Precursors of extreme increments

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We investigate precursors and predictability of extreme increments in a time series. The events we are focusing on consist in large increments within successive time steps. We are especially interested in understanding how the quality of the predictions depends on the strategy to choose precursors, on the size of the event and on the correlation strength. We study the prediction of extreme increments analytically in an AR(1) process, and numerically in wind speed recordings and long-range correlated ARMA data. We evaluate the success of predictions via receiver operator characteristics (ROC-curves). Furthermore, we observe an increase of the quality of predictions with increasing event size and with decreasing correlation in all examples. Both effects can be understood by using the likelihood ratio as a summary index for smooth ROC-curves.

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I. INTRODUCTION

Extreme value statistics [1] is a well established approach to predict the relative frequency of rare extreme events, but does not include forecasts of when the next event will occur. There have been many attempts to employ time series strategies for the latter purpose. These strategies usually investigate a record of historical data about the phenomenon under study and try to infer knowledge about the future. A standard approach is to search for precursors, i.e., typical signatures preceding an extreme event. Such precursors have been discussed, e.g., in the literature about earthquakes [2], epileptic seizures [3], and stock market crashes [4, 5, 6]. As the above listed examples illustrate, the definitions of what an extreme event is depends on the context. Frequently, one encounters extremely large values of some observable, or some drastic changes. It is the latter which is the focus of this paper where we discuss large increments motivated by stock markets or by turbulent gust in wind speed data.

One might expect that the more extreme an event is, the more difficult it is to predict it, simply because more extreme events are usually also much rarer. However, it has been reported in the literature of wind speed predictions [7], precipitation forecast [10], multi agent games [11] and earthquakes [12] that more extreme events are better predictable than small events. Therefore one particular goal of this contribution is to investigate how the predictability of large increments depends on the size of the increment.

In this contribution we study predictions in a simple autoregressive process of order 1 (AR(1) process) analytically in order to obtain a detailed understanding of some questions on precursors and predictions. The AR(1) process is a simple stationary stochastic model process, that might not reflect all features of more complex processes occurring in nature, but it admits a fully analytic treatment. Additionally, we study similar prediction procedures numerically in long-range correlated data and in wind speed data, verifying the same quantitative results. The questions, which we intend to answer are the following:

(Q1) How to choose a precursor in order to obtain good predictions?
(Q2) Are extreme increments the better predictable, the more extreme they are?
(Q3) How does the correlation of the data influence the predictability of extreme increments?

The paper is organized as follows. In Sec. II A we discuss two strategies which can be used to choose precursory structures and in Sec. II B we introduce a method to evaluate the predictive power of precursors. The extreme events we discuss in this contribution are defined in Sec. II C and we show how to obtain their joint PDFs analytically in Sec. II D. We apply these procedures to AR(1) correlated stochastic processes in Sec. III and to ARMA data Sec. IV. Conclusions appear in Sec. V.

II. DEFINITIONS AND SET-UP

The considerations in this introductory section are made for general dynamical systems with a complex time evolution. They might be purely deterministic, then high-dimensional and chaotic, or they might be stochastic. In any case we assume that the time evolution of the system cannot be easily modeled and hence one tries to extract information about the future from time series data. This means that through some experimental observation one can record a usually univariate time series, i.e., a set of measurements $x_n$ at discrete times $t_n$, where $t_n = t_0 + n \Delta$ with a sampling interval $\Delta$. The recording
should contain sufficiently many extreme events so that we are able to extract statistical information about them. We also assume that the event of interest can be identified on the basis of the observations, e.g., by the value of the observation function exceeding some threshold, by a sudden increase, or by its variance exceeding again some threshold.

A. The choice of the precursor

Ideally, a precursor is a typical signature in the data preceeding every individual event. Unfortunately the time evolution of most systems is usually too irregular to demand this, so one would call a precursor a data structure which is typically an event allowing deviations from the given structure, but also allowing events without preceding structure. This interpretation of a precursor allows to determine the specific values of the precursory structure by statistical considerations.

In order to predict an event occurring at the time \((n+1)\) we compare the last \(k\) observations \(x_{(n,k)} = (x_{n-k+1}, x_{n-k+2}, \ldots, x_n)\) with a specific precursory structure \(x_{\text{pre}} = (x_{n-k+1}^{\text{pre}}, x_{n-k+2}^{\text{pre}}, \ldots, x_n^{\text{pre}})\).

This precursory structure can be chosen according to different strategies. The two possible strategies which we address here, represent the most fundamental choices. They consist in using either the maximum of the a posteriori PDF or of the maximum of the likelihood \([14]\). In more applied examples one looks for precursors which minimize or maximize more sophisticated quantities, e.g., discriminant functions or loss matrices. These quantities are usually functions of the posterior PDF or the likelihood, but they take into account the additional demands of the specific problem, e.g., minimizing the loss due to a false prediction \([15]\). The two strategies studied in this contribution are thus fundamental in the sense that they enter into most of the more sophisticated quantities which are used for predictions and decision making.

The a posteriori PDF \(\rho(x_{(n,k)}|X)\) takes into account all events of size \(X\) and provides the probability density to find a specific precursory structure before an observed event.

(I) Hence strategy I consists in defining the precursors in a retrospective or a posteriori way: once the extreme event \(X\) has been identified, one asks for the signals right before it. Formally, this implies that the precursory structure consists of the global maxima in each component \((x_{n-k+1}^*, x_{n-k+2}^*, \ldots, x_{n-1}^*, x_n^*)\) of the a posteriori PDF.

The likelihood \(\rho(X|x_{(n,k)})\) takes into account all possible values of precursory structures, and provides the probability density that an event of size \(X\) will follow them. Note that the likelihood is thus not a density function with respect to the precursory structure, but with respect to the event size \(X\). The precursory structure enters into the likelihood only as a parameter.

(II) Strategy II consists in determining those values of each component \(x_i\) of the condition \(x_{(n,k)}\) for which the likelihood has a global maximum.

Note that the a posteriori PDF and the likelihood are linked via Bayes’s theorem

\[
\rho(x_{(n,k)}|X) = \rho(x_{(n,k)}) \rho(X|x_{(n,k)}) = \rho(x_{(n,k)}) \rho(X),
\]

where \(\rho(x_{(n,k)})\) represents the marginal PDF to find the precursory structure \(x_{(n,k)}\) and \(\rho(X)\) represents the marginal PDF to find events of size \(X\).

