D-brane on Poisson manifold and Generalized Geometry

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Abstract

The properties of the D-brane fluctuations are investigated using the two types of deformation of the Dirac structure, based on the $B$-transformation and the $\beta$-transformation, respectively. The former gives the standard gauge theory with 2-form field strength. The latter gives a non-standard gauge theory on the Poisson manifold with bivector field strength and the vector field as a gauge potential, where the gauge symmetry is a diffeomorphism generated by the Hamiltonian vector field. The map between the two gauge theories is also constructed with the help of Moser’s Lemma and the Magnus expansion. We also investigate the relation to the gauge theory on the noncommutative D-branes.
1 Introduction

Among the various dualities in string theory, the T-duality is the most characteristic and also interesting one from the viewpoint of stringy geometry. T-duality means that a closed string does not distinguish a pair of spaces, which are T-dual to each other. Since a string is the only object which observes the spacetime, this property should be incorporated into the axioms of the stringy geometry. It is also known that when we consider T-duality transformations under the existence of fluxes, many types of topologically nontrivial spaces appear. Among them, there are spaces which are not manifolds in a standard sense, generally called "non-geometric spaces". Since a closed string can travel in those "non-geometric spaces", they must be equally natural from the stringy geometry viewpoint.

Generalized geometry is a proposal by Hitchin to formulate a geometry where these duality properties are realized manifestly \cite{1, 2}. Especially the generalized complex structure is useful to classify possible supersymmetric compactified spaces with fluxes such as Calabi-Yau manifolds. Thus the generalized geometry is mostly applied to analyze the supergravity theories corresponding to the closed superstring theory \cite{3}.

However, it is also interesting how open strings and D-branes are characterized and behave in such a geometry \cite{1, 5}. For example, we know that a T-duality transformation changes the dimension of a D-brane, and thus we expect that the effective theory based on the generalized geometry will treat D-branes of different dimensions in an unified way.

Recently, a geometrical characterization of D-branes in the framework of the generalized geometry has been proposed \cite{6}. There, a D-brane including fluctuations is identified with a leaf of a foliation generated by a Dirac structure of the generalized tangent bundle. The scalar fields and vector fields on the D-brane are also unified as a generalized connection. From this geometrical setting, the richer symmetry of the D-brane in the target space becomes transparent and it was shown that the Dirac-Born-Infeld (DBI) action, the effective action of the low energy theory of a D-brane, realizes these symmetries nonlinearly.

The nonlinear realization of the symmetry can be understood as a phenomenon following from the spontaneous symmetry breaking of a larger symmetry in the generalized geometry. The symmetry of a Dirac structure in the generalized geometry is characterized by a foliation preserving generalized diffeomorphism, where a leaf is mapped to another leaf keeping the foliation structure. However, the existence of a D-brane chooses one of the leaves and thus breaks the symmetry to the leaf preserving generalized diffeomorphism, which keeps the leaf itself. Based on this picture, fields in the effective theory have been identified systematically as Nambu-Goldstone (NG) bosons associated with the broken symmetries, and the nonlinear realizations of the broken symmetries are derived. These wider symmetries restrict the action of the effective theory, and in the lowest order it is
shown that the result is the DBI action. Here the induced generalized metric appearing in the action is also understood as a generalized metric seen by the Dirac structure.

This result means that Dirac structures in the generalized geometry are the proper geometrical concepts to characterize D-branes. In this paper, we develop this picture further.

Here we investigate the Dirac structure corresponding to a D-brane with a non-trivial gauge flux, or equivalently, a bound state of D-branes of various dimensions. A typical property of such a situation is, as we explain below, that there are always two ways of describing the same Dirac structure. For example, one can define a Dirac structure from $TM$ by using a so-called $B$-transformation generated by a symplectic 2-form $\omega$, then a graph of the map corresponds to a Dirac structure which we call $L_\omega$. On the other hand, the same graph can be described by a so-called $\beta$ transformation from the cotangent bundle $T^*M$, which is generated by a Poisson bivector $\theta$, and its graph becomes also a Dirac structure which we call $L_\theta$. $L_\omega$ and $L_\theta$ are dual descriptions of the same Dirac structure. The first description $L_\omega$ is a direct generalization of the results given in a previous paper [6], and it fits to describe a D9-brane with a 2-form gauge flux $\omega$. Here we develop another formulation of a D-brane as a Dirac structure based on the second description $L_\theta$. As we will see, the proper language to formulate a gauge theory appearing naturally from this description is the differential calculus used in the Poisson cohomology, where the role of 1-forms and vector fields is exchanged.

Although the two Dirac structures $L_\omega$ and $L_\theta$ are equivalent, considering the fluctuations on the D-brane, we see that they are quite different. In our previous paper we have shown that, including fluctuations, the D-brane can still be characterized by a Dirac structure. In other words the fluctuation is identified with a deformation of the Dirac structure. Now the deformation can also be seen in two ways, either as a variation of the symplectic structure $\omega' = \omega + \tilde{F}$ or a variation of the Poisson structure $\theta' = \theta + \hat{F}$. Of course, the 2-form $\tilde{F}$ and the bivector $\hat{F}$ are related with each other. Then, the condition that the $L_\theta$ obtained by a deformation becomes again a Dirac structure is formulated by a Maurer-Cartan type equation for $\hat{F}$. As we will see, it is very natural to identify $\hat{F}$ with a kind of field strength and the Maurer-Cartan type equation with the Bianchi identity in a gauge theory. We will find a gauge potential $\Phi$ corresponding to this "field strength" $\hat{F}$ requiring that the Maurer-Cartan type equation is automatically satisfied, like the Bianchi identity is trivially satisfied if one writes the field strength $\tilde{F}$ by a gauge potential $a$ in a usual gauge theory.

The above observation is also consistent with the analysis on the non-linear realization of broken symmetry for Dirac structures developed in [6]. There, we found that the non-linearly realized symmetry leads to the conclusion that the 2-form $\tilde{F}$ can be written as
a field strength of a gauge potential \( a \), and it describes an ordinary gauge theory on the \( D \)-brane with \( U(1) \) gauge symmetry. By using the same analysis, we arrive at the interesting conclusion that the bivector \( \tilde{F} \) can also be considered as a kind of field strength, and the corresponding gauge potential is a vector field \( \Phi \), and the gauge symmetry is a diffeomorphism generated by a set of Hamiltonian vector fields.

The new gauge theory is quite different in shape but, of course, it should be equivalent to the ordinary \( U(1) \) gauge theory corresponding to \( \tilde{F} \). In fact, we find an explicit relation between the two gauge fields \( a \) and \( \Phi \), which is highly non-linear. It is also shown that the two gauge theories are gauge equivalent, although the structures of gauge symmetries look quite different. In the proof, Moser’s lemma \([7]\) which relates a deformation of a symplectic structure with a diffeomorphism, plays an important role. We show that Moser’s lemma is also explained in a natural way within the framework of generalized geometry. We show that a diffeomorphism can be seen either as a \( B \)-gauge transformation or as a \( \beta \)-gauge transformation up to a generalized diffeomorphism that preserves the Dirac structure. Another technique used to show this relation is the Magnus expansion \([8]\), which relates a time ordered exponential to an ordinary exponential.

One motivation to study \( L_\theta \) comes from a suggestion given in \([10]\), in which \( L_\theta \) is proposed to corresponds to a noncommutative description of a \( D9 \)-brane in the \( B \)-field background. Here we obtain in a quite simple way the two relations associated with a noncommutative D-brane: the Seiberg-Witten (SW) relation between so-called open and closed string variables and the SW map between commutative and noncommutative gauge field strengths, which is valid when all tensors are constant \([11, 12]\). These relations are, however, the result of two dual descriptions of the same Dirac structure and valid in a wider context than merely the original noncommutative D-brane.

The paper is organized as follows. In the next section, we review the basic facts about generalized geometry. In \( \S3 \), we study how the two Dirac structures \( L_\omega \) and \( L_\theta \) characterize a D-brane. We also reproduce two relations concerning noncommutative D-branes. In \( \S4 \), we focus on the differential geometry on the worldvolume. We show that the Dirac structure is isomorphic to a Lie algebroid of a Poisson manifold. In \( \S5 \), we argue that the two fluctuations \( \tilde{F} \) and \( \hat{F} \) are in fact field strengths, from the viewpoint of both the Maurer-Cartan type relation and the non-linear realization of broken symmetries. Then, we introduce the corresponding gauge potential and the gauge transformation. After recalling Moser’s lemma in \( \S6 \), we obtain the relation between the two gauge potentials \( a \) and \( \Phi \) of \( \tilde{F} \) and \( \hat{F} \), respectively in \( \S7 \). A short review on the Magnus expansion is also included there. \( \S8 \) is devoted to conclusions and discussions.
2 Generalized geometry

In this section, we present some basic facts about Lie algebroids [13] and the generalized geometry [1, 2, 14], with introducing the notations used in this paper.

2.1 Lie algebroids

First, we briefly introduce the notion of a Lie algebroid. A Lie algebroid \((A, \rho, [\cdot, \cdot]_A)\) consists of a vector bundle \(A \to M\) over a base manifold \(M\) together with a Lie bracket on its sections, \([\cdot, \cdot]_A : \Gamma(A) \times \Gamma(A) \to \Gamma(A)\), and an anchor map \(\rho : A \to TM\) such that the induced map \(\rho : \Gamma(A) \to \Gamma(TM)\) is a Lie-algebra homomorphism and the Leibniz identity

\[
[X, fY]_A = f[X, Y]_A + (\rho(X) \cdot f)Y,
\]

is satisfied for any \(X, Y \in \Gamma(A)\) and \(f \in C^\infty(M)\). Here \(\rho(X) \cdot f\) denotes the action of the vector field \(\rho(X)\) on \(f\). The tangent bundle \(TM\) itself is a Lie algebroid with the anchor \(\rho\) being the identity map.

2.1.1 Differential algebra

For a given Lie algebroid \((A, \rho, [\cdot, \cdot]_A)\), one can construct an algebra of associated \(A\)-differential forms \((\Gamma(\wedge A^*), \wedge, d_A)\) in general. Here the exterior differential \(d_A\) is a map from a \(k\)-form \(\omega \in \wedge^k A^*\) to a \((k + 1)\)-form \(d_A \omega\), which is defined by

\[
d_A \omega(X_1, \cdots, X_{k+1}) = \sum_i (-)^{i+1} \rho(X_i) \cdot \omega(X_1, \cdots, \hat{X}_i, \cdots, X_{k+1}) \\
+ \sum_{i<j} (-)^{i+j} \omega([X_i, X_j]_A, X_1, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_{k+1}).
\]

As usual, \(d_A\) is a graded derivation on a wedge product, and it satisfies \(d_A^2 = 0\).

For the tangent bundle \(A = TM\), it is the standard exterior differential algebra \((\Gamma(\wedge^* T^* M), \wedge, d)\), where \(d\) is the de Rham exterior differential.

2.1.2 Gerstenhaber algebra

Another algebra related to a Lie algebroid \(A\) is the Gerstenhaber algebra \((\Gamma(\wedge^* A), \wedge, [\cdot, \cdot]_A)\) of \(A\)-polyvectors. Here the Gerstenhaber bracket is defined as a derived bracket, extending the Lie bracket. For \(V, W \in \wedge^* A\), it is

\[
i_{[V, W]}_A \omega = (-)^{(|V|-1)(|W|-1)} i_V d_A i_W \omega - i_W d_A i_V \omega - (-)^{|W|-1} i_{[V,W]} A d_A \omega.
\]
The bracket is graded commutative

\[ [V, W]_A = -(-1)^{|V|-1}|W|-1[W, V]_A, \]  

(2.4)

and satisfies the graded Jacobi identity. Note that the bracket is defined in particular for \( \wedge^0 A = C^\infty(M) \). It is an extension of \( A \)-Lie derivative \( [X, f]_A = \rho(X) \cdot f \) for \( X \in A \) to any polyvector \( V \).

In the case of \( A = TM \), this bracket is called the Schouten(-Nijenhuis) bracket. For two polyvectors of the form \( V = X_1 \wedge \cdots \wedge X_k \) and \( W = Y_1 \wedge \cdots \wedge Y_l \) with \( X_i, Y_j \in TM \), the Schouten bracket in \( \Gamma(\wedge^\bullet TM) \) is also written as the sum of Lie brackets of vector fields as

\[ [V, W]_S = \sum_{i=1,j=1}^{k,l} (-1)^{i+j} [X_i, Y_j] \wedge X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots X_k \wedge Y_1 \wedge \cdots \hat{Y}_j \wedge \cdots Y_l. \]  

(2.5)

The bracket for \( \wedge^0 A = C^\infty(M) \) is given for example \( [X, f]_S = X \cdot f = X(df) \).

