Tales of 1001 Gluons

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Abstract

This report is centred around tree-level scattering amplitudes in pure Yang-Mills theories, the most prominent example is given by the tree-level gluon amplitudes of QCD. I will discuss several ways of computing these amplitudes, illustrating in this way recent developments in perturbative quantum field theory. Topics covered in this review include colour decomposition, spinor and twistor methods, off- and on-shell recursion, MHV amplitudes and MHV expansion, the Grassmannian and the amplituhedron, the scattering equations and the CHY representation. At the end of this report there will be an outlook on the relation between pure Yang-Mills amplitudes and scattering amplitudes in perturbative quantum gravity.
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1 Introduction

Each student of physics has encountered the harmonic oscillator during her or his studies, usually more than once. The harmonic oscillator is a simple system, familiar to the students, and can be used as a pedagogical example to introduce new concepts – just think about the introduction of raising and lowering operators for the quantum mechanical harmonic oscillator.

Another system encountered by any physics student is the hydrogen atom. Again, methods to solve this system apply to a wider context. For example, positronium and charmonium are close cousins of the hydrogen atom.

Within the context of quantum field theory the computation of tree-level scattering amplitudes in pure Yang-Mills theories constitutes another example of physical problems every student should know about: The tree-level scattering amplitudes in pure Yang-Mills theories can be computed in several way and each possibility illuminates certain mathematical structures underlying quantum field theories.

In this report we will take the tree-level scattering amplitudes in pure Yang-Mills theories as our reference object and explore in a pedagogical way the mathematical structures associated to it. A prominent special case are tree-level gluon amplitudes of QCD, corresponding to the choice $G = \text{SU}(3)$ as gauge group. Originally, the study of these amplitudes was motivated by finding efficient methods to compute multi-parton amplitudes, where the multiplicity ranges from 4 (corresponding to a $2 \rightarrow 2$ scattering process) to roughly 10 (corresponding to a $2 \rightarrow 8$ scattering process). This is relevant for hadron collider experiments. In particular at the LHC gluon-initiated processes constitute the bulk of scattering events. Although we focus on tree-level amplitudes in this review, many of the techniques discussed in this report have an extension towards higher orders in perturbation theory. This is useful for precision calculations. Finally, let me mention that the study of scattering amplitudes has developed in recent years in a field of its own, revealing hidden structures of quantum field theory which are not manifest in the conventional Lagrangian formulation.

There are other excellent review articles and lecture notes, covering some of the topics presented here, for example [1–4]. Furthermore the content of this report is not meant to cover the field completely. More on the contrary, by focusing solely on the tree-level amplitudes in Yang-Mills theory we try to explain the basic principles and leave advanced topics to the current research literature. Not included in this report are for example the extension of pure Yang-Mills theory to QCD, i.e. the inclusion of fermions in the fundamental representation of the gauge group, massless or massive. Furthermore, not included is the important field of loop calculations, this field alone would fill another report. Neither covered in detail are supersymmetric extensions of Yang-Mills theory, the most popular example of the latter would be $\mathcal{N} = 4$ supersymmetric Yang-Mills theory.

We will start with a review of the textbook method based on Feynman diagrams in section (2). Feynman diagrams allows us in principle to compute a tree-level amplitude for any number $n$ of external particles. However, in practice this method is highly inefficient.

In section (3) we will discuss efficient methods for the computation of tree-level scattering amplitudes in Yang-Mills theory. We will learn about colour decomposition, the spinor helicity method and off-shell recurrence relations. The combination of these three tools constitutes one
of the best (probably the best) methods to compute tree-level scattering amplitudes numerically. In section (3) we treat in addition the topics of colour-kinematics duality and twistors. These two topics are not directly targeted at efficiency. However, colour-kinematics duality is closely related to colour decomposition and twistors are closely related to the spinor helicity method. Therefore these topics are best discussed together.

Section (4) is centred around maximally helicity violating amplitudes. We present the Parke-Taylor formulæ and show that an arbitrary tree-level Yang-Mills amplitude may be computed from an effective scalar theory with an infinite tower of vertices, where the new interaction vertices are given by the Parke-Taylor formulæ.

In section (5) we present the on-shell recursion relations. On-shell recursion relations are one of the best (probably the best) methods to compute tree-level scattering amplitudes analytically. In addition, they are a powerful tool to prove the correctness of new representations of tree-level scattering amplitudes in Yang-Mills theory.

Section (6) is devoted to a geometric interpretation of a scattering amplitude as the volume of a generalised polytope (the amplituhedron) in a certain auxiliary space. We discuss Grassmannians and the representation of scattering amplitude in this formalism.

In section (7) we will learn about the Cachazo-He-Yuan (CHY) representation. Again, we look at some auxiliary space and associate to each external particle a variable \( z_j \in \mathbb{CP}^1 \). Of particular interest are the values of the \( z_j \)'s, which are solutions of the so-called scattering equations. We will present the scattering equations and show that the scattering amplitude can be expressed as a global residue at the zeros of the scattering equations.

In section (8) we explore tree-level amplitudes in perturbative quantum gravity. This may at first sight seem to be disconnected from the rest of this review, but we will see that there is a close relation between scattering amplitudes in Yang-Mills theory and gravity. The link is either provided by colour-kinematics duality, the CHY representation or the Kawai-Lewellen-Tye (KLT) relations. The latter are introduced in this section.

Finally, section (9) gives an outlook.

Let me add a word on the notation used in this review: Although it is desirable to have a unique notation throughout these notes, with the variety of topics treated in this course we face the challenge that one topic is best presented in one notation, while another topic is more clearly exposed in another notation. An example is given by a central quantity of this report, the tree-level primitive helicity amplitudes. These amplitudes depend on a set of external momenta \( p = (p_1, \ldots, p_n) \), a description of the spin states of all external particles, either given through a set of helicities \( (\lambda_1, \ldots, \lambda_n) \) or through a list of polarisation vectors \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \). Finally, these amplitudes depend as well on an external cyclic order, which may be specified by a permutation \( \sigma \in S_n \). Depending on the context we denote these amplitudes either by

\[
A_n^{(0)} \left( \lambda_{\sigma_1}, \ldots, \lambda_{\sigma_n} \right) \quad \text{or by} \quad A_n^{(0)} (\sigma, p, \varepsilon).
\]  

A second example is given by the notation used for Weyl spinors. Also here one finds in the literature various notations, either with dotted/undotted indices \( p_A, p_A \), as bra/ket-spinors \( \langle p \mid, \mid p \rangle \) or in a notation with square/angle brackets \( [p], \langle p \rangle \). Also here we introduce all commonly used notations, giving the reader a dictionary to translate between the various notations.
This review grew out of lectures given at the Saalburg summer school 2016. As such it was natural to include several exercises in the main text, with solutions to all exercises provided in the appendix. Students reading this report are encouraged to do these exercises. The structure of exercises with solutions in the appendix is kept in this report for the following reason: Many of the exercises provide detailed proofs of statements made in the main text. Leaving them out is not an option, as it will remove essential information for understanding the main concepts. Putting these detailed derivations into the appendix has the advantage of not disrupting the argumentation line in the main text.
2 The textbook method

We start our exploration of Yang-Mills theory with a short review of topics which can be found in modern textbooks. The presentation will be brief and concise, assuming that the reader is already familiar with the major part of the material presented in this section. Any gaps can be closed by consulting a textbook on quantum field theory. There are many excellent textbooks on quantum field theory, as an example let me just mention the books by Peskin and Schroeder [5], Srednicki [6] or Schwartz [7]. Topics covered in this section include the Lagrangian description of Yang-Mills theory, gauge-fixing and the computation of amplitudes through Feynman diagrams.

2.1 The Lagrangian of Yang-Mills theory

We start with the definition of Yang-Mills theory [8] in the notation usually used in physics textbooks. This is almost a synonym for “a notation with a lot of indices”. Our convention for the Minkowski metric is

$$g_{\mu\nu} = \text{diag}(1,-1,-1,-1).$$

(2)

Let $G$ be a Lie group, $\mathfrak{g}$ its Lie algebra and $T^a$ the generators of the Lie algebra where the index $a$ takes values from 1 to dim $G$. We use the conventions

$$[T^a, T^b] = if^{abc}T^c, \quad \text{Tr}(T^a T^b) = \frac{1}{2} \delta^{ab}.$$  

(3)

Yang-Mills theory consists of a gauge field $A^a_\mu(x)$. The action is given as usual by the integral

$$S = \int d^4x \mathcal{L}_{YM}.$$  

(4)

The Lagrange density for the gauge field reads

$$\mathcal{L}_{YM} = -\frac{1}{4} F^{a\mu\nu} F_{a\mu\nu},$$

(5)

where repeated indices are summed over and the field strength is given by

$$F^{a\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu.$$  

(6)

The coupling of Yang-Mills theory is denoted by $g$. The Lagrange density is invariant under the local transformations

$$T^a A^a_\mu(x) \rightarrow U(x) \left( T^a A^a_\mu(x) + \frac{i}{g} \partial_\mu \right) U^\dagger(x),$$  

(7)

with

$$U(x) = \exp\left( -iT^a \theta_a(x) \right).$$  

(8)
For later purpose let us define the **covariant derivative in the fundamental representation** by

\[
D_\mu = \partial_\mu - igT^a A^a_\mu,
\]

where we suppressed the indices \(i\) and \(j\) in \(T^a_{ij}\) and the Kronecker symbol \(\delta_{ij}\) accompanying \(\partial_\mu\).

The **covariant derivative in the adjoint representation** is given by

\[
D^{ab}_\mu = g^{ab} \partial_\mu - g f^{abc} A^c_\mu.
\]

### 2.2 Yang-Mills theory in terms of differential forms

Let us now phrase Yang-Mills theory in the language of differential geometry. Readers not familiar with differential forms, fibre bundles or the Hodge \(*\)-operator may either skip this paragraph or look up the basic concepts and definitions in a textbook. The books by Nakahara [9] or Isham [10] can be recommended for this purpose. The rest of this report will not depend on this sub-section. However, since we are interested in mathematical structures associated to Yang-Mills theory, the geometric fibre bundle description should not be omitted in this review.

We consider a **principal bundle** \(P(M, G)\) with a **connection one-form** \(\omega\). The base space \(M\) is the flat Minkowski space, the fibre consists of the gauge group \(G\). A connection one-form \(\omega\) is a Lie-algebra valued one-form on the total space \(P\), satisfying two requirements. We may identify a vector \(Y\) in the Lie algebra \(g\) of the Lie group \(G\) with a tangent vector \(Y\) at a point \(p\) in the total space \(P\). With this isomorphism the first requirement reads

\[
\omega_p(Y) = Y
\]

and states that the connection one-form \(\omega\) is a projection on the vertical sub-space at \(p\). There is a right action of an element \(g \in G\) on a point \(p \in P\). Within a local trivialisation a point \(p\) in the total space is given by \(p = (x, g')\) and the right action by \(g\) is given by \(pg = (x, g'g)\). The right action by \(g\) maps the point \(p\) to another point \(pg\) on the same fibre. The second requirement for the connection one-form \(\omega\) relates the evaluation of \(\omega\) on tangent vectors at different points \(p\) and \(pg\) on the same fibre:

\[
\omega_{pg}(R_{g*}Y) = g^{-1}\omega_p(Y)g,
\]

where \(R_{g*}Y\) denotes the push-forward of the tangent vector \(Y\) at the point \(p\) to the point \(pg\) by the right action of \(g\).

Let \(\sigma : M \to P\) be a (global) section and denote by \(A\) the pull-back of \(\omega\) to \(M\):

\[
A = \sigma^* \omega.
\]

In this report we are not interested in topologically non-trivial configurations, therefore a trivial fibre bundle and a global section are fine for us. The concepts of this paragraph are easily extended to the general case by introducing an atlas of coordinate patches and local sections glued together in the appropriate way. We use the notation

\[
A = A_\mu dx^\mu = \frac{g}{l} T^a A^a_\mu dx^\mu, \quad A_\mu = \frac{g}{l} T^a A^a_\mu,
\]
where $A^a_{\mu}$ is the gauge field already encountered in the previous paragraph. Let us now consider two sections $\sigma_1$ and $\sigma_2$, with associated pull-backs $A_1$ and $A_2$. We can always relate the two sections $\sigma_1$ and $\sigma_2$ by

$$\sigma_1(x) = \sigma_2(x)U(x),$$

(15)

where $U(x)$ is a $x$-dependent element of the Lie group $G$. Then we obtain for the pull-backs $A_1$ and $A_2$ of the connection one-form the relation

$$A_2 = UA_1U^\dagger + UdU^\dagger,$$

(16)

This is nothing else than a gauge transformation.

Exercise 1: Derive eq. (16) from eq. (15).

Hint: Recall that the action of the pull-back $A_2$ on a tangent vector is defined as the action of the original form $\omega$ on the push-forward of the tangent vector. Recall further that a tangent vector at a point $x$ can be given as a tangent vector to a curve through $x$. It is sufficient to show that the actions of $A_2$ and $UA_1U^\dagger + UdU^\dagger$ on an arbitrary tangent vector give the same result. In order to prove the claim you will need in addition the defining relations for the connection one-form $\omega$, given in eq. (11) and eq. (12).

The connection one-form defines the covariant derivative

$$D_A = d + A = d + A_\mu dx^\mu = d + \left(\frac{R}{2}\right) T^a A^a_\mu dx^\mu = d - igT^a A^a_\mu dx^\mu.$$  

(17)

If $A$ is a $g$-valued $p$-form and $B$ is a $g$-valued $q$-form, the commutator of the two is defined by

$$[A,B] = A \wedge B - (-1)^{pq} B \wedge A,$$

(18)

the factor $(-1)^{pq}$ takes into account that we have to permute $p$ differentials $dx^\mu$ from $A$ past $q$ differentials $dx^\nu$ from $B$. If $A$ and $B$ are both one-forms we have

$$[A,B] = A \wedge B + B \wedge A = [A_\mu, B_\nu] \, dx^\mu \wedge dx^\nu.$$  

(19)

In particular

$$A \wedge A = \frac{1}{2} [A,A] = \frac{1}{2} [A_\mu, A_\nu] \, dx^\mu \wedge dx^\nu.$$  

(20)

We define the curvature two-form of the fibre bundle by

$$F = D_A A = dA + A \wedge A = dA + \frac{1}{2} [A,A].$$

(21)

This definition is in close analogy with the definition of the Riemannian curvature tensor. Also the Riemannian curvature tensor can be calculated through the covariant derivative of the affine connection. We have

$$dA = \left(\frac{R}{2}\right) T^a A^a_\mu dx^\mu \wedge dx^\nu = \frac{1}{2} \left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) dx^\mu \wedge dx^\nu,$$

$$A \wedge A = \frac{1}{2} [A_\mu, A_\nu] \, dx^\mu \wedge dx^\nu.$$  

(22)
and therefore

\[ F = \frac{1}{2} \left( \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \right) dx^\mu \wedge dx^\nu = \frac{1}{2} F_{\mu \nu} dx^\mu \wedge dx^\nu. \]  

(23)

With the notation

\[ F_{\mu \nu} = \frac{g}{i} T^a F^a_{\mu \nu} \]  

(24)

we have

\[ F = \frac{1}{2} F_{\mu \nu} dx^\mu \wedge dx^\nu = \frac{1}{2} \frac{g}{i} T^a F^a_{\mu \nu} dx^\mu \wedge dx^\nu, \]  

(25)

where

\[ F^a_{\mu \nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A_b^\mu A^c_\nu, \]  

(26)

in agreement with the previous notation.

Exercise 2: Under a gauge transformation the pull-back \( A \) of the connection one-form transforms as

\[ A \rightarrow U A U^\dagger + U d U^\dagger. \]  

(27)

Show that the curvature two-form transforms as

\[ F \rightarrow U F U^\dagger. \]  

(28)

In this context we also introduce the dual field strength

\[ \tilde{F}_{\mu \nu} = \frac{1}{2} \varepsilon_{\mu \nu \rho \sigma} F^{\rho \sigma}, \quad F_{\mu \nu} = -\frac{1}{2} \varepsilon_{\mu \nu \rho \sigma} \tilde{F}^{\rho \sigma}, \]  

(29)

or equivalently, the dual field strength two-form

\[ \star F = \star \left( \frac{1}{2} F_{\mu \nu} dx^\mu \wedge dx^\nu \right) = \frac{1}{4} \varepsilon_{\mu \nu \rho \sigma} F^{\mu \nu} dx^\rho \wedge dx^\sigma = \frac{1}{2} \tilde{F}_{\mu \nu} dx^\mu \wedge dx^\nu. \]  

(30)

In Minkowski space the Hodge \( \star \)-operator applied to a two-form satisfies

\[ \star \star F = -F, \]  

(31)

therefore the eigenvalues are \( \pm i \). We call a two-form self dual, respectively anti-self dual, if

self dual : \( F = i \star F \),

anti-self dual : \( F = -i \star F \).  

(32)
Equivalently we have in the notation with indices

\[
\begin{align*}
\text{self dual} & : F_{\mu\nu} = i\tilde{F}_{\mu\nu}, \\
\text{anti-self dual} & : F_{\mu\nu} = -i\tilde{F}_{\mu\nu}.
\end{align*}
\]

Then the action can be written as

\[
S = -\frac{1}{4} \int d^4x \, F_{\mu\nu}^2 = \frac{1}{2g^2} \int d^4x \, \text{Tr} F_{\mu\nu} F_{\mu\nu} = \frac{1}{g^2} \int \text{Tr} \, F \wedge \ast F.
\]

With the Hodge inner product of two \(p\)-forms \(\eta_1\) and \(\eta_2\)

\[(\eta_1, \eta_2) = \int \eta_1 \wedge \ast \eta_2\]

the action can be written as

\[
S = \frac{1}{g^2} \text{Tr} \, (F, F).
\]

Let us summarise: The gauge potential corresponds to the connection of the fibre bundle, the field strength to the curvature of the fibre bundle.

### 2.3 Gauge fixing

Consider the path integral

\[
\int \mathcal{D}A_\mu^a(x) \exp \left( i \int d^4x \, \mathcal{L}_{YM} \right).
\]

The path integral is over all possible gauge field configurations, including ones which are just related by a gauge transformation. We will encounter similar situations later on in this report and we will therefore discuss this issue in more detail. Gauge-equivalent configuration describe the same physics and it is sufficient to count them only ones. Technically this is done as follows: Let us denote a gauge transformation by

\[
U(x) = \exp \left( -iT^b \theta_b(x) \right).
\]

The gauge transformation is therefore completely specified by the functions \(\theta_b(x)\). We denote by \(A_\mu^a(x, \theta_b)\) the gauge field configuration obtained from \(A_\mu^a(x)\) through the gauge transformation \(U(x)\):

\[
T^a A_\mu^a(x, \theta_b) = U(x) \left( T^a A_\mu^a(x) + \frac{i}{g} \partial_\mu \theta \right) U^\dagger(x),
\]

\(A_\mu^a(x, \theta_b)\) and \(A_\mu^a(x)\) are therefore \textbf{gauge-equivalent configurations}. From all gauge-equivalent configurations we are going to pick the one, which satisfies for a given \(G^\mu\) and \(B^a(x)\) the equation

\[
G^\mu A_\mu^a(x, \theta_b) = B^a(x).
\]
Let us assume that this equation gives a unique solution $\theta_b$ for a given $A^a$. (This is not necessarily always fulfilled, cases where a unique solution may not exist are known as the Gribov ambiguity \[11\]) We insert the identity \[12\]

$$
\int \prod_b \mathcal{D}\theta_b(x) \delta^n \left( G^\mu A^a_\mu(x, \theta_b(x)) - B^a(x) \right) \det M_G = 1 \tag{41}
$$

where

$$
(M_G(x,y))_{ab} = \frac{\delta G^\mu A^a_\mu(x, \theta_c(x))}{\delta \theta_b(y)} \tag{42}
$$

and $n$ is the number of generators of the Lie algebra $\mathfrak{g}$. The functional derivative is defined by

$$
\frac{\delta}{\delta \theta_b(y)} F(\theta_\alpha(x)) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[ F(\theta_\alpha(x) + \varepsilon \delta \theta_b \delta^4(x-y)) - F(\theta_\alpha(x)) \right]. \tag{43}
$$

**Exercise 3**: Prove the equivalent of eq. (41) in the finite dimensional case. Let $\vec{\alpha} = (\alpha_1, ..., \alpha_n)$ be a $n$-dimensional vector and let

$$
g_i = g_i(\alpha_1, ..., \alpha_n), \ i = 1, ..., n, \tag{44}
$$

be $n$ functions of the $n$ variables $\alpha_j$. Show that

$$
\int \left( \prod_{j=1}^n d\alpha_j \right) \left( \prod_{i=1}^n \delta(g_i(\alpha_1, ..., \alpha_n)) \right) \det \left( \frac{\partial g_i}{\partial \alpha_j} \right) = 1. \tag{45}
$$

From the finite gauge transformation

$$
(T^a A^a_\mu(x))^f = U(x) \left( T^a A^a_\mu(x) + i \frac{g}{\theta} \partial_\mu \delta^4(x-y) \right) U^\dagger(x), \tag{46}
$$

we obtain the infinitesimal version

$$
A^a_\mu' = A^a_\mu - f^{abc} A^b_\mu \theta^c - \frac{1}{g} \theta^a \partial_\mu. \tag{47}
$$

Therefore:

$$
\frac{\delta}{\delta \theta^a_\mu(y)} G^\mu A^a_\mu' = G^\mu \left[ f^{abc} \theta^c \delta^4(x-y) - \frac{1}{g} \delta^ab \partial_\mu \delta^4(x-y) \right] = -\frac{1}{g} G^\mu D^a_\mu \delta^4(x-y), \tag{48}
$$

where $D^a_\mu$ denotes the covariant derivative in the adjoint representation, defined in eq. (10). We now consider

$$
Z[0] = \int \mathcal{D}A^a_\mu(x) \exp \left[ i \int d^4x \mathcal{L}_{YM}(A^a_\mu(x)) \right] \tag{49}
$$
and insert eq. (41). Using the gauge invariance of the action and of the measure $\mathcal{D}A^a_\mu(x)$ one arrives at

$$\begin{align*}
Z[0] &= \left( \int \prod_b \mathcal{D}_b(x) \right) \\
&\times \int \mathcal{D}A^a_\mu(x) \det M_G(A^a_\mu(x)) \delta^n \left( G^\mu A^a_\mu(x) - B^a(x) \right) \exp \left[ i \int d^4x \mathcal{L}_{YM}(A^a_\mu(x)) \right].
\end{align*}$$

(50)

The integral over all gauge-transformations

$$\int \prod_b \mathcal{D}_b(x)$$

is just an irrelevant prefactor, which may be neglected. We then obtain

$$\begin{align*}
Z[0] &= \int \mathcal{D}A^a_\mu(x) \det M_G(A^a_\mu(x)) \delta^n \left( G^\mu A^a_\mu(x) - B^a(x) \right) \exp \left[ i \int d^4x \mathcal{L}_{YM}(A^a_\mu(x)) \right].
\end{align*}$$

(52)

This functional still depends on $B^a(x)$. As we are not interested in any particular choice of $B^a(x)$, we average over $B^a(x)$ with weight

$$\exp \left( -\frac{i}{2\xi} \int d^4x B^a(x) B_a(x) \right)$$

and obtain

$$\begin{align*}
\int \mathcal{D}B^a(x) Z[0] \exp \left( -\frac{i}{2\xi} \int d^4x B^a(x) B_a(x) \right) &= \\
= \int \mathcal{D}A^a_\mu(x) \det M_G \exp \left[ i \int d^4x \mathcal{L}_{YM}(A^a_\mu(x)) \right] \left( G^\mu A^a_\mu(x) \right) \left( G^\nu A^a_\nu(x) \right) .
\end{align*}$$

(54)

The determinant may be exponentiated with the help of Grassmann-valued ghost fields $\bar{c}^a$ and $c^b$. Ignoring a factor $1/g$ one obtains:

$$\det M_G = \int \mathcal{D}c^b(x) \mathcal{D}\bar{c}^a(x) \exp \left( i \int d^4x \bar{c}^a(x) \left( -G^\mu D^ab \right) c^b(x) \right).$$

(55)

Up to now we did not specify the quantity $G^\mu$. Different choices for $G^\mu$ correspond to different gauges. A popular gauge is Lorenz gauge, corresponding to

$$G^\mu = \partial^\mu.$$  

(56)

In Lorenz gauge the **gauge-fixing term** reads

$$\mathcal{L}_{GF} = -\frac{1}{2\xi} \left( \partial^\mu A^a_\mu \right) \left( \partial^\nu A^a_\nu \right) ,$$

(57)

the **Faddeev-Popov term** reads

$$\mathcal{L}_{FP} = -\epsilon^a \partial^\mu D^ab c^b.$$  

(58)

The Faddeev-Popov term contributes to loop amplitudes. In this report we are mainly concerned with tree amplitudes. For tree amplitudes we will only need the gauge-fixing term.
2.4 Feynman rules

With the inclusion of the gauge-fixing term and the Faddeev-Popov term we now have the effective Lagrange density

\[ \mathcal{L} = \mathcal{L}_{\text{YM}} + \mathcal{L}_{\text{GF}} + \mathcal{L}_{\text{FP}}. \]  

(59)

From the Lagrangian one obtains the Feynman rules. This is explained in detail in many textbooks of quantum field theory and we present here only the final cooking recipe. We first order the terms in the Lagrangian according to the number of fields they involve. From the terms bilinear in the fields one obtains the propagators, while the terms with three or more fields give rise to vertices. Note that a “normal” Lagrangian does not contain terms with just one or zero fields. Furthermore we always assume within perturbation theory that all fields fall off rapidly enough at infinity. Therefore we can use partial integration and ignore boundary terms. For the gauge-fixed Yang-Mills Lagrangian one obtains terms with two, three and four gauge fields plus the Faddeev-Popov term.

\[ \mathcal{L} = \frac{1}{2} A_{\mu}^{a}(x) \left[ \partial_\rho g^{\rho\nu} \delta^{ab} - \left( 1 - \frac{1}{\xi} \right) \partial^\mu \partial^\nu \delta^{ab} \right] A_\nu^{b}(x) \]
\[-g f^{abc} (\partial_\mu A_\nu^{a}(x)) A_\mu^{b}(x) A_\nu^{c}(x) - \frac{1}{4} g^2 f^{eab} f^{ecd} A_\mu^{a}(x) A_\nu^{b}(x) A_\mu^{c}(x) A_\nu^{d}(x) \]
\[+ \mathcal{L}_{\text{FP}}. \]  

(60)

Let’s see how this works out for the gluon propagator: The first line of eq. (60) gives the terms bilinear in the gluon fields. This defines an operator

\[ \sum_j P_{ij}(x) P_{jk}^{-1}(x-y) = \delta_{ik} \delta^4(x-y), \]  

(62)

and its Fourier transform by

\[ P_{ij}^{-1}(x) = \int \frac{d^4 p}{(2\pi)^4} e^{-i p x} \tilde{P}_{ij}^{-1}(p). \]  

(63)

Then the propagator is given by

\[ \Delta_F(p)_{ij} = i \tilde{P}_{ij}^{-1}(p). \]  

(64)

Let’s see how this works out for the gluon propagator: The first line of eq. (60) gives the terms bilinear in the gluon fields. This defines an operator

\[ P^{\mu\nu \, ab}(x) = \partial_\rho g^{\mu\nu} \delta^{ab} - \left( 1 - \frac{1}{\xi} \right) \partial^\mu \partial^\nu \delta^{ab}. \]  

(65)
For the propagator we are interested in the inverse of this operator

\[ \mathcal{P}^{\mu\sigma ac}(x) (\mathcal{P}^{-1})^{cb}_{\nu} (x-y) = g^\mu_\nu \delta^{ab} \delta^4(x-y). \]  

(66)

Working in momentum space we are more specifically interested in the Fourier transform of the inverse of this operator:

\[ (\mathcal{P}^{-1})^{ab}_{\mu\nu}(x) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} (\tilde{\mathcal{P}}^{-1})^{ab}_{\mu\nu}(p). \]  

(67)

The Feynman rule for the propagator is then given by \((\tilde{\mathcal{P}}^{-1})^{ab}_{\mu\nu}(p)\) times the imaginary unit. For the gluon propagator one finds the Feynman rule

\[ \mu, a \quad \quad \nu, b \quad = \quad \frac{i}{p^2} \left( -g_{\mu\nu} + (1 - \xi) \frac{p_{\mu} p_{\nu}}{p^2} \right) \delta^{ab}. \]  

(68)

**Exercise 4:** Derive eq. (68) from eq. (66) and eq. (67).

**Hint:** It is simpler to work directly in momentum space, using the Fourier representation of \(\delta^4(x-y)\).

Let us now consider a generic interaction term with \(n \geq 3\) fields. We may write this term as

\[ \mathcal{L}_{\text{int}}(x) = O_{i_1...i_n} (\partial_1, ..., \partial_n) \phi_{i_1}(x) ... \phi_{i_n}(x), \]  

(69)

with the notation that \(\partial_j\) acts only on the \(j\)-th field \(\phi_{i_j}(x)\). For each field we have the Fourier transform

\[ \phi_i(x) = \int \frac{d^Dp}{(2\pi)^D} e^{-ip\cdot x} \tilde{\phi}_i(p), \quad \tilde{\phi}_i(p) = \int d^Dx e^{ip\cdot x} \phi_i(x), \]  

(70)

where \(p\) denotes an in-coming momentum. We thus have

\[ \mathcal{L}_{\text{int}}(x) = \int \frac{d^Dp_1}{(2\pi)^D} ... \frac{d^Dp_n}{(2\pi)^D} e^{-i(p_1+...+p_n)x} O_{i_1...i_n} (-ip_1, ..., -ip_n) \tilde{\phi}_i(p_1) ... \tilde{\phi}_i(p_n). \]  

(71)

Changing to outgoing momenta we replace \(p_j\) by \(-p_j\). The vertex is then given by

\[ V = i \sum_{\text{permutations}} (-1)^{p_F} O_{i_1...i_n} (ip_1, ..., ip_n), \]  

(72)

where the momenta are taken to flow outward. The summation is over all permutations of indices and momenta of identical particles. In the case of identical fermions there is in addition a minus sign for every odd permutation of the fermions, indicated by \((-1)^{p_F}\).

Let us also work out an example here. We look as an example at the first term in the second line of eq. (60):

\[ \mathcal{L}_{ggg} = -gf^{abc} (\partial_\mu A^a_\nu(x)) A^b_\mu(x) A^c_\nu(x). \]  

(73)
This term contains three gluon fields and will give rise to the three-gluon vertex. We may rewrite this term as

$$\mathcal{L}_{ggg} = -gf^{abc}g^{\mu \rho} \partial_1^\nu A^a_\mu(x) A^b_\nu(x) A^c_\rho(x).\quad (74)$$

Thus

$$O^{abc, \mu \nu \rho}(\partial_1, \partial_2, \partial_3) = -gf^{abc}g^{\mu \rho} \partial_1^\nu, \quad O^{abc, \mu \nu \rho}(ip_1, ip_2, ip_3) = -gf^{abc}g^{\mu \rho} ip_1^\nu. \quad (75)$$

The Feynman rule for the vertex is given by the sum over all permutations of identical particles of the function $O^{abc, \mu \nu \rho}(ip_1, ip_2, ip_3)$ multiplied by the imaginary unit $i$. For the case at hand, we have three identical gluons and we have to sum over $3! = 6$ permutations. One finds

$$V_{ggg} = i \sum_{\text{permutations}} (-gf^{abc}g^{\mu \rho} ip_1^\nu) = -gf^{abc} [g^{\mu \nu} (p_1^\rho - p_2^\rho) + g^{\nu \rho} (p_2^\mu - p_3^\mu) + g^{\rho \mu} (p_3^\nu - p_1^\nu)]. \quad (76)$$

Note that we have momentum conservation at each vertex, for the three-gluon vertex this implies

$$p_1 + p_2 + p_3 = 0. \quad (77)$$

In a similar way one obtains the Feynman rules for the four-gluon vertex and the ghost-antighost-gluon vertex.

Let us summarise the Feynman rules for the propagators and the vertices of Yang-Mills theory: The gluon propagator (in Feynman gauge, corresponding to $\xi = 1$) and the ghost propagator are given by

$$\begin{align*}
\mu, a & \quad \begin{array}{cccc}
\circ & \circ & \circ & \circ & \circ
\end{array} & \nu, b & = \frac{-ig^{\mu \nu}}{p^2} \delta^{ab}, \\
a & \quad \begin{array}{ccc}
\circ & \circ & \circ & \circ & \circ
\end{array} & b & = \frac{i}{p^2} \delta^{ab}.
\end{align*} \quad (78)$$

The Feynman rules for the vertices are

$$\begin{align*}
p_1, \mu, a & \quad \begin{array}{cc}
\circ & \circ \circ \circ \circ \circ
\end{array} p_2, \nu, b & = g \left( if^{abc} \right) i [g^{\mu \nu} (p_1^\rho - p_2^\rho) + g^{\nu \rho} (p_2^\mu - p_3^\mu) + g^{\rho \mu} (p_3^\nu - p_1^\nu)], \\
p_3, \rho, c & \quad \begin{array}{ccc}
\circ & \circ & \circ \circ \circ \circ \circ
\end{array} p_2, \nu, b & = ig^2 \left[ \left( if^{abe} \right) \left( if^{cde} \right) (g^{\mu \rho} g^{\nu \sigma} - g^{\nu \rho} g^{\mu \sigma}) + \left( if^{bce} \right) \left( if^{ead} \right) (g^{\nu \rho} g^{\sigma \rho} - g^{\rho \mu} g^{\nu \sigma}) + \left( if^{cae} \right) \left( if^{ebd} \right) (g^{\rho \nu} g^{\mu \sigma} - g^{\rho \mu} g^{\nu \sigma}) \right],
\end{align*}$$

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In order to compute tree-level amplitudes will still need a rule for external particles. A gluon has
two spin states, which we may take as the two helicity states $+ \text{ and } -$. These are described by
polarisation vectors $\epsilon^\pm_{\mu}(p)$, where $p$ is the momentum of the gluon. The polarisation vectors
satisfy
\[
\epsilon^\pm_{\mu}(p) p^\mu = 0,
\]
and
\[
\epsilon^+ \cdot (\epsilon^+) = \epsilon^- \cdot (\epsilon^-) = -1, \quad \epsilon^+ \cdot (\epsilon^-) = 0.
\]
The Feynman rule for an external gluon is simply to include a polarisation vector $\epsilon^\pm_{\mu}(p)$.
These are all the Feynman rules we will need to compute tree-level amplitudes in Yang-Mills theory. For completeness let me mention that there are a few additional rules, relevant to amplitudes with loops and/or fermions. These additional rules are:
- There is an integration
\[
\int \frac{d^4 p}{(2\pi)^4}
\]
for each internal momentum not constrained by momentum conservation. Such an integration is called a “loop integration” and the number of independent loop integrations in a diagram is called the loop number of the diagram.
- A factor $(-1)$ for each closed fermion loop.
- Each diagram is multiplied by a factor $1/S$, where $S$ is the order of the permutation group of the internal lines and vertices leaving the diagram unchanged when the external lines are fixed.

2.5 Amplitudes and cross sections

Scattering amplitudes with $n$ external particles may be calculated within perturbation theory, assuming that the coupling $g$ is a small parameter. We write
\[
\mathcal{A}_n = \mathcal{A}^{(0)}_n + \mathcal{A}^{(1)}_n + \mathcal{A}^{(2)}_n + \mathcal{A}^{(3)}_n + \ldots,
\]
where $\mathcal{A}^{(l)}_n$ contains $(n - 2 + 2l)$ factors of $g$. In this expansion, $\mathcal{A}^{(l)}_n$ is an amplitude with $n$
external particles and $l$ loops. The recipe for the computation of $\mathcal{A}^{(l)}_n$ based on Feynman diagrams
is as follows:
Figure 1: The four Feynman diagrams contributing to the tree-level four-gluon amplitude $A_4^{(0)}$.

Algorithm 1. Calculation of scattering amplitudes from Feynman diagrams.

1. Draw all Feynman diagrams for the given number of external particles $n$ and the given number of loops $l$.

2. Translate each graph into a mathematical formula with the help of the Feynman rules.

3. The amplitude $A_n^{(l)}$ is then given as the sum of all these terms.

Tree-level amplitudes are amplitudes with no loops and are denoted by $A_n^{(0)}$. They give the leading contribution to the full amplitude. The computation of tree-level amplitudes involves only basic mathematical operations: Addition, multiplication, contraction of indices, etc. The above algorithm allows therefore in principle for any $n$ the computation of the corresponding tree-level amplitude. The situation is different for loop amplitudes $A_n^{(l)}$ (with $l \geq 1$). Here, the Feynman rules involve an integration over each internal momentum not constrained by momentum conservation. This constitutes an additional challenge, which we will not touch in this report. The interested reader is referred to textbooks [13] and review articles [14–17]. Let me stress that the calculation of loop amplitudes is important for precision predictions in particle physics.

Let us consider an example for the calculation of a tree-level amplitude. To compute the tree-level four-gluon amplitude $A_4^{(0)}$ we have to evaluate the four Feynman diagrams shown in fig. (1).

Exercise 5: Compute the amplitude $A_4^{(0)}$ from the four diagrams shown in fig. (1). Assume that all momenta are outgoing. The result will involve scalar products $2p_i \cdot p_j, 2p_i \cdot \epsilon_j$ and $2\epsilon_i \cdot \epsilon_j$ For a $2 \to 2$ process (more precisely for a $0 \to 4$ process, since we take all momenta to be outgoing), the Mandelstam variables are defined by

$$s = (p_1 + p_2)^2, \quad t = (p_2 + p_3)^2, \quad u = (p_1 + p_3)^2.$$  \hspace{1cm} (84)

The four momenta $p_1, p_2, p_3$ and $p_4$ are on-shell, $p_i^2 = 0$ for $i = 1, \ldots, 4$, and satisfy momentum conservation. Derive the Mandelstam relation

$$s + t + u = 0.$$ \hspace{1cm} (85)

This relation allows to eliminate in the result for $A_4^{(0)}$ one variable, say $u$. Furthermore we have from eq. (87) the relation $2p_i \cdot \epsilon_j = 0$. Combined with momentum conservation we may eliminate several scalar products $2p_i \cdot \epsilon_j$, such that for a given $j$ we only have $2p_{j-1} \cdot \epsilon_j$ and $2p_{j+1} \cdot \epsilon_j$, where the indices $(j-1)$
and \((j + 1)\) are understood modulo 4. You might want to use a computer algebra system to carry out the calculations. The open-source computer algebra systems FORM [18] and GiNaC [19] have their roots in particle physics and were originally invented for calculations of this type.

The scattering amplitude enters directly the calculation of a physical observable. Let us first consider the scattering process of two incoming elementary uncoloured spinless particles with four-momenta \(p'_a\) and \(p'_b\) and \((n - 2)\) outgoing particles with four-momenta \(p_1\) to \(p_{n-2}\). Let us further assume that we are interested in an observable \(O(p_1, ..., p_{n-2})\) which depends on the momenta of the outgoing particles. In general the observable depends on the experimental set-up and can be an arbitrary complicated function of the four-momenta. In the simplest case this function is just a constant equal to one, corresponding to the situation where we count every event with \((n - 2)\) particles in the final state. In more realistic situations one takes for example into account that it is not possible to detect particles close to the beam pipe. The function \(O\) would then be zero in these regions of phase space. Furthermore any experiment has a finite resolution. Therefore it will not be possible to detect particles which are very soft or which are very close in angle to other particles. We will therefore sum over the number of final state particles. In order to obtain finite results within perturbation theory we have to require that in the case where one or more particles become unresolved the value of the observable \(O\) has a continuous limit agreeing with the value of the observable for a configuration where the unresolved particles have been merged into “hard” (or resolved) pseudo-particles. Observables having this property are called infrared-safe observables. The expectation value for the observable \(O\) is given by

\[
\langle O \rangle = \frac{1}{2(p'_a + p'_b)^2} \sum_n \int d\phi_{n-2} O(p_1, ..., p_{n-2}) |A_n|^2,
\]

where \(1/(p'_a + p'_b)^2\) is a normalisation factor taking into account the incoming flux. The phase space measure is given by

\[
d\phi_n = \frac{1}{n!} \prod_{i=1}^n \frac{d^3p_i}{(2\pi)^3} \frac{1}{2E_i} 2(2\pi)^4 \delta^4 \left(p'_a + p'_b - \sum_{i=1}^n p_i\right). \tag{87}
\]

The quantity \(E_i\) is the energy of particle \(i\), for massless particles we have

\[
E_i = \sqrt{\not{p}_i^2} = p_i^0. \tag{88}
\]

We see that the expectation value of \(O\) is given by the phase space integral over the observable, weighted by the norm squared of the scattering amplitude. As the integrand can be a rather complicated function, the phase space integral is usually performed numerically by Monte Carlo integration.

Let us now look towards a more realistic theory relevant to LHC physics. As an example consider QCD consisting of quarks and gluons. Quarks and gluons are collectively called partons.
There are a few modifications to eq. (86). The master formula reads now

\[
\langle O \rangle = \sum_{f_a, f_b} \int dx_a f_{f_a}(x_a) \int dx_b f_{f_b}(x_b) \left( \frac{1}{2\hat{s}n_s(a)n_s(b)n_c(a)n_c(b)} \sum_n \int d\phi_{n-2} O(p_1, \ldots, p_{n-2}) \right) \sum_{\text{spins,colour}} |A_n|^2. 
\] (89)

The partons have internal degrees of freedom, given by the spin and the colour of the partons. In squaring the amplitude we sum over these degrees of freedom. For the particles in the initial state we would like to average over these degrees of freedom. This is done by dividing by the factors \(n_s(i)\) and \(n_c(i)\), giving the number of spin degrees of freedom (2 for quarks and gluons) and the number of colour degrees of freedom (3 for quarks, 8 for gluons). The second modification is due to the fact that the particles brought into collision are not partons, but composite particles like protons. At high energies the elementary constituents of the protons interact and we have to include a function \(f_{f_a}(x_a)\) giving us the probability of finding a parton of flavour \(f_a\) with momentum fraction \(x_a\) of the original proton momentum inside the proton. If the momenta of the incoming protons are \(P'_a\) and \(P'_b\), then the momenta of the two incoming partons are given by

\[
p'_a = x_a P'_a, \quad p'_b = x_b P'_b. 
\] (90)

\(\hat{s}\) is the centre-of-mass energy squared of the two partons entering the hard interaction. Neglecting particle masses we have

\[
\hat{s} = \left( p'_a + p'_b \right)^2 = x_a x_b \left( P'_a + P'_b \right)^2. 
\] (91)

In addition there is a small change in eq. (87). The quantity \((n!)\) is replaced by \((\prod n_j!\)\), where \(n_j\) is the number of times a parton of type \(j\) occurs in the final state.

It is very convenient to calculate the amplitude with the convention that all particles are outgoing. To this aim we set

\[
p_{n-1} = -p'_a, \quad p_n = -p'_b 
\] (92)

and calculate the amplitude for the momentum configuration

\[
\{p_1, \ldots, p_{n-2}, p_{n-1}, p_n\}. 
\] (93)

Momentum conservation reads

\[
p_1 + \ldots + p_{n-2} + p_{n-1} + p_n = 0. 
\] (94)

Note that the momenta \(p_{n-1}\) and \(p_n\) have negative energy components.

In eq. (89) the square of the amplitude, summed over spins and colours, enters. In order to perform the spin sum, we may use for each gluon the formula

\[
\sum_{\lambda \in \{+,-\}} \left( \epsilon^\lambda_{\mu}(p) \right)^* \epsilon^\lambda_{\nu}(p) = -g_{\mu\nu} + \frac{p_\mu p_\nu + n_\mu n_\nu}{p \cdot n} - n_\mu n_\nu \frac{p_\mu p_\nu}{(p \cdot n)^2}. 
\] (95)
Here $n^\mu$ is an arbitrary four vector. The dependence on $n^\mu$ cancels in gauge-invariant quantities. Efficient methods for the colour summation are discussed in section \ref{section:3.1}.

At LHC energies the incoming flux of gluons is rather high and the tree-level gluon amplitudes discussed in this review will give a significant contribution to the jet events observed at the LHC experiments.

\section{Singular limits of scattering amplitudes}

Let us take a small digression and discuss soft and collinear limits of tree-level gluon amplitudes. The scattering amplitudes show a universal factorisation behaviour in these limits. We will discuss single unresolved limits, corresponding to either one particle becoming soft or two particles becoming collinear. In this limit, the amplitude $A_{n+1}^{(0)}$ will not contribute to an $(n-1)$-particle observable, but to an $(n-2)$-particle observable. This gives a next-to-leading order contribution to an $(n-2)$-particle observable. Double unresolved limits and unresolved limits with even more unresolved partons contribute to higher orders in perturbation theory, as well as unresolved limits of loop amplitudes.

To make contact with the literature \cite{20,21} let us first introduce the colour charge operators $T_j$. The $n$-gluon amplitude $A_n$ has $n$ colour indices $a_j$, one for each gluon $j$. To make this explicit, let us write the amplitude temporarily as $A_{a_1 \ldots a_n}^{(0)}$. The colour charge operator $T_j$ acts on the gluon $j$ with colour index $a_j$ as

\begin{equation}
(T_j)^{b_1}_a \ldots ^{b_2}_a A_{a_1 \ldots a_j \ldots a_n}^{(0)} = \left( i f^{b_1 b_2}_j \right) A_{a_1 \ldots a_j \ldots a_n}^{a_1 \ldots a_j \ldots a_n}.
\end{equation}

In the sequel we will drop again the colour indices and simply write

\begin{equation}
T_j A_n = (T_j)^{b_1}_a \ldots ^{b_2}_a A_{a_1 \ldots a_j \ldots a_n}^{a_1 \ldots a_j \ldots a_n}.
\end{equation}

Colour conservation implies the relation

\begin{equation}
\sum_{j=1}^n T_j A_n = 0.
\end{equation}

Let us first consider the soft limit. We consider the case where particle $j$ becomes soft. In the soft limit we parametrise the momentum of the soft parton $p_j$ as

\begin{equation}
p_j = \lambda q
\end{equation}

and consider contributions to $|A_{n+1}^{(0)}|^2$ of the order $O(\lambda^{-2})$. Contributions to $|A_{n+1}^{(0)}|^2$ which are less singular than $\lambda^{-2}$ are integrable in the soft limit. In the soft limit a Born amplitude $A_{n+1}^{(0)}$ with $(n+1)$ partons behaves as

\begin{equation}
\lim_{p_j \to 0} A_{n+1}^{(0)} = g \epsilon_\mu(p_j) J^\mu A_{n}^{(0)}.
\end{equation}
The eikonal current is given by
\[ J^\mu = \sum_{i \neq j} T_i \frac{p_i^\mu}{p_i \cdot p_j} \quad (101) \]
The sum is over the remaining \( n \) hard momenta \( p_i \).

Let us now consider the collinear limit. We consider a splitting \( \bar{i} \to i + j \), where all particles are in the final state. (Formulæ for a collinear splitting involving initial-state particles can be worked out in a similar way.) In the collinear limit we parametrise the momenta of the two collinear final-state partons \( i \) and \( j \) as
\[ p_i = z p_{\bar{i}} + k_\perp - \frac{k_i^2}{2 p_{\bar{i}} \cdot n} n, \]
\[ p_j = (1 - z) p_{\bar{i}} - k_\perp - \frac{k_j^2}{1 - z} 2 p_{\bar{i}} \cdot n. \quad (102) \]

Here \( n \) is a massless four-vector and the transverse component \( k_\perp \) satisfies \( 2 p_{\bar{i}} \cdot k_\perp = 2 n \cdot k_\perp = 0 \). The four-vectors \( p_{\bar{i}}, p_i \) and \( p_j \) are on-shell:
\[ p_{\bar{i}}^2 = p_i^2 = p_j^2 = 0. \quad (103) \]

In the collinear limit we have to consider contributions to \( |A_n^{(0)}|^2 \) of the order \( \mathcal{O}(k_\perp^2) \). In this limit the Born amplitude factorises according to
\[ \lim_{p_i \parallel p_j} A_n^{(0)}(..., p_i, \lambda_i, ..., p_j, \lambda_j, ...) = g \sum_{\lambda_i} \text{Split}(p_{\bar{i}}, p_i, p_j, \lambda_i, \lambda_j) T_i A_n^{(0)}(..., p_i, \lambda_i, ...), \quad (104) \]
where the sum is over all polarisations of the intermediate particle. The variables \( \lambda_i \) and \( \lambda_j \) denote the polarisations of the particles \( i \) and \( j \), respectively. The splitting function \( \text{Split} \) is given by
\[ \text{Split}(p_{\bar{i}}, p_i, p_j, \lambda_i, \lambda_j) = \frac{2}{2 p_{\bar{i}} \cdot p_j} \left[ \epsilon^{\lambda_i}(p_{\bar{i}}) \cdot \epsilon^{\lambda_j}(p_i) p_i \cdot \epsilon^{\lambda_j}(p_{\bar{i}}) \right. \]
\[ + \epsilon^{\lambda_j}(p_j) \cdot \epsilon^{\lambda_i}(p_{\bar{i}}) p_j \cdot \epsilon^{\lambda_i}(p_j) - \epsilon^{\lambda_i}(p_i) \cdot \epsilon^{\lambda_j}(p_j) \]
\[ \left. p_i \cdot \epsilon^{\lambda_j}(p_j) \right]. \quad (105) \]

The factorisation formulæ for the soft limit and the collinear limit in eq. (100) and eq. (104) are very important for higher-order calculations. The phase space integration over these infrared regions diverges. It is therefore necessary to regulate this integration first. This is usually done by dimensional regularisation \([22–24]\). Within dimensional regularisation one replaces four-dimensional space-time with a \( D \)-dimensional space-time. Phase-space integrations and loop integrations are performed consistently in \( D \) dimensions. Setting \( D = 4 - 2 \epsilon \) one is interested in the behaviour of the result of these integrations as \( D \) approaches 4 (or equivalently as \( \epsilon \) approaches zero). Within dimensional regularisation the original divergences will show up as poles
Infrared divergences from the integration over the unresolved phase space cancel with infrared divergences from the loop integrations for infrared-safe observables. This is the content of the Kinoshita-Lee-Nauenberg theorem \cite{25, 26}. Ultraviolet divergences from the loop integrations are removed by renormalisation.

In this report we will not discuss higher-order calculations in perturbation theory. We included the discussion of soft and collinear limits for another reason: Suppose we have a new way or a new conjecture for the computation of the tree-level Yang-Mills amplitudes. If this conjecture is true, it has to give the same result as a calculation based on Feynman diagrams (or any other known method for the computation of these amplitudes). In particular, it has to satisfy the soft and collinear limits in eq. \eqref{100} and eq. \eqref{104}. Checking the correct soft and collinear limits is often simpler than a full proof that a new method computes the amplitude correctly. This allows us either to eliminate false conjectures quickly or gives us further evidence, that a conjecture might be correct. Of course, the correctness of the soft and collinear limits alone does not constitute a complete proof that two methods for the computation of an amplitude agree.

In this context it is also useful to consider the following limit: Consider an amplitude $\mathcal{A}_n^{(0)}$ with $n$ external particles $(1,2,...,n)$. Let $I_m = \{i_1,i_2,...,i_m\} \subset \{1,2,...,n\}$ be a subset with $3 \leq m \leq n - 3$. Denote by $J_{n-m} = \{j_1,...,j_{n-m}\}$ the remaining $(n-m)$ indices not belonging to the index set $I_m$. Let us denote

$$p = p_{i_1} + p_{i_2} + ... + p_{i_m}, \quad (106)$$

and consider the limit $p^2 \rightarrow 0$. This limit corresponds to the case where an internal propagator of the amplitude goes on-shell. In this limit the amplitude factorises as

$$\lim_{p^2 \rightarrow 0} \mathcal{A}_n^{(0)}(p_1, \lambda_1, a_1, ..., p_n, \lambda_n, a_n) = \quad (107)$$

$$\sum_{\lambda} \mathcal{A}_{m+1}^{(0)}(p_{i_1}, \lambda_{i_1}, a_{i_1}, ..., p_{i_m}, \lambda_{i_m}, a_{i_m}, -p, -\lambda, a) \frac{i \delta^{ab}}{p^2}$$

$$\times \mathcal{A}_{n-m+1}(p, \lambda, b, p_{j_1}, \lambda_{j_1}, a_{j_1}, ..., p_{j_{n-m}}, \lambda_{j_{n-m}}, a_{j_{n-m}}).$$

We may view the collinear limit as a special case of eq. \eqref{107}, corresponding to $m = 2$ or $m = n - 2$, where the appearance of the three-particle amplitude $\mathcal{A}_3^{(0)}$ softens the $1/p^2$-behaviour.
3 Efficiency improvements

In the previous section we presented a method to compute tree-level Yang-Mills amplitudes based on Feynman diagrams. Note that for tree-level amplitudes this algorithm involves only algebraic operations (contraction of indices, multiplication, summation, ...). Therefore Feynman diagrams allow us to compute for any $n$ the corresponding tree-level Yang-Mills amplitude $A^{(0)}_n$ within a finite number of steps. We might be tempted to call the computation of tree-level Yang-Mills amplitudes a solved problem. However, the situation is not as simple. Already for moderate values of $n$, the algorithm based on Feynman diagrams is highly inefficient and will produce large intermediate expressions. This can already be inferred from the rapid growth of the number of Feynman diagrams contributing to the amplitude $A^{(0)}_n$, shown for the first few values of $n$ in table (1). Thus it is practically impossible to compute with the help of Feynman diagrams the tree-level amplitude $A^{(0)}_{1001}$ for 1001 gluons.

The deficiency of the algorithm based on Feynman diagrams is analogue to the deficiency of the simplest algorithm to test if an integer $N$ is prime. Consider the following algorithm:

- For $2 \leq j \leq \sqrt{N}$ check if $j$ divides $N$.
- If such a $j$ is found, $N$ is not prime.
- Otherwise $N$ is prime.

This algorithm works in principle, but for large $N$ it does not work in practice.

Now scattering amplitudes with 1001 gluons are not really needed in particle physics phenomenology, but amplitudes with – say – up to roughly 10 gluons are useful to describe physics related to the LHC experiments. A driving force for the study of scattering amplitudes has been the quest to find efficient methods for the computation of these amplitudes. We will now discuss colour decomposition, the spinor-helicity method and off-shell recurrence relations. These efficiency improvements offer the opportunity to discuss some formal topics: The discussion of colour decomposition is followed by an exposition of colour-kinematics duality, the introduction of the spinor-helicity method by a discussion of twistors.

3.1 Colour decomposition

Let us consider colour first. The most important examples for the Lie group $G$ of Yang-Mills theory are the special unitary groups $SU(N)$ and the unitary groups $U(N)$. The unitary group $U(N)$ consists of all complex $N \times N$-matrices $U$ with

$$UU^\dagger = 1,$$

where $1$ denotes the unit matrix. The unitary group $U(N)$ is parametrised by $N^2$ real variables. This is also the number of generators of the associated Lie algebra.

The special unitary group $SU(N)$ consists of all complex $N \times N$-matrices $U$ with

$$UU^\dagger = 1 \text{ and } \det U = 1.$$

\[ (108) \]

\[ (110) \]
Table 1: The number of diagrams contributing to the amplitude $A_n^{(0)}$.

| $n$ | diagrams |
|-----|----------|
| 4   | 4        |
| 5   | 25       |
| 6   | 220      |
| 7   | 2485     |
| 8   | 34300    |
| 9   | 559405   |
| 10  | 10525900 |

The special unitary group SU($N$) is parametrised by $N^2 - 1$ real variables and has as many generators.

In both cases we denote the generators by $T^a$ with $a \in \{1, \ldots, N^2 - 1\}$ for SU($N$) and $a \in \{1, \ldots, N\}$ for U($N$). We take the generators to be hermitian matrices. We recall from eq. (5) our conventions for the generators:

\[
\left[ T^a, T^b \right] = i f^{abc} T^c, \quad \text{Tr}(T^a T^b) = \frac{1}{2} \delta^{ab}. \tag{111}
\]

A useful formula is the Fierz identity. For SU($N$) the Fierz identity reads

\[
T^a_{ij} T^a_{kl} = \frac{1}{2} \left( \delta_{il} \delta_{jk} - \frac{1}{N} \delta_{ij} \delta_{kl} \right), \tag{112}
\]

where a sum over $a$ is understood. The proof is rather simple: The matrices $T^a$ with $a = 1, \ldots, N^2 - 1$ and the unit matrix form a basis of the $N \times N$ hermitian matrices, therefore any hermitian matrix $A$ can be written as

\[
A = c_0 1 + c_a T^a. \tag{113}
\]

The constants $c_0$ and $c_a$ are determined using the normalisation condition and the fact that the $T^a$ are traceless:

\[
c_0 = \frac{1}{N} \text{Tr}(A), \quad c_a = 2 \text{Tr}(T^a A). \tag{114}
\]

Therefore

\[
A_{lk} \left( 2 T^a_{ij} T^a_{kl} + \frac{1}{N} \delta_{ij} \delta_{kl} - \delta_{il} \delta_{jk} \right) = 0. \tag{115}
\]

This has to hold for an arbitrary hermitian matrix $A$, therefore the Fierz identity follows.

Let $X$ and $Y$ be arbitrary matrix products of the generators $T^a$. An immediate corollary of the Fierz identity are the following formulæ involving traces:

\[
\text{Tr}(T^a X) \text{Tr}(T^a Y) = \frac{1}{2} \left[ \text{Tr}(XY) - \frac{1}{N} \text{Tr}(X) \text{Tr}(Y) \right],
\]

\[
\text{Tr}(T^a XT^a Y) = \frac{1}{2} \left[ \text{Tr}(X) \text{Tr}(Y) - \frac{1}{N} \text{Tr}(XY) \right]. \tag{116}
\]
For $U(N)$ the Fierz identity reads
\[ T_{ij}^a T_{kl}^a = \frac{1}{2} \delta_{il} \delta_{jk}, \tag{117} \]
and the analogue of eq. (116) reads
\begin{align*}
\text{Tr}(T^a X) \text{Tr}(T^a Y) &= \frac{1}{2} \text{Tr}(XY), \\
\text{Tr}(T^a X T^a Y) &= \frac{1}{2} \text{Tr}(X) \text{Tr}(Y). \tag{118}
\end{align*}

**Exercise 6:** Compute the traces $\text{Tr}(T^a T^b T^a T^b)$ and $\text{Tr}(T^a T^b T^a T^b)$ for $U(N)$ and $SU(N)$.

There is another relation, which is quite useful. From
\[ [T^a, T^b] = i f^{abc} T^c \tag{119} \]
one derives by multiplying with $T^d$ and taking the trace:
\[ i f^{abc} = 2 \left[ \text{Tr}(T^a T^b T^c) - \text{Tr}(T^b T^a T^c) \right] \tag{120} \]
This yields an expression of the structure constants $f^{abc}$ in terms of the matrices of the fundamental representation. Eq. (120) together with eq. (116) (for an $SU(N)$ gauge theory) or with eq. (118) (for an $U(N)$ gauge theory) can be used to convert any colour factors into traces of matrices in the fundamental representation of the gauge group. Let us consider the following diagram:

![Diagram](image)

The colour part of this diagram is
\begin{align*}
if^{a_1 a_2 b} f^{b a_3 a_4} &= 4 \left[ \text{Tr}(T^{a_1} T^{a_2} T^b) - \text{Tr}(T^{a_2} T^{a_1} T^b) \right] \left[ \text{Tr}(T^{a_3} T^{a_4} T^b) - \text{Tr}(T^{a_4} T^{a_3} T^b) \right] \\
&= 2 \text{Tr}(T^{a_1} T^{a_2} T^{a_3} T^{a_4}) - 2 \text{Tr}(T^{a_1} T^{a_2} T^{a_4} T^{a_3}) - 2 \text{Tr}(T^{a_2} T^{a_1} T^{a_3} T^{a_4}) + 2 \text{Tr}(T^{a_2} T^{a_1} T^{a_4} T^{a_3}). \tag{121}
\end{align*}

Note in this calculation we obtain the same result, independently if we use eq. (116) or eq. (118). The terms proportional to $-1/N$ drop out. This is a general feature of pure Yang-Mills theories. The $-1/N$-terms in the Fierz identity of $SU(N)$ subtract out the trace part of $U(N)$, generated
by the unit matrix. One says that the \(-1/N\)-terms correspond to the exchange of an \(U(1)\)-gluon. The unit matrix commutes with any other matrix, therefore \(f^{abc} = 0\), whenever one index refers to the \(U(1)\)-gluon and the \(U(1)\)-gluon does not couple to any other gauge boson. However, the \(U(1)\)-gluon couples to quarks. In theories with fermions in the fundamental representation of the gauge group the proper Fierz identities have to be used.