In summary the possible values of precursors are given by

\[
x_{\text{pre}} = \begin{cases} x_I, \\ x_{II}, \end{cases}
\]

where

\[
x_I := (x_{n-k+1}^*, x_{n-k+2}^*, \ldots, x_{n-1}^*, x_n^*),
\]

and

\[
x_{II} := (x_{n-k+1}^t, x_{n-k+2}^t, \ldots, x_{n-1}^t, x_n^t),
\]

where \(x_i^*\) are the points in which \(\rho(x_{(n,k)}|X)\) has a global maximum and \(x_i^t\) are the points in which \(\rho(X|x_{(n,k)})\) has its largest maximum, with \(n-k+1 \leq i \leq n\). In both cases the event size \(X\) is assumed to be fixed. Once the precursory structure \(x_{\text{pre}}\) is determined, we give an alarm for an extreme event when we find the last \(k\) observations \(x_{(n,k)}\) in the volume

\[
V_{\text{pre}}(\delta) = \left(x_{n-k+1}^{\text{pre}} + \frac{\delta}{2} - x_{n-k+1}^{\text{pre}} + \frac{\delta}{2}\right) \times \left(x_{n-k+2}^{\text{pre}} + \frac{\delta}{2} - x_{n-k+2}^{\text{pre}} + \frac{\delta}{2}\right) \times \ldots \times \left(x_n^{\text{pre}} + \frac{\delta}{2} - x_n^{\text{pre}} + \frac{\delta}{2}\right).
\]
nected, but apart from this the procedure of predicting should not be different. However, we restrict ourselves to unimodal PDFs in this contribution.

B. Testing for predictive power

A common method to verify a hypothesis or test the quality of a prediction is the receiver operating characteristic curve (ROC-plot) [11, 17]. The idea of the ROC-curve consists simply in comparing the rate of correctly predicted events \( r_c \) with the rate of false alarms \( r_f \) by plotting \( r_c \) vs. \( r_f \). The resulting curve in the unit-square of the \( r_f \)-\( r_c \) plane approaches the origin for \( \delta \to 0 \) and the point (1,1) in the limit \( \delta \to \infty \), where \( \delta \) accounts for the size of the precursor volume \( V_{\text{pre}}(\delta) \) (see Eq. (2)).

The shape of the curve characterizes the significance of the prediction. A curve above the diagonal reveals that the corresponding strategy of prediction is better than a random prediction which is characterized by the diagonal. Furthermore we are interested in curves which converge as fast as possible to \( r_c = 1 \) and \( r_f = 0 \), and apart from this the procedure of predicting rare and hence \( \rho(X) \), \( \rho(X| \mathbf{x}_{n,k}) \ll 1 \), the likelihood ratio is approximately given by

\[
m(\mathbf{x}_{n,k}, X) \sim \frac{\rho(X| \mathbf{x}_{n,k})}{\rho(X)} \equiv \frac{\rho(\mathbf{x}_{n,k}|X)}{\rho(\mathbf{x}_{n,k})} \quad (5)
\]

Eq. (5) already suggest an answers to questions (Q1) and (Q2), by considering \( m(\mathbf{x}_{n,k}, X) \) as a summary index.

Ad (Q1): This asymptotic form of the likelihood ratio allows us to compare different strategies of prediction. Looking for the maximum of \( \rho(\mathbf{x}_{n,k}|X) \) in \( \mathbf{x}_{n,k} \), according to strategy I, there is always the influence of the denominator \( \rho(\mathbf{x}_{n,k}) \) which will keep the likelihood ratio small, even if \( \rho(\mathbf{x}_{n,k}|X) \) in \( \mathbf{x}_{n,k} \) is maximized. This is due to the fact that \( \rho(\mathbf{x}_{n,k}|X) \) cannot be large without \( \rho(\mathbf{x}_{n,k}) \) being large. Strategy II, which uses the maximum of \( \rho(X| \mathbf{x}_{n,k}) \) in \( \mathbf{x}_{n,k} \) should thus be superior, since the denominator \( \rho(X) \) is independent of the chosen precursor. The examples which are studied in Sec. III Sec. IV and Sec. V support this idea.

Ad (Q2): According to Eq. (5), the likelihood ratio is larger than unity, if \( \rho(\mathbf{x}_{n,k}|X) > \rho(\mathbf{x}_{n,k})\rho(X) \), i.e., if \( \mathbf{x}_{n,k} \) and \( X \) are correlated. This condition can be also written as \( \rho(X| \mathbf{x}_{n,k}) > \rho(X) \) or as \( \rho(\mathbf{x}_{n,k}|X) > \rho(\mathbf{x}_{n,k}) \) using Bayes’s theorem. The latter expression states that the a posteriori PDF \( \rho(X| \mathbf{x}_{n,k}) \), i.e., the probability to find the precursor prior to an event should be larger than the probability to find the precursor prior to an arbitrary value. Thus, the condition is fulfilled by choosing the precursor in a reasonable way, e.g., using the maximum of \( \rho(X| \mathbf{x}_{n,k}) \) in \( \mathbf{x}_{n,k} \) or the maximum of \( \rho(\mathbf{x}_{n,k}|X) \).

C. Definition of Extreme Increments

In this contribution we will concentrate on extreme events which consist in a sudden increase (or decrease) of the observed variable within a few time steps. Examples of this kind of extreme events are the increases in wind speed in [9, 19], but also stock market crashes [4, 5] which consist in sudden decreases.

We define our extreme event by an increment \( x_{n+1} - x_n \) exceeding a given threshold \( d \)

\[
x_{n+1} - x_n \geq d, \quad (6)
\]

where \( x_n \) and \( x_{n+1} \) denote the observed values at two consecutive time steps.

D. Obtaining the analytic expression of the posterior PDFs

A mathematical expression for a filter, which selects the PDF of our extreme events out of the PDFs of the underlying stochastic process can be obtained through the Heaviside function \( \Theta(x_{n+1} - x_n - d) \). This filter is then applied to the joint PDF of a stochastic process.
Since only the time steps \( (x_n, x_{n+1}) \) are of relevance for the filtering, we can neglect all previous time steps and apply the filter simply to the joint PDF for \( (x_n, x_{n+1}) \), which has the form \( \rho(x_n, x_{n+1}) = \rho(x_n)\rho(x_{n+1}|x_n) \). This implies that we can regard all previous time-steps \( x_0, x_1, \ldots, x_{n-1} \), on which \( \rho_n \) and \( \rho_{n+1} \) might depend, as parameters.

