Supersymmetry Relations and MHV Amplitudes in Superstring Theory

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Abstract

We discuss supersymmetric Ward identities relating various scattering amplitudes in type I open superstring theory. We show that at the disk level, the form of such relations remains exactly the same, to all orders in $\alpha'$, as in the low–energy effective field theory describing the $\alpha' \to 0$ limit. This result holds in $D = 4$ for all compactifications, even for those that break supersymmetry. We apply SUSY relations to the computations of $N$–gluon MHV superstring amplitudes, simplifying the existing results for $N \leq 6$ and deriving a compact expression for $N = 7$. 
1. Introduction

For the last thirty years, multi-gluon amplitudes and their supersymmetric variants have been extensively studied in the framework of quantum relativistic theory of gauge fields. These amplitudes are very important from both theoretical and experimental points of view because they describe the scattering processes underlying hadronic jet production at high energy colliders. The modern tools for amplitude computations\(^1\) include helicity techniques, string-inspired color ordering and the use of supersymmetry [3,4,5] and recursion relations [6]. More recently, a substantial progress has been achieved by employing some elements of twistor theory [7]. In particular, a new type of recursion relations has been constructed, inspired by twistor string theory [8]. Here, similarly to the case of color ordering, string theory comes as a handy tool.

Multi-gluon scattering is also a process of considerable theoretical interest in the framework of full-fledged superstring theory, and would have a great experimental relevance if the string mass scale turned out to be low enough to be reached at LHC [9,10]. In this case, some effects due to the off- and on-shell propagation of the string Regge excitations could be detected in jet cross sections. The parameter that controls the size of such effects is the Regge slope \(\alpha'\). As pointed out in [9,10], a consistent low string mass (small \(\alpha'\)) scenario must necessarily involve large extra dimensions [11], therefore massive string excitations come together with Kaluza-Klein states. However, the effects of Regge states dominate over the effects of extra dimensions for a wide range of parameters, notably in the weak string coupling regime [12].

In two recent papers [13,14], we initiated a systematic study of multi-gluon scattering processes in open superstring theory, at the semiclassical, disk level of the string world-sheet. We focused on one particular, maximum helicity violating (MHV) gluon helicity configuration, because in the \(\alpha' \to 0\) limit, the respective amplitude is described by the well-known, simple formula [15] (written in the notation of [1,2]):

\[
A_{YM}(1^-, 2^-, 3^+, 4^+, \ldots, N^+) = (\sqrt{2} g_{YM})^{N-2} \text{Tr}(T^{a_1} \cdots T^{a_N}) M_{YM}^{(N)}
\]  

(1.1)

where \(g_{YM}\) is the properly normalized Yang-Mills coupling constant and

\[
M_{YM}^{(N)} = i \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \cdots \langle N1 \rangle}.
\]

(1.2)

For \(N \leq 6\) external gluons, we presented the amplitudes in a factorized form, as a product of the zero slope (Yang-Mills) MHV amplitude \(M_{YM}^{(N)}\) times a string “formfactor” involving \((N-3)!\) generalized hypergeometric functions of kinematic invariants. We argued that the soft and collinear factorization properties, combined with the Abelian limit, are completely

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\(^1\) For reviews, see [1] and [2].
sufficient to determine all $N$-gluon MHV amplitudes. Nevertheless, the increasing complexity of string formfactors has not yet allowed us writing a compact expression describing MHV string amplitudes with arbitrary numbers of gluons.

In this work, we show that the computations of $N$-gluon string amplitudes can be simplified by utilizing supersymmetric Ward identities, exactly in the same way as in $\alpha' = 0$ gauge theories [3]. In Section 1, we prove that at the disk level, the well-known supersymmetry (SUSY) relations [3] between amplitudes involving gluons, gluinos and gauge scalars are valid to all orders in $\alpha'$. Since the gluonic disk amplitudes are completely determined by the four-dimensional ($D = 4$) spacetime part of the underlying world-sheet superconformal field theory (SCFT), these relations can be used to simplify them even if supersymmetry is broken by the compactification. In particular, for any compactification of open superstring theory, the $N$-gluon MHV disk amplitude can be expressed in terms of an amplitude involving $N-4$ gluons and 4 scalars. In Section 2, we use this relation to rewrite the results of [13,14] in a non-factorized way, but in terms of “primitive” (hypergeometric) integrals. This non-factorized form of amplitudes looks more promising for constructing a string generalization of MHV rules [8] and it could potentially lead to some soluble recursion relations [16]. In Section 3, we demonstrate the power of SUSY relations by deriving a compact expression for the seven-gluon MHV string amplitude. In Section 4 we compute various four–point amplitudes involving scalars, gauginos and vectors and present an explicit example of a SUSY relation.

2. Spacetime supersymmetry relations and string amplitudes

In superstring theory, the vertices creating gauge bosons, gauginos and gauge scalars are related by supersymmetry transformations. Here, we use the operator product expansion (OPE) rules in order to evaluate the corresponding contour integrals, with SUSY transformations generated by the appropriate insertions of (spacetime) supercharge operators. We use these transformations in order to derive supersymmetric Ward identities for some amplitudes related to $N$-gluon scattering.

This Section is divided into four parts. In the first part, we use the so-called “doubling trick” [7] to rewrite the commutators of (spacetime) SUSY generators with open string vertices, inserted at the boundary of a disk world-sheet, as the contour integrals on the sphere. In the second part, we apply these contour integrals in order to determine the SUSY commutators for the vertices creating full gauge supermultiplet. In the third part, we explain how to use contour deformations in order to derive supersymmetric Ward identities relating various superstring scattering amplitudes. Finally, in the last part we construct the helicity basis for string vertex operators, and show that their SUSY transformations agree with the known results in the $\alpha' \to 0$ (Yang-Mills) limit.
2.1. Extended spacetime supersymmetry and SUSY variations on the disk

In the following, we establish SUSY transformation rules on the open string disk world-sheet. We consider type II or type I superstring compactifications on an internal six-dimensional manifold with $2\mathcal{N}$ or $\mathcal{N}$ extended supersymmetry charges in $D = 4$ spacetime, respectively. In type II superstring theory, one can construct, on a closed string world-sheet, two sets of $\mathcal{N}$ holomorphic charges $Q_{\alpha}^I$ and $\tilde{Q}_{\dot{\alpha}}^I$, $I = 1, \ldots, \mathcal{N}$, with independent actions on the left- and right-moving closed string modes, respectively. The supercharges $Q_{\alpha}^I$ of the left-moving sector are given by the contour integrals (fixed time lines on the closed string world-sheet)

$$Q_{\alpha}^I = \oint \frac{dz}{2\pi i} V_{\alpha}^I(z) , \quad \bar{Q}_{\dot{\alpha}}^I = \oint \frac{dz}{2\pi i} \bar{V}_{\dot{\alpha}}^I(z) ,$$

(2.1)

of the holomorphic supercurrents $V_{\alpha}^I(z)$ and $\bar{V}_{\dot{\alpha}}^I(z)$. In $D = 4$, in the $(-1/2)$-ghost picture [18], these are given by

$$V_{\alpha}^I(z) = \alpha'^{-1/4} e^{-\frac{1}{4} \phi(z)} S_{\alpha}(z) \Sigma^I(z) , \quad \bar{V}_{\dot{\alpha}}^I(z) = \alpha'^{-1/4} e^{-\frac{1}{4} \phi(z)} S_{\dot{\alpha}}(z) \bar{\Sigma}^I(z) ,$$

(2.2)

where $S_{\alpha}, S_{\dot{\alpha}}$ are the spin fields with the indices $\alpha$ (or $\dot{\alpha}$) denoting negative (positive) chirality in four dimensions. The Ramond fields $\Sigma^I$ belong to an internal SCFT with $c = 9$. Finally, $\phi$ is the scalar bosonizing the superghost system. The OPEs of the internal Ramond fields $\Sigma^I$ are [19] (cf. also [20])

$$\Sigma^I(z) \bar{\Sigma}^J(w) = (z - w)^{-3/4} \delta^I J^J(w) + \ldots , \quad \Sigma^I(z) \bar{\Sigma}^J(w) = (z - w)^{-1/4} \psi^{IJ}(w) + \mathcal{O}((z - w)^{3/4}) ,$$

(2.3)

with the dimension one currents $J^{IJ}$ and dimension 1/2 operators $\psi^{IJ}$. Additional material on OPEs is given in the Appendix A, where we also summarize some basic facts about type II SUSY algebra. The right-moving supercharges $\bar{Q}_{\dot{\alpha}}^I$ and $\tilde{Q}_{\dot{\alpha}}^I$ are constructed in the same way from the anti-holomorphic currents $\bar{V}_{\dot{\alpha}}^I(\bar{z})$ and $\tilde{V}_{\dot{\alpha}}^I(\bar{z})$, respectively.

In type I open superstring theory the left- and right-movers are tied together by the world-sheet boundaries and only the total charge

$$Q_{\alpha}^I = Q_{\alpha}^I + \tilde{Q}_{\dot{\alpha}}^I ,$$

(2.4)

acting on the open string modes, is preserved. The combination (2.4) enjoys then the open string SUSY algebra. On an open string world-sheet, some boundary conditions have to be imposed on the fields. On the disk, which is conformally equivalent to the upper half plane $\mathbb{H}_+$, the following boundary conditions are imposed on the supersymmetry currents:

$$V_{\alpha}^I(z) = \tilde{V}_{\dot{\alpha}}^I(\bar{z}) , \quad z = \bar{z} .$$

(2.5)

3
It is convenient to extend the definition of the currents $V_I^\alpha$ to the full complex plane by using the so-called doubling trick [17]:

$$V_I^\alpha(z) = \begin{cases} V_I^\alpha(z), & z \in \mathbb{H}_+ , \\ \tilde{V}_I^\alpha(z), & z \in \mathbb{H}_- . \end{cases}$$  \hspace{1cm} (2.6)

On the disk $\mathbb{H}_+$ a fixed time integral is represented by a half-circle around $z = 0$ in the upper half plane. Now, the line (fixed time) integrals on the disk $\mathbb{H}_+$ can be combined to full closed contour integrals on the sphere. In this way, the conserved supersymmetry charges (2.4) can be rewritten as

$$Q_I^\alpha = \int \frac{dz}{2\pi i} V_I^\alpha(z) + \int \frac{d\bar{z}}{2\pi i} \tilde{V}_I^\alpha(\bar{z}) := \oint \frac{dz}{2\pi i} V_I^\alpha(z) ,$$ \hspace{1cm} (2.7)

where on the l.h.s., the $z$ and $\bar{z}$ integrals are over semicircles in the upper and lower half-planes ($\mathbb{H}_+$ and $\mathbb{H}_-$), respectively, while the integral on the r.h.s. is over a circle on the full complex plane (Riemann sphere), see Fig. 1.

![Fig. 1 Fixed time lines C1 and C2 on the disk H+ and their extensions to a closed loop C on the sphere.](image)

Hence when converting from type II to type I we can use the fields from the left-moving closed string sector, extend their definition to the full complex plane in lines of (2.6) and consider contour integrals in the full complex plane as in Eq. (2.7). A similar procedure can be applied in the presence of D$p$-branes that can further reduce the amount of conserved supercharges, however for our purposes, $D = 4$ type I compactifications are completely sufficient.

In order to determine the variation of an arbitrary open string vertex operator $O(z)$ (inserted at the boundary $z = \bar{z}$) under infinitesimal SUSY transformations generated by

$$Q_I^\alpha(\eta_I, \bar{\eta}_I) = Q_I^\alpha(\eta_I) + Q_I^\alpha(\bar{\eta}_I) \quad \text{with} \quad Q_I^\alpha(\eta_I) = \bar{\eta}_I^\alpha Q_I^\alpha , \quad Q_I^\alpha(\bar{\eta}_I) = \bar{\eta}_{I\dot{\alpha}} Q_{\dot{\alpha}I}$$ \hspace{1cm} (2.8)

one needs the commutators

$$[Q_I^\alpha(\eta_I), O(z)] := \oint_{C_\varepsilon(z)} \frac{dw}{2\pi i} \eta_I^\alpha V^\alpha_I(w) O(z) ,$$ \hspace{1cm} (2.9)

where $C_\varepsilon(z)$ is a circle of radius $\varepsilon \to 0$ surrounding the point $z$ and $V$ is the supercurrent extended to the full complex plane according to Eq. (2.6). Thus the integral (2.9) effectively picks up the residuum at $w = z$, hence it can be determined by using OPE. One also needs similar commutators with $\tilde{Q}_I^\alpha(\bar{\eta}_I)$. 
2.2. SUSY transformations of open string vertices

For our purposes, it is completely sufficient to consider a maximally supersymmetric, toroidal compactification of type I (or type IIA/B) superstring, with \( N = 4 \) SUSY in \( D = 4 \). Therefore in the following, we shall specialize to the this case, i.e. \( I,J = 1,\ldots,4 \). For \( N = 4 \), the internal fields \( \Sigma^I \) have an explicit realization as pure exponentials

\[
\Sigma^1 = e^{\frac{i}{2}(H_1+H_2+H_3)} \ , \quad \Sigma^2 = e^{\frac{i}{2}(H_1-H_2-H_3)} \ , \quad \Sigma^3 = e^{\frac{i}{2}(-H_1+H_2-H_3)} \ , \quad \Sigma^4 = e^{\frac{i}{2}(-H_1-H_2+H_3)} \ .
\]

(2.10)

Furthermore, the fields \( \psi^{IJ} \) in (2.3) become complex fermions \( \Psi^I = e^{iH_I} \), \( \Psi^I = e^{-iH_I} \):

\[
\psi^{12} = \overline{\psi}^{34} = e^{iH_1} \ , \quad \psi^{13} = \overline{\psi}^{24} = e^{iH_2} \ , \quad \psi^{14} = \overline{\psi}^{23} = e^{iH_3} \ .
\]

(2.11)

The fields \( \psi^{IJ} \) are anti–symmetric w.r.t. to their internal indices \( I \) and \( J \).

The massless open string modes on the Dp–brane world–volume form an \( N = 4 \) gauge vector multiplet: three complex scalars \( \phi^i \), four gauginos \( \lambda^I \) and one vector \( A_\mu \) in the adjoint representation of the gauge group. The scalar vertex, in the \((-1)\)-ghost picture, reads

\[
V_{\phi^i,j,k}^{(-1)}(z,k) = g_\phi \ T^a \ e^{-\phi} \ \Psi^j \ e^{ikX} \ , \quad j = 1,\ldots,3 \ ,
\]

(2.12)

with the complex fermions \( \Psi^j = \frac{1}{\sqrt{2}} (\psi^{2j+2} \mp i\psi^{2j+3}) \). The gaugino vertex operators, in the \((-1/2)\)-ghost picture, are

\[
V_{\lambda^I,j,k}^{(-1/2)}(z,u,k) = g_\lambda \ T^a \ e^{-\phi/2} \ \bar{u}^\alpha \ S_\alpha \ \sum^I \ e^{ik_\mu X^\mu} \ ,
\]

\[
V_{\bar{\lambda}^I,j,k}^{(-1/2)}(z,u,k) = g_\lambda \ T^a \ e^{-\phi/2} \ \bar{u}^\alpha \ S_\alpha \ \sum^I \ e^{ik_\mu X^\mu} \ , \quad I = 1,\ldots,4 \ .
\]

(2.13)

The gauge boson vertex operator in the \((-1)\)-ghost picture reads:

\[
V_{A^\mu}^{(-1)}(z,\xi,k) = g_A \ T^a \ e^{-\phi} \ \xi^\mu \ \psi_\mu \ e^{ikX} \ .
\]

(2.14)

The open string vertex couplings are

\[
g_\phi = (2\alpha')^{1/2} \ g_{YM} \ , \quad g_\lambda = (2\alpha')^{1/2} \alpha'^{1/4} \ g_{YM} \ , \quad g_A = (2\alpha')^{1/2} \ g_{YM} \ (2.15)
\]

for the scalar, gaugino and vector, respectively \([21]\). The \( D = 4 \) gauge coupling \( g_{YM} \) can be expressed in terms of the ten–dimensional gauge coupling \( g_{10} \) and the dilaton field \( \phi_{10} \) through the relation \( g_{YM} = g_{10}e^{\phi_{10}/2} \) \([21]\). In the above definitions, \( T^a \) are the Chan–Paton factors accounting for the gauge degrees of freedom of the two open string ends. Furthermore, the on–shell constraints \( k^2 = 0 \), \( \bar{k}u = 0 \) are imposed.

