Study on Pata E-contractions

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Abstract

In this paper, we introduce the notion of an $\alpha-\tilde{\zeta}-E$-Pata contraction that combines well-known concepts, such as the Pata contraction, the $E$-contraction and the simulation function. Existence and uniqueness of a fixed point of such mappings are investigated in the setting of a complete metric space. An example is stated to indicate the validity of the observed result. At the end, we give an application on the solution of nonlinear fractional differential equations.

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1 Introduction

In 2015, Khojasteh et al. [1] initiated the concept of simulation functions.

Definition 1.1 ([1]) A mapping $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ is called a simulation function if the following conditions hold:

$(\zeta_1)$ $\zeta(x, y) < y - x$ for all $x, y > 0$;

$(\zeta_2)$ if $(x_n), (y_n)$ are sequences in $(0, \infty)$ such that $\lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n > 0$, then

$$\limsup_{n \to \infty} \zeta(x_n, y_n) < 0.$$  \hfill (1.1)

We denote by $\mathcal{Z}$ the family of all above simulation functions.

Let $(X, d)$ be a metric space and $\alpha : X \times X \to [0, \infty)$ be a function. A mapping $h : X \to X$ is called $\alpha$-orbital admissible if the following condition holds:

$$\alpha(v, h^2v) \geq 1 \implies \alpha(v, hv) \geq 1,$$  \hfill (1.2)

for all $v \in X$. Moreover, an $\alpha$-orbital admissible mapping is called triangular $\alpha$-orbital admissible if for all $v, \omega \in X$, we have

$$\alpha(v, \omega) \geq 1 \text{ and } \alpha(\omega, \kappa \omega) \geq 1 \implies \alpha(v, \kappa \omega) \geq 1.$$  \hfill (1.3)
Definition 1.2 A set $X$ is said to be regular with respect to a given function $\alpha : X \times X \rightarrow [0, \infty)$ if for each sequence $\{v_n\}$ in $X$ such that $\alpha(v_n, v_{n+1}) \geq 1$ for all $n$ and $v_n \rightarrow v \in X$ as $n \rightarrow \infty$, then $\alpha(v_n, v) \geq 1$ for all $n$.

The notion of $\alpha$-admissible $\mathcal{Z}$-contractions with respect to a given simulation function was merged and used by Karapinar in [2]. Using this new type of contractive mappings, he investigated the existence and uniqueness of a fixed point in standard metric spaces.

Definition 1.3 ([2]) Let $T$ be a self-mapping defined on a metric space $(X, d)$. If there exist a function $\zeta \in \mathcal{Z}$ and $\alpha : X \times X \rightarrow [0, \infty)$ such that

$$\zeta(\alpha(v, \omega)d(Tv, T\omega), d(v, \omega)) \geq 0 \quad \text{for all} \quad v, \omega \in X,$$

then we say that $T$ is an $\alpha$-admissible $\mathcal{Z}$-contraction with respect to $\zeta$.

Theorem 1.4 ([2]) Let $(X, d)$ be a complete metric space and let $T : X \rightarrow X$ be an $\alpha$-admissible $\mathcal{Z}$-contraction with respect to $\zeta$. Suppose that:

(a) $T$ is triangular $\alpha$-orbital admissible;
(b) there exists $v_0 \in X$ such that $\alpha(v_0, Tv_0) \geq 1$;
(c) $T$ is continuous.

Then there is $v_* \in X$ such that $Tv_* = v_*$. 

Remark 1.5 The continuity condition in Theorem 1.4 can be replaced by the “regularity” condition, which is considered in Definition 1.2.

We will consider the following set of functions:

$$\mathcal{Z} = \{\psi : [0, 1] \rightarrow [0, \infty) \mid \psi \text{ is continuous at zero with } \psi(0) = 0\}$$

and we denote

$$\|v\| = d(v, v_0), \quad \text{for an arbitrary but fixed } v_0 \in X.$$

Several interesting extensions and generalizations of the Banach contraction principle [3] appeared in the literature. For instance, see [4–10]. Among these generalizations, we cite the paper of Pata [11]. Since then, much work appeared in the same direction; see [12–15].

Theorem 1.6 ([11]) Let $(X, d)$ be a complete metric space and let $\Lambda \geq 0, \lambda \geq 1, \beta \in [0, \lambda]$ be fixed constants. The mapping $h : X \rightarrow X$ has a fixed point in $X$ if the inequality

$$d(hv, h\omega) \leq (1 - \varepsilon)d(v, \omega) + \Lambda(\varepsilon)\psi(\varepsilon)[1 + \|v\| + \|\omega\|]^\beta,$$

is satisfied for every $\varepsilon \in [0, 1]$ and $\psi \in \mathcal{Z}$.

Definition 1.7 Let $(X, d)$ be a metric space. We say that $h : X \rightarrow X$ is a Pata type Zamfirescu mapping if for all $v, \omega \in X$, $\psi \in \mathcal{Z}$ and for every $\varepsilon \in [0, 1]$, $h$, it satisfies the following inequality:

$$d(hv, h\omega) \leq (1 - \varepsilon)M(v, \omega) + \Lambda(\varepsilon)\psi(\varepsilon)[1 + \|v\| + \|\omega\| + \|hv\| + \|h\omega\|]^\beta,$$
where
\[ M(\nu, \omega) = \max \left\{ \frac{d(\nu, \omega)}{2}, \frac{d(\nu, \omega) + d(\omega, \nu) + d(\nu, h\omega) + d(\omega, h\nu)}{2} \right\} \]

and \( \Lambda \geq 0, \lambda \geq 1 \) and \( \beta \in [0, \lambda] \) are constants.

**Theorem 1.8** ([16]) Let \( (\mathcal{X}, d) \) be a complete metric space and let \( h : \mathcal{X} \to \mathcal{X} \) be a Pata type Zamfirescu mapping. Then \( h \) has a unique fixed point in \( \mathcal{X} \).

