SINGULAR MINIMAL TRANSLATION SURFACES IN EUCLIDEAN SPACES ENDOWED WITH SEMI-SYMMETRIC CONNECTIONS

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Abstract. In this paper, we study and classify singular minimal translation surfaces in a Euclidean space of dimension 3 endowed with a certain semi-symmetric (non-)metric connection.

1. INTRODUCTION

Let $\mathbb{R}^3$ denote a Euclidean space of dimension 3 with the canonical metric $\langle \cdot, \cdot \rangle$ and $(x, y, z)$ the rectangular coordinates of $\mathbb{R}^3$. A surface $M$ in $\mathbb{R}^3$ is called translation surface if it can be written as the sum of two curves. Then, a local parametrization $\sigma$ on $M$ follows $\sigma(s_1, s_2) = \gamma_1(s_1) + \gamma_2(s_2)$, for $\gamma_i : I_i \subset \mathbb{R} \to \mathbb{R}^3$, $i = 1, 2$ [8]. In the special case that the curves $\gamma_i$ lie in orthogonal planes, up to a change of coordinates, the surface $M$ can be locally expressed by the explicit form $z = f(x) + g(y)$ for smooth functions $f, g$ of single variable. If such a surface is minimal, i.e. its mean curvature vanishes identically, then it is either a plane or describes the Scherk surface [48].

While this sort of surfaces has been studied in classical manner since the former half of nineteenth century (see [4], [12]-[14], [17], [25]-[33], [36], [41], [48]-[55], [58], [59]), their complete classification/characterization in $\mathbb{R}^3$ imposing natural curvature conditions (e.g. minimality, flatness and having nonzero constant mean or Gaussian curvatures) have been recently found, see [20]-[22]. Yet, in higher dimensions and different ambient spaces, there are still numerous unsolved problems.

On the other hand, a semi-symmetric metric (resp. non-metric) connection on a Riemannian manifold were defined by Hayden [23] (resp. Agashe [1]) and since then has been studied by many authors. For example, see [2], [3], [7], [9], [15], [18], [24], [42]-[45], [56], [57], [60].

As can be seen from the great number of published studies, the notions of translation surfaces and semi-symmetric (non-) metric connection have great interest and very recently Wang [53] combined these separate research areas into one, which came up with a new perspective. In the cited paper, the author introduced and obtained minimal translation surfaces in 3-dimensional space forms endowed with a certain semi-symmetric (non-)metric connection.

In this study we mainly concern with singular minimal surfaces in $\mathbb{R}^3$, namely those surfaces satisfying an equation of mean curvature type (see [11]). The notion of singular minimal surface is a generalization of two-dimensional analogue of the
catenary which is known as a model for the surfaces with the lowest gravity center, in other words, one has minimal potential energy [6]. In this context, the present study aims to contribute to Wang’s perspective by considering singular minimal translation surfaces in $\mathbb{R}^3$ endowed with a certain semi-symmetric (non-)metric connection.

In order to explicitly initiate the notion of singular minimality, we begin with the one-dimensional case: let $\gamma : I \subset \mathbb{R} \to \mathbb{R}^2$ be a parametrized curve and $u \in \mathbb{R}^2$ a fixed unit vector and $\alpha$ some real constant. Then the curve $\gamma$ is called $\alpha$-catenary (see [11]) if the following holds

\begin{equation}
\kappa(s) = \alpha \frac{(\mathbf{n}(s), u)}{(\mathbf{\gamma}(s), u)},
\end{equation}

where $\kappa$ and $\mathbf{n}$ are the curvature and principle unit normal vector field of $\gamma$. Up to a change of coordinates, one can assumed as $u = (0, 1)$ and $\gamma$ a graph locally given by $\gamma(s) = (s, f(s))$, for $f : I \subset \mathbb{R} \to \mathbb{R}^+$. Thereby, for $\alpha = 1$, Eq. (1.1) writes

\begin{equation}
\frac{f''}{1 + (f')^2} = \frac{1}{f},
\end{equation}

for each $s \in I$. The solution of Eq. (1.2) is the catenary $f(s) = \frac{1}{\lambda} \cosh(\lambda s + \mu)$, $\lambda, \mu \in \mathbb{R}$, $\lambda \neq 0$, see [39].

As its two-dimensional analogue, Eq. (1.2) has a remarkable a physical point of view, which can be initiated as follows: let the direction of gravity be chosen as $y-$axis. Then Eq. (1.2) defines a configuration in which a uniform chain, whose two ends are fixed and hanged under its own weight, is in balance with the effect of the gravitational field. So, a catenary actually minimizes potential energy under the influence of gravity force, in other words has the lowest center of gravity (e.g. [16]).