Under the anchor map \( \rho : A \to TM \), its pullback \( \rho^* : \Gamma(\wedge^\bullet \Lambda^* M) \to \Gamma(\wedge^\bullet A^* ) \) is a morphism of differential algebras, that satisfies \( \rho^* \circ d = d_A \circ \rho^* \). On the other hand, its natural extension \( \wedge^\bullet \rho : \Gamma(\wedge^\bullet A) \to \Gamma(\wedge^\bullet TM) \) is a morphism of Gerstenhaber algebras.

2.2 Generalized tangent bundle

Let \( M \) be a \( d \)-dimensional smooth manifold, \( TM \) be the tangent and \( T^* M \) be the cotangent bundle over \( M \), respectively. The sum of these bundles, \( \mathbb{T}M = TM \oplus T^* M \) is called a generalized tangent bundle. A section of \( \mathbb{T}M \) is called a generalized vector field and will be represented by a sum \( X + \xi \) of a vector field \( X = X^\mu \partial_\mu \in \Gamma(TM) \) and a 1-form \( \xi = \xi_\mu dx^\mu \in \Gamma(T^* M) \). It is equipped with the following operations:

1. The anchor map \( \pi \) is a bundle map \( \mathbb{T}M \to TM \), the projection to the tangent bundle. It induces a map \( \Gamma(\mathbb{T}M) \to \Gamma(TM) \) given by \( \pi(X + \xi) = X \).

2. The canonical inner product is a fiberwise non-degenerate symmetric bilinear form \( \langle \cdot, \cdot \rangle : \Gamma(\mathbb{T}M) \times \Gamma(\mathbb{T}M) \to C^\infty(M) \), given by

\[ \langle X + \xi, Y + \eta \rangle = \frac{1}{2} (\xi Y + \eta X) = \frac{1}{2} \left( X^T \xi \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} Y \\ \eta \end{pmatrix} \right), \]  

(2.6)

where the last expression is written in terms of \( 2d \)-component vectors.

3. The Dorfman bracket is a map \( \Gamma(\mathbb{T}M) \times \Gamma(\mathbb{T}M) \to \Gamma(\mathbb{T}M) \) defined by

\[ [X + \xi, Y + \eta] = [X, Y] + \mathcal{L}_X \eta - \iota_Y d\xi, \]  

(2.7)
where \([X,Y] \in \Gamma(TM)\) is the ordinary Lie bracket of vector fields, \(\iota_X\) is the interior product, i.e. \(\iota_X \eta = X^\mu \eta_\mu\), and \(L_X\) is the Lie derivative along a vector field \(X\). It is not skew-symmetric, but satisfies Jacobi identity.

It is known that the bundle \(TM\) together with these structures is an example of Courant algebroids \([15, 16]\), which is a natural generalization of the Lie algebroid \(TM\). Note that the Courant bracket \([\cdot, \cdot]_C\), which is the skew-symmetrization of the Dorfman bracket (2.7), can also be used to define a Courant algebroid. In the following, we do not distinguish a bundle \(TM\) and its sections \(\Gamma(TM)\) if it is not confusing. In this paper, we do not consider the \(H\)-twisted Courant bracket.

### 2.2.1 Generalized Lie derivative

In the ordinary differential geometry, a diffeomorphism \(\varphi : M \to M\) induces an automorphism \(\varphi_* : TM \to TM\) of the tangent bundle \(TM\), so that the symmetry of the Lie algebroid \(TM\) is the diffeomorphism group \(\text{Diff}(M)\). An infinitesimal diffeomorphism is generated by a vector field \(\epsilon = \epsilon^\mu \partial_\mu\), and its action on \(TM\) is represented by the Lie derivative \(L_\epsilon\), as \(L_\epsilon = [\epsilon, X]\) for \(X \in TM\).

Similarly, the symmetry of the generalized tangent bundle is defined by the automorphism group of the Courant algebroid \(TM\). It is a semi-direct product \(\text{Diff}(M) \rtimes \Omega^2_{\text{closed}}(M)\) of the ordinary diffeomorphism group and an abelian group of B-field transformations \(\Omega^2_{\text{closed}}(M)\), where the action of a closed 2-form \(b\) on a section of \(TM\) is defined by \(X + \xi \to X + \xi + \iota_X b\). We call an element of this group as a generalized diffeomorphism.

Infinitesimally, the Courant automorphism group defines the algebra of derivations, whose element is a generalized Lie derivative \(L_{(\epsilon, b)}\) generated by a pair \((\epsilon, b)\) of a vector field and a closed 2-form, which acts on \(X + \xi \in TM\) as

\[
L_{(\epsilon, b)}(X + \xi) := L_\epsilon(X + \xi) + \iota_X b.
\]  

Especially when \(b\) is exact, \(b = -d\Lambda\), it reduces to the inner derivation with respect to the Dorfman bracket in eq.(2.7). We denote the generalized Lie derivative \(L_{\epsilon + \Lambda}\) in such a case for brevity, then and thus

\[
L_{\epsilon + \Lambda}(X + \xi) = L_\epsilon(X + \xi) - \iota_X d\Lambda = [\epsilon + \Lambda, X + \xi].
\]

### 2.3 Dirac structure

A Dirac structure is a subbundle \(L \subset TM\), such that it is involutive under the Dorfman bracket \([X + \xi, Y + \eta] \in L\) for \(X + \xi, Y + \eta \in L\), it is isotropic under the canonical inner product \(\langle X + \xi, Y + \eta \rangle = 0\) for \(X + \xi, Y + \eta \in L\), and it has the maximal rank. The
Dorfman bracket restricted on $L$ is antisymmetric and thus a Dirac structure is a Lie algebroid by definition.

It is immediate to see that a generalized diffeomorphism (2.9) generated by an element of $L$ is a symmetry of the Dirac structure $L$. In fact the action $\mathcal{L}_{\epsilon + \Lambda}$ for $\epsilon + \Lambda \in L$ on a section $X + \xi \in L$ lies again in $L$, $\mathcal{L}_{\epsilon + \Lambda}(X + \xi) \in L$, because $L$ is involutive. We call it as a $L$-diffeomorphism.

Trivial examples of the Dirac structure are $TM$ and $T^*M$. Less trivial examples are obtained by a $B$-transformation of $TM$ and a $\beta$-transformation of $T^*M$, which we describe below. They are the main objects studied in this paper.

### 2.3.1 $B$-transformation of $TM$

Given an arbitrary 2-form $\omega \in \wedge^2 T^*M$, a $B$-transformation of $TM$ defines a subbundle $L_{\omega} = e^{\omega}(TM)$,

$$
L_{\omega} = \{e^{\omega}(X) = X + \omega(X) | X \in TM\}. 
$$

(2.10)

Here the 2-form $\omega$ is considered as a map $TM \rightarrow T^*M$, defined by

$$
\omega(X) := \omega(X, \cdot) = \iota_X \omega = \omega_{\mu\nu} X^\mu dx^\nu,
$$

(2.11)

where the last term is a local expression written in the components of $\omega = \frac{1}{2} \omega_{\mu\nu} dx^\mu \wedge dx^\nu$. The subbundle $L_{\omega}$ is a Dirac structure if and only if $\omega$ is a closed 2-form, $d\omega = 0$. This is because the $B$-transformation generated by a closed 2-form is a symmetry of the generalized tangent bundle.

### 2.3.2 $\beta$-transformation of $T^*M$

Given an arbitrary bivector $\theta \in \wedge^2 TM$, a $\beta$-transformation of $T^*M$ defines a subbundle $L_{\theta} = e^{\theta}(TM)$,

$$
L_{\theta} = \{e^{\theta}(\xi) = \xi + \theta(\xi) | \xi \in T^*M\}. 
$$

(2.12)

Here, the bivector $\theta$ is considered as a map $T^*M \rightarrow TM$, defined by

$$
\theta(\xi) := \theta(\xi, \cdot) = \iota_\xi \theta = \theta^{\mu\nu} \xi_\mu \partial_\nu,
$$

(2.13)

where the last term is a local expression written in components of $\theta = \frac{1}{2} \theta^{\mu\nu} \partial_\mu \wedge \partial_\nu$. The subbundle $L_{\theta}$ is a Dirac structure if and only if $\theta$ is a Poisson bivector, i.e., $[\theta, \theta]_S = 0$ where $[\cdot, \cdot]_S$ is the Schouten bracket (2.5) of polyvector fields. The condition to be a Poisson bivector can be written as $\theta^{\mu\tau} \partial_\tau \theta^{\nu\rho} + \theta^{\nu\tau} \partial_\tau \theta^{\rho\mu} + \theta^{\rho\tau} \partial_\tau \theta^{\mu\nu} = 0$ in components,
which is the same condition for the Jacobi identity of the Poisson bracket \( \{ f, g \} = \theta(df, dg) \) for \( f, g \in C^\infty(M) \).

Note that a \( \beta \)-transformation is not a symmetry of the generalized tangent bundle \( TM \). As a result, \( L_\omega \) and \( L_\theta \) are different in many respects, that we study in this paper. See more information on \( L_\theta \) as a Lie algebroid in \( \S 4 \) and a proof of the condition in Appendix.

### 3 D-branes and Dirac structures

As we have addressed in the introduction, a D-brane can be characterized by a Dirac structures \([6]\). The worldvolume of the D-brane is identified with a leaf of foliation, an integral manifold determined by the image of the anchor map \( \pi : L \to TM \). The set of all possible leaves just means that of all possible positions of a D-brane (moduli space) obtained by transverse displacements. A spacetime with a D-brane is thus a foliated manifold that admit such a foliation. In particular, a Dirac structure \( L = TM \) corresponds to a spacetime filling D-brane, i.e. \( D9 \)-brane in superstring theory.

In this section, we assume that the Dirac structures \( L_\omega \) and \( L_\theta \) introduced in the previous section are also giving characterizations of certain kinds of D-branes. As a consequence, there are indications that \( L_\theta \) corresponds to a noncommutative D-brane.

#### 3.1 D-brane as a leaf of foliation

Let us start our discussion with a Dirac structure \( L_\omega = e^\omega(TM) \) with a closed 2-form \( \omega \). Since the anchor map \( \pi : L_\omega \to TM \) is surjective, we understand that the world volume of corresponding D-brane is the base manifold \( M \) and thus the D-brane is space-time filling. Since the Dirac structure \( L_\omega \) is a \( B \)-transformation of \( TM \), \( \omega \) is identified with a non-trivial \( U(1) \)-gauge flux on a \( D9 \)-brane. The existence of a \( U(1) \)-gauge flux generates lower D-brane charges (brane within brane), it is equivalent to a bound state of \( D9 \)-brane and lower dimensional D-branes.

The case where the Dirac structure is specified by \( L_\theta = e^\theta(T^*M) \) is more subtle. According to the identification above, the world volume is a symplectic leaf of a Poisson manifold \( (M, \theta) \), which is a symplectic manifold but whose dimension depends on the rank of the bivector \( \theta \) at that point. Then the spacetime is a collection of symplectic leaves with various dimensions. To avoid this complexity, we focus on the nondegenerate Poisson structure (full rank everywhere) in this paper. Correspondingly, \( L_\theta \) is identified with a kind of \( D9 \)-brane, whose worldvolume is a nondegenerate Poisson manifold \( (M, \theta) \).
3.2 Two ways to describe the same Dirac structure

The above assumption is equivalent to the requirement that the spacetime \((M, \omega)\) is a symplectic manifold where \(\omega\) is a symplectic structure (i.e., non-degenerate). In the generalized geometry, it means that there are two possibilities \(L_\theta\) and \(L_\omega\) which define the same Dirac structure, as we will see.

Any element of \(L_\omega\) can be represented by using a vector \(X \in TM\) as \(X + \omega(X)\), and any element of \(L_\theta\) can be represented by using 1-form \(\xi \in T^*M\) as \(\xi + \theta(\xi)\). If two Dirac structures are the same subbundle \(L_\theta = L_\omega\), there must be one to one correspondence between these two representations. Thus we have (see Fig.1)

\[
\xi + \theta(\xi) = X + \omega(X) .
\]

Comparing the vectors and forms in both sides, we get

\[
\xi = \omega(X) , \quad X = \theta(\xi).
\]

The first relation defines a 1-form \(\xi\) for a vector \(u\), and then the second equation gives a consistency

\[
X = \theta(\omega(X)),
\]

which should be satisfied for an arbitrary \(X\). In components this gives a relation of matrices \(\theta^{\mu\nu} = (\omega_{\mu\nu})^{-1}\). For brevity, we write it as

\[
\theta = \omega^{-1}.
\]

This is a simple demonstration of a trick which we are going to use frequently in the following sections. In this example, it gives a rather trivial statement that a symplectic
structure defines a Poisson bivector as its inverse. However this "trick" is useful to identify two subbundles defined by the different ways in more complicated cases.

We give two more such examples below, and interestingly, both of them suggest the connection between a Dirac structure $L_\theta$ and a noncommutative description of a D-brane.

