Nil$_*^*$-Noetherian rings

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Abstract

In this paper, we say a ring $R$ is Nil$_*^*$-Noetherian provided that any nil ideal is finitely generated. First, we show that the Hilbert basis theorem holds for Nil$_*^*$-Noetherian rings, that is, $R$ is Nil$_*^*$-Noetherian if and only if $R[x]$ is Nil$_*^*$-Noetherian, if and only if $R[[x]]$ is Nil$_*^*$-Noetherian. Then we discuss some Nil$_*^*$-Noetherian properties on idealizations and bi-amalgamated algebras. Finally, we give the Cartan-Eilenberg-Bass Theorem for Nil$_*^*$-Noetherian rings in terms of Nil$_*^*$-injective modules and Nil$_*^*$-FP-injective modules. Besides, some examples are given to distinguish Nil$_*^*$-Noetherian rings, Nil$_*^*$-coherent rings and so on.

Key Words: Nil$_*^*$-Noetherian ring; Hilbert basis theorem; Idealization; Bi-amalgamated algebra, Cartan-Eilenberg-Bass Theorem.

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Throughout this paper, all rings are commutative with identity and all modules are unitary. Let $R$ be a ring, we denote by $R[x]$ (resp., $R[[x]]$) the polynomial ring (resp., the formal series ring) in one variable over $R$. An element $r$ in $R$ is said to be nilpotent provided that $r^n = 0$ for some positive integer $n$. The set of all nilpotent elements in $R$ is denoted by Nil($R$). An ideal $I$ of $R$ is said to be a nil ideal provided that any element in $I$ is nilpotent.

Recall that a ring $R$ is called to be a Noetherian ring if every ideal of $R$ is finitely generated. The concept of Noetherian rings, which was originally due to the mathematician Emmy Noether, is one of the most important topics that is widely used in ring theory, commutative algebra and algebraic geometry. The importance of Noetherian property was first shown in Hilbert basis theorem: if a ring $R$ is a Noetherian ring, then $R[x]$ and $R[[x]]$ are also Noetherian. Noetherian rings also have many module-theoretic characterizations, such as the well-known Cartan-Eilenberg-Bass Theorem states that a ring $R$ is Noetherian if and only if every direct sum of injective $R$-modules is injective, if and only if every direct limit of injective $R$-modules over a directed set is injective (see [11, Theorem 4.3.4] for example). Some
new constructions of Noetherian rings, such as idealizations and bi-amalgamated algebras, are also considered by many algebraists (see [3, 9]). Several generalizations of Noetherian rings are introduced and studied by many algebraists. The famous generalization is the notion of coherent rings, i.e., rings in which any finitely generated ideal are finitely presented. For a further generalization, Xiang [13] introduced the notions of Nil\_\_\_\_\_coherent rings in terms of nil ideals in 2014. A ring \( R \) is said to be Nil\_\_\_\_coherent provided that any finitely generated nil ideal is finitely presented. Later in 2017, Alaoui Ismaili et al. [1] studied the Nil\_\_\_coherent properties via idealization and amalgamated algebras under several assumptions.

The main motivation of this paper is to introduce and study the Nil\_\_\_\_\_Noetherian property of rings. Comparing with the concepts of Noetherian rings and Nil\_\_\_\_\_coherent rings, we say a ring \( R \) is Nil\_\_\_\_\_Noetherian provided that any nil ideal is finitely generated. It is important that the Hilbert basis theorem also holds for Nil\_\_\_\_\_Noetherian rings, that is, a ring \( R \) is Nil\_\_\_\_\_Noetherian if and only if \( R[[x]] \) is Nil\_\_\_\_\_Noetherian, if and only if \( R[[x]] \) is Nil\_\_\_\_\_Noetherian (see Theorem [1, 9] and Theorem [1, 12]). Utilizing these results, we study the idealization properties of Nil\_\_\_\_\_Noetherian rings in Theorem [2, 1], and then find a Nil\_\_\_\_\_coherent ring that is not Nil\_\_\_\_\_Noetherian (see Example [2, 3]). Surprisingly, Nil\_\_\_\_\_Noetherian rings can also be not Nil\_\_\_coherent (see Example [2, 4]). By computing the nil-radical of the bi-amalgamated algebras under some assumptions (see Lemma [2, 6]), we characterize Nil\_\_\_\_\_Noetherian properties of bi-amalgamated algebras in Proposition [2, 7] and Theorem [2, 8]. Finally, we give the Cartan-Eilenberg-Bass Theorem for Nil\_\_\_\_\_rings in terms of Nil\_\_\_\_\_injective modules and Nil\_\_\_\_FP-injective modules (See Theorem [3, 3]). In surprise, we show that the direct limits of Nil\_\_\_\_\_injective modules need not be Nil\_\_\_\_\_inj-injective for Nil\_\_\_\_\_Noetherian rings (see Remark [3, 6]).

1. Basic Properties of Nil\_\_\_\_\_Noetherian Rings

We begin with the concept of Nil\_\_\_\_\_Noetherian rings.

**Definition 1.1.** A ring \( R \) is said to be a Nil\_\_\_\_\_Noetherian ring provided that any nil ideal is finitely generated.

Trivially, reduced rings and Noetherian rings are Nil\_\_\_\_\_Noetherian.

**Proposition 1.2.** Let \( R \) be a ring. Then the following assertions are equivalent:

1. \( R \) is a Nil\_\_\_\_\_Noetherian ring;
2. \( R \) satisfies the ascending chain condition on nil ideals;
3. Every non-empty set of nil ideals of \( R \) has a maximal element.
Proof. (1) $\Rightarrow$ (2) Let $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_j \subseteq \cdots$ be an ascending chain condition on nil ideals. Set $I = \bigcup_{j=1}^{\infty} I_j$. Then $I$ is also a nil ideal. Hence $I$ is finitely generated. Consequently, there exists $k \in \mathbb{Z}$ such that $I = I_k$.

(2) $\Rightarrow$ (3) Let $\Gamma$ be a non-empty set of nil ideals of $R$. On contrary, suppose $\Gamma$ has no maximal element. For any $I_1 \in \Gamma$, there exits $I_2 \in \Gamma$ such that $I_1 \subsetneq I_2$ as $I_1$ is not maximal. Continuing these steps, there is an strictly ascending chain condition on nil ideals $I_1 \subsetneq I_2 \subsetneq \cdots \subseteq I_n \subsetneq \cdots$, which contradicts (2).

