Local Petrovskii lacunas close to parabolic singular points of the wavefronts of strictly hyperbolic partial differential equations

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Abstract. We enumerate the local Petrovskii lacunas (that is, the domains of local regularity of the principal fundamental solutions of strictly hyperbolic PDEs with constant coefficients in $\mathbb{R}^N$) close to parabolic singular points of their wavefronts (that is, at the points of types $P_8^1$, $P_8^2$, $\pm X_9$, $X_9^1$, $X_9^2$, $J_{10}^1$ and $J_{10}^3$). These points form the next most difficult family of classes in the natural classification of singular points after the so-called simple singularities $A_k$, $D_k$, $E_6$, $E_7$ and $E_8$, which have been investigated previously.

Also we present a computer program which counts the topologically distinct morsifications of critical points of smooth functions, and hence also the local components of the complement of a generic wavefront at its singular points.

Bibliography: 22 titles.

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§ 1. Introduction

The lacunas of a hyperbolic PDE are the components of the complement of its wavefront such that the principal fundamental solution of this equation can be extended from any such component to a regular function in some neighbourhood of it. The theory of lacunas started with Petrovskii [18]. He related this regularity condition to the topology and geometry of algebraic manifolds and gave a criterion for it in terms of certain homology classes of complex projective algebraic manifolds defined by the principal symbol of the hyperbolic operator. This theory was further developed in numerous works including [6]–[10], [12], [17], [19]–[22]. An earlier work which was important in this regard is [16]. Most of these works also treat a local aspect of the problem, formulated explicitly in [7] in terms of local lacunas and a local version of the Petrovskii topological condition.

Any hyperbolic operator with constant coefficients in $\mathbb{R}^N$ admits a unique fundamental solution of the Cauchy problem, which has support in a proper cone in the half-space $\mathbb{R}^N_+$. This fundamental solution is regular (that is, locally it coincides with some smooth analytic functions) everywhere in $\mathbb{R}^N$ outside some conic

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semi-algebraic hypersurface in $\mathbb{R}^N_+$, called the *wavefront* of our operator. We consider only *strictly* hyperbolic operators, which means that the cone $A(P) \subset \mathbb{R}^N$ of zeros of the principal symbol of our operator $P$ is nonsingular away from the origin in $\mathbb{R}^N$. Here $\mathbb{R}^N$ is the dual space of *momenta* with coordinates $\eta_j \equiv \frac{1}{i} \frac{\partial}{\partial x_j}$, so that the operator $P$ is considered as a polynomial in these variables. In this case the wavefront $W(P) \subset \mathbb{R}^N_+$ is just the projectively dual cone of $A(P)$, that is, the union of the rays from the origin in $\mathbb{R}^N_+$ whose orthogonal hyperplanes in $\mathbb{R}^N$ are tangent to $A(P)$. Singular points of the wavefront (apart from the origin) correspond via projective duality to points of inflection of $A(P)$, that is, to points where the rank of the second fundamental form of this cone is less than $N - 2$. A deep classification of these singular points was developed in Arnold’s works, see [3], for example.

**Definition 1.** A *local $C^\infty$-lacuna* (respectively, a *holomorphic local lacuna*) close to some point of the wavefront is a component of the complement of the wavefront in a small neighbourhood of this point such that the restriction of the principal fundamental solution to this component can be extended to a $C^\infty$-smooth function on the closure of this component (to an analytic function in a neighbourhood of this point, respectively).

The same (global) component of the complement of the wavefront can be a local lacuna at some points on its boundary and not a local lacuna at other points.

The local lacunas occurring in neighbourhoods of all singularities of wavefronts from an initial segment of the Arnold classification (so-called *simple* singularities) were enumerated in [20] and [21]. In this work we study and enumerate the holomorphic local lacunas adjoining the singularities whose classes belong to the next natural segment of this classification.

### 1.1. Previous results on local lacunas

Nonsingular points of the wavefront of the operator $P$ correspond to points of the cone $A(P)$ at which its second fundamental form is maximally nondegenerate. The existence and number of local lacunas close to such points of the wavefront can be determined in terms of its differential geometry, see [10] and [8]. Namely, a component of the complement of the wavefront at such a point is a local lacuna if and only if the positive inertia index of the second fundamental form of the wavefront (with the normal directed into this component) is even. Davydova [10] proved the ‘only if’ part of this statement: if this signature condition is not satisfied, then the leading term of the asymptotics of the fundamental solution behaves as a half-integer (but not integer) power of the distance from the wavefront. Borovikov [8], by means of complicated analytic estimates, proved that otherwise we have a local lacuna, that is, all the terms of the asymptotic expansion of this solution in terms of this distance have integer powers, and the corresponding power series converges. His result was later explained in [7] as a corollary to the removable singularity theorem by moving into the complex domain.

All local lacunas adjoining the simplest singular points of the wavefronts, of types $A_2$ (cuspidal edges, see Figure 1) and $A_3$ (swallowtails), were counted in [12]. An interesting situation occurs close to a point on a cuspidal edge if $N$ is odd and the inertia indices, $i_\pm$, of the quadratic part of the *generating function* of our singular point (see §1.2 below) are also odd. For all other combinations of these numbers,
if a component of the complement of the wavefront close to the cuspidal edge is not a local lacuna, then the Davydova-Borovikov signature condition from the side of this component fails to hold at some nonsingular points of the wavefront arbitrarily close to the edge. However, in the case of odd $N$ and $i_{\pm}$ the Davydova-Borovikov condition from the side of the larger component (see Figure 1) is satisfied at all nearby nonsingular points of the wavefront, but nevertheless this component is not a local lacuna (nor is it for all other combinations of $N$ and $i_{\pm}$).

All local lacunas for all simple singularities of wavefronts (that is, singularities of classes $A_k$, $D_k$, $E_6$, $E_7$, $E_8$ in Arnold’s classification) were found in [20]. Table 1 gives a count of them.

Table 1. The number of local lacunas at simple singularities of wavefronts.

| Singularity class | $N$ even $i_+$ even | $N$ even $i_+$ odd | $N$ odd $i_+$ even | $N$ odd $i_+$ odd |
|-------------------|---------------------|---------------------|---------------------|---------------------|
| $A_1$             | 2                   | 0                   | 1                   | 1                   |
| $A_{2k}$, $k \geq 1$ | 0                   | 0                   | 1                   | 0                   |
| $\pm A_{2k+1}$, $k \geq 1$ | 0                   | 1                   | 1                   | 1                   |
| $D_4^-$            | 0                   | 3                   | 1                   | 1                   |
| $D_{2k}^+$, $k \geq 2$ | 0                   | 0                   | 1                   | 1                   |
| $D_{2k}^-$, $k \geq 3$ | 0                   | 2                   | 1                   | 1                   |
| $\pm D_{2k+1}$, $k \geq 2$ | 0                   | 0                   | 1                   | 1                   |
| $\pm E_6$          | 0                   | 0                   | 1                   | 1                   |
| $E_7$              | 0                   | 0                   | 1                   | 1                   |
| $E_8$              | 0                   | 0                   | 1                   | 1                   |

Atiyah, Bott and Gårding [7] introduced a local version of the homological Petrovskii criterion and proved that it implies that the corresponding local component of the complement of the wavefront is a holomorphic local lacuna. In [20] the reverse implication was proved for finite-type points of wavefronts (that is, for points corresponding by projective duality to only finitely many lines in the complexification of $A(P)$; this condition is satisfied for all singular points of wavefronts of generic operators). In [21], an easy geometric criterion for a component to be a local lacuna
of a simple singularity was proved. Namely, it follows from the facts given above that if a local component of the complement of the wavefront is a local lacuna, then the Davydova-Borovikov signature condition is satisfied at all nonsingular points of its boundary, and in addition our component is the ‘smaller’ component of the complement of the wavefront at all points of type $A_2$ (that is, cuspidal edges) of this boundary, see Figure 1. In [21] it was proved that if the singularity is simple, and some technical condition is satisfied (the versality of the generating family, which always holds for the wavefronts of generic operators), then this necessary condition is also sufficient; moreover, in this case the concepts of local $C^\infty$-lacunas and holomorphic local lacunas are equivalent.

