Accelerated and Inexact Soft-Impute for Large-Scale Matrix and Tensor Completion

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Abstract

Matrix and tensor completion aim to recover a low-rank matrix/tensor from limited observations and have been commonly used in applications such as recommender systems and multi-relational data mining. A state-of-the-art matrix completion algorithm is Soft-Impute, which exploits the special “sparse plus low-rank” structure of the matrix to allow efficient SVD in each iteration. Though Soft-Impute is a proximal algorithm, it is generally believed that acceleration destroys the special structure and is thus not useful. In this paper, we show that Soft-Impute can indeed be accelerated without comprising this structure. To further reduce the iteration time complexity, we propose an approximate singular value thresholding scheme based on the power method. Theoretical analysis shows that the proposed algorithm still enjoys the fast $O(1/T^3)$ convergence rate of accelerated proximal algorithms. We further extend the proposed algorithm to tensor completion with the scaled latent nuclear norm regularizer. We show that a similar “sparse plus low-rank” structure also exists, leading to low iteration complexity and fast $O(1/T^2)$ convergence rate. Extensive experiments demonstrate that the proposed algorithm is much faster than Soft-Impute and other state-of-the-art matrix and tensor completion algorithms.

1 Introduction

Matrices are common place in data mining applications. For example, in recommender systems, the ratings data can be represented as a sparsely observed user-item matrix [Adomavicius and Tuzhilin, 2005] [Koren, 2008]. In social networks, user interactions can be modeled by an adjacency matrix [Kim and Leskovec, 2011] [Chiang et al., 2014]. Matrices also appear in applications such as image processing [Liu et al., 2013] [Haefele et al., 2014] [Yao and Kwok, 2015b] and question answering [Zhao et al., 2015].

Due to limited feedback from users, these matrices are usually not fully observed. For example, users may only give opinions on very few items in a recommender system. As the rows/columns are usually related to each other, the low-rank matrix assumption is particularly useful to capture such relatedness, and low-rank matrix completion has become a powerful tool to predict missing values in these matrices. Sound recovery guarantees [Candes and Recht, 2009] and good empirical performance [Koren, 2008] have been obtained.

However, directly minimizing the matrix norm is NP-hard [Recht et al., 2010]. To alleviate this problem, the nuclear norm (which is the sum of singular values) is often used instead. It is known that the nuclear norm is the tightest convex lower bound of the rank [Recht et al., 2010]. Specifically, consider an $m \times n$ matrix $O$ (without loss of generality, we assume that $m \geq n$), with positions of the observed entries indicated by $\Omega \in \{0, 1\}^{m \times n}$, where $\Omega_{ij} = 1$ if $O_{ij}$ is observed, and 0 otherwise. The matrix completion problem can be formulated as

$$\min_X \frac{1}{2} \|P_\Omega (X - O)\|_F^2 + \lambda \|X\|_*,$$  (1)

where $[P_\Omega(A)]_{ij} = A_{ij}$ if $\Omega_{ij} = 1$, and 0 otherwise; and $\| \cdot \|_*$ is the nuclear norm. Though the nuclear norm is only a surrogate of the matrix rank, there are theoretical guarantees that the underlying matrix can be exactly recovered [Candes and Recht, 2009].

Computationally, though the nuclear norm is nonsmooth, problem (1) can be solved by various optimization tools. An early attempt is based on reformulating (1) as a semidefinite program (SDP) [Candes and Recht, 2009]. However, SDP solvers have large time and space complexities, and are only suitable for small data sets. For large-scale matrix completion, singular value thresholding (SVT) algorithm [Cai et al., 2010] pioneered the use of first-order methods. However, a singular value decomposition (SVD) is required in each SVT iteration. This takes $O(nnz^2)$ time and can be computationally expensive. In [Toh and Yun, 2010], this is reduced to a partial SVD by computing only the leading singular values/vectors using PROPACK (a variant of the Lanczos algorithm) [Larsen, 1998]. Another major breakthrough is made by the Soft-Impute algorithm [Mazumder et al., 2010], which utilizes a special “sparse plus low-rank” structure associated with the SVT to efficiently compute the SVD. Empirically, this allows Soft-Impute to perform matrix completion on the entire Netflix data set. The SVT algorithm
can also be viewed as a proximal algorithm \cite{Tibshirani, 2010}. Hence, it converges with a \(O(1/T)\) rate, where \(T\) is the number of iterations \cite{Beck and Teboulle, 2009, Nesterov, 2013}. Later, this is further “accelerated”, and the convergence rate is improved to \(O(1/T^2)\) \cite{Ji and Ye, 2009, Toh and Yun, 2010}. However, Tibshirani \cite{Tibshirani, 2010} suggested that this is not useful, as the special “sparse plus low-rank” structure crucial to the efficiency of Soft-Impute no longer exist. In other words, the gain in convergence rate is more than compensated by the increase in iteration time complexity.

In this paper, we show that accelerating Soft-Impute is indeed possible while still preserving the “sparse plus low-rank” structure. To further reduce the iteration time complexity, instead of computing SVT exactly using PROPACK \cite{Toh and Yun, 2010, Mazumder et al., 2010}, we propose an approximate SVT scheme based on the power method \cite{Halko et al., 2011}. Though the SVT obtained in each iteration is only approximate, we show that convergence can still be as fast as performing exact SVT. Hence, the resultant algorithm has low iteration complexity and fast \(O(1/T^2)\) convergence rate. To further boost performance, we propose a post-processing procedure by extending \cite{Mazumder et al., 2010}, and can handle any smooth convex loss function.

Besides matrices, tensors have also been commonly used to describe the linear and multilinear relationships in the data \cite{Kolda and Bader, 2009, Liu et al., 2013, Kazarinov et al., 2011, Shin et al., 2017}. For example, in remote sensing applications, a hyperspectral image with multiple bands can be naturally represented as a 3-dimensional tensor. A multidimensional social network can also be modeled as a 3-dimensional tensor, where the third mode may represent different type of relations. Higher-dimensional tensors are also useful. For example, a multi-mode social network (such as the DBLP network) with heterogeneous actors (papers, authors, terms and venues) can be represented by a 4-order tensor, and a relation can connect these four kinds of entities \cite{Tang et al., 2008}. Analogous to matrix completion, tensor completion attempts to recover a low-rank tensor that best approximates a partially observed data tensor \cite{Tomio et al., 2010, Liu et al., 2013}. For example, in hyperspectral imaging, as some bands may be partially missing due to sensor problems, tensor completion can be used to inpaint the incomplete image \cite{Xu et al., 2013}.

Analogous to matrix completion, tensor completion can also be solved by convex optimization algorithms. However, multiple expensive SVDS on large dense matrices are required \cite{Liu et al., 2013, Tomio et al., 2010}. To alleviate this problem, we demonstrate that a similar “sparse plus low-rank” structure also exists when the scaled latent nuclear norm \cite{Tomio et al., 2010, Wimalawarne et al., 2014} is used as the regularizer. We extend the proposed matrix-based algorithm to this tensor scenario. The resulting algorithm has low iteration cost and fast \(O(1/T^2)\) convergence rate. Experiments on matrix/tensor completion problems with both synthetic and real-world data sets show that the proposed algorithm outperforms state-of-the-art algorithms.

Preliminary results of this paper have been reported in a shorter conference version \cite{Yao and Kwok, 2015a}. While only the square loss is used in \cite{Yao and Kwok, 2015a}, here we consider more general smooth convex loss functions. Moreover, we extend the proposed algorithm to tensor completion. Besides, post-processing is proposed to boost the recovery performance for matrix/tensor completion.

The rest of the paper is organized as follows. Section 2 provides a brief review on the related work. The proposed accelerated inexact Soft-Impute algorithm is described in Section 3 and its extension to tensor completion in Section 4. Experimental results are presented in Section 5 and the last section gives some concluding remarks. All the proofs are in the appendix.

**Notation**

In the sequel, the transpose of vector/matrix is denoted by the superscript ‘\(^\top\)’, and tensors are denoted by boldface Euler. For a vector \(x\), \(\|x\|_1 = \sum_i |x_i|\) is its \(\ell_1\)-norm, and \(\|x\| = \sqrt{\sum_i x_i^2}\) its \(\ell_2\)-norm. For a matrix \(X\), \(\sigma_1(X) \geq \sigma_2(X) \geq \ldots \sigma_m(X)\) are its singular values, \(\sigma(X) = \sum_i X_{ii}\) is its trace, \(\|X\|_1 = \sum_{i,j} |X_{ij}|\), \(\|X\|_\infty\) is its maximum singular value, and \(\|X\|_F = tr(X^\top X)\) the Frobenius norm, \(\|X\|_1 = \sum_i \sigma_i(X)\) the nuclear norm, and \(span(X)\) is the column span of \(X\). Moreover, \(I\) denotes the identity matrix.

For tensors, we follow the notations in \cite{Kolda and Bader, 2009}. For a \(D\)-order tensor \(X \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_D}\), its \((i_1,i_2,\ldots,i_D)\)th entry is \(x_{i_1i_2\cdots i_D}\). Let \(I_{D \setminus d} = \prod_{j=1,j \neq d}^D I_j\), the mode-\(d\) matricizations \(X(d)\) of \(X\) is a \(I_d \times \prod_{j=1,j \neq d}^D I_j\) matrix with \((X(d))_{i_dj} = x_{i_1i_2\cdots i_{d-1}i_d\cdots i_D}\), and \(j = 1 + \sum_{l=1,l \neq d}^D (i_l - 1) \prod_{m=1,m \neq d}^{l-1} I_m\). Given a matrix \(A\), its mode-\(d\) tensorization \(A(d)\) is a tensor \(X\) with elements \(x_{i_1i_2\cdots i_D} = a_{i_dj}\), and \(j\) is as defined above. The inner product of two tensors \(X\) and \(Y\) is \(\langle X, Y \rangle = \sum_{i_1=1}^{I_1} \cdots \sum_{i_D=1}^{I_D} x_{i_1i_2\cdots i_D} y_{i_1i_2\cdots i_D}\), and the Frobenius norm of \(X\) is \(\|X\|_F = \sqrt{\langle X, X \rangle}\).

For a convex but nonsmooth function \(f\), the subgradient is \(g \in \partial f(x)\) where \(\partial f(x) = \{u : f(y) \geq f(x) + u^\top(y-x), \forall y\}\) is its subdifferential. When \(f\) is differentiable, we use \(\nabla f\) for its gradient.

