INTEGRAL REPRESENTATION FOR BESSEL’S FUNCTIONS OF THE FIRST KIND AND NEUMANN SERIES

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ABSTRACT. A Fourier-type integral representation for Bessel’s function of the first kind and complex order is obtained by using the Gegenbauer extension of Poisson’s integral representation for the Bessel function along with a trigonometric integral representation of Gegenbauer’s polynomials. This representation lets us express various functions related to the incomplete gamma function in series of Bessel’s functions. Neumann series of Bessel functions are also considered and a new closed-form integral representation for this class of series is given. The density function of this representation is simply the analytic function on the unit circle associated with the sequence of coefficients of the Neumann series. Examples of new closed-form integral representations of special functions are also presented.

1. INTRODUCTION

The Bessel function of the first kind and order $\nu$ is defined by the series [27, Eq. 8, p. 40]

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k}}{k! \Gamma(\nu + k + 1)}, \quad (z \in \mathbb{C}; \nu \in \mathbb{C}, \nu \neq -1, -2, \ldots),$$

which is convergent absolutely and uniformly in any closed domain of $z$ and in any bounded domain of $\nu$. It is the solution of Bessel’s equation

$$z^2 \frac{d^2 y}{dz^2} + z \frac{dy}{dz} + (z^2 - \nu^2) = 0,$$

which is nonsingular at $z = 0$. The function $J_\nu(z)$ is therefore an analytic function of $z$ for any $z$, except for the branch point $z = 0$ if $\nu$ is not an integer.

In this paper, a new integral representation for the Bessel functions of the first kind and order $\nu \in \mathbb{C}$, $\text{Re} \nu > -\frac{1}{2}$, is obtained. This representation, which is given in Section 3, generalizes to complex values of the order $\nu$ the classical Bessel’s integral

$$J_n(z) = (-i)^n \frac{\pi}{\pi} \int_0^\pi e^{iz \cos \theta} \cos n\theta \, d\theta \quad (n \in \mathbb{Z}),$$

which holds only for integral order values (see, e.g., formula (3.21)). The Bessel functions of the first kind are expressed in terms of an integral of Fourier-type, which involves the regularized incomplete gamma function $P\{\nu\}, z)$, with $\{\nu\}$ being the (complex) fractional part of $\nu$. This result is achieved by using the Poisson integral representation of $J_{\nu+n}(z)$ ($\text{Re} \nu > -\frac{1}{2}$, $\ell = 0, 1, 2, \ldots$) given in terms of
Gegenbauer polynomials and then by exploiting the integral representation of the latter polynomials that we prove in Section 2.

The Fourier form of the integral representation of \( J_{\nu+n}(z) \) motivates us to consider the Fourier inversion formula in Section 4. This analysis allows us to obtain, for either integer and half-integer orders \( \nu \), a trigonometric expansion of the lower incomplete gamma function, whose coefficients are related to (modified and unmodified) Bessel functions of the first kind. Thus, classical Bessel series expansions of various special functions which are connected with the incomplete gamma function, can be easily obtained in a unified form.

A Neumann series of Bessel functions is an expansion of the type

\[
(1.4) \sum_{n=0}^{\infty} a_n J_{\nu+n}(z) \quad (z, \nu \in \mathbb{C}),
\]

where \( \{a_n\}_{n=0}^{\infty} \) are given coefficients. Many special functions of the mathematical physics enjoy an expansion of this type, e.g., Kummer confluent hypergeometric’s, Lommel’s, Kelvin’s, Whittaker’s, and so on. In Section 5 we exploit the novel representation of Bessel’s functions presented in Section 3 in order to derive a simple closed-form integral representation of expansions (1.4). The set of coefficients \( \{a_n\}_{n=0}^{\infty} \), which characterizes the series (1.4), comes into this integral representation in a very simple way through the associated analytic function \( A(z) \) in the unit disk \( D \), i.e.: \( A(z) = \sum_{n=0}^{\infty} a_n z^n \) with a weak restriction on the sequence of coefficients \( \{a_n\} \), which is sufficient to guarantee the boundedness of \( A(z) \) on the unit circle. Examples of special functions with this kind of closed-form integral representation are also given.

2. Connection between the coefficients of Gegenbauer and Fourier expansions

The Gegenbauer (ultraspherical) polynomials \( C_\nu^\ell(x) \) of order \( \nu \) \((\nu > -\frac{1}{2}, \nu \neq 0)\) may be defined by means of the generating function \[23, Eq. (4.7.23)] (see also \[23, Eq. (4.7.6)]): \n
\[
(2.1) \sum_{\ell=0}^{\infty} C_\nu^\ell(x) t^\ell = (1 - 2xt + t^2)^{-\nu} \quad (x \in [-1, 1]; \ t \in (-1, 1)).
\]

For fixed \( \nu \), the Gegenbauer polynomials are orthogonal on the interval \([-1, 1]\) with respect to the weight function \( w^\nu(x) = (1 - x^2)^{\nu-1/2} \), that is:

\[
(2.2) \int_{-1}^{1} C_\nu^\ell(x) C_\nu^m(x) w^\nu(x) \, dx = h_\nu^\ell \delta_{\ell,m},
\]

where:

\[
(2.3) h_\nu^\ell = \frac{1}{(\ell + \nu) \frac{2^{1-2\nu} \pi \Gamma(\ell + 2\nu)}{\ell! \Gamma(\nu)^2}}.
\]

Particularly important special cases of Gegenbauer polynomials are obtained for \( \nu = 1/2 \), which gives the Legendre polynomials \( P_\ell(x) \), for \( \nu = 1 \) yielding the Chebyshev polynomials of the second kind \( U_\ell(x) \), and in a suitable limiting form for \( \nu \to 0 \) the Chebyshev polynomials of the first kind \( T_\ell(x) \) \[23\]: \( \lim_{\nu \to 0}(\ell!/(2\nu)_\ell)C_\nu^\ell(x) = T_\ell(x) \), where \((\cdot)_\ell\) denotes the Pochhammer symbol.
For $\Re \nu > 0$ they admit the following integral representation [26, Eq. (1), p. 559]:

$$C_{\nu}^{\ell}(x) = \frac{(\ell + \nu) \, h_{\nu}^{\ell}}{\pi} \int_{0}^{\pi} (x + i \sqrt{1 - x^2} \cos \eta)^{\ell} (\sin \eta)^{2\nu - 1} \, d\eta. \tag{2.4}$$

Now, our aim in this section is to present an integral representation of the Gegenbauer polynomials. To this end, we prove the following proposition (see also [6, 9]).