The next most important natural set of singularity classes of wavefronts is that of parabolic (or simple-elliptic) singularities, see [3]. It consists of seven one-parameter families of singularities, which are listed in the left-hand column of Table 2. The number of local lacunas at these singularities is shown in the remaining columns of this table: this, together with an explicit description of these lacunas, is the main result of our article, see Theorem 1 below. However, we need some preliminaries to describe these singularities and formulate this result accurately.

**Table 2. The number of local lacunas at parabolic singularities.**

| Singularity class | $N$ even $i_+\text{ even}$ | $N$ even $i_+\text{ odd}$ | $N$ odd $i_+\text{ even}$ | $N$ odd $i_+\text{ odd}$ |
|-------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|
| $P_8^1$           | $0_c$                       | $0_c$                       | $\geq 2$                    | $0$                         |
| $P_8^2$           | $0_c$                       | $0_c$                       | $\geq 2$                    | $0$                         |
| $\pm X_9$         | $1_c$                       | $0$                         | $\geq 2$                    | $0$                         |
| $X_9^1$           | $0$                         | $0$                         | $0$                         | $0$                         |
| $X_9^2$           | $0$                         | $\geq 4$                    | $0$                         | $0$                         |
| $J_1^3$           | $0_c$                       | $\geq 1$                    | $0$                         | $0$                         |
| $J_1^3$           | $0_c$                       | $0_c$                       | $0$                         | $0$                         |

### 1.2. Generating functions and generating families of wavefronts.

Given a point $x \in \mathbb{R}^N \setminus 0$ of the wavefront of a strictly hyperbolic operator $P$, the local geometry of this wavefront near this point (in particular the set of local components of its complement at this point) is determined by its *generating function*, which is just the function $f$ in the local equation

$$\xi_0 = f(\xi_1, \ldots, \xi_{N-2})$$

of the projectivization $A^*(P) \subset \mathbb{RP}^{N-1}$ of the set of zeros of the principal symbol of our operator. Here $\xi_0, \ldots, \xi_{N-2}$ are local affine coordinates in $\mathbb{RP}^{N-1}$ with the origin at the point of tangency of the hypersurface $A^*(P)$ and the hyperplane $L(x)$ orthogonal to the line containing $x$ and such that $L(x)$ has the equation $\xi_0 = 0$. In particular, $f$ has a critical point at the origin; the dual piece of the wavefront is smooth if this critical point is Morse.
Let $n = N - 2$ denote the number of variables of the generating functions of wavefronts in $\mathbb{R}^N$. The parabolic singularities studied in this work have generating functions which can be reduced to the following normal forms by a local diffeomorphism in $\mathbb{R}^n$. The functions in the class $P_8$ in appropriate (curvilinear) local coordinates have the expression $\varphi(x_1, x_2, x_3) + Q(x_4, \ldots, x_n)$, where $\varphi$ is a nondegenerate homogeneous cubic polynomial and $Q$ is a nondegenerate quadratic form in the remaining coordinates, for example $\pm x_4^2 \pm \cdots \pm x_n^2$. The projectivization of the zero set of the polynomial $\varphi$ can consist of one or two curves, therefore we obtain two subclasses, called $P^1_8$ and $P^2_8$, respectively. The remaining parabolic functions have the following normal forms (where the $Q$ are nondegenerate quadratic forms in the coordinates $x_3, \ldots, x_n$):

\[
\begin{align*}
\pm X_9 & : \pm (x_1^4 + \alpha x_1^2 x_2^2 + x_2^4 + Q), \quad \alpha > -2, \\
X_9^3 & : x_1 x_2 (x_1^2 + \alpha x_1 x_2 + x_2^2) + Q, \quad \alpha^2 < 4, \\
X_9^2 & : x_1 x_2 (x_1 + x_2) (x_1 + \alpha x_2) + Q, \quad \alpha \in (0, 1), \\
J^3_{10} & : x_1 (x_1 - x_2^2) (x_1 - \alpha x_2^2) + Q, \quad \alpha \in (0, 1), \\
J^1_{10} & : x_1 (x_1^2 + \alpha x_1 x_2^2 + x_2^4) + Q, \quad \alpha^2 < 4.
\end{align*}
\]

The index $i_+$ in Table 2 is the positive inertia index of the quadratic part $Q$ of the corresponding function.

Another important idea, which reduces the study of wavefronts to the context of critical points of functions, is that of generating families. In our case, this is the family of functions

\[ f_\lambda \equiv f(\xi_1, \ldots, \xi_{N-2}) - \lambda_0 - \lambda_1 \xi_1 - \cdots - \lambda_{N-2} \xi_{N-2}, \quad (1.1) \]

depending on the parameter $\lambda = (\lambda_0, \ldots, \lambda_{N-2}) \in \mathbb{R}^{N-1}$. It is natural to consider these parameters $\lambda_i$ as local affine coordinates in $\mathbb{R}^{N-1}$ close to the point $\{x\}$. In fact, any tuple $\lambda$ of these numbers defines the hyperplane $L(\lambda) \subset \mathbb{R}^{N-1}$ distinguished by

\[ \xi_0 = \lambda_0 + \lambda_1 \xi_1 + \cdots + \lambda_{N-2} \xi_{N-2}, \]

and hence the line in $\mathbb{R}^N$ orthogonal to it or a point in $\mathbb{R}P^{N-1}$. The projectivized wavefront close to our point in $\mathbb{R}P^{N-1}$ consists of all discriminant values of the parameters of the family $(1.1)$, that is, of those values of $\lambda$ for which the function $(1.1)$ has the critical value 0 (which is equivalent to the hypersurfaces $A^+(P)$ and $L(\lambda)$ being tangent).

Recall that a deformation of the function $f : \mathbb{R}^n \to \mathbb{R}$ is a function $F : \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}$ considered as a family of functions

\[ f_\lambda \equiv F(\cdot, \lambda) : \mathbb{R}^n \to \mathbb{R} \]

depending on the parameter $\lambda \in \mathbb{R}^l$, such that $f_0$ coincides with the function $f$ to be deformed. The discriminant variety of such a deformation is the set of values of its parameter $\lambda$ such that the corresponding function $f_\lambda$ has a critical point with zero critical value. In particular, the generating family of a wavefront is a deformation of its generating function, and the wavefront itself is the discriminant set of this deformation.
The concept of (holomorphic) local lacunas makes sense for arbitrary deformations of critical points of real functions (not necessarily related to wavefronts): a component of the complement of the discriminant set of such a deformation is a local lacuna if some homological condition concerning the level manifolds \( f_\lambda^{-1}(0) \) (that the local Petrovskii homology class described in §2 is trivial) is satisfied for the values of \( \lambda \) in this component. If our deformation is the generating family of the wavefront of a hyperbolic operator, then this concept turns out to be equivalent to the one described previously in terms of the fundamental solutions, see Proposition 2 in §2.

Among the deformations \( F(x, \lambda) \) of a function \( f \) with an isolated critical point we can distinguish a class of *versal deformations*, that is, of sufficiently ample deformations such that all other deformations can be reduced to them in some precise sense, see [3] and [22]. The number of parameters of a versal deformation cannot be smaller than the *Milnor number* \( \mu(f) \) of our critical point (which is the standard lower index in the notation for its class, like 8 for \( P^1_8 \)), on the other hand almost all deformations depending on \( \geq \mu(f) \) parameters satisfy this condition. An important consequence of the idea of versality is as follows: if the parameter space of some versal deformation of the function \( f \) contains no local lacunas, then the same is true for any other deformation of it.

**Theorem 1.** The number of holomorphic local lacunas in the parameter space of a versal deformation of a parabolic critical point of a function \( f_0: \mathbb{R}^{N-2} \to \mathbb{R} \) is equal to the number indicated in the corresponding cell of Table 2 or satisfies the inequality given in that table. (The subscript \( c \) in some cells means that in the corresponding case the upper bound on the number of local lacunas only has a computer proof).

