Interleaving & Reconfigurable Interaction: Separating Choice from Scheduling using Glue*

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Abstract. Reconfigurable interaction induces another dimension of nondeterminism in concurrent systems which makes it hard to reason about the different choices of the system from a global perspective. Namely, (1) choices that correspond to concurrent execution of independent events; and (2) forced interleaving (or scheduling) due to reconfiguration. Unlike linear order semantics of computations, partial order semantics recovers information about the interdependence among the different events for fixed interaction, but still is unable to handle reconfiguration. We introduce glued partial orders as a way to capture reconfiguration. Much like partial orders capture all possible choices for fixed systems, glued partial orders capture all possible choices alongside reconfiguration. We show that a glued partial order is sufficient to correctly capture all partial order computations that differ in forced interleaving due to reconfiguration. Furthermore, we show that computations belonging to different glued partial orders are only different due to non-determinism.

1 Introduction

Reconfigurable concurrent systems [3,5,4] are a class of computational systems, consisting of a collection of processes (or agents) that interact and exchange information in nontrivial ways. Agents interact using message-passing (or token-passing) and based on dynamic notions of connectivity where agents may only observe, inhibit or participate in interactions happening on links they are connected to. Agents may get connected or disconnected to links as side-effects of the interaction, and thus providing dynamic and sophisticated scoping mechanisms of interaction through reconfigurable interfaces.

Reconfiguration induces another dimension of nondeterminism in concurrent systems where it becomes hard to reason about the different choices of the system from a global perspective. It creates a situation where some events must be ordered with respect to sequences of other events dynamically during execution, and thus forcing interleaving in a non-trivial way. That is, from the point of an event, a sequence of other events is considered as a single block and can only happen before or after it. Note that reconfiguration is an internal event, and is totally hidden from the perspective of an external observer who may only observe message/token-passing. Indeed, messages or tokens can only indicate the occurrence of exchange but cannot help with noticing that a reconfiguration has happened and what are the consequences of reconfiguration. Knowing the reason why some event is scheduled before some others and the causal dependencies among the different events is crucial to facilitate reasoning about specific internal aspects from a global perspective. It also becomes very relevant when applying correct-by-construction techniques to synthesise such systems.

Clearly, linear order semantics of computations cannot be used to globally distinguish a system choice due to concurrent execution of independent events and a forced interleaving due to reconfiguration. It cannot be even used to recover information about the participants of an event and the interdependence of the different events. Therefore, a partial order semantics of computations is in-order. Existing approaches to partial order semantics (cf. Process semantics of Petri nets and Mazurkiewicz traces of Zielonka automata) proved useful in recovering information about the participants of events and independence of concurrent events. For instance, in the Process semantics of Petri nets, two concurrent events can be executed in any order or even simultaneously, and thus we can distinguish concurrent execution from mere nondeterminism. However, these formalisms have fixed interaction structures that define interdependence of events in a static way, and thus leads to a straightforward partial order semantics. Indeed, while the interdependence of events is statically defined based on the structure of a Petri net, it is also defined based on the domains of events of Zielonka automata which are fixed in advance.

* This work is funded by the ERC consolidator grant D-SynMA (No. 772459) and the Swedish research council grants: SynTM (No. 2020-03401) and VR project (No. 2020-04963).
In this paper, we propose a partial order semantics of computations under reconfiguration. In such settings, dependencies among events emerge dynamically as side-effects of interaction, and thus we handle these emergencies while ensuring that the semantics defines the actual behaviour of the system. Our approach consists of characterising reconfiguration points and their corresponding scheduling decisions in a single structure, while preserving a true-concurrent execution of independent events. Our semantics allows reasoning about the individual behaviour of agents composing the system and their interaction information. We test our results on Petri net with inhibitor arcs (PTI-nets), and Channelled Transition Systems (CTS) [65]. These modelling frameworks cover a wide range of interaction capabilities alongside reconfiguration from two different schools of concurrency. In fact, inhibitor arcs add a restricted form of reconfiguration to Petri nets while CTS can be considered as a generalisation of Zielonka automata, supporting rich interactions alongside reconfiguration.

Contributions. We define specialised partial orders, that we call labelled partial orders (LPO for short), to represent computations. An LPO is a representation of a specific computation. That is, given a system consisting of a set of agents, we can construct an LPO by only considering the local views of individual agents and their interaction information. An LPO defines how the individual computations of agents are related, and also how different events are related. In the spirit of Mazurkiewicz traces, the states of different agents are (strictly) incomparable, that is there is no notion of a global state. This way we can easily single out finite sequences of computation steps where an agent or a (small) group of agents execute independently. We can also distinguish individual events from joint ones. Despite the fact that an LPO may refer to reconfiguration points, it cannot fully characterise reconfiguration in a single structure. For this reason, we introduce glued labeled partial orders (g-LPO, for short), that is an extension of LPO with glue to separate a non-deterministic choice from forced scheduling due to reconfiguration. Intuitively, two elements are glued from the point of view of another element if they both happen either before or after said element. We show that a g-LPO is sufficient to represent LPO computations that differ in scheduling due to reconfiguration. We also show that LPO computations belonging to different g-LPO(s) are different due to nondeterministic selection of independent events.

The paper is organised as follows: In Sect. 2 we informally present our partial order semantics and in Sect. 3 we introduce the necessary background. In Sect. 4 we provide LPO semantics for PTI-nets and CTSs. In Sect. 5 we define glued partial orders and the corresponding extension to both PTI-nets and CTSs. In Sect. 6 we prove important results on g-LPO with respect to reconfiguration and nondeterminism. In Sect. 7 we present concluding remarks, related works, and future directions. All proofs are included in the appendix.

2 Labelled Partial Order Computations in a Nutshell

In this section, we use a fragment of a PTI-net to informally illustrate the LPO semantics under reconfiguration and the idea behind g-LPO.

We consider the PTI-net in Fig. 1(a), where we interpret reconfiguration and concurrency in the following way: each token represents an individual agent and the structure of the net defines the combined behaviour. The places of the net, denoted by circles, define the states of the different agents during execution. The transitions, denoted by squares, can either refer to synchronisation points (e.g., $t_4$) or individual computation steps (e.g., $t_3$ and $t_4$).

Arrows define which places require to have tokens to enable a transition and the places to put tokens after firing. In our examples all arrows consume/produce one token. For instance, transition $t_1$ may fire when there is at least one token in both $p_1$ and $p_2$. Transition firing induces removal of tokens from input places and addition of tokens in output places. Thus, when $t_1$ fires, one token is removed from $p_1$ and one from $p_2$ and one token is placed in $p_3$ and $p_4$, each. Sometimes a place can choose nondeterministically which transition to participate in (e.g., $t_4$ chooses $t_2$ or $t_3$). A place can inhibit the firing of some transition (e.g., $p_3$ inhibiting $t_4$) using an inhibitor arc $(p_3 \rightarrow t_4)$. While the place contains a token it inhibits the transition. We interpret this as the agent represented by the token (e.g., in $p_3$) starting to listen to the transition ($t_4$), but it cannot participate, and thus it inhibits its execution. In our example, in $p_1$ the agent is not listening to $t_4$, but once $t_1$ is executed the agent reconfigures its interaction interface and starts listening. This means that $t_4$ may only fire either before a token is placed in $p_3$ or after the token is removed. Clearly, this can only happen when $t_4$ either happens before $t_1$ or after $t_2$. Thus from the point of view of $t_4$ both $t_1$ and $t_2$ are considered as a single block, and their execution cannot be interrupted. Namely, the only viable sequences of execution (in case $t_2$ is scheduled later) are $t_4, t_1, t_2$ or $t_1, t_2, t_4$. Note that this is only from the point of view of $t_4$ and has no implications for other transitions. Indeed,
other transitions can have a different point of view (e.g., $t_5$). This creates a forced interleaving in a non-trivial way due to the occurrence of non-observable events (i.e., reconfiguration) that we cannot reason about from a global perspective. Furthermore, these dependencies among events emerge dynamically as side-effects of interaction, and thus put the correctness of partial order semantics at stake.

To handle this issue, we introduce a partial order semantics of computations under reconfiguration. We handle the above mentioned emergences by characterising reconfiguration points and their corresponding scheduling decisions in a single structure, while preserving a true-concurrent execution of independent events. Our semantics allows reasoning about the individual behaviour of agents composing the system and their interaction information.

We illustrate our LPO and g-LPO semantics in Fig. 1(b), which characterises all possible (maximal) computations of the net. Here, we use the arrow $\rightarrow$ to indicate a happen before relation.

The two figures succinctly encode three possible LPOs: (i) the LPO obtained from Fig. 1(b) left structure with the dashed arrow from $t_4$ to $t_1$; (ii) the LPO obtained from Fig. 1(b) left structure with the dashed arrow from $t_2$ to $t_4$; and (iii) the LPO obtained from Fig. 1(b) right structure with the dashed arrow from $t_4$ to $t_1$. LPOs (i) and (ii) agree that the token in $p_4$ nondeterministically chooses the transition $t_2$ while in (iii) the nondeterministic choice is $t_3$. All LPOs capture information about interaction and interdependence among events. Indeed, in all cases we see that both $p_1$ and $p_2$ synchronize through the transition $t_1$. Places that are not strictly ordered with respect to a common transition are considered concurrent. Thus, as in Mazurkiewicz traces there is no notion of a global state. Notice that LPOs (i) and (ii) differ only in the forced interleaving of $t_4$ with respect to the block $t_1, t_2$.