## 2 Related Work

### 2.1 Proximal Algorithms

Consider minimizing composite functions of the form:

\[
F(x) \equiv f(x) + g(x),
\]

where \(f, g\) are convex, and \(f\) is smooth but \(g\) is possibly nonsmooth. The proximal algorithm \cite{Parikh and Boyd, 2014} generates a sequence of estimates \(\{z_t\}\) as

\[
x_{t+1} = \text{prox}_{\mu g}(z_t) \equiv \arg \min_x \frac{1}{2} \|x - z_t\|^2_2 + \mu g(x),
\]

where

\[
z_t = x_t - \mu \nabla f(x_t),
\]

and \(\text{prox}_{\mu g}(\cdot)\) is the proximal operator. When \(f\) is \(\rho\)-Lipschitz smooth (i.e., \(\|\nabla f(x_1) - \nabla f(x_2)\| \leq \rho \|x_1 - x_2\|\)) and a fixed stepsize

\[
\mu \leq 1/\rho
\]
is used, the proximal algorithm converges to the optimal solution with a rate of $O(1/T)$, where $T$ is the number of iterations [Combettes and Wajs, 2005 Parikh and Boyd, 2014]. By replacing the update in (4) with

$$
y_t = (1 + \theta_t)x_t - \theta_tx_{t-1}, \quad (5)
$$

$$
z_t = y_t - \mu \nabla f(y_t), \quad (6)
$$

where $\theta_{t+1} = \frac{1}{t+1}$, it can be accelerated to a convergence rate of $O(1/T^2)$ [Beck and Teboulle, 2009 Nesterov, 2013]. This is also known to be the best possible rate for problems of the form in [Nesterov, 2013].

**Algorithm 1** Accelerated proximal gradient (APG) algorithm.

1: initialize $x_0 = x_1 = 0$ and stepsize $\mu \in (0, 1/\rho]$; 
2: for $t = 1, 2, \cdots$ do 
3: $\theta_t = \frac{1}{t+1}$;
4: $y_t = (1 + \theta_t)x_t - \theta_t x_{t-1}$;
5: $z_t = y_t - \mu \nabla f(y_t)$;
6: $x_{t+1} = \text{prox}_{\mu g}(z_t)$;
7: end for
8: return $x_{t+1}$.

Often, $g$ is “simple” in the sense that $\text{prox}_{\mu g}(\cdot)$ can be easily obtained. However, in more complicated problems such as overlapping group lasso [Jacob et al., 2009], $\text{prox}_{g}(\cdot)$ may be expensive to compute. To alleviate this problem, inexact proximal algorithm is proposed which allows two types of errors in standard/accelerated proximal algorithms [Schmidt et al., 2011]: (i) an error $e_t$ in computing $\nabla f(\cdot)$, and (ii) an error $\tilde{e}_t$ in the proximal step, i.e.,

$$
h_{\mu g}(x_{t+1}; z_t) \leq e_t + h_{\mu g}(\text{prox}_{\mu g}(z_t); z_t), \quad (7)
$$

where

$$
h_{\mu g}(x; z_t) \equiv \frac{1}{2} \|x - z_t\|^2 + \mu g(x) \quad (8)
$$

is the proximal step’s objective. Let the dual problem of $\min_x h_{\mu g}(x; z_t)$ be $\max_w D_{\mu g}(w)$. Note that $e_t$ is upper-bounded by the duality gap $\delta_t \equiv h_{\mu g}(x_{t+1}; z_t) - D(w_{t+1})$ where $w_{t+1}$ is the corresponding dual variable of $x_{t+1}$. Thus, (7) can be ensured by monitoring $\delta_t$. The following Proposition shows that by decreasing $e_t$ and $\tilde{e}_t$ sufficiently fast, the convergence rate remains at $O(1/T^2)$.

**Proposition 2.1** (Schmidt et al., 2011). If $\|e_t\|$ and $\sqrt{\delta_t}$ decrease as $O(1/t^{2+\delta})$ for some $\delta > 0$, the inexact accelerated proximal gradient algorithm converges with a rate of $O(1/T^2)$.

In the sequel, as our focus is on matrix completion, the variable $x$ in (1) will be a matrix $X$.

### 2.2 Soft-Impute

Soft-Impute [Mazumder et al., 2010] is a state-of-the-art algorithm for matrix completion (Algorithm 2). At iteration $t$, let the current iterate be $X_t$. The missing values in $O$ are filled in as

$$
Z_t = P_{\Omega}(O) + P_{\Omega^c}(X_t) = P_{\Omega}(O - X_t) + X_t, \quad (9)
$$

where $\Omega^c$ is the complement of $\Omega$ (i.e., $\Omega^c = 1 - \Omega_{ij}$). The next estimate $X_{t+1}$ is then generated by the singular value thresholding (SVT) operator Cai et al., 2010

$$
X_{t+1} = \text{SVT}_\lambda(Z_t) \equiv \text{arg min}_{\hat{X}} \frac{1}{2} \|X - Z_t\|_F^2 + \lambda \|X\|_*, \quad (10)
$$

which can be computed as follows.

**Lemma 2.2** ([Cai et al., 2010]). Let the SVD of a matrix $Z_t$ be $U \Sigma V^\top$. Then, $\text{SVT}_\lambda(Z_t) \equiv U(\Sigma - \lambda I)_+ V^\top$ where $[(A)_+]_{ij} = \max(A_{ij}, 0)$.

**Algorithm 2** Soft-Impute.

**Require**: Partially observed matrix $O$, parameter $\lambda$;

1: initialize $X_1 = 0$;
2: for $t = 1, 2, \cdots$ do
3: $Z_t = P_{\Omega}(O) + P_{\Omega^c}(X_t)$;
4: $X_{t+1} = \text{SVT}_\lambda(Z_t)$;
5: end for
6: return $X_{t+1}$.

Let $k_t$ be the number of singular values in $Z_t$ that are larger than $\lambda$. From Lemma 2.2, a rank-$k_t$ SVD, where $k_t \geq k_1$, is sufficient for computing $X_{t+1}$ in (10). In [Mazumder et al., 2010], this rank-$k_t$ SVD is obtained by the PROPACK algorithm [Larsen, 1998].

To make Soft-Impute efficient, an important observation in [Mazumder et al., 2010] is that $Z_t$ in (9) has a special “sparse plus low-rank” structure, namely that $P_{\Omega}(O - X_t)$ is sparse and $X_t$ is low-rank. The most expensive steps in computing the SVD are matrix-vector multiplications of the form $Z u^\top$ and $u^\top Z$, where $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^m$. Let the rank of $X_t$ be $r_t$, and its SVD be $U_t \Sigma_t V_t^\top$. $Z_t v$ can be computed as

$$
Z_t v = P_{\Omega}(O - X_t) v + U_t \Sigma_t (V_t v^\top) v. \quad (11)
$$

Constructing $P_{\Omega}(O - X_t)$ takes $O(r_t ||\Omega||_1)$ time, while computing the products $P_{\Omega}(O - X_t) u$ and $U_t \Sigma_t (V_t v^\top) u$ take $O(||\|\Omega||_1||_1)$ and $O(m r_t)$ time, respectively. Similarly, $u^\top Z_t$ can be computed as $u^\top P_{\Omega}(O - X_t) + (u^\top U_t) \Sigma_t V_t^\top$. Thus, to obtain the rank-$k$ SVD of $Z_t$, Soft-Impute needs only

$$
O(k_t ||\|\Omega||_1 + r_t k_t m) \quad (12)
$$

time, and one iteration costs

$$
O((r_t + k_t) ||\|\Omega||_1 + r_t k_t m) \quad (13)
$$

time. Since the solution is low-rank, $k_t, r_t \ll m$, and (13) is much faster than the $O(m n k_t)$ time for direct rank-$k_t$ SVD.

### 3 Accelerated Inexact Soft-Impute

In this section, we describe the proposed matrix completion algorithm. Tibshirani [Tibshirani, 2010] suggested that acceleration is not useful, as it destroys the essential “sparse plus low-rank” structure. However, we will show that it can indeed be preserved with acceleration. We also show that further speedup can be achieved by using approximate SVT.
3.1 Soft-Impute as a Proximal Algorithm

In [1], let

\[ f(X) = \frac{1}{2} \| P_{\Omega}(X - O) \|_F^2 = \sum_{(i,j) \in \Omega} \ell(X_{ij}, O_{ij}), \]

where \( \ell \) is the loss function, and \( g(X) = \lambda \|X\|_s \). The proximal step in the (unaccelerated) proximal algorithm is

\[ X_{t+1} = \text{prox}_{\mu g}(Z_t) = \arg \min_X \frac{1}{2} \|X - Z_t\|_F^2 + \mu \lambda \|X\|_s, \]

where \( Z_t = X_t - \mu P_{\Omega}(X_t - O) \). Note that the square loss \( \ell(X_{ij}, O_{ij}) \equiv \frac{1}{2}(X_{ij} - O_{ij})^2 \) in (11) is 1-Lipschitz smooth. The following shows that \( f \) in (14) is also 1-Lipschitz smooth. The proof can be found in Appendix A.1.

**Proposition 3.1.** If \( \ell \) is \( \rho \)-Lipschitz smooth, \( f \) in (14) is also \( \rho \)-Lipschitz smooth.

From (14), one can thus simply set \( \mu = 1 \) for (1). We then have

\[ X_{t+1} = \text{prox}_{\mu g}(Z_t) = \text{SVT}_\lambda(Z_t) \]

where \( X \) is the same as (10). Hence, interestingly, Soft-Impute is a proximal algorithm [Tibshirani, 2010], and thus converges at a rate of \( O(1/T) \) [Mazumder et al., 2010].

3.2 Accelerating Soft-Impute

Since Soft-Impute is a proximal algorithm, it is natural to accelerate it (Section 2.1). In this section, we show that the “sparse plus low-rank” structure can also be preserved.

