A study of relativistic corrections to $J/\psi \rightarrow p\bar{p}$ decay

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Abstract

We study relativistic corrections in exclusive $S$-wave charmonium decays into proton-antiproton final state. We calculate the NRQCD corrections to the dominant decay amplitude, which depend on the nucleon twist-3 light-cone distribution amplitudes only. It is shown that in this case the collinear factorisation is also valid beyond the leading-order approximation. Our numerical estimates show that relativistic correction of relative order $v^2$ provides large numerical impact.
1 Introduction

In recent years, the BESII and BESIII collaborations have obtained many accurate data on baryon-antibaryon decays of $S$-wave charmonia, see e.g. Refs. [1-8]. These data allows one to get information about various important hadronic parameters and provide an interesting possibility to study the QCD dynamics.

There are different approaches for a description of the baryon-antibaryon decays. In Refs. [9,10] it was proposed to use a phenomenological model based on an empirical effective Lagrangian with unknown parameters, which are subject to restrictions from $SU(3)$ flavour symmetry and experimental data.

Another way is to use effective field theory framework, combining nonrelativistic expansion and collinear factorisation [11-14]. Appropriate effective Lagrangian is described by NRQCD [15,16] and collinear copies of QCD [11,12,14].

Such approach allows one to built a systematic expansion in powers of $1/m_c$ describing decay amplitudes as a superposition of perturbative and universal nonperturbative functions. The perturbative part can be computed in pQCD at the scale of order $m_c$. The nonperturbative matrix elements are defined at soft hadronic scale $\Lambda \ll m_c$. They can be limited to other processes or evaluated using non-perturbative methods such as QCD sum rules or lattice QCD.

Since the real mass of the charm quark is not large enough, a systematic description requires careful analysis of the various corrections. Existing calculations of exclusive and inclusive reactions show that in many cases radiative corrections in $\alpha_s(m_c)$ and relativistic corrections can have a significant numerical effect, see for instance, reviews [17,18].

The first estimates for exclusive proton-antiproton decay based on the leading-order (LO) approximation were obtained already long time ago in Refs. [11,14]. Recently, this analysis has been extended in Refs. [19-21], where the power-law corrections $\sim \Lambda/m_c$ associated with the collinear expansion were taken into account for different amplitudes. The collinear power corrections are associated with the higher twist baryon light-cone distribution amplitudes (LCDAs). It has been established that the effect of such corrections is not very strong and one can describe various channels of baryon-antibaryon decay with good accuracy. However, relativistic and QCD radiative corrections for baryon–antibaryon decays have not yet been studied.

The available data for various baryons indicate the possible presence of effects from relativistic corrections. For example, the so-called “12% rule”

$$Q_B = \frac{Br[\psi(2S) \rightarrow BB]}{Br[J/\psi \rightarrow BB]} \approx \frac{Br[\psi(2S) \rightarrow e^+e^-]}{Br[J/\psi \rightarrow e^+e^-]} \approx 0.13, \quad (1)$$

must hold if the width is dominated by the LO NRQCD approximation. The available experimental data [22] show the following results: $Q_p = 0.1386(3)$, $Q_\Lambda = 0.146(1)$, $Q_\Sigma = 0.204(1)$, $Q_{\Sigma^+} = 0.21(3)$, $Q_{\Sigma^+} = 0.072(5)$ and $Q_{\Xi^+} = 0.276(5)$. Consequently, the data for baryons other than nucleon indicate a violation of this expectation.

For nucleon-antinucleon decays the rule (1) works quite well, but on the other hand, the values of the polar angular distribution coefficient $\alpha_p$ for ground and excited $S$-wave charmonium ($\psi(2S) \equiv \psi'$) are very different [3,7]

$$\alpha_p|_{J/\psi} = 0.595 \pm 0.012, \quad \alpha_p|_{\psi'} = 1.03 \pm 0.06. \quad (2)$$

This observation may also indicate a significant contribution of relativistic corrections.

The purpose of this work is to study the effects of relativistic corrections in proton-antiproton decays. The technique for calculating relativistic corrections is well known. Such corrections
were already studied in for different exclusive decays as \( J/\psi \to e^+e^- \) [23, 24], \( H \to \gamma J/\psi \) [25, 26], \( \chi_cJ \to \gamma\gamma \) [27, 28] and for exclusive production \( e^+e^- \to J/\psi\eta_c \) [29, 30]. However, relativistic corrections to exclusive hadronic decays have not yet been considered.

The feature of hadronic decays is that the hard kernels are not simple numbers but depend on light quark momentum fractions. Corresponding decay amplitude is described by the convolution integral over the momentum fractions of the hard kernel with LCDAs. In this case it is more convenient to perform the hard matching at the amplitude level. In some cases the convolution integrals have infrared divergencies that indicates about the violation of the collinear factorisation. This is often the case for next-to-leading power contributions in exclusive processes. However, for the dominant amplitude describing \( nS_1 \to B\bar{B} \) decay, the collinear factorisation is still applicable and this makes it possible to systematically study the relativistic effects.

Our paper is organised as follows. In Sec. 2 we give important definitions and kinematical notation. In Sec. 3 we calculate the hard kernels and perform the resummation of some class of relativistic effects. After that we provide a qualitative numerical analysis for different choices of the nucleon LCDAs. In Sec. 4 we discuss the obtained results and make conclusions. In Appendix we give important details about the nucleon twist-3 LCDAs and provide analytical expressions for the hard kernels.

2 Kinematics and decay amplitudes

In this work we use notation from Ref. [21]. We define the kinematics of \( J/\psi(P) \to p(k)\bar{p}(k') \) decay in the charmonium rest frame

\[
P = M_\psi \omega, \quad \omega = (1, 0, 0, 0).
\]  

The outgoing momenta \( k \) and \( k' \) are directed along the \( z \)-axis and read

\[
k = (M_\psi/2, 0, 0, M_\psi\beta/2), \quad k' = (M_\psi/2, 0, 0, -M_\psi\beta/2),
\]  

where \( m_N \) is the nucleon mass and \( \beta = \sqrt{1 - 4m_N^2/M_\psi^2} \). We also define auxiliary light-cone vectors

\[
n = (1, 0, 0, -1), \quad \bar{n} = (1, 0, 0, 1),
\]  

so that any four-vector \( V \) can be represented as

\[
V = V_+ \frac{n}{2} + V_- \frac{\bar{n}}{2} + V_\perp,
\]  

where \( V_+ \equiv (V \cdot n) = V_0 + V_3, \quad V_- \equiv (V \cdot \bar{n}) = V_0 - V_3. \)

The decay amplitude \( J/\psi \to p\bar{p} \) is defined as

\[
\langle k, k' | i\tilde{T} | P \rangle = (2\pi)^4 \delta(P - k - k') iM,
\]  

with

\[
M = \tilde{N}(k) \left\{ A_1 \epsilon_\psi + A_2 (\epsilon_\psi)_{\mu} (k' + k) \frac{i\sigma^{\mu\nu}}{2m_N} \right\} V(k'),
\]  

where \( \hat{p} = p_{\mu}\gamma^\mu \). The nucleon \( \tilde{N}(k) \) and antinucleon \( V(k') \) spinors have standard normalisation \( \tilde{N}N = 2m_N \) and \( \bar{V}V = -2m_N \). The charmonium polarisation vector \( \epsilon_\psi^\mu \equiv \epsilon_\psi^\mu(P, \lambda) \) satisfies

\[
\sum_\lambda \epsilon_\psi^\mu(P, \lambda) \epsilon_\psi^\nu(P, \lambda) = -g^{\mu\nu} + \frac{P_\mu P_\nu}{M_\psi^2}.
\]
The scalar amplitudes $A_1$ and $A_2$ describe the decay process. These amplitudes are computed in the effective field framework by expansion with respect to small relative heavy quark velocity $v$ and small ratio $\lambda^2 \sim \Lambda/m_Q$. The amplitude $A_2$ is suppressed as

$$A_2/A_1 \sim \lambda^2. \quad (10)$$

The decay width is conveniently described in terms of two combinations

$$G_M = A_1 + A_2, \quad G_E = A_1 + \frac{M_\psi^2}{4m_N^2} A_2, \quad (11)$$

and reads

$$\Gamma[J/\psi \to p\bar{p}] = \frac{M_\psi \beta}{12\pi} \left( |G_M|^2 + \frac{2m_N^2}{M_\psi^2} |G_E|^2 \right). \quad (12)$$

The ratio $|G_E| / |G_M|$ can be measured through the angular behaviour of the cross section $e^+e^- \to J/\psi \to p\bar{p}$, see e.g. Ref. [31]. Available data indicate that the contribution with the amplitude $G_E$ is quite small (about 10%) and the value of width is dominated by the amplitude $|G_M|^2$.