The vertices creating gauge bosons, gauginos and gauge scalars are related by \( N = 4 \) SUSY transformations \([2,3]\). The sixteen Grassmann parameters \( \eta^I, \bar{\eta}^I \) are chiral spinors
of both $SL(2,C)$ Lorentz group and the internal $SO(6)$ R-symmetry group. Here, we use the OPE rules in order to evaluate the corresponding contour integrals (2.9). As an example,

$$[\overline{Q}^I (\bar{\eta}_I), V_{\lambda^a,\tau} ] = g_A \, T^a \sum_{w=z} \frac{dw}{2\pi i} e^{-\frac{i}{2} \phi(w)} \overline{\eta}_I \sigma^\mu \Sigma^I (w) \Sigma^J (z) e^{ikX}$$

$$= g_A \sqrt{2} \, T^a \sum_{w=z} \frac{dw}{2\pi i} e^{-\phi(z)} (\overline{\eta}_I \sigma^\mu \Sigma^J (w) \Sigma^I (z) e^{ikX})$$

where we used the OPEs given in (2.3) and in Appendix A. All remaining commutators, obtained in a similar way, are collected below.

Applying the transformation (2.9) on the scalar vertex operators (2.12), we obtain:

$$[ Q^I (\eta_I), V_{\phi^a,\pm} ] = g_\lambda \, T^a \phi^\mu (\eta_I^\alpha \sigma^\mu \Sigma^J (z) e^{ikX}$$

$$[ \overline{Q}^I (\bar{\eta}_I), V_{\phi^a,\pm} ] = g_\lambda \, T^a \phi^\mu (\bar{\eta}_I \sigma^\mu \Sigma^J (w) \Sigma^J (z) e^{ikX}.$$

The respective internal indices $J$ for the Ramond fields (gaugino vertices) are given in Tables 1a and 1b.

| $Q^1$ | $\phi^1$ | $\phi^2$ | $\phi^3$ | $\phi^4$ | $\phi^5$ | $\phi^6$ | $\phi^7$ | $\phi^8$ |
|-------|----------|----------|----------|----------|----------|----------|----------|----------|
| 0     | 0        | 0        | 0        | 0        | 0        | 0        | 0        | 0        |
| $Q^2$ | $\Sigma^1$ | 0        | 0        | 0        | 0        | 0        | 0        | 0        |
| $Q^3$ | 0        | $\Sigma^2$ | 0        | 0        | 0        | 0        | 0        | 0        |
| $Q^4$ | 0        | 0        | $\Sigma^3$ | 0        | 0        | 0        | 0        | 0        |

**Table 1a:** Index structure $(I, i)$ and $J$ of the supersymmetry variations $[Q^I (\eta_I), V_{\phi^a,\pm}]$.

| $\overline{Q}^1$ | $\phi^1$ | $\phi^2$ | $\phi^3$ | $\phi^4$ | $\phi^5$ | $\phi^6$ | $\phi^7$ | $\phi^8$ |
|-----------------|----------|----------|----------|----------|----------|----------|----------|----------|
| $\Sigma^1$     | 0        | 0        | 0        | 0        | 0        | 0        | 0        | 0        |
| $\Sigma^2$     | 0        | $\Sigma^1$ | 0        | 0        | 0        | 0        | 0        | 0        |
| $\Sigma^3$     | 0        | 0        | $\Sigma^2$ | 0        | 0        | 0        | 0        | 0        |
| $\Sigma^4$     | 0        | 0        | 0        | $\Sigma^3$ | 0        | 0        | 0        | 0        |

**Table 1b:** Index structure $(I, i)$ and $J$ of the supersymmetry variations $[\overline{Q}^I (\bar{\eta}_I), V_{\phi^a,\pm}]$.

For gaugino vertices (2.13), we obtain

$$[ Q^I (\eta_I), V_{\ lambda,\tau} ] = g_\lambda \, T^a \phi^\mu (\eta_I^\alpha \sigma^\mu \Sigma^J (z) e^{ikX}$$

$$[ \overline{Q}^I (\bar{\eta}_I), V_{\ lambda,\tau} ] = g_\lambda \sqrt{2} \, T^a \phi^\mu (\bar{\eta}_I \sigma^\mu \Sigma^J (z) e^{ikX}$$

$$[ Q^I (\eta_I), V_{\ X^a,\tau} ] = g_A \, T^a \phi^\mu (\eta_I^\alpha \sigma^\mu \Sigma^J (z) e^{ikX}$$

$$[ \overline{Q}^I (\bar{\eta}_I), V_{\ X^a,\tau} ] = g_A \sqrt{2} \, T^a \phi^\mu (\bar{\eta}_I \sigma^\mu \Sigma^J (z) e^{ikX}$$

with the complex fermions $\psi^{\lambda J}$ defined in (2.11). Finally, for the vectors (2.14), we obtain

$$[ Q^I (\eta_I), V_{\ A^a} ] = g_A \, T^a \phi^\mu (\eta_I^\alpha \sigma^\mu \Sigma^J (z) e^{ikX}$$

$$[ \overline{Q}^I (\bar{\eta}_I), V_{\ A^a} ] = g_A \sqrt{2} \, T^a \phi^\mu (\bar{\eta}_I \sigma^\mu \Sigma^J (z) e^{ikX}$$
with \((\sigma^{\mu\nu})_\alpha^\beta = \frac{1}{2}(\sigma^{\mu}_{\alpha\dot{\alpha}}\sigma^{\nu\beta}_{\dot{\alpha}\beta} - \sigma^{\nu}_{\alpha\dot{\alpha}}\sigma^{\mu\beta}_{\dot{\alpha}\beta})\) and \((\overline{\sigma}^{\mu\nu})_{\dot{\alpha}^\beta} = \frac{1}{2}(\overline{\sigma}^{\mu\beta}_{\alpha\dot{\alpha}}\sigma^{\nu\alpha}_{\dot{\alpha}\beta} - \overline{\sigma}^{\nu\alpha}_{\alpha\dot{\alpha}}\sigma^{\mu\beta}_{\dot{\alpha}\beta})\).

The results (2.16)–(2.18) can be rewritten without referring to a particular ghost picture, in the following form:

\[
[ Q^I(\eta I), V_{\phi^\alpha,i^\pm}(z, k) ] = V_{\lambda_{\alpha},j}(z, \bar{\nu}, k) , \quad \bar{v}_{\beta} = k_\mu \bar{\eta}_{\beta}^\alpha \sigma^{\mu\alpha}_{\beta} \\
[ \overline{Q}^J(\bar{\eta} I), V_{\phi^\alpha,i^\pm}(z, k) ] = V_{\lambda_{\alpha},j}(z, \nu, k) , \quad v_{\beta} = k_\mu \bar{\eta}_{\beta}^\alpha \overline{\sigma}^{\mu\alpha}_{\beta} \\
[ Q^I(\eta I), V_{\lambda_{\alpha},j}(z, u, k) ] = (\eta I u) \ V_{\phi^\alpha,i^\pm}(z, k) , \\
[ \overline{Q}^J(\bar{\eta} I), V_{\lambda_{\alpha},j}(z, \bar{u}, k) ] = (\bar{\eta} I \bar{u}) \ V_{\phi^\alpha,i^\pm}(z, k) , \\
[ Q^I(\eta I), V_{\lambda_{\alpha},j}(z, \bar{u}, k) ] = \frac{1}{\sqrt{2}} \delta^{IJ} V_{A^\alpha}(z, \xi, k) , \quad \xi^\mu = \eta^\alpha_I \sigma^{\mu\alpha}_{\beta} \bar{u}^\beta , \\
[ \overline{Q}^J(\bar{\eta} I), V_{\lambda_{\alpha},j}(z, u, k) ] = \frac{1}{\sqrt{2}} \delta^{IJ} V_{A^\alpha}(z, \xi, k) , \quad \xi^\mu = \bar{\eta} I \dot{\alpha} \overline{\sigma}^{\mu\alpha}_{\beta} u^\beta , \\
[ Q^I(\eta I), V_{\lambda_{\alpha},j}(z, v, k) ] = \frac{1}{\sqrt{2}} V_{\lambda_{\alpha},j}(z, \nu, k) , \quad v_{\beta} = \xi_{\mu} \ k_\nu \ \eta_{I}^\alpha (\sigma^{\mu\nu})_{\alpha}^\beta , \\
[ \overline{Q}^J(\bar{\eta} I), V_{\lambda_{\alpha},j}(z, \bar{v}, k) ] = \frac{1}{\sqrt{2}} V_{\lambda_{\alpha},j}(z, \bar{u}, k) , \quad \bar{v}_{\beta} = \xi_{\mu} \ k_\nu \ \bar{\eta} I \dot{\alpha} (\overline{\sigma}^{\mu\nu})_{\dot{\alpha}}^\beta .
\]

(2.19)

### 2.3. Contour deformations and SUSY Ward identities

In this section we consider the action of the SUSY generator (2.1) on a correlation function of \(N\) open string states with vertex operators \(V_i(z_i)\) inserted at the boundary of the disk. We choose a contour \(C_\infty\) that surrounds all vertex positions \(z_i\) and consider the integral (see Fig. 2):

\[
\mathcal{W} := \oint_{C_\infty} \frac{dw}{2\pi i} \ \eta^\alpha_I \ (V^I_{\alpha}(w) \ V_1(z_1) \ V_2(z_2) \ldots V_N(z_N)) .
\]

(2.20)

The SUSY current \(V^I_{\alpha}(w)\) has conformal dimension one. Therefore at infinity the correlator must behave like \(\sim w^{-2}\). Since the contour \(C_\infty\) may be deformed to infinity, we conclude \(\mathcal{W} = 0\).
On the other hand, analyticity allows to deform the contour to a sum over contours $C_l$ encircling each of the points $z_l$ and we may rewrite (2.20) as:

$$W = \sum_{l=1}^{N} \langle V_1(z_1) \cdots V_{l-1}(z_{l-1}) \left[ \oint_{C_l} \frac{dw}{2\pi i} \eta_l^{\alpha} V_\alpha^I(w) V_l(z_l) \right] V_{l+1}(z_{l+1}) \cdots V_N(z_N) \rangle .$$

(2.21)

In the above expression, each SUSY variation (2.9) gives rise to a SUSY-transformed vertex operator (depending on $\eta_l^{\alpha}$) discussed in the previous subsection. Hence the above equation gives rise to non–trivial relations between different correlators:

$$\sum_{l=1}^{N} \langle V_1(z_1) \cdots V_{l-1}(z_{l-1}) \left[ \eta_l^{\alpha} Q_{\alpha}^I, V_l(z_l) \right] V_{l+1}(z_{l+1}) \cdots V_N(z_N) \rangle = 0 .$$

(2.22)

Since Eq.(2.22) holds for arbitrary spinors $\eta$, the coefficients of $\eta_1$ and $\eta_2$ on its l.h.s., as well as the coefficients of $\bar{\eta}_1$ and $\bar{\eta}_2$ in the relations involving $\bar{Q}(\bar{\eta})$, must vanish. In this way, SUSY Ward identities can be derived, relating various scattering amplitudes $[3,4,5]$. One of such Ward identities will be applied below to the computations of $N$-gluon amplitudes. We also use this equation in section 4.2 to relate various four–point string amplitudes.

2.4. Helicity basis and comparison with $\alpha' \to 0$ limit

It is well-known from field-theoretical computations that scattering amplitudes simplify in the so-called helicity basis, that is by considering processes with definite left- and right-handed polarizations states (helicity $-$ or $+$, respectively) of external particles. Furthermore, such a basis is very natural in the context of supersymmetry. In order to construct the string vertex operators describing helicity eigenstates, it is convenient to express the corresponding wave functions in terms of two-component, chiral spinors. We introduce the following shorthand notation for the wave functions of chiral fermions:

$$u^\alpha(k) = k^\alpha , \quad \bar{u}_{\dot{\alpha}}(k) = \bar{k}_{\dot{\alpha}} .$$

(2.23)

Then the on-shell momentum vectors $k$, with $k^2 = 0$, factorize as

$$k_\mu \sigma_\alpha^{\mu \dot{\alpha}} \equiv k_\alpha \bar{k}_{\dot{\alpha}} = k_\alpha \bar{k}_{\dot{\alpha}} , \quad k_\mu \bar{\sigma}_\dot{\alpha}^{\mu \alpha} \equiv k_{\dot{\alpha}} \bar{k}^{\dot{\alpha}} = \bar{k}^{\dot{\alpha}} k^{\alpha} .$$

(2.24)

Note that the inverse relation is

$$k_\mu = -\frac{1}{2} k_\alpha \bar{\sigma}_\mu^{\alpha \dot{\alpha}} = -\frac{1}{2} k_\dot{\alpha} \sigma^{\dot{\alpha} \alpha} .$$

(2.25)

The bracket products of two spinors associated to momenta $p$ and $q$ are defined as

$$\langle p | q \rangle = p^{\alpha} q_{\alpha} , \quad [p | q] = \bar{p}_{\dot{\alpha}} \bar{q}^{\dot{\alpha}} .$$

(2.26)
We also use the same symbols for products involving arbitrary spinors: \( \eta^\alpha p_\alpha \equiv \langle \eta p \rangle \) etc. In this notation, the polarization vectors of left- and right-handed bosons are:

\[
\xi^-_{\alpha\dot{\alpha}}(k, r) = \sqrt{2} \frac{k_\alpha \bar{r}_{\dot{\alpha}}}{|k r|}, \quad \xi^+_{\alpha\dot{\alpha}}(k, r) = \sqrt{2} \frac{r_\alpha \bar{k}_{\dot{\alpha}}}{|r k|},
\]

where \( r \) is an arbitrary reference vector.