Notice that both LPOs (i) and (ii) have information both on reconfiguration and nondeterminism, but each individually cannot be used to distinguish the hidden reconfiguration. In fact, $t_4 \rightarrow t_1$ in (i) indicates that $t_4$ happened before a reconfiguration caused by $t_1$, and $t_2 \rightarrow t_4$ in (ii) indicates that $t_4$ happened after the reconfiguration. In (iii), due to the different nondeterministic choice, the only possible case we have to consider is that of $t_4$ happening before $t_1$.

This suggests that we can actually isolate reconfiguration from nondeterminism by using a more sophisticated structure than LPO, and thus expose the difference in a way that allows reasoning about these hidden events from a global perspective. For this reason, we define g-LPO computations, that are an extension of LPO with a notion of glue.

For PTI-nets like fixed systems, a g-LPO simply drops strict ordering of events with respect to each other (like $t_4 \rightarrow t_1$ or $t_2 \rightarrow t_4$), and instead assigns each event a (possibly empty) glue relation defining the glued elements from the point of view of that event. The glue relation is defined based on reconfiguration points, and in case of Petri nets is based on inhibitor arcs. We will see later how this is defined in a more dynamic and compositional model like CTS, where structural information does not simply exist. There, the g-LPO has to account also to event-to-event ordering when sharing the same communication channel.

Consider now the structures in Fig. 1(b) without the dashed arrows and, now, with an explanation of the red arrows. These two structures are each a g-LPO. For the one on the left, since $p_3$ inhibits $t_4$ all existing incoming and outgoing edges from $p_3$ are glued to $p_3$. Thus, $t_4$’s glue relation includes these edges (in red). All other transitions have empty glue relations because they are not inhibited. As they are not inhibited, their interdependence is well-captured statically based on the structure of the net. Note that the glue relation is not required to be transitive and the glue only relates places and transitions. In
the structure on the right of the figure, \( t_1 \) is glued only to \( p_4 \). As \( t_3 \) is scheduled rather than \( t_2 \), then \( p_3 \) remains as a maximal element.

As we show later, a single g-LPO can be used to characterise reconfiguration and separate it from other sources of nondeterminism in the system.

### 3 Preliminaries: Labeled Partial Orders

We use partial orders to represent computations. We specialize notations to match our needs.

A **partial order** (PO, for short) is a binary relation \( \leq \) over a set \( O \) that is reflexive, antisymmetric, and transitive. We use \( a < b \) for \( a \leq b \) and \( a \neq b \). We use \( a \neq b \) for \( a \leq b \) and \( b \leq a \), i.e., \( a \) and \( b \) are incomparable.

A **labeled partial order** (LPO, for short) is \( (O, \rightarrow_c, \rightarrow_i, \Sigma, \gamma, \Lambda) \), where \( O = V \uplus E \) is a set of elements partitioned to nodes and edges, respectively, \( \rightarrow_c \) and \( \rightarrow_i \) are disjoint, anti-reflexive, anti-symmetric, and non-transitive communication and interleaving order relations over \( O \). We have \( \rightarrow_c \subseteq V \times E \cup E \times V \) and \( \rightarrow_i \subseteq E \times E \). When \( \rightarrow_i = \emptyset \) we omit it from the tuple. The relation \( \leq \) is the reflexive and transitive closure of the union of \( \rightarrow_c \) and \( \rightarrow_i \). We require that \( \leq \) is a partial order. Moreover, \( \Sigma \) is a node alphabet, \( \gamma \) is an edge alphabet, and \( \Lambda : O \rightarrow \Sigma \cup \gamma \) such that \( \Lambda(V) \subseteq \Sigma \) and \( \Lambda(E) \subseteq \gamma \) is the labelling function.

Intuitively, elements in \( V \) can denote states or execution histories of individual agents and elements in \( E \) denote transitions or events. Thus, a history belongs to an individual agent and a transition corresponds to either an individual computational step or a synchronisation point among multiple agents. The relation \( \rightarrow_c \) captures participation in communication and the relation \( \rightarrow_i \) captures order requirements.

We denote \( \rightarrow = \rightarrow_c \cup \rightarrow_i \). Given an element \( a \in O \) we write \( ^*a \) for \( \{ b \mid b \rightarrow a \} \) and \( a^* \) for \( \{ b \mid a \rightarrow b \} \).

### 4 LPO Semantics

In this section, we present Petri Nets with inhibitor arcs [16,12] and Channeled Transition Systems [6,5] and we provide each with a labelled partial order semantics. The labelled partial order semantics of Petri nets extends occurrence nets [21] with event-to-event connections that allow to capture reconfigurations. We include in appendix the labelled partial order semantics of asynchronous automata, which do not require the relation \( \rightarrow_i \) and, thus, show that the separation of results in this paper only make sense in reconfigurable systems.

#### 4.1 Petri Nets with Inhibitor Arcs (PTI-nets)

A Petri net \( N \) with inhibitor arcs is a bipartite directed graph \( N = (P, T, F, I) \), where \( P \) and \( T \) are the set of places and transitions such that \( P \cap T = \emptyset \), \( F : (P \times T) \cup (T \times P) \rightarrow N \) is the flow relation, and \( I \subseteq (P \times T) \) is the inhibiting relation. We write \( (s, s') \in F \) for \( F(s, s') > 0 \). We restrict attention to Petri nets where all transitions have a non-empty preset.

The configuration of a Petri net at a time instant is defined by means of a **marking**. Formally, let \( N \) be a Petri net with a set of places \( P = \{p_1, \ldots, p_k\} \). A marking is a function \( m : P \rightarrow \mathbb{N} \) and is defined as a vector \( \vec{m} = m[p_1], \ldots, m[p_k] \) where \( m[p_i] \) corresponds to the number of tokens in \( p_i \), for \( i = 1, \ldots, k \). Vectors can be added, subtracted, and compared in the usual way. We assume some initial marking \( m_0 \).

For \( p \in P \) let \( \vec{p} \) be the singleton vector \( \vec{p} : P \rightarrow \{0, 1\} \) such that \( \vec{p}(p) = 1 \) and \( \vec{p}(p') = 0 \) for every \( p' \neq p \).

For a transition \( t \in T \) we define the **pre-vector** of \( t \), denoted by \( ^*t \), to represent the vector \( ^*t[p_1], \ldots, ^*t[p_k] \), where \( ^*t[p_i] = F(p_i, t) \). Similarly, the **post-vector** of \( t \) is \( ^t = ^t[p_1], \ldots, ^t[p_k] \), where \( ^t[p_i] = F(t, p_i) \).

An inhibitor arc from a place to a transition means that the transition can only fire if no token is on that place. The inhibitor set of a transition \( t \) is the set \( ^t = \{ p \in P \mid (p, t) \in I \} \), and represents the places to be “tested for absence” of tokens. That is, an inhibiting place allows to prevent the transition firing.

A transition \( t \) is enabled at \( m \) if for every \( p \in ^*t \) we have \( m[p] \geq F(p, t) \) and all inhibitor places are empty, i.e., for every \( p \in ^t \) we have \( m[p] = 0 \). Note that if for some \( t \) and \( p \in ^t \) we have \( (p, t) \in F \) then \( t \) can never fire, thus it is called blocked.

A transition \( t \) enabled at marking \( m \) can fire and produce a new marking \( m' \) such that \( m' = m - ^*t + ^t \), denoted \( m[t]m' \). That is, for every place \( p \in P \), the firing transition \( t \) consumes \( F(p, t) \) tokens and produces \( F(t, p) \) tokens.

**Definition 1 (History).** We define the set of histories of a net \( N \) by induction.

We define a special transition \( t_c \) such that \( t_c^* = m_0 \). The pair \( (\emptyset, t_c) \) is a t-history. Note that \( t_c \) is not a transition in \( T \).
For a place $p$, let $h = (S, t)$ be a $t$-history such that $t^*(p) > 0$. Then we have $(h, p, t^*(p))$ is a $p$-history. That is, given a $t$-history $h$ ending in transition $t$, where $p$ is in $t^*$, then the combination of $h$, $p$, and the number of tokens that $t$ puts in form a $p$-history.

Consider a transition $t \in T$. A $t$-history is a pair $(S, t)$, where $S = \{(h_1, i_1), \ldots, (h_n, i_n)\}$ is a multiset satisfying the following. For every $j$ we have $h_j = (-, p, c_j)$ is a $p$-history, where $c_j \geq i_j$ and $t = \sum_j i_j \cdot \tau_j$. That is, the $t$-history identifies the set of $p$-histories from which $t$ takes tokens with the multiplicity of tokens taken from every $p$-history.

Let $\text{hist}(N)$ be the set of all histories of $N$ partitioned to $\text{hist}_p(N)$ and $\text{hist}(N)$ in the obvious way. Given a $t$-history $h = (S, t)$ and a $p$-history $h'$ we write $h(h')$ for the number of appearances of $h'$ in the multiset $S$.

Now, everything is in place to define the labelled partial order semantics of a PTI-net.

