HANKEL TENSOR DECOMPOSITIONS AND RANKS

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ABSTRACT. Hankel tensors are generalizations of Hankel matrices. This article studies the relations among various ranks of Hankel tensors. We give an algorithm that can compute the Vandermonde ranks and decompositions for all Hankel tensors. For a generic n-dimensional Hankel tensor of even order or order three, we prove that the the cp rank, symmetric rank, border rank, symmetric border rank, and Vandermonde rank all coincide with each other. In particular, this implies that the Comon’s conjecture is true for generic Hankel tensors when the order is even or three. Some open questions are also posed.

1. Introduction

1.1. Various ranks for tensors. For integers \( m, n > 0 \), denote by \( T^m(\mathbb{C}^n) \) the space of all \( n \)-dimensional complex tensors of order \( m \). A tensor \( A \in T^m(\mathbb{C}^n) \) is an array indexed by an integer tuple \((i_1, \ldots, i_m)\) in the range \( 1 \leq i_1, \ldots, i_m \leq n \), that is,
\[
A = (A_{i_1 \cdots i_m})_{1 \leq i_1, \ldots, i_m \leq n}.
\]
The tensor \( A \) is symmetric if \( A_{i_1 \cdots i_m} \) is invariant with respect to all permutations of \((i_1, \ldots, i_m)\). Denote by \( S^m(\mathbb{C}^n) \) the space of all \( n \)-dimensional complex symmetric tensors of order \( m \).

There are various types of ranks for tensors, which measure the complexity of tensor computations from different aspects. The typical ones are the classical rank \([27]\) (also called the candecomp-parafac (cp) rank), multilinear rank \([14, 28]\), tensor network rank \([67]\) and nuclear rank \([20, 40]\) for all tensors. The symmetric rank \([12]\) is defined for symmetric tensors. The Vandermonde rank \([56]\) is defined for Hankel tensors. The border rank for general tensors and symmetric border rank for symmetric tensors are also defined for studying algebraic properties. We refer to \([35, 41]\) for various definitions of tensor ranks. For convenience of reading, we shortly review them in the below.

All tensors can be expressed as linear combinations of outer products of vectors. For \( u_1, \ldots, u_m \in \mathbb{C}^n \), the outer product \( u_1 \otimes \cdots \otimes u_m \) is the tensor in \( T^m(\mathbb{C}^n) \) such that for all \( 1 \leq i_1, \ldots, i_m \leq n \)
\[
(u_1 \otimes \cdots \otimes u_m)_{i_1 \cdots i_m} = (u_1)_{i_1} \cdots (u_m)_{i_m}.
\]
The tensors of the form \( u_1 \otimes \cdots \otimes u_m \) are called rank-1 tensors. The cp rank of \( A \in T^m(\mathbb{C}^n) \) is defined as
\[
\text{rank}(A) := \min \left\{ r \left| \sum_{i=1}^{r} u_{i,1} \otimes \cdots \otimes u_{i,m} = A, \ u_{i,j} \in \mathbb{C}^n \right. \right\}.
\]

2010 Mathematics Subject Classification. 15A18, 15A69.

Key words and phrases. Hankel tensor, tensor rank, Vandermonde decomposition, Comon’s conjecture, Waring decomposition.
If \( \text{rank}(A) = r \), the decomposition in (1.1) is called a \textit{rank decomposition} of \( A \). The \textit{border rank} of \( A \) is then defined as

\[
\text{brank}(A) := \min \left\{ r \mid \lim_{p \to \infty} \sum_{i=1}^{r} u_i^{(p)} \otimes \cdots \otimes u_{i,m}^{(p)} = A, \ u_{i,j}^{(p)} \in \mathbb{C}^n \right\}.
\]

Clearly, it holds that \( \text{brank}(A) \leq \text{rank}(A) \).

For symmetric tensors, we are typically interested in their symmetric ranks. For \( A \in S^m(\mathbb{C}^n) \), its symmetric rank is defined as

\[
\text{rank}_S(A) := \min \left\{ r \mid \sum_{i=1}^{r} (u_i)^{\otimes m} = A, \ u_i \in \mathbb{C}^n \right\}.
\]

In the above, \((u_i)^{\otimes m} := u_i \otimes \cdots \otimes u_i\), where \( u_i \) is repeated \( m \) times. If \( \text{rank}_S(A) = r \), the decomposition in (1.2) is called a \textit{symmetric rank decomposition} of \( A \). A symmetric tensor \( A \in S^m(\mathbb{C}^n) \) can be uniquely represented by a homogeneous polynomial (i.e., a form) of degree \( m \) and in \((x_1, \ldots, x_n)\), which is

\[
A(x) := \sum_{1 \leq i_1, \ldots, i_m \leq n} A_{i_1 \cdots i_m} x_{i_1} \cdots x_{i_m}.
\]

A symmetric decomposition of \( A \) is equivalent to that \( A(x) \) is a sum of powers of linear forms. In the literature, the symmetric rank decomposition is also called a \textit{Waring decomposition}, and the symmetric rank is also called \textit{Waring rank} \([35, 49]\).

The symmetric rank of a form means the symmetric rank of the corresponding symmetric tensor. The \textit{symmetric border rank} of \( A \in S^m(\mathbb{C}^n) \) is then defined as

\[
\text{branks}(A) := \min \left\{ r \mid \lim_{p \to \infty} \sum_{i=1}^{r} u_i^{(p)} \otimes_{m} = A, \ u_i^{(p)} \in \mathbb{C}^n \right\}.
\]

For a symmetric tensor \( A \), it is straightforward to see that

\[
\text{brank}(A) \leq \text{branks}(A) \leq \text{rank}_S(A).
\]

Determining ranks and decompositions of tensors are fundamental questions in many applications, such as signal processing \([11, 59]\), multiway factor analysis \([14, 63]\), statistics \([44]\), computational complexity \([41, 66]\), chemometrics \([51]\) and psychometrics \([50, 61]\). A general survey about applications can be found in \([50]\). It is NP-hard to compute the ranks and decompositions of tensors \([25, 26]\). Even for symmetric tensors, the question of computing their symmetric ranks and Waring decompositions still remains NP-hard \([58]\). We refer to the work \([2, 15, 18, 38]\) for general tensor decompositions and refer to the work \([5, 3, 46, 49]\) for symmetric tensor decompositions. Other interesting questions about tensors include low rank approximations \([10, 29, 45]\), uniqueness of tensor decompositions \([9, 21, 31]\), symmetric rank of monomials \([34, 50]\), defining ideals of low rank tensors \([32, 33]\), and tensor eigenvalues \([13, 39, 47, 53, 54, 55]\).

For symmetric tensors, a challenging question of great importance is the Comon’s conjecture:

\textbf{Comon’s conjecture} \([38]\) The cp rank of a symmetric tensor is equal to its symmetric rank.
The Conon’s conjecture has been proved to be true for several classes of symmetric tensors \[1,12,19,68\]. In this paper, we will show that the Conon’s conjecture is also true for generic Hankel tensors of even order or order three.

1.2. Hankel tensors. A tensor \( H \in T^m(\mathbb{C}^n) \) is called Hankel if \( H_{i_1 \ldots i_m} \) is invariant whenever the sum \( i_1 + \cdots + i_m \) is a constant \[12,50\]. In other words, \( H \) is a Hankel tensor if and only if there exists a vector \( h := (h_0, h_1, \ldots, h_{(n-1)m}) \) such that

\[
H_{i_1 \ldots i_m} = h_{i_1} + \cdots + h_{i_m - m}.
\]

Clearly, each Hankel tensor is also symmetric. Denote by \( H_{\text{ Hankel}} \) whenever the sum \( i_1 + \cdots + i_m \) is also true for generic Hankel tensors of even order or order three.

As shown by Qi \[56\], \( H \) is a Hankel tensor if and only if it has a Vandermonde decomposition, i.e., for some \( \lambda_i, t_i \in \mathbb{C} \),

\[
H = \sum_{i=1}^{r} \lambda_i (1, t_i, \ldots, t_i^{n-1}) \otimes m.
\]

In this paper, we consider the homogenization of the above:

\[
H = \sum_{i=1}^{r} (a_i^{n-1}, a_i^{n-2} b_i, \ldots, a_i b_i^{n-2}, b_i^{n-1}) \otimes m.
\]

The smallest \( r \) in the above, denoted as \( \text{rank}_V(H) \), is called the Vandermonde rank (or just \( V \)-rank) of \( H \), and the corresponding decomposition is called the Vandermonde rank decomposition or just \( V \)-rank decomposition. If we consider \( H \in H^m(\mathbb{C}^n) \) as a tensor in \( T^m(\mathbb{C}^n) \), its cp rank \( \text{rank}(H) \) is defined in \[11\]; if we consider it as a tensor in \( S^m(\mathbb{C}^n) \), its symmetric rank \( \text{rank}_S(H) \) is defined in \[12\]. The border rank \( \text{brank}(H) \) and symmetric border rank \( \text{brank}_S(H) \) are defined in the same way\[1\].

For a Hankel tensor \( H \), it clearly holds that

\[
\text{brank}(H) \leq \text{rank}(H) \leq \text{rank}_S(H) \leq \text{rank}_V(H).
\]

For the relations among various ranks of a Hankel tensor \( H \), we have the following two simple facts:

1) The \( V \)-rank of \( H \) is 1 if and only if \( \text{rank}_S(H) = 1 \). Clearly, if \( \text{rank}_V(H) = 1 \), then \( H \neq 0 \) and \( \text{rank}_S(H) \geq 1 \), so they are the same by \[15\]. Conversely, if \( \text{rank}_S(H) = 1 \), then \( H = v \otimes m \), for some \( v = (v_1, \ldots, v_n) \in \mathbb{C}^n \). Since \( H \) is Hankel, \( v_1 \cdots v_m = v_j \cdots v_m \) for all \( i_1 + \cdots + i_m = j_1 + \cdots + j_m \).

In particular, for \( i_1 = \cdots = i_m = i \) and \( j_1 = \cdots = j_m = 1 \), we get \( v_{i-1} v_{i+1} = v_i^2 \) for \( i = 2, \ldots, m-1 \). Therefore, we can parametrize \( v \) as \( v = (a^{n-1}, a^{n-2} b, \ldots, b^{n-1}) \), so \( \text{rank}_V(H) = 1 \). For this case, all the ranks are the same by \[16\].

2) For the 2-dimensional case (i.e., \( n = 2 \)), we also have \( \text{rank}_S(H) = \text{rank}_V(H) \). This is because for every 2-dimensional vector \( v = (a, b) \), the tensor power \( v \otimes m \) is itself a Vandermonde decomposition. When \( n = 2 \), it holds that

\[
H^m(\mathbb{C}^2) = S^m(\mathbb{C}^2) \simeq \mathbb{C}^{m+1}.
\]

\[1\]The notion of border \( V \)-rank can be defined in the same way as for border rank and symmetric border rank. This paper does not discuss this type of rank.
Hankel tensors have broad applications. They were originally defined in signal processing [52] for studying the Harmonic Retrieval problem [42]. Moreover, Hankel tensors can also be used to solve the interpolation problem [64]. Qi [56] studied Vandermonde decompositions and complete/strong Hankel tensors. The inheritance properties and sum-of-squares decompositions for Hankel tensors are studied by Ding, Qi and Wei [17]. Extremal eigenvalues of Hankel tensors are discussed in Chen, Qi and Wang [8]. Some further results on Hankel tensors are done in Chen, Li and Qi [7].

1.3. Contributions. The $V$-rank decompositions of Hankel tensors are closely related to symmetric rank decompositions of binary forms. Let $d = (n - 1)m$ and $h$ be as in (1.3). The vector $h$ can be uniquely identified as a binary form of degree $d$:

$$h(x, y) := \sum_{j=0}^{d} \binom{d}{j} h_j x^j y^{d-j}.$$  

It can also be thought of as a symmetric binary tensor of order $d$. By writing $\text{rank}_S(h)$ (resp., $\text{brank}_S(h)$), we mean the symmetric rank (resp., border rank) when $h$ is regarded as the symmetric tensor represented by the binary form $h(x, y)$. Note that $\text{rank}_S(h)$ is just the Waring rank of the form $h(x, y)$. The Vandermonde decomposition (1.4) is equivalent to

$$h(x, y) = \sum_{i=1}^{r} (a_i x + b_i y)^d.$$  

The symmetric rank of $h$ is the smallest $r$ in the above. In Lemma 3.1, we will show that $\text{rank}_V(H) = \text{rank}_S(h)$.

This article focuses on various ranks of Hankel tensors. We mainly address the following two basic questions:

- How can we determine the Vandermonde rank and decomposition of a Hankel tensor?
- What are the relations among various ranks of a Hankel tensor?

First, we propose an algorithm (Algorithm 3.3) that can compute the Vandermonde rank and decomposition for all Hankel tensors. This will be done in Section 3.

Second, we show that the cp rank, symmetric rank, border rank, symmetric border rank and Vandermonde rank are the same for a generic $H \in \mathcal{H}^m(\mathbb{C}^n)$ when $m$ is even or $m = 3$. In particular, this implies that the Comon’s conjecture is true for generic Hankel tensors of even order or order three. Moreover, for a specifically given Hankel tensor, we give concrete conditions for determining these ranks. This will be done in Sections 4 and 5.

We give some preliminary results in Section 2 and conclude the paper with some open questions/conjectures in Section 6.