(3) $\Rightarrow$ (1) Let $I$ be a nil ideal of $R$. Denoted by $\Gamma$ the set of all finitely generated sub-ideal of $I$. Then there is a maximal element $J$ in $\Gamma$. We claim $J = I$. Indeed, if there is an element $r \in I - J$, then $J + Rr$ is a finitely generated sub-ideal of $I$ which is strictly lager than $J$, which is a contraction. So $I$ is finitely generated. $\square$

Lemma 1.3. Let $R$ be a Nil$_*\,$-Noetherian ring. If $I$ is a nil ideal of $R$, then $R/I$ is Nil$_*\,$-Noetherian.

Proof. Let $K$ be a nil ideal of $R/I$. Then $K = J/I$ for some $R$-ideal $J$ containing $I$. Then for any $j \in J$, there is an $n$ such that $j^n \in I$. Since $I$ is a nil ideal, $j^{nk} = 0$ for some $k$. Hence $J$ is a nil ideal of $R$, and so is finitely generated. Hence $K$ is finitely generated $R/I$-ideal. $\square$

Note that the condition “$I$ is a nil ideal of $R$” in Lemma 1.3 cannot be removed.

Example 1.4. Let $S = k[x_1, x_2, \cdots]$ be the polynomial ring over a field $k$ with countably infinite variables. Then $S$ is Nil$_*\,$-Noetherian. Set the quotient ring $R = S/\langle x_2^i | i \geq 1 \rangle$. Then $\text{Nil}(R) = \langle \overline{x_1}, \overline{x_2}, \cdots \rangle$ is infinitely generated, where $\overline{x_i}$ denotes the representative of $x_i$ in $R$ for each $i$. Hence $R$ is not Nil$_*\,$-Noetherian.

Proposition 1.5. A finite direct product of rings $R = R_1 \times \cdots \times R_n$ is Nil$_*\,$-Noetherian if and only if each $R_i$ is Nil$_*\,$-Noetherian $(i = 1, \cdots, n)$.

Proof. It follows by $\text{Nil}(R) = \text{Nil}(R_1) \times \cdots \times \text{Nil}(R_n)$ and $\text{Nil}(R)$ is a Noetherian $R$-module if and only each $\text{Nil}(R_i)$ is a Noetherian $R_i$-module $(i = 1, \cdots, n)$. $\square$

Remark 1.6. Suppose $R = \prod_{i \in \Lambda} R_i$ is an infinite direct product of rings $R_i$. If $R$ is Nil$_*\,$-Noetherian, then trivially each direct summand $R_i$ is also Nil$_*\,$-Noetherian. However, the converse does not hold in general. Indeed, let $R_i = \mathbb{Z}_{p^i}$ the residue rings modulo $p^i$ where $p$ is a prime and $i$ an positive integer. Then each $R_i$ is Noetherian, and thus Nil$_*\,$-Noetherian. However, the direct product $R = \prod_{i=1}^{\infty} R_i = \prod_{i=1}^{\infty} \mathbb{Z}_{p^i}$ is not Nil$_*\,$-Noetherian since the nil ideal $\bigoplus_{i=1}^{\infty} \text{Nil}(\mathbb{Z}_{p^i})$ is not finitely generated.
Proposition 1.7. Let $\phi : R \to S$ be a ring homomorphism making $R$ a module retract of $S$. If $S$ is $\text{Nil}_e$-Noetherian, then $R$ is also $\text{Nil}_e$-Noetherian.

**Proof.** We can assume that $\phi$ is an inclusion map. Let $\psi : S \to R$ be an $R$-homomorphism such that $\psi \circ \phi = \text{Id}_R$. Let $I$ be a nil ideal of $R$. Then $IS := \{ \sum_{i=1}^{t} r_i s_i' \mid r_i' \in I, s_i' \in S \}$ is a nil ideal of $S$, and thus is finitely generated, say by $\{s_1, \cdots, s_n\}$. We will show $I$ is generated by $\{\psi(s_1), \cdots, \psi(s_n)\}$ as an $R$-module.

Indeed, let $x$ be an element in $I$. Then $x = x1 = \sum_{i=1}^{n} r_is_i$ for some $r_i \in R$. Then $x = \psi \circ \phi(x) = \psi \circ \phi(\sum_{i=1}^{n} r_is_i) = \sum_{i=1}^{n} r_i\psi \circ \phi(s_i) = \sum_{i=1}^{n} r_i\psi(s_i)$. Hence $I$ is finitely generated. Consequently, $R$ is a $\text{Nil}_e$-Noetherian ring. $\square$

Next, we will focus on the $\text{Nil}_e$-Noetherian properties of polynomial rings.

**Lemma 1.8.** ([11] Theorem 1.7.7(2)] Let $R$ be a ring. An element $f = \sum_{i=0}^{n} a_i x^i \in R[x]$ is a nilpotent element in $R[x]$ if and only if each $a_i$ is a nilpotent element in $R$ ($i = 0, \cdots, n$).

The well-known Hilbert Basis Theorem states that a ring $R$ is a Noetherian ring if and only if $R[x]$ is a Noetherian ring (see [11] Theorem 4.3.15])

**Theorem 1.9.** (Hilbert Basis Theorem for $\text{Nil}_e$-Noetherian rings-1) Let $R$ be a ring. Then $R$ is a $\text{Nil}_e$-Noetherian ring if and only if $R[x]$ is a $\text{Nil}_e$-Noetherian ring.

**Proof.** Suppose $R[x]$ is a $\text{Nil}_e$-Noetherian ring. Let $I$ be a nil ideal of $R$. Then $[x]$ is a nil ideal of $R[x]$, so is finitely generated, say is generated by $\{h_1, \cdots, h_m\}$ as $R[x]$-ideal. Let $a_i$ be the constant of $h_i$, then it is easy to verify that $I$ is generated by $\{a_1, \cdots, a_m\}$.

Suppose $R$ is a $\text{Nil}_e$-Noetherian ring. Let $J$ be a nil ideal of $R[x]$. Set $I$ to be the set of leading coefficients of polynomials $f$ in $J$. Then $I$ is a nil ideal by Lemma [1.8] and hence finitely generated. Write $I = Ra_1 + \cdots + Ra_k$ for each $a_i \in R$ and let $f_i \in J$ with the leading coefficient $a_i$. Then $A = R[X]f_1 + \cdots + R[X]f_k \subseteq J$. Set $\deg(f_i) = n_i$ and $n = \max\{n_1, \cdots, n_k\}$. For any $f \in J$, write $f = aX^m + \cdots$.

