An analytic index for Lie groupoids

Paulo Carrillo Rouse

Abstract. For a Lie groupoid there is an analytic index morphism which takes values in the $K$-theory of the $C^*$-algebra associated to the groupoid. This is a good invariant but extracting numerical invariants from it, with the existent tools, is very difficult. In this work, we define another analytic index morphism associated to a Lie groupoid; this one takes values in a group that allows us to do pairings with cyclic cocycles. This last group is related to the compactly supported functions on the groupoid. We use the tangent groupoid to define our index as a sort of ”deformation”.

Mathematics Subject Classification (2000).
Primary 19-06; Secondary 19K56.

Keywords.
Lie groupoids, Tangent groupoid, K-theory, Index theory.

1. Introduction

The concept of groupoid is central in non commutative geometry. Groupoids generalize the concepts of spaces, groups and equivalence relations. In the late 70’s, mainly with the work of Alain Connes, it became clear that groupoids are natural substitutes of singular spaces. Furthermore, Connes showed that many groupoids and algebras associated to them appeared as ‘non commutative analogues’ of smooth manifolds to which many tools of geometry such as K-theory and Characteristic classes could be applied.

One classical way to obtain invariants is through the index theory in the sense of Atiyah-Singer. Given $\mathcal{G} \Rightarrow \mathcal{G}^{(0)}$ a Lie groupoid, it is possible to talk about Pseudodifferential calculus on it (See [2],[4],[6], [7]). Hence, if one has a $\mathcal{G}$-pseudodifferential operator $P$, then one can consider two elements associated to it

- a symbol class $[\sigma_P] \in K^0(A^{\ast}\mathcal{G})$,
- an index $\text{ind}(P) \in K_0(C^\infty_c(\mathcal{G}))$.

In fact, the index of a $\mathcal{G}$-pseudodifferential operator is usually considered as an element of $K_0(C^\ast(\mathcal{G}))$ (in the present work $C^\ast(\mathcal{G})$ stands for the reduced $C^\ast$-algebra) rather than an element of $K_0(C^\infty_c(\mathcal{G}))$. The idea is to take the image of $\text{ind}(P)$ through the $K$-theory morphism

$$K_0(C^\infty_c(\mathcal{G})) \xrightarrow{j} K_0(C^\ast(\mathcal{G}))$$
induced by the inclusion $C^\infty_c(G) \hookrightarrow C^*(\mathcal{G})$. Such an image, noted by $\text{ind}_a(P)$, is called "The analytic index of the Lie groupoid $\mathcal{G}$". The reason for considering the "$C^*$-index" $\text{ind}_a(P)$, is that it is a homotopy invariant of the $\mathcal{G}$-pseudodifferential elliptic operators while the index $\text{ind}(P)$ is not. Even more, the analytic index can be given as a group morphism (using the Connes tangent groupoid for example)

$$\text{ind}_a : K^0(A^*\mathcal{G}) \to K_0(C^*(\mathcal{G}))$$

that fits in the following commutative diagram:

$$\begin{array}{ccc}
\{ \mathcal{G} \text{ - pdoell.}\} & \xrightarrow{\text{ind}} & K_0(C^\infty_c(\mathcal{G})) \\
\downarrow \text{symbol} & & \downarrow j \\
K_0(A^*\mathcal{G}) & \xrightarrow{\text{ind}_a} & K_0(C^*(\mathcal{G})) \\
\end{array}$$

We cannot expect (1) to be injective even if it comes from the canonical inclusion. In [3], Alain Connes gives examples of the non injectivity of (1) and he discusses other reasons why it is not sufficient, in general, to stay with the index at the $C^\infty_c$-level. The analytic index $\text{ind}_a$ is a good invariant, however, extracting numerical invariants from it, with the existent tools, is very difficult. Moreover, roughly speaking, the fact that the morphism in (1) is not injective tells us that in some way we are loosing information.

In the present work we propose an intermediate group $K_0(C^\infty_c(\mathcal{G})) \to K^{h,k}_0(\mathcal{G}) \to K_0(C^*(\mathcal{G}))$ (one for each $k \in \mathbb{N}$) and we construct an "analytic index morphism"

$$\text{ind}^{h,k}_a : K_0(A^*\mathcal{G}) \to K^{h,k}_0(\mathcal{G})$$

that will allow us to obtain more explicit invariants. Let us briefly explain what is this group $K^{h,k}_0(\mathcal{G})$, why we call $\text{ind}^{h,k}_a$ an "analytic index", and why we state that it will allow to obtain more explicit invariants.

