KOSTANT’S GENERATING FUNCTIONS, EBELING’S THEOREM AND MCKAY’S OBSERVATION RELATING THE POINCARÉ SERIES

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Abstract. We generalize B. Kostant’s construction of generating functions to the case of multiply-laced diagrams and we prove for this case W. Ebeling’s theorem which connects the Poincaré series $[P_G(t)]_0$ and the Coxeter transformations. According to W. Ebeling’s theorem

$$[P_G(t)]_0 = \frac{X(t^2)}{X(t^2)},$$

where $X$ is the characteristic polynomial of the Coxeter transformation and $\tilde{X}$ is the characteristic polynomial of the corresponding affine Coxeter transformation.

We prove McKay’s observation relating the Poincaré series $[P_G(t)]_i$:

$$(t + t^{-1})[P_G(t)]_i = \sum_{i \rightarrow j} [P_G(t)]_j,$$

where $j$ runs over all vertices adjacent to $i$.

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1. Introduction

B. Kostant’s construction of a vector-valued generating function $P_G(t)$ appears in the context of the McKay correspondence [McK80] and gives a way to obtain multiplicities of indecomposable representations $\rho_i$ of the binary polyhedral group $G$ in the decomposition of $\pi_n|_G$, [Kos84].

Using B. Kostant’s construction we generalize to the case of multiply-laced diagrams W. Ebeling’s theorem which connects the Poincaré series $[P_G(t)]_0$ and the Coxeter transformations. According to W. Ebeling’s theorem [Ebl02]

$$[P_G(t)]_0 = \frac{X(t^2)}{X(t^2)},$$

where $X$ is the characteristic polynomial of the Coxeter transformation and $\tilde{X}$ is the characteristic polynomial of the corresponding affine Coxeter transformation.

We prove McKay’s observation [McK99] relating the Poincaré series $[P_G(t)]_i$:

$$(t + t^{-1})[P_G(t)]_i = \sum_{i \prec j} [P_G(t)]_j,$$

where $j$ runs over all vertices adjacent to $i$.

1.1. The McKay correspondence. Let $G$ be a finite subgroup of $SU(2)$, $\{\rho_0, \rho_1, \ldots, \rho_n\}$ be the set of all distinct irreducible finite dimensional complex representations of $G$, of which $\rho_0$ is the trivial one. Let $\rho : G \rightarrow SU(2)$ be a faithful representation. Then, for each group $G$, we can define a matrix $A(G) = (a_{ij})$, by decomposing the tensor products:

$$\rho \otimes \rho_j = \bigoplus_{k=0}^{r} a_{jk} \rho_k, \quad j = 0, 1, \ldots, r, \quad (1.1)$$

where $a_{jk}$ is the multiplicity of $\rho_k$ in $\rho \otimes \rho_j$. McKay [McK80] observed that

The matrix $2I - A(G)$ is the Cartan matrix of the extended Dynkin diagram $\tilde{\Delta}(G)$ associated to $G$. There is one-to-one correspondence between finite subgroups of $SU(2)$ and simply-laced extended Dynkin diagrams.

This remarkable observation, called the McKay correspondence, was based first on an explicit verification [McK80].

For the multiply-laced case, the McKay correspondence was extended by D. Happel, U. Preiser, and C. M. Ringel in [HPR80], and by P. Slodowy in [Sl80, App.III]. We consider P. Slodowy’s approach in [St85].

The systematic proof of the McKay correspondence based on the study of affine Coxeter transformations was given by R. Steinberg in [St85].

Other proofs of the McKay correspondence were given by G. Gonzalez-Sprinberg and J.-L. Verdier in [GS83], by H. Knörrer in [Kn85]. A nice review is given by J. van Hoboken in [Hob02].

1.2. The Slodowy generalization of the McKay correspondence. Let us fix a pair $H \triangleleft G$ from Table 1.1. We formulate now the essence of the Slodowy correspondence [Sl80, App.III].

