SOME COMBINATORIAL ASPECTS OF COMPOSITION OF A SET OF FUNCTIONS

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Abstract. In this paper we determine a number of meaningful compositions of higher order of a set of functions, which is considered in [2], in implicit and explicit form. Results which are obtained are applied to the vector analysis in order to determine the number of meaningful differential operations of higher order.

1. The composition of a set of functions

Main topic of consideration in this paper is the set of functions $A_n$, for $n=2,3,\ldots$, determined in the following form:

\begin{align}
A_n (n=2m): \quad \nabla_1: A_0 \to A_1 \\
\nabla_2: A_1 \to A_2 \\
\vdots \\
\nabla_i: A_i \to A_{i+1} \\
\vdots \\
\nabla_m: A_m \to A_{m-1} \\
\nabla_{m+1}: A_m \to A_{m-1} \\
\vdots \\
\nabla_{n-j}: A_{j+1} \to A_j \\
\vdots \\
\nabla_{n-1}: A_2 \to A_1 \\
\nabla_n: A_1 \to A_0,
\end{align}

Additional, we make an assumption that $A_i$ are non-empty sets, for $i=0,1,\ldots,m$, where $m=\lceil n/2 \rceil$. For each set of functions $A_n$ we determine the number of meaningful compositions of higher order in implicit and explicit form. Let us define a binary relation $\rho$ "to be in composition" with $\nabla_i \rho \nabla_j = 1$ iff the composition $\nabla_j \circ \nabla_i$ is meaningful for $i,j \in \{1,2,\ldots,n\}$. Let us form an adjacency matrix $A = [a_{ij}]$ of the graph, determined by relation $\rho$, with

\begin{align}
a_{ij} = \begin{cases} 
1 & : (j = i + 1) \vee (i + j = n + 1) \\
0 & : (j \neq i + 1) \wedge (i + j \neq n + 1)
\end{cases}
\end{align}

for $i,j \in \{1,2,\ldots,n\}$. Thus, on the basis of the article [2], some implicit formulas for the number of meaningful compositions are given by the following statement.

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Theorem 1.1 Let \( P_n(\lambda) = |A - \lambda I| = \alpha_0 \lambda^n + \alpha_1 \lambda^{n-1} + \ldots + \alpha_n \) be the characteristic polynomial of the matrix \( A = [a_{ij}] \), determined by (2), and \( v_n = [1 \cdots 1]_{1 \times n} \). If we denote by \( f(k) \) the number of meaningful composition of \( k \)-th order of functions from \( A_n \), then the following formulas are true:

\[
\begin{align*}
(3) & \quad f(k) = v_n \cdot A^{k-1} \cdot v_n^T \\
(4) & \quad \alpha_0 f(k) + \alpha_1 f(k - 1) + \ldots + \alpha_n f(k - n) = 0 \quad (k > n).
\end{align*}
\]

Remark 1.2 Generally, let a graph \( G \), with vertices \( \nu_1, \ldots, \nu_n \), be determined by adjacency matrix \( A \) and let \( P_n(\lambda) = |A - \lambda I| = \alpha_0 \lambda^n + \alpha_1 \lambda^{n-1} + \ldots + \alpha_n \) be the characteristic polynomial of the matrix \( A \). If we denote with \( a_{ij}^{(k)} \) the number of \( \nu_i, \nu_j \)-walks of length \( k \) in the graph \( G \), then for every choice of \( \nu_i \) and \( \nu_j \), the sequence \( a_{ij}^{(k)} \) satisfies the same recurrent relation (4). The previous statement is the first problem in the section 8.6, of the supplementary problems page, of the book [3].

2. Some explicit formulas for the number of composition

In this part we give some explicit formulas for the number of meaningful compositions of functions from the set \( A_n \). The following statements are true.

Lemma 2.1 The characteristic polynomial \( P_n(\lambda) \) of matrix \( A = [a_{ij}] \), determined by (2), fulfills the following recurrent relation

\[
(5) \quad P_n(\lambda) = \lambda^2 \left( P_{n-2}(\lambda) - P_{n-4}(\lambda) \right).
\]

Proof. Expanding the determinant \( P_n(\lambda) = |A - \lambda I| \) by first column we have

\[
(6) \quad P_n(\lambda) = -\lambda C_{n-1}(\lambda) + (-1)^{n+1} D_{n-1}(\lambda),
\]

where \( C_{n-1}(\lambda) \) and \( D_{n-1}(\lambda) \) are suitable minors of the elements \( a_{11} \) i \( a_{n1} \) of the determinant \( P_n(\lambda) \). Continuing the expansion of the determinant \( C_{n-1}(\lambda) \) by the ending row we can conclude that

\[
(7) \quad C_{n-1}(\lambda) = -\lambda P_{n-2}(\lambda).
\]

