HARMONIC SECTIONS OF RIEMANNIAN VECTOR BUNDLES, AND METRICS OF CHEEGER-GROMOLL TYPE

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Dedicated to Professors J Eells and J H Sampson.

Abstract. We study harmonic sections of a Riemannian vector bundle \( E \to M \) whose total space is equipped with a 2-parameter family of metrics \( h_{p,q} \) which includes both the Sasaki and Cheeger-Gromoll metrics. The restrictions of the \( h_{p,q} \) to the total space of any sphere subbundle \( SE(k) \) of \( E \) (where \( k > 0 \) is the radius) are essentially the same for all \((p,q)\), and it is shown that for every \( k \) there exists a unique \( p \) such that the harmonic sections of \( SE(k) \) are harmonic sections of \( E \) with respect to \( h_{p,q} \) for all \( q \). In both the compact and non-compact cases Bernstein regions of the \((p,q)\)-plane are identified, where the only harmonic sections of \( E \) with respect to \( h_{p,q} \) are parallel. Examples are constructed of compact vector fields which are harmonic sections of \( E = TM \) in the case where \( M \) has non-zero Euler characteristic.

1. Introduction

The aim of this paper is to introduce some new criteria for deciding which smooth vector fields on a smooth, oriented, connected (but not necessarily compact) Riemannian manifold \((M, g)\), or in general which smooth sections \( \sigma \) of a smooth oriented Riemannian vector bundle \((E, \langle , \rangle, \nabla) \to M\), qualify as “better than the rest”. In so doing we overcome some of the limitations of existing criteria, which we briefly review.

1. \( \nabla \sigma = 0 \). Since the fibre metric \( \langle , \rangle \) is holonomy-invariant, and \( M \) is connected, parallel sections have constant length. Therefore if the Euler class \( \chi(E) \neq 0 \) there are no non-trivial solutions. (The trivial solution is of course the zero section.) Since the existence of solutions is equivalent to reduction of the holonomy of \( \nabla \), amongst the many other necessary conditions is de Rham’s decomposition theorem: if \( E = TM \) and \( \langle , \rangle = g \) then the universal cover of \( M \) splits as a Riemannian product \( M' \times \mathbb{R} \). So

1991 Mathematics Subject Classification. 53C43 (53C07, 53C24, 53E15, 58E20, 58G30).

Key words and phrases. \((p,q)\)-harmonic section, Sasaki metric, Cheeger-Gromoll metric, (strictly) \( q \)-Riemannian section, Kato inequality, Bernstein region, Hopf vector field, conformal gradient field.

This research originated during the EDGE conference held at the Universidad de Granada in February 2004, and the authors would like to express their gratitude to the Departamento de Geometría y Topología for hosting and organizing the event.
whilst this criterion undeniably produces the “best” sections, its applicability is severely limited.

(2) $\Delta \sigma = 0$. Here $\mathcal{E} = TM$, $\langle , \rangle = g$, $\nabla$ is the Levi-Civita connection, and $\Delta$ is the Hodge-de Rham laplacian on 1-forms, dualized to act on vector fields. By Hodge’s theorem, if $M$ is compact then the solution space is isomorphic to $H^1(M, \mathbb{R})$; in particular if the first Betti number $\beta_1(M) = 0$ then there are no non-trivial solutions. Furthermore Bochner’s vanishing theorem informs us that when $M$ has positive Ricci curvature there are no non-parallel solutions.

(3) $\sigma$ is a harmonic section of $\mathcal{E}$ [13, 14]. Here one measures the vertical energy (or total bending [20]) of $\sigma$:

$$E^v(\sigma) = \frac{1}{2} \int_M |\nabla \sigma|^2 \text{vol}(g),$$

(1-1)

(assuming for convenience that $M$ is compact; otherwise one works over relatively compact domains), and looks for critical points with respect to smooth variations through sections of $\mathcal{E}$. The Euler-Lagrange equations are once again linear:

$$\nabla^* \nabla \sigma = 0,$$

(1-2)

where $\nabla^* \nabla$ is the rough Laplacian:

$$\nabla^* \nabla = - \text{Trace} \nabla^2$$

If $M$ is compact and (1-2) holds then integrating by parts:

$$0 = \int_M \langle \nabla^* \nabla \sigma, \sigma \rangle \text{vol}(g) = \int_M |\nabla \sigma|^2 \text{vol}(g),$$

so all harmonic sections of $\mathcal{E}$ are parallel. The same is true if $M$ is non-compact, provided $\sigma$ has constant length (see Lemma 3.4).

(4) $|\sigma| = k$ (constant) and $\sigma$ is a harmonic section of the radius-$k$ sphere bundle [22, 24]. Here the vertical energy functional (1-1) is restricted to sections of the subbundle $S\mathcal{E}(k) \to M$, where:

$$S\mathcal{E}(k) = \{ e \in \mathcal{E} : |e| = k \},$$

and the imposition of this constraint causes the Euler-Lagrange equations to become mildly non-linear (see Remark 3.8):

$$\nabla^* \nabla \sigma = \frac{1}{k^2} |\nabla \sigma|^2 \sigma$$

(1-3)
The solutions of (1-3) clearly include all parallel sections of length $k$ (if any), but when $\mathcal{E} = TM$ many additional solutions have been identified [1, 8, 9, 17, 19], which in turn may be examined for stability [2, 3, 4, 12, 24]. Unfortunately the theory is limited to bundles with $\chi(\mathcal{E}) = 0$.

Our new criteria remove the topological restriction $\chi(\mathcal{E}) = 0$, whilst retaining all solutions of the constrained variational problem (4) (Theorem A/4.1). The basic idea is to obtain interesting non-linear equations, such as (1-3), by altering the background metric data, rather than introducing constraints. We note first that definition (1-1) is equivalent to:

$$E^v(\sigma) = \frac{1}{2} \int_M |d^v\sigma|^2 \text{vol}(g),$$

(1-4)

where $d^v\sigma$ is the vertical component of the differential $d\sigma$ with respect to the connection $\nabla$ in $\mathcal{E} \to M$, and the norm in $T\mathcal{E}$ is that of the Sasaki metric $h$ on $\mathcal{E}$ [18]. The idea is to study the functional (1-4) when $h$ is generalized to a 2-parameter family of metrics $h_{p,q}$ on $\mathcal{E}$, for which $h_{0,0} = h$ and $h_{1,1}$ is the Cheeger-Gromoll metric [7, 15]. (Both the Sasaki and Cheeger-Gromoll metrics were originally defined for $\mathcal{E} = TM$, but generalize in a natural way; see Remark 2.1.) Other geometrically interesting metrics occur in this family; for example $h_{2,0}$ is the stereographic metric (Remark 2.2).

Actually the term “metric” is used somewhat informally. If $q \geq 0$ then $h_{p,q}$ is indeed a Riemannian metric. However if $q < 0$ then $h_{p,q}$ has varying signature and is consequently not even semi-Riemannian; it is Riemannian within the tubular neighbourhood of the zero section of radius $1/\sqrt{-q}$, Lorentzian on the interior of the complement, and positive semi-definite on the boundary. This behaviour may be viewed as a manifestation of Kato’s inequality [6]. A section whose image lies in the closure of this tubular neighbourhood is said to be $q$-Riemannian (see Remark 2.4). If $q < 0$ and $\sigma$ is not $q$-Riemannian then it is possible that $E^v(\sigma) < 0$. In any case, if $\sigma$ is stationary for (1-4) with respect to the metric $h_{p,q}$ on $\mathcal{E}$, and smooth variations through sections of $\mathcal{E}$ then we say that $\sigma$ is a $(p, q)$-harmonic section of $\mathcal{E}$. The Euler-Lagrange equations for $(p, q)$-harmonic sections are derived in §3 (Theorem 3.6), after a somewhat lengthy sequence of calculations. They are considerably more complicated than (1-2), to which of course they reduce when $(p, q) = (0, 0)$. However for all $(p, q)$ the parallel sections of $\mathcal{E}$ are always $(p, q)$-harmonic, and amongst $q$-Riemannian sections they comprise the absolute minima of $E^v$.

An interesting feature of the $h_{p,q}$ is that they restrict to essentially the same Riemannian metric on $SE(k)$, even when $q < 0$ and $k > 1/\sqrt{-q}$ (Remark 2.3). Hence $(p, q)$-harmonic sections of $SE(k)$ are characterised by equations (1-3) for all $(p, q)$, and may therefore be referred to simply as harmonic sections of $SE(k)$. For bundles with $\chi(\mathcal{E}) = 0$ we establish the following relationship between $(p, q)$-harmonic sections of $\mathcal{E}$ and harmonic sections of $SE(k)$:
**Theorem A.** Suppose that $|\sigma(x)| = k > 0$ for all $x \in M$.

(a) If $p \neq 1 + 1/k^2$ then $\sigma$ is a $(p, q)$-harmonic section of $E$ if and only if $\sigma$ is parallel.

(b) If $p = 1 + 1/k^2$ then $\sigma$ is a $(p, q)$-harmonic section of $E$ if and only if $\sigma$ is a harmonic section of $SE(k)$.

