Zero modes of six-dimensional Abelian vortices

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Abstract

We analyze the fluctuations of Nielsen-Olesen vortices arising in the six-dimensional Abelian-Higgs model. The regular geometry generated by the defect breaks spontaneously six-dimensional Poincaré symmetry leading to a warped space-time with finite four-dimensional Planck mass. As a consequence, the zero mode of the spin two fluctuations of the geometry is always localized but the graviphoton fields, corresponding to spin one metric fluctuations, give rise to zero modes which are not localized either because of their behaviour at infinity or because of their behaviour near the core of the vortex. A similar situation occurs for spin zero fluctuations. Gauge field fluctuations exhibit a localized zero mode.

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I. INTRODUCTION

In the absence of gravitational interactions, five-dimensional domain-wall solutions allow the localization of fermionic zero modes [1]. The increase of the dimensionality of space-time represents a tool in order to obtain an effective lower dimensional theory where chiral fermions may be successfully localized. The chiral fermionic zero modes is still present if the five-dimensional continuous space is replaced by a lattice [2]. Chiral symmetry can then be realized in different ways avoiding the known problem of doubling of fermionic degrees of freedom [3]. A crucial ingredient, in this context, was the use of the so-called Wilson-Ginzparg relation [4]. If we move from five to six dimensions, chiral fermions can still be localized [3, 4] and the structure of the zero modes gets more realistic.

If gravitational interactions are consistently included [7–10], infinite extra-dimensions do not only lead to the localization of chiral fermions but also to the localization of gravity itself [11, 12]. Gravitons can be localized in five dimensional AdS space-times with a brane source [11, 12] and explicit physical (thick) brane solutions, compatible with AdS$_5$ geometries, have been derived using scalar domain-walls [13–18] breaking five dimensional Poincaré invariance but leaving unbroken Poincaré group in four space-time dimensions. Fields of various spin, coming from the fluctuations of the geometry itself, can be classified, according to four-dimensional Poincaré transformations, into scalars, vectors and tensors. In the case of scalar domain-walls neither the scalar nor the vector modes of the geometry are localized [19, 20] if the four-dimensional Planck mass is finite. A field of given spin is localized if its the corresponding zero modes are normalizable as a function of the bulk coordinates describing the geometry in the internal space.

By going from five to six dimensions not only the volume of the internal space gets larger but also supplementary metric fluctuations arise. Thin strings [21–26], monopoles [27–29] or instantons [30–33] (in six, seven or eight dimensional space-times) have been studied mainly looking at the localization properties of the tensor modes of the geometry.

In six-dimensions, physical brane solutions (including background gauge fields) have been
recently analyzed [34] in the context of the Abelian-Higgs model and later generalized to the case when higher derivative terms are present in the gravity part of the action [35]. These solutions are the analog of the Nielsen-Olesen vortices [36] in a six-dimensional warped geometry where the two transverse dimensions play the rôle of the radial and angular coordinates defining the location of the Abelian string.

The purpose of the present investigation is the analysis of the zero modes of the vortices arising in the six-dimensional Abelian-Higgs model. While the spin two of the geometry are decoupled from the very beginning, the spin one fluctuations of the metric are coupled with the vector fluctuations arising in the gauge sector. Finally the scalar fluctuations of the metric are coupled with the spin zero fluctuations arising both from the Higgs sector and gauge sectors.

Following the formalism developed in [19,20], the invariance for infinitesimal coordinate transformations, can be used in order to address the problem in a coordinate independent way. This is the spirit of Bardeen formalism which was originally formulated in four-dimensions [37], later generalized to five-dimensional warped geometries [19,20] and now applied in order to discuss the zero modes of six-dimensional Abelian vortices.

The plan of our paper is then the following. In Section II the main features of the six-dimensional Abelian vortices will be summarized. Particular attention will be given to those properties of the background entering directly the discussion of the zero modes. In Section III the gauge-invariant fluctuations of the six-dimensional geometry will be introduced and classified. The general system of the fluctuations will be presented in Section IV. The tensor zero modes will be analyzed in Section V, while Section VI and VII will deal, respectively, with the gauge and scalar sectors. Section VIII contains our concluding remarks. In the Appendices various technical results have been summarized.
II. ABELIAN VORTICES IN SIX DIMENSIONS

The gravitating Abelian-Higgs model with cosmological constant in the bulk

\[ S = \int d^6x \sqrt{-G} \left[ -\frac{R}{2\kappa} - \Lambda + \frac{1}{2} G^{AB} (D_A \varphi)^* D_B \varphi - \frac{1}{4} F_{AB} F^{AB} - \frac{\lambda}{4} (\varphi^* \varphi - v^2)^2 \right], \quad (2.1) \]

will be studied in a six-dimensional warped geometry whose line element can be written as

\[ ds^2 = G_{AB} dx^A dx^B = M^2(\rho) \eta_{\mu \nu} dx^\mu dx^\nu - d\rho^2 - L(\rho)^2 d\theta^2, \quad (2.2) \]

where \( \rho \) is the bulk radius, \( \eta_{\mu \nu} \) is the four-dimensional Minkowski metric and \( M(\rho), L(\rho) \) are the warp factors. In Eqs. (2.1) \( v \) is the vacuum expectation value of the Higgs field, \( \lambda \) is the self-coupling constant and \( e \) is the gauge coupling. The metric of Eq. (2.2) breaks naturally Poincaré symmetry in six dimensions.

From the action of Eq. (2.1) the corresponding equations of motion can be derived and they are

\[ G^{AB} \nabla_A \nabla_B \varphi - e^2 A_A A^A \varphi - ie A_A \partial^A \varphi - i e \nabla_A (A^A \varphi) + \lambda (\varphi^* \varphi - v^2) \varphi = 0, \quad (2.3) \]

\[ \nabla_A F^{AB} = -e^2 A^B \varphi^* \varphi + \frac{i e}{2} (\varphi \partial^B \varphi^* - \varphi^* \partial^B \varphi), \quad (2.4) \]

\[ R_{AB} - \frac{1}{2} G_{AB} R = \kappa (T_{AB} + \Lambda G_{AB}), \quad (2.5) \]

where

\[ T_{AB} = \frac{1}{2} \left[ (D_A \varphi)^* D_B \varphi + (D_B \varphi)^* D_A \varphi \right] - F_{AC} F_B^C - G_{AB} \left[ \frac{1}{2} (D_A \varphi)^* D_A \varphi - \frac{1}{4} F_{MN} F^{MN} - \frac{\lambda}{4} (\varphi^* \varphi - v^2)^2 \right]. \quad (2.6) \]

The vortex ansatz, characterized by the winding \( n, \)

\[ ^1 \text{Notice that } D_A = \nabla_A - ie A_A \text{ is the gauge covariant derivative, while } \nabla_A \text{ is the generally covariant derivative. Latin (uppercase) indices run over the six dimensional space-time. Greek indices run over the four (Poincaré invariant) dimensions. Latin (lowercase) indices run over the two-dimensional transverse space.} \]
\[ \phi(\rho, \theta) = v f(\rho) e^{in\theta}, \]

\[ A_\theta(\rho, \theta) = \frac{1}{e} [n - P(\rho)] , \]  

(2.7)

breaks naturally the \( U(1) \) symmetry. Inserting Eqs. (2.2) and (2.7) into Eqs. (2.3)–(2.5), the resulting system only depends upon the bulk radius. It is useful, for practical purposes, to define the following set of dimension-less quantities

\[ \nu = \kappa v^2, \quad \alpha = \frac{e^2}{\lambda}, \quad \mu = \frac{\kappa \Lambda}{\lambda v^2}. \]  

(2.8)

whose specific numerical value select a given solution in the parameter space of the model. Using the rescalings of Eq. (2.8), it is natural to write Eqs. (2.3)–(2.5) in terms of the rescaled bulk radius \( x = m_H \rho / \sqrt{2} \equiv \sqrt{\lambda v} \rho \), and in terms of the derivatives of the logarithms of the warp factors

\[ H(x) = \frac{d \ln M(x)}{dx}, \quad F(x) = \frac{d \ln L(x)}{dx}, \]  

(2.9)

where

\[ L(x) = \sqrt{\lambda v} L(x). \]  

(2.10)

has been defined.

Eqs. (2.3)–(2.5) become then

\[ \frac{d^2 f}{dx^2} + (4H + F) \frac{df}{dx} + (1 - f^2) f - \frac{P^2}{L^2} f = 0, \]  

(2.11)

\[ \frac{d^2 P}{dx^2} + (4H - F) \frac{dP}{dx} - \alpha f^2 P = 0, \]  

(2.12)

\[ \frac{dH}{dx} + 3 \frac{dF}{dx} + F^2 + 6H^2 + 3HF = -\mu - \nu \tau_0, \]  

(2.13)

\[ 4 \frac{dH}{dx} + 10H^2 = -\mu - \nu \tau_\theta, \]  

(2.14)

\[ 4HF + 6H^2 = -\mu - \nu \tau_\rho. \]  

(2.15)

\(^2\text{Notice that the Higgs boson and vector masses are, in our definitions, } m_H = \sqrt{2\lambda} v \text{ and } m_V = ev.\)
Eqs. (2.11) and (2.12) correspond, respectively, to Eqs. (2.3)–(2.4). The other equations come, respectively, from the \((\mu, \nu)\), \((\rho, \rho)\) and \((\theta, \theta)\) components of the Einstein equations (2.5). The functions \(\tau_0\), \(\tau_\rho\) and \(\tau_\theta\) denote the components of the energy-momentum tensor:

\[
\tau_0(x) \equiv T^0_0 = T^i_i = \frac{1}{2} \left( \frac{df}{dx} \right)^2 + \frac{1}{4} \left( f^2 - 1 \right)^2 + \frac{1}{2\alpha L^2} \left( \frac{dP}{dx} \right)^2 + \frac{f^2 P^2}{2L^2},
\]

\[
\tau_\rho(x) \equiv T^\rho_\rho = -\frac{1}{2} \left( \frac{df}{dx} \right)^2 + \frac{1}{4} \left( f^2 - 1 \right)^2 - \frac{1}{2\alpha L^2} \left( \frac{dP}{dx} \right)^2 + \frac{f^2 P^2}{2L^2},
\]

\[
\tau_\theta(x) \equiv T^\theta_\theta = \frac{1}{2} \left( \frac{df}{dx} \right)^2 + \frac{1}{4} \left( f^2 - 1 \right)^2 - \frac{1}{2\alpha L^2} \left( \frac{dP}{dx} \right)^2 - \frac{f^2 P^2}{2L^2}.
\]

Fig. 1 is representative of a class of solutions whose parameter space is reported in Fig. 3 in terms of the dimension-less quantities defined in Eq. (2.8). Each point on the surface of Fig. 2 corresponds to a solution of the type of the one reported in Fig. 1. The solutions illustrated in Fig. 1 and 2 have been numerically obtained with the techniques described in [34,35]. From Fig. 1, it can be appreciated that the scalar field reaches, for large \(x\), its vacuum expectation value and close to the core of the string the Higgs and gauge fields are regular:

\[
f(0) = 0, \quad \lim_{x \to \infty} f(x) = 1,
\]
FIG. 2. The parameter space of the vortex solutions.

\[ P(0) = n, \quad \lim_{x \to \infty} P(x) = 0. \] (2.19)

The solution of Fig. 1 corresponds to the case of lowest winding, i.e. \( n = 1 \) in Eq. (2.7), but regular solutions with higher winding can be easily obtained [34,35].

The regularity of the geometry in the core of the string implies

\[ \left. \frac{dM}{dx} \right|_0 = 0, \quad L(0) = 0, \quad \left. \frac{dL}{dx} \right|_0 = 1, \] (2.20)

and \( M(0) = 1 \). At large distances from the core the behaviour of the geometry is AdS\(_6\) space characterized, in this coordinate system, by exponentially decreasing warp factors

\[ M(x) \sim e^{-cx}, \quad L(x) \sim e^{-cx} \] (2.21)

where \( c = \sqrt{-\mu/10} \). Since the defects corresponding to the solution of Fig. 1 are local, the corresponding energy-momentum tensor goes to zero for large \( x \). Hence, for large \( x \), the geometry is determined only by the value of the bulk cosmological constant which is related to the parameter \( \mu \). This feature is also illustrated by Fig. 4 where the components of the energy momentum tensors are reported in the case of the solution of Fig. 1.