To accelerate Soft-Impute, recall from (5) and (9) that we have to compute

\[ \text{prox}_{\mu g}(Z_t) = \text{SVT}_\lambda(Z_t) = \arg \min_X \frac{1}{2} \|X - Z_t\|_F^2 + \lambda \|X\|_s, \]

where

\[ Y_t = (1 + \theta_t)X_t - \theta_tX_{t-1}, \]

\[ \tilde{Z}_t = P_{(\Omega)}(O - Y_t) + (1 + \theta_t)X_t - \theta_tX_{t-1}. \]

Assume that \( X_t \) and \( Y_{t-1} \) have ranks \( r_t \) and \( r_{t-1} \), and their SVDs are \( U_t\Sigma_tV_t^T \) and \( U_{t-1}\Sigma_{t-1}V_{t-1}^T \), respectively. Similar to (11), for any \( v \in \mathbb{R}^n \), we have \( \tilde{Z}_tv = P_{(\Omega)}(O - Y_tv) + (1 + \theta_t)U_t\Sigma_t(V_t^Tv) - \theta_tU_{t-1}\Sigma_{t-1}(V_{t-1}^Tv) \). The first term takes \( O(\|\Omega\|) \) time while the last two terms take \( O((r_{t-1} + r_t)\text{time}) \), thus, a total of \( O(\|\Omega\| + (r_{t-1} + r_t)\text{time}) \). Similarly, for any \( u \in \mathbb{R}^m \), \( u^T\tilde{Z}_t \) takes \( O(\|\Omega\| + (r_{t-1} + r_t)\text{time}) \) time. The rank-\( k_t \) SVD of \( \tilde{Z}_t \) can be obtained using PROPACK in

\[ \text{SVT}(\|u\|_1 + (r_{t-1} + r_t)k_t) \]

time. As the target matrix is low-rank, \( r_{t-1} \) and \( r_t \) are much smaller than \( n \). Hence, (17) is much faster than the \( O(mnk_t) \) time required for a direct rank-\( k_t \) SVD.

The accelerated algorithm has a slightly higher iteration complexity than the unaccelerated one in (12). However, this is more than compensated by improvement in the convergence rate (from \( O(1/T) \) to \( O(1/T^2) \)), as will be empirically demonstrated in Section 5.1.

3.3 Approximating the SVT

Though acceleration preserves the “sparse plus low-rank” structure, the proposed algorithm (and Soft-Impute) can still be computationally expensive as the SVT in each iteration uses exact SVD. In this section, we show that further speedup is possible by using inexact SVD.

As SVT in (11) can be seen as a proximal step, one might want to perform inexact SVT by monitoring the duality gap as in Section 2.1. It can be shown that the dual of (15) is

\[ \max_{W} \text{tr}(W^T\tilde{Z}_t) - \frac{\lambda}{2} \|W\|_F^2, \]

where \( W \in \mathbb{R}^{m \times n} \) is the dual variable.

**Proposition 3.2** ([Parikh and Boyd, 2014]). Let the SVD of matrix \( \tilde{Z}_t \) be \( U\Sigma V^T \). The optimal solution of (18) is \( \tilde{W}_* = U \min(\Sigma, I)V^T \), where \([\min(A, B)]_{ij} = \min(A_{ij}, B_{ij}) \).

**Proposition 3.2** shows that a full SVD is required. This takes \( O(mnk_t) \) time and is even more expensive than directly using SVD (\( O(mnk_t) \) time). Instead, the proposed approximation is motivated by the following Proposition. The proof can be found in Appendix A.2.

**Proposition 3.3.** Let \( \tilde{k}_t \) be the number of singular values in \( \tilde{Z}_t \) larger than \( \lambda \), and \( Q \in \mathbb{R}^{m \times k_t} \), where \( k_t \geq \tilde{k}_t \), be orthogonal and contains the subspace spanned by the top \( \tilde{k}_t \) left singular vectors of \( \tilde{Z}_t \). Then, \( \text{SVT}_\lambda(\tilde{Z}_t) = \text{QSVT}_\lambda(Q^T\tilde{Z}_t) \).

Since a low-rank solution is desired, \( k_t \) can be much smaller than \( n \) [Mazumder et al., 2010]. Thus, once we identify the span of \( \tilde{Z}_t \)'s top left singular vectors, we only need to perform SVT on the much smaller \( Q^T\tilde{Z}_t \in \mathbb{R}^{k_t \times n} \) (instead of \( \tilde{Z} \in \mathbb{R}^{m \times n} \)). The question is how to find \( Q \). We adopt the power method (Algorithm 3) [Halko et al., 2011], which is more efficient than PROPACK [Wu and Simon, 2000]. Matrix \( R_t \) in Algorithm 3 is for warm-start.

**Algorithm 3** PowerMethod(\( \tilde{Z}_t, R_t, J \)).

Require: \( \tilde{Z}_t \in \mathbb{R}^{m \times n}, R_t \in \mathbb{R}^{n \times k_t} \), and the number of iterations \( J \);

1: initialize \( Q_0 = QR(\tilde{Z}_tR_t); // QR(\cdot) \) is QR factorization
2: for \( j = 1, 2, \ldots, J \) do
3: \( Q_j = QR(\tilde{Z}_t(Z_t^TQ_{j-1}); \)
4: end for
5: return \( Q_j \).

**Algorithm 3** shows the approximate SVT procedure. Step 1 approximates the top \( k_t \) left singular vectors of \( \tilde{Z}_t \) with \( Q \). In steps 2 to 5, a much smaller and less expensive (exact) SVT is performed on \( Q^T\tilde{Z}_t \). Finally, \( \text{SVT}_\lambda(\tilde{Z}_t) \) is recovered as \( X = (QU)\Sigma V^T \) using Proposition 3.3.
3.4 The Proposed Algorithm

We extend problem (1) by allowing the loss $\ell$ to be $\rho$-Lipschitz smooth (e.g., logistic loss and squared hinge loss): 

$$
\min_X F(X) \equiv \sum_{(i,j) \in \Omega} \ell(X_{ij}, O_{ij}) + \lambda \|X\|_*.
$$

(19)

Using (6),

$$
\hat{Z}_t = Y_t - \mu \nabla f(Y_t) = Y_t - \mu S_t,
$$

where $S_t$ is a sparse matrix with

$$
[S_t]_{ij} = \begin{cases} 
\frac{dt((Y_{ij})_t, O_{ij})}{dt(Y_{ij})} & \text{if } (i, j) \in \Omega \\
0 & \text{otherwise}
\end{cases}
$$

(20)

Using Proposition 3.1 and 3.4, the stepsize $\mu$ can be set as $1/\rho$. The whole procedure is shown in Algorithm 5.

The core steps are 6–8, which perform approximate SVT. As in [Hsieh and Olsen, 2014], $R_t$ is warm-started as $QR([V_t, V_{t-1}])$ at step 7. Moreover, as in [O’Donoghue and Candès, 2012; Nesterov, 2013], we restart the algorithm if $F(X)$ starts to increase (step 10). For further speedup, $\lambda$ is dynamically reduced (step 3) by a continuation strategy [Toh and Yun, 2010; Mazumder et al., 2010].

Algorithm 5 Accelerated Inexact Soft-Impute (AIS-Impute).

Require: partially observed matrix $O$, parameter $\lambda$.

1: initialize $c = 1$, $X_0 = X_1 = 0$, stepsize $\mu = 1/\rho$, $\lambda > \lambda$ and $\nu \in (0, 1)$;
2: for $t = 1, 2, \ldots, T$ do
3: $\lambda_t = (\lambda - \lambda)\nu_{t-1} + \lambda$;
4: $Y_t = X_t + \theta_t(X_t - X_{t-1})$, where $\theta_t = \frac{\nu_{t-1}}{\nu_{t-1} + 2}$;
5: $\hat{Z}_t = Y_t - \mu S_t$, with $S_t$ in (20);
6: $V_{t-1} = V_{t-1} - V_t(V_t^T V_{t-1})$, remove zero columns;
7: $R_t = QR([V_t, V_{t-1}])$;
8: $[U_{t+1}, \Sigma_{t+1}, V_{t+1}] = \text{approx-SVT} \left( \hat{Z}_t, R_t, \mu \lambda_t, J \right)$;
9: if $F(X_{t+1}) > F(X_t)$ then
10: $c = 1$;
11: else
12: $c = c + 1$;
13: end if
14: end for
15: return $U_{T+1}, \Sigma_{T+1}$ and $V_{T+1}$.

3.5 Convergence and Time Complexity

In the following, we will show that the proposed algorithm has a convergence rate of $O(1/T^2)$. Let $X_{t+1} = U_{t+1} \Sigma_{t+1} V_{t+1}^T$ be the output of approx-SVT at step 8. Since it only approximates $\text{SVT}_{\mu \lambda}(\hat{Z}_t)$, there is a difference ($\varepsilon_t$ in (7)) between the proximal objectives $h_{\mu \lambda \| \cdot \|_*}(X_{t+1}; \hat{Z}_t)$ and $h_{\mu \lambda \| \cdot \|_*}(\text{SVT}_{\mu \lambda}(\hat{Z}_t); \hat{Z}_t)$ after performing step 8, where $h_{\mu \lambda \| \cdot \|_*}(\cdot)$ is as defined in (6). The following shows that $\varepsilon_t$ decreases at a linear rate. The proof can be found in Appendix A.3.

Proposition 3.4. Assume that (i) $k_t \geq k_t$ for all $t$ and $J = J$ and (iii) $\{F(X_t)\}$ is upper-bounded. Then $\varepsilon_t$ decreases to zero linearly.

Using Propositions 3.1 and 3.4, convergence of the proposed algorithm is provided by the following Theorem. The proof can be found in Appendix A.4.

Theorem 3.5. The sequence $\{X_t\}$ generated from Algorithm 5 converges to the optimal solution with a $O(1/T^2)$ rate.

The basic operations in the power method are multiplications of the form $Z_t U$ and $V^T Z_t$. The tricks in Section 3.2 can again be used for acceleration, and computing the approximate SVT using Algorithm 4 takes only

$$
O(k_t \|\Omega\|_1 + (r_{t-1} + r_t)k_t m)
$$

(21)
time. This is slightly more expensive than (12), the time for performing exact SVD in Soft-Impute. However, Soft-Impute is not accelerated and has slower convergence than Algorithm 5 (Theorem 3.5). The complexity in (21) is also the same as (17). However, as will be demonstrated in Section 5.1, approximate SVT is empirically much faster.

Table 1 shows a breakdown of the iteration time complexity of Algorithm 5. As can be seen, it is only slightly more expensive than (13) for Soft-Impute.

Table 1: Iteration time complexity of Algorithm 5

| steps | complexity |
|-------|------------|
| 5 (construct $S_t$) | $O(r_t \|\Omega\|_1)$ |
| 6,7 (warm-start) | $O(nk_t^2)$ |
| 8 (approximate SVT) | $O(k_t \|\Omega\|_1 + (r_{t-1} + r_t)k_t m)$ |
| total | $O((r_t + k_t)\|\Omega\|_1 + (r_{t-1} + r_t + k_t)k_t m)$ |

Table 2 compares Algorithm 5 with some existing algorithms that will be empirically compared in Section 5.2. Overall, Algorithm 5 enjoys fast convergence and low iteration complexity.

