A characterization of colorless anonymous $t$-resilient task computability

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Abstract A task is a distributed problem for $n$ processes, in which each process starts with a private input value, communicates with other processes, and eventually decides an output value. A task is colorless if each process can adopt the input or output value of another process. Colorless tasks are well studied in the non-anonymous shared-memory model where each process has a distinct identifier that can be used to access a single-writer/multi-reader shared register. In the anonymous case, where processes have no identifiers and communicate through multi-writer/multi-reader registers, there is a recent topological characterization of the colorless tasks that are solvable when any number of asynchronous processes may crash.

In this paper we study the case where at most $t$ processes may crash, where $1 \leq t < n$. We prove that a colorless task is $t$-resilient solvable non-anonymously if and only if it is $t$-resilient solvable anonymously. This implies a complete characterization of colorless anonymous $t$-resilient asynchronous task computability.

Keywords: Distributed problems, Formal specifications, Tasks, Sequential specifications, Linearizability, Long-lived objects.

1 Introduction

The central result in distributed task computability is the asynchronous computability theorem (ACT) \cite{SST06}. It characterizes the tasks that are solvable in asynchronous shared-memory systems where $n$ processes that may fail by crashing communicate by writing and reading shared registers. It is sometimes called the wait-free characterization, because any number of processes may crash, and the processes are asynchronous (run at arbitrary speeds, independent from each other). The characterization is of an algebraic topology nature. A task is represented as a relation $\Delta$ between an input complex $I$ and an output complex $O$. Each simplex $\sigma$ in $I$ is a set that specifies the initial inputs to the processes in some execution. The processes communicate with each other, and eventually decide output values that form a simplex $\tau$ in $O$. The computation is correct, if $\tau$ is in $\Delta(\sigma)$. The complex $I$ (resp. $O$) is chromatic because each simplex specifies not only input values, but also which process gets which input (resp. output) value. Roughly, the ACT characterization states that the task is solvable if and only if there is a simplicial map $\delta$ from a chromatic subdivision of $I$ to $O$ respecting $\Delta$. The map $\delta$ is also chromatic, because it sends an input vertex corresponding to process $p_i$ to an output vertex corresponding to the same process $p_i$.

The ACT is the basis to obtain a characterization of distributed task computability in the case where at most $t$ asynchronous processes may crash, $1 \leq t < n$. Also, it is the basis to study other distributed computing models, parametrized by the failure, timing and communication model, and even mobile robot models \cite{Yan07}. There are basically two ways of extending the results from the wait-free model to other models. One by directly generalizing the algorithmic and topological techniques, and another by reduction to other models using simulations (either algorithmic \cite{Yan07} or topological \cite{SST06}). An overview of results in this area can be found in the book \cite{Yan07}.
to \( R_i \). However, often processes, while they know their ids, the number of possible ids \( N \) is much bigger than the number of processes, \( n \). In this situation, preallocating a register for each identifier would lead to a distributed algorithm with a very large space complexity, namely \( N \) registers. Instead, it is shown in [11] that \( n \) multi-writer/multi-reader (MWMR) registers are sufficient to solve any read-write wait-free solvable task.

However, in some distributed systems, processes are anonymous; they have no ids at all or they cannot make use of their identifiers (e.g. due to privacy issues). Processes run identical programs, and the means by which processes access the shared memory are identical to all processes. A process cannot have a private register to which only this process may write, and hence the shared memory consists only of MWMR registers. This anonymous shared memory model of asynchronous distributed computing has been studied since early on [3,25], in the case where processes do not fail.

Only recently a characterization of the tasks that are wait-free solvable in the anonymous model has been given [29]. The characterization implies that the anonymity does not reduce the computational power of the asynchronous shared-memory model as long as colorless tasks are concerned. Indeed, in an anonymous system, the task specification must be colorless in the sense that it cannot refer to which process has which input or output value. In consequence, the topological characterization is in terms of input and output complexes which are not chromatic. Furthermore, the anonymous wait-free characterization matches exactly the eponymous wait-free characterization for non-anonymous systems [20,22].

**Results** Our main result is an extension of the wait-free characterization of [29], to the case where at most \( t \) processes may crash, \( 1 \leq t < n \). We prove that a colorless task is \( t \)-resilient solvable anonymously if and only if it is \( t \)-resilient solvable non-anonymously.

Our main result is to show that if a colorless task is \( t \)-resilient solvable non-anonymously, then it is solvable anonymously using only \( n \) MWMR registers. The result is obtained through a series of reductions depicted in the figure below. We hope they provide useful basic tools to study further anonymous fault-tolerant computation. First, we design an anonymous non-blocking implementation of an atomic weak set object with \( n \) registers. The construction is based on the non-blocking atomic snapshot of [12,16]. Then we build a wait-free implementation of a safe agreement object for an arbitrary value set \( V \). Our implementation is a generalization of the anonymous consensus algorithm proposed in [3]. We then describe two ways of deriving the \( t \)-resilient anonymous solvability characterization. One way is through a novel anonymous implementation of the BG-simulation [5], which we use to simulate a non-anonymous system by an anonymous system, both \( t \)-resilient. The other way is to use the safe-agreement object to solve \( k \)-set agreement, and then do the topological style of analysis [21] and [29].

**Related work** Colorless tasks include many tasks such as consensus [13], set agreement [9], and loop agreement [19], and have been widely studied (in the non-anonymous case). The first part of the book [17] is devoted to colorless tasks. Colorless tasks were identified in [5] as the ones for which the BG-simulation works, and in [18] for the purpose of showing that they are undecidable in most distributed computing models. Not all tasks of interest are colorless though, and general tasks can be much harder to study, e.g. [8,20].

A characterization of the colorless tasks that are solvable in the presence of processes that can crash in a dependent way is provided in [20], and a characterization when several processes can run solo is provided in [22]. Both encompass the wait-free colorless task solvability characterization, and the former encompasses the \( t \)-resilient characterization that we use in this paper.

A certain kind of anonymity has been considered in [23] to establish the anonymous computability theorem. However, they allow SWMR registers while we assume a fully anonymous model with only MWMR registers.

Anonymous distributed computing remains an active research area since the shared-memory seminal papers [3,25] and the message-passing paper [11]. For some recent papers and references herein see, e.g. [7,15].
Closer to our paper is [16] where the anonymous asynchronous MWMR fault-tolerant shared-memory model is considered. Our weak atomic set object provides an enhanced atomic implementation of the weak set object supporting non-atomic operations presented in [10]. A set object that also supports a remove operation, but satisfies a weaker consistency condition, called per-element sequential consistency is presented in [4].

Organization In Section 2, we briefly recall some of the notions used in this paper, about the model of computation and the topology tools, both of which are standard. In Section 3, we present the anonymous implementation of an atomic weak set object from MWMR registers. In Section 4, we present the safe agreement implementation. In Section 5, we derive our anonymous characterization of the t-resilient solvability of colorless tasks.