To the leading-order approximation

$$G_M \simeq A_{10}^1 + O(\alpha_s) + O(v^2) + O(\lambda^2), \quad (13)$$

where in the rhs we indicate various possible corrections. These are next-to-leading (NLO) order contribution with respect to $\alpha_s$, relativistic corrections of relative order $v^2$ and power corrections associated with the higher twist light-cone distribution amplitudes, respectively. The NLO radiative QCD corrections and relativistic corrections are not known yet. The power suppressed contribution have been estimated in Ref. [21] and corresponding numerical effect is moderate, about 25%. In this paper we want to study the relativistic corrections to the amplitude $A_{10}^1$.

The amplitude $A_{10}^1$ is described by the sum of two different contributions: the hard and electromagnetic one, which correspond to the annihilation into three gluons or one hard photon as shown in Fig. 1. The electromagnetic amplitude is relatively small, however it can provide numerical impact through the interference with QCD amplitude Ref. [21]. In this paper, for simplicity, the electromagnetic contribution is not considered.

The leading-order factorisation formula for the amplitude $A_{10}^1$ reads [11,12]

$$A_{1}^{(0)} = \sqrt{2M_\psi} \left\langle 0 \right| O \left| J/\psi \right\rangle \frac{f_N^2}{m_\pi^2} (\pi \alpha_s)^3 \frac{10}{81} J_0, \quad (14)$$

where the NRQCD matrix element is defined in the standard way

$$\left\langle 0 \right| O \left| J/\psi \right\rangle = \left\langle 0 \right| \chi^4 \sigma \cdot \epsilon(\lambda) \psi \left| J/\psi \right(\lambda) \right\rangle = \sqrt{\frac{N_c}{2\pi}} R_{10}(0), \quad (15)$$
where \( R_{10}(0) \) is the radial charmonium wave function at the origin. The constant \( f_N \) has the dimension of the square of the mass and describes the normalization of the matrix element of the nucleon. The decay amplitude is sensitive to nucleon structure that is described by the dimensionless collinear convolution integral \( J_0 \), which reads

\[
J_0 = \frac{1}{4} \int Dx_i \int Dy_i \frac{1}{x_1 x_2 x_3} \frac{1}{y_1 y_2 y_3} \frac{y_1 x_3}{D_1 D_3} \times \left\{ \varphi_3(x_{123}) \varphi_3(x_{123}) + \frac{1}{2} \left( \varphi_3(y_{123}) + \varphi_3(y_{321}) \right) \left( \varphi_3(x_{123}) + \varphi_3(x_{321}) \right) \right\},
\]

where \( D_i = x_i + y_i - 2x_i y_i \) and we used short notation for the twist-3 nucleon LCDA \( \varphi_3(x_{123}) \equiv \varphi_3(x_1, x_2, x_3) \). This nonperturbative function depends on the quark momentum fractions \( 0 < x_i < 1 \) and the factorisation scale, which is not shown explicitly. The light-cone fractions \( x_i \) satisfy the momentum conservation condition \( x_1 + x_2 + x_3 = 1 \). Therefore the measure of the convolution integrals in Eq. (16) includes the \( \delta \)-function

\[
Dx_i = dx_1 dx_2 dx_3 \delta(1 - x_1 - x_2 - x_3).
\]

The nucleon LCDA \( \varphi_3 \) is defined as the matrix element of a 3-quark operator and is well known in the literature, see e.g. Refs. [12, 32]. For a convenience, the definition and some important details are given in Appendix.

Our task is to calculate the relativistic corrections associated with the higher order matrix elements in NRQCD

\[
\langle 0 | O_n | J/\psi \rangle = m_c E_{J/\psi} \langle 0 | O_0 | J/\psi \rangle \left( \frac{i}{2} \cdot \mathbf{D} \right)^{2n} \psi | J/\psi(\lambda) \rangle,
\]

where \( \langle 0 | O_0 | J/\psi \rangle \equiv \langle 0 | O | J/\psi \rangle \). Usually, it is convenient to describe the contribution of these matrix elements introducing the ratio

\[
\langle q^2 \rangle = \frac{\langle 0 | O_n | J/\psi \rangle}{\langle 0 | O_0 | J/\psi \rangle}.
\]

The set of the operators in Eq. (18) does not represent the complete set of the all possible NRQCD operators. The operator \( O_2 \) provides the correct description of the relativistic correction of relative order \( v^2 \). But to describe relativistic corrections of a higher order, many more different operators are used, for example, colour-octet ones. However, resumming the contributions of all orders associated with the subset (19) is useful because it allows one to study the convergence of the nonrelativistic expansion.

The value of the NLO matrix element (19) can be exactly found with the help of Kapustin-Gremm relation Ref. [33]

\[
\langle 0 | O_1 | J/\psi \rangle = m_c E_{J/\psi} \langle 0 | O_0 | J/\psi \rangle,
\]

where \( E_{J/\psi} \) denotes the binding energy. This gives

\[
\langle q \rangle = m_c E_{J/\psi}.
\]

In Ref. [34] it was found that the higher order operators satisfy approximate relation

\[
\langle q^{2n} \rangle = \langle q^2 \rangle^n + \mathcal{O}(v^2).
\]
Neglecting the higher order corrections in the rhs of Eq. (22) allows one to resumm the power series of the higher order terms with the matrix elements (18) to all orders. Such resummation includes those relativistic corrections that are contained in the quark-antiquark quarkonium wave function in the leading potential model for the wave function [30].

The resummation of the relativistic corrections associated with the matrix elements (19) is already considered for various processes in Refs. [23, 24, 30] and this technique can be easily adapted for calculations of exclusive hadronic decays.

3 Matching and all order resummation

In order to calculate the decay amplitude we use the covariant spin-projector technique [35]. The generalisation of this technique to higher orders in \( v^2 \) is considered in Refs. [23, 24]. Here we briefly describe the NRQCD matching and clarify some features of the present calculation.

Our task is to compute the hard coefficient functions in front of higher order operators (18). The matching can be conveniently done using \( \bar{Q}Q \)-state as initial state instead of charmonium state. The corresponding heavy quark and antiquark have the momenta \( p \) and \( p' \), respectively

\[
p = (E, q), \quad p' = (E, -q), \quad E = \sqrt{m_Q^2 + q^2}.
\]  

Then total and relative momenta reads

\[
P = p + p', \quad q = \frac{1}{2}(p - p').
\]  

In the rest frame

\[
P = (2E, 0), \quad q = (0, q).
\]  

Let the perturbative amplitude is described as

\[
A_{QQ} = \bar{v}_Q(p', s') \hat{A}_{QQ} u_Q(p, s),
\]  

where \( \bar{v}_Q \) and \( u_Q \) denote heavy quark spinors. The amplitude projected onto \( S \)-wave state of heavy quark-antiquark \( \bar{Q}Q(3S_1) \) reads

\[
A_{QQ}(3S_1) = \text{Tr} \left[ \Pi_1 \hat{A}_{QQ}(x_i, y_i) \right],
\]  

where the spin-triplet projector \( \Pi_1 \) is given by [23]:

\[
\Pi_1 = \frac{-1}{2\sqrt{2}E(E + m_Q)} \left( \frac{1}{2} P + \not{q} + m_Q \right) \frac{P + 2E}{4E} f \left( \frac{1}{2} P - \not{q} - m_Q \right) \otimes \frac{1}{\sqrt{N_c}}.
\]  

This projector is relativistically normalised

\[
\text{Tr}[\Pi_1 \Pi_1^\dagger] = 4E^2.
\]  