Theorem A may also be regarded as a first source of examples of $(p, q)$-harmonic sections of $E$, when $p > 1$ and $\chi(E) = 0$ (Example 4.2). In seeking non-trivial examples of $(p, q)$-harmonic sections of bundles with non-zero Euler class, we establish the following rather more complicated set of restrictions on $(p, q)$:

**Theorem B.** Suppose $M$ is compact, $\chi(E) \neq 0$, and $\sigma$ is a non-trivial section of $E$. For each $p \in \mathbb{R}$ there exists at most one $q \in \mathbb{R}$ such that $\sigma$ is a $(p, q)$-harmonic section of $E$, and:

(a) if $-4 \leq p \leq -1$ then $q < -1 - p$;

(b) if $-1 \leq p \leq 1$ then $q < 0$;

(c) if $1 < p \leq 2$ and $\|\sigma\|_{\infty} \leq 1/\sqrt{p - 1}$ then $q < 0$;

(d) if $2 \leq p$ and $\|\sigma\|_{\infty} \leq 1/\sqrt{p - 1}$ then $q < 1 - p/2$.

The appearance of $\|\sigma\|_{\infty}$ in Theorem B (c),(d) at first sight seems counter-intuitive, since it implies that $(p, q)$-harmonicity is not invariant under scaling (when $p > 1$). This reflects the non-linearity of the $(p, q)$-harmonic section equations. We do not know whether any restrictions on the location of $q$ exist when $p < -4$.

Theorem B is deduced from a Bernstein-type theorem (Theorem 4.6) and a uniqueness theorem (Theorem 4.8), which in fact yield a more general result valid for $(p, q)$-harmonic sections of non-constant length (Corollary 4.9). Analogous results are available for the non-compact case (Theorem 4.3, Theorem 4.5, Corollary 4.10), although in order to compensate for the unavailability of the global techniques used for Theorem B these are all qualified by the assumption that $|\sigma|^2: M \to \mathbb{R}$ is a harmonic function; they may therefore be viewed as generalizations of Theorem A. In both the compact and non-compact cases it transpires that no additional examples of harmonic sections of $E$ arise when the Sasaki metric is replaced by the Cheeger-Gromoll metric [16], or indeed any metric $h_{p,p}$ with $0 \leq p \leq 1$ (Remarks 4.4 and 4.7). In fact all metrics $h_{p,q}$ with $0 \leq p \leq 1$ and $q \geq 0$ exhibit this behaviour. In order to find non-trivial examples of $(p, q)$-harmonic sections of bundles with non-zero Euler class it is therefore necessary to explore more “remote” regions of the $(p, q)$-plane.

The non-applicability of standard existence theory for harmonic maps (for example [10]), and its generalization to harmonic sections (for example [21, 23]) necessitates a somewhat ad hoc approach to the construction of examples. In [3] it was shown that normalizing a conformal gradient field on $S^5, S^7, \ldots$ away from its (two) zeroes produces a singular unit vector field whose energy infinizes the energy functional when restricted to the space of smooth unit vector fields. (On $S^3$ the Hopf vector field is an absolute
energy minimizer [4,12].) We show (Theorem 5.2) that if \( \sigma \) is a conformal gradient field on \( M = S^n \) with \( n \geq 3 \) then \( \sigma \) is a \((p,q)\)-harmonic section of \( T M \) precisely when \( p = n + 1, q = 2 - n, \) and \( \|\sigma\|_\infty = 1/\sqrt{n-2} \). (Note that these values of \((p,q)\) are consistent with Theorem B.) Although \( q < 0 \), these \((p,q)\)-harmonic sections are \( q \)-Riemannian, but only just (Remark 5.3). This example suggests that in general a section should not be expected to be \((p,q)\)-harmonic for more than one metric \( h_{p,q} \), although the existence of a 1-parameter family of \((p,q)\) is not precluded by Theorem B. It also illustrates once again, in a dramatic way, the non-invariance of solutions of the \((p,q)\)-harmonic section equations under scaling. This suggests the following simple general ansatz: given a “trial” section \( \xi \), try to construct a \((p,q)\)-harmonic section by linearly rescaling \( \xi \). If this fails, try a conformal rescaling. Of course, the choice of \( \xi \) remains ad hoc. In this vein, we conclude by showing (Theorem 5.4) that when \( M \) is an odd-dimensional sphere the only \((p,q)\)-harmonic sections of \( T M \) obtained by conformally rescaling the Hopf vector field \( \xi \) are precisely those covered by Theorem A: namely \( \sigma = k\xi \) where \( k = \pm 1/\sqrt{p-1} \) and \( p > 1 \).

2. The Vertical \((p,q)\)-Energy Functional

Let \( (M,g) \) be a connected Riemannian \( n \)-manifold, and let \( \pi: \mathcal{E} \to M \) be a vector bundle with connection \( \nabla \) and holonomy-invariant fibre metric \( \langle \cdot, \cdot \rangle \).

**Remark.** For the most part, connectedness of \( M \) is simply a convenience which allows us to simplify the exposition (for example, a parallel section of \( \mathcal{E} \) then has constant length), and most of our results are true whether or not \( M \) is connected. The exceptions are Theorems 4.5 (b) and 4.6 (b) where connectedness is an essential hypothesis.

**Remark.** The holonomy-invariance of \( \langle \cdot, \cdot \rangle \) will be used in many of our calculations, usually without comment, and is essential to our results.

Let \( K: T\mathcal{E} \to \mathcal{E} \) be the connection map [11] for \( \nabla \):

\[
\begin{array}{ccc}
\mathcal{E} & \xleftarrow{K} & T\mathcal{E} \\
\pi \downarrow & & \downarrow d\pi \\
M & \longleftarrow & TM
\end{array}
\]

and let \( e \in \mathcal{E} \) and \( A, B \in T_e\mathcal{E} \). For any pair of parameters \( p, q \in \mathbb{R} \) we define a symmetric 2-covariant tensor \( h_{p,q} \) on \( \mathcal{E} \) as follows:

\[
h_{p,q}(A, B) = g(d\pi(A), d\pi(B)) + w^p(e)\left(\langle KA, KB \rangle + q\langle KA, e \rangle\langle KB, e \rangle\right),
\]

where:

\[
w(e) = \frac{1}{1 + |e|^2}
\]
If \( q \geq 0 \) then \( h_{p,q} \) is a Riemannian metric; however if \( q < 0 \) then \( h_{p,q} \) is a Riemannian metric only on the following tubular neighbourhood of the zero section:

\[
BE(1/\sqrt{-q}) = \{ e \in \mathcal{E} : |e|^2 < -1/q \}
\]

**Remark 2.1.** If \((p, q) = (0, 0)\) then \( h_{p,q} \) is the Sasaki metric [18]:

\[
h_{0,0}(A, B) = g(d\pi(A), d\pi(B)) + \langle KA, KB \rangle,
\]

whereas if \((p, q) = (1, 1)\) then \( h_{p,q} \) is the Cheeger-Gromoll metric [7]:

\[
h_{1,1}(A, B) = g(d\pi(A), d\pi(B)) + \frac{1}{1 + |e|^2} \left( \langle KA, KB \rangle + \langle KA, e \rangle \langle KB, e \rangle \right)
\]

In all cases the bundle projection \((\mathcal{E}, h_{p,q}) \to (M, g)\) is horizontally isometric; in particular, if \( q \geq 0 \) it is a Riemannian submersion.

**Remark 2.2.** Another way of thinking about \( h_{p,q} \) is as the horizontal lift of \( g \) to \( \mathcal{E} \), supplemented by the metric on the fibres induced by the following rotationally symmetric metric on Euclidean space:

\[
\frac{1}{(1 + |x|^2)^p} \left( \sum_i (dx^i)^2 + q \sum_{i,j} x_i x_j \, dx^i \, dx^j \right)
\] (2-2)

In particular, if \((p, q) = (2, 0)\) then (2-2) is the stereographic metric, up to homothety.

**Remark 2.3.** For any \( k > 0 \), if \( A, B \) are tangent to the total space of the sphere bundle \( S\mathcal{E}(k) \to M \) then \( \langle KA, e \rangle = 0 \) etc. It therefore follows from equation (2-1) that the restriction of \( h_{p,q} \) to \( S\mathcal{E}(k) \) is a Riemannian metric for all \((p, q)\), and these metrics are essentially the same: the restriction of \( h_{p,q} \) differs from that of \( h_{0,0} \) by the (constant) factor \( (1 + k^2)^{-p} \) in the vertical direction.