1) Let $\rho_i$, where $i = 1, \ldots, n$, be irreducible representations of $G$; let $\rho_i^H$ be the corresponding restricted representations of the subgroup $H$. Let $\rho$ be a faithful representation of $H$,
### Table 1.1. The pairs $H \triangleleft G$ of binary polyhedral groups

| Subgroup $H$ | Dynkin diagram $\Gamma(H)$ | Group $R$ | Dynkin diagram $\Gamma(G)$ | Index $[G : H]$ |
|-------------|-----------------------------|--------|-----------------------------|-----------------|
| $D_2$       | $D_4$                       | $T$    | $E_6$                       | 3               |
| $T$         | $E_6$                       | $O$    | $E_7$                       | 2               |
| $D_{r-1}$   | $D_{n+1}$                   | $D_{2(r-1)}$ | $D_{2n}$                 | 2               |
| $\mathbb{Z}/2r\mathbb{Z}$ | $A_{n-1}$ | $D_r$ | $D_{r+2}$ | 2 |

which may be considered as the restriction of the fixed faithful representation $\rho_f$ of $G$. Then the following decomposition formula makes sense

$$\rho \otimes \rho_i^\dagger = \bigoplus a_{ji} \rho_j^\dagger$$

(1.2)

and uniquely determines an $n \times n$ matrix $\tilde{A} = (a_{ij})$ such that

$$K = 2I - \tilde{A}$$

(1.3)

(see [Sl80, p.163]), where $K$ is the Cartan matrix of the corresponding folded extended Dynkin diagram given in Table 1.2.

2) Let $\tau_i$, where $i = 1, \ldots, n$, be irreducible representations of the subgroup $H$, let $\tau_i^\dagger$ be the induced representations of the group $G$. Then the following decomposition formula makes sense

$$\rho \otimes \tau_i^\dagger = \bigoplus a_{ij} \tau_j^\dagger,$$

(1.4)

i.e., the decomposition of the induced representation is described by the matrix $A^\vee = A^t$ which satisfies the relation

$$K^\vee = 2I - \tilde{A}^\vee$$

(1.5)

(see [Sl80, p.164]), where $K^\vee$ is the Cartan matrix of the dual folded extended Dynkin diagram given in Table 1.2.

We call matrices $\tilde{A}$ and $\tilde{A}^\vee$ the Slodowy matrices, they are analogs of the McKay matrix. The Slodowy correspondence is an analogue to the McKay correspondence for the multiply-laced case, so one can speak about the McKay-Slodowy correspondence.
### Table 1.2. The pairs $H \triangleleft G$ and folded extended Dynkin diagrams

| Groups $H \triangleleft G$ | Dynkin diagram $\tilde{\Gamma}(H)$ and $\tilde{\Gamma}(G)$ | Folded extended Dynkin diagram $\tilde{\Gamma}(H)\tilde{\Gamma}(G)$ |
|---------------------------|--------------------------------------------------|--------------------------------------------------|
| $D_2 \triangleleft T$    | $D_4$ and $E_6$                                   | $\tilde{G}_{21}$ and $\tilde{G}_{22}$             |
| $T \triangleleft O$      | $E_6$ and $E_7$                                   | $\tilde{F}_{41}$ and $\tilde{G}_{42}$             |
| $D_{r-1} \triangleleft D_{2(r-1)}$ | $D_{n+1}$ and $D_{2n}$                        | $\tilde{D}D_n$ and $\tilde{C}D_n$            |
| $\mathbb{Z}/2r\mathbb{Z} \triangleleft D_r$ | $A_{n-1}$ and $D_{r+2}$                            | $\tilde{B}_n$ and $\tilde{C}_n$                 |

1.3. The Kostant generating function and Poincaré series. Let $\text{Sym}(\mathbb{C}^2)$ be the symmetric algebra over $\mathbb{C}^2$, in other words, $\text{Sym}(\mathbb{C}^2) = \mathbb{C}[x_1, x_2]$. The symmetric algebra $\text{Sym}(\mathbb{C}^2)$ is a graded $\mathbb{C}$-algebra, see [Sp77], [Ben93]:

$$\text{Sym}(\mathbb{C}^2) = \bigoplus_{n=0}^{\infty} \text{Sym}^n(\mathbb{C}^2).$$ (1.6)

