CALCULATING THE CRITICAL EXPONENTS OF THE CHIRAL PHASE TRANSITION

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We calculate the critical exponents of the chiral phase transition at nonzero temperature using the thermal and chiral susceptibilities. We show that within a class of confining Dyson–Schwinger equation (DSE) models the transition is mean field, and that an accurate determination of the critical exponents requires extremely small values of the current-quark mass, several order of magnitude smaller than realistic up- and down-quark masses. In general, rainbow truncation models of QCD exhibit mean field exponents as a result of the gap equation’s fermion substructure.

1 Introduction

It is anticipated that the restoration of chiral symmetry, which accompanies the formation of a quark-gluon plasma at nonzero temperature $T$, is a second-order phase transition in QCD with 2 light flavours. Such transitions are characterised by two critical exponents: $(\beta, \delta)$, which describe the response of the chiral order parameters, $\mathcal{X}$, to changes in $T$ and in the current-quark mass, $m$. Denoting the critical temperature by $T_c$, and introducing the reduced-temperature $t := T/T_c - 1$ and reduced mass $h := m/T$, then

$$\mathcal{X} \propto (-t)^{\beta}, \quad t \to 0^-, \quad h = 0,$$

$$\mathcal{X} \propto h^{1/\delta}, \quad h \to 0^+, \quad t = 0. \tag{1}$$

Calculating the critical exponents is an important goal because of the notion of universality, which states that their values depend only on the symmetries and dimensions, but not on the microscopic details of the theory.

The success of the nonlinear $\sigma$-model in describing long-wavelength pion dynamics underlies a conjecture that chiral symmetry restoration at finite $T$ in 2-flavour QCD is in the same universality class as the 3-dimensional, $N = 4$ Heisenberg magnet ($O(4)$ model), with critical exponents $\beta_H = 0.38 \pm .01$, $\delta_H = 4.82 \pm .05$. However, recently it was argued that the compositeness of QCD’s mesons affects the nature of the phase transition and the Gross–Neveu model was presented as a counterexample to universality. Subsequent studies indicated that nontrivial $1/N$ corrections are important and that this model has the same critical exponents as the Ising model, as was argued on the notion of universality, but only in a scaling region of width $1/N$.

Calculating the exponents $\beta$ and $\delta$ directly from Eqs. (1) and (2) is often difficult because of numerical noise near the critical temperature. Another
method is to consider the chiral and thermal susceptibilities:
\[
\chi_{t}(t, h) := \frac{\partial X(t, h)}{\partial t} \bigg|_{t} , \quad \chi_{i}(t, h) := \frac{\partial X(t, h)}{\partial h} \bigg|_{h} .
\] (3)

At each \( h \), \( \chi_{i}(t, h) \), \( i = h, t \), are smooth functions of \( t \) with maxima \( \chi_{t}^{pe} \) at the pseudocritical points \( t_{pc}^{i} \). Near the critical point \( t = 0 = h \) we have
\[
t_{pc}^{h} \propto h^{1/(16)} \propto t_{pc}^{t} , \quad \chi_{h}^{pe} = \chi_{h}(t_{pc}^{h}, h) \propto h^{-z_{h}}, \quad z_{h} := 1 - \frac{1}{6} , \quad \chi_{t}^{pe} = \chi_{t}(t_{pc}^{t}, h) \propto h^{-z_{t}}, \quad z_{t} := \frac{1}{16} (1 - \beta) .
\] (4) (5) (6)

Therefore, by calculating the chiral and thermal susceptibilities and locating the pseudocritical points, one can determine \( T_{c} \) and the critical exponents.

2 Quark Dyson–Schwinger Equation

We have analysed \( \chi_{h}(t, h) \) and \( \chi_{i}(t, h) \) in a class of confining DSE models that underlies many successful phenomenological applications at both zero and finite-(\( T, \mu \)). The foundation of our study is the renormalised quark DSE
\[
S^{-1}(p, \omega_{k}) := i \gamma \cdot \vec{p} A(p^{2}, \omega_{k}) + i \gamma_{4} \omega_{k} C(p^{2}, \omega_{k}) + B(p^{2}, \omega_{k}) \quad (7)
\]
\[
= Z_{2}^{A} i \gamma \cdot \vec{p} + Z_{2}^{C} i \gamma_{4} \omega_{k} + Z_{4} m_{R}(\zeta) + \Sigma'(\vec{p}, \omega_{k}) .
\] (8)