Now let \( \sigma \) be a section of \( \mathcal{E} \). Throughout the paper it is convenient to abbreviate:

\[
F = \frac{1}{2} |\sigma|^2
\] (2-3)

If \( \{E_i\} \) is a local orthonormal tangent frame in \( M \) then by the defining properties [11] of the connection map, and holonomy-invariance of \( \langle , \rangle \) we have:

\[
|d^\sigma|^2 = \sum_i h\left( d^\sigma(E_i), d^\sigma(E_i) \right)
= \sum_i w^p(\sigma) \left( \langle K \circ d\sigma(E_i), K \circ d\sigma(E_i) \rangle + q \langle K \circ d\sigma(E_i), \sigma \rangle^2 \right)
= w^p(\sigma) \sum_i \left( \langle \nabla_{E_i} \sigma, \nabla_{E_i} \sigma \rangle + q \langle \nabla_{E_i} \sigma, \sigma \rangle^2 \right)
= w^p(\sigma) \left( |\nabla \sigma|^2 + q |\nabla F|^2 \right),
\] (2-4)
where $\nabla F$ is the gradient vector. In the Sasaki case (2-4) reduces to:

$$|d^v \sigma|^2 = |\nabla \sigma|^2,$$

(2-5)

whereas in the Cheeger-Gromoll case:

$$|d^v \sigma|^2 = \frac{1}{1 + |\sigma|^2} (|\nabla \sigma|^2 + |\nabla F|^2)$$

If $q < 0$ and $|\sigma(x)|^2 < -1/q$ for all $x \in M$ then the fact that $h_{p,q}$ is a Riemannian metric in $B\mathcal{E}(1/\sqrt{-q})$ (so that $|d^v \sigma|^2 \geq 0$), and $d^v \sigma = 0$ if and only if $\nabla \sigma = 0$, allows us to immediately deduce from (2-4) Kato’s inequality:

$$|\nabla \sigma|^2 + q |\nabla F|^2 \geq 0, \quad \text{with equality if and only if } \nabla \sigma = 0.$$

(2-6)

It is not hard to see that (2-6) remains true if $|\sigma|^2 \leq -1/q$.

**Definitions.** For any $q \in \mathbb{R}$, a smooth section $\sigma$ satisfying $q|\sigma(x)|^2 \geq -1$ for all $x \in M$ will be called $q$-Riemannian, and the set of all such $\sigma$ will be denoted $\mathcal{C}(\mathcal{E}, q)$. Smooth sections satisfying (2-6) will be called $q$-positive, and the set of all such will be denoted by $\mathcal{C}(\mathcal{E}, q^+)$.

**Remark 2.4.** If $q \geq 0$ then $\mathcal{C}(\mathcal{E}, q) = \mathcal{C}(\mathcal{E})$, the space of all smooth sections of $\mathcal{E}$, and if $q < 0$ then $\sigma$ is $q$-Riemannian precisely when its image lies in the closure of the “Riemannian tube” $B\mathcal{E}(1/\sqrt{-q})$. Although there are tangent vectors (to $\mathcal{E}$) at points on the boundary $S\mathcal{E}(1/\sqrt{-q})$ of $B\mathcal{E}(1/\sqrt{-q})$ which are $h_{p,q}$-null, since a $q$-Riemannian section which encounters $S\mathcal{E}(1/\sqrt{-q})$ does so tangentially it follows from Remark 2.3 that the restriction of $h_{p,q}$ to $\sigma(M)$ is indeed a Riemannian metric.

**Remark 2.5.** If $q_1 < q_2$ then $\mathcal{C}(\mathcal{E}, q_1) \subset \mathcal{C}(\mathcal{E}, q_2)$. Certainly $\mathcal{C}(\mathcal{E}, q) \subset \mathcal{C}(\mathcal{E}, q^+)$, but $\mathcal{C}(\mathcal{E}, q^+)$ also includes (for example) all sections of constant length.

We denote by $E^v_{p,q}$ the vertical energy functional with respect to $h_{p,q}$. By (1-4) and (2-4):

$$E^v_{p,q}(\sigma) = \frac{1}{2} \int_M w^p(\sigma) \left( |\nabla \sigma|^2 + q |\nabla F|^2 \right) \text{vol}(g),$$

(2-7)

for all $\sigma \in \mathcal{C}(\mathcal{E})$. We refer to $E^v_{p,q}(\sigma)$ as the vertical $(p,q)$-energy of $\sigma$. When $(p,q)$ are understood we simply write $E^v(\sigma)$.

**Remark.** Since $(\mathcal{E}, h_{p,q}) \to (M, g)$ is horizontally isometric for all $(p,q)$, $E^v(\sigma)$ differs from the full energy $E(\sigma)$ [10] by a positive constant, depending only on the dimension and volume of $M$. 
3. The First Variation Formula

Let $\Sigma: M \times \mathbb{R} \to \mathcal{E}; \Sigma(x,t) = \sigma_t(x)$ be a smooth variation of $\sigma = \sigma_0$ through sections of $\mathcal{E}$. Then it is natural to identify the variation field, which is necessarily vertical, with a family of sections $\rho_t$ of $\mathcal{E}$:

$$\rho_t(x) = K \circ \frac{d}{dt}(\sigma_t(x)) \in \pi^{-1}(x),$$

bearing in mind that the restriction of $K$ to the vertical distribution is canonical. Furthermore, $\Sigma$ may be viewed as a section of the pullback vector bundle $\pi_1^{-1}\mathcal{E} \to M \times \mathbb{R}$, where $\pi_1: M \times \mathbb{R} \to M; (x,t) \mapsto x$, and this allows the variation field to be expressed in terms of the pullback connection:

**Lemma 3.1.** If $\partial_t$ is the unit vector field on $M \times \mathbb{R}$ in the positive $\mathbb{R}$-direction then $\nabla_{\partial_t} \Sigma = \pi_1^{-1}\rho_t$ (the $\pi_1$-pullback of $\rho_t$), for all $t$.

*Proof.* In general, if $f: P \to M$ is any smooth map and $(p,e) \in f^{-1}\mathcal{E} \subset P \times \mathcal{E}$ (thus $f(p) = \pi(e)$) then the tangent space of $f^{-1}\mathcal{E}$ is the following subspace of $T_{(p,e)}(P \times \mathcal{E})$:

$$T_{(p,e)}(f^{-1}\mathcal{E}) = \{(Y,A) : Y \in T_pP, A \in T_e\mathcal{E}, df(Y) = d\pi(A)\},$$

using the natural identification of $T_{(p,e)}(P \times \mathcal{E})$ with $T_pP \oplus T_e\mathcal{E}$, and the connection map for the pullback connection is:

$$\tilde{K}(Y,A) = (p, KA)$$

Therefore, since as a section of $\pi_1^{-1}\mathcal{E}$ we have $\Sigma(x,t) = ((x,t),\sigma_t(x))$, it follows that:

$$\nabla_{\partial_t} \Sigma(x,t) = \tilde{K}(d\Sigma(\partial_t)) = ((x,t), K \circ \frac{d}{dt}[\sigma_t(x)]) = ((x,t), \rho_t(x))$$

Thus $\nabla_{\partial_t} \Sigma$ is the $\pi_1$-pullback of $\rho_t$. $\square$

We note also the following property of the curvature of the pullback connection (in a slightly more general setting):

**Lemma 3.2.** Let $\pi_1: M \times N \to M; (x,y) \mapsto x$ be the projection from any product with $M$, and let $\tilde{\mathcal{E}} = \pi_1^{-1}\mathcal{E} \to M \times N$. The curvature of the pullback connection satisfies:

$$\tilde{R}(X,Y) = 0, \quad \text{for all } X \in T_xM, Y \in T_yN,$$

regarding $T_xM, T_yN \subset T_{(x,y)}(M \times N)$ in the natural way.

*Proof.* Extend $X,Y$ to local vector fields in $M,N$ respectively; then $X,Y$ may also be regarded as local vector fields in $M \times N$. Suppose $\tilde{e} \in \tilde{\mathcal{E}}_{(x,y)}$; thus $\tilde{e} = ((x,y), e)$ where
Extend $e$ to a local section $\alpha$ of $E_x$, and then extend $\tilde{e}$ to the local section $\tilde{\alpha}(x, y) = ((x, y), \alpha(x))$ of $\tilde{E}$. Thus $\tilde{\alpha}$ is the $\pi_1$-pullback of $\alpha$, and so by a fundamental characterization of pullback connections:

\[
\nabla_X \tilde{\alpha} = ((x, y), \nabla_{d\pi_1(X)} \alpha) = \tilde{\nabla}_X \alpha, \tag{3-1}
\]

\[
\nabla_Y \tilde{\alpha} = ((x, y), \nabla_{d\pi_1(Y)} \alpha) = 0 \tag{3-2}
\]

Therefore, using the fact that the local vector field $X$ in $M \times N$ is $\pi_1$-adapted to its counterpart in $M$, and $Y$ is $\pi_1$-adapted to $0$:

\[
\tilde{R}(X, Y)\tilde{e} = \nabla_X \nabla_Y \tilde{\alpha} - \nabla_Y \nabla_X \tilde{\alpha} - \nabla_{[X, Y]} \tilde{\alpha}
\]

\[
= \nabla_X \nabla_Y \tilde{\alpha} - \nabla_Y \nabla_X \tilde{\alpha}, \quad \text{since } [X, Y] = 0,
\]

\[
= -\nabla_Y \tilde{\nabla}_X \alpha, \quad \text{by (3-1) and (3-2),}
\]

\[
= 0, \quad \text{by (3-2).} \quad \square
\]

Now it follows from (2-7) that:

\[
\frac{d}{dt}\bigg|_{t=0} E^v_{p, q}(\sigma_t) = \frac{1}{2} \int_M \frac{d}{dt}\bigg|_{t=0} w^p(\sigma_t) \left( |\nabla \sigma_t|^2 + q |\nabla F|^2 \right) \text{vol}(g)
\]

\[
+ \frac{1}{2} \int_M w^p(\sigma) \frac{d}{dt}\bigg|_{t=0} (|\nabla \sigma_t|^2 + q |\nabla F_t|^2) \text{vol}(g)
\]

\[
= V_1 + V_2, \quad \text{say.}
\]

