) r - k(s, x) \left[ \sqrt{1 + |r|^2} - 1 \right].
\]

For constants \( a_i \) (\( i = 1, 2, 3, 4 \)) satisfying \( 0 < a_3 < a_4 < 1 \) and \( 0 < a_1 < a_2 \), suppose

\[
a_3 < \frac{\theta(\cdot)}{k(\cdot, x)} \leq a_4, \quad -a_2 < k_x(\cdot, x) \leq -a_1.
\]

Given \( s, x \), we obtain \( r \mapsto \tilde{h}(s, x, \theta(s), r) \) is concave in \( \mathbb{R} \) with

\[
\tilde{h}_r(s, x, \theta(s), r) = -\theta(s) - \frac{k(s, x) r}{\sqrt{1 + r^2}}, \quad \tilde{h}_{rr}(s, x, \theta(s), r) = -\frac{k(s, x)}{(\sqrt{1 + r^2})^2}.
\]
Consequently, the supremum in (39) is reachable at \( r = \frac{-\theta(s)}{\sqrt{k^2(s, x) - \theta^2(s)}} \). Putting it back to \( \tilde{h}(s, x, \theta(s), r) \), we have

\[
G(s, x, \theta(s)) = -\left[ \sqrt{k^2(s, x) - \theta^2(s)} - k(s, x) \right].
\]

Therefore, we obtain a BSDE for \((\tilde{Y}, \tilde{Z})\) as follows,

\[
\tilde{Y}(t, x) = -x + \int_t^T \left[ \sqrt{k^2(s, x) - \theta^2(s)} - k(s, x) - \tilde{Z}(s, x)\theta(s) \right] ds
- \int_t^T \tilde{Z}(s, x)dW(s), \quad t \in [0, T].
\]

With \( \tilde{Z}(\cdot, x) \) satisfying (53), we define \( \bar{\pi}(\cdot, \cdot) \) as

\[
\bar{\pi}(s, x) := \left[ \frac{\theta(s)}{\sqrt{k^2(s, x) - \theta^2(s)}} + \bar{Z}(s, x) \right], \quad s \in [0, T], \quad x \in [0, b].
\]

Thus (40) is fulfilled. According to (41), the following

\[
\bar{\pi}(s, x) = \beta^{-1}(s) \left[ \frac{\theta(s)}{\sqrt{k^2(s, x) - \theta^2(s)}} + \bar{Z}(s, x) \right], \quad s \in [0, T], \quad x \in [a, b],
\]

is an optimal strategy. On the other hand, given above \( G \), it is a simple computation that,

\[
\frac{a_1}{\sqrt{1 - a_3}} \leq G_x(s, x, \theta(s)) = -k_x(s, x) \left( \frac{k(s, x)}{\sqrt{k^2(s, x) - \theta^2(s)}} - 1 \right) \leq \frac{a_2}{\sqrt{1 - a_4}}.
\]

If \( \frac{a_2}{\sqrt{1 - a_4}} T \leq \frac{a_1}{a_2} \) with \( a_i \) in (H4), then according to Theorem 3.2, the risk indifference price satisfies (46) with above defined \( G \).

**Remark 8.** As to coefficient \( k \), we emphasize two interesting results.

- Inspired by [3], we require it to depend on \( x \). Otherwise, \( \bar{\pi} \) becomes nothing to do with initial wealth which in some sense seems unrealistic.
- We consider \( k \) as a subjective risk aversion parameter representing investors’ reluctance towards risk. For example, suppose two investors \( A, B \) assess the risk of \( \xi \) as \( \rho^A(\xi), \rho^B(\xi) \). If \( k_A \leq k_B \), by comparison theorems of BSDEs, the risk on behalf of \( A \) is less than \( B \). In other words, \( A \) is more risk tolerant (less risk averse) than \( B \).

**Remark 9.** If \( \theta(\cdot) \) is deterministic, we present some conclusions on optimal strategy, which fit the common economic knowledge in the real life and existing papers (e.g., [4], [29]).

