AN ASYMPTOTIC FORMULA IN \( q \) FOR THE NUMBER OF \([n, k]\) \( q \)-ARY MDS CODES

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Abstract. We obtain an asymptotic formula in \( q \) for the number of MDS codes of length \( n \) and dimension \( k \) over a finite field with \( q \) elements.

1. Introduction

MDS codes of dimension \( k \) and length \( n \) over \( \mathbb{F}_q \) are codes for which the minimum distance equals the Singleton bound \( n - k + 1 \). Equivalently these are codes for which any \( k \times n \) generator matrix has the property that all its \( k \times k \) submatrices are nonsingular. Thus \([n, k]_q\) MDS codes can be identified with the set of row equivalence classes of \( k \times n \) matrices over \( \mathbb{F}_q \) having the property that any \( k \) columns are linearly independent. Let \( \gamma(k, n) \) denote the number of \( q \)-ary \([n, k]\) MDS codes. It is well known that the dual of an MDS code is also MDS, therefore \( \gamma(k, n) = \gamma(n-k, n) \). Consequently, we henceforth assume without loss of generality that \( k \leq n/2 \). Exact values of \( \gamma(k, n) \) are hard to determine and are known only for \( k = 1, 2 \) all \( n \), and for \( k = 3 \) and \( n \leq 9 \). The formula for \( \gamma(3,9) \) was found in the work [5] in 1995. Formulae for \( \gamma(3,10) \) and \( \gamma(4,8) \) for example, are still unknown.

The known exact formulae can be written in the asymptotic form:

\[
\gamma(k, n) = q^\delta + (1 - N) q^{\delta-1} + a_2 q^{\delta-2} + O(q^{\delta-3})
\]

where

\[
\delta := k(n-k) \quad \text{and} \quad N := \binom{n}{k}.
\]

We reproduce from the known exact formulae for \( \gamma(k, n) \) (see [5] §1) the corresponding values of \( a_2(k, n) \).

\[
\begin{align*}
a_2(1, n) & = \frac{n^2 - 3n + 2}{2} \\
a_2(2, n) & = \frac{3n^4 - 10n^3 + 9n^2 - 26n + 48}{24} \\
a_2(3, 6) & = 152 \\
a_2(3, 7) & = 506 \\
a_2(3, 8) & = 1360 \\
a_2(3, 9) & = 3158
\end{align*}
\]

Key words and phrases. MDS codes, Grassmannian.

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For general \((k, n)\), the following asymptotic formula was obtained in \([3]\):

\[
\gamma(k, n) = q^\delta + (1 - N) q^{\delta - 1} + O(q^{\delta - 2})
\]

(4)

Our main result is:

**Theorem 1.1.** The asymptotic formula \([1]\) holds for all \(k\) and \(n\), with

\[
a_2 = N k(n - k) \left( \frac{k^2 - nk + n + 3}{2(k + 1)(n - k + 1)} \right) + N^2/2 - 5N/2 + 2
\]

The formulae in \([3]\) agree with the theorem. The quantity

\[
\tilde{\gamma}(k, n) := \gamma(k, n)/(q - 1)^{n - 1}
\]

is always an integer, as it is the number of \(n\)-arcs in \(PG(k - 1; q)\) (see \([3]\) pp.219-220, \([5]\) Lemma 2)). We rewrite the theorem in terms of \(\tilde{\gamma}(k, n)\):

**Corollary 1.2.** The number \(\tilde{\gamma}(k, n)\) of \(n\)-arcs in \(PG(k - 1, q)\) is of the asymptotic form

\[
q^{\delta - n + 1} - b_1 q^{\delta - n + 2} + b_2 q^{\delta - n + 3} + O(q^{\delta - n + 4})
\]

where \(b_1 = N - n\), and

\[
b_2 = a_2 - (n - 1)(N - n) - (n^2 - 3n + 2)/2
\]

In particular:

\[
\tilde{\gamma}(3, 10) = q^{12} - 110q^{11} + 5561q^{10} + O(q^9)
\]

(5)

\[
\tilde{\gamma}(4, 8) = q^9 - 62q^8 + 1710q^7 + O(q^6)
\]

The ideas of the proof of Theorem \([\ldots]\) are as follows. Following \([3]\), we identify the set of MDS codes with a subset \(U(k, n)\) of the Grassmannian \(G(k, n)\) of \(k\)-dimensional subspaces of \(F_q^n\). The subset \(U(k, n)\) is an intersection of \(N\) Schubert cells of \(G(k, n)\), and hence its cardinality \(|U(k, n)|\) can be expressed using inclusion-exclusion principle as a sum

\[
E_1 - E_2 + E_3 - \cdots + (-1)^{N-1} E_N.
\]

For each \(r\), the quantity \(E_r\) is a sum of \(\binom{N}{r}\) terms each of which is of the form

\[
|G(k, n) \setminus L|\]

where \(L\) is a codimension \(r\) linear subspace of the Plücker space \(\mathbb{P}(\wedge^k F_q^n)\) (in which \(G(k, n)\) embeds). In \([3]\) asymptotic formulae of the form \(q^\delta + aq^{\delta - 1} + O(q^{\delta - 2})\) were obtained for each \(|G(k, n) \setminus L|\), and hence for the \(E_r\)’s to obtain the result given in equation \([\ldots]\). We study linear sections \(|G(k, n) \cap L|\) more closely in section \([\ldots]\) leading to Theorem \([\ldots]\) which gives an asymptotic formula of the form \(q^\delta + aq^{\delta - 1} + bq^{\delta - 2} + O(q^{\delta - 3})\) for \(|G(k, n) \setminus L|\). These formulae in turn imply the desired asymptotic formulae for the \(E_r\)’s and hence for \(\gamma(k, n)\).

2. MDS codes as a subset of the Grassmannian

Let \(V = F_q^n\) with standard basis \(\{e_1, \ldots, e_n\}\). Let

\[
I_{k, n} := \{(i_1, i_2, \ldots, i_k) : 1 \leq i_1 < \cdots < i_k \leq n\},
\]

and for a multi-index \(I = (i_1, \ldots, i_k) \in I_{k, n}\) let \(e_I := e_{i_1} \wedge \cdots \wedge e_{i_k} \in \wedge^k V\). The multivectors \(\{e_I : I \in I_{k, n}\}\) form the standard basis for \(\wedge^k V\). Let \(G(k, n)\) or \(G_k(V)\) denote the Grassmannian of \(k\) dimensional subspaces of \(F_q^n\). We recall the the Plücker embedding of \(G_k(V)\) into \(\mathbb{P}(\wedge^k V)\). For any \(\Lambda \in G_k(V)\) and \(k \times n\) matrix \(M\) whose rows \(b_1, \ldots, b_k\) form a basis for \(\Lambda\), let \(i(\Lambda) := [b_1 \wedge \cdots \wedge b_k] \in \mathbb{P}(\wedge^k V)\). It is
easy to see that \( i(\Lambda) \) depends only on the row equivalence class of the matrix \( M \), and hence only on \( \Lambda \). Therefore we have a function \( i : G_k(V) \to \mathbb{P}(\lambda^k V) \) which can also be shown to be injective. The image \( i(G_k(V)) \), which we will denote again by \( G_k(V) \) is cut out by the quadratic Plücker relations \((17)\), and hence \( i : G_k(V) \to \mathbb{P}(\lambda^k V) \) realizes \( G_k(V) \) as a projective variety. Expanding the expression \( b_1 \wedge \cdots \wedge b_k \) above in terms of the standard basis of \( \lambda^k V \) we obtain

\[
b_1 \wedge \cdots \wedge b_k = \sum_{I \in I_{k,n}} p_I(\Lambda) e_I
\]

where \( p_I(\Lambda) \) is the minor of the matrix \( M \) on the columns indexed by \( I \). It follows that \( p_I(\Lambda), I \in I_{k,n} \) regarded as homogeneous coordinates, depend only on \( \Lambda \) and are known as its Plücker coordinates.