**Definition 2 (LPO-computation).** A computation of $N$ is an LPO $(O, \rightarrow_c, \rightarrow_t, \Sigma, T, L)$, where $V \subseteq \text{hist}_p(N)$, $E \subseteq \text{hist}(N)$, $\Sigma = P$, $T = T$, for a $p$-history $v = (-, p, i)$ we have $L(v) = p$ and for a $t$-history $(S, t)$ we have $L(e) = t$, and such that:

1. The $t$-history $(\emptyset, t)$ is the unique minimal element according to $\leq$.
2. For a $p$-history $v = (e, p, i) \in V$ we have $e \leq E$ and $e$ is the unique edge such that $e \rightarrow_c v$.
3. For a $p$-history $v = (h, p, i) \in V$, let $c_1, \ldots, c_j$ be the $t$-histories such that $v \rightarrow_c c_j$. Then, for every $j$ we have $c_j(v) > 0$ and $\sum_j c_j(v) \leq i$. That is, $v$ leads to $t$-histories that contain it with the multiplicity of $v$ being respected.
4. For every $e \in E$, where $e = ((v_1, i_1), \ldots, (v_n, i_n)), t)$, all the following hold:
   (a) $\bullet \cap V = \{v_1, \ldots, v_n\}$ and $\bullet \cap V = \{(e, p, t^*(p)) \mid t^*(p) > 0\}$.
   (b) For every $v \in V$ such that $L(v) \in \delta(V)$ we have $e \leq v$ or $v \leq e$.
   (c) If $e \rightarrow v'$ then there is some $v$ such that either (i) $v \rightarrow_c e$ and $(L(v), L(v')) \in I$ or (ii) $v' \rightarrow_c v$ and $(L(v), L(e)) \in I$.

That is, a computation starts from the dummy transition $t$, which establishes the initial marking. Every other transition is a $t$-history that connects the $p$-histories that it contains. If a place inhibits a transition then either the transition happens before a token arrives to the place or after the token left that place. This is possible by adding direct interleaving dependencies ($\rightarrow_t$) between edges. Namely, if $p$ inhibits $t$ then either $t$ happens before the transition putting token in $p$ or after the transition taking the token from $p$.

### 4.2 Channelled Transition Systems (CTS)

A Channelled Transition System (CTS) is a tuple of the form $T = (C, A, B, S, S_0, R, L, \tau)$, where $C$ is a set of channels, including the broadcast channel ($\cdot$), $A$ is a state alphabet, $B$ is a transition alphabet, $S$ is a set of states, $S_0 \in S$ is an initial state, $R \subseteq S \times B \times S$ is a transition relation, $L : S \rightarrow A$ is a labelling function, and $\tau : S \rightarrow 2^C$ is a channel-listening function such that for every $s \in S$ we have $s \in \text{LS}(s)$. That is, a CTS is listening to the broadcast channel in every state. We assume that $B = B^+ \times \{!?, \} \times C$, for some set $B^+$. That is, every transition labeled with some $b \in B$ is either a message send ($!$) or a message receive ($?$) on some channel $c \in C$.

Given $(b^+, !, c) \in B$ we write $?(b^+, t, c)$ for $(b^+, ?, c)$ and $\text{ch}(b^+, -, c)$ for $c$. That is, $?/(b)$ is the corresponding receive transition of a send transition $b$ and $\text{ch}(b)$ is the channel of $b$.

For a receive transition $b = (b^+, ?, c)$ and a state $s \in S$ we write $s \rightarrow_b c$ if $c \in \text{LS}(s)$ and there is some $s'$ such that $(s, b, s') \in R$. That is, $s$ is listening on channel $c$ and can participate, i.e., has an outgoing receive transition for $b$. We write $s \not\rightarrow_b$ if $c \in \text{LS}(s)$ and it is not the case that $s \rightarrow_b$. That is, $s$ is listening on channel $c$ and is not able to participate.

A history $h = s_0, \ldots, s_n$ is a finite sequence of states such that $s_0 \in S_0$ and for every $0 \leq i < n$ we have that $(s_i, b_i, s_{i+1}) \in R$ for some $b_i \in B$. The length of $h$ is $n + 1$, denoted $|h|$. For convenience we generalise notations applying to states to apply to histories. For example, we write $c \in \text{LS}(h)$ when $c \in \text{LS}(s_i)$, $h \rightarrow_b$ when $s_i \rightarrow_b$ and $h \not\rightarrow_b$ for $s_i \not\rightarrow_b$. Similarly, if $h = s_0, \ldots, s_n$ and $b^! = b_0, \ldots, b_n$, then $(s_0, b_0, s_1, \ldots, s_{n+1}) \in R$, we write $(h, b_0, b^!') \in R$. Let $\text{hist}(T)$ be the set of all histories of $T$. An execution $\pi = s_0, b_0, s_1, \ldots$ is an infinite sequence such that for every $i \geq 0$ we have $(s_i, b_i, s_{i+1}) \in R$ and $b_i \in B$. Thus, every prefix of $\pi$ (projected on states) is a history.

The linear semantics for CTS is given by a parallel composition operator over a set of CTSs. We include the full definition in appendix and refer the reader to [6]. Intuitively, multicast channels are blocking. All agents who are listening to the channel must be able to participate in the communication.
in order for a send to be possible. The broadcast channel, on the other hand, is non-blocking. Agents always listen to the broadcast channel. However, if they cannot participate in a communication it still goes on without them.

The PTI-net in Fig. 1(a) can be modelled as the parallel composition of the CTSs in Fig. 2 where we label states with the listening function. Starting from the initial states, we have that either \((v_1, !, c)\) or \((v_2, !, d)\) can be sent. The former is an individual transition of agent \(T_1\) while the latter is a joint transition between \(T_2\) and \(T_3\) where \(T_2\) sends and \(T_3\) receives. Note that \(T_3\) is initially connected to channel \(d\). If \((v_2, !, d)\) is scheduled first then the listening function of both \(T_2\) and \(T_3\) is reconfigured where \(T_2\) starts listening to channel \(c\) and \(T_3\) starts listening to \(e\). This way, \((v_1, !, c)\) is blocked until \((v_2, !, e)\) is sent. It is not hard to see that a reconfiguration due to changes in the listening function is equivalent to token passing. However, here we can model a more interesting compositional interactions with meaningful message exchange.

Now, everything is in place to define the labelled partial order semantics of a CTS. Consider a system \(S = T_1 \parallel \cdots \parallel T_n\), where \(T_i = (C_i, A_i, B_i, S_i, S'_0, R_i, L, LS_i)\). We denote \(C = \bigcup C_i\) and \(B = \bigcup B_i\).

**Definition 3 (LPO-computation).** A computation of \(S\) is an LPO \((O, \rightarrow_e, \rightarrow_i, \Sigma, \mathcal{T}, L)\), where \(V \subseteq \bigcup \text{hist}(T_i), \Sigma = V, \rightarrow_e = \rightarrow_i \bigcup \rightarrow_r\) is the disjoint union of the send and receive relations, \(\mathcal{T} = \{(v, !, c) \in B\}\), and for \(h \in V\) we have \(L(h) = h\). In addition we require the following:

1. The edge \(e_{\neq} \) such that \(L(e_{\neq}) = (b, !, *)\) is the unique minimal element according to \(\leq\). For every \(i\), we have \(s^0_i \in V\) and \(e_{\neq} \rightarrow_r s^1_i\).
2. If \(h \in V \cap \text{hist}(T_i)\) there is a unique \(e \in E\) such that \(e \rightarrow_r h\). If \(|h| > 1\), there is also a unique \(h' \in V\) such that \(h' \rightarrow_r e\) and either \((h', L(e)), h) \in R_i\) or \((h', ?(L(e)), h) \in R_i\).
3. For every \(h \in V\) there is at most one \(e \in E\) such that \(h \rightarrow_r e\).
4. For every \(e \in E \setminus \{e_{\neq}\}\) there is \(I \subseteq [n]\) such that all the following hold:
   - (a) For every \(i \in I\) we have \(|e \cap \text{hist}(T_i)| = 1\) and \(|e \cap \text{hist}(T_i)| = 1\).
   - (b) There is a unique \(i \in I\) and \(h, h' \in V \cap \text{hist}(T_i)\) such that \((h, L(e)), h') \in R_i\) and \(h \rightarrow_s e \rightarrow_s h'\) and for every \(i' \in I \setminus \{i\}\) there are \(h'', h'''' \in V \cap \text{hist}(T_{i'}\) such that \(h'' \rightarrow_r e \rightarrow_r h''''\) and \((h'', ?(L(e)), h''''') \in R_{i'}\).
   - (c) If \(L(e) = (v, !, c)\) for \(c \neq *\) then for every \(h \in V\) such that \(c \in LS(h)\) we have \(h \leq e\) or \(e \leq h\).
   - (d) If \(L(e) = (v, !, *)\) then for every \(h \in V\) such that \(h \rightarrow_{\gamma(L(e))}\) we have \(h \leq e\) or \(e \leq h\).
5. For every \(e \neq e'\) such that \(ch(e) = ch(e')\) we have \(e \leq e'\) or \(e' \leq e\).
6. If \(e \rightarrow_r e'\) then there is some \(h \in s_0, \ldots, s_j\) such that one of the following holds:
   - (a) \(ch(e) = ch(e')\).
   - (b) \(L(e') = (v, !, c)\) for \(c \neq *\), \(h \rightarrow_e e\) and \(ch(L(e')) \in LS(h)\).
   - (c) \(L(e) = (v, !, c)\) for \(c \neq *\), \(e' \rightarrow_e h\) and \(ch(L(e')) \in LS(h)\).
   - (d) \(L(e') = (v, !, *), h \rightarrow_e e\) and \(h \rightarrow_{\gamma(L(e'))}\).
   - (e) \(L(e) = (v, !, *), e' \rightarrow_e h\) and \(h \rightarrow_{\gamma(L(e))}\).

Note that an LPO computation relates histories of individual CTSs, and thus allows to draw relations among finite sequences of individual computation steps of one CTS (or a group of CTSs) with respect to others; furthermore, a CTS is always listening to the broadcast channel, and thus, it becomes mandatory to order broadcast messages that enable/disable participation to each other.

