TYPE II BLOW UP SOLUTIONS WITH OPTIMAL STABILITY PROPERTIES FOR THE CRITICAL FOCUSING NONLINEAR WAVE EQUATION ON $\mathbb{R}^{3+1}$

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Abstract. We show that the finite time type II blow up solutions for the energy critical nonlinear wave equation

$$\Box u = -u^5$$

on $\mathbb{R}^{3+1}$ constructed in [26], [25] are stable along a co-dimension one Lipschitz manifold of data perturbations in a suitable topology, provided the scaling parameter $\lambda(t) = t^{-1-\nu}$ is sufficiently close to the self-similar rate, i.e. $\nu > 0$ is sufficiently small. This result is qualitatively optimal in light of the result of [23]. The paper builds on the analysis of [19].

1. Introduction

The critical focusing nonlinear wave equation on $\mathbb{R}^{3+1}$ given by

$$\Box u = -u^5, \Box = -c_t^2 + \Delta,$$  \hspace{1cm} (1.1)

has received a lot of attention recently as a key model for a critical nonlinear wave equation displaying interesting type II dynamics, the latter referring to energy class Shatah-Struwe type solutions $u(t, x)$ which have a priori bounded $H^1$ norm on their life-span $I$, i.e. with the property

$$\sup_{t \in I} \| \nabla_x u(t, \cdot) \|_{L^2} < \infty.$$  \hspace{1cm} (1.2)

Throughout the paper, we shall be interested exclusively in the case of radial solutions. In that case, a rather complete abstract classification theory for type II dynamics in terms of the ground state

$$W(x) = \frac{1}{(1 + |x|^2)^{\frac{3}{2}}}$$

has been developed in [11], see the discussion in [19]. On the other hand, the first ‘non-trivial’ type II dynamics, were constructed explicitly in [24], [26], [25], [5], [7]. As far as finite time type II blow up solutions are concerned, the issue of their stability properties has been shrouded in some mystery. The fact that there is a continuum of blow up rates in the works [26], [25], seemed to suggest that these solutions, and maybe also their analogues for critical Wave Maps and other models,
such as in [27], [28], are intrinsically less stable than ‘generic type II blow ups’, and that the requirement of optimal stability of some sort may in fact single out a more or less unique blow up dynamics for type II solutions. An example of ‘optimally stable’ type II blow up was exhibited in the context of the 4 + 1-dimensional critical NLW in the work [14], see also the brief historical comments in [19]. Note that the linearisation of (1.1) around the ground state $W$ has a unique unstable eigenmode $\phi_d$, and in accordance with this, [14] exhibits a co-dimension one manifold of data perturbations of $W$ (in the $4 + 1$-dimensional context) resulting in the stable blow up.

In this article we show that the solutions constructed in [26], [25], corresponding to $\lambda(t) = t^{-1-\nu}$ and with $\nu > 0$ small enough are also optimally stable in a suitable sense. In fact, from [23], it is known that any type II solution with data close to the ground state $W$ can be at best stable for perturbations of the data along a co-dimension one hyper surface in energy space. Now let $\phi_d$ be the unique positive $L^2$-normalized unstable eigenmode of the operator $\mathcal{L} := -\Delta - 5W^4$, restricted to operating on radial functions. We have

**Theorem 1.1.** Let $\nu_0 > 0$ be small enough. Then for any $0 < \nu \leq \nu_0$, there is a finite time type II blow up solution of the form

$$u(t, x) = W_{\lambda(t)}(x) + \epsilon(t, x), \lambda(t) = t^{-1-\nu}, \left(\epsilon(t, \cdot), \epsilon_t(t, \cdot)\right) \in H^{1+\frac{2}{\nu}-} \times H^{\frac{2}{\nu}-},$$

on a sufficiently small time interval $(0, t_0)$, $t_0 > 0$, constructed as in [26], [25] with suitable parameters there, and such that the following holds: there is a suitable Banach space $\tilde{S}$ (consisting of pairs of functions in $\phi_d^\perp$) with associated norm $\| \cdot \|_{\tilde{S}}$ (the same one as in [19]), such that for a suitable $\delta_1 = \delta_1(\nu) > 0$ small enough and $B_{\delta_1}$ the $\delta_1$-vicinity of $(0, 0) \in \tilde{S} \times \mathbb{R}$, there exists a Lipschitz function $\gamma_1 : B_{\delta_1} \to \mathbb{R}$, such that for any triple $(\epsilon_0, \epsilon_1, \gamma) \in B_{\delta_1}$, the initial data

$$u(0) + (\epsilon_0 + \gamma \phi_d, \epsilon_1 + \gamma \epsilon_0(\epsilon_0, \gamma) \phi_d)$$

lead to a type II blow up solution on $(0, t_0]$ of the form

$$\tilde{u}(t, \cdot) = W_{\tilde{\lambda}}(x) + \tilde{\epsilon}(t, \cdot), \left(\tilde{\epsilon}(t, \cdot), \tilde{\epsilon}_t(t, \cdot)\right) \in H^{1+\frac{2}{\nu}-} \times H^{\frac{2}{\nu}-},$$

where

$$\lim_{t \to 0} \frac{\tilde{\lambda}(t)}{\lambda(t)} = 1, \lim_{t \to 0} \int_{|x| \leq t} |\nabla_x \tilde{\epsilon}(t, \cdot)|^2 \, dx = 0.$$
Remark 1.2. We note that the technique developed in this paper as well as its precursor [19] should in no way only apply to the ‘rough kind’ of blow up solutions constructed here. In fact, the limited smoothness is simply a consequence of our choice of approximate solutions underlying these examples, and which have their origin in [25], [26]. In fact, given any family of approximate solutions rich enough to allow for an argument like the one below to fulfill the two vanishing conditions pivotal in [19], and with monotone scaling factor close enough to \( r^1 \), the same type of argument ought in principle to apply. This appears to furnish a method building stable blow up solutions of very limited regularity without any recourse to Morawetz/virial type identities.

The method of proof builds directly on the prequel [19], in which a conditional stability result was proved (with two additional co-dimensions imposed). More precisely, the result there shows that imposing a suitable co-dimension two condition on the perturbation \((\epsilon_0, \epsilon_1, \gamma)\) suffices to obtain blow up solutions with scaling parameter unchanged, i.e., \( \tilde{\lambda}(t) = \lambda(t) = r^{-1-\gamma} \) and \( \nu \) sufficiently small. The strategy of [19] can be roughly summarised as follows:

Letting \( u_\nu(t,x) \) with \( \nu > 0 \) sufficiently small be one of the blow up solutions constructed in [25], the goal is to build a perturbed solution of the form \( u(t,x) = u_\nu(t,x) + \epsilon(t,x) \) with \( \epsilon(t,x) \) small in a suitable sense, all on the same time interval \((0,t_0]\) on which \( u_\nu \) exists. To control \( \epsilon \), a translation to the Fourier side is effected, where the Fourier transform is with respect to the Schrödinger operator \( \mathcal{L} := -\Delta - 5W^4 \), where \( W \) denotes as usual the ground state. More precisely, introducing the variables \( R = \lambda(t)r, \tau = \int_0^t \lambda(s)\, ds \), and \( \tilde{\epsilon} = R\epsilon \), one first infers the equation

\[
(\partial_\tau + \tilde{\lambda}^{-1}R\partial_R)^2\tilde{\epsilon} - \beta_\nu(\tau)(\partial_\tau + \tilde{\lambda}^{-1}R\partial_R)\tilde{\epsilon} + \mathcal{L}\tilde{\epsilon} = \lambda^{-2}(\tau)RN_0(\epsilon) + \partial_\tau(\tilde{\lambda}^{-1})\tilde{\epsilon}; \beta_\nu(\tau) = \tilde{\lambda}(t)\lambda^{-1}(t),
\]

(1.2)

For this see (4.1) in [19]. Here \( \lambda(t) = r^{-1-\gamma} \), but in fact this formalism remains valid for arbitrary scaling laws. The strategy then is to express \( \tilde{\epsilon}(\tau,R) \) in terms of its distorted Fourier transform

\[
\tilde{\epsilon}(\tau,R) = x_d(\tau)\phi_d(R) + \int_0^\infty x(\tau,\xi)\phi(R,\xi)d\xi
\]

and derive a system for the Fourier coefficients \( x_d(\tau), x(\tau,\xi) \). This happens to be of the form

\[
(D^2_\tau + \beta_\nu(\tau)D_\tau + \xi^2)\hat{x}(\tau,\xi) = R(\tau,\xi) + f(\tau,\xi),
\]

(1.3)

where the operator \( D_\tau \) is essentially given by \( \partial_\tau - 2\beta_\nu(\tau)\xi\partial_\xi \), \( x \) is vector valued (containing both discrete and continuous spectral part), and \( R(\tau,\xi) \) stands for certain non-local integral operators, while \( f \) is the (distorted) Fourier transform of all the non-linear source terms, see (4.7) of [19]. The first step then is to consider the free transport equation

\[
(D^2_\tau + \beta_\nu(\tau)D_\tau + \xi^2)\hat{x}(\tau,\xi) = 0.
\]
and to infer conditions on the data such that its solutions don’t grow too fast in a suitable sense. This is accomplished in Lemma 2.1 and Proposition 3.1 in [19], resulting in a co-dimension one condition to prevent exponential growth from the unstable mode of \( L \) (a condition which is also reflected in Theorem [1.1]), as well as two additional vanishing conditions on the continuous spectral part of the data, \((x_0, x_1)\), and given by the formulae

\[
\begin{align*}
\int_0^{\infty} \frac{\rho^\frac{1}{2}(\xi)}{\xi^\frac{1}{2}} \cos[\nu \tau_0 \xi^\frac{1}{2}] d\xi &= 0, \\
\int_0^{\infty} \frac{\rho^\frac{1}{2}(\xi)}{\xi^\frac{1}{2}} \sin[\nu \tau_0 \xi^\frac{1}{2}] d\xi &= 0.
\end{align*}
\]

(1.4)

Here \( \tau_0 = \tau(t_0) \) is the initial time with respect to the re-scaled variables. These two ensure that the norm \( \|\tilde{z}_{(\tau, R)}\|_{L_\infty} \) (with \( \tilde{e} \) the function corresponding to Fourier transform \( \lambda \)) only grows linearly in time. Replacing \( \lambda(t) = t^{-1-\nu} \) by a more general scaling law means simply replacing the above vanishing conditions by the following analogous ones (where \( \lambda \) needs to be expressed as a function of the renormalised time variable \( \tau \))

\[
\begin{align*}
\int_0^{\infty} \frac{\rho^\frac{1}{2}(\xi)}{\xi^\frac{1}{2}} \cos[\lambda(\tau_0) \xi^\frac{1}{2}] \int_{\tau_0}^{\infty} \lambda^{-1}(u) du \ d\xi &= 0, \\
\int_0^{\infty} \frac{\rho^\frac{1}{2}(\xi)}{\xi^\frac{1}{2}} \sin[\lambda(\tau_0) \xi^\frac{1}{2}] \int_{\tau_0}^{\infty} \lambda^{-1}(u) du \ d\xi &= 0.
\end{align*}
\]

(1.5)

It was suggested in [19] that one may be able to force these two vanishing conditions by replacing \( \lambda(t) = t^{-1-\nu} \) by a suitably generalised scaling law, depending on two additional parameters. This we shall do in the next section. The key shall be to obtain a more general class of approximate blow up solutions \( u_{\gamma_1, \gamma_2}(t, x) \), constructed using the inductive ’renormalisation procedure’ of [26], [25], and depending on two parameters \( \gamma_1, \gamma_2 \). It is important to note here that we cannot use time or scaling invariance directly to force the two vanishing conditions. This is because one thereby replaces the profile of \( u_{\nu}(t, x) \) by one which is infinitely far removed in terms of the \( \|\cdot\|_3 \)-norm. In some sense, the ’shock behaviour along the light cone’ inherent in the solutions \( u_\nu \) (which gets more pronounced the smaller \( \nu > 0 \) is) results in a certain amount of rigidity of these solutions, forcing even suitably perturbed solutions to blow up in the same space time location.

2. CONSTRUCTION OF A TWO-PARAMETER FAMILY OF APPROXIMATE BLOW UP SOLUTIONS

Our goal here shall be the construction on \([0, t_0], 0 < t_0 \ll 1\), of approximate blow up solutions for \( \Box u = -u^5 \) of the form

\[
u_{\lambda}(t, x) = W_{\lambda(t)}(x) + \epsilon(t, x),
\]

where we have the asymptotic relation

\[
\lim_{t \to 0} \frac{\lambda(t)}{t^{1-\nu}} = 1,
\]

(2.1)
and \( \nu > 0 \), and such that \( \lambda(t) = \lambda_{\gamma_1, \gamma_2}(t) \) depends smoothly on two small parameters \( \gamma_{1,2} \in \mathbb{R} \) in such fashion that the mapping

\[
(\gamma_1, \gamma_2) \rightarrow u_{\lambda_{\gamma_1, \gamma_2}}[t_0]
\]

is 'non-degenerate' in the following precise sense: introduce the map

\[
\Phi(\gamma_1, \gamma_2) := (A(\gamma_1, \gamma_2), B(\gamma_1, \gamma_2)),
\]

with

\[
A(\gamma_1, \gamma_2) := \int_0^{\infty} x_1^{(\gamma_1,2)}(\xi) \frac{\rho^{\frac{1}{2}}(\xi)}{\xi^{\frac{1}{2}}} \sin[\lambda_{\gamma_1, \gamma_2}(\tau_0^{(\gamma_1,2)}) \xi] \frac{1}{\xi} \lambda_{\gamma_1, \gamma_2}^{-1}(u) \; du \; d\xi,
\]

\[
B(\gamma_1, \gamma_2) := \int_0^{\infty} x_0^{(\gamma_1,2)}(\xi) \frac{\rho^{\frac{1}{2}}(\xi)}{\xi^{\frac{1}{2}}} \cos[\lambda_{\gamma_1, \gamma_2}(\tau_0^{(\gamma_1,2)}) \xi] \frac{1}{\xi} \lambda_{\gamma_1, \gamma_2}^{-1}(u) \; du \; d\xi,
\]

and furthermore

\[
\tau_0^{(\gamma_1,2)} := \int_0^{\infty} \lambda_{\gamma_1, \gamma_2}(s) \; ds,
\]

while \( x_{0,1}^{(\gamma_1,2)} \) are the distorted Fourier coefficients of the spatial truncated data

\[
\chi_{r \in C_0} u_{\lambda_{\gamma_1, \gamma_2}}[I_0],
\]

as detailed in [19]. Then we need to ensure that \( \Phi \) is locally invertible around \( (\gamma_1, \gamma_2) = (0, 0) \). We shall now construct such a family of blow up solutions, restricting to \( 0 < \nu \leq \frac{1}{4} \), say. In fact, we shall stipulate the following ansatz for \( \lambda_{\gamma_1, \gamma_2}(t) \), where \( \langle x \rangle := \sqrt{1 + x^2} \):

\[
\lambda_{\gamma_1, \gamma_2}(t) = \left( 1 + \gamma_1 \cdot \frac{\rho^{1}}{\langle \rho \rangle^{\nu}} + \gamma_2 \log t \cdot \frac{\rho^{1}}{\langle \rho \rangle^{\nu}} \right)^{-1},
\]

which obviously satisfies (2.1). Here \( N \gg 1 \) is sufficiently large. The intuition here is that we replace the precise power law \( \lambda(t) = t^{-1-\nu} \) by one of the form

\[
C(t) t^{-1-\nu(t)},
\]

and impose \( \lim_{t \to 0} C(t) = 1 \), \( \lim_{t \to 0} \nu(t) = \nu \). In fact, these changed scaling functions are still monotone for small \( t_0 \). To assure the convergence of the integral in the definition of \( \tau_0^{(\gamma_1,2)} \) we cannot allow \( C(t) \) to grow too fast for large \( t \). However since \( 0 < t < t_0 \), up to error of high order, we in fact have

\[
\lambda_{\gamma_1, \gamma_2}(t) \approx \left( 1 + \gamma_1 \cdot \rho^{1} + \gamma_2 \log t \cdot \rho^{1} \right)^{-1}, \quad k_0 = \left[ N \nu^{-1} \right],
\]

The goal now is to apply the procedure in [26] leading to an approximate blow up solution to the preceding scaling function, and carefully analyse the dependence on \( \gamma_{1,2} \) of the resulting function, as well as the non-degeneracy of \( \Phi \).
2.1. Construction of an approximate solution with scaling law \( \lambda_{\gamma_1, \gamma_2}(t) \). In analogy to \([26], [25]\), we prove here the following result

**Theorem 2.1.** For fixed \( \gamma_{1,2} \), \( N \) as above and \( k_s = \left[ \frac{1}{2} N \nu^{-1} \right] \), there exists an approximate solution \( u_{\text{approx}} = u_{\text{approx}}^{(\gamma_{1,2})} \) for \( \Box u = -u^5 \) of the form (putting \( \lambda(t) := \lambda_{\gamma_{1,2}}(t) \) for simplicity)

\[
u_{\text{approx}}^{(\gamma_{1,2})} = \lambda_{\gamma_{1,2}}^1(t)[W(R) + \frac{c}{(At)^2}R^2(1 + R^2)^{-\frac{1}{2}} + O((At)^{-2} \log RR^2(1 + R^2)^{-\frac{1}{2}})],\]

such that the corresponding error

\[
u_{\text{approx}}^{(\gamma_{1,2})} = \Box \nu_{\text{approx}} + \nu_{\text{approx}}^5
\]

is of the form

\[
u^2 \nu_{\text{approx}} = \left[ |\gamma_1| + |\gamma_2| \right]
\[
O\left( \log t \frac{\nu_{\gamma_1}^1 R}{(At)^{k_s+1}} (1 + (1 - a)^{1+\nu_2}) \right)
\]

\[
+ O\left( \log t \frac{\nu_{\gamma_1}^1 R}{(At)^{k_s+2}} R^{-1} (1 + (1 - a)^{1+\nu_2}) \right)\]

and such that this relation may be formally differentiated. We use the notation \( R = \lambda t, a = \frac{2}{\nu} \). Furthermore, writing \( u_{\text{approx}}^{(\gamma_{1,2})} = u_{\text{approx}}^{(\gamma_{1,2})}(t, r, \gamma_{1,2}, \nu) \) we have the \( \gamma \)-dependence

\[
u_{\gamma_1} \nu_{\text{approx}}^{(\gamma_{1,2})} = O(t^{k_0} \nu^{\frac{1}{2}} \frac{R}{(At)^2}),
\]

with symbol type behaviour with respect to the \( \nu_{t,r} \) derivatives up to order two, and similarly for

\[
u_{\gamma_2} \nu_{\text{approx}}^{(\gamma_{1,2})} = O(t^{k_0} \log t \nu^{\frac{1}{2}} \frac{R}{(At)^2}),
\]

**Remark 2.1.** The key point here is the last part, which ensures that the \( \gamma \) dependent part of the solutions \( u_{\text{approx}}^{(\gamma_{1,2})} \) is smoother than the solutions themselves (they are only of class \( H^{1+\nu_1+\nu_2} \) regularity).

**Remark 2.2.** Observe from the preceding construction that \( e_{\text{approx}} = 0 \) provided \( \gamma_1 = \gamma_2 = 0 \). Thus in that case the function \( u_{\text{approx}}^{(0,0)} \) is an exact solution.

**Proof:** This follows closely the iterative schemes in \([26], [25]\), and exploits a certain flexibility in this scheme. The key point is the realisation that for the singular corrections improving the accuracy near the light cone, one may in fact utilise the leading singular behaviour

\[
u_{\gamma_1}^1(t) \frac{R_{0,0}(t)}{(A_{0,0}(t) \cdot t)^2} (1 - a)^{1+\nu_2}
\]

where we put \( R_{0,0} = t^{-1-\nu_2}, A_{0,0}(t) := t^{-1-\nu}, \) and \( c \) is a constant independent of \( \gamma_{1,2} \). We observe that any \( \gamma \)-dependence of this leading singularity would destroy
the strategy of this paper, as it would lead to perturbations of too rough character.

From now on we shall write \( u_{2k-1}^{(\gamma, 1, 2)} = u_{2k-1} \) for simplicity’s sake, keeping the parameters \( \gamma_{1, 2} \) fixed throughout. We construct \( u_{2k-1} = u_0 + \sum_{l=1}^{2k-1} v_l, u_0 = \lambda^2 W(R) \), as in [25] via a sequence of corrections, paralleling the steps there except that for us we use the scaling factor \( \lambda(t) = \lambda_{\gamma_{1, 2}}(t) \) for the main bulk term, while we stick to \( \lambda_{0,0}(t) = t^{-\gamma} \) to define the corrections \( v_j \).

Define the algebra of functions \( Q \) exactly as in [25], upon having fixed the parameter \( \nu \). Similarly, the space \( Q' \) is defined as in [25] via

\[
Q' = a^{-1}\delta_0 Q.
\]

Then almost as in [25] one introduces the function algebras \( S^{m}(R_{0,0}^k (\log R_{0,0})^l, Q) \) except that in addition to the variable \( b(t) = \mu_{0,0}(t)^{-1}, \mu_{0,0}(t) = \lambda_{0,0}(t) \) \( \cdot \) \( t \), we introduce an additional variable \( b_1 \), which will represent \( \frac{\log t}{\mu_{0,0}(t)} = \frac{\log t}{\ell_{0,0}(t)} \). Thus we use

**Definition 2.1.** (a) \( S^{m}(R_{0,0}^k (\log R_{0,0})^l, Q) \) is the class of analytic functions

\[
v : [0, \infty) \times [0, 1] \times [0, b_0] \times [0, b_0] \longrightarrow \mathbb{R}
\]

such that

- \( v \) is analytic as a function of \( R_{0,0}, b, b_1 \) and \( v : [0, \infty) \times [0, b_0] \times [0, b_0] \longrightarrow Q \).
- \( v \) vanishes of order \( m \) relative to \( R \), and \( R^{-m} v \) has an even Taylor expansion at \( R_{0,0} = 0 \).
- \( v \) has a convergent expansion at \( R_{0,0} = +\infty \).

\[
v(R_{0,0}, a, b, b_1) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_{ij}(a, b, b_1) R_{0,0}^{k-i} (\log R_{0,0})^j
\]

where the coefficients \( c_{ij}(\cdot, b) \in Q \) and \( c_{ij}(a, b, b_1) \) are analytic in \( b, b_1 \in [0, b_0] \) for all \( 0 \leq a \leq 1 \).