Any \( q \)-ary code of dimension \( k \) and length \( n \), being a \( k \) dimensional subspace of \( \mathbb{F}_q^n \), can be regarded as point \( \Lambda \) of \( G_k(V) \). The matrix \( M \) in the above discussion is precisely a generator matrix of the code \( \Lambda \). As observed in the introduction, the code \( \Lambda \) is \([n,k]_q\)-MDS, if and only if all \( k \times k \) submatrices of \( M \) are non-singular. Hence the set of \([n,k]_q\) MDS codes can be identified with the subset of \( G_k(V) \) defined as:

\[
U(k,n) := \{ \Lambda \in G(k,n) : p_I(\Lambda) \neq 0, \forall I \in I_{k,n} \}
\]

In particular we note

\[
\gamma(k,n) = |U(k,n)|
\]

For each of the \( N = \binom{n}{k} \) multi-indices \( I \in I_{k,n} \) let

\[
C_I := \{ \Lambda \in G(k,n) : p_I(\Lambda) \neq 0 \}
\]

For each \( \Lambda \in C_I \), there is a unique \( k \times n \) matrix \( M \) whose rows forms a basis for \( \Lambda \), and which satisfies the property that the \( k \times k \) submatrix of \( M \) on the columns indexed by \( I \) is the \( k \times k \) identity matrix. Conversely the row space of a matrix \( M \) which has the \( k \times k \) identity matrix as the submatrix on the columns indexed by \( I \), is a point of \( U(k,n) \). Therefore \( C_I \) can be identified with the set of all such matrices. Since the columns of \( M \) indexed by \( I \) are fixed and the remaining \( n-k \) columns free, we see that

\[
|C_I| = q^{k(n-k)} = q^\delta.
\]

It follows from the definitions \((7)\) and \((8)\), that

\[
U(k,n) = \bigcap_{I \in I_{k,n}} C_I
\]

For each \( r \) with \( 1 \leq r \leq N \), let \( I_{k,n}^r \) denote the subsets of \( I_{k,n} \) of cardinality \( r \). Let

\[
E_r := \sum_{\{I_1, \ldots, I_r\} \in I_{k,n}^r} |C_{I_1} \cup \cdots \cup C_{I_r}|
\]

It follows from the inclusion-exclusion principle applied to \((10)\) that

\[
|U(k,n)| = E_1 - E_2 + E_3 - \cdots + (-1)^{N-1} E_N
\]

The quantity \( E_r \) is a sum of \( \binom{N}{r} \) terms of the form \( |C_{I_1} \cup \cdots \cup C_{I_r}| \). The set \( G_k(V) \setminus (C_{I_1} \cup \cdots \cup C_{I_r}) \) consists of points of \( G_k(V) \) having Plücker coordinates
We call a codimension 1 subspace \( m \) a nonzero scalar multiple of the above discussion that each hyperplane corresponds to a unique \( \omega \) of \( V \) spanning by \( \{e_i : i \notin \{1, \ldots, L\}\} \), and \( L = \mathbb{P}\hat{L} \), then \( G_k(V) \setminus (C_{i_1} \cup \cdots \cup C_{i_L}) = G_k(V) \cap L \). The set \( G_k(V) \cap L \) is an example of a linear section of \( G_k(V) \) which we study in the next section.

3. LINEAR SECTIONS OF THE GRASSMANNIAN

We say \( L \subset \mathbb{P}(\wedge^k V) \) is a codimension \( r \) linear subspace if \( L = \mathbb{P}\hat{L} \) with \( \hat{L} \) is a codimension \( r \) linear subspace of \( \wedge^k V \). Corresponding to the standard basis \( e_1, \ldots, e_n \) of \( V \), the dual space \( V^* \) has the standard dual basis \( e^1, \ldots, e^n \) defined by \( \langle e^i, e_j \rangle = \delta_{ij} \). Here \( \langle, \rangle \) is the natural pairing between \( V^* \) and \( V \). Similarly the space \( \wedge^k V^* \) has the standard basis \( \{e^I : I \in I_{k,n}\} \) where \( e^I = e^{i_1} \wedge \cdots \wedge e^{i_k} \).

Given \( k \) elements \( \omega_1, \ldots, \omega_k \) of \( V^* \) and \( k \) elements \( v_1, \ldots, v_k \) of \( V \), the determinant of the \( k \times k \) matrix with entry in row \( i \) and column \( j \) being \( \langle \omega_i, v_j \rangle \), is multilinear and alternating in both \( (\omega_1, \ldots, \omega_k) \) and \( (v_1, \ldots, v_k) \). It therefore defines a bilinear pairing \( \langle, \rangle \) between \( \wedge^k V^* \) and \( \wedge^k V \). In terms of the standard bases we have \( \langle e^I, e_J \rangle = \delta_{IJ} \), which also shows that the pairing is non-degenerate and hence gives an isomorphism between \( \wedge^k V^* \) and \( (\wedge^k V)^* \). We refer to elements of \( \wedge^k V^* \) in short as \( k \)-forms.

Given a codimension \( r \) linear subspace \( L = \mathbb{P}(\hat{L}) \) of \( \mathbb{P}(\wedge^k V) \), let

\[ \operatorname{Ann}(\hat{L}) := \{ \omega \in \wedge^k V^* : \langle \omega, \xi \rangle = 0, \forall \xi \in \hat{L} \}, \]

and let \( \operatorname{Ann}(L) = \mathbb{P}(\operatorname{Ann}(\hat{L})) \). We note that \( \operatorname{Ann}(L) \) is a \( r - 1 \) dimensional linear subspace of \( \mathbb{P}(\wedge^k V^*) \). The correspondence \( L \leftrightarrow \operatorname{Ann}(L) \) allows us to identify codimension \( r \) linear subspaces of \( \mathbb{P}(\wedge^k V) \) with \( r - 1 \) dimensional linear subspaces of \( \mathbb{P}(\wedge^k V^*) \). We define

\[ ||L|| := |G_k(V) \setminus L| \]

We call a codimension 1 subspace \( H \) of \( \mathbb{P}(\wedge^k V) \), a hyperplane. It follows from the above discussion that each hyperplane corresponds to a unique \( k \)-form (up to a nonzero scalar multiple) \( \omega \). We write \( H = H_\omega \) to emphasize this, and we also define

\[ ||\omega|| := ||H_\omega|| = |G_k(V) \setminus H_\omega| \]

The main results of this section are Theorems \([5.10]\) and \([5.11]\) which appear at the end. In the remaining part of this section, we either recall or develop several results leading upto the main results.