Then $a = r_1 a_1 + \cdots + r_k a_k, r_i \in R$. If $m \geq n$, then $f' := f - \sum_{i=1}^{k} r_i X^{m-n_i} f_i \in J$ with $\deg(f') < m$. If $\deg(f') \geq n$, we continue this process. Hence there are polynomials $g \in A$ and $h \in R[X]$ with $\deg(h) < n$ such that $f = g + h$. Let $M[n]$ be an $R$-submodule of $R[X]$ generated by $1, X, \cdots, X^{n-1}$. Then $h = f - g \in M[n] \cap J$. Thus $J$ as an $R$-module is a sum of two $R$-submodules, that is, $J = A + J \cap M[n]$.

Claim that: the $R$-module $J \cap M[n]$ is finitely generated.
Indeed, note that \( J \cap M[n] = \{ f \in J \mid \deg(f) \leq n - 1 \} \cup \{0\}. \) we will show the claim by induction on \( n. \) If \( n = 1, \) then \( J \cap M[1] \) is a nil ideal of \( R, \) so is finitely generated. If the claim holds for \( n = k, \) let \( n = k + 1. \) Consider the following exact sequence:

\[
0 \to J \cap M[k] \to J \cap M[k + 1] \to L \to 0.
\]

It is easy to verify that \( L = J \cap M[k + 1]/J \cap M[k] \) is isomorphic to a nil ideal of \( R \) by Lemma \[L8\] so is finitely generated. Since \( J \cap M[k] \) is finitely generated by induction, we have \( J \cap M[k + 1] \) is also finitely generated, and the claim holds.

Hence \( J \cap M = Rg_1 + \cdots + Rg_t, \) where \( g_j \in J \cap M. \) Thus

\[
J \subseteq R[X]f_1 + \cdots + R[X]f_k + Rg_1 + \cdots + Rg_t
\]

\[
\subseteq R[X]f_1 + \cdots + R[X]f_k + R[X]g_1 + \cdots + R[X]g_t \subseteq J.
\]

Consequently, \( J = R[X]f_1 + \cdots + R[X]f_k + R[X]g_1 + \cdots + R[X]g_t, \) which is finitely generated. \( \square \)

Finally, we will focus on the \( \text{Nil}_*\text{-Noetherian} \) properties of formal series rings. Recall that an ideal \( I \) of \( R \) is said to be an SFT (strong finite type) ideal if there is a finitely generated sub-ideal \( F \) of \( I \) and an integer \( n \) such that \( a^n \in F \) for any \( a \in I. \) Trivially, every finitely generated ideal is an SFT ideal. For a ring \( R, \) we denote by \( R[[x]] \) the formal series ring over \( R. \) By [6, Theorem 11], an element \( f = \sum_{i=0}^{\infty} a_i x^i \in R[[x]] \) is a nilpotent element, then each \( a_i \) is nilpotent. However, the converse does not hold (see [6, Example 2]).

**Lemma 1.10.** [7, Proposition 2.3] Let \( R \) be a ring and an ideal \( I \subseteq \text{Nil}(R). \) Then the following assertions are equivalent:

1. \( I \) is an SFT ideal of \( R; \)
2. \( I[[x]] \subseteq \text{Nil}(R[[x]]); \)
3. there exists an integer \( k \) such that \( r^k = 0 \) for any \( r \in I; \)
4. \( I[[x]] \) is an SFT ideal of \( R[[x]]. \)

**Lemma 1.11.** [7, Corollary 2.4] Let \( R \) be a ring. Then the following assertions are equivalent:

1. \( \text{Nil}(R) \) is an SFT ideal of \( R; \)
2. \( \text{Nil}(R)[[x]] = \text{Nil}(R[[x]]); \)
3. there exists an integer \( k \) such that \( r^k = 0 \) for any \( r \in \text{Nil}(R); \)
4. \( \text{Nil}(R[[x]]) \) is an SFT ideal of \( R[[x]]. \)
It is also well-known that a ring $R$ is a Noetherian ring if and only if $R[[x]]$ is a Noetherian ring (see [11, Theorem 4.3.15]).

**Theorem 1.12. (Hilbert Basis Theorem for Nil$_s$-Noetherian rings-2)** Let $R$ be a ring. Then $R$ is a Nil$_s$-Noetherian ring if and only if $R[[x]]$ is a Nil$_s$-Noetherian ring.

**Proof.** Suppose $R[[x]]$ is a Nil$_s$-Noetherian ring. Then $	ext{Nil}(R[[x]])$ is a finitely generated ideal, thus an SFT ideal of $R[[x]]$. Then $	ext{Nil}(R)$ is an SFT ideal of $R$ by Lemma [1.11]. Let $I$ be a nil ideal of $R$. Then $I$ is also an SFT ideal of $R$. So $I[[x]]$ is a nil ideal of $R[[x]]$ by Lemma [1.10] and thus is finitely generated, say is generated by $\{h_1, \ldots, h_m\}$. Let $a_i$ be the constant of $h_i$, then it is easy to verify that $I$ is generated by $\{a_1, \ldots, a_m\}$.

On the other hand, suppose $R$ is a Nil$_s$-Noetherian ring. Let $J$ be a nil ideal of $R[[x]]$. We will prove $J$ is finitely generated. Write $J_r$ as an ideal of $R$ generated by the leading coefficients $a_r$ of $f = a_rx^r + a_{r+1}x^{r+1} + \cdots$ where $f \in J \cap x^rR[[x]]$. Note that each $a_r$ is nilpotent. So we have an increasing chain of nil ideals:

$$J_0 \subseteq J_1 \subseteq J_2 \subseteq \cdots \subseteq J_n \subseteq J_{n+1} \subseteq \cdots$$

Since $R$ is Nil$_s$-Noetherian, there is an integer $s$ such that $J_n = J_s$ for any $n \geq s$ and $J_i$ is finitely generated for each $i$ with $0 \leq i \leq s$ by Proposition [1.2]. Now for each $i$ with $0 \leq i \leq s$, take finitely many elements $a_{iv} \in R$ generating $J_i$, and take an element $g_{iv} \in J \cap x^iR[[x]]$ with $a_{iv}$ the leading coefficient of $g_{iv}$.