We abbreviate the variation field $\rho_0 = \rho$. It is also convenient to define an $E$-valued 1-form $\varphi$ on $M$ as follows:

\[
\varphi(Y) = \langle \nabla F, Y \rangle \sigma = \frac{1}{2} \langle \nabla |\sigma|^2, Y \rangle \sigma \tag{3-3}
\]

**Lemma 3.3.**

(i) \[
\frac{d}{dt}\bigg|_{t=0} w^p(\sigma_t) = -2pw^{p+1}(\sigma) \langle \sigma, \rho \rangle
\]

(ii) \[
\frac{d}{dt}\bigg|_{t=0} |\nabla \sigma_t|^2 = 2 \langle \nabla \rho, \nabla \sigma \rangle
\]

(iii) \[
\frac{d}{dt}\bigg|_{t=0} |\nabla F_t|^2 = 2 \langle \varphi, \nabla \rho \rangle + 2 \langle \nabla_{\nabla F} \sigma, \rho \rangle
\]

**Proof.**
1. We have:
\[
\frac{d}{dt} \left|_{t=0} \right. w^p(\sigma_t) = \frac{d}{dt} \left|_{t=0} \right. w(\sigma_t) = \frac{d}{dt} \left|_{t=0} \right. w^p(\sigma_t) = \frac{d}{dt} \left|_{t=0} \right. \frac{1}{1 + |\sigma|^2} |\sigma_t|^2
\]
\[
= -pw^{p+1}(\sigma) \frac{d}{dt} \left|_{t=0} \right. |\sigma_t|^2
\]
By Lemma 3.1:
\[
\frac{d}{dt} |\sigma_t|^2 = \frac{d}{dt} |\Sigma|^2 = 2 \langle \nabla_{\partial_t} \Sigma, \Sigma \rangle = 2 \langle \rho_t, \sigma_t \rangle \tag{3-4}
\]

2. Summing over \(i\):
\[
\frac{1}{2} \frac{d}{dt} |\nabla \sigma_t|^2 = \frac{1}{2} \frac{d}{dt} \langle \nabla_{E_i} \sigma_t, \nabla_{E_i} \sigma_t \rangle = \frac{1}{2} \frac{d}{dt} \langle \nabla_{E_i} \Sigma, \nabla_{E_i} \Sigma \rangle = \langle \nabla_{\partial_t} \nabla_{E_i} \Sigma, \nabla_{E_i} \Sigma \rangle, \quad \text{by Lemma 3.2,}
\]
\[
= \langle \nabla_{E_i} \nabla_{\partial_t} \Sigma, \nabla_{E_i} \Sigma \rangle, \quad \text{since} \ [\partial_t, E_i] = 0,
\]
\[
= \langle \nabla_{E_i} \rho_t, \nabla_{E_i} \sigma_t \rangle, \quad \text{by Lemma 3.1,}
\]
\[
= \langle \nabla \rho_t, \nabla \sigma_t \rangle
\]

3. Summing over \(i\):
\[
\frac{1}{2} \frac{d}{dt} \left|_{t=0} \right. |\nabla F_t|^2 = \frac{1}{2} \frac{d}{dt} \left|_{t=0} \right. \langle \nabla_{E_i} \sigma_t, \sigma_t \rangle^2 = \langle \nabla_{E_i} \sigma, \sigma \rangle \frac{d}{dt} \left|_{t=0} \right. \langle \nabla_{E_i} \sigma_t, \sigma_t \rangle
\]
\[
= \langle \nabla_{E_i} \sigma, \sigma \rangle \left( \langle \nabla_{\partial_t} \nabla_{E_i} \Sigma, \sigma \rangle + \langle \nabla_{E_i} \sigma, \nabla_{\partial_t} \Sigma \rangle \right)
\]
\[
= \frac{1}{2} \langle E_i, |\sigma|^2 \rangle \left( \langle \nabla_{E_i} \nabla_{\partial_t} \Sigma, \sigma \rangle + \langle \nabla_{E_i} \sigma, \nabla_{\partial_t} \Sigma \rangle \right), \quad \text{by Lemma 3.2,}
\]
\[
= \langle \nabla F, E_i \rangle \left( \langle \nabla_{E_i} \rho, \sigma \rangle + \langle \nabla_{E_i} \sigma, \rho \rangle \right), \quad \text{by Lemma 3.1,}
\]
\[
= \langle \nabla_{E_i} \rho, \varphi(E_i) \rangle + \langle \nabla_{\nabla F} \rho, \sigma \rangle, \quad \text{by (3-3)}
\]
\[
= \langle \varphi, \nabla \rho \rangle + \langle \nabla_{\nabla F} \rho, \sigma \rangle.
\]

It follows from Lemma 3.3 that:
\[
V_1 = -p \int_M w^{p+1}(\sigma) \left( (|\nabla \sigma|^2 + q |\nabla F|^2) \sigma \right) \text{vol}(g) \tag{3-5}
\]
\[
V_2 = \int_M w^p(\sigma) \left( \langle \nabla \sigma + q \varphi, \nabla \rho \rangle + \langle q \nabla_{\nabla F} \sigma, \rho \rangle \right) \text{vol}(g) \tag{3-6}
\]
The expression (3-6) is only partially in divergence form, a situation which we rectify with the following sequence of calculations. For any \( E \)-valued 1-form \( \beta \) on \( M \) we have (summing over \( i \)):

\[
\nabla^*(f \beta) = -\nabla_{E_i} (f \beta)(E_i) = -df(E_i) \beta(E_i) - f \nabla_{E_i} \beta(E_i)
\]

\[
= f \nabla^* \beta - \beta(\nabla f)
\]

In particular, if \( f = w^p(\sigma) \) then:

\[
df = pw^{-1}(\sigma) d(w(\sigma)) = pw^{-1}(\sigma) \frac{-2}{(1 + |\sigma|^2)^2} \langle \nabla \sigma, \sigma \rangle
\]

\[
= -2pw^{p+1}(\sigma) \langle \nabla \sigma, \sigma \rangle,
\]

and so, summing over \( i \):

\[
\nabla f = df(E_i)E_i = -2pw^{p+1}(\sigma) \langle \nabla_{E_i} \sigma, \sigma \rangle E_i = -pw^{p+1}(\sigma)(E_i |\sigma|^2) E_i
\]

\[
= -pw^{p+1}(\sigma) \nabla |\sigma|^2 = -2pw^{p+1}(\sigma) \nabla F
\]

Taking \( \beta = \nabla \sigma + q \varphi \) yields:

\[
\nabla^*(f \sigma) = w^p(\sigma)(\nabla^* \nabla \sigma + q \nabla^* \varphi) + 2pw^{p+1}(\sigma) (\nabla_{\nabla F} \sigma + q \varphi(\nabla F)) \quad (3-7)
\]

**Lemma 3.4.** \( \langle \nabla^* \nabla \sigma, \sigma \rangle = |\nabla \sigma|^2 + \Delta F \)

**Proof.** Abbreviating \( |\sigma|^2 = 2F \):

\[
\langle \nabla^* \nabla \sigma, \sigma \rangle = -\sum_i \langle \nabla_{E_i} \nabla_{E_i} \sigma - \nabla_{E_i E_i} \sigma, \sigma \rangle
\]

\[
= -\sum_i \left( E_i \langle \nabla_{E_i} \sigma, \sigma \rangle - \langle \nabla_{E_i} \sigma, \nabla_{E_i} \sigma \rangle - \langle \nabla_{E_i E_i} \sigma, \sigma \rangle \right)
\]

\[
= |\nabla \sigma|^2 - \sum_i \left( E_i \cdot E_i F - (\nabla_{E_i} E_i) F \right)
\]

\[
= |\nabla \sigma|^2 - \text{Trace} \nabla dF = |\nabla \sigma|^2 + \Delta F. \quad \square
\]

**Note.** Our sign convention for the Laplace-Beltrami operator is:

\[
\Delta F = -\text{Trace} \nabla dF
\]
Lemma 3.5. \( \nabla^* \varphi = (\Delta F)\sigma - \nabla_{\nabla F}\sigma \), where \( 2F = |\sigma|^2 \).