More precisely, C1 ensures that a unique broadcast initiates all the initial states of \(T_i\) for all \(i\) and that nothing happens before that. As expected, C2 and C3 ensure that an LPO defines a unique resolution of a nondeterministic choice in every single step. Moreover, C4 models interactions, where (a) and (b) model synchronisation while (c)-(f) model ordering due to schedule imposed by using global resources and
restrictions due to reconfiguration. First, communications on the same channel must be ordered. Then, a multicast must be ordered with respect to every individual history that listens to it. Furthermore, a broadcast must be ordered with respect to every individual history that can participate in it. Clearly, the last two requirements are crucial to preserve the blocking semantics of multicasts and the input enabledness of broadcasts.

Thus, for a multicast, if a history \( h \) blocks the multicast execution then either \( h \) can be extended so that the multicast is released or the multicast happens directly before \( h \) is reached. We solve this by adding a strict ordering between multicasts. The same holds for a broadcast, but in this case we handle input enabledness of broadcast rather.

We will use \( \text{comp}(S) \) for \( S \) being a Petri net or CTS, to denote the set of LPO computations of \( S \).

5 Partial Order with Glue

In this section we extend labeled partial orders with glue. Intuitively, two elements are glued from the point of view of another element if they both happen either before or after said element.

Definition 4 (Glue relation). A Glue over a set \( O \) and a relation \( \rightarrow_c \subseteq O \times O \) is a relation \( R \subseteq \rightarrow_c \).

Intuitively, a glue relation \( R \) over the set \( O \) and a relation \( \rightarrow_c \) defines pairs of elements that are glued together.

Definition 5 (Glued LPO). A glued labeled partial order (gLPO, for short) is \( \text{LPG} = (P, G, E) \), where \( P = (O = V \cup E, \rightarrow_c, \rightarrow_g, \Sigma, \mathcal{T}, L) \) is an LPO, \( G = \{G_1, \ldots, G_k\} \) is a set of Glue relations over \( O \) and \( \rightarrow_c \), and \( E : \mathcal{T} \rightarrow G \) labels elements in \( E \) (through their edge labels) by glue relations.

Definition 6 (gLPO-refinement). An LPO \( \text{LPO} = (O, \rightarrow_c, \rightarrow_g, \Sigma, \mathcal{T}, L) \) over \( O = V \cup E \) refines a gLPO \( \text{LPG} = (P_g, G, E) \), denoted \( \text{LPO} \preceq \text{LPG} \), where \( P_g = (O, \rightarrow_c, \rightarrow_g, \Sigma, \mathcal{T}, L) \) if the following conditions hold:

- For every \( e \in E \) and \( (a, b) \in \mathcal{E}(L)(e) \) we have \( e \leq a \) or \( b \leq e \).
- \( \rightarrow_g \subseteq \rightarrow_c \) and \( (e, e') \in (\rightarrow_c \setminus \rightarrow_g) \) implies \( (e', e) \in \mathcal{E}(L)(e') \) for some \( v \) or \( (v, e) \in \mathcal{E}(L)(e) \) for some \( v \).

That is, the two share the relation \( \rightarrow_c \), the relation \( \rightarrow_g \) is preserved and extended by extra interleaving to capture the glue. In order to respect the glue, an edge that is glued to a pair \( (a, b) \) must happen either before or after \( a \) or \( b \).

We show now that gLPOs enable to remove parts of the interleaving order relation for both PTI-nets and CTSs. gLPOs capture better reconfiguration by combining multiple order choices due to the same reconfiguration into the same g-computation.

5.1 Glue Computations for PTI-nets

Let \( N = (P, T, F, I) \) be a PTI-net and \( m_0 \) its initial marking. We now define a g-computation. The differences from the definition of LPO (Definition 2) are highlighted with a “∗”.

Definition 7 (g-computation). A g-computation of \( N \) is a gLPO \( (P, G, E) \), where \( P = (O, \rightarrow_c, \rightarrow_g, \Sigma, \mathcal{T}, L) \) is a LPO and \( \rightarrow_c \), the components \( V, E, \Sigma, \mathcal{T}, \) and \( L \) are as for LPO, and the following holds.

N1. The t-history \((0, t_0)\) is the unique minimal element according to \( \leq \).

N2. For a p-history \( v = (e, p, i) \in V \) we have \( e \in E \) and \( e \) is the unique edge such that \( e \rightarrow_c v \).

N3. For a p-history \( v = (h, p, i) \in V \), let \( e_1, \ldots, e_j \) be the t-histories such that \( v \rightarrow e_j \). Then, for every \( j \) we have \( e_j(v) > 0 \) and \( \sum_j e_j(v) \leq i \). That is, \( v \) leads t-histories that contain it with the multiplicity of \( v \) being respected.

*N4. For every \( e \in E \), where \( e = \{ (v_1, i_1), \ldots, (v_n, i_n) \} \) the following holds:

\* \( (a)^* = \{ v_1, \ldots, v_n \} \) and \( \star = \{ (e, p, \star(p)) \mid \star(p) > 0 \} \).

*N5. For every \( t \in T \) we have:

\( \mathcal{E}(t) = \{ (v, e) \mid v \rightarrow_c e \) and \( (L(v), t) \in I \} \) \cup \n
\( \{ (e, v) \mid e \rightarrow_c v \) and \( (L(v), t) \in I \} \)

That is, we drop \( \rightarrow_i \) and assign each inhibited event (or transition) with a glue relation. Namely, for every transition \( t \) add all existing ingoing and outgoing transitions of places that inhibit \( t \).

We use \( \text{comp}_g(N) \) to denote the set of g-computations of Petri net \( N \).

Theorem 1. Given a PTI-net \( N \), \( \text{comp}(N) = \{ \pi \mid \pi \leq \pi_g \) and \( \pi_g \in \text{comp}_g(N) \} \).
5.2 Glue Computations for CTSs

Consider a system $S = \mathcal{T}_1 \parallel \cdots \parallel \mathcal{T}_n$, where $\mathcal{T}_i = \langle C_i, A_i, B_i, S_i, S'_i, R_i, L_i, \text{LS}_i \rangle$. We denote $C = \bigcup_i C_i$ and $B = \bigcup_i B_i$.

We now define a $g$-computation for CTS. As before, the differences from the definition of LPO (Definition 3) are highlighted with a “$\ast$”.

**Definition 8 ($g$-computation).** A $g$-computation of $S$ is a $g$-LPO $\mathcal{P} = (P, \mathcal{G}, \mathcal{E})$, where $P = (O, \rightarrow_i, \rightarrow_c, \Sigma, \mathcal{T}, L_V, L_E)$ and $V$, $E$, $\Sigma$, $\mathcal{T}$, and $L$ are as before, $\rightarrow_c = \rightarrow_\ast \bigcup \rightarrow_r$, and in addition:

1. The edge $e_i$ such that $L(e_i) = (b, \text{!, } \ast)$ is the unique minimal element according to $\leq$. For every $i$, we have $s^0_i \in V$ and $e_i \rightarrow_r s^0_i$.
2. If $h \in V \cap \text{hist}(\mathcal{T}_i)$ there is a unique $e \in E$ such that $e \rightarrow_c h$. If $|h| > 1$, there is also a unique $h' \in V$ such that $h' \rightarrow_c e$ and either $(h', L(e), h) \in R_i$ or $(h', \ast, L(e), h) \in R_i$.
3. For every $h \in V$ there is at most one $e \in E$ such that $h \rightarrow_c e$.

* C4. For every $e \in E \setminus \{e_i\}$ there is $I \subseteq \{i\}$ such that all the following hold:
   1. For every $i \in I$ we have $|e \cap \text{hist}(\mathcal{T}_i)| = 1$ and $|e \cap \text{hist}(\mathcal{T}_i)| = 1$.
   2. There is a unique $i \in I$ and $h, h' \in V \cap \text{hist}(\mathcal{T}_i)$ such that $(h, L(e), h') \in R_i$ and $h \rightarrow_c e \rightarrow_s h'$ and for every $e' \in I \setminus \{i\}$ there are $h'', h''' \in V \cap \text{hist}(\mathcal{T}_e)$ such that $h'' \rightarrow_r e \rightarrow_r h'''$ and $(h'', \ast, L(e), h''') \in R_i$.
5. For every $e \neq e'$ such that $ch(e) = ch(e')$ we have $e \leq e'$ or $e' \leq e$.

* C6. If $e \rightarrow e'$ then the following holds:
   1. $ch(e) = ch(e')$.
   2. For every $(v_i, !, c) \in B$ then $E((v_i, !, c)) = \{(h, e) \mid e \neq \ast, h \rightarrow_c e \text{ and } e \in \text{LS}(h)\} \cup \{(e, h) \mid e \neq \ast, e \rightarrow_c h \text{ and } e \in \text{LS}(h)\} \cup \{(h, c) \mid c = \ast, h \rightarrow_c e \text{ and } e \rightarrow_i (v_i, !, c)\} \cup \{(e, c) \mid c = \ast, e \rightarrow_r h \text{ and } h \rightarrow_i (v_i, g, c)\}$

We drop from the interleaving relation all order relations that correspond to reconfiguration and keep only those that correspond to the usage of a common resource. Furthermore, we assign each broadcast and multicast message with a glue relation. Namely, for every multicast $m$ add all existing ingoing and outgoing messages of histories that blocks $m$ execution; for every broadcast $b$ add all existing ingoing and outgoing messages of histories that may participate in $m$. Note that, for the case of broadcast, the rationale is that if such histories can participate in a broadcast then they cannot be enabled independently from the broadcast. Notice that * C6 adds one glue for every multicast channel but one for every broadcast message.

We use $\text{comp}_g(S)$ to denote the set of $g$-computations of CTS $S$.