**Definition 3.1.** A linear subspace of \( G_k(V) \) is a linear subspace of \( \mathbb{P}(\wedge^k V) \) which is entirely contained in \( G_k(V) \).

For each \( \alpha \in G_{k-1}(V) \) and each \( \gamma \in G_{k+1}(V) \), we define:

\[ \pi_\alpha := \{ \beta \in G_k(V) : \beta \supset \alpha \} \]
\[ \pi_\gamma := \{ \beta \in G_k(V) : \beta \subset \gamma \} \]

Since \( \pi_\alpha \) is projectively isomorphic to \( \mathbb{P}(V/\alpha) \), it is a linear subspace of \( G_k(V) \) of dimension \( n - k \). Similarly \( \pi_\gamma \) is projectively isomorphic to \( \mathbb{P}(\gamma^*) \), and hence it is a linear subspace of \( G_k(V) \) of dimension \( k \). It is a classical fact \([1, \S 21.1, \text{7, Proposition 3.2}]\) that, for \( k \geq 2 \), the \( \pi_\alpha \)'s and the \( \pi_\gamma \)'s are the maximal linear subspaces of \( G_k(V) \). Some facts about these spaces that we will need are as follows. They easily follow from the definitions of \( \pi_\alpha \) and \( \pi_\gamma \) (also see \([2\text{ p.}88]\)).
Fact 3.2.  
(1) For $\alpha \neq \alpha' \in G_{k-1}(V)$ the intersection $\pi_\alpha \cap \pi_{\alpha'}$ is empty if $\dim(\alpha + \alpha') > k$ and consists of the single point $\alpha + \alpha'$ if $\dim(\alpha + \alpha') = k$.

(2) For $\gamma \neq \gamma' \in G_{k+1}(V)$ the intersection $\pi^- \cap \pi^-'$ is empty if $\dim(\gamma \cap \gamma') < k$ and consists of the single point $\gamma \cap \gamma'$ if $\dim(\gamma \cap \gamma') = k$.

(3) The intersection $\pi_\alpha \cap \pi^-$ is empty if $\alpha \not\subset \gamma$, and it is the line $\{\beta : \alpha \subset \beta \subset \gamma\}$ if $\alpha \subset \gamma$.

(4) For any pair $\beta, \beta' \in \pi_\alpha$ the intersection $\beta \cap \beta'$ is $\alpha$, and for any pair $\beta, \beta' \in \pi^- \cap \pi^-$ the vector space sum $\beta + \beta'$ is $\gamma$.

We recall the definition of the interior multiplication operator: for $\xi \in \wedge^f V$, $\zeta \in \wedge^n V$, and $\omega \in \wedge^{k+m} V^*$, $\iota_\xi \omega \in \wedge^m V^*$ is defined by:

$$\langle \iota_\xi \omega, \zeta \rangle = \langle \omega, \xi \wedge \zeta \rangle$$

For any nonzero $\omega \in \wedge^k V^*$, we define subspaces $V_\omega \subset V$ and $U_\omega \subset V^*$ by

$$V_\omega := \{ v \in V : \iota_v \omega = 0 \}$$

$$U_\omega := \{ \theta \in V^* : \langle \theta, v \rangle = 0 \quad \forall v \in V_\omega \}$$

Fact 3.3. We recall [4, p.210] that $\omega$ is decomposable if and only if $\dim(V_\omega) = n-k$ or equivalently $\dim(U_\omega) = k$. We also note that if $\omega$ is indecomposable then $\dim(U_\omega) \geq k+2$ because $\omega \in \wedge^k U_\omega$ and every element of $\wedge^k \mathbb{F}^{k+1}$ is decomposable.

The next lemma will be used in the proof of Theorem 3.10.

Lemma 3.4. Let $k \geq 2$. A $k+1$-form $\omega \in \wedge^{k+1} V^*$ is decomposable if and only if $\iota_{\omega} \omega$ is decomposable for all $v \in V$.

Proof. If $\omega$ is decomposable, then so is $\iota_v \omega$. Conversely if $\iota_v \omega$ is decomposable for all $v \in V$, then

$$S_\omega := \mathbb{P}\{ \iota_v \omega : v \in V \} \subset G_k(V^*) \subset \mathbb{P}(\wedge^k V^*)$$

is a linear subspace of $G_k(V^*)$. Therefore $S_\omega$ is contained either in a $\pi_\alpha$ for some $\alpha \in G_{k-1}(V^*)$, or a $\pi^- \omega$ for some $\gamma \in G_{k+1}(V^*)$. Suppose $S_\omega \subset \pi_\alpha$. Let $f_1, f_2, \ldots, f_n$ be a basis of $V$ such that $f_1, f_2, \ldots, f_n$ is a basis of $V_\omega$ or equivalently $f^{n+1}, \ldots, f^n$ is a basis of $U_\omega$ where $f^1, \ldots, f^n$ is the dual basis. The condition $n-\ell = k+1$ for $\omega$ to be decomposable can be written as $n-\ell \leq k+1$ since $n-\ell$ equals $\dim(U_\omega) \geq k+1$. The expression for $\omega$ in terms of the basis vectors $\{f^I : I \in I_{k+1,n}\}$ of $\wedge^k V^*$ involves only those basic vectors $f^I$ for which all indices of $I$ are strictly greater than $\ell$. For each $i \geq \ell + 1$, the fact that $\iota_{f_i} \omega \neq 0$ implies that there is at least one $j \neq i$ with $j \geq \ell + 1$ such that $\iota_{f_j} \wedge \omega \neq 0$. Now let $\beta = \iota_{f_i} \omega$ and $\beta' = \iota_{f_j} \omega$. Then $\iota_{f_i} \wedge \beta = -\iota_{f_j} \wedge \beta' = \iota_{f_i} \beta$ represents a $k-1$ dimensional space contained in the $k$ dimensional spaces $\beta$ and $\beta'$. It follows that $\iota_{f_i} \wedge f_j \omega$ represents $\beta \wedge \beta'$. By part 4) of Fact 3.2 we conclude that the expression for $\alpha$ in terms of the basis vectors $\{f^I : I \in I_{k-1,n}\}$ of $\wedge^{k-1} V^*$ involves only those basis vectors $f^I$ for which no index of $I$ equals $i$. Since $i \geq \ell + 1$ was arbitrary, the expression for $\alpha$ involves only those basic vectors $f^I$ for which all indices of $I$ are less than or equal to $\ell$. This however contradicts the fact that the expression for $\omega$ and hence $\iota_{f_i} \wedge \omega$ involves only those $f^I$ for which the indices of $I$ are strictly greater than $\ell$. This contradiction shows that $S_\omega \subset \pi^- \omega$ for some $\gamma \in G_{k+1}(V^*)$. Since $\dim(\pi^-) = k$ and $\dim(S_\omega) = n-\ell - 1$ we get $n-\ell \leq k+1$ as desired. \qed

The next lemma will be used in the proof of Proposition 3.8.
Lemma 3.5. If \( P \) is a subset of \( G_k(V) \) such that for every pair \( P, Q \in P \), the line \( PQ \) joining them is contained in \( G_k(V) \), then the linear subspace of \( \mathbb{P}(\wedge V) \) generated by \( P \) is completely contained in \( G_k(V) \).

