TRIANGULATED CATEGORIES OF RELATIVE 1-MOTIVES

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Abstract. We construct and study a candidate for the standard motivic $t$-structure on the triangulated category $\mathbf{DA}^1(S, \mathbb{Q})$ of relative cohomological 1-motives with rational coefficients over a noetherian finite dimensional scheme $S$. This $t$-structure is defined as a generated $t$-structure, and we show it is non-degenerate. We relate its heart $\mathbf{MM}^1(S)$ with Deligne 1-motives over $S$; in particular, when $S$ is regular, the category of Deligne 1-motives embeds in $\mathbf{MM}^1(S)$ fully faithfully. We also study the inclusion of $\mathbf{DA}^1(S)$ into the larger category $\mathbf{DA}^{coh}(S)$ of relative cohomological motives on $S$, and prove that its right adjoint $\omega^1$ preserves compact objects.

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Introduction

We now have at our disposal a mature theory of triangulated categories of motivic sheaves with rational coefficients over general base schemes. Here are some of its highlights.

- Given a noetherian finite-dimensional scheme \( S \), there is a tensor triangulated category \( \DA(S) \).
- There are realisation functors from \( \DA(S) \) to classical triangulated categories of coefficients: derived categories of abelian sheaves for the classical topology in the Betti setting \([9]\) and derived categories of \( \ell \)-adic sheaves in the \( \ell \)-adic setting \([11, 27]\).
- The assignment \( S \mapsto \DA(S) \) has a rich functoriality leading to a “formalism of Grothendieck operations” \([7, 8]\) (including nearby and vanishing cycles) which is compatible via realisation functors to the classical Grothendieck operations for constructible sheaves in the Betti setting and for \( \ell \)-adic sheaves in the \( \ell \)-adic setting \([9, \text{Theoreme 3.19}], [11, \text{Theoreme 9.7}]\).
- Morphisms groups in \( \DA(S) \) are related to rational algebraic \( K \)-theory for \( S \) regular \([25, \text{Corollary 14.2.14}] \) and to Bloch’s higher Chow groups when \( S \) is smooth over a field \([25, \text{Example 11.23}]\).
- A natural finiteness condition leads to a subcategory \( \DA_c(S) \) of “constructible” motivic sheaves, which is stable under Grothendieck operations and maps to the constructible derived categories of classical coefficients via realisations functors \([11, 8]\) \([25, 15]\).
- The category \( \DA(S) \) can be constructed in several ways, each of which captures important aspects of the theory: motives without transfers \([11]\), Beilinson motives \( \DM(S) \) \([25]\), motives with transfers \( \DM(S) \) \([25]\), \( h \)-motives \( \DM_h(S) \) \([60]\) \([25]\) \([27]\), etc. In each of those cases, it is constructed as the homotopy category of a stable combinatorial dg-model category, hence \( \DA(S) \) admits natural enhancements as a stable dg-category, a stable derivator and a stable \((\infty,1)\)-category.
- When \( S \) is the spectrum of a perfect field \( k \), the category \( \DA(k) \) is in particular equivalent to \( \DM(k) \), which gives access to Voevodsky’s cancellation theorem \([61]\) and to the theory of homotopy invariant sheaves with transfers \([62]\) \([48]\).

In view of those achievements, a major open question is the existence of the motivic \( t \)-structure on \( \DA(S) \), whose heart would provide an abelian category of mixed motivic sheaves realising the conjectures of Beilinson \([43]\). Here is one possible statement in terms of the \( \ell \)-adic realisation.

Conjecture 0.1. Let \( S \) be a noetherian finite-dimensional scheme and \( \ell \) a prime invertible on \( S \).

- The \( \ell \)-adic realisation functor \( R_\ell : \DA_c(S) \to D^b_c(S_{\text{ét}}, \Q_\ell) \) is conservative.
- There exists a non-degenerate \( t \)-structure \( t_{\MM} \) on \( \DA_c(S) \) such that if we equip \( D^b_c(S_{\text{ét}}, \Q_\ell) \) with its standard \( t \)-structure, the functor \( R_\ell \) is \( t \)-exact.

The \( t \)-structure \( t_{\MM} \) is uniquely determined by the compatibility with \( R_\ell \) if it exists, because we include conservativity of realisations in the statement.

The case where \( S \) is the spectrum of a field (say of characteristic 0) is already extremely interesting: the conjecture in that case implies the Beilinson-Soulé vanishing conjecture for \( K \)-theory, Grothendieck’s standard conjectures on algebraic cycles \([17]\) and the Bloch-Beilinson-Murre conjectures on the structure of Chow groups of smooth projective varieties \([43]\). Moreover, a theorem of Bondarko \([20, \text{Theorem 3.1.4}] \) shows that for a large class of schemes, if \( t_{\MM}(K) \) exists for any residue field \( K \) of such a scheme \( S \), then the perverse analogue \( pt_{\MM}(S) \) of \( t_{\MM}(S) \) exists, and one can then presumably reconstruct the standard motivic \( t \)-structure from the perverse one as in \([55, 4.6.2]\).

Since the general conjecture seems inaccessible, one looks for subcategories of \( \DA(S) \) where one can hope to construct the restriction of the conjectural \( t \)-structure. For \( n \in \N \), we introduce the subcategory \( \DA_n(S) \) of homological \( n \)-motives, i.e., the subcategory generated by homological motives of smooth \( S \)-schemes of relative dimension less than or equal to \( n \). It seems reasonable to conjecture further that \( t_{\MM} \) should restrict to a \( t \)-structure \( t_{\MM,n} \) on \( \DA_{n,c}(S) \). For \( n \geq 2 \), we
have no idea how to construct $t_{\text{MM},n}$ even when $S$ is a field. Our goal is to provide a reasonable candidate for $t_{\text{MM},0}$ and $t_{\text{MM},1}$.

For a perfect field $k$, the structure of $DA_1(k)$ and $t_{\text{MM},1}$ has already been extensively studied. Here is a summary of the main results, transferred from the set-up of $\text{DM}^{\text{eff}}$ in the original papers to $DA$ via the cancellation theorem and the comparison theorem of [25, Corollary 16.2.22] (for details on these results, we refer the reader to Sections 3.2 and 4.3).

Theorem 0.2 (Voevodsky, Orgogozo [53], Barbieri-Viale-Kahn [16], Ayoub-Barbieri-Viale [14], Ayoub [10]). Let $k$ be a perfect field and $\ell$ a prime different from $\text{char}(k)$.

(i) There exists a non-degenerate $t$-structure $t_{\text{MM},1}$ on $DA_1(k)$ which restricts to $DA_{1,c}(k)$.

(ii) There is an equivalence of $t$-categories

$$DA_{1,c}(k) \simeq D^b(M_1(k))$$

where $M_1(k)$ is the abelian category of Deligne 1-motives over $k$ with rational coefficients [31].

(iii) The $\ell$-adic realisation functor $R_! : DA_{1,c}(k) \to D(k_{\ell\text{-rig}}, \mathbb{Q}_\ell)$ is conservative and $t$-exact.

(iv) The inclusion of $DA_1(k)$ into the category $DA_{\text{hom}}(k)$ of all homological motives admits a left adjoint, the "motivic Albanese functor" $\text{LAlb} : DA_{\text{hom}}(k) \to DA_1(k)$, which sends constructible objects to constructible objects, and whose value on the motive of a smooth $k$-variety $X$ is closely related to its semi-abelian Albanese variety.

Our work builds on these results and the six operations formalism to produce a similar picture for $DA_1(S)$.

The most natural approach to a motivic $t$-structure on $DA_1(S)$ would proceed by combining the $t$-structures on $DA_1(s)$ provided by the previous theorem for all points $s$ of $S$ to a $t$-structure on $DA_1(S)$, i.e., proving that the subcategories $DA_1(S)_{\geq 0} := \{M \in DA_1(S) \mid \forall s \in S, s^*S \in DA_1(s)_{\geq 0}\}$ and $DA_1(S)_{\leq 0} := \{M \in DA_1(S) \mid \forall s \in S, s^*S \in DA_1(s)_{\leq 0}\}$ form a $t$-structure, which would then automatically be compatible with standard $t$-structures on target categories of realisation functors when they are defined. We do not know how to prove this in general, even when restricting to subcategories of compact objects; the gluing arguments of [20, §3.2] are tailored for "perverse" $t$-structures and cannot be applied directly. We refer however to [58] for a different approach to the motivic $t$-structure for 1-motives via gluing.

We thus implement an alternative approach, which is inspired by another description [10, Proposition 3.7] of $t_{\text{MM},1}(k)$ for a perfect field $k$, as a generated $t$-structure in the sense of [7, Definition 2.1.71]. This leads to a $t$-structure $t_{\text{MM},1}(S)$ on $DA_1(S)$ (Definition 4.10). Let us write $MM_1(S)$ for its heart.

Theorem 0.3 (4.29, 4.21, 4.30, 4.23). Let $S$ be a noetherian finite-dimensional scheme.

(i) If $S$ is the spectrum of a field $k$, the $t$-structure $t_{\text{MM},1}(k)$ coincides with the $t$-structure of Theorem 0.2.

(ii) The $t$-structure $t_{\text{MM},1}(S)$ is non-degenerate.

(iii) Write $M_1(S)$ for the $\mathbb{Q}$-linear category of Deligne 1-motives over $S$ with rational coefficients. The natural functor $\Sigma^\infty : M_1(S) \to DA(S)$ factors through $MM_1(S)$, and is fully faithful if $S$ is regular.

(iv) Let $G$ be a smooth commutative group scheme with connected fibres. Then the motive $\Sigma^\infty G_{\mathbb{Q}}[-1]$ is in $MM_1(S)$.

The result on $\Sigma^\infty G_{\mathbb{Q}}[-1]$ was announced in [6]; there, this motive appeared as the $H_1$ piece in a "K"unneth-type" decomposition of the homological motive $M_S(G)$ [6, Theorem 3.3].

In the relative situation, it is unclear whether the left adjoint $\text{LAlb}$ of the inclusion $DA_1(S) \to DA_{\text{hom}}(S)$ actually exists. We can however define a motivic analogue of the Picard scheme. We have a category $DA^1(S)$ of cohomological 1-motives (resp. $DA^{\text{co}}(S)$ of cohomological motives) and it turns out that $DA^1(S) = DA_1(S)[-1]$ (Proposition 1.27), so that $DA^1(S)$ also has a motivic $t$-structure $t_{\text{MM},1} = t_{\text{MM},1}(S)[-1]$ which satisfies analogues of the theorems above. The inclusion $DA^1(S) \to DA^{\text{co}}(S)$ admits a right adjoint $\omega : DA^{\text{co}}(S) \to DA^1(S)$, as a corollary of Neeman’s version of Brown representability for compactly generated categories; however, unlike most right adjoints constructed this way, $\omega$ satisfies a strong finiteness property.
Theorem 0.4 (3.21). Let $S$ be a noetherian finite-dimensional excellent scheme satisfying resolution of singularities by alterations. Then $\omega^1$ sends compact objects to compact objects.

The main step of the proof is to compute $\omega^1$ in a special case, namely $\omega^1(f, \mathbb{Q}_X)$ with $S$ regular and $f : X \to S$ smooth projective “Pic-smooth” (Definition 2.30). In this case, Theorem 3.15 shows that

$$\omega^1(f, \mathbb{Q}_X) \simeq \Sigma^\infty P(X/S)(-1)[-2]$$

where $P(X/S)$ is the Picard complex of $f$, an object closely related to the Picard scheme of $X$ over $S$. Another approach to prove Theorem 3.21 is discussed in [58].

The results above on $t_{\text{MM},1}$ and $\omega^1$ also have (simpler) counterparts for the category $\text{DA}_0(S)$ of 0-motives and for the functor $\omega^0$, which we establish along the way.

The two main questions which this work leaves open are whether the $t$-structure $t_{\text{MM},1}$ restricts to compact objects and whether the resulting $t$-structure on $\text{DA}_{1,c}(S)$ satisfies the analogue of Conjecture 0.1, i.e., whether the $t$-adic realisation on $\text{DA}_{1,c}(S)$ is then $t$-exact. The first question has been answered in the preprint [58].

We have chosen to work with motives with rational coefficients. The theory of triangulated categories of motives with integral coefficients naturally splits in two: a “Nisnevich” version and an “étale” version, depending on what topology we want to have descent for. Voevodsky already observed that the Nisnevich category $\text{DM}_{\text{Nis}}(S, \mathbb{Z})$ does not admit a motivic $t$-structure, even over a field. Let $A$ be a ring of coefficients. Then, if we make the assumption that every prime is invertible either in $A$ or in $\mathcal{O}_S$, the category $\text{DA}_A(S, A)$ is rather well understood. The key statement is the relative rigidity theorem of Ayoub [11, Theoreme 4.1] which roughly tells us that the category of étale motives with torsion coefficients coincides with the derived category of torsion étale sheaves. Building on this, one can show that with these hypotheses, the motivic $t$-structure on $\text{DA}_{A_1}(S, A)$ exists if and only if it exists for $\text{DA}_A(S, A \otimes \mathbb{Q})$. More specifically for 1-motives over a perfect field $k$ with exponential characteristic $p$, the motivic $t$-structure on $\text{DA}_{A_1}(k, \mathbb{Z}_p[1])$ was constructed in [16, Remark 2.7.2]. It seems likely that the ideas of [16] on 1-motives with torsion and the relative rigidity theorem can be combined with the methods of this paper to give a satisfactory theory of $t_{\text{MM},1}$ and $\omega^1$ for relative étale motives.

Structure of this paper. Let $S$ be a finite dimensional noetherian scheme. In Section 1, we introduce the categories $\text{DA}_n(S)$ of homological $n$-motives (resp. $\text{DA}^n(S)$ of cohomological $n$-motives) which are full subcategories of $\text{DA}(S)$ generated as triangulated categories with small sums by homological (resp. cohomological) motives of smooth (resp. proper) $S$-schemes of relative dimension less or equal to $n$ (Definition 1.1). We then study their permanence properties under Grothendieck operations (Propositions 1.10 to 1.17) and prove that the homological and cohomological variants are closely related (Proposition 1.27).

In Section 2, we study the motives associated to smooth commutative group schemes over $S$ and prove that they live in $\text{DA}_1(S)$ (Proposition 2.11). We also study motives attached to Deligne 1-motives. Finally, we introduce a motive attached to what we call the Picard complex $P(X/S)$ of a morphism of schemes $f : X \to S$. It is an object in a derived category of sheaves which packages together information about the relative connected components of $f$ and the Picard scheme of $X/S$; in some cases, $P(X/S)$ yields a motive in $\text{DA}_{1,c}(S)$ (Corollary 2.43).

In Section 3, we introduce and study the right adjoint $\omega^1 : \text{DA}^{\text{coh}}(S) \to \text{DA}^1(S)$ to the embedding of cohomological 1-motives into cohomological motives. We first establish a number of relatively formal results involving its commutation properties with the six operations (Proposition 3.3). The main result is then that $\omega^1$ preserves constructibility (Theorem 3.21). This relies on combining techniques from [15] with a computation of $\omega^1(f, \mathbb{Q}_X)$ in a favorable situation: the precise statement involves the motive of the Picard complex from the previous section.

In Section 4, we finally introduce a candidate for the motivic $t$-structure on $\text{DA}_1(S)$ and $\text{DA}^1(S)$, using the formalism of generated $t$-structures. A number of equivalent generating families can be used for this purpose (see Definition 4.4). We prove some basic exactness properties for the six operations. The main result we show is that motives attached to Deligne 1-motives lie in the heart $\text{MM}_1(S)$ of the $t$-structure on $\text{DA}_1(S)$, and that the category $\mathcal{M}_1(S)$ embeds fully faithfully into $\text{MM}_1(S)$ for $S$ regular.
Appendix A provide technical results about Deligne 1-motives over a general base. Appendix B gathers some computations of motivic cohomology groups for \( \mathbb{Q}(0) \) and \( \mathbb{Q}(1) \) which are used at several places in the text.

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Background, conventions and notations

We collect here several conventions and pieces of notation which will be used throughout this paper.

Homological algebra in abelian and triangulated categories. When discussing complexes in abelian categories and t-structures on triangulated categories, we consistently use homological indexing conventions.

Let \( F : \mathcal{T} \to \mathcal{T}' \) be a triangulated functor between triangulated categories with t-structures. We say that \( F \) is t-positive or right t-exact (resp. t-negative or left t-exact) if \( F(T_{\leq 0}) \subset T'_{\geq 0} \) (resp. \( F(T_{\leq 0}) \subset T'_{\leq 0} \)).

Let \( \mathcal{T} \) be a triangulated category, and \( \mathcal{G} \) be a family of objects of \( \mathcal{T} \). We introduce a number of subcategories of \( \mathcal{T} \) generated in various ways by \( \mathcal{G} \). Recall that a triangulated subcategory is said to be thick (resp. localizing) if it is stable by direct factors (resp. small sums).

We denote by \( \langle \mathcal{G} \rangle \) (resp. \( \langle \mathcal{G} \rangle^+, \langle \mathcal{G} \rangle^- \)) the smallest thick triangulated subcategory of \( \mathcal{T} \) (resp. the smallest subcategory stable by extensions, positive shifts and direct factors, the smallest subcategory stable by extensions, negative shifts and direct factors) containing \( \mathcal{G} \). Assume now that \( \mathcal{T} \) admits small sums; by convention, this includes the hypothesis that small sums of distinguished triangles are distinguished triangles. We denote by \( \langle \langle \mathcal{G} \rangle \rangle \) (resp. \( \langle \langle \mathcal{G} \rangle \rangle^+, \langle \langle \mathcal{G} \rangle \rangle^- \)) the smallest localizing triangulated subcategory of \( \mathcal{T} \) (resp. the smallest subcategory stable by extensions, small sums and \([+1] \), the smallest subcategory stable by extensions, small sums and \([-1] \)) containing \( \mathcal{G} \). Note that \( \langle \langle \mathcal{G} \rangle \rangle \subset \langle \langle \mathcal{G} \rangle \rangle \) by [7, Lemme 2.1.17].

In the constructions above, we refer informally to \( \mathcal{G} \) as the generating family and to objects of \( \mathcal{G} \) as generators. In each case, these subcategories can be defined by an induction (transfinite in the \( \langle \langle \rangle \rangle \) cases): start with the full subcategory with objects \( \mathcal{G}[\mathbb{Z}] \); to pass to a successor ordinal, introduce, depending on the case, cones of all morphisms and direct factors of all objects, just the cones and direct factors, just the cocones and direct factors, the cones, direct factors and small sums, etc.; finally, to pass to a limit ordinal, take the union over all previous subcategories. These subcategories do not change if one replaces \( \mathcal{T} \) by a triangulated subcategory containing \( \mathcal{G} \) (and stable under small sums for the \( \langle \langle \rangle \rangle \) variants), so that we will in general not need to specify the ambient triangulated category.

We adopt the notational convention that functors between triangulated categories are triangulated by default, i.e., we write \( f_* \) for \( Rf_* \), \( f^* \) for \( Lf^* \), \( \otimes \) for \( \otimes^L \), \( a_l \) for \( L_{a_l} \), etc. In the few cases where we need to refer to the "underived functor", that is, the underlying Quillen functor at the level of model categories, we underline the notation, i.e., we write \( \underline{L}_*, \underline{L}^*, \underline{\otimes} \), etc.

Schemes and group schemes. Unless specified, all schemes are noetherian and finite dimensional and all smooth morphisms are assumed to be separated of finite type. The notation \( \text{Sm}/S \) (resp. \( \text{Sch}/S \)) denotes the category of all smooth \( S \)-schemes (resp. all separated finite type \( S \)-schemes), usually considered as a site with the étale topology.

Definition 0.5. We say that a scheme \( S \) admits the resolution of singularities by alterations if for any separated \( S \)-scheme \( X \) of finite type and any nowhere dense closed subset \( Z \subset X \), there is a projective alteration \( g : X' \to X \) with \( X' \) regular and such that \( g^{-1}(Z) \) is a strict normal crossing divisor.
The best result available in this direction is due to Temkin [57, Theorem 1.2.4]: any $S$ which is of finite type over a quasi-excellent scheme of dimension $\leq 3$ satisfies resolution of singularities by alterations.

Let us recall basic terminology and facts about exact sequences of group schemes. Let
\[(C): 0 \rightarrow G' \xrightarrow{i} G \xrightarrow{p} G'' \rightarrow 0\]
be a sequence of commutative group schemes over a scheme $S$. We say that $(C)$ is exact if it induces an exact sequence of fppf sheaves on $\mathbf{Sch}/S$. If $(C)$ is exact, then $G'$ is the scheme-theoretic kernel of $p$ and $p$ is a surjective morphism of schemes. In the other direction, if $p$ is an fppf morphism and $G'$ is its scheme-theoretic kernel, then $(C)$ is exact. Moreover, if the group scheme $G'$ is smooth over $S$, then one obtains an equivalent definition by replacing the fppf topology with the étale topology.

**Triangulated categories of motives.** We work in the context of the stable homotopical 2-functor $DA^{et}(-, \mathbb{Q})$ considered in [11, §3]. Most results in the paper are still valid, with the same proofs, for $DA^{et}(-, R)$ with $R$ a $\mathbb{Q}$-algebra; however, we stick to $R = \mathbb{Q}$ for simplicity.

Since we only consider the étale topology and rational coefficients, we simplify the notation and write $DA(S)$ for $DA^{et}(S, \mathbb{Q})$. The category $DA(S)$ is equivalent to several other constructions of triangulated categories of motives with rational coefficients, e.g. Beilinson motives [25]; see [25, §16] for various comparison theorems.

By [7], the functor $DA(-)$ admits the functoriality of the Grothendieck six operations. In particular, for any quasi-projective morphism $f : S \rightarrow T$ of schemes, Ayoub constructs adjoint pairs
\[
\begin{align*}
    f^* : DA(T) &\leftrightarrows DA(S) : f_* \\
    f_! : DA(S) &\leftrightarrows DA(T) : f^!
\end{align*}
\]
and when $f$ is smooth
\[
f_2 : DA(S) \leftrightarrows DA(T) : f^*.
\]

There is a morphism of functors $f_1 \rightarrow f_*$, which is an isomorphism for $f$ projective.

Note that for those operations, as well as for the pullbacks and pushforwards functors on derived categories of sheaves on $\mathbf{Sm}/-$, the notation $f^*$, $f_*$, $\ldots$ stands for the triangulated or derived functor. When we want to use the underived functor, we underline the functor: $\underline{f}^*, \underline{f}_*, \ldots$.

In the definitions of the Grothendieck operations, one can relax the condition that $f$ is quasi-projective in the following ways.

(i) As observed in [13, Appendice 1.A], one can define $f^*$ and $f_*$ for any morphism $f$ (without any finiteness hypothesis), and prove for instance that proper base change [11, Proposition 3.5], the $\text{Ex}^n_*$ isomorphism [11, Proposition 3.6] and “regular base change” [13, Corollaire 1.A.4] still hold.

(ii) As observed in [25, Theorem 2.2.14], one can define the exceptional functors $f_!$ and $f^!$ for any $f$ separated of finite type, and prove that all the properties in [7] still hold (in particular with $f_! \simeq f_*$ for any $f$ proper).

We freely use these more general constructions and results.

The six operations for $DA(-)$ satisfy a large number of properties and compatibilities (see [11, Proposition 3.2], [7, Scholie 1.4.2]). For results which come up repeatedly in this thesis, we introduce the following terminology. Let
\[
\begin{array}{ccc}
Z & \xrightarrow{\tilde{g}} & X \\
\downarrow f & & \downarrow f \\
W & \xrightarrow{g} & Y
\end{array}
\]
be a cartesian square of morphisms of schemes.

- By the $\text{Ex}^n_*$ isomorphism (resp. the $\text{Ex}^n_!$ isomorphism, the $\text{Ex}^n_!$ isomorphism), we mean the natural isomorphism $\tilde{f}_!\tilde{g}^* \xrightarrow{\sim} g^*f_*$ for $f$ smooth (resp. the natural isomorphism $\tilde{f}_!\tilde{g}^* \xrightarrow{\sim} g^!f_*$, the natural isomorphism $g^*f_! \xrightarrow{\sim} \tilde{f}_!\tilde{g}^*$).
- By “smooth base change”, we mean the natural isomorphism $\tilde{f}_!\tilde{g}^* \xrightarrow{\sim} g^*f_*$ for $g$ smooth.
• By “proper base change”, we mean the natural isomorphism $g^*f_* \xrightarrow{\sim} j_!g^*$ for $f$ proper.
• Let $i : Z \to X$ be a closed immersion and $j : U \to X$ be the complementary open immersion. When we write “by localisation”, we mean the use of the distinguished triangle of functors
  
  $$j_!i^* \to \text{id} \to i_*j^* \xrightarrow{\sim}.$$  

  Dually, when we write “by colocalisation”, we mean the use of the distinguished triangle of functors
  
  $$i_*j^* \to \text{id} \to j_!i^* \xrightarrow{\sim}.$$  

• By “relative purity”, we mean the fact that for any smooth morphism $f : S \to T$ of relative dimension $d$, there are isomorphism of functors $f_! \simeq f_!(d)[2d]$ and $f^! \simeq f^!(−d)[−2d]$.
• By “the separation property for $DA$”, we mean the fact that for any surjective morphism of finite type (resp. any finite surjective radicial morphism) $f : S \to T$, the functor $f^* : DA(T) \to DA(S)$ is conservative (resp. an equivalence of categories) [11, Théorème 3.9].
• By “absolute purity”, we mean the fact that for any closed immersion $i : S \to T$ of codimension $d$ with $S$, $T$ regular schemes, we have $i^!Q_T \simeq Q_S(−d)[−2d]$ ([11, Corollaire 7.5] and [11, Remarque 11.2]).
• By “cohomological $h$-descent”, we mean the fact that for any finite type morphism $f : S \to T$ of quasi-excellent schemes and any hypercovering $\pi_\bullet : T \to S$ in Voevodsky’s $h$-topology, [25, Theorem 14.3.4], the natural morphism of functors

  $$f_*f^!(−) \to f_*(\pi_\bullet)_*\pi_\bullet^*f^!(−)$$

  (which is part of the algebraic derivator structure for $DA(−)$) is an isomorphism. More precisely, we always use this in the case $f = \text{id}$ and through the induced descent spectral sequence for morphisms groups in $DA(−)$; namely, for such an hypercover $\pi_\bullet$ and for any motives $M, N \in DA(S)$, there is a cohomological spectral sequence

  $$E_1^{p,q} = DA(S_p)(\pi_\bullet^*M, \pi_\bullet^*N[q]) \Rightarrow DA(S)(M, N[p + q]).$$

  Note that this spectral sequence is only contained a priori in the right half-plane and so is not guaranteed to converge in general.

We also need some functoriality properties for categories of (effective) motives with transfers. For any noetherian finite dimensional scheme $S$, we have tensor triangulated categories $DM^{\text{eff}}(S)$ and $DM(S)$. By [25, §11.1.a.], when $S$ vary, these acquire the structure of a “premotivic category” in the sense of loc. cit.; in particular, for any morphism $f : T \to S$, there are adjunctions

  $$f^* : DM^{\text{eff}}(S) \leftrightarrows DM^{\text{eff}}(T) : f_*$$

  and

  $$f^* : DM(S) \leftrightarrows DM(T) : f_*$$

  and, when $f$ is smooth, there are adjunctions

  $$f^\sharp : DM^{\text{eff}}(T) \leftrightarrows DM^{\text{eff}}(S) : f^\flat$$

  and

  $$f^\sharp : DM^{\text{eff}}(T) \leftrightarrows DM^{\text{eff}}(S) : f^\flat.$$  

  These satisfy a smooth base change and a smooth projection formula.

1. Triangulated categories of $n$-motives

Categories of motives are naturally filtered by the dimension of “geometric generators”, and such filtrations have been studied in various motivic contexts [18] [10] [14]. We give definitions in the context of $DA(−)$ and prove a number of basic results. Since such a treatment does not appear in the literature, we study a more general situation than is necessary for the rest of the paper; outside of this section, we are concerned with the special case of (co)homological 0- and 1-motives.

Note that some of our results on the operations for cohomological motives are also discussed in [59, §3.1].
1.1. Definitions. We fix a base scheme $S$ and an integer $n \geq 0$ for the remainder of this section.

**Definition 1.1.** The category $DA^{\text{coh}}(S)$ (resp. $DA_{\text{hom}}(S)$) of cohomological motives (resp. homological motives) is the full subcategory of $DA(S)$ defined as

$$DA^{\text{coh}}(S) = \langle f_*Q_X | f : X \to S \text{ proper morphism} \rangle$$

(resp.

$$DA_{\text{hom}}(S) = \langle f_*Q_X | f : X \to S \text{ smooth morphism} \rangle$$)

The category $DA^n(S)$ (resp. $DA_{n}(S)$) of cohomological $n$-motives (resp. homological $n$-motives) is the full subcategory of $DA(S)$ defined as

$$DA^n(S) = \langle f_*Q_X | f : X \to S \text{ proper morphism of relative dimension } \leq n \rangle$$

(resp.

$$DA_{n}(S) = \langle f_*Q_X | f : X \to S \text{ smooth morphism of relative dimension } \leq n \rangle$$).

**Remark 1.2.** As we will see in Proposition 1.27, the categories $DA_{n}(S)$ and $DA^n(S)$ are in fact equivalent as triangulated categories, so that many questions about $DA^n(S)$ can be reduced to $DA_{n}(S)$. In the special cases $n = 0, 1$, this is a crucial ingredient for several results in this paper. However to establish Proposition 1.27 we need to study $DA_{n}$ and $DA^n$ in parallel.

We have subcategories of smooth and geometrically smooth objects. Recall that an object $X$ in a symmetric monoidal category is said to be strongly dualizable if there exists an object $X'\otimes$ together with morphisms $\epsilon : 1 \to X \otimes X'$ and $\eta : X \otimes X' \to 1$ satisfying the classical adjunction triangle laws.

**Definition 1.3.** The category $DA^{\text{gsm}}(S)$ (resp. $DA^{\text{coh}}_{\text{gsm}}(S)$, $DA^{\text{gsm}}_{\text{hom}}(S)$) of geometrically smooth motives (resp. geometrically smooth cohomological motives resp. of geometrically smooth homological motives) is the full subcategory of $DA(S)$ defined as

$$DA^{\text{gsm}}(S) = \langle f_*Q_X(-n) | f : X \to S \text{ proper smooth morphism, } n \in \mathbb{Z} \rangle$$

(resp.

$$DA^{\text{coh}}_{\text{gsm}}(S) = \langle f_*Q_X | f : X \to S \text{ smooth morphism} \rangle$$,  

$$DA^{\text{gsm}}_{\text{hom}}(S) = \langle f_*Q_X | f : X \to S \text{ smooth morphism} \rangle$$).

The category $DA^{\text{sm}}(S)$ (resp. $DA^{\text{coh}}_{\text{sm}}(S)$, $DA^{\text{sm}}_{\text{hom}}(S)$) of smooth motives (resp. smooth cohomological motives, smooth homological motives) is defined as

$$DA^{\text{sm}}(S) = \langle M \in DA(S) | M \text{ strongly dualizable } \rangle$$

(resp.

$$DA^{\text{coh}}_{\text{sm}}(S) = \langle M \in DA^{\text{coh}}(S) | M \text{ strongly dualizable in } DA(S) \rangle$$,  

$$DA^{\text{sm}}_{\text{hom}}(S) = \langle M \in DA^{\text{hom}}(S) | M \text{ strongly dualizable in } DA(S) \rangle$$).

We then define

$$DA^{\text{gsm}}_{n}(S) = DA^n(S) \cap DA^{\text{coh}}_{\text{gsm}}(S),$$  

$$DA^{\text{gsm}}_{n}(S) = DA_{n}(S) \cap DA^{\text{gsm}}_{\text{hom}}(S),$$  

$$DA^{\text{sm}}_{n}(S) = DA_{n}(S) \cap DA^{\text{coh}}_{\text{sm}}(S),$$  

and $$DA^{\text{sm}}_{n}(S) = DA_{n}(S) \cap DA^{\text{sm}}_{\text{hom}}(S).$$

**Remark 1.4.** In the definition of $DA^{\text{gsm}}_{n}(S)$ and $DA^{\text{sm}}_{n}(S)$, we make the choice not to impose the geometric smoothness to come from proper smooth morphisms of relative dimension $\leq n$. This more restrictive definition seems too strong for what we can actually establish about geometrically smooth 1-motives, as the proof of Corollary 2.14 below shows. Deciding whether the two possible definitions are equivalent seems difficult.

**Lemma 1.5.** Geometrically smooth objects are smooth: we have $DA^{\text{gsm}}(S) \subset DA^{\text{sm}}(S)$, $DA^{\text{gsm}}_{\text{hom}}(S) \subset DA^{\text{sm}}_{\text{hom}}(S)$, etc.

**Proof.** This follows from relative purity and the projection formula, see e.g. [27, Lemma 4.2.8].
Remark 1.6. Proposition 1.25 below shows that when \( S \) is the spectrum of a field, any motive is (geometrically) smooth.

The converse of the above lemma is not known and it is not clear if one should expect it. Informally, when \( S \) is a discrete valuation ring, it would mean that a “motive with good reduction” is always realisable in the cohomology of a variety with good reduction.

There is a further reasonable definition of a smooth-like object in \( \text{DA}_{c}(S) \), namely a motive whose realisations have cohomology sheaves which are local systems (in the appropriate sense, e.g. lisse \( \ell \)-adic sheaves). This is conjecturally equivalent to being strongly dualizable; this equivalence would follow from the conservativity of realisation functors.

An important property of smooth compact objects is that they satisfy a form of absolute purity.

Proposition 1.7. Let \( i : Z \to S \) be a regular immersion of codimension \( c \). For \( M \in \text{DA}_{c}^{\text{sm}}(S) \) (i.e., \( M \) strongly dualisable), there is a purity isomorphism

\[
i^{*}M \cong i^{*}M(c)[2c]
\]

which is functorial in \( M \), in the sense that for any \( f : M \to N \in \text{DA}_{c}^{\text{sm}}(S) \) the diagram

\[
\begin{array}{ccc}
i^{*}M & \xrightarrow{i^{*}(f)} & i^{*}N \\
\downarrow & & \downarrow \\
i^{*}(M(c))[2c] & \xrightarrow{i^{*}(f)(c)[2c]} & i^{*}(N(c))[2c]
\end{array}
\]

commutes.

Proof. The idea is to use dualisability to reduce to the usual absolute purity property for the unit object. The functor \( i^{*} \) is monoidal, hence preserves strongly dualisable objects and sends strong duals to strong duals. By biduality, this provides a natural isomorphism

\[
i^{*}M \cong \text{Hom}(i^{*}M^{\vee}, Q_{Z})
\]

By absolute purity, this last group is isomorphic to \( \text{Hom}(i^{*}M^{\vee}, i^{!}Q_{Z}(c)[2c]) \), which is itself naturally isomorphic to \( i^{!}\text{Hom}(M^{\vee}, Q(c)[2c]) \) by [7, Proposition 2.3.55]. Since \( Q(c)[2c] \) is invertible and \( M \) is strongly dualisable, \( i^{!}\text{Hom}(M^{\vee}, Q(c)[2c]) \) \( \cong i^{!}\text{Hom}(M^{\vee}, Q(c)[2c][2c] \cong i^{!}M(c)[2c] \). The composition gives the required isomorphism. Each step of the construction is functorial in \( M \). \( \square \)

Lemma 1.8. Let \( T \) be one of \( \text{DA}_{\text{hom}}(S), \text{DA}_{\text{coh}}(S), \text{DA}_{n}(S), \text{DA}^{\cdot}(S) \) or their subcategories of smooth or geometrically smooth objects. Then the triangulated category \( T \) is compactly generated by its generating family, and an object of \( T \) is compact if and only if it is compact in \( \text{DA}(S) \).

Proof. Write \( G \) for the generating family of \( T \). By the fact that strongly dualizable objects in a symmetric monoidal triangulated category are automatically compact (for the \( \text{DA}^{\text{sm}}(S) \) case) and [11, Proposition 3.20, Proposition 8.5] (for the other cases), we see that all objects of \( G \) are compact. This means that \( T \) is compactly generated by \( G \). Write \( T_{c} \) for the full subcategory of objects of \( T \) which are compact in \( T \). By [51, Lemma 4.4.5], \( T_{c} = \langle G \rangle \). In particular any object of \( T_{c} \) is compact in \( \text{DA}(S) \); the converse implication is clear. \( \square \)

Definition 1.9. We write \( \text{DA}_{\text{coh}}^{c}(S), \text{DA}_{\text{hom},c}(S) \), etc. for the full subcategories of compact objects of \( \text{DA}_{\text{coh}}^{c}(S), \text{DA}_{\text{hom}}^{c}(S) \), etc.

1.2. Permanence properties. The subcategories we have introduced are each stable under certain Grothendieck operations. We start with the compatibilities with the monoidal structure.

Proposition 1.10. Let \( S \) be a base scheme.

(i) \( \text{DA}_{\text{coh}}^{c}(S) \) is stable by tensor products and negative Tate twists.

(ii) For all \( m, n \geq 0 \), we have \( \text{DA}^{m}(S) \otimes \text{DA}^{n}(S) \subset \text{DA}^{m+n}(S) \).

(iii) For all \( m, n \geq 0 \), we have \( \text{DA}^{m}(S)(-n) \subset \text{DA}^{m+n}(S) \).

(iv) \( \text{DA}_{\text{hom}}^{c}(S) \) is stable by tensor products and positive Tate twists.

(v) For all \( m, n \geq 0 \), we have \( \text{DA}_{m}(S) \otimes \text{DA}_{n}(S) \subset \text{DA}_{m+n}(S) \).

(vi) For all \( m, n \geq 0 \), we have \( \text{DA}_{m}(S)(n) \subset \text{DA}_{m+n}(S) \).

The same properties hold for the smooth and geometrically smooth versions of those subcategories.
\textbf{Proof.} First, note that \( \otimes \) commutes with small sums in both variables, being a left adjoint. This reduces the proof to checking the result for generators.

Let us prove point (i). Recall that we have a projection formula for \( f \) and \( f^* \) from [7, Theoreme 2.3.40], i.e., for any finite type separated morphism \( f : S \to T \) and any \( M \in \text{DA}(S), N \in \text{DA}(T) \), we have a natural isomorphism

\[
f_!(M \otimes f^* N) \simeq f_! M \otimes N.
\]

Let \( g : X \to S \) and \( h : Y \to S \) be proper morphisms. Let \( Z = X \times_S Y \) and let \( g' : Z \to Y \) and \( h' : Z \to X \) be the two projections. We have a sequence of isomorphisms

\[
g_* \mathcal{Q}_X \otimes h_* \mathcal{Q}_Y \simeq g_! \mathcal{Q}_X \otimes h_! \mathcal{Q}_Y
\]

where the first and fourth isomorphisms follows from properness, the second is the projection formula and the third is the \( \text{Ex}_!^* \) isomorphism. This shows that \( g_* \mathcal{Q}_X \otimes h_* \mathcal{Q}_Y \) is cohomological. The negative Tate twist \( \mathcal{Q}_S(-n) \) is cohomological, as it is a direct factor of \( (\mathbb{P}^n_S \to S)_! \mathcal{Q} \). This finishes the proof of (i). The same proof, combined with the fact that relative dimension is stable by base change and adds up in compositions, gives (ii) and (iii).

For the proof of point (iv), we use a parallel argument; we combine the projection formula for \( f_! \) and \( f^* \) of [8, Proposition 4.5.17] with the \( \text{Ex}_*^* \) isomorphism and the fact that \( \mathcal{Q}_S(n) \) is a direct factor of \( (\mathbb{P}^n_S \to S)_! \mathcal{Q} \) by the projective bundle formula. The same proof, combined with the fact that relative dimension is stable by base change and adds up in compositions, gives (v) and (vi).

Finally, the analogous statement for smooth and geometrically smooth versions follow from similar arguments together with the fact that a tensor product of strongly dualizable objects is strongly dualizable. \( \square \)

\textbf{Proposition 1.11.} Let \( f : S \to T \) be a morphism of schemes. The following operations preserve the subcategories \( \text{DA}^\text{coh}(\cdot) \).

\begin{enumerate}[(i)]
  \item \( f^* \) for any \( f \).
  \item \( f_* \) when \( f \) is separated of finite type and \( S \) admits the resolution of singularities by alterations.
  \item \( f_! \) when \( f \) is separated of finite type.
  \item \( f^! \) when \( f \) is quasi-finite separated and \( S \) admits the resolution of singularities by alterations.
\end{enumerate}

Moreover, they also preserve \( \text{DA}_c^\text{coh}(\cdot) \) (with the assumption that the schemes involved are quasi-excellent for points (ii)-(iv)).

\textbf{Proof.} The results for \( \text{DA}_c^\text{coh}(\cdot) \) follow from the ones for \( \text{DA}^\text{coh}(\cdot) \) together with the constructibility theorem for Grothendieck operations ([11, Theoreme 8.10], which actually holds for quasi-excellent instead of excellent since the hypothesis is only used through Gabber’s theorem) and Lemma 1.8. We thus focus on \( \text{DA}^\text{coh}(\cdot) \). We prove the results in a slightly different order than in the statement: we first establish (i), (iii) (which contains the special case of (ii) for proper morphisms), (iv) for closed immersions, (ii) and finally (iv) in all generality. In each case, we first check that the functor commutes with small sums, and then compute its action on generators of \( \text{DA}^\text{coh}(\cdot) \).

\textbf{Proof of (i):} the functor \( f^* \) is a left adjoint hence commutes with small sums. Moreover proper base change implies that \( f^* \) sends generators of \( \text{DA}^\text{coh}(T) \) to generators of \( \text{DA}^\text{coh}(S) \).

\textbf{Proof of (iii):} the functor \( f_! \) is a left adjoint hence commutes with small sums. Let \( g : X \to S \) be a proper morphism. We need to show that \( f_! g_* \mathcal{Q}_X \simeq (f \circ g)_! \mathcal{Q}_X \) is in \( \text{DA}^\text{coh}(T) \). Since \( f \) is assumed to be separated of finite type, the same holds for \( f \circ g \). Nagata’s theorem [50] [28] implies that \( f \circ g \) admits a compactification, i.e., that there exists a factorisation \( f \circ g = f \circ j \) with \( j : X \to \bar{X} \) an open immersion and \( f : \bar{X} \to T \) a proper morphism. Let \( i : Z \to \bar{X} \) be a complementary closed immersion to \( j \). By localisation, we have a distinguished triangle

\[
j_! \mathcal{Q}_X \to \mathcal{Q}_{\bar{X}} \to i_* \mathcal{Q}_Z \to
\]
which after applying $\bar{f}_* \simeq f_*$ yields

$$\bar{f}_*j^!Q_X \simeq f_*g_*Q_X \to \bar{f}_!Q_X \to (\bar{f}_!)_!Q_Z \implies .$$

By definition, the second and third terms in this triangle are in $\text{DA}^{\text{coh}}(T)$. This implies that the first is as well.

**Proof of (iv) for $f = i$ closed immersion:**

The functor $i^!$ has a left adjoint $i_!$ which sends compact objects to compact objects by [11, Proposition 8.5]. By [7, Lemme 2.1.28] this implies that $i^!$ commutes with small sums.

The blueprint for this proof is taken from Section 2.2.2 of [7].

Lemma 1.12 below, applied to $i : S \to T$, shows that it is enough to prove that, for any $g : X \to T$ with $X$ connected regular and $g^{-1}(S)$ equal to either $X$ or a normal crossing divisor, the motive $i^!g_*Q_X$ is compact. Form the cartesian square

$$\begin{array}{ccc}
Y & \xrightarrow{\iota'} & X \\
\downarrow{g'} & & \downarrow{g} \\
S & \xrightarrow{i} & T.
\end{array}$$

We have an $\text{Ex}^1$ isomorphism $i^!g_*Q_X \simeq g'_*i'^!Q_X$. By point (iii), it is enough to show that $i'^!Q_X$ is in $\text{DA}^{\text{coh}}(X)$. By assumption, $Y$ is either equal to $X$ or is a normal crossing divisor; only the second case requires a proof. By [7, Lemme 2.2.31] applied to the branches and point (iii) for closed immersions, we reduce to the case of a regular immersion, which then follows from absolute purity and Proposition 1.10 (i).

**Proof of (ii):**

Using Nagata’s theorem and the proper case of point (iii), it suffices to show that $j_*Q_S$ is in $\text{DA}^{\text{coh}}(T)$ for $j : S \to T$ open immersion. This now follows from colocalisation and point (iv) for the complementary closed immersion.

**Proof of (iv) for $f$ quasi-finite general:**

By the same argument as above, using the $\text{Ex}^1$ isomorphism, it is enough to show that $f'_*Q_T$ is in $\text{DA}^{\text{coh}}(S)$. Using Zariski's main theorem [38, Théorème 8.12.6], the fact that $j'_! \simeq j^*$ for $j$ open immersion and point (i), we are reduced to the case of finite morphisms.

If $f$ is finite étale, then $f'_! \simeq f^*$ again and we are done by point (i). If $f$ is finite and purely inseparable, then a corollary of the separation property of $\text{DA}$ is that $f'_! \simeq f^*$ is an equivalence of categories [7, Corollaire 2.1.164]. In general, we proceed by induction on the dimension of $T$. The proof for the 0-dimensional case follows the same pattern as the inductive step, so we treat both in parallel. If $T$ is 0-dimensional, or generically on $T$, say above an everywhere dense open set $j : U \to T$, the morphism $f$ is the composite of a finite étale morphism followed by a finite purely inseparable morphism. Let $l : V \to S$ be $j \times_T S$ and $k : W \to S$ be a complementary closed immersion (take $W$ empty in the 0-dimensional case). Then $l^!f'_*Q_T \simeq f'_UQ_V$ is in $\text{DA}^{\text{coh}}(V)$ by combining the arguments for finite étale and finite purely inseparable morphisms above. By point (ii), we get that $l_*l^!f'_*Q_T$ is in $\text{DA}^{\text{coh}}(S)$. This concludes the proof for $\dim(T) = 0$. In general, by the inductive hypothesis and point (iii), we get that $k_!k^!f'_*Q_T$ lies in $\text{DA}^{\text{coh}}(S)$. The colocalisation triangle then shows that $f'_*Q_T$ lies in $\text{DA}^{\text{coh}}(S)$. This completes the proof.

**Lemma 1.12.** Let $S$ be a scheme admitting the resolution of singularities by alterations, $f : X \to S$ a finite type morphism and $T \subseteq X$ closed. Then $\text{DA}^{\text{coh}}(X)$ is compactly generated by motives of the form $g_*Q_X$, with $g : X' \to X$ a projective morphism and $X'$ connected regular and $g^{-1}(T)$ equal either to $X'$ or to a normal crossing divisor.

**Proof.** The reference [7, Proposition 2.2.27] specialized to the $\mathbb{Q}$-linear, separated, homotopical 2-functor $\text{DA}(\_)$ proves a similar statement for the category of constructible objects $\text{DA}(S)$ (with added positive Tate twists of the generators, and restriction to quasi-projective morphisms). Once one removes the Tate twists and the restriction to quasi-projective morphisms, one notices that using Statement (iii) of Proposition above instead of Corollaire 2.2.21 in loc.cit, the proof of loc.cit [7, Proposition 2.2.27] then applies verbatim. \qed
Proposition 1.13. Let \( f : S \to T \) be a morphism of schemes. The following operations preserve the subcategories \( \DA_{\hom}(-) \) and \( \DA_{\hom,c}(-) \).

(i) \( f^* \) for any \( f \).
(ii) \( f_* \) when \( f \) is smooth.
(iii) \( f! \) for any quasi-finite separated morphism \( f \).

Remark 1.14. In the proof of point (iii), we use results from Sections 1.3 and 1.4. The reader can check that we do not use the reference 1.13 (iii) in between. We feel this break from logical order is justified by the commodity of stating these properties together.