- In this case, the term \( \bar{Z}(\cdot, x) \), arising from the randomness of \( \theta \), becomes null and

\[
\bar{\pi}(s, x) = \frac{\theta(s)}{\sqrt{k^2(s, x) - \theta^2(s)}} \beta^{-1}(s), \quad s \in [0, T], \quad x \in [0, b].
\]

- From (56), there is a negative monotonic link between \( k \) and \( |\bar{\pi}| \). With a higher risk aversion parameter, people are so conservative such that they put less money into risky assets.
- If \( \beta > 0 \), \( \bar{\pi} \) has positive monotonicity with market price of risk \( \theta \). If \( \alpha > 0 \), \( \bar{\pi} \) is increasing in initial wealth \( x \) since \( k_x \) is assumed to be negative.
Example 3.2. Given $l : [0; T] \times [a, b] \to \mathbb{R}^+$, we consider

$$h(s, x, z) = l(s, x) \cdot \ln \left( \frac{1}{2} (1 + e^{-z}) \right), \quad s \in [0, T], \; z \in \mathbb{R}, \; x \in [0, b].$$

For constants $a_i$ $(i = 1, 2, 3, 4)$ satisfying $0 < a_1 < a_2$, $1 < a_3 < a_4$, suppose that

$$\frac{(2a_3 - 1)}{2a_3} < \frac{\theta(s)}{l(s, x)} < \frac{(2a_4 - 1)}{2a_4}, \quad -a_2 < l_x(\cdot, x) < -a_1.$$

First, by (39),

$$\tilde{h}(s, x, \theta(s), r) = -\theta(s) r - l(s, x) \cdot \ln \left( \frac{1}{2} (1 + e^{-r}) \right).$$

Given $s, x$, we obtain $r \mapsto \tilde{h}(s, x, \theta(s), r)$ is concave in $\mathbb{R}$ with

$$\tilde{h}_r(s, x, \theta(s), r) = -\theta(s) + \frac{l(s, x)}{1 + e^r}, \quad \tilde{h}_{rr}(s, x, \theta(s), r) = -\frac{l(s, x) e^r}{(1 + e^r)^2}.$$

Consequently, the supremum is reachable at $r = \ln \left( \frac{l(s, x)}{\theta(s)} - 1 \right)$. Putting it back to $\tilde{h}(s, x, \theta(s), r)$, we have

$$G(s, x, \theta) = \theta(s) \ln \left( \frac{\theta(s)}{l(s, x) - \theta(s)} \right) - l(s, x) \ln \left( \frac{l(s, x)}{2(l(s, x) - \theta(s))} \right).$$

With $\bar{Z}(\cdot, x)$ satisfying the following,

$$\bar{Y}(0) = -x + \int_0^T \left[ l(s, x) \ln \left( \frac{l(s, x)}{2(l(s, x) - \theta(s))} \right) - \theta(s) \ln \left( \frac{\theta(s)}{l(s, x) - \theta(s)} \right) \right] ds - \int_0^T \bar{Z}(s, x) dW(s),$$

we define $\bar{p}(s, x)$ as

$$\bar{p}(s, x) := \left[ \bar{Z}(s, x) - \ln \left( \frac{l(s, x)}{\theta(s)} - 1 \right) \right], \quad s \in [0, T], \; x \in [0, b].$$

Thus (40) is fulfilled. According to (41), the following $\bar{\pi}$ is an optimal strategy

$$\bar{\pi}(s, x) = \beta^{-1}(s) \ln \left( \frac{\theta(s)}{l(s, x) - \theta(s)} \right) + \beta^{-1}(s) \bar{Z}(s, x), \quad s \in [t, T].$$

By the form of $G$, we see that

$$a_1 \ln a_3 \leq G_x(s, x, \theta(s)) = -\ln \left( \frac{l(s, x)}{2(l(s, x) - \theta(s))} \right) \cdot l_x(s, x) \leq a_2 \ln a_4.$$

If $a_2 \ln a_4 T \leq \frac{a_1}{a_2}$, from Theorem 3.2 we see risk indifference price in (46) with above $G$.