Let $\pi_n$ be the representation of $SU(2)$ in $\text{Sym}^n(\mathbb{C}^2)$ induced by its action on $\mathbb{C}^2$. The set $\{\pi_n\}$, where $n = 0, 1, ...$ is the set of all irreducible representations of $SU(2)$, see, e.g., [Zhe73] §37. Let $G$ be any finite subgroup of $SU(2)$. B. Kostant in [Kos84] considered the question:

**how does $\pi_n|G$ decompose for any $n \in \mathbb{N}$?**

The answer — the decomposition $\pi_n|G$ — is as follows:

$$\pi_n|G = \sum_{i=0}^{r} m_i(n) \rho_i,$$ (1.7)

where $\rho_i$ are irreducible representations of $G$, considered in the context of McKay correspondence, see [Kos84]). Thus, the decomposition (1.7) reduces the question to the following one:

**what are the multiplicities $m_i(n)$ equal to?**

B. Kostant in [Kos84] obtained the multiplicities $m_i(n)$ by means the orbit structure of the Coxeter transformation on the highest root of the corresponding Lie algebra. For further details concerning this orbit structure and the multiplicities $m_i(n)$, see [3].
Note, that multiplicities $m_i(n)$ in (1.7) are calculated as follows:

$$m_i(n) = \langle \pi_n | G, \rho_i \rangle,$$

(for the definition of the inner product $\langle \cdot, \cdot \rangle$, see (1.21)).

**Remark 1.3.1.** For further considerations, we extend the relation for multiplicity (1.8) to the cases of restricted representations $\rho_i^\updownarrow := \rho_i \downarrow_H^G$ and induced representations $\rho_i^\uparrow := \rho_i \uparrow_H^G$, where $H$ is any subgroup of $G$ (see [St05, Section A.5.1]):

$$m_i^\updownarrow(n) = \langle \pi_n | H, \rho_i^\updownarrow \rangle, \quad m_i^\uparrow(n) = \langle \pi_n | G, \rho_i^\uparrow \rangle.$$  (1.9)

We do not have any decomposition like (1.7) neither for restricted representations $\rho_i^\updownarrow$ nor for induced representations $\rho_i^\uparrow$. Nevertheless, we will sometimes denote both multiplicities $m_i^\updownarrow(n)$ and $m_i^\uparrow(n)$ in (1.9) by $m_i(n)$ as in (1.7).

**Remark 1.3.2.** 1) A representation $\rho : G \to GL_k(V)$ defines a $k$-linear action $G$ on $V$ by

$$gv = \rho(g)v.$$  (1.10)

The pair $(V, \rho)$ is called a $G$-module. The case where $\rho(g) = Id_V$ is called the trivial representation in $V$. In this case

$$gv = v \text{ for all } g \in V.$$  (1.11)

In (1.7), the trivial representation $\rho_0$ corresponds to a particular vertex (see McK80), which extends the Dynkin diagram to the extended Dynkin diagram.

2) Let $\rho_0(H)$ (resp. $\rho_0(G)$) be the trivial representation of any subgroup $H \subset G$ (resp. of group $G$). The trivial representation $\rho_0(H) : H \to GL_k(V)$ coincides with the restricted representation $\rho_0 \downarrow_H^G : G \to GL_k(V)$, and the trivial representation $\rho_0(G) : G \to GL_k(V)$ coincides with the induced representation $\rho_0 \uparrow_H^G : H \to GL_k(V)$.

Since there is one-to-one correspondence between the $\rho_i$ and the vertices of the Dynkin diagram, we can define (see Kos84 p.211) the vectors $v_n$, where $n \in \mathbb{Z}_+$, as follows:

$$v_n = \sum_{i=0}^{r} m_i(n)\alpha_i, \text{ where } \pi_n|G = \sum_{i=0}^{r} m_i(n)\rho_i,$$  (1.12)

where $\alpha_i$ are simple roots of the corresponding extended Dynkin diagram. Similarly, for the multiply-laced case, we define vectors $v_n$ to be:

$$v_n = \sum_{i=0}^{r} m_i^\updownarrow(n)\alpha_i \quad \text{or} \quad v_n = \sum_{i=0}^{r} m_i^\uparrow(n)\alpha_i,$$  (1.13)