The form of the solutions in the vicinity of the core of the vortex can be studied by expressing the metric functions, together with the scalar and gauge fields, as a power series
FIG. 3. The components of the energy-momentum tensor reported in the case of the solution of Fig. 1.

in $x$, i.e. the dimensionless bulk radius. Inserting the power series into Eqs. (2.11)–(2.15) and requiring that the series obeys, for $x \to 0$, the boundary conditions of Eqs. (2.19), the form of the solutions can be determined as a function of the parameters of the model:

\[
\begin{align*}
  f(x) &\simeq Ax + \frac{A}{8} \left( \frac{2\mu}{3} + \frac{\nu}{6} + \frac{2A^2\nu}{3} - 1 + 2B + \frac{4B^2\nu}{3\alpha} \right) x^3, \\
  P(x) &\simeq 1 + Bx^2, \\
  M(x) &\simeq 1 + \left( \frac{\mu}{8} - \frac{\nu}{32} + \frac{\nu B^2}{4\alpha} \right) x^2, \\
  \mathcal{L}(x) &\simeq x + \left[ \frac{\mu}{12} + \nu \left( \frac{1}{48} - \frac{5B^2}{6\alpha} - \frac{A^2}{6} \right) \right] x^3.
\end{align*}
\]

In Eq. (2.25) $A$ and $B$ are two arbitrary constants which cannot be determined by the local analysis of the equations of motion. These constants are to be found by studying, simultaneously, the boundary conditions for $f(x)$ and $P(x)$ at infinity and in the origin. This analysis can be achieved by looking at the behaviour of the string tensions.

By studying the relations among the string tensions [34,35], the specific value of $B$ required in order to have AdS$_6$ at infinity and regular geometry in the origin can be obtained. Indeed, for all the solutions of the family defined by the fine-tuning surface of Fig. 2 we
FIG. 4. The curvature invariants computed for the solution reported in Fig. (1).

have that

$$-\nu \alpha \frac{dP}{dx} \bigg|_0 = 1.$$  \hspace{1cm} (2.26)

According to Eq. (2.25), for $x \to 0$, $P \sim 1 + Bx^2$. Using Eq. (2.26) the expression for $B$ can be exactly computed

$$B = -\frac{\alpha}{2\nu}.$$  \hspace{1cm} (2.27)

The solutions of the type of Fig. 1 are regular everywhere, not only in the origin or at infinity. For this purpose, the behaviour of the curvature invariants [i.e. $R_{ABCD}R^{ABCD}$, $C_{ABCD}C^{ABCD}$, $R_{AB}R^{AB}$, $R^2$] is reported in Fig. 4 for the solution of Fig. 1. Far from the core the Higgs and gauge fields approach their boundary values in an exponential way. Taking into account that, for large $x$, the warp factors decrease exponentially, the approach of the gauge and Higgs fields to their boundary values can be analytically obtained by inserting

$$P(x) = \mathcal{P} + \delta P(x), \quad \mathcal{P} \sim 0$$

$$f(x) = \mathcal{F} - \delta f(x), \quad \mathcal{F} \sim 1$$  \hspace{1cm} (2.28)
into Eqs. (2.11)–(2.15). The background equations imply that

\[ \delta P(x) \sim e^{\sigma_1 x}, \quad \sigma_1 = \frac{3c}{2}[1 \pm \sqrt{1 + \frac{4\alpha}{9c^2}}], \]

\[ \delta f(x) \sim e^{\sigma_2 x}, \quad \sigma_2 = \frac{5c}{2}[1 \pm \sqrt{1 + \frac{8}{25c^2}}], \]  

(2.29)

If \( 4\alpha \gg 9c^2 \) (the limit of small bulk cosmological constant) the solution is compatible with
the gauge field decreasing asymptotically as \( \delta P \sim e^{-\sqrt{\alpha}x} \). If \( 25c^2 < 8 \) the the perturbed
solution goes as \( \delta f \sim e^{-\sqrt{\alpha}x} \).

Consider now the difference between the \((0,0)\) and \((\theta, \theta)\) components of the Einstein
equations, i.e. (2.13) and (2.14):

\[ \frac{d(H - F)}{dx} + (F + 4H)(F - H) = -\nu(\tau_0 - \tau_\theta) \]  

(2.30)

Multiplying both sides of this equation by \( \sqrt{-G} = M^4\mathcal{L} \) and integrating from 0 to \( x \), the
following relation

\[ (H - F) = \frac{\nu}{\alpha \mathcal{L}^2} \frac{dP}{dx} - \left(1 + \frac{\nu}{\alpha \mathcal{L} \frac{dP}{dx} \big|_0} \right) \frac{1}{M^4\mathcal{L}}, \]  

(2.31)

is obtained. According to Eqs. (2.26) and (2.27) the boundary term in the core disappears
and the resulting equation will be

\[ F = H - \frac{\nu}{\alpha \mathcal{L}^2} \frac{dP}{dx}. \]  

(2.32)

Eq. (2.32) holds for all the family of solutions describing vortex configurations (2.7) with
AdS\(_6\) behaviour at infinity.

The solutions presented in this Section satisfy everywhere (not only for large bulk radius)
the corresponding equations of motion obtained from the action (2.1). In this sense, the
following analysis in not general since it relies on the specific brane action of the Abelian-
Higgs model. Sometimes (see for instance [38]) solutions of this sort are postulated without
assuming a specific brane action and by further postulating that the gauge-Higgs background
is absent. This procedure is not appropriate in the present case since, as we shall see, the
fluctuations of gauge and Higgs fields become sources of the fluctuations of the geometry.
III. GAUGE-INvariant Fluctuations of Abelian Vortices

The fluctuations of the geometry and of the gauge-Higgs sources around the fixed vortex background will now be discussed. In [19,20] gauge-invariant techniques were used in order to address the fluctuations of five-dimensional domain-wall solutions breaking spontaneously five-dimensional Poincaré invariance. The main physical difference between the scalar domain walls in five dimensions and Abelian vortices in six dimensions is given by the presence of a gauge field background whose fluctuations mix with the graviphoton fields coming from the geometry. The invariance of the fluctuations of the six-dimensional metric for an infinitesimal coordinate transformation around the vortex background guarantees that the obtained fluctuations (and their related evolution equations) are independent on the specific coordinate system and therefore free of spurious gauge modes [37].

A. Basic considerations

In analogy with what customarily done in the five-dimensional situation [12,19,20], the six-dimensional line element of Eq. (2.2) can be written as

$$ds^2 = M^2(w)[dt^2 - d\vec{x}^2] - L^2(w)[dw^2 + d\theta^2].$$

Eqs. (2.2) and (3.1) are connected by the usual differential relation

$$d\rho = L(w)dw,$$

which can be also written as

$$dx = L(w)dw,$$

if we recall that, according to Eq. (2.10), \(x = \sqrt{\lambda}v\rho\) and \(L(w) = \sqrt{\lambda}vL(w)\).

A generic fluctuation \(J(x^\mu, w, \theta)\) of the six-dimensional Abelian vortex will admit derivatives with respect to the ordinary four-dimensional coordinates, but also with respect to the two-dimensional transverse space. Hence, the following notation will be employed:
\[ j' = \frac{\partial J}{\partial w}, \quad j = \frac{\partial J}{\partial \theta}. \] (3.4)

Using Eq. (3.1) into Eqs. (2.3)–(2.5) together with the vortex ansatz

\[ \varphi = v e^{i n \theta} f(w), \quad A_\theta(w) = \frac{n - P(w)}{e}, \] (3.5)

the equations of the background for the Higgs and gauge field are, respectively,

\[ f'' + 4\mathcal{H} f' + \mathcal{L}^2 f(1 - f^2) - P^2 f = 0, \] (3.6)

\[ P'' + (4\mathcal{H} - 2\mathcal{F}) P' - \alpha f^2 P^2 = 0, \] (3.7)

while Eq. (2.5) leads to

\[ 3\mathcal{H}' + \mathcal{F}' + 6\mathcal{H}^2 = -\mu \mathcal{L}^2 - \nu \left[ \frac{f^2}{2} + \frac{1}{4} (f^2 - 1)^2 \mathcal{L}^2 + \frac{P^2}{2\alpha \mathcal{L}^2} + \frac{f^2 P^2}{2} \right], \] (3.8)

\[ 6\mathcal{H}^2 + 4\mathcal{H}\mathcal{F} = -\mu \mathcal{L}^2 - \nu \left[ -\frac{f^2}{2} + \frac{1}{4} (f^2 - 1)^2 \mathcal{L}^2 - \frac{P^2}{2\alpha \mathcal{L}^2} + \frac{f^2 P^2}{2} \right], \] (3.9)

\[ 4\mathcal{H}' - 4\mathcal{H}\mathcal{F} + 10\mathcal{H}^2 = -\mu \mathcal{L}^2 - \nu \left[ \frac{f^2}{2} + \frac{1}{4} (f^2 - 1)^2 \mathcal{L}^2 - \frac{P^2}{2\alpha \mathcal{L}^2} + \frac{f^2 P^2}{2} \right]. \] (3.10)

The quantities

\[ \mathcal{H} = \frac{\partial \ln M}{\partial w}, \quad \mathcal{F} = \frac{\partial \ln L}{\partial w}. \] (3.11)

have been defined in full analogy with Eqs. (2.11)–(2.15).

Summing up Eqs. (3.8) and (3.10) we get

\[ 4\mathcal{H}' + 16\mathcal{H}^2 = -2\mu \mathcal{L}^2 - \nu \left[ \frac{1}{2} (f^2 - 1)^2 \mathcal{L}^2 - \frac{P^2}{\alpha \mathcal{L}^2} \right]. \] (3.12)

Subtracting Eq. (3.10) from Eq. (3.8)

\[ \mathcal{F}' - \mathcal{H}' + 4\mathcal{H}(\mathcal{F} - \mathcal{H}) = -\nu \left[ \frac{P^2}{\alpha \mathcal{L}^2} + f^2 P^2 \right], \] (3.13)

which is the analog of Eq. (2.30) obtained in the \( x \)-parametrization. In the limit \( w \to \infty \)
\( P' \to 0 \) and \( P \to 0 \). Hence from Eq. (3.13), the limit \( P' \to 0 \) and \( P \to 0 \) implies \( \mathcal{H} \to \mathcal{F} \),

namely the AdS\(_6\) space-time.

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Integrating Eq. (3.13) with the boundary conditions fixed by the vortex configuration and by the relations among the string tensions the following relation is obtained

\[ \mathcal{H} - \mathcal{F} = \frac{\nu}{\alpha L^2} PP', \]  

(3.14)

which is the analog of Eq. (2.32) obtained in the \( x \)-parametrization.

Recalling that, for \( x \to 0 \), \( L(x) \simeq x \) and using Eq. (3.3), the limit \( x \to 0 \) corresponds, in the \( w \) parametrization, to the limit \( w \to -\infty \). For \( w \to -\infty \)

\[ \mathcal{L}(w) \simeq e^w + \mathcal{O}(e^{3w}), \]
\[ M(w) \simeq 1 + \mathcal{O}(e^{2w}), \]
\[ f(w) \simeq A e^w + \mathcal{O}(e^{3w}), \]
\[ P(w) \simeq 1 + \mathcal{O}(e^{2w}), \]  

(3.15)

From Eq. (3.3), the limit \( x \to +\infty \) corresponds to \( w \to +\infty \) which implies, according to Eq. (2.21),

\[ M(w) \simeq M_0 \frac{1}{cw}, \]
\[ \mathcal{L}(w) \simeq L_0 \frac{1}{cw}. \]  

(3.16)

As usual, \( c = \sqrt{-\mu/10} \) is related to the inverse of the AdS_6 radius and it is determined by the negative cosmological constant dominating the solutions far from the core. The asymptotic behaviour of the gauge and Higgs field are, in the same limit,

\[ f(w) \simeq (cw)^{-\gamma_f}, \]  

(3.17)
\[ P(w) \simeq (cw)^{-\gamma_P}, \]  

(3.18)

where \( \gamma_f > 0 \) and \( \gamma_P > 0 \). According to Eqs. (2.29)–(2.29) \( 4\alpha \gg 9c^2 \) and \( 25c^2 \ll 8 \). Hence, \( \gamma_f \gg 1 \) and \( \gamma_P \gg 1 \):

\[ \gamma_f = \sqrt{2}/c \equiv \sqrt{-\frac{20}{\mu}} > 1, \quad \gamma_P = \sqrt{\alpha}/c \equiv \sqrt{-\frac{10\alpha}{\mu}} > 1, \]  

(3.19)

implying that \( f(w) \) and \( P(w) \) reach their boundary values at infinity faster than \( M(w) \) and \( \mathcal{L}(w) \).
B. Scalar, vector and tensor modes of the geometry

Since four-dimensional Poincaré symmetry is unbroken by the presence of the vortex background, the fluctuation of the metric can be decomposed in terms of scalar vector and tensor modes

$$\delta G_{AB}(x^\mu, w, \theta) = \delta G^{(S)}_{AB}(x^\mu, w, \theta) + \delta G^{(V)}_{AB}(x^\mu, w, \theta) + \delta G^{(T)}_{AB}(x^\mu, w, \theta). \quad (3.20)$$

with respect to Poincaré transformations along the four physical dimensions:

$$\delta G_{AB} = \begin{pmatrix} 2M^2 H_{\mu\nu} & LMG_\mu & LMB_\mu \\ LMG_\mu & 2L^2 \xi & L^2 \pi \\ LMB_\mu & L^2 \pi & 2L^2 \phi \end{pmatrix}, \quad (3.21)$$

where

$$H_{\mu\nu} = h_{\mu\nu} + \frac{1}{2}(\partial_\mu f_\nu + \partial_\nu f_\mu) + \eta_{\mu\nu}\psi + \partial_\mu \partial_\nu E,$$

$$G_\mu = D_\mu + \partial_\mu C,$$

$$B_\mu = Q_\mu + \partial_\mu P, \quad (3.22)$$

with

$$\partial_\mu h^{\mu}_\nu = 0, \quad h^{\mu}_\mu = 0,$$

$$\partial_\mu f^{\mu} = 0, \quad \partial_\mu D^{\mu} = 0, \quad \partial_\mu Q^{\mu} = 0. \quad (3.23)$$

The tensor $h_{\mu\nu}$ has five independent components, while $Q_\mu$, $f_\mu$ and $D_\mu$ have, overall nine independent components. The scalars $C$, $P$, $\psi$, $\phi$, $\xi$, $E$ and $\pi$ correspond to seven scalar degrees of freedom.

