Some evaluation of cubic Euler sums

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Abstract P. Flajolet and B. Salvy [15] prove the famous theorem that a nonlinear Euler sum $S_{i_1 i_2 \cdots i_r, q}$ reduces to a combination of sums of lower orders whenever the weight $i_1 + i_2 + \cdots + i_r + q$ and the order $r$ are of the same parity. In this article, we develop an approach to evaluate the cubic sums $S_{1^m, p}$ and $S_{1^l, l_2, l_3}$. By using the approach, we establish some relations involving cubic, quadratic and linear Euler sums. Specially, we prove the cubic sums $S_{1^m, m}$ and $S_{1(2l+1)^2, 2l+1}$ are reducible to zeta values, quadratic and linear sums. Moreover, we prove that the two combined sums involving multiple zeta values of depth four

$$\sum_{\{i,j\} \in \{1,2\}, i \neq j} \zeta(m_i, m_j, 1, 1) \quad \text{and} \quad \sum_{\{i,j,k\} \in \{1,2,3\}, i \neq j \neq k} \zeta(m_i, m_j, m_k, 1)$$

can be expressed in terms of multiple zeta values of depth $\leq 3$, here $2 \leq m_1, m_2, m_3 \in \mathbb{N}$. Finally, we evaluate the alternating cubic Euler sums $S_{1^3, 2r+1}$ and show that it are reducible to alternating quadratic and linear Euler sums. The approach is based on Tornheim type series computations.

Keywords Harmonic number; polylogarithm function; Euler sum; Tornheim type series; Riemann zeta function, multiple zeta value.

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1 Introduction

In response to a letter from Goldbach, Euler considered sums of the form (see Berndt [4] for a discussion)

\[
S_{p,q} := \sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^q},
\]

and was able to give explicit values for certain of these sums in terms of the Riemann zeta function, where \( p, q \) are positive integers with \( q \geq 2 \), and \( w := p + q \) denotes the weight of linear sums \( S_{p,q} \). These kind of sums are called the linear Euler sums (for short linear sums) today. Here \( H_n^{(p)} \) denotes the harmonic number which is defined by

\[
H_n^{(k)} := \sum_{j=1}^{n} \frac{1}{j^k} \quad \text{and} \quad H_0^{(k)} := 0.
\]

When \( k = 1 \), then \( H_n := H_n^{(1)} \), which is called the classical harmonic number.

In their famous paper [15], Flajolet and Salvy introduced the following generalized series

\[
S_{S,q} := \sum_{n=1}^{\infty} \frac{H_n^{(s_1)}H_n^{(s_2)} \cdots H_n^{(s_r)}}{n^q},
\]

which is called the generalized (nonlinear) Euler sums. Here \( S := (s_1, s_2, \ldots, s_r) \) \((r, s_i \in \mathbb{N}, i = 1, 2, \ldots, r)\) and \( q \geq 2 \). The quantity \( w := s_1 + \cdots + s_r + q \) is called the weight and the quantity \( r \) is called the degree. As usual, repeated summands in partitions are indicated by powers, so that for instance

\[
S_{1^22^4,q} = S_{112224,q} = \sum_{n=1}^{\infty} \frac{H_n^{(2)}H_n^{(2)}H_n^{(4)}}{n^q}.
\]

It has been discovered in the course of the years that many Euler sums admit expressions involving finitely the “zeta values”, that is to say of the Riemann zeta function [2],

\[
\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}, \Re(s) > 1
\]

with positive integer arguments. For example, the linear sums \( S_{p,q} \) can be evaluated in terms of zeta values in the following cases: \( p = 1, p = q, p + q \) odd and \( p + q = 6 \) with \( q \geq 2 \) (for more details, see [3, 5, 15]). In 1994, Bailey et al. [3] proved that all Euler sums of the form \( S_{1^p,q} \) for weights \( p + q \in \{3, 4, 5, 6, 7, 9\} \) are reducible to \( \mathbb{Q} \)-linear combinations of zeta values by using the experimental method. In [28, 29], we proved that all Euler sums of weight \( \leq 7 \) are reducible to \( \mathbb{Q} \)-linear combinations of single zeta monomials. For weight 9, all Euler sums of the form \( S_{s_1 \cdots s_r,q} \) with \( q \in \{4, 5, 6, 7\} \) are expressible polynomially in terms of zeta values. Very recently, Wang et al [25] shown that all Euler sums of weight nine are reducible to zeta values.

However, there are also many nonlinear Euler sums which need not only zeta values but also linear sums. Namely, many nonlinear Euler sums are reducible to polynomials in zeta values and to linear sums (see [3, 5, 15, 18, 27, 30]). For instance, in 1995, Borwein et al. [5] showed that the quadratic sums \( S_{1^2, q} \) can reduce to linear sums \( S_{2,q} \) and polynomials in zeta values. In 1998, Flajolet and Salvy [15] used the contour integral representations and residue computation
to show that the quadratic sums $S_{p_1p_2,q}$ are reducible to linear sums and zeta values when the weight $p_1 + p_2 + q$ is even and $p_1, p_2 > 1$. In [28], we proved that all Euler sums with weight eight are reducible to zeta values and linear sum $S_{2,6}$.

Hence, a good deal of work on Euler sums has focused on the problem of determining when ‘complicated’ sums can be expressed in terms of ‘simpler’ sums. Thus, researchers are interested in determining which sums can be expressed in terms of other sums of lesser degree. Besides the works referred to above, there are many other researches devoted to the Euler sums. For example, please see [1, 6, 10, 12, 13, 16, 18] and references therein.

Multiple zeta values (also called Zagier sums) are the natural generalizations of classical Euler sums from double sums to more general $k$-fold Euler sums defined by [7, 8, 28]

$$
\zeta(S) \equiv \zeta(s_1, s_2, \cdots, s_k) := \sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_k^{s_k}},
$$

where $s_1, s_2, \ldots, s_k$ are positive integers with $s_1 > 1$ for the sake of convergence. The numbers $w := s_1 + \cdots + s_k$ and $k$ are called the weight and depth of $\zeta(s_1, s_2, \cdots, s_k)$ respectively. For convenience, we let $\{a\}_k$ be the $k$ repetitions of a such that

$$
\zeta(5, 3, \{1\}_4, 2) = \zeta(5, 3, 1, 1, 1, 1, 2).
$$

Many papers use the opposite convention, with the $n_i$’s ordered by $n_1 < n_2 < \cdots < n_k$ or $n_1 \leq n_2 \leq \cdots \leq n_k$, see [17, 35].

From the definitions of Euler sums and multiple zeta values, we can find the relations

$$
S_{p,q} = \zeta(q, p) + \zeta(q + p),
$$

$$
S_{p_1p_2,q} = \zeta(q, p_1, p_2) + \zeta(q, p_2, p_1) + \zeta(q + p_1, p_2)
+ \zeta(q + p_2, p_1) + \zeta(q, p_1 + p_2) + \zeta(q + p_1 + p_2).
$$

Multiple zeta values were introduced and studied by Euler in the old days. The multiple zeta values have attracted considerable interest in recent years. In the past two decades, many authors have studied multiple zeta values, and a number of relations among them have been found, see [7–9, 14, 17, 20–22, 24, 28, 34–36] and references therein. For example, some results on triple zeta values and quadruple zeta values were evaluated in [9, 13, 14, 20, 21]. Tsumura [24] proved in 2004 that the multiple zeta value $\zeta(s_1, s_2, \cdots, s_k)$ can be expressed as a rational linear combination of products of multiple zeta values of lower depth provided that the sum of weight and depth, $w + k$, is odd. Jonathan M. Borwein, David M. Bradley and David J. Broadhurst [7] proved the following duality relation

$$
\zeta \left( m_1 + 2, \{1\}_{n_1}, \ldots, m_p + 2, \{1\}_{n_p} \right) = \zeta \left( n_p + 2, \{1\}_{m_p}, \ldots, n_1 + 2, \{1\}_{m_1} \right).
$$

Zagier [35] proved that the multiple zeta values $\zeta(\{2\}_a, 3, \{2\}_b)$ are reducible to polynomials in zeta values, $a, b \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$, and gave explicit formulae.