The joint PDF of the extreme events \( \rho^\mathcal{O}(x_{n+1}, x_n, d) \) can then be obtained by multiplication with \( \Theta(x_{n+1} - x_n - d) \). If the resulting expression is non-zero, the condition of the extreme event (6) is fulfilled and for \( x_{n+1} \) and \( x_n \) the following relation holds:

\[
x_{n+1} = x_n + d + \gamma \quad (\gamma \in \mathbb{R}, \gamma \geq 0)
\]  

(7)

Hence it is possible to express the joint probability density in terms of \( x_n \) or \( x_{n+1} \) with the new random variable \( \gamma \). We can then use the integral representation of the Heaviside function with appropriate substitutions to obtain:

\[
f^\mathcal{O}(x_{n+1}, x_n, d) = \rho(x_n) \int_0^\infty \rho(x_n + d + \gamma|x_n) \delta((x_{n+1} - x_n - d) - \gamma) \, d\gamma.
\]

(8)

By normalizing this expression with the total probability \( \rho_\mathcal{O}(d) \) to find extreme events of size \( d \) or larger we obtain the joint PDF \( \rho^\mathcal{O}(x_n, x_{n+1}, d) \) of all values of \( x_n \) and \( x_{n+1} \) which are part of an extreme event. Integrating the resulting joint PDF \( \rho^\mathcal{O}(x_n, x_{n+1}, d) \) over \( x_{n+1} \) we find the following expression for the marginal distribution, i.e., the a posteriori PDF:

\[
\rho(x_n|X(d)) = \frac{\rho(x_n)}{\rho_\mathcal{O}(d)} \int_0^\infty d\gamma \rho(x_n + d + \gamma|x_n).
\]

(9)

Analogously \( \rho(x_n|X(d)) \) denotes the a posteriori PDF to observe the value \( x_n \) before an non-event, i.e., before an increment which is smaller than \( d \):

\[
\rho(x_n|\overline{X(d)}) = \frac{\rho(x_n)}{(1-\rho_\mathcal{O}(d))} \int_{-\infty}^\infty dx_{n+1} \left( 1 - \Theta(x_{n+1} - x_n - d) \right) \rho_{n+1}(x_{n+1}|x_n).
\]

(10)

If for a given process the joint PDF of two consecutive events is known, we can hence analytically determine \( \rho(x_n|X(d)) \), \( \rho(x_n|\overline{X(d)}) \) and \( \rho_\mathcal{O}(d) \).

III. EXTREME INCREMENTS IN THE AR(1) MODEL

A. AR(1) model

We assume that the time-series \( \{x_n\} \) is generated by an auto-regressive model of order 1 (AR(1)) (see e.g., [7])

\[
x_{n+1} = ax_n + \xi_n.
\]

(11)

where \( \xi_n \) are uncorrelated Gaussian random numbers with unit variance and \(-1 < a < 1 \) is a constant which represents the coupling strength. The size and the sign of the coupling strength sets whether successive values of \( x_n \) are clustered or spread, as illustrated in Fig. 1.

In the case \( a = 0 \) the process reduces to uncorrelated random numbers with mean \( \mu = 0 \) and variance \( \sigma^2 = 1 \), whereas generally the process is exponentially correlated \( \langle x_n x_{n+k} \rangle = a^k < 1 \) and has the marginal PDF

\[
\rho(x_n) = \sqrt{\frac{1-a^2}{2\pi}} \exp\left(-\frac{1-a^2}{2} x_n^2 \right).
\]

(12)

Since the size of the events is naturally measured in units of the standard deviation \( \sigma(a) \), we introduce a new scaled variable \( \eta = \frac{a}{\sigma(a)} = d\sqrt{1-a^2} \).

Applying the filter mechanism developed in Sec. [1] we obtain the following expressions for the posterior PDF of extreme events and the posterior PDF of non-extreme events

\[
\rho(x_n|X(\eta), a) = \frac{\sqrt{1-a^2}}{2\sqrt{2\pi}a^3} \exp\left(-\frac{1-a^2}{2} x_n^2 \right) \ \text{erfc} \left( \frac{(1-a)x_n}{\sqrt{2}} + \frac{\eta}{\sqrt{2}\sqrt{1-a^2}} \right).
\]

(13)

\[
\rho(x_n|\overline{X(\eta)}, a) = \frac{\sqrt{1-a^2}}{2\sqrt{2\pi}(1-a^3)} \exp\left(-1-a^2 x_n^2 \right) \left(1 + \text{erf} \left( \frac{(1-a)x_n}{\sqrt{2}} + \frac{\eta}{\sqrt{2}\sqrt{1-a^2}} \right) \right).
\]

(14)
B. Determining the precursor value

Because of the Markov-property of the AR(1) model the probability for an event at time \( n + 1 \) depends only on the last value \( x_n \), hence \( k = 1 \) in Eq. (1). Thus, we give an alarm for an extreme event when an observed value \( x_n \) is in an interval \( V_{\text{pre}} = [x_{\text{pre}} - \delta/2, x_{\text{pre}} + \delta/2] \); around the precursor value \( x_{\text{pre}} \). We compute the precursor values \( x_I \) and \( x_{II} \) defined by Eq. (1) according to the strategies described in Sec. II A.

The maximum \( x_I \) of \( \rho(x_n | X(\eta), a) \) is given by the solution of the transcendental equation

\[
x_I(\eta) = \frac{\sqrt{2}}{\sqrt{\pi} (1 + a)} \exp \left( -\frac{1}{2} \left( \frac{1}{1-a} x_I + \frac{\eta}{\sqrt{1-a}} \right)^2 \right) \text{erfc} \left( \frac{1-a x_I}{\sqrt{2}} + \frac{\eta}{\sqrt{2 \sqrt{1-a}}} \right).
\]

Inserting the asymptotic expansion for large arguments of the complementary error function

\[
erfc(z) \sim \frac{\exp(-z^2)}{\sqrt{\pi} z} \left( 1 + \sum_{m=1}^{\infty} (-1)^m \frac{1 \cdot 3 ... (2m-1)}{(2z^2)^m} \right), \quad \left( z \to \infty, |\arg z| < \frac{3\pi}{4} \right)
\]

which can be found in [20] we obtain:

\[
x_I(\eta) \sim \frac{-\eta}{2 \sqrt{1-a^2} \left( 1 + \mathcal{O} \left( \frac{1}{\eta^2} \right) \right)}, \quad (\eta \to \infty).
\]

Fig. 2 shows the posterior PDFs \( \rho(x_n | X(\eta), a) \) according to Eq. (13) for different values of \( a \) and \( \eta \). One can see that the maximum of \( \rho(x_n | X(\eta), a) \) moves towards \(-\infty\) with increasing size of \( \eta \) and \( a \to 1 \). Although we can always formally define the maximum \( x_I \) and the mean \( \langle x_n \rangle \) as precursor values, one can argue that the maximum of the distribution has no predictive power if \( a \to 1 \). Since the variance of the posterior PDF increases immensely in this limit, the value of \( \rho(x_n | X(\eta), a) \) in its maximum does not considerably differ from the values in any other point.

