Weyl fermions in a family of Gödel-type geometries with a topological defect

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Abstract

In this paper we study Weyl fermions in a family of Gödel-type geometries in Einstein general relativity. We also consider that these solutions are embedded in a topological defect background. We solve the Weyl equation and find the energy eigenvalues and eigenspinors for all three cases of Gödel-type geometries where a topological defect is passing through them. We show that the presence of a topological in these geometries contributes to modification of the spectrum of energy. The energy zero modes for all three cases of the Gödel geometries are discussed.

Keywords: Weyl-Dirac Equation, Landau Quantization, Gödel-type Geometries
I. INTRODUCTION

The Gödel [1] solution of the Einstein equation is the first example of cosmological universe with rotating matter. It was obtained considering a cylindrically symmetric stationary solution of the equations of the general relativity. The solutions of this class are characterized by the presence of closed timelike curves (CTCs). Hawking [2] investigated the presence of CTCs and had conjectured that the presence of CTCs is physically inconsistent. Rebouças et al [3–5] investigated the Gödel-type solution in general relativity and Riemann-Cartan theory of gravity. They presented a detailed study of the problem of causality in this family of spacetimes, and then, established the following set of classes of solutions: (i) solutions where there are no CTCs; (ii) solutions where there is a sequence of alternating causal and non-causal regions; finally, the case (III) where a class of solutions involves only one non-causal region. In Ref. [6], Dabrowski had investigated the criteria necessary for the existence of CTCs, to obtain this criteria he have made use of two quantities, called the super-energy and super-momentum. Recently, the appearance of CTCs in the Gödel-type spacetime was investigated in the framework of string theory [7, 8] in Ref. [9]. Others interesting properties of the Gödel solution were investigated by Barrow in the Ref. [10]. This family of solutions is given by the line element:

\[ ds^2 = - \left( dt + \frac{H(r)}{D(r)} d\phi \right)^2 + \frac{1}{D^2(r)} (dr^2 + J^2(r)d\phi^2) + dz^2, \]  

(1)

where for each case, we have different functions \( H(r), J(r) \) and \( D(r) \): (i) for the case of Som-Raychaudhuri spacetime we have \( D(r) = 1, \ H(r) = \alpha \Omega r^2 \) and \( J(r) = \alpha r \). (ii) we have \( D(r) = \left( 1 + \frac{r^2}{4R^2} \right), \ H(r) = \alpha \Omega r^2 \) and \( J(r) = \alpha r \), for the case of Gödel-type spacetime with spherical symmetry. (iii) finally, for hyperbolic Gödel-type spacetime we have \( D(r) = (1 - l^2r^2), \ H(r) = \alpha \Omega r^2 \) and \( J(r) = \alpha r \). The system of coordinates \((t, r, \phi, z)\) are defined in the ranges: \( 0 \leq r < \infty, 0 \leq \phi \leq 2\pi \) and \(-\infty < (z,t) < \infty\). The parameters \( \Omega \) and \( \alpha \) are related with the vorticity and angular deficit arising from the topological defect, respectively, since it is associated with the deficit of angle \( \alpha = (1 - 4\Theta) \), with \( \Theta \) being the mass per unit length of the cosmic string, and it assumes values in the range \( 0 < \alpha < 1 \). In geometric theory of defect in condensed matted for a negative disclination, which are characterized by angular excess where the parameter \( \alpha \) can assumes values \( \alpha > 1 \). Note if you make the following change of variable \( \theta = \alpha \phi \) the theta are defined in the range \( 0 < \theta < 2\pi \alpha \). the introduction of defects have topological nature and it presence
in this geometry do not removed by simple coordinate transformation. Several studies of
the physical problems involving Gödel-type spacetimes have been developed in recent years.
These geometries have been studied from the point of view of the equivalence problem
techniques in the Riemannian Gödel-type spacetimes \cite{3} and for Riemann-Cartan Gödel-type
spacetimes \cite{11, 13}. Recently, a large number of issues related to rotating Gödel solutions
in general relativity as well as in alternative theories of gravitation have been studied, for
example: the hybrid metric-Palatini gravity \cite{14}, the Chern-Simons modified gravity theory
\cite{15, 16}, the Horava-Lifshitz theory of gravity \cite{17, 18} and the Brans-Dick theory of gravity
\cite{19}. In a recent paper \cite{20}, the electronic properties of spherical symmetric carbon molecule –
Buckminsterfullerene – were analyzed employing a geometric model based on the spherical
Gödel spacetime. Figueiredo et al \cite{21} investigated the scalar and spin-1/2 particles in
Gödel spacetimes with positive, negative and zero curvatures. The relationship between
the Klein-Gordon solution in a class of Gödel solutions in general relativity with Landau
levels in a curved spaces \cite{22, 23} was investigated by Drukker et al \cite{24, 25}. This analogy
was also observed by Das and Gegenberg \cite{26} within studying the quantum dynamics of
scalar particles in the Som-Raychaudhuri spacetime (Gödel flat solution) and compared with
the Landau levels in the flat space. The quantum dynamics of a scalar quantum particle
in a class of the Gödel solutions with a presence of a cosmic string and the solutions of
Klein-Gordon equation in the Som-Raychaudhuri spacetime have also been studied in Refs.
\cite{27, 28}. In Ref. \cite{29} Villalba had obtained solutions of Weyl equation in a non-stationary
Gödel-type cosmological universe. The Weyl equation was studied for a family of metrics
of the Gödel-type by Pimentel and collaborators \cite{30}. They have solved the Weyl equation
for a specific case of the Gödel solution. In the recent article \cite{31} Fernandes et al. have
solved the Klein-Gordon equation for a particle confined in two concentric thin shells in
Gödel, Kerr-Newman and FRW spacetimes with the presence of a topological defect passing
through them. Havare and Yetkin \cite{32} have studied the solutions of photon equation in
stationary Gödel-type and Gödel spacetimes. Several equations for different spins have been
studied in the Gödel universe \cite{33, 34, 35}.