In order to project the amplitude onto state with \( L = 0 \) the expression in Eq. (27) must be averaged over the angles of the momentum \( q \)

\[
A_{QQ}(q^2)|_{3S_1, \ L=0} = \bar{A}_{QQ}(q^2) = \frac{1}{4\pi} \int d\Omega \text{Tr} \left[ \Pi_1 \hat{A}_{QQ}(x_i, y_i) \right].
\]  

In NRQCD the amplitude reads
\[ A_{\text{NRQCD}} = \sqrt{2M_\psi} \sum_{n \geq 0} c_n \langle 0 | O_n | J/\psi \rangle, \] (31)
where \( \sqrt{2M_\psi} \) arises from relativistic normalisation. The coefficients \( c_n \) can be obtained from the amplitude (30) using that
\[ \langle 0 | O_n | J/\psi \rangle = \langle 0 | Q\bar{Q} \rangle \bar{A}_{QQ}(q^2) = \sum_n c_n \sqrt{2N_c} 2E q^{2n}. \] (32)
Using this one finds
\[ c_n = \frac{1}{\sqrt{2N_c}} \frac{1}{n!} \frac{\partial}{\partial q^{2n}} \bar{A}_{QQ}(q^2). \] (34)
Substituting this into Eq. (31) and using Eq. (22) one finds
\[ A_{\text{NRQCD}} \sim \sqrt{2M_\psi} \langle 0 | O | J/\psi \rangle \sum_n \frac{1}{n!} (q^2)^n \frac{\partial^n}{\partial q^{2n}} \bar{A}_{QQ}(3S_1) \sqrt{2N_c} 2E \] (35)
\[ = \sqrt{2M_\psi} \langle 0 | O | J/\psi \rangle \bar{A}_{QQ}(\langle q^2 \rangle) \sqrt{2N_c} 2E. \] (36)

The specific of our calculation is that the amplitude \( \bar{A}_{QQ}(3S_1) \) also includes the nonperturbative matrix elements describing the couplings with the nucleons. The amplitude \( Q\bar{Q} \rightarrow pp \) is described by the diagrams in Fig. 2. Each diagram can be described as consisting of a heavy quark subdiagram \( \Gamma_Q \) describing the \( Q\bar{Q} \) annihilation into three gluons, and a residual part describing the transition of three virtual gluons to the \( pp \) state. The expression for each diagram can be written in the following form
\[ A^{ij}_{QQ} = \int Dx_i \int Dy_i \Delta_1^{\mu_1 \mu_i'} \Delta_2^{\mu_2 \mu_i'} \Delta_3^{\mu_3 \mu_i'} B_{\mu_1 \mu_2 \mu_3} (x_i, y_i) \bar{v}_Q(p', s') \delta^{\mu_i' \mu_i} u_Q(p, s), \] (37)
where \( \bar{v}_Q \) and \( u_Q \) denote heavy quark spinors, \( \Delta_i^{\mu_1 \mu_i'} \) denotes the gluon propagator (we use Feynman gauge and do not write the colour indices for simplicity)
\[ \Delta_i^{\mu_1 \mu_i'} = \frac{(-i)}{(k + k_i')^2}, \quad k_i = x_i k, \quad k_i' = y_i k', \] (38)

Figure 2: The diagrams, which describe \( Q\bar{Q} \rightarrow pp \) subprocess.
where the proton and antiproton momenta reads

\[ k \simeq E \, n, \quad k' \simeq E \, \bar{n}, \quad k^2 = k'^2 = 0. \]  

(39)

The set of indices \( \{ \mu_i, \mu_c, \mu_j \} \) describes six possible transpositions of the indices \( \{ \mu_1, \mu_2, \mu_3 \} \), which correspond to the different diagrams: \( \{ \mu_1, \mu_2, \mu_3 \}, \{ \mu_3, \mu_2, \mu_1 \}, \{ \mu_2, \mu_1, \mu_3 \}, \{ \mu_3, \mu_1, \mu_2 \}, \{ \mu_1, \mu_3, \mu_2 \}, \{ \mu_2, \mu_3, \mu_1 \} \). Therefore each diagram can be coded by the set \( i c j \) and the sum of these diagrams gives the total result.

The term \( B_{\mu_1 \mu_2 \mu_3} \) in Eq. (37) describes the trace, which occur after contractions of the Dirac indices \( \alpha_i \) and \( \alpha'_i \) of the proton and antiproton light-cone matrix elements with the \( \gamma \)-matrices of the quark-gluon vertices

\[ \langle 0 | O_{tw3} | \bar{p}(k') \rangle \big( [i g] \gamma_{\mu_1} \alpha'_1 \alpha_1 \big) \big( [i g] \gamma_{\mu_2} \gamma_{\mu_3} \big) | p(k) \rangle = FT \big[ B_{\mu_1 \mu_2 \mu_3} \delta(x_i, y_i) \big]. \]  

(42)

The symbol “FT” denotes Fourier transformations, which occur from the light-cone matrix elements (the definition of the light-cone operator \( O_{tw3} \) is given in Appendix A). The nucleon matrix elements are described by the convenient combinations of LCDAs \( V_1, A_1 \) and \( T_1 \), which are defined as

\[ V_1(x_{123}) = \frac{1}{2} (\varphi_3(x_{123}) + \varphi_3(x_{213})), \]  

(43)

\[ A_1(x_{123}) = \frac{1}{2} (\varphi_3(x_{123}) - \varphi_3(x_{213})), \]  

(44)

\[ T_1(x_{123}) = \frac{1}{2} (\varphi_3(x_{132}) + \varphi_3(x_{231})). \]  

(45)

The \( B \)-term (42) is universal for all diagrams. The calculation of the Dirac contractions yields

\[ B_{\mu_1 \mu_2 \mu_3} = \bar{N}_n \gamma^\rho V_n (i g)^{\frac{3}{2}} \frac{1}{4} (k \cdot k') \quad T_{\mu_1 \mu_2 \mu_3} \]  

with

\[ T_{\mu_1 \mu_2 \mu_3} = g^\perp_{\mu_1 \mu_2} g^\perp_{\mu_3} \{ V_1(x_{123}) V_1(y_{123}) + A_1(x_{123}) A_1(y_{123}) \} \]  

(47)

\[ - i \varepsilon^\perp_{\mu_1 \mu_2} i \varepsilon^\perp_{\mu_3 \rho} \{ V_1(y_{123}) A_1(x_{123}) + A_1(y_{123}) V_1(x_{123}) \} \]  

\[ + G^\perp_{\mu_1 \mu_2 \sigma} G^\perp_{\rho \rho_3} T_1(y_{123}) T_1(x_{123}). \]

where we used the following short notations

\[ g^\perp_{\alpha \beta} = g_{\alpha \beta} - \frac{1}{2} (n_\alpha \bar{n}_\beta + n_\beta \bar{n}_\alpha), \quad i \varepsilon^\perp_{\alpha \beta} = \frac{1}{2} i \varepsilon_{\alpha \beta \lambda n_\sigma \bar{n}_\lambda}, \]  

(48)

\[ G^\perp_{\alpha \sigma \rho} = g^\perp_{\alpha \sigma} g^\perp_{\beta \rho} + g^\perp_{\alpha \rho} g^\perp_{\beta \sigma} - g^\perp_{\sigma \rho} g^\perp_{\alpha \beta}. \]  

(49)

For the projected nucleon spinors \( \bar{N}(k) \) and are \( V(k') \) we used

\[ \bar{N}(k) \frac{\gamma^\rho}{4} = \bar{N}_n, \quad \frac{\gamma^\rho}{4} V(k') = V_n. \]  