2 Preliminaries

We assume a standard anonymous asynchronous shared-memory model [16] consisting of n sequential processes that have no identifiers and execute an identical code. We assume that at most t of the processes may fail by crashing, where 1 ≤ t < n. Processes are asynchronous, i.e., they run at arbitrary speeds, independent from each other. The processes communicate via multi-writer/multi-reader (MWMR) registers. Let R[0...m – 1] denote an array of m registers. The read operation, denoted by read, returns the state of R[i]. The write operation, denoted by write(i, v), changes the state of R[i] to v and returns ack. The registers are assumed to be atomic (linearizable) [24]. We assume that the registers are initialized to some default value. We sometimes refer to the processes by unique names p₀,...,pₙ₋₁ for the convenience of exposition, but processes themselves have no means to access these names. Let us write Π = {p₀,...,pₙ₋₁}.

A complex K on a finite set V(K) of vertices is a family of nonempty subsets of V(K), called simplices, such that {v} ∈ K for every v ∈ V(K). Also, K is closed under containment, meaning that if s ∈ K then s′ ∈ K, for every s′ ⊆ s. A subset of a simplex s is called a face of s. The dimension of s is #s – 1, where #V denotes the cardinal of a set V. A map φ : V(K) → V(L), where K and L are complexes, is said to be a simplicial map, if φ(σ) ∈ L for every σ ∈ K. We can associate any complex K to the corresponding topological space |K| ⊆ Rd, for a sufficiently large integer d, by embedding vertices of each simplex into Rd in an affine independent way and taking convex hull of these vertices.

The barycentric subdivision of a complex K, denoted by baryK, is the complex such that V(K) = K and a set {s₀,...,sᵣ} ⊆ K is a simplex of baryK if and only if s₀,...,sᵣ are totally ordered by containment. The b-iterated barycentric subdivision of a complex K, denoted by baryᵇK, is defined by bary(baryᵇ⁻¹K), where K₀ = K. The k-skeleton of a complex K, denoted by skelᵏK, is the complex whose simplices are the simplices of K of dimension less than or equal to k.

Let I and O be complexes. A carrier map from I to O is a mapping Δ : I → 2⁰ such that, for each s ∈ I, Δ(s) is a subcomplex of O and s′ ⊆ s implies Δ(s′) ⊆ Δ(s). If a continuous map f : |I| → |O| satisfies f(σ) ⊆ Δ(σ) for all σ ∈ I, we say that f is carried by Δ. If a simplicial map δ : baryᵇI → O satisfies δ(baryᵇσ) ⊆ Δ(σ) for all σ ∈ I, we say that δ is carried by Δ. As an immediate consequence of Lemma 3.7.8. of [17], the following lemma holds.

Lemma 1. If Δ : I → 2⁰ is a carrier map and f : |I| → |O| is a continuous map carried by Δ, then there is a non-negative integer b and a simplicial map δ : baryᵇI → O carried by Δ.

A colorless task is a triple T = (I, O, Δ), where I and O are simplicial complexes and Δ is a carrier map. A colorless task T is solvable, if for each input simplex s ∈ I, whenever each process pᵢ starts with input value vᵢ ∈ s (different processes may start with the same value), eventually it decides an output value vᵢ', such that the set of output values form a simplex s' ∈ Δ(s). The colorless tasks that are fundamental to the present paper are the b-iterated barycentric agreement and the k-set agreement. The b-iterated barycentric agreement is a colorless task T = (I, baryᵇI, baryᵇ), where we write by baryᵇ the carrier map that maps s ∈ I to baryᵇs for an abuse of notation. The k-set agreement is a colorless task Tₖ = (I, skelᵏI, skelᵏ), where skelᵏ denotes the carrier map that maps a simplex s ∈ I to the subcomplex skelᵏI.

3 Atomic weak set

Here, we present an anonymous implementation of an atomic weak set object on an arbitrary value set V.
3.1 Specification and Algorithm

An atomic weak set object, denoted by SET, is an atomic object used for storing values. The object supports only two operations, add and get, and has no remove operation, which is why it is called “weak.” The add operation, denoted by \( \text{ADD}(v) \), takes an argument \( v \in V \) and returns ACK. The get operation, denoted by \( \text{GET()} \), takes no argument and returns the set of values that have appeared as arguments in all the \( \text{ADD()} \) operations preceding the \( \text{GET()} \) operation. We assume that \( \text{SET} \) initially holds no values, i.e., it is \( \emptyset \).

We assume that a non-blocking atomic snapshot object is available. An implementation in an anonymous setting with \( n \) registers is described in \([12,16]\). The snapshot object exports two operations, \( \text{UPD}() \) and \( \text{SCAN}() \). Informally, a \( \text{scan()} \) returns an array of \( n \) values, which are contained in the array of \( n \) MWMR registers at some point in time between the invocation and the response of the \( \text{scan}() \) operation.

We propose an anonymous non-blocking implementation of the atomic weak set object on \( n \) MWMR registers. The pseudocode of the implementation appears in Fig. 1. If \( S \) is an array of \( n \) cells, we denote \( \text{vals}(S) = \{S[i]\}i \in \{1, ..., n\} \). The idea of the algorithm is as follows. To execute an \( \text{ADD}(v) \) operation, the algorithm repeatedly tries to store the value \( v \) in each one of the \( n \) components of the snapshot object, using an \( \text{UPD}() \) operation (line 5) until it detects that \( v \) appears in all the components. In each iteration, the algorithm deposits in the snapshot object not only \( v \) appearing in all the components of the snapshot object, the \( \text{ADD}(v) \) terminates. The \( \text{GET()} \) operation is similar, except that now the \( V \)iew of the process has to appear in all the components of the snapshot for the operation to terminate. Intuitively, once a value \( v \) (or a set of values) appears in all \( n \) components of the snapshot object, it cannot be overwritten and go unnoticed by other processes, because the other processes can be covering (about to write) at most \( n - 1 \) components.

```
Shared variable:
array of n MWMR-register : R[0...n-1]

Code for a process
Local variable:
array of n set of Values Snap[0...n-1]
set of Values View init ∅
integer next

Macro:
vals(Snap) = ∪Snap[i]

ADD(v):
  next = 0
  Snap = R.scan()
  View = View ∪ vals(Snap) ∪ {v}
  while (#r[v in Snap[r]] < n)
    R.update(next, View)
    next = (next + 1) mod n
  Snap = R.scan()
  View = vals(Snap) ∪ View
  return ACK

GET:
  next = 0
  Snap = R.scan()
  View = vals(Snap) ∪ View
  while (#r[View = Snap[r]] < n)
    R.update(next, View)
    next = (next + 1) mod n
  Snap = R.scan()
  View = vals(Snap) ∪ View
  return View
```

Fig. 1. non-blocking implementation of atomic weak set for \( n \) processes.
3.2 Correctness of weak set object implementation

Safety Given an operation op, invoc(op) denotes its invocation and resp(op) its response.

Let $H$ be a history of the algorithm as defined in [24]. $H_{seq}$ denotes the sequential history in which each operation of $H$ appears as if it has been executed at a single point (the linearization) of the time line. We have to define linearization points and prove that:

- the linearization point of each operation GET() and ADD() appear between the beginning and the end of this operation;
- the sequential history that we get with these points respect the sequential specification of the weak set.