We state the following useful known lemma.

**Lemma 1.9** Let \( (\mathcal{X}, d) \) be a complete metric space and \( \{u_n\} \) be a sequence in \( \mathcal{X} \) such that
\[ \lim_{n \to \infty} d(u_{n+1}, u_{n+1}) = 0. \]
If the sequence \( \{u_n\} \) is not Cauchy, then there exist \( e > 0 \) and subsequences \( \{u_{n_l}\} \) and \( \{u_{m_l}\} \) of \( \{u_n\} \) such that
\[ \lim_{n \to \infty} d(u_{n_l+1}, u_{m_l+1}) = e \]
and
\[ \lim_{n \to \infty} d(u_{n_l}, u_{m_l}) = \lim_{n \to \infty} d(u_{n_l+1}, u_{m_l}) = \lim_{n \to \infty} d(u_{n_l}, u_{m_l+1}) = e. \]

In this paper, we combine the concepts of simulation functions and \( \alpha \)-admissibility to give a generalized Pata type fixed point result. At the end, we present an application on fractional calculus.

### 2 Main results

We denote by \( \tilde{\mathcal{Z}} \) the set of all functions \( \tilde{\zeta} : [0, \infty) \times [0, \infty) \to \mathbb{R} \) satisfying the following condition:
\[ (\tilde{\zeta}_1) \quad \tilde{\zeta}(x, y) \leq y - x \text{ for all } x, y > 0. \]

**Definition 2.1** Let \( (\mathcal{X}, d) \) be a metric space and \( \phi \in \Phi \). Let \( \Lambda \geq 0, \lambda \geq 1 \) and \( \beta \in [0, \lambda] \) be fixed constants. A triangular \( \alpha \)-orbital admissible mapping \( h : \mathcal{X} \to \mathcal{X} \) is called an \( \alpha \)-\( \tilde{\zeta} \)-\( \mathcal{E} \)-Pata contraction if there exists a function \( \tilde{\zeta} \in \tilde{\mathcal{Z}} \) such that, for every \( \epsilon \in [0, 1] \), the following condition is satisfied:
\[ \tilde{\zeta}(\alpha(x, y)d(hx, hy), (1 - \epsilon)\mathcal{E}(x, y) + S(x, y)) \geq 0 \]
for all \( x, y \in \mathcal{X} \), where
\[ \mathcal{E}(x, y) = \max \left\{ \frac{d(x, y) + |d(x, hy) - d(\omega, h\omega)|}{2}, \frac{d(x, hy) + d(\alpha(x, y)d(hx, hy), (1 - \epsilon)\mathcal{E}(x, y) + S(x, y))}{2} \right\} \]
and
\[ S(x, y) = \Lambda \epsilon^2 \psi(\epsilon) \left[ 1 + \|x\| + \|y\| + \|h\omega\| \right]^\beta. \]
Remark 2.2. It is clear that any Pata type Zamfirescu mapping is also an $\alpha-\xi-E$-Pata mapping. Indeed, letting $\alpha(v, \omega) = 1$ and $\xi(x, y) = y - x$, the inequality (2.1) becomes

$$d(hv, hw) \leq (1 - \varepsilon)E(v, \omega) + S(v, \omega)$$

$$= (1 - \varepsilon)E(v, \omega) + \Lambda \varepsilon \psi(\varepsilon)[1 + \|v\| + \|\omega\| + \|hv\| + \|hw\|]^\beta.$$

Moreover, note that $M(v, \omega) \leq E(v, \omega)$ for all $v, \omega \in X$.

Theorem 2.3 Every $\alpha-\xi-E$-Pata contraction $h$ on a complete metric space $(X, d)$ possesses a fixed point if

(i) there exists $u_0 \in X$ such that $\alpha(u_0, hu_0) \geq 1$;
(ii) $h$ is triangular $\alpha$-orbital admissible;
(iii) either $h$ is continuous, or the set $X$ is regular.

If in addition we assume that the following condition is satisfied:

(iv) $\alpha(z^*, v^*) \geq 1$ for all $z^*, v^* \in \text{Fix}_X(h)$,

then such a fixed point of $h$ is unique.

Proof Let $u_0 \in X$ be a point such that $\alpha(u_0, hu_0) \geq 1$. On account of the assumption that $h$ is a triangular $\alpha$-orbital admissible mapping, we derive that

$$\alpha(u_0, hu_0) \geq 1 \Rightarrow \alpha(hu_0, h^2 u_0) \geq 1,$$

and iteratively we find

$$\alpha(h^n u_0, h^{n+1} u_0) \geq 1 \quad \text{for every } n \in \mathbb{N}. \quad (2.4)$$

Moreover, by (2.4) together with (1.3), we have

$$\alpha(u_0, hu_0) \geq 1 \quad \text{and} \quad \alpha(hu_0, h^2 u_0) \geq 1 \Rightarrow \alpha(u_0, h^2 u_0) \geq 1.$$

Again, iteratively, one writes

$$\alpha(u_0, h^n u_0) \geq 1 \quad \text{for every } n \in \mathbb{N}. \quad (2.5)$$

Starting from this point $u_0 \in X$, we build an iterative sequence $\{u_n\}$ where $u_n = hu_{n-1} = h^n u_0$ for $n = 1, 2, 3, \ldots$. We can presume that any two consequent terms of this sequence are distinct. Indeed, if, on the contrary, there exists $i_0 \in \mathbb{N}$ such that

$$u_{i_0} = u_{i_0 + 1} = hu_{i_0},$$

then $u_{i_0}$ is a fixed point. To avoid this, we will assume in the following that for all $n \in \mathbb{N}$

$$u_n \neq u_{n+1} \quad \Leftrightarrow \quad d(hu_{n-1}, hu_n) = d(u_n, u_{n+1}) > 0.$$

We mention that (2.4) can be rewritten as

$$\alpha(u_n, u_{n+1}) \geq 1,$$  \quad (2.6)
respectively,

\[ \alpha(u_0, u_0) \geq 1, \quad (2.7) \]

for any \( n \in \mathbb{N} \). In the sequel, we will denote \( d(v, u_0) = \|v\| \) for all \( v \in X \).