Under infinitesimal coordinates transformations around the background geometry fixed by the vortex solution

$$x^A \to \tilde{x}^A = x^A + \epsilon^A, \quad (3.24)$$

the twenty one degrees of freedom of the perturbed six-dimensional metric do transform as
\[ \delta \tilde{G}_{AB} = \delta G_{AB} - \nabla_A \epsilon_B - \nabla_B \epsilon_A, \] 

(3.25)

where the Lie derivative is computed using the background metric (3.1) and where

\[ \epsilon_A = (M^2 \epsilon_\mu, -L^2 \epsilon_w, -L^2 \epsilon_\theta). \] 

(3.26)

The shift along the longitudinal coordinates has a pure vector part and a scalar part \(^3\)

\[ \epsilon_\mu = \partial_\mu \epsilon + \zeta_\mu. \] 

(3.27)

The infinitesimal diffeomorphisms preserving the scalar nature of the fluctuation involve \( \epsilon, \epsilon_\theta \) and \( \epsilon_w \), whereas the coordinate transformations preserving the vector nature of the fluctuation only involve \( \zeta_\mu \). Apart from the transverse and traceless tensors, which do not change for infinitesimal gauge transformations (i.e. \( \tilde{h}_{\mu\nu} = h_{\mu\nu} \)), from Eq. (3.25) the vector transform as

\[ \tilde{f}_\mu = f_\mu - \zeta_\mu, \] 

(3.28)

\[ \tilde{D}_\mu = D_\mu - \frac{M}{L} \zeta_\mu', \] 

(3.29)

\[ \tilde{Q}_\mu = Q_\mu - \frac{M}{L} \dot{\zeta}_\mu, \] 

(3.30)

while the scalars transform as

\[ \tilde{E} = E - \epsilon, \] 

(3.31)

\[ \tilde{\psi} = \psi - \mathcal{H} \epsilon_w, \] 

(3.32)

\[ \tilde{C} = C - \frac{M}{L} \dot{\epsilon} + \frac{L}{M} \epsilon_w, \] 

(3.33)

\[ \tilde{\xi} = \xi + \mathcal{F} \epsilon_w + \dot{\epsilon}_w, \] 

(3.34)

\[ \tilde{P} = P + \frac{L}{M} \epsilon_\theta - \frac{M}{L} \dot{\epsilon}, \] 

(3.35)

\[ \tilde{\pi} = \pi + \dot{\epsilon}_\theta + \dot{\epsilon}_w, \] 

(3.36)

\[ \tilde{\phi} = \phi + \dot{\epsilon}_\theta + \mathcal{F} \epsilon_w. \] 

(3.37)

\(^3\)Notice that in order to write down this decomposition, the gauge functions should be regular enough (in order to guarantee the existence of \( \Box^{-1} \)) and, in any case, not singular.
Since there are twenty one independent components of the perturbed metric and six gauge functions we can define fifteen gauge-invariant degrees of freedom. These fifteen degrees of freedom are decomposed into four gauge-invariant scalars,

$$\tilde{\Psi} = \tilde{\psi} + \mathcal{H} \left[ \frac{M}{L} \tilde{C} - \frac{M^2}{L^2} \tilde{E}' \right],$$

$$\tilde{\Xi} = \tilde{\xi} - \frac{1}{L} \left[ \mathcal{L} \left( \frac{M}{L} \tilde{C} - \frac{M^2}{L^2} \tilde{E}' \right) \right]' ,$$

$$\tilde{\Phi} = \tilde{\phi} - \left[ \frac{M}{L} \mathcal{P} - \frac{M^2}{L^2} \tilde{E} \right]' - \mathcal{F} \left[ \frac{M}{L} \tilde{C} - \frac{M^2}{L^2} \tilde{E}' \right] ,$$

$$\tilde{\Pi} = \tilde{\pi} - \left[ \frac{M}{L} \tilde{P} - \frac{M^2}{L^2} \tilde{E}' \right]' - \left[ \frac{M}{L} \tilde{C} - \frac{M^2}{L^2} \tilde{E}' \right] ,$$

two gauge-invariant divergence-less vectors (corresponding to six degrees of freedom)

$$\tilde{V}_\mu = \tilde{D}_\mu - \frac{M}{L} \tilde{f}_\mu ,$$

$$\tilde{Z}_\mu = \tilde{Q}_\mu - \frac{M}{L} \tilde{f}_\mu ,$$

supplemented by the divergence-less and trace-less tensor degrees of freedom $h_{\mu\nu}$.

C. Gauge-invariant fluctuations of the sources

As for the fluctuations of the geometry, the fluctuations of the Higgs and gauge fields will also transform for infinitesimal diffeomorphisms. Following the conventions

$$\varphi(x^\mu, w, \theta) = \varphi(w, \theta) + \chi(x^\mu, w, \theta),$$

$$A_M(x^\mu, w, \theta) = A_M(w) + \delta A_M(x^\mu, w, \theta),$$

we have that

$$\delta \tilde{A}_A = \delta A_A - A^C \nabla_A e_C - \epsilon^B \nabla_B A_A ,$$

$$\tilde{\chi} = \chi - \epsilon^A \partial_A \varphi,$$

$$\tilde{\chi}^* = \chi - \epsilon^A \partial_A \varphi^* .$$

Using the convenient notation
\[ \delta A_\mu(x^\mu, w, \theta) = \frac{1}{e} a_\mu(x^\mu, w, \theta), \quad (3.47) \]
\[ \delta \varphi(x^\mu, w, \theta) = \chi(x^\mu, w, \theta), = v e^{i\theta} g(x^\mu, w, \theta) \quad (3.48) \]
\[ \delta \varphi(x^\mu, w, \theta)^* = \chi(x^\mu, w, \theta)^* = v e^{-i\theta} g(x^\mu, w, \theta)^*, \quad (3.49) \]

the gauge variation reads, in explicit terms,
\[
\begin{align*}
\tilde{a}_w &= a_w - (n - P) \dot{\epsilon}_\theta, \quad (3.50) \\
\tilde{a}_\theta &= a_\theta - (n - P) \dot{\epsilon}_\theta + \epsilon_w P', \\
\tilde{a} &= a - (n - P) \epsilon_\theta, \\
\tilde{A}_\mu &= A_\mu. \quad (3.51)
\end{align*}
\]

where the vector fluctuation \( a_\mu \)
\[ a_\mu = A_\mu + \partial_\mu a, \quad (3.52) \]

has been decomposed in a divergence-less part \( A_\mu \) (i.e. \( \partial_\mu A_\mu = 0 \)), transforming as a pure Poincaré vector, and a divergence full part \( a \) transforming as a scalar. Thanks to the symmetry of the background solutions (whose only non-vanishing component is \( A_\theta \)), the pure vector fluctuation, i.e. \( A_\mu \), is automatically invariant for infinitesimal coordinate transformations as implied by Eq. (3.45).

The other components (i.e. \( a, a_w \) and \( a_\theta \)) do change for infinitesimal coordinate transformations. Recalling the explicit form of gauge variation of the metric components given in Eqs. (3.31)–(3.37), Eqs. (3.50)–(3.51) lead to the following gauge-invariant quantities
\[
\begin{align*}
\tilde{A}_w &= \tilde{a}_w + (n - P) \left[ \frac{M}{L} \tilde{P} - \frac{M^2}{L^2} \dot{\tilde{E}} \right], \\
\tilde{A}_\theta &= \tilde{a}_\theta + (n - P) \left[ \frac{M}{L} \tilde{P} - \frac{M^2}{L^2} \dot{\tilde{E}} \right] + P' \left( \frac{M^2}{L^2} \tilde{E}' - \frac{M}{L} \tilde{C} \right), \\
\tilde{A} &= \tilde{a} + (n - P) \left[ \frac{M}{L} \tilde{P} - \frac{M^2}{L^2} \dot{\tilde{E}} \right]. \quad (3.53, 3.54, 3.55)
\end{align*}
\]

The same type of construction can be carried on in the case of the fluctuations of the Higgs field. In this case the explicit variation for infinitesimal coordinate transformations turns out to be
\[ \tilde{\chi} = \chi - \epsilon_w \phi' - \epsilon_\theta \dot{\varphi}, \]
\[ \tilde{\chi}^* = \chi^* - \epsilon_w \phi'^* - \epsilon_\theta \dot{\varphi}^*. \]  
(3.56)

Recalling, again, Eqs. (3.31)–(3.37), the gauge-invariant fluctuation corresponding to the Higgs field becomes
\[ \tilde{X} = \tilde{\chi} + \left( \frac{M}{L} \tilde{C} - \frac{M^2}{L^2} \tilde{E}' \right) \phi' + \left[ \frac{M}{L} \tilde{P} - \frac{M^2}{L^2} \tilde{E} \right] \dot{\varphi}, \]  
(3.57)

and analogously for the complex conjugate field. Using Eqs. (3.48)–(3.49), the gauge invariant combination corresponding to \( g \) will then be, from Eq. (3.57)
\[ \tilde{\Delta} = \tilde{g} + \left( \frac{M}{L} \tilde{C} - \frac{M^2}{L^2} \tilde{E}' \right) f' + i \left[ \frac{M}{L} \tilde{P} - \frac{M^2}{L^2} \tilde{E} \right] \dot{f}. \]  
(3.58)

D. Gauge choices

Before analyzing the explicit form of the evolution equations for the fluctuations it is appropriate to mention that specific gauge choices can be made in full analogy with what happens in the five-dimensional case [19,20]. In spite of the fact that our analysis will be gauge-invariant, the physical interpretation of a given result is more transparent in a specific gauge. Of particular relevance is the longitudinal gauge where the physical interpretation of the gauge-invariant fluctuations becomes particularly simple
\[ \tilde{E} = 0, \quad \tilde{P} = 0, \quad \tilde{C} = 0, \quad \tilde{f}_\mu = 0. \]  
(3.59)

By solving Eqs. (3.59) with respect to \( \epsilon, \epsilon_w, \epsilon_\theta \) and \( \zeta_\mu \), and by recalling Eqs. (3.28)–(3.30) and (3.31)–(3.37) we get
\[ \epsilon = E, \]
\[ \epsilon_w = \left( \frac{M^2}{L^2} \tilde{E}' - \frac{M}{L} \tilde{C} \right), \]
\[ \epsilon_\theta = \frac{M^2}{L^2} \tilde{E} - \frac{M}{L} \tilde{P}, \]
\[ \zeta_\mu = \tilde{f}_\mu. \]  
(3.60)
This gauge fixing was called *longitudinal* \([13,20]\) since the off-diagonal (scalar) fluctuations vanish and the perturbed form of the metric given in Eq. \((3.21)\) contains, in its off-diagonal entries, only pure vector fluctuations. Interestingly enough, using Eqs. \((3.59)–(3.60)\) we have that, in this gauge, the gauge-invariant fluctuations defined above greatly simplify. For the vectors we have that

\[
V_\mu = D_\mu, \quad Z_\mu = Q_\mu, \tag{3.61}
\]

whereas for the scalars

\[
\Psi = \psi, \quad \Phi = \phi, \quad \Xi = \xi, \quad \Pi = \pi, \tag{3.62}
\]

and similarly for the sources. This shows, as stated before, that in the longitudinal gauge, the off-diagonal elements of the metric are pure vectors.

