Many-body localization in the Bose-Hubbard model: Evidence for mobility edge

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Motivated by recent experiments on interacting bosons in a quasi-one-dimensional optical lattice [M. Rispoli et al., Nature (London) \textbf{573}, 385 (2019)], we analyze theoretically properties of the system in the crossover between delocalized and localized regimes. Comparison of time dynamics for uniform and density-wave-like initial states enables a demonstration of the existence of the mobility edge. To this end we define a different observable, the mean speed of transport at long times. It gives us an efficient estimate of the critical disorder for the crossover. We also show that the mean velocity growth of occupation fluctuations close to the edges of the system carries similar information. Using the quantum quench procedure, we show that it is possible to probe the mobility edge for different energies.

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I. INTRODUCTION

Many-body localization (MBL), despite numerous efforts over the last 15 years (for some reviews see [1–4]), is still a phenomenon not fully understood. Recently, even its very existence in the thermodynamic limit was questioned [5], which created a vivid debate [6–9]. Simulations of large systems dynamics are also not fully conclusive [10–12]. It seems that, as suggested by [8], present-day computer resources prevent us from drawing a definite conclusion on this point. The issue may be addressed in experiments via a quantum simulator approach [13], although the required precision may also be prohibitive.

Experimental studies of MBL are much less numerous. Early work [14] provided indications of MBL in a large fermionic system in an optical lattice. Subsequent studies showing the lack of thermalization and memory of the initial configuration in time evolution were reported for interacting fermions in quasiperiodic disorder in an optical quasi-one-dimensional (quasi-1D) lattice [15]. Experiments also considered rather large systems with either fermions [16] or bosons [17] (the latter in two dimensions). Only recently have quite small systems attracted experimental attention for interacting photons [18], for which even an attempt at level spacing statistics measurement was made, as well as for bosonic atoms, for which logarithmic spreading of entanglement was observed [19] and long-range correlations were analyzed [20].

On the theory side bosonic systems were not the prime choice for the analysis for a simple reason. While for spin 1/2 (or a spinless fermion) the local Hilbert space dimension is 2, twice that for spinful fermions, for bosons the strict limit is set by the total number of particles in the system. This severely limits the number of sites that can be included in any simulation performed within exact-diagonalization-type studies. In effect only a few papers addressed MBL with bosons. By far the most notable is a courageous attempt to treat bosonic systems in two dimensions using the approximate tensor network approach [21] (still restricting the model to maximally double occupations of sites). Earlier treatment [22] considered both small and large systems in one dimension, revealing the existence of the so-called reverse mobility edge in the energy spectra. With assumed 3/2 filling of the system it was shown that higher-lying energy states are localized for lower amplitude of the disorder for sufficiently strong interactions. Many-body localization with superconducting circuits were discussed in [23], while the role of doublons for thermalization was discussed in [24]. Very recently, the same system was discussed also at half-filling [25], confirming the existence of the mobility edge. Let us mention for completeness also works on MBL with randomly interacting bosons [26,27], in which case no on-site disorder is necessary to observe localization.

In the present work, inspired by recent experiments [19,20], we present numerical evidence of the existence of the mobility edge in the system. To this end, rather than looking at standard spectral properties such as the gap ratio statistics [28,29], which is hardly accessible for convincing experimental analysis, we consider the time dynamics of correlations in the system, addressing its transport properties for different initial states. The analysis is based on the comparison of the system response for different initial states prepared as Fock separable states (as typically done in MBL experiments [15,19,20], as well as in earlier treatments [22]), and also we apply the idea of quantum quench spectroscopy [30], devised originally for spin systems.

This paper is organized as follows. After a brief treatment of spectral properties in Sec. II we recall the definition and present a numerical study of the transport distance time dynamics, showing its characteristic time behavior for the small systems studied. This allows us to analyze the mean long-time transport distance, which, we argue, shows evidence of
the mobility edge (Sec. III). In Sec. IV we introduce some observables, the mean transport velocity $V_t$ and the mean fluctuation velocity $V_F$, and we provide compelling evidence that their behavior as a function of disorder amplitude shows significant differences for different initial states (different energies), which provides convincing evidence that the onset of MBL is energy (state) dependent: the system possesses a mobility edge. Section V is devoted to the quantum quench spectroscopy, which allows us to analyze the onset of MBL smoothly changing the energy probed. Finally, the conclusions are followed by the Appendixes, which contain additional supporting data.