Since \( h \) is an \( \alpha-\xi-E \)-Pata contraction, we have

\[ \tilde{\xi}(\alpha(u_{n-1}, u_0) d(hu_{n-1}, hu_0), (1 - \varepsilon)E(u_{n-1}, u_0) + S(u_{n-1}, u_0)) \geq 0. \]

Thus, taking into account \( \tilde{\xi} \), together with (2.6) we get

\[ d(u_n, u_{n+1}) = d(hu_{n-1}, hu_0) \]

\[ \leq \alpha(u_{n-1}, u_0) d(hu_{n-1}, hu_0) \]

\[ \leq (1 - \varepsilon)E(u_{n-1}, u_0) + S(u_{n-1}, u_0), \quad (2.8) \]

where

\[
E(u_{n-1}, u_n) = \max \left\{ \frac{d(u_{n-1}, u_n) + \{d(u_{n-1}, hu_{n-1}) - d(u_0, hu_0)\}}{2} \right\} = \max \left\{ \frac{d(u_{n-1}, u_n) + \{d(u_{n-1}, u_0) - d(u_0, u_{n+1})\}}{2} \right\} = \max \left\{ \frac{d(u_{n-1}, u_n) + \{d(u_{n-1}, u_0) - d(u_0, u_{n+1})\}}{2} \right\} = \max \left\{ \frac{d(u_{n-1}, u_n) + \{d(u_{n-1}, u_0) - d(u_0, u_{n+1})\}}{2} \right\}
\]

and

\[
S(u_{n-1}, u_n) = \Lambda e^\beta \psi(\varepsilon) \left[ 1 + \|u_{n-1}\| + \|u_n\| + \|hu_{n-1}\| + \|hu_n\| \right]^\beta
\]

\[ = \Lambda e^\beta \psi(\varepsilon) \left[ 1 + \|u_{n-1}\| + \|u_n\| + \|u_{n+1}\| \right]^\beta
\]

Denoting by \( \gamma_n = d(u_{n-1}, u_n) \), we have

\[ E(u_{n-1}, u_n) \leq \max \left\{ \gamma_n + |\gamma_n - \gamma_{n+1}|, \frac{\gamma_n + \gamma_{n+1} + |\gamma_n - \gamma_{n+1}|}{2} \right\}. \]

Thus, (2.8) becomes

\[ \gamma_{n+1} \leq (1 - \varepsilon) \max \left\{ \gamma_n + |\gamma_n - \gamma_{n+1}|, \frac{\gamma_n + \gamma_{n+1} + |\gamma_n - \gamma_{n+1}|}{2} \right\} + \Lambda e^\beta \psi(\varepsilon) \left[ 1 + \|u_{n-1}\| + 2\|u_n\| + \|u_{n+1}\| \right]^\beta. \quad (2.9) \]
We claim that the sequence \( \{ \gamma_n \} \) is non-increasing. Indeed, if we suppose the contrary that, for some \( p, \gamma_p < \gamma_{p+1} \), and so \( \max \{ \gamma_p, \gamma_{p+1} \} = \gamma_{p+1} \), then we have \( |\gamma_p - \gamma_{p+1}| = \gamma_{p+1} - \gamma_p \).

\[
\mathcal{E}(u_n, u_{n+1}) \leq \gamma_n+1.
\] (2.10)

Consequently, from (2.9), we get, for such an integer \( p \),

\[
\gamma_{p+1} \leq (1 - \varepsilon) \gamma_p + \varepsilon \left( 1 + \| u_p \| + 2 \| u_p \| + \| u_{p+1} \| \right) \delta.
\] (2.11)

The above inequality is true for all \( \varepsilon \in [0, 1] \). In particular, for \( \varepsilon = 0 \), we get \( \gamma_{p+1} \leq \gamma_p \), which clearly is a contradiction. In this case, we find that the sequence \( \{ \gamma_n \} \) is non-increasing. So we can find a non-negative real number \( \gamma \) such that

\[
\lim \limits_{n \to \infty} d(u_{n-1}, u_n) = \lim \limits_{n \to \infty} \gamma_n = \gamma.
\]

We claim that \( \gamma = 0 \). In order to prove this, we have to show that the sequence \( \{ \kappa_n \} \) is bounded, where \( \kappa_n = \| u_n \| = d(u_n, u_0) \). Since the sequence \( \{ d(u_n, u_{n+1}) \} \) is non-increasing, we have

\[
d(u_n, u_{n+1}) = \gamma_n \leq \kappa_n = d(u_1, u_0).
\]

By the triangle inequality, we get

\[
\kappa_n = d(u_n, u_0) \leq d(u_n, u_{n+1}) + d(u_{n+1}, u_1) + d(u_1, u_0)
\]

\[
= d(u_n, u_{n+1}) + d(hu_n, hu_0) + \kappa_1 \leq d(hu_n, hu_0) + 2\kappa_1.
\] (2.12)

On account of (2.5), regarding that \( h \) is an \( \alpha\beta - \xi \)-Pata-\( \mathcal{E} \) contraction, we have

\[
0 \leq \xi \left( \alpha(u_0, u_n) d(hu_0, hu_n), (1 - \varepsilon) \mathcal{E}(u_0, u_n) + S(u_0, u_n) \right)
\]

\[
\leq (1 - \varepsilon) \mathcal{E}(u_0, u_n) + S(u_0, u_n) - \alpha(u_0, u_n) d(hu_0, hu_n).
\]