## IV. EVOLUTION EQUATIONS FOR THE FLUCTUATIONS

Denoting with \(\delta\) the first order fluctuation of the corresponding tensor, the perturbed Einstein equations can be written as

\[
\delta R_{AB} = \kappa \delta \tau_{AB}, \tag{4.1}
\]

where

\[
\delta R_{AB} = \partial_C \delta \Gamma^C_{AB} - \partial_B \delta \Gamma^C_{AC} + \Gamma^C_{AB} \delta \Gamma^D_{CD} + \delta \Gamma^C_{AB} \Gamma^D_{CD} - \delta \Gamma^D_{BC} \Gamma^C_{AD} - \Gamma^D_{BC} \delta \Gamma^C_{AD}. \tag{4.2}
\]

In Eq. \((4.2)\), \(\Gamma^C_{AB}\) are the background values of the Christoffel connections [computed from Eq. \((3.1)\)] and \(\delta \Gamma^C_{AB}\) their first order fluctuations. In Appendix A all the explicit form of the Ricci fluctuations in the perturbed metric \((3.21)\) are reported [see Eqs. \((A.3)–(A.8)\)] together with the explicit expressions of the perturbed Christoffel connection [see Eqs. \((A.2)\)] and together with the perturbed components of the energy-momentum tensor [see Eqs. \((A.9)–(A.14)\)] whose general expression is
\[ \delta \tau_{AB} = \left[ -\frac{\Lambda}{2} + \frac{1}{8} F_{MN} F^{MN} - \frac{\lambda}{8} (\varphi^* \varphi - v^2)^2 \right] \delta G_{AB} + \frac{ie}{2} \delta A_A (\varphi^* \partial_B \varphi - \varphi \partial_B \varphi^*) + \frac{ie}{2} \delta A_B (\varphi^* \partial_A \varphi - \varphi \partial_A \varphi^*) \\
+ \frac{ie}{2} A_A (\chi^* \partial_B \varphi + \varphi^* \partial_B \chi - \chi \partial_B \varphi^* - \varphi \partial_B \chi^*) + \frac{ie}{2} A_B (\chi^* \partial_A \varphi + \varphi^* \partial_A \chi - \chi \partial_A \varphi^* - \varphi \partial_A \chi^*) \\
+ e^2 \delta A_A A_B \varphi^* \varphi + e^2 A_A \delta A_B \varphi^* \varphi + e^2 A_A A_B (\chi^* \varphi + \varphi^* \chi) - \delta F_{AC} F_{BD} G^{DC} - F_{AC} \delta F_{BD} G^{DC} - F_{AC} F_{BD} \delta G^{DC} \\
+ \frac{G_{AB}}{8} \left[ \delta F_{MN} F_{CD} G^{CM} G^{DN} + F_{MN} \delta F_{CD} G^{CM} G^{DN} + F_{MN} F_{CD} \delta G^{CM} G^{DN} + F_{MN} F_{CD} \delta G^{CM} G^{DN} \right] - \frac{\lambda}{4} (\varphi^* \varphi - v^2) (\chi^* \varphi + \varphi^* \chi) G_{AB} \\
+ \frac{1}{2} \left[ \partial_A \chi^* \partial_B \varphi + \partial_A \varphi^* \partial_B \chi + \partial_B \chi^* \partial_A \varphi + \partial_B \varphi^* \partial_A \chi \right] , \quad (4.3) \]

where \( \delta F_{CD} = \partial_C \delta A_D - \partial_D \delta A_C \).

Eq. (1.1) is supplemented by the perturbed version of Eqs. (2.3)–(2.4). The first order fluctuation of Eq. (2.3) gives

\[ \delta G^{AB} \left( \partial_A \partial_B \varphi - \Gamma^C_{AB} \partial_C \varphi \right) \\
G^{AB} \left( \partial_A \partial_B \chi - \Gamma^C_{AB} \partial_C \chi - \delta \Gamma^C_{AB} \partial_C \varphi \right) = e^2 \delta G^{AB} A_A A_B \varphi \\
- e^2 G^{AB} \delta A_A A_B \varphi - e^2 G^{AB} A_A \delta A_B \varphi - e^2 G^{AB} A_A A_B \chi \\
- i e \delta G^{AB} A_A \partial_B \varphi - i e G^{AB} A_A \partial_B \chi - i e G^{AB} \delta A_A \partial_B \varphi \\
- i e \delta G^{AB} \left[ \partial_A (A_B \varphi) - \Gamma^C_{AB} A_C \varphi \right] \\
- i e G^{AB} \left[ \partial_A (\delta A_B \varphi) + \partial_A (A_B \chi) - \delta \Gamma^C_{AB} A_C \varphi - \Gamma^C_{AB} \delta A_C \varphi - \Gamma^C_{AB} A_C \chi \right] \]
\[ + \lambda (\varphi^* \varphi - v^2) \chi + \lambda (\varphi^* \chi + \varphi^* \chi) \varphi = 0. \quad (4.4) \]

Finally, the first order fluctuation of Eq. (2.4) leads to

\[ \delta G^{AC} \left[ \partial_C F_{AB} - \Gamma^D_{AC} F_{DB} - \Gamma^D_{BC} F_{AD} \right] + e^2 \delta A_B \varphi^* \varphi + e^2 A_B \left[ \chi^* \varphi + \varphi^* \chi \right] \\
G^{AC} \left[ \partial_C \delta F_{AB} - \delta \Gamma^D_{AC} F_{DB} - \Gamma^D_{AC} \delta F_{DB} - \delta \Gamma^D_{BC} F_{AD} - \Gamma^D_{BC} \delta F_{AD} \right] \\
- \frac{ie}{2} \left[ \chi \partial_B \varphi^* + \varphi \partial_B \chi^* - \chi^* \partial_B \varphi - \varphi^* \partial_B \chi \right] = 0. \quad (4.5) \]
Three separate gauge-invariant problems then emerge naturally within the construction developed up to now. Depending upon the transformation properties of the given fluctuation, the tensor problem involves the evolution equations for the spin two fields coming from the geometry, namely $h_{\mu\nu}$. The vector problem involves all the pure vectors coming both from the geometry and from the fluctuations of the gauge fields, i.e. all the spin one fields. The relevant gauge-invariant degrees of freedom are, in this case, defined according to Eqs. (3.42)–(3.43) and (3.51)
\[ V_\mu, \quad Z_\mu, \quad A_\mu. \] (4.6)

Finally, the scalar problem involves all the gauge invariant degrees of freedom carrying spin zero, namely, according to Eqs. (3.38)–(3.41), (3.53)–(3.55) and (3.58)
\[ \Psi, \quad \Phi, \quad \Pi, \quad \Xi, \quad A_w, \quad A_\theta, \quad \Delta, \quad \Delta^*. \] (4.7)

Since no pure tensor source is present in the Abelian vortex, the tensor fluctuations will be decoupled from the very beginning. The gauge-invariant vector fluctuations of the geometry will be coupled with the vector fluctuations of the vortex. Finally, the gauge-invariant scalar fluctuations will mix both with the fluctuations of the Higgs and with the scalar fluctuations coming from the gauge sector.

**V. Spin Two Fluctuations: The Tensor Problem**

The explicit expression for the evolution equation of $h_{\mu\nu}$ is
\[ \ddot{h}_{\mu\nu} + h_{\mu\nu}'' + 4\mathcal{H}h_{\mu\nu} - \frac{L^2}{M^2}\partial_\alpha\partial^\alpha h_{\mu\nu} = 0, \] (5.1)
is obtained from the tensor part component of Eq. (4.1)
\[ \delta R_{\mu\nu} = \kappa\delta\tau_{\mu\nu} \] (5.2)
with the use of the explicit expressions of the Ricci tensors of Appendix A and of the background equations of (3.8)–(3.10) (allowing to eliminate the dependence upon the rescaled cosmological constant).
For the zero mode, the solution of Eq. (5.1) is, for each polarization,

$$h = K,$$

where $K$ is an arbitrary constant. By looking at the normalization of the kinetic term of $h_{\mu\nu}$ in the action, the canonical fluctuation is exactly

$$v_{\mu\nu} = ML h_{\mu\nu}.$$ 

Hence, the normalization condition for the canonical zero mode is

$$K^2 \int_{-\infty}^{+\infty} M^2(w)L^2(w)dw.$$ 

But the four-dimensional Planck mass is finite and the integral

$$M_P^2 \sim \frac{M_6^4}{m_H^2} \int_0^\infty dx M^2(x)L(x) \equiv \frac{M_6^4}{2} \int_{-\infty}^{+\infty} M^2(w)L^2(w)dw$$ 

is always convergent both for $w \to -\infty$ (going as $e^{2w}$) and for $w \to +\infty$ going as $(cw)^{-4}$. Since the integral defining the normalization of the tensor zero mode is the same integral appearing in the expression of the four-dimensional Planck mass, we can conclude that the tensor zero mode is always localized. Notice that in deriving this conclusion no specific vortex solution has been used, but only the background equations together with the asymptotics which are common to the whole class of vortex backgrounds discussed in Section II.

VI. SPIN ONE FLUCTUATIONS: THE VECTOR PROBLEM

From Eq. (4.3), taking into account Eq. (A.2), the pure vector component of the perturbed gauge field equation leads to the following gauge-invariant expression

$$\frac{L^2}{M^2} \partial_\alpha \partial^\alpha A_\mu - \ddot{A}_\mu - A''_\mu - 2\mathcal{H}A'_\mu + P \frac{M}{L} Z'_\mu - (\mathcal{H} - \mathcal{F})Z_\mu - \dot{V}_{\mu} + \alpha A_\mu L^2 f^2 = 0,$$ 

where Eqs. (3.42)–(3.43) have been used.

From Eqs. (4.1), the equations mixing the vectors coming from the metric and the vector coming from the source are
\[ \delta R_{\mu \nu} = \kappa \delta \tau_{\mu \nu}, \quad (6.2) \]
\[ \delta R_{\mu w} = \kappa \delta \tau_{\mu w}, \quad (6.3) \]
\[ \delta R_{\mu \theta} = \kappa \delta \tau_{\mu \theta}. \quad (6.4) \]

Using Eqs. (3.42) and (3.43) into Eqs. (6.2)–(6.4) and bearing in mind the vector part of Eqs. (A.3)–(A.5) and (A.9)–(A.11), the following equations are obtained

\[ V'_{\mu} + (3\mathcal{H} + F)V_{\mu} + \dot{Z}_{\mu} = 0, \quad (6.5) \]
\[ \frac{M}{2L} \left[ \ddot{V}_{\mu} - \frac{L^2}{M^2} \partial_{\alpha} \partial^\alpha V_{\mu} - \dot{Z}'_{\mu} + \left( \mathcal{H} - F \right) \dot{Z}_{\mu} \right] - \frac{P'}{\alpha L^2} \dot{A}_{\mu} = 0, \quad (6.6) \]
\[ Z''_{\mu} + 4\mathcal{H}Z'_{\mu} + (F' - \mathcal{H}' + 6\mathcal{H}F - 5\mathcal{H}^2 - F^2)Z_{\mu} - \frac{L^2}{M^2} \partial_{\alpha} \partial^\alpha Z_{\mu} - \frac{\nu P'}{\alpha L^2} A'_{\mu} = 0. \quad (6.7) \]

Eqs. (6.1) together with (6.5)–(6.7) form a system determining the coupled evolution of the vector fluctuations coming from the gauge and metric sector. In order to find the zero modes of the system, define first, the background function

\[ \varepsilon = \frac{L}{M}, \quad (6.8) \]

whose derivatives satisfy

\[ \frac{\varepsilon'}{\varepsilon} = \left( F - \mathcal{H} \right) = -\frac{\nu P}{\alpha L^2} \frac{P'}{P}, \quad (6.9) \]

as dictated by Eqs. (3.14) and as a consequence of the relations among the string tensions. Define then, the following combination of the gauge-invariant graviphoton fields

\[ \mu_{\alpha} = \varepsilon \dot{V}_{\alpha} - \left( \varepsilon Z_{\alpha} \right)', \quad (6.10) \]

Inserting now Eq. (6.10) into Eq. (6.1) and Eqs. (6.6)–(6.7), the system becomes

\[ \ddot{A}_{\alpha} + A''_{\alpha} + 2\mathcal{H}A'_{\alpha} - \varepsilon^2 \Delta A_{\alpha} + \frac{P'}{\varepsilon^2} \mu_{\alpha} - \alpha L^2 f^2 A_{\alpha} = 0, \quad (6.11) \]
\[ \dot{\mu}_{\alpha} - \varepsilon^3 \dot{V}_{\alpha} = 2\varepsilon^2 \frac{\nu P'}{\alpha L^2} \dot{A}_{\alpha}, \quad (6.12) \]
\[ \dot{\mu}'_{\alpha} + (4\mathcal{H} - 2\varepsilon \frac{\nu}{\varepsilon} ) \mu_{\alpha} + \varepsilon^3 \Delta Z_{\alpha} = 2\varepsilon^2 \left[ \nu P f^2 A_{\alpha} + \frac{\nu P'}{\alpha L^2} A'_{\alpha} \right], \quad (6.13) \]
subjected to the constraint

$$V'_\alpha + \left(4\mathcal{H} + \frac{\dot{\varepsilon}'}{\varepsilon}\right)V_\alpha + \dot{Z}_\alpha = 0. \quad (6.14)$$