We introduce the following notation for the vertices describing \(-\) and \(+\) helicity states of the \( \mathcal{N} = 4 \) gauge vector multiplet:

\[
\phi^i_-(z, k) = V_{\phi^i-}(z, k) \quad \lambda^J_-(z, k) = V_{\lambda^J-}(z, u^\alpha, k) \quad g^-(z, k) = V_A(z, \xi^\mu, k)
\]

\[
\phi^i_+(z, k) = V_{\phi^i+}(z, k) \quad \lambda^J_+(z, k) = V_{\lambda^J+}(z, \bar{u}_{\dot{\alpha}}, k) \quad g^+(z, k) = V_A(z, \xi^{+\mu}, k).
\]

Next, we consider the gaugino transformations (2.17). We will prove that

\[
\bar{\eta}^\dot{\alpha} u_\alpha^\mu(k) \sigma_{\alpha\dot{\alpha}} \psi_\mu = \sqrt{2} \frac{k_\alpha}{|k r|} \langle k \eta \rangle \langle p k \rangle\langle r q \rangle = \| \eta q \| \langle p k \rangle\langle r q \rangle,
\]

with the provision that, as in our case, \( \psi_\mu \) belongs to the gluon vertex operator (2.14) inserted in an on-shell scattering amplitude. Then the contractions involving \( \psi_\mu \) amount to replacements \( \sigma_{\alpha\dot{\alpha}} \psi_\mu \rightarrow p_\alpha \bar{q}_{\dot{\alpha}} \), where \( p \) and \( q \) are some light-like vectors. Then the l.h.s. of Eq.(2.31) becomes

\[
\bar{\eta}^\dot{\alpha} k_\alpha \sigma_{\alpha\dot{\alpha}} \psi_\mu = [\eta q] \langle p k \rangle.
\]

On the other hand, after substituting the polarization vector (2.27) into the r.h.s. of Eq.(2.31), we obtain

\[
\sqrt{2} \frac{k_\alpha}{|k r|} \langle k \eta \rangle \langle p k \rangle\langle r q \rangle = [\eta q] \langle p k \rangle + \frac{|r \eta|}{|k r|} \langle p k \rangle \langle k q \rangle,
\]

where we used Schouten's identity. After reinstating \( \psi_\mu \) in the second term on the r.h.s. of Eq.(2.33), \( \langle p k \rangle \langle k q \rangle \rightarrow k_\mu \psi_\mu \), one finds that it is equivalent to a contribution of the

\[\text{Here, we skip the gauge indices.}\]
longitudinal part of a polarization vector, therefore it does not contribute to the amplitude. Then Eq.(2.31) follows from Eqs.(2.32) and (2.33). Similarly, one finds
\[
\eta^\alpha \bar{u}^\dot{\alpha}(k) \sigma_{\alpha \dot{\alpha}}^\mu \psi_\mu = -\sqrt{2}(\eta k) \xi^{+\mu} \psi_\mu \tag{2.34}
\]
Finally, we consider the gluon transformations (2.18). The term
\[
\xi_\mu^k k_\nu(\eta^\alpha \sigma_{\alpha \beta}^\nu S^\beta) = \frac{1}{2}(\eta^\alpha k_\alpha \bar{k}_\dot{\alpha} \xi^{\dot{\alpha} \beta} S_\beta - \eta^\alpha \xi_{\alpha \dot{\alpha}} \bar{k}^\dot{\alpha} k^\beta S_\beta) \tag{2.35}
\]
vanishes for the right-handed polarization vector \( \xi^+ \), c.f. Eq.(2.27), while for the left-handed polarization
\[
\xi_{\mu}^- k_\nu(\eta^\alpha \sigma_{\alpha \beta}^\nu S^\beta) = \sqrt{2}(\eta k) u^\alpha(k) S_\alpha \tag{2.36}
\]
Similarly, one finds
\[
\xi_{\mu}^+ k_\nu(\bar{\eta}^\dot{\alpha} \bar{\sigma}_{\dot{\alpha} \beta}^\mu S^\beta) = -\sqrt{2}[\eta k] \bar{u}^{\dot{\alpha}}(k) S^{\dot{\alpha}} \tag{2.37}
\]
Note that all transformations involve one of the factors
\[
\Gamma^-(\eta, k) = \langle \eta k \rangle \ , \ \Gamma^+(\eta, k) = [\eta k] . \tag{2.38}
\]
After collecting all above formulae, Eq.(2.13) can be rewritten as
\[
\begin{align*}
[ Q^I(\eta_I), \phi^{i\pm}(z, k) ] &= \Gamma^-(\eta_I, k) \lambda^J(z, \bar{u}, k) \\
[ \bar{Q}^I(\bar{\eta}_I), \phi^{i\pm}(z, k) ] &= \Gamma^+(\eta_I, k) \lambda^J(z, u, k) \\
[ Q^I(\eta_I), \lambda_{iJ}^-(z, k) ] &= \Gamma^-(\eta_I, k) \phi^{j\pm}(z, k) \\
[ \bar{Q}^I(\bar{\eta}_I), \lambda_{iJ}^+(z, k) ] &= \Gamma^+(\eta_I, k) \phi^{j\mp}(z, k) \\
[ Q^I(\eta_I), g^{i\pm}(z, k) ] &= -\Gamma^-(\eta_I, k) \delta^{IJ} g^J(z, \xi, k) \\
[ \bar{Q}^I(\bar{\eta}_I), g^{i\pm}(z, k) ] &= \Gamma^+(\eta_I, k) \delta^{IJ} g^J(z, \xi, k) \\
[ Q^I(\eta_I), g^-(z, k) ] &= -\Gamma^-(\eta_I, k) \lambda^J(z, k) \\
[ \bar{Q}^I(\bar{\eta}_I), g^+(z, k) ] &= -\Gamma^+(\eta_I, k) \lambda^J(z, k) ,
\end{align*}
\tag{2.39}
\]
where the l.h.s. and r.h.s. indices are paired according to Eq.(2.11) and Table 1, keeping in mind that \( \Sigma \) and \( \overline{\Sigma} \) are associated to \( - \) and \( + \) gaugino helicities, respectively, while \( \Psi = \Psi^- \) to \( - \) and \( + \) scalars.

As far as practical applications of SUSY relations to the computations of gluon amplitudes are concerned, there is no need to employ the full \( \mathcal{N} = 4 \) algebra. It is often sufficient to consider the gauge multiplet of \( \mathcal{N} = 2 \) SUSY generated by \( Q^K = (Q^1, iQ^2) \) \[3\]. Such \( \mathcal{N} = 2 \) SUSY transformations, parameterized by a Dirac spinor \((\eta_K, \bar{\eta}_K)\), are generated by:
\[
Q^K(\eta_K, \bar{\eta}_K) = \eta_K^\alpha Q^K_\alpha + \bar{\eta}_\dot{\alpha} K\bar{Q}^{\dot{\alpha}K} . \tag{2.40}
\]
The $\mathcal{N} = 2$ gauge multiplet consists of the gluon $A$, two gauginos $\lambda^L = (\lambda^1, i\lambda^2)$ and the scalar $\phi^1 \equiv \phi$. The SUSY transformations of the respective vertex operators can be extracted from Eq.(2.39). They are:

$$[Q^K(\eta_K, \bar{\eta}_K), \phi^\pm(z, k)] = \pm i \varepsilon^{KL} \Gamma^\mp(\eta_K, k) \lambda^L(z, k),$$

$$[Q^K(\eta_K, \bar{\eta}_K), \lambda^L(z, k)] = \mp \delta^{KL} \Gamma^\pm(\eta_K, k) g^\pm(z, k) \mp i \varepsilon^{KL} \Gamma^\pm(\eta_K, k) \phi^\pm(z, k), \quad (2.41)$$

$$[Q^K(\eta_K, \bar{\eta}_K), g^\pm(z, k)] = \mp \Gamma^\pm(\eta_K, k) \lambda^K(z, k).$$

The field-theoretical relations that should be compared with our SUSY transformations of string vertices are the transformations of the creation and annihilation operators, summarized in Ref.[3]. Indeed, Eq.(2.41) agree with Eq.(4) of [3], to all orders in $\alpha'$. Since the contour manipulations described at the beginning of this Section are equivalent to applying SUSY Ward identities, all field-theoretical ($\alpha' = 0$) SUSY relations between various amplitudes remain valid in full-fledged superstring theory.

The main focus of this work are the $\mathcal{N}$-gluon MHV disk amplitudes. We will use the well-known relation [3,1,2]

$$A(1^-, 2^-, 3^+, 4^+, \ldots, N^+) =$$

$$= \frac{(12)^2}{(34)^2} A[\phi^-(k_1), \phi^-(k_2), \phi^+(k_3), \phi^+(k_4), g^+(k_5), \ldots, g^+(k_N)], \quad (2.42)$$

a consequence of Eqs.(2.41), now guaranteed to hold to all orders in $\alpha'$. It allows replacing four gluons by four scalars, which is expected to yield a considerable simplification because the scalar vertices are much easier to handle than the gluonic ones. We would like to stress that at the disk level, the $\mathcal{N}$-gluon MHV amplitude on the l.h.s. of Eq.(2.42) is universal to all compactifications while the amplitude on the r.h.s. involves the scalar member of $\mathcal{N} = 2$ gauge multiplet, and can be evaluated in an arbitrary $\mathcal{N} \geq 2$ compactification. For our purposes, it is most convenient to use the toroidal, $\mathcal{N} = 4$ compactifications, with the gauge scalar associated to one of the three complex planes.

We will be first considering the so-called partial amplitudes, associated to one particular gauge group (Chan-Paton) factor $\text{Tr}(T^{a_1} \cdots T^{a_N})$, as in Eq.(1.1), and later explain how to obtain full amplitudes.

3. Disk scattering of scalars and gluons

In this section, we compute the string amplitudes for the scattering of four scalars and $N - 4$ gluons at the disk level. All vertex operators are inserted at the boundary of the
Chan-Paton factor takes the form:
\[
A(\phi^{a_1}, \ldots, \phi^{a_N}) = V^{-1}_{CKG} \int_{z_1 \ldots z_N} \prod_{k=1}^{N} dz_k \int \frac{d^{2N-1}}{4\pi} \partial \quad \text{for} \quad z_1 \ldots z_N
\]

In the notation of Refs.\[13,14\], the partial amplitude associated to \( \text{Tr}(T^{a_1} \ldots T^{a_N}) \) is:
\[
A(\phi^{a_1}, \ldots, \phi^{a_N}) = V^{-1}_{CKG} \int_{z_1 \ldots z_N} \prod_{k=1}^{N} dz_k \int \frac{d^{2N-1}}{4\pi} \partial \quad \text{for} \quad z_1 \ldots z_N
\]

The vertex operators for the scalars, in \((-1)-\) and zero–ghost pictures, are (cf. also (2.12))
\[
V^{(-1)}_{\phi^{a_-}}(z, k) = g_\phi T^a \ e^{-\phi} \ e^{ik_\mu X^\mu(z)} \\
V^{(0)}_{\phi^{a_-}}(z, k) = \frac{g_\phi}{(2\alpha')^{1/2}} T^a [ i\partial Z + 2\alpha' (k\psi) ] \ e^{ik_\mu X^\mu(z)} ,
\]

respectively. The vertex operator for the gauge boson has been already written before in the \((-1)-\)ghost picture in Eq.(2.14), and in the zero–ghost picture it takes the form:
\[
V^{(0)}_{A^{a}}(z, \xi, k) = \frac{g_A}{(2\alpha')^{1/2}} T^a \xi_\mu [ i\partial X^\mu + 2\alpha' (k\psi) ] \ e^{ik_\mu X^\mu(z)} .
\]

In order to cancel the background ghost charge on the disk, two vertices in the correlator (3.1) will be inserted in the \((-1)-\)ghost picture, with the remaining ones in the zero–ghost picture. Furthermore, in Eq.(3.1), the factor \( V_{CKG} \) accounts for the volume of the conformal Killing group of the disk after choosing the conformal gauge. It will be canceled by fixing three vertex positions and introducing the respective \( c \)-ghost contractions. The correlator of vertex operators in (3.1) is evaluated by performing all possible Wick contractions. It decomposes into products of the two–point functions on the boundary of the disk:
\[
\langle \partial X^\mu(z_1) X^\nu(z_2) \rangle = -\frac{2\alpha' \delta^{\mu\nu}}{z_{12}}, \quad \langle \partial X^\mu(z_1) \partial X^\nu(z_2) \rangle = -\frac{2\alpha' \delta^{\mu\nu}}{z_{12}^2} \\
\langle e^{ik_\mu X^\mu(z_1)} e^{ik_\nu X^\nu(z_2)} \rangle = |z_{12}|^{2\alpha' k_1 k_2}, \quad \langle e^{-\phi(z_1)} e^{-\phi(z_2)} \rangle = \frac{1}{z_{12}}, \quad \langle \psi^\mu(z_1) \psi^\mu(z_2) \rangle = \frac{\delta^{\mu\nu}}{z_{12}} \\
\langle \bar{\Psi}(z_1) \Psi(z_2) \rangle = \frac{1}{z_{12}}, \quad \langle \bar{\Psi}(z_1) \bar{\Psi}(z_2) \rangle = 0, \\
\langle \partial Z(z_1) \partial \bar{Z}(z_2) \rangle = -\frac{2\alpha' \delta^{\mu\nu}}{z_{12}^2}, \quad \langle \partial Z(z_1) \partial Z(z_2) \rangle = 0.
\]

(3.4)
Because of the $\text{PSL}(2, \mathbb{R})$ invariance on the disk, we can fix three positions of the vertex operators. A convenient choice respecting the integration region $z_1 < \ldots < z_N$ is

$$z_1 = -z_\infty = -\infty \quad , \quad z_2 = 0 \quad , \quad z_3 = 1 , \quad (3.5)$$

which implies the ghost factor \langle c(z_1) c(z_2) c(z_3) \rangle = -z_\infty^2$. The remaining $N - 3$ vertex positions $z_4, \ldots z_N$ take arbitrary values inside the integration domain $1 < z_4 < \ldots < z_N < \infty$.

In order to correctly normalize the amplitudes, some additional factors have to be taken into account. They stem from determinants and Jacobians of certain path integrals. On the disk, the net result of those contributions is an additional factor of

$$C_{D_2} = \frac{1}{2g_Y^2 \alpha' \ell^2} , \quad (3.6)$$

which must be included in all disk correlators [21].

3.1. Five-point amplitudes

For $N = 5$, Eq. (3.1) yields the following correlator:

$$\langle V_{\phi a_{1,-}}^{(0)}(z_1, k_1) V_{\phi a_{2,-}}^{(0)}(z_2, k_2) V_{\phi a_{3,+}}^{(-1)}(z_3, k_3) V_{\phi a_{4,+}}^{(0)}(z_4, k_4) V_{\phi a_{5}}^{(-1)}(z_5, \xi_5, k_5) \rangle . \quad (3.7)$$

Proceeding as outlined above, we obtain

$$A(\phi a_{1,-}, \phi a_{2,-}, \phi a_{3,+}, \phi a_{4,+}, A^{a_5}) = 2(2\alpha') \ g_Y^3 \ \text{Tr}(T^{a_1} T^{a_2} T^{a_3} T^{a_4} T^{a_5}) \ V_{C kg}^{-1}$$

$$\times \int_{z_1 < \ldots < z_5} \left( \prod_{k=1}^{5} dz_k \right) \left( \prod_{i<j} |z_{ij}|^{s_{ij}} \right) \frac{1}{z_{35}} \left\{ \frac{\xi_5 k_4}{z_{45}} \frac{s_{12}}{z_{24} z_{13}} \frac{z_{34}}{z_{14} z_{23}} \right\} , \quad (3.8)$$

with $s_{ij} = 2\alpha' k_i k_j$ [13][14]. We set $z_1, z_2, z_3$ as in (3.5) (taking into account the ghost factor) and use the parameterization

$$z_4 = x^{-1} \quad , \quad z_5 = (x y)^{-1} , \quad (0 < x , y < 1) , \quad (3.9)$$

with the corresponding Jacobian $\text{det}(\frac{\partial (z_4, z_5)}{\partial (x, y)}) = x^{-3} y^{-2}$. Then Eq. (3.8) becomes

$$A(\phi a_{1,-}, \phi a_{2,-}, \phi a_{3,+}, \phi a_{4,+}, A^{a_5}) = 2(2\alpha') \ g_Y^3 \ \text{Tr}(T^{a_1} T^{a_2} T^{a_3} T^{a_4} T^{a_5}) \$$

$$\times \left[ (\xi_5 k_1) K_1 + (\xi_5 k_2) K_2 + (\xi_5 k_4) K_3 \right] , \quad (3.10)$$

13
with the three functions

\[
K_1 = \int_0^1 dx \int_0^1 dy \left( 1 - s_{24} + \frac{s_{24}}{x} \right) \frac{\mathcal{I}(x,y)}{y(1-xy)} , \\
K_2 = \int_0^1 dx \int_0^1 dy \left( \frac{1 - s_{14}}{x} + s_{14} \right) \frac{\mathcal{I}(x,y)}{(1-xy)} , \\
K_3 = s_{12} \int_0^1 dx \int_0^1 dy \left( \frac{1-x}{x(1-y)} \right) \frac{\mathcal{I}(x,y)}{(1-xy)} ,
\]

where

\[
\mathcal{I}(x,y) = x^{s_2} y^{s_5} (1-x)^{s_3} (1-y)^{s_4} (1-xy)^{s_1-s_3-s_4} ,
\]

and \( s_i \equiv \alpha'(k_i + k_{i+1})^2 \), subject to the cyclic identification \( i + 5 \equiv i \). The functions (3.11) integrate to Gaussian hypergeometric functions \( _3F_2[14] \). Actually, \( K_3 \) may be expressed in terms of \( K_1 \) and \( K_2 \):

\[
K_3 = -\frac{s_5}{s_4} K_1 + \frac{s_1 - s_3 + s_5}{s_4} K_2 .
\]

Thus as expected, \( \alpha' \)-dependence of the amplitude describing the scattering of four-scalars and one gluon is determined by two independent functions, exactly as many as in the five-gluon case \[22,23,14\].

