On Solutions to the Twisted Yang-Baxter equation

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Dedicated to L.D. Faddeev on his 60th birthday

Abstract. Solutions to the twisted Yang-Baxter equation arising from intertwiners for cyclic representations of $U_q(\hat{sl}_n)$ are described via two coupled the lattice current algebras.

Introduction

In this note I describe a few solutions to the twisted (generalized) Yang-Baxter equation

$$S(q, r)\tilde{S}(p, r)S(p, q) = g(p, q, r)\tilde{S}(p, q)S(p, r)\tilde{S}(q, r).$$

Recently, such an equation was introduced explicitly in [KS]. It was shown there that starting from its solution one can make a solution to the usual Yang-Baxter equation (cf. “box” construction), and thus a quantum integrable model (a solvable model in statistical mechanics) can be defined. An example of a solution for the twisted Yang-Baxter equation related to cyclic representations of $U_q(\hat{sl}_3)$ ($sl_3$ chiral Potts model) was also given therein.

On the other side it was shown in [DJMM], [T] that intertwiners for cyclic representations of $U_q(\hat{sl}_n)$ for any $n$ give solutions to the twisted Yang-Baxter equation. In this paper I clear up the simple algebraic base of these solutions. The algebra $A$, which comes into being, constitutes of two coupled the lattice current algebras [FV]. The representation theory of this new algebra depends on the residue of the number of generators modulo 3 (instead of the residue modulo 2 for the lattice current algebra).

The paper is organized as follows. In the first section I give necessary definitions and formulate the basic theorem (Theorem 1.5). The represen-
tation theory for the algebra $\mathcal{A}$ is described in the second section. The third section contains a proof of Theorem 1.5.

1. Basic solution

Let $\omega$ be a primitive $N$-th root of unity and $q^N = (-1)^{N-1}$.

**Definition 1.1.** An algebra $\mathcal{A}$ is an associative algebra with unit generated by elements $J_i, \tilde{J}_i, \ i \in \mathbb{Z}_n$ subject to the relations

$$
\begin{align*}
J_iJ_{i+1} &= \omega J_{i+1}J_i, & \tilde{J}_i\tilde{J}_{i+1} &= \omega \tilde{J}_{i+1}\tilde{J}_i, \\
J_i\tilde{J}_i &= \omega^{-1} \tilde{J}_iJ_i, & J_{i+1}\tilde{J}_i &= \omega \tilde{J}_iJ_{i+1}, \\
J_iJ_j &= J_jJ_i, & \tilde{J}_i\tilde{J}_j &= \tilde{J}_j\tilde{J}_i \\
J_i\tilde{J}_j &= \tilde{J}_j\tilde{J}_i, & J_{i+1}\tilde{J}_j &= \omega \tilde{J}_j\tilde{J}_{i+1} \\
J_1\ldots J_n &= \omega^{-1}, & \tilde{J}_1\ldots \tilde{J}_n &= \omega^{-1}.
\end{align*}
$$

Let $M = \mathbb{C}P^{2n-1}$ with homogeneous coordinates $a_1, \ldots, a_n, b_1, \ldots, b_n$ and $\hat{M} = \mathbb{C}P^{2n-1}$ with homogeneous coordinates $\hat{a}_1, \ldots, \hat{a}_n, \hat{b}_1, \ldots, \hat{b}_n$. Consider a covering

$$
\hat{\cdot} : M \to \hat{M},
$$

$$
p = (a_1, \ldots, b_n) \mapsto \hat{p} = (a_1^N, \ldots, b_n^N).
$$

Let $\Gamma \subset M$ be a preimage of a projective line $\hat{\Gamma} = \mathbb{C}P^1 \subset \hat{M}$ for this covering. Suppose that for any $i, k$

$$
\frac{\partial (\hat{a}_i, \hat{b}_i)}{\partial (\hat{a}_k, \hat{b}_k)} \neq 0, \quad \frac{\partial (\hat{a}_i, \hat{b}_{i+1})}{\partial (\hat{a}_k, \hat{b}_{k+1})} \neq 0
$$