**Proposition 2.1.** For $\Re \nu > 0$, the following integral representation for the Gegenbauer polynomials $C_{\nu}^{\ell}(\cos u)$ ($u \in [0, \pi], \ell \in \mathbb{N}_0$) holds:

$$C_{\nu}^{\ell}(\cos u) = \frac{(\ell + \nu) \, h_{\nu}^{\ell}}{\pi} \int_{0}^{2\pi} e^{i(\ell + \nu) \tau} (\cos u - \cos t)^{\nu - 1} \, dt. \tag{2.5}$$

**Proof.** In representation (2.4) we first introduce the variable $u$, defined by $x = \cos u$ ($u \in [0, \pi]$):

$$C_{\nu}^{\ell}(\cos u) = \frac{(\ell + \nu) \, h_{\nu}^{\ell}}{\pi} \int_{0}^{\pi} (\cos u + i \sin u \cos \eta)^{\ell} (\sin \eta)^{2\nu - 1} \, d\eta. \tag{2.6}$$

Then, in the integral in (2.6) we substitute to $\eta$ the complex integration variable $\tau$ defined by

$$e^{i\tau} = \cos u + i \sin u \cos \eta.$$

It can be checked that

$$2e^{i\tau}(\cos \tau - \cos u) = (e^{i\tau} - e^{iu})(e^{i\tau} - e^{-iu}) = \sin^2 u \sin^2 \eta. \tag{2.7}$$
Now, since $e^{i\tau}d\tau = -\sin u \sin \eta \, d\eta$, the integrand on the right-hand side of (2.6) can be written as follows:

$$
-(\sin u)^{\nu-1} \, e^{i(\ell+1)\tau} \left[ (e^{i\tau} - e^{-iu})(e^{i\tau} - e^{-iu}) \right]^{(\nu-1)} \, d\tau
$$

$$(2.8) \quad = -(\sin u)^{\nu-1} \, e^{i(\ell+\nu)\tau} \left[ 2(\cos \tau - \cos u) \right]^{(\nu-1)} \, d\tau.
$$

In order to determine the integration path, consider the intermediate step where $e^{i\tau}$ is chosen as integration variable; the original path (corresponding to $\eta \in [0, \pi]$) is the (oriented) linear segment $\delta_0(u)$ starting at $e^{iu}$ and ending at $e^{-iu}$. Since (as shown by (2.8)) the integrand is an analytic function of $e^{i\tau}$ in the disk $|e^{i\tau}| < 1$ (since $\ell \in \mathbb{N}_0$), the integration path $\delta_0(u)$ can be replaced by the circular path $\delta_{\pm}(u) = \{ e^{i\tau} : \tau = t, u \leq t \leq 2\pi - u \}$ (see Fig. 1). Moreover, by using the fact that $[(e^{i\tau} - e^{iu})(e^{i\tau} - e^{-iu})]^{\nu-1}$ is positive for $e^{i\tau} \in \delta_0(u) \cup \mathbb{R}$ and therefore at $e^{i\tau} = e^{i\pi}$, we conclude from the left equality in (2.7) that in the r.h.s. of (2.8) the following specification holds (for $\tau = t; u \leq t \leq 2\pi - u$):

$$
2(\cos t - \cos u)^{\nu-1} = (\cos u - \cos t)^{\nu-1}.
$$

Finally, accounting for this latter expression, the integral representation (2.5) follows directly from (2.6).

**Remark 1.** In the proof of Proposition 2.2 we could even choose the integration path $\delta_{\pm}(u) = \{ e^{i\tau} : \tau = t, -u \leq t \leq u \}$, oriented as shown in Fig. 1. Following the same arguments given above, it is easily seen that this option yields the following alternative integral representation of the Gegenbauer polynomials $[6, 8]$:

$$
C_{\ell}^{\nu}(\cos u) = \frac{(\ell + \nu)}{\pi} \frac{2^{\nu-1}}{(\sin u)^{2\nu-1}} \int_{-u}^{u} e^{i(\ell+\nu)t} \left( \cos t - \cos u \right)^{\nu-1} \, dt.
$$

3. **Fourier-type integral representation of Bessel functions of the first kind**

It is well-known that the Bessel functions of the first kind of integral order $n$ can be represented by Bessel’s trigonometric integral (1.3) (see also [11, Eq. 10.9.2]). For $n \in \mathbb{Z}$, $J_n(x)$ satisfies the symmetry relation: $J_{-n}(x) = (-1)^n J_n(x)$. This latter relation no longer holds when $n$ is not an integral number since, for $\nu \notin \mathbb{Z}$, $J_\nu(x)$ and $J_{-\nu}(x)$ are linearly independent solutions of the Bessel equation of order $\nu$. Numerous formulæ express the Bessel functions of the first kind as definite integrals, which can be exploited to obtain, for instance, approximations and asymptotic expansions (see [27, Chapter VI], [12], and the website [11, Sect. 10.9] for a useful collection of formulæ). Our goal in what follows is to generalize the trigonometric representation (1.3) in order to obtain a Fourier-type integral representation for the Bessel functions of the first kind with holds for complex order $\nu$ with Re $\nu > -\frac{1}{2}$. We can prove the following theorem.

**Theorem 3.1.** Let $\ell \in \mathbb{N}_0 = \{ 0, 1, 2, \ldots \}$ be a non-negative integer, and $\nu$ any complex number such that Re $\nu > -\frac{1}{2}$. Then, the following integral representation for the Bessel functions of the first kind $J_{\nu+\ell}(z)$ holds:

$$
J_{\nu+\ell}(z) = (-1)^{\ell} \int_{-\pi}^{\pi} \mathcal{J}_{\nu}^{(\nu)}(\theta) \, e^{i\ell\theta} \, d\theta \quad (\ell \in \mathbb{N}_0, \text{Re} \, \nu > -\frac{1}{2}),
$$

$$
(3.1)
$$
where the $2\pi$-periodic function $\overline{\mathcal{J}}_\nu^\nu(\theta)$ is given by

$$
(3.2) \quad \overline{\mathcal{J}}_\nu^\nu(\theta) = \frac{\nu}{2\pi} e^{i\nu[\theta-\pi \text{sgn}(\theta)]} e^{iz \cos \theta} P(\nu, -iz (1 - \cos \theta)) \quad (\text{Re } \nu > -\frac{1}{2}),
$$

and $\text{sgn}(\cdot)$ is the sign function, $\{\nu\}$ is the complex fractional part of $\nu$, $P(\nu, w) = \gamma(\nu, w)/\Gamma(\nu)$ denotes the regularized incomplete gamma function, $\gamma(\nu, w)$ being the lower incomplete gamma function.