3.6 Post-Processing

Recall that the nuclear norm penalizes all singular values equally. This may over-penalize the more important leading singular values. To alleviate this problem, we post-process the solution as in [Mazumder et al., 2010]. However, only the
The overlapped nuclear norm regularizer penalizes nuclear norms on all modes. When only several modes are low-rank, decomposition with the scaled latent nuclear norm has better generalization [Tomioka et al., 2010]. In this paper, we focus on the scaled latent nuclear norm regularizer.

Given a partially observed tensor \( \mathcal{O} \in \mathbb{R}^{I_1 \times \ldots \times I_D} \), with the observed entries indicated by \( \Omega \in \{0, 1\}^{I_1 \times \ldots \times I_D} \). The tensor completion problem can be formulated as

\[
\min_{\mathcal{X}^1, \ldots, \mathcal{X}^D} \mathcal{F} \left( \mathcal{X}^1, \ldots, \mathcal{X}^D \right) \quad \text{(23)}
\]

\[
= \sum_{(i_1, \ldots, i_D) \in \Omega} \ell \left( \sum_{d=1}^{D} \mathcal{X}^d_{i_1 \ldots i_D}, \mathcal{O}_{i_1 \ldots i_D} \right) + \sum_{d=1}^{D} \lambda_d \| \mathcal{X}^d_{(d)} \|_*. \]

The recovered tensor is \( \mathcal{X} = \sum_{d=1}^{D} \mathcal{X}^d \). In [Tomioka et al., 2010] [Liu et al., 2013], problem (23) is solved using ADMM [Boyd et al., 2011]. However, the ADMM update involves SVD in each iteration, which takes \( O \left( \prod_{d=1}^{D} I_d \sum_{d=1}^{D} I_d \right) \) time and can be expensive.

### 4.2 Generalizing SVT

In (23), let

\[
f(\mathcal{X}^1, \ldots, \mathcal{X}^D) = \sum_{(i_1, \ldots, i_D) \in \Omega} \ell \left( \sum_{d=1}^{D} \mathcal{X}^d_{i_1 \ldots i_D}, \mathcal{O}_{i_1 \ldots i_D} \right) \quad \text{(24)}
\]

\[
g(\mathcal{X}^1, \ldots, \mathcal{X}^D) = \sum_{d=1}^{D} \lambda_d \| \mathcal{X}^d_{(d)} \|_* \quad \text{(25)}
\]

The iterates in Algorithm 5 are generated by SVT. As there are multiple nuclear norms in (25), the following extends SVT for this case.

As in (25) is separable w.r.t. \( \mathcal{X}^i \) ’s, one can compute the proximal step for each \( \mathcal{X}^i \) separately [Parikh and Boyd, 2014]. Updates (5), (6) in the accelerated proximal gradient algorithm (Algorithm 1) become

\[
y^t_{d} = (1 + \theta_d)\mathcal{X}^d_{t-1} - \theta_d \mathcal{X}^d_{t-1}, \quad \mathcal{Z}^d_{t} = y^t_{d} - \mu \mathcal{S}_{t} = (1 + \theta_d)\mathcal{X}^d_{t} - \theta_d \mathcal{X}^d_{t-1} - \mu \mathcal{S}_{t}, \quad \text{for } d = 1, \ldots, D, \text{ where } \mathcal{S}_{t} \text{ is a sparse tensor with}
\]

\[
\langle \mathcal{S}_{t} \rangle_{i_1 \ldots i_D} = \begin{cases} d(\mathcal{Y}_{i_1 \ldots i_D}) & \text{if } (i_1, \ldots, i_D) \in \Omega, \\ 0 & \text{otherwise} \end{cases}
\]

Table 2: Comparison of AIS-Impute (Algorithm 5) and other algorithms. The algorithms in active subspace selection, TR and boost involve solving some optimization subproblems iteratively, and \( T_n \) is the number of iterations used. Moreover, integer \( T_n \) and \( c \in (0, 1) \) are some constants.

| Method                        | Iteration complexity    | Rate         |
|-------------------------------|-------------------------|--------------|
| SSGD [Avron et al., 2012]    | \( O(mk^2) \)          | \( O(1/\sqrt{T}) \) |
| active subspace selection [Hsieh and Olsen, 2014] | \( O(\|\Omega\|_1 k^2 T_n) \) | \( O(c^{1-2c}) \) |
| LMaFit [Wen et al., 2012]    | \( O(\|\Omega\|_1 k_i + mk_i) \) | ---          |
| boosting [Zhang et al., 2012] | \( O(\|\Omega\|_1 T_n) \) | \( O(1/T) \) |
| TR [Mishra et al., 2013]     | \( O(\|\Omega\|_1 T_n) \) | ---          |
| ALT-Impute [Hastie et al., 2015] | \( O((r_i + k_i)\|\Omega\|_1 + (r_i + k_i + k_i)mk_i) \) | \( O(1/T) \) |
| AIS-Impute                    | \( O((r_i + k_i)\|\Omega\|_1 + (r_i + k_i)mk_i) \) | \( O(1/T^2) \) |
and \( \hat{y}_t = \sum_{d=1}^{D} y_t^d \). Lemma 2.2 is also extended to
\[
[X_1^1, \ldots, X_1^D] = \arg\min_{X_1} \left\{ \sum_{d=1}^{D} \|X_1^d\|_{\Omega} + k_d^1 (I_d + I_{D\setminus d})(r_d^1 + r_{d-1}^1) \right\}
\]
as follows. The proof can be found in Appendix A.5.

**Proposition 4.1.** (\( X_{t+1}^d \) vs SVT) \( \mu_d\|\| \) \( (\hat{Z}_t^d) \).

The stepsize rule in (4) depends on the modulus of Lipschitz smoothness of \( f \), which is given in Proposition 3.6. The proof can be found in Appendix A.6.

**Proposition 4.2.** If \( \ell \) is \( \rho \)-Lipschitz smooth, \( f \) in (24) is \( \sqrt{D}\rho \)-Lipschitz smooth.

Proposition 3.3 can be used to reduce the size of \( (\hat{Z}_t^d) \) in Proposition 4.1 and Algorithm 3 can be used to approximate the underlying SVD. However, this is still not fast enough. Assume that \( k_t^d \) singular values in \( (\hat{Z}_t^d) \) are larger than \( \mu_{\lambda_d} \), and rank-\( k_t^d \) SVD, where \( k_t^d \geq k_t^d \), is performed. SVT on \( (\hat{Z}_t^d) \) takes \( O \left( k_t^d \prod_{d=1}^{D} I_d \right) \) time. As SVT has to be performed on each mode, one iteration of Algorithm 1 takes \( O \left( \prod_{d=1}^{D} I_d \sum_{d=1}^{D} k_t^d \right) \) time, which is expensive.

### 4.3 Fast Approximate SVT with Special Structure

In Section 3.2, the special “sparse plus low-rank” structure can greatly reduce the time complexity of matrix multiplications. As \( X_{t-1}^d \), \( X_t^d \) are low-rank tensors and \( S_t \) is sparse, \( \hat{Z}_t^d \) in (26) also has the “sparse plus low-rank” structure. However, to generate \( X_{t+1}^d \) using Proposition 4.1 we need to perform matrix multiplications of the form \( \langle \hat{Z}_t^d \rangle v \), where \( v \in \mathbb{R}^{I_{D-d}} \), and \( u^T \hat{Z}_t^d \), where \( u \in \mathbb{R}^{I_d} \). Unfolding \( \hat{Z}_t^d \) takes \( O \left( \prod_{d=1}^{D} I_d \right) \) time and can be expensive. In the following, we show how this can be avoided.

To generate \( (X_{t+1}^d) \), it can be seen from Proposition 4.1 and (26) that \( X_t^d \) and \( X_{t-1}^d \) only need to be unfolded along their rank margins. Hence, instead of storing them as tensors, we store \( (X_t^d) \) as its rank-\( r_t^d \) SVD \( U_t^d \sum_{d=1}^{D} V_t^d v^T \), and \( (X_{t-1}^d) \) as its rank-\( r_{t-1}^d \) SVD \( U_{t-1}^d \sum_{d=1}^{D} V_{t-1}^d v^T \). For any \( v \in \mathbb{R}^{I_{D-d}} \),

\[
(\hat{Z}_t^d) v = (1 + \theta_t) U_t^d \sum_{d=1}^{D} (V_t^d v^T) - \theta_t U_{t-1}^d \sum_{d=1}^{D} (V_{t-1}^d v^T) - \mu(S_t^d) v.
\]

The first two terms can be computed in \( O(Dd) \) time. As \( S_t \) is sparse, the last term takes \( O(Dd) \) time. Thus, \( \langle \hat{Z}_t^d \rangle v \) can be obtained in \( O(Dd) \) time. Similarly, for any \( u \in \mathbb{R}^{I_d} \),

\[
u^T(\hat{Z}_t^d) \]

Thus, performing approximate SVT on \( \hat{Z}_t^d \), with rank \( k_t^d \), using Algorithm 4 takes \( O(k_t^d(Dd)) \) time. Using Proposition 4.1 solving the proximal step \( \text{prox}_{\mu_g}(\hat{Z}_t^1, \ldots, \hat{Z}_t^D) \) takes a total

\[
O \left( \sum_{d=1}^{D} k_t^d \|\|_{\Omega} + k_t^d (I_d + I_{D\setminus d})(r_d^1 + r_{d-1}^1) \right)
\]
time. As the target tensor is low-rank, \( r_d^1, k_t^1 \ll I_d \) for \( d = 1, \ldots, D \). Hence, (28) is much faster than directly using Proposition 4.1 \( (O(Dd)) \) time.

### 4.4 The Proposed Algorithm

The whole procedure is shown in Algorithm 6. Unlike, Algorithm 5 \( D \) SVTs have to be computed (steps 5-11) in each iteration.