Let us now consider the smooth immersion $\sigma : M \to \mathbb{R}_+^3(u)$ of an oriented surface $M$ in the halfspace

\[ \mathbb{R}_+^3(u) = \{p \in \mathbb{R}^3, \langle p, u \rangle > 0\}, \]

for a fixed unit vector $u \in \mathbb{R}^3$. Then, the potential $\alpha-$energy of $\sigma$ in the direction of $u$ is defined by ([37, 38])

\begin{equation}
E(\varphi) = \int_M \langle \sigma(q), u \rangle^\alpha dM, \quad q \in M,
\end{equation}

where $dM$ refers to the measure on $M$ with respect to the induced metric tensor from $\mathbb{R}^3$. If $\sigma$ is a critical point of Eq. (1.3), it then follows

\begin{equation}
2H = \alpha \frac{\langle \xi, u \rangle}{\langle \sigma, u \rangle},
\end{equation}

where $H$ and $\xi$ are the mean curvature and unit normal vector field on $M$. A surface in $\mathbb{R}^3$ fulfilling Eq. (1.4) is called singular minimal surface or $\alpha-$minimal surface see [10], [11].

Eq. (1.4) is clearly an equation of mean curvature type and reduces to the classical minimal surface equation when $\alpha = 0$ [34, p. 17]. If we take $u = (0, 0, 1)$ and $\alpha = 1$ in Eq. (1.4), then the surface $M$ is said to be two-dimensional analogue of the catenary [6].

López [37] proved that a singular minimal translation surface in $\mathbb{R}^3$ with respect to a horizontal or a vertical direction is a $\alpha-$catenary cylinder, a generalized
cylinder (see [19, p. 439]) whose the base curve is a \( \alpha \)-catenary. This result was generalized to higher dimensions by the present authors [5].

Noting that the mean curvature \( H \) in Eq. (1.4) is given with respect to the Levi-Civita connection on \( \mathbb{R}^3 \), we modify it by considering the mean curvatures arising via special semi-symmetric metric and non-metric connections \( \nabla \) and \( D \) given by Eqs. (2.1) and (2.2). We call the modified concepts \( \nabla^{-} \) and \( D^{-} \) singular minimality and these allow us non-trivial and new problems. One of the problems is to find \( \nabla^{-} \) and \( D^{-} \) singular minimal translation surfaces with respect to a horizontal direction and we solve it completely.

2. PRELIMINARIES

Let \( (\tilde{M}, g) \) be a Riemannian manifold of dimension 3 and \( \tilde{\nabla} \) an affine connection on \( \tilde{M} \). Let us denote the set of sections of a vector bundle \( E \rightarrow \tilde{M} \) by \( \Gamma (E) \) and the set of tensor fields of type \( (r, s) \) on \( \tilde{M} \) by \( \Gamma (T\tilde{M}(r,s)) \). Then the torsion tensor field \( T \in \Gamma (T\tilde{M}^{(1,2)}) \) of \( \tilde{\nabla} \) is defined by

\[
T(X, Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y],
\]

for \( X, Y \in \Gamma (T\tilde{M}) \). Then the connection \( \tilde{\nabla} \) is called (see [42],[57])

(1) a (resp. non-) symmetric connection if \( T \) (resp. does not) vanishes identically;

(2) a (resp. non-) metric connection if \( g \) is (resp. not) parallel;

(3) a semi-symmetric connection if the following holds

\[
T(X, Y) = \pi(X) Y - \pi(Y) X,
\]

for \( \pi(X) = g(X, W), \pi \in \Gamma (T\tilde{M}^{(0,1)}), W \in \Gamma (T\tilde{M}); \)

(4) a Levi-Civita connection if it is both symmetric and metric.

Let \( \tilde{M} = \mathbb{R}^3 \) and \( \{ \partial_x, \partial_y, \partial_z \} \) the standard basis on \( \mathbb{R}^3 \). Consider the certain semi-symmetric metric and non-metric connections, respectively [53]

\[
(2.1) \quad \nabla_X Y = \nabla^L_X Y + g(Y, \partial_z) X - g(X, Y) \partial_z
\]

and

\[
(2.2) \quad D_X Y = \nabla^L_X Y + g(Y, \partial_z) X,
\]

where \( \nabla^L \) is the Levi-Civita connection on \( \mathbb{R}^3 \) and \( X, Y \in \Gamma (T\mathbb{R}^3) \). For Eqs. (2.1) and (2.2), the nonzero derivatives are given by

\[
\nabla_{\partial_x} \partial_z = -\partial_z, \quad \nabla_{\partial_x} \partial_y = \partial_x, \quad \nabla_{\partial_y} \partial_z = \partial_y,
\]

and

\[
D_{\partial_x} \partial_z = \partial_x, \quad D_{\partial_y} \partial_z = \partial_y, \quad D_{\partial_z} \partial_z = \partial_z.
\]