Thanks to four-dimensional Poincaré invariance the four-dimensional D’Alembertian can be replaced by $-m^2$ where $m$ denotes the mass eigenvalue of the corresponding fluctuations. Furthermore, a generic fluctuation can be expanded in a Fourier series in $\theta$. For instance

$$V_\mu(x^\nu, w, \theta) = \sum_{\ell=-\infty}^{\infty} V_\mu^{(\ell)}(x^\nu, w)e^{i\ell\theta}. \quad (6.15)$$

Consider now the situation where $V_\mu$ is massless, i.e. $\Box V_\mu = 0$. From Eq. (6.12)

$$\mu_\alpha = 2\varepsilon^2 \frac{P'}{\alpha L^2} A_\alpha. \quad (6.16)$$

Inserting Eq. (6.16) into Eq. (6.11) the following decoupled equation is easily obtained by using, simultaneously, Eq. (3.7) together with Eq. (3.14)

$$\ddot{A} + A''_\alpha + 2\mathcal{H}A'_\alpha - \varepsilon^2 \Box A_\alpha - \left[\frac{P''}{P} + 2\mathcal{H}\frac{P''}{P}\right]A_\alpha = 0, \quad (6.17)$$

which can also be written as

$$N''_\alpha - \left[\frac{(MP)''}{MP}\right]N_\alpha + \left[\dot{N}_\alpha - \left(\frac{L}{M}\right)^2 \Box N_\alpha\right] = 0, \quad (6.18)$$

where $N_\alpha = MA_\alpha$. Eq. (6.18) holds for any mass or angular momentum eigenstates of the vector $A_\alpha$. Inserting now Eq. (6.16) into Eq. (6.13) and using the background equations we get

$$\Box Z_\alpha = 0, \quad (6.19)$$

implying that also $Z_\alpha$ should be massless.

From Eq. (6.18) the solution for the zero mode can be obtained

$$A_\alpha = K_{1,\alpha}P(w) + K_{2,\alpha}P(w) \int_{-\infty}^{w} \frac{dw'}{M(w')^2P(w')^2}. \quad (6.20)$$

Since $M(w')P(w') \to 1$ for $w \to -\infty$, the integral appearing in the second solution diverges in the lower limit of integration. The gauge zero mode is then given by
\[ A_\alpha = K_{1,\alpha} P(w). \quad (6.21) \]

In order to check for the normalizability of the zero mode the gauge action has to be perturbed to second order. By doing so we find that the corresponding kinetic term appears in the action for the fluctuations as

\[ \int d^4x \, dw \, d\theta [L^2 \partial_\mu A_\alpha \partial_\nu A_\beta \eta^{\mu\nu} \eta^{\alpha\beta}]. \quad (6.22) \]

Hence, the canonical zero mode is \( K_{1,\alpha} L(w) P(w) \) and the normalization condition reads

\[ K_{1,\alpha}^2 \int_{-\infty}^{+\infty} L^2(w) P(w)^2 = 1. \quad (6.23) \]

From Eqs. (3.15)–(3.18), the asymptotic behavior of the integrand of Eq. (6.23) in the two limits of integration is, respectively,

\[ L(w)^2 P(w)^2 \simeq e^{2w}, \quad w \to -\infty, \]
\[ L(w)^2 P(w)^2 \simeq (cw)^{-2-2\gamma_P}, \quad w \to +\infty. \quad (6.24) \]

Since \( \gamma_P \gg 1 \), the integral converges in both limits and the gauge zero mode is normalized. Furthermore, the appropriate boundary conditions for the zero mode are satisfied. In fact, from Eq. (6.21)

\[ A'_\alpha(-\infty) = A'_\alpha(+\infty) = 0. \quad (6.25) \]

Consider now the zero modes of the gauge-invariant vector fluctuations of the geometry, namely \( V_\mu \) and \( Z_\alpha \). For the lowest angular momentum eigenstates \( \dot{V}_\mu = 0 \) and \( \dot{Z}_\alpha = 0 \). Hence, from Eq. (6.14) the zero mode of \( V_\mu \) is simply obtained and it is

\[ \]

\(^4\)According to [39], localized gauge zero modes are present in five-dimensions, provided the four-dimensional Planck mass is not finite. In the present case, the localization of the gauge zero mode occurs in a six-dimensional geometry (leading to a finite four-dimensional Planck mass) and in the presence of a gauge field background (which is absent in the case of [39]).
\[ V_\alpha(w) = \frac{C_{1,\alpha}}{M(w)^3 L(w)}, \]  

(6.26)

where \( C_{1,\alpha} \) is the integration constant. Using Eq. (6.21) into Eq. (6.16) the zero mode of \( Z_\alpha \) is obtained recalling Eq. (6.10):

\[ Z_\alpha(w) = K_{1,\alpha} \frac{L}{M} + C_{2,\alpha} \frac{M}{L}, \]  

(6.27)

where \( C_{2,\alpha} \) is the further integration constant while \( K_{1,\alpha} \) is the same constant appearing in Eq. (6.21) and which is determined from Eq. (6.23).

In order to assess the localization of the vector fluctuations of the metric, the appropriate normalization of the kinetic term of the fluctuations has to be deduced by perturbing the action to second order. It is better to perturb to second order the Einstein-Hilbert action directly in the form

\[ G^{AB} \left( \Gamma^D_{AC} \Gamma^C_{BD} - \Gamma^C_{AB} \Gamma^D_{CD} \right) \]  

(6.28)

where the total derivatives are absent. The kinetic terms appear in the action as

\[ \int d^4x \, dw \, d\theta M^2 L^2 \left[ \partial_\alpha Z_\mu \partial_\beta Z_\nu \eta^{\alpha\beta} \eta^{\mu\nu} \right], \]

\[ \int d^4x \, dw \, d\theta M^2 L^2 \left[ \partial_\alpha V_\mu \partial_\beta V_\nu \eta^{\alpha\beta} \eta^{\mu\nu} \right] \]  

(6.29)

From Eqs. (6.20) and (6.27) the normalization condition for \( V_\mu \) and \( Z_\mu \) lead, respectively, to the following two integrals

\[ C_1^2 \int_{-\infty}^{\infty} \frac{dw}{M(w)^4}, \]  

(6.30)

and

\[ \int_{-\infty}^{\infty} \left[ K_{1,\alpha} L(w)^2 + C_{2,\alpha} M(w)^2 \right]^2 dw. \]  

(6.31)

For \( w \to +\infty, M(w)^{-4} \to (cw)^4 \), as implied by the AdS\(_6\) nature of the geometry, in this limit. The normalization integral of Eq. (6.30), corresponding to the \( V_\mu \) fluctuation, diverges at infinity. Consequently the zero mode of \( V_\mu \) is not localized.
In Eq. (6.31) \( K_{1,\alpha} \) is determined from the normalization of the gauge zero mode according to Eq. (6.23). The integrand appearing in Eq. (6.31) has three terms: two going, respectively, as \( L^4 \) and \( M^2 L^2 \) and one going as \( M^4 \). The term going as \( M^4 \) makes the integral divergent. For \( w \to -\infty \), \( M(w)^2 \to 1 \) and, hence, the integral appearing in Eq. (6.31) will be linearly divergent for \( w \to -\infty \).

The situation can then be summarized by saying that while the gauge zero mode is localized, the vector modes of the geometry are never localized. One of them, \( V_\mu \), because of the behavior at infinity, the other, \( Z_\mu \), because of the behavior in the core of the defect.

It is interesting, in this context, to repeat the calculation described in the present Section in the coordinate system defined by the line element of Eq. (2.2). In this case the evolution equations of the fluctuations are less symmetric but, in spite of this, the same conclusions can be reached through a slightly different algebraic procedure. The results of this exercise are reported in Appendix B.

Two final remarks are in order. To get to the correct conclusion is crucial to discuss the full system describing the fluctuations of the geometry together with the fluctuation of the source. Consider then the case where \( P' \to 0 \), \( P \to 1 \) and the gauge field fluctuations are absent, i.e. \( A_\alpha = 0 \). If \( P' \to 0 \), then, \( \varepsilon \to 1 \) and \( H \to F \). In this case the space-time is \( \text{AdS}_6 \) and the evolution of the fluctuations follow from Eqs. (6.5)–(6.7):

\[
\begin{align*}
V'_\mu + 4\mathcal{H}V_\mu + \dot{Z}_\mu &= 0, \tag{6.32} \\
\tilde{V}_\mu - \Box V_\mu - \tilde{Z}'_\mu &= 0 \tag{6.33} \\
Z''_\mu + 4\mathcal{H}Z'_\mu - \Box Z_\mu - [\dot{V}'_\mu + 4\mathcal{H}\dot{V}_\mu] &= 0. \tag{6.34}
\end{align*}
\]

Using Eq. (6.32) into Eq. (6.34) the following decoupled equation can be obtained

\[
\dot{Z}_\mu + Z''_\mu - \Box Z_\mu + 4\mathcal{H}Z'_\mu = 0, \tag{6.35}
\]

which implies that the canonically normalized zero mode, i.e. \( M^2 Z_\mu \), behaves as \( M(w)^2 \). The canonically normalized zero mode related to \( V_\mu \), i.e. \( M^2 V_\mu \) behaves, on the contrary, as \( M(w)^{-2} \). Therefore, we would erroneously conclude that while \( V_\mu \) is not normalizable, \( Z_\mu \) is
normalizable. This argument overlooks the behaviour of the wavefunctions close to the core of the vortex which implies that also $Z_\mu$ is not normalizable.

Eq. (6.18) can be written as

$$\mathcal{N}_{p,\ell}'' + \left[ m^2 - \ell^2 - \frac{(MP)''}{MP} \right] \mathcal{N}_{p,\ell} = 0,$$

(6.36)

where the polarization index has been suppressed in order to leave room for the eigenvalues indices. Eq. (6.36) can be studied using the usual techniques of supersymmetric quantum mechanics [40] by associating the appropriate superpotential to the Schrödinger-like potential $(MP)''/(MP)$. Since the effective potential goes to zero for $w \to \infty$ the spectrum will probably be continuous. Moreover, stability requires that $m > \ell$. Finally using the explicit solutions of Eq. (6.36) in the different physical limits the massive eigenstates of the system can be analyzed by inserting the obtained solutions back into Eq. (5.11)–(5.13).

VII. SPIN ZERO FLUCTUATIONS: THE SCALAR PROBLEM

The gauge-invariant metric fluctuations of spin zero couple both to the scalar fluctuations generated by the Higgs field and to the scalar fluctuations coming from the gauge field. Eq. (4.4) leads, respectively, to the following expressions for the real and imaginary parts of the Higgs fluctuations:

$$\varepsilon^2 \Box \Delta_1 - \Delta_1'' - \Delta_1' - 4\mathcal{H}\Delta_1' + \left[ P^2 + \mathcal{L}^2(2f^2 - 1) \right] \Delta_1$$

$$+ \mathcal{L}^2 f^2 \Delta_1 - 4f\Psi' + 2P^2 f\Phi + f'\Phi' - 2(f'' + 4\mathcal{H}f')\Xi - f'\Xi'$$

$$- f'\Pi - 2P fA_\theta + 2P \Delta_2 = 0,$$

(7.1)

and

$$\varepsilon^2 \Box \Delta_2 - \Delta_2'' - \Delta_2' - 4\mathcal{H}\Delta_2' + \left( P^2 + \mathcal{L}^2(2f^2 - 1) \right) \Delta_2$$

$$- \mathcal{L}^2 f^2 \Delta_2 - 4P f\Psi + Pf(\dot{\Xi} - \dot{\Phi}) - (4\mathcal{H}P f + P' f + 2f'P)\Pi - fP\Pi'$$

$$+ f \left[ -\varepsilon^2 \partial_\alpha \partial^\alpha A + A'_w + \dot{A}_\theta \right] + (2f' + 4\mathcal{H}f)A_w = 0,$$

(7.2)
where, according to the notations of Eq. (3.58) we wrote

\[ \Delta = \Delta_1 + i\Delta_2. \] (7.3)

From the scalar components of Eq. (4.5) the following set of equations is obtained

\[ A'' + \ddot{A} - A' + 2\mathcal{H}(A' - A_w) = \alpha L^2 f^2 A - \alpha L^2 f \Delta, \] (7.4)

\[ \varepsilon^2 \left[ \Box(A_w - A') - \ddot{A_w} + \ddot{A}_\theta - P' \left( \dot{\Phi} + \dot{\Xi} + (4\mathcal{H} - 2\mathcal{F})\Pi + 4\dot{\Psi} \right) + P'' \Pi + \alpha L^2 f^2 \Delta + \alpha L^2 \left( f\Delta_2 - f'\Delta_2 \right) \right] = 0, \] (7.5)

\[ \varepsilon^2 \Box(\dot{A}_\theta - \dot{A}) + 2 \left[ P'' + 2P' (2\mathcal{H} - \mathcal{F}) \right] \Xi + P' \left( \Phi' + \Xi' + 4\Psi' \right) + (4\mathcal{H} - 2\mathcal{F})(\dot{A}_w - \dot{A}'_\theta) + \dot{A}_w' - \dot{A}_\theta' \] + \[ \alpha L^2 f^2 A_\theta - 2\alpha L^2 f \Delta_1 - \alpha L^2 f \Delta_2 = 0. \] (7.6)

Eq. (7.4) represents the divergence-full part of Eq. (4.5). Eqs. (7.5) and Eq. (7.6) are the \(w\) and \(\theta\) component of Eq. (4.5).

