ORIENTABLE 4-DIMENSIONAL POINCARÉ COMPLEXES HAVE REDUCIBLE SPIVAK FIBRATIONS

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Abstract. We show that the Spivak normal fibration of an orientable 4-dimensional Poincaré complex has a vector bundle reduction.

1. Introduction

A Poincaré complex (PD-complex), as introduced by Wall [10, p. 214], is a (connected) finitely dominated CW complex $X$ equipped with:

(i) a homomorphism $w : \pi_1(X) \to \{\pm 1\}$ defining a twisted $\Lambda := \mathbb{Z}\pi_1(X)$ module structure $\mathbb{Z}^\ell$ on $\mathbb{Z}$.

(ii) an integer $n$ and a class $[X] \in H_n(X; \mathbb{Z}^\ell)$ such that

(iii) for all integers $r \geq 0$, cap product with $[X]$ induces an isomorphism $[X] \cap : H^r(X; \Lambda) \to H_{n-r}(X; \Lambda \otimes \mathbb{Z}^\ell)$.

The integer $n = \dim X$ is called the dimension of $X$. It follows from the foundational results of Kirby and Siebenmann [5, Annex 3] that every closed topological $n$-manifold has the homotopy type of a Poincaré complex of dimension $n$ (see the discussion in Wall [11, §17B]). In the manifold case, the homomorphism $w : \pi_1(X) \to \{\pm 1\}$ is given by the first Stiefel-Whitney class. Accordingly, a PD-complex $X$ is called orientable if its homomorphism $w$ is trivial.

Spivak [9] discovered that every simply-connected PD-complex $X$ with $\dim X = n$ has an associated spherical fibration, denoted $\nu_X$, which is unique up to stable fibre homotopy equivalence. It is constructed by embedding $X$ in a high-dimensional Euclidean space $\mathbb{R}^{n+k}$ ($k \gg n$), and considering the fibration homotopic to the projection map $p : \partial N \to X$ from the boundary of a regular neighbourhood $N \subset \mathbb{R}^{n+k}$. The duality properties of $X$ imply that the fibres of $p$ are homotopy equivalent to $S^{k-1}$. The definition and the uniqueness statement were generalized by Wall [10, §3] to all PD-complexes, and $\nu_X$ is now called the Spivak normal fibration of $X$.

In the smooth manifold case, $\nu_X$ is the spherical fibration associated to the sphere bundle of the (stable) normal $k$-vector bundle of $X$. For topological manifolds, the corresponding notion is the (stable) normal $\mathbb{R}^k$-bundle ($k \gg n$), and its sub-bundle with fibres $\mathbb{R}^k - \{0\} \cong S^{k-1}$.

After the further development of geometric surgery theory, due to Browder, Milnor, Novikov, Sullivan and Wall, the normal structures on PD-spaces and manifolds were re-expressed via classifying spaces (see [11, §10 and §17B], [3], [8], [4]). One outcome was
the construction of a sequence of classifying spaces

\[ BO \to BPL \to BTOP \to BG \]

relating smooth, PL, and topological bundles to spherical fibrations. In particular, the (stable) Spivak normal fibre space \( \nu_X \) is classified by a map \( \nu_X : X \to BG \).

**Definition 1.1.** We say that PD-complex \( X \) has a reducible Spivak normal fibration if the classifying map \( \nu_X : X \to BG \) lifts to a map \( \tilde{\nu}_X : X \to BTOP \).

Similarly, we say that the Spivak normal fibre space is reducible to a vector bundle if \( \nu_X \) lifts to a map \( \tilde{\nu}_X : X \to BO \). The lifting obstruction is given by the image of \( \nu_X \) in \( [X, B(G/TOP)] \) or \( [X, B(G/O)] \), respectively. In dimensions \( \geq 5 \), these are different obstruction groups, but if \( \dim X \leq 4 \) these two obstruction groups are the same because

\[ [X, B(G/O)] = [X, B(G/PL)] = [X, B(G/O)] \cong H^3(X; \mathbb{Z}/2), \quad \text{if } \dim X \leq 4. \]

This is explained in Kirby-Taylor [6, §2]. In other words, the obstruction to reducibility for the Spivak normal fibration of a PD-complex \( X \) in dimensions \( \leq 4 \) is a single characteristic class \( k_3(X) \in H^3(X; \mathbb{Z}/2) \).

**Theorem A.** Let \( X \) be an Poincaré complex. If \( \dim X \leq 3 \), or \( \dim X = 4 \) and \( X \) is orientable, then the Spivak normal fibration of \( X \) is reducible to a vector bundle.

**Remark 1.2.** The dimension 4 case was known to the experts (see the statements in Spivak [9, p. 95] and Kirby-Taylor [6, p. 10]), but Land [7] pointed out the lack of a proof in the literature, and provided his own argument. For dimensions \( \leq 2 \) the result is immediate, and the dimension 3 cases follow easily from the dimension 4 statement. In general, non-oriented PD-complexes in dimensions \( \geq 4 \) do not have reducible Spivak normal fibrations (see Hambleton and Milgram [4] for explicit examples in every even dimension \( \geq 4 \)). The first non-reducible orientable example occurs in dimension 5 (see Gitler and Stasheff [3]).

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## 2. The proof of Theorem A

Here is a short argument to show that an orientable 4-dimensional Poincaré complex has a reducible Spivak normal fibration. The proof is essentially contained in [4].

1. Suppose that \( X \) is an orientable 4-dimensional PD-complex such that \( \nu_X \) is not reducible. Then by Poincaré duality there is a class \( e \in H^1(X; \mathbb{Z}/2) \) such that

\[ \langle k_3(X) \cup e, [X] \rangle \neq 0, \]

where \( k_3(X) \) denotes the pullback to \( X \) of the first exotic characteristic class.

2. Let \( f : X \to RP^\infty \) represent the cohomology class \( e \in H^1(X; \mathbb{Z}/2) \). Then the element \( 0 \neq (X, f) \in \mathcal{A}^{PD}_4(RP^\infty) \) has Arf invariant \( A(X, f) = 1 \) (see [4], Corollary 4.2, Corollary 5.3, and Theorem 5.6).
3. By low-dimensional surgery, we may assume that $\pi_1(X) = \mathbb{Z}/2$ and that $f : X \to \mathbb{R}P^\infty$ classifies its universal covering $\tilde{X} \to X$ (see Wall [10, Corollary 2.3.2] to justify this much Poincaré surgery).

4. The form $B(a, b) = \langle a \cup T^*b, [X] \rangle$ is a symmetric unimodular bilinear form on $H^2(\tilde{X}, \mathbb{Z})$, where $T$ denotes the non-trivial covering involution. The form $B$ is even (see Bredon [1, Chap VII, Theorem 7.4]).

5. The invariant $A(X, f)$ is the Arf invariant associated to the Browder-Livesay quadratic map $q$ (see [2, §4], and [11, Theorem 1.4]), which refines the mod 2 reductions of $B$. By [2, Lemma 4.6], we have

$$q(a) \equiv \frac{B(a, a)}{2} \pmod{2}$$

since $T : \tilde{X} \to \tilde{X}$ is orientation preserving. But $B$ is an even unimodular symmetric bilinear form, so the Arf invariant obtained in this way is zero, and we have a contradiction. □

Remark 2.1. To obtain the reducibility results for 3-dimensional PD-complexes, one can make an appropriate circle bundle construction (which does not affect reducibility) resulting in orientable 4-dimensional PD-complexes, and then apply Theorem A.

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