The low–energy behavior of the amplitude (3.10) is determined, up to the order \( \mathcal{O}(\alpha'^2) \), by the following expansions:

\[
K_1 = \frac{1}{s_2} - \frac{s_3}{s_2 s_5} - \zeta(2) \left( s_1 + s_3 - \frac{s_3 s_4}{s_2} + \frac{s_4 s_5}{s_2} - \frac{s_3^2}{s_5} \right) + \ldots , \\
K_2 = \frac{1}{s_2} - \zeta(2) \left( s_1 + s_3 + \frac{s_4 s_5}{s_2} \right) + \ldots , \\
K_3 = \frac{s_1}{s_2 s_4} - \zeta(2) \left( \frac{s_1 s_5}{s_2} + \frac{s_1^2}{s_4} \right) + \ldots .
\]

The functions \( \{K_1,K_2\} \) can be expressed in terms of another two-element basis, previously used in \[13,14\]. There, we defined two functions:

\[
f_1 = \int_0^1 dx \int_0^1 dy \ x^{-1} y^{-1} \mathcal{I}(x,y) , \quad f_2 = \int_0^1 dx \int_0^1 dy \ (1-xy)^{-1} \mathcal{I}(x,y) .
\]

Expressed in terms of \( \{f_1,f_2\} \), the functions \( K_1 \) and \( K_2 \) read:

\[
K_1 = (s_5 - s_3) f_1 - s_1 f_2 , \quad K_2 = s_5 f_1 - s_1 f_2 .
\]
The result (3.10) can be further simplified by choosing $k_4$ as the reference vector $r$ for the polarization vector $\xi_5$, so that $\xi_5 k_4 = 0$, see Eq.(2.27). Then, for the positive polarization of the gluon,

$$\xi_5^+ k_1 = - \frac{1}{\sqrt{2}} \langle 41 \rangle [15] , \quad \xi_5^+ k_2 = - \frac{1}{\sqrt{2}} \langle 42 \rangle [25] .$$

(3.17)

As a result of combining Eq.(3.10) with the SUSY relation (2.42), after factorizing out $(\sqrt{2} g_{YM})^3 \text{Tr}(T^{a_1} \cdots T^{a_5})$, we obtain the five-gluon MHV amplitude:

$$A(1^-, 2^-, 3^+, 4^+, 5^+) = \alpha' \frac{(12)^2}{(34)^2} \langle 41 \rangle [15] K_1 + \langle 42 \rangle [25] K_2 .$$

(3.18)

The above result should be compared with the factorized form derived in [14]

$$A(1^-, 2^-, 3^+, 4^+, 5^+) = [ V^{(5)}(s_j) - 2i P^{(5)}(s_j) \epsilon(1, 2, 3, 4) ] M^{(5)}_{YM} ,$$

(3.19)

where:

$$V^{(5)}(s_i) = s_2 s_5 f_1 + \frac{1}{2} (s_2 s_3 + s_4 s_5 - s_1 s_2 - s_3 s_4 - s_1 s_5) f_2 , \quad P^{(5)}(s_i) = f_2 .$$

(3.20)

Indeed, it is a matter of simple spinor manipulations to show that Eqs.(3.18) and (3.19) agree upon using the relations (3.16) between the two sets of basis functions, $\{K_1, K_2\}$ and $\{f_1, f_2\}$. Unlike the factorized amplitude (3.19), the new form (3.18) is expressed directly in terms of “primitive” integrals (3.11), without resorting to the definitions of “formfactor” functions $V$ and $P$, see Eq.(3.20). Although for five gluons it is not a dramatic simplification, we will see that for six gluons and more, the non-factorized form is more suitable.

3.2. Six–point amplitudes

Now we turn to $N = 6$. For that case in (3.1), we compute the following correlator:

$$\langle V^{(5)}_{\phi^{a_1}, -}(z_1, k_1) V^{(0)}_{\phi^{a_2}, -}(z_2, k_2) V^{(6)}_{\phi^{a_3}, +}(z_3, k_3) V^{(0)}_{\phi^{a_4}, +}(z_4, k_4) V^{(-1)}_{A^{a_5}}(z_5, k_5) V^{(-1)}_{A^{a_6}}(z_6, k_6) \rangle .$$

(3.21)

With the choice (3.3) and the parameterization

$$z_4 = x^{-1} , \quad z_5 = (x y)^{-1} , \quad z_6 = (x y z)^{-1} ,$$

the Jacobian $\det(\frac{\partial (z_4, z_5, z_6)}{\partial (x, y, z)}) = x^{-4} y^{-3} z^{-2}$. The amplitude (3.21) becomes

$$A(\phi^{a_1}, -, \phi^{a_2}, -, \phi^{a_3}, +, \phi^{a_4}, +, A^{a_5}, A^{a_6}) = 2(2\alpha')^2 g_{YM}^4 \text{Tr}(T^{a_1} T^{a_2} T^{a_3} T^{a_4} T^{a_5} T^{a_6})$$

$$\times \left[ (\xi_5 k_3)(\xi_5 k_2) K_1 + (\xi_5 k_2)(\xi_6 k_3) K_2 + (\xi_5 k_1)(\xi_6 k_2) K_3 + (\xi_5 k_1)(\xi_6 k_3) K_4$$

$$+ (\xi_5 k_2)(\xi_6 k_1) K_5 + (\xi_5 k_3)(\xi_6 k_1) K_6 + (\xi_5 k_3)(\xi_6 k_2) K_7 + (\xi_5 k_4)(\xi_6 k_3) K_8$$

$$+ (\xi_5 k_4)(\xi_6 k_4) K_9 + (\xi_5 k_4)(\xi_6 k_2) K_{10} + (\xi_5 k_1)(\xi_6 k_4) K_{11} + (\xi_5 k_4)(\xi_6 k_1) K_{12}$$

$$+ (\xi_5 k_6) K_{13} \right] ,$$

(3.23)
where the integrals

\[
K_1 = \int_0^1 dx \int_0^1 dy \int_0^1 dz \left( \frac{1 - s_{14}}{x} + s_{14} \right) \frac{\mathcal{I}(x, y, z)}{(1 - z)(1 - xy)},
\]

\[
K_2 = -\int_0^1 dx \int_0^1 dy \int_0^1 dz \left( \frac{1 - s_{14}}{x} + s_{14} \right) \frac{\mathcal{I}(x, y, z)}{(1 - z)(1 - xyz)},
\]

\[
K_3 = s_{34} \int_0^1 dx \int_0^1 dy \int_0^1 dz \frac{\mathcal{I}(x, y, z)}{xy(1 - z)},
\]

\[
K_4 = -\int_0^1 dx \int_0^1 dy \int_0^1 dz \left( 1 - s_{24} + \frac{s_{24}}{x} \right) \frac{\mathcal{I}(x, y, z)}{y(1 - z)(1 - xyz)},
\]

\[
K_5 = -s_{34} \int_0^1 dx \int_0^1 dy \int_0^1 dz \frac{\mathcal{I}(x, y, z)}{xyz(1 - z)},
\]

\[
K_6 = \int_0^1 dx \int_0^1 dy \int_0^1 dz \left( 1 - s_{24} + \frac{s_{24}}{x} \right) \frac{\mathcal{I}(x, y, z)}{yz(1 - z)(1 - xy)},
\]

and the remaining seven integrals are displayed in Eq.(B.1) of Appendix B. All integrands contain the common factor

\[
\mathcal{I}(x, y, z) = x^{s_{12}} y^{t_{12}} z^{s_{3}} (1 - x)^{s_{3}} (1 - y)^{s_{4}} (1 - z)^{s_{5}}
\times (1 - xy)^{t_{3} - s_{3} - s_{4}} (1 - yz)^{t_{1} - s_{4} - s_{5}} (1 - xyz)^{s_{1} + s_{4} - t_{1} - t_{3}},
\]

(3.25)

where \(s_i = \alpha'(k_i + k_{i+1})^2\) and \(t_j = \alpha'(k_j + k_{j+1} + k_{j+2})^2\), subject to the cyclic identification \(i + 6 \equiv i\).

The integrals (3.24) and (B.1) integrate to multiple Gaussian hypergeometric functions, more precisely they represent triple hypergeometric functions [23]. Although the result (3.23) uses thirteen functions, only six of them are linearly independent, exactly as many as in the six-gluon case [23]. For our purposes, it is convenient to choose \(\{K_1, \ldots, K_6\}\) as the basis. The remaining seven functions are then expressed in terms of these functions in Eq.(B.2)

The low–energy behavior of the amplitude (3.10) is determined, up to the order \(\mathcal{O}(\alpha'^2)\),

16
by the following expansions:

\[
K_1 = \frac{1}{s_{2s5}} + \zeta(2) \left( \frac{1 - \frac{s_4 + s_5 + s_6 - t_1 - t_2}{s_2}}{s_5} - \frac{s_1 + s_3}{s_{2s5}} - \frac{t_1 t_2}{s_{2s5}} \right) + \ldots ,
\]

\[
K_2 = -\frac{1}{s_{2s5}} + \zeta(2) \left( \frac{s_4 + s_5 + s_6 - t_1 - t_2}{s_2} + \frac{s_1 + s_3}{s_5} + \frac{t_1 t_2}{s_{2s5}} \right) + \ldots ,
\]

\[
K_3 = \frac{s_3}{s_{2s5} t_2} + \zeta(2) \left( \frac{s_3 - s_3 t_1 - \frac{s_3^2}{s_{5t2}} - \frac{s_3 s_6}{s_{2t2}}}{s_{2s5}} \right) + \ldots ,
\]

\[
K_4 = -\frac{1}{s_{2s5}} + \frac{s_3}{s_{2s5} t_2} + \zeta(2) \left( \frac{s_3 + s_6 - t_2}{s_2} + \frac{s_1 + s_3}{s_5} - \frac{s_3 t_1}{s_{2s5}} - \frac{s_3^2}{s_{5t2}} - \frac{s_3 s_6}{s_{2t2}} + \frac{t_1 t_2}{s_{2s5}} \right) + \ldots ,
\]

\[
K_5 = -\frac{s_3}{s_{2s5} t_2} - \frac{s_3}{s_{2s6} t_2} + \zeta(2) \left( -\frac{s_3 + s_4 + s_3 t_1}{s_2} + \frac{s_3^2}{s_{2s5}} + \frac{s_3 t_2}{s_{2t2}} + \frac{s_3 s_5}{s_{2t2}} + \frac{s_3^2}{s_{2t2}} + \frac{s_3 s_6}{s_{2t2}} \right) + \ldots ,
\]

\[
K_6 = \frac{1}{s_{2s5}} + \frac{1}{s_{2s6}} - \frac{s_3}{s_{2s5} t_2} - \frac{s_3}{s_{2s6} t_2} + \zeta(2) \left( -\frac{s_3 + s_5 + s_6 + t_2}{s_2} - \frac{s_1 + s_3}{s_5} - \frac{s_3 + t_3}{s_6} \right.
\]

\[
+ \frac{s_3 t_1}{s_{2s5}} + \frac{s_3^2}{s_{5t2}} + \frac{s_3 s_3}{s_{2t2}} + \frac{s_3^2}{s_{2t2}} + \frac{s_3 s_6}{s_{2t2}} - \frac{s_4 t_1}{s_{2s6}} - \frac{t_1 t_2}{s_{2s5}} \bigg) + \ldots .
\]

(3.26)

At this point, we specify the result (3.23) to the case of two positive-helicity gluons, as in the amplitude entering the SUSY relation (2.42). We choose \(k_4\) as the reference vector \(r\) for both polarization vectors, so that \(\xi_i k_4 = 0\), see Eq.(2.27). Then:

\[
\xi_i^+ \xi_j^+ = 0 \quad , \quad \xi_i^+ k_j = -\frac{1}{\sqrt{2}} \frac{\langle 4j | 4i \rangle}{\langle 4i \rangle} .
\]

(3.27)

We also define:

\[
\tau(a, b) = 2 \left( \zeta_5^+ k_a \right) \left( \zeta_6^+ k_b \right) = \frac{\langle 4a | 4b \rangle [a^5] [b^6]}{\langle 45 \rangle [46]} .
\]

(3.28)

As a result of combining Eqs.(1.23) and (2.42), after factorizing out \((\sqrt{2} g_{YM})^4 \text{Tr}(T^{a_1} \cdots T^{a_6})\), we obtain the six-gluon MHV amplitude:

\[
A(1^-, 2^-, 3^+, 4^+, 5^+, 6^+) = \alpha^2 \left[ \frac{12}{(34)^2} \right] \left[ \tau(3, 2) K_1 + \tau(2, 3) K_2 + \tau(1, 2) K_3 
\right.

\[
+ \tau(1, 3) K_4 + \tau(2, 1) K_5 + \tau(3, 1) K_6 \bigg] .
\]

(3.29)

The above result should be compared with the factorized form derived in [14]:

\[
A(1^-, 2^-, 3^+, 4^+, 5^+, 6^+) = \left[ V^{(6)}(s_i, t_i) - 2i \sum_{k=1}^{k=5} \epsilon_k F_k^{(6)}(s_i, t_i) \right] \mathcal{M}_{YM}^{(6)} \quad (3.30)
\]

where

\[
\epsilon_1 = \epsilon(2, 3, 4, 5) \quad \epsilon_2 = \epsilon(1, 3, 4, 5) \quad \epsilon_3 = \epsilon(1, 2, 4, 5) \quad \epsilon_4 = \epsilon(1, 2, 3, 5) \quad \epsilon_5 = \epsilon(1, 2, 3, 4).
\]

(3.31)
The functions $V^{(6)}$ and $P_{k}^{(6)}, k = 1, \ldots, 5$, are certain (complicated) linear combinations of:

\[ F_1 = \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \frac{I(x, y, z)}{xyz} \quad F_2 = \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \frac{I(x, y, z)}{z(1-xy)} \]

\[ F_3 = \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \frac{I(x, y, z)}{1-xyz} \quad F_4 = \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \frac{yI(x, y, z)}{(1-xy)(1-yz)} \]

\[ F_5 = \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \frac{I(x, y, z)}{(1-xy)(1-xyz)} \quad F_6 = \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \frac{I(x, y, z)}{(1-yz)(1-xyz)} \] (3.32)

weighted by coefficients which are polynomial in the kinematic invariants.

In order to compare Eqs. (3.29) and (3.30), one needs relations between $K$- and $F$-functions. These can be obtained by integrating by parts combined with other manipulations described in \([13,14]\). One finds, for instance,

\[ s_2 s_5 \quad K_3 = s_2s_3s_6 \quad F_1 + s_3s_6(s_4 + s_5 - t_1) \quad F_2 + s_3(s_4 + s_5 - t_1)(s_1 + s_3 - s_5 - t_3) \quad F_6 \]

\[ + s_3[-s_1s_2 - (s_4 + s_5 - t_1)(s_3 - s_5 + s_6 + t_1 - t_3) + s_2(s_5 + t_3)] \quad F_3 \]

\[ + s_3(s_4 + s_5 - t_1)(s_3 - s_5 + s_6 + t_1 - t_3) \quad F_4 + s_3(s_4 + s_5 - t_1)(-s_1 + s_3 - s_5 + t_1) \quad F_5. \]

(3.33)

The remaining five elements of the $K$-basis are expressed in terms of \(\{F_1, F_2, F_3, F_4, F_5, F_6\}\) in Appendix B. After some tedious algebra one finds that the new expression (3.29) does indeed agree with the results of \([13,14]\). The use of $K$-basis imposed by SUSY relations leads to a much simpler expression, as a linear combination of “primitive” integrals (3.24) weighted by single-term twistor-like coefficients.