(a) The group $K^{h,k}_0(\mathcal{G})$ is simply the group $K_0(C^k_c(\mathcal{G}))$ modulo the following homotopy equivalence relation:

Given $x, y \in K_0(C^k_c(\mathcal{G}))$ we say that $x$ and $y$ are homotopic, $x \sim_h y$, if and only if there exists $z \in K_0(C^k_c(\mathcal{G} \times [0, 1]))$ such that $e_0(z) = x$ and $e_1(z) = y$ where $e_t$ is the morphisms in $K$-theory induced by the respective evaluations. One proves that this gives an equivalence relation and $K_0(C^k_c(\mathcal{G}))/\sim_h$ is a group under the natural operation $[x]_h + [y]_h = [x + y]_h$.

(b) We call $\text{ind}^{h,k}_a$ an "analytic index" because it fits in the following commutative diagram:
An analytic index for Lie groupoids

\[ \{ \mathcal{G} - \text{pdoell.} \} \xrightarrow{\text{ind}} K_0(C_c^\infty(\mathcal{G})) \]

\[ \xrightarrow{\text{symb}} K_0^{h,k}(\mathcal{G}) \]

\[ \xrightarrow{\text{ind}_{h,k}} K_0(A^*\mathcal{G}) \xrightarrow{\text{ind}_a} K_0(C_r^*(\mathcal{G})) \]

(c) We would like to obtain information from \( K \)-theory elements through the "Chern-Weil theories". In the classical case of spaces (manifolds, locally compact spaces, etc.) this is done by a Chern character. Now, Connes developed in the early 80’s a way to generalize this to more general "spaces". In particular he introduced the Cyclic cohomology for general algebras and he defined pairings between \( K \)-theory elements and cyclic cocycles. There is also the notion of Chern character for the non commutative case. Now, the cyclic homology theory is only interesting if it is applied to "smooth" algebras, typically algebras as \( C^\infty(M) \) (for \( M \) a manifold) or \( C_c^\infty(\mathcal{G}) \). In our case, it is possible to pair cyclic cocycles with the index \( \text{ind}(P) \in K_0(C_c^\infty(\mathcal{G})) \) but it is not always possible to extend this action to the \( K_0(C^*(\mathcal{G})) \).

What we are proposing has the advantages of both indices (\( \text{ind} \) and \( \text{ind}_a \)), in one hand it is a well defined group morphism and it defines a homotopy invariant, and on the other hand it is possible to extract numerical invariants using the existent tools.

The article is organized as follows:

In the second section we recall the basic facts about Lie groupoids. We explain very briefly how to define the index \( \text{ind}(P) \in K_0(C_c^\infty(\mathcal{G})) \). For a complete exposition see [7]. In the third section we explain the “deformation to the normal cone” construction associated to an injective immersion. Even if this could be considered as classical material we do it in some detail since we will use in the sequel very explicit descriptions. A particular case of this construction is the tangent groupoid associated to a Lie groupoid. This last example will be fundamental in our construction of the index.

In the fourth section we will construct an algebra of functions over the tangent groupoid that satisfy a rapid decay condition at zero and are compactly supported elsewhere. This algebra is the main point of this work and its \( K \)-theory allows us to define the index \( \text{ind}_{a,k} \) as a “deformation” (in the same spirit as in [3], [5], [6]). The last section is devoted to the construction of the analytic index.

All the results of the present work are part of the author’s PHD thesis. Complete proofs and details may be found in [1] and will be published elsewhere.

I want to thank my PHD advisor, Georges Skandalis, for all the ideas that he shared with me. I would like also to thank him for all the comments and remarks he made to the present work.
2. Lie groupoids

Let us recall what a groupoid is:

**Definition 2.1.** A groupoid consists of the following data: two sets $G$ and $G^{(0)}$, and maps

- $s, r : G \rightarrow G^{(0)}$ called the source and target map respectively,
- $m : G^{(2)} \rightarrow G$ called the product map
  (where $G^{(2)} = \{(\gamma, \eta) \in G \times G : s(\gamma) = r(\eta)\}$),
- $u : G^{(0)} \rightarrow G$ the unit map and
- $i : G \rightarrow G$ the inverse map

such that, if we note $m(\gamma, \eta) = \gamma \cdot \eta, u(x) = x$ and $i(\gamma) = \gamma^{-1}$, we have

1. $\gamma \cdot (\eta \cdot \delta) = (\gamma \cdot \eta) \cdot \delta, \forall \gamma, \eta, \delta \in G$ when this is possible.
2. $\gamma \cdot x = \gamma$ and $x \cdot \eta = \eta, \forall \gamma, \eta \in G$ with $s(\gamma) = x$ and $r(\eta) = x$.
3. $\gamma \cdot \gamma^{-1} = u(r(\gamma))$ and $\gamma^{-1} \cdot \gamma = u(s(\gamma)), \forall \gamma \in G$.
4. $r(\gamma \cdot \eta) = r(\gamma)$ and $s(\gamma \cdot \eta) = s(\eta)$.

Generally, we denote a groupoid by $G \Rightarrow G^{(0)}$ where the parallel arrows are the source and target maps and the other maps are given.