2. Preliminaries

**Notation.** The symbol $\mathbb{N}$ (resp., $\mathbb{R}$, $\mathbb{C}$) denotes the set of nonnegative integers (resp., real, complex numbers). The symbol $\mathbb{C}[x] := \mathbb{C}[x_1, \ldots, x_n]$ denotes the ring of polynomials in $x := (x_1, \ldots, x_n)$ over the complex field $\mathbb{C}$. For any $t \in \mathbb{R}$, $\lfloor t \rfloor$
(resp., \(|t|\)) denotes the smallest integer not smaller (resp., the largest integer not bigger) than \(t\). The cardinality of a set \(S\) is denoted as \(|S|\). For a matrix \(M\), its null space is denoted as \(\ker M\).

2.1. Elementary algebraic geometry. For basics in algebraic geometry, we refer to \cite{24, 57}. A set \(X \subseteq \mathbb{C}^n\) is an algebraic variety if there exist polynomials \(f_1, \ldots, f_s \in \mathbb{C}[x]\) such that

\[X = \{a \in \mathbb{C}^n : f_1(a) = \cdots = f_s(a) = 0\}.
\]

The Zariski topology on \(\mathbb{C}^n\) is the topology whose open sets are of the form

\[U_f = \{a \in \mathbb{C}^n : f(a) \neq 0\},
\]

with \(f \in \mathbb{C}[x]\). That is, a subset \(X \subseteq \mathbb{C}^n\) is closed in the Zariski topology if and only if \(X\) is an algebraic variety. An algebraic variety is called irreducible if it is not a union of two distinct algebraic varieties. We need the notion of a generic point in an irreducible algebraic variety \(V\). For a property \(P\) on \(V\), we say that a generic point in \(V\) has the property \(P\) if the set of points in \(V\) which do not satisfy \(P\) is contained in a proper closed subset of \(V\) in the Zariski topology. For instance, a generic point \((x, y) \in \mathbb{C}^n \times \mathbb{C}^n\) uniquely determines a line \(L \subset \mathbb{C}^n\) such that \(x, y \in L\). This is because lines passing through \(x, y\) are not unique if and only if \(x = y\) and \(\{(x, x) : x \in \mathbb{C}^n\}\) is a proper closed subset of \(\mathbb{C}^n \times \mathbb{C}^n\). If the property \(P\) is clear from the context, we just say “a generic point” without mentioning \(P\).

The projective space \(\mathbb{P}^n\) consists of all lines in \(\mathbb{C}^{n+1}\), or equivalently, \(\mathbb{P}^n\) is the set of equivalence classes of \([a_0 : \cdots : a_n] = \{(\lambda a_0, \ldots, \lambda a_n) \in \mathbb{C}^{n+1} : (a_0, \ldots, a_n) \neq 0, \lambda \neq 0\}\).

A subset \(X \subseteq \mathbb{P}^n\) is a projective variety if there exist homogeneous polynomials \(f_1, \ldots, f_s \in \mathbb{C}[x_0, \ldots, x_n]\) such that

\([a] \in X\) if and only if \(f_1(a) = \cdots = f_s(a) = 0\).

2.2. Multilinear algebra. Let \(B\) be a vector space of dimension \(n\) and let \(\{b_1, \ldots, b_n\}\) be a basis for it. For an integer \(0 < p \leq n\), the \(p\)-th exterior power of \(B\), denoted as \(\bigwedge^p B\), is the vector space spanned by the \(\binom{n}{p}\) vectors \(b_{i_1} \wedge \cdots \wedge b_{i_p}\) \((1 \leq i_1 < \cdots < i_p \leq n)\), where the wedge product \(v_1 \wedge \cdots \wedge v_p\) is defined as

\[v_1 \wedge \cdots \wedge v_p := \sum_{\sigma \in \mathfrak{S}_p} \mathrm{sgn}(\sigma)v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(p)}.
\]

In the above, \(\mathfrak{S}_p\) is the permutation group on \(p\) elements and \(\mathrm{sgn}(\sigma)\) is the sign of the permutation \(\sigma \in \mathfrak{S}_p\). Clearly, \(\bigwedge^p B\) is a linear subspace of \(B^\otimes n\), with dimension \(\binom{n}{p}\). Moreover,

\[v_1 \wedge \cdots \wedge v_p = \mathrm{sgn}(\sigma)v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(p)}, \quad \forall \sigma \in \mathfrak{S}_p.
\]

The exterior power \(\bigwedge^p B\) is a generalization of skew-symmetric matrices. Indeed, if \(p = 2\), then \(\bigwedge^2 B\) is simply the vector space of all \(n \times n\) skew symmetric matrices.

Let \(A, B, C\) be vector spaces of dimensions \(m, n, q\) respectively. Every tensor \(T \in A \otimes B \otimes C\) can be regarded as a linear map \(\varphi_T : A^* \rightarrow B \otimes C\). Choose bases \(\{a_1, \ldots, a_m\}\), \(\{b_1, \ldots, b_n\}\) and \(\{c_1, \ldots, c_q\}\) for \(A, B, C\) respectively. Then we can write \(T\) as

\[T = \sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^q t_{ijk} a_i \otimes b_j \otimes c_k.
\]
Let \( \{ \alpha_1, \ldots, \alpha_m \} \) be the dual basis of \( A^* \), then \( \varphi_T \) is given by

\[
\varphi_T(\alpha_i) = \sum_{j=1}^{n} \sum_{k=1}^{q} t_{ijk} b_j \otimes c_k.
\]

For each integer \( 0 < p < n \), define \( \varphi_T^p \) to be the linear map

\[
\varphi_T^p : \left( \bigwedge^p B \right) \otimes A^* \to \left( \bigwedge^{p+1} B \right) \otimes C,
\]

which is obtained by tensoring \( \varphi_T \) with the identity map \( \text{Id}_{\bigwedge^pB} : \bigwedge^p B \to \bigwedge^p B \) and projecting the image of \( \varphi_T \otimes \text{Id}_{\bigwedge^pB} \) in \( \left( \bigwedge^p B \right) \otimes B \otimes C \) onto \( \left( \bigwedge^{p+1} B \right) \otimes C \). Here, the projection of \( \bigwedge^p B \otimes B \otimes C \) onto \( \bigwedge^{p+1} B \otimes C \) is determined by

\[
(b_1 \wedge \cdots \wedge b_p) \otimes b_{p+1} \otimes c \mapsto (b_1 \wedge \cdots \wedge b_p \wedge b_{p+1}) \otimes c.
\]

In a word, the linear map \( \varphi_T^p \) is defined such that

\[
(b_1 \wedge \cdots \wedge b_p) \otimes \alpha_i \mapsto \sum_{j=1}^{n} \sum_{k=1}^{q} t_{ijk} (b_1 \wedge \cdots \wedge b_p \wedge b_j) \otimes c_k.
\]

The rank of \( \varphi_T^p \) gives a lower bound for the border rank of \( T \).

**Theorem 2.1.** [37] Let \( T \) be a tensor in \( A \otimes B \otimes C \) and \( 0 < p < n \) be an integer. If \( \varphi_T^p \) is the linear map defined in (2.1), then

\[
\text{rank}(T) \geq \text{rank} \frac{\varphi_T^p}{\left( \begin{array}{c} n-1 \end{array} \right)}.
\]

**2.3. Symmetric ranks of binary forms.** A binary form \( h(x, y) \) is a homogeneous polynomial in two variables \( x, y \). Every binary form of degree \( d \) can be regarded as a symmetric binary tensor of order \( d \), so \( S^d(\mathbb{C}^2) \simeq \mathbb{C}^{d+1} \). The symmetric rank of the symmetric tensor represented by \( h(x, y) \) is also called the symmetric rank of \( h(x, y) \). We can write \( h(x, y) = \sum_{i=0}^{d} \binom{d}{i} h_i x^i y^{d-i} \). For convenience, we denote

\[
h := (h_0, h_1, \ldots, h_d).
\]

For \( 0 \leq r \leq d \), denote the Hankel matrix

\[
C_{d-r,r}(h) := \begin{bmatrix} h_0 & h_1 & \cdots & h_r \\ h_1 & h_2 & \cdots & h_{r+1} \\ \vdots & \vdots & \ddots & \vdots \\ h_{d-r} & h_{d-r+1} & \cdots & h_d \end{bmatrix}.
\]

The symmetric rank \( \text{rank}_S(h) \) of \( h(x, y) \) can be determined as follows.

**Theorem 2.2.** [10] Let \( h(x, y) = \sum_{i=0}^{d} h_i \binom{d}{i} x^i y^{d-i} \) be a binary form of degree \( d \). Then, we have the following:

- The symmetric border rank of \( h \) is \( (s = \lfloor d/2 \rfloor) \)

\[
r := \text{rank} C_{d-s,s}(h).
\]

- If \( d \) is even and \( r = d/2 + 1 \), then \( \text{rank}_S(h) = r \).
If $d$ is odd, or if $r < d/2 + 1$, then $\text{rank}_S(h) = r$ or $d - r + 2$. Let $(f_0, f_1, \ldots, f_r) \neq 0$ be a vector from $\ker C_{d-r,r}(h)$, which is unique up to scaling. If the polynomial

$$f(x, y) := f_0 x^r + f_1 x^{r-1} y + \cdots + f_r y^r$$

has no multiple roots, then $\text{rank}_S(h) = r$; otherwise, $\text{rank}_S(h) = d - r + 2$.

Once we know the symmetric rank $k := \text{rank}_S(h)$, Sylvester’s method can be applied to compute the Waring decomposition of the binary form $h(x, y)$. By the above theorem, either $k = r$ or $k = d - r + 2$. Select a generic vector $0 \neq (g_0, g_1, \ldots, g_k) \in \ker C_{d-k,k}(h)$. Then, the binary form

$$(2.4) \quad g(x, y) := g_0 x^k + g_1 x^{k-1} y + \cdots + g_k y^k$$

has $k$ distinct complex roots, say, $(a_1, b_1), \ldots, (a_k, b_k)$, in the projective space $\mathbb{P}^1$. Moreover, there exist scalars $\lambda_1, \ldots, \lambda_k$ satisfying

$$h(x, y) = \lambda_1 (a_1 x + b_1 y)^d + \cdots + \lambda_k (a_k x + b_k y)^d.$$ 

The above is justified by the following theorem of Sylvester.

Theorem 2.3 (Sylvester [62, 63]). A binary form $h(x, y) = \sum_{i=0}^d h_i(x^i y^{d-i})$ of degree $d$ has the decomposition $h(x, y) = \sum_{i=1}^k \lambda_i (a_i x + b_i y)^d$ for $\lambda_1, \ldots, \lambda_k \neq 0$ and $(a_1, b_1), \ldots, (a_k, b_k) \in \mathbb{C}^2$ pairwise linearly independent if and only if there exists $0 \neq (g_0, g_1, \ldots, g_k) \in \ker C_{d-k,k}(h)$ such that the binary form $g(x, y)$ as in (2.4) has the $k$ distinct complex roots in $\mathbb{P}^1$.

3. Vandermonde ranks and decompositions

For a Hankel tensor $\mathcal{H} \in \mathbb{H}^m(\mathbb{C}^n)$, let $d = (n - 1)m$ and $h = (h_0, h_1, \ldots, h_d)$ be the vector as in (2.2). We think of $h$ as the symmetric binary tensor in $S^d(\mathbb{C}^2)$ that is represented by the binary form $h(x, y) := \sum_{i=0}^d h_i(x^i y^{d-i})$. By writing $\text{rank}_S(h)$ (resp., $\text{rank}_{br}(h)$), we mean the symmetric rank (resp., the symmetric border rank) of the tensor represented by $h(x, y)$. Recall that $C_{d-r,r}(h)$ is defined as in (2.2).

First, we show that the symmetric rank of $h(x, y)$ is equal to the Vandermonde rank of $\mathcal{H}$.

Lemma 3.1. Let $\mathcal{H}, h$ be as above, then $\text{rank}_V(\mathcal{H}) = \text{rank}_S(h)$, $\text{rank}_{br}(h) \geq \text{rank}_S(\mathcal{H})$.

Proof. Consider the linear map $\pi : \mathbb{H}^m(\mathbb{C}^n) \to S^d(\mathbb{C}^2)$ defined as $\pi(\mathcal{H}) = h$.

It is a bijection between $\mathbb{H}^m(\mathbb{C}^n)$ and $S^d(\mathbb{C}^2)$. Note that $\text{rank}_V(\mathcal{H}) = 1$ if and only if $\text{rank}_S(h) = 1$. This is because that every Hankel tensor $\mathcal{H}$ with $\text{rank}_V(\mathcal{H}) = 1$ can be written as $\mathcal{H} = (a_{n-1}, a_{n-2} b_1, \ldots, b_n) \otimes \mathbb{C}^m$.

The above is equivalent to that $h(x, y) = (ax + by)^d$, so $\text{rank}_V(\mathcal{H}) = \text{rank}_S(h)$. Let $r := \text{rank}_{br}(h)$, then there exists a sequence $(a_{i,p}, b_{i,p})$ $(i = 1, \ldots, r$, $p = 1, 2, \ldots)$ such that

$$h(x, y) = \lim_{p \to \infty} \sum_{i=1}^r (a_{i,p} x + b_{i,p} y)^d.$$
The relation $\pi(H) = h$ implies that

$$H = \pi^{-1}(h) = \lim_{p \to \infty} \sum_{i=1}^{r} \left( (a_{i,p})^{n-1}, (a_{i,p})^{n-2}b_{i,p}, \ldots, (b_{i,p})^{n-1} \right)^{\otimes m}.$$ 

Hence, $\text{brank}_S(H) \leq r = \text{rank}_S(h)$. \hfill $\square$

For a generic Hankel tensor, its Vandermonde rank is given by a formula in the dimension and order.