Continuing these studies, in this article we investigate the Weyl equation in a family
of Gödel-type metrics in the presence of a topological defect. We obtain the solutions of
the Weyl equation in this set of Gödel geometries. Recently, a series of studies have been
made with the purpose of investigating the quantum and classical dynamics of particles in
curved spaces with topological defects and the possible detection of this defect. In Ref. [37] the quantum dynamics of Dirac fermions in Gödel-type solution in gravity with torsion was investigated, and we have observed that the presence of torsion in the space-time yields new contributions to the relativistic spectrum of energies of massive Dirac fermion, and that the presence of the topological defect modifies the degeneracy of energy levels. Another conclusion is that the torsion effect on the allowed energies corresponds to the splitting of each energy level in a doublet. The purpose of this contribution is to investigate the influence of curvature, rotation of Gödel-type metrics and topology introduced by the topological defect in the quantum dynamics of Weyl fermions. We investigated influence of Gödel-type geometries of positive, negative and flat curvature that contain a topological defect in the eigenvalues and eigenfunctions of Weyl fermions. The possible application in Condensed matter systems are discussed. In this system the quasiparticle has behaviour similar the Weyl fermions an it travel by the ”speed of light”. Ins this system the model are described by Weyl fermion where Fermi velocity play de role of the light velocity in this effective theory. Recently in condensed matter physics there are systems that quasi-particles behave with a massless fermions or Weyl fermions. Systems such as graphene, Weyl semi-metals and topological insulator. In this way, the study carried out in this article can be used to investigate the influence of rotation, curvature and topology in the condensed matter systems described by massless fermions. The results obtained for the present case of Weyl fermions in a family of Gödel-type metrics in Einstein relativity theory are quite different from the previous result obtained for Dirac fermions with torsion in [37]. We claim that the studies of these problems in the present paper can be used to investigate the influence of the topological defect in the Gödel-type background metrics. The approach applied in this paper can be used to investigate the influence of disclinations (cosmic strings) in condensed matter systems as well as to investigate the Hall effect in spherical droplets with rotation and with the presence of disclinations. Recently, in ref. [20] one of us have used a similar approach to investigate the influence of rotation in fullerene molecule where the in this model was the rotation is introduced via a three-dimensional Gödel-type metric. In this way, we can study using this approach proposed here in condensed matter system described by curved geometries with rotation. In this paper, we analyse the relativistic quantum dynamics of a massless fermion in the presence of a topological defect in a class of Gödel-type metrics. We solve Weyl equation in Som-Raychaudhury, spherical and hyperbolic
background metrics pierced by a topological defect. We found the eigenvalues of energy in all three cases and observe their similarity with Landau levels for a massless spin-1/2 particle. We also observe that presence of the topological defect breaks the degeneracy of the relativistic energy levels, and the eigenfunctions depend on the parameter that characterizes the presence of the topological defect in these background metrics. The possibility of zero mode for the eigenvalues of the Weyl spinor are discussed and the physical implications are analysed for for all case three class of geometries investigated in this paper.

This contribution is organized as follows: in section II we present the Weyl equation in a geometry of Gödel-type metric pierced by a topological defect in an Einstein theory of relativity. The Weyl equation in the background of the Gödel-type metric is written for the flat space in the section III, for the spherical one in the section IV and for the hyperbolic one in the section V. The relativistic bound state solutions to the Weyl equation in Gödel-type background metric are investigated; finally, in section VI we present the conclusions. In this paper, we shall use natural units (ℏ = c = G = 1).

II. WEYL FERMIONS IN GÖDEL-TYPE SOLUTIONS IN GENERAL RELATIVITY

In this section we will introduce the Dirac equation in Weyl representation on a curved background. Following the theory of spinors in curved spacetime [39–41], to do it, one must extend the partial derivative $\partial_\mu$ up to a covariant derivative $\nabla_\mu = \partial_\mu + \Gamma_\mu(x)$. So that, we can write the equations for massless spin-$\frac{1}{2}$ field in the follow way,

$$i\gamma^\mu \nabla_\mu \psi = 0; \quad (2)$$

$$\left(1 + \gamma^5\right) \psi = 0; \quad (3)$$

here we have that $\gamma^\mu = \gamma^a e^a_\mu(x)$ are the gamma matrices in Weyl representation and $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$.

Let us consider fermions in a Gödel-type spacetime of a general form. The metric of this curved spacetime is given by equation (1). If we want to study the relativistic dynamics of fermions in the Som-Raychaudhuri spacetime, we must introduce spinors in a curved spacetime [39–41]. For this purpose, we will define the spinors via a noncoordinate basis
\( \hat{\theta}^a = e^a_\mu(x)dx^\mu \), where the components of tetrad \( e^a_\mu(x) \) must obey the following relation
\( g_{\mu\nu} = e^a_\mu(x)e^b_\nu(x)\eta_{ab} \). The tensor \( \eta_{ab} = \text{diag}(-+++ \) is the Minkowski tensor. We still can define the inverse tetrad from the relation \( dx^\mu = e^a_\mu(x)\hat{\theta}^a \), so that \( e^a_\mu(x)e^\mu_b(x) = \delta^a_b \) and \( e^a_\mu(x)e^\nu_a(x) = \delta^\nu_\mu \). Thus, for the metric (1), the tetrad and its inverse are defined as
\[
e^a_\mu(x) = \begin{pmatrix} 1 & 0 & \frac{H(r)}{J(r)} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{J(r)}{D(r)} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} ; \quad e^\mu_a(x) = \begin{pmatrix} 1 & 0 & \frac{H(r)}{J(r)} & 0 \\ 0 & D(r) & 0 & 0 \\ 0 & 0 & \frac{D(r)}{J(r)} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (4)
\]

From solutions of Maurer-Cartan structures \( d\hat{\theta}^a + \omega^a_b \wedge \hat{\theta}^b = 0 \), we can obtain the 1-forms connections \( \omega^a_b(x) = \omega^a_\mu(x)dx^\mu \), with \( d \) is exterior derivative and the symbol \( \wedge \) is the wedge product. The object \( \omega^a_\mu(x) \) is the spinorial connection. Thus, the non-zero components of spinorial connections are:
\[
\omega^0_1(x) = -\omega^1_0(x) = \frac{D}{2} \frac{d}{dr} \left( \frac{H}{D} \right) ;
\omega^0_2(x) = -\omega^2_0(x) = \frac{D}{2J} \frac{d}{dr} \left( \frac{H}{D} \right) ;
\omega^1_2(x) = -\omega^2_1(x) = -\frac{D^2}{2J} \frac{d}{dr} \left( \frac{H}{D} \right) ;
\omega^1_0(x) = -\omega^0_1(x) = -\left[ \frac{DH}{2J} \frac{d}{dr} \left( \frac{H}{D} \right) + \frac{D}{2J} \frac{d}{dr} \left( \frac{J}{D} \right) \right] .
\quad (5)
\]