(b) \( IS^{m}(R_{0,0}^k (\log R_{0,0})^l, Q) \) is the class of analytic functions \( w \) on the cone \( C_0 \) which can be represented as

\[
w(r, t) = v(R_{0,0}, a, b, b_1), v \in S^{m}(R_{0,0}^k (\log R_{0,0})^l, Q), b = \frac{1}{\mu_{0,0}(t)}, b_1 = \frac{\log t}{\mu_{0,0}(t)}, \mu_{0,0}(t) = t \cdot \lambda_{0,0}(t).
\]

(c) Denote by \( Q_{\text{smooth}} \) the algebra of continuous functions \( q : [0, 1] \longrightarrow \mathbb{R} \) with the following properties:

- \( q \) is analytic in \( [0, 1] \) with an even expansion at 0 and with \( q(0) = 0 \).
- Near \( a = 1 \) we have an expansion of the form

\[
q(a) = q_0(a) + \sum_{i=1}^{\infty} (1 - a)^{\beta(i)+1} \sum_{j=0}^{\infty} q_{ij}(a) (\log (1 - a))^j
\]
with analytic coefficients \( q_0, q_{ij} \). The \( \beta(i) \) are of the form

\[
\sum_{k \in K, k \geq [Nv^{-1}]} a_k ((k - \frac{1}{2}) \nu - \frac{1}{2})
\]

where \( K \) consist of finite sets of natural numbers and \( a_k \in \mathbb{N} \). Only finitely many of the \( q_{ij}(a) \) are non-zero.

Then define \( S^m(R_{0,0}^k (\log R_{0,0})^l, Q_{\text{smooth}}), IS^m(R_{0,0}^k (\log R_{0,0})^l, Q_{\text{smooth}}) \) as in (a), (b) above. We shall also use the notation \( IS^m(R_{0,0}^k (\log R_{0,0})^l) \) to denote functions analytic in \( b, b_1, R_{0,0} \) with the indicated vanishing and decay properties.

We emphasise that throughout we set

\[
R_{0,0} = \lambda_{0,0}(t) r = t^{-1-\nu} r,
\]

which corresponds to the variable \( R \) used in [25]. We shall reserve the variable \( R \) later on for \( R = \lambda(t) \cdot r \), which will then be dependent on \( \gamma_{1,2} \). The theorem will then be a consequence of the following

**Lemma 2.2.** For any \( k_* := \left\lfloor \frac{1}{2} Nv^{-1} \right\rfloor \geq k \geq 1 \) there exist corrections \( v_{2k}, v_{2k-1} \) such that the approximations \( u_{2k-1} = u_0 + \sum_{j=1}^{2k-1} v_j, u_{2k} = u_0 + \sum_{j=1}^{2k} v_j \) generate errors \( e_{2k-1}, e_{2k} \) as below:

\[
v_{2k-1} = \frac{1}{\mu_{0,0}(t)^{2k}} IS^2(R_{0,0} (\log R_{0,0})^{m_k}, Q)
\]

\[
t^2 e_{2k-1} = \frac{1}{\mu_{0,0}(t)^{2k}} IS^0(R_{0,0} (\log R_{0,0})^{m_k}, Q')
\]

\[
v_{2k} = \frac{1}{\mu_{0,0}(t)^{2k+2}} IS^2(R_{0,0}^3 (\log R_{0,0})^{m_k}, Q)
\]

\[
t^2 e_{2k} = \frac{1}{\mu_{0,0}(t)^{2k+2}} [IS^0(R_{0,0}^{-1} (\log R_{0,0})^{q_k}, Q) + b^2 IS^0(R_{0,0} (\log R_{0,0})^{q_k}, Q')]
\]
Here the functions \(v_{2k-1}, v_{2k}\) are independent of \(\gamma_{1,2}\), but not the errors \(e_{2k-1}, e_{2k}\). Furthermore, we may pick two more corrections \(v_{\text{smooth},1}, v_{\text{smooth},2}\), such that

\[
\begin{align*}
\tilde{v}_{\gamma_1, v_{\text{smooth},1}} &\in \frac{\lambda^\frac{1}{2}_{0,0}}{\mu_{0,0}(t)k_0+2} IS^2 (R_{0,0}, Q_{\text{smooth}}), \\
\tilde{v}_{\gamma_2, v_{\text{smooth},1}} &\in \log t \frac{\lambda^\frac{1}{2}_{0,0}}{\mu_{0,0}(t)k_0+2} IS^2 (R_{0,0}, Q_{\text{smooth}}), \\
\tilde{v}_{\gamma_1, v_{\text{smooth},2}} &\in \frac{\lambda^\frac{1}{2}_{0,0}}{\mu_{0,0}(t)k_0+4} IS^2 (R^3_{0,0}, Q_{\text{smooth}}), \\
\tilde{v}_{\gamma_2, v_{\text{smooth},2}} &\in \log t \frac{\lambda^\frac{1}{2}_{0,0}}{\mu_{0,0}(t)k_0+4} IS^2 (R^3_{0,0}, Q_{\text{smooth}}),
\end{align*}
\]

such that the final error generated by \(u_{\text{prelim}} := u_0 + \sum_{j=1}^{2k_*-1} v_j + \sum_{a=1,2} v_{\text{smooth},a}\) satisfies

\[
t^2 \tilde{e}_{\text{prelim}} := t^2 (\Box u_{\text{prelim}} + u'^5_{\text{prelim}})
\]

\[
\in \gamma_1 \frac{\lambda^\frac{1}{2}_{0,0}}{\mu_{0,0}(t)k_0+2} \left[ IS^0 (R_{0,0}^{-1}, Q) + b^2 IS^0 (R_{0,0}, Q) \right]
\]

\[
+ \gamma_2 \log t \frac{\lambda^\frac{1}{2}_{0,0}}{\mu_{0,0}(t)k_0+2} \left[ IS^0 (R_{0,0}^{-1}, Q) + b^2 IS^0 (R_{0,0}, Q) \right] + t^2 \tilde{e}_{\text{prelim}},
\]

where the remaining error \(t^2 \tilde{e}_{\text{prelim}}\) does not depend on \(\gamma_{1,2}\) and resides in \(IS^0 (R_{0,0}, (\log R_{0,0})e_{k_*}, Q')\).

**Proof.** We follow closely the procedure in [25], section 2. The only novelty is that we perturb around \(u_0 = \lambda^\frac{1}{2}(t) W(\lambda(t)r)\) as opposed to \(\lambda^\frac{1}{2}_{0,0}(t) W(\lambda(0,t)r)\), which will generate additional error terms during the construction of the \(v_j\), \(1 \leq j \leq 2k_* - 1\). We relegate these to the end of the procedure, and use the final two corrections \(v_{\text{smooth},a}\) to decimate this remaining error, leaving only \(e_{\text{prelim}}\).

**Step 0:** We put \(u_0(t, r) = \lambda^\frac{1}{2}(t) W(R), R = \lambda(t)r, \lambda(t) = \lambda_{\gamma_1, \gamma_2}(t)\). Then (with \(D = f + R\partial_R\))

\[
e_0 := L_{\text{quintic}} u_0 = \lambda^\frac{1}{2}(t) \left[ \left( \frac{\lambda'}{\lambda} \right) (D^2 W)(R) + \left( \frac{\lambda'}{\lambda} \right)' (D W)(R) \right]
\]

\[
r^2 e_0 := \lambda^\frac{1}{2}_{0,0}(t) \left[ \omega_1 \frac{1 - R^2_{0,0}/3}{(1 + R^2_{0,0}/3)^2} + \omega_2 \frac{9 - 30R^2_{0,0} + R^4_{0,0}}{(1 + R^2_{0,0}/3)^2} \right] + e_0 =: t^2 e_0^0 + e_0
\]
where we have
\[
\epsilon_0 \in \gamma_1 \frac{\lambda_{0,0}^{\frac{1}{2}}}{\mu_{0,0}(t)^{k_0}} I S^{0}(R_{0,0}^{-1}) + \gamma_2 \frac{\lambda_{0,0}^{\frac{1}{2}}}{\mu_{0,0}(t)^{k_0}} \log I S^{0}(R_{0,0}^{-1})
\]

Further, importantly the constants \(\omega_{1,2}\) do not depend on \(\gamma_{1,2}\). We shall then treat \(\epsilon_0\) as a lower order error which can be neglected in the first \(k_0\) stages of the iteration process.

**Step 1** Here we choose the first correction \(v_1\) exactly as in section 2.3 in [25]. Introduce the operator
\[
L_0 := \frac{\partial_{R_0,0}^2 \lambda_{R_0,0}^{\frac{1}{2}}}{\lambda_{R_0,0}^{\frac{1}{2}}} + \frac{2}{R_0,0} \partial_{R_0,0} + 5W^4(R_{0,0})
\]
Then we solve
\[
\gamma_2^2 \partial_{R_0,0}(t) v_1 = t^2 \epsilon_0^2, \quad v_1(0) = v'_1(0) = 0
\]
Following the method in [25], we infer that
\[
v_1(t, r) = \lambda_{0,0}^{\frac{1}{2}}(t) \mu_{0,0}^{-2}(t) \left( \omega_1 f_1(R_{0,0}) + \omega_2 f_2(R_{0,0}) \right) =: \lambda_{0,0}^{\frac{1}{2}}(t) \mu_{0,0}^{-2}(t) f(R_{0,0})
\]
where further
\[
f_j(R_{0,0}) = R_{0,0} \left( b_{1j} + b_{2j} R_{0,0}^{-1} + R_{0,0}^{-2} \log R_{0,0} \varphi_1(R_{0,0}^{-2}) + R_{0,0}^{-2} \varphi_2(R_{0,0}^{-1}) \right)
\]
\[
= R_{0,0} \left( F_j(\rho) + \rho^2 G_j(\rho^2) \log \rho \right)
\]
where \(\varphi_{1j}, \varphi_{2j}\) and \(F_j, G_j\) are analytic around zero, with \(\rho := R_{0,0}^{-1}\). Moreover, the coefficients of these analytic functions do not depend on \(\gamma_{1,2}\).

**Step 2** Here we analyse the error \(e_1\) generated by the approximate solution \(u_1 = u_0 + v_1\), which equals
\[
e_1 = \partial_{v_1}^2 v_1 - 10u_0^5 v_1^2 - 10u_0^2 v_1^3 - 5u_0 v_1^4 - v_1^5 + 5 \lambda_{0,0}^2(t) \left( \frac{\lambda^2(t)}{\lambda_{0,0}^2(t)} W^4(R) - W^4(R_{0,0}) \right) v_1
\]
\[
+ \epsilon_0.
\]
\[
(2.19)
\]
Inserting the preceding formula for $v_1(t, R_{0,0})$, this becomes

$$
t^2 e_1 = \lambda_0^2(t) \mu_0^2(t) \left( 10 W^3(R_{0,0}) f^2(R_{0,0}) + 10 W^2(R_{0,0}) \mu_0^2(t) f^3(R_{0,0}) \right)
$$

$$
+ 5 W(R_{0,0}) \mu_0^2(t) f^5(R_{0,0}) + \mu_0^3(t) f^5(R_{0,0})
$$

$$
+ \lambda_{0,0}^2(t) \left( \left( \t + \frac{\lambda_0^2(t)}{\lambda_{0,0}^2(t)} D \right)^2 - \left( \t + \frac{\lambda_0(t)}{\lambda_{0,0}^2(t)} D \right) \right) w_1(t, R_{0,0})
$$

(2.20)

$$
+ 5 \mu_0^2(t) \left( \lambda_0^2(t) W^4(R) - W^4(R_{0,0}) \right) v_1
$$

where $w_1(t, R_{0,0}) = \mu_0^2(t) f(R_{0,0})$. Observe that we have the following identity for the last line

$$
\lambda_{0,0}^2(t) \left( \left( \t + \frac{\lambda_0^2(t)}{\lambda_{0,0}^2(t)} D \right)^2 - \left( \t + \frac{\lambda_0(t)}{\lambda_{0,0}^2(t)} D \right) \right) w_1(t, R_{0,0})
$$

(2.21)

$$
= \lambda_{0,0}^2(t) \mu_0^2(t) \left[ (2 \nu - (1 + \nu) D) - (2 \nu - (1 + \nu) D) \right] f(R_{0,0}).
$$

On the other hand, for the principal term we may write

$$
\lambda_{0,0}^2(t) \mu_0^2(t) \left[ (2 \nu - (1 + \nu) D) - (2 \nu - (1 + \nu) D) \right] f(R_{0,0}) = \lambda_{0,0}^2(t) \mu_0^2(t) g(R_{0,0}),
$$

where $g(R_{0,0})$ has the same structure as $f(R_{0,0})$ before, in particular, its expansion coefficients do not depend on $\gamma_{1,2}$. On the other hand, sum of the last four difference terms in (2.20) does depend on $\gamma_{1,2}$, and can be placed into

$$
\gamma_1 \frac{\lambda_{0,0}^2(t)}{\lambda_{0,0}(t)} I S^0(R_{0,0}^{-1}) + \gamma_2 \frac{\lambda_{0,0}^2(t)}{\lambda_{0,0}(t)} I S^0(R_{0,0}^{-1})
$$

We shall deal with it when we define $v_{smooth,a}$. At any rate, the error $e_1$ satisfies (2.6) for $k = 1$.

**Step 3** Choice of second correction $v_2$. The key in this step shall be to ensure that the singular part of $v_2$ will be independent of $\gamma_{1,2}$. This we can achieve since by our preceding construction the principal part of the error $e_1$ is independent of $\gamma_{1,2}$. Write

$$
e_1 = e_1^0 + t^{-2} e_1, e_1 := (\text{sum of the last four difference terms in (2.20)}) + e_0.$$

Then from [25] we infer the leading behaviour of the term $e^0_1$ (where we change the notation with respect to [25]), as follows:

$$t^2 e^{00}_1(t, r) := \lambda_0^{1/2}(t)\mu_0^{-1}(t)(c_1a + c_2b)$$

(2.22)

where we have $a = \xi, b = b(t) = \frac{1}{\mu_0(t)}$, and as remarked before the coefficients $c_j$ do not depend on $\gamma_{1,2}$. Also, recall

$$\mu_0(t) = (\lambda_0(t) \cdot t).$$

The second correction will then be obtained by neglecting the effect of the potential term, and setting

$$t^2 (v_{2,rr} - \frac{2}{r}v_{2,r}) = -t^2 e^{00}_1$$

(2.23)

To solve this we make the ansatz

$$v_2(t, r) = \lambda_0(t)^{1/2}(\mu_0^{-1}(t)q_1(a) + \mu_0^{-2}(t)q_2(a))$$

(2.24)

In fact, proceeding exactly as in [25], we then infer the equations

$$L_{\lambda_2} q_1 = c_1 a, \quad L_{\lambda_2} q_2 = c_2,$$

(2.25)

where we set

$$L_\beta := (1 - a^2)\tilde{\gamma}_a^2 + (2\beta - 1)a + 2a^{-1}\tilde{\gamma}_a - \beta^2 + \beta.$$

(2.26)

In fact, our $\lambda_0, \mu_0$ are exactly the $\lambda, \mu$ in [25]. To uniquely determine $q_{1,2}$, we impose the vanishing conditions

$$q_j(0) = q'_j(0) = 0, \quad j = 1, 2.$$ (2.27)

As in [25], one can then write (using $a = \frac{R_{0,0}}{\mu_0(t)}$ where $R_{0,0} := R_0$)

$$v_2 = \frac{\lambda_0(t)^{1/2}}{\mu_0^2(t)}(R_{0,0}q_1(a) + q_2(a)),$$

where now $\tilde{q}_1, q_2$ both have even power expansions around $a = 0$. In order to ensure the necessary parity of exponents in the power series expansions around $R_{0,0} = 0$ imposed by the definition of $Q$, we sacrifice some accuracy in the approximation, relabel the preceding expression $v_2(t, r)$ (as in [25]), and then use for the true correction $v_2$ the formula

$$v_2 = \frac{\lambda_0(t)^{1/2}}{\mu_0^2(t)}(R_{0,0}^{-1}q_1(a) + q_2(a), R_{0,0} = \sqrt{R_{0,0}^2 + 1}.$$

(2.28)

Again by construction $\tilde{q}_1, q_2$ and hence $v_2$ do not depend on $\gamma_{1,2}$. 


Step 4 Here we analyse the error generated by the approximate solution $u_2 = u_0 + v_1 + v_2$, which is given by the expression

$$
e_2 = \varepsilon_1 - \varepsilon_1^{00} - 5u_1^4v_2 - 10u_1^3v_2^2 - 10u_1^2v_2^3 - 5u_1v_2^4 - v_2^5$$

$$+ (\partial_n - \partial_{rr} - \frac{2}{r}\partial_r)(v_2 - v_2^0)$$

Then according to the preceding we have

$$\ell^2 (e_1 - e_0^{00}) - e_0$$

$$\in O(R_{0,0}^{-1}\lambda_{0,0}(t)\frac{4}{3}\mu_{0,0}^{-2}(t)) + \gamma_1 \frac{\lambda_1^4(t)}{\mu_{0,0}^{2+2}(t)} IS^0(R_{0,0},0) + \gamma_2 \frac{\lambda_2^4}{\mu(t)\lambda_0^{k_0+2}} \log t IS^0(R_{0,0}),$$

where the first term $O(R_{0,0}^{-1}\lambda_{0,0}(t)\frac{4}{3}\mu_{0,0}^{-2}(t))$ is independent of $\gamma_{1,2}$. The sum of the last two terms on the right will then be deferred until the last stage, when we define $v_{\text{smooth,0}}$. Next, consider

$$\ell^2 [- 5u_1^4v_2 - 10u_1^3v_2^2 - 10u_1^2v_2^3 - 5u_1v_2^4 - v_2^5 + (\partial_n - \partial_{rr} - \frac{2}{r}\partial_r)(v_2 - v_2^0)]$$

Here the interaction terms $u_1^{s-j}v_2^j$, $j \leq 4$, are only of the smoothness implied by $Q$, but do depend on $\gamma_{1,2}$ on account of $u_1 = u_0 + v_1$ and the $\gamma$-dependence of $u_0$. However, writing

$$u_1 = [u_0 - \lambda_{0,0}^4 W(\lambda_{0,0}r)] + [v_1 + \lambda_{0,0}^4 W(\lambda_{0,0}r)],$$

and expanding out $u_1^{s-j}$, we can place any term of the form

$$\ell^2[u_0 - \lambda_{0,0}^4 W(\lambda_{0,0}r)]l_1 [v_1 + \lambda_{0,0}^4 W(\lambda_{0,0}r)]v_2^j, \sum l_j = 5,$$

and with $l_1 \geq 1, l_3 \geq 1$ into

$$\gamma_1 \frac{\lambda_1^4}{\mu_{0,0}^{2+2}(t)} [IS^0(R_{0,0},Q)] + b^2 IS^0(R_{0,0},Q)]$$

$$+ \gamma_2 \log \frac{\lambda_1^4}{\mu_{0,0}^{2+2}(t)} [IS^0(R_{0,0},Q)] + b^2 IS^0(R_{0,0},Q)],$$

and so this can be placed into $\ell^2 e_{\text{prelim}}$. Finally, the preceding also implies (2.8) for $k = 1$.

Step 5 The inductive step. Here we again follow closely, but need to carefully keep track of various parts of $e_k$. First consider the case of even indices, i.e. assume $e_{2k-2}$, $2 \leq k \leq k_s$, satisfies (2.8) with $k$ replaced by $k - 1$, and more precisely, that we can decompose

$$e_{2k-2} = e_{2k-2}^1 + e_{2k-2}^2 + e_{2k-2}^3,$$  \hspace{1cm} \text{(2.27)}
where we have

\[ t^2 e_{2k-2}^1 = \frac{\lambda_{2,0}^1}{\mu_{0,0}(t)^{2k-2}} \left[ IS^0(R_{0,0}^{-1} \log R_{0,0})^{q_{k,1}} + b^2 IS^0(R_{0,0} (\log R_{0,0})^{q_{k,1}}, Q') \right], \]

\[ t^2 e_{2k-2}^2 = \gamma_1 \frac{\lambda_{2,0}^1}{\mu_{0,0}(t)^{2}} \left[ IS^0(R_{0,0}^{-1}) + \gamma_2 \frac{\lambda_{2,0}^1}{\mu_{0,0}(t)^{2}} \log t IS^0(R_{0,0}^{-1}) \right], \]

the term \( e_{2k-2}^1 \) being independent of \( \gamma_{1,2} \), while for the third term we have

\[ t^2 e_{2k-2}^3 = \gamma_1 \frac{\lambda_{2,0}^1}{\mu_{0,0}(t)^{2k_0+2}} \left[ IS^0(R_{0,0}^{-1}, Q) + b^2 IS^0(R_{0,0}, Q) \right] \]

\[ + \gamma_2 \log t \frac{\lambda_{2,0}^1}{\mu_{0,0}(t)^{2k_0+2}} \left[ IS^0(R_{0,0}^{-1}, Q) + b^2 IS^0(R_{0,0}, Q) \right]. \]

We have verified such a structure for the case \( k = 2 \) in the preceding step. Then we introduce the correction \( v_{2k-1} \) in order to improve the error \( e_{2k-1}^1 \), exactly mirroring Step 1 in section 2.7 of [25]. We completely forget about \( e_{2k-2}^3 \) as it can be moved into the final error \( e_{\text{prelim}} \), while we shall deal with the intermediate term \( e_{2k-2}^2 \) when introducing \( v_{\text{smooth,a}} \). Returning to \( v_{2k-1} \), and proceeding just as in Step 1, we see that \( v_{2k-1} \) will satisfy (2.5), and moreover be independent of \( \gamma_{1,2} \). The error \( e_{2k-1} \) generated by the approximation \( u_0 + \sum_{j=1}^{2k-1} v_j \) will be mostly independent of \( \gamma_{1,2} \), and satisfy (2.6), except for the cross interaction terms of \( v_{2k-1} \) and \( u_0 \), of the form \( u_0^{j-1} v_{j-1} \), \( 1 \leq j \leq 4 \). However, splitting

\[ u_0 = [u_0 - \lambda_{0,0}^1 W(\lambda_{0,0}(t)r)] + [\lambda_{0,0}^1 W(\lambda_{0,0}(t)r)], \]

we may replace \( u_0 \) by \( u_0 - \lambda_{0,0}^1 W(\lambda_{0,0}(t)r) \), and then the corresponding cross interactions, multiplied by \( t^2 \), can again be seen to be in

\[ \gamma_1 \frac{\lambda_{2,0}^1}{\mu_{0,0}(t)^{2k_0+2}} \left[ IS^0(R_{0,0}^{-1}, Q) + b^2 IS^0(R_{0,0}, Q) \right] \]

\[ + \gamma_2 \log t \frac{\lambda_{2,0}^1}{\mu_{0,0}(t)^{2k_0+2}} \left[ IS^0(R_{0,0}^{-1}, Q) + b^2 IS^0(R_{0,0}, Q) \right], \]

whence these error terms may be placed into \( e_{\text{prelim}} \) and discarded.