Taking into account (2.7), this is equivalent to

\[
d(hu_n, hu_0) = d(hu_0, hu_n) \leq \alpha(u_0, u_n) d(hu_0, hu_n)
\]

\[
(1 - \varepsilon) \mathcal{E}(u_0, u_n) + S(u_0, u_n)
\]

\[
= (1 - \varepsilon) \max \left\{ \frac{d(u_n, u_0)}{d(u_{n+1}, u_0)} + \frac{|d(u_n, hu_0) - d(u_0, hu_0)|}{d(u_{n+1}, u_0)} + \frac{2}{2} \right\}
\]

\[
+ \lambda \varepsilon \beta \mathcal{E}(u_0, u_n) \left[ 1 + \| u_n \| + \| u_0 \| + \| hu_n \| + \| hu_0 \| \right] \delta
\]

\[
= (1 - \varepsilon) \max \left\{ \frac{d(u_n, u_0)}{d(u_{n+1}, u_0)} + \frac{|d(u_n, u_{n+1}) - d(u_0, u_1)|}{d(u_{n+1}, u_0)} + \frac{2}{2} \right\}
\]

\[
+ \lambda \varepsilon \beta \mathcal{E}(u_0, u_n) \left[ 1 + \| u_n \| + \| u_0 \| + \| u_{n+1} \| + \| u_1 \| \right] \delta
\]
\[
\begin{align*}
&\leq (1 - \varepsilon) \max \left\{ \kappa_n + |\gamma_n - K_1|, \frac{|\gamma_n + 1 - K_1|}{2}, \frac{|\gamma_n - 1|}{2} \right\} \\
&+ \Lambda \varepsilon^\lambda \psi(\varepsilon)[1 + \kappa_n + \gamma_n + K_1]^\beta \leq (1 - \varepsilon) \max\{ \kappa_n + \kappa_1 + \gamma_n, \kappa_n + K_1 \} \\
&+ \Lambda \varepsilon^\lambda \psi(\varepsilon)[1 + 2\kappa_n + 2\kappa_1]^\beta.
\end{align*}
\]

Using (2.12) and the above inequality, we get

\[
\kappa_n \leq \delta(hu_n, hu_0) + 2\kappa_1 \leq (1 - \varepsilon) \max\{ \kappa_n + \kappa_1 + \gamma_n, \kappa_n + \kappa_1 \} + \Lambda \varepsilon^\lambda \psi(\varepsilon)[1 + 2\kappa_n + 2\kappa_1]^\beta + 2\kappa_1 \leq (1 - \varepsilon)(\kappa_n + \kappa_1) + \Lambda \varepsilon^\lambda \psi(\varepsilon)[1 + 2\kappa_n + 2\kappa_1]^\beta + 2\kappa_1.
\]

Moreover, since \( \beta \leq \lambda \), we have

\[
\varepsilon \kappa_n \leq (3 - \varepsilon)\kappa_1 + \Lambda \varepsilon^\lambda \psi(\varepsilon)[1 + 2\kappa_n + 2\kappa_1]^\beta \leq (3 - \varepsilon)\kappa_1 + \Lambda \varepsilon^\lambda \psi(\varepsilon)[1 + 2\kappa_n + 2\gamma_1]^\lambda \leq (3 - \varepsilon)\kappa_1 + \Lambda \varepsilon^\lambda \psi(\varepsilon)(1 + 2\kappa_n)^\lambda \left[ 1 + \frac{2\kappa_1}{1 + 2\kappa_n} \right]^\lambda \leq 3\kappa_1 + \Lambda \varepsilon^\lambda \psi(\varepsilon)2^\lambda \kappa_n^\lambda \left( 1 + \frac{1}{2\kappa_n} \right)^\lambda (1 + 2\kappa_1)^\lambda.
\]

Now, supposing that the sequence \( \{\kappa_n\} \) is not bounded, there exists a subsequence \( \{\kappa_{n_l}\} \) of \( \{\kappa_n\} \) such that \( \kappa_{n_l} \to \infty \) as \( l \to \infty \). In this case, letting \( \varepsilon = \varepsilon_l = \frac{1 + 3\kappa_1}{\kappa_{n_l}} (\in [0, 1]) \), the above inequality yields

\[
1 \leq \Lambda 2^\lambda \left[ \varepsilon \kappa_{n_l}^\lambda \right](1 + 2\kappa_1)^\lambda \left( 1 + \frac{1}{2\kappa_{n_l}} \right)^\lambda \psi(\varepsilon_l) \leq \Lambda 2^\lambda (3\kappa_1)^\lambda (1 + 2\kappa_1)^\lambda \left( 1 + \frac{1}{2\kappa_{n_l}} \right)^\lambda \psi(\varepsilon_l) \leq \Lambda 2^\lambda (3\kappa_1)^{2\lambda} \left( 1 + \frac{1}{2\kappa_{n_l}} \right)^\lambda \psi(\varepsilon_l) \to 0 \quad \text{as} \quad l \to \infty.
\]

This is a contradiction. Thus, we conclude that our presumption is false and then the sequence \( \{\kappa_n\} \) is bounded. Furthermore, there exists \( K > 0 \) such that \( \kappa_n \leq K \) for all \( n \in \mathbb{N} \).

Let us go back now and prove that \( \gamma = 0 \) (where \( \gamma = \lim_{n \to \infty} \gamma_n \)). In view of (2.10) and the fact that the sequence \( \{\gamma_n\} \) is non-increasing, one writes

\[
\mathcal{E}(u_{n-1}, u_n) \leq 2\gamma_n - \gamma_{n+1}.
\]

Recall that

\[
\mathcal{S}(u_{n-1}, u_n) \leq \Lambda \varepsilon^\lambda \psi(\varepsilon)[1 + \|u_{n-1}\| + 2\|u_n\| + \|u_{n+1}\|]^\beta \leq \Lambda \varepsilon^\lambda \psi(\varepsilon)[1 + 4\kappa]^\beta.
\]
Taking into account that \( h \) is an \( \alpha \tilde{\zeta} - \varepsilon \alpha \) contraction, keeping in mind (2.6) and using (\( \xi_{1} \)), we have

\[
0 \leq \tilde{\zeta} \left( \alpha(u_{n-1}, u_{n}) d(hu_{n-1}, hu_{n}), (1 - \varepsilon)E(u_{n-1}, u_{n}) + S(u_{n-1}, u_{n}) \right)
\]

\[
\leq (1 - \varepsilon)(1 - \varepsilon)E(u_{n-1}, u_{n}) + Z(u_{n-1}, u_{n}) - \alpha(u_{n-1}, u_{n}) d(hu_{n-1}, hu_{n}).
\]