where the multiplicities $m_i^\updownarrow(n)$ and $m_i^\uparrow(n)$ are defined by (1.9). The vector $v_n$ belongs to the root lattice generated by simple roots. Following B. Kostant, we define the generating function $P_G(t)$ for cases (1.12) and (1.13) as follows:

$$P_G(t) = ([P_G(t)]_0, [P_G(t)]_1, \ldots, [P_G(t)]_r)^t := \sum_{n=0}^{\infty} v_n t^n,$$  (1.14)

the components of the vector $P_G(t)$ being the following series

$$[P_G(t)]_i = \sum_{n=0}^{\infty} \tilde{m}_i(n)t^n,$$  (1.15)
where $i = 0, 1, \ldots, r$ and $\tilde{m}_i(n)$ designates $m_i(n), m_i^+(n)$ or $m_i^-(n)$. In particular, for $i = 0$, we have

$$[P_G(t)]_0 = \sum_{n=0}^{\infty} m_0(n)t^n, \quad (1.16)$$

where $m_0(n)$ is the multiplicity of the trivial representation $\rho_0$ (Remark 1.3.2) in $\text{Sym}^n(\mathbb{C}^2)$. The algebra of invariants $R^G$ is a subalgebra of the symmetric algebra $\text{Sym}(\mathbb{C}^2)$. Thanks to (1.11), we see that $R^G$ coincides with $\text{Sym}(\mathbb{C}^2)$, and $[P_G(t)]_0$ is the Poincaré series of the algebra of invariants $\text{Sym}(\mathbb{C}^2)^G$, i.e.,

$$[P_G(t)]_0 = P(\text{Sym}(\mathbb{C}^2)^G, t). \quad (1.17)$$

(see [Kos84] p.221, Rem.3.2].)

Remark 1.3.3. According to Remark 1.3.2 heading 2), we have

$$[P_H(t)]_0 = P(\text{Sym}(\mathbb{C}^2)^{\rho_0 \otimes \bar{G}}, t), \quad [P_G(t)]_0 = P(\text{Sym}(\mathbb{C}^2)^{\rho_1 \otimes \bar{G}}, t). \quad (1.18)$$

We will need this fact in the proof of W. Ebeling’s theorem for the multiply-laced case in [2] see Remark 1.3.1.

The following theorem gives a remarkable formula for calculating the Poincaré series for the binary polyhedral groups. The theorem is known in different forms, see Kostant [Kos84], Knörrer [Kn85], Gonsales-Sprinberg, Verdier [GV83]. B. Kostant in [Kos84] shows it in the context of the Coxeter number $h$.

Theorem 1.3.4. The Poincaré series $[P_G(t)]_0$ can be calculated as the following rational function:

$$[P_G(t)]_0 = \frac{1 + t^h}{(1 - t^a)(1 - t^b)}, \quad (1.19)$$

where

$$b = h + 2 - a, \text{ and } ab = 2|G|. \quad (1.20)$$

For a proof, see Theorem 1.4 and Theorem 1.8 from [Kos84], [Kn85] p.185], [GV83] p.428].

We call the numbers $a$ and $b$ the Kostant numbers. They can be easily calculated, see Table 1.3, compare also with [St05] Table A.2]. Note that $a = 2d$, where $d$ is the maximal coordinate of the nil-root vector from the kernel of the Tits form, [Kac80].

1.4. The characters and the McKay operator. Let $\chi_1, \chi_2, \ldots, \chi_r$ be all irreducible $\mathbb{C}$-characters of a finite group $G$ corresponding to irreducible representations $\rho_1, \rho_2, \ldots, \rho_r$, and let $\chi_1$ correspond to the trivial representation, i.e., $\chi_1(g) = 1$ for all $g \in G$.

All characters constitute the character algebra $C(G)$ of $G$ since $C(G)$ is also a vector space over $\mathbb{C}$. An hermitian inner product $< \cdot, \cdot >$ on $C(G)$ is defined as follows. For characters $\alpha, \beta \in C(G)$, let

$$< \alpha, \beta > = \frac{1}{|G|} \sum_{g \in G} \alpha(g)\overline{\beta(g)} \quad (1.21)$$

Sometimes, we will write inner product $< \rho_i, \rho_j >$ of the representations meaning actually the inner product of the corresponding characters $< \chi_{\rho_i}, \chi_{\rho_j} >$. Let $z_{ijk} = < \chi_i, \chi_j, \chi_k >$, where $\chi_i \chi_j$ corresponds to the representation $\rho_i \otimes \rho_j$. It is known that $z_{ijk}$ is the multiplicity of the representation $\rho_k$ in $\rho_i \otimes \rho_j$ and $z_{ijk} = z_{jik}$. The numbers $z_{ijk}$ are integer and are called the structure constants, see, e.g., [Kar92] p.765.