We may now repeat the exercise for any Feynman diagram contributing to an amplitude \(A_n^0(g_1, g_2, \ldots, g_n)\) and collect all terms proportional to a specific trace \(\text{Tr}(T^{a_{\sigma(1)}} \ldots T^{a_{\sigma(n)}})\). We thus arrive at the colour decomposition of the tree-level Yang-Mills amplitudes \([27–33]\):

\[
A_n^0(g_1, g_2, \ldots, g_n) = g^{n-2} \sum_{\sigma \in S_n/Z_n} 2 \text{Tr}(T^{a_{\sigma(1)}} \ldots T^{a_{\sigma(n)}}) A_n^0(g_{\sigma(1)}, \ldots, g_{\sigma(n)}),
\]

where the sum is over all non-cyclic permutations of \(\{1, 2, \ldots, n\}\). The quantities \(A_n^{(0)}(g_{\sigma(1)}, \ldots, g_{\sigma(n)})\) accompanying the colour factor \(2 \text{Tr}(T^{a_{\sigma(1)}} \ldots T^{a_{\sigma(n)}})\) are called partial amplitudes. Partial amplitudes are gauge-invariant. Closely related are primitive amplitudes, which for tree-level Yang-Mills amplitudes are calculated from planar diagrams with a fixed cyclic ordering of the external legs and cyclic-ordered Feynman rules \([31, 34–40]\). Primitive amplitudes are gauge invariant as well. For tree-level Yang-Mills amplitudes the notions of partial amplitudes and primitive amplitudes coincide. However, this is no longer true if one considers amplitudes with quarks and/or amplitudes with loops. The most important features of a primitive amplitude are gauge invariance and a fixed cyclic ordering of the external legs. (For amplitudes with quarks and/or loops there will be some additional requirements, which are not relevant here.) Partial amplitudes are defined as the kinematic coefficients of the independent colour structures. Partial amplitudes are also gauge invariant, but not necessarily cyclic ordered. The leading contributions in an \(1/N\)-expansion (with \(N\) being the number of colours) are usually cyclic ordered, the subleading parts are in general not.

Let us now give the cyclic-ordered Feynman rules. The gluon propagator in Feynman gauge is given by

\[
\mu \quad Q \quad Q \quad Q \quad Q \quad Q \quad v = \frac{-i g^{\mu \nu}}{p^2},
\]

the cyclic-ordered Feynman rules for the three-gluon and the four-gluon vertices are

\[
\begin{align*}
&\mu_1 \mu_2 \mu_3 \mu_4 \mu_5 \\
&= i \left[ g^{\mu_1 \mu_2} (p_1^\mu_3 - p_2^\mu_3) + g^{\mu_2 \mu_3} (p_2^\mu_1 - p_3^\mu_1) + g^{\mu_3 \mu_4} (p_3^\mu_2 - p_4^\mu_2) \right], \\
&\mu_1 \mu_2 \mu_3 \\
&= i \left[ 2 g^{\mu_1 \mu_3} g^{\mu_2 \mu_4} - g^{\mu_1 \mu_2} g^{\mu_3 \mu_4} - g^{\mu_1 \mu_3} g^{\mu_2 \mu_3} \right].
\end{align*}
\]
The cyclic-ordered Feynman rules are obtained from the standard Feynman rules by extracting from each formula the coupling constant and by taking the coefficient of the cyclic-ordered colour part. Note that the Feynman rule for the cyclic-ordered four-gluon vertex is considerably simpler than the Feynman rule for the full four-gluon vertex.

The primitive amplitude $A_n^{(0)} (g_1, g_2, \ldots, g_n)$ – or $A_n^{(0)} (1, 2, \ldots, n)$ for short – with the external order $(1, 2, \ldots, n)$ is calculated from planar diagrams with this external order. There are fewer diagrams contributing to $A_n^{(0)}$ than to $\mathcal{A}_n^{(0)}$. For the case of four gluons there are three cyclic-ordered Feynman diagrams contributing to the primitive amplitude $A_4^{(0)} (1, 2, 3, 4)$. These are shown in fig. (2). For the next few values of $n$ we list the number of cyclic-ordered diagrams in table (2) and compare them with the number of Feynman diagrams contributing to the full amplitude $\mathcal{A}_n^{(0)}$. The number of diagrams contributing to the cyclic-ordered primitive amplitude $A_n^{(0)}$ is significantly smaller than the number of Feynman diagrams contributing to the full amplitude $\mathcal{A}_n^{(0)}$.

The colour decomposition in eq. (122) allows us to perform the summation over the colour degrees of freedom when calculating $|\mathcal{A}_n^{(0)}|^2$. We have

$$
\sum_{\text{colour}} \left| \mathcal{A}_n^{(0)} \right|^2 = g^{2n-4} \sum_{\sigma \in S_n/Z_n} \sum_{\pi \in S_n/Z_n} 4 \text{Tr} \left( T^{a_{\sigma(1)}} \ldots T^{a_{\sigma(n)}} \right) \text{Tr} \left( T^{a_{\pi(n)}} \ldots T^{a_{\pi(1)}} \right) A_n^{(0)} (\sigma(1), \ldots, \sigma(n)) A_n^{(0)*} (\pi(1), \ldots, \pi(n))
$$

The traces can be evaluated with the help of the formulæ in eq. (116) or eq. (118).
Let us now consider the primitive amplitude $A_n^{(0)}$ for a fixed set of external momenta, denoted by $p = (p_1, ..., p_n)$ and a fixed set of external polarisations, denoted by $\varepsilon = (\varepsilon_1^\lambda, ..., \varepsilon_n^\lambda)$ with $\lambda_j \in \{+, -\}$. In addition, the primitive amplitude will depend on the cyclic order of the external legs, specified by $\sigma = (\sigma_1, ..., \sigma_n)$. An obvious question related to the colour decomposition is: How many independent primitive amplitudes are there for $n$ external particles? For a fixed set of external momenta and a fixed set of polarisations the primitive amplitudes are distinguished by the permutation specifying the order of the external particles. For $n$ external particles there are $n!$ permutations and therefore $n!$ different orders. However, there are relations among primitive amplitudes with different external order. The first set of relations is rather trivial and given by cyclic invariance:

$$A_n^{(0)}(1, 2, ..., n) = A_n^{(0)}(2, ..., n, 1)$$

Cyclic invariance is the statement that only the external cyclic order matters, not the point, where we start to read off the order. Cyclic invariance reduces the number of independent primitive amplitudes to $(n-1)!$.

The first non-trivial relations are the Kleiss-Kuijf relations [41]. Let

$$\vec{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_j), \quad \vec{\beta} = (\beta_1, \beta_2, ..., \beta_{n-2-j})$$

be two ordered sequences of numbers, such that

$$\{1\} \cup \{\alpha_1, ..., \alpha_j\} \cup \{\beta_1, ..., \beta_{n-2-j}\} \cup \{n\} = \{1, ..., n\}.$$  \hspace{1cm} (129)

We further set $\vec{\beta}^T = (\beta_{n-2-j}, ..., \beta_2, \beta_1)$. The Kleiss-Kuijf relations read

$$A_n^{(0)}(1, \alpha_1, ..., \alpha_j, n, \beta_1, ..., \beta_{n-2-j}) = (-1)^{n-2-j} \sum_{\sigma \in \vec{\alpha} \cup \vec{\beta}^T} A_n^{(0)}(1, \sigma_1, ..., \sigma_{n-2}, n).$$  \hspace{1cm} (130)

Here, $\vec{\alpha} \cup \vec{\beta}^T$ denotes the set of all shuffles of $\vec{\alpha}$ with $\vec{\beta}^T$, i.e. the set of all permutations of the elements of $\vec{\alpha}$ and of the elements $\vec{\beta}^T$, which preserve the relative order of the elements of $\vec{\alpha}$ and of the elements $\vec{\beta}^T$.
of $\vec{B}^T$. The Kleiss-Kuijf relations reduce the number of independent primitive amplitudes to $(n-2)!$.

The Kleiss-Kuijf relations follow from the anti-symmetry of the colour-stripped trivalent vertices. Let us first consider a theory with only trivalent vertices, which are anti-symmetric under the exchange of any two legs. Consider as an example the situation shown in fig. (3), corresponding to a specific contribution to the left-hand side of eq. (130). Only currents consisting of particles from $\alpha_1, ..., \alpha_j$ or $\beta_1, ..., \beta_{n-2-j}$ couple to the line from 1 to $n$. No mixed currents, involving particles from $\alpha_1, ..., \alpha_j$ and $\beta_1, ..., \beta_{n-2-j}$, couple to the line from 1 to $n$. Next consider the right-hand side of eq. (130). Now all particles from $\{\alpha_1, ..., \alpha_j\} \cup \{\beta_1, ..., \beta_{n-2-j}\}$ are above the line from 1 to $n$. Flipping the particles from the set $\{\beta_1, ..., \beta_{n-2-j}\}$ will give the sign $(-1)^{n-2-j}$. The shuffle product will cancel all contributions from mixed currents coupling to the line from 1 to $n$. To see this, consider a mixed current contributing to the right-hand side of eq. (130). Such a current will necessarily contain two sub-currents, one made out entirely of particles from $\{\alpha_1, ..., \alpha_j\}$, the other made out entirely of particles from $\{\beta_1, ..., \beta_{n-2-j}\}$ and coupled together through a trivalent vertex. The shuffle product ensures that both cyclic orderings at this vertex contribute. Since the trivalent vertex is anti-symmetric under the exchange of two legs, these contributions cancel. Note that these arguments apply to any theory with anti-symmetric trivalent vertices only.

It remains to show that Yang-Mills theory can be cast into a form involving only anti-symmetric trivalent vertices. The cyclic ordered three-gluon vertex is clearly anti-symmetric under the exchange of any two legs:

$$1 \quad 2 \quad 3 \quad \Rightarrow \quad - \quad 1 \quad 2 \quad 3$$ (131)

We have to eliminate the four-gluon vertex. This can be done by introducing an auxiliary tensor field $[42, 43]$. From eq. (60) we read off that the four-gluon vertex part of the Lagrangian is

$$\mathcal{L}_{gggg} = \frac{1}{g^2} \Tr \left[ A_\mu, A_\nu \right] \left[ A^\mu, A^\nu \right].$$ (132)

Let us introduce an auxiliary tensor field

$$B_{[\mu\nu]} = \frac{g}{i} T^a B^a_{[\mu\nu]},$$ (133)

anti-symmetric in the indices $\mu$ and $\nu$ with the Lagrange density

$$\mathcal{L}_{aux} = \frac{2}{g^2} \Tr \left( \frac{1}{2} B_{[\mu\nu]} B^{[\mu\nu]} - \frac{i}{\sqrt{2}} B_{[\mu\nu]} \left[ A^\mu, A^\nu \right] \right).$$ (134)

The auxiliary field $B_{[\mu\nu]}$ occurs at the most quadratically and can be integrated out

$$\int \mathcal{D}B_{[\mu\nu]} \exp \left( i \int d^4x \, \mathcal{L}_{aux} \right) = \mathcal{N} \exp \left( i \int d^4x \, \mathcal{L}_{gggg} \right),$$ (135)
where the (irrelevant) prefactor $N$ is given by

$$N = \int DB_{\mu\nu} \exp \left( \frac{i}{g^2} \int d^4x \, \text{Tr} \, B_{\mu\nu} B^{\mu\nu} \right).$$

**Exercise 7:** Let us investigate in more detail the procedure of “integrating out” a field $\phi$, which appears at the most quadratically in the Lagrangian. Consider the path integral

$$\int \mathcal{D}\phi \, \exp i \int d^4x \, \text{Tr} \left\{ \frac{1}{2} \phi \partial \phi + \phi K \right\}.$$  

Assume that $P$ is a pseudo-differential operator of even degree and independent of the other fields. $K$ on the other hand may depend on some other fields. Show that under the substitution

$$\phi \rightarrow \phi + P^{-1}K,$$

where $P^{-1}$ denotes the inverse pseudo-differential operator, one obtains

$$\int \mathcal{D}\phi \, \exp i \int d^4x \, \text{Tr} \left\{ \frac{1}{2} \phi \partial \phi - \frac{1}{2} K P^{-1} K \right\}. $$

The irrelevant factor

$$\int \mathcal{D}\phi \, \exp i \int d^4x \, \text{Tr} \left\{ \frac{1}{2} \phi \partial \phi \right\}$$

may be neglected, leaving

$$\exp i \int d^4x \, \text{Tr} \left( -\frac{1}{2} K P^{-1} K \right)$$

as the result of integrating out the field $\phi$.

From the Lagrangian in eq. (134) we may derive the Feynman rules for this tensor field. The colour-stripped “propagator” for the tensor field is given by

$$[\mu\nu] [\rho\sigma] = \frac{-i}{2} \left( g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho} \right).$$

The cyclic-ordered vertex of the tensor field with the gluon fields is given by

$$1, \mu \rightarrow 2, \nu \rightarrow 3, [\rho\sigma] = \frac{i}{\sqrt{2}} \left( g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho} \right).$$

This vertex is anti-symmetric;

$$1 \rightarrow 3 \rightarrow 2 \rightarrow 1.$$  

**Exercise 8:** Show that at the level of cyclic-ordered Feynman rules we have
where arbitrary sub-graphs may be attached to the legs 1-4 (but always the same sub-graph for a specific leg). This verifies at the level of diagrams that the four-gluon vertex can be cast into trivalent vertices with the help of the auxiliary tensor field.

Let us continue with our investigation on the number of independent primitive amplitudes. Apart from cyclic invariance and the Kleiss-Kuijf relations there are in addition the Bern-Carrasco-Johansson relations (BCJ relations) [44]. The fundamental BCJ relations read

\[
\sum_{i=2}^{n-1} \left( \sum_{j=i+1}^{n} 2p_2 p_j \right) A_n^{(0)}(1,3,\ldots,i,2,i+1,\ldots,n-1,n) = 0. \tag{145}
\]

Cyclic invariance allows us to fix one external particle at a specified position, say position 1. The Kleiss-Kuijf relations allow us to fix a second external particle at another specified position, say position \( n \). The BCJ relations allow us to fix a third external particle at a third specified position, say position 2. The BCJ relations reduce the number of independent primitive amplitudes to \((n-3)!\). The full set of relations among primitive tree amplitudes in pure Yang-Mills theory is given by cyclic invariance, Kleiss-Kuijf relations, and the fundamental BCJ relations. Therefore a basis of independent primitive amplitudes consists of \((n-3)!\) elements.

The BCJ relations have been first conjectured by Bern, Carrasco and Johansson in [44]. They have been proven first with methods from string theory [45, 46]. It should be mentioned that the BCJ relations do not rely on string theory and a proof within quantum field theory exists [47].

We will present this proof later in this report in section 5, once we are familiar with on-shell recurrence relations.

### 3.2 Colour-kinematics duality

A Lie group \( G \) is a group which is also an analytic manifold such that the mapping \((a,b) \rightarrow ab^{-1}\) of the product manifold \( G \times G \) into \( G \) is analytic.

A Lie algebra \( g \) over a field \( F \) is a vectorspace \( A \) together with a mapping \( x \otimes y \rightarrow [x,y] \) such that for \( x,y,z \in A \):

\[
[x,x] = 0, \\
[x,y],z + [y,z],x + [z,x],y = 0. \tag{146}
\]

**Exercise 9:** Show that \([x,x] = 0\) implies the anti-symmetry of the Lie bracket \([x,y] = -[y,x]\). Show further that also the converse is true, provided \( \text{char } F \neq 2 \). Explain, why the argument does not work for
char $F = 2$.

The second line in eq. (146) is called the Jacobi identity. In physics we are usually concerned with Lie groups which are matrix groups. The corresponding Lie algebra is then an algebra of matrices and the Lie bracket corresponds to the commutator of matrices:

$$\left[ T^a, T^b \right] = T^a \cdot T^b - T^b \cdot T^a. \quad (147)$$

It is an easy exercise to verify that the relations in eq. (146) are satisfied. The Jacobi identity translates to

$$\left[ \left[ T^a, T^b \right], T^c \right] + \left[ \left[ T^b, T^c \right], T^a \right] + \left[ \left[ T^c, T^a \right], T^b \right] = 0, \quad (148)$$

or equivalently

$$
\begin{align*}
&\left( if^{abc} \right) \left( if^{ecd} \right) + \left( if^{bce} \right) \left( if^{ead} \right) + \left( if^{cae} \right) \left( if^{ebd} \right) = 0. \quad (149)
\end{align*}
$$

A product in an algebra takes two elements of the algebra as an input and gives one element of the algebra as an output. This may be represented graphically by a trivalent vertex with two input lines on the top and one output line at the bottom. The anti-symmetry of the Lie bracket is expressed through

$$1 \quad 2 \quad 3 \quad = \quad 2 \quad 3 \quad 1 \quad (150)$$

and the Jacobi identity is visualised through

$$1 \quad 2 \quad 3 \quad 4 \quad + \quad 2 \quad 3 \quad 1 \quad 4 \quad + \quad 3 \quad 1 \quad 2 \quad 4 \quad = \quad 0. \quad (151)$$

In particular these two relations hold true, if we assume that each vertex of these graphs corresponds to a factor $\left( if^{abc} \right)$. Let us now focus on the Jacobi identity. We may attach subgraphs to the external legs 1, 2, 3 and 4. The Jacobi identity remains true, as long as we always attach the same subgraph to leg 1, another subgraph to all occurrences of leg 2, etc..

Let us consider a set of $n$ external momenta $p_1, p_2, ..., p_n$ and an internal edge $e$ of some tree graph. Assume that the particles $\{\alpha_1, \alpha_2, ..., \alpha_j\}$ are attached on the one side of the internal edge.
while the remaining $(n - j)$ particles $\{\beta_1, \ldots, \beta_{n - j}\}$ are attached to the other side of the internal edge $e$. We set

$$s_e = (p_{\alpha_1} + \ldots + p_{\alpha_j})^2 = (p_{\beta_1} + \ldots + p_{\beta_{n - j}})^2. \quad (152)$$

We are now in a position to discuss colour-kinematics duality [48]: Colour-kinematics duality states that Yang-Mills amplitudes can be brought into a form

$$A_n^{(0)} = ig^{n-2} \sum_{\text{trivalent graphs } G} \frac{C(G) N(G)}{D(G)}, \quad D(G) = \prod_{\text{edges } e} s_e, \quad (153)$$

where the summation is over all trivalent graphs $G$ (i.e. the sum does not involve graphs with four-valent vertices), $D(G)$ is the product of Lorentz invariants $s_e$ corresponding to the internal edges of the graph $G$ (i.e. up to factors of $i$, the quantity $1/D(G)$ is the product of the scalar propagators of the graph $G$ and the quantity $C(G)$ is the colour factor of the graph (i.e. a factor $i f_{abc}$ for every vertex of the graph $G$). The quantity $N(G)$ is called the kinematic numerator of the graph $G$. We have seen that for graphs $G_1$, $G_2$ and $G_3$ as in eq. (151) the colour factors $C(G_1)$, $C(G_2)$ and $C(G_3)$ satisfy the Jacobi relation. The non-trivial point of the representation of the Yang-Mills amplitude in the form of eq. (153) is, that kinematic numerators $N(G)$ can be found, such that the kinematical numerators $N(G)$ satisfy anti-symmetry and Jacobi-like relations, whenever the corresponding colour factors $C(G)$ do:

$$C(G_1) + C(G_2) = 0 \Rightarrow N(G_1) + N(G_2) = 0,$$
$$C(G_1) + C(G_2) + C(G_3) = 0 \Rightarrow N(G_1) + N(G_2) + N(G_3) = 0. \quad (154)$$

The anti-symmetry of the kinematic numerators follows trivially from the anti-symmetry of the trivalent vertices. The non-trivial part are the Jacobi-like relations for the kinematic numerators. Note that colour-kinematics duality does not state that for any trivalent graph expansion as in eq. (153) the kinematic numerators satisfy Jacobi-like relations. Colour-kinematics duality states, that there exists kinematic numerators such that eq. (153) and eq. (154) are fulfilled. In general, the kinematic numerators are not unique, changing the kinematic numerators while leaving eq. (153) and eq. (154) intact is referred to as a generalised gauge transformation. The kinematic numerators may be found by solving a linear system of equations. The basic idea is as follows: If the vertices are anti-symmetric, we may rewrite eq. (151) as

$$\begin{align*}
\begin{array}{c}
\quad 2 & \quad 3 \\
\quad 1 & \quad 4
\end{array}
\Rightarrow & \quad \begin{array}{c}
\quad 2 & \quad 3 \\
\quad 1 & \quad 4
\end{array}
\quad -
\begin{array}{c}
\quad 3 & \quad 2 \\
\quad 1 & \quad 4
\end{array}.
\end{align*} \quad (155)$$

Eq. (155) is called a STU-relation. If eq. (155) holds we may reduce any tree graph with $n$ external legs and containing only trivalent vertices to a multi-peripheral form with respect to 1 and $n$. We say that a graph is multi-peripheral with respect to 1 and $n$, if all other external legs
connect directly to the line from 1 to \( n \), i.e. there are no non-trivial sub-trees attached to this line. A graph in multi-peripheral form can be drawn as

\[
1 \rightarrow \sigma_2 \rightarrow \sigma_3 \rightarrow \cdots \rightarrow \sigma_{n-1} \rightarrow n
\]

Repeated use of eq. (155) reduces any tree graph with non-trivial sub-trees attached to the line from 1 to \( n \) to a multi-peripheral form. We may therefore express any kinematic numerator as a linear combination of the \((n-2)!\) kinematic numerators corresponding to the multi-peripheral graphs with leg 1 and \( n \) fixed. Let us then consider cyclic-ordered amplitudes with legs 1, 2 and \( n \) fixed. On the one hand we may express these amplitudes in terms of the kinematic numerators corresponding to the multi-peripheral graphs. On the other hand, we may simply calculate these amplitudes by any method discussed in this report (for example Feynman diagrams will do). This gives us a set of \((n-3)!\) linear equations for \((n-2)!\) unknown kinematic numerators, which may be solved by linear algebra, determining \((n-3)!\) kinematic numerators and leaving \((n-2)!-(n-3)!\) “free” kinematic numerators. The kinematic numerators are not unique, we may give the “free” kinematic numerators any value. In particular we may set them to zero. The non-uniqueness of the kinematic numerators is referred to as generalised gauge-invariance. The existence of kinematic numerators satisfying anti-symmetry and Jacobi-like relations implies the BCJ relations. This can be seen by expressing an amplitude with legs 1 and \( n \) fixed, but 2 arbitrary in terms of kinematic numerators, which in turn may be expressed in terms of amplitudes with legs 1, 2 and \( n \) fixed and a priori the “free” kinematic numerators. One then observes that the dependence on the “free” kinematic numerators drops out in this relation.

### 3.3 The spinor helicity method

Let us now return to the spin- and colour-summed matrix element squared,

\[
\sum_{\text{spins}, \text{colour}} \left| A_{(0)}^{(0)} \right|^2,
\]

entering in leading order the formula (89) for the calculation of an observable. Assume that \( A_{(0)}^{(0)} \) is given as the sum of \( N_{\text{terms}} \) terms. Given the number of diagrams in table (1) and the fact that the Feynman rules for the vertices in eq. (79) imply that each diagram will contribute several terms, we anticipate that \( N_{\text{terms}} \) can be a rather large number. Squaring the amplitude and summing over the spins will result in \( \mathcal{O}(N_{\text{terms}}^2) \) terms. To be precise, we obtain by using the spin sum of eq. (95) with a light-like reference vector \( n^\mu \) in the squared expression \( 3n^\mu \) terms. The factor 3 comes from the three terms in the expression of the spin sum for a light-like reference vector \( n^\mu \) with \( n^2 = 0 \):

\[
\sum_{\kappa \in \{+, -\}} \left( \epsilon^{\lambda \kappa}_{\gamma \nu}(p) \right)^* \epsilon^{\lambda \kappa}_{\gamma \nu}(p) = -g_{\mu \nu} + \frac{p_\mu p_\nu}{p \cdot n} + \frac{n_\mu p_\nu}{p \cdot n}.
\]
This \( \mathcal{O}(N_{\text{terms}}^2) \)-behaviour is highly prohibitive. Can we do better? Yes, we can. Up to now we treated the polarisation vectors for the external gluons as abstract objects, satisfying eqs. (80)-(81) and eq. (95). We may however use explicit expressions for the polarisation vectors. Given numerical values for the external momenta \((p_1, ..., p_n)\) and – for a specific helicity configuration \((\lambda_1, ..., \lambda_n)\) – numerical values for the polarisation vectors \((\varepsilon_1, ..., \varepsilon_n)\), we may evaluate the \(N_{\text{terms}}\) terms of the amplitude and obtain a complex number, ignoring colour for the moment. Taking the norm of a complex number is a \(\mathcal{O}(1)\)-operation. We repeat this for every helicity configuration and sum up the individual contributions. There are \(2^n\) helicity configurations. In total we have to evaluate \(2^nN_{\text{terms}}\) expressions and we arrive at an \(\mathcal{O}(N_{\text{terms}})\)-behaviour. In practice one even avoids the prefactor \(2^n\) by Monte Carlo sampling over the helicity configurations or a Monte Carlo integration over helicity angles [42, 49–51].

The spinor-helicity method [52–57] gives us explicit expression for the polarisation vectors. As the name suggests, it involves spinors. This may seem at first sight a little bit strange, as we are dealing with particles which are bosons with spin 1. However, as we will soon see, the use of spinors is an elegant way to implement the transversality conditions of the polarisation vectors. Let us therefore discuss spinors in more detail.

### 3.3.1 The Dirac equation

The Lagrange density for a Dirac field depends on four-component spinors \(\psi_\alpha(x)\) \((\alpha = 1, 2, 3, 4)\) and \(\bar{\psi}_\alpha(x) = (\psi^\dagger(x)\gamma^\mu)_\alpha\):

\[
\mathcal{L}(\psi, \bar{\psi}, \partial_\mu \psi) = i\bar{\psi}(x)\gamma^\mu \partial_\mu \psi(x) - m\bar{\psi}(x)\psi(x)
\]

Here, the \((4 \times 4)\)-Dirac matrices satisfy the anti-commutation rules

\[
\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}1, \quad \{\gamma^\mu, \gamma_5\} = 0, \quad \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{i}{24}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma.
\]

The Dirac equations read

\[
(i\gamma^\mu \partial_\mu - m) \psi(x) = 0, \quad \bar{\psi}(x) \left( i\gamma^\mu \partial_\mu + m \right) = 0.
\]

For computations it is useful to have an explicit representation of the Dirac matrices. There are several widely used representations. A particular useful one is the **Weyl representation** of the Dirac matrices:

\[
\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

Here, the 4-dimensional \(\sigma^\mu\)-matrices are defined by

\[
\sigma^\mu_{AB} = (1, -\bar{\sigma}), \quad \bar{\sigma}^{\mu AB} = (1, \bar{\sigma}).
\]

and \(\bar{\sigma} = (\sigma_x, \sigma_y, \sigma_z)\) are the standard Pauli matrices:

\[
\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]
The $\sigma^\mu$-matrices satisfy the Fierz identities
\[
\sigma^\mu_{AA} \sigma^B_{\mu} = 2\delta_A^B \delta^B_A, \quad \sigma^\mu_{AA} \sigma_{\mu BB} = 2\epsilon_{AB} \epsilon^A_B, \quad \sigma^\mu_{\alpha A} \sigma^B_{\mu} = 2\epsilon^A_B \epsilon^{AB}.
\] (165)

Let us now look for plane wave solutions of the Dirac equation. We make the ansatz
\[
\psi(x) = \begin{cases} 
  u(p)e^{-ipx}, & p^0 > 0, \quad p^2 = m^2, \quad \text{incoming fermion}, \\
  v(p)e^{ipx}, & p^0 > 0, \quad p^2 = m^2, \quad \text{outgoing anti-fermion}.
\end{cases}
\] (166)

$u(p)$ describes incoming particles, $v(p)$ describes outgoing anti-particles. Similar,
\[
\bar{\psi}(x) = \begin{cases} 
  \bar{u}(p)e^{ipx}, & p^0 > 0, \quad p^2 = m^2, \quad \text{outgoing fermion}, \\
  \bar{v}(p)e^{-ipx}, & p^0 > 0, \quad p^2 = m^2, \quad \text{incoming anti-fermion},
\end{cases}
\] (167)

where
\[
\bar{u}(p) = u^\dagger(p)\gamma^0, \quad \bar{v}(p) = v^\dagger(p)\gamma^0.
\] (168)

$\bar{u}(p)$ describes outgoing particles, $\bar{v}(p)$ describes incoming anti-particles. Then
\[
(p' - m)u(p) = 0, \quad (p' + m)v(p) = 0,
\]
\[
\bar{u}(p)(p' - m) = 0, \quad \bar{v}(p)(p' + m) = 0,
\] (169)

There are two solutions for $u(p)$ (and the other spinors $\bar{u}(p), v(p), \bar{v}(p)$). We will label the various solutions with $\lambda$. The degeneracy is related to the additional spin degree of freedom. We require that the two solutions satisfy the orthogonality relations
\[
\bar{u}(p, \lambda)u(p, \lambda) = 2m\delta_{\lambda\lambda},
\]
\[
\bar{v}(p, \lambda)v(p, \lambda) = -2m\delta_{\lambda\lambda},
\]
\[
\bar{u}(p, \lambda)\bar{v}(p, \lambda) = \bar{v}(\lambda)u(\lambda) = 0,
\] (170)

and the completeness relations
\[
\sum_\lambda u(p, \lambda)\bar{u}(p, \lambda) = p' + m, \quad \sum_\lambda v(p, \lambda)\bar{v}(p, \lambda) = p' - m.
\] (171)

### 3.3.2 Massless spinors in the Weyl representation

Let us now try to find explicit solutions for the spinors $u(p), v(p), \bar{u}(p)$ and $\bar{v}(p)$. The simplest case is the one of a massless fermion:
\[
m = 0.
\] (172)

In this case the Dirac equation for the $u$- and the $v$-spinors are identical and it is sufficient to consider
\[
p' u(p) = 0, \quad \bar{u}(p)p' = 0.
\] (173)
In the Weyl representation \( p/ \) is given by

\[
\begin{pmatrix}
0 \\
p_\mu \sigma^\mu \\
p_\mu \sigma^\mu \\
0
\end{pmatrix},
\]

therefore the 4 \( \times \) 4-matrix equation for \( u(p) \) (or \( \bar{u}(p) \)) decouples into two 2 \( \times \) 2-matrix equations.

We introduce the following notation: Four-component Dirac spinors are constructed out of two Weyl spinors as follows:

\[
u(p) = \begin{pmatrix} |p+\rangle \\ |p-\rangle \end{pmatrix} = \begin{pmatrix} p_A \\ p_B \end{pmatrix} = \begin{pmatrix} u_+(p) \\ u_-(p) \end{pmatrix},
\]

\[(175)\]

Bra-spinors are given by

\[
\bar{\nu}(p) = \begin{pmatrix} \langle p-| \\ \langle p+| \end{pmatrix} = \begin{pmatrix} p^A \\ p^B \end{pmatrix} = \begin{pmatrix} \bar{u}_-(p) \\ \bar{u}_+(p) \end{pmatrix}.
\]

In the literature there exists various notations for Weyl spinors. Eq. (175) and eq. (176) show four of them and the way how to translate from one notation to another notation. By a slight abuse of notation we will in the following not distinguish between a two-component Weyl spinor and a Dirac spinor, where either the upper two components or the lower two components are zero. If we define the chiral projection operators

\[
P_+ = \frac{1}{2} (1 + \gamma_5) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_- = \frac{1}{2} (1 - \gamma_5) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},
\]

\[(177)\]

then (with the slight abuse of notation mentioned above)

\[
u_\pm(p) = P_\pm u(p), \quad \bar{u}_\pm(p) = \bar{u}(p) P_\mp .
\]

The two solutions of the Dirac equation

\[
p/ \nu(p, \lambda) = 0
\]

are then

\[
u(p, +) = u_+(p), \quad u(p, -) = u_-(p).
\]

We now have to solve

\[
p_\mu \sigma^\mu |p+\rangle = 0, \quad p_\mu \sigma^\mu |p-\rangle = 0,
\]

\[
\langle p+| p_\mu \sigma^\mu = 0, \quad \langle p-| p_\mu \sigma^\mu = 0.
\]

\[(181)\]

It it convenient to express the four-vector \( p^\mu = (p^0, p^1, p^2, p^3) \) in terms of light-cone coordinates:

\[
p^+ = \frac{1}{\sqrt{2}} (p^0 + p^3), \quad p^- = \frac{1}{\sqrt{2}} (p^0 - p^3), \quad p^\perp = \frac{1}{\sqrt{2}} (p^1 + ip^2), \quad p^{\perp*} = \frac{1}{\sqrt{2}} (p^1 - ip^2).
\]
Note that $p^\perp*$ does not involve a complex conjugation of $p^1$ or $p^2$. For null-vectors one has
\[ p^\perp* p^\perp = p^+ p^- \]  
(182)

Then the equation for the ket-spinors becomes
\[
\begin{pmatrix} p^- & -p^\perp* \\
-p^\perp & p^+ \end{pmatrix} |p+\rangle = 0, \quad \begin{pmatrix} p^+ & p^\perp* \\
p^\perp & p^- \end{pmatrix} |p-\rangle = 0,
\]  
(183)
and similar equations can be written down for the bra-spinors. This is a problem of linear algebra.

Solutions for ket-spinors are
\[
|p+\rangle = p_A = c_1 \begin{pmatrix} p^\perp* \\
p^- \end{pmatrix}, \quad |p-\rangle = p_A = c_2 \begin{pmatrix} p^- \\
p^\perp \end{pmatrix},
\]  
(184)
with some yet unspecified multiplicative constants $c_1$ and $c_2$. Solutions for bra-spinors are
\[
\langle p+ | = p_A = c_3 \begin{pmatrix} p^\perp, p^- \end{pmatrix}, \quad \langle p-| = p_A = c_4 \begin{pmatrix} p^-, p^\perp* \end{pmatrix},
\]  
(185)
with some further constants $c_3$ and $c_4$. Let us now introduce the 2-dimensional antisymmetric tensor:
\[
\varepsilon_{AB} = \begin{pmatrix} 0 & 1 \\
-1 & 0 \end{pmatrix}, \quad \varepsilon_{BA} = -\varepsilon_{AB}
\]  
(186)
Furthermore we set
\[
\varepsilon^{AB} = \varepsilon^{BA} = \varepsilon_{AB} = \varepsilon_{AB}.
\]  
(187)
Note that these definitions imply
\[
\varepsilon^{AC} \varepsilon_{BC} = \delta^A_B, \quad \varepsilon^{AC} \varepsilon_{BC} = \delta^A_B.
\]  
(188)
We would like to have the following relations for raising and lowering a spinor index $A$ or $B$:
\[
p^{\dot{A}} = \varepsilon^{AB} p_B, \quad p^{\dot{A}} = \varepsilon^{BA} p_B, \\
p_B = \varepsilon^{\dot{A}B} p_{\dot{A}}, \quad p_B = \varepsilon^{\dot{A}B} p_{\dot{A}}.
\]  
(189)
Note that raising an index is done by left-multiplication, whereas lowering is performed by right-multiplication. Postulating these relations implies
\[
c_1 = c_4, \quad c_2 = c_3.
\]  
(190)
In addition we normalise the spinors according to
\[
\langle p^\pm |p^\pm \rangle = 2 p^\mu.
\]  
(191)
This implies
\[ c_1 c_3 = \frac{\sqrt{2}}{p}, \quad c_2 c_4 = \frac{\sqrt{2}}{p}. \]  
(192)

Eq. (189) and eq. (191) determine the spinors only up to a scaling
\[ p_A \to \lambda p_A, \quad p_\dot{A} \to \frac{1}{\lambda} p_\dot{A}. \]  
(193)

This scaling freedom is referred to as \textbf{little group scaling}. Keeping the scaling freedom, we define the spinors as
\[
|p^+\rangle = p_A = \frac{\lambda_p 2^{\frac{1}{2}}}{\sqrt{p}} \left( \begin{array}{c} p^{\perp*} \\ p_\perp \end{array} \right), \quad |p^-\rangle = p_\dot{A} = \frac{2^{\frac{1}{2}}}{\lambda_p \sqrt{p}} \left( \begin{array}{c} p^- \\ -p_\perp \end{array} \right), \\
\langle p^+| = p_\dot{A} = \frac{2^{\frac{1}{2}}}{\lambda_p \sqrt{p}} \left( p_\perp, p^- \right), \quad \langle p^-| = p_A = \frac{\lambda_p 2^{\frac{1}{2}}}{\sqrt{p}} \left( p^-, -p^{\perp*} \right). 
\]  
(194)

Popular choices for \( \lambda_p \) are
\[
\lambda_p = 1 \quad \text{: symmetric,} \\
\lambda_p = 2^{\frac{1}{4}} \sqrt{p} \quad \text{: } p_A \text{ linear in } p^\mu, \\
\lambda_p = \frac{1}{2^{\frac{1}{4}} \sqrt{p}} \quad \text{: } p_\dot{A} \text{ linear in } p^\mu. 
\]  
(195)

Note that all formulae in this sub-section (3.3.2) work not only for real momenta \( p^\mu \) but also for complex momenta \( p^\mu \). This will be useful later on, where we encounter situations with complex momenta. However there is one exception: The relations \( p_\dot{A} = p_A \) and \( p_\dot{A} = p_\dot{A} \) (or equivalently \( \bar{u}(p) = u(p)p_\dot{A} \) encountered in previous sub-sections are valid only for real momenta \( p^\mu = (p^0, p^1, p^2, p^3) \). If on the other hand the components \( (p^0, p^1, p^2, p^3) \) are complex, these relations will in general not hold. In the latter case \( p_A \) and \( p_\dot{A} \) are considered to be independent quantities. The reason, why the relations \( p_\dot{A} = p_A \) and \( p_\dot{A} = p_\dot{A} \) do not hold in the complex case lies in the definition of \( p^{\perp*} \): We defined \( p^{\perp*} = (p^1 - i p^2)/\sqrt{2} \), and not as \( (p^1 - i(p^2)^*)/\sqrt{2} \). With the former definition \( p^{\perp*} \) is a holomorphic function of \( p^1 \) and \( p^2 \). There are applications where holomorphicity is more important than nice properties under hermitian conjugation.

**Exercise 10:** The helicity operator \( h \) for particles is defined by
\[
h = \frac{\vec{p} \cdot \vec{S}}{|\vec{p}|}, 
\]  
(196)

where the components of \( \vec{S} = (S_1, S_2, S_3) \) are given by
\[
S_i = \frac{1}{2} \epsilon_{ijk} S^{jk} 
\]  
(197)
with

\[ S^{\alpha\nu} = \frac{i}{4} [\gamma^\nu, \gamma^\alpha]. \]  

(198)

One finds in the Weyl representation

\[ h = \frac{1}{2|\vec{p}|} \left( \vec{p} \cdot \vec{\sigma} 0 0 \vec{p} \cdot \vec{\sigma} \right). \]  

(199)

(The helicity operator for anti-particles is defined by \( h = -\vec{p} \cdot \vec{S} / |\vec{p}| \). A particle with eigenvalue \( h = +1/2 \) is called right-handed, a particle with \( h = -1/2 \) is called left-handed. Assume that \( p^\mu \) is a real four-vector with positive energy \( (p^0 > 0) \). Show that the spinors \( |p^+\rangle = p_A \) and \( \langle p^+ | = p^A \) have helicity \( h = +1/2 \), while the spinors \( |p^-\rangle = p^A \) and \( \langle p^- | = p_A \) have helicity \( h = -1/2 \).

3.3.3 Spinor products

Let us now make the symmetric choice \( \lambda_p = 1 \). Spinor products are defined by

\[ \langle pq \rangle = \langle p - q^+ \rangle = p^A q_A = \frac{\sqrt{2}}{\sqrt{p^0} \sqrt{q^0}} \left( p^- q^{+\ast} - q^- p^{+\ast} \right), \]

\[ [qp] = \langle q + p^- \rangle = q^A p^A = \frac{\sqrt{2}}{\sqrt{p^0} \sqrt{q^0}} \left( p^- q^{\ast} - q^- p^{\ast} \right), \]  

(200)

where the last expression in each line used the choice \( \lambda_p = \lambda_q = 1 \). We have

\[ \langle pq \rangle [qp] = 2pq. \]

(201)

If \( p^\mu \) and \( q^\mu \) are real we have

\[ [qp] = \langle pq \rangle^\ast \operatorname{sign}(p^0) \operatorname{sign}(q^0). \]  

(202)

The spinor products are anti-symmetric

\[ \langle qp \rangle = -\langle pq \rangle, \quad [pq] = -[qp]. \]  

(203)

From the Schouten identity for the 2-dimensional antisymmetric tensor

\[ \varepsilon_{AB} \varepsilon_{CD} + \varepsilon_{BC} \varepsilon_{AD} + \varepsilon_{CA} \varepsilon_{BD} = 0. \]  

(204)

one derives

\[ \langle p_1 p_2 \rangle \langle p_3 p_4 \rangle + \langle p_2 p_3 \rangle \langle p_1 p_4 \rangle + \langle p_3 p_1 \rangle \langle p_2 p_4 \rangle = 0, \]

\[ [p_1 p_2] [p_3 p_4] + [p_2 p_3] [p_1 p_4] + [p_3 p_1] [p_2 p_4] = 0. \]  

(205)

The Fierz identity reads

\[ \langle p_1 + |\gamma_\mu| p_2 \rangle \langle p_3 - |\gamma^\nu| p_4 \rangle = 2 [p_1 p_4] \langle p_3 p_2 \rangle. \]  

(206)
Note that with our slight abuse of notation we identify a two-component Weyl spinor with a Dirac spinor, where the other two components are zero. Therefore

\[ \langle p_1 + |\gamma_\mu|p_2 + \rangle = \langle p_1 + |\sigma_\mu|p_2 + \rangle, \quad \langle p_3 - |\gamma_\mu|p_4 - \rangle = \langle p_3 - |\sigma^\mu|p_4 - \rangle. \quad (207) \]

We further have the reflection identities

\[ \langle p \pm |\gamma^{\mu_1}...\gamma^{\mu_{2n+1}}|q \pm \rangle = \langle q \mp |\gamma^{\mu_{2n+1}}...\gamma^{\mu_1}|p \mp \rangle, \]
\[ \langle p \pm |\gamma^{\mu_1}...\gamma^{\mu_{2n}}|q \mp \rangle = -\langle q \pm |\gamma^{\mu_{2n}}...\gamma^{\mu_1}|p \mp \rangle. \quad (208) \]

**Exercise 11:** The field strength in spinor notation: For a rank-2 tensor \( F_{\mu\nu} \) we define the spinor representation by

\[ F_{AB\dot{A}\dot{B}} = F_{\mu\nu}\sigma^\mu_A\sigma^\nu_B. \quad (209) \]

On the other hand we may decompose \( F_{\mu\nu} \) into a self dual part and an anti-self dual part,

\[ F_{\mu\nu} = F_{\mu\nu}^{\text{self dual}} + F_{\mu\nu}^{\text{anti-self dual}}, \quad (210) \]

with

\[ F_{\mu\nu}^{\text{self dual}} = \frac{1}{2}(F_{\mu\nu} + i\tilde{F}_{\mu\nu}), \quad F_{\mu\nu}^{\text{anti-self dual}} = \frac{1}{2}(F_{\mu\nu} - i\tilde{F}_{\mu\nu}). \quad (211) \]

The dual field strength \( \tilde{F}_{\mu\nu} \) was defined in eq. (29). Show that the spinor representations of the self dual part / anti-self dual part may be written as

\[ F_{AB\dot{A}\dot{B}}^{\text{self dual}} = \varepsilon_{AB}\bar{\phi}_{\dot{A}\dot{B}}, \quad F_{AB\dot{A}\dot{B}}^{\text{anti-self dual}} = \bar{\phi}_{AB}\varepsilon_{\dot{A}\dot{B}}, \quad (212) \]

where \( \phi_{AB} \) and \( \bar{\phi}_{\dot{A}\dot{B}} \) satisfy

\[ \phi_{AB} = \phi_{BA}, \quad \bar{\phi}_{\dot{A}\dot{B}} = \bar{\phi}_{\dot{B}\dot{A}}. \quad (213) \]

### 3.3.4 Polarisation vectors

It was a major break-through, when it was realised that also gluon polarisation vectors can be expressed in terms of two-component Weyl spinors [52-57]. The polarisation vectors of external gluons can be chosen as

\[ \varepsilon_+^{\mu}(p,q) = -\frac{\langle p + |\gamma_\mu|q + \rangle}{\sqrt{2}\langle p - |q + \rangle}, \quad \varepsilon_-^{\mu}(p,q) = \frac{\langle p - |\gamma_\mu|q - \rangle}{\sqrt{2}\langle p + |q - \rangle}, \quad (214) \]

where \( p \) is the momentum of the gluon and \( q \) is an arbitrary light-like reference momentum. The dependence on the reference spinors \(|q + \rangle\) and \(|q - \rangle\) which enters through the gluon polarisation vectors will drop out in gauge invariant quantities. We have the relations

\[ \varepsilon_\pm^{\mu}(p,q)p^{\mu} = 0, \quad \varepsilon_\pm^{\mu}(p,q)q^{\mu} = 0, \quad \varepsilon_+^{\mu}(p_1,q)\varepsilon_+^{\mu}(p_2,q) = 0, \quad \varepsilon_-^{\mu}(p_1,q)\varepsilon_-^{\mu}(p_2,q) = 0. \quad (215) \]
The polarisation sum is
\[
\sum_{\lambda \in \{+,-\}} \varepsilon^\lambda_\mu(p, q)\varepsilon^{-\lambda}_\nu(p, q) = -g_{\mu\nu} + 2\frac{P_\mu q_\nu + q_\mu P_\nu}{2pq}.
\]  
(216)

If \( p^\mu \) and \( q^\mu \) are real we have
\[
\left(\varepsilon^\lambda_\mu(p, q)\right)^* = \varepsilon^{-\lambda}_\mu(p, q).
\]
(217)

Changing the reference momentum will give a term proportional to the momentum of the gluon:
\[
\varepsilon^+(p, q_1) - \varepsilon^+(p, q_2) = \sqrt{2}\frac{\langle q_1 q_2 \rangle}{\langle q_1 p \rangle \langle p q_2 \rangle} p_\mu.
\]
(218)

Eq. (214) provides the desired explicit expression for the polarisation vectors. For each external particle \( i \) we may choose a light-like reference momentum \( q_i \). When calculating gauge-invariant quantities like a primitive amplitude the reference momentum \( q_i \) of two different external legs need not be equal. When calculating several gauge-invariant quantities we may change the reference momentum \( q_i \) of the external leg \( i \) from one calculation to the other. However we may not change the reference momentum \( q_i \) of the external leg \( i \) within one calculation of a gauge-invariant quantity. Let \( \lambda = (\lambda_1, \ldots, \lambda_n) \) be a helicity configuration. We denote the primitive helicity amplitude with cyclic order \((1, \ldots, n)\) by
\[
A_n^{(0)}(1^{\lambda_1}, 2^{\lambda_2}, \ldots, n^{\lambda_n}).
\]
(219)

For the spin- and colour summed matrix element squared we have then
\[
\sum_{\text{spins, colour}} |\mathcal{A}_n|^2 = g^{2n-4} \sum_{\sigma \in S_n/Z_n} \sum_{\pi \in S_n/Z_n} \sum_{\lambda_1 \in \{+,-\}} \ldots \sum_{\lambda_n \in \{+,-\}} \sum_{\nu} \sum_{\mu} 4 \text{Tr} (T^{\sigma_1 \sigma_2} \ldots T^{\sigma_n}) \text{Tr} (T^{\sigma_1 \sigma_2} \ldots T^{\sigma_n}) A_n^{(0)}(\sigma_1^{\lambda_1}, \ldots, \sigma_n^{\lambda_n}) A_n^{(0)}(\pi_1^{\lambda_1}, \ldots, \pi_n^{\lambda_n})^*.
\]
(220)

Let us shortly discuss the singular limit of cyclic-ordered helicity amplitudes. In the limit where one gluon \( j \) becomes soft, the primitive amplitudes behave as
\[
\lim_{p_j \to 0} A_n(p_1, \ldots, p_j^+, \ldots, p_n) = \sqrt{2} \frac{\langle p_{j-1} p_{j+1} \rangle}{\langle p_{j-1} p_j \rangle \langle p_{j+1} p_j \rangle} A_{n-1}(p_1, \ldots, p_{j-1}, p_{j+1}, \ldots, p_n),
\]
\[
\lim_{p_j \to 0} A_n(p_1, \ldots, p_j^-, \ldots, p_n) = \sqrt{2} \frac{[p_{j+1} p_{j-1}]}{[p_{j+1} p_j] [p_{j-1} p_j]} A_{n-1}(p_1, \ldots, p_{j-1}, p_{j+1}, \ldots, p_n).
\]
(221)

For a singular collinear limit the two collinear particles have to be adjacent in the cyclic order. Let us assume that the particles \( i \) and \( i + 1 \) become collinear as in eq. (102). We then have
\[
\lim_{p_i \parallel p_{i+1}} A_n^{(0)}(\ldots, p_i^{\lambda_i}, p_{i+1}^{\lambda_{i+1}}, \ldots) = \sum_{\lambda_i} \text{Split}(p_i, p_i, p_j, \lambda_i, \lambda_j) A_{n-1}^{(0)}(\ldots, p_i^{\lambda_i}, \ldots).
\]
(222)
For a singular contribution from the factorisation on propagator poles we have to divide the external particles into two consecutive subsets. Without loss of generality we may take these subsets as \( I_m = \{1, 2, \ldots, m\} \) and \( I_{n-m} = \{m+1, m+2, \ldots, n\} \). We set \( p = p_1 + p_2 + \ldots, p_m \). In the limit, where \( p^2 \) vanishes we have

\[
\lim_{p^2 \to 0} A_n^{(0)} (p_1, \ldots, p_n) = \sum_{\lambda} A_{m+1}^{(0)} (p_1, \ldots, p_m, -p^{-\lambda}) \frac{i}{p^2} A_{n-m+1}^{(0)} (p, p_{m+1}, \ldots, p_n). \tag{223}
\]

**Exercise 12:** Calculate the primitive helicity amplitude \( A_{4}^{(0)} (1^-, 2^+, 3^+, 4^-) \).

### 3.3.5 Massive spinors

As in the massless case, a massive spinor satisfying the Dirac equation has a two-fold degeneracy. We will label the two different eigenvectors by “+” and “-”. Let \( p \) be a massive four-vector with \( p^2 = m^2 \), and let \( q \) be an arbitrary light-like four-vector. With the help of \( q \) we can construct a light-like vector \( p^\gamma \) associated to \( p \):

\[
p^\gamma = p - \frac{p^2}{2p \cdot q}. \tag{224}
\]

We define \([55, 58, 59]\)

\[
u(p, +) = \frac{1}{\langle p^\gamma + | q^\gamma \rangle} \langle p^\gamma + | q^\gamma \rangle | q^\gamma \rangle \tag{225}
\]

\[
u(p, -) = \frac{1}{\langle p^\gamma - | q^\gamma \rangle} \langle p^\gamma - | q^\gamma \rangle | q^\gamma \rangle \tag{226}
\]

For the conjugate spinors we have

\[
\bar{\nu}(p, +) = \frac{1}{\langle q^\gamma - | p^\gamma + \rangle} \langle q^\gamma - | p^\gamma + \rangle | q^\gamma + \rangle \tag{225}
\]

\[
\bar{\nu}(p, -) = \frac{1}{\langle q^\gamma + | p^\gamma - \rangle} \langle q^\gamma + | p^\gamma - \rangle | q^\gamma - \rangle \tag{226}
\]

These spinors satisfy the Dirac equations of eq. (169), the orthogonality relations of eq. (170) and the completeness relations of eq. (171). We further have

\[
\bar{u}(p, \tilde{\lambda}) \gamma^\mu u(p, \lambda) = 2p^\mu \delta_{\lambda\tilde{\lambda}}, \quad v(p, \tilde{\lambda}) \gamma^\mu v(p, \lambda) = 2p^\mu \delta_{\lambda\tilde{\lambda}}. \tag{227}
\]

In the massless limit the definition reduces to

\[
u(p, +) = v(p, -) = | p^\gamma + \rangle, \quad \bar{u}(p, +) = \bar{v}(p, -) = \langle p^\gamma + |, \quad u(p, -) = v(p, +) = | p^\gamma - \rangle, \quad \bar{u}(p, -) = \bar{v}(p, +) = \langle p^\gamma - | \tag{228}
\]

and the spinors are independent of the reference spinors \( | q^\gamma + \rangle \) and \( \langle q^\gamma - | \).

**Exercise 13:** Show that the spinors defined in eq. (225) and in eq. (226) satisfy the Dirac equations of eq. (169), the orthogonality relations of eq. (170) and the completeness relations of eq. (171).
3.3.6 The Majorana representation

The Weyl spinors involve the light-cone coordinates \( p^\perp = (p^1 + ip^2)/\sqrt{2} \) and \( p^{\perp*} = (p^1 - ip^2)/\sqrt{2} \). For real momenta \( p^\mu = (p^0, p^1, p^2, p^3) \) the light-cone coordinates are in general complex numbers. When we compute helicity amplitudes numerically this forces us to carry out all operations (like addition, multiplication) with complex numbers. It would be faster, if we could devise a method, which allows us to carry out a significant fraction of the operations with real numbers only. This can be achieved in the Majorana representation. The Majorana representation is obtained from the Weyl representation by a unitary transformation

\[
\gamma_\mu^\text{Majorana} = U \gamma_\mu^\text{Weyl} U^\dagger, \tag{229}
\]

where

\[
U = \frac{1}{2} \begin{pmatrix} 1 + \sigma_2 & -i(1 - \sigma_2) \\ i(1 - \sigma_2) & 1 + \sigma_2 \end{pmatrix}. \tag{230}
\]

The unitary matrix \( U \) has the additional property of being hermitian as well. Thus we have

\[
U^\dagger = U^{-1} = U. \tag{231}
\]

Let us now write down the Dirac matrices in the Majorana representation explicitly. We have

\[
\gamma_0^\text{Majorana} = i \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \gamma_1^\text{Majorana} = i \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{232}
\]

\[
\gamma_2^\text{Majorana} = i \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \gamma_3^\text{Majorana} = i \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\]

The matrix \( \gamma_5 \) is given in the Majorana representation by

\[
\gamma_5^\text{Majorana} = i \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{233}
\]

Note that all matrices are purely imaginary. Let us now consider massless spinors in the Majorana representation. In general we have

\[
\bar{u}_\text{Majorana} = \bar{u}_\text{Weyl} U^\dagger, \quad u_\text{Majorana} = U u_\text{Weyl}. \tag{234}
\]

As a basis for the two independent helicity states we could choose \( u_+ \) and \( u_- \). However, it will be more convenient to choose a basis given by

\[
(\bar{u}_a, \bar{u}_b) = (\bar{u}_+, \bar{u}_-)^T S^\dagger, \quad \begin{pmatrix} u_a \\ u_b \end{pmatrix} = S \begin{pmatrix} u_+ \\ u_- \end{pmatrix}, \tag{235}
\]

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where

\[
S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}.
\]

The matrix \( S \) is unitary:

\[
S^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} = S^\dagger.
\]

We then obtain (for \( \lambda_p = 1 \))

\[
u^{\text{Majorana}}_a = \frac{2^\frac{1}{2} i}{2\sqrt{p}} \begin{pmatrix} -p^0 - p^2 + p^3 \\ -p^1 \\ p^0 - p^2 - p^3 \end{pmatrix},
\]

\[
u^{\text{Majorana}}_b = \frac{2^\frac{1}{2} i}{2\sqrt{p}} \begin{pmatrix} -p^0 - p^2 + p^3 \\ p^0 - p^2 - p^3 \\ -p^1 \end{pmatrix},
\]

For the \( \bar{u} \)-spinors we find

\[
\bar{\nu}^{\text{Majorana}}_a = \frac{2^\frac{1}{2}}{2\sqrt{p}} \begin{pmatrix} p^0 - p^2 - p^3, -p^1, p^0 + p^2 - p^3 \end{pmatrix},
\]

\[
\bar{\nu}^{\text{Majorana}}_b = \frac{2^\frac{1}{2}}{2\sqrt{p}} \begin{pmatrix} -p^1, -p^0 + p^2 + p^3, -p^0 - p^2 + p^3, p^1 \end{pmatrix}.
\]

We recall that in the Majorana representation the Dirac matrices are purely imaginary. In addition we may choose a spinor basis, where the \( u \)-spinors are purely imaginary and the \( \bar{u} \)-spinors are real. The imaginary unit occurs only as a prefactor. We may separate this prefactor from the rest, perform numerically the calculation with real numbers and multiply by the appropriate power of imaginary units \( i \) in the end. In this way the number of operations with complex numbers is reduced significantly.

### 3.4 Twistors

Twistors are closely related to spinors. Although this is a more formal topic and not related to efficiency issues, it is best discussed together with spinors. We first delve directly into twistors in momentum space and the way they are used in connection with scattering amplitudes. We then discuss twistors in position space and the twistor transform.

