Toric arrangement and discrete truncated power

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Abstract

Discrete truncated power is very useful to study the number of nonnegative integer solutions of linear Diophantine equations. In this paper, by using the Laplace transform and the theory of toric arrangement, we show that discrete truncated power is a periodic piecewise polynomial on the shifted integral lattice cone. Based on the toric reduction method in the real field, we give a toric arrangement method to compute discrete truncated power.

Key words: Toric arrangement, discrete truncated power, Laplace transform

1 Introduction

Let \( \mathbb{Z}^s \) be the collection of \( s \)-dimensional vectors whose components are integers, and \( \mathbb{R}_+ \) be the collection of all nonnegative real numbers. Let \( X \) be a multiset (i.e. the set whose elements can be same) of \( n \) vectors \( a_1, \ldots, a_n \), where \( a_i \in \mathbb{Z}^s \setminus \{0\}, i = 1, \ldots, n \), such that for \( x_1, \ldots, x_n \in \mathbb{R}_+ \), the equation \( \sum_{i=1}^n x_i a_i = 0 \) does not have nonzero solutions with respect to \( a_i, i = 1, \ldots, n \). Thus we see that

\[
t_X(x) := \#\{\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n : \sum_{i=1}^n \beta_i a_i = x\}, \tag{1}
\]

is finite for every \( x \in \mathbb{Z}^s \), where \# denotes the cardinality of the set \( A \), and \( \mathbb{N}^n \) denotes the collection of \( n \)-dimensional vectors whose components are nonnegative integers. Then \( t_X \) is called the discrete truncated power or the

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partition function. Discrete truncated power was first introduced by Dahmen and Michelli [9]. They exploit the relationship between discrete truncated power and multivariate splines, and propose some important properties of discrete truncated power [10]. In particular, they present a recursive method to compute discrete truncated power, which is based on the following recursive property:

\[ t_X(\alpha) = \sum_{j=0}^{\infty} t_{X\setminus \{a_i\}}(\alpha - ja_i), \quad i = 1, \ldots, n. \]

On the other hand, Laplace transform is found to be a useful tool to study the discrete truncated power. To see this, we let

\[ \Pi_X(\alpha) := \{ x \in \mathbb{R}^n | \sum_{i=1}^{n} x_i a_i = \alpha, \quad x_i \geq 0 \} \]

be a polytope. Then the value of \( t_X(\alpha) \) is the number of integral points in \( \Pi_X(\alpha) \), and the Laplace transform of discrete truncated power is given by:

\[ \sum_{h \in \mathbb{Z}^s} e^{-\langle h, x \rangle} t_X(h) = \prod_{a \in X} \frac{1}{1 - e^{-\langle a, x \rangle}}. \]  

(2)

That motivates some scholars to study discrete truncated power in another way. Szene and Vergne establish the relation between discrete truncated power and Jeffrey-Kirwan residues, and propose a residue method to compute discrete truncated power [1]. Concini and Procesi present the relation between discrete truncated power and toric arrangement, and show that discrete truncated power is a quasi polynomial on some rational lattice [3,4].

In this paper, we investigate the discrete truncated power by using toric arrangement. Our idea comes from two aspects: the relation between multivariate spline and hyperplane arrangement [3], and the Laplace transform of the discrete truncated power. Although the second technique was used in [13], within the complex field, our argument and result are simple and clear, as the discussion is based on the real field. Moreover, we prove that the discrete truncated power \( t_X \) is a periodic piecewise polynomial on the integral lattice cone spanned by the vectors in \( X \). This is more precise than the result of Concini and Procesi in [3,1]. Moreover, we propose a toric arrangement method to compute the discrete truncated power, which is able to give the explicit expression of \( t_X \).

The paper is organized as follows. In Section 2, we briefly review the conclusion given by Concini and Procesi, and introduce some useful definitions
and notations. In Section 3, we propose a new method to show the relation between discrete truncated power and toric arrangement, by using factor decompositions in the real field. We can see that our method can be used to compute the explicit expression of the discrete truncated power, which is also illustrated in two examples.