**Algorithm 6** AIBS-Impute (tensor case).

**Require:** partially observed tensor \( \mathcal{O} \), parameter \( \ell \).

1: initialize \( c = 1, \mathcal{X}_0^1 = \ldots = \mathcal{X}_0^D = 0, \mathcal{X}_1^1 = \ldots = \mathcal{X}_1^D = 0 \), step-size \( \mu = 1/(\sqrt{D}\rho) \), \( \lambda \geq \max_{d=1,\ldots,D} \lambda_d \) and \( \mu \in (0,1) \);
2: for \( t = 1, 2, \ldots, T \) do
3: construct the sparse observed tensor \( \mathcal{S}_t \) from (27);
4: for \( d = 1, \ldots, D \) do
5: \( \hat{Z}_t^d = (1 + \theta_t) \mathcal{X}_t^d - \theta_t \mathcal{X}_{t-1}^d + \mu \mathcal{S}_t^d \) \( V_{t-1}^d = V_{t-1}^d - V_t^d (V_t^d)^T V_{t-1}^d \), remove zero columns;
6: \( \mathcal{X}_t^d = \text{approx-SVT}(\hat{Z}_t^d, R_t^d, \mu(\lambda_d), J) \);
7: \( \mathcal{X}_t^d = U_{t+1}^d \sum_{d=1}^{D} (V_{t+1}^d)^T \) \( \mathcal{X}_t^d = \mathcal{X}_t^d \sum_{d=1}^{D} (V_{t+1}^d)^T \) \( \mathcal{X}_t^d = \mathcal{X}_t^d \sum_{d=1}^{D} (V_{t+1}^d)^T \);
8: end for
9: if \( F(\mathcal{X}_t^1, \ldots, \mathcal{X}_t^D) \) then \( c = 1 \);
10: else \( c = c + 1 \);
11: end if
12: return \( U_{t+1}^d, \sum_{d=1}^{D} (V_{t+1}^d)^T \) where \( d = 1, \ldots, D \).

Analogous to Theorem 3.5, we have the following. The proof can be found in Appendix A.7.

**Theorem 4.3.** Assume that (i) \( k_t^d \geq k_t^d \) for \( d = 1, \ldots, D \), all \( t \) and \( J = t \); (ii) \( F(\mathcal{X}_1^1, \ldots, \mathcal{X}_t^D) \) is upper bounded. The sequence \( \{\mathcal{X}_1^1, \ldots, \mathcal{X}_t^D\} \) generated from Algorithm 6 converges to the optimal solution with a \( O(1/T^2) \) rate.

### 4.5 Post-Processing

As in Section 3.6, the nuclear norm regularizer in (23) may over-penalize top singular values. To undo such shrinkage and boost recovery performance, we also adopt post-processing here. Let the tensor output from Algorithm 6 be \( \mathcal{X} = \sum_{d=1}^{D} \mathcal{X}_d \), where \( \mathcal{X}_d = U_d^d \Sigma_d^d (V_d^d)^T \) has rank \( k_d \).
In this section, we perform experiments on matrix completion (Sections 5.1, 5.3) and tensor completion (Sections 5.4, 5.5). Experiments are performed on a PC with Intel Xeon E5-2695 CPU and 256GB RAM. All algorithms are implemented in Matlab, with operations on $\Omega$ written in C.

5 Experiments

In this section, we perform experiments on matrix completion using synthetic data. The ground-truth matrix has a rank of 5, and is generated as $O = UV \in \mathbb{R}^{m \times m}$, where the entries of $U \in \mathbb{R}^{m \times 5}$ and $V \in \mathbb{R}^{5 \times m}$ are sampled i.i.d. from the standard normal distribution $\mathcal{N}(0, 1)$. Noise, sampled from $\mathcal{N}(0, 0.05)$, is then added. We randomly choose $15m \log(m)$ of the entries in $O$ as observed. Half of them are used for training, and the other half as validation set for parameter tuning. Testing is performed on the unobserved (missing) entries. We vary $m$ in the range {500, 1000, 2000}.

The following proximal algorithms are compared:

1. accelerated proximal gradient algorithm (denoted “APG”) [Toh and Yun, 2010]: It uses PROPACK to obtain singular values that are larger than $\lambda$;
2. Soft-Impute [Mazumder et al., 2010];
3. AIS-Impute (the proposed Algorithm 5);
4. AIS-Impute (exact): This is a variant of the proposed algorithm with exact SVT step (computed using PROPACK).

Let $X$ be the recovered matrix. For performance evaluation, we use the (i) normalized mean squared error NMSE = $||P_{\Omega^c}(X - UV)||_F / ||P_{\Omega^c}(UV)||_F$, and (ii) rank of $X$. To reduce statistical variability, experimental results are averaged over 5 repetitions.

Results are shown in Table 3. As can be seen, all algorithms have similar NMSE performance, with Soft-Impute being slightly worse. The objective of this section is to demonstrate the performance of the proposed algorithms on synthetic data.

| m = 500 | sparsity: 18.6% | APG | NMSE | rank \\
|---------|----------------|-----|------|-----|
|         |                | Soft-Impute | 17.1 ± 0.1 | 5 \\
|         |                | AIS-Impute (exact) | 16.8 ± 0.1 | 5 \\
|         |                | AIS-Impute | 16.9 ± 0.1 | 5 \\
| m = 1000 | sparsity: 10.4% | APG | 16.7 ± 0.1 | 5 \\
|         |                | Soft-Impute | 17.5 ± 0.1 | 5 \\
|         |                | AIS-Impute (exact) | 16.8 ± 0.1 | 5 \\
|         |                | AIS-Impute | 16.7 ± 0.1 | 5 \\
| m = 2000 | sparsity: 5.7% | APG | 14.3 ± 0.1 | 5 \\
|         |                | Soft-Impute | 14.9 ± 0.1 | 5 \\
|         |                | AIS-Impute (exact) | 14.2 ± 0.1 | 5 \\
|         |                | AIS-Impute | 14.2 ± 0.1 | 5 |

Table 4 shows the NMSE results with post-processing in Section 5.6. Compared with the time used by the main algorithm (Figure 1), the post-processing time is small and can be ignored. Thus, post-processing are always be performed in Sections 5.2 and 5.3.

5.2 Recommender System

In this section, we consider the standard matrix completion problem in Figure 1. Experiments are performed on two well-known benchmark data sets, MovieLens (Section 5.2) and Netflix (Section 5.2).

MovieLens

The MovieLens data set[1] (Table 5) contains ratings in {1, 2, 3, 4, 5} of different users on movies. It has been commonly used in matrix completion experiments [Mazumder et al., 2010] [Hsieh and Olsen, 2014]. We randomly use 50% of the observed ratings for training, 25% for validation and the rest for testing.

Besides the proximal gradient algorithms in Section 5.1, we also compare with the following state-of-the-art non-proximal matrix completion algorithms:

1. http://www.math.nus.edu.sg/~mattohkc/NNLS.html
2. http://www.mit.edu/~rahulmaz/software.html
3. http://grouplens.org/datasets/movielens/
Figure 1: Convergence of objective value on the synthetic matrix data. Top: vs CPU time (in seconds); Bottom: vs number of iterations.

(a) $m = 500$. (b) $m = 1000$. (c) $m = 2000$.

Figure 2: Testing RMSE vs CPU time (in seconds) on the MovieLens data sets.

(a) 100K. (b) 1M. (c) 10M.

1. active subspace selection (denoted “active”) \cite{Hsieh_2014}: In each iteration, this algorithm uses the power method to identify the active row and column subspaces, and then reduces the nuclear norm optimization problem to a smaller problem;
2. boosting \cite{Zhang_2012}, a variant of the Frank-Wolfe algorithm \cite{Frank_1956} for matrix completion. To speedup convergence local optimization (using L-BFGS) is performed in each iteration;
3. second-order trust-region algorithm (denoted “TR”) \cite{Mishra_2013}, which alternates between fixed-rank optimization and rank-one updates \cite{Mishra_2013};
4. ALT-Impute \cite{Hastie_2015}, a speedup variant of Soft-Impute that avoids SVD by alternating least squares. Its convergence rate is $O(1/T)$;
5. SSGD \cite{Avron_2012}, a stochastic algorithm for the nuclear norm regularized matrix completion problem; and
6. a fixed-rank approach solving by alternative minimization (denoted “LMaFit”) \cite{Wen_2012}. Over-relaxation is used for further speedup \cite{Wen_2012}.

Except LMaFit, all the other algorithms solve \eqref{eq:1} or its equivalent problem. Let $X$ be the recovered matrix,
Table 4: Matrix completion results on synthetic data, with post-processing. NMSE is scaled by $10^{-3}$. The lowest and comparable NMSEs (according to the pairwise t-test with 95% confidence) are highlighted.

| m   | NMSE  | post-proc time (sec) |
|-----|-------|----------------------|
| 500 | APG   | 9.8±0.1              |
|     | Soft-Impute | 9.9±0.1              |
|     | AIS-Impute (exact) | 9.8±0.1              |
|     | AIS-Impute | 9.8±0.1              |
| 1000| APG   | 9.1±0.1              |
|     | Soft-Impute | 9.9±0.1              |
|     | AIS-Impute (exact) | 9.4±0.1              |
|     | AIS-Impute | 9.3±0.1              |
| 2000| APG   | 8.5±0.1              |
|     | Soft-Impute | 9.3±0.1              |
|     | AIS-Impute (exact) | 8.4±0.1              |
|     | AIS-Impute | 8.4±0.1              |

Table 5: MovieLens data sets used in the experiments.

| #users | #movies | # observed ratings |
|--------|--------|--------------------|
| 100K   | 1,682  | 100,000            |
| 1M     | 3,449  | 999,714            |
| 10M    | 69,878 | 10,000,054         |

and the testing ratings $\{\hat{O}_{ij}\}$ be indexed by the set $\hat{\Omega}$. For performance evaluation, as in [Hsieh and Olsen, 2014, Mazumder et al., 2010], we use (i) the testing root mean squared error RMSE $= \sqrt{\|P(O(X - \hat{O}))\|_F^2/\|\hat{\Omega}\|_1}$; and (ii) rank of $X$. The experiment is repeated 5 times and the average performance is reported.

Results are shown in Table 6. As can be seen, AIS-Impute is consistently the fastest and has the lowest RMSE. On MovieLens-10M, TR, SSGD and APG are not as fast as they are slower. Figure 2 shows the testing RMSE with CPU time. As can be seen, Boost, TR, SSGD and APG are all very slow. Boost and TR need to solve an expensive subproblem in each iteration; SSGD has slow convergence; while APG requires SVT and are faster than Soft-Impute. However, their nonconvex formulations have slow convergence, and are thus slower than AIS-Impute. Overall, AIS-Impute is the fastest, as it combines cheap iteration and fast convergence.

Netflix

In this Section, we demonstrate the speedup of AIS-Impute over other algorithms solving the nuclear norm regularized problem (1) on the Netflix data set. It contains ratings of 53,819 users on 24,000 movies. 3% of the ratings matrix are observed. We randomly sample 50% of the observed ratings for training, and the rest for testing.