#### 3.4.1 Momentum twistors

Let \( p = (p_1, \ldots, p_n) \) be a momentum configuration for the amplitude \( A_n^{(0)} \). In physical applications all momenta are real, but nothing stops us to consider the amplitude \( A_n^{(0)} \) as a function of complex momenta \( p_j^\mu \). A momentum configuration is a set of \( n \) real or complex four-vectors, subject to the constraints of momentum conservation

\[
p_1 + p_2 + \ldots + p_n = 0
\]
and the on-shell conditions
\[ p_j^2 = 0, \quad j \in \{1, \ldots, n\}. \quad (240) \]

Thus there are only \((3n - 4)\) independent variables. It is sometimes convenient to have a parametrisation of the momentum configurations of an \(n\)-point amplitude free of constraints. Momentum twistors allow us to do that. Let us first focus on the on-shell conditions. The \(\sigma^\mu\)- and \(\bar{\sigma}^\mu\)-matrices from the Weyl representation can be used to convert any four-vector to a bispinor representation:
\[ p_{\dot{A}B} = p_\mu \bar{\sigma}^{\mu}_{B\dot{A}}, \quad \text{or} \quad p_{AB} = p_\mu \sigma^{\mu}_{AB}. \quad (241) \]

The inverse equations read
\[ p_\mu = \frac{1}{2} p_{\dot{A}B} \sigma^{\mu}_{B\dot{A}}, \quad \text{or} \quad p_\mu = \frac{1}{2} p_{AB} \bar{\sigma}^{\mu}_{AB}. \quad (242) \]

If \(p\) is light-like, the bispinor factorises into a dyad
\[ p_{\dot{A}B} = p_\dot{A} p_\dot{B}, \quad p_{AB} = p_A p_B \quad (243) \]

and we have
\[ p_\mu = \frac{1}{2} p_\dot{A} \sigma^{\mu}_{\dot{A}B} p_\dot{B} = \frac{1}{2} p_A \bar{\sigma}^{\mu}_{AB} p_B. \quad (244) \]

Thus giving for each external particle a pair of spinors \(p_A = |p+\rangle\) and \(p_A = \langle p+|\) allow us to reconstruct the four-vector \(p_\mu\) with the help of eq. (244). Furthermore, this four-vector will be automatically on-shell (i.e. light-like). For an arbitrary pair of spinors \(|p+\rangle\) and \(\langle p+|\) the resulting four-vector will be in general complex. We see that the use of spinor variables \((p_a, p_{\dot{A}}) = (|p+\rangle, \langle p+|)\) for each external leg trivialises the on-shell constraints.

We still have to think about the constraint of momentum conservation. Here, momentum twistors enter the game \([60–63]\). We denote a momentum twistor by
\[ Z_\alpha = (p_A, \mu_\dot{A}) = (|p+\rangle, \langle \mu+|)\quad (245) \]

The index \(\alpha\) takes the values \(\alpha \in \{1, 2, \dot{1}, \dot{2}\}\). The scaling behaviour of a momentum twistor is
\[ (p_A, \mu_\dot{A}) \rightarrow (\lambda p_A, \lambda \mu_\dot{A}), \quad (246) \]

and therefore \(Z_\alpha \in \mathbb{CP}^3\). As with momentum vectors, we will not always write the index \(\alpha\) explicitly. Let us now consider \(n\) momentum twistors \(Z_1, Z_2, \ldots, Z_n\). The ordered set \((Z_1, Z_2, \ldots, Z_n)\) defines a configuration of \(n\) momentum vectors with associated spinors as follows: We first define the spinors by
\[ |p_i+\rangle = |p_i+\rangle, \quad \langle p_i+| = -\frac{\langle p_{i+1}p_i \rangle}{\langle p_{i-1}p_i \rangle} \langle \mu_{i-1}+| \quad (247) \]

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Figure 4: Relation between the ordered set of $n$ four-vectors $(y_1, \ldots, y_n)$ and the ordered set of $n$ four-vectors $(p_1, \ldots, p_n)$ constrained by $p_1 + \ldots + p_n = 0$ plus an additional four-vector $Q$. Given $(y_1, \ldots, y_n)$ we obtain $Q$ as $y_n$ and the $p_i$’s as $y_i - y_{i+1}$. Given the $(p_1, \ldots, p_n)$ alone we cannot construct the $(y_1, \ldots, y_n)$, we need in addition the four-vector $Q$.

where indices are understood modulo $n$. The momentum vector $p^\mu_i$ is then given by

\[ p^\mu_i = \frac{1}{2} \langle p_i + |\sigma^\mu| p_i+ \rangle. \tag{248} \]

Each momentum vector $p^\mu_i$ is massless:

\[ p_i^2 = 0. \tag{249} \]

The configuration of $n$ momentum vectors satisfies momentum conservation:

\[ \sum_{i=1}^{n} p^\mu_i = 0. \tag{250} \]

We see that the use of momentum twistors trivialises the on-shell constraints and the constraint from momentum conservation. Therefore we may start from a momentum twistor configuration $(Z_1, \ldots, Z_n)$, where all momentum twistor variables can be chosen freely without any constraints. We then obtain the associated momentum configuration with the help of eq. (247) and eq. (248). This momentum configuration will automatically satisfy momentum conservation and the on-shell conditions.

Let us look into this construction in more detail: Starting from an ordered set $(p_1, p_2, \ldots, p_n)$ of light-like four-vectors satisfying momentum conservation $p_1 + p_2 + \ldots + p_n = 0$ and an arbitrary four-vector $Q$ we define four-vectors $y_j$ by

\[ y_j = Q + \sum_{i=1}^{j} p_i. \tag{251} \]

In the reverse direction we have

\[ p_i = y_i - y_{i-1}, \quad Q = y_n. \tag{252} \]
The geometric situation is shown in fig. (4). We then define
\[ \langle \mu_i+ \rangle = \langle p_i- | y_i = \langle p_i- | y_{i-1}. \] (253)

In the reverse direction we have given \(|p_i+\rangle \) and \(\langle \mu_i+ \rangle\)
\[ |p_{i+1}+\rangle \langle \mu_i+ | - |p_i+\rangle \langle \mu_{i+1}+ | = (|p_{i+1}+\rangle \langle p_i- | - |p_i+\rangle \langle p_{i+1}- |) y_i = \langle p_{i+1} | y_i, \] (254)

and therefore
\[ y_i = \frac{1}{\langle p_i p_{i+1} \rangle} (|p_{i+1}+\rangle \langle \mu_i+ | - |p_i+\rangle \langle \mu_{i+1}+ |). \] (255)

From
\[ \dot{y}_i = y_i - y_{i-1}, \] (256)

and writing the resulting expression as a dyad one finds the expression given in eq. (247) for
\[ \langle p_i+ | \) in terms of \(|p_i+\rangle \) and \(\langle \mu_i+ \rangle\).

For momentum twistors one defines a four-bracket through
\[ \langle Z_1, Z_2, Z_3, Z_4 \rangle = \epsilon^{\alpha \beta \gamma \delta} Z_1,_{\alpha} Z_2,_{\beta} Z_3,_{\gamma} Z_4,_{\delta} \]
\[ = - \langle p_1 p_2 \rangle [\mu_3, \mu_4] - \langle p_1 p_3 \rangle [\mu_4, \mu_2] - \langle p_1 p_4 \rangle [\mu_2, \mu_3] \]
\[ - \langle p_3 p_4 \rangle [\mu_1, \mu_2] - \langle p_4 p_2 \rangle [\mu_1, \mu_3] - \langle p_2 p_3 \rangle [\mu_1, \mu_4], \] (257)

with \(\epsilon^{1234} = 1\). The four-bracket equals the determinant of the 4 \times 4-matrix, where the rows (or equivalently the columns) are given by the momentum twistors \(Z_j\):
\[ \langle Z_1, Z_2, Z_3, Z_4 \rangle = \det (Z_1, Z_2, Z_3, Z_4). \] (258)

Note that the spinor products \(\langle pq \rangle \) and \([qp] \) can also be viewed as determinants of 2 \times 2-matrices:
\[ \langle pq \rangle = - \epsilon^{AB} p_A q_B = - \det (p_A, q_B), \]
\[ [qp] = \epsilon^{AB} q_A p_B = \det (q_A, p_B). \] (259)

We defined the momentum twistors as
\[ Z_{\alpha} = (p_A, \mu_{\dot{A}}) = (|p+\rangle, \langle \mu+ |), \] (260)
which scales as \((p_A, \mu_{\dot{A}}) \rightarrow (\lambda p_A, \lambda \mu_{\dot{A}})\). Of course there is a second possibility, transforming \(p_A\) and keeping \(p_{\dot{A}}\):
\[ W_{\alpha} = (\mu_{\dot{A}}, p_A) = (|\mu+\rangle, \langle p+ |), \] (261)
To recover the spinors we now have

\[ |p_{i+}\rangle = -\frac{[p_{i+1}p_i]}{[p_{i+1}p_i][p_ip_{i-1}]} |\mu_{i-1}\rangle - \frac{[p_{i-1}p_{i+1}]}{[p_{i+1}p_i][p_ip_{i-1}]} |\mu_i+\rangle - \frac{[p_ip_{i-1}]}{[p_{i+1}p_i][p_ip_{i-1}]} |\mu_{i+1}\rangle, \]

\[ \langle p_{i+} | = \langle p_{i+}. \]  

(262)

The $W_\alpha$-momentum twistor scales as

\[ (\mu_A, p_A) \rightarrow (\lambda^{-1}\mu_A, \lambda^{-1}p_A). \]

(263)

For the four-bracket we now have

\[ \langle W_1, W_2, W_3, W_4 \rangle = \varepsilon^{\alpha\beta\gamma\delta} W_{1,\alpha} W_{2,\beta} W_{3,\gamma} W_{4,\delta} \]

\[ = -\langle \mu_1 \mu_2 \rangle [p_3, p_4] - \langle \mu_1 \mu_3 \rangle [p_4, p_2] - \langle \mu_1 \mu_4 \rangle [p_2, p_3] \]

\[ - \langle \mu_3 \mu_4 \rangle [p_1, p_2] - \langle \mu_4 \mu_3 \rangle [p_1, p_3] - \langle \mu_2 \mu_3 \rangle [p_1, p_4]. \]

(264)

**Exercise 14:** Show that the momenta defined by eq. (262) satisfy momentum conservation.

### 3.4.2 The twistor transform

Let us do a small excursion and discuss the original ideas of the twistor programme initiated by R. Penrose. This paragraph is more mathematical. Readers not familiar with Grassmannians or flag varieties may skip this paragraph in a first reading. We will present a definition of Grassmannians and flag varieties in section 6.1. You may look up the definitions there. The rest of this report will not depend on this paragraph.

In the previous paragraph we considered momentum twistors. Originally, twistors were introduced in position space with the aim of relating residues of functions in projective twistor space to solutions of free field equations. This relation is provided by the twistor transform.

Let us now review this construction [64, 65]. The twistor transform starts with a double fibration of a space $F$ over two spaces $P$ and $M$.

\[ \begin{user Equation} F \end{user Equation} \]

\[ \pi_P \]

\[ P \]

\[ \pi_M \]

\[ M \]

(265)

$F$ is called the **correspondence space**. We may think of the space $P$ as **projective twistor space** and of the space $M$ as **Minkowski space**. To be precise, $M$ corresponds to a compactification of complex Minkowski space. The two projections are denoted by $\pi_P$ and $\pi_M$. In order to describe the correspondence space $F$, let us first introduce yet another space. We set $T = \mathbb{C}^4$ and call $T$ **twistor space**. Sub-spaces of $T$ are denoted by $T_j$. We define the correspondence space to be

\[ F = \{ (T_1, T_2) | T_1 \subset T_2 \subset T, \dim_{\mathbb{C}} (T_1) = 1, \dim_{\mathbb{C}} (T_2) = 2 \}. \]

(266)
The space $F$ is a flag variety. Elements of $F$ are called flags and are given by pairs $(T_1, T_2)$, where $T_2$ is a two-dimensional sub-space of $T$ and $T_1$ is a one-dimensional sub-space of $T_2$ (and of course of $T$).

We define the projections $\pi_P$ and $\pi_M$ by

$$
\pi_P(T_1, T_2) = T_1, \quad \pi_M(T_1, T_2) = T_2.
$$

Thus $P$ consists of all lines through the origin in $T$ and $M$ consists of all planes through the origin in $T$. In other words, we have

$$
P \cong \text{Gr}_{1,4}(\mathbb{C}) = \mathbb{CP}^3, \quad M \cong \text{Gr}_{2,4}(\mathbb{C}).
$$

On $T$ we choose coordinates

$$
Z_\alpha = (\lambda_A, \pi_B).
$$

We apologise for the slightly confusing notation: $\pi_1$ and $\pi_2$ (both with a dot) denote two of the four coordinates on $T$, while $\pi_P$ and $\pi_M$ denote projections. We mentioned that $M$ can be considered as Minkowski space. In order to see this, consider an affine chart of $\text{Gr}_{2,4}(\mathbb{C})$ with coordinates

$$
\begin{pmatrix}
  x_{11} & x_{12} \\
  x_{21} & x_{22} \\
  0 & -1 \\
  1 & 0
\end{pmatrix}.
$$

The two columns of eq. (270) correspond to two linearly independent vectors, which span a plane $T_2$ in $T$. We have the usual isomorphisms

$$
x_{AB} = x_{\mu} \sigma^{\mu}_{AB}, \quad x^\mu = \frac{1}{2} x_{AB} \bar{\sigma}^{\mu BA},
$$

giving us the relation between a point with coordinates $(x_{11}, x_{12}, x_{21}, x_{22})$ of $\text{Gr}_{2,4}(\mathbb{C})$ and a point $x^\mu$ in Minkowski space. Let us now consider a point $T_2 \in \text{Gr}_{2,4}(\mathbb{C})$ with coordinates as in eq. (270). Note that a point of $\text{Gr}_{2,4}(\mathbb{C})$ is a two-dimensional plane in $T$. A point $(\lambda_A, \pi_B) \in T$ belongs to the plane $T_2$ if and only if

$$
\lambda_A = x_{AB} \epsilon^{BC} \pi_C \quad \text{or} \quad \lambda_A = x_{AB} \pi^B.
$$

Eq. (272) is called the **incidence relation**.

Let us now consider the one-dimensional sub-spaces $T_1$ of the plane $T_2$. Given a two-dimensional plane $T_2 \subset T$ the set of all one-dimensional lines $T_1 \subset T_2$ through the origin is isomorphic to $\mathbb{CP}^1$. We may use $[\pi_B]$ as homogeneous coordinates for $\mathbb{CP}^1$. Putting everything together we have for the correspondence space $F$

$$
F = \text{Gr}_{1,2}(\mathbb{C}) \times \text{Gr}_{2,4}(\mathbb{C}) = \mathbb{CP}^1 \times \text{Gr}_{2,4}(\mathbb{C}).
$$
We have \( \dim \mathbb{C} F = 5 \). For a point \( w \in F \) we use the local coordinates

\[
    w = ([\pi_A], x_{BC}).
\]

The projections are then

\[
    \pi_M : F \to \text{Gr}_{2,4}(\mathbb{C}),
    \pi_M(w) = x_{BC},
\]

and

\[
    \pi_P : F \to \mathbb{CP}^3
    \pi_P(w) = [x_{AB} \pi^B, \pi^C].
\]

For \( x_{AB} \in M \) the fibre is

\[
    \pi^{-1}_M(x_{AB}) \cong \mathbb{CP}^1.
\]

For \( [\lambda_A : \pi_B] \in \mathbb{CP}^3 \) the fibre is

\[
    \pi^{-1}_P([\lambda_A : \pi_B]) \cong \mathbb{CP}^2.
\]

This can be seen as follows: A point \( [\lambda_A : \pi_B] \in \mathbb{CP}^3 \) defines a one-dimensional sub-space \( T_1 \) of \( T \). The fibre consists of all two-dimensional sub-spaces \( T_2 \) of \( T \), which contain \( T_1 \). These can be specified by choosing an independent vector not in \( T_1 \). As only the direction matters, we see that the set of all possible choices corresponds to \( \mathbb{CP}^2 \).

Now let \( f(\lambda_A, \pi_B) \) be a function on \( T \), homogeneous of degree \((-2)\) and holomorphic on \( U_1 \cap U_2 \), where \( U_B \) is the chart with \( \pi_B \neq 0 \). We consider the contour integral

\[
    \phi(x) = \frac{1}{2\pi i} \oint_{\mathcal{C}} d\pi^E \pi_E f\left( x_{AB} \pi^B \pi^C, \pi^D \right),
\]

where the contour \( \mathcal{C} \) is a closed curve in \( U_1 \cap U_2 \). Since \( f \) is required to be homogeneous of degree \((-2)\), the integral is actual an integral on projective twistor space \( P \) and is called the twistor transform (or the Penrose transform) of \( f \). The field \( \phi \) is a scalar field (obviously with helicity 0) and satisfies

\[
    \Box \phi(x) = 0.
\]

The field \( \phi \) is therefore a free field. Eq. \(280\) is easily verified: With

\[
    \Box = \partial_{\rho} \partial^{\rho} = \frac{1}{2} \varepsilon_{AC} \varepsilon_{BD} \frac{\partial}{\partial x_{AB}} \frac{\partial}{\partial x_{CD}},
\]

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we have

\[ \Box \phi(x) = \frac{1}{2} \varepsilon_{AC} \varepsilon_{BD} \frac{1}{2\pi i} \oint d\pi^E \pi^E \frac{\partial^2 f}{\partial \lambda_A \partial \lambda_C} \varepsilon^{BF} \pi^F \varepsilon^{DG} \pi^G \]

\[ = \frac{1}{2} \frac{1}{2\pi i} \oint d\pi^E \pi^E \left( \varepsilon_{AC} \frac{\partial^2 f}{\partial \lambda_A \partial \lambda_C} \right) \] \[ \pi\pi = 0. \] \hspace{1cm} (282)

Note that in this expressions we obtain a product of two zeros: One factor of zero comes from the spinor product \([\pi\pi]\), the other from the second derivative, which is symmetric and vanishes when contracted into the anti-symmetric tensor \(\varepsilon_{AC}\).

We may generalise this construction to free fields with non-zero helicities as follows: Let us first consider

\[ \phi_{\dot{A}_1 \ldots \dot{A}_n}(x) = \frac{1}{2\pi i} \oint d\pi^E \pi^E \pi_{\dot{A}_1} \ldots \pi_{\dot{A}_n} f \left( x_{AB} \varepsilon^{BC} \pi_C, \pi_D \right), \] \hspace{1cm} (283)

where \(f\) is homogeneous of degree \((-2-n)\). This defines a field of helicity \(+n/2\). The field \(\phi_{\dot{A}_1 \ldots \dot{A}_n}\) is symmetric in all indices \(\{\dot{A}_1, \ldots, \dot{A}_n\}\). The field satisfies

\[ \frac{\partial}{\partial x_{A\dot{A}_1}} \phi_{\dot{A}_1 \ldots \dot{A}_n}(x) = 0, \] \hspace{1cm} (284)

due to \([\pi\pi] = 0\). Let us further consider

\[ \phi^{A_1 \ldots A_n}(x) = \frac{1}{2\pi i} \oint d\pi^E \pi^E \frac{\partial}{\partial \lambda_{A_1}} \ldots \frac{\partial}{\partial \lambda_{A_n}} f \left( x_{AB} \varepsilon^{BC} \pi_C, \pi_D \right), \] \hspace{1cm} (285)

where \(f\) is now homogeneous of degree \((-2+n)\). This defines a field of helicity \(-n/2\). Again, the field \(\phi^{A_1 \ldots A_n}\) is symmetric in all indices \(\{A_1, \ldots, A_n\}\). We now have

\[ \varepsilon_{A\dot{A}_1} \frac{\partial}{\partial x_A} \phi^{A_1 \ldots A_n}(x) = 0, \] \hspace{1cm} (286)

due to

\[ \varepsilon_{A\dot{A}_1} \frac{\partial^2 f}{\partial \lambda_A \partial \lambda_{A_1}} = 0. \] \hspace{1cm} (287)

### 3.5 Off-shell recurrence relations

Let us now return to the master formula for an observable, given in eq. (89). For physical particles the momenta are real and a phase space point (i.e. a momentum configuration of \(n\) four-vectors, where two four-vectors correspond to the given incoming momenta, all four-vectors are real and
This will evaluate to a complex number. In this paragraph we will review an efficient algorithm to compute this complex number, avoiding Feynman diagrams. Without loss of generality we will take the cyclic order to be \((1, \ldots, n)\).

Off-shell recurrence relations \([70]\) build primitive amplitudes from smaller building blocks, usually called cyclic-ordered off-shell currents. **Off-shell currents** are objects with \(j\) on-shell legs and one additional leg off-shell, with \(j\) ranging from 1 to \((n - 1)\). Momentum conservation is satisfied. It should be noted that off-shell currents are not gauge-invariant objects. Recurrence relations relate off-shell currents with \(j\) legs to off-shell currents with fewer legs. The recursion starts with \(j = 1\):

\[
J_{\mu}(p_{1}^{\lambda_{1}}) = \varepsilon_{\mu}^{\lambda_{1}}(p_{1}, q_{1}). \tag{289}
\]

\(\varepsilon_{\mu}^{\lambda_{1}}\) is the polarisation vector of the gluon with momentum \(p_{1}\) and helicity \(\lambda_{1}\). \(q_{1}\) an arbitrary light-like reference momentum. The recursive relation states that a gluon couples to other gluons only via the three- or four-gluon vertices. Off-shell currents with \(j \geq 2\) are computed as

\[
J_{\mu}(p_{1}^{\lambda_{1}}, \ldots, p_{j}^{\lambda_{j}}) = \frac{-i}{p_{i,j}^{2}} J_{\mu}^{\text{trunc}}(p_{1}^{\lambda_{1}}, \ldots, p_{j}^{\lambda_{j}}), \tag{290}
\]

where

\[
p_{i,j} = p_{i} + p_{i+1} + \ldots + p_{j} \tag{291}
\]

and \(J_{\mu}^{\text{trunc}} = g_{\mu\nu} J_{\nu}^{\text{trunc}}\) is given by

\[
J_{\mu}^{\text{trunc}}(p_{1}^{\lambda_{1}}, \ldots, p_{j}^{\lambda_{j}}) = \sum_{k=1}^{j-1} V_{3}^{\mu\nu\rho}(-p_{1,j}, p_{1,k}, p_{k+1,j})J_{\nu}(p_{1}^{\lambda_{1}}, \ldots, p_{k}^{\lambda_{k}})J_{\rho}(p_{k+1}^{\lambda_{k+1}}, \ldots, p_{j}^{\lambda_{j}}) \tag{292}
\]

\[
+ \sum_{k=1}^{j-2} \sum_{l=k+1}^{j-1} V_{4}^{\mu\nu\rho\sigma} J_{\nu}(p_{1}^{\lambda_{1}}, \ldots, p_{k}^{\lambda_{k}})J_{\rho}(p_{k+1}^{\lambda_{k+1}}, \ldots, p_{l}^{\lambda_{l}})J_{\sigma}(p_{l+1}^{\lambda_{l+1}}, \ldots, p_{j}^{\lambda_{j}}),
\]

where \(V_{3}\) and \(V_{4}\) are the cyclic-ordered three-gluon and four-gluon vertices

\[
V_{3}^{\mu\nu\rho}(p_{1}, p_{2}, p_{3}) = i \left[ g^{\mu\nu}(p_{1}^{\rho} - p_{2}^{\rho}) + g^{\nu\rho}(p_{2}^{\mu} - p_{3}^{\mu}) + g^{\rho\mu}(p_{3}^{\nu} - p_{1}^{\nu}) \right], \tag{293}
\]

\[
V_{4}^{\mu\nu\rho\sigma} = i \left( 2g^{\mu\rho}g^{\nu\sigma} - g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho} \right).
\]
Figure 5: Off-shell recurrence relation: In an off-shell current particle $n+1$ is kept off-shell. This allows to express an off-shell current with $n$ on-shell legs in terms of currents with fewer legs.

The recurrence relation is shown pictorially in fig. 5. The gluon current $J_{\mu}$ is conserved:

$$\left( \sum_{k=1}^{j} p_{k}^{\mu} \right) J_{\mu} (p_1, \ldots, p_j) = 0.$$ (294)

The gluon current further satisfies the photon decoupling relation

$$J_{\mu} (p_1, p_2, p_3, \ldots, p_j) + J_{\mu} (p_2, p_1, p_3, \ldots, p_j) + \ldots + J_{\mu} (p_2, p_3, \ldots, p_j, p_1) = 0$$ (295)

and the reflection identity

$$J_{\mu} (p_1, p_2, p_3, \ldots, p_j) = (-1)^{j+1} J_{\mu} (p_j, \ldots, p_3, p_2, p_1).$$ (296)

With the help of the off-shell currents we obtain the following algorithm for the computation of a primitive amplitude:

**Algorithm 2.** Calculation of a primitive helicity amplitude $A_n^{(0)}(1^{\lambda_1}, \ldots, n^{\lambda_n})$ from off-shell currents.

1. For $j = 1, 2, \ldots, (n-1)$ compute all off-shell currents for the ordered sequence of on-shell legs $(1, \ldots, n-1)$, starting from the one-currents $J_{\mu} (p_1^{\lambda_1})$, $J_{\mu} (p_2^{\lambda_2})$, ..., $J_{\mu} (p_{n-1}^{\lambda_{n-1}})$, then the two-currents $J_{\mu} (p_1^{\lambda_1}, p_2^{\lambda_2})$, ..., $J_{\mu} (p_{n-2}^{\lambda_{n-2}}, p_{n-1}^{\lambda_{n-1}})$, up to the $(n-1)$-current $J_{\mu} (p_1^{\lambda_1}, \ldots, p_{n-1}^{\lambda_{n-1}})$. At each step re-use the results for the already computed lower-point currents.

2. The primitive amplitude is given by

$$A_n^{(0)}(1^{\lambda_1}, \ldots, n^{\lambda_n}) = \varepsilon_{\mu}^{\lambda_n} (p_n, q_n) J_{\mu, \text{trunc}} (p_1^{\lambda_1}, \ldots, p_{n-1}^{\lambda_{n-1}}).$$ (297)
It can be shown that the scaling behaviour of this algorithm with the number of external particles \( n \) is \( n^4 \). This polynomial behaviour is much better than the factorial growth of the algorithm based on Feynman diagrams. The re-use of the results for the already computed lower-point currents is essential in achieving this polynomial behaviour. We may even improve this scaling behaviour further. Computing the contributions from the four-gluon vertex in the recurrence relations involves the most operations and dominates the scaling behaviour. We can reduce the scaling behaviour from \( n^4 \) down to \( n^3 \) by eliminating the four-gluon vertex with the help of the auxiliary tensor field introduced in section (3.1). Let us denote by

\[
V_{4}^{\mu\nu[\rho\sigma]} = \frac{i}{\sqrt{2}} (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho})
\]  

the cyclic-ordered gluon-gluon-tensor vertex. For \( j \geq 2 \) we introduce a tensor current \( J_{[\mu\nu]} \) through

\[
J_{[\mu\nu]}(p_1^{\lambda_1}, \ldots, p_j^{\lambda_j}) = -ig_{\mu\rho}g_{\nu\sigma} \sum_{k=1}^{j-1} V_{4}^{\alpha\beta[\rho\sigma]} J_{\alpha}(p_1^{\lambda_1}, \ldots, p_k^{\lambda_k})J_{\beta}(p_{k+1}^{\lambda_{k+1}}, \ldots, p_j^{\lambda_j}).
\]  

This equation is shown pictorially in fig. (6). With the help of the auxiliary tensor current we may compute the gluon current as

\[
J_{\mu,\text{trunc}}(p_1^{\lambda_1}, \ldots, p_j^{\lambda_j}) = \sum_{k=1}^{j-1} V_{3}^{\mu\nu\rho} (-p_{1,k}, p_{1,k}, p_{k+1,j})J_{\nu}(p_1^{\lambda_1}, \ldots, p_k^{\lambda_k})J_{\rho}(p_{k+1}^{\lambda_{k+1}}, \ldots, p_j^{\lambda_j})
\]  

\[
+ \sum_{k=2}^{j-2} V_{4}^{\mu\nu[\rho\sigma]} J_{\nu}(p_1^{\lambda_1}, \ldots, p_k^{\lambda_k})J_{[\rho\sigma]}(p_{k+1}^{\lambda_{k+1}}, \ldots, p_j^{\lambda_j})
\]  

\[
+ \sum_{k=2}^{j-1} V_{4}^{\nu\rho[\mu\sigma]} J_{[\rho\sigma]}(p_1^{\lambda_1}, \ldots, p_k^{\lambda_k})J_{\nu}(p_{k+1}^{\lambda_{k+1}}, \ldots, p_j^{\lambda_j}).
\]  

Eq. (300) is shown pictorially in fig. (7). Now all currents involve only trivalent vertices, giving us a \( n^3 \)-scaling behaviour.
Exercise 15: Recurrence relations are also useful for the following problem: Count the number of Feynman diagrams contributing to the cyclic-ordered amplitude $A_n^{(0)}$.

Hint: Find a recurrence relation for the number of diagrams. Consider a particular diagram: Follow the leg $n$ inwards into the diagram until you hit the first vertex (either a three-gluon vertex or a four-gluon vertex). Attached to this vertex are sub-graphs with fewer legs.


4 MHV amplitudes and MHV expansion

This section is centred around maximally helicity violating amplitudes. These amplitudes have for any number of external particles an astonishingly simple formula.

4.1 The Parke-Taylor formulæ

Tree-level primitive amplitudes for specific helicity combinations are remarkably simple or vanish altogether. Parke and Taylor conjectured [71] that for \( n \geq 4 \)

\[
A_n^{(0)}(1^+,2^+,\ldots,n^+) = 0, \\
A_n^{(0)}(1^+,2^+,\ldots,j^-,\ldots,n^+) = 0, \\
A_n^{(0)}(1^+,2^+,\ldots,j^-,\ldots,k^-,\ldots,n^+) = i\left(\sqrt{2}\right)^{n-2}\frac{\langle jk\rangle^4}{\langle 12\rangle\ldots\langle n1\rangle}. 
\]

(301)

The primitive amplitudes where all gluons have positive helicities vanish and so do the primitive amplitudes where all gluons except one have positive helicities. The first non-vanishing result is obtained for the amplitudes with \( n-2 \) gluons of positive helicity and 2 gluons of negative helicity. It is given by a remarkable simple formula. Note that this formula holds for all \( n \). An amplitude with \( n-2 \) gluons of positive helicity and 2 gluons of negative helicity is called a maximally helicity violating amplitude (MHV amplitude). As it depends only on the spinors \( p_A = |p+\rangle \) (and not on \( p_A \)) one says that this amplitude is holomorphic. The name “maximally helicity violating amplitude” stems from the following fact: We use the convention that all particles are treated as outgoing. Crossing a particle to the initial state will reverse the helicity. Thus the the amplitude \( A_n^{(0)}(1^+,2^+,\ldots,n^+) \) corresponds – when 1 and 2 are crossed to the initial state – to the process \((-p_1)^- + (-p_2)^- \rightarrow p_3^+ + \ldots + p_n^+\). In this process all final state particles have the opposite helicity of the two initial state particles. This process would violate helicity conservation maximally. Actually the amplitude for this process is zero and so is the amplitude for the process \((-p_1)^- + (-p_2)^- \rightarrow p_3^- + p_4^+ + \ldots + p_n^+\). The first non-zero amplitude is obtained for the process \((-p_1)^- + (-p_2)^- \rightarrow p_3^- + p_4^- + p_5^+ + \ldots + p_n^+\). In this process we have – starting with two negative helicity particles in the initial state – the maximal number of positive helicity particles in the final state such that the amplitude is non-zero. The corresponding amplitude is therefore called a maximally helicity violating amplitude.

Obviously, we find similar formulæ if we exchange all positive and negative helicities:

\[
A_n^{(0)}(1^-,2^-,\ldots,n^-) = 0, \\
A_n^{(0)}(1^-,2^-,\ldots,j^+,\ldots,n^-) = 0, \\
A_n^{(0)}(1^-,2^-,\ldots,j^+,\ldots,k^+,\ldots,n^-) = i\left(\sqrt{2}\right)^{n-2}\frac{|kj|^4}{[1n][n(n-1)]\ldots[21]}. 
\]

(302)

Primitive amplitudes with \( n-2 \) gluons of negative helicity and 2 gluons of positive helicity are called anti-MHV amplitudes. These amplitudes depend only on the spinors \( p_A = \langle p+\rangle \) (and not on \( p_A \)) and are called anti-holomorphic.
We may extend these formulæ to $n = 3$, by defining the amplitude with one positive helicity and two negative helicities according to the MHV formula in eq. (301) and the amplitude with one negative helicity and two positive helicities according to the anti-MHV formula in eq. (302). Thus we have

\begin{align*}
A_3^{(0)}(1^-, 2^-, 3^+) &= i\sqrt{2}\frac{(12)^3}{(23)(31)}, \\
A_3^{(0)}(1^+, 2^+, 3^-) &= i\sqrt{2}\frac{[21]^3}{[13][32]}.
\end{align*}

(303)

The three-point amplitudes, where all helicities are equal vanishes.

\begin{align*}
A_3^{(0)}(1^+, 2^+, 3^+) &= A_3^{(0)}(1^-, 2^-, 3^-) = 0.
\end{align*}

(304)

Eq. (301) and eq. (302) are called the Parke-Taylor formulæ. Before we proceed to a proof of these formulæ, let us first look at the simple case $n = 4$. Amplitudes with four positive helicities or four negative helicities vanish, as do amplitudes with three helicities of one type and one helicity of the other type. The only non-vanishing four-gluon helicity amplitudes are the ones with two positive helicities and two negative helicities. These are at the same time MHV amplitudes and anti-MHV amplitudes, or phrased differently at the same time holomorphic and anti-holomorphic. As an example let us consider the amplitude $A_4^{(0)}(1^-, 2^-, 3^+, 4^+)$. The MHV representation (or holomorphic representation) is given by

\begin{align*}
A_4^{(0)}(1^-, 2^-, 3^+, 4^+) &= 2i\frac{(12)^4}{(12)(23)(34)(41)}. \\
\end{align*}

(305)

The anti-MHV representation (or anti-holomorphic representation) is given by

\begin{align*}
A_4^{(0)}(1^-, 2^-, 3^+, 4^+) &= 2i\frac{[34]^4}{[12][23][34][41]}.
\end{align*}

(306)

Of course, these two expressions have to be equal. Let’s see how this works out: From momentum conservation it follows that $2p_1p_2 = 2p_3p_4$ and therefore

\begin{align*}
\frac{(12)}{(34)} &= \frac{[34]}{[12]},
\end{align*}

(307)

Momentum conservation implies also that $p_2 = -p_1 - p_3 - p_4$ and therefore

\begin{align*}
\frac{(12)}{(41)} &= -\frac{[34]}{[23]}.
\end{align*}

(308)

A similar relation follows from $p_1 = -p_2 - p_3 - p_4$:

\begin{align*}
\frac{(12)}{(23)} &= -\frac{[34]}{[41]}.
\end{align*}

(309)

Therefore, the two representations are equivalent.
Now let us turn to the proof of the Parke-Taylor formulæ: The proof of the Parke-Taylor formulæ has been given by Berends and Giele \([70]\) and uses the off-shell currents discussed in section (3.5). In some cases the recurrence relations can be solved in closed form. If all the gluons have the same helicity one obtains

\[
J_\mu(p_1^+, p_2^+, \ldots, p_n^+) = \left(\sqrt{2}\right)^{n-2} \left\langle q - |\gamma_\mu p_1 n| q^+ \right\rangle \left\langle q_1 \right\rangle \left\langle 12 \right\rangle \ldots \left\langle (n-1)n \right\rangle \left\langle nq \right\rangle,
\]

(310)

if a common reference momentum \(q\) is chosen for all gluons. The backslash notation stands for contraction with \(\gamma_\mu\), e.g.

\[
p_1 n = \gamma_\nu (p_1^\nu + \ldots + p_n^\nu).
\]

(311)

**Exercise 16:** Prove eq. (310) by induction.

If one gluon has opposite helicity, let’s say gluon 1, one finds

\[
J_\mu(p_1^-, p_2^+, \ldots, p_n^+) = \left(\sqrt{2}\right)^{n-2} \left\langle 1 - |\gamma_\mu p_2 n| 1^+ \right\rangle \sum_{m=3}^{n} \left\langle 1 - |\gamma^m p_2 n| 1_m^+ \right\rangle \frac{p_1^2}{p_{1,m-1} p_{1,m}},
\]

(312)

where the reference momentum choice is \(q_1 = p_2, q_2 = \ldots = q_n = p_1\).

From the off-shell currents in eq. (310) and in eq. (312) we obtain the amplitudes \(A^{(0)}_{n+1}\) by multiplying with \(ip_1^2\) and contracting with \(\epsilon^\lambda_{n+1}(p_{n+1})\). From eq. (310) we see immediately that the amplitudes with all plus helicities and the amplitudes with one minus helicity and the rest plus helicities must vanish, as the off-shell current \(J_\mu(p_1^+, p_2^+, \ldots, p_n^+)\) does not have a pole at \(p_{1,n}^2 = 0\). From eq. (312) we obtain the MHV amplitude where the two negative helicity particles are adjacent by multiplying with \(ip_1^2\) and contracting with \(\epsilon^\mu_{n+1}(p_{n+1})\). Proving the correctness of the Parke-Taylor formula for two non-adjacent negative helicities will be an exercise in section (5.5), after we introduced on-shell recursion relations.

### 4.2 The CSW construction

The Parke-Taylor formulæ give us compact expressions for amplitudes with zero, one or two gluons of negative helicity and all remaining gluons having positive helicity. An obvious question is what happens if we go to amplitudes with a higher number of negative helicity gluons. Amplitudes with three gluons of negative helicity are called next-to-maximally helicity violating amplitudes (or NMHV amplitudes for short), amplitudes with four gluons of negative helicity are called next-to-next-to-maximally helicity violating amplitudes (or \(N^2\)MHV amplitudes for short). In general, a \(N^{k-2}\)MHV amplitude contains \(k\) gluons with negative helicity.

After the discovery of the Parke-Taylor formulæ in 1986 it took almost twenty years (until 2004) to find the pattern for the general case. This generalisation triggered a very rapid development of the field following the years after 2004. We will now review the Cachazo-Svrcek-Witten (CSW) construction \([72]\). The basic idea of the CSW construction is to obtain tree amplitudes in...
Yang-Mills theory from tree graphs in which the vertices are tree-level MHV scattering amplitudes, continued off-shell in a particular fashion.

Let us first discuss the off-shell continuation. Let \( q \) be a light-like four-vector, which will be kept fixed throughout the discussion. Using \( q \), any massive vector \( p \) can be written as a sum of two light-like four-vectors \( p^\flat \) and \( q \) [73]:

\[
p = p^\flat + \frac{p^2}{2pq}q.
\]  

(313)

Obviously, if \( p^2 = 0 \), we have \( p = p^\flat \). Note further that \( 2pq = 2p^\flat q \). This construction appeared already in section (3.3.5). Using eq. \((313)\) we may associate a massless four-vector \( p^\flat \) to any four-vector \( p \). Using the projection onto \( p^\flat \) we define the off-shell continuation of Weyl spinors as

\[
|p^\pm\rangle \rightarrow |p^\flat^\pm\rangle,
\]

\[
\langle p^\pm| \rightarrow \langle p^\flat^\pm|.
\]

(314)

The basic building blocks of the CSW construction are the off-shell continued MHV amplitudes, which serve as new vertices:

\[
V_n(1^+, \ldots, j^-, \ldots, k^-, \ldots, n^+) = i \left( \sqrt{2} \right)^{n-2} \frac{\langle j^\flat k^\flat \rangle^4}{\langle 1^\flat 2^\flat \rangle \ldots \langle n^\flat 1^\flat \rangle}.
\]

(315)

Each MHV vertex has exactly two lines carrying negative helicity and at least one line carrying positive helicity. Each internal line has a positive helicity label on one side and a negative helicity label on the other side. The propagator for each internal line is the propagator of a scalar particle:

\[
\frac{i}{p^2}
\]

(316)

The number of MHV vertices is related to the number of negative helicity gluons. To see this, consider a tree diagram with \( v \) vertices. This diagram

- has \( (v - 1) \) propagators,

- has in total \( 2v \) negative helicity labels (two per vertex),

- has \( (v + 1) \) external negative helicity label (since \( (v - 1) \) labels are used by the internal propagators).

Therefore an amplitude with \( k \) negative helicity gluons has \( (k - 1) \) MHV vertices.

Let us now consider an example. The NMHV amplitude \( A_6(1^-, 2^-, 3^-, 4^+, 5^+, 6^+) \) has three gluons of positive helicity and three gluons of negative helicity and is one of the first non-trivial amplitudes, which are non-zero and which are not MHV amplitudes. It will be given by diagrams with two MHV vertices. Fig. 8 shows the six MHV diagrams contributing to this amplitude. We obtain the set of MHV diagrams by first considering the stripped diagrams with only external
Let us consider the first MHV diagram of fig. (8). This diagram yields

$$\left[ i\sqrt{2} \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 2 (-p_{12})^3 \rangle \langle (-p_{12}) \rangle 1} \right] \frac{i}{p_{12}^4} \left[ i \left( \sqrt{2} \right)^3 \frac{\langle 3p_{12}^4 \rangle^4}{\langle 34 \rangle \langle 45 \rangle \langle 56 \rangle \langle 6p_{12} \rangle} \langle p_{12}^4 \rangle \right] ,$$

and similar expressions are obtained for the five other diagrams. In this expression, $p_{12} = p_1 + p_2$.
is the momentum flowing through the internal line and \( p_{12}^\lambda \) is the projection onto a light-like four-vector as in eq. (313).

Bena, Bern and Kosower [74] derived a recursive formulation, which allows to obtain vertices with more gluons of negative helicity from simpler building blocks:

\[
V_n(p_1^\lambda_1, \ldots, p_n^\lambda_n) = \frac{1}{(k - 2)} \sum_{j=1}^{n} \sum_{l=j+1}^{n-j-2} \frac{i}{p_{j,l}^2} V_{l-(j+2)} \mod n(p_j^\lambda_1, \ldots, p_l^\lambda_j, (p_{j,l})^\perp) \times V_{l-(j+2)} \mod n(p_{l+1}^\lambda_{l+1}, \ldots, p_{j-1}^\lambda_{j-1}, (p_{l+1}, (j-1))^\perp),
\]

where \( k \) is the number of negative helicity gluons. The recursion stops if \( k \) is less or equal than two. For \( k = 0 \) or \( k = 1 \) the quantity \( V_n(p_1^\lambda_1, \ldots, p_n^\lambda_n) \) vanishes. For \( k = 2 \) it is given by eq. (315).

The primitive amplitude \( A_n^{(0)} \) coincides with \( V_n \), if all gluons are on-shell:

\[
A_n^{(0)}(p_1^\lambda_1, \ldots, p_n^\lambda_n) = V_n(p_1^\lambda_1, \ldots, p_n^\lambda_n).
\]

This gives us the following algorithm:

**Algorithm 3. Calculation of the primitive helicity amplitude** \( A_n^{(0)}(p_1^\lambda_1, \ldots, p_n^\lambda_n) \) with \( k \) gluons of negative helicity from MHV vertices:

1. Calculate recursively the \( N^{k-2} \) MHV vertex \( V_n(p_1^\lambda_1, \ldots, p_n^\lambda_n) \), using eq. (317).

2. The \( N^{k-2} \) MHV amplitude \( A_n^{(0)}(p_1^\lambda_1, \ldots, p_n^\lambda_n) \) is then given by

\[
A_n^{(0)}(p_1^\lambda_1, \ldots, p_n^\lambda_n) = V_n(p_1^\lambda_1, \ldots, p_n^\lambda_n).
\]

Note that this algorithm over-counts each contribution \((k - 2)\) times. This over-counting is compensated by the explicit factor \( 1/(k - 2) \) in front.

### 4.3 The Lagrangian of the MHV expansion

Let us now consider a proof of the MHV expansion. There are various possibilities to prove the MHV expansion. The first approach makes use of a canonical transformation in the field variables [75–80]. A second approach starts from an action in twistor space [81–86]. The action in twistor space has an extended gauge symmetry. The conventional Lagrangian and the MHV Lagrangian are then obtained from the action in twistor space for different gauge choices. A third approach proves the MHV expansion with the help of on-shell recursion relations [87], which will be discussed in section (5).

Here, we will discuss the approach based on a canonical transformation. The exposition follows [80]. The derivation proceeds through four steps. We will be working in light-cone coordinates. We defined contra-variant light-cone coordinates in eq. (182). For the co-variant light-cone coordinates we use the convention

\[
x_+ = \frac{1}{\sqrt{2}} (x_0 + x_3), \quad x_- = \frac{1}{\sqrt{2}} (x_0 - x_3), \quad x_\perp = \frac{1}{\sqrt{2}} (x_1 + ix_2), \quad x_{\perp*} = \frac{1}{\sqrt{2}} (x_1 - ix_2).
\]
With these conventions we have
\[ p^\mu x_\mu = p^+ x_+ + p^- x_- + p^\perp_\pm x^\perp_\pm + p^\perp x^\perp. \] (321)

For the vector \( \vec{x} = (x_-, x_\perp, x^\perp_\pm) \) we set in this section
\[ \vec{p} \cdot \vec{x} = p^- x_- + p^\perp_\pm x^\perp_\pm + p^\perp x^\perp. \] (322)

We define the spinors as in eq. (194) with \( \lambda_p = 1 \). This definition applies to all four-vectors \( p^\mu \). If the four-vector \( p^\mu \) is light-like, the spinors are the eigenstates of the Dirac equation with eigenvalue zero. If the four-vector \( p^\mu \) is not light-like, eq. (194) defines the off-shell continuation of the spinors.

**Step 1: Light-cone gauge.** Our starting point is the Lagrangian of Yang-Mills theory, given in eq. (5). We can re-write this Lagrangian as
\[ \mathcal{L}_{\text{YM}} = \frac{1}{2 g^2} \text{Tr} F_{\mu \nu} F^{\mu \nu} = \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4, \] (323)
such that \( \mathcal{L}_2 \) contains all terms bilinear in the gauge fields. Terms with three or four gauge fields are collected in \( \mathcal{L}_3 \) and \( \mathcal{L}_4 \), respectively. We choose the light-cone gauge
\[ A_- = 0. \] (324)

In this gauge we have
\[ \mathcal{L}_2 = \frac{1}{g^2} \text{Tr} \left[ A_+ \partial^2_+ A_+ - 2A_+ \partial_+ \partial_\perp A_\perp + 2A_+ \partial_- \partial_\perp A_\perp + A_+ \partial^2_\perp A_\perp + A_\perp \partial^2_\perp A_\perp \right. \]
\[ + 2A_\perp \left( 2 \partial_- \partial_+ - \partial_+ \partial_\perp \right) A_\perp, \]
\[ \mathcal{L}_3 = \frac{2}{g^2} \text{Tr} \left[ (\partial_\perp A_\perp^\pm) [A_\perp^\pm, A_\pm] + (\partial_- A_\perp) [A_\perp, A_\perp^\pm] - (\partial_+ A_\perp^\pm) [A_\perp^\pm, A_\pm] \right], \]
\[ \mathcal{L}_4 = - \frac{1}{g^2} \text{Tr} [A_\perp A_\perp^\pm] [A_\perp^\pm, A_\pm]. \] (325)

**Step 2: Integrating out \( A_+. \)** We observe that the field \( A_+ \) occurs only quadratically or linearly in eq. (325). We can therefore integrate this field out, similar to way we treated the field \( B_{\mu \nu} \) in section (3.1). After integrating out \( A_+ \) we can write the Lagrange density as
\[ \mathcal{L}_{\text{YM}} = \mathcal{L}_{++} + \mathcal{L}_{++} + \mathcal{L}_{++} + \mathcal{L}_{++}, \] (326)
with
\[ \mathcal{L}_{--} = \frac{4}{g^2} \text{Tr} A_\perp \left( \partial_- \partial_+ - \partial_+ \partial_\perp \right) A_\perp, \]
\[ \mathcal{L}_{++} = \frac{4}{g^2} \text{Tr} \left( \partial_\perp A_\perp \right) \partial_+^{-1} [A_\perp, \partial_- A_\perp^\pm], \]
\[ \mathcal{L}_{+-} = \frac{4}{g^2} \text{Tr} \left( \partial_\perp A_\perp^\pm \right) \partial_+^{-1} [A_\perp^\pm, \partial_- A_\perp], \]
\[ \mathcal{L}_{++} = - \frac{4}{g^2} \text{Tr} \left[ A_\perp^\pm, \partial_- A_\perp \right] \partial_+^{-2} [A_\perp, \partial_- A_\perp]. \] (327)
The Lagrange density contains now only the transverse degrees of freedom for the field $A$.

**Step 3: Canonical transformation.** In the third step we eliminate the non-MHV vertices contained in $\mathcal{L}_{++-}$ by a canonical change of the field variables.

To motivate the canonical transformation we treat the variable $x_+$ as a time variable and collect the remaining three variables in a vector $\vec{x} = (x_-, x_\perp, x_{\perp*})$. In order to simplify the notation we will suppress the dependence of the fields on $x_+$ and write $A(\vec{x})$ instead of $A(x_+, \vec{x})$. We will denote the new field after the canonical transformation with a tilde, e.g.

$$ A \rightarrow \tilde{A}.$$ 

(328)

Now let us look again at eq. (326) and eq. (327). The “momentum” conjugate to $A^a_\perp$ is

$$ \frac{\delta \mathcal{L}_{YM}}{\delta \partial_+ A^a_\perp} = 2\partial_- A^a_\perp.$$ 

(329)

We look for a canonical transformation, where the generating function of the transformation depends on the new “coordinates” $\tilde{A}_\perp$ and the old “momenta” $\partial_- A_{\perp*}$:

$$ G[\tilde{A}_\perp, \partial_- A_{\perp*}] = \int d^3 y A^a_\perp [\tilde{A}_\perp(\vec{y})] \partial_- A^a_{\perp*}(\vec{y})$$ 

(330)

The new “momenta” are then given by

$$ \partial_- \tilde{A}^a_{\perp*}(\vec{x}) = \int d^3 y \frac{\delta A^b_\perp(\vec{y})}{\delta \tilde{A}^a_\perp(\vec{x})} \partial_- A^b_{\perp*}(\vec{y}).$$ 

(331)

The transformation should eliminate the unwanted $\mathcal{L}_{++-}$ term, therefore we require

$$ \mathcal{L}_{+-} [\tilde{A}_\perp, \tilde{A}_{\perp*}] = \mathcal{L}_{+-} [A_\perp, A_{\perp*}] + \mathcal{L}_{++-} [A_\perp, A_{\perp*}].$$ 

(332)

The fact that the transformation is canonical implies

$$ \int d^3 x 2 (\partial_- A^a_{\perp*}) (\partial_+ A^a_\perp) = \int d^3 x 2 (\partial_- \tilde{A}^a_{\perp*}) (\partial_+ \tilde{A}^a_\perp).$$ 

(333)

We then plug the expressions in eq. (331) into eq. (332) and use eq. (333). It is convenient to introduce the following two differential operators

$$ \omega = \frac{\partial_+ \partial_{\perp*}}{\partial_-}, \quad \zeta = \frac{\partial_{\perp*}}{\partial_-}.$$ 

(334)

From the coefficient of $\partial_- A_{\perp*}$ we find the integro-differential equation

$$ \omega A^a_\perp(\vec{x}) - gf^{abc} (\zeta A^b_\perp(\vec{x})) A^c_\perp(\vec{x}) = \int d^3 y \frac{\delta A^a_\perp(\vec{x})}{\delta \tilde{A}^b_\perp(\vec{y})} \omega A^b_\perp(\vec{y}).$$ 

(335)
The solution to the integro-differential equation (335) is given by

\[ A^a_{\perp}(\vec{x}) = \sum_{n=1}^{\infty} 2 \text{Tr}(T^a T^{a_1} \ldots T^{a_n}) \int \frac{d^3 p_1}{(2\pi)^3} \ldots \frac{d^3 p_n}{(2\pi)^3} e^{-i(\vec{p}_1 + \ldots + \vec{p}_n) \cdot \vec{x}} \]  
(336)

The coefficient functions are given by

\[ Y(\vec{p}_1, \ldots, \vec{p}_n) = \frac{\sqrt{2g}}{\sqrt{\langle p_1 p_2 \rangle \ldots \langle p_{n-1} p_n \rangle}} \frac{p_1 + \ldots + p_n}{p_1 p_n}, \]

\[ \Xi_r(\vec{p}_1, \ldots, \vec{p}_n) = \left( \frac{p_r}{p_1 + \ldots + p_n} \right)^2 Y(\vec{p}_1, \ldots, \vec{p}_n). \]  
(337)

**Exercise 17:** Derive eq. (336) together with eq. (337) from eq. (335).

We remark that the field \( A^a_{\perp}(\vec{x}) \) is expressed in terms of the fields \( \tilde{A}^a_{\perp}(\vec{p}) \) alone, while the field \( A^a_{\perp*}(\vec{x}) \) involves the fields \( \tilde{A}^a_{\perp*}(\vec{p}) \) and \( \tilde{A}^a_{\perp}(\vec{p}) \). The new fields agree with the old fields to leading order in \( g \):

\[ A^a_{\perp}(\vec{x}) = \tilde{A}^a_{\perp}(\vec{x}) + O(g), \quad A^a_{\perp*}(\vec{x}) = \tilde{A}^a_{\perp*}(\vec{x}) + O(g). \]  
(338)

**Step 5: Assembling the pieces.** We are now in a position to put all the pieces together. Inserting the solutions (336) of the canonical transformation into the Lagrange density (326) one finds that the Lagrange density can be written in the following form:

\[ \mathcal{L}_{YM} = \mathcal{L}_{kin} + \sum_{n=3}^{\infty} \mathcal{L}^{(n)}. \]  
(339)

The first term \( \mathcal{L}_{kin} \) is rather simple and contains the kinetic term:

\[ \mathcal{L}_{kin} = -\tilde{A}^a_{\perp*}(x) \square \tilde{A}^a_{\perp}(x). \]  
(340)

We further obtain an ascending tower of interaction vertices. Each interaction vertex is most conveniently expressed with the help of the Fourier transforms. One finds for \( \mathcal{L}^{(n)} \)

\[ \mathcal{L}^{(n)} = \frac{1}{2} \sum_{j=2}^{n} \int \frac{d^4 p_1}{(2\pi)^4} \ldots \frac{d^4 p_n}{(2\pi)^4} e^{-i(p_1 + \ldots + p_n) \cdot x} \alpha_j(p_1, \ldots, p_n) \]

\[ 2 \text{Tr}(\tilde{A}_{\perp*}(p_1)\tilde{A}_\perp(p_2) \ldots \tilde{A}_\perp(p_{j-1})\tilde{A}_{\perp*}(p_j)\tilde{A}_\perp(p_{j+1}) \ldots \tilde{A}_\perp(p_n)). \]  
(341)
The vertex function \( \alpha_j(p_1, \ldots, p_n) \) is given

\[
\alpha_j(p_1, \ldots, p_n) = -\frac{1}{g^2} \left( i\sqrt{2} \right)^{n-2} \frac{\langle p_1 p_j \rangle^4}{\langle p_1 p_2 \rangle \langle p_2 p_3 \rangle \ldots \langle p_{n-1} p_n \rangle \langle p_n p_1 \rangle},
\]

and corresponds exactly to the MHV formula. Note that the vertex function depends only on the light-cone coordinates \( p_-^* \) and \( p_- \), but not on \( p_\perp \) and \( p_+ \). Each vertex contains two fields \( \tilde{A}_\perp \) with indices 1 and \( j \) and an arbitrary number of fields \( \tilde{A}_\perp^* \). Since the trace is cyclic, we have

\[
\text{Tr} (T^{a_1} \ldots T^{a_{j-1}} T^{a_j} \ldots T^{a_n}) = \text{Tr} (T^{a_j} \ldots T^{a_n} T^{a_1} \ldots T^{a_{j-1}}).
\]

The factor \( 1/2 \) takes into account that we are summing twice over identical traces.

Finally let us comment on a subtle point: We are interested in the scattering amplitudes involving the original fields \( A^a_\perp \) and \( A^a_\perp^* \). Let us denote these amplitudes by \( A_n^{(0)}(p_1, \ldots, p_n) \) and the amplitudes involving the new fields \( \tilde{A}^a_\perp \) and \( \tilde{A}^a_\perp^* \) by \( \tilde{A}_n^{(0)}(p_1, \ldots, p_n) \). In order to obtain the amplitude \( A_n^{(0)}(p_1, \ldots, p_n) \) from the theory with the new fields \( \tilde{A}^a_\perp \) and \( \tilde{A}^a_\perp^* \) we have to go one step back in quantum field theory and recall that \( A_n^{(0)} \) is obtained from un-amputated Green’s function through the LSZ reduction procedure. At tree-level the LSZ reduction procedure corresponds to a multiplication with the inverse propagator \( \left( -i p_j^2 \right) \) for each external leg \( j \). In the un-amputated Green’s functions we replace the original fields \( A^a_\perp \) and \( A^a_\perp^* \) by the new fields \( \tilde{A}^a_\perp \) and \( \tilde{A}^a_\perp^* \), using eq. (336). We have seen in eq. (338) that the new fields and the old fields agree to leading order in \( g \). The additional higher-order terms introduced through eq. (336) are called MHV completion vertices. For \( n \geq 4 \) and non-exceptional kinematics the terms with MHV completion vertices miss at least one singular propagator and are therefore killed by the LSZ reduction procedure. In this case we simply have \( A_n^{(0)} = \tilde{A}_n^{(0)} \).

This argumentation fails if the coefficient functions in eq. (337) become singular. This may happen for the three-point amplitudes. In this case terms with MHV completion vertices may give a finite non-zero result after the LSZ reduction procedure. In particular this is the way how the three-point anti-MHV amplitude \( A_3^{(0)}(1^+, 2^+, 3^-) \) is obtained from the Lagrangian in eq. (339).

This mechanism is sometimes called evasion of the S-matrix equivalence theorem [78].
5 On-shell recursion

In section 3.5 we discussed off-shell recurrence relations, which allow us to compute an amplitude recursively from off-shell currents. The off-shell currents are gauge-dependent objects. There is nothing wrong with the fact that at intermediate stages we deal with gauge-dependent objects, even in a Feynman diagram based calculation the evaluation of an individual Feynman diagrams gives in general a gauge-dependent expression. The amplitude itself is gauge-independent and all gauge-dependence cancels in the expression for the amplitude.

Nevertheless we may ask, if there is a way to deal at all stages with on-shell gauge-invariant objects only. This is indeed possible and we will now discuss on-shell recursion relations. On-shell recursion relations compute the primitive tree amplitude $A_n^{(0)}$ with $n$ external particles recursively from primitive tree amplitudes with fewer legs. Obviously, the primitive tree amplitudes with fewer legs entering this calculation satisfy momentum conservation and the on-shell conditions. However we do not require that the momenta of the external particles are real. We will allow complex external momenta. This may seem strange at first sight, but a tree-level primitive amplitude is a rational function of the external momenta and nothing stops us to evaluate this function for complex external momenta. Allowing complex momenta opens the door to the tools of complex analysis and Cauchy’s theorem in particular.

A special role is played by the three-point amplitudes. The equal helicity ones vanish $A_3^{(0)}(1^+, 2^+, 3^+) = A_3^{(0)}(1^-, 2^-, 3^-) = 0$, the ones with mixed helicities are given by

$$A_3^{(0)}(1^-, 2^-, 3^+) = i\sqrt{2} \frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 31 \rangle}, \quad A_3^{(0)}(1^+, 2^-, 3^-) = i\sqrt{2} \frac{[21]^3}{[13][32]},$$

and cyclic permutation thereof. Momentum conservation and the on-shell conditions read

\[ p_1 + p_2 + p_3 = 0, \quad p_1^2 = p_2^2 = p_3^2 = 0. \]

Therefore

\[ 2p_1 \cdot p_2 = (p_1 + p_2)^2 = p_3^2 = 0. \]

On the other hand we have

\[ 2p_1 \cdot p_2 = \langle p_1 p_2 \rangle [p_2 p_1], \]

and it follows that either $\langle p_1 p_2 \rangle = 0$ or $[p_1 p_2] = 0$. The same conclusion is reached for the other spinor products. For real momenta we have in addition

\[ |\langle p_1 p_2 \rangle| = ||[p_2 p_1]|, \]

and therefore it follows that $\langle p_1 p_2 \rangle = 0$ and $[p_1 p_2] = 0$. In the real case all three-point amplitudes vanish. A rough argument uses the fact, that in eq. (344) there are three spinor products in the numerator, but only two in the denominator. A more careful analysis uses a parametrisation of the momenta like in eq. (102) and shows that the amplitudes vanish in the limit $k_\perp \to 0$.