II. THE BOSE-HUBBARD MODEL: SPECTRAL PROPERTIES

The tight-binding model describing the bosons in the optical lattice potential is given by the 1D Bose-Hubbard Hamiltonian:

$$H = -J \sum_{j} \hat{b}_j^\dagger \hat{b}_{j+1} + \text{H.c.} + \frac{U}{2} \sum_{j} \hat{n}_j (\hat{n}_j - 1) + \sum_{j} \mu_j \hat{n}_j,$$

where $\hat{b}_j$ and $\hat{b}_j^\dagger$ are bosonic annihilation (creation) operators on site $i$ with $[\hat{b}_i, \hat{b}_j^\dagger] = \delta_{ij}$ and $\hat{n}_i = \hat{b}_i^\dagger \hat{b}_i$. We assume uniform interactions $U$ across the lattice, while the on-site energy (chemical potential) $\mu_i$ is site dependent. While [22,25] considered a random uniform on-site disorder, following the experiments [15,19,20], from now on we assume the quasiperiodic disorder $\mu_i = W \cos(2\pi \beta i + \phi)$, where, for a given realization, $\phi$ is fixed and disorder averaging corresponds to an average over uniformly distributed $\phi \in [0, 2\pi)$. It is known that the localization properties of the system strongly depend on the value of $\beta$ [31,32]; from now on we take $\beta = 1/1.618$ and unit filling following [19,20] and work with open boundary conditions. We consider sufficiently deep optical lattices, so (1) may be used for the shallow, quasiperiodic potential case (see [33]). From now on we assume $J = 1$, taking it as an energy unit.

With the model defined, we study first its spectral properties. The localization may be probed by looking at the mean gap ratio $\bar{r}$ [28], defined as an average of $r_n$, the minimum of the ratio of consecutive level spacings $s_n = E_{n+1} - E_n$:

$$r_n = \min \left\{ \frac{s_{n+1}}{s_n}, \frac{s_n}{s_{n+1}} \right\}. \quad (2)$$

$\bar{r}$ is a simple, dimensionless probe of level statistics [28], with $\bar{r} \approx 0.38$ for Poisson statistics (corresponding to the localized, integrable case) and $\bar{r} \approx 0.53$ for the Gaussian orthogonal ensemble of random matrices describing statistically well the ergodic case [34,35]. Diagonalizing the model for eight bosons on eight sites and calculating $\bar{r}$ as a function of the disorder amplitude $W$ and the relative energy $\epsilon_n = \frac{E_n - E_{\min}}{E_{\max} - E_{\min}}$, we obtain the plot represented in Fig. 1. To obtain good statistics we use 200 disorder realizations for each $W$. Observe that while for $U = 1$ the energy dependence of $\bar{r}$ seems only weakly dependent on energy, for $U = 2.87$ a clear inverted mobility edge emerges: at the same disorder values, states of higher energies seem localized, while the low-lying states have $\bar{r}$ corresponding to a characteristic of the extended regime. This is in agreement with earlier studies [22,25] at different densities. The particular value of $U = 2.87$ taken follows the experimental realization [20]; at this value the ground state of the system is superfluid.

With colored lines in Fig. 1 we denote the mean scaled energies corresponding to two states, the dynamics of which we shall study in detail. The red line corresponds to the uniform (UN) density state, which is a product of unit occupations at sites $|\text{UN}\rangle = \prod_i |\hat{b}_i^\dagger \hat{b}_i \rangle_{\text{vac}}$, where $|\text{vac}\rangle$ denotes the bosonic vacuum. The green line represents the scaled energy of the density wave (DW) state with double occupation of odd sites $|\text{DW}\rangle = \prod_i |\hat{b}^\dagger_{i,\text{odd}} |^2 / \sqrt{2} |\text{vac}\rangle$. For $U = 2.87$ the crossover to the localized regime occurs for the DW state at lower $W$ than for the UN state, lying lower in energy.