In this paper, we develop an approach to evaluation of Euler sums. The approach is based on Tornheim type series computations. The purpose of this paper is to establish some explicit formulas of Euler sums with weight even and degree three in terms of zeta values, linear and quadratic sums by using the method of Tornheim type series computations. Moreover, by using the approach, we can obtain some relations between Euler sums of degree three and multiple zeta values of depth four.
2 Main Theorems and Corollaries

The main theorems and corollaries of this paper can be stated as follows.

Theorem 2.1 For positive integers \( m, p \geq 2 \), then the following identity holds:

\[
S_{12m,p} + S_{1p,m} + S_{2m,p} + S_{2p,m} \\
= 2(-1)^{p-1} \sum_{i=1}^{p-2} (-1)^{i-1} S_{m,i+1} \cdot S_{1,p-i} + 2(-1)^{m-1} \sum_{i=1}^{m-2} (-1)^{i-1} S_{p,i+1} \cdot S_{1,m-i} \\
- 2\zeta (2) (\zeta (m) \zeta (p) + \zeta (m + p)) + 2S_{1m,p+1} + 2S_{1p,m+1} \\
+ 2(-1)^{m+p} S_{1^2,m+p} - 2(-1)^{m+p} S_{1,p} \cdot S_{1,m} \\
- 2(-1)^{p-1} \zeta (m + 1) S_{1,p} - 2(-1)^{m-1} \zeta (p + 1) S_{1,m} \\
- 2(-1)^{p-1} \sum_{j=1}^{m-1} (-1)^{j-1} \zeta (m + 1 - j) (S_{1j,p} - S_{1,p+j}) \\
- 2(-1)^{m-1} \sum_{j=1}^{p-1} (-1)^{j-1} \zeta (p + 1 - j) (S_{1j,m} - S_{1,m+j}). \tag{2.1}
\]

Theorem 2.2 If \( l_1 + l_2 + l_3 \) is odd, and \( l_1, l_2, l_2 \geq 2 \) are positive integers, then the cubic combination \( S_{1l_1,l_2,l_3} + S_{1l_1,l_3,l_2} + S_{1l_2,l_3,l_1} \) is expressible in terms of quadratic, linear sums and zeta values. We have

\[
S_{1l_1,l_2,l_3} + S_{1l_1,l_3,l_2} + S_{1l_2,l_3,l_1} \\
= (-1)^{l_3} \sum_{i=1}^{l_2-2} (-1)^{i-1} S_{l_3,i+1} \cdot S_{l_1,l_2-i} + S_{l_1,l_1} (\zeta (l_2) \zeta (l_3) + \zeta (l_2 + l_3)) \\
+ (-1)^{l_2} \sum_{i=1}^{l_1-2} (-1)^{i-1} S_{l_2,i+1} \cdot S_{l_3,l_1-i} + S_{l_1,l_2} (\zeta (l_1) \zeta (l_3) + \zeta (l_1 + l_3)) \\
+ (-1)^{l_1} \sum_{i=1}^{l_3-2} (-1)^{i-1} S_{l_1,i+1} \cdot S_{l_2,l_3-i} + S_{l_1,l_3} (\zeta (l_1) \zeta (l_2) + \zeta (l_1 + l_2)) \\
+ (-1)^{l_1-1} \sum_{j=1}^{l_1-1} (-1)^{j-1} \zeta (l_1 + 1 - j) (S_{l_3,j,l_2} + S_{l_2,j,l_3} - S_{l_3,l_2+j} - S_{l_3,l_3+j}) \\
+ (-1)^{l_2-1} \sum_{j=1}^{l_2-1} (-1)^{j-1} \zeta (l_2 + 1 - j) (S_{l_3,j,l_1} + S_{l_1,j,l_3} - S_{l_3,l_1+j} - S_{l_3,l_3+j}) \\
+ (-1)^{l_3-1} \sum_{j=1}^{l_3-1} (-1)^{j-1} \zeta (l_3 + 1 - j) (S_{l_1,j,l_2} + S_{l_2,j,l_1} - S_{l_1,l_2+j} - S_{l_2,l_3+j}) \\
+ (-1)^{l_1-1} \zeta (l_1 + 1) (\zeta (l_2) \zeta (l_3) + \zeta (l_2 + l_3)) \\
+ (-1)^{l_2-1} \zeta (l_2 + 1) (\zeta (l_1) \zeta (l_3) + \zeta (l_1 + l_3)) \\
+ (-1)^{l_3-1} \zeta (l_3 + 1) (\zeta (l_1) \zeta (l_2) + \zeta (l_1 + l_2)) \\
- S_{1(l_1+l_2),l_3} - S_{1(l_1+l_3),l_2} - S_{1(l_2+l_3),l_1}. \tag{2.2}
\]
Theorem 2.3 For positive integers $m_1, m_2 \geq 1$ and $m_3 > 1$, then the following relation holds:

\[
S_{1m_1m_2,m_3} = (-1)^{m_3-1} \sum_{i=1}^{m_3-2} (-1)^{i-1} \zeta (m_3 - i) S_{m_1m_2,i+1}
\]

\[
+ (-1)^{m_3} (S_{m_1,m_2+1} + S_{m_2,m_1+1} - \zeta (m_1 + m_2 + 1)) \zeta (m_3)
\]

\[
+ (-1)^{m_3} \left\{ \sum_{i=1}^{m_2-1} (-1)^{i-1} S_{m_1,m_2+1-i} \zeta (m_3, i) + \sum_{i=1}^{m_1-1} (-1)^{i-1} S_{m_2,m_1+1-i} \zeta (m_3, i) \right\}
\]

\[
+ (-1)^{m_3} \left\{ (-1)^{m_2-1} \zeta (m_1 + 1) \zeta (m_2, m_3) + (-1)^{m_1-1} \zeta (m_2 + 1) \zeta (m_3, m_1) \right\}
\]

\[
+ (-1)^{m_3} \left\{ (-1)^{m_2-1} m_1 \sum_{i=1}^{m_1-1} (-1)^{i-1} \zeta (m_1 + 1 - i) \zeta (m_3, m_2, i) \right\}
\]

\[
+ (-1)^{m_3} \left\{ (+1)^{m_1-1} \sum_{i=1}^{m_2-1} (-1)^{i-1} \zeta (m_2 + 1 - i) \zeta (m_3, m_1, i) \right\}
\]

\[
+ (-1)^{m_1+m_2+m_3} \left\{ \zeta (m_3, m_2, m_1, 1) + \zeta (m_3, m_1, m_2, 1) \right\}
\]

\[
\sum_{i=1}^{m_3-1} (-1)^{i-1} \zeta (m_1 + m_2 + 1 - i) \zeta (m_3, i).
\]

(2.3)

Letting $p = m$ in Theorem 2.1 and $l_1 = l_2 = l_3 = 2l + 1$ in Theorem 2.2, we can get the following two corollaries.