(Jacobians are calculated on $\hat{\Gamma}$ and the periodicity conditions $a_{n+i} = a_i, b_{n+i} = b_i$ are assumed.) Introduce $\Phi_i, \tilde{\Phi}_i, \ i = 1, \ldots, n$ such that

$$
\frac{\partial (\hat{a}_i, \hat{b}_i)}{\partial (\hat{a}_k, \hat{b}_k)} = \Phi_i^N \Phi_k^{-N}, \quad \frac{\partial (\hat{a}_i, \hat{b}_{i+1})}{\partial (\hat{a}_k, \hat{b}_{k+1})} = \tilde{\Phi}_i^N \tilde{\Phi}_k^{-N}.
$$
Definition 1.2. Let $\mathbf{s} \in \mathbb{Z}^n$ be subject to the inequalities $s_1 \leq \ldots \leq s_n$, $s_0$ – an integer, such that $s_0 \leq s_1$, $s_0 = s_n \pmod{N}$ and $p, p' \in \Gamma$. Set

$$W(p, p', \mathbf{s}) = \left( \frac{\Phi_i^N}{b_i^N a_i^N - b_i^N a_i^N} \right)^{s_n-s_n} \prod_{i=1}^{n} \prod_{j=1}^{s_i-s_{i+1}} b_i a_i^{\omega - a_i b_i^{\omega}} \prod_{i=1}^{s_{i-1}-s_{i+1}} \frac{b_i a_i^{\omega - a_i b_i^{\omega}}}{\Phi_i},$$

$$\tilde{W}(p, p', \mathbf{s}) = \left( \frac{\tilde{\Phi}_i^N}{b_i^N a_i^N - b_i^N a_i^N} \right)^{s_n-s_n} \prod_{i=1}^{n} \prod_{j=1}^{s_i-s_{i+1}} b_i a_i^{\omega - a_i b_i^{\omega}} \prod_{i=1}^{s_{i-1}-s_{i+1}} \frac{b_i a_i^{\omega - a_i b_i^{\omega}}}{\tilde{\Phi}_i}.$$ 

Evidently, the right hand side in the above formulae do not depend on a choice of $s_0$, and the first factors therein actually do not depend on $i$ due to equalities (1.2).

Definition 1.3. For any $\mathbf{s} \in \mathbb{Z}^n$ set

$$J(\mathbf{s}) = \omega^{s_1 s_n} \prod_{i=1}^{n} \omega^{(1-s_i)s_i/2} J_{s_1}^{s_n},$$

$$\tilde{J}(\mathbf{s}) = \omega^{s_1 s_n} \prod_{i=1}^{n} \omega^{(1-s_i)s_i/2} \tilde{J}_{s_1}^{s_n}.$$ 

Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the canonical base in $\mathbb{Z}^n$. The functions $W(p, p', \mathbf{s})$ and $\tilde{W}(p, p', \mathbf{s})$ enjoy the following relations

$$\frac{W(p, p', \mathbf{s} + \mathbf{e}_i)}{W(p, p', \mathbf{s})} = \Phi_i^{-1} \frac{b_i a_i^{\omega - a_i b_i^{\omega}}}{b_i a_i^{\omega - a_i b_i^{\omega}}} \prod_{i=1}^{s_{i-1}-s_{i+1}} \frac{b_i a_i^{\omega - a_i b_i^{\omega}}}{\Phi_i},$$

$$\frac{\tilde{W}(p, p', \mathbf{s} + \mathbf{e}_i)}{\tilde{W}(p, p', \mathbf{s})} = \tilde{\Phi}_i^{-1} \frac{b_i a_i^{\omega - a_i b_i^{\omega}}}{b_i a_i^{\omega - a_i b_i^{\omega}}} \prod_{i=1}^{s_{i-1}-s_{i+1}} \frac{b_i a_i^{\omega - a_i b_i^{\omega}}}{\tilde{\Phi}_i},$$