Let \( M \) be an oriented immersed surface into \( \mathbb{R}^3 \). For any \( X, Y \in \Gamma (T\mathbb{R}^3) \) and \( \xi \in \Gamma (T\mathbb{R}^{3^+}) \), the Gauss formulae with respect to \( \nabla \) and \( D \) follow

\[
\nabla_X Y = (\nabla_X Y)^T + h^\nabla (X, Y) \xi,
\]

and

\[
D_X Y = (\nabla_X Y)^T + h^D (X, Y) \xi,
\]
where $\tau$ is the projection on the tangent bundle of $M$ and $h^\nabla$ and $h^D$ are so-called the second fundamental forms with respect to $\nabla$ and $D$, respectively. Let $\{e_1, e_2\}$ be an orthonormal tangent frame on $M$. Then the mean curvatures of $M$ with respect to $\nabla$ and $D$ are defined by

$$H^\nabla = \frac{1}{2} \left[ h^\nabla(e_1, e_1) + h^\nabla(e_2, e_2) \right]$$

and

$$H^D = \frac{1}{2} \left[ h^D(e_1, e_1) + h^D(e_2, e_2) \right].$$

The surface $M$ is said to be minimal with respect to $\nabla$ (resp. $D$) if $H^\nabla$ (resp. $H^D$) vanishes, respectively.

Let $g_{ij}, 1 \leq i, j \leq 2$, denote the components of the induced metric tensor on $M$ from the canonical metric. Then the mean curvatures $H^\nabla$ and $H^D$ are respectively given by

$$H^\nabla = \frac{g_{22}h^\nabla_{11} - g_{12}(h^\nabla_{12} + h^\nabla_{21}) + g_{11}h^\nabla_{22}}{2 \det g_{ij}}$$

and

$$H^D = \frac{g_{22}h^D_{11} - g_{12}(h^D_{12} + h^D_{21}) + g_{11}h^D_{22}}{2 \det g_{ij}},$$

where $h^\nabla_{ij} = \langle \nabla f_i, f_j, \xi \rangle$ and $h^D_{ij} = \langle D f_i, f_j, \xi \rangle$, for some basis $\{f_1, f_2\}$ of $\Gamma(TM)$.

We utterly enable to introduce the notion of singular minimality with respect to the connections $\nabla$ and $D$: let $\sigma : M \to \mathbb{R}^3$ be a smooth immersion of an oriented surface $M$ and $\mathbf{u} \in \mathbb{R}^3$ a fixed unit vector. Let $H^\nabla$ and $H^D$ denote the mean curvatures with respect to $\nabla$ and $D$, respectively. The surface $M$ is called $\nabla$–singular minimal surface with respect to the vector $\mathbf{u}$ if it holds

$$2H^\nabla = \alpha \langle \xi, \mathbf{u} \rangle \langle \sigma, \mathbf{u} \rangle,$$

where $\xi$ is the unit normal vector field on $M$ and $\alpha$ a real constant. Accordingly the surface $M$ is called $D$–singular minimal surface with respect to the vector $\mathbf{u}$ if it holds

$$2H^D = \alpha \langle \xi, \mathbf{u} \rangle \langle \sigma, \mathbf{u} \rangle.$$

It is obvious that these notions coincide with the usual minimality when $\alpha = 0$ and so $\alpha \neq 0$ is assumed throughout the study.

3. $\nabla$–Singular Minimal Translation Surfaces

In this section, we characterize $\nabla$–singular minimal translation surfaces in $\mathbb{R}^3$. As the translation property of the surface changes, Eq. (2.5) generates different tasks. Because there are three types of translation surfaces as follows

$$z = f(x) + g(y), y = f(x) + g(z), x = f(y) + g(z),$$

we state three separate results.

**Theorem 1.** A $\nabla$–singular minimal translation surface in $\mathbb{R}^3$ of type $z = f(x) + g(y)$ with respect to a horizontal vector $\mathbf{u}$ is a generalized cylinder and one of the following occurs
Then Eq. (2.5) gives

\[
\frac{\partial \xi}{\partial y} = \cos (2y + c_2) + c_3,
\]

which gives the second statement of the theorem. Then the proof finishes if we show that Eq. (3.3) has no solution for \( f' = 0 \). By contradiction, assume that \( f'' \neq 0 \). Then taking partial derivative Eq. (3.2) with respect to \( y \) leads to

\[
\frac{\partial f''}{\partial y} g' g'' + \left[ 1 + (g')^2 \right] g''' = -2 \alpha g' g'' f'.
\]

That \( g' = g_0, g_0 \in \mathbb{R} \), is clearly a solution of Eq. (3.3). Then Eq. (3.2) reduces to

\[
\left[ -2 - 2 g_0^2 + (1 + g_0^2) f'' - 2 (f')^2 \right] x + \alpha f' \left[ 1 + g_0^2 + (f')^2 \right] = 0,
\]

or

\[
f'' + \frac{\alpha}{(1 + g_0^2) x} (f')^3 - \frac{2}{1 + g_0^2} (f')^2 + \frac{\alpha}{x} f' - 2 = 0,
\]

which gives the second statement of the theorem. Then the proof finishes if we show that Eq. (3.3) has no solution for \( f'' g'' \neq 0 \). By contradiction, assume that \( f'' g'' \neq 0 \). Dividing Eq. (3.3) with \( 2y g'' \), we get

\[
\frac{g''}{2y g''} f'' + \frac{\alpha f'}{x} - 2 = 0,
\]

yielding

\[
g'' = 2c f' g'', \ c \in \mathbb{R}.
\]
Integrating Eq. (3.5) gives
\[ g'' = c(g')^2 + d, \quad d \in \mathbb{R}. \]