Corollary 2.4 For positive integer $m > 1$, then the cubic sums

\[
S_{1^2m,m} = \sum_{n=1}^{\infty} \frac{H_n^2 H_n^{(m)}}{n^m}
\]

are reducible to quadratic and linear sums. We have

\[
S_{1^2m,m} = 2(-1)^{m-1} \sum_{i=1}^{m-2} (-1)^{i-1} S_{m,i+1} \cdot S_{1,m-i} - \zeta (2) \left( \zeta^2 (m) + \zeta (2m) \right)
\]

\[
- 2(-1)^{m-1} \sum_{j=1}^{m-1} (-1)^{j-1} \zeta (m + 1 - j) (S_{1j,m} - S_{1,m+j})
\]

\[
+ 2S_{1m,m+1} + S_{1^2,2m} - S_{1,m}^2 - 2(-1)^{m-1} \zeta (m + 1) S_{1,m} - S_{2m,m}.
\]

(2.4)

Corollary 2.5 For positive integer $l$, then the cubic sums

\[
S_{1(2l+1)^2,(2l+1)} = \sum_{n=1}^{\infty} \frac{H_n \left( H_n^{(2l+1)} \right)^2}{n^{2l+1}}
\]

are reducible to quadratic and linear sums. We have

\[
S_{1(2l+1)^2,(2l+1)} = \zeta (2l + 2) \left( \zeta^2 (2l + 1) + \zeta (4l + 2) \right) + \left( \zeta (4l + 2) + \zeta^2 (2l + 1) \right) S_{1,2l+1}
\]

\text{5}
\[-(-1)^i S_{2l+1,i+1} - S_{1(4l+2),2l+1} - 2 \sum_{i=1}^{l} (-1)^{i-1} S_{2l+1,i+1} \cdot S_{2l+1,2l+1-i} \]
\[+ 2 \sum_{j=1}^{2l} (-1)^{j-1} \zeta(2l + 2 - j) (S_{j(2l+1),2l+1} - S_{2l+1,2l+j+1}) \cdot (2.5)\]

3 Proofs of Theorem 2.1, 2.2 and 2.3

In this section, we will use the Tornheim type series \( T \) which is also called the partial sum of polylogarithm function. For \( \zeta \), \( l, m, p \) are defined by

\[\frac{1}{k^p (n + k)}\]

\[\sum_{k,n=1}^{\infty} \frac{H_k^l H_n^m}{k^p (n + k)} \quad \text{and} \quad \sum_{k,n=1}^{\infty} \frac{H_k^l H_n^m}{k^p (n + k)} \quad (l, m, p \in \mathbb{N}). \quad (3.1)\]

In order to prove the main theorems, we need the following lemmas.

**Lemma 3.1** ([29]) Let \( m, k \) be positive integers, then

\[\sum_{n=1}^{\infty} \frac{H_k^m}{n(n + k)} = \frac{1}{k} \left\{ \zeta(m + 1) + \sum_{j=1}^{m-1} (-1)^{j-1} \zeta(m + 1 - j) H_k^{(j)} + (-1)^{m-1} \sum_{i=1}^{k-1} \frac{H_i}{j^m} \right\}. \quad (3.2)\]

**Lemma 3.2** ([23]) For \( l_1, l_2, l_3 \in \mathbb{N} \) and \( x, y, z \in [-1, 1] \), we have the following relation

\[\sum_{n=1}^{\infty} \frac{\zeta_n(l_1; x) \zeta_n(l_2; y) z^n}{n^{l_3}} + \sum_{n=1}^{\infty} \frac{\zeta_n(l_1; x) \zeta_n(l_3; y) y^n}{n^{l_2}} + \sum_{n=1}^{\infty} \frac{\zeta_n(l_2; y) \zeta_n(l_3; z) x^n}{n^{l_1}} = \sum_{n=1}^{\infty} \frac{\zeta_n(l_1; x) \zeta_n(l_2; y) z^n}{n^{l_3 + l_2}} + \sum_{n=1}^{\infty} \frac{\zeta_n(l_1; x) \zeta_n(l_3; y) y^n}{n^{l_2 + l_3}} + \sum_{n=1}^{\infty} \frac{\zeta_n(l_2; y) \zeta_n(l_3; z) x^n}{n^{l_1 + l_3}} + \text{Li}_{l_3}(z) \text{Li}_{l_1}(x) \text{Li}_{l_2}(y) - \text{Li}_{l_1+l_2+l_3}(xyz), \quad (3.3)\]

where \( \text{Li}_p \) denotes the polylogarithm function defined by

\[\text{Li}_p(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^p}, \Re(p) > 1, \ |x| \leq 1, \]

with \( \text{Li}_1(x) = - \ln(1 - x), \ x \in [-1, 1] \).

The partial sum \( \zeta_n(l; x) \) is defined by

\[\zeta_n(l; x) := \sum_{k=1}^{n} \frac{x^k}{k^l}, \quad (l > 0, x \in [-1, 1]), \]

which is also called the partial sum of polylogarithm function.
3.1 Proof of Theorem 2.1

From the definition of Tornheim type series $T_1(l, m; p)$, we can rewrite the right hand side of (3.1) as

$$T_1(l, m; p) = \sum_{k=1}^{\infty} \frac{H^{(l)}_k}{k^p} \sum_{n=1}^{\infty} \frac{H^{(m)}_n}{n(n+k)}$$

$$= \sum_{n=1}^{\infty} \frac{H^{(m)}_n}{n} \sum_{k=1}^{\infty} \frac{H^{(l)}_k}{k^p(n+k)}.$$  (3.4)

Then with the help of the following partial fraction decomposition formula

$$\frac{1}{k^p(n+k)} = \sum_{i=1}^{p-1} \frac{(-1)^{i-1}}{n^{i}} \frac{1}{k^{p-i}} + \frac{(-1)^{p-1}}{n^{p-1}} \frac{1}{k(n+k)},$$

and using Lemma 3.1, we deduce that

$$(-1)^{m-1} \sum_{n=1}^{\infty} \frac{H^{(l)}_n}{n^{p+1}} \left( \sum_{k=1}^{n} \frac{H^{(m)}_k}{k^m} \right) - (-1)^{p+i} \sum_{n=1}^{\infty} \frac{H^{(m)}_n}{n^{p+1}} \left( \sum_{k=1}^{n} \frac{H^{(l)}_k}{k^p} \right)$$

$$= \sum_{i=1}^{p-1} (-1)^{i-1} S_{m,i+1} \cdot S_{l,p+1-i} + (-1)^{p-1} \zeta(l+1) S_{m,p+1}$$

$$+ (-1)^{p-1} \sum_{j=1}^{l-1} (-1)^{j-1} \zeta(l+1-j) \{ S_{j,m,p+1} - S_{m,p+j+1} \}$$

$$- (-1)^{p+i} S_{1,m,p+i+1} + (-1)^{m-1} S_{l,p+m+1} - \zeta(m+1) S_{l,p+1}$$

$$- \sum_{j=1}^{m-1} (-1)^{j-1} \zeta(m+1-j) \{ S_{j,l,p+1} - S_{l,p+j+1} \}.$$  (3.5)