The case of odd indices, i.e. departing from \( e_{2k-1} \), \( k \leq k_a \), is handled just the same.

Repeating this procedure leads to the \( v_j \), \( 1 \leq j \leq 2k_a - 1 \). Moreover, each of the errors generated satisfies a decomposition analogous to (2.27), replacing (2.8) by (2.6) for odd indices.

**Step 6** Choice of \( v_{\text{smooth,a}} \), \( a = 1, 2 \). Here we depart from the approximation \( u_{2k_a-1} = u_0 + \sum_{j=1}^{2k_a-1} v_j \), which generates an error \( e_{2k_a-1} \) satisfying (2.6) for
$k = k_*$, as well as a decomposition

$$e_{2k_*-1} = \sum_{j=1}^{3} e_{2k_*-1}^j$$

(2.28)

analogous to (2.27). Importantly, the first error

$$r^2 e_{2k_*-1}^1 = \frac{\lambda_{0,0}^j}{\mu_{0,0}(t)^{k_0}} |S_0^0(R_{0,0}, \log R_{0,0})^{p_{k_*}}, Q'$$

is independent of $\gamma_{1,2}$, and the last error $e_{2k_*-1}^3$ may be placed into $e_{\text{prelim}}$, and so it remains to deal with the middle error which for technical reasons is still too large. Recall that the middle error satisfies

$$r^2 e_{2k_*-1}^2 = \gamma_1 \lambda_{0,0}^j |S_0^0(R_{0,0}) + \gamma_2 \frac{\lambda_{0,0}^j}{\mu_{0,0}(t)^{k_0}} \log |S_0^0(R_{0,0})|,$$

and in particular is $C^\infty$-smooth. Then set

$$\mu_{0,0}^2(t) L_0 v_{\text{smooth,1}} = r^2 e_{2k_*-1}^2,$$

leading to

$$v_{\text{smooth,1}} = \gamma_1 \lambda_{0,0}^j |S_0^0(R_{0,0}) + \gamma_2 \frac{\lambda_{0,0}^j}{\mu_{0,0}(t)^{k_0+2}} \log |S_0^0(R_{0,0})|.$$

Then all errors generated by $v_{\text{smooth,1}}$ by interaction with the bulk part $u_{2k_*-1}$ can be placed into $e_{\text{prelim}}$. On the other hand, the error $r^2 \partial_t^2 v_{\text{smooth,1}}$ is of the same form as $v_{\text{smooth,1}}$. We next construct $v_{\text{smooth,2}}$, proceeding in analogy to Step 3, to improve the error generated by $\partial_t^2 v_{\text{smooth,1}}$. The key here is that on the account of the rapid temporal decay of this term, the method of [25] applied to it results in a term of sufficient smoothness, to be acceptable for a correction depending on $\gamma_{1,2}$. Specifically, we write the leading order term of $r^2 \partial_t^2 v_{\text{smooth,1}}$ in the form

$$(c_1 + c_3 \log t) \frac{\lambda_{0,0}^j}{\mu_{0,0}(t)^{k_0+2}} R_{0,0} + (c_2 + c_4 \log t) \frac{\lambda_{0,0}^j}{\mu_{0,0}(t)^{k_0+2}}.$$

and then set (where the coefficients $c_{1,2}$ depend on $\gamma_{1,2}$)

$$r^2 (\partial_t^2 v_{\text{smooth,2}} - \partial_t^2 v_{\text{smooth,2}} - \frac{2}{r} \partial_r v_{\text{smooth,2}})$$

$$= (c_1 + c_3 \log t) \frac{\lambda_{0,0}^j}{\mu_{0,0}(t)^{k_0+2}} R_{0,0} + (c_2 + c_4 \log t) \frac{\lambda_{0,0}^j}{\mu_{0,0}(t)^{k_0+2}}.$$

Making the correct ansatz as in [25] this is solved by

$$v_{\text{smooth,2}} = \frac{\lambda_{0,0}^j}{\mu_{0,0}(t)^{k_0+4}} |S_0^2(R_{0,0}, Q_{\text{smooth}}) + \log t \frac{\lambda_{0,0}^j}{\mu_{0,0}(t)^{k_0+4}} |S_0^2(R_{0,0}, Q_{\text{smooth}}).$$
The effect of this correction is that we replace the middle term in (2.28) by one in $e_{\text{prelim}}$, i.e. our final approximate solution

$$u_{\text{prelim}} := u_0 + \sum_{j=1}^{2k_\gamma-1} v_j + \sum_{a=1}^{2} v_{\text{smooth},a}$$

generates an error $e_{\text{prelim}}$ as claimed in the lemma.

In order to complete the proof of the Theorem 2.1, we need to improve the approximate solution obtained in the preceding lemma a bit in order to replace the generated error $e_{\text{prelim}}$ by one which is smoother. More precisely, we need to get rid of the rough part of the error $\tilde{e}_{\text{prelim}}$. For this, we replace $u_{\text{prelim}}$ by $u_{\text{approx}} := u_{\text{prelim}} + v$,

where $v$ solves the equation

$$\Box v + 5\tilde{u}_{\text{prelim}}^4 v + \sum_{2 \leq j \leq 5} \binom{5}{j} v^j \tilde{u}_{\text{prelim}}^{5-j} = -\tilde{e}_{\text{prelim}},$$

where

$$\tilde{u}_{\text{prelim}} = u_{\text{prelim}} - v_{\text{smooth}} + \lambda_{0,0} W(\lambda_{0,0}(t)r) - \lambda_{1}^4 W(\lambda(t)r), \quad v_{\text{smooth}} = \sum_{a=1}^{2} v_{\text{smooth},a}$$

is the $\gamma$-independent part of $u_{\text{prelim}}$. Also, we shall impose vanishing of $v$ at $t = 0$. Then it is clear that $v$ will not depend on $\gamma_{1,2}$. The fact that such a $v$ can be computed with the required smoothness and bounds, provided $N$ is chosen large enough, follows exactly as in [26], see the discussion there after equation (2.1). Also, we have for any $t \in (0, t_0]$

$$\|\nabla_{t,x} v(t)\|_{H^2_x} \leq t^{-3}$$

Then we arrive at the error

$$\Box u_{\text{approx}} + u_{\text{approx}}^5 = \Box u_{\text{prelim}} + u_{\text{prelim}}^5 + \sum_{2 \leq j \leq 5} \binom{5}{j} v^j u_{\text{prelim}}^{5-j} + \Box v + 5\tilde{u}_{\text{prelim}}^4 v + 5(-\tilde{u}_{\text{prelim}}^4 + u_{\text{prelim}}^4)v$$

It follows that

$$e_{\text{approx}} = e_{\text{prelim}} - \tilde{e}_{\text{prelim}} + \sum_{2 \leq j \leq 5} \binom{5}{j} v^j [u_{\text{prelim}}^{5-j} - \tilde{u}_{\text{prelim}}^{5-j}] + 5(-\tilde{u}_{\text{prelim}}^4 + u_{\text{prelim}}^4)v,$$

This remaining error is easily seen to satisfy the claimed properties of the theorem.

$\Box$
3. Modulation theory: the choice of the parameters $\gamma_{1,2}$ for a perturbation

3.1. Change of scale and the space $\tilde{S}$. Assume that the function $\tilde{e}(R)$ is given in terms of its distorted Fourier transform by

$$
\tilde{e}(R) = \int_0^{\infty} x(\xi) \phi(R, \xi) d\xi + x_d \phi_d(R),
$$

$$
x(\xi) = \mathcal{F}(\tilde{e})(\xi) = \int_0^{\infty} \tilde{e}(R) \phi(R, \xi) dR,
$$

$$
x_d = \int_0^{\infty} \tilde{e}(R) \phi_d(R) dR.
$$

For a quick development of the Fourier transform associated with the operator $-\Delta - 5W^4$ we refer the [26], in particular, the precise definition and asymptotic expansions of the Fourier basis $\phi(R, \xi)$. We measure the size of the function $\tilde{e}(R)$ in terms of the norm $\|x\|_{\tilde{S}_1} + \|x_d\|_{\tilde{S}_2}$. We quickly recall from [19] the definition of the norms $\|\cdot\|_{\tilde{S}_1}, \|\cdot\|_{\tilde{S}_2}, \|\cdot\|_{\tilde{S}_2}$. For a pair of functions $(x_0(\xi), x_1(\xi)), \xi \in (0, \infty)$, we set

$$
\|(x_0, x_1)\|_{\tilde{S}} := \|x_0\|_{\tilde{S}_1} + \|x_1\|_{\tilde{S}_2} = \|\langle\xi\rangle^{-\frac{1}{2}} x_0\|_{L^2_{\tilde{S}}} + \|\langle\xi\rangle^{-\frac{1}{2}} x_1\|_{L^2_{\tilde{S}}}.
$$

(3.1)

For later reference (Proposition 4.2) we also use the norms

$$
\|(x_0, x_1)\|_{\tilde{S}} := \|x_0\|_{\tilde{S}_1} + \|x_1\|_{\tilde{S}_2} = \|\langle\xi\rangle^{1+\delta} x_0\|_{L^2_{\tilde{S}}} + \|x_1\|_{\tilde{S}_2}.
$$

(3.2)

The precise choices of the coefficients $\frac{1}{2} + + = \frac{1}{2} + 2\delta_0, 1 + + = 1 + 2\delta_0, 1- - = \frac{1}{2} - \delta_0, 0 - = -\delta_0$, where $\delta_0 > 0$ is a small fixed constant (only depending on $\nu$) are exactly as in Proposition 3.1 in [19]. In the sequel, we shall sometimes have to change the scaling, i.e. replace $\tilde{e}(R)$ by $\tilde{e}(\varepsilon R)$ for some small $\varepsilon \in \mathbb{R}$. The question how this affects $\|x\|_{\tilde{S}_1}$ is then nontrivial as we cannot translate the re-scaling on the $R$-side to a re-scaling on the $\xi$ side, as is the case (up to a multiple) for the standard flat Fourier transform. Nonetheless, up to an error which is described in terms of an operator analogous to $\mathcal{K}_{cc}$ discussed in [26], changing the scale with respect to $R$ translates into a 'dual change of scale' with respect to the Fourier variable $\xi$.

**Lemma 3.1.** Assume $\tilde{e}$ has the Fourier representation given above. Then we have the formula

$$
\mathcal{F}(\tilde{e}(e^{-x}R)) (\xi) = x(e^{2x}\xi) + \kappa \cdot \widetilde{K}_k x + O(\|x\|_{\tilde{S}} (\|x\|_{\tilde{S}_1} + |x_d|))
$$

where $\widetilde{K}_k$ has the same properties as the operator $\mathcal{K}_{cc}$ discussed in section 5 of [26]. In particular, we have

$$
\|\mathcal{F}(\tilde{e}(e^R))\|_{\tilde{S}_1} \leq \tau_0 \kappa (\|x\|_{\tilde{S}_1} + |x_d|).
$$

and more precisely, we have

$$
\|\mathcal{F}(\tilde{e}(e^R)) - (\mathcal{F}(\tilde{e})) (e^{2x}\xi)\|_{\tilde{S}_1} \leq \tau_0 \kappa (\|x\|_{\tilde{S}_1} + |x_d|).
$$
as well as

\[ \| \mathcal{F}(\hat{e}(e^R)) \|_{\mathcal{S}_1} \lesssim (1 + \tau_0 \kappa) \| \mathcal{F}(\hat{e}(R)) \|_{\mathcal{S}_1} + \kappa |x_d|. \]

**Proof.** This is entirely analogous to the proof of Theorem 5.1 in [26]; in effect the latter deals with the 'infinitesimal version' of the current situation. Consider the expression

\[ (\Xi_\kappa x)(\eta) := \langle \int_0^\infty x(\xi) \phi(e^{-s}R, e^{2s} \xi) \rho(\xi) \, d\xi, \phi(R, \eta) \rangle, \]

where \( x \in C_0^\infty(0, \infty) \). Under the latter restriction the integral converges absolutely. Then proceeding as in [26], see in particular Lemma 4.6 and the proof of Theorem 5.1 for the definition and properties of the function \( a(\xi) \), we get

\[ (\Xi_\kappa x)(\xi) = \frac{|a(e^{2s} \xi)|^2}{|a(\xi)|^2} x(\xi) + \int_0^\infty f^\ast(\xi, \eta) x(\eta) \, d\eta. \]

Here in order to determine the kernel \( f^\ast \) of the 'off-diagonal' operator at the end, we use

\[ (\eta - \xi) f^\ast(\xi, \eta) = (\xi - \eta) f^\ast(\eta, \xi), \]

\[ = \langle \int_0^\infty x(\xi) 5[e^{-2s}W^4(e^{-s}R) - W^4(R)] \phi(e^{-s}R, e^{2s} \xi) \rho(\xi) \, d\xi, \phi(R, \eta) \rangle \]

Then by following the argument of [26], proof of Theorem 5.1, one infers that

\[ f^\ast(\xi, \eta) = \kappa \cdot \frac{\rho(\eta) F_\kappa(\xi, \eta)}{\xi - \eta}, \]

with \( F_\kappa \) having the same asymptotic and vanishing properties as the kernel \( F(\xi, \eta) \) in [26], uniformly in \( \kappa \in [0, 1] \), say. It remains to translate the properties of \( \Xi_\kappa \) to those of the re-scaling operator. Let \( \Psi \) be the operator which satisfies

\[ \mathcal{F}(\Psi(\hat{e}))(\xi) = e^{-2s} \frac{\rho(\xi)}{\rho(\hat{e})} x(\hat{e}) e^{2s} \]

and leaves the discrete spectral part invariant, while \( S_{e^{-s}}(\hat{e})(R) = \hat{e}(\hat{e}) \) is the scaling operator. Then we have

\[ (\Xi_\kappa x)(\xi) = \mathcal{F}(S_{e^{-s}}(\hat{e}))(\xi) + O(\kappa |x_d|). \]

We conclude that

\[ \mathcal{F}(S_{e^{-s}}(\hat{e}))(\xi) = \Xi_\kappa (\mathcal{F}(\Psi^{-1}(\hat{e}))) + O(\kappa |x_d|). \]

It follows that we can write

\[ \mathcal{F}(S_{e^{-s}}(\hat{e}))(\xi) = x(e^{2s} \xi) + \left[ e^{2s} \frac{|a(e^{2s} \xi)|^2}{|a(\xi)|^2} \cdot \frac{\rho(e^{2s} \xi)}{\rho(\xi)} - 1 \right] x(e^{2s} \xi) \]

\[ + \int_0^\infty f^\ast(\xi, \eta) x(\eta) \, d\eta + O(\kappa |x_d|), \]

where we put

\[ f^\ast(\xi, \eta) := f^\ast(\xi, \eta) \cdot \frac{\rho(\eta)}{\rho(\xi e^{2s})}. \]
functionals, which are hence given by

\[ A(\gamma_1, \gamma_2) = 0, B(\gamma_1, \gamma_2) = 0, \]

where

\[ \lambda_{0,0}(t) = t^{-1-\nu}, \lambda(t) = \lambda_{\gamma_1, \gamma_2}(t) = (1 + \gamma_1 t^{10\nu} + \gamma_2 \log t \cdot t^{10\nu})t^{-1-\nu}, \]

In a first approximation, we use the versions \( A := A(0, 0), B := B(0, 0) \) for these functionals, which are hence given by

\[
A = \int_0^\infty \frac{x_1(\xi)\rho^{\frac{1}{2}}(\xi)}{\xi^{\frac{1}{2}}} \sin[n\tau_0 \xi^{\frac{1}{2}}] d\xi, \quad B = \int_0^\infty \frac{x_0(\xi)\rho^{\frac{1}{2}}(\xi)}{\xi^{\frac{1}{2}}} \cos[n\tau_0 \xi^{\frac{1}{2}}] d\xi
\]

where \( \tau_0 = \nu^{-1} r^{-\nu} \).

The basic setup for the construction of a family of stable blow up solutions is now the following: Starting with the approximate blow up solution corresponding to \( (\gamma_1, \gamma_2) = (0, 0) \), which we denote \( u^{(0,0)}_{\text{approx}} \), we consider perturbed data

\[ u^{(0,0)}_{\text{approx}}[t_0] + (\epsilon_1, \epsilon_2). \]

Here we think of the perturbations \( \epsilon_{1,2} \) as functions of \( R_{0,0} = rt_0^{-1-\nu} \), and we shall measure them by using the distorted Fourier transform with respect to \( R_{0,0} \). As the perturbation will not satisfy the required vanishing conditions in general, we shall then pass to the proper reference frame by writing

\[
u^{(0,0)}_{\text{approx}}[t_0] + (\epsilon_1, \epsilon_2) = u^{(\gamma_1, \gamma_2)}_{\text{approx}}[t_0] + (\eta_1, \eta_2),
\]

where we now think of \( \eta_{1,2} \) as functions of \( R = r \lambda_{\gamma_1, \gamma_2}(t_0) \). More precisely, to stay in the required function spaces, we shall tacitly truncate \( u^{(0,0)}_{\text{approx}}[t_0], u^{(\gamma_1, \gamma_2)}_{\text{approx}}[t_0] \) smoothly to a dilate \( r \leq C t_0 \) of the light cone. Correspondingly, we have the distorted Fourier transform

\[
x^{(\gamma_1, \gamma_2)}_0(\xi) = \int_0^\infty \phi(R, \xi) R \mathcal{E}_1(R) dR,
\]

and we define the corresponding 'temporal' \( x^{(\gamma_1, \gamma_2)}_1(\xi) \) by analogy to formula (4.3) in [19], i.e. we put

\[
x^{(\gamma_1, \gamma_2)}_1(\xi) = - \lambda_{\gamma_1, \gamma_2}^{-1} \int_0^\infty \phi(R, \xi) R \mathcal{E}_2(R) dR - \frac{\lambda_{\gamma_1, \gamma_2}}{\lambda_{\gamma_1, \gamma_2}} \mathcal{K}_{cc} x^{(\gamma_1, \gamma_2)}_0(\xi)
\]

\[
- \frac{\lambda_{\gamma_1, \gamma_2}}{\lambda_{\gamma_1, \gamma_2}} \mathcal{K}_{cd} x^{(\gamma_1, \gamma_2)}_0(\xi)\]

This implies the claims of the lemma.

\[ \square \]

3.2. The effect of scaling the bulk part. Here we investigate how changing the bulk part from \( \lambda_{1,0}^\frac{1}{2} W(\lambda_{1,0} r) \) to \( \lambda_{\gamma_1, \gamma_2}^\frac{1}{2} W(\lambda r) \) affects the functionals appearing in the vanishing conditions (recall the expressions (2.3), (2.4))

\[ A(\gamma_1, \gamma_2) = 0, B(\gamma_1, \gamma_2) = 0, \]
Here, we make the following remarks: first,
\[ x_{0d}^{(\gamma_1, \gamma_2)} = \int_0^{\infty} \phi_d(R) R \tau_1(R) \, dR \]
is the unstable spectral part, with respect to the coordinate \( R \). Second, \( \lambda_{\gamma_1, \gamma_2} \) in the preceding is thought of as function of the new time variable
\[ \tau := \int_0^t \lambda_{\gamma_1, \gamma_2}(s) \, ds, \]
which in the formula for \( x_1^{(\gamma_1, \gamma_2)}(\xi) \) gets equated with the time \( t_0^{(\gamma_1, \gamma_2)} = \int_0^\infty \lambda_{\gamma_1, \gamma_2}(s) \, ds \).

In order to measure the perturbation \((\epsilon_1, \epsilon_2)\), it is natural to use \( x_0^{(0,0)} := x_0, x_1^{(0,0)} = x_1 \). Moreover, we also set \( x_{0d}^{(0,0)} := x_{0d}, l = 0, 1 \). We shall strive to have no condition other than smallness in a suitable sense for \((x_0, x_1)\), while \((x_{0d}, x_{1d})\) shall be restrained by a co-dimension one condition like the one in Lemma 2.1 in [19]. We now have the setup to formulate the modulation step:

**Proposition 3.2.** Given a fixed \( \nu \in (0, \nu_0], \nu_0 \in (0, 1] \), there is a \( \delta_1 = \delta_1(\nu, \nu_0) > 0 \) small enough such that for any perturbation \((\epsilon_1, \epsilon_2)\) satisfying
\[ \| (x_0, x_1) \|_S + |x_{0d}| < \delta_1, \]
there is a unique pair \( \gamma_{1,2} \) with \( |\gamma_1| + |\gamma_2| \leq \nu_0 \| (x_0, x_1) \|_S \) and a unique parameter \( x_{1d} \) satisfying \( |x_{1d}| \leq \nu |x_{0d}| \) such that
\[ A(\gamma_1, \gamma_2) = B(\gamma_1, \gamma_2) = 0, \]
and the discrete spectral part \((x_{0d}^{(\gamma_1, \gamma_2)}, x_{1d}^{(\gamma_1, \gamma_2)})\) satisfies the vanishing property of Lemma 2.1 in [19] with respect to the scaling law \( \lambda = \lambda_{\gamma_1, \gamma_2} \). We have the precise bound
\[ |\gamma_1 \lambda_{0,0,0}^{\frac{1}{2}}| + |\gamma_2 \lambda_{0,0}^{\frac{1}{2}} \log t_0^{\frac{1}{2}}| \leq \tau_0 \log \tau_0 (\| (x_0, x_1) \|_S + |x_{0d}|). \]

Finally, we have the bound
\[ \| x_0^{(\gamma_1, \gamma_2)} - \lambda_{0,0}^{\frac{1}{2}} S \frac{\partial}{\partial \xi} x_0 \|_{L^2} + \| x_1^{(\gamma_1, \gamma_2)} - \lambda_{0,0}^{\frac{1}{2}} S \frac{\partial}{\partial \xi} x_1 \|_{L^2} \leq \tau_0 \tau_0^{0+} (\| (x_0, x_1) \|_S + |x_{0d}|). \]
where \( S \frac{\partial}{\partial \xi} x_1(\xi) = x_1(\frac{\partial}{\partial \xi} \xi) \) is the scaling operator.