Most of the details of the proofs are in the appendix.

Consider a history $H$, let $v$ be a value or a set of values, define time $\tau_v$ as the first time, if any, that $v$ belongs to all registers of $R$. When there is no such time, $\tau_v$ is $\bot$.

**Lemma 2.** If the operation $\text{ADD}(v)$ terminates, then before the end of this operation $v$ belongs to all registers of $R$. If the operation $\text{GET}()$ terminates and returns $V$, then before the end of this operation $V$ belongs to all registers of $R$.

By Lemma 2, $\tau_v$ is not $\bot$ for each operation $\text{ADD}(v)$ that terminates and $\tau_V$ is also not $\bot$ for each operation $\text{GET}()$ that terminates and returns $V$.

**Linearization points for operations ADD() and GET():**

- $op = \text{ADD}(v)$: If $\tau_v \neq \bot$, the linearization point $\tau_{op}$ of an operation $op = \text{ADD}(v)$ is $\max\{\tau_v, \text{invoc}(op)\}$. If $\tau_v = \bot$, the operation $op$ does not terminate and is not linearized.
- $op = \text{GET}()$: The linearization point $\tau_{op}$ of an operation $op = \text{GET}()$ that returns $V$ is $\max\{\tau_V, \text{invoc}(op)\}$. A GET() operation that does not terminate is not linearized.

Directly from the definition and Lemma 2, the linearization point $\tau_{op}$ appears between the invocation and the response of $op$.

**Lemma 3.** Let $op$ be an operation of $H$. If $\tau_{op}$ is defined, then $\tau_{op}$ belongs to $\{\text{invoc}(op), \text{resp}(op)\}$. If $\tau_{op}$ is undefined, then $op$ does not terminate and is not linearized.

In the following, we consider the next snapshot operation on $R$ of each process. This next snapshot operation is either a SCAN() or an UPDATE() or there is no next snapshot operation (when the process is about to satisfy the termination loop condition: $\#(r|v \in \text{Snap}[r]) \geq n$ for an ADD() and $\#(r|\text{View} = \text{Snap}[r]) \geq n$ for an UPDATE()).

Consider any time $\tau$ and define $r_v(\tau)$, $w_v(\tau)$, $c_v(\tau)$ and $\alpha_v(\tau)$ as follows:

- $r_v(\tau)$ is the number of processes for which, after time $\tau$, the next snapshot operation is a SCAN();
- $w_v(\tau)$ is the number of processes such that (1) $v \in \text{View}$ at time $\tau$ and for which after time $\tau$ the next snapshot operation is an UPDATE(), or (2) there is no next snapshot operation for that process (the process has finished -or is going to finish- its main loop or it takes no more steps);
- $c_v(\tau)$ is the number of registers that contains $v$ at time $\tau$;
- $\alpha_v(\tau)$ is defined by $\alpha_v(\tau) = r_v(\tau) + w_v(\tau) + c_v(\tau)$.

As soon as $\alpha_v(\tau) > n$, $\alpha_v(\tau)$ is not decreasing:

**Lemma 4.** Assume $\alpha_v(\tau) > n$ and the next step in $H$, is made at time $\tau' \geq \tau$, then we have $\alpha_v(\tau) \leq \alpha_v(\tau')$.

And by an easy induction on the steps of $H$ we get:

**Lemma 5.** If $\alpha_v(\tau) > n$ then for all $\tau'$, such that $\tau \leq \tau'$, $\alpha_v(\tau) \leq \alpha_v(\tau')$.

When $\tau_v$ is defined, we can verify that $\alpha_v(\tau) > n$ then:

**Lemma 6.** If $\tau_v \neq \bot$ then for all $\tau$, such that $\tau \leq \tau$, $\alpha_v(\tau_v) \leq \alpha_v(\tau)$.

From the previous Lemmas we may deduce:

**Lemma 7.** $H_{seq}$ satisfies the sequential specification of the weak set.
Liveness We prove that the algorithm is non-blocking, namely, if processes perform operations forever, an infinite number of operations terminates.

By contradiction, assume that there is only a finite number of operations \( \text{GET}() \) and \( \text{ADD}() \) and some operations made by correct processes do not terminate.

Operations \( \text{ADD}() \) or \( \text{GET}() \) may not terminate because the termination conditions of the while loop are not satisfied (Lines 11 or 13): for an \( \text{ADD}(v) \) operation, in each \( \text{SCAN}() \) made by the process, \( v \) is not in at least one of the registers of \( R \), and for a \( \text{GET}() \) operation, in each \( \text{SNAP}() \), all the registers are not equal to the view of the process.

There is a time \( \tau_0 \) after which there is no new process crash and all processes that terminate \( \text{GET}() \) or \( \text{ADD}() \) operations in the run have already terminated. Consider the set \( N \) of processes alive after time \( \tau_0 \) that do not terminate operations in the run. Note that after time \( \tau_0 \) only processes in \( N \) take steps and as no process in \( N \) may crash each process in \( N \) makes an infinite number of steps.

We notice that all values in variables \( \text{View} \) have been proposed by some \( \text{GET}() \). If there is a finite number of operations, then all variables \( \text{View} \) are subsets of a finite set of values. Moreover, considering the inclusion \( \subseteq \), the views of each process are increasing, then there is a time \( \tau_1 > \tau_0 \) after which the view of each process \( p \) in \( N \) converges to a stable view \( S\text{View}_p \): after time \( \tau_1 \) forever the view of \( p \) is \( S\text{View}_p \). In the following \( SV \) denotes \( \{S\text{View}_p | p \in N \} \) the set of all stable views for processes in \( N \). Observe that:

**Observation 1** If \( p \) does not terminate an \( \text{ADD}(v) \) then \( v \in S\text{View}_p \).

After time \( \tau_1 \) processes only update \( R \) with their stables views \( S\text{View} \), and as each process in \( N \) updates infinitely often each register of \( R \) with its \( S\text{View} \) there is a time \( \tau_2 \) after which all registers in \( R \) contain only stable views of processes in \( N \):

**Observation 2** After time \( \tau_2 \) for all \( i \), \( R[i] \in SV \).

Among stable views consider any minimal view \( S\text{View}_0 \) for inclusion, i.e. for all \( S \in SV \), \( S \subseteq S\text{View}_0 \) implies \( S\text{View}_0 = S \).