Now, a Lie groupoid is a groupoid in which every set and map involved in the last definition is $C^\infty$ (possibly with borders), and the source and target maps are submersions. For $A, B$ subsets of $G^{(0)}$ we use the notation $G_A^B$ for the subset $\{\gamma \in G : s(\gamma) \in A, r(\gamma) \in B\}$.

All along this paper, $G \Rightarrow G^{(0)}$ is going to be a Lie groupoid.

We recall how to define an algebra structure in $C^\infty_c(G)$ using smooth Haar systems.

**Definition 2.2.** A smooth Haar system over a Lie groupoid consists of a family of measures $\mu_x$ in $G_x^{(0)}$ for each $x \in G^{(0)}$ such that,

- for $\eta \in G_x^{(0)}$ we have the following compatibility condition:
  $$\int_{G_x} f(\gamma) d\mu_x(\gamma) = \int_{G_y} f(\gamma \circ \eta) d\mu_y(\gamma)$$

- for each $f \in C^\infty_c(G)$ the map
  $$x \mapsto \int_{G_x} f(\gamma) d\mu_x(\gamma)$$

belongs to $C^\infty_c(G^{(0)})$.
An analytic index for Lie groupoids

A Lie groupoid always has a smooth Haar system. In fact, if we fix a smooth (positive) section of the 1-density bundle associated to the Lie algebroid we obtain a smooth Haar system in a canonical way. The advantage of using 1-densities is that the measures are locally equivalent to the Lebesgue measure. We suppose for the rest of the paper that the smooth Haar systems are given by 1-densities (for complete details see [8]). We can now define a convolution product on $C^\infty_c(G)$: Let $f, g \in C^\infty_c(G)$, we set

$$(f * g)(\gamma) = \int_{\mathcal{G}_{s}(\gamma)} f(\gamma \cdot \eta^{-1})g(\eta)\mu_{s(\gamma)}(\eta)$$

This gives a well defined associative product.

**Remark 2.3.** There is a way to avoid Haar systems when one works with Lie groupoids, using the half densities (see Connes book [3]).

As we have mentioned in the introduction, we are going to consider some elements in the $K$-theory group $K_0(C^\infty_c(G))$. We recall how this elements are usually defined (See [7] for complete details):

First we recall what a $G$-Pseudodifferentiel operator is:

**Definition 2.4 ($G$-PDO).** A $G$-Pseudodifferential operator is a familiy of pseudodifferential operators $\{P_x\}_{x \in \mathcal{G}(0)}$ acting in $C^\infty_c(G_x)$ such that if $\gamma \in \mathcal{G}$ and $U_{s(\gamma)} : C^\infty_c(\mathcal{G}_{s(\gamma)}) \to C^\infty_c(\mathcal{G}_{r(\gamma)})$ the induced operator, then we have the following compatibility condition

$$P_{r(\gamma)} \circ U_{s(\gamma)} = U_{r(\gamma)} \circ P_{s(\gamma)}$$

There is also a differentiability condition with respect to $x$ that can be found in [7].

Let $P$ be a $\mathcal{G}$-Pseudodifferential elliptic operator. By definition this means that there exists a parametrix, i.e., a $\mathcal{G}$-Pseudodifferential operator $Q$ such that $PQ - 1$ and $QP - 1$ belong to $C^\infty_c(\mathcal{G})$ (where we are identifying $C^\infty_c(\mathcal{G})$ with $\Psi^{-\infty}(\mathcal{G})$ as in [7]). The last data defines a quasi-isomorphism in $(\Psi^\infty_c(C^\infty_c(\mathcal{G}))$ and then an element in $K_0(C^\infty_c(\mathcal{G}))$ that we call the index $\text{ind}(P)$. Similarly to the classical case, a $\mathcal{G}$-PDO operator has a principal symbol that defines an element in the $K$-theory group $K^0(A^*\mathcal{G})$.

### 3. Deformation to the normal cone

Let $M$ be a $C^\infty$ manifold and $X \subset M$ be a $C^\infty$ submanifold. We denote by $\mathcal{N}_X^M$ the normal bundle to $X$ in $M$, i.e., $\mathcal{N}_X^M := T_XM/TX$.

We define the following set

$$\mathcal{G}_X^M := \mathcal{N}_X^M \times 0 \bigcup M \times (0, 1]$$

(2)
The purpose of this section is to recall how to define a $C^\infty$-structure with boundary in $\mathcal{D}^M_X$. This is more or less classical, for example it was extensively used in [5]. Here we are only going to do a sketch.

Let us first consider the case where $M = \mathbb{R}^n$ and $X = \mathbb{R}^p \times \{0\}$ (where we identify canonically $X = \mathbb{R}^p$). We denote by $q = n - p$ and by $\mathcal{D}^n_p$ for $\mathcal{D}^\mathbb{R}_n^p$ as above.