Substituting Eq. (3.6) to Eq. (3.2) leads to
\[ f'' + \left[ 1 + (f')^2 \right] \left( -2 + d + \alpha \frac{f'}{x} \right) = 0. \]

From Eqs (3.4) and (3.7), we conclude two equations
\[ f'' = \frac{\left( -2 + \alpha \frac{f'}{x} \right) \left( -2 + d + \alpha \frac{f'}{x} \right)}{2 + c - d - \alpha \frac{f'}{x}} \]
and
\[ (-2 - c + d) (f')^2 + \alpha \frac{(f')^3}{x} + d - c = 0. \]

Notice that the denominator in Eq. (3.8) is not zero because the contradiction \( f'' = 0 \) is obtained otherwise. Taking derivative of Eq. (3.9) leads to
\[ \left( 2 \left( -2 - c+d \right) + \frac{3 \alpha f'}{x} \right) f'' = \alpha \left( \frac{f'}{x} \right)^2. \]

Considering Eq. (3.8) to Eq. (3.10) gives a polynomial equation of \( \frac{f'}{x} \)
\[ \alpha^2 \left( 1 + 3 \alpha \right) \left( \frac{f'}{x} \right)^3 + \alpha [\alpha (-16 - 2c + 5d) - 2 - c + d] \left( \frac{f'}{x} \right)^2 \]
\[ 2 \alpha [(-4 + d) (-2 - c + d) - 3(-2 + d)] \frac{f'}{x} - 4 (-2 - c + d) (-2 + d) = 0, \]
in which the fact that the leading coefficient vanish yields \( \alpha = -\frac{1}{3} \) and therefore Eq. (3.11) reduces
\[ \frac{10+2c+2d}{9} \left( \frac{f'}{x} \right)^2 - \frac{2}{3} [(-4 + d) (-2 - c + d) - 3(-2 + d)] \frac{f'}{x} \]
\[ -4 (-2 - c + d) (-2 + d) = 0. \]

The constant term must be zero and for this reason assume that \( d = 2. \) Then Eq. (3.12) gives the contradiction \( c = 6 \) and \( c = 0. \) This means that \( d \neq 2 \) and \( -2 - c + d = 0 \) and therefore the coefficient of the term of degree 1 cannot vanish, a contradiction. \( \square \)

The ODE (3.1) admits a reduction of order
\[ u' = \frac{-\alpha}{(1 + c_3^2) x} u^3 + \frac{2}{1 + c_3^2} u^2 - \frac{\alpha}{x} u + 2, \]
where \( u = u(x) = f' \) and \( u'(x) = f''. \) Due to \( \alpha \neq 0, \) the ODE (3.13) is an Abel equation of the first kind given by
\[ u' = p_3 (x) u^3 + p_2 (x) u^2 + p_1 (x) u + p_0 (x) \]
and no admits a general solution in terms of known functions, excepting very special cases depending on the functions \( p_0 (x) , \ldots, p_3 (x). \) We refer to [46, 47] for more details. Notice also that the surface given by the first statement of Theorem 1 is both \( \nabla \)-singular minimal and \( \nabla \)-minimal.
Theorem 2. A \( \nabla - \) singular minimal translation surface in \( \mathbb{R}^3 \) of type \( y = f(x) + g(z) \) with respect to a horizontal vector \( \mathbf{u} \) is either a plane parallel to \( \mathbf{u} \) or a generalized cylinder satisfying one of the following

1. \( f(x) = c_1 \) and

\[
g(z) = \pm \frac{1}{2} \arctan \left( \frac{1}{c_2} \sqrt{e^{2z} - c_2^2} \right) + c_3,
\]

2. \( g(z) = \alpha + c_5 z \) and \( f \) is a solution of the ODE

\[
f'' = -\frac{\alpha}{(1 + c_5^2)x} (f')^3 - 2c_5 \frac{(f')^2}{1 + c_5^2} - \frac{\alpha}{x} f' - 2c_5,
\]

where \( c_1, \ldots, c_5 \in \mathbb{R}, c_2 \neq 0, \) and \( f' = \frac{df}{dx}, \) etc.

Proof. The unit normal vector field and mean curvature are given by

\[
\xi = \frac{f' \partial_x - \partial_y + g' \partial_z}{\sqrt{1 + (f')^2 + (g')^2}}
\]

and

\[
2H^\nabla = -\left[ \frac{1 + (g')^2}{1 + (f')^2 + (g')^2} f'' + \frac{2 + (g')^2}{1 + (f')^2 + (g')^2} g'' + 2 \frac{1 + (f')^2 + (g')^2}{1 + (f')^2 + (g')^2} g' \right],
\]

where \( g' = \frac{dg}{dz} \) and so. Without loss of generality we may assume that \( \mathbf{u} = \partial_z \).

