DISCONTINUOUS SOLUTIONS
FOR THE GENERALIZED SHORT PULSE EQUATION

GIUSEPPE MARIA COCLITE*

Dipartimento di Meccanica, Matematica e Management
Politecnico di Bari
via E. Orabona 4, 70125 Bari, Italy

LORENZO DI RUVO

Dipartimento di Matematica
Università di Bari
via E. Orabona 4, 70125 Bari, Italy

(Communicated by Vladimir Gueorguiev)

Abstract. The generalized short pulse equation is a non-slowly-varying envelope approximation model that describes the physics of few-cycle-pulse optical solitons. This is a nonlinear evolution equation. In this paper, we prove the wellposedness of the Cauchy problem associated with this equation within a class of discontinuous solutions.

1. Introduction. In past years, intense ultrashort light pulses comprising merely a few-optical cycles became routinely available; for a review of the various techniques of their production and measurement as well as relevant theoretical methods used to model their unique features (see [7]). These intense ultrashort optical pulses have various applications in the field of light-matter interactions, high-order harmonic generation, extreme nonlinear optics (see [52]), and attosecond physics (see [43]). Several theoretical approaches have been considered thus far to describe the physics of few-cycle-pulse optical solitons; chiefly we have three classes of governing models: the first one is the full quantum approach (see [36, 39, 40, 46]), the second one is the refinements of envelope nonlinear Schrödinger type equations, in the framework of the slowly-varying envelope approximation (SVEA) (see [6, 49, 51]) and the third one is the non-SVEA models (see [3, 4, 27]).

A non-SVEA model is represented by the following equation (see [2, 4, 5, 29, 30, 31]):

\[ \partial_t u + au^2 \partial_x u + b \partial_{xxx} u = 0, \quad a, b \in \mathbb{R}, \]

(1)

known as the modified Korteweg-de Vries equation [16, 22, 26, 42, 47]. In [26], the Cauchy problem for (1) is studied, while, in [16, 42], the convergence of the solution

* Corresponding author: G. M. Coclite.

2000 Mathematics Subject Classification. Primary: 35G15, 35L65, 35L05; Secondary: 35A05.

Key words and phrases. Existence, uniqueness, stability, entropy solutions, conservation laws, generalized short pulse equation, Cauchy problem.

The authors are members of the Gruppo Nazionale per l’Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).
of (1) to the unique entropy solution of the following scalar conservation law
\[ \partial_t u + au^2 \partial_x u = 0, \]
(2)
is proven.

One additional non-SVEA model is (see [1, 4, 27, 38, 41, 45, 50]):
\[ \partial_x (\partial_t u + a \partial_x u^3) = cu \quad a, c \in \mathbb{R}. \]
(3)
It was introduced by Kozlov and Sazonov [27] as a model equation describing the nonlinear propagation of optical pulses of a few oscillations duration in dielectric media, and by Schäfer and Wayne [41] as a model equation describing the propagation of ultra-short light pulses in silica optical fibers. An interpretation of (3) in the context of Maxwell equations is given in [38].

Recently, wellposedness results for the Cauchy problem of (3) are proven in the context of energy spaces (see [24, 37, 45]). A similar result is proven in [14, 19] in the context of the entropy solution, while, in [15, 20], the wellposedness of the homogeneous initial boundary value problem is studied. Finally, the convergence of a finite difference scheme is studied in [9].

The papers [29, 30, 32] show that the following evolution equation
\[ \partial_x (\partial_t u + a \partial_x u^3 + b \partial^3_{xxx} u) = cu + d \partial_x \left( \int_x^\infty u dy \right)^3, \quad a, b, c, d \in \mathbb{R}, \]
(4)
known as the generalized short pulse equation, is the most general of all approximate non-SVEA models for FCPs, and in fact contains all of them.

Observe that, if \(d = 0\), (4) reads
\[ \partial_x (\partial_t u + a \partial_x u^3 + b \partial^3_{xxx} u) = cu, \quad a, b, c \in \mathbb{R}. \]
(5)
It was derived by Costanzino, Manukian and Jones [23] in the context of the nonlinear Maxwell equations with high-frequency dispersion. Kozlov and Sazonov [27] show that (5) is a more general equation than (3) to describe the nonlinear propagation of optical pulses of a few oscillations duration in dielectric media.

Mathematical properties of (5) are studied in many different contexts, including the local and global wellposedness in energy spaces [23, 37] and stability of solitary waves [23, 34].

Observe that letting \(b \to 0\) in (5), we obtain (3). Hence, following [11, 33, 42], in [18, 19], we study the convergence of the solution of (5) to the unique entropy solution of (3).

In this paper, we assume
\[ d \neq 0, \]
and investigate the wellposedness of classes of discontinuous functions for (4) with \(b = 0\), that is
\[ \partial_x (\partial_t u + a \partial_x u^3) = cu + d \partial_x \left( \int_x^\infty u dy \right)^3. \]
(6)
Integrating (6) in \(x\), we gain the integro-differential formulation of (6)
\[ \partial_t u + a \partial_x u^3 = c \int_x^\infty u dy + d \left( \int_x^\infty u dy \right)^3, \]
(7)
that is equivalent to
\[ \partial_t u + a \partial_x u^3 = cP + dP^3, \quad \partial_x P = u. \]
(8)
The presence of the cubic term in the first equation of (8) makes the analysis of such equation more subtle than the one of (3).

We are interested in the Cauchy problem for this equation, thus we augment with the initial condition

$$ u(0, x) = u_0(x), \quad x \in \mathbb{R}. $$

(9)

The assumption, which we make on (9), depends on the conserved quantities of (8). In fact, one of the main issues during the analysis of (8) is that the equation does not preserve the $L^1$ norm, while, in general, the unique conserved quantity is (see Lemmas 2.2):

$$ t \mapsto -\int_{\mathbb{R}} u^2(t, x) \, dx. $$

(10)

We consider only the case

$$ a \neq 0, \quad \frac{c}{d} \geq 0, $$

(11)

being the one $c/d < 0$ similar. On the initial condition $u_0$ we assume that

$$ u_0(x) \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}). $$

(12)

Following [12, 14, 19], on the function

$$ P_0(x) = \int_{-\infty}^{x} u_0(y) \, dy, $$

(13)

we assume that

$$ \|P_0\|_{L^2(\mathbb{R})}^2 = \int_{\mathbb{R}} \left( \int_{-\infty}^{x} u_0(y) \, dy \right)^2 \, dx < \infty, $$

$$ \|P_0\|_{L^4(\mathbb{R})}^4 = \int_{\mathbb{R}} \left( \int_{-\infty}^{x} u_0(y) \, dy \right)^4 \, dx < \infty, $$

(14)

in order to get the boundedness of $u$.

The following Cauchy problem

$$ \begin{cases}
\partial_t u + a \partial_x u^3 = c \int_0^x u \, dy + d \left( \int_0^x u \, dy \right)^3, & t > 0, \quad x \in \mathbb{R}, \\
u(0, x) = u_0(x), & x \in \mathbb{R},
\end{cases} $$

(15)

can be rewritten as follows

$$ \begin{cases}
\partial_t u + a \partial_x u^3 = c P + d P^3, & t > 0, \quad x \in \mathbb{R}, \\
\partial_x P = u, & t > 0, \quad x \in \mathbb{R}, \\
P(t, 0) = 0, & t > 0, \quad x \in \mathbb{R}, \\
u(0, x) = u_0(x), & x \in \mathbb{R}.
\end{cases} $$

(16)

Due to the regularizing effect of the $P$ equation in (16), we have that

$$ u \in L^\infty((0, T) \times \mathbb{R}) \implies P \in L^\infty(0, T; W^{1, \infty}(\mathbb{R})), \quad T > 0. $$

(17)

Following [12, 14], we give the following definition of solution.