For every $i \in \{1, \ldots, r\}$, there exists some $\hat{i} \in \{1, \ldots, r\}$ such that

$$\chi_{\hat{i}}(g) = \chi_i(g) \text{ for all } g \in G. \quad (1.22)$$
Table 1.3. The binary polyhedral groups (BPG) and the Kostant numbers $a, b$

| Dynkin diagram | Order of BPG | Coxeter number | $a$ | $b$ |
|----------------|--------------|----------------|-----|-----|
| $A_{n-1}$      | $n$          | $\mathbb{Z}/n\mathbb{Z}$ | $n$ | $n$ |
| $D_{n+2}$      | $4n$         | $D_n$          | $2n+2$ | $2n$ |
| $E_6$          | 24           | $\mathcal{T}$  | 12  | 6   | 8   |
| $E_7$          | 48           | $\mathcal{O}$  | 18  | 8   | 12  |
| $E_8$          | 120          | $\mathcal{J}$  | 30  | 12  | 20  |

The character $\chi_i$ corresponds to the contragredient representation $\rho_i$ determined from the relation

$$\rho_i(g) = \rho_i(g)^{-1}.$$  \hspace{2cm} (1.23)

We have

$$< \chi_i \chi_j, \chi_k > = < \chi_i, \chi_j^i \chi_k >$$ \hspace{2cm} (1.24)

since

$$< \chi_i \chi_j, \chi_k > = \frac{1}{|G|} \sum_{g \in G} \chi_i(g) \chi_j(g) \overline{\chi_k(g)} =$$

$$\frac{1}{|G|} \sum_{g \in G} \chi_i(g) \chi_j(g) \overline{\chi_k(g)} = < \chi_i, \chi_j \chi_k > .$$

Remark 1.4.1. The group $SU(2)$ is the set of all unitary unimodular $2 \times 2$ matrices $u$, i.e.,

$$u^* = u^{-1} \text{ (unitarity)},$$

$$\text{det}(u) = 1 \text{ (unimodularity)} .$$
The matrices \( u \in SU(2) \) have the following form:
\[
\begin{pmatrix}
a & b \\
-b^* & a^*
\end{pmatrix}, \quad \text{and} \quad u^* = \begin{pmatrix} a^* & -b \\ b^* & a \end{pmatrix}, \quad \text{where} \quad aa^* + bb^* = 1,
\]
(1.25)
see, e.g., [Ha89] Ch.9, §6. The mutually inverse matrices \( u \) and \( u^{-1} \) are
\[
u = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{and} \quad u^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \quad \text{where} \quad ad - bc = 1.
\]
(1.26)
Set
\[
s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \text{then} \quad s^{-1} = s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]
(1.27)
For any \( u \in SU(2) \), we have
\[
su s^{-1} = u^{'-1}.
\]
(1.28)
The element \( s \) is called the Weyl element.

According to (1.23) and (1.28) we see that every finite dimensional representation of the group \( SL(2, \mathbb{C}) \) (and hence, of \( SU(2) \)) is equivalent to its contragredient representation, see [Zhe73] §37, Rem.3. Thus by (1.24), for representations \( \rho \) of any finite subgroup \( G \subset SU(2) \), we have
\[
< \chi_i \chi_j, \chi_k > = < \chi_i, \chi_j \chi_k >.
\]
(1.29)
Relation (1.29) holds also for characters of restricted representations \( \chi^\downarrow := \chi \downarrow^G_H \) and induced representations \( \chi^\uparrow := \chi \uparrow^G_H \), see Remark 1.3.1:

\[
< \chi_i \chi^\downarrow_j, \chi_k >_H = < \chi_i, \chi^\downarrow_j \chi_k >_H,
\]
\[
< \chi_i \chi^\uparrow_j, \chi_k >_G = < \chi_i, \chi^\downarrow_j \chi_k >_G.
\]
(1.30)
Indeed, every restricted representation \( \chi^\downarrow_j \) (resp. induced representation \( \chi^\uparrow_j \)) is decomposed into the direct sum of irreducible characters \( \chi_s \in \text{Irr}(H) \) (here, \( \text{Irr}(H) \) is the set of irreducible characters of \( H \)) with some integer coefficients \( a_s \), for example:
\[
\chi^\downarrow_j = \sum_{\chi_s \in \text{Irr}(H)} a_s \chi_s,
\]
(1.31)
and since \( \overline{a_s} = a_s \) for all \( \chi_s \in \text{Irr}(H) \), we have
\[
< \chi_i \chi^\downarrow_j, \chi_k >_H = \sum_{\chi_s \in \text{Irr}(H)} < a_s \chi_i \chi_s, \chi_k >_H = \sum_{\chi_s \in \text{Irr}(H)} a_s < \chi_i, \chi_s \chi_k >_H = \sum_{\chi_s \in \text{Irr}(H)} a_s \chi_s \chi_k >_H = < \chi_i, \sum_{\chi_s \in \text{Irr}(H)} a_s \chi_s \chi_k >_H = < \chi_i, \chi^\downarrow_j \chi_k >_H.
\]
(1.32)

The matrix of multiplicities \( A := A(G) \) from (1.1) was introduced by J. McKay in [McK80]; it plays the central role in the McKay correspondence, see [McK80]. We call this matrix — or the corresponding operator — the McKay matrix or the McKay operator.

Similarly, let \( \tilde{A} \) and \( \tilde{A}' \) be matrices of multiplicities (1.2), (1.4). These matrices were introduced by P. Slodowy [Sl80] by analogy with the McKay matrix for the multiply-laced case. We call these matrices the Slodowy operators.

The following result of B. Kostant [Kos84], which holds for the McKay operator holds also for the Slodowy operators.
Proposition 1.4.2. If $B$ is either the McKay operator $A$ or the Slodowy operator $\tilde{A}$ or $\tilde{A}^{\vee}$, then
\[ Bv_n = v_{n-1} + v_{n+1}. \] (1.33)

Proof. From now on
\[ \rho_i = \begin{cases} \rho_i & \text{for } B = A, \\ \rho_i^{\downarrow} & \text{for } B = \tilde{A}, \\ \rho_i^{\uparrow} & \text{for } B = \tilde{A}^{\vee}, \end{cases} \quad m_i(n) = \begin{cases} m_i(n) & \text{for } B = A, \\ m_i^{\downarrow}(n) & \text{for } B = \tilde{A}, \\ m_i^{\uparrow}(n) & \text{for } B = \tilde{A}^{\vee}. \end{cases} \] (1.34)

By (1.12), (1.13), and by definition of the McKay operator (1.1) and by definition of the Slodowy operator (1.2), (1.4), we have
\[ Bv_n = B \left( \begin{array}{c} m_0(n) \\ \vdots \\ m_r(n) \end{array} \right) = \left( \begin{array}{c} \sum a_0 m_i(n) \\ \vdots \\ \sum a_r m_i(n) \end{array} \right) = \left( \begin{array}{c} \sum a_0 < \rho_i, \pi_n > \\ \vdots \\ \sum a_r < \rho_i, \pi_n > \end{array} \right). \] (1.35)

By (1.29), (1.32) we obtain
\[ Bv_n = \left( \begin{array}{c} < \rho_0, \pi_0, \pi_n > \\ \vdots \\ < \rho_r, \pi_0, \pi_n > \end{array} \right). \] (1.36)

By Clebsch-Gordan formula we have
\[ \pi_1 \otimes \pi_n = \pi_{n-1} \oplus \pi_{n+1}, \] (1.39)

where $\pi_{-1}$ is the zero representation, see [Sp77, exs. 3.2.4] or [Ha89, Ch. 5, §6, §7]. From (1.38) and (1.39) we have (1.35). \[ \square \]

For the following corollary, see [Kos84, p. 222] and also [Sp77, §4.1].