Using the equations (2) and (3), we can use the chiral representation [49], so that the spinorial connection will be \( \Gamma_\mu(x) = \frac{i}{8}\omega_{\mu ab}(x) [\sigma^a, \sigma^b] \), where \( \sigma^a \) are the Pauli matrices, and \( \omega_{\mu ab}(x) \) are the 1-form connections related with the curvature of manifold given by equations (5). Thus, the covariant derivative will be
\[
\nabla_\mu = \partial_\mu + \frac{1}{8}\omega_{\mu ab}(x) [\sigma^a, \sigma^b] .
\quad (6)
\]

As a result, the Weyl equation will assume the following form:
\[
i\sigma^a e^\mu_a (\partial_\mu + \Gamma_\mu) \psi = 0 \quad (7)
\]

Then, using the equations (5) and (7), we obtain the Weyl equation in Gödel-type space-time in the following form:
\[
i\sigma^0 \frac{\partial \psi}{\partial t} + i\sigma^1 \left( D(r) \frac{\partial}{\partial r} - \frac{D(r)}{J(r)} \frac{\omega^1_2}{2} + \frac{H(r)}{J(r)} \frac{\omega^1_2}{2} \right) \psi +
+ i\sigma^2 \left( \frac{D(r)}{J(r)} \frac{\partial}{\partial \phi} - \frac{H(r)}{J(r)} \frac{\partial}{\partial t} \right) \psi + i\sigma^3 \left( \frac{\partial}{\partial z} - i\nu A_z \right) \psi = 0 .
\quad (8)
\]
We solve this equation for all three possibilities for functions $H(r)$, $J(r)$ and $D(r)$ for Gödel-type spacetimes. After we have described the Weyl equation in a Gödel-type spacetime context, we are looking for the energy levels of a massless fermion in a Gödel-type spacetime. We will solve the Weyl equation for three classes of Gödel-type metric: in the Som-Raychaudhuri spacetime, the Gödel-type spacetime with spherical symmetry and the hyperbolic Gödel-type spacetime.

Note that the axial vector coupled with the $z$-component of the momentum in equation (8), for massless fermions, is a gauge field, with $A_z = \Omega$ is a constant field, and $\nu = \frac{1}{2}$ is the minimal coupling parameter for the 3 cases of Gödel-type metric. This means that the Dirac equation in the Weyl representation is invariant under a gauge transformation of this kind: $\psi' = e^{f(z)} \psi$ and $A'_z = A_z - \nu^{-1} \partial f(z)$. From the Dirac’s phase method, we can write the solution for Dirac equation in the following way:

$$\psi = \exp \left[ -i \nu \int_{z_0}^{z} A_z dz \right] \psi_0,$$

(9)

the exponential term corresponds to relativistic phase acquired by wave function of the massless fermion, and $\psi_0$ is the solution of Dirac equation:

$$i \sigma^0 \frac{\partial \psi_0}{\partial t} + i \sigma^1 \left( D(r) \frac{\partial}{\partial r} - \frac{D(r) \omega^1_2}{2} + \frac{H(r) \omega^2_2}{2} \right) \psi_0 +$$

$$+ i \sigma^2 \left( \frac{D(r) \partial}{J(r) \partial \phi} - \frac{H(r) \partial}{J(r) \partial t} \right) \psi_0 + i \sigma^3 \frac{\partial \psi_0}{\partial z} = 0.$$

(10)

Now we obtain the Hamiltonian of this system. It’s possible to rewrite the Weyl equation in the follow way:

$$i \frac{\partial \psi_0}{\partial t} = \left[ \sigma^1 \hat{\pi}_r + \sigma^2 \hat{\pi}_\phi + \sigma^3 \hat{\pi}_z \right] \psi_0 = \hat{H} \psi_0,$$

(11)

where $\hat{H}$ is the Hamiltonian of Weyl particle and the conjugated momentum are given by,

$$\hat{\pi}_r = -i \left( D(r) \frac{\partial}{\partial r} - \frac{D(r) \omega^1_2}{2} + \frac{H(r) \omega^2_2}{2} \right);$$

$$\hat{\pi}_\phi = -i \left( \frac{D(r) \partial}{J(r) \partial \phi} - \frac{H(r) \partial}{J(r) \partial t} \right);$$

$$\hat{\pi}_z = -i \frac{\partial}{\partial z}.$$

(12)

Analysing the symmetry of the Hamiltonian $\hat{H}$ to solve the Weyl equation associate with $\hat{H}$, we can choose the following ansatz :

$$\psi_0 (t, r, \phi, z) = \begin{pmatrix} \psi_1(r) \\ \psi_2(r) \end{pmatrix} e^{-i (Et - \ell \phi - k z)},$$

(13)
where $\psi_1$ and $\psi_2$ are two-component spinors, and through the Pauli matrices:

$$
\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};
$$

(14)

then we obtain two coupled differential equations. In next section we solve the eigenvalues and eigenfunction associated to Hamiltonian $H$ for flat, spherical and hyperboluc Gödel space-times witha topological defect.

III. THE SOLUTION OF WEYL EQUATION IN THE SOM-RAYCHAUDHURI GEOMETRY

In this section we solve the Weyl equation for flat Gödel metric. Let us start with the solution of Weyl equation in Som-Raychaudhuri spacetime. For this case, we have that $D(r) = 1$, $H(r) = \alpha \Omega r^2$ and $J(r) = \alpha r$. Therefore, the Weyl equation (10) will assume the following form,

$$
i \sigma^0 \frac{\partial \psi_0}{\partial t} + i \sigma^1 \left( \frac{\partial}{\partial r} + \frac{1}{2r} \right) \psi_0 + i \sigma^2 \left( \frac{\partial}{\partial \phi} - \alpha \Omega r^2 \frac{\partial}{\partial t} \right) \psi_0 + i \sigma^3 \frac{\partial \psi_0}{\partial z} = 0.
$$

(15)