In the case of nonversal deformations this statement remains true for all the cells in Table 2 containing 0 or 0. If the deformation contains all functions \( f_0 + \text{const} \) (which holds for all generating families of wavefronts) then both entries \( 1_c \) (in the second column) and \( \geq 2 \) (in the fourth column) for the singularity \( \pm X_9 \) can be replaced by \( \geq 1 \).

**Remark 1.** Some of these results were obtained previously, see [5] and [22], for example. The new results are as follows:

- the singularity \( P^2_8 \) is investigated for the first time;
- for \( \pm X_9 \), even \( N \) and even \( i_+ \) the estimate \( \geq 1 \) is replaced by the exact value \( 1_c \);
- for \( X^2_9 \), even \( N \) and odd \( i_+ \) the inequality \( \geq 2 \) is replaced by \( \geq 4 \);
- for \( J^3_1 \), even \( N \) and even \( i_+ \) the absence of local lacunas is proved;
- for \( J^3_1 \), even \( N \) and even or odd \( i_+ \) the absence of local lacunas is proved in both cases.

All local lacunas implied in nonzero cells of Table 2 will be produced in §5. All zeros (but not the entries \( 0_c \)) in this table follow from a topological obstruction described in §3. All the entries \( 0_c \) have been proved by means of a combinatorial Fortran program, which enumerates all possible topological types of morsifications of given critical points and checks the local Petrovskii condition for them. This program has also found for the first time the local lacunas for singularities \( P^1_8 \), \( X^2_9 \) and \( J^3_1 \), which are given below, as well as one of the local lacunas for \( \pm X_9 \),
see Proposition 9 in § 5.2. This program is described in § 6. The upper bound \( 1_e \) for the singularity \( \pm X_0 \) with even \( N \) and \( i_\lambda \) will be proved in § 8.

**Conjecture 1.** In all the cells in Table 2 (except maybe for the case \( X_0^2 \)) the inequalities can be replaced by equalities.

Here is some information supporting this conjecture. Given a Morse perturbation \( f_\lambda \) of a function \( f \) with a complicated critical point at the origin, let \( \chi(\lambda) \) denote the number of real critical points it has with negative critical values and even Morse index minus the number of critical points which also have negative values but which have odd Morse index.

**Proposition 1.** No perturbations of a parabolic singularity \( f \) which belong to local lacunas can have values of \( \chi(\lambda) \) distinct from those for the perturbations presented in § 5.

This fact is also proved by our program.

### § 2. Local Petrovskii classes and their properties

Let \( f: (\mathbb{C}^n, \mathbb{R}^n, 0) \to (\mathbb{C}, \mathbb{R}, 0) \) be a holomorphic function with an isolated critical point at 0, let \( \mu(f) \) be its Milnor number (see [3]), \( B_\varepsilon \subset \mathbb{C}^n \) a ball centred at 0 with small radius \( \varepsilon \). Let \( f_\lambda \) be a very small (with respect to \( \varepsilon \)) perturbation of \( f \), such that 0 is not a critical value of the \( f_\lambda \) in \( B_\varepsilon \). Consider the corresponding Milnor fibre \( V_\lambda \equiv f_\lambda^{-1}(0) \cap B_\varepsilon \). It is a smooth \((2n - 2)\)-dimensional manifold with boundary \( \partial V_\lambda \equiv V_\lambda \cap \partial B_\varepsilon \). By Milnor’s theorem it is homotopy equivalent to the wedge of \( \mu(f) \) spheres \( S^{n-1} \). In particular, \( \tilde{H}_{n-1}(V_\lambda) \simeq \mathbb{Z}^{\mu(f)} \simeq \tilde{H}_{n-1}(V_\lambda, \partial V_\lambda) \); here \( \tilde{H}_*(V_\lambda) \) denotes the homology group reduced modulo a point and \( \tilde{H}_*(V_\lambda, \partial V_\lambda) \) the relative homology group reduced additionally modulo the fundamental cycle. Furthermore, \( \mu(f) \) is equal to the number of critical points of \( f_\lambda \) in \( B_\varepsilon \) if the function \( f_\lambda \) is Morse and is sufficiently close to \( f \).

We fix an orientation of \( \mathbb{R}^n \), and assume that the differential form \( dx_1 \wedge \cdots \wedge dx_n \) is positive with respect to this orientation.

There are two important elements in the group \( \tilde{H}_{n-1}(V_\lambda, \partial V_\lambda) \), the even and odd Petrovskii classes. The first of these, \( P_{ev}(\lambda) \), is given by the cycle of real points \( \mathbb{R}^n \cap V_\lambda \) oriented by the differential form \( (dx_1 \wedge \cdots \wedge dx_n)/df_\lambda \). The definition of the second class, \( P_{odd} \), is a bit more complicated. First we consider the \( n \)-dimensional cycle \( \Pi(\lambda) \) in \( B_\varepsilon \setminus V_\lambda \) represented by two copies of the canonically oriented \( \mathbb{R}^n \), slightly moved in a small neighbourhood of the submanifold \( \mathbb{R}^n \cap V_\lambda \) in \( B_\varepsilon \) so that they streamline \( V_\lambda \) from two different sides in the complex domain: for the case \( n = 1 \) see the left-hand part of Figure 2, where the set \( V_\lambda \) is marked by thick dots.

The odd Petrovskii class \( P_{odd} \) is defined as the preimage of the homology class of this cycle under the Leray tube operator

\[
\tilde{H}_{n-1}(V_\lambda, \partial V_\lambda) \to \tilde{H}_n(B_\varepsilon \setminus V_\lambda, \partial B_\varepsilon),
\]

which sends any relative cycle in the submanifold \( V_\lambda \) to the union of boundaries of the fibres of the tubular neighbourhood of this submanifold over the points of this cycle. This operator is conjugate via the Poincaré-Lefschetz isomorphisms to the boundary isomorphism \( \tilde{H}_n(B_\varepsilon, V_\lambda) \to \tilde{H}_{n-1}(V_\lambda) \) and also is an isomorphism.
Let $F: \left( \mathbb{R}^n \times \mathbb{R}^l, 0 \right) \rightarrow (\mathbb{R}, 0)$ be a deformation of the function $f$ and $\Sigma(F) \subset \mathbb{R}^l$ the set of discriminant values of the parameter $\lambda$.

**Definition 2.** A local (close to the point $0 \in \mathbb{R}^l$) connected component of the set $\mathbb{R}^l \setminus \Sigma(F)$ is called an *even* (*odd*) local lacuna of the deformation $F$ if for any value of $\lambda$ in this component the element $P_{ev}(\lambda) \ (P_{odd}(\lambda))$ of the group $\tilde{H}_{n-1}(V_{\lambda}, \partial V_{\lambda})$ related to the corresponding perturbation $f_{\lambda}$ of $f$, is equal to 0.

This definition is consistent with Definition 1 for the following reason.

**Proposition 2.** Suppose that $x \in \mathbb{R}^N \setminus 0$ is a point on the wavefront of a strictly hyperbolic operator, the critical point of its generating function $f$ is isolated, and $y$ is a point outside the wavefront but very close to $x$. Let $y^* \in \mathbb{R}^{N-1}$ be the direction of the line containing $y$, $\lambda = (\lambda_0, \ldots, \lambda_{N-2})$ the local coordinates of the point $y^*$ in accordance with §1.2, and $f_{\lambda}$ the perturbation of $f$ with these values of $\lambda_i$ given by (1.1). Then $y$ belongs to a local (close to $x$) holomorphic lacuna of the hyperbolic operator if and only if $\lambda$ belongs to an even local lacuna of the corresponding generating family (if $N$ is even) or to an odd local lacuna (if $N$ is odd).

The ‘if’ part of this proposition was essentially proved in [7], and ‘only if’ was conjectured there and proved in the translator’s note to the Russian translation of [7], see also [20].

This proposition reduces the study of local lacunas to a problem on deformations of real critical points of functions, namely that of calculating local Petrovskii classes of perturbations of these functions and looking for perturbations for which these classes vanish.