For large values of \( \eta \) we can also assume that the maximum and the mean of \( \rho(x_n | X(\eta), a) \) nearly coincide, i.e.,

\[
\langle x_n \rangle \approx x_I \sim \frac{-\eta}{2 \sqrt{1-a^2} \left( 1 + \mathcal{O} \left( \frac{1}{\eta^2} \right) \right)}, \quad (\eta \to \infty), \quad (18)
\]

provided that \( \rho(x_n | X(\eta), a) \) is not too asymmetric (i.e., \( a \) is not close to \(-1\)). In the numerical tests in Sec. III C we will hence use the mean of the posterior PDF as a precursor for strategy I, since it can be calculated explicitly by evaluating the corresponding integral.

In order to determine \( x_{II} \), the precursor for strategy II, we have to find the maximum in \( x_n \) of the likelihood

\[
\rho(X(\eta) | x_n, a) = \frac{1}{2} \text{erfc} \left( \frac{(1-a)x_n}{\sqrt{2}} + \frac{\eta}{\sqrt{2 \sqrt{1-a}}} \right).
\]

Since the complementary error function is a monotonously decreasing function of \( x_n \) we see that we do not have a well defined maximum \( x_{II} \), (we will thus denote \( x_{II} : -\infty \)) and that the interval \( V_- = (-\infty, x_-) \) with the upper limit \( x_- \) represents the interval for raising alarms according to strategy II.

C. Testing the Performance of the Precursors

In order to test for the predictive power of the precursors specified above, we used two different methods to create ROC-curves (see Sec. III B). The first method consists in evaluating the integrals which lead to the rate of correct and false predictions

\[
r_c(x_{\text{pre}}, \eta, \delta) = \int_{V(\delta)} dx_n \rho(x_n | X(\eta), a), \quad (20)
\]

\[
r_f(x_{\text{pre}}, \eta, \delta) = \int_{V(\delta)} dx_n \rho(x_n | X(\eta), a). \quad (21)
\]

The second method consists in simply performing predictions on a time series of \( 10^7 \) AR(1) data and counting the number of extreme increments, which could be predicted by using the precursors specified above. For different values of the correlation coefficients the data sets contained the following numbers of extreme increments:
In all cases, where the AR(1) correlated data sets contain increments, the empirically determined rates comply very well with the rates obtained via the evaluation of Eqs. (20) and (21). For those values of \(a\) and \(\eta\), which were not accessible for the numerical test, we evaluated the integrals in Eqs. (20) and (21).

In the numerical tests for both strategies and also for the evaluation of the integrals in Eqs. (20) and (21) according to strategy I, the size of the precursory volume ranged from 10 to 4, measured in size of the standard deviation of the marginal PDF of the AR(1) process \(\sigma(a) = \sqrt{1 - a^2}\). As precursors according strategy I we used the means of the a posteriori PDF. For the empirically created ROC-plots according to strategy II we used the smallest values of the data sets as precursors.

The evaluation of the integrals in Eqs. (20) and (21) was done in a slightly different way for strategy II. Since there were no events in the data sets for certain value of \(a\) and \(d\) (as indicated in the table above), one could argue that the data sets also did not contain any precursor. From the previous section, we know that the theoretical precursor value according to strategy II should be \(x_{\text{II}} = -\infty\). Thus, we used a sufficiently small value as a precursor and adjusted the size of the prediction interval in order to capture all events. However, the resulting ROC-curves for strategy II coincided with the curves obtained empirically, as far as they were available.

The resulting ROC-curves in Fig. 3 display the following properties:

- **ad (Q1):** The predictions according to strategy II are better than the predictions according to strategy I for all values of \(a\) and \(\eta\).
- **ad (Q2):** The ROC-curves display an increase of the quality of our prediction with increasing size of the events \(\eta\).
- **ad (Q3):** The ROC-curves in Fig. 3 show that the quality of the predictions increases with decreasing correlation strength \(a\). Especially for \(a = 0\), when the predictions were made within completely uncorrelated random numbers, the ROC curves are far better than ROC curves for any random prediction. This is in agreement with results reported in Fig. 2 for the prediction of signs of increments in uncorrelated random numbers, i. e., the case \((a = 0, \eta = 0)\).

Intuitively, the result for (Q3) can be understood easily by considering that increments are not independent from the last observation. More precisely \(x_{n+1} = (a-1)x_n + \xi_n\), so that the known part of the increment \((a-1)x_n\) is the larger, the smaller \(a\). In other words:

\[
\eta \begin{array}{c|c|c|c|c} a & \eta \geq 0 & \eta \geq 2 & \eta \geq 4 & \eta \geq 8 \\ \hline -0.99 & 5000059 & 1579103 & 222858 & 310 \\ -0.75 & 5000563 & 1425146 & 162405 & 107 \\ 0 & 5000417 & 786355 & 23370 & 0 \\ 0.75 & 5000818 & 23377 & 0 & 0 \\ 0.99 & 5001081 & 0 & 0 & 0 \\
\end{array}
\]

if we consider a very small value of \(x_n\) (small compared to the mean) in an uncorrelated process, the probability that the next value will be closer to the mean and hence lead to a large increment is high. Positive correlation hinders this effect, since it causes successive values to be closer to each other.

A formal explanation of the results (Q1)-(Q3) is also given by an asymptotic expression for the slope \(m(a, \eta, x_{\text{pre}})\) in the following section.

**D. Analytical discussion of the Precursor Performance**

In this section, we will try to understand the effects shown by the ROC-curves in the previous section more detailed. Thus, we evaluate the asymptotic structure of the likelihood ratio as defined by Eq. (3) for different scenarios.