Note that by a simple calculation, we obtain a simple reflection formula

$$\sum_{k=1}^{n} A_k b_k + \sum_{k=1}^{n} B_k a_k = \sum_{k=1}^{n} a_k b_k + A_n B_n,$$  (3.6)

where

$$A_n = \sum_{k=1}^{n} a_k, B_n = \sum_{k=1}^{n} b_k.$$  

Hence, from (3.6), the following identities are easily derived

$$\sum_{k=1}^{n} \frac{H_k}{k} = \frac{1}{2} \left\{ H^{(2)}_n + H^{(2)}_n \right\},$$  (3.7)

$$\sum_{n=1}^{\infty} \frac{H_n}{n^p} \left( \sum_{k=1}^{n} \frac{H_k}{k^m} \right) + \sum_{n=1}^{\infty} \frac{H_n}{n^m} \left( \sum_{k=1}^{n} \frac{H_k}{k^p} \right) = S_{1,p} S_{1,m} + S_{1,2,p+m}, \quad 2 \leq m, p \in \mathbb{N}.$$  (3.8)

Thus, letting $l = 1$ and replacing $p$ by $p - 1$ in (3.5), then combining formulas (3.7) with (3.8), by a direct calculation, we may easily deduce the result (2.1). \qed
3.2 Proof of Theorem 2.2

Letting \( x, y \to 1 \) in (3.3), then multiplying it by \((1-z)^{-1}\) and integrating over the interval \((0,1)\), we can find that

\[
\sum_{n=1}^{\infty} \frac{H_n H_n^{(l_1)} H_n^{(l_2)}}{n^{l_3}} + \sum_{n=1}^{\infty} \frac{H_n^{(l_1)} \left( \sum_{k=1}^{n} H_k \right)}{n^{l_2}} + \sum_{n=1}^{\infty} \frac{H_n^{(l_2)} \left( \sum_{k=1}^{n} H_k \right)}{n^{l_1}}
\]

\[
= \sum_{n=1}^{\infty} \left( \frac{n}{k^{l_3}} \right) + \sum_{n=1}^{\infty} \frac{H_n H_n^{(l_1)}}{n^{l_3} + l_2} + \sum_{n=1}^{\infty} \frac{H_n H_n^{(l_2)}}{n^{l_3} + l_1}
\]

\[+ \zeta(l_1) \zeta(l_2) \left( \sum_{n=1}^{\infty} \frac{H_n}{n^{l_3}} \right) - \left( \sum_{n=1}^{\infty} \frac{H_n}{n^{l_3} + l_1 + l_2} \right). \tag{3.9}\]

From [15, 29], we derive the following identity

\[
\sum_{n=1}^{\infty} \frac{1}{n^p} \left( \sum_{k=1}^{n} \frac{H_k}{k^{m}} \right) = \sum_{n=1}^{\infty} \frac{H_n}{n^{m+p}} + \zeta(p) \sum_{n=1}^{\infty} \frac{H_n}{n^m} - \sum_{n=1}^{\infty} \frac{H_n H_n^{(p)}}{n^m}. \tag{3.10}\]

Substituting (3.10) into (3.9), then

\[
S_{1l_12l_23} + \sum_{n=1}^{\infty} \frac{H_n^{(l_1)}}{n^{l_2}} \left( \sum_{k=1}^{n} \frac{H_k}{k^{l_3}} \right) + \sum_{n=1}^{\infty} \frac{H_n^{(l_2)}}{n^{l_1}} \left( \sum_{k=1}^{n} \frac{H_k}{k^{l_3}} \right)
\]

\[= S_{1l_12l_23} + S_{1l_21l_34} - S_{1(l_1+l_2),l_4} + \zeta(l_1 + l_2) S_{1,l_3} + \zeta(l_1) \zeta(l_2) S_{1,l_3} \tag{3.11}\]

Hence, we can easily deduce from (3.11) that

\[
S_{1l_12l_23} + S_{1l_13l_2} + S_{1l_23l_1}
\]

\[+ \sum_{n=1}^{\infty} \left\{ \frac{H_n^{(l_1)}}{n^{l_2}} \left( \sum_{k=1}^{n} \frac{H_k}{k^{l_3}} \right) + \frac{H_n^{(l_3)}}{n^{l_2}} \left( \sum_{k=1}^{n} \frac{H_k}{k^{l_1}} \right) \right\}
\]

\[+ \sum_{n=1}^{\infty} \left\{ \frac{H_n^{(l_2)}}{n^{l_1}} \left( \sum_{k=1}^{n} \frac{H_k}{k^{l_3}} \right) + \frac{H_n^{(l_1)}}{n^{l_1}} \left( \sum_{k=1}^{n} \frac{H_k}{k^{l_2}} \right) \right\}
\]

\[+ \sum_{n=1}^{\infty} \left\{ \frac{H_n^{(l_1)}}{n^{l_3}} \left( \sum_{k=1}^{n} \frac{H_k}{k^{l_2}} \right) + \frac{H_n^{(l_2)}}{n^{l_3}} \left( \sum_{k=1}^{n} \frac{H_k}{k^{l_1}} \right) \right\}
\]

\[= 2S_{1l_12l_23} + 2S_{1l_21l_34} + 2S_{1l_3l_1} - S_{1(l_1+l_2),l_3} - S_{1(l_1+l_3),l_2} - S_{1(l_2+l_3),l_1}
\]

\[+ \left( \zeta(l_1 + l_2) + \zeta(l_1) \zeta(l_2) \right) S_{1,l_3}
\]

\[+ \left( \zeta(l_1 + l_3) + \zeta(l_1) \zeta(l_3) \right) S_{1,l_2}
\]

\[+ \left( \zeta(l_2 + l_3) + \zeta(l_2) \zeta(l_3) \right) S_{1,l_1}. \tag{3.12}\]

Letting \( l = l_1, p = l_2 - 1, m = l_3 \) and \( l_1 + l_2 + l_3 = 2r + 1 \) \((r \in \mathbb{N})\) with \( l_1, l_2, l_3 \geq 2 \) in (3.5), we obtain

\[
\sum_{n=1}^{\infty} \left\{ \frac{H_n^{(l_1)}}{n^{l_2}} \left( \sum_{k=1}^{n} \frac{H_k}{k^{l_3}} \right) + \frac{H_n^{(l_3)}}{n^{l_2}} \left( \sum_{k=1}^{n} \frac{H_k}{k^{l_1}} \right) \right\}
\]

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Thus, combining formulas (3.12) and (3.13) and noting that the symmetry relations
\[ S_{p,q} + S_{q,p} = \zeta(p + q) + \zeta(p) \zeta(q), \quad (p, q > 1), \]
we deduce the desired result. This completes the proof of Theorem 2.2.