A more qualitative analysis might be carried out with finite-size scaling (FSS) [29]. Recent works have indicated [36–38], however, that MBL transition may be of Kosterlitz-Thouless type, rendering symmetric standard FSS questionable. In view of that we show $\bar{r}$ as a function of disorder for different system sizes, with crossing of the curves being an indicator of the crossover disorder value $W_c$ (compare the left panels of Fig. 2). Here for each of 200 disorder realizations we take for the analysis 200 eigenvalues around a chosen relative energy $\epsilon = 0.15$ ($\epsilon = 0.30$) for the top (bottom) panel. We may observe that the transition for $U = 2.87$ clearly depends on the initial-state energy, suggesting the existence of the mobility edge. The crossing between $L = 8$ and $L = 9$ data occurs for $W_c = 6.5$ ($W_c = 4.0$) for $\epsilon$ corresponding to the UN (DW) state. On the other hand, for $U = 1$, $W_c \approx 4.6$ for both UN and DW energies (see Appendix A for the corresponding plot). The estimates for the critical disorder strength are slightly different when a standard FSS is performed (see the Appendix A). The conclusion that critical disorders are close
The curves correspond to straight-line behavior in log-log plots with the slope well below 0.5. Two states, which reveals subdiffusive growth on the localized side (for other studies of the related quantity for spin systems see [39,40]), defined as [20]

\[
\text{FIG. 2. Uniform (top) versus density wave (bottom) situation.}
\]

Left: Mean gap ratio \( \gamma \) reveals that the transition to MBL, as indicated by crossing of the curves for different system sizes (indicated in the figure), occurs for energies corresponding to a uniform state [shown in (a), \( \epsilon \approx 0.15 \)] for much stronger disorder than for \( \epsilon = 0.3 \), corresponding to the density wave initial state, both for \( U = 2.87 \).

Right: This is manifested by the transport distance \( \Delta x \) (see text for the definition) as obtained from time dynamics starting from those two states, which reveals subdiffusive growth on the localized side (straight-line behavior in log-log plots with the slope well below 0.5). The curves correspond to \( W = 1, 3, 4, 5, 6, 7, 8 \) from black (top) to cyan (bottom). The rapid ballisticlike growth at \( W = 1 \) in both cases saturates at intermediate times due to a finite system size; the time dependence is shown for \( L = 8 \).

For different energies for \( U = 1 \) but significantly differ for \( U = 2.87 \) also holds for FSS results.

III. TRANSPORT DISTANCE ANALYSIS

We now pose a question: Can the mobility edge manifest itself in experiments? Since individual levels are hardly accessible, we shall seek manifestations of the mobility edge in observables accessible to measurements in time dynamics. Instead of the imbalance, which can be used for different DW-type states [22], we shall consider the transport distance (for other studies of the related quantity for spin systems see [39,40]), defined as [20]

\[
\Delta x = 2 \sum_d d \langle G_z^{(2)}(i, i + d) \rangle_i, \tag{3}
\]

i.e., the modulus of the disorder-averaged (as denoted by the overbar) and site-averaged (as denoted by a subscript \( i \)) second-order correlation function of the density

\[
G_z^{(2)}(i, i + d) = \langle \hat{h}_i \hat{h}_{i+d} \rangle - \langle \hat{h}_i \rangle \langle \hat{h}_{i+d} \rangle. \tag{4}
\]

We concentrate mostly on the \( U = 2.87 \) case, which reveals the mobility edge in the above spectral analysis, and compare the results with data obtained for \( U = 1 \) when no mobility edge evidence is expected for UN or DW states.

The time dependence of the transport distance obtained for the \( L = 8 \) system is shown in Fig. 2 (right panels) for different values of the disorder. Data are obtained using the exact diagonalization approach in a full appropriate Hilbert space. On the delocalized side one observes a fast, almost ballistic growth (for \( W = 1 \)), followed by a saturation when the finite size of the system dominates the dynamics. Upon approaching the transition the power law growth becomes apparent for intermediate times, with the corresponding power \( \gamma = d \ln \Delta x / d \ln t \) decreasing smoothly within the interval of \([0,0.35]\), indicating a subdiffusive dynamics for both initial states considered. For sufficiently large \( W \) the motion freezes with very small \( \gamma \). The data are averaged over 200 disorder realizations; the residual fluctuations apparent in the figures form an estimate of the statistical error. Observe that bigger fluctuations seem to be present for larger \( W \). This is due to the fact that the initial state may be expanded on fewer eigenstates in the localized regime; individual disorder realizations lead to pronounced oscillations which are averaged out partially by the finite number of disorder realizations taken for averaging.