### 3.3 Metric seen by $L_\theta$

Here, we want to discuss a metric structure on the D-brane in the present formulation. For this purpose we need to introduce a generalized Riemannian structure \[2\]. The generalized Riemannian structure $C_+ \subset TM$ is a subbundle, which is defined as a set of positive-definite generalized vectors $V = X + \xi$, s.t. $\langle V, V \rangle > 0$, and can be represented by a graph of a map of the generalized metric $E = g + b : TM \to T^*M$, i.e.,

$$C_+ = \{X + E(X) \mid X \in TM\}. \tag{3.5}$$

where $g$ and $B$ are the Riemannian metric and the $B$-field, respectively. Note that the generalized metric tensor $E = g + b$ comes from the closed string background, and it is independent of the Dirac structure related with a D-brane which originates from the open strings.

As argued in \[6\], we can represent the same $C_+$ by a graph of a different map $t : L \to L^*$, from an arbitrary Dirac structure $L$ to its dual $L^*$, instead of the map from $TM$ to $T^*M$. In the case of Dirac structures treated in \[6\], the map $t$ is identified with the induced generalized metric on the Dp-brane, corresponding to $L$. We call such $t$ as a metric seen from $L$.

We consider here a map $t : L_\theta \to L^*_\theta = TM$, which maps any element $\xi + \theta(\xi)$ in $L_\theta$ to a vector $t(\xi)$ and define a graph of the Riemannian structure $C_+$. Then any section of $C_+$ is given by

$$C_+ = \{\xi + \theta(\xi) + t(\xi) \mid \xi \in T^*M\}. \tag{3.6}$$

Since the above two graphs define the same Riemannian structure $C_+$, there must be one to one correspondence (see Fig\[2\])

$$X + E(X) = \xi + \theta(\xi) + t(\xi), \tag{3.7}$$

or equivalently,

$$X = \theta(\xi) + t(\xi), \quad E(X) = \xi. \tag{3.8}$$

As a result, we have a relation

$$E^{-1} = \theta + t. \tag{3.9}$$
If we write \( t = (G + \Phi)^{-1} \), where \( G \) is a symmetric tensor and \( \Phi \) is an anti-symmetric tensor, then (3.9) is nothing but the relation given in [11]:

\[
\frac{1}{g + B} = \frac{1}{G + \Phi} + \theta. \tag{3.10}
\]

This is usually called the Seiberg-Witten relation, or the open-closed relation, in the case that all tensors are constant. In the right hand side, \( \theta \) is a noncommutative parameter of the D-brane, \( G \) is a metric seen by the noncommutative D-brane, and \( \Phi \) is a background 2-form. This suggests that the Dirac structure \( L_\theta \) corresponds to a noncommutative D-brane [10]. Note that in this formulation, all the tensors are not restricted to be constant. Note also that it is natural to regard the Poisson bivector \( \theta \) as a free parameter in the relation, and it is simply associated with a choice of a Dirac structure \( L_\theta \), i.e., a choice of the description of the D-brane. On the other hand, the remaining tensors \( g + B \) and \( G + \Phi \) are background fields coming from closed strings.

The generalized Riemannian structure \( C_+ \) can also be seen from \( L_\omega \). In this case, the relation corresponding to (3.10) is rather trivial \( g + B = t + \omega \), where \( t = g + B - \omega \) is the generalized metric seen by \( L_\omega \). Here, the 2-form \( \omega \) is a free parameter.

### 3.4 Fluctuations and SW map

If we regard both structures \( L_\theta = L_\omega \) as a kind of D9-branes, there are two natural candidate of D-brane fluctuations corresponding to the choice \( L_\omega \) and \( L_\theta \), respectively. The validity of these fluctuations from a gauge theory point of view will be studied concretely in the next sections. Here we focus on the relation of the two fluctuations using the same method in the previous subsection.
Starting from \( L_\omega \), its \( B \)-transformation by an arbitrary 2-form \( \tilde{F} = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu \in \wedge^2 T^* M \) defines a new subbundle \( L_{\omega + \tilde{F}} \), in which the element has a form

\[
e^\tilde{F}(X + \omega(X)) = X + (\omega + \tilde{F})(X).
\] (3.11)

This is a natural way to add a fluctuation for the D-brane characterized by \( L_\omega \).

Another way is obtained by a \( \beta \)-transformation of \( L_\theta \) by a bivector \( \hat{F} = \frac{1}{2} \hat{F}^{\mu\nu} \partial_\mu \wedge \partial_\nu \in \wedge^2 TM \). It defines a new subbundle \( L_{\theta + \hat{F}} \), in which an element has a form

\[
e^{\hat{F}}(\xi + \theta(\xi)) = \xi + (\theta + \hat{F})(\xi).
\] (3.12)

We want to consider that the above two deformed subbundles are the same D-brane system including the fluctuation and thus the two subbundles should be equivalent \( L = L_{\omega + \tilde{F}} = L_{\theta + \hat{F}} \), and thus their sections are relating each other as (see Fig.3)

\[
\xi + (\theta + \hat{F})(\xi) = X + (\omega + \tilde{F})(X),
\] (3.13)

or equivalently,

\[
(\theta + \hat{F})(\xi) = X, \quad \xi = (\omega + \tilde{F})(X).
\] (3.14)

This condition leads to a matrix relation \((\theta + \hat{F})(\omega + \tilde{F}) = 1\), corresponding to the relation \( \theta \omega = 1 \) for the case without fluctuation. Therefore, we have

\[
\theta + \hat{F} = \frac{1}{\omega + \tilde{F}} = \frac{1}{1 + \theta \tilde{F}}
\] (3.15)

and thus

\[
\hat{F} = -\frac{1}{1 + \theta \hat{F}} \theta \tilde{F} \theta.
\] (3.16)
This relation is the same as the Seiberg-Witten map between commutative and noncommutative $U(1)$ field strength, if the 2-form $\omega \hat{F} \omega$ is identified with a noncommutative field strength, and in the case that all of $\theta, \tilde{F},$ and $\hat{F}$ are constant tensors [11]. It suggests again a connection to noncommutative gauge theories [10, 12]. However, our relation is valid for more general tensors. Moreover, it is not yet clear whether $\tilde{F}$ and $\hat{F}$ are considered to be field strengths of some gauge theories. In the remainder of this paper, we elaborate on the latter question. First, in the next section, we study $L_\omega$ and $L_\theta$ themselves, focusing on the differential geometry on the worldvolume of the D-brane. Then, in the subsequent section, we argue that $\tilde{F}$ and $\hat{F}$ are indeed gauge field strengths.

4 Differential geometry on worldvolume

A Dirac structure is automatically a Lie algebroid, and there is a proper differential calculus associated with a Lie algebroid. In this section, we focus on such calculi associated with $L_\omega$ and $L_\theta$, as a preparation for formulating gauge theories. In particular, we see that $L_\theta$ is the same as a Lie algebroid of a Poisson manifold, and thus the calculus is that used for the Poisson cohomology.

4.1 Lie algebroid of Poisson manifold

First we introduce a Lie algebroid associated with a Poisson manifold, following [13]. Let $(M, \theta)$ be a Poisson manifold, with a Poisson bivector $\theta$. Then a Lie algebroid $A = (T^*M)_\theta$, or more precisely, $(T^*M, \theta, [\cdot, \cdot])$ is defined as follows. The anchor map $\rho : T^*M \to TM$ of the algebroid $(T^*M)_\theta$ is given by the Poisson bivector $\theta$. Here the Poisson bivector $\theta = \frac{1}{2} \theta^{\mu\nu} \partial_\mu \wedge \partial_\nu \in \wedge^2 T^*M$ defines a map $\theta : T^*M \to TM$ by $\theta(\xi) = i_\xi \theta = \theta^{\mu\nu} \xi_\mu \partial_\nu$ for $\xi \in T^*M$. The Lie bracket of $A$ is given by so-called Koszul bracket, which is defined for $\xi, \eta \in T^*M$ by

$$[\xi, \eta]_K := L_{\theta(\xi)} \eta - L_{\theta(\eta)} \xi - d(\theta(\xi, \eta))$$

$$= i_{\theta(\xi)} d\eta - i_{\theta(\eta)} d\xi + d(\theta(\xi, \eta))$$

(4.1)

and in components

$$[\xi, \eta]_K = \theta^{\mu\nu} \xi_\mu \partial_\nu \eta_\rho dx^\rho + \theta^{\mu\nu} (\partial_\mu \xi_\nu) \eta_\rho dx^\rho + (\partial_\mu \theta^{\rho\nu}) \xi_\mu \eta_\nu dx^\rho.$$

(4.2)

The Koszul bracket reduces for exact 1-forms to the Poisson bracket as

$$[df, dg]_K = d(\theta(df, dg)) = d\{f, g\}.$$  

(4.3)

We refer to this as the Lie algebroid of the Poisson manifold [13].
As explained in §2, for a given Lie algebroid \((A, \rho, [\cdot, \cdot])\), one can construct the exterior differential algebra \((\Gamma(\wedge^\bullet A^*), \wedge, d_A)\) and the Gerstenhaber algebra \((\Gamma(\wedge^\bullet A), \wedge, [\cdot, \cdot])_A\).

In our case of \(A = (T^*M)_\theta\), the differential algebra \((\Gamma(\wedge^\bullet T^*M), \wedge, d_\theta)\) is that of polyvector fields. For a vector \(X \in T^*M\), we have

\[
d_\theta X(\xi, \eta) = \theta(\xi)(X(\eta)) - \theta(\eta)(X(\xi)) - X([\xi, \eta]_K)
= (\mathcal{L}_\theta(\xi)X)(\eta) - (\mathcal{L}_\theta(\eta)X)(\xi) + X(d(\theta(\xi, \eta)))
= [\theta, X]_S(\xi, \eta),
\]

where in the last line we used the Schouten bracket for polyvector fields. In general,

\[
d_\theta = [\theta, \cdot]_S.
\]

It is nilpotent \(d_\theta^2 = \frac{1}{2}[[\theta, \theta]_S, \cdot]_S = 0\) because \(\theta\) is a Poisson structure \([\theta, \theta]_S = 0\). The cohomology group defined with respect to \(d_\theta\) is called the Poisson cohomology.

On the other hand, the Gerstenhaber bracket in the algebra \(\Gamma(\wedge^\bullet T^*M)\) is the extension of the Koszul bracket of 1-forms defined as in (4.5).

### 4.2 Lie algebroid of Poisson manifold as Dirac structure

Here we see that the Lie algebroid \(A = (T^*M)_\theta\) is isomorphic to the Dirac structure \(L_\theta = e^\theta(T^*M)\), as shown in [10].

Since any Dirac structure is a Lie algebroid, the Courant bracket of the Dirac structure \(L_\theta\) is automatically a Lie bracket (This is also true for the Dorfman bracket.). The Courant bracket for two elements in \(L_\theta\) is

\[
[\xi + \theta(\xi), \eta + \theta(\eta)]_C = [\theta(\xi), \theta(\eta)] + \mathcal{L}_{\theta(\xi)}\eta - \mathcal{L}_{\theta(\eta)}\xi - \frac{d}{2}(i_{\theta(\xi)}\eta - i_{\theta(\eta)}\xi)
= [\theta(\xi), \theta(\eta)] + \mathcal{L}_{\theta(\xi)}\eta - \mathcal{L}_{\theta(\eta)}\xi - d(\theta(\xi, \eta)).
\]

The \(T^*M\)-part is nothing but the Koszul bracket \([\xi, \eta]_K\) of the Lie algebroid \(A = (T^*M)_\theta\), defined in (4.1). Similarly, the \(TM\)-part can be rewritten as (see Appendix for the proof.)

\[
[\theta(\xi), \theta(\eta)] = \theta([\xi, \eta]_K),
\]

and thus the Courant bracket of the two elements in \(L_\theta\) is again of the form

\[
[\xi + \theta(\xi), \eta + \theta(\eta)]_C = \theta([\xi, \eta]_K) + [\xi, \eta]_K.
\]

Thus, two Lie algebroids \((T^*M)_\theta\) and \(L_\theta\) can be identified with each other. More precisely, a bundle map \(\iota : A \to L_\theta\) such that \(\iota(\xi) = \xi + \theta(\xi)\) is an isomorphism of Lie algebroids.
4.3 Worldvolume of the D-brane

As was explained already, the D-brane worldvolume associated with the Dirac structure $L_\theta$ is a Poisson manifold $(M, \theta)$. From the isomorphism above, we may say that the worldvolume is also associated with the Lie algebroid $(T^*M)_\theta$. The difference of the representations can be interpreted as follows: the Lie algebroid $(T^*M)_\theta$ corresponds to a worldvolume before embedded into a spacetime, and the Dirac structure $L_\theta$ is a image of the embedding map. In any case, it is clear that a natural differential geometry on the worldvolume, to formulate a gauge theory on it, is the differential algebra $(\Gamma(\wedge \cdot T M), \wedge, d_\theta)$ as well as the Gerstenhaber algebra $(\Gamma(\wedge \cdot T^* M), \wedge, [\cdot, \cdot]_S)$. In particular, in the differential geometry based on $(T^*M)_\theta$, the roles of 1-forms and vector fields are exchanged, as compared to the ordinary differential geometry based on $TM$. As a consequence, a gauge theory on a Poisson manifold $(M, \theta)$ is naturally described in terms of polyvectors. This is indeed the case, which will become clear in the next section.