2 Toric arrangement and discrete truncated power

In this section, we introduce some definitions and notations, and previous results regarding the relation between toric arrangement and discrete truncated power. We make the stipulation that in this paper, the set associated with the discrete truncated power is multiset.

Let $\Lambda$ be an integral lattice in $\mathbb{R}^s$ and

$$C[\Lambda] := \{e^{<a, x>}, \ a \in \Lambda\}$$

be a collection of multivariate exponential functions. Let $v$ be an integral vector in $\mathbb{R}^s$ and $\tau_v$ be a translation operator such that

$$\tau_v f(x) = f(x + v)$$

for any function $f$ on $\mathbb{R}^s$. Then we define

$$\tau[\Lambda] = \{\tau_v, \ v \in \Lambda\}$$

to be the collection of translation operators associated with $\Lambda$. Denote $\mathcal{L}$, $\mathcal{L}^{-1}$ to be the operators of the Laplace transform and the inverse Laplace transform respectively. Then the elements in $C[\Lambda]$ and $\tau[\Lambda]$ have the following relations:

$$\mathcal{L}(\tau_v) = e^{<v, x>}, \ \mathcal{L}^{-1}(e^{<v, x>}) = \tau_v.$$

Let $x^\alpha = x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_s^{\alpha_s}$ be a monomial in $s$ variables, where $\alpha_1, \alpha_2, \cdots, \alpha_s$ are nonnegative integers, and denote

$$P(s)(x) := \{\sum_\alpha a_\alpha x^\alpha, \ \alpha \in \mathbb{N}^s, \ a_\alpha \in \mathbb{R}\}$$

to be the space of all the polynomials in $s$ variables. Let $D^\alpha = D_{x_1}^{\alpha_1}D_{x_2}^{\alpha_2}\cdots D_{x_s}^{\alpha_s}$ be a difference operators in $s$ variables, and denote

$$P(s)(D) := \{\sum_\alpha a_\alpha D^\alpha, \ \alpha \in \mathbb{N}^s, \ a_\alpha \in \mathbb{R}\}$$
to be the space of all difference operators in $s$ variables. Similarly, the elements in $P^{(s)}(x)$ and $P^{(s)}(D)$ have the following relations:

$$\mathcal{L}(x^\alpha) = (-D)^\alpha,$$
$$\mathcal{L}^{-1}(D^\alpha) = (-x)^\alpha.$$

Because the Laplace transform of $t_X$: Eq.(2), has the form

$$\prod_{a \in X} \frac{1}{1 - e^{-\langle a, x \rangle}},$$

the following theorem is given.

**Theorem 1**

Let

$$S_X = C[\Lambda]P^{(s)}(D) \prod_{a \in X} (1 - e^{-\langle a, x \rangle})^{-1}, \quad T_X = \tau[\Lambda]P^{(s)}(x)t_X$$

be two collections of functions. Then under the Laplace transform, $T_X$ is mapped isomorphically onto $S_X$.

This theorem means that for any element $\tau_cp(x)t_X(x)$ in $T_X$, there exists an element $e^{(c,x)}p(D) \prod_{a \in X} (1 - e^{-\langle a, x \rangle})^{-1}$ in $S_X$ such that

$$\mathcal{L}(\tau_c(p(x)t_X(x))) = p(x + c)t_X(x + c) = e^{(c,x)}p(D) \prod_{a \in X} (1 - e^{-\langle a, x \rangle})^{-1}.$$

Contrarily, for any element $e^{(c,x)}p(D) \prod_{a \in X} (1 - e^{-\langle a, x \rangle})^{-1}$ in $S_X$, there exists an element $\tau_c p(x)t_X(x)$ in $T_X$ such that

$$\mathcal{L}^{-1}(e^{(c,x)}p(D) \prod_{a \in X} (1 - e^{-\langle a, x \rangle})^{-1}) = p(x + c)t_X(x + c) = \tau_c(p(x)t_X(x)).$$

To study the properties of $t_X$ on the lattice, Concini and Procesi propose a toric reduction method to study the discrete truncated power. The result is given in the following theorem.