To solve the equation (15), we must use the ansatz (13), and the Pauli matrices (14). Then we obtain two coupled differential equations:

$$
[E - k] \psi_1 = -i \left[ \frac{\partial}{\partial r} + \frac{1}{2r} - \frac{\ell}{\alpha r} - E \Omega r \right] \psi_2; \quad (16)
$$

$$
[E + k] \psi_2 = -i \left[ \frac{\partial}{\partial r} + \frac{1}{2r} + \frac{\ell}{\alpha r} + E \Omega r \right] \psi_1. \quad (17)
$$

Now it is possible to decouple these two differential equations. From equations (16) and (17), we are capable to convert these two differential equations of first order to two differential equations of second order. Thereby,

$$
\frac{d^2 \psi_i}{dr^2} + \frac{1}{r} \frac{d \psi_i}{dr} - \left[ E^2 \Omega^2 r^2 + \frac{\zeta_i^2}{r^2} - 4 E \Omega \beta_i \right] \psi_i = 0, \quad (18)
$$

with $i = 1, 2$ label each spinor component. And the parameter $\zeta_i$ and $\beta_i$ is given by,

$$
\zeta_1 = \left( \frac{|\ell|}{\alpha} + \frac{1}{2} \right), \quad \zeta_2 = \left( \frac{|\ell|}{\alpha} - \frac{1}{2} \right),
$$

(19)

and,

$$
\beta_1 = \frac{E}{4 \Omega} - \frac{k^2}{4 E \Omega} + \frac{1}{4} - \frac{\ell}{2 \alpha}; \quad (20)
$$

$$
\beta_2 = \frac{E}{4 \Omega} - \frac{k^2}{4 E \Omega} - \frac{1}{4} - \frac{\ell}{2 \alpha}. \quad (21)
$$
Let us do a change of variables in the equation (18), where \( \varsigma = E \Omega r^2 \). Thus,

\[
\varsigma \frac{d^2 \psi_i}{d\varsigma^2} + \frac{d\psi_i}{d\varsigma} + \left[ \beta_i - \frac{\varsigma}{4} - \frac{\varsigma^2}{4\varsigma} \right] \psi_i = 0.
\]  

(22)

The solution of equation (22) is

\[
\psi_i(\varsigma) = e^{-\frac{\varsigma}{2}} \varsigma^{\frac{|\zeta_i|}{2}} F(\varsigma),
\]

(23)

substituting this solution in equation (22), we get

\[
\varsigma \frac{d^2 F}{d\varsigma^2} + (|\zeta_i| + 1 - \varsigma) \frac{dF}{d\varsigma} + \left( \beta_i - \frac{1}{2} - \frac{|\zeta_i|}{2} \right) F = 0.
\]  

(24)

This equation is the confluent hypergeometric differential equation, whose solution is the confluent hypergeometric function \( F(\varsigma) = F\left(-\left[\beta_i - \frac{1}{2} - \frac{|\zeta_i|}{2}\right], |\zeta_i| + 1; \varsigma\right) \). Therefore, the first parameter of confluent hypergeometric function is

\[
\nu = \beta_i - \frac{1}{2} - \frac{|\zeta_i|}{2},
\]

(25)

so that the energy levels for both spinors are given by,

\[
E_{\nu, \ell} = 2\Omega \left[ \nu + \frac{|\ell|}{2\alpha} + \frac{\ell}{2\alpha} + \frac{1}{2} \right] \pm 2\Omega \sqrt{\left[ \nu + \frac{|\ell|}{2\alpha} + \frac{\ell}{2\alpha} + \frac{1}{2} \right]^2 + \frac{\kappa^2}{4\Omega^2}},
\]

(26)

where \( \nu = 0, 1, 2, 3, \ldots \) and \( \ell \) is half-integer. In this way, the expression (26) gives the eigenvalues of energy of a Weyl fermion in the Som-Raychaudhuri geometry pierced by a topological defect. In the limit \( \alpha = 1 \), we obtain the same results for energy spectrum of Weyl fermions arisen in the Som-Raychaudhuri metric, note that we do not obtain from the Eq. (26) the spectrum found in the Ref. [37], due to the dependence of torsion coupling obtained in energy spectrum of Ref. [37] even considering the limit case of the massless fermions. We conclude that the energy spectrum (26) is similar to the the relativistic Landau levels for fermions and is characterized by an infinite degeneracy for \( \alpha = 1 \). Note that the degeneracy of the eigenvalues (26) is broken due to the presence of topological defect (\( \alpha \neq 1 \)) in this background. Now we analyze the zero mode energy for this case. Now we analyse the zero mode \( E_{\nu=0} = 0 \) energy for this case. From equation (26) we can derive the zero modes of the Weyl spinor in this Gödel-type metric with the topological defect considering \( \nu = 0 \). The zero mode occurs when the particle is confined to plane, i. e., when \( k = 0 \).
Thus, the eigenspinor for the Weyl particle in the Som-Raychaudhuri geometry with the topological defect is given by

\[ \psi_0(t, \varsigma, \phi, z) = C_{\nu, \ell} e^{-\frac{\varsigma}{2} e^{-i[E t - \ell \phi - k z]}} \times \]

\[ \times \begin{pmatrix} \varsigma^{\ell + \frac{1}{2}} F_1\left(-\nu, -\frac{\ell}{\alpha} + \frac{3}{2}; \varsigma\right) \\ \varsigma^{-\ell - \frac{1}{2}} F_1\left(-\nu, -\frac{\ell}{\alpha} + \frac{3}{2}; \varsigma\right) \end{pmatrix}, \]

where \( C_{\nu, \ell} \) is a constant spinor, and \( \ell \geq \frac{1}{2} \). The wave function for case \( \ell \leq -\frac{1}{2} \) is

\[ \psi_0(t, \varsigma, \phi, z) = C_{\nu, \ell} e^{-\frac{\varsigma}{2} e^{-i[E t - \ell \phi - k z]}} \times \]

\[ \times \begin{pmatrix} \varsigma^{-\ell - \frac{1}{2}} F_1\left(-\nu, -\frac{\ell}{\alpha} + \frac{1}{2}; \varsigma\right) \\ \varsigma^{-\ell + \frac{1}{2}} F_1\left(-\nu, -\frac{\ell}{\alpha} + \frac{1}{2}; \varsigma\right) \end{pmatrix}, \]

here \( C_{\nu, \ell} \) is a constant spinor.