**Proof.** We start from Gegenbauer’s generalization of Poisson’s integral representation of the Bessel functions of the first kind [27, §3.32, Eq. (1), p. 50]:

$$
(3.3) \quad J_{\nu+\ell}(z) = \frac{(-i)^\ell \Gamma(2\nu) \ell!}{\Gamma(\nu + \frac{1}{2}) \Gamma(\nu + \ell)} \left(\frac{z}{2}\right)^\nu \int_0^\pi e^{iz \cos u} (\sin u)^{2\nu} C_{\nu+\ell}^\nu(\cos u) \, du,
$$

where $C_{\nu}^\nu(\ell)$ is the Gegenbauer polynomial of order $\nu$ and degree $\ell$. Formula (3.3) holds for $\ell \in \mathbb{N}_0$ and $\text{Re } \nu > -\frac{1}{2}$. Now, we plug (2.5) into (3.3) and, using the Legendre duplication formula for the gamma function, we obtain:

$$
(3.4) \quad J_{\nu+\ell}(z) = \frac{(-i)^\ell e^{-i\pi\nu} z^\nu}{2\pi \Gamma(\nu)} \int_0^\pi du \sin u e^{iz \cos u} \int_0^{2\pi-u} e^{i(\ell+\nu)\theta} (\cos u - \cos \theta)^{\nu-1} \, d\theta.
$$

Interchanging the order of integration, (3.4) can be written as follows:

$$
(3.5) \quad J_{\nu+\ell}(z) = \frac{(-i)^\ell e^{-i\pi\nu} z^\nu}{2\pi \Gamma(\nu)} \left[ \int_0^\pi d\theta e^{i(\ell+\nu)\theta} \int_0^{\theta} e^{iz \cos u} (\cos u - \cos \theta)^{\nu-1} \sin u \, du 
+ \int_0^{2\pi} d\theta e^{i(\ell+\nu)\theta} \int_0^{2\pi-\theta} e^{iz \cos u} (\cos u - \cos \theta)^{\nu-1} \sin u \, du \right].
$$

Next, changing the integration variables: $\theta - 2\pi \to \theta$ and $u \to -u$, the second integral on the r.h.s. of (3.5) becomes:

$$
e^{-i\pi\nu} \int_0^\pi d\theta e^{i(\ell+\nu)\theta} \int_0^{\theta} e^{iz \cos u} (\cos u - \cos \theta)^{\nu-1} \sin u \, du,
$$

which, inserted in (3.5), yields:

$$
(3.6) \quad J_{\nu+\ell}(z) = \frac{(-i)^\ell z^\nu}{2\pi \Gamma(\nu)} \left[ e^{i\pi\nu} \int_0^\pi d\theta e^{i(\ell+\nu)\theta} \int_0^{\theta} e^{iz \cos u} (\cos u - \cos \theta)^{\nu-1} \sin u \, du
+ e^{-i\pi\nu} \int_0^\pi d\theta e^{i(\ell+\nu)\theta} \int_0^{\theta} e^{iz \cos u} (\cos u - \cos \theta)^{\nu-1} \sin u \, du \right].
$$

Formula (3.6) can then be written as follows:

$$
(3.7) \quad J_{\nu+\ell}(z) = (-i)^\ell \int_{-\pi}^\pi \overline{\mathcal{J}}_\nu^\nu(\theta) e^{i\ell \theta} \, d\theta \quad (\ell \in \mathbb{N}_0, \text{Re } \nu > 0),
$$

where

$$
(3.8) \quad \overline{\mathcal{J}}_\nu^\nu(\theta) = \frac{z^\nu}{2\pi \Gamma(\nu)} e^{i\nu[\theta-\pi \text{sgn}(\theta)]} \int_{-\cos \theta}^1 e^{iz(t - \cos \theta)^{\nu-1}} \, dt \quad (\text{Re } \nu > 0).
$$
Then, changing in (3.8) the integration variable $-iz(t - \cos \theta) \to t$ and recalling that for $\Re \nu > 0$, $\gamma(\nu, w) = \int_0^w t^{\nu-1} e^{-t} \, dt$, $\mathcal{J}_z^{(\nu)}(\theta)$ can be rewritten as follows:

$$\mathcal{J}_z^{(\nu)}(\theta) = \frac{i^\nu}{2\pi} e^{i\nu[\theta - \pi \sgn(\theta)]} e^{iz \cos \theta} P(\nu, -iz (1 - \cos \theta)) \quad (\Re \nu > 0).$$

The regularized gamma function $P(\nu, w)$ enjoys the following recurrence relation [11 Eq. 8.8.11]:

$$P(\nu, w) = P(\nu - k, w) - e^{-w} \sum_{j=1}^{k} \frac{w^{\nu-j}}{\Gamma(\nu - j + 1)} \quad (k = 1, 2, \ldots).$$

Let us set $k$ as the integer part of $\Re \nu$, i.e., $k = \lfloor \Re \nu \rfloor$, where

$$\lfloor x \rfloor \doteq \begin{cases} \lceil x \rceil & \text{for } x \geq 0, \\ \lfloor x \rfloor & \text{for } x < 0. \end{cases}$$

Moreover, we denote by $\{ \nu \}$ the complex fractional part of $\nu$, defined as: $\{ \nu \} \doteq \nu - \lfloor \Re \nu \rfloor$, with $-1 < \Re \{ \nu \} < 1$. We can then insert formula (3.10) into (3.9) and obtain:

$$\mathcal{J}_z^{(\nu)}(\theta) = \frac{i^\nu}{2\pi} e^{i\nu[\theta - \pi \sgn(\theta)]} e^{iz \cos \theta} \left[ P(\nu, -iz (1 - \cos \theta)) - e^{iz} \sum_{1 \leq j \leq \lfloor \Re \nu \rfloor} \frac{(-iz)^{\nu-j}}{\Gamma(\nu - j + 1)} (1 - \cos \theta)^{\nu-j} \right].$$

When plugged into (3.7) the second term in the squared brackets of (3.12) (which is different from zero only if $\lfloor \Re \nu \rfloor \geq 1$) contributes with a (finite) linear combination of the following integrals:

$$\int_{-\pi}^\pi e^{i\nu[\theta - \pi \sgn(\theta)]}(1 - \cos \theta)^{\nu-j} e^{j\theta} \, d\theta,$$

which are null for $\ell \in \mathbb{N}_0$ and $\Re \nu - j \geq 0$. Therefore, $\mathcal{J}_z^{(\nu)}(\theta)$ finally reads

$$\mathcal{J}_z^{(\nu)}(\theta) = \frac{i^\nu}{2\pi} e^{i\nu[\theta - \pi \sgn(\theta)]} e^{iz \cos \theta} P(\nu, -iz (1 - \cos \theta)) \quad (\Re \nu > 0).$$