However for complex momenta we do not have eq. (348) and the three-point amplitudes may be non-zero.
5.1 BCFW recursion

Let us now derive the on-shell recursion relation. This relation is also known as Britto-Cachazo-Feng-Witten (BCFW) recursion relation \[88, 89\]. We consider a tree-level primitive amplitude \( A^{(0)}(p_1^{\lambda_1}, \ldots, p_n^{\lambda_n}) \). The basic idea is to pick two momenta, say \( p_i \) and \( p_j \), and to deform them as

\[
\hat{p}_i^\mu = p_i^\mu - zn^\mu, \quad \hat{p}_j^\mu = p_j^\mu + zn^\mu, \tag{349}
\]

where \( z \) is a complex variable and \( n \) a light-like four-vector \((n^2 = 0)\). It is clear that this deformation respects momentum conservation. If in addition we require

\[
2p_i \cdot n = 0, \quad 2p_j \cdot n = 0, \tag{350}
\]

also the on-shell conditions \( \hat{p}_i^2 = \hat{p}_j^2 = 0 \) are satisfied. The first question to be asked is the following: Does such a light-like four-vector \( n^\mu \) satisfying eq. (350) exist? It does, a possible choice (not the only one) is given by

\[
n^\mu = \frac{1}{2} \langle i + |\gamma^j| j+ \rangle. \tag{351}
\]

The four-vector \( n^\mu \) is light-like

\[
n^2 = \frac{1}{4} \langle i + |\gamma^j| j+ \rangle \langle i + |\gamma_\mu| j+ \rangle = 0, \tag{352}
\]

due to the Fierz identity. The four-vector \( n^\mu \) satisfies in addition the conditions in eq. (350):

\[
2p_i \cdot n = \langle i + |p_1^i| j+ \rangle = 0,
2p_j \cdot n = \langle i + |p_j^j| j+ \rangle = 0. \tag{353}
\]

Let us now consider the amplitude with the two deformed momenta. We may view this amplitude as a function of \( z \):

\[
A(z) = A^{(0)}(p_1^{\lambda_1}, \ldots, \hat{p}_i(z)^{\lambda_i}, \ldots, \hat{p}_j(z)^{\lambda_j}, \ldots, p_n^{\lambda_n}). \tag{354}
\]

\( A(z) \) is a rational function of \( z \), since tree amplitudes are rational functions of the momenta variables. Let us assume that \( A(z) \) falls off at least like \( 1/|z| \) for \( |z| \to \infty \). We will discuss the detailed conditions for this to happen in the next paragraph. With this assumption we have

\[
\frac{1}{2\pi i} \oint dz \frac{A(z)}{z} = 0, \tag{355}
\]

where the contour is a large circle at \( |z| = \infty \) oriented counter-clockwise. On the other hand we may evaluate this integral with the help of Cauchy’s residue theorem. There is one residue at \( z = 0 \) due to the explicit factor of \( 1/|z| \) in the integrand. This residue gives

\[
A(0) = A^{(0)}(p_1^{\lambda_1}, \ldots, p_i^{\lambda_i}, \ldots, p_j^{\lambda_j}, \ldots, p_n^{\lambda_n}). \tag{356}
\]
Figure 10: Contribution to the residues from an on-shell propagator. For a \( z \)-dependent momentum flow through the indicated internal propagator, particle \( i \) has to be on the one side and particle \( j \) has to be on the other side.

which is the undeformed amplitude we want to calculate. There cannot be any residue coming from the explicit representation of the polarisation vectors. This would depend on the reference momenta \( q_i \), however the amplitude is gauge-invariant and therefore independent of the choice of the reference momenta \( q_i \). Therefore all other residues come from internal propagators of the amplitude. We have to consider only the propagators, which are \( z \)-dependent. Let us consider a situation as shown in fig. (10). It is clear that in order to have a \( z \)-dependent propagator particle \( i \) has to be on the one side and particle \( j \) on the other side. Let us denote the set of external legs on the one side by \( I \) and the set of external legs on the other side by \( J \). Let us further assume that \(|I| = m \) and \(|J| = n - m \). We set

\[
p_J = \sum_{j \in J} p_j. \tag{357}
\]

The momentum flowing through the internal propagator is

\[
\hat{p}_J = p_J + zn. \tag{358}
\]

The internal propagator goes on-shell for

\[
\hat{p}_J^2 = 0 \Rightarrow z_J = -\frac{p_J^2}{2p_J \cdot n}. \tag{359}
\]

In this limit the amplitude factorises as in eq. (223) and the residue is given by

\[
-\sum_{\lambda} A_{m+1}^{(0)} \left( \ldots, \hat{p}_i(z_J)^{\lambda_i}, \ldots, \hat{p}_J(z_J)^{\lambda} \right) \frac{i}{p_J} A_{n-m+1}^{(0)} \left( \ldots, \hat{p}_J(z_J)^{\lambda_j}, \ldots, -\hat{p}_J(z_J)^{-\lambda} \right). \tag{360}
\]

Summing over all residues we obtain the on-shell recursion relation:

\[
A_n^{(0)} \left( p_1^{\lambda_1}, \ldots, p_i^{\lambda_i}, \ldots, p_j^{\lambda_j}, \ldots, p_n^{\lambda_n} \right) = \sum_{\text{partitions } (I,J)} \sum_{\lambda} A_{m+1}^{(0)} \left( \ldots, \hat{p}_i(z_J)^{\lambda_i}, \ldots, \hat{p}_J(z_J)^{\lambda} \right) \frac{i}{p_J} A_{n-m+1}^{(0)} \left( \ldots, \hat{p}_J(z_J)^{\lambda_j}, \ldots, -\hat{p}_J(z_J)^{-\lambda} \right), \tag{361}
\]
where the sum is over all partitions \((I,J)\) such that particle \(i \in I\) and particle \(j \in J\). The momentum \(\hat{p}_J(z_J)\) is again on-shell \((\hat{p}_J(z_J))^2 = 0\), the momentum \(p_J\) appearing in the denominator is in general not on-shell \((p_J^2 \neq 0)\). Eq. (361) allows us to compute the \(n\)-particle amplitude \(A_n^{(0)}\) recursively through on-shell amplitudes with fewer external legs. In order to apply this recursion relation we have to ensure that the amplitude \(A(z)\) vanishes as \(|z| \to \infty\).

### 5.2 Momenta shifts and the behaviour at infinity

Let us now consider the momenta shifts and the behaviour at \(|z| = \infty\) in more detail. Usually we view the amplitude \(A_n^{(0)}\) as a function of the four-momenta \(p_j^\mu\). However, we may replace each four-vector by a pair of two-component Weyl spinors. In detail this is done as follows: Each four-vector \(p_\mu\) has a bi-spinor representation, given by

\[
p_{AB} = p_\mu \sigma_\mu^{AB}, \quad p_\mu = \frac{1}{2} p_{AB} \sigma_\mu^{BA}.
\]

(362)

For light-like vectors this bi-spinor representation factorises into a dyad of Weyl spinors:

\[
p_\mu p^\mu = 0 \iff p_{AB} = p_A p_B.
\]

(363)

The equations (362) and (363) allow us to convert any light-like four-vector into a dyad of Weyl spinors and vice versa. Therefore the partial amplitude \(A_n^{(0)}\), being originally a function of the momenta \(p_j\) and helicities \(\lambda_j\), can equally be viewed as a function of the Weyl spinors \(p^j_A, p^j_B\) and the helicities \(\lambda_j\):

\[
A_n^{(0)} \left( p_1^\lambda, \ldots, p_n^\lambda \right) = A_n^{(0)} \left( p_A^1, p_B^1, \lambda_1, \ldots, p_A^n, p_B^n, \lambda_n \right).
\]

(364)

Note that for an arbitrary pair of Weyl spinors, the corresponding four-vector will in general be complex-valued. The shift defined in eq. (349) combined with eq. (351) reads

\[
\hat{p}_i^\mu = p_i^\mu - \frac{1}{2} z \langle i + |\gamma^\mu| j^+ \rangle, \quad \hat{p}_j^\mu = p_j^\mu + \frac{1}{2} z \langle i - |\gamma^\mu| j^+ \rangle,
\]

(365)

and corresponds to the following shift in the Weyl spinors:

\[
\hat{p}_A^i = p_A^i - z p_A^i, \quad \hat{p}_B^i = p_B^i, \\
\hat{p}_A^j = p_A^j, \quad \hat{p}_B^j = p_B^j + z p_B^j.
\]

(366)

Eq. (366) deforms the spinors \(p_A^i\) and \(p_B^j\). The spinors \(p_B^i\) and \(p_A^j\) remain unchanged. Let us now study the effect of the shift on the polarisation vectors of the gluons. We have

\[
\epsilon_\mu^+(p, q) = - \frac{\langle p + |\gamma_\mu| q^+ \rangle}{\sqrt{2} \langle p - |q^+ \rangle} = - \frac{p_A \sigma_\mu^{AB} q_B}{\sqrt{2} p e^{CD} q_D}, \quad \epsilon_\mu^-(p, q) = \frac{\langle p - |\gamma_\mu| q^- \rangle}{\sqrt{2} \langle p + |q^- \rangle} = \frac{q_A \sigma_\mu^{AB} p_B}{\sqrt{2} p e^{CD} q_D}.
\]
We may now read-off the large-$z$ behaviour of the polarisation vectors:

\[
\begin{align*}
\varepsilon^+_{\mu}(\hat{p}_i,q_i) & : z^{-1}, & \varepsilon^+_{\mu}(\hat{p}_j,q_j) & : z, \\
\varepsilon^-_{\mu}(\hat{p}_i,q_i) & : z, & \varepsilon^-_{\mu}(\hat{p}_j,q_j) & : z^{-1}.
\end{align*}
\] (367)

Let us now consider the amplitude

\[
A_n^{(0)}\left(\ldots,\hat{p}_i^+,\ldots,\hat{p}_j^-,\ldots\right)
\] (368)

For this helicity configuration $(\lambda_i,\lambda_j) = (+,-)$ and the shift as in eq. (365) each individual Feynman diagram vanishes for $z \to \infty$. In order to see this consider the flow of the $z$-dependence in a particular diagram. The most dangerous contribution comes from a path, where all vertices are three-gluon-vertices. For a path made of $n$ propagators we have $n+1$ vertices and the product of propagators and vertices behaves therefore like $z$ for large $z$. This statement remains true for a path containing only one vertex and no propagators. The polarisation vectors for the helicity combination $(\lambda_i,\lambda_j) = (+,-)$ contribute a factor $1/z^2$, therefore the complete diagram behaves like $1/z$ and vanishes therefore for $z \to \infty$.

Exercise 18: Consider the MHV amplitude $A_n^{(0)}(\ldots,\hat{p}_i^+,\ldots,\hat{p}_j^-,\ldots)$, where the particles $j$ and $k$ have negative helicity, while all other particles have positive helicity. Consider the shift of eq. (365), which deforms the momenta of the particles $i$ and $j$. What is the large-$z$ behaviour of the amplitude?

What about the other helicity configurations? There is an alternative approach to study the large-$z$ behaviour. Physically, the large-$z$ limit corresponds to a hard particle moving through a soft background [90]. In this limit we may use eikonal approximations to study the large-$z$ behaviour and one finds that under the shift defined in eq. (365) the amplitude $A_n^{(0)}(\ldots,\hat{p}_i^{\lambda_i},\ldots,\hat{p}_j^{\lambda_j},\ldots)$ behaves at least as follows for the various helicity configurations

\[
\begin{align*}
\lambda_i,\lambda_j & = (+,+): z^{-1}, & \lambda_i,\lambda_j & = (+,-): z^{-1}, \\
\lambda_i,\lambda_j & = (-,+): z^3, & \lambda_i,\lambda_j & = (-,-): z^{-1}.
\end{align*}
\] (369)

We see that we may use the momentum shift of eq. (365) for all helicity configurations except $(\lambda_i,\lambda_j) = (-,+)$.  

What about the helicity configuration $(\lambda_i,\lambda_j) = (-,+)$? The answer is simple: Instead of the momentum shift of eq. (365) consider the shift

\[
\hat{p}_i^\mu = p_i^\mu - \frac{1}{2}z \langle j|\gamma^\mu |i^+\rangle, \quad \hat{p}_j^\mu = p_j^\mu + \frac{1}{2}z \langle j|\gamma^\mu |i^+\rangle.
\] (370)

This shift corresponds to the following shift in the Weyl spinors:

\[
\begin{align*}
\hat{p}_A & = p_A, & \hat{p}_B & = p_B - z p_B^i, \\
\hat{p}_A^j & = p_A^j + z p_A^j, & \hat{p}_B^j & = p_B^j.
\end{align*}
\] (371)
We now shift the spinors $p^i_B$ and $p^j_A$ and leave the spinors $p^i_A$ and $p^j_B$ untouched. The large-$z$ behaviour of the polarisation vectors is now given by

\[
\begin{align*}
\varepsilon^+ (\hat{p}_i, q_i) &\colon z, & \varepsilon^+ (\hat{p}_j, q_j) &\colon z^{-1}, \\
\varepsilon^- (\hat{p}_i, q_i) &\colon z^{-1}, & \varepsilon^- (\hat{p}_j, q_j) &\colon z,
\end{align*}
\]  

and one finds that the amplitude behaves for large $z$ at least as

\[
\begin{align*}
(\lambda_i, \lambda_j) = (+, +) &\colon z^{-1}, & (\lambda_i, \lambda_j) = (+, -) &\colon z^3, \\
(\lambda_i, \lambda_j) = (-, +) &\colon z^{-1}, & (\lambda_i, \lambda_j) = (-, -) &\colon z^{-1}.
\end{align*}
\]  

Therefore we may use the shift of eq. (370) for all helicity configurations except $(\lambda_i, \lambda_j) = (+, -)$ and we see that for each helicity configuration there is at least one possible momenta shift, such that the amplitude vanishes for $|z| \to \infty$.

Before we close this paragraph let us mention that there are more possibilities of deforming the momenta, while keeping the on-shell conditions and momentum conservation. Up to now we only considered shifts involving two particles. As an alternative, we may deform the momenta (or spinors) of three particles. Consider for example the three-particle shift

\[
\begin{align*}
\hat{p}_i^A &= p^A_i - z [jk] \eta_A, \\
\hat{p}_j^A &= p^A_j - z [ki] \eta_A, \\
\hat{p}_k^A &= p^A_k - z [ij] \eta_A,
\end{align*}
\]  

where $\eta_A$ is an arbitrary spinor. The deformed spinors correspond to the following on-shell momenta

\[
\begin{align*}
\hat{p}_i^\mu &= p_i^\mu - \frac{1}{2} z [jk] \langle i + |\gamma^\mu| \eta^+ \rangle, \\
\hat{p}_j^\mu &= p_j^\mu - \frac{1}{2} z [ki] \langle j + |\gamma^\mu| \eta^+ \rangle, \\
\hat{p}_k^\mu &= p_k^\mu - \frac{1}{2} z [ij] \langle k + |\gamma^\mu| \eta^+ \rangle.
\end{align*}
\]  

Momentum conservation is satisfied due to the Schouten identity. Such a shift can be advantageous in situations, where particles $i, j$ and $k$ have positive helicities. The polarisation vectors contribute then a factor $1/z^3$ to the large-$z$ behaviour. We will see an application of this shift later on.

Shifts, which deform the momenta of more than two or three particles are also possible. Ref. [91] contains an extensive discussion of generalised shifts. As a rule of thumb, deforming more momenta may help to improve the large-$z$ behaviour.

5.3 The on-shell recursion algorithm

Let us now look at an example. We consider again the amplitude $A_6(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$. The non-vanishing diagrams from the on-shell recursion approach are shown in fig. 11. Note that we
Figure 11: Diagrams contributing to the tree-level six-gluon amplitude $A_6(1^-,2^-,3^-,4^+,5^+,6^+)$ in the on-shell recursive approach. The vertices are on-shell amplitudes.

Table 3: The number of diagrams contributing to the cyclic-ordered six-gluon amplitude $A_6(1^-,2^-,3^-,4^+,5^+,6^+)$ using various methods.

| method                     | diagrams |
|----------------------------|----------|
| brute force approach       | 220      |
| colour-ordered amplitudes  | 38       |
| MHV vertices               | 6        |
| on-shell recursion         | 2        |

may stop the recursion as soon as a sub-amplitude is a MHV amplitude and use the Parke-Taylor formula for this sub-amplitude. One obtains as a result for the amplitude

$$A_6(1^-,2^-,3^-,4^+,5^+,6^+) = \frac{\langle 6 + |1+2|3+ \rangle^3}{[61][12][34][45]s_{126}(2+|1+6|5+)} + \frac{\langle 4 + |5+6|1+ \rangle^3}{[23][34][56][61]s_{156}(2+|1+6|5+)} ,$$

where we used the notation

$$s_{ijk} = (p_i + p_j + p_k)^2, \quad \langle i + |j+k|l+ \rangle = \langle p_i + |j|j+k|p_j+p \rangle .$$

Note that there are only two diagrams, which need to be calculated. We recall from table 2 that a brute force approach would require the calculation of 220 Feynman diagrams. Restricting ourselves to a primitive amplitude with a fixed cyclic order reduces this number to 38 diagrams. In the approach based on MHV vertices there are only six diagrams. The on-shell recursion method brings the number of diagrams further down to two diagrams. Table 3 shows a comparison of the number of diagrams contributing to the cyclic-ordered six-gluon amplitude in the various approaches.

Let us formulate an algorithm based on the on-shell recursion approach. We consider the primitive $A^{(0)}_n$ with $n_+$ gluons of positive helicity and $n_- = n - n_+$ gluons of negative helicity. We discuss the case of mostly plus helicities $n_- \leq n_+$, the case mostly minus case ($n_- > n_+$).
can be obtained be exchanging dotted with un-dotted indices and $+ \leftrightarrow -$. We have the following algorithm:

**Algorithm 4.** Calculation of the primitive helicity amplitude $A^{(0)}_n(p_1^{\lambda_1}, \ldots, p_n^{\lambda_n})$ from on-shell recursion relations.

1. If $n_- \in \{0, 1, 2\}$, the amplitude $A^{(0)}_n$ is given by the Parke-Taylor formulæ.

2. Pick a pair of adjacent particles, such that the helicities of this pair in the cyclic order are not $(+, -)$. Label these adjacent particles $n$ and $1$. It is always possible to find such a pair, since on-shell amplitudes, where all gluons have the same helicity, vanish. We may therefore assume that $(\lambda_n, \lambda_1) \neq (+, -)$.

3. Consider the shift
   \[
   \hat{p}_A^1 = p_A^1 - z p_A^n, \quad \hat{p}_B^n = p_B^n + z p_B^1.
   \] (378)

4. The amplitude is given by the recurrence relation
   \[
   A^{(0)}_n(p_1^{\lambda_1}, \ldots, p_n^{\lambda_n}) = \sum_{j=2}^{n-2} \sum_{\lambda=\pm} A^{(0)}_{n-j+1}(\hat{p}_1(z_j)^{\lambda_j} p_2^{\lambda_2}, \ldots, \hat{p}_j(z_j)^{-\lambda_j}, \ldots, \hat{p}_{n-j+1}(z_j)^{-\lambda_{n-j+1}} p_{n-j+2}^{\lambda_{n-j+2}} \ldots p_n^{\lambda_n}),
   \] (379)

   where
   \[
   \hat{p}_{1,j}^\mu(z) = \sum_{k=1}^{j} p_{1,k}^\mu - \frac{z}{2} \langle 1 + |\gamma^\mu|n^+ \rangle,
   \] (380)

   and
   \[
   z_j = \frac{p_{1,j}^2}{\langle 1 + |\gamma_{1,j}^\mu|n^+ \rangle}.
   \] (381)

   The four-vector $\hat{p}_{1,j}(z_j)$ is on-shell, the corresponding Weyl spinors are given by
   \[
   |\hat{p}_{1,j}^+\rangle = \frac{\hat{p}_{1,j}^\mu|1-\rangle}{\sqrt{\langle 1 + |\gamma_{1,j}^\mu|n^+ \rangle}}, \quad \langle \hat{p}_{1,j}^+| = \frac{\langle n^-|\gamma_{1,j}^\mu|n^+ \rangle}{\sqrt{\langle 1 + |\gamma_{1,j}^\mu|n^+ \rangle}}.
   \] (382)

The on-shell recursion algorithm is a very powerful tool to obtain compact analytical results for helicity amplitudes and $-$ as we will see $-$ to facilitate proofs. However, when used numerically, the on-shell recursion algorithm does not beat the off-shell recursion algorithm discussed in section 3.5, both in speed and numerical stability [43, 92, 93]. This can be understood as follows: The on-shell recursion algorithm introduces at each step new momenta (or pairs of Weyl spinors). The off-shell algorithm operates on the same set of momenta from the very beginning. Furthermore, we see from eq. (376) that the on-shell algorithm introduces spurious singularities like $1/\langle 2 + |1 + 6|5^+ \rangle$, which affect the numerical stability.
5.4 Three-point amplitudes and on-shell constructible theories

The on-shell algorithm of the previous paragraph uses the Parke-Taylor formula as soon as an MHV amplitude is encountered. This is practical and efficient, since the MHV amplitudes are given by the compact Parke-Taylor formula. On the other hand it is in principle possible to use the BCFW recursion relation also for MHV amplitudes. This possibility implies that all primitive tree-level Yang-Mills amplitudes can be brought down to three-point amplitudes. In other words, the three-point amplitudes together with the on-shell recursion relations determine the full set of primitive tree-level Yang-Mills amplitudes. The same holds true for any theory, whose tree-level amplitudes are constructible through on-shell recursion relations. For massless particles there aren’t too many theories of this type, since Lorentz invariance and locality put tight constraints on the possible three-point amplitudes [94]. Let us denote the three-point amplitudes

\[ \langle p_1 p_2 \rangle = \langle p_2 p_3 \rangle = \langle p_3 p_1 \rangle = 0, \]  

(383)

or

\[ [p_1 p_2] = [p_2 p_3] = [p_3 p_1] = 0. \]  

(384)

A mixed case, where two spinor products of one type and one spinor product of the other type are zero, implies that the third spinor product of the type where already two are zero, is zero as well. It follows that the three-point amplitude is a sum of two functions, one which depends only on the spinor products \( \langle p_1 p_2 \rangle, \langle p_2 p_3 \rangle \) and \( \langle p_3 p_1 \rangle \), while the other function depends only on the spinor products \([p_1 p_2], [p_2 p_3]\) and \([p_3 p_1]\). Under little group scaling \( p_A^j \rightarrow \lambda p_A^j, p_A^\dot{j} \rightarrow \lambda^{-1} p_A^\dot{j} \) the amplitude scales as \( \lambda^{-2h_j} \) for a particle of helicity \( h_j \). This implies that the holomorphic function must be proportional to

\[ \langle p_1 p_2 \rangle^{h_3-h_1-h_2} \langle p_2 p_3 \rangle^{h_1-h_2-h_3} \langle p_3 p_1 \rangle^{h_2-h_3-h_1}, \]  

(385)

while the anti-holomorphic one must be proportional to

\[ [p_1 p_2]^{-h_3+h_1+h_2} [p_2 p_3]^{-h_1+h_2+h_3} [p_3 p_1]^{-h_2+h_3+h_1}. \]  

(386)

If we now restrict ourselves to theories where all particles have helicity \( \pm h \), with \( h \) being a positive integer, one finds

\[ \mathcal{M}_3^{(0)}(1_a^{-h}, 2_b^{-h}, 3_c^{-h}) = i\kappa_{abc} \left( \frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 31 \rangle} \right)^h, \]

\[ \mathcal{M}_3^{(0)}(1_a^{+h}, 2_b^{+h}, 3_c^{-h}) = i\kappa_{abc} \left( \frac{[21]^3}{[13][32]} \right)^h, \]

\[ \mathcal{M}_3^{(0)}(1_a^{-h}, 2_b^{+h}, 3_c^{+h}) = i\kappa'_{abc} \left( \langle 12 \rangle \langle 23 \rangle \langle 31 \rangle \right)^h, \]

\[ \mathcal{M}_3^{(0)}(1_a^{+h}, 2_b^{+h}, 3_c^{+h}) = i\kappa'_{abc} \left( [32][21][13] \right)^h. \]

The indices \( a, b, c \) denote other quantum numbers of the particles (like colour). The couplings \( \kappa \) and \( \kappa' \) may depend on these. Since the amplitude \( \mathcal{M}_3^{(0)} \) must be symmetric under the exchange
of two particles, it follows that $\kappa_{abc}$ and $\kappa'_{abc}$ are anti-symmetric if $h$ is odd and symmetric if $h$ is even.

**Exercise 19:** *Show that a term of the form*

$$\langle p_1 p_2 \rangle^{y_3} \langle p_2 p_3 \rangle^{y_1} \langle p_3 p_1 \rangle^{y_2}$$

*and contributing to $\mathcal{M}_3^{(0)}(1^{h_1}, 2^{h_2}, 3^{h_3})$ must have the exponents as in eq. (385).*

5.5 **Application: Proof of the fundamental BCJ relations**

Let us come back to the fundamental BCJ relation, stated in eq. (145):

$$n-1 \sum_{i=2}^{n-1} \left( \sum_{j=i+1}^{n} 2p_2 p_j \right) A_n^{(0)} (1, 3, \ldots, i, 2, i+1, \ldots, n-1, n) = 0. \quad (388)$$

We promised a proof of this relation based on on-shell recursion relations [47, 95, 96]. The exposition follows [97]. We prove the fundamental BCJ relations by induction. For $n = 3$ the fundamental BCJ-relation reduces to

$$2p_2 p_3 A_3^{(0)} (1, 2, 3) = 0. \quad (389)$$

For generic external momenta $A_3^{(0)} (1, 2, 3)$ is finite and

$$2p_2 p_3 = (p_2 + p_3)^2 = p_1^2 = 0. \quad (390)$$

For the induction step let us consider a three-particle shift, where we deform the momenta of the particles 1, 2 and $n$:

$$\hat{p}_1(z), \quad \hat{p}_2(z), \quad \hat{p}_n(z). \quad (391)$$

We require

$$\hat{p}_1(0) = p_1, \quad \hat{p}_2(0) = p_2, \quad \hat{p}_n(0) = p_n. \quad (392)$$

For $j \neq 1, 2, n$ we simply set $\hat{p}_j(z) = p_j$. Let us consider the quantity

$$I_n(z) = \sum_{i=2}^{n-1} \left( \sum_{j=i+1}^{n} 2\hat{p}_2 \hat{p}_j \right) A_n^{(0)} (\hat{1}, 3, \ldots, i, \hat{2}, i+1, \ldots, n-1, \hat{n}). \quad (393)$$

For $z = 0$ the expression $I_n(z)$ reduces to the left-hand-side of eq. (388). $I_n(z)$ is clearly a rational function of $z$. We have to show that

$$I_n(0) = 0, \quad (394)$$
or equivalently

\[
\frac{1}{2\pi i} \oint_{z=0} \frac{dz}{z} I_n(z) = 0,
\]

where the contour is a small counter-clockwise circle around \(z = 0\). We may assume that \(I_j(0) = 0\) for \(j < n\). Let us assume that \(I_n(z)\) falls off for large \(z\) at least with \(1/z\), we will show in a minute that we can always deform the momenta in such a way to achieve this. Deforming the contour to a large circle at infinity and the residues at the finite poles \(z_\alpha \neq 0\) we obtain

\[
I_n(0) = -\sum_\alpha \text{res} \left( \frac{I_n(z)}{z} \right)_{z_\alpha}.
\]

Since we assumed that \(I_n(z)\) falls off for large \(z\) at least with \(1/z\), there is no contribution from the large circle at infinity. It will be convenient to introduce the following notation for the various factorisation channels:

\[
A_n^{(0)}(\hat{1}, 2, \ldots, k, \hat{p} | -\hat{p}, k + 1, \ldots, n - 1, \hat{n}) = \sum_\lambda A_{k+1}^{(0)}(\hat{1}, 2, \ldots, k, \hat{p}) \frac{i}{p^2} A_{n-k+1}^{(0)}(-\hat{p}, k + 1, \ldots, n - 1, \hat{n}),
\]

(397)

together with the convention that the hatted quantities are evaluated at \(z = z_\alpha\). The sum is over the helicity of the intermediate particle. Let us look at the \(z\)-momentum flow for a three-particle BCFW-shift. For each diagram we may divide the \(z\)-dependent propagators into three segments. Each segment starts at the common vertex, where the \(z\)-dependent momentum flow meets and goes outwards towards the particles 1, 2 and \(n\). We may use these segments to divide the finite residues into three groups and we write

\[
I_n(0) = R_1 + R_2 + R_n,
\]

(398)

with

\[
R_1 = \sum_{i=2}^{n-1} \left( \sum_{j=i+1}^{n} 2\hat{p}_2 \hat{p}_j \right) \sum_{k=3}^{n} A_n^{(0)}(\hat{1}, 3, \ldots, k, \hat{p} | -\hat{p}, k + 1, \ldots, i, \hat{2}, i + 1, \ldots, n - 1, \hat{n}),
\]

\[
R_2 = \sum_{i=2}^{n-1} \left( \sum_{j=i+1}^{n} 2\hat{p}_2 \hat{p}_j \right) \times \sum_{k=2}^{i} \sum_{l=i}^{n-1} A_n^{(0)}(k + 1, \ldots, i, \hat{2}, i + 1, \ldots, l, \hat{p} | -\hat{p}, l + 1, \ldots, n - 1, \hat{n}, \hat{1}, 3, \ldots, k),
\]

\[
R_n = \sum_{i=2}^{n-1} \left( \sum_{j=i+1}^{n} 2\hat{p}_2 \hat{p}_j \right) \sum_{k=i}^{n-2} A_n^{(0)}(\hat{1}, 3, \ldots, i, \hat{2}, i + 1, \ldots, k, \hat{p} | -\hat{p}, k + 1, \ldots, n - 1, \hat{n}).
\]

(399)
Let us first look at $R_1$. We may exchange the summation over $i$ and $k$ as

$$
\sum_{i=2}^{n-1} \sum_{k=3}^{i} f(i,k) = \sum_{k=3}^{n-1} \sum_{i=k}^{n-1} f(i,k).
$$

One obtains

$$
R_1 = \sum_{k=3}^{n-1} \sum_{i=k}^{n-1} \left( \sum_{j=i+1}^{n} 2\hat{p}_2 \hat{p}_j \right) A_n^{(0)} (\hat{1}, 3, \ldots, k, \hat{p} - \hat{p}, k + 1, \ldots, i, \hat{2}, i + 1, \ldots, n - 1, \hat{n}) \tag{401}
$$

We recognise the fundamental BCJ relation for $(n - k + 2) < n$ external particles. We may therefore use the induction hypothesis and we conclude

$$
R_1 = 0. \tag{402}
$$

The argument for $R_n$ is very similar. By exchanging the summation over $i$ and $k$ and by using momentum conservation in the sum over $j$ one shows $R_n = 0$.

For the contribution from $R_2$ we have to work a little bit more. Exchanging the summation indices for $R_2$ one obtains

$$
R_2 = \sum_{k=3}^{n-1} \sum_{i=k}^{n-1} \sum_{l=k+1}^{i} \left( \sum_{j=i+1}^{n} 2\hat{p}_2 \hat{p}_j \right) A_n^{(0)} (k + 1, \ldots, i, \hat{2}, i + 1, \ldots, l, \hat{p} - \hat{p}, l + 1, \ldots, n - 1, \hat{n}, \hat{1}, \ldots, k) \tag{403}
$$

We may split the sum over $j$ as

$$
\sum_{j=i+1}^{n} 2\hat{p}_2 \hat{p}_j = \sum_{j=i+1}^{l} 2\hat{p}_2 \hat{p}_j + \sum_{j=l+1}^{n} 2\hat{p}_2 \hat{p}_j \tag{404}
$$

The terms of type A vanish again by the induction hypothesis

$$
\sum_{i=k}^{l-1} \left( \sum_{j=i+1}^{l} 2\hat{p}_2 \hat{p}_j \right) A_{l-k+2}^{(0)} (\hat{p}, k + 1, \ldots, i, \hat{2}, i + 1, \ldots, l) = 0. \tag{405}
$$

Note that the sum over $i$ extends only to $(l - 1)$, the case $i = l$ contributes only to the terms of type $B$.

For the terms of type $B$ the sum over $j$ is independent of $i$ and may be taken outside the sum over $i$. The sum over $i$ vanishes then due to the $U(1)$-decoupling relation.

$$
\left( \sum_{j=i+1}^{n} 2\hat{p}_2 \hat{p}_j \right) \sum_{i=k}^{l} A_{l-k+2}^{(0)} (\hat{p}, k + 1, \ldots, i, \hat{2}, i + 1, \ldots, l) = 0. \tag{406}
$$
The $U(1)$-decoupling relation is a special case of the Kleiss-Kuijf relations given in eq. (130), corresponding to the case where $\vec{\beta}$ consists only of a single element. We therefore conclude that $R_2 = 0$. Putting the partial results for $R_1$, $R_2$ and $R_n$ together we find that

$$I_n(0) = 0.$$  \hspace{1cm} (407)

It remains to show that we can always find a momenta shift such that $I_n(z)$ falls off for large $z$ at least with $1/z$. Our strategy is as follows: We will choose a deformation such that the external polarisation vectors contribute a factor $z^{-3}$. The contribution from internal propagators and vertices is at worst $z$. Together with another factor $z^1$ from the prefactor $(2\hat{p}_2\hat{p}_j)$ we then achieve a total $1/z$-behaviour for $I_n(z)$.

The explicit expressions for the momenta shifts will depend on the helicities of the particles 1, 2 and $n$. For $(\lambda_1, \lambda_2, \lambda_n) = (+, +, +)$ we may choose the shift defined in eq. (374). For $(\lambda_1, \lambda_2, \lambda_n) = (+, +, -)$ we may choose

$$\hat{p}_A^1 = p_A^1 - zy_1 p_A^n, \quad \hat{p}_B^2 = p_B^2 + zy_1 p_B^1 + zy_2 p_B^n,$$
$$\hat{p}_A^2 = p_A^2 - zy_2 p_A^n,$$  \hspace{1cm} (408)

where $y_1$ and $y_2$ are two non-zero constants. The cases $(\lambda_1, \lambda_2, \lambda_n) = (+, -, +)$ and $(\lambda_1, \lambda_2, \lambda_n) = (-, +, +)$ may be obtained from eq. (408) by permutation. Finally, the shifts for the helicity configurations

$$(-, -, -), \; (-, -, +), \; (-, +, -), \; (+, -, -)$$  \hspace{1cm} (409)

can be obtained from the helicity configurations

$$(+, +, +), \; (+, +, -), \; (+, -, +), \; (-, +, +)$$  \hspace{1cm} (410)

by exchanging holomorphic and anti-holomorphic spinors. This completes the proof of the fundamental BCJ relations.

On-shell recursion relations may also be used to prove the MHV expansion discussed in section (4.2), the interested reader is referred to [87].

As another application of the on-shell recursion relations consider the following exercise:

**Exercise 20:** In section (4.1) we used off-shell recurrence relations to prove the Parke-Taylor formula for two adjacent negative helicities. Consider now the general case of two non-adjacent negative helicities. Prove the Parke-Taylor formula for two non-adjacent negative helicities from the formula of the adjacent case and by using on-shell recursion relations.
6 Grassmannian geometry

In this section we explore the relation between scattering amplitudes and Grassmannian mani-
folds. We first give a short introduction to projective spaces, Grassmannian manifolds and flag
varieties and present then the link representation of a primitive tree-level amplitude as a residue
on a Grassmannian manifold. In addition we introduce the notation related to the $\mathcal{N} = 4$ super-
symmetric extension of Yang-Mills theory and discuss in this context the scattering amplitude as
a volume of the amplituhedron.

6.1 Projective spaces, Grassmannians and flag varieties

The complex projective space $\mathbb{CP}^n$ is the set of lines through the origin in $\mathbb{C}^{n+1}$. Let us consider
points $(x_0,x_1,...,x_n) \in \mathbb{C}^{n+1}\{0\}$. We call two points of $\mathbb{C}^{n+1}\{0\}$ equivalent, if there is a $\lambda \neq 0$
such that

$$(x_0,x_1,...,x_n) = (\lambda y_0,\lambda y_1,...,\lambda y_n).$$

We have

$$\mathbb{CP}^n = \left(\mathbb{C}^{n+1}\{0\}\right)/\mathbb{C}^*,$$

where $\mathbb{C}^* = \mathbb{C}\{0\}$. Points in $\mathbb{CP}^n$ will be denoted in homogeneous coordinates by

$$[z_0:z_1:...:z_n].$$

Affine charts are defined by

$$U_j = \left\{ (z_0,...,z_{j-1},1,z_{j+1},...,z_n) \right\}, \quad 0 \leq j \leq n.$$

The cell decomposition of $\mathbb{CP}^n$ is

$$\mathbb{CP}^n = \mathbb{C}^n \sqcup \mathbb{C}^{n-1} \sqcup ... \sqcup \mathbb{C}^1 \sqcup \mathbb{C}^0,$$

where $\sqcup$ denotes the disjoint union. This can be seen as follows: Consider $\mathbb{CP}^n$ with homoge-
neous coordinates $[z_0:z_1:...:z_n]$. Divide the space into the regions $z_n \neq 0$ and $z_n = 0$. The first
region is homeomorphic to $\mathbb{C}^n$, the second region is homeomorphic to $\mathbb{CP}^{n-1}$. We therefore have
the recursion

$$\mathbb{CP}^n = \mathbb{C}^n \sqcup \mathbb{CP}^{n-1},$$

from which the result in eq. (415) follows immediately.

Projective spaces have a generalisation: In order to obtain the projective space, we consid-
ered lines through the origin in $\mathbb{C}^{n+1}$. Instead of lines we may consider higher dimensional sub-spaces. This brings us to the definition of a Grassmannian manifold. The Grassmannian manifold $\text{Gr}_{k,n}(\mathbb{C})$ is the set of $k$-dimensional planes in $\mathbb{C}^n$. Let $M_{k,n}(\mathbb{C})$ be the set of $k \times n$
matrices of rank $k$ (with $k \leq n$). An element of $M_{k,n}(\mathbb{C})$ may be written as $k$ linearly independent row vectors

$$
\begin{pmatrix}
a_{11} & \ldots & a_{1n} \\
\vdots & \ddots & \vdots \\
a_{k1} & \ldots & a_{kn}
\end{pmatrix}.
$$

Two matrices $A_1, A_2 \in M_{k,n}(\mathbb{C})$ are called equivalent, if there is a $\Lambda \in \text{GL}(k, \mathbb{C})$ such that

$$
A_1 = \Lambda A_2.
$$

The Grassmannian is then defined as

$$
\text{Gr}_{k,n}(\mathbb{C}) = M_{k,n}(\mathbb{C})/\text{GL}(k, \mathbb{C}).
$$

The projective space $\mathbb{CP}^n$ is a special case of a Grassmannian manifold:

$$
\mathbb{CP}^n = \text{Gr}_{1,n+1}(\mathbb{C}).
$$

Let us discuss affine charts for Grassmannian manifolds. Let $I = (i_1, i_2, \ldots, i_k)$ be a set of $k$ column indices (with $1 \leq i_1 < i_2 < \ldots < i_k \leq n$), such that the corresponding $k \times k$-minor $A_I$ of $A \in M_{k,n}(\mathbb{C})$ has $\det A_I \neq 0$. We may then use $\Lambda \in \text{GL}(k, \mathbb{C})$ to convert $A_I$ to the $k \times k$ unit matrix. This defines an affine chart of $\text{Gr}_{k,n}(\mathbb{C})$. For example, in the case $I = (n-k+1, n-k+2, \ldots, n)$ the elements of $\text{Gr}_{k,n}(\mathbb{C})$ in this chart are parametrised as

$$
\begin{pmatrix}
a_{11} & \ldots & a_{1(n-k)} & 1 & 0 & \ldots & 0 \\
a_{21} & \ldots & a_{2(n-k)} & 0 & 1 & \ldots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
a_{k1} & \ldots & a_{k(n-k)} & 0 & 0 & \ldots & 1
\end{pmatrix}.
$$

This shows that

$$
\dim \mathbb{C} \text{Gr}_{k,n}(\mathbb{C}) = k(n-k).
$$

A Grassmannian manifold may be embedded as a sub-manifold in a higher dimensional complex projective space. Set

$$
d = \begin{pmatrix} n \\ k \end{pmatrix}.
$$

This is the number of different sets of $k$ indices $I = (i_1, i_2, \ldots, i_k)$. The Plücker embedding is the map

$$
\text{Gr}_{k,n}(\mathbb{C}) \to \mathbb{CP}^{d-1},
$$

$$
A \to [\det A_{I_1} : \det A_{I_2} : \ldots : \det A_{I_d}].
$$

The numbers $\det A_{I_1}, \det A_{I_2},$ etc. are referred to as Plücker coordinates.
Up to now we considered complex Grassmannian manifolds. There is an analogue definition for real Grassmannian manifolds:

\[
\text{Gr}^k_{k,n}(\mathbb{R}) = M_{k,n}(\mathbb{R})/\text{GL}(k,\mathbb{R}).
\] (425)

Over the real numbers we may define the positive Grassmannian \(\text{Gr}^+_{k,n}(\mathbb{R})\) by the condition that all Plücker coordinates are positive. Note that we required the set of column indices \((i_1, i_2, \ldots, i_k)\) to be ordered: \(1 \leq i_1 < i_2 < \ldots < i_k \leq n\). This reduces the definition of positivity of points of the Grassmannian \(\text{Gr}_{k,n}(\mathbb{R})\) to the definition of positivity of points in the projective space \(\mathbb{RP}^{d-1}\).

Let us return to the complex case. We learned about projective spaces and Grassmannians, but we haven’t reached the end of sophistication yet. Consider a plane through the origin in \(\mathbb{C}^4\). The plane is isomorphic to \(\mathbb{C}^2\) and within this plane we may again consider lines through the origin. This brings us to the definition of a flag. A flag is an increasing sequence of sub-spaces

\[
\{0\} \subset V_1 \subset V_2 \subset \ldots \subset V_k = \mathbb{C}^n.
\] (426)

The motivation for the name “flag” is shown in fig. (12). We set

\[
d_j = \dim V_j.
\] (427)

The signature of the flag is the sequence

\[
(d_1, d_2, \ldots, d_k).
\] (428)

Note that by definition we always have \(d_k = n\). The flag variety \(F_{d_1, d_2, \ldots, d_k}(\mathbb{C})\) is the space of all flags of signature \((d_1, d_2, \ldots, d_k)\) in the vector space \(\mathbb{C}^n\). Note that we have \(d_k = n\). For \(k = 2\) this definition reduces to the Grassmannian manifold:

\[
F_{d,n}(\mathbb{C}) = \text{Gr}_{d,n}(\mathbb{C}).
\] (429)

A flag is called a complete flag if \(k = n\) and consequently \((d_1, d_2, \ldots, d_n) = (1, 2, \ldots, n)\), otherwise it is called a partial flag. A partial flag can be obtained from a complete flag by deleting some of the sub-spaces.
the sub-spaces $V_j \neq V_n$. Let us first consider the complete flag variety. Let $(e_1, e_2, \ldots, e_n)$ be an ordered basis of $C^n$. The standard flag associated to this basis is given by the vector spaces

$$V_j = \langle e_1, \ldots, e_j \rangle.$$  \hfill (430)

The group $GL(n, C)$ acts transitively on $F_{1,2,\ldots,n}(C)$. The stabiliser group of the standard flag is the group of all non-singular upper triangular matrices. Multiples of the identity matrix act trivially on all flags and we can restrict ourselves to $SL(n, C)$ and the group of upper triangular matrices of determinant 1. We denote the latter by $B_n$. (The letter $B$ stands for Borel sub-group.) The complete flag variety is given as the homogeneous space

$$F_{1,2,\ldots,n}(C) = SL(n, C)/B_n.$$  \hfill (431)

We have

$$\dim_C F_{1,2,\ldots,n}(C) = \frac{1}{2} (n-1) n.$$  \hfill (432)

Let us now discuss partial flag varieties. As an example of a partial flag variety we consider the Grassmannian $Gr_{d,n}(C)$. The stabiliser group of $V_d = \langle e_1, \ldots, e_d \rangle$ consists of $A \in SL(n, C)$ of the form

$$A = \begin{pmatrix}
  a_{1,1} & \cdots & a_{1,d} & a_{1,d+1} & \cdots & a_{1,n} \\
  \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
  a_{d,1} & \cdots & a_{d,d} & a_{d,d+1} & \cdots & a_{d,n} \\
  0 & \cdots & 0 & a_{d+1,d+1} & \cdots & a_{d+1,n} \\
  \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
  0 & \cdots & 0 & a_{n,d+1} & \cdots & a_{n,n}
\end{pmatrix}.$$  \hfill (433)

Matrices of this form are called block upper triangular matrices. For a flag variety with signature $(d_1, d_2, \ldots, d_k)$ the stabiliser group of the standard flag consists of block upper triangular matrices of unit determinant, where the sizes of the blocks on the diagonal are $n_i = d_i - d_{i-1}$ (with the convention that $d_0 = 0$). These block upper triangular matrices define a sub-group $P$ of $SL(n, C)$. (The letter $P$ stands for parabolic sub-group.) The partial flag variety is then given as the homogeneous space

$$F_{d_1,d_2,\ldots,d_k}(C) = SL(n, C)/P.$$  \hfill (434)

The dimension of $F_{d_1,d_2,\ldots,d_k}(C)$ is (again with the convention $d_0 = 0$) \footnote{99}

$$\dim_C F_{d_1,d_2,\ldots,d_k}(C) = \sum_{1 \leq i < j \leq k} (d_i - d_{i-1}) (d_j - d_{j-1}).$$  \hfill (435)

### 6.2 The link representation

We may express momentum conservation in various ways:

$$\sum_{j=1}^{n} p_j^\mu = 0 \Leftrightarrow \sum_{j=1}^{n} p_A^j p_A^j = 0.$$  \hfill (436)
Let us now consider the $n$-dimensional vectors
\begin{align*}
P_1 &= (p^1_1, p^2_1, \ldots, p^n_1), & P_1 &= (p^1_1, p^2_1, \ldots, p^n_1), \\
P_2 &= (p^1_2, p^2_2, \ldots, p^n_2), & P_2 &= (p^1_2, p^2_2, \ldots, p^n_2).
\end{align*}
(437)

The last equation in eq. (436) can be interpreted as an orthogonality condition: The plane spanned by the vectors $(P_1, P_2)$ is orthogonal to the plane spanned by the vectors $(P_1, P_2)$. Now consider a $k$-dimensional plane $C$ in $\mathbb{C}^n$, together with its $(n-k)$-dimensional orthogonal complement $C^\perp$. Assume now that $P_1$ and $P_2$ are orthogonal to $C$ and that $P_1$ and $P_2$ are orthogonal to $C^\perp$. In formulæ:
\[ C^\perp \cdot P_A = 0, \quad C \cdot P_A = 0. \]
(438)

Of course, these conditions imply $P_A \in C$ and $P_A \in C^\perp$. In general, there are many planes $C \in \text{Gr}_{k,n}(\mathbb{C})$ subject to the orthogonality condition in eq. (438). The idea is now to integrate over all possibilities with a suitable measure. This idea leads us to the link representation for the primitive tree amplitude $A^{(0)}_n$ and ties the amplitude to Grassmannian geometry.

Let us see how this is done: Consider the primitive tree-amplitude $A^{(0)}_n(1^{\lambda_1}, \ldots, n^{\lambda_n})$ with the cyclic order $(1,2,\ldots,n)$. Assume that $k$ gluons have negative helicities, while the remaining $(n-k)$ gluons have positive helicity. Let us denote the indices of the $k$ negative helicity gluons by $J = (j_1, \ldots, j_k)$ with $j_1 < j_2 < \ldots < j_k$. The ordered set $J$ defines a chart of the Grassmannian $\text{Gr}_{k,n}(\mathbb{C})$: We gauge-fix the GL($k$)-redundancy by requiring that for $C \in \text{Gr}_{k,n}(\mathbb{C})$ we have
\[ C_{ij_l} = \delta_{il}, \quad l = 1, \ldots, k. \]
(439)

Thus, the $j_l$-th column of $C$ is given by the $l$-th unit vector. We denote the remaining $(n-k)$ indices not in $J$ by $K = (k_1, \ldots, k_{n-k})$. The **link representation** of the (non-supersymmetric) helicity amplitude multiplied by the momentum-conserving delta-function is given by $[100, 104]$

\[
A^{(0)}_n \delta^4 \left( \sum_{j=1}^n p^j_A p^j_A \right) = \frac{i \left( \sqrt{2} \right)^{n-2}}{(2\pi i)^{(k-2)(n-k-2)}} \int_{\mathcal{M}} d^{k \times n} \delta^k \left( C_{ij_l} - \delta_{il} \right) \prod_{l=1}^k \delta^2 \left( p^j_A + C_{j_k} p^k_A \right) \prod_{m=1}^{n-k} \delta^2 \left( p^m_A - C_{j_m} p^m_A \right)
\]
(440)

where $(i_1, \ldots, i_k)$ denotes the $k \times k$-minor of the matrix $C$ made out of the columns $i_1, \ldots, i_k$:
\[ (i_1, \ldots, i_k) = \det C_{(i_1, \ldots, i_k)} = \epsilon^{h_1 \ldots h_k} C_{h_1 i_1} \ldots C_{h_k i_k}. \]
(441)

The first product of delta-functions fixes the gauge according to eq. (439). The other delta-functions ensure
\begin{align*}
p^k_A &= C_{j_k} p^l_A, & k &\in K, \\
p^j_A &= -C_{j_k} p^k_A, & j &\in J.
\end{align*}
(442)
These equations imply momentum conservation:

\[
\sum_{j=1}^{n} p^j_A p_A^j = \sum_{k \in K} p^k_A p_A^k + \sum_{j \in J} p^j_A p_A^j = \sum_{j \in J} \sum_{k \in K} C_{jk} p^j_A p_A^k - \sum_{j \in J} \sum_{k \in K} C_{jk} p^j_A p_A^k = 0. (443)
\]

There are \(k \cdot n\) integrations in eq. (440), the delta-functions remove \(k^2 + 2k + 2(n - k) - 4\) integrations, since four delta-functions should remain for the four-dimensional delta-function expressing momentum conservation on the left-hand-side of eq. (440). This leaves us with

\[
k n - k^2 - 2k - 2(n - k) + 4 = (k - 2)(n - k - 2) (444)
\]

integrations, which are understood as residues. Multi-variable residues are defined as follows: Consider a map

\[
p : \mathbb{C}^n \to \mathbb{C}^n, \quad z = (z_1, ..., z_n) \to (p_1(z), ..., p_n(z)). (445)
\]

Let us further assume that the system of equations \(p(z) = 0\) has as solutions a finite number of isolated points \(z^{(j)} = (z_1^{(j)}, ..., z_n^{(j)})\), where \(j\) labels the individual solutions. Let us further consider a function \(g : \mathbb{C}^n \to \mathbb{C}\), regular at the solutions \(z^{(j)}\). We define the \textbf{local residue} of \(g\) with respect to \(p_1, ..., p_n\) at \(z^{(j)}\) by

\[
\text{Res}_{\{p_1, ..., p_n\}}(g, z^{(j)}) = \frac{1}{(2\pi i)^n} \oint_{\Gamma_\delta} \frac{g(z) \, dz_1 \wedge ... \wedge dz_n}{p_1(z) ... p_n(z)}. (446)
\]

The integration in eq. (446) is around a small \(n\)-torus

\[
\Gamma_\delta = \{ (z_1, ..., z_n) \in \mathbb{C}^n \mid |p_i(z)| = \delta \}, (447)
\]

encircling \(z^{(j)}\) with orientation

\[
d \arg p_1 \wedge d \arg p_2 \wedge ... \wedge d \arg p_n \geq 0. (448)
\]

The \textbf{global residue} of \(g\) with respect to \(p_1, ..., p_n\) is defined as

\[
\text{Res}_{\{p_1, ..., p_n\}}(g) = \sum_{\text{solutions } j} \text{Res}_{\{p_1, ..., p_n\}}(g, z^{(j)}). (449)
\]

In order to evaluate a local residue it is advantageous to perform a change of variables

\[
z_i' = p_i(z), \quad i = 1, ..., n. (450)
\]

Let us denote the Jacobian of this transformation by

\[
J(z) = \frac{1}{\det \left( \frac{\partial (p_1, ..., p_n)}{\partial (z_1, ..., z_n)} \right)}. (451)
\]
The local residue at \( z^{(j)} \) is then given by
\[
\text{Res}_{\{p_1, \ldots, p_n\}}(g, z^{(j)}) = \frac{1}{(2\pi i)^n} \oint_{\Gamma_0} \frac{g(z) \, dz_1 \wedge \ldots \wedge dz_n}{p_1(z) \ldots p_n(z)} = J(z^{(j)}) \, g(z^{(j)}),
\] (452)
and the global residue as the sum over all local residues.

Let us now return to eq. (440). There are \((k-2)(n-k-2)\) integrations and in order to specify the integration contour we have to give a map \( F : \mathbb{C}^{(k-2)(n-k-2)} \to \mathbb{C}^{(k-2)(n-k-2)} \), or equivalently to give \((k-2)(n-k-2)\) maps \( f^j_l : \mathbb{C}^{(k-2)(n-k-2)} \to \mathbb{C} \) for \( j = 1, \ldots, (k-2) \) and \( l = 3, \ldots, n \). The appropriate maps have been derived in [105] and are given as a product of three minors
\[
f^j_l = \left( \sigma^j_l, l-2, l-1, l \right) \left( \sigma^j_l, l, j, j+1 \right) \left( \sigma^j_l, j+1, j+2, l-2 \right),
\] (453)
where \( \sigma^j_l \) denotes the \((k-3)\) columns \((1, \ldots, j-1) \cup (j+l-k, \ldots, l-3) \). It can be shown that we may re-write the integrand of eq. (440) in a form, where the product
\[
\prod_{l=k+3}^{n} \left( \prod_{j=1}^{k-2} f^j_l \right)
\] (454)
appears in the denominator and a regular function in the numerator. The contour integration is then defined as a global residue of the zero locus \( f^j_l = 0 \). The link representation may be proved with the help of the on-shell recursion relations.

Let us look at an example. We consider the \( n \)-point MHV amplitude \( A^{(0)}_n(1^-, 2^-, 3^+, \ldots, n^+) \). In this case we have \((k-2)(n-k-2) = 0\) and there are no residues to be taken. We have \( J = (1, 2) \) and \( K = (2, 3, \ldots, n) \). Thus we have
\[
A^{(0)}_n(1^-, 2^-, 3^+, \ldots, n^+) \, \delta^4 \left( \sum_{j=1}^{n} p^j_A p^j_A \right) = i \left( \sqrt{2} \right)^{n-2}
\] (455)
\[
\int_{\mathcal{C}} d^{2\times n}C \prod_{i,l=1}^{2} \delta(C_{il} - \delta_{il}) \prod_{l=1}^{2} \delta^2 \left( p^j_A + C_{lk} p^k_A \right) \prod_{m=3}^{n} \delta^2 \left( p^m_A - p^j_A C_{jm} \right)
\]
\[
\frac{1}{(1, 2) (2, 3) \ldots (n, 1)},
\]
with \( j \in J \) and \( k \in K \). For the case at hand it is the simplest to perform a \( \text{GL}(2, \mathbb{C}) \)-variable transformation
\[
C' = g \cdot C,
\] (456)
with
\[
g = \begin{pmatrix} p^1_1 & p^2_1 \\ p^2_1 & p^2_2 \end{pmatrix}, \quad g^{-1} = \frac{1}{(12)} \begin{pmatrix} p^2_1 & -p^2_2 \\ -p^2_2 & p^1_1 \end{pmatrix}, \quad \det g = -\langle 12 \rangle.
\] (457)
We then have
\[
A_n^{(0)} (1^-, 2^-, 3^+, ..., n^+) \delta^4 \left( \sum_{j=1}^{n} p_A^j p_A^j \right) = i \left( \sqrt{2} \right)^{n-2} \langle 12 \rangle^4
\]  
(458)
\[
d^{2\times n} \mathcal{C} \prod_{A, I = 1}^{2} \delta^2 \left( (C'_{AI} - g_{AI}) \prod_{B=1}^{2} \delta^2 \left( g_{BJ} p_A^j + C'_{BK} p_A^k \right) \right) \prod_{m=3}^{n} \delta^2 \left( p_m^m - C'_{Am} \right)
\]
(459)
where the minors are now with respect to the matrix \( C' \). The matrix \( C' \) is completely fixed by the delta-functions:
\[
C' = \begin{pmatrix}
p_1^1 & p_2^1 & p_3^1 & ... & p_n^1 \\
p_1^2 & p_2^2 & p_3^2 & ... & p_n^2
\end{pmatrix}
\]
(460)
The minors are then
\[
(i, j) = - \langle ij \rangle.
\]
(461)
We further have
\[
\prod_{B=1}^{2} \delta^2 \left( g_{BJ} p_A^j + C'_{BK} p_A^k \right) = \delta^4 \left( \sum_{j=1}^{n} p_A^j p_A^j \right)
\]
(462)
giving the momentum-conserving delta-function. Thus
\[
A_n^{(0)} (1^-, 2^-, 3^+, ..., n^+) \delta^4 \left( \sum_{j=1}^{n} p_A^j p_A^j \right) = i \left( \sqrt{2} \right)^{n-2} \frac{\langle 12 \rangle^4}{\langle 12 \rangle \cdots \langle n1 \rangle} \delta^4 \left( \sum_{j=1}^{n} p_A^j p_A^j \right).
\]
(463)

### 6.3 Supersymmetric notation

Up to now we considered individual helicity amplitudes. Is there a simple way to write a formal expression, covering all possible helicity configurations in a single formula? Yes, there is a possibility. Let’s look at one specific external gluon. This gluon can have either helicity +1 or −1. The amplitude is linear in the polarisation vector for this particle. Thus we may consider to substitute
\[
\varepsilon^+ (p) + \eta \varepsilon^- (p)
\]
(464)
for the polarisation vector. One recovers the result for \( h = +1 \) by setting \( \eta = 0 \), and the one for \( h = -1 \) by first differentiating with respect to \( \eta \) and setting afterwards \( \eta = 0 \). (Since eq. [464] is linear in \( \eta \), the last operation has no effect.) This will be the basic idea. We are going to use a fancy version of this basic idea: Instead of one number \( \eta \) we are going to use a product of four numbers \( \eta_1 \eta_2 \eta_3 \eta_4 \). We will write these numbers as \( \eta_l \) with \( l = 1, ..., 4 \). Instead of being ordinary numbers, we will take the \( \eta_l \)’s to be anti-commuting Grassmann numbers. Finally,
instead of just considering a field with the two helicity states \( h = \pm 1 \), we will be considering a field with one state of helicity \(+1\), four states of helicity \(+1/2\), six states of helicity \(0\), four states of helicity \(-1/2\) and one state of helicity \(-1\). Thus we write for the field \([106, 107]\)

\[
\Phi = |+1\rangle + \eta |+1/2\rangle_I + \frac{1}{2} \varepsilon_{IJKL} \eta^I \eta^J \eta^K \eta^L \rangle - 1/2\rangle_L + \frac{1}{24} \varepsilon_{IJKL} \eta^I \eta^J \eta^K \eta^L |1\rangle.
\]

Of course, this is just the description of the \( \mathcal{N} = 4 \) supersymmetric Yang-Mills theory \([108]\). The index \( I \) refers to the four supersymmetric generators. We are not going into detail on supersymmetric generalisations of Yang-Mills theory, which we will use in this report merely as a book-keeping device. The \(|+1\rangle\)-state is projected out by setting all \( \eta_I = 0 \), the \(|-1\rangle\)-state is projected out by applying first the differential operator

\[
\frac{\partial}{\partial \eta_1} \frac{\partial}{\partial \eta_2} \frac{\partial}{\partial \eta_3} \frac{\partial}{\partial \eta_4} = \frac{\partial}{\partial \eta_4} \frac{\partial}{\partial \eta_3} \frac{\partial}{\partial \eta_2} \frac{\partial}{\partial \eta_1}
\]

and then setting all \( \eta_I \)'s to zero. Note that derivatives with respect to Grassmann numbers anticommute. The primitive tree amplitudes with all external particles having helicities \( \pm 1 \) agree between non-supersymmetric Yang-Mills theory and \( \mathcal{N} = 4 \) supersymmetric Yang-Mills theory. This statement is no longer true if we go to loop amplitudes. The reason is that in tree amplitudes with all external states restricted to be spin-one states, the additional states of spin \( 1/2 \) or \( 0 \) present in \( \mathcal{N} = 4 \) theory but not in the non-supersymmetric version cannot propagate internally. In loop amplitudes they may propagate in closed loops.