**Proposition 3.2.** If $H \in \mathbb{H}^m(\mathbb{C}^n)$ is generic, then

$$\text{rank}_V(H) = \left\lfloor \frac{(n-1)m+1}{2} \right\rfloor.$$ 

**Proof.** Let $s = \lfloor (d-1)/2 \rfloor$ and $r = \text{rank} C_{d-s,s}$. By Theorem 2.2, $\text{rank}_S(h) = r$ or $d-r+2$. Since $d-s \geq s$, $s+1 \geq \text{rank} C_{d-s,s} = r$. When $H$ is generic, the vector $h$ is also generic, so rank $C_{d-s,s} = s+1$. The polynomial $f(x,y)$ in Theorem 2.2 does not have a multiple root, when $h$ is generic. Again, by Theorem 2.2, $\text{rank}_S(h) = s+1 = \lfloor (d+1)/2 \rfloor$, which equals the Vandermonde rank of $H$ by Lemma 3.1. \hfill $\square$

From the proof of Proposition 3.2, we can see that

- When $d$ is even, if rank $C_{s,s}(h) = d/2 + 1$, then rank$_V(H) = d/2 + 1$.
- When $d$ is odd, if rank $C_{d,s,s} = (d+1)/2$ and $f \in \ker C_{d-s-1,s+1}(h)$ has no multiple roots, then rank$_V(H) = (d+1)/2$.

The Vandermonde rank of a Hankel tensor can be determined by Theorem 2.2 and Lemma 3.1. The Vandermonde rank decomposition can be obtained by Sylvester’s method.

**Algorithm 3.3.** (Vandermonde rank decompositions for Hankel tensors.) For a given $H \in \mathbb{H}^m(\mathbb{C}^n)$, let $d = (n-1)m$ and $h$ be as in (1.3). Do the following:

1. Let $s = \lfloor d/2 \rfloor$ and form the matrix $C_{d-s,s}(h)$ as in (2.2).
2. Let $r := \text{rank} C_{d-s,s}(h)$. Find $\not= (f_0, f_1, \ldots, f_r) \in \ker C_{d-r,r}(h)$. Set $f(x,y) = f_0 x^r + f_1 x^{r-1}y + \cdots + f_r y^r$.
3. If $r = d/2 + 1$ or $f(x,y)$ has no multiple roots, then rank$_V(H) = \text{rank}_S(h) = r$; otherwise, rank$_V(H) = \text{rank}_S(h) = d-r+2$.
4. Let $k := \text{rank}_V(H)$, which is either $r$ or $d-r+2$. If $k = r$, compute the $r$ distinct roots $(a_1, b_1), \ldots, (a_r, b_r)$ of the binary form $f(x,y)$. If $k = d-r+2$, select a generic $0 \not= (g_0, g_1, \ldots, g_k) \in \ker C_{d-k,k}$ and compute the $k$ distinct roots $(a_1, b_1), \ldots, (a_k, b_k)$ of the binary form $g(x,y) := g_0 x^k + g_1 x^{k-1}y + \cdots + g_k y^k$.
5. Determine the scalars $\lambda_1, \ldots, \lambda_k$ such that

$$H = \sum_{i=1}^{k} \lambda_i (a_i^{n-1}, a_i^{n-2}b_i, \ldots, b_i^{n-1})^{\otimes m}.$$ 

The above is equivalent to $h(x,y) = \sum_{i=1}^{k} \lambda_i (a_i x + b_i y)^d$.

For symmetric tensors of generic rank or subgeneric rank, the uniqueness of the rank decompositions is studied in Chiantini et al. [9] and Galuppi et al. [21]. By Lemma 3.1, we can get similar uniqueness results about Vandermonde rank decompositions. If $H$ is a generic Hankel tensor of odd order, or if $H$ is a generic
Hankel tensor of Vandermonde rank \( r < [\lfloor (d + 1)/2 \rfloor] \), then it has a unique Vandermonde rank decomposition. However, if \( m \) is even and \( \text{rank}_V(\mathcal{H}) = [\lfloor (d + 1)/2 \rfloor] \), then there are infinitely many Vandermonde rank decompositions. Indeed, we can get concrete conditions, instead of genericity, which ensure the uniqueness.

**Theorem 3.4.** Let \( \mathcal{H} \in \mathbb{H}^m(\mathbb{C}^n) \) be a Hankel tensor. In Algorithm 3.3, if the rank of \( C_{d-s,s}(h) \) is less than \( d/2 + 1 \) and the binary form \( f(x, y) \) has no multiple roots, then the Vandermonde rank decomposition of \( \mathcal{H} \) is unique.

**Proof.** The condition rank \( C_{d-s,s}(h) < d/2 + 1 \) implies that either \( d \) is odd, or \( d \) is even but rank \( C_{d/2,d/2}(h) < d/2 + 1 \). In either case, we have rank \( C_{d-r,s}(h) = r \) with \( r = \text{rank} \ C_{d-r,r}(h) \). Since the null space \( \ker C_{d-r,r}(h) \) is one-dimensional, the vector \( (f_0, \ldots, f_r) \) is unique up to scaling. When the binary form \( f(x, y) \) has no multiple roots, \( \text{rank}_V(\mathcal{H}) = \text{rank}_S(\mathcal{H}) = r \). By Theorem 3.4, the Waring decomposition of \( h(x, y) \) is uniquely determined by the roots of \( f(x, y) \). Equivalently, this means that the Vandermonde rank decomposition of \( \mathcal{H} \) is unique. \( \square \)

In the following, we give some examples for the Vandermonde rank decomposition of a Hankel tensor \( \mathcal{H} \). We remark that:

1. It is possible that \( \text{rank}_V(\mathcal{H}) > \text{rank}_S(\mathcal{H}) \).
2. The symmetric rank decomposition of \( \mathcal{H} \) is not necessarily a Vandermonde rank decomposition, even if \( \text{rank}_V(\mathcal{H}) = \text{rank}_S(\mathcal{H}) \).

**Example 3.5.** Consider the Hankel matrix \( \mathcal{H} \in \mathbb{H}^2(\mathbb{C}^3) \):

\[
\mathcal{H} = (\mathcal{H}_{ij})^3_{i,j=1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},
\]

We have \( d = 4, s = 2 \) and \( h = (0, 0, 1, 0, 0) \). By Algorithm 3.3, we get \( \text{rank}_V(\mathcal{H}) = \text{rank}_S(h) = 3 \) and

\[
h(x, y) = x^2 y^2 = \frac{1}{6} \sum_{i=1}^{3} \lambda_i (\alpha_i x + \beta_i y)^4,
\]

where (denote \( i := \sqrt{-1} \))

\[
(\lambda_1, \lambda_2, \lambda_3) = \left( \frac{1}{3}, -\frac{1}{3} - \frac{\sqrt{3}}{6} i, -\frac{1}{3} + \frac{\sqrt{3}}{6} i \right),
\]

\[
(\alpha_1, \alpha_2, \alpha_3) = \left( \frac{1}{2}, \frac{1 + \sqrt{3} i}{2}, \frac{1 - \sqrt{3} i}{2} \right),
\]

\[
(\beta_1, \beta_2, \beta_3) = (1, -1, -1).
\]

So, \( \mathcal{H} \) has the Vandermonde rank decomposition

\[
\mathcal{H} = \frac{1}{6} \sum_{i=1}^{3} \lambda_i (\alpha_i^2, \alpha_i \beta_i, \beta_i^2) \otimes^2.
\]

On the other hand, \( \mathcal{H} \) also has the symmetric rank decomposition

\[
(3.1) \quad \mathcal{H} = \frac{1}{2} (e_1 - e_3) \otimes^2 - \frac{1}{2} (e_1 + e_3) \otimes^2 + e_2 \otimes^2,
\]

where \( \{e_1, e_2, e_3\} \) is the standard unit basis of \( \mathbb{C}^3 \). Indeed, \( \text{rank}_S(\mathcal{H}) = 3 \), because \( \mathcal{H} \) is a matrix and all the tensor ranks are the same.
Example 3.6. Let $\mathcal{H} \in \mathbb{H}^3(\mathbb{C}^3)$ be the Hankel tensor such that

$$\mathcal{H}_{ijk} = \begin{cases} 
1, & \text{if } i + j + k = 7, \\
0, & \text{otherwise}.
\end{cases}$$

The vector $h = (0,0,0,0,1,0)$. In Algorithm\textsuperscript{133} $s = 3$ and rank $C_{3,3}(h) = 3$. The unique (up to scaling) vector from the null space of $C_{3,3}(h)$ is $f = (1,0,0,0)$. The equation $f(x,y) = x^3 = 0$ has a triple root. So, rank$V(\mathcal{H}) = d - 3 + 2 = 5$. The polynomial associated to $\mathcal{H}$ is $(xz + y^2)z$, which has the Waring decomposition:

$$\frac{1}{6} \left[-\frac{1}{2}(x-z)^3 + \frac{1}{3}(x-2z)^3 + \frac{1}{6}(x+z)^3 + (y+z)^3 - (y-z)^3\right].$$

So, rank$\mathcal{S}(\mathcal{H}) \leq 5$. In fact, rank$\mathcal{S}(\mathcal{H}) = 5$ by \textsuperscript{34} Table 1. However, (3.2) is not a Vandermonde rank decomposition.

Example 3.7. Consider the Hankel tensor $\mathcal{H} \in \mathbb{H}^m(\mathbb{C}^3)$ be such that

$$\mathcal{H}_{i_1\ldots i_m} = \begin{cases} 
1, & \text{if } i_1 + \cdots + i_m = m + 1 \text{ or } 3m \\
0, & \text{otherwise}.
\end{cases}$$

The polynomial associated to $\mathcal{H}$ is $x^{m-1}y + z^m$, $d = 2m$ and

$$h_l = \begin{cases} 
1, & \text{if } l = 1 \text{ or } 2m \\
0, & \text{otherwise}.
\end{cases}$$

One can check that rank $C_{d-s,s}(h) = 3$ and for $f \in \ker C_{2m-s,3}(h)$ the polynomial $f(x,y)$ has a multiple root. Hence we have rank$V(\mathcal{H}) = 2m - 1$. On the other hand, by \textsuperscript{34} Theorem 10.2], we know $m \leq \text{rank}_S(\mathcal{H}) \leq m + 1$. Therefore, if $m \geq 3$, we have rank$\mathcal{S}(\mathcal{H}) \leq \text{rank}_V(\mathcal{H})$.

4. Rank relations when the order is even

For a Hankel tensor $\mathcal{H} \in \mathbb{H}^m(\mathbb{C}^n)$, it has the cp rank, symmetric rank, Vandermonde rank, border rank and symmetric border rank. This section studies their relations when the order $m$ is even.

Recall that $h$ is determined by $\mathcal{H}$ as in (1.3) and $C_{d-r,r}(h)$ is the Hankel matrix determined by $h$ as in (2.2). If $m$ is even, then $d = (n-1)m$ is even and $s = (n-1)m/2$. Let $r = \text{rank } C_{s,s}(h)$. If $r = s + 1$, then rank$V(\mathcal{H}) = r$ by Theorem 2.2. If $r < s + 1$, then there exists a unique (up to scaling) vector $0 \neq f \in \ker C_{d-r,r}$. The rank relations are summarized as follows.

Theorem 4.1. Suppose $m = 2m_0$ is even. For $\mathcal{H} \in \mathbb{H}^m(\mathbb{C}^n)$, let $h, d, s, r, f$ be as above. Then, we have:

(i) If $r = s + 1$, or if $r < s + 1$ and $f$ has no multiple roots, then

$$\text{rank}_V(\mathcal{H}) = \text{rank}_S(\mathcal{H}) = \text{rank}(\mathcal{H}) = \text{brank}_S(\mathcal{H}) = \text{brank}(\mathcal{H}) = r,$$

(ii) If $r < s + 1$ and $f$ has a multiple root, then

$$d - r + 2 = \text{rank}_V(\mathcal{H}) \geq \text{rank}_S(\mathcal{H}) \geq \text{brank}_S(\mathcal{H}) = \text{brank}(\mathcal{H}) = r.$$
So, $\mathcal{H}$ has a Vandermonde rank decomposition, say,

$$\mathcal{H} = \sum_{j=1}^{r} \mathcal{H}_j \text{ where } \mathcal{H}_j = (a_j^{n-1}, a_j^{n-2} b_j, \ldots, b_j^{n-1}) \otimes m.$$  \hspace{1cm} (4.3)

Consider the flattening of $\mathcal{H}$, which is the matrix $\text{Flat}(\mathcal{H})$ indexed by $I = (i_1 \ldots i_m)$ and $J = (i_{m_0+1}, \ldots, i_m)$ such that

$$\text{Flat}(\mathcal{H})_{I,J} = \mathcal{H}_{i_1 \ldots i_{m_0} i_{m_0+1} \ldots i_m}.$$  

The flattening $\text{Flat}(\mathcal{H})$ is a $n^{m_0} \times n^{m_0}$ matrix. Let $F(\mathcal{H})$ be the submatrix of $\text{Flat}(\mathcal{H})$ whose row index $I = (i_1, \ldots, i_{m_0})$ is such that

$$i_1 \leq \cdots \leq i_{m_0}, \quad m_0 \leq i_1 + \cdots + i_{m_0} \leq n m_0,$$

and whose column index $J = (i_{m_0+1}, \ldots, i_m)$ is such that

$$i_{m_0+1} \leq \cdots \leq i_m, \quad m_0 \leq i_{m_0+1} + \cdots + i_m \leq n m_0.$$