(50)
Therefore the expression for the amplitude (30) reads

\[ iA_{QQ}(q^2) = \frac{1}{24\pi} \int d\Omega \sum_{\{ij\}} \int Dx_i \int Dy_i \ T_{\mu_1\nu_2\mu_3\nu_4} \Delta_{\mu_1\mu_2}^{\nu_1\nu_2} \Delta_{\mu_3\mu_4}^{\nu_3\nu_4} \text{Tr} \left[ \Pi_1 \tilde{A}_{\nu_1\nu_2\nu_3\nu_4} \right], \quad (51) \]

where we also show explicitly the symmetry factor 1/2. Calculating the trace and contracting the indices we obtain the following result:

\[ A_{QQ}(q^2) = \sqrt{2N_c} \frac{2 f_N^2}{m_Q (m_Q^2 + q^2)^2} (\pi \alpha_s)^3 \frac{10 m_Q + E}{81 m_Q} J(q^2/m_Q^2), \quad (52) \]

where the dimensionless collinear integral reads

\[ J(v^2) = \frac{1}{32} \int \frac{Dx_i}{x_1 x_2 x_3} \int Dy_i \frac{1}{y_1 y_2 y_3} \left\{ \frac{A(x_i, y_i)}{D_1 D_3} + \frac{B(x_i, y_i)}{D_1 D_2} + \frac{C(x_i, y_i)}{D_2 D_3} \right\}. \quad (53) \]

The integrand in Eq.\,(53) has the following structure

\[ D_i = x_i + y_i - 2x_i y_i = x_i (1 - y_i) + y_i (1 - x_i) \geq 0. \quad (54) \]

The functions \( A, B \) and \( C \) also depend on \( v^2 = q^2/m_Q^2 \), which is not explicitly shown for simplicity. They can be presented in the following way

\[ A(x_{123}, y_{123}) = \sum_{i=0}^{4} A_k(x_{123}, y_{123}) I_k(1, 3), \quad (55) \]

\[ B(x_{123}, y_{123}) = \sum_{i=0}^{4} B_k(x_{123}, y_{123}) I_k(1, 2), \quad (56) \]

\[ C(x_{123}, y_{123}) = \sum_{i=0}^{4} C_k(x_{123}, y_{123}) I_k(2, 3). \quad (57) \]

These expressions include the angular integrals \( I_k \) which appears due to the integration over \( d\Omega = d\varphi \ d\cos \theta \). The polar integration can be easily computed and the remaining azimuth integrals read

\[ I_k[ij] = \frac{1}{2} \int_{-1}^{1} d\eta \frac{|v|^k \eta^k}{[1 + |v| a_i \eta] [1 - |v| a_j \eta]}, \quad (58) \]

where we used \( \eta \equiv \cos \theta, \ |v| = |q|/m_Q \) and convenient short notation

\[ a_i = \frac{m_Q x_i - y_i}{E D_i} < 1. \quad (59) \]

The numerator in (58) arises from the trace and contractions in Eq.\,(51). The denominators \( [1 + |v| a_i \eta] [1 - |v| a_j \eta] \) occur from the heavy quark propagators. The angular dependence occurs from the scalar products

\[ (qk) = -E_{q_k} = -E |q| \cos \theta, \quad (qk') = E_{q_k'} = E |q| \cos \theta. \quad (60) \]

From the definition (58) it follows that

\[ I_0 \sim \mathcal{O}(1), \quad I_{1, 2} \sim \mathcal{O}(v^2), \quad I_{3, 4} \sim \mathcal{O}(v^4). \quad (61) \]

\footnote{We used the package Feyncalc \cite{36}.}
If one neglects by the contribution form the denominators setting \( a_i = a_j = 0 \) then

\[
I_0[ij] = 1, \quad I_1[ij] = 0, \quad I_2[ij] = \frac{v^2}{3}, \quad I_3[ij] = 0, \quad I_4 = \frac{v^2}{5}.
\]  

(62)

One can also easily find that

\[
I_k[ij] = (-1)^k I_k[ij].
\]  

(63)

These integrals can be computed analytically that gives

\[
I_0[ij] = \frac{1}{2} \frac{1}{a_i + a_j} \left( \frac{1}{|a_i|} \ln \frac{1 + |a_i|}{1 - |a_i|} + \ln \frac{1 + |a_j|}{1 - |a_j|} \right)
\]

\[
= \frac{1}{a_i + a_j} \sum_{n=0}^{\infty} \frac{|a|^n}{n + 1} \left( a_j^{n+1} + (-1)^n a_i^{n+1} \right) \frac{1}{2} [1 + (-1)^n],
\]  

(64)

and for \( k > 0 \)

\[
I_k[ij] = \frac{1}{2} \frac{1}{a_i + a_j} \frac{(-1)^k}{a_i^{k-1}} \frac{1}{a_j^{k-1}} \left( \frac{1}{|a_i|} \ln \frac{1 + |a_i|}{1 - |a_i|} - \sum_{l=0}^{k-1} |a|^l a_i^{1-l} \right)
\]

\[
+ \frac{1}{2} \frac{1}{a_i + a_j} \frac{1}{a_i^{k-1}} \frac{1}{a_j^{k-1}} \left( \frac{1}{|a_i|} \ln \frac{1 + |a_i|}{1 - |a_i|} - \sum_{l=0}^{k-1} |a|^l a_j^{1-l} \right)
\]

\[
= \frac{|a|^k}{a_i + a_j} \sum_{n=0}^{\infty} \frac{|a|^n}{n + 1 + k} \left( a_j^{n+1} + (-1)^n a_i^{n+1} \right) \frac{1}{2} [1 + (-1)^{n+k}].
\]  

(65)

The convolution integral \( J \) in Eq.(53) is well defined that can be easily understood taking into account the general behaviour of the LCDAs

\[
X_k(x_{123}, y_{123}) = [X_k]_{VV} \{ V_1(x_{123}) V_1(y_{123}) + A_1(x_{123}) A_1(y_{123}) \}
\]

\[
+ [X_k]_{AV} \{ A_1(x_{123}) V_1(y_{123}) + V_1(x_{123}) A_1(y_{123}) \}
\]

\[
+ [X_k]_{TT} \{ T_1(x_{123}) T_1(y_{123}) \},
\]  

(68)

where coefficients \([X_k]_{VV,AV,TT}\) are polynomial functions of the momentum fractions \( x_i \) and \( y_i \). Their analytical expressions are presented in Appendix B

The convolution integral \( J \) in Eq.(53) is well defined that can be easily understood taking into account the general behaviour of the LCDAs

\[
Z(x_{123}) = x_1 x_2 x_3 \times \text{(polynomial function in } x_i), \quad Z = \{ V_1, A_1, T_1 \}.
\]  

(69)

Using the expression in Eq.(52) one finds the NRQCD amplitude \( A_1 \) defined in Eq.(8)

\[
(A_1)_{NRQCD} = \sqrt{2 M_\psi} \langle 0 | O | J/\psi \rangle \frac{f_N^2}{m_c^2 (1 + \langle v^2 \rangle)^2} \left( \pi \alpha_s \right)^3 \frac{10}{81} \left( 1 + \sqrt{1 + \langle v^2 \rangle} \right)^2 \frac{1}{2} J(\langle v^2 \rangle).
\]  

(70)

In the limit \( \langle v^2 \rangle \to 0 \) this result reproduces the well known leading-order approximation in Eq.(14). The obtained expression depends on the power of heavy quark mass \( m_c^{-6} \) as required by the scale properties of the amplitude. The charm mass in the factor \( \sqrt{2 M_\psi} \) is usually calculated as \( M_\psi \simeq 2 m_c \sqrt{1 + \langle v^2 \rangle} \). The factor \((1 + \langle v^2 \rangle)^{-2}\) is naturally provided by the hard propagators in the diagrams. The essential part of the relativistic corrections is included in the
convolution integral, which depends on the twist-3 LCDAs, which describe the nonperturbative overlaps with outgoing nucleons. Therefore the total effect of the relativistic corrections also depends on the nonperturbative structure of nucleon.

To see numerical impact of the obtained relativistic corrections we need to compute the convolution integral $J$. This integral is not simple and, in the general case, can only be calculated numerically. In order to understand better the dependence of the integral on the model of the LCDA we consider two different models. As an example of relatively simple case, consider the asymptotic LCDA, which is defined as

$$V_{1}^{as} = T_{1}^{as} = 120 \ f_{N} \ x_{1} x_{2} x_{3}, \quad A_{1}^{as} = 0.$$  \hfill (71)

Appropriate analysis allows better understanding numerical values of various convolution integrals. For a more realistic description we use ABO-model [37], which provides a reliable description of various data.