In the case of the AR(1) process the slope of the ROC-curve in the vicinity of the origin is given by

\[
m(a, \eta, x_{\text{pre}}) \sim \frac{(1 - \rho_\Theta(\eta))}{\rho_\Theta(\eta)} r(x_{\text{pre}}, \eta),
\]

with

\[
r(x_{\text{pre}}, \eta) = \frac{\text{erfc}(\frac{(1-a)x_{\text{pre}}}{\sqrt{2}} + \frac{\sqrt{\eta}}{\sqrt{2\sqrt{1-a}}})}{1 + \text{erf}(\frac{(1-a)x_{\text{pre}}}{\sqrt{2}} + \frac{\sqrt{\eta}}{\sqrt{2\sqrt{1-a}}})}.
\]

- **ad (Q1):** We will first consider the behavior of the precursor according to strategy II. As we saw in Sec. III B the optimal precursor value of strategy II is the limiting case \(x_{\text{II}} = -\infty\). Since \(\lim_{x_{\text{pre}} \to -\infty} r(x_{\text{pre}}, \eta) = \infty\) we find \(\lim_{x_{\text{pre}} \to -\infty} m(a, \eta, x_{\text{II}}) = \infty\). Thus, we should ex-
for the total probability to observe extreme events \( I \). For the following calculations we use an approximation of

\[ x \to -\infty \]

alarms, \( \rho \). The posterior PDF \( P(x_n|x) \) serve extreme events to the vertical axis of the curve and hence represent an ideal predictability for all sizes of events and all possible correlation strengths. However, for any finite precursor value of strategy I and strategy II we find non-ideal ROC-curves.

Another way to understand the superiority of strategy II is to analyze the asymptotic behavior of the rate of correct predictions \( \rho(x_n|x) \), which is used as precursor, moves towards \(-\infty\) with increasing \( \eta \) since \( x_I \sim -\eta/(2\sqrt{(1-a^2)}) \). Because the maximum of the failure posterior PDF \( \rho(x_n|x) \) remains at the origin, the values of \( \rho(x_n|x) \) which are observed at the precursor value \( x_I \) decrease according to the decrease of \( \rho(x_n|x) \) as \( x_n \to -\infty \).

Hence the value of \( \rho(x_n|x) \) at the precursor value approaches a constant for large \( \eta \), whereas the values of \( \rho(x_n|x) \) decrease exponentially in this limit. Fig. 4 illustrates this effect for the case \( a = -0.75 \). The maximum of the failure PDF remains at the origin for \( \eta \to \infty \). Thus the values of this PDF which are observed at the decreasing precursor value \( x_I \sim \frac{\eta}{\sqrt{1-a^2}} \) decrease according to the shape of the distribution. This explains also the success of strategy II. Since the precursor value obtained by strategy II is the smallest possible value, strategy II seems to focus on the minimization of the failure rate. Note that by "minimization of the failure rate", we understand here a minimization of the integrand in Eq. (21), while the alarm interval of size \( \delta \) remains constant. The fact that in this point the corresponding value of \( \rho(x_n|x) \) is also far away from the maximum of \( \rho(x_n|x) \) does apparently not influence the outcome of the prediction.

ad (Q2): In the following calculation we will obtain the asymptotic form of the likelihood ratio for large events. Inserting the asymptotic form of the probability \( \rho_0(\eta,a) \) provided by Eq. (A3) and using the asymptotic expansion of the complementary error function Eq. (10), the likelihood ratio reads

\[
\frac{m(a, \eta, x_{pre})}{\eta \exp\left(\frac{\eta^2}{4(1-a)}\right)} \to \frac{1}{2\sqrt{1-a}} \frac{\eta \exp\left(\frac{\eta^2}{4(1-a)} - z(\eta,a)^2\right)\left(1 + \mathcal{O}\left(\frac{1}{\eta^2}\right)\right)}{z(\eta,a) + \mathcal{O}\left(\exp(-z(\eta,a)^2)\right)} + \mathcal{O}\left(\frac{\exp(-z(\eta,a)^2)}{z}\right),
\]

with

\[
z(\eta,a) = \frac{(1-a)}{2\sqrt{1-a^2}} x_{pre} + \frac{\eta}{\sqrt{2\sqrt{1-a^2}}},
\]

\( \eta \to \infty \), if the argument of the exponential function in

which is derived in Appendix A.

Inserting the asymptotic expression for \( \rho_0(\eta,a) \), the approximation of \( x_I \) in Eq. (A3) and the asymptotic expansion of the complementary error function Eq. (10) into Eqs. (13) and (14), we find the following expressions

\[
\begin{align*}
\rho(x_I|x) &\sim \frac{1 + \mathcal{O}\left(\frac{1}{\eta^2}\right)}{1 + a + \mathcal{O}\left(\frac{1}{\eta^2}\right)} \left(\frac{1 - a^2}{1 + a} \right)^{\frac{\eta^2}{4(1-a)}}
\end{align*}
\]

\( \eta \to \infty \).

\[
\begin{align*}
\rho(x_I|x) &\sim \frac{1 + \mathcal{O}\left(\frac{1}{\eta^2}\right)}{1 + a + \mathcal{O}\left(\frac{1}{\eta^2}\right)} \left(\frac{1 - a^2}{1 + a} \right)^{\frac{\eta^2}{4(1-a)}}
\end{align*}
\]

\( \eta \to \infty \).

Note that the limit \( \rho(x_n|x) \to \infty \) corresponds to the limit \( \eta \to \infty \) in the context of (Q2), but we can also interpret it as the limit \( a \to \pm 1 \) in the context of (Q3) if \( \eta \neq 0 \).

The expression in Eq. (27) tends to infinity in the limit

FIG. 4: (Color online) \( \rho(x_n|x) \) and \( \rho(x_n|x) \) for \( a = -0.75 \). The maximum of the posterior PDF to observe extreme events \( \rho(x_n|x) \) which is used as precursor, moves towards \(-\infty\) with increasing \( \eta \) since \( x_I \sim -\eta/(2\sqrt{(1-a^2)}) \). Because the maximum of the failure posterior PDF \( \rho(x_n|x) \) remains at the origin, the values of \( \rho(x_n|x) \) which are observed at the precursor value \( x_I \) decrease according to the decrease of \( \rho(x_n|x) \) as \( x_n \to -\infty \).
is positive. This is indeed the case for every precursor value \(x_{\text{pre}} < 0\). Therefore, for both strategies of prediction, the slope \(m(x_{\text{pre}}, a, \eta)\) increases as a squared exponential with increasing size of the events \(\eta\) according to Eq. (27). Hence, the considerations of Sec. II B hold for our example, according to which an event is the better predictable the more rare it is.