**Theorem 2**

Let $X = \{a_1, a_2, \ldots, a_N\}$ be a set of $N$ points on $\Lambda$, and $\tilde{X}$ be all the vectors in $X$ with positive rational multiples. Then the function

$$\prod_{a \in X} \frac{1}{1 - e^{-\langle a, x \rangle}}$$

can be written as a linear combination of the form $\prod_{a \in A} \frac{1}{(1 - e^{-\langle a, x \rangle})^{k_a}}$ with constant coefficients, where $A$ is a linearly independent set of elements in $\tilde{X}$.

By using the preceding theorem, Concini and Procesi give the following theorem, which describes the general structure of $t_X$.

**Theorem 3**

Let $A$ be a linearly independent set of elements in $\tilde{X}$, such that the function $\prod_{a \in X} \frac{1}{1 - e^{-\langle a, x \rangle}}$ can be written as a linear combination of the
form $\prod_{a \in A} \frac{1}{1-e^{-\langle a, x \rangle} y_a}$ with constant coefficients. On the closure of the cone spanned by $A$, the partition function $t_X$ coincides with a quasi polynomial for some rational lattice $\Lambda/n$.

The idea of Concini and Procesi gives us a new understanding of discrete truncated power. Based on this idea, we get our main result in the next section (Theorem 6).

3 Main results

We present our results in this section. First of all, we give a useful lemma.

Let $y_i$, $\alpha_i$, $i = 1, \ldots, s$ be $2s$ functions from $\mathbb{R}^s$ to $\mathbb{R}$, such that $y_i$ is a nonzero function for each $i$. Then we have the following lemma.

**Lemma 4** Let $y_0 = \alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_s y_s$ be a function and $h_i$ be a positive integer, $i = 1, \ldots, s$. Then $1/(y_0 \prod_{i=1}^{s} y_i^{h_i})$ can be written as a linear combination of elements of the form $f/ \prod_{i=0}^{s} y_i^{t_i}$, where $f$ is the product of some $\alpha_i^{m_i}$, $1 \leq k \leq s$, $t_i$, $m_i$ are integers, and $\sum_{i=0}^{s} t_i = 1 + \sum_{i=1}^{s} h_i$.

**PROOF.** Assume that $\alpha_i$ is a nonzero function. Then we get the following algorithm to decompose $1/(y_0 \prod_{i=1}^{s} y_i^{h_i})$:

$$
\frac{1}{y_0 \prod_{i=1}^{s} y_i^{h_i}} = \frac{(\alpha_1 y_1 + \cdots + \alpha_s y_s)}{y_0 \prod_{i=1}^{s} y_i^{h_i}} = \sum_{j} \frac{\alpha_j}{y_0 \prod_{i=1}^{s} y_i^{h_i} \cdot y_j^{h_j-1}}.
$$

Then $1/(y_0 \prod_{i=1}^{s} y_i^{h_i})$ can be decomposed into a sum of $s$ terms. The numerator of each term is $\alpha_i$. In the denominator, there exists $y_i$ whose power decreases by one after decomposition, and the sum of powers of all $y_i$ is also $1 + \sum_{i=1}^{s} h_i$. 
By using the algorithm again, we have:

\[
\frac{\alpha_j}{y_0 \prod_{i=1}^{s} y_i^{h_i}} = \frac{\alpha_j}{y_0 \prod_{i=1, i \neq j}^{s} y_i^{h_i} y_j^{h_j-1}} = \frac{\alpha_j}{y_0 \prod_{i=1, i \neq j}^{s} y_i^{h_i} \prod_{k=1, k \neq j}^{s} y_k^{h_k-1}} + \sum_{k=1}^{s} \frac{\alpha_j \alpha_k}{y_0 \prod_{i=1, i \neq j, k}^{s} y_i^{h_i}}.
\]  