The $\nu$-domain of representation (3.7) can be analytically extended to the half-plane $\Re \nu > -\frac{1}{2}$, where, for any $z \in \mathbb{C} \setminus (-\infty, 0]$, the integrand is analytic on $\theta \in (-\pi, \pi]$ and the integral defines a function which is locally bounded on every compact subsets of $\Re \nu > -\frac{1}{2}$. To see this, it is useful to make explicit the singularity brought by the regularized gamma function $P(\nu, w)$ by writing the latter in terms of Tricomi’s form $\gamma^*(\nu, w)$ of the incomplete gamma function, i.e.: $P(\nu, w) = w^\nu \gamma^*(\nu, w)$, where

$$\gamma^*(\nu, w) = e^{-w} \sum_{n=0}^{\infty} \frac{w^n}{\Gamma(\nu + n + 1)}$$
is an entire function in \( w \) as well as in \( \nu \) [24, Eq. (2)]. From (3.7) and (3.14) we thus obtain:

\[
J_{\nu+\ell}(z) = \frac{\Gamma(\nu+1)\nu}{2\pi i} \int_{-\infty}^{\infty} e^{iz\cos \theta} \left(1 - \cos \theta\right)^{\nu} \gamma^*(\{\nu\}, -iz(1 - \cos \theta)) e^{i(\ell+\nu)\theta} d\theta.
\]  

(3.16)

For negative values of Re\( \nu \), the integrand in (3.16) has a singularity in \( \theta = 0 \) due to the term \( (1 - \cos \theta)^{\nu} \), which is integrable for Re\( \nu > -\frac{1}{2} \). Hence, the integral (3.16) defines an analytic function of \( z \) on \( C \) and representation (3.1) holds true for Re\( \nu > -\frac{1}{2} \).

**Note 1.** Alternatively, if in the Abel-type integral in (3.8) we change the integration variable \( 2z(t - \cos \theta)/\pi^\nu \rightarrow t \) the function \( J_z^{(\nu)}(\theta) \) can be written in terms of Fresnel-type integrals \( \mathcal{F}_\lambda(z) \), i.e:

\[
J_z^{(\nu)}(\theta) = \frac{\pi^{\nu-1}}{2^{\nu+1}\Gamma(\nu+1)} e^{i\nu[\theta - \pi \text{sgn}(\theta)]} e^{iz \cos \theta} \mathcal{F}_\nu \left( \frac{2z(1 - \cos \theta)}{\pi} \right)^\nu,
\]

(3.17)

where

\[
\mathcal{F}_\lambda(w) = \int_0^w \exp \left( \frac{i}{2} \lambda^2 s^2 \right) ds,
\]

(3.18)

and the classical Fresnel integrals \( C(w) \) and \( S(w) \) can be obtained as real and imaginary part of (3.18) with \( \lambda = 2 \), respectively.

Representation (3.1) can be reformulated as follows:

\[
i^\nu J_{\nu+\ell}(z) = i^\nu \int_{-\infty}^{\infty} e^{iz\cos \theta} P(\{\nu\}, -iz(1 - \cos \theta)) e^{i(\ell+\nu)\theta + \pi\nu} d\theta
\]

(3.19)

\[+
\int_0^\pi e^{iz\cos \theta} P(\{\nu\}, -iz(1 - \cos \theta)) e^{i(\ell+\nu)\theta - \pi\nu} d\theta
\]

\[= \frac{i^\nu}{\pi} \int_0^\pi e^{iz\cos \theta} P(\{\nu\}, -iz(1 - \cos \theta)) \cos(\ell + \nu(\theta - \pi)) d\theta.
\]

In the last integral we now change the integration variable: \( \theta \rightarrow \pi - \theta \), and obtain for \( \ell \in \mathbb{N}_0, \text{Re} \nu > -\frac{1}{2} \):

\[
J_{\nu+\ell}(z) = i^{\nu+\ell} / \pi \int_0^\pi e^{-iz\cos \theta} P(\{\nu\}, -iz(1 - \cos \theta)) \cos(\ell + \nu(\theta - \pi)) d\theta.
\]

(3.20)

Now, we can put \( \mu = \ell + \nu \), with Re\( \mu > -\frac{1}{2} \). Since \{\nu\} = \{\nu\}, from (3.20) we therefore obtain the following integral representation of the Bessel function of the first kind:

\[
J_{\mu}(z) = i^{\mu} / \pi \int_0^\pi e^{-iz\cos \theta} P(\{\mu\}, -iz(1 - \cos \theta)) \mu \cos \theta d\theta \quad (\text{Re} \mu > -\frac{1}{2}).
\]

(3.21)

As mentioned earlier, formula (3.21) (and formulae (3.1) and (3.2) as well) generalizes to complex values of \( \mu \) (Re\( \mu > -\frac{1}{2} \)) the classical Bessel integral (1.3), which holds for integral values of \( m \). In fact, if in (3.21) we put \( \mu \equiv m \) integer, then \{\mu\} = 0 and, since \( \gamma^*(0, w) = 1 \) (see [14, Eq. (2.2)]), formula (1.3) readily follows.

In view of the relation: \( I_{\mu}(w) = (-1)^\mu J_{\mu}(iw) \) for \(-\pi \leq \text{ph} w \leq \frac{\pi}{2} \) [11, Eq. 10.27.6], we have also the following integral representation for the modified Bessel function of the first kind \( I_{\mu}(z) \).
Corollary 3.2. Let $\mu$ be any complex number such that $\Re \mu > -\frac{1}{2}$. Then the following integral representation for the modified Bessel function of the first kind holds ($-\pi \leq \phi z \leq \frac{\pi}{2}$):

\begin{equation}
I_\mu(z) = \frac{1}{\pi} \int_0^\pi e^{z \cos \theta} P(\{\mu\}, z(1 + \cos \theta)) \cos \mu \theta \, d\theta \quad (\Re \mu > -\frac{1}{2}).
\end{equation}

Formula (3.22) generalizes to complex values of $\mu$, $\Re \mu > -\frac{1}{2}$, the well-known representation \[11, \text{Eq. 10.32.3}]:

\begin{equation}
I_m(z) = \frac{1}{\pi} \int_0^\pi e^{z \cos \theta} \cos m \theta \, d\theta \quad (m \in \mathbb{Z}),
\end{equation}

which holds for integral values of $m$.