Proof. Summing over \( i \):

\[
\nabla^* \varphi = -\nabla_{E_i} \varphi(E_i) = -\nabla_{E_i} \varphi(E_i), \quad \text{if } \{E_i\} \text{ is at the centre of a normal chart},
\]

\[
= -\nabla_{E_i} (\langle \nabla F, E_i \rangle \sigma) = -\nabla_{E_i} (\langle \nabla F, E_i \rangle \sigma) = -\nabla_{E_i} ((E_i, F)\sigma)
\]

\[
= -\nabla_{E_i} (\langle \nabla_{E_i} \sigma, \sigma \rangle \sigma) = \langle -\nabla_{E_i} \nabla_{E_i} \sigma, \sigma \rangle \sigma - |\nabla \sigma|^2 \sigma - \langle \nabla_{E_i} \sigma, \sigma \rangle \nabla_{E_i} \sigma
\]

\[
= (\Delta F)\sigma - \nabla_{\nabla F}\sigma, \quad \text{by (2-3) and Lemma 3.4.} \quad \square
\]

Theorem 3.6. For any 1-parameter smooth variation \( \sigma_t \) of \( \sigma \) through sections we have:

\[
\frac{d}{dt} \bigg|_{t=0} E_{p,q}^v(\sigma_t) = \int_M (w^p(\sigma) \langle \nabla^* \nabla \sigma + q \Delta F \sigma, \rho \rangle
\]

\[
+ pw^{p+1}(\sigma) \langle 2 \nabla_{\nabla F}\sigma + q |\nabla F|^2 \sigma - |\nabla \sigma|^2 \sigma, \rho \rangle) \operatorname{vol}(g),
\]

where \( 2F = |\sigma|^2 \). For all \((p, q)\), \( \sigma \) is a \((p, q)\)-harmonic section of \( \mathcal{E} \) if and only if:

\[
T_p(\sigma) = \phi_{p,q}(\sigma) \sigma,
\]

where:

\[
T_p(\sigma) = (1 + 2F)\nabla^* \nabla \sigma + 2p \nabla_{\nabla F}\sigma,
\]

\[
\phi_{p,q}(\sigma) = p |\nabla \sigma|^2 - pq |\nabla F|^2 - q(1 + 2F)\Delta F.
\]

Proof. Applying Lemma 3.5 to equation (3-7), and noting the cancellation of terms involving \( \pm q \nabla_{\nabla F}\sigma \), yields:

\[
V_2 = \int_M (w^p(\sigma) \langle \nabla^* \nabla \sigma + q(\Delta F)\sigma, \rho \rangle
\]

\[
+ 2pw^{p+1}(\sigma) \langle \nabla_{\nabla F}\sigma + q |\nabla F|^2 \sigma, \rho \rangle) \operatorname{vol}(g)
\]

The first variation formula is then obtained by combining this with (3-5). \quad \square

Remark 3.7. By (2-6) and (2-7), \( E^v(\sigma) \geq 0 \) for all \( q \)-positive sections \( \sigma \), and the zeroes of \( E^v \) in \( \mathcal{C}(\mathcal{E}, q+) \) are precisely the parallel sections, which are therefore the absolute minima of the restriction of \( E^v \) to \( \mathcal{C}(\mathcal{E}, q+) \). In particular, any parallel \( \sigma \) is a \((p, q)\)-harmonic section of \( \mathcal{E} \) for all \((p, q)\) with \( q|\sigma|^2 > -1 \). However, it follows from Theorem 3.6 that parallel sections are in fact \((p, q)\)-harmonic for all \((p, q)\).
Remark. In the Sasaki case, when \((p, q) = (0, 0)\), we recover the Euler-Lagrange equations (1-2). In fact we get:
\[
\frac{d}{dt} \bigg|_{t=0} E^v(\sigma_t) = \int_M \langle \nabla^* \nabla \sigma, \rho \rangle \text{vol}(g)
\]  
(3-8)

Remark 3.8. If \(|\sigma| = k\) (constant) then \(\nabla F = 0\) and it follows from (2-7) that \(E^v(\sigma)\) is a constant multiple (depending only on \(k\) and \(p\)) of the Sasaki vertical energy. Therefore if \(k > 0\) then \(\sigma\) is a \((p, q)\)-harmonic section of \(SE(k)\) (i.e. a critical point of \(E^v\) with respect to \(h_{p,q}\) and smooth variations through sections of \(SE(k)\)) if and only if \(\sigma\) is a harmonic section of \(SE(k)\) with respect to the Sasaki metric, so we simply say that \(\sigma\) is a harmonic section of \(SE(k)\). Differentiating the constraint equation \(|\sigma_t|^2 = k^2\) with respect to \(t\) yields \(\langle \sigma, \rho \rangle = 0\), by (3-4), and it therefore follows from (3-8) that \(\sigma\) is a harmonic section of \(SE(k)\) if and only if \(\nabla^* \nabla \sigma = \lambda \sigma\) for some smooth \(\lambda : M \to \mathbb{R}\). It then follows from Lemma 3.4 that:
\[
k^2 \lambda = \langle \nabla^* \nabla \sigma, \sigma \rangle = |\nabla \sigma|^2
\]
Thus \(\sigma\) is a harmonic section of \(SE(k)\) if and only if \(\sigma\) satisfies equation (1-3).

4. Main Theorems

Our first result is Theorem A of the Introduction.

Theorem 4.1. Suppose \(\sigma\) has constant length \(k > 0\). Then \(\sigma\) is a \((p, q)\)-harmonic section of \(\mathcal{E}\) if and only if \(\sigma\) is parallel, except when \(p = 1 + 1/k^2\) in which case \(\sigma\) is a \((p, q)\)-harmonic section of \(\mathcal{E}\) if and only if \(\sigma\) is a harmonic section of \(SE(k)\).

Proof. If \(|\sigma| = k\) then:
\[
T_p(\sigma) = (1 + k^2) \nabla^* \nabla \sigma \quad \text{and} \quad \phi_{p,q}(\sigma) = p|\nabla \sigma|^2
\]
Hence the \((p, q)\)-harmonic section equations reduce to:
\[
\nabla^* \nabla \sigma = \frac{p}{1 + k^2} |\nabla \sigma|^2 \sigma,
\]
which by Lemma 3.4 implies:
\[
\frac{(p - 1)k^2 - 1}{k^2 + 1} |\nabla \sigma|^2 = 0
\]
Therefore \(\nabla \sigma = 0\), except when \(p = 1 + 1/k^2\), in which case the \((p, q)\)-harmonic section equations become:
\[
\nabla^* \nabla \sigma = (p - 1)|\nabla \sigma|^2 \sigma
\]
(4-1)
But (4-1) is precisely the equation (1-3) for \(\sigma\) to be a harmonic section of \(SE(k)\) when \(k = 1/\sqrt{p - 1}\). \(\square\)
Example 4.2. If \( \xi \) is a harmonic section of the unit sphere bundle \( SE(1) \) then it follows from equation (1-3) that \( \sigma = k\xi \) is a harmonic section of \( SE(k) \) for all \( k > 0 \). It follows from Theorem 4.1 that \( \sigma \) is a \((p, q)\)-harmonic section of \( E \) for \( p = 1+1/k^2 \) and all \( q \). Thus given a harmonic section of \( SE(1) \) and any \((p, q)\) with \( p > 1 \) it is possible to construct a \((p, q)\)-harmonic section of \( E \). For example, let \( M = S^{2m+1} \) and let \( \xi \) be the standard Hopf vector field: \( \xi(x) = ix \) (where \( i = \sqrt{-1} \), thinking of \( x \in S^{2m+1} \subset \mathbb{R}^{2m+2} \cong \mathbb{C}^{m+1} \)). Then \( \xi \) is a harmonic section of \( SE(1) \) where \( E = \mathcal{T}M \) [22], and so \( \sigma = \xi/\sqrt{p-1} \) is a \((p, q)\)-harmonic section of \( E \). In particular, this shows that for all \((p, q)\) with \( p > 1 \) there exist examples of \((p, q)\)-harmonic sections of constant length which are not parallel.

For non-compact \( M \), sections of constant length are generalized by those for which \( |\sigma|^2: M \to \mathbb{R} \) is a harmonic function. Lemma 3.4 shows that if such a section is \((0, 0)\)-harmonic then it is parallel. We now investigate this “Bernstein phenomenon” for other values of \((p, q)\). For this it is convenient to identify the following regions of the \((p, q)\)-plane, for any \( \mu > 0 \):

\[
\mathcal{F}_-(\mu) = \{(p, q) : p < 0, \mu q \leq 2p\}, \quad \mathcal{F}_0 = \{(p, q) : 0 \leq p \leq 1\},
\]

\[
\mathcal{F}_1(\mu) = \{(p, q) : p > 1, \mu q < 1 - p\},
\]

and then define:

\[
\mathcal{F}(\mu) = \mathcal{F}_-(\mu) \cup \mathcal{F}_0 \cup \mathcal{F}_1(\mu)
\]

Theorem 4.3. Suppose that \( \mu \geq 1/2 \) and \((p, q) \in \mathcal{F}(\mu)\), \( \sigma \) is a \( \mu q \)-Riemannian section of \( E \), and \( |\sigma|^2 \) is a harmonic function. Then \( \sigma \) is a \((p, q)\)-harmonic section if and only if \( \sigma \) is parallel.

Proof. From Theorem 3.6 and Lemma 3.4:

\[
\langle T_p(\sigma), \sigma \rangle = (1 + 2F) (|\nabla \sigma|^2 + \Delta F) + 2p |\nabla F|^2 \\
\langle \phi_{p,q}(\sigma) \sigma, \sigma \rangle = 2pF |\nabla \sigma|^2 - 2pqF |\nabla F|^2 - 2qF(1 + 2F)\Delta F
\]

Therefore:

\[
\langle T_p(\sigma) - \phi_{p,q}(\sigma) \sigma, \sigma \rangle = (1 + 2(1-p)F)|\nabla \sigma|^2 + (1 + 2qF)(1 + 2F)\Delta F \\
+ 2p(1 + qF)|\nabla F|^2 \\
= C_1 |\nabla \sigma|^2 + C_2 \Delta F + C_3 |\nabla F|^2, \quad \text{say.} \quad (4-2)
\]

Thus if \( \sigma \) is \((p, q)\)-harmonic and \( F \) is a harmonic function then by Theorem 3.6:

\[
0 = C_1 |\nabla \sigma|^2 + C_3 |\nabla F|^2 \quad (4-3)
\]
We now consider each subregion of \( F(\mu) \) separately.