**IV. SOLUTION OF WEYL EQUATION IN THE SPHERICALLY SYMMETRIC GÖDEL-TYPE GEOMETRY**

Now we investigate the Weyl equation in the spherically symmetric Gödel-type spacetime. So, we will resolve the Weyl equation (10) for \( D(r) = \left(1 + \frac{r^2}{4R^2}\right), H(r) = \alpha \Omega r^2 \) and \( J(r) = \alpha r \):

\[
i\sigma^0 \frac{\partial \psi_0}{\partial t} + i\sigma^1 \left[ \left(1 + \frac{r^2}{4R^2}\right) \frac{\partial}{\partial r} + \left(1 - \frac{r^2}{4R^2}\right) \frac{1}{2} \right] \psi_0 + \frac{i\sigma^2}{\alpha r} \left(1 + \frac{r^2}{4R^2}\right) \frac{\partial \psi_0}{\partial \phi} - i\Omega r \sigma^2 \frac{\partial \psi_0}{\partial t} + i\sigma^3 \frac{\partial \psi_0}{\partial z} = 0, \]

Using the spinor \( \mathbf{13} \) to solve the Dirac equation, we get two coupled equations:

\[
\begin{align*}
[E - k] \psi_1 &= i \left[ \left(1 + \frac{r^2}{4R^2}\right) \frac{\partial}{\partial r} + \left(1 - \frac{r^2}{4R^2}\right) \frac{1}{2r} - \left(1 + \frac{r^2}{4R^2}\right) \frac{\ell}{\alpha r} - E\Omega r \right] \psi_2; \\
[E + k] \psi_2 &= i \left[ \left(1 + \frac{r^2}{4R^2}\right) \frac{\partial}{\partial r} + \left(1 - \frac{r^2}{4R^2}\right) \frac{1}{2r} + \left(1 + \frac{r^2}{4R^2}\right) \frac{\ell}{\alpha r} + E\Omega r \right] \psi_1.
\end{align*}
\]

It is possible to convert these equations, so that we obtain two second order equations for each spinor, so that we have

\[
\left(1 + \frac{r^2}{4R^2}\right)^2 \left[ \frac{d^2 \psi_1}{dr^2} + \frac{1}{r} \frac{d\psi_1}{dr} \right] - \left[ a_1 r^2 + \frac{b_1}{r^2} - c_1 \right] \psi_1 = 0,
\]

\[
\left(1 + \frac{r^2}{4R^2}\right)^2 \left[ \frac{d^2 \psi_2}{dr^2} + \frac{1}{r} \frac{d\psi_2}{dr} \right] - \left[ a_2 r^2 + \frac{b_2}{r^2} - c_2 \right] \psi_2 = 0.
\]
for \( i = 1, 2 \) representing each spinor. The parameters \( a'_i \) are given by:

\[
a'_1 = \frac{a'_1}{16R^4} = \frac{1}{16R^4} \left( \frac{\ell}{a} - \frac{1}{2} + 4R^2E\Omega \right)^2; \tag{33}
\]

\[
a'_2 = \frac{a'_2}{16R^4} = \frac{1}{16R^4} \left( \frac{\ell}{a} + \frac{1}{2} + 4R^2E\Omega \right)^2, \tag{34}
\]

and \( b_i \) by

\[
b_1 = \left( \frac{\ell}{a} + \frac{1}{2} \right)^2; \tag{35}
\]

\[
b_2 = \left( \frac{\ell}{a} - \frac{1}{2} \right)^2. \tag{36}
\]

Finally,

\[
c_1 = E^2 - k^2 - \frac{3}{8R^2} + E\Omega - \frac{\ell^2}{2a^2R^2} - \frac{2E\Omega}{\alpha} \tag{37}
\]

\[
c_2 = E^2 - k^2 - \frac{3}{8R^2} - E\Omega - \frac{\ell^2}{2a^2R^2} - \frac{2E\Omega}{\alpha}. \tag{38}
\]

Now, let us solve the differential equation \( (32) \). We should introduce the new coordinate \( \theta \), through a change of variables \( r = 2R \tan \theta \). Thereby, the equation \( (32) \) will be

\[
\frac{d^2 \psi_i}{d\theta^2} + \left( \frac{1}{\cos \theta \sin \theta} - \frac{2 \sin \theta}{\cos \theta} \right) \frac{d\psi_i}{d\theta} - \left[ a^2_i \frac{\sin^2 \theta}{\cos^2 \theta} + b^2_i \frac{\cos^2 \theta}{\sin^2 \theta} - 4R^2c_i \right] \psi_i = 0, \tag{39}
\]

and we still can do two more changes of variables, first, \( x = \cos \theta \), and then, \( \varsigma = 1 - x^2 \), hence we get

\[
\varsigma (1 - \varsigma) \frac{d^2 \psi_i}{d\varsigma^2} + (1 - 2\varsigma) \frac{d\psi_i}{d\varsigma} - \left[ \frac{a^2_i}{4} \frac{\varsigma}{(1 - \varsigma)} + \frac{b^2_i}{4} \frac{(1 - \varsigma)}{\varsigma} - R^2c_i \right] \psi_i = 0. \tag{40}
\]

Studying the asymptotic limits of \( (40) \) we obtain \( \lambda_i = \frac{|a_i|}{2} \) and \( \delta_i = \frac{|b_i|}{2} \), in this way, the solution of Eq. \( (40) \) can be written as

\[
\psi_i (\varsigma) = \varsigma^{\delta_i} (1 - \varsigma)^{\lambda_i} \bar{F}_i (\varsigma), \tag{41}
\]

where \( \bar{F}_i (\varsigma) \) are unknown functions. By substituting the solution \( (41) \) into \( (40) \), we obtain the following equations for functions \( \bar{F}_1 (\varsigma) \) and \( \bar{F}_2 (\varsigma) \),

\[
\varsigma (1 - \varsigma) \frac{d^2 \bar{F}_i}{d\varsigma^2} + [2\delta_i + 1 - 2\varsigma (\delta_i + \lambda_i + 1)] \frac{d\bar{F}_i}{d\varsigma} - \left[ \lambda_i + 2\lambda_i\delta_i + \delta_i - R^2c_i \right] \bar{F}_i = 0. \tag{42}
\]