In this case we clearly have that $\mathcal{D}^n_p = \mathbb{R}^p \times \mathbb{R}^q \times [0,1]$ (as a set). Consider the bijection $\Psi : \mathbb{R}^p \times \mathbb{R}^q \times [0,1] \to \mathcal{D}^n_p$ given by

$$\Psi(x, \xi, t) = \begin{cases} (x, \xi, 0) & \text{if } t = 0 \\ (x, t\xi, t) & \text{if } t > 0 \end{cases}$$

which inverse is given explicity by

$$\Psi^{-1}(x, \xi, t) = \begin{cases} (x, \xi, 0) & \text{if } t = 0 \\ (x, \frac{t}{t}\xi, t) & \text{if } t > 0 \end{cases}$$

We can consider the $C^\infty$-structure with border on $\mathcal{D}^n_p$ induced by this bijection.

In the general case, let $(\mathcal{U}, \phi)$ be a local chart in $M$ and suppose it is an $X$-slice, so that it satisfies

1) $\phi : \mathcal{U} \to U \subset \mathbb{R}^p \times \mathbb{R}^q$

2) If $\mathcal{U} \cap X = \mathcal{V}$, $\mathcal{V} = \phi^{-1}(U \cap \mathbb{R}^p \times \{0\})$ (we note $V = U \cap \mathbb{R}^p \times \{0\}$)

With this notation we have that $\mathcal{D}^U_\mathcal{V} \subset \mathcal{D}^n_p$ is an open subset. We may define a function

$$\tilde{\phi} : \mathcal{D}^\mathcal{V}_\mathcal{U} \to \mathcal{D}^U_\mathcal{V}$$

in the following way: For $x \in \mathcal{V}$ we have $\phi(x) \in \mathbb{R}^p \times \{0\}$. If we write $\phi(x) = (\phi_1(x), 0)$, then

$$\phi_1 : \mathcal{V} \to V \subset \mathbb{R}^p$$

is a diffeomorphism, where $V = U \cap (\mathbb{R}^p \times \{0\})$. We set $\tilde{\phi}(u, \xi, 0) = (\phi_1(v), d_N\phi_v(\xi), 0)$ and $\tilde{\phi}(u, t) = (\phi(u), t)$ for $t \neq 0$. Here $d_N\phi_v : N_v \to \mathbb{R}^q$ is the normal component of the derivate $d\phi_v$ for $v \in \mathcal{V}$. It is clear that $\tilde{\phi}$ is also a bijection (in particular it induces a $C^\infty$ structure with border over $\mathcal{D}^\mathcal{V}_\mathcal{U}$).

Now, let us consider an atlas $\{(\mathcal{U}_\alpha, \phi_\alpha)\}_{\alpha \in \Delta}$ of $M$ consisting of $X$-slices. Then we have the following proposition:

**Proposition 3.1.** The collection $\{(\mathcal{D}^\mathcal{U}_\mathcal{V}_\alpha, \tilde{\phi}_\alpha)\}_{\alpha \in \Delta}$ is a $C^\infty$-atlas with border over $\mathcal{D}^M_X$.

**Definition 3.2 (DNC).** Let $X \subset M$ be as above. The set $\mathcal{D}^M_X$ provided with the $C^\infty$ structure with border induced by the atlas described in the last proposition is called “The deformation to the normal cone associated to $X \subset M$”. We will often write DNC instead of Deformation to the normal cone.

**Remark 3.3.** Following the same steps, we can define the deformation to the normal cone associated to an injective immersion $X \hookrightarrow M$. 

---

Paulo Carrillo Rouse
The most important feature about the DNC construction is that it is in some sense functorial. More explicitly, let \((M, X)\) and \((M', X')\) be \(C^\infty\)-couples as above and let \(F : (M, X) \to (M', X')\) be a couple morphism, i.e., a \(C^\infty\) map \(F : M \to M'\), with \(F(X) \subset X'\). We define \(\mathcal{D}(F) : \mathcal{D}_X^M \to \mathcal{D}_X^{M'}\) by the following formulas:

\[
\mathcal{D}(F)(x, \xi, 0) = (F(x), dNF_x(\xi), 0) \quad \text{and} \quad \mathcal{D}(F)(m, t) = (F(m), t) \text{ for } t \neq 0,
\]

where \(dNF_x\) is by definition the map \((\mathcal{A}_X^M)_x \xrightarrow{dNF_x} (\mathcal{A}_X^{M'})_{F(x)}\) induced by \(T_xM \xrightarrow{dF_x} T_{F(x)}M'\).

We have the following proposition.