We associate to each external particle four Grassmann numbers \( \eta^I_j \) with \( I = 1, \ldots, 4 \) and \( j = 1, \ldots, n \). We may specify the momentum of each particle by two spinors \( p^I_A \) and \( \dot{p}^I_A \). We call for a given \( j \) the quantity

\[
p^I_A \dot{\eta}^I_j
\]

the **fermionic part of the super-momentum**, the bosonic part is as usual

\[
p^I_\mu = \frac{1}{2} p^I_A \sigma_\mu^A p^I_A.
\]

Momentum conservation for the fermionic part of the super-momentum reads

\[
\sum_{j=1}^n p^I_A \dot{\eta}^I_j = 0.
\]

Note that these are eight equations, since \( A \in \{1, 2\} \) and \( I \in \{1, 2, 3, 4\} \).

Let us define a fermionic delta-function. We first consider a single Grassmann variable \( \eta \). Recall that the Taylor expansion of a function \( f(\eta) \) stops after the linear term: \( f(\eta) = f_0 + f_1 \eta \). Integration over Grassmann variables is very simple and defined by the two rules

\[
\int d\eta = 0, \quad \int d\eta \cdot \eta = 1.
\]
Thus we have
\[ \int d\eta \, \delta (\eta) \, f (\eta) = f_0 = \int d\eta \, \eta \, f (\eta) \]  \hspace{1cm} (470)
and it follows that \( \delta (\eta) = \eta \). Conservation of super-momentum is enforced by four bosonic and eight (= \( 2N \)) fermionic delta-functions:
\[ \delta^8 \left( \sum_{j=1}^{n} p_A^j \eta^j \right) \]
\[ \times \delta^4 \left( \sum_{j=1}^{n} p_A^j \eta^j \right). \]  \hspace{1cm} (471)
It is convenient to set
\[ Q_{AI} = \sum_{j=1}^{n} p_A^j \eta^j. \]  \hspace{1cm} (472)
Within a product of fermionic delta-functions the order matters. We define the eight-fold fermionic delta-function as
\[ \delta^8 (Q_{AI}) = \delta (Q_{11}) \delta (Q_{12}) \delta (Q_{13}) \delta (Q_{14}) \delta (Q_{21}) \delta (Q_{22}) \delta (Q_{23}) \delta (Q_{24}) \]
\[ = \delta (Q_{11}) \delta (Q_{21}) \delta (Q_{12}) \delta (Q_{22}) \delta (Q_{13}) \delta (Q_{23}) \delta (Q_{24}) \delta (Q_{14}) \delta (Q_{24}). \]  \hspace{1cm} (473)
With the help of the supersymmetric notation we may write the three-point MHV amplitudes as
\[ A_3^{(0)} (1, 2, 3) = i \sqrt{2} \frac{\delta^8 \left( p_A^1 \eta^1 + p_A^2 \eta^2 + p_A^3 \eta^3 \right)}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle}. \]  \hspace{1cm} (474)
The three-point anti-MHV amplitudes are
\[ A_3^{(0)} (1, 2, 3) = i \sqrt{2} \frac{\delta^4 \left( [32] \eta^1 + [13] \eta^2 + [21] \eta^3 \right)}{[32][21][13]}. \]  \hspace{1cm} (475)
Exercise 21: Re-derive the expression for the MHV amplitude \( A_3^{(0)} (1^-, 2^-, 3^+) \) and the anti-MHV amplitude \( A_3^{(0)} (1^+, 2^+, 3^-) \) given in eq. (303) from eq. (474) and eq. (475), respectively.

The link representation of the \( N = 4 \) supersymmetric primitive tree-level \( N^{k-2} \)MHV amplitude is
\[ A_n^{(0)} \delta^4 \left( \sum_{j=1}^{n} p_A^j \eta^j \right) \delta^8 (Q_{AI}) = \frac{i \left( \sqrt{2} \right)^{n-2}}{(2\pi i)^{k-2(n-k-2)}} \]
\[ \times \int_{\mathcal{J}} d^{k \times n} C \prod_{l=1}^{k} \delta^k \left( C_{ij} - \delta_{il} \right) \prod_{l=1}^{k} \delta^2 \left( p_A^j + C_{ji,k} p_A^k \right) \prod_{m=1}^{n-k} \delta^2 \left( p_A^{km} - C_{ji,k} \eta^j \right) \]
\[ \times \prod_{o=1}^{k} \delta^4 \left( \eta^j_o + C_{ji,k} \eta^j_i \right). \]
Eq. (476) differs from eq. (440) only by an additional product of $4k$ fermionic delta-functions. Note that eq. (476) gives all helicity configurations in the $N^{k-2}_{\text{MHV}}$ sector. This implies that the gauge fixing is arbitrary, we may choose any set of $k$ columns $J = (j_1, \ldots, j_k)$. In particular we may compute a specific non-supersymmetric helicity amplitude with any gauge fixing. However, projecting out the specific helicity amplitude from the super-amplitude is trivial if we choose $J = (j_1, \ldots, j_k)$ to be the set of negative helicity states. For other gauge-fixing conditions the projection operation is more involved. Of course it will give identical results.

6.4 The amplituhedron

We would like to close this chapter by giving an introduction to recent ideas, which express the scattering amplitude as the volume of a geometric object, called the amplituhedron [109–114].

The development of this method originated from a study of spurious singularities appearing in the on-shell recursion relations. In section (5) we gave in eq. (376) an example for a helicity amplitude calculated with the help of the on-shell recursion relations:

$$A_6(1^-, 2^-, 3^-, 4^+, 5^+, 6^+) =$$

$$4i \frac{\langle 6 + 1 + 2|3+\rangle^3}{[61][12][34]\langle 45\rangle s_{126}(2 + |1 + 6|5+)} + \frac{\langle 4 + |5 + 6|1+\rangle^3}{[23][34][56]\langle 61\rangle s_{156}(2 + |1 + 6|5+)}.$$ 

Let us look at the denominators: The spinor products like $[12]$ or $\langle 34 \rangle$ correspond to collinear singularities, the Lorentz invariants $s_{126}$ and $s_{156}$ to three-particle poles. These are all physical. However, the singularity at $\langle 2 + |1 + 6|5+ \rangle = 0$ is spurious and cancels between the two terms. Had we used in the recursion relation a different momentum shift, we would have obtained a representation with different spurious singularities. It turns out, that the NMHV amplitude in eq. (477) corresponds to a volume of a polytope [62], and the two terms in eq. (477) to the volumes of two smaller building blocks in a decomposition of the polytope. The situation is similar to the one shown in fig. (13): The area of the quadrangle can be computed from a triangulation as the sum over the areas of the triangles. Different triangulations yields the same area, but the expressions may look different.

Let us now formalise this idea: In this paragraph we consider $\mathcal{N} = 4$ supersymmetric primitive tree amplitudes of Grassmann weight $4k$. These will give us the non-supersymmetric Yang-Mills amplitudes with $k$ negative helicity gauge bosons. It is convenient to factor out the Parke-
Taylor amplitude and to consider the remainder:

\[
A_{n,k}^{(0)}(1,...,n) \delta^4 \left( \sum_{j=1}^{n} p_A^j p_A^{j'} \right) \delta^8(Q_{AI}) = \frac{i \left( \sqrt{2} \right)^{n-2} \delta^4 \left( \sum_{j=1}^{n} p_A^j p_A^{j'} \right) \delta^8(Q_{AI})}{\langle 12 ... n1 \rangle} B_{n,k}^{(0)}(1,...,n).
\]

The remainder \( B_{n,k}^{(0)} \) has Grassmann weight \( 4(k-2) \). We will also assume in this paragraph that the signature of the space-time metric is \( (2, 2) \). This has the advantage that spinors and twistors can be chosen as real variables. Working with real numbers allows us to talk about positive and negative numbers. In particular we have the notion of the positive Grassmannian \( \text{Gr}_{k,n}^+(\mathbb{R}) \), defined by the condition that all Plücker coordinates (i.e. all ordered \( k \times k \)-minors) are positive. We are actually interested in the closure of this object, allowing some coordinates to be zero. This defines the non-negative Grassmannian \( \text{Gr}_{k,n}^{+\mathbb{R}} \), where all Plücker coordinates are non-negative. In a similar way we may introduce positive \( k \times n \)-matrices \( M \in \text{M}_{k,n}^+(\mathbb{R}) \), defined by the condition that all ordered minors are positive, the difference with the Grassmannian \( \text{Gr}_{k,n}^+(\mathbb{R}) \) being that for \( \text{M}_{k,n}^+(\mathbb{R}) \) the \( \text{GL}(k, \mathbb{R}) \)-action has not been modded out. At the end of day, when an analytic formula has been obtained, we may relax the restriction to real numbers and continue to complex numbers.

The **tree amplituhedron** is a region in the Grassmannian \( \text{Gr}_{k-2,D+k-2}(\mathbb{R}) = \text{Gr}_{k-2,k+2}(\mathbb{R}) \), where \( D = 4 \) denotes the dimension of space-time, obtained as the image of the non-negative Grassmannian \( \text{Gr}_{k-2,n}^{+\mathbb{R}}(\mathbb{R}) \) under the map

\[
\text{Gr}_{k-2,n}^{+\mathbb{R}}(\mathbb{R}) \xrightarrow{Z} \text{Gr}_{k-2,k+2}(\mathbb{R}).
\]

This map is given for \( Y \in \text{Gr}_{k-2,k+2}(\mathbb{R}), C \in \text{Gr}_{k-2,n}^{+\mathbb{R}}(\mathbb{R}) \) and \( Z \in \text{M}_{k+2,n}^+(\mathbb{R}) \) by

\[
Y = C \cdot Z^T.
\]

The \( (k+2) \times n \)-matrix \( Z \) depends on the external data for the scattering process. This matrix is made out of \( n \) column vectors \( Z^j \), one for each particle \( j = 1,...,n \). The vectors \( Z^j \) are \( (k+2) \)-dimensional, the first four entries are the momentum twistor variables for particle \( j \), the remaining \( (k-2) \) entries are denoted by \( c^j_i \),

\[
Z^j = \begin{pmatrix}
p_1^j \\
p_2^j \\
\mu_1^j \\
\mu_2^j \\
c^j_1 \\
\vdots \\
c^j_{k-2}
\end{pmatrix}.
\]
The \( c_i^j \)'s should be thought of as infinitesimal variables, in the sense that some power of them can be neglected. This can be achieved by writing them as

\[
c_i^j = \sum_{I=1}^{4} \phi_{i,I} \eta_I^j,
\]

(482)

where \( \phi_{i,I} \) and \( \eta_I^j \) are Grassmann numbers. We then have \((c_i^j)^5 = 0\). Note that the \( c_i^j \)'s have even Grassmann degree and commute like ordinary numbers. In the end we will integrate over the auxiliary Grassmann numbers \( \phi_{i,I} \). The dimension of the Grassmannian \( \text{Gr}_{k-2,k+2}(\mathbb{R}) \) is

\[
\dim_{\mathbb{R}} \text{Gr}_{k-2,k+2}(\mathbb{R}) = 4(k-2).
\]

(483)

Up to now we defined the region in \( \text{Gr}_{k-2,k+2}(\mathbb{R}) \), which makes up the amplituhedron. We still need a volume form, which we can integrate over this region to give the volume. Let us start with a brief discussion of top forms on a Grassmannian manifold \( \text{Gr}_{k,n}(\mathbb{R}) \). These are differential forms of degree \( k(n-k) \). We start with the projective space \( \mathbb{P}^{n-1} = \text{Gr}_{1,n}(\mathbb{R}) \) with homogeneous coordinates \( x = [x_1 : \ldots : x_n] \). Any top form on \( \mathbb{P}^{n-1} \) can be written as

\[
\Omega = \frac{\omega}{f(x)},
\]

(484)

where the differential \((n-1)\)-form \( \omega \) is given by

\[
\omega = (n-1)! \sum_{j=1}^{n} (-1)^{j-1} x_j \, dx_1 \wedge \ldots \wedge \hat{dx}_j \wedge \ldots \wedge dx_n.
\]

(485)

The hat indicates that the corresponding term is omitted. The factor \((n-1)!\) appears in order to have a consistent notation with later generalisations. The function \( f(x) \) must be homogeneous of degree \( n \):

\[
f(\lambda x) = \lambda^n f(x).
\]

(486)

This ensures that \( \Omega \) is invariant under the \( \text{GL}(1,\mathbb{R}) \)-action.

Let us now move on to the Grassmannian manifold \( \text{Gr}_{k,n}(\mathbb{R}) \) and consider \( C \in \text{Gr}_{k,n}(\mathbb{R}) \). Any top form on \( \text{Gr}_{k,n}(\mathbb{R}) \) can be written as

\[
\Omega = \frac{\omega}{f(C)},
\]

(487)

where \( f(C) \) is a function of the \( k \times k \)-minors of \( C \) and homogeneous under scaling as

\[
f(\lambda C) = \lambda^{k} f(C).
\]

(488)

The \( k(n-k) \)-form \( \omega \) is given by

\[
\omega = \left\langle C^1, \ldots, C^k, (dC^1)^{(n-k)} \right\rangle \wedge \ldots \wedge \left\langle C^1, \ldots, C^k, (dC^k)^{(n-k)} \right\rangle,
\]

(489)
and
\[
\left\langle C^1, \ldots, C^k, (dC^i)^{(n-k)} \right\rangle = \epsilon^{a_1 a_2 \ldots a_n} C_{1 a_1} \ldots C_{ka_k} dC_{i,a_{k+1}} \wedge \ldots \wedge dC_{i,a_n}.
\] (490)

For example,
\[
\Omega = \frac{\omega}{(1,2,\ldots,k)(2,3,\ldots,k+1)\ldots(n,1,\ldots,(k-1))}
\] (491)

would be a top form on $\text{Gr}_{k,n}(\mathbb{R})$. We encountered this form in the link representation discussed in section (6.2). Now consider the top-cell $\text{Gr}^+_{k,n}(\mathbb{R})$. It can be shown that this cell has $n$ co-dimension-one boundaries, corresponding to the cases where one of the $n$ consecutive minors vanishes. Thus we may associate to the top-cell the differential form $\Omega$, given in eq. (491). This differential form is characterised by the fact that it has logarithmic singularities on the boundary of the cell.

**Exercise 22:** Show that for $k = 1$ the form $\omega$ given in eq. (489) reduces to the expression given in eq. (485).

Now let us return to the problem of finding a suitable volume form for the amplituhedron. Consider a cell decomposition of the amplituhedron. In each cell $\Gamma$ associated with positive coordinates $\alpha_{1\Gamma}, \ldots, \alpha_{(k-2)\Gamma}$ we may associate a differential form with logarithmic singularities on the boundary on the cell:
\[
\Omega_{n,k} = \prod_{i=1}^{4(k-2)} \frac{d\alpha_{i\Gamma}}{\alpha_{i\Gamma}^4}.
\] (492)

Consider now a collection $T$ of cells covering the amplituhedron. The volume form for the amplituhedron is then
\[
\Omega_{n,k} = \sum_{\Gamma \in T} \Omega_{n,k}^\Gamma.
\] (493)

By a $\text{GL}(k+2,\mathbb{R})$ transformation on the input data $Z$ we may send $Y$ to
\[
y_0 = \left(0_{(k-2)\times 4}, 1_{(k-2)\times (k-2)} \right).
\] (494)

We now have all ingredients to give an expression for $B_{n,k}^{(0)}$ from the amplituhedron. $B_{n,k}^{(0)}$ is obtained by localising $Y$ to $Y_0$ and by integrating over the Grassmann variables $\phi_{i,l}$:
\[
B_{n,k}^{(0)} = \int d^N \phi_1 \ldots \int d^N \phi_{k-2} \int \Omega_{n,k} \delta^{4(k-2)}(Y;Y_0).
\] (495)

The delta-function is defined by
\[
\delta^{4(k-2)}(Y;Y_0) = \int d^{(k-2)(k-2)} g_{ij} \det(g)^4 \delta^{4(k-2)(k+2)}(Y - g Y_0).
\] (496)
Note that there are in eq. (495) as many (bosonic) delta-functions as there are (bosonic) integrations. In addition one integrates over $\mathcal{N}(k-2) = 4(k-2)$ auxiliary Grassmann variables $\phi_i$, leaving a homogeneous function of degree $4(k-2)$ in the Grassmann variables $\eta_j$. Therefore all integrations are trivial. The non-trivial information is contained in the differential form $\Omega_{n,k}$. Let us illustrate this with an example. Since the dimension of the amplituhedron is $4(k-2)$, we have for $k = 2$ a zero-dimensional object, for $k = 3$ a four-dimensional object. The former is trivial, the latter already too complicated to draw. We will therefore consider the case, where

$$Y \in \text{Gr}_{1,3}(\mathbb{R}), \quad C \in \text{Gr}_{1,4}^+(\mathbb{R}) \quad \text{and} \quad Z \in \mathbb{M}_{3,4}^+(\mathbb{R}).$$

(497)

This case corresponds to $D = 2, n = 4$ and $k = 3$. As external data we take as an example

$$Z = \begin{pmatrix}
2 & 8 & 6 & 3 \\
2 & 3 & 8 & 7 \\
1 & 1 & 1 & 1
\end{pmatrix}.$$  

(498)

It is easily checked that all ordered minors, obtained by deleting one column, are positive. Thus we have $Z \in \mathbb{M}_{3,4}^+(\mathbb{R})$. The Grassmannian $\text{Gr}_{1,4}(\mathbb{R})$ is identical to $\mathbb{RP}^3$ and points in the Grassmannian $\text{Gr}_{1,4}(\mathbb{R})$ can be denoted by homogeneous coordinates $[c_1 : c_2 : c_3 : c_4]$. The non-negative Grassmannian $\text{Gr}_{1,4}^+(\mathbb{R})$ is the set of points, whose homogeneous coordinates satisfy

$$c_j \geq 0 \quad \text{for} \quad j = 1, \ldots, 4.$$ 

(499)

This is a three-dimensional simplex, whose corners are given by the four points

$$[1 : 0 : 0 : 0], \quad [0 : 1 : 0 : 0], \quad [0 : 0 : 1 : 0], \quad [0 : 0 : 0 : 1].$$

(500)

For $Y \in \text{Gr}_{1,3}(\mathbb{R})$ we will use the homogeneous coordinates $Y = [y_1 : y_2 : y_3]$. The amplituhedron is then the set of points with the coordinates

$$y_1 = 2c_1 + 8c_2 + 6c_3 + 3c_4,$$

$$y_2 = 2c_1 + 3c_2 + 8c_3 + 7c_4, \quad c_j \geq 0,$$

$$y_3 = c_1 + c_2 + c_3 + c_4.$$ 

(501)

Let us look at the images of the four corner points given in eq. (500). These are mapped to the four points

$$Y_1 = [2 : 2 : 1], \quad Y_2 = [8 : 3 : 1], \quad Y_3 = [6 : 8 : 1], \quad Y_4 = [3 : 7 : 1].$$

(502)

We may now draw the amplituhedron in the chart $y_3 = 1$. The result is shown in fig. (13). Thus for this example the amplituhedron is a quadrangle. How do we obtain the differential form $\Omega_{4,3}$? Let us first assume that the amplituhedron would be the non-negative Grassmannian $\text{Gr}_{1,3}^+(\mathbb{R})$, parametrised with homogeneous coordinates

$$[\bar{y}_1 : \bar{y}_2 : \bar{y}_3], \quad \bar{y}_j \geq 0.$$ 

(503)
Geometrically, this is a triangle. We have to find a differential two-form with logarithmic singularities on the boundary. For the non-negative Grassmannian $\text{Gr}^+_1(\mathbb{R})$, we know the answer from eq. (484):

$$\tilde{\Omega} = \frac{\tilde{y}_1 d\tilde{y}_2 \wedge d\tilde{y}_3 + \tilde{y}_2 d\tilde{y}_3 \wedge d\tilde{y}_1 + \tilde{y}_3 d\tilde{y}_1 \wedge d\tilde{y}_2}{\tilde{y}_1 \tilde{y}_2 \tilde{y}_3}.$$  \hfill (504)

In the chart $\tilde{y}_3 = 1$ this form reduces to

$$\tilde{\Omega} = \frac{d\tilde{y}_1 \wedge d\tilde{y}_2}{\tilde{y}_1 \tilde{y}_2}.$$  \hfill (505)

In order to find the form $\Omega_{4,3}$ for amplituhedron defined in eq. (501) we first triangulate the quadrangle. There are two possible triangulations, both are shown in fig. (13). Let us take for concreteness the triangulation

$$\Gamma_1 = \text{triangle} (Y_1, Y_2, Y_3), \quad \Gamma_2 = \text{triangle} (Y_1, Y_3, Y_4).$$  \hfill (506)

We may map each of these two triangles to the triangle of the non-negative Grassmannian $\text{Gr}^+_1(\mathbb{R})$ defined in eq. (503). Thus we have maps

$$\varphi_i : \Gamma_i \rightarrow \text{Gr}^+_1(\mathbb{R}), \quad i = 1, 2.$$  \hfill (507)

On $\text{Gr}^+_1(\mathbb{R})$ we have a form with logarithmic singularities on the boundaries, given in eq. (504). Pulling this form back with $\varphi_1$ or $\varphi_2$ gives us a form on $\Gamma_1$ or $\Gamma_2$ with logarithmic singularities on the boundaries:

$$\Omega_{4,3}^{\Gamma_1} = \varphi_1^* \tilde{\Omega}, \quad \Omega_{4,3}^{\Gamma_2} = \varphi_2^* \tilde{\Omega},$$  \hfill (508)

We therefore have

$$\Omega_{4,3} = \Omega_{4,3}^{\Gamma_1} + \Omega_{4,3}^{\Gamma_2} = \varphi_1^* \tilde{\Omega} + \varphi_2^* \tilde{\Omega}.$$  \hfill (509)

The virtue of the representation of $B_{n,k}^{(0)}$ in eq. (495) in terms of the amplituhedron lies in the fact that although it might take a while to digest this representation, the basic ingredients of this representation are of simple geometric origin. In this paragraph we discussed the tree amplituhedron. It is important to mention that the geometric concepts underlying the amplituhedron generalise to loop amplitudes [109].

### 6.5 Infinity twistors

In this paragraph we would like to motivate the occurrence of $Y_0$ and of the Grassmann numbers $\phi_{i,J}$ in the formula for the amplituhedron in eq. (495). This will involve a discussion of infinity twistors.
We have given the cell decomposition of \( \mathbb{C}P^n \) in eq. 415, and a similar cell decomposition holds for \( \mathbb{R}P^n \). Let us consider \( \mathbb{R}P^1 \) and let us denote a point \( p \in \mathbb{R}P^1 \) by homogeneous coordinates \([x_1 : x_2] \). For \( \mathbb{R}P^1 \) we have the cell decomposition

\[
\mathbb{R}P^1 = \mathbb{R}^1 \sqcup \mathbb{R}^0.
\]  

(510)

More concretely we may write

\[
\mathbb{R}P^1 = \{ [x_1 : 1] \mid x_1 \in \mathbb{R} \} \sqcup \{ [1 : 0] \}.
\]  

(511)

The point \([1 : 0] \) is called the **point at infinity**. We may think of this point as being obtained from \([x_1 : 1] \) in the limit \( x_1 \to \infty \), where we may neglect the second entry 1 against the large number \( x_1 \).

Let us now consider momentum twistors

\[
Z_{\alpha j} = \left( p^j_A, \mu^j_A \right),
\]  

(512)

where the index \( \alpha \) takes the values \( \alpha \in \{1, 2, \dot{1}, \dot{2} \} \) and \( j \in \{1, \ldots, n\} \). In addition we define two infinity twistors, corresponding to the two cases where either \( \mu_2 \) or \( \mu_1 \) is large compared to all other components. We may take these two infinity twistors as \( I_{\alpha 3} = (0, 0, 0, -1) \) and \( I_{\alpha 4} = (0, 0, 1, 0) \). Note that since we are in projective space, only the direction matters, not the sign. If we pack these two infinity twistors in a \( 4 \times 2 \)-matrix we have

\[
\begin{pmatrix}
0 & 0 \\
0 & 0 \\
0 & 1 \\
-1 & 0 \\
\end{pmatrix}
= 
\begin{pmatrix}
\epsilon_{AB} \\
0_{2 \times 2}
\end{pmatrix},
\]  

(513)

which motivates the specific choice of the two infinity twistors. Let us now look at a four-bracket with two infinity twistors. We have

\[
\langle Z_1, Z_2, I_3, I_4 \rangle = \epsilon^{\alpha \beta \gamma \delta} Z_{\alpha 1} Z_{\beta 2} I_{\gamma 3} I_{\delta 4} = -\langle p_1 p_2 \rangle.
\]  

(514)

Thus the four-bracket reduces to the spinor product.

We may proceed in complete analogy for the momentum twistors

\[
W_{\alpha j} = \left( \mu^j_A, p^j_A \right).
\]  

(515)

We define two infinity twistors \( J_{\alpha 1} = (0, -1, 0, 0) \) and \( J_{\alpha 2} = (1, 0, 0, 0) \). If we pack them again into a \( 4 \times 2 \)-matrix we now have

\[
\begin{pmatrix}
0 & 1 \\
-1 & 0 \\
0 & 0 \\
0 & 0 \\
\end{pmatrix}
= 
\begin{pmatrix}
\epsilon_{AB} \\
0_{2 \times 2}
\end{pmatrix}.
\]  

(516)
For a four-bracket with the two infinity twistors $J_1$ and $J_2$ we obtain

$$\langle J_1, J_2, W_3, W_4 \rangle = \varepsilon^{\alpha\beta\gamma\delta} J_{\alpha 1} J_{\beta 2} W_{\gamma 3} W_{\delta 4} = [p_3, p_4].$$  

(517)

Again, we see that the four-bracket reduces to a spinor product.

Let us now go to higher dimensions: If we have five five-dimensional vectors $Z^i_\alpha$ with $\alpha = 1, \ldots, 5$ and $j = 1, \ldots, 5$, we may construct a five-bracket by taking the determinant of the matrix formed by the five vectors $Z_1, \ldots, Z_5$:

$$\langle Z^1, Z^2, Z^3, Z^4, Z^5 \rangle = \varepsilon^{\alpha\beta\gamma\delta\epsilon} Z_\alpha^1 Z_\beta^2 Z_\gamma^3 Z_\delta^4 Z_\epsilon^5.$$

(518)

More generally one defines a $n$-bracket for $n$ vectors $Z^1, \ldots, Z^n$ of dimension $n$ by

$$\langle Z^1, Z^2, \ldots, Z^n \rangle = \varepsilon^{\alpha_1 \alpha_2 \ldots \alpha_n} Z^1_{\alpha_1} Z^2_{\alpha_2} \cdots Z^n_{\alpha_n}.$$

(519)

Suppose now, that our initial input data lies in a four-dimensional subspace. Thus we may assume that $Z^j_5 = 0$ for all $j = 1, \ldots, 5$. The five-bracket vanishes in this case. There are two possible ways out to obtain a non-vanishing five-bracket. The first option is to introduce an additional infinity vector

$$I^0_\alpha = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$  

(520)

A five-bracket with four of the five vectors $Z^j$ from above will in general now be non-zero. It will not depend on the fifth component (i.e. the component with $\alpha = 5$) of any $Z^j$, since $I^0_\alpha = 0$ for $\alpha = 1, \ldots, 4$. Of course, this five-bracket will depend only on four out of the five vectors $Z^j$ and will be equal to the four-bracket of these four vectors.

In order to obtain an expression, which depends on all five vectors, let us consider a second option: For each vector, we introduce a fifth component as in eq. \[482\]

$$Z^j_5 = \sum_{i=1}^{4} \phi_i \eta^j_i.$$  

(521)

Let us now assume, that the fifth components of the $Z^j$‘s are given by eq. \[521\]. Let us now consider the expression

$$\left[ Z^1, Z^2, Z^3, Z^4, Z^5 \right] = \frac{1}{4!} \int d^4 \phi \frac{1}{\langle 1, 2, 3, 4, 5 \rangle^4} \langle 1, 2, 3, 4, 5 \rangle \langle 0, 1, 2, 3, 4 \rangle \langle 0, 2, 3, 4, 5 \rangle \langle 0, 3, 4, 5, 1 \rangle \langle 0, 4, 5, 1, 2 \rangle \langle 0, 5, 1, 2, 3 \rangle,$$

with

$$\langle 1, 2, 3, 4, 5 \rangle = \langle Z^1, Z^2, Z^3, Z^4, Z^5 \rangle, \quad \langle 0, i, j, k, l \rangle = \langle I^0, Z^i, Z^j, Z^k, Z^l \rangle.$$  

(523)
Note that in eq. (522) no Grassmann number occurs in the denominator. Eq. (522) is equivalent to

\[
\mathbf{Z} = \delta^4 \left( \langle 1, 2, 3, 4 \rangle \eta_5 + \text{cyclic} \right) / \langle 1, 2, 3, 4 \rangle \langle 2, 3, 4, 5 \rangle \langle 3, 4, 5, 1 \rangle \langle 4, 5, 1, 2 \rangle \langle 5, 1, 2, 3 \rangle.
\]  

Expressions of this type are encountered when one evaluates NMHV amplitudes with the help of the link representation in eq. (476) or with the help of the amplituhedron in eq. (495).

**Exercise 23:** Derive the expression in eq. (524) from eq. (522).

Can we give a geometric meaning to the integrand in eq. (522)? We will now show that it corresponds to the volume of a polytope. Let us start with the simplest polytope, a triangle. Suppose that the corners of the triangles are given in \( \mathbb{RP}^2 \) by

\[
X_1^\alpha = \begin{pmatrix} X_1^1 \\ X_1^2 \\ 1 \end{pmatrix}, \quad X_2^\alpha = \begin{pmatrix} X_2^1 \\ X_2^2 \\ 1 \end{pmatrix}, \quad X_3^\alpha = \begin{pmatrix} X_3^1 \\ X_3^2 \\ 1 \end{pmatrix}.
\]  

The “volume” (i.e. area) of the triangle is then

\[
\text{vol} (\text{triangle}) = \frac{1}{2} \left| \begin{array}{ccc} X_1^1 & X_2^1 & X_3^1 \\ X_1^2 & X_2^2 & X_3^2 \\ 1 & 1 & 1 \end{array} \right|
\]  

\[= \frac{1}{2} \left( (X_1^2 - X_1^1) (X_2^3 - X_2^1) - (X_2^2 - X_2^1) (X_3^1 - X_1^1) \right).
\]  

(526)

This is most easily seen by first considering the area of parallelogram spanned by the two vectors \((X_2 - X_1)\) and \((X_3 - X_1)\) and then taking half of this area. The area of the parallelogram is

\[
\left( \vec{X}^2 - \vec{X}^1 \right) \times \left( \vec{X}^3 - \vec{X}^1 \right) = (X_2^2 - X_1^1) (X_3^3 - X_2^1) - (X_2^2 - X_2^1) (X_3^1 - X_1^1).
\]  

(527)

Instead of specifying the triangle by its corners, we may equally well specify the boundary lines. Each boundary line can be characterised by a vector orthogonal to it. Let’s call this vector \(Z_j^\alpha\) for the boundary line which does not contain the corner \(j\). Points \(X_\alpha\) on the boundary line \(j\) satisfy the incidence relation

\[
\sum_{\alpha=1}^3 Z_j^\alpha X_\alpha = 0.
\]  

(528)

Given the \(Z^j\)’s, we may reconstruct the \(X^\alpha\)’s as

\[
X_\alpha^1 = \epsilon_{\alpha\beta\gamma} Z_\beta^2 Z_\gamma^3, \quad X_\alpha^2 = \epsilon_{\alpha\beta\gamma} Z_\beta^3 Z_\gamma^1, \quad X_\alpha^3 = \epsilon_{\alpha\beta\gamma} Z_\beta^1 Z_\gamma^2.
\]  

(529)

In terms of the \(Z^j\)’s, the volume of the triangle is given by

\[
\text{vol} (\text{triangle}) = \frac{1}{2} \frac{\langle Z^1, Z^2, Z^3 \rangle^2}{\langle I^0, Z^1, Z^2 \rangle \langle I^0, Z^2, Z^3 \rangle \langle I^0, Z^3, Z^1 \rangle}.
\]  

(530)
where \( I^0 \) is an infinity vector

\[
I^0 = \begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix}.
\] (531)

We may generalise this to an \( n \)-simplex in \( \mathbb{R}P^n \), specified by \( (n + 1) \) vectors \( Z^j_\alpha \), with \( \alpha = 1, \ldots, (n + 1) \) and \( j = 1, \ldots, (n + 1) \). These vectors define the \( (n + 1) \) faces (or boundaries) of the simplex. A point \( X_\alpha \) lies on the \( j \)-th boundary if

\[
\sum_{\alpha=1}^{n+1} Z^j_\alpha X_\alpha = 0.
\] (532)

The volume of the simplex is given by

\[
[Z^1, \ldots, Z^{n+1}] = \frac{1}{n! \langle I^0, Z^1, \ldots, Z^n \rangle \langle I^0, Z^2, \ldots, Z^{n+1} \rangle \ldots \langle I^0, Z^{n+1}, Z^1, \ldots, Z^{n-1} \rangle} \langle Z^1, \ldots, Z^{n+1} \rangle^n.
\] (533)

where \( I^0 \) is again an infinity vector or reference vector, which we may take as \( I^0_\alpha = 0 \) for \( \alpha = 1, \ldots, n \) and \( I^0_{n+1} = 1 \). We recognise in this formula the integrand of eq. (522).
7 The CHY representation

In this section we return to pure (non-supersymmetric) Yang-Mills theory and introduce the CHY representation for the primitive tree-level amplitudes. This representation expresses the amplitude as a global residue at the zeros of the scattering equations and separates the information on the external cyclic order and on the polarisations in two different functions. The formulæ in this section are not specific to four space-time dimensions and we may take Minkowski space to be \( D \) dimensional.

7.1 The scattering equations

We denote by \( \mathbb{C}M \) the complexified Minkowski space, i.e. a complex vector space of dimension \( D \). The Minkowski metric \( g_{\mu\nu} = \text{diag}(1,-1,-1,-1,...) \) is extended by linearity. We further denote by \( \Phi_n \) the (momentum) configuration space of \( n \) external massless particles:

\[
\Phi_n = \{ (p_1, p_2, \ldots, p_n) \in (\mathbb{C}M)^n \mid p_1 + p_2 + \ldots + p_n = 0, p_1^2 = p_2^2 = \ldots = p_n^2 = 0 \}. \tag{534}
\]

In other words, a \( n \)-tuple \( p = (p_1, p_2, \ldots, p_n) \) of momentum vectors belongs to \( \Phi_n \) if this \( n \)-tuple satisfies momentum conservation and the mass-shell conditions \( p_i^2 = 0 \) for massless particles. It will be convenient to use the notation \( p \) without any index to denote such an \( n \)-tuple. In the same spirit we denote the \( n \)-tuple of external polarisations by \( \varepsilon = (\varepsilon_1^\lambda, \ldots, \varepsilon_n^\lambda) \) with \( \lambda \in \{+,-\} \). A primitive tree amplitude \( A_n^{(0)} \) has a fixed cyclic order of the external legs, which can be specified by a permutation \( \sigma = (\sigma_1, \ldots, \sigma_n) \). In this section it will be convenient to specify a primitive tree amplitude by the three \( n \)-tuples \( \sigma, p \) and \( \varepsilon \):

\[
A_n^{(0)}(\sigma, p, \varepsilon) = A_n^{(0)}(p_{\sigma_1}^{\lambda_{\sigma_1}}, \ldots, p_{\sigma_n}^{\lambda_{\sigma_n}}). \tag{535}
\]

We denote by \( \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \). The space \( \hat{\mathbb{C}} \) is equivalent to the complex projective space \( \mathbb{CP}^1 \). For amplitudes with \( n \) external particles we consider the space \( \hat{\mathbb{C}}^n \). Points in \( \hat{\mathbb{C}}^n \) will be denoted by \( z = (z_1, z_2, \ldots, z_n) \). Again we use the convention that \( z \) without any index denotes an \( n \)-tuple.

We set for \( 1 \leq i \leq n \)

\[
f_i(z, p) = \sum_{j=1, j \neq i}^n \frac{2p_i \cdot p_j}{z_i - z_j}. \tag{536}
\]

The scattering equations read \([115][117]\)

\[
f_i(z, p) = 0, \quad i \in \{1, \ldots, n\}. \tag{537}
\]

For a fixed \( p \in \Phi_n \) a solution of the scattering equation is a point \( z \in \hat{\mathbb{C}}^n \), such that the scattering equations in eq. (537) are satisfied.

The scattering equations are invariant under the projective special linear group \( \text{PSL}(2, \mathbb{C}) = \text{SL}(2, \mathbb{C})/\mathbb{Z}_2 \). Here, \( \mathbb{Z}_2 \) is given by \( \{1, -1\} \), with 1 denoting the unit matrix. Let

\[
g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \in \quad \text{PSL}(2, \mathbb{C}). \tag{538}
\]
Each \( g \in \text{PSL}(2, \mathbb{C}) \) defines an automorphism of \( \hat{\mathbb{C}} \) as follows:

\[
g \cdot z_i = \frac{az_i + b}{cz_i + d}, \quad z_i \in \hat{\mathbb{C}}.
\]  

(539)

Conversely, since every automorphism of \( \hat{\mathbb{C}} \) is of the form above, it follows that each automorphism of \( \hat{\mathbb{C}} \) defines an element \( g \in \text{PSL}(2, \mathbb{C}) \). We therefore have a group isomorphism

\[
\text{PSL}(2, \mathbb{C}) \rightarrow \text{Aut}(\hat{\mathbb{C}}),
\]

(540)

where we denoted by \( \text{Aut}(\hat{\mathbb{C}}) \) the automorphism group of \( \hat{\mathbb{C}} \). The transformation in eq. (539) is called a Möbius transformation. We further set

\[
g \cdot (z_1, z_2, \ldots, z_n) = (g \cdot z_1, g \cdot z_2, \ldots, g \cdot z_n).
\]

(541)

If \((z_1, z_2, \ldots, z_n)\) is a solution of eq. (537), then also \((z'_1, z'_2, \ldots, z'_n) = g \cdot (z_1, z_2, \ldots, z_n)\) is a solution.

**Exercise 24:** Let \( g \in \text{PSL}(2, \mathbb{C}) \) and let \((z_1, z_2, \ldots, z_n)\) be a solution of the scattering equations. Show that \((z'_1, z'_2, \ldots, z'_n) = g \cdot (z_1, z_2, \ldots, z_n)\) is also a solution of the scattering equations.

We call two solutions which are related by a PSL\((2, \mathbb{C})\)-transformation equivalent solutions. The \( n \) scattering equations in eq. (537) are not independent, only \((n - 3)\) of them are. The Möbius invariance implies the relations

\[
\sum_{i=1}^{n} f_i(z, p) = 0, \quad \sum_{i=1}^{n} z_i f_i(z, p) = 0, \quad \sum_{i=1}^{n} z_i^2 f_i(z, p) = 0.
\]

(542)

**Exercise 25:** Prove the relations in eq. (542).

There is an alternative formulation of the scattering equations in polynomial form\[\text{[118]}\]. We set \( I = \{1, 2, \ldots, n\} \) and for any subset \( S \subseteq I \) we define

\[
p_S = \sum_{i \in S} p_i, \quad z_S = \prod_{i \in S} z_i.
\]

(543)

For the empty set \( \emptyset \) we define \( p_{\emptyset} = 0 \) and \( z_{\emptyset} = 1 \). For \( 0 \leq m \leq n \) we define the polynomials \( h_m(z, p) \) in the variables \( z = (z_1, \ldots, z_n) \) by

\[
h_m(z, p) = \sum_{S \subseteq I, |S| = m} p_S^2 z_S.
\]

(544)

The polynomial \( h_0 \) is trivially zero, due to \( p_0 = 0 \). The polynomials \( h_1, h_{n-1} \) and \( h_n \) vanish trivially due to momentum conservation and the on-shell conditions:

\[
h_1(z, p) = \sum_{j=1}^{n} p_j^2 z_j = 0,
\]

\[
h_{n-1}(z, p) = \sum_{j=1}^{n} p_j^2 z_1 \cdots z_{j-1} z_{j+1} \cdots z_n = 0,
\]

\[
h_n(z, p) = (p_1 + \ldots + p_n)^2 z_1 \cdots z_n = 0.
\]

(545)
The non-vanishing polynomials $h_m$ are homogeneous polynomials in the variables $z_1, z_2, \ldots, z_n$ of degree $m$. They are linear in each variable $z_j$. The scattering equations are equivalent to the set of equations

$$h_m(z, p) = 0, \quad 2 \leq m \leq n - 2. \quad (546)$$

Let us denote by

$$C^{n-3} = \hat{C}^n / \text{Aut}(\hat{C}), \quad (547)$$

i.e. $\hat{C}^n$ modulo the diagonal action of $\text{Aut}(\hat{C})$. $C^{n-3}$ is a space of complex dimension $(n - 3)$. The polynomials in eq. (544) define an algebraic variety, called scattering variety [119]:

$$V_n(p) = \{ z \in C^{n-3} | h_m(z, p) = 0; \ 2 \leq m \leq n - 2 \}. \quad (548)$$

Since there are $(n - 3)$ non-vanishing polynomials $h_m(z, p)$, the algebraic variety $V_n$ is of dimension zero. In other words, $V_n$ consists of individual points. There is a theorem in algebraic geometry by Bézout [120] which tells us that $V_n$ consists of $(n - 3)!$ points. Therefore there are $(n - 3)!$ different solutions of the scattering equations not related by a $\text{PSL}(2, \mathbb{C})$-transformation. We will denote a solution by

$$z^{(j)} = (z_1^{(j)}, \ldots, z_n^{(j)}) \quad (549)$$

and a sum over the $(n - 3)!$ inequivalent solutions by

$$\sum_{\text{solution } j} \quad (550)$$

Let us add the following remark: Consider a Riemann sphere (i.e. an algebraic curve of genus zero) with $n$ distinct marked points. The moduli space of genus 0 curves with $n$ distinct marked points is denoted by

$$\mathcal{M}_{0,n} = \left\{ z \in (\mathbb{CP}^1)^n : z_i \neq z_j \right\} / \text{PSL}(2, \mathbb{C}). \quad (551)$$

$\mathcal{M}_{0,n}$ is an affine algebraic variety of dimension $(n - 3)$. We see that any solution of the scattering equations with $z_i \neq z_j$ for all $i \neq j$ corresponds to a point $z \in \mathcal{M}_{0,n}$.

**Exercise 26:** Consider the Koba-Nielsen function

$$U(z, p) = \prod_{i<j} (z_i - z_j)^{2p_i \cdot p_j} \quad (552)$$

Show

$$U^{-1} \frac{\partial}{\partial z_i} U = f_i(z, p). \quad (553)$$

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7.2 Polarisation factors and Parke-Taylor factors

There are two essential ingredients for the CHY representation of $A_n^{(0)}$: A polarisation factor $E(p, \varepsilon, z)$ and a cyclic factor (or Parke-Taylor factor) $C(\sigma, z)$, which we will both define in this paragraph. We start by defining a $(2n) \times (2n)$ anti-symmetric matrix $\Psi$ through

$$
\Psi = \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix}
$$

with

$$
A_{ab} = \begin{cases} 
\frac{2p_a \cdot p_b}{z_a - z_b} & a \neq b, \\
0 & a = b, 
\end{cases} \quad B_{ab} = \begin{cases} 
\frac{2\varepsilon_a \cdot \varepsilon_b}{z_a - z_b} & a \neq b, \\
0 & a = b, 
\end{cases}
$$

and

$$
C_{ab} = \begin{cases} 
\frac{2\varepsilon_a \cdot p_b}{z_a - z_b} & a \neq b, \\
-\sum_{j=1, j \neq a} \frac{2\varepsilon_a \cdot p_j}{z_a - z_j} & a = b.
\end{cases}
$$

Let $1 \leq i < j \leq n$. One denotes by $\Psi_{ij}^{ij}$ the $(2n-2) \times (2n-2)$-matrix where the rows and columns $i$ and $j$ of $\Psi$ have been deleted. $\Psi_{ij}^{ij}$ has a non-vanishing Pfaffian and one sets as polarisation factor

$$
E(p, \varepsilon, z) = \frac{(-1)^{i+j}}{2(z_i - z_j)} \text{Pf} \Psi_{ij}^{ij}.
$$

$E(p, \varepsilon, z)$ is independent of the choice of $i$ and $j$ on the solutions of the scattering equations.

Let us now consider a permutation $\sigma = (\sigma_1, ..., \sigma_n)$. The cyclic factor (or Parke-Taylor factor) is given by

$$
C(\sigma, z) = \frac{1}{(z_{\sigma_1} - z_{\sigma_2})(z_{\sigma_2} - z_{\sigma_3})... (z_{\sigma_n} - z_{\sigma_1})}.
$$

The cyclic factor depends on the cyclic order $\sigma$, but not on the polarisations $\varepsilon$. On the other hand, the polarisation factor depends on the polarisations $\varepsilon$, but not on the cyclic order $\sigma$. Under a PSL(2, $\mathbb{C}$) transformation we have

$$
E(p, \varepsilon, g \cdot z) = \left\{ \prod_{j=1}^{n} (cz_j + d)^2 \right\} E(p, \varepsilon, z),
$$

$$
C(\sigma, g \cdot z) = \left\{ \prod_{j=1}^{n} (cz_j + d)^2 \right\} C(\sigma, z).
$$
7.3 The two forms of the CHY representation

There are two equivalent forms of the Cachazo-He-Yuan (CHY) representation [115–117]. The first form expresses the primitive tree amplitude $A_n^{(0)}$ as a \textbf{multidimensional complex contour integral} in the auxiliary space $\mathcal{C}^n$:

$$A_n^{(0)}(\sigma, p, \varepsilon) = i \oint_{\mathcal{C}^n} d\Omega_{\text{CHY}} C(\sigma, z) E(z, p, \varepsilon), \quad (560)$$

where the cyclic factor $C(\sigma, z)$ and the polarisation factor $E(z, p, \varepsilon)$ have been defined in the previous paragraph. The measure $d\Omega_{\text{CHY}}$ is defined by

$$d\Omega_{\text{CHY}} = \frac{1}{(2\pi i)^{n-3}} \frac{d^n z}{d\omega} \prod'_{1/f_a(z, p)} \frac{1}{f_a(z, p)}, \quad (561)$$

where

$$\prod'_{1/f_a(z, p)} = (-1)^{i+j+k} (z_i - z_j) (z_j - z_k) (z_k - z_i) \prod_{a \neq i, j, k} \frac{1}{1/f_a(z, p)}; \quad (562)$$

and

$$d\omega = (-1)^{p+q+r} \frac{dz_p dz_q dz_r}{(z_p - z_q) (z_q - z_r) (z_r - z_p)}. \quad (563)$$

The primed product of $1/f_a$ is independent of the choice of $i, j, k$ and takes into account that only $(n - 3)$ of the scattering equations are independent. The quantity $d\omega$ is independent of the choice of $p, q, r$ and corresponds to the invariant measure on $\text{PSL}(2, \mathbb{C})$. Dividing by $d\omega$ ensures that from each class of equivalent solutions only one representative is taken. The integration contour $\mathcal{C}$ encircles the inequivalent solutions of the scattering equations. Let us investigate how the various pieces transform under a $\text{PSL}(2, \mathbb{C})$-transformation $z' = g \cdot z$. We have

$$z'_i - z'_j = \frac{z_i - z_j}{(cz_i + d) (cz_j + d)} \quad \text{and} \quad dz'_j = \frac{dz_j}{(cz_j + d)^2}. \quad (564)$$

The measure $d^n z$ transforms as

$$d^n z' = \left( \prod_{j=1}^{n} \frac{1}{(cz_j + d)^2} \right) d^n z. \quad (565)$$

The measure $d\omega$ is invariant:

$$d\omega' = d\omega. \quad (566)$$

The primed product of $1/f_a$ transforms as

$$\prod'_{1/f_a(z', p)} = \left( \prod_{j=1}^{n} \frac{1}{(cz_j + d)^2} \right) \left( \prod'_{1/f_a(z, p)} \right). \quad (567)$$
With the transformation properties of $C(\sigma, z)$ and $E(z, p, \epsilon)$, given in eq. (559), we see that the integrand is PSL$(2, \mathbb{C})$-invariant. In eq. (560) there are $(n - 3)$ contour integrations encircling the poles given by the zeros of the independent scattering equations. We may equally well express the quantity of eq. (560) as a sum over the inequivalent zeros of the scattering equations. Doing so will introduce a Jacobian factor $J(z, p)$ and we find

$$A_n^{(0)}(\sigma, p, \epsilon) = i \sum_{\text{solutions } j} J(z^{(j)}, p) \ C(\sigma, z^{(j)}) \ E(z^{(j)}, p, \epsilon).$$

(568)

The Jacobian is obtained as follows: We define a $n \times n$-matrix $\Phi$ with entries

$$\Phi_{ab} = \frac{\partial f_a}{\partial z_b} = \begin{cases} \frac{2p_a \cdot p_b}{(z_a - z_b)^2} & a \neq b, \\ - \sum_{j=1, j \neq a}^{n} \frac{2p_a \cdot p_j}{(z_a - z_j)^2} & a = b. \end{cases}$$

(569)

Let $\Phi_{rst}^{ijk}$ denote the $(n - 3) \times (n - 3)$-matrix, where the rows $\{i, j, k\}$ and the columns $\{r, s, t\}$ have been deleted. We set

$$\det' \Phi = (-1)^{i+j+k+r+s+t} \frac{\left| \Phi_{rst}^{ijk} \right|}{(z_i z_j z_k z_l)}.$$  

(570)

where we used the abbreviation $z_{ab} = z_a - z_b$. With the above sign included, the quantity $\det' \Phi$ is independent of the choice of $\{i, j, k\}$ and $\{r, s, t\}$ on the solutions of the scattering equations. We then have

$$J(z, p) = \frac{1}{\det' \Phi}.$$ 

(571)

Under PSL$(2, \mathbb{C})$-transformations the Jacobian $J(z, p)$ transforms as

$$J(g \cdot z, p) = \left( \prod_{j=1}^{n} \frac{1}{(cz_j + d)^4} \right) J(z, p)$$

(572)

and each summand in eq. (568) is PSL$(2, \mathbb{C})$-invariant. Let us briefly look at the mass dimensions of the various ingredients. The dimension of $A_n^{(0)}$ is

$$\dim A_n^{(0)} = 4 - n.$$ 

(573)

The ingredients of eq. (568) have the mass dimensions

$$\dim J(z, p) = -2(n - 3), \quad \dim C(\sigma, z) = 0, \quad \dim E(z, p, \epsilon) = n - 2,$$

(574)

adding up to $4 - n = \dim A_n^{(0)}$ as they should.

Eq. (560) and eq. (568) are the two forms of the CHY representation of the primitive tree amplitude $A_n^{(0)}$. Note that the cyclic order enters only through the Parke-Taylor factor $C(\sigma, z),$. 

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whereas the helicity configuration enters only through the polarisation factor $E(z, p, \epsilon)$. The CHY representation separates the information on the cyclic order from the information on the helicity configuration.

How do we know that eq. (560) or eq. (568) actually compute the primitive tree amplitude $A_n^{(0)}$? We can prove this by induction. The induction step uses once again the powerful method of on-shell recursion relations and is given in [121].

Let us now look at a few examples. We first discuss the case $n = 3$. In general we have $(n-3)!$ inequivalent solutions of the scattering equations. For $n = 3$ we thus have $0! = 1$ inequivalent solution. Due to Möbius invariance we may fix in general three $z$-variables at some chosen values. For $n = 3$ we have in total only three variables $(z_1, z_2, z_3)$. Therefore we may take without loss of generality as representative for the one inequivalent solution

$$z_1^{(1)} = 0, \quad z_2^{(1)} = 1, \quad z_3^{(1)} = \infty.$$ (575)

Let us consider the cyclic order $\sigma = (1, 2, 3)$. We find for the Jacobian, the Parke-Taylor factor and the polarisation factor

$$J(z, p) = (z_1 - z_2)^2 (z_2 - z_3)^2 (z_3 - z_1)^2;$$

$$C(\sigma, z) = \frac{1}{(z_1 - z_2)(z_2 - z_3)(z_3 - z_1)};$$

$$E(z, p, \epsilon) = \frac{2[(\epsilon_1 \cdot \epsilon_2)(\epsilon_3 \cdot p_1) + (\epsilon_2 \cdot \epsilon_3)(\epsilon_1 \cdot p_2) + (\epsilon_3 \cdot \epsilon_1)(\epsilon_2 \cdot p_3)]}{(z_1 - z_2)(z_2 - z_3)(z_3 - z_1)}. \quad (576)$$

Putting everything together we obtain for $A_3^{(0)}$

$$A_3^{(0)}(\sigma, p, \epsilon) = iJ(z^{(1)}, p) C(\sigma, z^{(1)}) E(z^{(1)}, p, \epsilon) \quad (577)$$

$$= 2i[(\epsilon_1 \cdot \epsilon_2)(\epsilon_3 \cdot p_1) + (\epsilon_2 \cdot \epsilon_3)(\epsilon_1 \cdot p_2) + (\epsilon_3 \cdot \epsilon_1)(\epsilon_2 \cdot p_3)]$$

$$= i \left[ g_{\mu_1 \mu_2} (p_1 - p_2)_{\mu_3} + g_{\mu_2 \mu_3} (p_2 - p_3)_{\mu_1} + g_{\mu_3 \mu_1} (p_3 - p_1)_{\mu_2} \right] \epsilon_1^{\mu_1} \epsilon_2^{\mu_2} \epsilon_3^{\mu_3},$$

which is the correct result.

Let us also look at the case $n = 4$. We consider the cyclic order $\sigma = (1, 2, 3, 4)$. The scattering equations read

$$-\frac{s}{z_1 - z_2} + \frac{u}{z_1 - z_3} + \frac{t}{z_1 - z_4} = 0,$$

$$-\frac{s}{z_1 - z_2} + \frac{t}{z_2 - z_3} + \frac{u}{z_2 - z_4} = 0,$$

$$-\frac{u}{z_1 - z_3} + \frac{t}{z_2 - z_3} + \frac{s}{z_3 - z_4} = 0,$$

$$-\frac{t}{z_1 - z_4} + \frac{u}{z_2 - z_4} + \frac{s}{z_3 - z_4} = 0. \quad (578)$$
We immediately obtain the following cross-ratios:
\[
\frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)} = -\frac{s}{t}, \quad \frac{(z_1 - z_4)(z_2 - z_3)}{(z_1 - z_3)(z_2 - z_4)} = -\frac{t}{u}, \quad \frac{(z_1 - z_3)(z_4 - z_2)}{(z_1 - z_2)(z_4 - z_3)} = -\frac{u}{s}. \tag{579}
\]

There is one solution to the scattering equations, which can be taken without loss of generality as
\[
z_1^{(1)} = -\frac{s}{t}, \quad z_2^{(1)} = 0, \quad z_3^{(1)} = 1, \quad z_4^{(1)} = \infty. \tag{580}
\]

For \( n = 4 \) we have as usual the Mandelstam relation
\[
s + t + u = 0. \tag{581}
\]

In addition, we have from the polynomial form of the scattering equations the relation
\[
(z_1 z_2 + z_3 z_4)s + (z_2 z_3 + z_1 z_4)t + (z_1 z_3 + z_2 z_4)u = 0. \tag{582}
\]

There are several equivalent forms for \( J(z, p) \), due to the choice of rows and columns to be deleted from the matrix \( \Phi \). All these forms can be related with the help of eq. (581) and eq. (582). We find
\[
J(z, p) = -\frac{1}{s} (z_1 - z_2)^2 (z_1 - z_3) (z_1 - z_4) (z_2 - z_3) (z_2 - z_4) (z_3 - z_4)^2. \tag{583}
\]

The cyclic factor is given by
\[
C(\sigma, z) = \frac{1}{(z_1 - z_2) (z_2 - z_3) (z_3 - z_4) (z_4 - z_1)}. \tag{584}
\]

For the polarisation factor we find
\[
E(z, p, \epsilon) = \frac{1}{z_1 z_2 z_3 z_4} \left[ 4 (\epsilon_1 \cdot \epsilon_2) (\epsilon_3 \cdot p_2) (\epsilon_4 \cdot p_1) - 4 (\epsilon_1 \cdot \epsilon_4) (\epsilon_3 \cdot p_2) (\epsilon_2 \cdot p_1) 
- 4 (\epsilon_3 \cdot \epsilon_4) (\epsilon_1 \cdot p_2) (\epsilon_2 \cdot p_3) + 4 (\epsilon_2 \cdot \epsilon_3) (\epsilon_1 \cdot p_2) (\epsilon_4 \cdot p_3) + 4 (\epsilon_2 \cdot \epsilon_4) (\epsilon_1 \cdot p_2) (\epsilon_3 \cdot p_2) 
+ t (\epsilon_1 \cdot \epsilon_2) (\epsilon_3 \cdot \epsilon_4) + s (\epsilon_2 \cdot \epsilon_3) (\epsilon_1 \cdot \epsilon_4) \right] 
+ \frac{1}{z_2 z_3 z_4 z_1} \left[ 4 (\epsilon_2 \cdot \epsilon_3) (\epsilon_1 \cdot p_3) (\epsilon_4 \cdot p_2) - 4 (\epsilon_2 \cdot \epsilon_4) (\epsilon_1 \cdot p_3) (\epsilon_3 \cdot p_2) 
- 4 (\epsilon_1 \cdot \epsilon_4) (\epsilon_2 \cdot p_3) (\epsilon_3 \cdot p_1) + 4 (\epsilon_3 \cdot \epsilon_4) (\epsilon_1 \cdot p_3) (\epsilon_2 \cdot p_3) + 4 (\epsilon_3 \cdot \epsilon_4) (\epsilon_2 \cdot p_3) (\epsilon_1 \cdot p_3) 
+ u (\epsilon_2 \cdot \epsilon_3) (\epsilon_1 \cdot \epsilon_4) + t (\epsilon_3 \cdot \epsilon_1) (\epsilon_2 \cdot \epsilon_4) \right] 
+ \frac{1}{z_3 z_4 z_1 z_2} \left[ 4 (\epsilon_3 \cdot \epsilon_1) (\epsilon_2 \cdot p_1) (\epsilon_4 \cdot p_3) - 4 (\epsilon_3 \cdot \epsilon_4) (\epsilon_2 \cdot p_1) (\epsilon_1 \cdot p_3) 
- 4 (\epsilon_2 \cdot \epsilon_4) (\epsilon_3 \cdot p_1) (\epsilon_1 \cdot p_2) + 4 (\epsilon_1 \cdot \epsilon_4) (\epsilon_3 \cdot p_1) (\epsilon_4 \cdot p_2) + 4 (\epsilon_1 \cdot \epsilon_4) (\epsilon_3 \cdot p_1) (\epsilon_2 \cdot p_1) 
+ s (\epsilon_3 \cdot \epsilon_1) (\epsilon_2 \cdot \epsilon_4) + u (\epsilon_1 \cdot \epsilon_2) (\epsilon_3 \cdot \epsilon_4) \right].
\]

Let us now specialise to the helicity configuration \( 1^-, 2^-, 3^+, 4^+ \). Putting everything together we obtain
\[
A_4^{(0)}(1^-, 2^-, 3^+, 4^+) = iJ(z^{(1)}, p) C(\sigma, z^{(1)}) E(z^{(1)}, p, \epsilon) = 2i \frac{(12)^4}{(12)(23)(34)(41)}. \tag{586}
\]
7.4 Global residues

The CHY representation has a mathematical interpretation as a global residue [122,123]. This allows us to compute the primitive tree amplitude $A_n^{(0)}$ from the CHY representation without the need to know the solutions of the scattering equations. Note that we already encountered global and local residues in the context of the link representation in section (6.2).