Further, let us remark that determinant \( D_n(\lambda) \) has minor \( P_{n-3}(\lambda) \) as follows

\[
(8) \quad D_n(\lambda) = \begin{vmatrix}
1 & 0 & 0 & \ldots & 0 & 0 & 0 & 1 \\
-\lambda & 1 & 0 & \ldots & 0 & 1 & 0 & 0 \\
0 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
1 & 0 & 0 & \ldots & 0 & 0 & 1 & -\lambda
\end{vmatrix}.
\]

If in the previous determinant we multiply the first row by \(-1\) and add it to the \( n^{th} \)-row and then, if in the next step, we expand determinant by ending column, we can conclude

\[
(9) \quad D_n(\lambda) = (-1)^{n-1} \lambda^2 P_{n-3}(\lambda).
\]
On the basis of expansion (6) and formulas (7), (9) it is true that
\[ P_n(\lambda) = \lambda^n(P_{n-2}(\lambda) - P_{n-4}(\lambda)). \]

**Lemma 2.2** Characteristic polynomial \( P_n(\lambda) \) of the matrix \( \mathcal{A} = [a_{ij}] \), determined by (2), has the following explicit representation
\[
P_n(\lambda) = \begin{cases} 
\sum_{k=1}^{\left\lceil \frac{n+1}{2} \right\rceil} (-1)^{k-1} \left( \frac{n}{2} - k \right)^2 \lambda^{n-k+2} : n = 2m, \\
\sum_{k=1}^{\left\lceil \frac{n+2}{2} \right\rceil} (-1)^{k-1} \left( \frac{n+3}{2} - k \right) \lambda^{n-k} : n = 2m+1. 
\end{cases}
\]

**Proof.** Let us determine a few initial characteristic polynomials in the following forms:
\[
P_2(\lambda) = \lambda^2 - 1 = \sum_{k=1}^{2} (-1)^{k-1} \left( \frac{3}{2} - k \right) \lambda^{4-2k},
\]
\[
P_4(\lambda) = \lambda^4 - 2 \lambda^2 = \sum_{k=1}^{2} (-1)^{k-1} \left( \frac{4}{2} - k \right) \lambda^{6-2k},
\]
and
\[
P_3(\lambda) = \lambda^3 - \lambda^2 - \lambda = \sum_{k=1}^{3} (-1)^{k-1} \left( \frac{3}{2} - k \right) \lambda^{5-2k} + \left( \frac{3}{2} - k \right) \lambda^{6-2k},
\]
\[
P_5(\lambda) = \lambda^5 - \lambda^4 - 2 \lambda^3 + \lambda^2 = \sum_{k=1}^{3} (-1)^{k-1} \left( \frac{4}{2} - k \right) \lambda^{7-2k} + \left( \frac{4}{2} - k \right) \lambda^{8-2k}.
\]

Then the statement of this lemma follows by mathematical induction on the basis of the recurrent relation (5). \( \blacksquare \)

From theorem 1.1 and lemma 2.2 the following statement follows.

**Theorem 2.3** Let \( \mathcal{A} = [a_{ij}] \) be the matrix determined by (2). Then the number of meaningful composition of \( k^{th} \)-order of functions from \( \mathcal{A}_n \) fulfills the recurrent relation (4), whereas \( \alpha_i \) (\( i = 0, 1, \ldots, n \)) are coefficients of the characteristic polynomial \( P_n(\lambda) \) determined by (11).

Further, the following general statement is true.