Therefore, Eq. (2.5) implies

\[
f'' \left[ 1 + (g')^2 \right] + g'' \left[ 1 + (f')^2 \right] + 2 \frac{1 + (f')^2 + (g')^2}{1 + (f')^2 + (g')^2} g' = -\frac{\alpha f'}{x}.
\]

To solve Eq. (3.14), we distinguish several cases: the first case is that \( f(x) = f_0 \in \mathbb{R} \). Then Eq. (3.14) reduces to \( g'' = -2 \left[ 1 + (g')^2 \right] g' \). That \( g' = 0 \) is a trivial solution of this equation and leads the surface to a plane parallel to \( \mathbf{u} \). Otherwise, \( g' \neq 0 \), after solving it, we obtain the first statement of the theorem. The second case is that \( f' = f_0 \neq 0 \). Then Eq. (3.14) reduces to a polynomial equation of \( x \) in the form

\[
\left\{ \left[ 1 + f_0^2 \right] g'' + 2 \left[ 1 + f_0^2 + (g')^2 \right] g' \right\} x + \alpha f_0 \left[ 1 + f_0^2 + (g')^2 \right] = 0,
\]

which implies that this case is false because the constant term of the polynomial equation cannot be zero. The last case is that \( f'' \neq 0 \). Taking partial derivative Eq. (3.14) with respect to \( z \), we get

\[
6 (g')^2 g'' + 2 \left( f'' + \alpha \frac{f'}{x} \right) g' g'' + \left[ 1 + (f')^2 \right] (2g'' + g') = 0.
\]

That \( g' = g_0 \in \mathbb{R} \) is clearly a solution to Eq. (3.15). So, Eq. (3.14) reduces to

\[
f'' + \frac{\alpha}{(1 + g_0^2)x} (f')^3 + \frac{2g_0}{1 + g_0^2} (f')^2 + \frac{\alpha}{x} f' + 2g_0 = 0,
\]

which gives the second statement of the theorem. Then the proof finishes if we show that Eq. (3.15) has no solution for \( f'' g'' \neq 0 \). By contradiction, assume that
\[ f''g'' \neq 0. \] Dividing Eq. (3.15) with \( 2g'g'' \), we have
\[
(3.16) \quad \left( \frac{g'''}{2g'g''} + \frac{1}{g'} \right) \left[ 1 + (f')^2 \right] + f'' + \frac{\alpha f'}{x} + 3g' = 0.
\]
Taking partial derivative of Eq. (3.16) with respect to \( x \) and \( z \) leads to
\[
\frac{f'f''}{2g'g''} \left( \frac{g'''}{2g'g''} + \frac{1}{g'} \right) = 0
\]
or, owing to \( f'f'' \neq 0 \), the term \( g''/ (2g'g'') + 1/g' \) becomes a constant. Therefore, the partial derivative of Eq. (3.16) with respect to \( z \) gives \( g'' = 0 \), a contradiction \( \square \)

Note that, as in previous result, the surface given by the first statement of Theorem 2 is both \( \nabla^- \) singular minimal and \( \nabla^- \) minimal.

**Theorem 3.** A \( \nabla^- \) singular minimal translation surface in \( \mathbb{R}^3 \) of type \( x = f(y) + g(z) \) with respect to a horizontal vector \( u \) is a generalized cylinder and one of the following occurs

1. \( f(y) = c_1 \) and \( g \) is a solution of the autonomous ODE
   \[
   g'' = \left( \frac{\alpha}{c_1 + g} - 2g' \right) \left[ 1 + (g')^2 \right], \quad g'' \neq 0;
   \]
2. \( g(z) = c_2 \) and
   \[
   y = \pm \int \left[ c_3 (f + c_2)^{2\alpha} + c_4 \right]^{-1/2} df,
   \]
   for \( c_1, ..., c_4 \in \mathbb{R}, \ c_3 \neq 0, \) and \( g' = \frac{dg}{dz}, \) etc.

**Proof.** The unit normal vector field and mean curvature are
\[
\xi = \frac{\partial_x - f' \partial_y - g' \partial_z}{\sqrt{1 + (f')^2 + (g')^2}}
\]
and
\[
2H^\nabla = \frac{\left[ 1 + (g')^2 \right] f'' + \left[ 1 + (f')^2 \right] g'' + 2 \left[ 1 + (f')^2 + (g')^2 \right] g'}{\left[ 1 + (f')^2 + (g')^2 \right]^{3/2}},
\]
where \( f' = \frac{df}{dy} \) and so. Without of loss of generality we may assume that \( u = \partial_x \).