Remark 10. Similar as Remark 8, we identify $l(\cdot, x)$ as a subjective risk aversion parameter.

If $\theta(\cdot)$ is deterministic, $\beta(\cdot) > 0$, $\alpha(\cdot) > 0$, optimal strategy has positive correlation with market price of risk, initial wealth, while it keeps negative monotonicity with respect to risk aversion parameter, the conclusions of which coincide with those in e.g., [4], [29].

In Example 3.2, we have $G_x(\cdot, x, \theta(\cdot)) \geq 0$, which leads to $q(\cdot) \leq q_0$. Moreover, if $l_{xx}(s, x) \leq 0$, then $G_{xx}(\cdot, x, \theta(\cdot)) \geq 0$, which implies $q(\cdot)$ is decreasing with respect to $x$. 
Example 3.3. We consider a risk measure via (37) where the generator is

$$h(s, x, z) = \left[ z - \frac{1}{2\gamma(x)} \right] I_{\left[ 1, \infty \right)}(z) + \frac{\gamma(x)}{2} \left| z \right|^2 I_{[0, \frac{1}{\gamma(x)}]}(z)$$

(60)

Here $\gamma(\cdot)$ is a decreasing positive function. The corresponding BSDE is regarded as the continuous time analogue of the discrete Gini principle. Given $i = 1, 2$, $\delta_i > 0$, suppose $1 > |\theta(\cdot)| > \delta_1$ and $-\frac{\gamma(x)}{2\gamma(x)} > \delta_2$.

First, by (39),

$$\tilde{h}(s, x, \theta(s), r) = \begin{cases} -(\theta(s) + 1)r + \frac{1}{2\gamma(x)}, & r \geq \frac{1}{\gamma(x)}; \\ r(1 - \theta(s)) + \frac{1}{2\gamma(x)}, & r \leq -\frac{\gamma(x)}{2}; \\ -\theta(s)r - \frac{\gamma(x)}{2}, & -\frac{1}{\gamma(x)} \leq r \leq \frac{1}{\gamma(x)}. \end{cases}$$

If $r \geq \frac{1}{\gamma(x)}$, the supremum is $-\frac{\theta(s)}{\gamma(x)} - \frac{1}{2\gamma(x)}$, which is achieved at $r = \frac{1}{\gamma(x)}$. If $r \leq -\frac{\gamma(x)}{2}$, the supremum is $\frac{\theta(s)}{\gamma(x)} - \frac{1}{2\gamma(x)}$, which is achieved at $r = \frac{1}{\gamma(x)}$. If $-\frac{1}{\gamma(x)} \leq r \leq \frac{1}{\gamma(x)}$, the supremum is $\frac{\theta(s)}{\gamma(x)}$, which is achieved at $r = \frac{1}{\gamma(x)}$.

Since $\frac{\theta(s)}{2\gamma(x)} \geq \frac{|\theta(s)|}{\gamma(x)}$, we see that the supremum is reachable at $r = -\frac{\theta(s)}{\gamma(x)}$. Putting it back to $\tilde{h}(s, x, \theta(s), r)$, we have

$$G(s, x, \theta) = \frac{\theta^2(s)}{2\gamma(x)}.$$ 

With $\tilde{Z}(\cdot, x)$ satisfying the following BSDE,

$$\tilde{Z}(r, s, x) = -x - \int_r^T \left( \tilde{Z}(s, x) \theta(s) + \frac{\theta^2(s)}{2\gamma(x)} \right) ds - \int_r^T \tilde{Z}(s, x) dW(s).$$

we define $\bar{p}(\cdot, \cdot)$ as

$$\bar{p}(s, x) := \tilde{Z}(s, x) + \frac{\theta(s)}{\gamma(x)}, \quad s \in [0, T], \quad x \in [0, b].$$