Similarly, the D-brane worldvolume associated with the Dirac structure $L_\omega$ is a symplectic manifold $(M, \omega)$. In this case, the similar isomorphism between the Lie algebroid $TM$ and $L_\omega$ is given by the map $X \rightarrow X + \omega(X)$. We can use the ordinary differential calculus based on the de Rham differential forms. It means that we can apply the same calculus for the Dirac structure $L_\omega$ corresponding to a bound state of D-branes and for the Dirac structure $TM$ corresponding to a single D9-brane.

As we stated, we can take both $L_\omega$ and $L_\theta$ as a Dirac structure to describe the same D-brane. However, the differential calculus is quite different in the two representation. For the case with the symplectic form $\omega$, we can put $\omega$ on the worldvolume $M$ as a field and the differential calculus is not modified. On the other hand, for the case with the bivector $\theta$, the existence of the bivector is essential for the differential geometry on the worldvolume. Thus, when $\theta = \omega^{-1}$, we have the two quite different descriptions of the same D-brane bound state.

We recall that in the paper [12], polyvectors were used to describe a gauge theory on a Poisson manifold, as a first step to formulate a noncommutative gauge theory. However, as we will see, the gauge field in this paper is different from theirs.

5 Fluctuations from broken symmetries

In the analysis performed in [9], the symmetry preserved by the Dirac structure and the symmetry spontaneously broken by putting a D-brane are important to identify the correct fluctuations. We take here the same strategy for the Dirac structure $L_\theta$ as well as $L_\omega$. In particular for $L_\theta$, an exotic type of a gauge potential will be obtained, corresponding to the differential geometry on a Poisson manifold.
5.1 The case with $L = TM$

First we briefly formulate the case of $L = TM$ \[6\]. Among the total symmetry $\text{Diff}(M) \ltimes \Omega^2_{\text{closed}}(M)$ of the generalized tangent bundle (Courant algebroid) $TM$, the symmetry preserved by the Dirac structure $TM$ consists of the diffeomorphism generated by $\epsilon = \epsilon^\mu \partial_\mu$ and the $B$-field gauge transformation generated by closed 1-form $\Lambda$ ($d\Lambda = 0$). In the case of $M = \mathbb{R}^D$, the latter means that $\Lambda = d\lambda$ for a function $\lambda$, so that the unbroken symmetry is $\text{Diff}(M) \ltimes U(1) \ni \epsilon + d\lambda$. On the other hand, the broken symmetry consists of $B$-field gauge transformations generated by non-closed 1-form $A = A_\mu dx^\mu$, which is in general a Nambu-Goldstone (NG) boson. It acts on $X \in TM$ by the generalized Lie derivative $L_A$ as

$$e^{-L_A}X = e^{dA}X = X + F(X), \quad F = dA. \tag{5.1}$$

Thus $A$ produces a new Dirac structure $L = L_F$, and is regarded as a gauge-field fluctuation on the D9-brane. The above $U(1)$ symmetry is nothing but the gauge transformation $A \rightarrow A + d\lambda$.

5.2 Ordinary gauge field

Now we consider the case of $L_\omega$ of the form

$$L_\omega = \{X + \omega(X) | X \in TM\}. \tag{5.2}$$

Note that its dual bundle is $L^*_\omega = T^*M$ and thus, we may write $TM \oplus T^*M = L_\omega \oplus L^*_\omega$.

Correspondingly, let us take a parametrization of total symmetry $\text{Diff}(M) \ltimes \Omega^2_{\text{exact}}(M)$ as $\epsilon + \omega(\epsilon) + \Lambda$, where $\epsilon \in TM$ and $\Lambda \in T^*M$. It acts on a section $X + \omega(X) \in L_\omega$ as a generalized Lie derivative as

$$\mathcal{L}_{\epsilon + \omega(\epsilon) + \Lambda}(X + \omega(X)) = \mathcal{L}_\epsilon X + \omega(\mathcal{L}_\epsilon X) - i_X d\Lambda. \tag{5.3}$$

Clearly, a transformation generated by a generalized vector $\epsilon + \omega(\epsilon)$ is an unbroken symmetry for any $\epsilon \in TM$, since it preserves $L_\omega$. We call this symmetry as $L_\omega$-diffeomorphism ($L_\omega$-diff. for short). In addition, a $B$-field gauge transformation generated by a closed 1-form $d\Lambda = 0$ is also a symmetry. This includes $U(1)$ with $\Lambda = d\lambda$ for any $\lambda \in C^\infty(M)$. Therefore, the unbroken symmetry of $L_\omega$ is $\text{Diff}(M) \ltimes U(1)$ for $M = \mathbb{R}^D$, which is the same as that of $TM$.

On the other hand, the broken symmetry is a $B$-field gauge transformation generated by a non-closed 1-form $a = a_\mu dx^\mu \in T^*M$, which is a NG-boson. It is again a $U(1)$ gauge potential with a gauge transformation $a \rightarrow a + d\lambda$. Note that $a \in L^*_\omega$ is regarded as a
generalized connection on \( L_\omega \). This broken symmetry produces a new Dirac structure \( L = L_{\omega + \tilde{F}} \) as
\[
e^{-\mathcal{L}_a}(X + \omega(X)) = X + (\omega + \tilde{F})(X),
\]
where \( \tilde{F} = da \in \wedge^2 T^* M \) is the corresponding \( U(1) \) field strength. This is nothing but the fluctuation \( \tilde{F} \) discussed in §3.4. The condition for the Dirac structure \( d\tilde{F} = 0 \) is nothing but the Bianchi identity from the gauge theory point of view, and is trivially satisfied by \( \tilde{F} = da \) (of course, we can also consider a non-trivial \( U(1) \) gauge flux \( \tilde{F} \) by replacing \( a \) with a set of locally defined 1-forms, as usual.). We can say that the new symplectic structure \( \omega' = \omega + \tilde{F} \) is in the same class as \( \omega \) in the second de Rham cohomology group \([\omega'] = [\omega] \in H^2_{\text{dR}}(M)\).

### 5.3 New type of gauge field

Now let us consider the Dirac structure \( L_\theta \). The definition is
\[
L_\theta = \{ \xi + \theta(\xi) | \xi \in T^* M \}. \tag{5.5}
\]
The dual bundle is \( L^*_\theta = TM \) and thus, we can write \( TM \oplus T^* M = L_\theta \oplus L^*_\theta \).

Correspondingly, we adopt a parametrization of the total symmetry \( \text{Diff}(M) \times \Omega^2_{\text{exact}}(M) \) as \( \epsilon + a + \theta(a) \), where \( \epsilon \in TM \) and \( a \in T^* M \). The action of an infinitesimal transformation on a section \( \xi + \theta(\xi) \in L_\theta \) can be represented by a generalized Lie derivative as
\[
\mathcal{L}_{\epsilon + a + \theta(a)}(\xi + \theta(\xi)) = (\mathcal{L}_{\epsilon + a}(\xi) - i_{\theta(\xi)}da) + \theta(\mathcal{L}_{\epsilon + a}(\xi) - i_{\theta(\xi)}da) + (\mathcal{L}_\epsilon \theta)(\xi). \tag{5.6}
\]
From this expression, it is evident that the \( L_\theta \)-diff., the transformation generated by \( a + \theta(a) \) is an unbroken symmetry for any \( a \in T^* M \). In addition, as we see from the last term in the r.h.s., a \( \theta \)-preserving diffeomorphism generated by a vector field \( \epsilon \in TM \) such that \( \mathcal{L}_\epsilon \theta = 0 \) is also a symmetry. This class of transformations includes the diffeomorphism generated by the Hamiltonian vector fields \( \epsilon = X_f = \theta(df) \) for any \( f \in C^\infty(M) \).

On the other hand, the rest of the diffeomorphisms is broken. Denote a generator of those diffeomorphisms as \( \Phi = \Phi^\mu \partial_\mu \in TM \). It is a NG-boson according to the general argument of the broken symmetries. The vector field \( \Phi \) can be also seen as a kind of the gauge field, since it is a generalized connection on \( L_\theta \), i.e., \( \Phi \in TM = L^*_\theta \). The infinitesimal action of the vector field \( \Phi \) on \( L_\theta \) is
\[
\mathcal{L}_\Phi(\xi + \theta(\xi)) = \mathcal{L}_\Phi \xi + \theta(\mathcal{L}_\Phi \xi) + (\mathcal{L}_\Phi \theta)(\xi), \tag{5.7}
\]
and the finite action is
\[
e^{-\mathcal{L}_\Phi}(\xi + \theta(\xi)) = e^{-\mathcal{L}_\Phi} \xi + (e^{-\mathcal{L}_\Phi} \theta)(e^{-\mathcal{L}_\Phi} \xi). \tag{5.8}
\]
By defining $\xi' = e^{L\Phi}\xi$, and $\theta' = e^{-L\Phi}\theta$, the r.h.s. of (5.8) is rewritten as $\xi' + \theta'(\xi')$, which is a section of a new subbundle $L_{\theta'}$. Therefore, the diffeomorphism by $\Phi$ causes a map of subbundles $L_{\theta} \rightarrow L_{\theta'}$, which is effectively seen as a $\beta$-transformation generated by a bivector $\hat{F} \in \wedge^2 TM$ defined by

$$\theta' = \theta + \hat{F}. \tag{5.9}$$

(We will return to more systematic analysis on the relation between diffeomorphisms and $\beta$-transformations in §6.) In other words, $\hat{F}$ is represented by the field $\Phi$ as

$$\hat{F} = e^{-L\Phi}\theta - \theta. \tag{5.10}$$

We now argue that $\hat{F}$ is a generalized field strength, corresponding to a generalized connection $\Phi$.

### 5.3.1 $\hat{F}$ as field strength

Let us consider the $\beta$-transformation $e^{\hat{F}}$ of the Dirac structure $L_{\theta}$, which defines a new subbundle $L_{\theta'}$ with $\theta' = \theta + \hat{F}$. The deformed subbundle $L_{\theta'}$ stays in a Dirac structure if and only if $\theta' = \theta + \hat{F}$ is a Poisson structure. This requires

$$0 = [\theta', \theta']_S = [\theta + \hat{F}, \theta + \hat{F}]_S = [\theta, \theta]_S + 2[\theta, \hat{F}]_S + [\hat{F}, \hat{F}]_S \tag{5.11}$$

Thus, $\hat{F}$ should satisfy the Maurer-Cartan-type equation [10]:

$$d_\theta \hat{F} + \frac{1}{2}[\hat{F}, \hat{F}]_S = 0. \tag{5.12}$$

Here we used the notion of $d_\theta$ and the Schouten bracket in the differential geometry discussed in the previous section.

When we consider $\hat{F}$ as a field strength, this condition play the role of the Bianchi identity. It means that (5.12) should be trivially satisfied, when $\hat{F}$ is written in terms of the corresponding gauge potential. In fact, for an arbitrary vector field $\Phi \in TM$, the $\hat{F}$ in (5.10) satisfies (5.12) automatically. It is easily shown by noting

$$d_\theta \hat{F} = d_\theta(e^{-L\phi}\theta),$$

$$\frac{1}{2}[\hat{F}, \hat{F}]_S = -[\theta, e^{-L\phi}\theta]_S + \frac{1}{2}[e^{-L\phi}\theta, e^{-L\phi}\theta]_S$$

$$= -d_\theta(e^{-L\phi}\theta) + \frac{1}{2}e^{-L\phi}[\theta, \theta]_S$$

$$= -d_\theta(e^{-L\phi}\theta), \tag{5.13}$$
where $d\theta = [\theta, \theta]_S = 0$ is used. Thus, we can regard $\Phi$ as a gauge potential. The field strength $\hat{F}$ is expanded with respect to $\Phi$ as

$$\hat{F} = \sum_{n=1}^{\infty} \frac{1}{n!} (-\mathcal{L}_\Phi)^n \theta = \frac{e^{-\mathcal{L}_\Phi} - \text{id}}{\mathcal{L}_\Phi} d\Phi \Phi$$

$$= d\Phi - \frac{1}{2} [\Phi, d\Phi]_S + \frac{1}{3!} [\Phi, [\Phi, d\Phi]_S]_S \cdots. \tag{5.14}$$

where $\mathcal{L}_\Phi \theta = [\Phi, \theta]_S = -d\theta \Phi$ is used. The first term $d\Phi$ is similar to the abelian gauge field strength, but in addition to this term, there are infinitely many non-linear corrections.

