Least energy positive solutions for $d$-coupled Schrödinger systems with critical exponent in dimension three

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Abstract

In the present paper, we consider the coupled Schrödinger systems with critical exponent:

$$
\begin{aligned}
-\Delta u_i + \lambda_i u_i &= \sum_{j=1}^{d} \beta_{ij} |u_j|^3 |u_i| \quad \text{in } \Omega, \\
|u_i| &\in H_0^1(\Omega), \quad i = 1, 2, \ldots, d.
\end{aligned}
$$

Here, $\Omega \subset \mathbb{R}^3$ is a smooth bounded domain, $d \geq 2$, $\beta_{ii} > 0$ for every $i$, and $\beta_{ij} = \beta_{ji}$ for $i \neq j$. We study a Brézis-Nirenberg type problem: $-\lambda_1(\Omega) < \lambda_1, \cdots, \lambda_d < -\lambda^*(\Omega)$, where $\lambda_1(\Omega)$ is the first eigenvalue of $-\Delta$ with Dirichlet boundary conditions and $\lambda^*(\Omega) \in (0, \lambda_1(\Omega))$. We acquire the existence of least energy positive solutions to this system for weakly cooperative case ($\beta_{ij} > 0$ small) and for purely competitive case ($\beta_{ij} \leq 0$) by variational arguments. The proof is performed by mathematical induction on the number of equations, and requires more refined energy estimates for this system. Besides, we present a new nonexistence result, revealing some different phenomena comparing with the higher-dimensional case $N \geq 5$. It seems that this is the first paper to give a rather complete picture for the existence of least energy positive solutions to critical Schrödinger system in dimension three.

Key words: Schrödinger system; Critical exponent; Dimension three; Least energy positive solutions; Variational arguments.

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1 Introduction

Consider the following elliptic system with $d \geq 2$ equations

$$
\begin{aligned}
-\Delta u_i + \lambda_i u_i &= \sum_{j=1}^{d} \beta_{ij} |u_j|^p |u_i|^{p-2} u_i \quad \text{in } \Omega, \\
|u_i| &\in H_0^1(\Omega), \quad i = 1, 2, \ldots, d.
\end{aligned}
$$

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where \( N \geq 3, 2p \in (2, 2^*], 2^* = \frac{2N}{N-2} \) is the Sobolev critical exponent, \( \beta_{ii} > 0 \) for every \( i \), \( \beta_{ij} = \beta_{ji} \) when \( i \neq j \). System (1.1) appears when looking for standing wave solutions \( \Psi_i(x,t) = e^{i\lambda_i t} u_i(x) \) of time-dependent coupled nonlinear Schrödinger system

\[
i\partial_t \Psi_i + \Delta \Psi_i + \sum_{j=1}^{d} \beta_{ij} |\Psi_j|^p |\Psi_i|^{p-2} \Psi_i = 0, \quad i = 1, \ldots, d,
\]

where \( i \) is the imaginary unit. This system originates from many physical models; for example, system (1.1) can be used to explain Bose-Einstein condensation (see [34]). In quantum mechanics, the solutions \( \Psi_i(i = 1, \ldots d) \) are the corresponding condensate amplitudes, \( \beta_{ii} \) represent self-interactions within the same component, while \( \beta_{ij} (i \neq j) \) describe the strength and type of interactions between different components \( u_i \) and \( u_j \). Furthermore, \( \beta_{ij} > 0 \) means the interaction is cooperative, while \( \beta_{ij} < 0 \) represents the interaction is competitive.

Set \( H_d := (H^1_0(\Omega))^d \). Note that \( \beta_{ij} = \beta_{ji} \), then solutions of (1.1) correspond to the critical points of the \( C^2 \)- energy functional \( J : H_d \to \mathbb{R} \) defined by

\[
J(u) = \frac{1}{2} \sum_{i=1}^{d} \|u_i\|^2 - \frac{1}{6} \sum_{i,j=1}^{d} \int_{\Omega} \beta_{ij} |u_i|^p |u_j|^p,
\]

where \( u = (u_1, \ldots, u_d) \) and \( \|u_i\|^2 := \int_{\Omega} (|\nabla u_i|^2 + \lambda_i u_i^2) \).

We say a solution is trivial if all its components are vanishing. We say a solution is semi-trivial if there exist at least one (but not all) vanishing component. We say a solution is nontrivial if all its components are nontrivial. However, we are interested in the existence of positive solutions, i.e., \( u \) solving (1.1) such that \( u_i > 0 \) for every \( i \). In particular, we mainly focus on the existence of positive least energy solutions (or positive ground state), which attain

\[
C_{LES} := \inf \{ J(u) : u \text{ is a solution of (1.1) such that } u_i > 0 \text{ for all } i = 1, 2, \ldots, d \}.
\]

Since the system may admit many semi-trivial solutions, we will also consider

\[
\inf \{ J(u) : J'(u) = 0, \quad u \in H^1_0(\Omega; \mathbb{R}^d), \quad u \neq 0 \}.
\]

We call a solution \( u \neq 0 \) is a generalized ground state solution if it achieves (1.3).

In the last twenty years, for the subcritical case \( 2 < 2p < 2^* \), the existence of solutions to (1.1) has been investigated extensively. For the two equations case \( d = 2 \), where there is only one interaction constant \( \beta = \beta_{12} = \beta_{21} \), see [4, 12, 22, 30, 35] and reference therein. For an arbitrary number of equations \( d \geq 3 \), starting from Lin and Wei [21], where the authors presented the nonexistence of least energy positive solutions for the purely competitive case and the existence of least energy positive solutions for the purely cooperative case with some additional conditions. In [17] the authors studied the existence and nonexistence of positive ground state solutions to system (1.1) for the purely cooperative case. For the mixed case, that is, the existence of at least two pairs, \((i_1,j_1)\) and \((i_2,j_2)\), such that \( i_1 \neq j_1, i_2 \neq j_2, \beta_{i_1,j_1} > 0 \) and \( \beta_{i_2,j_2} < 0 \), the existence of solutions has attracted great interest, see [7, 8, 17, 20, 28, 29].
Different from the subcritical case, we are more concerned about the critical equation in this article, i.e., \(2p = 2^*\). For the single equation case \(d = 1\), the system (1.1) turns into the classical Brézis-Nirenberg problem \([5]\), where the existence of a positive ground state solution is shown for \(-\lambda_1(\Omega) < \lambda_i < 0\) when \(N \geq 4\). However, in sharp contrast to the high-dimensional situation, from the pioneering paper \([5]\) we learn that there exist essential differences and difficulties in the three-dimensional case \((N = 3)\).

When \(N = 3\), system (1.1) reduces to the following problem

\[
−\Delta u + \lambda_i u = \beta_i |u|^4 u, \quad u \in H^1_0(\Omega).
\]

In \([5]\), the authors proved that (1.4) has a least energy positive solution \(\omega_i \in C^2(\Omega) \cap C^1(\overline{\Omega})\) if \(\lambda_i \in (-\lambda_1(\Omega), -\lambda^*(\Omega))\), where \(\lambda^*(\Omega) = \frac{\pi^2}{4R_0^2}\) with \(R_0 = \sup \{R|x \in \Omega, B_R(x) \subset \Omega\}\), and moreover

\[
m_i := \frac{1}{2} \int_\Omega (|\nabla \omega_i|^2 + \lambda_i \omega_i^2) - \frac{1}{6} \int_\Omega \mu_i |\omega_i|^6 = \frac{1}{3} \|\omega_i\|_1^2 < \frac{1}{3} \beta_i^{−\frac{1}{2}} S_i^{\frac{1}{2}},
\]

where \(S_i\) is the Sobolev best constant of \(D^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)\). In particular, when \(\Omega\) is a ball \(B\) in \(\mathbb{R}^3\), the authors \([5]\) presented that (1.4) admits a least energy positive solution if and only if \(\lambda_i \in (-\lambda_1(B), -\lambda_1(B))\).

In \([11]\), the authors proved that (1.4) has a ground state solution with \(-\lambda_{n+1}(\Omega) < \lambda_i < -\lambda_{n+1}(\Omega) + S_i^{\frac{1}{2}}\), where \(n \geq 1\) and \(\lambda_{n+1}(\Omega)\) is the \(n + 1\)-th Dirichlet eigenvalue of \((-\Delta, \Omega)\) with multiplicity. For more results related to the Brézis-Nirenberg problem, see \([9, 13, 25, 27]\).

When \(N = 4\) and the system (1.1) consists of two equations (that is, \(d = 2\)), Chen and Zou \([10]\) considered the problem under the assumption that there is only one interaction constant \(\beta = \beta_{12} = \beta_{21}\). Then they proved that there exist \(0 < \beta_1 < \beta_2\) such that system (1.1) has a least energy positive solution if \(\beta \in (-\infty, \beta_1) \cup (\beta_2, +\infty)\) when \(N = 4\). Subsequently, Chen and Zou \([13]\) showed that system (1.1) has a least energy positive solution for any \(\beta \neq 0\) when \(N \geq 5\).

When the number of the system (1.1) is \(d \geq 3\) and the dimension \(N \leq 4\), Guo, Luo and Zou \([18]\) studied the pure cooperative system defined on a bounded smooth domain of \(\mathbb{R}^N\), they obtained the existence and classification of the least energy positive solutions to (1.1) under the hypotheses \(-\lambda_1(\Omega) < \lambda_1 = \cdots = \lambda_d < 0\) and some additional technical conditions on the coupling coefficients. We remark that when \(N = 2, 3\) the system considered in \([18]\) is subcritical.

When \(N \geq 4\) and \(2p = 2^*\), in \([16, 37]\) the authors obtained existence of least energy positive solutions in a bounded smooth domain of \(\mathbb{R}^N\) for the purely competitive cases. While for the purely cooperative case and \(N \geq 5\), Yin and Zou \([39]\) obtained the existence of positive ground state solutions to (1.1). Recently, Tavares and You \([31]\) dealt with the existence of least energy positive solutions for the mixed case in a bounded smooth domain of \(\mathbb{R}^4\). Afterwards, Tavares, You and Zou \([33]\) established the existence of least energy positive solutions for the mixed case with \(N \geq 5\). For the other topics regarding critical system, see \([15, 19, 23, 24]\).

All the papers that deal with system (1.1) with \(d \geq 2\) in the critical case \((2p = 2^*)\) mainly focus on the higher dimensional case \((N \geq 4)\). To the best of our knowledge, there are only three papers \([20, 38, 40]\) studying system (1.1) for the critical case on a smooth bounded domain with \(N = 3\) and \(d = 2\) in the literature. In \([20, 38]\) the authors proved that there exists \(\beta > 0\) such that system (1.1) has a least energy positive solution if \(\beta_{12} > \beta\). Recently, You and Zou \([40]\) proved that system (1.1) has a least energy positive solution for \(\beta_{12} > 0\) small. Those papers mentioned above only deal with the purely cooperative case \((\beta_{12} > 0)\).
As far as we know, there is no paper considering the existence of least energy positive solutions of (1.1) with $N = 3$ and $2p = 2^*$ for the purely competitive cases ($\beta_{12} < 0$) or multi equation coupling case ($d \geq 3$). The present paper makes a first contribution in this direction. We consider the following critical system

\begin{equation}
\begin{cases}
-\Delta u_i + \lambda_i u_i = \sum_{j=1}^{d} \beta_{ij} |u_j|^3|u_i| & \text{in } \Omega \subset \mathbb{R}^3, \\
u_i \in H_{0}^{1}(\Omega), & i = 1, 2, ..., d.
\end{cases}
\end{equation}

Throughout this text we always work under the following assumptions

\begin{equation}
-\lambda_1(\Omega) < \lambda_1, ..., \lambda_d < -\lambda^*(\Omega), \quad \Omega \text{ is a bounded smooth domain of } \mathbb{R}^3,
\end{equation}

and

\begin{equation}
\beta_{ii} > 0 \quad \forall i = 1, 2, ..., d, \quad \beta_{ij} = \beta_{ji} \quad \forall i, j = 1, 2, ..., d, i \neq j,
\end{equation}

where $\lambda_1(\Omega)$ denotes the first eigenvalue of $-\Delta$ with Dirichlet boundary conditions. We note $\lambda^*(\Omega) \in (0, \lambda_1(\Omega))$.

### 1.1 Main results

Consider the Nehari type set

$$\mathcal{N} = \left\{ u \in \mathbb{H}_{d} : u_i \neq 0, \|u_i\|^2 = \sum_{j=1}^{d} \int_{\Omega} \beta_{ij} |u_i|^3|u_j|^3 \text{ for every } i = 1, 2, ..., d \right\},$$

and the infimum of $J$ on the set $\mathcal{N}$

$$\mathcal{C} := \inf_{u \in \mathcal{N}} J(u) = \inf_{u \in \mathcal{N}} \frac{1}{3} \sum_{i=1}^{d} \|u_i\|^2.$$

It is easy to see that $\mathcal{C}_{LES} = \mathcal{C}$ if $\mathcal{C}$ is attained on $\mathcal{N}$, where $\mathcal{C}_{LES}$ is defined in (1.2).

Our first result of this paper is the following

**Theorem 1.1.** Assume that (1.7) and (1.8) hold. There exists a constant $K = K(\Omega, \{\lambda_i\}_{i=1}^{d}, \{\beta_{ii}\}_{i=1}^{d}) > 0$ such that if

$$0 < \beta_{ij} < K \quad \forall i, j = 1, 2, ..., d, i \neq j,$

then $\mathcal{C}$ is achieved by a positive $u \in \mathcal{N}$, and the system (1.6) has a positive least energy solution.

**Remark 1.1.** Theorem 1.1 shows that the system (1.6) has a least energy positive solution when the interactions between different components are weakly cooperative. We mention that $K$ is only dependent on $\Omega, \lambda_i, \beta_{ii}, i = 1, \ldots, d$. In particular, we will see (2.23) ahead for the explicit expression of $K$.

**Remark 1.2.** When $N = 4$ and $p = 2$, in [31] the authors have proved that the system (1.1) has a least energy positive solution for the weakly cooperative case, see [31] Corollary 1.6.
In [31], because of lack of compactness, the authors established some precise energy estimates and compared the least energy level to (1.1) with that of some kinds of limit system (\( \Omega = \mathbb{R}^N \) and \( \lambda_i = 0 \)) and appropriate subsystem. In order to obtain the corresponding energy estimate, the authors in [31] took a cutoff function \( \xi \) such that \( \xi U \in H^1_0(\Omega) \), where \( U \) is the Aubin-Talenti bubble (see (3.4)). Unlike the higher dimensional case \( N \geq 4 \), the Aubin-Talenti bubble \( U \) decays slowly in dimension three. Therefore, \( \xi \) can only be chosen as some particular functions for \( N = 3 \) (this fact has been implicitly pointed out by [5]). So it is difficult to acquire the corresponding energy estimates for the system (1.6) with \( d \geq 3 \) by using the method in [31]. Thus we need to introduce new ideas to deal with this problem. In this paper, we compare the least energy level to (1.6) with that of single equation \( (d = 1) \) and appropriate subsystem, and establish new energy estimates (see Proposition 2.1 and Proposition 2.2).