One can verify that

$$F(\mathcal{H}_j) = \begin{bmatrix} a_j^0 & a_j^{-1} b_j & a_j^{-2} b_j^2 & \cdots & a_j^{-s} b_j^s \\ a_j^1 b_j & a_j^{-2} b_j^2 & a_j^{-3} b_j^3 & \cdots & a_j^{-s} b_j^{s+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_j^{s-1} b_j^{s+1} & a_j^{-s} b_j^{s+2} & \cdots & b_j^d \end{bmatrix}.$$  

The decomposition $\mathcal{H} = \sum_{j=1}^{r} \mathcal{H}_j$ implies that

$$h(x, y) = \sum_{j=1}^{r} (a_j x + b_j y)^d,$$

so

$$\sum_{j=1}^{r} F(\mathcal{H}_j) = \begin{bmatrix} h_0 & h_1 & h_2 & \cdots & h_s \\ h_1 & h_2 & h_3 & \cdots & h_{s+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_s & h_{s+1} & h_{s+2} & \cdots & h_d \end{bmatrix} = C_{d-s,s}(h).$$

By the linearity of flattening,

$$\text{Flat}(\mathcal{H}) = \sum_{j=1}^{r} \text{Flat}(\mathcal{H}_j), \quad F(\mathcal{H}) = \sum_{j=1}^{r} F(\mathcal{H}_j) = C_{d-s,s}(h).$$

It is well-known that the border rank of a tenor is always greater than or equal to the rank of its flattening \cite{1435.15003}, so

$$\text{brank}(\mathcal{H}) \geq \text{rank} \text{Flat}(\mathcal{H}) \geq \text{rank} F(\mathcal{H}) = \text{rank} C_{d-s,s}(h) = r.$$  

Moreover, we also have

$$\text{rank}_S(\mathcal{H}) \geq \text{rank} F(\mathcal{H}) = \text{rank} C_{d-s,s}(h) = r.$$  

Since we have already shown $r \geq \text{rank}_S(\mathcal{H})$, all the ranks must be the same and the equalities in \cite{1435.15003} hold.

(ii) If $r < d/2 + 1$ and $f$ has a multiple root, then, by Theorem \cite{2022.03835} and Lemma \cite{885.0314}, we have

$$\text{rank}_V(\mathcal{H}) = \text{rank}_S(h) = d - r + 2 > r.$$
Note that the symmetric border rank of \( h \) is \( r \), by Theorem 2.2. Then, Lemma 3.1 implies that
\[
r = \text{brank}_S(h) \geq \text{brank}_S(H) \geq \text{rank}(H).
\]

As in the proof of (i), we can also prove that
\[
\text{brank}_S(H) \geq \text{rank}(\text{Flat}(H)) \geq \text{rank}(F(H)) = r.
\]
Hence, (4.2) is true, because \( \text{rank}_S(H) \geq \text{brank}_S(H) \).
\[\square\]

Theorem 4.1 implies that the Comon’s conjecture is true for generic Hankel tensors of an even order.

Corollary 4.2. If \( H \in \mathbb{H}^m(\mathbb{C}^n) \) is generic and \( m \) is even, then its cp rank, symmetric rank, border rank, symmetric border rank, and Vandermonde rank are the same, which is \( 1 + (n - 1)m/2 \). In particular, the Comon’s conjecture is true for generic Hankel tensors of even orders.

Proof. When \( H \) is generic, the vector \( h \) is also generic and \( \text{rank}_{C_s,s}(h) = d/2 + 1 \). By Theorem 4.1(i), all the ranks are equal. \[\square\]

In particular, Theorem 4.1 also implies the following:

Corollary 4.3. For a generic Hankel tensor of an even order, Algorithm 3.3 produces a Vandermonde decomposition that achieves its cp rank, symmetric rank, border rank, symmetric border rank and Vandermonde rank simultaneously.

In the following, we give some examples to show applications of Theorem 4.1.

In particular, we would like to remark that:

1) It is possible that \( \text{rank}_V(H) > \text{rank}_S(H) \), even if the order is even.

2) We may have \( \text{rank}_V(H) = \text{rank}_S(H) > \text{brank}_S(H) \).

Example 4.4. Consider the Hankel tensor \( H \) in Example 3.7. We have
\[
\text{rank}_V(H) = 2m - 1, \quad m \leq \text{rank}_S(H) \leq m + 1, \quad \text{brank}_S(H) = 3.
\]
If \( m \geq 4 \), then \( \text{brank}_S(H) < \text{rank}_S(H) < \text{rank}_V(H) \).

Example 4.5. Consider the Hankel tensor \( H \in \mathbb{H}^m(\mathbb{C}^3) \) such that
\[
H_{i_1\ldots,i_m} = \begin{cases} 
1, & \text{if } i_1 + \cdots + i_m = m + 2 \\
0, & \text{otherwise}.
\end{cases}
\]
The polynomial associated to \( H \) is \( x^{m-2}y^2 + x^{m-1}z \). By Algorithm 4.3 and Theorem 4.1, \( \text{rank}_V(H) = 2m - 1 \). By [34] Theorem 10.2,
\[
\text{brank}_S(H) = 3, \quad m \leq \text{rank}_S(H) \leq 2m - 1.
\]
If \( m \geq 4 \), then \( \text{brank}_S(H) < \text{rank}_S(H) \).

5. Rank relations when the order is odd

This section studies the relations among various ranks of Hankel tensors when the order is odd. We first discuss the case that \( m \) is odd but the ranks are small, and then discuss the case that \( m = 3 \) but the Hankel tensor is generic.
5.1. The case of general odd orders. Recall that \( C_{d-r,r}(h) \) is the Hankel matrix determined by \( h \) as in (2.2). The rank relations for Hankel tensors of odd orders are summarized as follows.

**Theorem 5.1.** Assume \( m = 2m_0 + 1 \) is odd. Let \( d = (n - 1)m \) and \( s = \lfloor d/2 \rfloor \). For \( H \in \mathbb{H}^m(\mathbb{C}^n) \), let \( h \) be as in (1.3) and \( r := \text{rank } C_{d-s,s}(h) \). Let \( f = (f_0, \ldots, f_r) \in \mathbb{C}^{r+1} \) be the unique vector (up to scaling) in \( \ker C_{d-r,r} \). Suppose that \( r \leq 1 + (n - 1)m_0 \). Then, we have:

(i) If \( f(x, y) \) has no multiple roots, then

\[
\text{rank}_V(H) = \text{rank}_S(H) = \text{rank}(H) = \text{brank}_S(H) = \text{brank}(H) = r,
\]

(ii) If \( f(x, y) \) has a multiple root, then

\[
d - r + 2 = \text{rank}_V(H) \geq \text{rank}_S(H) \geq \text{brank}_S(H) = \text{brank}(H) = r.
\]

**Proof.** We follow the same approach as in the proof of Theorem 4.1. The difference is that we need the condition \( r \leq 1 + (n - 1)m_0 \) when the order \( m \) is odd.

(i) When \( f \) has no multiple roots, by Theorem 2.2 and Lemma 3.1, we also have \( r = \text{rank}_S(h) = \text{rank}_V(H) \geq \text{rank}_S(H) \). Consider the Vandermonde rank decomposition of \( H \) as in (1.3). When \( m \) is odd, the flattening \( \text{Flat}(H) \) is slightly different. It is the matrix indexed by \( I = (i_1, \ldots, i_{m_0 + 1}) \) and \( J = (i_{m_0 + 2}, \ldots, i_m) \) such that

\[
\text{Flat}(H)_{I,J} := H_{i_1, \ldots, i_{m_0 + 1}, i_{m_0 + 2}, \ldots, i_m}.
\]

The matrix \( \text{Flat}(H) \) is \( n^{m_0 + 1} \times n^{m_0} \). Let \( F(H) \) be the submatrix of \( \text{Flat}(H) \) whose row index \( I = (i_1, \ldots, i_{m_0}) \) is such that

\[
i_1 \leq \cdots \leq i_{m_0 + 1}, \quad m_0 + 1 \leq i_1 + \cdots + i_{m_0 + 1} \leq n(m_0 + 1),
\]

and whose column index \( J = (i_{m_0 + 2}, \ldots, i_m) \) is such that

\[
i_{m_0 + 2} \leq \cdots \leq i_m, \quad m_0 \leq i_{m_0 + 2} + \cdots + i_m \leq nm_0.
\]

Let \( l := (n - 1)m_0 \). Note that

\[
H_j = (a_j^{n-1}, a_j^{n-2}b_j, \ldots, b_j^{n-1})^\otimes m.
\]

One can check that

\[
F(H_j) = \begin{bmatrix}
    a_j^d & a_j^{d-1}b_j & a_j^{d-2}b_j^2 & \cdots & a_j^{d-l}b_j^{l+1} \\
    a_j^{d-1}b_j & a_j^{d-2}b_j^2 & a_j^{d-3}b_j^3 & \cdots & a_j^{d-l-1}b_j^{l+1} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    a_j^{d-l+n-1}b_j^{n-1} & a_j^{d-l-n}b_j^n & a_j^{d-l-n-1}b_j^{n+1} & \cdots & b_j^d 
\end{bmatrix}.
\]

The Vandermonde rank decomposition of \( H \) in (1.3) implies that

\[
F(H) = \sum_{j=1}^r F(H_j) = \begin{bmatrix}
    h_0 & h_1 & h_2 & \cdots & h_l \\
    h_1 & h_2 & h_3 & \cdots & h_{l+1} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    h_{l+n-1} & h_{l+n} & h_{l+n+1} & \cdots & h_d 
\end{bmatrix} = C_{d-l,l}(h).
\]

In the next, we show that

\[
\text{rank } C_{d-l,l}(h) = r.
\]

Note that \( r \leq 1 + l, \text{rank } C_{d-s,s}(h) = r, \) and \( \text{brank}_S(h) = r \) by Theorem 2.2.

- When \( r \leq l, \) (5.3) is true by Proposition 5 of (10), since \( \text{brank}_S(h) = r \).
When \( r = 1 + l \), rank \( C_{d-l,l}(h) \leq r \). If rank \( C_{d-l,l}(h) < r \), then \( \text{rank}_S(h) < r \) by Proposition 5 of [10], which is a contradiction. So, (5.3) is also true.

The \( \text{F}(\mathcal{H}) \) is a submatrix of \( \text{Flat}(\mathcal{H}) \), so

\[
\text{rank}(\mathcal{H}) \geq \text{rank} \text{Flat}(\mathcal{H}) \geq \text{rank} \text{F}(\mathcal{H}) = \text{rank} C_{d-l,l}(h) = r.
\]

Also note that \( \text{rank}(\mathcal{H}) \leq \text{rank}_S(\mathcal{H}) \leq \text{rank}(\mathcal{H}) \). Since \( \text{rank}_V(\mathcal{H}) = r \), the relation (1.5) and the above imply that all the ranks must be the same.

(ii) The proof is the same as for item (ii) of Theorem 4.1.

When \( \mathcal{H} \) is generic and \( m = 2m_0 + 1 \), for \( n > 2 \), we have

\[
r = (n - 1)m_0 + \lceil n/2 \rceil > 1 + (n - 1)m_0.
\]

Hence, the rank relations in Theorem 5.1 are not guaranteed any more. However, we can still get a lower and upper bound for those ranks.

**Proposition 5.2.** If \( \mathcal{H} \in \mathcal{H}^m(\mathbb{C}^n) \) is generic with odd order \( m = 2m_0 + 1 \), then \( \text{rank}_V(\mathcal{H}) = m_0(n - 1) + \lceil n/2 \rceil \) and

\[
m_0(n - 1) + 1 \leq \text{rank}(\mathcal{H}) \leq \text{rank}_S(\mathcal{H}) \leq \text{rank}_V(\mathcal{H}).
\]

**Proof.** By Proposition 3.2 we know that \( \text{rank}_V(\mathcal{H}) = m_0(n - 1) + \lceil n/2 \rceil \). The latter three inequalities are obvious. It is enough to prove the first one. We follow the proof of item (i) in Theorem 5.1. For all \( \mathcal{H} \), we always have \( (l = (n - 1)m_0) \):

\[
\text{rank}(\mathcal{H}) \geq \text{rank} \text{Flat}(\mathcal{H}) \geq \text{rank} \text{F}(\mathcal{H}) = \text{rank} C_{d-l,l}(h).
\]

When \( \mathcal{H} \) is generic, \( h \) is also generic and so \( \text{rank} C_{d-l,l}(h) = 1 + l \), which completes the proof.

\[\square\]

5.2. **The case of order three.** When the order \( m = 3 \), we are able to get better rank relations, in addition to those given in Theorem 5.1. In particular, we can show that the Comon’s conjecture is true for generic Hankel tensors of order 3.