**Proposition 3.4.** The map \(\mathcal{D}(F) : \mathcal{D}_X^M \to \mathcal{D}_X^{M'}\) is \(C^\infty\).

**Remark 3.5.** If we consider the category \(\mathcal{C}_2^\infty\) of \(C^\infty\) pairs given by a \(C^\infty\) manifold and a \(C^\infty\) submanifold, and pair morphisms as above, we can reformulate the proposition and say that we have a functor \(\mathcal{D} : \mathcal{C}_2^\infty \to \mathcal{C}^\infty\) where \(\mathcal{C}^\infty\) denote the category of \(C^\infty\) manifolds with border.

### 3.1. The tangent groupoid.

**Definition 3.6 (Tangent groupoid).** Let \(\mathcal{G} \Rightarrow \mathcal{G}^{(0)}\) be a Lie groupoid. The tangent groupoid associated to \(\mathcal{G}\) is the groupoid that has \(\mathcal{D}^{\mathcal{G}}\) as the set of arrows and \(\mathcal{G}^{(0)} \times [0, 1]\) as the units, with:

- \(s^T(x, \eta, 0) = (x, 0)\) and \(r^T(x, \eta, 0) = (x, 0)\) at \(t = 0\).
- \(s^T(\gamma, t) = (s(\gamma), t)\) and \(r^T(\gamma, t) = (r(\gamma), t)\) at \(t \neq 0\).
  - The product is given by \(m^T((x, \eta, 0), (x, \xi, 0)) = (x, \eta + \xi, 0)\) et \(m^T((\gamma, t), (\beta, t)) = (m(\gamma, \beta), t)\) if \(t \neq 0\) and if \(r(\beta) = s(\gamma)\).
  - The unit map \(u^T : \mathcal{G}^{(0)} \to \mathcal{G}^T\) is given by \(u^T(x, 0) = (x, 0)\) and \(u^T(x, t) = (u(x), t)\) for \(t \neq 0\).

We denote \(\mathcal{G}^T := \mathcal{D}^{\mathcal{G}}\).

As we have seen above, \(\mathcal{G}^T\) can be considered as a \(C^\infty\) manifold with border. As a consequence of the functoriality of the DNC construction we can show that
the tangent groupoid is in fact a Lie groupoid. Indeed, it is easy to check that if we identify in a canonical way $\mathcal{G}^{(2)}_{\mathcal{G}(0)}$ with $(\mathcal{G}^{(2)})$, then

$$m^T = \mathcal{G}(m), \quad s^T = \mathcal{G}(s), \quad r^T = \mathcal{G}(r), \quad u^T = \mathcal{G}(u)$$

where we are considering the following pair morphisms:

$$m : (\mathcal{G}^{(2)}_{\mathcal{G}(0)}, \mathcal{G}(0)) \rightarrow (\mathcal{G}_{\mathcal{G}(0)}, \mathcal{G}(0)),$$

$$s, r : (\mathcal{G}_{\mathcal{G}(0)}, \mathcal{G}(0)) \rightarrow (\mathcal{G}(0), \mathcal{G}(0)),$$

$$u : (\mathcal{G}(0), \mathcal{G}(0)) \rightarrow (\mathcal{G}_{\mathcal{G}(0)}).$$

Finally, if $\{\mu_x\}$ is a smooth Haar system on $\mathcal{G}$, then, setting

$$\cdot \mu_{(x,0)} := \mu_x \text{ at } (\mathcal{G}^{(2)}_{(x,0)}) = T_x \mathcal{G}$$

and

$$\cdot \mu_{(x,t)} := e^{-t \cdot \mu_x} \text{ at } (\mathcal{G}^{(2)}_{(x,t)}) = \mathcal{G}_x \text{ for } t \neq 0, \text{ where } q = \dim \mathcal{G}_x,$$

one obtains a smooth Haar system for the Tangent groupoid (details may be found in [8]).

4. An algebra for the Tangent groupoid

In this section we will show how to construct an algebra for the tangent groupoid which consist of $C^\infty$ functions that satisfy a rapid decay condition at zero while out of zero they satisfy a compact support condition. This algebra is the main construction in this work.

4.1. Schwartz type spaces for Deformation to the normal cone manifolds. Our algebra for the Tangent groupoid will be a particular case of a construction associated to any deformation to the normal cone.

Before giving the general definition, let us start locally. Let $p, q \in \mathbb{N}$ and $U \subset \mathbb{R}^p \times \mathbb{R}^q$ an open subset, and let $V = U \cap (\mathbb{R}^p \times \{0\})$. We set

$$\Omega^U_V := \{(x, \xi, t) \in \mathbb{R}^p \times \mathbb{R}^q : (x, t \cdot \xi) \in U\},$$

which is an open subset of $\mathbb{R}^p \times \mathbb{R}^q \times [0, 1]$ and is diffeomorphic to $\mathcal{D}_V^U$ via the restriction of the map $\Psi$ used to define the $C^\infty$ structure on $\mathcal{D}_p^n$ (where, as above, $n = p + q$). We can now give the following definition.