To study the existence of ground state solutions of (1.6), we consider the following Nehari manifold

\[
M := \left\{ u \in H^d : u \neq 0, \sum_{i=1}^{d} \| u_i \|^2 = \sum_{i,j=1}^{d} \int_{\Omega} \beta_{ij} |u_i|^3 |u_j|^3 \right\},
\]

and the level

\[
A := \inf \{ J(u) : u \in M \}.
\]

Observe that a solution \( u \neq 0 \) is a generalized ground state solution if it achieves \( A \). It follows from \( \mathcal{N} \subset M \) that \( A \leq C \).

Theorem 1.1 shows that the system (1.6) has a positive least energy solution when the interactions between different components are weakly cooperative. While the next theorem shows that the system (1.6) does not have any nontrivial generalized ground state solution for the weakly cooperative case.

**Theorem 1.2.** Assume that (1.7), (1.8) hold, and \( \beta_{ij} \geq 0 \) for any \( i \neq j \). Then \( A \) is attained and system (1.6) has a generalized ground state solution. However, if \( \beta_{ij} \equiv b \), for any \( i \neq j \), and

\[
0 < b < 2^{3-d} \sqrt{\max_{1 \leq i \leq d} \{ \beta_{ii} \}} \min_{1 \leq i \leq d} \{ \beta_{ii} \},
\]

then system (1.6) has no nontrivial generalized ground state solutions, i.e., strictly we have \( A < C \) and \( A \) is attained only by a semi-trivial element.

**Remark 1.3.** In [32, 39] the authors proved that system (1.1) with \( 2p = 2^* \) and \( N \geq 5 \) has a positive generalized ground state solution for any \( \beta_{ij} \geq 0 \), \( i \neq j \) (the purely cooperative case). Hence, the case of \( N = 3 \) is quite different from the higher-dimensional case \( N \geq 5 \) about the critical system (1.1).

In subcritical case, when \( d = 2 \) and \( p \geq 2 \), the author in [22] showed that system (1.1) does not have any nontrivial generalized ground state solutions if \( \beta_{12} \in (0, b_0) \), where \( b_0 > 0 \); When \( d \geq 3 \) and \( p = 2 \), the authors in [17] proved that system (1.1) does not have any nontrivial generalized ground state solutions if \( \beta_{ij} = b \in (0, b_1) \) (see [17 Theorem 1.7]), where \( b_1 > 0 \).

In subcritical case, it is easy to obtain the existence of ground state solutions. However, lack of compactness makes system (1.6) very complicated. In this paper, we acquire the existence of ground state solutions by establishing a new energy estimate (see Lemma 3.2). Then inspired from [17] we show that the generalized ground state is semi-trivial for the weakly cooperative case. But the authors in [17] take full use of the fact \( p = 2 \), and the method can not be used directly to deal with system (1.6) (the case \( p = 3 \)). Thus, we need some important modifications for our proof.
For the purely competitive case, we have the following theorem.

**Theorem 1.3.** Assume that (1.7), (1.8) hold and
\[ \beta_{ij} \leq 0, \text{ for any } i \neq j. \]
Then system (1.6) has a positive least energy solution.

**Remark 1.4.** Up to our knowledge, Theorem 1.3 is the first result to deal with the existence of positive least energy solutions of system (1.6) for the purely competitive case.

We recall the paper [16], where the authors established energy estimate by induction on the number of equation, then they obtained the existence of least energy positive solutions for the purely competitive case when \( N \geq 4 \). Here, we established the corresponding energy estimate by using this idea for \( N = 3 \) (see Proposition 4.1). However, we need more precise estimates due to the nature of the three-dimensional Brézis-Nirenberg type problem. To the best of our knowledge, it is first time to establish this energy estimate (see Proposition 4.1) of system (1.6) for the purely competitive case.

### 1.2 Structure of the paper

Section 2 is devoted to the proof of Theorem 1.1, in subsection 2.1, we present a uniform energy estimate and some preliminary results; in subsection 2.2, we establish new energy estimates, see Proposition 2.1 and Proposition 2.2; in subsection 2.3, we will give the proof of Theorem 1.1 by the method of induction on the number of equations. Section 3 is devoted to the proof of Theorem 1.2, in subsection 3.1, we introduce the limit system; in subsection 3.2, we give the proof of Theorem 1.2. Section 4 is devoted to the proof of Theorem 1.3.

### 1.3 Further notations

- The \( L^p(\Omega) \) norms will be denoted by \( | \cdot |_p, 1 \leq p \leq \infty \).
- Set \( (\mathbb{R}^+)^d = \{ x = (x_1, ..., x_d) : x_i > 0, \text{ for every } i = 1, 2, ..., d \} \).

For a vector \( X = (x_1, ..., x_d) \in \mathbb{R}^d \), denote the transpose of \( X \) by \( X^T \) and define the norm by
\[ |X| = \sqrt{x_1^2 + \cdots + x_d^2}. \]

- For a subset \( I \subset \{1, \cdots, d\} \) with \( |I| = q \), we denote the number of elements in set \( I \) by \( |I| \) and define \((u_i)_{i \in I} = (u_{i_1}, \cdots, u_{i_q})\), where \( I = \{i_1, \cdots, i_q\} \) and \( i_1 < i_2 < \cdots < i_q \).
- Let \( \tilde{S} \) be the Sobolev best constant of \( \mathcal{D}^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3) \),
\begin{equation}
\tilde{S} = \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^3) \setminus \{0\}} \frac{\int_{\mathbb{R}^3} |\nabla u|^2}{(\int_{\mathbb{R}^3} |u|^6)^{\frac{1}{6}}},
\end{equation}
where \( \mathcal{D}^{1,2}(\mathbb{R}^3) = \{ u \in L^2(\mathbb{R}^3) : |\nabla u| \in L^2(\mathbb{R}^3) \} \) with norm \( \|u\|_{\mathcal{D}^{1,2}} := (\int_{\mathbb{R}^3} |\nabla u|^2)^{\frac{1}{2}} \).
- Let
\begin{equation}
S := \inf_{i=1,2,\ldots,d} \inf_{u \in \mathcal{H}_0^1(\Omega) \setminus \{0\}} \frac{\|u\|}{(\int_{\Omega} |u|^6)^{\frac{1}{6}}}.
\end{equation}
Moreover, since $\lambda_i \in (-\lambda_1(\Omega), -\lambda^*(\Omega))$ we have
\[
S|u_0|^2 \leq \|u\|_i^2 \leq |\nabla u|^2_2 \quad \forall u \in H^1_0(\Omega).
\]

- We use “$\rightharpoonup$” and “$\to$” to denote the strong convergence and weak convergence in corresponding space respectively.
- The capital letter $C$ will appear as a constant which may vary from line to line, and $C_1, C_2, C_3$ are fixed constants.

## 2 Least energy positive solutions for the weakly cooperative case

In this section, we present the proof of Theorem 1.1. Given $I \subseteq \{1, 2, \ldots, d\}$ with $|I| = q, 1 \leq q \leq d$, we consider the following subsystem

\[
(2.1) \begin{cases}
-\Delta u_i + \lambda_i u_i = \sum_{j \in I} \beta_{ij} |u_j|^3 |u_i| u_i & \text{in } \Omega, \ i \in I, \\
u_i \in H^1_0(\Omega), \ i \in I,
\end{cases}
\]

and define
\[
J_I(u_I) = \frac{1}{2} \sum_{i \in I} \|u_i\|_i^2 - \frac{1}{6} \sum_{i,j \in I} \int_{\Omega} \beta_{ij} |u_i|^3 |u_j|^3,
\]

\[
N_I = \left\{ u_I \in H^q : u_i \not\equiv 0 \text{ and } \sum_{j \in I} \int_{\Omega} \beta_{ij} |u_i|^3 |u_j|^3 = 0, \ i \in I \right\},
\]

\[
C_I = \inf_{u_I \in N_I} J_I(u_I) = \inf_{u_I \in N_I} \frac{1}{3} \sum_{i \in I} \|u_i\|_i^2.
\]

Obviously, we have $C = C_{\{1, \ldots, d\}}$.

### 2.1 Preliminary results

In this subsection, we present some preliminary lemmas, which are used to prove Theorem 1.1. Firstly, we show a uniform energy estimate for level $C_I$.

**Lemma 2.1.** Take

\[
\overline{C} = \frac{d}{3} \max_{1 \leq i \leq d} \left\{ \frac{1}{\sqrt[\beta_{ii}]} \right\} S^2,
\]

then for every $I \subseteq \{1, 2, \ldots, d\}$, there holds

\[
C_I \leq \overline{C}.
\]

**Proof.** We follow the arguments of Lemma 2.1 in [28] and Lemma 3.1 in [31] to prove this lemma.

For every $I \subseteq \{1, 2, \ldots, d\}$ with $|I| = q$, we take $\hat{u}_i \not\equiv 0, i \in I$ such that $\hat{u}_i \cdot \hat{u}_j \equiv 0$, whenever $i \neq j$. Denote $u_i = t_i \hat{u}_i$, where
\[
t_i = \frac{\|\hat{u}_i\|_i^{\frac{2}{\beta_{ii}}} \hat{u}_i}{(\beta_{ii})^{\frac{1}{2}} |\hat{u}_i|_i^2} \quad \text{for every } i \in I,
\]
then \( \tilde{u} \neq 0 \) and \((\tilde{u}_i)_{i \in I} \in N_I \). Thus, since \( \lambda_i < -\lambda^*(\Omega) < 0 \), we infer that

\[
C_I \leq J_I((\tilde{u}_i)_{i \in I}) = \frac{1}{3} \sum_{i \in I} \|\tilde{u}_i\|^2 = \frac{1}{3} \sum_{i \in I} t_i^2 \|\tilde{u}_i\|^2
\]

\[
< \frac{1}{3} \sum_{i \in I} \frac{1}{\sqrt{\beta_{ii}}} \|\tilde{u}_i\|^2
\]

\[
\leq \frac{1}{3} \max_{i \in I} \left\{ \frac{1}{\sqrt{\beta_{ii}}} \right\} \sum_{i \in I} \|\tilde{u}_i\|^2.
\]

Notice the choice of \( \hat{u}_i \), we have

\[
C_I \leq \frac{1}{3} \max_{i \in I} \left\{ \frac{1}{\sqrt{\beta_{ii}}} \right\} \sum_{i \in I} \tilde{S}^2(\Omega_i).
\]

On the other hand, for every open subset \( \Omega' \) of \( \mathbb{R}^3 \), by [36, Proposition 1.43] we have,

\[
\tilde{S}(\Omega') = \inf_{u \in H^1_0(\Omega')} \{0\} \int_{\Omega'} |\nabla u|^2 \leq \frac{1}{3} \sum_{i \in I} \|\hat{u}_i\|^2,
\]

where \( \tilde{S} \) is defined by (1.9). Therefore,

\[
C_I \leq \frac{1}{3} \max_{i \leq d} \left\{ \frac{1}{\sqrt{\beta_{ii}}} \right\} \tilde{S}^2 \leq \frac{d}{3} \max_{i \leq d} \left\{ \frac{1}{\sqrt{\beta_{ii}}} \right\} \tilde{S}^2,
\]

which yields that \( C_I \leq \overline{C} \), where \( \overline{C} \) is defined in (2.2).

Proof. For any \( u \in N_I \) with \( J_I(u) \leq 2\overline{C} \), we have

\[
\sum_{i \in I} \|u_i\|^2 \leq 6\overline{C}.
\]

Therefore,

\[
S\left( \int_{\Omega} |u_i|^6 \right)^{\frac{1}{6}} \leq \|u_i\|^2 \leq \sum_{i \in I} \|u_i\|^2 \leq 6\overline{C},
\]

Define

\[
(2.3) \quad K_1 = \frac{7S^3}{12(6\overline{C})^2},
\]

where \( S \) is defined in (1.10).

**Lemma 2.2.** If

\[
\beta_{ii} > 0 \quad \forall i = 1, 2, ..., d, \quad 0 < \beta_{ij} < K_1 \quad \forall i, j = 1, 2, ..., d, i \neq j,
\]

and for every \( I \subseteq \{1, 2, ..., d\} \), \( u \in N_I \) with \( J_I(u) \leq 2\overline{C} \), then there exists constant \( C_2 > C_1 > 0 \) dependent only on \( K_1, \lambda_i, \beta_{ii} \), such that

\[
C_1 \leq \int_{\Omega} |u_i|^6 \leq C_2 \quad \text{for every } i \in I.
\]

Proof. For any \( u \in N_I \) with \( J_I(u) \leq 2\overline{C} \), we have

\[
\sum_{i \in I} \|u_i\|^2 \leq 6\overline{C}.
\]

Therefore,

\[
S\left( \int_{\Omega} |u_i|^6 \right)^{\frac{1}{6}} \leq \|u_i\|^2 \leq \sum_{i \in I} \|u_i\|^2 \leq 6\overline{C},
\]

which yields that \( C_I \leq \overline{C} \), where \( \overline{C} \) is defined in (2.2).
that is $\int_{\Omega} |u_i|^6 \leq C_2$. On the other hand, we have

$$S \left( \int_{\Omega} |u_i|^6 \right)^{\frac{1}{6}} \leq \|u_i\|_2^2 = \sum_{j \in I} \int_{\Omega} \beta_{ij} |u_i|^3 |u_j|^3 \leq d \max_{i=1,2,...,d} \{ K_1, \beta_{ii} \} \left( \frac{6C}{S} \right)^{\frac{1}{6}} \left( \int_{\Omega} |u_i|^6 \right)^{\frac{1}{6}},$$

which yields that $\int_{\Omega} |u_i|^6 \geq C_1$. \qed

Before proceeding, we introduce some notations. For every $I \subseteq \{1, 2, ..., d\}$ with $|I| = q$, we define the matrix $A_I(u) = (a_{ij}(u))_{(i,j) \in I^2}$ by

$$a_{ii}(u) = 4 \int_{\Omega} \beta_{ii} |u_i|^6 + \sum_{j \in I, j \neq i} \int_{\Omega} \beta_{ij} |u_i|^3 |u_j|^3, \quad i \in I,$$

(2.4)

$$a_{ij}(u) = 3 \int_{\Omega} \beta_{ij} |u_i|^3 |u_j|^3, \quad i, j \in I, i \neq j.$$

Set

$$\Gamma_I = \{ u \in \mathbb{H}_q : A_I(u) \text{ is strictly diagonally dominant} \}.$$

**Remark 2.1.** (1) For any $u \in \Gamma_I$, we know that $A_I(u)$ is positive definite by Gershgorin circle theorem.

(2) Since there holds the embedding $H^1_I(\Omega) \hookrightarrow L^6(\Omega)$, it is not difficult to verify that $\Gamma_I$ is open in $\mathbb{H}_q$.

(3) The set $N_I \cap \Gamma_I$ is not empty. In fact, following the proof of Lemma 2.1, it is easy to see that $(\tilde{u}_i)_{i \in I} \in N_I \cap \Gamma_I$.

