Explicit formulas for the Vassiliev knot invariants.

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1 Knots and singular knots.

Let $K : S^1 \rightarrow \mathbb{R}^3$ be an oriented knot and $K_{n}^{\text{sing}} : S^1 \rightarrow \mathbb{R}^3$ be a singular knot with $n$ double points.

Denote by $K$ the space of knots and by $K_{n}^{\text{sing}}$ the space of singular knots.

2 The Vassiliev invariants.

Let $V : K \rightarrow \mathbb{Q}$ be a knot invariant. It may be extended from ordinary knots to singular knots. Define $i$-th derivative of invariant $V$ $V^{(i)} : K_{n}^{\text{sing}} \rightarrow \mathbb{Q}$ inductively via the Vassiliev skein-relation:

$$V^{(0)} = V,$$

$$V^{(i)}(\begin{array}{c} \ast \\ \ast \end{array}) = V^{(i-1)}(\begin{array}{c} \ast \\ \ast \end{array}) - V^{(i-1)}(\begin{array}{c} \ast \\ \ast \end{array}).$$

**Definition.** A knot invariant $V : K \rightarrow \mathbb{Q}$ is said to be Vassiliev invariant of order less than or equal to $n$, if it's $(n+1)$-st derivative vanishes identically:

$$V^{(n+1)} \equiv 0.$$

We will denote Vassiliev invariant of order less than or equal to $n$ by $V_{n}$ and the space of Vassiliev invariants over $\mathbb{Q}$ by $\mathcal{V}_{n}$. The spaces of Vassiliev invariants have following dimensions:

| $n$ | 0 | 1 | 2 | 3 | 4 |
|-----|---|---|---|---|---|
| dim $\mathcal{V}_{n}$ | 1 | 1 | 2 | 3 | 6 |
3 Chord diagrams.

A chord diagram $D_n$ of order $n$ is a circle with a distinguished set of $n$ unordered pairs of points connected by chords regarded up to orientation preserving diffeomorphisms of the circle. Denote by $D_n$ the space generated by chord diagrams of order $n$ over $\mathbb{Q}$. For example, there exist only two non-equivalent chord diagrams of order 2: $D_2 = \{ \begin{array}{c} \bigcirc \\ \bigcirc \end{array} \}$ and space $D_3$ contains 5 diagrams: $D_3 = \{ \begin{array}{c} \bigcirc \\ \bigcirc \end{array} , \begin{array}{c} \bigcirc \\ \bigcirc \end{array} , \begin{array}{c} \bigcirc \\ \bigcirc \end{array} , \begin{array}{c} \bigcirc \\ \bigcirc \end{array} \}$. For any singular knot $K^{\text{sing}}$ we construct it’s chord diagram. A chord diagram of the singular knot is the circle with pre-images of double points connected with chords.

4 Weight systems.

**Definition.** A linear function $W : D_n \to \mathbb{Q}$ is called a weight system of order $n$ if it satisfies next relations:

1-term relation: $W_n( \begin{array}{c} \bigcirc \\ \bigcirc \end{array} ) = 0$

(this diagram has $n$ chords, one of which is isolated) and

4-term relation: $W_n( \begin{array}{c} \bigcirc \\ \bigcirc \end{array} ) - W_n( \begin{array}{c} \bigcirc \\ \bigcirc \end{array} ) + W_n( \begin{array}{c} \bigcirc \\ \bigcirc \end{array} ) - W_n( \begin{array}{c} \bigcirc \\ \bigcirc \end{array} ) = 0$

these diagrams have $n$ chords, $(n - 2)$ of which are not drown here and 2 chords are positioned as shown.

Denote by $W_n$ the space of weight systems of order $n$ over $\mathbb{Q}$. The spaces $W_n$ have following dimensions:

| $n$ | 0 | 1 | 2 | 3 | 4 |
|-----|---|---|---|---|---|
| dim $W_n$ | 1 | 0 | 1 | 3 | 3 |

Examples of weight systems.

$W_2( \begin{array}{c} \bigcirc \\ \bigcirc \end{array} ) = 1$

$W_2( \begin{array}{c} \bigcirc \\ \bigcirc \end{array} ) = 0$

$W_3( \begin{array}{c} \bigcirc \\ \bigcirc \end{array} ) = 2$

$W_3( \begin{array}{c} \bigcirc \\ \bigcirc \end{array} ) = 1$

$W_3 = 0$ in other cases.

Elsewhere $W_1^4 = W_2^4 = W_3^4 = 0$.

5 Symbol of Vassiliev invariant.

**Definition.** The symbol of Vassiliev invariant of order $n$ is the restriction of this invariant to singular knots with exactly $n$ double points:

$\nabla^n V_n = V_n|_{K^{\text{sing}}}.$

The symbol of Vassiliev invariant of order $n$ $\nabla^n V_n$ depends only on chords diagrams of singular knots. So the symbol is a linear function on the space $D_n$:

$\nabla^n V_n : D_n \to \mathbb{Q}.$

One can prove that the symbol of Vassiliev invariant $\nabla^n V_n$ satisfies 1-term and 4-term relations.

**Kontsevich theorem.** Over $\mathbb{Q}$ each weight system $W_n$ is the symbol of an appropriate Vassiliev invariant $V_n$ and there exist next exact sequence

$0 \to V_{n-1} \to V_n \to W_n \to 0,$

$V_n/V_{n-1} \simeq W_n.$

2
From the Kontsevich theorem follows that we have
– one basic Vassiliev invariant of order 2,
– one basic Vassiliev invariant of order 3,
– three basic Vassiliev invariants of order 4.
And it is easy fact that Vassiliev knot invariants of order 0 and 1 are the constant maps.