**Definition 4.1.** Let $p, q \in \mathbb{N}$ and $U \subset \mathbb{R}^p \times \mathbb{R}^q$ an open subset, and let $V = U \cap (\mathbb{R}^p \times \{0\})$.

1. Let $K \subset U \times [0, 1]$ be a compact subset. We say that $K$ is a *conic compact* subset of $U \times [0, 1]$ relative to $V$ if

$$K_0 = K \cap (U \times \{0\}) \subset V$$
(2) Let $g \in C^\infty(\Omega^U_V)$. We say that $f$ has \textit{conic compact support} $K$, if there exists a conic compact $K$ of $U \times [0, 1]$ relative to $V$ such that if $t \neq 0$ and $(x, t\xi, t) \notin K$ then $g(x, \xi, t) = 0$.

(3) We denote by $\mathcal{S}_{r,c}(\Omega^U_V)$ the set of functions $g \in C^\infty(\Omega^U_V)$ that have compact conic support and that satisfy the following condition:

$$(s_1) \; \forall \; k, m \in \mathbb{N}, \; l \in \mathbb{N}^p \; \text{and} \; \alpha \in \mathbb{N}^a \; \text{there exists} \; C_{(k, m, l, \alpha)} > 0 \; \text{such that}$$

$$
(1 + \|\xi\|^2)^k \|\partial^l_x \partial^m_{\xi} g(x, \xi, t)\| \leq C_{(k, m, l, \alpha)}
$$

Now, the spaces $\mathcal{S}_{r,c}(\Omega^U_V)$ are invariant under diffeomorphisms. More precisely, let $F : U \to U'$ be a $C^\infty$ diffeomorphism where $U \subset \mathbb{R}^p \times \mathbb{R}^a$ and $U' \subset \mathbb{R}^p \times \mathbb{R}^a$ are open subsets. We write $F = (F_1, F_2)$ and we suppose that $F_2(x, 0) = 0$. Then the function $	ilde{F} : \Omega^U_V \to \Omega^{U'}_{V'}$ defined by

$$
\tilde{F}(x, \xi, t) = \begin{cases} 
(F_1(x, 0), \frac{\partial F_2}{\partial x}(x, 0) \cdot \xi, 0) & \text{if } t = 0 \\
(F_1(x, t\xi), \frac{\partial F_2}{\partial t}(x, t\xi), t) & \text{if } t > 0
\end{cases}
$$

is a $C^\infty$ map and one proves the next proposition.

**Proposition 4.2.** Let $g \in \mathcal{S}_{r,c}(\Omega^U_V)$, then $\tilde{g} := g \circ \tilde{F} \in \mathcal{S}_{r,c}(\Omega^{U'}_{V'})$.

With the last compatibility result in hand we are ready to give the main definition in this work.

**Definition 4.3.** Let $g \in C^\infty(\mathcal{D}^M_X)$.

(a) We say that $g$ has \textit{conic compact support} $K$, if there exists a compact subset $K \subset M \times [0, 1]$ with $K_0 := K \cap (M \times \{0\}) \subset X$ (conic compact relative to $X$) such that if $t \neq 0$ and $(m, t) \notin K$ then $g(m, t) = 0$.

(b) We say that $g$ is \textit{rapidly decaying at zero} if for every $(\mathcal{U}, \phi)$ $X$-slice chart and for every $\chi \in C^\infty(\mathcal{U} \times [0, 1])$, the map $g_{\chi} \in C^\infty(\Omega^U_V)$ given by

$$
g_{\chi}(x, \xi, t) = (g \circ \varphi^{-1})(x, \xi, t) \cdot (\chi \circ p \circ \varphi^{-1})(x, \xi, t)
$$

is in $\mathcal{S}_{r,c}(\Omega^U_V)$, where $p$ is the projection $p : \mathcal{D}^M_X \to M \times [0, 1]$ given by $(x, \xi, 0) \mapsto (x, 0)$, and $(m, t) \mapsto (m, t)$ for $t \neq 0$, and $\varphi$ is simply the composition $\mathcal{D}^U_V \xrightarrow{\phi} \mathcal{D}^U_{V'} \xrightarrow{\varphi^{-1}} \Omega^U_{V'}$.

Finally, we denote by $\mathcal{S}_{r,c}(\mathcal{D}^M_X)$ the set of functions $g \in C^\infty(\mathcal{D}^M_X)$ that are rapidly decaying at zero with conic compact support.

We are going to state an important property of the last construction that we use in the next section. First of all, let us recall the notion of Schwartz space associated to a vector bundle.
**Definition 4.4.** Let \((E, p, X)\) be a smooth vector bundle over a \(C^\infty\) manifold \(X\). We define the Schwartz space \(\mathcal{S}(E)\) as the set of \(C^\infty\) functions \(g \in C^\infty(E)\) such that \(g\) is a Schwartz function at each fiber (uniformly) and \(g\) has compact support in the direction of \(X\), i.e., if there exists a compact subset \(K \subset X\) such that \(g(E_x) = 0\) for \(x \notin K\).