The following lemma shows that $N_I \cap \Gamma_I$ is a natural constraint for the weakly cooperative case.

**Lemma 2.3.** Assume that

$$\beta_{ii} > 0 \quad \forall i = 1, 2, ..., d, \quad 0 < \beta_{ij} < K_1 \quad \forall i, j = 1, 2, ..., d, i \neq j,$$

then for every $I \subseteq \{1, 2, ..., d\}$, the set $N_I \cap \Gamma_I$ is a smooth manifold. Moreover, the constrained critical points of $J_I$ on $N_I \cap \Gamma_I$ are free critical points of $J_I$. In other words, $N_I \cap \Gamma_I$ is a natural constraint.

**Proof.** For every $I \subseteq \{1, 2, ..., d\}$ with $|I| = q$, we take $u = (u_i)_{i \in I} \in N_I \cap \Gamma_I$, we define

$$G_i(u) := \|u_i\|^2 - \sum_{j \in I} \int_{\Omega} \beta_{ij} |u_i|^3 |u_j|^3, \quad i \in I.$$  \hspace{1cm} (2.5)

By a direct computation, for every $v = (u_i)_{i \in I} \in \mathbb{H}_q$ we obtain

$$G'_i(u)v = 2 \int_{\Omega} (\nabla u_i \cdot \nabla v_i + \lambda_i u_i v_i) - 3 \sum_{j \in I} \left( \int_{\Omega} \beta_{ij} |u_i|^3 u_i v_i + \int_{\Omega} \beta_{ij} |u_j|^3 u_j v_j \right).$$

We claim that the set $N_I \cap \Gamma_I$ is a smooth manifold of codimension $q$ in a neighborhood of $u$ in $\mathbb{H}_q$. To verify this, we take $u = (u_i)_{i \in I} \in N_I \cap \Gamma_I$ and prove the map $\hat{G}_u : \mathbb{H}_q \to \mathbb{R}^q$ is a surjective as linear operator, where

$$\hat{G}_u(v) = (G'_i(u)v)_{i \in I}.$$
Note that \( u \in \mathcal{N}_I \). Then take \( v_i = -t_i u_i \), we have

\[
G'_i(u)v = \left( -2\|u_i\|_T^2 + 3 \sum_{j \in I} \int_\Omega \beta_{ij} |u_i|^3 |u_j|^3 \right) t_i + 3 \sum_{j \in I} \left( \int_\Omega \beta_{ij} |u_i|^3 |u_j|^3 \right) t_j
\]

\[
= \left( 4 \int_\Omega \beta_{ii} |u_i|^6 + \sum_{j \in I, j \neq i} \int_\Omega \beta_{ij} |u_i|^3 |u_j|^3 \right) t_i + 3 \sum_{j \in I, j \neq i} \left( \int_\Omega \beta_{ij} |u_i|^3 |u_j|^3 \right) t_j.
\]

Hence, we see that

\[
(G'_i(u)v)^T_{i \in I} = A_I(u)t,
\]

where \( A_I(u) \) is defined in (2.4), \( t = (t_i)_{i \in I} \in \mathbb{R}^q \) and \( v = (-t_i u_i)_{i \in I} \). Since \( u \in \Gamma_I \), the matrix \( A_I(u) \) is strictly diagonally dominant and from Remark 2.1 (1) that \( A_I(u) \) is positive definite. Hence, \( A_I(u) \) is non-singular. Then for any \( h = (h_i)_{i \in I} \in \mathbb{R}^q \), there exists \( v' = (-s_i u_i)_{i \in I} \) such that

\[
(G'_i(u)v')^T_{i \in I} = h,
\]

where \( s = A_I(u)^{-1} h \). Therefore, the claim is true.

Finally, we show that \( \mathcal{N}_I \cap \Gamma_I \) is a natural constraint. Assume that \( G_I \) is achieved by \( u = (u_i)_{i \in I} \in \mathcal{N}_I \cap \Gamma_I \). By Remark 2.1 (2), the constraint \( \mathcal{N}_I \cap \Gamma_I \) is an open subset of \( \mathcal{N}_I \) in the topology of \( \mathbb{H}_q \). Thus the function \( u \) is an inner critical point of \( J_I \) in an open subset of \( \mathcal{N}_I \), and in particular it is a constrained critical point of \( J_I \) on \( \mathcal{N}_I \). Since the set \( \mathcal{N}_I \cap \Gamma_I \) is a smooth manifold of codimension \( q \) in a neighborhood of \( u \) in \( \mathbb{H}_q \), then by the Lagrange multipliers rule there exists \( \mu_i \in \mathbb{R} \), \( i \in I \) such that

\[
J'_I(u) - \sum_{i \in I} \mu_i G'_i(u) = 0,
\]

where \( G_i(u) \) is defined in (2.5). Testing (2.6) by \((0, \ldots, u_i, \ldots, 0)\) for \( i \in I \) and thanks to \( G_i(u) = 0 \) for every \( i \in I \), we have

\[
\left( 4 \int_\Omega \beta_{ii} |u_i|^6 + \sum_{j \in I, j \neq i} \int_\Omega \beta_{ij} |u_i|^3 |u_j|^3 \right) \mu_i + 3 \sum_{j \in I, j \neq i} \left( \int_\Omega \beta_{ij} |u_i|^3 |u_j|^3 \right) \mu_j = 0.
\]

Hence, we have \( A_I(u) \mu = 0 \), \( \mu = (\mu_i)_{i \in I} \). From the above arguments, we know that the matrix \( A_I(u) \) is non-singular, then \( \mu_i = 0 \) for all \( i \in I \). Combining this with (2.6), we have \( J'_I(u) = 0 \). That is, \( u \) is a free critical point of \( J_I \) on \( \mathbb{H}_q \), which means that \( \mathcal{N}_I \cap \Gamma_I \) is a natural constraint.

**Lemma 2.4.** Assume that

\[
\beta_{ii} > 0 \quad \forall i = 1, 2, \ldots, d, \quad 0 < \beta_{ij} < K_1 \quad \forall i, j = 1, 2, \ldots, d, i \neq j,
\]

then we have

\[
\mathcal{N}_I \cap \{ u \in \mathbb{H}_q : J_I(u) \leq 2C \} \subset \Gamma_I.
\]

Moreover, the constrained critical points of \( J_I \) on \( \mathcal{N}_I \) satisfying \( J_I(u) \leq 2C \) are free critical points of \( J_I \).
Proof. Take \( u \in N_f \cap \{ u \in H_q : J_f(u) \leq 2C \} \). We will prove that \( A_f(u) \) is strictly diagonally dominant, that is
\[
4 \int_{\Omega} \beta_{ii}|u_i|^6 + \sum_{j \in I, j \neq i} \int_{\Omega} \beta_{ij}|u_i|^3|u_j|^3 - 3 \sum_{j \in I, j \neq i} \int_{\Omega} \beta_{ij}|u_i|^3|u_j|^3 > 0, \quad i \in I.
\]
Notice that \( \beta_{ij} > 0 \) and \( u \in N_f \), we only need to show
\[
4\|u_i\|_p^2 - 6 \sum_{j \in I, j \neq i} \int_{\Omega} \beta_{ij}|u_i|^3|u_j|^3 > 0, \quad i \in I.
\]
In fact, thanks to the choice of \( K_1 \), we have
\[
6 \sum_{j \in I, j \neq i} \int_{\Omega} \beta_{ij}|u_i|^3|u_j|^3 \leq \frac{6K_1}{S^3} \sum_{j \in I, j \neq i} \|u_i\|_p^3 \|u_j\|_p^3 \leq \frac{6K_1}{S^3}(6C)^2\|u_i\|_p^2 \leq \frac{7}{2}\|u_i\|_p^2.
\]
Thus, by Lemma 2.2 we have
\[
4\|u_i\|_p^2 - 6 \sum_{j \in I, j \neq i} \int_{\Omega} \beta_{ij}|u_i|^3|u_j|^3 \geq \frac{1}{2}\|u_i\|_p^2 \geq \frac{1}{2}S\left(\int_{\Omega}|u_i|^6\right)^{\frac{1}{2}} \geq \frac{1}{2}SC_1^\frac{1}{2}.
\]
It follows that
\[
4 \int_{\Omega} \beta_{ii}|u_i|^6 + \sum_{j \in I, j \neq i} \int_{\Omega} \beta_{ij}|u_i|^3|u_j|^3 - 3 \sum_{j \in I, j \neq i} \int_{\Omega} \beta_{ij}|u_i|^3|u_j|^3 \geq \frac{1}{2}SC_1^\frac{1}{2} > 0,
\]
which means that \( A_f(u) \) is strictly diagonally dominant. Therefore,
\[
N_f \cap \{ u \in H_q : J_f(u) \leq 2C \} \subset \Gamma_f,
\]
and so
\[
N_f \cap \{ u \in H_q : J_f(u) \leq 2C \} \subset N_f \cap \Gamma_f.
\]
By Lemma 2.3 we know that the constrained critical points of \( J_f \) on \( N_f \) satisfying \( J_f(u) \leq 2C \) are free critical points of \( J \). This completes the proof. \( \square \)

Next, we construct a Palais-Smale sequence at level \( C_f \).

Lemma 2.5. (Existence of Palais-Smale sequence) Assume that
\[
\beta_{ii} > 0 \quad \forall i = 1, 2, \ldots, d, \quad 0 < \beta_{ij} < K_1 \quad \forall i, j = 1, 2, \ldots, d, i \neq j.
\]
Then for every \( I \subseteq \{1, 2, \ldots, d\} \), there exists a sequence \( \{u_n\} \subset N_f \) satisfying
\[
\lim_{n \to \infty} J_f(u_n) = C_f, \quad \lim_{n \to \infty} J'_f(u_n) = 0.
\]
Proof. By the definition of \( C_f \), there exists a minimizing sequence \( \{u_n\} \subset N_f \) with \( u_n = (u_{i,n})_{i \in I} \) satisfying
\[
J_f(u_n) \to C_f, \quad J'_f(u_n) - \sum_{i \in I} \mu_{i,n}G'_{i}(u_n) = o(1)
\]
(2.7)

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where
\[ G_i(u) = \|u_i\|^2 - \sum_{j \in I} \beta_{ij} |u_i u_j|^3. \]

By Lemma 2.1 we can assume that \( J_I(u_n) \leq 2C \) for \( n \) large enough, then following lemma 2.4 we have
\[ 4\beta_{ii} |u_{i,n}|^6 + \sum_{j \in I, j \neq i} \int_{\Omega} \beta_{ij} |u_{i,n}|^3 |u_{j,n}|^3 - 3 \sum_{j \in I, j \neq i} \left| \int_{\Omega} \beta_{ij} |u_{i,n}|^3 |u_{j,n}|^3 \right| \geq \frac{1}{2} SC_i^2 \text{ for } i \in I. \]

Suppose that \( \nu_n \) is the minimum eigenvalues of \( A_I(u_n) \). By Gershgorin circle theorem and (2.8) we have
\[ \nu_n \geq \frac{1}{2} SC_i^2, \]
where \( C_i \) is independent on \( n \).

Note that \( u_n \in N_I \), then test the second equation in (2.7) with \((0, \ldots, u_{i,n}, \ldots, 0)\), \( i \in I \) and multiply by \( \mu_n = (\mu_{i,n})_{i \in I} \), by (2.9) we have
\[ o(1) |\mu_n| \geq \mu_n A_I(u_n) \mu_n^T \geq \nu_n |\mu_n|^2 \geq \frac{1}{2} SC_i^2 |\mu_n|^2, \]
where \( A_I(u_n) \) is defined in (2.4). It follows that \( \mu_{i,n} \to 0 \) as \( n \to \infty \). Since for every \( \varphi \in H_0^1(\Omega) \), \( G'_I(u_n) \varphi \) is uniformly bounded, we have \( J'_I(u_n) \varphi = o(\|\varphi\|) \), which yields that \( J'_I(u_n) \to 0 \) in \( H^{-1}(\Omega) \). Therefore, \( \{u_n\} \) is a standard Palais-Smale sequence.

We conclude this section by introducing the Brézis-Lieb lemma (see [6]) for two components, and its proof is referred to [13, p.447].

**Lemma 2.6.** Assume that \( u_n \rightharpoonup u, v_n \rightharpoonup v \) in \( H_0^1(\Omega) \) as \( n \to \infty \) and \( 1 < p < +\infty \). Then, up to subsequence, there holds
\[ \lim_{n \to \infty} \int_{\Omega} (|u_n|^p|v_n|^p - |u_n - u|^p|v_n - v|^p - |u|^p|v|^p) = 0. \]

### 2.2 Energy estimates

In this subsection, we present two crucial energy estimates, which are important to prove Theorem 1.1. The first one is the following proposition, which plays a key role in showing that the limit of Palais-Smale sequence is not zero. Define
\[ K_2 = \min_{1 \leq i \leq d} \{ \sqrt{\beta_{ii} m_i} \} \]
(2.10)

Then we have

**Proposition 2.1.** Assume that there holds
\[ \beta_{ii} > 0 \ \forall i = 1, 2, \ldots, d, \ 0 < \beta_{ij} < K_2 \ \forall i, j = 1, 2, \ldots, d, i \neq j, \]
then we have
\[ C_I \leq \sum_{i \in I} m_i \ \forall I \subseteq \{1, \ldots, d\}. \]
Proof. Without loss of generality, we only prove that

\[ C \leq \sum_{i=1}^{d} m_i. \]

We will prove this statement in three steps. To begin with, we recall that \( \omega_i \) is a least energy positive solution of the Brézis-Nirenberg problem with energy \( m_i = \frac{1}{4} \| \omega_i \|_2^2 = \frac{1}{4} \beta_i |\omega_i|^6 \) (see (1.5)).

**Step1:** We claim that the matrix \( (\int_{\Omega} \beta_{ij} |\omega_i|^3 |\omega_j|^3)_{d \times d} \) is positive definite.

For every \( 1 \leq i \leq d \),

\[
\begin{align*}
\int_{\Omega} \beta_{ii} |\omega_i|^6 - \sum_{j=1 \atop j \neq i}^{d} \left| \int_{\Omega} \beta_{ij} |\omega_i|^3 |\omega_j|^3 \right| & \geq 3 m_i - K_2 \sum_{j=1 \atop j \neq i}^{d} \left( \int_{\Omega} |\omega_i|^6 \right)^{\frac{1}{6}} \left( \int_{\Omega} |\omega_j|^6 \right)^{\frac{1}{6}} \\
& \geq 3 m_i - K_2 \sqrt{3 m_i} \sum_{j=1}^{d} \sqrt{\frac{3 m_i}{\beta_{ii}}} \geq \frac{3}{2} m_i > 0.
\end{align*}
\]

This implies that the matrix \( (\int_{\Omega} \beta_{ij} |\omega_i|^3 |\omega_j|^3)_{d \times d} \) is strictly diagonally dominant. Since the diagonal elements are positive, then this matrix is positive definite.