The gauge transformation of the potential $\Phi$ is defined as an insertion of a diffeomorphism generated by a Hamiltonian vector field $d\Phi h \in TM$ for $h \in C^\infty(M)$ as

$$e^{-\mathcal{L}_\Phi} \rightarrow e^{-\mathcal{L}_{\Phi'}} = e^{-\mathcal{L}_\Phi} e^{-\mathcal{L}_{d\Phi h}}. \tag{5.15}$$

Apparently, $\hat{F}$ in (5.10) is invariant by this transformation, since $\mathcal{L}_{d\Phi h} \theta = -d\Phi h = 0$ holds. Due to the Baker-Campbell-Hausdorff (BCH) formula, the existence of such $\mathcal{L}_{\Phi'}$ is guaranteed. The first few terms in $\Phi'$ are obtained by the BCH-formula as

$$\Phi' = \Phi + d\Phi h - \frac{1}{2} [\Phi, d\Phi h]_S + \frac{1}{12} [\Phi, [\Phi, d\Phi h]_S]_S - \frac{1}{12} [d\Phi h, [\Phi, d\Phi h]_S]_S + \cdots, \tag{5.16}$$

which is again a non-linear extension of the abelian gauge transformation. It is also possible to exchange the order in (5.15) as

$$e^{-\mathcal{L}_\Phi} e^{-\mathcal{L}_{d\Phi h}} = e^{-\mathcal{L}_{d\Phi h'}} e^{-\mathcal{L}_\Phi}, \tag{5.17}$$

where $h' = e^{-\mathcal{L}_\Phi} h$. To show this, we first use a variant of the BCH formula to obtain $e^{-\mathcal{L}_\Phi} e^{-\mathcal{L}_{d\Phi h}} = e^{-\mathcal{L}_W} e^{-\mathcal{L}_\Phi}$ with $W = e^{-\mathcal{L}_\Phi} d\Phi h$. Next, use the relation:

$$d\Phi h' = [\theta + \hat{F}, h']_S$$

$$= [e^{-\mathcal{L}_\Phi} \theta, e^{-\mathcal{L}_\Phi} h]_S$$

$$= e^{-\mathcal{L}_\Phi} [\theta, h]_S$$

$$= e^{-\mathcal{L}_\Phi} d\Phi h. \tag{5.18}$$

In the latter form of the gauge transformation in (5.17), the field strength $\hat{F}$ is invariant, since $d\Phi h'$ is a Hamiltonian vector field with respect to $\theta'$:

$$\hat{F} \rightarrow e^{-\mathcal{L}_{d\Phi h'}} e^{-\mathcal{L}_\Phi} \theta - \theta$$

$$= e^{-\mathcal{L}_{d\Phi h'}} \theta' - \theta$$

$$= \theta' - \theta$$

$$= \hat{F}. \tag{5.19}$$
Therefore, the gauge transformation associated with the gauge field $\Phi$ preserves both $L_{\theta}$ and $L_{\theta'}$ and is generated by a Hamiltonian diffeomorphism. Note that the set of Hamiltonian vector fields (and thus diffeomorphisms) forms a Lie algebra

$$\mathcal{L}_{\phi}, \mathcal{L}_{\phi'}, \mathcal{L}_{\phi} = \mathcal{L}_{\phi},$$

where $\{\cdot, \cdot\}$ is a Poisson bracket with respect to $\theta$. This shows that the gauge symmetry is not abelian.

In summary, we obtain a gauge theory where the field strength is given by the bivector $\hat{F}$, and identified the gauge transformations. This justifies our discussion on the fluctuation in §3.4. The field strength can be written by the gauge potential, a vector field $\Phi$, of which the gauge transformation law is non-linear. Interestingly, the Bianchi identity of this gauge theory is the Maurer-Cartan-type relation, which guarantees that the $\beta$-transformed subbundle $L_{\theta+\hat{F}}$ becomes again the Dirac structure. Such kind of gauge theory is new (at least to the authors), but it is natural from the viewpoint of the differential geometry associated with the Lie algebroid $(T^*M)_\theta$.

Here, we have seen just necessary conditions in constructing a gauge theory. It is interesting to consider a action functional for $\hat{F}$, but because a Yang-Mills type action needs a induced (generalized) metric on the D-brane, we left this problem in the future publication. In this paper, we concentrated on to establish the relation between the two gauge theories corresponding to $\hat{F}$ and $\hat{F}$.

### 5.3.2 Poisson cohomology

Finally, we comment on a related mathematical argument. In [13], infinitesimal deformations of a Poisson structures are studied. They consider a deformation

$$\theta(\epsilon) = \theta + \epsilon\theta_1 + \epsilon^2\theta_2 + \cdots,$$

as a formal power series in an infinitesimal parameter $\epsilon$, and the condition for $\theta(\epsilon)$ to be a Poisson structure is obtained order by order in $\epsilon$:

$$d_\theta \theta_1 = 0, \quad d_\theta \theta_2 + \frac{1}{2}[\theta_1, \theta_1]_S = 0, \cdots$$

They are set of recursive equations and related to the second Poisson cohomology group $H^2_\theta$. In our case, if we insert the parameter $\epsilon$ by replacing $\Phi \rightarrow \epsilon\Phi$, the expansion in (5.14) directly gives a power series in $\epsilon$. Then the first condition in (5.22) is trivially solved by $\theta_1 = d_\theta \Phi$, which means that the deformation is trivial. The second condition in (5.22) is also solved by $\theta_2 = -\frac{1}{2}[\Phi, d_\theta \Phi]_S$. It means that the trivector $[\theta_1, \theta_1]_S$ is $d_\theta$-exact and thus trivial in $H^3_\theta$, which captures the obstructions to continuing infinitesimal deformations.
The same considerations to higher order equations are needed in this approach. Instead, we know that our $\theta' = e^{-\mathcal{L}_{\Phi}}\theta$ is a solution to all orders, because $\theta'$ is a Poisson structure automatically. That is, we get an expression for finite trivial deformation $[\theta'] = [\theta] \in H^2_\theta$.

To summarize this section, by identifying the broken symmetries as fluctuations in two ways, we found the two kinds of gauge potentials $a$ and $\Phi$ and associated gauge symmetries. If they are the two different descriptions of the same Dirac structure of a corresponding D-brane, these two gauge fluctuations should also be equivalent. So our next task is to find a relation between the two gauge fields $a$ and $\Phi$. To this end, the diffeomorphism appearing in the Moser’s lemma plays an important role and thus in the next section we reformulate the Moser’s lemma in the framework of generalized geometry.

6 Moser’s Lemma in Generalized Geometry

Here we show that the Moser’s lemma [7] and its Poisson version [12] are understood quite naturally within the generalized geometry framework. As a result, we obtain another expression for the bivector field strength $\tilde{F}$.

6.1 Moser’s lemma

We start with a quick review of Moser’s lemma. Suppose we have a symplectic form $\omega \in \wedge^2 T^* M$, i.e. a non-degenerate closed 2-form on $M$, and consider a deformation $\omega' = \omega + \tilde{F}$ by adding an exact 2-form $\tilde{F} = da$. Moser’s lemma states that there exists a diffeomorphism $\rho_a : M \to M$ which realize the deformation as $\omega' = \rho_a^* \omega$.

Such a diffeomorphism $\rho_a$ is constructed in the following way. Let us define one-parameter family of symplectic forms by

$$\omega_t = \omega + tda = \omega + t\tilde{F}, \quad (6.1)$$

where $t$ is a time parameter $t \in [0, 1]$, and the boundary conditions are $\omega_0 = \omega$ and $\omega_1 = \omega'$. If we define a vector field $X_a(t)$ such that

$$\omega_t(X_a(t)) = i_{X_a(t)} \omega_t = a, \quad (6.2)$$

then the Lie derivative generated by this vector field satisfies the differential equation

$$\mathcal{L}_{X_a(t)} \omega_t = di_{X_a(t)} \omega_t = da = \tilde{F} = \dot{\omega}_t. \quad (6.3)$$

Integrating this equation, we obtain the flow generated by $X_a(t)$ which relates $\omega$ and $\omega + \tilde{F}$. 

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In [12], its Poisson version is considered. They considered a one-parameter deformation of Poisson structure $\theta_t$, which is defined by a flow equation

$$\dot{\theta}_t = \mathcal{L}_{\theta_t(a)}\theta_t = d\theta_t \theta_t(a) = -\theta_t \tilde{F} \theta_t,$$

(6.4)

where $\tilde{F} = da$ and the last expression is understood as a matrix multiplication. Its solution (with initial condition $\theta_0 = \theta$) is given by

$$\theta_t = \theta_1 \frac{1}{1 + t\tilde{F} \theta}.$$  

(6.5)

Since the equation (6.4) defines a flow generated by a vector field $\theta_t(a)$, by integrating, we obtain a diffeomorphism $\rho_a$ such that $\rho_a^* \theta_1 = \theta_0$. It is a Poisson map (symplectomorphism):

$$\rho_a^* \{f, g\}_\theta = \rho_a^* (\theta_1(df, dg)) = \theta_0(d(\rho_a^* f), d(\rho_a^* g)) = \{\rho_a^* f, \rho_a^* g\}_\theta.$$  

(6.6)

Note that $X_a(t) = \theta_t(a)$ in the symplectic case where $\theta = \omega^{-1}$.

The Moser’s lemma relates a change of a symplectic (Poisson) structure to a one-parameter diffeomorphisms. We will see below that it originates from more general relations between $B(\beta)$-transformations and diffeomorphisms.

### 6.2 Moser’s diffeomorphism

First, we show that an arbitrary diffeomorphism acting on a Dirac structure $L_\omega = e^\omega(TM)$ equals to a $B$-field gauge transformation up to $L_\omega$-diffeomorphism (symmetry).

To this end, we decompose an infinitesimal diffeomorphism transformation of a section $X + \omega(X) \in L_\omega$ generated by a vector field $\epsilon \in TM$ as

$$X + \omega(X) \xrightarrow{\mathcal{L}_\epsilon} X + \omega(X) + \mathcal{L}_\epsilon(X + \omega(X)) = X' + \omega(X') + (\mathcal{L}_\epsilon \omega)(X'),$$  

(6.7)

up to $O(\epsilon^2)$, where $X' = X + \mathcal{L}_\epsilon X$. The term $X' + \omega(X') \in L_\omega$ is obtained by a $L_\omega$-diffeo. generated by $\epsilon + \omega(\epsilon) \in L_\omega$. The term $(\mathcal{L}_\epsilon \omega)(X')$ is a result of a $B$-gauge transformation. By noting $\mathcal{L}_{\mathcal{L}_\epsilon \omega} = di\omega = d(\omega(\epsilon))$, it is also written by a generalized Lie derivative $-\mathcal{L}_{\omega(\epsilon)}$ generated by a 1-form $\omega(\epsilon) \in T^*M$. Thus, a diffeomorphism can be seen as two-step transformations,

$$X + \omega(X) \xrightarrow{\mathcal{L}_{\epsilon + \omega(\epsilon)}} X' + \omega(X') \in L_\omega \xrightarrow{-\mathcal{L}_{\omega(\epsilon)}} X' + \omega(X') + (\mathcal{L}_\epsilon \omega)(X'),$$  

(6.8)
corresponding to the decomposition $\epsilon = (\epsilon + \omega(\epsilon)) - \omega(\epsilon)$ or equivalently $\mathcal{L}_\epsilon = \mathcal{L}_{\epsilon + \omega(\epsilon)} - \mathcal{L}_{\omega(\epsilon)}$. A graphical explanation is given in Fig. 4 where the diffeomorphism represented by the arrow $P \to Q$ is decomposed into the two arrows $P \to S$ and $S \to Q$. The arrow $P \to S$ corresponds to a $L_\omega$-diffeomorphism preserving the Dirac structure $L_\theta = L_{\omega}$. On the other hand, the arrow $S \to Q$ is a $B$-gauge transformation.

Figure 4: Two decompositions of an infinitesimal diffeomorphism acting on $L_\theta_t = L_{\omega_t}$

Only the $B$-gauge transformation contributes to the net transformation of $L_\omega$ as a set of sections. As a result, the diffeomorphism can be seen as a map $L_\omega \to L_{\omega + L_\epsilon \omega}$ of Dirac structures.