Consider any process \( p \in N \) having the \( S\text{View}_0 \) as stable view, eventually \( p \) makes a scan of the memory \( R \) (Line 2 or 4 for \( \text{ADD}() \), Line 11 or 13 for \( \text{GET}() \)). Let \( \text{Snap} \) be the array returned by the \( \text{scan} \). \( \text{Snap} \) is the value of the array of registers \( R \) at some time after \( \tau_2 \). Then \( p \) adds \( \bigcup_{1 \leq i \leq n} \text{Snap}[i] \) to its view \( S\text{View}_0 \). The \( S\text{View}_0 \) being stable we have \( \bigcup_{1 \leq i \leq n} \text{Snap}[i] \subseteq S\text{View}_0 \) and then for all \( i \), \( \text{Snap}[i] \subseteq S\text{View}_0 \). But by Observation 2 \( R[i] = \text{Snap}[i] \) is a stable view \( S \in SV \). Then by the minimality of \( S\text{View}_0 \), for all \( i \) we have \( \text{Snap}[i] = S\text{View}_0 \). Then consider the two following cases:

- if \( p \) is performing an \( \text{ADD}(v) \), as \( v \in S\text{View}_p = S\text{View}_0 \), for all \( i \), \( v \in \text{Snap}[i] \) and the loop condition \( \#\{r | \text{Snap}[r] < n \} \) (Line 4) is false and \( p \) terminates \( \text{ADD}(v) \) — A contradiction
- if \( p \) is performing an \( \text{GET}() \), then the loop continuation condition \( \#\{r | \text{View} = \text{Snap}[r] \} < n \) is false and \( p \) terminates operation \( \text{GET}() \) — A contradiction

We deduce that there is no minimal stable view proving that \( SV = \emptyset \) and also \( N = \emptyset \).

4 Safe Agreement Object

An **safe agreement object** \( S \) provides two operations, \( \text{PROPOSE}() \) and \( \text{RESOLVE}() \). A \( \text{PROPOSE}() \) operation takes an argument \( v \in V \) and returns \( \text{ACK} \). A \( \text{RESOLVE}() \) operation takes no argument and returns \( u \in V \) or \( \bot \). The safe agreement object is one-shot, i.e., each process may perform at most one \( \text{PROPOSE}() \) operation on each object, while an arbitrary number of \( \text{RESOLVE}() \) operation can be invoked on a single object. The object satisfies the following five conditions:

**Validity** Any non-\( \bot \) value returned by a \( \text{RESOLVE}() \) operation is an argument of some \( \text{PROPOSE}() \) operation;

**Agreement** If two \( \text{RESOLVE}() \) operations return non-\( \bot \) values \( v \) and \( v' \), then \( v = v' \);

**Termination** Every operation performed by a non-faulty process eventually terminates;

**Nontriviality** If more than one \( \text{PROPOSE}() \) operations are performed and no process fails while performing these \( \text{PROPOSE}() \) operations, every \( \text{RESOLVE}() \) operation started after some time instance returns non-\( \bot \) value.
A characterization of colorless anonymous $t$-resilient task computability

The above specification of the safe agreement object is based on [26].

We propose an anonymous wait-free implementation, presented in Fig. 2 of the safe agreement object for an arbitrary value set $V$. The implementation makes use of $n$-array of weak set objects $SET[0 \ldots n-1]$. Each process firstly sets its input value to a local variable $view$. The process repeats the following procedure for $i = 0, \ldots, n-1$: it adds $view$ to the $SET[i]$; if $SET[i]$ holds a set of cardinality more than 2 and $view$ is the minimum value of the set, it waits until it gets a non-empty set from the $SET[n-1]$; otherwise, it sets the minimum value of the set to $view$. When a process gets a non-empty set from the $SET[n-1]$, it returns the minimum value in the set.

Our implementation is a generalization of the anonymous consensus algorithm proposed by Attiya et al. [3]. Bouzid and Corentin [6] have proposed an anonymous implementation of the safe agreement object for the case of $V = \{0, 1\}$, where their implementation is also based on [3]. However, their implementation is not immediately extended to the case of the infinite value set.

**Shared variable:**
- array of atomic weak set objects: $SET[0 \ldots n-1]$

**Code for a process**

Local variable:
- Value view init \( \perp \)
- Integer $i$ init 0
- set of Values Snap init \( \emptyset \)

**operation** propose($v$):
1. $view = v$
2. for $i = 0, \ldots, n-1$
3. $SET[i].ADD(view)$
4. $Snap = SET[i].GET()$
5. if #$Snap \geq 2$ && $view == \min(Snap)$ then
6. return $ACK$
else
7. $view = \min(Snap)$
8. return $ACK$

**operation** resolve():
1. $Snap = SET[n-1].GET()$
2. if $Snap \neq \emptyset$ then
3. return $\min(Snap)$
else
4. return $\perp$

Fig. 2. Anonymous implementation of safe agreement object

We now prove the correctness of the algorithm of Fig. 2. Recall that, although we refer to the processes by unique names $p_0, \ldots, p_{n-1}$, processes themselves have no means to access these names. These set of names is denoted by $\Pi = \{p_0, \ldots, p_{n-1}\}$.

**Lemma 8.** Fix an execution of the algorithm of Fig. 2. Let $V_i$ be the set of all the values that are added to $SET[i]$ in the execution. Then, $V_i \supseteq V_{i+1}$ holds for all $i = 0, \ldots, n-2$.

**Proof.** Every value added to $SET[i+1]$ is a value held in $SET[i]$ by Lines 4 and 8 of the algorithm. Thus, the lemma follows.

**Lemma 9 (Validity).** The algorithm of Fig. 2 satisfies the validity condition.

**Proof.** Fix an execution and define $V_i$ in the same way as Lemma 8. By Line 1, every value in $V_0$ is an argument of some propose() operation. By Lemma 8, $V_0 \supseteq \cdots \supseteq V_{n-1}$ and thus every value in $V_{n-1}$ is also an argument of some propose() operation. This completes the proof because every non-$\perp$ value returned by a resolve() operation is a value held in the object $SET[n-1]$.
Lemma 10. Fix an execution of the algorithm of Fig. 2 in which more than one \texttt{propose()} operations are performed. Let $V_i$ be the set of all values that are added to \texttt{SET}[i] in the execution. Let

$$
\Pi_i = \{ p \in \Pi | p \text{ performs propose()} \text{ and adds } v \in V_i \setminus \{ \min V_i \} \text{ to } \text{SET}[i] \}.
$$

Then, $\Pi_i \supseteq \Pi_{i+1}$ holds for all $i = 0, \ldots, n - 2$.

Proof. If $p \notin \Pi_i$, there are two cases: the process $p$ does not execute the $i$-th iteration of the for loop; the process $p$ executes the $i$-th iteration and adds $\min V_i$ to \texttt{SET}[i]. In the former case, the process does not execute the $(i+1)$-th iteration and thus $p \notin \Pi_{i+1}$. In the latter case, the process adds $\min V_i$ to \texttt{SET}[i+1] (if it does not fail) and thus $p \notin \Pi_{i+1}$ because $\min V_i \leq \min V_{i+1}$ by Lemma 8. In either case, $p \notin \Pi_i$ implies $p \notin \Pi_{i+1}$. We obtain $\Pi_i \supseteq \Pi_{i+1}$ by taking contrapositive.

Lemma 11 (Agreement). The algorithm of Fig. 2 satisfies the agreement condition.

Proof. If no \texttt{propose()} operation are performed in an execution, it is clear to see that every \texttt{resolve()} operation returns $\perp$. Thus, the lemma trivially holds for this case.