3.3 Proof of Theorem 2.3

In this subsection, we consider the Tornheim type series \( T_2(m_1, m_2; p) \) to prove the Theorem 2.3. First, from [32], we have

\[
\sum_{n=1}^{\infty} \frac{H_n^{(m_1)} H_n^{(m_2)}}{n(n+k)} = \frac{1}{k} \sum_{n=1}^{\infty} \left\{ \frac{H_n^{(m_1)}}{n^{m_2+1}} + \frac{H_n^{(m_2)}}{n^{m_1+1}} \right\} - \sum_{n=1}^{\infty} \frac{H_n^{(m_1+m_2)}}{n(n+k)} + \frac{1}{k} \sum_{j=1}^{k-1} \sum_{n=1}^{\infty} \left\{ \frac{H_n^{(m_1)}}{n^{m_2}(n+j)} + \frac{H_n^{(m_2)}}{n^{m_1}(n+j)} \right\}, \quad m_1, m_2, k \in \mathbb{N}. \tag{3.14}
\]

Then, applying the formula (3.2) into (3.14), by a simple calculation we obtain the following result

\[
\sum_{n=1}^{\infty} \frac{H_n^{(m_1)} H_n^{(m_2)}}{n(n+k)} = \frac{1}{k} \left\{ \begin{array}{l}
S_{m_1,m_2+1}^{m_2-1} + S_{m_2,m_1+1}^{m_1-1} - \zeta(m_1 + m_2 + 1) \\
+ \sum_{i=1}^{m_2-1} (-1)^{i-1} S_{m_1,m_2+1-i}^{m_2-1} \zeta_{k-1}(i) \\
+ \sum_{i=1}^{m_1-1} (-1)^{i-1} S_{m_2,m_1+1-i}^{m_1-1} \zeta_{k-1}(i) \\
+ (-1)^{m_2-1} \zeta(m_1 + 1) \zeta_{k-1}(m_2) \\
+ (-1)^{m_1-1} \zeta(m_2 + 1) \zeta_{k-1}(m_1) \\
+ (-1)^{m_2-1} \zeta(m_1 + 1 - i) \zeta_{k-1}(m_2, i) \\
+ (-1)^{m_1-1} \zeta(m_2 + 1 - i) \zeta_{k-1}(m_1, i) \\
+ (-1)^{m_1+m_2} \zeta_{k-1}(m_2, m_1, 1) + \zeta_{k-1}(m_2, m_1 + 1) \\
+ (-1)^{m_1+m_2} \zeta_{k-1}(m_1, m_2, 1) + \zeta_{k-1}(m_1, m_2 + 1) \\
+ (-1)^{m_1+m_2} \zeta_{k-1}(m_1 + m_2, 1) + \zeta_{k-1}(m_1 + m_2 + 1) \\
- \sum_{i=1}^{m_1+m_2} (-1)^{i-1} \zeta(m_1 + m_2 + 1 - i) \zeta_{k-1}(i),
\end{array} \right\}, \tag{3.15}
\]
where $\zeta_n(s_1, s_2, \cdots, s_k)$ denotes the multiple harmonic sum (also called the partial sum of multiple zeta value) defined by [19, 28]

$$\zeta_n(s_1, s_2, \cdots, s_k) := \sum_{n \geq n_1 > n_2 > \cdots > n_k \geq 1} \frac{1}{n_1^{s_1}n_2^{s_2} \cdots n_k^{s_k}},$$

when $n < k$, then $\zeta_n(s_1, s_2, \cdots, s_k) = 0$, and $\zeta_n(0) = 1$. The integers $k$ and $w := s_1 + \cdots + s_k$ are called the depth and the weight of a multiple harmonic sum. It is obvious that

$$\zeta_n(s) = H_n^{(s)}, \quad s \in \mathbb{N}.$$

By the definition of $T_2(m_1, m_2; p)$ we can find that

$$T_2(m_1, m_2; p) = \sum_{k=1}^{\infty} \frac{1}{kp} \sum_{n=1}^{\infty} \frac{H_n^{(m_1)}H_n^{(m_2)}}{n(n+k)} = \sum_{n=1}^{\infty} \frac{H_n^{(m_1)}H_n^{(m_2)}}{n} \sum_{k=1}^{\infty} \frac{1}{kp(n+k)}.$$

Hence, multiplying (3.15) by $\frac{1}{kp}$ and taking the summation from $k = 0$ to $\infty$, by a similar argument as in the proofs of Theorem 2.1 and 2.2, we may easily deduce the desired result.

Taking $m_1 = m_2 = 1, m_3 = 2$ in (2.3) yields

$$S_{1^3,2} = 4\zeta(2)\zeta(3) + 2\zeta(2, \{1\}_3) + 2\zeta(2, 1, 2) + \zeta(2, 2, 1) + \zeta(2, 3).$$

### 4 Some Examples and Corollaries

From Theorem 2.1, 2.2 and Corollary 2.4, 2.5 in section 2, we can get the following some connections between Euler sums and zeta values.

$$S_{1^2,2} = \frac{41}{12} \zeta(6) + 2\zeta^2(3),$$
$$S_{1^2,3} = \frac{9}{2} \zeta(3) \zeta(5) + \frac{3}{2} \zeta(2) \zeta^2(3) - \frac{443}{288} \zeta(8) - \frac{23}{4} S_{2,6},$$
$$S_{1^3,2} = \frac{883}{20} \zeta(10) - 26\zeta^2(5) - \frac{31}{4} \zeta(3) \zeta(7) - 8\zeta(2) \zeta(3) \zeta(5) + \frac{3}{4} \zeta^2(3) \zeta(4) + 9\zeta(2) S_{2,6} - \frac{21}{4} S_{2,8},$$
$$S_{1^2,4} = \frac{7749}{160} \zeta(10) - 16\zeta^2(5) - \frac{125}{8} \zeta(3) \zeta(7) - 14\zeta(2) \zeta(3) \zeta(5) + \frac{3}{2} \zeta^2(3) \zeta(4) + 11 S_{2,8} + \frac{5}{2} \zeta(2) S_{2,6},$$
$$S_{1^2,3} + 2S_{1^2,3,2} = -\frac{2225}{96} \zeta(8) + \frac{73}{2} \zeta(3) \zeta(5) - \frac{5}{2} \zeta(2) \zeta^2(3) - \frac{31}{4} S_{2,6},$$
$$S_{1^2,5} + S_{1^2,5,2} = \frac{1235}{72} \zeta(9) - 2\zeta^3(3) - \frac{101}{8} \zeta(2) \zeta(7) + \frac{185}{24} \zeta(3) \zeta(6) + \frac{13}{4} \zeta(4) \zeta(5),$$
$$S_{1^3,4} + S_{1^3,4,3} = \frac{761}{36} \zeta(9) + \zeta^3(3) - \frac{3}{8} \zeta(2) \zeta(7) - 11\zeta(3) \zeta(6) - \frac{21}{4} \zeta(4) \zeta(5),$$
$$S_{1^2,5} + 2S_{1^2,5,2} = -\frac{2403}{160} \zeta(10) + \frac{69}{4} \zeta^2(5) + \frac{491}{8} \zeta(3) \zeta(7) - 34 \zeta(2) \zeta(3) \zeta(5).$$
S_{123,4} + S_{124,3} + S_{134,2} = -12889 \frac{320}{16} \zeta(10) + 85 \frac{3}{4} \zeta^2(5) + 151 \frac{11}{8} \zeta(3) \zeta(7) + 25 \frac{5}{2} \zeta(2) \zeta(3) \zeta(5) - 9 \frac{1}{8} \zeta^2(3) \zeta(4) - 3 \frac{3}{2} \zeta(2) S_{2,6} + 4S_{2,8}, \tag{4.8}

S_{1^{2}2,6} + S_{1^{2}6,2} + S_{2^{2},6} + S_{26,2} = \frac{5837}{160} \zeta(10) - 20 \zeta^2(5) - 189 \frac{1}{8} \zeta(3) \zeta(7) + 7 \zeta(2) \zeta(3) \zeta(5) + \frac{1}{4} \zeta^2(3) \zeta(4) - 3 \frac{3}{2} \zeta(2) S_{2,6} + 10S_{2,8}, \tag{4.9}

S_{1^{2}3,5} + S_{1^{2}5,3} + S_{23,5} + S_{25,3} = -4559 \frac{60}{32} \zeta(10) + 103 \frac{2}{5} \zeta^2(5) + 337 \frac{4}{2} \zeta(3) \zeta(7) - 29 \zeta(2) \zeta(3) \zeta(5) - 3 \frac{3}{2} \zeta^2(3) \zeta(4) + 9 \zeta(2) S_{2,6} - \frac{59}{2} S_{2,8}. \tag{4.10}