We have

\[
\gamma_{n} = d(u_{n}, u_{n+1}) \leq \alpha(u_{n-1}, u_{n}) d(hu_{n-1}, hu_{n})
\]

\[
\leq (1 - \varepsilon)E(u_{n-1}, u_{n}) + S(u_{n-1}, u_{n})
\]

\[
\leq (1 - \varepsilon)(2\gamma_{n} - \gamma_{n+1}) + \Lambda \varepsilon^{\beta} \psi(\varepsilon)[1 + 4\mathcal{A}]^{\beta}.
\] (2.13)

Letting \( n \to \infty \) in the previous inequality, we obtain

\[
\gamma \leq (1 - \varepsilon)\gamma + \Lambda \varepsilon^{\beta}(1 + 4\mathcal{A})^{\beta} \psi(\varepsilon),
\]

which is equivalent to

\[
\gamma \leq \Lambda \varepsilon^{\beta} - 1 (1 + 4\mathcal{A})^{\beta} \psi(\varepsilon).
\]

When \( \varepsilon \to 0 \), we get \( \gamma \leq 0 \). Therefore,

\[
\gamma = \lim_{\varepsilon \to 0} d(u_{n}, u_{n+1}) = 0.
\] (2.14)

As a next step, we claim that \( \{u_{n}\} \) is a Cauchy sequence. On the contrary, assuming that the sequence is not Cauchy, it follows from Lemma 1.9 that there exist \( \varepsilon > 0 \) and subsequences \( \{u_{n_{k}}\} \) and \( \{u_{m_{j}}\} \) such that (1.7) and (1.8) hold. Replacing \( v = u_{n_{k}} \) and \( \omega = u_{m_{j}} \) in (2.1), we have

\[
0 \leq \tilde{\zeta} \left( \alpha(u_{n_{k}}, u_{m_{j}}) d(hu_{n_{k}}, hu_{m_{j}}), (1 - \varepsilon)E + S(u_{n_{k}}, u_{m_{j}}) \right)
\]

\[
\leq (1 - \varepsilon)E(u_{n_{k}}, u_{m_{j}}) + S(u_{n_{k}}, u_{m_{j}}) - \alpha(u_{n_{k}}, u_{m_{j}}) d(hu_{n_{k}}, hu_{m_{j}}),
\] (2.15)

where

\[
\mathcal{E}(u_{n_{k}}, u_{m_{j}}) = \max \left\{ \frac{d(u_{n_{k}}, u_{m_{j}}) + d(u_{n_{k}}, hu_{m_{j}}) - d(u_{m_{j}}, hu_{n_{k}})}{d(u_{n_{k}}, hu_{n_{k}}) + d(u_{m_{j}}, hu_{m_{j}}) - d(u_{m_{j}}, hu_{n_{k}})}, \frac{d(u_{n_{k}}, hu_{n_{k}}) + d(u_{m_{j}}, hu_{m_{j}}) - d(u_{m_{j}}, hu_{n_{k}})}{2} \right\}
\]

\[
= \max \left\{ \frac{d(u_{n_{k}}, u_{m_{j}}) + d(u_{n_{k}}, u_{m_{j}+1}) - d(u_{m_{j}}, u_{m_{j}+1})}{d(u_{m_{j}+1}, u_{m_{j}+1}) + d(u_{m_{j}}, u_{m_{j}+1}) - d(u_{m_{j}}, u_{m_{j}+1})}, \frac{d(u_{m_{j}+1}, u_{m_{j}+1}) + d(u_{m_{j}}, u_{m_{j}+1}) - d(u_{m_{j}}, u_{m_{j}+1})}{2} \right\}.
\]

The triangular \( \alpha \)-orbital admissibility of \( h \) shows that \( \alpha(u_{n_{k}}, u_{m_{j}}) \geq 1 \). Thus,

\[
d(u_{n_{k}+1}, u_{m_{j}+1}) \leq (1 - \varepsilon)\mathcal{E}(u_{n_{k}}, u_{m_{j}}) + S(u_{n_{k}}, u_{m_{j}}),
\] (2.16)
Letting $l \to \infty$ and taking into account (2.14) and Lemma 1.9, we have

$$\lim_{l \to \infty} \mathcal{E}(u_{n_l}, u_{n_l}) = \epsilon.$$ (2.17)

At the same time, one writes

$$S(u_{n_l}, u_{n_l}) = \Lambda \varepsilon^\lambda \psi(\varepsilon) \left[1 + \|u_{n_l}\| + \|u_{n_l}\| + \|\hat{h} u_{n_l}\| + \|\hat{h} u_{n_l}\|\right]^\beta$$

$$\leq \Lambda \varepsilon^\lambda \psi(\varepsilon)[1 + 4 \mathcal{X}]^\beta.$$ 

Denoting by $a_l = d(u_{n_l+1}, u_{n_l+1})$ and $b_l = (1 - \varepsilon)\mathcal{E}(u_{n_l}, u_{n_l}) + S(u_{n_l}, u_{n_l})$, by Lemma 1.9, it follows that

$$a_l \to \epsilon \quad \text{and} \quad \limsup_{l \to \infty} b_l \leq (1 - \varepsilon)\epsilon + \Lambda \varepsilon^\lambda \psi(\varepsilon)[1 + 4 \mathcal{X}]^\beta.$$ 

Thus, passing to the limit as $l \to \infty$ in (2.16), we get

$$\epsilon = \limsup_{l \to \infty} a_l \leq \limsup_{l \to \infty} b_l \leq \Lambda \varepsilon^\lambda \psi(\varepsilon)[1 + 4 \mathcal{X}]^\beta.$$ 

Furthermore,

$$\epsilon \leq (1 - \varepsilon)\epsilon + \Lambda \varepsilon^\lambda \psi(\varepsilon)[1 + 4 \mathcal{X}]^\beta,$$

i.e.,

$$\epsilon \leq \Lambda \varepsilon^{\lambda-1} \psi(\varepsilon)[1 + 4 \mathcal{X}]^\beta.$$ 

That is, $\epsilon = 0$. Therefore, $\{u_n\}$ is a Cauchy sequence in the complete metric space. For this reason, there exists $v^* \in \mathcal{X}$ such that $u_n \to v^*$, as $n \to \infty$.