Fix an execution, in which more than one \texttt{propose()} operations are performed and define $V_i$ and $\Pi_i$ in the same way as Lemma 10. Note that $\#\Pi_0 \leq n - 1$ by definition. It is sufficient to prove that $\#V_{n-1} < 2$ for any execution, because every non-$\perp$ value returned by a \texttt{resolve()} is a value held in the object \texttt{SET}[n-1]. We prove this by contradiction.

Assume that $\#V_{n-1} \geq 2$ holds. This implies $\#V_i \geq 2$ for all $i = 0, \ldots, n - 1$ by Lemma 8. Let us define

$$
\Sigma_i = \{ p \in \Pi | p \text{ performs propose()} \text{ and adds } \min V_i \text{ to } \text{SET}[i] \}.
$$

Note that $\Pi_i \cap \Sigma_i = \emptyset$.

We now prove that $\Pi_i \supseteq \Pi_{i+1}$ for all $i = 0, \ldots, n - 2$. We consider the following two cases:

Case 1: Suppose that some process $p \in \Pi_i$ sees $\min V_i$ at Line 4 in the $i$-th iteration. Then, the process $p$ assigns $\min V_i$ to its view and adds $\min V_i$ to \texttt{SET}[i+1] in the $(i+1)$-th iteration. This leads $p \in \Sigma_{i+1}$ and thus $p \notin \Pi_{i+1}$ because $\min V_i \leq \min V_{i+1}$ by Lemma 8.

Case 2: Suppose that no process in $\Pi_i$ sees $\min V_i$ at Line 4 in the $i$-th iteration. Then, the processes in $\Sigma_i$ perform \texttt{add()} operations on \texttt{SET}[i] after the processes in $\Pi_i$ perform \texttt{get()} operations on \texttt{SET}[i]. This leads that all the processes in $\Sigma_i$ sees $V_i$ in the $i$-th iteration and jump to the while loop and thus $\min V_i < \min V_{i+1}$ by Lemma 8. Thus, at least one process in $\Pi_i$ adds $\min V_{i+1}$ to \texttt{SET}[i+1] and is not in $\Pi_{i+1}$.

In either case, $\Pi_i \supseteq \Pi_{i+1}$ holds by Lemma 10. $\#\Pi_0 \leq n - 1$ and $\Pi_0 \supseteq \cdots \supseteq \Pi_{n-1}$ imply $\Pi_{n-1} = \emptyset$. Thus, only $\min V_{n-1}$ is added to the object $WS_{n-1}$ and $\#V_{n-1} = 1$. This leads contradiction.

It is clear to see that the algorithm satisfies the termination condition:

Lemma 12 (Termination). The algorithm of Fig. 2 satisfies the termination condition.

Lemma 13 (Nontriviality). The algorithm of Fig. 2 satisfies the nontriviality condition.

Proof. Fix an execution, in which more than one \texttt{propose()} operation are performed and no process fails while performing these \texttt{propose()} operations. Define $V_i$ in the same way as in Lemma 8. It is enough to show that $V_i \neq \emptyset$ for all $i = 0, \ldots, n - 1$. We prove this by induction on $i$.

Basis: Every process that performs a \texttt{propose()} operation adds its argument to \texttt{SET}[0] and thus $V_0 \neq \emptyset$.

Induction step: Suppose that $V_i \neq \emptyset$ holds. If $\#V_i < 2$, every process that executes the $i$-th iteration of the for loop proceeds to the $(i+1)$-th iteration because the condition “$\#\text{Snap} \geq 2$” at Line 5 is never satisfied. If $\#V_i \geq 2$, the process that adds $\max V_i$ to \texttt{SET}[i] never satisfies the condition of the if statement and proceeds to the $(i+1)$-th iteration, because $\max V_i$ cannot be the minimum of any subset of $V_i$ of cardinality more than 1. In either case, $V_{i+1} \neq \emptyset$.

By Lemmas 9, 10, 11, and 12, we obtain the following theorem.

Theorem 3. The algorithm of Fig. 2 is an anonymous wait-free implementation of safe agreement object.

Note that the space complexity of the implementation of the safe agreement object is $n$ atomic registers, because an arbitrary finite number of atomic weak set objects are simulated on top of a single atomic weak set object.
5 \textit{t}-Resilient Solvable Colorless Tasks

We now give a characterization of the \textit{t}-resilient solvability of colorless tasks in the anonymous model.

\textbf{Theorem 4.} A colorless task is \textit{t}-resilient solvable by \textit{n} anonymous processes with atomic weak set objects if and only if it is \textit{t}-resilient solvable by \textit{n} non-anonymous processes with atomic snapshot objects. Moreover, if a colorless task is \textit{t}-resilient solvable by \textit{n} anonymous processes, it is solvable with \textit{n} atomic registers.

The only if part of the theorem is immediate because every anonymous protocol can be executed by non-anonymous processes. We prove the if part by two different approaches, topological one and operational one.

5.1 Topological Approach

We prove the if part of Theorem 4 by a topological argument.

We first show that the \((t+1)\)-set agreement is \textit{t}-resilient solvable by \textit{n} anonymous processes. An algorithm of Fig. 3 presents an anonymous \textit{t}-resilient protocol for the \((t+1)\)-set agreement. In the protocol, each process first proposes its input value to \(SA[i]\) for \(i = 0, \ldots, n - 1\). Then, the process repeatedly performs a resolve operation to all \(SA[i]\) in the round-robin manner until it gets non-\(\perp\) value. Once the process gets non-\(\perp\) value, the process returns the value.

\begin{algorithm}
\caption{Anonymous \textit{t}-resilient \((t+1)\)-set agreement protocol}
\begin{algorithmic}
\State \textbf{Shared variable :}
\State \quad array of safe agreement objects : \(SA[0\ldots t]\)
\State \textbf{Code for a process}
\State \textbf{Local variable:}
\State \quad Integer \(i\) init 0
\State \quad Value \(result\) init \(\perp\)
\Function{setagree(v)}{
\For{\(i = 0, \ldots, t\)}
\State \(SA[i].\text{propose(v)}\)
\State \(i = 0\)
\State \While{\(result = \perp\)}
\State \(result = SA[i].\text{resolve()}\)
\State \(i = i + 1 \mod t + 1\)
\EndWhile
\EndFor
\State \Return{result}
\EndFunction
\end{algorithmic}
\end{algorithm}

\textbf{Theorem 5.} The algorithm of Fig. 3 is a \textit{t}-resilient anonymous protocol for the \((t+1)\)-set agreement.

\textbf{Proof.} \textit{Termination:} In the protocol, each process performs propose operations to \(SA[0], \ldots, SA[n-1]\) sequentially. Thus, even if \(t\) processes fail, there is at least one safe agreement object such that no process fails while performing a propose operation on the object. By the nontriviality property of safe agreement objects, after some time instance, resolve operations on some safe agreement object return non-\(\perp\) value and thus the while loop of Line 4-6 eventually terminates.