Proof. An element \( \lambda \in \wedge^k V \) is decomposable if and only if it satisfies the Plücker relations (\cite{4}, pp.210-211)

\[
(\iota_\xi \lambda) \wedge \lambda = 0 \quad \forall \xi \in \wedge^{k-1} V^* 
\]

Let \( \lambda_1, \lambda_2, \ldots, \lambda_n \) be decomposable elements of \( \wedge^k V \) representing points \( P_1, P_2, \ldots, P_n \) of \( P \). For each \( \xi \in \wedge^{k-1} V^* \), let \( a_i(\xi) := (\iota_\xi \lambda_i) \wedge \lambda_i \) for \( 1 \leq i \leq n \), and let \( a_{ij}(\xi) := (\iota_\xi \lambda_i) \wedge \lambda_j + (\iota_\xi \lambda_j) \wedge \lambda_i \) for \( 1 \leq i < j \leq n \). The condition for \( \lambda = \sum t_i \lambda_i \) to be decomposable is

\[
\sum_i t_i^2 a_i(\xi) + \sum_{i<j} t_i t_j a_{ij}(\xi) = 0, \quad \forall \xi \in \wedge^{k-1} V^* 
\]

Using (17), \( a_i(\xi) = 0 \) because \( \lambda_i \) is decomposable, and \( a_{ij}(\xi) = 0 \) because \( PQ \subset G_k(V) \).

Some of the results we will need about the cardinalities \(|L|\) (as defined in (13)) appear in the literature in the context of linear codes associated to the embedding \( G(k, n) \hookrightarrow \mathbb{P}(\wedge^k \mathbb{F}_q^n) \). We briefly describe the construction of these Grassmann codes \( C(k, n) \), the details of which can be found in [6]. These are linear codes of length \( \tilde{n} \) and dimension \( \tilde{k} \) where:

\[
\tilde{n} = |G(k, n)| = \frac{(q^n - 1)(q^{n-1} - 1)\cdots(q^{n-k+1} - 1)}{(q^k - 1)(q^{k-1} - 1)\cdots(q - 1)} 
\]

\[
\tilde{k} = \binom{\tilde{n}}{k} 
\]

Let \( P_1, P_2, \ldots, P_{\tilde{n}} \) denote representatives in \( \wedge^k \mathbb{F}_q^n \) of the \( \tilde{n} \) points of \( G(k, n) \) arranged in some order. We also arrange in some order, the \( \tilde{k} \) basic vectors \( \{e_I : I \in I_{k,n}\} \) of \( \wedge^k \mathbb{F}_q^n \). A \( \tilde{k} \times \tilde{n} \) generator matrix for the code \( C(k, n) \) has for its \((i, j)\)-th entry, the \( i \)-th Plücker coordinate \( p_i(P_j) \). The fact that the Plücker embedding is non-degenerate (i.e., no hyperplane of \( \mathbb{P}(\wedge^k \mathbb{F}_q^n) \) contains \( G(k, n) \)) is equivalent to the fact that the generator matrix above has full row rank. A codeword of \( C(k, n) \) is of the form \( \sum_I a_I p_I(P_1), \ldots, \sum_I a_I p_I(P_{\tilde{n}}) \). If \( \omega \) is the \( k \)-form \( \sum_I a_I e^I \), then the above codeword can be expressed as \( (\omega(P_1), \ldots, \omega(P_{\tilde{n}})) \). In this way we identify codewords of \( C(k, n) \) with \( k \)-forms on \( V = \mathbb{F}_q^n \). Moreover, the Hamming weight of the above codeword clearly coincides with the expression \(|\omega|\) as defined in (13).

More generally, let \( \tilde{L} \) be a codimension \( r \) subspace of \( \wedge^k \mathbb{F}_q^n \), and let \( \{\omega_1, \ldots, \omega_r\} \) be a basis for \( \text{Ann}(\tilde{L}) \). Evaluating elements of \( \text{Ann}(\tilde{L}) \), i.e., forms \( \sum_{i=1}^r a_i \omega_i \) on \( P_1, \ldots, P_{\tilde{n}} \) gives a \( r \) dimensional subcode of \( C(k, n) \). Moreover, the Hamming weight of this subcode is clearly \(|L|\) as defined in (13) (where \( L = \mathbb{P}(\tilde{L}) \)). The minimum distance \( d_1(C(k, n)) \) of the code \( C(k, n) \) equals minimum value of \(|\omega|\) over all nonzero \( k \)-forms on \( V \), and the higher weight \( d_i(C(k, n)) \) equals the minimum value of \(|L|\) over all codimension \( r \) subspaces of \( \mathbb{P}(\wedge^k V) \).

Using the well known formula (see [6], p.7)

\[
||D|| = \frac{1}{q^r - q^{r-1}} \sum_{c \in D} ||c||
\]
relating the Hamming weight of a \( r \) dimensional subcode \( D \) to the Hamming weights of its constituent codewords, we get the corresponding formula

\[
||L|| = \frac{1}{q^{r-1}} \sum_{[\omega] \in \text{Ann}(L)} ||\omega|| = \frac{1}{q^{r-1}} \sum_{H \supset L} ||H||
\]

where the sum \( H \supset L \) is over all hyperplanes \( H \) containing \( L \).

The next theorem summarizes the information we will need about the minimum distance, higher weights and the weight spectrum of the code \( C(k, n) \).

**Theorem 3.6** (Nogin). \([6]\)

1. The minimum distance \( d_{\min}(C(k, n)) \) equals \( q^\delta \). The codewords of minimum weight correspond precisely with decomposable \( k \)-forms.

2. More generally, for \( 1 \leq r \leq n - k + 1 \), the higher weight

\[
d_r(C(k, n)) = q^\delta + q^{\delta - 1} + \cdots + q^{\delta - r + 1}.
\]

The \( r \) dimensional subcodes with minimum weight correspond exactly to codimension \( r \) subspaces \( L \) of \( \mathbb{P}(\wedge^k V) \) such that \( \text{Ann}(L) \) is a linear subspace of \( G_k(V^*) \).