**Definition 1.1.** We say that $u \in L^\infty((0, T) \times \mathbb{R})$ is an entropy solution of the initial value problem (15) if

i) $u$ is a distributional solution of (16);
for every convex function $\eta \in C^2(\mathbb{R})$ the entropy inequality
\[
\partial_t \eta(u) + \partial_x q(u) - c \eta'(u) P - d \eta'(u) P^3 \leq 0, \quad q(u) = 3 \int_0^u \xi^2 \eta'(\xi) d\xi, \quad (18)
\]
holds in the sense of distributions in $(0, \infty) \times \mathbb{R}$.

The main result of this section is the following theorem.

**Theorem 1.2.** Assume (11), (12) and (14). The initial value problem (15) possesses an unique entropy solution $u$ in the sense of Definition 1.1. Moreover, if $u$ and $v$ are two entropy solutions of (6) in the sense of Definition 1.1, the following inequality holds
\[
\|u(t, \cdot) - v(t, \cdot)\|_{L^1(-R,R)} \leq C(T) \|u(0, \cdot) - v(0, \cdot)\|_{L^1(-R-C(T)h,R+C(T)h)}, \quad (19)
\]
for almost every $0 < t < T$, $R > 0$, and some suitable constant $C(T) > 0$.

This paper is organized into two sections. In Section 2, we prove Theorem 1.2.

2. **Proof of Theorem 1.2.** Our existence argument is based on passing to the limit in a vanishing viscosity approximation of (16).

Fix a small number $\varepsilon > 0$ and let $u_\varepsilon = u_\varepsilon(t,x)$ be the unique classical solution of the following mixed problem [17, 21]:
\[
\begin{align*}
\partial_t u_\varepsilon + a \partial_x u_\varepsilon^3 &= c P_\varepsilon + d P_\varepsilon^3 + \varepsilon \partial_{xx} u_\varepsilon, & t > 0, \ x \in \mathbb{R}, \\
\partial_x P_\varepsilon &= u_\varepsilon, & t > 0, \ x \in \mathbb{R}, \\
P_\varepsilon(t,0) &= 0, & t > 0, \ x \in \mathbb{R}, \\
u_\varepsilon(0,x) &= u_{\varepsilon,0}(x), & x \in \mathbb{R},
\end{align*}
\quad (20)
\]
where $u_{\varepsilon,0}$ is a $C^\infty_c$ approximation of $u_0$ such that
\[
\begin{align*}
\|u_{\varepsilon,0}\|_{L^\infty(\mathbb{R})} &\leq \|u_0\|_{L^\infty(\mathbb{R})}, & \|u_{\varepsilon,0}\|_{L^2(\mathbb{R})} &\leq \|u_0\|_{L^2(\mathbb{R})}, \\
\|u_\varepsilon\|_{L^1(\mathbb{R})} &\leq \|u_0\|_{L^1(\mathbb{R})}, & \int_{\mathbb{R}} u_{\varepsilon,0}(x) dx &= 0, \\
\|P_\varepsilon\|_{L^2(\mathbb{R})} &\leq \|P_0\|_{L^2(\mathbb{R})}, & \|P_{\varepsilon,0}\|_{L^1(\mathbb{R})} &\leq \|P_0\|_{L^1(\mathbb{R})}.
\end{align*}
\quad (21)
\]
Clearly, (20) is equivalent to the integro-differential problem
\[
\begin{align*}
\partial_t u_\varepsilon + a \partial_x u_\varepsilon^3 &= c \int_0^x u_\varepsilon(t,y) dy + d \left( \int_0^x u_\varepsilon(t,y) dy \right)^3 + \varepsilon \partial_{xx} u_\varepsilon, & t > 0, \ x \in \mathbb{R}, \\
u_\varepsilon(0,x) &= u_{\varepsilon,0}(x), & x \in \mathbb{R}.
\end{align*}
\quad (22)
\]
Let us prove some a priori estimates on $u_\varepsilon$ and $P_\varepsilon$, denoting with $C_0$ the constants which depend only on the initial data, and $C(T)$ the constants which depend also on $T$.

Following [13, Lemma 2.1], we prove the following result.

**Lemma 2.1.** Assume (11). For each $t \in (0, \infty)$, we have that
\[
P_\varepsilon(t, -\infty) = P_\varepsilon(t, \infty) = 0,
\quad (23)
\]
that is
\[
\int_0^{-\infty} u_\varepsilon(t,x) dx = \int_0^\infty u_\varepsilon(t,x) dx = 0.
\quad (24)
\]
In particular, we have that
\[ \int_{\mathbb{R}} u_\varepsilon(t, x) dx = 0. \] (25)

**Proof.** We begin by proving that
\[ P_\varepsilon(t, -\infty) = 0 \] (26)

By (20), we have that
\[ 0 = \lim_{x \to -\infty} \left( \p_t u_\varepsilon + a \p_x u_\varepsilon^3 - \varepsilon \p_{xx}^2 u_\varepsilon \right) = c P_\varepsilon(t - \infty) + d P_\varepsilon^3(t, -\infty). \]

Consequently,
\[ P_\varepsilon(t, -\infty) \left( c + d P_\varepsilon^2(t, -\infty) \right) = 0, \] (27)

and due to (11), we gain (26).

In a similar way, we can prove that
\[ P_\varepsilon(t, \infty) = 0. \]

We continue by proving (24).

Integrating the second equation of (20), again by (20), we have that
\[ P_\varepsilon(t, x) = \int_0^x u_\varepsilon(t, y) dy. \] (28)

(24) follows from (23) and (28).

Finally, we prove (25). We begin by observing that, by (24),
\[ \int_{-\infty}^0 u_\varepsilon(t, x) dx = 0. \] (29)

Therefore, by (24) and (29),
\[ \int_{-\infty}^0 u_\varepsilon(t, x) dx + \int_0^\infty u_\varepsilon(t, x) dx = \int_{\mathbb{R}} u_\varepsilon(t, x) dx = 0, \]

that is (25).  

\[ \square \]

**Lemma 2.2.** Assume (11). For each \( t \in (0, \infty) \),
\[ \| u_\varepsilon(t, \cdot) \|^2_{L^2(\mathbb{R})} + 2 \varepsilon \int_0^t \| \p_x u_\varepsilon(s, \cdot) \|^2_{L^2(\mathbb{R})} ds \leq \| u_0 \|^2_{L^2(\mathbb{R})}. \] (30)

**Proof.** Multiplying the first equation of (20) by \( 2 u_\varepsilon \), from the second equation of (20), an integration on \( \mathbb{R} \) gives
\[ \frac{d}{dt} \| u_\varepsilon(t, \cdot) \|^2_{L^2(\mathbb{R})} = 2 \int_{\mathbb{R}} u_\varepsilon \p_t u_\varepsilon dx \]
\[ = -6a \int_{\mathbb{R}} u_\varepsilon^3 \p_x u_\varepsilon dx + 2c \int_{\mathbb{R}} P_\varepsilon u_\varepsilon dx + 2d \int_{\mathbb{R}} P_\varepsilon^3 u_\varepsilon dx + 2 \varepsilon \int_{\mathbb{R}} u_\varepsilon^2 \p_{xx}^2 u_\varepsilon dx \] (31)
\[ = 2c \int_{\mathbb{R}} P_\varepsilon \p_x P_\varepsilon dx + 2d \int_{\mathbb{R}} P_\varepsilon^3 \p_x P_\varepsilon dx - 2 \varepsilon \| \p_x u_\varepsilon(t, \cdot) \|^2_{L^2(\mathbb{R})}. \]