Claim that: **these finitely many elements $g_{iv}$ generate $J$.**

Indeed, for each $f \in J$, we can take a linear combination $g_0$ of $g_{0v}$ with coefficients nilpotent in $R$ such that $f - g_0 \in J \cap xR[[x]]$. Then take a linear combination $g_1$ of the $g_{1v}$ with coefficients nilpotent in $R$ such that $f - g_0 - g_1 \in J \cap x^2R[[x]]$. Continuing these steps, we get $f - g_0 - g_1 - \cdots - g_s \in J \cap x^{s+1}R[[x]]$. Since $J_{s+1} = J_s$, we can take a linear combination $g_{s+1}$ of $xg_{sv}$ with coefficients nilpotent in $R$ such that $f - g_0 - g_1 - \cdots - g_s - g_{s+1} \in J \cap x^{s+2}R[[x]]$. Now, we proceed in the same way to get $g_{s+2}, g_{s+3}, \cdots$ For $i \leq s$, each $g_i$ is a linear combination of $g_{iv}$, and for each $i > s$, a combination of elements $x^{i-s}g_{sv}$. For each $i \geq s$, we write $g_i = \sum_{v} a_{iv}x^{i-s}g_{sv}$, and then for each $v$, we write $h_v = \sum_{i=s}^{\infty} a_{iv}x^{i-s}$. So

$$f = g_0 + \cdots + g_{s-1} + \sum_{v} h_v g_{sv}.$$
2. Nil$_{\ast}$-Noetherian properties on some ring constructions

Some non-reduced rings are constructed by the idealization $R(+)M$ where $M$ is an $R$-module (see [3]). Set $R(+)M = R \oplus M$ as an $R$-module, and then define

1. $(r, m) + (s, n) = (r + s, m + n)$,
2. $(r, m)(s, n) = (rs, sm + rn)$.

Under this construction, $R(+)M$ becomes a commutative ring with identity $(1, 0)$. Note that $(0, m)^2 = 0$ for any $m \in M$. Now we characterize when $R(+)M$ is a Nil$_{\ast}$-Noetherian ring.

**Theorem 2.1.** Let $R$ be a ring and $M$ an $R$-module. Then $R(+)M$ is a Nil$_{\ast}$-Noetherian ring if and only if $R$ is a Nil$_{\ast}$-Noetherian ring and $M$ is a finitely generated $R$-module.

**Proof.** For necessity, since $R \cong R(+)M/0(+)M$ and $0(+)M$ is a nil ideal, $R$ is Nil$_{\ast}$-Noetherian by Lemma 1.3. Since $0(+)M$ is a nil ideal, then $0(+)M$ is finitely generated $R(+)M$-module. It is easy to check that $M$ is also a finitely generated $R$-module.

For sufficiency, suppose $M$ is generated by $n$ elements. Set $K = \ker(R^n \to M)$. Then we have a short exact sequence $0 \to 0(+)K \to R(+)R^n \to R(+)M \to 0$ as $R(+)R^n$-modules. By [3, Proposition 2.2], $R(+)R^n \cong R[x_1, \ldots, x_n]/\langle x_i^2 \mid i = 1, \ldots, n \rangle$ is a Nil$_{\ast}$-Noetherian by Theorem 1.9 and Lemma 1.3. Hence $R(+)M \cong R(+)R^n/0(+)K$ is also a Nil$_{\ast}$-Noetherian by Lemma 1.3 again. \hfill \Box

First, we give an example of Nil$_{\ast}$-Noetherian rings which are neither reduced nor Noetherian.

**Example 2.2.** Let $S = \prod_k^\infty k$ be a countable copies of direct product of a field $k$, $e_i$ is an element in $S$ with the $i$-th component 1 and others 0. Set $R = S(+)Se_i$. Then $R$ is neither reduced nor Noetherian. However, since $S$ is a reduced ring and $\Nil(R) = 0(+)Se_i$ is a simple ideal, $R$ is Nil$_{\ast}$-Noetherian by Theorem 2.1.

Recall from [13] that a ring $R$ is called Nil$_{\ast}$-coherent provided that any finitely generated ideal in $\Nil(R)$ is finitely presented. Similar to the classical case, Nil$_{\ast}$-coherent rings are not Nil$_{\ast}$-Noetherian in general.

**Example 2.3.** Let $D$ be a non-field Noetherian GCD domain and $Q$ its quotient field. Set $R = D(+)Q/D$. Then by Theorem 2.1, $R$ is not Nil$_{\ast}$-Noetherian since $Q/D$ is not finitely generated. We will show $R$ is Nil$_{\ast}$-coherent. Indeed, let $I$ be a finitely generated nil ideal of $R$. Since $\Nil(R) = 0(+)Q/D$, we can assume $I$ is generated by $\{(0, s_i/D), \ldots, (0, s_n/D)\}$ with all $\gcd(s_i, t_i) = 1$ and $s_i \neq 1$. Consider the
short exact sequence $0 \to K_n \to R^n \to I \to 0$. We will show $K_n$ is finitely generated by induction on $n$. Indeed, for each $l \geq 1$, set $I_l = \langle (0, \frac{t}{s_1} + D), \cdots, (0, \frac{t}{s_l} + D) \rangle$. Suppose $n = 1$, then $K_1 = (0 :_R \frac{t}{s_1} + D) = \langle s_1 \rangle (+) Q/D$ which is generated by $(s_1, 0)$. Suppose it hold for $n = k \geq 1$. That is there is a natural exact sequence $0 \to K_k \to R^k \to I_k \to 0$. If $n = k + 1$, set $a = (0, \frac{t_{k+1}}{s_{k+1}} + D)$. There is an $R$-module $K_{k+1}$ such that the following commutative diagram have exact rows and columns:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & K_k & \longrightarrow & R^k & \longrightarrow & I_k & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_{k+1} & \longrightarrow & R^{k+1} & \longrightarrow & I_k + Ra & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (I_k :_R Ra) & \longrightarrow & R & \longrightarrow & (I_k + Ra)/I_k & \longrightarrow & 0,
\end{array}
\]

Since $(s_{k+1}, 0) \in (I_k :_R Ra) - 0(+) Q/D$, we have $(I_k :_R Ra) \supseteq 0(+) Q/D$, and so $(I_k :_R Ra) = J'(+) Q/D$ for some nonzero finitely generated ideal $J'$ of $D$ by [3] Corollary 3.4. Hence $J'(+) Q/D$ is finitely generated by [14] Proposition 2.6. Consequently, $K_{k+1}$ is also finitely generated. It follows that $I$ is finitely presented, and thus $R$ is $\text{Nil}_s$-coherent.

Surprisingly, different with the classical case, $\text{Nil}_s$-Noetherian rings can also be non-$\text{Nil}_s$-coherent.