\textit{Validity:} Every argument of a propose operation is a proposed value. Because of the validity property of safe agreement objects, a non-\(\perp\) value returned by some resolve operation is one of the arguments of propose operations. Thus, the validity condition holds.

\textit{k-Agreement:} There are \(t+1\) distinct safe agreement objects. Thus, by the agreement property of safe agreement objects, at most \(t+1\) distinct values are decided.

As the \textit{b}-iterated barycentric agreement task is wait-free solvable by anonymous processes [30], the following theorem holds.
Lemma 14. Let \( T = (I, O, \Delta) \) be a colorless task. If there exists a continuous map \( f : \text{skel' } I \rightarrow |O| \) carried by \( \Delta \), \( T \) is \( t \)-resilient solvable by \( n \) anonymous processes.

Proof. By Lemma 1 there is an integer \( b \) and a simplicial map \( \delta : \text{bary } b \text{skel' } I \rightarrow O \) that satisfies \( \delta(\text{bary } b \sigma) \subseteq \Delta(\sigma) \) for every \( \sigma \in \text{skel' } I \).

The following anonymous protocol solves the colorless task. Suppose that the set of all inputs to the processes is \( s \in I \). Execute first the anonymous \((t+1)\)-set agreement protocol, and then the \( b \)-iterated barycentric agreement protocol (for sufficiently large value of \( b \)). Each process chooses a vertex of \( \text{bary } b \text{skel' } \sigma \). Finally, each process determines its output by applying \( \delta \) to the vertex it chose.

The if part of Theorem 4 follows from Lemma 14 and the following theorem by Herlihy and Rajsbaum:

Theorem 6 ([20, Theorem 4.3]). A colorless task \( T = (I, O, \Delta) \) is \( t \)-resilient solvable by \( n \) non-anonymous processes if and only if there exists a continuous map \( f : \text{skel' } I \rightarrow |O| \) carried by \( \Delta \).

Note that the protocol appeared in the proof of Lemma 14 only makes use of a finite number of atomic weak set objects, which are constructed on top of a single atomic weak set object. Thus, every colorless task that is \( t \)-resilient solvable by \( n \) anonymous processes is solved with \( n \) atomic registers. The space complexity lower bound of Theorem 4 follows.

5.2 Simulation-Based Approach

We now prove the if part of Theorem 4 by a simulation, which is an anonymous variant of the BG-simulation [5]. More precisely, we show that \( n \) anonymous \( t \)-resilient processes with atomic weak set objects can simulate \( n \) non-anonymous \( t \)-resilient processes with atomic snapshot objects. We write anonymous simulators by \( p_0, \ldots, p_{n-1} \) and non-anonymous simulated processes by \( P_0, \ldots, P_{n-1} \). Without loss of generality, we may assume that non-anonymous processes communicate via a single \( n \)-ary atomic snapshot object and execute a full-information protocol. In the protocol, the process \( P_i \) repeatedly writes its local state to the \( i \)-th component of the array, takes a snapshot of the whole array and update its state by the result of the snapshot until it reaches a termination state. When the process reaches the termination state, it decides on the value obtained by applying some predefined function \( f \) to the state.

Our simulation algorithm for each simulator is presented in Fig. 4. The algorithm makes use of a two dimensional array of safe agreement object \( \text{SA}[0 \ldots n-1][0 \ldots n-1] \), where the column \( \text{SA}[0 \ldots n-1][i] \) is for storing simulated states of the process \( P_i \). The local variables \( \text{view}_i \) and \( \text{round}_i \) stand for the current simulated state and the current simulated round of \( P_i \), respectively. The function \( \text{latest_views} \) maps a set of tuples consists of a process name, its simulated state, and its simulated round to the array whose \( i \)-th component is the simulated view of \( P_i \) associated with the latest simulated round number of \( P_i \). The function \( \text{latest_rounds} \) maps a set of the same kind to the latest round number of \( P_i \).

In the algorithm, each simulator first proposes its input value to \( \text{SA}[0][i] \) for all \( P_0, \ldots, P_{n-1} \). Then, the simulator repeats the following procedure for \( P_0, \ldots, P_{n-1} \) in the round-robin manner until one of \( P_0, \ldots, P_{n-1} \) reach a termination state: it performs \( \text{resolve}() \) operation on \( \text{SA}[\text{round}_i][i] \); if the return value of the \( \text{resolve}() \) operation is \( \perp \), the simulator adds the return value, with the name \( P_i \) and its current simulated round, to \( \text{SET} \), updates simulated state and round, and proposes the new simulated state of \( P_i \) to \( \text{SA}[\text{round}_i][i] \).

By the use of safe agreement objects, simulators can agree on the return value of each simulated snapshot. Note that there is no need to use a safe agreement object on each simulated update because each value to be updated is deterministically determined by the return value of the preceding simulated snapshot. In the algorithm of Fig. 4, each simulator performs \( \text{propose}() \) operations sequentially. Thus, even though \( t \) simulators crash, they block at most \( t \) simulated processes by the nontriviality property of the safe agreement object. By these observations, we establish the following lemma:

Lemma 15. If a colorless task \( t \)-resilient solvable by \( n \) non-anonymous processes with atomic snapshot objects, it is also \( t \)-resilient solvable by \( n \) anonymous processes with atomic weak set objects.

The proof of the lemma is similar to the proof of Theorem 5 in [5].

The space complexity of the simulation of Fig. 4 is exactly \( n \) atomic registers because a single atomic weak set object can simulate an arbitrary finite number of atomic weak set objects and safe agreement objects in the non-blocking manner. This establishes the space complexity lower bound of Theorem 4.
A characterization of colorless anonymous $t$-resilient task computability

Shared variable:
- atomic weak set: $SET$
- array of safe agreement objects: $SA[0...][0...n-1]$

Code for a process

Local variable:
- Value $view_i$ init ⊥ for $i = 0, \ldots, n-1$
- Integer $round_i$ init 0 for $i = 0, \ldots, n-1$
- Integer $i$ init 0
- Value $snap$ init ⊥

Simulation($v$):
1. for $i = 0, \ldots, n-1$ do
2. $SA[0][i].PROPOSE(v)$
3. while true do
4. for $i = 0, \ldots, n-1$ do
5. $view_i = SA[round_i][i].RESOLVE()$
6. if $view_i$ is a termination state of $P_i$ then
7. return $f(view_i)$
8. elseif $view_i \neq ⊥$ then
9. $SET.ADD((P_i,view_i,round_i))$
10. $snap = SET.GET()$
11. $view_i = latest_views(snap)$
12. $round_i = latest_round_i(snap) + 1$
13. $SA[round_i][i].PROPOSE(view_i)$

Fig. 4. $n$ anonymous processes simulates $n$ non-anonymous processes

6 Conclusion

The theory of distributed computing in [17] has been successful in characterizing task computability in a variety of shared-memory, message passing and mobile robot models using combinatorial topology. For the case of shared-memory models, it assumes that the processes, $p_0, \ldots, p_{n-1}$, communicate using SWMR registers. Recently a characterization of wait-free colorless task solvability has been derived for the anonymous case, where processes have no identifiers and communicate through multi-writer/multi-reader registers [29]. In this paper we have extended this result to the case where at most $t$ processes may crash, $1 \leq t < n$. Furthermore, we have shown that any $t$-resilient solvable colorless task can be $t$-resilient solvable anonymously using only $n$ MWMR registers.