It is known that the quadratic Euler sums of even weight \( S_{p_{1}p_{2},q} := \sum_{n=1}^{\infty} H_{n}^{(p_{1})} H_{n}^{(p_{2})} n^{-q} \) can be expressed by linear sums and zeta values at integer arguments if \( p_{1} + p_{2} + q \) is even and \( p_{1}, p_{2}, q > 1 \), see [15]. We can deduce the following results

\[
S_{2^{2},6} = \frac{2697}{40} \zeta(10) - 41 \zeta^2(5) - 63 \zeta(3) \zeta(7) + 16 \zeta(2) \zeta(3) \zeta(5) + 4 \zeta^2(3) \zeta(4) + \frac{23}{2} S_{2,8} + 2 \zeta(2) S_{2,6}, \tag{4.11}
\]

\[
S_{26,2} = -\frac{2997}{80} \zeta(10) + 23 \zeta^2(5) + 35 \zeta(3) \zeta(7) - 8 \zeta(2) \zeta(3) \zeta(5) - 2 \zeta^2(3) \zeta(4) - \frac{13}{2} S_{2,8} - \zeta(2) S_{2,6}, \tag{4.12}
\]

\[
S_{23,5} = -\frac{2227}{32} \zeta(10) + \frac{89}{2} \zeta^2(5) + 56 \zeta(3) \zeta(7) - 15 \zeta(2) \zeta(3) \zeta(5) - 10 S_{2,8} - \frac{5}{2} \zeta(2) S_{2,6}, \tag{4.13}
\]

\[
S_{25,3} = \frac{3223}{160} \zeta(10) - \frac{17}{4} \zeta^2(5) + \frac{63}{2} \zeta(3) \zeta(7) - 32 \zeta(2) \zeta(3) \zeta(5) - 2 \zeta^2(3) \zeta(4) - \frac{19}{4} S_{2,8} + \frac{25}{2} \zeta(2) S_{2,6}. \tag{4.14}
\]

Hence, we obtain the following closed form of two combined sums involving two cubic Euler sums

\[
S_{1^{2}2,6} + S_{1^{2}6,2} = \frac{1043}{160} \zeta(10) - 2 \zeta^2(5) + \frac{35}{8} \zeta(3) \zeta(7) - \zeta(2) \zeta(3) \zeta(5) - \frac{7}{4} \zeta^2(3) \zeta(4) - \frac{5}{2} \zeta(2) S_{2,6} + 5S_{2,8}, \tag{4.15}
\]

\[
S_{1^{2}3,5} + S_{1^{2}5,3} = -\frac{398}{15} \zeta(10) + \frac{45}{4} \zeta^2(5) - \frac{13}{4} \zeta(3) \zeta(7) + 18 \zeta(2) \zeta(3) \zeta(5) + \frac{1}{2} \zeta^2(3) \zeta(4) - \zeta(2) S_{2,6} - \frac{59}{4} S_{2,8}. \tag{4.16}
\]

In [28], we prove that for integers \( m, p > 0 \) and \( r > 1 \), then

\[
\sum_{n=1}^{\infty} s(n, m) H_{n}^{(r)} \frac{1}{n! n^p} = \zeta(r) \zeta(m + 1, \{1\}_{p-1}) - \zeta(m + 1, \{1\}_{p-1}, 2, \{1\}_{r-2}). \tag{4.17}
\]
Here \( s(n, k) \) denotes the (unsigned) Stirling number of the first kind [11], and we have
\[
\begin{align*}
    s(n, 1) &= (n - 1)!, \\
    s(n, 2) &= (n - 1)!H_{n-1}, \\
    s(n, 3) &= \frac{(n - 1)!}{2} \left[ H_{n-1}^2 - H_{n-1}^{(2)} \right], \\
    s(n, 4) &= \frac{(n - 1)!}{6} \left[ H_{n-1}^3 - 3H_{n-1}H_{n-1}^{(2)} + 2H_{n-1}^{(3)} \right], \\
    s(n, 5) &= \frac{(n - 1)!}{24} \left[ H_{n-1}^4 - 6H_{n-1}^{(4)} - 6H_{n-1}^2H_{n-1}^{(2)} + 3(H_{n-1}^{(2)})^2 + 8H_{n-1}H_{n-1}^{(3)} \right].
\end{align*}
\]
Hence, taking \( m = 3 \) in (4.18) and using the duality relation (1.6), we obtain
\[
S_{1^{2r,p+1}} = S_{2r,p+1} + 2S_{1r,p+2} - 2S_{r,p+3} + 2\zeta(r)\zeta(4, \{1\}_{p-1}) - 2\zeta(r, p + 1, 1, 1) \quad (4.19)
\]
Since the multiple zeta value \( \zeta(m + 1, \{1\}_{k-1}) \) can be represented as a polynomial of zeta values with rational coefficients [28]. For example:
\[
\begin{align*}
    \zeta(2, \{1\}_m) &= \zeta(m + 2), \\
    \zeta(3, \{1\}_m) &= \frac{m + 2}{2}\zeta(m + 3) - \frac{1}{2}\sum_{k=1}^{m} \zeta(k + 1)\zeta(m + 2 - k).
\end{align*}
\]
Thus, from Theorem 2.1, 2.2 and 2.3 with the help of the formulas (1.5) and (4.19), we obtain the following description of multiple zeta values of depth four.

**Corollary 4.1** For positive integers \( m_1, m_2, m_3 > 1 \), then the combined sums
\[
\zeta(m_1, m_2, 1, 1) + \zeta(m_2, m_1, 1, 1), \quad \zeta(m_1, 1, m_2, 1) + \zeta(m_2, 1, m_1, 1)
\]
and
\[
\zeta(m_1, m_2, m_3, 1) + \zeta(m_1, m_3, m_2, 1) + \zeta(m_2, m_3, m_1, 1) \\
+ \zeta(m_2, m_1, m_3, 1) + \zeta(m_3, m_2, m_1, 1) + \zeta(m_3, m_1, m_2, 1)
\]
can be expressed as a rational linear combination of products of single, double or triple zeta values.

Hence, from Corollary 4.1, we know that for any positive integer \( m > 1 \), then the three quadruple zeta values
\[
\zeta(m, m, 1, 1), \zeta(m, 1, m, 1) \quad \text{and} \quad \zeta(m, m, m, 1)
\]
can be expressed in terms of other multiple zeta values of depth \( \leq 3 \). Note that the conclusion was also proved in [14] by another method.