**Example 2.4.** Let $S = k[x_1, x_2, \cdots]$ be the polynomial ring over a field $k$ with countably infinite variables. Set $R = S/\langle x_i x_i \mid i \geq 1 \rangle$. Then $\text{Nil}(R) = \langle \overline{x_i} \rangle$ is the only non-trivial nil ideal of $R$, where $\overline{x_i}$ denotes the representative of $x_i$ in $R$ for each $i$. So $R$ is $\text{Nil}_s$-Noetherian. However, since $(0 :_R \overline{x_i}) = \langle \overline{x_1}, \overline{x_2}, \cdots \rangle$ is infinitely generated, $\text{Nil}(R)$ is not finitely presented. Hence $R$ is not $\text{Nil}_s$-coherent.

We recall the bi-amalgamated algebras constructed in [9]. Let $f : A \to B$ and $g : A \to C$ be two ring homomorphisms and let $J$ and $J'$ be two ideals of $B$ and $C$, respectively, such that $f^{-1}(J) = g^{-1}(J')$. Denote by $I_0 := f^{-1}(J) = g^{-1}(J')$. The bi-amalgamated algebra of $A$ with $(B, C)$ along $(J, J')$ with respect to $(f, g)$ is the subring of $B \times C$ given by:

$$A \bowtie^{f,g} (J, J') := \{(f(a) + j, g(a) + j') \mid a \in A, (j, j') \in J \times J'\}.$$ 

The bi-amalgamation is determined by the following pull-back:

$$\begin{array}{ccc}
A \bowtie^{f,g} (J, J') & \longrightarrow & f(A) + J \\
\downarrow & & \downarrow \alpha \\
g(A) + J' & \longrightarrow & A/I_0,
\end{array}$$
with \( \alpha(f(a) + j) = \beta(g(a) + j') = \overline{a} \) for any \( a \in A \). The following proposition is useful for continuation.

**Proposition 2.5.** [9] Proposition 4.1] Let \( f : A \to B \) and \( g : A \to C \) be two ring homomorphisms and let \( J \) and \( J' \) be two ideals of \( B \) and \( C \), respectively, such that \( f^{-1}(J) = g^{-1}(J') := I_0 \). Let \( I \) be an ideal of \( A \). Then the following statements hold.

1. \( \frac{A \otimes f^{-1}(J), J')}{I \otimes f^{-1}(J), J'} \cong \frac{A}{I + I_0} \).
2. \( \frac{A \otimes f^{-1}(J), J')}{0 \otimes J} \cong \frac{f(A) + J}{J} \) and \( \frac{A \otimes f^{-1}(J), J')}{J \otimes 0} \cong \frac{g(A) + J'}{J} \).
3. \( \frac{A}{I_0} \cong \frac{A \otimes f^{-1}(J), J')}{J \otimes J} \cong \frac{f(A) + J}{J} \cong \frac{g(A) + J'}{J} \).

**Lemma 2.6.** Let \( f : A \to B \) and \( g : A \to C \) be two ring homomorphisms and let \( J \) and \( J' \) be two ideals of \( B \) and \( C \), respectively, such that \( f^{-1}(J) = g^{-1}(J') := I_0 \). Suppose one of the following cases holds:

1. \( I_0 \) is nil.
2. \( J \subseteq \text{Im}(f) \) and \( \text{Ker}(f) \) is nil.

Then

\[
\text{Nil}(A \otimes f^{-1} (J, J')) = \text{Nil}(A) \otimes f^{-1} (J \cap \text{Nil}(f(A) + J), J' \cap \text{Nil}(g(A) + J')).
\]

**Proof.** (1) Let \( \xi := (f(a) + j, g(a) + j') \in \text{Nil}(A \otimes f^{-1} (J, J')) \). Then \( f(a) + j \in \text{Nil}(f(A) + J) \) and \( g(a) + j' \in \text{Nil}(g(A) + J') \). So \( f(a^n) \in J \) for some positive integer \( n \). So \( a^n \in I_0 \), and thus is nilpotent. Hence \( a \in \text{Nil}(A) \). So \( f(a) \in \text{Nil}(f(A)) \subseteq \text{Nil}(f(A) + J) \), and hence \( j = (f(a) + j) - f(a) \in J \cap \text{Nil}(f(A) + J) \). Similarly, \( j' \in J' \cap \text{Nil}(g(A) + J') \). Consequently, \( \xi \in \text{Nil}(A) \otimes f^{-1} (J \cap \text{Nil}(f(A) + J), J' \cap \text{Nil}(g(A) + J')) \).

On the other hand, let \( a \in \text{Nil}(A), j \in J \cap \text{Nil}(f(A) + J) \) and \( j' \in J' \cap \text{Nil}(g(A) + J') \). Then there is \( k, m, n \) such that \( a^k = j^m = j'^n = 0 \). Set \( \xi := (f(a) + j, g(a) + j') \in A \otimes f^{-1} (J, J') \). Then \( \xi^{kmn} = 0 \), and hence \( \xi \in \text{Nil}(A \otimes f^{-1} (J, J')) \).

(2) Let \( \xi := (f(a)+j, g(a)+j') \in \text{Nil}(A \otimes f^{-1} (J, J')) \). Then \( f(a)+j \in \text{Nil}(f(A)+J) \) and \( g(a)+j' \in \text{Nil}(g(A)+J') \). Since \( J \subseteq \text{Im}(f) \), there is an \( x \in I_0 \) such that \( f(x) = j \).

Note that \( g(x) \notin J' \). So \( \xi := (f(a+x), g(a+x) + j' - g(x)) \). Since \( \text{Ker}(f) \) is nil, we have \( a+x \) is nilpotent. Now, we claim that \( j' - g(x) \) is nilpotent. Indeed, since \( a+x \) is nilpotent, \( g(a)+g(x) \) is nilpotent. So \( j' - g(x) = (g(a)+j') - (g(a)+g(x)) \) is nilpotent as \( g(a)+j' \) is nilpotent. Hence \( \xi \in \text{Nil}(A \otimes f^{-1} (J \cap \text{Nil}(f(A)+J), J' \cap \text{Nil}(g(A)+J'))) \).

The other hand is the same as (1). \( \square \)

It was proved in [9] Proposition 4.2] that a ring \( A \otimes f^{-1} (J, J') \) is Noetherian if and only if \( f(A) + J \) and \( g(A) + J' \) are Noetherian. The main purpose of the rest section is to study Noetherian properties on bi-amalgamated algebras.
Proposition 2.7. Let $f : A \to B$ and $g : A \to C$ be two ring homomorphisms and let $J$ and $J'$ be two ideals of $B$ and $C$, respectively, such that $f^{-1}(J) = g^{-1}(J') := I_0$ is nil. Then $A \bowtie^{f, g} (J, J')$ is $\Nil_*$-Noetherian if and only if $f(A) + J$ and $g(A) + J'$ are $\Nil_*$-Noetherian.