Proof. The results about \( \DA_{\hom,c}(-) \) follow from the ones for \( \DA_{\hom}(-) \) together with the constructibility result of [11, Proposition 8.5] and Lemma 1.8. We thus focus on \( \DA_{\hom}(-) \).

Proof of (i): The functor \( f^* \) is a left adjoint so commutes with small sums. Moreover the \( \text{Ex}_T^* \) isomorphism implies that \( f^* \) sends generators of \( \DA_{\hom}(T) \) to generators of \( \DA_{\hom}(S) \).

Proof of (ii): The functor \( f_* \) is a left adjoint so commutes with small sums. The fact that generators are sent to homological motives follows directly from the definition.

Proof of (iii): The functor \( f_! \) is a left adjoint so preserves small sums. Using Zariski’s Main theorem [38, Théorème 8.12.6] and (ii), we see that it is enough to treat the case of \( f \) finite.

We first do the case of closed immersions. The next lemma is proved using Mayer-Vietoris distinguished triangles.

Lemma 1.15. Let \( T \) be a scheme and \( U = \{ j_k : U_k \to T \}_{k=1}^n \) be a finite Zariski open covering of \( T \). Let \( M \in \DA_{\hom}(S) \) Then
\[
M \in \DA_{\hom}(S) \iff \text{for all } 1 \leq k \leq n, \text{ we have } j_k^* M \in \DA_{\hom}(S).
\]

Let \( i : Z \to X \) be a closed immersion and \( g : U \to Z \) be a smooth morphism. We need to show that \( i_* g_! U \in \DA_{\hom}(X) \). There exists a finite open affine cover \( \{ U_k = \text{Spec}(A_k) \}_{1 \leq k \leq n} \) of \( U \) and a finite open affine cover \( \{ Z_k = \text{Spec}(R_k) \}_{1 \leq k \leq n} \) of \( Z \) with \( g(U_k) \subset Z_k \) and such that via \( g_k := g|_{U_k} \), the ring \( A_k \) takes the form:
\[
A_k = R_k[x_1, \ldots, x_{n_k}]/(f_1^k, \ldots, f_{c_k}^k)
\]
with \( \det(g_{ij}^k) \) invertible in \( A_k \) (i.e. \( g_k \) is a standard smooth map). We can choose an open affine cover \( \{ W_k \} \) of \( X \) such that \( W_k \cap Z = Z_k \). Applying Lemma 1.15 to the open cover \( W_k \) and using base change for closed immersions and smooth base change, we can suppose that \( g \) itself is a standard smooth map and that \( X = \text{Spec}(R) \) is affine.

In this situation, we can lift the equations \( f_j \) to \( \tilde{f}_j \in \text{Spec}(R[x_1, \ldots, x_n]/(f_1, \ldots, f_n)) \) \( \to X \) is standard smooth contains \( Z \), and \( \tilde{g} \) extends \( g \). We have a localisation triangle
\[
(W \setminus Z \to W)_{\mathcal{I}} g_! Q \to \tilde{g}_! Q \to i_* g_! Q_{U} \xrightarrow{\sim}
\]
where the first two terms are in \( \DA_{\hom}(X) \). We deduce that \( i_* g_! Q_U \in \DA_{\hom}(X) \) as wanted.

For a general quasi-finite \( f : T \to S \), using localisation, the case of closed immersions and an induction on the dimension of \( S \), we see that we can replace \( S \) by any everywhere dense open subset. The case of closed immersions also ensures that we can assume \( S \) is reduced. By continuity for \( \DA_{\hom}(-) \) (proven in Proposition 1.22 below; the proof does not use permanence properties of \( \DA_{\hom}(-) \) besides (i)), we see that we can even replace \( S \) by any of its generic points. We are thus reduced to the case of a finite field extension, which follows from the more precise Lemma 1.26 below.

Proposition 1.16.

(i) Let \( f \) be any morphism of schemes. Then \( f^* \) preserves the subcategories \( \DA^n(-) \) and \( \DA^n_c(-) \).

(ii) Let \( f : S \to T \) be separated of finite type and of relative dimension \( m \). Then \( f^! \) sends \( \DA^n(S) \) (resp. \( \DA^n_c(S) \)) to \( \DA^{n+m}(T) \) (resp. \( \DA^n_{c+m}(T) \)). In particular, if \( f \) is quasi-finite, then \( f^! \) preserves the subcategories \( \DA^n(-) \) and \( \DA^n_c(-) \).
Lemma 1.20. Let $\text{DA}^n(-)$ together with Lemma 1.8 and the constructibility result of [11, Proposition 8.5]. Consequently, we only treat the case of $\text{DA}^n(-)$.

Statement (i) follows from the fact that $f^*$, being a left adjoint, commutes with small sums, proper base change and the fact that being of relative dimension $\leq n$ is stable by base change.

The proof of (ii) is the same as that of Proposition 1.11 (iii), keeping track of the relative dimensions involved.

\begin{proof}

The statements for the subcategories of compact objects follow from those for $\text{DA}^n(-)$ together with Lemma 1.8 and the constructibility result of [11, Proposition 8.5]. Consequently, we only treat the case of $\text{DA}^n(-)$.

Statement (i) follows from the fact that $f^*$, being a left adjoint, commutes with small sums, from the $\text{Ex}^*_f$ isomorphism and the fact that being of relative dimension $\leq n$ is stable by base change.

The proof of (ii) is the same as that of Proposition 1.13 (iii), keeping track of the relative dimensions involved.

\end{proof}

\begin{proposition}

(i) Let $f$ be any morphism of schemes. Then $f^*$ preserves the subcategories $\text{DA}_n(-)$ and $\text{DA}_{n,c}(-)$.

(ii) Let $f : S \to T$ be separated of finite type and of relative dimension $m$. Then $f_!$ sends $\text{DA}_n(S)$ (resp. $\text{DA}_{n,c}(S)$) to $\text{DA}_{n+m}(T)$ (resp. $\text{DA}_{n+m,c}(T)$). In particular, if $f$ is quasi-finite, then $f_!$ preserves the subcategories $\text{DA}_n(-)$ and $\text{DA}_{n,c}(-)$.

\end{proposition}

\begin{proof}

The statements for the subcategories of compact objects follow from those for $\text{DA}_n(-)$ together with Lemma 1.8 and the constructibility result of [11, Proposition 8.5]. Consequently, we only treat the case of $\text{DA}_n(-)$.

Statement (i) follows from the fact that $f^*$, being a left adjoint, commutes with small sums, from the $\text{Ex}^*_f$ isomorphism and the fact that being of relative dimension $\leq n$ is stable by base change.

The proof of (ii) is the same as that of Proposition 1.13 (iii), keeping track of the relative dimensions involved.

\end{proof}

We list some useful corollaries of the results above.

\begin{corollary}

Let $\mathcal{T}(-)$ be one of $\text{DA}^{\text{coh}}(-)$, $\text{DA}_{\text{hom}}(-)$, $\text{DA}^{n}(-)$, $\text{DA}_{n,c}(-)$ or one of their subcategories of compact objects.

(i) The system $\mathcal{T}(-)$ localises in the following sense: for $M \in \text{DA}(S)$, $i : Z \to S$ and $j : U \to S$ are complementary closed and open immersions, we have $M \in \mathcal{T}(S)$ if and only if $i^*M \in \mathcal{T}(Z)$ and $j^*M \in \mathcal{T}(U)$.

(ii) Let $f : T \to S$ be a finite surjective purely inseparable morphism (e.g. a nil-immersion), $M \in \text{DA}(S)$, $N \in \text{DA}(T)$. Then we have $M \in \mathcal{T}(S)$ if and only if $f^*M \in \mathcal{T}(T)$, and we have $N \in \mathcal{T}(T)$ if and only if $f_*N \in \mathcal{T}(S)$.

\end{corollary}

\begin{proof}

Statement (i) follows directly from localisation and the permanence properties above. Similarly, statement (ii) follows directly from [7, Proposition 2.1.163] (which applies because $\text{DA}(-)$ is separated) and the permanence properties.

\end{proof}

Finally, let us discuss what happens with internal Homs and duality.

\begin{corollary}

The internal Hom satisfies $\text{Hom}(\text{DA}_{\text{hom},c}(S), \text{DA}_{c}^{\text{coh}}(S)) \subset \text{DA}_{c}^{\text{coh}}(S)$. In particular, if $S$ is regular and we take $Q_S$ as dualizing object, then Verdier duality $\mathbb{D}_S := \text{Hom}(-, Q_S)$ sends compact homological motives to compact cohomological motives.

\end{corollary}

\begin{proof}

If $M \in \text{DA}(S)$ is compact, then $\text{Hom}(M, -)$ commutes with small sums. This shows that we can restrict to generators of $\text{DA}_{c}^{\text{coh}}(S)$ in the second variable. Using [51, Lemma 4.4.5], we see that we can restrict to generators of $\text{DA}_{\text{hom},c}(S)$ in the first variable. The result then follows from [7, Proposition 2.3.51-52], the $\text{Ex}^*_f$ isomorphism and Proposition 1.11 (ii).

\end{proof}

\begin{lemma}

Let $S$ be a regular scheme. Write $\mathbb{D}_S := \text{Hom}(-, Q_S) : \text{DA}(S)^{op} \to \text{DA}(S)$ for the Verdier duality functor. We have $\mathbb{D}_S(\text{DA}_{\text{hom},c}(S)) \subset \text{DA}_{c}^{\text{coh}}(S)$ and $\mathbb{D}_S$ restricts to anti-equivalences of categories $\mathbb{D}_S : \text{DA}_{\text{gsm},c}^{\text{gsm}}(S) \sim \text{DA}_{c}^{\text{gsm},c}(S)$ and $\mathbb{D}_S : \text{DA}_{n,c}^{\text{gsm}}(S) \sim \text{DA}_{n,c}^{n}(S)$.

\end{lemma}
Proof. For a separated scheme $X$ of finite type over $S$, consider the more general Verdier duality functor $\mathbb{D}_{X/S} := \text{Hom}(-, \pi_X^* \mathbb{L}) : \text{DA}(X)^{op} \to \text{DA}(X)$. By [11, Théorèmes 8.12-8.14], this functor preserves compact objects and its restriction to $\text{DA}_c(X)$ is involutive, i.e. an anti-autoequivalence which is its own quasi-inverse.

The behaviour of $\mathbb{D}_{X/S}$ with respect to the four operations is explained in [7, Théorème 2.3.75]: informally, Verdier duality exchanges $f_*$ and $f^!$, and $f^*$ and $f^!$. Moreover, recall that, for $f$ smooth, relative purity provides an isomorphism $f_* f^* \simeq f f^!$. This allows to compute the action of $\mathbb{D}_{X/S}$ on generating families. For instance, we have, for any $f$ smooth, $\mathbb{D}_S (f_* f^* \mathbb{L}) \simeq \mathbb{D}_S (f f^! \mathbb{L}) \simeq f_* f^* \mathbb{L}$, which is in $\text{DA}^{\text{coh}}(S)$ by Proposition 1.11(ii). This proves the first inclusion. For the equalities for geometrically smooth subcategories, note that if $f$ is smooth projective (resp. smooth projective of relative dimension $\leq n$), the same computation shows that $\mathbb{D}_S (f_* f^* \mathbb{L})$ is in $\text{DA}_c^{\text{coh}}(S)$ (resp. $\text{DA}_c^{\text{sm}}(S)$). This proves one inclusion of the equalities, and the other follows by the involutivity of $\mathbb{D}$. \hfill \Box

Remark 1.21. Even on a regular scheme, the categories of constructible homological and cohomological motives are not anti-equivalent through Verdier duality with dualizing object $\mathbb{Q}$ (see, however, Proposition 1.25 below for the field case). Indeed, assume $S$ regular of dimension $d > 0$, let $i : x \to S$ be the inclusion of a closed point $x$ and $j : U \to S$ be the complementary open immersion. Then by colocalisation and absolute purity, $j_* \mathbb{Q}_U \in \text{DA}^{\text{coh}}(S)$ sits in a triangle

\[ i_* \mathbb{Q}(-d)[-2d] \to \mathbb{Q}_S \to j_* \mathbb{Q}_U \to. \]

In particular, the motive $j_* \mathbb{Q}_U$ is cohomological. On the other hand, we have $\mathbb{D}_S (\mathbb{Q}_S) \simeq \mathbb{Q}_S \in \text{DA}^{\text{coh}}(S)$ and $\mathbb{D}_S (i_! \mathbb{Q}_S) \simeq i_* \mathbb{Q}_S \in \text{DA}^{\text{coh}}(S)$, so that by taking the Verdier dual of the triangle above, we have $\mathbb{D}_S (j_! \mathbb{Q}_U) \in \text{DA}^{\text{coh}}(S)$.

If Verdier duality exchanged homological and cohomological motives, we would have $j_* \mathbb{Q}_U \in \text{DA}_{\text{hom}}^{\text{coh}}(S) \cap \text{DA}^{\text{coh}}(S)$ which is equal to $\text{DA}_S$ by Corollary 3.9(ii) below. We would then also have $i_* \mathbb{Q}(-\ell)[-2d] \in \text{DA}_S$; hence, $\ell i_* \mathbb{Q}(-\ell) \simeq \mathbb{Q}(-\ell) \in \text{DA}_S$. This is not the case, as can be seen in a number of ways: for instance, in the proof of Corollary 3.9(iv) we will show that for all $M \in \text{DA}_S$ and $d > 0$, we have $\text{Hom}(M, \mathbb{Q}(-\ell)) = 0$.

1.3. Continuity. We have a continuity result for subcategories of compact objects.

Proposition 1.22. Let $I$ be a cofiltering small category and $(X_i)_{i \in I} \in \text{Sch}^I$ with affine transition morphisms. Let $\mathbb{X} = \lim_{i \in I} X_i$, which we assume to be noetherian and finite-dimensional. Then $\text{DA}^{\text{coh}}_c(X_i)$ (resp. $\text{DA}_{\text{hom},c}(X_i)$, $\text{DA}^{\text{coh}}_{c}(X_i)$, $\text{DA}_{\text{eff},c}(X_i)$) is equal to the 2-colimit of the $\text{DA}^{\text{coh}}_c(X_i)$ (resp. $\text{DA}_{\text{hom},c}(X_i)$, $\text{DA}^{\text{coh}}_{c}(X_i)$, $\text{DA}_{\text{eff},c}(X_i)$) via the pullback functors $(X \to X_i)^*$.

Proof. Using the continuity result for morphisms in $\text{DA}$ from [11, Proposition 3.19] and the arguments from [13, Corollaire 1.A.3], it is enough to prove the following lemma (which extends [13, Lemme 1.A.2]).

Lemma 1.23. With the notation of the proposition, let $Y$ be an $X$-scheme of finite presentation. Then there exists an $i \in I$ and an $X_i$-scheme $Y_i$ of finite presentation such that $Y \simeq Y_i \times_{X_i} X$. Moreover, if $Y/X$ is smooth (resp. of relative dimension $\leq n$, smooth of relative dimension $\leq n$), then $Y_i$ can be chosen smooth (resp. of relative dimension $\leq n$, smooth of relative dimension $\leq n$).

Proof. The first part is well known (see [38, §8]). For the second part, the arguments of the proof of [13, Lemme 1.A.2] cover the case of smooth and smooth of relative dimension $\leq n$. The case of morphisms of relative dimension $\leq n$ (without smoothness assumption) is treated in [56, Tag 05M5].

We deduce a useful punctual characterisation of compact $n$-motives:

Proposition 1.24. Let $S$ be a scheme and $M \in \text{DA}_c(S)$. Then the following are equivalent.

(i) $M \in \text{DA}^{\text{coh}}_c(S)$ (resp. $\text{DA}_{\text{hom},c}(S)$, $\text{DA}^{\text{coh}}_{c}(S)$, $\text{DA}_{\text{eff},c}(S)$).

(ii) For all $s \in S$, we have $s^* M \in \text{DA}^{\text{coh}}_c(s)$ (resp. $\text{DA}_{\text{hom},c}(s)$, $\text{DA}^{\text{coh}}_{c}(s)$, $\text{DA}_{\text{eff},c}(s)$).
Proof. The direction (i)⇒(ii) follows from the stability by pullbacks for all these subcategories established above. In the other direction, we can assume $S$ is reduced by Corollary 1.18 (ii). We then proceed by noetherian induction. The case of generic points is settled by the hypothesis, we then use Proposition 1.22 to spread-out the property to an open set. We conclude by using Corollary 1.18 (i) and the induction hypothesis. □

1.4. Over a field. When the base is a field, several of the notions we have introduced coincide.

Proposition 1.25. Let $k$ be any field; then we have the following equalities.

\[
DA_{\text{hom}}(k) = DA_{\text{sm}}^{\text{hom}}(k) = DA_{\text{gsm}}^{\text{hom}}(k).
\]

\[
DA_{\text{coh}}^{\text{hom}}(k) = DA_{\text{sm}}^{\text{coh}}(k) = DA_{\text{gsm}}^{\text{coh}}(k).
\]

\[
DA_{\text{n}}(k) = DA_{\text{n},c}^{\text{sm}}(k) = DA_{\text{n}}^{\text{gsm}}(k).
\]

The same equalities hold for the subcategories of compact objects, and $D_k$ restricts to anti-equivalences of categories:

\[
D_k : DA_{\text{hom,c}}(k) \simto DA_{\text{c}}^{\text{coh}}(k) : D_k
\]

\[
D_k : DA_{\text{n,c}}(k) \simto DA_{\text{n}}^{\text{gsm}}(k) : D_k
\]

Proof. In each case, we prove equality by showing that the generating family on each side lies in the other. The generating families used in the definitions of these categories are formed of compact objects, hence it suffices to prove the equalities for the subcategories of compact objects. By Lemma 1.5, we need only prove the inclusions

\[
DA_{\text{hom,c}}(k) \subset DA_{\text{gsm}}^{\text{hom,c}}(k),
\]

\[
DA_{\text{c}}^{\text{coh}}(k) \subset DA_{\text{gsm}}^{\text{coh}}(k),
\]

\[
DA_{\text{n,c}}(k) \subset DA_{\text{gsm}}^{\text{n,c}}(k)
\]

and

\[
DA_{\text{n}}^{\text{c}}(k) \subset DA_{\text{gsm}}^{\text{n,c}}(k).
\]

The key is to prove the following claim: for all $n \in \mathbb{N}$, we have $D_k(DA_{\text{gsm}}^{n}(k)) \subset DA_{\text{n,c}}^{\text{gsm}}(k)$.

Indeed, assume this claim for the next three paragraphs. Then by looking at generators we also get $D_k(DA_{\text{c}}^{\text{coh}}(k)) \subset DA_{\text{hom,c}}^{\text{coh}}(k)$. By applying $D_k$ again and the equivalence of categories of Lemma 1.20, we get inclusions $DA_{\text{n}}^{n}(k) \subset DA_{\text{n,c}}^{\text{gsm}}(k)$ and $DA_{\text{c}}^{\text{coh}}(k) \subset DA_{\text{gsm}}^{\text{coh}}(k)$. By applying $D_k$ to the inclusion $D_k(DA_{\text{hom,c}}^{\text{c}}(k)) \subset DA_{\text{c}}^{\text{coh}}(k)$ of Lemma 1.20, we also obtain $DA_{\text{hom,c}}^{\text{c}}(k) \subset DA_{\text{gsm}}^{\text{hom,c}}(k)$. It remains to see that $DA_{\text{n,c}}^{\text{c}}(k) \subset DA_{\text{n}}^{\text{c}}(k)$, which is slightly less clear.

Let $f : X \to k$ smooth of relative dimension $i \leq n$ (we can reduce to this case by considering connected components of $X$). By relative purity, we have $f_!f^!Q_X(-n) \simeq f_!Q_X(i-n)[2i]$ which is in $DA_{\text{n}}^{i}(k)$ by Proposition 1.16 and 1.10. This shows that $DA_{\text{n,c}}^{i,n}(k)(-n) \subset DA_{\text{n}}^{i,n}(k) = DA_{\text{n}}^{\text{gsm}}(k)$ (the last equality having just been established in the previous paragraph). Applying Verdier duality, we get $D_k(DA_{\text{n,c}}^{i,n}(k))(-n) \subset D_k(DA_{\text{gsm}}^{i,n}(k))$. By the previous lemma, this implies that $D_k(DA_{\text{n,c}}^{i,n}(k))(-n) \subset DA_{\text{n}}^{\text{gsm}}(k)$.

Another application of relative purity shows that $DA_{\text{n,c}}^{i,n}(k) \simeq DA_{\text{n}}^{i,n}(k)$. Putting everything together, we have $D_k(\text{DA}_{\text{n,c}}^{n}(k)) \subset DA_{\text{gsm}}^{n}(k) = D_k(\text{DA}_{\text{n,c}}^{n}(k))$ so by the involutivity of $D$ we get the missing inclusion $DA_{\text{n}}^{n}(k) \subset DA_{\text{gsm}}^{n}(k)$. This finishes the proof of the proposition modulo the claim.

The rest of the proof is devoted to show that $D_k(\text{DA}_{\text{n,c}}^{n}(k)) \subset DA_{\text{gsm}}^{n}(k)$. To simplify notations, we write $\pi^Y : Y \to k$ for the structural morphism of a $k$-scheme $Y$. Using the generating families, we reformulate the claim as follows: for $\pi_X : X \to k$ proper of relative dimension $\leq n$, we have $D_k(\pi_{X,k}^*Q_X) \simeq \pi_X^*\pi^*_XQ_X$ in $DA_{\text{n,c}}^{\text{gsm}}(k)$. Let $i : X_{\text{red}} \to X$. Then by localisation we have $\pi_{X_{\text{red}}}^*Q_X \simeq \pi_{X_{\text{red}}}^*\pi_X^*Q_X \simeq \pi_{X_{\text{red}}}^*\pi_X^*\pi^*_XQ_X$. Consequently, we can assume that $X$ is reduced.

We first treat the case of a perfect field $k$. We proceed by induction on the dimension of $X$. When $X$ is 0-dimensional, we see that $\pi_X$ is finite étale because $k$ is perfect and $X$ is reduced, so that $\pi_{X_{\text{red}}}^*\pi_X^*Q_X \simeq \pi_{X_{\text{red}}}^*\pi_X^*Q_X$ and we are done. For the induction step, we apply De Jong's resolution of singularities by alterations [29, Theorem 4.1 and following remark]. We obtain an alteration $h : \tilde{X} \to X$ with $\tilde{X}/k$ a smooth projective variety (smoothness can be achieved because $k$ is
perfect). Recall that $h$ is proper surjective and generically finite. We choose a diagram of schemes with cartesian squares

$$
\begin{array}{ccc}
V & \xrightarrow{j} & \tilde{X} \\
\downarrow h_U & & \downarrow h \\
U & \xrightarrow{i} & T
\end{array}
$$

with the following properties.

- $T$ is a nowhere dense closed subset of $X$ and $U$ is its open complement.
- $h_U$ can be written as the composite of a purely inseparable finite morphism followed by a finite étale morphism.

Starting from the distinguished colocalisation triangle for the pair $(X, U)$ and applying $\pi_{X!}$, we obtain a triangle

$$
\pi_{X!} j^! \pi_X^! \mathcal{Q}_k \rightarrow \pi_{X!} \pi_X^! \mathcal{Q}_k \rightarrow \pi_{X!} j^! \pi_X^! \mathcal{Q}_k \xrightarrow{\pi_{T!} \pi_{T,U}^! \mathcal{Q}_k}.
$$

that we rewrite as

$$
\pi_{T!} \pi_{T,U}^! \mathcal{Q}_k \rightarrow \pi_{X!} \pi_X^! \mathcal{Q}_k \rightarrow \pi_{X!} j^! \pi_X^! \mathcal{Q}_k \xrightarrow{\pi_{T!} \pi_{T,U}^! \mathcal{Q}_k}.
$$

The left-hand term of this last triangle is in $\text{DA}_{n,c}^{gm}(k)$ by induction. To prove that the middle term is in $\text{DA}_{n,c}^{gm}(k)$, it remains to prove the same for the right-hand term. Since $h_U$ is finite and the composite of a purely inseparable morphism followed by an étale morphism, the separation property of $\text{DA}$ [11, Theorem 3.9] together with [7, Corollaire 2.1.164] implies that there is a natural isomorphism of functors:

$$
h_U \circ h_U^* \simeq h_U, h_U^*.
$$

Now, [7, Lemma 2.1.165] implies that $\pi_Y^! \mathcal{Q}_k$ is a direct factor of $(h_U)^* h_U^* \pi_X^! \mathcal{Q}_k$. Applying the isomorphism just above, we conclude that $\pi_Y^! \mathcal{Q}_k$ is a direct factor of $(h_U)^* h_U^* \pi_X^! \mathcal{Q}_k$. This last motive is isomorphic to $(h_U)^* \pi_Y^! \mathcal{Q}_k \simeq (h_U)^* j^* \pi_X^! \mathcal{Q}_k$, because $h_U$ is proper and $\tilde{j}$ is étale. We get that $\pi_{X!} j^! \pi_Y^! \mathcal{Q}_k$ is a direct factor of $\pi_{X!} j^* \pi_X^! \mathcal{Q}_k \simeq \pi_{X!} h_* \pi_X^! \mathcal{Q}_k$. We have $\pi_{X!} j^* \pi_X^! \mathcal{Q}_k$ is a direct factor of $\pi_{X!} j^* \pi_X^! \mathcal{Q}_k$ since $h$ is proper, hence we deduce that $\pi_{X!} j^! \pi_Y^! \mathcal{Q}_k$ is a direct factor of $\pi_{X!} j^* \pi_X^! \mathcal{Q}_k$. Applying localisation to the pair $(\tilde{X}, V)$, the fact that $\tilde{X}/k$ is smooth projective and the induction hypothesis for $Z$ shows that this last object is in $\text{DA}_{n,c}^{gm}(k)$. This concludes the proof when $k$ is perfect.

We now treat the case of a general field $k$. By the perfect field case and continuity for $\text{DA}_{n,c}^{gm}(-)$ (Proposition 1.22) applied to the spectrum of the perfect closure of $k$, we see that there exists a finite purely inseparable extension $l/k$ with $(l/k)^* \pi_{X!} \pi_X^! \mathcal{Q}_k$ in $\text{DA}_{n,c}^{gm}(l)$. By the separation property, we have an isomorphism of functors $id \simeq (l/k)^* (l/k)^*$, so that it is enough to show Lemma 1.26 below. This completes the proof of the claim, hence of the chains of equalities in the proposition.

Finally, the Verdier duality statement is just a restatement of Lemma 1.20 in the light of these chains of equalities.

**Lemma 1.26.** For a finite field extension $l/k$ and $g : Y \rightarrow \text{Spec}(l)$ a smooth projective morphism of relative dimension $\leq n$, there exists a smooth projective variety $g' : Y' \rightarrow k$ of dimension $\leq n$ such that $(l/k)_{g'!} \mathcal{Q}_{Y'} \simeq g'_{!} \mathcal{Q}_{Y'} \in \text{DA}_{n,c}^{gm}(k)$.

**Proof.** We immediately reduce to the case of a purely inseparable extension $l/k$. By treating separately the connected components of $Y$, we can assume that $Y$ is of dimension $n$. Let $F : \text{Spec}(l) \rightarrow \text{Spec}(l)$ be a high enough power of the Frobenius of $l$ that factors through $k$. We denote again by $F$ the induced morphism $\text{Spec}(k) \rightarrow \text{Spec}(l)$ and its natural lift $\text{Spec}(k) \rightarrow \text{Spec}(l)$ (the corresponding power of $\text{Fr}_k$). We have the following diagram of schemes, where the upper square is cartesian:

$$
\begin{array}{ccc}
Y'' & \xrightarrow{F_Y} & Y' \\
\downarrow \pi_{Y''} & & \downarrow \pi_{Y'} \\
\text{Spec}(k) & \xrightarrow{F} & \text{Spec}(l) \\
\downarrow F & & \downarrow (l/k) \\
\text{Spec}(k).
\end{array}
$$
By base change, the $k$-scheme $Y'$ is smooth projective and the morphism $F_Y$ is finite purely inseparable. By the separation property of $DA$, we have
\[(l/k)_*(\pi_Y)_*\mathbb{Q}_Y \cong (l/k)_*(\pi_Y)_*(F_Y)_*\mathbb{Q}_{Y'} \cong (l/k)_*F_*(\pi_Y)_*\mathbb{Q}_{Y'} \cong F_*(\pi_Y)_*\mathbb{Q}_{Y'} .\]
Let $Fr_{Y'}$ be the corresponding power of the absolute Frobenius on $Y'$. By naturality of the absolute Frobenius, we have $\pi_{Y'} \circ Fr_{Y'} = F \circ \pi_{Y'} : Y' \to \text{Spec}(k)$. We deduce that
\[F_*(\pi_Y)_*\mathbb{Q}_{Y'} \cong (\pi_{Y'})_*(Fr_{Y'})_*(\pi_{Y'})_*\mathbb{Q}_{Y'} \cong (\pi_{Y'})_*\mathbb{Q}_{Y'} \in DA^{n}_{\text{gm}}(k),\]
where the last isomorphism follows by separation. By relative purity and the projection formula, we deduce that
\[(l/k)_*(\pi_Y)_*\mathbb{Q}_Y \cong (l/k)_*(\pi_Y)_*(\pi_Y \otimes \mathbb{Q}_k(n))[2n] \cong (\pi_{Y'})_*\mathbb{Q}_{Y'} \otimes \mathbb{Q}_k(n)[2n] \cong (\pi_{Y'})_*(\pi_Y)_*\mathbb{Q}_{Y'} .\]
This completes the proof of the lemma.

1.5. Homological vs cohomological motives.

Proposition 1.27. Let $S$ be a scheme and $n \geq 0$. We have
\[DA^{n}_{(c)}(S) = DA_{n,(c)}(S)(-n)\]
In particular, we have $DA^{0}_{(c)}(S) = DA_{0,(c)}(S)$.

Proof. In both directions, it is enough to check the inclusion for a family of compact generators.

Let $f : X \to S$ be a smooth morphism of relative dimension $i \leq n$ (we can reduce to this case by considering connected components of $S$ and $X$). By relative purity, we have
\[f_!\mathbb{Q}_X(-n) \cong f_!\mathbb{Q}_X(i-n)[2i]\]
which is in $DA^n(S)$ by Propositions 1.16 and 1.10.

The other inclusion is true for smooth cohomological $n$-motives by the same relative purity argument. For general compact cohomological $n$-motives (which include the generating family), we argue as follows. By Corollary 1.18 (ii), we can assume $S$ reduced. We then proceed by noetherian induction. Let $M \in DA^n(S)$. The restriction of $M$ to any generic point of $S$ is smooth by Proposition 1.25. There we can apply the smooth case and see that $\eta^*M \in DA_{n,(c)}(S)(-n)$ for any generic point $\eta$ of $S$. Then we apply continuity for compact homological $n$-motives (Proposition 1.22) to find a dense open immersion $j : U \to S$ with $j^*M \in DA_{n,(c)}(U)(-n)$. Applying the induction hypothesis, localisation and the fact that $i_*$ preserves homological $n$-motives for $i$ closed immersion (Proposition 1.17 (ii)) completes the proof.

1.6. Nearby cycles. To conclude this section, we prove a result about the nearby cycles functor and $n$-motives.

Let $R$ be an excellent henselian discrete valuation ring and let $S = \text{Spec}(R)$, $\eta$ be the generic fibre and $\sigma$ be the closed fibre. Fix a separable closure $K_{\text{sep}}$ of $K = \text{Frac}(R)$ and let $\sigma$ be the spectrum of its residue field. Let $B$ be an $S$-scheme and $f : X \to B$ a morphism. There is a tame nearby motive functor $\Psi^\text{mod}_f : DA(X_0) \to DA(X)$ and a nearby motive functor $\Psi_f : DA(X) \to DA(X)$ (see [11, Section 10, Définition 10.14]); the functor $\Psi^\text{mod}_f$ is part of a specialisation system in the sense of [8, Definition 3.1.1]. An important case is when $X = S$ and $f = \pi$ is a uniformizer of $R$.

Lemma 1.28. With the above notations, the functors $\Psi^\text{mod}_f$ and $\Psi_f$ commute with infinite sums.

Proof. The fact that $\Psi^\text{mod}_f$ commutes with infinite sums follows from [8, Lemma 3.2.10], which applies because of the definition of the specialisation system $\Psi^\text{mod}$ as $R \bullet \xi$ with $R$ a certain diagram of schemes and $\xi = r^*j_*$ the canonical specialisation system, which itself commutes with infinite sums [8, Definition 3.5.6]. If the residual characteristic of $R$ is 0, then $\Psi_!$ is simply defined as the composition $(X_{\sigma}/X_{\sigma})^*\Psi^\text{mod}_f$ and we are done. Let us assume that the residual characteristic is $p > 0$. 

Recall that the functor $\Psi_f$ of [11, Définition 10.14] is then constructed from $\Psi^\text{mod}$ via an homotopy colimit along all the finite extensions of $K$ contained in a fixed maximum $p$-primary extension inside $K^\text{sep}$ of the maximal unramified extension $K^\text{nr}$ (such extensions exist by the theorem of Schur-Zassenhaus).

We make this construction slightly more explicit using the algebraic derivator structure of $\text{DA}(\mathcal{E})$, as suggested in [11, Remark 10.15]. If $M_{\mathcal{E}}/K^\text{nr}$ is such a fixed maximum $p$-primary extension, let $\mathcal{L}$ be the poset of all the finite subextensions $K \subset L \subset M_{\mathcal{E}}$ ordered by the reverse of inclusion. Then there is a diagram of schemes $(T_L, \mathcal{L})$ where $T_L$ is the normalisation of $S$ inside $L/K$, along with diagrams $(\eta_L, \mathcal{L})$ and $(\sigma_L, \mathcal{L})$ of generic and special fibres. We have a morphism $\gamma: (T_L, \mathcal{L}) \rightarrow S$ and we pullback the diagram of schemes over $S$ used to compute $\Psi^\text{mod}_f$ along this morphism. We also use the notation $\tilde{L} = L \times \Delta \times (\mathbb{N}')^\times$ where $\Delta$ is the simplicial category and $(\mathbb{N}')^\times = \{ n \in \mathbb{N}' | \text{char}(\sigma) \mid n \}$. Altogether, we get a commutative diagram of diagrams of schemes with cartesian squares:

\[
\begin{array}{ccc}
(R'_L, \tilde{L}) & \xrightarrow{j} & (X_{\eta_L}, \tilde{L}) \\
\downarrow j_{\mathcal{L}} & & \downarrow j_{\mathcal{L}} \\
(R''_L, \tilde{L}) & \xrightarrow{j} & ((G_m)T_L, \tilde{L}) \\
\downarrow j_{\mathcal{L}} & & \downarrow j_{\mathcal{L}} \\
& & (T_L, \tilde{L})
\end{array}
\]

We also have morphisms $\tilde{\pi}: (X_0, \mathcal{E}) \rightarrow (X_{\sigma_L}, \mathcal{E})$, $p_{\mathcal{L}}: (X_0, \mathcal{E}) \rightarrow (X_0, \mathcal{E})$ and $\gamma_{\eta_L}: (X_{\eta_L}, \mathcal{E}) \rightarrow X_\eta$. We can finally define:

\[
\Psi_f := (p_{\mathcal{L}})_* \tilde{\pi}_* (p_{\Delta \times (\mathbb{N})^\times})_! i_* (j_{\mathcal{L}})^*(\theta^R_{\mathcal{L}})^* (\theta^R_{\mathcal{L}})^*(\theta^R_{\mathcal{L}})^* (\theta^R_{\mathcal{L}})^* \gamma_{\eta_L}.
\]

To prove that $\Psi_f$ commutes with infinite sums from this formula, we adapt the proof of [8, Lemma 3.2.10]. The functors $(p_{\mathcal{L}})_* \tilde{\pi}_* (p_{\Delta \times (\mathbb{N})^\times})_! i_* (j_{\mathcal{L}})^*$, $(\theta^R_{\mathcal{L}})^*$, $(\theta^R_{\mathcal{L}})^*$ and $\gamma_{\eta_L}$ are all left adjoints, hence they commute with small sums. To show that $j_*$ and $(\theta^R_{\mathcal{L}})^*$ commute with small sums, we use the fact that the corresponding morphisms of diagrams of schemes are “of geometric type”, i.e. the underlying morphism of small categories is an isomorphism. This implies that the commutuation with small sums can be checked after restriction to a vertex in the corresponding diagram (because $\text{DA}(\mathcal{E})$ is an algebraic derivator, the family of such functors for a given diagram of schemes is compatible), and then we find ordinary pushforward functors in $\text{DA}(\mathcal{E})$ which do commute with small sums. This concludes the proof. □

**Proposition 1.29.** With the notations above, the functor $\Psi^\text{mod}_\pi$ (resp. $\Psi_\pi$):

(i) sends $\text{DA}^{\text{coh}}(\eta)$ to $\text{DA}^{\text{coh}}(\sigma)$ (resp. to $\text{DA}^{\text{coh}}(\hat{\sigma})$),

(ii) sends $\text{DA}^n(\eta)$ to $\text{DA}^n(\sigma)$ (resp. to $\text{DA}^n(\hat{\sigma})$) for any $n \geq 0$,

(iii) sends $\text{DA}_{\text{hom}}(\eta)$ to $\text{DA}_{\text{hom}}(\sigma)$ (resp. to $\text{DA}_{\text{hom}}(\hat{\sigma})$), and

(iv) sends $\text{DA}_n(\eta)$ to $\text{DA}_n(\sigma)$ (resp. to $\text{DA}_n(\hat{\sigma})$).

Similar results hold for the subcategories of compact objects.

**Proof.** By Lemma 1.28, the functors $\Psi^\text{mod}_\pi$ and $\Psi_\pi$ commute with small sums. This shows that we only need to establish the results for compact objects. Using the fact that nearby cycles commute with Verdier duality on constructible objects [11, Théorème 10.20] together with the fact that Verdier duality exchanges $\text{DA}_{\text{c}}^n(k)$ and $\text{DA}_{n,c}(k)$ (Lemma 1.20) allows us to deduce (iii) and (iv) from (i) and (ii).

It remains to show the property (i) (resp. (ii)) for the compact generators of $\text{DA}^{\text{coh}}(\eta)$ of the form $f_* \mathcal{Q}_X$ with $f: X \rightarrow \eta$ proper (resp. for the compact generators of $\text{DA}^n(\eta)$ of the form $f_* \mathcal{Q}_X$ with $f: X \rightarrow \eta$ a proper morphism of relative dimension $\leq n$). Using Mayer-Vietoris for closed covers, we can further assume that $X$ is irreducible. Moreover, since these objects are constructible, the fact that $\Psi_\pi$ is computed as $\Psi^\text{mod}_\pi$ after a large enough finite extension [11, Théorème 10.13, Remarque 10.15] reduces the proof to the case of $\Psi^\text{mod}_\pi$.

Let $f^0: X^0 \rightarrow \eta$ be proper of dimension $\leq n$. We show show that $\Psi^\text{mod}_\pi f^0_* \mathcal{Q}_{X^0}$ is in $\text{DA}^n(\sigma)$ by induction on $n$: this is enough to prove both (i) and (ii). In the case $n = 0$, we reduce by localisation to $X^0 = \eta$ and the result is immediate from $\Psi^\text{mod}_\pi \mathcal{Q}_{\eta} \simeq \mathcal{Q}_{\sigma}$. 
In general, choose a proper flat morphism $f : X \to S$ such that $X^0 = X_\eta$ is irreducible and $f^0 = f^0_\eta$.

The special fibre $X_\sigma$ is also of relative dimension $\leq n$. Because $\Psi^\text{mod}$ is a specialisation system [8, Definition 3.1.1] and $f$ is proper, we have an isomorphism

$$\beta : \Psi^\text{mod}(f_\eta)_* \mathbb{Q}_{X_\eta} \overset{\sim}{\longrightarrow} (f_\eta)_* \Psi^\text{mod} \mathbb{Q}_{X_\eta}.$$ 

To simplify the notation, for any $S$-scheme $g : W \to S$, we write

$$\Psi_W := (g_\sigma)_* \Psi^\text{mod}(W_{\eta})_* \mathbb{Q}_{X_\eta}.$$ 

Now we want to reduce to a situation with a better behaved special fibre. We apply De Jong’s theorem on semi-stable reduction by alterations [29, Theorem 4.5]. There exists an henselian discrete valuation ring $\tilde{S}$ finite over $S$ and a commutative square

$$\hspace{1cm} \tilde{S} \xrightarrow{\pi} S \xrightarrow{\tilde{X}} X \xrightarrow{g} W \xrightarrow{f} S \hspace{1cm}$$

such that

- $\tilde{X}$ is regular and strictly semi-stable over $\tilde{S}$ (in the sense of [29, 2.16]), and
- $p$ is an alteration.

Let $V$ be an open set contained in $\tilde{X}_\eta$ such that $p_V$ is the composition of a finite flat purely inseparable morphism followed by a finite étale morphism, and consider $Z = \tilde{X} \setminus V$ with its reduced scheme structure. We have a commutative diagram (not necessarily cartesian, but with $U := p(V)$ open by flatness and $T := p(Z) = X \setminus U$ by surjectivity)

$$\begin{array}{ccc}
V & \xrightarrow{j} & \tilde{X} & \xleftarrow{i} & Z \\
\downarrow{p_V} & & \downarrow{p} & & \downarrow{p_Z} \\
U & \xrightarrow{j} & X & \xleftarrow{i} & T.
\end{array}$$

We have distinguished triangles in $\textbf{DA}(X_\eta)$,

$$(j_\eta)_* \mathbb{Q}_{U_\eta} \to \mathbb{Q}_{X_\eta} \to (i_\eta)_* \mathbb{Q}_{T_\eta} \overset{\sim}{\rightarrow}$$

and

$$(p_V)_*(j_\eta)_* \mathbb{Q}_{V_\eta} \to (p_\eta)_* \mathbb{Q}_{X_\eta} \to (i_\eta)_* (p_Z)_* \mathbb{Q}_{Z_\eta} \overset{\sim}{\rightarrow}.$$ 

After applying $\Psi^\text{mod}_f$, pushing forward to $\sigma$ (we forget temporarily the $\tilde{S}$-scheme structure of $X'$, $V$ and $Z$) and using properness of $p$ and $i$, we get distinguished triangles

$$(f_\sigma)_* \Psi^\text{mod}_f (j_\eta)_* \mathbb{Q}_{U_\eta} \to \Psi_X \to \Psi_T \overset{\sim}{\rightarrow}$$

and

$$(f_\sigma)_* \Psi^\text{mod}_f (p_V)_* (j_\eta)_* \mathbb{Q}_{V_\eta} \to \Psi_X \to \Psi_Z \overset{\sim}{\rightarrow}$$

in $\textbf{DA}(\sigma)$.

Since $\dim(T_\eta), \dim(Z_\eta) < n$, the motives $\Psi_T$ and $\Psi_Z$ are in $\textbf{DA}^n(\sigma)$ by the induction hypothesis. On the other hand, since $p_V$ is the composition of a finite purely inseparable morphism and a finite étale morphism, the separation property of $\textbf{DA}(\cdot)$ together with [7, Lemme 2.1.165] imply that $(j_\eta)_* \mathbb{Q}_{U_\eta}$ is a direct factor of $(p_V)_*(j_\eta)_* \mathbb{Q}_{V_\eta}$. It is thus enough to prove that $\Psi_X$ is in $\textbf{DA}^n(\sigma)$.

We will prove that $(g_\sigma)_* \Psi^\text{mod}_{g_*} \mathbb{Q}_{X_\eta} \in \textbf{DA}^n(\tilde{S})$. Since $\pi : \tilde{S} \to S$ is finite, this will imply the same for $\Psi_X$ by Proposition 1.16 (ii).

Write $\tilde{X}_\eta = \bigcup_{k \in I} D_k$ as a union of its irreducible components. For each $I \subset [1, m]$ write $D_I = \bigcap_{k \in I} D_k$ for the scheme-theoretic intersection. For $I \subset J$, write $(i^I_J) : D_J \to D_I$ for the corresponding closed immersion. For any $I$, write

$$D^j_I := D_I \setminus \bigcup_{i \notin I} D_i.$$
and $j_I : D^n_I \to D_I$ for the corresponding open immersion. By a Mayer-Vietoris argument [7, Lemme 2.2.31], it is enough to prove that for any $I \neq \emptyset$, we have

$$(f_\sigma)_*(ii)_*i_I^!\Psi^\text{mod}_g Q_{X^n} \in \text{DA}^n(\sigma).$$

Let $I$ be such an index set, and let $k \in I$. By absolute purity for the regular codimension 1 closed immersion $i_k$ and the "localisation" formula for specialisation systems in [8, Theorem 3.3.43] (which applies to the $\mathbb{Q}$-linear separated homotopical 2-functor $\text{DA}^d(\cdot, \mathbb{Q})$), we have

$$i_I^!\Psi_g Q_{X^n} \simeq (i_I^!)^i \Psi_g Q_{X^n} \simeq (i_I^!)^i \Psi_g Q_{X^n} (1)[2]$$

Because $D_k$ is smooth over $\sigma$ by semi-stability, we have $(j_k)^*(i_k)^! \Psi_g Q_{X^n} \simeq (i_k \circ j_k)^! \Psi_g Q_{X^n} \simeq (i_k \circ j_k)^! \Psi_g Q_{X^n} \simeq Q_{D_k}$ by smooth base change for specialisation systems [8, Definition 3.1.1] and [11, Theorem 10.6] (with $e = 1$). So we are reduced to computing $(i_I^!)^i (j_k)^*(j_k)^! \Psi_g Q_{X^n} (1)[2]$.

This completes the proof. \hfill \Box

2. Commutative group schemes and motives

Several motives of interest for this paper are obtained from group schemes or complexes of group schemes. The main examples we are interested in are smooth commutative group schemes (Section 2.1), Deligne 1-motives (Section 2.2), and the smooth Picard complex (Section 2.3).

2.1. Motives of commutative group schemes. In this section, we introduce the relevant definitions and reformulate results from [6] and [53] in this language. For the rest of the section, fix a noetherian finite-dimensional scheme $S$.

In [6, Thm D.1], we constructed a functorial cofibrant resolution of the sheaf $G \otimes \mathbb{Q}$ for $G$ a smooth (locally of finite type) commutative group scheme over $S$. Let us recall the statement.

Lemma 2.1. [6, Thm D.1] Let $(S, \tau)$ be a Grothendieck site. We denote $\mathbb{Z}(\_)$ the "free abelian group sheaf" functor (the sheafication of the sectionwise free abelian group functor).

There is a functor:

$$A : \text{Sh}_\tau(S, \mathbb{Z}) \to \text{Cpl}_{\geq 0} \text{Sh}_\tau(S, \mathbb{Z})$$

together with a natural transformation

$$r : A \to (-)[0]$$

satisfying the following properties.

1. For any $G \in \text{Sh}_\tau(S, \mathbb{Z})$ and $i \geq 0$, the sheaf $A(G)$, is of the form $\bigoplus_{j \in \mathbb{N}} \mathbb{Z}(G^d(j.i))$ for some $d(i), a(i, j) \in \mathbb{N}$.

2. There is a natural transformation $\tilde{a} : \mathbb{Z}(-)[0] \to A$ which lifts the addition map $a : \mathbb{Z}(-) \to \text{id}$; that is, one has $a[0] = \tilde{a}a$.