3. For the code \( C(2, n) \), up to a code-isomorphism any codeword corresponds to one of the \( 2 \)-forms

\[
\omega_r = e^1 \wedge e^2 + e^3 \wedge e^5 + \cdots + e^{2r - 1} \wedge e^{2r},
\]

where \( 1 \leq r \leq \lfloor n/2 \rfloor \). Moreover

\[
||\omega_r|| = q^\delta + q^{\delta - 2} + \cdots + q^{\delta - 2r + 2}
\]

**Proof.** We refer to [6] for the proofs of parts 1) and 3). We give a quick proof of part 2) of the theorem. For a codimension \( r \) subspace \( L \subset \mathbb{P}(\wedge^k V) \), there are \( 1 + q + q^2 + \cdots + q^{r - 1} \) elements in \( \text{Ann}(L) \), and each \( [\omega] \in \text{Ann}(L) \) satisfies \( ||\omega|| \geq q^\delta \) (by the first part of the theorem). The formula (19) gives

\[
||L|| \geq \frac{q^\delta(1 + q + \cdots + q^{r - 1})}{q^{r - 1}} = q^\delta + q^{\delta - 1} + \cdots + q^{\delta - r + 1}
\]

Moreover, equality holds above if and only if each \( [\omega] \) in \( \text{Ann}(L) \) satisfies \( ||\omega|| = q^\delta \), or equivalently is decomposable. In other words equality holds if and only if \( \text{Ann}(L) \) is a \( r - 1 \) dimensional linear subspace of \( G_k(V^*) \). Since linear subspaces of \( G_k(V^*) \) exist only in dimensions less than or equal to \( n - k \), taking such subspaces for \( \text{Ann}(L) \) we obtain the desired formula for \( d_r(C(k, n)) \) when \( 1 \leq r \leq n - k + 1 \). \( \square \)

The next lemma relates the weight of a \( C(k, n) \) codeword to certain codewords of \( C(k - 1, n - 1) \) associated with it. For a nonzero \( \omega \in \wedge^k V^* \) and a vector \( u \not\in V_\omega \), let \( \omega_u \) denote \( \iota_u \omega \) regarded as a \( k - 1 \) form on the quotient vector space \( V/\langle \mathbb{F}_q u \rangle \). The weight \( ||\omega|| \) can be expressed in terms of the weights \( ||\omega_u|| \) for \( u \not\in V_\omega \) as follows.

**Lemma 3.7.**

\[
||\omega|| = \frac{1}{q^k - 1} \sum_{u \not\in V_\omega} ||\omega_u||
\]
Proof. Let us denote the cardinality of the general linear group $GL(m, \mathbb{F}_q)$ by

$$[m]_q := q^{m(m-1)/2} (q^m - 1)(q^{m-1} - 1) \cdots (q - 1).$$

Let $[v_1, v_2, \ldots, v_k]$ denote an ordered set of $k$ vectors of $V$. By definition of $||\omega|| = |G_k(V) \setminus \mathcal{H}_\omega|$ we get:

$$[k]_q \cdot ||\omega|| = \sum_{u \in \mathcal{V}_\omega} |\{[v_1, v_2, \ldots, v_k] : \langle \omega, v_1 \wedge \cdots \wedge v_k \rangle \neq 0\}|.$$

Let $\hat{v}_i$ for $2 \leq i \leq k$ denote the class of $v_i$ in $V/(\mathbb{F}_q u)$. There are $q^{k-1}$ ordered sets $[v_2, \ldots, v_k]$ whose projection under $V \to V/(\mathbb{F}_q u)$ is a given ordered set $[\hat{v}_2, \ldots, \hat{v}_k]$. Therefore,

$$\frac{[k]_q \cdot ||\omega||}{q^{k-1}} = \sum_{u \in \mathcal{V}_\omega} |\{[\hat{v}_2, \ldots, \hat{v}_k] : \langle \omega_u, \hat{v}_2 \wedge \cdots \wedge \hat{v}_k \rangle \neq 0\}| = [k - 1]_q \sum_{u \in \mathcal{V}_\omega} ||\omega_u||.$$

The lemma now follows by noting that

$$[k]_q = q^{k-1}(q^k - 1)[k - 1]_q.$$

\hfill \Box

Proposition 3.8. For $1 \leq m \leq n - k + 1$, let $L$ be an $m$ dimensional subspace of $\mathbb{F}(\wedge^k V)$ such that $L$ is not contained in $G_k(V)$. Suppose $L_1 \subset L$ is an $m - 1$ dimensional subspace which is contained in $G_k(V)$. Then we have:

$$|L \cap G_k(V)| - |L_1| \leq \begin{cases} 1 & \text{if } m = 1, \\ q & \text{if } m = 2, \\ q^2 & \text{if } m \geq 3. \end{cases} \tag{20}$$

Proof. If there is no point $P \in L \cap G_k(V)$ which is not in $L_1$, then $|L \cap G_k(V)| - |L_1|$ being zero, clearly satisfies the stated bounds. So we assume such a point $P$ exists. Let $L_2$ be the subset of $L_1$ defined by

$$L_2 := \{Q \in L_1 : \overline{PQ} \subset G_k(V)\}$$

where, for any two points $P_1 \neq P_2 \in G_k(V)$, $\overline{P_1 P_2}$ denotes the line joining $P_1$ and $P_2$. By extending a basis of $P_1 \cap P_2$ to a basis of $P_1$ and $P_2$, it is easy to see that

$$\overline{P_1 P_2} \cap G_k(V) = \begin{cases} \overline{P_1 P_2} & \text{if } \dim(P_1 \cap P_2) = k - 1, \\ \{P_1, P_2\} & \text{if } \dim(P_1 \cap P_2) < k - 1. \end{cases}$$

This together with the fact that $L = \bigcup_{P' \in L_1} \overline{PP'}$, gives:

$$|L \cap G_k(V)| - |L_1| = 1 + (q - 1)|L_2| \tag{21}$$

Let $\mathcal{P} = L_2 \cup \{P\}$. Given $P_1, P_2 \in \mathcal{P}$, if $P_1$ or $P_2$ equals $P$, then by definition of $L_2$ the line $\overline{P_1 P_2}$ is contained in $G_k(V)$. Otherwise $P_1, P_2 \in L_2 \subset L_1$, and since $L_1 \subset G_k(V)$ is a linear subspace we again get $\overline{P_1 P_2} \subset G_k(V)$. Therefore we can apply Lemma 3.5 to $\mathcal{P}$ to conclude that the linear subspace $L(\mathcal{P})$ generated by $\mathcal{P}$ is contained in $G_k(V)$. Let $\mathcal{P}'$ be a maximal linear subspace of $G_k(V)$ containing $L(\mathcal{P})$. If $Q, Q' \in L_2$ and $Q'' \in \overline{QQ'}$, then $\overline{QQ''} \in L(\mathcal{P})$, and hence $\overline{QQ''} \subset G_k(V)$. This shows that $Q'' \in L_2$, and hence that $L_2$ is a linear subspace of $G_k(V)$.
Let \( \pi \) be a maximal linear subspace of \( G_k(V) \) containing \( L_1 \). We note that \( P \notin \pi \), for otherwise all lines joining \( P \) to \( L_1 \) are contained in \( G_k(V) \), and hence \( L \) itself, being the union of these lines, is contained in \( G_k(V) \), which is not true by hypothesis. Therefore \( \pi \neq \pi' \). By parts 1)-3) of Fact 3.2, \( \pi \cap \pi' \) is either a line, a point or empty. Since \( L_2 \) is a linear subspace of \( \pi \cap \pi' \), we conclude that \( |L_2| \in \{0, 1, 1 + q\} \). Using this in (21) we get:

\[
|L \cap G_k(V)| - |L_1| \in \{1, q, q^2\}.
\]

The fact that \( L \notin G_k(V) \) implies \( |L \cap G_k(V)| - |L_1| < q^m \). Therefore, in the case \( m = 1 \) we get \( |L \cap G_k(V)| - |L_1| = 1 \), and in the case \( m = 2 \), we get \( |L \cap G_k(V)| - |L_1| \in \{1, q\} \).