**ad (Q3):** One can also calculate the asymptotic behavior of the likelihood ratio for \(a \to \pm 1\). The limit \(z(\eta, a) \to \infty\), which is relevant for the asymptotic form in Eq. (27), can also be interpreted as the limit \(a \to \pm 1\). We assume that \(\eta\) is big enough, e.g., \(\eta > 2\), such that Eq. (A3), which enters into Eq. (27), is a useful approximation. One can now discuss again the argument of the exponential function in Eq. (28).

Inserting the precursor of strategy I (as given by Eq. ref), one obtains \(f(x, a, \eta) = \eta^2/8\), hence

\[
m(a, \eta, x) \to 2 \sqrt{1 + a} \exp \left( \frac{\eta^2}{8} \right), \quad (z(\eta, a) \to \infty).
\]

(29)

As \(a \to 1\), this expression converges to \(\exp (\eta^2/8)\). As \(a \to -1\), this expression approaches infinity as \(m(1, \eta, x) \sim 1/\sqrt{1 + a}\). Fig. (a) illustrates this behavior. Fig. (b) shows that the asymptotic expression in Eq. (28) becomes better in the limit \(\eta \to \infty\), since in this limit the higher order terms of the approximation vanish even faster.

For the theoretical precursor of strategy II \(x_{\text{II}} = -\infty\) the slope would be independent of the value of the coupling strength if the exact precursor of strategy II could be used. For any real precursor value of strategy II \(x_{\text{II}} = \text{const.} < 0\), Eq. (28) reads

\[
f(x_{\text{II}}, a, \eta) \sim \frac{\eta^2}{2(1 - a)} \left( \frac{1}{2} - \frac{1}{1 + a} \right) + O((1 - a)), \quad (a \to 1).
\]

(30)

This expression approaches a small negative value close to zero in the point \(a = 1\). Hence, we find \(m(a, \eta, x_{\text{II}}) \sim 1\), as \(a \to 1\).

In the limit \(a \to -1\) and for any finite precursor value \(x_{\text{II}} = \text{const.} < 0\), Eq. (28) reads

\[
f(x_{\text{II}}, a, \eta) \sim \frac{\eta^2}{4} \left( \frac{1}{2} - \frac{1}{1 - a^2} \right) - \frac{2x_{\text{II}} \eta}{\sqrt{1 - a^2}} - 2x_{\text{II}}^2
\]

\[
\sim -\frac{1}{1 - a^2} \frac{\eta^2}{4} - \frac{2x_{\text{II}} \eta}{\sqrt{1 - a^2}} - 2x_{\text{II}}^2.
\]

(31)

If the precursor is sufficiently small, e.g. \(x_{\text{II}} < -\eta/(4\sqrt{1 - a^2})\), this expression is positive and hence \(m(a, \eta, x_{\text{II}}) \to \infty\), as \(a \to -1\). Hence, the asymptotic expressions of the likelihood ratio are able to describe the behavior of the ROC-curves, shown in the previous section. Fig. (a) combines the dependence of the likelihood ratio on the event size and the correlation strength. One can see that the influence of the event size on the likelihood ratio is dominating, as long as one does not approach the singularity at \(a \to -1\).

**IV. APPLICATION: WIND SPEED MEASUREMENTS**

As an illustration of the preceding considerations and also in order to demonstrate the usefulness of the benchmarks derived for AR(1) processes, we study here time series data of wind speed measurements. The data are recorded at 30m above ground by a cup anemometer with a sampling rate of 8 Hz in the Lammefjord site of the
Risø research center [23]. Wind speed data are evidently non-stationary and strongly correlated, so that, e.g., the principle of persistence yields surprisingly accurate forecasts: the very simple prediction scheme \( \hat{x}_{n+1} = x_n \) is almost as accurate as an AR(20) model fitted on moving windows (in order to take non-stationarity into account) or order-10 Markov chains [10]. The amplitude of the fluctuations around a time local mean value are proportional to this mean value, i.e., there is statistical evidence that the noise in this process is multiplicative. However, when subtracting the time local mean (more precisely, performing a high-pass filtering with a Gaussian kernel with a standard deviation of 75 time steps), we receive data for which it is reasonable to fit an AR(1) process. When doing so, we find a coefficient \( a \approx 0.94 \).

Turbulent gusts, i.e., sudden increases of the wind speed, are relevant events, e.g., for the save operation of wind turbines, for aircrafts during take-off and landing, and for all wind-driven sports activities. In previous work [4], we were therefore concerned with their prediction, where we were studying the performance of a Markov chain model. Here, we will restrict ourselves to the simpler (and less appropriate) AR(1)-philosophy: The current state of the process generating the wind time series is assumed to be fully specified by the last observation \( x_n \), and the event is assumed to be characterized by the upward jump of the wind speed in a single time step by more than \( g \) m/s.

A. Determining the precursor value

If we extract from the data set all subsequences of data where such a jump is present, then we can, in principle, construct empirically the distribution \( p(x_n|g) \), which corresponds to \( \rho(x_{n,k}|X) \) of strategy I. In Fig. 7 we show instead the mean value of \( p(x_{n+k}|g) \) for \( k = -20, \ldots, 20 \), i.e., we show the mean profile of gusts of strength \( g \). Otherwise said, this is an average of all those time series segments, which (in shifted time) fulfill \( x_1 - x_0 > g \), so that the part of these segments with \( k \leq 0 \) is what one would call naively a precursor of a gust event. This has to be compared to the values \( x_{n+k} \) which we find when we focus on the maximum \( x_{II} \) in \( x_n \) of \( p(g|x_n) \) which corresponds to the conditional probability \( \rho(X|x_n) \) of strategy II. More specifically, in Fig. 8 we show the profiles \( (x_{n+k})|x_n=x_{II} \), where \( x_{II} \) is defined by \( p(g|x_{II}) = \max_{x_n} \). In even different words, the value plotted at \( k = 0 \) is the value \( x_n \) for which \( p(g|x_n) \) is maximal, and at the preceeding and succeeding time steps we show the average over all time series segments which fulfill \( x_n = x_{II} \) is some precision. These profiles differ from the precursors shown before, as we have to expect for an AR(1)-model: In a perfect AR(1) process, the precursors equivalent to those in Fig. 7 would show a jump larger than \( g \) from \( k = 0 \) to \( k = 1 \), with \( x_0 = -x_1 \), and with \( x_k = a^k x_0 \) for \( k < 0 \), and \( x_k = a^k x_1 \) for \( k > 1 \). For the same idealized process, one expects Fig. 8 to show curves given by \( x_k = a^k x_{II} \) for all \( k \). Evidently, the wind data show a qualitatively very similar behavior, whereas, however, additional correlations are visible.