An integration on \((0, t)\) (21) and (23) give (30).  

\[ \square \]
Lemma 2.3. Assume (11). For each \( t \in (0, \infty) \), we have that
\[
c \int_0^{-\infty} P_\varepsilon(t, x) \, dx + d \int_0^{-\infty} P^3_\varepsilon(t, x) \, dx = \gamma_\varepsilon(t),
\]
\[
c \int_0^{\infty} P_\varepsilon(t, x) \, dx + d \int_0^{\infty} P^3_\varepsilon(t, x) \, dx = \gamma_\varepsilon(t),
\]
where
\[
\gamma_\varepsilon(t) := -au_\varepsilon^2(t, 0) + \varepsilon \partial_x u_\varepsilon(t, 0).
\]
Moreover,
\[
c \int_\mathbb{R} P_\varepsilon(t, x) \, dx + d \int_\mathbb{R} P^3_\varepsilon(t, x) \, dx = 0.
\]
Proof. We begin by observing that, differentiating (28) with respect to \( t \), we have
\[
\partial_t P_\varepsilon(t, x) = \frac{d}{dt} \int_0^x u_\varepsilon(t, y) \, dy = \int_0^x \partial_t u_\varepsilon(t, y) \, dy.
\]
Integrating the first equation of (20) on \((0, x)\), by (36), we get
\[
c \int_0^x P_\varepsilon(t, y) \, dy + d \int_0^x P^3_\varepsilon(t, y) \, dy
\]
\[
= \partial_t P_\varepsilon(t, x) + au_\varepsilon^2(t, x) - au_\varepsilon^2(t, 0) - \varepsilon \partial_x u_\varepsilon(t, x) + \varepsilon \partial_x u_\varepsilon(t, 0).
\]
Since \( u_\varepsilon \) is a smooth solution of (20),
\[
\lim_{x \to -\infty} \left( au_\varepsilon^2(t, x) - \varepsilon \partial_x u_\varepsilon(t, x) \right) = 0.
\]
Moreover, by (24) and (36),
\[
\partial_t P_\varepsilon(t, -\infty) = \int_0^{-\infty} \partial_t u_\varepsilon(t, x) \, dx = \frac{d}{dt} \int_0^{-\infty} u_\varepsilon(t, x) \, dx = 0.
\]
(37), (38) and (39) give (32).

We prove (33). Again by the smoothness of \( u_\varepsilon \), we have that
\[
\lim_{x \to -\infty} \left( au_\varepsilon^2(t, x) - \varepsilon \partial_x u_\varepsilon(t, x) \right) = 0,
\]
while, from (24) and (36),
\[
\partial_t P_\varepsilon(t, \infty) = \int_0^{\infty} \partial_t u_\varepsilon(t, x) \, dx = \frac{d}{dt} \int_0^{\infty} u_\varepsilon(t, x) \, dx = 0.
\]
(33) follows from (37), (40) and (41).

Finally, (35) follows from (32) and (33).

Lemma 2.4. Assume (11). Let \( T > 0 \). There exists a constant \( C(T) > 0 \), independent on \( \varepsilon \), such that
\[
\| P_\varepsilon \|_{L^\infty((0, T) \times \mathbb{R})} \leq C(T).
\]
In particular, for every $0 \leq t \leq T$, we have
\[
\|u_\varepsilon(t, \cdot)\|_{L^4(\mathbb{R})} \leq C(T),
\]
\[
\frac{2\varepsilon}{d} \|P_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})} \leq C(T),
\]
\[
\|P_\varepsilon(t, \cdot)\|_{L^4(\mathbb{R})} \leq C(T),
\]

(43)
\[
\varepsilon \int_0^t \|P_\varepsilon(s, \cdot)u_\varepsilon(s, \cdot)\|^2_{L^2(\mathbb{R})} \, ds \leq C(T),
\]
\[
\varepsilon \int_0^t \|u_\varepsilon(s, \cdot)\partial_x u_\varepsilon(s, \cdot)\|^2_{L^2(\mathbb{R})} \, ds \leq C(T).
\]

Proof. Let $0 \leq t \leq T$. Multiplying the first equation of (20) by $4u_\varepsilon^2$, an integration on $\mathbb{R}$ gives
\[
\frac{d}{dt} \|u_\varepsilon(t, \cdot)\|^4_{L^4(\mathbb{R})} = 4 \int_\mathbb{R} u_\varepsilon^3 \partial_t u_\varepsilon \, dx
\]
\[
= -12a \int_\mathbb{R} u_\varepsilon^5 \partial_x u_\varepsilon \, dx + 4c \int_\mathbb{R} P_\varepsilon u_\varepsilon^3 \, dx + 4d \int_\mathbb{R} P_\varepsilon^2 u_\varepsilon^3 \, dx + 4\varepsilon \int_\mathbb{R} u_\varepsilon^3 \partial_x^2 u_\varepsilon \, dx
\]
\[
= 4c \int_\mathbb{R} P_\varepsilon u_\varepsilon^3 \, dx + 4d \int_\mathbb{R} P_\varepsilon^2 u_\varepsilon^3 \, dx - 12\varepsilon \|u_\varepsilon(t, \cdot)\partial_x u_\varepsilon(t, \cdot)\|^2_{L^2(\mathbb{R})}.
\]
Hence, we have that
\[
\frac{d}{dt} \|u_\varepsilon(t, \cdot)\|^4_{L^4(\mathbb{R})} + 12\varepsilon \|u_\varepsilon(t, \cdot)\partial_x u_\varepsilon(t, \cdot)\|^2_{L^2(\mathbb{R})}
\]
\[
= 4c \int_\mathbb{R} P_\varepsilon u_\varepsilon^3 \, dx + 4d \int_\mathbb{R} P_\varepsilon^2 u_\varepsilon^3 \, dx.
\]

(44)

Thanks to (32), we can consider the following function
\[
F_\varepsilon(t, x) = c \int_{-\infty}^x P_\varepsilon(t, y) \, dy + d \int_{-\infty}^x P_\varepsilon^2(t, x) \, dx.
\]

(45)

Integrating the second equation of (20), by (23), we get
\[
P_\varepsilon(t, x) = \int_{-\infty}^x u_\varepsilon(t, y) \, dy.
\]

(46)

Differentiating (46) with respect to $t$, we obtain that
\[
\partial_t P_\varepsilon(t, x) = \frac{d}{dt} \int_{-\infty}^x u_\varepsilon(t, y) \, dy = \int_{-\infty}^x \partial_t u_\varepsilon(t, y) \, dy.
\]

(47)

Integrating the first equation of (20) on $(-\infty, x)$, from (45) and (47), we have
\[
\partial_t P_\varepsilon(t, x) = -au_\varepsilon^2(x) + F_\varepsilon(t, x) + \varepsilon \partial_x u_\varepsilon(t, x).
\]

(48)

Multiplying (48) by $cP_\varepsilon + dP_\varepsilon^3$, an integration on $\mathbb{R}$ give
\[
\int_\mathbb{R} (cP_\varepsilon + dP_\varepsilon^3) \partial_t P_\varepsilon \, dx
\]
\[
= -ac \int_\mathbb{R} P_\varepsilon u_\varepsilon^3 \, dx - ad \int_\mathbb{R} P_\varepsilon^2 u_\varepsilon^3 \, dx + \int_\mathbb{R} (cP_\varepsilon + dP_\varepsilon^3) F_\varepsilon \, dx
\]
\[
+ c\varepsilon \int_\mathbb{R} P_\varepsilon \partial_x u_\varepsilon \, dx + d\varepsilon \int_\mathbb{R} P_\varepsilon^2 \partial_x u_\varepsilon \, dx.
\]

