Bianchi type-I string cosmological model in the presence of a magnetic field: classical versus loop quantum cosmology approaches

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Abstract

A Bianchi type-I cosmological model in the presence of a magnetic flux along a cosmological string is considered. The first objective of this study is to investigate Einstein equations using a tractable assumption usually accepted in the literature. Quantum effects of the present cosmological model are examined in the framework of loop quantum cosmology. Finally we draw a parallel between the classical and quantum approaches.

Pacs: 95.30.Sf; 98.80.Jk; 04.60.Pp
Key words: Bianchi type-I model, cosmological string, magnetic field, loop quantum cosmology

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1 Introduction

The Bianchi models, which describe homogeneous but anisotropic spacetimes, have been extensively discussed in the literature, motivated in part by attempts to explain small but significant anisotropies in the cosmic microwave background (CMB) [1, 2] and large structures [3].

Among the anisotropic Bianchi models, the simplest ones are Bianchi type-I (BI) models whose spatial sections are flat but the expansion or contraction rates are direction dependent.

Primordial magnetic fields can have a significant impact on the CMB anisotropy. Also the presence of strong magnetic fields raises interesting problems like the formation of galaxies in the Universe. The BI models are appropriate for the investigation of a Universe which is permeated by a large scale, homogeneous magnetic field.

In the early stages of the evolution of the Universe it is expected that topological defects could have formed naturally during phase transitions followed by spontaneous broken symmetries. Cosmic strings are linear topological defects, have very interesting properties and might play an important role in structure formation [4, 5].

In the first part of the paper we shall investigate the evolution of a BI model in presence of a cloud of strings and magnetic field. In order to solve Einstein equations we resort to a tractable assumption concerning a relation between the rest energy and tension density of the system of strings [6].

Loop Quantum Gravity (LQG) [7, 8] represents one of the most compelling attempt towards a complete non perturbative quantum theory for the gravitational interaction. The cosmological application of LQG was developed in terms of invariant connections [9] and this model was denoted by Loop Quantum Cosmology (LQC). LQC takes the ingredients of LQG and applies them to expanding Universe or black hole models. LQC permits the exploration of the effects of quantum physics and quantum geometry in gravitation [10, 11].

In order to test the robustness of the LQC it is necessary to apply the methodology to some concrete situations and one of the most favorable model is represented by the simplest of anisotropic models, namely BI cosmologies. The detailed formulation for LQC in the BI models [12, 13, 14, 15, 16] reveals the fact that gravity can behave repulsively at Planckian energy densities leading to the replacement of the big bang singularity with a big bounce.

The plan of the paper is as follows: In Section 2 we outline the classical
equations for a BI string cosmological model in the presence of a magnetic field. In Section 3 we describe the effective loop dynamics for the present BI cosmological model. In the next section we present some numerical simulations and compare the classical and LQC approaches. Some conclusions and open problems are discussed in the last Section.

2 Classical equations

The line element of a BI Universe is

\[ ds^2 = -dt^2 + a_1^2dx^2 + a_2^2dy^2 + a_3^2dz^2, \]  

with three scale factors \( a_i \) (\( i = 1, 2, 3 \)) which are functions of time \( t \) only and consequently three expansion rates. In principle all these scale factors could be different and it is useful to express the mean expansion rate in terms of the average Hubble rate:

\[ H = \frac{1}{3}(H_1 + H_2 + H_3) = \frac{1}{3}\left(\frac{\dot{a}_1}{a_1} + \frac{\dot{a}_2}{a_2} + \frac{\dot{a}_3}{a_3}\right) = \frac{1}{3}\frac{\dot{V}}{V}, \]  

where we have defined a new function

\[ V = \sqrt{-g} = a_1a_2a_3, \]  

which is in fact the volume scale of the BI space-time. \( H_i \) are the so-called directional Hubble parameters:

\[ H_i = \frac{\dot{a}_i}{a_i}. \]  

In (2), (4) and further over-dot means differentiation with respect to \( t \).