**Step2:** We claim that there exists \( (a_1, \ldots, a_d) \in (\mathbb{R}^+)^d \) such that \( (a_1 \omega_1, \ldots, a_d \omega_d) \in \mathcal{N} \).

We define the polynomial function \( F : (\mathbb{R}^+)^d \rightarrow \mathbb{R} \)

\[
F(t_1, \ldots, t_d) = J(t_1 \omega_1, \ldots, t_d \omega_d) = \frac{1}{2} \sum_{i=1}^{d} t_i^2 \| \omega_i \|_i^2 - \frac{1}{6} \sum_{i,j=1}^{d} t_i^3 t_j \int_{\Omega} \beta_{ij} |\omega_i|^3 |\omega_j|^3,
\]

where \( (\mathbb{R}^+)^d = \{ x = (x_1, \ldots, x_d) : x_i > 0 \text{ for } i = 1, \cdots, d \} \). By using the conclusion of the Step1, there exists a constant \( C \) such that

\[
F(t_1, \ldots, t_d) \leq \frac{1}{2} \sum_{i=1}^{d} t_i^2 \| \omega_i \|_i^2 - C \sum_{i=1}^{d} t_i^6 = \frac{3}{2} \sum_{i=1}^{d} m_i t_i^2 - C \sum_{i=1}^{d} t_i^6 \rightarrow -\infty \text{ as } |t| \rightarrow +\infty.
\]

Thus, the polynomial \( F(t_1, \ldots, t_d) \) has a global maximum in \( \overline{(\mathbb{R}^+)^d} \).

Assume the global maximum points \( t = (a_1, \ldots, a_d) \) belongs to \( \partial(\mathbb{R}^+)^d \). Without loss of generality, we assume that \( a_1 = 0 \) and \( a_i > 0, \forall i = 2, \ldots, d \), then

\[
F(0, a_2, \ldots, a_d) = \frac{1}{2} \sum_{i=2}^{d} a_i^2 \| \omega_i \|_i^2 - \frac{1}{6} \sum_{i,j=2}^{d} a_i^3 a_j \int_{\Omega} \beta_{ij} |\omega_i|^3 |\omega_j|^3.
\]

For \( s > 0 \) small enough, we have

\[
F(s, a_2, \ldots, a_d) - F(0, a_2, \ldots, a_d) = \frac{1}{2} s^2 \| \omega_1 \|_1^2 - \frac{1}{6} s^6 \int_{\Omega} \beta_{11} |\omega_1|^6 - \frac{1}{3} s^3 \sum_{j=2}^{d} a_j^3 \int_{\Omega} \beta_{1j} |\omega_1|^3 |\omega_j|^3 > 0,
\]

which contradicts to the fact that \( (0, a_2, \ldots, a_d) \) is a global maximum of \( (\mathbb{R}^+)^d \). Thus, the global maximum point of \( F(t_1, \ldots, t_d) \) can not belong to \( \partial(\mathbb{R}^+)^d \), which implies that the global maximum point of \( F(t_1, \ldots, t_d) \)
is a interior point in \((\mathbb{R}^+)^d\). Moreover, the global maximum point \((a_1, ..., a_d) \in (\mathbb{R}^+)^d\) is a critical point, which means that 

\[
\frac{\partial F}{\partial t_i}(a_1, ..., a_d) = 0, \quad \text{for every } i = 1, 2, ..., d.
\]

Therefore, 

\[(a_1\omega_1, ..., a_d\omega_d) \in \mathcal{N}.
\]

**Step 3:** We claim that 

\[C \leq \sum_{i=1}^{d} m_i.
\]

By the definition of \(\omega_i\) and \(\beta_{ij} > 0\) for any \(i \neq j\) we see that 

\[
C \leq J(a_1\omega_1, ..., a_d\omega_d) = \frac{1}{2} \sum_{i=1}^{d} a_i^2 \|\omega_i\|_i^2 - \frac{1}{6} \sum_{i,j=1}^{d} a_i^3 a_j^3 \int_{\Omega} \beta_{ij} |\omega_i|^{3} |\omega_j|^{3}
\]

\[
\leq \frac{1}{2} \sum_{i=1}^{d} a_i^2 \|\omega_i\|_i^2 - \frac{1}{6} \sum_{i,j=1}^{d} a_i^6 \int_{\Omega} \beta_{ij} |\omega_i|^{6}
\]

\[
= \sum_{i=1}^{d} \left( \frac{3}{2} a_i^2 - \frac{1}{2} a_i^6 \right) m_i \leq \sum_{i=1}^{d} m_i.
\]

This completes the proof. \(\square\)

The following proposition will play a critical role in proving that \(C\) is achieved by a solution with \(d\) non-trivial components. Define

\[
(2.11) \quad K_3 = \min \left\{ K_1, \frac{S^3}{4(6C)^2}, \frac{S^2 C_{1}^{\frac{1}{3}}}{4(6C)^2 \sum_{i=1}^{d} \sqrt{\frac{3m_i}{\beta_i}}, \frac{\sqrt{\frac{S}{K_1}} \min_{1 \leq i \leq d} \sqrt{\beta_i} m_i}{\sqrt{\frac{S}{K_1}}}, \frac{\sqrt{\frac{S}{K_1}} \sum_{i=1}^{d} \sqrt{\frac{3m_i}{\beta_i}}}{\sqrt{\frac{S}{K_1}}}} \right\},
\]

where \(K_1\) is defined in (2.3). Then we have the following energy estimate.

**Proposition 2.2.** Assume that there holds

\[
\beta_{ii} > 0 \quad \forall i = 1, 2, ..., d, \quad 0 < \beta_{ij} < K_3 \quad \forall i, j = 1, 2, ..., d, i \neq j.
\]

Given \(I \subseteq \{1, 2, ..., d\}\), suppose that \(C_Q\) is achieved by \(u_Q\) for every \(Q \subseteq I\), then 

\[
C_I \leq \min \left\{ C_Q + \sum_{i \in I \setminus Q} m_i : Q \subseteq I \right\}.
\]

Next, we present the proof of this proposition. Without loss of generality, we fix \(1 \leq q \leq d - 1\) and prove that 

\[
(2.12) \quad C \leq C_{1, \ldots, q} + \sum_{i=q+1}^{d} m_i,
\]

where we use the notation \(J_{1, \ldots, q}, N_{1, \ldots, q}, C_{1, \ldots, q}\) instead of \(J_{\{1, \ldots, q\}}, N_{\{1, \ldots, q\}}, C_{\{1, \ldots, q\}}\) for simplicity, and the other inequalities can be proved in the same way. Before proving (2.12), let us firstly prove the following Lemma 2.7 and Lemma 2.8.
Lemma 2.7. Assume that there holds
\[ \beta_{ii} > 0 \quad \forall i = 1, 2, \ldots, d, \quad 0 < \beta_{ij} < K_3 \quad \forall i, j = 1, 2, \ldots, d, i \neq j. \]

Given \( 1 \leq q \leq d - 1 \), if \( C_{1, \ldots, q} \) is achieved by \( u_q = (u_1, \ldots, u_q) \in \mathcal{N}_{1, \ldots, q} \), then
\[
\max_{t_1, \ldots, t_q > 0} f_q(t_1, \ldots, t_q) = f_q(1, \ldots, 1) = C_{1, \ldots, q}.
\]

Proof. Notice that \( C_{1, \ldots, q} \) is achieved by \( u_q = (u_1, \ldots, u_q) \in \mathcal{N}_{1, \ldots, q} \), then by Lemma 2.1 we have \( J_{1, \ldots, q}(u_q) = C_{1, \ldots, q} < 2C \). Consider the polynomial function \( f_q : (\mathbb{R}^+)^q \to \mathbb{R} \)
\[
f_q(t_1, \ldots, t_q) = J_{1, \ldots, q}(t_1 u_1, \ldots, t_q u_q) := \frac{1}{2} \sum_{i=1}^{q} t_i^2 \|u_i\|^2 - \frac{1}{6} \sum_{i,j=1}^{q} t_i^3 t_j^3 \int_{\Omega} \beta_{ij} |u_i|^3 |u_j|^3.
\]
Define the matrix \( B_q(u_q) = (b_{ij}(u_q)) \) by
\[
b_{ij}(u_q) = \int_{\Omega} \beta_{ij} |u_i|^3 |u_j|^3, \quad i, j = 1, 2, \ldots, q.
\]
We claim that the matrix \( B_q(u_q) \) is positive definite. We will prove that \( B_q(u) \) is strictly diagonally dominant, that is
\[
\int_{\Omega} \beta_{ii} |u_i|^6 - \sum_{j=1, j \neq i}^{q} \left| \int_{\Omega} \beta_{ij} |u_i|^3 |u_j|^3 \right| > 0.
\]
Note that \( u_q \in \mathcal{N}_{1, \ldots, q} \), then the inequality (2.15) is true if we show
\[
\|u_i\|^2 - 2 \sum_{j=1, j \neq i}^{q} \int_{\Omega} \beta_{ij} |u_i|^3 |u_j|^3 > 0.
\]
By the definition of \( K_3 \) we have
\[
2 \sum_{j=1, j \neq i}^{q} \int_{\Omega} \beta_{ij} |u_i|^3 |u_j|^3 \leq \frac{2K_3}{S^3} \sum_{j=1, j \neq i}^{q} \|u_i\|^3 \|u_j\|^3 \leq \frac{2K_3}{S^3} (6C)^2 \|u_i\|^2 \leq \frac{1}{2} \|u_i\|^2.
\]
Thus,
\[
\|u_i\|^2 - 2 \sum_{j=1, j \neq i}^{q} \int_{\Omega} \beta_{ij} |u_i|^3 |u_j|^3 \geq \frac{1}{2} \|u_i\|^2 \geq \frac{1}{2} SC_1^4.
\]
Therefore, \( B_q(u) \) is strictly diagonally dominant, and so \( B_q(u) \) is positive definite. It follows that there exists a constant \( C > 0 \) such that
\[
f_q(t_1, \ldots, t_q) = \frac{1}{2} \sum_{i=1}^{q} t_i^2 \|u_i\|^2 - \frac{1}{6} \sum_{i,j=1}^{q} t_i^3 t_j^3 \int_{\Omega} \beta_{ij} |u_i|^3 |u_j|^3 \leq \frac{1}{2} \sum_{i=1}^{q} t_i^2 \|u_i\|^2 - C \frac{q}{6} \sum_{i=1}^{q} t_i^6 \to -\infty, \quad \text{as} \; |t| \to +\infty.
\]
which implies that \( f_q(t_1, ..., t_q) \) has a global maximum in \( \overline{\mathbb{R}}^q \). Here, \( t = (t_1, ..., t_q) \). Similar to the proof of Step2 in proposition 2.1, we can get that the global maximum point of \( f_q(x_1, ..., x_q) \) can not belong to \( \partial(\overline{\mathbb{R}}^q)^q \), which implies that the global maximum point of \( f_q(x_1, ..., x_q) \) is a interior point in \( (\mathbb{R}^+)^q \). Therefore, the global maximum point of \( f_q \) is a critical point. Next, we will show that \( f_q \) has a unique critical point.

For convenience of calculations, we consider

\[
\tilde{f}_q(t_1, ..., t_q) = \frac{1}{2} \sum_{i=1}^{q} t_i^2 \|u_i\|^2 - \frac{1}{6} \sum_{i,j=1}^{q} t_i t_j \int_\Omega \beta_{ij} |u_i|^3 |u_j|^3.
\]

By a direct calculation,

\[
\frac{\partial \tilde{f}_q}{\partial t_i}(t_1, ..., t_q) = \frac{1}{3} t_i \|u_i\|^2 - \frac{1}{3} \sum_{j=1}^{q} t_j \int_\Omega \beta_{ij} |u_i|^3 |u_j|^3, \quad 1 \leq i \leq q,
\]

\[
\frac{\partial^2 \tilde{f}_q}{\partial t_i \partial t_j}(t_1, ..., t_q) = -\frac{1}{9} t_i \|u_i\|^2 - \frac{1}{9} \int_\Omega \beta_{ij} |u_i|^6, \quad 1 \leq i, j \leq q, \quad i \neq j,
\]

Thus the Hessian matrix of \( \tilde{f}_q \) is

\[
H(\tilde{f}_q) = -\frac{1}{9} \begin{pmatrix} t_1 \|u_1\|^2 & \cdots & b_{1q}(u_q) \\ \vdots & \ddots & \vdots \\ t_q \|u_q\|^2 & \cdots & b_{qq}(u_q) \end{pmatrix} - \frac{1}{3} \begin{pmatrix} b_{11}(u_q) & \cdots & b_{1q}(u_q) \\ \vdots & \ddots & \vdots \\ b_{q1}(u_q) & \cdots & b_{qq}(u_q) \end{pmatrix}
\]

\[= -\frac{1}{9} B(t) - \frac{1}{3} B_q(u_q),\]

where \( t = (t_1, ..., t_q) \in (\mathbb{R}^+)^q \) and \( b_{ij}(u) \) is defined in (2.14). We already know the matrix \( B_q(u_q) \) is positive definite and it is easy to see the matrix \( B(t) \) is also positive definite, thus the Hessian matrix of \( f_q \) is negative definite, which implies that \( \tilde{f}_q \) has a unique critical point. Therefore, the critical point must be the global maximum point. Notice that \( \tilde{f}_q(t_1^3, ..., t_q^3) = f_q(t_1, ..., t_q) \), thus \( \tilde{f}_q \) has a unique critical point and the critical point must be the global maximum point. Since \( u_q \in N_{1, ..., q} \), then by a direct calculation we have

\[
\frac{\partial f_q}{\partial t_i}(1) = \|u_i\|^2 - \sum_{j=1}^{q} \int_\Omega \beta_{ij} |u_i|^3 |u_j|^3 = 0 \quad \text{for every } 1 \leq i \leq q,
\]

which implies that \( 1 = (1, ..., 1) \in (\mathbb{R}^+)^q \) is a critical point. As a consequence, \( 1 = (1, ..., 1) \) is a maximum point of \( f_q \). In other words,

\[
\max_{t_1, ..., t_q > 0} f_q(t_1, ..., t_q) = f(1, ..., 1) = C_{1, ..., q}.
\]

This completes the proof. \(\square\)

**Lemma 2.8.** Assume that there holds

\[
\beta_{ii} > 0 \quad \forall i = 1, 2, ..., d, \quad 0 < \beta_{ij} < K_3 \quad \forall i, j = 1, 2, ..., d, i \neq j.
\]
Given $1 \leq q \leq d - 1$, if $C_{1, q}$ is attained by $u_q = (u_1, \ldots, u_q) \in N_{1, \ldots, q}$, then there exists $\bar{t}_i > 0, i = 1, \ldots, d$, such that

$$(t_1 u_1, \ldots, t_q u_q, \bar{t}_{q+1} \omega_{q+1}, \ldots, \bar{t}_d \omega_d) \in N,$$

where $\omega_i$ is a least energy positive solution of (1.4).