**Theorem 2.** Given a CTS $\mathcal{T}$, $\text{comp}_g(\mathcal{T}) = \{\pi \mid \pi \preceq \pi_g \land \pi_g \in \text{comp}_g(\mathcal{T})\}$.

6 Separating Choice and Reconfiguration-Forced Interleaving

We show that $g$-LPOs capture the differences between nondeterministic choice, which corresponds to different $g$-LPOs, and interleaving choices due to reconfiguration, which correspond to different ways to refer to glue. For both PTI-nets and CTSs we show that distinct $g$-LPOs contain different nondeterministic or order choices.

6.1 Choice vs Interleaving in PTI-nets

A choice is a situation where a set of tokens have exactly the same history and they do a different exchange.

We show that every two distinct $g$-computation of the same net have a set of tokens that are different. That is, they participate in a different transition in the two $g$-computation. This includes the option of tokens in one $g$-computation participating in a transition and tokens in the other $g$-computation not continuing.

**Theorem 3.** Given a Petri net $P$ and two different $g$-LPOs $G_1, G_2 \in \text{comp}_g(N)$ then there exists a set of nodes $v_1, \ldots, v_n$ appearing in both $G_1$ and in $G_2$ such that one of the following holds:
1. There is a node $v_i$ such that the number of tokens not taken from $v_i$ in $G_1$ and $G_2$ is different.
2. There is a set of $p$-histories $v_1, \ldots, v_n$ that participate in some transition $t$ in $G_i$ but not in $G_{3-i}$.

Notice that item 2 includes the case where the transition $t$ happens in both $G_1$ and $G_2$ but takes a different number of tokens from every node. This difference is indeed significant as the nodes communicate via the identified transition and share the knowledge about the difference.

Theorem 3 is not true for LPOs. This is already shown by the very simple examples in Figure 1(b). Indeed, in the two LPOs corresponding to each of the dashed arcs in the figure all sets of nodes participate in exactly the same transitions.

We note that by the proof of Theorem 1 all the LPOs that disagree only on forced interleavings are refined by the same g-LPO.

6.2 Choice vs Interleaving in CTSs

We now proceed with CTS. Here, a choice is either a situation where all the agents have exactly the same history and at least one agent participates in a different communication or communications on the same channel are ordered in a different way. Notice that as channels are global resources, the case that changing the order of communications on a channel does not have side effects is accidental. Indeed, such a change of order could have side effects and constitutes a different choice.

We show that every two distinct g-computations of the same CTS have a joint history of some agent that “sees the difference” or a channel that transfers messages in a different order. Difference for a history is either maximality in one and not the other or extension by different communications in the two same channels.

**Theorem 4.** Given a CTS $T$ and two different g-LPOs $G_1, G_2 \in \text{comp}_g(N)$ then one of the following holds:

1. For some agent $i$ there exists a history $h_i$ in both $G_1$ and $G_2$ such that either $h_i$ is maximal in $G_1$ and not maximal $G_{3-i}$;
2. For some agent $i$ there exists a history $h_i$ in both $G_1$ and $G_2$ such that the edges $e_1$ and $e_2$ such that $h_1 \rightarrow e_1 h_1$ and $h_1 \rightarrow e_2 h_2$ we have $L^E_1(e_1) \neq L^E_2(e_2)$;
3. or; There is a pair of agents $i$ and $i'$ and histories $h_i$ and $h_{i'}$ in both $G_1$ and $G_2$ such that the order between the communications of $i$ and $i'$ is different in $G_1$ and $G_2$.

As for PTI-nets, Theorem 3 is not true for LPOs. This does not hold as shown by the LPOs and g-LPO of the CTS in Figure 2. Recall, that this CTS has the same LPOs and g-LPOs depicted in Figure 1(b).

We note that by the proof of Theorem 2 all the LPOs that disagree only on forced interleavings are refined by the same g-LPO.

7 Concluding Remarks

In this paper, we laid down the basis to reason about reconfiguration in concurrent systems from a global perspective. We showed how to isolate forced interleaving decisions of the system due to reconfiguration, and other decisions due to standard concurrent execution of independent events. To test our results, we considered PTI-nets [10,12] and CTS [5,6] which cover a wide range of interaction capabilities alongside reconfiguration from two different schools of concurrency. We proposed, for both, a partial order semantics, named LPO, of computations under reconfiguration. An LPO extends occurrence nets [27] with event-to-event connections that allows to refer to reconfiguration points. Moreover, to fully characterise reconfiguration in a single structure, we proposed a glued LPO semantics, named g-LPO. The latter is able to fully isolate scheduling decisions due to reconfiguration from the ones due to standard concurrency. We show that any LPO computation is only a refinement of some g-LPO of the same system. Finally, we prove important results on g-LPO with respect to reconfiguration and nondeterminism.

For future work, we would like to exploit g-LPO semantics to verify properties about reconfiguration and interaction in general. Namely, we would like to define a specification logic that considers g-LPO computations as the underlying structure rather than the standard linear computations. Clearly, logics over linear structures easily distinguish different interleavings of the same LPO. However, different linearisations of the same LPO are either all computations of a system or none of them is. Similarly, a logic defined over LPOs would easily distinguish different schedules that relate to reconfiguration. Again, these
different LPOs are either all computations of a system or none of them is. By considering g-LPOs as the underlying structure we can create specifications that do not distinguish between different schedules that correspond to the same choices of the system. Our view is that such a specification language that incorporates elements of Strategy logic [14] and LTOL [5] would not only allow us to reason about interaction and reconfiguration, but also to reason about the local views of agents as well as their combined behaviour.

Related works The prevalent approach to semantics of reconfigurable interactions is based on linear order semantics (cf. Pi-calculus [23,15], Mobile Ambients [13], Applied Pi-calculus [1], Psi-calculus [11,8], concurrent constraint programming [25,18], fusion calculus [28], the \( AbC \) calculus [31], ReCiPe [5] etc.). This semantics cannot distinguish the different choices of the system from a global perspective, and thus does not facilitate reasoning about reconfiguration from an external observer’s point of view. It also hides information about interactions and possible interdependence among events. In fact, linear order semantics ignores the possible concurrency of events, which can be important e.g. for judging the temporal efficiency of the system [27]. However, it still provides a correct abstraction of the system behaviour, while hiding such details.

Partial order semantics (cf. Process semantics of Petri nets [24,27] and Mazurkiewicz traces of Zielonka automata [29,41]), on the other hand, is able to refer to the interaction and event dependencies, but does not deal very well with reconfiguration. This is because the latter formalisms have fixed interaction structures, and thus the interdependence of events is defined structurally. Reconfiguration, on the other hand, enforces reordering of events dynamically in non-trivial ways, and thus makes defining correct partial order semantics very challenging. As shown in [19], some aspects of concurrency are almost impossible to tackle in both linear-order and partial-order causality-based models, and one of them is PTI-nets [16]. In fact, reconfiguration increases the expressive power of the formalism, e.g., adding inhibitor arcs to Petri nets makes them Turing Powerful [2]. However, this expressive power does not come without expenses. In fact, it prevents most analysis techniques for standard Petri nets [12].

To the best of our knowledge, the closest to our LPO semantics is Relational Structures [19]. In order to capture inhibition they add an additional “not later than” relation to partial orders. Much like our LPOs, this allows to represent the different forced interleavings separately. The emphasis in [19] is on providing a general semantic framework for concurrent systems. Thus, relational structures handle issues like priority and error recovery, which we do not handle. However, relational structures are not concerted directly with separation of choice from interleaving as we are. So the two works serve different purposes and it would be interesting to investigate mutual extensions.

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A Proofs for Section 4 (LPO Semantics)

A.1 Asynchronous automata

We include the LPO semantics of asynchronous automata. As asynchronous automata do not have reconfigurations of communication we only need the communication relation and do not use the interleaving order relation. This makes the notion of glue not relevant for asynchronous automata.

A process is $P = (\text{Act}, S, s^0, \delta)$, where $\text{Act}$ is a finite non-empty alphabet, $S$ is a finite and non-empty set of states, $s^0 \in S$ is an initial state, and $\delta \subseteq S \times \text{Act} \times S$. We also write $\delta : S \times \text{Act} \rightarrow 2^S$ when convenient.

A history $h = s_0, \ldots, s_n$ is a finite sequence such that $s_0 = s^0$ and for every $0 \leq i < n$ we have $s_{i+1} \in \delta(s_i, a_i)$ for some $a_i \in \text{Act}$. The length of $h$ is $n + 1$, denoted $|h|$. For convenience, if $h_1 = s_0, \ldots, s_n$ and $h_2 = s_0, \ldots, s_n, s_{n+1}$ such that $s_{n+1} \in \delta(s_n, a_n)$ we write $h_2 \in \delta(h_1, a_n)$ or $\delta(h_1, a_n, h_2)$. We define $\text{hist}(P)$ to be the set of histories of $P$.

A finite asynchronous automaton $A$ with $n$ processes is $A = (P_1, \ldots, P_n)$ such that each $P_i = (\text{Act}_i, s^0_i, \delta_i)$ is a process. Let $\text{Act} = \bigcup \text{Act}_i$.