Let us apply this decomposition to the Moser’s situation. In this case, the input is a family of symplectic structures (6.1). They define a family of Dirac structures $L_{\omega_t} = e^{\omega_t}(TM)$. The time evolution from $L_{\omega_t}$ to $L_{\omega_{t+\delta t}}$ is governed by the B-field gauge transformation generated by the difference $\omega_{t+\delta t} - \omega_t = \delta t da$. Then, corresponding to this B-field gauge transformation, a vector field $\epsilon_t$ at time $t$ will be determined through the decomposition (6.8). (In Fig 4, $P$ and $S$ locate in $L_{\omega_t}$ and $Q$ lies in $L_{\omega_{t+\delta t}}$. Then, what we have performed here is that for a given $B$-transformation of $S \to Q$, we determine the arrow $P \to Q$.) Therefore, $\mathcal{L}_\epsilon \omega$ in (6.8) should be $\mathcal{L}_\epsilon \omega_t = d(\omega_t(\epsilon_t)) = da$, and we obtain the Moser’s diffeomorphism $\epsilon_t = \theta_t(a)$ (up to exact 1-form). In other words, Moser’s diffeomorphism $\mathcal{L}_{\theta_t(a)}$ is decomposed into $\mathcal{L}_{\theta_t(a)} = L_{\theta_t(a)} + a - L_a$, which is effectively a B-field gauge transformation $-L_a$ from the Dirac structure viewpoint. Here $\theta_t$ is considered as the inverse a map of $\omega_t : TM \to T^*M$, for each $t$. In component matrix, it is nothing but (6.5):

$$\theta_t = \frac{1}{\omega_t} = \frac{1}{\omega + t F'} = \frac{\theta}{1 + t F' \theta}.$$  

(6.9)
6.3 Evolution equation on $\theta_t$

Next, we show that the same diffeomorphism $L_\epsilon$ can also be decomposed into a $\beta$-transformation and a $L_\theta$-diffeomorphism. For a section $\xi + \theta(\xi) \in L_\theta = L_\omega$ in another parametrization, $L_\epsilon$ acts as

$$\xi + \theta(\xi) \xrightarrow{L_\epsilon} \xi + \theta(\xi) + L_\epsilon(\xi + \theta(\xi)) = \xi' + \theta(\xi') + (L_\epsilon \theta)(\xi'),$$

(6.10)

up to $O(\epsilon^2)$, where $\xi' = \xi + L_\epsilon \xi$. Similar to the previous argument, the term $\xi' + \theta(\xi')$ is obtained by a $L_\theta$-diffeomorphism $L_{\epsilon + \omega(\epsilon)}$ while the term $(L_\epsilon \theta)(\xi')$ is a result of a $\beta$-transformation $e^\beta$ with $\beta = L_\epsilon \theta$. Thus, the transformation by $\epsilon$ is decomposed as

$$\xi + \theta(\xi) \xrightarrow{L_{\epsilon + \omega(\epsilon)}} \xi' + \theta(\xi') \in L_\omega$$

(6.11)

and

$$\xi' + \theta(\xi') \xrightarrow{e^L_\epsilon} \xi' + \theta(\xi') + (L_\epsilon \theta)(\xi').$$

(6.12)

It is also represented graphically in Fig. 4, where the diffeomorphism $P \rightarrow Q$ is decomposed into $P \rightarrow R$ and $R \rightarrow Q$.

Again, the diffeomorphism can effectively be seen as a $\beta$-transformation $L_\theta \rightarrow L_{\theta + L_\epsilon \theta}$ from the Dirac structure viewpoint, as a map from a Dirac structure to the other. This fact was already used in §5, to find the gauge potential $\Phi$ corresponding to the bivector field strength $\hat{F}$.

Let us now apply the above decomposition to the Moser’s situation, again. For the Moser’s diffeomorphism $\epsilon_t = \theta_t(a)$, the parameter $\beta$ is written as

$$\beta = L_{\theta_t(a)} \theta_t = (\theta_t \wedge \theta_t)(da) = -\theta_t \tilde{F} \theta_t,$$

(6.13)

where $\theta_t \tilde{F} \theta_t$ denotes a bivector, the component of which is given by the matrix product $\theta_t^{\mu \nu} \tilde{F}_{\nu \lambda} \theta_t^{\lambda \rho}$. The time derivative of the equation (6.5) gives

$$\dot{\theta}_t = \frac{d}{dt} \omega_t^{-1} = -\omega_t^{-1} \dot{\omega}_t \omega_t^{-1} = -\theta_t \tilde{F} \theta_t.$$

(6.14)

Thus, we obtain the flow equation

$$\dot{\theta}_t = \beta = L_{\theta_t(a)} \theta_t$$

(6.15)

which is the desired equation (6.4). In particular, $\dot{\theta}_t$ is understood as the parameter for the infinitesimal $\beta$-transformation at time $t$, like in the previous section, the parameter for the $B$-transformation was $\dot{\omega}_t = \tilde{F}$. In Fig 4, it gives a relation between an arrow $S \rightarrow Q$ and an arrow $R \rightarrow Q$. 

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In summary, the diffeomorphism defined in the Moser’s lemma can be identified either $B$-transformation or $\beta$ transformation up to the generalized diffeomorphism preserving the Dirac structure. The differential equation shows that the flow is generated by these infinitesimal $B$- or $\beta$-transformations, respectively.

### 6.4 $\hat{F}$ from $B$-field gauge transformation

We see that the Moser’s lemma is used to relate a $B$-field gauge transformation and a $\beta$-transformation acting on $L_\theta$. By integrating (6.4) iteratively, we obtain the formal solution

$$\theta' = T e^{\int_0^1 dt \mathcal{L}_{\theta_t(a)}(\theta)}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n!} \int_0^1 dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \mathcal{L}_{\theta_{t_1}(a)} \mathcal{L}_{\theta_{t_2}(a)} \cdots \mathcal{L}_{\theta_{t_n}(a)} \theta. \quad (6.16)$$

where $T$ denotes the symbol of the time ordered product and defined in the second line. It says that $\theta'$ is obtained as a chain of infinitesimal diffeomorphisms, graphically seen in Fig.5. Each small triangle in Fig.4 corresponds to a decomposition of the infinitesimal diffeomorphism in Fig.4. Since this results in a change of Poisson structures $\theta \rightarrow \theta'$, the difference $\theta' - \theta$ must give a bivector $\hat{F}$,

$$\hat{F} = T e^{\int_0^1 dt \mathcal{L}_{\theta_t(a)} dt} - \theta. \quad (6.17)$$

In this way, the original $B$-gauge transformation with the parameter $\tilde{F} = da$ is converted into a $\beta$-transformation $\hat{F}$ through the Moser’s diffeomorphism.

![Figure 5: A chain of Moser’s diffeomorphisms](image-url)
7 Relation between Two Gauge Fields

In the previous sections, the two different expressions of the bivector field strength have been obtained:

\[ \hat{F} = e^{-L_{\Phi} \theta} - \theta, \quad \text{and} \quad \hat{F} = T e^{\int_0^1 L_{\theta(t)} dt} \theta - \theta. \]  (7.1)

The purpose of this section is to find a relation between the two gauge potentials \( \Phi \) and \( a \) from the equivalence of these expressions. To this end, we need to know the relation between the ordinary and the time-ordered exponentials. This problem is solved in more wider context, known as the Magnus expansion [8].

7.1 Magnus expansion

We summarize necessary information. See [8] and [9] for more detail and for applications to other physics.

Consider a differential equation

\[ \frac{d}{dt} Y(t) = A(t) Y(t), \]  (7.2)

with initial condition \( Y(0) = Y_0 \), where \( A(t) \) is a matrix (or more generally an operator) valued function of time \( t \), and \( Y(t) \) is a vector valued function to be solved. Its formal solution is given by the time-ordered exponential as

\[ Y(t) = T e^{\int_0^1 A(s) ds} Y_0. \]  (7.3)

Magnus found another representation of the solution of the form

\[ Y(t) = e^{\Omega(t)} Y_0, \]  (7.4)

that is, a true matrix exponential. Here \( \Omega(t) \) satisfies

\[ \dot{\Omega} = \frac{\text{ad}_\Omega}{e^{\text{ad}_\Omega} - I} (A) = \sum_{n=0}^{\infty} \frac{B_n}{n!} \text{ad}_\Omega^n (A), \]  (7.5)

where the dot denotes the time derivative, \( \text{ad}_\Omega = [\Omega(t), \cdot] \) and \( B_n \) are the Bernoulli numbers. By integrating it, \( \Omega(t) \) is obtained in terms of \( A(t) \).

To show that \( \Omega(t) \) gives the solution, we use an identity of the derivative of matrix exponentials

\[ \frac{de^{\Omega}}{dt} = \frac{e^{\text{ad}_\Omega} - I}{\text{ad}_\Omega} (\dot{\Omega}) e^{\Omega} \left( = \int_0^1 e^{\int_0^1 \dot{\Omega} e^{(1-s)\Omega(t)} ds} \right). \]  (7.6)
Then, the l.h.s. of the equation (7.2) is written as
\[ \frac{d}{dt} Y(t) = \frac{d e^{\Omega}}{d\Omega} Y_0 = \frac{e^{ad\Omega} - I}{ad\Omega} (\dot{\Omega}) e^{\Omega} Y_0 = \frac{e^{ad\Omega} - I}{ad\Omega} (\dot{\Omega}) Y(t) \] (7.7)

Using the equation (7.5), we obtain
\[ \frac{d}{dt} Y(t) = \frac{e^{ad\Omega} - I}{ad\Omega} \left( \frac{ad\Omega}{e^{ad\Omega} - I}(A) \right) Y(t) = A(t)Y(t). \] (7.8)

The exponent \( \Omega(t) \) is obtained as an expansion with respect to the order of \( A(t) \) as
\[ \Omega(t) = \sum_{n=1}^{\infty} \Omega_n(t), \] (7.9)
and is called the Magnus expansion. The first few terms are obtained by substituting the Bernoulli numbers in (7.5), as follows
\[ \dot{\Omega}(t) = A(t) - \frac{1}{2} [\Omega(t), A(t)] + \frac{1}{12} [\Omega(t), [\Omega(t), A(t)]] + \cdots. \] (7.10)

By comparing the same order in both sides, it leads to the relations
\[ \dot{\Omega}_1 = A, \quad \dot{\Omega}_2 = -\frac{1}{2} [\Omega_1, A], \quad \dot{\Omega}_3 = -\frac{1}{2} [\Omega_2, A] + \frac{1}{12} [\Omega_1, [\Omega_1, A]], \] (7.11)
which can be integrated as (we use a notation \( A(t_1) = A_1 \))
\[ \Omega_1(t) = \int_0^t dt_1 A_1, \]
\[ \Omega_2(t) = \frac{1}{2} \int_0^t dt_1 \int_0^{t_1} dt_2 [A_1, A_2], \]
\[ \Omega_3(t) = \frac{1}{6} \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 ([A_1, [A_2, A_3]] + [[A_1, A_2], A_3]). \] (7.12)

See Appendix for the proof, and for a direct check of the equivalence of (7.3) and (7.4) at a first non-trivial order.

### 7.2 Relation using Magnus expansion

In our context, the Magnus expansion gives an expression of an operator \( \mathcal{L}_\Phi \) in terms of a given time-dependent operator \( \mathcal{L}_{\theta(a)} \) defined by the equation
\[ e^{-\mathcal{L}_\Phi} = Te^{\int_0^t \mathcal{L}_{\theta(a)} dt}. \] (7.13)

As we saw above, the relation is given by the iterated integrals of multiple commutators of operators \( \mathcal{L}_{\theta(a)} \) as in (7.12). It leads directly to an expression of the vector field \( \Phi \)
in terms of \( \theta_t(a) \), since the commutator of Lie derivatives corresponds to the Lie bracket (Schouten bracket) of vector fields thanks to \([\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]}\). By expanding the gauge field \( \Phi \) as

\[
\Phi = \sum_{n=1}^{\infty} \Phi_n, \tag{7.14}
\]

with respect to the order of \( a \), each \( \Phi_n \) is determined in principle at any order. First few terms are read off from (7.12) as

\[
\Phi_1 = -\int_0^1 dt \theta_t(a), \\
\Phi_2 = \frac{1}{2} \int_0^1 dt_1 \int_0^{t_1} dt_2 [\theta_t(a), \theta_t(a)], \\
\Phi_3 = -\frac{1}{6} \int_0^1 dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 ([\theta_t(a), [\theta_t(a), \theta_t(a)]] + [[\theta_t(a), \theta_t(a)], \theta_t(a)]). \tag{7.15}
\]

Thus, the gauge potential \( \Phi \) is highly non-linear in \( a \).