**Proof:** The strategy shall be to first fix the discrete spectral part to \((x_{0d}, x_{1d})\) while choosing \( \gamma_{1,2} \), and at the end finalising the choice of \( x_{1d} \) to satisfy the required co-dimension one condition.

Observe that from our definition and the structure of \( \nu_{\text{approx}}^{(\gamma_1, \gamma_2)} \), we can write
\[ \mathcal{E}_1 = \lambda_{0,0}^{\frac{1}{2}} W(\lambda_{0,0} r) - \lambda_{\gamma_1, \gamma_2}^{\frac{1}{2}} W(\lambda_{\gamma_1, \gamma_2} r) + \nu_{\text{smooth}} + \epsilon_1, \]
(3.5)
as well as
\[ \mathcal{E}_2 = \mathcal{E}_1 \left[ \lambda_{0,0}^{\frac{1}{2}} W(\lambda_{0,0} r) - \lambda_{\gamma_1, \gamma_2}^{\frac{1}{2}} W(\lambda_{\gamma_1, \gamma_2} r) \right] + \mathcal{E}_1 \nu_{\text{smooth}} + \epsilon_2, \]
(3.6)
where we have introduced the notation $v_{\text{smooth}} = \sum_{\ell=1,2} v_{\text{smooth},\ell}$. Also, it is implied that the expressions gets evaluated at $t = t_0$. To begin with, observe that

\[
\dot{x}_0^{(\gamma_1, \gamma_2)}(\xi) = \int_{0}^{\infty} \phi(R, \xi) R e_1(R_0,0) \, dR, \quad \dot{x}_{0d}^{(\gamma_1, \gamma_2)} = \int_{0}^{\infty} \phi_d(R) R e_1(R_0,0) \, dR
\]

\[
\dot{x}_1^{(\gamma_1, \gamma_2)}(\xi) = -\lambda_{\gamma_1, \gamma_2}^{-1} \int_{0}^{\infty} \phi(R, \xi) R e_2(R_0,0) \, dR - \frac{\lambda_{\gamma_1, \gamma_2}}{\lambda_{\gamma_1, \gamma_2}} (K_c d^{\gamma_1, \gamma_2}) (\xi)
\]

then using Lemma 3.1 we have

\[
\| \dot{x}_0^{(\gamma_1, \gamma_2)}(\xi) - \frac{\lambda}{\lambda_{0,0}} x_0(\frac{A_2^2}{\lambda_{0,0}^2})\| \|_S 1 \leq \tau_0 |y_1 t_0^{kv} + \gamma_2 \log t_0 \cdot t_0^{kv} | \|_S + |x_{0d}|,
\]

while we directly infer the bound

\[
|\dot{x}_{0d}^{(\gamma_1, \gamma_2)} - x_{0d}| \leq |y_1 t_0^{kv} + \gamma_2 \log t_0 \cdot t_0^{kv} | (\tau_0 \|x_0\|_S) + |x_{0d}|.
\]

Similarly, we obtain

\[
\| \dot{x}_1^{(\gamma_1, \gamma_2)}(\xi) - \frac{\lambda}{\lambda_{0,0}} x_1(\frac{A_2^2}{\lambda_{0,0}^2})\| \|_S 2 \leq |y_1 t_0^{kv} + \gamma_2 \log t_0 \cdot t_0^{kv} | \|_S + |x_{1d}| + |x_{0d}|.
\]

Then denoting by $A(\gamma_1, \gamma_2)$, resp. $B(\gamma_1, \gamma_2)$ the quantity defined like $A(\gamma_1, \gamma_2)$, $B(\gamma_1, \gamma_2)$, but with $x_j^{(\gamma_1, \gamma_2)}$ replaced by $\dot{x}_j^{(\gamma_1, \gamma_2)}$, $j = 1, 0$, we infer after a change of variables that

\[
A(\gamma_1, \gamma_2) = A + O(|y_1 t_0^{kv} + \gamma_2 \log t_0 \cdot t_0^{kv} | \|x_1\|_S + \tau_0^{-1} \|x_0\|_S + |x_{1d}| + |x_{0d}|),
\]

\[
B(\gamma_1, \gamma_2) = B + O(|y_1 t_0^{kv} + \gamma_2 \log t_0 \cdot t_0^{kv} | \|x_0\|_S + \tau_0^{-1} |x_{0d}|).
\]

Here of course $A, B$ are independent of $\gamma_{1,2}$, while the error terms are of quadratic character and hence negligible. Recalling the relations (3.5), (3.6), we conclude that denoting the contributions of the bulk parts there by

\[
\tilde{\dot{x}}_0^{(\gamma_1, \gamma_2)}(\xi) = \int_{0}^{\infty} \phi(R, \xi) R [\frac{A_0}{A_{0,0}} W(A_{0,0} R) - \lambda_{\gamma_1, \gamma_2} \frac{A_0}{A_{0,0}} W(A_{\gamma_1, \gamma_2} R) + v_{\text{smooth}}] \, dR
\]

and their contributions to $A(\gamma_1, \gamma_2)$ by $\tilde{A}(\gamma_1, \gamma_2)$ etc, we can write

\[
0 = A(\gamma_1, \gamma_2) = \tilde{A}(\gamma_1, \gamma_2) + \tilde{\dot{A}}(\gamma_1, \gamma_2),\quad 0 = B(\gamma_1, \gamma_2) = \tilde{B}(\gamma_1, \gamma_2) + \tilde{\dot{B}}(\gamma_1, \gamma_2),
\]

and so

\[
\tilde{A}(\gamma_1, \gamma_2) = -A + O(|y_1 t_0^{kv} + \gamma_2 \log t_0 \cdot t_0^{kv} | \|x_1\|_S + \tau_0^{-1} \|x_0\|_S + \tau_0^{-1} |x_{0d}|),
\]

\[
\tilde{\dot{B}}(\gamma_1, \gamma_2) = -B + O(|y_1 t_0^{kv} + \gamma_2 \log t_0 \cdot t_0^{kv} | \|x_0\|_S + \tau_0^{-1} |x_{0d}|).
\]
It remains to compute $\tilde{\mathcal{B}}(\gamma_1, \gamma_2)$ in terms of $\gamma_{1,2}$, which we now do: note that
\[
\int_0^\infty \phi(R, \xi) R X_R \cdot C_0 \left( \lambda_{A_{0,0}} W(\lambda_{0,0} r) - \lambda_{A_{1,1}} W(\lambda_{1,1} r) \right) dR = |\gamma_1 k_{0,0} + \gamma_2 \log t_0 k_{0,0}| \left[ O_N \left( \frac{1}{\lambda_{A_{0,0}}} \frac{C r_0}{(C r_{0,0})^N} \right) + O_N (\lambda_{A_{0,0}}) \right]
\]
and we also have the important relation
\[
\lim_{R \to 0} R^{-1} X_R \cdot C_0 \left( \lambda_{A_{0,0}} W(\lambda_{0,0} r) - \lambda_{A_{1,1}} W(\lambda_{1,1} r) \right) = \frac{1}{2} \lambda_{A_{0,0}}^\xi \left[ |\gamma_1 k_{0,0} + \gamma_2 \log t_0 k_{0,0}| + O_N \left( \frac{1}{\lambda_{A_{0,0}}} \left[ |\gamma_1 k_{0,0} + \gamma_2 \log t_0 k_{0,0}|^2 \right) \right. \right.
\]
As for the contribution of $v_{smooth}$, we get from its construction that
\[
\lim_{R \to 0} R^{-1} X_R \cdot C_0 v_{smooth}(R) = 0,
\]
and furthermore
\[
\int_0^\infty \phi(R, \xi) R X_R \cdot C_0 v_{smooth}(R) |_{R = t_0} dR = |\gamma_1 k_{0,0} + \gamma_2 \log t_0 k_{0,0}| \left[ O_N \left( \frac{1}{\lambda_{A_{0,0}}} \frac{C r_0}{(C r_{0,0})^N} \right) + O_N (\lambda_{A_{0,0}}) \right].
\]
Thus we get roughly the same asymptotics as for the contribution of the bulk part. We conclude that (for a suitable constant $c > 0$)
\[
\tilde{\mathcal{B}}(\gamma_1, \gamma_2) = \int_0^\infty \frac{\tilde{\mathcal{B}}(\gamma_1, \gamma_2)}{2} \cos[\nu \tau_0 \xi_0^\delta] d\xi \approx \gamma_1 k_{0,0} + \gamma_2 \log t_0 k_{0,0} \left[ O_N \left( \frac{1}{\lambda_{A_{0,0}}} \frac{C r_0}{(C r_{0,0})^N} \right) + O_N (\lambda_{A_{0,0}}) \right].
\]
The last term on the right is essentially quadratic and negligible in the sequel. The second and third terms are also negligible on account of the asymptotics from before for the Fourier transform of the bulk part as well as $v_{smooth}$: for the second term, we get (for suitable $c > 0$)
\[
|\int_0^\infty \tilde{\mathcal{B}}(\gamma_1, \gamma_2) \left( \frac{1}{2} \cos[\nu \tau_0 \xi_0^\delta] - c \rho(\xi) \right) d\xi | \leq \frac{1}{\lambda_{A_{0,0}}}^\gamma_1 k_{0,0} + \gamma_2 \log t_0 k_{0,0} |.$
while the third term becomes small upon choosing $C$ sufficiently large:

$$\left| \int_0^\infty \tilde{\zeta}^{(\gamma_1, \gamma_2)}_0 \tilde{x}_0(\xi) \rho(\xi) (\cos[\nu_0 \xi^2] - 1) \, d\xi \right|$$

$$\leq C^{-1} \frac{1}{\lambda_0} \left| \gamma_1 t_0^{k_{0y}} + \gamma_2 \log t_0 t_0^{k_{0y}} \right|$$

Finally, for the first term above, we have according to the earlier limiting relations

$$\lim_{R \to 0} R^{-1} \int_0^\infty \phi(R, \xi) \tilde{\zeta}^{(\gamma_1, \gamma_2)}_0(\xi) \rho(\xi) \, d\xi$$

$$= \frac{1}{2} \lambda_0^{1/2} \left[ \gamma_1 t_0^{k_{0y}} + \gamma_2 \log t_0 t_0^{k_{0y}} \right] + O(\lambda_0^{-1/2} \left| \gamma_1 t_0^{k_{0y}} + \gamma_2 \log t_0 t_0^{k_{0y}} \right|)$$

Summarizing the preceding observations, we have obtained the first relation determining $\gamma_{1,2}$, given by

$$B = - \frac{1}{2} \lambda_0^{1/2} \left[ \gamma_1 t_0^{k_{0y}} + \gamma_2 \log t_0 t_0^{k_{0y}} \right] + O(\lambda_0^{-1/2} \left| \gamma_1 t_0^{k_{0y}} + \gamma_2 \log t_0 t_0^{k_{0y}} \right|)$$

To derive the second equation determining $\gamma_{1,2}$, we recall the formula for $x_1^{(\gamma_1, \gamma_2)}$, which hinges on $\bar{\epsilon}_2$. Then from (3.6) recall that we have (using the notation $\Lambda := \frac{1}{2} + R\hat{c}_R$)

$$\bar{\mathcal{R}}_2 = \hat{c}_1 \left[ \left( t_0^{k_{0y}} \gamma_1 + \log t_0 t_0^{k_{0y}} \gamma_2 \right) \lambda_0^{1/2} \phi(R, 0) + R\nu_{smooth} \right]_{t=0} + R\bar{\epsilon}_2$$

$$+ O(\lambda_0^{1/2} t_0^{k_{0y}-1} \log t_0 \left( \sum |\gamma_j| \right) R^{-2})$$

$$= c_1 t_0^{-1} \left( t_0^{k_{0y}} \gamma_1 + \log t_0 t_0^{k_{0y}} \gamma_2 \right) \lambda_0^{1/2} \phi(R, 0)$$

$$+ c_2 t_0^{-1} \left( t_0^{k_{0y}} \gamma_1 + \log t_0 t_0^{k_{0y}} \gamma_2 \right) \lambda_0^{1/2} \left( \Lambda^2 W \right)(R)$$

$$+ \gamma_2 t_0^{k_{0y}-1} \lambda_0^{1/2} \phi(R, 0) + O(\lambda_0^{1/2} t_0^{k_{0y}-1} \log t_0 \left( \sum |\gamma_j| \right) R^{-2})$$

Then recalling the relation

$$x_1^{(\gamma_1, \gamma_2)}(\xi) = - \lambda_0^{-1} \int_0^\infty \phi(R, \xi) \bar{\mathcal{R}}_2(R) \, dR - \frac{1}{\lambda_0} \left( \mathcal{K}_{1} x_0^{(\gamma_1, \gamma_2)}(\xi) \right)$$

$$- \frac{1}{\lambda_0} \left( \mathcal{K}_{cd} x_0^{(\gamma_1, \gamma_2)} \right)(\xi)$$,
as well as the corresponding relation for $x_1 = x_1^{(0,0)}$, we deduce
\[
\begin{align*}
  x_1^{(\gamma_1,\gamma_2)}(\xi) &= \\
  &- c_1(t_0 \Lambda_{\gamma_1,\gamma_2})^{-1}(t_0^{k_0} \gamma_1 + \log t_0 \cdot t_0^{k_0} \gamma_2) 
  \int_0^{\infty} \phi(R, \xi) \chi_{R \leq \xi \tau_0} \lambda_{0,0}^{\frac{1}{2}} \phi(R, 0) dR \\
  &- c_2(t_0 \Lambda_{\gamma_1,\gamma_2})^{-1}(t_0^{k_0} \gamma_1 + \log t_0 \cdot t_0^{k_0} \gamma_2) 
  \int_0^{\infty} \phi(R, \xi) \chi_{R \leq \xi \tau_0} \lambda_{0,0}^{\frac{1}{2}} (A^2 W)(R, 0) dR \\
  &- \gamma_2 \tau_0^{-1} \lambda_{\gamma_1,\gamma_2}^{-1} 
  \int_0^{\infty} \phi(R, \xi) \chi_{R \leq \xi \tau_0} \lambda_{0,0}^{\frac{1}{2}} \phi(R, 0) dR \\
  &- \lambda_{\gamma_1,\gamma_2}^{-1} \int_0^{\infty} \phi(R, \xi) \chi_{R \leq \xi \tau_0} R \xi \nu \text{smooth } \nu dR + \frac{A}{\lambda_{0,0}} x_1(\frac{\nu^2}{\lambda_{0,0}}) \\
  &+ O_N((t_0 \Lambda_{\gamma_1,\gamma_2})^{-1} \frac{\tau_0}{(C \tau_0 \xi)^{\frac{1}{2}}} \tau_0^{-2} \lambda_{0,0}^{\frac{1}{2}} |\gamma_1 t_0^{k_0} + \gamma_2 \log t_0 \cdot t_0^{k_0} v|) \\
  &+ O(\lambda_{0,0}^{\frac{1}{2}} |\gamma_1 t_0^{k_0} + \gamma_2 \log t_0 \cdot t_0^{k_0} v|) \\
  &+ \gamma_2 \tau_0^{-1} \gamma_1 t_0^{k_0} + \gamma_2 \log t_0 \cdot t_0^{k_0} \tau_0 ([x_1]_{\tilde{S}_1} + \tau_0^{-1} \|x_0\|_{\tilde{S}_2} + |x_{1d}| + \tau_0^{-1} |x_{0d}|)).
\end{align*}
\]

We substitute this expression into $A(\gamma_1, \gamma_2)$, and proceeding in analogy to $B(\gamma_1, \gamma_2)$, we infer
\[
A = \gamma_2 \lambda_{0,0}^{\frac{1}{2}} t_0^{k_0} \gamma_1 + c_3 \lambda_{0,0}^{\frac{1}{2}} (t_0^{k_0} \gamma_1 + \log t_0 \cdot t_0^{k_0} \gamma_2) \\
+ O(\lambda_{0,0}^{\frac{1}{2}} \tau_0^{-1} |\gamma_1 t_0^{k_0} + \log t_0 \cdot t_0^{k_0} \gamma_2|) \\
+ O(\lambda_{0,0}^{\frac{1}{2}} \tau_0 |\gamma_1 t_0^{k_0} + \gamma_2 \log t_0 \cdot t_0^{k_0} v| ([x_1]_{\tilde{S}_1} + \tau_0^{-1} \|x_0\|_{\tilde{S}_2} + |x_{1d}| + \tau_0^{-1} |x_{0d}|)).
\]

In conjunction with the earlier relation for $B$ above, we now have a system of equations uniquely determining the quantities
\[
\lambda_{0,0}^{\frac{1}{2}} (t_0^{k_0} \gamma_1 + \log t_0 \cdot t_0^{k_0} \gamma_2), \gamma_2 \lambda_{0,0}^{\frac{1}{2}} t_0^{k_0}.
\]

On account of the easily verified bounds
\[
|A| \leq \tau_0 \|x_1\|_{\tilde{S}_1}, |B| \leq \tau_0 \|x_0\|_{\tilde{S}_2},
\]
we then infer
\[
|\gamma_1 \lambda_{0,0}^{\frac{1}{2}} t_0^{k_0} + |\gamma_2 \lambda_{0,0}^{\frac{1}{2}} \log t_0^{k_0} v| \leq (\log \tau_0) \cdot \tau_0 \|x_0, x_1\|_{\tilde{S}}.
\]

Recall that throughout the preceding discussion we kept the discrete spectral parts $(x_{0d}, x_{1d})$ of the initial perturbation $(\epsilon_1, \epsilon_2)$ fixed. If instead we allow $x_{1d}$ to vary, we can think of $\gamma_{1,2}$ as functions of $x_{1d}$, and moreover one easily checks that
\[
\varphi_{1d}^{(\gamma_1, \gamma_2)} = x_{1d} + O([x_0]_{\tilde{S}_1} + \|x_1\|_{\tilde{S}_2} + |x_{0d}| + |x_{1d}|)^2).
\]

with a corresponding Lipschitz bound. It follows that there is a unique choice of $x_{1d}$ such that (for given $x_0, x_1, x_{0d}$) the pair $(\varphi_{1d}^{(\gamma_1, \gamma_2)}, \varphi_{1d}^{(\gamma_1, \gamma_2)})$ satisfies the linear compatibility relation from Lemma 2.1 in [19] with respect to the scaling parameter $\lambda = \lambda_{\gamma_1,\gamma_2}$.
The last bound of the proposition follows from the preceding formulas for \(x_1^{(\gamma_1, \gamma_2)}\), as well as \(x_0^{(\gamma_1, \gamma_2)}\) in terms of \(x_1, x_0\). 

For later purposes, we also mention the following important Lipschitz continuity properties, which follow easily from the preceding proof:

**Lemma 3.3.** Let \((\overline{\gamma}_1, \overline{\gamma}_2)\) the parameters associated with data \((\overline{x}_0, \overline{x}_1)\) \(\in \tilde{S}\). Then using the notation from before and putting

\[
\overline{\lambda} = \lambda(\overline{\gamma}_1, \overline{\gamma}_2),
\]

we have

\[
\left| (\gamma_1 - \overline{\gamma}_1) \lambda_{x,0,0}^1 \right| + \left| (\gamma_2 - \overline{\gamma}_2) \lambda_{x,0,0}^1 \right| \leq \tau_0 \log \tau_0 \left\| (x_0 - \overline{x}_0, x_1 - \overline{x}_1) \right\| \mathcal{S} + \left\| (x_0, x_1) \right\| \mathcal{S} \left\| x_{0d} - \overline{x}_{0d} \right\|.
\]

\[
\left\| x_0^{(\gamma_1, \gamma_2)} - x_0^{(\overline{\gamma}_1, \overline{\gamma}_2)} \right\| + \left\| x_1^{(\gamma_1, \gamma_2)} - x_1^{(\overline{\gamma}_1, \overline{\gamma}_2)} \right\| \leq \log \tau_0 \cdot \tau_0^+ \cdot \left\| (x_0 - \overline{x}_0, x_1 - \overline{x}_1) \right\| \mathcal{S} + \left\| (x_0, x_1) \right\| \mathcal{S} \left\| x_{0d} - \overline{x}_{0d} \right\|.
\]

Finally, we have the bound

\[
\left| (x_{1d}^{(\gamma_1, \gamma_2)} - x_{1d}) - (\overline{x}_{1d}^{(\gamma_1, \gamma_2)} - \overline{x}_{1d}) \right| \leq \left\| (x_0 - \overline{x}_0, x_1 - \overline{x}_1) \right\| \mathcal{S} + \left\| x_{0d} - \overline{x}_{0d} \right\| \cdot \left\| (x_0, x_1) \right\| \mathcal{S} + \left\| x_{0d} \right\|.
\]

4. **Iterative construction of blow up solution ’almost matching’ the perturbed initial data**

As in the preceding section, consider data

\[
u_{0,0}^{(0)}(\ell_0) + (\epsilon_1, \epsilon_2).
\]

Here we shall only impose the co-dimension one condition arising from the unstable mode (as in the preceding proposition), i.e. \(x_{1d}\) is a function of \((x_0, x_1, x_{0d})\), and otherwise, assume that

\[
\left\| (x_0, x_1) \right\| \mathcal{S} + \left\| x_{0d} \right\| \leq \delta_1
\]

is sufficiently small, and that \(\tau_0 > 0\) is also sufficiently small. According to the preceding section, we can then uniquely determine coefficients \(\gamma_{1,2}\) such that

\[
u_{0,0}^{(0)}(\ell_0) + (\epsilon_1, \epsilon_2) = u_{approx}^{(\gamma_{1,2})}(\ell_0) + (\overline{\gamma}_1, \overline{\gamma}_2),
\]

and such that the distorted Fourier coefficients \(x_{0,1}^{(\gamma_{1,2})}\) associated with \(\overline{\gamma}_{1,2}\) in the sense of the preceding sections and with respect to the variable \(R = \lambda_{\gamma_{1,2}}(\ell_0) r\) satisfy the required vanishing conditions

\[
A(\gamma_1, \gamma_2) = B(\gamma_1, \gamma_2) = 0.
\]
We can now essentially verbatim re-peat the iterative construction in [19], to arrive at the desired singular solution, whose data 'almost' match

\[ u^{(0,0)}_{\text{approx}}[t_0] + (\epsilon_1, \epsilon_2) \]

at time \( t = t_0 \). We commence by translating the problem to the Fourier side.