Definition 9 (computation). A computation of $A$ is an LPO $(O, \rightarrow_c, \Sigma, \Upsilon, L)$, where $V \subseteq \bigcup \text{hist}(P_i)$, $\Sigma = V$, $L(h) = h$, and $T = \text{Act}$ such that:

1. The edge $e_i$ such that $L(e_i) = a$ for some $a$ is the unique minimal element according to $\leq$. For every $i$, we have $s_0^i \in V$ and $e_i \rightarrow_c s_0^i$.
2. If $h \in V$ there is a unique $e \in E$ such that $e \rightarrow_c h$. If $|h| > 1$, there is also a unique $h' \in V$ such that $h \in \delta(h', L(e))$ and $h' \rightarrow_c e$.
3. For every $h \in V$ there is at most one $e \in E$ such that $h \rightarrow_c e$.
4. For every $e \in E$ there is $I \subseteq [n]$ such that all the following hold:
   (a) $L_{\|_E}(e) \in \bigcap_{i \in I} \text{Act}_i \setminus \bigcup_{i \notin I} \text{Act}_i$
   (b) $e, e^* \in \bigcup_{i \in I} \text{hist}(P_i)$
   (c) For every $i \in I$ we have $|e \cap \text{hist}(P_i)| = 1$ and $|e^* \cap \text{hist}(P_i)| = 1$

That is, a computation starts from an arbitrary joint edge that leads to the initial states of all processes. For every transition, the set of participating processes is all those having the transition’s label in their alphabet. Each participating process has a history that is a predecessor of the transition and a history that is a successor of the transition. The pair of histories that belong to one process satisfy the transition relation of that process.

A.2 Composition for Channeled Transition Systems

We include the definition of the parallel composition operator over CTS. A parallel composition of CTSs is again a CTS.

Definition 10 (Parallel Composition). Given two CTS $\mathcal{T}_1 = (C_1, A_1, B_1, S_1, s_0, R_1, L_1, LS_1)$, where $i \in \{1, 2\}$ their composition $\mathcal{T}_1 || \mathcal{T}_2$ is the following CTS $\mathcal{T} = (C, A, B, S, s_0, R, L, LS)$, where the components of $T$ are:

- $C = C_1 \cup C_2$
- $A = A_1 \times A_2$
- $B = B_1 \cup B_2$
- $S = S_1 \times S_2$
- $LS_1(s_1, s_2) = LS_1(s_1) \cup LS_2(s_2)$
- $R =$

\[
\begin{align*}
\left\{(s_1, s_2, (\_!, \_!, c), (s'_1, s'_2)) \middle| (s_1, (\_!, \_!, c), s'_1) \in R_1, c \in LS_2(s_2) \text{ and } (s_2, (\_!, \_!, c), s'_2) \in R_2 \text{ or } \\
(s_1, (\_!, \_!, c), s'_1) \in R_1, c \in LS_2(s_2), \text{ and } (s_2, (\_!, \_!, c), s'_2) \in R_2 \text{ or } \\
(s_1, (\_!, \_!, c), s'_1) \in R_1, c \not\in LS_2(s_2), \text{ and } s_2 = s'_2 \text{ or } \\
(s_1, (\_!, \_!, c), s'_1) \not\in LS_1(s_1), s_1 = s'_1, \text{ and } (s_2, (\_!, \_!, c), s'_2) \in R_2 \text{ and } (s_2, (\_!, \_!, c), s'_2) \not\in R_2 \text{ or } \\
c \in LS_1(s_1), (s_1, (\_!, \_!, c), s'_1) \in R_1, c \in LS_2(s_2) \text{ and } (s_2, (\_!, \_!, c), s'_2) \in R_2 \text{ or } \\
(s_1, (\_!, \_!, c), s'_1) \in R_1, c \not\in LS_2(s_2), \text{ and } s_2 = s'_2 \text{ or } \\
c \not\in LS_1(s_1), s_1 = s'_1, \text{ and } (s_2, (\_!, \_!, c), s'_2) \in R_2 \text{ and } (s_2, (\_!, \_!, c), s'_2) \not\in R_2 \text{ or }
\right\}
\end{align*}
\]
The transition relation $R$ of the composition defines two modes of interactions, namely multicast and broadcast. In both interaction modes, the composition $T$ sends a message $(v, !, c)$ on channel $c$ (i.e., $((s_1, s_2), (v, !, c), (s'_1, s'_2)) \in R$) if either $T_1$ or $T_2$ is able to generate this message, i.e., $(s_1, (v, !, c), s'_1) \in R_1$ or $(s_2, (v, !, c), s'_2) \in R_2$.

Consider the case of a multicast channel. A multicast is blocking. Thus, a multicast message is sent if either it is received or the channel it is sent on is not listened to. Suppose that a message originates from $T_1$, i.e., $(s_1, (v, !, c), s'_1) \in R_1$. Then, $T_2$ must be able to either receive the message or, in the case that $T_2$ does not listen to the channel, discard it. CTS $T_2$ receives if $(s_2, (v, ?, c), s'_2) \in R_2$. It discards if $c \notin s^2(s_2)$ and $s_2 = s'_2$. The case of $T_2$ sending is dual. Note that $T_2$ might be a composition of other CTS(s), say $T_2 = T_3 \parallel T_4$. In this case, $T_2$ listens to channel $c$ if at least one of $T_3$ or $T_4$ is listening. That is, it could be that either $c \in (L(s_3) \cap L(s_4))$, $c \in (L(s_2) \setminus L(s_3))$, or $c \in (L(s_2) \setminus L(s_4))$. In the first case, both must receive the message. In the latter cases, the listener receives and the non-listener discards. According to when a message is sent by one system, it is propagated to all other connected systems in a joint transition. A multicast is indeed blocking because a connected system cannot discard an incoming message on a channel it is listening to. More precisely, a joint transition $((s_1, s_2), (v, !, c), (s'_1, s'_2))$ where $c \in L(s_2)$ requires that $(s_2, (v, ?, c), s'_2)$ is supplied. In other words, message sending is blocked until all connected receivers are ready to participate in the interaction. Clearly, the latter correspond to inhibition arcs in Petri nets.

Consider now a broadcast. A broadcast is non-blocking. Thus, a broadcast message is either received or discarded. Suppose that a message originates from $T_1$, i.e., $(s_1, (v, !, c), s'_1) \in R_1$. If $T_2$ is receiving, i.e., $(s_2, (v, ?, c), s'_2) \in R_2$ the message is sent. However, by definition, we have that $\ast \in L(s)$ for every $s$ in a CTS. Namely, a system may not disconnect the broadcast channel $\ast$. For this reason, the last part of the transition relation $R$ defines a special case for handling (non-blocking) broadcast. Accordingly, a joint transition $((s_1, s_2), (v, \gamma, \ast), (s'_1, s'_2)) \in R$ where $\gamma \in \{!, ?\}$ is always possible and may not be blocked by any receiver. In fact, if $(\gamma = !)$ and $(s_1, (v, !, \ast), s'_1) \in R_1$ then the joint transition is possible whether $(s_2, (v, ?, \ast), s'_2) \in R_2$ or not. In other words, a broadcast can happen even if there are no receivers. Furthermore, if $(\gamma = ?)$ and $(s_1, (v, ?, \ast), s'_1) \in R_1$ then also the joint transition is possible regardless of the other participants. In other words, a broadcast is received only by interested participants.

### B Proofs for Section 5 (Partial Order with Glue)

**Lemma 1.** Given an LPO $\pi \in \text{comp}(N)$, there exists a corresponding g-LPO $[\pi] \in \text{comp}_g(N)$ such that $\pi \preceq [\pi]$.

**Proof.** Let $[\pi]$ be the g-LPO obtained from $\pi$ by using $\rightarrow_c$ of $\pi$, setting $\rightarrow^c_\emptyset = \emptyset$, and adding the glue relations according to Definition 4.

We have to show that the conditions of Definition 5 hold. Note that by construction both $\pi$ and $[\pi]$ agree on $\rightarrow_c \subseteq V \times E \cup E \times V$ and only disagree in terms of $\rightarrow_\emptyset \subseteq E \times E$ and the glue.

Consider some $t \in T$ and $(a, b) \in E(t)$. We have to show that $e \preceq a$ or $b \preceq e$. By definition we know that $a \rightarrow_e b$. We have the following cases.

- If $a \in V$ and $b \in E$ then $(L(a), t) \in I$. By $N4(b)$ in definition 2 we have that either $e \preceq a$ or $a \preceq e$. If $e \preceq a$ we are done. If $a \preceq e$ then from $a \rightarrow_e b$ it follows that either $e = b$ or $b < e$.

- If $a \in E$ and $b \in V$ then by definition $(L(a), t) \in I$. By $N4(b)$ in definition 2 we have that either $e \preceq b$ or $b \preceq e$. If $e \preceq b$ we are done. If $b \preceq e$ then from $a \rightarrow_e b$ it follows that either $e = a$ or $e < a$.

Consider some $(e, e') \in \rightarrow_\emptyset$. We have to show that either $(e', v) \in E(L(e))$ for some $v$ or $(e, v) \in E(L(e'))$. By definition, we have that there exists $v \in V$ such that one of the following holds.

- $(L(v), L(e')) \in I$ and $v \rightarrow_e e'$. By *N5 in Definition 4* we have that $(v, e) \in E(L(e'))$ as required.
- $(L(v), L(e)) \in I$ and $e' \rightarrow_v e$. By *N5 in Definition 4* we have that $(e', v) \in E(L(e))$ as required.

**Lemma 2.** Given a g-LPO $\pi_1 \in \text{comp}_g(N)$ and an LPO $\pi_2$ such that $\pi_2 \preceq \pi_1$ then $\pi_2 \in \text{comp}(N)$.
Proof. Given that $\pi_2 \leq \pi_1$, it follows that both $\pi_1$ and $\pi_2$ agree on $\rightarrow_c \subseteq V \times E \cup E \times V$ and only disagree in terms of $\rightarrow_i \subseteq E \times E$ and the glue.

It is sufficient to prove that $N_4$, items (b) and (c) in Definition 2 hold for $\pi_2$. Consider some $e \in E$. We have the following cases.