(4)

If the denominator of a term in the right hand side of Eq.(4) is the form of \(\prod_{i=0}^{s} y_i^{t_i}, k \in \{1, \ldots, s\}\), we stop using the algorithm on this term. Then we can express the numerator of this factor in the form of \(\prod_{i=1}^{s} \alpha_i^{m_i} / \prod_{i=0}^{s} y_i^{t_i}\), where \(\sum_{i=1}^{s} t_i = 1 + \sum_{i=1}^{s} h_i\).

If some \(\alpha_i \equiv 0\), then we denote \(\sigma := \{i|\alpha_i = 0\}\) to be the collection of indices such that \(\alpha_i \equiv 0\). As

\[
\frac{1}{y_0 \prod_{i=0}^{s} y_i^{h_i}} = \frac{1}{\prod_{i \in \sigma} y_i^{h_i}} \cdot \frac{1}{y_0 \prod_{i \in \{1, \ldots, s\} \setminus \sigma} y_i^{h_i}}.
\]

We consider the decomposition of \(1/(y_0 \prod_{i \in \{1, \ldots, s\} \setminus \sigma} y_i^{h_i})\). By using the similar argument, we can get the result. The proof is finished. \(\Box\)

We can see the result of Lemma 4 is similar with the reduction of hyperplane arrangement. However, our result generalizes \(\alpha_i\) from a number to a function.

Let \(X \subset \mathbb{Z}^s \setminus \{0\}\) be a set of integer vectors in \(\mathbb{R}^s\) such that \(\dim X = s\). Denote \(\bar{X}\) to be the collection of nonnegative integral multiples of the vectors in \(X\). Then we can get the following reduction theorem which is different from Theorem 2. We shall explain the difference later.

**Theorem 5** The function \(\prod_{a \in X} \frac{1}{1-e^{-\langle c, x \rangle}}\) can be written as a combination of elements of the form \(\prod_{a \in A} \frac{1}{(1-e^{-\langle c, x \rangle})^h_a}\), with \(h_a \in \mathbb{N}\), such that

\[
\sum_{a \in X} h_a = \#X,
\]

where \(A\) is a linearly independent set of \(s\) elements in \(\bar{X}\) and \(c\) is a linear combination of elements in \(X\).
PROOF. We use the mathematical induction on the number of elements in \( X \). If \( X \) is linearly independent, the result is obvious. Otherwise, we assume that the conclusion is valid for \( X \) with \( N - 1 \) elements, where \( N \geq s + 1 \). Let \( X = \{a_1, a_2, \ldots, a_N\} \) be a set of \( N \) vectors. As \( X \) is not independent, there is an element, say \( a_N \), which depends on the rest of the system \( X' = \{a_1, a_2, \ldots, a_{N-1}\} \).

From the assumption, the function \( \prod_{a \in X'} \frac{1}{1 - e^{-\langle a, x \rangle}} \) can be represented as a linear combination of elements of the form

\[
\prod_{a \in A'} \frac{e^{\langle c', x \rangle}}{(1 - e^{-\langle a, x \rangle}) h_a},
\]

where \( h_a \in \mathbb{N} \), such that \( \sum_{a \in A'} h_a = \#X' \), \( c' = \sum_{a \in X'} a \) (here \( a \) can be repeated) and \( A' \) is a linearly independent set of \( s \) elements in \( \widetilde{X}' \) where \( \widetilde{X}' = \{na | a \in X', n \in \mathbb{N}\} \). Then for the function \( \prod_{a \in X} \frac{1}{1 - e^{-\langle a, x \rangle}} \), we have

\[
\prod_{a \in X} \frac{1}{1 - e^{-\langle a, x \rangle}} = \frac{1}{(1 - e^{-\langle a_N, x \rangle})} \sum_{a \in A'} \prod_{a \in A'} \frac{e^{\langle c', x \rangle}}{(1 - e^{-\langle a, x \rangle}) h_a},
\]

where \( \sum \) denotes the summation of all the terms with the form \( \prod_{a \in A'} \frac{e^{\langle c', x \rangle}}{(1 - e^{-\langle a, x \rangle}) h_a} \).