A careful visual inspection of individual \( \Delta x(t) \) curves in the right panels of Fig. 2 reveals that the transition to localization, as monitored by \( \Delta x \) growth, appears for lower \( W \) values for the DW initial state (bottom panel) than for the uniform initial state. That correlates well with the mean gap ratio dependence on \( W \) shown in the left panels of Fig. 2. This qualitative observation may be put on a more qualitative basis by looking at the power growth coefficient \( \gamma(W) \). The
drawback of such an approach is that the fitted values depend on the time interval in which fitting is performed. This interval cannot be made too large because of the finite-size effect: saturation of $\Delta x$ for lower values of $W$. On the other hand, for larger disorder, the above-mentioned larger fluctuations affect the fitting. Fortunately, as we shall describe below, there are other observables that more clearly allow us to pin down the transition to localization.

First, let us, however, take a look at the dynamics at $U = 1$, where no mobility edge effects are expected. Indeed, as shown in Figs. 3(a) and 3(c), the transport distance behaves almost identically for the same $W$ values, showing a similar effect of disorder. On the localized side, both the growth and the values reached are quite small, allowing for identification of the localized regime. It might therefore be useful to consider mean transport distance $\Delta x_L$ at some experimentally reachable time of a few hundred tunneling times. We find the mean value of the transport distance in the interval $t \in [220, 250]$. As shown in Fig. 3(c), for such long times $\Delta x$ changes relatively slowly; the average over time will wash out the remaining fluctuations visible in Fig. 3(c) resulting from the finite number of disorder realizations. Importantly, for later considerations, for long times, as shown in Fig. 3(d), the remaining growth of $\Delta x$ may be, to good precision, approximated by the linear growth.

Let us first consider the mean transport distance defined above $\Delta x_L$ (again, we take $t \in [220, 250]$) and show its dependence on $W$ as well as on the system size. For the latter, we consider not only $L = 8$ but also larger systems with $L = 10, 12$. For such systems the propagation based on exact diagonalization of the Hamiltonian is not practical due to the memory requirements; instead, we use the Chebyshev propagation scheme [41,42], which allows us to propagate "exactly" without introducing any cutoff of the Hilbert space. The obtained data are presented in Fig. 4. The numerical values depend, of course, on the chosen initial and final times (here 220 and 250) since apart from strongly localized regime the mean distance still grows [compare again Fig. 3(c)]. Still, the shape of the disorder dependence allows us to estimate the critical disorder value corresponding to the transition as the place where $\Delta x_L$ practically vanishes (reaches, say, 1/1000 of the maximal value); then the error of $\Delta x_L$ is very small. For small disorder values, e.g., $W = 3$, the error comes not from the disorder-induced fluctuations (which are the main reason for averaging over a finite-time interval) but from the remaining growth. Still, this error for $W = 3$, which may be estimated from Fig. 3(b) to be about 0.05, is less than 1% of the mean value in this interval $\Delta x_L \approx 6.1$ [compare Fig. 4(a)].

With all that in mind we observe in Fig. 4 that for $U = 1$ (left column) the uniform [Fig. 4(a)] and DW [Fig. 4(c)] initial states lead to similar $\Delta x_L$ dependence as a function of $W$ with a rapid drop around $W = 3$. When MBL sets in, the transport distance reached eventually vanishes. The situation is markedly different for the $U = 2.87$ case; the transport distance of the uniform initial state starts a significant drop for much larger values of $W$ than those corresponding to the DW initial state. That correlates well with gap ratio statistics, confirming that the transition to MBL depends on the energy of the system.

### IV. THE MOBILITY EDGE

While the results shown in Fig. 4 already indicate the existence of the mobility edge, a more spectacular demonstration of its existence is obtained taking the mean time derivative of $\Delta x(t)$ over some large interval. Such a velocity of transport $V_x$ can be readily obtained as $V_x = (\Delta x(t_2) - \Delta x(t_1))/(t_2 - t_1)$. It is shown for $t_1 = 100$ and $t_2 = 250$ in Fig. 5 as a function of the disorder amplitude. Observe the spectacular maxima of the derivative that are more pronounced the larger the system size is. On the delocalized side $V_x$ is small as $\Delta x$...
saturates due to the finite sizes considered. On the localized side, \( V_x \) practically vanishes as transport for long times is prohibited. Thus, the pronounced maximum of the derivative in the critical transition regime is a robust and \textit{a posteriori} expected phenomenon.