Here \( \omega_{k} = (2k + 1) \pi T \) is the fermion Matsubara frequency and \( m_{R}(\zeta) \) is the current quark mass at the renormalisation point \( \zeta \). The self-energy is
\[
\Sigma'(\vec{p}, \omega_{k}) = T \sum_{l = -\infty}^{\infty} \int \frac{d^{3}q}{(2\pi)^{3}} 4 g^{2} D_{\mu\nu}(\vec{p} - \vec{q}, \Omega_{k-l}) \gamma_{\mu} S(\vec{q}, \omega_{l}) \Gamma_{\nu} , \quad (9)
\]
with \( D_{\mu\nu}(k, \Omega_{j}) \) the renormalised dressed-gluon propagator and \( \Gamma_{\nu} \) the renormalised dressed-quark-gluon vertex. In renormalising the DSE we require
\[
S^{-1}(\vec{p}, \omega_{0})\big|_{p^{2} + \omega_{0}^{2} = c^{2}} = i \gamma \cdot \vec{p} + i \gamma_{4} \omega_{0} + m_{R} .
\] (10)

Equations (8)-(10) define the exact QCD gap equation.

We use the rainbow truncation for the vertex, \( \Gamma_{\nu} = \gamma_{\nu} \), which is the leading term in a \( 1/N_{c} \)-expansion of the vertex, and consider three models in which the long-range part of the interaction is an integrable infrared singularity motivated by \( T = 0 \) studies of the gluon DSE.

\[
g^{2} D_{\mu\nu}(\vec{k}, \Omega_{j}) = P_{\mu\nu}(\vec{k}, \Omega_{j}) D(\vec{k}, \Omega_{j}; m_{g}) + P_{\mu\nu}^{T}(\vec{k}, \Omega_{j}) D(\vec{k}, \Omega_{j}; 0) ,
\]
\[
D(\vec{k}, \Omega_{j}; m_{g}) := 2\pi^{2} D \frac{2\pi}{T} \delta_{0 j} \delta^{(4)}(\vec{k}) + D_{M}(k^{2} + \Omega_{j}^{2} + m_{g}^{2}) ,
\] (11)
where $P_{44}^T = P_{4i}^T = 0$, $P_{ij}^T = \delta_{ij} - k_i k_j / k^2$, $P_{\mu\nu}^L = \delta_{\mu\nu} - k_\mu k_\nu / (k^2 + \Omega^2) - P_{\mu\nu}^T$, $m_\eta$ is a Debye mass, and $D$ is a mass-parameter fitted to $m_\eta$ and $f_\pi$ at $T = 0$. We compare the results for 3 different models, denoted by $\mathcal{D}_M$, $M = A, B, C$.

One order parameter for dynamical chiral symmetry breaking is the quark condensate $\langle \bar{q} q \rangle_0^\zeta$. There are other, equivalent order parameters and in calculating the chiral and thermal susceptibilities we employ

$$X := B(p^2 = 0, \omega_0), \quad \chi_C := \frac{B(p^2 = 0, \omega_0)}{C(p^2 = 0, \omega_0)}.$$ (12)

They should be equivalent and, as we will see, the onset of that equivalence is a good way to determine the $h$-domain on which Eqs. (1)- (2) are valid. Further, we have verified numerically that in the chiral limit ($m = 0$) and for $t \sim 0$: $f_\pi \propto \langle \bar{q} q \rangle \propto X(t, 0)$; i.e., that these quantities are all equivalent, bona fide order parameters. It thus follows from the pseudoscalar mass formula:

$$f_\pi^2 m_\pi^2 \zeta = 2 m_\eta \langle \bar{q} q \rangle_0^\zeta,$$

that $m_\pi$ increases with temperature.

3 Results

The first model we consider is an infrared dominant model with $\mathcal{D}_A(s) = 0$, and the mass-scale $D = 0.56 \text{ GeV}^2$ fixed by fitting $\pi$- and $\rho$-meson masses at $T = 0$. A current-quark mass of $m = 12 \text{ MeV}$ yields $m_\pi = 140 \text{ MeV}$. The quark DSE obtained with $\mathcal{D}_A$ is an algebraic equation. Chiral symmetry restoration is therefore easy to analyse and either directly, via Eqs. (1) and (2), or using the susceptibilities and Eqs. (5) and (6), it is straightforward to establish that this model has mean field critical exponents and to determine the critical temperature in Table 1. The exponents are unchanged and $T_c$ reduced by $< 2\%$ upon the inclusion of some higher-order $1/N_c$-corrections to the dressed-quark-gluon vertex.

3.1 Model B: QED-like tail

To improve the ultraviolet behaviour, we consider a model with

$$\mathcal{D}_B(s) = \frac{16}{9} s^2 \frac{1 - e^{-s/(4m_t^2)}}{s},$$ (13)

and $D = (8/9) m_t^2$. The mass-scale $m_t = 0.69 \text{ GeV} = 1/0.29 \text{ fm}$ marks the boundary between the perturbative and nonperturbative domains, and was fixed by requiring a good description of $\pi$- and $\rho$-meson properties at $T = 0$.

