SACCHERI’S QUADRILATERALS

PRODROMOS FILIPPIDIS AND CHRISTOS FILIPPIDIS

Abstract. We study Saccheri’s three hypotheses on a two right-angled isosceles quadrilateral, under certain assumptions, with respect of the independence of the euclidean parallel postulate. We also trace the historical circumstances under which, the development of arguments for the consistency of non-Euclidean geometries occurred; indicating an important shift in mathematicians’ attitude towards the fifth Euclidean postulate.

1. Introduction

Euclid’s fifth postulate (called also the eleventh or twelfth axiom) states: ”If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines if produced indefinitely meet on that side on which are the angles less than two right angles.” The earliest commentators found fault with this statement as being not self-evident [1]. Euclid uses terms such as “indefinitely” and makes logical assumptions that had not been proven or stated. Thus the parallel postulate seemed less obvious than the others.

For two thousand years, many attempts were made to prove the parallel postulate using Euclid’s first four postulates. Finally the mathematical society have proven the consistency of non-euclidean plane geometry and the independence of the euclidean parallel postulate; and extend that in stating that it would be impossible to prove Euclid’s Parallel Postulate from the other assumptions made by him, since this would involve the denial of the Parallel Postulate of Bolyai and Lobatschewsky [2]. This conclusions mainly derived from Beltrami’s (1868) seminal papers concerning the existence of non-Euclidean objects and by the arguments offered by Jules Houel in 1860-1870 for the unprovability of the parallel postulate and for the existence of non-Euclidean geometries. The final step towards rigorous consistency proofs is taken in the 1880s by Henri Poincare.

What Houel sees in Beltrami’s putative demonstration that regions of pseudo-spheres are complete Bolyai-Lobatschewsky planes is the presentation of an instance in which all of the properties of the euclidean plane are present except for the property of unique parallelism [3]. A trivial example will make the principle evident. If we wish to show that the property of upright posture in mammals does not imply intelligence (in this sense of ‘implication’), it is sufficient to show that there are some mammals in which upright posture is present without intelligence (kangaroos), even though there may be others in which upright posture occurs together with intelligence (humans). The application of the result here to the human case,

2010 Mathematics Subject Classification. Primary 51-02; Secondary 51-03.

Key words and phrases. quadrilaterals, independence of the euclidean parallel postulate, Beltrami, invariant relations.
e.g. if we want to say that upright posture in humans does not imply their intelligence, turns on referring to the same properties in the case of both humans and kangaroos (Scanlan 1988).

It’s obvious that the independence of the euclidean parallel postulate is well formed, but a detailed looked at the writings will show that some claims may are overstatement or misinterpreted. The most striking example of a non-logical use of the word “independent” is in Bolyai’s title of his famous appendix, one of the founding texts of hyperbolic geometry, Absolute Geometry: Independent of the Truth or Falsity of Euclids Axiom XI (which can never be decided a priori). Bolyai does not, of course, claim that the Parallel Postulate is independent from the rest of Euclidean geometry in the sense required to show that the Parallel Postulate is unprovable (though he does claim that it cannot be proven). He means simply that he can prove some theorems of geometry without relying on either the Parallel Postulate or its negation [4].

Sometimes, it was simply stated that the attempt to prove the Parallel Postulate had gone on too long and that repeated lack of success shows that it is impossible to prove the Parallel Postulate. On the other hand the geometers working in order to support the existence of non-Euclidean objects were in the pursue of some very impressive results. That’s why Beltrami’s work had a profound impact on Houel. Houel immediately after he saw it, announced that Beltrami had shown that it is impossible to prove the Parallel Postulate. Strikingly, he made this announcement in eight different journals. In reality they wanted to strongly support and consolidate the initial hypothesis of the acute angle. The geometers had made just another logical assumption, believing that by finding a contradiction to Saccheri’s quadrilateral would weaken the foundations of non-Euclidean geometries.

We can also realize that some statements wasn’t very well formed or may were just some logical assumptions and we have to treat them with the same way we did with the logical assumptions that Euclid did. H. S. CARSLAW (1909) states: “How ever far the hyperbolic geometry were developed, no contradictory results could be obtained“. Stating ”How ever far” isn’t it equivalent with the term ”indefinitely” that Euclid uses?