3.3. Seven-point amplitudes

Finally, we consider the case of $N = 7$. We evaluate the following correlator:

\[ \langle V^{(0)}_{\phi^{a_1} \phi^{a_2} \cdots} (z_1) V^{(0)}_{\phi^{a_3} \cdots} (z_2) V^{(0)}_{\phi^{a_4} \cdots} (z_3) V^{(0)}_{\phi^{a_5} \cdots} (z_4) V^{(-1)}_{\phi^{a_6} \cdots} (z_5) V^{(-1)}_{\phi^{a_7} \cdots} (z_6) V^{(0)}_{\phi^{a_8} \cdots} (z_7) \rangle . \] (3.34)

Here again, we make the choice (3.5) and use the parameterization

\[ z_4 = x^{-1}, \quad z_5 = (x y)^{-1}, \quad z_6 = (x y z)^{-1}, \quad z_7 = (x y z w)^{-1} , \] (3.35)

with the corresponding Jacobian $\det(\partial (x_4, x_5, x_6, x_7)/\partial (x, y, z, w)) = x^{-5}y^{-4}z^{-3}w^{-2}$. Even after using momentum conservation to eliminate the scalar products $(\xi_5 k_6), (\xi_6 k_7)$ and $(\xi_7 k_5)$,

\[ \xi_5 k_6 = -\xi_5 k_1 - \xi_5 k_2 - \xi_5 k_3 - \xi_5 k_4 - \xi_5 k_7 , \]

\[ \xi_6 k_7 = -\xi_6 k_1 - \xi_6 k_2 - \xi_6 k_3 - \xi_6 k_4 - \xi_6 k_5 , \]

\[ \xi_7 k_5 = -\xi_7 k_1 - \xi_7 k_2 - \xi_7 k_3 - \xi_7 k_4 - \xi_7 k_6 , \] (3.36)

18
the amplitude (3.34) still involves many kinematic factors, each of them multiplied by a certain Euler integral or multiple hypergeometric function. However, all integrals can be expressed in terms of the 24-element basis \( \{ K_1, \ldots, K_{24} \} \) written in Appendix C. As we will see below, similarly to the six-gluon case, this basis appears naturally in the MHV part of the amplitude (3.34).

At this point, we specify the correlator (3.34) to the case of three positive–helicity gluons, as in the amplitude entering the SUSY relation (2.42). We choose \( k_4 \) as the reference vector \( r \) for all polarization vectors, so that \( \xi_i k_4 = 0 \), see Eq. (2.27). Then:

\[
\xi^+_i \xi^+_j = 0 \quad , \quad \xi^+_i k_j = -\frac{1}{\sqrt{2}} \frac{\langle 4 j \rangle \langle j i \rangle}{\langle 4 i \rangle} .
\]

We also define:

\[
\tau(a,b,c) = -2\sqrt{2} (\xi_5 k_a) (\xi_6 k_b) (\xi_7 k_c) = \frac{\langle 4 a \rangle \langle 4 b \rangle \langle 4 c \rangle \langle a 5 \rangle \langle b 6 \rangle \langle c 7 \rangle}{\langle 4 5 \rangle \langle 4 6 \rangle \langle 4 7 \rangle} .
\]

For \( \xi_i k_4 = 0 \), the correlator (3.34) gives rise to the partial amplitude

\[
A(\phi^{a_1,-}, \phi^{a_2,-}, \phi^{a_3,+}, \phi^{a_4,+}, A^{a_5}, A^{a_6}, A^{a_7}) = 2(2\alpha')^3 g^5_Y M 
\times \text{Tr}(T^{a_1} T^{a_2} T^{a_3} T^{a_4} T^{a_5} T^{a_6} T^{a_7}) \sum_{I=1}^{24} K_I K_I ,
\]

with 24 kinematics \( K_I \) and integrals \( K_I \) to be specified below. After using the SUSY relation (2.42) and factorizing out \((\sqrt{2} g_Y M)^5 \text{Tr}(T^{a_1} \cdots T^{a_7})\), the correlator (3.34) yields the seven–gluon MHV amplitude

\[
A(1^-, 2^-, 3^+, 4^+, 5^+, 6^+, 7^+) = \alpha'^3 \frac{(12)^2}{(34)^2} \sum_{I=1}^{24} K_I K_I ,
\]

\[\text{3 After applying (3.36) the amplitude (3.34) involves 111 different kinematical factors. Each of them is multiplied by a certain integral or hypergeometric function. However, some of those kinematical factors appear with the same function. In fact, only 73 different integrals or functions appear in (3.34). These 73 functions may be expressed in terms of a basis, whose dimension is 24. As in the five or six–point case this basis is completely specified by the MHV part of the amplitude (3.34).}\]
where the kinematic factors $\mathcal{K}_I$ are as follows:

$$\mathcal{K}_1 = \tau(1,2,1) + \tau(7,2,1), \quad \mathcal{K}_2 = \tau(1,3,1) + \tau(7,3,1), \quad \mathcal{K}_3 = \tau(2,2,1) + \tau(2,5,1),$$

$$\mathcal{K}_4 = \tau(3,3,1) + \tau(3,5,1), \quad \mathcal{K}_5 = \tau(2,1,2) + \tau(7,1,2), \quad \mathcal{K}_6 = \tau(2,3,2) + \tau(7,3,2),$$

$$\mathcal{K}_7 = \tau(1,1,2) + \tau(1,5,2), \quad \mathcal{K}_8 = \tau(3,3,2) + \tau(3,5,2), \quad \mathcal{K}_9 = \tau(3,1,3) + \tau(7,1,3),$$

$$\mathcal{K}_{10} = \tau(3,2,3) + \tau(7,2,3), \quad \mathcal{K}_{11} = \tau(1,1,3) + \tau(1,5,3), \quad \mathcal{K}_{12} = \tau(2,2,3) + \tau(2,5,3),$$

$$\mathcal{K}_{13} = \tau(2,1,1) + \tau(2,1,6), \quad \mathcal{K}_{14} = \tau(3,1,1) + \tau(3,1,6), \quad \mathcal{K}_{15} = \tau(1,2,2) + \tau(1,2,6),$$

$$\mathcal{K}_{16} = \tau(3,2,2) + \tau(3,2,6), \quad \mathcal{K}_{17} = \tau(1,3,3) + \tau(1,3,6), \quad \mathcal{K}_{18} = \tau(2,3,3) + \tau(2,3,6),$$

$$\mathcal{K}_{19} = \tau(3,2,1), \quad \mathcal{K}_{20} = \tau(2,3,1), \quad \mathcal{K}_{21} = \tau(3,1,2),$$

$$\mathcal{K}_{22} = \tau(1,3,2), \quad \mathcal{K}_{23} = \tau(2,1,3), \quad \mathcal{K}_{24} = \tau(1,2,3).$$

(3.41)

The 24 basis functions $K_i$ are given as Euler integrals, e.g.:

$$K_1 = -s_{34} \int_0^1 dx \int_0^1 dy \int_0^1 dz \int_0^1 dw \frac{I(x,y,z,w)}{xyw(1-z)(1-wz)},$$

(3.42)

with

$$I(x,y,z,w) = x^{s_2} y^{t_2} z^{t_6} w^{s_7}(1-x)^{s_3}(1-y)^{s_4}(1-z)^{s_5}(1-w)^{s_6} \times (1-xy)^{t_3-s_3-s_4}(1-yz)^{t_4-s_4-s_5}(1-xz)^{t_5-s_5-s_6}(1-xyz)^{t_6}. $$

(3.43)

$s_i = \alpha'(k_i + k_{i+1})^2$ and $t_j = \alpha'(k_j + k_{j+1} + k_{j+2})^2$ subject to the cyclic identification $i + 7 \equiv i$. All 24 integrals are displayed in Appendix C. The 18 functions $K_1 \ldots K_{18}$ seem like straightforward generalizations of the six-point functions (3.24). Indeed, they are related to them in certain soft-boson limits: for example, in the $k_7 \to 0$ limit, $K_1 \to -\frac{1}{s_7}K_3$ of $N = 6$. On the other hand, the remaining functions $K_{19}, \ldots, K_{24}$ seem to have a different character.

3.4. From partial to full amplitudes

So far, we focussed only on one partial amplitude, associated to the color factor $\text{Tr}(T^{a_1} \cdots T^{a_N})$. In Ref. [14], we gave a prescription for constructing any partial amplitude $A_\sigma(1^-, 2^-, 3^+, 4^+, \ldots, N^+)$, associated the color factor $\text{Tr}(T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(N)}})$ with the indices $a_i$ permuted by an arbitrary permutation $\sigma$. One simply factorizes out $\langle 1 2 \rangle^4$ and applies $\sigma$ to all momenta inside the remainder:

$$A_\sigma(1^-, 2^-, 3^+, 4^+, \ldots, N^+) = \langle 1 2 \rangle^4 \times \left( \frac{A(1^-, 2^-, 3^+, 4^+, \ldots, N^+)}{\langle 1 2 \rangle^4} \bigg|_{k_i \rightarrow k_{\sigma(i)}} \right).$$

(3.44)

For an amplitude written as in [14], as the product of the zero-slope MHV amplitude $\mathcal{M}_{YM}^{(N)}(\mathbb{L}2)$ times a string “formfactor”, this is a completely trivial operation, however with the non-factorized form of amplitudes presented in this section, it requires more care.
4. Four–point string amplitudes and supersymmetry relations

In the previous section we have seen, that string amplitudes may be written much simpler by making use of the SUSY Ward identities (2.22). In this Section we demonstrate this for the four–point string amplitude involving scalars, gauginos and vectors from the $\mathcal{N} = 4$ vector multiplet. After computing the latter we apply the results of Section 2.3 to generate relations among them.

4.1. Four–point disk scattering of scalars, gauginos and gluons

Here we compute in $D = 4$ the tree–level four–point string amplitude involving scalars, gauginos and vectors from the $\mathcal{N} = 4$ vector multiplet. The world–sheet of the string $S$–matrix is described by a disk with all external states $\Phi^a$ created through vertex operators $V_{\Phi^a}$ at the boundary of the disk. The color–ordered part of the amplitude of interest takes the form:

$$A(\Phi^{a_1}, \Phi^{a_2}, \Phi^{a_3}, \Phi^{a_4}) = V_{CKG}^{-1} \int_{z_1 < \ldots < z_4} \prod_{k=1}^{4} dz_k \langle V_{\Phi^{a_1}}(z_1) \ V_{\Phi^{a_2}}(z_2) \ V_{\Phi^{a_3}}(z_3) \ V_{\Phi^{a_4}}(z_4) \rangle.$$  \hspace{1cm} (4.1)

The vertex operators for the gauginos are given in (2.13), while the vertex operators for the scalars and vectors are given in (3.2). The four–point correlator in the integrand of (4.1) is evaluated by performing all possible Wick contractions. Some of the relevant correlators are

$$\langle S_\alpha(z_1) S_\beta(z_2) \rangle = \epsilon_{\alpha\beta} \ z_{12}^{-1/2} \ , \ \langle S_\alpha(z_1) S_\beta(z_2) \rangle = 0 \ ,$$

$$\langle S_\alpha(z_1) S_\beta(z_2) \psi^\mu(z_3) \rangle = \frac{1}{\sqrt{2}} \ \sigma^\mu_{\alpha\beta} \ z_{13}^{-1/2} \ z_{23}^{-1/2} \ ,$$

$$\langle S_\alpha(z_1) S_\beta(z_2) \psi^\mu(z_3) \rangle = 0 \ ,$$

and:

$$\langle e^{-\frac{i}{2} \phi(z_1)} e^{-\frac{i}{2} \phi(z_2)} e^{-\frac{i}{2} \phi(z_3)} e^{-\frac{i}{2} \phi(z_4)} \rangle = (z_{12} \ z_{13} \ z_{14} \ z_{23} \ z_{24} \ z_{34})^{-1/4}.$$ \hspace{1cm} (4.3)

Again, $PSL(2, \mathbb{R})$ invariance on the disk allows to fix three vertex positions according (3.3), which implies the ghost factor $\langle c(z_1) c(z_2) c(z_3) \rangle = -z_{\infty}^2$. The remaining vertex position $z_4 := 1/x$ takes arbitrary values inside the integration domain $0 < x < 1$ respecting the integration region $z_1 < \ldots < z_4$ in (4.1).

Four gauginos:

For the four–gaugino amplitude involving four gauginos of the same chirality in (4.1) we compute the correlator

$$\langle V_{\lambda^{a_1}, i}^{(-1/2)}(z_1, u_1, k_1) \ V_{\lambda^{a_2}, j}^{(-1/2)}(z_2, u_2, k_2) \ V_{\lambda^{a_3}, k}^{(-1/2)}(z_3, u_3, k_3) \ V_{\lambda^{a_4}, L}^{(-1/2)}(z_4, u_4, k_4) \rangle.$$ \hspace{1cm} (4.4)
with the vertices (2.13). We need the following space–time fermion correlator:

\[
\langle S_\alpha(z_1)S_\beta(z_2)S_\gamma(z_3)S_\delta(z_4) \rangle = \frac{\epsilon_{\alpha\beta} \epsilon_{\gamma\delta} (z_{13} z_{14} - \epsilon_{\alpha\gamma} \epsilon_{\beta\delta} z_{12} z_{14} + \epsilon_{\alpha\delta} \epsilon_{\beta\gamma} z_{12} z_{13})}{(z_{12} z_{13} z_{14} z_{23} z_{24} z_{34})^{1/2}}.
\]

(4.5)

In the last step we have applied the Fierz identity \( \epsilon_{\alpha\gamma} \epsilon_{\beta\delta} = \epsilon_{\alpha\beta} \epsilon_{\gamma\delta} + \epsilon_{\alpha\delta} \epsilon_{\beta\gamma} \). Furthermore, the correlator for the internal Ramond fields \( \Sigma \) is:

\[
\langle \Sigma^I(z_1) \Sigma^J(z_2) \Sigma^K(z_3) \Sigma^L(z_4) \rangle = (z_{12} z_{13} z_{14} z_{23} z_{24} z_{34})^{-1/4}, \quad I \neq J \neq K \neq L.
\]

(4.6)

With the choice (3.5), the correlators (4.3), (4.9) and (4.10) we arrive at the partial amplitude of four gauginos:

\[
A(\lambda^I \lambda^J \lambda^K \lambda^L) = (2\alpha')g_{YM}^2 \frac{\Gamma(s) \Gamma(u)}{\Gamma(1 + s + u)} \left[ u \left( u_1 u_2 \right) (u_3 u_4) - s \left( u_1 u_4 \right) (u_2 u_3) \right],
\]

(4.7)

with the Mandelstam invariants \( s = 2\alpha' k_1 k_2 \), \( t = 2\alpha' k_1 k_3 \) and \( u = 2\alpha' k_1 k_4 \). On the other hand, for the string amplitude with two gauginos of opposite chirality

\[
\langle V_{\alpha_1, t}(-1/2)(z_1, u_1, k_1) V_{\alpha_2, j}(-1/2)(z_2, u_2, k_2) V_{\lambda_3, K}(-1/2)(z_3, \bar{u}_3, k_3) V_{\lambda_4, L}(-1/2)(z_4, \bar{u}_4, k_4) \rangle,
\]

(4.8)

we need the correlator

\[
\langle S_\alpha(z_1)S_\beta(z_2)S_\gamma(z_3)S_\delta(z_4) \rangle = (z_{12} z_{34})^{-1/2} \epsilon_{\alpha\beta} \epsilon_{\gamma\delta}.
\]