6 Diagrams of knots.

Let $K$ be a knot with marked base-point $a$ and $x$ be double point of it’s planar projection.

Numerate branches in neighbourhood of $x$ according to the order of their passing. Define function $\delta_x$ by next rule:

$$
\begin{align*}
\delta_x &= 0 \\
\delta_x &= 1
\end{align*}
$$

(the orientations of branches are not important).

Define function $\varepsilon_x$ as follows:

$$
\begin{align*}
\varepsilon_x &= +1 \\
\varepsilon_x &= -1
\end{align*}
$$

As above, a chord diagram of the singular knot is the circle with pre-images of double points connected with chords.

To obtain the analogous diagram of an ordinary knot (that is called an arrow diagram) from the chord diagram of corresponding singular knot we must give the information on overpasses and underpasses. Each chord is oriented from the upper branch to the lower one and equipped with the sign (the local writh number of corresponding double point of planar projection of the knot).

7 Formulas for Vassiliev invariants of orders 2, 3 and 4.
7.1 Formulas of Lannes.

We present formulas of Lannes in new form.

\[ V_2(K) = \frac{1}{2} \sum_{\{x,y\} \in P_2} (-1)^{\delta_x + \delta_y} W_2(\{x,y\}) \varepsilon_x \varepsilon_y [\delta_x (1 - \delta_y) + \delta_y (1 - \delta_x)], \]

\[ V_3(K) = \frac{1}{2} \sum_{\{x,y,z\} \in P_3} (-1)^{\delta_x + \delta_y + \delta_z} W_3(\{x,y,z\}) \varepsilon_x \varepsilon_y \varepsilon_z [\delta_y (1 - \delta_x) (1 - \delta_z) - \delta_x \delta_z (1 - \delta_y)], \]

where the sum is taken over all unordered pairs (triplets) of double points of planar projection, \( W_2(\{x,y\}) \) (\( W_3(\{x,y,z\}) \)) is weight of chord diagram corresponding to pair (triplet) of double points.

7.2 Formulas of Viro-Polyak.

Denote by \( \langle A, G \rangle \) algebraic number of subdiagrams of given combinatorial type \( A \subset G \) and let \( \sum_i n_i A_i, G > = \sum_i n_i < A_i, G >, \ n_i \in \mathbb{Q} \) by definition. Then

\[
\begin{align*}
V_2(K) &= \langle \quad , G >, \\
V_3(K) &= \langle [ \quad ] + \frac{1}{2} [ \quad ], G >, \\
V_4(K) &= \langle \begin{array}{c}
\begin{array}{c}
\text{ mostly diagrams}
\end{array}
\end{array} + 2 \begin{array}{c}
\begin{array}{c}
\text{ mostly diagrams}
\end{array}
\end{array} + 6 \begin{array}{c}
\begin{array}{c}
\text{ mostly diagrams}
\end{array}
\end{array} + 2 \begin{array}{c}
\begin{array}{c}
\text{ mostly diagrams}
\end{array}
\end{array} - \begin{array}{c}
\begin{array}{c}
\text{ mostly diagrams}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\text{ mostly diagrams}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\text{ mostly diagrams}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\text{ mostly diagrams}
\end{array}
\end{array} + 2 \begin{array}{c}
\begin{array}{c}
\text{ mostly diagrams}
\end{array}
\end{array} + \\
\begin{array}{c}
\begin{array}{c}
\text{ mostly diagrams}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\text{ mostly diagrams}
\end{array}
\end{array} + 2 \begin{array}{c}
\begin{array}{c}
\text{ mostly diagrams}
\end{array}
\end{array} + 3 \begin{array}{c}
\begin{array}{c}
\text{ mostly diagrams}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\text{ mostly diagrams}
\end{array}
\end{array}, G >
\end{array}
\end{align*}
\]

7.3 New formula.

\[
\begin{align*}
V_4(K) &= \frac{1}{2} W_4( \begin{array}{c}
\begin{array}{c}
\text{ mostly diagrams}
\end{array}
\end{array} ) < \begin{array}{c}
\begin{array}{c}
\text{ mostly diagrams}
\end{array}
\end{array} , G > + \frac{1}{2} W_4( \begin{array}{c}
\begin{array}{c}
\text{ mostly diagrams}
\end{array}
\end{array} ) < \begin{array}{c}
\begin{array}{c}
\text{ mostly diagrams}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\text{ mostly diagrams}
\end{array}
\end{array} , G > + \\
\frac{1}{2} W_4( \begin{array}{c}
\begin{array}{c}
\text{ mostly diagrams}
\end{array}
\end{array} ) < \begin{array}{c}
\begin{array}{c}
\text{ mostly diagrams}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\text{ mostly diagrams}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\text{ mostly diagrams}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\text{ mostly diagrams}
\end{array}
\end{array} , G > + \frac{1}{2} W_4( \begin{array}{c}
\begin{array}{c}
\text{ mostly diagrams}
\end{array}
\end{array} ) < \begin{array}{c}
\begin{array}{c}
\text{ mostly diagrams}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\text{ mostly diagrams}
\end{array}
\end{array} , G >
\end{align*}
\]

for any weight system \( W_4 \) such that \( W_4(\begin{array}{c}
\begin{array}{c}
\text{ mostly diagrams}
\end{array}
\end{array} ) = 0 \)

References

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