The Einstein’s gravitational field equation has the form

\[ \frac{\ddot{a}_2}{a_2} + \frac{\ddot{a}_3}{a_3} + \frac{\dot{a}_2\dot{a}_3}{a_2a_3} = -\kappa T_1^0, \]  

\[ \frac{\ddot{a}_3}{a_3} + \frac{\dot{a}_1}{a_1} + \frac{\dot{a}_3\dot{a}_1}{a_3a_1} = -\kappa T_2^0, \]  

\[ \frac{\ddot{a}_1}{a_1} + \frac{\ddot{a}_2}{a_2} + \frac{\dot{a}_1\dot{a}_2}{a_1a_2} = -\kappa T_3^0, \]  

\[ \frac{\dot{a}_1\dot{a}_2}{a_1a_2} + \frac{\dot{a}_2\dot{a}_3}{a_2a_3} + \frac{\dot{a}_3\dot{a}_1}{a_3a_1} = -\kappa T_0^0, \]

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where $\kappa$ is the gravitational constant.

The energy momentum tensor for a system of cosmic strings and magnetic field in a comoving coordinate is given by

$$ T_{\mu}^{\nu} = \rho_{\text{string}} u_{\mu} u^{\nu} - \lambda x_{\mu} x^{\nu} + E_{\mu}^{\nu}, \quad (6) $$

where $\rho_{\text{string}}$ is the rest energy density of strings with massive particles attached to them and can be expressed as $\rho_{\text{string}} = \rho_p + \lambda$, where $\rho_p$ is the rest energy density of the particles attached to the strings and $\lambda$ is the tension density of the system of strings $[6, 17, 18]$ which may be positive or negative. Here $u_i$ is the four velocity and $x_i$ is the direction of the string, obeying the relations

$$ u_i u^i = -x_i x^i = -1, \quad u_i x^i = 0. \quad (7) $$

In (6) $E_{\mu\nu}$ is the electromagnetic field given by Lichnerowich [19]. In our case the electromagnetic field tensor $F_{\mu\nu}$ has only one non-vanishing component, namely

$$ F_{23} = h, \quad (8) $$

where $h$ is assumed to be constant. For the electromagnetic field $E_{\mu}^{\nu}$ one gets the following non-trivial components

$$ E_0^0 = E_1^1 = -E_2^2 = -E_3^3 = \frac{h^2}{2\bar{\mu}a_2^2a_3^2} \equiv \frac{1}{2} \frac{\beta^2}{a_2^2}, \quad (9) $$

where $\bar{\mu}$ is a constant characteristic of the medium and called the magnetic permeability. Typically $\bar{\mu}$ differs from unity only by a few parts in $10^5$ ($\bar{\mu} > 1$ for paramagnetic substances and $\bar{\mu} < 1$ for diamagnetic).

Choosing the string along $x^1$ direction and using comoving coordinates we have the following components of energy momentum tensor $[20]$:

$$ T_0^0 + \rho_{\text{string}} = T_1^1 + \lambda = -T_2^2 = -T_3^3 = \frac{\beta^2 a_1^2}{2 V^2}. \quad (10) $$

Taking into account the conservation of the energy-momentum tensor, i.e., $T_{\mu}^{\nu} = 0$, after a little manipulation of (10) one obtains $[21, 22]$:

$$ \dot{\rho}_{\text{string}} + \frac{\dot{V}}{V} \rho_{\text{string}} - \frac{\dot{a}_1}{a_1} \lambda = 0. \quad (11) $$

Here we take into account that the conservation law for magnetic field fulfills identically.
It is customary to assume a relation between \( \rho_{\text{string}} \) and \( \lambda \) in accordance with the state equations for strings. The simplest one is a proportionality relation [6]:

\[
\rho_{\text{string}} = \alpha \lambda .
\]  
(12)

The most usual choices of the constant \( \alpha \) are

\[
\alpha = \begin{cases} 
1 & \text{geometric string} \\
1 + \omega & \omega \geq 0, \ p \text{ string or Takabayasi string} \\
-1 & \text{Reddy string}.
\end{cases}
\]  
(13)

From eq. (11) with (12) we get

\[
\rho_{\text{string}} = Ra_1^{\frac{1-\alpha}{\alpha}} a_2^{-1} a_3^{-1},
\]  
(14)

with \( R \) a constant of integration.