Finally, let us consider the following: "for every diameter of \( \gamma \) let us imagine another point ‘at infinity’ such that all the Euclidean lines parallel in the Euclidean sense to this diameter meet in this point at infinity, just as railroad tracks appear to meet at horizon. These points at infinity will also be called ultra ideal" [5]. By imagining that the Euclidean lines meet to a point ‘at infinity’ we managed to show that these lines are parallel to the diameter \( \gamma \), but are not parallel to each other. Nevertheless, in a specific area, by construction, we have lines that are parallel to each other due to the fact that they are parallel to a third line, something that we can not succeed without using the Euclidean sense of parallelism.

Finding a contradiction on the three hypotheses made by Saccheri does not mean that the non-euclidean geometries aren’t consistent. It means that we can apply Euclid’s proposition at these geometries without deconstruct them, because the independence of the Euclidean parallel postulate is bidirectional. It means that non-euclidean geometries are independent from the Euclidean parallel postulate and that the postulate, itself, is also independent from the geometries that exist in the Euclidean field. This concludes, that we can build relations without using it, but in the same time we can apply it anywhere we choose, without deconstructing
our work. In reality Euclid did exactly the same thing, using the parallel axiom after the first 28 propositions. Thus was Euclid "vindicated" in an unexpected manner. Knowingly or not, the wise Greek had stated the case correctly, and only his followers had been at fault in their efforts for improvement [1].

The paper is organized in the following way, at section 2 we present the used Definitions, postulates and theorems. At section 3 we prove that the angles adjacent to the opposite sides of a quadrilateral, which are perpendicular to the base, are shorter or greater, if they are adjacent, respectively, to the greater or shorter sides. And they are equal if they adjacent to equal sides. At section 4 we prove that the bisector of the vertex angle of an isosceles triangle and the perpendicular bisector of the congruent sides of this triangle are intersecting. At section 5 we prove the rejection of the acute-angled and obtuse-angled quadrilaterals.

2. Used definitions, postulates and theorems

(1) The first four Euclid’s postulates, the euclidean common notions and the theorems deduced by them.

(2) The plane separation postulate.

(3) Theorems 1 and 2. We shall deal with them, in the course of our work.

3. Theorem 1

. The angles adjacent to the opposite sides of a quadrilateral, which are perpendicular to the base, are shorter or greater, if they are adjacent, respectively, to the greater or shorter sides. And they are equal if they adjacent to equal sides (corresponds to the Lemma I of G. Saccheri).

proof: In quadrilateral $AB\Gamma\Delta$ (figure 1) we are given that angles $A\Delta\Gamma$ and $B\Gamma\Delta$ are right angles and sides $A\Delta$ and $B\Gamma$ are equal. Drawing the diagonals $A\Gamma$ and $B\Delta$ two congruent triangles $A\Delta\Gamma$ and $B\Gamma\Delta$ are formed, we prove this equality easily using the Side, Angle, Side (SAS) postulate. We may therefore conclude by the corresponding parts of the above congruent triangles, that the sides $A\Gamma=B\Delta$ and the angles $A2=B2, \Delta1=\Gamma1$. Since the angles $\Delta2$ and $\Gamma2$ are complement of equal angles $\Delta1$ and $\Gamma1$ they are themselves equal, that is, $\Delta2=\Gamma2$. Using again the SAS postulate we also conclude that the triangles $A\Delta B$ and $B\Gamma A$ are congruent, therefore by the corresponding parts of these triangles we conclude that the angles $\Delta AB$ and $\Gamma BA$ are equal.

. In quadrilateral $AB\Gamma\Delta$ (figure 2) we are given that angles $A\Delta\Gamma$ and $B\Gamma\Delta$ are right angles, sides $A\Delta$ and $B\Gamma$ are equal and angles $\Delta AB$ and $\Gamma BA$ are equal too, as it has been proved above. On the line segment $B\Gamma$ we take any point $E$ between $\Gamma$ and $B$, then by joining the points $A$ and $E$ the quadrilateral $AE\Gamma\Delta$ is formed. This new quadrilateral has as above, the angles $A\Delta\Gamma$ and $E\Gamma\Delta$ right angles, but its opposite sides $A\Delta$ and $E\Gamma$ are not equal now, that is, $A\Delta > E\Gamma$. Since the angle $AE\Gamma$, as an exterior angle of the triangle $ABE$, is greater in measure than the interior remote angle $ABE$, the angles $ABE$ and $\Delta AB$ are equal, and the angle $\Delta AB$ is greater in measure than the angle $\Delta AE$, it follows that the angle $AE\Gamma > \Delta AE$. Now the converse of the above statement. The greater in measure angle of a quadrilateral, which has the consecutive angles of its base right angles, is adjacent to its shorter side and vice versa.