Proof. Let $v = (v_1, \ldots, v_d) = (u_1, \ldots, u_q, \omega_{q+1}, \ldots, \omega_d)$ and

$$\Phi(t_1, \ldots, t_d) = J(t_1 v_1, \ldots, t_d v_d) = \frac{1}{2} \sum_{i=1}^{q} t_i^2 \|v_i\|^2 - \frac{1}{6} L_t M(v) L_t^T$$

where $L_t = (t_1^3, \ldots, t_d^3)$ is a vector in $\mathbb{R}^d$ and $M(v) = (M_{ij}(v))_{d \times d}$ is a symmetric matrix with

$$M_{ij}(v) = \int_{\Omega} \beta_{ij} |u_i|^3 |u_j|^3, \quad \text{for every } 1 \leq i, j \leq q$$

$$M_{ij}(v) = \int_{\Omega} \beta_{ij} |u_i|^3 |\omega_j|^3, \quad \text{for every } 1 \leq i \leq q, q + 1 \leq j \leq d$$

$$M_{ij}(v) = \int_{\Omega} \beta_{ij} |\omega_i|^3 |\omega_j|^3, \quad \text{for every } q + 1 \leq i, j \leq d$$

We will show that the matrix $M(v)$ is strictly diagonally dominant. We separate the proof into two cases.

For the case $1 \leq i \leq q$, we want to show that

$$\int_{\Omega} \beta_{ii} |u_i|^6 - \sum_{j=1}^{q} \int_{\Omega} \beta_{ij} |u_i|^3 |u_j|^3 \geq 1 \frac{SC_i^4}{2}. $$

In fact, by Lemma 2.7 we know

$$\int_{\Omega} \beta_{ii} |u_i|^6 - \sum_{j=1}^{q} \int_{\Omega} \beta_{ij} |u_i|^3 |u_j|^3 \geq 1 \frac{SC_i^4}{2}. $$

Moreover, under the assumptions of $\beta_{ij}$ and the definition of $m_i$, we have

$$\sum_{j=q+1}^{d} \int_{\Omega} \beta_{ij} |u_i|^3 |\omega_j|^3 \leq K_3 \sum_{j=q+1}^{d} \left( \int_{\Omega} |u_i|^6 \right)^{\frac{1}{2}} \left( \int_{\Omega} |\omega_j|^6 \right)^{\frac{1}{2}} \leq K_3 \sum_{j=q+1}^{d} \frac{3 m_j}{\beta_{jj}} \left( \int_{\Omega} |u_i|^6 \right)^{\frac{1}{2}} \leq K_3 \frac{3 (6C)^{\frac{1}{2}}}{\beta_{jj}} \left( \int_{\Omega} |u_i|^6 \right)^{\frac{1}{2}} \leq \frac{1}{4} SC_i^4,$$

which implies that

$$(2.16) \quad \int_{\Omega} \beta_{ii} |u_i|^6 - \sum_{j=1}^{q} \int_{\Omega} \beta_{ij} |u_i|^3 |u_j|^3 \geq 1 \frac{SC_i^4}{4} > 0.$$
For the case $q + 1 \leq i \leq d$: We want to show

\[
\int \beta_{ii} |\omega_i|^6 - \sum_{j=1}^{q} \int_{\Omega} \beta_{ij} |\omega_i|^3 |u_j|^3 - \sum_{j=q+1}^{d} \int_{\Omega} \beta_{ij} |\omega_i|^3 |\omega_j|^3 > 0.
\]

By a direct calculation, we have

\[
\int \beta_{ii} |\omega_i|^6 - \sum_{j=1}^{q} \int_{\Omega} \beta_{ij} |\omega_i|^3 |u_j|^3 - \sum_{j=q+1}^{d} \int_{\Omega} \beta_{ij} |\omega_i|^3 |\omega_j|^3 \\
\geq 3m_i - K_3 \sum_{j=1}^{q} \left( \int_{\Omega} |\omega_i|^6 \right)^{\frac{1}{2}} \left( \int_{\Omega} |u_j|^6 \right)^{\frac{1}{2}} - K_3 \sum_{j=q+1}^{d} \left( \int_{\Omega} |\omega_i|^6 \right)^{\frac{1}{2}} \left( \int_{\Omega} |\omega_j|^6 \right)^{\frac{1}{2}}
\]

(2.17)

\[
\geq 3m_i - K_3 \sum_{j=1}^{q} \frac{\|u_j\|^3}{S^2} - K_3 \sum_{j=q+1}^{d} \frac{3m_i}{\beta_{ii}} \sqrt{\frac{3m_i}{\beta_{jj}}}
\]

\[
\geq 3m_i - K_3 \sum_{j=1}^{q} \frac{\left( \frac{6C}{S} \right)^{\frac{1}{2}}}{\beta_{ii}} + \sum_{j=q+1}^{d} \sqrt{\frac{3m_i}{\beta_{jj}}} \geq \frac{3}{2} m_i.
\]

We deduce from (2.16) and (2.17) that $M(\nu)$ is strictly diagonally dominant, then $M(\nu)$ is positive definite. Then there exists $C > 0$ such that

\[
\Phi(t_1, \ldots, t_d) = \frac{1}{2} \sum_{i=1}^{q} t_i^2 \|v_i\|^2 - \frac{1}{6} L_i M(\nu) L_i^T \\
\leq \frac{1}{2} \sum_{i=1}^{q} \left( t_i^2 \|v_i\|^2 - \frac{C}{6} t_i^6 \right) \to -\infty \quad \text{as} \ |t| \to +\infty.
\]

Therefore, $\Phi(t_1, \ldots, t_d)$ has a global maximum $(\tilde{t}_1, \ldots, \tilde{t}_d)$ in $(\mathbb{R}^+)^d$. By a similar argument as used in Lemma 2.7, Step 1, the global maximum point $(\tilde{t}_1, \ldots, \tilde{t}_d)$ can not belong to $\partial(\mathbb{R}^+)^d$, and it must be a critical point. Therefore, $(\tilde{t}_1 u_1, \ldots, \tilde{t}_q u_q, \tilde{t}_{q+1} \omega_{q+1}, \ldots, \tilde{t}_d \omega_d) \in \mathcal{N}$. 

**Proof of proposition 2.2** Without loss of generality, we prove that

\[
C \leq C_{1, \ldots, q} + \sum_{i=q+1}^{d} m_i.
\]

Assume that $C_{1, \ldots, q}$ is achieved by $u_q = (u_1, \ldots, u_q)$. By Lemma 2.8, there exists $(\tilde{t}_1, \ldots, \tilde{t}_d)$ such that $(\tilde{t}_1 u_1, \ldots, \tilde{t}_q u_q, \tilde{t}_{q+1} \omega_{q+1}, \ldots, \tilde{t}_d \omega_d) \in \mathcal{N}$. Note that $\beta_{ij} > 0$ for any $i \neq j$, then by a direct calculation we have

\[
J(\tilde{t}_1 u_1, \ldots, \tilde{t}_q u_q, \tilde{t}_{q+1} \omega_{q+1}, \ldots, \tilde{t}_d \omega_d) \leq \frac{1}{2} \sum_{i=1}^{q} \tilde{t}_i^{-2} \|u_i\|^2 - \frac{1}{6} \sum_{i,j=1}^{q} \tilde{t}_i^{-3} \tilde{t}_j^{-3} \int \beta_{ij} |u_i|^3 |u_j|^3 \\
+ \frac{1}{2} \sum_{i=q+1}^{d} \tilde{t}_i^{-2} \|\omega_i\|^2 - \frac{1}{6} \sum_{i=q+1}^{d} \tilde{t}_i^{-6} \int \beta_{ii} |\omega_i|^6
\]

(2.18)

\[
= f(\tilde{t}_1, ..., \tilde{t}_q) + g(\tilde{t}_{q+1}, ..., \tilde{t}_d),
\]

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where \( f(t_1, ..., t_q) \) is defined in \((2.13)\) and
\[
g(t_{q+1}, ..., t_d) := \frac{1}{2} \sum_{i=q+1}^{d} t_i \| \omega_i \|^2 - \frac{1}{6} \sum_{i=q+1}^{d} t_i^6 \int_{\Omega} \beta_{ij} |\omega_i|^6.
\]
Notice that \( \| \omega_i \|^2 = \int_{\Omega} \beta_{ij} |\omega_i|^6 = 3m_i \), it is easy to show that
\[
(2.19) \quad g(\tilde{t}_{q+1}, ..., \tilde{t}_d) \leq \max_{t_{q+1}, ..., t_d > 0} g(t_{q+1}, ..., t_d) = \sum_{i=q+1}^{d} m_i.
\]
By Lemma 2.7 we get that
\[
(2.20) \quad f(\tilde{t}_1, ..., \tilde{t}_q) \leq \max_{t_1, ..., t_q > 0} f(t_1, ..., t_q) = f(1, ..., 1) = C_1, ..., q.
\]
We deduce from \((2.18)\), \((2.19)\) and \((2.20)\) that
\[
C \leq J(\tilde{t}_1u_1, ..., \tilde{t}_q u_q, \tilde{t}_{q+1} \omega_{q+1}, ..., \tilde{t}_d \omega_d) \leq f(\tilde{t}_1, ..., \tilde{t}_q) + g(\tilde{t}_{q+1}, ..., \tilde{t}_d) \leq C_1, ..., q + \sum_{i=q+1}^{d} m_i.
\]
This completes the proof of Proposition 2.2.

2.3 Proof of Theorem 1.1

In this subsection, we present the proof of Theorem 1.1. Recall that \( m_i < \frac{1}{3} \beta_{ii}^{-\frac{2}{3}} S^{\frac{3}{2}} \) (see \((1.5)\)) for every \( 1 \leq i \leq d \). Set
\[
(2.21) \quad \delta = \frac{1}{2} \min_{1 \leq i \leq d} \{ \beta_{ii}^{-1} S^{\frac{3}{2}} - (3m_i)^2 \} > 0,
\]
then we have
\[
(2.22) \quad (3m_i)^2 < \beta_{ii}^{-1} S^{\frac{3}{2}} - \delta, \quad 1 \leq i \leq d.
\]
Denote
\[
(2.23) \quad K_4 = \min_{1 \leq i \leq d} \left\{ \frac{\beta_{ii} S^{\frac{3}{2}}}{(6C)^2 S^{\frac{3}{2}}} \delta \right\} \quad \text{and} \quad K = \min \{ K_1, K_2, K_3, K_4 \},
\]
where \( K_1 \) is defined in \((2.3)\), \( K_2 \) is defined in \((2.10)\), \( K_3 \) is defined in \((2.11)\), \( \delta \) is fixed in \((2.21)\). From now on, we assume that \( \beta_{ij} \) satisfies \( 0 < \beta_{ij} < K \) for any \( i \neq j \).

Conclusion of the proof of Theorem 1.1 We will proceed by mathematical induction on the number of the equations in the subsystem. Set \( |I| = M \), that is \( M \) the number of the equations in the subsystem, and \( M = 1, ..., d \).

When \( M = 1 \), system \((1.6)\) reduces to the following problem
\[
-\Delta u + \lambda_i = \beta_{ii} |u|^4 u, \quad u \in H_0^1(\Omega),
\]
and by \([5]\) we see that Theorem 1.1 is true.
We suppose by induction hypothesis that Theorem 1.1 holds true for every level $C_I$ with $|I| \leq M$ for some $1 \leq M \leq d - 1$. We need prove Theorem 1.1 for $C_I$ with $|I| = M + 1$. Without loss of generality, we will present the proof for $I = \{1, \ldots, M + 1\}$. By induction hypothesis we know that Proposition 2.2 is true for $C_I$. By Lemma 2.5 there exists a sequence $\{u_n\} \subset N_I$ satisfying
\[
\lim_{n \to \infty} J_I'(u_n) = 0,
\]
then $\{u_{i,n}\}$ is uniformly bounded in $H^1_0(\Omega)$, $i = 1, 2, \ldots, M + 1$. Passing to subsequence, we may assume that
\[
(2.24) \quad u_{i,n} \rightharpoonup u_i \text{ weakly in } H^1_0(\Omega), \quad u_{i,n} \to u_i \text{ strongly in } L^2(\Omega).
\]
It is standard to see that $J_I'(u) = 0$ and
\[
\|u_i\|^2 = \sum_{j=1}^{M+1} \int_{\Omega} \beta_{ij} |u_i|^3 |u_j|^3 \text{ for every } i = 1, 2, \ldots, M + 1.
\]
Denote $\sigma_{i,n} = u_{i,n} - u_i$, $i = 1, 2, \ldots, M + 1$, and so
\[
\sigma_{i,n} \rightharpoonup 0 \text{ weakly in } H^1_0(\Omega).
\]
We deduce from (2.24) that
\[
(2.25) \quad \int_{\Omega} |\nabla u_{i,n}|^2 = \int_{\Omega} |\nabla \sigma_{i,n}|^2 + \int_{\Omega} |\nabla u_i|^2 + o(1),
\]
and by Lemma 2.6 we have
\[
(2.26) \quad \int_{\Omega} |u_{i,n}|^3 |u_{j,n}|^3 = \int_{\Omega} |\sigma_{i,n}|^3 |\sigma_{j,n}|^3 + \int_{\Omega} |u_i|^3 |u_j|^3 + o(1).
\]
By (2.25) and (2.26) we have
\[
J_I(u_n) = J_I(u) + \frac{1}{3} \sum_{i=1}^{M+1} \int_{\Omega} |\nabla \sigma_{i,n}|^2 + o(1).
\]
Passing to subsequence, we may assume that
\[
\lim_{n \to \infty} \int_{\Omega} |\nabla \sigma_{i,n}|^2 = k_i \geq 0, \quad i = 1, 2, \ldots, M + 1.
\]
Thus,
\[
(2.27) \quad 0 \leq J_I(u) \leq J_I(u) + \frac{1}{3} \sum_{i=1}^{M+1} k_i = \lim_{n \to \infty} J_I(u_n) = C_I.
\]
Next, we will show that all $u_i \not\equiv 0$ for every $1 \leq i \leq M + 1$ by using a contradiction argument.