4. Inversion of the Fourier Representation

Representation (3.1) shows that, for fixed $z$, the function $i^\ell J_{\ell + \nu}(z)$ coincides with the $\ell$th Fourier coefficient ($\ell \geq 0$) of the $2\pi$-periodic function $\tilde{f}_z^{(\nu)}(\theta)$. We are thus prompted to consider the following trigonometrical series:

\begin{equation}
\tilde{f}_z^{(\nu)}(\theta) = \frac{1}{2\pi} \sum_{\ell = -\infty}^{\infty} \tilde{f}_\ell^{(\nu)}(z) e^{-i\ell \theta},
\end{equation}

where $\tilde{f}_\ell^{(\nu)}(z)$ denote, for fixed $z$, the $\ell$th Fourier coefficients of $\tilde{f}_z^{(\nu)}(\theta)$:

\begin{equation}
\tilde{f}_\ell^{(\nu)}(z) = \int_{-\pi}^{\pi} \tilde{f}_z^{(\nu)}(\theta) e^{i\ell \theta} \, d\theta \quad (\ell \in \mathbb{Z}).
\end{equation}

However, the Fourier sum representation (4.1) of $\tilde{f}_z^{(\nu)}(\theta)$ can actually be written explicitly only in some specific cases for the lack of knowledge of the Fourier coefficients with $\ell < 0$. Indeed, equation (3.1) states that $\tilde{f}_\ell^{(\nu)}(z) = i^\ell J_{\ell + \nu}(z)$ only for $\ell \geq 0$. However, from equation (3.2) we see that (for fixed $z$) $\tilde{f}_z^{(\nu)}(\theta)$ enjoys the following symmetry:

\begin{equation}
\tilde{f}_z^{(\nu)}(-\theta) = e^{-i2\nu \theta - \pi \text{sgn}(\theta)} \tilde{f}_z^{(\nu)}(\theta),
\end{equation}

which, substituted in (4.2), gives

\begin{equation}
\tilde{f}_\ell^{(\nu)}(z) = \int_{-\pi}^{\pi} \tilde{f}_z^{(\nu)}(\theta) e^{i2\nu \theta \text{sgn}(\theta)} e^{-i(\ell + 2\nu) \theta} \, d\theta.
\end{equation}

Formula (4.4) can thus induce a $\ell$-index symmetry on the Fourier coefficients $\tilde{f}_\ell^{(\nu)}(z)$ only if $2\nu$ is integer, i.e.: $\nu$ can be either integer or half-integer ($\nu > -\frac{1}{2}$).

Case $\nu$ integer. If $\nu \equiv n$ is a nonnegative integer, formula (4.4) yields:

\begin{equation}
\tilde{f}_\ell^{(n)}(z) = \tilde{f}_{-\ell - 2n}^{(n)}(z) \quad (\ell \in \mathbb{Z}; n = 0, 1, 2, \ldots).
\end{equation}

For $n = 0$ all the Fourier coefficients $\tilde{f}_\ell^{(n)}(z)$, $\ell \in \mathbb{Z}$, can be determined directly from (3.1) since $\tilde{f}_0^{(n)}(z) = \tilde{f}_0^{(n)}(z)$. For $n \neq 0$, the Fourier coefficients $\tilde{f}_\ell^{(n)}(z)$ with $-2n + 1 \leq \ell \leq -1$ cannot be obtained by means of the above symmetry relation, but can nevertheless be expressed as (linear) functionals of the set $\{\tilde{f}_\ell^{(n)}(z); \ell \in \mathbb{N}_0\}$.\[6]
Since \( \{ \nu \} = 0 \) and \( P(0, w) = 1 \), from \( (3.2) \) we have: \( 2\pi \tilde{\mathcal{H}}^{(n)}_1(\theta) = (-i)^n e^{iz \cos \theta} \cos n \theta \), which inserted in \( (4.1) \), and using \( (4.5) \), gives:

\[
(-i)^n e^{iz \cos \theta} = \tilde{\mathcal{H}}^{(n)}_1(z) + 2 \sum_{\ell = -n+1}^{\infty} \tilde{\mathcal{H}}^{(n)}_\ell(z) \cos (\ell + n) \theta,
\]

where \( \tilde{\mathcal{H}}^{(n)}_\ell(z) = i^\ell J_{n+\ell}(z) \) for \( \ell \geq 0 \). If we put \( n = 0 \), formula \( (4.6) \) yields:

\[
e^{iz \cos \theta} = J_0(z) + 2 \sum_{\ell = 1}^{\infty} i^\ell J_\ell(z) \cos \ell \theta,
\]

which represents the well-known Jacobi-Anger expansion of a plane wave into a series of cylindrical waves \([11, \text{Eq. 10.12.2}]\).

**Case \( \nu \) half-integer.** If \( \nu \) is a positive half-integer, \( \nu \equiv n + \frac{1}{2}, \) \( n = 0, 1, 2, \ldots, \) the symmetry formula \( (4.4) \) reads:

\[
\tilde{\mathcal{H}}^{(n+\frac{1}{2})}_\ell(z) = -\tilde{\mathcal{H}}^{(n+\frac{1}{2})}_{-\ell+2n-1}(z) \quad (\ell \in \mathbb{Z}; n = 0, 1, 2, \ldots).
\]

For \( n = 0 \) the Fourier coefficients with \( \ell < 0 \) can be obtained from those with \( \ell \geq 0 \) from the symmetry relation: \( \tilde{\mathcal{H}}^{(n+\frac{1}{2})}_\ell(z) = -\tilde{\mathcal{H}}^{(n+\frac{1}{2})}_{-\ell+2n-1}(z) \) \( (\ell \in \mathbb{Z}) \). Instead, as in the case of integral \( \nu \), if \( n \neq 0 \) the Fourier coefficients \( \tilde{\mathcal{H}}^{(n+\frac{1}{2})}_\ell(z) \) with \( -2n \leq \ell \leq -1 \) are expressible as linear functionals of the coefficients with \( \ell \geq 0 \).