In the given quadrilateral $A\Gamma\Delta$ (figure 2) let angle $AE\Gamma$ adjacent to the side $E\Gamma$ be greater in measure than the angle $\Delta AE$ which is adjacent to the side $A\Delta$, 

then according to the above argument, it follows that the side $A\Delta > E\Gamma$, since if $A\Delta < E\Gamma$ or $A\Delta = E\Gamma$ then it must be angle $\Delta\AE > AE\Gamma$ or angle $\Delta\AE = AE\Gamma$. This last contradicts the original assumption that state angle $AE\Gamma < \Delta\AE$.

From now on, we shall call the above mentioned two right-angled isosceles quadrilaterals, which have the angles adjacent to the opposite and perpendicular to the base sides congruent, acute-angled quadrilaterals, right-angled quadrilaterals, or obtuse-angled quadrilaterals, if the pair of the congruent angles are, respectively, acute, right or obtuse angles.

3.1. **The common property of the acute-angled, right-angled and obtuse-angled quadrilaterals (the main property of Saccheri’s quadrilaterals).**

The perpendicular bisector to the base of the acute-angled, right-angled and obtuse-angled quadrilaterals is also perpendicular bisector to the opposite the base side of them.
Proof: In acute-angled quadrilateral $AB\Delta\Gamma$ (figure 3) we are given that angles $\Delta A\Gamma$ and $B\Gamma A$ are right angles, $\Delta A\Gamma$ and $\Gamma B\Delta$ are equal acute angles and sides $\Delta \Delta = \Gamma B$. We draw the line segments $Z\Delta$ and $Z\Gamma$, joining the midpoint $Z$ of the side $AB$, with the endpoints $\Delta$ and $\Gamma$ of the side $\Delta \Gamma$ of the above quadrilateral $AB\Delta$. We notice that the formed triangles $ZA\Delta$ and $ZB\Gamma$ are equal, we can prove this equality using the side, angle, side postulate. We may therefore conclude by the corresponding parts of the above congruent triangles, that the angles $Z3=Z4$ and the sides $Z\Delta=Z\Gamma$. Since the triangle $\Delta Z\Gamma$ is isosceles it follows that the median $ZE$ is also bisector of the vertex angle $\Delta Z \Gamma$ and perpendicular bisector to the base $\Delta \Gamma$ of the above triangle, hence the angles $Z1$ and $Z2$ are equal. Since the angles $Z3$, $Z1$, $Z2$ and $Z4$ are adjacent angles lie at the same straight line and the sums of the angles measures $Z3+Z1$ and $Z2+Z4$ are equal, it follows that the line segment $ZE$, which is perpendicular bisector to the base $\Delta \Gamma$ of the acute-angled quadrilateral $AB\Delta$, is also perpendicular bisector to the opposite the base side, in which the acute angles of the quadrilateral are adjacent. By the same argument we prove that this property holds for the right-angled and obtuse-angled quadrilaterals too. Equivalent we can state that; in a two right-angled isosceles quadrilateral: the line segment which joins the midpoint of the base with the midpoint of the summit is perpendicular bisector to both of them.

4. Theorem 2

The bisector of the vertex angle of an isosceles triangle and the perpendicular bisector of the congruent sides of this triangle are intersecting.

Case A.: In the isosceles triangle $AB\Gamma$ (figure 4) we are given that sides $AB$ and $B\Gamma$ are equal and the half of its base length $A\Gamma$ equals to the length of the bisector $B\Delta$, of the vertex angle $AB\Gamma$, that is $A\Delta = B\Delta$. In this case the bisector of the vertex angle $AB\Gamma$ and the perpendicular bisector to the side $AB$ are intersecting at the midpoint $\Delta$ of the base $A\Gamma$ of the above triangle.

Proof: Since the side $AB$ of the triangle $AB\Gamma$ is now base for the isosceles triangle $A\Delta B$, it will be that the line segment $E\Delta$, which joins the midpoint of the line
segment AB and the vertex ∆ of the isosceles triangle A∆B, must be perpendicular bisector to the side AB.