Using the results of Appendix A, the \((\mu \neq \nu)\), \((\mu = \nu)\), \((\mu, w)\) and \((\mu, \theta)\) components of Eq. (4.1) lead, respectively, to

\[ \Phi + \Xi - 2\Psi = 0, \] (7.7)

\[ \Psi'' + \ddot{\Psi} - \varepsilon^2 \Box \Psi + 8\mathcal{H}\Psi' + \mathcal{H}\Xi' + 2(\mathcal{H}' + 4\mathcal{H}^2)\Xi - \mathcal{H}(\Phi' - \dot{\Pi}) = \] \[ \frac{\nu}{2\alpha L^2} \left[ P' (\dot{A}_w - \dot{A}_\theta') + P'' (\Phi + \Xi) \right] - 2\frac{\nu L^2}{4} f(f^2 - 1)\Delta, \] (7.8)

\[ \Phi' + (\mathcal{F} - \mathcal{H})\Phi - \frac{1}{2} (3\mathcal{H} + \mathcal{F})\Xi - 3\Psi' = \nu f' \Delta_1 + \frac{\nu P'}{\alpha L^2} (\dot{A} - \dot{A}_\theta), \] (7.9)

\[ \dot{\Xi} - \frac{1}{2} \Pi' - (\mathcal{H} + \mathcal{F})\Pi - 3\dot{\Psi} = \nu f P \Delta_2 + \frac{\nu P'}{\alpha L^2} (A_w - A') - \nu f^2 P A, \] (7.10)

whereas the \((w, w)\), \((w, \theta)\) and \((\theta, \theta)\) components of Eq. (4.1) give

\[ - \varepsilon^2 \Box \Xi + \ddot{\Xi} - (4\mathcal{H} + \mathcal{F})\Xi' + \left[ \mu L^2 + \frac{\nu L^2}{4} (f^2 - 1)^2 \right] \Xi + \Phi'' + \mathcal{F}' \Phi - \frac{3}{2} \frac{P''}{\alpha L^2} \Phi \] \[ - \dot{\Pi}' - \mathcal{F}\dot{\pi} - 4\Psi'' + 4(\mathcal{F} - 2\mathcal{H})\Psi' = 2\frac{\nu L^2}{4} (f^2 - 1) f \Delta_1. \]
\[ + 2\nu f'\Delta_1' + \frac{3\nu P'}{2\alpha\mathcal{L}^2} (\hat{A}_w - A'_w), \quad (7.11) \]

\[ \frac{1}{2}\varepsilon^2 \Box \Pi + \nu P^2 f^2 \Pi + 4\mathcal{H}\ddot{\Xi} + 4\dot{\Psi}' + 4(\mathcal{H} - \mathcal{F})\dot{\Psi} = \nu P f^2 A_w \]

\[ - \nu f' (\hat{\Delta}_1 - P \hat{\Delta}_2) - \nu P f \Delta_1', \quad (7.12) \]

\[ - \varepsilon^2 \Box \Phi + \Phi'' + (4\mathcal{H} + \mathcal{F})\dot{\Phi}' - \nu \left( \frac{3P'^2}{2\alpha\mathcal{L}^2} + 2P^2 f^2 \right) \Phi - \dot{\Pi}' \]

\[ - (4\mathcal{H} + \mathcal{F})\dot{\Pi} + \ddot{\Xi} - \mathcal{F}\Xi' - 2 \left( 4\mathcal{H}\mathcal{F} + \mathcal{F}' + \frac{3\nu P'^2}{4\alpha\mathcal{L}^2} \right) \Xi - 4(\ddot{\Psi} + \mathcal{F}\Psi') \]

\[ = \frac{3\nu P'}{2\alpha\mathcal{L}^2} (\hat{A}_w - A'_w) + \left[ \frac{\nu \mathcal{L}^2}{2} (f^2 - 1)f + P^2 \nu f \right] \Delta_1 - 2P\nu f^2 A_\theta + 2\nu P f \Delta_2. \quad (7.13) \]

The lowest angular momentum eigenstates are recovered by setting to zero the derivatives with respect to \(\theta\). The system of scalar fluctuations separates then into two coupled sets of equations. In the first set the off-diagonal scalar \(\Pi\) couples with the imaginary part of the Higgs field \(\Delta_2\) together with \(A\) and \(A_w\). In the second set \(\Psi\) and \(\Xi\) couple with \(A_\theta\) and with the real part of the Higgs fluctuation \(\Delta_1\).

Using the variables,

\[ q = A' - A_w, \]

\[ \mathcal{T} = \frac{\Delta_2}{f} - A, \quad (7.14) \]

Eqs. (7.4)–(7.5) give

\[ \varepsilon^2 \Box q + 2\nu Pf^2 \left( P\Pi + q + \mathcal{T}' \right) = 0, \quad (7.15) \]

\[ q' + 2\mathcal{H}q + \alpha\mathcal{L}^2 f^2 \mathcal{T} = 0. \quad (7.16) \]

Inserting Eqs. (7.14) into Eqs. (7.10) and (7.12) we obtain

\[ \Pi' + 2(\mathcal{H} + \mathcal{F})\Pi + 2\nu f^2 P\mathcal{T} - 2\nu P' \frac{P'}{\alpha\mathcal{L}^2} q = 0, \quad (7.17) \]

\[ \varepsilon^2 \Box \Pi + 2\nu Pf^2 \left( P\Pi + q + \mathcal{T}' \right) = 0. \quad (7.18) \]

Finally, inserting Eq. (7.14) into Eq. (7.2):

\[ \mathcal{T}'' + 4\mathcal{H}\mathcal{T}' - \left[ P^2 + \mathcal{L}^2 (f^2 - 1) \right] \mathcal{T} - \varepsilon \Box \mathcal{T} \]

\[ - f q' - 2(f' + 2\mathcal{H} f) - (4\mathcal{H} P + P' f + 2f' P) \Pi = 0. \quad (7.19) \]
Eq. (7.15) leads to

\[ q = -\mathcal{T}' - P\Pi - \frac{\varepsilon}{2\nu Pf^2} q, \]  

(7.20)

and inserting Eq. (7.20) into Eq. (7.19), the following relation holds

\[ \varepsilon^2 \Box \left( \mathcal{T} + \frac{q'}{\alpha L^2 f^2} \right) = 0. \]  

(7.21)

Summing up Eqs. (7.17) and (7.16)

\[ \mathcal{R}' + 2(\mathcal{H} + \mathcal{F}) \mathcal{R} = 0, \]  

(7.22)

whereas subtracting Eqs. (7.18) and (7.15)

\[ \varepsilon^2 \Box \mathcal{R} = 0, \]  

(7.23)

where

\[ \mathcal{R} = \Pi - \frac{2
\nu P}{\alpha L^2} q. \]  

(7.24)

While Eq. (7.23) implies that \( \mathcal{R} \) is a massless combination, the solution of Eq. (7.22) gives the evolution of the zero mode

\[ \mathcal{R}(w) = \Pi - \frac{2\nu P}{\alpha L^2} \simeq \frac{k_0}{L^2}. \]  

(7.25)

where \( k_0 \) is the integration constant. The zero mode of \( \Pi \) can be obtained by using Eq. (7.20) into eq. (7.17) in the case where the mass of \( q \) vanishes. The obtained relation can be written as

\[ \frac{\partial}{\partial w} \left[ M^4 \Pi + 2\frac{\nu}{\alpha} P' \mathcal{T} \frac{M^4}{\mathcal{L}^2} \right] = 0, \]  

(7.26)

whose integral gives

\[ \Pi = -\frac{k_1}{M^4} - 2\frac{\nu}{\alpha L^2} P' \mathcal{T}, \]  

(7.27)

where \( k_1 \) is an integration constant. From Eqs. (3.30) and (7.27), \( \Pi \) can be eliminated and a relation between \( q \) and \( \mathcal{T} \) is obtained. Inserting the obtained relation back into Eq. (7.16) the decoupled equation for \( \mathcal{T} \) is found and it can be expressed as
\[
\left(\frac{L^2}{M^2 P} T \right)' = S(w) \tag{7.28}
\]

where
\[
S(w) = \frac{\alpha}{2\nu P^2 M^4} \left\{ \frac{k_0}{L^2} P' + \frac{k_1}{M^2} \left[ 2(H - F) + \frac{P'}{P} \right] \right\} \approx \frac{m^2_H}{2\nu P} \frac{k_0 + k_1}{M^2}. \tag{7.29}
\]

The second equality in Eq. (7.29) follows in the limit where \( M \approx L \) (i.e. \( H = F \)), namely in the regime far from the core when the space-time is AdS. The final expressions for the various zero modes is then
\[
T(w) = k_2 \frac{M^2}{L^2} P + \frac{M^2}{L^2} P \int_w^\infty dw' S(w'), \tag{7.30}
\]
\[
q(w) = -k_2 P' \frac{M^2}{L^2} - \frac{\alpha}{2\nu M^2} \left[ \frac{k_0}{L^2} + \frac{k_1}{M^2} \right] - \frac{M^2}{L^2} P' \int_w^\infty dw' S(w'), \tag{7.31}
\]
\[
\Pi(w) = -2k_2(H - F) \frac{M^2}{L^2} - \frac{k_1}{M^4} + \frac{M^2}{L^2} P' \int_w^\infty dw' S(w'). \tag{7.32}
\]

Notice that in the limit of \( w \to \infty \), \( \Pi \) diverges as \( M^{-4} \sim w^4 \). Since the canonically normalized fluctuation related to \( \Pi \) is, for large \( w \), \( M^2 \Pi \), we can also deduce that the gauge-invariant fluctuation \( \Pi \) is not localized.

The evolution of the zero modes of \( \Phi \) and \( \Xi \) is very complicated. The cumbersome expressions which can be obtained should anyway evaluated in some limit. The zero modes of \( \Phi \) and \( \Xi \) can be obtained by reminding that far from the core of the vortex the background solutions are determined by the cosmological constant and nothing else. In this limit \( H = F \), \( H^2 = H' \) while \( f = 1 \) and \( P = 0 \). In order to take the limit consistently it should also be borne in mind that, according to the relations among the string tensions, \( PP'/L^2 \propto (H - F) \to 0 \). The resulting equations are still coupled and they can be written, for the zero modes, as
\[
\Psi'' + 6H \Psi' + 2H \Xi' + 2(H' + 4H^2) \Xi = 0, \tag{7.33}
\]
\[
\Xi'' + 2\Psi'' + 2H \Psi' + 2(H' + 4H^2) \Xi = 0, \tag{7.34}
\]
\[
2\Psi'' - \Xi'' + 6H \Psi' - 6H \Xi' - 2(H' + 4H^2) \Xi = 0, \tag{7.35}
\]
\[
\Psi' + \Xi' + 4H \Xi = 0. \tag{7.36}
\]
In order to get Eqs. (7.33)–(7.36), Eq. (7.4) has been used in order to eliminate $\Phi$ in Eqs. (7.8)–(7.9) and in Eqs. (7.11)–(7.13).