4.1. **Formulation of the perturbation problem on Fourier side.** We seek a solution of the form

\[ u(t, x) = u^{(\gamma_1, \gamma_2)}_{\text{approx}}(t, x) + \epsilon(t, x) \]

Then working with the variable

\[ R = \lambda(t) r, \quad \tau = \int_{t}^{\infty} \lambda(s) ds, \quad \lambda(t) = \lambda_{\gamma_1, \gamma_2}(t), \]

and setting

\[ \tilde{\epsilon}(\tau, R) := \text{Re}(t(\tau), R(\tau, R)) \]

we find the equation

\[ (\tilde{\epsilon}_t + \lambda \lambda^{-1} R \tilde{\epsilon}_R)^2 \tilde{\epsilon} - \beta(\tau)(\tilde{\epsilon}_t + \lambda \lambda^{-1} R \tilde{\epsilon}_R) \tilde{\epsilon} + \mathcal{L} \tilde{\epsilon} = \lambda^{-2}(\tau) R[N_{\text{approx}}(\epsilon) + \epsilon_{\text{approx}}] + \tilde{\epsilon}_t (\lambda \lambda^{-1}) \tilde{\epsilon}; \quad \beta(\tau) = \dot{\lambda}(\tau) \lambda^{-1}(\tau), \quad (4.1) \]

in direct analogy to [19]. We use the notation

\[ R N_{\text{approx}}(\epsilon) = 5(u_{\text{approx}}^4 - u_0^4) \tilde{\epsilon} + R N(u_{\text{approx}}, \tilde{\epsilon}), \]

\[ R N(u_{\text{approx}}, \tilde{\epsilon}) = R(u_{\text{approx}} + \frac{\tilde{\epsilon}}{R})^5 - Ru_{\text{approx}}^5 - 5u_{\text{approx}}^4 \tilde{\epsilon} \]

Here \( u_{\text{approx}} = u^{(\gamma_1, \gamma_2)}_{\text{approx}} \). We note that we always may and shall when needed include a spatial cutoff \( \chi_{R \leq C \tau} \) in front of these expressions. This is because it suffices to construct a solution within the light cone \( r \leq t, 0 < t \leq t_0 \).

Ideally we will want to match

\[ \epsilon[t_0] = (\tau_1, \tau_2), \]

but we shall have to deviate from this by a small error. In order to solve (4.1), we pass to the distorted Fourier transform of \( \tilde{\epsilon} \), by using the representation

\[ \tilde{\epsilon}(\tau, R) = x_d(\tau) \phi_d(R) + \int_{0}^{\infty} x(\tau, \xi) \phi(R, \xi) p(\xi) d\xi. \]

Writing

\[ \chi(\tau, \xi) := \left( \begin{array}{c} x_d(\tau) \\ x(\tau, \xi) \end{array} \right), \quad \xi := \left( \begin{array}{c} \xi_d \\ \xi \end{array} \right), \]

we infer

\[ (\mathcal{D}_r^2 + \beta(\tau) \mathcal{D}_r + \xi) x(\tau, \xi) = \mathcal{R}(\tau, \chi) + f(\tau, \xi), \quad f := \left( \begin{array}{c} f_d \\ f \end{array} \right), \quad (4.2) \]

where we have

\[ \mathcal{R}(\tau, \chi)(\xi) = \left( -4\beta(\tau) \mathcal{K} \mathcal{D}_r \chi - \beta^2(\tau)(\mathcal{K}^2 + [\mathcal{A}, \mathcal{K}] + \mathcal{K} + \beta' \beta^{-2} \mathcal{K}_d) \right)(\xi) \quad (4.3) \]
Finally, we have energy decay within the light cone: the description from [26], [25].

Theorem 4.1. Let \( x_0^{(\gamma_1, \gamma_2)} \), \( x_1^{(\gamma_1, \gamma_2)} \) \( \in \widetilde{S} \), \( x_{0d}^{(\gamma_1, \gamma_2)} \), \( l = 0, 1 \), be as in Proposition 3.2 and assume \( t_0 \) is sufficiently small, or analogously, \( \tau_0 \) is sufficiently large. Then there exist corrections

\[
\left( \Delta \dot{x}_0^{(\gamma_1, \gamma_2)}, \Delta \dot{x}_1^{(\gamma_1, \gamma_2)} \right), \left( \Delta x_{0d}^{(\gamma_1, \gamma_2)}, \Delta x_{1d}^{(\gamma_1, \gamma_2)} \right)
\]

satisfying

\[
\left\| \left( \Delta \dot{x}_0^{(\gamma_1, \gamma_2)}, \Delta \dot{x}_1^{(\gamma_1, \gamma_2)} \right) \right\|_S \ll \left\| (x_0, x_1) \right\|_S + \left\| x_{0d} \right\|,
\]

\[
\left| \Delta x_{0d}^{(\gamma_1, \gamma_2)} \right| + \left| \Delta x_{1d}^{(\gamma_1, \gamma_2)} \right| \ll \left\| (x_0, x_1) \right\|_S + \left\| x_{0d} \right\|,
\]

and such that the \( \left( \Delta \dot{x}_0^{(\gamma_1, \gamma_2)} , \Delta \dot{x}_1^{(\gamma_1, \gamma_2)} \right), \left( \Delta x_{0d}^{(\gamma_1, \gamma_2)} , \Delta x_{1d}^{(\gamma_1, \gamma_2)} \right) \) depend in Lipschitz continuous fashion on \( (x_0, x_1, x_{0d}) \) with respect to \( \left\| \cdot \right\|_S + \left\| \cdot \right\| \) with Lipschitz constant \( \ll 1 \), and such that the equation (4.2) with data

\[
(x(\tau_0, \xi), (D_x x)(\tau_0, \xi)) = (x_0^{(\gamma_1, \gamma_2)} + \Delta x_0^{(\gamma_1, \gamma_2)}, x_1^{(\gamma_1, \gamma_2)} + \Delta x_1^{(\gamma_1, \gamma_2)})
\]

\[
(x_{0d}(\tau_0), \dot{c}_t x_{0d}(\tau_0)) = (x_{0d}^{(\gamma_1, \gamma_2)} + \Delta x_{0d}^{(\gamma_1, \gamma_2)}, x_{1d}^{(\gamma_1, \gamma_2)} + \Delta x_{1d}^{(\gamma_1, \gamma_2)})
\]

admits a solution \( \dot{x}(\tau, \xi) \) for \( \tau \geq \tau_0 \) corresponding to \( \dot{\varepsilon}(\tau, R) \in H^{\frac{3}{2} + \epsilon} \) where

\[
\dot{\varepsilon}(\tau, R) = x_{0d}(\tau) \phi_{0d}(R) + \int_{0}^{\tau} x(\tau, \xi) \phi(R, \xi) \rho(\xi) d\xi.
\]

Finally, we have energy decay within the light cone:

\[
\lim_{r \to 0} \int_{|x| \leq r} \frac{1}{2} \left| \nabla_{t,x} \varepsilon \right|^2 dx = 0
\]

where we recall \( \varepsilon = R^{-1} \dot{\varepsilon} \).
Remark 4.1. In fact, the Fourier coefficients \( (\Delta x_0^{(\gamma_1, \gamma_2)}, \Delta x_1^{(\gamma_1, \gamma_2)}) \) will have a very specific form, which makes them well-behaved with respect to re-scalings (which hence don’t entail smoothness loss). This shall be important when reverting to the original coordinates \( R_{0,0} \) at time \( t = t_0 \), which were used to specify the perturbation \((x_0, x_1)\) to begin with.

4.2. The proof of Theorem 4.1. It is divided into two parts: the existence part for the solution, which follows essentially verbatim the scheme in [19], and the more delicate verification of Lipschitz dependence of the solution on the data \((x_0, x_1, x_{0d})\). Here the issue is the fact that there are re-scalings involved, and the very parametrix used to solve (4.2), as well as the source terms there, depend implicitly on \( \gamma_{1,2} \), which in turn depend on \((x_0, x_1, x_{0d})\).

4.2.1. Setup of the iteration scheme; the zeroth iterate. Proceeding in close analogy to [19], we shall obtain the final solution \( x(\tau, \xi) \) of (4.2) as the limit of a sequence of iterates \( x^{(j)}(\tau, \xi) \). To begin with, we introduce the zeroth iterate in the following proposition. The somewhat complicated estimates are of the exactly same form as those in [19], and they are motivated and explained there. In particular, below we use the same notation as that used in [19]; thus \( \delta_0 \) is a small constant (depending on \( \nu \)), and we set \( \kappa = 2(1 + \nu^{-1})\delta_0 \) throughout.

**Proposition 4.2.** There is a pair \( (\Delta \tilde{x}_0^{(0)}, \Delta \tilde{x}_1^{(0)}) \in \tilde{S} \), satisfying the bounds
\[
\| (\Delta \tilde{x}_0^{(0)}, \Delta \tilde{x}_1^{(0)}) \|_{\tilde{S}} \leq \tau_0^{-2} \| (x_0, x_1) \|_{\tilde{S}}.
\]
and such that if we set for the continuous spectral part
\[
x^{(0)}(\tau, \xi) := \int_{\tau_0}^{\tau} \Delta^2(\tau, \xi) \frac{\partial^2}{\partial \tau_0^2} \left( \frac{\partial^2}{\partial \tau^2} \xi \right) \sin[\lambda(\tau) \xi^2] \int_{\tau_0}^{\tau} \lambda^{-1}(u) \, du \, \chi(\xi^2) \, d\tau\!
\]
\[
\quad + S(\tau) \left( x_0^{(\gamma_1, \gamma_2)} + \Delta \tilde{x}_0^{(0)} + x_1^{(\gamma_1, \gamma_2)} + \Delta \tilde{x}_1^{(0)} \right),
\]
then the following conclusions obtain, analogous to the estimates in [19]: writing
\[
\tilde{x}^{(0)}(\tau, \xi) := x^{(0)}(\tau, \xi) - S(\tau) \left( x_0^{(\gamma_1, \gamma_2)} + x_1^{(\gamma_1, \gamma_2)} \right), \quad \kappa = 2(1 + \nu^{-1})\delta_0,
\]
we have the high frequency bound
\[
\sup_{\tau \geq \tau_0} \left( \frac{\tau}{\tau_0} \right)^{-k} \| x^{(0)}(\tau, \xi) \|_{S_1} + \sup_{\tau \geq \tau_0} \left( \frac{\tau}{\tau_0} \right)^{k} \| x^{(0)}(\tau, \xi) \|_{S_2}
\]
\[
+ \left( \sum_{N \geq \tau_0} \sup_{\tau \approx N} \left( \frac{\lambda(\tau)}{\lambda(\tau_0)} \right)^{k_0} \| D \xi \xi^4 (\tilde{x}^{(0)}(\tau, \xi) - S(\tau) (\Delta \tilde{x}_0^{(0)} + \Delta \tilde{x}_1^{(0)}) \right) ^2_{\tilde{L}_2^2(\ell_1 > 1)} \right) ^{\frac{1}{2}}
\]
\[
\leq \tau_0^{-1} \| (x_0, x_1) \|_{\tilde{S}} + |x_{0d}|.
\]
For low frequencies \( \xi < 1 \), there is a decomposition

\[
\tilde{x}^{(0)}(\tau, \xi) = \Delta_{> \tau} \tilde{x}^{(0)}(\tau, \xi) + S(\tau) \left( \Delta \tilde{x}^{(0)}_{0}(\xi), \Delta \tilde{x}^{(0)}_{1}(\xi) \right)
\]

where the data \( (\Delta \tilde{x}^{(0)}_{0}(\xi), \Delta \tilde{x}^{(0)}_{1}(\xi)) \) satisfy the vanishing conditions

\[
\int_0^\infty \rho (\xi) \frac{\Delta \tilde{x}^{(0)}_{0}(\xi)}{\xi^\frac{1}{2}} \cos[\lambda(\tau_0)\xi^\frac{1}{2}] \int_{\tau_0}^\infty \lambda^{-1}(u) \, du \, d\xi = 0,
\]

\[
\int_0^\infty \rho (\xi) \frac{\Delta \tilde{x}^{(0)}_{1}(\xi)}{\xi^\frac{3}{2}} \sin[\lambda(\tau_0)\xi^\frac{1}{2}] \int_{\tau_0}^\infty \lambda^{-1}(u) \, du \, d\xi = 0,
\]

and such that we have the bound

\[
\| (\Delta \tilde{x}^{(0)}_{0}(\xi), \Delta \tilde{x}^{(0)}_{1}(\xi)) \|_S + \sup_{\tau > \tau_0} \| \chi _{\xi < 1} \Delta_{> \tau} \tilde{x}^{(0)}(\tau, \xi) \|_{S_1} + \left( \sum_{N \geq \tau_0} \sup_{\tau \sim N} \frac{\lambda(\tau)}{\lambda(\tau_0)} \right)^{\frac{1}{2}} \| \delta \|_{\dot{S}^0} \lesssim \tau_0^{-\frac{1}{2}} \| (x_0, x_1) \|_S + |x_{0d}|.
\]

Furthermore, letting \( \Delta \tilde{x}^{(0)}_j, \Delta \tilde{x}^{(0)}_j \), \( j = 1, 2 \), be the corrections corresponding to two initial perturbation pairs

\( (x_0, x_1), (\overline{x}_0, \overline{x}_1) \),

we have

\[
\| (\Delta \tilde{x}^{(0)}_0 - \Delta \tilde{x}^{(0)}_0, \Delta \tilde{x}^{(0)}_1 - \Delta \tilde{x}^{(0)}_1) \|_S \lesssim \tau_0^{-\frac{1}{2}} \| (x_0 - \overline{x}_0, x_1 - \overline{x}_1) \|_S.
\]

For the discrete spectral part, setting

\[
\Delta x^{(0)}_d(\tau) := \int_{\tau_0}^\infty H_d(\tau, \sigma) \langle \lambda^{-2}(\sigma) Re_{\text{approx}} \phi_d(R) \rangle \, d\sigma,
\]

we have the bound

\[
\tau^2 \| \Delta x^{(0)}_d(\tau) \| + \| \partial_\tau \Delta x^{(0)}_d(\tau) \| \lesssim \| (x_0, x_1) \|_S + |x_{0d}|.
\]

We also have the difference bound

\[
\tau^2 \| \Delta x^{(0)}_d(\tau) - \Delta \overline{x}^{(0)}_d(\tau) \| + \| \partial_\tau \Delta x^{(0)}_d(\tau) - \partial_\tau \Delta \overline{x}^{(0)}_d(\tau) \| \lesssim \| (x_0 - \overline{x}_0, x_1 - \overline{x}_1) \|_S + |x_{0d} - \overline{x}_{0d}|.
\]

We shall then set

\[
x^{(0)}_d(\tau) := x^{(0)}_{d_{\gamma_1, \gamma_2}}(\tau) + \Delta x^{(0)}_d(\tau),
\]

where \( x^{(0)}_{d_{\gamma_1, \gamma_2}}(\tau) \) is the ‘free evolution’ of the discrete spectral part constructed as in Lemma 2.1 in [19] with data \( (x^{(0)}_{0d}, \overline{x}^{(0)}_{1d}) \).
Proof. This follows the procedure in [19], and more specifically the proof of Proposition 8.1 there, except for the last statement about Lipschitz continuous dependence. Naturally the precise structure of $e_{\text{approx}}$ comes into play here. We proceed in a number of steps:

**Step 1:** Proof of the high frequency bound. Due to Lemma 7.2 in [19], it suffices to consider the contribution of

$$\tilde{x}^{(0)}(\tau, \xi) - S(\tau)(\Delta \tilde{x}^{(0)}_0, \Delta \tilde{x}^{(0)}_1);$$

we shall prove the somewhat more delicate square-sum type bound, the remaining bounds being more of the same. Recalling (2.29), we consider two cases:

The contribution of $e_{\text{prelim}} - \tilde{e}_{\text{prelim}}$. Write

$$\Xi_1(\tau, \xi) := \int_{\tau_0}^\tau \frac{\lambda^2(\tau)}{\lambda^2(\sigma)} \frac{\rho^2(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)} \xi)}{\rho^2(\xi)} \sin[\lambda(\tau)\xi^\frac{1}{2} \int_0^\sigma \lambda^{-1}(u) \, du] \xi^\frac{3}{2}$$

$$\cdot \mathcal{F} \left( \lambda^{-2}(\sigma) R(e_{\text{prelim}} - \tilde{e}_{\text{prelim}}) \right) (\sigma, \frac{\lambda^2(\tau)}{\lambda^2(\sigma)} \xi) \, d\sigma$$

Then we need to bound (with $\frac{1}{2} + \frac{1}{2} + \delta_0$)

$$\left( \sum_{N \geq \tau_0}^\infty \sup_{\tau \sim N} \left( \frac{\lambda(\tau)}{\Lambda(\tau_0)} \right)^{4\delta_0} \| \xi^\frac{1}{2} \mathcal{D}_\tau \Xi_1(\tau, \xi) \|_{L^2(\xi > 1)} \right) \frac{\lambda^2(\tau)}{\lambda^2(\sigma)} \xi > 1,$$

We observe that on account of

$$\frac{\lambda^2(\tau)}{\lambda^2(\sigma)} \rho^2(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)} \xi)}{\rho^2(\xi)} \sim \frac{\lambda^2(\tau)}{\lambda^2(\sigma)}, \xi > 1,$$

we get

$$\| \xi^\frac{1}{2} \mathcal{D}_\tau \Xi_1(\tau, \xi) \|_{L^2(\xi > 1)} \leq \int_{\tau_0}^\tau \frac{\lambda^2(\tau)}{\lambda^2(\sigma)} \| \xi^\frac{1}{2} + \mathcal{F} \left( \lambda^{-2}(\sigma) R(e_{\text{prelim}} - \tilde{e}_{\text{prelim}}) \right) (\sigma, \frac{\lambda^2(\tau)}{\lambda^2(\sigma)} \xi) \|_{L^2(\xi > 1)} \, d\sigma$$

Furthermore, on account of the properties of the distorted Fourier transform, we have

$$\frac{\lambda^2(\tau)}{\lambda^2(\sigma)} \| \xi^\frac{1}{2} + \mathcal{F} \left( \lambda^{-2}(\sigma) R(e_{\text{prelim}} - \tilde{e}_{\text{prelim}}) \right) (\sigma, \frac{\lambda^2(\tau)}{\lambda^2(\sigma)} \xi) \|_{L^2(\xi > 1)} \leq (\frac{\lambda^2(\tau)}{\lambda^2(\sigma)})^{-\delta_0 - \frac{1}{2}} \| \xi^\frac{1}{2} + \mathcal{F} \left( \lambda^{-2}(\sigma) R(e_{\text{prelim}} - \tilde{e}_{\text{prelim}}) \right) (\sigma, \cdot) \|_{L^2_{\mathfrak{d}}}$$

$$\leq (\frac{\lambda^2(\tau)}{\lambda^2(\sigma)})^{-\delta_0 - \frac{1}{2}} \| \lambda^{-2}(\sigma) R(e_{\text{prelim}} - \tilde{e}_{\text{prelim}}) \|_{H^{\frac{1}{2}}_{\mathfrak{d}}}. $$
Moreover, the fine structure of $\epsilon^{\text{prelim}} \sim \tilde{\epsilon}^{\text{prelim}}$ from Lemma 2.2 as well as Proposition 3.2 give the bound
\[
\left\| \tilde{\lambda}^{-2}(\sigma) R(\epsilon^{\text{prelim}} - \tilde{\epsilon}^{\text{prelim}}) \right\|_{H_{tk}^{1+}} 
\lesssim \sigma^{-2} \cdot \sigma^{2(1 + \nu) - k_0 - 2 +} \cdot \log \tau_0 \tau_0^{-k_0 + \frac{1}{2}(1 + \nu) -} \cdot \sigma^{-\frac{1}{2}} \cdot \left\| (x_0, x_1) \right\| + \left| x_0d \right|.
\]
We conclude that
\[
\left\| \tilde{\epsilon}^{j+} D_t \tilde{\xi}_1(\tau, \xi) \right\|_{L_{\tilde{\xi}}^{j+} (\tilde{\xi} > 1)} \lesssim \tau_0^{-\frac{3}{2}} \cdot \left( \frac{\tilde{\lambda}^2(\tau)}{\lambda(t_0)} \right)^{\frac{1}{2}} \left\| (x_0, x_1) \right\| + \left| x_0d \right|.
\]
It follows that
\[
\left( \sum_{N \gg \tau_0} \sup_{N \text{ dyadic}} \left( \frac{\lambda(t)}{\lambda(t_0)} \right)^{4d_0} \left\| \tilde{\epsilon}^{j+} D_t \tilde{\xi}_1(\tau, \xi) \right\|_{L_{\tilde{\xi}}^{j+} (\tilde{\xi} > 1)} \right)^{\frac{1}{2}} \lesssim \tau_0^{-\frac{3}{2}} \left\| (x_0, x_1) \right\| + \left| x_0d \right|.
\]

The contribution of the remaining source terms
\[
\sum_{2 \leq j \leq 5} \left( \begin{array}{c} 5 \end{array} \right)^{j'} \left( \begin{array}{c} 5 \end{array} \right)^{(5 - j')} v^{j'} [\tilde{\tau}^{5-j'} - \tilde{u}^{5-j'}]^{(5 - \tilde{u}^{4} + u^{4})} v^{j}
\]
Here we use the crude bound
\[
\left\| v(\tau, R) \right\|_{H_{tk}^{1+}} \lesssim \tau^2(2 + \nu) - 2k_*.
\]
Then setting
\[
\tilde{\xi}_2(\tau, \xi) :=
\int_{\tau_0}^{\tau} \frac{\lambda^{-\frac{1}{2}}(\sigma) \rho^{-\frac{1}{2}}(\tilde{\xi}) \sigma^{-\frac{1}{2}}(\tilde{\xi})}{\tilde{\lambda}^2(\sigma)} \sin[\lambda(\tau) \tilde{\xi} \sigma] \tilde{\lambda}^{-1}(u) du
\cdot \mathcal{F}(\lambda^{-2}(\sigma) R \left[ \sum_{2 \leq j \leq 5} \left( \begin{array}{c} 5 \end{array} \right)^{j'} \left( \begin{array}{c} 5 \end{array} \right)^{(5 - j')} v^{j'} [\tilde{\tau}^{5-j'} - \tilde{u}^{5-j'}]^{(5 - \tilde{u}^{4} + u^{4})} v^{j} \right])(\sigma, \lambda^{-2}(\sigma) \tilde{\xi}) d\sigma,
\]
and arguing as for the preceding term, we easily infer the desired bound
\[
\left( \sum_{N \gg \tau_0} \sup_{N \text{ dyadic}} \left( \frac{\lambda(t)}{\lambda(t_0)} \right)^{4d_0} \left\| \tilde{\epsilon}^{j+} D_t \tilde{\xi}_1(\tau, \xi) \right\|_{L_{\tilde{\xi}}^{j+} (\tilde{\xi} > 1)} \right)^{\frac{1}{2}} \lesssim \tau_0^{-1} \left\| (x_0, x_1) \right\| + \left| x_0d \right|.
\]
on account of our choice of $k_*$. This concludes Step 1.