Hence Eq. (2.5) turns to
\[
(3.17) \quad \frac{\left[ 1 + (g')^2 \right] f'' + \left[ 1 + (f')^2 \right] g'' + 2 \left[ 1 + (f')^2 + (g')^2 \right] g'}{\left[ 1 + (f')^2 + (g')^2 \right]^{3/2}} = \alpha \frac{1}{f + g},
\]
in which both \( f \) and \( g \) cannot be constant simultaneously because the situation \( \alpha = 0 \) is obtained otherwise. We distinguish the remaining cases: the first case is that \( f(y) = f_0 \in \mathbb{R} \). Eq. (3.17) writes
\[
(3.18) \quad g'' + 2 \left[ 1 + (g')^2 \right] g' = \alpha \frac{1 + (g')^2}{f_0 + g}.
\]
in which $g$ must be non-linear because the situation $\alpha = 0$ is obtained otherwise. This concludes the first statement of the theorem. The second case is that $f(y) = d + cy$, $c, d \in \mathbb{R}$, $c \neq 0$. Then Eq. (3.17) turns to

$$
\frac{(1 + c^2)g''}{1 + c^2 + (g')^2} + 2g' = \alpha \frac{1}{d + cy + g}
$$

By taking partial derivative of Eq. (3.19) with respect to $y$, we obtain the contradiction $ac = 0$. The third case is that $f'' \neq 0$ and $g = g_0 \in \mathbb{R}$. Then Eq. (3.17) reduces to

$$
\frac{f''}{1 + (f')^2} = \alpha \frac{1}{f + g_0}.
$$

By multiplying Eq. (3.20) with $2f'$ and taking first integral, we obtain

$$
f' = \pm \sqrt{c(f + g_0)^{2a} - 1}, \quad c \in \mathbb{R}, \ c \neq 0.
$$

Taking derivative of Eq. (3.21), we can conclude

$$f'' = ac (f + g_0)^{2a-1},$$

which is known as Emden–Fowler equation (see [47]) and the solution follows

$$y = \pm \int \left[c(f + g_0)^{2a} + d\right]^{-1/2} df + e, \quad d, e \in \mathbb{R},$$

which proves the second statement of the theorem. The fourth case is that $f'' \neq 0$ and $g(z) = d + cz$, $c, d \in \mathbb{R}$, $c \neq 0$. Then Eq. (3.17) reduces to

$$
\frac{[1 + c^2]f''}{1 + c^2 + (f')^2} + 2c = \alpha \frac{1}{d + cz + f}.
$$

By taking partial derivative of Eq. (3.22) with respect to $z$, we obtain the contradiction $ac = 0$. Therefore the proof finishes if we show that Eq. (3.17) has no solution for $f''g'' \neq 0$. By contradiction, assume that $f''g'' \neq 0$. Then Eq. (3.17) can be rewritten as

$$
(f + g) \left\{1 + (g')^2 \right\} + \left[1 + (f')^2\right]g'' + 2 \left[1 + (f')^2 + (g')^2\right]g' = \alpha \left[1 + (f')^2 + (g')^2\right].
$$

If we take partial derivative of Eq. (3.23) with respect to $y$ and $z$ and afterwards divide the generated equation with $f'f''g'g''$, we can deduce

$$
4 + \frac{f''f'''}{f''} \left[\frac{1 + (g')^2}{g''} + 2g\right] + \left(\frac{f'''}{g''} + \frac{2}{g}\right) \left[\frac{1 + (f')^2}{f''} + 2f\right]
$$

$$+ 2f'f''' + \frac{2gg''}{g''} + \frac{4a}{g} + \frac{4c}{g'} + \frac{6c}{f'} = 0.
$$

The partial derivative of Eq. (3.24) with respect to $y$ and $z$ leads to

$$
\left(\frac{f'''}{f''f'''} + 2g\right) \left[\frac{1 + (g')^2}{g''} + 2g\right] + \left(\frac{g''}{g'y' + g''} + \frac{2}{g}'\right) \left[\frac{1 + (f')^2}{f''} + 2f\right] - \frac{6gg''}{(f'')^2} = 0.
$$

We have subcases:
(1) $f'' = f_0 \in \mathbb{R}$, $f_0 \neq 0$. Then the partial derivative of Eq. (3.24) with respect to $y$ leads to

$$\frac{g'''}{g'g''} + \frac{2}{g'} = 0.$$  \hspace{1cm} (3.26)

Therefore Eq. (3.24) reduces to

$$2 + \frac{2g'}{g''} + \frac{3g'}{f_0} = 0.$$  \hspace{1cm} (3.27)

On the other hand, a first integration of Eq. (3.26) gives

$$g'' = -2g' + c, \quad c \in \mathbb{R}.$$  \hspace{1cm} (3.28)

By substituting Eq. (3.28) into Eq. (3.27), we obtain a polynomial equation of $g'$ whose the leading coefficient coming from the term $(g')^2$ is $-\frac{6}{f_0}$, a contradiction.

(2) $f''' = 2cf''', \quad c \in \mathbb{R}, \quad c \neq 0$. A first integration yields

$$f'' = c(f')^2 + d, \quad d \in \mathbb{R}.$$  \hspace{1cm} (3.25)

Then Eq. (3.25) reduces to

$$\left(\frac{g'''}{g'g''} + \frac{2}{g'}\right) = \frac{2d}{c} - 1 + (f')^2 - 6g'' = 0,$$

which implies \(\left(\frac{g'''}{g'g''} + \frac{2}{g'}\right) = 0\). This gives from Eq. (3.25) the contradiction $g'' f''' = 0$.