**Case 1:** $u_i \equiv 0$ for every $1 \leq i \leq M + 1$.

Firstly, we claim that $k_i > 0$, $i = 1, 2, \ldots, M + 1$. By contradiction, without loss of generality, we assume that $k_1 = 0$, notice that $\sigma_{1,n} = u_{1,n}$, then we know that $\sigma_{1,n} \to 0$ strongly in $H^1_0(\Omega)$ and $u_{1,n} \to 0$ strongly in $H^1_0(\Omega)$. Hence, by Sobolev inequality we have
\[
\lim_{n \to \infty} \int_{\Omega} |u_{1,n}|^6 = 0.
\]
On the other hand, by Lemma 2.2, we see that
\[
\lim_{n \to \infty} \int_{\Omega} |u_{1,n}|^6 \geq C_1 > 0,
\]
which is a contradiction. Therefore, \( k_i > 0 \), \( i = 1, 2, \ldots, M + 1 \). Notice that \( J_I(u_n) \leq 2C_I \leq 2\overline{C} \) for \( n \) large enough, thus
\[
\int_{\Omega} |\nabla \sigma_{i,n}|^2 \leq \sum_{j=1}^{M+1} \int_{\Omega} |\nabla \sigma_{j,n}|^2 + 3J_I(u) + o(1) = 3J_I(u_n) \leq 6\overline{C}.
\]
Hence,
(2.28) \[ 0 < k_i \leq 6\overline{C}. \]
Since \( u_n \in \mathcal{N}_I \) and \( \sigma_{i,n} = u_{i,n} \), then we have \( \sum_{i=1}^{M+1} ||\sigma_{i,n}||^2_i = \sum_{i=1}^{M+1} ||u_{i,n}||^2 \leq 6\overline{C} \). Therefore,
\[
\int_{\Omega} |\nabla \sigma_{i,n}|^2 = \beta_{ii} \int_{\Omega} |\sigma_{i,n}|^6 + \sum_{j=1}^{M+1} \beta_{ij} \int_{\Omega} |\sigma_{i,n}|^3|\sigma_{j,n}|^3 \\
\leq \beta_{ii} \overline{S}^{-3} \left( \int_{\Omega} |\nabla \sigma_{i,n}|^2 \right)^3 + K \sum_{j=1}^{M+1} \left( \int_{\Omega} |\sigma_{i,n}|^6 \right)^\frac{1}{3} \left( \int_{\Omega} |\sigma_{j,n}|^6 \right)^\frac{1}{3} \\\n\leq \beta_{ii} \overline{S}^{-3} \left( \int_{\Omega} |\nabla \sigma_{i,n}|^2 \right)^3 + KS^{-3} ||\sigma_{i,n}||^3_i \sum_{j=1}^{M+1} ||\sigma_{j,n}||^3_j \\\n\leq \beta_{ii} \overline{S}^{-3} \left( \int_{\Omega} |\nabla \sigma_{i,n}|^2 \right)^3 + KS^{-3}(6\overline{C})^\frac{3}{2} \left( \int_{\Omega} |\nabla \sigma_{i,n}|^2 + o(1) \right)^\frac{3}{2}.
\]
Let \( n \to \infty \), we have
\[
k_i \leq \beta_{ii} \overline{S}^{-3} k_i^3 + KS^{-3}(6\overline{C})^\frac{3}{2} k_i^2.
\]
Combining this with (2.28), we get
\[
1 \leq \beta_{ii} \overline{S}^{-3} k_i^2 + KS^{-3}(6\overline{C})^2.
\]
Then by the definition of \( K, K_4 \) and (2.22) we get
\[
k_i^2 \geq \beta_{ii}^{-1} \overline{S}^{-3} - K \frac{\overline{S}^3(6\overline{C})^2}{\beta_{ii} \overline{S}^3} \geq \beta_{ii}^{-1} \overline{S}^{-3} - \delta > (3m_i)^2,
\]
which implies
\[
k_i > 3m_i.
\]
By proposition 2.1 and (2.27) we have
\[
\sum_{i=1}^{M+1} m_i \geq C_I = \lim_{n \to \infty} J_I(u_n) = J_I(u) + \frac{1}{3} \sum_{i=1}^{M+1} k_i = \frac{1}{3} \sum_{i=1}^{M+1} k_i > \sum_{i=1}^{M+1} m_i,
\]
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that is a contradiction. Therefore, Case 1 is impossible.

**Case 2:** Only one component of $u$ is not zero.

Without loss of generality, we assume that $u_1 \neq 0$, and $u_i \equiv 0$, $2 \leq i \leq p + 1$. Similarly to Case 1, we can prove that $k_i > 3m_i > 0$ for every $2 \leq i \leq M + 1$. Notice that $(u_1, 0, \ldots, 0)$ is a solution of (1.6). Then $J(u_1, 0, \ldots, 0) \geq m_1$. Combining this with Proposition 2.1, we know that

$$
\sum_{i=1}^{M+1} m_i \geq C_I = \lim_{n \to \infty} J_I(u_n) = J_I(u_1, 0, \ldots, 0) + \frac{1}{3} \sum_{i=1}^{M+1} k_i \geq m_1 + \frac{1}{3} \sum_{i=2}^{M+1} k_i \geq \sum_{i=1}^{M+1} m_i,
$$

that is a contradiction. Therefore, Case 2 is impossible.

**Case 3:** There are $q$ components of $u$ that are not zero, $2 \leq q \leq M$.

Without loss of generality, we may assume that $u_1, \ldots, u_q \neq 0$, and $u_{q+1}, \ldots, u_{M+1} \equiv 0$. Similarly to Case 1, we have $k_i > 3m_i$, $q + 1 \leq i \leq M + 1$. Note that $(u_1, u_2, \ldots, u_q, 0, \ldots, 0)$ is a solution of subsystem and $(u_1, u_2, \ldots, u_q) \in N_{1, \ldots, q}$, then $J_I(u) \geq C_{1, \ldots, q}$. Combining this with Proposition 2.2, we have

$$
C_{1, \ldots, q} + \sum_{i=q+1}^{M+1} m_i \geq C_I = \lim_{n \to \infty} J_I(u_n) = J_I(u) + \frac{1}{3} \sum_{i=1}^{M+1} k_i > C_{1, \ldots, q} + \sum_{i=q+1}^{M+1} m_i,
$$

that is a contradiction. Therefore, Case 3 is impossible.

Since Case 1, Case 2 and Case 3 are impossible, then we get that all components of $u = (u_1, \ldots, u_{M+1})$ are not zero. Therefore $u \in N_I$. Combining this with (2.27), we see that

$$
C_I \leq J_I(u) \leq J_I(u) + \frac{1}{3} \sum_{i=1}^{M+1} k_i = \lim_{n \to \infty} J_I(u_n) = C_I,
$$

which yields that $J_I(u) = C_I$. Obviously,

$$
\tilde{u} = (|u_1|, \ldots, |u_{M+1}|) \in N_I \text{ and } J_I(\tilde{u}) = C_I.
$$

It follows from Lemma 2.1 and 2.4 that $\tilde{u}$ is a nonnegative critical point of $J_I$, and $(|u_1|, \ldots, |u_{M+1}|)$ is a nonnegative solution of system (1.6). By the maximum principle, we know that $|u_i| > 0$ in $\Omega$, $1 \leq i \leq M + 1$. Therefore, $\tilde{u}$ is a least energy positive solution of subsystem (2.1) with $I = \{1, \cdots, M + 1\}$. We proceed by repeating this step, then we obtain a least energy positive solution of subsystem (2.1) with $I = \{1, \cdots, d\}$. This completes the proof.
3 Ground state solutions for the weakly cooperative case

In this section, we show the proof of Theorem 1.2.

3.1 limit system

Since the problem (1.6) has a critical nonlinearity and critical coupling terms, the existence of nontrivial ground state solutions of (1.6) strongly depend on the existence of the ground state solutions of the following limit system

\[
\begin{aligned}
-\Delta u_i &= \sum_{j=1}^{d} \beta_{ij} |u_j|^3 u_i \quad \text{in } \mathbb{R}^3, \\
u_i &\in D^{1,2}(\mathbb{R}^3), \quad i = 1, 2, \ldots, d,
\end{aligned}
\]

where \( D^{1,2}(\mathbb{R}^3) = \{ u \in L^2(\mathbb{R}^3) : |\nabla u| \in L^2(\mathbb{R}^3) \} \) with norm \( \| u \|_{D^{1,2}} := (\int_{\mathbb{R}^3} |\nabla u|^2)^{\frac{1}{2}}. \) Define \( D := (D^{1,2}(\mathbb{R}^3))^d \) and \( C^1 \) functional \( E : D \to \mathbb{R} \) as follows

\[
E(u) := \frac{1}{2} \sum_{i=1}^{d} \int_{\mathbb{R}^3} |\nabla u_i|^2 - \frac{1}{6} \sum_{i,j=1}^{d} \int_{\mathbb{R}^3} \beta_{ij} |u_i|^3 |u_j|^3.
\]

We consider the set

\[
\mathcal{M}' = \left\{ u \in D \setminus \{0\} : \sum_{i=1}^{d} \int_{\mathbb{R}^3} |\nabla u_i|^2 = \sum_{i,j=1}^{d} \int_{\mathbb{R}^3} \beta_{ij} |u_i|^3 |u_j|^3 \right\}
\]

Then any nontrivial solution of (3.1) belongs to \( \mathcal{M}' \). We set

\[
\mathcal{M} = \left\{ u : \int_{\mathbb{R}^3} \left( \frac{1}{2} \sum_{i=1}^{d} \int_{\mathbb{R}^3} |\nabla u_i|^2 - \frac{1}{6} \sum_{i,j=1}^{d} \int_{\mathbb{R}^3} \beta_{ij} |u_i|^3 |u_j|^3 \right) \right\}
\]

For \( \varepsilon > 0 \) and \( y \in \mathbb{R}^3 \), we consider the Aubin-Talenti bubble (3.3) \( U_{\varepsilon,y} \in D^{1,2}(\mathbb{R}^3) \) defined by

\[
U_{\varepsilon,y}(x) = \frac{(3\varepsilon^2)^\frac{1}{2}}{(\varepsilon^2 + |x-y|^2)^\frac{1}{2}}.
\]

Then \( U_{\varepsilon,y} \) solves the equation

\[-\Delta u = u^5 \text{ in } \mathbb{R}^3,
\]

and

\[
\int_{\mathbb{R}^3} |\nabla U_{\varepsilon,y}|^2 = \int_{\mathbb{R}^3} |U_{\varepsilon,y}|^6 = \tilde{S}^\frac{2}{3},
\]

where \( \tilde{S} \) is the Sobolev best constant of \( D^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3) \). Furthermore, \( \{ U_{\varepsilon,y} : \varepsilon > 0, y \in \mathbb{R}^3 \} \) contains all positive solutions of the equation \(-\Delta u = u^5 \) in \( \mathbb{R}^3 \). To simplify the notation, we denote

\[
U_\varepsilon(x) := U_{\varepsilon,0}(x) = \frac{(3\varepsilon^2)^\frac{1}{2}}{(\varepsilon^2 + |x|^2)^\frac{1}{2}}.
\]
Thanks to [19], we can get the existence and classification results for ground state solutions of system (3.1), which is used to prove the existence of ground state solution in the next subsection. Before proceeding, we need introduce some notations.

Consider the polynomial function $P : \mathbb{R}^d \to \mathbb{R}$ defined by

$$P(x) = \sum_{i,j=1}^{d} \beta_{ij} |x_i|^3 |x_j|^3$$

and denote by $X$ the set of solutions to the maximization problem

$$P(\tau) = \max_{|X|=1} P(X) = P_{\text{max}}, \; \tau = (\tau_1, \cdots, \tau_d), \; |\tau| = 1.$$  

**Lemma 3.1.** Assume that (1.8) holds, then the level $B$ is attained by a solution of system (3.1). Moreover, any of such minimizers has the form $(\tau_1 P^{-\frac{1}{4}} \max U_{\varepsilon,y}, \cdots, \tau_d P^{-\frac{1}{4}} \max U_{\varepsilon,y})$, where $\tau = (\tau_1, \cdots, \tau_d) \in X$ and $y \in \mathbb{R}$, $\varepsilon > 0$.

### 3.2 Proof of Theorem 1.2

In this subsection, we start to prove Theorem 1.2. Recall the Nehari manifold

$$\mathcal{M} = \left\{ u \in H_d : u \neq 0, \sum_{i=1}^{d} |u_i|^2 = \sum_{i,j=1}^{d} \int_{\Omega} \beta_{ij} |u_i|^3 |u_j|^3 \right\},$$

and the level of $J$

$$\mathcal{A} = \inf \{ J(u) : u \in \mathcal{M} \},$$

which are defined in the Introduction. Define

$$\tilde{C} := \inf_{\gamma \in \Gamma, t \in [0,1]} \max J(\gamma(t)), $$

where $\Gamma = \{ \gamma \in C([0,1], \mathbb{H}_d) : \gamma(0) = 0, J(\gamma(1)) < 0 \}$. It is easy to see that

$$\tilde{C} = \inf_{u \in H_d \setminus \{0\}} \max_{t>0} J(tu) = \inf_{u \in \mathcal{M}} J(u) = \mathcal{A}.$$  

Next, we present an energy estimate for the level $\mathcal{A}$, which plays a critical role in showing that the limit of Palais-Smale sequence is not zero.

**Lemma 3.2.** Assume that (1.7) and (1.8) hold. Then we have $\mathcal{A} < B$, where $B$ is defined in (3.3).

**Proof.** Without loss of generality, we assume that $0 \in \Omega$ and $B_{R_0}(0)$ is the largest ball contained in $\Omega$, then we take $\lambda^*(\Omega) = \frac{\pi^2}{4R_0^2}$ and

$$\varphi(x) = \begin{cases} \cos \left( \frac{\pi |x|}{2R_0} \right), & x \in B_{R_0}(0), \\ 0, & x \in \Omega \setminus B_{R_0}(0). \end{cases}$$

Set

$$w_\varepsilon(x) := U_\varepsilon(x) \varphi(x).$$
and
\[ V^\varepsilon_i(x) = \varphi(x)V^\varepsilon(x) = \tau_i P_{\max}^{-\frac{1}{3}} w_\varepsilon(x). \]
where $U_\varepsilon$ is defined in (3.4). By a standard argument (see Lemma 1.3 in [5]), we get
\begin{equation}
\int_\Omega |\nabla w_\varepsilon|² = \tilde{S}^\frac{2}{3} + \frac{\sqrt{3}}{2R_0} \pi^3 \varepsilon + O(\varepsilon^2),
\end{equation}
\begin{equation}
\int_\Omega |w_\varepsilon|^6 = \tilde{S}^\frac{2}{3} + O(\varepsilon^2),
\end{equation}
\begin{equation}
\int_\Omega |w_\varepsilon|^2 = 2\sqrt{3}\pi R_0 + O(\varepsilon^2).
\end{equation}

By Lemma 3.4 we know that $(V^\varepsilon_1, \cdots, V^\varepsilon_d) = (\tau_1 P_{\max}^{-\frac{1}{3}} U_\varepsilon, \cdots, \tau_d P_{\max}^{-\frac{1}{3}} U_\varepsilon)$ is a ground state solution of system (3.1), where $P_{\max}, \tau_i$ are defined in (3.5) and $U_\varepsilon$ is defined in (3.4). Then
\[ B = E(V^\varepsilon_1, \cdots, V^\varepsilon_d) = \frac{1}{3} \sum_{i,j=1}^d \int_{\mathbb{R}^3} \beta_{ij} |V^\varepsilon_i|^3 |V^\varepsilon_j|^3 = \frac{1}{3} \left( \sum_{i,j=1}^d \beta_{ij} |\tau_i|^3 |\tau_j|^3 \right) P_{\max}^{-\frac{2}{3}} \int_{\mathbb{R}^3} |U_\varepsilon(x)|^6 = \frac{1}{3} P_{\max}^{-\frac{2}{3}} \tilde{S}^\frac{2}{3}.
\]

On the other hand, we deduce from $|\tau| = 1$ that there must exist $1 \leq i_0 \leq d$ such that $\tau_{i_0}^2 > 0$. Based on this fact, for $\varepsilon > 0$ small, by (3.6), (3.7) and (3.8) we have
\[ \max_{t>0} J(tV^\varepsilon_1(x), \cdots, tV^\varepsilon_d(x)) \]
\[ = \frac{1}{3} \left( \sum_{i=1}^d \int_{\Omega} |\nabla V^\varepsilon_i(x)|^2 + \lambda_i |V^\varepsilon_i(x)|^2 dx \right)^{\frac{1}{3}} \]
\[ = \frac{1}{3} \left( \sum_{i,j=1}^d \beta_{ij} |V^\varepsilon_i(x)|^3 |V^\varepsilon_j(x)|^3 \right)^{\frac{1}{3}} \]
\[ = \frac{1}{3} \left[ \sum_{i,j=1}^d \tau_i^2 P_{\max}^{-\frac{2}{3}} \left( \int_{\Omega} |\nabla w_\varepsilon(x)|^2 dx + \lambda_i \int_{\Omega} |w_\varepsilon(x)|^2 dx \right) \right]^{\frac{1}{3}} \]
\[ \leq \frac{1}{3} \left( \tilde{S}^\frac{2}{3} + \frac{2\sqrt{3}\pi R_0 \tau_i^2}{\lambda_i + \frac{\tau_i^2}{4}} + O(\varepsilon^2) \right)^{\frac{2}{3}} \]
\[ = \frac{1}{3} \left( \tilde{S}^\frac{2}{3} + O(\varepsilon^2) \right)^{\frac{2}{3}} \]
\[ < \frac{1}{3} P_{\max}^{-\frac{2}{3}} \tilde{S}^\frac{2}{3} = B.
\]
As a consequence,
\[ \mathcal{A} = \inf_{u \in \mathbb{H}_d \setminus \{0\}} \max_{t>0} J(tu) \leq \max_{t>0} J(tV^\varepsilon_1(x), \cdots, tV^\varepsilon_d(x)) < B.
\]
This completes the proof. \qed
Proposition 3.1. Suppose that $\beta_{ij} \geq 0$ for any $i \neq j$. Then $A$ is attained on $\mathcal{M}$.