Case B:. In the isosceles triangle ABΓ (figure 5) we are given that sides AB and BΓ are equal and the half of its base length AΓ is shorter than the length of the altitude B∆, that is A∆ < B∆. In this case the bisector B∆ of the vertex angle ABΓ and the perpendicular bisectors to the congruent sides AB and BΓ are intersecting.

proof: Since the side A∆ of the right triangle AB∆, is shorter in length than the side B∆, it results that the angle ∆AB is greater in measure than the angle AB∆. Furthermore by the equivalence of the plane separation postulate and Pasch’s postulate [6] it follows that the perpendicular bisector ΠT except with the side AB intersects a side of the triangle AB∆. We suppose that the perpendicular bisector ΠT to the side AB of the triangle AB∆ also intersects the side A∆ of the above triangle at a point M, since every point on the perpendicular bisector ΠT is equidistant from the endpoints of the line segment AB, it must be that the sides AM=MB and the angles BAM and ABM are equal, but this is impossible since the angle BAM=BA∆ is greater in measure than the angle AB∆ and angle AB∆ is greater in measure than the angle ABM. Therefore the perpendicular bisector ΠT except the side AB intersects always the bisector of the vertex angle ABΓ too.
Case $\Gamma$: In the isosceles triangle $\Delta \Gamma \Gamma$ (figure 6) we are given that sides $\Delta \Gamma$ and $\Gamma \Gamma$ are equal and $\Delta \Gamma > \Gamma \Delta$, then the bisector of the vertex angle $\Delta \Gamma \Gamma$ and the perpendicular bisector of the side $\Delta \Gamma$ are intersecting.

Proof: We are given that the line segment $\Delta \Gamma$, and the perpendicular bisector of it($\Sigma \Gamma$) are intersecting at a point $\Delta$. Using the point $\Delta$ as a center and the line segment $\Delta \Gamma$, which is equal in length to the half of the line segment $\Delta \Gamma$, as a radius we draw the circle($\Delta \Delta$, $\Delta \Gamma$), this circle intersects the perpendicular bisector $\Sigma \Gamma$ at the points $\Gamma$ and $\Pi$. Using the point $\Gamma$ as a vertex and the line segment $\Delta \Gamma$ as a base we form the isosceles triangle $\Delta \Pi \Gamma$, we notice that its altitude $\Pi \Delta$ equals in length to the half of the base $\Delta \Gamma$. Using the point of intersection $\Pi$ of the above circle($\Delta \Delta$, $\Delta \Gamma$) with the perpendicular bisector $\Sigma \Gamma$ as a center and the cord...
of the same circle $\Delta, A\Delta$) $EA$ as a radius, we draw the circle $(E, EA)$ this second circle intersects the perpendicular bisector $\Sigma Z$ at the points $\Theta$ and $N$. Since in the circle $(\Delta, A\Delta)$ the cord $EA$ is shorter in length than the diameter $EH$, it follows that the point $N$ lies between the points $H$ and $\Delta$, and the altitude of the formed isosceles triangle $AN\Gamma$ is shorter in length than the half of the common base $A\Gamma$. By repeating the above used procedure we can form isosceles triangles, which have the given line segment $A\Gamma$ as a base and an altitude in length, in comparison with the half of its base $A\Gamma$, as short as we want. This last takes place, since by the repetition of the above referred procedure, the arc of the given cord $A\Gamma$ continuously diminished tend to coincide with it. Since the above described isosceles triangles are inscribed in circles, it follows that the perpendicular bisectors to their equal sides intersect the common bisector of the vertex angles of them $\Sigma Z$.

Let $AM\Gamma$ be any isosceles triangle in which the length of the altitude $M\Delta$ is shorter than the half of the base length $A\Gamma$. We remind here that for any isosceles triangle $AM\Gamma$, with an altitude $M\Delta$ we can form another inscribed isosceles triangle $AN\Gamma$ with the same base, collinear vertex angle bisector and with an altitude in length $N\Delta < M\Delta$. Then the perpendicular bisectors to its equal sides will intersect the bisector $\Sigma Z$, of the vertex angle $AM\Gamma$, in respect of the following arguments. Given that the line segment $\theta M$ is greater in length than $\theta N$ and $\theta N$ is greater than $\theta A$, since $\theta N$ and $\theta A$ are, respectively, diameter and cord in the same circle $(E, EA)$ it implies that $\theta M > \theta A$. From this we can conclude that in the triangle $A\theta M$ the angle $\theta AM$ is greater in measure than the angle $\theta MA$. This last ensure that the perpendicular bisectors to the equal sides of the isosceles triangle $AM\Gamma$ will intersect the bisector of the vertex angle of it. Since the opposite assumption, which states that the perpendicular bisector to the side $AM$ will intersect the side $A\theta$ in the triangle $A\theta M$, given the Pasch’s postulate, implys that the angle $\theta MA$ is greater in measure than the angle $\theta AM$ which is a contradiction.