(4.9)

which may be derived by using the identity \( \sigma_{\alpha\beta}^\mu \delta^\gamma_\mu = -2 \delta^\gamma_\alpha \delta_\beta^\delta \) and the correlator:

\[
\langle \Sigma^I(z_1) \Sigma^J(z_2) \Sigma^K(z_3) \Sigma^L(z_4) \rangle = \left( \frac{z_{13} z_{14} z_{23} z_{24}}{z_{12} z_{34}} \right)^{1/4} \left( -\frac{\delta^{IK} \delta^{JL}}{z_{13} z_{24}} + \frac{\delta^{IL} \delta^{JK}}{z_{14} z_{23}} \right).
\]

(4.10)

With the choice (3.3), the correlators (1.3), (1.9) and (4.10) the partial amplitude becomes:

\[
A(\lambda^I \lambda^J \lambda^K \lambda^L) = - (2\alpha')g_{YM}^2 \left( u_1 u_2 \right) (\bar{u}_3 \bar{u}_4) \frac{\Gamma(s) \Gamma(u)}{\Gamma(1 + s + u)} \left( u \delta^{IK} \delta^{JL} + t \delta^{IL} \delta^{JK} \right).
\]

(4.11)

**Two gauginos and two scalars:**

Now we compute the amplitude of two gauginos and two scalars by calculating in (1.1) the correlator

\[
\langle V_{\alpha_1, t}(-1/2)(z_1, u_1, k_1) V_{\alpha_2, j}(-1/2)(z_2, k_2) V_{\lambda_3, K}(-1/2)(z_3, \bar{u}_3, k_3) V_{\lambda_4, L}(0)(z_4, k_4) \rangle.
\]

(4.12)
Here the index \( j \) includes the two cases \( j^+ \) and \( j^- \). After deriving the set of correlators

\[
\langle \Sigma^I (z_1) \Sigma^J (z_3) \Psi^{-j}(z_2) \Psi^{j+}(z_4) \rangle = \frac{1}{z_24 \ z_13^{3/4}} \left( \frac{z_{14} z_{32}}{z_{12} z_{34}} \right)^{1/2},
\]

\( (a) : \quad (j, I) \in \{(1,3), (1,4), (2,2), (2,4), (3,2), (3,3)\} \),

\[
\langle \Sigma^I (z_1) \Sigma^J (z_3) \Psi^{-j}(z_2) \Psi^{j+}(z_4) \rangle = \frac{1}{z_24 \ z_13^{3/4}} \left( \frac{z_{12} z_{34}}{z_{14} z_{32}} \right)^{1/2},
\]

\( (b) : \quad (j, I) \in \{(1,1), (1,2), (2,1), (2,3), (3,1), (3,4)\} \),

\[
\langle \Sigma^I (z_1) \Sigma^K (z_3) \Psi^{-j}(z_2) \Psi^{I+}(z_4) \rangle = z_{13}^{1/4} \left( z_{12} z_{14} z_{23} z_{34} \right)^{-1/2},
\]

\( (c) : \quad (j-, l, I, K) \in \{(1,2,3,2), (1,3,4,2), (2,3,4,3), (2,1,2,3), (3,1,2,4), (3,2,3,4)\} \),

\( (j+, l, I, K) \in \{(1,2,1,4), (1,3,1,3), (2,3,1,2), (2,1,1,4), (3,1,1,3), (3,2,1,2)\} \), \( (4.13) \)

with (4.12) the resulting partial amplitude becomes

\[
A(\lambda^I \phi^{j+} \lambda^K \phi^{I+}) = (2\alpha^i) g_Y^2 \frac{\Gamma(s)}{\Gamma(1+s+u)} \frac{\Gamma(u)}{\Gamma(1+s+u)} \frac{k_{4\mu}}{(u_1 \sigma^\mu_{\alpha \beta} \frac{\delta^\beta}{\delta^\mu} \Sigma_{3})} \times \left\{ \begin{array}{ll}
u, & \text{case (a)}, \\ -s, & \text{case (b)}, \\ t, & \text{case (c)}, \end{array} \right. \]

(4.14)

for the three cases in (4.13).

**Two gauginos, one vector and one scalar:**

Next, we compute the four–point amplitude of two gauginos, one vector and one scalar. In (4.11) we compute the correlator:

\[
\langle V^{(-1/2)}_{\lambda_1, i} (z_1, u_1, k_1) V^{(-1/2)}_{\lambda_2, j} (z_2, u_2, k_2) V^{(1)}_{\lambda_3} (z_3, \xi_3, k_3) V^{(0)}_{\phi_4, l^+} (z_4, k_4) \rangle. \]

(4.15)

For the space–time fermions we need the correlator:

\[
\langle S_\alpha (z_1) S_\beta (z_2) \psi^\mu (z_3) \psi'^\mu (z_4) \rangle = z_{34}^{-1} \left( z_{12} z_{13} z_{14} z_{23} z_{24} \right)^{-1/2} 
\times \left( \epsilon_{\alpha \beta}^{\delta, \mu \nu} z_{13} z_{14} + \frac{1}{2} \sigma_{\alpha \gamma}^{\mu \nu} \epsilon_{\gamma \delta}^{\mu \nu} z_{12} z_{14} + \frac{1}{2} \sigma_{\alpha \gamma}^{\mu \nu} \epsilon_{\gamma \delta}^{\mu \nu} z_{12} z_{13} \right) 
\frac{\sigma_{\mu \nu}^{\mu \nu} z_{12}^{1/2}}{z_{13} z_{14} z_{23} z_{24}^{1/2}}. \]

(4.16)

The last step follows from the relations \( \sigma_{\mu \nu}^{\mu \nu} = \epsilon_{\alpha \beta}^{\mu \nu} \epsilon_{\beta \gamma}^{\mu \nu} \sigma_{\gamma \delta}^{\mu \nu} \) and \( \sigma_{\beta \delta}^{\mu \nu} \sigma_{\gamma \delta}^{\mu \nu} = -2 \delta_{\mu \nu} \delta_{\gamma \delta} \). The necessary correlator for the internal fields is:

\[
\langle \Sigma^I (z_1) \Sigma^J (z_2) \Psi^{I+} (z_4) \rangle = z_{12}^{-1/4} \left( z_{14} z_{24} \right)^{-1/2}, \quad (I, J, l^+) \in \mathcal{I}, \quad (4.17)
\]

23
with $I = \{(1, 2, 1), (1, 3, 2), (1, 4, 3), (2, 1, 1), (3, 1, 2), (4, 1, 3)\}$. With (4.16) and (4.17) the partial amplitude of (4.1) becomes

$$A(\lambda^I \lambda^J A^{(I+}) = \sqrt{2} \alpha' g_{YM}^2 \frac{\Gamma(s) \Gamma(u)}{\Gamma(1 + s + u)} \left( u - t \right) (u_1 u_2)(\xi_3 k_4) - s \left( u_1^\alpha \sigma_{\alpha\beta} u_2^\beta \right) \xi_3 \mu_4 k_{4\nu} ,$$

(4.18)

with $(I, J, l+) \in I$.

**Four scalars**

Finally we compute the string $S$–matrix of four scalars. With the vertex operators of (2.12) in (4.1) we compute the correlator:

$$\langle V^{(0)}_{\phi^a_{i-}}(z_1, k_1) V^{(0)}_{\phi^b_{j-}}(z_2, k_2) V^{(-1)}_{\phi^c_{k+}}(z_3, k_3) V^{(-1)}_{\phi^d_{l+}}(z_4, k_4) \rangle .$$

(4.19)

After some algebra and using (4.6) we find the following expression for the partial amplitude:

$$A(\phi^a_{i-} \phi^b_{j-} \phi^c_{k+} \phi^{l+}) = -2 g_{YM}^2 \frac{\Gamma(s) \Gamma(u)}{\Gamma(1 + s + u)} \left( u \delta^{ik} \delta^{jl} + t \delta^{il} \delta^{jk} \right) s .$$

(4.20)

**4.2. Supersymmetry relations of string amplitudes**

By applying the results of Subsection 2.3 in this part we derive relations between the string amplitudes we have computed in the previous Subsection. Inspection of those amplitudes shows that they all have the same prefactor encoding the $\alpha'$–dependence of the amplitude. Indeed here we show that they all related through supersymmetry transformations. With (2.16) we may derive a SUSY relation between a four gaugino amplitude and two amplitudes involving two gauginos and two scalars.

In analogy to (2.20) we start with the contour integral

$$\oint_{C_{\infty}} \frac{dw}{2 \pi i} \eta_M^{\alpha} \langle V^\alpha_M(w) V^{(-1/2)}_{\lambda_{\alpha_1,i}}(z_1, u_1, k_1) V^{(-1/2)}_{\lambda_{\alpha_2,j}}(z_2, u_2, k_2) V^{(-1/2)}_{\lambda_{\alpha_3,k}}(z_3, \bar{u}_3, k_3) V^{(0)}_{\phi^d_{l+}}(z_4, k_4) \rangle ,$$

(4.21)

where $C_{\infty}$ is a closed contour in the complex plane encircling all four vertex positions $z_1 \ldots, z_4$. With the arguments of Subsection 2.3 the integral (4.21) vanishes. On the other hand, by analyticity in (4.21) we may deform the contour to the other four vertex operators with the SUSY operator acting on one vertex operator, respectively (c.f. Eq. (2.19)):

$$[Q_M(\eta_M), V^{(-1/2)}_{\lambda_{\alpha_1,i}}(z_1, u_1, k_1)] = \langle \eta_M u_1 \rangle V^{(-1)}_{\phi^a_{i-}}(z_1, u_1, k_1) ,$$

$$[Q_M(\eta_M), V^{(-1/2)}_{\lambda_{\alpha_2,j}}(z_2, u_2, k_2)] = \langle \eta_M u_2 \rangle V^{(-1)}_{\phi^b_{j-}}(z_2, u_2, k_2) ,$$

$$[Q_M(\eta_M), V^{(-1/2)}_{\lambda_{\alpha_3,k}}(z_3, \bar{u}_3, k_3)] = \frac{1}{\sqrt{2}} \delta^{MK} V^{(-1)}_{A^a_{\alpha_3}}(z_3, \xi_3) ,$$

$$[Q_M(\eta_M), V^{(0)}_{\phi^a_{i+}}(z_4, k_4)] = V^{(-1/2)}_{\lambda_{\alpha_4,l}}(z_4, \bar{u}_4, k_4) ,$$

$$[Q_M(\eta_M), V^{(0)}_{\phi^a_{j+}}(z_4, k_4)] = V^{(-1/2)}_{\lambda_{\alpha_5,l}}(z_4, \bar{u}_4, k_4) ,$$

$$\bar{u}^a_{\lambda_{\alpha_4,l}} = k_{4\rho} \eta_M^\alpha \sigma^\rho_{\alpha\beta} ,$$

(4.22)
Inserting these results into $(2.22)$ gives a sum of the following four correlators
\[
\langle \eta_M u_1 \rangle \langle V_{\phi, a, l}^{(-1)} (z_1, u_1, k_1) V_{\phi, a, l}^{(-1/2)} (z_2, u_2, k_2) V_{\phi, a, l}^{(-1/2)} (z_3, \overline{u}_3, k_3) V_{\phi, a, l}^{(0)} (z_4, k_4) \rangle
\]
\[
= -2\alpha' g_Y^2 \frac{\Gamma(s) \Gamma(u)}{\Gamma(1+s+u)} k_{4\mu} \langle \eta_M u_1 \rangle (u_2^\alpha \sigma_{\alpha \beta}^\mu \overline{u}_3^\beta) \left( u \delta^{IK} \delta^{JL} + t \delta^{IL} \delta^{JK} \right),
\]
where we have used $(4.14)$. Similarly, the second correlator becomes:
\[
\langle \eta_M u_2 \rangle \langle V_{\phi, a, l}^{(-1/2)} (z_1, u_1, k_1) V_{\phi, a, l}^{(-1/2)} (z_2, u_2, k_2) V_{\phi, a, l}^{(-1/2)} (z_3, \overline{u}_3, k_3) V_{\phi, a, l}^{(0)} (z_4, k_4) \rangle
\]
\[
= 2\alpha' g_Y^2 \frac{\Gamma(s) \Gamma(u)}{\Gamma(1+s+u)} k_{4\mu} \langle \eta_M u_2 \rangle (u_1^\alpha \sigma_{\alpha \beta}^\mu \overline{u}_3^\beta) \left( u \delta^{IK} \delta^{JL} + t \delta^{IL} \delta^{JK} \right).
\]
With $(4.18)$ the third correlator gives:
\[
\frac{1}{\sqrt{2}} \delta^{MK} \langle V_{\phi, a, l}^{(-1/2)} (z_1, u_1, k_1) V_{\phi, a, l}^{(-1/2)} (z_2, u_2, k_2) V_{\phi, a, l}^{(-1)} (z_3, \overline{u}_3, k_3) V_{\phi, a, l}^{(0)} (z_4, k_4) \rangle
\]
\[
= \alpha' g_Y^2 \frac{\Gamma(s) \Gamma(u)}{\Gamma(1+s+u)} (u \delta^{IK} \delta^{JL} + s(u_1^\alpha \sigma_{\alpha \beta}^\mu u_2^\beta) \xi_3 k_4^\nu) \delta^{MK}, \quad \xi_3 = \eta_M^\alpha \sigma_{\alpha \beta}^\mu \overline{u}_3^\beta,
\]
\[
= 2\alpha' g_Y^2 \frac{\Gamma(s) \Gamma(u)}{\Gamma(1+s+u)} k_{4\mu} \left( u_1 u_2 (u_1^\alpha \sigma_{\alpha \beta}^\mu \overline{u}_3^\beta) - s \left( \eta_M u_2 \right) (u_1^\alpha \sigma_{\alpha \beta}^\mu \overline{u}_3^\beta) \right) \delta^{MK}, \quad (I, J, l+) \in \mathcal{I}.
\]
Finally, with $(4.11)$ the last correlator becomes:
\[
\langle V_{\phi, a, l}^{(-1/2)} (z_1, u_1, k_1) V_{\phi, a, l}^{(-1/2)} (z_2, u_2, k_2) V_{\phi, a, l}^{(-1/2)} (z_3, \overline{u}_3, k_3) V_{\phi, a, l}^{(-1/2)} (z_4, \overline{u}_4, k_4) \rangle
\]
\[
= -2\alpha' g_Y^2 \frac{\Gamma(s) \Gamma(u)}{\Gamma(1+s+u)} (u_1 u_2 [u_3 u_4] (u \delta^{IK} \delta^{JL} + t \delta^{IL} \delta^{JK}) \overline{u}_4^\beta = k_{4\rho} \eta_M^\alpha \sigma_{\alpha \beta}^\rho \overline{u}_3^\beta.
\]
For $M \neq K$ the third correlator $(4.25)$ vanishes. In that case summing up $(4.23)$, $(4.24)$ and $(4.26)$ gives
\[
2\alpha' g_Y^2 \frac{\Gamma(s) \Gamma(u)}{\Gamma(1+s+u)} (u \delta^{IK} \delta^{JL} + t \delta^{IL} \delta^{JK})
\]
\[
\times \left[ -\langle \eta_M u_1 \rangle (u_2^\alpha \sigma_{\alpha \beta}^\mu \overline{u}_3^\beta) + \langle \eta_M u_2 \rangle (u_1^\alpha \sigma_{\alpha \beta}^\mu \overline{u}_3^\beta) - \langle u_1 u_2 \rangle (\eta_M^\alpha \sigma_{\alpha \beta}^\mu \overline{u}_3^\beta) \right],
\]
which vanishes due to the Fierz relation:
\[
\langle \eta_M u_2 \rangle (u_1^\alpha \sigma_{\alpha \beta}^\mu \overline{u}_3^\beta) = \langle \eta_M u_1 \rangle (u_2^\alpha \sigma_{\alpha \beta}^\mu \overline{u}_3^\beta) + \langle \eta_M^\alpha \sigma_{\alpha \beta}^\mu \overline{u}_3^\beta \rangle \langle u_1 u_2 \rangle.
\]
To conclude, to all orders in $\alpha'$ we have proven the following SUSY identity
\[
\langle \eta_M u_1 \rangle A(\phi^I \lambda^I \overline{\lambda} \phi^I) \langle \eta_M u_2 \rangle A(\lambda^I \phi^I \overline{\lambda} \phi^I) + A(\lambda^I \phi^I \overline{\lambda} \phi^I) + A(\lambda^I \lambda^I \overline{\lambda} \lambda^I) = 0 \frac{1}{k_{4\beta} = k_{4\rho} \eta_M^\alpha \sigma_{\alpha \beta}^\rho},
\]
(4.28)
for an arbitrary spinor $\eta_M$ and $M \neq K$. Table 2 shows the combination of indices, for which all the three correlators (4.23), (4.24) and (4.26) are non–vanishing, but (4.24) vanishing due to $M \neq K$.