Proof. Note that since $I_0$ is nil, $J$ and $J'$ are both nil. Then $J \times 0$ and $0 \times J'$ are also nil ideals of $A \bowtie^{f, g} (J, J')$. Suppose $A \bowtie^{f, g} (J, J')$ is $\Nil_*$-Noetherian. Hence $f(A) + J$ and $g(A) + J'$ are $\Nil_*$-Noetherian by Lemma 1.3 and Proposition 2.5 (2). On the other hand, suppose $f(A) + J$ and $g(A) + J'$ are $\Nil_*$-Noetherian. Let $I$ be a nil ideal of $A \bowtie^{f, g} (J, J')$ generated by $\{f(a_i) + j_i, g(a_i) + j'_i \mid i \in \Lambda\}$. Then the ideals $\langle f(a_i) + j_i \mid i \in \Lambda \rangle \subseteq \Nil(f(A) + J)$ and $\langle g(a_i) + j'_i \mid i \in \Lambda \rangle \subseteq \Nil(g(A) + J')$ by Lemma 2.6. So there is a finitely subset $\Lambda_0 \subseteq \Lambda$ such that $\langle f(a_i) + j_i \mid i \in \Lambda \rangle = \langle f(a_i) + j_i \mid i \in \Lambda_0 \rangle$ and $\langle g(a_i) + j'_i \mid i \in \Lambda \rangle = \langle g(a_i) + j'_i \mid i \in \Lambda_0 \rangle$. Hence $I$ is a nil ideal of $A \bowtie^{f, g} (J, J')$ generated by $\{f(a_i) + j_i, g(a_i) + j'_i \mid i \in \Lambda_0\}$. So $A \bowtie^{f, g} (J, J')$ is $\Nil_*$-Noetherian. \qed

Theorem 2.8. Let $f : A \to B$ and $g : A \to C$ be two ring homomorphisms and let $J$ and $J'$ be two ideals of $B$ and $C$, respectively, such that $f^{-1}(J) = g^{-1}(J')$. If $J \subseteq \Im(f)$, $J' \subseteq \Im(g)$, $\Ker(f)$ and $\Ker(g)$ are nil, then $A \bowtie^{f, g} (J, J')$ is $\Nil_*$-Noetherian if and only if $f(A)$ and $g(A)$ are $\Nil_*$-Noetherian.

Proof. Since $J \subseteq \Im(f)$, we have $f(A) + J = f(A)$. Similarly, $g(A) + J' = g(A)$. Suppose $A \bowtie^{f, g} (J, J')$ is $\Nil_*$-Noetherian. Let $I$ be a nil ideal of $f(A)$ generated by $\{f(a_i) \mid a_i \in \Lambda\}$. Since $\Ker(f)$ is nil, each $a_i$ is also nilpotent. Consider the ideal $K$ of $A \bowtie^{f, g} (J, J')$ generated by $\{(f(a_i), g(a_i)) \mid a_i \in \Lambda\}$. Then $K$ is nil and thus finitely generated. So there is a finite subset $\Lambda_0 \subseteq \Lambda$ such that $K$ is generated by $\{(f(a_i), g(a_i)) \mid a_i \in \Lambda_0\}$. Thus $I$ is generated by $\{f(a_i) \mid a_i \in \Lambda_0\}$. Hence $f(A)$ is $\Nil_*$-Noetherian. Similarly, we have $g(A)$ is $\Nil_*$-Noetherian. Suppose $f(A)$ and $g(A)$ are $\Nil_*$-Noetherian. Let $I$ be a nil ideal of $A \bowtie^{f, g} (J, J')$ generated by $\{(f(a_i) + j_i, g(a_i) + j'_i) \mid i \in \Lambda\} \subseteq f(A) \times g(A)$. Then the ideals $\langle f(a_i) + j_i \mid i \in \Lambda \rangle \subseteq \Nil(f(A) + J) = \Nil(f(A))$ and $\langle g(a_i) + j'_i \mid i \in \Lambda \rangle \subseteq \Nil(g(A) + J') = \Nil(g(A))$ with each $a_i \in \Nil(A)$ by Lemma 2.6. So there is a finitely subset $\Lambda_0 \subseteq \Lambda$ such that $\langle f(a_i) + j_i \mid i \in \Lambda \rangle = \langle f(a_i) + j_i \mid i \in \Lambda_0 \rangle$ and $\langle g(a_i) + j'_i \mid i \in \Lambda \rangle = \langle g(a_i) + j'_i \mid i \in \Lambda_0 \rangle$. Hence $I$ is a nil ideal of $A \bowtie^{f, g} (J, J')$ generated by $\{(f(a_i) + j_i, g(a_i) + j'_i) \mid i \in \Lambda_0\}$. So $A \bowtie^{f, g} (J, J')$ is $\Nil_*$-Noetherian. \qed

Recall from [4] that, by setting $g = \Id_A : A \to A$ to be the identity homomorphism of $A$, we denote by $A \bowtie^f J = A \bowtie^{f, \Id_A} (J, f^{-1}(J))$ and call it the amalgamated algebra of $A$ with $B$ along $J$ with respect to $f$. Note that in this situation, the ring
homomorphism \( i : A \to A \bowtie_f J \), defined by \( i(a) = (a, f(a)) \) for any \( a \in A \), is an \( A \)-module retract.

**Proposition 2.9.** Let \( f : A \to B \) be a ring homomorphism and \( J \) an ideal of \( B \). If \( A \bowtie_f J \) is \( \Nil_* \)-Noetherian, so is \( A \). Moreover, if \( J \subseteq \text{Im}(f) \) and \( \text{Ker}(f) \) is nil, or \( f^{-1}(J) \) is nil, then \( A \bowtie_f J \) is \( \Nil_* \)-Noetherian if and only if \( A \) is \( \Nil_* \)-Noetherian.

**Proof.** Suppose \( A \bowtie_f J \) is \( \Nil_* \)-Noetherian. Then \( A \) is \( \Nil_* \)-Noetherian by Proposition 1.7. Since \( f(A) \cong A/\text{Ker}(f) \), the equivalence follows by Lemma 1.3, Theorem 2.8 and Proposition 2.7 respectively. \( \square \)

Recall from [5] that, by setting \( f = \text{Id}_A : A \to A \) to be the identity homomorphism of \( A \), we denote by \( A \bowtie J = A \bowtie_{\text{Id}_A} J \) and call it the amalgamated algebra of \( A \) along \( J \). By Proposition 2.9 we obviously have the following result.