Thus (40) is fulfilled. According to (41), the following $\bar{\pi}$ is an optimal strategy

$$\bar{\pi}(s, x) = \beta^{-1}(s) \left[ \tilde{Z}(s, x) + \frac{\theta(s)}{\gamma(x)} \right].$$

(61)

In our scenario,

$$G(\cdot, x, \theta(\cdot)) = \frac{\theta^2(s)}{2\gamma(x)} \quad G_x(s, x, \theta(s)) = -\frac{\theta^2(s)\gamma(x)}{2\gamma^2(x)}.$$

By the conditions imposed on $\theta$ and $\gamma$, if $\frac{\delta^2\delta_x}{2} T \leq \frac{\alpha_1}{\alpha_2}$, we see the result of $q(\cdot)$ by Theorem 3.2.

Remark 11. Similar as above, we identify $\gamma(\cdot)$ as a risk aversion function, and discuss the relations among optimal strategy, initial wealth, risk aversion parameter and market price of risk under proper conditions.

By the imposed conditions, $G_x \geq 0$, and hence $q(\cdot) \leq q_0$. In addition, if $\gamma_{xx}(\cdot) \leq 0$, then $G_{xx} \geq 0$, which indicates that $q(\cdot)$ is decreasing function.
Remark 12. To illustrate the assumptions of $\gamma(\cdot)$ in Example 3.3, let $\gamma(x) := \frac{c}{x}$, $c > 0$. This kind of risk aversion parameter was used in e.g., [3]. Then

$$\gamma'(x) = -\frac{c}{x^2}, \quad \frac{\gamma'(x)}{2\gamma(x)} = \frac{1}{2c}, \quad \pi(s, x) = \frac{\theta(s)x}{\beta(s)c},$$

$$q(x) = \frac{E_{\pi_x} F}{1 + \frac{1}{2c} E_{\pi_x} \int_0^T \theta^2(s)ds}, \quad x \in [0, b].$$

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REFERENCES

[1] P. Barrieu and N. El Karoui, Pricing, hedging and optimally designing derivatives via minimization of risk measures, *Indifference Pricing: Theory and Applications*. Princeton University Press, (2008), Princeton, USA.

[2] S. Basak and G. Chabakauri, Dynamic mean-variance asset allocation, *Rev. Finan. Stud.*, 23 (2010), 2970–3016.

[3] T. Björk, A. Murgoci and X. Y. Zhou, Mean-variance portfolio optimization with state-dependent risk aversion, *Math. Finance*, 24 (2014), 1–24.

[4] C. Borell, Monotonicity properties of optimal investment strategies for log-Brownian asset prices, *Math. Finance*, 17 (2007), 143–153.

[5] L. Delong, Optimal investment for insurance company with exponential utility and wealth-dependent risk aversion coefficient, *Math. Method. Oper. Res.* 89 (2019), 73–113.

[6] L. Delong and P. Imkeller, Backward stochastic differential equations with time delayed generators-results and counterexamples, *Ann. Appl. Probab.*, 20 (2010), 1512–1536.

[7] L. Delong and P. Imkeller, On Malliavin’s differentiability of BSDEs with time delayed generators driven by Brownian motions and Poisson random measures, *Stochastic Process. Appl.* 120 (2010), 1748–1775.

[8] Y. Dong and R. Sircar, Time-Inconsistent Portfolio Investment Problems, In: Crisan D., Hambley B., Zariphopoulou T. (eds) *Stochastic Analysis and Applications* 2014, 239–281. Springer Proceedings in Mathematics Statistics, vol 100, 2014, Springer, Cham.

[9] N. El Karoui, S. Peng and M. C. Quenez, Backward stochastic differential equations in finance, *Math. Finance*, 7 (1997), 1–71.