Proof. It is easy to see that the functional $J$ has a mountain pass structure, by the mountain pass theorem (see [36]), there exists $\{u_n\} \subset H_d$ such that
\[ \lim_{n \to \infty} J(u_n) = A, \quad \lim_{n \to \infty} J'(u_n) = 0, \]
where $u_n = (u_{1,n}, \ldots, u_{d,n})$. By a standard argument it is easy to see that $\{u_n\}$ is bounded in $H_d$. Up to subsequence, we may assume that
\[ u_{i,n} \rightharpoonup u_i \text{ weakly in } H_0^1(\Omega), \quad u_{i,n} \to u_i \text{ strongly in } L^2(\Omega). \]

It is standard to show $J'(u) = 0$. Set $\sigma_{i,n} = u_{i,n} - u_i$, by Lemma 2.6 we have
\[ \int_\Omega |u_{i,n}|^6 = \int_\Omega |\sigma_{i,n}|^6 + \int_\Omega |u_i|^6 + o(1), \]
and for $i \neq j$,
\[ \int_\Omega |u_{i,n}|^3 |u_{j,n}|^3 = \int_\Omega |\sigma_{i,n}|^3 |\sigma_{j,n}|^3 + \int_\Omega |u_i|^3 |u_j|^3 + o(1). \]
We deduce from (3.9) that
\[ \int_\Omega |\nabla u_{i,n}|^2 = \int_\Omega |\nabla \sigma_{i,n}|^2 + \int_\Omega |\nabla u_i|^2 + o(1). \]
Note that $u_n \in \mathcal{M}$ and $J'(u_n) \to 0$, by a direct calculation we have
\[ \int_\Omega |\nabla \sigma_{i,n}|^2 = \sum_{j=1}^d \beta_{ij} |\sigma_{i,n}|^3 |\sigma_{j,n}|^3 + o(1), \]
and
\[ J(u_n) = J(u) + \frac{1}{3} \sum_{i=1}^d \int_\Omega |\nabla \sigma_{i,n}|^2 + o(1), \]
which implies that $\int_\Omega |\nabla \sigma_{i,n}|^2$ is uniformly bounded for every $i = 1, 2, \ldots, d$ and $n \in \mathbb{N}$. Passing to subsequence, we may assume that
\[ \lim_{n \to \infty} \int_\Omega |\nabla \sigma_{i,n}|^2 = b_i \geq 0, \quad 1 \leq i \leq d. \]
Thus,
\[ 0 \leq J(u) \leq J(u) + \frac{1}{3} \sum_{i=1}^d b_i = \lim_{n \to \infty} J(u_n) = A. \]

Next, we will show that $u \neq 0$. By using a contradiction argument we assume that all components of $u$ are zero, i.e., $u_i \equiv 0$, $i = 1, 2, \ldots, d$. By (3.11), we see that $b_1 + b_2 + \cdots + b_d = 3A > 0$. Then we may assume
that \((\sigma_{1,n}, \cdots, \sigma_{d,n}) \neq 0\) for \(n\) large. Recall the definition of \(\mathcal{M}'\) in (3.2) and by (3.10), it is easy to check that there exist \(t_n\) such that \((t_n\sigma_{1,n}, \cdots, t_n\sigma_{d,n}) \in \mathcal{M}'\) and \(t_n \to 1\) as \(n \to \infty\). Then by (3.11), we have

\[
A = \frac{1}{3} \sum_{i=1}^{d} b_i = \lim_{n \to \infty} E(\sigma(1,n, \cdots, \sigma_{d,n}) = \lim_{n \to \infty} E(t_n\sigma_{1,n}, \cdots, t_n\sigma_{d,n}) \geq B,
\]

which is a contradiction to Lemma 3.2. Therefore, we have \(\mathcal{u} \neq 0\). Since \(J'(\mathcal{u}) = 0\), we have \(\mathcal{u} \in \mathcal{M}\). Combing this with (3.11) we have

\[
A = \inf_{v \in \mathcal{M}} J(v) \leq J(\mathcal{u}) \leq A,
\]

that is \(J(\mathcal{u}) = A\). This completes the proof.

The following proposition shows that the minimizer of \(A\) is semi-trivial for the weakly cooperative case.

**Proposition 3.2.** If \(\beta_{ij} \equiv b\), for any \(i \neq j\), and

\[
0 < b < 2^{\frac{d}{2}} \sqrt{\max_{1 \leq i \leq d} \{\beta_{ii}\} \min_{1 \leq i \leq d} \{\beta_{ii}\}},
\]

then the system (1.6) has no nontrivial ground state solutions.

**Proof.** Based on Proposition 3.1 we know that \(A\) can be attained by a \(\mathcal{u} = (u_1, \cdots, u_d) \in \mathcal{M}\), and \(\mathcal{u} = (u_1, \cdots, u_d)\) is a ground state solution of system (1.6). Assume now that the ground state solution \(\mathcal{u}\) is nontrivial. Notice that

\[
(tu_1, 0, 0, \cdots, 0) \in \mathcal{M} \iff t^4 = \frac{\|u_1\|^2}{\beta_{11}|u_1|^6},
\]

and \(\mathcal{u}\) is a ground state solution, we have

\[
J(u_1, ..., u_d) \leq J(tu_1, 0, \cdots, 0) \iff \frac{1}{3} \left( \sum_{i=1}^{d} \beta_{ii}|u_i|^6 + b \sum_{i,j=1, j \neq i}^{d} |u_i u_j|^3 \right) \leq \frac{1}{3} \left( \frac{\|u_1\|^3}{\beta_{11}|u_1|^6} \right)^{\frac{2}{3}}
\]

(3.12)

\[
\iff \beta_{11}|u_1|^6 \left( \sum_{i=1}^{d} \beta_{ii}|u_i|^6 + b \sum_{i,j=1, j \neq i}^{d} |u_i u_j|^3 \right)^{\frac{2}{3}} \leq \left( \beta_{11}|u_1|^6 + b \sum_{j=2}^{d} |u_j|^3 \right)^{\frac{3}{3}}.
\]

In this proof, to simplify the notations, we take

\[
A = \beta_{11}|u_1|^6, B = \sum_{i=2}^{d} \beta_{ii}|u_i|^6, C = \sum_{i,j=1, j \neq i}^{d} |u_i u_j|^3, D = \sum_{j=2}^{d} |u_j|^3, E = \sum_{i,j=2, j \neq i}^{d} |u_i u_j|^3.
\]

Notice \(C = 2D + E\), then (3.12) is equivalent to

\[
A(A + B + 2bD + bE)^2 \leq (A + bD)^3.
\]

(3.13)
By a direct calculation, (3.13) implies that
\[ AB^2 + 2A^2B \leq b^3D^3. \]

Define
\[ a = \frac{2^{d-2}}{\beta_{11}} \max_{2 \leq i \leq d} \left\{ \frac{1}{\beta_{ii}} \right\}. \]

Notice that
\[ D^2 = \left( \sum_{j=2}^{d} |u_1 u_j|^{3/2} \right)^2 \leq \left( |u_1|^{3/2} \sum_{j=2}^{d} |u_j|^{3/2} \right)^2 \leq 2^{d-1} |u_1|^{6} \sum_{j=2}^{d} |u_j|^{6} \leq aAB. \]

Thus
\[ 2\sqrt{2}A^2 B^{3/2} \leq A^2 B + 2AB^2 \leq b^3 D^3 \leq b^3 a^{3/2} A^{3/2} B^{3/2}. \]

From this inequality, we obtain that
\[ b \geq \sqrt{2a^{-1/2}} = 2^{3d/2} \beta_{11}^{1/2} \min_{2 \leq i \leq d} \{ \beta_{ii} \}^{1/2}. \]

By interchanging the roles of \( \beta_{11} \) and \( \beta_{jj}, j \geq 2 \), we get that for all \( j \in \{1, \cdots, d\} \),
\[ b \geq \sqrt{2a^{-1/2}} = 2^{3d/2} \beta_{jj}^{1/2} \min_{1 \leq i \leq d} \{ \beta_{ii} \}^{1/2}. \]

In particular,
\[ b \geq \sqrt{2a^{-1/2}} = 2^{3d/2} \max_{1 \leq i \leq d} \{ \beta_{jj} \}^{1/2} \min_{1 \leq i \leq d} \{ \beta_{ii} \}^{1/2}. \]

Thus, if
\[ b < \sqrt{2a^{-1/2}} = 2^{3d/2} \max_{1 \leq i \leq d} \{ \beta_{jj} \}^{1/2} \min_{1 \leq i \leq d} \{ \beta_{ii} \}^{1/2} \]
holds, then the system (1.6) has no nontrivial ground state solution. \( \square \)

**Conclusion of the proof of Theorem 1.2.** By Proposition 3.1 and Proposition 3.2, we know that Theorem 1.2 is true. \( \square \)

### 4 Existence for the purely competitive case

In this section, we consider the purely competitive case \( \beta_{ij} \leq 0 \) and present the proof of theorem 1.3. Recall the definitions of \( N_I, J_I \) and \( C_I \) in section 2, where \( I \subseteq \{1, 2, \ldots, d\} \). To prove theorem 1.3, we need the following several fundamental lemmas.

**Lemma 4.1.** Given \( I \subseteq \{1, 2, \ldots, d\} \). Assume that \( \beta_{ij} \leq 0 \) for all \( i \neq j \), then there exist \( C_3 > 0 \) such that for any \( u_I \in N_I \) there holds
\[ \int_{\Omega} |u_i|^6 \geq C_3, \quad i \in I. \]
The crucial step is to show that the inequality (2.8) holds. Since the remainder of the argument is analogous to that in Lemma 2.5, we omit it.

\[ \text{Lemma 4.4. Given } I \subseteq \{1, 2, \ldots, d\}, N_I \text{ is a smooth manifold. Moreover, the constrained critical points of } J_I \text{ on } N_I \text{ are free critical point of } J_I. \]

**Proof.** The proof of this lemma can be completed by the method analogous to that used in Lemma 2.5. Recall the proof of Lemma 2.5, the key point is to show that \( A_I(u) \) is strictly diagonally dominant. However, in the purely competitive case, the result is straightforward. Since \( \beta_{ij} < 0 \) and \( u_i \neq 0 \), we have

\[
a_{ii}(u) - \sum_{j \in I, j \neq i} |a_{ij}(u)| = 4\beta_{ii}|u_i|^6 + \sum_{j \in I, j \neq i} \beta_{ij}|u_i u_j|^3 - 3 \sum_{j \in I, j \neq i} |\beta_{ij}| |u_i u_j|^3
\]

\[
= 4\beta_{ii}|u_i|^6 + 4 \sum_{j \in I, j \neq i} \beta_{ij}|u_i u_j|^3 = 4||u_i||^2 > 0, \quad i \in I,
\]

which implies that \( A_I(u) \) is strictly diagonally dominant in the purely competitive case. Then using the same arguments as in the proof of Lemma 2.5, we can easily carry out the proof of this lemma.

\[ \text{Lemma 4.3. Given } I \subseteq \{1, \ldots, d\}, \text{ there exists a sequence } \{u_n\} \subset N_I \text{ satisfying}
\]

\[
\lim_{n \to \infty} J_I(u_n) = C_I, \quad \lim_{n \to \infty} J'_I(u_n) = 0 \text{ in } H^{-1}(\Omega).
\]

**Proof.** The proof is similar to that of Lemma 2.5 so we only sketch it. Recall the proof of Lemma 2.5, the crucial step is to show that the inequality (2.8) holds. Since \( \beta_{ij} \leq 0 \) and \( u_n \in N_I \), by Lemma 4.1 we have

\[
4\beta_{ii}|u_{i,n}|^6 + \sum_{j \in I, j \neq i} \int_{\Omega} \beta_{ij}|u_{i,n}|^3|u_{j,n}|^3 - 3 \sum_{j \in I, j \neq i} \int_{\Omega} \beta_{ij}|u_{i,n}|^3|u_{j,n}|^3
\]

\[
= 4||u_{i,n}||^2 \geq 4S|u_{i,n}|^2 \geq 4SC_{1/3}^2 > 0 \quad \text{for every } i \in I.
\]

The remainder of the argument is analogous to that in Lemma 2.5.

**Lemma 4.4. Given } I \subseteq \{1, \ldots, d\}, \text{ if}
\]

\[ C_I < \min \left\{ C_{\Gamma} + \frac{1}{3} \sum_{i \in I \setminus \Gamma} \beta_{ii}^{-2/3} \frac{S_{\Gamma}^2}{S^{2/3}} : \text{ for any } \Gamma \subseteq I \right\}, \]

then \( C_I \) is attained by \( J_I \) on \( N_I \).