### 2.1. Important example.

If the function $f$ has a minimum (maximum) point at 0, then the function $f + \tau \ (f - \tau)$ belongs to its even local lacuna for sufficiently small positive $\tau$.

In fact, in this case the set of real points of the corresponding Milnor fibre is empty.
Conjecture 2. If a function $f(x_1, x_2)$ has an isolated non-Morse critical point at 0, then its deformations have no even local lacunas unless $f$ has an extremum at the origin; in this case all critical values at real critical points of all its small perturbations which belong to such a lacuna are positive (if $f$ has a minimum point at the origin) or negative (if $f$ has a maximum).

2.2. Explicit calculation of local Petrovskii classes. The group $\tilde{H}_{n-1}(V_\lambda, \partial V_\lambda)$ is Poincaré dual to $\tilde{H}_{n-1}(V_\lambda)$, therefore any of its elements is completely characterized by its intersection indices with the basis elements of the latter group. For these basis elements we can take the vanishing cycles corresponding to critical points of $f_\lambda$ (see [4], [15] and [22], for example). In [17] and [20] explicit formulae for these intersection indices of both local Petrovskii classes with cycles vanishing at real critical points were calculated: these indices are expressed in the terms of the Morse indices of these critical points and the intersection indices of vanishing cycles. We do not give these large formulae here, but refer the reader to §V.1.6 in [22] or §5.1.4 in [5].

In particular, these formulae describe the Petrovskii classes completely if all the $\mu(f)$ critical points of the perturbation $f_\lambda$ are real and we know their Morse indices and the intersection indices of the corresponding vanishing cycles.

Remark 2. In this and other related calculations it is important to use the orientations of these vanishing cycles that are compatible with the fixed orientation of $\mathbb{R}^n$. Fortunately, the methods for calculating the intersection indices developed in [13], [1] and [11] use exactly these orientations. These methods give us the desired data for appropriate perturbations of all parabolic singularities except $P^2_8$. We solve the analogous problem in the latter case in §7.

2.3. Stabilization.

Definition 3 (see [3]). Two functions $f, \tilde{f} : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0)$ with critical points at 0 are equivalent if they can be taken one into the other by a germ of a diffeomorphism $G : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$, that is, $f \equiv \tilde{f} \circ G$. Two functions with critical points (maybe depending on a different number of variables) are stably equivalent if they become equivalent after adding on nondegenerate quadratic forms depending on additional variables.

For example, the functions $f(x) = x^k$, $f_1(x, y) = x^k + y^2$, $f_2(x, y, z) = x^k - yz$ and $f_3(x, y) = -y^2 + (x - y)^k$ are stably equivalent to one another, but they are not stably equivalent to the function $f_1(x, y) = x^k$.

If $F(x, \lambda)$ is a deformation of a function $f(x)$, $x = (x_1, \ldots, x_n)$, then the family of functions $F(x_1, \ldots, x_n, \lambda) \pm x_{n+1}^2 \pm \cdots \pm x_{n+m}^2$ depending on $n + m$ variables is a deformation of the stabilization $f(x) \pm x_{n+1}^2 \pm \cdots \pm x_{n+m}^2$ of $f(x)$; the latter deformation is versal if and only if $F(x, \lambda)$ is.

Proposition 3 (see [22]). The following conditions are equivalent:

1) a perturbation $f_\lambda$ of a function $f : (\mathbb{C}^n, \mathbb{R}^n, 0) \rightarrow (\mathbb{C}, \mathbb{R}, 0)$ belongs to an even (odd) local lacuna;

2) the perturbation $f_\lambda + x_{n+1}^2 + x_{n+2}^2$ of $f + x_{n+1}^2 + x_{n+2}^2 : (\mathbb{C}^{n+2}, \mathbb{R}^{n+2}, 0) \rightarrow (\mathbb{C}, \mathbb{R}, 0)$ belongs to an even (odd) local lacuna;
3) the perturbation \( f_\lambda - x_{n+1}^2 - x_{n+2}^2 \) of \( f - x_{n+1}^2 - x_{n+2}^2 : (\mathbb{C}^{n+2}, \mathbb{R}^{n+2}, 0) \rightarrow (\mathbb{C}, \mathbb{R}, 0) \) belongs to an even (odd) local lacuna;

4) the perturbation \( f_\lambda + x_{n+1}^2 - x_{n+2}^2 \) of \( f + x_{n+1}^2 - x_{n+2}^2 : (\mathbb{C}^{n+2}, \mathbb{R}^{n+2}, 0) \rightarrow (\mathbb{C}, \mathbb{R}, 0) \) belongs to an odd (even) local lacuna.

So adding a positive or negative definite quadratic form in an even number of additional variables takes even (odd) local lacunas to lacunas of the same type; adding a quadratic function of signature \((1,1)\) in two additional variables takes even lacunas to odd and vice versa. Therefore, any stable equivalence class of functions splits into four subclasses, depending on the parities of \( n \) and an arbitrary (say, positive) inertia index of the quadratic part of the Taylor expansion of these functions. The sets of local lacunas (of the same parity) of versal deformations of functions from any of these four subclasses are in a one to one correspondence with each other.

**Corollary 1.** If \( f(x_1, \ldots, x_n) \) has a minimum (maximum) point at 0, then \( f(x_1, \ldots, x_n) - x_{n+1}^2 + \tau \) \( (f(x_1, \ldots, x_n) + x_{n+1}^2 - \tau, \text{ respectively}) \) belongs to an odd local lacuna for a sufficiently small \( \tau > 0 \).

In the first case the function \( f - x_{n+1}^2 + x_{n+2}^2 - x_{n+3}^2 + \tau \) belongs to an even local lacuna in view of 3) in Proposition 3 and the example in §2.1; it remains to use 4) in the same proposition.

2.4. Multiplication by \(-1\). It follows immediately from the definitions of Petrovskii classes that the perturbation \(-f_\lambda\) of the function \(-f\) belongs to a local lacuna if and only if the perturbation \(f_\lambda\) of \(f\) does.

2.5. Another form of the odd Petrovskii cycle. It is easy to see that the cycle on the left of Figure 2 is homological in \( \mathbb{C}^1 \setminus V_\lambda \) (modulo the complement of \( B_\varepsilon \)) to the sum of small circles encircling all nonreal points of \( V_\lambda \), as shown on the right of this figure. So the pre-image of its homology class under the Leray tube operator, that is, the odd Petrovskii class, is represented by the sum of these points taken with appropriate signs. The same construction allows us to realize this class in the case of an arbitrary odd \( n \).

So, we choose a point \( x \in \mathbb{R}^n \setminus V_\lambda \). Let \( S(x) \sim S^{n-1} \) be the space of oriented affine lines in \( \mathbb{R}^n \) through \( x \), let \( \phi : E(x) \rightarrow S(x) \) be the tautological line bundle and \( \phi_C : E_C(x) \rightarrow S(x) \) its complexification, so that \( E_C \) is the union of pairs \((l, x) \in S(x) \times \mathbb{C}^n \) such that \( x \) belongs to the complexification of the line \( \{l\} \subset \mathbb{R}^n \). The manifold \( E \) is obviously orientable and subdivided into two parts by the section of the line bundle which consists of all points \((l, x)\). The forgetful map \( \Psi : (l, x) \mapsto x \) takes each of these parts diffeomorphically to \( \mathbb{R}^n \setminus x \). In the case of odd \( n \) the orientations of these parts induced from the fixed orientation of \( \mathbb{R}^n \) belong to the same orientation of the whole of \( E \).