| $M$ | $(I, J, K, L, i, j, l)$ | $M$ | $(I, J, K, L, i, j, l)$ | $M$ | $(I, J, K, L, i, j, l)$ |
|-----|-------------------------|-----|-------------------------|-----|-------------------------|
| 2   | $(2, 2, 2, 1, -1, -1, 1)$ | 2   | $(3, 4, 3, 4, 2, -3, -3, 3)$ | 2   | $(1, 3, 3, 1, -1, 3, +, 1)$ |
| 2   | $(2, 3, 3, 2, -1, 2, -1, 1)$ | 2   | $(4, 3, 3, 4, 2, -3, -3, 3)$ | 2   | $(1, 4, 4, 1, 1, 2, 1, 1, 1)$ |
| 2   | $(2, 3, 2, 3, 1, -2, 2, 2)$ | 2   | $(4, 4, 4, 3, 3, -3, -3, 3)$ | 2   | $(3, 1, 3, 1, 1, 2, 2, 1, 1)$ |
| 2   | $(2, 4, 4, 2, 1, -3, -1, 1)$ | 2   | $(4, 2, 4, 2, 3, -1, 1, 1)$ | 2   | $(4, 1, 4, 1, 2, 1, 1, 1)$ |
| 2   | $(4, 2, 4, 1, -3, -3, 3)$ | 1   | $(4, 2, 4, 3, 1, -1, 3, 3)$ | 2   | $(1, 2, 2, 1, 2, 3, 3, 2)$ |
| 2   | $(3, 2, 3, 2, -1, 1, 1)$ | 1   | $(4, 3, 4, 3, -2, -2, 2)$ | 2   | $(1, 4, 4, 1, 1, 2, 1, 1)$ |
| 2   | $(3, 3, 3, 3, 2, -2, 2, 2)$ | 3   | $(2, 1, 1, 1, 1, -1, 1, +1)$ | 2   | $(2, 1, 2, 1, -2, 2, 1)$ |
| 3   | $(1, 1, 1, 1, -2, -2, 2)$ | 3   | $(1, 1, 1, 1, 2, -2, 2)$ | 3   | $(4, 1, 4, 1, 1, 1, 2, 1)$ |
| 3   | $(3, 2, 3, 2, 1, -2)$ | 4   | $(1, 1, 1, 1, 2, -2, 2)$ | 3   | $(4, 1, 4, 1, 2, 1, 1)$ |
| 3   | $(3, 2, 3, 2, -1, 1, 2)$ | 4   | $(1, 1, 1, 1, 2, -2, 2)$ | 4   | $(1, 3, 3, 1, -1, +1, 3)$ |
| 3   | $(3, 4, 4, 3, 2, -3, -3, 2)$ | 4   | $(1, 1, 1, 1, 1, -1, 1, 1)$ | 4   | $(2, 1, 2, 1, +3, -3, 2)$ |
| 4   | $(1, 1, 1, 1, 3, -3, 3)$ | 4   | $(1, 1, 1, 1, 1, 3, -3, 3)$ | 4   | $(3, 1, 3, 1, 1, 3, -3, 3)$ |

Table 2: Index structure for the SUSY relation (4.28)

On the other hand, in the case $M = K$ the third correlator (4.25) contributes in (2.22). However, in that case one of the three correlators (4.23), (4.24) or (4.26) vanishes because the corresponding SUSY transformation (4.22) gives a vanishing contribution. In either case the three non–vanishing residua sum up to zero due to (4.27). E.g. for $M = K = 2, I = 1, J = 2$ and $l = 1$ the SUSY transformations (1.22) yield $i = 1, L = 1$ but vanishing SUSY transformation for the variation of $V_{\lambda_2, J}$. Hence the total contribution of residual to (4.21) is:

$$2\alpha' g_{YM}^2 \frac{\Gamma(s) \Gamma(u)}{\Gamma(1 + s + u)} k_{4\mu} \left[ -s \langle \eta_M u_1 \rangle (u^\alpha_2 \sigma^\mu_{\alpha\beta} \bar{u}^\beta_3) - t \langle u_1 u_2 \rangle (\eta_M^\alpha \sigma^\mu_{\alpha\beta} \bar{u}^\beta_3) \right. \n\left. - u \langle u_1 u_2 \rangle (\eta_M^\alpha \sigma^\mu_{\alpha\beta} \bar{u}^\beta_3) + s \langle \eta_M u_2 \rangle (u^\alpha_1 \sigma^\mu_{\alpha\beta} \bar{u}^\beta_3) \right] ,$$

which vanishes according to (4.27). Eventually (2.22) gives rise to the following identity:

$$\langle \eta_{12} u_1 \rangle A(\phi^{1+1} \lambda^{2} \bar{\lambda}^{2} \phi^{1+1}) + \frac{1}{\sqrt{2}} A(\lambda^{1} \lambda^{2} A \phi^{1+1}) + A(\lambda^{1} \lambda^{2} \bar{\lambda}^{2} \lambda^{1}) = 0 \quad (4.29)$$

5. Concluding Remarks

The main result of this paper is the extension of the well-known SUSY Ward identities relating the scattering amplitudes of particles with different spin but belonging to the same supermultiplet, to type I open superstring theory at the disk level. For arbitrary
compactifications, the form of such relations remains exactly the same as in the low-energy
effective field theory describing the $\alpha' \to 0$ limit. Although we studied explicitly only the
case of gauge supermultiplets it is clear that our results extend to “matter” supermultiplets.

Here, we focussed on one particular application of SUSY relations: to the computa-
tions of $N$-gluon MHV amplitudes. In this case, four gluons can be replaced by scalars,
leading to significant simplifications. By using this method, we were able to reproduce all
known results for $N \leq 6$ and to derive a compact expression for the seven-gluon MHV
amplitude. It is very interesting that the use of supersymmetry dictates certain choice
of $(N-3)!$ elements of the basis of boundary integrals over the vertex positions, such
that MHV amplitudes can be represented as linear combinations of these basis functions
weighted by very simple twistor-like coefficients. It seems that such a representation is
more natural than the factorized form discussed previously in [13,14]. We believe that it
is also more suitable for studying the recursive structure for arbitrary $N$.

There are several extensions and applications of our results that deserve further stud-
ies. In particular, it would be interesting to see if SUSY relations lead to a novel rep-
resentation of superstring amplitudes also for non-MHV configurations (that appear for
$N \geq 6$) [16]. Furthermore, a generalization of SUSY relations to world-sheets with string
loops would be useful for understanding how supersymmetry is realized at the level of
quantum-corrected, or perhaps even exact, scattering amplitudes.

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Appendix A. Operator product expansions and extended supersymmetry

In this appendix we present the $N$ holomorphic spacetime supersymmetry currents and their operator product expansions. We adapt to the notation and spinor algebra of the book of Wess and Bagger. In particular, spinor indices are raised and lowered with the anti-symmetric tensors $\epsilon_{\alpha\beta}$ and $\hat{\epsilon}^{\dot{\alpha}\dot{\beta}}$. Besides spinor products are defined to be $\chi\eta = \chi^\alpha\epsilon_{\alpha\beta}\eta^\beta \ (\overline{\eta} = \overline{\chi}\hat{\epsilon}^{\dot{\alpha}\dot{\beta}}\overline{\eta}_{\dot{\beta}})$ for some spinors $\chi, \eta \ (\overline{\chi}, \overline{\eta})$. The $D = 4$ supersymmetry currents in the $(-1/2)$-ghost picture have already been given in (2.2). The internal Ramond fields $\Sigma^I$, which belong to an internal superconformal field theory with $c = 9$, have conformal dimensions $3/8$, while the space–time spin fields $S_\alpha$ have conformal dimensions $1/4$ w.r.t. the holomorphic stress tensor $T(z)$. The supersymmetry currents in the $(+1/2)$-ghost picture take the form:

$$V^I_\alpha(z) = i\alpha^{I-1/4} \frac{e^{\frac{1}{2} \phi}}{(2\alpha')^{1/2}} \left( \frac{1}{\sqrt{2}} \sigma^\mu_{\alpha\beta} S^{\dot{\beta}} \partial X_\mu \Sigma^I + S_\alpha \Sigma^I \right) . \quad (A.1)$$

The dimension $11/8$ internal conformal field $\Sigma^I$ appears in the operator product expansion (OPE) of the field $\Sigma^I$ with the internal supercurrent: $\Sigma^I(z) T^\text{int}_F(w) = i(2\alpha')^{-1/2}(z - w)^{-1/2} \Sigma^I(w) + \ldots$. The full supercurrent is $T_F = i(2\alpha')^{-1/2} \partial X_\mu \psi_\mu + T^\text{int}_F$. The supersymmetry currents $V^I_\alpha$ have conformal dimension one. The relevant OPEs for the spin fields are in $D = 4$ [24]:

$$e^{q_1 \phi(z)} e^{q_2 \phi(w)} = (z - w)^{-q_1q_2} e^{(q_1+q_2)\phi(w)} + \ldots ,$$

$$S_\alpha(z) S_\beta(w) = \frac{1}{\sqrt{2}} (z - w)^0 \sigma^\mu_{\alpha\beta} \psi_\mu(w) + \ldots , \quad S^{\dot{\alpha}}(z) S^{\dot{\beta}}(w) = \frac{1}{\sqrt{2}} (z - w)^0 \overline{\sigma^{\dot{\alpha}\dot{\beta}}} \psi_\mu(w) + \ldots ,$$

$$S_\alpha(z) S_\beta(w) = (z - w)^{-1/2} \epsilon_{\alpha\beta} + \ldots , \quad S^{\dot{\alpha}}(z) S^{\dot{\beta}}(w) = -(z - w)^{-1/2} \hat{\epsilon}^{\dot{\alpha}\dot{\beta}} + \ldots ,$$

$$\psi^\mu(z) S_\alpha(w) = \frac{1}{\sqrt{2}} (z - w)^{-1/2} \sigma^\mu_{\alpha\beta} S^{\dot{\beta}}(w) - \frac{1}{\sqrt{2}} (z - w)^{1/2} \psi^\mu(w) \psi_\nu(w) \sigma^\nu_{\alpha\beta} S^{\dot{\beta}}(w) + \ldots ,$$

$$\psi^\mu(z) S^{\dot{\alpha}}(w) = \frac{1}{\sqrt{2}} (z - w)^{-1/2} \overline{\sigma^{\dot{\mu}\dot{\alpha}}} S_\beta(w) - \frac{1}{\sqrt{2}} (z - w)^{1/2} \psi^\mu(w) \psi_\nu(w) \overline{\sigma^{\dot{\nu}\dot{\alpha}}} S_\beta(w) + \ldots . \quad (A.2)$$

Furthermore we need the following two OPEs:

$$S_\alpha(z) \psi^\mu(w) \psi_\nu(w) = \frac{1}{2} (z - w)^{-1} (\sigma^{\mu\nu})_\alpha^\beta S_\beta(z) + \ldots , \quad (A.3)$$

$$S^{\dot{\alpha}}(z) \psi^\mu(w) \psi_\nu(w) = \frac{1}{2} (z - w)^{-1} (\overline{\sigma}^{\mu\nu})_\dot{\alpha}\dot{\beta} S^{\dot{\beta}}(z) + \ldots .$$

It is convenient to represent the spin fields $S_\alpha, S^{\dot{\alpha}}$ as exponentials of two free bosons $H^1, H^2$: $S_\alpha = e^{i\alpha H}, S^{\dot{\alpha}} = e^{i\dot{\alpha} H}$, with $\alpha = (\pm\frac{1}{2}, \pm\frac{1}{2})$ and $\dot{\alpha} = (\pm\frac{1}{2}, \mp\frac{1}{2})$, respectively. 

\footnote{In these bosonized expressions we neglect cocycle factors which are required to obtain $SO(4)$ covariant correlation functions [25].}
Their corresponding OPEs are:
\[ e^{is_1H^i(z)} e^{is_2H^j(w)} = \delta^{ij} (z - w)^{s_1s_2} e^{i(s_1+s_2)H^i(w)} + \mathcal{O}((z - w)^{s_1s_2+1}) \, . \tag{A.4} \]
The OPEs of two supercharges (2.1)
\[
\mathcal{Q}_\alpha^I(z) \overline{\mathcal{Q}}_\beta^J(w) = \frac{\alpha'^{-1/2}}{\sqrt{2(2\pi i)^2}} \oint \oint (z - w)^{-1/4} \frac{i\partial X_\mu}{(2\alpha')^{1/2}} \sigma^{\alpha\beta}_\mu \Sigma^I(z) \Sigma^J(w) + \ldots , 
\tag{A.5}
\]
and the OPEs (2.3) of the internal Ramond fields \( \Sigma^I \) reproduce the spacetime supersymmetry algebra
\[
\{ \mathcal{Q}_\alpha^I, \overline{\mathcal{Q}}_\beta^J \} = \delta^{IJ} \sigma^{\alpha\beta}_\mu P_\mu \, , \quad \{ \mathcal{Q}_\alpha^I, \mathcal{Q}_\beta^J \} = \epsilon_{\alpha\beta} Z^{IJ} \, , \tag{A.6}
\]
with \( P_\mu = i \oint \frac{dw}{2\pi i} \frac{\partial X_\mu}{2\alpha'} \) and the central charges \( Z^{IJ} = \alpha'^{-1/2} \oint \frac{dw}{2\pi i} e^{-\phi(w)} \psi^{IJ}(w) \) (in the \((-1)-\)ghost picture) of the extended supersymmetry algebra. The latter correspond to the compactified fields \( \partial Z^i \).