Now let us consider the factor

\[
\frac{1}{(1 - e^{-\langle a_N, x \rangle})} \prod_{a \in A'} \frac{e^{\langle c', x \rangle}}{(1 - e^{-\langle a, x \rangle}) h_a}.
\]

If the factor (5) can be written as the linear combination of the form

\[
\prod_{a \in A} \frac{e^{\langle c, x \rangle}}{(1 - e^{-\langle a, x \rangle}) h_a}
\]

where \( c \) is a linear combination of elements in \( X \) and \( A \) is a linearly independent set of \( s \) elements in \( \{na | a \in X', n \in \mathbb{N}\} \), then the conclusion is valid. Without loss of generality, we let \( A' = \{a_1, a_2, \ldots, a_s\} \) (then \( h_a = h_{a_1} \)). And let \( h_i = h_{a_i} \) for simplicity. Because

\[
A' \cup \{a_N\} \subset X \subset \mathbb{Z}^s,
\]

there exists some \( m_i \in \mathbb{N}, i \in \{1, 2, \ldots, s, N\} \), such that

\[
m_N a_N = \sum_{i \in \sigma} m_i a_i - \sum_{i \in \{1, \ldots, s\} \setminus \sigma} m_i a_i,
\]

where \( \sigma \subset \{1, \ldots, s\} \).
Assume $m_i \neq 0$, $i = 1, \ldots, s, N$. By reordering the elements in $A$:

$$m_N a_N = \sum_{i=1}^{k} m_i a_i - \sum_{i=k+1}^{s} m_i a_i,$$

we have

$$1 - e^{-\langle n_N a_N, x \rangle} = 1 - e^{-\langle n_1 a_1, x \rangle} \prod_{i=k+1}^{s} e^{\langle n_i a_i, x \rangle} = 1 - \prod_{i=1}^{k} e^{-\langle n_i a_i, x \rangle} \prod_{i=k+1}^{s} e^{\langle n_i a_i, x \rangle}$$

$$= \prod_{i=1}^{k} e^{-\langle n_i a_i, x \rangle} \prod_{i=k+1}^{s} e^{\langle n_i a_i, x \rangle} (1 - e^{\langle n_1 a_1, x \rangle}) + \cdots + \prod_{i=1}^{k} e^{-\langle n_i a_i, x \rangle} \prod_{i=k+1}^{s} e^{\langle n_i a_i, x \rangle} (1 - e^{\langle n_{k+1} a_{k+1}, x \rangle})$$

$$+ \prod_{i=1}^{k} e^{-\langle n_i a_i, x \rangle} \prod_{i=k+1}^{s} e^{\langle n_i a_i, x \rangle} (1 - e^{\langle n_k a_k, x \rangle}) + \cdots + (1 - e^{-\langle n_1 a_1, x \rangle})$$

To give simple argument, we give the following notations:

$$\alpha_t = \begin{cases} 1 & t = 1, \\ \prod_{i=1}^{t-1} e^{-\langle n_i a_i, x \rangle} & 2 \leq t \leq k + 1, \\ \prod_{i=1}^{k} e^{-\langle n_i a_i, x \rangle} \prod_{i=k+1}^{t-1} e^{\langle n_i a_i, x \rangle} & k + 2 \leq t \leq s, \end{cases}$$

$$\beta_t = \begin{cases} \sum_{i=0}^{l-1} e^{\langle ia, x \rangle} & 1 \leq l \leq k, l = N, \\ e^{\langle la, x \rangle} \sum_{i=0}^{l-1} e^{-\langle ia, x \rangle} & k + 1 \leq l \leq s, \end{cases}$$

$$y_d = \begin{cases} 1 - e^{\langle d a, x \rangle} & 1 \leq d \leq s, \\ 1 - e^{-\langle d a_d, x \rangle} & d = N. \end{cases}$$