**Proposition 3.9.** Let \( L \) be an \( \ell \) dimensional linear subspace of \( \mathbb{P}(\wedge^k V) \) which is not contained in \( G_k(V) \). Also suppose \( \ell \geq 3 \). Then:

\[
|L \cap G_k(V)| \leq 1 + q + 2q^2 + q^3 + \cdots + q^{\ell-1}
\]

**Proof.** Let \( L'' \) be a subspace of \( L \) which is maximal with respect to the property of being contained in \( G_k(V) \). If \( L'' = \emptyset \) then \( |L \cap G_k(V)| = 0 \) satisfies the asserted bound. So we assume \( \dim(L'') = \mu \geq 0 \) and let

\[
\mathcal{F} := \{L' \subset L : \dim(L') = \mu + 1, L' \supset L''\}
\]

We note that \( |\mathcal{F}| = q^{\ell-\mu-1} \). Each point of \( L \) is contained in some \( L' \in \mathcal{F} \), and any two distinct elements of \( \mathcal{F} \) intersect in \( L'' \). Therefore

\[
|L \cap G_k(V)| = |L''| + \sum_{L' \in \mathcal{F}} (|L' \cap G_k(V)| - |L''|)
\]

The pair of spaces \( L'' \subset L' \) satisfies the hypothesis of Proposition 3.8, therefore \( |L' \cap G_k(V)| - |L''| \) satisfies the bounds of (20). Consequently,

\[
\sum_{L' \in \mathcal{F}} (|L' \cap G_k(V)| - |L''|) \leq \begin{cases} 1 + q + \cdots + q^{\ell-1} & \text{if } \mu = 0, \\ q + q^2 + \cdots + q^\ell - 1 & \text{if } \mu = 1, \\ q^2 + q^3 + \cdots + q^{\ell+\mu+1} & \text{if } \mu \geq 2 \end{cases}
\]

Adding \( |L''| = 1 + q + \cdots + q^\mu \) to the above equation we get:

\[
(22) \quad |L \cap G_k(V)| \leq \begin{cases} 2 + q + \cdots + q^{\ell-1} & \text{if } \mu = 0 \\ 1 + 2q + q^2 + \cdots + q^{\ell-1} & \text{if } \mu = 1 \end{cases}
\]

and if \( \mu \geq 2 \) then

\[
(23) \quad |L \cap G_k(V)| \leq 1 + q + 2 \left(q^2 + \cdots + q^{\min(\mu, \ell-\mu+1)}\right) + \left(q^{\min(\mu+1, \ell-\mu+2)} + \cdots + q^{\max(\mu, \ell-\mu+1)}\right)
\]

Consider the quantity:

\[
A := 1 + q + 2q^2 + q^3 + \cdots + q^{\ell-1}
\]

and let \( B \) denote the right hand side of (22) if \( \mu = 0 \) or \( \mu = 1 \), and the right hand side of (23) if \( 2 \leq \mu \leq \ell - 1 \). The assertion in the proposition statement that needs to be established is \( |L \cap G_k(V)| \leq A \). Hence it suffices to show \( A \geq B \). If \( \mu = 0 \),
then \( A - B = q^2 - 1 > 0 \) and if \( \mu = 1 \) then \( A - B = q^2 - q > 0 \). If \( \mu = 2 \) or \( \ell - 1 \), then \( A = B \). In the remaining cases \( 3 \leq \mu \leq \ell - 2 \), we get:

\[
\frac{(q - 1)(A - B)}{q^3} = (q^{\ell-1-\mu} - 1)(q^{\mu-2} - 1)
\]

Since \( 3 \leq \mu \leq \ell - 2 \) implies \( \ell - 1 - \mu \geq 1 \) and \( \mu - 2 \geq 1 \), we obtain \( A > B \).

**Theorem 3.10.** Let \( \omega \in \wedge^k V^* \) be a nonzero \( k \)-form. If \( \omega \) is decomposable then \( ||\omega|| = q^\delta \), and if \( \omega \) is indecomposable then

\[
||\omega|| = q^\delta + q^{\delta-2} + O(q^{\delta-3})
\]

(where \( \delta(k, n) = k(n - k) \) and \( k \leq n/2 \))

**Proof.** If \( \omega \) is decomposable then \( ||\omega|| = q^\delta \) by part 1) of Theorem 3.6. If \( \omega \) is indecomposable and \( k = 2 \), then the desired result (24) holds by part 3) of Theorem 3.6 for all \( n \geq 4 \). So, we assume \( k \geq 3 \) and assume inductively that the result holds for all \( k - 1 \) forms \( \omega \) on a vector space of dimension \( n \geq 2(k - 1) \). We now use Lemma 3.4 and the notation therein. Let \( W_\omega \) be a complement of \( V_\omega \) in \( V \), so that every element of \( V \setminus V_\omega \) can be written as \( u + v \) with \( u \in W_\omega \setminus \{0\} \) and \( v \in V_\omega \).

We let \( \dim(W_\omega) = \dim(U_\omega) = k + s \) where \( s \geq 2 \) as noted in Fact 3.3. We get:

\[
||\omega|| = \frac{q^{n-k-s}}{q^k - 1} \sum_{u \in W \setminus \{0\}} ||\omega_u||
\]

For any \( u \in W_\omega \), the form \( \omega_u \in \wedge^{k-1}(V/Fu)^* \) is decomposable if and only if \( \iota_u \omega \) is decomposable. This easily follows by working with a basis for \( V = W_\omega \oplus V_\omega \) that extends \( u \). Let \( L_\omega \) denote the \( k + s - 1 \) dimensional subspace of \( \mathbb{P}(\wedge^{k-1}U_\omega) \) defined by

\[
L_\omega = \mathbb{P}\{\iota_u \omega : u \in W_\omega\}
\]

Since \( \omega \) is indecomposable, it follows by Lemma 3.4 that \( L_\omega \) is not contained in \( G_{k-1}(U_\omega) \). Hence, by Proposition 3.3, we get

\[
|L_\omega \cap G_{k-1}(U_\omega)| \leq 1 + q + 2q^2 + q^3 + \cdots + q^{k+s-2}.
\]