(49)
Observe that, by (45),
\[\partial_x F_\varepsilon(t, x) = cP_\varepsilon(t, x) + dP_\varepsilon^3(t, x).\]  
(50)

Consequently, from (35) and (50),
\[\int_R (cP_\varepsilon + dP_\varepsilon^3) F_\varepsilon dx = \int_R F_\varepsilon \partial_x F_\varepsilon dx = \frac{1}{2} F_\varepsilon^2(t, \infty) - \frac{1}{2} F_\varepsilon^2(t, -\infty)\]
\[= \frac{1}{2} F_\varepsilon^2(t, \infty) = \frac{1}{2} \left( c \int_R P_\varepsilon dx + d \int_R P_\varepsilon^3 dx \right)^2 = 0.\]  
(51)

Moreover, by (20) and (23),
\[c \varepsilon \int_R P_\varepsilon \partial_x u_\varepsilon dx = -c \varepsilon \int_R u_\varepsilon \partial_x P_\varepsilon dx = -c \varepsilon \|u_\varepsilon(t, \cdot)\|^2_{L^2(R)},\]
\[d \varepsilon \int_R P_\varepsilon^3 \partial_x u_\varepsilon dx = -3d \varepsilon \int_R P_\varepsilon^2 \partial_x P_\varepsilon u_\varepsilon dx = -3d \varepsilon \|P_\varepsilon(t, \cdot)u_\varepsilon(t, \cdot)\|^2_{L^2(R)}.\]  
(52)

Since
\[c \int_R P_\varepsilon \partial_t P_\varepsilon dx = \frac{c}{2} \frac{d}{dt} \|P_\varepsilon(t, \cdot)\|^2_{L^2(R)},\]
\[d \int_R P_\varepsilon^3 \partial_t P_\varepsilon dx = \frac{d}{4} \frac{d}{dt} \|P_\varepsilon(t, \cdot)\|^4_{L^4(R)},\]

it follows from (49), (51) and (52) that
\[\frac{d}{dt} \left( \frac{c}{2} \|P_\varepsilon(t, \cdot)\|^2_{L^2(R)} + \frac{d}{4} \|P_\varepsilon(t, \cdot)\|^4_{L^4(R)} \right)\]
\[+ c \varepsilon \|u_\varepsilon(t, \cdot)\|^2_{L^2(R)} + 3d \varepsilon \|P_\varepsilon(t, \cdot)u_\varepsilon(t, \cdot)\|^2_{L^2(R)}\]
\[= -ac \int_R P_\varepsilon u_\varepsilon^2 dx - ad \int_R P_\varepsilon^3 u_\varepsilon^2 dx.\]  
(53)

Dividing (53) by \(\frac{d}{4}\), from (11), we get
\[\frac{d}{dt} \left( \frac{2c}{d} \|P_\varepsilon(t, \cdot)\|^2_{L^2(R)} + \|P_\varepsilon(t, \cdot)\|^4_{L^4(R)} \right)\]
\[+ \frac{4c}{d} \varepsilon \|u_\varepsilon(t, \cdot)\|^2_{L^2(R)} + 12 \varepsilon \|P_\varepsilon(t, \cdot)u_\varepsilon(t, \cdot)\|^2_{L^2(R)}\]
\[= -\frac{4ac}{d} \int_R P_\varepsilon u_\varepsilon^2 dx - 4a \int_R P_\varepsilon^3 u_\varepsilon^2 dx.\]  
(54)

Adding (44) and (54), we get
\[\frac{dG(t)}{dt} + \frac{4c}{d} \varepsilon \|u_\varepsilon(t, \cdot)\|^2_{L^2(R)}\]
\[+ 12 \varepsilon \|P_\varepsilon(t, \cdot)u_\varepsilon(t, \cdot)\|^2_{L^2(R)} + 12 \varepsilon \|u_\varepsilon(t, \cdot)\|_{L^2(R)}\]
\[= \frac{4c(d - a)}{d} \int_R P_\varepsilon u_\varepsilon^2 dx + 4(d - a) \int_R P_\varepsilon^3 u_\varepsilon^2 dx,\]  
(55)

where
\[G(t) := \|u_\varepsilon(t, \cdot)\|^4_{L^4(R)} + \frac{2c}{d} \|P_\varepsilon(t, \cdot)\|^2_{L^2(R)} + \|P_\varepsilon(t, \cdot)\|^4_{L^4(R)}.\]  
(56)
Due to (30) and the Young inequality,

\[ 4 \left| \frac{c(d-a)}{d} \right| \int_{\mathbb{R}} |P^3_\varepsilon| |u_\varepsilon|^3 dx = 4 \int_{\mathbb{R}} \left| \frac{c(d-a)}{d} P^3_\varepsilon u_\varepsilon \right| u_\varepsilon^2 dx \]

\[ \leq 2c^2(d-a)^2 \int_{\mathbb{R}} P^2_\varepsilon u_\varepsilon^2 dx + 2 \|u_\varepsilon(t, \cdot)\|_{L^4(\mathbb{R})}^4 \]

\[ \leq 2c^2(d-a)^2 \|P_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})}^2 \|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2 \|u_\varepsilon(t, \cdot)\|_{L^4(\mathbb{R})}^4 \]

\[ \leq 2c^2(d-a)^2 \|P_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})}^2 \|u_0\|_{L^2(\mathbb{R})}^2 + 2 \|u_\varepsilon(t, \cdot)\|_{L^4(\mathbb{R})}^4. \]

It follows from (55) that

\[ \frac{dG(t)}{dt} + \frac{4c}{d} \varepsilon \|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \]

\[ + 12 \varepsilon \|P_\varepsilon(t, \cdot)u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 12 \varepsilon \|u_\varepsilon(t, \cdot)\|_{L^4(\mathbb{R})}^4 \]

\[ \leq 4 \|u_\varepsilon(t, \cdot)\|_{L^4(\mathbb{R})}^4 + \frac{2c^2(d-a)^2}{d^2} \|u_0\|_{L^2(\mathbb{R})}^2 \|P_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})}^2 \]

\[ + 2(d-a)^2 \|u_0\|_{L^2(\mathbb{R})}^2 \|P_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})}^6. \]

Observe that, by (23), we have

\[ P^3_\varepsilon(t, x) = 3 \int_{-\infty}^x P^2_\varepsilon(t, y) \partial_x P_\varepsilon(t, y) dy. \]

Thanks to (20), (30), (58) and the Hölder inequality,

\[ |P^3_\varepsilon(t, x)|^3 = 3 \left| \int_{-\infty}^x P^2_\varepsilon(t, y) \partial_x P_\varepsilon(t, y) dy \right| \leq 3 \int_{-\infty}^x P^2_\varepsilon(t, y) \partial_x P_\varepsilon(t, y) dy \]

\[ \leq 3 \int P^2_\varepsilon(t, x) \partial_x P_\varepsilon(t, x) dx \leq 3 \|P_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \|\partial_x P_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})} \]

\[ = 3 \|P_\varepsilon(t, \cdot)\|_{L^4(\mathbb{R})}^2 \|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})} \leq 3 \|P_\varepsilon(t, \cdot)\|_{L^4(\mathbb{R})}^2 \|u_0\|_{L^2(\mathbb{R})}. \]

Therefore,

\[ \|P_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})}^6 \leq 9 \|u_0\|_{L^2(\mathbb{R})}^2 \|P_\varepsilon(t, \cdot)\|_{L^4(\mathbb{R})}^4. \]