Furthermore, in the case that $\hat{h}$ is a continuous mapping, we get $\hat{h} v^* = v^*$, that is, $v^*$ is a fixed point of $\hat{h}$.

Now, suppose that $\mathcal{X}$ is regular. From (2.1), one writes

$$\bar{\zeta} \left( \alpha(u_n, v^*) d(h u_n, \hat{h} v^*), (1 - \varepsilon)\mathcal{E}(u_n, v^*) + S(u_n, v^*) \right).$$ (2.18)

Using the regularity of $\mathcal{X}$ and $(\bar{\zeta})$, we get

$$d(h u_n, \hat{h} v^*) \leq \alpha(u_n, v^*) d(h u_n, \hat{h} v^*) \leq (1 - \varepsilon)\mathcal{E}(u_n, v^*) + S(u_n, v^*)$$ (2.19)

where

$$\mathcal{E}(u_n, v^*) = \max \left\{ \frac{d(u_n, v^*) + |d(u_n, h u_n) - d(v^*, \hat{h} v^*)|}{d(u_n, h u_n) + d(v^*, \hat{h} v^*) + d(u_n, h u_n) - d(v^*, \hat{h} v^*)}, \frac{d(u_n, h u_n) + d(v^*, \hat{h} v^*) - d(u_n, h u_n) - d(v^*, \hat{h} v^*)}{2} \right\}$$

$$= \max \left\{ \frac{d(u_n, v^*) + |d(u_n, u_{n+1}) - d(v^*, \hat{h} v^*)|}{d(u_n, u_{n+1}) + d(v^*, \hat{h} v^*) + d(u_n, u_{n+1}) - d(v^*, \hat{h} v^*)}, \frac{d(u_n, v^*) + d(v^*, \hat{h} v^*) - d(u_n, v^*)}{2} \right\}.$$
and
\[ S(u_n, v) = \Lambda \varepsilon^\lambda \psi(\varepsilon) \left[ 1 + \| u_n \| + \| v \| + \| \dot{u} v \| \right]^\beta \]
\[ = \Lambda \varepsilon^\lambda \psi(\varepsilon) \left[ 1 + \| u_n \| + \| v \| + \| u_{n+1} \| + \| \dot{u} v \| \right]^\beta \]
\[ = \Lambda \varepsilon^\lambda \psi(\varepsilon) \left[ 1 + \kappa_n + \| v \| + \kappa_{n+1} + \| \dot{u} v \| \right]^\beta. \]

Taking into account the boundedness of the sequence \( \{\kappa_n\} \), we have
\[ S(u_n, v) \leq \Lambda \varepsilon^\lambda \psi(\varepsilon) \left[ 1 + 2 \kappa + \| v \| + \| \dot{u} v \| \right]^\beta. \]

On the other hand,
\[ \lim_{n \to \infty} E(u_n, v) = d(v, \dot{u} v). \]

Letting \( n \to \infty \) in the inequality (2.19), we find
\[ d(v, \dot{u} v) \leq (1 - \varepsilon) d(v, \dot{u} v) + \Lambda \varepsilon^\lambda \psi(\varepsilon) \left[ 1 + 2 \kappa + \| v \| + \| \dot{u} v \| \right]^\beta, \]
which is equivalent to
\[ d(v, \dot{u} v) \leq \Lambda \varepsilon^\lambda \psi(\varepsilon) \left[ 1 + 2 \kappa + \| v \| + \| \dot{u} v \| \right]^\beta. \]

Obviously, we obtain for \( \varepsilon = 0 \) that \( d(v, \dot{u} v) \leq 0 \), so \( v = \dot{u} v \). Thus, \( v \) is a fixed point of \( \dot{u} \). Finally, to prove the uniqueness of the fixed point, we suppose that there exist two fixed points \( v^*, \omega^* \in \text{Fix}_{\chi}(\dot{u}) \) such that \( v^* \neq \omega^* \). We have
\[ 0 \leq \xi(\alpha(v^*, \omega^*) d(\dot{u} v^*, \dot{u} \omega^*)) (1 - \varepsilon) E(v^*, \omega^*) + S(v^*, \omega^*) \]
\[ \leq (1 - \varepsilon) E(v^*, \omega^*) + S(v^*, \omega^*) - \alpha(v^*, \omega^*) d(\dot{u} v^*, \dot{u} \omega^*). \]

Taking into account (iv), we obtain
\[ d(v^*, \omega^*) \leq \alpha(v^*, \omega^*) d(\dot{u} v^*, \dot{u} \omega^*) \leq (1 - \varepsilon) E(v^*, \omega^*) + S(v^*, \omega^*) \]
\[ = (1 - \varepsilon) d(v^*, \omega^*) + \Lambda \varepsilon^\lambda \psi(\varepsilon) \left[ 1 + 2 \| v^* \| + 2 \| \omega^* \| \right]^\beta, \]
which leads to
\[ d(v^*, \omega^*) \leq \Lambda \varepsilon^\lambda \psi(\varepsilon) \left[ 1 + 2 \| v^* \| + 2 \| \omega^* \| \right]^\beta. \]