Therefore, \( |L_\omega \setminus G_{k-1}(U_\omega)| \geq q^{k+s-1} - q^2 \). On the other hand \( |L_\omega \setminus G_{k-1}(U_\omega)| \leq |L_\omega| \). Putting these inequalities together we get:

\[
|L_\omega \setminus G_{k-1}(U_\omega)| = q^{k+s-1} - q^2 + O^+(q^{k+s-2}) = q^{k+s-1} + O(q^{k+s-2})
\]

where \( O^+(q^m) \) denotes a positive quantity which is \( O(q^m) \). Let

\[
\delta' := \delta(k - 1, n - 1) = (k - 1)(n - k).
\]

We note that \( q^{\delta'} = q^\delta / q^{n-k} \). The weight \( ||\omega_u|| \) equals \( q^{\delta'} \) if \( \iota_u \omega \) is decomposable, and if \( \iota_u \omega \) is indecomposable, then

\[
||\omega_u|| = q^{\delta'} + q^{\delta'-2} + O(q^{\delta'-3}),
\]

by the inductive hypothesis. Using this in (25), we get:

\[
\frac{q^{\delta'}(q^{k-1})}{q-1} ||\omega|| = (q^{\delta-2} + O(q^{\delta-3}))|L_\omega \setminus G_{k-1}(U_\omega)| + q^\delta |\mathcal{P}(W_\omega)|
\]

\[
= (q^{\delta-2} + O(q^{\delta-3}))(q^{k+s-1} + O(q^{k+s-2})) + q^\delta \frac{q^{k+s-1} - q^2}{q-1}
\]
Simplifying this, we get:

$$||\omega|| = q^d + q^{d-s} \frac{q^s - 1}{q^k - 1} + (q^{d-2} + O(q^{d-3})) \frac{q^k + O(q^{k-1})}{q^k - 1}$$

$$= q^d + O(q^{d-k}) + q^{d-2} + O(q^{d-3})$$

(27)

$$= q^d + q^{d-2} + O(q^{d-3})$$

\[\square\]

**Theorem 3.11.** Let $L$ be a codimension $r$ subspace of $\mathbb{P}(\wedge^k V)$. If $\text{Ann}(L) \subseteq G_k(V^*)$ then

$$||L|| = q^d + q^{d-1} + \cdots + q^{d-r+1}.$$

If $\text{Ann}(L) \not\subseteq G_k(V^*)$ and $r \geq 3$ then:

$$||L|| = q^d + q^{d-1} + 2q^{d-2} + O(q^{d-3})$$

If $\text{Ann}(L) \not\subseteq G_k(V^*)$ and $r = 2$ then:

$$||L|| = q^d + q^{d-1} + q^{d-2} + O(q^{d-3})$$

**Proof.** When $\text{Ann}(L) \subseteq G_k(V^*)$, the assertion is a restatement of part 2) of Theorem 3.10. We now assume $\text{Ann}(L) \not\subseteq G_k(V^*)$. Using the formula (19) and Theorem 3.10, we get:

$$q^{r-1} ||L|| = |\text{Ann}(L) \cap G_k(V^*)| q^d + |\text{Ann}(L) \setminus G_k(V^*)| (q^d + q^{d-2} + O(q^{d-3}))$$

$$= q^d (1 + q + \cdots + q^{r-1}) + |\text{Ann}(L) \setminus G_k(V^*)| (q^d + q^{d-2} + O(q^{d-3}))$$

Since $\text{Ann}(L) \not\subseteq G_k(V^*)$ and $r \geq 3$, we use Proposition 3.9 to obtain (as in the proof of Theorem 3.10):

$$|\text{Ann}(L) \setminus G_k(V^*)| = q^{r-1} + O(q^{r-2})$$

Using this in (28), we get:

$$||L|| = q^d + q^{d-1} + \cdots + q^{d-r+1} + (q^{d-2} + O(q^{d-3}))$$

$$= q^d + q^{d-1} + 2q^{d-2} + O(q^{d-3})$$

In the case when $r = 2$, $\text{Ann}(L)$ is a line in $\mathbb{P}(\wedge^k V^*)$. As mentioned in the proof of Proposition 3.8 if a line is not contained in $G_k(V^*)$, then it meets $G_k(V^*)$ in at most two points. Hence $|\text{Ann}(L) \setminus G_k(V^*)|$ being $q, q-1$ or $q-2$, can be written as $q + O(1)$. Using this in (28), we get

$$||L|| = \frac{1}{q} \left[ q^d (1 + q) + (q + O(1))(q^{d-2} + O(q^{d-3})) \right]$$

$$= q^d + q^{d-1} + q^{d-2} + O(q^{d-3})$$

\[\square\]

4. Proof of the main result

We recall from (12) the formulas (11) and (12)

$$\gamma(k, n) = E_1 - E_2 + E_3 - \cdots + (-1)^{N-1} E_N$$

$$E_r = \sum_{\{I_1, \ldots, I_r\} \in T_{k,n}} |C_{I_1} \cup \cdots \cup C_{I_r}|$$
From (29), we get

\[ E_1 = N q^\delta \]

If \( L \) is the codimension \( r \) subspace of \( \mathbb{P}(\wedge^k V) \) defined by

\[ \text{Ann}(L) = \mathbb{P} \left( \text{Span}(e^{I_1}, \ldots, e^{I_r}) \right) \]

then \( |C_{I_1} \cup \cdots \cup C_{I_r}| = ||L|| \). If \( r = 2 \), the line \( \text{Ann}(L) \) of \( \mathbb{P}(\wedge^k V^*) \) joining \( e^I \) and \( e^J \) is contained in \( G_k(V^*) \) if and only if, the \( k \)-dimensional subspaces of \( V^* \) represented by \( e^I \) and \( e^J \) have a \( k-1 \) dimensional intersection. In other words there is a multi-index \( K \in I_{k-1,n} \) with \( I \) and \( J \) of the form \( I = K \cup \{i\} \) and \( J = K \cup \{j\} \). Clearly there are

\[ \binom{n}{k-1} \binom{n-k+1}{\frac{1}{2}} = N \delta / 2 \]

such elements of \( I_{k,n}^2 \). Therefore, by Theorem 3.11 we get:

\[ E_2 = \frac{N \delta}{2} (q^\delta + q^{\delta-1}) + \left( \binom{n}{2} - N \delta / 2 \right) (q^\delta + q^{\delta-1} + q^{\delta-2} + O(q^{\delta-3})) \]

\[ = \left( \binom{n}{2} \right) (q^\delta + q^{\delta-1}) + \frac{N^2 - N \delta}{2} q^{\delta-2} + O(q^{\delta-3}) \]

Now, let \( r \geq 3 \). The subspace \( \text{Ann}(L) \) of (29) is contained in \( G_k(V^*) \) if and only if it is contained in a maximal linear subspace \( \pi_\alpha \) or \( \pi_\gamma \) for some \( \alpha \in G_{k-1}(V^*) \) or some \( \gamma \in G_{k+1}(V^*) \). Moreover, these two cases are disjoint, because \( \text{Ann}(L) \) is \( r-1 \) dimensional and the intersection of a \( \pi_\alpha \) and a \( \pi_\gamma \) is at most one dimensional (part 3) of Fact 3.2. Suppose \( \text{Ann}(L) \subset \pi_\alpha \). By part 4) of Fact 3.2 it follows that \( \alpha \) is the intersection of the points of \( G_k(V^*) \) represented by \( e^{I_\mu} \) and \( e^{J_\nu} \) for any pair \( 1 \leq \mu < \nu \leq r \). In other words, there is a \( J \in I_{k-1,n} \) common to all the multi-indices \( I_1, \ldots, I_r \). Clearly there are

\[ c_1(r) := \begin{cases} \binom{n}{k-1} \binom{n-k+1}{\frac{1}{r}} = \frac{kN}{n-k+1} \binom{n-k+1}{\frac{1}{r}} & \text{if } r \leq n-k+1 \\ 0 & \text{if } r > n-k+1 \end{cases} \]

such elements of \( I_{k,n}^r \).