Under the relation (7.13) between the two gauge potentials, it can be shown that the two gauge theories are gauge equivalent. That is, given a \( U(1) \)-gauge transformation \( a \to a + d\lambda \) with a gauge parameter (function) \( \lambda \), there is a gauge parameter \( h \) in the other theory which defines a gauge transformation (5.15). Such a gauge parameter \( h \) can be defined by

\[
e^{-\mathcal{L}_\Phi} e^{-\mathcal{L}_h} = T e^{\int_0^1 \mathcal{L}_{\theta_t(a+d\lambda)} dt}. \tag{7.16}
\]

As is shown in the proof below, \( h \) is also explicitly determined by the Magnus expansion

\[
h = \sum_{n=1}^{\infty} h_n, \tag{7.17}
\]

with respect to the order of \( \lambda \). The first few terms are

\[
h_1 = -\int_0^1 dt \tilde{\lambda}(t), \\
h_2 = \frac{1}{2} \int_0^1 dt_1 \int_0^{t_1} dt_2 [\tilde{\lambda}(t_1), d\theta \tilde{\lambda}(t_2)], \\
h_3 = -\frac{1}{6} \int_0^1 dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \left( [\tilde{\lambda}(t_1), [d\theta \tilde{\lambda}(t_2), d\theta \tilde{\lambda}(t_3)]] + [[d\theta \tilde{\lambda}(t_1), d\theta \tilde{\lambda}(t_2)], \tilde{\lambda}(t_3)] \right), \tag{7.18}
\]

where

\[
\tilde{\lambda}(t) = e^{\mathcal{L}_{\Phi(t)}} \lambda, \tag{7.19}
\]
with $\Phi(t)$ defined by the Magnus expansion of $Te^\int_0^t L_{\theta_s(a)}ds = e^{-L_{\Phi(t)}}$.

Before we give the proof, a few remarks are in order. First, the existence of such $h$ is natural since $U(1)$-gauge transformation preserves $\tilde{F}$ as well as $(5.15)$ preserves $\tilde{F}$. However, it is not necessarily trivial that $h$ non-linearly depends not only on $\lambda$ but also on the $U(1)$-gauge potential $a$ through $\Phi$. Note also that the equivalence of $(7.13)$ is stronger condition than the equivalence of the two expression in $(7.1)$. This means that there are ambiguities in relating $\Phi$ and $a$ coming from $\theta$-preserving diffeomorphisms, in addition to the gauge ambiguities. It is interesting to note that the similarity of these properties with the Seiberg-Witten map, i.e., a non-linear relation between the two gauge fields, a gauge equivalence [11], and its ambiguities [17].

**Proof:** We use an identity valid for any operator-valued function $A(t)$ and $B(t)$ of time $t$:

$$Te^\int_0^1 (A(t)+B(t))dt = Te^\int_0^1 A(t)dt Te^\int_0^1 B^{(A)}(t)dt,$$

where

$$B^{(A)}(t) = Te^{-\int_0^1 A(s)ds} B(t) Te^\int_0^t A(s)ds .$$

(the proof of this identity is given in Appendix.) We apply this identity to the r.h.s. of $(7.16)$ by setting $A(t) = \mathcal{L}_{\theta_t(a)}$ and $B(t) = \mathcal{L}_{\theta_t(d\lambda)}$. Then, we have

$$B^{(A)}(t) = Te^{-\int_0^1 \mathcal{L}_{\theta_s(a)}ds} \mathcal{L}_{\theta_t(d\lambda)} Te^\int_0^t \mathcal{L}_{\theta_s(a)}ds = e^{\mathcal{L}_{\Phi(t)}} \mathcal{L}_{\theta_t(d\lambda)} e^{-\mathcal{L}_{\Phi(t)}} = \mathcal{L}_{e^{\mathcal{L}_{\Phi(t)}} \theta_t(d\lambda)}$$

(7.22)

Here we used the same $\Phi(t)$ appeared in $(7.19)$. Note that $\theta_t = e^{-\mathcal{L}_{\Phi(t)}} \theta$ is corresponding to the Moser’s diffeomorphism integrated up to time $t$. By using the relation $\theta_t(d\lambda) = d\theta_t \lambda = [\theta_t, \lambda]_S$, the vector field inside the Lie derivative in the last line becomes

$$e^{\mathcal{L}_{\Phi(t)}} \theta_t(d\lambda) = e^{\mathcal{L}_{\Phi(t)}} [\theta_t, \lambda]_S$$

$$= [e^{\mathcal{L}_{\Phi(t)}} \theta_t, e^{\mathcal{L}_{\Phi(t)}} \lambda]_S$$

$$= [\theta, e^{\mathcal{L}_{\Phi(t)}} \lambda]_S$$

$$= d\theta(e^{\mathcal{L}_{\Phi(t)}} \lambda)$$

(7.23)

Thus, by defining $\tilde{\lambda}(t)$ as $(7.19)$, the right hand side of $(7.16)$ is written as

$$Te^\int_0^1 \mathcal{L}_{\theta_t(a+d\lambda)}dt = Te^\int_0^1 \mathcal{L}_{\theta_t(a)}dt Te^\int_0^1 \mathcal{L}_{d\lambda_t}dt .$$

(7.24)

Of course, the first term is written as $e^{-\mathcal{L}_\Phi}$ by the Magnus expansion. By applying the Magnus expansion to the second term, we can also find a vector field $Y$ such that

$$Te^\int_0^1 \mathcal{L}_{d\lambda_t}dt = e^{-\mathcal{L}_Y} .$$

(7.25)
First few terms in the expansion \( Y = \sum_{n=1}^{\infty} Y_n \) w.r.t. the order in \( \tilde{\lambda}(t) \) are read off from (7.12) as

\[
Y_1 = - \int_0^1 dt d\theta \tilde{\lambda}(t),
\]

\[
Y_2 = \frac{1}{2} \int_0^1 dt_1 \int_0^{t_1} dt_2 [d\theta \tilde{\lambda}(t_1), d\theta \tilde{\lambda}(t_2)],
\]

\[
Y_3 = -\frac{1}{6} \int_0^1 dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \left( [d\theta \tilde{\lambda}(t_1), [d\theta \tilde{\lambda}(t_2), d\theta \tilde{\lambda}(t_3)]] + [[d\theta \tilde{\lambda}(t_1), d\theta \tilde{\lambda}(t_2)], d\theta \tilde{\lambda}(t_3)]\right).
\]

(7.26)

As easily seen, all the entries inside the Lie brackets in \( Y_n \) (\( n \geq 2 \)) are \( d\theta \)-exact. Thus, by using the derivation property \( d\theta [X, Y] = [d\theta X, Y] + [X, d\theta Y] \) and \( d\theta^2 = 0 \) repeatedly, we can always extract one \( d\theta \) from each \( Y_n \). As a result, \( Y \) is \( d\theta \)-exact, that is, \( Y = d\theta h \) for some \( h \). By construction, the parameter \( h \) is also determined as the sum \( h = \sum_n h_n \), and first few terms are already given in (7.18).

It is also shown that the relation between \( a \) and \( \Phi \) obtained in this section is compatible with the relation between \( \tilde{F} \) and \( \hat{F} \) discussed in §3.4. To this end, recall that the Moser’s diffeomorphism relates two symplectic structures as \( \omega' = \omega + \tilde{F} = Te^{\int_0^1 L_{\theta t}(\phi) dt} \omega \). By the Magnus expansion, it can be also written as \( \omega' = e^{-\mathcal{L}_\phi} \omega \). Now, by applying \( e^{-\mathcal{L}_\phi} \) on both sides of \( \theta \omega = 1 \), we have

\[
(e^{-\mathcal{L}_\phi} \theta)(e^{-\mathcal{L}_\phi} \omega) = e^{-\mathcal{L}_\phi} 1
\]

\[
\Rightarrow (\theta + \hat{F})(\omega + \hat{F}) = 1,
\]

(7.27)

which is nothing but the relation obtained graphically in §3.4.

### 7.3 Expression using 1-form

As a final remark in this section, we point out that the vector field \( \Phi \) can also be represented by a 1-form \( \hat{A} \in T^*M \) as

\[
\Phi = \theta(\hat{A}).
\]

(7.28)

To this end, we first write \( \theta_t \) in (6.5) as \( \theta_t = \theta Z_t \) with

\[
Z_t = \frac{1}{1 + t\hat{F}\theta},
\]

(7.29)

which is a map \( Z_t : T^*M \to T^*M \) or equivalently a (1,1)-tensor. Then, we have for example,

\[
\theta_t(a) = \theta(Z_t(a)),
\]

\[
[\theta_{t_1}(a), \theta_{t_2}(a)] = \theta([Z_{t_1}(a), Z_{t_2}(a)]_K),
\]

(7.30)
where $[\cdot, \cdot]_K$ is the Koszul bracket with respect to $\theta$. As a result, we find $\hat{A}$, by replacing all the Lie bracket in the Magnus expansion (7.15) with the Koszul brackets as

$$
\hat{A}_1 = - \int_0^1 dt Z_t(a), \\
\hat{A}_2 = \frac{1}{2} \int_0^1 dt_1 \int_0^{t_1} dt_2 [Z_{t_1}(a), Z_{t_2}(a)]_K, \\
\hat{A}_3 = - \frac{1}{6} \int_0^1 dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 ([Z_{t_1}(a), [Z_{t_2}(a), Z_{t_3}(a)]_K]_K + [[Z_{t_1}(a), Z_{t_2}(a)]_K, Z_{t_3}(a)]_K).
$$

(7.31)

In this expression, the new gauge theory based on $L_\theta$ is also written by 1-form gauge potential. This is a more similar situation as the relation between commutative and noncommutative gauge theories.

The field strength $\hat{F}$ is also written by $\hat{A}$. By definition,

$$
\hat{F} = e^{-L_{\theta(A)}} \theta - \theta = \sum_{n=1}^{\infty} (-1)^n \frac{n!}{n!} L_{\theta(A)}^n \theta.
$$

(7.32)

Here the first term is rewritten as

$$
L_{\theta(A)} \theta = (\theta \wedge \theta)(d\hat{A}) = -\theta N \theta
$$

(7.33)

where $N = d\hat{A}$ is a 2-form and we used the matrix multiplication of tensors in the last expression. Similarly, the second term becomes

$$
L_{\theta(A)}^2 \theta = L_{\theta(A)}(\theta \wedge \theta)(d\hat{A})
= (L_{\theta(A)} \theta \wedge \theta)(d\hat{A}) + (\theta \wedge L_{\theta(A)} \theta)(d\hat{A})
= ((\theta \wedge \theta)(d\hat{A}) \wedge \theta)(d\hat{A}) + (\theta \wedge (\theta \wedge \theta)(d\hat{A}))(d\hat{A})
= 2\theta N \theta N \theta,
$$

(7.34)

where $L_{\theta(A)} d\hat{A} = 0$ is used. It is easy to find similar expressions for all $n$, and we have

$$
\hat{F} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} (-1)^n n! \theta N \theta N \cdots N \theta
= \theta \left( \sum_{n=0}^{\infty} (N \theta)^n \right) N \theta
= \theta \left( \frac{1}{1 - N \theta} \right) N \theta.
$$

(7.35)

Thus, the bivector field strength $\hat{F}$ is always sandwiched by $\theta$. The 2-form field strength $\frac{1}{1 - N \theta} N$ is basically given by the $U(1)$ field strength $N = d\hat{A}$, but dressed with non-linear corrections.
8 Conclusion and Discussion

As we have stressed, there are always two ways to specify the same Dirac structure in the generalized geometry. Since a Dirac structure gives the geometrical characterization of a D-brane, these two possibilities gives two types of geometrical descriptions of the D-brane, which are equivalent but with quite different appearances. In this paper, we have considered two such descriptions of the Dirac structure, the one based on the $B$-transformation and the other based on the $\beta$-transformation.

The characterization of D-branes by Dirac structures is very useful for analyzing the fluctuation of D-branes, since a D-brane with fluctuation is also described by a Dirac structure. Thus, to analyze the fluctuation we can use the whole machinery of the deformation theory of the Dirac structure in the generalized geometry. In [6], we have already used a part of this theory to reveal the full symmetry of D-brane effective theory, the DBI action. In the present paper, we have made use of this machinery to analyze the gauge symmetry of the new representation of the D-brane fluctuation.

We have shown that when considering the bound state of D-branes by assigning the Dirac structure $L_\omega = L_\theta$, we obtain two gauge theories to describe the fluctuations, corresponding to the deformation $L_{\omega+\tilde{F}} = L_{\theta+\hat{F}}$ based on the $B$-transformation or the $\beta$-transformation. The former turns out to be the standard $U(1)$ gauge theory with the 2-form field strength $\tilde{F}$ and a 1-form gauge potential $a$. On the other hand, the latter is a non-standard gauge theory with a bivector field strength $\hat{F}$ with the potential of a vector field $\Phi$, and the gauge symmetry is diffeomorphisms generated by Hamiltonian vector fields.

Since the two gauge theories are constructed from two different deformations of the Dirac structure, they show quite different properties. However, when they describe the same fluctuation of the same D-brane, they should be equivalent. We found the map between the two gauge fields, $a$ and $\Phi$, by using Moser’s lemma and the Magnus expansion. The relation between the field strengths, $\tilde{F}$ and $\hat{F}$, can be found easily by applying a graphical representation of the fluctuation. On the other hand, to derive the relation between the gauge potentials and the gauge symmetries we need to solve the flow equation explicitly. The resulting relations are non-linear and formulated as an infinite series. We gave a general formula and the concrete relation for the lower order of the expansion.