Then we have

$$y_N = \sum_{i=1}^{s} \alpha_i y_i$$
and
\[
\frac{1}{(1 - e^{-(a_N \cdot x)})} \prod_{a \in A'} e^{(c', x)} = e^{(c', x)} \prod_{i=1}^{s} \frac{\alpha_i \beta_i}{y_i^{h_i}} = (e^{(c', x)} \prod_{i=1}^{s} \alpha_i \beta_i) \frac{1}{y_N \prod_{i=1}^{s} y_i^{h_i}}.
\]

According to Lemma 4, \(1/(y_N \prod_{i=1}^{s} y_i^{h_i})\) can be represented as the linear combination of elements of the form \(\prod_{i=1}^{s} \frac{\alpha_i^{m_i}}{y_i^{t_i}}\), where \(t_N + \sum_{i \neq k} t_i = 1 + \sum_{i=1}^{h_i}\).

Because each \(m_i\) is a nonzero integer in Eq. (6),
\[\{y_i : i = 1, \ldots, k - 1, k + 1, \ldots, s, N\}\]
are linearly independent. We observe that
\[(e^{(c', x)} \prod_{i=1}^{s} \alpha_i \beta_i)(\prod_{i=1}^{s} \alpha_i^{m_i})\] (7)
is obviously a sum of the form of \(e^{(c,x)}\), where \(c\) is a linear combination of elements in \(X\).

If some \(m_i\)'s are equal to 0 in Eq. (6). We let
\[m_n = \sum_{i=1}^{d} m_i a_i,
\]
where integer \(m_i \neq 0\) for \(i = 1, \ldots, d\). Then we consider the decomposition of
\[
\frac{1}{(1 - e^{-(a_N \cdot x)})} \prod_{i=1}^{d} e^{(c', x)} = (e^{(c', x)} \prod_{i=1}^{d} \alpha_i \beta_i) \frac{1}{y_N \prod_{i=1}^{d} y_i^{h_i}}.
\]

With the similar discussion, we also can conclude that the hypotheses is right for \(X = \{a_1, \ldots, a_N\}\). Then the proof is finished. \(\square\)

With Theorem 1 and Theorem 5 in hands, we can get the following theorem, which is different from but more precise than Theorem 3. Before proposing the theorem, we introduce some definitions. For a matrix \(X = \{a_1, \ldots, a_N\} \subset Z^s\), we denote by
\[
\Lambda_{X} := \{ \sum_{i=1}^{s} \alpha_i a_i | \alpha_i \in N \}
\]
a lattice cone and
\[
\Lambda_{X}^{+}(c) := \{ c + \sum_{i=1}^{s} \alpha_i a_i | \alpha_i \in N \}
\]
a shift of \(\Lambda_{X}\). Let \(C(X)\) be the cone
\[C(X) := \{ \sum_{i=1}^{s} \alpha_i a_i | \alpha_i \in R_+ \}.
\]
Then the following theorem is established.
**Theorem 6** Let \( X = \{a_1, \ldots, a_N\} \) be a set of integral vectors in \( \mathbb{Z}^s \), then \( t_X \) is a sum of some periodic piecewise polynomials on the shifted sublattice cone of \( \Lambda_X^+ \).