It follows from (57) and (59) that

\[ \frac{dG(t)}{dt} + \frac{4c}{d} \varepsilon \|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \]

\[ + 12 \varepsilon \|P_\varepsilon(t, \cdot)u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 12 \varepsilon \|u_\varepsilon(t, \cdot)\|_{L^4(\mathbb{R})}^4 \]

\[ \leq 4 \|u_\varepsilon(t, \cdot)\|_{L^4(\mathbb{R})}^4 + \frac{2c^2(d-a)^2}{d^2} \|u_0\|_{L^2(\mathbb{R})}^2 \|P_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})}^2 \]

\[ + 18(d-a)^2 \|u_0\|_{L^2(\mathbb{R})}^2 \|P_\varepsilon(t, \cdot)\|_{L^4(\mathbb{R})}^4. \]
Consequently, by (56) and (60), we have

\[
\frac{dG(t)}{dt} + \frac{4c}{\delta} \|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 12\varepsilon \|P_\varepsilon(t, \cdot)u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 12\varepsilon \|u_\varepsilon(t, \cdot)\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq C_0 \left( \|u_\varepsilon(t, \cdot)\|_{L^4(\mathbb{R})}^4 + \|P_\varepsilon(t, \cdot)\|_{L^4(\mathbb{R})}^4 + \|P_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})}^2 \right)
\]

\[
\leq C_0 \left( \|u_\varepsilon(t, \cdot)\|_{L^4(\mathbb{R})}^4 + \|P_\varepsilon(t, \cdot)\|_{L^4(\mathbb{R})}^4 + \frac{2c}{\delta} \|P_\varepsilon(t, \cdot)\|^2_{L^2(\mathbb{R})} + \|P_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})}^2 \right) = C_0 G(t) + C_0 \|P_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})}^2,
\]

that is

\[
\frac{dG(t)}{dt} - C_0 G(t) + \frac{4c}{\delta} \|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq C_0 \|P_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})}^2.
\]

The Gronwall Lemma (21) and (56) give

\[
\|u_\varepsilon(t, \cdot)\|_{L^4(\mathbb{R})}^4 + \frac{2c}{\delta} \|P_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \|P_\varepsilon(t, \cdot)\|_{L^4(\mathbb{R})}^4 + \frac{4c}{\delta} C_0 t \varepsilon \int_0^t e^{-C_0 s} \|u_\varepsilon(s, \cdot)\|_{L^2(\mathbb{R})}^2 \, ds + 12\varepsilon \|P_\varepsilon(t, \cdot)u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\varepsilon \|P_\varepsilon(t, \cdot)\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq \|u_{\varepsilon,0}\|_{L^4(\mathbb{R})}^4 + \frac{2c}{\delta} \|P_0\|_{L^2(\mathbb{R})}^2 + \|P_0\|_{L^4(\mathbb{R})}^4 + C_0 C_0 t \varepsilon \int_0^t e^{-C_0 s} \|P_\varepsilon(s, \cdot)\|_{L^\infty(\mathbb{R})}^2 \, ds \leq \|u_{\varepsilon,0}\|_{L^4(\mathbb{R})}^4 + \frac{2c}{\delta} \|P_0\|_{L^2(\mathbb{R})}^2 + \|P_0\|_{L^4(\mathbb{R})}^4 + C_0 C_0 t \|P_\varepsilon\|_{L^\infty(0, T) \times \mathbb{R}}^2 \leq C(T) \left( 1 + \|P_\varepsilon\|_{L^\infty((0, T) \times \mathbb{R})}^2 \right).
\]

Following [10, Lemma 2.4], or [12, Lemma 2.3], we prove (42). By (59) and (61), we have that

\[
\|P_\varepsilon\|_{L^\infty((0, T) \times \mathbb{R})}^6 \leq C(T) \left( 1 + \|P_\varepsilon\|_{L^\infty((0, T) \times \mathbb{R})}^2 \right),
\]

that is

\[
\|P_\varepsilon\|_{L^\infty((0, T) \times \mathbb{R})}^6 - C(T) \|P_\varepsilon\|_{L^\infty((0, T) \times \mathbb{R})}^2 - C(T) \leq 0.
\]

Consider the following function

\[
h(X) = X^6 - C(T)X^2 - C(T).
\]

We observe that

\[
\lim_{X \to -\infty} h(X) = \infty, \quad h(0) = -C(T).
\]
THE GENERALIZED SHORT PULSE EQUATION

Since \( h'(X) = 6X^5 - 2C(T)X \), we have that
\[
h(X) \text{ is increasing in } [E_1(T), 0], [E_2(T), \infty),
\]
where
\[
E_1(T) = -\frac{\sqrt[3]{C(T)}}{3}, \quad E_2(T) = \frac{\sqrt[3]{C(T)}}{3}.
\]
Thus,
\[
h(E_1(T)) < h(0) < 0, \quad h(E_2(T)) < h(0) < 0. \tag{65}
\]
Moreover,
\[
\lim_{X \to \infty} h(X) = \infty. \tag{66}
\]
Then, it follows from (64), (65) and (66) that the function \( h \) has only two zeros \( D(T) < 0 < C(T). \) Therefore, the inequality
\[
X^6 - C(T)X^2 - C(T) \leq 0,
\]
gives that
\[
D(T) \leq X \leq C(T).
\]

Taking \( X = \|P_\varepsilon\|_{L^\infty((0,T) \times \mathbb{R})} \), we have (42).

Finally, (43) follows from (42) and (61).

Lemma 2.5. Let \( T > 0 \). Then,
\[
\|u_\varepsilon\|_{L^\infty((0,T) \times \mathbb{R})} \leq C(T). \tag{67}
\]

Proof. Due to (20) and (42),
\[
\partial_t u_\varepsilon + a \partial_x u_\varepsilon^3 - \varepsilon \partial^2_x u_\varepsilon \leq |c| \|P_\varepsilon(t,x)\| + |d| \|P_\varepsilon(t,x)\|^3
\]
\[
\leq |c| \|P_\varepsilon\|_{L^\infty((0,T) \times \mathbb{R})} + |d| \|P_\varepsilon\|^3_{L^\infty((0,T) \times \mathbb{R})} \leq C(T).
\]
Since the map
\[
F(t) := \|u_0\|_{L^\infty(0,\infty)} + \gamma C(T)t,
\]
solves the equation
\[
\frac{dF}{dt} = C(T)
\]
and
\[
\max\{u_\varepsilon(0,x), 0\} \leq F(t), \quad (t, x) \in (0, T) \times \mathbb{R},
\]
the comparison principle for parabolic equations implies that
\[
u_\varepsilon(t,x) \leq F(t), \quad (t, x) \in (0, T) \times \mathbb{R}.
\]
In a similar way we can prove that
\[
u_\varepsilon(t,x) \geq -F(t), \quad (t, x) \in (0, T) \times \mathbb{R}.
\]
Therefore,
\[
|u_\varepsilon(t,x)| \leq \|u_0\|_{L^\infty(\mathbb{R})} + \gamma C(T)t \leq \|u_0\|_{L^\infty(\mathbb{R})} + C(T) \leq C(T),
\]
that is (67).