which define these functions for all $\mathbf{s} \in \mathbb{Z}^n$. $W(p, p', \mathbf{s}), \tilde{W}(p, p', \mathbf{s}), J(\mathbf{s}), \tilde{J}(\mathbf{s})$ are invariant under the translations $\mathbf{s} \mapsto \mathbf{s} + N \mathbf{e}_i$; further on they are considered as functions on $\mathbb{Z}_N^n$. $(W(p, p', \mathbf{s}), \tilde{W}(p, p', \mathbf{s}), J(\mathbf{s}), \tilde{J}(\mathbf{s})$ are also invariant under the translation $\mathbf{s} \mapsto \mathbf{s} + \sum_{i=1}^{n} \mathbf{e}_i$, but this property is not employed explicitly in the paper).

Definition 1.4. Set

$$S(p, p') = \sum_{\mathbf{s} \in \mathbb{Z}_N^n} W(p, p', \mathbf{s}) J(\mathbf{s}), \quad \tilde{S}(p, p') = \sum_{\mathbf{s} \in \mathbb{Z}_N^n} \tilde{W}(p, p', \mathbf{s}) \tilde{J}(\mathbf{s}).$$
Theorem 1.5. The following equations hold

\[ S(p, p') S(p', p) = N^{n+1} \prod_{i=1}^{n} \frac{b_i a_i' - a_i b_i'}{b_i a_i'^N - a_i b_i'^N} \cdot \prod_{i=1}^{n} \frac{b_i a_i'^N - a_i b_i'^N}{n} \prod_{i=1}^{n} b_i a_i - \prod_{i=1}^{n} a_i b_i', \]

\[ \tilde{S}(p, p') \tilde{S}(p', p) = N^{n+1} \prod_{i=1}^{n} \frac{b_{i+1} a_i' - a_i b_{i+1}'}{b_{i+1} a_i'^N - a_i b_{i+1}'^N} \cdot \prod_{i=1}^{n} \frac{b_{i+1} a_i'^N - a_i b_{i+1}'^N}{n} \prod_{i=1}^{n} b_{i+1} a_i - \prod_{i=1}^{n} a_i b_{i+1}' \]

\[ S(p', p'') \tilde{S}(p, p') S(p, p') = \varrho(p, p', p'') \tilde{S}(p, p') S(p, p'') \tilde{S}(p', p'') \]

where \( \varrho(p, p', p'') \) is a scalar factor.

The proof is given in the last section.

2. Representations of the algebra \( \mathcal{A} \)

Definition 2.1. An algebra \( \mathcal{W} \) is an associative algebra with unit generated by elements \( Z, X \) subject to the relations

\[ ZX = \omega XZ, \quad Z^N = 1, \quad X^N = 1. \]

Let \( \tilde{\mathcal{V}} \) be a unique irreducible representation of \( \mathcal{W} \) and \( e_1, \ldots, e_N \) – its natural base:

\[ Ze_i = \omega^i e_i, \quad Xe_i = e_{i+1} \]

where \( e_{N+1} = e_1 \).

Theorem 2.2.

a) \( \mathcal{A} \) is isomorphic to \( \mathcal{W}^{\otimes (n-1)} \) if \( n \neq 0 \) (mod 3);

b) \( \mathcal{A} \) is isomorphic to \( \mathcal{W}^{\otimes (n-2)} \otimes \mathbb{C}[\mathbb{Z}_N^2] \) if \( n = 0 \) (mod 3).