(3) \((f''' f'')' \neq 0\). Eq. (3.25) can be rewritten by dividing $g'' (f''' f'')'$ as

$$A(y) B(z) = C(y) + D(z),$$  \hspace{1.5cm} (3.29)

where

$$A(y) = \frac{\left\{1 + (f')^2\right\} / f'' + 2f}{(f''' f'')'}, \quad B(z) = (g'' / g'g'' + 2g') / g''$$

and

$$C(y) = 6 \frac{f'''}{(f''' f'')'^2}, \quad D(z) = - \left\{ \left\{1 + (g')^2\right\} / g'' + 2g' \right\} / g''.$$  \hspace{1cm} (3.30)

The functions $A, B, C, D$ from Eq. (3.29) must be all constant. Let us put $B(z) = B_0$ and $D(z) = -D_0, B_0, D_0 \in \mathbb{R}$. Therefore, we get

$$\left[\frac{g'''}{g'g''} + \frac{2}{g'}\right] = B_0 g''$$

and

$$4g' - \left[1 + (g')^2\right] \frac{g''}{(g'')^2} = D_0 g''$$  \hspace{1cm} (3.31)

A first integration of Eq. (3.30) yields

$$\frac{g'''}{g'g''} + \frac{2}{g'} = B_0 g' + d_1.$$  \hspace{1.5cm} (3.32)

for $d_1 \in \mathbb{R}$. Multiplying Eq. (3.32) with $g'g''$ and then taking first integration the generated equation gives

$$g'' = \frac{B_0}{3} (g')^3 + \frac{d_1}{2} (g')^2 - 2g' + d_2.$$  \hspace{1cm} (3.33)
for $d_2 \in \mathbb{R}$. By considering Eq. (3.33) into Eq. (3.32), we deduce that

\begin{equation}
\frac{B_0^2}{3} (g')^5 + \text{rest terms.}
\end{equation}

(3.34)

Substituting Eqs. (3.33) and (3.34) into Eq. (3.31), we have a polynomial equation of $g'$ whose the coefficient coming from $(g')^7$ is $\frac{B_0^2}{3}$, yielding $B_0 = 0$. It follows from Eq. (3.32)

\begin{equation}
g''' = (-2 + d_1 g') g''
\end{equation}

and plugging it into Eq. (3.31)

\begin{equation}
4g'g'' - \left[1 + (g')^2\right] (-2 + d_1 g') = D_0 (g'')^2.
\end{equation}

(3.35)

Considering Eq. (3.33) into Eq. (3.35) leads to a polynomial equation of $g'$ whose the coefficient coming from $(g')^3$ is $d_1$ which must vanish. Therefore Eqs. (3.33) and (3.35) reduce to

\begin{equation}
g'' = -2g' + d_2
\end{equation}

and

\begin{equation}
4g'g'' + 2 (g')^2 + 2 = D_0 (g'')^2,
\end{equation}

respectively. From these two equations, we can conclude a polynomial equation of $g'$

\begin{equation}
(4D_0 + 6) (g')^2 - 4 (D_0 + d_2) g' + D_0 d_2^2 - 2 = 0,
\end{equation}

which gives a contradiction because the constant term cannot vanish.

\[ \square \]

4. $D-$Singular Minimal Translation Surfaces

As in previous section, we characterize translation surfaces in $\mathbb{R}^3$ of each type to be $D-$singular minimal through the following results.

**Theorem 4.** A $D-$singular minimal translation surface in $\mathbb{R}^3$ of type $z = f(x) + g(y)$ with respect to a horizontal vector $u$ is either a plane parallel to $u$ or a generalized cylinder satisfying $g(y) = c_1 + c_2 y$ and

\[ f(x) = \pm |c_3| \sqrt{1 + c_2^2} \int \left( x^{2\alpha} - c_3^2 \right)^{-1/2} dx, \]

where $c_1, c_2, c_3 \in \mathbb{R}$, $c_3 \neq 0$.

**Proof.** The unit normal vector field and mean curvature follow

\[ \xi = \frac{-f' \partial_x - g' \partial_y + \partial_z}{\sqrt{1 + (f')^2 + (g')^2}} \]

and

\[ H^D = \frac{f'' \left[ 1 + (g')^2 \right] + g'' \left[ 1 + (f')^2 \right]}{2 \left[ 1 + (f')^2 + (g')^2 \right]^{3/2}}, \]
where \( f' = \frac{df}{dx}, \ g' = \frac{dg}{dy} \) and so. Without loss of generality we may assume that \( u = \partial_x \). Hence Eq. (2.6) turns to

\[
(4.1) \quad \left[ \frac{1 + (g')^2}{1 + (f')^2 + (g')^2} \right] f'' + \left[ \frac{1 + (f')^2}{1 + (f')^2 + (g')^2} \right] g'' = -\frac{\alpha}{x}.
\]