We only compare with active subspace selection, ALT-Impute and Soft-Impute; while methods including boosting, TR, SSGD, APG are slow and not compared. LMaFit solves an a different optimization problem based on matrix factorization, and has worse recovery performance than AIS-Impute. Thus, it is also not compared. As in [Mazumder et al., 2010], several choices of $\lambda$ are experimented.

Results are shown in Table 7. As in previous experiments, the RMSEs and ranks obtained by the various algorithms are similar. Figure 3 shows the detailed comparison on testing RMSE versus CPU time. As can be seen, AIS-Impute is again much faster.

5.3 Link Prediction

Given a graph with $m$ nodes and an incomplete adjacency matrix $O \in \{\pm 1\}^{m \times m}$, link prediction aims to recover a low-rank matrix $X \in \mathbb{R}^{m \times m}$ such that the signs of $X_{ij}$’s and $O_{ij}$’s agree on most of the observed entries. This is a binary matrix completion problem [Chiang et al., 2014], and we use the logistic loss $\ell(X_{ij}, O_{ij}) \equiv \log(1 + \exp(-X_{ij}O_{ij}))$ in (19).

Experiments are performed on the Epinions and Slashdot data sets [Chiang et al., 2014] (Table 9). Each row/column of the matrix $O$ corresponds to a user (users with fewer than two observations are removed). For Epinions, $O_{ij} = 1$ if user $i$ trusts user $j$, and $-1$ otherwise. Similarly for Slashdot, $O_{ij} = 1$ if user $i$ tags user $j$ as friend, and $-1$ otherwise. As can be seen from previous sections, Boost, TR, SSGD, APG and Soft-Impute are all slow, and thus

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https://snap.stanford.edu/data/
Table 7: Results on the Netflix data set. The regularization parameter $\lambda$ in (1) is set as $\lambda_0/c$, where $\lambda_0 = \|P_2(O)\|_F$. The lowest and comparable RMSEs (according to the pairwise t-test with 95% confidence) are highlighted. Soft-Impute with $c = 30$ is not run as it is very slow.

| $c = 10$ | RMSE     | rank |
|-----------|----------|------|
| active    | 0.894±0.001 | 3    |
| ALT-Impute | 0.900±0.006 | 3    |
| Soft-Impute | 0.893±0.001 | 3    |
| AIS-Impute | 0.893±0.001 | 3    |

| $c = 20$ | RMSE     | rank |
|-----------|----------|------|
| active    | 0.847±0.001 | 14   |
| ALT-Impute | 0.850±0.001 | 14   |
| Soft-Impute | 0.847±0.001 | 14   |
| AIS-Impute | 0.847±0.001 | 14   |

| $c = 30$ | RMSE     | rank |
|-----------|----------|------|
| active    | 0.820±0.001 | 116  |
| ALT-Impute | 0.825±0.001 | 116  |
| AIS-Impute | 0.820±0.001 | 116  |

Table 8: Datasets used for link prediction.

| Dataset | #rows | #columns | #signs |
|---------|-------|----------|--------|
| Epinions | 84,601 | 48,091   | 505,074 |
| Slashdot | 70,284 | 32,188   | 324,745 |

Results are shown in Table 9 and Figure 4 shows the testing accuracy with CPU time. As can be seen, active and AIS-Impute have slightly better accuracies than AltMin, and AIS-Impute is the fastest.

Table 9: Performance for link prediction in social network. The lowest and comparable RMSEs (according to the pairwise t-test with 95% confidence) are highlighted.

| Dataset   | accuracy | rank |
|-----------|----------|------|
| Epinions  | 0.939±0.002 | 12   |
| AltMin    | 0.936±0.002 | 41   |
| AIS-Impute | 0.940±0.001 | 12   |
| Slashdot  | 0.844±0.001 | 16   |
| AltMin    | 0.839±0.002 | 39   |
| AIS-Impute | 0.843±0.001 | 16   |

5.4 Tensor Completion: Synthetic Data

In this section, we perform tensor completion experiments using synthetic data. The ground-truth data tensor (of size $m \times m \times 3$) is generated as $O = C_1 A_1 \times_2 A_2 \times_3 A_3$, where the elements of $A_1 \in \mathbb{R}^{m \times 3}$, $A_2 \in \mathbb{R}^{m \times 3}$, $A_3 \in \mathbb{R}^{3 \times 3}$ and the core tensor $C \in \mathbb{R}^{3 \times 3}$ are all sampled i.i.d. from the standard normal distribution $N(0, 1)$, and $\times_k$ is the $k$-mode product. Thus, $O$ is low-rank on the first two mode but not on the third one. Noise $\xi$, with its elements sampled i.i.d. from the normal distribution $N(0, 0.05)$, is then added. A total number of $\Omega = 45m \log(m)$ random elements in $O$ are observed. Half of them are used for training, and the other half for validation. On testing, we perform evaluation on the unobserved entries and use the same criteria as in Section 5.1 i.e., NMSE and recovered rank on each mode.

Similar to Section 5.1, we compare the following algorithms: (i) APG; (ii) the proposed algorithm with exact SVD (AIS-Impute(exact)), and (iii) the proposed algorithm which uses power method to approximate SVT (AIS-Impute). Soft-Impute has not been extended to tensor completion, and is thus not compared. Besides, we also compare with the ADMM approach in [Tomioka et al., 2010] (denoted “ADMM(scaled)”).

The $k$-mode product of a tensor $X$ and a matrix $A$ is defined as $X \times_k A = \{X_{(k)} A_{(k)}\}_{(k)}$ [Kolda and Bader, 2009].
As suggested in [Wimalawarne et al., 2014], we set \((\lambda_1, \lambda_2, \lambda_3)\) in the scaled latent nuclear norm to \((1, 1, \sqrt{m/3})\). Thus, the only tunable parameter is \(\lambda\), which is obtained by grid search using the validation set. We also vary \(m\) in \{500, 1000, 2000\}. Experimental results are averaged over 5 repetitions.

Results on NMSE and rank are shown in Table 10. As can be seen, APG, AIS-Impute(exact) and AIS-Impute have comparable performance. The comparison of objective vs time and iterations are shown in Figure 5. In terms of iterations, APG, AIS-Impute(exact) and AIS-Impute have similar behavior as they all have \(O(1/T^2)\) convergence rate. These also agree with the matrix case in Section 5.1. They are faster than ADMM(scaled), which only has a slower \(O(1/T)\) rate [He and Yuan, 2012]. In terms of time, as APG does not utilize the “sparse plus low-rank” structure, it is slower than AIS-Impute(exact) and AIS-Impute. ADMM(scaled) is even slower than APG, as it only has slower convergence rate compared with APG. AIS-Impute is the fastest, as it has both fast \(O(1/T^2)\) convergence rate and low per-iteration complexity.

Performance with post-processing in Section 4.5 is shown in Table 11. As can be seen, it is very efficient and improves NMSE. Thus, we always perform post-processing in Section 5.5.

### 5.5 Multi-Relational Link Prediction

In this section, we perform experiments on the YouTube data set[^11] [Lei et al., 2009]. It contains 15,088 users, and describes five types of user interactions: contact, number of shared friends, number of shared subscriptions, number of shared subscribers, and the number of shared favorite videos. Thus, it forms a \(15088 \times 15088 \times 5\) tensor, with a total of \(27,257,790\) nonzero elements. Following [Chiang et al., 2014], we formulate multi-relational link prediction as a tensor completion problem. As the observations are real-valued, we use the square loss in (23). Besides AIS-Impute (Algorithm 6), we also compare with the following state-of-the-art non-proximal-based tensor completion algorithms:

| m = 500 | APG | AIS-Impute(exact) | AIS-Impute | ADMM(scaled) |
|---------|-----|------------------|-------------|--------------|
| NMSE    | 17.3 ± 1.4 | 17.3 ± 1.8 | 17.2 ± 1.7 | 17.5 ± 1.8 |
| rank of mode | 3 | 3 | 3 | 3 |

| m = 1000 | APG | AIS-Impute(exact) | AIS-Impute | ADMM(scaled) |
|----------|-----|------------------|-------------|--------------|
| NMSE     | 17.2 ± 1.3 | 17.1 ± 1.5 | 17.0 ± 1.4 | 17.0 ± 2.3 |
| rank of mode | 3 | 3 | 3 | 3 |

| m = 2000 | APG | AIS-Impute(exact) | AIS-Impute | ADMM(scaled) |
|----------|-----|------------------|-------------|--------------|
| NMSE     | 18.1 ± 2.3 | 18.0 ± 1.8 | 18.0 ± 1.7 | 18.3 ± 2.3 |
| rank of mode | 3 | 3 | 3 | 3 |

Table 10: Tensor completion results on synthetic data. No post-processing is performed and NMSE is scaled by \(10^{-3}\). The lowest and comparable NMSEs (according to the pairwise t-test with 95% confidence) are highlighted.

| m = 500 | APG | AIS-Impute(exact) | AIS-Impute | ADMM(scaled) |
|---------|-----|------------------|-------------|--------------|
| NMSE    | 10.5 ± 0.4 | 10.5 ± 0.4 | 10.5 ± 0.5 | 10.5 ± 0.6 |
| time (sec) | 0.1 | 0.1 | 0.1 | 0.1 |

| m = 1000 | APG | AIS-Impute(exact) | AIS-Impute | ADMM(scaled) |
|----------|-----|------------------|-------------|--------------|
| NMSE     | 11.2 ± 0.4 | 11.2 ± 0.5 | 11.2 ± 0.4 | 11.2 ± 0.7 |
| time (sec) | 0.1 | 0.1 | 0.1 | 0.1 |

| m = 2000 | APG | AIS-Impute(exact) | AIS-Impute | ADMM(scaled) |
|----------|-----|------------------|-------------|--------------|
| NMSE     | 11.1 ± 0.6 | 11.1 ± 0.6 | 11.2 ± 0.6 | 11.3 ± 0.8 |
| time (sec) | 0.5 | 0.4 | 0.4 | 0.1 |

Table 11: Tensor completion results on synthetic data (with post-processing). NMSE is scaled by \(\times 10^{-3}\). The lowest and comparable NMSEs (according to the pairwise t-test with 95% confidence) are highlighted.

[^11]: [http://socialcomputing.asu.edu/datasets/](http://socialcomputing.asu.edu/datasets/)
Let $X$ be the recovered tensor, and the testing ratings $\hat{O}_{ij}$ be indexed by the set $\hat{\Omega}$. For performance evaluation, we use (i) the testing root mean squared error $\text{RMSE} = \sqrt{\frac{\|P_{\hat{\Omega}}(X - \hat{O})\|_F^2}{\|\hat{\Omega}\|_1}}$; and (ii) rank of the unfolded matrix in each mode. The experiments are repeated five times.