(i) \( (p, q) \in F_-(\mu) \)

Since \( C_1 > 0 \) and \( \sigma \) is \( \mu q \)-Riemannian, applying Kato’s inequality (2-6) to (4-3) yields:

\[
0 \geq (C_3 - \mu q C_1) |\nabla F|^2 \\
= (2p - \mu q + 2((\mu + 1)p - \mu q))|\nabla F|^2 \\
= (A + B q F)|\nabla F|^2, \quad \text{say.} \quad \tag{4-4}
\]

We have \( A \geq 0 \) and:

\[
B q F = 2(\mu(p - 1) + p)q F > 0,
\]

since \( \mu > 0 \) and \( p, q < 0 \). Hence \( \nabla F = 0 \) identically, and so \( F \) is constant. But then \( \sigma \) is parallel, by Theorem 4.1.

(ii) \( (p, q) \in F_0 \)

We have \( C_1 > 0 \) in (4-3). If \( q \geq 0 \) then \( C_3 \geq 0 \), and therefore \( \nabla \sigma = 0 \). If \( q < 0 \) then since \( C_1 > 0 \) the inequality (4-4) still holds. For \( p \leq \mu / (\mu + 1) \) we have \( B q F \geq 0 \), and since \( A > 0 \) it follows as in (i) that \( \sigma \) is parallel. If \( p > \mu / (\mu + 1) \) then \( B > 0 \) and since \( \sigma \) is \( \mu q \)-Riemannian we have:

\[
B q F \geq B \left( -\frac{1}{2\mu} \right) = 1 - \left( \frac{\mu + 1}{\mu} \right) p
\]

Therefore (4-4) may be strengthened:

\[
0 \geq \left( 2p - \mu q + 1 - \left( \frac{\mu + 1}{\mu} \right) p \right) |\nabla F|^2 = \left( \frac{\mu(p + 1) - p}{\mu} - \mu q \right) |\nabla F|^2
\]

Since \( \mu(p + 1) - p \geq 0 \) for all \( p \in [0, 1] \) if and only if \( \mu \geq 1/2 \), the coefficient of \( |\nabla F|^2 \) is strictly positive. It therefore follows as in (i) that \( \sigma \) is parallel.

(iii) \( (p, q) \in F_1(\mu) \)

Since \( \mu q < 1 - p \) and \( \sigma \) is \( \mu q \)-Riemannian, it follows that \( C_1 > 0 \) in (4-3). Furthermore since \( \mu \geq 1/2 \) we have \( \mu q \leq q/2 \), hence \( \sigma \) is \( (q/2) \)-Riemannian. Therefore \( C_3 \geq 0 \) in (4-3). Hence \( \nabla \sigma = 0 \).

\[\square\]

**Remark.** Of the \((p, q)\)-harmonic sections addressed by Theorem 4.3, only those with \( \mu \geq 1 \) are necessarily \( q \)-Riemannian. Thus the “Bernstein phenomenon” described by Theorem 4.3 is not necessarily confined to \((p, q)\)-harmonic sections whose image lies in the region of \( E \) where \( h_{p, q} \) is a Riemannian metric.

**Remark.** If \( |\sigma| = k > 0 \) and \( \sigma \) is \( \mu q \)-Riemannian then \( 1/k^2 \geq -\mu q \). Hence if in addition \( p = 1 + 1/k^2 \) then \( 1 - p \leq \mu q \). It therefore follows from Theorem 4.1 that \( F_1(\mu) \) is the best possible “Bernstein region” when \( p > 1 \) in Theorem 4.3.
Remark 4.4. It follows from Theorem 4.3 that if $\sigma$ is a harmonic section with respect to the Riemannian metric $h_{p,p}$ for any $0 \leq p \leq 1$ and $|\sigma|^2$ is a harmonic function, then $\sigma$ is parallel. In particular, this is the case for the Cheeger-Gromoll metric ($p = 1$).

Remark 4.4 illustrates a surprising feature of Theorem 4.3, which is that for all $(p, q)$ in the following vertical strip:

$$\mathcal{F}_0^+ = \{(p, q) : 0 \leq p \leq 1, q \geq 0\},$$

any $(p, q)$-harmonic section $\sigma$ with $|\sigma|^2$ harmonic is necessarily parallel, without any a priori bound on its length. By placing an alternative bound on $|\sigma|^2$ when $q \geq 0$ and $p > 1$ it is possible to extend $\mathcal{F}_0^+$ rightwards into the following adjacent region:

$$\mathcal{G}_1(\nu) = \{(p, q) : p > 1, q \geq 2\nu(1-p)\},$$

for any $\nu > 0$.

Definition. A $q$-Riemannian section $\sigma$ is strictly $q$-Riemannian if $q|\sigma(x)|^2 > -1$ for some $x \in M$.

Remark. For this condition to have any force it is clearly necessary for $M$ to be connected. In our next two results (Theorems 4.5 (b) and 4.6 (b)) connectedness of $M$ is therefore essential.

Theorem 4.5. Suppose $(p, q) \in \mathcal{G}_1(\nu)$ and either:

(a) $\nu > 1$ and $\sigma$ is a $\nu(1-p)$-Riemannian section of $\mathcal{E}$;
(b) $\nu = 1$ and $\sigma$ is a strictly $(1-p)$-Riemannian section of $\mathcal{E}$.

Suppose also that $|\sigma|^2$ is a harmonic function on $M$. Then $\sigma$ is a $(p, q)$-harmonic section of $\mathcal{E}$ if and only if $\sigma$ is parallel.

Proof. Since $2\nu(1-p)F \geq -1$ and $q \geq 2\nu(1-p)$ we have $qF \geq -1$; hence $C_3$ in (4-3) is non-negative. Furthermore since $\nu \geq 1$ and $\sigma$ is $\nu(1-p)$-Riemannian, $\sigma$ is $(1-p)$-Riemannian (Remark 2.5); hence $C_1$ in (4-3) is also non-negative. Now consider $U = \{x \in M : C_1(x) > 0\}$. It follows from (4-3) that $\nabla \sigma$ vanishes on $U$. If $\nu > 1$ we have $U = M$ and hence $\sigma$ is parallel. If $\nu = 1$ we have in particular that $|\sigma|$ is constant on $U$, hence $U$ is closed. But $U$ is open, and non-empty by hypothesis. Therefore $U = M$, since $M$ is connected. Thus $\sigma$ is parallel. \qed

Remark. The $(p, q)$-harmonic sections addressed by Theorem 4.5 are $q$-Riemannian if $q \geq \nu(1-p)$, but not necessarily so if $2\nu(1-p) \leq q < \nu(1-p)$.

Remark. For all $\mu \geq 1/2$ and $\nu \geq 1$ we have:

$$\mathcal{F}_1(\mu) \cup \mathcal{G}_1(\nu) = \{(p, q) : p > 1\},$$
and:

\[ \mathcal{F}_1(\mu) \cap \mathcal{G}_1(\nu) = \{ (p, q) : p > 1, 2\mu\nu(1-p) \leq \mu q < 1-p \} = \mathcal{W}(\mu, \nu), \text{ say.} \]

In particular, the union is disjoint precisely when \( \mu = 1/2 \) and \( \nu = 1 \). Furthermore, for all \((p, q) \in \mathcal{W}(\mu, \nu)\) every \( \mu q \)-Riemannian section is strictly \((1-p)\)-Riemannian, so that Theorem 4.5 is consistent with Theorem 4.3 in the region where both theorems apply, namely \( \mathcal{W}(\mu, 1) \). In addition, it follows from Theorem 4.5 that Theorem 4.3 extends to the closure of \( \mathcal{F}(\mu) \) for all sections which are strictly \( \mu q \)-Riemannian.

We now consider the compact case. To identify a “Bernstein region” of the \((p, q)\)-plane it is convenient to introduce the following monotone decreasing piecewise-linear cut-off function:

**Definition.** For any \( \nu > 0 \) define \( \varrho_\nu : [-4, \infty) \to \mathbb{R} \) by:

\[
\varrho_\nu(p) = \begin{cases} 
-1 - p, & \text{if } -4 \leq p \leq -1, \\
0, & \text{if } -1 \leq p \leq 2, \\
\nu(2-p)/2, & \text{if } p \geq 2.
\end{cases}
\]

**Theorem 4.6.** Suppose \( M \) is compact, \( p \geq -4, q \geq \varrho_\nu(p) \) and either:

(a) \( \nu > 1 \) and \( \sigma \) is a \( \nu(1-p) \)-Riemannian section of \( \mathcal{E} \);

(b) \( \nu = 1 \) and \( \sigma \) is a strictly \((1-p)\)-Riemannian section of \( \mathcal{E} \).

Then \( \sigma \) is \((p, q)\)-harmonic if and only if \( \sigma \) is parallel.