Proof. In the first case the algebra \( \mathcal{A} \) is obviously generated by elements \( J_i, \tilde{J}_i, \quad i = 1, \ldots, n-1 \). In the second case it is generated by \( J_i, \tilde{J}_i, \quad i = 1, \ldots, n-2 \) and two central elements

\[ C_1 = \prod_{k=1}^{n/3} J_{3k-2} \tilde{J}_{3k-1}^{-1}, \quad C_2 = \prod_{k=1}^{n/3} J_{3k-1} \tilde{J}_{3k}^{-1} \]

obeying the conditions \( C_1^N = 1, \quad C_2^N = 1 \). Hence, it suffices to show that for any \( m < n \), such that \( m \neq 2 \) (mod 3), the subalgebra generated by
$J_i, \tilde{J}_i, \ i = 1, \ldots, m$, is isomorphic to $\mathcal{W}^\otimes m$. The isomorphism can be written explicitly on generators:

\begin{align*}
J_i &\mapsto qZ_{i-1}Z_i, & \tilde{J}_i &\mapsto qX_i^{-1}Z_{i+1}, & i &\equiv 1 \pmod{3}, \\
J_i &\mapsto qX_{i-1}X_i^{-1}, & \tilde{J}_i &\mapsto qZ_{i-1}Z_{i+1}, & i &\equiv 2 \pmod{3}, \\
J_i &\mapsto Z_{i-2}X_{i-1}X_i^{-1}, & \tilde{J}_i &\mapsto qX_iX_{i+1}^{-1}, & i &\equiv 0 \pmod{3}
\end{align*}

where $i = 1, \ldots, m$ and

\begin{align*}
Z_i &= 1^{\otimes(i-1)} \otimes Z \otimes 1^{\otimes(m-i)}, & X_i &= 1^{\otimes(i-1)} \otimes X \otimes 1^{\otimes(m-i)}, \\
Z_0 &= 1, & Z_{m+1} &= 1, & X_{m+2} &= 1.
\end{align*}

To interpret $S(p, p')$ and $\tilde{S}(p, p')$ in terms of integrable models we are looking for representations of the algebra $A$ subject to the restrictions:

a) The representation space is equal to $V^\otimes 3$ for some vector space $V$;

b) Generators $J_i$’s act as an identity operator in the third tensor factor $V$;

c) Generators $\tilde{J}_i$’s act as an identity operator in the first tensor factor $V$.

Let us give examples of such representations for the algebra $A$.

**Example 1.** Let $\nu : A \to \text{End}(V)$ be a representation of the algebra $A$. Fix two commutative sets $\{Q_i \in \text{End}(V)\}_{i=1}^n$ and $\{\tilde{Q}_i \in \text{End}(V)\}_{i=1}^n$. Then a map

\begin{align*}
J_i &\mapsto Q_i \otimes \nu(J_i) \otimes 1, \\
\tilde{J}_i &\mapsto 1 \otimes \nu(\tilde{J}_i) \otimes \tilde{Q}_i
\end{align*}

(2.3)

gives a required representation of $A$. The most important case is $\nu$ being a unique irreducible representation of $A$ ($V = \mathbb{C}^N^{\otimes(n-1)}$ for $n \neq 0 \pmod{3}$ and $V = \mathbb{C}^N^{\otimes(n-2)}$ for $n = 0 \pmod{3}$). It also should be noted that the choice of operators $Q_i, \tilde{Q}_i$ is essential for getting concrete matrix solutions to the twisted Yang-Baxter equation.

Consider the case $n = 3$ in more details. Change slightly the construction given above and introduce a homomorphism $A \to \mathcal{W}^\otimes 3$

\begin{align*}
J_1 &\mapsto Z^{-1}X \otimes Z \otimes 1, & \tilde{J}_1 &\mapsto 1 \otimes X^{-1} \otimes ZX^{-1}, \\
J_2 &\mapsto 1 \otimes Z^{-1}X \otimes 1, & \tilde{J}_2 &\mapsto 1 \otimes Z \otimes Z^{-1}X, \\
J_3 &\mapsto ZX^{-1} \otimes X^{-1} \otimes 1, & \tilde{J}_3 &\mapsto 1 \otimes Z^{-1}X \otimes 1.
\end{align*}