In eq. (560) we presented the primitive tree amplitude $A_n^{(0)}$ as a multidimensional complex contour integral:

$$A_n^{(0)}(\sigma, p, \epsilon) = \frac{-1}{(2\pi i)^{n-3}} \oint \frac{d^m z}{\omega} \frac{\prod_{a \neq i,j,k} (z_i - z_j)}{\prod_{m=2}^{n-2} h_m(z, p)} C(\sigma, z) E(z, p, \epsilon).$$

Switching to the polynomial form of the scattering equations we have

$$A_n^{(0)}(\sigma, p, \epsilon) = \frac{-1}{(2\pi i)^{n-3}} \oint \frac{d^m z}{\omega} \prod_{i<j} (z_i - z_j) \prod_{m=2}^{n-2} h_m(z, p) C(\sigma, z) E(z, p, \epsilon),$$

where the polynomials $h_2(z, p), \ldots, h_{n-2}(z, p)$ have been defined in eq. (544). Let us now use the PSL$(2, \mathbb{C})$-invariance and gauge-fix three variables

$$z_1 = 0, \quad z_{n-1} = 1, \quad z_n = \infty.$$  

We then have

$$A_n^{(0)}(\sigma, p, \epsilon) = \frac{1}{(2\pi i)^{n-3}} \oint \frac{R(z, \sigma, p, \epsilon) \ dz_2 \wedge \ldots \wedge dz_{n-2}}{h_2'(z, p) \ldots h_{n-2}'(z, p)},$$

with

$$R(z, \sigma, p, \epsilon) = -z_n^4 \left( \prod_{i<j<n} z_{ij} \right) C(\sigma, z) E(z, p, \epsilon) \bigg|_{z_1=0, z_{n-1}=1, z_n=\infty}$$

and

$$h_m'(z, p) = \left. \frac{dh_m(z, p)}{dz_n} \right|_{z_n=0}.$$  

The polynomials $h_m(z, p)$ are linear in each variable $z_j$. Therefore $h_m'(z, p)$ gives the coefficient of $z_n$ in the polynomial $h_m(z, p)$. In the following we will simply write $R(z) = R(\sigma, p, \epsilon)$ and $h_m'(z) = h_m'(z, p)$. The quantity $R(z)$ is a rational function of the variables $(z_2, \ldots, z_{n-2})$.

Let $f(z) = f(z_2, \ldots, z_{n-2})$ be a meromorphic function, regular at the solutions $z^{(j)}$ of the scattering equations. We define the local residue at $z^{(j)}$ with respect to the divisors $h_2', \ldots, h_{n-2}'$ as

$$\text{Res}_{h_2' \ldots h_{n-2}'} \left( f(z^{(j)}) \right) = \frac{1}{(2\pi i)^{n-3}} \oint \frac{f(z) \ dz_2 \wedge \ldots \wedge dz_{n-2}}{h_2'(z) \ldots h_{n-2}'(z)}.$$  

The integration in eq. (593) is around a small \((n-3)\)-torus
\[
\Gamma_{\delta} = \{ (z_2, \ldots, z_{n-2}) \in \mathbb{C}^{n-3} | |h'_m(z)| = \delta \},
\]
encircling \(z^{(j)}\) with orientation
\[
d \arg h_2 \wedge d \arg h_3 \wedge \ldots \wedge d \arg h_{n-2} \geq 0.
\]
The **global residue** with respect to the divisors \(h'_2, \ldots, h'_{n-2}\) is defined as
\[
\text{Res}_{\{h'_2, \ldots, h'_{n-2}\}}(f) = \sum_{\text{solutions } j} \text{Res}_{\{h'_2, \ldots, h'_{n-2}\}}(f, z^{(j)}).
\]
If it is clear which divisors are meant, we will simply write \(\text{Res}(f) = \text{Res}_{\{h'_2, \ldots, h'_{n-2}\}}(f)\). Thus
\[
A_n^{(0)}(\sigma, p, \varepsilon) = i \text{Res} ( R(z, \sigma, p, \varepsilon) ) .
\]
Let us now consider the ring \(R = \mathbb{C}[z_2, \ldots, z_{n-2}]\) and the ideal \(I = \langle h'_2, \ldots, h'_{n-2} \rangle\). The zero locus of \(h'_2 = \ldots = h'_{n-2} = 0\) is a zero-dimensional variety. For the case at hand the zero locus consists of \((n-3)!\) points. It follows that he quotient ring \(R/I\) is a finite-dimensional \(\mathbb{C}\)-vectorspace. Let \(\{e_i\}\) be a basis of this vectorspace and \(P_1, P_2 \in R/I\) two polynomials (i.e. vectors) in this vectorspace. A theorem of algebraic geometry states that the global residue defines a **symmetric non-degenerate inner product**:
\[
\langle P_1, P_2 \rangle = \text{Res} ( P_1 \cdot P_2 ).
\]
Since the inner product is non-degenerate there exists a **dual basis** \(\{\Delta_i\}\) with the property
\[
\langle e_i, \Delta_j \rangle = \delta_{ij}.
\]
To compute the global residue of a polynomial \(P(z)\) we therefore obtain the following method: We express \(P\) in the basis \(\{e_i\}\) and 1 in the dual basis \(\{\Delta_i\}\):
\[
P = \sum_i \alpha_i e_i, \quad 1 = \sum_i \beta_i \Delta_i, \quad \alpha_i, \beta_i \in \mathbb{C}.
\]
We then have
\[
\text{Res} \left( P \right) = \text{Res} \left( P \cdot 1 \right) = \sum_i \sum_j \alpha_i \beta_j \langle e_i, \Delta_j \rangle = \sum_i \alpha_i \beta_i.
\]
Given a basis \(\{e_i\}\) and the associated dual basis \(\{\Delta_i\}\), eq. (601) allows us to compute the global residue of a polynomial \(P\) without knowing the solutions of the scattering equations. Eq. (601) simplifies, if the dual basis contains a constant polynomial \(\Delta_{i_0} = c\). We then have
\[
\text{Res} \left( P \right) = \frac{\alpha_{i_0}}{c}.
\]
Eq. (601) is not yet directly applicable to our problem, since in eq. (597) we have a rational function \( R(z) \), not a polynomial. We write \( R(z) = P(z)/Q(z) \). We may assume that \( \{ h'_2, \ldots, h'_{n-2}, Q \} \) have no common zeros, otherwise we would have in the contour integrals a double pole. Hilbert’s Nullstellensatz guarantees that there exist polynomials \( p_2, \ldots, p_{n-2}, \tilde{Q} \in \mathbb{R} \), such that

\[
p_2 h'_2 + \ldots + p_{n-2} h'_{n-2} + \tilde{Q} Q = 1.
\] (603)

We call \( \tilde{Q} \) the polynomial inverse of \( Q \) with respect to \( \langle h'_2, \ldots, h'_{n-2} \rangle \). For the global residue we have

\[
\text{Res} \left( R \right) = \text{Res} \left( \frac{P}{Q} \right) = \text{Res} \left( P \tilde{Q} \right).
\] (604)

Note \( P \tilde{Q} \) is a polynomial. We have therefore reduced the case of a rational function \( R(z) \) to the polynomial case \( P(z) \tilde{Q}(z) \).

The above calculations can be carried out with the help of a Gröbner basis for the ideal \( I \) [122]. However, the determination of a Gröbner basis is by itself computationally intensive and actually not needed. It is sufficient to use a coarser analogue, known as a Macaulay H-basis. A set of polynomials \( \{ b_1, \ldots, b_k \} \in I \) is an H-basis for the ideal \( I \), if for all \( P \in I \) there exists polynomials \( q_1, \ldots, q_k \in \mathbb{R} \) with \( \deg q_j \leq \deg P - \deg b_j \) and

\[
P = \sum_{j=1}^{k} q_j b_j.
\] (605)

The essential requirement is that the degrees of the polynomials \( q_j \) are bounded. There is an alternative definition of an H-basis: For any polynomial \( P \in \mathbb{R} \) one denotes by \( \text{in}(P) \) the homogeneous part of \( P \) of degree \( \deg P \). The polynomial \( \text{in}(P) \) is called the initial form of \( P \). To give an example

\[
\text{in} \left( x_1^2 x_2^3 + 5 x_3^5 + x_2 x_3 + x_1 \right) = x_1^2 x_2^3 + 5 x_3^5.
\] (606)

A set of polynomials \( \{ b_1, \ldots, b_k \} \in I \) is an H-basis for the ideal \( I \), if

\[
\langle \text{in}(I) \rangle = \langle \text{in}(b_1), \ldots, \text{in}(b_k) \rangle.
\] (607)

Bosma, Søgaard and Zhang [123] have shown that the polynomials \( \{ h'_2, \ldots, h'_{n-2} \} \) form an H-basis for the ideal \( I \). Furthermore, any polynomial \( P \in \mathbb{R} \) may be reduced with respect to \( \{ h'_2, \ldots, h'_{n-2} \} \) as

\[
P = \sum_{j=2}^{n-2} q_j h'_j + \tilde{P},
\] (608)

where the remainder \( \tilde{P} \) is bounded in degree by

\[
\deg \tilde{P} \leq d^* = \frac{1}{2} (n-4) (n-3).
\] (609)
It is obvious that the terms proportional to $h'_j$ do not contribute to the global residue. It is less obvious that neither the terms in $\tilde{P}$ of degree strictly less than $d^*$ contribute to the global residue. Thus

$$\text{Res}\{h'_2,\ldots,h'_{n-2}\}(P) = \text{Res}\{h'_2,\ldots,h'_{n-2}\}(\tilde{P}) = \text{Res}\{h'_2,\ldots,h'_{n-2}\}(\text{in}(\tilde{P})).$$  \hspace{1cm} (610)

The last equality follows from the proper map theorem for H-bases.

Let us now look at the vectorspace $V_{d^*}$ of homogeneous polynomials $P \in \mathbb{R}$ with degree $d^*$. It can be shown that

$$\dim\mathbb{C}V_{d^*} - \dim\mathbb{C}(V_{d^*} \cap \langle \text{in}(I) \rangle) = 1.$$  \hspace{1cm} (611)

Therefore we may choose a basis of $V_{d^*}$, where only a single element is not in $\langle \text{in}(I) \rangle$. Such an element is easily found:

$$M = \prod_{m=2}^{n-2} z_m^{n-2}.$$  \hspace{1cm} (612)

Thus we may write

$$\text{in}(\tilde{P}) = \alpha M + \sum_{j=2}^{n-2} q'_j \text{in}(h'_j),$$  \hspace{1cm} (613)

with $\alpha \in \mathbb{C}$ and $q'_j \in \mathbb{R}$. We then have

$$\text{Res}\{h'_2,\ldots,h'_{n-2}\}(P) = \alpha \text{Res}\{h'_2,\ldots,h'_{n-2}\}(M).$$  \hspace{1cm} (614)

The global residue of $M$ is easily computed:

$$\text{Res}(M) = \frac{(-1)^{(n-4)(n-3)}}{\prod_{j=2}^{n-2} (p_j n - p_n)^2}.$$  \hspace{1cm} (615)

This gives us the following algorithm:

**Algorithm 5.** Calculation of a primitive helicity amplitude $A^{(0)}_{n}(\sigma, p, \varepsilon)$ as a global residue.

1. Start with the rational function $R(z)$ given in eq. (591):

$$R(z, \sigma, p, \varepsilon) = -z_n^4 \left( \prod_{i<j<n} z_{ij} \right) C(\sigma, z) E(z, p, \varepsilon) \bigg|_{z_1=0, z_{n-1}=1, z_n=\infty}$$  \hspace{1cm} (616)

and write $R(z) = P(z)/Q(z)$, where $P(z)$ and $Q(z)$ are polynomials with $\gcd(P, Q) = 1$.  

2. Determine the polynomial inverse $\tilde{Q}(z)$ of $Q(z)$ with respect to $\{h'_2, \ldots, h'_{n-2}\}$ and set

$$P_1(z) = P(z)\tilde{Q}(z).$$

3. Reduce $P_1(z)$ with respect to $h'_2, \ldots, h'_{n-2}$:

$$P_1(z) = p_2h'_2 + \ldots + p_{n-2}h'_{n-2} + \tilde{P}(z). \quad (617)$$

The remainder $\tilde{P}(z)$ is a polynomial of degree at most $(n-3)(n-4)/2$.

4. Express the initial form in $\tilde{P}$ as

$$\text{in}(\tilde{P}) = \alpha M + \sum_{j=2}^{n-2} q'_j \text{in}(h'_j). \quad (618)$$

5. The amplitude is given by

$$A_n^{(0)} = \frac{i(-1)^{(n-4)(n-3)}}{\prod_{j=2}^{n-2} (p_j n - 2 + p_n)^2} \alpha. \quad (619)$$

Let us look at an example. In order to keep things simple we will replace in the CHY representation the polarisation factor with another copy of the Parke-Taylor factor. We consider

$$m_0^{(0)}(\sigma, \bar{\sigma}, p) = \frac{i}{\mathcal{C}} \int d\Omega_{\text{CHY}} C(\sigma, z) C(\bar{\sigma}, z). \quad (620)$$

In the next section we will discuss amplitudes of this type in more detail. Let us now take $n = 4$ and $\sigma = \bar{\sigma} = (1, 2, 3, 4)$. We gauge-fix $z_1 = 0, z_3 = 1$ and $z_4 = \infty$. The rational function $R(z)$ is then given by

$$R(z) = \frac{1}{z_2(1-z_2)}. \quad (621)$$

The polynomial $h'_2$ is given by

$$h'_2 = uz_2 + s. \quad (622)$$

The degree zero polynomial $\tilde{Q} = u^2/(st)$ (the degree refers to the variable $z_2$) is a polynomial inverse to $z_2(1-z_2)$ with respect to $h'_2$, since

$$\frac{u^2}{st} z_2(1-z_2) + \left(\frac{u}{st} z_2 + \frac{1}{s}\right) h'_2 = 1. \quad (623)$$

We further have $M = 1$ and $\alpha = u^2/(st)$. Hence

$$m_0^{(0)}(\sigma, \bar{\sigma}, p) = \frac{i\alpha}{(p_2+p_4)^2} = \frac{i}{st} u = -i \left(\frac{1}{s} + \frac{1}{t}\right). \quad (624)$$
8 Perturbative quantum gravity

Up to now we only considered Yang-Mills theory. Let us now turn to a second theory: In this section we will discuss gravity. At the classical level gravity is described by Einstein’s theory of general relativity. Einstein’s equations (for the case without any matter) are derived from the Einstein-Hilbert Lagrange density

\[ \mathcal{L}_{EH} = -\frac{2}{\kappa^2} \sqrt{-g} \left( R + 2\Lambda \right). \]  

(625)

We continue to use the convention that

\[ c = 1, \quad \hbar = 1. \]  

(626)

The parameter \( \kappa \) is a dimensionful constant of dimension mass\(^{-1}\) given, by \( \kappa = \sqrt{8G_N} \) in natural units, where \( G_N \) is Newton’s constant. In the Gauss unit system we have \( \kappa = \sqrt{32\pi G_N} \). The Planck mass is defined by \( m_{Pl} = 1/\sqrt{G_N} \). We denote the full metric in general relativity by \( g_{\mu\nu} \) and write

\[ g = \det (g_{\mu\nu}). \]  

(627)

The quantity \( R \) denotes the scalar curvature and is given by

\[ R = g^{\mu\nu} \text{Ric}_{\mu\nu} = g^{\mu\nu} R^\lambda_{\mu\lambda\nu}, \]  

(628)

where \( \text{Ric}_{\mu\nu} \) denotes the Ricci tensor and \( R^\lambda_{\mu\lambda\nu} \) the Riemann curvature tensor. The parameter \( \Lambda \) denotes the cosmological constant.

Let us denote the metric of flat Minkowski space by

\[ \eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \]  

The metric \( \eta_{\mu\nu} \) of flat Minkowski space is a solution of Einstein’s fields equations without a cosmological constant,

\[ \text{Ric}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0. \]  

(629)

Note that \( \eta_{\mu\nu} \) is not a solution of Einstein’s field equations for a non-zero value of the cosmological constant.

We will now discuss perturbative quantum gravity [124–127]. We consider small fluctuations of the gravitational field around the flat Minkowski metric and weight every fluctuation with \( \exp(iS) \), where \( S \) is the action. We further assume that the coupling \( \kappa/4 \) is small. Within the energy range accessible at current collider experiments (i.e. up to TeV scales), the gravitational coupling is incredible small. This set-up leads us directly to a perturbative description of
quantum gravity and allows us to describe gravitational waves and their scattering. Perturbative quantum gravity is an effective quantum field theory, the “true” quantum theory of gravity has to agree with perturbative quantum gravity at low energies. They may differ at higher energies, where perturbative quantum gravity is expected to break down.

We write
\[ g_{\mu \nu} = \eta_{\mu \nu} + \kappa h_{\mu \nu} \]  
and treat the term \( \kappa h_{\mu \nu} \) as a small perturbation. The tensor \( h_{\mu \nu} \) describes the field of a graviton. Expanding around the flat Minkowski metric \( \eta_{\mu \nu} \) and requiring that the background metric is a solution of the field equations, implies that we are considering the case of a vanishing cosmological constant. In the sequel we assume therefore \( \Lambda = 0 \). The Einstein-Hilbert action without a cosmological constant is given by
\[ S_{\text{EH}} = \int d^4x \mathcal{L}_{\text{EH}}, \quad \mathcal{L}_{\text{EH}} = -\frac{2}{\kappa^2} \sqrt{-g} R. \]  

8.1 Gauge invariance of gravity

The Einstein-Hilbert action in eq. (631) is invariant under general coordinate transformations
\[ x'^\mu = f^\mu (x). \]  
In fact, one of Einstein’s original motivations was to find a theory invariant under these transformations. We may view the general coordinate transformations in eq. (632) as (generalised) gauge transformations. We write an infinitesimal general coordinate transformation as
\[ x'^\mu = x^\mu - \varepsilon \xi^\mu (x). \]  
The minus sign has no particular importance and is just a convention. The infinitesimal inverse transformation is given by
\[ x^\mu = x'^\mu + \varepsilon \xi^\mu (x') + \mathcal{O} (\varepsilon^2). \]  

Let us now work out the metric in the transformed system:
\[ g'_{\mu' \nu'} (x') = \frac{\partial x^\mu}{\partial x'^\mu} \frac{\partial x^\nu}{\partial x'^\nu} g_{\mu \nu} (x(x')). \]
\[ = \left( \delta^\mu_{\mu'} + \varepsilon \partial_{\mu'} \xi^\mu (x') \right) \left( \delta^\nu_{\nu'} + \varepsilon \partial_{\nu'} \xi^\nu (x') \right) \left( g_{\mu \nu} (x') + \varepsilon \xi^\rho (x') \partial_\rho g_{\mu \nu} (x') \right) + \mathcal{O} (\varepsilon^2) \]
\[ = g_{\mu' \nu'} (x') + \varepsilon \left[ (\partial_{\mu'} \xi^\mu (x')) g_{\nu \rho'} (x') + (\partial_{\nu'} \xi^\nu (x')) g_{\mu \rho'} (x') + \xi^\rho (x') \partial_\rho g_{\mu' \nu'} (x') \right] + \mathcal{O} (\varepsilon^2). \]  

We may write this in a shortened form as
\[ g'_{\mu \nu} = g_{\mu \nu} + \varepsilon \left[ (\partial_{\mu} \xi^\rho) g_{\rho \nu} + (\partial_{\nu} \xi^\rho) g_{\mu \rho} + \xi^\rho \partial_\rho g_{\mu \nu} \right] + \mathcal{O} (\varepsilon^2). \]  

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Let us now specialise to an expansion around the flat Minkowski metric. With

\[ g_{\mu\nu}(x) = \eta_{\mu\nu} + \kappa h_{\mu\nu}(x), \] (637)

we find for \( h'_{\mu\nu} \):

\[ h'_{\mu\nu} = h_{\mu\nu} + \frac{\epsilon}{\kappa} \left[ (\partial_\mu \xi_\rho) \eta_{\rho\nu} + (\partial_\nu \xi_\rho) \eta_{\mu\rho} \right] + \epsilon \left[ (\partial_\mu \xi_\rho) h_{\rho\nu} + (\partial_\nu \xi_\rho) h_{\mu\rho} + \xi_\rho \partial_\rho h_{\mu\nu} \right] + \mathcal{O} (\epsilon^2). \] (638)

This expression can be simplified and we find

\[ h'_{\mu\nu} = h_{\mu\nu} + \frac{\epsilon}{\kappa} \left[ \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \right] + \mathcal{O} (\epsilon^2), \] (639)

where \( \xi_\mu = g_{\mu\nu} \xi^\nu = \eta_{\mu\nu} \xi^\nu + \kappa h_{\mu\nu} \xi^\nu \). We may view the transformation from \( h_{\mu\nu} \) to \( h'_{\mu\nu} \) as an infinitesimal gauge transformation.

### 8.2 Gravity amplitudes from Feynman diagrams

We may use the textbook formalism of quantum field theory to compute perturbatively scattering amplitudes for gravitons. In the previous section we have seen that the action is invariant under general coordinate transformations. In the path integral formalism we would like to restrict the path integral to configurations not equivalent by general coordinate transformations. As in section (2.3) this can be achieved with the technique of gauge fixing and the Faddeev-Popov method.

Let us now consider for \( h_{\mu\nu} \) the effective (not necessarily renormalisable) quantum field theory described by the generating functional

\[ Z \left[ J^{\mu\nu} \right] = \int \mathcal{D} h_{\mu\nu} \exp \left[ i \int d^4x \left( \mathcal{L}_{EH} + \mathcal{L}_{GF} + \mathcal{L}_{FP} + J^{\mu\nu} h_{\mu\nu} \right) \right], \] (640)

where \( \mathcal{L}_{GF} \) denotes the gauge-fixing term and \( \mathcal{L}_{FP} \) the corresponding Faddeev-Popov term. We will give an expression for the gauge-fixing term later on. The Faddeev-Popov term will again only contribute to loop amplitudes. We will treat the quantum field theory defined by eq. (640) perturbatively. Our first goal is the expansion of the Lagrange density in powers of \( h_{\mu\nu} \) (or equivalently in powers of \( \kappa \)). Let us introduce the following abbreviations:

\[ (\eta h)_{\mu\nu} = \eta^{\mu\mu_1} h_{\mu_1\mu_2} \eta^\mu_2 \eta^\nu, \]
\[ (\eta h h)_{\mu\nu} = \eta^{\mu\mu_1} h_{\mu_1\mu_2} \eta^{\mu_2\mu_3} h_{\mu_3\mu_4} \eta^\mu_4, \]
\[ (\eta h h h)_{\mu\nu} = \eta^{\mu\mu_1} h_{\mu_1\mu_2} \eta^{\mu_2\mu_3} h_{\mu_3\mu_4} \eta^{\mu_4\mu_5} h_{\mu_5\mu_6} \eta^\mu_6. \] (641)

With the help of these abbreviations we may express the inverse metric tensor \( g_{\mu\nu} \) through \( h_{\mu\nu} \):

\[ g_{\mu\nu} = \eta_{\mu\nu} - \kappa (\eta h)_{\mu\nu} + \kappa^2 (\eta h h)_{\mu\nu} - \kappa^3 (\eta h h h)_{\mu\nu} + \mathcal{O} (\kappa^4). \] (642)
The inverse metric tensor is an infinite power series in $\kappa$. Let us now turn to the determinant $g = \det(g_{\mu\nu})$. Also here we introduce a few abbreviations:

\[
\begin{align*}
(\eta h) &= \eta^{\mu_1 \mu_2} h_{\mu_2 \mu_1}, \\
(\eta h \eta h) &= \eta^{\mu_1 \mu_2} h_{\mu_2 \mu_1} \eta^{\nu_1 \nu_2} h_{\nu_2 \nu_1}, \\
(\eta h \eta h \eta h) &= \eta^{\mu_1 \mu_2} h_{\mu_2 \mu_1} \eta^{\nu_1 \nu_2} h_{\nu_2 \nu_1} \eta^{\rho_1 \rho_2} h_{\rho_2 \rho_1}, \\
(\eta h \eta h \eta h \eta h) &= \eta^{\mu_1 \mu_2} h_{\mu_2 \mu_1} \eta^{\nu_1 \nu_2} h_{\nu_2 \nu_1} \eta^{\rho_1 \rho_2} h_{\rho_2 \rho_1} \eta^{\sigma_1 \sigma_2} h_{\sigma_2 \sigma_1}.
\end{align*}
\]

(643)

We then find for the determinant:

\[
\begin{align*}
-\det (g_{\mu\nu}) &= \\
&= 1 + \kappa (\eta h) + \kappa^2 \left[ \frac{1}{2} (\eta h)^2 - \frac{1}{2} (\eta h \eta h) \right] \eta^3 \left[ \frac{1}{6} (\eta h)^3 - \frac{1}{2} (\eta h \eta h) (\eta h) + \frac{1}{3} (\eta h \eta h \eta h) \right] \\
&+ \kappa^4 \left[ \frac{1}{24} (\eta h)^4 - \frac{1}{4} (\eta h \eta h) (\eta h)^2 + \frac{1}{8} (\eta h \eta h)^2 + \frac{1}{3} (\eta h \eta h \eta h) (\eta h) - \frac{1}{4} (\eta h \eta h \eta h \eta h) \right].
\end{align*}
\]

(644)

Note that this expression is a polynomial in $\kappa$ and terminates with the $\kappa^4$-term. However, by taking the square root of this expression we again obtain an infinite power series in $\kappa$:

\[
\sqrt{-g} = 1 + \frac{\kappa}{2} (\eta h) + \frac{\kappa^2}{8} \left[ (\eta h)^2 - 2 (\eta h \eta h) \right] + \frac{\kappa^3}{48} \left[ (\eta h)^3 - 6 (\eta h \eta h) (\eta h) + 8 (\eta h \eta h \eta h) \right] \\
+ O(\kappa^4).
\]

(645)

In order to find the expression for the scalar curvature $R$ let us first consider the Christoffel symbols

\[
\Gamma_{\kappa\nu\lambda} = \frac{1}{2} \left( \partial_\lambda g_{\kappa\nu} + \partial_\nu g_{\lambda\kappa} - \partial_\kappa g_{\nu\lambda} \right) = \frac{\kappa}{2} \left( \partial_\lambda h_{\kappa\nu} + \partial_\nu h_{\lambda\kappa} - \partial_\kappa h_{\nu\lambda} \right).
\]

(646)

Here we used $\partial_\alpha \eta_{\beta\gamma} = 0$. The Riemann curvature tensor is then given by

\[
R_{\kappa\lambda\mu\nu} = \frac{\kappa}{2} \left( \partial_\lambda \partial_\mu h_{\kappa\nu} - \partial_\lambda \partial_\nu h_{\kappa\mu} + \partial_\kappa \partial_\nu h_{\lambda\mu} + \partial_\lambda \partial_\mu h_{\nu\kappa} - \partial_\kappa \partial_\mu h_{\nu\lambda} \right) + g^{\xi\eta} \left( \Gamma_{\xi\kappa\lambda} \Gamma_{\eta\mu\nu} - \Gamma_{\xi\kappa\mu} \Gamma_{\eta\lambda\nu} \right).
\]

(647)

The first term is linear in $h_{\mu\nu}$, while the second term is at least quadratic in $h_{\mu\nu}$. For the scalar curvature we have then

\[
R = g^{\kappa\mu} g^{\lambda\nu} R_{\kappa\lambda\mu\nu}.
\]

(648)

Since both $g^{\mu\nu}$ and $\sqrt{-g}$ are infinite power series in $\kappa$ we obtain for the Lagrange density an infinite power series in $\kappa$ as well. We write

\[
L_{EH} + L_{GF} = \sum_{j=1}^{\infty} L^{(j)},
\]

(649)

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where the term $\mathcal{L}^{(j)}$ contains the field $h_{\mu\nu}$ exactly $j$ times. In this way we obtain a theory with an infinite tower of vertices, ordered by the number of the fields. The term $\mathcal{L}^{(1)}$ is given by

$$\mathcal{L}^{(1)} = -\frac{2}{\kappa} \eta^{\kappa\mu} \eta^{\lambda\nu} \partial_\lambda (\partial_\mu h_{\kappa\nu} - \partial_\nu h_{\kappa\mu}).$$

(650)

This term is a total derivative and vanishes in the action after partial integration:

$$-\frac{2}{\kappa} \eta^{\kappa\mu} \eta^{\lambda\nu} \int d^4x \partial_\lambda (\partial_\mu h_{\kappa\nu} - \partial_\nu h_{\kappa\mu}) = 0.$$  

(651)

We may therefore ignore this term and start the expansion of the Lagrange density in powers of $\kappa$ with the term quadratic in $h_{\mu\nu}$.

Let us add the following remark: If we would have expanded naively the Einstein-Hilbert action with a cosmological constant $\Lambda \neq 0$ around the flat Minkowski metric $\eta_{\mu\nu}$, we would have picked up an additional term

$$-\frac{2\Lambda}{\kappa} \eta^{\mu\nu} h_{\mu\nu}.$$  

(652)

contributing to $\mathcal{L}^{(1)}$, coming from the expansion of $\sqrt{-g}$. This additional term is not a total derivative and does not vanish. Terms of this type are called tadpoles and indicate that we expanded around the wrong background field.

Let us now return to the case $\Lambda = 0$. We consider the term $\mathcal{L}^{(2)}$, bilinear in $h_{\mu\nu}$. The gauge-fixing term $\mathcal{L}_{GF}$ gives a contribution to $\mathcal{L}^{(2)}$. A popular gauge choice for gravity is de Donder gauge. This gauge is defined by

$$\mathcal{L}_{GF} = \frac{1}{\kappa^2} C_\mu \eta^{\mu\nu} C_\nu,$$

(653)

where $C_\mu$ is given by

$$C_\mu = \eta^{\alpha\beta} \Gamma_{\mu\alpha\beta} = \frac{\kappa}{2} \eta^{\alpha\beta} (\partial_\alpha h_{\beta\mu} + \partial_\beta h_{\alpha\mu} - \partial_\mu h_{\alpha\beta}) = \kappa \eta^{\alpha\beta} \left( \partial_\alpha h_{\beta\mu} - \frac{1}{2} \partial_\mu h_{\alpha\beta} \right).$$

(654)

In this gauge one finds

$$\mathcal{L}^{(2)} = \frac{1}{2} h_{\mu_1\mu_2} \left( \frac{1}{2} \eta^{\mu_1\mu_2} \eta^{v_1 v_2} - \frac{1}{2} \eta^{\mu_1 v_1} \eta^{\mu_2 v_2} - \frac{1}{2} \eta^{v_1 v_2} \eta^{\mu_1 \mu_2} \right) \Box h_{v_1 v_2}.$$  

(655)

Here, we symmetrised the expression in the bracket in $(\mu_1, \mu_2)$ and $(v_1, v_2)$. We are free to do this, since $h_{\mu\nu}$ is symmetric under an exchange of $\mu$ and $v$. Let us first consider the tensor structure (in $D$ space-time dimensions). For

$$M^{\mu_1\mu_2 v_1 v_2} = \frac{1}{2} \eta^{\mu_1 v_1} \eta^{\mu_2 v_2} + \frac{1}{2} \eta^{\mu_1 v_2} \eta^{\mu_2 v_1} - \frac{1}{2} \eta^{\mu_1\mu_2} \eta^{v_1 v_2},$$

$$N_{\mu_1\mu_2 v_1 v_2} = \frac{1}{2} \left( \eta_{\mu_1 v_1} \eta_{\mu_2 v_2} + \eta_{\mu_1 v_2} \eta_{\mu_2 v_1} - \frac{2}{D-2} \eta_{\mu_1\mu_2} \eta_{v_1 v_2} \right),$$

(656)
we have
\[ M^{\mu_1\mu_2\rho_1\rho_2}N_{\rho_1\rho_2 v_1 v_2} = \frac{1}{2} (\delta^\mu_1_2 \delta^\rho_1_2 + \delta^\mu_1_3 \delta^\rho_1_2). \] (657)

The propagator of the graviton is therefore given by
\[ \frac{1}{2} \left( \eta_{\mu_1 v_1} \eta_{\mu_2 v_2} + \eta_{\mu_1 v_2} \eta_{\mu_2 v_1} - \frac{2}{D-2} \eta_{\mu_1 \mu_2} \eta_{v_1 v_2} \right) \frac{i}{p^2}. \] (658)

Let us now turn to the three-graviton vertex. The three-graviton vertex is determined by \( \mathcal{L}^{(3)} \). After a longer calculation and by using integration-by-parts one finds
\[
\mathcal{L}^{(3)} = \kappa \left[ -\frac{1}{4} \eta_{\mu_1 v_1} \eta_{\mu_2 v_2} \eta_{\mu_3 v_3} \eta_{\rho_1 \rho_2} + \frac{1}{4} \eta_{\mu_1 v_1} \eta_{\mu_2 v_3} \eta_{\mu_3 v_2} \eta_{\rho_1 \rho_2} + \eta_{\mu_1 v_2} \eta_{\mu_2 v_1} \eta_{\mu_3 v_3} \eta_{\rho_1 \rho_2} \\
- \eta_{\mu_1 v_2} \eta_{\mu_2 v_3} \eta_{\mu_3 v_1} \eta_{\rho_1 \rho_2} + \frac{1}{2} \eta_{\mu_1 \mu_2} \eta_{v_1 v_2} \eta_{\mu_3 v_3} \eta_{\rho_1 \rho_2} - \frac{1}{2} \eta_{\mu_1 \mu_2} \eta_{v_1 v_2} \eta_{\mu_3 v_3} \eta_{\rho_1 \rho_2} \\
+ 2 \eta_{\mu_1 \mu_2} \eta_{v_1 v_2} \eta_{\mu_3 v_3} \eta_{\rho_1 \rho_2} - \eta_{\mu_1 \mu_2} \eta_{v_3 v_1} \eta_{\rho_1 \rho_2} - \frac{1}{2} \eta_{\mu_1 \mu_2} \eta_{v_3 v_1} \eta_{\rho_1 \rho_2} + \frac{1}{2} \eta_{\mu_1 \mu_2} \eta_{v_3 v_1} \eta_{\rho_1 \rho_2} \\
+ \frac{1}{2} \eta_{\mu_1 \mu_2} \eta_{v_3 v_1} \eta_{\rho_1 \rho_2} \right] h_{\mu_1 v_1} \left( \partial_{\rho_2} h_{\mu_2 v_2} \right) \left( \partial_{\rho_3} h_{\mu_3 v_3} \right). \] (659)

Let us write \( \mathcal{L}^{(3)} \) as
\[
\mathcal{L}^{(3)} = O^{\mu_1 \mu_2 \mu_3} \left( \partial_{\mu_1} \partial_{\mu_2} \partial_{\mu_3} \right) h_{\mu_1 v_1} h_{\mu_2 v_2} h_{\mu_3 v_3}, \] (660)

where \( O^{\mu_1 \mu_2 \mu_3} \left( \partial_{\mu_1} \partial_{\mu_2} \partial_{\mu_3} \right) \) is defined by comparison with the previous equation \(659\). The notation \( \partial_j \) denotes a derivative acting on the field \( h_{\mu_j v_j} \). The Feynman rule for the three-graviton vertex is then
\[
V_{\mu_1 \mu_2 \mu_3}^{v_1 v_2 v_3} (p_1, p_2, p_3) = i \sum_{\sigma \in S_3} O^{\mu_1 \mu_2 \mu_3} v_{\sigma(1)} v_{\sigma(2)} v_{\sigma(3)} (i p_{\sigma(1)} + i p_{\sigma(2)} + i p_{\sigma(3)}). \] (661)

The explicit expression for \( V_{\mu_1 \mu_2 \mu_3}^{v_1 v_2 v_3} \) is rather long and not given here. However, one interesting property should be mentioned: The three-graviton vertex can be written as
\[
V_{\mu_1 \mu_2 \mu_3}^{v_1 v_2 v_3} (p_1, p_2, p_3) = i \frac{\kappa}{4} V_{\mu_1 \mu_2}^{\mu_3} (p_1, p_2, p_3) V_{\nu_1 v_1}^{v_2 v_3} (p_1, p_2, p_3) + \ldots, \] (662)

where the dots denote terms, which vanish in the on-shell limit. The expression \( V_{\mu_1 \mu_2}^{\mu_3} (p_1, p_2, p_3) \) is the Feynman rule for the colour-stripped cyclic-order three-gluon vertex (see eq. \(124\)), given by
\[
V_{\mu_1 \mu_2}^{\mu_3} (p_1, p_2, p_3) = i \left[ g_{\mu_1 \mu_2} (p_1^{\mu_3} - p_2^{\mu_3}) + g_{\mu_2 \mu_3} (p_2^{\mu_3} - p_3^{\mu_3}) + g_{\mu_3 \mu_1} (p_3^{\mu_3} - p_1^{\mu_3}) \right]. \] (663)

We see that the three-graviton vertex in the on-shell limit is given (up to a prefactor involving the coupling) as the square of the cyclic-ordered three gluon vertex. This relates gravity with non-abelian gauge theories. We will explore this connection in more detail in the next sub-section.
In principle it is possible to derive from the Lagrange density in eq. (649) systematically the additional Feynman rules for vertices with four, five, ..., \( n \) gravitons. In addition we need a rule for the external graviton states. This rule is rather simple. A graviton is a spin 2 particle with two polarisation states, corresponding to the helicities \( h = +2 \) and \( h = -2 \). We label these states by \( ++ \) and \( -- \). We may describe the polarisation tensor of an external graviton by a product of two polarisation vectors for gauge bosons:

\[
\varepsilon_{\mu\nu}^{++}(p) = \varepsilon_{\mu}^{+}(p)\varepsilon_{\nu}^{+}(p), \quad \varepsilon_{\mu\nu}^{--}(p) = \varepsilon_{\mu}^{-}(p)\varepsilon_{\nu}^{-}(p).
\]

Let us denote the tree-level scattering amplitude for \( n \) gravitons by

\[
\mathcal{M}_n^{(0)}(p_1^{\lambda_1\tilde{\lambda}_1}, \ldots, p_n^{\lambda_n\tilde{\lambda}_n}),
\]

with \((\lambda_j, \tilde{\lambda}_j)\) either \((+, +)\) or \((- , -)\). It will be convenient to factor of the gravitational coupling and we define \( M_n^{(0)} \) by

\[
\mathcal{M}_n^{(0)}(p_1^{\lambda_1\tilde{\lambda}_1}, \ldots, p_n^{\lambda_n\tilde{\lambda}_n}) = \left(\frac{\kappa}{4}\right)^{n-2} M_n^{(0)}(p_1^{\lambda_1\tilde{\lambda}_1}, \ldots, p_n^{\lambda_n\tilde{\lambda}_n}).
\]

For the calculation of the scattering amplitude \( M_n^{(0)} \) with \( n \) gravitons we will need all vertices with up to \( n \) gravitons. The scattering amplitude may then be computed through Feynman diagrams. However, this approach is rather tedious. In the next sub-section we will discuss more efficient methods for the computation of the \( n \)-graviton amplitude.

### 8.3 Gravity amplitudes from the CHY representation

In this paragraph we discuss the CHY representation of tree-level graviton amplitudes. This will give us a first method to compute the \( n \)-graviton amplitude without recourse to Feynman diagrams. We recall that in the context of the CHY representation for primitive gauge amplitudes we used an alternative notation for the arguments of the amplitude:

\[
A_n^{(0)}(\sigma, p, \varepsilon) = A_n^{(0)}(p^{\sigma_1} \ldots, p^{\sigma_n}),
\]

where \( \sigma = (\sigma_1, \ldots, \sigma_n) \) denotes the external cyclic order, \( p = (p_1, \ldots, p_n) \) the \( n \)-tuple of the external momenta and \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \) the \( n \)-tuple of the external polarisation vectors. We have seen in eq. (664) that the polarisation of an external graviton is described by a product of two polarisation vectors

\[
\varepsilon_{\mu\nu}^{\lambda_j\tilde{\lambda}_j}(p_j) = \varepsilon_{\mu}^{\lambda_j}(p_j)\varepsilon_{\nu}^{\tilde{\lambda}_j}(p_j),
\]

therefore we may describe the polarisation configuration of \( n \) external gravitons by two \( n \)-tuples

\[
\varepsilon = (\varepsilon_1^{\lambda_1}, \ldots, \varepsilon_n^{\lambda_n}), \quad \tilde{\varepsilon} = (\varepsilon_1^{\tilde{\lambda}_1}, \ldots, \varepsilon_n^{\tilde{\lambda}_n}),
\]
where for each graviton the $n$-tuple $\varepsilon$ contains one polarisation vector and the $n$-tuple $\tilde{\varepsilon}$ the other polarisation vector. Of course, since either $(\lambda_j, \tilde{\lambda}_j) = (+, +)$ or $(\lambda_j, \tilde{\lambda}_j) = (−, −)$ we have $\varepsilon = \tilde{\varepsilon}$ for gravitons. This motivates us to introduce also for the graviton amplitudes an alternative notation:

$$M_n(0)^{(p, \varepsilon, \tilde{\varepsilon})} = M_n(0)^{(p^{\lambda_1} \tilde{\lambda}_1}, ..., p^{\lambda_n} \tilde{\lambda}_n)}.$$  \hspace{1cm} (670)

Let us recall the CHY representation for gauge amplitudes. We may express the primitive tree-level gauge amplitude either as a multidimensional contour integral around the zeros of the scattering equations or as a sum over the solutions of the scattering equations:

$$A_n(0)^{(\sigma, p, \varepsilon)} = i \oint_{\mathbb{C}} d\Omega_{\text{CHY}} C(\sigma, z) E(z, p, \varepsilon)$$

$$= i \sum_{\text{solutions } j} J(z^{(j)}, p) C(\sigma, z^{(j)}) E(z^{(j)}, p, \varepsilon).$$  \hspace{1cm} (671)

What happens if we replace the Parke-Taylor factor $C(\sigma, z)$ by another copy of $E(z, p, \varepsilon)$? It turns out that this gives the $n$-graviton amplitude $M_n^{(0)}$. We thus arrive at the CHY representation of the $n$-graviton amplitude

$$M_n(0)^{(p, \varepsilon, \tilde{\varepsilon})} = i \oint_{\mathbb{C}} d\Omega_{\text{CHY}} E(z, p, \varepsilon) E(z, p, \tilde{\varepsilon})$$

$$= i \sum_{\text{solutions } j} J(z^{(j)}, p) E(z^{(j)}, p, \varepsilon) E(z^{(j)}, p, \tilde{\varepsilon}).$$  \hspace{1cm} (672)

Note that for gravitons we have $\varepsilon = \tilde{\varepsilon}$, which derives from the fact that the external graviton states are described by a product of equal-helicity polarisation vectors $\varepsilon^{\pm}_{\mu} \varepsilon^{\pm}_{\nu}$. This brings up two questions: What states do the opposite helicity combinations $\varepsilon^{\pm}_{\mu} \varepsilon^{\mp}_{\nu}$ describe and what does eq. (672) compute, if we evaluate this formula with some opposite helicity combinations? The answer is the following: The opposite helicity combinations correspond to a linear combination of a dilaton and an anti-symmetric tensor field.

$$\varepsilon^{\text{dilaton}}_{\mu\nu} = \frac{1}{\sqrt{2}} \left( \varepsilon^{+}_{\mu} \varepsilon^{-}_{\nu} + \varepsilon^{-}_{\mu} \varepsilon^{+}_{\nu} \right), \quad \varepsilon^{\text{anti-symm}}_{\mu\nu} = \frac{1}{\sqrt{2}} \left( \varepsilon^{+}_{\mu} \varepsilon^{-}_{\nu} - \varepsilon^{-}_{\mu} \varepsilon^{+}_{\nu} \right).$$  \hspace{1cm} (673)

The anti-symmetric tensor field can be related through Hodge duality to an axion field. If some opposite helicity combinations are plugged into eq. (672), this formula computes the corresponding tree amplitude in a theory with a graviton, a dilaton and an anti-symmetric tensor field. If we restrict ourselves to $\varepsilon = \tilde{\varepsilon}$ we obtain the pure graviton amplitude. It can be shown for tree amplitudes, that in the case where all external particles are gravitons, the dilaton and anti-symmetric tensor modes in the extended theory do not propagate internally. Thus the tree graviton amplitudes coincide in Einstein gravity and in a theory consisting of Einstein gravity plus a dilaton plus an anti-symmetric tensor field. This statement is no longer true for loop amplitudes.
In going from eq. (671) to eq. (672) we replaced the cyclic factor $C(\sigma, z)$ with another copy of the polarisation factor $E(z, p, \varepsilon)$. What happens, if we do the reverse, i.e. if we replace the polarisation factor $E(z, p, \varepsilon)$ with another copy of a Parke-Taylor factor $C, \tilde{\sigma}, z$? Let us consider

$$m_n^{(0)} (\sigma, \tilde{\sigma}, p) = i \oint d\Omega_{\text{CHY}} C(\sigma, z) C(\tilde{\sigma}, z)$$

$$= i \sum_{\text{solutions } j} J(z^{(j)}, p) C(\sigma, z^{(j)}) C(\tilde{\sigma}, z^{(j)}).$$

(674)

The quantity $m_n^{(0)} (\sigma, \tilde{\sigma}, p)$ depends on two external orders $\sigma$ and $\tilde{\sigma}$. It turns out that $m_n^{(0)} (\sigma, \tilde{\sigma}, p)$ computes the double-ordered amplitudes of a bi-adjoint scalar theory with trivalent vertices [117, 128]. This theory consists of a scalar field $\phi^{ab}$ in adjoint representation of two Lie groups $G$ and $\tilde{G}$. We will denote indices referring to $G$ by $a$, indices referring to $\tilde{G}$ by $b$. The theory is described by the Lagrange density

$$L_{\text{bi-adjoint scalar}} = \frac{1}{2} \left( \partial_\mu \phi^{ab} \right) \left( \partial^\mu \phi^{ab} \right) - \frac{\lambda}{3!} f_a^{a_1a_2a_3} f_b^{b_1b_2b_3} \phi^{a_1b_1} \phi^{a_2b_2} \phi^{a_3b_3}.$$  

(675)

The Feynman rule for the propagator is

$$\frac{i}{p^2} \delta^{a_1a_2} \delta^{b_1b_2},$$

(676)

the Feynman rule for the trivalent vertex is

$$i\lambda \left( i f^{a_1a_2a_3} \right) \left( i \tilde{f}^{b_1b_2b_3} \right).$$

(677)

Amplitudes in this theory have a double colour decomposition, similar to the (single) colour decomposition of gauge amplitudes in eq. (122):

$$m_n^{(0)} (p) = \lambda^{n-2} \sum_{\sigma \in S_n/Z_n} \sum_{\tilde{\sigma} \in S_n/Z_n} 2 \text{Tr}(T^{a_1(\sigma)} \ldots T^{a_n(\sigma)}) 2 \text{Tr}(\tilde{T}^{b_1(\tilde{\sigma})} \ldots \tilde{T}^{b_n(\tilde{\sigma})}) m_n^{(0)} (\sigma, \tilde{\sigma}, p).$$

(678)

The double-ordered amplitudes $m_n^{(0)} (\sigma, \tilde{\sigma}, p)$ are computed by eq. (674). Of course we may also compute the amplitudes $m_n^{(0)} (\sigma, \tilde{\sigma}, p)$ from Feynman diagrams, which for a scalar theory is not so complicated. We denote by $\mathcal{T}_n(\sigma)$ the set of all ordered tree diagrams with trivalent vertices and external order $\sigma$. The number of diagrams in this set is given by

$$|\mathcal{T}_n(\sigma)| = \frac{(2n-4)!}{(n-2)!(n-1)!}.$$  

(679)

Two diagrams with different external orders are considered to be equivalent, if we can transform one diagram into the other by a sequence of flips. Under a flip operation one exchanges at a vertex two branches. We denote by $\mathcal{T}_n(\sigma) \cap \mathcal{T}_n(\tilde{\sigma})$ the set of diagrams compatible with the
external orders $\sigma$ and $\tilde{\sigma}$ and by $n_{\text{flip}}(\sigma, \tilde{\sigma})$ the number of flips needed to transform any diagram from $T_n(\sigma) \cap T_n(\tilde{\sigma})$ with the external order $\sigma$ into a diagram with the external order $\tilde{\sigma}$. The number $n_{\text{flip}}(\sigma, \tilde{\sigma})$ will be the same for all diagrams from $T_n(\sigma) \cap T_n(\tilde{\sigma})$. For a diagram $G$ we denote by $E(G)$ the set of the internal edges and by $s_e$ the Lorentz invariant corresponding to the internal edge $e$. We then have

$$m^{(0)}_n(\sigma, \tilde{\sigma}, p) = i(-1)^{n-3+n_{\text{flip}}(\sigma, \tilde{\sigma})} \sum_{G \in T_n(\sigma) \cap T_n(\tilde{\sigma})} \prod_{e \in E(G)} \frac{1}{s_e}.$$  \hspace{1cm} (680)

The prefactor $i(-1)^{n-3}$ collects all the factors of $i$ from the vertices and propagators: We have $(n - 3)$ factors of $i$ from the propagators and $(n - 2)$ factors of $i$ from the vertices. Let us give an example: For $n = 4$ and $\sigma = (1, 2, 3, 4)$ and $\tilde{\sigma} = (1, 3, 2, 4)$ we have

$$m^{(0)}_n(\sigma, \sigma, p) = -i \left( \frac{1}{1} + \frac{1}{1} \right), \quad m^{(0)}_n(\sigma, \tilde{\sigma}, p) = i \frac{1}{1}.$$  \hspace{1cm} (681)

We may equally well express the full amplitude $m^{(0)}_n(p)$ as a sum over Feynman diagrams. To this aim we denote by $\mathcal{U}_n$ the set of all unordered tree diagrams with trivalent vertices. The number of diagrams in this set is given by

$$|\mathcal{U}_n| = (2n - 5)!!.$$  \hspace{1cm} (682)

We have

$$m^{(0)}_n(p) = i(-1)^{n-3} \lambda^{n-2} \sum_{G \in \mathcal{U}_n} \frac{C(G) \tilde{C}(G)}{D(G)},$$  \hspace{1cm} (683)

where the factor in the denominator $D(G)$ and the colour factors $C(G)$ and $\tilde{C}(G)$ are as in eq. (153). Note that the symbol $G$ in $C(G)$ and $\tilde{C}(G)$ refers to the graph $G$. The quantity $C(G)$ gives the colour factor corresponding to the “$a$”-indices, $\tilde{C}(G)$ the one corresponding to the “$b$”-indices.

### 8.4 Gravity amplitudes from colour-kinematics duality

Let us now discuss a second alternative method for computing the $n$-graviton amplitude, based on colour-kinematics duality [48,129]. We recall from section (3.2) that there exists kinematic numerators $N(G)$, such that they satisfy anti-symmetry and Jacobi-like relations, whenever the corresponding colour factors $C(G)$ do

$$C(G_1) + C(G_2) = 0 \quad \Rightarrow \quad N(G_1) + N(G_2) = 0,$$

$$C(G_1) + C(G_2) + C(G_3) = 0 \quad \Rightarrow \quad N(G_1) + N(G_2) + N(G_3) = 0,$$  \hspace{1cm} (684)

and the full tree amplitude in Yang-Mills theory is given by

$$A^{(0)}_n(p, \epsilon) = ig^{n-2} \sum_{G \in \mathcal{U}_n} \frac{C(G)N(G)}{D(G)},$$  \hspace{1cm} (685)
where \( \mathcal{U}_n \) denotes as in the previous paragraph the set of all of all (unordered) tree diagrams with trivalent vertices. Suppose we have a set of kinematic numerators \( N(G) \), then we may compute the \( n \)-graviton amplitude by squaring the kinematic numerators:

\[
M_n^{(0)}(p, \varepsilon, \tilde{\varepsilon}) = i (-1)^{n-3} \left( \frac{\lambda}{4} \right)^{n-2} \sum_{G \in \mathcal{U}_n} \frac{N(G)\tilde{N}(G)}{D(G)}.
\]

(686)

For pure graviton amplitudes we take \( N(G) = \tilde{N}(G) \). Again there is a generalisation towards a theory consisting of Einstein gravity plus dilaton plus anti-symmetric tensor field. In this theory, the kinematic numerators \( N(G) \) correspond to the set of polarisation vectors \( \varepsilon \), while the kinematic numerators \( \tilde{N}(G) \) correspond to \( \tilde{\varepsilon} \). It is worth pointing out that the kinematic numerators are not unique, they may be changed by a generalised gauge transformation. However it should be stressed, that eq. (686) is invariant under generalised gauge transformation.

In going from eq. (685) to eq. (686) we replaced the colour numerator \( C(G) \) by another copy of the kinematic numerator \( N(G) \). Again we may ask what happens if we do the reverse, i.e. replace the kinematic numerator \( N(G) \) by a second copy of a colour numerator \( \tilde{C}(G) \)? From eq. (683) we already know the answer. This is nothing than the full amplitude \( m_n^{(0)}(p) \) in the bi-adjoint scalar theory with trivalent vertices:

\[
m_n^{(0)}(p) = i (-1)^{n-3} \lambda^{n-2} \sum_{G \in \mathcal{U}_n} \frac{C(G)\tilde{C}(G)}{D(G)}.
\]

(687)

### 8.5 Gravity amplitudes from the KLT relations

As a third alternative method to compute the \( n \)-graviton amplitude we consider Kawai-Lewellen-Tye (KLT) relations [130]. These relations express the tree-level \( n \)-graviton amplitude as products of tree-level primitive gauge amplitudes. Originally the KLT relations were derived from relations between open and closed strings. However, there is a modern formulation of these relations, which does not rely on string theory and which we present here.

Let us recall from section (3.1) that there are \((n - 3)!\) independent primitive tree-level amplitudes in Yang-Mills theory. Using cyclic-invariance, the Kleiss-Kuijf relations and the BCJ relations we may fix three external particles at specified positions. A basis of the independent cyclic orders is then for example given by

\[
B = \{ \sigma = (\sigma_1, \ldots, \sigma_n) \in S_n \mid \sigma_1 = 1, \sigma_2 = 2, \sigma_n = n \}.
\]

(688)

Clearly,

\[
|B| = (n - 3)!. \quad (689)
\]

Let us now define a \((n - 3)! \times (n - 3)!\)-dimensional matrix \( m_{\sigma\tilde{\sigma}} \), indexed by permutations \( \sigma \) and \( \tilde{\sigma} \) from \( B \). We set

\[
m_{\sigma\tilde{\sigma}} = m_n^{(0)}(\sigma, \tilde{\sigma}, p).
\]

(690)
The entries of the matrix $m_{\sigma\bar{\sigma}}$ are the double-ordered primitive amplitudes for the bi-adjoint scalar theory with trivalent vertices encountered in the previous paragraphs. The matrix $m_{\sigma\bar{\sigma}}$ is invertible and we set

$$S_{\sigma\bar{\sigma}} = (m^{-1})_{\sigma\bar{\sigma}}.$$  \hfill (691)

The KLT relations read \cite{115,130-137}

$$M_n^{(0)}(p,\varepsilon,\bar{\varepsilon}) = \sum_{\sigma,\bar{\sigma}\in B} A_n^{(0)}(\sigma, p, \varepsilon) S_{\sigma\bar{\sigma}} A_n^{(0)}(\bar{\sigma}, p, \bar{\varepsilon}),$$  \hfill (692)

where the sum runs over a basis of cyclic orders.

Let us look at an example: For $n = 4$ particles there is only one independent cyclic order, which we may take as $\sigma = (1,2,3,4)$. Hence, the set $B$ has only one element $B = \{\sigma\}$. The matrix $m_{\sigma\sigma}$ is a $1 \times 1$-matrix, given by

$$m_{\sigma\sigma} = m_4^{(0)}(\sigma, p) = -i \left( \frac{1}{s} + \frac{1}{t} \right) = i \frac{u}{st}. \hfill (693)$$

The $1 \times 1$-matrix $S_{\sigma\sigma}$, being the inverse of $m_{\sigma\sigma}$, is then

$$S_{\sigma\sigma} = -i \frac{st}{u}. \hfill (694)$$

We therefore have

$$M_4^{(0)}\left(p_1^{\lambda_1}, p_2^{\lambda_2}, p_3^{\lambda_3}, p_4^{\lambda_4}\right) = -i \frac{st}{u} \left[ A_4^{(0)}\left(p_1^{\lambda_1}, p_2^{\lambda_2}, p_3^{\lambda_3}, p_4^{\lambda_4}\right) \right]^2. \hfill (695)$$

This is visualised in fig. (14). Note that the amplitude $A_4^{(0)}(p_1, p_2, p_3, p_4)$ has poles in $s$ and $t$, but not in $u$. The prefactor $(st)$ cancels the double poles in $s$ and $t$. Therefore $M_4^{(0)}$ has only single poles in $s$, $t$ and $u$, as it should. For four particles the four-graviton amplitude is given as the square of the four-gluon amplitude divided by a scalar amplitude. For $n > 4$ the basis of independent cyclic orders of the primitive gluon amplitudes has more than one element and the KLT relations turn into matrix equations.

**Exercise 27**: Consider the case $n = 5$ and determine the matrix $S_{\sigma\sigma}$ for the basis $B$ given in eq. (688).
9 Outlook

In this report we considered primitive tree-level amplitudes in Yang-Mills theory. These amplitudes may be computed with the textbook method of Feynman diagrams, involving only simple algebraic operations like addition, multiplication, contraction of indices, etc. One may ask, why we did not stop with this review shortly after the introduction? The answer is two-fold. First of all, the method based on Feynman diagrams is highly inefficient as we increase the number of external particles. We would like to have efficient methods to compute these scattering amplitudes. If we opt for a numerical approach, one of the best methods is based on colour-decomposition, the spinor-helicity method and off-shell recurrence relations. If on the other hand we are interested in analytical results, one of the best methods in this case is based on on-shell recursion relations.