**PROOF.** By Theorems 1 and 5 we have

\[
\mathcal{L}(t_X) = \frac{1}{\prod_{a \in X} (1 - e^{-ax})} = \sum_{\alpha \in A} \frac{e^{\langle c, x \rangle}}{\prod_{a \in A} (1 - e^{-\langle a, x \rangle})h_a},
\]

where \( A = \{a_1, \ldots, a_s\} \) is a basis formed by vectors in \( \tilde{X} := \{na | a \in X, n \in \mathbb{N}\} \), \( c \) is a linear combination of elements in \( X \) and \( \sum_{a \in A} h_a = \#X \). Let \( h_i = h_{a_i} \) for simplicity. Now, let us compute the inverse Laplace transform of \( \frac{e^{\langle c, x \rangle}}{\prod_{a \in A} (1 - e^{-\langle a, x \rangle})h_a} \).

Let \( A_i^+ \) be a vector who is perpendicular to all the vectors in \( A \) except for \( a_i \). Then we have

\[
\frac{e^{\langle c, x \rangle}}{\prod_{a \in A} (1 - e^{-\langle a, x \rangle})h_a} = e^{\langle c, x \rangle} \prod_{i=1}^{s} \frac{(-1)^h_i (e^{\langle a_i, x \rangle} D_{A_i^+})^{h_i-1}}{(h_i - 1)! (\langle A_i^+, a_i \rangle)^{h_i-1}} \cdot \frac{1}{\prod_{i=1}^{s} (1 - e^{-\langle a, x \rangle})}, \quad (8)
\]

where \( D_a \) is a difference operator along vector \( a \). Through Eq. (8) and Theorem 1, we can show that the inverse Laplace transform of \( \frac{e^{\langle c, x \rangle}}{\prod_{a \in A} (1 - e^{-\langle a, x \rangle})h_a} \) is

\[
t_A \prod_{i=1}^{s} \frac{(\tau_{h_i}(A_i^{\perp}, x))^{h_i-1}}{(h_i - 1)! (\langle A_i^{\perp}, a_i \rangle)^{h_i-1}} t_A = \prod_{i=1}^{s} \prod_{j=1}^{h_i-1} \frac{(A_i^{\perp}, x + ja_i + c)}{(h_i - 1)! (\langle A_i^{\perp}, a_i \rangle)^{h_i-1}} t_A(x + \sum_{k=1}^{s} (h_k - 1)a_k + c).
\]

It is clearly that

\[
t_A(x + \sum_{k=1}^{s} (h_k - 1)a_k + c) = t_A(x + c) \equiv 1
\]

on the lattice \( \Lambda_A^+(c) \). Thus the inverse Laplace transform of \( \frac{e^{\langle c, x \rangle}}{\prod_{a \in A} (1 - e^{-\langle a, x \rangle})h_a} \) is a polynomial of degree \( \#X - s \) on the lattice \( \Lambda_A^+(c) \). It is obvious that \( \Lambda_A^+ \in C(X) \) is a sublattice cone of \( \Lambda_X^+ \). The theorem is proved. \( \square \)

Now let us analysis the difference between Theorem 3 and Theorem 6. The main difference is their domain, i.e. the lattice. For discrete truncated power \( t_X \), the lattice

\[
\Lambda_X = \{\Sigma a_i | a_i \in \mathbb{Z}\}
\]

in Theorem 3 is obviously smaller than the lattice \( \Lambda_X/n \) in Theorem 3. This means that \( t_X \) makes no sense on the point in \( (\Lambda_X/n) \setminus \Lambda_X \). The reason comes from the different definitions of “\( X_n \)” in two toric reduction theorems: Theorem
and Theorem 2. As \( t_X \) is a function in \( \mathbb{Z}^s \), and our discussion in Theorem 5 is based on \( \mathbb{R}^s \), while Theorem 2 is discussed in \( \mathbb{C}^s \) (see [3]). Hence our results: Theorems 5, 6 are the improvements of Theorems 2, 3 respectively.

From Theorem 6, we see that \( t_X \) is a sum of some periodic piecewise polynomials on the shifted sublattice cone of \( \Lambda_X^+ \). This property is shown in the following two examples.