5. The Rejection of the Acute-Angled and Obtuse-Angled Quadrilaterals

In the acute-angled quadrilateral $AEZ\Delta$ (figure 7) we are given that the line segment $\Gamma M$ is perpendicular bisector to the sides $\Delta Z$ and $AE$, because it joins the midpoints of them, see the common property of the acute-angled, right-angled and obtuse-angled quadrilaterals.
Since the right angle $\angle AM\Gamma = EM\Gamma$ is greater in measure than the congruent acute angles $\angle AM$ and $\angle ZEM$ it results, according to the theorem 1, that in the quadrilaterals $AM\Gamma\Delta$ and $MEZ\Gamma$ the sides $A\Delta = EZ > M\Gamma$. Following, we extend the line segment $\Gamma M$, we take on it a point $B$, so that $A\Delta = B\Gamma = EZ$, we draw the line segments $AB$ and $BE$ and then we notice that the formed triangle $ABE$ is an isosceles triangle, since the line segment $B\Gamma$, which now is bisector of the angle $ABE$, is perpendicular bisector to the side $AE$. By repeating the above procedure we take the line segment $O\Pi$ perpendicular bisector to the sides $\Delta \Gamma$ and $AB$ of the acute-angled quadrilateral $AB\Gamma\Delta$, the $AB\Gamma\Delta$ is an acute-angled quadrilateral, since the angles $A\Delta\Gamma$ and $B\Gamma\Delta$ are right angles, the sides $A\Delta$ and $B\Gamma$ are equal, and the angles $BA\Delta$ and $AB\Gamma$ are congruent acute angles, because these are adjacent to the equal sides $A\Delta$ and $B\Gamma$, and the angle $ABM = AB\Gamma$ is a non right angle of the right triangle $ABM$. Similarly we take the line segment $\Sigma T$ perpendicular bisector to the sides $\Gamma Z$ and $BE$, for the same mentioned above reasons. From the above analysis we can conclude that the perpendicular bisectors $O\Pi$ and $\Sigma T$ to the congruent sides of the isosceles triangle $ABE$, as well the bisector of the vertex angle $ABE$ are not intersected, since they are perpendiculars to the same side $\Delta Z$, and the perpendicular to a line through a point not on the line is unique. But this conclusion contradicts to the results of the theorem 2.

Using the same procedure, and taking into account that in the obtuse-angled quadrilateral $AE\Gamma\Delta$ (figure 8) the perpendicular bisector $\Gamma M$, to the base $\Delta Z$ and to the summit $AE$, is longer than the congruent sides $A\Delta$ and $EZ$, because the equal obtuse angles $\angle AM\Gamma$ and $\angle ZEM$ are greater in measure than the right angles $AM\Gamma$ and $EM\Gamma$, we conclude again that the perpendicular bisectors to the equal sides of the isosceles triangle $ABE$, as well the bisector of the vertex angle $ABE$ are not intersected. But this conclusion also contradicts to the results of the theorem 2.

6. Conclusions

Since from the above conclusions we have to reject the hypotheses that the congruent angles adjacent to the equal opposite sides of the quadrilaterals, in which their base angles are right angles, are acute angles or obtuse angles, since these contradicts the results of the theorem 2, we have to accept that the quadrilaterals in which their base angles are right angles and their opposite sides are equal to each other and perpendicular to the base are only rectangle quadrilaterals.
References

[1] Florence P. Lewis, History of the Parallel Postulate, The American Mathematical Monthly, Vol. 27, No. 1 (Jan., 1920), pp. 16-23
[2] Professor H. S. CARSLAW, The Bolyai - Lobatschewsky Non-Euclidean Geometry: an Elementary Interpretation of this Geometry, and some Results which follow from this Interpretation (March 1910).
[3] J. Scanlan Michael (1988): Beltrami’s model and the independence of the parallel postulate, History and Philosophy of Logic, 9:1, 13-34
[4] David J. Stump (2007): The independence of the parallel postulate and development of rigorous consistency proofs, History and Philosophy of Logic, 28:1, 19-30
[5] Marvin J. Greenberg, Euclidean and non-Euclidean geometries: development and history, p 240.
[6] Marvin J. Greenberg, Euclidean and non-Euclidean geometries: development and history, p 80.

Glyka Nerα Attiki 15354, Greece
E-mail address: prodromos.filippidis@gmail.com
E-mail address: filippidis@inp.demokritos.gr