The second part to the answer of the question raised above is as follows: We would have had no impetus to reveal hidden structures in quantum field theory. We learned about colour-kinematics duality, twistor methods, Grassmannian geometry, the scattering equations and global residues. These mathematical developments culminate in relations between scattering amplitudes in Yang-Mills theory and gravity. This is a very interesting aspect, relating the three forces described by the Standard Model of particle physics to the one force not covered by the Standard Model.

In this review we focused on primitive tree-level amplitudes in Yang-Mills theory, presenting new developments in quantum field theory in this setting. Although many of the themes discussed in this report provide foundational background to current research directions, it is on the other hand unavoidable that the focus on tree-level amplitudes in Yang-Mills theory leaves some important topics uncovered.

The theory of the strong interactions, quantum chromodynamics (QCD), is based on the (unbroken) non-abelian gauge group $SU(3)$ and all results for gauge amplitudes discussed in this report directly apply to gluon amplitudes. However, QCD does not only involve gluons, but also quarks. Although we did not discuss fermions in detail in this review, many of the methods described in this report extend to fermions.

A second important topic not covered at all in this report are loop amplitudes. These are needed for precision calculations in particle physics. Again, many of the techniques discussed in this review have an extension towards loop level. This is a wide field of research and it is impossible to give a complete list of references. Therefore we limit us here to a few suggestions for further reading related to multiple polylogarithms and recent elliptic generalisations.

There are a few mathematical topics, which we did not discuss or only mentioned briefly. The first topic is centred around supersymmetric extensions of Yang-Mills theory, and $N = 4$ supersymmetric Yang-Mills theory in particular. One may argue that $N = 4$ supersymmetric Yang-Mills theory is simpler than normal (non-supersymmetric) Yang-Mills theory.

We had no time to discuss Yangian symmetries or cluster algebras.

The last section on the relation between gauge theories and gravity merely touched the tip of an iceberg, there is certainly more which can be said and even more waits to be revealed.
It is the hope that this report covers the foundations on which the readers may start their own research, interesting research directions are not missing.

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A Solutions to the exercises

Exercise 1: Derive eq. (16) from eq. (15).

Solution: Let $\gamma : [a, b] \to M$ be a curve in $M$ with $\gamma(0) = x$. A tangent vector at $x$ is given by

$$X = \frac{d}{dt} \gamma(t) \bigg|_{t=0}$$

(696)

In order to keep the notation simple we will suppress maps between a manifold and an appropriate coordinate chart. We have to show

$$A_2(X) = \left( UA_1 U^\dagger + U dU^\dagger \right)(X),$$

(697)

where $A_1$ and $A_2$ are defined by

$$A_1 = \sigma_1^* \omega, \quad A_2 = \sigma_2^* \omega,$$

(698)

and the sections $\sigma_1$ and $\sigma_2$ are related by

$$\sigma_2 = \sigma_1 U^\dagger.$$

(699)

Let us choose a local trivialisation $(x, g)$ of $P(M, G)$ and work out $\sigma_2^* X$. With $U_0 = U(\gamma(0))$ we have

$$\sigma_2^* X = \sigma_2^* \left( \frac{d}{dt} \gamma(t) \bigg|_{t=0} \right) = \frac{d}{dt} (\gamma(t), \sigma_2(\gamma(t))) \bigg|_{t=0} = \frac{d}{dt} (\gamma(t), \sigma_1(\gamma(t)) U(\gamma(t))^\dagger) \bigg|_{t=0}$$

$$= \left( X, \frac{d}{dt} \sigma_1(\gamma(t)) \bigg|_{t=0} U_0^\dagger + \sigma_1(\gamma(0)) \frac{d}{dt} U(\gamma(t))^\dagger \bigg|_{t=0} \right)$$

$$= R_{U_0^\dagger}(\sigma_1^* X) + \left( 0, \sigma_2(\gamma(0)) U_0 \frac{d}{dt} U(\gamma(t))^\dagger \bigg|_{t=0} \right).$$

(700)

We then have, using eq. (10) and eq. (12),

$$A_2(X) = \omega(\sigma_2^* X) = \omega_{(x, \sigma_2(\gamma(0)))} \left( R_{U_0^\dagger}(\sigma_1^* X) \right) + \omega_{(x, \sigma_1(\gamma(0)))} \left( U_0 \frac{d}{dt} U(\gamma(t))^\dagger \bigg|_{t=0} \right)$$

$$= U_0 \left( \omega_{(x, \sigma_1(\gamma(0)))} (\sigma_1^* X) \right) U_0^\dagger + (U_0 dU^\dagger)(X)$$

$$= \left( UA_1 U^\dagger + U dU^\dagger \right)(X).$$

(701)

Exercise 2: Under a gauge transformation the pull-back $A$ of the connection one-form transforms as

$$A \to UAU^\dagger + U dU^\dagger.$$

(702)

Show that the curvature two-form transforms as

$$F \to UFU^\dagger.$$

(703)
Solution: $F$ is given in terms of $A$ by

$$F = dA + A \wedge A.$$  \hspace{1cm} (704)

Let

$$A' = UAU^\dagger + UdU^\dagger$$  \hspace{1cm} (705)

be the gauge-transformed gauge field. We have

$$F' = dA' + A' \wedge A'$$

$$= d(UAU^\dagger + UdU^\dagger) + (UAU^\dagger + UdU^\dagger) \wedge (UAU^\dagger + UdU^\dagger)$$

$$= (dU) \wedge AU^\dagger + U(dA)U^\dagger - UA \wedge dU^\dagger + (dU) \wedge (dU^\dagger) + UA \wedge AU^\dagger + UA \wedge dU^\dagger$$

$$+ U(dU^\dagger) \wedge UAU^\dagger + U(dU^\dagger) \wedge UdU^\dagger.$$  \hspace{1cm} (706)

From $UU^\dagger = 1$ we have $d(UU^\dagger) = 0$ and hence

$$UdU^\dagger = -(dU)U^\dagger.$$  \hspace{1cm} (707)

Using this relation we obtain

$$F' = U(dA)U^\dagger + UA \wedge AU^\dagger = UFU^\dagger.$$  \hspace{1cm} (708)

Exercise 3: Let $\bar{\alpha} = (\alpha_1, ..., \alpha_n)$ be a $n$-dimensional vector and let

$$g_i = g_i(\alpha_1, ..., \alpha_n), \ i = 1, ..., n,$$  \hspace{1cm} (709)

be $n$ functions of the $n$ variables $\alpha_j$. Show that

$$\int \left( \prod_{j=1}^{n} d\alpha_j \right) \left( \prod_{i=1}^{n} \delta(g_i(\alpha_1, ..., \alpha_n)) \right) \det \left( \frac{\partial g_i}{\partial \alpha_j} \right) = 1.$$  \hspace{1cm} (710)

Solution: We change the variables from $\alpha_j$ to

$$\beta_i = g_i(\alpha_1, ..., \alpha_n).$$  \hspace{1cm} (711)

Then

$$\int \left( \prod_{j=1}^{n} d\alpha_j \right) \left( \prod_{i=1}^{n} \delta(g_i(\alpha_1, ..., \alpha_n)) \right) \det \left( \frac{\partial g_i}{\partial \alpha_j} \right) = \int \left( \prod_{j=1}^{n} d\beta_j \right) \left( \prod_{i=1}^{n} \delta(\beta_i) \right) = 1.$$  \hspace{1cm} (712)

Exercise 4: Derive eq. (68) from eq. (66) and eq. (67).

Solution: With

$$p^{\mu \nu ab}(x) = \partial_\rho \partial^\rho g^{\mu \nu} \delta^{ab} - \left( 1 - \frac{1}{\xi} \right) \partial^\mu \partial^\nu \delta^{ab}$$  \hspace{1cm} (713)
and

\[(P^{-1})_{\mu \nu}^{ab}(x) = \int \frac{d^4 p}{(2\pi)^4} e^{-ip \cdot x} (\bar{P}^{-1})_{\mu \nu}^{ab}(p).\] (714)

we have

\[p_{\sigma}^{ac}(x) (P^{-1})_{\sigma \nu}^{cb}(x - y) = \int \frac{d^4 p}{(2\pi)^4} e^{-ip \cdot (x - y)} p^2 \left[ -g_{\mu \sigma} + \left( 1 - \frac{1}{\xi} \right) \frac{p^\mu p^\sigma}{p^2} \right] \delta^{ac} (\bar{P}^{-1})_{\sigma \nu}^{cb}(p).\]

This should be equal to

\[g_{\mu \nu} \delta^{ab} \delta^4(x - y) = \int \frac{d^4 p}{(2\pi)^4} e^{-ip \cdot (x - y)} g_{\mu \nu} \delta^{ab}.\] (715)

We have for

\[M^{\mu \nu} = -g_{\mu \nu} + \left( 1 - \frac{1}{\xi} \right) \frac{p^\mu p^\nu}{p^2} \quad \text{and} \quad N_{\mu \nu} = -g_{\mu \nu} + \left( 1 - \frac{1}{\xi} \right) \frac{p^\mu p^\nu}{p^2}\] (716)

the relation

\[M^{\mu \sigma} N_{\sigma \nu} = g_{\nu}^\nu.\] (717)

Therefore

\[\mu, a, \ ... \ ... \ b, \ v, b = \frac{i}{p^2} \left( -g_{\mu \nu} + \left( 1 - \frac{1}{\xi} \right) \frac{p_{\mu} p_{\nu}}{p^2} \right) \delta^{ab}.\] (718)

**Exercise 5:** Compute the amplitude \(A_4^{(0)}\) from the four diagrams shown in fig. 7. Assume that all momenta are outgoing. Derive the Mandelstam relation

\[s + t + u = 0.\] (719)

**Solution:** Let us start with the Mandelstam relation. Using momentum conservation and the on-shell relations we have

\[0 = p_4^2 = (p_1 + p_2 + p_3)^2 = 2p_1 \cdot p_2 + 2p_2 \cdot p_3 + 2p_1 \cdot p_3 = (p_1 + p_2)^2 + (p_2 + p_3)^2 + (p_1 + p_3)^2 = s + t + u.\] (720)

We then turn to the computation of the amplitude. Let us first examine the colour factors. The first diagrams has a colour factor \(C_s = (if^{a_1a_2b})(if^{b_3a_4})\). The second diagram we may equally well draw with legs 1 and 4 exchanged. The colour factor is then given by \(C_t = (if^{a_2a_3b})(if^{b_1a_4}).\) (If we read off the colour factor directly from diagram 2, we find \((if^{a_2a_3b})(if^{b_1a_4}) = -(if^{a_2a_3b})(if^{b_1a_4}).\) The minus sign cancels with another minus sign from the kinematic part.) The third diagram has the colour factor \(C_u = (if^{a_3a_4b})(if^{b_2a_1}).\) The fourth diagram with the four-gluon vertex gives three terms, one contributing to each colour structure. We may therefore write the amplitude as

\[A_4^{(0)} = ig^2 \left[ \frac{(if^{a_1a_2b})(if^{b_3a_4})}{s} N_s + \frac{(if^{a_2a_3b})(if^{b_1a_4})}{t} N_t + \frac{(if^{a_3a_4b})(if^{b_2a_1})}{u} N_u \right].\] (721)
\[ N_s \text{ is given by} \]

\[ N_s = \left\{ \left[ g^{\mu_1 \mu_2} (p_1^\nu - p_2^\nu) + g^{\mu_2 \nu} (p_2^{\mu_1} - p_3^{\mu_1}) + g^{\nu_1} (p_3^{\mu_1} - p_1^{\mu_1}) \right] g_{\nu_2} g^{\nu_3 \nu_4} (p_3^\rho - p_4^\rho) + g^{\mu_4 \rho} (p_4^\mu - p_{12}^\mu) \\
+ g^{\nu_2} (p_{12}^{\mu_1} - p_3^{\mu_1}) \right\} 2p_1 \cdot p_2 (g^{\mu_1 \nu_1} g^{\mu_2 \nu_4} - g^{\mu_2 \nu_3} g^{\mu_1 \nu_4}) \right\} \varepsilon_{\mu_1} \varepsilon_{\mu_2} \varepsilon_{\mu_3} \varepsilon_{\mu_4}, \]

\[ (722) \]

where we used the notation \( p_{ij} = p_i + p_j \). Using momentum conservation and \( p_j \cdot \varepsilon_j = 0 \) we may simplify this expression to

\[ N_s = \left\{ \left[ g^{\mu_1 \mu_2} (p_1^\nu - p_2^\nu) + 2g^{\mu_2 \nu} p_2^{\mu_1} - 2g^{\nu_1} p_1^{\mu_1} \right] g_{\nu_2} \left[ g^{\nu_3 \nu_4} (p_3^\rho - p_4^\rho) + 2g^{\mu_4 \rho} p_4^{\mu_1} - 2g^{\mu_1 \rho} p_{12}^\mu \right] \right\} \varepsilon_{\mu_1} \varepsilon_{\mu_2} \varepsilon_{\mu_3} \varepsilon_{\mu_4}. \]

\[ (723) \]

The contraction of indices for long expressions is best done with the help of a computer algebra program. Here is a short FORM program, which performs the contractions in eq. (723):

* Example program for FORM

```
V p1,p2,p3,p4, e1,e2,e3,e4;
I mu1,mu2,mu3,mu4,nu,rho;

L Ns = ((d_(mu1,mu2)*(p1(nu)-p2(nu)) + 2*d_(mu2,nu)*p2(mu1) - 2*d_(nu,mu1)*p1(mu2))* \\
  d_(nu,rho)*) \\
  (d_(mu3,mu4)*(p3(rho)-p4(rho)) + 2*d_(mu4,rho)*p4(mu3) - 2*d_(rho,mu3)*p3(mu4)) \\
  +2*p1(nu)*p2(nu)*(d_(mu1,mu3)*d_(mu2,mu4)-d_(mu2,mu3)*d_(mu1,mu4)))* \\
  e1(mu1)*e2(mu2)*e3(mu3)*e4(mu4);
```

print;

.end

The same can be done in C++, using the GiNaC library:

```
#include <iostream>
#include <string>
#include <sstream>
#include <ginac/ginac.h>
using namespace std;
using namespace GiNaC;

string itos(int arg)
{
    ostringstream buffer;
    buffer << arg;
    return buffer.str();
}

int main()
{
    varidx mu1(symbol("mu1"),4), mu2(symbol("mu2"),4), \\
                mu3(symbol("mu3"),4), mu4(symbol("mu4"),4),
```
After a few additional simplifications one finds

\[
N_s = 4(p_1 \cdot e_2)(p_3 \cdot e_4)(e_1 \cdot e_3) - 4(p_1 \cdot e_2)(p_4 \cdot e_3)(e_1 \cdot e_4) + 4(p_2 \cdot e_1)(p_4 \cdot e_3)(e_2 \cdot e_4)
- 4(p_2 \cdot e_1)(p_3 \cdot e_4)(e_2 \cdot e_3) + 4[(p_1 \cdot e_3)(p_2 \cdot e_4) - (p_1 \cdot e_4)(p_2 \cdot e_3)](e_1 \cdot e_2)
+ 4[(p_3 \cdot e_1)(p_4 \cdot e_2) - (p_3 \cdot e_2)(p_4 \cdot e_1)](e_3 \cdot e_4) + 2(p_1 \cdot p_2)(e_1 \cdot e_3)(e_2 \cdot e_4)
- 2(p_1 \cdot p_2)(e_1 \cdot e_4)(e_2 \cdot e_3) - 2(p_2 \cdot p_3 - p_1 \cdot p_3)(e_1 \cdot e_2)(e_3 \cdot e_4).
\]

(724)

The numerator \(N_t\) is obtained from the numerator \(N_s\) by the substitution \((1, 2, 3) \rightarrow (2, 3, 1)\), the numerator \(N_u\) is obtained from the numerator \(N_s\) by the substitution \((1, 2, 3) \rightarrow (3, 1, 2)\).

In view of section (3.2), where we discuss colour kinematics duality let us add the following remark: The colour factors satisfy (obviously) the Jacobi identity

\[
C_s + C_t + C_u = 0.
\]

(725)
It is an easy exercise to check, that the numerators $N_s$, $N_t$ and $N_u$, as determined above, satisfy the Jacobi-like identity

$$N_s + N_t + N_u = 0.$$  \hfill (726)

**Exercise 6:** Compute the traces $\text{Tr} \left( T^a T^b T^b T^a \right)$ and $\text{Tr} \left( T^a T^b T^a T^b \right)$ for $U(N)$ and $\text{SU}(N)$.

**Solution:** Using the Fierz identities for $\text{SU}(N)$ one finds

$$\text{Tr} \left( T^a T^b T^b T^a \right) = \frac{1}{2} \left[ \text{Tr} \left( T^a T^a \right) \text{Tr} \left( 1 \right) - \frac{1}{N} \text{Tr} \left( T^a T^a \right) \right] = \frac{N^2 - 1}{2N} \text{Tr} \left( T^a T^a \right) = \frac{(N^2 - 1)^2}{4N},$$

$$\text{Tr} \left( T^a T^b T^a T^b \right) = -\frac{1}{2N} \text{Tr} \left( T^a T^a \right) = -\frac{N^2 - 1}{4N}.$$  \hfill (727)

For $U(N)$ one obtains

$$\text{Tr} \left( T^a T^b T^b T^a \right) = \frac{1}{2} \text{Tr} \left( T^a T^a \right) \text{Tr} \left( 1 \right) = \frac{N}{4} \text{Tr} \left( 1 \right) \text{Tr} \left( 1 \right) = \frac{N^3}{4},$$

$$\text{Tr} \left( T^a T^b T^a T^b \right) = \frac{1}{2} \text{Tr} \left( T^a \right) \text{Tr} \left( T^a \right) = \frac{1}{4} \text{Tr} \left( 1 \right) = \frac{N}{4}.$$  \hfill (728)

**Exercise 7:** Consider the path integral

$$\int \mathcal{D} \phi \exp i \int d^4 x \text{Tr} \left( \frac{1}{2} \phi P \phi + \phi K \right).$$  \hfill (729)

Assume that $P$ is a pseudo-differential operator of even degree and independent of the other fields. $K$ on the other hand may depend on some other fields. Show that under the substitution

$$\phi \rightarrow \phi + P^{-1} K,$$  \hfill (730)

where $P^{-1}$ denotes the inverse pseudo-differential operator, one obtains

$$\int \mathcal{D} \phi' \exp i \int d^4 x \text{Tr} \left( \frac{1}{2} \phi' P \phi' - \frac{1}{2} K P^{-1} K \right).$$  \hfill (731)

**Solution:** Using the shift $\phi = \phi' - P^{-1} K$ one obtains

$$\int \mathcal{D} \phi' \exp i \int d^4 x \text{Tr} \left( \frac{1}{2} \left( \phi' - P^{-1} K \right) P \left( \phi' - P^{-1} K \right) + \left( \phi' - P^{-1} K \right) K \right) =$$

$$= \int \mathcal{D} \phi' \exp i \int d^4 x \text{Tr} \left( \frac{1}{2} \phi' P \phi' - \frac{1}{2} \left( P^{-1} K \right) P \phi' + \frac{1}{2} \left( P^{-1} K \right) K \phi' + \frac{1}{2} \phi' K - \left( P^{-1} K \right) K \right).$$

Using partial integration for $P^{-1}$ one obtains the result

$$\int \mathcal{D} \phi' \exp i \int d^4 x \text{Tr} \left( \frac{1}{2} \phi' P \phi' - \frac{1}{2} K P^{-1} K \right).$$  \hfill (733)
**Exercise 8:** Show that at the level of cyclic-ordered Feynman rules we have

\[
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{feynman_rule.png}
\end{array}
\]

**Solution:** We calculate the left-hand side with the cyclic-ordered Feynman rules for the auxiliary tensor field \(B_{[pq]}\):

\[
\begin{align*}
1.\text{h.s.} & = \frac{i}{\sqrt{2}} (g^{\mu_1 \rho_1} g^{\mu_2 \sigma_1} - g^{\mu_1 \sigma_1} g^{\mu_2 \rho_1}) \left( -\frac{i}{2} \right) (g_{\rho_1 \sigma_2} g_{\sigma_1 \rho_2} - g_{\rho_1 \sigma_2} g_{\sigma_1 \rho_2}) \frac{i}{\sqrt{2}} (g^{\mu_1 \rho_2} g^{\mu_3 \sigma_2} - g^{\mu_1 \sigma_2} g^{\mu_3 \rho_2}) \\
& + \frac{i}{\sqrt{2}} (g^{\mu_1 \rho_1} g^{\mu_2 \sigma_1} - g^{\mu_1 \sigma_1} g^{\mu_2 \rho_1}) \left( -\frac{i}{2} \right) (g_{\rho_1 \sigma_2} g_{\sigma_1 \rho_2} - g_{\rho_1 \sigma_2} g_{\sigma_1 \rho_2}) \frac{i}{\sqrt{2}} (g^{\mu_1 \rho_1} g^{\mu_2 \sigma_1} - g^{\mu_1 \sigma_1} g^{\mu_2 \rho_1}) \\
& = i (g^{\mu_1 \mu_2} g^{\mu_3 \mu_4} - g^{\mu_1 \mu_4} g^{\mu_3 \mu_2}) + i (g^{\mu_2 \mu_1} g^{\mu_3 \mu_4} - g^{\mu_2 \mu_4} g^{\mu_3 \mu_1}) \\
& = i (2g^{\mu_1 \mu_2} g^{\mu_3 \mu_4} - g^{\mu_1 \mu_4} g^{\mu_3 \mu_2} - g^{\mu_1 \mu_2} g^{\mu_3 \mu_4}) \quad \text{(734)}
\end{align*}
\]

which is the cyclic-ordered Feynman rule for the four-gluon vertex.

**Exercise 9:** Show that \([x,x] = 0\) implies the anti-symmetry of the Lie bracket \([x,y] = -[y,x]\). Show further that also the converse is true, provided \(\text{char } F \neq 2\). Explain, why the argument does not work for \(\text{char } F = 2\).

**Solution:** Starting from \([x,x] = 0\) for all \(x\) we have

\[
0 = [x+y,x+y] = [x,y] + [y,x] \quad \text{(735)}
\]

and therefore \([x,y] = -[y,x]\). Now let us consider the other direction. Assuming \([x,y] = -[y,x]\) we have for \(x = y\) the relation \([x,x] = -[x,x]\) or equivalently

\[
2 [x,x] = 0. \quad \text{(736)}
\]

For \(\text{char } F \neq 2\) it follows that \([x,x] = 0\). For \(\text{char } F = 2\) we have \(2 = 0 \mod 2\) and eq. (736) does not give any constraint on \([x,x]\).

**Exercise 10:** The helicity operator \(h\) for particles is defined by

\[
h = \frac{\vec{p} \cdot \vec{S}}{|\vec{p}|}. \quad \text{(737)}
\]

Assume that \(p^\mu\) is a real four-vector with positive energy \((p^0 > 0)\). Show that the spinors \(|p^+\rangle = p_A\) and \(\langle p^+| = p_A\) have helicity \(h = +1/2\), while the spinors \(|p^-\rangle = p_A^\dagger\) and \(\langle p^-| = p_A^\dagger\) have helicity \(h = -1/2\).
**Solution:** In the Weyl representation we have

\[
h = \frac{1}{2|\vec{p}|} \begin{pmatrix} \vec{p} \cdot \vec{\sigma} & 0 \\ 0 & \vec{p} \cdot \vec{\sigma} \end{pmatrix}.
\]

We further have

\[
p_\mu \sigma_\mu = p^0 \cdot 1 - \vec{p} \cdot \vec{\sigma}, \quad p_\mu \sigma_\mu = p^0 \cdot 1 + \vec{p} \cdot \vec{\sigma}.
\]

The Weyl spinors \(|p^\pm\rangle\) satisfy

\[
p_\mu \sigma_\mu |p^+\rangle = 0, \quad p_\mu \sigma_\mu |p^-\rangle = 0,
\]

and hence

\[
h |p^+\rangle = \frac{1}{2} \frac{p^0}{|\vec{p}|} |p^+\rangle, \quad h |p^-\rangle = -\frac{1}{2} \frac{p^0}{|\vec{p}|} |p^-\rangle.
\]

For \(p^0 > 0\) we thus have

\[
h |p^+\rangle = \frac{1}{2} |p^+\rangle, \quad h |p^-\rangle = -\frac{1}{2} |p^-\rangle.
\]

For the bra-spinors it follows from

\[
\langle p^+| p_\mu \sigma_\mu = 0, \quad \langle p^-| p_\mu \sigma_\mu = 0,
\]

that

\[
\langle p^+| h = \frac{1}{2} \frac{p^0}{|\vec{p}|} \langle p^+|, \quad \langle p^-| h = -\frac{1}{2} \frac{p^0}{|\vec{p}|} \langle p^-|,
\]

and for \(p^0 > 0\)

\[
\langle p^+| h = \frac{1}{2} \langle p^+|, \quad \langle p^-| h = -\frac{1}{2} \langle p^-|.
\]

**Exercise 11:** The field strength in spinor notation: For a rank-2 tensor \(F_{\mu\nu}\) we define the spinor representation by

\[
F_{A\bar{A}B\bar{B}} = F_{\mu\nu} \sigma^{\mu}_A \sigma^{\nu}_{B\bar{B}}.
\]

On the other hand we may decompose \(F_{\mu\nu}\) into a self dual part and an anti-self dual part,

\[
F_{\mu\nu} = F^\text{self dual}_{\mu\nu} + F^\text{anti-self dual}_{\mu\nu},
\]

with

\[
F^\text{self dual}_{\mu\nu} = \frac{1}{2} (F_{\mu\nu} + i\tilde{F}_{\mu\nu}), \quad F^\text{anti-self dual}_{\mu\nu} = \frac{1}{2} (F_{\mu\nu} - i\tilde{F}_{\mu\nu}).
\]
The dual field strength $\tilde{F}_{\mu\nu}$ was defined in eq. (29). Show that the spinor representations of the self dual part / anti-self dual part may be written as

\[
F^{\text{self dual}}_{AB\dot{A}\dot{B}} = \varepsilon_{AB} \bar{\phi}_{AB}, \quad F^{\text{anti-self dual}}_{AB\dot{A}\dot{B}} = \phi_{AB} \bar{\phi}_{AB},
\]

(749)

where $\phi_{AB}$ and $\bar{\phi}_{AB}$ satisfy

\[
\phi_{AB} = \phi_{BA}, \quad \bar{\phi}_{AB} = \bar{\phi}_{BA}.
\]

(750)

**Solution:** From the anti-symmetry $F_{\mu\nu} = -F_{\nu\mu}$ it follows

\[
F_{AB\dot{A}\dot{B}} = -F_{BA\dot{B}\dot{A}}.
\]

(751)

We can therefore write

\[
F_{AB\dot{A}\dot{B}} = \frac{1}{2} (F_{AB\dot{A}\dot{B}} - F_{AB\dot{B}\dot{A}}) + \frac{1}{2} (F_{AB\dot{B}\dot{A}} - F_{AB\dot{A}\dot{B}}).
\]

(752)

We set

\[
F_{AB[\dot{A}\dot{B}]} = \frac{1}{2} (F_{AB\dot{A}\dot{B}} - F_{BA\dot{B}\dot{A}}), \quad F_{[A,B]\dot{A}\dot{B}} = \frac{1}{2} (F_{AB\dot{B}\dot{A}} - F_{AB\dot{A}\dot{B}}).
\]

(753)

The Schouten identity

\[
\varepsilon_{AB} \varepsilon_{CD} + \varepsilon_{BC} \varepsilon_{AD} + \varepsilon_{CA} \varepsilon_{BD} = 0.
\]

(754)

can also be written as

\[
\varepsilon_{AB} \varepsilon_{CD} = \varepsilon_{A}^{\ C} \varepsilon_{B}^{\ D} - \varepsilon_{A}^{\ D} \varepsilon_{B}^{\ C}.
\]

(755)

Contraction with $\chi_{CD}$ yields

\[
\varepsilon_{AB} \chi_{C} = \chi_{AB} - \chi_{BA}.
\]

(756)

Therefore we have

\[
F_{AB[\dot{A}\dot{B}]} = \frac{1}{2} \varepsilon_{AB} F_{AB\dot{C}}, \quad F_{[A,B]\dot{A}\dot{B}} = \frac{1}{2} \varepsilon_{AB} F_{\dot{C}BA}.
\]

(757)

We set

\[
\phi_{AB} = \frac{1}{2} F_{AB\dot{C}}, \quad \bar{\phi}_{AB} = \frac{1}{2} F_{\dot{C}BA}.
\]

(758)

Note that we have

\[
F_{[A,B]\dot{A}\dot{B}} = F_{[A,B]BA}, \quad F_{AB[\dot{A}\dot{B}]} = F_{BA[\dot{A}\dot{B}]},
\]

(759)

and therefore

\[
\phi_{AB} = \phi_{BA}, \quad \bar{\phi}_{AB} = \bar{\phi}_{BA}.
\]

(760)
We thus arrive at

$$F_{ABAB} = \phi_{AB} \varepsilon_{AB} + \varepsilon_{AB} \bar{\phi}_{AB}. \quad (761)$$

Let us now consider the dual field strength

$$\tilde{F}_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} \quad (762)$$

in the spinor representation

$$\tilde{F}_{ABAB} = \tilde{F}_{\mu\nu} \sigma^\mu_{AA} \sigma^\nu_{BB}. \quad (763)$$

Using $2g^{\mu\nu} = \sigma^\mu_{AB} \sigma^\nu_{BA}$ we have

$$F^{\rho\sigma} = \frac{1}{4} \sigma^{CC}_{\rho} \sigma^{DD}_{\sigma} F_{CDCD},$$

$$\tilde{F}_{ABAB} = \frac{1}{8} \sigma^\mu_{AA} \sigma^\nu_{BB} \sigma^{CC}_{\rho} \sigma^{DD}_{\sigma} \varepsilon_{\mu\nu\rho\sigma} F_{CDCD}. \quad (764)$$

Using in addition

$$4i\varepsilon_{\mu\nu\rho\sigma} = \begin{pmatrix} P_+ \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma \end{pmatrix} - \begin{pmatrix} P_- \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma \end{pmatrix},$$

$$\varepsilon_{\mu\nu\rho\sigma} = -\frac{i}{4} \begin{pmatrix} \sigma_{\mu\nu} \sigma^{FG}_v \sigma_{\rho\sigma} \sigma^{HE}_{\mu} - \sigma_{\mu\nu} \sigma_{\rho\sigma} \sigma^{FG}_{\mu} \sigma^{HE}_{\nu} \end{pmatrix}, \quad (765)$$

and the Fierz identities one finds

$$\tilde{F}_{ABAB} = \frac{i}{2} \left( \delta^D_A \delta^C_B \varepsilon_{AB} \varepsilon^{CD} - \varepsilon_{AB} \varepsilon^{CD} \delta^D_A \delta^C_B \right) F_{CDCD}. \quad (766)$$

Finally one obtains

$$\tilde{F}_{ABAB} = i \left( \phi_{AB} \varepsilon_{AB} - \varepsilon_{AB} \bar{\phi}_{AB} \right). \quad (767)$$

Therefore $\varepsilon_{AB} \bar{\phi}_{AB}$ is self dual, while $\phi_{AB} \varepsilon_{AB}$ is anti-self dual.

**Exercise 12:** Calculate the primitive helicity amplitude $A^{(0)}_4(1^-, 2^+, 3^+, 4^-)$.

**Solution:** As reference momenta for the polarisation vectors we choose

$$q_1 = p_2, \quad q_2 = p_1, \quad q_3 = p_1, \quad q_4 = p_2. \quad (768)$$

The polarisation vectors are then

$$\varepsilon^\mu_1 = \frac{1 - |\gamma| 2^-}{\sqrt{2} [12]}, \quad \varepsilon^\mu_2 = \frac{2 + |\gamma| 1^+}{\sqrt{2} [21]}, \quad \varepsilon^\mu_3 = \frac{3 + |\gamma| 1^+}{\sqrt{2} [31]}, \quad \varepsilon^\mu_4 = \frac{4 - |\gamma| 2^-}{\sqrt{2} [42]} \quad (769)$$

From the Fierz identity we have

$$\varepsilon_1 \cdot \varepsilon_2 = \varepsilon_1 \cdot \varepsilon_3 = \varepsilon_1 \cdot \varepsilon_4 = \varepsilon_2 \cdot \varepsilon_3 = \varepsilon_2 \cdot \varepsilon_4 = 0, \quad (770)$$
and only $\varepsilon_3 \cdot \varepsilon_4$ is non-zero. We have to calculate the three diagrams shown in fig. (2). In the diagram with the four-gluon vertex all polarisation vectors are contracted with each other. In particular $\varepsilon_1$ is contracted into some other polarisation vector, giving zero. Therefore the four-gluon diagrams does not contribute. The s-channel diagram gives

$$D_s = \frac{ig}{s} \varepsilon_{1\mu_1} \varepsilon_{2\mu_2} \varepsilon_{3\mu_3} \varepsilon_{4\mu_4} \left[ g^{\mu_1\mu_2} \left( p_1^\rho - p_2^\rho \right) + g^{\mu_3\rho} \left( p_2^{\mu_1} - p_3^{\mu_1} \right) + g^{\mu_4\rho} \left( p_3^{\mu_2} - p_4^{\mu_2} \right) \right]$$

$$\times \left[ g^{\mu_1\nu_1} \left( p_3^\sigma - p_4^\sigma \right) + g^{\mu_3\nu_3} \left( p_4^{\mu_1} - p_1^{\mu_1} \right) + g^{\mu_4\nu_4} \left( p_1^{\mu_2} - p_2^{\mu_2} \right) \right]$$

$$= \frac{i}{s} \varepsilon_{1\mu_1} \varepsilon_{2\mu_2} \left( \varepsilon_3 \cdot \varepsilon_4 \right) \left[ \left( p_2^{\mu_1} - p_3^{\mu_1} \right) \left( p_3^{\mu_2} - p_4^{\mu_2} \right) + \left( p_3^{\mu_1} - p_4^{\mu_1} \right) \left( p_4^{\mu_2} - p_3^{\mu_2} \right) \right]. \quad (771)$$

Now by momentum conservation and our choice of reference momenta

$$\varepsilon_{1\mu_1} \left( p_2^{\mu_1} - p_3^{\mu_1} \right) = 2 \varepsilon_1 \cdot p_2 = 0,$$

$$\varepsilon_{2\mu_2} \left( p_3^{\mu_2} - p_4^{\mu_2} \right) = -2 \varepsilon_2 \cdot p_1 = 0,$$  

and therefore this diagram does not contribute either. It remains to compute the t-channel diagram

$$D_t = \frac{ig}{t} \varepsilon_{1\mu_1} \varepsilon_{2\mu_2} \varepsilon_{3\mu_3} \varepsilon_{4\mu_4} \left[ g^{\mu_1\nu_1} \left( p_2^\sigma - p_3^\sigma \right) + g^{\mu_3\nu_3} \left( p_4^{\mu_1} - p_1^{\mu_1} \right) + g^{\mu_4\nu_4} \left( p_1^{\mu_2} - p_2^{\mu_2} \right) \right]$$

$$\times \left[ g^{\mu_1\nu_1} \left( p_3^\sigma - p_4^\sigma \right) + g^{\mu_3\nu_3} \left( p_4^{\mu_1} - p_1^{\mu_1} \right) + g^{\mu_4\nu_4} \left( p_1^{\mu_2} - p_2^{\mu_2} \right) \right]$$

$$= \frac{i}{t} \varepsilon_{1\mu_1} \varepsilon_{2\mu_2} \left( \varepsilon_3 \cdot \varepsilon_4 \right) \left( p_2^{\mu_1} - p_3^{\mu_1} \right) \left( p_3^{\mu_2} - p_4^{\mu_2} \right)$$

$$= -4 \frac{i}{t} \left( \varepsilon_1 \cdot p_4 \right) \left( \varepsilon_2 \cdot p_3 \right) \left( \varepsilon_3 \cdot \varepsilon_4 \right). \quad (773)$$

This is the only non-vanishing contribution and we obtain

$$A_4^{(0)} \left( 1^-, 2^+, 3^+, 4^- \right) = -\frac{2i}{t} \left( \varepsilon_1 \cdot p_4 \right) \left( \varepsilon_2 \cdot p_3 \right) \left( \varepsilon_3 \cdot \varepsilon_4 \right)$$

$$= -\frac{4i}{(23) [32] (13)} \langle 24 \rangle \langle 41 \rangle \langle 13 \rangle \langle 32 \rangle \langle 21 \rangle \langle 12 \rangle \langle 13 \rangle \langle 24 \rangle$$

$$= 4i \frac{(14)^2 [23]}{\langle 12 \rangle \langle 23 \rangle [21]}. \quad (774)$$

We may still simplify this expression a little bit. Using $[23] \langle 34 \rangle = -\langle 21 \rangle \langle 14 \rangle$ one arrives at the Parke-Taylor formula for $n = 4$:

$$A_4^{(0)} \left( 1^-, 2^+, 3^+, 4^- \right) = 4i \frac{(14)^4 \langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle}. \quad (775)$$

**Exercise 13:** Show that the spinors defined in eq. (225) and in eq. (226) satisfy the Dirac equations of eq. (169), the orthogonality relations of eq. (170) and the completeness relations of eq. (171).

**Solution:** We start with the Dirac equation. For the spinor $u(p,+)$ we have

$$(p' - m) u(p,+) = \frac{1}{[p' q]} (p' - m) (p' + m) \langle q^- \rangle = \frac{1}{[p' q]} (p^2 - m^2) \langle q^- \rangle = 0, \quad (776)$$
due to \( p^2 = m^2 \). The argument is similar for all other spinors. Let us now look at the orthogonality relations. We have

\[
\bar{u}(p,+)u(p,+) = \frac{1}{\langle q p^\nu \rangle \langle p^\nu q \rangle} \langle q - |(\not\! p + m)(\not\! p + m)|q- \rangle = \frac{2m}{\langle q p^\nu \rangle \langle p^\nu q \rangle} \langle q - |\not\! p |q- \rangle
\]

\[
= 2m \frac{2p \cdot q}{2p \cdot q} = 2m,
\]

\[
\bar{u}(p,+)u(p,-) = \frac{1}{\langle q p^\nu \rangle \langle p^\nu q \rangle} \langle q - |(\not\! p + m)(\not\! p + m)|q+ \rangle = - \frac{2m^2}{\langle q p^\nu \rangle \langle p^\nu q \rangle} \langle q q \rangle = 0. \quad (777)
\]

The argumentation for the other combinations is similar. Finally, let us consider the completeness relation:

\[
\sum_{\lambda} u(p,\lambda)\bar{u}(p,\lambda) = u(p,+)\bar{u}(p,+) + u(p,-)\bar{u}(p,-)
\]

\[
= \frac{1}{\langle p^\nu q \rangle \langle q p^\nu \rangle} \langle (\not\! p + m)|q- \rangle \langle q - |(\not\! p + m)(\not\! p + m)|q+ \rangle \langle q + |(\not\! p + m) \rangle
\]

\[
= \frac{1}{2p \cdot q} (\not\! p + m)q (\not\! p + m) = \frac{\not\! p \cdot q + m(\not\! p \cdot q + q \not\! p) + m^2 q}{2p \cdot q}. \quad (778)
\]

With

\[
\not\! p \cdot q = (2p \cdot q) \not\! p - p^2 q \quad \text{and} \quad \not\! p \cdot q + q \not\! p = 2p \cdot q
\]

we obtain

\[
\sum_{\lambda} u(p,\lambda)\bar{u}(p,\lambda) = \not\! p + m. \quad (780)
\]

The proof of the completeness relation for the v-spinors is obtained by the substitution \( m \rightarrow -m \).

**Exercise 14:** Show that the momenta defined by eq. \( \text{(262)} \) satisfy momentum conservation.

**Solution:** We have

\[
\sum_{i=1}^{n} p_i^\mu \quad = \quad - \frac{1}{2} \left( \frac{[p_{i+1}p_i]}{[p_i+1][p_i]} \langle p_i + |\gamma^\mu|\mu_{i+1} \rangle + \frac{[p_i-1p_{i+1}]}{[p_i+1][p_i]} \langle p_i + |\gamma^\mu|\mu_i \rangle + \frac{[p_{i+1}p_i]}{[p_i+1][p_i]} \langle p_i + |\gamma^\mu|\mu_{i+1} \rangle \right), \quad (781)
\]

where all indices are understood mod \( n \). Let’s reorder the sum:

\[
\sum_{i=1}^{n} p_i^\mu \quad = \quad - \frac{1}{2} \left( \frac{[p_{i+2}p_{i+1}]}{[p_{i+2}p_{i+1}]p_{i+1}} \langle p_{i+1} + |\gamma^\mu|\mu_{i+2} \rangle + \frac{[p_{i+1}p_i]}{[p_{i+1}][p_{i+1}]} \langle p_i + |\gamma^\mu|\mu_i \rangle + \frac{[p_{i-1}p_{i-2}p_{i-1}]}{[p_{i-1}p_{i-2}]} \langle p_{i-1} + |\gamma^\mu|\mu_{i+1} \rangle \right). \quad (782)
\]

Now let’s use the Schouten identity for the second term,

\[
[p_{i-1}p_i] (p_i+) = - [p_{i+1}p_i] (p_{i-1}+) - [p_{i+1}p_{i-1}] (p_{i+1}+), \quad (783)
\]

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and the result follows:

\[ \sum_{i=1}^{n} p_i^\mu = 0. \] (784)

**Exercise 15:** Count the number of Feynman diagrams contributing to the cyclic-ordered amplitude \( A_n^{(0)} \).

**Solution:** Let us denote by \( f(n) \) the number of Feynman diagrams contributing to the \( A_n^{(0)} \). For \( f(n) \) we have the recurrence relation

\[ f(n) = \sum_{k=1}^{j-1} f(k)f(n-k) + \sum_{k=1}^{j-2} \sum_{l=k+1}^{j-1} f(k)f(l-k)f(n-l). \] (785)

Compare this equation with eq. (292). \( f(n) \) counts the number of Feynman diagrams contributing to the off-shell current \( J_\mu \) with \( n \) on-shell legs and one off-shell leg. We start with \( f(1) = 1 \). The first few values are easily computed

\[
\begin{align*}
  f(2) &= f(1)f(1) = 1, \\
  f(3) &= f(1)f(2) + f(2)f(1) + f(1)f(1)f(1) = 3, \\
  f(4) &= f(1)f(3) + f(2)f(2) + f(3)f(1) + f(1)f(1)f(1)f(1) + f(1)f(2)f(1)f(1) + f(2)f(1)f(1) \\
        &= 3 + 1 + 3 + 1 + 1 + 1 = 10. \\
\end{align*}
\] (786)

At each step we have to consider all ordered partitions of \( n \) into two or three positive integers. For \( n = 5 \) we have the ordered partitions

\[ (1,4), (2,3), (3,2), (4,1), (1,1,3), (1,3,1), (3,1,1), (1,2,2), (2,1,2), (2,2,1), \] (787)

giving us

\[ f(5) = 10 + 3 + 3 + 10 + 3 + 3 + 1 + 1 + 1 = 38. \] (788)

For \( n = 6 \) we have the partitions into two numbers

\[ (1,5), (2,4), (3,3), (4,2), (5,1) \] (789)

contributing

\[ 38 + 10 + 9 + 10 + 38 = 105 \] (790)

diagrams. For the partitions into three numbers we have the three permutations of \( (1,1,4) \), six permutations of \( (1,2,3) \) and one permutation of \( (2,2,2) \), giving

\[ 3 \cdot 10 + 6 \cdot 3 + 1 \cdot 1 = 49. \] (791)

In total we obtain \( f(5) = 105 + 49 = 154 \) diagrams.

**Exercise 16:** Prove eq. (310) by induction.
Solution: We start with eq. (310). For \( n = 1 \) we have

\[
J_\mu (p_1^+) = \frac{\langle q - \gamma_\mu 1 \rangle}{\sqrt{2} \langle q 1 \rangle} = \frac{\langle q - \gamma_\mu 1 q^+ \rangle}{\sqrt{2} \langle q 1 \rangle \langle 1 q \rangle},
\]

which agrees with eq. (310). Let us now assume that eq. (310) is correct for off-shell currents with less than \( n \) on-shell legs. The contraction of two sub-currents with less than \( n \) on-shell legs due to the common choice \( q \) as reference momentum. Therefore the four-gluon vertex in the recurrence relation does not give a contribution and the recurrence relation reduces to

\[
J_\mu (p_1^+, \ldots, p_n^+) = \frac{g_{\mu \nu}}{p_{1,n}^2} \sum_{j=1}^{n-1} \left( 2 g_{\nu \mu 1} p_{j+1,n}^\mu - 2 g_{\nu \mu 1} p_{j,j}^\mu \right) J_{\mu_1} (p_1^+, \ldots, p_j^+) J_{\mu_2} (p_{j+1}^+, \ldots, p_n^+). \tag{793}
\]

Using the induction hypothesis one obtains

\[
J_\mu (p_1^+, \ldots, p_n^+) = \left( \sqrt{2} \right)^{n-2} \frac{1}{\langle q 1 \rangle \langle q 2 \rangle \ldots (n-1) \langle q n \rangle} \frac{1}{p_{1,n}^2} \sum_{j=1}^{n-1} \frac{\langle j (j+1) \rangle}{\langle jq \rangle \langle q (j+1) \rangle} \langle q - \gamma_{j+1,n} \gamma_{1,j} q^+ \rangle - \langle q - \gamma_{j+1,n} \gamma_{1,j} q^+ \rangle \langle q - \gamma_{j+1,n} \gamma_{1,j} q^+ \rangle.
\]

\[
\left( \sqrt{2} \right)^{n-2} \frac{1}{\langle q 1 \rangle \langle q 2 \rangle \ldots (n-1) \langle q n \rangle} \frac{1}{p_{1,n}^2} \sum_{j=1}^{n-1} \frac{\langle j (j+1) \rangle}{\langle jq \rangle \langle q (j+1) \rangle} \langle q - \gamma_{j+1,n} \gamma_{1,j} q^+ \rangle.
\]

It remains to show that

\[
\frac{1}{p_{1,n}^2} \sum_{j=1}^{n-1} \frac{\langle j (j+1) \rangle}{\langle jq \rangle \langle q (j+1) \rangle} \langle q - \gamma_{j+1,n} \gamma_{1,j} q^+ \rangle = 1.
\tag{795}
\]

In order to prove eq. (795) we first show that

\[
\sum_{j=k}^{l-1} \frac{\langle j (j+1) \rangle}{\langle jq \rangle \langle q (j+1) \rangle} = \frac{\langle kl \rangle}{\langle kq \rangle \langle ql \rangle}. \tag{796}
\]

The eikonal identity in eq. (796) is trivially true for \( l - k = 1 \). Assuming that the identity is true for sums with \( (l - k - 1) \) summands, we show that it is also true for \( (l - k) \) summands:

\[
\sum_{j=k}^{l-2} \frac{\langle j (j+1) \rangle}{\langle jq \rangle \langle q (j+1) \rangle} + \frac{\langle (l-1) \rangle}{\langlejq \rangle \langle q (j+1) \rangle} = \frac{\langle kl \rangle}{\langle kq \rangle \langle ql \rangle}.
\]

The last step used the Schouten identity. Let us now consider

\[
\sum_{j=1}^{n-1} \frac{\langle j (j+1) \rangle}{\langle jq \rangle \langle q (j+1) \rangle} \langle q - \gamma_{j+1,n} \rangle = \sum_{j=1}^{n-1} \sum_{k=1}^{j} \frac{\langle j (j+1) \rangle}{\langle jq \rangle \langle q (j+1) \rangle} \langle q - \gamma_{k} \rangle = \sum_{k=1}^{n} \sum_{j=k}^{n} \frac{\langle j (j+1) \rangle}{\langle jq \rangle \langle q (j+1) \rangle} \langle q - \gamma_{k} \rangle = \sum_{k=2}^{n} \frac{\langle 1k \rangle}{\langle 1q \rangle} \langle q - \gamma_{k} \rangle = \frac{1 - \gamma_{1,n} \gamma_{1,q}}{\langle 1q \rangle}.
\tag{798}
\]
We may now prove eq. (795):

\[
\frac{1}{p_{1,n}^2} \sum_{j=1}^{n-1} \frac{\langle j \vert (j+1) \rangle}{\langle jq \vert (j+1) \rangle} (q - \vert H_{j+1,n}^1 \rangle \langle q^+ \rangle) = \frac{1}{p_{1,n}^2} \sum_{j=1}^{n-1} \frac{\langle j \vert (j+1) \rangle}{\langle jq \vert (j+1) \rangle} (q - \vert H_{j+1,n}^1 \rangle \langle q^+ \rangle) = \frac{p_{1,n}^2}{p_{1,n}^2} \frac{\langle 1 \vert q^+ \rangle}{\langle 1q \rangle} = 1.
\]

(799)

Exercise 17: Derive eq. (336) together with eq. (337) from eq. (335).

Solution: We would like to solve the integro-differential equation, given in eq. (335):

\[
\omega A^a_\perp (\vec{x}) - g f^{abc} (\zeta A^b_\perp (\vec{x})) A^c_\perp (\vec{x}) = \int d^3y \frac{\delta A^a_\perp (\vec{x})}{\delta A^b_\perp (\vec{y})} \omega_A^b (\vec{y}).
\]

(800)

We make the ansatz

\[
A^a_\perp (\vec{x}) = \tilde{A}^a_\perp (\vec{x}) + \sum_{n=2}^{\infty} 2 \text{Tr} (T^a T^{a_1} ... T^{a_n}) \int d^3x_1 ... d^3x_n \mathcal{Y} (\vec{x}, \vec{x}_1, ..., \vec{x}_n) \tilde{A}^{a_1}_\perp (\vec{x}_1) ... \tilde{A}^{a_n}_\perp (\vec{x}_n).
\]

We introduce the Fourier transforms

\[
\tilde{A}^a_\perp (\vec{x}) = \int \frac{d^3p}{(2\pi)^3} e^{-i\vec{p} \cdot \vec{x}} A^a_\perp (\vec{p}),
\]

\[
\mathcal{Y} (\vec{x}, \vec{x}_1, ..., \vec{x}_n) = \int \frac{d^3p_1}{(2\pi)^3} ... \frac{d^3p_n}{(2\pi)^3} e^{-i\vec{p}_1 \cdot (\vec{x} - \vec{x}_1) - ... - i\vec{p}_n \cdot (\vec{x} - \vec{x}_n)} \mathcal{Y} (\vec{p}_1, ..., \vec{p}_n).
\]

(801)

Expressed in terms of the Fourier transforms we obtain:

\[
A^a_\perp (\vec{x}) = \sum_{n=1}^{\infty} 2 \text{Tr} (T^a T^{a_1} ... T^{a_n}) \int \frac{d^3p_1}{(2\pi)^3} ... \frac{d^3p_n}{(2\pi)^3} e^{-i(\vec{p}_1 + ... + \vec{p}_n) \cdot \vec{x}} \mathcal{Y} (\vec{p}_1, ..., \vec{p}_n) \tilde{A}^{a_1}_\perp (\vec{p}_1) ... \tilde{A}^{a_n}_\perp (\vec{p}_n),
\]

(802)

with \( \mathcal{Y} (\vec{p}) = 1 \). The functional derivative is calculated to

\[
\frac{\delta A^a_\perp (\vec{x})}{\delta A^b_\perp (\vec{y})} = \sum_{n=1}^{\infty} \sum_{r=1}^{n} 2 \text{Tr} (T^a T^{a_1} ... T^{a_{r-1}} T^b T^{a_{r+1}} ... T^{a_n}) \int \frac{d^3p_1}{(2\pi)^3} ... \frac{d^3p_n}{(2\pi)^3} e^{-i(\vec{p}_1 + ... + \vec{p}_n) \cdot \vec{x} + i\vec{p}_r \cdot \vec{y}} \mathcal{Y} (\vec{p}_1, ..., \vec{p}_n) \tilde{A}^{a_1}_\perp (\vec{p}_1) ... \tilde{A}^{a_{r-1}}_\perp (\vec{p}_{r-1}) \tilde{A}^{a_{r+1}}_\perp (\vec{p}_{r+1}) ... \tilde{A}^{a_n}_\perp (\vec{p}_n).
\]

(803)

We then plug these expressions into eq. (800). The coefficient of each trace \( \text{Tr} (T^a T^{a_1} ... T^{a_n}) \) has to vanish separately. This leads to the following equation

\[
(\omega_{p_1} + ... + \omega_{p_n}) \mathcal{Y} (\vec{p}_1, ..., \vec{p}_n) = 0
\]

\[
\omega_{p_1} + ... + \omega_{p_n} \mathcal{Y} (\vec{p}_1, ..., \vec{p}_n) + ig \sum_{r=1}^{n-1} (\zeta_{p_1 + ... + p_r} - \zeta_{p_{r+1} + ... + p_n}) \mathcal{Y} (\vec{p}_1, ..., \vec{p}_r) \mathcal{Y} (\vec{p}_{r+1}, ..., \vec{p}_n).
\]

(804)
In order to simplify the notation we have set
\[ \omega_p = e^{i\vec{p}_x / \hbar} e^{-i\vec{p}_x / \hbar} = -i\hat{p}_x \frac{\hat{p}_+}{p^+}, \quad \zeta_p = e^{i\vec{p}_x / \hbar} e^{-i\vec{p}_x / \hbar} = \frac{\hat{p}_-}{p^-}. \] (805)

Eq. (804) is a recursion relation for the coefficient functions \( \Upsilon(\vec{p}_1, \ldots, \vec{p}_n) \):
\[ \Upsilon(\vec{p}_1, \ldots, \vec{p}_n) = \frac{\omega_{p_1} - \omega_{p_1} - \omega_{p_2} + \omega_{p_2}}{\omega_{p_1} - \omega_{p_2}} \Upsilon(\vec{p}_1, \ldots, \vec{p}_r) \Upsilon(\vec{p}_{r+1}, \ldots, \vec{p}_n), \] (806)
with \( \Upsilon(\vec{p}) = 1 \). This recursion relation has the solution:
\[ \Upsilon(\vec{p}_1, \ldots, \vec{p}_n) = \frac{(\sqrt{2g})^{n-1}}{(p_{1p_2} \ldots p_{n-1} p_n)} \frac{p^- + \ldots + p_n}{\sqrt{p_1 p_n}}, \] (807)
which is easily verified by inserting the solution into the recursion relation. We note that the coefficients \( \Upsilon \) satisfy a decoupling identity:
\[ \Upsilon(\vec{p}_n, \vec{p}_1, \ldots, \vec{p}_n) + \Upsilon(\vec{p}_1, \vec{p}_n, \vec{p}_2, \ldots, \vec{p}_n) + \ldots + \Upsilon(\vec{p}_1, \ldots, \vec{p}_n, \vec{p}_n) = 0. \] (808)

In addition we have to express the “old” conjugated fields \( A_{\perp s}^a(\tilde{x}) \) in terms of the “new” fields. The relevant equation to be solved is given in eq. (337):
\[ \partial_\perp \tilde{A}_{\perp s}^a(\tilde{x}) \equiv \int d^3\gamma \frac{\delta A_b^b(\gamma)}{\delta A_{\perp s}^a(\tilde{x})} \partial_\perp A_b^a(\gamma). \] (809)

We make the ansatz
\[ A_{\perp s}^a(\tilde{x}) = \sum_{n=1}^{\infty} \sum_{r=1}^{n} 2 \text{Tr} (T^a T^{a_1} \ldots T^{a_n}) \int d^3 x_1 \ldots d^3 x_n \] (810)
\[ \Xi_r(\tilde{x}, \tilde{x}_1, \ldots, \tilde{x}_n) \tilde{A}_{\perp}^{a_1}(\tilde{x}_1) \ldots \tilde{A}_{\perp}^{a_{r-1}}(\tilde{x}_{r-1}) \tilde{A}_{\perp}^{a_r}(\tilde{x}_{r}) \tilde{A}_{\perp}^{a_{r+1}}(\tilde{x}_{r+1}) \ldots \tilde{A}_{\perp}^{a_n}(\tilde{x}_n). \] (811)

In Fourier space we have
\[ A_{\perp s}^a(\tilde{x}) = \sum_{n=1}^{\infty} \sum_{r=1}^{n} 2 \text{Tr} (T^a T^{a_1} \ldots T^{a_n}) \int \frac{d^3 p_1}{(2\pi)^3} \ldots \frac{d^3 p_n}{(2\pi)^3} e^{-i(\vec{p}_1 + \ldots + \vec{p}_n) \cdot \vec{x}} \] (812)
\[ \Xi_r(\vec{p}_1, \ldots, \vec{p}_n) \tilde{A}_{\perp}^{a_1}(\vec{p}_1) \ldots \tilde{A}_{\perp}^{a_{r-1}}(\vec{p}_{r-1}) \tilde{A}_{\perp}^{a_r}(\vec{p}_r) \tilde{A}_{\perp}^{a_{r+1}}(\vec{p}_{r+1}) \ldots \tilde{A}_{\perp}^{a_n}(\vec{p}_n), \] (813)
with \( \Xi_1(\vec{p}) = 1 \). We then insert this ansatz into eq. (809). For \( n > 1 \) we obtain the equation
\[ 0 = \sum_{i_1=1}^{r} \sum_{i_2=r+1}^{n} (p_{i_1}^- + \ldots + p_{i_2-1}^-) \Xi_{r-i_1+1}(\vec{p}_{i_1}, \ldots, \vec{p}_{i_2-1}) \Upsilon(\vec{p}_{i_2}, \ldots, \vec{p}_{n-1}, - \sum_{i=1}^{n-1} \vec{p}_i, \vec{p}_1, \ldots, \vec{p}_{i_1-1}). \] (814)

This is again a recursion relation for the coefficients \( \Xi_r(\vec{p}_1, \ldots, \vec{p}_n) \). We can rewrite this equation as
\[ \Xi_r(\vec{p}_1, \ldots, \vec{p}_n) = - \sum_{i_1=2}^{r} \sum_{i_2=r+1}^{n} \frac{p_{i_1}^- + \ldots + p_{i_2-1}^-}{p_{i_1}^- + \ldots + p_n} \Xi_{r-i_1+1}(\vec{p}_{i_1}, \ldots, \vec{p}_{i_2-1}) \Upsilon(\vec{p}_{i_2}, \ldots, \vec{p}_{n}, - \sum_{i=1}^{n} \vec{p}_i), \] (815)
The solution is given by
\[ \Xi (\vec{p}_1, ..., \vec{p}_n) = \left( \frac{p_r}{p_1 + ... + p_n} \right)^2 \Upsilon (\vec{p}_1, ..., \vec{p}_n). \] (816)

**Exercise 18:** Consider the MHV amplitude \( A_n^{(0)} (\vec{p}_1^+, ..., \vec{p}_j^-, ..., \vec{p}_k^-, ...) \), where the particles \( j \) and \( k \) have negative helicity, while all other particles have positive helicity. Consider the shift of eq. (365), which deforms the momenta of the particles \( i \) and \( j \). What is the large-\( z \) behaviour of the amplitude?