We construct a solution by passing to the limit in a sequence \( \{u_\varepsilon\}_{\varepsilon > 0} \) of viscosity approximations (20). We use the compensated compactness method [48].
Lemma 2.6. Let $T > 0$. There exists a subsequence $\{u_{\varepsilon_k}\}_{k \in \mathbb{N}}$ of $\{u_{\varepsilon}\}_{\varepsilon > 0}$ and a limit function $u \in L^\infty((0,T) \times \mathbb{R})$ such that

$$u_{\varepsilon_k} \to u \text{ a.e. and in } L^p_{loc}((0,T) \times \mathbb{R}), \quad 1 \leq p < \infty.$$  

(68)

Moreover, we have

$$P_{\varepsilon_k} \to P \text{ a.e. and in } L^p_{loc}(0,\infty; W^{1,p}_{loc}({\mathbb{R}))}, \quad 1 \leq p < \infty,$$

(69)

where

$$P(t,x) = \int_0^x u(t,y) dy, \quad t \geq 0, \quad x \in \mathbb{R}.$$  

(70)

Proof. Let $\eta : \mathbb{R} \to \mathbb{R}$ be any convex $C^2$ entropy function, and $q : \mathbb{R} \to \mathbb{R}$ be the corresponding entropy flux defined by $q(u) = 3au^2\eta'(u)$. By multiplying the first equation in (20) with $\eta'(u_{\varepsilon})$ and using the chain rule, we get

$$\partial_t \eta(u_{\varepsilon}) + \partial_x q(u_{\varepsilon}) = \epsilon \eta''(u_{\varepsilon}) (\partial_x u_{\varepsilon})^2 + \eta'(u_{\varepsilon}) P_{\varepsilon} + dy(u_{\varepsilon}) P_{\varepsilon}^3,$$

where $\mathcal{L}_{1,\varepsilon}, \mathcal{L}_{2,\varepsilon}, \mathcal{L}_{3,\varepsilon}$ and $\mathcal{L}_{4,\varepsilon}$ are distributions. Let us show that $\mathcal{L}_{1,\varepsilon} \to 0$ in $H^{-1}((0,T) \times \mathbb{R})$, $T > 0$ as $\varepsilon \to 0$.

Since

$$\varepsilon \partial_{xx} \eta(u_{\varepsilon}) = \partial_x (\varepsilon \eta'(u_{\varepsilon}) \partial_x u_{\varepsilon}),$$

by (30) and Lemma 2.5,

$$\|\varepsilon \eta'(u_{\varepsilon}) \partial_x u_{\varepsilon}\|_{L^2((0,T) \times \mathbb{R})}^2 \leq \varepsilon \|\eta''\|_{L^\infty(-C(T),C(T))} \left( \int_0^T \|\partial_x u_{\varepsilon}(s,\cdot)\|_{L^2(\mathbb{R})}^2 ds \right)$$

$$\leq \varepsilon \|\eta''\|_{L^\infty(-C(T),C(T))} \|u_0\|_{L^2(\mathbb{R})}^2 \to 0.$$  

We claim that $\{\mathcal{L}_{2,\varepsilon}\}_{\varepsilon > 0}$ is uniformly bounded in $L^1((0,T) \times \mathbb{R})$, $T > 0$. Again by (30) and Lemma 2.5,

$$\|\varepsilon \eta''(u_{\varepsilon}) (\partial_x u_{\varepsilon})^2\|_{L^1((0,T) \times \mathbb{R})} \leq \|\eta''\|_{L^\infty(-C(T),C(T))} \varepsilon \left( \int_0^T \|\partial_x u_{\varepsilon}(s,\cdot)\|_{L^2(\mathbb{R})}^2 ds \right)$$

$$\leq \|\eta''\|_{L^\infty(-C(T),C(T))} \|u_0\|_{L^2(\mathbb{R})}^2.$$  

We have that $\{\mathcal{L}_{3,\varepsilon}\}_{\varepsilon > 0}$ is uniformly bounded in $L^1_{loc}((0,\infty) \times \mathbb{R})$. Let $K$ be a compact subset of $(0,T) \times \mathbb{R}$. Using (42) and Lemma 2.5,

$$|c| \|\eta'(u_{\varepsilon}) P_{\varepsilon}\|_{L^1(K)} = |c| \int_K |\eta'(u_{\varepsilon})||P_{\varepsilon}| dx dt$$

$$\leq |c| \|\eta''\|_{L^\infty(-C(T),C(T))} \|P_{\varepsilon}\|_{L^\infty((0,T) \times \mathbb{R})} |K|.$$  

We have that $\{\mathcal{L}_{4,\varepsilon}\}_{\varepsilon > 0}$ is uniformly bounded in $L^1_{loc}((0,\infty) \times \mathbb{R})$. Let $K$ be a compact subset of $(0,T) \times \mathbb{R}$. Again by (42) and Lemma 2.5,

$$|d| \|\eta'(u_{\varepsilon}) P_{\varepsilon}\|_{L^1(K)} = |d| \int_K |\eta'(u_{\varepsilon})||P_{\varepsilon}|^3 dx dt$$

$$\leq |d| \|\eta''\|_{L^\infty(-C(T),C(T))} \|P_{\varepsilon}\|_{L^\infty((0,T) \times \mathbb{R})}^2 |K|.$$  


Therefore, Murat’s lemma [35] implies that
\[
\{ \partial_t \eta(u_\varepsilon) + \partial_x q(u_\varepsilon) \}_{\varepsilon > 0} \text{ lies in a compact subset of } H_{loc}^{-1}((0, T) \times \mathbb{R}). \tag{71}
\]
The \(L^\infty\) bound stated in Lemma 2.5, (71) and the Tartar’s compensated compactness method [48] give the existence of a subsequence \(\{u_{\varepsilon_k}\}_{k \in \mathbb{N}}\) and a limit function \(u \in L^\infty((0, T) \times \mathbb{R}), T > 0,\) such that (68) holds.

Finally, (69) follows from (68), the Hölder inequality and the identity
\[
P_{\varepsilon_k} = \int_0^x u_{\varepsilon_k} \, dy, \quad \partial_x P_{\varepsilon_k} = u_{\varepsilon_k}. \]

\[\square\]

Now, we prove Theorem 1.2.

**Proof of Theorem 1.2.** Lemma 2.6 and the Dominated Convergence Theorem give the existence of an entropy solution \(u\) for (15), or equivalently (16).

We prove that \(u\) is unique and (19) holds. Fix \(T > 0.\) Let \(u\) and \(v\) be two entropy solution of (15), or (16). Consequently, we have that
\[
|u^3 - v^3| \leq C(T)|u - v|, \tag{72}
\]
where
\[
C(T) = \frac{3}{|a|} \max \left\{ \|u\|^2_{L^\infty((0,T) \times \mathbb{R})}, \|v\|^2_{L^\infty((0,T) \times \mathbb{R})} \right\}. \tag{73}
\]

We define
\[
P_u = \int_0^x u \, dy, \quad P_v = \int_0^x v \, dy. \tag{74}
\]

Thanks to (72), following [8, 25, 28], we can prove that
\[
\partial_t (|u - v|) + \partial_x [(au^3 - av^3) \text{sign}(u - v)]
- \text{sign}(u - v) \left[ c(P_u - P_v) + d(P^3_u - P^3_v) \right] \leq 0,
\]
holds in sense of distributions in \((0, \infty) \times \mathbb{R},\) and
\[
\|u(t, \cdot) - v(t, \cdot)\|_{L^1(I(t))} \leq \|u_0 - v_0\|_{L^1(I(0))}
+ c \int_0^t \int_{I(s)} \text{sign}(u - v) (P_u - P_v) \, ds \, dx \tag{75}
+ d \int_0^t \int_{I(s)} \text{sign}(u - v) (P^3_u - P^3_v) \, ds \, dx,
\]
where
\[
I(s) = [\mathcal{R} - C(T)(t - s), \mathcal{R} + C(T)(t - s)]. \tag{76}
\]

Since
\[
|I(s)| = 2\mathcal{R} + 2C(T)(t - s) \leq 2\mathcal{R} + 2C(T)t \leq C(T), \tag{77}
\]
due to (74),
\[
c \int_0^t \int_{I(s)} \text{sign}(u - v) (P_u - P_v) \, ds \, dx
\leq |c| \int_0^t \int_{I(s)} |P_u - P_v| \, ds \, dx \tag{78}
\leq |c| \int_0^t \int_{I(s)} \left( \int_0^x |u - v| \, dy \right) \, ds \, dx.
Consider the following function