In the limit \( \varepsilon \to 0 \), we get \( d(v^*, \omega^*) \leq 0 \), that is, \( v^* = \omega^* \), which is a contradiction. Therefore, the fixed point of \( \dot{u} \) is unique. \( \square \)

In the following, we present an example that supports our statement, that is, Theorem 2.3 is a generalization of Theorem 1.8.
Example 2.4 Take $\mathcal{X} = A \times A$, where $A = [0,11]$ and $d : \mathcal{X} \times \mathcal{X} \to [0,\infty)$ is the usual distance. Define the mapping $h : \mathcal{X} \to \mathcal{X}$ by

$$h v = \begin{cases} (2,0), & \text{if } v \in B, \\ (11,9), & \text{if } v = (11,0), \\ (5,0), & \text{otherwise}, \end{cases}$$

where $B = \{(x,0) | x \in [0,11]\}$. For $v_1 = (11,0)$ and $v_2 = (2,0)$, we have

$$d(v_1,v_2) = 9, \quad d(h(v_1), h(v_2)) = d((11,9), (2,0)) = 9\sqrt{2},$$

$$d(v_2, h(v_2)) = d(v_2, v_2) = 0, \quad d(v_1, h(v_1)) = d((11,0), (11,9)) = 9,$$

and

$$M(v_1, v_2) = \max \left\{ \frac{d(v_1,v_2)}{2}, \frac{d(v_1,h(v_1)) + d(v_2, h(v_2))}{2}, \frac{d(v_1, h(v_1)) + d(v_2, h(v_2))}{2} \right\}$$

$$= \max \left\{ 9, \frac{9(1 + \sqrt{2})}{2} \right\} = \frac{9(1 + \sqrt{2})}{2}.$$

Thus,

$$d(h(v_1), h(v_2)) = 9\sqrt{2} > \frac{9(1 + \sqrt{2})}{2} = M(v_1, v_2),$$

so that the inequality (1.6) does not hold for $\varepsilon = 0$. That is, $h$ is not a Pata type Zamfirescu mapping.

Consider the function $\alpha : \mathcal{X} \times \mathcal{X} \to [0,\infty)$ given as

$$\alpha(v, \omega) = \begin{cases} 2, & \text{if } v, \omega \in B, \\ 1, & \text{if } v = (11,0), \omega = (2,0), \\ 0, & \text{otherwise}. \end{cases}$$

Since the assumptions (i)–(iv) are obviously satisfied, we have to prove that $h$ is an $\alpha;x;\tau;E$-Pata contraction. Take $\alpha = \beta = 1$, $\Lambda = 6$ and the functions $\Psi(t) = \frac{t}{2}$, $\tilde{\tau}(x,y) = y - x$.

For $v, \omega \in B$, we have $d(h(v), h(\omega)) = 0$, so that (2.1) holds.

For $v = (11,0)$ and $\omega = (2,0)$ we have

$$\alpha(v, \omega) d(h(v), h(\omega))$$

$$= 9\sqrt{2} \leq \frac{3}{4} \cdot 18 = \frac{3}{4} (9 + |9 - 0|)$$

$$= \frac{3}{4} \left( d(v, \omega) + d(v, h(v) - d(\omega, h\omega)) \right)$$

$$\leq 1 - \varepsilon) \left( d(v, \omega) + d(v, h(v) - d(\omega, h\omega)) \right)$$

$$+ \left( \frac{3}{4} + \varepsilon - 1 \right) \left( d(v, \omega) + d(v, h(v) - d(\omega, h\omega)) \right)$$
≤ (1 − ε)(\varepsilon - 1)\bigg(1 \cdot \frac{d'(v, \omega)}{3} + \frac{d'(v, \omega)}{3} + \frac{h\omega}{3} + \frac{h\omega}{3}\bigg) \\
≤ (1 − ε)|\varepsilon|\bigg(1 + 4(\varepsilon - 1)^3\bigg)\left(\frac{1}{2}d'(v, \omega) + \frac{1}{2}d'(v, \omega) + \frac{1}{2}d'(v, \omega) + \frac{1}{2}d'(v, \omega)\right) \\
≤ (1 − ε)|\varepsilon|\bigg(1 + \|v\| + \|\omega\| + \|h\omega\|\bigg) \\
= (1 − ε)|\varepsilon|\bigg(1 + \|v\| + \|\omega\| + \|h\omega\|\bigg) \\
= (1 − ε)|\varepsilon|\|v\| + |\omega\| + \|h\omega\|.

Due to the way the function \( \alpha \) was defined, we omit the other cases.

3 An application on a fractional boundary value problem

In this section, we ensure the existence of a solution of a nonlinear fractional differential equation (for more related details, see [17–23]). Denote by \( \mathcal{X} = C[0, 1] \) the set of all continuous functions defined on \([0, 1]\). We endow \( \mathcal{X} \) with the metric given as

\[ d(\rho, \omega) = \|\rho - \omega\|_{\infty} = \max_{s \in [0, 1]} |\rho(s) - \omega(s)|, \]

Consider the fractional differential equation

\[ \text{^{c}D}^{\mu}\rho(t) = f(t, \rho(t)), \quad 0 < t < 1, \quad 1 < \mu \leq 2, \tag{3.1} \]

with boundary conditions

\[ \begin{cases} 
\rho(0) = 0, \\
I_{\mu}^{\rho}(1) = \rho'(0).
\end{cases} \tag{3.2} \]

Here, \( \text{^{c}D}^{\mu} \) corresponds for the Caputo fractional derivative of order \( \mu \), given as

\[ D^{\mu}f(t) = \frac{1}{\Gamma(n - \mu)} \int_{0}^{1} (t - s)^{n-\mu-1}f^n(s) \, ds, \tag{3.3} \]

where \( n - 1 < \mu < n \) and \( n = \lceil \mu \rceil + 1 \), and \( I_{\mu}^{\rho} \) is the Riemann–Liouville fractional integral of order \( \mu \) of a continuous function \( f \), defined by

\[ I_{\mu}^{\rho}f(t) = \frac{1}{\Gamma(\mu)} \int_{0}^{t} (t - s)^{\mu-1}f(s) \, ds, \quad \mu > 0. \tag{3.4} \]