The quark DSE obtained with this model can be solved numerically and $\chi_{ph}^p(h)$ and $\chi_{ph}^p(h)$ are depicted in Fig. 1(a). Following Eqs. (5) and (6), the
critical exponents can be determined by defining a local critical exponent as a function of the reduced mass $h$ for each of the equivalent order parameters:

$$z_i(h) := -h \frac{\partial \ln \chi_{pc}^i}{\partial h},$$

see Fig. 1(b). $h$ lies in the scaling region when $z_i$ is independent of the order parameter. This shows that the scaling relations are not valid until

$$\log_{10}(h/h_u) < -7,$$

where $h_u = m_R/T_c$ corresponds to the current-quark mass that gives $m_\pi = 140\text{ MeV}$ in this model. The values of $z_h$ and $z_t$ in Table 1 are obtained by a Padé fit to the five smallest $h$-values in Fig. 1(b), and extrapolating to $h \to 0^+$. The critical temperature is obtained using Eq. (4); its value is insensitive to whether $t_{pc}^h$ or $t_{pc}^t$ is used and to which of the equivalent order parameters is used.

| \(T_c\) (MeV) | mean field | A | B | C |
|------------|------------|---|---|---|
|            | 169        | 174 | 120 |   |
| \(z_h\)   | 2/3        | 0.666 | 0.67 ± 0.01 | 0.667 ± 0.001 |
| \(z_t\)   | 1/3        | 0.335 | 0.33 ± 0.02 | 0.333 ± 0.001 |

Table 1. Critical temperature for chiral symmetry restoration and critical exponents characterising the second-order transition in the three exemplary models.
3.2 Model C: Logarithmic tail

Finally, we consider the finite-$T$ extension of a model which further improves the ultraviolet behaviour, via the inclusion of the one-loop $\ln$-suppression at $s \gg \Lambda_{\text{QCD}}^2$. Again, the parameters are fixed at $T = 0$ by requiring a good fit to a range of $\pi^-, K$-meson properties. Recent calculations show that the vector mesons are also described well in this model. The study of chiral symmetry restoration in this model is very similar to the previous study, with the additional $\ln[s]$-suppression in the ultraviolet making the numerical analysis easier. The critical temperature and exponents are presented in Table 1. Also in this case the scaling relations are only valid for very small current-quark masses: $\log_{10}(h/h_u) < -5$. These results are qualitatively, and for the critical exponents quantitatively, independent of the parameters in this model. Direct calculation of the critical exponents using Eqs. (1) and (2) are in good agreement with the critical exponents found using the susceptibilities.

4 Conclusions

It is clear from Table 1 that each of these models is mean field in nature. In hindsight that may be not surprising because the long-range part of the interaction is identical. However, the models differ by the manner in which the interactions approach their long-range limits, and our numerical demonstration of their equivalence required extremely small values of the current-quark mass, Eq. (15). This might also be true in QCD; i.e., while $T_c$ is relatively easy to determine, very small current-quark masses may be necessary to accurately calculate the critical exponents from the susceptibilities. In that case, calculation of $\beta$ and $\delta$ via lattice-QCD will not be easy. The discrepancies found in recent lattice calculations could be a signal of this difficulty.

The class of models we have considered can describe the long-wavelength dynamics of QCD very well in terms of mesons that are quark-antiquark composites. The characteristic feature is the behaviour of the confining interaction. It provides a driving term in the quark DSE proportional to the dressed-quark propagator, which means that boson Matsubara zero-modes do not influence the critical behaviour determined from the gap equation. The class of Coulomb gauge models also describes mesons as composite particles and it too exhibits mean field critical exponents. The long-range part of the interaction in that class of models corresponds to the regularised Fourier amplitude of a linearly rising potential. Hence it is not equivalent to ours in any simple way, except insofar as zero modes do not influence the gap equation.

The quark DSE is the QCD gap equation and the many equivalent chiral
order parameters are directly related to properties of its solution. We have observed that several classes of models exhibit the same (mean field) critical exponents. Only in our simplest confining model did we consider the effect of $1/N_c$-corrections to the quark-gluon vertex, and in that case the critical exponents were unchanged. These results suggest that mean field exponents are a feature of the essential fermion substructure in the gap equation. It can likely only be false if nonperturbative corrections to the vertex are large in the vicinity of the transition. In this context the role of mesonic bound states, which can appear as nonperturbative contributions in the dressed-quark-gluon vertex, has to be studied in more detail. This might also give a nontrivial dependence on the number of fermion flavours, as is anticipated on universality arguments.

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