**Solution:** The shift of eq. (365) transforms the spinors
\[ \hat{p}_A^i = p_A^i - zp_A^j, \quad \hat{p}_B^j = p_B^j + zp_B^i, \] (817)
all other spinors are left untouched. The MHV amplitude is given by
\[ A_n^{(0)} (\vec{p}_1^+, ..., \vec{p}_j^-, ..., \vec{p}_k^-, ...) = i \left( \sqrt{2} \right)^{n-2} \frac{\langle jk \rangle^d}{\langle 12 \rangle ... \langle (i-1)i \rangle \langle i(i+1) \rangle ... \langle n1 \rangle}, \] (818)
and involves only the holomorphic spinors. Therefore only the shift \( \hat{p}_A^i = p_A^i - zp_A^j \) is relevant. This spinor appears twice in the denominator. We now have to distinguish two cases. The first case is given when the particles \( i \) and \( j \) are adjacent, i.e. \( j = i - 1 \) or \( j = i + 1 \). Let’s assume \( j = i - 1 \). Then the spinor product
\[ \langle (i-1)i \rangle = \langle ji \rangle - z\langle jj \rangle = \langle ji \rangle, \] (819)
is \( z \)-independent, while the other one grows linearly with \( z \):
\[ \langle i(i+1) \rangle = \langle i(i+1) \rangle - z\langle j(i+1) \rangle. \] (820)
Therefore the amplitude scales like \( 1/z \) in this case.

If the particles \( i \) and \( j \) are not adjacent, the amplitude behaves as \( 1/z^2 \) for large \( z \).

**Exercise 19:** Show that a term of the form
\[ \langle p_1 p_2 \rangle^{v_3} \langle p_2 p_3 \rangle^{v_1} \langle p_3 p_1 \rangle^{v_2} \] (821)
and contributing to \( M_3^{(0)} (1^{h_1}, 2^{h_2}, 3^{h_3}) \) must have the exponents as in eq. (385).

**Solution:** Let us consider little group scaling. If we scale \( p_1^i \rightarrow \lambda p_1^i \), the function in eq. (821) scales as \( \lambda^{v_2+v_3} \). The amplitude and every term in the amplitude scales as \( \lambda^{-2h_1} \). This gives us the equation
\[ v_2 + v_3 = -2h_1. \] (822)
Repeating the argument for the other particles, we obtain the system of equations
\[ v_2 + v_3 = -2h_1, \quad v_1 + v_3 = -2h_2, \quad v_1 + v_2 = -2h_3. \] (823)
Solving this linear system of equation yields

\[ \nu_1 = h_1 - h_2 - h_3, \quad \nu_2 = h_2 - h_3 - h_1, \quad \nu_3 = h_3 - h_1 - h_2. \quad (824) \]

**Exercise 20:** Prove the Parke-Taylor formula for two non-adjacent negative helicities from the formula of the adjacent case and by using on-shell recursion relations.

**Solution:** Let us denote the two negative helicity legs 1 and \( j \). The adjacent case is given by \( j = n \) or \( j = 2 \) and we may assume that the Parke-Taylor formula is already proven in this case. We will prove the general non-adjacent case by induction in the variable \( n' = n - j \). The induction start is given by the adjacent case \( n' = 0 \). Let us now consider the induction step. We may assume that the Parke-Taylor formula is correct for \( n' = 1 \) and we have to show that it holds for \( n' \). We consider

\[ A_n^{(0)} \left( \begin{array}{c} 1^- \, 2^+ \, \ldots \, (n-1)^- \, n^+ \end{array} \right). \quad (825) \]

For the helicity configuration \( (1^-, n^+) \) we use the shift

\[ \hat{p}_B = p_B - zp_B^n, \quad \hat{p}_A = p_A^n + zp_A^1, \quad (826) \]

In the on-shell recursion relation we obtain sub-amplitudes with one negative helicity leg. These vanish except for the case with three external legs. This simplifies the recursion relation significantly. We distinguish the cases \( j = n - 1 \) and \( j < n - 1 \). In the former case we find from the on-shell recursion relations

\[ A_n^{(0)} \left( \begin{array}{c} 1^- \, 2^+ \, \ldots \, (n-1)^- \, n^+ \end{array} \right) = A_n^{(0)} \left( \begin{array}{c} \hat{1}^- \, 2^+ \, \ldots \, (n-2)^- \, \hat{1}^{n+} \end{array} \right) + A_n^{(0)} \left( \begin{array}{c} \hat{1}^- \, 2^+ \, \ldots \, \hat{1}^{n-1} \, \hat{1}^+ \end{array} \right), \quad (827) \]

where we used the notation of eq. (825). In the case \( j < n - 1 \) the on-shell recursion relations read

\[ A_n^{(0)} \left( \begin{array}{c} 1^- \, 2^+ \, \ldots \, j^- \, \ldots \, n^+ \end{array} \right) = A_n^{(0)} \left( \begin{array}{c} \hat{1}^- \, 2^+ \, \ldots \, \hat{1}^{n-2} \, \hat{1}^+ \end{array} \right) + A_n^{(0)} \left( \begin{array}{c} \hat{1}^- \, 2^+ \, \ldots \, j^- \, \ldots \, \hat{1}^+ \end{array} \right), \quad (828) \]

In both case the z-values of the two factorisation channels \( p_{1,n-2}^2 = 2p_{n-1}p_n \) and \( p_{1,2}^2 = 2p_1p_2 \) are

\[ z_{1,n-2} = -\frac{2p_{n-1}p_n}{\langle n + |p_{n-1}| 1^+ \rangle} = \frac{\langle (n-1)n \rangle}{\langle 1(n-1) \rangle}, \quad z_{1,2} = \frac{2p_1p_2}{\langle n + |p_2| 1^+ \rangle} = -\frac{12}{2n}. \quad (829) \]

From eq. (827) we obtain

\[ A_n^{(0)} \left( \begin{array}{c} 1^- \, 2^+ \, \ldots \, (n-1)^- \, n^+ \end{array} \right) = \]

\[ i \left( \sqrt{2} \right)^{n-2} \left[ \frac{1}{\langle 12 \ldots (n-2) \hat{1}^{n-2} \hat{1}^{1} \rangle}\cdot\frac{\hat{1}^{n-2} \hat{1}^{1}}{2p_{n-1}p_n} \mid \hat{1}^{n-2} \hat{1}^{1} \mid (n-1) \rangle\mid \hat{1}^{n-2} \hat{1}^{1} \mid \hat{1}^{n-2} \hat{1}^{1} \rangle \right]_{z = z_{1,n-2}} + i \left( \sqrt{2} \right)^{n-2} \left[ \frac{1}{\langle \hat{1}^{12} \rangle}\cdot\frac{\hat{1}^{12}}{2p_1p_2} \mid \hat{1}^{12} \rangle\mid (n-1) \rangle\mid \hat{1}^{12} \rangle \right]_{z = z_{1,2}}. \quad (830) \]
From eq. (828) we obtain

$$A_n^{(0)}(1^-,2^+,\ldots,j^-,\ldots,n^+) =$$

$$i \left( \sqrt{2} \right)^{n-2} \frac{1}{\langle 12 \rangle \cdots \langle (n-2) \hat{p}_{1,n-2} \hat{p}_{1,n-2} \rangle} \frac{1}{2p_{n-1}p_n} \frac{[\hat{n}(n-1)]^4}{\langle \hat{n}(n-1) \hat{p}_{1,n-2} \rangle \langle \hat{n}(n-1) \hat{n} \rangle} \bigg|_{z=z_{1,n-2}}$$

$$+ i \left( \sqrt{2} \right)^{n-2} \frac{[\hat{p}_{1,2}^4]}{[\hat{p}_{1,2}^2][21][1\hat{p}_{1,2}]} \frac{1}{2p_{1}p_{2} \langle \hat{p}_{1,2}^3 \rangle} \langle (n-1) \hat{n} \rangle \bigg|_{z=z_{1,2}}.$$

(831)

It is easy to see that eq. (831) reduces for $j = n - 1$ to eq. (830), therefore it is sufficient to consider in the following only eq. (831). Let us look at the second term of eq. (831). With $z_{1,2}$ given in eq. (829) we find

$$[\hat{p}_{1,2}]|_{z=z_{1,2}} = [2\hat{1}]|_{z=z_{1,2}} = [\hat{1}]|_{z=z_{1,2}} = 0,$$

(832)

and therefore the three-point amplitude $A_{3}^{(0)}(1^-,2^+,-\hat{p}_{1,2}^+) \rangle$ contained in this term vanishes. Thus the second term of eq. (831) does not give a contribution. Let us now look at the first term of eq. (831). We have $|1\hat{1}+| = |1+\rangle$ and $|\hat{n}+| = |n+\rangle$. Therefore we obtain

$$A_n^{(0)}(1^-,2^+,\ldots,j^-,\ldots,n^+) =$$

$$i \left( \sqrt{2} \right)^{n-2} \frac{\langle 1j \rangle^4}{\langle 12 \rangle \cdots \langle n1 \rangle} \left( \frac{\langle (n-2)(n-1) \rangle \langle n1 \rangle}{\langle (n-2) \hat{p}_{1,n-2} \rangle \langle \hat{p}_{1,n-2} \rangle} \frac{[\hat{n}(n-1)]^2}{\langle (n-1) \hat{p}_{1,n-2} \rangle \langle \hat{n} \rangle} \right) \bigg|_{z=z_{1,n-2}}.$$

(833)

It remains to show that the term in the bracket equals one:

$$\frac{\langle (n-2)(n-1) \rangle \langle n1 \rangle}{\langle (n-2) \hat{p}_{1,n-2} \rangle \langle \hat{p}_{1,n-2} \rangle} \frac{[\hat{n}(n-1)]^2}{\langle (n-1) \hat{p}_{1,n-2} \rangle \langle \hat{n} \rangle} \bigg|_{z=z_{1,n-2}} =$$

$$\frac{\langle (n-2)(n-1) \rangle \langle n1 \rangle}{\langle (n-2) \hat{f}_{1,n-2} \rangle \langle \hat{f}_{1,n-2} \rangle} \frac{1}{\langle (n-1) \hat{f}_{1,n-2} \rangle \langle \hat{f}_{1,n-2} \rangle} \bigg|_{z=z_{1,n-2}} =$$

$$\frac{\langle (n-2)(n-1) \rangle \langle n1 \rangle}{\langle (n-2) \hat{p}_{1,n-2} \rangle \langle \hat{p}_{1,n-2} \rangle} \frac{1}{\langle (n-1) \hat{p}_{1,n-2} \rangle \langle \hat{p} \rangle} \bigg|_{z=z_{1,n-2}} =$$

$$\frac{\langle (n-2)(n-1) \rangle \langle n1 \rangle}{\langle (n-2) \hat{f}_{1,n-2} \rangle \langle \hat{f}_{1,n-2} \rangle} \frac{1}{\langle (n-1) \hat{f}_{1,n-2} \rangle \langle \hat{f} \rangle} \bigg|_{z=z_{1,n-2}} =$$

$$1.$$

(834)

**Exercise 21:** Re-derive the expression for the MHV amplitude $A_{3}^{(0)}(1^-,2^-,3^+)$ and the anti-MHV amplitude $A_{3}^{(0)}(1^+,2^+,3^-)$ given in eq. (803) from eq. (474) and eq. (475), respectively.

**Solution:** Let us start with the anti-MHV amplitude $A_{3}^{(0)}(1^+,2^+,3^-)$. In order to project out this helicity configuration from the super-Yang-Mills amplitude

$$A_{3}^{(0)}(1,2,3) = i \sqrt{2} \delta^4 \left( [32]\eta_1 + [13]\eta_2 + [21]\eta_3^2 \right),$$

we set $\eta_1 = \eta_2 = 0$ and apply the differential operator

$$\frac{\partial}{\partial \eta_1^2} \frac{\partial}{\partial \eta_3^2} \frac{\partial}{\partial \eta_4^2} \frac{\partial}{\partial \eta_3^2} \frac{\partial}{\partial \eta_4^2} \bigg|_{\eta_1 = \eta_2 = 0}.$$

(836)
We have
\[
\frac{\partial}{\partial \eta_1^i} \frac{\partial}{\partial \eta_2^i} \frac{\partial}{\partial \eta_3^i} \frac{\partial}{\partial \eta_4^i} \delta([21] \eta_1^i) \delta([21] \eta_2^i) \delta([21] \eta_3^i) \delta([21] \eta_4^i) = [21]^4,
\]
and therefore
\[
A_3^{(0)} (1^+, 2^+, 3^-) = i \sqrt{2} \frac{[21]^4}{[32][21][13]}.
\] (837)

Let us now consider the MHV amplitude \(A_3^{(0)} (1^-, 2^-, 3^+)\). The super-Yang-Mills amplitude is
\[
A_3^{(0)} (1, 2, 3) = i \sqrt{2} \delta^8(p_1^1 \eta_1^1 + p_2^2 \eta_2^2 + p_3^3 \eta_3^3)
\]
(12)(23)(31),
and in order to project out the helicity configuration \((1^-, 2^-, 3^+)\) we set \(\eta_i^3 = 0\) and we work out
\[
\frac{\partial}{\partial \eta_1^i} \frac{\partial}{\partial \eta_2^i} \frac{\partial}{\partial \eta_3^i} \frac{\partial}{\partial \eta_4^i} \delta(p_1^1 \eta_1^i + p_2^2 \eta_2^i) = \left[ \frac{\partial}{\partial \eta_1^i} \frac{\partial}{\partial \eta_2^i} \delta(p_1^1 \eta_1^i + p_2^2 \eta_2^i) \right]^4 = (p_2^1 p_1^1 - p_1^1 p_2^1)^4 = \langle 12 \rangle^4.
\] (840)

Therefore
\[
A_3^{(0)} (1^-, 2^-, 3^+) = i \sqrt{2} \frac{\langle 12 \rangle^4}{
\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle}.
\] (841)

**Exercise 22:** Show that for \(k = 1\) the form \(\omega\) given in eq. (489) reduces to the expression given in eq. (485).

**Solution:** For \(k = 1\) we write for an element \(C \in \text{Gr}_{1,n}(\mathbb{R}) = \mathbb{R}^{n-1}\)
\[
C = \begin{pmatrix} C_{11} & \cdots & C_{1n} \end{pmatrix} = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}.
\] (842)

The form \(\omega\), as defined by eq. (489), reads
\[
\omega = \left\langle C^1, (dC^1)^{(n-1)} \right\rangle = \varepsilon^{i_1 \ldots i_n} x_{a_1} dC_{1,i_1} \wedge \ldots \wedge dC_{1,i_n} = \varepsilon^{i_1 \ldots i_n} x_{a_1} dx_{a_1} \wedge \ldots \wedge dx_{a_n}.
\] (843)

Consider now the case \(a_1 = 1\). The sum over the \((n-1)!\) permutations of the remaining indices \((2, \ldots, n)\) gives
\[
\varepsilon^{1a_2 \ldots a_n} x_1 dx_{a_2} \wedge \ldots \wedge dx_{a_n} = (n-1)! x_1 dx_{a_2} \wedge \ldots \wedge dx_{a_n},
\] (844)
the minus signs for odd permutations from the totally anti-symmetric tensor \(\varepsilon^{1a_2 \ldots a_n}\) compensate exactly the minus signs needed to bring the differentials \(dx_{a_2} \wedge \ldots \wedge dx_{a_n}\) into the order \(dx_2 \wedge \ldots \wedge dx_n\). For \(a_2 = 2\) we obtain
\[
\varepsilon^{2a_3 \ldots a_n} x_2 dx_{a_3} \wedge \ldots \wedge dx_{a_n} = -(n-1)! x_3 dx_{a_2} \wedge \ldots \wedge dx_{a_n},
\] (845)
Thus we have
\[ \varepsilon^{213\ldots n} = -\varepsilon^{123\ldots n}. \] (846)

In general we find for \( a_1 = j \)
\[ \varepsilon^{i_2\ldots i_n} x_j dx_{i_2} \wedge \ldots \wedge dx_{i_n} = (-1)^{j-1} (n-1)! x_j dx_1 \wedge \ldots \wedge dx_j \wedge \ldots \wedge dx_n. \] (847)

Thus we have
\[ \omega = (n-1)! \sum_{j=1}^{n} (-1)^{j-1} x_j dx_1 \wedge \ldots \wedge dx_j \wedge \ldots \wedge dx_n, \] (848)

which is the expression given in eq. (485).

**Exercise 23**: Derive the expression in eq. (524) from eq. (522).

**Solution**: We start from eq. (522):
\[
[Z^1, Z^2, Z^3, Z^4, Z^5] = \frac{1}{4!} \int d^4 \phi \frac{\langle 1, 2, 3, 4, 5 \rangle^4}{\langle 0, 1, 2, 3, 4 \rangle \langle 0, 2, 3, 4, 5 \rangle \langle 0, 3, 4, 5, 1 \rangle \langle 0, 4, 5, 1, 2 \rangle \langle 0, 5, 1, 2, 3 \rangle}
\]

Let us look at the denominator first: Since \( I_0^\alpha = 0 \) for \( \alpha = 1, \ldots, 4 \) and \( I_5^0 = 1 \), the five-brackets in the denominator reduce to four-brackets:
\[
\langle 0, i, j, k, l \rangle = \langle i, j, k, l \rangle.
\] (849)

Let us now look at the five-bracket \( \langle 1, 2, 3, 4, 5 \rangle \) appearing in the numerator. Expanding the determinant by Laplace’s formula yields
\[
\langle 1, 2, 3, 4, 5 \rangle = \langle 1, 2, 3, 4 \rangle \phi_1 \eta_5^5 + \langle 2, 3, 4, 5 \rangle \phi_2 \eta_5^5 + \langle 3, 4, 5, 1 \rangle \phi_3 \eta_5^5 + \langle 4, 5, 1, 2 \rangle \phi_4 \eta_5^5 + \langle 5, 1, 2, 3 \rangle \phi_5 \eta_5^5.
\]

We have four factors of this expression in the numerator. Now let us perform the integration over \( \phi_1 \). This will produce a factor
\[
\langle 1, 2, 3, 4 \rangle \eta_1^5 + \langle 2, 3, 4, 5 \rangle \eta_1^5 + \langle 3, 4, 5, 1 \rangle \eta_1^5 + \langle 4, 5, 1, 2 \rangle \eta_1^5 + \langle 5, 1, 2, 3 \rangle \eta_1^5 = \\
= \delta^4 \left( \langle 1, 2, 3, 4 \rangle \eta_1^5 + \langle 2, 3, 4, 5 \rangle \eta_1^5 + \langle 3, 4, 5, 1 \rangle \eta_1^5 + \langle 4, 5, 1, 2 \rangle \eta_1^5 + \langle 5, 1, 2, 3 \rangle \eta_1^5 \right).
\] (850)

We have four possibilities to produce this factor. Now let us proceed to the integration over \( \phi_2 \). This will give us a Grassmann delta-function similar to the one above, with \( \eta_1^5 \) replaced by \( \eta_2^5 \). We now have three possibilities to produce this delta-function. Proceeding in this way with the integration over \( \phi_3 \) and \( \phi_4 \) we see that the prefactor \( 1/4! \) gets cancelled by the total number of possibilities and we obtain
\[
[Z^1, Z^2, Z^3, Z^4, Z^5] = \frac{\delta^4 \left( \langle 1, 2, 3, 4 \rangle \eta_1^5 + \text{cyclic} \right)}{\langle 1, 2, 3, 4 \rangle \langle 2, 3, 4, 5 \rangle \langle 3, 4, 5, 1 \rangle \langle 4, 5, 1, 2 \rangle \langle 5, 1, 2, 3 \rangle}.
\] (851)

**Exercise 24**: Let \( g \in \text{PSL}(2, \mathbb{C}) \) and let \( (z_1, z_2, \ldots, z_n) \) be a solution of the scattering equations. Show that \( (z_1', z_2', \ldots, z_n') = g \cdot (z_1, z_2, \ldots, z_n) \) is also a solution of the scattering equations.
Solution: We have

\[ \frac{1}{z_i' - z_j'} = \frac{A}{z_i - z_j} + B, \quad (852) \]

where

\[ A = \frac{(cz_i + d)^2}{ad - bc} = (cz_i + d)^2, \quad B = -\frac{c(cz_i + d)}{ad - bc} = -c(cz_i + d). \quad (853) \]

A and B are independent of \( z_j \). When eq. (852) is inserted into the scattering equations, the terms proportional to A vanish due to our assumption that the unprimed variables \( (z_1, z_2, \ldots, z_n) \) are a solution of the scattering equations

\[ \sum_{j=1, j \neq i}^n 2p_i \cdot p_j \frac{A}{z_i - z_j} = A \sum_{j=1, j \neq i}^n 2p_i \cdot p_j = 0. \quad (854) \]

The terms proportional to B vanish due to momentum conservation and the on-shell condition \( p_i^2 = 0 \):

\[ \sum_{j=1, j \neq i}^n (2p_i \cdot p_j) B = 2p_i \cdot \left( \sum_{j=1, j \neq i}^n p_j \right) B = -2p_i^2 B = 0. \quad (855) \]

Exercise 25: Prove the relations in eq. (854).

Solution: We start with the first relation:

\[ \sum_{i=1}^n f_i(z, p) = \sum_{i=1}^n \sum_{j=1, j \neq i}^n \frac{2p_i \cdot p_j}{z_i - z_j} = 0, \quad (856) \]

due to the antisymmetry of the denominators \( 1/(z_i - z_j) \). Let us now turn to the second relation: We use \( z_i = (z_i - z_j) + z_j \), momentum conservation together with the on-shell conditions and obtain

\[ \sum_{i=1}^n z_i f_i(z, p) = \sum_{i=1}^n z_i \sum_{j=1, j \neq i}^n \frac{2p_i \cdot p_j}{z_i - z_j} = \sum_{i=1}^n \sum_{j=1, j \neq i}^n 2p_i \cdot p_j + \sum_{i=1}^n \sum_{j=1, j \neq i}^n \frac{2p_i \cdot p_j}{z_i - z_j} \]

\[ = -\sum_{i=1}^n z_i \sum_{j=1, j \neq i}^n \frac{2p_i \cdot p_j}{z_i - z_j} = -\sum_{i=1}^n z_i f_i(z, p). \quad (857) \]

In the second line we re-labeled the summation indices \( i \leftrightarrow j \). It follows

\[ \sum_{i=1}^n z_i f_i(z, p) = 0. \quad (858) \]

The third relation is proven as follows:

\[ \sum_{i=1}^n z_i^2 f_i(z, p) = \sum_{i=1}^n z_i^2 \sum_{j=1, j \neq i}^n \frac{2p_i \cdot p_j}{z_i - z_j} = \frac{1}{2} \sum_{i=1}^n \sum_{j=1, j \neq i}^n \left( z_i^2 - z_j^2 \right) \frac{2p_i \cdot p_j}{z_i - z_j} \]

\[ = \frac{1}{2} \sum_{i=1}^n \sum_{j=1, j \neq i}^n 2p_i \cdot p_j (z_i + z_j) = \sum_{i=1}^n z_i \sum_{j=1, j \neq i}^n 2p_i \cdot p_j = 0. \quad (859) \]
Exercise 26: Consider the Koba-Nielsen function

\[ U(z, p) = \prod_{i<j}(z_i - z_j)^{2p_i \cdot p_j}. \]  

(860)

Show

\[ U^{-1} \frac{\partial}{\partial z_i} U = f_i(z, p). \]  

(861)

Solution: We have

\[
\frac{\partial U}{\partial z_i} = \frac{\partial}{\partial z_i} \prod_{j=1}^{n-1} \prod_{k=j+1}^{n} (z_j - z_k)^{2p_j \cdot p_k} = \sum_{k=i+1}^{n} \frac{2p_i \cdot p_k}{z_i - z_k} U - \sum_{j=1}^{i-1} \frac{2p_j \cdot p_i}{z_j - z_i} U
\]

\[
= \sum_{j \neq i} \frac{2p_i \cdot p_j}{z_i - z_j} U = f_i(z, p) U.
\]  

(862)

Hence

\[ U^{-1} \frac{\partial}{\partial z_i} U = f_i(z, p). \]  

(863)

Exercise 27: Consider the case \( n = 5 \) and determine the matrix \( S_{\sigma\tilde{\sigma}} \) for the basis \( B \) given in eq. (688).

Solution: We take all momenta as outgoing and set \( p = (p_1, p_2, p_3, p_4, p_5) \). For \( n = 5 \) the basis \( B \) consists of \( (n-3)! = 2 \) elements, which with the choice as in eq. (688) are given by

\[ B = \{(1, 2, 3, 4, 5), (1, 2, 4, 3, 5)\}. \]  

(864)

We set \( \rho = (1, 2, 3, 4, 5) \) and \( \tau = (1, 2, 4, 3, 5) \). The matrix \( m_{\sigma\tilde{\sigma}} \) (with indices \( \sigma, \tilde{\sigma} \in \{\rho, \tau\} \)) is given by

\[ m = \begin{pmatrix} m^{(0)}_{\rho, \rho, p} & m^{(0)}_{\rho, \tau, p} \\ m^{(0)}_{\tau, \rho, p} & m^{(0)}_{\tau, \tau, p} \end{pmatrix}. \]  

(865)

The entries are given by

\[
m^{(0)}_{\rho, \rho, p} = i \left( \frac{1}{s_{12} s_{34}} + \frac{1}{s_{23} s_{45}} + \frac{1}{s_{15} s_{34}} + \frac{1}{s_{45} s_{12}} + \frac{1}{s_{23} s_{15}} \right),
\]

\[
m^{(0)}_{\rho, \tau, p} = m^{(0)}_{\tau, \rho, p} = i \left( -\frac{1}{s_{12} s_{34}} - \frac{1}{s_{15} s_{34}} \right),
\]

\[
m^{(0)}_{\tau, \tau, p} = i \left( \frac{1}{s_{12} s_{34}} + \frac{1}{s_{24} s_{35}} + \frac{1}{s_{15} s_{34}} + \frac{1}{s_{35} s_{12}} + \frac{1}{s_{24} s_{15}} \right),
\]  

(866)
where we used the notation $s_{ij} = (p_i + p_j)^2 = 2p_i \cdot p_j$. We calculate $\det m$ and find

$$
\det m = \frac{s_{13} s_{14} s_{25}}{s_{12} s_{15} s_{23} s_{24} s_{34} s_{35} s_{45}}.
$$

Thus

$$
S = \frac{s_{12} s_{15} s_{23} s_{24} s_{34} s_{35} s_{45}}{s_{13} s_{14} s_{25}} \left( \begin{array}{cc}
m_5^{(0)}(\tau, \tau, p) & -m_5^{(0)}(\rho, \tau, p) \\
-m_5^{(0)}(\tau, \rho, p) & m_5^{(0)}(\rho, \rho, p) \end{array} \right).
$$
References

[1] M. L. Mangano and S. J. Parke, Phys. Rept. **200**, 301 (1991), arXiv:hep-th/0509223.

[2] L. J. Dixon, Calculating scattering amplitudes efficiently, in *QCD and beyond. Proceedings, Theoretical Advanced Study Institute in Elementary Particle Physics, TASI-95, Boulder, USA, June 4-30, 1995*, pp. 539–584, 1996, arXiv:hep-ph/9601359.

[3] H. Elvang and Y.-t. Huang, *Scattering Amplitudes* (Cambridge University Press, 2015), arXiv:1308.1697.

[4] J. M. Henn and J. C. Plefka, *Scattering Amplitudes in Gauge Theories* (Springer, Lecture Notes in Physics 883, 2014).

[5] M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory* (Perseus Books, 1995).

[6] M. Srednicki, *Quantum field theory* (Cambridge University Press, 2007).

[7] M. D. Schwartz, *Quantum Field Theory and the Standard Model* (Cambridge University Press, 2014).

[8] C.-N. Yang and R. L. Mills, Phys. Rev. **96**, 191 (1954).

[9] M. Nakahara, *Geometry, Topology and Physics* (Institute of Physics Publishing, 2003).

[10] C. J. Isham, *Modern differential geometry for physicists* (World Scientific, 1999).

[11] V. N. Gribov, Nucl. Phys. **B139**, 1 (1978).

[12] L. D. Faddeev and V. N. Popov, Phys. Lett. **B25**, 29 (1967).

[13] V. A. Smirnov, *Feynman integral calculus* (Springer, Berlin, 2006).

[14] S. Weinzierl, Fields Inst. Commun. **50**, 345 (2006), hep-ph/0604068.

[15] S. Weinzierl, Introduction to Feynman Integrals, in *Geometric and Topological Methods for Quantum Field Theory: Proceedings, 6th Summer School, Villa de Leyva, Colombia, 6-23 Jul 2009*, pp. 144–187, 2013, arXiv:1005.1855.

[16] C. Duhr, Mathematical aspects of scattering amplitudes, in *Theoretical Advanced Study Institute in Elementary Particle Physics: Journeys Through the Precision Frontier: Amplitudes for Colliders (TASI 2014) Boulder, Colorado, June 2-27, 2014*, arXiv:1411.7538.

[17] J. M. Henn, J. Phys. **A48**, 153001 (2015), arXiv:1412.2296.

[18] J. A. M. Vermaseren, (2000), math-ph/0010025.

[19] C. Bauer, A. Frink, and R. Kreckel, J. Symbolic Computation **33**, 1 (2002), cs.sc/0004015.

[20] A. Bassetto, M. Ciafaloni, and G. Marchesini, Phys. Rept. **100**, 201 (1983).

[21] S. Catani and M. H. Seymour, Nucl. Phys. **B485**, 291 (1997), hep-ph/9605323.
[22] G. ’t Hooft and M. J. G. Veltman, Nucl. Phys. B44, 189 (1972).
[23] C. G. Bollini and J. J. Giambiagi, Nuovo Cim. B12, 20 (1972).
[24] G. M. Cicuta and E. Montaldi, Nuovo Cim. Lett. 4, 329 (1972).
[25] T. Kinoshita, J. Math. Phys. 3, 650 (1962).
[26] T. D. Lee and M. Nauenberg, Phys. Rev. 133, B1549 (1964).
[27] P. Cvitanovic, P. G. Lauwers, and P. N. Scharbach, Nucl. Phys. B186, 165 (1981).
[28] F. A. Berends and W. Giele, Nucl. Phys. B294, 700 (1987).
[29] M. L. Mangano, S. J. Parke, and Z. Xu, Nucl. Phys. B298, 653 (1988).
[30] D. Kosower, B.-H. Lee, and V. P. Nair, Phys. Lett. B201, 85 (1988).
[31] Z. Bern and D. A. Kosower, Nucl. Phys. B362, 389 (1991).
[32] V. Del Duca, L. J. Dixon, and F. Maltoni, Nucl. Phys. B571, 51 (2000), hep-ph/9910563.
[33] F. Maltoni, K. Paul, T. Stelzer, and S. Willenbrock, Phys. Rev. D67, 014026 (2003), hep-ph/0209271.
[34] Z. Bern, L. J. Dixon, and D. A. Kosower, Nucl. Phys. B437, 259 (1995), hep-ph/9409393.
[35] R. K. Ellis, W. Giele, Z. Kunszt, K. Melnikov, and G. Zanderighi, JHEP 0901, 012 (2009), arXiv:0810.2762.
[36] R. K. Ellis, Z. Kunszt, K. Melnikov, and G. Zanderighi, Phys.Rept. 518, 141 (2012), arXiv:1105.4319.
[37] H. Ita and K. Ozeren, JHEP 1202, 118 (2012), arXiv:1111.4193.
[38] S. Badger, B. Biedermann, P. Uwer, and V. Yundin, Comput.Phys.Commun. 184, 1981 (2013), arXiv:1209.0100.
[39] C. Reuschle and S. Weinzierl, Phys.Rev. D88, 105020 (2013), arXiv:1310.0413.
[40] T. Schuster, Phys. Rev. D89, 105022 (2014), arXiv:1311.6296.
[41] R. Kleiss and H. Kuijf, Nucl. Phys. B312, 616 (1989).
[42] P. Draggiotis, R. H. P. Kleiss, and C. G. Papadopoulos, Phys. Lett. B439, 157 (1998), hep-ph/9807207.
[43] C. Duhr, S. Hoche, and F. Maltoni, JHEP 08, 062 (2006), hep-ph/0607057.
[44] Z. Bern, J. J. M. Carrasco, and H. Johansson, Phys. Rev. D78, 085011 (2008), arXiv:0805.3993.
[45] N. Bjerrum-Bohr, P. H. Damgaard, and P. Vanhove, Phys.Rev.Lett. 103, 161602 (2009), arXiv:0907.1425.
[46] S. Stieberger, (2009), arXiv:0907.2211.
[47] B. Feng, R. Huang, and Y. Jia, Phys.Lett. B695, 350 (2011), arXiv:1004.3417.
[48] Z. Bern, J. J. M. Carrasco, and H. Johansson, Phys.Rev.Lett. 105, 061602 (2010), arXiv:1004.0476.
[49] D. Götz, C. Schwan, and S. Weinzierl, Phys.Rev. D85, 116011 (2012).
[50] M. Czakon, C. G. Papadopoulos, and M. Worek, JHEP 08, 085 (2009), arXiv:0905.0883.
[51] S. Dittmaier, A. Kabelschacht, and T. Kasprzik, Nucl. Phys. B800, 146 (2008), arXiv:0802.1405.
[52] F. A. Berends, R. Kleiss, P. De Causmaecker, R. Gastmans, and T. T. Wu, Phys. Lett. B103, 124 (1981).
[53] P. De Causmaecker, R. Gastmans, W. Troost, and T. T. Wu, Nucl. Phys. B206, 53 (1982).
[54] J. F. Gunion and Z. Kunszt, Phys. Lett. B161, 333 (1985).
[55] R. Kleiss and W. J. Stirling, Phys. Lett. B179, 159 (1986).
[56] Z. Xu, D.-H. Zhang, and L. Chang, Nucl. Phys. B291, 392 (1987).
[57] R. Gastmans and T. T. Wu, The Ubiquitous photon: Helicity method for QED and QCD (Clarendon Press, 1990).
[58] C. Schwinn and S. Weinzierl, JHEP 05, 006 (2005), hep-th/0503015.
[59] G. Rodrigo, JHEP 09, 079 (2005), hep-ph/0508138.
[60] A. P. Hodges, (2005), arXiv:hep-th/0503060.
[61] A. P. Hodges, (2005), arXiv:hep-th/0512336.
[62] A. Hodges, JHEP 05, 135 (2013), arXiv:0905.1473.
[63] L. J. Mason and D. Skinner, JHEP 01, 064 (2010), arXiv:0903.2083.
[64] M. G. Eastwood, R. Penrose, and R. Wells, Commun.Math.Phys. 78, 305 (1981).
[65] R. Penrose and M. A. H. MacCallum, Phys. Rept. 6, 241 (1972).
[66] E. Byckling and K. Kajantie, Particle Kinematics (John Wiley & Sons, 1973).
[67] R. Kleiss, W. J. Stirling, and S. D. Ellis, Comput. Phys. Commun. 40, 359 (1986).
[68] A. van Hameren and C. G. Papadopoulos, Eur. Phys. J. C25, 563 (2002), hep-ph/0204055.
[69] A. van Hameren, (2010), arXiv:1003.4953.
[70] F. A. Berends and W. T. Giele, Nucl. Phys. B306, 759 (1988).
[71] S. J. Parke and T. R. Taylor, Phys. Rev. Lett. 56, 2459 (1986).
[72] F. Cachazo, P. Svrcek, and E. Witten, JHEP 09, 006 (2004), hep-th/0403047.
[73] D. A. Kosower, Phys. Rev. D71, 045007 (2005), hep-th/0406175.
[74] I. Bena, Z. Bern, and D. A. Kosower, Phys. Rev. D71, 045008 (2005), hep-th/0406133.
[75] A. Gorsky and A. Rosly, JHEP 01, 101 (2006), hep-th/0510111.
[76] P. Mansfield, JHEP 03, 037 (2006), hep-th/0511264.
[77] J. H. Ettle and T. R. Morris, JHEP 08, 003 (2006), hep-th/0605121.
[78] J. H. Ettle, C.-H. Fu, J. P. Fudger, P. R. W. Mansfield, and T. R. Morris, JHEP 05, 011 (2007), hep-th/0703286.
[79] J. H. Ettle, T. R. Morris, and Z. Xiao, JHEP 08, 103 (2008), arXiv:0805.0239.
[80] S. Buchta and S. Weinzierl, JHEP 09, 071 (2010), arXiv:1007.2742.
[81] L. J. Mason and D. Skinner, Phys. Lett. B636, 60 (2006), hep-th/0510262.
[82] L. J. Mason, JHEP 10, 009 (2005), hep-th/0507269.
[83] R. Boels, L. Mason, and D. Skinner, Phys. Lett. B648, 90 (2007), hep-th/0702035.
[84] R. Boels, L. Mason, and D. Skinner, JHEP 02, 014 (2007), hep-th/0604040.
[85] R. Boels, Phys. Rev. D76, 105027 (2007), arXiv:hep-th/0703080.
[86] R. Boels, K. J. Larsen, N. A. Obers, and M. Vonk, JHEP 11, 015 (2008), arXiv:0808.2598.
[87] K. Risager, JHEP 12, 003 (2005), hep-th/0508206.
[88] R. Britto, F. Cachazo, and B. Feng, Nucl. Phys. B715, 499 (2005), hep-th/0412308.
[89] R. Britto, F. Cachazo, B. Feng, and E. Witten, Phys. Rev. Lett. 94, 181602 (2005), hep-th/0501052.
[90] N. Arkani-Hamed and J. Kaplan, JHEP 0804, 076 (2008), arXiv:0801.2385.
[91] C. Cheung, C.-H. Shen, and J. Trnka, JHEP 06, 118 (2015), arXiv:1502.05057.
[92] M. Dinsdale, M. Ternick, and S. Weinzierl, JHEP 03, 056 (2006), hep-ph/0602204.
[93] S. Badger et al., Phys. Rev. D87, 034011 (2013), arXiv:1206.2381.
[94] P. Benincasa and F. Cachazo, (2007), arXiv:0705.4305.
[95] Y. Jia, R. Huang, and C.-Y. Liu, Phys.Rev. D82, 065001 (2010), arXiv:1005.1821.
[96] Y.-X. Chen, Y.-J. Du, and B. Feng, JHEP 1102, 112 (2011), arXiv:1101.0009.
[97] L. de la Cruz, A. Kniss, and S. Weinzierl, JHEP 09, 197 (2015), arXiv:1508.01432.
[98] A. Postnikov, (2013), Lecture notes.
[99] M. Brion, (2004), math/0410240.
[100] N. Arkani-Hamed, F. Cachazo, C. Cheung, and J. Kaplan, JHEP 03, 110 (2010), arXiv:0903.2110.
[101] N. Arkani-Hamed, F. Cachazo, C. Cheung, and J. Kaplan, JHEP 03, 020 (2010), arXiv:0907.5418.
[102] N. Arkani-Hamed, J. Bourjaily, F. Cachazo, and J. Trnka, JHEP 01, 108 (2011), arXiv:0912.3249.
[103] N. Arkani-Hamed, J. Bourjaily, F. Cachazo, and J. Trnka, JHEP 01, 049 (2011), arXiv:0912.4912.
[104] N. Arkani-Hamed et al., Scattering Amplitudes and the Positive Grassmannian (Cambridge University Press, 2012), arXiv:1212.5605.
[105] J. L. Bourjaily, J. Trnka, A. Volovich, and C. Wen, JHEP 01, 038 (2011), arXiv:1006.1899.
[106] V. P. Nair, Phys. Lett. B214, 215 (1988).
[107] N. Arkani-Hamed, F. Cachazo, and J. Kaplan, JHEP 1009, 016 (2010), arXiv:0808.1446.
[108] J. Wess and J. Bagger, Supersymmetry and supergravity (Princeton University Press, 1992).
[109] N. Arkani-Hamed and J. Trnka, JHEP 10, 030 (2014), arXiv:1312.2007.
[110] N. Arkani-Hamed and J. Trnka, JHEP 12, 182 (2014), arXiv:1312.7878.
[111] N. Arkani-Hamed, A. Hodges, and J. Trnka, JHEP 08, 030 (2015), arXiv:1412.8478.
[112] Y. Bai and S. He, JHEP 02, 065 (2015), arXiv:1408.2459.
[113] S. Franco, D. Galloni, A. Mariotti, and J. Trnka, JHEP 03, 128 (2015), arXiv:1408.3410.
[114] Z. Bern, E. Herrmann, S. Litsey, J. Stankowicz, and J. Trnka, JHEP 06, 098 (2016), arXiv:1512.08591.
[115] F. Cachazo, S. He, and E. Y. Yuan, Phys.Rev. D90, 065001 (2014), arXiv:1306.6575.
[116] F. Cachazo, S. He, and E. Y. Yuan, Phys.Rev.Lett. 113, 171601 (2014), arXiv:1307.2199.
[117] F. Cachazo, S. He, and E. Y. Yuan, JHEP 1407, 033 (2014), arXiv:1309.0885.
[118] L. Dolan and P. Goddard, JHEP 1407, 029 (2014), arXiv:1402.7374.
[119] Y.-H. He, C. Matti, and C. Sun, JHEP 1410, 135 (2014), arXiv:1403.6833.
[120] P. Griffiths and J. Harris, Principles of Algebraic Geometry (John Wiley & Sons, New York, 1994).
[121] L. Dolan and P. Goddard, JHEP 1405, 010 (2014), arXiv:1311.5200.
[122] M. Søgaard and Y. Zhang, Phys. Rev. D93, 105009 (2016), arXiv:1509.08897.
[123] J. Bosma, M. Søgaard, and Y. Zhang, Phys. Rev. D94, 041701 (2016), arXiv:1605.08431.
[124] B. S. DeWitt, Phys. Rev. 160, 1113 (1967).
[125] B. S. DeWitt, Phys. Rev. 162, 1195 (1967).
[126] B. S. DeWitt, Phys. Rev. 162, 1239 (1967).
[127] M. J. G. Veltman, Quantum Theory of Gravitation, in Methods in Field Theory: Proceedings, 28th Les Houches Summer School, France, July 28-September 6, 1975, pp. 265–327, 1975.

[128] C. D. White, Phys. Lett. B763, 365 (2016), arXiv:1606.04724.

[129] Z. Bern, T. Dennen, Y.-t. Huang, and M. Kiermaier, Phys.Rev. D82, 065003 (2010), arXiv:1004.0693.

[130] H. Kawai, D. Lewellen, and S. Tye, Nucl.Phys. B269, 1 (1986).

[131] Z. Bern, A. De Freitas, and H. L. Wong, Phys. Rev. Lett. 84, 3531 (2000), arXiv:hep-th/9912033.

[132] N. E. J. Bjerrum-Bohr and K. Risager, Phys. Rev. D70, 086011 (2004), arXiv:hep-th/0407085.

[133] N. Bjerrum-Bohr, P. H. Damgaard, B. Feng, and T. Sondergaard, Phys.Rev. D82, 107702 (2010), arXiv:1005.4367.

[134] B. Feng and S. He, JHEP 09, 043 (2010), arXiv:1007.0055.

[135] P. H. Damgaard, R. Huang, T. Sondergaard, and Y. Zhang, JHEP 08, 101 (2012), arXiv:1206.1577.

[136] L. de la Cruz, A. Kniss, and S. Weinzierl, JHEP 11, 217 (2015), arXiv:1508.06557.

[137] L. de la Cruz, A. Kniss, and S. Weinzierl, Phys. Rev. Lett. 116, 201601 (2016), arXiv:1601.04523.

[138] N. Bjerrum-Bohr, P. H. Damgaard, R. Monteiro, and D. O’Connell, JHEP 1206, 061 (2012), arXiv:1203.0944.

[139] R. Monteiro and D. O’Connell, JHEP 1107, 007 (2011), arXiv:1105.2565.

[140] C. R. Mafra, O. Schlotterer, and S. Stieberger, JHEP 1107, 092 (2011), arXiv:1104.5224.

[141] S. G. Naculich, JHEP 1409, 029 (2014), arXiv:1407.7836.

[142] S. G. Naculich, JHEP 05, 050 (2015), arXiv:1501.03500.

[143] S. G. Naculich, JHEP 09, 122 (2015), arXiv:1506.06134.

[144] C. Kalousios, J.Phys. A47, 215402 (2014), arXiv:1312.7743.

[145] M. Tolotti and S. Weinzierl, JHEP 07, 111 (2013), arXiv:1306.2975.

[146] S. Weinzierl, JHEP 1404, 092 (2014), arXiv:1402.2516.

[147] C. S. Lam, Phys. Rev. D91, 045019 (2015), arXiv:1410.8184.

[148] R. Monteiro and D. O’Connell, JHEP 1403, 110 (2014), arXiv:1311.1151.

[149] F. Cachazo and H. Gomez, JHEP 04, 108 (2016), arXiv:1505.03571.

[150] C. Baadsgaard, N. E. J. Bjerrum-Bohr, J. L. Bourjaily, and P. H. Damgaard, JHEP 09, 129 (2015), arXiv:1506.06137.
[151] C. Baadsgaard, N. E. J. Bjerrum-Bohr, J. L. Bourjaily, P. H. Damgaard, and B. Feng, JHEP 11, 080 (2015), arXiv:1508.03627.

[152] C. Baadsgaard, N. E. J. Bjerrum-Bohr, J. L. Bourjaily, and P. H. Damgaard, JHEP 09, 136 (2015), arXiv:1507.00997.

[153] R. Huang, J. Rao, B. Feng, and Y.-H. He, JHEP 12, 056 (2015), arXiv:1509.04483.

[154] C. Cardona, B. Feng, H. Gomez, and R. Huang, JHEP 09, 133 (2016), arXiv:1606.00670.

[155] H. Gomez, JHEP 06, 101 (2016), arXiv:1604.05373.

[156] C. Cardona and H. Gomez, JHEP 06, 094 (2016), arXiv:1605.01446.

[157] C. Cardona and H. Gomez, JHEP 10, 116 (2016), arXiv:1607.01871.

[158] N. E. J. Bjerrum-Bohr, P. H. Damgaard, P. Tourkine, and P. Vanhove, Phys.Rev. D90, 106002 (2014), arXiv:1403.4553.

[159] L. Mason and D. Skinner, JHEP 1407, 048 (2014), arXiv:1311.2564.

[160] N. Berkovits, JHEP 1403, 017 (2014), arXiv:1311.4156.

[161] H. Gomez and E. Y. Yuan, JHEP 1404, 046 (2014), arXiv:1311.4156.

[162] T. Adamo, E. Casali, and D. Skinner, JHEP 1404, 104 (2014), arXiv:1312.3828.

[163] Y. Geyer, A. E. Lipstein, and L. J. Mason, Phys.Rev.Lett. 113, 081602 (2014), arXiv:1404.6219.

[164] E. Casali and P. Tourkine, JHEP 04, 013 (2015), arXiv:1412.3787.

[165] Y. Geyer, L. Mason, R. Monteiro, and P. Tourkine, Phys. Rev. Lett. 115, 121603 (2015), arXiv:1507.00321.

[166] Y. Geyer, L. Mason, R. Monteiro, and P. Tourkine, Phys. Rev. D94, 125029 (2016), arXiv:1607.08887.

[167] B. U. W. Schwab and A. Volovich, Phys.Rev.Lett. 113, 101601 (2014), arXiv:1404.7749.

[168] N. Afkhami-Jeddi, (2014), arXiv:1405.3533.

[169] M. Zlotnikov, JHEP 1410, 148 (2014), arXiv:1407.5936.

[170] C. Kalousios and F. Rojas, JHEP 01, 107 (2015), arXiv:1407.5982.

[171] C. White, Phys.Lett. B737, 216 (2014), arXiv:1406.7184.

[172] C. Cardona and C. Kalousios, Phys. Lett. B756, 180 (2016), arXiv:1511.05915.

[173] L. Dolan and P. Goddard, JHEP 10, 149 (2016), arXiv:1511.09441.

[174] R. W. Brown and S. G. Naculich, JHEP 10, 130 (2016), arXiv:1608.04387.

[175] R. W. Brown and S. G. Naculich, JHEP 11, 060 (2016), arXiv:1608.05291.
[176] F. Cachazo and G. Zhang, (2016), arXiv:1601.06305.

[177] F. Cachazo, S. Mizera, and G. Zhang, (2016), arXiv:1609.00008.

[178] C. Baadsgaard et al., Phys. Rev. Lett. 116, 061601 (2016), arXiv:1509.02169.

[179] N. E. J. Bjerrum-Bohr, J. L. Bourjaily, P. H. Damgaard, and B. Feng, Nucl. Phys. B913, 964 (2016), arXiv:1605.06501.

[180] N. E. J. Bjerrum-Bohr, J. L. Bourjaily, P. H. Damgaard, and B. Feng, JHEP 09, 094 (2016), arXiv:1608.00006.

[181] H. Elvang et al., JHEP 12, 181 (2014), arXiv:1410.0621.

[182] A. Hodges, JHEP 08, 051 (2013), arXiv:1004.3323.

[183] C. Schwinn and S. Weinzierl, JHEP 04, 072 (2007), hep-ph/0703021.

[184] L. J. Dixon, J. M. Henn, J. Plefka, and T. Schuster, JHEP 01, 035 (2011), arXiv:1010.3991.

[185] T. Melia, Phys.Rev. D88, 014020 (2013), arXiv:1304.7809.

[186] T. Melia, Phys.Rev. D89, 074012 (2014), arXiv:1312.0599.

[187] T. Melia, JHEP 12, 107 (2015), arXiv:1509.03297.

[188] S. Weinzierl, JHEP 1503, 141 (2015), arXiv:1412.5993.

[189] H. Johansson and A. Ochirov, JHEP 01, 170 (2016), arXiv:1507.00332.

[190] A. B. Goncharov, Math. Res. Lett. 5, 497 (1998).

[191] J. M. Borwein, D. M. Bradley, D. J. Broadhurst, and P. Lisonek, Trans. Amer. Math. Soc. 353:3, 907 (2001), math.CA/9910045.

[192] J. A. M. Vermaseren, Int. J. Mod. Phys. A14, 2037 (1999), hep-ph/9806280.

[193] E. Remiddi and J. A. M. Vermaseren, Int. J. Mod. Phys. A15, 725 (2000), hep-ph/9905237.

[194] A. V. Kotikov, Phys. Lett. B254, 158 (1991).

[195] A. V. Kotikov, Phys. Lett. B267, 123 (1991).

[196] E. Remiddi, Nuovo Cim. A110, 1435 (1997), hep-th/9711188.

[197] T. Gehrmann and E. Remiddi, Nucl. Phys. B580, 485 (2000), hep-ph/9912329.

[198] A. B. Goncharov, (2001), math.AG/0103059.

[199] S. Moch, P. Uwer, and S. Weinzierl, J. Math. Phys. 43, 3363 (2002), hep-ph/0110083.

[200] F. Brown, C. R. Acad. Sci. Paris 342, 949 (2006).

[201] F. Brown, Commun. Math. Phys. 287, 925 (2008), arXiv:0804.1660.
[202] E. Panzer, Comput. Phys. Commun. 188, 148 (2014), arXiv:1403.3385.
[203] M. Argeri and P. Mastrolia, Int. J. Mod. Phys. A22, 4375 (2007), arXiv:0707.4037.
[204] S. Müller-Stach, S. Weinzierl, and R. Zayadeh, Commun.Math.Phys. 326, 237 (2014), arXiv:1212.4389.
[205] J. M. Henn, Phys. Rev. Lett. 110, 251601 (2013), arXiv:1304.1806.
[206] J. Ablinger et al., Comput. Phys. Commun. 202, 33 (2016), arXiv:1509.08324.
[207] S. Müller-Stach, S. Weinzierl, and R. Zayadeh, Commun. Num. Theor. Phys. 6, 203 (2012), arXiv:1112.4360.
[208] L. Adams, C. Bogner, and S. Weinzierl, J. Math. Phys. 54, 052303 (2013), arXiv:1302.7004.
[209] S. Bloch and P. Vanhove, J. Numb. Theor. 148, 328 (2015), arXiv:1309.5865.
[210] L. Adams, C. Bogner, and S. Weinzierl, J. Math. Phys. 55, 102301 (2014), arXiv:1405.5640.
[211] L. Adams, C. Bogner, and S. Weinzierl, J. Math. Phys. 56, 072303 (2015), arXiv:1504.03255.
[212] L. Adams, C. Bogner, and S. Weinzierl, J. Math. Phys. 57, 032304 (2016), arXiv:1512.05630.
[213] E. Remiddi and L. Tancredi, Nucl.Phys. B880, 343 (2014), arXiv:1311.3342.
[214] S. Bloch, M. Kerr, and P. Vanhove, (2016), arXiv:1601.08181.
[215] E. Remiddi and L. Tancredi, Nucl. Phys. B907, 400 (2016), arXiv:1602.01481.
[216] L. Adams, C. Bogner, A. Schweitzer, and S. Weinzierl, J. Math. Phys. 57, 122302 (2016), arXiv:1607.01571.
[217] R. Bonciani et al., JHEP 12, 096 (2016), arXiv:1609.06685.
[218] J. Broedel, C. R. Mafra, N. Matthes, and O. Schlotterer, JHEP 07, 112 (2015), arXiv:1412.5535.
[219] G. P. Korchemsky and E. Sokatchev, Nucl. Phys. B829, 478 (2010), arXiv:0907.4107.
[220] L. F. Alday, B. Eden, G. P. Korchemsky, J. Maldacena, and E. Sokatchev, JHEP 09, 123 (2011), arXiv:1007.3243.
[221] Z. Bern, J. J. M. Carrasco, L. J. Dixon, H. Johansson, and R. Roiban, Phys. Rev. D82, 125040 (2010), arXiv:1008.3327.
[222] D. A. Kosower, R. Roiban, and C. Vergu, Phys. Rev. D83, 065018 (2011), arXiv:1009.1376.
[223] L. J. Dixon, J. M. Drummond, and J. M. Henn, JHEP 06, 100 (2011), arXiv:1104.2787.
[224] Z. Bern, J. J. M. Carrasco, H. Johansson, and R. Roiban, Phys. Rev. Lett. 109, 241602 (2012), arXiv:1207.6666.
[225] B. Eden, P. Heslop, G. P. Korchemsky, V. A. Smirnov, and E. Sokatchev, Nucl. Phys. B862, 123 (2012), arXiv:1202.5733.
[226] T. Bargheer, Y.-t. Huang, F. Loebbert, and M. Yamazaki, Phys. Rev. D91, 026004 (2015), arXiv:1407.4449.

[227] L. Ferro, T. Łukowski, and M. Staudacher, Nucl. Phys. B889, 192 (2014), arXiv:1407.6736.

[228] L. J. Dixon, J. M. Drummond, C. Duhr, and J. Pennington, JHEP 06, 116 (2014), arXiv:1402.3300.

[229] S. Caron-Huot, L. J. Dixon, A. McLeod, and M. von Hippel, Phys. Rev. Lett. 117, 241601 (2016), arXiv:1609.00669.

[230] D. Chicherin et al., JHEP 03, 031 (2016), arXiv:1506.04983.

[231] C. R. Mafra, O. Schlotterer, S. Stieberger, and D. Tsimpis, Phys.Rev. D83, 126012 (2011), arXiv:1012.3981.

[232] N. J. MacKay, Int. J. Mod. Phys. A20, 7189 (2005), arXiv:hep-th/0409183.

[233] J. M. Drummond, J. Henn, G. P. Korchemsky, and E. Sokatchev, Nucl. Phys. B795, 52 (2008), arXiv:0709.2368.

[234] J. M. Drummond, J. Henn, G. P. Korchemsky, and E. Sokatchev, Nucl. Phys. B828, 317 (2010), arXiv:0807.1095.

[235] J. M. Drummond, J. M. Henn, and J. Plefka, JHEP 05, 046 (2009), arXiv:0902.2987.

[236] J. M. Drummond and L. Ferro, JHEP 12, 010 (2010), arXiv:1002.4622.

[237] J. M. Drummond and L. Ferro, JHEP 07, 027 (2010), arXiv:1001.3348.

[238] N. Beisert, J. Henn, T. McLoughlin, and J. Plefka, JHEP 04, 085 (2010), arXiv:1002.1733.

[239] L. Ferro, (2011), arXiv:1107.1776.

[240] B. Keller, (2008), arXiv:0807.1960.

[241] A. B. Goncharov, M. Spradlin, C. Vergu, and A. Volovich, Phys. Rev. Lett. 105, 151605 (2010), arXiv:1006.5703.

[242] J. Golden, A. B. Goncharov, M. Spradlin, C. Vergu, and A. Volovich, JHEP 01, 091 (2014), arXiv:1305.1617.

[243] J. Golden, M. F. Paulos, M. Spradlin, and A. Volovich, J. Phys. A47, 474005 (2014), arXiv:1401.6446.

[244] J. M. Drummond, G. Papathanasiou, and M. Spradlin, JHEP 03, 072 (2015), arXiv:1412.3763.

[245] Y.-X. Chen, Y.-J. Du, and B. Feng, JHEP 01, 081 (2011), arXiv:1011.1953.

[246] Z. Bern and T. Dennen, Phys.Rev.Lett. 107, 081601 (2011), arXiv:1103.0312.

[247] Z. Bern, S. Davies, T. Dennen, A. V. Smirnov, and V. A. Smirnov, Phys. Rev. Lett. 111, 231302 (2013), arXiv:1309.2498.
[248] R. Monteiro, D. O’Connell, and C. D. White, JHEP 12, 056 (2014), arXiv:1410.0239.
[249] F. Cachazo, S. He, and E. Y. Yuan, JHEP 01, 121 (2015), arXiv:1409.8256.
[250] F. Cachazo, S. He, and E. Y. Yuan, JHEP 07, 149 (2015), arXiv:1412.3479.
[251] H. Johansson and A. Ochirov, JHEP 11, 046 (2015), arXiv:1407.4772.
[252] M. Chiodaroli, M. Gunaydin, H. Johansson, and R. Roiban, Phys. Rev. Lett. 117, 011603 (2016), arXiv:1512.09130.
[253] M. Chiodaroli, Q. Jin, and R. Roiban, JHEP 01, 152 (2014), arXiv:1311.3600.
[254] N. E. J. Bjerrum-Bohr, J. F. Donoghue, B. R. Holstein, L. Planté, and P. Vanhove, Phys. Rev. Lett. 114, 061301 (2015), arXiv:1410.7590.
[255] N. E. J. Bjerrum-Bohr, J. F. Donoghue, B. R. Holstein, L. Plante, and P. Vanhove, (2016), arXiv:1609.07477.
[256] D. Nandan, J. Plefka, O. Schlotterer, and C. Wen, JHEP 10, 070 (2016), arXiv:1607.05701.
[257] L. de la Cruz, A. Kniss, and S. Weinzierl, (2016), arXiv:1607.06036.
[258] S. Stieberger and T. R. Taylor, Phys. Lett. B739, 457 (2014), arXiv:1409.4771.
[259] S. Stieberger and T. R. Taylor, Phys. Lett. B744, 160 (2015), arXiv:1502.00655.
[260] S. Stieberger and T. R. Taylor, Phys. Lett. B750, 587 (2015), arXiv:1508.01116.
[261] S. Stieberger and T. R. Taylor, Nucl. Phys. B903, 104 (2016), arXiv:1510.01774.
[262] S. Stieberger and T. R. Taylor, Nucl. Phys. B913, 151 (2016), arXiv:1606.09616.
[263] M. Chiodaroli, M. Günaydin, H. Johansson, and R. Roiban, JHEP 01, 081 (2015), arXiv:1408.0764.
[264] M. Chiodaroli, M. Gunaydin, H. Johansson, and R. Roiban, (2015), arXiv:1511.01740.
[265] M. Chiodaroli, (2016), arXiv:1607.04129.