Unfortunately, the disorder at which the maximum occurs cannot be taken as an estimate of the critical disorder. The maximal \( V_x \) occurs in the critical regime where the transport distance considerably grows. The critical disorder may be read out as the point where \( V_x \) practically vanishes (being, say, 1% of the maximal value). Of course, there is a certain ambiguity in such a definition. Still, we consider a small system in which there is no hope to observe a genuine phase transition; we observe only a signature of the crossover. Thus, we can identify only the border of this crossover when \( V_x \) vanishes on the large disorder side.

For \( U = 1 \) (Fig. 5, left column) the uniform \([\text{Fig. 5(a)}]\) and DW \([\text{Fig. 5(c)}]\) initial states lead to similar \( V_x \) dependence as a function of \( W \). \( V_x \) approaches zero and becomes almost independent of the system size for \( W \approx 5 \pm 0.5 \), which nicely matches the critical disorder values obtained from the gap ratio analysis. For the \( U = 2.87 \) case, the positions of the peaks of \( V_x \) for UN \([\text{Fig. 5(b)}]\) and DW \([\text{Fig. 5(d)}]\) states strongly differ. Using the same criterion of vanishing \( V_x \) as in the localized regime, we get the critical disorder \( W_c \approx 7 \) for the UN state and \( W_d \approx 6 \) for the DW state (the latter value is a bit too large with respect to gap ratio data).

While, as we admit above, the determination of the critical disorder value for a given initial state has a certain degree of ambiguity, what really counts as evidence of the mobility edge’s existence is really the difference between the curves for \( V_x(W) \) obtained for the UN and DW states. Here the difference in the positions of maxima for systems of the same size could equally well be considered evidence.

The choice of the interval in which \( V_x \) is evaluated, \([t_1, t_2] \), is not important, as long as we consider times sufficiently large so that the ballistic transport results in reaching the system boundaries. Thus, \( V_x \) on the delocalized side of the crossover are entirely dominated by the finite-size effect. This usual drawback appears as a necessary important ingredient in our method. For example, the mean velocities obtained for the \([120,200]\) time interval are shown in Fig. 6. Similarly, instead of a linear fit of \( \Delta x(t) \) with equal weights, one could use the weighted linear fit in which squared deviations are weighted by a Gaussian centered at the center of the interval considered with a standard deviation \( \sigma = 25 \). Transport velocities obtained with such a procedure for the center at \( t = 175 \) in the same interval yield essentially the same data (within the size of the symbols) as in Fig. 6. This indicates that the long-time mean transport velocity is a robust measure of system properties.

While the evidence for the mobility edge’s existence apparent in Fig. 5 is quite strong, determination of \( \Delta x \) requires measurements of all second-order correlations. As it turns out, this may not be necessary. Consider on-site fluctuations \( F_i = \langle n_i^2 \rangle - \langle n_i \rangle^2 \). While [30] considered fluctuations in the middle of the chain, it is advantageous to concentrate on \( F = (F_1 + F_d)/2 \), i.e., averaged fluctuation on the edges. This follows the argument that edges are the least sensitive to the system size [43]. While already the fluctuation \( F \) at long times shows indications of the crossover to the localized phase we present in Fig. 7 its derivative \( V_F = dF/dt \) averaged over “long times” in a manner completely analogous to \( V_x \). Remarkably, it also reveals the mobility edge when varying \( W \) in a manner similar to \( V_x \) plotted just above.

**V. QUANTUM QUENCH SPECTROSCOPY**

Our analysis up to now has been restricted to initial Fock-like states and the corresponding energies. It was suggested, however, that different energy regimes may be addressed by “quantum quench spectroscopy” [30] to reveal the mobility edge energy dependence. The proposed scheme assumes
VF
Vs
statistics for corresponding
crossover to MBL as a function of
Vs
times
states of different energies
final values
disorder amplitudes
entropy as well as on-site density fluctuations.
value and observing the time dependence of the entanglement
in [30] on the spinless fermion case by quenching the disorder
this parameter. By changing the initial parameter value one
wave packet with excess energy dependent on the change in
final investigated value transfers the system into an excited
of a parameter; then the fast quench of that parameter to the
preparation of the ground state of the system at some value
order amplitude to a desired final value
quenching.
effect and limit the accuracy of the final energy obtained by
noted in [30], the resulting error diminishes with increasing
φ
results obtained. The error bars shown in Fig.8 are due to this
problem (appdix A).