Performance is shown in Table 12 and Figure 6(a) shows the time comparison. ADMM(overlap) and FaLRTC have similar recovery performance, but are all very slow due to usage of SVD. As the overlapping nuclear norm is smoothed in FaLRTC, it cannot exactly recover a low-rank tensor. TMac is fast, but has the worst recovery performance. AIS-Impute enjoys fast speed and good recovery performance.

Table 12: Results on the YouTube subset. The lowest and comparable RMSEs (according to the pairwise t-test with 95% confidence) are highlighted.

| Method          | RMSE        | rank of mode |
|-----------------|-------------|--------------|
| GeomCG          | 0.672±0.050 | 7 7 5        |
| ADMM(overlap)   | 0.690±0.030 | 142 142 5    |
| FaLRTC          | 0.672±0.032 | 1000 1000 5  |
| TMac            | 0.786±0.027 | 4 4 0        |
| AIS-Impute      | 0.616±0.029 | 33 33 0      |

Full YouTube Data

Next, we perform experiments on the full YouTube data set with the same setup. As ADMM(overlap) and FaLRTC are too slow, we only compare GeoCG, TMac and AIS-Impute. Experiments are repeated five times.
Figure 6: Testing RMSE vs CPU time on the Youtube data set.

Results are shown in Table 13 and Figure 6(b) shows the time. TMac has much worse performance than GeomCG and AIS-Impute. GeomCG is based on the (nonconvex) Turker decomposition, and its convergence rate is unknown. Moreover, its iteration time complexity has a worse dependency on the tensor rank than AIS-Impute (\(\prod_{i=1}^{D} r_i^d\) vs \(\sum_{i=1}^{D} r_i^d\)), and thus GeomCG becomes very slow when the tensor rank is large. Overall, AIS-Impute has fast speed and good recovery performance.

Table 13: Results on the full YouTube dataset. The lowest and comparable RMSEs (according to the pairwise t-test with 95% confidence) are highlighted.

|       | RMSE     | rank of mode |
|-------|----------|--------------|
|       |          | 1 | 2 | 3 | 4 |
| GeomCG | 0.388±0.001 | 51 | 51 | 5 | 5 |
| TMac   | 0.611±0.007 | 10 | 10 | 0 | 0 |
| AIS-Impute | 0.369±0.006 | 70 | 70 | 0 | 0 |

6 Conclusion

In this paper, we show that Soft-Impute, as a proximal algorithm, can be accelerated without losing the “sparse plus low-rank” structure crucial to the efficiency of Soft-Impute. To further reduce the per-iteration time complexity, we proposed an approximate-SVT scheme based on the power method. Theoretical analysis shows that the proposed algorithm still enjoys the fast \(O(1/T^2)\) convergence rate. We also extend the proposed algorithm to handle low-rank tensor completion with the scaled latent nuclear norm as regularizer. We show that it is possible to preserve the “sparse plus low-rank” structure and fast \(O(1/T^2)\) convergence. Extensive experiments on both synthetic and real-world data sets show that the proposed algorithm is much faster than the state-of-the-art.

As for future work, it will be interesting to develop parallel and distributed versions of the proposed algorithms for better speedup and scalability. Moreover, recommender systems usually have rich side information, such as the users’ relationships, movies’ content [Koren, 2008] [Ma et al., 2011] [Zhao et al., 2015]. It will be useful to incorporate these information into the algorithm for improved performance.

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### A Proofs

#### A.1 Proposition 3.1

**Proof.** For any $X, Y \in \mathbb{R}^{m \times n}$,

$$
\| \nabla f(X) - \nabla f(Y) \|_F^2
= \sum_{(i,j) \in \Omega} \left( \frac{d\ell(X_{ij}, O_{ij})}{dX_{ij}} - \frac{d\ell(Y_{ij}, O_{ij})}{dY_{ij}} \right) ^2
\leq \sum_{(i,j) \in \Omega} \rho^2 (X_{ij} - Y_{ij})^2
\leq \rho^2 \| X - Y \|_F^2,
$$

where (30) follows from the fact that $\ell$ being $\rho$-Lipschitz smooth. Thus, $f(X)$ is $\rho$-Lipschitz smooth.

#### A.2 Proposition 3.3

**Proof.** Let the SVD of $Z_t$ be $U \Sigma V^T$, and

$$
\tilde{Z}_t = [U_{k_t}; U_{\perp}] \begin{bmatrix} \Sigma_{k_t} & & \\ & \Sigma_{\perp} & \\ & & [V_{k_t}; V_{\perp}]^T \end{bmatrix},
$$

where $U_{k_t}$ contains the top $k_t$ columns of $U$, and $U_{\perp}$ contains the remaining columns (and similarly for $(\Sigma_{k_t}, \Sigma_{\perp})$ in $\Sigma$ and $(V_{k_t}, V_{\perp})$ in $V$).

**Lemma A.1.** The SVD of $Q^T \tilde{Z}_t$ is $(Q^T U_{k_t}) \Sigma_{k_t} V_{k_t}^T$.

**Proof.** As $Q^T U_{\perp} = 0$,

$$
Q^T \tilde{Z}_t = [Q^T U_{k_t}; Q^T U_{\perp}] \begin{bmatrix} \Sigma_{k_t} & & \\ & \Sigma_{\perp} & \\ & & [V_{k_t}; V_{\perp}]^T \end{bmatrix}
= [Q^T U_{k_t}; 0] \begin{bmatrix} \Sigma_{k_t} & & \\ & \Sigma_{\perp} & \\ & & [V_{k_t}; V_{\perp}]^T \end{bmatrix} = (Q^T U_{k_t}) \Sigma_{k_t} V_{k_t}^T.
$$

Moreover, as $\text{span}(Q) = \text{span}(U_{k_t})$, we have

$$(Q^T U_{k_t})^T (Q^T U_{k_t}) = U_{k_t}^T Q Q^T U_{k_t} = U_{k_t}^T U_{k_t} U_{k_t}^T U_{k_t} = I.$$

Thus, $Q^T U_{k_t}$ is orthogonal.

From Lemma 2.2 and (31),

$$
\text{SVT}_t(\tilde{Z}_t) = [U_{k_t}; U_{\perp}] \begin{bmatrix} \Sigma_{k_t} & & \\ & \Sigma_{\perp} & \\ & & [V_{k_t}; V_{\perp}]^T \end{bmatrix} = U_{k_t} (\Sigma_{k_t} - \lambda I) + V_{k_t}^T
\leq U_{k_t} (\Sigma_{k_t} - \lambda I) + V_{k_t}^T
= \biggr[ U_{k_t} U_{k_t}^T \biggr] U_{k_t} (\Sigma_{k_t} - \lambda I) + V_{k_t}^T
= Q \biggr[ (Q^T U_{k_t}) (\Sigma_{k_t} - \lambda I) + V_{k_t} \biggr]^T
= Q \text{SVT}_t(Q^T \tilde{Z}_t).
$$

Here, (32) follows from $k_t \geq \tilde{k}$ and that $\tilde{k}$ singular values of $\tilde{Z}_t$ are larger than $\lambda$; (33) follows from $\text{span}(Q) = \text{span}(U_{k_t})$; and (34) follows from Lemma A.1.

#### A.3 Proposition 3.4

**Proof.** Before proof of Proposition 3.4, we first introduce some Lemmas (Lemma A.2, A.4, A.3 and A.7) and Propositions (Proposition A.5 and A.6).

**Lemma A.2 (Combettes and Wajs, 2005).** For any matrices $A$ and $B$, $\| \text{SVT}_t(A) - \text{SVT}_t(B) \|_F \leq \| A - B \|_F$.

Let $Z_t := \text{SVT}_t(\tilde{Z}_t)$, $\beta_t := \| \tilde{Z}_t \|_F$ and $\eta_t = \frac{\sigma_{k+1}(\tilde{Z}_t)}{\sigma_k(\tilde{Z}_t)}$.

**Lemma A.3 (Arbenz, 2010).** Let the input to Algorithm 3 be $\tilde{Z}_t$ and its top $k$ left singular vectors be contained in $U_{k_t}$.

Then, for $j = 0, 1, 2, \ldots$,

$$
\| Q_j Q_j^T - U_{k_t} U_{k_t}^T \|_F \leq \eta_t^j \alpha_t,
$$

where $\alpha_t = \| Q_0 Q_0^T - U_{k_t} U_{k_t}^T \|_F$ and $Q_0$ is the span of $\tilde{Z}_t R_t$.
Lemma A.4. For output $\hat{X} = (QU)\Sigma V^\top$ from Algorithm 4 we have $\|\hat{X} - Z_t^*\|_F \leq \|U_kU_k^\top - QQ^\top\|_F \beta_t$.

Proof. From Proposition A.3
\[
Z_t^* - \hat{X} = \text{SVT}_{\mu\lambda}(\tilde{Z}_t) - Q\text{SVT}_{\mu\lambda}(Q^T \tilde{Z}_t) = \text{SVT}_{\mu\lambda}(U_kU_k^\top \tilde{Z}_t) - \text{SVT}_{\mu\lambda}(QQ^T \tilde{Z}_t).
\]
Using Lemma A.2 and the Cauchy’s inequality,
\[
\|\hat{X} - Z_t\|_F = \|\text{SVT}_{\mu\lambda}(U_kU_k^\top \tilde{Z}_t) - \text{SVT}_{\mu\lambda}(QQ^T \tilde{Z}_t)\|_F \leq \|U_kU_k^\top - QQ^T\|\|\tilde{Z}_t\|_F \leq \|U_kU_k^\top - QQ^T\|_F \beta_t,
\]
and result follows.

Proposition A.5. Let $G_t \in \partial h_{\mu\lambda\|\|_t}(\tilde{X}; \tilde{Z}_t)$, then $\|G_t\|_F$ is upper-bounded by a constant $\gamma_t$.