Corollary 1.4.3. Let $x = P_G(t)$ be given by (1.14). Then
\[ tBx = (1 + t^2)x - v_0, \] (1.40)

where $B$ is either the McKay operator $A$ or the Slodowy operators $\tilde{A}$, $\tilde{A}^{\vee}$. 
Proof. From (1.33) we obtain
\[
Bx = \sum_{n=0}^{\infty} Bv_n t^n = \sum_{n=0}^{\infty} (v_{n-1} + v_{n+1}) t^n = \sum_{n=0}^{\infty} v_{n-1} t^n + \sum_{n=0}^{\infty} v_{n+1} t^n = \\
\sum_{n=1}^{\infty} v_{n-1} t^{n-1} + t^{-1} \sum_{n=0}^{\infty} v_{n+1} t^{n+1} = tx + t^{-1} \left( \sum_{n=0}^{\infty} v_n t^n - v_0 \right) = \\
tx + t^{-1} x - t^{-1} v_0. \quad \square
\]

2. The Poincaré series and W. Ebeling’s theorem

W. Ebeling in [Ebl02] makes use of the Kostant relation (1.33) and deduces a new remarkable fact about the Poincaré series, a fact that shows that the Poincaré series of a binary polyhedral group (see (1.19)) is the quotient of two polynomials: the characteristic polynomial of the Coxeter transformation and the characteristic polynomial of the corresponding affine Coxeter transformation, see [Ebl02, Th.2].

We show W. Ebeling’s theorem also for the multiply-laced case, see Theorem 2.0.1. The Poincaré series for the multiply-laced case is defined by (1.18).

**Theorem 2.0.1** (generalized W.Ebeling’s theorem [Ebl02]). Let \(G\) be a binary polyhedral group and \([P_G(t)]_0\) the Poincaré series (1.17) of the algebra of invariants \(\text{Sym}(\mathbb{C}^2)^G\). Then
\[
[P_G(t)]_0 = \frac{\det M_0(t)}{\det M(t)},
\]
where
\[
\det M(t) = \det \left| t^2 I - C_a \right|, \quad \det M_0(t) = \det \left| t^2 I - C \right|,
\]
\(C\) is the Coxeter transformation and \(C_a\) is the corresponding affine Coxeter transformation.

**Proof.** By (1.40) we have
\[
[(1 + t^2) I - tB] x = v_0, \quad (2.3)
\]
where \(x\) is the vector \(P_G(t)\) and by Cramer’s rule the first coordinate \(P_G(t)\) is
\[
[P_G(t)]_0 = \frac{\det M_0(t)}{\det M(t)}, \quad (2.4)
\]
where
\[
\det M(t) = \det \left( (1 + t^2) I - tB \right),
\]
and \(M_0(t)\) is the matrix obtained by replacing the first column of \(M(t)\) by \(v_0 = (1, 0, \ldots, 0)^t\). The vector \(v_0\) corresponds to the trivial representation \(\pi_0\), and by the McKay correspondence, \(v_0\) corresponds to the particular vertex which extends the Dynkin diagram to the extended Dynkin diagram, see Remark 1.3.2 and (1.12). Therefore, if \(\det M(t)\) corresponds to the affine Coxeter transformation, and
\[
\det M(t) = \det \left| t^2 I - C_a \right|,
\]
then \(\det M_0(t)\) corresponds to the Coxeter transformation, and
\[
\det M_0(t) = \det \left| t^2 I - C \right|.
\]
So, it suffices to prove (2.6), i.e.,
\[
\det[(1 + t^2) I - tB] = \det \left| t^2 I - C_a \right|.
\]
Table 2.1. The characteristic polynomials $\mathcal{X}$, $\tilde{\mathcal{X}}$ and the Poincaré series