Using Eq. (7.36) into Eq. (7.34) the following equation for $\Xi$ is obtained:

$$\Xi'' + 4H\Xi' + 6H'\Xi = 0,$$

implying that the zero mode goes as $\Xi \sim M^{-2}$. The canonically normalized field is $M^2\Xi$. This implies that $\Xi$ is not localized since at infinity the normalized zero mode goes as a constant and the integral is linearly divergent in $w$ or, exponentially divergent in $x$. Using Eq. (7.36) the zero mode of $\Psi$ can be also obtained. It corresponds to $\Psi \sim \text{constant}$. In this case the canonical zero mode is normalized, at infinity. Finally, from Eq. (7.7), it can be deduced that $\Phi$ is not normalizable since $\Xi$ is not localized. The last thing which should be determined is the behaviour of $\Psi$ for $w \to -\infty$. In fact $\Psi$ is normalizable at infinity. If it would also be normalizable in the origin then, the zero mode of $\Psi$ would be localized. The analysis of the equations in the limit $w \to -\infty$ lead to the conclusion that $\Psi \sim \text{constant}$ is also a solution. However, since the canonical fluctuation related to $\Psi$ is given by $M^2\Psi$, then also this scalar mode is not localized. In fact, the normalization integral will go as $w$ for $w \to -\infty$.

**VIII. CONCLUDING REMARKS**

In this paper the zero modes of six-dimensional vortex solutions have been analyzed in the framework of the gravitating Abelian-Higgs model with a (negative) cosmological constant in the bulk. The vortices lead to regular geometries (i.e. free of curvature singularities) with finite four-dimensional Planck mass. Far from the core of the vortex the geometry is determined by the value of the bulk cosmological constant leading to a AdS$_6$ space-time.

The analysis of the fluctuations of the vortex has been performed in order to check for the localization of the corresponding zero modes. A given fluctuation is localized if it is normalizable with respect to the bulk coordinates describing the geometry of the vortex in
the transverse space. In order to obtain a realistic low-energy theory from higher dimensions it is important to localize fields of various spin whose low energy dynamics could lead to the known interactions of the standard model.

The ambiguity deriving from the change of the fluctuations for infinitesimal coordinate transformations around the (fixed) vortex background has been resolved by resorting to a fully gauge-invariant approach. While the tensor zero modes are localized on the vortex neither the graviphotons fields (spin one) coming from the geometry nor the scalar fluctuations of the metric are localized. The gauge field fluctuation leads to a localized zero mode.

In the model discussed in the present paper fermionic degrees of freedom are absent. It is therefore difficult to understand how the gauge zero mode will interact with them. Using recent results [3,4,11] concerning the localization of fermionic zero modes on six-dimensional vortices, it would be interesting to understand if the gauge zero mode discussed in the present paper could mediate electromagnetic interactions. It is in fact unclear if the gauge zero mode is charged under an Abelian (local) symmetry which could be interpreted as the ordinary electromagnetism.

In spite of the absence of fermionic degrees of freedom in the set-up, the vector field coming from the geometry cannot mediate any type of electromagnetic interaction. In fact it has been shown in general terms that the spin one fields associated with the fluctuations of the six-dimensional geometry are not localized on the vortex if the four-dimensional Planck mass is finite and if the geometry of the vortex is regular.

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APPENDIX A: EXPLICIT EXPRESSIONS OF THE FLUCTUATIONS

Since we ought to obtain gauge-independent evolution equations for the fluctuations, it is important to obtain, as a first step, general expressions for the fluctuations of the Ricci tensors. Subsequently, the evolution equations for the fluctuations can be written without specifying the gauge but re-expressing the fluctuations in terms of the gauge-invariant potentials discussed before. The values of the background Christoffel connections are

\[\Gamma^\mu_{\nu\lambda} = \frac{M^2}{L^2} \mathcal{H} \eta_{\nu\lambda},\]
\[\Gamma^\beta_{\alpha\lambda} = \mathcal{H} \delta_{\alpha}^{\beta},\]
\[\Gamma^b_{a\lambda} = \mathcal{F} \delta_{a}^{b},\]
\[\Gamma^w_{ab} = \mathcal{F} \eta_{ab},\] (A.1)

[a, b run over the two transverse dimensions and the Greek indices run over the four spacetime dimensions].

Using the explicit form of the line element \((3.1)\) together with the generic form of the perturbed metric given in eq. \((3.21)\), the eighteen perturbed Christoffel connections

\[\delta \Gamma^w_{\mu\nu} = \frac{M^2}{L^2} (H'_{\mu\nu} + 2 \mathcal{H} H_{\mu\nu}) + 2 \frac{M^2}{L^2} \mathcal{H} \xi_{\mu\nu} - \frac{M}{2L} \mathcal{G}_{\nu} \mathcal{G}_{\mu},\]
\[\delta \Gamma^w_{\mu\nu} = \frac{M}{L} \mathcal{G}_{\mu} - \partial_{\mu} \xi,\]
\[\delta \Gamma^w_{\mu\nu} = -\xi',\]
\[\delta \Gamma^w_{\mu\nu} = \frac{M}{2L} \left( \mathcal{B}'_{\mu} + (\mathcal{F} + \mathcal{H}) \mathcal{B}_{\mu} \right) - \frac{1}{2} \partial_{\mu} \pi - \frac{M}{2L} \dot{\mathcal{G}}_{\mu},\]
\[\delta \Gamma^w_{\mu\theta} = -\dot{\xi} + \mathcal{F} \pi,\]
\[\delta \Gamma^w_{\mu\theta} = (\mathcal{F} + \mathcal{H}) \mathcal{B}_{\mu},\]
\[\delta \Gamma^w_{\theta\theta} = \phi' - \dot{\pi} + 2 \mathcal{F} (\phi - \xi),\]
\[\delta \Gamma^w_{\mu\nu} = \frac{M^2}{L^2} \left( \mathcal{H} \pi \eta_{\mu\nu} + H_{\mu\nu} \right) - \frac{M}{2L} (\partial_{\mu} \mathcal{B}_{\nu} + \partial_{\nu} \mathcal{B}_{\mu}),\]
\[\delta \Gamma^w_{\mu\nu} = -\partial_{\mu} \phi,\]
\[\delta \Gamma^w_{\theta\theta} = -\phi' + \mathcal{F} \pi,\]
\[\delta \Gamma^w_{\mu\nu} = \frac{M}{2L} \left( -\mathcal{B}'_{\mu} + (\mathcal{F} - \mathcal{H}) \mathcal{B}_{\mu} \right) + \frac{M}{2L} \dot{\mathcal{G}}_{\mu} - \frac{1}{2} \partial_{\mu} \pi,\]
perturbed Ricci tensors, lead, through the repeated use of the Palatini identities in Eq. (4.2) to the following six perturbed Ricci tensors,

\[
\delta \Gamma^\theta_{\theta\mu} = -\dot{\phi}', \\
\delta \Gamma^\theta_{\mu w} = \dot{\xi} - \dot{\pi}' - \mathcal{F}\pi, \\
\delta \Gamma^\mu_{\alpha\beta} = -\frac{M}{L} \mathcal{H} \mathcal{G}^{\alpha} \eta_{\alpha\beta} + ( -\partial^\mu \mathcal{H}_{\alpha \beta} + \partial_{\beta} \mathcal{H}^\mu_{\alpha} + \partial_{\alpha} \mathcal{H}^\mu_{\beta} ), \\
\delta \Gamma^\mu_{\alpha w} = \mathcal{H}^\mu_{\alpha} + \frac{L}{2M} (\partial_{\alpha} \mathcal{G}^\mu - \partial^\mu \mathcal{G}_{\alpha} ), \\
\delta \Gamma^\mu_{\mu w w} = \frac{L}{M} ( \mathcal{G}^{\mu'} + \mathcal{H} \mathcal{G}^\mu ) - \frac{L^2}{M^2} \partial^\mu \xi, \\
\delta \Gamma^\mu_{\mu \theta} = \frac{L}{M} ( \mathcal{F} \mathcal{G}^\mu + \dot{\mathcal{B}}^\mu ) - \frac{L^2}{M^2} \partial^\mu \phi, \\
\delta \Gamma^\mu_{\alpha \theta} = \frac{L}{2M} ( \partial_{\alpha} \mathcal{B}^\mu - \partial^\mu \mathcal{B}_{\alpha} ) + \dot{\mathcal{H}}^\mu, \\
\delta \Gamma^\mu_{\theta w} = \frac{L}{2M} \left( \mathcal{B}^{\mu'} + ( \mathcal{H} - \mathcal{F} ) \mathcal{B}_\mu - \partial^\mu \pi \right) + \frac{L}{2M} \dot{\mathcal{G}}^\mu. 
\] (A.2)

These equations lead, through the repeated use of the Palatini identities in Eq. (4.2) to the following six perturbed Ricci tensors,
From Eq. (4.3) and using, again, Eqs. (A.2) the explicit form of the perturbed energy-

$$\delta R_{\theta\mu} = \frac{L}{M} (\partial_{\alpha} B^\alpha) + \frac{L}{M} F \partial_{\alpha} G^\alpha$$

$$+ \phi'' + \phi'(4H + F) + 2(F' + 4HF)\phi - \frac{L^2}{M^2} \partial_{\alpha} \partial^\alpha \phi$$

$$+ \dot{\xi} - 2(F' + 4HF)\xi - \dot{F} \xi - \ddot{\pi} - \dddot{\pi} (F + 4H) - \dddot{H} - \dot{H} - \dot{F} \dot{H}$$

$$= \frac{L}{2M} \left[ \delta \mu \right] + (F - H) \partial_{\alpha} B^\alpha - \frac{L^2}{2M^2} \partial_{\alpha} \partial^\alpha \pi + \frac{L}{2M} (\partial_{\alpha} \dot{G}^\alpha)$$

$$+ (F' + 4HF) \pi - \dot{H} \pi + \dot{H} (F - H) - 4H \dot{\xi}.$$  \hspace{1cm} (A.8)

In Eqs. (A.3) and using, again, Eqs. (A.2) the explicit form of the perturbed energy-
momentum tensor is obtained:

$$\kappa \delta \tau_{\mu\nu} = 2M^2 \left[ -\frac{\mu}{2} L^2 - \frac{\nu}{8} (f^2 - 1)^2 L^2 + \frac{\nu P^2}{4 \alpha L^2} \right] H_{\mu\nu}$$

$$+ \frac{M^2}{L^2} \eta_{\mu\nu} \left\{ \frac{\nu P^2}{2 \alpha L^2} (\phi + \xi) - \frac{\nu P'}{2 \alpha L^2} (a_\theta - \dot{a}_\theta) - \frac{\nu}{4} f (f^2 - 1) L^2 (g + g^*) \right\},$$  \hspace{1cm} (A.9)

$$\kappa \delta \tau_{\mu\beta} = \frac{M}{L} B_{\mu} \left[ -\frac{\mu}{2} L^2 - \frac{\nu}{8} (f^2 - 1)^2 L^2 + \frac{\nu P^2}{4 \alpha L^2} \right] - \frac{\nu P'}{\alpha L^2} A_{\mu} - \nu Pf^2 A_{\mu}$$

$$+ \partial_{\mu} \left[ \frac{i\nu}{2} Pf (g - g^*) + \frac{\nu P'}{\alpha L^2} (a_\theta - \dot{a}_\theta - f P) \right],$$  \hspace{1cm} (A.10)

$$\kappa \delta \tau_{\omega\nu} = 2\xi \left[ -\frac{\mu}{2} L^2 - \frac{\nu}{8} L^2 (f^2 - 1)^2 \right] + \frac{3}{2} \nu P^2$$

$$+ \frac{3}{2} \nu \frac{P'}{\alpha L^2} (a_\omega - \dot{a}_\omega) + \nu f'(g + g^*) + \frac{\nu}{4} L^2 f (f^2 - 1) (g + g^*)$$

$$- 2\nu a_\theta Pf^2$$

$$+ \frac{3}{2} \nu \frac{P'}{\alpha L^2} (a_\omega - \dot{a}_\omega) - i\nu Pf (\dot{g} - \dot{g}^*) + \nu [P^2 f + \frac{L^2}{4} f (f^2 - 1)] (g + g^*)$$

$$\kappa \delta \tau_{\omega\theta} = \left[ -\frac{\mu}{2} L^2 - \frac{3 \nu P^2}{4 \alpha L^2} - \frac{\nu}{8} L^2 (f^2 - 1)^2 \right] \pi$$

$$+ \frac{\nu}{2} f'(g + g^*) - \frac{i\nu}{2} Pf (g' - g'^*) - f'(g - g^*) - \nu Pf^2 a_\omega.$$  \hspace{1cm} (A.14)

In Eqs. (A.3)–(A.14) the notations of Eqs. (3.47)–(3.49) have been used. Both Eqs. (A.3)–

(A.8) and Eqs. (A.9)–(A.14) are written in general terms and no gauge-fixing has been

invoked. The gauge-invariant equations of motion for the fluctuations are then obtained
by writing, component by component, Eq. (4.1) using the explicit expressions reported in this Appendix. Then, the gauge-invariant fluctuations obtained in Eqs. (3.38)-(3.41), Eqs. (3.42)-(3.43), Eqs. (3.53)-(3.55) and Eq. (3.58) are inserted in the various components of Eq. (4.1). The final results of this procedure are Eqs. (5.1) [see Section V], Eqs. (6.5)-(6.7) [see Section VI] and Eqs. (7.7)-(7.13) [see Section VII]. The same procedure, using the explicit expressions of Eqs. (A.2) has to be carried on in the case of Eqs. (4.4) and (4.5), namely the evolution equations for the Higgs and gauge field fluctuations whose explicit expressions are also reported in the bulk of the paper.