Let us also write the other features of BI metric such as expansion and shear. The expansion for the BI metric takes the form

\[
\dot{\vartheta} = \frac{\dot{a}_1}{a_1} + \frac{\dot{a}_2}{a_2} + \frac{\dot{a}_3}{a_3} = \frac{\dot{V}}{V},
\]  
(15)

while the nonzero components for the shear tensor read

\[
\sigma_i \equiv \sigma_i^j = \frac{\dot{a}_i}{a_i} - \frac{1}{3} \vartheta.
\]  
(16)

In (16) and henceforth there is no summation over repeated index "\( i \)". The shear energy density in given by

\[
\Sigma^2 = \frac{1}{2} \sigma_{\mu\nu} \sigma^{\mu\nu} = \frac{1}{6} \left( (H_1 - H_2)^2 + (H_2 - H_3)^2 + (H_3 - H_1)^2 \right).
\]  
(17)

### 3 Effective loop quantum dynamics

In the loop quantum cosmology approach we shall use a Hamiltonian framework where the degrees of freedom of the Bianchi type-I model are encoded in the triad components \( p_i \) and momentum components \( c_i \) as follows:

\[
p_1 = a_2 a_3, \quad p_2 = a_1 a_3, \quad p_3 = a_1 a_2, \quad c_i = \gamma \dot{a}_i.
\]  
(18)
Here $\gamma$ is the Barbero-Immirzi parameter and represents a quantum ambiguity of loop quantum gravity which is a non-negative real valued parameter.

In terms of these variables, the total Hamiltonian of the model is

$$\mathcal{H} = \mathcal{H}_{\text{grav}} + \mathcal{H}_{\text{matter}}$$

where $\rho_M$ is the matter energy density. In our model $\rho_M$ comprises the contribution of cosmological string density $\rho_{\text{string}}$ given by (14):

$$\rho_{\text{string}} = R p_1^{-\frac{\alpha + 1}{2\alpha}} (p_2 p_3)^{\frac{\alpha - p_1}{2\alpha}},$$

and the energy density of the magnetic field (9) [16]

$$\rho_{\text{mag}} = \frac{1}{2} \frac{\beta^2}{(a_2 a_3)^2} = \frac{1}{2} \frac{\beta^2}{p_1^2}.$$ (21)

Einstein’s equations are derived from Hamilton’s equations:

$$\dot{p}_i = \{p_i, \mathcal{H}\} = -\kappa \gamma \frac{\partial \mathcal{H}}{\partial c_i}, \quad \dot{c}_i = \{c_i, \mathcal{H}\} = \kappa \gamma \frac{\partial \mathcal{H}}{\partial p_i}.$$ (22)

On the other hand, the total Hamiltonian $\mathcal{H}$ is of constrained type whereby it vanishes identically for solutions of Einstein’s equations

$$\mathcal{H} = 0.$$ (23)

Using the explicit form of the Hamiltonian $\mathcal{H}$ we have for $p_1$ the following equation

$$\frac{dp_1}{dt} = \frac{p_1}{\gamma \sqrt{p_1 p_2 p_3}} (c_2 p_2 + c_3 p_3),$$ (24)

and similar equations for $p_2$ and $p_3$. For the evolution of $c_1$ we get:

$$\frac{dc_1}{dt} = -\frac{c_1}{\gamma \sqrt{p_1 p_2 p_3}} (c_2 p_2 + c_3 p_3)$$

$$+ \frac{1}{2 \gamma p_1 \sqrt{p_1 p_2 p_3}} (c_2 c_3 p_2 p_3 + c_1 c_3 p_1 p_3 + c_1 c_2 p_1 p_2)$$

$$+ \frac{\kappa \gamma}{p_1} \left[ -\frac{1}{2 \alpha} R \left( \frac{p_2 p_3}{p_1^3} \right)^{\frac{1}{2\alpha}} - \frac{3}{4} \beta^2 \left( \frac{p_2 p_3}{p_1^3} \right)^{\frac{3}{2}} \right],$$ (25a)
\[
\frac{dc_2}{dt} = - \frac{c_2}{\gamma \sqrt{p_1 p_2 p_3}}(c_1 p_1 + c_3 p_3)
+ \frac{1}{2 \gamma p_2 \sqrt{p_1 p_2 p_3}}(c_2 c_3 p_2 p_3 + c_1 c_3 p_1 p_3 + c_1 c_2 p_1 p_2)
+ \frac{\kappa \gamma}{p_2} \left[ \frac{1}{2 \alpha} R \left( \frac{p_2 p_3}{p_1} \right)^{\frac{1}{\alpha}} + \frac{1}{4} \beta^2 \left( \frac{p_2 p_3}{p_1^3} \right)^{\frac{3}{2}} \right],
\]
(25b)

\[
\frac{dc_3}{dt} = - \frac{c_3}{\gamma \sqrt{p_1 p_2 p_3}}(c_1 p_1 + c_2 p_2)
+ \frac{1}{2 \gamma p_3 \sqrt{p_1 p_2 p_3}}(c_2 c_3 p_2 p_3 + c_1 c_3 p_1 p_3 + c_1 c_2 p_1 p_2)
+ \frac{\kappa \gamma}{p_3} \left[ \frac{1}{2 \alpha} R \left( \frac{p_2 p_3}{p_1} \right)^{\frac{1}{\alpha}} + \frac{1}{4} \beta^2 \left( \frac{p_2 p_3}{p_1^3} \right)^{\frac{3}{2}} \right].
\]
(25c)