**Step 2: Choice of the corrections $(\Delta^{(0)}_0, \Delta^{(0)}_1)$.** In analogy to [19], we shall pick these corrections in the specific form
\[
\Delta^{(0)}_0(\xi) = \alpha \mathcal{F} \left( \chi_{R \in \mathcal{C}_1} \phi(R, 0) \right), \Delta^{(0)}_1(\xi) = \beta \mathcal{F} \left( \chi_{R \in \mathcal{C}_1} \phi(R, 0) \right),
\]
and we need to determine the parameters $\alpha, \beta$ in order to force the required vanishing conditions for $\Delta x^{(0)}_0(\xi), \Delta x^{(0)}_1(\xi)$, which in turn are determined by

$$
\Delta x^{(0)}_0(\xi) = \int_{t_0}^{\infty} \frac{\lambda^2(\tau_0)}{\lambda^2(\sigma)} \frac{\rho^2(\xi)}{\rho^2(\xi)} \sin[\lambda(\tau_0)\xi^3 \int_{\tau_0}^{\sigma} \lambda^{-1}(u) du] \mathcal{F}(\lambda^{-2}(\sigma)Re_{approx})(\sigma, \frac{\lambda^2(\tau_0)}{\lambda^2(\sigma)} \xi) d\sigma + \Delta \tilde{x}^{(0)}_0(\xi),
$$

$$
\Delta x^{(0)}_1(\xi) = \int_{t_0}^{\infty} \frac{\lambda^2(\tau_0)}{\lambda^2(\sigma)} \frac{\rho^2(\xi)}{\rho^2(\xi)} \cos[\lambda(\tau_0)\xi^3 \int_{\tau_0}^{\sigma} \lambda^{-1}(u) du] \mathcal{F}(\lambda^{-2}(\sigma)Re_{approx})(\sigma, \frac{\lambda^2(\tau_0)}{\lambda^2(\sigma)} \xi) d\sigma + \Delta \tilde{x}^{(0)}_1(\xi),
$$

Thus writing $\Delta x^{(0)}_j(\xi) := \Delta x^{(0)}_j(\xi) - \Delta \tilde{x}^{(0)}_j(\xi), j = 0, 1$, we need the following simple

**Lemma 4.3.** We have the bounds

$$
| \int_{t_0}^{\infty} \frac{\rho^2(\xi)}{\xi^4} \Delta x^{(0)}_0(\xi) \sin[\lambda(\tau_0)\xi^3 \int_{\tau_0}^{\sigma} \lambda^{-1}(u) du] d\xi | \leq \tau_0^{-1-\frac{1}{n}} \| (x_0, x_1) \|_S + |x_0d|,
$$

$$
| \int_{t_0}^{\infty} \frac{\rho^2(\xi)}{\xi^4} \Delta x^{(0)}_1(\xi) \cos[\lambda(\tau_0)\xi^3 \int_{\tau_0}^{\sigma} \lambda^{-1}(u) du] d\xi | \leq \tau_0^{-1-\frac{1}{n}} \| (x_0, x_1) \|_S + |x_0d|,
$$

**Proof.** (lemma) This is accomplished by checking the contributions of the various terms comprising $e_{approx}$. We consider here the contribution of

$$
e_{approx}^\text{prelim} - \tilde{e}_{approx}^\text{prelim},$$

the remaining terms being treated similarly. We distinguish between three frequency regimes:

(i): $\xi < 1$. Here we get

$$
|\Delta x^{(0)}_0(\xi)| \leq \xi^{-\frac{1}{n}+} \int_{t_0}^{\infty} \frac{\lambda(\tau_0)}{\lambda(\sigma)} |\mathcal{F}(\lambda^{-2}(\sigma)Re_{approx})(\sigma, \frac{\lambda^2(\tau_0)}{\lambda^2(\sigma)} \xi)| d\sigma
$$

$$
\leq \xi^{-\frac{1}{n}+} \int_{t_0}^{\infty} \frac{\lambda(\tau_0)}{\lambda(\sigma)} \cdot \tau_0^{-1} \log \tau_0 \| (x_0, x_1) \|_S + |x_0d| \cdot \int_{t_0}^{\infty} \frac{\lambda(\tau_0)}{\lambda(\sigma)} \cdot \sigma^{-1} \cdot \frac{\lambda^2(\tau_0)}{\lambda^2(\sigma)} \frac{\tau_0}{\sigma} \kappa + 2 \sigma^{-\frac{1}{n}+} d\sigma
$$

$$
\leq \xi^{-\frac{1}{n}+} \tau_0^{-1-\frac{1}{n}} \| (x_0, x_1) \|_S + |x_0d|,
$$

where $\kappa = \kappa(n, \mathcal{H}, \mathcal{K}, \mathcal{P})$. The remaining terms are estimated similarly.
We conclude that
\[
\int_0^1 \frac{\rho^\frac{1}{2}(\xi)}{\xi^\frac{1}{2}} \cos[\lambda(a \tau_0)\xi^\frac{1}{2}] \int_0^\infty \lambda^{-1}(u) \, du \, d\xi
\]
\[
\leq \tau_0^{-(1-\frac{1}{2})} \left[ \| (x_0, x_1) \|_{\mathcal{S}} + |x_0d| \right] \int_0^1 \xi^{-(1-\frac{1}{2})} \, d\xi
\]
\[
\leq \tau_0^{-(1-\frac{1}{2})} \left[ \| (x_0, x_1) \|_{\mathcal{S}} + |x_0d| \right].
\]

(ii): \(1 \leq \xi \leq \frac{e^2}{\lambda^2(\tau_0)}\). Call the contribution to \(\tilde{\Delta}x^{(0)0}_1\) under this restriction \(\tilde{\Delta}x^{(0)0}_1\).

Arguing as in the preceding case, we obtain here
\[
\left| \tilde{\Delta}x^{(0)0}_1(\xi) \right| \leq \tau_0^{-(1-\frac{1}{2})} \left[ \| (x_0, x_1) \|_{\mathcal{S}} + |x_0d| \right],
\]
which in turn implies
\[
\left| \int_1^\infty \frac{\rho^\frac{1}{2}(\xi)}{\xi^\frac{1}{2}} \cos[\lambda(a \tau_0)\xi^\frac{1}{2}] \int_0^\infty \lambda^{-1}(u) \, du \, d\xi \right| \leq \tau_0^{-(1-\frac{1}{2})} \left[ \| (x_0, x_1) \|_{\mathcal{S}} + |x_0d| \right].
\]

(iii): \(\xi > \frac{e^2}{\lambda^2(\tau_0)}\). Here we use that for the corresponding contribution to \(\tilde{\Delta}x^{(0)0}_2\), which we call \(\tilde{\Delta}x^{(0)0}_2\), we have
\[
\| \xi \tilde{\Delta}x^{(0)0}_2(\xi) \|_{L_{\mathcal{S}}^2} \leq \int_0^\infty \| \xi^\frac{1}{2} \frac{\lambda^2(\tau_0)}{\lambda^2(\sigma)} \mathcal{F}(\lambda^{-2}(\sigma)R(e_{\text{prelim}} - \bar{e}_{\text{prelim}}))(\sigma, \lambda^2(\sigma)\xi) \|_{L_{\mathcal{S}}^2(\xi^\frac{1}{2} e^2(\sigma) / \lambda^2(\tau_0))} \, d\sigma
\]
\[
\leq \int_0^\infty \| \xi^\frac{1}{2} \mathcal{F}(\lambda^{-2}(\sigma)R(e_{\text{prelim}} - \bar{e}_{\text{prelim}}))(\sigma, \cdot) \|_{L_{\mathcal{S}}^2} \, d\sigma
\]
\[
\leq \int_0^\infty \left( \mathcal{F}(\lambda^{-2}(\sigma)R(e_{\text{prelim}} - \bar{e}_{\text{prelim}}))(\sigma, \cdot) \right) \|_{\mathcal{S}^1} \, d\sigma
\]
\[
\leq \tau_0^{-\frac{1}{2}} \left[ \| (x_0, x_1) \|_{\mathcal{S}} + |x_0d| \right].
\]

We conclude by Cauchy-Schwarz that
\[
\left| \int_1^\infty \frac{\rho^\frac{1}{2}(\xi)}{\xi^\frac{1}{2}} \cos[\lambda(a \tau_0)\xi^\frac{1}{2}] \int_0^\infty \lambda^{-1}(u) \, du \, d\xi \right| \leq \| \xi \tilde{\Delta}x^{(0)0}_2(\xi) \|_{L_{\mathcal{S}}^2} \leq \tau_0^{-\frac{1}{2}} \left[ \| (x_0, x_1) \|_{\mathcal{S}} + |x_0d| \right].
\]

The contributions of the remaining terms forming \(e_{\text{approx}}\) are handled similarly, as is the second estimate of the lemma involving \(\tilde{\Delta}x^{(0)1}_1\).

We can now conclude Step 2 by observing that
\[
\int_0^\infty \frac{\rho^\frac{1}{2}(\xi)}{\xi^\frac{1}{2}} \mathcal{F}(\chi_{R \leq C \phi(R, 0)})(\xi) \cos[\lambda(a \tau_0)\xi^\frac{1}{2}] \int_0^\infty \lambda^{-1}(u) \, du \, d\xi \sim 1,
\]
\[ \int_0^\infty \rho_\tau^\frac{1}{2}(\xi) \mathcal{F}\left(\chi_{R<\epsilon}\phi(R,0)\right)(\xi) \sin[\lambda(\tau_0)\xi^\frac{1}{2}] \int_{\tau_0}^\infty \lambda^{-1}(u) \, du \, d\xi | \sim \tau_0, \]

see the proof of Proposition 4.2 while we also have

\[ \|\mathcal{F}\left(\chi_{R<\epsilon}\phi(R,0)\right)\|_{\mathcal{S}_1} \lesssim \tau_0^{-(1-\epsilon)}, \quad \|\mathcal{F}\left(\chi_{R<\epsilon}\phi(R,0)\right)\|_{\mathcal{S}_2} \lesssim \tau_0^{0+}. \]

**Step 3:** *Proof of the low frequency bounds.* From [19], see the quantity A in step 3 of the proof of Prop. 7.0.6 there, we infer the definition of \( \Delta_{x^0}(\tau,\xi) \) as follows:

\[ \Delta_{x^0}(\tau,\xi) = -\int_\tau^\infty \frac{\lambda^2(\tau)}{\lambda^2(\sigma)} \frac{\rho_\tau^\frac{1}{2}(\xi)}{\rho_\tau^\frac{1}{2}(\sigma)} \sin[\lambda(\tau)\xi^\frac{1}{2}] \frac{\lambda^{-1}(\sigma) \sigma^2}{\xi^\frac{1}{2}} \cdot \mathcal{F}\left(\lambda^{-2}(\sigma)R(e_{approx})\right)(\sigma, \frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi) \, d\sigma \]

Then we estimate the undifferentiated expression by

\[ \|\xi^{-0+}\Delta_{x^0}(\tau,\xi)\|_{L^2_{\mathcal{S}}(\xi<1)} \lesssim \tau_0^{0+} \int_\tau^\infty \frac{\lambda(\tau)}{\lambda(\sigma)} \|\mathcal{F}\left(\lambda^{-2}(\sigma)R(e_{approx})\right)(\sigma, \frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)\|_{L^\infty_{\mathcal{S}}} \, d\sigma \]

Then as usual we distinguish between the different parts of \( e_{approx} \). For example, for the contribution of the principal part \( e_{\text{prelim}} - \tilde{e}_{\text{prelim}} \), we get by arguing as in (i) of the proof of the preceding lemma

\[ \tau_0^{0+} \int_\tau^\infty \frac{\lambda(\tau)}{\lambda(\sigma)} \|\mathcal{F}\left(\lambda^{-2}(\sigma)R(e_{\text{prelim}} - \tilde{e}_{\text{prelim}})\right)(\sigma, \frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)\|_{L^\infty_{\mathcal{S}}} \, d\sigma \leq \tau_0^{0+} \cdot \frac{k_0+1}{\lambda_{\text{prelim}}^1(\tau_0)} \cdot \int_\tau^\infty \frac{\lambda(\tau)}{\lambda(\sigma)} \sigma^{-k_0-3} \lambda_{\text{prelim}}^1(\sigma) \, d\sigma \cdot \|\mathbf{x}_0, x_1\|_{\mathcal{S}} + |x_0| \]

This is even better than what we need, since we have omitted the weight \((\frac{\tau}{\tau_0})^{-\epsilon}\).

The remaining terms in \( e_{approx} \) lead to similar contributions, and the square sum norm in the low frequency estimate in Proposition 4.2 is also estimated similarly.

**Step 4:** *Control over the data for the free part in the low frequency regime, i.e. \( \Delta x^0(\tau,\xi) \).* In light of the low frequency bound established in the preceding step, it suffices to establish the high-frequency bound, i.e. restrict to \( \xi > 1 \). Thus we need to bound

\[ \|\xi^{1+}\Delta x^0(\tau,\xi)\|_{L^2_{\mathcal{S}}(\xi>1)} + \|\xi^{\frac{1}{2}+}\Delta x^0(\tau,\xi)\|_{L^2_{\mathcal{S}}(\xi>1)}. \]
We shall restrict to bounding the contribution of the term \(e_{\text{prelim}} - \tilde{e}_{\text{prelim}}\), whence replace \(\Delta x^{(0)}(\xi)\) by

\[
- \int_{\tau_0}^{\infty} \frac{\hat{1}^* (\tau_0) \hat{\rho}^\frac{1}{2} (\frac{\lambda^2 (\tau_0)}{\lambda^2 (\sigma)} \xi)}{\hat{\rho}^\frac{1}{2} (\xi)} \sin[\lambda (\tau_0) \xi^\frac{1}{2} \int_{\tau_0}^{\sigma} \lambda^{-1} (u) \, du] \left[ F (\lambda^{-2} (\sigma) R (e_{\text{prelim}} - \tilde{e}_{\text{prelim}})) (\sigma, \frac{\lambda^2 (\tau_0)}{\lambda^2 (\sigma)} \xi) \right] d\sigma
\]

and analogously for \(\Delta x^{(0)}_j (\xi)\). In light of the bounds for \(\Delta x^{(0)}_j\), \(j = 0, 1\), it then suffices to bound rather crudely

\[
\| \xi^\frac{1}{2} + \int_{\tau_0}^{\infty} \frac{\lambda^2 (\tau_0)}{\lambda^2 (\sigma)} \xi \left[ F (\lambda^{-2} (\sigma) R (e_{\text{prelim}} - \tilde{e}_{\text{prelim}})) (\sigma, \frac{\lambda^2 (\tau_0)}{\lambda^2 (\sigma)} \xi) \right] d\sigma \| L^2_{\text{def}} (\xi > 1)
\]

Then the first term on the right (intermediate frequencies) is bounded by

\[
\int_{\tau_0}^{\infty} \frac{\lambda^2 (\tau_0)}{\lambda^2 (\sigma)} \xi \left[ F (\lambda^{-2} (\sigma) R (e_{\text{prelim}} - \tilde{e}_{\text{prelim}})) (\sigma, \frac{\lambda^2 (\tau_0)}{\lambda^2 (\sigma)} \xi) \right] d\sigma
\]

The second term above (large frequencies) in turn is bounded by

\[
\int_{\tau_0}^{\infty} \frac{\lambda^2 (\tau_0)}{\lambda^2 (\sigma)} \xi \left[ F (\lambda^{-2} (\sigma) R (e_{\text{prelim}} - \tilde{e}_{\text{prelim}})) (\sigma, \frac{\lambda^2 (\tau_0)}{\lambda^2 (\sigma)} \xi) \right] d\sigma
\]
The contributions of the remaining parts of $e_{\text{approx}}$ to $\Delta \tilde{x}_0(\xi)$ as well as the second term $\Delta \tilde{x}_1(\xi)$ are handled similarly.

**Step 5:** Lipschitz continuity of the corrections $(\Delta \tilde{x}_0(\xi), \Delta \tilde{x}_1(\xi))$ with respect to the original perturbations $(x_0, x_1)$. Here we prove the final assertion of the proposition. We note that on account of our construction of $(\Delta \tilde{x}_0(\xi), \Delta \tilde{x}_1(\xi))$ in step 2, their dependence on $(x_0, x_1)$ comes solely through the coefficients $\alpha, \beta$. We consider the first of these, the second being treated similarly. Then recall that we have

$$\alpha = \frac{-\int_0^\infty \frac{1}{\xi} \frac{1}{\xi} \cos[\lambda(\tau_0)\xi^\frac{1}{2} \int_0^\infty \lambda^{-1}(u) du] \, d\xi}{\int_0^\infty \frac{1}{\xi} \frac{1}{\xi} \cos[\lambda(\tau_0)\xi^\frac{1}{2} \int_0^\infty \lambda^{-1}(u) du] \, d\xi}.$$ 

Here recall that $\lambda = \lambda_{1,2}$ depends implicitly on the perturbation $(x_0, x_1)$ via the parameters $\gamma_{1,2}$. We then also need to analyse the dependence of $\Delta \tilde{x}_0(\xi)$ on $(x_0, x_1)$, via $\gamma_{1,2}$. Recall that by construction we can write

$$e_{\text{approx}} = e_{\text{approx}}(\tau_{0,0}, R_{0,0}, \gamma_{1,2}),$$

where we use the $\gamma$-independent variables

$$\tau_{0,0}(t) := \int_t^\infty s^{-1-\nu} ds, R_{0,0} = \lambda_{0,0}(t)r = t^{-1-\nu}r,$$

which are to be contrasted with the variables $\tau, R$ that are defined by

$$\tau(t) = \int_t^\infty \lambda_{1,2}(s) ds, R = \lambda_{1,2}(t)r.$$

Thus committing abuse of notation and setting

$$e_{\text{approx}}(\tau, R, \gamma_{1,2}) = e_{\text{approx}}(\tau_{0,0}(\tau, \gamma_{1,2}), R_{0,0}(\tau, R, \gamma_{1,2}), \gamma_{1,2}),$$

we infer

$$\partial_\tau e_{\text{approx}}(\tau, R, \gamma_{1,2}) = \partial_\tau \tau_{0,0} \cdot \partial_\tau e_{\text{approx}} + \partial_\tau \left(\frac{\lambda_{0,0}}{\lambda}\right) \cdot R \partial_{R_{0,0}} e_{\text{approx}},$$

$$\partial_R e_{\text{approx}}(\tau, R, \gamma_{1,2}) = \frac{\lambda_{0,0}}{\lambda} \cdot \partial_{R_{0,0}} e_{\text{approx}}.$$

Further, we have

$$\partial_\gamma e_{\text{approx}}(\tau, R, \gamma_{1,2}) = \partial_\gamma \tau_{0,0} \cdot \partial_\gamma e_{\text{approx}} + \partial_\gamma \left(\frac{\lambda_{0,0}}{\lambda}\right) R \cdot \partial_{R_{0,0}} e_{\text{approx}} + \partial_\gamma e_{\text{approx}}.$$
for \( j = 1, 2 \). It follows that

\[
\partial_j \varphi e_{\text{approx}}(\tau, R, \gamma_{1,2}) = \\
\frac{A}{\lambda_{0,0}} \partial_j \left( \frac{A_{0,0}}{A} \right) R \cdot \partial R e_{\text{approx}} + \partial_j \tau_{0,0}(\partial \tau_{0,0})^{-1} \left[ \partial \tau e_{\text{approx}} - \frac{A}{\lambda_{0,0}} \partial \left( \frac{A_{0,0}}{A} \right) \cdot R \partial e_{\text{approx}} \right] \\
+ \partial_j e_{\text{approx}} \\
= A(\tau, \gamma_{1,2}) R \partial e_{\text{approx}} + B(\tau, \gamma_{1,2}) \partial \tau e_{\text{approx}} + \partial_j e_{\text{approx}}.
\]

Next, recall that

\[
\Delta x(0) e(\xi) = \\
\int_{\tau_0}^{\infty} \frac{A^2(\tau_0)}{\lambda^2(\sigma)} \frac{\rho^2(\xi)}{\lambda^2(\xi)} \sin[\lambda(\tau_0)\xi^2 \int_{\tau_0}^{\sigma} \lambda^{-1}(u) du] \frac{\lambda^2(\tau_0)}{\lambda^2(\sigma)} \xi^2 \frac{F(\lambda^{-2}(\sigma) Re_{\text{approx}})(\sigma, \lambda^2(\tau_0) \xi)}{\xi^2} d\sigma
\]

where the time \( \tau_0 \) also depends on \( \gamma_{1,2} \) via the equation

\[
\tau_0 = \int_{\tau_0}^{\infty} \lambda_{1,2}(s) ds.
\]

Then we directly check from the definitions that \( j = 1, 2 \)

\[
\left| \partial_j \left( \frac{A^2(\tau_0)}{\lambda^2(\sigma)} \frac{\rho^2(\xi)}{\lambda^2(\xi)} \sin[\lambda(\tau_0)\xi^2 \int_{\tau_0}^{\sigma} \lambda^{-1}(u) du] \right) \right| \\
\leq \log \tau_0 \tau_0^{k_0} \frac{A^2(\tau_0)}{\lambda^2(\sigma)} \frac{\rho^2(\xi)}{\lambda^2(\xi)} (\tau_0 + \frac{\sin[\lambda(\tau_0)\xi^2 \int_{\tau_0}^{\sigma} \lambda^{-1}(u) du]}{\xi^2})
\]

On the other hand, when the derivative falls on the Fourier coefficient, we shall take advantage of Lemma 3.1 (more precisely, we use an infinitesimal version of it here) in order to obtain terms which can be integrated by parts with respect to
Specifically, we set for the first iterate

\[ \mathcal{F}(\lambda^{-2}(\sigma)Re_{\text{approx}})(\sigma, \frac{\lambda^2(\tau_0)}{\lambda^2(\sigma)} \xi) \]

\[ = \tau_0^{-k_0} \mathcal{F}(\lambda^{-2}(\sigma)Re_{\text{approx}})(\sigma, \frac{\lambda^2(\tau_0)}{\lambda^2(\sigma)} \xi) \]

\[ + \tau_0^{-k_0}(\xi \partial_{\xi})[\mathcal{F}(\lambda^{-2}(\sigma)Re_{\text{approx}})(\sigma, \frac{\lambda^2(\tau_0)}{\lambda^2(\sigma)} \xi)] \]

\[ + \tau_0^{-k_0}[K \mathcal{F}(\lambda^{-2}(\sigma)Re_{\text{approx}})](\sigma, \frac{\lambda^2(\tau_0)}{\lambda^2(\sigma)}) \]

\[ + \tau_0^{-k_0}(\sigma \partial_{\sigma})[\mathcal{F}(\lambda^{-2}(\sigma)Re_{\text{approx}})](\sigma, \frac{\lambda^2(\tau_0)}{\lambda^2(\sigma)} \xi) \]

\[ + \mathcal{F}(\lambda^{-2}(\sigma)R \partial_{\gamma}Re_{\text{approx}})(\sigma, \frac{\lambda^2(\tau_0)}{\lambda^2(\sigma)} \xi) \]

with a similar relation for \( j = 2 \) but including an extra logarithm \( \log \tau_0 \). Using these relations to evaluate \( \partial_{\gamma} \Delta x^{(0)}(\xi) \) and performing integrations by parts with respect to \( \sigma \) or \( \xi \) as needed and also using Proposition 3.2 allows us to infer the bound

\[ \| \int_0^\infty \rho^\xi(\xi) \partial_{\xi} \Delta x^{(0)}(\xi) \| \xi^{\frac{1}{2}} \leq \lambda^{\frac{1}{2}}(\tau_0) \tau_0^{-k_0-2} + O_{r_0}(\| (x_0, x_1) \| S + |x_0d|) \]

It is then easily checked that denoting by \( \alpha, \alpha \) the coefficients corresponding to perturbations \( (x_0, x_1) \) respectively \( (\overline{x}_0, \overline{x}_1) \) (as in Step 2), we get

\[ | \alpha - \overline{\alpha} | \leq \tau_0^{-1} \| (x_0 - \overline{x}_0, x_1 - \overline{x}_1) \| S \]

provided \( \| (x_0, x_1) \| S + |x_0d| \) is sufficiently small depending on \( \tau_0 \). The preceding inequality in turn implies the desired bound

\[ \| \Delta \tilde{x}^{(0)}(0) - \Delta \tilde{x}_0^{(0)} \| S \leq \tau_0^{-(1)} \| (x_0 - \overline{x}_0, x_1 - \overline{x}_1) \| S \]

The bound for the difference \( \Delta \tilde{x}_1^{(0)}(0) - \Delta \tilde{x}_1^{(0)}(0) \) is similar.