**Appendix B. Material for six–point function**

In this Appendix we present some additional material for the six–point function (3.23), which has been computed in Subsection 3.2. In addition to the integrals (3.24) the remaining seven integrals \( K_i \) determining the complete seven–point function (3.23) are:

\[
K_7 = \int_0^1 dx \int_0^1 dy \int_0^1 dz \frac{s_{12} (1 - x)}{x(1 - z)(1 - xy)(1 - yz)} \mathcal{I}(x, y, z) ,
\]
\[
K_8 = -\int_0^1 dx \int_0^1 dy \int_0^1 dz \frac{s_{12} (1 - x)}{x(1 - y)(1 - z)(1 - xyz)} \mathcal{I}(x, y, z) ,
\]
\[
K_9 = -\int_0^1 dx \int_0^1 dy \int_0^1 dz \left( 1 - s_{13} + \frac{s_{13}}{x} \right) \frac{\mathcal{I}(x, y, z)}{(1 - z)(1 - yz)} ,
\]
\[
(\text{B.1})
\]
\[
K_{10} = \int_0^1 dx \int_0^1 dy \int_0^1 dz \left( 1 - s_{13} + \frac{s_{13}}{x} \right) \frac{\mathcal{I}(x, y, z)}{(1 - y)(1 - z)} ,
\]
\[
K_{11} = -\int_0^1 dx \int_0^1 dy \int_0^1 dz \left( \frac{1 - s_{23}}{x} + s_{23} \right) \frac{\mathcal{I}(x, y, z)}{xy(1 - z)(1 - yz)} ,
\]
\[
K_{12} = \int_0^1 dx \int_0^1 dy \int_0^1 dz \left( \frac{1 - s_{23}}{x} + s_{23} \right) \frac{\mathcal{I}(x, y, z)}{xyz(1 - y)(1 - z)} ,
\]

29
In (3.33) we have expressed the function
\[
K_{13} = - \int_0^1 dx \int_0^1 dy \int_0^1 dz \left[ (1 - s_{13})(1 - s_{24}) + \frac{(1 - s_{14})(1 - s_{23})}{x^2} + \frac{s_{12}s_{34} - s_{14}s_{23} - s_{13}s_{24}}{x} \right] \frac{T(x, y, z)}{y(1 - z)^2} .
\]

The six–point amplitude (3.28) is completely specified by the basis of six functions \(\{K_1, \ldots, K_6\}\), introduced in (3.24). Therefore the above integrals (B.1) can be expressed in terms of this basis:

\[
(s_4 + s_5 - t_1) t_1 K_7 = (s_1 + s_6 - t_3)(s_3 + s_4 - t_1 - t_3) K_1 + (s_1 + s_4 - t_1 - t_3)(s_4 + s_6 - t_2 - t_3) K_2 + (s_5 + s_6 - t_2) \\
\times (s_1 + s_6 - t_3) K_3 - (s_5 + s_6 - t_2)(s_1 + s_4 - t_1 - t_3) K_4 + (s_6 + s_3 + s_6 - t_2 - t_3) K_5 - s_6(s_3 + s_4 - t_1 - t_3) K_6 ,
\]

\[
s_4 t_1 K_8 = -(s_3 + s_4 - t_3)(s_1 + s_6 - t_3) K_1 - (s_1 + s_4 - t_3)(s_3 + s_6 - t_2 - t_3) K_2 - (s_5 + s_6 - t_2) \\
\times (s_1 + s_6 - t_3) K_3 + (s_5 + s_6 - t_2)(s_1 + s_4 - t_3) K_4 - s_6(s_3 + s_6 - t_2 - t_3) K_5 + s_6(s_3 + s_4 - t_3) K_6 ,
\]

\[
(s_4 + s_5 - t_1) t_1 K_9 = (s_3 + s_4 - t_3)(s_1 + s_6 - t_3) K_1 + (s_1 + s_4 - t_1 - t_3)(s_3 + s_6 + t_1 - t_2 - t_3) K_2 + (s_5 + s_6 - t_2) \\
\times (s_1 + s_6 - t_3) K_3 - (s_5 + s_6 - t_2)(s_1 + s_4 - t_1 - t_3) K_4 + (s_6 + s_3 + s_6 + t_1 - t_2 - t_3) K_5 - s_6(s_3 + s_4 - t_3) K_6 ,
\]

\[
s_4 t_1 K_{10} = -(s_3 + s_4 - t_3)(s_1 + s_6 - t_1 - t_3) K_1 - (s_1 + s_4 - t_1 - t_3)(s_3 + s_6 - t_2 - t_3) K_2 - (s_5 + s_6 - t_2) \\
\times (s_1 + s_6 - t_1 - t_3) K_3 + (s_5 + s_6 - t_2)(s_1 + s_4 - t_1 - t_3) K_4 - s_6(s_3 + s_6 - t_2 - t_3) K_5 + s_6(s_3 + s_4 - t_3) K_6 ,
\]

\[
(s_4 + s_5 - t_1) t_1 K_{11} = (s_3 + s_4 - t_3)(s_1 + s_6 - t_3) K_1 + (s_1 + s_4 - t_1 - t_3)(s_3 + s_6 - t_2 - t_3) K_2 + (s_5 + s_6 - t_1 - t_2) \\
\times (s_1 + s_6 - t_3) K_3 - (s_5 + s_6 - t_1 - t_2)(s_1 + s_4 - t_1 - t_3) K_4 + (s_6 + s_3 + s_6 - t_2 - t_3) K_5 - s_6(s_3 + s_4 - t_3) K_6 ,
\]

\[
s_4 t_1 K_{12} = -(s_3 + s_4 - t_3)(s_1 + s_6 - t_3) K_1 - (s_1 + s_4 - t_1 - t_3)(s_3 + s_6 - t_2 - t_3) K_2 - (s_5 + s_6 - t_2)(s_1 + s_6 - t_3) K_3 \\
+ (s_5 + s_6 - t_2)(s_1 + s_4 - t_1 - t_3) K_4 - (s_6 + t_1)(s_3 + s_6 - t_2 - t_3) K_5 + (s_6 + t_1)(s_3 + s_4 - t_3) K_6 ,
\]

\[
t_1 K_{13} = -(s_3 + s_4 - t_3)(s_1 + s_6 - t_3) K_1 - (s_1 + s_4 - t_1 - t_3)(s_3 + s_6 - t_2 - t_3) K_2 - (s_5 + s_6 - t_2) \\
\times (s_1 + s_6 - t_3) K_3 + (s_5 + s_6 - t_2)(s_1 + s_4 - t_1 - t_3) K_4 - s_6(s_3 + s_6 - t_2 - t_3) K_5 + s_6(s_3 + s_4 - t_3) K_6 .
\]

(B.2)

In (3.33) we have expressed the function \(K_3\) w.r.t. to the basis \(\{F_1, F_2, F_3, F_4, F_5, F_6\}\), introduced in (3.32). Similarly the remaining five basis elements are expressed in terms of the basis \(\{F_1, F_2, F_3, F_4, F_5, F_6\}\):

\[
s_2 s_5 K_1 = s_2 s_6 t_2 \left[F_1 + s_6(-s_1 s_2 + s_2 s_5 - (s_4 + s_5 - t_1)(s_5 - t_2)) \right] F_2 + (s_4 + s_5 - t_1)(s_5 - t_2) (-s_1 - s_3 + s_5 + t_3) F_6 \\
+ \left\{s_2 s_6 t_2 (-s_4 + s_5 - t_1 - t_2)(s_3 - s_5 - t_2) + s_4 s_5 - t_1 (s_3 - s_5 + s_6 - t_1 + t_3)] F_3 \\
- (s_4 + s_5 - t_1)[s_1 s_2 + (s_5 - t_2)(s_3 + s_5 - t_1 + t_3)] F_4 \\
+ \left\{s_2 s_6 t_2 (-s_4 + s_5 - t_1)(s_3 - s_5 + t_1)(s_5 - t_2) + s_4 (s_4 + s_5 - t_1)(s_5 - t_2) + s_2 (s_4 - t_1 - t_3)] F_5 \right\}
\]

(B.3)
\[ s_2 \ s_5 \ K_2 = -s_2 s_6 t_2 \ F_1 + s_6 [s_1 s_2 + (s_4 + s_5 - t_1) (s_5 - t_2)] F_2 + (s_4 + s_5 - t_1) (s_5 - t_2) (s_1 + s_3 - s_5 - t_2) F_6 \\
+ \left\{ s_1^2 s_2 + s_3^2 - s_3 (s_4 + s_5 - t_1) (s_5 - t_2) - t_1 (s_6 + t_1) t_2 + s_1 s_2 (s_4 - t_1 + t_2 - t_3) - s_2^2 (s_6 + 2 t_1 + t_2 - t_3) \right\} \ F_3 \\
- (s_2 - t_1) t_2 t_3 + s_4 (s_5 - t_2) (s_5 - s_6 - t_1 + t_3) + s_5 [s_6 (t_1 + t_2) - (t_1 + t_2) t_3 + s_2 (-s_6 + t_3)] \right\} \ F_3 \]

\[ + (s_4 + s_5 - t_1) [s_1 s_2 + (s_5 - t_2) - s_3 + s_5 - t_1) t_3 \ F_4 \\
+ \left[ -s_1^2 s_2 + (s_4 + s_5 - t_1) (s_3 - s_5 + t_1) (s_5 - t_2) + s_1 s_2 (s_4 + t_1 - t_2 + t_3) - s_2^2 (s_6 - t_3) + t_3) \right\} \ F_4 \]

\[ s_2 \ s_5 \ K_4 = s_2 s_6 (s_3 - t_2) F_1 + s_6 [s_1 s_2 + (s_4 + s_5 - t_1) (s_3 - t_2)] F_2 + (s_4 + s_5 - t_1) (s_3 - t_2) (s_1 + s_3 - s_5 - t_3) F_6 \\
+ \left\{ s_1^2 s_2 - s_3^2 (s_4 + s_5 - t_1) + s_3 (s_4 + s_5 - t_1) (s_5 - s_6 - t_1 + t_2) + s_3 (s_4 + s_5 - t_1) t_3 \right\} \ F_3 \\
- (s_4 + s_5 - t_1) t_2 (s_5 - s_6 - t_1 + t_3) - s_1 s_2 (s_4 - s_1 + t_1 - t_2 + t_3) - s_2 [s_5 (s_6 - t_3) + t_3)] \right\} \ F_3 \\
+ (s_4 + s_5 - t_1) [s_1 s_2 - (s_3 - t_2) (s_3 - s_5 + t_1) - t_3] \ F_4 \\
+ \left[ (s_4 + s_5 - t_1) (s_3 - s_5 + t_1) (s_3 - t_2) - s_1^2 s_2 - s_1 (s_4 + s_5 - t_1) (s_2 + s_3 - t_2) + s_1 s_2 \right\} \ F_5 \\
+ s_3 [s_1 s_2 + (s_4 + s_5 - t_1) (s_3 - s_5 + s_6 + t_1 - t_3) - s_2 (s_5 - t_3)] \ F_3 + s_3 (s_1 - s_3 + s_5 - t_1) (s_4 + s_5 - t_1) \ F_5 \]

\[ s_2 s_5 \ K_6 = -s_2 (s_5 + s_6) (s_3 - t_2) F_1 + [s_6 (s_4 + s_5 - t_1) (t_2 - s_3) - s_1 s_2 s_6 + s_2 s_5 (s_6 - t_3)] F_2 \\
+ (s_4 + s_5 - t_1) (s_3 - t_2) (s_5 + s_6 - s_1 - s_3) F_6 - (s_4 + s_5 - t_1) [s_1 s_2 - (s_3 - t_2) (s_5 + s_6 + t_1 - t_3)] \ F_4 \\
+ \left\{ s_1 s_2 (s_4 + s_5 - t_1) - t_2 - t_3 - s_2 s_2 + (s_4 - s_5 - t_1) (s_3 - s_5 + s_6 + t_1 - t_3) - s_2 (s_5 + t_3)] \right\} \ F_3 \\
+ s_3 + s_2 (s_4 + s_5 - t_1) (s_3 - s_5 + t_1) (s_3 - t_2) + s_3 (s_4 + s_5 - t_1) (s_5 - s_6 - t_3) + s_4 + s_5 - t_1) \ F_5 \]

Appendix C. Material for seven-point function

In this Appendix we present some additional material for the seven-point function \((3.40)\), which has been computed in Subsection 3.3. The complete amplitude is specified by 24 integrals \(\{K_1, \ldots, K_{24}\}\), whose integrands contain the common factor \((3.43)\). The integrals may be written in the following way

\[ K_i = \int_0^1 dx \int_0^1 dy \int_0^1 dz \int_0^1 dw \ P_i \ I(x, y, z, w) \]

with the following 24 polynomials:

\[ P_1 = -s_{34} \ xyw (1 - z) (1 - zw) \ , \ P_2 = (1 - s_{24} + s_{24} \ xyw (1 - z) (1 - zw) (1 - x) \ ] , \ P_3 = \]
\[ s_{34} \ xzw (1 - z) \ , \ P_4 = (1 - s_{24} + s_{24} \ xzw (1 - z) (1 - w) (1 - xy) \ , \]
\[ P_5 = s_{34} \ xy (1 - z) (1 - zw) \ , \ P_6 = \frac{1}{(1 - z) (1 - zw) (1 - x) y} \ , \]
\[ P_7 = s_{34} \ x (1 - z) (1 - w) \ , \ P_8 = \frac{1}{(1 - z) (1 - w) (1 - xy) } \]

\[ (C.1) \]
\[\begin{align*}
P_9 &= -\left(1 - s_{24} + \frac{s_{24}}{x}\right) \frac{1}{y(1 - z)(1 - zw)(1 - xyzw)}, \\
P_{10} &= -\left(\frac{1 - s_{14}}{x} + s_{14}\right) \frac{z}{(1 - z)(1 - wz)(1 - xyzw)}, \\
P_{11} &= -\left(1 - s_{24} + \frac{s_{24}}{x}\right) \frac{1}{y(1 - z)(1 - w)(1 - xyzw)}, \\
P_{12} &= -\left(\frac{1 - s_{14}}{x} + s_{14}\right) \frac{1}{(1 - z)(1 - w)(1 - xyzw)}, \\
P_{14} &= \left(1 - s_{24} + \frac{s_{24}}{x}\right) \frac{1}{yz(1 - w)(1 - xy)(1 - zw)}, \\
P_{16} &= \left(\frac{1 - s_{14}}{x} + s_{14}\right) \frac{1}{(1 - w)(1 - xy)(1 - zw)}, \\
P_{17} &= -\left(1 - s_{24} + \frac{s_{24}}{x}\right) \frac{1}{y(1 - w)(1 - zw)(1 - xyz)}, \\
P_{18} &= -\left(\frac{1 - s_{14}}{x} + s_{14}\right) \frac{1}{(1 - w)(1 - zw)(1 - xyz)}, \\
\end{align*}\]

and:

\[\begin{align*}
P_{19} &= \left[\left(1 - s_{24} + \frac{s_{24}}{x}\right) \frac{1}{yz(1 - w)} - \left(\frac{1 - s_{14}}{x} + s_{14}\right)\right] \frac{1}{1 - wz} + \frac{(1 - x)s_{47}}{x(1 - yzw)} \frac{1}{w(1 - z)(1 - xy)}, \\
P_{20} &= \left[\left(\frac{1 - s_{14}}{x} + s_{14}\right) \frac{1}{(1 - zw)(1 - xy)} - \frac{s_{34}}{(1 - w)xyz} - \frac{(1 - x)s_{47}}{x(1 - yzw)(1 - xyz)}\right] \frac{1}{w(1 - z)}, \\
P_{21} &= \left[\left(\frac{1 - s_{14}}{x} + s_{14}\right) \frac{1}{(1 - w)} - \left(1 - s_{24} + \frac{s_{24}}{x}\right)\right] \frac{1}{y(1 - zw)} - \frac{(1 - x)s_{47}}{x(1 - yzw)} \frac{1}{(1 - z)(1 - xy)}, \\
P_{22} &= \left[\left(1 - s_{24} + \frac{s_{24}}{x}\right) \frac{z}{y(1 - wz)(1 - xyz)} + \frac{s_{34}}{xy(1 - w)} - \frac{z(1 - x)s_{47}}{x(1 - yzw)(1 - xy)}\right] \frac{1}{(1 - z)}, \\
P_{23} &= \left[\left(\frac{1 - s_{14}}{x} + s_{14}\right) \frac{1}{(1 - w)} + \frac{s_{34} - s_{34}}{xy(1 - w)}\right] \frac{1}{1 - zw} - \frac{(1 - x)s_{47}}{x(1 - yzw)} \frac{(-1)}{(1 - z)(1 - xyzw)}, \\
P_{24} &= \left[\left(1 - s_{24} + \frac{s_{24}}{x}\right) \frac{1}{y(1 - w)} - \left(s_{34} - \frac{s_{34}}{xy}\right)\right] \frac{z}{1 - zw} - \frac{(s_{47} - \frac{s_{47}}{x})}{x} \frac{z}{(1 - yzw)} \times \frac{(-1)}{(1 - z)(1 - xyzw)}. \\
\end{align*}\]

(C.2)
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