3. The functor $A$ and the transformations $r$ and $\tilde{a}$ are compatible with pullbacks by morphisms of sites.

4. The map $r \otimes \mathbb{Q}$ is a quasi-isomorphism.

Let us make more explicit the statement in 3. Recall that we use underlines to denote underived functors between categories of complexes. For a morphism of sites $F : S' \to S$, and $G$ as in the theorem, we assert that there exists an isomorphism of complexes $b_{F,G} : \bigoplus_{i=-1}^1 A(G) \to A(F^{-1}(G))$.
which is termwise compatible with the standard isomorphisms \( E^{-1}Z(G^{a(i,j)}) \simeq Z(E^{-1}G^{a(i,j)}) \) and which makes the diagram

\[
\begin{array}{ccc}
E^{-1}(A(G)) & \xrightarrow{E^{-1}(r(G))} & E^{-1}G \\
b_{FG} & \sim & r(E^{-1}G) \\
A(E^{-1}G) & \xrightarrow{g} & 
\end{array}
\]

commute.

**Proposition 2.2.** Let \( K_* \) be a bounded complex of smooth commutative group schemes over \( S \) and \( f : T \rightarrow S \) a morphism of schemes. We have an isomorphism

\[
R_f : f^{-1}(K_* \otimes \mathbb{Q}) \xrightarrow{\sim} \underline{f}^{-1}(K_* \otimes \mathbb{Q})
\]

in \( D(\text{Sm}/S) \) which is natural in \( K_* \). Moreover, \( R_f \) is compatible with further pullbacks: for \( g : U \rightarrow T \), the diagram

\[
\begin{array}{ccc}
g^{-1}f^{-1}(K_* \otimes \mathbb{Q}) & \xrightarrow{R_f} & (fg)^{-1}(K_* \otimes \mathbb{Q}) \\
g^{-1} \xrightarrow{\sim} & R_g & \xrightarrow{\sim} \\
g^{-1} \underline{f}^{-1}(K_* \otimes \mathbb{Q}) & \xrightarrow{R_g} & g^{-1} \underline{f}^{-1}(K_* \otimes \mathbb{Q})
\end{array}
\]

commutes.

**Proof.** We apply Lemma 2.1 to the individual sheaves \( K_n \), and use the functoriality of the construction. This yields a double complex \( A(K_n) \) together with a map \( r : A(K_n) \rightarrow K_n \). We then tensor by \( \mathbb{Q} \) and take the total complex along the second index. This yields a complex \( B_Q(K_n) \) of sheaves of \( \mathbb{Q} \)-vector spaces on \( (\text{Sm}/S)_{\text{et}} \) together with a map \( r_Q(K_n) : B_Q(K_n) \rightarrow K_n \) with the following properties.

(i) For all \( i \in \mathbb{Z} \), the sheaf \( B_Q(K_i) \) is of the form \( Q(H_i) \) for some smooth commutative group scheme \( H_i \) over \( S \) (a fibre product of various copies of the \( K_n \)'s); therefore, \( B_Q(K_n) \) is a projective object in \( \text{Cpl}(\text{Sh}_{\text{et}}(\text{Sm}/S, \mathbb{Q})) \).

(ii) The map \( r_Q(K_n) \) is a quasi-isomorphism, hence a projective resolution of \( K_n \otimes \mathbb{Q} \).

(iii) The formation of \( B_Q(K_n) \) and \( r_Q(K_n) \) is compatible with (underived) pullback, in the sense that, for any morphism \( f : T \rightarrow S \), there exists an isomorphism of complexes \( b_{f,K_n} : \underline{f}^{-1}(B_Q(K_n)) \rightarrow B_Q(\underline{f}^{-1}K_n) \) which makes the following diagram in \( \text{Cpl}(\text{Sh}_{\text{et}}(\text{Sm}/T, \mathbb{Q})) \)

\[
\begin{array}{ccc}
\underline{f}^{-1}(B_Q(K_n)) & \xrightarrow{\underline{f}^{-1}(r_Q(K_n))} & \underline{f}^{-1}(K_n \otimes \mathbb{Q}) \\
b_{f,K_n} & \sim & r_Q(\underline{f}^{-1}K_n) \\
B_Q(\underline{f}^{-1}(K_n)) & \xrightarrow{r_Q(\underline{f}^{-1}K_n)} & 
\end{array}
\]

commute.

As \( r_Q \) is a projective resolution, we have an isomorphism in \( D(\text{Sm}/S) \) given by

\[
f^{-1}(K_n \otimes \mathbb{Q}) \xrightarrow{\sim} \underline{f}^{-1}(B_Q(K_n)) \simeq \underline{f}^{-1}(B_Q(K_n)).
\]

We define \( R_f \) as the composition

\[
f^{-1}(K_n \otimes \mathbb{Q}) \xrightarrow{\sim} \underline{f}^{-1}(B_Q(K_n)) \xrightarrow{\underline{f}^{-1}(r_Q(K_n))} \underline{f}^{-1}(K_n \otimes \mathbb{Q}).
\]

It remains to check the compatibility with further pullbacks. Let \( g : U \rightarrow T \) be a morphism of schemes. The reader is invited to contemplate the following diagram in \( D(\text{Sm}/S) \) (where the unlabelled maps are either cocycle isomorphisms for the pullbacks - derived and not - or isomorphisms
of the form $h^{-1}(C) \simeq \mathbb{A}^{-1}(C)$ for $C$ projective).

$$g^{-1}f^{-1}(K_\ast \otimes \mathbb{Q}) \xrightarrow{\sim} g^{-1}f^{-1}B_\mathbb{Q}(K_\ast) \xrightarrow{\sim} g^{-1}f^{-1}B_\mathbb{Q}(K_\ast) \xrightarrow{\sim} g^{-1}f^{-1}B_\mathbb{Q}(f^{-1}(K_\ast \otimes \mathbb{Q}))$$

The quadrangles $(A)$ and $(F)$ commute because of the naturality of the cocycle isomorphisms for pullbacks. The triangle $(B)$ and the quadrangle $(E)$ commute trivially. The triangles $(C)$ and $(G)$ commute because of property (iii) above. Finally, the quadrangle $(D)$ commutes because the cocycle isomorphisms for derived and underived pullbacks are compatible.

**Corollary 2.3.** Let $K_\ast$ be a bounded complex of smooth commutative group schemes over $S$ and $f : T \to S$ be a morphism of schemes. We have natural isomorphisms

$$R_f : f^\ast K_\ast \otimes \mathbb{Q} \xrightarrow{\sim} f^\ast(K_\ast \otimes \mathbb{Q})$$

in $DA^{\text{eff}}(S)$ and

$$R_f : f^\ast \Sigma^\infty(K_\ast \otimes \mathbb{Q}) \xrightarrow{\sim} \Sigma^\infty f^\ast(K_\ast \otimes \mathbb{Q})$$

in $DA(S)$. These isomorphisms are compatible with further pullbacks in the same way as in the previous proposition.

**Proof.** The first isomorphism follows directly from Proposition 2.2. The second follows from the first together with the commutation of $f^\ast$ and $\Sigma^\infty$. $\square$

For some arguments, we need to use motives with transfers of commutative group schemes. Let $S$ be an excellent scheme and $G$ a smooth (locally of finite type) commutative group scheme over $S$. Recall that the \v{e}tale sheaf $G \otimes \mathbb{Q}$ on $\text{Sm}/S$ admits a canonical structure of sheaf with transfers, which is functorial in $G$. We write $G^\text{tr}_\mathbb{Q}$ for the resulting sheaf with transfers. Recall that there are adjunctions

$$\alpha_{\text{tr}} : DA^{\text{eff}}(S) \rightleftarrows DM^{\text{eff}}(S) : \delta^{\text{tr}}$$

which relate motives with and without transfers. By construction, we have $\delta^{\text{tr}}G^\text{eff}_\mathbb{Q} = G^\text{tr}_\mathbb{Q}$, and $\delta^{\text{tr}}$ preserves $\mathbb{A}^1$-equivalences [12, Lemme 2.111].

**Proposition 2.4.** [6, Proposition 2.10] Let $S$ be an excellent scheme and $G$ a smooth commutative group scheme over $S$. Then the counit morphisms

$$\alpha_{\text{tr}}\delta^{\text{tr}}M \xrightarrow{\sim} G^\text{tr}_\mathbb{Q}$$

in $DM^{\text{eff}}(S)$ and

$$\alpha_{\text{tr}}\delta^{\text{tr}}\Sigma^\infty G^\text{tr}_\mathbb{Q} \xrightarrow{\sim} \Sigma^\infty G^\text{tr}_\mathbb{Q}$$

in $DM(S)$ are isomorphisms.

An important consequence for us is the following computation, which consists of translating a classical result of Voevodsky to our context, and which we will generalize later on.
Proposition 2.5. Let $k$ be a field and $C/k$ be a smooth projective geometrically connected curve. There exists a direct sum decomposition

$$M(C) \simeq \mathbb{Q} \oplus \Sigma^\infty \text{Jac}(C)_\mathbb{Q} \oplus \mathbb{Q}(1)[2]$$

in $\text{DA}(k)$.

Proof. We first assume that $k$ is perfect. For a smooth projective connected curve $C$ over $k$ with a rational point, Voevodsky has computed the motive $M_{\text{tr}}(C) \in \text{DM}^{\text{eff}}(k)$ (see e.g. [16, Proposition 2.5.5]) and shown that

$$M_{\text{tr}}(C) \simeq \mathbb{Q} \oplus (\text{Jac}(C)_\mathbb{Q}) \oplus \mathbb{Q}(1)[2].$$

The role of the rational point in this argument can be played by a 0-cycle of degree 1 as long as $C$ is geometrically connected; such a cycle exists with rational coefficients on any geometrically connected smooth projective curve. By Proposition 2.4 and the remarks preceding it, we have

$$J(C)^{\text{tr}} \simeq a^{\text{tr}} J(C)^{\text{eff}} \simeq a^{\text{tr}} J(C) \simeq a^{\text{tr}} J(C).$$

Applying $\Sigma^\infty$ and using that $a^{\text{tr}}$ commutes with suspension, we get an isomorphism

$$M_{\text{tr}}(C) \simeq \mathbb{Q} \oplus a^{\text{tr}} \Sigma^\infty (\text{Jac}(C)_\mathbb{Q}) \oplus \mathbb{Q}(1)[2]$$

in $\text{DM}(k)$. The adjunction $a^{\text{tr}} : \text{DA}(k) \rightleftarrows \text{DM}(k) : a^{\text{tr}}$ is an equivalence of categories by [25, Corollary 16.2.22]. This implies that $a^{\text{tr}} M_{\text{tr}}(C) \simeq a^{\text{tr}} a^{\text{tr}} M(C) \simeq M(C)$ and similarly $a^{\text{tr}} \mathbb{Q} \simeq \mathbb{Q}$ and $a^{\text{tr}} \mathbb{Q}(1)[2] \simeq \mathbb{Q}(1)[2]$. Applying $a^{\text{tr}}$ to the isomorphism above, we thus get an isomorphism

$$M(C) \simeq \mathbb{Q} \oplus \Sigma^\infty (\text{Jac}(C)_\mathbb{Q}) \oplus \mathbb{Q}(1)[2]$$

as required.

Let $k$ be an arbitrary field, and $k^{\text{perf}}$ a perfect closure of $k$. Write $\xi : \text{Spec}(k^{\text{perf}}) \rightarrow \text{Spec}(k)$. The field extension $k^{\text{perf}}/k$ is a filtered union of finite purely inseparable field extensions. By the separation and continuity properties of $\text{DA}(\_)$, the pullback functor $\xi^* : \text{DA}_c(k) \rightarrow \text{DA}_c(k^{\text{perf}})$ is an equivalence of categories with inverse $\xi_*$. We thus have

$$M(C) \simeq \xi_* \xi^* M(C)$$

$$\simeq \xi_* M(C_{k^{\text{perf}}})$$

$$\simeq \xi_* \xi^* \mathbb{Q} \oplus \xi_* \xi^* \Sigma^\infty \text{Jac}(C_{k^{\text{perf}}})_\mathbb{Q} \oplus \xi_* \xi^* \mathbb{Q}(1)[2]$$

$$\simeq \xi_* \xi^* \mathbb{Q} \oplus \xi_* \xi^* \Sigma^\infty \text{Jac}(C)_\mathbb{Q} \oplus \xi_* \xi^* \mathbb{Q}(1)[2]$$

$$\simeq \mathbb{Q} \oplus \Sigma^\infty \text{Jac}(C)_\mathbb{Q} \oplus \mathbb{Q}(1)[2]$$

where the second isomorphism follows from the perfect field case and the third isomorphism uses the base change property of the Jacobian of a curve and Proposition 2.2. □

We also need an alternative description of the motive $\Sigma^\infty (\mathbb{G}_m \otimes \mathbb{Q})$ (a relative, rational version of the standard description of the motivic complex $\mathbb{Z}(1)$).

Proposition 2.6. There is a canonical isomorphism

$$u_S : \Sigma^\infty (\mathbb{G}_m \otimes \mathbb{Q}) \sim \mathbb{Q}_S(1)[1]$$

in $\text{DA}(S)$. The isomorphism $u_S$ is compatible with pullbacks and the isomorphisms $R_f$ of Corollary 2.3: for $f : T \rightarrow S$, the diagram

$$\begin{array}{ccc}
    f^* \Sigma^\infty (\mathbb{G}_{m,S} \otimes \mathbb{Q}) & \xrightarrow{R_f} & \Sigma^\infty (\mathbb{G}_{m,T} \otimes \mathbb{Q}) \\
    u_S & \sim & u_T \\
    f^*(\mathbb{Q}_S(1)[1]) & \xrightarrow{\sim} & \mathbb{Q}_T(1)[1]
\end{array}$$

commutes.

Proof. By Theorem [6, Theorem 3.3] in the special case $G = \mathbb{G}_m$ (with the “Kimura dimension” $\text{kd}(\mathbb{G}_m/S)$ of the statement equal to 1), there is an isomorphism

$$\Psi := \Psi_{\mathbb{G}_m/S} : M_S(\mathbb{G}_m) \simeq \mathbb{Q} \oplus \Sigma^\infty (\mathbb{G}_m \otimes \mathbb{Q}).$$

It is compatible with pullbacks and the isomorphisms $R_f$ of Corollary 2.3 (This is the precise meaning of “compatible with pullbacks” in loc.cit). By definition, $\mathbb{Q}_S(1)[1]$ is the reduced motive
of $M_S(G_m)$ pointed at the unit section of $G_m$, and it follows from the naturality of $\Psi_{G/S}$ applied to the neutral section in $G$ that the direct factor $Q_S(1)[1] \cong \Sigma^\infty(G_m \otimes Q)$. This yields an isomorphism $\tilde{\Psi} : Q_S(1)[1] \cong \Sigma^\infty(G_m \otimes Q)$, and we put $u_S := \Psi^{-1}$.

**Remark 2.7.** Various results and constructions in this paper could be simplified if we knew the effective analogue of Proposition 2.3, i.e., that the natural map $Q(1) \to G_m[-1] \otimes Q$ in $DA^\text{eff}(S)$ is an isomorphism. The corresponding statement in $DM^\text{eff}(S)$ is known if $S$ is normal [25, Proposition 11.2.11], hence in $DA^\text{eff}(S)$ for $S$ normal scheme of finite type over a field of characteristic 0 by [12, Théorème B.1].

We also need a version with transfers of this statement.

**Corollary 2.8.** Let $S$ be an excellent scheme. There is a canonical isomorphism

$$ u^S_S : \Sigma^\infty(G_m^\text{tr} \otimes Q) \cong Q_S(1)[1]. $$

It is compatible with pullbacks in the same way as in Proposition 2.3. Modulo the isomorphism of Proposition 2.4, we have in fact

$$ a_{tr}u_S = u^S_S. $$

**Proof.** For our purposes, it is enough to define $u^S_S$ as $a_{tr}u_S$ modulo the isomorphism of Proposition 2.4. The claim then follow from Proposition 2.3.

**Corollary 2.9.** Let $T/S$ be a torus, and $X_s(T)$ its cocharacter lattice. There is an isomorphism

$$ \Sigma^\infty(T_\mathbb{Q}) \cong \Sigma^\infty X_s(T)(1)[1]. $$

In particular, if $S$ is geometrically unibranch, the motive $\Sigma^\infty(T_\mathbb{Q})$ is in $DA^\text{gsm}_{1,c}(S)$.

**Proof.** In this proof, we distinguish between derived and underived tensor products for clarity.

There is a natural morphism $X_s(T) \otimes G_m \to T$ of étale sheaves on $S^\text{m}/S$, which is an isomorphism (this can be checked étale locally, hence for a split torus, where it is obvious). Since the functor $\Sigma^\infty$ is monoidal, we have $\Sigma^\infty(X_s(T) \otimes (G_m \otimes Q)) \cong \Sigma^\infty(X_s(T) \otimes (G_m \otimes Q)) \cong \Sigma^\infty X_s(T)(1)[1]$ (by Proposition 2.3). It remains to check that the tensor product $X_s(T) \otimes G_m$ coincides with the derived tensor product; this follows from the fact that the lattice $X_s(T)$ is étale locally free, thus flat.

If $S$ is geometrically unibranch, $X_s(T)_\mathbb{Q}$ is a direct factor of the sheaf $\mathbb{Q}(V)$ for $V/S$ finite étale by Lemma A.2, so it is geometrically smooth.

**Remark 2.10.** For more precise (integral) results on motives attached to tori over a field, see [42, §7].

We can now give a result which is our main source of compact homological 1-motives.

**Proposition 2.11.** Let $G$ be a smooth (not necessarily of finite type) commutative group scheme over $S$. Then $\Sigma^\infty G_\mathbb{Q}$ lies in $DA^\text{gsm}_{1,c}(S)$.

**Proof.** Write $M = \Sigma^\infty G_\mathbb{Q}$. By [6, Theorem 3.3.(3)], $M$ is a compact motive. It remains to show that $M$ is an homological 1-motive. The proof of [6, Theorem 3.3.(3)] essentially establishes this as well, but we provide an argument for convenience. By compactness and Proposition 1.24, it is enough to show that for all $s \in S$, the motive $s^*M$ is in $DA^\text{gsm}_{1}(s)$.

Let $\kappa(s)^{\text{perf}}$ a perfect closure of $\kappa(s)$, and write $\xi : \text{Spec}(\kappa(s)^{\text{perf}}) \to \text{Spec}(\kappa(s))$. The field extension $\kappa(s)^{\text{perf}}/\kappa(s)$ is a filtered union of finite purely inseparable field extensions. By continuity for $DA^\text{gsm}_{1,c}(-)$ and Corollary 1.18 (ii), it is enough to show that $\xi^*s^*M \in DA^\text{gsm}_{1}(\kappa(s)^{\text{perf}})$. By Proposition 2.2, we have $\xi^*s^*M \cong \Sigma^\infty(G_\kappa(s)^{\text{perf}})_\mathbb{Q}$. We are thus reduced to the case where $S$ is the spectrum of a perfect field $k$.

The group scheme $G$ over the field $k$ has a neutral component $G^0$ which is smooth and of finite type. The quotient group scheme $G/G^0$ is a discrete group scheme, hence can be written as a filtered colimit of discrete group schemes, i.e., étale locally constant and of finite type as an abelian group. Since we are working with rational coefficients, we can assume that those discrete group schemes are in fact lattices. Using Lemma A.2, we then conclude that the motive $\Sigma^\infty(G/G^0)_\mathbb{Q}$ lies in $DA_{g_{\mathbb{Q}}}(k) \subset DA_{1}(k)$. In the case of a smooth commutative connected algebraic group, we reduce by a standard dévissage to the cases of unipotent algebraic groups, tori and abelian varieties.
A unipotent algebraic group over a perfect field has a composition series with $G_m$ factors, and the motive $\Sigma^\infty G_m \otimes \mathbb{Q}$ is trivial by [5, Lemma 7.4.5] (proved in $\text{DM}^{\text{eff}}(k)$), which yields the result in $\text{DA}(k)$ by applying $\Sigma^\infty_{\text{oT}}$. If $G = T$ is a torus, let $e : \text{Spec}(l) \to \text{Spec}(k)$ be a finite étale morphism with $T_l$ split. Then $e^* \Sigma^\infty(T \otimes \mathbb{Q}) \simeq \Sigma^\infty(T_l \otimes \mathbb{Q})$. By a transfer argument using [7, Lemma 2.1.165] and Proposition 1.13 (ii), this reduces us to the case of split tori, and then by direct sum to the case of $G_m$, which follows from Proposition 2.3. If $G$ is an abelian variety, using [45, Theorem 11] we reduce us to the case of a Jacobian $J(C)$ of a smooth projective curve $C/k$ with a rational point. The fact that $\Sigma^\infty(J(C) \otimes \mathbb{Q})$ is in $\text{DA}_1(k)$ then follows from Proposition 2.5. \hfill $\square$

We now lay the groundwork for the study of the motivic Picard functor in Section 3.3. Let $n \in \mathbb{N}$. Recall that the adjunction “suspension-evaluation” at the level of spectra induces derived adjunctions

\[ \text{Sus}^n : \text{DA}^{\text{eff}}(S) \rightleftarrows \text{DA}(S) : \text{Ev}_n \]

with $\text{Sus}^0 = \Sigma^\infty$ and, for $M \in \text{DA}^{\text{eff}}(S)$ and $N \in \text{DA}(S)$, canonical isomorphisms

\[ \text{Sus}^n(M) \simeq \Sigma^\infty M(-n)[-2n] \in \text{DA}(S), \]

\[ \text{Ev}_n(N) \simeq \text{Ev}_0(M(n)[2n]). \]

Using the map $u_S : \Sigma^\infty(G_m \otimes \mathbb{Q}) \to \mathbb{Q}_S(1)[1]$, we get a map

\[ \text{Sus}^1(G_m \otimes \mathbb{Q}[1]) \to \mathbb{Q}_S \]

which by adjunction corresponds to a map

\[ u_S : G_m \otimes \mathbb{Q}[1] \to \text{Ev}_1(\mathbb{Q}_S). \]

Over an excellent scheme $S$, there is an analogous construction for motives with transfers (using the map $u^S$ instead of $u_S$), resulting in a map

\[ u^S : G_m^{\text{tr}} \otimes \mathbb{Q}[1] \to \text{Ev}_1^t(\mathbb{Q}_S^{\text{tr}}) \]

in $\text{DM}^{\text{eff}}(S)$.

Let $f : X \to S$ be a morphism of schemes. To state the compatibility of $u_S$ with base change, we introduce the composition

\[ d_f : f^* \text{Ev}_1^t \mathbb{Q}_S \longrightarrow f^* \text{Ev}_1 \text{Sus}_1 f^* \text{Ev}_1 \mathbb{Q}_S \simeq \text{Ev}_1 f^* \text{Sus}_1 \text{Ev}_1 \mathbb{Q}_S \longrightarrow \text{Ev}_1 f^* \mathbb{Q}_S \simeq \text{Ev}_1 f^* \mathbb{Q}_X \]

where the isomorphism in the middle is the canonical isomorphism $\text{Sus}_1 f^* \simeq f^* \text{Sus}_1$.

**Lemma 2.12.** Let $S$ be a noetherian finite-dimensional scheme. If $f : X \to S$ is any morphism of finite type, the following diagram

\[
\begin{array}{ccc}
  f^*(G_m \otimes \mathbb{Q}[1]) & \xrightarrow{R_f} & G_m \otimes \mathbb{Q}[1] \\
  f^*w_S \sim \downarrow & & \downarrow w_X \\
  f^* \text{Ev}_1 \mathbb{Q}_S & \xrightarrow{d_f} & \text{Ev}_1 \mathbb{Q}_X
\end{array}
\]

commutes.

**Proof.** Going through the definitions of $w_S$ and $d_f$, we see that the diagram in (i) is obtained from the commutative diagram of Proposition 2.3 via the adjunction $\text{Sus}_1 \dashv \text{Ev}_1$ and the commutation of $\text{Sus}_1$ and $f^*$. \hfill $\square$

The following result is not used in the rest of the paper, but seems of independent interest.

**Proposition 2.13.**

(i) Assume $S$ is regular. Then the morphism $w_S$ is an isomorphism.

(ii) If $f : X \to S$ is a morphism of finite type with $X$ and $S$ regular, then $d_f$ is an isomorphism.

**Proof.** Statement (ii) follows from the combination of (i) and Lemma 2.12, so we are left with proving (i).

Since $\text{DA}^{\text{eff}}(S)$ is generated as a triangulated category by objects of the form $M^\text{eff}_S(X)[n]$ for $f : X \to S$ smooth morphism and $n \in \mathbb{Z}$, it is enough to show that for such an object, the induced map

\[ \text{DA}^{\text{eff}}(S)(M^\text{eff}_S(X)[n], G_m \otimes \mathbb{Q}[1]) \xrightarrow{w_S} \text{DA}^{\text{eff}}(S)(M^\text{eff}_S(X)[n], \text{Ev}_1(\mathbb{Q}_S)) \]
is an isomorphism. The idea is to compare both sides to similar morphism groups in the derived category $D(\text{Sm}/S)$. Consider the following diagram.

$$
\begin{array}{cccc}
D(\text{Sm}/S)(\mathbb{Q}_S(X)[n], \mathbb{G}_m[1]) & \overset{(\alpha)}{\longrightarrow} & \mathcal{D}^\text{eff}(S)(M^\text{eff}_S(X)[n], \mathbb{G}_m[1]) & \overset{\sim}{\longrightarrow} & \mathcal{D}^\text{eff}(S)(M^\text{eff}_S(X)[n], \text{Ev}_1(\mathbb{Q}_S)) \\
\sim & \overset{\text{adj}}{\longrightarrow} & \mathcal{D}(S)(M_S(X)[n], \Sigma^\infty(\mathbb{G}_m)[1]) & \overset{\sim}{\longrightarrow} & \mathcal{D}(S)(M_S(X)[n], \mathbb{Q}_S(1)[2]) \\
D(\text{Sm}/X)(\mathbb{Q}_X[n], f^*\mathbb{G}_m[1]) & \overset{\sim}{\longrightarrow} & \mathcal{D}(X)(\mathbb{Q}_X[n], f^*\Sigma^\infty\mathbb{G}_m[1]) & \overset{(\beta)}{\longrightarrow} & \mathcal{D}(X)(\mathbb{Q}_X[n], \mathbb{Q}_X(1)[2]) \\
\sim & \overset{R_f}{\longrightarrow} & \mathcal{D}(X)(\mathbb{Q}_X[n], \Sigma^\infty\mathbb{G}_m[1]) & \overset{\sim}{\longrightarrow} & \mathcal{D}(X)(\mathbb{Q}_X[n], \mathbb{Q}_X(1)[2]) \\
\end{array}
$$

The square $(A)$ commutes because the isomorphisms $R_f$ in the derived category and in $\mathcal{D}A$ are compatible by construction. The square $(B)$ commutes by construction of $w_S$ and $w_S$. The square $(C)$ commutes by naturality of the adjunction. Finally, the square $(D)$ commutes by Proposition 2.3.

To complete the proof that $w_{S*}$ is an isomorphism, it remains to see that the maps $(\alpha)$ and $(\beta)$ are isomorphisms as well. For $(\beta)$, this is precisely the statement of Proposition B.6 (ii)-(iv). Let us prove that $(\alpha)$ is an isomorphism.

Since $S$ is regular, all smooth $S$-schemes are regular. They are in particular reduced, which implies that $\mathbb{G}_m$ is $\mathbb{A}^1$-invariant on $\text{Sm}/S$, and normal, which implies that $\text{Pic} = H^1(-, \mathbb{G}_m)$ is $\mathbb{A}^1$-invariant. The higher cohomology groups $H^i(-, \mathbb{G}_m)$ for $i \geq 2$ are torsion on regular schemes by [40, Proposition 1.4]. All together, this shows that the sheaf $\mathbb{G}_m \otimes \mathbb{Q}$ is $\mathbb{A}^1$-local in the model category underlying $\mathcal{D}^\text{eff}(S)$. We deduce that the morphism $(\alpha) : D(\text{Sm}/S)(\mathbb{Q}_S(X)[n], \mathbb{G}_m \otimes \mathbb{Q}[1]) \rightarrow \mathcal{D}^\text{eff}(M^\text{eff}_S(X), \mathbb{G}_m \otimes \mathbb{Q})$ is an isomorphism. This completes the proof that $w_S$ is an isomorphism.

2.2. Motives of Deligne 1-motives. We relate the category $\mathcal{M}_1(S)$ of Deligne 1-motives with rational coefficients (Appendix A) to $\mathcal{D}A(S)$. Let $\mathcal{M} = [L \rightarrow G] \otimes \mathbb{Q}$ be in $\mathcal{M}_1(S)$. Then by viewing $\mathcal{M}$ as a complex of étale sheaves of $\mathbb{Q}$-vector spaces on $\text{Sm}/S$, we can associate to $\mathcal{M}$ an object in $\mathcal{D}^\text{eff}(S)$, which we also denote by $\mathcal{M}$.

**Corollary 2.14.** Let $\mathcal{M} \in \mathcal{M}_1(S)$. Then $\Sigma^\infty\mathcal{M}$ lies in $\mathcal{D}A^1_{\text{c}}(S)$. If $S$ is moreover assumed to be geometrically unibranch, then the motive $\Sigma^\infty\mathcal{M}$ is also geometrically smooth, thus lies in $\mathcal{D}A^1_{\text{c,sm}}(S)$.

**Proof.** Let $\mathcal{M} = [L \rightarrow G] \otimes \mathbb{Q}$. We apply Proposition 2.11 to the distinguished triangle

$$
\Sigma^\infty\mathcal{G}_Q[-1] \rightarrow \Sigma^\infty\mathcal{M} \rightarrow \Sigma^\infty\mathcal{L}_Q \rightarrow
$$

which proves the first part. Assume now that $S$ is geometrically unibranch. We have a further distinguished triangle

$$
\Sigma^\infty\mathcal{T}_Q \rightarrow \Sigma^\infty\mathcal{G}_Q \rightarrow \Sigma^\infty\mathcal{A}_Q \rightarrow
$$

The motives $\Sigma^\infty\mathcal{T}_Q$ and $\Sigma^\infty\mathcal{L}_Q$ are geometrically smooth by Corollary 2.9 and its proof. The motive $\Sigma^\infty\mathcal{A}_Q$ is a direct factor of the motive of $\mathcal{A}$ by Theorem [6, Theorem 3.3], so it is geometrically smooth. This completes the proof.

From Corollary 2.3 and the definition of $\Sigma^\infty$, we deduce the following.

**Corollary 2.15.** Let $f : T \rightarrow S$ be a morphism of schemes. There is an isomorphism of functors

$$
R_f : f^*\Sigma^\infty \sim \rightarrow \Sigma^\infty f^{-1} : \mathcal{M}_1(S) \rightarrow \mathcal{D}A(T),
$$

which is compatible with further pullbacks.
As explained in Section A.3, we have also a covariant functoriality for finite étale morphisms, coming from Weil restrictions of scalars. Here is how this relates to pushforwards of motives.

Lemma 2.16. Let \( f : T \to S \) be a finite étale morphism of schemes. There is an isomorphism of functors

\[
\Sigma_\infty^\infty \overset{\sim}{\to} f_* \Sigma^\infty_\infty \colon \mathcal{M}_1(S) \to \mathcal{D}A(T)
\]

Proof. Because of the definition of pushforwards in \( \mathcal{M}_1(-) \) (Definition A.17), it is enough to show the following: for \( G/T \) smooth (not necessarily of finite type) commutative group scheme, there is an isomorphism \( f_* \Sigma_\infty^\infty G_\mathbb{Q} \cong \Sigma^\infty_\infty (\text{Res}_T G)_\mathbb{Q} \), functorial in \( G \) (note that we do not claim that the sheaf \( \text{Res}_T G \) is representable in this generality). We have a sequence of functorial isomorphisms

\[
\begin{align*}
\Sigma^\infty_\infty f_* G_\mathbb{Q} & \cong \Sigma^\infty_\infty f_* G_\mathbb{Q} \\
& \cong \Sigma^\infty_\infty f_* G_\mathbb{Q} \\
& \cong \Sigma^\infty_\infty f_* G_\mathbb{Q} \\
& \cong \Sigma^\infty_\infty (\text{Res}_T G)_\mathbb{Q}
\end{align*}
\]

where the first and second isomorphisms follow from the fact that \( f \) is finite étale, the second comes from the commutation between \( \Sigma^\infty \) and \( f_* \), the fourth follows from the fact that \( f_* \) in \( \mathcal{D}A^{\text{eff}}(-) \) preserves \((\mathbb{A}^1, \text{ét})\)-equivalences for \( f \) finite (an argument can be found in Part A of the proof of [12, Lemme B.7]), and the last is the definition of Weil restriction. This completes the proof. \( \square \)

2.3. Picard complex. Recall the following classical fact, which follows from the fact that sheaf cohomology in a coherent topos commutes with filtered colimits of sheaves.

Lemma 2.17. Let \( S \) be a scheme, and \( F \) a sheaf of abelian group on one of the sites \((\text{Sm}/S)_{\text{ét}}\) or \((\text{Sch}/S)_{\text{ét}}\). Then the canonical morphism

\[
H^1_{\text{ét}}(S, F) \otimes \mathbb{Q} \to H^1_{\text{ét}}(S, F \otimes \mathbb{Q})
\]

is an isomorphism.

Classically the Picard functor of a morphism of schemes \( f \) is defined as \( R^1 f_* \mathbb{G}_m \). We introduce a variant of this construction which includes information about relative connected components.

Definition 2.18. Let \( f : X \to S \) be a finite type morphism of schemes. The Picard complex \( P(X/S) \) of \( X \) over \( S \) is the object \( \tau_{\geq 0} f_* (\mathbb{G}_m \otimes \mathbb{Q}[1]) \in D_{[0,1]}(\text{Sm}/S) \).

Remark 2.19. Recall from [3, Exposé XVIII §1.4] that there is an equivalence of categories between the category of commutative group stacks over a site \( S \) (with morphisms taken up to 2-isomorphisms) and the category \( D_{[0,1]}(\text{Sh}(S, \mathbb{Z})) \). The Picard complex corresponds via this equivalence to the smooth Picard stack, i.e., the version for \( \text{Sm}/S \) of the usual Picard stack (see e.g. [24]). This point of view will not be used in the rest of this paper.

We will also need a version with transfers.

Definition 2.20. Let \( f : X \to S \) be an excellent scheme, \( f : X \to S \) a finite type morphism of schemes. The Picard complex with transfers \( P^{\text{tr}}(X/S) \) of \( X \) over \( S \) is the object \( \tau_{\geq 0} f_* (\mathbb{G}_m^{\text{tr}} \otimes \mathbb{Q}[1]) \in D_{[0,1]}(\text{Cor}/S) \).

There is a canonical map

\[
a^{\text{tr}}(P(X/S)) \to P^{\text{tr}}(X/S)
\]

coming from adjunction and [6, Proposition 2.10] (which applies since \( X \) is also excellent).

Since we only work with rational coefficients, the following result will be useful.

Lemma 2.21. Let \( f : X \to S \) be a smooth morphism with \( S \) regular. Then for \( i \geq 1 \), the sheaf \( R^i f_* (\mathbb{G}_m \otimes \mathbb{Q}[1]) \cong R^{i+1} f_* (\mathbb{G}_m \otimes \mathbb{Q}) \) is trivial. As a consequence, we have

\[
P^{\text{tr}}(X/S) \cong f_* (\mathbb{G}_m^{\text{tr}} \otimes \mathbb{Q}[1]).
\]

Proof. This follows from Lemma 2.17 together with the fact that for a regular scheme \( T \) and \( i \geq 2 \), the étale cohomology groups \( H^i(T, \mathbb{G}_m) \) are torsion [40, Proposition 1.4]. \( \square \)
Lemma 2.24. Let $f : X \to S$ be a morphism of schemes or algebraic spaces. We say that $f$ is cohomologically flat in degree 0 if the construction of $f^\ast \mathcal{O}_X$ commutes with arbitrary base change.

Proof. For any $U \to S$ smooth, we have $\bigoplus_m \mathcal{O}_m(U) = \mathcal{O}^\times(X \times_U U) \cong \mathcal{O}^\times(\pi_0(X) \times U/U) \cong \mathcal{O}^\times(\pi_0(X) \times S/U)$.

This implies the claim. \qed

Next, we look at the Picard sheaf $\mathcal{P}_{ic_{X/S}} := R^1f_\ast\mathcal{G}_m \in \mathbf{Sh}(\mathbf{Sch}/S)_{et, Z}$ and its smooth analogue $\mathcal{P}_{ic_{X/S}}^{sm} \in \mathbf{Sh}(\mathbf{Sm}/S)_{et, Z}$ defined by the same formula on the smooth site. By exactness of $\mathcal{G}_m$, we have $\mathcal{P}_{ic_{X/S}} \cong \mathcal{P}_{ic_{X/S}}^{sm}$.

Because $f$ has sections locally in the étale topology by [36, 17.16.3 (ii)], the Leray spectral sequence follows, for all $T \in \mathbf{Sm}/S$, a short exact sequence

\[(L) \quad 0 \to \text{Pic}(\pi_0(X_T/T)) \to \text{Pic}(X_T) \to \mathcal{P}_{ic_{X/S}}^{sm}(T) \to 0.\]

Lemma 2.23. Let $f : X \to S$ be a smooth proper morphism. The sheaf $\bigoplus_m \mathcal{G}_m$ is representable by a torus, the Weil restriction $\mathcal{P}_{ic_{X/S}}^{sm}$ (see Definition A.12).

Proof. For any $U \to S$ smooth, we have $\bigoplus_m \mathcal{O}_m(U) = \mathcal{O}^\times(X \times_U U) \cong \mathcal{O}^\times(\pi_0(X) \times U/U) \cong \mathcal{O}^\times(\pi_0(X) \times S/U)$.

This implies the claim. \qed

We proceed to analyse the structure of $\mathcal{P}(X/S)$, following closely the standard structure theory for the Picard scheme [47] and the Picard stack [24]. We will see that restricting to the smooth site leads to simpler results than in the classical case.

In the sequel, we consider étale sheaves of abelian groups and $\mathbb{Q}$-vector spaces on the two sites $(\mathbf{Sch}/S)_{et}$ and $(\mathbf{Sm}/S)_{et}$. We have a morphism of sites $\zeta : \mathbf{Sch}/S \to \mathbf{Sm}/S$. The restriction functor $\zeta_* : \mathbf{Sh}(\mathbf{Sch}/S) \to \mathbf{Sh}(\mathbf{Sm}/S)$ is exact, since an étale scheme over a smooth $S$-scheme is a smooth $S$-scheme. We have $\zeta_*\mathcal{G}_m \cong \mathcal{G}_m$. The functor $\zeta_*$ commutes with $f_\ast$ and $\bigoplus_m$, in the sense that there are natural isomorphisms of functors $\zeta_*f^\ast \mathcal{G}_m \cong f^\ast\zeta_*\mathcal{G}_m$ and $\zeta_*f^\ast \cong f^\ast\zeta_*$. By abuse of terminology, we say that a sheaf of sets on $\mathbf{Sm}/S$ is representable if it is isomorphic to the functor $\zeta_*X$ for $X$ a (not necessarily smooth) $S$-scheme; such a scheme is then not uniquely determined up to isomorphism.

Definition 2.22. Let $f : X \to S$ be a morphism of schemes or algebraic spaces. We say that $f$ is cohomologically flat in degree 0 if the construction of $\bigoplus_m \mathcal{O}_X$ commutes with arbitrary base change.

Recall that by [36, 7.8.6], a smooth proper morphism $f$ has a Stein factorisation

$$f : X \xrightarrow{\pi_0} \mathcal{P}(\pi_0(X/S)) := \mathbf{Spec}(\bigoplus_m \mathcal{O}_m)^{\pi_0(f)} S$$

with $\pi_0(f)$ finite étale. Moreover, $f$ is cohomologically flat in degree 0, and hence the construction of $\pi_0(f)$ commutes with arbitrary base change.

Lemma 2.24. Let $\pi : S' \to S$ a morphism of schemes.

(i) There are natural isomorphisms

$$v_\pi : \pi^{-1}\mathcal{P}_{ic_{X/S}} \cong \mathcal{P}_{ic_{X \times_S S'/S'}}.$$ 

(ii) There are natural morphisms

$$v_\pi^{sm} : \pi^{-1}\mathcal{P}_{ic_{X/S}}^{sm} \to \mathcal{P}_{ic_{X \times_S S'/S'}}^{sm},$$

which are isomorphisms if $\pi$ is smooth.

Proof. The sheaf $\mathcal{P}_{ic_{X/S}}$ is the étale sheaf associated with the “naive” Picard functor $\mathcal{P}_{ic_{X/S}} : T \mapsto \text{Pic}(X \times_T T)$. We have, for any $S'$-scheme $T'$,

$$(\pi^{-1}\mathcal{P}_{ic_{X/S}})(T') = \text{Pic}(X \times_T T') = \text{Pic}((X \times_S S') \times_{S'} T') = \mathcal{P}_{ic_{X \times_S S'/S'}}^{sm}(T')$$

This equality is functorial in $T'$. After passing to associated sheaves, we get the isomorphism $v_\pi$. This proves the conclusion of (i).

We now turn to $\mathcal{P}_{ic_{X/S}}^{sm}$. This sheaf is also the étale sheaf associated with the “naive” Picard functor $\mathcal{P}_{ic_{X/S}}^{sm}$ on $\mathbf{Sm}/S$. We have, for any smooth $S'$-scheme $T'$,

$$(\pi^{-1}\mathcal{P}_{ic_{X/S}}^{sm})(T') = \text{Colim}_{T \in (T' \setminus \mathbf{Sm}/S)} \text{Pic}(X \times_S T) \to \text{Pic}(X \times_S T') = \mathcal{P}_{ic_{X \times_S S'/S'}}(T')$$

and this defines the morphism $v_\pi^{sm}$. If $\pi$ is smooth, then the category $T' \setminus \mathbf{Sm}/S$ has an initial object $T' \to S' \to S$ and we get isomorphisms. This proves (ii). \qed
Let us also define a natural base change map for \( P(X/S) \). Consider a cartesian diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{\bar{\pi}} & X \\
\downarrow{f} & & \downarrow{f} \\
S' & \xrightarrow{\pi} & S
\end{array}
\]

with \( \pi \) any morphism of schemes. The following composition

\[
\pi^{-1}P(X/S) = \pi^{-1}\tau_{\geq 0}(f_*\mathbb{G}_m \otimes \mathbb{Q}[1]) \to \pi^{-1}f_*\mathbb{G}_m \otimes \mathbb{Q}[1] \to \tilde{f}_*\pi^{-1}\mathbb{G}_m \otimes \mathbb{Q}[1]
\]

factors through the truncation \( \tau_{\geq 0}(\tilde{f}_*\mathbb{G}_m \otimes \mathbb{Q}[1]) = P(X_{S'/S'}) \). We denote the resulting morphism by

\[
V_{\pi} : \pi^{-1}P(X/S) \to P(X_{S'/S'}).
\]

In general, the construction of \( \mathcal{P}ic^{sm} \) and \( P(X/S) \) does not commute with arbitrary base change, i.e., \( v_{\pi}^{sm} \) and \( V_{\pi} \) are not always isomorphisms.

Representability results for \( \mathcal{P}ic \) are subtle in general. Let \( f : X \to S \) be a smooth projective morphism. It is in particular proper, flat and cohomologically flat in degree 0. By [21, 8.3/1], \( \mathcal{P}ic_{X/S} \) is represented by a group algebraic space \( \mathcal{P}ic_{X/S} \) over \( S \). Note that if \( f \) has in addition geometrically connected fibres, then \( \mathcal{P}ic_{X/S} \) is in fact a group scheme, separated and of finite presentation [21, 8.2/1]; since we do not want to restrict to this case, we use algebraic spaces as a technical crutch. Finally, if \( S \) is the spectrum of a field \( k \), then \( \mathcal{P}ic_{X/k} \) is represented by a group scheme which is locally of finite type over \( k \), regardless of whether \( f \) has geometrically connected fibres or not [21, 8.2/3].

We want to discuss the identity component \( \mathcal{P}ic^{0}_{X/S} \) of \( \mathcal{P}ic_{X/S} \). Before we introduce it, let us recall some basic facts about connected components of locally of finite type group schemes over a field. Let \( k \) be an arbitrary field and \( G \) be a \( k \)-group scheme which is locally of finite type. Since \( G \) is locally noetherian, its connected components are open and closed in \( G \), and we denote by \( G^{0} \) the connected component containing the identity.

**Definition 2.25.** Let \( f : X \to S \) be a smooth projective morphism. The group algebraic space \( \mathcal{P}ic_{X/S} \) comes with a natural subfunctor \( \mathcal{P}ic^{0}_{X/S} \), its relative identity component. Given a point \( s \in S \), through Lemma 2.24, we can restrict a section of \( \mathcal{P}ic_{X/S} \) to a section of \( \mathcal{P}ic_{X,s} \), which is represented by a group scheme \( \mathcal{P}ic_{X,s} \) over \( s \), locally of finite type, which has therefore an identity component \( \mathcal{P}ic^{0}_{X,s} \). A section of \( \mathcal{P}ic_{X/S} \) lies in \( \mathcal{P}ic^{0}_{X/S} \) if for all \( s \in S \), its restriction to \( \mathcal{P}ic_{X,s} \) lies in \( \mathcal{P}ic^{0}_{X,s} \).

We then also define \( \mathcal{P}ic^{sm,0}_{X/S} \) as \( \mathcal{P}ic^{0}_{X/S} \).

We have a further result on base change.

**Lemma 2.26.** Let \( \pi : S' \to S \) a morphism of schemes.

(i) There are natural isomorphisms

\[
v_{\pi} : \pi^{-1}\mathcal{P}ic^{0}_{X/S} \simeq \mathcal{P}ic^{0}_{X \times_{S}S'/S'}.
\]

(ii) There are natural morphisms

\[
v_{\pi}^{sm} : \pi^{-1}\mathcal{P}ic^{sm,0}_{X/S} \to \mathcal{P}ic^{sm,0}_{X \times_{S}S'/S'},
\]

which are isomorphisms if \( \pi \) is smooth.

**Proof.** The maps in the lemma are obtained by restriction from Lemma 2.24. We thus have only have to check that the subfunctor \( \mathcal{P}ic^{0}_{X/S} \) is mapped into \( \mathcal{P}ic^{0}_{X \times_{S}S'/S'} \). Let \( T \in \text{Sch}/S' \) and \( \alpha \in \mathcal{P}ic^{0}_{X/S}(T) \). Let \( s' \) be a point in \( S' \), with corresponding morphism \( i' : s' \to S' \). Write \( i = \pi i : s' \to S \). It is easy to see from the construction of the base change map that we have \( v_t = v_{i'} \circ (i')^{-1}v_{\pi} : (i')^{-1}\pi^{-1}\mathcal{P}ic_{X/S} \simeq i^{-1}\mathcal{P}ic_{X,S'/S'} \to \mathcal{P}ic_{X,s'/S'} \). By definition, we have \( v_{\pi}(i^{-1}\alpha) \in \mathcal{P}ic_{X,s'/S'}(T_{s'}) \), hence \( v_{\pi}(v_{i'}(i')^{-1}\alpha) \in \mathcal{P}ic_{X,s'/S'}(T_{s'}) \). Since this holds for all \( s' \in S' \), we conclude that \( v_{\pi}(\alpha) \) lies in \( \mathcal{P}ic^{0}_{X \times_{S}S'/S'} \). \( \square \)

**Lemma 2.27.** \( \mathcal{P}ic^{0}_{X/S} \) is representable by a proper group algebraic space \( \mathcal{P}ic^{0}_{X/S} \) over \( S \).
Proof. By the discussion of the previous paragraph, it is enough to show that the functor \( Pr^0_{X/S} \rightarrow Pic_{X/S} \) is relatively representable by a closed immersion and that the resulting group algebraic spaces \( Pic^0_{X/S} \) is of finite type.

By [41, Corollaire 2.3], this is the case for \( Pic^0_{X/S} \) under the additional assumptions that the geometric fibres of \( f \) are integral (or equivalently, connected); note that in this case, \( Pic_{X/S} \) is representable by a scheme. We are going to reduce to this case by étale descent.