For our consideration, we use $m_{c} = 1.4 \text{ GeV}$ for the pole heavy quark mass and use the estimate for the binding energy $J/\psi$ from the Ref. [24]

$$E_{J/\psi} = 0.306 \text{ GeV},$$  \hfill (72)

that gives

$$\langle v^2 \rangle_{J/\psi} \simeq 0.225.$$  \hfill (73)

The value $\langle v^2 \rangle_{\psi(2S)} \equiv \langle v^2 \rangle_{\psi'}$ can be estimated as

$$\langle v^2 \rangle_{\psi'} = \frac{E_{\psi'} - M_{J/\psi} + E_{J/\psi}}{(M_{J/\psi} - E_{J/\psi})/2} = 0.64,$$  \hfill (74)

which is quite large due to the large difference $M_{\psi'} - M_{J/\psi} \simeq 589 \text{ MeV}$.

3.1 Relativistic corrections with asymptotic nucleon LCDA

The asymptotic DA (71) is not a realistic model, but the use of this approximation makes it possible to simplify analytical expressions and study the properties of convolution integrals. At first step we also calculate the correction of relative order $v^2$ only.

The results in Eqs. (64) and (67) yield

$$I_{0}[ij] = 1 + \langle v^2 \rangle \frac{1}{3} (a_{i}^{2} - a_{i} a_{j} + a_{j}^{2}) + \mathcal{O}(v^4) \simeq 1 + \langle v^2 \rangle I_{0}^{(1)}[ij],$$  \hfill (75)

$$I_{1}[ij] = \frac{\langle v^2 \rangle}{3} (a_{j} - a_{i}) + \mathcal{O}(v^4) \simeq I_{1}^{(1)}[ij],$$  \hfill (76)

$$I_{2}[ij] = \frac{\langle v^2 \rangle}{3} + \mathcal{O}(v^4), \quad I_{3,4} \sim \mathcal{O}(v^4),$$  \hfill (77)

where $a_{i}$ is defined in Eq.(59). Therefore the contributions with $I_{3,4}$ can be neglected. It is convenient to define

$$J_{as}(\langle v^2 \rangle) = \frac{120}{32} \sum_{k=0}^{4} J_{as}^{(k)}(\langle v^2 \rangle),$$  \hfill (78)
where the coefficient in front of the sum is chosen in order to get a convenient normalisation. In what follows we are going to discuss the values of the integrals

\[ J_{as}^{(k)}(v) = \frac{1}{120^2 f_N} \int Dx_i \int Dy_i \left\{ \frac{A_k^{as} I_k[13]}{D_1 D_3} + \frac{B_k^{as} I_k[12]}{D_1 D_2} + \frac{C_k^{as} I_k[23]}{D_2 D_3} \right\}, \] (79)

and their dependence on the parameter \( \langle v^2 \rangle \).

For the coefficients \( A_k^{as} \) we get

\[ A_k^{as} = \left[ A_k \right]_{VV} \{ V_1^{as}(x_{123}) V_1^{as}(y_{123}) \} + \left[ A_k \right]_{TT} \{ T_1^{as}(x_{123}) T_1^{as}(y_{123}) \} \] (80)

\[ = f_N^2 120^2 (x_1 x_2 x_3) (y_1 y_2 y_3) \left( \left[ A_k \right]_{VV} + \left[ A_k \right]_{TT} \right) = f_N^2 120^2 (x_1 x_2 x_3) (y_1 y_2 y_3) A_k^{as}, \] (81)

with

\[ \bar{A}_k^{as} = \left[ A_k \right]_{VV} + \left[ A_k \right]_{TT}, \] (82)

and similarly for \( B_k \) and \( C_k \). Substituting this in Eq. (79) we obtain

\[ J_{as}^{(k)}(v) \simeq \int Dx_i \int Dy_i \left\{ \frac{\bar{A}_k^{as} I_k[13]}{D_1 D_3} + \frac{\bar{B}_k^{as} I_k[12]}{D_1 D_2} + \frac{\bar{C}_k^{as} I_k[23]}{D_2 D_3} \right\}. \] (83)

Substituting the explicit expressions for the coefficients \( \bar{A}_k^{as}, \bar{B}_k^{as} \) and \( \bar{C}_k^{as} \) and neglecting the higher order terms we get

\[ \sum_{k=0}^{2} J_{as}^{(k)}(\langle v^2 \rangle) = \int Dx_i \int Dy_i \frac{x_1 y_2}{D_1 D_2} \left\{ 24 x_1 y_2 - 8 \langle v^2 \rangle + 24 \langle v^2 \rangle x_1 y_2 (I_{0}^{(1)}[12] + I_{1}^{(1)}[12]) + \langle v^2 \rangle \left( 2 x_1 y_2 - \frac{1}{6} (x_1 + x_2 + y_1 + y_2) \right) \right\}. \] (86)

The integrals, which enter in these formulae can be easily calculated numerically

\[ \text{LO: } \int Dx_i \int Dy_i \frac{x_1 y_2}{D_1 D_2} = 0.140. \] (87)

\[ \text{NLO: } 1^{st} = \int Dx_i \int Dy_i \frac{1}{D_1 D_2} = 2.029, \] (88)

\[ \text{NLO: } 2^{nd} = \int Dx_i \int Dy_i \frac{x_1 y_2}{D_1 D_2} (I_{0}^{(1)}[12] + I_{1}^{(1)}[12]) = 0.012, \] (89)

\[ \text{NLO: } 3^{rd} = \int Dx_i \int Dy_i \frac{1}{D_1 D_2} \left( 2 x_1 y_2 - \frac{1}{6} (x_1 + x_2 + y_1 + y_2) \right) = -0.121. \] (90)

This gives

\[ \sum J_{as}^{(k)}(\langle v^2 \rangle) = 24 \ast 0.140 - 8 \langle v^2 \rangle \ast (2.029)_{1^{st}} + 24 \langle v^2 \rangle (0.012)_{2^{nd}} + \langle v^2 \rangle (-0.121)_{3^{rd}} \] (91)

\[ = 3.36_{1^{st}} - 16.232 \langle v^2 \rangle_{1^{st}} + 0.167 \langle v^2 \rangle_{2^{nd}} + 0.36 - 16.40 \langle v^2 \rangle. \] (92)
Substituting the numerical values for $\langle v^2 \rangle_{J/\psi}$ and $\langle v^2 \rangle_{\psi'}$ [73, 74] we obtain

$$\sum J^{(k)}_{as}(\langle v^2 \rangle_{J/\psi}) = -0.33, \quad (93)$$
$$\sum J^{(k)}_{as}(\langle v^2 \rangle_{\psi'}) = -7.14. \quad (94)$$

These results show that the relativistic correction is negative and provides about 100% and 300% numerical effect for 1S and 2S-states, respectively. To better understand the origin of such a large numerical impact, consider the structure of the expression in the equation. [91]

From this result it follows that relativistic correction is dominated by the terms which appear from the integral [88]. This integral is about an order of magnitude larger than the leading-order integral [87]. This has the natural mathematical explanation: the both integrands in Eqs.(87) and (88) are positive, but the leading-order integrand in (87) has in the numerator small factor $x_1y_2$ that reduces the value of the integral [87] comparing to one in [88]. The actual values of the NRQCD parameter $\langle v^2 \rangle$ for charmonium states can not compensate the numerical enhancement of the subleading integral. This indicates about the relative smallness of the charm quark mass.