We have the following result:

**Proposition 4.5.** The evaluation at zero \(\mathcal{S}_{r,c}(\mathcal{G}_X) \xrightarrow{e_0} \mathcal{S}(\mathcal{M}_X)\) is a surjective linear map.

### 4.2. Schwartz type algebra for the Tangent groupoid

In this section we define an algebra structure on \(\mathcal{S}_{r,c}(\mathcal{G}_T)\). In order to do it, we are going to use the functoriality of the construction \(\mathcal{S}_{r,c}(\mathcal{G}_X)\) that we have seen above.

We start by defining a function \(m_{r,c} : \mathcal{S}_{r,c}(\mathcal{G}_T^{(2)}) \to \mathcal{S}_{r,c}(\mathcal{G}_T)\) by the following formulas:

For \(F \in \mathcal{S}_{r,c}(\mathcal{G}_T^{(2)})\)

\[
m_{r,c}(F)(x, \xi, 0) = \int_{T_x \mathcal{G}_x} F(x, \xi - \eta, \eta, 0) d\mu_x(\eta)
\]

and

\[
m_{r,c}(F)(\gamma, t) = \int_{\mathcal{G}_s(\gamma)} F(\gamma \circ \delta^{-1}, \delta, t) t^{-q} d\mu_s(\gamma)(\delta).
\]

We have the following proposition:

**Proposition 4.6.** \(m_{r,c} : \mathcal{S}_{r,c}(\mathcal{G}_T^{(2)}) \to \mathcal{S}_{r,c}(\mathcal{G}_T)\) is a well defined linear map.

Here we are again identifying \(\mathcal{G}_T^{(2)}\) with \(\mathcal{G}_T^{(2)}\) and the map above is nothing else than the integration along the fibers of \(m_T : (\mathcal{G}_T^{(2)}) \to \mathcal{G}_T\).

We are ready now to define the product in \(\mathcal{S}_{r,c}(\mathcal{G}_T)\).

**Definition 4.7.** Let \(f, g \in \mathcal{S}_{r,c}(\mathcal{G}_T)\), we define a function \(f * g\) in \(\mathcal{G}_T\) by

\[
(f * g)(x, \xi, 0) = \int_{T_x \mathcal{G}_x} f(x, \xi - \eta, \eta, 0) g(x, \eta, 0) d\mu_x(\eta)
\]

and

\[
(f * g)(\gamma, t) = \int_{\mathcal{G}_s(\gamma)} f(\gamma \circ \delta^{-1}, \delta, t) t^{-q} g(\gamma \circ \delta^{-1}, \delta, t) d\mu_s(\gamma)(\delta)
\]

for \(t \neq 0\).

**Proposition 4.8.** The product \(*\) is well defined and associative.

Thanks to the proposition 4.5 we have an exact sequence of algebras

\[
0 \to J \to \mathcal{S}_{r,c}(\mathcal{G}_T) \xrightarrow{e_0} \mathcal{S}(\mathcal{G}_T) \to 0,
\]

where \(J = Ker(e_0)\) by definition.
5. Analytic index of order $k$

This last section is devoted to the construction of the index announced in the introduction. The main reason to construct the algebra $S_{r,c}(G^T)$ is that the "Schwartz algebras" have in general the good $K$-theory groups. For example, we are interested in the symbols of $G$-PDO and more precisely in their homotopy classes in $K$-theory, that is, we are interested in the group $K_0(A^*G) = K_0(C_0(A^G))$. Here it would not be enough to take the $K$-theory of $C_0^\infty(A^G) (\text{see for example } [3])$, however it is enough to consider the Schwartz algebra $S(A^*G)$. Indeed, the Fourier transform shows that this last algebra is stable under holomorphic calculus on $C_0^\infty(A^G)$ and so it has the "good" $K$-theory, meaning that $K_0(A^*G) = K_0(S(A^*G))$. By applying $K$-theory to the exact sequence (3) we obtain

$$K_0(J) \rightarrow K_0(\mathcal{J}_{r,c}(G^T)) \xrightarrow{\epsilon_0} K_0(S(A^*G)) \rightarrow 0.$$  

(4)

The surjectivity of $K_0(\mathcal{J}_{r,c}(G^T)) \xrightarrow{\epsilon_0} K_0(S(A^*G))$ comes from the existence of a Pseudodifferential calculus on $G^T$.