We can assume \( S \) to be connected, and then that \( f \) has non-empty fibres. Write \( f : X \rightarrow S \) for the Stein factorisation of \( f \). The morphism \( \pi_0(f) \) is finite étale surjective. Let \( p : \Pi \rightarrow S \) be an étale Galois covering through which \( \pi_0(f) \) factors. We thus have that \( \Pi' := \pi_0(X/S) \times_S \Pi \simeq \prod_{i \in I} \Pi_i \) for some finite set \( I \) and \( \Pi_i \simeq \Pi \). Write \( Y = X \times_S \Pi \overset{g}{\rightarrow} \Pi \). The Stein factorisation of the morphism \( g \) is \( Y \rightarrow \Pi' \rightarrow \Pi \). Since \( Y \rightarrow \Pi' \) is smooth with connected geometric fibres, we have \( Y = \prod Y_i \) with \( g'(Y_i) \subset \Pi_i \), and each morphism \( Y_i \rightarrow \Pi \) is smooth projective with geometrically connected fibres. We have

\[
p^{-1}Pr^0_{X/S} \simeq Pr^0_{Y/\Pi} \simeq \prod_{i \in I} Pr^0_{Y_i/\Pi}
\]

and, by the beginning of the proof, each of the factors in this product is representable by a proper group algebraic space. The same argument applies over \( \Pi \times S \Pi \). Using that étale descent for algebraic spaces is effective, we conclude that \( Pr^0_{X/S} \) is representable by a proper group algebraic space.

\[\square\]

Proposition 2.28. Let \( S \) be a \( \mathbb{Q} \)-scheme and \( f : X \rightarrow S \) a smooth projective morphism. The algebraic group space \( Pic^0_{X/S} \) is in fact an abelian scheme over \( S \).

Proof. Under these hypotheses, and assuming furthermore that \( f \) has geometrically connected fibres, \( Pic^0_{X/S} \) (which is then a group scheme) is smooth, as explained in [47, Remark 5.21]. Let us now drop the assumption on the fibres of \( f \). By the étale descent argument from Lemma 2.27 and the fact that smoothness can be checked étale locally, we deduce that the algebraic group space \( Pic^0_{X/S} \) is smooth and proper. Moreover, by definition of \( Pic^0_{X/S} \), it has geometrically connected fibres. But a smooth proper algebraic group space with geometrically connected fibres is an abelian scheme [34, Theorem 1.9].

\[\square\]

In general, when \( S \) is not a \( \mathbb{Q} \)-scheme, \( Pic^0_{X/S} \) can have non-reduced fibres. We need a condition under which we can “extract” an abelian scheme from \( Pic^0 \). Here is a result in that direction.

Proposition 2.29. [23, Proposition 2.15] Let \( S \) be a noetherian scheme, and \( G \) a group \( S \)-algebraic space which is proper, flat and cohomologically flat. Then there exists an abelian scheme \( A/S \) and a finite flat group scheme \( F/S \) such that \( G \) fits into a unique exact sequence

\[
0 \rightarrow A \rightarrow G \rightarrow F \rightarrow 0
\]

of \( S \)-group schemes. In particular, \( G \) is a scheme.

The uniqueness in the previous proposition is not stated in [23], but follows from the proof as \( F \) is shown to be the affinisation \( \text{Spec}_{\mathbb{Q}}((G \rightarrow S)_*, \mathcal{O}_G) \) of \( G \).

This motivates the following definition.

Definition 2.30. A morphism \( f : X \rightarrow S \) is Pic-smooth if it is smooth projective and if the algebraic space \( Pic^0_{X/S} \) is flat and cohomologically flat in degree 0.

By Proposition 2.28, this is automatic if \( S \) is of characteristic 0.

Lemma 2.31. Let \( f : X \rightarrow S \) a Pic-smooth morphism, and \( T \rightarrow S \) be any morphism of schemes. Then \( f \times_S T \) is Pic-smooth.

Proof. This follows from Lemma 2.24 together with the facts that flatness and cohomological flatness in degree 0 are stable by base change.

\[\square\]

Proposition 2.32. Let \( f : X \rightarrow S \) be a smooth projective morphism. Assume \( S \) is reduced. Then there is a dense open set \( U \subset S \) such that \( f \times_S U \) is Pic-smooth.
that this morphism is surjective, with kernel exactly equal to $G$.

Since $S$ is reduced, generic flatness for morphisms of algebraic spaces \cite[Corollaire 6.9.3]{EGA} provides a dense open subset $V \subset U$ of $S$ over which $g \times_S V$ is flat.

By restricting $V$ further, we can assume that $V$ is affine a disjoint union of its irreducible components, and thus reduced to the case $V = \text{Spec}(A)$ affine and integral. Cohomological flatness in degree 0 on a dense open set of $V$ then follows from \cite[Corollaire 7.3.9]{EGA}, applied to the homological functor $T_*(\cdot): \text{Mod}(A) \to \text{Mod}(A)$, $N \mapsto R^ng_*g^{-1}N$, which takes values in finitely generated $A$-modules by properness of $g$.

**Proposition 2.33.** Let $f: X \to S$ be a Pic-smooth morphism. Then $\mathcal{Pic}_{X/S}^\text{sm,0}$ is representable by an abelian scheme.

**Proof.** By Proposition 2.29, we have a short exact sequence of group schemes

$$0 \to \text{Pic}_{X/S}^\text{0,red} \to \text{Pic}_{X/S}^\text{0} \to F \to 0$$

with $F$ a finite flat group scheme with connected fibres (since the fibres of $\text{Pic}_{X/S}^\text{0}$ are connected by definition), and $\text{Pic}_{X/S}^\text{0}$ represents $\mathcal{Pic}_{X/S}^\text{0}$.

Let us show, more generally, that if $G$ is a group scheme fitting in an exact sequence

$$0 \to A \to G \xrightarrow{\pi} F \to 0$$

with $A$ an abelian scheme and $F$ a finite flat group scheme with connected fibres, then the restriction to $\text{Sm}/S$ of the functor of points of $G$ is representable by $A$.

Let $T \in \text{Sm}/S$ and $h: T \to G$ an $S$-morphism. Let $s \in S$. Then $F_s$ is a finite flat connected group scheme over $\kappa(s)$, hence we have $(F_s)^\text{red} = \text{Spec}(\kappa(s))$. Hence the morphism $T_s \to F_s$ factors through the identity section of $F_s$. Using the smoothness of $S$, this implies that the morphism $\pi \circ h: T \to F$ also factors through the identity section. By the exactness in the middle of

$$0 \to A(T) \to G(T) \to F(T)$$

we see that $h$ comes from $A(T)$. This proves the result.

**Definition 2.34.** Let $f: X \to S$ be a Pic-smooth morphism. We denote by $\text{Pic}_{X/S}^{\text{0,red}}$ the abelian scheme representing $\mathcal{Pic}_{X/S}^{\text{sm,0}}$.

**Lemma 2.35.** Let $f: X \to S$ be a Pic-smooth morphism. For any morphism $\pi: S' \to S$, the morphism $v_{\pi}^\text{sm}$ induces an isomorphism

$$v_{\pi}^\text{sm}: \text{Pic}_{X/S}^{\text{0,red}} \times_S S' \simeq \text{Pic}_{X'/S'}^{\text{0,red}}.$$  

**Proof.** First, the morphism $X' \to S'$ is still Pic-smooth by Lemma 2.31; hence the statement makes sense. By combining Lemma 2.24 and Proposition 2.33, we get a morphism $v_{\pi}^\text{sm}: \text{Pic}_{X/S}^{\text{0,red}} \times_S S' \to \text{Pic}_{X'/S'}^{\text{0,red}}$. By the uniqueness of the short exact sequence in Proposition 2.29, we see that, on the other hand, we have a base change isomorphism $\text{Pic}_{X/S}^{\text{0,red}} \times_S S' \simeq \text{Pic}_{X'/S'}^{\text{0,red}}$, and it is not difficult to see that it coincides with $v_{\pi}^\text{sm}$.

We now turn to the study of the Néron-Severi groups in families.

**Definition 2.36.** We define the Néron-Severi sheaf as the étale quotient sheaf

$$\mathcal{N}S_{X/S}^{\text{sm}} := \mathcal{Pic}_{X/S}^{\text{sm}}/\mathcal{Pic}_{X/S}^{\text{sm,0}}.$$  

This definition works well over a general scheme $S$. Let us explain an equivalent, group-theoretic perspective, when $S$ is the spectrum of a perfect field $k$. Let $G$ be a $k$-group scheme, locally of finite type. The $k$-algebra admits a maximal étale $k$-subalgebra (not necessarily of finite type), whose spectrum we denote by $\pi_0(G)$. We thus have a morphism $G \to \pi_0(G)$. A standard result is that this morphism is surjective, with kernel exactly equal to $G^0$. In other words, we have a short exact sequence of $k$-group schemes

$$0 \to G^0 \to G \to \pi_0(G) \to 0.$$
Since $k$ is perfect, $G_{\text{red}}$ is a closed subgroup scheme of $G$. Then $G_{\text{red}}$, being a reduced group scheme over a perfect field, is geometrically reduced hence smooth. We have $(G_{\text{red}})^0 = (G^0)_{\text{red}}$, which we denote in this situation by $G^0_{\text{red}}$. Since the maximal étale $k$-subalgebra of $\mathcal{O}(G_{\text{red}}) = \mathcal{O}(G)_{\text{red}}$ is sent isomorphically to the maximal étale sub-$k$-algebra of $\mathcal{O}(G)$, we have an isomorphism $\pi_0(G_{\text{red}}) \simeq \pi_0(G)$. So we get an exact sequence

$$0 \to G^0_{\text{red}} \to G_{\text{red}} \to \pi_0(G) \to 0$$

and since $G^0_{\text{red}}$ is smooth, this sequence gives rise to an exact sequence of étale sheaves (both on $\text{Sch}/k$ and $\text{Sm}/k$). But, as sheaves on $\text{Sm}/k$, we have $G = G_{\text{red}}$ and $G^0 = G^0_{\text{red}}$. Hence we get $G/G^0 \simeq \pi_0(G)$ as étale sheaves on $\text{Sm}/k$.

In this level of generality generality, it is not the case that $\pi_0(G)$ is constructible as a sheaf of abelian groups. However, suppose that, for a fixed algebraic closure $\bar{\pi}$ of $\pi$, we have $G_0(\bar{\pi})$ a finitely generated abelian group. Then $\pi_0(G)$ is constructible as a sheaf of abelian groups.

In the special case of Pic, using the fact that the geometric Neron-Severi group is finitely generated, we get the following conclusion.

**Lemma 2.37.** With the notations above (in particular, we assume that $k$ perfect), we have an isomorphism

$$NS_{X/k} \simeq \pi_0(\text{Pic}\_X/k)$$

as étale sheaves on $\text{Sm}/k$, and $NS_{X/k}$ is locally constant and constructible as a sheaf of abelian groups.

We now return to a general base scheme $S$. The following lemma comes out directly from the exactness of $\zeta$, and from Lemma 2.24.

**Lemma 2.38.** We have a canonical isomorphism $\zeta_*NS_{X/S} \simeq NS_{X/S}^{0}$, and the construction of $NS_{X/S}$ (resp. $NS_{X/S}^{0}$) commutes with base change by an arbitrary morphism (resp. by a smooth morphism).

We have also a useful simplification on a regular base.

**Lemma 2.39.** Let $f : X \to S$ be smooth projective Pic-smooth with $S$ regular. Then for all $T \in \text{Sm}/S$, the natural map

$$(\text{Pic}_{X/S}^{0}(T)/\text{Pic}_{X/S}^{0,0}(T)) \otimes \mathbb{Q} \longrightarrow NS_{X/S}^{0} \otimes \mathbb{Q}(T).$$

is an isomorphism.

**Proof.** It suffices to prove that in this situation, the cohomology group $H^1_\text{ét}(T, \text{Pic}_{X/S}^{0,0} \otimes \mathbb{Q})$ vanishes. By Proposition 2.29, we have a short exact sequence

$$0 \to \text{Pic}_{X/S}^{0,\text{red}} \to \text{Pic}_{X/S}^{0} \to F \to 0$$

where $\text{Pic}_{X/S}^{0,\text{red}}$ is an abelian scheme and $F$ is a finite flat commutative group scheme, thus $\text{Pic}_{X/S}^{0} \otimes \mathbb{Q} \simeq \text{Pic}_{X/S}^{0,\text{red}} \otimes \mathbb{Q}$ as étale sheaves.

Since $T$ is noetherian and regular, [54, Proposition XIII 2.6.(ii)] and [54, Proposition XIII 2.3.(ii)] imply that torsors under the abelian scheme $\text{Pic}_{X/S}^{0,\text{red}}$ are torsion, which shows that $H^1(T, \text{Pic}_{X/S}^{0,\text{red}}) \otimes \mathbb{Q} = 0$. By Lemma 2.17, we deduce that $H^1(T, \text{Pic}_{X/S}^{0,\text{red}} \otimes \mathbb{Q}) = 0$.

We have a morphism of sites $\alpha : \text{Sm}/S \to \text{Et}/S$ where $\text{Et}/S$ is the small étale site of $S$. Put $\gamma = \alpha \circ \zeta : \text{Sch}/S \to \text{Et}/S$. We say that a sheaf $F$ on $\text{Sm}/S$ (resp. $\text{Sch}/S$) is constructible if it is in the essential image of the fully faithful functor $\alpha^{-1}$ (resp. $\gamma^{-1}$), or equivalently if the counit morphism $\alpha^{-1}\alpha_*F \to F$ (resp. $\gamma^{-1}\gamma_*F \to F$) is an isomorphism and $\alpha_*F$ (resp. $\gamma_*F$) is constructible (as a sheaf of $\mathbb{Z}$-modules, i.e., we do not require fibres to be finite abelian groups, but only to be finitely generated).

It is well known that the sheaf $NS_{X/S}$ is far from being constructible, even for $f$ smooth projective and $S$ regular in characteristic $0$; in particular, the rank of the geometric fibres (which are finitely generated abelian groups by [2, Exp XIII, Thm 5.1]) is not a constructible function [21, 8.4 Remark 8]. We are going to see that $NS_{X/S}$ is better behaved in some cases.

Let $S$ be a regular scheme and $f : X \to S$ a smooth projective morphism. We want to define a locally constant étale sheaf over $S$; for this it is enough to work connected component by connected
component, so that we can assume $S$ to be irreducible, with generic point $η$. Fix a geometric point $η$ over $η$. A locally constant sheaf on $S_η$ can be identified with a representation of the étale fundamental group $π^e(S, η)$. Since $S$ is regular, this fundamental group is the maximal quotient of $Gal(κ(η)/κ(η))$ which is unramified at every codimension 1 point.

By [2, Exp XIII, Thm 5.1], the geometric Néron-Severi group $NS(X_0)$ of line bundles modulo algebraic equivalence is finitely generated. It comes with a continuous action of the Galois group $Gal(η/η)$ (which thus factors through a finite quotient). The $ℓ$-adic first Chern class yield a Galois-equivariant morphism $c_1 : NS(X_0) → H^2(X_η, Q_ℓ(1))$ which is injective after tensoring by $Q_ℓ$, thus also injective after tensoring by $Q$. Moreover, since $f$ is smooth and projective, the proper and smooth base change theorems for $ℓ$-adic cohomology imply that, for any codimension 1 point $s ∈ S$, the Galois representation on $H^2(X_η, Q_ℓ(1))$ is unramified at $s$. By [2, Exp X, 7.13.10], the construction of $c_1$ commutes with specialisation; this implies that $NS(X_η)_{η}$ is also unramified at $s$.

**Definition 2.40.** Let $S$ be a regular scheme and $f : X → S$ a smooth projective morphism. The locally constant constructible étale sheaf of $Q$-vector spaces attached to $NS(X_0)$ is the Néron-Severi lattice $N_{X/S}$ of $X$ over $S$.

Let $π : S' → S$ be a universally open morphism between regular schemes. Then any generic point of an irreducible component of $S'$ is sent to the generic point of an irreducible component of $S$. Let $f : X → S$ be a smooth projective morphism, and write $f' : X' → S'$ for its base change along $π$. We are going to define a base change isomorphism $v_{X/S} : π_{1}^{et}(S', η') → π_{1}^{et}(S, η)$. Let us assume for simplicity that $S$, $S'$ are regular (the general case is easily obtained from this, given that $S$, $S'$ are regular). Write $η'$, $η$ for the generic points; we have $π(η') = η$ by assumption. Choose compatible geometric points $η'$, $η$ above them. The morphism $π$ induces compatible morphisms of absolute Galois groups and étale fundamental groups (which are unramified quotients of the former): $π_* : Gal(κ(η')/κ(η)) → Gal(κ(η)/κ(η))$.

The locally constant sheaf $π_{1}^{et}(S, η)$ is attached to the representation of $π_{1}^{et}(S', η')$ obtained from the one of $π_{1}^{et}(S, η)$ on $NS(X_0)$ by restriction along $π_*$. We also have an induced isomorphism $NS(X_0) ≃ NS(X_0')$ which is equivariant for the Galois actions (via $π_*$); note that it is an isomorphism because the geometric Néron-Severi group is invariant under extension of algebraically closed field. Combined with the previous observation, this provides the desired isomorphism $v_{X/S} : π_{1}^{et}(S, η) → π_{1}^{et}(S, η)$. With the same hypotheses, let us now define a morphism $ε_S : α_* NS_{X/S}^{sm} → N_{X/S}$.

We first define a morphism $ε_S : α_* Pic_{X/S}^{et} → N_{X/S}$. Recall that $α_* Pic_{X/S}^{et}$ is the étale sheaf associated to the presheaf $Pic_{X/S}^{sm, psh} : V → Et/S → Pic(X ×_S V)$. Since $N_{X/S}$ is an étale sheaf, defining $ε_S$ is equivalent to writing down a morphism $Pic_{X/S}^{sm, psh} → N_{X/S}$. Let $V ∈ Et/S$, which we can assume to be connected, and $L$ be a line bundle on $X ×_S V$. Choose a factorisation $η → V_η → η$, which induces a morphism $π_1(V, η) → π_1(S, η)$. Using this factorisation, lift $L_η$ to a class in $NS(X_η) ≃ NS(X_{V_η} ×_S η)$ which by construction is fixed by $π_1(V, η)$, so gives a section in $N_{X/S}(V)$. This is the required class $ε_S([L])$. The morphism $ε_S$ is trivial on $α_* Pic_{X/S}^{sm, 0}$ since algebraic equivalence over $V$ implies algebraic equivalence over $η$. So $ε_S$ induces a morphism $ε_{X/S} : α_* NS_{X/S}^{sm} → N_{X/S}$ as required.

By going through the definitions of the various base change maps, one can show the following compatibility.
Lemma 2.41. Let \( \pi : S' \to S \) be a universally open morphism between regular schemes. The base change morphisms of Lemma 2.21 induce a morphism
\[
\e^\NS_{S'/S'} : \pi^{-1}\NS_{X/S} \to \NS_{X'/S'}
\]
The diagram of étale sheaves
\[
\begin{array}{ccc}
\pi^{-1}\alpha_*\NS_{X/S} & \xrightarrow{\pi^{-1}\e_{X/S}} & \pi^{-1}\NS_{X/S}.
\end{array}
\]
is commutative.

Proposition 2.42. Let \( f : X \to S \) be a Pic-smooth morphism with \( S \) regular. Then
(i) \( \e_{X/S} : \alpha_*\NS_{X/S} \otimes \Q \to \NS_{X/S} \) is an isomorphism, and
(ii) the counit morphism \( \alpha^{-1}\alpha_*\NS_{X/S} \otimes \Q \to \NS_{X/S} \otimes \Q \) is an isomorphism.
In particular, the sheaf \( \NS_{X/S} \) is locally constant constructible sheaf of \( \Q \)-vector spaces.

Proof. The morphism in (i) (resp. (ii)) is a morphism of étale sheaves on \( \Et / S \) (resp. \( \Sm / S \)). To show that it is an isomorphism, it suffices to check on stalks at geometric points of \( S \) (resp. at geometric points of all smooth \( S \)-schemes).

We first compute the stalks of \( \NS_{X/S} \) on \( \Et / S \). Let \( s \) be a geometric point of \( S \). Write \( \U \) for the projective system of all étale neighbourhoods of \( s \) in \( S \), i.e. the system of all pairs \((U, \bar{u})\) with \( U \) an étale \( S \)-scheme and \( \bar{u} \) a lift of \( s \) to \( U \). Write \( S^{sh}_s \) for the projective limit of \( U \), the spectrum of a strict henselisation of the local ring \( \O_{S,s} \), and \( X^{sh}_s = X \times_S S^{sh}_s \). Let us write \( \nu \) for the generic point of \( S^{sh}_s \). By definition of \( \NS_{X/S} \) and exactness of stalks, we have an exact sequence of stalks
\[
0 \to (\Pic^{\sm,0}_{X/S})_s \to (\Pic^{\sm}_{X/S})_s \to (\NS^{\sm}_{X/S})_s \to 0.
\]
The stalks of a higher direct image is easily computed; e.g., by [56, Tag 03Q7], we have
\[
(\Pic^{\sm}_{X/S})_s \simeq (R^1f_*\Gm)_s \simeq H^1(X^{sh}_s, \Gm) \simeq \Pic(X^{sh}_s).
\]
Moreover, for \((V, \bar{v}) \in \U \), it is easy to see that the composition
\[
\Pic(X \times_S V) \to \Pic^{\sm}_{X/S}(V) \to \Pic(X^{\sh}_s)
\]
coincides with the pullback map on Picard groups. Let us denote \( \Pi^0(X^{\sh}_s) \) for the subgroup of \( \Pic(X^{\sh}_s) \) consisting of isomorphism classes of line bundles \( L \) on \( X^{\sh}_s \) which are such that for all geometric points \( r \) of \( S^{sh}_s \), we have \( L_r = 0 \in \NS(X_r) \). Note that it is equivalent to check this at one such geometric point of the connected scheme \( S^{\sh}_s \), as being algebraic to 0 is a deformation invariant of line bundles. Then we have
\[
(\Pic^{\sm,0}_{X/S})_s \simeq \Pi^0(X^{\sh}_s)
\]
as subgroups of \( (\Pic^{\sm}_{X/S})_s \simeq \Pic(X^{\sh}_s) \). We thus have
\[
(\NS^{\sm}_{X/S})_s \simeq \Pi(\Pic(X^{\sh}_s))/\Pi^0(X^{\sh}_s).
\]

Let us now prove that the map (i) is an isomorphism. It suffices to prove that the induced morphism on étale stalks at \( s \) geometric point of \( S \) is an isomorphism. The sheaf \( \NS_{X/S} \) is locally constant, hence all its stalks at geometric points are canonically identified. By construction, its stalk at a geometric generic point \( \eta \) is canonically isomorphic to \( \NS(X_\eta) \otimes \Q \). We choose a
geometric generic point with a factorisation \( \bar{\eta} \to \nu \to \eta \) through the generic point of \( S^\text{sh}_s \). Hence we can identify \( (e_{X/S})_s \) with a morphism

\[
(Pic(X^\text{sh}_s)/Pic^0(X^\text{sh}_s)) \otimes \mathbb{Q} \to \text{NS}(X_s) \otimes \mathbb{Q}
\]

and by going through the definitions, it is easy to see that this map is induced by the composition

\[
\text{Pic}(X^\text{sh}_s) \to \text{Pic}(X^\text{sh}_s \times_{S^\text{sh}_s} \bar{\eta}) = \text{Pic}(X \times_S \bar{\eta}) \to \text{NS}(X_\eta)
\]

where the first map is the pullback map on Picard groups. Let us show that this map is an isomorphism, at least \( \otimes \mathbb{Q} \).

We first show the injectivity. Let \([L] \in \text{Pic}(X^\text{sh}_s)\) with \( \overline{L_\eta} = 0 \in \text{NS}(X_\eta) \). We have to show that for all geometric points \( \bar{\xi} \) of \( S^\text{sh}_s \), we have \( \overline{L_\xi} = 0 \in \text{NS}(X_\xi) \); but this follows from the deformation invariance of the property of being algebraically equivalent to 0 for line bundles.

We now prove the surjectivity after tensoring by \( \mathbb{Q} \). Let \([\bar{L}] \in \text{NS}(X_\eta)\) with \( L \) a line bundle on \( X_\eta \). Let \( \kappa(\nu) \subset K \subset \kappa(\bar{\eta}) \) be a finite extension such that \( L \) is obtained by base change from a line bundle \( L_K \) on \( X_K \). Write \( \pi : X_K \to X_\eta \) for the natural morphism. Since \( X_K \) is regular, we have an isomorphism \( c_1 : \text{Pic}(X_K) \simeq H^1(X_K) \). Let \( c_1(L_K) \in \text{CH}^1(X_K) \). We have morphisms \( \pi^* : \text{CH}^1(X_K) \to \text{CH}^1(X_\nu) = \pi_* \), with \( \pi^* \pi^* = [K : \kappa(\nu)] \); the pullback morphism \( \pi_* \) is compatible with \( c_1 \) and the pullback of line bundles.

We check that for any \( \alpha \in \text{CH}^1(X_K) \), the class of \( (K, K(\nu))^{\pi_*} \pi_* \alpha - \alpha \) in \( \text{NS}(X_\nu) \otimes \mathbb{Q} \) is 0. Indeed, by Matsusaka’s theorem, this holds if and only if the class is numerically equivalent to 0, i.e., if for every curve class \( C \in \text{CH}_1(X_\nu) \) we have \( (K, K(\nu))^{\pi_*} \pi_* \alpha \cdot C = \alpha \cdot C \) (where we have implicitly pulled back to \( X_\nu \). But \( C \) also descends to a finite extension \( K'/K \), and intersection numbers can be computed after pushforward by a finite morphism via the same degree formula as above.

We conclude that it suffices to show that \( (K, K(\nu))^{\pi_*} c_1(L_K) \in \text{CH}^1(X_\nu) \otimes \mathbb{Q} \) is in the image of the restriction map to the generic fiber \( \text{Pic}(X^\text{sh}_s) \otimes \mathbb{Q} \simeq \text{CH}^1(X^\text{sh}_s) \otimes \mathbb{Q} \to \text{CH}^1(X_\nu) \otimes \mathbb{Q} \) (we use the fact that \( X^\text{sh}_s \) is regular). The cycle \( (K, K(\nu))^{\pi_*} c_1(L_K) \) extends to some dense open subset \( U \subset X^\text{sh}_s \), and the restriction map \( \text{CH}^1(X^\text{sh}_s) \to \text{CH}^1(U) \) is surjective (with an explicit section given by closing up a cycle in \( U \) in \( X^\text{sh}_s \)). This concludes the proof of surjectivity.

Let us finally prove that (ii) is an isomorphism. Let \( T \in \mathfrak{Sm}/S \) and \( \bar{t} \) a geometric point of \( T \). We must show that the morphism

\[
(\alpha^{-1} \alpha_* \mathcal{NS}^\text{sm}_{X/S})_T \otimes \mathbb{Q} \to (\mathcal{NS}^\text{sm}_{X/S})_T \otimes \mathbb{Q}
\]

is an isomorphism. By replacing \( T \) by a neighbourhood of the image of \( \bar{t} \), we can assume that \( T \) is integral. By composition, \( \bar{t} \) determines a geometric point \( s \) of \( S \), and we have \( (\alpha^{-1} \alpha_* \mathcal{NS}^\text{sm}_{X/S})_T \simeq (\mathcal{NS}^\text{sm}_{X/S})_s \).

Since, for any étale \( T \)-scheme \( W \), we have

\[
\mathcal{NS}^\text{sm}_{X/S}(W) = \mathcal{NS}^\text{sm}_{X \times_S T/T}(W)
\]

we conclude that

\[
(\mathcal{NS}^\text{sm}_{X/S})_T \simeq (\mathcal{NS}^\text{sm}_{X \times_S T})_T
\]

In the proof of (i) above, we have seen that for any morphism \( f : X \to S \) satisfying the hypothesis of the proposition and with \( S \) integral, we have \( (\mathcal{NS}^\text{sm}_{X/S})_T \otimes \mathbb{Q} \simeq \text{NS}(X_\eta) \), with \( \bar{\eta} \) a geometric point above the generic point \( \eta \) of \( S \). By Lemma 2.31, the morphism \( X \times_S T \to T \) satisfies these assumptions. Choose \( \bar{\theta} \) a geometric generic point of \( T \), compatible with \( \bar{\eta} \). We deduce that the morphism we are interested in coincides with the natural morphism

\[
\text{NS}(X_\eta) \to \text{NS}(X_\theta)
\]

which is an isomorphism since the geometric Neron-Severi group is invariant by base change to an algebraically closed extension. This completes the proof of (ii), and of the proposition.

\[\Box\]

This result has several useful corollaries.

**Corollary 2.43.** Assume \( S \) is regular. Let \( f : X \to S \) be a smooth projective Pic-smooth morphism of schemes. Then we have natural distinguished triangles

\[
\Sigma^\infty(f_* \mathcal{G}_m \otimes \mathbb{Q})[1] \to \Sigma^\infty P(X/S) \to \Sigma^\infty(Pic^\text{sm}_{X/S} \otimes \mathbb{Q}) \quad \Delta
\]
and
\[ \Sigma^\infty (\mathcal{P} i_{X/S}^{\text{sm},0} \otimes \mathbb{Q}) \to \Sigma^\infty (\mathcal{P} i_{X/S}^{\text{sm}} \otimes \mathbb{Q}) \to \Sigma^\infty \mathcal{N} S_{X/S}^{\text{sm}} \otimes \mathbb{Q} \]
and the motive \( \Sigma^\infty \mathcal{P}(X/S) \) lies in \( \text{DA}_{1,c}^{\text{sm}}(S) \). Moreover, these two distinguished triangulations admit (non-canonical) splittings, so that we have
\[ \Sigma^\infty \mathcal{P}(X/S) \simeq \Sigma^\infty \left( f_* \mathcal{G}_m \otimes \mathbb{Q}\right)[1] \oplus \Sigma^\infty (\mathcal{P} i_{X/S}^{\text{sm},0} \otimes \mathbb{Q}) \oplus \Sigma^\infty \mathcal{N} S_{X/S}^{\text{sm}} \otimes \mathbb{Q} \]

**Proof.** The first distinguished triangle is obtained from the truncation triangle for \( \mathcal{P}(X/S) \) for the standard t-structure on \( D((\text{Sm}/S)_{et}, \mathbb{Q}) \). The second one follows from the short exact sequence of sheaves
\[ 0 \to \mathcal{P} i_{X/S}^{\text{sm},0} \otimes \mathbb{Q} \to \mathcal{P} i_{X/S}^{\text{sm}} \otimes \mathbb{Q} \to \mathcal{N} S_{X/S}^{\text{sm}} \otimes \mathbb{Q} \to 0 \]
The sheaf \( f_* \mathcal{G}_m \simeq \text{Res}_{\pi_0(f)} \mathcal{G}_m \) is representable by a torus, the sheaf \( \mathcal{P} i_{X/S}^{\text{sm},0} \) is representable by the abelian scheme \( \text{Pic}^{0,\text{red}} \) because \( f \) is Pic-smooth (Proposition 2.33), and the sheaf \( \mathcal{N} S_{X/S}^{\text{sm}} \) is representable by a lattice by Proposition 2.42. From Corollary 2.14, we conclude that \( \Sigma^\infty \mathcal{P}(X/S) \) is in \( \text{DA}_{1,c}^{\text{sm}}(S) \).

To show the triangles split, it is enough to show that the connecting morphisms vanish. Given the representability results for the various pieces, this follows from Lemma 2.44 below.

**Lemma 2.44.** Let \( S \) be a regular scheme. Let \( L \) be a lattice over \( S \), \( T \) be an \( S \)-torus and \( A \) be an \( S \)-abelian scheme.

(i) \( \text{DA}(S)(\Sigma^\infty L_Q, \Sigma^\infty T_Q[2]) = 0 \)

(ii) \( \text{DA}(S)(\Sigma^\infty A_Q, \Sigma^\infty T_Q[2]) = 0 \)

(iii) \( \text{DA}(S)(\Sigma^\infty L_Q, \Sigma^\infty A_Q[1]) = 0 \)

**Proof.** Since \( S \) is regular, and in particular geometrically unibranch, Lemma A.2 together with Proposition implies that there exists \( e : T \to S \) finite étale such that \( \Sigma^\infty L_Q \) is a direct factor of \( e_* \mathcal{O}_T \) and that \( \Sigma^\infty T_Q \) is a direct factor of \( e_* \mathcal{O}_T(1)[1] \). By adjunction and Proposition 2.2, we then reduce to the case \( L \), \( T \) trivial. Point (ii) then says that \( \text{DA}(S)(Q, Q(1)[3]) = 0 \), which is proved in Proposition B.6 (iv) (or in Proposition B.3). By [6, ], writing \( \pi : A \to S \) for the structure morphism, we see that \( \Sigma^\infty A_Q \) is a direct factor of \( \pi_* Q_A \). By adjunction, we have
\[ \text{DA}(S)(\pi_* Q_A, Q(1)[3]) \simeq \text{DA}(A)(Q_A, Q_A[1][3]) \]
which vanishes, again by Proposition B.6 (iv). This proves (iii). Write \( d \) for the fibre dimension of \( A/S \). We have \( \pi_* Q_A \simeq \pi_* Q_A(d)[2d] \), hence by adjunction
\[ \text{DA}(S)(Q, \pi_* A_Q[1]) \simeq \text{DA}(A)(Q, Q(d)[2d + 1]) \]
This last group vanishes by Proposition B.3.

We turn to the question of base change for \( \mathcal{P}(X/S) \). Let us define a natural base change map for \( \mathcal{P}(X/S) \). Consider a cartesian diagram
\[
\begin{array}{ccc}
X' & \xrightarrow{\pi} & X \\
\downarrow f & & \downarrow f \\
S' & \xrightarrow{\pi} & S 
\end{array}
\]
with \( \pi \) any morphism of schemes. We have the following composition
\[ \pi^{-1} \mathcal{P}(X/S) = \pi^{-1} \tau_{\geq 0}(f_* \mathcal{G}_m \otimes \mathbb{Q}[1]) \to \pi^{-1} f_* \mathcal{G}_m \otimes \mathbb{Q}[1] \to \tilde{f}_* \pi^{-1} \mathcal{G}_m \otimes \mathbb{Q}[1] \cong \tilde{f}_* \mathcal{G}_m \otimes \mathbb{Q}[1] \]
and this morphism factors through the truncation \( \tau_{\geq 0}(\tilde{f}_* \mathcal{G}_m \otimes \mathbb{Q}[1]) = \mathcal{P}(X_{S'/S'}) \). We define \( v_\pi \) as the resulting morphism \( \pi^{-1} \mathcal{P}(X/S) \to \mathcal{P}(X_{S'/S'}) \).

**Corollary 2.45.** Let \( S, S' \) be regular schemes, and \( f : X \to S \) be a Pic-smooth morphism. Let \( \pi : S' \to S \) be a universally open morphism of schemes. Then the base change map
\[ v_\pi : \pi^{-1} \mathcal{P}(X/S) \to \mathcal{P}(X'/S') \]
is an isomorphism.
Proof. By Corollary 2.43, we know how to compute \( P(X/S) \) and \( P(X'/S') \). From the commutation of \( \pi_0(X/S) \) with arbitrary base change and Proposition 2.33, we see that \( v_\pi \) is an isomorphism if and only if \( v^{NS}_{\pi S} : \pi^{-1} N^{sm}_X/S \to N^{sm}_X/S \) is. By Lemma 2.41 and Proposition 2.42, we see that this is the case since \( v^{N}_{\pi} \) is an isomorphism by construction.

Another important corollary is the comparison with the theory with transfers.

**Corollary 2.46.** Let \( S \) be a regular excellent scheme and \( f : X \to S \) a smooth projective Pic-smooth morphism. We have distinguished natural triangles

\[
\left( \mathcal{L} \mathcal{G}_{m, tr} \right) \otimes \mathbb{Q} \to P(X/S)_G^\triangleright \to (\mathcal{P}tr_{X/S}^{sm, tr} \otimes \mathbb{Q}) \to \mathcal{V} \to (\mathcal{G}_{m, tr} \otimes \mathbb{Q}) \to \mathbb{Q} \to \mathbb{Q} \to .
\]

and these triangles are (non-canonically) split. Moreover, the natural map

\[
a^\triangleright P(X/S) \to P^\triangleright(X/S)
\]

is an isomorphism.

Proof. The distinguished triangles follow from the same arguments as for \( P(X/S) \). For \( G/S \) smooth commutative group scheme, the natural map \( a^\triangleright G \otimes \mathbb{Q} \to G^\triangleright \otimes \mathbb{Q} \) is an isomorphism by [6, Proposition 2.10]. Since each term of the triangles is represented by a smooth commutative group scheme, we deduce that the map \( a^\triangleright P(X/S) \to P^\triangleright(X/S) \) is an isomorphism.

Finally, we look more closely at the case of a relative smooth projective curve, where things are simpler.

**Proposition 2.47.** Let \( f : C \to S \) be a smooth projective relative curve (for \( S \) arbitrary). Then \( f \) is Pic-smooth, and \( N^{sm}_{C/S} \) is represented by a lattice canonically isomorphic to \( \mathbb{Q}[\pi_0(C/S)] \). In particular, for any \( g : T \to S \), the morphism \( v_g : g^{-1} P(C/S) \to P(C_T/T) \) is an isomorphism.

Proof. When \( f \) has connected fibres, this is contained in the computation of relative Picard schemes for smooth projective curves in [21, Theorem 9.3.1]. Since \( \pi_0(C/S) \) is finite étale, the general case follows by étale descent. The addendum comes from Corollary 2.45 and the fact that the construction of \( \pi_0(C/S) \) commutes with arbitrary base change.

We adopt a special notation in this case.

**Notation 2.48.** Let \( f : C \to S \) be a smooth projective relative curve. We call the abelian scheme \( \text{Pic}^{0, \text{red}}(C/S) \) the (relative) Jacobian of \( C \) over \( S \), and we denote it by \( \text{Jac}(C/S) \).

Let \( f : X \to S \) be a finite type morphism of schemes. We introduce a morphism \( \Theta_f : \Sigma^\infty P(X/S)(-1)[-2] \to f_* Q_X \) which plays a key role in the computation of the motivic Picard functor in the next section.

We start with the adjunction morphism

\[
\text{Sus}^1 \text{Ev}_1 f_* Q_X \xrightarrow{\eta} f_* Q_X.
\]

The functors \( \text{Ev}_1 \) and \( f_* \) commute, because they are right derived functors of right Quillen functors which commute at the model category level. We thus have a canonical isomorphism

\[
f_* \text{Ev}_1 \simeq \text{Ev}_1 f_* : \text{DA}^{\text{eff}}(X) \to \text{DA}(S).
\]

By composition we obtain a map

\[
\text{Sus}^1 f_* \text{Ev}_1 (Q_X) \to f_* Q_X.
\]

We then use the morphism \( w_S \) described at the end of Section 2.1 to construct a map

\[
\text{Sus}^1 f_* (G_m \otimes \mathbb{Q}[1]) \to f_* Q_X.
\]

Recall that \( \text{Sus}^1 \simeq \Sigma^\infty (-)(-1)[-2] \) so that we get a morphism

\[
\Sigma^\infty f_* (G_m \otimes \mathbb{Q}[1])(-1)[-2] \to f_* Q_X.
\]

Then by composing with the counit morphism \( \tau_{\geq 0}(-) \to \text{id} \), we obtain the desired morphism

\[
\Theta_f : \Sigma^\infty P(X/S)(-1)[-2] \to f_* Q_X.
\]
We can do the same construction in $\text{DM}(-)$ using $w^r_3$ and the pushforward operations in $\text{DM}(-)$, resulting in a morphism
\[ \Theta_f^\text{tr} : \Sigma^\infty P^\text{tr}(X/S)(-1)[-2] \to f_! Q^\text{tr}_X \]
in $\text{DM}(S)$. Later on, we will need an alternative description of the map $\Theta_f^\text{tr}$ at the effective level. Recall from the conventions section that $\text{DM}^\text{eff}(-)$ has its own functoriality, in the form of a premotivic category as in [25, §11.1.a].

**Proposition 2.49.** Let $S$ be a regular scheme and $f$ be a Pic-smooth morphism.

(i) The natural morphism $a^\text{tr} : a^\text{tr} \Sigma^\infty P(X/S) \simeq \Sigma^\infty a^\text{tr} P(X/S) \to \Sigma^\infty P^\text{tr}(X/S)$ is an isomorphism by Corollary 2.46, and the natural morphism $a^\text{tr} f_! Q_X \to f_! Q^\text{tr}_X$ in $\text{DM}(S)$ is an isomorphism because of the comparison theorem between $\text{DA}$ and $\text{DM}$ on geometrically unibranch schemes [25, 16.2.22]. Modulo these identifications, we have
\[ a^\text{tr} \Theta_f = \Theta_f^\text{tr}. \]

(ii) The morphism $\Theta_f^\text{tr}$ admits the following alternative description. The morphism $a^\text{eff,tr}_G : \Sigma^\infty Q^{1}(1)[1] \to \Sigma^\infty Q^{1} \otimes \Sigma^\infty Q$ is an isomorphism in $\text{DM}^\text{eff}(X)$ by [25, Proposition 11.2.1] since $X$ is normal, and we denote by $u^\text{eff,tr}_X$ its inverse, so that we have $\Sigma^\infty u^\text{eff,tr}_X = u^\text{tr}_X$ (they are inverses to the same map). Then $\Theta_f^\text{tr}$ is the composition
\[ \text{Sus}^1 \tau_{\geq 0}(f_! (\Sigma^\infty Q^{1}[1])) \xrightarrow{\text{Sus}^1 f_! u^\text{eff,tr}_X} \text{Sus}^1 f_!(Q^{1}(1)[2]) \simeq (\Sigma^\infty g_! (Q^{1}(1)[2])(-1)[-2]) \to f_! Q^\text{tr} \]
where the last morphism is induced by the natural transformation $\Sigma^\infty f_* \to f_! \Sigma^\infty$ which is constructed by adjunction from the natural isomorphism $\Sigma^\infty f^* \simeq f^* \Sigma^\infty$.

**Proof.** Statement (i) translates into proving the commutativity of the outer square in the following diagram.

\[
\begin{array}{ccc}
\text{Sus}^1 f_* G_m \otimes Q[1] & \xrightarrow{w^X} & \text{Sus}^1 f_* G_m \otimes Q[1] \\
\downarrow & & \downarrow \\
\text{Sus}^1 f_* f_! G_m \otimes Q[1] & \xrightarrow{w^X} & \text{Sus}^1 f_* f_! G_m \otimes Q[1] \\
\end{array}
\]

\[
\begin{array}{ccc}
a^\text{tr} & & a^\text{tr} \\
\downarrow & & \downarrow \\
\text{Sus}^1 f_* f_! G_m \otimes Q[1] & \xrightarrow{w^X} & \text{Sus}^1 f_* f_! G_m \otimes Q[1] \\
\end{array}
\]

\[
\begin{array}{ccc}
a^\text{tr} & & a^\text{tr} \\
\downarrow & & \downarrow \\
\text{Sus}^1 f_* f_! G_m \otimes Q[1] & \xrightarrow{w^X} & \text{Sus}^1 f_* f_! G_m \otimes Q[1] \\
\end{array}
\]

All squares in this diagram commute either by naturality of adjunctions or because of the commutation $\text{Sus}^1 a^\text{tr} \simeq a^\text{tr} \text{Sus}^1$.

For Statement (ii), we observe that $\Theta_f^\text{tr}$ is defined as the composition
\[
\text{Sus}^1 f_* G_m[1] \xrightarrow{\text{Sus}^1 f_* f_! G_m[1]} \text{Sus}^1 f_* f_! G_m \otimes Q[1] \xrightarrow{w^X} \text{Sus}^1 f_* \text{Ev}_1 Q^{1} \simeq \text{Sus}^1 f_* \text{Ev}_1 Q^{1} \xrightarrow{\eta} f_* Q^\text{tr}
\]
(we have expanded the definition of $w^X_3$), whereas the map of the statement is the composition
\[
\text{Sus}^1 f_* G_m[1] \xrightarrow{\text{Sus}^1 f_* f_! G_m[1]} \text{Sus}^1 f_* Q^{1}(1)[2] \xrightarrow{\text{Sus}^1 f_* \text{Ev}_1 \text{Sus}^1 f_* g_! Q^{1}(1)[2]} \text{Sus}^1 f_* Q^{1}(1)[2] \xrightarrow{\eta} f_* Q^\text{tr}
\]
(we have expanded the definition of the map $\Sigma^\infty f_* \to f_* \Sigma^\infty$). The equality of those two compositions follows from the naturality of the $(\text{Sus}^1, \text{Ev}_1)$ adjunction and the equality
\[ \text{Sus}^1 u^\text{eff,tr}_X = \Sigma^\infty u^\text{eff,tr}_X(1)[-2] = u^\text{tr}_X(-1)[-2] \]
\[ \blacksquare \]

We finish with a study of the compatibility of the map $\Theta_f$ with base change.
Proposition 2.50. Let \( f : X \to S \) be a smooth projective Pic-smooth morphism of schemes. Let \( g : T \to S \) be any morphism. Let \( f' : X_T \to T \) be the pullback (which is still smooth projective Pic-smooth by Lemma 2.31). The diagram

\[
g^* \Sigma^\infty P(X/S)(-1)[-2] \xrightarrow{\nu_{g^*R^s}} g^* f_* \mathbb{Q}_X \\
\Sigma^\infty P(X_T/T)(-1)[-2] \xrightarrow{\nu_{f'_*}} f'_* \mathbb{Q}_{X_T}
\]

commutes in \( \text{DA}(S) \).

Proof. The first observation is that, using the natural transformation \( g^* \tau_{\geq 0} \to \tau_{\geq 0} g^* \), we can reduce to the same commutation for the full \( f_* \mathbb{G}_m \otimes \mathbb{Q}[1] \) instead of \( P(X/S) \).