B. Testing for predictive power

The ROC-curves for the two prediction strategies are shown in Fig. 9 and 10. As expected, the minimization of false alarms (strategy II) is here superior, as strategy I has no predictive power. The latter is consistent with the observed value \( a \approx 0.94 \) and the results for the AR(1) process.

In order to compute the ROC-curves we use the following numerically expensive but theoretically best justified algorithm: In theory, we want to generate an alarm if the current observation \( x_n \) lies in an interval \( V \) which is defined by the subset of the \( \mathbb{R} \) where either \( p(g|x_n) \) or...
\(p(x_n | g)\) exceeds some threshold \(0 \leq p_c \leq 1\). We assume that both conditional PDFs are smooth in \(x_n\).

We can locally approximate \(p(g | x_n)\) by searching all similar states \(x_j\), with \(|x_n - x_j| < \epsilon\) and counting the relative number of events in this set of states. When this number exceeds \(p_c\), we give the alarm and can see whether it is a hit or a false alarm.

In order to evaluate \(p(x_n | g)\) we first create the set of all states \(x_c\) which are preceeding an event, and then compute the fraction of these which is \(\epsilon\)-close to the current state \(x_n\). Since this fraction evidently depends on the value of \(\epsilon\), we should introduce a normalization. However, in order to create the ROC statistics we just have to introduce a threshold which runs from 0 to the largest value thus found. Both schemes can be straightforwardly generalized to situations where the current state of the process is defined by a sequence \(x_{(n,k)}\) of \(k\) past measurements \((x_{n-k+1}, x_{n-k+2}, \ldots, x_{n-1}, x_n)\), e.g., for an AR(2) model \(k = 2\), whereas in Fig. 9 we were using \(k = 10\) for a Markov chain of order 10.

Since the wind speed data are strongly correlated, \(a \approx 0.94\), it is not possible to predict the increments of the data sufficiently well. This corresponds to the previously derived results for the AR(1) model in the limit \(a \to 1\). However, we also find deviations from the theoretical ROC-curve for \(a = 0.94\), which is additionally plotted in Figs. 9 and 10. These deviations show that the AR(1) model is not able to describe the wind data completely.

The wind data also show the increase of predictability with increasing event size. This suggests that this effect is more general and not limited to the class of AR(1) models. Again, we also observe that strategy II is superior to strategy I.

\[C_x(t) = \left\{x_n x_{n+t}\right\} = \frac{1}{N-t} \sum_{n=1}^{N-t} x_n x_{n+t} \sim t^{-\gamma_c} \quad (32)\]

The correlation coefficient \(\gamma_c\) is controlling, how fast the correlations decay.

We study the predictability of increments numerically by applying the prediction strategies described in Sec. II A. The data used for this numerical study were generated as described in [24] and used in [25]: Imposing a power-law decay on the Fourier spectrum,

\[f_x(k) \propto k^{-\beta} \quad (33)\]

with \(0 < \beta < 0.5\) and choosing phase angles at random one obtains through an inverse Fourier transform the long-range correlated time series in \(x\) with \(\gamma_c = 1 - 2 \beta\).

The data are Gaussian distributed with \(\langle x \rangle = 0, \sigma = 1\). Having specified the power spectrum or, correspondingly, the autocorrelation function for sequences of Gaussian random numbers means to have fixed all parameters of a linear stochastic process. Hence, in principle the coefficients of an autoregressive or moving average process

\[\begin{align*}
\begin{array}{c}
\text{FIG. 9: (Color online) The ROC curves using strategy I,}\n\text{exploiting } p(x_n | X) \text{ and maximizing the hit rate. Evidently,}\n\text{the rate of false alarms exceeds the hit rate.}
\end{array}
\end{align*}\]

\[\begin{align*}
\begin{array}{c}
\text{FIG. 10: (Color online) The ROC-curves for the prediction of}\n\text{jumps of amplitude larger than } g \text{ for the wind data. Strategy}\n\text{II exploits } p(X | x_n) \text{ which minimizes the false alarm rate and}\n\text{performs the better the larger } g.
\end{array}
\end{align*}\]
can be uniquely determined, where, due to the power-law nature of the spectrum and autocorrelation function the order of either of these models have to be infinite \[7, 8\]. Thus, the effects which we observed for this ARMA(\(\infty, \infty\)) model should be valid for the whole class of linear long-term correlated processes. The ROC-curves in Fig. 11, which are generated from the long-range correlated data are very similar to the ones for the AR(1) process in terms of the question we want to study.

ad (Q1): The ROC-curves obtained by using strategy II are superior to the curves resulting from strategy I.

ad (Q2) and (Q3): The quality of the prediction also increases with increasing event size and decreasing correlation. Hence we observe the same effects which we described before for the AR(1) process and the wind speed data in a long range correlated ARMA(\(\infty, \infty\)) process.

VI. CONCLUSIONS

We studied the predictability of extreme increments in an AR(1) correlated process, in wind speed data and in a long-range correlated ARMA process. To measure the quality of the prediction we used the ROC-curve and additionally the slope of the ROC-curve in the vicinity of the origin as a summary index. This so called likelihood ratio, characterizes particularly the behavior in the limit of low false-alarm rates.

In the case of the AR(1) process we could construct the posterior PDF and the likelihood analytically from a given joint PDF and hence we were able to obtain the asymptotic behavior of the likelihood ratio analytically. In the case of the two other examples, we constructed the posterior PDFs numerically. The resulting distributions were then used to determine precursors according to two different strategies of prediction.