Let us observe that from equations for \(p_i\) and \(c_i\) we have the following relation:
\[
\frac{d}{dt}(p_i c_i) = \kappa \gamma \sqrt{p_1 p_2 p_3} \left( \frac{1}{2} \rho_M + p_i \frac{\partial \rho_M}{\partial p_i} \right).
\]
(26)

The directional Hubble rates now reads
\[
H_i = \frac{\dot{a}_i}{a_i} = \frac{\sqrt{p_i c_i}}{\gamma \sqrt{p_j p_k}}, \quad i \neq j \neq k = 1, 2, 3.
\]
(27)

Hamiltonian (19) on account of the vanishing condition (23) and (27) leads to
\[
H_1 H_2 + H_1 H_3 + H_2 H_3 = \kappa \rho_M,
\]
(28)
which is in fact the zero-zero component (5d) of the Einstein system of equations (5).

Taking into account the symmetry of the density \(\rho_M\) with respect to variables \(p_2\) and \(p_3\) we have
\[
\frac{d}{dt}(p_2 c_2 - p_3 c_3) = 0.
\]
(29)

This means that for the directional Hubble parameters \(H_2, H_3\) we have
\[
H_2 - H_3 = \frac{\alpha_{23}}{a_1 a_2 a_3} = \frac{\alpha_{23}}{\sqrt{p_1 p_2 p_3}},
\]
(30)
with $\alpha_{23}$ a constant.

The quantum effects in loop quantum cosmology arise in the effective Hamiltonian constructed from the classical one by replacing the classical $c_i$ terms with sine functions:

$$c_i \rightarrow \frac{\sin(\bar{\mu}'_i c_i)}{\bar{\mu}'_i}.$$  

(31)

where $\bar{\mu}_i$ are real valued functions of the triad coefficients $p_i$.

The effective Hamiltonian is given by:

$$H_{\text{eff}} = \frac{-1}{\kappa \gamma^2 \sqrt{p_1p_2p_3}} \left\{ \frac{\sin(\bar{\mu}'_2 c_2) \sin(\bar{\mu}'_3 c_3)}{\bar{\mu}'_2 \bar{\mu}'_3} p_2p_3 + \text{cyclic terms} \right\} + \sqrt{p_1p_2p_3} \rho_M,$$

(32)

and the Hamilton’s equations for $p_i$ and $c_i$ will be modified accordingly.

It is quite evident that in the limit $\bar{\mu}_i \rightarrow 0$, the classical Hamiltonian $H$ is recovered. The expression of the parameters $\bar{\mu}'_i$ as functions of the triad components $p_i$ represent an ambiguity of the quantization. Two most preferable constructions are discussed in [13, 14, 15].

In what follows we shall adopt the $\bar{\mu}'$-scheme in which the parameters $\bar{\mu}'_i$ are chosen as follows:

$$\bar{\mu}'_1 = \sqrt{\frac{p_1 \Delta}{p_2p_3}}, \bar{\mu}'_2 = \sqrt{\frac{p_2 \Delta}{p_1p_3}}, \bar{\mu}'_3 = \sqrt{\frac{p_3 \Delta}{p_1p_2}},$$

(33)

with $\Delta$ a constant related to the minimum area gap in LQG. For the numerical simulations it is assumed that $\Delta = O(1)$ in Planck units.

Let us remark that from the vanishing of the Hamiltonian (23) we have the bound:

$$p_1p_2p_3 \rho_M \leq \frac{1}{\kappa \gamma^2} \left\{ \frac{p_2p_3}{\bar{\mu}_2 \bar{\mu}_3} + \frac{p_1p_3}{\bar{\mu}_1 \bar{\mu}_3} + \frac{p_1p_2}{\bar{\mu}_1 \bar{\mu}_2} \right\}.$$  

(34)

In particular, in the $\bar{\mu}'$ scheme the total density is bounded by the critical value:

$$\rho_{M \text{ crit}} = 3(\kappa \gamma^2 \Delta)^{-1},$$

(35)

implying that the classical singularity is never approached. Indeed the total energy of the matter must be below this value and the classical collapse is replaced by a bounce.
4 Numerical simulations and a comparison of the approaches

The complexity of the equations does not allow for analytic solutions and imposes numerical simulations. The classical equations of motion given by (24)-(25) and the corresponding ones for quantum effects in LQC approach with the replacement (31) can be solved once the initial values $p_i(t = t_0)$ and $c_i(t = t_0)$ are given.