Consider the linear map \( \varphi_T \) and \( \varphi_T^p \) defined in the subsection 2.2 for the vector space \( A = B = C = \mathbb{C}^n \). We use the standard unit vector basis \( \{e_1, \ldots, e_n\} \) for \( \mathbb{C}^n \). Note that \( \{e_1, \ldots, e_n\} \) is also a dual basis for itself. A tensor \( T \in \mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n \) can be written as

\[
T = \sum_{i,j,k=1}^{n} t_{ijk} e_i \otimes e_j \otimes e_k, \quad t_{ijk} \in \mathbb{C}.
\]

Let \( 0 < p < n \) be an integer. Recall from 2.1 that the linear map

\[
\varphi_T^p : \left( \bigwedge^p \mathbb{C}^n \right) \otimes \left( \mathbb{C}^n \right)^* \to \left( \bigwedge^{p+1} \mathbb{C}^n \right) \otimes \mathbb{C}^n
\]

is defined by setting

\[
(e_{k_1} \wedge \cdots \wedge e_{k_p}) \otimes e_i \mapsto \sum_{j,k=1}^{n} t_{ijk} (e_{k_1} \wedge \cdots \wedge e_{k_p} \wedge e_j) \otimes e_k
\]

and extending it linearly. By Theorem 2.1 we have

\[
\text{rank}(T) \geq \text{rank} \varphi_T^p / \binom{n-1}{p}.
\]
We construct the representing matrix $M := M^p_T$ for the linear map $\varphi^p_T$ under the standard basis. The set
$$
\{(e_{k_1} \wedge \cdots \wedge e_{k_p}) \otimes e_i : 1 \leq k_1 < \cdots < k_p \leq n, 1 \leq i \leq n\}
$$
is a basis of $\left( \bigwedge^p \mathbb{C}^n \right) \otimes (\mathbb{C}^n)^*$ and
$$
\{(e_{k_1} \wedge \cdots \wedge e_{k_{p+1}}) \otimes e_k : 1 \leq k_1 < \cdots < k_{p+1} \leq n, 1 \leq k \leq n\}
$$
is a basis of $\left( \bigwedge^{p+1} \mathbb{C}^n \right) \otimes \mathbb{C}^n$. The representing matrix $M := M^p_T$ for $\varphi^p_T$ is $n(n) \times n(n)$. We label the rows of $M$ by
$$(J, i) := (j_1 < \cdots < j_p, i)$$
and label the columns of $M$ by
$$(J', k) := (j'_1 < \cdots < j'_{p+1}, k).$$
The entry of $M$ on the $(J, i)$-th row and $(J', k)$-th column is
$$
M_{(J, i), (J', k)} = \epsilon_{J, J', j} t_{ijk},
$$
where
$$
(5.4) \quad \epsilon_{J, J', j} = \begin{cases} \begin{array}{ll}
(-1)^{p-q}, & \text{if } j'_1 = j_1, \ldots, j'_{q-1} = j_{q-1}, \text{ and } \noalign{\vspace{1em}}
0, & \text{otherwise.}
\end{array} \end{cases}
$$

The following is an example of $M^p_T$ when $T$ is a cubic Hankel tensor.

**Example 5.3.** Consider the linear map $\varphi^1_H : \mathbb{C}^3 \otimes (\mathbb{C}^3)^* \to \left( \bigwedge^2 \mathbb{C}^3 \right) \otimes \mathbb{C}^3$ for a Hankel tensor $H \in S^3(\mathbb{C}^3)$. Note that
$$
\varphi^1_H(e_j \otimes e_i) = \sum_{j', k=1}^3 \mathcal{H}_{ij'k} (e_j \wedge e_{j'}) \otimes e_k.
$$
Let $h$ be the vector as in (1.3), then
$$
M^1_H = \begin{bmatrix}
h_2 & h_3 & h_4 & h_5 & 0 & 0 & 0 \\
h_1 & -h_2 & -h_3 & 0 & 0 & h_4 & h_5 \\
0 & 0 & -h_1 & -h_2 & -h_3 & -h_4 & -h_5 \\
h_3 & h_4 & h_5 & h_6 & 0 & 0 & 0 \\
h_2 & -h_3 & -h_4 & 0 & 0 & h_5 & h_6 \\
0 & 0 & -h_2 & -h_3 & -h_4 & -h_5 & -h_6 \\
h_4 & h_5 & h_6 & h_7 & 0 & 0 & 0 \\
-h_3 & -h_4 & -h_5 & 0 & 0 & h_6 & h_7 \\
0 & 0 & -h_3 & -h_4 & -h_5 & -h_6 & -h_7
\end{bmatrix}.
$$

One can verify that $\text{rank } M^1_H = 8$ when $\mathcal{H}$ is generic. Indeed, the sum of the third and the seventh column is equal to the fifth column of $M^1_H$, which implies that $\text{rank } M^1_H \leq 8$. Moreover, it is easy to verify that the submatrix obtained by removing the seventh column from $M^1_H$ has full rank 8. Hence, $\text{rank}(\mathcal{H}) \geq 4$ by Theorem 2.1. We can also use $\varphi^2_H$ to get a lower bound for $\text{rank}(\mathcal{H})$. However, $\text{rank } M^2_H \leq 3$, which is worse than the bound by using $\varphi^1_H$.

\footnote{One can also take the transpose to obtain an $n(p+1) \times n(n)$ matrix. Since we only concern the rank, both matrices are okay for the proof.}
Theorem 5.4. For a generic Hankel tensor $H \in S^3(\mathbb{C}^n)$, its border rank is at least $\left\lfloor \frac{3n^2}{2} \right\rfloor$.

Proof. Let $r := \left\lfloor \frac{3n^2}{2} \right\rfloor$ and $p := \left\lfloor \frac{n}{2} \right\rfloor$. By Theorem [2.1], it is sufficient to prove that the rank of the linear map $\varphi_H^p : (\bigwedge^p \mathbb{C}^n) \otimes (\mathbb{C}^n)^* \to (\bigwedge^{p+1} \mathbb{C}^n) \otimes \mathbb{C}^n$ has rank $(n-1)r$. By the lower semi-continuity of the matrix rank function, it is sufficient to prove that $\text{rank}(M_H^p) \geq (n-1)r$ for some order three Hankel tensor $H$. To do this, we take $H = (H_{ijk})$, where

$$H_{ijk} = \begin{cases} 1, & \text{if } i + j + k - 2 = r, \\ 0, & \text{otherwise.} \end{cases}$$

The tensor $(e_j, \cdots \otimes e_{j_p}) \otimes e_i$ is mapped by $\varphi_H^p$ to

$$\sum_{j=1}^{n} (-1)^{p-q}(e_{j_1} \otimes \cdots \otimes e_{j_q} \otimes e_j \otimes e_{j_{q+1}} \otimes \cdots \otimes e_{j_p}) \otimes e_{r+2-i-j},$$

where the summation is over

$$1 \leq j_1 < \cdots < j_q < j < j_{q+1} < \cdots < j_p \leq n.$$

We set $e_{r+2-i-j} = 0$ if $i + j \geq r + 2$ or $i + j \leq r - n + 1$. Hence, the summand in (5.5) is non-zero if and only if

$$\max\{1, r - i - n + 2\} \leq j \leq \min\{n, r - i + 1\} \text{ and } j \notin \{j_1, \ldots, j_p\}.$$

For given $j_1 < \cdots < j_q < j < j_{q+1} < \cdots < j_p$, denote

$$J := (j_1 < \cdots < j_p), \quad J' := (j_1 < \cdots < j_q < j < j_{q+1} < \cdots < j_p).$$

By (5.5), the matrix $M_H^p$ has a block that is

$$T_{J,J'} := \epsilon_{J,J'} T_{J-(r-n+2)} =$$

$$\begin{pmatrix}
(J',n) & (J',n-1) & \cdots & (J',r-j+2) & (J',r-j+1) & (J',r-j) & \cdots & (J',1)
\end{pmatrix}
\begin{pmatrix}
(J,1) & 0 & 0 & \cdots & 0 & (-1)^{p-q} & 0 & \cdots & 0 \\
(J,2) & 0 & 0 & \cdots & 0 & 0 & (-1)^{p-q} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
(J,r-j+1) & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & (-1)^{p-q} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
(J,n-1) & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
(J,n) & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0
\end{pmatrix},$$

where $\epsilon_{J,J'} = (-1)^{p-q}$ and $T_{J-(\lfloor (n+1)/2 \rfloor)}$ is the $n \times n$ Toeplitz matrix whose $(i,k)$-th entry $t_{ik}$ is defined by

$$t_{ik} = \begin{cases} 1, & \text{if } k - i = j - (\lfloor (n+1)/2 \rfloor), \\ 0, & \text{otherwise.} \end{cases}$$

---

To be more precise, if $M(x_1, \ldots, x_k)$ is an $m \times n$ matrix polynomial and rank $M(a_1, \ldots, a_k) = r$ for some $(a_1, \ldots, a_k) \in \mathbb{C}^k$, then there exists a Zariki open dense subset $U$ of $\mathbb{C}^k$ such that rank $M(b_1, \ldots, b_k) \geq r$ for all $(a_1, \ldots, a_k) \in U$. 


Clearly, we have
\[
\text{rank}(T_{J,J'}) = \begin{cases} 
  r - j + 1, & \text{if } r - j + 1 \leq n, \\
  2n - (r - j + 1), & \text{otherwise}. 
\end{cases}
\]

In particular, if \( J' = J \cup \{(n+1)/2\} \), then \( \text{rank}(T_{J,J'}) = n \), i.e., the matrix \( T_{J,J'} \) is the identity matrix up to a sign. Let \( C_0 \) be the set of sequences \( J' = (j_1 < \cdots < j_{p+1}) \) such that \( [(n+1)/2] \in J' \) and let \( R_0 \) be the set of sequences \( J = (j_1 < \cdots < j_p) \) such that \( [(n+1)/2] \notin J \). For each \( J \in R_0 \), there is a unique \( J' \in C_0 \) such that \( J' = J \cup \{(n+1)/2\} \). We also let \( C \) be the set of sequences \( J' = (j_1 < \cdots < j_{p+1}) \) such that \( [(n+1)/2] \notin J' \) and let \( R \) be the set of sequences \( J = (j_1 < \cdots < j_p) \) such that \( [(n+1)/2] \in J \). Then the matrix \( M_{\mathcal{H}} \) may be visualized as follows:

\[
M_{\mathcal{H}} = R_0 \begin{bmatrix} C_0 & C \end{bmatrix},
\]

where \( I \) is the submatrix of \( M_{\mathcal{H}} \) obtained by taking \( J \)'s from \( R_0 \) and \( J' \)'s from \( C_0 \) and \( T_1, T_2 \) are defined in the same way. Since for each \( J \in R_0 \) there exists the unique \( J' \in C_0 \) such that \( J' = J \cup \{(n+1)/2\} \), we see that \( I \) is actually a block diagonal matrix where each diagonal block is of the form \( T_{J,J'} \), which is the \( n \times n \) identity matrix (up to a sign) because \( J' = J \cup \{(n+1)/2\} \). Moreover, the cardinality of both \( C_0 \) and \( R_0 \) is equal to

\[
\#C_0 = \#R_0 = \binom{n-1}{p},
\]

which implies that \( I \) is a full rank \( \binom{n-1}{p} \times \binom{n-1}{p} \) matrix.

Next, we apply column operations to \( M_{\mathcal{H}} \) to make it a triangular matrix. More precisely, we compute

\[
M_{\mathcal{H}} = \begin{bmatrix} I_{(n-1)p} & -I_{(n-1)p} T_1 \\
0 & I_{(n+1)p} 
\end{bmatrix} = \begin{bmatrix} I_{p} & 0 \\
T_2 & -T_2 I_{p}^{-1} T_1 
\end{bmatrix} = \begin{bmatrix} I_{p} & 0 \\
T_2 & -T_2 I T_1 
\end{bmatrix}.
\]

Here \( I_m \) is the \( m \times m \) identity matrix and the second equality follows from \( I = T_1^{-1} \), since \( I \) is a diagonal matrix whose diagonal entries are 1 or \(-1\). We denote by \( M \) the matrix \(-T_2 I^{-1} T_1 \) and by Lemma 5.3 we see that

\[
\text{rank } M = \binom{n-1}{p+1} (p+1).
\]

Therefore, \( M_{\mathcal{H}} \) has the rank

\[
\text{rank } M_{\mathcal{H}} = \binom{n-1}{p} n + \binom{n-1}{p+1} (p+1) = \binom{n-1}{p} r.
\]

\[\square\]

**Lemma 5.5.** Let \( \mathcal{H}, M_{\mathcal{H}}, T_1, T_2, I \) and \( M \) be as in the proof of Theorem 5.4. The rank of \( M \) is equal to \( \binom{n-1}{p+1} (p+1) \).

The proof of Lemma 5.5 will be given in the Appendix.

**Theorem 5.6.** For a generic Hankel tensor \( \mathcal{H} \in \mathcal{H}(\mathbb{C}^n) \), we have

\[
\text{brank}(\mathcal{H}) = \text{rank}_S(\mathcal{H}) = \text{rank}(\mathcal{H}) = \text{rank}_S(\mathcal{H}) = \text{rank}_V(\mathcal{H})
\]
which equals $\left\lfloor \frac{3n-1}{2} \right\rfloor$. In particular, the Comon’s conjecture is true for a generic Hankel tensor of order three.

**Proof.** By Theorem 3.2, we have seen that $\text{rank}_V(\mathcal{H})$ is

$$\left\lceil \frac{3n-2}{2} \right\rceil = \left\lceil \frac{3n}{2} \right\rceil - 1 = \left\lfloor \frac{3n-1}{2} \right\rfloor.$$ 

Theorem 5.4 implies that $\text{brank}(\mathcal{H}) \geq \left\lfloor \frac{3n-1}{2} \right\rfloor$ when $\mathcal{H}$ is generic. Since we always have

$$\text{brank}(\mathcal{H}) \leq \text{rank}(\mathcal{H}) \leq \text{rank}_S(\mathcal{H}) \leq \text{rank}_V(\mathcal{H}),$$

the conclusion follows directly. \qed

Theorem 5.6 clearly implies the following.