Now, we can plug formula \( (3.2) \) for \( J^{(n+\frac{1}{2})}_\ell(\theta) \) into \( (4.1) \) and, in view of \( (4.8) \), obtain the following expansion for the incomplete gamma function of order \( \frac{1}{2} \):\( (4.9) \)

\[
\gamma\left(\frac{1}{2}, -iz(1 - \cos \theta)\right) = 2\sqrt{\pi} i^n \text{sgn}(\theta) e^{-iz \cos \theta} \sum_{\ell = -n}^{\infty} \tilde{\mathcal{H}}^{(n+\frac{1}{2})}_\ell(z) \sin (\ell + n + \frac{1}{2}) \theta.
\]

Now, putting in \( (4.9) \) \( n = 0, \varphi = \theta/2 \) and \( w^2 = -2iz \) we obtain:

\[
(4.10) \quad \gamma\left(\frac{1}{2}, w^2 \sin^2 \varphi\right) = 2\sqrt{\pi} \text{sgn}(\varphi) e^{\frac{1}{2}w^2 \cos 2\varphi} \sum_{\ell = 0}^{\infty} (-1)^\ell I_{n+\frac{1}{2}}(w^2/2) \sin(2\ell + 1) \varphi,
\]

where we used the modified Bessel functions of the first kind \( I_\nu(z) = (-i)^\nu J_\nu(iz) \). Finally, recalling that \( \text{erf}(z) = \pi^{-\frac{1}{2}} \gamma\left(\frac{1}{2}, z^2\right) \), where \( \text{erf}(z) \) denotes the error function \([11, \text{Eq. 7.2.1}]\), we obtain the following expansion:

\[
(4.11) \quad \text{erf}(w \sin \varphi) = 2 e^{\frac{1}{2}w^2 \cos 2\varphi} \sum_{\ell = 0}^{\infty} (-1)^\ell I_{n+\frac{1}{2}}(w^2/2) \sin(2\ell + 1) \varphi \quad (-\frac{\pi}{2} < \text{ph}w \leq \frac{\pi}{2}).
\]

A similar representation of the error function in terms of modified Bessel functions of the first kind of integer order is given by Luke in \([19, \text{Eq. (2.11)}]\) through the expansion of the confluent hypergeometric function in series of Bessel functions. As a particular case of \( (4.11) \), we first put \( \varphi = \pi/2 \) and obtain (see also \([20, \text{Eq. (20)}]\)):

\[
(4.12) \quad \text{erf}(w) = 2 e^{-w^2/2} \sum_{\ell = 0}^{\infty} I_{\ell+\frac{1}{2}}(w^2/2).
\]
Similarly, if in (4.11) we put \( \varphi = \pi/4 \) and \( w \to \sqrt{2}w \), we get the expansion:

\[
\text{erf}(w) = \sqrt{2} \sum_{\ell=0}^{\infty} (-1)^{\ell} \left[ I_{2\ell+\frac{1}{2}}(w^2) - I_{2\ell+\frac{3}{2}}(w^2) \right],
\]

which is equivalent to [11, Eq. 7.6.8] (see also [20, Eq. (1), p. 122] and [25, Eqs. (33) and (34)]).

Finally, as a last example of application of expansion (4.1), we can use the form (3.17) of \( J_{\nu+\frac{1}{2}}(z) \) given in terms of Fresnel integrals. We can set \( \nu = \frac{1}{2} \) in (3.17) and (4.1) and put \( w = 2(z/\pi)^{\frac{1}{2}}, \ a = \cos(\theta - \pi)/2, \) then we obtain the following representation of Fresnel’s integral \( F_2(aw) \):

\[
F_2(aw) = we^{i\pi/4}w^2(2a^2-1) \sum_{\ell=0}^{\infty} (-i)^{\ell} T_{2\ell+1}(a) j_{\ell}(\pi w^2/4),
\]

where \( T_k(\cdot) \) denotes the Chebyshev polynomials of the first kind and \( j_{\ell}(w) \) are the spherical Bessel functions of the first kind: \( j_{\ell}(w) = \sqrt{\frac{\pi}{2w}} J_{\ell+\frac{1}{2}}(w) \).

5. Neumann series of Bessel functions

Neumann series of Bessel functions are defined by [27, Chapter XVI]

\[
\mathcal{R}_\nu(z) \sim \sum_{n=0}^{\infty} a_n J_{\nu+n}(z),
\]

where \( z \) is in general a complex variable, and \( \nu \) and \( \{a_n\}_{n=0}^{\infty} \) are constants. This kind of series have been investigated extensively in view of their relevance in a number of physical problems (see the Introduction of Ref. [21] for a brief summary of these problems). In addition, Neumann series have been shown to be a useful mathematical tool to the solution of classes of differential and mixed differences equations (see, e.g., [18] and [27, p. 530]). The domain of convergence of series (5.1) is a disk whose radius evidently depends on the asymptotic behaviour of the sequence of coefficients \( a_n \) and can be determined by the condition

\[
\lim_{n \to \infty} \left| \frac{a_n (z/2)^{\nu+n}}{\Gamma(\nu+n+1)} \right|^{1/n} < 1.
\]

Integral representations are powerful tools to deduce properties of series. In the case of expansions of type (5.1), examples of integral representations have been given by Wilkins [28], Rice [22] and, more recently, by Pogany, Suli and coworkers [3, 15, 21]. In this section we give an integral representation of the function \( \mathcal{R}_\nu(z) \), which the Neumann series converges to. This is achieved by exploiting the representation (3.1) of \( J_{\nu+n}(z) \) given in Section 3 and assuming the condition on the asymptotic behavior of the sequence \( \{a_n\} \) which basically guarantees the boundedness of trigonometric series. The Fourier-type integral representation of \( J_{\nu+n}(z) \) allows us to write \( \mathcal{R}_\nu(z) \) in integral form with a kernel (the kernel associated with Neumann expansions) which is proportional to the regularized incomplete gamma function \( P(\{\nu\}, \cdot) \) and is independent of the coefficients \( a_n \). The density function of the integral representation, instead, depends only upon the given input sequence of coefficients \( a_n \), and is simply the analytic function, on the unit circle, associated with the sequence of coefficients \( \{a_n\}_{n=0}^{\infty} \). This representation easily leads
to closed-form integral representations of Neumann expansions of Bessel functions whenever the closed-form of the analytic function associated with the sequence of coefficients is known. As examples of application of this result, new integral representations for bivariate Lommel’s functions and Kelvin’s functions will then be given. More examples of this type of integral representations of special functions of the mathematical physics will be presented and analyzed in a forthcoming paper [10]. Now, we can prove the following theorem.

**Theorem 5.1.** Let \( \{a_n\}_{n=0}^{\infty} \subset \mathbb{C} \) be a null sequence such that

\[
\sum_{n=0}^{\infty} |a_n| < \infty.
\]

Then, for \( \Re \nu > -\frac{1}{2} \), the series \((5.1)\) has the following integral representation in the slit domain \( z \in \mathbb{C} \setminus (-\infty, 0] \) if \( \nu \notin \mathbb{N}_0 \) or in \( z \in \mathbb{C} \) if \( \nu \in \mathbb{N}_0 \):

\[
\mathcal{N}_\nu(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} K_z^{(\nu)}(\theta) A(e^{i(\theta - \pi/2)}) \, d\theta \quad (\Re \nu > -\frac{1}{2}),
\]

where

\[
A(e^{i\theta}) = \sum_{n=0}^{\infty} a_n e^{in\theta},
\]

and the kernel \( K_z^{(\nu)}(\theta) \) is given by

\[
K_z^{(\nu)}(\theta) = i^{\nu} e^{i\nu[\theta - \pi \text{sgn}(\theta)]} e^{iz \cos \theta} P(\{\nu\}, -iz (1 - \cos \theta)),
\]

\( P(\cdot) \) denoting the regularized gamma function.