- Consider some $v \in V$ and $e \in E$ such that $L(v) \in \triangleleft L(e)$. In order to show that $\pi_2 \in \text{comp}(N)$ we have to show that $e \leq v$ or $v \leq e$. Let $e'$ be the edge such that $e' \rightarrow_c v$. By Definition 4 (*N5) we have that $(e', v) \in E(L(e'))$. By refinement, we have that either $e \leq e'$, which implies $e \leq v$, or $v \leq e$ as required.

- Consider some $e' \in E$ such that $e \rightarrow_i e'$. By refinement, we have one of the following cases holds.
  - $(e', v) \in E(L(e'))$ for some $v$. By Definition 4 (*N5), we have that $e' \rightarrow_c v$ and $(L(v), L(e')) \in I$ as required.
  - $(v, e) \in E(L(e'))$ for some $v$. By Definition 4 (*N5), we have that $v \rightarrow_c e$ and $(L(v), L(e')) \in I$ as required.

**Theorem 1.** Given a PTI-net $N$, \(\text{comp}(N) = \{\pi \mid \pi \leq \pi_g \land \pi_g \in \text{comp}_g(N)\}\).

Proof. The proof follows directly from Lemma 1 and Lemma 2.

**Lemma 3.** Given an LPO $\pi \in \text{comp}(T)$, there exists a corresponding $g$-LPO $[\pi] \in \text{comp}_g(T)$ such that $\pi \leq [\pi]$.

Proof. Let $[\pi]$ be the g-LPO obtained from $\pi$ by using the partial order induced by $\rightarrow_c$ of $\pi$, by $\rightarrow_i$ of $\pi$ whenever $e \rightarrow_i e'$ implies $ch(e) = ch(e')$, and adding the glue relations according to Definition 8.

Note that by construction both $\pi$ and $[\pi]$ agree on $\rightarrow_c \subseteq V \times E \cup E \times V$ and only agree on $\rightarrow_i \subseteq E \times E$ whenever $e \rightarrow_i e'$ implies $ch(e) = ch(e')$. We have to show that the conditions of Definition 6 hold.

Consider some $e \in E$ and $(a, b) \in E(L(e))$. We have to show that $e \leq a$ or $b \leq e$. By definition we know that $a \rightarrow_i e$. We show that either $e \leq a$ or $b \leq e$. We have the following cases.

- $ch(L(e))$ is a multicast channel:
  - If $a \in V$ and $b \in E$ then by definition $ch(L(a)) \in LS(a)$. By C4(c) in definition 3 we have that either $e \leq a$ or $a \leq e$. If $a \leq e$ then from $a \rightarrow_i b$ it follows that either $e = b$ or $b \leq e$.
  - If $a \in E$ and $b \in V$ then by definition $ch(L(e)) \in LS(b)$. By C4(c) in definition 3 we have that either $e \leq b$ or $b \leq e$. If $b \leq e$ we are done. If $e \leq b$ then from $a \rightarrow_i b$ it follows that either $e = a$ or $e \leq a$.

- $ch(L(e))$ is the broadcast channel:
  - If $a \in V$ and $b \in E$ then $a \rightarrow_{\gamma(L(e))}$. By C4(d) in definition 3 we have that either $e \leq a$ or $a \leq e$. If $e \leq a$ we are done. If $e \leq a$ then from $a \rightarrow_i b$ it follows that either $e = b$ or $b < e$.
  - If $a \in E$ and $b \in V$ then $b \rightarrow_{\gamma(L(e))}$. By C4(d) in definition 3 we have that either $e \leq b$ or $b \leq e$. If $b \leq e$ we are done. If $e \leq b$ then from $a \rightarrow_i b$ it follows that either $e = a$ or $e < a$.

Consider $e, e' \in E$ such that $(e, e') \rightarrow_i \rightarrow_j$. By construction we have that $(e, e') \rightarrow_i \rightarrow_j$, and thus $\rightarrow_i \rightarrow_j \subseteq \rightarrow_i \rightarrow_j \subseteq \rightarrow_i \rightarrow_j$. Consider $e, e' \in E$ such that $(e, e') \in (\rightarrow_i \setminus \rightarrow_j)$. By C6(b) – (e) in Definition 3 there exists $v \in E$ such that one of the following holds.

- $ch(L(e')) \neq \star$, $ch(L(e')) \in LS(v)$ and $v \rightarrow e$. By C7 in Definition 8 we have that $(v, e) \in E(L(e'))$ as required.
- $ch(e) \neq \star$, $ch(L(e)) \in LS(v)$ and $v \rightarrow e$. By C7 in Definition 8 we have that $(v, e) \in E(L(e))$ as required.
- $ch(e') \neq \star$, $v \rightarrow \gamma(L(e'))$ and $v \rightarrow e$. By C7 in Definition 8 we have that $(v, e) \in E(L(e'))$ as required.
- $ch(e) = \star$, $v \rightarrow \gamma(L(e))$ and $e' \rightarrow e$. By C7 in Definition 8 we have that $(v, e) \in E(L(e))$ as required.

**Lemma 4.** Given a g-LPO $\pi_1 \in \text{comp}_g(T)$ and an LPO $\pi_2$ such that $\pi_2 \leq \pi_1$ then $\pi_2 \in \text{comp}(T)$.

Proof. Given that $\pi_2 \leq \pi_1$, it follows that both $\pi_2$ and $\pi_2$ agree on $\rightarrow_c \subseteq V \times E \cup E \times V$ and only agree on $\rightarrow_i \subseteq E \times E$ whenever $e \rightarrow_i e'$ implies $ch(e) = ch(e')$. Hence, it is sufficient to prove that $C4(c) - (d)$ and C6(b) - (e) in Definition 8 hold for $\pi_2$.

We prove $C4(c) - (d)$. Consider some $e \in E$. We have the following cases.

- $ch(L(e))$ is a multicast channel:
  - Consider some $v \in V$ such that $ch(L(e)) \in LS(v)$. We have to show that $e \leq v$ or $v \leq e$. By Definition 8 (*C7), there is some e' such that one of the following cases holds.


• \((v, e') \in \mathcal{E}(L(e))\) where \(v \to_c e'\). By refinement, we have that if \((v, e') \in \mathcal{E}(L(e))\) then either \(e \leq v\) as required or \(e' \leq e\), which implies that \(v \leq e\).

• \((e', v) \in \mathcal{E}(L(e))\) where \(e' \to_v e\). By refinement, we have that if \((e', v) \in \mathcal{E}(L(e))\) then either \(v \leq e\) as required or \(e' \leq e\), which implies that \(e \leq v\).

− \(ch(L(e))\) is a broadcast channel:

Consider some \(v \in V\) such that \(v \to_{\gamma(L(e))} \cdot e\). We have to show that \(e \leq v\) or \(v \leq e\). By Definition 8 (*C7), there is some \(e'\) such that one of the following cases holds.

• \((v, e') \in \mathcal{E}(L(e))\) where \(v \to_e e'\). By refinement, we have that if \((v, e') \in \mathcal{E}(L(e))\) then either \(e \leq v\) as required or \(e' \leq e\), which implies that \(e \leq v\).

• \((e', v) \in \mathcal{E}(L(e))\) where \(e' \to_v e\). By refinement, we have that if \((e', v) \in \mathcal{E}(L(e))\) then either \(v \leq e\) as required or \(e' \leq e\), which implies that \(e \leq v\).

We prove \(C6(b) \implies (e, e') \in \to_i\) such that \((e, e') \in (\to_i \setminus \to_i^0)\). By refinement, we have one of the following cases hold.

− \(ch(L(e))\) is a multicast channel:

• \((e', v) \in \mathcal{E}(L(e))\) for some \(v\). By Definition 8 (*C7), we have that \(e' \to_v e\) and \(ch(L(e))\) \(\in \text{LS}(v)\) as required.

• \((v, e) \in \mathcal{E}(L(e'))\) for some \(v\). By Definition 8 (*C7), we have that \(v \to_e e\) and \(ch(L(e))\) \(\in \text{LS}(v)\) as required.

− \(ch(L(e))\) is a broadcast channel:

• \((e', v) \in \mathcal{E}(L(e))\) for some \(v\). By Definition 8 (*C6), we have that \(e' \to_v e\) and \(v \to_{\gamma(L(e))} \cdot e\) as required.

• \((v, e) \in \mathcal{E}(L(e'))\) for some \(v\). By Definition 8 (*C6), we have that \(v \to_e e\) and \(v \to_{\gamma(L(e'))} \cdot e\) as required.

Theorem 2. Given a CTS \(T\), \(\text{comp}(T) = \{\pi \mid \pi \leq \pi_g \land \pi_g \in \text{comp}_g(T)\}\).

Proof. The proof follows by Lemma 3 and Lemma 4.

C Proofs for Section 6 (Separating Choice and Reconfiguration-Forced Interleaving)

Theorem 3. Given a Petri net \(P\) and two different g-LPOs \(G_1, G_2 \in \text{comp}_g(N)\) then there exists a set of nodes \(v_1, \ldots, v_n\) appearing in both \(G_1\) and \(G_2\) such that one of the following holds:

1. There is a node \(v_i\) such that the number of tokens not taken from \(v_i\) in \(G_1\) and \(G_2\) is different.
2. There is a set of p-histories \(v_1, \ldots, v_n\) that participate in some transition \(t\) in \(G_i\) but not in \(G_{3-i}\).

Proof. We define the depth of a history to be the maximal number of transitions taken by some token in the history. Formally, the depth \((\emptyset, t_n)\) is 0. The depth of a p-history \((h, p, j)\) is \(\text{depth}(h) + 1\). For a t-history \(e \in E\), let \(h^e = \{e_1, \ldots, e_n\}\), then the depth of \(e\) is \(\max_j \text{depth}(e_j)\). Notice, that a t-history \(e\) could have other edges in its preset.