**Corollary 2.10.** Let \( J \) be an ideal of \( A \). Then \( A \bowtie J \) is \( \Nil_* \)-Noetherian if and only if \( A \) is \( \Nil_* \)-Noetherian.

3. **Module-theoretic characterizations of \( \Nil_* \)-Noetherian rings**

We begin this section with the following two conceptions.

**Definition 3.1.** An \( R \)-module \( M \) is said to be

(1) \( \Nil_* \)-injective, provided that \( \text{Ext}^1_R(R/I, M) = 0 \) for any nil ideal \( I \);
(2) \( \Nil_* \)-FP-injective, provided that \( \text{Ext}^1_R(R/I, M) = 0 \) for any finitely generated nil ideal \( I \).

Trivially, injective modules are \( \Nil_* \)-injective; \( \Nil_* \)-injective modules and FP-injective modules are \( \Nil_* \)-FP-injective. The class of \( \Nil_* \)-injective modules is closed under direct products; the class of \( \Nil_* \)-FP-injective is is closed under direct sums and direct products.

The following result shows that non-trivial nil ideals can never be projective.

**Lemma 3.2.** [11, Proposition 6.7.12] Let \( I \) be a non-zero nil ideal of a ring \( R \). Then \( I \) is not projective.

We first characterize rings over which all modules are \( \Nil_* \)-injective or \( \Nil_* \)-FP-injective.

**Proposition 3.3.** Let \( R \) be a ring. Then the following statements are equivalent.

(1) \( R \) is a reduced ring.
(2) Every \( R \)-module is \( \Nil_* \)-injective.
(3) Every \( R \)-module is \( \Nil_* \)-FP-injective.
Proof. (1) ⇒ (2) ⇒ (3): Trivial.

(3) ⇒ (1): Let $I$ be a finitely generated nil ideal of $R$. Consider the exact sequence $0 \to I \to R \to R/I \to 0$. Then $I$ is \textit{Nil}-FP-injective by (3). So the exact sequence splits. Then $I$ is projective. So each finitely generated nil ideal $I$, and thus $\text{Nil}(R)$ is equal to 0 by Lemma 3.2.

We characterize \textit{Nil}-coherent rings in terms of \textit{Nil}-FP-injective modules.

\textbf{Proposition 3.4.} Let $R$ be a ring. Then $R$ is \textit{Nil}-coherent if and only if any direct limit of injective $R$-modules is \textit{Nil}-FP-injective.

\textbf{Proof.} Let $I$ be a finitely generated nil ideal of $R$ and \{$M_i$\}$_{i \in I}$ a direct system of injective $R$-modules. Then $\lim \mathrm{Ext}^1_R(R/I, M_i) = 0$ for each $i \in I$. Consider the short exact sequence $0 \to I \to R \to R/I \to 0$, we have the following commutative diagram with rows exact:

$$
\begin{array}{cccccccc}
\longrightarrow & \lim \mathrm{Hom}_R(R, M_i) & \longrightarrow & \lim \mathrm{Hom}_R(I, M_i) & \longrightarrow & \lim \mathrm{Ext}^1_R(R/I, M_i) & \longrightarrow & 0 \\
\varphi_R & \Phi_I & \varphi_{R/I} & & & & & \\
\longrightarrow & \mathrm{Hom}_R(R, \lim M_i) & \longrightarrow & \mathrm{Hom}_R(I, \lim M_i) & \longrightarrow & \mathrm{Ext}^1_R(R/I, \lim M_i) & \longrightarrow & 0.
\end{array}
$$

Since $R$ is a \textit{Nil}-coherent ring, $I$ is a finitely presented ideal, then $\varphi_I$ is an isomorphism. Since $\varphi_R$ is an isomorphism, then $\varphi_{R/I}$ is also an isomorphism. Consequently, $\lim M_i$ is a \textit{Nil}-FP-injective $R$-module.

Let $I$ be a finitely generated nil ideal, \{$M_i$\}$_{i \in I}$ a direct limit of $R$-modules. Suppose $\alpha : I \to \lim M_i$ is an $R$-homomorphism. For any $i \in I$, $E(M_i)$ is the injective envelope of $M_i$. Then $\alpha$ can be extended to be $\beta : R \to \lim E(M_i)$. So there exists $j \in I$ such that $\beta$ can factor through $R \to E(M_j)$. Since the composition $I \to R \to E(M_j) \to E(M_j)/M_j$ becomes to be 0 in the direct limit. We can assume $I \to R \to E(M_j)$ factors through $M_j$. Then $\alpha$ factor through $M_j$. So the natural homomorphism $\lim \mathrm{Hom}_R(I, M_i) \to \mathrm{Hom}_R(I, \lim M_i)$ is an epimorphism. So $I$ is a finitely presented ideal. \qed

\textbf{Theorem 3.5. (Cartan-Eilenberg-Bass Theorem for \textit{Nil}-rings)} Let $R$ be a ring. Then the following assertions are equivalent:

1. $R$ is \textit{Nil}-Noetherian;
2. any direct sum of \textit{Nil}-injective $R$-modules is \textit{Nil}-injective;
3. any direct sum of injective $R$-modules is \textit{Nil}-injective;
4. any direct union of \textit{Nil}-injective $R$-modules is \textit{Nil}-injective;
5. any direct union of injective $R$-modules is \textit{Nil}-injective;
(6) any Nil$_*$-FP-injective $R$-module is Nil$_*$-FP-injective.

Proof. (1) $\Rightarrow$ (6), (4) $\Rightarrow$ (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5) $\Rightarrow$ (3) : Trivial.

(1) $\Rightarrow$ (4) : Let $\{M_i, f_{i,j}\}_{i,j\in A}$ be a direct system of Nil$_*$-injective $R$-modules, where each $f_{i,j}$ is an inclusion map. Set by $\lim M_i$ the direct limit. Let $I$ be nil ideal of $R$, then $I$ is finitely generated. So $R/I$ is finitely presented. Consider the following commutative diagram with exact rows:

$$
\begin{array}{cccccc}
\lim \text{Hom}_R(R, M_i) & \longrightarrow & \lim \text{Hom}_R(I, M_i) & \longrightarrow & \lim \text{Ext}^1_R(R/I, M_i) & \longrightarrow & 0 \\
\varphi_R & & & & \varphi_I & & \varphi_{R/I} \\
\longrightarrow \text{Hom}_R(R, \lim M_i) & \longrightarrow & \text{Hom}_R(I, \lim M_i) & \longrightarrow & \text{Ext}^1_R(R/I, \lim M_i) & \longrightarrow & 0.
\end{array}
$$

By [12] Theorem 24.10, $\varphi_R$ and $\varphi_I$ are isomorphisms. So $\varphi_{R/I}$ is also an isomorphism. Hence, $\lim M_i$ is Nil$_*$-injective.