Suppose \( \text{Ann}(L) \subset \pi_\gamma \). By part 4) of Fact 3.2 it follows that for any pair \( 1 \leq \mu < \nu \leq r \), \( \gamma \) is the vector space sum of the points of \( G_k(V^*) \) represented by \( e^{I_\mu} \) and \( e^{J_\nu} \). In other words, there is a \( J \in I_{k+1,n} \) containing all the multi-indices \( I_1, \ldots, I_r \). Clearly there are

\[ c_2(r) := \begin{cases} \binom{n}{k+1} \binom{k+1}{\frac{1}{r}} = \frac{N(n-k)}{k+1} \binom{k+1}{\frac{1}{r}} & \text{if } r \leq k+1 \\ 0 & \text{if } r > k+1 \end{cases} \]

such elements of \( I_{k,n}^r \). (A derivation of \( c_1(r) \) and \( c_2(r) \) can also be found in [3 Corollary 4.4].) Therefore, by Theorem 3.11 we get:

\[ \sum_{r=3}^{N} (-1)^{r-1} E_r = \sum_{r=3}^{N} (-1)^{r-1} \left[ (c_1(r) + c_2(r))(q^\delta + q^{\delta-1} + \ldots + q^{\delta-r+1}) \right. \\
+ \left. \left( \binom{N}{r} - c_1(r) - c_2(r) \right)(q^\delta + q^{\delta-1} + 2q^{\delta-2} + O(q^{\delta-3})) \right] \]
which simplifies to

\[
\sum_{r=3}^{N} (-1)^{r-1} E_r = O(q^\delta^{-3}) + \left( q^\delta + q^\delta^{-1} \right) \sum_{r=3}^{N} (-1)^{r-1} \binom{N}{r}
\]

\[+ q^{\delta - 2} \left[ \frac{N_k}{n-k+1} \sum_{r=3}^{n-k+1} (-1)^{r} \binom{n-k+1}{r} + N(a-k) \sum_{r=3}^{k} (-1)^{r} \binom{k+1}{r} + 2 \sum_{r=3}^{N} (-1)^{r-1} \binom{N}{r} \right] \]

Adding \( E_1 - E_2 \) from formulas (28) and (30) to the above expression and simplifying we get:

\[
\gamma(k, n) = O(q^{\delta - 3}) + q^{\delta} \left( \sum_{r=1}^{N} (-1)^{r-1} \binom{N}{r} \right) + q^{\delta - 1} \left( \sum_{r=2}^{N} (-1)^{r-1} \binom{N}{r} \right)
\]

\[+ q^{\delta - 2} \left[ \frac{N^2}{2(n-k+1)(k+1)} \left( k^2 - nk + n + 3 \right) + 2 - \frac{5N}{2} + \frac{N^2}{2} \right] \]

\[= q^{\delta} + (1 - N) q^{\delta - 1} + a_2(k, n) q^{\delta - 2} + O(q^{\delta - 3}) \]

\[\square\]

5. Conclusion

Since the problem of determining the number \( \gamma(k, n; q) \) of \([n, k]\) \( q \)-ary MDS codes is very difficult and complicated, we have studied the problem of determining asymptotic formulae in \( q \) for \( \gamma(k, n; q) \). We have improved the known formula \( \gamma(k, n) = q^\delta - (N - 1)q^{\delta - 1} + O(q^{\delta - 2}) \) by by the formula \( \gamma(k, n) = q^\delta - (N - 1)q^{\delta - 1} + a_2(k, n)q^{\delta - 2} + O(q^{\delta - 3}) \) where \( a_2(k, n) \) is given by \( \delta \). The main tool is a closer study of cardinalities of linear sections of the Grassmannian (Theorems 3.10 and 3.11). The problem of improving this to a formula \( \gamma(k, n) = q^\delta - (N - 1)q^{\delta - 1} + a_2(k, n)q^{\delta - 2} + a_3(k, n)q^{\delta - 3} + O(q^{\delta - 4}) \) is more challenging, and will be studied in future work. On a different note, Theorem 6.10 can be significantly strengthened: In a future work we will show that for an indecomposable \( k \)-form \( \omega \), we in fact have

\[||\omega|| = q^{\delta} + q^{\delta - 2} + O^+(q^{\delta - 3}) \]

where \( O^+(q^{\delta - 3}) \) is positive quantity which is \( O(q^{\delta - 3}) \). This enables us to determine some of the as yet unknown higher weights of the Grassmann code \( C(k, n) \).

References

[1] W.-L. Chow. On the geometry of algebraic homogeneous spaces. Ann. of Math. (2), 50:32–67, 1949.
[2] Sudhir R. Ghorpade and Krishna V. Kaipa. Automorphism groups of Grassmann codes. Finite Fields Appl., 23:80–102, 2013.
[3] Sudhir R. Ghorpade and Gilles Lachaud. Hyperplane sections of Grassmannians and the number of MDS linear codes. Finite Fields Appl., 7(4):468–506, 2001.
[4] P. Griffiths and J. Harris. Principles of algebraic geometry. Wiley-Interscience [John Wiley & Sons], New York, 1978.
[5] Anna V. Iampolskaia, Alexei N. Skorobogatov, and Evgenii A. Sorokin. Formula for the number of \([9, 3]\) MDS codes. IEEE Trans. Inform. Theory, 41(6, part 1):1667–1671, 1995. Special issue on algebraic geometry codes.
[6] D. Yu. Nogin. Codes associated to Grassmannians. In Arithmetic, geometry and coding theory (Luminy, 1993), pages 145–154. de Gruyter, Berlin, 1996.
[7] M. Pankov. Grassmannians of Classical Building. World Scientific, Hackensack, NJ, 2010.
[8] Alexei N. Skorobogatov. Linear codes, strata of Grassmannians, and the problems of Segre. In *Coding theory and algebraic geometry (Luminy, 1991)*, volume 1518 of *Lecture Notes in Math.*, pages 210–223. Springer, Berlin, 1992.

[9] Michael Tsfasman, Serge Vlăduț, and Dmitry Nogin. *Algebraic geometric codes: basic notions*, volume 139 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2007.