The mechanism of the enhancement of the subleading integral does not depend on the model of LCDA of the nucleon. The structure of leading- and subleading integrals is closely related to the structure of the numerators of the Feynman diagrams and can be interpreted as a specific feature of the hard perturbative subprocess. In order to see this consider the leading-order approximation with $q = 0$, then the trace in Eq.(51) reads

$$\text{Tr} \left[ \Pi^{(lo)}_1 \Gamma_Q^{\mu'_i \mu'_j}(x_i, y_i) \right] \sim \text{Tr} \left[ (\phi + 1) \gamma^c \gamma^\perp (\not{P}/2 + m_Q + k_i + k'_j) \gamma^\perp \right. \left. \times (\not{P}/2 + k_j + k'_j + m_Q) \gamma^j \right] \quad (95)$$

where we used expression from Eq.(40) and

$$\Pi^{(lo)}_1 = \frac{-1}{2\sqrt{2}} (\phi + 1) \gamma^c. \quad (96)$$

Notice that all Dirac matrices with open Lorentz indices in Eq.(95) are transvers, because of contraction with transverse tensor $T_{\mu_1 \mu_2 \rho_3 \rho_4}$ in Eq.(47). Remind, that the heavy quark velocity

$$\omega^\mu = \delta^{\mu 0} = \frac{1}{2}(n^\mu + n^\mu). \quad (97)$$

If one neglects collinear momenta $k_i$, $k'_j$ or $k_j$, $k'_j$ in the trace Eq.(95) then the trace vanishes, for instance

$$\text{Tr} \left[ (\phi + 1) \gamma^c \gamma^\perp (\not{P}/2 + m_Q) \gamma^\perp \gamma^\perp \right] = m_Q \text{Tr} \left[ (\phi + 1) \gamma^c \gamma^\perp (\not{P}/2 + m_Q) \gamma^\perp \gamma^\perp \right] = m_Q \text{Tr} \left[ (\phi + 1) \gamma^c \gamma^\perp \gamma^\perp \right] = 0, \quad (98, 99, 100)$$

where we used $[\phi, \gamma^\perp] = [\phi, \gamma^c] = 0$. Therefore the nontrivial result is obtained only from the term with collinear momenta in both quark propagators

$$\text{Tr} \left[ \Pi^{(lo)}_1 \Gamma_Q^{\mu'_i \mu'_j}(x_i, y_i) \right] \sim \text{Tr} \left[ (\phi + 1) \gamma^c \gamma^\perp (k_i + k'_j) \gamma^\perp \right. \left. \times (k_j + k'_j) \gamma^j \right] \sim x_iy_j + x_jy_i. \quad (101)$$
This explains the structure of the leading-order integral in Eq. (87).

The calculation of the subleading correction involves the relative momentum $q$ that introduces the traces with two insertions of $q$ instead of collinear momenta, for instance

$$\text{Tr} \left[ \Pi Q^{\mu_1 \nu_1} \gamma_\perp \mu_2 \gamma_\perp \nu_2 \right] \sim \text{Tr} \left[ (\psi + 1) f_\psi \gamma_\perp \mu_1 \gamma_\perp \nu_1 \right] + \ldots . \quad (102)$$

Such subleading contributions give terms of relative order $v^2$ but the corresponding integral has no momentum fractions in the numerator, see Eq. (88). As a result, the large numerical contributions are generated by the integrals $J_{as}^{(0)}$ and $J_{as}^{(2)}$ only, which can only get appropriate contributions from the trace and contractions in Eq. (51). Parametrically such integrals are suppressed by small velocity, however, in reality such suppression is not sufficiently strong in order to provide a small numerical correction with respect to the leading-order contribution.

The enhanced integral also takes place in the relative order $v^4$, but the corresponding numerical effect is already suppressed by additional factor $\langle v^2 \rangle$ and for $J/\psi$ such a contribution can be estimated at about $20 - 30\%$. But for the excited state such a contribution still remains quite large because of large value of $\langle v^2 \rangle$.

In order to study the more general situation and the possible numerical effects of higher order terms in $v^2$, we present in Table 1 the results for the integrals $J_{as}^{(k)}$, which are calculated with the resummed relativistic corrections. Remind, that the leading-order approximation is given by the integral $J_{as}^{(0)}(\langle v^2 \rangle = 0) = 3.36$.

Table 1: Numerical results for integrals $J_{as}^{(k)}$ for $J/\psi$ and $\psi'$, upper and bottom tables, respectively.

| $\langle v^2 \rangle_{J/\psi}$ | $J_{as}^{(0)}$ | $J_{as}^{(1)}$ | $J_{as}^{(2)}$ | $J_{as}^{(3)}$ | $J_{as}^{(4)}$ | $\sum J_{as}^{(k)}$ |
|-----------------------------|----------------|----------------|----------------|----------------|----------------|----------------|
| NLO                         | -1.84          | -0.18          | 1.89           | -              | -              | -0.13          |
| all orders                  | -1.00          | -0.19          | 1.66           | -0.01          | 0.13           | 0.59           |

| $\langle v^2 \rangle_{\psi'}$ | $J_{as}^{(0)}$ | $J_{as}^{(1)}$ | $J_{as}^{(2)}$ | $J_{as}^{(3)}$ | $J_{as}^{(4)}$ | $\sum J_{as}^{(k)}$ |
|------------------------------|----------------|----------------|----------------|----------------|----------------|----------------|
| NLO                          | -11.68         | -0.50          | 5.37           | -              | -              | -6.81          |
| all orders                   | -6.78          | -0.69          | 4.53           | -0.14          | 0.98           | -2.10          |

The largest numerical effects are provided by the NLO corrections in integrals $J_{as}^{(0,2)}$ for the same reason as discussed above. The numerical effect from the the resummation is most visible for $J_{as}^{(0)}$ but corresponding effects are not sufficiently large and do not change the qualitative picture discussed for NLO approximation. Therefore we can conclude that relativistic expansion for $J/\psi$ is well convergent. The large numerical impact is only generated by the correction of relative order $v^2$ and associated with the numerical enhancement of the NLO convolution integral.

In the Table 1 (bottom part) we also show the results for the 2S-state, which has much larger relativistic corrections. In this case the general structure remains quite similar: the largest numerical effect is provided by the NLO contribution in $J_{as}^{(0,2)}$ but the higher order power corrections also make a significant numerical contribution, reducing the total sum by a factor of three. This clearly illustrates the expected conclusion: the relativistic expansion for excited charmonium has very large relativistic corrections and converges rather slowly.
3.2 Relativistic corrections with realistic nucleon LCDA

The value of the collinear integral also depends on the model of the LCDA $\varphi_3$, which describes a distribution of the quark longitudinal momenta at zero transverse separation. To illustrate the possible effect of the nucleon structure, we calculate in this section the convolution integrals with a realistic LCDA model. For that purpose we use the ABOI-model from Ref. [37]. This model gives a reliable description of the electromagnetic form factor data within the light-cone sum rules [38] and also provides a good description of $J/\psi \rightarrow p\bar{p}$ decay data in the leading-order approximation [21]. The expression for the model reads

$$\varphi^{\text{ABO}}_3(x_i) = 120 x_1 x_2 x_3 \{ 1 + \varphi_{10} P_{10}(x_i) + \varphi_{11} P_{11}(x_i) + \varphi_{20} P_{20}(x_i) + \varphi_{21} P_{21}(x_i) + \varphi_{22} P_{22}(x_i) \}, \quad (103)$$

where the orthogonal polynomials $P_{ij}(x_i)$ read

$$P_{10}(x_i) = 21(x_1 - x_3), \quad P_{11}(x_i) = 7(x_1 - 2x_2 + x_3), \quad (104)$$

$$P_{20}(x_i) = \frac{63}{10} \left[ 3(x_1 - x_3)^2 - 3x_2(x_1 + x_3) + 2x_2^2 \right], \quad (105)$$

$$P_{21}(x_i) = \frac{63}{2} (x_1 - 3x_2 + x_3)(x_1 - x_3), \quad (106)$$

$$P_{22}(x_i) = \frac{9}{5} \left[ x_1^2 + 9x_2(x_1 + x_3) - 12x_1x_3 - 6x_2^2 + x_3^2 \right]. \quad (107)$$

The moments $\varphi_{ij} \equiv \varphi_{ij}(\mu)$ are multiplicatively renormalisable, more details about properties of the polynomials $P_{ij}$ and about the evolution of the moments can be found in Ref. [39]. In Appendix A we also provide some useful details. In our calculation we fix the relatively low normalisation scale $\mu^2 = 1.5$ GeV following to Ref. [21]. Then the values of the twist-3 moments read

$$\varphi_{10} = 0.051, \quad \varphi_{11} = 0.052, \quad \varphi_{20} = 0.078, \quad \varphi_{21} = -0.028, \quad \varphi_{22} = 0.179. \quad (108)$$