Let us apply the same method to the current problem.
We assume that the ground state is prepared at different
disorder amplitudes \( W_i \) and the rapid quench brings the dis-
order amplitude to a desired final value \( W_f \). Changing \( W_r \),
we may change the energy \( \epsilon \) of the prepared wave packet,
as shown in Fig. 8(a). Note that to reach significant excitation
Novt we start with “negative” amplitude \( W_i \). In this way we
can reach different final \( W \) values. The obtained energies are
characterized by, unfortunately, quite significant error. For a
given prepared initial \( W_i \) different realizations of the disorder
(different phase \( \phi \)) lead to different excitation energies. As
noted in [30], the resulting error diminishes with increasing
system size; here we consider a small system \( L = 8 \) mostly
because then the gap ratio statistics allows us to verify the
results obtained. The error bars shown in Fig. 8 are due to this
effect and limit the accuracy of the final energy obtained by
quenching.

The initial state is then evolved in time, determining \( \Delta x(t) \)
and its mean derivative \( \dot{\epsilon}_f = d\Delta x/dt \) exactly as before. At
different final disorder values a transition from delocalized to
localized regimes is observed when “smoothly” changing \( \epsilon \)
[compare Fig. 8(b)]. Again, the tail of \( \dot{\epsilon}_f \), when its value gets
close to zero, is an indicator of the MBL regime. For compar-
ison, Fig. 8(c) shows the corresponding gap ratio statistics at
these disorder values. The agreement is spectacular, showing
that the quantum quench spectroscopy combined with the
transport distance measurements allows us to continuously
monitor the mobility edge in our system. The drawback of
the method is the inherent limitation of the energy \( \epsilon \) resolution
discussed above.

To complete the picture, Fig. 8(d) presents the edge-
fluctuation derivative \( V_F = dF/dx \) averaged over long times.
As for the UN and DW states in Fig. 7, it also reveals the
mobility edge when varying final disorder strength \( W \).
This is quite promising for possible applications of quench
spectroscopy to systems of larger size for which measure-
ments leading to transport distance (and \( V_x \)) may become
costly while measurements of edge fluctuations require site
resolution at only the edges.

VI. CONCLUSIONS

To conclude, by considering transport properties in the
transition between extended and localized states in the Bose-
Hubbard Hamiltonian describing bosons in an optical lattice
with diagonal quasiperiodic disorder we have shown that
observables directly accessible to the experiment [20] re-
veal the existence of the mobility edge in the system. The
mobility edge has been convincingly shown to exist for the
finite-size Heisenberg chain [29] by studying different spectra
measures (most notably, the gap ratio statistics) as well as
using quantum quench spectroscopy and time dependence
of the entanglement and the number of fluctuations [30]. It
has also been postulated to exist for a Bose-Hubbard system
with a larger density by considering different density wave
states [22]. Here we showed that within the realm of state-
of-the-art, current experiment [20] the mobility edge may
be verified experimentally via readily accessible observables.
We considered different initial states, either prepared as Fock
states with different density patterns or obtained via quantum
quench of the disorder amplitude. The latter approach allowed
us (modulo inherent uncertainties) to follow the mobility edge
in energy. Both the global trends and values of the critical
disorder obtained by us quantitatively agree across different
observables taken for analysis and agree with the spectral gap
ratio analysis.

Note added. Recently, we learned that energy-resolved
MBL was also considered in a spin quantum simulator [44].