We omit the simpler proof for the estimates on the discrete spectral part. \( \square \)

4.2.2. Setup of the iteration scheme: the higher iterates. We next add a sequence of corrections \( \Delta x^{(j)}(\tau, \xi) \) to the zeroth iterate in order to arrive at a solution of (4.2). Specifically, we set for the first iterate

\[ \left( D_\tau^2 + \beta(\tau)D_\tau + \xi \Delta x^{(1)}(\tau, \xi) = R(\tau, \Delta x^{(0)}(\tau, \xi) + \Delta x^{(0)}(\tau, \xi) \right), \quad (4.5)\]
where we recall (4.3) and further use the notation
\[
\Delta f^{(0)}(\tau, \xi) = \mathcal{F} \left( \lambda^{-2}(\tau) \left[ 5(u_{\text{approx}}^4 - u_0^4) \tilde{e}^{(0)} + RN(u_{\text{approx}}, \tilde{e}^{(0)}) \right] \right)(\xi),
\]
\[
\Delta f_d^{(0)}(\tau) = \langle \lambda^{-2}(\tau) \left[ 5(u_{\text{approx}}^4 - u_0^4) \tilde{e}^{(0)} + RN(u_{\text{approx}}, \tilde{e}^{(0)}), \phi_d(R) \rangle.
\]
and we naturally set
\[
\tilde{e}^{(0)}(\tau, R) = \int_0^\infty \phi(R, \xi)x^{(0)}(\tau, \xi)\rho(\xi)\,d\xi.
\]
For the higher iterates \(\Delta x^{(j)}, j \geq 2\), we set correspondingly
\[
(D_\tau^2 + \beta(\tau)D_\tau + \xi^2)\Delta x^{(j)}(\tau, \xi) = \mathcal{R}(\tau, \Delta x^{(j-1)}) + \Delta f^{(j-1)}(\tau, \xi), \quad (4.6)
\]
and we use the definitions
\[
\Delta f^{(j-1)}(\tau, \xi) = \mathcal{F} \left( \lambda^{-2}(\tau) \left[ 5(u_{\text{approx}}^4 - u_0^4) \Delta \tilde{e}^{(j-1)} + RN(u_{\text{approx}}, \Delta \tilde{e}^{(j-1)}) \right] \right)(\xi),
\]
\[
\Delta f_d^{(j-1)}(\tau) = \int_0^\infty \lambda^{-2}(\tau) \left[ 5(u_{\text{approx}}^4 - u_0^4) \Delta \tilde{e}^{(j-1)} + RN(u_{\text{approx}}, \Delta \tilde{e}^{(j-1)}) \right]\phi_d(R)\,dR
\]
where we set
\[
\Delta \tilde{e}^{(j-1)}(\tau, R) = \int_0^\infty \phi(R, \xi)\Delta x^{(j-1)}(\tau, \xi)\rho(\xi)\,d\xi + \Delta x_d^{(j-1)}(\tau)\phi_d(R), \quad j \geq 2.
\]
The fact that upon using suitable initial conditions these equations yield in fact iterates which rapidly converge to zero in a suitable sense follows exactly as in [19], and so we formulate the corresponding result, which is a summary of Propositions 9.1 - 9.6 and most importantly Corollary 12.2, Corollary 12.3 in [19]:

**Proposition 4.4.** For each \(j \geq 1\), there exists a pair \((\Delta \tilde{x}_0^{(j)}, \Delta \tilde{x}_1^{(j)}) \in \tilde{S}\), and such that if we set up the inductive scheme
\[
\Delta x^{(j)}(\tau, \xi) =
\]
\[
\int_{\tau_0}^\tau \frac{\lambda^2(\sigma)}{\lambda^2(\tau)} \frac{\rho^4(\xi)}{\rho^4(\sigma)} \sin[\lambda(\tau)\xi^2 \int_{\tau_0}^\sigma \lambda^{-1}(u)\,du] \xi^2 \cdot \left[ \mathcal{R}(\tau, \Delta \tilde{x}^{(j-1)}) + \Delta f^{(j-1)}(\tau, \xi) \right] d\sigma \quad (4.7)
\]
for the continuous spectral part, while we set
\[
\Delta_d x^{(j)}(\tau) = \int_{\tau_0}^\infty H_d(\tau, \sigma) \cdot \left[ \mathcal{R}_d(\tau, \Delta \tilde{x}^{(j-1)}) + \Delta_d f^{(j-1)}(\tau, \xi) \right](\sigma)\,d\sigma, \quad (4.8)
\]
then we obtain control over the iterates in the following precise sense: there is a splitting
\[
\Delta x^{(j)}(\tau, \xi) = \Delta x^{(j)}(\tau, \xi) + S(\tau)(\Delta \tilde{x}_0^{(j)}, \Delta \tilde{x}_1^{(j)})
\]
in which $\Delta^{(j)}_{\theta}, \Delta^{(j)}_{\xi}$ satisfy the vanishing conditions

$$\int_0^\infty \rho^{\frac{\gamma}{2}}(\xi) \Delta^{(j)}_{\theta}(\xi) \cos[\lambda(\tau_0)] \int_0^\infty \lambda^{-1}(u) \, du \, d\xi = 0 \quad (4.9)$$

and such that if we set

$$\widetilde{\Delta^{(j)}_{\theta}}(\tau, \xi) = \int_0^\tau \frac{\rho^{\frac{\gamma}{2}}(\xi) \sin[\lambda(\tau)\xi^{\frac{1}{2}} \int_0^\tau \lambda^{-1}(u) \, du]}{\rho^{\frac{\gamma}{2}}(\xi)} \cdot [R(\tau, \Delta^{(j-1)}) + \Delta^{(j-1)}(\tau, \xi)](\sigma, \frac{\lambda^2(\tau)}{\lambda^2(\sigma)} \xi) \, d\sigma$$

and introduce the quantities (with $\kappa = 2(1 + \nu^{-1})\delta_0$)

$$\Delta A_j := \sup_{\tau \geq \tau_0} \left( \int_{\tau}^{\tau_0} \lambda^{\frac{\gamma}{2}}(\xi) \Delta^{(j)}_{\theta}(\tau, \xi) \right)_{S^1} + \left( \sum_{N \geq \tau_0} \left[ \sup_{\tau \geq \tau_0} \left( \int_{\tau}^{\tau_0} \lambda^{\frac{\gamma}{2}}(\xi) \Delta^{(j)}_{\theta}(\tau, \xi) \right)_{S^2} \right]^2 \right)^{\frac{1}{2}}$$

then we have exponential decay

$$\Delta A_j \lesssim \delta \left[ \left\| (x_0, x_1) \right\|_{S^2} + |x_0d| \right]$$

for any given $\delta > 0$, provided $\tau_0$ is sufficiently large (or equivalently, $t_0$ is sufficiently small). In particular, the series

$$\xi(\tau, \xi) = \xi^{(0)}(\tau, \xi) + \sum_{j \geq 1} \Delta^{(j)}_{\theta}(\tau, \xi),$$

converges with

$$\sup_{\tau \geq \tau_0} \left( \int_{\tau}^{\tau_0} \lambda^{\frac{\gamma}{2}}(\xi) \Delta^{(j)}_{\theta}(\tau, \xi) \right)_{S^1} \leq \sup_{\tau \geq \tau_0} \left( \int_{\tau}^{\tau_0} \lambda^{\frac{\gamma}{2}}(\xi) \Delta^{(j)}_{\theta}(\tau, \xi) \right)_{S^2} \leq \left\| (x_0, x_1) \right\|_{S^2} + |x_0d|.$$
such that \( \overline{x}_0, \overline{x}_1 \) satisfy the natural analogues of (4.9), (4.10), and we have the bounds
\[
\sup_{\tau \geq \tau_0} \left( \frac{\tau_0}{\tau} \right)^{\epsilon} \| x(\tau, \xi) \|_{L^2_{\tau\xi}(\epsilon < 1)} + \sup_{\tau \geq \tau_0} \left( \frac{\tau_0}{\tau} \right)^{-\epsilon} \| x(\tau, \xi) \|_{L^2_{\tau\xi}(\epsilon < 1)}
\]
+ \( \| (\overline{x}_0, \overline{x}_1) \|_{\overline{S}} \leq \| (x_0, x_1) \|_{\overline{S}} + |x_{0d}|. \)

Finally, we also have
\[
\sup_{\tau \geq \tau_0} \tau^{1-\epsilon} |x_{d}(\tau) - x_{d}^{(0)}(\tau)| \leq \| (x_0, x_1) \|_{\overline{S}} + |x_{0d}|.
\]

The function
\[
u(\tau, R) = u_{\text{approx}}(\tau, R) + \tilde{\epsilon}(\tau, R)
\]
with
\[
\tilde{\epsilon}(\tau, R) := x_{d}(\tau)\phi_{d}(R) + \int_{0}^{\infty} \phi(\tau, \xi) x(\tau, \xi) \rho(\xi) \, d\xi
\]
is then the desired solution of (4.11), satisfying the properties in terms of its Fourier transform specified in Theorem 4.1. In fact, we set
\[
\Delta x_{k}(\gamma_1, \gamma_2) = \sum_{j \geq 1} \Delta x_{k}^{(j)}, \Delta x_{d}^{(\gamma_1, \gamma_2)} = \sum_{j \geq 1} \Delta x_{d}^{(j)} x_{d}^{(j)} |_{\tau = \tau_0}, \beta = 0, 1.
\]

In fact, all of the assertions in the preceding long proposition follow exactly from the arguments in [19] (the only difference being the slightly different scaling law \( \lambda(\tau) \)), and this will easily establish almost all of Theorem 4.1 except its last statement concerning the Lipschitz continuous dependence of the initial data perturbation with respect to the initial perturbation \((x_0, x_1)\). This is a somewhat delicate point which requires a special argument, analogous to the one given for the corresponding assertion in Proposition 4.2. We formulate this as a separate proposition at the level of the iterative corrections:

**Proposition 4.5.** If \((\Delta \overline{x}_0^{(j)}, \Delta \overline{x}_1^{(j)}), (\Delta \overline{x}_0^{(j)}, \Delta \overline{x}_1^{(j)})\), \(j \geq 1\), are as in the preceding proposition and with respect to perturbations \((x_0, x_1) \in \overline{S}\) respectively \((\overline{x}_0, \overline{x}_1) \in \overline{S}\), then for any given \( \delta > 0 \) we have the Lipschitz bound
\[
\| (\Delta \overline{x}_0^{(j)} - \Delta \overline{x}_0^{(j)}), \Delta \overline{x}_1^{(j)} - \Delta \overline{x}_1^{(j)} \|_{\overline{S}} \leq \delta \tau_0^{(1-\delta)} \| (x_0 - \overline{x}_0, x_1 - \overline{x}_1) \|_{\overline{S}} + |x_{0d} - \overline{x}_{0d}|,
\]
provided \( \tau_0 \) is sufficiently large compared to \( \delta \), and
\[
\| (x_0, x_1) \|_{\overline{S}} + \| (\overline{x}_0, \overline{x}_1) \|_{\overline{S}} + |x_{0d}| + |\overline{x}_{0d}|
\]
is sufficiently small depending on \( \tau_0 \).

To begin the proof, we observe from the proofs of Proposition 7.1, 8.1, 9.1 in [19] that the profiles of the corrections \( \Delta \overline{x}_k^{(j)}, \kappa = 0, 1 \), are fixed up to a multiplication parameter, and more precisely we set
\[
\Delta \overline{x}_0^{(j)} = a^{(j)}F(\chi_{R \in C_{\tau_0} \phi(R, 0)}), \Delta \overline{x}_1^{(j)} = b^{(j)}F(\chi_{R \in C_{\tau_0} \phi(R, 0)}),
\]
whence the only dependence of the corrections \( \Delta x^{(j)} \) on the data \( x_{0,1} \) reside in the coefficients \( \alpha^{(j)}, \beta^{(j)} \). The latter, however, depend in a complex manner on the iterative functions \( \Delta x^{(j)}, \Delta_\delta x^{(j)} \), and so we cannot get around analysing the (Lipschitz)-dependence of the latter on \( x_{0,1} \). This latter task is rendered somewhat cumbersome by the fact that in each iterative step we use a parametrix which rescales the ingredients (via the factors \( \frac{L^2(\tau)}{L^2(\sigma)} \)), which depend on \( \gamma_{1,2} \) whence on \( x_{0,1} \), and so differentiating with respect to \( \gamma_j \) will result in a loss of smoothness. What saves things here is the fact that the coefficients \( \alpha^{(j)}, \beta^{(j)} \) are given by certain integrals, which are well-behaved with respect to inputs with lesser regularity, as already seen in Step 5 of the proof of Proposition 4.2: differentiating the term 

\[
\tau_0^{-k_0(\xi)} \left( \mathcal{F} \left( \lambda^{-2}(\sigma) \text{Re}_{\text{approx}} \right) \left( \sigma, \frac{L^2(\tau_0)}{L^2(\sigma)} \xi \right) \right)
\]

which is of lesser regularity with respect to \( \xi \), but the corresponding contribution to \( \partial_\gamma \Delta x^{(0)}(\xi) \) and thence to the integral

\[
\int_0^\infty \rho_0^*(\xi) \partial_\gamma \Delta x^{(0)}(\xi) \cos \left( \lambda(\tau_0) \xi^4 \right) \int_0^\infty \lambda^{-1}(u) du \ d\xi
\]

is then handled by integration by parts with respect to \( \xi \).

The exact same type of observation applies to the higher order corrections \( \Delta x^{(j)}(\tau, \xi) \) as well.

To render this intuition precise, we first need to exhibit a functional framework which will be preserved by the iterative steps and which adequately describes the \( \gamma_j \) differentiated corrections \( \Delta x^{(j)} \). To begin with, we introduce two types of norms:

**Definition 4.1.** Call a pair of functions \( (\Delta y(\tau, \xi), \Delta y_\delta(\tau)) \) strongly bounded, provided there exist \( (\Delta \tilde{y}_0(\xi), \Delta \tilde{y}_1(\xi)) \in \tilde{S} \), as well as \( (\Delta \tilde{y}_0(\xi), \Delta \tilde{y}_1(\xi)) \in \tilde{S} \), the latter satisfying the vanishing conditions \( (4.9), (4.10) \), such that if we set

\[
\begin{align*}
\Delta y(\tau, \xi) &= \Delta_{\tau} y(\tau, \xi) + S(\tau) \left( \Delta \tilde{y}_0(\xi), \Delta \tilde{y}_1(\xi) \right), \\
\Delta y_\delta(\tau, \xi) &= \Delta \tilde{y} \left( (\xi, \xi) \right) + S(\tau) \left( \Delta \tilde{y}_0(\xi), \Delta \tilde{y}_1(\xi) \right)
\end{align*}
\]
then we have

\[ + \infty > \| (\Delta y(\tau, \xi), \Delta y_d(\tau)) \|_{S_{\text{strong}}} := \]

\[ \sup_{\tau \geq \tau_0} \left( \frac{\tau_0}{\tau} \right)^\kappa \| \chi_{\varepsilon > 1} \Delta y(\tau, \xi) \|_{S_1} + \left( \sum_{N \geq \tau_0} \left[ \sup_{\tau \sim N} \left( \frac{\tau}{\tau_0} \right)^\kappa \| \chi_{\varepsilon > 1} D_\tau \Delta y(\tau, \xi) \|_{S_2} \right] \right)^{\frac{1}{2}} \]

\[ + \sup_{\tau \geq \tau_0} \left( \frac{\tau_0}{\tau} \right)^\kappa \| \chi_{\varepsilon < 1} \Delta > \tau \Delta y(\tau, \xi) \|_{S_1} + \left( \sum_{N \geq \tau_0} \left[ \sup_{\tau \sim N} \left( \frac{\tau}{\tau_0} \right)^\kappa \| \chi_{\varepsilon < 1} D_\tau \Delta > \tau \Delta y(\tau, \xi) \|_{S_2} \right] \right)^{\frac{1}{2}} \]

\[ + \| (\Delta \tilde{y}_0, \Delta \tilde{y}_1) \|_S + \| (\Delta \tilde{y}_0, \Delta \tilde{y}_1) \|_S + \sup_{\tau \geq \tau_0} \tau^{(1-\kappa)} |\Delta y_d(\tau)| + \sup_{\tau \geq \tau_0} \tau^{(1-\kappa)} |\partial_\tau \Delta y_d(\tau)|. \]

We call a pair of functions \((\Delta z(\tau, \xi), \Delta z_d(\tau))\) weakly bounded, provided there exist \((\Delta \tilde{z}_0(\xi), \Delta \tilde{z}_1(\xi)) \in \tilde{S}\) as well as \((\Delta \tilde{z}_0(\xi), \Delta \tilde{z}_1(\xi)) \in \tilde{S}\) not necessarily satisfying any vanishing conditions, such that if we set

\[ \Delta z(\tau, \xi) = \Delta > \tau z(\tau, \xi) + S(\tau) (\Delta \tilde{z}_0(\xi), \Delta \tilde{z}_1(\xi)), \]

\[ \Delta z(\tau, \xi) = \Delta \tilde{z}(\tau, \xi) + S(\tau) (\Delta \tilde{z}_0(\xi), \Delta \tilde{z}_1(\xi)) \]

then we have

\[ + \infty > \| (\Delta z(\tau, \xi), \Delta z_d(\tau)) \|_{S_{\text{weak}}} := \]

\[ \tau_0^{-1} \left[ \sup_{\tau \geq \tau_0} \left( \frac{\tau_0}{\tau} \right) \frac{\lambda(\tau_0)}{\lambda(\tau)} \right]^{2\delta_0 + 1} \left( \chi_{\varepsilon > 1} \Delta z(\tau, \xi) \right)_{S_1}^{\frac{1}{2}} + L_{d_0}^{\frac{1}{2}} \]

\[ + \left( \sum_{N \geq \tau_0} \left[ \sup_{\tau \sim N} \left( \frac{\tau}{\tau_0} \right) \right] \lambda(\tau) \right)^{1-\delta_0} \| \chi_{\varepsilon > 1} D_\tau \Delta \tilde{z}(\tau, \xi) \|_{S_2}^{\frac{1}{2}} \]

\[ + \sup_{\tau \geq \tau_0} \left( \frac{\tau_0}{\tau} \right) \lambda(\tau_0) \left[ \Delta z(\tau, \xi) \right]_{S_1}^{\frac{1}{2}} + \left( \sum_{N \geq \tau_0} \left[ \sup_{\tau \sim N} \left( \frac{\tau}{\tau_0} \right) \right] \right)^{\frac{1}{2}} \| \chi_{\varepsilon > 1} D_\tau \Delta > \tau \Delta z(\tau, \xi) \|_{S_2} \]

\[ + \| (\Delta \tilde{z}_0, \Delta \tilde{z}_1) \|_S + \| (\Delta \tilde{z}_0, \Delta \tilde{z}_1) \|_S + \sup_{\tau \geq \tau_0} \frac{\tau_0}{\tau} \lambda(\tau_0) \left[ \Delta z_d(\tau) \right] + \left| \partial_\tau \Delta z_d(\tau) \right|. \]

Observe that by comparison to \( \| \cdot \|_{S_{\text{strong}}} \), the norm \( \| \cdot \|_{S_{\text{weak}}} \) loses \( \xi^{-\frac{1}{2}} \) in terms of decay for large \( \xi \), and we lose a factor \( \tau_0 \frac{\lambda(\tau_0)}{\lambda(\tau)} \) in terms of temporal decay.