(3) $\Rightarrow$ (1) : On contrary, suppose $R$ is not Nil$_*$-Noetherian. Hence there is a non-finitely generated nil ideal $I$. So there is a nilpotent element $a_0 \in I$ such that $a_0 R \neq I$. Take $0 \neq a_1 \in I - a_0 R$, then the nil ideal $a_1 R + a_0 R \neq I$. Take $a_2 \in I - (a_1 R + a_0 R)$, then the nil ideal $a_1 R + a_2 R + a_0 R \neq I$. Repeat these steps, we can get a strictly increasing chain of nil ideals:

$$a_0 R + a_1 R \subseteq a_0 R + a_1 R + a_2 R \subseteq \cdots \subseteq a_0 R + \cdots + a_n R \subset \cdots$$

Set $A_i = \sum_{j=0}^{i} a_j R$. According to [2] Corollary 10.5, for any $A_i$, there exists maximal sub-ideal $C_i$ of $A_i$ satisfying

$$C_1 \subset A_1 \subset C_2 \subset \cdots \subset C_n \subset \cdots$$

Set $E(A_i/C_i)$ the injective envelope of $A_i/C_i$. Write $E = \bigoplus_{i=1}^{\infty} E(A_i/C_i)$. Then $E$ is Nil$_*$-injective by (2). Set $A = \bigcup_{i=1}^{\infty} A_i$. Then $A$ is a nil ideal. Set $\pi_i : A_i \rightarrow A_i/C_i$ the natural epimorphism and $\rho_i : A_i/C_i \hookrightarrow E(A_i/C_i)$ the inclusion map. Then there are homomorphism $f_i : A \rightarrow E(A_i/C_i)$ which is an extension of $\rho_i \circ \pi_i$. Let $a \in A$ and $f : A \rightarrow E$ satisfies $f(a) = (f_i(a))_{i=1}^{\infty}$. Since $E$ is Nil$_*$-injective, then $f$ can be extended to be $g : R \rightarrow E$ with $g(r) = g(1)r$. Set $g(1) = (c_1, c_2, \cdots, c_n, 0, \cdots)$. Take $a \in A$ such that $a \in A_{n+1} - C_{n+1}$. Then $\rho_{n+1}\pi_{n+1}(a) \neq 0$, and so $f_{n+1}(a) \neq 0$. However,

$$f(a) = g(a) = g(1)a = (c_1a, c_2a, \cdots, c_na, 0, \cdots),$$

which is a contradiction. Hence, $R$ is Nil$_*$-Noetherian.
(6) ⇒ (2): Follows by any direct sum of Nil$_*$-FP-injective $R$-modules is Nil$_*$-FP-injective. □

**Remark 3.6.** We must remark that a ring $R$ is Nil$_*$-Noetherian if and only if any direct union of Nil$_*$-injective $R$-modules is Nil$_*$-injective in Theorem 3.5. So if any direct limit of Nil$_*$-injective $R$-modules is Nil$_*$-injective, then $R$ is a Nil$_*$-Noetherian ring. However, the converse does not hold in general. Indeed, let $R$ is a Nil$_*$-Noetherian ring which is not Nil$_*$-coherent (see Example 3.4). Then the class of Nil$_*$-FP-injective $R$-modules is equal to that of Nil$_*$-FP-injective modules. However, by Proposition 3.4, there exists a direct system of Nil$_*$-injective $R$-modules, whose direct limit is not Nil$_*$-injective.

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**References**

[1] K. Alaoui Ismaili, D. E. Dobbs and N. Mahdou, *Commutative rings and modules that are Nil$_*$-coherent or special Nil$_*$-coherent*, J. Algebra Appl, **16** (2017) 1750187.

[2] F. W. Anderson, K. R. Fuller, *Rings and Categories of Modules*, second edition. Springer: Berlin, Germany; New York, NY, USA, 1992.

[3] D. D. Anderson, M. Winders, *Idealization of a module*, J. Commut. Algebra **1** (2009) 3-56.

[4] M. D’Anna, C. Finocchiaro and M. Fontana, *Amalgamated algebras along an ideal*, in: M. Fontana, S. Kabbaj, B. Olberding, I. Swanson (Eds.), Commutative Algebra and its Applications, Walter de Gruyter, Berlin, (2009) 155-172.

[5] M. D’Anna and M. Fontana, *An amalgamated duplication of a ring along an ideal: the basic properties*, J. Algebra Appl. **6**(3) (2007) 443-459.

[6] J. W. Brewer, *Power series over commutative rings*, Lecture Notes in Pure Appl. Math. 64, Marcel Dekker, Inc., New York, 1981.

[7] S. Hizem, A. Benhissi, *Nonnil-Noetherian rings and SFT property*, Rocky Mountain J. Math. **41** (2011), 1483-1500.

[8] J. A. Huckaba, *Commutative rings with Zero Divisors*, Monographs and Textbooks in Pure and Applied Mathematics, **117**, Marcel Dekker, Inc., New York, 1988.

[9] S. Kabbaj, K. Louartiti, and M. Tamekkante, *Bi-amalgamated algebras along ideals*, J. Commut. Algebra **9** (2017) 65-87.

[10] B. Stenström, *Rings of Quotients*, Die Grundlehren Der Mathematischen Wissenschaften, Berlin: Springer-verlag, 1975.

[11] F. G. Wang, H. Kim, *Foundations of Commutative rings and Their Modules*, Singapore: Springer, 2016.

[12] R. Wisbauer, *Foundations of Module and Ring Theory*, Algebra, Logic and Applications, vol. 3, Amsterdam: Gordon and Breach, 1991.
[13] Y. Xiang, L. Ouyang, *Nil*-coherent rings, Bull. Korean Math. Soc. 51 (2014) 579-594.
[14] X. L. Zhang, *A homological characterization of Pr"ufer v-multiplicaiton rings*, Bull. Korean Math. Soc. 59(1) (2022) 213-226