On the other hand, we note that H.N. Minh and M. Petitot [22] gave all closed form of multiple zeta values of weight \( \leq 9 \). By using their results and combining (4.19), we give the following explicit formulas of cubic Euler sums
\[
\begin{align*}
    S_{1^{23,4}} &= \sum_{n=1}^{\infty} \frac{H_{n}^2H_{n}^{(3)}}{n^4} = \frac{3895}{72}\zeta(9) - \frac{5}{8}\zeta(2)\zeta(7) - \frac{227}{24}\zeta(3)\zeta(6) - \frac{75}{2}\zeta(4)\zeta(5) + \zeta^3(3), \\
    S_{1^{24,3}} &= \sum_{n=1}^{\infty} \frac{H_{n}^2H_{n}^{(4)}}{n^4} = -\frac{791}{24}\zeta(9) + \frac{1}{4}\zeta(2)\zeta(7) - \frac{37}{24}\zeta(3)\zeta(6) + \frac{129}{4}\zeta(4)\zeta(5), \\
    S_{1^{25,2}} &= \sum_{n=1}^{\infty} \frac{H_{n}^2H_{n}^{(5)}}{n^4} = \frac{679}{18}\zeta(9) - \frac{61}{8}\zeta(2)\zeta(7) - \frac{55}{12}\zeta(3)\zeta(6) - \frac{59}{4}\zeta(4)\zeta(5) + \zeta^3(3).
\end{align*}
\]
5 Alternating Cubic Euler Sums

Let

\[ H_n^{(p)} := \sum_{j=1}^{n} \frac{(-1)^{j-1}}{j^p}, \quad H_n := H_n^{(1)} \]

denote the alternating harmonic numbers, \( p \in \mathbb{N} \). In this section, we will prove that the alternating cubic Euler sums

\[ S_{1^r,2^{r+1}} := \sum_{n=1}^{\infty} \frac{H_n^3}{n^{2r+1}}, \quad (r \in \mathbb{N}) \]

are reducible to alternating quadratic and linear Euler sums. The generalized alternating Euler sums are defined by

\[
S_{P_l \bar{R}_k,q} := \sum_{n=1}^{\infty} \frac{H_n^{(p_1)} \cdots H_n^{(p_l)} \bar{H}_n^{(r_1)} \cdots \bar{H}_n^{(r_k)}}{n^q}, \quad S_{P_l \bar{R}_k,q} := \sum_{n=1}^{\infty} \frac{H_n^{(p_1)} \cdots H_n^{(p_l)} \bar{H}_n^{(r_1)} \cdots \bar{H}_n^{(r_k)}}{n^q} (-1)^{n-1},
\]

where the \( P_l := (p_1, \ldots, p_l) \in (\mathbb{N}_0)^l \), \( R_k := (r_1, \ldots, r_k) \in (\mathbb{N}_0)^k \), \( \bar{R}_k := (\bar{r}_1, \ldots, \bar{r}_k) \) and the quantity \( w := p_1 + \cdots + p_l + r_1 + \cdots + r_k + q \) being called the weight and the quantity \( l + k \) being the degree, \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \). Since repeated summands in partitions are indicated by powers, we denote, for example, the sums

\[
S_{1^3 2^1 3,5} = \sum_{n=1}^{\infty} \frac{H_n^3 H_n^{(2)} H_n^{(3)} H_n^{(4)}}{n^5}, \quad S_{1^2 3^1 3,4} = \sum_{n=1}^{\infty} \frac{H_n^2 H_n^{(3)} \bar{H}_n^{(3)} \bar{H}_n^{(4)}}{n^4} (-1)^{n-1}.
\]

In [15], P. Flajolet and B. Salvy gave explicit reductions to zeta values for all linear sums

\[
S_{p,q} = \sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^q}, \quad S_{p,q} = \sum_{n=1}^{\infty} \frac{\bar{H}_n^{(p)}}{n^q}, \quad S_{p,q} = \sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^q} (-1)^{n-1}, \quad S_{p,q} = \sum_{n=1}^{\infty} \frac{\bar{H}_n^{(p)}}{n^q} (-1)^{n-1}
\]

when \( p + q \) is an odd weight. The evaluation of linear sums in terms of values of Riemann zeta function and polylogarithm function at positive integers is known when \( (p, q) = (1, 3), (2, 2) \), or \( p + q \) is odd (see [5, 15, 27, 31]). In [31], the author jointly with Y. Yang and J. Zhang proved that the alternating quadratic Euler sums

\[
S_{1^2 2^m} = \sum_{n=1}^{\infty} \frac{H_n^2}{n^{2m}} (-1)^{n-1}, \quad S_{1^2 2^m} = \sum_{n=1}^{\infty} \frac{H_n \bar{H}_n}{n^{2m}} (-1)^{n-1}, \quad S_{1^2 2^m} = \sum_{n=1}^{\infty} \frac{H_n \bar{H}_n}{n^{2m}} (-1)^{n-1},
\]

are reducible to polynomials in zeta values and to linear sums, and are able to give explicit values for certain of these sums in terms of the Riemann zeta function and the polylogarithm function, here \( m \) is a positive integer. In [3], D.H. Bailey, J.M. Borwein and R. Girgensohn considered sums

\[
S_{1^p,q} = \sum_{n=1}^{\infty} \frac{H_n^p}{n^q}, \quad S_{1^p,q} = \sum_{n=1}^{\infty} \frac{\bar{H}_n^p}{n^q} (-1)^{n-1}, \quad S_{1^p,q} = \sum_{n=1}^{\infty} \frac{H_n^p}{n^q} (-1)^{n-1}
\]

and obtained a number of experimental identities using the experimental method, where \( p \) and \( q \) are positive integers with \( p + q \leq 5 \).

Now we state our main results in this section. First, we need the following lemma.
Lemma 5.1 For positive integers \( m \) and \( k \), then

\[
\sum_{n=1}^{\infty} \frac{\bar{H}_n^{(m)}}{n(n+k)} = \frac{1}{k} \left\{ \frac{\bar{\zeta}(m+1) + \sum_{j=1}^{m-1} (-1)^{j-1} \bar{\zeta}(m+1-j) H_{k-1}^{(j)}}{\bar{\zeta} \ln 2} \right\}, \tag{5.1}
\]

where \( \bar{\zeta}(s) \) denotes the alternating Riemann zeta function defined by

\[
\bar{\zeta}(s) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}, \quad \Re(s) \geq 1.
\]

Proof. By the definition of polylogarithm function and using Cauchy product formula, we can find that

\[
-\frac{\text{Li}_m(-x)}{1-x} = \sum_{n=1}^{\infty} \bar{H}_n^{(m)} x^n, \quad x \in (-1,1).
\]

Multiplying (5.2) by \( x^{r-1} - x^{k-1} \) and integrating over \((0,1)\), we obtain

\[
\int_0^1 \left( x^{k-1} - x^{r-1} \right) \frac{\text{Li}_m(-x)}{1-x} dx = (k-r) \sum_{n=1}^{\infty} \bar{H}_n^{(m)} \frac{H_n^{(m)}}{(n+r)(n+k)} \quad (0 \leq r < k, \ r, k \in \mathbb{N}). \tag{5.3}
\]

We now evaluate the integral on the left side of (5.3). Noting that the integral of (5.3) can be written as

\[
\int_0^1 \left( x^{k-1} - x^{r-1} \right) \frac{\text{Li}_m(-x)}{1-x} dx = \sum_{i=1}^{k-r} (-1)^{r+i} \int_0^1 x^{r+i-2} \text{Li}_m(x) dx. \tag{5.4}
\]

Then using integration by parts we have

\[
\int_0^x t^{n-1} \text{Li}_q(t) dt = \sum_{i=1}^{q-1} (-1)^{i-1} \frac{x^n}{n^{i+1}} \text{Li}_{q-i+1}(t) + \frac{(-1)^{q}}{n^q} \ln(1-x)(x^n-1) - \frac{(-1)^{q}}{n^q} \left( \sum_{k=1}^{n} \frac{x^k}{k} \right). \tag{5.5}
\]

Thus, taking \( r = 0 \) in (5.3) and (5.4), then substituting (5.5) into (5.3) and (5.4), we deduce the formula (5.1).