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APPENDIX A: GAP RATIO ANALYSIS AND FINITE
SIZE SCALING

We did not show the results of finite-size scaling analysis
in the main text, although it has been quite customary to do
FIG. 9. $U = 1$ (left) and $U = 2.87$ (right) in the scaled energy regions corresponding to two interesting states. The top row corresponds to (a) $\epsilon = 0.30$ and (b) $0.15$ energy regions appropriate for the uniform initial state for $L = 8$ (compare to Fig. 1 in the main text). The bottom row shows scaling for (c) $\epsilon = 0.40$ and (d) $0.25$, appropriate for density wave initial state (for $L = 8$). The obtained values of the critical disorder amplitude and critical exponent are indicated. The inset shows unscaled data for comparison. Data are obtained for 2000 disorder realizations.

so when analyzing spectral properties in the context of MBL, while several authors used a simple, single-parameter scaling of the form $W \rightarrow (W - W_c)^{1/\nu}$ to extract the critical disorder value $W_c$, this procedure has recently been critiqued. Already, [29] noticed that the obtained values of $W_c$ and $\nu$ dangerously depend on system sizes taken for the analysis. Moreover, the exponent $\nu$ appears to be close to unity, which violates the so-called Harris bound (for details see [29] and references therein). Having at our disposal systems with $L = 7, 8, 9$, we may perform such an FSS at energies corresponding to the uniform (UN) and density wave (DW) states considered; the results are shown in Fig. 9. The difference in $W_c$ for $U = 2.87$ for $\epsilon$ values corresponding to UN and DW states is significant, giving an argument for the existence of the mobility edge. On the other hand, variations of $\nu$ are large (20%), while there is no reason to believe that the character of the transition changes. Moreover, $\nu$ values, as for spin systems, violate the Harris bound.

Recent works on spin systems tend to describe the MBL transition as being of Kosterlitz-Thouless (KT) type [37,38] and perform FSS with the appropriate KT correlation function. We refrain from doing so because for bosons much smaller system sizes are available and such a procedure would also be doubtful. For that reason we use a simple crossing of $\bar{\rho}$ curves for different system sizes as a reasonable estimate of the characteristic disorder when MBL sets in. One should also keep in mind that for such small sizes (as also used in the experiment [20]) the crossover region must take a finite range of disorder amplitudes $W$, so determining a precise number for $W_c$ in the thermodynamic limit is not needed really.

APPENDIX B: QUANTUM QUENCH—ADDITIONAL RESULTS

We provide here an additional discussion of the quantum quench scenario.

In the quantum quench scenario [30] the system is prepared in the ground state at some disorder amplitude $W_i$ which is, at $t = 0$, switched abruptly to the desired $W$ value. Thus, at $t = 0$ the system is prepared in some wave packet of energy $\epsilon$ (using scaled units as defined in the main text),
with the value of \( \epsilon \) being dependent on the initial \( W_i \) and the particular disorder realization. Scanning \( W_i \) allows us to scan \( \epsilon \) as shown in Fig. 8(a) for \( L = 8 \) and in this way realize a “vertical cut” of Fig. 1 at some \( W \) value. By time evolving this state we may measure, like for initial Fock-like states, the transport distance \( \Delta x(t) \) as well as fluctuations on edges \( F_i(t) \). Those are presented in Fig. 10 for two exemplary disorder values (and several initial \( W_i \)). Since the initial state is not strictly separable, \( \Delta x(0) \) does not strictly vanish; however, it is very small as amplitudes \( W_i \) needed to excite the system must be quite large (in absolute values). As seen in the left panels of Fig. 10, the evolution resembles that for Fock states with the initial rapid growth on the scale of a few tunneling times and the power law subdiffusive growth for larger times. To probe a broad range of excitation energies it turns out that \( W_i \) should have a sign opposite to \( W \) assumed for time evolution; the smaller \( W - W_i \) is, the smaller \( \epsilon \) is. Looking back at Fig. 1, localized states correspond to large \( \epsilon \) and thus large \( |W_i| \), and for those cases the growth of \( \Delta x(t) \) practically stops. The significant difference between the \( W = 1 \) and \( W = \beta \) curves is another indicator of the mobility edge. Note that edge fluctuations (right panels) behave in quite a similar manner; thus, site fluctuations bring similar information to the transport distance (but do not require two-point correlations).

This is further visualized in Fig. 11, where the mean site fluctuation \( F_i \) (averaged over \( t \in [220, 250] \) in a manner similar to that for \( \Delta x(t) \)) is plotted for different disorder values \( W \). Note that already, this observable confirms the sensitivity of the crossover to the excitation energy at different disorder values. Of course, similar to the edge-fidelity derivative, evaluation of the fidelity does not require a two-point correlation function and involves only local measurement, so it may be the method of choice for larger systems. Importantly, the fluctuations in the center of the chain are much less sensitive and do not reveal any transition in a consistent way.

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