[10] R. J. Elliott and T. K. Siu, Risk-based indifference pricing under a stochastic volatility model, *Commun. Stoch. Anal.*, 4 (2010), 51–73.

[11] R. J. Elliott and T. K. Siu, A BSDE approach to a risk-based optimal investment of an insurer, *Automatica*, 47 (2011), 253–261.

[12] V. Henderson and D. Hobson, Utility indifference pricing—an overview, In *Volume on Indifference Pricing*, (ed. R. Carmona), (2004), Princeton University Press.

[13] E. Karni, Risk aversion for state-dependent utility functions: Measurement and applications, *Int. Econ. Review*, 24 (1983), 637–647.

[14] S. Klöppel and M. Schweizer, Dynamic Utility Indifference Valuation via Convex Risk Measures, Nccr Finrisk working paper No. 209, (2005), ETH Zürich.

[15] S. Klöppel and M. Schweizer, Dynamic indifference valuation via convex risk measures, *Math. Finance* 17 (2007), 599–627.

[16] D. Kramkov and M. Sirbu, Sensitivity analysis of utility based prices and risk-tolerance wealth processes, *Ann. Appl. Probab.*, 16 (2006), 2140–2194.

[17] D. Kramkov and M. Sirbu, Asymptotic analysis of utility-based hedging strategies for small number of contingent claims, *Stochastic Process Appl.*, 117 (2007), 1606–1620.

[18] J. P. Lepeltier and J. San Martin, Asymptotic analysis of utility-based hedging strategies for small number of contingent claims, *Stochastic Process Appl.*, 117 (2007), 1606–1620.

[19] J. Li and Q. Wei, Optimal control problem of fully coupled FBSDEs and viscosity solutions of Hamilton-Jacobi-Bellman equations, *SIAM J. Control Optim.* 52 (2014), 1622–1662.

[20] B. Øksendal and A. Sulem, Risk indifference pricing in jump diffusion markets, *Math. Finance* 19 (2009), 619–637.
[21] B. Øksendal and A. Sulem, Portfolio optimization under model uncertainty and BSDE games, *Quant. Finance* 11 (2011), 1665–1674.
[22] E. Pardoux and S. G. Peng, Adapted solution of a backward stochastic differential equation, *Systems Control Lett.* 14 (1990), 55–61.
[23] E. Rosazza Gianin, Risk measures via g-expectations, *Insurance Math. Econom.*, 39 (2006), 19–34.
[24] H. Wang, J. Sun and J. Yong, Recursive utility processes, dynamic risk measures and quadratic backward stochastic Volterra integral equations, to appear in *Appl. Math. Optim.* (2020).
[25] T. Wang and J. Yong, Comparison theorems for some backward stochastic Volterra integral equations, *Stochastic Process. Appl.* 125 (2015), 1756–1798.
[26] T. Wang and H. Zhang, Optimal control problems of forward-backward stochastic Volterra integral equations with closed control regions, *SIAM J. Control Optim.*, 55 (2017), 2574–2602.
[27] T. Wang and J. Yong, Backward stochastic Volterra integral equations–representation of adapted solutions, *Stochastic Process. Appl.*, 129 (2019), 4926–4964.
[28] Z. Wu and Z. Yu, Probabilistic interpretation for a system of quasilinear parabolic partial differential equations combined with algebra equations, *Stochastic Process. Appl.* 124 (2014), 3921–3947.
[29] J. Xia, Risk aversion and portfolio selection in a continuous-time model, *SIAM J. Control Optim.*, 49 (2011), 1916–1937.
[30] M. Xu, Risk measure pricing and hedging in incomplete markets, *Ann. Financ.* 2 (2006), 51–71.
[31] J. Yong, Continuous-time dynamic risk measures by backward stochastic Volterra integral equations, *Appl. Anal.*, 86 (2007), 1429–1442.
[32] J. Yong, Well-posedness and regularity of backward stochastic Volterra integral equations, *Probab. Theory. Related Fields*, 142 (2008), 21–77.

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