Extend the map \( \Psi \) to the analogous forgetful map \( \Psi_C : E_C \rightarrow \mathbb{C}^n \). For any oriented line \( l \in S(x) \) the set \( \Psi_C^{-1}(V_\lambda) \cap \{l\} \subset \mathbb{C}^n \) is finite and symmetric with respect to the real line \( \{l\} \subset \{l\} \). Define the relative cycle \( \tilde{P}(x) \subset E_C \cap \Psi_C^{-1}(B_\varepsilon \setminus V_\lambda) \) as the union (over all points \( l \in S(x) \)) of the real lines \( \{l\} \)

\[ \text{In the case } n = 1, \text{ which is responsible for Figure 2, the role of the orientation of the base is played by the choice of (different) signs of the two points of the 0-dimensional sphere } S(x). \] So the canonical orientation of the line over the negative point must be reversed.
slightly moved in their complexifications \( \{l_C\} \) close to all points in \( \Psi^{-1}(V_\lambda) \) in such a way that they bypass these points from the left with respect to the canonical orientation of \( \{l\} \). The homology class of the cycle \( \Pi(\lambda) \) from the construction of the odd Petrovskii class can be realized as the direct image of this cycle \( \tilde{P}(x) \) under the map \( \Psi_C \). (In fact, \( \tilde{P}(x) \) is a kind of ‘blowup’ of the cycle \( \Pi(\lambda) \) at the point \( x \).

For any \( l \) the 1-dimensional cycle obtained in \( \{l_C\} \setminus \Psi^{-1}(V_\lambda) \) is homological (within the upper half-plane and modulo the intersection with \( \Psi^{-1}(\mathbb{C}^n \setminus B_\varepsilon) \)) to the union of small circles about all imaginary points in \( \Psi^{-1}(V_\lambda \cap B_\varepsilon) \) in this half-plane. These homologies can be performed simultaneously and consistently over all \( l \) and sweep out a homology between the cycle \( \tilde{P}(x) \) and a cycle which is a Leray tube around the union (over all \( l \in S(x) \)) of all such imaginary points with positive imaginary parts in the fibres \( \{l_C\} \). Thus, in the case of odd \( n \) the odd Petrovskii class can be realized as the direct image of the cycle formed by this union under the map \( \Psi_C \).

This is essentially the original definition of an odd Petrovskii cycle, see [18]. Unlike an even cycle, it depends on the choice of the point \( x \), but its homology class does not.

\section*{§ 3. An obstacle to the existence of local lacunas}

Each of the two local Petrovskii classes related to a nondiscriminant point \( \lambda \in \mathbb{R}^l \) is an element of the relative homology group \( \tilde{H}_{n-1}(V_\lambda, \partial V_\lambda) \) of the corresponding Milnor fibre \( V_\lambda = f_\lambda^{-1}(0) \cap \partial B_\varepsilon \). The boundary operator of the exact sequence of the pair \( (V_\lambda, \partial V_\lambda) \) sends this class to some element of the group \( \tilde{H}_{n-2}(\partial V_\lambda) \).

For any deformation \( F(x, \lambda) \) of the function \( f(x) \), the spaces \( \partial V_\lambda \) form a locally trivial (and hence trivializable) fibre bundle over a neighbourhood of the origin in the parameter space of our deformation (including the discriminant values of \( \lambda \)). Therefore, the homology groups \( \tilde{H}_{n-2}(\partial V_\lambda) \) over all values of \( \lambda \) are naturally identified one with another. It follows easily from the construction of the Petrovskii classes that the boundaries of all cycles \( P_{ev}(\lambda) \) (or \( P_{odd}(\lambda) \)) over all nondiscriminant values of \( \lambda \) are mapped into one another by this identification. Therefore, if for some value \( \lambda \in \mathbb{R}^l \) this boundary is not homologous to zero, then the same is true for all other values of \( \lambda \); in particular, the local Petrovskii classes of the same parity are nontrivial for all \( \lambda \) and we have no local lacunas of this parity for the corresponding singularity.

All the zeros in the cells of Table 2 (but not the entries \( 0_c \)), with the exception of the last column for \( P_{c2}^2 \), follow from this obstacle and from the explicit calculation of the Petrovskii classes mentioned in § 2.2. Conversely, in the case of \( P_{c2}^2 \) the calculation of the intersection indices of vanishing cycles in § 7 will be based on the calculation of this boundary, which we will do in § 3.1.

\textbf{Remark 3.} The group \( \tilde{H}_*(\partial V_\lambda) \) can only be nontrivial in dimension \( n - 1 \) or \( n - 2 \), and its structure can be obtained from the intersection form in \( \tilde{H}_{n-1}(V_\lambda) \). Indeed, by Milnor’s theorem the only nontrivial segment of the exact sequence of the pair \( (V_\lambda, \partial V_\lambda) \) is

\begin{equation}
0 \rightarrow \tilde{H}_{n-1}(\partial V_\lambda) \rightarrow \tilde{H}_{n-1}(V_\lambda) \xrightarrow{\beta} \tilde{H}_{n-1}(V_\lambda, \partial V_\lambda) \rightarrow \tilde{H}_{n-2}(\partial V_\lambda) \rightarrow 0. \tag{3.1}
\end{equation}
If we fix Poincaré dual frames in the two middle groups in this sequence (which are isomorphic to $\mathbb{Z}^{\mu(f)}$), then the homomorphism $j$ will be given by the intersection matrix of basis elements of $\tilde{H}_{n-1}(V_\lambda)$. It determines both the marginal groups $\tilde{H}_i(\partial V_\lambda)$ completely.

3.1. The boundary of the even Petrovskii class for $P_8$ singularities.

**Proposition 4.** For each singularity in the class $P_8^1$ or $P_8^2$ given by a homogeneous function $f(x, y, z)$ of degree 3 and any nondiscriminant perturbation $f_\lambda$ of $f$ the boundary of the even local Petrovskii class is nontrivial in $\tilde{H}_1(\partial V_\lambda)$.

**Proof.** The isomorphism class of the intersection form of the singularities $P_8$ is well-known; for example, see [11]. For any nondiscriminant (and hence for any) perturbation $f_\lambda$ of this function $f$ the group $\tilde{H}_1(\partial V_\lambda)$ can easily be calculated from (3.1) and is equal to $\mathbb{Z}^2 \oplus \mathbb{Z}_3$.

We can assume that the coordinates $x, y, z$ are chosen so $f$ has the Newton-Weierstrass normal form $x^3 + axz^2 + bz^3 - y^2z$. The Hopf bundle projection $S^5 \to \mathbb{C}P^2$ maps $\partial V_0$ to the elliptic curve $\{f = 0\}$. If we consider the submanifolds $f^{-1}(\tau) \cap B_\varepsilon \cap \mathbb{R}^n$, $0 < \tau \ll \varepsilon$, which realize the classes $P_{ev}(f - \tau)$, and let $\tau$ tend to 0, we see that the class of $\partial P_{ev}(0)$ is mapped by this Hopf projection into twice the class of the real part of this elliptic curve, oriented as the boundary of the part of the affine chart $\{z = 1\}$ in $\mathbb{R}P^2$ in which $f$ takes negative values. This last class is never homological to zero in the elliptic curve. (In the case of $P_8^2$, when this real part consists of two components, some sum of these components is homological to zero, but the orientations of these components in this sum must be coordinated in a different way.)

3.2. Boundary of $P_{ev}$ for critical points of functions of two variables.

Denote the real part $B_\varepsilon \cap \mathbb{R}^n$ of the ball $B_\varepsilon$ by $D_\varepsilon$. The real zero set of a function $f(x_1, x_2)$ with a critical point at 0 consists of several irreducible curves passing through 0. Any such curve intersects the circle $\partial D_\varepsilon$ in two points. The corresponding chord diagram is the graph consisting of the circle $\partial D_\varepsilon$ and the chords connecting the endpoints of each of these components.

**Proposition 5.** The boundary $\partial P_{ev} \in \tilde{H}_0(V_\lambda)$ is trivial for some (and therefore all) nondiscriminant perturbations $f_\lambda$ of $f$ if and only if each chord in this chord diagram intersects an even number of other chords.