Theorem 5.2 For positive integers \( l, m \) and \( p \), we have

\[
(-1)^m \sum_{n=1}^{\infty} \frac{\bar{H}_n^{(l)}}{n^{p+1}} \left( \sum_{i=1}^{n} \frac{\bar{H}_i}{i^{m-1}} \right) + (-1)^{p+l} \sum_{n=1}^{\infty} \frac{\bar{H}_n^{(m)}}{n^{p+1}} \left( \sum_{i=1}^{n} \frac{\bar{H}_i}{i^{l-1}} \right) = \sum_{i=1}^{p-1} (-1)^{i-1} S_{m,i+1} \cdot S_{l,p+1-i} + (-1)^{p-1} \bar{\zeta}(l+1) S_{m,p+1} - \bar{\zeta}(m+1) S_{l,p+1}
\]

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\[ + (-1)^{p-1} \sum_{j=1}^{l-1} (-1)^{j-1} \zeta (l + 1 - j) \left( S_{m,j,p+1} - S_{m,p+j+1} \right) \]
\[ - \sum_{j=1}^{m-1} (-1)^{j-1} \zeta (m + 1 - j) \left( S_{l,j,p+1} - S_{l,p+j+1} \right) \]
\[ + (-1)^{p+l} \ln 2 \left( S_{n,l,p+1} + S_{m,l,p+1} - S_{m,p+l+1} \right) \]
\[ + (-1)^{m} \ln 2 \left( S_{l,m,p+1} + S_{l,m,p+1} - S_{l,m+p+1} \right) \]
\[ + (-1)^{p+l} S_{1,n,p+l+1} + (-1)^{m} S_{l,m+p+m+1}. \]

**Proof.** Similarly as in the proof of formula (3.5), we consider the Tornheim type series

\[ T (l, m; p) := \sum_{k,n=1}^{\infty} \frac{\tilde{H}_k^{(l)} \tilde{H}_n^{(m)}}{k^p \ln (n+k)}, \quad (l, m, p \in \mathbb{N}). \]

Then with the help of formula (5.1) we may easily deduce the result. \( \square \)

Taking \( l = m = 1, p = 2r \) in (5.6) and noting that

\[ \sum_{i=1}^{n} \tilde{H}_i (-1)^{i-1} = \frac{\tilde{H}_n^2 + H_n^{(2)}}{2}, \]

then (5.6) is equal to

\[ S_{1,2r+1} + S_{1,2r+1} = 2 \zeta (2) S_{1,2r+1} + 2 \ln 2 \left( S_{1,1,2r+1} + S_{1,2,2r+1} \right) \]
\[ - 2 \ln 2 \left( S_{1,2r+2} + S_{1,2r+2} \right) + 2 S_{1,2,2r+2} \]
\[ - 2 \sum_{i=1}^{r} (-1)^{i-1} S_{1,i+1} \cdot S_{1,2r+1-i} + (-1)^{r-1} S_{1,2r+1}. \]

Thus, we obtain the conclusion that the alternating cubic Euler sums \( S_{1,2r+1} \) are reducible to alternating quadratic and linear Euler sums.

Next, we give a simple case. From [3, 26, 31], we know that

\[ S_{1,3} = \sum_{n=1}^{\infty} \frac{\tilde{H}_n \tilde{H}_n^{(2)}}{n^3} \]
\[ = \frac{29}{8} \zeta (2) \zeta (3) \ln 2 - \frac{93}{32} \zeta (5) \ln 2 - \frac{1855}{128} \zeta (6) + \frac{17}{16} \zeta^2 (3) \]
\[ - S_{1,5} + S_{2,4} + 4 S_{2,3} + 8 S_{1,5}, \]

(5.8)

\[ S_{1,4} = \sum_{n=1}^{\infty} \frac{\tilde{H}_n^2}{n^4} (-1)^{n-1} \]
\[ = \frac{15}{4} \ln 2 \zeta (4) + \frac{9}{4} \zeta (2) \zeta (3) \ln 2 - \frac{93}{16} \zeta (5) \ln 2 + \frac{35}{64} \zeta (6) \]
\[ - \frac{15}{16} \zeta^2 (3) + S_{2,4}, \]

(5.9)
\begin{align}
S_{12,3} &= \sum_{n=1}^{\infty} \frac{\bar{H}_n^2}{n^3} \\
&= 4\text{Li}_5 \left( \frac{1}{2} \right) - \frac{1}{30} \ln^5 2 + \frac{7}{4} \zeta(3) \ln^2 2 - \frac{167}{32} \zeta(5) + \frac{1}{3} \zeta(2) \ln^3 2 \\
&\quad + \frac{3}{4} \zeta(2) \zeta(3) + \frac{19}{8} \zeta(4) \ln 2,
\end{align}
(5.10)

\begin{align}
S_{11,3} &= \sum_{n=1}^{\infty} \frac{H_n \bar{H}_n}{n^3} \\
&= 2\text{Li}_5 \left( \frac{1}{2} \right) - \frac{1}{60} \ln^5 2 - \frac{193}{64} \zeta(5) - \frac{7}{8} \zeta(3) \ln^2 2 + \frac{1}{6} \zeta(2) \ln^3 2 \\
&\quad + 4 \zeta(4) \ln 2 + \frac{3}{8} \zeta(2) \zeta(3) .
\end{align}
(5.11)

Setting \( r = 1 \) in above equation (5.7) and combining (5.8)-(5.11), we can get the cubic sum

\begin{align}
S_{13,3} &= \sum_{n=1}^{\infty} \frac{\bar{H}_n^3}{n^3} \\
&= \frac{1925}{128} \zeta(6) - 3 \zeta^2(3) + 12\text{Li}_5 \left( \frac{1}{2} \right) \ln 2 - \frac{155}{8} \zeta(5) \ln 2 + \frac{27}{8} \zeta(2) \zeta(3) \ln 2 \\
&\quad + \frac{57}{8} \zeta(4) \ln 2 + \frac{7}{4} \zeta(3) \ln^3 2 + \zeta(2) \ln^4 2 - \frac{1}{10} \ln^6 2 \\
&\quad + S_{1,5} - S_{2,4} - 2S_{2,1} - 8S_{1,5} .
\end{align}
(5.12)

From [33], we can find the following relation

\[ \zeta^* (\bar{4}, 2) + 4 \zeta^* (\bar{5}, 1) = \zeta (\bar{4}, 2) + 4 \zeta (\bar{5}, 1) + 5 \zeta (6) = \frac{1105}{192} \zeta (6) + \frac{3}{4} \zeta^2 (3) . \]
(5.13)

By the definitions of alternating linear Euler sums and alternating double zeta values, we have

\[ S_{p,\bar{q}} = \zeta^* (\bar{q}, \bar{p}) , S_{\bar{p},\bar{q}} = -\zeta^* (\bar{q}, p) , S_{\bar{p},q} = -\zeta^* (q, \bar{p}) . \]

Hence, the formula (5.12) can be rewritten as

\begin{align}
S_{13,3} &= \frac{13855}{384} \zeta(6) - \frac{3}{2} \zeta^2(3) + 12\text{Li}_5 \left( \frac{1}{2} \right) \ln 2 - \frac{155}{8} \zeta(5) \ln 2 + \frac{27}{8} \zeta(2) \zeta(3) \ln 2 \\
&\quad + \frac{57}{8} \zeta(4) \ln 2 + \frac{7}{4} \zeta(3) \ln^3 2 + \zeta(2) \ln^4 2 - \frac{1}{10} \ln^6 2 + \zeta^* (\bar{5}, 1) + \zeta^* (4, 2) .
\end{align}
(5.14)

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