It follows from (75), (78) and (83) that

\[ 
|P_u^3 - P_v^3| = |P_u - P_v||P_u^2 + P_uP_v + P_v^2| 
\leq (P_u^2 + |P_u||P_v| + P_v^2)|P_u - P_v|. \]

It follows from (74), (79) and (80) that

\[ 
|P_u^3 - P_v^3| \leq C(T)x^2|\int_0^x |u - v|dy| \leq C(T)x^2 \int_0^x |u - v|dy \leq C(T)x^2. \]

Moreover, from (76),

\[ 
\int_{I(s)} x^2dx = \int_0^{R+C(T)(t-s)} x^2dx = \frac{2}{3} (R + C(T)(t-s))^3 \leq C(T). \]

Consequently, by (81) and (82),

\[ 
\frac{d}{dt} \int_{I(s)} \text{sign} (u - v) (P_u^3 - P_v^3)ds 
\leq |d| \int_0^t \int_{I(s)} |P_u^3 - P_v^3|ds 
\leq C(T) \int_0^t \int_{I(s)} \left( x^2 \int_0^x |u - v|dy \right)ds 
\leq C(T) \int_0^t \int_{I(s)} x^2 \int_{I(s)} |u - v|dyds 
\leq C(T) \int_0^t \|u(s, \cdot) - v(s, \cdot)\|_{L^1(I(s))} ds. \]

Consider the following function

\[ G_1(t) = \|u(t, \cdot) - v(t, \cdot)\|_{L^1(I(t))}, \quad t \geq 0. \]

It follows from (75), (78) and (83) that

\[ G_1(t) \leq G_1(0) + C(T) \int_0^t G_1(s)ds. \]
The Gronwall inequality and (84) give
\[ \|u(t, \cdot) - v(t, \cdot)\|_{L^1(-R,R)} \leq e^{C(T)t} \|u_0 - v_0\|_{L^1(-R-C(T)t,R+C(T)t)}, \]
that is (19). \( \square \)

REFERENCES

[1] S. Amiranashvili, A. G. Vladimirov and U. Bandelow, Solitary-wave solutions for few-cycle optical pulses, Phys. Rev. A, 77 (2008), 063821, URL https://link.aps.org/doi/10.1103/PhysRevA.77.063821.

[2] N. Belashenkov, A. Drozdov, S. Kozlov, Y. Shpolyanski and A. Tsypkin, Phase modulation of femtosecond light pulses whose spectra are superbroadened in dielectrics with normal group dispersion, J. Opt. Technol., 75 (2008), 611–614, URL http://jot.osa.org/abstract.cfm?URI=jot-75-10-611.

[3] A. N. Berkovsky, S. A. Kozlov and Y. A. Shpolyanskiy, Self-focusing of few-cycle light pulses in dielectric media, Phys. Rev. A, 72 (2005), 043821, URL https://link.aps.org/doi/10.1103/PhysRevA.72.043821.

[4] V. G. Bespalov, S. A. Kozlov, Y. A. Shpolyanskiy and I. A. Walmsley, Simplified field wave equations for the nonlinear propagation of extremely short light pulses, Phys. Rev. A, 66 (2002), 013811, URL https://link.aps.org/doi/10.1103/PhysRevA.66.013811.

[5] V. Bespalov, S. Kozlov, Y. Shpolyanskiy and A. N. Sutyagin, Spectral superbroadening of high-power femtosecond laser pulses and their time compression down to one period of the light field, Journal of Optical Technology, 65 (1998), 823–825.

[6] T. Brabec and F. Krausz, Nonlinear optical pulse propagation in the single-cycle regime, Phys. Rev. Lett., 78 (1997), 3282–3285, URL https://link.aps.org/doi/10.1103/PhysRevLett.78.3282.

[7] T. Brabec and F. Krausz, Intense few-cycle laser fields: Frontiers of nonlinear optics, Rev. Mod. Phys., 72 (2000), 545–591, URL https://link.aps.org/doi/10.1103/RevModPhys.72.545.

[8] G. M. Coclite, L. di Ruvo and K. H. Karlsen, Some wellposedness results for the Ostrovsky-Hunter equation, in Hyperbolic Conservation Laws and Related Analysis with Applications, vol. 49 of Springer Proc. Math. Stat., Springer, Heidelberg, 2014, 143–159, URL https://doi.org/10.1007/978-3-642-39007-4_7.

[9] G. M. Coclite, J. Ridder and N. H. Risebro, A convergent finite difference scheme for the Ostrovsky-Hunter equation on a bounded domain, BIT, 57 (2017), 93–122, URL https://doi.org/10.1007/s10543-016-0625-x.

[10] G. M. Coclite and L. di Ruvo, Convergence of the Ostrovsky equation to the Ostrovsky-Hunter one, J. Differential Equations, 256 (2014), 3245–3277, URL https://doi.org/10.1016/j.jde.2014.02.001.

[11] G. M. Coclite and L. di Ruvo, Dispersive and diffusive limits for Ostrovsky-Hunter type equations, NoDEA Nonlinear Differential Equations Appl., 22 (2015), 1733–1763, URL https://doi.org/10.1007/s00030-015-0342-1.

[12] G. M. Coclite and L. di Ruvo, Oleinik type estimates for the Ostrovsky-Hunter equation, J. Math. Anal. Appl., 423 (2015), 162–190, URL https://doi.org/10.1016/j.jmaa.2014.09.033.

[13] G. M. Coclite and L. di Ruvo, Well-posedness of bounded solutions of the non-homogeneous initial-boundary value problem for the Ostrovsky-Hunter equation, J. Hyperbolic Differ. Equ., 12 (2015), 221–248, URL https://doi.org/10.1142/S021989161550006X.

[14] G. M. Coclite and L. di Ruvo, Well-posedness results for the short pulse equation, Z. Angew. Math. Phys., 66 (2015), 1529–1557, URL https://doi.org/10.1007/s00033-014-0478-6.

[15] G. M. Coclite and L. di Ruvo, Wellposedness of bounded solutions of the non-homogeneous initial boundary for the short pulse equation, Boll. Unione Mat. Ital., 8 (2015), 31–44, URL https://doi.org/10.1007/s40574-015-0023-3.

[16] G. M. Coclite and L. di Ruvo, Convergence of the solutions on the generalized Korteweg–de Vries equation, Math. Model. Anal., 21 (2016), 239–259, URL https://doi.org/10.3846/13926922.2016.1150358.

[17] G. M. Coclite and L. di Ruvo, Well-posedness of the Ostrovsky-Hunter equation under the combined effects of dissipation and short-wave dispersion, J. Evol. Equ., 16 (2016), 365–389, URL https://doi.org/10.1007/s00028-015-0306-2.
[18] G. M. Coclite and L. di Ruvo, Convergence of the regularized short pulse equation to the short pulse one, *Math. Nachr.*, **291** (2018), 774–792, URL https://doi.org/10.1002/mana.201600301.

[19] G. M. Coclite and L. di Ruvo, Well-posedness and dispersive/diffusive limit of a generalized Ostrovsky-Hunter equation, *Milan J. Math.*, **86** (2018), 31–51, URL https://doi.org/10.1007/s00032-018-0278-0.

[20] G. M. Coclite, L. di Ruvo and K. H. Karlsen, The initial-boundary-value problem for an Ostrovsky-Hunter type equation, in *Non-linear Partial Differential Equations, Mathematical Physics, and Stochastic Analysis*, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2018, 97–109.

[21] G. M. Coclite, H. Holden and K. H. Karlsen, Wellposedness for a parabolic-elliptic system, *Discrete Contin. Dyn. Syst.*, **13** (2005), 659–682, URL https://doi.org/10.3934/dcds.2005.13.659.