**Remark.** If \( p \leq 1 \) then all sections are strictly \((1-p)\)-Riemannian.

**Proof.** Referring to identity (4-2), we note first that repeated use of the Divergence Theorem yields the following integral formula for the term involving the Laplacian (where all integrals are taken over \( M \), with respect to the Riemannian volume element):

\[
\int C_2 \Delta F = \int (1 + 2F)\Delta F + 2q \int F(1 + 2F)\Delta F
\]

\[
= 2 \int |\nabla F|^2 + 2q \int \{|\nabla F|^2 + 2g(\nabla(\nabla F^2), \nabla F)\}
\]

\[
= 2(1 + q) \int |\nabla F|^2 + 8q \int F |\nabla F|^2
\]

Therefore if \( \sigma \) is \((p, q)\)-harmonic then integration of (4-2) yields:

\[
0 = \int C_1 |\nabla \sigma|^2 + 2(p + q + 1) \int |\nabla F|^2 + 2(p + 4)q \int F |\nabla F|^2 \quad (4-5)
\]

Since \( \sigma \) is \( \nu(1-p) \)-Riemannian and \( \nu \geq 1, \sigma \) is \((1-p)\)-Riemannian (Remark 2.5); hence \( C_1 \geq 0 \). Define \( U = \{ x \in M : C_1(x) > 0 \} \). If \( p \in [-4, 2] \) then \( q \geq \varrho_\nu(p) \) if and only if
$p + q + 1 \geq 0$ and $(p + 4)q \geq 0$. It therefore follows from (4-5) that $C_1 |\nabla \sigma|^2$ vanishes identically, and so $\nabla \sigma$ vanishes on $U$. If $p \geq 2$ then since $\nu \geq 1$:

$$\nu(1 - p) < \nu \left(1 - \frac{p}{2}\right) = \varrho_i(p) \leq q$$  \hspace{1cm} (4-6)

Therefore, since $\sigma$ is $\nu(1 - p)$-Riemannian, $\sigma$ is also $q$-Riemannian. It then follows from (4-5) that:

$$0 \geq \int C_1 |\nabla \sigma|^2 + (p + 2q - 2) \int |\nabla F|^2 \geq \int C_1 |\nabla \sigma|^2,$$

since by (4-6):

$$2q \geq \nu(2 - p) \geq 2 - p, \text{ because } \nu \geq 1.$$

We therefore deduce again that $\nabla \sigma$ vanishes on $U$. If $\nu > 1$ then $U = M$. If $\nu = 1$ then the connectedness argument of Theorem 4.5 may be used to conclude the proof. \hfill \square

**Remark.** All the $(p, q)$-harmonic sections addressed by Theorem 4.6 are in fact $q$-Riemannian (as shown during the proof).

**Remark.** The re-scaled Hopf vector fields described in Example 4.2 show that Theorem 4.6 is false if $\sigma$ is merely $(1 - p)$-Riemannian.

**Remark 4.7.** It follows from Theorem 4.6 that if $0 \leq p \leq 1$ then any $(p, p)$-harmonic section of a compact vector bundle is parallel. In particular, this is true of both the Sasaki ($p = 0$) and Cheeger-Gromoll ($p = 1$) metrics [16] (cf. Remark 4.4).

It follows from Theorem 4.1 that if a section $\sigma$ of constant length is $(p, q)$-harmonic then $\sigma$ is $(p, r)$-harmonic for all $r \in \mathbb{R}$. However the following result shows that this is exceptional.

**Theorem 4.8.** Suppose $M$ is compact, and $\sigma$ is a section of $\mathcal{E}$ whose length is not constant. Then for each $p \in \mathbb{R}$ there exists at most one $q \in \mathbb{R}$ such that $\sigma$ is $(p, q)$-harmonic.

**Proof.** If $\sigma$ is both $(p, q)$-harmonic and $(p, r)$-harmonic then it follows from Theorem 3.6 that:

$$0 = \phi_{p, r}(\sigma) - \phi_{p, q}(\sigma) = p(q - r)|\nabla F|^2 + (q - r)(1 + 2F)\Delta F$$

$$= (q - r) \left(p |\nabla F|^2 + (1 + 2F)\Delta F\right) \hspace{1cm} (4-7)$$

Then by the Divergence Theorem:

$$0 = (q - r) \int (p |\nabla F|^2 + (1 + 2F)\Delta F) = (q - r)(p + 2) \int |\nabla F|^2$$
If \( q \neq r \) then since \( F \) is not constant it follows that \( p = -2 \), in which case (4-7) reduces to:

\[
0 = \Delta F + 2F\Delta F - 2|\nabla F|^2 = \Delta(F + F^2),
\]

where we have used the formula for the Laplacian of a product:

\[
\Delta(f_1f_2) = f_1\Delta f_2 + f_2\Delta f_1 - 2g(\nabla f_1, \nabla f_2)
\]

(4-8)

But this implies that \( F^2 + F \), and hence \( F \), is constant.

\( \square \)

**Remark.** It follows directly from (4-7) that Theorem 4.8 also holds if \( M \) is non-compact provided \( |\sigma|^2 \) is a non-constant harmonic function.

Combining Theorems 4.6 and 4.8 yields the following result, which may be regarded as a partial complement to Theorem 4.1:

**Corollary 4.9.** Suppose \( M \) is compact, and \( \sigma \) is a section of \( E \) whose length is not constant. Then for each \( p \in \mathbb{R} \) there exists at most one \( q \in \mathbb{R} \) such that \( \sigma \) is \((p, q)\)-harmonic, and if either \(-4 \leq p \leq 1\), or \( p > 1 \) and \( \|\sigma\|_\infty^2 \leq 1/(p-1) \) for some \( \nu \geq 1 \), then \( q < 2\nu(p) \).

Theorem B is now a simple consequence of Corollary 4.9 (with \( \nu = 1 \)). Combining Theorem 4.3 (with \( \mu = 1/2 \)), Theorem 4.5 (with \( \nu = 1 \)) and Theorem 4.8 yields the following non-compact analogue of Corollary 4.9:

**Corollary 4.10.** Suppose \( \sigma \) is a section of \( E \) for which \( |\sigma|^2 \) is a non-constant harmonic function. Then for each \( p \in \mathbb{R} \) there exists at most one \( q \in \mathbb{R} \) such that \( \sigma \) is \((p, q)\)-harmonic, and:

(a) if \( p < 0 \) and \( q \leq 4p \) then \( |\sigma(x)|^2 > -2/q \) for some \( x \in M \);
(b) if \( 0 \leq p \leq 1 \) then \( q < 0 \) and \( |\sigma(x)|^2 > -2/q \) for some \( x \in M \);
(c) if \( p > 1 \) and \( q < 2(1-p) \) then \( |\sigma(x)|^2 > -2/q \) for some \( x \in M \);
(d) if \( p > 1 \) and \( q \geq 2(1-p) \) then \( |\sigma(x)|^2 > 1/(p-1) \) for some \( x \in M \).

5. Example: Vector Fields on Spheres

Let \( M = S^n \subset \mathbb{R}^{n+1} \) be the unit sphere, with the induced Riemannian metric \( g \), and let \( E = TM \) with \( \langle , \rangle = g \). We look for \((p, q)\)-harmonic sections of \( E \) amongst the class of conformal gradient fields on \( M \). Thus, let \( a \in \mathbb{R}^{n+1} \setminus \{0\} \), and let \( \lambda: S^n \to \mathbb{R} \) be the restriction of the linear functional on \( \mathbb{R}^{n+1} \) dual to \( a \). Define \( \sigma = \nabla \lambda \). We refer to \( a \) as the axial vector of \( \sigma \), and note that \( |a| = \|\sigma\|_\infty \). We say that \( \sigma \) is standard if \( |a| = 1 \). The following is a collation of data required to compute the \((p, q)\)-harmonic section equations.
Lemma 5.1. Suppose the axial vector of $\sigma$ has length $c$. Then:

(i) $\nabla_X \sigma = -\lambda X$
(ii) $\nabla^* \nabla \sigma = \sigma$
(iii) $2F = c^2 - \lambda^2$
(iv) $\nabla F = -\lambda \sigma$
(v) $\Delta F = c^2 - (n + 1)\lambda^2$

Proof. (i) and (ii) are well-known; see for example [25].

(iii) By definition:

$$\sigma(x) = a - \lambda(x)x = a - (a \cdot x)x,$$

where $a \cdot x$ is the Euclidean dot product, and it therefore follows that:

$$2F(x) = |\sigma(x)|^2 = |a|^2 - 2(a \cdot x)^2 + (a \cdot x)^2 = c^2 - \lambda(x)^2$$

(iv) From (iii):

$$\nabla F = \nabla F = -\frac{1}{2} \nabla \lambda^2 = -\lambda \nabla \lambda = -\lambda \sigma$$

(v) We have:

$$\Delta F = -\text{div} \nabla F = -\text{div} \nabla F = \text{div}(\lambda \sigma), \quad \text{by (iv)},$$

$$= \langle \nabla \lambda, \sigma \rangle + \lambda \text{div} \sigma = |\sigma|^2 - n\lambda^2, \quad \text{by (i)},$$

$$= c^2 - \lambda^2 - n\lambda^2, \quad \text{by (iii)}.$$

$\square$

Theorem 5.2. Let $\sigma$ be a conformal gradient field on $M = S^n$. Then $\sigma$ is a $(p,q)$-harmonic section of $TM$ if and only if $n \geq 3$ and:

$$p = n + 1, \quad q = 2 - n, \quad \|\sigma\|_\infty = 1/\sqrt{-q}.$$

Proof. From Lemma 5.1 it follows that:

$$|\nabla \sigma|^2 = n\lambda^2, \quad \nabla_{\nabla F} \sigma = \lambda^2 \sigma, \quad |\nabla F|^2 = \lambda^2(c^2 - \lambda^2)$$