Some of the avenues for future research are the following. It would be interesting to look for lower bound on the number of MWMR registers needed to solve specific colorless tasks. Also, to investigate which non-colorless tasks are solvable in the anonymous setting. We have derived our result through a series of reductions that seem interesting in themselves, to study further anonymous computability, especially for long-lived objects (as opposed to tasks) and uniform solvability (instead of a fixed number of processes $n$). For this, it may be useful to extend our non-blocking implementation of the weak set object to be wait-free. Also, to eliminate our assumption of finite input complexes.

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7 Appendix

7.1 Correctness Proofs of Weak Set Implementation

We give the correctness proofs of the weak free implementation.

Safety Given an operation \( op, \text{invoc}(op) \) denotes its invocation and \( \text{resp}(op) \) its response.

Let \( H \) be an history of the algorithm. \( H_{\text{seq}} \) denotes the sequential history in which each operation of \( H \) appears as if it has been executed at a single point (the linearization) of time line. We have to define linearization points and prove that:

\begin{itemize}
  \item the linearization point of each operation \( \text{GET} \) and \( \text{ADD} \) appear between the beginning and the end of this operation, and
  \item the sequential history that we get with these points respect the sequential specification of the weak set.
\end{itemize}

Most of the details of the proofs are in the appendix.

Consider an history \( H \), let \( v \) be a value or a set of values, define time \( \tau_v \) as the first time, if any, that \( v \) belongs to all registers of \( R \). When there is no such time, \( \tau_v \) is \( \bot \).

**Lemma 1.** (Lemma[2] If the operation \( \text{ADD}(v) \) terminates then before the end of this operation \( v \) belongs to all registers of \( R \). If the operation \( \text{GET}() \) terminates and returns \( V \) then before the end of this operation \( V \) belongs to all registers of \( R \).

**Proof.** If the \( \text{ADD}(v) \) terminates then the condition Line[2] is false. \( v \) belongs to all cells of \( \text{Snap} \). \( \text{Snap} \) comes from the snapshot Line[2] or Line[7] When the process executes this line, \( v \) belongs to all registers of \( R \).

If the \( \text{GET}() \) terminates and returns \( V \) then the condition Line[13] is false. \( V \) belongs to all cells of \( \text{Snap} \). \( \text{Snap} \) comes from the snapshot Line[11] or Line[16] When the process executes this line, \( V \) belongs to all registers of \( R \).

We prove that the algorithm is non-blocking, namely, if processes perform operations forever, an infinite number of operations terminates.

By contradiction, assume that there is only a finite number of operations \( \text{GET} \) and \( \text{ADD} \) and some operations made by correct processes do not terminate.

Operations \( \text{ADD} \) or \( \text{GET} \) may not terminate because the termination conditions of the while loop are not satisfied (Lines[4] or [13]): for an \( \text{ADD}(v) \) operation, in each scan made by the process, \( v \) is not in at least one of the registers of \( R \), and for a \( \text{GET}() \) operation, in each snap, all the registers are not equal to the view of the process.

There is a time \( \tau_0 \) after which there is no new process crash and all processes that terminate \( \text{GET} \) or \( \text{ADD} \) operations in the run have already terminated. Consider the set \( N \) of processes alive after time \( \tau_0 \) that do not terminate operations in the run. Note that after time \( \tau_0 \) only processes in \( N \) take steps and as no process in \( N \) may crash each process in \( N \) makes an infinite number of steps.

From observation[3] if there is a finite number of operations, then all variables \( \text{View} \) are subsets of a finite set of values. Moreover from observation[5] the views of each process are increasing, then there is a time \( \tau_1 > \tau_0 \) after which the view of each process \( p \) in \( N \) converges to a stable view \( \text{SView}_p \): after time \( \tau_1 \) forever the view of \( p \) is \( \text{SVview}_p \). In the following \( \text{SV} \) denotes \( \{\text{SVview}_p | p \in N\} \) the set of all stable views for processes in \( N \). Observe that:

**Observation 1** (Observation[1] If \( p \) does not terminate an \( \text{ADD}(v) \) then \( v \in \text{SVview}_p \).

After time \( \tau_1 \) processes only update \( R \) with their stables views \( \text{SVview} \), and as each process in \( N \) updates infinitely often each register of \( R \) with its \( \text{SVview} \) there is a time \( \tau_2 \) after which all registers in \( R \) contain only stable views of processes in \( N \):

**Observation 2** (Observation[2] After time \( \tau_2 \) for all \( i \), \( R[i] \in \text{SV} \).
Lemma 2. Let \( \text{op} \) be an operation of \( H \). If \( \text{op} \) is defined then \( \tau_{op} \) belongs to \( \{ \text{invoc}(\text{op}), \text{resp}(\text{op}) \} \). If \( \text{op} \) is undefined then \( \tau_{op} \) does not terminate and is not linearized.

In the following we consider the next snapshot operation on \( R \) of each process. This next snapshot operation is either a scan or an update or there is no next snapshot operation (when the process is about to satisfy the termination loop condition: \( \text{card}(r|v \in \text{Snap}[r]) \geq n \) for an \( \text{add} \) and \( \text{card}(r|\text{View} = \text{Snap}[r]) \geq n \) for an \( \text{update} \)).

Consider any time \( \tau \). Define \( r_v(\tau), w_v(\tau), c_v(\tau) \) and \( \alpha_v(\tau) \):

- \( r_v(\tau) \) is the number of processes for which, after time \( \tau \), the next snapshot operation is a scan.
- \( w_v(\tau) \) is the number of processes such that (1) \( v \in \text{View} \) at time \( \tau \) and for which after time \( \tau \) the next snapshot operation is an update, or (2) there is no next snapshot operation for that process (the process has finished -or is going to finish- its main loop or it takes no more steps).
- \( c_v(\tau) \) is the number of registers that contains \( v \) at time \( \tau \).
- \( \alpha_v(\tau) \) is defined: \( \alpha_v(\tau) = r_v(\tau) + w_v(\tau) + c_v(\tau) \)

We make first some easy observations.

For a process the next snapshot operation is either a scan or an update or nothing:

Observation 3 \( r_v(\tau) + w_v(\tau) \leq n \).

Observation 4 All values in variables \( \text{View} \) have been proposed by some \( \text{get} \)

Observation 5 Considering the inclusion \( \subseteq \), for each process, variable \( \text{View} \) is not decreasing

Observation 6 Each update made by a process is followed for this process by a scan or the process stops to take step.

Observation 7 \( \alpha_v(\tau) \) may only be modified by a scan (Lines 2, 7, 11 and 16) or an update (Lines 5 and 14).