(2.4)
(Cf. (2.2)). For the irreducible representation \( \hat{V} \) of \( W \) (cf. (2.1)) \( S(p, p') \) and \( \tilde{S}(p, p') \) are represented in \( \text{End}(\hat{V} \otimes \hat{V}) \) by the following matrices

\[
\langle m_1, m_2 | S(p, p') | n_1, n_2 \rangle = \omega^{(m_1-n_1)(m_2-m_1)-m_2(n_2-1)} W(p, p', s),
\]

\[
s = (m_1 - n_1, m_2 - n_2, 0),
\]

\[
\langle m_1, m_2 | \tilde{S}(p, p') | n_1, n_2 \rangle = \omega^{(m_2-n_2)(m_1-m_2)-m_1(n_1-1)} \tilde{W}(p, p', \tilde{s}),
\]

\[
\tilde{s} = (0, m_2 - n_2, m_1 - n_1).
\]

Another matrix solution can be obtained from a homomorphism

\[
J_1 \mapsto Z^{-1} X \otimes Z \otimes 1,
\]

\[
J_2 \mapsto 1 \otimes Z^{-1} X \otimes 1,
\]

\[
J_3 \mapsto Z X^{-1} \otimes X^{-1} \otimes 1,
\]

(only the second column differs from (2.4)). Now \( S(p, p') \) is represented by the same matrix as before. But for \( \tilde{S}(p, p') \) we have

\[
\langle m_1, m_2 | \tilde{S}(p, p') | n_1, n_2 \rangle = q^{n_1-m_1\omega^{(m_2-n_2)(n_1-1)}-m_1(n_1-1)/2} \tilde{W}(p, p', -s).
\]

One can easily give more matrix solutions to the twisted Yang-Baxter equation in a similar way.

**Example 2.** Consider a matrix \( \epsilon \) with integer entries such that \( \epsilon_{ij} + \epsilon_{ji} = 1 + \delta_{ij} \). (\( \delta_{ij} \) is the Kronecker symbol). Set \( \omega_{ij} = \omega^{\epsilon_{ij}} \).

**Definition 2.3.** An algebra \( \mathcal{F} \) is an associative algebra with unit generated by elements \( F_i, G_i, \ i \in \mathbb{Z}_n \) subject to the relations

\[
F_i F_j = F_j F_i, \quad \omega_{ij} F_i G_j = \omega_{i,j+1} G_j F_i, \quad \omega_{ij} G_i G_j = \omega_{i+1,j+1} G_j G_i,
\]

\[
F_i^N = 1, \quad G_1^N = 1, \quad G_i = (-1)^{N-1} \quad \text{for} \ i = 2, \ldots, n
\]

\[
F_1 \ldots F_n G_1^{-1} \ldots G_i^{-1} = 1.
\]

**Lemma 2.4.** There exist integers \( m_{ij}, \ i, j \in \mathbb{Z}_n \) such that

\[
m_{i,l+1} - m_{il} - m_{i,l+1} + m_{li} = \epsilon_{i+1,l+1} - \epsilon_{il}.
\]

**Proof.** The right hand side of the equality above is antisymmetric with respect to permutation of \( i, l \). Hence, one can find integers \( n_{il} \) such that \( n_{il} = \epsilon_{i+1,l+1} - \epsilon_{il} \). Set \( n_{il} \) to obey \( \sum_{l=1}^n n_{il} = 0 \). Now \( m_{ij} \) can be certainly found from \( m_{i,l+1} - m_{il} = n_{il} \). \[\square\]
Lemma 2.5. The map
\[ F_i \mapsto \prod_{l=1}^{n} Z_{i,l}^{e_{il}}, \quad G_i \mapsto c_i X_i^{-1} X_{i+1} \prod_{l=1}^{n} Z_{i,l}^{m_{il}} \]
extends to a homomorphism of algebras \( F \mapsto W^{\otimes(n-1)} \). Here
\[ c_1 = \prod_{l=1}^{n-1} \omega^{m_{i,i+1}-m_{i+1}} c_{i+1}, \quad c_i = q^{m_{i,i+1}-m_{i+1}} \quad \text{for} \quad i = 2, \ldots, n, \]
and
\[ Z_i = 1^{\otimes(i-1)} \otimes Z \otimes 1^{(n-i-1)}, \quad X_i = 1^{\otimes(i-1)} \otimes X \otimes 1^{(n-i-1)}, \]
\[ Z_n = Z_1^{-1} \cdots Z_{n-1}^{-1}, \quad X_n = 1. \]