In [24], it is showed that the problem (3.1) and (3.2) can be written in the following integral form:

\[ \rho(t) = \frac{1}{\Gamma(\mu)} \int_{0}^{t} (t - s)^{\mu-1}f(s, \rho(s)) \, ds + \frac{2\varepsilon}{\Gamma(\mu)} \int_{0}^{1} \int_{0}^{s} (s - r)^{\mu-1}f(r, \rho(r)) \, dr \, ds. \tag{3.5} \]

**Theorem 3.1** Assume that

1. \( f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R} \) is continuous;
2. for all \( \rho, \omega \in \mathcal{X} \), we have
\[
|f(s, \rho(s)) - f(s, \omega(s))| \leq \frac{\varepsilon^2}{4} \Gamma(\mu + 1)|\rho(s) - \omega(s)|,
\]
(3.6)
for each \( s \in [0,1] \), where \( \varepsilon \in [0,1] \).

Then the problem 3.1 and 3.2 possesses a unique solution.

Proof Consider the functional
\[
T\rho(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} f(s, \rho(s)) \, ds + \frac{2t}{\Gamma(\mu)} \int_0^1 \int_0^s (s-r)^{\mu-1} f(r, \rho(r)) \, dr \, ds,
\]
(3.7)
Note that a solution of (3.5) is also a fixed point of \( T \). We mention that \( T \) is well posed. For all \( \rho, \omega \in \mathcal{X} \) and \( s \in [0,1] \), we have
\[
|T\rho(t) - T\omega(t)|
\]
\[
= \left| \frac{1}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} f(s, \rho(s)) \, ds - \frac{1}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} f(s, \omega(s)) \, ds \right|
\]
\[
- \frac{1}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} f(s, \omega(s)) \, ds
\]
\[
+ \frac{2t}{\Gamma(\mu)} \int_0^1 \int_0^s (s-r)^{\mu-1} f(r, \rho(r)) \, dr \, ds - \frac{2t}{\Gamma(\mu)} \int_0^1 \int_0^s (s-r)^{\mu-1} f(r, \omega(r)) \, dr \, ds
\]
\[
\leq \frac{1}{\Gamma(\mu)} \left| \int_0^t (t-s)^{\mu-1} f(s, \rho(s)) \, ds - \int_0^t (t-s)^{\mu-1} f(s, \omega(s)) \, ds \right|
\]
\[
+ \frac{2t}{\Gamma(\mu)} \int_0^1 \int_0^s (s-r)^{\mu-1} f(r, \rho(r)) \, dr \, ds - \int_0^1 \int_0^s (s-r)^{\mu-1} f(r, \omega(r)) \, dr \, ds
\]
\[
\leq \frac{\varepsilon^2 \Gamma(\mu + 1)}{4 \Gamma(\mu)} \int_0^1 (t-s)^{\mu-1} |\rho(s) - \omega(s)| \, ds
\]
\[
+ \frac{2\varepsilon^2 \Gamma(\mu + 1)}{4 \Gamma(\mu)} \int_0^1 \int_0^s (s-r)^{\mu-1} |\rho(r) - \omega(r)| \, dr \, ds
\]
\[
\leq \frac{\varepsilon^2 \Gamma(\mu + 1)}{4 \Gamma(\mu)} \Gamma(\mu + 1) d(\rho, \omega)
\]
\[
+ \frac{2\varepsilon^2 B(\mu + 1, 1)}{4 \Gamma(\mu)} \Gamma(\mu + 1) d(\rho, \omega)
\]
\[
\leq \frac{\varepsilon^2}{4} d(\rho, \omega) + \frac{\varepsilon^2}{2} d(\rho, \omega)
\]
\[
\leq \varepsilon^2 d(\rho, \omega).
\]
where $B$ is the beta function. Consequently, one has

$$d(T\rho, T\omega) \leq \varepsilon^2 d(\rho, \omega)$$

$$= \varepsilon d(\rho, \omega) - \varepsilon^2 d(\rho, \omega) + 2\varepsilon^2 d(\rho, \omega)$$

$$\leq (1 - \varepsilon)E(\rho, \omega) + 2\varepsilon^2 d(\rho, \omega)$$

$$\leq (1 - \varepsilon)E(\rho, \omega) + 2\varepsilon^2 [d(\rho, 0) + d(0, \omega)]$$

$$= (1 - \varepsilon)E(\rho, \omega) + 2\varepsilon^2 \| \rho \| + \| \omega \|$$

$$\leq (1 - \varepsilon)E(\rho, \omega) + \Lambda \varepsilon^2 \psi(\varepsilon) [1 + \| \rho \| + \| \omega \| + \| T\rho \| + \| T\omega \|]^{\beta},$$

where $\psi(\varepsilon) = \varepsilon$, $\beta = \lambda = 1$ and $\Lambda = 2$. Applying Theorem 2.3, the functional $T$ admits a unique fixed point, that is, the problem (3.1) and (3.2) possesses a unique solution. 

4 Conclusion and remarks

Our results merged from and generalized several existing results in the related literature. First of all, as underlined in Remark 2.2, the main result of [16] is a consequence of our given theorem. On the other hand, by choosing the auxiliary functions in a proper way, we may state a long list of corollaries. More precisely, by choosing the mapping $\alpha$ in a proper way, we can get the analogue of our result in the setting of partially ordered metric spaces, or in the set-up of cyclic mappings. Note that, if we take $\alpha(x, y) = 1$ for all $x, y$, we get the standard fixed point theorems in the context of complete metric spaces; see [25–29]. In addition, by choosing the appropriate simulation function, one can get several more results; see [30–35].

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