In all examples we studied the aspects: (Q1) Which is the best strategy to choose precursors? (Q2) How does the predictability depend on the event size? (Q3) And how does the predictability depend on the correlation? The results can be summarized as follows:

ad (Q1): Strategy I, the a posteriori approach, maximizes the rate of correct predictions, while strategy II focuses on the minimization of the rate of false alarms. For the example of the AR(1) process one can show that strategy II is the optimal strategy to make predictions. For other stochastic processes, it is not in general clear which of the two strategies leads to a better predictability. However, the application to the prediction of wind speeds and the numerical study within long-range correlated data reveals that also for these examples better results are obtained by predicting according to strategy II.

ad (Q2): For all examples studied, we observe an increase of predictability with increasing size of the events. This phenomenon which is also reported in the literature \[9, 10, 11\], can be better studied by investigating the asymptotic behavior of our summary index. In the case of the AR(1) process we showed explicitly that the likelihood ratio increases as a squared exponential with increasing event size. In Sec. IV we discussed for a general stochastic process that this effect appears, if the PDFs of the studied process fulfill certain conditions.

ad (Q3): For the AR(1) process and the long-range correlated data we observe that the correlation of the data is inversely proportional to the quality of the predictions. The ROC-curves for the wind data, which we assume to be a strongly correlated AR(1) process with correlation strength \(\gamma = 0.94\), display also a bad predictability. This effect is due to the special definition of the events as increments. The asymptotic expression for the likelihood ratio in Eq. 27 provides us also with a formally understanding of the \(\gamma\)-dependence.

All the considerations made in this contribution are made for a very simple but general method. In order to make predictions, we use the largest maximum of the a posterior PDF or the likelihood. For multimodal distributions, one can think about more sophisticated methods, which take into account also other maxima of the distribution. Furthermore, we investigate only stationary processes in these contributions. It remains to be studied, whether the answers, obtained to the questions (Q1)-(Q3) are also valid for non-stationary processes or multimodal distributions.

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APPENDIX A: OBTAINING AN ASYMPTOTIC FORM OF THE TOTAL PROBABILITY TO FIND INCREASEMENTS OF SIZE $\eta$

The total probability $\rho(\eta, a)$ to find increments of size $\eta$ can be obtained by integrating the pre-form of the posterior probability in Eq. (8). For the example of the AR(1) process the corresponding integral reads

$$\rho(\eta, a) = \int_{-\infty}^{\infty} \frac{\sqrt{1 - a^2}}{2\sqrt{2\pi}a} \exp \left( -\frac{1 - a^2}{2} x_n^2 \right) \text{erf} \left( \frac{(1 - a) x_n}{\sqrt{2}} + \frac{\eta}{\sqrt{2(1 - a^2)}} \right) \, dx_n.$$  

(A1)

In the special case $\eta = 0$ one can find the analytical form of the total probability $\rho(0, a)$ using again an integral identity from \cite{21}. The resulting value $\rho(0, a) = 1/2$ corresponds to the intuitive expectation one would have, since for $\eta = 0$ the condition of our extreme event is always fulfilled if $x_{n+1}$ is larger than $x_n$. This special case of predicting the sign of increments in uncorrelated data is discussed in \cite{22}.

For $\eta \neq 0$, we can find an asymptotic form of the total probability $\rho(\eta, a)$ via evaluating the mean of the posterior PDF. An analytic expression of the mean can be obtained using an integral representation from \cite{21}

$$\langle x_n \rangle = \frac{−\exp \left( −\frac{\eta^2}{2(1−a)} \right)}{2\sqrt{\pi} \sqrt{1−a^2} \rho(\eta, a)}.$$  

(A2)

For large values of $\eta$ we can also assume that the maximum and the mean of $\rho(x_n | X(\eta), a)$ nearly coincide, i.e.,

$$\langle x_n \rangle \simeq x_1 \simeq \frac{-\eta}{2\sqrt{1-a^2} \left( 1 + O \left( \frac{1}{\eta^2} \right) \right)}, \quad (\eta \to \infty),$$  

(A3)

provided that $\rho(x_n | X(\eta), a)$ is not too asymmetric (i.e., $a$ is not close to $−1$). Using this approximation, we find the following asymptotic form of the total probability to find increments of size $\eta$

$$\rho(\eta, a) \sim \frac{\sqrt{1-a}}{\sqrt{\pi} \eta} \exp \left( −\frac{\eta^2}{4(1-a)} \right) \left( 1 + O \left( \frac{1}{\eta^2} \right) \right), \quad (\eta \to \infty).$$  

(A4)

APPENDIX B: TRANSFORMATION OF EXTREME INCREMENTS INTO EXTREME VALUES

We show how to relate the results obtained using the definition of extreme events as extreme increments $(x_{n+1} - x_n \geq d$ as in Eq. (6)) to the case when extreme events are defined as extreme values $(y_{n+1} \geq d)$ which exceed a certain threshold $d$, for ARMA(p,q) processes. An ARMA(p,q) model is defined as \cite{7}

$$\Phi(B)x_n = \theta(B)\xi_n,$$  

(B1)

where $\{\xi\}$ correspond to white noise and

$$\Phi(B) = 1 - \Phi_1 B - \Phi_2 B^2 - \ldots - \Phi_p B^p,$$

$$\theta(B) = 1 + \theta_1 B + \theta_2 B^2 + \ldots + \theta_q B^q,$$

with $B^j x_n = x_{n-j}$. Searching for extreme increments in a time series $\{x\}$ is equivalent to search for extreme values in the time series $\{y\}$, defined through the transformation

$$y_{n+1} = x_{n+1} - x_n.$$  

(B2)

Assuming that $\{x\}$ is described by an ARMA(p,q) process defined by Eq. (B1), and inserting Eq. (B2) in Eq. (B1), one obtains that $\{y\}$ is described by an ARMA(p,q) model with the following transformed coefficients

$$\Phi_i^\dagger = \Phi_i \quad i = 1, 2, \ldots, p \quad ,$$

$$\theta_i^\dagger = \theta_i - \theta_{i-1} \quad i = 1, 2, \ldots, q \quad ,$$

$$\theta_q^\dagger = \theta_q \quad .$$  

(B3)

Due to the transformation (B2) the precursory structure equivalent to the one used in Sec. III is obtained choosing\cite{26}

$$y_{pre} = \sum_{j=0}^{n} y_j - x_0 = x_n.$$  

(B4)

With this choice of precursory structure and the corresponding transformation of the process (Eq. (B2)), the results obtained for extreme increments can be transferred to the case of extreme values. In particular, for the case of AR(1) processes (which corresponds to an ARMA(1,0)) discussed in Sec. III all results are also valid for an ARMA(1,1) process with the precursor given by (B4) and events defined as extreme values. E.g the alarm strategies consist in this case in raising an alarm whenever $y_{pre}$ falls near the precursor values given in Eq. (1).
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