**Proof.** Take the function $f - \tau$, $0 < \tau \ll \varepsilon$, as the perturbation $f_\lambda$, so that its Milnor fibre is the set $f^{-1}(\tau) \cap B_\varepsilon$. The geometric boundary of the set of real points of this fibre is in the obvious one-to-one correspondence with the set of endpoints of chords in the chord diagram. Points in this boundary belong to the same component of the manifold $\partial V_\lambda$ if and only if they correspond to endpoints of the same chord. It is easy to calculate that two points corresponding to the endpoints of some chord are counted with the same sign in the homological boundary of the even Petrovskii cycle if and only if on the circle $\partial D_\varepsilon$ they are separated by an odd number of other endpoints.
§ 4. Invariants of components of the complement of the real discriminant

We choose \( \Delta > 0 \) small enough so that all varieties \( f^{-1}(t) \) with \( t \in [-\Delta, \Delta] \) are transversal to \( \partial D_\varepsilon \).

Let \( \Lambda \subset \mathbb{R}^l \) be a very small neighbourhood of the origin in the space of parameters \( \lambda \) such that the same transversality condition is satisfied not only for \( f \), but also for all functions \( f_\lambda, \lambda \in \Lambda \), and in addition all real critical values of these functions \( f_\lambda \) in \( D_\varepsilon \) belong to the interval \((-\Delta, \Delta)\). Denote the sets of lower values \( f^{-1}_\lambda((\infty, -\Delta]) \cap D_\varepsilon, f^{-1}_\lambda((\infty, 0]) \cap D_\varepsilon \) and \( f^{-1}_\lambda((\infty, \Delta]) \cap D_\varepsilon \) by \( M_-(\lambda), M_0(\lambda) \) and \( M_+(\lambda) \), respectively.

The diagrams of spaces

\[
M_- (\lambda) \subset M_+ (\lambda) \subset D_\varepsilon \\
M_- (\lambda) \cap \partial D_\varepsilon \subset M_0 (\lambda) \cap \partial D_\varepsilon \subset \partial D_\varepsilon
\]

form a locally trivial (and hence trivializable) fibre bundle over the neighbourhood \( \Lambda \). Therefore we can fix a family of homeomorphisms (depending continuously on \( \lambda \)) mapping all of them to the same diagram, corresponding to some distinguished value \( \lambda_0 \) of \( \lambda \), say, to \( \lambda_0 = 0 \). For this distinguished value we denote the spaces \( M_- (\lambda_0) \) and \( M_+ (\lambda_0) \) simply by \( M_- \) and \( M_+ \). Given an arbitrary \( \lambda \), composing the embedding \( M_0 (\lambda) \to D_\varepsilon \) with this unifying homeomorphism we obtain the diagram of spaces

\[
M_- \subset M_0 (\lambda) \subset M_+ \subset D_\varepsilon \\
M_- \cap \partial D_\varepsilon \subset M_0 (\lambda) \cap \partial D_\varepsilon \subset M_+ \cap \partial D_\varepsilon \subset \partial D_\varepsilon
\]

**Proposition 6.** If two points \( \lambda, \lambda' \in \Lambda \) belong to the same connected component of the set of nondiscriminant perturbations of \( f \), then the corresponding diagrams (4.2) are isotopic to one another via an isotopy of the pair \((D_\varepsilon, \partial D_\varepsilon)\) which is constant on \( M_- \) and \( D_\varepsilon \setminus M_+ \).

In particular, all homological invariants of isotopy classes of such diagrams are also invariants of components of the complement of the discriminant, and we get the following corollary.

**Proposition 7.** The following objects are the same for all \( \lambda \) in the same component of the complement of the discriminant:

1) the isomorphism classes of groups

\[
H_* (M_0(\lambda)), \quad H_* (M_0(\lambda), \partial D_\varepsilon), \quad H_* (M_0(\lambda), M_-(\lambda)), \\
H_* (M_0(\lambda), (M_- \cup \partial D_\varepsilon)), \quad H_* (M_+, M_0(\lambda)), \quad H_* (M_+, (M_0(\lambda) \cup \partial D_\varepsilon));
\]

2) the images of the boundary operators

\[
\partial : H_* (M_0(\lambda), M_-) \to H_* (M_-), \quad \partial : H_* (M_0(\lambda), M_- \cup \partial D_\varepsilon) \to H_* (M_- \cup \partial D_\varepsilon);
\]
3) the kernels of the operators defined by the inclusions

\[ H_+ (M_-) \to H_+ (M_0 (\lambda)), \quad H_+ (M_- \cup \partial D_\epsilon) \to H_+ (M_0 (\lambda) \cup \partial D_\epsilon), \]
\[ H_+ (M_+, M_-) \to H_+ (M_+, M_0 (\lambda)), \]

and so on.

This is a direct consequence of the construction.

For example, the invariant \( \chi (\lambda) \) used in Proposition 1 is just the Euler characteristic of the third group mentioned in item 1) of Proposition 7.

§ 5. Realization of the local lacunas promised in Theorem 1

5.1. The classes \( P^1_8 \) and \( P^2_8 \).

**Proposition 8.** If \( f(x_1, x_2, x_3) \) is a nondegenerate homogeneous polynomial of degree 3 (so that it belongs to either \( P^1_8 \) or \( P^2_8 \)) then the polynomials \( f_{\pm \tau} \equiv f \pm (\tau (x_1^2 + x_2^2 + x_3^2) - \tau^3) \), for sufficiently small \( \tau > 0 \), belong to odd local lacunas.

Moreover, the perturbations \( f_{\tau} \) and \( f_{-\tau} \) belong to different odd local lacunas.

**Proof.** The odd Petrovskii cycle of \( f_{\pm \tau} \), realized as in §2.5 with the central point \( x \) at the origin, is empty. In fact, any complex line through 0 that is the complexification of a real line intersects \( V_{\pm \tau} \) in at least two real points. The set of nonreal intersection points is its own complex conjugate and consists of one point at most since the degree of \( f_{\pm \tau} \) is equal to 3.

The perturbations \( f_{\tau} \) and \( f_{-\tau} \) are distinguished by an invariant from Proposition 7, part 1). Namely, the relative homology group \( H_* (M_0 (\tau), (M_- \cup \partial D_\epsilon)) \) coincides with the homology group of a single point, and the group \( H_* (M_0 (-\tau), (M_- \cup \partial D_\epsilon)) \) is isomorphic to \( H_* (S^2, \text{pt}) \).

5.2. The class \( \pm X_9 \). We will only consider the singularity class \( +X_9 \), since the class \( -X_9 \) can be reduced to it, see §2.4.

The local lacuna for a singularity in the class \( +X_9 \) implied in the second column of Table 2 is described in §2.1. One of two lacunas implied in the fourth column is described in Corollary 1 in §2.3 and is represented by the function \( \varphi (x_1, x_2) - x_3^2 + \tau, \) where \( \tau > 0 \) is small enough and \( \varphi \) has a minimum point at 0.

**Proposition 9.** If a function \( \varphi (x_1, x_2) \) in the class \( +X_9 \) is a nonnegative homogeneous polynomial of degree 4, then the function \( f_{\tau} \equiv \varphi (x_1, x_2) - \tau (x_1^2 + x_2^2) - x_3^2 + \tau^3 \) belongs to the odd local lacuna of \( \varphi (x_1, x_2) - x_3^2 \) for sufficiently small \( \tau > 0 \). This lacuna is distinct from the second lacuna indicated in the previous paragraph.

**Proof.** By the Morse lemma, changing the local coordinate \( x_3 \) slightly (which certainly does not change the values of the Petrovskii classes) we can replace the function \( -x_3^2 \) of one variable by \( -x_3^2 + x_3^4 \), and hence the function \( f_{\tau} \) by \( f_{\tau} + x_3^4 \).