Again, we define the integral as the sum

$$J^{(k)}_{\text{ABO}}(\langle v^2 \rangle) = \frac{120^2}{32} \sum_{k=0}^{4} J^{(k)}_{\text{ABO}}(\langle v^2 \rangle), \quad (109)$$

where the integrals $J^{(k)}_{\text{ABO}}$ are defined as

$$J^{(k)}_{\text{ABO}}(\langle v^2 \rangle) = \frac{1}{120^2 f_N^2} \int \frac{Dx_i}{x_1 x_2 x_3} \int \frac{Dy_i}{y_1 y_2 y_3} \left\{ \frac{A_k^{\text{ABO}} I_{k}[13]}{D_1 D_3} + \frac{B_k^{\text{ABO}} I_{k}[12]}{D_1 D_2} + \frac{C_k^{\text{ABO}} I_{k}[23]}{D_2 D_3} \right\}. \quad (110)$$

The value of the leading order integral reads

$$J^{(0)}_{\text{ABO}}(\langle v^2 \rangle = 0) = 5.08. \quad (111)$$

The numerical results for the different integrals $J^{(k)}_{\text{ABO}}$ for $J/\psi$ and $\psi'$ are presented in Table 2.

The qualitative picture remains the same as described above, the sum of the integrals $J^{(0)}_{\text{ABO}} + J^{(2)}_{\text{ABO}}$ provides the largest numerical impact but the values of the all integrals are larger. We can conclude that the described mechanism of the large numerical effect also works for the
Table 2: Numerical results for the integrals $J_{ABO}^{(k)}$ for $J/\psi$ and $\psi'$, upper and bottom tables, respectively.

|       | $\langle v^2 \rangle_{J/\psi}$ = 0.225 | $J_{ABO}^{(0)}$ | $J_{ABO}^{(1)}$ | $J_{ABO}^{(2)}$ | $J_{ABO}^{(3)}$ | $\sum J_{ABO}^{(k)}$ |
|-------|-----------------------------------|----------------|----------------|----------------|----------------|------------------|
| NLO   | -5.18                             | -0.27          | 3.66           | -              | -              | -1.79            |
| all orders | -3.65                         | -0.35          | 3.30           | -0.03          | 0.22           | -0.51            |

|       | $\langle v^2 \rangle_{\psi'}$ = 0.64 | $J_{ABO}^{(0)}$ | $J_{ABO}^{(1)}$ | $J_{ABO}^{(2)}$ | $J_{ABO}^{(3)}$ | $\sum J_{ABO}^{(k)}$ |
|-------|-----------------------------------|----------------|----------------|----------------|----------------|------------------|
| NLO   | -24.09                            | -0.75          | 10.42          | -              | -              | -14.42           |
| all orders | -15.69                        | -1.39          | 9.31           | -0.30          | 1.78           | -6.29            |

realistic model of LCDAs. In case of $J/\psi$ the total sum is negative, which indicates that the negative relativistic correction is somewhat larger than the LO contribution.

Additional numerical impact is also provided by $\langle v^2 \rangle$-dependence of the coefficient in front of the convolution integral in Eq. (70). The dominant numerical effect is provided by the factor $1/(1 + \langle v^2 \rangle)^2$, which occurs from the hard propagators. It seems that this factor can be understood as an indication that the charmonium mass $M_{\psi}^2 \approx 4m_c^2(1 + \langle v^2 \rangle)$ is a more natural scale for the hard gluon propagators in the Feynman diagrams. The corresponding numerical effect is smaller in comparison with one in the convolution integral discussed above. For $J/\psi$ the modification of the coefficient in Eq. (70) gives a reduction about 37%. For $\psi'$, a similar effect is already 67%.

4 Discussion

We presented the first study of relativistic corrections in exclusive $S$-wave charmonium decays into proton-antiproton final state. The relativistic corrections are calculated within the NRQCD and collinear factorisation frameworks. We only consider the helicity conserving amplitude $A_1$, which provides the dominant numerical contribution to the decay width. In this case the baryon non-perturbative light-cone matrix elements depend on the twist-3 LCDAs only and the formula for the amplitude with relativistic corrections includes the collinear convolution integral, which is free from any infrared singularities.

The result obtained provides a full correction of the relative order $v^2$, and also includes the summation of all orders of relativistic corrections associated with the quark-antiquark-quarkonia wave function in the potential model [30].

The largest numerical impact is provided by the NLO relativistic correction with the matrix element, which can be computed using equation of motion and is proportional to the binding energy. Due to a specific structure of the LO and NLO hard scattering kernels the value of the NLO collinear integral is about an order of magnitude larger than the value of LO one. The charmonium parameter $\langle v^2 \rangle$ is not small enough to compensate for this effect, and the magnitude of the resulting relativistic correction is numerically of the same order as the LO contribution. For the case of $J/\psi$ the resulting NLO correction is negative and therefore, the two contributions almost cancel out. The calculated higher order corrections are relatively small and their resummation shows that relativistic expansion converges sufficiently well. The numerical effect also depends on the model of the nucleon LCDA however this dependence does not change the main qualitative conclusions about the large relativistic corrections.

A description of excited state $\psi'$ is more challenging because of relatively large value of
\( \langle \psi^2 \rangle_{\psi'} \). In this case the numerical effect from relativistic corrections is much larger than the LO contribution and is more sensitive to the higher order contributions. Therefore a description of \( \psi' \) baryonic decays suffers from large uncertainties, which are associated with the higher-order contributions of the relativistic expansions.

Taking into account these large relativistic corrections, one cannot expect the 12% rule in Eq. (1) to hold in the general case. The rather good agreement for the nucleon channel is probably an accidental consequence of different numerical cancellations.

The large effect from relativistic corrections raises the question about a phenomenological description of baryon decays in the effective field theory framework. Various existing phenomenological estimates for \( J/\psi \) decay width and angular behaviour are based on the LO approximation and provide a qualitative reliable estimates \[21\]. However such qualitative picture is violated by the large value of relativistic corrections. The large negative correction almost cancel the LO contribution, which greatly reduces the amplitude value and, therefore, makes the description problematic. A possible solution of this problem might be associated with the large and positive NLO radiative correction, which is not yet known. Cancellation of radiative and relativistic corrections could resolve the situation. The large NLO radiative corrections are already observed in the exclusive production \( e^+ e^- \rightarrow J/\psi \eta_c \), see e.g. Ref. \[40\]. Probably a similar situation also arises in the baryonic decays. Therefore, the calculation of the NLO radiative correction is necessary in order to better understand the hadronic decay dynamics.

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**Appendix**

**A Definition and properties of nucleon LCDA**

In this Appendix we provide definition of the light-cone matrix element for proton state. The formulae for the antiproton state can be obtained by charge conjugation. In order to simplify formulas, we also use the light-cone gauge

\[
 n \cdot A(x) = 0.
\]  

(A.1)

In this case the light-cone three-quark operator which we need for the proton state can be defined as \((i,j,k \text{ are the colour indices})\)

\[
 O_{tw3} = \varepsilon^{ijk} \xi^i(z_{1-}) \xi^j(z_{2-}) \xi^k(z_{3-}),
\]  

(A.2)

where the projected quark field \( \xi(z) \equiv \frac{d}{dz} \bar{\psi}(z) \), the arguments of the fields read

\[
 z_{i-} = (\bar{n} \cdot z_i) \frac{n^2}{2}.
\]  

(A.3)