**Proof.** We first insert the integral representation \((3.1)\) of \( J_{\nu+n}(z) \) into sum \((5.1)\):

\[
\sum_{n=0}^{\infty} a_n J_{\nu+n}(z) = \sum_{n=0}^{\infty} (-i)^n a_n \int_{-\pi}^{\pi} \mathcal{J}_z^{(\nu)}(\theta) e^{in\theta} \, d\theta \quad (\Re \nu > -\frac{1}{2}),
\]

and then swap sum and integral to give:

\[
\int_{-\pi}^{\pi} d\theta \mathcal{J}_z^{(\nu)}(\theta) \sum_{n=0}^{\infty} (-i)^n a_n e^{in\theta} = \frac{1}{2\pi} \int_{-\pi}^{\pi} K_z^{(\nu)}(\theta) A(e^{i(\theta - \pi/2)}) \, d\theta,
\]

where \( K_z^{(\nu)}(\theta) \) is given by \((5.6)\) and \( A(e^{i\theta}) \) by \((5.5)\). By the Fubini-Tonelli theorem, exchanging sum and integral is legitimate if

\[
\int_{-\pi}^{\pi} \left| K_z^{(\nu)}(\theta) A(e^{i(\theta - \pi/2)}) \right| \, d\theta < \infty.
\]

For \( \theta \in (-\pi, \pi] \) the kernel \( K_z^{(\nu)}(\theta) \) in \((5.6)\) can be rewritten as follows:

\[
K_z^{(\nu)}(\theta) = i^{\Re \nu} z^{(\nu)} e^{i\nu[\theta - \pi \text{sgn}(\theta)]} e^{iz \cos \theta} (1 - \cos \theta)^{\nu} \gamma^* \{\nu\}, -iz (1 - \cos \theta),
\]

where \( \gamma^*(a, w) \) denotes the Tricomi form of the incomplete gamma function. Recall that \( \gamma^*(a, w) \) is an entire function either in \( a \) and in \( w \). Consequently, on compact domains in \( \Omega \doteq (z \in \mathbb{C} \setminus (-\infty, 0]) \times (-\frac{1}{2} < \{\nu\} < 1) \) the lower incomplete gamma function \( \gamma^*(\{\nu\}, -iz(1 - \cos \theta)) \) can be bounded uniformly in \( \theta \), i.e.: \( \gamma^*(\{\nu\}, -iz(1 - \cos \theta)) \leq c_{z,\nu} \), the constant \( c_{z,\nu} \) depending only on \( z \) and \( \nu \). \([14, 24]\) From \((5.10)\) it is clear that, for fixed \( \theta \), the multivaluedness of the kernel
$K_z^{(\nu)}(\theta)$ is brought by the fractional power $z^{(\nu)}$ (when $\{\nu\} \neq 0$). From (5.10) we have:

\[(5.11) \quad \left| K_z^{(\nu)}(\theta) \right| \leq c_{z,\nu} e^{3\pi} |z| |z|^{\Re\{\nu\}} |1 - \cos \theta|^{\{\nu\}} = C_{\nu,z} |1 - \cos \theta|^{\{\nu\}},\]

$C_{\nu,z}$ being a finite constant on compacta in $\Omega$. From (5.9) and (5.11) it then follows:

\[(5.12) \quad |N_{\nu}(z)| = \left| \int_{-\pi}^{\pi} K_z^{(\nu)}(\theta) A(e^{i(\theta - \pi/2)}) \, d\theta \right| \leq \int_{-\pi}^{\pi} \left| K_z^{(\nu)}(\theta) \right| \left| \sum_{n=0}^{\infty} a_n e^{i(\theta - \pi/2)} \right| \, d\theta < \infty,\]

if $\Re\{\nu\} > -\frac{1}{2}$ and in view of (5.3). Statement (5.4) then follows. □

**Remark 2.** From (5.11) we see that for $\Re\nu \geq 0$ the kernel $K_z^{(\nu)}(\theta)$ is bounded uniformly in $\theta$ on compacta in $\Omega$. Hence, assumption (5.3) of boundedness for $A(e^{i\theta})$ can be substituted by a less restrictive assumption of integrability (see (5.12)). Explicitly, this is guaranteed for instance (see [5, Theorem 1, p. 54]) by replacing (5.3) with the weaker asymptotic condition

\[(5.13) \quad \sum_{n=1}^{\infty} \frac{|a_n|}{n} < \infty,\]

along with the following additional regularity assumption on the sequence $\{a_n\}$:

\[(5.14) \quad \sum_{n=1}^{\infty} \left( \frac{\sum_{k=n}^{\infty} |\Delta a_k|^p}{n} \right)^{1/p} < \infty \quad \text{for some integer } p > 1,\]

where $\Delta a_k = (a_k - a_{k+1})$.

**Remark 3.** The integral representation (5.4) can also be written in the following form:

\[(5.15) \quad \mathcal{N}_{\nu}(z) = \frac{e^{iz}}{\pi} \int_{0}^{\pi} \mathcal{K}^{(\nu)}(-iz(1 + \cos \theta)) \, A^{(\nu)}(\theta) \, d\theta,\]

where the kernel is

\[(5.16) \quad \mathcal{K}^{(\nu)}(w) = e^w P\{\nu\}, w,\]

and the function associated with the sequence of coefficients $\{a_n\}$ is:

\[(5.17) \quad A^{(\nu)}(\theta) = \sum_{n=0}^{\infty} i^{n+\nu} a_n \cos(n + \nu)\theta.\]

### 5.1. Lommel’s functions of two variables.

As first example of application of the previous analysis, we now consider the bivariate Lommel functions $U_{\nu}(w, z)$ and $V_{\nu}(w, z)$ of order $\nu$. These functions can be regarded as particular solutions of the inhomogeneous Bessel equation (see [17, Section 27] for a summary of their properties) and were first introduced by Lommel in the analysis of diffraction problems [4, p. 438]. They have found applications also, e.g., in soliton theory [13] and in the theory of incomplete cylindrical functions [1, Chapter 4]. Lommel’s functions...
of two variables can be defined for unrestricted values of \( \nu \) by the Neumann series [27] Section 16.5, Eqs. (5) and (6); p. 537, 538]:

\[
U_\nu(w, z) \doteq \sum_{n=0}^{\infty} (-1)^n \left( \frac{w}{z} \right)^\nu J_{\nu+2n}(z),
\]