In what follows we report some numerical studies on the behavior of $\rho_M$, $V$ and the anisotropic shear $\sigma_{\mu\nu}$. Taking into account that only diagonal components of shear tensor are non-zero, in the new variables they now read

$$\sigma_i = \frac{\sqrt{p_i c_i}}{\gamma \sqrt{p_j p_k}} - \frac{1}{3\gamma} \left( \frac{\sqrt{p_1 c_1}}{\sqrt{p_2 p_3}} + \frac{\sqrt{p_2 c_2}}{\sqrt{p_3 p_1}} + \frac{\sqrt{p_3 c_3}}{\sqrt{p_1 p_2}} \right), \quad i \neq j \neq k = 1, 2, 3, \quad (36)$$

and the shear energy density is

$$\Sigma^2 = \frac{1}{6\gamma^2 p_1 p_2 p_3} \left[ (p_1 c_1 - p_2 c_2)^2 + (p_2 c_2 - p_3 c_3)^2 + (p_3 c_3 - p_1 c_1)^2 \right]. \quad (37)$$

In doing so we considered a number of cases that helps us to clarify the role of various parameters. To begin with we examined both positive and negative $\alpha$. In particular we considered the case with $\alpha = 2$ and $\alpha = -2$ and it was found that the value or sign of $\alpha$ leaves the overall picture qualitatively unchanged. As a second consideration we set a large value of $R$ and small value of $\beta$ and vice versa, for example, $R = 18.24$, $\beta = 1$ and $R = 0.90$, $\beta = 6.5$, respectively. It was established that for both cases the overall picture remains qualitatively unaltered. Finally we consider the case setting different initial values for $c_i = c_0$.

In Figs. 1, 3, 5, 7 we have illustrated the evolution of volume scale $V$ (red solid line), energy density $\rho_M$ (blue dash line) and shear energy $\Sigma^2$ (black dot line), whereas in Figs. 2, 4, 6, 8 we have plotted the evolution of the components of shear tensor $\sigma_1$ (red solid line), $\sigma_2$ (blue dash line) and $\sigma_3$ (black dot line).

In the classical case the initial condition $c_0 < 0$ leads to a collapsing Universe [cf. Figs. 1, 2], while $c_0 > 0$ gives rise to an expanding one [cf. Figs. 3, 4].

In the LQC approach, even for $c_0 < 0$ the energy density remains bounded below the critical energy (35) as expected from analytical considerations [cf.
Fig. 5. After the bounce, the regime is an expanding one and the Universe isotropizes [cf. Fig. 6]. On the other hand, for $c_0 > 0$ we have only expanding phase of the Universe, the shear energy density $\Sigma^2$ remains finite [cf. Fig. 7] and again $\sigma_i$ tend to zero [cf. Fig. 8].

5 Conclusions

In this report within the scope of Bianchi type-I cosmological model we investigate the role of cosmic string and magnetic field on the evolution of the Universe. In doing so we employ both classical and LQC approaches. It is found that the qualitative picture of evolution does not depend on the cosmic string ($\alpha$), though the value of $\alpha$ leads to quantitative changes. On the other hand magnetic field together with cosmic string (given by the pair $\beta$ and $R$) also leave the qualitative feature unaltered. Only initial value of $c_i$’s, i.e., initial rate of change of the metric functions $a_i$’s play essential role.

In the classical approach the initial condition $c_0 < 0$ corresponds to a classically collapsing Universe, while $c_0 > 0$ is associated with an expansion. In the LQC approach for $c_0 < 0$ the singularity is avoided via a bounce. After the bounce the Universe enters an expansion phase with an asymptotic isotropization. For positive $c_0$ we get always expansion and isotropization.

At the classical level other Bianchi models have richer phenomenology than the cosmological BI model. From this point of view the extension of the string cosmological model in the presence of electromagnetic fields to other types of anisotropies and comparisons between the classical approach and LQC one deserve further studies.