In the rest of the proof, we need notations for natural transformations

\[
(\alpha_f) : f^* \text{Sus}^1 \xrightarrow{\sim} \text{Sus}^1 f^* \\
(\beta_f) : f_* \text{Ev}_1 \xrightarrow{\sim} \text{Ev}_1 f_*
\]

and

\[
(\gamma_f) : f^* \text{Ev}_1 \xrightarrow{\sim} \text{Ev}_1 f^*.
\]

The natural isomorphisms \((\alpha), (\beta)\) are derived versions of isomorphisms at the level of model categories of spectra. The natural transformation \((\gamma)\) can be defined in two different ways, one using \((\alpha)\) and one using \((\beta)\); namely, as the two equal compositions

\[
f^* \text{Ev}_1 \xrightarrow{\sim} \text{Ev}_1 \text{Sus}^1 f^* \xrightarrow{\alpha_f^{-1}} \text{Ev}_1 f^* \text{Sus}^1 \xrightarrow{\eta} \text{Ev}_1 f^*
\]

and

\[
f^* \text{Ev}_1 \xrightarrow{\sim} g^* \text{Ev}_1 g_* g^* \xrightarrow{(\beta_f)^{-1}} g^* g_* \text{Ev}_1 g^* \xrightarrow{\eta} \text{Ev}_1 f^*
\]

Writing down the definition of the maps in the square, we see that we have to show the commutation of the outer square in the following diagram (when an arrow is obtained from another one by a clear functoriality, we omit the functor from the notation as well; for instance the first vertical arrow in the top left should be named \( g^* \text{Sus}^1 f_* w_S \)).

\[
\begin{array}{c}
g^* \text{Sus}^1 f_* \mathbb{G}_m[1] \xrightarrow{w_S} g^* \text{Sus}^1 f_* \mathbb{Q} \xrightarrow{(\beta_f)} g^* \text{Ev}_1 \mathbb{Q} \xrightarrow{\alpha_f} g^* \text{Ev}_1 f_* \mathbb{Q} \xrightarrow{\eta} g^* f_* \mathbb{Q} \\
\sim \xrightarrow{(\alpha_f)} \sim \xrightarrow{(\alpha_f)} \sim \xrightarrow{(\alpha_f)} \\
\text{Sus}^1 g^* f_* \mathbb{G}_m[1] \xrightarrow{w_S} \text{Sus}^1 g^* f_* \mathbb{Q} \xrightarrow{(\beta_f)} \text{Sus}^1 \text{Ev}_1 \mathbb{Q} \xrightarrow{\alpha_f} \text{Sus}^1 \text{Ev}_1 f_* \mathbb{Q} \xrightarrow{\eta} \\
\sim \xrightarrow{\text{Ev}_s} \text{Ev}_s \xrightarrow{(\gamma_f)} \\
\text{Sus}^1 f'_* g^* \mathbb{G}_m[1] \xrightarrow{w_S} \text{Sus}^1 f'_* g^* \mathbb{Q} \xrightarrow{(+)} \text{Sus}^1 \text{Ev}_1 g^* f_* \mathbb{Q} \\
\sim \xrightarrow{R_{f'}} \text{Ev}_s \xrightarrow{(\gamma'_{f'})} \text{Ev}_s \xrightarrow{(\gamma_f)} \\
\text{Sus}^1 f'_* \mathbb{G}_m[1] \xrightarrow{w_S} \text{Sus}^1 f'_* \mathbb{Q} \xrightarrow{(\beta_{f'})} \text{Sus}^1 \text{Ev}_1 f'_* \mathbb{Q} \xrightarrow{\eta} f'_* \mathbb{Q}
\end{array}
\]

The commutation of the three squares in the top left corner and of the bottom right corner follows directly by naturality of various natural transformations. The bottom left square commutes by Proposition 2.13. The top right square commutes by the first description of \((\gamma)\).
It remains to show the commutation of $(\ast)$. By expanding the second description of $(\gamma)$, we see that we have to show the commutativity of the outer square in the following diagram.

$$
\begin{array}{cccc}
g^* f_! \text{Ev}_1 Q & \overset{\beta_f}{\longrightarrow} & g^* \text{Ev}_1 f_* Q \\
\text{Ex}^* & & & \\
f'_! g^* \text{Ev}_1 Q & \overset{\epsilon_g}{\longrightarrow} & g^* f_! \text{Ev}_1 Q & \overset{\beta_f}{\longrightarrow} & g^* \text{Ev}_1 f_* Q \\
\end{array}
$$

The commutation of each of the subdiagrams follow from naturality properties of various natural transformations and from the definition of the exchange maps $\text{Ex}^*_n$. This completes the proof. □

3. Motivic Picard functor

We introduce and study the motivic Picard functor $\omega^1$, which is a (mixed motivic, relative) generalisation of the Picard variety of a smooth projective variety over a field. We also study in parallel the 0-motivic analogue $\omega^0$, which was first introduced in [15].

3.1. Definition and elementary properties.

**Definition 3.1.** Let $n \geq 0$. The full embedding $\iota^n : \text{DA}^n(S) \hookrightarrow \text{DA}^{\text{coh}}(S)$ preserves small sums, thus by Neeman’s version of Brown representability for compactly generated triangulated categories (see e.g. [51, Theorem 8.3.3]), $\iota^n$ admits a right adjoint $\omega^n : \text{DA}^{\text{coh}}(S) \to \text{DA}^n(S)$. We also write $\omega^n$ for the functor $\text{DA}^{\text{coh}}(S) \to \text{DA}^{\text{coh}}(S)$ obtained by postcomposing with $\iota^n$. We write $\delta^n : \omega^n \to \text{id}$ for the natural transformation induced by the counit.

**Remark 3.2.** The definition above can be extended to the whole of $\text{DA}(S)$, but the resulting functors are not well-behaved; in particular, they do not respect compactness. Here is the simplest example of this phenomenon. Let $k$ be an algebraically closed field. It is easy to see that the category $\text{DA}_0,0(k)$ is equivalent to the bounded derived category of the category of finite dimensional $Q$-vector spaces. In particular homomorphisms groups in this category are finite dimensional. On the other hand, $\text{DA}(k)[\mathbb{Q}_k, \mathbb{Q}_k(1)[1]] \simeq k^\infty \otimes Q$ (Proposition B.6) is not finite dimensional in general. This shows that $\omega^0(\mathbb{Q}(1))$ is not compact.

We start by giving some general formal properties of all the $\omega^n$ functors.

**Proposition 3.3.** Let $S$ be a noetherian finite-dimensional scheme.

(i) Let $M \in \text{DA}^n(S)$. Then we have an isomorphism $\delta^n(M) : \omega^n(M) \simeq M$ and the natural transformation $\delta^n(\omega^n) : \omega^n \circ \omega^n \to \omega^n$ is invertible.

(ii) Let $f : T \to S$ be any morphism of schemes. There is a natural transformation $\alpha_f^n : f^* \omega^n \to \omega^n f^*$ making the triangles

$$
\begin{array}{ccc}
f^* \omega^n & \overset{\alpha_f^n}{\longrightarrow} & \omega^n f^* \\
\delta^n(f^*) & & \delta^n(f^*) \\
f^* & & f^*
\end{array}
$$

and

$$
\begin{array}{ccc}
\omega^n f^* \omega^n & \overset{\delta^n(f^* \omega^n)}{\longrightarrow} & f^* \omega^n \\
\omega^n f^* & \overset{\alpha_f^n}{\longrightarrow} & f^* \omega^n
\end{array}
$$

(iii) Let $f : T \to S$ be any morphism of schemes. The natural transformation $\omega^n f_*(\delta^n)$ is invertible. Moreover there is a natural transformation $\beta_f^n : \omega^n f_* \to f_* \omega^n$ such that

...
a) the following triangles
\[
\begin{array}{cccc}
\omega^n f_* & \beta^n_j & f_* \omega^n & \\
\downarrow & & \downarrow & \\
\delta^n(f_*) & & f_* (\delta^n) & \\
\end{array}
\quad \text{and} \quad
\begin{array}{cccc}
\omega^n f_* & \omega^n(f_* \delta^n) & \omega^n f_* & \\
\downarrow & & \downarrow & \\
\delta^n(f_*) & & f_* (\delta^n) & \\
\end{array}
\]
are commutative,
b) \(\omega^n(\beta^n_j)\) is invertible for any \(f\), and
c) \(\beta^n_j\) is invertible for \(f\) finite.

(iv) Let \(e : T \to S\) be a quasi-finite morphism of schemes. There exists a natural transformation \(\eta^n_e : e_! \omega^n \to \omega^n e_!\) such that
a) the following triangles
\[
\begin{array}{cccc}
e_! \omega^n & \eta^n_e & \omega^n e_! & \\
\downarrow & & \downarrow & \\
e_!(\delta^n) & & e_!(\delta^n) & \\
\end{array}
\quad \text{and} \quad
\begin{array}{cccc}
\omega^n e_! & \omega^n(\eta^n_e) & \omega^n e_! & \\
\downarrow & & \downarrow & \\
(\omega^n e_!)(\delta^n) & & \omega^n e_!(\delta^n) & \\
\end{array}
\]
commute, and
b) when \(e\) is finite, \(\eta^n_e\) is invertible and coincides with \(\beta^{-1}_{e_!}\) modulo the natural isomorphism \(e_! \simeq e_*\).

(v) Let \(e : T \to S\) be a quasi-finite morphism. The natural transformation \(\omega^n e_! (\delta^n)\) is invertible. Moreover, there is a natural transformation \(\gamma^n_e : \omega^n e_! \to e_! \omega^n\) such that
a) the following triangles
\[
\begin{array}{cccc}
\omega^n e_! & \gamma^n_e & e_! \omega^n & \\
\downarrow & & \downarrow & \\
\omega^n e_!(\delta^n) & & \omega^n e_! (\delta^n) & \\
\end{array}
\quad \text{and} \quad
\begin{array}{cccc}
\omega^n e_! & \omega^n e_! (\gamma^n_e) & \omega^n e_! & \\
\downarrow & & \downarrow & \\
\omega^n e_! (\gamma^n_e)(\delta^n) & & \omega^n e_! (\gamma^n_e)(\delta^n) & \\
\end{array}
\]
are commutative,
b) \(\omega^n(\gamma^n_e)\) is invertible for any \(e\) quasi-finite, and
c) \(\gamma^n_e\) is invertible for \(e\) \(\acute{e}tale\).

(vi) Let \(j : U \to S\) and \(i : Z \to S\) be complementary open and closed immersions. Let \(M \in DA^{\text{coh}}(S)\) with \(j^* M \in DA^n(S)\). Then the morphism \(i^* \omega^n M \to \omega^n i^* M\) is invertible.

Remark 3.4. The formulation of Proposition 3.3 follows closely the one of [15, Proposition 2.16] about \(\omega^0\). More precisely, it is a direct generalisation to all \(\omega^n\) and to more general base schemes of all statements of loc. cit., except the assertions that \(\alpha^n_f\) is invertible for \(f\) smooth and that \(\omega^0\) preserves compact objects. Unlike the others, these properties of \(\omega^0\) are not formal. We study their generalisation to more general base schemes and higher \(n\)'s below.

Proof. We can apply verbatim the proof of [15, Proposition 2.16] up to the sentence “To complete the proof (…)” on page 319. Notice that the rest of the proof after that sentence establishes the non-formal assertions described in the previous remark, which are exactly the points we are not claiming.

More precisely, up to that sentence, the proof of loc. cit. uses only general properties of \(DA\), the definition of \(\omega^0\) as right adjoint, and the following permanence properties of cohomological 0-motives under the six operations.

- For all morphisms \(f\), the functor \(f^*\) preserves \(DA^0\).
- For all finite morphisms \(f\), the functor \(f_*\) preserves \(DA^0\).
- For all quasi-finite morphism \(e\), the functor \(e_!\) preserves \(DA^0\).

The generalisation of these properties to \(DA^n\) are established in the necessary generality in Proposition 1.16. \(\square\)

Here are other useful common properties of the \(\omega^n\)'s.
Lemma 3.5. Let $S$ be a noetherian finite dimensional scheme and $n \in \mathbb{N}$. The functor $\omega^n : \text{DA}^\text{coh}(S) \to \text{DA}^n(S)$ commutes with small sums.

Proof. The inclusion functor $\text{DA}^n(S) \to \text{DA}^\text{coh}(S)$ sends compact objects to compact objects by Lemma 1.8; hence by [7, Lemme 2.1.28], its right adjoint $\omega^n$ commutes with infinite sums. \hfill \Box

Lemma 3.6. Let $h : S' \to S$ be a finite purely inseparable morphism of schemes, and $n \in \mathbb{N}$. The natural transformation $\alpha_h : h^*\omega^1 \to \omega^1h^*$ is an isomorphism.

Proof. This follows directly from the separation property of $\text{DA}(-)$ and Corollary 1.18 (ii). \hfill \Box

We now come to the less formal properties of $\omega^0$.

Proposition 3.7. Let $S$ be noetherian of finite dimension.

(i) Let $f : X \to S$ be a smooth proper morphism of schemes. Let $X \xrightarrow{f^*} \pi_0(X/S) \xrightarrow{\pi_0(f)} S$ be its Stein factorisation, with $\pi_0(f)$ finite étale. Then there is a natural isomorphism $\pi_0(f)^*, \mathbb{Q}_{\pi_0(X/S)} \xrightarrow{\sim} \omega^0(f_*\mathbb{Q}_X)$.

(ii) The functor $\omega^0$ preserves geometrically smooth objects. More precisely, it sends $\text{DA}_{\text{gsm}}^\text{coh}(S)$ to $\text{DA}_{\text{gsm}}^0(S)$ and $\text{DA}_{\text{gsm},c}^\text{coh}(S)$ to $\text{DA}_{\text{gsm},c}^0(S)$. Moreover, for any $M \in \text{DA}_{\text{gsm}}^\text{coh}(S)$ and any morphism $f : T \to S$, the natural morphism $\alpha^0_f(M) : f^*\omega^0M \to \omega^0f^*M$ is an isomorphism.

(iii) The morphism $\alpha^0_f$ is invertible for $f$ smooth.

(iv) The functor $\omega^0$ preserves compact objects. More precisely, it sends $\text{DA}^\text{coh}_c(S)$ to $\text{DA}^0_c(S)$.

Remark 3.8. These results were proved in [15, §2] under the assumption that $S$ is quasi-projective over a field $k$ and $f$ is projective; they were also generalized, in a slightly different terminology, to the case of $S$ separated of finite type over a field in [59, §3.1-2].

Proof. It is easy to see from the definition of geometrically smooth motives and the fact that $\pi_0$ commutes with base change that point (ii) follows from (i). We now notice that the end of the proof of [15, Proposition 2.16] (starting at “To complete the proof (...”), which deduces (iii) and (iv) in the situation of loc. cit. from [15, Proposition 2.11], applies verbatim and reduce Statements (ii)-(iv) to the sole Statement (i).

To prove Statement (i), it is enough by the Yoneda lemma to establish that for all $N \in \text{DA}^0(S)$, the natural map $\pi_0(f)^*, \mathbb{Q} \to f_*\mathbb{Q}_X$ induces an isomorphism $\text{DA}(S)(N, \pi_0(f)^*, \mathbb{Q}) \xrightarrow{\sim} \text{DA}(S)(N, f_*\mathbb{Q}_X)$. By Proposition 1.27, we have $\text{DA}^0(S) = \text{DA}_0(S)$. It is thus enough to show that for all $e : U \to S$ étale and $n \in \mathbb{Z}$, we have an isomorphism $\text{DA}(S)(e^*\mathbb{Q}_U[-n], \pi_0(f)^*, \mathbb{Q}) \xrightarrow{\sim} \text{DA}(S)(e^*\mathbb{Q}_U[-n], f_*\mathbb{Q}_X)$.

By the $(e^*, e^*)$ adjunction, proper base change, and the fact that $\pi_0$ commutes with smooth base change, we see that we can assume $e = \text{id}$. We are thus left to prove that for all $n \in \mathbb{Z}$, we have $\text{DA}(\pi_0(X/S))(\mathbb{Q}, \mathbb{Q}[n]) \xrightarrow{\sim} \text{DA}(X)(\mathbb{Q}, \mathbb{Q}[n])$ where the morphism is induced by pullback by $f^*$. The morphism $f^*$ is smooth proper with geometrically connected fibres, so this follows from Proposition B.5 (iv). \hfill \Box

Here are some corollaries of Proposition 3.7.

Corollary 3.9. Let $S$ be a noetherian finite-dimensional scheme.

(i) Let $M$ be in $\text{DA}_{\text{hom}}(S)$ and $N$ be in $\text{DA}^\text{coh}(S)$. Then the morphism $\delta^0(N)$ induces an isomorphism $\text{DA}(S)(M, \omega^0N) \xrightarrow{\delta^0(N)^*} \text{DA}(S)(M, N)$.

(ii) We have $\text{DA}_{\text{hom}}(S) \cap \text{DA}^\text{coh}(S) = \text{DA}^0(S)$.

(iii) For all $N \in \text{DA}^\text{coh}(S)$ we have $\omega^0(N(-1)) \simeq 0$.

(iv) For all $N \in \text{DA}^\text{coh}(S)$ and $d \geq 1$, we have $\omega^1(N(-d)) \simeq \begin{cases} (\omega^0N)(-1), & d = 1 \\ 0, & d \geq 2. \end{cases}$.
In particular, by Proposition 1.27, we have \( DA ) \). Hence by the Yoneda lemma, it is enough to show that for all
\[ \delta \]
Since this is a compact object, we can similarly assume that 
\[ \delta \]
Proposition 3.3 (ii) shows that we have a commutative square
\[ \delta \]
Since \( g \) is smooth, the left vertical map is an isomorphism by Proposition 3.7 (ii): the bottom map is an isomorphism because \( Q_X[n] \) is a cohomological 0-motive. Putting this together with the previous commutative square concludes the proof of (i).

Statement (ii) follows directly from (i) applied to the identity map of an object in \( DA^{coh}(S) \cap DA_{hom}(S) \).

To prove Statement (iii), we must show that for all \( M \in DA^0(S) \), we have \( DA(S)(M, N(-1)) = 0 \). Since \( DA^0(S) = DA_0(S) \) by Proposition 1.27 and \( DA_{hom}(S) \) is stable by positive twists by Proposition 1.10 (iv), the motive \( M(1) \) is homological. By (i), this implies that \( DA(S)(M(1), N) \cong DA(S)(M, N(0)) \). In other words, we can assume that both \( M \) and \( N \) are 0-motives. The statement to be proven is triangulated and commutes with infinite sums in \( M \), so that we can assume that \( M \) is a generator of the form \( e_q Q_U[n] \) for \( e : U \to S \) an étale morphism and \( n \in \mathbb{Z} \).

Since this is a compact object, we can similarly assume that \( N \) is a generator of \( DA^0(S) \), of the form \( f_q Q_V[m] \) for \( f : V \to S \) a finite morphism. We then have
\[ DA(S)(M, N(-1)) \cong DA(U \times_S V)(Q, Q(-1)[m - n]) \]
This group vanishes by Proposition B.2.

By (iii), we only need to establish (iv) in the case \( d = 1 \). The motive \( \omega^0(N)(-1) \) is in \( DA^1(S) \) by Proposition 1.10 (ii). Hence by the Yoneda lemma, it is enough to show that for all \( M \in DA^1(S) \), the map \( \delta^0(N) \) induces an isomorphism
\[ DA(S)(M, \omega^0 N)(-1) \overset{\sim}{\longrightarrow} DA(S)(M, N(-1)) \]
By Proposition 1.27, we have \( DA^1(S) = DA_1(S)(-1) \). Write \( M = M'(-1) \) with \( M' \in DA_1(S) \). In particular, \( M' \) is an homological motive. We have a commutative square
\[ \delta \]
The bottom map is an isomorphism by (i), and this concludes the proof in case \( d = 1 \).

We now compute \( \omega^0 \) for some motives attached to commutative group schemes.

**Proposition 3.10.**

(i) Let \( G \) be an abelian scheme or a lattice over \( S \); then \( \omega^0(\Sigma^\infty G_Q)(-1) \cong 0 \).

(ii) Let \( T \) be a torus over \( S \). Let \( X_* (T) \) be the cocharacter lattice of \( T \). Then \( \Sigma^\infty T_Q(-1) \cong \Sigma^\infty X_*(T)_Q \) is in \( DA_{0,c}(S) \).

(iii) Let \( M \in M_1(S) \) and \( W_{-2} M \) be its toric part. Then \( \omega^0(\Sigma^\infty M)(-1) \cong \Sigma^\infty X_*(W_{-2} M)_Q \).
Proof. First of all, we note that the objects to which we wish to apply $\omega^0$ are in $\text{DA}^1(S) \subset \text{DA}^{\text{coh}}(S)$ by Corollary 2.14 and Proposition 1.27.

We first prove (i). We first treat the case of an abelian scheme $A$. By the conservativity of the family of pullbacks to points [11, Proposition 3.23], it is enough to show that for any $s \in S$, the restriction $s^*\omega^0 M \simeq 0$. We know from [6, Theorem 3.3] (essentially, in this case, the theorem of Deninger and Murre on Chow motives of abelian schemes over a regular base) that the motive $\Sigma^\infty G_\mathbb{Q}$ is geometrically smooth. By Proposition 3.7 (ii) and Proposition 2.2, we see that $s^*\omega^0 M \simeq \omega^0 s^* M \simeq \omega^0 \Sigma^\infty (A_s)_{\mathbb{Q}}(-1)$. We are thus reduced to the case where $S$ is the spectrum of a field $k$. We have to show that, for every 0-motive $N$ over $\text{Spec}(k)$, we have

$$\text{DA}(k)(N, \Sigma^\infty \text{Jac}(C)_k[n]) = 0.$$ 

The category $\text{DA}_0(k)$ is generated, as a localizing subcategory, by motives of the form $g_\ell \mathbb{Q}_L$ with $g : \text{Spec}(L) \to \text{Spec}(k)$ with $L/k$ finite étale. By adjunction, we are then reduced to the case $N = \mathbb{Q}_k[-n]$ for some $n \in \mathbb{Z}$.

We write $A$ as direct factor of the Jacobian of a smooth projectively geometrically connected curve $f : C \to \text{Spec}(k)$ [45, Theorem 11]. By Proposition 2.5 and relative purity, we have

$$\mathbb{Q}(-1)[-2] \oplus \Sigma^\infty \text{Jac}(C)_\mathbb{Q}(-1)[-2] \oplus \mathbb{Q} \simeq f_* \mathbb{Q}_C.$$

We have $\text{DA}(k)(\mathbb{Q}_k, \mathbb{Q}_k(-1)[n]) = 0$ for all $n$ (Proposition B.2). By adjunction, we have

$$\text{DA}(k)(\mathbb{Q}_k, f_* \mathbb{Q}_C[n]) \simeq \text{DA}(C)(\mathbb{Q}_C, \mathbb{Q}_C[n])$$

which is isomorphic to $\mathbb{Q}$ for $n = 0$ and 0 otherwise (Proposition B.5). Similarly, we have $\text{DA}(k)(\mathbb{Q}_k, \mathbb{Q}_k[n])$ is isomorphic to $\mathbb{Q}$ for $n = 0$ and 0 otherwise. Putting everything together, for any $n$ we deduce that $\text{DA}(k)(\mathbb{Q}_k, \Sigma^\infty \text{Jac}(C)_k[n]) = 0$, as required.

We now turn to the lattice case. Again by an adjunction argument, we immediately reduce to show that, for all $n \in \mathbb{Z}$, we have

$$\text{DA}(S)(\mathbb{Q}_S, \Sigma^\infty L_\mathbb{Q}[n]) = 0.$$

If $S$ is geometrically unibranch, using Lemma A.2, write $M$ as a direct factor $f_* \mathbb{Q}$ for $f$ finite étale, and we are done by adjunction and Proposition B.2.

Unfortunately, if the base is not geometrically unibranch, it is not clear that $M$ is geometrically smooth, and we cannot directly reduce to the field case. However, iterating the construction of the normalisation, it is easy to see that $S$ admits a proper hypercovering $\pi_* : S_* \to S$ with normal terms. By cohomological $h$-descent for $\text{DA}(\text{-})$ and Proposition 2.2, we get a spectral sequence

$$E_{p,q}^1 = \text{DA}(S_p)(\mathbb{Q}_{S_p}, \Sigma^\infty (L_{S_p} \otimes \mathbb{Q})[q]) \Rightarrow \text{DA}(S)(\mathbb{Q}_S, \Sigma^\infty (L \otimes \mathbb{Q})[p + q]).$$

By the geometrically unibranch case, the $E_{p,q}^1$ term of the spectral sequence vanishes completely. This ensures the convergence and finishes the proof of (i).

We prove (ii). Let $T$ be a torus. We have $\Sigma^\infty T_\mathbb{Q}(-1) \simeq \Sigma^\infty X_*(T)_{\mathbb{Q}}$ by Corollary 2.9. The motive $\Sigma^\infty X_*(T)_{\mathbb{Q}}$ lies in $\text{DA}_0(S)$: this can be tested pointwise by Proposition 1.24, and over a field a lattice is a direct factor of the motive of a finite étale morphism by Lemma A.2. This concludes the proof.

Finally, (iii) follows immediately from the two previous points by a devissage of a Deligne 1-motive along its weight filtration. \hfill \Box

Corollary 3.11. Assume $S$ regular. Let $f : X \to S$ be a smooth projective Pic-smooth morphism of schemes. Then there is an isomorphism

$$\omega_0(\Sigma^\infty P(X/S)_\mathbb{Q}(-1)[-2]) \simeq \pi_0(f)_* \mathbb{Q}$$

Proof. First, by Corollary 2.43, Proposition 2.11 and Proposition 1.27, $\Sigma^\infty P(X/S)_\mathbb{Q}(-1)[-2]$ is in $\text{DA}^1(S)$, and it makes sense to apply $\omega^0$. More precisely, Corollary 2.43 together with Proposition 3.10 shows that there is an isomorphism

$$\omega_0(\Sigma^\infty P(X/S)_\mathbb{Q}(-1)[-2]) \simeq \Sigma^\infty X_* (\text{Res}_{\pi_0(f)_* \mathbb{Q}} \mathcal{G}_m)_\mathbb{Q}.$$

The cocharacter lattice of the Weil restriction $\text{Res}_{\pi_0(f)_* \mathbb{Q}} \mathcal{G}_m$ is the permutation lattice associated to $\pi_0(f)_*$; hence, $\Sigma^\infty X_* (\text{Res}_{\pi_0(f)_* \mathbb{Q}} \mathcal{G}_m)_\mathbb{Q} \simeq \pi_0(f)_* \mathbb{Q}$ as required. \hfill \Box
3.2. The functors $\omega^n$ over a perfect field. In this short section, we explain how, for $S$ the spectrum of a perfect field $k$, the functors $\omega^n$ and $\omega^1$ are related to the functors $L\pi_0$ and LAlb studied in [16] and [14].

We need to connect our setup with the categories of effective motives with transfers over $k$. First, we define for every $n \in \mathbb{N}$ the category $\text{DM}^{\text{eff}}_{n,(c)}(k)$ in a similar as $\text{DA}^{\text{eff}}_{n,(c)}(k)$, replacing $\text{DA}(k)$ with $\text{DM}^{\text{eff}}(k)$ and $f_2\mathbb{Q}_X$ with $M_k^{\text{eff},\text{tr}}(X)$ for $f : X \to \text{Spec}(k)$ smooth. We also define the category $\text{DM}^{\text{hom}}_{n,(c)}(k)$ (resp. $\text{DM}^{\text{coh}}_{n,(c)}(k)$) in a similar way as $\text{DA}^{\text{hom}}_{n,(c)}(k)$ (resp. $\text{DA}^{\text{coh}}_{n,(c)}(k)$).

Recall that by construction of $\text{DM}(k)$ as homotopy category of spectra, there is an adjunction

$$\Sigma^n_{\text{tr}} : \text{DM}^{\text{eff}}(k) \rightleftarrows \text{DM}(k) : \Omega^n_{\text{tr}}.$$

**Lemma 3.12.** Let $k$ be a perfect field and $n \in \mathbb{N}$. The adjoint pairs $\Sigma^n_{\text{tr}} \dashv \Omega^n_{\text{tr}}$ and $a_{\text{tr}} \dashv o^\text{tr}$ restrict to equivalences of categories

$$\text{DA}_{n,(c)}(k) \xrightarrow{a_{\text{tr}}}_{o^\text{tr}} \text{DM}^{\text{eff}}_{n,(c)}(k),$$

$$\text{DA}^{\text{hom}}_{n,(c)}(k) \xrightarrow{a_{\text{tr}}}_{o^\text{tr}} \text{DM}^{\text{eff}}_{n,(c)}(k),$$

$$\text{DA}^{\text{coh}}_{n,(c)}(k) \xrightarrow{a_{\text{tr}}}_{o^\text{tr}} \text{DM}^{\text{eff}}_{n,(c)}(k),$$

and $\text{DA}^n_{n,(c)}(k) \xrightarrow{a_{\text{tr}}}_{o^\text{tr}} \text{DM}^n_{n,(c)}(k)$.

**Proof.** The argument is essentially the same for the four series of equivalences; we only give the details for the first one. Recall that $a_{\text{tr}} : \text{DM}(S) \rightarrow \text{DA}(S)$ is an equivalence of categories for all $S$ geometrically unibranch [25, Corollary 16.2.22], hence in particular for all $X$ smooth over $k$. By construction of $a_{\text{tr}}$, we have $a_{\text{tr}}f_2 \simeq f_2a_{\text{tr}}$ for $f$ smooth.

Let us show that the functors $a_{\text{tr}}$, $o^\text{tr}$ both commute with small sums and preserve compact objects. Indeed, $a_{\text{tr}}$ is a left adjoint, it preserves constructible objects by construction ($a_{\text{tr}}M_k(X) \simeq M_k^{\text{eff}}(X)$) and compact = constructible holds in $\text{DM}(k)$ as well as in $\text{DA}(k)$. Furthermore, $a_{\text{tr}}$ has a left adjoint which preserves compact objects hence it commutes with small sums, and it preserves constructible objects because $o^\text{tr}M_k^{\text{eff}}(X) \simeq o^\text{tr}a_{\text{tr}}M_k(X) \simeq M_k(X)$ since we have an equivalence of categories.

Put together, these facts imply that $a_{\text{tr}} \dashv o^\text{tr}$ restricts to an equivalence of categories between $\text{DA}_{n,(c)}$ and $\text{DM}^{\text{eff}}_{n,(c)}$.

By Voevodsky’s cancellation theorem [61], the functor $\Sigma^n_{\text{tr}} : \text{DM}^{\text{eff}} \rightarrow \text{DM}(k)$ is fully faithful. We have $\Sigma^n_{\text{tr}}f_2 \simeq f_2\Sigma^n_{\text{tr}}$ for $f : X \rightarrow \text{Spec}(k)$ smooth; this shows that $\text{DM}_{n,(c)}(k)$ lies in the essential image of $\text{DM}^{\text{eff}}_{n,(c)}(k)$. Since $\Sigma^n_{\text{tr}}$ and $\Omega^n_{\text{tr}}$ both commute with small sums and preserve compact objects (this follows from the same line of argument as for $a_{\text{tr}}$, $o^\text{tr}$ above), this implies further that $\Sigma^n_{\text{tr}} \dashv \Omega^n_{\text{tr}}$ restricts to an equivalence of categories between $\text{DM}^{\text{hom}}_{n,(c)}(k)$ and $\text{DM}^{\text{eff}}_{n,(c)}(k)$. This completes the proof. \qed

By [14, Theorem 2.4.1] specialized to the case of $\mathbb{Q}$-coefficients, we have a functor

$$L\pi_0 : \text{DM}^{\text{eff}}(k) \rightarrow \text{DM}^{\text{eff}}_0(k)$$

(respectively

$$\text{LAlb} : \text{DM}^{\text{eff}}(k) \rightarrow \text{DM}^{\text{eff}}_1(k)$$

which is a left adjoint to the inclusion $\text{DM}^{\text{eff}}_0(k) \rightarrow \text{DM}^{\text{eff}}(k)$ (resp. $\text{DM}^{\text{eff}}_1(k) \rightarrow \text{DM}^{\text{eff}}(k)$) and restricts by [14, Proposition 2.3.3] (resp. [14, Proposition 2.4.7]) to a functor

$$L\pi_0 : \text{DM}^{\text{eff}}_{n,c}(k) \rightarrow \text{DM}^{\text{eff}}_{n,c}(k)$$

(respectively

$$\text{LAlb} : \text{DM}^{\text{eff}}_{n,c}(k) \rightarrow \text{DM}^{\text{eff}}_{1,c}(k))$$

To be more precise, our notation differs from loc. cit. in the following way. The functor $L\pi_0$ (resp. $\text{LAlb}$) in loc. cit. has as target category $D(\text{HI}_{\leq 0}(k))$ (resp. $D(\text{HI}_{\leq 1}(k))$, the derived category.
of the abelian category \( \text{Sh}((\text{Spec}(k))_{\ell}, \mathbb{Q}) \) (resp. \( \text{HI}_{\leq 1}(k) \)) of 1-motivic sheaves [14, Definition 1.1.20]), which is equivalent by [14, Lemma 2.3.1] (resp.[14, Theorem 2.4.1.(i)]) to \( \text{DM}_{\text{eff}}^{0}(k) \) (resp. \( \text{DM}_{\text{eff}}^{n}(k) \)), and the functor we call \( L\pi_{0} \) (resp. \( \text{LAlb} \)) is obtained by composing the functor of loc. cit. with this equivalence.

**Proposition 3.13.** Let \( k \) be a perfect field. The functors \( \omega^{0} \) and \( \omega^{1} \) restrict to compact objects. Moreover, when restricting to compact objects, we have isomorphisms of functors

\[
\omega^{0} \simeq \mathbb{D}_{k}^{} o_{\Sigma_{+}^{\infty}}^{} P_{\pi_{0}}^{\infty} a_{\pi_{1}}^{\infty} D_{k} : \text{DA}_{c}^{\text{coh}}(k) \to \text{DA}_{c}^{0}(k)
\]

and

\[
\omega^{1} \simeq \mathbb{D}_{k}^{} o_{\Sigma_{+}^{\infty}}^{} \text{LAlb} \Omega_{+}^{\infty} a_{\pi_{1}}^{\infty} D_{k} : \text{DA}_{c}^{\text{coh}}(k) \to \text{DA}_{c}^{1}(k).
\]

**Proof.** By Proposition 1.25, the duality functor \( \mathbb{D}_{k} \) restricts to anti-equivalences of categories \( \text{DA}_{c}^{\text{coh}}(k)^{\text{op}} \simeq \text{DA}_{\text{hom},c}(k) \) and \( \text{DA}_{c}^{n}(k)^{\text{op}} \simeq \text{DA}_{n,c}(k) \) for any \( n \in \mathbb{N} \). By Lemma 3.12, this implies that the inclusion \( \text{DA}_{0,c}(k) \to \text{DA}_{c}(k) \) (resp. \( \text{DA}_{1,c}(k) \to \text{DA}_{c}(k) \)) admits as right adjoint the composition

\[
\mathbb{D}_{k}^{} o_{\Sigma_{+}^{\infty}}^{} \text{LAlb} \Omega_{+}^{\infty} a_{\pi_{1}}^{\infty} D_{k} : \text{DA}_{c}^{\text{coh}}(k) \to \text{DA}_{c}^{0}(k)
\]

(resp. \( \mathbb{D}_{k}^{} o_{\Sigma_{+}^{\infty}}^{} \text{LAlb} \Omega_{+}^{\infty} a_{\pi_{1}}^{\infty} D_{k} : \text{DA}_{c}^{\text{coh}}(k) \to \text{DA}_{c}^{1}(k) \)).

In the case \( n = 0 \), we already know that the functor \( \omega^{0} \) restricts to compact objects by Proposition 3.7 (iv), so that this right adjoint and the restriction of \( \omega^{0} \) (which we also denote by \( \omega^{0} \)) coincide. In the case \( n = 1 \), this is also the case by the following observation. Write temporarily \( \tilde{\omega}^{1} := \mathbb{D}_{k}^{} o_{\Sigma_{+}^{\infty}} \text{LAlb} \Omega_{+}^{\infty} a_{\pi_{1}}^{\infty} D_{k} \). Let \( M \in \text{DA}_{c}^{\text{coh}}(k) \). There is a morphism \( \tilde{\omega}^{1} M \to M \) in \( \text{DA}_{c}^{\text{coh}}(k) \), which by the adjunction property of \( \omega^{1} \) factors through a morphism \( \tilde{\omega}^{1} M \to \omega^{1} M \) in \( \text{DA}^{1}(k) \). The category \( \text{DA}^{1}(k) \) is compactly generated, hence to show that this is an isomorphism, it is enough to show that for every \( N \in \text{DA}^{1}_{c}(k) \), the induced morphism \( \text{DA}^{1}_{c}(k)(N, \tilde{\omega}^{1} M) \to \text{DA}^{1}_{c}(k)(N, \omega^{1} M) \) is an isomorphism; however, this follows from the adjunction properties of both functors. We deduce that \( \omega^{1} \) restricts to compact objects, and that restriction is related to \( \text{LAlb} \) by the formula above.

Finally, we use another result of [14] to show that the \( \omega^{n} \)‘s for \( n \geq 2 \) are not well-behaved, at least over “large” fields.

**Proposition 3.14.** Let \( n \geq 2 \) and \( k \) be an algebraically closed field of infinite transcendence degree over \( \mathbb{Q} \), e.g. \( k = \mathbb{C} \). Then \( \omega^{n} : \text{DA}_{c}^{\text{coh}}(k) \to \text{DA}^{n}(k) \) does not preserve compact objects.

**Proof.** We prove this by contradiction. Assume that \( \omega^{n} \) preserves compact object and write again \( \omega^{n} : \text{DA}_{c}^{\text{coh}}(k) \to \text{DA}^{n}(k) \) for the restriction. By Proposition 1.25, the duality functor \( \mathbb{D}_{k} \) restricts to anti-equivalences of categories \( \text{DA}_{c}^{\text{coh}}(k)^{\text{op}} \simeq \text{DA}_{\text{hom},c}(k) \) and \( \text{DA}^{n}(k)^{\text{op}} \simeq \text{DA}_{n,c}(k) \). This implies that the composition \( \mathbb{D}_{k}^{} o_{\Sigma_{+}^{\infty}}^{} \omega^{n} \circ \mathbb{D}_{k} : \text{DA}_{\text{hom},c}(k) \to \text{DA}_{n,c}(k) \) provides a left adjoint to the inclusion \( \text{DA}_{n,c}(k) \to \text{DA}_{\text{hom},c}(k) \).

By Lemma 3.12, this also provides a left adjoint to \( \text{DM}_{n,c}^{\text{eff}}(k) \to \text{DM}_{c}^{\text{eff}}(k) \), which does not exists by [14, §2.5] (note that the assumption there is the existence of a left adjoint to \( \text{DM}_{n}^{\text{eff}}(k) \to \text{DM}_{0}^{\text{eff}}(k) \) but the proof only uses the existence of the adjoint on compact objects). This contradiction finishes the proof.

### 3.3. Computation and finiteness of \( \omega^{1} \)

We can now compute \( \omega^{1} \) in an important special case.

**Theorem 3.15.** Let \( f : X \to S \) be a smooth projective Pic-smooth morphism, with \( S \) regular. The morphism \( \Theta_{f} : \Sigma_{\infty}^{c} P(X/S)(-1)[-2] \to f_{*}Q_{X} \) of Section 2.3 induces an isomorphism

\[
\omega^{1} f_{*}Q_{X} \simeq \Sigma_{\infty}^{c} P(X/S)(-1)[-2].
\]

In particular, the motive \( \omega^{1} f_{*}Q_{X} \) is compact.

**Proof.** Assume \( S \) is a regular scheme. First of all, the motive \( \Sigma_{\infty}^{c} P(X/S) \) lies in \( \text{DA}_{1,c}(S) \) by Corollary 2.43. By Proposition 1.27, this implies that \( \Sigma_{\infty}(P(X/S) \otimes \mathbb{Q})(-1)[-2] \) lies in \( \text{DA}_{1}(S) \). We deduce that \( \Theta_{f} \) induces a morphism \( \Sigma_{\infty}(P(X/S) \otimes \mathbb{Q})(-1)[-2] \to \omega^{1} f_{*}Q_{X} \). It remains to show that this is an isomorphism. We have also observed that \( (\Sigma_{\infty} P(X/S))(-1)[-2] \) is compact, so this will also establish the last claim.
We first treat the case when $S$ is the spectrum of a perfect field $k$. The proof proceeds by reduction to a computation in the category of effective Voevodsky motives $DM^{\text{eff}}(k)$. By Proposition 1.27, the category $DA^1(k)$ is compactly generated by motives of the form $g_\omega Q(X)$ for a smooth curve $g : C \to k$. We thus have to show that for all such $g$ and all $n \in \mathbb{Z}$, the map

$$f : DA(k)(g_\omega Q(X)[-n], S_{N}^{\infty} P(X/k)(-1)[-2]) \xrightarrow{\Theta_f} DA(k)(g_\omega Q(X)[-n], f_* Q_X)$$

induced by $\Theta_f$ is an isomorphism (this turns out to hold for any smooth $C$, not only for curves, as the argument below shows).

First, using that $a^U \Theta_f = \Theta_f^{\text{tr}}$ modulo a certain isomorphism (Proposition 2.49), this is equivalent to the morphism

$$DM(k)(g_\omega Q(X)^{\vee}, S_{N}^{\infty} P^{\vee}(X/k)(-1)[-2]) \xrightarrow{(\Theta_f)^{\vee}} DM(k)(g_\omega Q(X)^{\vee}, f_* Q_X^{\vee}(1)[n])$$

being an isomorphism. By Lemma 3.16, we have a commutative diagram.

$$\begin{array}{ccc}
DM(k)(g_\omega Q(X)^{\vee}, S_{N}^{\infty} f_* Q_X^{\vee}(1)[n]) & \xrightarrow{\Sigma^\infty} & DM(k)(g_\omega Q(X)^{\vee}, f_* Q_X^{\vee}(1)[n]) \\
\Sigma^\infty & \sim & \Lambda \\
DM^{\text{eff}}(k)(g_\omega Q(X), f_* Q_X^{\vee}(1)[n]) & \xleftarrow{\Lambda^{\text{tr}}} & DM^{\text{eff}}(k)(g_\omega Q(X) \otimes f_* Q_X^{\vee}, Q_X^{\vee}(1)[n]) \\
\Sigma^{\text{eff}} & \sim & \Sigma^\infty \\
DM^{\text{eff}}(k)(g_\omega Q(X) \otimes f_* Q_X^{\vee}, Q_X^{\vee}(1)[n]) & \xrightarrow{\sim} & DM^{\text{eff}}(k)(g_\omega Q(X)^{\vee} \otimes f_* Q_X^{\vee}, Q_X^{\vee}(1)[n])
\end{array}$$

Using the alternative description of $\Theta_f^{\text{tr}}$ from Proposition 2.49 with $\nu^{\text{eff}}$ and the fact that $\nu^{\text{eff}, \text{tr}}$ is an isomorphism, we see that we have to show that the top morphism in this diagram is an isomorphism.

The maps induced by $\Sigma^\infty$ are isomorphisms because of the Cancellation theorem [61] (this is where we use the hypothesis $k$ perfect), hence the top morphism is an isomorphism. This concludes the proof in the case $k$ perfect.

We now turn to the case of $S = \text{Spec}(k)$ with $k$ an arbitrary field. Let $k^{\text{perf}}$ be a perfect closure of $k$ and $h : \text{Spec}(k^{\text{perf}}) \to \text{Spec}(k)$ be the canonical morphism. By Proposition 2.50 and applying $\omega^1$, we have a commutative diagram

$$h^* S_{N}^{\infty} P(X/S)(-1)[-2] \xrightarrow{\nu_h R_h} \omega^1(h^* f_* Q_X)$$

By Corollary 2.45, to show that $V_h$ is an isomorphism, it is enough to show that $V_h^N$ is an isomorphism. Let $k^s$ be a separable closure of $k$ and $\check{k} = k^s k^{\text{perf}}$. Looking at the proof of Proposition 2.42, we find that $\mathcal{N} S(X/k)$ is represented by the Gal($k^s/k$)-module $NS(X_{k^s}/k^s)$ while $\mathcal{N} S(X_{k^{\text{perf}}}/k^{\text{perf}})$ is represented by the Gal($k/k^{\text{perf}}$)-module $NS(X_{k}/k)$. Those two groups are canonically isomorphic, and we conclude that $V_h^N$ is an isomorphism. Since $R_h$ is an isomorphism, we see that the left vertical map in the diagram is an isomorphism. Moreover, since $h$ is finite and purely inseparable, by the separation property of $DA$ andLemma 1.18 (ii), we see that the natural morphism $\alpha_h : h^* \omega^1 \to \omega^1 h^*$ is an isomorphism. Together with the commutative diagram above, this shows that we are reduced to the perfect field case, which was treated above.

We now do the general case. We can assume $S$ is connected, and so integral. The statement of the theorem is equivalent to the following: for all $M \in DA^1(S)$, the map $\Theta_f$ induces an isomorphism

$$DA(S)(M, S_{N}^{\infty} P(X/S)(-1)[-2]) \xrightarrow{\Theta_f} DA(S)(M, f_* Q_X).$$

We first make a series of reformulations. By Proposition 1.27 and the definition of $DA^1(S)$, the category $DA^1(S)$ is compactly generated by objects of the form $g_\omega Q(X)$ for a smooth curve $g : C \to S$. We can thus state the theorem as follows: for every smooth curve $g : C \to S$ and all
Let $f' : X_C \to C$ be the pullback of $f$ along $g$. The morphism $f'$ is Pic-smooth by Lemma 2.31 and $C$ is regular. By Proposition 2.50 and the fact that $v_0$ is an isomorphism because $g$ is smooth (Lemma 2.24), the morphism $(g^* \Theta)\ast$ above is an isomorphism if and only the morphism

$$\text{DA}(C)(Q_C, \Sigma^\infty P(X/C)[n] \to \text{DA}(C)(Q_C, f'^* Q_X(1)[n])$$

is an isomorphism. In other words, since $f'$ still satisfies all the hypotheses of the theorem, we can assume that $g = \text{id}$.

By adjunction, the right-hand side is isomorphic to the motivic cohomology group $H^1_{\text{mot}}(X)$. Because $S$ is regular, we know from Proposition B.6 how to compute it: it is zero for $\neq 1, 2$, and we have explicit morphisms relating it to $\mathcal{O}(X)Q$ if $n = 1$ (resp. Pic$(X)Q$ if $n = 2$). The idea of the rest of the proof is to apply a similar localisation argument to the proof of Proposition B.6.

For concision, we introduce the ad hoc notation

$$\text{HP}^{n-2}(X/S) := \text{DA}(S)(Q_S, (\Sigma^\infty P(X/S))[n-2]).$$

Let $j : U \to S$ be a non-empty open set and $i : Z \to S$ its reduced closed complement. Then by applying colocalisation, we get a commutative diagram

$$\cdots \to \text{DA}(Z)(Q_Z, i_1^* \Sigma^\infty P(X/S)[n-2]) \to \text{HP}^{n-2}(X/S) \to \text{HP}^{n-2}(X_U/U) \to \cdots$$

Choose a stratification $Z = Z_0 \subset Z_1 \subset \cdots \subset Z_d = \emptyset$ in such a way that for all $k$, the scheme $Z_k \setminus Z_{k+1}$ is regular of codimension $d_k$ in $Z$ and in such a way that $(Z \setminus Z_1)$ contains all points of codimension 1 of $Z$ in $S$ (so that $d_k \geq 2$ for $k \geq 1$). Let $i_k : Z_k \setminus Z_{k+1} \to S$ be the corresponding regular locally closed immersion.