Our results have a lot of similarity with the typical properties of noncommutative D-branes, such as Seiberg-Witten relation for open and closed string metric, the SW map for constant fields, although our formulas are valid for more general situations. Note that the existence of such a map between two different gauge fields and their gauge equivalence are rather natural physically, since they are the properties for any two theories, which shares the same S-matrix. To proceed, it is important to verify the equivalence at the action
level. In this paper, we did not discuss about a particular action functional for a gauge potential $\Phi$, such as a DBI-type action for $\hat{F}$. For instance, an Yang-Mills type action functional, which is quadratic in $\Phi$, is highly non-linear in $a$, and infinitely many higher derivative terms appear as in the SW map [12]. It is also interesting to study whether the equivalence still holds as quantum field theories.

For simplicity, we assumed nondegenerate Poisson structures for $L_\theta$ in this paper. However, we can take more general Poisson structures and can construct a gauge theory with a gauge potential $\Phi$ and a field strength $\hat{F}$ also in that case. It would correspond to ordinary $U(1)$ gauge theories on symplectic leaves of various dimensions. The interpretation as D-brane bound state in that case is interesting to study.

One of our original motivations is the understanding of noncommutative description of a D-brane in the framework of generalized geometry. This work is considered as an intermediate step to this end. What we need to develop is the understanding of the relation between the noncommutative worldvolume and a leaf of foliation, where the latter is embedded into a commutative spacetime. In the description of Seiberg and Witten [11], the noncommutativity results from the string worldsheet theory in the constant $B$-field background. For an arbitrary Poisson manifold, Cattaneo and Felder [13] shows the deep connection of the worldsheet theory with the Kontsevich’s theory of deformation quantization [19]. In Wess et.al. [12], noncommutative gauge theories are constructed starting from any Poisson manifold. There, the noncommutativity is realized simply by applying the Kontsevich’s formality map to gauge theories on Poisson manifolds. Moreover, they argued that the SW map is related to a quantum version of the Moser’s diffeomorphism. These previous works already clarify the mechanism of the noncommutativity, however we would like to understand it more geometrically within the framework of the generalized geometry.

Note that the formulation of bivector gauge field strength $\hat{F}$ is possible only for the Dirac structure $L_\theta$ with non-zero $\theta$. We do not have any such description on the cotangent bundle $T^*M$. On the other hand, the Dirac structure $L = T^*M$ corresponds to a single $D$-instanton, where its worldvolume is a point. It is known that infinitely many $D$-instantons can describe a noncommutative D-brane [20,21]. Therefore, the problem is also related to the understanding of a reliable method to treat multiple D-branes in generalized geometry.

Finally, in this paper, we restrict ourselves to the case of vanishing flux and applied to the D-brane, i.e., to the open string sector. However the description of Dirac structures developed in this paper would also help to analyze a $H$-flux background, or more general non-geometric flux backgrounds in the closed string sector [22,23,24]. Especially, we would like to investigate the role of this new type of gauge theory in this context. We hope to report on this subject in the near future.
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A Proof of the formula (4.7)

Here we prove the identity

\[ [\theta(\xi), \theta(\eta)] = \theta([\xi, \eta]_K) + \frac{1}{2} [\theta, \theta]_S(\xi, \eta), \tag{A.1} \]

for an arbitrary bivector \( \theta \in \wedge^2 TM \). In particular, if \( \theta \) is a Poisson bivector, (4.7) holds.

Here we demonstrate it locally in terms of components. The l.h.s. of (A.1) is written locally by using vector fields \( \theta^\mu := \theta^\mu_\nu \partial_\nu \) as

\[ [\theta(\xi), \theta(\eta)] = [\theta^{\mu\nu} \xi_\mu \partial_\nu, \theta^{\rho\tau} \eta_\rho \partial_\tau] = [\xi_\mu \theta^\mu, \eta_\rho \theta^\rho] \]

\[ = \xi_\mu \theta^\mu (\eta_\rho) \theta^\rho + \xi_\mu \eta_\rho [\theta^\mu, \theta^\rho] - \theta^\mu (\xi_\mu) \eta_\rho \theta^\rho, \tag{A.2} \]

where (4.2) is used. Thus, the difference is obtained as

\[ [\theta(\xi), \theta(\eta)] = \theta^\rho \partial_\rho \theta^\mu \xi_\mu \partial_\nu \eta_\rho + \theta^\rho \partial_\rho \theta^\mu \xi_\rho \partial_\nu \eta_\mu + \theta^\mu \partial_\rho \theta^\rho \partial_\nu \theta^{\mu\nu} \partial_\tau \]

\[ = \xi_\mu \theta^\mu (\eta_\rho) \theta^\rho - \theta^\rho (\xi_\rho) \eta_\mu \theta^\mu + \xi_\mu \eta_\rho \theta^\rho \partial_\nu \theta^{\mu\nu} \partial_\tau \]

\[ = \frac{1}{2} [\theta, \theta]_S(\xi, \eta), \tag{A.3} \]

which is the desired result (A.1).

B On Magnus expansion

In this section, we prove (7.12) in the Magnus expansion, first. Then, we check the equivalence of (7.3) and (7.4).
The proof of (7.12) The first two equations in (7.12) are obvious. In the third one, we calculate as

\[ \Omega_3(t) = \int_0^t dt_1 \left( -\frac{1}{2} \Omega_2(t_1), A_1 \right) + \frac{1}{12} \left[ \Omega_1(t_1), [\Omega_1(t_1), A_1] \right] \]

\[ = -\frac{1}{4} \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \left[ A_2, A_3, A_1 \right] + \frac{1}{12} \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \left[ A_2, [A_3, A_1] \right] \]

\[ = -\frac{1}{4} \int_0^t \int_0^{t_1} dt_1 dt_2 dt_3 \Theta_{123} \left[ [A_2, A_3, A_1] \right] + \frac{1}{12} \int_0^t \int_0^{t_1} \int_0^{t_2} dt_1 dt_2 dt_3 \Theta_{123} \left( [A_2, [A_3, A_1]] + [A_3, [A_2, A_1]] \right) \]

\[ = \int_0^t \int_0^{t_1} \int_0^{t_2} dt_1 dt_2 dt_3 \Theta_{123} \left( \frac{1}{4} [A_1, [A_2, A_3]] + \frac{1}{12} ([A_3, [A_2, A_1]] - [A_1, [A_2, A_3]] + [A_3, [A_2, A_1]]) \right) \]

Here \( \Theta_{123} = \Theta(t_1 - t_2) \Theta(t_2 - t_3) \) and \( \Theta_{12} = \Theta(t_1 - t_2) \) is the Heviside step function. Note that \( \Theta_{12} \Theta_{23} = \Theta_{12} \Theta_{13} \Theta_{23} \). In the second term of third line, we convert the ordinary double integral into the time-ordered one using the identity

\[ \int dt_1 A_1 \int dt_2 A_2 \cdots \int dt_n A_n = \int \cdots \int dt_1 dt_2 \cdots dt_n \Theta_{12 \cdots n} \sum_{\sigma \in S_n} A_{\sigma(1)} A_{\sigma(1)} \cdots A_{\sigma(n)}, \quad (B.2) \]

where \( S_n \) is the permutation group. In the fourth line, the Jacobi identity \([A_2, [A_3, A_1]] = [A_3, [A_2, A_1]] - [A_1, [A_2, A_3]] \) is used.

Equivalence of (7.3) and (7.4) With respect to the Magnus expansion, matrix exponential is also expanded as

\[ e^{\Omega} = e^{\Sigma_n \Omega_n} = 1 + \Omega_1 + \left( \Omega_2 + \frac{1}{2} \Omega_1^2 \right) + \cdots, \quad (B.3) \]

according to the order of \( A(t) \) in the time-ordered exponential. The equivalence up to the second order is easy to show. In the third order, we explicitly show the equivalence:

\[ \Omega_3 + \frac{1}{2} (\Omega_1 \Omega_2 + \Omega_2 \Omega_1) + \frac{1}{6} \Omega_1^3 = \int_0^t \int_0^{t_1} \int_0^{t_2} dt_1 dt_2 dt_3 \Theta_{123} A_1 A_2 A_3. \quad (B.4) \]

First by using (B.2), the term \( \Omega_1^3 \) in (B.4) is rewritten as

\[ \Omega_1^3 = \int_0^t \int_0^{t_1} \int_0^{t_2} dt_1 dt_2 dt_3 \Theta_{123} \left( A_1 \{ A_2, A_3 \} + A_2 \{ A_3, A_1 \} + A_3 \{ A_1, A_2 \} \right). \quad (B.5) \]

On the other hand, the integrand of \( \Omega_3 \) in (B.1) is

\[ [A_1, [A_2, A_3]] + [[A_1, A_2], A_3] \]

\[ = 2 (A_1 A_2 A_3 + A_3 A_2 A_1) - (A_1 A_3 A_2 + A_2 A_3 A_1 + A_2 A_1 A_3 + A_3 A_1 A_2). \quad (B.6) \]
Thus, the sum of these two terms in (B.4) is simply
\[ \Omega_3 + \frac{1}{6} \Omega_1^3 = \frac{1}{2} \int_0^t \int_0^t \int_0^t dt_1 dt_2 dt_3 \Theta_{123}(A_1 A_2 A_3 + A_3 A_2 A_1). \] (B.7)

Next, we rewrite the second term in (B.4). Using (B.2) again, we have
\[ \Omega_2 \Omega_1 = \frac{1}{2} \int_0^t \int_0^t \int_0^t dt_1 dt_2 dt_3 \Theta_{123}(\Theta_{12}[A_1, A_2]A_3 + \Theta_{23}[A_2, A_3]A_1 \cdots) \]
\[ = \frac{1}{2} \int_0^t \int_0^t \int_0^t dt_1 dt_2 dt_3 \Theta_{123}([A_1, A_2]A_3 + [A_2, A_3]A_1 + [A_1, A_3]A_2) \] (B.8)
where the half of terms in the summation are dropped due to the product of two step functions. The \( \Omega_2 \Omega_1 \) can be represented similarly. Then, we have
\[ \frac{1}{2}(\Omega_1 \Omega_2 + \Omega_2 \Omega_1) \]
\[ = \frac{1}{4} \int_0^t \int_0^t \int_0^t dt_1 dt_2 dt_3 \Theta_{123}([A_1, A_2], A_3) + ([A_2, A_3], A_1) + ([A_1, A_3], A_2) \]
\[ = \frac{1}{2} \int_0^t \int_0^t \int_0^t dt_1 dt_2 dt_3 \Theta_{123}(A_1 A_2 A_3 - A_3 A_2 A_1), \] (B.9)
where
\[ [A_1, A_2], A_3] + [A_2, A_3], A_1] + [A_1, A_3], A_2] = A_1 A_2 A_3 - A_3 A_2 A_1. \] (B.10)

(B.4) is now obvious.

C Time-ordered BCH formula

Here we show the identity used in (7.20)
\[ T e^{\int_0^t (A(t) + B(t)) dt} = T e^{\int_0^t A(t) dt} T e^{\int_0^t B^{(A)}(t) dt}, \quad B^{(A)}(t) = T e^{-\int_0^t A(s) ds} B(t) T e^{\int_0^t A(s) ds}, \] (C.1)
which is a time-ordered version of the BCH formula.

For convenience, we denote \( T_M(t) \) as
\[ T_M(t) = T e^{\int_0^t ds M(s)}, \] (C.2)
for a time-dependent operator \( M(s) \). This is a formal solution of the differential equation
\[ \frac{d}{dt} T_M(t) = M(t) T_M(t), \] (C.3)
with \( T_M(0) = 1 \). With this notation, the formula (C.1) is expressed as
\[ T_{A+B}(1) = T_A(1) T_B^{(A)}(1), \] (C.4)
and we prove this below. First, by using the differential equation (C.3), we have

\[ \frac{d}{dt} \left[ T_A^{-1}(t)T_{A+B}(t) \right] = \left[ -T_A^{-1}(t)A(t) \right] T_{A+B}(t) + T_A^{-1}(t) \left[ (A(t) + B(t))T_{A+B}(t) \right] = T_A^{-1}(t)B(t)T_{A+B}(t) \]

\[ = \left[ T_A^{-1}(t)B(t)T_A(t) \right] T_A^{-1}(t)T_{A+B}(t) = B^{(A)}(t) \left[ T_A^{-1}(t)T_{A+B}(t) \right]. \] (C.5)

Since this is again the form of (C.3), we conclude

\[ T_A^{-1}(t)T_{A+B}(t) = T_{B^{(A)}}(t), \] (C.9)

and this gives (C.4) if we put \( t = 1 \).

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