The corresponding odd Petrovskii cycle, described in §2.5 for \( x = 0 \), is empty since any real line through 0 intersects the zero set of this function \( f_{\tau} + x_3^4 \) in four points. The last statement of the proposition follows immediately from Proposition 7.
5.3. The remaining lacunas for corank 2 parabolic singularities. By Proposition 3 all the remaining local lacunas implied in the nonzero cells of Table 2 can be regarded as odd local lacunas for some functions of two variables. We realize these lacunas in the following way. As in [13] and [1] we produce a perturbation $f_\lambda(x_1, x_2)$ of the corresponding function $f$ all of whose $\mu(f)$ critical points are real, all the critical values at saddlepoints are equal to 0 and all the critical values at minima (maxima) are negative (positive, respectively). Using a further, very small perturbation of $f_\lambda$ we can obtain a function $f_\bar{\lambda}$ which is arbitrarily close to $f_\lambda$ but whose critical values at all saddlepoints are moved away from 0 to arbitrarily prescribed sides; in particular, $f_\bar{\lambda}$ is nondiscriminant. In Figures 3 and 4 we draw the zero sets of the preliminary perturbations $f_\lambda$, and indicate by black (white) circles the saddlepoints at which the values should be moved away from 0 in a negative (positive) direction.

![Figure 3. Lacunas for $X_9^2$.](image)

![Figure 4. Lacuna for $J_{10}^3$.](image)

**Proposition 10.** If a function $f(x_1, x_2)$ in the class $X_9^2$ is represented by a homogeneous polynomial of degree 4 vanishing on four distinct real lines, then

1) it has a perturbation $f_\lambda$ whose zero set is as shown on either side of Figure 3;
2) the further nondiscriminant perturbations $f_\bar{\lambda}(x_1, x_2)$ shown in these pictures by black and white circles belong to odd local lacunas of $f$;
3) these two local lacunas are distinct;
4) rotating both pictures in Figure 3 through an angle $\pi/2$ we obtain the pictures of two other perturbations of $f$, which belong to two additional local lacunas, distinct from the previous two.
Proof. Statement 1) is obvious, 2) follows from the calculation of odd Petrovskii cycles mentioned in §2.2, and statements 3) and 4) follow from Proposition 7: indeed, the images of the boundary operators \( H_1(M_0(\lambda), M_-) \to H_0(M_-) \simeq \mathbb{Z}^4 \) for these four cases are four distinct subgroups of the latter group.

**Proposition 11.** If a function \( f(x_1, x_2) \) belongs to the class \( J_{10}^3 \), then

1) it has a perturbation \( f_{\lambda} \) whose zero set is homeomorphic to the one shown in Figure 4;

2) the further nondiscriminant perturbation \( f_{\lambda}(x_1, x_2) \) which is described in this figure using black and white circles belongs to an odd local lacuna of \( f \).

Proof. This proposition follows immediately from the normal form of critical points of type \( J_{10}^3 \) and from the calculation of odd Petrovskii classes discussed in §2.2.

§6. A program for counting the topologically distinct morsifications of critical points of real functions

This program has two versions, one for singularities of corank \( \leq 2 \) (see the file at https://www.hse.ru/mirror/pubs/share/185895886; it currently contains the starting data for the singularity class \( J_{10}^3 \)), and the other for singularities of arbitrary corank (see https://www.hse.ru/mirror/pubs/share/185895827, currently with initial data for \( P_{18}^1 \)). The further versions of the program will be published at the bottom of the web page https://www.hse.ru/en/org/persons/1297545#sci.

For a description of the program see §V.8 in [22]; however the web reference given there links to an obsolete version of the program.

The starting data for the program are the topological characteristics of some morsification \( f_{\lambda} \) of \( f \) all of whose critical points are real and their critical values are distinct and distinct from 0. Namely, these data include the Morse indices of all critical points in increasing order of their critical values (in the program for corank 2 singularities), or just the parities of these indices (in the program for the general case) and the intersection indices of the corresponding vanishing cycles in \( \tilde{H}_{n-1}(V_{\lambda}) \) which are defined by a certain canonical system of paths and have canonical orientations compatible with the orientation of \( \mathbb{R}^n \). One additional element of the data is the number of negative critical values of \( f_{\lambda} \). This information is sufficient to calculate both the Petrovskii classes of our morsification \( f_{\lambda} \) and of all its stabilizations.

Our program is modelling (on the level of such sets of topological data) all potentially possible topological surgeries of the initial morsification, namely, the jumps of critical values through 0, collisions of real critical values (which can either bypass one another or undergo a Morse surgery and go into the imaginary domain), the converse operations (that is, a collision of two complex conjugate critical values at a real point), and also rotations of imaginary critical values around one another. Knowing our topological data before any of these surgeries is enough to predict the data after it.

In general, it is not certain that any sequence of such operations over sets of topological invariants can actually be realized by a path in the parameter space of the deformation, so we consider their results to be virtual morsifications, that is, some admissible collections of our topological data, including the Petrovskii classes.
However, any actual morsification is surely represented by a virtual one, which will be found by our algorithm sooner or later (provided it has enough memory and time). In particular, if the program has enumerated all the possible virtual morsifications of a singularity class and has found that their Petrovskii classes never vanish, then we can put the symbol $0_c$ in the corresponding cell in Table 2.

On the other hand, the majority of the real local lacunas described in §5 have been discovered by this program. More precisely, it found suspicious virtual morsifications with vanishing Petrovskii classes and printed out their topological data; after that, in all our cases it was easy to find real morsifications with these data by hand.

The numbers of topologically distinct virtual morsifications found by our program are: 6503 for $P^1_8$, 9174 for $P^2_8$, 16928 for $\pm X_9$, 96960 for $X^2_9$, 549797 for $J^1_{10}$, and 77380 for $J^3_{10}$.

§7. Initial data for the singularity $P^2_8$

The initial data for our program (that is, appropriate morsifications having only real critical points and intersection indices of their vanishing cycles) for all real parabolic singularities except for $P^2_8$ can easily be calculated using the methods of [13] and [1] (for critical points whose quadratic part has corank $\leq 2$) or [11] (for the class $P^1_8$, which has a suitable representative $x^3 + y^3 + z^3$). It is important for our algorithm that the orientations of these vanishing cycles, which define the signs of the intersection indices, should be compatible with the fixed orientation of $\mathbb{R}^n$; fortunately, all these methods satisfy this condition. In this section we solve a similar problem for the remaining case $P^2_8$.

We choose a function in this class whose Newton-Weierstrass normal form is $f = x^3 - xz^2 + y^2 z$. Consider a small perturbation of it, $f_1 = f + \varepsilon z^2$, $\varepsilon > 0$; dilating the coordinates and the function we can assume that $\varepsilon = 1$. The perturbation $f_1$ has a critical point of type $E_6$ at the origin. Making a slight local change of coordinates (with linear part the identity) at the origin this function can be reduced to the form $\bar{x}^3 - y^4 + \bar{z}^2$. In addition, $f_1$ has two real Morse critical points, $(1, 0, \sqrt{3})$ and $(1, 0, -\sqrt{3})$, with the same critical value 1; their Morse indices are equal to 2 and 1 respectively. It was shown in [13], Example 3 that we can slightly perturb our function $f_1$ so that its $E_6$-type critical point splits into six real Morse critical points, and the new function $f_2$ has the form $\varphi(\bar{x}, \bar{y}) + \bar{z}^2$ in the corresponding neighbourhood of the origin, where the zero level set of $\varphi$ in $\mathbb{R}^2$ looks like Figure 5. The crossing points of this set correspond to the Morse critical points of $f_2$ with

![Figure 5. A suitable morsification for $E_6$.](image-url)
critical value 0 and Morse index 1, and any of the three bounded domains contains a point with a slightly greater critical value and Morse index 2. Let $f_3$ be an additional very small perturbation of the Morse function $f_2$ making it a strictly Morse function, that is, separating all the 8 critical values. We can do this in such a way that the order in $\mathbb{R}^1$ of these values will be as indicated by the numbers in Figure 5. We choose a real value $A$ which is greater than all the six critical values obtained from the perturbation of the $E_6$-type critical point but smaller than the critical values at the two Morse critical points obtained from the two points with critical value 1 and define the basis of vanishing cycles in $H_2(f_3^{-1}(A))$ by the system of paths connecting $A$ to all eight critical values within the upper half-plane in $\mathbb{C}^1$. We number the first six basis cycles $\Delta_i$ in the order of the corresponding critical values, see Figure 5; let the 7th and the 8th cycles be the ones arising from the critical points with value $\approx 1$ and Morse indices 2 and 1, respectively. All these cycles must be oriented in accordance with the fixed orientation of $\mathbb{R}^n$ (see p. 177 in [22], particularly formula (V.6) there). Our purpose is to calculate the matrix of intersection indices of these eight cycles: this will give us a set of initial data for the perturbation $f_3 - A$ of the initial function $f$. This matrix is symmetric since $n - 1 = 2$ is even.