By Corollary 2.43, the motive $\Sigma^\infty P(X/S)(-1)$ is in $\text{DA}_{\text{ram}}^1(S)$. By absolute purity in the form of Proposition 1.7, for any $k$, we have $i_k^* \Sigma^\infty P(X/S) \simeq i^* P(X/S)(-d_k)[2d_k]$. In particular, by Corollary 3.9 (iii), we have $\omega^0(i_k^* \Sigma^\infty P(X/S)) \simeq 0$ for $k \geq 2$. This shows that by inductively applying absolute purity and colocalisation, we get a commutative diagram

$$\cdots \to \text{DA}(Z)[Q_{Z \setminus Z_k}, i_k^* \Sigma^\infty P(X/S)(-1)[n-4]) \to \text{HP}^{n-2}(X/S) \to \text{HP}^{n-2}(X_U/U) \to \cdots$$

Write $Z' = Z \setminus Z_1$. The motive $i^* \Sigma^\infty P(X/S)(-1)[n-4]$ lies in $\text{DA}_{\text{coh}}(Z)$, so that

$$\text{DA}(Z')(Q_{Z'}, i^* \Sigma^\infty P(X/S)(-1)[n-4]) \simeq \text{DA}(Z')(Q_{Z'}, \omega^0(i^* \Sigma^\infty P(X/S)(-1)[n-4]).$$

Using Corollary 2.43, we apply Proposition 3.7 (ii) to get an isomorphism

$$\omega^0(i^* \Sigma^\infty P(X/S)(-1)[n-4]) \simeq i^* \omega^0(\Sigma^\infty P(X/S)(-1)[n-4]).$$

By Corollary 3.11, we then have

$$\omega^0(\Sigma^\infty P(X/S)(-1)[n-4]) \simeq \pi_0(f)_Q[n-2].$$

We deduce that

$$\text{DA}(Z')(Q_{Z'}, i^* \Sigma^\infty P(X/S)[n-2]) \simeq H^{n-2,0}(\pi_0(X_{Z'}/Z')).$$
We rewrite this into the previous commutative diagram to get
\[
\begin{array}{c}
\ldots \rightarrow H^{n-2,0}((\pi_0(X'_{Z'})/Z')) \rightarrow H^{n-2}(X/S) \rightarrow H^{n-2}(X_U/U) \rightarrow \ldots \\
\downarrow (\pi_0)^* \\
\ldots \rightarrow H^{n-2,0}(X'_{Z'}) \rightarrow H^{n,1}_M(X) \rightarrow H^{n,1}_M(X_U) \rightarrow \ldots
\end{array}
\]
By Proposition B.5, since \(X'_Z\), and \(\pi_0(X'_{Z'})\) are both regular and have the same set of connected components, the map \((\pi_0)^*\) is an isomorphism for all \(n\), and the groups \(H^{n-2,0}(X'_{Z'})\) vanish for \(\neq 2\). As a consequence, we see that the pullback map \(H^{n-2}(S) \rightarrow H^{n-2}(U)\) is an isomorphism for \(\neq 1, 2\), and there is a commutative diagram
\[
\begin{array}{c}
0 \rightarrow H^{-1}(X/S) \rightarrow H^{-1}(X_U/U) \rightarrow \mathbb{Q}^\text{tr}(X'_{Z'}) \rightarrow H^0(X/S) \rightarrow H^0(X_U/U) \rightarrow 0 \\
\downarrow \\
0 \rightarrow H^{-1,1}_M(X_S) \rightarrow H^{-1,1}_M(X_U) \rightarrow \mathbb{Q}^\text{tr}(X'_{Z'}) \rightarrow H^1(X/S) \rightarrow H^1(X_U/U) \rightarrow 0.
\end{array}
\]
We then pass to the limit over all non-empty closed subsets \(Z\) and use continuity for \(\text{DA}\). We obtain that \(H^{n-2}(S) \rightarrow H^{n-2}(\kappa(S))\) is an isomorphism for \(\neq 1, 2\), and we have a commutative diagram
\[
\begin{array}{c}
0 \rightarrow H^{-1}(X/S) \rightarrow H^{-1}(X_{\kappa(S)}/\kappa(S)) \rightarrow \Pi \rightarrow H^0(X/S) \rightarrow H^0(X_{\kappa(S)}/\kappa(S)) \rightarrow 0 \\
\downarrow \\
0 \rightarrow H^{-1,1}_M(X_S) \rightarrow H^{-1,1}_M(X_{\kappa(S)}) \rightarrow \Pi \rightarrow H^1(X_S) \rightarrow H^1(X_{\kappa(S)}) \rightarrow 0
\end{array}
\]
with \(\Pi\) a group which can be expressed in terms of the sheaf \(\pi_0(X,S)\), but which we do not need to know explicitly. Applying the already established result in the field case (for the function field \(\kappa(S)\)), we see that the second and fifth vertical maps are isomorphisms. By the five lemma, we conclude that the first and fourth one are as well. This concludes the proof.

The following lemma, which relates Grothendieck operations in the effective and non-effective settings, was used in the proof above. Since we have only discussed the functoriality of \(\text{DA}\) and not \(\text{DM}\) or \(\text{DM}^\text{eff}\), we need to specify what we mean here.

**Lemma 3.16.** Let \(S\) be a field, \(f : X \rightarrow \text{Spec}(k)\) a smooth \(k\)-variety, and \(M, N \in \text{DM}^\text{eff}(k)\). There exists natural isomorphisms
\[
\Lambda^\text{eff}_{M,N} : \text{DM}^\text{eff}(k)(M \otimes f_!Q, N) \simeq \text{DM}^\text{eff}(k)(M, f_*f^*N)
\]
such, for \(M, N \in \text{DM}^\text{eff}(k)\), the diagram
\[
\begin{array}{c}
\text{DM}(\Sigma^\infty_M, \Sigma^\infty f_*f^*N) \rightarrow \text{DM}(\Sigma^\infty_M, f_!f^*\Sigma^\infty N) \\
\Sigma^\infty \sim \quad \Lambda \sim \\
\text{DM}^\text{eff}(k)(M, f_*f^*N) \quad \text{DM}(\Sigma^\infty_M \otimes f_!Q^\text{eff}_X, \Sigma^\infty N) \\
\Lambda^\text{eff} \sim \quad \Sigma^\infty \sim \\
\text{DM}^\text{eff}(k)(M \otimes f_!Q^\text{eff}_X, N) \rightarrow \text{DM}^\text{eff}(k)(M \otimes f_!Q^\text{eff}_X, N)
\end{array}
\]
commutes.

**Proof.** Let \(A, B \in \text{DM}^\text{eff}(k)\). As part of the premotivic category structure of \(\text{DM}^\text{eff}(k)\), we have a smooth projection formula
\[
\text{sp}^\text{eff} : f_!(f^*A \otimes B) \simeq A \otimes f_!B
\]
The functor \(\Sigma^\infty_M : \text{DM}^\text{eff}(-) \rightarrow \text{DM}(-)\) is a morphism of premotivic categories in the sense of [25], which implies that we have natural isomorphisms \(\Sigma^\infty_M f_! \simeq f_!\Sigma^\infty_M\) and \(\Sigma^\infty f_* \simeq f_*\Sigma^\infty\), and that modulo these natural isomorphisms, we have \(\Sigma^\infty\text{sp}^\text{eff} = \text{sp}\).
We now define $\Lambda$ as
\[
DM^{\text{eff}}(M \otimes f_!Q, N)^{(sp^{\text{eff}})} \simeq DM^{\text{eff}}(k)(f_!f^*M, N) \simeq DM^{\text{eff}}(M, f_*f^*N)
\]
where the second map is given by the two adjunctions $(f_!, f^*)$ and $(f^*, f_*)$.

The commutation of the diagram then follows as $\Sigma^{\infty}sp^{\text{eff}} = sp$. \hfill \Box

**Remark 3.17.** In view of the non-canonical decomposition of $P(X/S)$ from Corollary 2.43, this result can be interpreted as a 1-motivic analogue of Deligne’s decomposition theorem for smooth projective morphisms.

**Remark 3.18.** In the special case of $S = \text{Spec}(k)$ with $k$ a perfect field, the theorem is closely related to computations of $\text{LA}_{\text{eff}}$ and $\text{RPic}$ from [16, §9]. Let us sketch this connection. Let $f : X \to \text{Spec}(k)$ be a smooth projective variety. Then $X$ is automatically Pic-smooth, and the morphism
\[
(\Sigma^{\infty}P(X/k))(-1)[-2] \to \omega^1 f_!Q_X,
\]
induced by $\Theta_f$ is an isomorphism. By Proposition 3.13, we have the following isomorphisms.

\[
\omega^1 f_!Q_X \simeq \mathbb{D}_k \omega^{\text{tr}1}(\text{LA}_{\text{eff}}\Omega^{\text{tr}}_{1} \otimes_{k} \mathbb{D}_k f_!Q_X) \simeq \omega^{\text{tr}1}(\text{LA}_{\text{eff}}M^{\text{eff, tr}}_k(X))
\]

where we have used the same arguments as in Section 3.2 to pass from $\text{DA}$ to $\text{DM}^{\text{eff}}$. Moreover, by Lemma 3.19 below, we can write

\[
\omega^{\text{tr}1}(\text{LA}_{\text{eff}}M^{\text{eff, tr}}_k(X)) \simeq \omega^{\text{tr}1}(\Sigma^{\infty}_1 \text{Hom}_{\text{eff}}(\text{LA}_{\text{eff}}M^{\text{eff, tr}}_k(X), Q(1))(-1) \simeq \omega^{\text{tr}1}(\Sigma^{\infty}_1 \text{RPic}(X))(-1)
\]

where $\text{RPic}(X)$ is the motive introduced in [16, Definition 8.3.1] and we have used the duality between $\text{LA}_{\text{eff}}(X)$ and $\text{RPic}(X)$ coming from [16, §4.5]. At this point, we have an isomorphism
\[
\omega^{\text{tr}1}(\Sigma^{\infty}_1 \text{RPic}(X))(-1) \simeq (\Sigma^{\infty}P(X/k))(-1)[-2].
\]

We now apply $\text{a}^{\text{tr}}$, use the isomorphism $\text{a}^{\text{tr}}\text{tr}^{\infty} \simeq \Sigma^{\infty}_1 \text{a}^{\text{tr}}$, Corollary 2.46, and the cancellation theorem: this yields an isomorphism
\[
\text{RPic}(X) \simeq \text{P}^{\text{tr}}(X/k)[-2].
\]

We are now in position to connect with the results of [16]: modulo this isomorphism, the distinguished triangles of Corollary 2.46 for $\text{P}^{\text{tr}}(X/k)$ give an alternative proof of [16, Corollary 9.6.1] in the special case where $X$ is smooth projective and we have $Q$-coefficients.

**Lemma 3.19.** Let $k$ be a perfect field. We have for $M \in DM_{n,c}^{\text{eff}}(k)$ a natural isomorphism
\[
\mathbb{D}_k^{\infty} \Sigma_{1} M \simeq (\Sigma^{\infty}_1 \text{Hom}_{\text{eff}}(M, Q(n)))(-n).
\]

**Proof.** Let $N, M \in DM_{n,c}^{\text{eff}}(k)$. By adjunction, monoidality of $\Sigma^{\infty}_1$ and the cancellation theorem, there is a sequence of natural isomorphisms
\[
DM^{\text{eff}}(k)(N, \Omega^{\infty}_1 \mathbb{D}_k^1(M(-n))) \simeq DM(k)(\Sigma^{\infty}_1 N, \mathbb{D}_k^1(M(-n))) \simeq DM(k)(\Sigma^{\infty} N \otimes \Sigma^{\infty}(M(-n)), Q) \simeq DM(k)(\Sigma^{\infty}(N \otimes M), Q(n)) \simeq DM^{\text{eff}}(k)(N \otimes M, Q(n)) \simeq DM^{\text{eff}}(k)(N, \text{Hom}_{\text{eff}}(M, Q(n)))
\]

which provides, by the Yoneda lemma, a natural isomorphism $\text{a}^{\text{tr}} \Sigma^{\infty}_1 \mathbb{D}_k^1(M(-n)) \simeq \text{Hom}_{\text{eff}}(M, Q(n))$. We apply $\Sigma^{\infty}_1$ to get an isomorphism $\Sigma^{\infty}_1 \Omega^{\infty}_1 \mathbb{D}_k^1(M(-n)) \simeq \Sigma^{\infty}_1 \text{Hom}_{\text{eff}}(M, Q(n))$.

Moreover, the motive $\mathbb{D}_k^1(M(-n))$ lies in $DM_{n, c}^{\text{hom}}(k)$ by Proposition 1.27, Lemma 3.12 and Proposition 1.25. Because of the cancellation theorem, the counit $\Sigma^{\infty}_1 \mathbb{D}_k^1 \to \text{id}$ is an isomorphism on $DM^{\text{hom}}(k)$; hence $\Sigma^{\infty}_1 \Omega^{\infty}_1 \mathbb{D}_k^1(M(-n)) \simeq \mathbb{D}_k^1(M(-n)) \simeq (\mathbb{D}_k^1 M)(n)$. Combining this with the previous paragraph completes the proof. \hfill \Box
In the special case of a relative curve, because the Neron-Severi rank is constant, we can remove the regularity hypothesis on the base. This yields a general computation of the motive of a smooth projective curve.

**Corollary 3.20.** Let $f : C \to S$ be a smooth projective curve.

(i) The morphism

$$\Theta_f : \Sigma^\infty P(C/S)(-1)[-2] \to f_*Q_C$$

is an isomorphism, and induces an isomorphism

$$\Sigma^\infty P(C/S) \simeq M_S(C).$$

(ii) If $S$ is regular, we then have (non-canonical) isomorphisms

$$f_*Q_C \simeq M_S(\pi_0(C/S)) \oplus \Sigma^\infty \text{Jac}(C/S) \oplus M_S(\pi_0(C/S))(1)[2]$$

and

$$f_*Q_C \simeq M_S(\pi_0(C/S)) \oplus \Sigma^\infty \text{Jac}(C/S)(-1)[-2] \oplus M_S(\pi_0(C/S))(-1)[-2].$$

(iii) If $f$ has geometrically connected fibres and a section $\sigma : S \to C$, we have canonical isomorphisms

$$f_*Q_C \simeq Q_S \oplus \Sigma^\infty \text{Jac}(C/S) \oplus Q_S(1)[2]$$

and

$$f_*Q_C \simeq Q_S \oplus \Sigma^\infty \text{Jac}(C/S)(-1)[-2] \oplus Q_S(-1)[-2].$$

**Proof.** Let us show that $\Theta_f$ is an isomorphism. By [11, Proposition 3.24], it is enough to show that $s^*\Theta_f$ is an isomorphism for any $s \in S$. By Proposition 2.50 and Proposition 2.47, we are then reduced to the case when $S$ is the spectrum of a field. The fact that $\Theta_f$ is then an isomorphism is a special case of Theorem 3.15.

The claims in (i) and (ii) then follow from 2.43 and Proposition 2.47.

Let us assume further that $f$ has geometrically connected fibres and a section $\sigma : S \to C$. The section $\sigma$ yields sections of the lattices $Q(\pi_0(C/S))$ and $NS^m_{C/S} \simeq Q(\pi_0(C/S))$ (Proposition 2.47), which can be used to produce splittings of the distinguished triangles computing $\Sigma^\infty P(C/S)(-1)[-2]$ in Proposition 2.43. From this and (i), we get the decompositions in (iii). \(\square\)

As an application of the computation, we can now prove a fundamental finiteness result for $\omega^1$.

**Theorem 3.21.** Let $S$ be a noetherian finite-dimensional excellent scheme. Assume that $S$ admits resolution of singularities by alterations. Then the functor $\omega^1 : DA^{coh}(S) \to DA^1(S)$ preserves compact objects.

**Proof.** We follow the argument of [15, Proposition 2.14 (vii)] for the case of $\omega^0$, with minor changes.

By Corollary 1.18 (ii) we can assume that $S$ is reduced. We prove the result by noetherian induction on $S$. Let $M$ be in $DA^{coh}_c(S)$. Since $M$ is compact and cohomological, Lemma 1.8, Proposition 1.25 and continuity implies that there exists a dense open set $V \subset S$ and a finite family $\{f_i\}_{i=1}^n$ of smooth projective morphisms $f_i : X_i \to V$ such that $M_V$ lies in the triangulated subcategory generated by the motives $f_i_*Q_{X_i}$. By Proposition 2.32, there exists an everywhere dense open subset $U \subset V$ such that $f_i \times_S U$ is Pic-smooth for every $i$. We can moreover assume that $U$ is regular. Write $j : U \to S$ for the open immersion and $i : Z \to S$ for the complementary reduced closed immersion. By Proposition 1.11, because of the hypothesis of resolution of singularities by alterations for $S$, the colocalisation triangle

$$i_*i^!M \to M \to j_*j^*M \xrightarrow{\Delta}$$

lies in $DA^{coh}(S)$. We apply $\omega^1$ and use Proposition 3.3 (iii) to obtain a distinguished triangle

$$i_*\omega^1(i^!M) \to M \to \omega^1(j_*j^*M) \xrightarrow{\Delta}.$$

By induction, we know that $\omega^1(i^!M)$ is compact, so it is enough to show that $\omega^1(j_*j^*M)$ is as well. By Proposition 3.3 (iii), we have an isomorphism $\omega^1(j_*j^*M) \simeq \omega^1(j_*\omega^1(j^*M))$. Put $N = j_*\omega^1(j^*M)$; we have to show that $\omega^1(N)$ is compact. The motive $j^*M$ lies in the triangulated subcategory generated by the motives $(f_i \times_S U)_*Q$ with $f_i \times_S U$ Pic-smooth and $U$ regular; hence by Theorem 3.15 we have $\omega^1(j^*M)$ compact. This implies that $N$ is compact, with $j^*N \in DA^1(U)$. 
In particular, we have \( j_! j^* N \in \mathcal{DA}_1(S) \). Thus applying \( \omega^1 \) to the localisation triangle for \( N \) and using Proposition 3.3 (iii) yield a distinguished triangle

\[
j_! j^* N \to \omega^1 N \to i_* \omega^1 i^* N \
\]

By Proposition 3.3 (vi), we have \( i^* \omega^1(N) \simeq \omega^1(i^* N) \), which is compact by induction. This completes the proof that \( \omega^1 N \) is compact, and the proof of the theorem. \( \square \)

4. Motivic \( t \)-structures

We introduce the motivic \( t \)-structures on \( \mathcal{DA}_1(S) \) and \( \mathcal{DA}_1(S) \) and study how Deligne 1-motives relates to its heart.

4.1. Conservativity of realisations. As we have explained in the introduction, in our approach to the motivic \( t \)-structure conjecture, the conservativity of realisation functors is a necessary first step to ensure uniqueness. Recall from [9] that for \( k \) a field of characteristic 0 with a fixed complex embedding \( \sigma : k \to \mathbb{C} \) and \( S \) scheme of finite type over \( k \), there is a covariant Betti realisation functor

\[
R_{B,\sigma} : \mathcal{DA}(S) \to D(S^{an}, \mathbb{Q})
\]

with target the derived category of sheaves of \( \mathbb{Q} \)-vector spaces on the complex analytic space \( S^{an} \). Note that in [9] this functor is only defined in the case \( S \) quasi-projective over \( k \), but this is due solely to quasi-projectivity hypotheses in the theory of the six operations for \( \mathcal{DA}(S) \) in [7] which can be removed systematically using results from [25].

Similarly, we fix a prime \( \ell \), and let \( S \) be a \( \mathbb{Z}[\ell] \)-scheme. Let \( D_c(S, \mathbb{Q}_\ell) \) be subcategory of complexes with constructible cohomology in the derived category of \( \mathbb{Q}_\ell \)-sheaves \( S \) in the sense of Ekedahl [33]. By [11, Section 9], there is a covariant functor

\[
R_{\ell} : \mathcal{DA}_{c}(S) \to D_c(S, \mathbb{Q}_\ell).
\]

Unfortunately, the “unbounded” \( \ell \)-adic realisation with source \( \mathcal{DA}(S) \), presumably with a natural target category defined using the pro-\( \acute{e} \)tal topology of [19], has not been constructed yet.

**Proposition 4.1.** With the notations and hypotheses above, the functors \( R_{B,\sigma} \) and \( R_{\ell} \), restricted to either of \( \mathcal{DA}_{0,c}(S) \), \( \mathcal{DA}_{1,c}(S) \) or \( \mathcal{DA}_{1,c}(S) \) are conservative.

**Proof.** Since \( \mathcal{DA}_{0,c}(S) \subset \mathcal{DA}_{1,c}(S) \) and \( \mathcal{DA}_{1,c}(S) = \mathcal{DA}_{1,c}(S)(-1) \), it is enough to treat the case of \( \mathcal{DA}_{1,c}(k) \).

If \( k \) is a perfect field of characteristic \( p \neq \ell \), we have an equivalence of triangulated categories with \( t \)-structures \( \Sigma^\infty : D^b_c(M_1(k)) \simeq \mathcal{DA}_{1,c}(k) \). By Lemma 4.2 below, we only have to check that the induced functor \( R^\Sigma \) from \( M_1(k) \) to either \( \mathbb{Q} \) or \( \mathbb{Q}_\ell \)-vector spaces is conservative. Using the weight filtration on Deligne 1-motives, it is enough to show that if \( M \) is a pure object in \( M_1(k) \) with trivial realisation, it is itself 0. This follows from the computation of the realisation of such a motive in [6, Proposition 5.1.(2)] (for \( R_B \)) and [6, 5.2] (for \( R_{\ell} \)).

We now do the general case. Let \( M \in \mathcal{DA}_{1,c}(S) \) and \( R \in \{ R_{B,\sigma}, R_{\ell} \} \). Assume that \( R(M) = 0 \). By the commutation of realisations with pullbacks ([9, Theoreme 3.19] for \( R_{B,\sigma} \), [11, Theoreme 9.7] for \( R_{\ell} \)), for any \( i: \mathcal{S} \to S \) geometric point, we have \( R(i_*^* M) = 0 \). By the perfect field case above, we have \( i_*^* M = 0 \). By [6, Lemma A.6], the family of such pullback functors is conservative, and we conclude that \( M = 0 \). This concludes the proof of conservativity. \( \square \)

**Lemma 4.2.** Let \( F : T \to T' \) be a \( t \)-exact functor between triangulated categories equipped with \( t \)-structures. Assume that the \( t \)-structure on \( T \) is non-degenerate and that the induced functor \( F^\Sigma : T^\Sigma \to T'^\Sigma \) is conservative. Then \( F \) is conservative.

**Proof.** Let \( M \in T \). Assume that \( F(M) = 0 \). Then \( H_n F(M) = F(H_n M) = 0 \) for all \( n \in \mathbb{Z} \). Since \( F^\Sigma \) is conservative, we have \( H_n M = 0 \) for all \( n \in \mathbb{Z} \). Since the \( t \)-structure on \( T \) is bounded, we deduce that \( M = 0 \). \( \square \)
4.2. Generated t-structures. We fix a (noetherian, finite dimensional) base scheme $S$ for the rest of this section. We want to define t-structures by generators and relations. This is possible in the context of compactly generated triangulated categories.

**Proposition 4.3.** ([7, Lemme 2.1.69, Proposition 2.1.70] Let $\mathcal{T}$ be a compactly generated triangulated category and $\mathcal{G}$ be a family of compact objects in $\mathcal{T}$. Define $\mathcal{T}_{\geq 0} = \langle \mathcal{G} \rangle_+$ and $\mathcal{T}_{< 0}$ as the right orthogonal of $\mathcal{G}[n]$, i.e., the full subcategory of all objects $N$ with

$$\forall n \in \mathbb{N}, \forall G \in \mathcal{G}, \text{Hom}(G, N[-n]) = 0.$$ 

Then $(\mathcal{T}_{\geq 0}, \mathcal{T}_{< 0})$ is a t-structure on $\mathcal{T}$, which we denote by $t(\mathcal{G})$ and call the t-structure generated by $\mathcal{G}$ on $\mathcal{T}$.

We can now introduce our candidate generating families. The definition uses Deligne 1-motives over a base: for definitions and notations, we refer to the first section of Appendix A.

**Definition 4.4.** We define classes of objects in $\text{DA}(S)$ as follows. We put

$$\mathcal{J}_{\mathcal{G}}(S) = \left\{ e|_T \Sigma^\infty(K \otimes \mathbb{Q}) : e: U \to S \text{ étale}, K = \text{one of } \mathbb{Z}, \mathcal{G}_m[-1], \text{ or } \text{Jac}(C/U)[-1], \begin{array}{l} \text{for } C/U \text{ smooth projective curve} \\ \text{with geometrically connected fibres} \\ \text{and a section } U \to C \end{array} \right\};$$

and

$$\mathcal{D}_{\mathcal{G}}(S) = \left\{ e|_T \Sigma^\infty(M) : e: U \to S \text{ étale}, M \in \mathcal{M}_1(U) \right\}.$$

We call objects $\mathcal{J}_{\mathcal{G}}(S)$ (resp. $\mathcal{D}_{\mathcal{G}}(S)$) Jacobian generators (resp. Deligne generators).

**Lemma 4.5.** (i) Let $f: T \to S$ be a morphism of schemes. Then we have $f^* \mathcal{J}_{\mathcal{G}}(S) \subset \mathcal{J}_{\mathcal{G}}(T)$ and $f^* \mathcal{D}_{\mathcal{G}}(S) \subset \mathcal{D}_{\mathcal{G}}(T)$.

(ii) Let $e: T \to S$ be an étale morphism. Then $e_\sharp \mathcal{J}_{\mathcal{G}}(S) \subset \mathcal{J}_{\mathcal{G}}(S)$ and $e_\sharp \mathcal{D}_{\mathcal{G}}(S) \subset \mathcal{D}_{\mathcal{G}}(S)$.

**Proof.** Point (i) follows from the Ex$_\sharp$ isomorphism and Corollary 2.3. Point (ii) follows directly from the definition.

**Lemma 4.6.** Let $S$ be a noetherian finite dimensional scheme. We have $\mathcal{J}_{\mathcal{G}}(S) \subset \mathcal{D}_{\mathcal{G}}(S)$ and $\langle \mathcal{J}_{\mathcal{G}}(S) \rangle_{(+)} \subset \langle \mathcal{G}_{\mathcal{S}} \rangle_{(+)}$.

**Proof.** The first statement follows immediately from the definition. We turn to the second one. We only need to treat the + variant, since for any family $\mathcal{G}$ we have $\langle \mathcal{G} \rangle = \langle \langle \mathcal{G} \rangle_{(+)} \rangle$.

Let $e: U \to S$ be an étale morphism. The motive $e_\sharp \mathbb{Q}$ is clearly in $\mathcal{G}_{\mathcal{U}}$. Consider the smooth curve $f: \mathcal{G}_m \times U \to U$; by Proposition 2.3, we have $M_{(+)}(\mathcal{G}_m \times U)[-1] \simeq \Sigma^\infty \mathcal{G}_m \otimes \mathbb{Q}[-1]$, which shows that $\Sigma^\infty \mathcal{G}_m \otimes \mathbb{Q}[-1]$ is in $\mathcal{G}_{\mathcal{U}}$. Let $e: U \to S$ be an étale morphism and be $f: C \to U$ a smooth projective curve. By Corollary 2.20, we have an isomorphism $M_{(C)} \simeq \Sigma^\infty \text{P}(C/U)$. By Corollary 2.43, the Picard complex of the curve $C$ fits into distinguished triangles

$$R_{\pi_0(f)} \mathcal{G}_m \otimes \mathbb{Q} \to \text{P}(C/U)_{\mathbb{Q}} \to \text{Pic}^{\text{sm}}_{C/U} \otimes \mathbb{Q} \to$$

and

$$\text{Jac}(C/U) \otimes \mathbb{Q} \to \text{Pic}^{\text{sm}}_{C/U} \otimes \mathbb{Q} \to \pi_0(C/U)_{\mathbb{Q}} \to.$$ 

Moreover, the map $M_{(C)} \simeq M_{(U)}(\pi_0(C/U))$ coincides modulo the isomorphism above with the map $\Sigma^\infty \text{P}(C/U)_{\mathbb{Q}} \to \Sigma^\infty \mathbb{Q}[\pi_0(C/U)]$. This gives us a distinguished triangle

$$M_{(+)}(C)[-1] \to \Sigma^\infty \text{Jac}(C/U) \otimes \mathbb{Q}[-1] \to \Sigma^\infty R_{\pi_0(f)} \mathcal{G}_m \otimes \mathbb{Q} \to.$$ 

The motive $\pi_0(f)_{\sharp} \mathbb{Q}$ is in $\mathcal{G}_{\mathcal{U}}$ by definition. Using the compatibility between Weil restriction and pushforward (Lemma 2.16), we have $\Sigma^\infty R_{\pi_0(f)} \mathcal{G}_m \otimes \mathbb{Q} \simeq \pi_0(f)_{\sharp} \Sigma^\infty \mathcal{G}_m \otimes \mathbb{Q}$ which is in $\langle \mathcal{G}_{\mathcal{U}} \rangle_{(+)}$ as we have shown earlier in the proof. All together, this shows $\text{Jac}(C/U) \otimes \mathbb{Q}[-1]$ is in $\langle \mathcal{G}_{\mathcal{U}} \rangle_{(+)}$. Finally, in all three cases, we apply $e_\sharp$ and use the previous lemma to see that the result lies in $\langle \mathcal{G}_{\mathcal{S}} \rangle_{(+)}$. This shows that $\mathcal{J}_{\mathcal{G}}(S) \subset \langle \mathcal{G}_{\mathcal{S}} \rangle_{(+)}$, as required.

We now come to a more difficult stability property.

**Proposition 4.7.** Let $i: Z \to S$ be a closed immersion. Then

$$i_*(\langle \mathcal{J}_{\mathcal{G}}(Z) \rangle_{(+)} \subset \langle \mathcal{J}_{\mathcal{G}}(S) \rangle_{(+)}.$$
Proof. Let \( r : Z_{\text{red}} \rightarrow Z \) be the canonical closed immersion. Localisation implies that \( \text{id} \simeq r_* r^* \). Since \( r^* \) preserves \( \mathcal{J} \mathcal{G} \) by Lemma 4.5, we see that it is enough to show the property for \( i \circ r \). We can thus assume \( Z \) reduced.

We proceed by induction on the dimension of \( Z \). If \( \dim(Z) = 0 \), because \( Z \) is reduced, it is a disjoint union of closed points of \( S \). Then \( i_* \) is canonically the direct sum of the corresponding push-forwards, so we can assume that \( Z \) is a single closed point \( s \in S \).

There are three different types of generators in \( \mathcal{J} \mathcal{G}_S \). Fix \( e : V \rightarrow s \) an étale morphism. Since \( s \) is a point, \( e \) is actually finite étale. By standard spreading out results [38, §8], there exists an open neighbourhood \( s \in U \hookrightarrow S \) and a finite étale morphism \( \tilde{e} : \tilde{V} \rightarrow U \) extending \( e \), in the sense that we have a commutative diagram of schemes

\[
\begin{array}{ccc}
\tilde{V} & \xrightarrow{j} & \tilde{V} \\
\downarrow{\tilde{e}} & & \downarrow{\tilde{e}} \\
U \setminus s & \xrightarrow{i} & U \setminus s \\
\end{array}
\]

with cartesian squares.

We first consider the case of a generator \( e_2 \mathbb{Q} \). By localisation, we have a distinguished triangle

\[
(j_2^* e_2 \mathbb{Q}) \rightarrow e_1 \mathbb{Q} \rightarrow (i_* e_2 \mathbb{Q}) \rightarrow \\
\]

to which we apply \( e_1 \) and then rewrite as

\[
(c_2^2 e_2 \mathbb{Q}) \rightarrow e_1 e_2 \mathbb{Q} \rightarrow i_* e_2 \mathbb{Q} \rightarrow \\
\]

The motives \( (c_2^2 e_2 \mathbb{Q}) \) and \( e_1 e_2 \mathbb{Q} \) are in \( \mathcal{J} \mathcal{G}_S \), so this triangle shows that \( i_* e_2 \mathbb{Q} \) lies in \( \langle \mathcal{J} \mathcal{G}_S \rangle^+ \).

The case of a generator of the form \( e_1 \Sigma^\infty G_m \otimes \mathbb{Q} \simeq e_2 \mathbb{Q}(1)[1] \) (cf. Proposition 2.3) follows from essentially the same proof, twisting by \( \mathbb{Q}(1)[1] \).

We now do the case of a generator of the form \( e_2 \Sigma^\infty \text{Jac}(C/V) \mathbb{Q} \) with \( f : C \rightarrow V \) a smooth projective curve with geometrically connected fibres and a distinguished section \( \sigma : V \rightarrow C \). We have an isomorphism

\[
i_* e_2 \Sigma^\infty \text{Jac}(C/V) \mathbb{Q} \simeq (i_* e_1) \Sigma^\infty \text{Jac}(C/V) \mathbb{Q} \simeq (\tilde{e} i_1) \Sigma^\infty \text{Jac}(C/V) \mathbb{Q} \simeq \tilde{e} i_1 \Sigma^\infty \text{Jac}(C/V) \mathbb{Q}
\]

which reduces us to show that \( i_* \Sigma^\infty \text{Jac}(C/V) \mathbb{Q} \) lies in \( \langle \mathcal{J} \mathcal{G}_V \rangle^+ \). Since \( V \) is finite étale over \( s \), it has finitely many closed points \( v \), and a simple argument shows that, up to restricting \( \tilde{V} \) we can assume that \( V \) and \( \tilde{V} \) are connected, with \( V \) consisting of a single point \( v \).

We use standard results from the deformation theory of curves. Namely, by [4, Théorème 7.3, Corollaire 7.4], the curve \( C \) can be deformed to a smooth projective curve \( \tilde{C} \) on the complete trait \( \text{Spec}(\tilde{\mathcal{O}}_{\tilde{V}, v}) \). By the Artin approximation theorem, one can in fact deform \( C \) to a smooth projective curve \( C^h \) on \( \text{Spec}(\mathcal{O}_{V, v}^h) \) where \( \mathcal{O}_{V, v}^h \) is the henselian local ring of \( \tilde{V} \) at \( v \). By smoothness of \( C^h \) and Hensel’s lemma, we can lift the section \( \sigma \) to a section \( \sigma^h \) of \( C^h \). Using spreading out results from [38, §8], we arrive at the following situation. We have a pointed étale neighbourhood \( (e : W \rightarrow \tilde{V}, v) \) of \( (\tilde{V}, v) \) and a smooth projective curve \( \tilde{f} : \tilde{C} \rightarrow W \) which extends \( C \), together with a section \( \tilde{\sigma} \) (which extends \( \sigma \)). By openness of geometric connectivity, we can also assume that the curve \( \tilde{C} \) has geometrically connected fibres.

We form the following diagram of schemes with cartesian squares

\[
\begin{array}{ccc}
\tilde{C} & \xrightarrow{j} & \tilde{C} \\
\downarrow{\tilde{f}} & & \downarrow{\tilde{f}} \\
C & \xrightarrow{f} & C \\
\end{array}
\]

\[
\begin{array}{ccc}
W & \xrightarrow{j} & W \\
\downarrow{\tilde{e}} & & \downarrow{\tilde{e}} \\
V & \xrightarrow{i} & V \\
\end{array}
\]

We have a localisation triangle

\[
(j_2^* \Sigma^\infty \text{Jac}(\tilde{C}/W) \mathbb{Q}) \rightarrow \Sigma^\infty \text{Jac}(\tilde{C}/W) \mathbb{Q} \rightarrow (i_* \Sigma^\infty \text{Jac}(\tilde{C}/W) \mathbb{Q}) \rightarrow \\
\]
to which we apply $e_i$ and rewrite using Corollary 2.3 to obtain

$$(e^o)_2 \Sigma^\infty \text{Jac}(\tilde{C}/W)^{\circ} \to e_2 \Sigma^\infty \text{Jac}(\tilde{C}/W) \to i_* \Sigma^\infty \text{Jac}(C/v) \to$$

The first two terms of this complex are in $\mathcal{JG}_v$, so this shows that $i_* \text{Jac}(C/v)$ is in $\langle \mathcal{JG}_v \rangle_+$. This concludes the proof in the case $\dim(Z) = 0$.

We now come to the induction step. Let $M \in \mathcal{JG}_Z$. Write for the moment $M = e_2 \Sigma^\infty G \otimes \mathbb{Q}$ with $G$ one of the three possible types and $e : U \to S$ an étale morphism.

Let $k : W \to Z$ be a dense open irreducible subset such that $e_W$ is finite étale. Let $l : T \to Z$ be the complementary reduced closed immersion; let further $k' : W' \to S$ be an open immersion with $W' \cap Z = W$ and $l' : T' \to Z$ be the complementary reduced closed immersion. Write $m : W \to W'$ and $n : T \to R$ for the induced closed immersions.

We have a localisation triangle for $k, l$ to which we apply $i_l$ and get

$$i_l k^* M \to i_* M \to i_! l^* M \to$$

which can be rewritten as

$$k' m k^* M \to i_* M \to (l' \circ n)_* l^* M \to .$$

By Lemma 4.5 (i), we have $k^* M \in \mathcal{JG}_W$ and $l^* M \in \mathcal{JG}_Z$. We have $\dim(T) < \dim(Z)$ so that by induction the third term of this triangle is in $\langle \mathcal{JG}_S \rangle_+$. Moreover $k'_!$ preserves $\langle \mathcal{JG} \rangle_+$ by Lemma 4.5 (i). Together, this means that to show that $i_* M$ is in $\langle \mathcal{JG}_S \rangle_+$, we need only show that $m k^* M$ is in $\langle \mathcal{JG}_W \rangle_+$. We are thus reduced to the case where $Z$ is irreducible (with generic point $\eta$) and $e$ is a finite étale morphism. The rest of the induction step consists of applying the same type of spreading out and deformation arguments we used in the $\dim(Z) = 0$ case to $G_\eta$. Since the three cases are similar and the case of $G = \text{Jac}(C/S)$ with $f : C \to S$ smooth projective curve is the most complicated, we only spell out that one. First, by extending the étale morphism $e$ to the whole of $S$, we can essentially assume $e = \text{id}$ and $V = S$, which we do here for simplicity of notation.

By the same deformation argument as in the dimension $0$ case, which applies to the non-closed point $\eta$ as well, we can find an open étale neighbourhood $(e : W \to S, x \to \eta)$ of $(S, \eta)$ and a smooth projective curve $\tilde{f} : \tilde{C} \to W$ (with geometrically connected fibres and a section) which extends $C_\eta$.

Let $V = \{x\} \subset W$ be the closure of $x$. By spreading out, there exists an open neighbourhood $V^{\circ} \subset V$ of $x$ and a dense open subset $Z^{\circ} \subset Z$ such that $\tilde{f}$ induces an isomorphism $V^{\circ} \simeq Z^{\circ}$ (since it is an isomorphism above $\eta$). By localisation and the induction hypothesis, we can assume that $Z^{\circ} = Z$. We now have a smooth projective curve above an open set of $S$ (with geometrically connected fibres and a section) which extends $f$, and we can then conclude by localisation as in the end of the proof of the $\dim(Z) = 0$ case. This finishes the proof. \hfill $\square$

The deformation theory argument in the proof is the reason why we have introduced an arbitrary étale morphism in the definitions of $\mathcal{DG}$ and $\mathcal{JG}$, instead of say an open immersion. A simplification of the same proof yields the following 0-motivic analogue.

Lemma 4.8. Let $i : Z \to S$ be a closed immersion. Then

$$i_* (e_2 \mathbb{Q}) e : U \to Z \text{ étale } \langle + \rangle \subset \langle f_1 \mathbb{Q} \rangle f : V \to S \text{ étale } \langle + \rangle.$$

We can now exhibit new generating families for $\text{DA}_1(S)$ and $\text{DA}^1(S)$.

Proposition 4.9. Let $S$ be a noetherian finite-dimensional scheme.

(i) $\langle \mathcal{JG}_S \rangle_+ = \langle \mathcal{DG}_S \rangle_+$.

(ii)

$$\text{DA}_1(S) = \langle \mathcal{JG}_S \rangle = \langle \mathcal{DG}_S \rangle$$

and

$$\text{DA}_1(S) = \langle \mathcal{JG}_S \rangle = \langle \mathcal{DG}_S \rangle.$$

(iii)

$$\text{DA}^1(S) = \langle \mathcal{JG}_S(-1) \rangle = \langle \mathcal{DG}_S(-1) \rangle$$

and

$$\text{DA}_1(S) = \langle \mathcal{JG}_S(-1) \rangle = \langle \mathcal{DG}_S(-1) \rangle.$$
Proof. Let us prove Point (i). Using Lemma 4.5 and localisation, we can assume that $S$ is reduced. Lemma 4.5 already implies that $\langle JG_S \rangle^{(+)} \subset \langle DG_S \rangle^{(+)}$. We prove the reverse inclusion by noetherian induction on $S$. Since $\langle \langle G \rangle \rangle^{(+)} = \langle G \rangle$ for any family $G$, it is enough to treat the + version. Let $M$ be in $DG_S$. By Proposition 4.7, Lemma 4.5 and localisation, to proceed with the induction, it is enough to show that there exists a non-empty open set $j : V \to S$ such that $j^* M$ lies in $\langle JG_V \rangle^{(+)}$.

A lattice (resp. a torus) on a reduced scheme is generically a direct factor of a permutation lattice (resp. torus) by [1, Exp. X 6.2], while an abelian scheme on $S$ is generically and up to isogeny a direct factor of the relative Jacobian of a smooth projective curve with a rational point by [45, Theorem 11] applied at a generic point and a spreading out argument. This implies that for any $M \in DG_S$, there exists a non-empty open $j : V \to S$ such that $j^* M$ is a direct factor of a motive in $JG_V$. This completes the proof of Point (i).

For Point (ii), it is enough to show that $DA_{1,c}(S) = \langle DG_S \rangle$. Over an arbitrary field $k$, we have that $DA_{1,c}(k)$ is generated by motives of smooth projective curves by Proposition 1.25, and those lies in $\langle DG_k \rangle$ by Proposition 2.5. In the other direction, it is enough to show that the image by $\Sigma^\infty$ of pure Deligne 1-motives over $k$ lie in $DA_{1,c}(k)$, and this follows from the standard trick of writing them as direct factors of motives coming from permutation lattices and tori or Jacobians, which are then taken care of by Proposition 2.5. By continuity for both sides, we can apply noetherian induction, localisation and use the stability by $\gamma$ of both sides (Proposition 1.17 for $DA_{1,c}$, Proposition 4.7 for $\langle DG \rangle$). This finishes the proof. □

We come to the main definition of this paper.

Definition 4.10. The motivic t-structure $t_{MM,1}(S)$ on $DA_1(S)$ (resp. $t_{MM}^0(S)$ on $DA^1(S)$) is the t-structure $t(CG_S)$ (resp. $t(CG_S(-1))$). The heart of $t_{MM,1}$ (resp. $t_{MM}^0$) is the abelian category $MM_1(S)$ (resp. $MM^0(S)$).

The two abelian categories $MM_1(S)$ and $MM^0(S)$ are isomorphic via Tate twists, but embedded differently in $DA(S)$. From Proposition 4.9 we immediately get the following statement.

Corollary 4.11. $t_{MM,1} = t(JG_S) = t(DG_S)$ (resp. $t_{MM}^0 = t(JG_S(-1)) = t(DG_S(-1))$).

We introduce a parallel definition for 0-motives.

Definition 4.12. The motivic t-structure $t_{MM,0}(S)$ on $DA_0(S) = DA^0(S)$ is the t-structure generated by the family of objects of the form $e_\mathbb{Q}$ with $e : T \to S$ étale. The heart of $t_{MM,0}^0$ is the abelian category of 0-motivic sheaves $MM^0(S)$.

Remark 4.13. The t-structure $t_{MM,0}(S)$ is somewhat similar to the homotopy t-structure on the whole of $DA(S)$, which we define, following [7, Definition 2.2.41], as the t-structure generated by the objects $f_\mathbb{Q}(n)[n]$ for all $f : T \to S$ smooth and $n \in \mathbb{Z}$. We conjecture that the homotopy t-structure restricts to $DA_0(S)$ and that its restriction is $t_{MM,0}(S)$.

We now discuss some elementary exactness properties of Grothendieck operations with respect to the motivic t-structures.

Proposition 4.14. The following properties hold for $t_{MM,1}$, $t_{MM}^1$ and $t_{MM,0}$.

(i) Let $f$ be a morphism of schemes; then $f^*$ is t-positive.
(ii) Let $f$ be a quasi-finite separated morphism; then $f_!$ is t-positive.
(iii) Let $e$ be an étale morphism; then $e^*$ is t-exact.
(iv) Let $f$ be a finite morphism; then $f^*$ is t-exact.

Let $e \in \{0,1\}$; the following properties hold for $t_{MM}^e$.

(v) Let $f$ be a morphism of schemes; then $\omega^e f_*$ is t-negative.
(vi) Let $f$ be a quasi-finite separated morphism of schemes; then $\omega^e f^*$ is t-negative.

Proof. By Proposition 1.17 (resp. 1.16) and the very definition of $\omega^0$ and $\omega^1$, all the operations above are well-defined. We prove the proposition for $t_{MM,1}$; the proof for the corresponding statements for $t_{MM}^1$ is then obtained by twisting by $Q(-1)$, and the proof for $t_{MM,0}$ is completely analogous (using Lemma 4.8 instead of Proposition 4.7).

Let $f : S \to T$ be any morphism of schemes. Then $f^*$, $f_!$ both commute with small sums since they are left adjoints. By [7, Lemme 2.1.78], to prove statements (i), (ii), (iii), it remains to show that $f^*DG_T \subset DA_1(S)_{\geq 0}$ and that when $f$ is quasi-finite, $f_!DG_S \subset DA_1(S)_{\geq 0}$.
In the case of $f^\ast$, we deduce from the $\text{Ex}_1^\ast$ isomorphism and Proposition 2.2 that we have the inclusion $f^\ast \mathcal{D}_T \subset \mathcal{D}_S$. This proves (i).

For the case of $f_i$, we proceed in several steps. If $e$ is an étale morphism, we have $e_i \mathcal{D}_S \subset \mathcal{D}_T$ by definition. If $i$ is a closed immersion, we have $i \mathcal{D}_S \subset (\mathcal{D}_T)^+$ by Proposition 4.9. Let $f$ be an arbitrary quasi-finite morphism. At this point, we know that for an open immersion $j$ (resp. a closed immersion $i$), the functors $j_!$ and $i^\ast$ (resp. the functors $i$ and $i^\ast$) are t-positive. This shows that to prove that an object $M$ is t-positive, one can proceed by localisation. A noetherian induction together with the étale case above then reduce us to the case where $f$ is finite surjective inseparable, and allows us further to restrict to an arbitrary dense open set of the base. Using continuity, this reduces us to the field case, where we can apply Lemma 1.26.

Let $f$ be an étale morphism (resp. a finite morphism). We have seen above that $f_i^\ast$ (resp. $f_i^\ast \simeq f_i$) is t-positive. Moreover, since $e_i \simeq e_1$ (resp. $f_i^\ast$) is t-positive, its right adjoint $e_1^\ast$ (resp. $f_i$) is t-negative. This proves (iii) (resp. (iv)).

Let $f : S \to T$ be a morphism (resp. a quasi-finite separated morphism). We have seen above that $f^\ast : \mathcal{D}_A^1(T) \to \mathcal{D}_A^1(S)$ (resp. $f_i : \mathcal{D}_A^1(S) \to \mathcal{D}_A^1(T)$) is t-positive, so its right adjoint $\omega^\ast f_i$ (resp. $\omega^\ast f_i$) is t-negative. This proves (v) (resp. (vi)).

From the definition, we also get a partial result about the Betti and ℓ-adic realisation functors.

**Proposition 4.15.**

- Let $k$ be a field with a fixed complex embedding $\sigma$ and $S$ be a scheme of finite type over $k$. The functor $R_{\sigma, r}$, restricted to either $\mathcal{D}_A^0(S)$, $\mathcal{D}_A^1(S)$ or $\mathcal{D}_A^1(S)$ is t-positive with respect to the motivic t-structure and the standard t-structure.

- Let $\ell$ be a prime, and let $S$ be a $\mathbb{Z}[[\ell]]$-scheme. The functor $R_{\ell}$, restricted to either $\mathcal{D}_A^0(S)$, $\mathcal{D}_A^1(S)$ or $\mathcal{D}_A^1(S)$, sends compact $t_{\text{MM}}$-positive objects to positive objects in the standard t-structure.

**Proof.** Because of the definition of the motivic t-structures above, and the structure of t-positive and compact objects in a generated t-structure, it is enough to show that the image of the compact generators is t-positive for the standard t-structure. The three cases being similar, let us treat the one of $\mathcal{D}_A^1(S)$. Let $e : U \to S$ be an étale morphism, $M = [L \to G] \in \mathcal{M}_1(U)$ and $M = e_1 \Sigma^\infty M \in \mathcal{D}_U$ (recall that $e_2 \simeq e_1$ as $e$ is étale).

Write $R$ for either $R_{\sigma}$ or $R_{\ell}$ (with the appropriate hypothesis on $S$). Then $RM \simeq e_1 R(\Sigma^\infty M)$ with $e_1$ the corresponding Grothendieck operations on derived categories of sheaves (by [9, Théorème 3.19] for $R = R_{\sigma}$ and [11, Théorème 9.7] for $R = R_{\ell}$). Since the functor $e_1$ is then t-exact for the standard t-structures, we only need to show that $R(\Sigma^\infty M)$ is t-positive. Let us show that it is in fact in the heart of the standard t-structure. We can show this separately for $M = [L \to 0]$ and $M = [0 \to G]$, i.e., we need to compute $R(\Sigma^\infty L)$ and $R(\Sigma^\infty G[-1])$.