Lemma 2.6. The map
\[ J_i \mapsto F_{i+1}^{-1} G_i \otimes G_i^{-1} F_i \otimes 1, \quad \tilde{J}_i \mapsto 1 \otimes F_{i+1}^{-1} G_i \otimes G_i^{-1} F_i \]
extends to a homomorphism of algebras \( A \mapsto F \).

The last two Lemmas can be proved by direct calculations.

Now taking a representation of the algebra \( W \) in a space \( V \) we get a required representation of the algebra \( A \) in \( V^{\otimes 3} \). As in the previous example the case of a unique irreducible representation of \( W \) is the most important.

Example 3. Let us consider the special case \( n = 3 \).

Lemma 2.7. The map
\[ J_1 \mapsto X Z \otimes Z^{-1} \otimes 1, \quad \tilde{J}_1 \mapsto 1 \otimes X Z \otimes Z^2, \]
\[ J_2 \mapsto Z^{-2} \otimes X Z^2 q^{-1} \otimes 1, \quad \tilde{J}_2 \mapsto 1 \otimes Z \otimes X Z^2 q^{-1}, \]
\[ J_3 \mapsto X^{-1} Z \otimes X^{-1} Z^{-1} q \otimes 1, \quad \tilde{J}_3 \mapsto 1 \otimes X^{-1} Z^{-2} \otimes X^{-1} Z^{-4} q \omega^3 \]
extends to a homomorphism of algebras \( A \mapsto W^{\otimes 3} \).

Taking \( \omega = \exp(2\pi i/N) \), \( q = -\exp(-\pi i/N) \) and the irreducible representation \( \hat{V} \) of \( W \) (cf. (2.1)) we reproduce the solution to the the twisted Yang-Baxter equation described in [KS] up to notations (in particular \( S, \tilde{S}, \omega \) in this paper correspond to \( \overline{S}, S, \omega^{-1} \) in [KS]).
3. Proof of Theorem 1.5

Consider again Example 2. Let $\hat{\mathcal{V}}$ be the irreducible representation (2.1) of the algebra $\mathcal{W}$. Employing Theorem 2.2 one can see that the representation $\hat{\mathcal{V}} \otimes \mathcal{A}$ of the algebra $\mathcal{A}$ is faithful. Therefore, it is enough to prove the equalities in this representation. In slightly different notations it has been done in [T].

Using explicit formulae for $S(p, p')$ and $\tilde{S}(p, p')$ one can get the following expressions for the factor $\varrho(p, p', p'')$ in Theorem 1.5:

\begin{align}
\varrho(p, p', p'') &= N^{-n-1} \sum_{s, t \in \mathbb{Z}_N^n} \frac{W(p, p', s)W(p', p'', t)}{W(p, p'', s + t)} \omega^{-\langle s, t \rangle}, \\
&(3.1) \\
&= \sum_{s \in \mathbb{Z}_N^n} W(p, p', s)W(p', p'', -s)\omega^{\langle s, s \rangle}, \\
&(3.2) \\
&= \sum_{s \in \mathbb{Z}_N^n} W(p, p', s)/W(p', p'', s), \\
&(3.3) \\
\varrho^{-1}(p, p', p'') &= N^{-n-1} \sum_{s, t \in \mathbb{Z}_N^n} \frac{\tilde{W}(p, p', s)\tilde{W}(p', p'', t)}{\tilde{W}(p, p'', s + t)} \omega^{-\langle s, t \rangle}, \\
&(3.4)
\end{align}

where $\langle s, t \rangle = \sum_{i=1}^n s_i(t_{i+1} - t_i)$, $t_{n+1} = t_1$. (Cf. also (A.23) in [KS]).

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