(5.18)

\[
V_\nu(w, z) \doteq \cos \left( \frac{w^2 + z^2}{2w} + \frac{\nu \pi}{2} \right) + U_{-\nu+2}(w, z).
\]

(5.19)

The function \((w/z)^{-\nu} U_\nu(w, z)\) has therefore the structure (5.1) with coefficients:

\[
a_n(w, z) = \begin{cases} 
(-1)^{n/2}(w/z)^n & \text{if } n \text{ is even}, \\
0 & \text{if } n \text{ is odd}.
\end{cases}
\]

(5.20)

Condition (5.3) then reads

\[
\sum_{n=0}^{\infty} |a_n(w, z)| = \sum_{n=0}^{\infty} |w/z|^{2n} < \infty,
\]

if \(|w/z| < 1\). Therefore, assuming the condition \(|w/z| < 1\) and using (5.20), the analytic function \(A_{w,z}(e^{i\theta})\) associated with the bivariate Lommel function \(U_\nu(w, z)\) then reads (see (5.5)):

\[
A_{w,z}(e^{i\theta}) = \sum_{n=0}^{\infty} (-1)^n \left( \frac{w}{z} \right)^{2n} e^{i2n\theta},
\]

(5.22)

which is a power series converging for \(|w/z| < 1\) and any \(\theta \in (-\pi, \pi]\) to

\[
A_{w,z}(e^{i\theta}) = \frac{1}{1 + (w/z)^2e^{i2\theta}}.
\]

(5.23)

Finally, from (5.4) the integral representation of the Lommel function \(U_\nu(w, z)\) reads explicitly for \(\Re \nu > -1/2\) and \(|w/z| < 1\):

\[
U_\nu(w, z) = \frac{i^\nu}{\pi} \left( \frac{w}{z} \right)^\nu \int_{\theta}^{\pi} \frac{P(\nu), -iz(1 + \cos \theta)) [\cos \nu \theta - (w/z)^2 \cos (\nu - 2) \theta]}{1 - 2(w/z)^2 \cos 2\theta + (w/z)^4} e^{-iz \cos \theta} d\theta.
\]

(5.24)

In view of the restriction \(\Re \nu > -1/2\), a similar integral representation can be obtained from (5.19) for the Lommel function \(V_\nu(w, z)\) for \(\Re \nu < \frac{1}{2}\).

5.2. Kelvin’s functions. Kelvin’s functions of the first kind \(\text{ber}_\nu(x)\) and \(\text{bei}_\nu(x)\) are, respectively, real and imaginary part of the solution (the symbols stand indeed for Bessel-real and Bessel-imaginary), regular in the origin, of the differential equation [2]

\[
x^2y'' + xy' - (\nu^2 + 1)x^2y = 0,
\]

(5.25)

and emerge as solutions of various physics and engineering problems occurring in the theory of electrical currents [17], magnetism [16], fluid mechanics and elasticity [2]. Kelvin’s functions of order \(\nu\) are related to the Bessel functions as follows [11]:

\[
\text{ber}_\nu(x) + i \text{bei}_\nu(x) = e^{i\pi \nu} J_{\nu}(xe^{-i\pi/4}) \quad (x \geq 0, \nu \in \mathbb{R}).
\]

(5.26)
Integral representations are known: for instance, when \( \nu \in \mathbb{R} \), Schl"afli’s representation of Bessel’s function \( J_\nu(x) \) leads to the following representation of Kelvin’s functions [2]:

\[
\text{ber}_\nu(\sqrt{2} x) = \frac{1}{\pi} \int_0^\pi \left[ \cos \pi \nu \cos(x \sin t - \nu t) \cosh(x \sin t) - \sin \pi \nu \sin(x \sin t - \nu t) \sinh(x \sin t) \right] dt
\]

(5.27)

\[
\text{bei}_\nu(\sqrt{2} x) = \frac{1}{\pi} \int_0^\pi \left[ \cos \pi \nu \sin(x \sin t - \nu t) \sinh(x \sin t) + \sin \pi \nu \cos(x \sin t - \nu t) \cosh(x \sin t + \pi \nu) \right] dt
\]

(5.28)

which, when \( \nu \equiv n \) is a nonnegative integer, reduce to:

\[
\text{ber}_n(\sqrt{2} x) = \left(-1\right)^n \frac{\pi}{\pi} \int_0^\pi \cos(x \sin t - nt) \cosh(x \sin t) dt,
\]

(5.29a)

\[
\text{bei}_n(\sqrt{2} x) = \left(-1\right)^n \frac{\pi}{\pi} \int_0^\pi \sin(x \sin t - nt) \sinh(x \sin t) dt.
\]

(5.29b)

A novel integral representation for \( \text{ber}_\nu(x) + i \text{bei}_\nu(x) \) can be readily obtained from the results of Theorem 5.1 (see (5.4)) observing that, for \( x \in \mathbb{R} \) and \( \nu \in \mathbb{C} \), the following expansion in series of Bessel functions holds [11, Eq. 10.66.1]:

\[
\text{ber}_\nu(x) + i \text{bei}_\nu(x) = e^{i3\pi\nu/4} \sum_{k=0}^{\infty} \frac{x^k}{2^{k/2} k!} e^{ik\pi/4} J_{\nu+k}(x).
\]

(5.30)

Referring to (5.1), we set

\[
a_k(x) = \frac{x^k}{2^{k/2} k!} e^{ik\pi/4} \quad (k \geq 0).
\]

(5.31)

Then, from (5.5) the analytic function associated with Kelvin’s functions reads:

\[
A^{(K)}_x(\theta) = \sum_{k=0}^{\infty} \frac{x^k}{2^{k/2} k!} e^{i(k\theta+\pi/4)} = \exp(x e^{i(\theta+\pi/4)}/\sqrt{2}),
\]

(5.32)

the series converging for any \( x \in \mathbb{R} \) and \( \theta \in (-\pi, \pi] \). Finally, using (5.4), (5.6) and (5.32) we have for \( \text{Re} \nu > -1/2 \) the following integral representation for Kelvin’s functions:

\[
\text{ber}_\nu(\sqrt{2} x) + i \text{bei}_\nu(\sqrt{2} x) = e^{i3\pi\nu/4} \int_0^\pi e^{-cx \cos \theta} P(\{\nu\}, -i\sqrt{2}x(1 + \cos \theta)) \cos [\nu \theta - \pi x \sin \theta] d\theta,
\]

with \( c = \exp(i\pi/4) \) and the bar standing for complex conjugate. Formula (5.33) represents the generalization to \( \text{Re} \nu > -1/2 \) of formulae (5.29).
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