Therefore:

$$T_p(\sigma) = (1 + 2F)\nabla^* \nabla \sigma + 2p \nabla_{\nabla F} \sigma$$

$$= (1 + c^2 - \lambda^2)\sigma + 2p\lambda^2 \sigma = (1 + c^2 + (2p - 1)\lambda^2)\sigma,$$

$$\phi_{p,q}(\sigma) = p|\nabla \sigma|^2 - pq|\nabla F|^2 - q(1 + 2F)\Delta F$$

$$= np\lambda^2 - pq\lambda^2(c^2 - \lambda^2) - q(1 + c^2 - \lambda^2)(c^2 - (n + 1)\lambda^2)$$

$$= p(n + q)\lambda^2 - q(1 + c^2 - \lambda^2)(c^2 - (n - p + 1)\lambda^2)$$
Thus the harmonic section equations are polynomial in $\lambda$. Since $\lambda$ is a continuous function on $M$, which vanishes only on the great hypersphere orthogonal to $a$, this polynomial is identically zero if and only if the coefficients of the various powers of $\lambda$ vanish:

$$q = -1/c^2,$$  \hspace{1cm} (5-1)

$$2p - 1 = p(n + q) + qe^2 + q(1 + c^2)(n - p + 1),$$  \hspace{1cm} (5-2)

$$p = n + 1,$$  \hspace{1cm} (5-3)

Substituting (5-1) and (5-3) into (5-2) yields:

$$2(n + 1) = q(n + 1) + n(n + 1),$$

and hence $q = 2 - n$. \hfill \Box

**Remark 5.3.** The $(p, q)$-harmonic sections $\sigma$ in Theorem 5.2 are clearly $q$-Riemannian, but not $r$-Riemannian for any $r < q$. Indeed a simple calculation using Lemma 5.1 yields:

$$|\nabla \sigma|^2 + q |\nabla F|^2 = (n - 1)\lambda^2 + (n - 2)\lambda^4,$$

and this is strictly positive except when $\lambda = 0$, which occurs precisely on the great hypersphere orthogonal to the axial vector; in particular $E^\nu(\sigma) \geq 0$. However, since:

$$1 - p = -n < 2 - n = q,$$

it follows that $\sigma$ is not $(1 - p)$-Riemannian, and hence not $\nu(1 - p)$-Riemannian for any $\nu \geq 1$. Therefore $\sigma$ is not amenable to Theorem 4.6. In fact $q < \varrho(p)$ if $n > 3$, but $q = \varrho(p)$ when $M = S^3$.

From Theorem 5.2, the only instance where standard conformal gradient fields are $(p, q)$-harmonic sections of $TM$ is when $M = S^3$, in which case $(p, q) = (4, -1)$. The fact that in higher dimensions $(p, q)$-harmonic sections are obtained by scaling the standard fields by a constant factor (viz. $1/\sqrt{n - 2}$) is a manifestation of the non-linearity of the $(p, q)$-harmonic section equations when $p \neq 0$. We now show that when $n = 2m + 1$ the only functional multiples of the Hopf vector field which are $(p, q)$-harmonic sections of $TM$ are those described in Example 4.2.

**Theorem 5.4.** Suppose $\sigma = f\xi$, where $\xi$ is the Hopf vector field on $M = S^{2m+1}$ and $f: M \to \mathbb{R}$ is any smooth function. Then $\sigma$ is a non-trivial $(p, q)$-harmonic section of $TM$ if and only if $p > 1$ and $f = \pm1/\sqrt{p-1}$.
Proof. Initially, suppose $\xi$ is any unit vector field on any manifold $M$ with $\chi(M) = 0$. Then:

$$2F = |\sigma|^2 = |f \xi|^2 = f^2, \quad \nabla F = \frac{1}{2} \nabla (f^2) = f \nabla f$$

$$\nabla_{\nabla F}\sigma = \nabla_{\nabla F}(f \xi) = (\nabla F.f)\xi + f \nabla_{\nabla F}\xi = f |\nabla f|^2 \xi + f^2 \nabla_{\nabla f}\xi$$

$$\nabla^* \nabla \sigma = \nabla^* \nabla (f \xi) = f \nabla^* \nabla \xi + (\Delta f)\xi - 2 \nabla_{\nabla f}\xi$$

It follows that:

$$T_p(\sigma) = (1 + 2F) \nabla^* \nabla \sigma + 2p \nabla_{\nabla F}\sigma$$

$$= (1 + f^2)f \nabla^* \nabla \xi + 2((p - 1)f^2 - 1) \nabla_{\nabla f}\xi + (2pf|\nabla f|^2 + (1 + f^2)\Delta f)\xi$$

We also have:

$$|\nabla \sigma|^2 = \sum_i |\nabla_{E_i}(f \xi)|^2 = \sum_i |(E_i.f)\xi + f \nabla_{E_i}\xi|^2$$

$$= \sum_i \left(|E_i.f|^2 + 2f(E_i.f)\langle \xi, \nabla_{E_i}\xi \rangle + f^2 |\nabla_{E_i}\xi|^2 \right)$$

$$= |\nabla f|^2 + f^2 |\nabla \xi|^2, \quad \text{since } |\xi| = 1,$$

$$|\nabla F|^2 = f^2 |\nabla f|^2,$$

$$\Delta F = \frac{1}{2} \Delta (f^2) = f \Delta f - |\nabla f|^2,$$

where to compute $\Delta (f^2)$ we have used formula (4-8). It follows that:

$$\phi_{p,q}(\sigma) = p|\nabla \sigma|^2 - q(1 + 2F)\Delta F - pq|\nabla F|^2$$

$$= pf^2 |\nabla \xi|^2 + (p + q + (1 - p)qf^2)|\nabla f|^2 - q(1 + f^2)f \Delta f$$

Therefore:

$$T_p(\sigma) - \phi_{p,q}(\sigma)\sigma = (1 + f^2)f \nabla^* \nabla \xi + 2((p - 1)f^2 - 1) \nabla_{\nabla f}\xi$$

$$+ ((1 + f^2)(1 + rf^2)\Delta f + ((p - 1)rf^2 + p - q)f |\nabla f|^2 - pf^3 |\nabla \xi|^2)\xi$$

Now suppose that $\xi$ is a harmonic section of $SE(1)$, and therefore satisfies equation (1-3) with $k = 1$. Then:

$$T_p(\sigma) - \phi_{p,q}(\sigma)\sigma = 2((p - 1)f^2 - 1) \nabla_{\nabla f}\xi$$

$$+ ((1 + f^2)(1 + qf^2)\Delta f + ((p - 1)qf^2 + p - q)f |\nabla f|^2$$

$$+ (1 + (1 - p)f^2)f |\nabla \xi|^2)\xi$$

(5-4)
Since $\nabla \xi$ is orthogonal to $\xi$, because $|\xi| = 1$, it follows from (5-4) that a necessary condition for $\sigma$ to be $(p, q)$-harmonic is:

$$\nabla_{\nabla f} \xi = 0 \quad (5-5)$$

Finally suppose that $\xi$ is the Hopf vector field on $M = S^{2m+1}$ (Example 4.2). Then:

$$\nabla_X \xi = \begin{cases} iX, & \text{if } X \perp \xi, \\ 0, & \text{if } X \parallel \xi, \end{cases} \quad (5-6)$$

which implies that (5-5) holds if and only if $\nabla f = \mu \xi$ for some smooth $\mu : M \to \mathbb{R}$, or equivalently:

$$df = \mu \xi^b, \quad (5-7)$$

where $\xi^b$ is the 1-form metrically dual to $\xi$. It also follows from (5-6) that $d\xi^b = \omega$, the restriction of the Kähler form of $\mathbb{C}^{m+1}$, for:

$$2d\xi^b(X, Y) = \nabla_X \xi^b(Y) - \nabla_Y \xi^b(X) = \langle \nabla_X \xi, Y \rangle - \langle \nabla_Y \xi, X \rangle$$

$$= \langle iX, Y \rangle - \langle iY, X \rangle, \quad \text{for all } X, Y$$

$$= 2\langle iX, Y \rangle = 2\omega(X, Y)$$

Exterior differentiation of (5-7) therefore yields the following differential equation for $\mu$:

$$0 = d\mu \wedge \xi^b + \mu \omega \quad (5-8)$$

In particular, if $A \perp \xi$ then $iA \perp \xi$ also and:

$$\omega(A, iA) = \langle iA, iA \rangle = |A|^2$$

Now:

$$2d\mu \wedge \xi^b(X, Y) = d\mu(X)\langle \xi, Y \rangle - d\mu(Y)\langle \xi, X \rangle,$$

and so:

$$d\mu \wedge \xi^b(A, iA) = 0$$

Therefore the only solution of (5-8) is $\mu = 0$, and it follows that $f$ is constant. But then (5-4) reduces to:

$$T_p(\sigma) - \phi_{p,q}(\sigma)\sigma = \left(2m(1 + (1 - p)f^2)f\right)\xi,$$

so if $\sigma$ is $(p, q)$-harmonic then either $f = 0$ or $f^2 = 1/(p - 1)$.
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