We have a set of two decoupled hypergeometric differential equations in the Eqs. \( (42) \), and the functions \( \bar{F}_i (x) = F_1 (A, B, C_i; \varsigma) \) are defined with the coefficients \( A \) and \( B \) given by

\[
A = \left[ \left( \frac{\ell}{\alpha} + \frac{1}{2} \right) + 2R^2E\Omega \right] + \sqrt{4R^4\Omega^2E^2 + R^2 [E^2 - k^2]}; \tag{43}
\]

\[
B = \left[ \left( \frac{\ell}{\alpha} + \frac{1}{2} \right) + 2R^2E\Omega \right] - \sqrt{4R^4\Omega^2E^2 + R^2 [E^2 - k^2]};
\]
Next, by imposing that the hypergeometric series truncates, becoming a polynomial of the degree $n$, then, we obtain (for both spinors)

$$E_{\nu, \ell} = 2\Omega \left[ \nu + \frac{|\ell|}{2\alpha} + \frac{\ell}{2\alpha} + \frac{1}{2} \right]$$

$$\pm 2\Omega \left\{ \left[ \nu + \frac{|\ell|}{2\alpha} + \frac{\ell}{2\alpha} + \frac{1}{2} \right]^2 + \frac{1}{4R^2 \Omega^2} \left[ \nu + \frac{|\ell|}{2\alpha} + \frac{\ell}{2\alpha} + \frac{1}{2} \right]^2 + \frac{k^2}{4\Omega^2} \right\}^{1/2}, \quad (44)$$

where $\nu = 0, 1, 2, 3, \ldots$ and $-\nu + \frac{1}{2} \leq \frac{\ell}{\alpha} \leq 4\Omega R^2 E + \frac{1}{2}$. Note that, in this case we not observe a possibility of zero mode, for case where $R \rightarrow \infty$, we obtain the flat case found in previous section for Som-Raychaudhuri metric. Note that the curvature and the rotation introduce a mass term in the Weyl fermion energy levels. Note that in the limit where $\alpha = 1$ we obtain the eigenvalues for Weyl fermions in the background of the spherical Gödel metric. We can also observe that the results obtained in this section differ from results obtained in the zero mass limit in Ref. [37] for Dirac fermions in a spherical Gödel spacetime with torsion.

The corresponding eigenspinor for a Weyl particle in the spherical Gödel-type geometry with a topological defect is given by

$$\psi_0(t, \varsigma, \phi, z) = \bar{C}_{\nu,m} e^{-i[Et - \ell\phi - kz]} \times$$

$$\times \left( (1 - \varsigma)^{\frac{|a_1|}{2}} \varsigma^{\frac{|b_1|}{2}} \binom{1}{2F1} (A, B, \frac{\ell}{\alpha} + \frac{3}{2}; \varsigma) \right) \times$$

$$\left( (1 - \varsigma)^{\frac{|a_2|}{2}} \varsigma^{\frac{|b_2|}{2}} \binom{1}{2F1} (A, B, \frac{\ell}{\alpha} + \frac{1}{2}; \varsigma) \right), \quad (45)$$

where $\bar{C}_{\nu,m}$ is a constant spinor and the parameters $A$ and $B$ of the hypergeometric functions have been defined in Eq. (43) and $\frac{1}{2} \leq \frac{\ell}{\alpha} \leq 4\Omega R^2 E + \frac{1}{2}$. And we have

$$\psi_0(t, \varsigma, \phi, z) = \bar{C}_{\nu,m} e^{-i[Et - \ell\phi - kz]} \times$$

$$\times \left( (1 - \varsigma)^{\frac{|a_1|}{2}} \varsigma^{\frac{|b_1|}{2}} \binom{1}{2F1} (1 - A, 1 - B, -\frac{\ell}{\alpha} + \frac{3}{2}; \varsigma) \right) \times$$

$$\left( (1 - \varsigma)^{\frac{|a_2|}{2}} \varsigma^{\frac{|b_2|}{2}} \binom{1}{2F1} (1 - A, 1 - B, -\frac{\ell}{\alpha} + \frac{1}{2}; \varsigma) \right), \quad (46)$$

where $\bar{C}_{\nu,m}$ is a constant spinor and $-\nu + \frac{1}{2} \leq \frac{\ell}{\alpha} \leq -\frac{1}{2}$. 

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V. SOLUTION OF WEYL EQUATION IN THE HYPERBOLIC GÖDEL-TYPE GEOMETRY

Finally we will analyse the third case with use of the equation (10) with \( D(r) = (1 - l^2 r^2) \), \( H(r) = \alpha \Omega r^2 \) and \( J(r) = \alpha r \),

\[
i\sigma^0 \frac{\partial \psi_0}{\partial t} + i\sigma^1 \left[ (1 - l^2 r^2) \frac{\partial}{\partial r} + (1 + l^2 r^2) \frac{1}{2r} \right] \psi_0 + i\sigma^2 \frac{\partial \psi_0}{\partial \phi} \alpha \left( 1 - l^2 r^2 \right) \frac{\partial \psi_0}{\partial \phi} - i \Omega r \sigma^2 \frac{\partial \psi_0}{\partial t} + i\sigma^3 \frac{\partial \psi_0}{\partial z} = 0,
\]

(47)

and one more time we use the eigenstate given by (13) to obtain two coupled differential equations:

\[
[E - k] \psi_1 = i \left[ (1 - l^2 r^2) \frac{\partial}{\partial r} + (1 + l^2 r^2) \frac{1}{2r} - (1 - l^2 r^2) \frac{\ell}{\alpha r} - E \Omega r \right] \psi_2; \quad (48)
\]

\[
[E + k] \psi_2 = i \left[ (1 - l^2 r^2) \frac{\partial}{\partial r} + (1 + l^2 r^2) \frac{1}{2r} + (1 - l^2 r^2) \frac{\ell}{\alpha r} + E \Omega r \right] \psi_1. \quad (49)
\]

We can convert these two coupled differential equation of first order in two decoupled differential equation of second order, as we made earlier. Therefore, we have two equations written in a compact way:

\[
(l^2 r^2 - 1)^2 \left[ \frac{d^2 \psi_i}{dr^2} + \frac{1}{r} \frac{d \psi_i}{dr} \right] - \left[ a'_i r^2 + \frac{b^2_i}{r^2} - c_i \right] \psi_i = 0,
\]