**Lemma 2.4** Let \( \mathcal{A} = [a_{ij}] \) be the \( k^{th} \)-power of the matrix \( \mathcal{A} = [a_{ij}] \in \mathbb{C}^{n \times n} (k \in \mathbb{N}) \) and let
\[ P_n(\lambda) = |\mathcal{A} - \lambda I| = \alpha_0 \lambda^n + \alpha_1 \lambda^{n-1} + \ldots + \alpha_n, \]
is characteristic polynomial \( P_n(\lambda) \) of the matrix \( \mathcal{A} \). If for each pair of indexes \( (i, j) \in \{1, 2, \ldots, n\}^2 \) the sequence \( g_{ij}(m) \), for \( m > n \), is determined as a solution of the recurrent relation
\[ \alpha_0 g_{ij}(m) + \alpha_1 g_{ij}(m-1) + \ldots + \alpha_n g_{ij}(m-n) = 0, \]
on the basis of initial values \( g_{ij}(1) = a_{ij}^{(1)}, g_{ij}(2) = a_{ij}^{(2)}, \ldots, g_{ij}(n) = a_{ij}^{(n)} \), then matrix \( \mathcal{G}_m = [g_{ij}(m)] \in \mathbb{C}^{n \times n} \) is the \( m^{th} \)-power of the matrix \( \mathcal{A} \) (\( m \in \mathbb{N} \)).
The well-known fact that \( \Omega \) of basis elements. Consequently, previously introduced functions are isomorphisms order. Let us remark that, in comparison to \([2]\) by forming

\[
(20) \quad \varphi_i : \Omega^i(\mathbb{R}^n) \to A_i (0 \leq i \leq m) \quad \text{and} \quad \varphi_{n-i} : \Omega^{n-i}(\mathbb{R}^n) \to A_i (0 \leq i < n-m),
\]

by forming \( \binom{n}{i} \)-tuple of coefficients with respect to the basis elements in the given order. Let us remark that, in comparison to \([2]\), we additionally consider the order of basis elements. Consequently, previously introduced functions are isomorphisms from the space of differential forms into the space of vector functions. Next, we use the well-known fact that \( \Omega^i(\mathbb{R}^n) \) and \( \Omega^{n-i}(\mathbb{R}^n) \) are spaces of the same dimension.
three-dimensional vector analysis. Let \( (\Omega^r)_{i=0}^{n} \) for \( i = 0, 1, \ldots, m \). They can be identified with \( A_i \), using corresponding isomorphism (20). Let’s define differential operations of the first order via exterior differentiation operator \( d \) as follows

\[
\nabla_r = \varphi_r \circ d \circ \varphi_r^{-1} \quad (1 \leq r \leq n).
\]

Thus the following diagrams commute:

\[
\begin{array}{ccc}
\Omega_{i-1} & \xrightarrow{d} & \Omega_i \\
\varphi_{i-1} & \xrightarrow{\nabla} & \varphi_i \\
A_i & (1 \leq i \leq m) & \end{array}
\quad
\begin{array}{ccc}
\Omega_{n-i} & \xrightarrow{d} & \Omega_{n-1} \\
\varphi_{n-i} & \xrightarrow{\nabla} & \varphi_{n-1} \\
A_{n-i} & (0 \leq i \leq n-m) & \end{array}
\]

Hence, the differential operations \( \nabla_r \) determine functions so that (1) is fulfilled. Let us define differential operations of the higher order as meaningful compositions of higher order of functions from the set \( A_n = \{ \nabla_1, \ldots, \nabla_n \} \). Let us consider, in the next sections, concrete dimensions \( n = 3, 4, 5, \ldots, 10 \).

**Three-dimensional vector analysis.** In the real three-dimensional space \( \mathbb{R}^3 \) we consider the following sets

\[
(23) \quad A_0 = \{ f : \mathbb{R}^3 \rightarrow \mathbb{R} \mid f \in C^\infty(\mathbb{R}^3) \} \text{ and } A_1 = \{ \tilde{f} : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \mid \tilde{f} \in C^\infty(\mathbb{R}^3) \}.
\]

Let \( dx, dy, dz \) respectively be the basis vectors of the space of 1-forms and let \( dy \wedge dz, dz \wedge dx, dx \wedge dy \) respectively be the basis vectors of the space 2-forms. Thus over the sets \( A_0 \) and \( A_1 \) there exist \( m = 3 \) differential operations of the first order

\[
\nabla_1 f = \text{grad} f : A_0 \rightarrow A_1,
\]

\[
\nabla_2 \tilde{f} = \text{curl} \tilde{f} : A_1 \rightarrow A_1,
\]

\[
\nabla_3 \tilde{f} = \text{div} \tilde{f} : A_1 \rightarrow A_0.
\]