Eq. (4.1) has no solution if \( g'' \neq 0 \), see [37, Theorem 5]. Therefore, we have \( g' = g_0 \in \mathbb{R} \) and Eq. (4.1) reduces to

\[
(4.2) \quad \left[ \frac{1 + (g_0^2)}{1 + (f')^2 + (g')^2} \right] f'' = -\frac{\alpha}{x},
\]

where being \( f \) a constant is a trivial solution, implying \( M \) is a plane parallel to the vector \( u \). Assume that \( f \) is no constant and then we can easily infer from Eq. (4.2) that \( f'' = 0 \), due to \( \alpha \neq 0 \). Hence, a first integration of Eq. (4.2) yields

\[
(4.3) \quad f' = \pm |c| \sqrt{1 + g_0^2}, \quad c \in \mathbb{R}, \ c \neq 0,
\]

or

\[
(4.3) \quad f' = \pm |c| \sqrt{1 + g_0^2} \frac{x^{\alpha}}{x^{2\alpha} - c^2}.
\]

A first integration of Eq. (4.3) completes the proof. \( \square \)

For a translation surface of type \( y = f(x) + g(z) \), the unit normal vector field and mean curvature follow

\[
\xi = \frac{f' \partial_x - \partial_y + g' \partial_z}{\sqrt{1 + (f')^2 + (g')^2}}
\]

and

\[
H^D = \frac{f'' \left[ 1 + (g')^2 \right] + g'' \left[ 1 + (f')^2 \right]}{2 \left[ 1 + (f')^2 + (g')^2 \right]^{\frac{3}{2}}},
\]

where \( f' = \frac{df}{dx}, \ g' = \frac{dg}{dz} \) and so. Then, with respect to the horizontal vector \( u = \partial_x \), the \( D \)-singular minimality equation is similar to Eq. (4.1) up to a sign. Thereby, for such a surface, we can state a similar result to Theorem 4 without proof by replacing \( \alpha \) with \( -\alpha \).

**Theorem 5.** A \( D \)-singular minimal translation surface in \( \mathbb{R}^3 \) of type \( y = f(x) + g(z) \) with respect to a horizontal vector \( u \) is either a plane parallel to \( u \) or a generalized cylinder satisfying \( g(z) = c_1 + c_2 z \) and

\[
f(x) = \pm |c_3| \sqrt{1 + c_2^2} \left( \int_0^x \left( 1 - c_2^2 x^{2\alpha} \right)^{-1/2} dx, \right.
\]

where \( c_1, c_2, c_3 \in \mathbb{R}, \ c_3 \neq 0. \)
Theorem 6. A $D$-singular minimal translation surface in $\mathbb{R}^3$ of type $x = f(y) + g(z)$ with respect to a horizontal vector $\mathbf{u}$ is a generalized cylinder satisfying $f(y) = c_1$ and

\begin{equation}
 z = \pm \int \left[ c_2^2 (c_1 + g)^{2\alpha} - 1 \right]^{-1/2} dg,
\end{equation}

for $c_1, c_2 \in \mathbb{R}$, $c_2 \neq 0$.

Proof. Without loss of generality we may assume that $\mathbf{u} = \partial_x$. Then Eq. (2.6) implies

\begin{equation}
 \frac{f''}{1 + (f')^2 + (g')^2} = \frac{\alpha}{f + g},
\end{equation}

in which the roles of $f$ and $g$ are symmetric. Eq. (4.4) has a solution provided that $f$ or $g$ is a constant, see [35, Theorem 4.1]. Thanks to the symmetry, we can assume $f = f_0 \in \mathbb{R}$. Thereby, Eq. (4.5) reduces to

\begin{equation}
 \frac{g''}{1 + (g')^2} = \frac{\alpha}{f_0 + g}.
\end{equation}

Put

\begin{equation}
 g' = q, \quad q' = \frac{dq}{dg}, \quad q'' = \frac{d}{dz} = \frac{g''}{g}, \quad q = q(g).
\end{equation}

By considering Eq. (4.7) into Eq. (4.6) we get

\begin{equation}
 \frac{qq'}{1 + q^2} = \frac{\alpha}{f_0 + g}.
\end{equation}

A first integration of Eq. (4.8) with respect to $g$ yields $1 + q^2 = c^2 (f_0 + g)^{2\alpha}$ or

\[ dz = \pm \frac{dg}{\sqrt{c^2 (f_0 + g)^{2\alpha} - 1}} \]

and the proof is completed by a first integration. \qed

5. Conclusions

In this study, new perspectives on the singular minimality of immersed surfaces in $\mathbb{R}^3$ were introduced. These, in particular the $\nabla-$singular minimality, provided us non-trivial and different results from the obtained one with respect to the Levi-Civita connection on $\mathbb{R}^3$, see [35, 37]. Our results were found with respect to a horizontal vector and it will not make big difference when the vector is assumed to be vertical. Yet, the problem of finding $\nabla-$ and $D-$singular minimal translation surfaces in $\mathbb{R}^3$ with respect to an arbitrary vector is open.

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