(50)

each spinor is represented by \( i = 1, 2 \). And the parameters of equation (50) are,

\[
a'_1 = l^4 a_1^2 = l^4 \left( \frac{E}{\alpha} - \frac{1}{2} - \frac{E \Omega}{r^2} \right)^2; \quad (51)
\]

\[
a'_2 = l^4 a_2^2 = l^4 \left( \frac{E}{\alpha} + \frac{1}{2} - \frac{E \Omega}{r^2} \right)^2, \quad (52)
\]

and \( b_i \),

\[
b_1^2 = \left( \frac{E}{\alpha} + \frac{1}{2} \right)^2 \quad (53)
\]

\[
b_2^2 = \left( \frac{E}{\alpha} - \frac{1}{2} \right)^2. \quad (54)
\]

Finally,

\[
c_1 = E^2 - k^2 + \frac{3l^2}{2} + E \Omega + \frac{2l^2 \ell^2}{\alpha^2} - \frac{2 \ell E \Omega}{\alpha} \quad (55)
\]

\[
c_2 = E^2 - k^2 + \frac{3l^2}{2} - E \Omega + \frac{2l^2 \ell^2}{\alpha^2} - \frac{2 \ell E \Omega}{\alpha}. \quad (56)
\]
To solve the equation (50), we make the change of the variable \( r = \frac{\tanh(l \theta)}{l} \) and get

\[
\frac{d^2 \psi_i}{d\theta^2} + \left( \frac{2l \sinh (l \theta)}{\cosh (l \theta)} + \frac{l}{\cosh (l \theta) \sinh (l \theta)} \right) \frac{d\psi_i}{d\theta} - \left[ a_i^2 l^2 \frac{\sinh^2 (l \theta)}{\cosh^2 (l \theta)} + b_i^2 l^2 \frac{\cosh^2 (l \theta)}{\sinh^2 (l \theta)} - c_i \right] \psi_i = 0, \tag{57}
\]

afterward, we make the following change of variables: \( y = \cosh (l \theta) \) and \( \varsigma = y^2 - 1 \), in this way we obtain the following equation

\[
\varsigma (1 + \varsigma) \frac{d^2 \psi_i}{d\varsigma^2} + (1 + 2\varsigma) \frac{d\psi_i}{d\varsigma} - \left[ \frac{a_i^2}{4} \frac{\varsigma}{1 + \varsigma} + \frac{b_i^2}{4} \frac{(1 + \varsigma)}{\varsigma} - \frac{\epsilon^2 - 1}{4} \right] \psi_i = 0, \tag{58}
\]

where \( \epsilon^2 - 1 = \frac{c_i}{4l^2} \). We choose \( \lambda_i = \frac{|a_i|}{2} \) and \( \delta_i = \frac{|b_i|}{2} \). Therefore, considering the critical points of equation we can write the solution of Eq. (58) in the form

\[
\psi_i (\varsigma) = \varsigma^{\delta_i} (1 + \varsigma)^{\lambda_i} \tilde{F}_i (\varsigma), \tag{59}
\]

where \( \tilde{F}_i (\varsigma) \) is an unknown functions. Thus, we obtain the following equation for \( \tilde{F}_i (x) \):

\[
\varsigma (1 - \varsigma) \frac{d^2 \tilde{F}_i}{d\varsigma^2} + [2\delta_i + 1 - 2\varsigma (\delta_i + \lambda_i + 1)] \frac{d\tilde{F}_i}{d\varsigma} - [\lambda_i + 2\lambda_i \delta_i + \delta_i + \frac{c_i}{4l^2}] \tilde{F}_i = 0. \tag{60}
\]

It results in two hypergeometric differential equations, where the functions \( \tilde{F}_i (\varsigma) = \, _2F_1 (A, B, C; \varsigma) \) and the parameters \( A \) and \( B \) given by

\[
A = \left[ \left( \frac{\ell}{\alpha} + \frac{1}{2} \right) \right] - \frac{E \Omega}{2l^2} \right] + \sqrt{\frac{E^2 \Omega^2}{4l^4} - \frac{1}{4l^2} \left( E^2 - k^2 \right)} \tag{61}
\]

\[
B = \left[ \left( \frac{\ell}{\alpha} + \frac{1}{2} \right) \right] - \frac{E \Omega}{2l^2} \right] - \sqrt{\frac{E^2 \Omega^2}{4l^4} - \frac{1}{4l^2} \left( E^2 - k^2 \right)}
\]

As in the previous section, we impose that the hypergeometric series must truncate, reducing to a polynomial of degree \( \nu \), then, we obtain the following allowed energies for both spinors:

\[
E_{\nu, \ell} = 2\Omega \left[ \nu + \frac{|\ell|}{2\alpha} + \frac{j}{2\alpha} + \frac{1}{2} \right] \pm 2\Omega \left\{ \left[ \nu + \frac{|\ell|}{2\alpha} + \frac{\ell}{2\alpha} + \frac{1}{2} \right] - \frac{l^2}{2\Omega} \left[ n + \frac{|\ell|}{2\alpha} + \frac{j}{2\alpha} + \frac{1}{2} \right]^2 + \frac{k^2}{4\Omega^2} \right\}^{1/2}, \tag{62}
\]

where \( \nu = 0, 1, 2, 3, \ldots \) and \( -\nu + \frac{1}{2} \leq \frac{\ell}{\alpha} \leq \infty \). Note that in the \( \alpha = 1 \) limit we obtain the eigenvalues for Weyl fermions in the background of hyperbolic Gödel-type metric. We
can also observe that in present case the results obtained in this section is different of the obtained in the limit where mass is zero in Ref. [37] for Dirac fermions in a hyperbolic Gödel spacetime with torsion. Now we verify the existence of zero mode for this Weyl spinor in hyperbolic Gödel-type geometry. When we impose the condition for (62) produces a eigenvalues with null energy. For this case, we obtain the following condition for zero modes that are given by

\[ k = \pm l; \quad (63) \]

when \( \ell \leq 0 \), or

\[ k = \pm 2l \left( \frac{\ell}{\alpha} + \frac{1}{2} \right); \quad (64) \]

for \( \ell \geq 0 \). In this case we observe that the condition (64) depends on the \( \alpha \) parameter.