**Example 7** Let \( X = \{1, 1, 2\} \) be a set of real numbers. Obviously that

\[
\mathcal{L}(t_X) = \frac{1}{(1 - e^{-2x})(1 - e^{-x})^2}.
\]

As \( 1 - e^{-2x} = (1 + e^{-x})(1 - e^{-x}) \), we have

\[
\frac{1}{(1 - e^{-2x})(1 - e^{-x})^2} = \frac{(1 + e^{-x})^2(1 - e^{-x})^2}{(1 - e^{-2x})^3(1 - e^{-x})^2} = \frac{(1 + e^{-x})^2}{(1 - e^{-2x})^3} = \frac{(1 + e^{-x})^2}{2!(1 - e^{-x})^2} \cdot 1 - e^{-2x}
\]

So the inverse Laplace transform of \( \frac{1}{(1 - e^{-2x})(1 - e^{-x})^2} \) is equal to

\[
t_X = (1 + \tau_{-1})^2 \frac{(x + 2)(x + 4)}{8} t_2(x + 4) = (1 + \tau_{-1})^2 \frac{(x + 2)(x + 4)}{8} t_2(x)
\]

on the lattice cone \( \Lambda_X^+ \). Therefore, \( t_X \) can be expressed in the following simple form:

\[
t_X(x) = \begin{cases} 
1 & x = 0 \\
\frac{(x + 2)(x + 4)}{4} & x \in \Lambda^+_X(1) \\
\frac{(x + 2)^2}{4} & x \in \Lambda^+_X(2)
\end{cases}
\]

**Example 8** Let \( X = \{(1,0), (0,1), (-1,2)\} \) be a set of real numbers. Observe that

\[
\mathcal{L}(t_X) = \frac{1}{(1 - e^{-x})(1 - e^{-y})(1 - e^{-(x+2y)})},
\]

we have

\[
\frac{1}{(1 - e^{-x})(1 - e^{-y})(1 - e^{-(x+2y)})} = \frac{(1 + e^{-y})(1 - e^{-y}) - e^{-2y}e^{x-2y}(1 - e^{-(x+2y)})}{(1 - e^{-x})^2(1 - e^{-y})(1 - e^{-(x+2y)})} = \frac{(1 + e^{-y})\frac{e^{x}D_{(2,1)}}{-2} - (e^{-x})\frac{e^{y}D_{(1,0)}}{-1}}{(1 - e^{-x})(1 - e^{-(x+2y)}) - (1 - e^{-x})},
\]

where we use the identity

\[
\]
\[ 1 - e^{-x} = (1 - e^{-(2y-(x+2y))}) = (1 - e^{-2y} + e^{-2y}(1 - e^{-x+2y}) \]
\[ = (1 + e^{-y})(1 - e^{-y}) - e^{-2y}e^{-x+2y}(1 - e^{-(x+2y)}). \]

Let \( A_1 = \{(1,0), (0, -1)\} \) and \( A_2 = \{(1,0), (1,1)\} \), then the inverse Laplace transform of \( \frac{1}{(1-e^{-x})(1-e^{-y})(1-e^{-(x+2y)})} \) is equal to

\[ t_X = (1 + \tau_{(0,-1)}) \frac{2x + y + 2}{2} t_{A_1}(x + 1, y) - \tau_{(-1,0)}(x + 1)t_{A_2}(x + 1, y) \]
on the lattice cone \( \Lambda^+ \). Therefore,

\[ t_X = \frac{2x + y + 2}{2} t_{A_1}(x, y) + \frac{2x + y + 1}{2} t_{A_1}(x, y - 1) - xt_{A_2}(x, y). \]

Acknowledgements

The work was partly supported by the National Natural Science Foundation of China (Grant Nos. 60533060, 10801024, and U0935004) and the Innovation Foundation of the Key Laboratory of High-Temperature Gasdynamics of CAS, China.

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