**Corollary 5.7.** For a generic Hankel tensor of order three, Algorithm 3.3 gives a decomposition that achieves its cp rank, symmetric rank, border rank, symmetric border rank, and Vandermonde rank.

In Theorem 5.6, we can get concrete conditions for the equalities there to hold. In Algorithm 3.3 by (1.5) and Theorem 2.1, we know (5.6) holds if

- $\text{rank} M^p_H \geq [(3n-1)/2] (n-1)/p$;
- when $n$ is odd, $\text{rank} C_{d-s,s}(h) = 1 + s$;
- when $n$ is even, $\text{rank} C_{d-s,s}(h) = 1 + s$ and the binary form $f(x, y)$ has no multiple roots.

In the following, we give some examples that the conclusion of Theorem 5.6 may not hold for non-generic Hankel tensors.

**Example 5.8.** Consider the Hankel tensor $\mathcal{H} \in H^3(\mathbb{C}^3)$ such that

$$\mathcal{H}_{ijk} = \begin{cases} 1, & \text{if } i+j+k = 8, \\ 0, & \text{otherwise}. \end{cases}$$

The polynomial associated to $\mathcal{H}$ is $yz^2$. We have $d = 6, s = 3, h = (0,0,0,0,0,0,1,0)$ and $r = \text{rank} C_{d-s,s}(h) = 2$. By Algorithm 3.3 we get $\text{rank}_V(\mathcal{H}) = 6$. However, $\text{rank}_S(\mathcal{H}) = 3$ and $\text{brank}_S(\mathcal{H}) = 2$ by [34, Table 1] or [50, Theorem 1.1 & 1.2].

$$\frac{3n-1}{2} = 4 > \text{rank}(\varphi_1^\mathcal{H}) = 2.$$ 

Hence, $\text{brank}(\mathcal{H}) = 2$ and $\text{rank}(\mathcal{H}) = 2$ or 3. In fact, $\text{rank}(\mathcal{H}) = 3$. If otherwise $\text{rank}(\mathcal{H}) = 2$, then $\mathcal{H}$ has a decomposition

$$\mathcal{H} = u_1 \otimes v_1 \otimes w_1 + u_2 \otimes v_2 \otimes w_2, \quad u_i, v_i, w_i \in \mathbb{C}^3, i = 1, 2.$$ 

One may use Macaulay2 [22] to verify that such a decomposition does not exist.

**Example 5.9.** Let $\mathcal{H}$ be the Hankel tensor as in Example 5.7 for $m = 3$. We know $\text{rank}_V(\mathcal{H}) = 5$. By [34, Table 1],

$$\text{rank}_S(\mathcal{H}) = 4, \quad \text{brank}_S(\mathcal{H}) = 3.$$ 

Moreover, $\text{rank}(\varphi_1^\mathcal{H}) = 3$, hence

$$\text{brank}(\mathcal{H}) = 3, \quad \text{rank}(\mathcal{H}) = 3 \text{ or } 4.$$
Since the monomials \(x^2y\) and \(z^3\) do not share a common variable, by \(\text{[35, Sec. 9.1.4]}\), we have
\[
\text{rank}(H) = \text{rank}(x^2y) + \text{rank}(z^3) = 3 + 1 = 4 = \text{rank}_S(H).
\]

**Example 5.10.** Consider the special case of Example 4.5 with \(m = 3\). We have seen that \(\text{rank}_V(H) = 5\), \(\text{brank}_S(H) = 3\). By \(\text{[34, Theorem 10.2]}\), \(\text{rank}_S(H) = 5\).

One can check that \(\text{rank}(\varphi^1_H) = 3\), hence
\[
3 = \text{brank}(H) = \text{brank}_S(H) < \text{rank}_V(H) = 5.
\]
This implies that \(\text{rank}(H) = 3, 4 \text{ or } 5\). Indeed, we may verify again by Macaulay2 \([22]\) that \(\text{rank}(H) = 5 = \text{rank}_S(H)\).

6. **Conclusions and Open Questions**

The major results of this article are:

1) We give an algorithm (Algorithm 3.3) for computing Vandermonde rank decompositions for all Hankel tensors. In particular, the Vandermonde rank of a generic \(H \in H^m(\mathbb{C}^n)\) is \(\left\lceil \frac{m(n-1)+1}{2} \right\rceil\) (Proposition 3.2).

2) We can determine the cp rank, symmetric rank, border rank and symmetric border rank of a Hankel tensor, under some concrete conditions (Theorem 4.1, Theorem 5.1).

3) We prove that the cp rank, symmetric rank, border rank, symmetric border rank and Vandermonde rank are all the same for a generic Hankel tensor of order even or three (Corollary 4.2, Theorem 5.6). In particular, the Comon’s conjecture is true for generic Hankel tensors of these orders.

However, we do not know much about the rank relations for generic Hankel tensors of odd order \(m \geq 5\). Naturally, we pose the following question:

**Question 6.1.** For an odd order \(m \geq 5\) and for a generic Hankel tensor \(H \in H^m(\mathbb{C}^n)\), do we have
\[
\text{rank}(H) = \text{rank}_S(H) = \text{brank}(H) = \text{brank}_S(H) = \text{rank}_V(H)\?
\]

We point out that the answer to Question 6.1 is “no” if we replace “generic” by “all”, as we have already seen in the earlier examples. However, we conjecture that the answer to Question 6.1 is yes.

**Conjecture 6.2.** The answer to Question 6.1 is yes.

Finally, we conjecture that the Comon’s conjecture remains true at least for Hankel tensors.

**Conjecture 6.3.** For all \(H \in H^m(\mathbb{C}^n)\), \(\text{rank}(H) = \text{rank}_S(H)\).

**Acknowledgement** The authors would like to thank Lek-Heng Lim for fruitful discussions on this manuscript. Jiawang Nie was partially supported by the NSF grants DMS-1417985 and DMS-1619973.

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\(^4\) \text{[34, Theorem 10.2]} states that \(m \leq \text{rank}_S(H) \leq 2m - 1\) in general, but by the remark after Theorem 10.2, \(\text{rank}_S(H)\) attains the upper bound 5 if \(m = 3\).
APPENDIX A. THE PROOF OF LEMMA 5.5

In this Appendix, we give the proof of Lemma 5.5, which is used in the proof of Theorem 5.4. We will also work out some examples to illustrate the idea of the proof. Readers are recommended to read these examples to better understand the proof.

Proof of Lemma 5.5. By definition of $R$ and $C$, we have

$$\# R = \binom{n-1}{p-1} \geq \# C = \binom{n-1}{p+1}.$$  

Then $I$ is a $(p-1)n \times (n-1)n$ matrix, $T_1$ is a $(n-1)n \times (n-1)n$ matrix and $T_2$ is a $(p-1)n \times (n-1)n$ matrix. This implies that $M = -T_2IT_1$ is a $(n-1)n \times (p+1)n$. For each $J' \in C$ and $1 \leq k \leq n$, we denote by $v_{J',k}$ the $(J',k)$-th column vector of $M$. We will prove that the matrix $M$ has rank $(n-1)(p+1)$ in the following steps.

Step 1: We describe blocks $M_{J,J'}$ of the matrix $M$. We notice that $T_1$ is a $(p-1)n \times (p+1)n$ matrix, we may partition $T_1$ by blocks of size $n \times n$ and index them by elements in $R_0$ and $C$. To be more precise, for each $J \in R_0$ and $J' \in C$ we denote by $T_{1,J,J'}$ the submatrix obtained by taking rows $(J,1), \ldots, (J,n)$ and columns $(J',1), \ldots, (J',n)$. Similarly, we may also partition $T_2$ (resp. $I$) by blocks of size $n \times n$ and index them by elements in $R$ (resp. $C_0$) and $C_0$ (resp. $R_0$). We denote these blocks by $T_{2,J,J'}, J \in R, J' \in C_0$ (resp. $I_{J,J'}, J \in R, J' \in C_0$). Since $M = -T_2IT_1$, we may partition $M$ in the same fashion and denote these blocks by $M_{J,J'}, J \in R, J' \in C$. Moreover,

$$M_{J,J'} = -\sum_{J_0' \in C_0, J_0 \in R_0} T_{2,J,J'} I_{J_0',J_0} T_{1,J_0,J'B}.$$ 

We notice that

- $I_{J_0,J_0'} \neq 0$ if and only if $J_0' = J_0 \cup \{(n+1)/2\} \cap C_0, J_0 \in R_0$.
- $T_{2,J,J'} \neq 0$ only if $J \subset J_0'$, $J \in R, J_0' \in C_0$.
- $T_{1,J,J'} \neq 0$ only if $J_0 \subset J'$, $J_0 \in R_0, J' \in C$.

Therefore, $M_{J,J'} \neq 0$ only if there exists $j < \lfloor (n+1)/2 \rfloor < j'$ such that

$$J' = J \setminus \{\lfloor (n+1)/2 \rfloor \} \cup \{j, j'\},$$

and

$$M_{J,J'} = -(T_{2,J,J_0 \cup \{j\}} I_{J_0 \cup \{j\}, J' \setminus \{j\}} T_{1,J_0 \cup \{j\}, J'},$$

$$+ T_{2,J,J_{0'} \cup \{j'\}} I_{J_{0'} \cup \{j'\}, J' \setminus \{j\}} T_{1,J_{0'} \cup \{j\}, J'}).$$

On the other hand, we recall from the proof of Theorem 5.4 that

$$I_{J_0 \cup \{\lfloor (n+1)/2 \rfloor \}, J} = \epsilon_{J_0 \cup \{\lfloor (n+1)/2 \rfloor \}} I_{n}, \quad T_{s,J_0 \cup \{j\}, J} = \epsilon_{J_0 \cup \{j\}} T_{J - \{\lfloor (n+1)/2 \rfloor \}},$$

$s = 1, 2$. Here $T_l$ is the Toeplitz matrix $(t_{ij})$ defined by

$$t_{ij} = \begin{cases} 1, & \text{if } j - i = l, \\ 0, & \text{otherwise.} \end{cases}$$

\[\text{Here we remark that according to the definition of } I, \text{ we should denote each block by } I_{J,J'} \text{ where } J \in R_0, J' \in C_0, \text{ but we switch } J \text{ with } J' \text{ to simplify our notation. This is valid as } \# R_0 = \# C_0 \text{ and } I^T = I.\]
Thus we obtain
\[
T_{2,i,j\cup\{j\}} = \epsilon_{j,i\cup\{j\}} T_{j-(n+1)/2},
\]
\[
I_{j,i\cup\{j\},j\setminus\{j'\}} = \epsilon_{j\setminus\{j'\},j'\setminus\{j\}} T_{j-(n+1)/2} \Id_n,
\]
\[
T_{1,j\setminus\{j'\}},j' = \epsilon_{j\setminus\{j'\},j'} T_{j-(n+1)/2},
\]
\[
T_{2,j\cup\{j\}},j' = \epsilon_{j,j\cup\{j\}} T_{j-(n+1)/2},
\]
\[
I_{j\cup\{j'\},j\setminus\{j\}} = \epsilon_{j\setminus\{j\},j\setminus\{j\}} T_{j-(n+1)/2} \Id_n,
\]
\[
T_{1,j\setminus\{j\}},j' = \epsilon_{j\setminus\{j\},j'} T_{j-(n+1)/2}.
\]

We set
\[
\delta_{j,j'} = \delta_{j,j'} = \epsilon_{j,j\cup\{j'\}} \epsilon_{j\setminus\{j'\},j\setminus\{j\}} \epsilon_{j\setminus\{j\}}.
\]
\[
\delta_{j,j',j} = \epsilon_{j,j\cup\{j\}} \epsilon_{j\setminus\{j\},j\setminus\{j\}} \epsilon_{j\setminus\{j\}}.
\]
and it is straightforward to verify that \( \delta_{j,j',j} = -\delta_{j,j',j} \). Hence we may write
\[
(M_{j,j'})_{i,k} = \begin{cases}
\delta_{j,j'}, & \text{if } j' - [(n+1)/2] \geq k \geq j + j' - 2[(n+1)/2] + 1, \\
-\delta_{j,j'}, & \text{if } n \geq k \geq 2n - r + j, \\
0, & \text{otherwise}.
\end{cases}
\]
\[
(M_{j,j'})_{i,k} = \begin{cases}
\delta_{j,j'}, & \text{if } j' - [(n+1)/2] \geq k \geq 1, \\
-\delta_{j,j'}, & \text{if } j + j' - 2r + 3n - 2 \geq k \geq j - r + 2n, \\
0, & \text{otherwise}.
\end{cases}
\]

In particular, we have
\[
(M_{j,j'})_{i,[n+1]/2} = (M_{j,j'})_{[n+1]/2,j,k} = 0
\]
for any \( J \in R, J' \in C \) and \( 1 \leq i \leq n \). By (A.2) and (A.3), we see that \( (M_{j,j'})_{i,k} = 0 \) unless
\[
\cdot \quad \text{if } j' \in R, J' \in C \text{ and } 1 \leq i \leq n.
\]

\[
\cdot \quad \text{if } j' - [(n+1)/2] < k < j',
\]
\[
\cdot \quad \text{if } k - i = (j + j') - 2[n+1]/2,
\]
\[
\cdot \quad \text{if } j + j' - 2[n+1]/2 \geq 0,
\]
\[
\cdot \quad \text{if } j + j' - 2[n+1]/2 < 0,
\]

\[
k \in [j + j' - 2[n+1]/2 + 1, j' - (n+1)/2] \cup [2n - r + j, n],
\]
\[
\text{if } j + j' - 2[n+1]/2 < 0,
\]
\[k \in [1, j' - (n+1)/2] \cup [j - r + 2n, j + j' - 2r + 3n - 2].\]
In particular, if both \((M_{j,j'})_{i,k}\) and \((M_{j',j})_{i,k}\) are nonzero, then we must have

(A.5) \[
\bar{k} - k = \sum_{i=1}^{p+1} (\bar{j}'_i - \bar{j}'_i) = (\bar{j} + \bar{j}') - (j + j'),
\]

where \(J' = (j'_1 < \cdots < j'_{p+1}) = J \setminus \{(n+1)/2\} \cup \{j, j'\} \) and \(\bar{J}' = (\bar{j}'_1 < \cdots < \bar{j}'_{p+1}) = J \setminus \{(n+1)/2\} \cup \{\bar{j}, \bar{j}'\} \). Similarly, if both \((M_{j,j'})_{i,k}\) and \((M_{j',j})_{i,k}\) are nonzero, then we must have

(A.6) \[
\bar{i} - i = (\bar{j} + \bar{j}') - (j + j'),
\]

where \(J \setminus \{(n+1)/2\} \cup \{j, j'\} = J' = \bar{J} \setminus \{(n+1)/2\} \cup \{\bar{j}, \bar{j}'\} \).