We order the elements in a g-LPO by increasing depth. In addition, elements of the same depth are ordered so that edges appear before vertices and there is some arbitrary order between edges of the same depth and between vertices of the same depth. Clearly, in this order every element appears after all the elements that are smaller than it according to \(\leq\). Indeed, if \(a \to_a b\) or \(a \to b\), then the depth of \(b\) is at least the depth of \(a\) plus one. As every element has a finite depth and there is a finite number of elements in every depth, it follows that this order is some linearisation of all the elements in the g-LPO.

We prove the theorem by induction according to the order mentioned above. We are going to mark nodes and edges that appear in both \(G_1\) and \(G_2\). Nodes are marked by the number of tokens in them that we have not handled yet. When this number is 0 the node is called closed. Otherwise, it is open. Edges are simply marked (or unmarked). For all marked nodes, we “handle” tokens that are participating in the same transitions in \(G_1\) and \(G_2\). Nodes could have tokens that do not participate in transitions. As we “handle” tokens we mark transitions continuing from the node as not forming part of the difference between \(G_1\) and \(G_2\). Once we mark nodes as closed they are also equivalent in \(G_1\) and \(G_2\). As we go through the nodes in \(G_1\) in induction order either we find a difference or, if not, the induction proves that \(G_1\) and \(G_2\) are equivalent in contradiction to the assumption.
Both $G_1$ and $G_2$ have the t-history $h = (\emptyset, t_e)$ as minimal element. Mark it as closed. The p-histories of the form $(h, p, m_0(p))$ such that $m_0(p) \neq 0$ are marked by $m_0(p)$. Clearly, as both $G_1$ and $G_2$ start from the initial marking $m_0$ both $G_1$ and $G_2$ have the same nodes marked and they have the same positive number of tokens.

Assume that we have marked a prefix of $G_1$ and $G_2$ such that all closed nodes have all their outgoing transitions marked. Furthermore, the number marking a node is sufficient for all unmarked transitions existing from the node. Clearly, this is true of the marking of the minimal nodes.

Suppose that there are some open nodes. Choose the minimal open node $v$ according to the induction order. If there are no unmarked edges connected to $v$ in both $G_1$ and $G_2$ then mark $v$ as closed. If there is no unmarked edge connected to $v$ in $G_1$ and there is some unmarked edge connected to $v$ in $G_2$ then we have found a difference as the number of tokens “left” in $v$ in $G_1$ is larger than in $G_2$. In this case, we have identified the difference between $G_1$ and $G_2$. Similarly for the other way around.

The remaining case is when both in $G_1$ and $G_2$ there are unmarked edges connected to $v$. Let $e$ be the minimal unmarked edge connected to $v$ in $G_1$. If $e$ is not connected to $v$ in $G_2$ we are done. Indeed, the preset of $e$ either participate in $e$ in $G_1$ and not in $G_2$ or participate in a transition $L_E(e)$ in different ways in $G_1$ and $G_2$.

Otherwise, $e$ is connected to $v$ both in $G_1$ and $G_2$. By its construction as a multiset of place histories, $e$ “takes” the same number of tokens from $v$ in $G_1$ and $G_2$. As $e$ is unmarked, all the other nodes that $e$ takes tokens from have a sufficient number of unhandled tokens. Again, by $e$’s structure as a pair of a multiset and a transition, $e$ connects to exactly the same nodes in $G_1$ and $G_2$ in the same way. Reduce the marking of all predecessors of $e$ by the number of tokens taken by $e$ from them. If some of them are reduced to 0 then they are closed. Mark $e$ as well.

If there are no open nodes, then both $G_1$ and $G_2$ are finite and equivalent. Otherwise, continue handling open nodes by induction.

**Theorem 4.** Given a CTS $T$ and two different g-LPOs $G_1, G_2 \in \mathsf{comp}_g(N)$ then one of the following holds:

1. For some agent $i$ there exists a history $h_i$ in both $G_1$ and $G_2$ such that either $h_i$ is maximal in $G_1$ and not maximal $G_{3-i}$.
2. For some agent $i$ there exists a history $h_i$ in both $G_1$ and $G_2$ such that the edges $e_1$ and $e_2$ such that $h_i \rightarrow e_1 e_1$ and $h_i \rightarrow e_2 e_2$ we have $L_E(v_1) \neq L_E(v_2)$;
3. or; There is a pair of agents $i$ and $i'$ and histories $h_i$ and $h_{i'}$ in both $G_1$ and $G_2$ such that the order between the communications of $i$ and $i'$ is different in $G_1$ and $G_2$.

**Proof.** As before, we define the depth of elements in a partial order as their distance from a minimal element. Formally, the depth of the minimal element is 0 and all the initial states (runs of length 1) have depth of 1. The depth of a non-minimal element $o$ is $\max_{o' < o} \text{depth}(o') + 1$.

As before, we order the elements in a g-LPO by increasing depth. In addition, elements of the same depth are ordered so that edges appear before vertices and there is some arbitrary order between edges of the same depth and between vertices of the same depth. In this order, every element appears after all the elements that are smaller than it according to $\leq$. As before, every element has a finite depth and there is a finite number of elements in every depth. Hence, if we follow this order constitutes a linearization of the elements of the g-LPO.

We prove the theorem by induction according to the order mentioned above. As before, we are going to mark elements in the partial order as “equivalent” in both $G_1$ and $G_2$. The marking here is simpler (immediately closed marking).

Consider the minimal element edges in $G_1$ and $G_2$ and their post-sets of runs of length 1 (depth 1). By definition, these correspond to the initial states of the different agents. It follows that they are the same. Mark all of them.

Assume that we have marked up to a point in $G_1$ and $G_2$ according to the induction order. We build the marking so that the maximal marked elements according to $\leq$ are all nodes. Obviously, all maximal (according to $\leq$) marked elements are incomparable. It follows that we maintain the minimal unmarked element (in induction order) as an edge. Clearly, this is true for the marking of the minimal nodes.

Consider the set of unmarked edges in $G_1$ and $G_2$. If both are empty, then $G_1$ and $G_2$ are the same. Suppose that the set of unmarked edges in (wlog) $G_1$ is empty and $G_2$ is not empty. Consider the sender participating in the communication of the first unmarked edge in $G_2$. It must be the case that we have found an agent $i$ and a history $h_i$ that is maximal in $G_1$ and not maximal in $G_2$. The remaining case is that both $G_1$ and $G_2$ have unmarked edges.
Consider the g-LPO $G_1$. Let $e$ be the minimal unmarked edge in $G_1$ according to the induction order. Let $h_1, \ldots, h_n$ be $\bullet e$ in $G_1$ with $h_1$ being the sender. As all elements of smaller depth than $e$ have been marked, it follows that $h_1, \ldots, h_n$ have been marked and that they appear also in $G_2$.

Consider a history $h_i \in \bullet e$. If $h_i$ is maximal in $G_2$ we are done. Otherwise, let $e_i$ be the edge such that $h_i \rightarrow_e e_i$. If $L_{E_i}(e) \neq L_{E_i}(e_i)$ we are done as $h_i$ does something different in $G_1$ and $G_2$. The same holds for every $j \in \{1, \ldots, m\}$. Hence, for every $j$ we have $e_j$ exists and $L_{E_i}(e_j) = L(e)$.

Suppose that $G_1$ and $G_2$ are different here. This can only happen if there are at least two agents $j$ and $j'$ for which $e_j$ and $e_{j'}$ are distinct edges labeled by the same communication. In particular, $n \geq 2$ and the agents in histories $h_i$ for $i > 1$ are listening to channel $ch(L_{E_i}(e))$.

However, for $e_j$ and $e_{j'}$ each, there is a unique sender. If $h_1$ is not sending in $G_2$ then $h_1$ does something different in $G_1$ and $G_2$ and we are done. Wlog, assume that $h_1$ is the sender of $e_j$. Consider the following options.

- Suppose that one of the agents $h_i$ for $i > 1$ is the sender of $e_{j'}$. Then, $h_i$ is a history that receives in $G_1$ and sends in $G_2$. Thus, $h_i$ does something different in $G_1$ and $G_2$.
- Suppose that there exists an additional agent $k$ and a history $h_k$ such that $h_k$ is the sender for $e_{j'}$. In order not to find a difference between $G_1$ and $G_2$, it must be the case that $h_k$ is a sender of $e_{j'}$ also in $G_1$ and the set of agents that participate in $e_j$ and $e_{j'}$ together is the same and they have the same roles. That is, every agent that is a receiver in $G_1$ is a receiver in $G_2$ and vice versa. However, as we assumed that $G_1$ and $G_2$ are different, there are again two options:
  
  - Either the order between $e_j$ and $e_{j'}$ in $G_1$ and $G_2$ is reversed. This matches the difference 3 where the senders are the agents witnessing the difference.
  - Or the order between $e_j$ and $e_{j'}$ is the same in both $G_1$ and $G_2$. Then, the matching between senders and receivers in $G_1$ and $G_2$ to $e_j$ and $e_{j'}$ is different. Consider a receiver that moved from listening to (wlog) $e_j$ to $e_{j'}$. It follows that this agent participates in an early communication in $G_1$ ($e_j$) and a later communication in $G_2$ ($e_{j'}$). This receiving agent and the sender of $e_j$ see a different order of the communication they participate in (from equal to one before the other).

By induction, unless this process terminates prematurely by finding a difference, it will visit all of $G_1$ and $G_2$ and show that they are, in fact, equivalent.