**Proof.** Based on Lemma 4.3 there exists a sequence \( \{u_n\} \subset N_I \) satisfying

\[
\lim_{n \to \infty} J_I(u_n) = C_I, \quad \lim_{n \to \infty} J'_I(u_n) = 0 \text{ in } H^{-1}(\Omega).
\]

Thus, \( \{u_n\} \) is bounded in \( (H_0^1(\Omega))^{\lfloor I \rfloor} \). So, after passing to subsequence, we may assume

\[ u_{i,n} \rightharpoonup u_i \text{ weakly in } H_0^1(\Omega), \quad u_{i,n} \to u_i \text{ strongly in } L^2(\Omega). \]

By a standard argument, \( u = (u_i)_{i \in I} \) is a solution to the subsystem (2.1). We assert that \( u \) is nontrivial.
If the assertion is false, then we may assume that some components of \( u \) are trivial. Let \( \Gamma := \{ i \in I : u_i \equiv 0 \} \). Then, for each \( i \in \Gamma \), we have \( u_{i,n} \to 0 \) strongly in \( L^2(\Omega) \). By Lemma 4.1 and Sobolev inequality we see that

\[
\int_\Omega |\nabla u_{i,n}|^2 \geq C,
\]

where \( C \) is independent on \( n \). As \( u_n \in N_I \) and \( \beta_{ij} \leq 0 \), we get that

\[
|\nabla u_{i,n}|^2 + o(1) = |\nabla u_{i,n}|^2 \leq \beta_{ii}|u_{i,n}|^2 \leq \beta_{ii}S^{-3/2}|\nabla u_{i,n}|^2 + o(1).
\]

Combining this with (4.1) we know that

\[
\beta_{ii}S^{-3/2} \leq |\nabla u_{i,n}|^2 + o(1) \quad \text{for every } i \in \Gamma.
\]

Since \( u \) solves (2.1), we obtain

\[
C_{\ell} = \lim_{n \to \infty} J_I(u_n) = \lim_{n \to \infty} \frac{1}{3} \left( \sum_{i \in I} |u_{i,n}|^2 + \sum_{i \in I} |\nabla u_{i,n}|^2 \right)
\]

\[
\geq \lim \inf_{n \to \infty} \frac{1}{3} \sum_{i \notin I} |u_{i,n}|^2 + \frac{1}{3} \sum_{i \in I} \beta_{ii}S^{3/4} \]

\[
\geq \frac{1}{3} \sum_{i \notin I} |u_i|^2 + \frac{1}{3} \sum_{i \in I} \beta_{ii}S^{3/4} = J_{I \setminus \Gamma}(u) + \frac{1}{3} \sum_{i \in I} \beta_{ii}S^{3/4} \]

which leads to a contradiction. Therefore, \( u \) is nontrivial. This implies that \( u \in N_I \), and

\[
C_I \leq J(u) \leq \lim \inf_{n \to \infty} J(u_n) = C_I.
\]

Hence, \( J(u) = C_I \). This completes the proof.

The following proposition will play an important role in proving that \( C \) is achieved by a solution with \( d \) nontrivial components. Our approach is inspired by [16].

**Proposition 4.1.** Suppose that \( \beta_{ij} \leq 0 \) for all \( i \neq j \), then

\[
C < \min \left\{ C_I + \frac{1}{3} \sum_{i \notin I} \beta_{ii}S^{3/4} : I \subseteq \{1, \cdots, d\} \right\}.
\]

**Proof.** We proceed to prove this statement by induction on the number of equations.

For the case \( d = 1 \), this statement was proved by Brézis and Nirenberg in [5].

Assume that the statement is true for every subsystem with \( |I| \leq d - 1 \). Then the statement (4.2) reduces to

\[
C < \min \left\{ C_I + \frac{1}{3} \sum_{i \notin I} \beta_{ii}S^{3/4} : |I| = d - 1 \right\}.
\]

Without loss of generality, we may assume that \( I = \{1, \cdots, d - 1\} \). By Lemma 4.4 and our induction hypothesis, there exists a least energy positive solution \((u_1, \cdots, u_{d-1})\) to the corresponding subsystem with \( I = \{1, \cdots, d - 1\} \) and \( J(u_1, \cdots, u_{d-1}) = C_I \).
Moreover, we may assume \( 0 \in \Omega \) and \( B_{R_0}(0) \) is the largest ball contained in \( \Omega \), then we take
\[
\phi(x) = \begin{cases} 
\cos \left( \frac{\pi |x|}{2R_0} \right), & x \in B_{R_0}(0), \\
0, & x \in \Omega \setminus B_{R_0}(0),
\end{cases}
\]
and set
\[
w_\varepsilon(x) := U_\varepsilon(x)\phi(x).
\]
where \( U_\varepsilon \) is defined in (3.4). Similarly to Lemma 3.2 we get
\[
\int_{\Omega} |\nabla w_\varepsilon|^2 = \tilde{S}^2 + \frac{\sqrt{3}}{2R_0} \pi^3 \varepsilon + O(\varepsilon^2), \quad \int_{\Omega} |w_\varepsilon|^6 = \tilde{S}^2 + O(\varepsilon), \tag{4.4}
\]

(4.5)
\[
\int_{\Omega} |w_\varepsilon|^2 = 2\sqrt{3} \pi R_0 + O(\varepsilon^2).
\]
Moreover,
\[
\int |w_\varepsilon|^3 \leq \int_{B_{R_0}(0)} U_\varepsilon^3 = C \int_{B_{R_0}(0)} \left( \frac{\varepsilon}{\varepsilon^2 + |x|^2} \right)^{\frac{2}{3}} dx = C \varepsilon^\frac{2}{3} \ln \varepsilon + O(\varepsilon^2).
\]
Note that by the standard regularity theory we have \( u_i \in C^0(\bar{\Omega}) \). Therefore,
\[
\int_{\Omega} |u_i|^3 |w_\varepsilon|^3 \leq \left( \max_{x \in \Omega} |u_i(x)|^3 \right) \int_{\Omega} |w_\varepsilon|^3 = C \varepsilon^\frac{2}{3} \ln \varepsilon + O(\varepsilon^2). \tag{4.6}
\]
To show that (4.3) holds, we need the following claim.

Claim: There exist \( r, R > 0 \) independent of \( \varepsilon \) and \( t_\varepsilon, 1, \ldots, t_\varepsilon, d \in [r, R] \) such that
\[
u_\varepsilon = (t_\varepsilon, u_1, \ldots, t_\varepsilon, d, u_d) \in \mathcal{N}.
\]
Assume now that this claim is true (we will prove this later). Then
\[
C \leq J(\nu_\varepsilon) \leq \sum_{i=1}^{d-1} t_{\varepsilon, i}^2 |u_i|^2 - \sum_{i=1}^{d-1} \frac{C}{6} \sum_{i,j \neq i} t_{\varepsilon, i}^3 |u_i u_j|^3 + \frac{1}{6} \sum_{i,j \neq i} t_{\varepsilon, i}^3 \sum_{i,j \neq i} t_{\varepsilon, j}^3 |u_i u_j|^3
\]
\[
+ \frac{1}{2} r^2 |w_\varepsilon|^2 - \frac{1}{6} t_{\varepsilon, d}^6 |w_\varepsilon|^6 + \frac{1}{3} \sum_{i=1}^{d-1} R_{\varepsilon, d}^2 |u_i w_\varepsilon|^3
\]
\[
\Phi(t_{\varepsilon, 1}, \ldots, t_{\varepsilon, d-1}) = \Psi(t_{\varepsilon, 1}, \ldots, t_{\varepsilon, d-1}) = J(1, \ldots, 1, u_1, \ldots, u_{d-1}) = C_I. \tag{4.3}
\]
As \( (u_1, \ldots, u_{d-1}) \) is a least energy positive solution to the corresponding subsystem, then \( (1, \ldots, 1) \) is a critical point of \( \Psi \). By [16] Lemma 2.2] we get that the critical point is unique and
\[
\max_{t_1, \ldots, t_{d-1} > 0} \Phi(t_1, \ldots, t_{d-1}) = \Psi(1, \ldots, 1) = J(1, \ldots, 1, u_1, \ldots, u_{d-1}) = C_I.
\]
By (4.4), (4.5), (4.6) and \( R \) is independent of \( \varepsilon \), we know that
\[
\frac{1}{3} \sum_{i=1}^{d-1} R_{\varepsilon, d}^2 |\beta_{i d}| |u_i w_\varepsilon|^3 = o(\varepsilon) \quad \text{for} \ \varepsilon \ \text{small enough},
\]
and so
\[ \Phi(t) = \frac{1}{2} \left( \tilde{S}^2 + 2\sqrt{3}\pi R_0(\lambda_d + \frac{\pi^2}{4 R_0^2})\varepsilon + o(\varepsilon) + O(\varepsilon^2) \right) t^2 - \frac{1}{6} \left( \beta_{dd} \tilde{S}^2 + O(\varepsilon^2) \right) t^6. \]

Since \( \lambda_d \in (-\lambda_1(\Omega), -\lambda^*(\Omega)) \), where \( \lambda^*(\Omega) = \frac{2^2}{4 R_0^2} \), it is standard to see that
\[ \max_{t > 0} \Phi(t) < \frac{1}{3} \beta_{dd} \tilde{S}^2 \quad \text{for}\ \varepsilon \ \text{small enough}. \]

It follows that
\[ C \leq \max_{t_1, \ldots, t_{d-1} > 0} \Psi(t_1, \ldots, t_{d-1}) + \max_{t > 0} \Phi(t) < C_I + \frac{1}{3} \beta_{dd} \tilde{S}^2 \quad \text{for}\ \varepsilon \ \text{small enough}. \]

Therefore, if we assume here that the claim is true, the proof is completed. It remains to prove this claim.

Consider the polynomial function \( J : (0, +\infty)^d \to \mathbb{R} \)
\[ J(t) := J(t_1, u_1, \ldots, t_d u_d) = \sum_{i=1}^d a_i t_i^2 - \sum_{i=1}^d b_i t_i^6 + \sum_{i,j=1, j \neq i}^d c_{ij} t_i^3 t_j^3, \]
where \( t = (t_1, \ldots, t_d) \) and
\[ a_i = \frac{1}{2} \| u_i \|^2, \quad b_i = \frac{1}{6} \beta_i |u_i|^6, \quad c_{ij} = -\frac{1}{6} \beta_{ij} |u_i u_j|^3, \quad i, j = 1, \ldots, d - 1, \]
\[ a_d = \frac{1}{2} \| w_\varepsilon \|^2, \quad b_d = \frac{1}{6} \beta_{dd} |w_\varepsilon|^6, \quad c_{id} = c_{di} = -\frac{1}{6} \beta_{id} |u_i w_\varepsilon|^3, \quad i = 1, \ldots, d - 1. \]

Then, for any \( i = 1, \ldots, d \),
\[ \partial_i J(t) = 2a_i t_i - 6b_i t_i^5 + 6 \sum_{j=1, j \neq i}^d c_{ij} t_i^3 t_j^3. \]

For \( \varepsilon \) small enough, by (4.6) we have
\[ 6b_i - \sum_{j=1, j \neq i}^d 6c_{ij} = \beta_{ii} |u_i|^6 + \sum_{j=1, j \neq i}^{d-1} \beta_{ij} \int_{\Omega} |u_i|^3 |u_j|^3 + \beta_{id} \int_{\Omega} |u_i|^3 |w_\varepsilon|^3 \]
\[ = \| u_i \|^2 + \beta_{ii} \int_{\Omega} |u_i|^3 |w_\varepsilon|^3 \geq \frac{1}{2} \| u_i \|^2 > 0 \quad \text{for every} \ 1 \leq i \leq d - 1, \]
\[ 6b_i - \sum_{j=1, j \neq i}^d 6c_{ij} = \beta_{dd} |w_\varepsilon|^6 + \sum_{j=1}^{d-1} \beta_{dj} \int_{\Omega} |w_\varepsilon|^3 |u_j|^3 > \frac{1}{2} \beta_{dd} \tilde{S}^2 > 0, \]
this implies that
\[ 6b_i - \sum_{j=1, j \neq i}^d 6c_{ij} > 0 \quad \text{for every} \ 1 \leq i \leq d. \]

For \( \varepsilon \) small enough, by (4.4) - (4.6) we have for \( 1 \leq i \leq d - 1 \)
\[ \frac{\| w_\varepsilon \|^2}{\beta_{dd} |w_\varepsilon|^6} > \frac{1}{2 \beta_{dd}}, \quad \frac{\| w_\varepsilon \|^2}{\beta_{dd} |w_\varepsilon|^6 + \sum_{j=1}^{d-1} \beta_{dj} |w_\varepsilon u_j|^3} < \frac{1}{\beta_{dd}}, \quad \frac{\| u_i \|^2}{\| u_i \|^2 + \beta_{id} |u_i w_\varepsilon|^3} < 2. \]
Take
\[ r^4 = \min_{1 \leq i \leq d-1} \left\{ \frac{\|u_i\|_2^2}{\beta_{ii} |u_i|_d^2} \right\}, \quad R^4 = \max \left\{ 2, \frac{1}{\beta_{dd}} \right\}. \]

Then we have
\[ 2a_i t - \left( 6b_i - 6 \sum_{j=1, j \neq i}^d c_{ij} \right) t^5 < 0 \quad \text{if } t \in (R, \infty), \]
and
\[ 2a_i t - 6b_i t^5 > 0 \quad \text{if } t \in (0, r). \]

Take \( s = (s_1, \cdots, s_d) \in (0, \infty)^d \), we assume that \( s_i = \max \{s_1, \cdots, s_d\} \). Then if \( s_i > R \), then
\[ \partial_i \mathcal{J}(s) \leq 2a_i s_i - \left( 6b_i - 6 \sum_{j=1, j \neq i}^d c_{ij} \right) s_i^5 < 0. \]

On the other hand, if \( s_i < r \), then
\[ \partial_i \mathcal{J}(s) \geq 2a_i s_i - 6b_i s_i^5 > 0. \]

This fact implies that
\[ \max_{t \in (0, \infty)^d} \mathcal{J}(t) = \max_{t \in [r, R]^d} \mathcal{J}(t). \]

In particular, \( \mathcal{J} \) attains its maximum on \((0, \infty)^d\), and this maximum point \((t_{\varepsilon, 1}, \cdots, t_{\varepsilon, d})\) must be a critical point. Then it is easy to check \((t_{\varepsilon, 1} u_1, \cdots, t_{\varepsilon, d-1} u_{d-1}, t_{\varepsilon, d} u_\varepsilon) \in N\) and the claim is true. This completes the proof. \( \square \)

**Conclusion of the proof of theorem 1.3.** Following directly from Lemma 4.2, Lemma 4.4 and Proposition 4.1, we get that \( u = (u_1, \cdots, u_d) \) is a nontrivial solution of system (1.6) and \( J(u) = C \). Set \( \hat{u} = (|u_1|, \cdots, |u_d|) \), then \( \hat{u} \) is a nonnegative solution of system (1.6) and \( J(\hat{u}) = C \). By the maximum principle, we see that \( \hat{u} \) is a least energy positive solution of system (1.6). The proof is completed. \( \square \)

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