Acknowledgments

This work is supported in part by a joint Romanian-LIT, JINR, Dubna Research Project, theme no. 05-6-1060-2005/2013. M.V. is partially supported by program PN-II-ID-PCE-2011-3-0137, Romania.

References

[1] J. Donkley et al. [WMAP Collaboration], Astrophys. J. Suppl. 180, 306 (2009).
Figure 1: Classical: Evolution of $V$, $\rho_M$ and $\Sigma^2$ for $\alpha = -2$, $R = 18.24$, $\beta = 1$, $c_0 = -1$. Here and further the red solid line corresponds to $V$, blue dash line to $\rho_M$ and black dot line to $\Sigma^2$.

Figure 2: Classical: Evolution of $\sigma_i$'s for $\alpha = -2$, $R = 18.24$, $\beta = 1$, $c_0 = -1$. Here and further the red solid line corresponds to $\sigma_1$, blue dash line to $\sigma_2$ and black dot line to $\sigma_3$.

Figure 3: Classical: Evolution of $V$, $\rho_M$ and $\Sigma^2$ for $\alpha = -2$, $R = 18.24$, $\beta = 1$, $c_0 = 1$.

Figure 4: Classical: Evolution of $\sigma_i$'s for $\alpha = -2$, $R = 18.24$, $\beta = 1$, $c_0 = 1$. 

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Figure 5: LQC: Evolution of $V$, $\rho_M$ and $\Sigma^2$ for $\alpha = -2$, $R = 6.20$, $\beta = 1$, $c_0 = -1$.

Figure 6: LQC: Evolution of $\sigma_i$'s for $\alpha = -2$, $R = 6.20$, $\beta = 1$, $c_0 = -1$.

Figure 7: LQC: Evolution of $V$, $\rho_M$ and $\Sigma^2$ for $\alpha = -2$, $R = R = 6.20$, $\beta = 1$, $c_0 = 1$.

Figure 8: LQC: Evolution of $\sigma_i$'s for $\alpha = -2$, $R = 6.20$, $\beta = 1$, $c_0 = 1$. 

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[2] E. Komatsu et al. [WMAP Collaboration], *Astrophys. J. Suppl.* **192**, 17 (2011).

[3] M. Tegmark et al. [SDSS Collaboration], *Phys. Rev. D* **69**, 103501 (2004).

[4] A. Vilenkin, E. P. S. Shellard, *Cosmic Strings and Other Topological Defects*, Cambridge University Press, Cambridge (1994).

[5] M. B. Hindmarsh, T. W. B. Kible, *Rep. Prog. Phys.* **58**, 477 (1995).

[6] P. S. Letelier, *Phys. Rev. D* **28**, 2414 (1983).

[7] C. Rovelli, *Quantum Gravity*, Cambridge University Press, Cambridge (2004).

[8] T. Thiemann, *Modern Canonical Quantum General Relativity*, Cambridge University Press, Cambridge (2006).

[9] M. Bojowald, *Class. Quantum Grav.* **17**, 1489 (2000).

[10] A. Ashtekar, *Gen. Relativ. Grav.* **41**, 707 (2009).

[11] M. Bojowald, *Class. Quantum Grav.* **26**, 075020 (2009).

[12] D.-W. Chiou, *Phys. Rev. D* **75**, 024029 (2007).

[13] D.-W. Chiou and K. Vandersloot, *Phys. Rev. D* **76**, 084015 (2007).

[14] D. W. Chiou, *Phys. Rev. D* **76**, 124037 (2007).

[15] A. Ashtekar and E. Wilson-Ewing, *Phys. Rev. D* **80**, 123532 (2009).

[16] R. Maartens and K. Vandersloot, *arXiv: 0812.1889 [gr-qc]*.

[17] A. Pradhan, A. K. Yadav, R. P. Singh, V. K. Singh, *Astrophys. Space Sci.* **312**, 145 (2007).

[18] G.S. Khadekar and S.D. Tade, *Astrophys. Space Sci.* **310**, 47 (2007).

[19] A. Lichnerowicz, *Relativistic Hydrodynamics and Magnetohydrodynamics*, Benjamin, New York (1967).

[20] B. Saha, *Astrophys. Space Sci.* **299**, 149 (2005).
[21] B. Saha and M. Visinescu, Astrophys. Space Sci. 315, 99 (2008).

[22] B. Saha, V. Rikhvitsky and M. Visinescu, Cent. Eur. J. Phys. 8, 113 (2010).