Using the preceding terminology, we can now introduce the proper norm to measure the expressions arising upon applying \( \partial_\gamma \) to the corrections \( \Delta x^{(j)}(\tau, \xi) \).

To emphasise that we want to measure the differences of functions, we introduce the symbol \( \Delta \tilde{S} \) for the relevant space:
Definition 4.2. We define $\Delta \tilde{S}$ as the space of pairs of functions $(\Delta x(\tau, \xi), \Delta x_d(\tau))$ which admit a decomposition

$$\Delta x(\tau, \xi) = (\xi \partial_\tau)\Delta y(\tau, \xi) + \Delta z(\tau, \xi), \Delta x_d(\tau) = \Delta y_d(\tau) + \Delta z_d(\tau)$$

such that $\Delta y$ is strongly bounded and $\Delta z$ is weakly bounded, and we then set

$$\| (\Delta x(\tau, \xi), \Delta x_d(\tau)) \|_{\Delta \tilde{S}} := \inf \left( \| (\Delta y(\tau, \xi), \Delta y_d(\tau)) \|_{S_{\text{strong}}} + \| (\Delta z(\tau, \xi), \Delta z_d(\tau)) \|_{S_{\text{weak}}} \right)$$

where the infimum is over all decompositions into differentiated strongly bounded and weakly bounded functions.

We use the norm $\| \cdot \|_{\Delta \tilde{S}}$ to measure the pair quantities $(\partial_\gamma \Delta x^{(j)}(\tau, \xi), \partial_\gamma \Delta x_d^{(j)}(\tau))$, where $\gamma = 1, 2$. To achieve this for all the corrections, we need an inductive step which infers the required bound for the next iterate, as well as rapid decay of these quantities. Correspondingly, we have the following two lemmas:

Lemma 4.6. Provided the $(\Delta x^{(j)}, \Delta x_d^{(j)})$ are constructed as in Proposition 4.4 and assuming the bounds there, we have

$$\| (\partial_\gamma \Delta x^{(j)}(\tau, \xi), \partial_\gamma \Delta x_d^{(j)}(\tau)) \|_{\Delta \tilde{S}} \leq \tau_0^{-k_0^{(\gamma)}} \| (\Delta x^{(j-1)}, \Delta x_d^{(j-1)}) \|_{S_{\text{strong}}} + \tau_0^{0^+} \| (\partial_\gamma \Delta x^{(j-1)}(\tau, \xi), \partial_\gamma \Delta x_d^{(j-1)}(\tau)) \|_{\Delta \tilde{S}},$$

$\gamma = 1, 2$.

Lemma 4.7. For any $\delta > 0$, there is $\tau_* = \tau_*(\delta)$ large enough such that if $\tau_0 \geq \tau_*$, then we have

$$\| (\partial_\gamma \Delta x^{(j)}(\tau, \xi), \partial_\gamma \Delta x_d^{(j)}(\tau)) \|_{\Delta \tilde{S}} \leq \delta \tau_0^{-k_0^{(\gamma)}} \| (x_0, x_1) \|_{\tilde{S}} + |x_{0\beta}|.$$  

The proofs of these lemmas follow very closely the arguments in [19], and we shall only indicate the outlines:

Outline of proof of Lemma 4.6. One may assume a decomposition

$$(\partial_\gamma \Delta x^{(j-1)}(\tau, \xi), \partial_\gamma \Delta x_d^{(j-1)}(\tau)) = (\xi \partial_\tau)\Delta y^{(j-1)}(\tau, \xi) + \Delta z^{(j-1)}(\tau, \xi), \Delta y_d^{(j-1)}(\tau) + \Delta z_d^{(j-1)}(\tau))$$

with, say,

$$\| (\Delta y^{(j-1)}, \Delta y_d^{(j-1)}) \|_{S_{\text{strong}}} + \| (\Delta z^{(j-1)}, \Delta z_d^{(j-1)}) \|_{S_{\text{weak}}} \leq \| (\partial_\gamma \Delta x^{(j-1)}(\tau, \xi), \partial_\gamma \Delta x_d^{(j-1)}(\tau)) \|_{\Delta \tilde{S}}$$

Now let the operator $\partial_\gamma$ fall on the expression for $\Delta x^{(j)}(\tau, \xi)$ in Proposition 4.4 given by the parametrix (4.7). Then if $\partial_\gamma$ acts on the scaling factor in

$$[R(\tau, \Delta x^{(j-1)}) + \Delta f^{(j-1)}(\tau, \xi)](\sigma, \frac{\lambda^2(\tau)}{\lambda^2(\sigma)} \xi),$$
Then writing \( \Delta x_0^{(j)} \), \( \Delta x_1^{(j)} \) outside of the parametrix. The situation is slightly more delicate provided one can incorporate the corresponding term into \( (\xi \partial_\tau) \Delta x_1^{(j)}(\tau, \xi) \). On the other hand, if \( \partial_\gamma \) falls on the parametrix factors

\[
\frac{\lambda^2(\tau) \rho^1(\lambda^2(\tau_0)(\xi))}{\lambda^2(\tau_0) \rho^1(\xi)} \sin[\lambda(\tau) \xi \int_0^\tau \lambda^{-1}(u) \ du],
\]

or on one of the \( \gamma_\kappa \)-dependent factors \( u_{\text{approx}} - u_0, u_{\text{approx}} \in \mathcal{N}_{\text{approx}}(\epsilon^{(j-1)}) - N_{\text{approx}}(\epsilon^{(j-2)}) \) (recalling (4.6)). The required bounds follow essentially directly from the proofs of Proposition 7.1, 8.1, 9.1, 9.6 in [19].

One notices that one can ‘essentially’ move the operator \( (\xi \partial_\tau) \) past the non-local operator \( \mathcal{R} \) modulo better errors which can be placed into \( \Delta x^{(j)} \), and further to the outside of the parametrix. The situation is slightly more delicate provided \( \partial_\gamma \) falls on a factor \( \Delta x^{(j)} \) in \( \Delta f^{(j-1)} \), again recalling (4.6) and the definition of \( \Delta f^{(j-1)} \). Then writing

\[
\Delta f^{(j)}(\tau, R) = \Delta x_*^{(j)}(\tau) f_*(R) + \int_0^\infty f(R, \xi) \Delta x^{(j)}(\tau, \xi) \rho(\xi) \ d\xi,
\]

we exploit the spatial localisation \( \mathcal{S} \) of the nonlinear source terms (to a ball \( R \leq C \tau \)) in order to perform an integration by parts, provided

\[
\partial_\gamma \Delta x^{(j)} = (\xi \partial_\xi) \Delta y^{(j)}.
\]

Thus write

\[
\chi_{R \leq C \tau} \int_0^\infty f(R, \xi) (\xi \partial_\xi) \Delta y^{(j)}(\tau, \xi) \rho(\xi) \ d\xi
\]

\[
= -\chi_{R \leq C \tau} \int_0^\infty (\xi \partial_\xi) f(R, \xi) \rho(\xi) \Delta y^{(j)}(\tau, \xi) \ d\xi,
\]

\[\text{[\text{Recall that we may always include a spatial cutoff in front of the nonlinearity, see the comment after (4.7).]}\]
and then use the bound
\[
\sup_{\tau \geq \tau_0} \tau^{-1} \| R^{-1} X_{\tau R} \|_{L^\infty} = \| R^{-1} X_{\tau R} \|_{S_{\text{strong}}}.
\]
If we assume
\[
\partial_{\gamma} \Delta x^{(j)} = \Delta z^{(j)} \in S_{\text{weak}},
\]
we have the weaker estimate
\[
\sup_{\tau \geq \tau_0} \frac{A(\tau_0)}{A(\tau)} \| R^{-1} X_{\tau R} \|_{L^\infty} \leq \| \Delta z^{(j)} \|_{S_{\text{weak}}}.
\]
Using these and arguing just as in the proof of Proposition 9.6 in [19] yields the desired bound for the corresponding contribution of \( \partial_{\gamma} \Delta f^{(j-1)} \) to \( \Delta x^{(j)}(\tau, \xi) \).

Next, consider the effect of \( \partial_{\gamma} \) on the free term, when it falls on the source term \( (\Delta \tilde{x}_0^{(j)}, \Delta \tilde{z}_0^{(j)}) \). In light of the choice of these terms, see the paragraph after the statement of Proposition 4.5 we have
\[
\partial_{\gamma} \Delta z^{(j)}_0 = (\partial_{\gamma} \alpha^{(j)}) \tilde{F}(\chi_{R \leq C \rho} \phi(R, 0)), \quad \partial_{\gamma} \Delta \tilde{x}_1^{(j)} = (\partial_{\gamma} \beta^{(j)}) \tilde{F}(\chi_{R \leq C \rho} \phi(R, 0)),
\]
and we have
\[
\partial_{\gamma} \alpha^{(j)} \sim \partial_{\gamma} \int_0^\infty \frac{\rho^{\frac{1}{2}}(\xi) \tilde{\Delta} \tilde{x}_0^{(j)}(\xi)}{\xi^{\frac{1}{2}}} \cos[\lambda(\tau_0) \xi^{\frac{1}{2}}] \int_{\tau_0}^\infty \lambda^{-1}(u) \, du \, d\xi,
\]
where
\[
\tilde{\Delta} \tilde{x}_0^{(j)}(\xi) = \int_0^{\tau_0} \frac{\lambda^2(\tau_0) \rho^{\frac{1}{2}}(\sigma) \xi^{\frac{1}{2}}}{\lambda^2(\sigma) \xi^{\frac{1}{2}}} \sin[\lambda(\tau_0) \xi^{\frac{1}{2}}] \int_{\tau_0}^\sigma \lambda^{-1}(u) \, du \, d\sigma \frac{H(\sigma, \lambda^2(\tau_0) \xi)}{\lambda^2(\sigma) \xi^{\frac{1}{2}}} \, d\sigma,
\]
and
\[
H(\sigma, \xi) := [R(\tau, \Delta x^{(j-1)}) + \Delta f^{(j-1)}(\tau, \xi)](\sigma, \xi).
\]
The performing integration by parts with respect to \( \xi \) if necessary, one checks that
\[
\left| \int_0^\infty \frac{\rho^{\frac{1}{2}}(\xi) \partial_{\gamma} \Delta \tilde{x}_0^{(j)}(\xi)}{\xi^{\frac{1}{2}}} \cos[\lambda(\tau_0) \xi^{\frac{1}{2}}] \int_{\tau_0}^\infty \lambda^{-1}(u) \, du \, d\xi \right| \leq \tau_0^{\frac{1}{4}} \| \Delta x^{(j-1)} \|_{S_{\text{strong}}} + \| \partial_{\gamma} \Delta x^{(j-1)} \|_{\Delta S_{\text{weak}}}.
\]
This implies the required bound for \( \partial_{\gamma} \Delta \tilde{x}_0^{(j)} \), and the bound for \( \partial_{\gamma} \Delta \tilde{x}_1^{(j)} \) is similar. One then places
\[
S(\tau)(\partial_{\gamma} \Delta \tilde{x}_0^{(j)}, \partial_{\gamma} \Delta \tilde{x}_1^{(j)})
\]
into \( S_{\text{weak}} \).

**Outline of proof of Lemma 4.7** This follows in analogy to the arguments in sections 11 and 12 in [19], a key being re-iterating the iterative step leading from \( \partial_{\gamma} \Delta x^{(j-1)} \) to \( \partial_{\gamma} \Delta x^{(j-1)} \) by differentiating 4.7.
Completion of proof of Proposition 4.5. Recalling Lemma 3.3, we obtain the schematic relation
\[
\| (\Delta \hat{x}^{(j)}_0 - \Delta \hat{\xi}^{(j)}_0, \Delta \hat{x}^{(j)}_1 - \Delta \hat{\xi}^{(j)}_1) \|_\bar{S} \\
\lesssim \delta \| (x_0 - x_0, x_1 - x_1) \|_\bar{S} + \| x_{0d} - \bar{x}_{0d} \| \\
\cdot \left( \frac{\tau_0^{k_0+1} \log \tau_0}{\lambda_{\delta,0}(\tau_0)} \right) \sum_{\kappa=1,2} \| (\partial_{\gamma} \Delta x^{(j)}_0, \partial_{\gamma} \Delta x^{(j)}_d) \|_{\Delta \bar{S}} + \delta^j \tau_0^{-(1-\delta)} \log \tau_0 \tau_0^0 
\]
for any \( \delta > 0 \), provided \( \tau_0 \) is chosen sufficiently large. Further taking advantage of Lemma 4.7, we finally infer
\[
\| (\Delta \hat{x}^{(j)}_0 - \Delta \hat{\xi}^{(j)}_0, \Delta \hat{x}^{(j)}_1 - \Delta \hat{\xi}^{(j)}_1) \|_\bar{S} \\
\lesssim \delta \| (x_0 - x_0, x_1 - x_1) \|_\bar{S} + O_{\tau_0} \left( \| (x_0, x_1) \|_\bar{S} + \| \bar{x}_{0d} \| \right) \\
\cdot \left( \frac{\tau_0^{k_0+1} \log \tau_0}{\lambda_{\delta,0}(\tau_0)} \right) \sum_{\kappa=1,2} \| (\partial_{\gamma} \Delta x^{(j)}_0, \partial_{\gamma} \Delta x^{(j)}_d) \|_{\Delta \bar{S}} + \delta^j \tau_0^{-(1-\delta)} \log \tau_0 \tau_0^0, 
\]
with a similar bound for the discrete spectral part corrections \( (\Delta x_0^{(j)}, \Delta x_0^{(j)}) \). This implies Proposition 4.5.

4.2.3. Proof of Theorem 4.1. This is a consequence of Proposition 4.5. Recalling Proposition 4.2, it suffices to set
\[
(\Delta_0 x_{0d}^{(\gamma_1, \gamma_2)}, \Delta_1 x_{0d}^{(\gamma_1, \gamma_2)}) = \sum_{j=0}^{\infty} \left( \Delta \hat{x}^{(j)}_0, \Delta \hat{x}^{(j)}_1 \right) \\
(\Delta_0 x_{0d}^{(\gamma_1, \gamma_2)}, \Delta_1 x_{0d}^{(\gamma_1, \gamma_2)}) = \sum_{j=0}^{\infty} \left( \Delta x^{(j)}_d (\tau_0), \partial_{\tau_{\gamma}} \Delta x^{(j)}_d (\tau_0) \right)
\]
Then the correction \( \tilde{v}(\tau, R) \) is given by its Fourier coefficients
\[
\tilde{v}(\tau, \xi) = \tilde{v}^{(0)}(\tau, \xi) + \sum_{j=1}^{\infty} \Delta \hat{v}^{(j)}(\tau, \xi)
\]
The decaying bounds over \( \| (\Delta x^{(j)}, \Delta x^{(j)}_d) \|_{\text{strong}} = \Delta A_j \) imply that
\[
\tilde{v}(\tau, R) = x_d(\tau) \phi_d(R) + \int_0^\infty \phi(R, \xi) x(\tau, \xi) \rho(\xi) \, d\xi \in H_d^{\frac{1}{2}+}
\]
for any \( \tau \geq \tau_0 \), as desired.

4.3. Translation to original coordinate system. In the preceding sections, we have obtained a singular solution of the form (the sum of the first four terms on the right representing \( v_{\text{approx}}^{(\gamma_1, \gamma_2)} \))
\[
u(\tau, R) = \lambda^\frac{1}{2}(\tau) W(R) + \sum_{j=1}^{2k-1} \nu_j(\tau, R) + \sum_{a=1,2} v_{\text{smooth},a}(\tau, R) + v(\tau, R) + R^{-1} \tilde{v}(\tau, R),
\]
with the error term $\tilde{\varepsilon}(\tau, R)$ given by the Fourier expansion

$$\tilde{\varepsilon}(\tau, R) = \int_0^\infty \phi(R, \xi) [x^{(0)}(\tau, \xi) + \sum_{j=1}^\infty \Delta x^{(j)}(\tau, \xi)] \rho(\xi) \, d\xi.$$ 

At initial time $\tau = \tau_0$, setting $x(\tau, \xi) := x^{(0)}(\tau, \xi) + \sum_{j=1}^\infty \Delta x^{(j)}(\tau, \xi)$, we have from our construction

$$(x(\tau_0, \xi), D_t x(\tau_0, \xi)) = (x_0^{(\gamma_1, \gamma_2)} + \Delta x_0^{(\gamma_1, \gamma_2)}, x_1^{(\gamma_1, \gamma_2)} + \Delta x_1^{(\gamma_1, \gamma_2)}),$$

where we recall

$$\Delta x^{(\gamma_1, \gamma_2)}(\xi) = \sum_{j=1}^\infty \Delta \tilde{x}^{(j)}(\xi),$$

$$\Delta x_0^{(\gamma_1, \gamma_2)}(\tau_0) = \sum_{j=0}^\infty \Delta x_0^{(j)}(\tau_0)$$

The fact that we have added on the correction terms $\Delta x^{(\gamma_1, \gamma_2)}(\xi)$ means that the data

$$(R^{-1} \tilde{\varepsilon}(\tau_0, R), \tilde{\varepsilon}(\tau_0, R))$$

will no longer match the original data $(\overline{\varepsilon}_1, \overline{\varepsilon}_2)$, and we need to precisely quantify this correction at the level of the Fourier variables associated with the old radial variable $R_{0,0}$. Doing so requires recalling (3.3), (3.4) as well as Lemma 3.1. Assume that our construction has replaced the data $(\overline{\varepsilon}_1, \overline{\varepsilon}_2)$ by $(\overline{\varepsilon}_1 + \Delta \overline{\varepsilon}_1, \overline{\varepsilon}_2 + \Delta \overline{\varepsilon}_2)$, we have the relations

$$R\Delta e_1(R) = \int_0^\infty \phi(R, \xi) \Delta x_0^{(\gamma_1, \gamma_2)}(\xi) \rho(\xi) \, d\xi + \Delta x_1^{(\gamma_1, \gamma_2)}(\xi),$$

$$\Delta x_1^{(\gamma_1, \gamma_2)}(\xi) = -\lambda^{-1}(\tau_0) \int_0^\infty \phi(R, \xi) R \Delta e_2 \, dR - \frac{\lambda}{\lambda} \mathcal{K}_{cc} \Delta x_0^{(\gamma_1, \gamma_2)} - \frac{\lambda}{\lambda} \mathcal{K}_{cd} \Delta x_d^{(\gamma_1, \gamma_2)},$$

where we recall that $\lambda = \lambda_{\gamma_1, \gamma_2}$. Recalling the relation

$$u^{(0,0)}_{\text{approx}}[t_0] + (\epsilon_1, \epsilon_2) = u^{(\gamma_1, \gamma_2)}_{\text{approx}}[t_0] + (\overline{\varepsilon}_1, \overline{\varepsilon}_2)$$

for the initial data, we see that the initial data perturbation $(\epsilon_1, \epsilon_2)$ has been replaced by

$$(\epsilon_1 + \Delta \epsilon_1, \epsilon_2 + \Delta \epsilon_2),$$

and so, in light of the fact that the corresponding Fourier variables $(x_0, x_1)$ were computed from $(\epsilon_1, \epsilon_2)$ via (3.3), (3.4) with $\gamma_{1,2} = 0$, we infer that the perturbed data (4.12) correspond to Fourier variables (with respect to the physical radial variable $R_{0,0}$) given by $(x_0 + \Delta x_0, x_1 + \Delta x_1)$ for the continuous part and $x_d + \Delta x_d$ for
the discrete part, where we have
\[ \Delta x_0(\xi) = \int_0^\infty \phi(R_{0,0}, \xi) R_{0,0} \Delta e_1(R(R_{0,0})) \, dR_{0,0}, \]
\[ \Delta x_d = \int_0^\infty \phi_d(R_{0,0}) R_{0,0} \Delta e_1(R(R_{0,0})) \, dR_{0,0} \]
\[ \Delta x_1(\xi) = -\lambda_{0,0}^{-1}(\tau_0) \int_0^\infty \phi(R_{0,0}, \xi) R_{0,0} \Delta e_2 \, dR_{0,0} - \frac{\lambda_{0,0}}{\lambda_{0,0}^c} \Delta x_0 - \frac{\lambda_{0,0}}{\lambda_{0,0}^c} \Delta x_d. \]

Then using Lemma 3.1 we easily infer
\[ \|\Delta x_0(\xi)\|_{\tilde{S}_1} \leq \|\Delta x_0^{(y_1,y_2)}\|_{\tilde{S}_1} + \|\Delta x_0^{(y_1,y_2)}\| \leq \tau_0^{-1-} \| (x_0, x_1) \|_{\tilde{S}} + |x_0d|, \]
and similarly
\[ \|\Delta x_1(\xi)\|_{\tilde{S}_2} \leq \tau_0^{-1-} \| (x_0, x_1) \|_{\tilde{S}} + |x_0d|. \]
For the discrete part of the correction, we get
\[ |\Delta x_{0d}| = \int_0^\infty \phi_d(R_{0,0}) R_{0,0} \Delta e_1(R(R_{0,0})) \, dR_{0,0} \]
\[ \leq \tau_0^{-1-} |x_{0d}| + \| (x_0, x_1) \|_{\tilde{S}} + |x_0d|^2. \]

Finally, we observe that the discrete spectral part of \( \epsilon_2 + \Delta \epsilon_2 \) with respect to the radial variable \( R_{0,0} \) is completely determined in terms of \( (x_0, x_1), x_{0d} \) and in fact a Lipschitz function of these. To conclude this discussion, we note that our precise choice of \( \Delta \epsilon_l, l = 1, 2 \), as well as Theorem 4.1 imply that the mapping
\[ (x_0, x_1, x_{0d}) \rightarrow (\Delta x_0, \Delta x_1, \Delta x_{0d}) \]
is Lipschitz with respect to the norm \( \| (\cdot, \cdot) \|_{\tilde{S}} + |\cdot| \), with Lipschitz constant \( \ll 1 \).

5. **Proof of Theorem 1.1**

This is immediate from the preceding discussion: the implicit function theorem guarantees that the mapping
\[ (x_0, x_1, x_{0d}) \rightarrow (x_0 + \Delta x_0, x_1 + \Delta x_1, x_{0d} + \Delta x_{0d}) \]
is invertible on a sufficiently small open neighbourhood of the origin in \( \tilde{S} \times \mathbb{R} \). Moreover, the second discrete spectral component \( x_{1d} + \Delta x_{1d} \) is then uniquely determined as a Lipschitz function of
\[ (x_0 + \Delta x_0, x_1 + \Delta x_1, x_{0d} + \Delta x_{0d}) \].

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