In this way, the presence of topological defect modify the condition for existence of zero mode. In Refs [21, 24, 27] it was demonstrated that eigenvalues of energy in the hyperbolic geometry have two contributions: the first is discrete spectrum contribution given by Eq. (62), and the other continuous energy spectrum contribution, whose lower bounds limit is obtained by the relation \(-\nu + \frac{1}{2} \leq \frac{\ell}{\alpha} \leq -\frac{1}{2}\). In this form, in the hyperbolic Gödel-type metric with a topological defect, the eigenvalues of energy have two parts, one is discrete and other is continuous, and they are determined by the parameter \( \varepsilon \) through the relation

\[ \varepsilon^2 = \frac{(\Omega^2 - l^2)}{4l^2} E^2 + \frac{k^2}{4l^2}; \quad (65) \]

in this way, we have discrete eigenvalues of energy for \( \varepsilon^2 \geq 1 \) and a continuous eigenvalues for \( \varepsilon^2 < 1 \).

Let us study the eigenvalues of energy (62) considering the parameters \( \Omega \) and \( l \). Thus, we consider three cases:

1. The case \( \Omega^2 > l^2 \), we have of Eq. (65) a discrete set of eigenvalues of the following condition for energy given by

\[ E^2 \geq \frac{l^2}{\Omega^2 - l^2} \left[ 4l^2 - k^2 \right]. \quad (66) \]

Otherwise, the relativistic spectrum of energy is continuous.
2. The second case: considering $\Omega^2 = \ell^2$, we have from Eq. (65) a discrete set of eigenvalues of energy from the following condition

$$k^2 \geq 4\ell^2.$$  

(67)

Otherwise, the relativistic eigenvalues spectrum is continuous.

3. Finally, in the third case we consider the condition $\Omega^2 < \ell^2$ and obtain from Eq. (65) a discrete set of eigenvalues of energy from the condition

$$E^2 \leq \frac{\ell^2}{\ell^2 - \Omega^2} [k^2 - 4\ell^2].$$  

(68)

Otherwise, we also have that the eigenvalues for Weyl particle of energy is continuous.

Thus, the spinor for the Weyl particle in the hyperbolic case is given by

$$\psi(t, \varsigma, \phi, z) = \tilde{C}_{\nu, m} e^{-i [Et - \ell \phi - k z]} \times$$

$$\times \left( (1 + \varsigma)^{\frac{|a_1|}{2}} \varsigma^{\frac{|b_1|}{2}} \frac{2F_1}{2} \left( \mathcal{A}, \mathcal{B}, \frac{\ell}{\alpha} + \frac{3}{2}; \varsigma \right) \right),$$  

(69)

where $\tilde{C}_{\nu, \ell}$ is a constant spinor and the parameters $\mathcal{A}$ and $\mathcal{B}$ of the hypergeometric functions have been defined in Eq. (61), and $\frac{1}{2} \leq \frac{\ell}{\alpha} \leq \infty$. We have the spinor given by

$$\psi(t, \varsigma, \phi, z) = \tilde{C}_{\nu, m} e^{-i [Et - \ell \phi - k z]} \times$$

$$\times \left( (1 + \varsigma)^{\frac{|a_2|}{2}} \varsigma^{\frac{|b_2|}{2}} \frac{2F_1}{2} \left( 1 - \mathcal{A}, 1 - \mathcal{B}, -\frac{\ell}{\alpha} + \frac{3}{2}; \varsigma \right) \right),$$  

(70)

with $\tilde{C}_{\nu, \ell}$ as constant spinor and $-\nu + \frac{1}{2} \leq \frac{\ell}{\alpha} \leq -\frac{1}{2}$.

VI. CONCLUSIONS

We have solved the Weyl equation for a family of Gödel-type geometries pieced by a topological defect. We obtained the corresponding Weyl equations in the Som-Raychaudhury, spherical and hyperbolic Gödel metrics containing a topological defect that passes along $z$-axis, and solved them exactly. In the case of the Som-Raychaudhury metric with a topological defect, we have obtained the allowed energies for this relativistic quantum system
and have shown an analogy between the relativistic energy levels and the Landau levels, with the rotation plays the role of the uniform magnetic field along the $z$-direction. We have also seen that the presence of the topological defect breaks the degeneracy of the Weyl particle energy levels. We have discussed a possibility of energy zero mode and we obtain the condition for this possibility.

In the case of the spherically symmetric Gödel-type metric with a cosmic string, we have also obtained the corresponding Weyl equation and solved it analytically. We have obtained the eigenvalues of energies of this Weyl fermions system. We observe that the presence of topological defect breaks the degeneracy of system. For the spherical symmetry case we demonstrated the absence of possibility of zero mode, but in the limit of $R \rightarrow \infty$ we obtain the condition of the Som-Raychaudhury case.

We have also obtained and solved analytically the Weyl equation in the hyperbolic Gödel-type geometry with a topological defect. We have shown that the eigenvalues of energies for the system can be both discrete and continuous. Moreover, the eigenvalues are similar to the Landau levels in a hyperbolic space, where we observe that the presence of the topological defect breaks the degeneracy of the eigenvalues of energy. We also investigate the possibility of a zero mode and we obtain condition for existence of these zero modes.

The results obtained in this contribution for quantum dynamics of Weyl fermions in family of Gödel-type metric in general relativity are similar to Landau levels for Weyl fermions in a curved space. The energy levels obtained for the class of spacetimes has properties different from the energy levels obtained in the case of quantum dynamics of Dirac fermions in the presence of Gödel-type background in theory relativity with torsion [37].

Note that the rotation of Gödel spacetime introduce a contribution in the Weyl energy levels, for all case analysed here, similar to a mass term. This geometric mass term reduces a possibility of an energy zero mode. We have obtained in the flat and hyperbolic case a condition of existence of the zero mode for Weyl particle in Gödel-type metric The present study can be important to investigate the Hall effect for Weyl fermions in the three-sphere $S^3$ [45, 46] with presence of topological defect and the higher dimensional quantum Hall effects and to study the A-class topological insulators with emphasis on the noncommutative geometry [47]. We claim the analogy between energy levels for Weyl fermions in a family of Gödel-type metrics and Landau levels in curved spaces can be used to investigate the Weyl semimetals [48] in curved spaces. The systems investigated here can be used to describe
condensed matter systems in curved geometries on the influence of rotation described by massless fermions\[20, 50 \, 53\]

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