[22] J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao, Sharp global well-posedness for KdV and modified KdV on $\mathbb{R}$ and $\mathbb{T}$, *J. Amer. Math. Soc.*, **16** (2003), 705–749, URL https://doi.org/10.1090/S0894-0347-03-00421-1.

[23] N. Costanzino, V. Manukian and C. K. R. T. Jones, Solitary waves of the regularized short pulse and Ostrovsky equations, *SIAM J. Math. Anal.*, **41** (2009), 2088–2106, URL https://doi.org/10.1137/080734327.

[24] M. Davidson, Continuity properties of the solution map for the generalized reduced Ostrovsky equation, *J. Differential Equations*, **252** (2012), 3797–3815, URL https://doi.org/10.1016/j.jde.2011.11.013.

[25] L. di Ruvo, Discontinuous solutions for the Ostrovsky–Hunter equation and two phase flows, *Phd Thesis, University of Bari*.

[26] C. E. Kenig, G. Ponce and L. Vega, Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle, *Comm. Pure Appl. Math.*, **46** (1993), 527–620, URL https://doi.org/10.1002/cpa.3160460405.

[27] S. A. Kozlov and S. V. Sazonov, Nonlinear propagation of optical pulses of a few oscillations duration in dielectric media, *Journal of Experimental and Theoretical Physics*, **84** (1997), 221–228, URL https://doi.org/10.1134/1.558109.

[28] S. N. Kružkov, First order quasilinear equations with several independent variables, *Mat. Sb. (N.S.)*, **81(123)** (1970), 228–255.

[29] H. Leblond and D. Mihalache, Few-optical-cycle solitons: Modified korteweg-de vries equation versus other non–slowly-varying-envelope-approximation models, *Phys. Rev. A*, **79** (2009), 063835, URL https://link.aps.org/doi/10.1103/PhysRevA.79.063835.

[30] H. Leblond and F. Sanchez, Models for optical solitons in the two-cycle regime, *Phys. Rev. A*, **67** (2003), 013804, URL https://link.aps.org/doi/10.1103/PhysRevA.67.013804.

[31] H. Leblond, S. V. Sazonov, I. V. Mel’nikov, D. Mihalache and F. Sanchez, Few-cycle nonlinear optics of multicomponent media, *Phys. Rev. A*, **74** (2006), 063815, URL https://link.aps.org/doi/10.1103/PhysRevA.74.063815.

[32] P. G. LeFloch and R. Natalini, Conservation laws with vanishing nonlinear diffusion and dispersion, *Nonlinear Anal.*, **36** (1999), 213–230, URL https://doi.org/10.1016/S0362-546X(98)00012-1.

[33] Y. Liu, D. Pelinovsky and A. Sakovich, Wave breaking in the short-pulse equation, *Dyn. Partial Differ. Equ.*, **6** (2009), 291–310, URL https://doi.org/10.4310/DPDE.2009.v6.n4.a1.

[34] F. Murat, L’injection du cône positif de $H^{-1}$ dans $W^{-1,q}$ est compacte pour tout $q < 2$, *J. Math. Pures Appl. (9)*, **60** (1981), 309–322.

[35] A. Nazarkin, Nonlinear optics of intense attosecond light pulses, *Phys. Rev. Lett.*, **97** (2006), 163904, URL https://link.aps.org/doi/10.1103/PhysRevLett.97.163904.

[36] D. Pelinovsky and A. Sakovich, Global well-posedness of the short-pulse and sine-gordon equations in energy space, *Communications in Partial Differential Equations*, **35** (2010), 613–629, URL https://doi.org/10.1080/03605300903509104.
[38] D. Pelinovsky and G. Schneider, Rigorous justification of the short-pulse equation, *Nonlinear Differential Equations and Applications NoDEA*, 20 (2013), 1277–1294, URL https://doi.org/10.1007/s00030-012-0208-8.

[39] N. N. Rosanov, V. E. Semenov and N. V. Vysotina, Few-cycle dissipative solitons in active nonlinear optical fibres, *Quantum Electronics*, 38 (2008), 137, URL http://stacks.iop.org/1063-7818/38/i=2/a=A08.

[40] N. N. Rosanov, V. E. Semenov and N. V. Vysotina, Collisions of few-cycle dissipative solitons in active nonlinear fibers, *Laser Physics*, 17 (2007), 1311, URL https://doi.org/10.1134/S1054660X07110072.

[41] T. Schäfer and C. Wayne, Propagation of ultra-short optical pulses in cubic nonlinear media, *Physica D: Nonlinear Phenomena*, 196 (2004), 90–105, URL http://www.sciencedirect.com/science/article/pii/S0167278904002064.

[42] M. E. Schonbek, Convergence of solutions to nonlinear dispersive equations, *Comm. Partial Differential Equations*, 7 (1982), 959–1000, URL https://doi.org/10.1080/03605308208820242.

[43] A. Scrinzi, M. Y. Ivanov, R. Kienberger and D. M. Villeneuve, Attosecond physics, *Journal of Physics B: Atomic, Molecular and Optical Physics*, 39 (2006), R1, URL http://stacks.iop.org/0953-4075/39/i=1/a=R01.

[44] S. A. Skobelev, D. V. Kartashov and A. V. Kim, Few-optical-cycle solitons and pulse self-compression in a kerr medium, *Phys. Rev. Lett.*, 99 (2007), 203902, URL https://link.aps.org/doi/10.1103/PhysRevLett.99.203902.

[45] A. Stefanov, Y. Shen and P. G. Kevrekidis, Well-posedness and small data scattering for the generalized Ostrovsky equation, *J. Differential Equations*, 249 (2010), 2600–2617, URL https://doi.org/10.1016/j.jde.2010.05.015.

[46] X. Tan, X. Fan, Y. Yang and D. Tong, Time evolution of few-cycle pulse in a dense v-type three-level medium, *Journal of Modern Optics*, 55 (2008), 2439–2448, URL https://doi.org/10.1080/09500340802130670.

[47] T. Tao, *Nonlinear Dispersive Equations*, vol. 106 of CBMS Regional Conference Series in Mathematics, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2006, URL https://doi.org/10.1090/cbms/106, Local and global analysis.

[48] L. Tartar, Compensated compactness and applications to partial differential equations, in *Nonlinear Analysis and Mechanics: Heriot-Watt Symposium, Vol. IV*, vol. 39 of Res. Notes in Math., Pitman, Boston, Mass.-London, 1979, 136–212.

[49] M. V. Tognetti and H. M. Crespo, Sub-two-cycle soliton-effect pulse compression at 800 nm in photonic crystal fibers, *J. Opt. Soc. Am. B*, 24 (2007), 1410–1415, URL http://josab.osa.org/abstract.cfm?URI=josab-24-6-1410.

[50] N. Tsitsas, T. Horikis, Y. Shen, P. Kevrekidis, N. Whitaker and D. Frantzeskakis, Short pulse equations and localized structures in frequency band gaps of nonlinear metamaterials, *Physics Letters A*, 374 (2010), 1384–1388, URL http://www.sciencedirect.com/science/article/pii/S0375960110000150.

[51] A. A. Voronin and A. M. Zheltikov, Soliton-number analysis of soliton-effect pulse compression to single-cycle pulse widths, *Phys. Rev. A*, 78 (2008), 063834, URL https://link.aps.org/doi/10.1103/PhysRevA.78.063834.

[52] M. Wegener, *Extreme Nonlinear Optics*, Advanced Texts in Physics, Springer-Verlag, Berlin, 2005, An introduction.

Received July 2018; revised January 2019.

E-mail address: giuseppemaria.coclite@poliba.it
E-mail address: lorenzo.diruvo77@gmail.com