**Step 3:** We may write \(J \in R\) as

\[ J = (j_1 < \cdots < j_s < (n+1)/2 < j_{s+1} < \cdots < j_p) \]

and write \(J' \in C\) as

\[ J' = (j'_1 < \cdots < j'_s < (n+1)/2 < j'_{s+1}) \]

where \(j'_s < (n+1)/2 < j'_{s+1}\).

By (A.1), we see that in particular, \(M_{j,j'} = 0\) if \(s + 1 \neq t\). This implies that the matrix \(M\) is a block diagonal matrix \(M = \text{Diag}\{M_1, \ldots, M_p\}\) where \(M_s, 1 \leq s \leq (n-1)/2\) is the submatrix of \(M\) obtained by taking \(J = (j_1 < \cdots < j_p) \in R\) and \(J' = (j'_1 < \cdots < j'_{p+1}) \in C\) such that

\[ j_s = [(n+1)/2], \quad j'_{s+1} = [(n+1)/2] < j'_{s+2}. \]

We define \(R_s\) to be the subset of \(R\) consisting of all \(J = (j_1 < \cdots < j_p) \in R\) such that \(j_s = [(n+1)/2]\) and define \(C_s\) to be the subset of \(C\) consisting of all \(J' = (j'_1 < \cdots < j'_{p+1}) \in C\) such that \(j'_s < [(n+1)/2] < j'_{s+1}\).

For each \(1 \leq s \leq (n-1)/2\) and \(J \in R_s, J' \in C_s\), we remove \((J,i)\)-th row from \(M_s\) where \(p-s+2 \leq i \leq n-s\) and we remove \((J',k)\)-th column from \(M_s\) where \(p-s+2 \leq k \leq n-s\). We still denote the new matrix by \(M_s\). We denote by \(v_{j',k}\) the \((J',k)\)-th column vector of \(M_s\) if \(J' \in C_s\). We use the same notation to denote column vectors of \(M\) before, but since \(M\) is a block diagonal matrix with diagonal blocks \(M_0, \ldots, M_p\), this abuse of the notation should cause no confusion.

We remark that the matrix \(M_s\) is of size

\[
\binom{n-p-1}{s-1} \binom{p}{s} (p+1) \times \binom{n-p-1}{s-1} \binom{p}{s} (p+1)
\]

and that

\[
\binom{n-p-1}{s-1} \binom{p}{s} (p+1) \geq \binom{n-p-1}{s-1} \binom{p}{s} (p+1).
\]

Hence it suffices to prove that the matrix \(M_s\) has the full rank, or equivalently, the set

(A.7) \[
S_s = \{v_{j',k} : J' \in C_s, k = 1, \ldots, p-s+1, n-s+1, \ldots, n\}
\]

is a linearly independent set.

**Step 4:** Let \(s\) be an integer such that \(1 \leq s \leq [(n-1)/2]\) and let \(S_s\) be the set defined in (A.7). To prove that \(S_s\) is a linearly independent set, we consider

(A.8) \[
\sum_{J' \in C_s, k=1, \ldots, p-s+1, n-s+1, \ldots, n} x_{j',k} v_{j',k} = 0,
\]
where \( x_{J',k} \)'s are unknowns and we want to prove that \( x_{J',k} = 0 \) for all \( J' \in C_s \) and \( k = 1, \ldots, p-s+1, n-s+1, \ldots, n \). Since \( v_{J',k} \) is the \((J',k)\)-th column vector of \( M_s \), the \((J,i)\)-th entry \( v_{J,k}^{i,i} \) of \( v_{J',k} \) is equal to the \((i,k)\)-th entry \((M_{J,J'})_{i,k}\) of \( M_{J,J'} \) defined by (A.2) and (A.3). Hence from (A.3) we have for each \( J \in R_s \) and \( 1 \leq i \leq n \) the following linear equation:

\[
(M_{J,J'})_{i,k} x_{J',k} = 0. \tag{A.9}
\]

We notice that if \((M_{J,J'})_{i,k} \neq 0\) and \( k \leq p-s+1 \) (resp. \( k \geq n-s+1 \)), then whenever \((M_{J,J'})_{i,k} \neq 0\), we must also have \( \tilde{k} \leq p-s+1 \) (resp. \( \tilde{k} \geq n-s+1 \)). This can be seen from (A.5) and (A.4). Hence (A.9) can be simplified as:

\[
\sum_{J' \in C_s, k=1, \ldots, p-s+1, n-s+1, \ldots, n} (M_{J,J'})_{i,k} x_{J',k} = 0 \quad \text{or} \quad \sum_{J' \in C_s, k=n-s+1, \ldots, n} (M_{J,J'})_{i,k} x_{J',k} = 0. \tag{A.10}
\]

We denote by \( X^t_j \) the set of variables \( x_{J',k} \) whose coefficient \((M_{J,J'})_{i,k} \neq 0\). Let \( A = \{(J_1, i_1), \ldots, (J_m, i_m)\} \) be a maximal set such that for any \((J_a, i_a) \in A\), there exists some \((J_b, i_b) \in A\), \( a \neq b \),

\[
X^t_0 \cap X^t_b, X^t_0 \not= \emptyset.
\]

If \( x_{J',k} \not\in \cup_{(J,i) \in A} X^t_j \) then we see that \( x_{J',k} \) is independent on \( x_{J',k} \in \cup_{(J,i) \in A} X^t_j \). Therefore, it is sufficient to prove that the solution of the linear system:

\[
\sum_{J' \in C_s, k=1, \ldots, p-s+1, n-s+1, \ldots, n} (M_{J,J'})_{i,k} x_{J',k} = 0, \quad t = 1, \ldots, m. \tag{A.12}
\]

is zero. By (A.10) and (A.11), we see that

- either \( k \leq p-s+1 \) for all \( x_{J',k} \in \cup_{(J,i) \in A} X^t_j \),
- or \( k \geq n-s+1 \) for all \( x_{J',k} \in \cup_{(J,i) \in A} X^t_j \).

Similarly, we also have

- either \( i \leq \lfloor (p-s+1) \rfloor \) for all \((J,i) \in A\),
- or \( i \geq n-s+1 \) for all \((J,i) \in A\).

If there exists \( x_{J',k} \in \cup_{(J,i) \in A} X^t_j \) such that \( k \geq n-s+1 \) and \( i \leq p-s+1 \), then \( k - i \geq n-p \). However, we have

\[
k - i = (j + j') - 2\left\lfloor \frac{n+1}{2} \right\rfloor,
\]

this implies that

\[
j + j' \geq n + 2\left\lfloor \frac{n+1}{2} \right\rfloor - p \geq n + \left\lfloor \frac{n+1}{2} \right\rfloor,
\]

which contradicts the assumption that \( j < \lfloor (n+1)/2 \rfloor < j' \leq n \). Similarly, we may prove that if \( k \leq p-s+1 \) and \( i \geq n-s+1 \), then

\[
j + j' \leq p - n + 2\left\lfloor \frac{n+1}{2} \right\rfloor \leq p + 1 \leq \left\lfloor \frac{n+1}{2} \right\rfloor + 1,
\]
which contradicts the assumption that \(1 \leq j < \lfloor (n+1)/2 \rfloor < j'\). Hence if \((J, i) \in A\) and \(i \leq p - s + 1\) (resp. \(i \geq n - s + 1\)), then all \(x_{j', k} \in \cup_{(J, i) \in A} X_{\tilde{j}_i}^s\) must have \(k \leq p - s + 1\) (resp. \(k \geq n - s + 1\)). We denote by \(E_1\) the set of integers \(1, 2, \ldots, p - s + 1\) and by \(E_2\) the set of integers \(n - s + 1, n - s + 2, \ldots, n\).

According to (A.6) and (A.3), we may describe the set \(A\) as follows: if \((J, i) \in A, i \in E_s, s = 1, 2\), then \((J, i) \in A\) if and only if

\[
\tilde{i} - i = \sum_{t=1}^{p} (\tilde{j}_t - j_t), \quad \tilde{i} \in E_s.
\]

The set \(\cup_{(J, i) \in A} X_{\tilde{j}_i}^s\) can be described as in a similar way: if \(x_{j', k} \in \cup_{(J, i) \in A} X_{\tilde{j}_i}^s, i, k \in E_s, s = 1, 2\), then \(x_{\tilde{j}', \tilde{k}} \in \cup_{(J, i) \in A} X_{\tilde{j}_i}^s\) if and only if

\[
\tilde{k} - k = \sum_{t=1}^{p+1} (\tilde{j'}_t - j'_t), \quad \tilde{k} \in E_s.
\]

**Step 5:** \(A = \{(J_1, i_1), \ldots, (J_m, i_m)\}\) be a maximal set such that for any \((J_a, i_a) \in A\) there exists some \((J_b, i_b) \in A\) such that

\[
X_{\tilde{j}_a}^s \cap X_{\tilde{j}_b}^s \neq \emptyset, \quad a \neq b.
\]

We prove that every \(x_{j', k} \in \cup_{(J, i) \in A} X_{\tilde{j}_i}^s\) is equal to zero. To see this, it is sufficient to prove that the solution to the linear system (A.12) over the field \(\mathbb{Z}_2 = \{0, 1\}\) must be trivial, i.e., \(x_{j', k} = 0\). Indeed, if (A.12) has a nontrivial solution over \(\mathbb{C}\), then it also has a nontrivial solution over \(\mathbb{Z}\) since coefficients of (A.12) are \(-1, 1\) or 0 and hence in particular are integers. Moreover, among these nontrivial integer solutions, there must be a solution \((a_{J', k})_{x_{j', k} \in \cup_{(J, i) \in A} X_{\tilde{j}_i}^s}\) such that

\[
a_{J', k} \equiv 1 \pmod{2}
\]

for some \((J', k)\). Equivalently, (A.12) has a nontrivial solution over \(\mathbb{Z}_2\).

**Step 6:** We denote by \(M_{s, A}\) the coefficient matrix of the system (A.12) and we suppose that \(M_{s, A}\) is an \(m \times l\) matrix. By the construction of \(M_{s, A}\) we know that \(m \geq l\). We define an order on column indices \(\{(J'_1, k_1), \ldots, (J'_l, k_l)\}\) of \(M_{s, A}\) by \((J'_a, k_a) > (J'_b, k_b)\) if

\[
\sum_{t=s+1}^{p+1} j'_a t - \sum_{t=1}^{s} j'_a t > \sum_{t=s+1}^{p+1} j'_b t - \sum_{t=1}^{s} j'_b t
\]

or

\[
\sum_{t=s+1}^{p+1} j'_a t - \sum_{t=1}^{s} j'_a t = \sum_{t=s+1}^{p+1} j'_b t - \sum_{t=1}^{s} j'_b t \text{ and } k_a > k_b,
\]

where

\[
J'_a = (j'_a, 1 < j'_a, s < j'_a, s+1 < \cdots < j'_a, p+1) \in C_s,
\]

\[
J'_b = (j'_b, 1 < j'_b, s < j'_b, s+1 < \cdots < j'_b, p+1) \in C_s.
\]

Similarly, we may define an order on the set \(A = \{(J_1, i_1), \ldots, (J_m, i_m)\}\) by \((J_a, i_a) > (J_b, i_b)\) if

\[
\sum_{t=s+1}^{p} j_{at} t - \sum_{t=1}^{s-1} j_{at} t > \sum_{t=s+1}^{p} j_{bt} t - \sum_{t=1}^{s-1} j_{bt} t
\]
or
\[
\sum_{t=s+1}^{p} j_{at} - \sum_{t=1}^{s-1} j_{at} = \sum_{t=s+1}^{p} j_{bt} - \sum_{t=1}^{s-1} j_{bt} \quad \text{and} \quad i_a > i_b,
\]

where
\[
J_a = (j_{a,1} < \cdots < j_{a,s-1} < j_{a,s} = [(n+1)/2] < j_{a,s+1} < \cdots < j_{a,p}) \in R_s,
\]
\[
J_b = (j_{b,1} < \cdots < j_{b,s-1} < j_{b,s} = [(n+1)/2] < j_{b,s+1} < \cdots < j_{b,p}) \in R_s.
\]

With the order defined above, we may reorder \((J_1, i_1), \ldots, (J_m, i_m)\) and \((J'_1, k_1), \ldots, (J'_l, k_l)\) respectively so that
\[
(J_a, i_a) \leq (J_b, i_b), \quad 1 \leq a < b \leq m,
\]
\[
(J'_α, k_α) \leq (J'_β, k_β), \quad 1 \leq α < β \leq l.
\]

To prove that \((A.12)\) only has a trivial solution over \(\mathbb{Z}_2\), it suffices to prove that

- there exists \((J, i) \in A\) such that the \((J, i)\)-th row vector \(w\) of \(M_{s,A}\) has exactly one nonzero entry indexed by \((J', k)\) and
- The submatrix obtained by removing \((J, i)\)-th row and \((J', k)\)-th column from \(M_{s,A}\) has the full rank.