The proton flavour structure implies that \( O_{tw3} = u u d \).
The twist-3 light-cone matrix element is defined as
\[
\langle 0 | O_{\text{tw3}}(z_1, z_2, z_3) | p(k) \rangle = \frac{1}{4} \left[ \gamma_i C_{\alpha\beta} [\gamma_5 N_n]_{\sigma} \ FT [V_1(y_i)] + \frac{1}{4} \left[ k^\gamma C_{\alpha\beta} [N_{\sigma}]_{\gamma} + \frac{1}{4} k^\gamma [i\sigma_{\mu\nu} C_{\alpha\beta} [\gamma_5 N_n]_{\sigma} \ FT [T_1(y_i)] \right] \right],
\] (A.4)
where the projected nucleon spinor reads
\[
N_n = \frac{i\beta_N}{4} N(k),
\] (A.5)
and \( C \) is charge conjugation matrix. The Fourier transformation “FT” is defined as
\[
\text{FT} [F(y_i)] = \int D y_i \ e^{-iy_1 k - z_1 / 2 - iy_2 k - z_2 / 2 - iy_3 k - z_3 / 2} F(y_1, y_2, y_3),
\] (A.6)
with the integration measure
\[
D y_i = dy_1 dy_2 dy_3 \delta(1 - y_1 - y_2 - y_3).
\] (A.7)

Three LCDAs \( V_1, A_1 \) and \( T_1 \) satisfy the following properties
\[
V_1(y_2, y_1, y_3) = V_1(y_1, y_2, y_3),
\] (A.8)
\[
A_1(y_2, y_1, y_3) = -A_1(y_1, y_2, y_3),
\] (A.9)
\[
T_1(y_2, y_1, y_3) = T_1(y_1, y_2, y_3).
\] (A.10)
The isospin symmetry allows one to get the following relation, see details in Refs. [32,41]
\[
T_1(y_1, y_2, y_3) = \frac{1}{2} (V_1 - A_1) (y_1, y_3, y_2) + \frac{1}{2} (V_1 - A_1) (y_2, y_3, y_1).
\] (A.11)
It is convenient to define the following combination
\[
f_N \varphi_3(y_1, y_2, y_3) = V_1(y_1, y_2, y_3) - A_1(y_1, y_2, y_3),
\] (A.12)
which allows one to describe the set of three LCDAs \( V_1, A_1 \) and \( T_1 \) in terms of the one function. The coupling \( f_N \) describes the normalisation so that \( \varphi_3 \) is dimensionless and normalised as
\[
\int D y_i \ \varphi_3(y_1, y_2, y_3) = 1.
\] (A.13)

The LCDA \( \varphi_3 \) also depends on the factorisation scale, which is not shown for simplicity
\[
\varphi_3(y_1, y_2, y_3) \equiv \varphi_3(y_1, y_2, y_3; \mu).
\] (A.14)
The evolution properties of this LCDA were studied in Ref. [39]. The moments in Eq.(103) are multiplicatively renormalisable and can be calculated as
\[
\phi_{ij}(\mu) = \phi_{ij}(\mu_0) \left( \frac{\alpha_s(\mu)}{\alpha_s(\mu_0)} \right)^{\gamma_{ij}/\beta_0},
\] (A.15)
where \( \beta_0 = 11 - 2n_f/3 \) and \( \gamma_{ij} \) are the corresponding anomalous dimensions
\[
\gamma_{10} = \frac{20}{9}, \ \gamma_{11} = \frac{8}{3}, \ \gamma_{20} = \frac{32}{9}, \ \gamma_{21} = \frac{40}{9}, \ \gamma_{22} = \frac{14}{3}.
\] (A.16)
B Analytical expressions for the coefficients $A_k$, $B_k$ and $C_k$ defined in Eqs. (55)-(57)

Here we provide the explicit expressions for the coefficients $[X_k]_{VV,AV,TT}$ defined in Eq. (68). These coefficients are functions of the momentum fractions

$$[X_k]_{VV,AV,TT} \equiv [X_k]_{VV,AV,TT}(x_{123},y_{123}).$$  \hfill (B.1)

Below we also use following short notation

$$\bar{m} = \frac{m_c}{E} = \frac{1}{\sqrt{1 + \nu^2}}, \quad C_{ij} = x_i y_j + x_j y_i. \hfill (B.2)$$

The coefficients $C_k$ can be obtained using the simple relations

$$[C_k]_{VV,TT}(x_{123},y_{123}) = [A_k]_{VV,TT}(x_{213},y_{213}), \hfill (B.3)$$

$$[C_k]_{AV}(x_{123},y_{123}) = - [A_k]_{AV}(x_{213},y_{213}). \hfill (B.4)$$

The other coefficients read

$$[A_0]_{VV} = 2C_{13} - (1 - \bar{m})(x_3 + y_3) - 2\bar{m}(1 - \bar{m}), \hfill (B.5)$$

$$[A_0]_{AV} = \frac{2}{1 + \bar{m}}(-2C_{13} + (1 - \bar{m}^2)(x_1 - x_2 + y_1 - y_2)), \hfill (B.6)$$

$$[A_0]_{TT} = -4(1 - \bar{m})(\bar{m} + 1 - x_3 - y_3). \hfill (B.7)$$

$$[B_0]_{VV} = -(1 - \bar{m})(2\bar{m} + x_3 + y_3), \hfill (B.8)$$

$$[B_0]_{AV} = 2(1 - \bar{m})(x_1 - x_2 + y_1 - y_2), \hfill (B.9)$$

$$[B_0]_{TT} = 4(2C_{12} + (1 - \bar{m})(x_3 + y_3) + \bar{m}^2 - 1). \hfill (B.10)$$

$$[A_1]_{VV} = \frac{\bar{m}}{1 + \bar{m}}(4(x_1 y_3 - x_3 y_1) + (1 - \bar{m})(x_3 - x_1 + y_1 - y_3)), \hfill (B.11)$$

$$[A_1]_{AV} = - [A_1]_{VV}, \hfill (B.12)$$

$$[A_1]_{TT} = 0. \hfill (B.13)$$

$$[B_1]_{VV} = 0, \hfill (B.14)$$

$$[B_1]_{AV} = 0, \hfill (B.15)$$

$$[B_1]_{TT} = \frac{4\bar{m}}{1 + \bar{m}}(4(x_1 y_2 - x_2 y_1) + (1 - \bar{m})(x_2 - x_1 + y_1 - y_2)). \hfill (B.16)$$

$$[A_2]_{VV} = \frac{\bar{m}^2}{(1 + \bar{m})^2}(2C_{13} + (1 + \bar{m})(2 + x_3 + y_3) - 3(1 - \bar{m}^2)), \hfill (B.17)$$

$$[A_2]_{AV} = - \frac{\bar{m}^2}{1 + \bar{m}}(4(x_1 + y_1 - 1) + 2(x_3 + y_3) + \bar{m} - 1), \hfill (B.18)$$

$$[A_2]_{TT} = \frac{4\bar{m}^2}{1 + \bar{m}}(1 - x_3 - y_3 + \bar{m}). \hfill (B.19)$$

$$[B_2]_{VV} = \frac{\bar{m}^2}{1 + \bar{m}}(x_3 + y_3 + 2\bar{m}), \hfill (B.20)$$

$$[B_2]_{AV} = \frac{2\bar{m}^2}{1 + \bar{m}}(x_2 - x_1 + y_2 - y_1), \hfill (B.21)$$

$$[B_2]_{TT} = \frac{4\bar{m}^2}{(1 + \bar{m})^2}(2C_{12} + (1 - \bar{m})(2 - x_3 - y_3) - 2(1 - \bar{m}^2)). \hfill (B.22)$$
\[ [A_3]_{VV} = \frac{\bar{m}^3}{(1 + \bar{m})^2}(2(x_1 - y_1) + y_3 - x_3), \]  
\[ [A_3]_{AV} = 0, \]  
\[ [A_3]_{TT} = -\frac{4\bar{m}^3}{(1 + \bar{m})^2}(x_3 - y_3). \]  
\[ [B_3]_{VV} = \frac{\bar{m}^3}{(1 + \bar{m})^2}(x_1 - x_2 - y_1 + y_2), \]  
\[ [B_3]_{AV} = 0, \]  
\[ [B_3]_{TT} = \frac{4\bar{m}^3}{(1 + \bar{m})^2}(x_1 - x_2 - y_1 + y_2). \]

\[ [A_4]_{VV} = [B_4]_{VV} = \frac{2\bar{m}^4}{(1 + \bar{m})^2}, \]  
\[ [A_4]_{AV} = [B_4]_{AV} = 0, \]  
\[ [A_4]_{TT} = [B_4]_{TT} = \frac{4\bar{m}^4}{(1 + \bar{m})^2}. \]

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