Our main result is the following:

**Theorem 5.1.** There is a well defined group morphism

$$\text{ind}^{h,k}_a : K_0(A^*G) \rightarrow K_0^{h,k}(G)$$

satisfying $\text{ind}^{h,k}_a \circ \epsilon_0 = e_1^{h,k}$ where $\epsilon_0 : K_0(\mathcal{J}_{r,c}(G^T)) \rightarrow K_0(S(A^*G))$ and $e_1^{h,k}$ is the composition

$$K_0(\mathcal{J}_{r,c}(G^T)) \xrightarrow{\epsilon_0} K_0(C^\infty_c(G)) \rightarrow K_0(C^k_c(G)) \xrightarrow{\pi} K_0^{h,k}(G).$$

Moreover, we have a commutative diagram

\[
\begin{array}{c}
\{ \mathcal{G} - \text{pdoell.} \} \xrightarrow{\text{ind}} K_0(C^\infty_c(G)) \\
\downarrow \text{symb} \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quan
The main point of the theorem is that \( \text{ind}_{h,k} \) is well defined, because it will evidently be a group morphism. Hence, in order to prove that it is well defined, we would like to have the following property:

(P) If \( w \in K_0(\mathcal{J}_{r,c}(\mathcal{G}^T)) \) with \( w \in \text{Ker}(e_0) \), then \( e_1^k(w) \sim_h 0 \) in \( K_0(C^*_c(\mathcal{G})) \).

First, consider the canonical projection \( t : \mathcal{G}^T \to [0,1] \). Thanks to the condition about \( \partial \) that we imposed in the definition of \( \mathcal{J}_{r,c}(\mathcal{G}^T) \) we have that (as an easy consequence of the Taylor development)

\[
J = t \cdot \mathcal{J}_{r,c}(\mathcal{G}^T).
\]

We are not going to work directly with \( K_0(J) \) but with the K-theory groups of powers \( J^N \), for \( N \in \mathbb{N} \).

Let \( k \in \mathbb{N} \) and \( q := \text{dim} \mathcal{G}_r \), we define

\[
\varphi_k : J^{k+q} \to C^*_c(\mathcal{G} \times [0,1])
\]

by the formula (expressing \( J^{k+q} = t^{k+q} \cdot \mathcal{J}_{r,c}(\mathcal{G}^T) \)):

\[
\varphi_k(t^{k+q} \cdot f)(\gamma, t) = \begin{cases} 
0 & \text{if } t = 0 \\
t^k f(\gamma, t) & \text{if } t \neq 0
\end{cases}
\]

Then we can prove:

**Proposition 5.3.** With the above definition we have an algebra morphism

\[
\varphi_k : J^{k+q} \to C^*_c(\mathcal{G} \times [0,1])
\]

It satisfies \( e_0 \circ \varphi_k = 0 \) by construction.

We note also by \( \varphi_k : K_0(J^{k+q}) \to K_0(C^*_c(\mathcal{G})) \) the morphism induced in K-theory by \( \varphi_k \). Now, using the fact that \( J/J^N \) is a nilpotent ring we check very easily that the morphism \( K_0(J^N) \xrightarrow{i} K_0(J) \) induced in K-theory by the inclusion \( J^N \hookrightarrow J \) is surjective and so we obtain the following commutative diagram:
We immediately get the desired property (P). Finally, the commutativity of
the diagram cited in the theorem is a consequence of the possibility of doing Pseu-
dodifferential calculus on the tangent groupoid and the fact that the $C^*$-analytic
index is also obtained as a "deformation" through the tangent groupoid, as was
shown by Monthubert and Pierrot in [6].

6. References

[1] Carrillo, P. Indices analytiques pour des groupoïdes de Lie et accouplements avec des
cocycles cycliques. preprint (2006)
[2] Connes, A. Sur la théorie non commutative de l'intégration. Algèbres d'opérateurs
(Sém., Les Plans-sur-Bex) (1978) 19-143.
[3] Connes, A. Non commutative geometry. Academic Press, Inc, San Diego, CA (1994)
[4] Connes, A. and Skandalis, G. The longitudinal index theorem for foliations. In
Publ. Res. Inst. Math. 6 (1984), 1139-1183.
[5] Hilsum, M. and Skandalis, G. Morphismes $K$-orientés d’espaces de feuilles et fon-
citorialité en théorie de Kasparov (d’après une conjecture d’A. Connes) Ann. Sci. École
Norm. Sup. (4) 20 (1987), no. 3, 325-390.
[6] Monthubert, B. and Pierrot, F. Indice analytique et groupoïdes de Lie. C.R. Acad. Sci.
Paris 325 (1997) no.2, 193-198.
[7] Nistor, V., Weinstein, A. and Xu, P. Pseudodifferential operators on differential
groupoids. Pacific J. Math 189 (1999), no.1, 117-152.
[8] Paterson, A. Groupoids and inverse semigroups, and their operator algebras. Progress
in Mathematics, 170 Birkhäuser Boston, Inc., Boston, MA, (1999).

Paulo Carrillo Rouse
Projet d’algèbres d’opérateurs
Université de Paris 7
175, rue de Chevaleret
Paris, France
E-mail: carrillo@math.jussieu.fr