Under the previous choice of order of basis vectors of spaces of 1-forms and 2-forms, the previously defined operations of the first order coincide with differential operations of first order in the classical vector analysis. Next, as a well-known fact, there are \( m = 5 \) differential operations of the second order. In the article [1] it is proved that there exists \( m = 8 \) differential operations of the third order. Further, in the article [2], it is proved that there exist \( F_{k+3} \) differential operations of the \( k \)th-order, where \( F_k \) is Fibonacci’s number of order \( k \). Here we give the proof of the previous statement, on the basis of results from the second part of this paper. Namely, using theorems 2.3 and 2.5, matrix \( \mathbf{A}^k \) has the following explicit form

\[
(25) \quad \mathbf{A}^k = \begin{bmatrix}
F_{k-1} & F_k & F_k \\
F_{k-1} & F_k & F_k \\
F_{k-2} & F_{k-1} & F_{k-1}
\end{bmatrix}.
\]

Hence, using (3), the number of differential operation of the \( k \)th-order is determined by

\[
(26) \quad f(k) = v_3 \cdot \mathbf{A}^{k-1} \cdot v_3^T = F_{k-3} + 4F_{k-2} + 4F_{k-1} = F_{k+3}.
\]
Multidimensional vector analysis. In the real $n$-dimensional space $\mathbb{R}^n$ the number of differential operations is determined by corresponding recurrent formulas, which for dimension $n = 3, 4, 5, \ldots, 10$, we cite according to [2]:

| dimension: | recurrent relations for the number of meaningful operations: |
|-----------|----------------------------------------------------------|
| $n = 3$   | $f(i + 2) = f(i + 1) + f(i)$                           |
| $n = 4$   | $f(i + 2) = 2f(i)$                                      |
| $n = 5$   | $f(i + 3) = f(i + 2) + 2f(i + 1) - f(i)$                |
| $n = 6$   | $f(i + 4) = 3f(i + 2) - f(i)$                          |
| $n = 7$   | $f(i + 5) = f(i + 3) + 3f(i + 2) - 2f(i + 1) - f(i)$    |
| $n = 8$   | $f(i + 4) = 4f(i + 2) - 3f(i)$                          |
| $n = 9$   | $f(i + 5) = f(i + 4) + 4f(i + 3) - 3f(i + 2) - 3f(i + 1) + f(i)$ |
| $n = 10$  | $f(i + 6) = 5f(i + 4) - 6f(i + 2) + f(i)$               |

For dimensions $n = 3$, as we have shown in the previous consideration, the numbers of differential operations of higher order are determined via Fibonacci numbers. Also, this is true for dimension $n = 6$. Namely, using theorems 2.3 and 2.5, matrix $A^k$ has following explicit form

$$(27) A^k = \begin{cases} 
F_{2p-1} & 0 & F_{2p} & 0 & F_{2p} & 0 \\
0 & F_{2p} & 0 & F_{2p-1} & 0 & F_{2p} \\
F_{2p-2} & 0 & F_{2p-1} & 0 & F_{2p-1} & 0 \\
0 & F_{2p} & 0 & F_{2p-1} & 0 & F_{2p} \\
F_{2p-1} & 0 & F_{2p} & 0 & F_{2p} & 0 \\
0 & F_{2p-1} & 0 & F_{2p-2} & 0 & F_{2p-1} 
\end{cases} : \quad k = 2p,
$$

$$(27) A^k = \begin{cases} 
0 & F_{2p+1} & 0 & F_{2p} & 0 & F_{2p+1} \\
F_{2p} & 0 & F_{2p+1} & 0 & F_{2p+1} & 0 \\
0 & F_{2p} & 0 & F_{2p-1} & 0 & F_{2p} \\
F_{2p} & 0 & F_{2p-1} & 0 & F_{2p+1} & 0 \\
0 & F_{2p+1} & 0 & F_{2p} & 0 & F_{2p+1} \\
F_{2p-1} & 0 & F_{2p} & 0 & F_{2p} & 0 
\end{cases} : \quad k = 2p + 1.
$$

Hence, using (3), the number of differential operations of the $k^{th}$-order is determined as follows

$$(28) \quad f(k) = 2 \cdot F_{k+3}. $$

For other dimensions $n = 4, 5, 7, 8, 9, 10$ the roots of suitable characteristic polynomials are not related to Fibonacci numbers.

Finally, let us outline that for all dimensions $n = 3, 4, 5, 6, 7, 8, 9, 10$ the values of the function $f(k)$, for initial values of the argument $k$, are given in [4] as sequences $A020701 \ (n = 3), A090989 \ (n = 4), A090990 \ (n = 5), A090991 \ (n = 6), A090992 \ (n = 7), A090993 \ (n = 8), A090994 \ (n = 9), A090995 \ (n = 10)$ respectively.
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