Note that because of the commutation of $R$ with the six operations, localisation and the t-exactness of $j_! j^\ast$ and $i_! i^\ast$ for the standard t-structures, we can always restrict to a non-empty open set of $U$ and argue by noetherian induction. We can then assume $U_{\text{red}}$ to be normal, and then write $L$ as a direct factor of $h_! Q_T$ for $h : T \to U$ finite étale using Lemma A.2. Applying again the commutation of $R$ with $h_!$ and the t-exactness of $h_!$ for the standard t-structures, we conclude that $R(\Sigma^\infty L)$ is in the heart.

In the case of $\Sigma^\infty G[-1]$, our claim follows from the computation of the realisation of such a motive in [6, Proposition 5.1.2)] (for $R_{\sigma}$) and [6, 5.2] (for $R_{\ell}$). This completes the proof. □

There are simple connections between the t-structures for 0 and 1-motives.

**Proposition 4.16.** Let $S$ be a noetherian finite-dimensional scheme.

1. The inclusion of $\mathcal{D}_A^0(S)$ into $\mathcal{D}_A^1(S)$ is t-exact.
2. The t-structure $t_{\text{MM}, 1}(S)$ restricts to $\mathcal{D}_A^1(S)$, and its restriction coincide with $t_{\text{MM}, 1}(S)$.

**Proof.** Let us prove Statement (i). The inclusion functor commutes with small sums. The generators $e_1 \Omega = (e : U \to S \text{ étale})$ of $t_{\text{MM}}^0$ are also t-positive for $t_{\text{MM}, 1}$; this implies that the inclusion is t-positive.

Let us now show the inclusion $\mathcal{D}_A^1(S)$ is t-negative. Let $N \in \mathcal{D}_A^0(S)$. Using Proposition 4.9 (i), we see that we have to show that for every étale morphism $e : U \to S$, $M = [L \to \mathbb{Q}^\ast_L(1)]$.
$G \in \mathcal{M}(U)$, and $n \in \mathbb{N}^*$, we have $\text{DA}(S)(e_\ell(\Sigma^\infty M)[n], N) = 0$. Using the $e_\ell \dashv e^*$ adjunction and the fact that $e^*$ is $t$-negative (Proposition 4.14 (iii)), we reduce to the case $e = \text{id}$. We have a distinguished triangle

$$\Sigma^\infty G_\mathbb{Q}[\cdot \cdot -1] \to \Sigma^\infty M \to \Sigma^\infty L_\mathbb{Q} \to .$$

Let us first show that, for all $P \in \text{DA}^0(S)$, we have $\text{DA}(S)(\Sigma^\infty G_\mathbb{Q}, P) = 0$. By $\Sigma^\infty G$ is compact, this vanishing statement can be checked on compact generators, so we can assume that $P$ is of the form $a, Q_X[m]$ for some $a : X \to S$ finite and $m \in \mathbb{Z}$. Using the $a^* \dashv a_*$ adjunction and Proposition 2.2, we see that we can assume $a = \text{id}$, so we have to show that $\text{DA}(S)(\Sigma^\infty G_\mathbb{Q}, [m]) = 0$. By [6, Theorem 3.3], $\Sigma^\infty G_\mathbb{Q}$ is a direct factor of $M_S(G)$, characterised as the $n$-eigenspace for the morphism induced by $[n]_G$ for any $n \neq 1$, and that $M_S(G)$ has also a direct factor $\mathbb{Q}_S$, characterised as the 1-th eigenspace for $[n]_G$. We have $\text{DA}(S)(M_S(G), Q[m]) \simeq H^{m,n}_M(G)$; since $\pi : G \to S$ is smooth surjective with connected fibres, we deduce by Proposition 2.5 (iv) that $\pi^* : H^{n,0}_M(S) \to H^{n,0}_M(S)$ is an isomorphism. Looking at the action of $[n]_G$, this shows that all the weight 0 motivic cohomology of $G$ comes from the direct factor $\mathbb{Q}_S$ of $M_S(G)$, and accordingly we deduce that $\text{DA}(S)(\Sigma^\infty G_\mathbb{Q}, Q[m]) = 0$ as claimed. This shows that $\text{DA}(S)(\Sigma^\infty M[n], N) \simeq \text{DA}(S)(\Sigma^\infty L_\mathbb{Q}[n], N)$.

On the other hand, the motive $\Sigma^\infty L(-1)$ is in $\text{DA}^0(S)$ and $t_{\text{MM},0}$-positive; this would be clear for $S$ normal since $L$ is then a direct factor of a permutation lattice, in general this can be checked by noetherian induction starting from a non-empty open set $V \subset U$ with $V_{\text{red}}$ normal, using localisation, Proposition 2.2 and Proposition 4.14. Since by hypothesis $N$ is $t_{\text{MM},0}$-negative, we have $\text{DA}(S)(\Sigma^\infty L_\mathbb{Q}[n], N) = 0$. This completes the proof that $DA_0(S) \to DA_1(S)$ is $t$-negative, hence exact.

We now prove Statement (ii). Write $a_{\geq 0}$ and $1_{\geq 0}$ for the truncation functors of $t_{\text{MM},0}$ and $t_{\text{MM},1}$. We have to show that for every $M \in \text{DA}_0(S)$, we have $1_{\geq 0} M \in \text{DA}_0(S)$ and $1_{\geq 0} M \simeq 0_{\geq 0} M$. But this follows immediately from the $t$-exactness of the inclusion, proved above. \qed

Remark 4.17. We also conjecture that Proposition 4.16 holds for $t_{\text{MM}}(S)$; this seems to require more delicate vanishing results.

4.3. The $t$-structures over a field. In this short section, we compare our $t$-structures for homological 0 and 1-motives with the existing work on $t$-structures for $\text{DM}^0_\text{eff}(k)$ and $\text{DM}^1_\text{eff}(k)$ with $k$ a perfect field [53] [10], and we extend the results from these references to a possibly imperfect field.

For clarity, let us treat the simpler case of 0-motives. Let $k$ be a perfect field. We reformulate the treatement in [53, §2]. There is a functor $\text{Sh}_{\text{et}}(k, Q) \to \text{DM}^0_\text{eff}(k)$ (any sheaf of $Q$-vector spaces on the small étale site has a canonical extension as an étale sheaf with transfers on $\text{Sm}(k)$ which extends to a triangulated functor $D(\text{Sh}_{\text{et}}(k, Q)) \to \text{DM}^0_\text{eff}(k, Q)$. This factors through $\text{DM}^0_\text{eff}(k)$, and the resulting functor is an equivalence of categories $R^0_\text{eff} : D(\text{Sh}_{\text{et}}(k)) \simeq \text{DM}^0_\text{eff}(k)$.

Another approach consists in first introducing the homotopy $t$-structure on $\text{DM}^0_\text{eff}(k)$; this is the $t$-structure induced on $\text{DM}^0_\text{eff}(k)$ from the standard $t$-structure on $D(\text{Sh}((\text{Cor})/k, Q))$, but for our purposes it is best described as the $t$-structure on the compactly generated triangulated category $\text{DM}^0_\text{eff}(k)$ by the family of objects of the form $M_\text{eff}^\text{tr}(X)$ for all $X \in \text{Sm}(k) [10, \text{Proposition 3.3}]$. We claim that the homotopy $t$-structure restricts to $\text{DM}^0_\text{eff}(k)$, and that the restriction coincides with the $t$-structure generated by the family of objects of the form $M_\text{eff}^\text{tr}(Y)$ for all $Y/k$ finite étale. To do this, it suffices to show that the inclusion functor $\text{DM}^0_\text{eff}(k) \to \text{DM}^\text{eff}(k)$ is $t$-exact for those two $t$-structures; it is $t$-positive because of the inclusion of generators, and $t$-negative because its left adjoint $L\pi_0$ is $t$-positive since $L\pi_0(M_\text{eff}^\text{tr}(X)) \simeq M_\text{eff}^\text{tr}(\pi_0(X/k))$ for any $X/k$ smooth.

It is easy to see that the $t$-structures on $\text{DM}^0_\text{eff}(k)$ introduced in the two previous paragraphs coincide. Moreover, through the equivalence of categories of Lemma 3.12, we get an equivalence of categories $R^0 : D(\text{Sh}_{\text{et}}(k, Q)) \to \text{DA}_0(k)$, and this is an equivalence of $t$-categories when we equip $\text{DA}_0(k)$ with $t_{\text{MM},0}$.

Finally, these $t$-structures on $\text{DM}^0_\text{eff}(k)$ and $\text{DA}_0(k)$ restrict to compact objects; more precisely, there are equivalences of categories $D^b(\text{Sh}_{\text{et}}(k, Q)) \simeq \text{DM}^0_\text{eff}(k) \simeq \text{DA}_0(k)$ and the restriction of the $t$-structure coincides with the standard $t$-structure on the bounded derived category.

\[ \text{DM}^0_\text{eff}(k) \text{ and } \text{DA}_0(k) \text{ restrict to compact objects; more precisely, there are equivalences of categories } D^b(\text{Sh}_{\text{et}}(k, Q)) \simeq \text{DM}^0_\text{eff}(k) \simeq \text{DA}_0(k) \text{ and the restriction of the } t \text{-structure coincides with the standard } t \text{-structure on the bounded derived category.} \]
Let now $k$ be a general field and let $h : \text{Spec}(k^{\text{perf}}) \to \text{Spec}(k)$ be a perfect closure. We have a commutative diagram

$$
\begin{array}{ccc}
D(\text{Sh}_k(k)) & \xrightarrow{\mathcal{R}^0} & DA_0(k) \\
\sim & & \sim \\
D(\text{Sh}_k(k^{\text{perf}}, Q)) & \xrightarrow{\mathcal{R}^0} & DA_0(k^{\text{perf}})
\end{array}
$$

where the bottom horizontal functor is an equivalence by the case of a perfect field, the left vertical functor is an equivalence because the étale sites of $k$ and $k^{\text{perf}}$ are canonically isomorphic via $h$, and the right vertical functor is an equivalence by the separation property of $DA(-)$ and Corollary 0.18 (ii). Moreover, the functor $h^* : D(\text{Sh}_k(k)) \to D(\text{Sh}_k(k^{\text{perf}}))$ is clearly $t$-exact, the functor $h^* : DA_0(k) \to DA_0(k^{\text{perf}})$ is $t$-exact because it is a quasi-inverse of the $t$-exact functor $h_*$ (Proposition 4.14 (iv)), and $\mathcal{R}^0(k^{\text{perf}})$ is $t$-exact by the perfect field case. This proves that the top arrow is also an equivalence of $t$-categories. There is a similar diagram in the compact case which we will not spell out. Let us summarise the results so far.

**Proposition 4.18.** Let $k$ be a field. The $t$-structure $t_{\text{MM}, 0}$ restricts to compact objects, and we have equivalences of $t$-categories

$$
\mathcal{R}^0 : (D(\text{Sh}_k(k, \mathbb{Q})), \text{std}) \xrightarrow{\sim} (DA_0(k), t_{\text{MM}, 0})
$$

$$
\mathcal{R}^0 : (D^b(\text{Sh}_{k,c}(k, \mathbb{Q})), \text{std}) \xrightarrow{\sim} (DA_0(k), t_{\text{MM}, 0}).
$$

We now turn to the case of 1-motives. Assume again momentarily that $k$ is a perfect field. By [16, Lemma 1.4.4], for any commutative locally of finite type $k$-group scheme $G$, the sheaf represented by $G$ on $\text{Sm}/k$ has a canonical structure of étale sheaf with transfers. Write $G^{\text{tr}}$ for this sheaf with transfers, with $G^{\text{tr}} G^{\text{tr}} \simeq G$.

Applying this construction at the level of complexes, Orgogozo defines in [53, 3.3.2] a functor which we will denote by

$$
\mathcal{R}^0_{\text{eff, tr}} : M_1(k) \to DM_{1, c}^{\text{eff}}(k).
$$

The category $M_1(1)$ is in this situation an abelian category [53, Lemme 3.2.2] and this functor can in fact be extended to a functor

$$
\mathcal{R}^0_{\text{eff, tr}} : D^b(M_1(k)) \to DM_{1, c}^{\text{eff}}(k).
$$

This functor factors through $DM_{1, c}^{\text{eff}}(k)$ (denoted as $d_{1} DM_{1,c}^{\text{eff}}(\eta)$ in loc. cit.) and the resulting functor is then an equivalence of categories [53, Theorem 3.4.1]. In particular, this provides a $t$-structure on $DM_{1, c}^{\text{eff}}(k)$, which we will denote by $t_{\text{tr}}^{\text{eff}}(k)$. By the equivalence between $DM_{1, c}^{\text{eff}}(k)$ and $DA_{1, c}(k)$, we get a $t$-structure on $DA_{1, c}(k)$ which we also denote by $t_{\text{tr}}^{\text{eff}}(k)$. Moreover, by comparing $\mathcal{R}^0_{\text{eff, tr}}$ with $\Sigma^\infty$, we get that the functor

$$
\Sigma^\infty : D^b(M_1(1)) \to DA_{1, c}(k)
$$

is an equivalence of $t$-categories. By [53, Proposition 3.3.3] and [53, Proposition 3.2.4], we have the following computation of morphisms groups in $DA_{1, c}(k)$.

**Proposition 4.19.** Let $k$ be a field, $M_1, M_2 \in M_1(k)$ and $n \in \mathbb{Z}$. Then

$$
DA(k)(\Sigma^\infty M_1, \Sigma^\infty M_2[n]) \simeq \text{Ext}_{M_1(k)}^n(M_1, M_2) \simeq 0, \ n \neq 0, 1.
$$

We can now show the following basic result.

**Proposition 4.20.** Let $k$ be a field and $k^{\text{perf}}$ a perfect closure. The $t$-structure $t_{\text{MM}, 1}$ restricts to compact objects, and we have an equivalence of $t$-categories

$$
\Sigma^\infty : (D^b(M_1(k^{\text{perf}})), \text{std}) \to (DA_{1, c}(k), t_{\text{MM}, 1}).
$$

**Proof.** We first assume that $k$ is perfect. Let us show that the $t$-structure $t_{\text{MM}, 1}(k)$ on $DA_1(k)$ restricts to $DA_{1, c}(k)$, and that its restriction is $t_{\text{tr}}^{\text{eff}}(k)$. For this, it is enough to show that if $M \in DA_{1, c}(k)$ is $t_{\text{tr}}^{\text{eff}}(k)$-positive (resp. negative), it is $t_{\text{MM}, 1}(k)$-positive (resp. negative). Using the equivalence $\Sigma^\infty$, it is clearly enough to show this for $M = \Sigma^\infty(M)$ with $M \in M_1(k)$. By
construction of \( t_{\text{MM},1}(k) = t(\Sigma^\infty (M_1(k))) \), we see that \( M \) is \( t_{\text{MM},1}(k) \)-positive. It remains to show that \( M \) is \( t_{\text{MM},1}(k) \)-negative, i.e., that for all \( N \in M_1(k) \) and \( k > 0 \), we have

\[
\mathbf{DA}(k)(\Sigma^\infty \mathbb{N}[k], \Sigma^\infty M) = 0.
\]

This is a special case of Proposition 4.19.

Let now \( k \) be a general field and \( h : \text{Spec}(k_{\text{perf}}) \to \text{Spec}(k) \) be a perfect closure. The functor \( h^*: (\text{DA}_1(k), t_{\text{MM},1}) \to (\text{DA}_1(k_{\text{perf}}), t_{\text{MM},1}) \) is an equivalence of \( t \)-categories by the separation property of \( \mathbf{DA}(-) \), Corollary 1.18 (ii), and Proposition 4.14 (iv). It then follows from the perfect case above that \( t_{\text{MM},1}(k) \) restricts to compact objects. \( \Box \)

4.4. Deligne 1-motives and the heart. In this section, we compute certain morphism groups between objects in \( \text{DA}_1(S) \) and \( \text{DA}^1(S) \) and deduce various properties of the motivic \( t \)-structure.

The following theorem shows the advantage of the Deligne generating family: it lies in the heart of the motivic \( t \)-structure.

**Theorem 4.21.** Let \( S \) be a noetherian finite-dimensional scheme. We have \( \mathcal{D}G_S \subset \mathbb{M}_1(S) \) (resp. \( \mathcal{D}G_S(-1) \subset \mathbb{M}^1(S) \)).

**Proof.** We have shown in Proposition 4.9 that the generators are \( t \)-positive, it remains to show that they are \( t \)-negative.

Using the generating family \( J\mathcal{G}_S \), this translates into the following vanishing statement. Let \( S \) be a noetherian finite dimensional scheme. Let \( e : U \to S \) be an étale morphism, and \( N = e_! \Sigma^\infty K \otimes \mathbb{Q} \) be a Jacobian generator. Let \( P = f! \Sigma^\infty M \in \mathcal{D}G_S \) (i.e., \( f : V \to S \) étale, \( M \in \mathcal{M}_1^\text{pure}(V) \)). Then we show, for all \( n < 0 \), that

\[
(\mathcal{V}_n(P)) = 0.
\]

By the \((e_!, e^!)\) adjunction and Proposition 2.2, we can assume that \( e = \text{id} \). By localisation and Proposition 2.2, we can assume that \( S \) is reduced. By Zariski’s main theorem, there exists a factorisation \( f = f \circ j \) with \( f : \bar{V} \to S \) finite and \( j : V \to \bar{V} \) an everywhere dense open immersion; we can assume \( \bar{V} \) is reduced as well. Combining this with the \((f^*, f_*)\) adjunction, and Proposition 2.2, we see that we can assume \( f = j \) is an everywhere dense open immersion. We write \( i : Z \to S \) for the complementary reduced closed immersion.

We want to prove \((\mathcal{V}_n(P))\) by induction on the dimension of \( S \). In each case, to treat the case of \( \dim(S) = 0 \), we reduce immediately to the case of \( \text{Spec}(k) \) for \( k \) a field and apply Proposition 4.19. We are thus left with the induction step.

First, we do a general reduction. Let \( I : W \to S \) an everywhere dense open immersion with \( W \subset V \) and \( k : Y \to S \) the complementary reduced closed immersion. Then by localisation we have exact sequences

\[
\mathbf{DA}(S)(N, P[n]) \to \mathbf{DA}(S)(N, P[n]) \to \mathbf{DA}(N, k^* P[n])
\]

and in both cases the right term vanishes for \( n < 0 \) by adjunction and the induction hypothesis (since \( \dim(Z) < \dim(S) \)). This means we can replace \( P \) with

\[
l_i^{t*} P \simeq l_i^{t*} j_! \Sigma^\infty M \simeq (W \to S)(W \to V)^* \Sigma^\infty M \simeq (W \to S)\Sigma^\infty M_W
\]

where we have used the \( \text{Ex}^* \) isomorphism and Corollary 2.3. In other words, we can replace the dense open subscheme \( V \) by any smaller dense open \( W \).

There are three types of Deligne generators and three types of Jacobian generators, which lead to a distinction in nine cases. We index these cases by weights: for instance, the case where \( M = L \to 0 \) and \( K = G_n \) will be labelled \((0, 0)\).

Cases \((0, 0)\):

Let \( M = [L \to 0] \otimes \mathbb{Q} \) with \( L \) a lattice on \( V \).

Using this reduction, we can assume \( V \) to be normal. This allows us by Lemma A.2 to write \( \Sigma^\infty M \) as a direct factor of \( e_* \mathcal{Q} \) for a finite étale morphism \( e : T \to V \). Applying Zariski’s main theorem to the morphism \( j \circ e : T \to S \) and adjunction, we reduce to the case \( P = \mathcal{Q} V \). The motive \( P \) then extends to a motive on \( S \), namely \( Q_S \). By localisation, we have an exact sequence

\[
\mathbf{DA}(S)(N, i_* \mathcal{Q}[n]) \to \mathbf{DA}(S)(N, j_* \mathcal{Q}[n]) \to \mathbf{DA}(S)(N, \mathcal{Q}[n])
\]

and in both cases the left term vanishes for \( n < 0 \) by adjunction and induction. This means we can assume \( V = S \).
If we are in case (0, 0) (resp. (0, −2)), then we have \( N = \mathbb{Q}_S \) (resp. \( N = \mathbb{Q}_S(1) \)). By adjunction and Proposition B.5(i) (resp. Proposition B.2), we get \( \text{DA}(S)(N, \mathbb{Q}[n]) = 0 \) for \( n < 0 \).

It remains to treat the case \( (0, -1) \). Let \( C \to S \) be a smooth projective with geometrically connected fibres and a section \( \sigma \). We have \( N = \Sigma^\infty \text{Jac}(C/S)\mathbb{Q}[-1] \), which by Corollary 3.20 is a direct factor of \( M(C)[−1] \). By adjunction, we thus have that \( \text{DA}(S)(N, \mathbb{Q}_S[n]) \) is a direct factor of \( \text{DA}(C)(\mathbb{Q}_C, \mathbb{Q}_C[n + 1]) \). For \( n < -1 \), this group vanishes by Proposition B.5(i). For \( n = 0 \), we apply Proposition B.5(ii) and get \( \mathbb{Q}^{\pi_0}(S) \cong \text{DA}(S)(\mathbb{Q}_S, \mathbb{Q}_S) \to \text{DA}(C)(\mathbb{Q}_C, \mathbb{Q}_C) \cong \mathbb{Q}^{\pi_0}(C) \). The map on \( \pi_0 \) is an isomorphism since \( C \) has geometrically connected fibres. This shows that the contribution of the direct factor \( \Sigma^\infty \text{Jac}(C/S)\mathbb{Q}[-1] \) is 0, and proves the case \( n = -1 \). This finishes the treatment of the cases \((0, *)\).

Cases \((-2, *)\):

Let now \( M \) be of the form \([0 \to T] \otimes \mathbb{Q} \) with \( T \) a torus on \( V \). As in the proof for a lattice, we can replace the dense open \( V \) by any smaller dense open. Again, this lets us assume that \( V \) is normal, hence reduce to a permutation torus using Lemma A.2, then finally to \( T = G_m \). Then \( \Sigma^\infty M \cong V_\infty(1) \) extends to a motive on \( S \), namely \( \mathbb{Q}_S(1) \). By localisation, we have an exact sequence

\[
\text{DA}(S)(N, i_*\mathbb{Q}(1)[n - 1]) \to \text{DA}(S)(N, j_*\mathbb{Q}(1)[n]) \to \text{DA}(S)(N, \mathbb{Q}(1)[n])
\]

and the left term vanishes for \( n < 0 \) by adjunction and induction. This means we can assume \( V = S \).

If we are in case (0, 0) (resp. (0, −2)), then we have \( N = \mathbb{Q}_S \) (resp. \( N = \mathbb{Q}_S(1) \)). By adjunction and Proposition B.6(i) (resp. Proposition B.5(i)), we get \( \text{DA}(S)(N, \mathbb{Q}(1)[n]) = 0 \) for \( n < 0 \).

It remains to treat the case \( (0, -1) \). Let \( C \to S \) be a smooth projective with geometrically connected fibres and a section \( \sigma \). We have \( N = \Sigma^\infty \text{Jac}(C/S)\mathbb{Q}[-1] \), which by Corollary 3.20 is a direct factor of \( M(C)[−1] \). By adjunction, we thus have that \( \text{DA}(S)(N, \mathbb{Q}_S(1)[n]) \) is a direct factor of \( \text{DA}(C)(\mathbb{Q}_C, \mathbb{Q}_C(1)[n + 1]) \). For all \( n < 0 \), this group vanishes by Proposition B.6(i).

Cases \((-1, +1)\):

Let \( M \) finally be of the form \([0 \to A] \otimes \mathbb{Q} \) with \( A \) an abelian scheme on \( V \). As in the two previous cases, we can replace the dense open \( V \) by any smaller dense open. Using [45, Theorem 11] and continuity, this lets us assume that there exists a smooth projective curve \( f : D \to V \) with geometrically connected fibres together with a section \( s : V \to D \) such that the \( \Sigma^\infty [0 \to A] \) is a direct factor of \( \Sigma^\infty [0 \to \text{Jac}(D/V)] \). In the following, we replace \( A \) by \( \text{Jac}(D/V) \).

Unlike in the two previous cases, we cannot ensure that the curve \( D \) extends to a smooth projective curve over \( S \), so we have to work around this. From Corollary 3.20, we have an isomorphism \( f_2\mathbb{Q}_D \cong \mathbb{Q}_V \otimes \Sigma^\infty \text{Jac}(D/V)\mathbb{Q} \otimes \mathbb{Q}_V(1)[2] \); hence \( \Sigma^\infty M \cong \Sigma^\infty \text{Jac}(D/V)\mathbb{Q}[-1] \) is a direct factor of \( f_2\mathbb{Q}_D[-1] \). By relative purity, we have \( f_2\mathbb{Q}_D[-1] \cong f_1\mathbb{Q}_D(1)[1] \).

We apply Nagata's theorem [50] [28] to compactify \( f \) over \( S \): there exists an open immersion \( j : D \to \overline{D} \) and a proper morphism \( f : \overline{D} \to S \) with \( j \circ f = f \circ j \). Write \( i : Y \to \overline{D} \) for the complementary closed immersion; note that because \( f \) was proper over \( V \), we can choose the compactification \( \overline{D} \) so that \( Y \) lies entirely over \( Z \), and we have a commutative diagram with cartesian squares.

\[
\begin{array}{ccc}
D & \xrightarrow{j} & \overline{D} \\
\downarrow f & & \downarrow i \\
V & \xrightarrow{j} & S \\
\downarrow & & \downarrow \\
& & Z
\end{array}
\]

This implies that \( j_! f_! \cong f_! j_! \cong f_* j_* \); hence \( j_! f_! \mathbb{Q}_D(1)[1] \cong f_* j_* \mathbb{Q}_D(1)[1] \). The motive \( j_* \mathbb{Q}_D(1)[1] \) extends to a motive on \( \overline{D} \), namely \( \mathbb{Q}_{\overline{D}(1)[1]} \). By localisation, we have an exact sequence

\[
\text{DA}(\overline{D})(f^*N, i_*\mathbb{Q}(1)[n]) \to \text{DA}(\overline{D})(f^*N, j_*\mathbb{Q}(1)[n + 1]) \to \text{DA}(\overline{D})(f^*N, \mathbb{Q}(1)[n + 1]) \to \text{DA}(\overline{D})(f^*N, i_*\mathbb{Q}(1)[n + 1])
\]

The left term is isomorphic to \( \text{DA}(Y)(f_!\mathbb{Q}(1)[n]) \). Since \( f_!\mathbb{Q}(1) \) is a Jacobian generator on \( Y \), the vanishing of this group was proved in Case \((-2, +1)\). Similarly, for \( n < -1 \), the right term vanishes by Cases \((-2, *)\). We can thus assume \( n = -1 \) in the end of the proof, so that we are looking at the exact sequence

\[
0 \to \text{DA}(\overline{D})(f^*N, j_*\mathbb{Q}(1)) \to \text{DA}(\overline{D})(f^*N, \mathbb{Q}(1)) \to \text{DA}(\overline{D})(f^*N, i_*\mathbb{Q}(1))
\]
and we need to show that the leftmost term vanishes.

If we are in case \((-1, 0)\), we have \(N = \mathbb{Q}_S\), hence \(\pi^* N = \mathbb{Q}_{\mathcal{P}}\), and the group \(\text{DA}(\mathcal{D})((\mathbb{Q}, \mathbb{Q}(1))\) vanishes by Proposition B.6(i). This concludes the proof for \((-1, 0)\).

If we are in case \((-1, -1)\), we have \(N = \Sigma^\infty \text{Jac}(C/S) \otimes [\mathcal{P}]\) with \(C \to S\) a smooth projective with geometrically connected fibres and a section \(\sigma\). Hence \(\pi^* N \simeq \Sigma^\infty \text{Jac}(C \times_S \mathcal{P}/\mathcal{D}) \otimes [\mathbb{Q}[-1]]\). The morphism group \(\text{DA}(\mathcal{D})((\Sigma^\infty \text{Jac}(C \times_S \mathcal{P}/\mathcal{D}) \otimes [\mathbb{Q}[-1]], \mathbb{Q}(1))\) vanishes by Lemma 4.22 below; this concludes the proof for \((-1, -1)\).

If we are in case \((-1, 2)\), we have \(N = \mathbb{Q}_S(1)\), hence \(\pi^* N = \mathbb{Q}_{\mathcal{P}(1)}\). We have

\[
\mathbb{Q}^\pi(\mathcal{D}) \simeq \text{DA}(\mathcal{D})(\mathbb{Q}_{\mathcal{P}(1)}, \mathbb{Q}(1)) \to \text{DA}(\mathcal{D})(\mathbb{Q}_{\mathcal{P}(1)}, \mathbb{Q}(1)) \simeq \mathbb{Q}^\pi(\mathcal{Y})
\]

by Proposition B.5 (ii), hence

\[
\text{DA}(\mathcal{D})(\pi^* N, \mathbb{Q}(1)) \simeq \ker(\mathbb{Q}^\pi(\mathcal{D}) \to \mathbb{Q}^\pi(\mathcal{Y})).
\]

On the other hand, we have, by the same argument

\[
\text{DA}(\mathcal{S})(\mathbb{P}, \mathbb{P}(1)) \simeq \ker(\mathbb{Q}^\pi(\mathcal{S}) \to \mathbb{Q}^\pi(\mathcal{Z})).
\]

Since \(Y \simeq \mathcal{P} \times_S \mathcal{Z}\), we have \(\pi_0(Y) \simeq \pi_0(\mathcal{P}) \times_{\pi_0(S)} \pi_0(\mathcal{Z})\), hence the map

\[
\text{DA}(\mathcal{D})(\pi^* N, \mathbb{P}(1)) \to \text{DA}(\mathcal{S})(\mathbb{P}, \mathbb{P}(1))
\]

is an isomorphism. By looking at the direct factor decomposition of \(f^* \mathbb{Q}_D\), we conclude that

\[
\text{DA}(\mathcal{S})(\mathbb{P}, \mathbb{P}(1)) \simeq \ker(\mathbb{Q}^\pi(\mathcal{D}) \to \mathbb{Q}^\pi(\mathcal{Y})).
\]

which finishes the proof of the case \((-1, -2)\). \(\square\)

**Lemma 4.22.** Let \(S\) be a noetherian finite dimensional scheme. Let \(A\) be an abelian scheme and \(T\) be a torus. Let \(n \leq 0\). Then

\[
\text{DA}(\mathcal{S})(\Sigma^\infty A \otimes \mathbb{Q}, \Sigma^\infty (T \otimes \mathbb{Q})[n]) = 0
\]

**Proof.** Let \(\pi_* : \mathcal{S}_* \to S\) be a \(h\)-hypercovering. Write \(A_p\) (resp. \(T_p\)) for \(A \times_S \mathcal{S}_p\) (resp. \(T \times_S \mathcal{S}_p\)). We have a descent spectral sequence

\[
E^p,q = \text{DA}(\mathcal{S}_p)(\Sigma^\infty A_p \otimes \mathbb{Q}, \Sigma^\infty T_p \otimes \mathbb{Q}[q]) \Rightarrow \text{DA}(\mathcal{S})(\Sigma^\infty A \otimes \mathbb{Q}, \Sigma^\infty T \otimes \mathbb{Q}[p+q]).
\]

Let \(\theta : A \to S\) be the structure morphism of \(A\). The motive \(\Sigma^\infty A \otimes \mathbb{Q}\) is a direct factor of \(\theta^* \mathbb{Q}A\), functorially in \(S\). We see that

\[
\text{DA}(\mathcal{S}_p)(\Sigma^\infty A_p \otimes \mathbb{Q}, \Sigma^\infty (T_p \otimes \mathbb{Q})[n])
\]

is a direct factor of

\[
\text{DA}(A_{\mathcal{S}_p})(\mathbb{Q}A_{\mathcal{P}}, \Sigma^\infty (T_p \times_S A_p) \otimes \mathbb{Q}[n]).
\]

We apply the previous spectral sequence for \(\pi_*\), the Cech covering associated to an étale covering trivializing \(T\); since \(\text{DA}(A_{\mathcal{S}_p})(\mathbb{Q}A, \mathbb{Q}(1)[1][n])\) vanishes for \(n < 0\) by Proposition B.6 (i), the corresponding spectral sequence converges and we have \(\text{DA}(\mathcal{S})(\Sigma^\infty A \otimes \mathbb{Q}, \Sigma^\infty T \otimes \mathbb{Q}[n]) = 0\) for \(n < 0\).

It remains to treat the case \(n = 0\) with \(T = \mathcal{G}_m\). We apply the previous spectral sequence for \(\pi_*\), the Cech covering associated to an affine Zariski cover of \(S\); by the previous paragraph, this spectral sequence converges, so that we are reduced to the case when \(S = \text{Spec}(R)\) is affine. By a continuity argument, we reduce to the case where \(R\) is of finite type of a Dedekind ring, in particular satisfying resolution of singularities by alterations. Then \(S\) admits an \(h\)-hypercovering \(\pi : \mathcal{S}_* \to S\) with regular terms. By the descent spectral sequence, which again converges by the previous paragraph, it is then enough to show the result for \(S\) regular, \(n = 0\) and \(T = \mathcal{G}_m\).

Again, we write \(\Sigma^\infty A \otimes \mathbb{Q}\) as a direct factor of \(\theta^* \mathbb{Q}A\). Since \(S\) and \(A\) are regular, Proposition B.6 (ii) implies that

\[
\text{DA}(\mathcal{S})(\mathbb{Q}, \mathbb{Q}(1)[1]) \simeq \mathcal{O}_S^\times
\]

and

\[
\text{DA}(\mathcal{A})(\mathbb{Q}, \mathbb{Q}(1)[1]) \simeq \mathcal{O}_A^\times.
\]

Since the induced morphism \(\theta\) is proper with geometrically connected fibres, the map \(\mathcal{O}_S^\times \to \mathcal{O}_A^\times\) is an isomorphism. This implies that

\[
\text{DA}(\mathcal{S})(\Sigma^\infty A \otimes \mathbb{Q}, \mathbb{Q}(1)[1]) = 0
\]
and concludes the proof.

\textbf{Corollary 4.23.} Let $S$ be a noetherian finite dimensional scheme. Let $G$ be a smooth commutative group scheme with connected fibres. Then the motive $\Sigma^\infty G[-1]$ lies in $\MM_1(S)$.

\textit{Proof.} By noetherian induction and localisation, we can assume $S$ reduced and it is enough to show that there exists $j : U \to S$ a dense open immersion such that $j_!\Sigma^\infty G_U[-1]$ is in $\MM_1(S)$. We can also assume $S$ to be irreducible; let $\eta$ be its generic point and $h : \eta_{\text{perf}} \to \eta$ a perfect closure. Then $G_\eta$ is a smooth commutative connected algebraic group over a perfect field, hence there exists an exact sequence

$$0 \to U \to G_{\text{purt}} \to H \to 0$$

with $U$ unipotent connected and $H$ a semi-abelian variety. Since $\eta_{\text{perf}}$ is perfect, the motive $\Sigma^\infty U \otimes \QQ$ is trivial (apply [5, Lemma 7.4.5] to a composition series), hence the morphism $h^!\Sigma^\infty G_{\QQ} \simeq \Sigma^\infty G_{\QQ} \to \Sigma^\infty H_{\QQ}$ is an isomorphism.

By Lemma 4.24, there is an abelian variety $H'$ over $\eta$ such that

$$\Sigma^\infty (H'_{\text{purt}} \otimes \QQ) \simeq \Sigma^\infty (H \otimes \QQ).$$

Applying the separation property of $DA(-)$, we get an isomorphism $\Sigma^\infty G_{\eta, \QQ} \simeq \Sigma^\infty H'_{\QQ}$. By spreading out arguments, we can arrange for such an isomorphism to hold over a dense open set $U$ of $S$. We then have $j_!\Sigma^\infty G_{U, \QQ}[-1] \simeq j_!\Sigma^\infty H'_{\QQ}[-1]$ and this last motive is in $\MM_1(S)$ by Theorem 4.21. □

\textbf{Lemma 4.24.} Let $l/k$ be a purely inseparable field extension, and $H$ a semi-abelian variety over $l$. Then there exists a semi-abelian variety $H'$ over $k$ such that $\Sigma^\infty ((H'_l) \otimes \QQ) \simeq \Sigma^\infty (H \otimes \QQ)$ in $DA(l)$.

\textit{Proof.} We can clearly assume $\text{char}(k) = p > 0$. By [22, Lemma 3.10], there exists a (smooth) commutative algebraic group $G'$ over $k$ and an epimorphism $f : H \to G'_l$ such that $\text{Ker}(f)$ is infinitesimal (in particular, killed by a power of $p$). By [22, Corollary 2.13], which applies over any field of positive characteristic, there exists an epimorphism of commutative algebraic $k$-groups $g : G' \to H' \times U'$ with $H'$ semi-abelian, $U'$ split unipotent and $\text{Ker}(g)$ finite (in particular, killed $\otimes \QQ$). We deduce that

$$H \otimes \QQ \simeq (H' \times_k U')_1 \otimes \QQ.$$

But the motive $\Sigma^\infty U'_l \otimes \QQ$ is trivial since $U'$ is split unipotent (apply [5, Lemma 7.4.5] to a composition series). We deduce that

$$\Sigma^\infty (H \otimes \QQ) \simeq \Sigma^\infty (H'_l \otimes \QQ).$$

□

In the course of the proof of Theorem 4.21 (in the lattice case), we also established the vanishing statements necessary to prove the following lemma.

\textbf{Lemma 4.25.} Let $S$ be a noetherian finite-dimensional scheme. Let $e : U \to S$ be an étale morphism. Then $e_! \QQ \in \MM_0(S)$.

Using the same strategy as in the proof of the abelian scheme case (reduction to Jacobian, extension of the curve), one can also prove the following related vanishing result.

\textbf{Proposition 4.26.} Let $S$ be a noetherian finite-dimensional scheme. Let $e : U \to S$ be an étale morphism and $A/U$ be an abelian scheme. Then for all $n \in \ZZ$, we have

$$DA(S)(\QQ, e_!\Sigma^\infty A(-1)[n]) = 0.$$

We deduce an additional compatibility relation between the motivic $t$-structures on 0 and 1-motives.

\textbf{Corollary 4.27.} Let $S$ be a noetherian finite-dimensional scheme. The functor

$$\omega^1 : (DA^1(S), l^1_{\MM}) \to (DA^0(S), l^0_{\MM})$$

is $t$-exact.
Proof. The functor $\omega^0 : \DA^1(S) \to \DA^0(S)$, defined as the restriction of $\omega^0$ to $\DA^1(S)$, is the right adjoint to the inclusion $\DA^0(S) \to \DA^1(S)$. This inclusion is $t$-positive by looking at generators, which implies that its right adjoint $\omega^0$ is $t$-negative.

It remains to show $\omega^0$ is $t$-positive. By Lemma 3.5, $\omega^0$ commutes with small sums. It is thus enough to show that a family of compact generators of $\DA^1(S)$ is sent to $t$-positive objects. By Proposition 4.9, $\DA^1(S)$ is compactly generated by $DG_S(\mathbb{A},\mathbb{Q})$. Let $s : U \to S$ be an étale morphism and $M = [L \to G] \in M_1(U)$, and let $e_2(\Sigma^\infty M)(-1) \in DG_S(\mathbb{A})$. We have to be careful because $\omega^0$ and $e_2$ do not commute in general and we cannot apply directly Proposition 3.10 (iii). However, we have distinguished triangles

\[
e_2 \Sigma^\infty T(-1) \to e_2 \Sigma^\infty M(-1) \to e_2 \Sigma^\infty W_{\geq -1}M(-1) \to \]

and

\[
e_2 \Sigma^\infty A(-1) \to e_2 \Sigma^\infty W_{\geq -1}M(-1) \to e_2 \Sigma^\infty L(-1) \to \]

The motive $(e_2\Sigma^\infty L)(-1)$ is in $\DA^0(S)(-1)$, which by Corollary 3.9 (iii) implies that its $\omega^0$ vanishes. Let us show that we have $\omega^0(e_2\Sigma^\infty A(-1)) \geq 0$. Using the generating family, we have to show that, for all $f : W \to S$ étale and all $n \in \mathbb{Z}$, we have $\DA(S)(f_\mathbb{Q}[-n], e_2\Sigma^\infty A(-1)) = 0$. By adjunction, the $\Ex^n_*$ isomorphism and Proposition 2.2, we can assume $f = \id$ and apply Proposition 4.26.

All together, this means that $\omega^0(e_2\Sigma^\infty M(-1)) \simeq \omega^0(e_2\Sigma^\infty T(-1)) \simeq e_2\Sigma^\infty X_*(T)$ (Proposition 3.10 (ii)) which is $t$-positive for $t_{MM,0}(S)$. This completes the proof. \qed

Notice that at this point we do not know if the motivic $t$-structures restricts to compact objects. A weaker result in that direction is the following result.

**Corollary 4.28.** Let $S$ be a noetherian finite-dimensional scheme. Any compact object in either $\DM_{0}(S)$, $\DM_{1}(S)$ or $\DM_{1}(S)$ is bounded for the motivic $t$-structure, i.e., it has only finitely many non-zero homology objects.

**Proof.** The argument is the same for the three categories, let us explain it for $\DA_{1}(S)$. Let $M \in \DA_{1,0}(S)$. Since $\DA_{1,0}(S) = (DG_S)$, the motive $M$ is obtained by a finite number of steps from objects of $G_S$ by taking cones of morphisms, shifts and direct factors. Since $DG_S$ lies in the heart, this implies immediately that $M$ is bounded. \qed

The proof of the following result is also very similar to the proof of Theorem 4.21, hence we include it here.

**Proposition 4.29.** Let $S$ be a noetherian finite-dimensional quasi-excellent scheme. The $t$-structures $t_{MM,0}(S)$, $t_{MM,1}(S)$ and $t_{MM}(S)$ are non-degenerate.

**Proof.** Since $t_{MM} = t_{MM,1}(-1)$, it is enough to treat the cases of $t_{MM,0}$ and $t_{MM,1}$. These $t$-structures are defined as generated $t$-structures. By [8, Proposition 2.1.73], to show that a $t$-structure of the form $(\mathbb{G})$ on a triangulated category $\mathbb{T}$ for a family of compact objects $G$ is non-degenerate, it is enough to check that $\mathbb{T} = (\mathbb{G})$ and that for $A \in G$, there exists an integer $d_A \geq 0$ such that for all $B \in G$, $\Hom(A, B[n]) = 0$ for $n \geq d_A$.

Let us check these conditions for $t_{MM,0}$, using the generating family $G_0 = \{e_2\mathbb{Q}|e : U \to S \text{ étale}\}$. By definition, we have $\DA_0(S) = (G_0)$. Let $e : U \to S$ and $h : V \to S$ be étale morphisms. We will prove that

$$\forall n > \dim(S), \ DA(S)(e_2\mathbb{Q}, h_\mathbb{Q}[n]) = 0.$$  

By the $(e_2, e^*)$ adjunction, we can assume $e = \id$. Using Zariski’s main theorem, we compactify $h$ into $h = h \circ j$ with $j : V \to V$ a dense open immersion and $h : V \to S$ a finite morphism. Using the $(h^*, h_*)$ adjunction, we see that we can assume $h = j$ a dense open immersion. Notice that through these reductions, the dimension of the base does not increase. Write $i : Z \to S$ for the complementary closed immersion to $j$. We have $\dim(Z) \leq \dim(S) - 1$. By localisation and adjunction, we have an exact sequence

$$DA(Z)(\mathbb{Q}, \mathbb{Q}[n - 1]) \to DA(S)(\mathbb{Q}, ji\mathbb{Q}[n]) \to DA(S)(\mathbb{Q}, \mathbb{Q}[n])$$

The two outer group vanish because of Proposition B.3 (noticing that $n - 1 > \dim(S) - 1 \geq \dim(Z)$), and this completes the proof that $t_{MM,0}$ is non-degenerate.
We now look at the case of $t_{\text{MM},1}$. Again by [8, Proposition 2.1.73] applied to the generating family $\mathcal{J}G_s$, it suffices to prove that

$$\forall M, N \in \mathcal{J}G_s, \forall n > \dim(S) + 4, \text{DA}(S)(M, N[n]) = 0.$$ 

Let us first concentrate on $M$. We have an étale morphism $e : U \to S$ and $M$ has the form $e_2^2 \Sigma^\infty(K \otimes Q)$ with $K$ one of $\mathbb{Z}$, $\mathbb{G}_m[-1]$ or $\text{Jac}(C/U)$ with $C$ a smooth projective curve with geometrically connected fibres and a section. We have $\dim(U) \leq \dim(S)$, hence by adjunction and Lemma 4.5, we can assume $e = \text{id}$. Then, using Proposition 2.3 and Corollary 3.20, in every case, we can write $M$ as a direct factor of the motive $M_S(C')[\epsilon]$ with $C'/S$ a smooth curve and $\epsilon \in \{0, -1\}$. By adjunction again, and taking into account that $\dim(C') \leq \dim(S) + 1$, we are reduced to showing

$$\forall N \in \mathcal{J}G_s, \forall n > \dim(S) + 3, \text{DA}(S)(M, N[n]) = 0.$$ 

We now go into the case distinction for $N$. Let $e : U \to S$ be an étale morphism. The motive $N$ is of one of the following forms: $e_2^2 Q_1$ or $e_2 \text{Jac}(C/U)[-1]$ for $\pi : C \to U$ a smooth projective curve with geometrically connected fibres and a section. By Zariski's main theorem, localisation and adjunction, we can assume $e = j$ is an open immersion (this does not change the dimension). In the first two cases, we apply the same argument as for $t_{\text{MM},0}$: by localisation, we can assume $e = \text{id}$ and then apply Proposition B.3. Let us focus on the Jacobian case. We write $\text{Jac}(C/U)[-1]$ as direct factor of the motive $M_S(C)[-1]$ by Corollary 3.20, then compactify $j \circ \pi = \bar{\pi} \circ j$ with $\bar{\pi} : C \to \overline{C}$ dense open immersion and $\bar{\pi} : \overline{C} \to S$ a proper morphism using Zariski's main theorem. Writing $\bar{\pi} : Z \to \overline{C}$ for the complementary closed immersion to $j$ and using localisation and relative purity, we have an exact sequence

$$\text{DA}(Z)(QZ, Q1)[n] \to \text{DA}(S)(Q_S, j_! M_S(C)[-1]) \to \text{DA}(\overline{C})(Q, Q1)[n + 1]).$$

We have $\dim(Z), \dim(\overline{C}) \leq \dim(S) + 1$, hence the two outer groups vanish for $n > \dim(S) + 3$ by Proposition B.3. This completes the proof that $t_{\text{MM},1}$ is non-degenerate. \hfill $\square$

Finally, we compute more precisely the morphisms between Deligne 1-motives over a regular base.

**Theorem 4.30.** Let $S$ be a regular scheme, $M_1, M_2 \in \mathcal{M}_1(S)$ and $n \in \mathbb{Z}$. Then

$$\text{DA}(S)(\Sigma^\infty M_1, \Sigma^\infty M_2[n]) \simeq \begin{cases} M_1(S)(M_1, M_2), & n = 0 \\ 0, & n < 0 \\ 0, & n > 3. \end{cases}$$

In particular, the functor $\Sigma^\infty : \mathcal{M}_1(S) \to \mathcal{MM}_1(S)$ is fully faithful.