Proof. Let the reduced SVD of $\tilde{X}$ be $U\Sigma V^\top$ (only positive singular values are considered). By the definition of subgradient of the nuclear norm (Candes and Recht, 2009),
\[
\partial h_{\mu\lambda\|\|_t}(\tilde{X}; \tilde{Z}_t) = \tilde{X} - \tilde{Z}_t + \mu\lambda(UV^T + W),
\]
where
\[
W^\top U = 0, WV = 0, \text{ and } \|W\|_\infty \leq 1. \tag{35}
\]
Thus,
\[
\|G_t\|_F = \|\tilde{X} - \tilde{Z}_t + \mu\lambda(UV^T + W)\|_F 
\leq \|\tilde{X} - \tilde{Z}_t\|_F + \mu\lambda\|UV^T + W\|_F. \tag{36}
\]
For the first term in (36),
\[
\|\tilde{X} - \tilde{Z}_t\|_F 
\leq \|\tilde{X} - Z_t^* + Z_t^* - \tilde{Z}_t\|_F 
\leq \|\tilde{X} - Z_t^*\|_F + \|Z_t^* - \tilde{Z}_t\|_F 
\leq \|Z_t^* - \tilde{Z}_t\|_F + \|Z_t^* - Q\text{SVT}_{\mu\lambda}(Q^T \tilde{Z}_t)\|_F 
\leq \|Z_t^* - \tilde{Z}_t\|_F + \|U_kU_k^\top - QQ^T\|_F \beta_t 
\leq \|Z_t^* - \tilde{Z}_t\|_F + \alpha_t\beta_t. \tag{37}
\]
Here, (37) follows from Lemma A.4 and (38) from Lemma A.3. As $\|W\|_\infty \leq 1$ from (35), thus
\[
\|W\|_F = \sqrt{\sum_{i=1}^{m} \sigma_i(W)} \leq \sqrt{m}.
\]
For the second term in (38), then
\[
\|UV^T + W\|_F \leq \sqrt{\text{tr}(U^\top UV^\top V) + \|W\|_F} \leq \sqrt{k_t + \sqrt{m}} \leq 2\sqrt{m}. \tag{39}
\]
Combining (38) and (39), by Lemma A.3
\[
\|G_t\|_F \leq 2\mu\lambda\sqrt{m} + \|Z_t^* - \tilde{Z}_t\|_F + \alpha_t\beta_t. \tag{40}
\]
Since $Z_t^*$ is independent of $X$, $\|Z_t^* - \tilde{Z}_t\|_F$ is a constant. Hence, $\|G_t\|_F$ is upper bounded by
\[
\gamma_t = 2\mu\lambda\sqrt{m} + \|Z_t^* - \tilde{Z}_t\|_F + \alpha_t\beta_t,
\]
which a constant.

Proposition A.6. Assume that $k_t \geq \tilde{k}_t$. Let $h_{\mu\lambda\|\|_t}(\tilde{X}; \tilde{Z}_t)$ be as defined in (8). Then, for Algorithm 4 we have
\[
\|\tilde{X}_t\|_F \leq \|\tilde{X}_t\|_F + \alpha_t\beta_t\eta_t^d. \tag{41}
\]
Proof. As $h$ is convex,
\[
\|\tilde{X}_t\|_F \leq \|\tilde{X}_t\|_F + \text{tr}((\tilde{X}_t - Z_t^*)^\top G_t) \leq \gamma_t\|\tilde{X}_t - Z_t^*\|_F \leq \gamma_t\beta_t\|QQ^T - U_kU_k^\top\|_F \leq \eta_t^d(\alpha_t\beta_t\gamma_t). \tag{42}
\]
Here, (42) follows from Proposition A.5, (43) from Lemma A.4, and (44) from Lemma A.3. Result follows on combining (41) and (44).

Lemma A.7. If $\{F(X_t)\}$ is upper-bounded where $F$ is the objective at (19), then $\|X_t\|_F$ from Algorithm 5 is upper-bounded.

Proof. As $\{F(X_t)\}$ is upper bounded and note that $F(X) \to +\infty \iff \|X\|_F \to +\infty$ for (19), then $\{\|X_t\|_F\}$ is also upper bounded.

Now, we are ready to prove Proposition A.3. As $\alpha_t$, $\beta_t$ and $\gamma_t$ only depend on $X_t$, from Lemma A.7 they are all upper bounded. Let $q = \sup \{\alpha_t\beta_t\gamma_t\}$, and $q < \infty$ is a constant. Then by Proposition A.6 and note that Algorithm 4 is run for $t$ iterations at $t$th loop of Algorithm 5. Let $\eta_t = \max_t \eta_t \in (0, 1)$, we have
\[
\|\tilde{X}_{t+1}\|_F \leq \|\tilde{X}_t\|_F + \varepsilon_t. \tag{45}
\]
Hence, $\varepsilon_t = q\eta_t^d$ decays at a linear rate.

A.4 Theorem 3.5
Proof. From Proposition A.4, $\varepsilon_t$ decays at a linear rate. Moreover, there is no error on the computation of gradient. Thus, conditions in Proposition 2.7 are satisfied, and Algorithm 5 converges with a rate of $O(1/T^2)$.

A.5 Proposition 4.1
Proof. Note that
\[
\min_{X^1, \ldots, X^D} \frac{1}{2} \|X^1, \ldots, X^D - (\tilde{Z}_t^1, \ldots, \tilde{Z}_t^D)\|_F^2 + \mu \sum_{d=1}^{D} \lambda_d \|X^d_{(d)}\|_* = \frac{1}{2} \|X_{d}^d - (\tilde{Z}_t^d)\|_F^2 + \mu \lambda_d \|X^d_{(d)}\|_* = \frac{1}{2} \|X_{d}^d - (\tilde{Z}_t^d)\|_F^2 + \mu \lambda_d \|X^d_{(d)}\|_* \tag{45}
\]
The $X^d_{(d)}$s in (45) are independent of each other, and $
\langle \text{SVT}_{\mu\lambda}(\tilde{Z}^d_{(d)}) \rangle_{(d)} = \arg \min_{X^d} \frac{1}{2} \|X^d_{(d)} - \tilde{Z}^d_{(d)}\|_F^2 + \mu \lambda_d \|X^d_{(d)}\|_*\] and thus result follows.
A.6 Proposition 4.2

Proof. For any \( X_1, \ldots, X^D, Y_1, \ldots, Y^D \), and let \( \tilde{X} = \sum_{d=1}^D X^d \) and \( \tilde{Y} = \sum_{d=1}^D Y^d \).

\[
\| \nabla f([X_1, \ldots, X^D]) - \nabla f([Y_1, \ldots, Y^D]) \|_F^2 \\
= \sum_{(i_1, \ldots, i_D) \in \Omega} \left[ \frac{df(X_{i_1 \ldots i_D}, O_{i_1 \ldots i_D})}{dx_{i_1 \ldots i_D}} - \frac{df(Y_{i_1 \ldots i_D}, O_{i_1 \ldots i_D})}{dy_{i_1 \ldots i_D}} \right]^2 \\
\leq \sum_{(i_1, \ldots, i_D) \in \Omega} \rho^2 \left( \| X_{i_1 \ldots i_D} - Y_{i_1 \ldots i_D} \|_F \right)^2,
\]

where the first inequality comes from the \( \rho \)-Lipschitz smoothness of \( f \). Note that

\[
\| \tilde{X} - \tilde{Y} \|_F^2 \leq D \sum_{d=1}^D \| X^d - Y^d \|_F^2 \\
= D \| [X^1, \ldots, X^D] - [Y^1, \ldots, Y^D] \|_F^2.
\]

We have

\[
\| \nabla f([X_1, \ldots, X^D]) - \nabla f([Y_1, \ldots, Y^D]) \|_F \\
\leq \sqrt{D} \rho \| [X^1, \ldots, X^D] - [Y^1, \ldots, Y^D] \|_F,
\]

and thus \( f \) is \( \sqrt{D} \rho \)-Lipschitz smooth. \( \square \)

A.7 Theorem 4.3

Proof. From the definition of \( h \) in (8),

\[
h_{\mu g} \left( [X^1_{t+1}, \ldots, X^D_{t+1}]; [\tilde{Z}^1_t, \ldots, \tilde{Z}^D_t] \right) \\
= \sum_{d=1}^D \frac{1}{2} \left( \| (X^d_{t+1})_d - (\tilde{Z}^d_t)_d \|_F^2 + \mu \lambda_d \| (X^d_{t+1})_d \|_*, \right) \\
= \sum_{d=1}^D h_{\mu \lambda_d \| \cdot \|_*} \left( (X^d_{t+1})_d; (\tilde{Z}^d_t)_d \right). \tag{46}\n\]

As proximal step is inexact in Algorithm 6 using Proposition A.6 on (46),

\[
h_{\mu \lambda_d \| \cdot \|_*} \left( (X^d_{t+1})_d; (\tilde{Z}^d_t)_d \right) \\
\leq h_{\mu \lambda_d \| \cdot \|_*} \left( (W^d_t)_d; (\tilde{Z}^d_t)_d \right) + (\alpha_d)_t (\beta_d)_t (\gamma_d)_t \eta_d)_t,
\]

where \( (W^d_t)_d = \text{SVT}_{\mu \lambda_d \| \cdot \|_*} \left( (\tilde{Z}^d_t)_d \right) \), and \( \alpha_d, \beta_d, \gamma_d, \eta_d \) are constants depending on \( (\tilde{Z}^d_t)_d \). As \( J = t \),

\[
h_{\mu g} \left( [X^1_{t+1}, \ldots, X^D_{t+1}]; [\tilde{Z}^1_t, \ldots, \tilde{Z}^D_t] \right) \tag{47}\n\leq h_{\mu g} \left( [W^1_*, \ldots, W^D_*]; [\tilde{Z}^1_*, \ldots, \tilde{Z}^D_*] \right) + \sum_{d=1}^D (c_d)_t (\eta_d)_t.
\]

As \( F([X^1, \ldots, X^D]) \) is upper-bounded and

\[
\lim_{\| X^d \|_F \to \infty} F([X^1, \ldots, X^D]) = \infty,
\]

for any \( d = 1, \ldots, D \). Then, \( \| X^d_t \|_F \) for \( d = 1, \ldots, D \) are also upper-bounded. Thus,

\[
q = \sup_{t} \sum_{d=1}^D (c_d)_t < \infty.
\]

Let \( \eta = \max_{t,d} ((\eta_d)_t) < 1. \) Together with (47), we have

\[
h_{\mu g} \left( [X^1_{t+1}, \ldots, X^D_{t+1}]; [\tilde{Z}^1_t, \ldots, \tilde{Z}^D_t] \right) \\
\leq h_{\mu g} \left( [W^1_*, \ldots, W^D_*]; [\tilde{Z}^1_*, \ldots, \tilde{Z}^D_*] \right) + \varepsilon_t,
\]

and the approximation error \( \varepsilon_t = q \eta^t \) decays at a linear rate. Moreover, there is no error on the computation of gradient. Thus, the conditions in Proposition 2.1 are satisfied, and Algorithm 6 converges with a rate of \( O(1/T^2) \). \( \square \)