$$
\begin{pmatrix}
-2 & 0 & 0 & 1 & 0 & 1 & X & -\frac{Z + W}{2} \\
0 & -2 & 0 & 0 & 1 & 1 & X & -\frac{Z + W}{2} \\
0 & 0 & -2 & 0 & 0 & 1 & Y & -\frac{W}{2} \\
1 & 0 & 0 & -2 & 0 & 0 & 0 & Z \\
0 & 1 & 0 & 0 & -2 & 0 & 0 & Z \\
1 & 1 & 1 & 0 & 0 & -2 & 0 & W \\
X & X & Y & 0 & 0 & 0 & -2 & 0 \\
-Z + W & -Z + W & -\frac{W}{2} & Z & Z & W & 0 & -2 \\
\end{pmatrix}
$$

(7.1)

**Lemma 1.** The desired intersection matrix has the form (7.1) for some values of $X$, $Y$, $Z$ and $W$.

**Proof.** The intersection indices of the first six cycles can be calculated using the method in [13] and [1] and are as shown in the upper left-hand $6 \times 6$ corner of the matrix (7.1). The intersection index $\langle \Delta_7, \Delta_8 \rangle$ is equal to 0 because these cycles arise from critical points which are far apart but whose critical values almost coincide. The cycles $\Delta_4$, $\Delta_5$ and $\Delta_6$ are invariant under complex conjugation in $f_3^{-1}(A)$, therefore their intersection indices with $\Delta_7$ (which is anti-invariant) are equal to 0, as indicated in the 7th row of (7.1). Also, the perturbation $f_1$ of the original function $f$ is invariant under reflection in the hyperplane $y = 0$, and we can make a further perturbation $f_2$ in such a way that it keeps this symmetry. This reflection preserves the basis cycles in $f_2^{-1}(A)$ that are close to the cycles $\Delta_7$ and $\Delta_8$, only changing their canonical orientations; on the other hand it interchanges the
cycles close to $\Delta_1$ and $-\Delta_2$, and also the cycles close to $\Delta_4$ and $-\Delta_5$. Therefore $\langle \Delta_1, \Delta_7 \rangle = \langle -\Delta_2, -\Delta_7 \rangle$ and $\langle \Delta_4, \Delta_8 \rangle = \langle \Delta_5, \Delta_8 \rangle$. This is why the corresponding cells of our matrix (7.1) are filled by the same letters ($X$ in the first case and $Z$ in the second).

For any $i = 1, 2, 3$ let $\Delta_i$ denote the vanishing cycle in $f_3^{-1}(A)$ obtained from the $i$th critical point by the path connecting the corresponding critical value with $A$ in the lower half-plane of $\mathbb{C}^1$. It is easy to see that the cycle $\Delta_i + \Delta_i$ is anti-invariant under complex conjugation and $\Delta_8$ is invariant, therefore $\langle \Delta_i + \Delta_i, \Delta_8 \rangle = 0$. But $\Delta_i$ can be considered as the image of $\Delta_i$ under the Picard-Lefschetz monodromy operator along a loop $L$ starting and ending at the point $A$ and enclosing all critical values placed between the $i$th one and $A$. This image is equal to $\Delta_i + \text{Var}_{L}(\Delta_i)$, therefore the previous equation gives us $-2\langle \Delta_i, \Delta_8 \rangle = \langle \text{Var}_{L}(\Delta_i), \Delta_8 \rangle$. By the Picard-Lefschetz formula this last number is equal to $\sum_{j=4}^{6} \langle \Delta_i, \Delta_j \rangle \langle \Delta_j, \Delta_8 \rangle$, which gives us the expressions for the intersection indices $\langle \Delta_i, \Delta_8 \rangle$ through $Z \equiv \langle \Delta_4, \Delta_8 \rangle = \langle \Delta_5, \Delta_8 \rangle$ and $W \equiv \langle \Delta_6, \Delta_8 \rangle$; see the first three cells of the last row of (7.1). Lemma 1 is proved.

It remains to calculate the numbers $X$, $Y$, $Z$ and $W$ in this matrix. We know that it is the matrix of a $P_8$-type bilinear form in some basis in $Z^8$. This form is well-known; see [11], for example. In particular it is easy to check that the image of the lattice $Z^8$ in the dual lattice $\hat{Z}^8$ under the map defined by this bilinear form coincides with the image of an arbitrary $E_6$-sublattice of it. Therefore, the last two rows of the desired matrix are integer linear combinations of the first six. Writing these rows in the form of such linear combinations with indeterminate coefficients, we get 16 equations in the 16 unknowns $a_1, \ldots, a_6, b_1, \ldots, b_6, X, Y, Z$ and $W$. Two of these equations follow from the others, but the Diophantine system of the remaining 14 equations is easily solved and has exactly four different integer solutions, which imply four possible combinations of the coefficients of the matrix (7.1): $\{X = 0, Y = \pm 1, Z = 0, W = \pm 2\}$. Now we choose the correct version.

Both local Petrovskii classes of a morsification all of whose critical points are real can be calculated explicitly from the Morse indices of these critical points and the intersection indices of all (properly oriented) vanishing cycles, see §2.2. Substituting all four hypothetical combinations of intersection indices into these calculations, in three cases we get a contradiction with the previously obtained results on these classes for $P_8^2$ singularities, namely, with the following statements:

a) in the case of odd $n$ and an even index $i_+$ (for example for $n = 3$) these singularities have local lacunas, therefore the homological boundary of the odd Petrovskii class is equal to 0 for all nondiscriminant perturbations of $f$, see §5.1;

b) the analogous homological boundary of the even Petrovskii class is distinct from 0 for the same values of $n$ and $i_+$, see §3.1.

The unique case which remains gives us $Y = 1$ and $W = -2$, and the intersection matrix is completely calculated. Plugging it into the initial data of our program, we get the message from it that no local lacunas exist close to the $P_8^2$ singularities in the cases of even $n$, whether $i_+$ is even or odd. This proves the first two zeros in Table 2 for $P_8^2$. 


§ 8. No extra lacunas for $\pm X_9$ singularities

The symbol $1_c$ in the second column of Table 2 for $\pm X_9$ (that is, the fact that this singularity has no local lacunas in addition to the one mentioned in §5.2) is proved by our program with the help of the following fact.

**Proposition 12.** If the function $f$ has a minimum point at the origin, then all sufficiently small perturbations $f_\lambda$ of it, such that all the real critical values of $f_\lambda$ are positive, belong to the same component of the complement of the discriminant of an arbitrary versal deformation of it.

**Proof.** The property of functions described here is preserved under inducing and equivalence of deformations, therefore it is enough to prove our proposition for one arbitrary versal deformation of $f$. In particular we can assume that together with any perturbation $f_\lambda$ of $f$ this deformation contains also all perturbations $f_\lambda + c$, where the constant $c$ runs through some interval containing 0. Choose some small $c > 0$ in this interval; then there exists $\delta > 0$ such that the $\delta$-neighbourhood of the point $\{f + c\}$ in the space $R^l$ of parameters of this deformation is separated away from the discriminant. For any point $\lambda$ in the $\delta$-neighbourhood of the origin in $R^l$ such that $f_\lambda$ satisfies the hypothesis of the proposition, the whole interval consisting of functions $f_\lambda + \tau$, $\tau \in [0, c]$ belongs to the complement of the discriminant, and the last point in it, $f_\lambda + c$, belongs to the $\delta$-neighbourhood of the point $f + c$ mentioned above.

Therefore, we asked our program to check that the functions of type $+X_9$ have no virtual morsifications with trivial even Petrovskii class and at least one negative critical value. Its affirmative answer justifies the symbol $1_c$ in the cell in question.

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