In fact, the row vector \(w\) is one of the following:

- \(w\) is the \((J_1, i_1)\)-th row of \(M_{s,A}\).
- \(w\) is the \((J_m, i_m)\)-th row of \(M_{s,A}\).
- \(w\) is the \((J, i)\)-th row of \(M_{s,A}\), where \((J, i)\) is maximal among those such that \((M_{s,A})(J,i),(J'_1,k_1)\).
- \(w\) is the \((J, i)\)-th row of \(M_{s,A}\), where \((J, i)\) is minimal among those such that \((M_{s,A})(J,i),(J'_1,k_1)\).
- \(w\) is the \((J, i)\)-th row of \(M_{s,A}\), where \((J, i)\) is maximal among those such that \((M_{s,A})(J,i),(J'_l,k_l)\).
- \(w\) is the \((J, i)\)-th row of \(M_{s,A}\), where \((J, i)\) is minimal among those such that \((M_{s,A})(J,i),(J'_l,k_l)\).

We denote by \(M_{s,A}^1\) the matrix obtained by removing the \((J, i)\)-th row and \((J', k)\)-th column from \(M_{s,A}\). Then \(M_{s,A}^1\) has a row vector which contains exactly one nonzero entry. Indeed, we can find this row vector in the same way as we find the row vector \(w\) for \(M_{s,A}\). Hence by induction, we see that \(M_{s,A}\) must have the full rank and this completes the proof. \(\square\)

We illustrate the proof of Lemma \(\ref{lem:example}\) by some examples.

**Example A.1.** By the definition of \(M_{s,A}\), we see that for any positive integer \(n\), if \(s = 1\) or \([ (n-1)/2 ] \), then \(M_{s,A}\) is simply a \(p \times p\) triangular matrix whose diagonal entries are all 1. In this case, we see that \(M_{s,A}\) obviously has the full rank.

**Example A.2.** Let \(n = 6\) and \(s = 2\), then \(p = [\frac{n+1}{2}] = 3\),
\[
\lfloor \frac{n+1}{2} \rfloor = 3, \quad p + 1 - s = 2, \quad n - s + 1 = 5.
\]

We also have
\[
C_s = \{(1, 2, 4, 5), (1, 2, 4, 6), (1, 2, 5, 6)\},
\]
\[
R_s = \{(1, 3, 4), (1, 3, 5), (1, 3, 6), (2, 3, 4), (2, 3, 5), (2, 3, 6)\}.
\]
The matrix $M_s$ is

$$
M_s = \begin{bmatrix}
(1,3,4) & (1,2,4,5) & (1,2,4,6) & (1,2,5,6) \\
(1,3,5) & T_{-1,2} & T_{-1,3} & 0 \\
(1,3,6) & 0 & T_{-1,3} & T_{-1,2} \\
(2,3,4) & T_{-2,2} & T_{-2,3} & 0 \\
(2,3,5) & T_{-2,1} & 0 & T_{-2,3} \\
(2,3,6) & 0 & T_{-2,1} & T_{-2,2}
\end{bmatrix},
$$

where $T_{a,b}$ is the submatrix obtained by removing columns $k = 3, 4$ and rows $i = 3, 4$ from $T_aT_b - T_bT_a$ over $\mathbb{Z}_2$. Here $T_a$ is the $n \times n$ Toeplitz matrix whose $(u,v)$-th entry is one if $v - u = a$ and is zero otherwise. For example, $T_{-1,2}$ and $T_{-2,2}$ are

$$
T_{-1,2} = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad T_{-2,2} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.
$$

If we take

$$
A = \{(2,3,4), 1\}, ((2,3,5), 2), ((1,3,5), 1), ((1,3,6), 2)\},
$$

then the matrix $M_{s,A}$ is

$$
M_{s,A} = \begin{bmatrix}
((2,3,4), 1) & ((1,2,4,5), 1) & ((1,2,4,6), 2) \\
((2,3,5), 2) & 1 & 1 \\
((1,3,5), 1) & 1 & 0 \\
((1,3,6), 2) & 0 & 0
\end{bmatrix}.
$$

It is straightforward to verify that the $((1,3,5), 1)$-th row of $M_{s,A}$ has only one nonzero entry, which is indexed by $(1,2,4,5), 1)$. We delete the $((1,3,5), 1)$-th row and the $((1,2,4,5), 1)$-th column of $M_{s,A}$ to obtain $M_{s,A}^1 = [1]$, which has the full rank.

**Example A.3.** We let $n = 7, s = 2$ then $p = 3$,

$$
\left\lfloor \frac{n + 1}{2} \right\rfloor = 4, \quad p + 1 - s = 2, \quad n - s + 1 = 6.
$$

We take

$$
A = \{(2,4,5), 1\}, ((3,4,5), 2), ((1,4,6), 1), ((2,4,6), 2), ((1,4,7), 2)\}.
$$

The matrix $M_{s,A}$ is

$$
M_{s,A} = \begin{bmatrix}
((3,4,5), 2) & ((2,3,5,6), 2) & ((1,3,5,6), 1) & ((1,2,5,7), 2) \\
((2,4,5), 1) & 1 & 0 & 0 \\
((2,4,6), 2) & 0 & 0 & 0 \\
((1,4,6), 1) & 0 & 1 & 0 \\
((1,4,7), 2) & 0 & 0 & 1
\end{bmatrix}.
$$
We see that the $((2, 4, 6), 2)$-th row has the unique nonzero entry indexed by $((1, 2, 6, 7), 2)$. We obtain $M^{1}_{s, A}$ by removing $((2, 4, 6), 2)$-th row and $((1, 2, 6, 7), 2)$-th column:

\[
M^{1}_{s, A} = \begin{pmatrix}
((3, 4, 5), 2) & ((2, 3, 5, 6), 2) & ((1, 3, 5, 6), 1) & ((1, 3, 5, 7), 2) & ((1, 2, 5, 7), 1) \\
1 & 1 & 1 & 0 \\
((2, 4, 5), 1) & 1 & 0 & 0 & 1 \\
((1, 4, 6), 1) & 0 & 1 & 0 & 0 \\
((1, 4, 7), 2) & 0 & 0 & 0 & 1 \\
\end{pmatrix}.
\]

By the same procedure, we obtain

\[
M^{2}_{s, A} = \begin{pmatrix}
((3, 4, 5), 2) & ((2, 3, 5, 6), 2) & ((1, 3, 5, 6), 1) & ((1, 3, 5, 7), 2) \\
1 & 1 & 1 & 1 \\
((2, 4, 5), 1) & 1 & 0 & 0 \\
((1, 4, 6), 1) & 0 & 1 & 0 \\
\end{pmatrix}
\]

by removing $((1, 4, 7), 1)$-th row and $((1, 2, 5, 7), 1)$-th column of $M^{1}_{s, A}$ and we obtain

\[
M^{3}_{s, A} = \begin{pmatrix}
((3, 4, 5), 2) & ((1, 3, 5, 6), 1) & ((1, 3, 5, 7), 2) \\
1 & 1 & 0 \\
((2, 4, 5), 1) & 0 & 1 \\
((1, 4, 6), 1) & 1 & 0 \\
\end{pmatrix},
\]

\[
M^{4}_{s, A} = ((3, 4, 5), 2) \begin{pmatrix}
1 \\
\end{pmatrix},
\]

by removing $((2, 4, 5), 1)$-th row and $((2, 3, 5, 6), 2)$-th column of $M^{2}_{s, A}$ and removing $((1, 4, 6), 1)$-th row and $((1, 3, 5, 6), 1)$-th column of $M^{3}_{s, A}$, respectively.

**Example A.4.** Let $n = 8, s = 2$ then $p = \lfloor n/2 \rfloor = 4$ and

\[
\lfloor \frac{n + 1}{2} \rfloor = 4, \quad p + 1 - s = 3, \quad n - s + 1 = 7.
\]

We consider

\[
A = ((3, 4, 5, 6), 1), ((3, 4, 5, 7), 2), ((3, 4, 6, 7), 3), ((2, 4, 5, 7), 1), ((3, 4, 5, 8), 3), ((2, 4, 5, 8), 2), ((1, 4, 6, 7), 1), ((2, 4, 6, 8), 3), ((1, 4, 5, 8), 2), ((1, 4, 7, 8), 3).
\]

The corresponding $M_{s, A}$ is

\[
M_{s, A} = \begin{pmatrix}
((2, 3, 5, 6, 7), 2) & ((1, 3, 5, 6, 7), 1) & ((2, 3, 5, 6, 8), 3) & ((1, 3, 5, 6, 8), 2) & ((1, 2, 5, 6, 8), 1) & ((1, 3, 5, 7, 8), 3) & ((1, 2, 5, 7, 8), 2) & ((1, 2, 6, 7, 8), 3) \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
((3, 4, 5, 7), 2) & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
((2, 4, 6, 7), 3) & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
((2, 4, 5, 7), 1) & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
((3, 4, 5, 8), 3) & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
((2, 4, 5, 8), 2) & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
((1, 4, 6, 7), 1) & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
((1, 4, 6, 8), 3) & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
((1, 4, 7, 8), 3) & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

and we remove $((1, 4, 7, 8), 3)$-th row and $((1, 2, 5, 7, 8), 2)$-th column of $M_{s, A}$ to obtain

\[
M^{1}_{s, A} = \begin{pmatrix}
((3, 4, 5, 6), 1) & ((3, 4, 5, 7), 2) & ((3, 4, 6, 7), 3) & ((2, 4, 5, 7), 1) & ((3, 4, 5, 8), 3) & ((2, 4, 5, 8), 2) & ((1, 4, 6, 7), 1) & ((1, 4, 6, 8), 3) & ((1, 4, 7, 8), 3) \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
((3, 4, 5, 7), 2) & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
((3, 4, 6, 7), 3) & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
((2, 4, 5, 7), 1) & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
((3, 4, 5, 8), 3) & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
((2, 4, 5, 8), 2) & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
((1, 4, 6, 7), 1) & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
((1, 4, 6, 8), 3) & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
((1, 4, 7, 8), 3) & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{pmatrix}.
\]
We remove \(((2, 4, 5, 7), 1)\)-th row and \(((2, 3, 5, 6, 7), 2)\)-th column of \(M_{s,A}^1\) to obtain

\[
M_{s,A}^2 = \begin{bmatrix}
((1, 3, 5, 6, 7), 1) & ((2, 3, 5, 6, 8), 3) & ((1, 3, 5, 6, 8), 2) & ((1, 2, 5, 6, 8), 1) & ((1, 3, 5, 7, 8), 3) & ((1, 2, 6, 7, 8), 3)
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
((1, 3, 5, 6, 7), 1) \\
((2, 3, 5, 6, 8), 3) \\
((1, 3, 5, 7, 8), 3) \\
((1, 2, 5, 6, 8), 1) \\
((1, 3, 5, 6, 7), 1) \\
((1, 4, 6, 8), 2)
\end{bmatrix}.
\]

We remove \(((3, 4, 6, 7), 3)\)-th row and \(((1, 3, 5, 6, 7), 1)\)-th column of \(M_{s,A}^3\) to obtain

\[
M_{s,A}^3 = \begin{bmatrix}
((1, 3, 5, 6, 7), 1) & ((1, 3, 5, 6, 8), 3) & ((1, 2, 5, 6, 8), 1) & ((1, 3, 5, 7, 8), 3) & ((1, 2, 6, 7, 8), 3)
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
((1, 3, 5, 6, 7), 1) \\
((1, 3, 5, 6, 8), 3) \\
((1, 2, 5, 6, 8), 2) \\
((1, 4, 6, 7), 1) \\
((1, 4, 6, 8), 3) \\
((1, 4, 6, 8), 2)
\end{bmatrix}.
\]

We remove \(((1, 4, 6, 7), 1)\)-th row and \(((1, 2, 6, 7, 8), 3)\)-th column of \(M_{s,A}^3\) to obtain

\[
M_{s,A}^4 = \begin{bmatrix}
((1, 3, 5, 6, 7), 1) & ((1, 3, 5, 6, 8), 3) & ((1, 2, 5, 6, 8), 1) & ((1, 3, 5, 7, 8), 3) & ((1, 2, 6, 7, 8), 3)
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
((1, 3, 5, 6, 7), 1) \\
((1, 3, 5, 6, 8), 3) \\
((1, 2, 5, 6, 8), 2) \\
((2, 4, 6, 8), 3) \\
((1, 4, 6, 8), 2) \\
((1, 4, 6, 8), 2)
\end{bmatrix}.
\]

It is easy to determine \(M_{s,A}^5, M_{s,A}^6\) and lastly \(M_{s,A}^7 = [1, 0, 0]^T\).

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