Due to this observation in the following we consider only steps of processes that are scan or update.
Lemma 3. (Lemma 4) Assume $\alpha_v(\tau) > n$ and the next step in $H$, is made at time $\tau' \geq \tau$, then we have $\alpha_v(\tau') \leq \alpha_v(\tau)$.

Proof. Consider $H$ an history and some time $\tau$ and assume that at time $\tau$, we have $\alpha_v(\tau) > n$. Consider the next snapshot operation in $H$ in algorithm of Fig. 1. Let $p$ the process that executes this operation.

- The next step is a scan (Lines 2, 7, 11 and 16), then by Observation 3 and the hypothesis $\alpha_v(\tau) > n$, we have

$\alpha_v(\tau') = r_v(\tau) - 1 + w_v(\tau) + c_v(\tau) = \alpha_v(\tau)$

In the last case $w_v(\tau') = w_v(\tau)$, and $r_v(\tau') = r_v(\tau)$. Hence at time $\tau'$ we have $c_v(\tau') = c_v(\tau)$. Then

$\alpha_v(\tau') = r_v(\tau) + w_v(\tau) + c_v(\tau) = \alpha_v(\tau)$

- The next step is an update (Lines 5 and 14). By Observation 6, either (1) the next snapshot operation of $p$ is a scan then $r_v(\tau') = r_v(\tau) + 1$, or (2) there is no next snapshot operation for $p$ then $r_v(\tau') = r_v(\tau)$. Consider the first case, and the two following subcases:

  - That update is an update($-, V$), with $v \in V$ or $v \subseteq V$, then $w_v(\tau') = w_v(\tau) - 1$, as the update write $V$ in $R$, $c_v(\tau') \geq c_v(\tau)$ and by Observation 6 $r_v(\tau') = r_v(\tau) + 1$. Hence:

    $\alpha_v(\tau') \geq r_v(\tau) + 1 + w_v(\tau) - 1 + c_v(\tau) = \alpha_v(\tau)$

  - That update does not contain $v$, then $w_v(\tau') = w_v(\tau)$, the update may erase at most one element of $R$ then $c_v(\tau') \geq c_v(\tau) - 1$ and by Observation 6 $r_v(\tau') = r_v(\tau) + 1$. Hence:

    $\alpha_v(\tau') \geq r_v(\tau) + 1 + w_v(\tau) + c_v(\tau) - 1 = \alpha_v(\tau)$

Consider the second case and the two following subcases:

  - That update is an update($-, V$), with $v \in V$ or $v \subseteq V$, then $w_v(\tau') = w_v(\tau)$. As the update write $V$ in $R$, $c_v(\tau') \geq c_v(\tau)$.

  - That update does not contain $v$, then $w_v(\tau') = w_v(\tau) + 1$, the update may erase at most one element of $R$ then $c_v(\tau') \geq c_v(\tau) - 1$. Hence:

    $\alpha_v(\tau') \geq r_v(\tau) + w_v(\tau) + c_v(\tau) - 1 = \alpha_v(\tau)$

By an easy induction on the steps of $H$ we get:

Lemma 4. (Lemma 5) If $\alpha_v(\tau) > n$ then for all $\tau'$, such that $\tau \leq \tau'$, $\alpha_v(\tau) \leq \alpha_v(\tau')$.

Lemma 5. (Lemma 6) If $\tau_v \neq \bot$ then for all $\tau$, such that $\tau_v \leq \tau$, $n < \alpha_v(\tau_v) \leq \alpha_v(\tau)$.

Proof. At time $\tau_v$, $v$ is in all registers then $c_v(\tau_v) = n$ and $\alpha_v(\tau) \geq n$. Consider the update made just before time $\tau_v$, and the process that made this update, the next snapshot operation for this process is a scan or the process terminates with $v$ in its view, we have $r_v(\tau) + w_v(\tau) \geq 1$ and then just after time $\tau_v$, $\alpha_v(\tau) > n$. Hence by Lemma 4 and an easy induction we deduce the Lemma.

Lemma 6. If $\tau_v \neq \bot$, for all $\tau \geq \tau_v$, $c_v(\tau) \geq 1$

Proof. From Lemma 5 $\alpha_v(\tau) = r_v(\tau) + w_v(\tau) + c_v(\tau) > n$ and by observation 3 that $r_v(\tau) + w_v(\tau) \leq n$. Hence $c_v(\tau) \geq 1$.

Lemma 7. (Lemma 7) $H_{seq}$ satisfies the sequential specification of the weak set.
Proof. Let \( op \) be a \text{get} operation, consider the set \( A \) of all \text{add} operations linearized before (such that \( \text{add}(x) \in A \) if \( \tau_{\text{add}(x)} \leq \tau_{op} \)). If \text{get} does not terminate, there is no linearization. If \text{get} operation \( op \) terminates it returns with, say, view \( V \).

Consider \( x \in V \). By observation \[4] then at least one process invoked an \text{add}(x).

Due to the termination loop condition of Line \[13\] of Algorithm of Fig. \[1\] at the time \( \tau \) of the last \text{scan} made by any process returning \( V \), if \( x \in V \), then \( x \) belongs to all registers proving that \( \tau_x < \tau \). Then if \( x \in V \), then there exists \text{add}(x) in \( A \).

Assume that there is a \( v \) such that \( \tau_v < \tau_{op} \), then by Lemma \[6\] after time \( \tau_v \), \( v \) belongs forever to at least one register in \( R \). Then all \text{scan} made after time \( \tau_v \) contains \( v \). Due to the termination loop condition Line \[13\] if any \text{scan} of a \text{get} contains \( v \), \( v \) is in the view returned by that \text{get}. Then if \( op \) performs a \text{scan} after time \( \tau_v \), the view returned contains \( v \). Then if \text{add}(v) \in A \) then \( v \in V \).

Hence we have \( V = \{x|\text{add}(x) \in A \} \).

### 7.2 \( b \)-Iterated barycentric agreement protocol

To make the present paper self-contained, we present, in Fig. \[5\] the \( b \)-iterated barycentric agreement protocol, which is a verbatim copy of one appeared in \[30\].

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**Shared variable:**

- array of atomic weak set objects: \( \text{SET}[0 \ldots b - 1] \)

**Code for a process**

**Local variable:**

- Integer \( i \) init 0
- Value \( \text{view} \) init \( \perp \)

**BARYAGREE\(_b\)(v):**

1. \( \text{view} = v \)
2. \( \text{for } i = 0, \ldots, b - 1 \text{ do} \)
3. \( \text{SET}[i].\text{ADD}(\text{view}) \)
4. \( \text{view} = \text{SET}[i].\text{GET}() \)
5. \( \text{return } \text{view} \)

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**Fig. 5.** Anonymous \( b \)-iterated barycentric agreement protocol

In the protocol, each process starts with its private input value and assigns it to the local variable \( \text{view} \) (line 1). Thereafter, the process iterates from \( i = 0 \) to \( b \), the operation of adding its view to \( \text{SET}[i] \) and updating its view by the result of a get operation to \( \text{SET}[i] \) (line 2–4). At last, the process outputs the value held in its local variable \( \text{view} \) (Line 5).