**Proof.** By considering the connected components, we reduce to the case where $S$ is irreducible. The idea of the proof is that in the range we are considering, i.e., for $n \neq 1, 2$, everything happens at the generic point $\eta$ of $S$. Let $j : U \to S$ be an open immersion with $U \neq \emptyset$. The restriction functor $j^* : \mathcal{M}_1(S) \to \mathcal{M}_1(U)$ is fully faithful by Proposition A.11. Moreover the category $\mathcal{M}_1(\eta)$ is the 2-colimit of the $\mathcal{M}_1(U)$ for $U$ running through all non-empty open sets of $S$ by Proposition A.10. This implies that $\mathcal{M}_1(S)(M_1, M_2) \simeq \mathcal{M}_1(\eta)(\eta^* M_1, \eta^* M_2)$. On the $\text{DA}(\_)$ side, by continuity and Proposition 2.2, we have that $\text{DA}(\eta)(\eta^* \Sigma^\infty M_1, \eta^* \Sigma^\infty M_2[n]) \simeq \text{Colim}_U \text{DA}(U)(j^* \Sigma^\infty M_1, j^* \Sigma^\infty M_2[n])$. Furthermore, by Proposition 4.19, we have an isomorphism

$$\text{DA}(\eta)(\Sigma^\infty \eta^* M_1, \Sigma^\infty \eta^* M_2[n]) \simeq \text{Ext}^n_{\mathcal{M}_1(\eta)}(M_1, M_2 \neq 0, 1)$$

for $n \neq 0, 1$.

Putting everything together, we see that the statement of the proposition follows from the claim that $j^* : \text{DA}(S)(\Sigma^\infty M_1, \Sigma^\infty M_2[n]) \to \text{DA}(U)(j^* \Sigma^\infty M_1, j^* \Sigma^\infty M_2[n])$ is bijective for $n \neq 1, 2$. Write $i : Z \to S$ for the reduced complementary closed immersion of $U$ in $S$. Consider the localisation exact sequence

$$\cdots \to \text{DA}(Z)(i^* \Sigma^\infty M_1, i^* \Sigma^\infty M_2[n]) \to \text{DA}(S)(\Sigma^\infty M_1, \Sigma^\infty M_2[n]) \xrightarrow{j^*} \text{DA}(U)(j^* \Sigma^\infty M_1, j^* \Sigma^\infty M_2[n])$$

for $n \neq 0, 1$.
We have to prove the vanishing of $\text{DA}(Z)(i^*\Sigma^\infty M_1, i^*\Sigma^\infty M_2[n+1])$ for $n \neq 2$. By Proposition 2.3, we have $i^*\Sigma^\infty M_1 \simeq \Sigma^\infty M_{1,Z}$. Stratifying $Z$ by regular constructible subschemes and applying further localisations, we can reduce to the case where $Z$ is also regular of some codimension $1 + e$ with $e \geq 0$. By absolute purity in the form of Proposition 1.7, which applies by Corollary 2.14, we then have $i^*\Sigma^\infty M_2[n+1] \simeq i^*\Sigma^\infty M_2(-1)\Sigma^\infty (n-1-2e) \simeq \Sigma^\infty M_{2,Z}(-1)\Sigma^\infty (n-1-2e)$. We know, again from Corollary 2.14, that the motive $\Sigma^\infty M_{1,Z}(-1)$ lies in $\text{DA}^1(S)$, hence we have an isomorphism

$$\text{DA}(Z)(\Sigma^\infty M_{1,Z}, \Sigma^\infty M_{2,Z}(-1)\Sigma^\infty (n-1-2e)) \simeq \text{DA}(Z)(\Sigma^\infty M_{1,Z}(-1), \omega^1(\Sigma^\infty M_{2,Z}(-1)(n-1-2e)))$$

The motive $\Sigma^\infty M_{2,Z}(-1)$ is cohomological, so by Corollary 3.9 the group on the right-hand side vanishes unless $e = 0$. If $e = 0$, we have further $\omega^1(\Sigma^\infty M_{2,Z}(-1)) \simeq \omega^0(\Sigma^\infty M_{2,Z}(-1))$. This motive was computed in Proposition 3.10 (iii) and we get

$$\omega^0(\Sigma^\infty M_{2,Z}(-1)) \simeq \Sigma^\infty X_e(W_{-2b_{2,Z}})(-1).$$

To sum up, we are reduced to showing that for $S$ regular, $M \in M_4(S)$ and $L$ lattice over $S$, the morphism group $\text{DA}(S)(\Sigma^\infty M, \Sigma^\infty L_Q[n-1])$ is 0 for $n \neq 2$. Since $S$ is normal, the motive $\Sigma^\infty L_Q$ is a direct factor of $e \otimes Q$ for $e : T \to S$ finite étale (Lemma A.2). By adjunction, we are then reduced to the case $L = Z$. Write $M = [N \to G]$ with $N$ a lattice and $G$ a semi-abelian scheme. We have a distinguished triangle

$$\Sigma^\infty [0 \to G] \to \Sigma^\infty M \to \Sigma^\infty [N \to 0] \to$$

which shows that we can treat separately the cases $M = [N \to 0]$ and $M = [0 \to G]$.

In the case $M = [N \to 0]$, we again write $N$ as a direct factor of a permutation lattice, which implies that $\Sigma^\infty M$ is a direct factor of $e'_Q$ with $e' : T' \to S$ finite étale. By adjunction, we are then reduced to a computation of weight zero motivic cohomology on a regular scheme, which vanishes exactly for $n \neq 2$ by Propositions B.2 and B.5.

In the second case, we have $\Sigma^\infty M = \Sigma^\infty G_Q[-1]$, which by [6, Theorem 3.3] is a direct factor of $M_S(G)$. We are then done using the $((G \to S)_2, (G \to S)^*)$ adjunction and Propositions B.2 and B.5. 

\section*{Appendix A. Deligne 1-motives}

We gather necessary results on Deligne 1-motives [31, §10] over general base schemes which we could not find in the literature. Useful references besides Deligne’s original work are [44], [16, Appendix C].

\subsection*{A.1. Definitions.}

\textbf{Definition A.1.} Let $S$ be a scheme. We say that a group scheme $G/S$ is

(i) \textit{discrete} if it is étale locally constant finitely generated.

(ii) \textit{a lattice} if it is discrete and torsion free.

Before we come to the definition of Deligne 1-motives, let us discuss a recurrent technical point about lattices and tori over general schemes. In general, it is not the case that a discrete group scheme is isotriivial in the étale topology. However, we have the following useful lemma.

\textbf{Lemma A.2.} Let $S$ be a locally noetherian, geometrically unibranch scheme. Let $L$ be a lattice over $S$ (resp. $T$ be a torus over $S$).

(i) $L$ (resp. $T$) is isotriivial, i.e., it becomes split after passing to a finite étale cover of $S$.

(ii) The sheaf $L \otimes Q \in \text{Sh}(\text{Sm}/S)$ (resp. $T \otimes Q \in \text{Sh}(\text{Sm}/S)$) is a direct factor of the sheaf $f_*Q$ (resp. $f_*(G_m \otimes Q)$) for $f : V \to S$ a finite étale cover.

\textbf{Proof.} Point (i) for lattices follows from the discussion in [1, Exp. X 6.2]. For tori, it is precisely [1, Exp. X Théorème 5.16].

We now prove Point (ii). Let $L$ be a lattice over $S$. By (i), we can find a finite étale cover $g : V \to S$ such that $g^*L$ is split, say $g^*L \simeq V'$. Because $g$ is finite étale, $L$ becomes a direct factor of $g_*g^*L$ after inverting $\text{deg}(f)$ by a transfer argument. We thus have that $L \otimes Q$ is a direct factor of $g_*g^*L \otimes Q \simeq g_*Q'$. Write $f : V' \to V$ for the coproduct of $r$ copies of $g$. Then $g_*Q' \simeq f_*Q$. This concludes the proof of (ii) for lattices. The case of tori follows by the same argument.
Definition A.3. Let $S$ be a scheme. A 2-term complex of commutative $S$-group schemes

$$M = [L \to G]$$

is called a Deligne 1-motive over $S$ if $L$ is a lattice and $G$ is a semi-abelian scheme. A morphism of Deligne 1-motives is a morphism of complexes of group schemes, or equivalently a morphism of complex of the associated representable sheaves on $(\text{Sm}/S)_{\text{et}}$. We denote by $\mathcal{M}_1(S, \mathbb{Z})$ the category of Deligne 1-motives. It is a pseudo-abelian additive category, with biproducts induced by fibre products of $S$-group schemes.

A Deligne 1-motive $M = [L \to G]$ comes with a 3-term functorial weight filtration, defined as

$$W_{-2}M = [0 \to T]$$

$$W_{-1}M = [0 \to G]$$

$$W_0M = M.$$

Notation A.4. Let $f : [L \to G] \to [L' \to G']$ be a morphism of Deligne 1-motives. We use the notation $f_L$, $f_G$, $f_A$, $f_T$ for the induced maps $Gr_{k}^{W}f : L \to L'$, $W_{0}f : G \to G'$, $Gr_{k}^{W}f : A \to A'$, $Gr_{k}^{W}f : T \to T'$.

Definition A.5. Let $f : S' \to S$ be any morphism of schemes. Then pullback of $S$-group schemes along $f$ induces an additive functor

$$f^* : \mathcal{M}_1(S, \mathbb{Z}) \to \mathcal{M}_1(S', \mathbb{Z}).$$

We are not so much interested in 1-motives per se as in the objects they define in the derived category of sheaves with rational coefficients.

Lemma A.6. Any morphism in $\mathcal{M}_1(S, \mathbb{Z})$ which induces a quasi-isomorphism of complexes of abelian sheaves on $(\text{Sm}/S)_{\text{et}}$ is an isomorphism.

Proof. Let $f = (f_L, f_G) : [L_1 \to G_1] \to [L_2 \to G_2]$ be a quasi-isomorphism of complexes of sheaves. By a diagram chase, this is equivalent to $\text{Ker}(f_L) \simeq \text{Ker}(f_G)$ and $\text{Coker}(f_L) \simeq \text{Coker}(f_G)$. Since $\text{Ker}(f_L)$ is locally constant finitely generated free and $\text{Ker}(f_G)$ is a group scheme whose identity component is semi-abelian and with finite $\pi_0$, they must be both trivial. Similarly, $\text{Coker}(f_L)$ is discrete and $\text{Coker}(f_G)$ has connected fibres, so they must be both trivial. Hence $f$ is an isomorphism. \qed

We can consequently think of $\mathcal{M}_1(S, \mathbb{Z})$ as a full subcategory of $D(\text{Cpl}(\mathbb{Sh}((\text{Sm}/S)_{\text{et}}, \mathbb{Z})))$.

Definition A.7. Let $S$ be a noetherian scheme. We write $\mathcal{M}_1(S)$ for the idempotent completion of the $\mathbb{Q}$-linear category $\mathcal{M}_1(S, \mathbb{Z}) \otimes \mathbb{Q}$. We say that a morphism in $\mathcal{M}_1(S)$ is integral if it comes from $\mathcal{M}_1(S, \mathbb{Z})$. For $f : S' \to S$ morphism of schemes, we still write $f^*$ for the induced additive functor $\mathcal{M}(S) \to \mathcal{M}(S')$.

By the above, we can and do view $\mathcal{M}_1(S)$ as a full subcategory of $D(\text{Cpl}(\mathbb{Sh}((\text{Sm}/S)_{\text{et}}, \mathbb{Q})))$. In practice, the idempotent completion in the definition does not affect anything that we do in this paper, and we will allow ourselves statements of the form “Let $M = [L \to G] \otimes \mathbb{Q}$ be an object in $\mathcal{M}_1(S)$” without spelling out the immediate reduction to that case.

A.2. Continuity and smoothness. We think of Deligne 1-motives as “1-motivic local systems” over the base $S$. The results in this section, which have classical analogues for local systems and lisse étale sheaves, justify in part this intuition.

We start with a lemma about discrete group schemes.

Lemma A.8. Let $S$ be a locally noetherian japanese scheme, $\eta$ its scheme of generic points. Then the category of discrete group schemes on $\eta$ is the 2-colimit of the categories of discrete group schemes on dense open subschemes of $S$. The same statement holds for the category of lattices.

Proof. The statement is equivalent to the following results.

(i) For $L/\eta$ discrete group scheme, there exists $U \subset S$ dense open and $L'/U$ discrete such that $L \simeq \eta^*L'$. Moreover, if $L$ is a lattice, one can choose $L'$ to be a lattice as well.

(ii) For $U \subset S$ dense open, $L, L'/U$ discrete, we have

$$\text{Hom}(\eta^*L, \eta^*L') \simeq \text{Colim}_{V \subset U} \text{Hom}((V \to U)^*L, (V \to U)^*L').$$
We first make some reductions which apply both to (i) and (ii). By the topologically invariance of the étale site, we can assume \( S \) to be reduced. Since \( S \) is locally noetherian japanese and reduced, the normal locus of \( S \) is open and non-empty [37, Proposition 6.13.2]. So any small enough open set \( U \) in \( S \) is normal, and in particular geometrically unibranch. By the discussion in [1, Exp. X 6.2], discrete group schemes on geometrically unibranch schemes are split by finite étale covers. Moreover, for any small enough open set \( U \), the set of connected components (open by local noetherianness) of \( U \) and of \( \eta \) coincide. We can thus reduce to the case where \( \eta \) is connected (i.e., \( S \) irreducible).

We prove (i). Since \( \eta \) itself is normal, there is a finite étale Galois cover \( \tilde{\eta}/\eta \) such that \( L_{\tilde{\eta}} \) is constant. In other words, \( L \) corresponds to a representation \( \rho \) of \( \text{Gal}(\tilde{\eta}/\eta) \) on a finitely generated abelian group \( F \). By [38, Théorème 8.8.2, Théorème 8.10.5] and [39, Théorème 17.7.8] there exists a \( U \subset S \) dense open and \( \tilde{U}/U \) finite étale such that \( \tilde{U} \times_U \eta \simeq \tilde{\eta} \). Up to shrinking \( U \), one can assume \( U \) to be normal. By [38, Théorème 8.8.2] applied to the finite group \( \text{Gal}(\tilde{\eta}/\eta) \), up to shrinking \( U \) one can assume that \( \text{Aut}(\tilde{U}/U) \simeq \text{Gal}(\tilde{\eta}/\eta) \) (in particular \( \tilde{U}/U \) is Galois). Then the representation of \( \text{Gal}(\tilde{U}/U) \) on \( F \) corresponding to \( \rho \) via this isomorphism defines a discrete group scheme \( L'/U \) such that \( L \simeq \eta^*L' \) as required. The addendum about lattices follows from the construction, i.e., \( L' \) is a lattice if \( L \) is.

We now prove (ii). Let \( U \subset S \) be a dense open subset, \( L, L'/U \) discrete group schemes. We can shrink \( U \) and assume it is normal. Let \( \tilde{V}/V \) be a finite étale Galois covering trivializing \( L \) and \( L' \).

We thus get two finitely generated abelian groups \( F, F' \) with representations \( \rho, \rho' \) of \( \text{Gal}(\tilde{V}/V) \). Let \( \tilde{\eta} := \tilde{V} \times_U \eta \). Then \( \tilde{\eta}/\eta \) is Galois with \( G := \text{Gal}(\tilde{V}/V) \simeq \text{Gal}(\tilde{\eta}/\eta) \). Then the system in the right-hand side of (ii) is constant and both sides of (ii) are in bijection with \( \text{Hom}_G(\rho, \rho') \). This concludes the proof.

Remark A.9. It is not clear to the author how to extend this result to a more general continuity result for discrete group schemes on a projective limit of schemes with affine transition morphisms.

We deduce from this a continuity result for Deligne 1-motives.

**Proposition A.10.** Let \( S \) be a locally noetherian japanese scheme, \( \eta \) its scheme of generic points. Then the category \( \mathcal{M}_1(\eta, \mathbb{Z}) \) (resp. \( \mathcal{M}_1(\eta) \)) is the 2-colimit of the categories \( \mathcal{M}_1(U, \mathbb{Z}) \) (resp. \( \mathcal{M}_1(U) \)) for all dense open subsets \( U \subset S \).

**Proof.** The case of \( \mathcal{M}_1(\cdot) \) follows directly from the one of \( M_1(\cdot, \mathbb{Z}) \). We have to show that

(i) for all \( M \in \mathcal{M}_1(\eta, \mathbb{Z}) \), there exists \( U \subset S \) dense open and \( M' \in \mathcal{M}_1(U, \mathbb{Z}) \) such that \( M \simeq \eta^*M' \), and that

(ii) for all \( U \subset S \) dense open and all \( M, N \in \mathcal{M}_1(U, \mathbb{Z}) \),

\[
\mathcal{M}_1(\eta, \mathbb{Z})((\eta^*M, \eta^*N)) \simeq \text{Colim}_{V \subset U} \mathcal{M}_1(V, \mathbb{Z})((V \to U)^*M, (V \to U)^*N).
\]

We prove (i). Let \( M = [L \to G] \in \mathcal{M}_1(\eta, \mathbb{Z}) \) with the extension \( 0 \to T \to G \to A \to 0 \).

By [38, Théorème 8.8.2.(ii), Scholie 8.8.3, Théorème 8.10.5.(xii)] and [39, Proposition 17.7.8], we can find an \( U \subset S \) and a smooth group scheme \( G'/U \) such that \( G \simeq G' \times_U \eta \). Recall that an abelian scheme is by definition a smooth proper group scheme with connected fibres, hence by [38, Théorème 8.8.2.(ii), Scholie 8.8.3, Théorème 8.10.5.(xii)] and [39, Proposition 17.7.8], we can shrink \( U \) and find an abelian scheme \( A'/U \) such that \( A \simeq A' \times_U \eta \). By Lemma A.8 and the duality between lattices and tori, we can shrink \( U \) and assume that there exists a lattice \( L' \) and a torus \( T' \) over \( U \) such that \( L \simeq L' \times_U \eta \) and \( T \simeq T' \times_U \eta \).

We have spread out the pure pieces of \( M \), it remains to glue them together. By [38, Théorème 8.8.2.(i)], up to shrinking \( U \), we have morphisms \( A' \to G' \to T' \) which restrict to the extension defining \( G \). By a standard argument based on [38, Théorème 8.10.5], up to shrinking \( U \), this is in fact an exact sequence of group schemes. Finally, we have to spread out the morphism \( L \to G \). This can be done by the same Galois descent argument as in the end of the proof of Lemma A.8.

Let us now prove (ii). In \( \mathcal{M}_1(\cdot, \mathbb{Z}) \), the components of a morphism are morphisms of (group) schemes. It is enough to spread them out one by one because the resulting diagram will commute by schematic density of \( \eta \) in \( S \). We have treated morphisms of lattices in Lemma A.8. The case of morphisms of semi-abelian schemes (which are in particular of finite presentation) is a direct application of [38, Théorème 8.8.2.(i)].
When the base scheme is noetherian excellent and reduced (resp. normal), we can say more.

**Proposition A.11.** Let $S$ be a noetherian excellent scheme, $i : \eta \to S$ its scheme of generic points.

(i) Suppose $S$ reduced. Then the pullback functor $\eta^* : M_1(S, \mathbb{Z}) \to M_1(\eta, \mathbb{Z})$ (resp. $\eta^* : M_1(S) \to M_1(\eta)$) is faithful.

(ii) Suppose moreover that $S$ is normal. Then $\eta^*$ is fully faithful.

**Proof.** Let us prove (i). By Proposition A.10 this is equivalent to the faithfulness of the functor $j^*$ for all $j : U \to V$ dense open immersions. It is enough to show faithfulness of $j^*$ separately for morphisms of discrete group schemes and semi-abelian schemes, and in both cases it follows from the "reduced to separated" uniqueness criterion [35, Lemme 7.2.2.1].

We now prove (ii). By Proposition A.10, it is enough to prove fullness for the functor $j^*$ for all dense open immersions $j : U \to V$. Let $M = [L \to G]$, $M' = [L' \to G'] \in M_1(V, \mathbb{Z})$ and $f_U = (f_U^L, f_U^G) : j^*M \to j^*M'$. First, using the normality of $V$ and [4, Exposé I Corollaire 10.3], the morphism $f_U^L$ extends uniquely to a morphism $f^L : L \to L'$. Second, using the normality of $V$ and [34, Proposition 2.7], the morphism $f_U^G$ extends uniquely to a morphism $f^G : G \to G'$. The uniqueness ensures that $(f^L, f^G)$ is a morphism $M \to M'$ which extends $f_U$. \hfill $\Box$

A.3. Pushforward and Weil restriction. Let $g : S' \to S$ be a finite étale morphism. We want to define a pushforward functor $g_* : M_1(S') \to M_1(S)$ using Weil restriction of scalars. Recall the following definition.

**Definition A.12.** Let $g : S' \to S$ be a morphism of schemes and $X/S'$ be a $S'$-scheme. The Weil restriction $R_g X$ is the presheaf of sets on $\text{Sch}/S$ defined for any $S$-scheme $U$ by

$$R_g X(U) = X(U \times_S S') .$$

If $X/S'$ is a commutative group scheme (or more generally an fppf sheaf of abelian groups on $\text{Sch}/S$), then $R_g X$ is naturally an fppf sheaf of abelian groups on $\text{Sch}/S$. Moreover, the construction of $R_g X$ is functorial and compatible with base change. We summarise results on the representability of $R_g X$ from the literature.

**Proposition A.13.** Let $g : S' \to S$ be a morphism of schemes and $X/S'$ be a $S'$-scheme.

(i) [52, Theorem 1.5] Assume that $g$ is proper flat of finite presentation. Then $R_g X$ is representable by an algebraic space (note that we will only need the case $g$ finite flat, which is presumably easier, but we could not find a reference).

(ii) [21, 7.6/5] Assume that $g$ is finite flat. If $X$ is smooth (resp. of finite presentation) then $R_g X$ (which exists at least as an algebraic space by (i)) is smooth (resp. of finite presentation).

(iii) [21, 7.6/5] Assume that $g$ is finite étale. If $X$ is proper then $R_g X$ (which exists at least as an algebraic space by (i)) is proper.

(iv) [21, 7.6/2] Let $h : X \to Y$ be a closed immersion of $S'$-schemes. Then $R_g h : R_g X \to R_g Y$ is a closed immersion of presheaves. As a corollary, if $X/S$ if affine, then $R_g X$ is representable by an affine scheme.

We now use the results above to analyse Weil restriction of pure 1-motives. We are spared from having to consider algebraic spaces by the following result.

**Proposition A.14.** Let $g : S' \to S$ be finite flat.

(1) Let $T/S'$ be a torus (resp. $L/S'$ be a lattice). Then $R_g T$ is a torus (resp. $R_g L$ is a lattice).

(2) Let $A/S'$ be an abelian scheme. Assume that $g$ is étale. Then $R_g A$ is an abelian scheme.

**Proof.** By Proposition A.13 (iv), we know that $R_g T$ and $R_g L$ are representable by affine $S'$-group schemes. Moreover, because of the compatibility with base change and étale descent, it is enough to consider the case of a split torus and a constant lattice over $S'$, in which case the Weil restrictions are directly seen to be a split torus or a constant lattice over $S$.

By Proposition A.13 (i)-(iii), we know that $R_g A$ is representable by a proper smooth algebraic group space over $S$. By [34, Theorem 1.9], this implies that $R_g A$ is an abelian scheme. \hfill $\Box$

Now we tackle the case of semi-abelian schemes.
Lemma A.15. Let \( g : S' \to S \) be a morphism of schemes.

(i) When restricted to fppf sheaves of abelian groups, the functor \( \text{Res}_g \) is left exact.

(ii) Assume that \( g \) is finite flat. Let \( f : G \to H \) be a smooth and surjective morphism between commutative groups schemes of finite presentation. Then the morphism of algebraic group spaces \( \text{Res}_g f : \text{Res}_g G \to \text{Res}_g H \) is smooth and surjective.

(iii) Assume \( g \) is finite flat. Let \( 0 \to G' \xrightarrow{\cdot} G \xrightarrow{p} G'' \to 0 \) be an exact sequence of smooth commutative \( S \)-group schemes with \( G \to G'' \) flat (and hence smooth). The sequence

\[
0 \to \text{Res}_g G' \to \text{Res}_g G \to \text{Res}_g G'' \to 0
\]

is exact.

Proof. Point (i) is clear from the definition. We turn to point (ii). The fact that \( \text{Res}_g f \) is smooth follows from the infinitesimal criterion of smoothness (and does not require that we are working with group schemes). The surjectivity can be tested pointwise on \( S \), so that by compability of \( \text{Res}_g \) with base change we can assume that \( S \) is the spectrum of a field \( k \). Surjectivity is a geometric property, so that we can assume \( k \) to be algebraically closed as well. We then have to check the surjectivity of the induced map \( \text{Res}_g G(k) = G(S') \to \text{Res}_g H(k) = H(S') \) on \( k \)-points. Since \( S'/k \) is finite flat, it is a product of finite local algebras. Surjectivity then follows from the surjectivity of \( f \), the fact that \( k \) is algebraically closed, and the formal smoothness of \( f \). Note that if \( g \) is finite étale, we do not need \( f \) smooth.

For (iii), it is enough to check that \( \text{Res}_g G' \) is the scheme-theoretic kernel of \( \text{Res}_g p \) and that \( \text{Res}_g p \) is an fppf morphism. The first assertion follows from (i), and the second from (ii). \( \square \)

Proposition A.16. Let \( g : S' \to S \) be finite étale and \( G/S \) be an semi-abelian scheme. Then \( \text{Res}_g G \) is an semi-abelian scheme.

Proof. The result follows directly from Proposition A.14 and Lemma A.15 (iii). \( \square \)

Definition A.17. Let \( g : S' \to S \) be a finite étale morphism. We define the Weil restriction of a Deligne 1-motive \( M = [L \xrightarrow{u} G] \in \mathcal{M}_1^p(S') \) as \( \text{Res}_g M = [\text{Res}_g L \xrightarrow{\text{Res}_g u} \text{Res}_g G] \) which is in \( \mathcal{M}_1(S) \) by Propositions A.14 and A.16. This induces a functor

\[
g_* : \mathcal{M}_1^p(S') \to \mathcal{M}_1^p(S).
\]

Appendix B. Motivic cohomology in degrees \((*, \leq 1)\)

We gather here some computations of rational motivic cohomology groups which are used at various places in this paper. Most of the following is present, explicitly or implicitly, in [11, §11] and in the K-theoretic interpretation of rational motivic cohomology provided by the comparison with Beilinson motives [25, §14].

Notation B.1. Let \( S \) be a noetherian finite dimensional scheme. For \( p, q \in \mathbb{Z} \), we write \( H_{\mathcal{M}}^{p,q}(S) := \text{DA}(S)(\mathbb{Q}_S, \mathbb{Q}_S(q)[p]) \).

Proposition B.2. [11, Proposition 11.1 (b)] Let \( S \) be a noetherian finite-dimensional scheme. For all \( w < 0 \) and \( n \in \mathbb{Z} \), we have \( H_{\mathcal{M}}^{w,n}(S) \simeq 0 \).

Proposition B.3. Let \( S \) be a noetherian finite dimensional quasi-excellent scheme (respectively, a regular and finite dimensional scheme). For all \( i \in \mathbb{N} \) and \( n > \dim(S) + 2i \) (resp. \( n > 2i \)), we have \( H_{\mathcal{M}}^{n,i}(S) \simeq 0 \).

Proof. The group \( H_{\mathcal{M}}^{n+2i,i}(S) \simeq \text{DA}(S)(\mathbb{Q}[n], \mathbb{Q}(i)[2i]) \) is a direct factor of \( \text{DA}(S)(\mathbb{Q}[n], \sum_{i \in \mathbb{Z}} \mathbb{Q}(i)[2i]) \). By Theorem [25, 16.2.18], this group is isomorphic to \( \text{DM}_{\mathbb{L}}(S)(\mathbb{Q}[n], \sum_{i \in \mathbb{Z}} \mathbb{Q}(i)[2i]) \) where \( \text{DM}_{\mathbb{L}}(S) \) is the triangulated category of Beilinson motives. By Corollary [25, 14.2.17], we have \( \sum_{i \in \mathbb{Z}} \mathbb{Q}(i)[2i] \simeq \text{KGL}_{\mathbb{Q},S} \), where the last object is the \( \mathbb{Q} \)-localisation of the motivic spectrum \( \text{KGL}_S \). This implies that

\[
\text{DM}_{\mathbb{L}}(S)(\mathbb{Q}[n], \text{KGL}_{\mathbb{Q},S}) \simeq \text{SH}(S)(\sum^n \Sigma^n \Sigma^n(S_+), \text{KGL}) \otimes \mathbb{Q}.
\]

By [26, Théorème 2.20], this last group is isomorphic to \( \text{KH}_n(S) \otimes \mathbb{Q} \), where \( \text{KH} \) is homotopy-invariant \( K \)-theory. The negative homotopy-invariant \( K \)-theory of a regular scheme vanishes, and this implies the resp. case. Finally, by the main step in the proof of [46, Theorem 3.5], under our
hypotheses on $S$ (including quasi-excellent), the group $\KH_n(S) \otimes \Q$ vanishes for $n < -\dim(S)$. This completes the proof. 

Remark B.4. For the cases $i = 0, 1$, it is likely that there is a non-$K$-theoretic proof, combining results below on $H^{n,0}_\M, H^{n,1}_\M$ of regular schemes with an ingenious use of resolution of singularities by alterations as in the proof of [46, Theorem 3.5].

Let $S$ be a scheme. Then we have $D(\Sm/(S, \Q_S)) \simeq \Q^n(S)$ (with $\pi_0(S)$ the set of connected components of $S$). This provides a morphism

$$\nu^{0,0} : \Q^n(S) \xrightarrow{\sim} D(\Sm/(S, \Q_S)) \rightarrow DA(S)/(S, \Q_S) = H^{n,0}_\M(S).$$

More generally, we have for all $n \in \Z$ a morphism

$$\nu^{n,0} : D(S)/(\Q_S, \Q_S[n]) \rightarrow H^{n,0}_\M(S)$$

coming from the construction of $DA^{(eff)}$.

Let $f : T \rightarrow S$ be any morphism of schemes. Then it is easy to see that the diagram

$$\begin{array}{ccc}
D(\Sm/(S, \Q_S, \Q_S[n])) & \xrightarrow{f^*} & H^{n,0}_\M(S) \\
\downarrow_{\nu^{0,0}} & & \downarrow_{f^*} \\
D(\Sm/(T, \Q_T, \Q_T[n])) & \xrightarrow{f^*} & H^{n,0}_\M(T)
\end{array}$$

is commutative. We will use this fact without comment in the proof below.

Proposition B.5.

(i) For all $n < 0$, we have $H^{n,0}_\M(S) \simeq 0$.

(ii) The morphism $\nu^{0,0}$ induces an isomorphism $H^{0,0}_\M(S) \simeq \Q^n(S)$.

(iii) Assume $S$ regular. For all $n > 0$, we have $H^{n,0}_\M(S) \simeq 0$.

(iv) Let $f : T \rightarrow S$ be a smooth surjective morphism with geometrically connected fibres. Then for all $n \in \Z$, we have $f^* : H^{n,0}_\M(S) \rightarrow H^{n,0}_\M(T)$.

Proof. Statement (i) and (ii) are proved in [11, Proposition 11.1 (a)].

Let us prove Statement (iii). Fix $n > 0$. We can assume that $S$ is connected with generic point $\eta$. By the argument at the beginning of the proof of [11, Corollaire 11.4], combining absolute purity and localisation with the vanishing of negative motivic cohomology B.2, one can deduce that for any dense open set $U$ in $S$, the restriction map $H^{n,0}_\M(S) \rightarrow H^{n,0}_\M(U)$ is injective. By the continuity property of [11, Proposition 3.20], we deduce that the restriction map $H^{n,0}_\M(S) \rightarrow H^{n,0}_\M(\eta)$ is injective. So we are reduced to the case where $S$ is the spectrum of a field $k$.

By separation, we can assume that $k$ is perfect. By [25, Corollary 16.2.22], we reduce to compute $DM(k, \Q)/(Q_k, Q_k[n])$. By the cancellation theorem [61], we reduce to compute $DM^{(eff)}(k, \Q)/(Q_k, Q_k[n])$. Since the sheaf with transfers $Q_k$ is both cofibrant and $A^1$-local, this coincides with the same Hom group computed in the derived category of étale sheaves with transfers over $\Sm/S$, which vanishes. This concludes the proof of (iii).

Let us prove Statement (iv). By Mayer-Vietoris, we can assume $S$ to be affine. By a limit argument using the continuity property of $DA$, we can then assume that $S$ is of finite type over a Dedekind ring. Using [30, Corollary 5.15] applied to the irreducible components of the normalisation of $S$ and then iterating, we build a proper hypercovering $\pi_n : \tilde{S}_n \rightarrow S$ with all $\tilde{S}_n$ regular. We pullback $\pi_\ast$ to obtain a proper hypercovering $\pi_\ast : \tilde{T}_\ast \rightarrow T$. Since $f$ is smooth, all $\tilde{T}_n$ are regular as well. By cohomological descent for the h-topology [25, Theorem 14.3.4], we have $Q_S \simeq \pi_\ast Q_{\tilde{S}_n}$ and $Q_T \simeq \pi_\ast Q_{\tilde{T}_n}$. We deduce that $H^{n,0}_\M(S) \simeq DA(\tilde{S}_n)/(Q_{\tilde{S}_n}, Q_{\tilde{S}_n}[n])$ and $H^{n,0}_\M(T) \simeq DA(\tilde{T}_n)/(Q_{\tilde{T}_n}, Q_{\tilde{T}_n}[n])$. By (i), (ii) and (iii), we have for every $k, m \in \Z$ that $DA(\tilde{S}_k)/(Q_{\tilde{S}_k}, Q_{\tilde{S}_k}[n])$ is isomorphic to $Q^{\pi_\ast(\tilde{S}_n)}$ if $m = 0$ and 0 otherwise; a similar formula holds for $\tilde{T}$.

A morphism of topological spaces which is open and has connected fibres induces an isomorphism on sets of connected components. The map $f$ and all its pullbacks are smooth with geometrically connected fibres, hence are open with connected fibres. This implies that the map $f$ and its
pullbacks induce isomorphisms $\pi_0(S_k) \simeq \pi_0(T_k)$ on sets of connected components. This implies the result.

Let $S$ be a scheme. We have $D(\text{Sm}/S)(Q_S, G_m \otimes Q) \simeq H^0(S_{et}, G_m \otimes Q) \simeq O^\times(S) \otimes Q$ and $D(\text{Sm}/S)(Q_S, G_m \otimes Q[1]) \simeq H^1(S_{et}, G_m \otimes Q) \simeq \text{Pic}(S) \otimes Q$. Combining these isomorphisms with Proposition 2.3, this induces morphisms

$$\nu^{1,1} : O^\times(S) \longrightarrow H^1_{\mathcal{M}}(S)$$

and

$$\nu^{2,1} : \text{Pic}(S)_Q \longrightarrow H^2_{\mathcal{M}}(S).$$

More generally, for any $n \in \mathbb{Z}$, we have an induced morphism

$$\nu^{n,1} : D(\text{Sm}/S)(Q_S, G_m[n - 1]) \rightarrow H^{n,1}_{\mathcal{M}}(S).$$

**Proposition B.6.**

(i) For all $n \leq 0$, we have $H^{n,1}_{\mathcal{M}}(S) \simeq 0$.

(ii) Assume $S$ regular. The morphism $\nu^{1,1}$ induces an isomorphism $H^{1,1}_{\mathcal{M}}(S) \simeq O^\times(S)_Q$.

(iii) Assume $S$ regular. The morphism $\nu^{2,1}$ induces an isomorphism $H^{2,1}_{\mathcal{M}}(S) \simeq \text{Pic}(S)_Q$.

(iv) Assume $S$ regular. For all $\neq 1, 2$, we have $H^{n,1}_{\mathcal{M}}(S) \simeq 0$. We have also

$$D(\text{Sm}/S)(Q_S, G_m[n - 1]) \simeq 0,$$

so that the morphism $\nu^{n,1}$ is an isomorphism.

**Proof.** Statement (i) for $S$ regular and a weaker version of (ii) (without specifying the isomorphism) are proved in [11, Corollaire 11.4].

To pass from (i) for $S$ regular to a general $S$, we apply resolution of singularities by alterations and cohomological $h$-descent for a proper regular hypercovering (which induces a descent spectral sequence for $H^{n,1}(-)$). To be more precise, one has to reduce to a situation where one can apply De Jong’s theorem, e.g. $S$ is of finite type over a Dedekind ring: for this, one uses Mayer-Vietoris to first reduce to $S$ affine, and then uses continuity. The argument is the same as in the proof of Lemma 4.22, so we do not spell out the details.

We revisit and make more precise the argument in [11, Corollaire 11.4] to establish (ii), (iii) and (iv).

Let us first treat the case where $S$ is the spectrum of a field. In that case, for $\neq 1$, both the source and target of $\nu^{n,1}$ are 0, so the only interesting case is $n = 1$. We have to show that the map

$$\nu^{1,1}_k : k^\times \otimes Q \rightarrow H^{1,1}_{\mathcal{M}}(k)$$

is an isomorphism. By the definition of $\nu^{1,1}$, we have to show that the map

$$k^\times \otimes Q \simeq \text{DA}^{\text{eff}}(k)(Q, G_m \otimes Q) \rightarrow \text{DA}(k)(Q, \Sigma^\infty(G_m \otimes Q))$$

induced by $\Sigma^\infty$ is an isomorphism.

Let $k^{\text{perf}}$ be a perfect closure of $k$ and $h : \text{Spec}(k^{\text{perf}}) \rightarrow \text{Spec}(k)$ be the canonical morphism. In the diagram

$$\begin{array}{cccc}
\text{DA}^{\text{eff}}(k)(Q, G_m \otimes Q) & \longrightarrow & \text{DA}(k)(Q, \Sigma^\infty(G_m \otimes Q)) \\
\downarrow h^* & & \downarrow k^* \\
\text{DA}^{\text{eff}}(k^{\text{perf}})(Q, G_m \otimes Q) & \longrightarrow & \text{DA}(k^{\text{perf}})(Q, h^* \Sigma^\infty(G_m \otimes Q)) \xrightarrow{\sim} \text{DA}(k)(Q, \Sigma^\infty(G_m \otimes Q))
\end{array}$$

the left square commutes because of the natural isomorphism $h^* \Sigma^\infty \simeq \Sigma^\infty h^*$. The left vertical arrow is an isomorphism because $k^\times \otimes Q \simeq (k^{\text{perf}})^\times \otimes Q$ (any element of $k^{\text{perf}}$ has a power in $k$), and the right vertical arrow is an isomorphism by separation for $\text{DA}$.

We are now reduced to the case when $k$ is perfect. Then we can follow a familiar pattern: comparison with $\text{DM}(k)$ using [6, Theorem 2.8, Proposition 2.10], then with $\text{DM}^{\text{eff}}(k)$ using Voevodsky’s cancellation theorem (this is where we need $k$ perfect), and finally the classical computation of weight one effective motivic cohomology [49, Lecture 4].
We now do the general case. We can assume $S$ connected, hence integral. Let $j : U \to S$ be a non-empty open set and $Z$ its closed complement. We stratify $Z = Z_0 \subset Z_1 \subset \ldots \subset Z_k = \emptyset$ in such a way that for all $i$, the scheme $(Z_i \setminus Z_{i+1})_{\text{red}}$ is regular and in such a way that $(Z \setminus Z_1)$ contains all points of codimension 1 in $Z$. Then by applying inductively localization, absolute purity (for the regular pair $(S, (Z_i \setminus Z_{i+1})_{\text{red}})$) and the vanishing result Proposition B.5 (i) and (ii) we see that

- the map $u^{0,0} : \mathbb{Q}^{\nu_0}(\mathbb{Z} \setminus Z_1) \to H_{M}^{0,0}(Z \setminus Z_1)$ is an isomorphism,
- the pullback map $H_{M}^{n,1}(S) \to H_{M}^{n,1}(U)$ is an isomorphism for $\neq 1, 2$, and
- there is a short exact sequence

$$0 \to H_{M}^{1,1}(S) \to H_{M}^{1,1}(U) \to H_{M}^{0,0}(Z \setminus Z_1) \to H_{M}^{2,1}(S) \to H_{M}^{2,1}(U) \to 0.$$}

Putting this together with the localization sequence for $\mathcal{O}^\times$ and Pic, we get a diagram

$$0 \to \mathcal{O}_S^\times \otimes \mathbb{Q} \to \mathcal{O}_U^\times \otimes \mathbb{Q} \overset{\nu_{0,0}}{\to} \mathbb{Q}^{\nu_0}(\mathbb{Z} \setminus Z_i) \simeq \bigoplus_{z \in \mathbb{Z}^{(1)}} \mathbb{Q}[z] \to \text{Pic}(S) \otimes \mathbb{Q} \to \text{Pic}(U) \otimes \mathbb{Q} \to 0$$

$$(A) \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad (B)$$

$$0 \to H_{M}^{1,1}(S) \to H_{M}^{1,1}(U) \to H_{M}^{0,0}(Z \setminus Z_1) \to H_{M}^{2,1}(S) \to H_{M}^{2,1}(U) \to 0.$$}

We claim that the diagram above is commutative. For the two outer squares, this follows from the commutation of pullbacks in Proposition 2.3.

For the commutation of diagrams (A) and (B), we have to do more work, since one arrow is defined explicitly using valuations and line bundle attached to a divisor while the other is defined via the absolute purity isomorphism. Instead of giving a long explicit computation, we prefer to use the case of a base field treated above, we see that $\mathcal{O}^\times(S)_{Q} \subset K_1(S) \otimes \mathbb{Q}$ (resp. $\text{Pic}(S)_{Q} \subset K_0(S) \otimes \mathbb{Q}$ for $S$ regular), and that the Chern character maps coincide with the maps $\nu_{0,1}$ modulo this identification.

Passing to the limit in the previous commutative diagram over all non-empty open sets, using continuity both for motivic cohomology and for the étale cohomology of $\mathbb{G}_m$, we get a commutative diagram

$$0 \to \mathcal{O}_S^\times \otimes \mathbb{Q} \to \kappa(S)^\times \otimes \mathbb{Q} \overset{\nu_{0,1}}{\to} \bigoplus_{z \in S^{(1)}} \mathbb{Q}[z] \to \text{Pic}(S) \otimes \mathbb{Q} \to \text{Pic}(\kappa(S)) \to 0$$

$$(A) \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad (B)$$

$$0 \to H_{M}^{1,1}(S) \to H_{M}^{1,1}(\kappa(S)) \to \bigoplus_{z \in S^{(1)}} \mathbb{Q}[z] \to H_{M}^{2,1}(S) \to H_{M}^{2,1}(\kappa(S)) \to 0.$$}

Using the case of a base field treated above, we see that

- the group $H_{M}^{n,1}(S)$ vanishes for $\neq 1, 2$, and
- there is a short exact sequence

$$0 \to H_{M}^{1,1}(S) \to \kappa(S)^\times \otimes \mathbb{Q} \overset{\nu_{0,1}}{\to} \bigoplus_{z \in S^{(1)}} \mathbb{Q}[z] \to H_{M}^{2,1}(S) \to 0.$$}

Using the normality (resp. regularity) of $S$, this implies $H_{M}^{1,1}(S) \simeq \mathcal{O}(S)_{Q}^\times$ and $H_{M}^{2,1}(S) \simeq \text{Pic}(S)_{Q}$ and finishes the proof.

We finish by giving an example which shows that even for weight zero motivic cohomology on normal (but not regular) schemes, the result can differ from étale cohomology.

**Proposition B.7.** Let $S$ be a normal excellent surface. Let $\pi : \tilde{S} \to S$ be a resolution of singularities of $S$, with $D = \pi^{-1}(p)$ simple normal crossing divisor in $\tilde{S}$. Let $\Gamma = (V, E)$ be the resolution graph of $D$. Then

$$H_{M}^{n,0}(S) \simeq \begin{cases} \mathbb{Q}, & n = 0 \\ H_{M}^{1}(\Gamma, \mathbb{Q}), & n = 2 \\ 0, & \neq 0, 2 \end{cases}$$
while on the other hand
\[ D(\text{Sm}/S)(Q_S, Q_S[n]) \simeq \begin{cases} Q, & n = 0 \\ 0, & n \neq 0 \end{cases}. \]

**Proof.** The last statement comes from the fact that the étale cohomology of a normal scheme with \( \mathbb{Q} \)-coefficients is trivial. So we concentrate on the first. For \( n \leq 0 \), the result follows from B.5, so we assume \( n > 0 \).

We have the cartesian diagram of schemes:

\[
\begin{array}{ccc}
U & \xrightarrow{j} & S \\
\uparrow & & \uparrow \\
\tilde{S} & \xrightarrow{i} & D \\
\downarrow & & \downarrow \\
p & \xrightarrow{\pi} & \pi_p
\end{array}
\]

Localisation yields the long exact sequence:

\[
\mathbf{DA}(S)(Q_S, Q_S[n-1]) \xrightarrow{DA(U)(Q_U, Q_U[n-1])} \mathbf{DA}(p)(Q_p, i^! Q_S[n]) \xrightarrow{DA(S)(Q_S, Q_S[n])} \mathbf{DA}(U)(Q_U, Q_U[n]).
\]

By Proposition B.5, this yields an isomorphism \( \mathbf{DA}(p)(Q_p, i^! Q_S[n]) \simeq \mathbf{DA}(S)(Q_S, Q_S[n]) \).

Write \( \{D_v\}_{v \in V} \) for the set of irreducible components of \( D \) and \( p_v \) for the intersection points \( D_v \cap D_{v'} \) for \( vv' \in E \). We set \( Z = \bigcup_{v \in E} \{p_v\} \) and \( D = D \setminus Z \). Write \( k : \tilde{D} \to E \), \( l : Z \to D \). Localisation gives a distinguished triangle

\[
l_*([i \circ l]_! Q_{\tilde{S}}) \to [\tilde{l}_! Q_{\tilde{S}}] \to k_*([i \circ k]_! Q_{\tilde{S}}) \xrightarrow{\sim}.
\]

By the relative purity theorem for \( \mathbf{DA} \) (see [7, 1.6.1] and [11, Corollaire 3.10]) applied to the regular immersions \( i \circ l \) and \( i \circ k \), this triangle takes the form:

\[
l_* Q_Z(-2)[-4] \to [\tilde{l}_! Q_{\tilde{S}}] \to k_* Q_D(-1)[-2] \xrightarrow{\sim}.
\]

So we get the exact sequence:

\[
\mathbf{DA}(\tilde{D})(Q_{\tilde{D}}, Q_{\tilde{D}}(-2)[n-4]) \to \mathbf{DA}(D)(Q_D, [\tilde{l}_! Q_{\tilde{S}}][n]) \to \mathbf{DA}(Z)(Q_Z, Q_Z(-1)[n-2]).
\]

By Proposition B.2, the groups on the left and on the right are zero for all \( n \in \mathbb{Z} \), so we conclude that \( \mathbf{DA}(D)(Q_D, [\tilde{l}_! Q_{\tilde{S}}][n]) = 0 \) for all \( n \in \mathbb{Z} \).

Now, the fact that \( \pi_U \) is an isomorphism, colocalisation and base change for immersions (see [7, 1.4.6]) implies that \( \text{Cone}(i^! Q_S \to \pi_{p*} i^! Q_S) \simeq \text{Cone}(Q_p \to \pi_{p*} Q_D) \). Combining with the previous result, we get that for all \( n \in \mathbb{Z} \):

\[
\mathbf{DA}(S)(Q_S, Q_S[n]) \simeq \mathbf{DA}(p)(Q_p, \text{Cone}(Q_p \to \pi_{p*} Q_D)[n-1]) \simeq \mathbf{DA}(p)(Q_p, \pi_{p*} Q_D[n-1])
\]

(where the last isomorphism follows as \( n > 1 \)).

Using Čech descent for closed covers and Proposition B.2, it is then easy to see that this last group is isomorphic to \( \mathbb{Q} \) if \( n = 0 \) (note that \( \Gamma \) is connected by normality of \( S \)), isomorphic to \( H^1(\Gamma, \mathbb{Q}) \) if \( n = 1 \), and 0 otherwise.

\[
\square
\]

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