**APPENDIX B: LOCALIZATION OF THE VECTOR MODES IN THE POLAR COORDINATE SYSTEM**

In this Appendix the evolution of the vector modes of the geometry will be studied in the coordinate system defined by the line element

\[ ds^2 = M^2(\rho)[dt^2 - d\vec{x}^2] - d\rho^2 - L^2(\rho)d\theta^2. \]  

(B.1)

Moreover, in order to check for the consistency of our result, it will also be done in a specific gauge, namely the gauge where \( \zeta_\mu = f_\mu \).

Consider the perturbed line element for vector fluctuations in the form 5

\[
\delta G_{AB} = \begin{pmatrix}
0 & MD_\mu & LMQ_\mu \\
MD_\mu & 0 & 0 \\
LMQ_\mu & 0 & 0
\end{pmatrix}.
\]  

(B.2)

The perturbed line element of Eq. (B.2) can be obtained by considering the pure vector modes of Eq. (3.21) in the gauge \( f_\mu = \zeta_\mu \) and by recalling that \( d\rho = L(w)dw \).

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5 In order not to make the formulas too heavy only in this Appendix we denoted by the prime the derivation with respect to \( x = \sqrt{\lambda}v\rho \). On the other hand, in the bulk of the paper the prime denotes the derivation with respect to \( w \).
In this set-up the evolution of the vector modes is given by the following equations:

\[ D_\mu' + (3H + F)D_\mu + \frac{\dot{Q}_\mu}{\mathcal{L}} = 0, \] (B.3)

\[ \frac{\ddot{D}_\mu}{\mathcal{L}^2} - \frac{2}{m_H^2 M^2} \Box D_\mu - \frac{\dot{Q}'_\mu}{\mathcal{L}} + (H - F) \frac{\dot{Q}_\mu}{\mathcal{L}} = 2 \frac{L}{M} \frac{\nu P'}{\alpha \mathcal{L}^2} \dot{a}_\mu, \] (B.4)

\[ Q''_\mu + (4H + F)Q'_\mu + (F' - H' + 5HF - 5H^2)Q_\mu - \frac{2}{m_H^2 M^2} \Box Q_\mu \]

\[ - \frac{1}{\mathcal{L}} \left[ \dot{D}'_\mu + (5H - F) \dot{D}_\mu \right] + 2 \frac{L}{M} \left( \frac{\nu P'}{\alpha} a'_\mu + \frac{\nu P f^2}{\mathcal{L}^2} a_\mu \right) = 0, \] (B.5)

\[ a''_\mu + \frac{\ddot{a}_\mu}{\mathcal{L}^2} - \frac{2}{m_H^2 M^2} \Box a_\mu + (2H + F) a'_\mu \]

\[ - \frac{P'}{L} \left[ Q'_\mu - (H - F) Q_\mu - \frac{\dot{D}_\mu}{\mathcal{L}} \right] - \alpha a_\mu f^2 = 0. \] (B.6)

In Eqs. (B.3)–(B.6), \( H = d \ln M/dx \) and \( F = d \ln L/dx \) and \( a_\mu \) is the gauge field fluctuation.

Defining now the following variables

\[ u_\mu = \varepsilon \frac{\dot{D}_\mu}{\mathcal{L}} - (\varepsilon Q_\mu)', \] (B.7)

where \( \varepsilon = L/M \) eqs. (B.3)–(B.6) can be written in the following form

\[ D'_\mu + (4H + \frac{\varepsilon'}{\varepsilon}) D_\mu + \frac{\dot{Q}_\mu}{\mathcal{L}} = 0, \] (B.8)

\[ \frac{\ddot{u}_\mu}{\mathcal{L}} - \frac{2}{m_H^2 M^2} \Box D_\mu = 2 \frac{\varepsilon^2 \nu P'}{\alpha} \dot{a}_\mu, \] (B.9)

\[ u'_\mu + (5H - \frac{\varepsilon'}{\varepsilon}) u_\mu + 2 \frac{\varepsilon}{m_H^2 M^2} \Box Q_\mu = 2 \varepsilon^2 \left[ \frac{\nu}{\alpha} P' a'_\mu + \frac{\nu P f^2}{\mathcal{L}^2} a_\mu \right], \] (B.10)

\[ a''_\mu + \frac{\ddot{a}_\mu}{\mathcal{L}^2} - \frac{2}{m_H^2 M^2} \Box a_\mu + (2H + F) a'_\mu + \frac{P'}{\varepsilon^2} u_\mu - \alpha a_\mu f^2 = 0. \] (B.11)

Consider now the case where the masses of \( D_\mu \) and \( Q_\mu \) are both vanishing. Then we have that

\[ u_\mu = 2 \varepsilon^2 \frac{\nu}{\alpha} \frac{P'}{\mathcal{L}^2} a_\mu. \] (B.12)

Correspondingly the equation for the gauge field fluctuation can be simplified

\[ b''_\mu + \frac{\ddot{b}_\mu}{\mathcal{L}^2} - \frac{2}{m_H^2 M^2} \Box b_\mu - \frac{(M \sqrt{\mathcal{L}} P)''}{M \sqrt{\mathcal{L}} P} b_\mu = 0, \] (B.13)

where \( b_\mu = M \sqrt{\mathcal{L}} a_\mu \). The zero modes of the system are
\[ a_\mu = k_{1,\mu} P, \]
\[ D_\mu = \frac{c_{1,\mu}}{M^3 L}, \]  \hspace{1em} (B.14)
\[ Q_\mu = k_{1,\mu} \frac{L}{M} + c_{2,\mu} \frac{M}{L}. \]  \hspace{1em} (B.15)

By now perturbing the action to second order the correct canonical normalization of the fields can be deduced. The kinetic terms of \( a_\mu, D_\mu \) and \( Q_\mu \) appear in the action in the following canonical form

\[ \overline{a}_\mu = \sqrt{L} a_\mu, \]
\[ \overline{D}_\mu = M \sqrt{L} D_\mu, \]
\[ \overline{Q}_\mu = M \sqrt{L} Q_\mu. \]  \hspace{1em} (B.16)

Hence, the normalization integrals which should converge are

\[ |k_{1,\mu}|^2 \int_0^\infty L(x) P^2(x) dx, \]  \hspace{1em} (B.17)
\[ |c_{1,\mu}|^2 \int_0^\infty \frac{dx}{M^4(x) L(x)}, \]  \hspace{1em} (B.18)
\[ \int_0^\infty \left[ |k_{1,\mu}|^2 L^3(x) + |c_{2,\mu}|^2 \frac{M^4(x)}{L(x)} + 2 k_{1,\mu} c_{1,\mu} M^2(x) L(x) \right] dx. \]  \hspace{1em} (B.19)

Since for \( x \to 0 \) \( L(x) \simeq x \), the integrand of Eq. (B.17) converges in the core. It converges also at infinity where, in this coordinate system, \( P(x) \sim e^{-\sqrt{\alpha} x} \) and \( L(x) \sim e^{-c x} \). Since for \( x \to \infty \) the warp factors are exponentially decreasing the integrand of Eq. (B.18) diverges. Finally the second term of the integrand of Eq. (B.19) diverges as \( 1/x \) for \( x \to 0 \) leading to an integral which is logarithmically divergent in the same limit.

Recalling that \( dx = \mathcal{L}(w) dw \) (where \( \mathcal{L} = \sqrt{\lambda} v L \)) and transforming, accordingly, the limits of integration, we get the same result obtained in the bulk of the paper taking into account that, in our gauge, \( A_\mu = a_\mu \), \( V_\mu = D_\mu \) and \( Z_\mu = Q_\mu \). With this observation we clearly see that the normalization integrals of Eqs. (B.17)–(B.19) become the same as the ones of Eqs. (6.23)–(6.30) and (6.31). This consistency check shows also that the canonical normalizations have been correctly derived in the two parametrizations of the background geometry.
APPENDIX C: LONGITUDINAL SYSTEM

For the lowest angular momentum eigenstate Eq. (7.9) allows to express $A_\theta$ as a function of the other fluctuations:

$$\Phi' + (F - H)\Phi - (3H + F)\Xi - 3\Psi' = \nu f' \Delta_1 - \frac{\nu P'}{\alpha L^2} A_\theta.$$  \hspace{1cm} (C.1)

Using Eq. (7.7), Eq. (C.1) can be expressed as

$$\frac{\nu P'}{\alpha L^2} A_\theta = \Xi' + \Psi' + 2(\mathcal{H} + F) \Xi \nu f' \Delta_1.$$ \hspace{1cm} (C.2)

Inserting Eq. (C.2) into Eqs. (7.1), (7.8), (7.11) and (7.13) we get

$$\Delta_1'' + 4\mathcal{H} \Delta_1' - \left[ P^2 + \mathcal{L}^2(3f^2 - 1) - \frac{2\alpha L^2 f f'}{P'} \right] \Delta_1$$

$$+ 2 \left( f' + \frac{\alpha L^2 f}{\nu P'} \right) (\Psi' + \Xi') + 2 \left[ f'' + 4\mathcal{H} f' + f P^2 + \frac{2\alpha L^2 f}{\nu P'} (\mathcal{H} + F) \right] \Xi = 0, \hspace{1cm} (C.3)$$

for the real part of the Higgs equation and

$$\Xi'' + 3\Psi'' + \left( 18\mathcal{H} - 2F - \frac{\alpha L^2 f^2 P}{P'} \right) \Psi' + \left( 10\mathcal{H} + 2F - F - \frac{\alpha L^2 f^2 P}{P'} \right) \Xi'$$

$$+ 2 \left[ 3\mathcal{H}' + F' + 4\mathcal{H}(3H + F) - (\mathcal{H} + F) \frac{\alpha L^2 f^2 P}{P'} \right] \Xi + \nu f' \Delta_1'$$

$$+ \nu \left[ f'' + \left( 4\mathcal{H} - \frac{\alpha L^2 f^2 P}{P'} \right) f' + \mathcal{L}^2 f (f^2 - 1) \right] \Delta_1 = 0, \hspace{1cm} (C.4)$$

$$\Xi'' - \Psi'' + \left[ 2\mathcal{H} + 6F - 2\frac{\alpha L^2 f^2 P}{P'} \right] \Psi' + \left( 10\mathcal{H} + 2F - 2\mathcal{L}^2 f^2 P \right) \Xi$$

$$+ 2 \left[ 3\mathcal{H}' + F' + 4\mathcal{H}(3H + F) - 2\nu P^2 f^2 - 3(\mathcal{H} + F) \frac{\alpha L^2 f^2 P}{P'} \right] \Xi - \nu f' \Delta_1'$$

$$+ \nu \left[ 3f'' + 3 \left( 4\mathcal{H} - \frac{\alpha L^2 f^2 P}{P'} \right) f' - \mathcal{L}^2 f (f^2 - 1) \right] \Delta_1 = 0, \hspace{1cm} (C.5)$$

$$7\Psi'' + \Xi'' + \left( 34\mathcal{H} - 10F + \frac{\alpha L^2 f^2 P}{P'} \right) \Psi' + \left( 10\mathcal{H} + 2F + \frac{\alpha L^2 f^2 P}{P'} \right) \Xi'$$

$$+ 2 \left[ 3\mathcal{H}' + F' + 4\mathcal{H}(3H + F) + 2\nu P^2 f^2 + (\mathcal{H} + F) \frac{\alpha L^2 f^2 P}{P'} \right] \Xi + 3\nu f' \Delta_1'$$

$$- \nu \left[ f'' + \left( 4\mathcal{H} - \frac{\alpha L^2 f^2 P}{P'} \right) f' - 3\mathcal{L}^2 f (f^2 - 1) \right] \Delta_1 = 0, \hspace{1cm} (C.6)$$

for the remaining components of the perturbed Einstein equations. These equations can be used in order to study the asymptotic behaviour of the zero mode of $\Psi$ near the core of the
vortex. By using the asymptotic expressions for \( w \to -\infty \) it is found, as expected, that \( \Psi \simeq \text{constant}, \Xi \simeq e^{-2w}, \text{and } \Delta_1 \simeq e^w. \)
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