| Dynkin diagram | Coxeter transformation $\mathcal{X}$ | Affine Coxeter transformation $\tilde{\mathcal{X}}$ | Quotient $p(\lambda) = \frac{\mathcal{X}}{\tilde{\mathcal{X}}}$ |
|----------------|--------------------------------------|--------------------------------------------------|--------------------------------------------------|
| $D_4$          | $(\lambda + 1)(\lambda^3 + 1)$       | $(\lambda - 1)^2(\lambda + 1)^3$                | $\frac{\lambda^3 + 1}{(\lambda^2 - 1)^2}$       |
| $D_{n+1}$      | $(\lambda + 1)(\lambda^n + 1)$       | $(\lambda^{n-1} - 1)(\lambda - 1)(\lambda + 1)^2$ | $\frac{\lambda^n + 1}{(\lambda^{n-1} - 1)(\lambda^2 - 1)}$ |
| $E_6$          | $\frac{(\lambda^6 + 1)(\lambda^3 - 1)}{(\lambda^2 + 1)(\lambda - 1)}$ | $(\lambda^3 - 1)^2(\lambda + 1)$                | $\frac{\lambda^6 + 1}{(\lambda^4 - 1)(\lambda^3 - 1)}$ |
| $E_7$          | $\frac{(\lambda + 1)(\lambda^9 + 1)}{(\lambda^5 + 1)(\lambda^3 + 1)}$ | $(\lambda^4 - 1)(\lambda^3 - 1)(\lambda + 1)$ | $\frac{\lambda^9 + 1}{(\lambda^4 - 1)(\lambda^6 - 1)}$ |
| $E_8$          | $\frac{(\lambda^{15} + 1)(\lambda + 1)}{(\lambda^5 + 1)(\lambda^3 + 1)}$ | $(\lambda^5 - 1)(\lambda^3 - 1)(\lambda + 1)$ | $\frac{\lambda^{15} + 1}{(\lambda^{10} - 1)(\lambda^6 - 1)}$ |
| $B_n$          | $\lambda^n + 1$                      | $(\lambda^{n-1} - 1)(\lambda^2 - 1)$            | $\frac{\lambda^n + 1}{(\lambda^{n-1} - 1)(\lambda^2 - 1)}$ |
| $C_n$          | $\lambda^n + 1$                      | $(\lambda^n - 1)(\lambda - 1)$                  | $\frac{\lambda^n + 1}{(\lambda^n - 1)(\lambda - 1)}$ |
| $F_4$          | $\frac{\lambda^6 + 1}{\lambda^2 + 1}$ | $(\lambda^2 - 1)(\lambda^3 - 1)$                | $\frac{\lambda^6 + 1}{(\lambda^4 - 1)(\lambda^3 - 1)}$ |
| $G_2$          | $\frac{\lambda^3 + 1}{\lambda + 1}$ | $(\lambda - 1)^2(\lambda + 1)$                  | $\frac{\lambda^3 + 1}{(\lambda^2 - 1)^2}$        |
| $A_n$          | $\frac{\lambda^{n+1} - 1}{\lambda - 1}$ | $(\lambda^{n-k+1} - 1)(\lambda^k - 1)$          | $\frac{\lambda^{n+1} - 1}{(\lambda - 1)(\lambda^{n-k+1} - 1)(\lambda^k - 1)}$ |
| $A_{2n-1}$     | $\frac{\lambda^{2n} - 1}{\lambda - 1}$ | $(\lambda^n - 1)^2$ for $k = n$                 | $\frac{\lambda^{n} + 1}{(\lambda^n - 1)(\lambda - 1)}$ |
If $B$ is the McKay operator $A$ given by (1.1), then

$$B = 2I - K = \begin{pmatrix} 2I & 0 \\ 0 & 2I \end{pmatrix} - \begin{pmatrix} 2I & 2D \\ 2Dt & 2I \end{pmatrix} = \begin{pmatrix} 0 & -2D \\ -2Dt & 0 \end{pmatrix},$$

where $K$ is a symmetric Cartan matrix \([St05, (3.2)]\). If $B$ is the Slodowy operator $\tilde{A}$ or $\tilde{A}^\vee$ given by (1.2), (1.4), then

$$B = 2I - K = \begin{pmatrix} 2I & 0 \\ 0 & 2I \end{pmatrix} - \begin{pmatrix} 2I & 2D \\ 2Dt & 2I \end{pmatrix} = \begin{pmatrix} 0 & -2D \\ -2Dt & 0 \end{pmatrix},$$

where $K$ is the symmetrizable but not symmetric Cartan matrix \([St05, (3.4)]\). Thus, in the generic case

$$M(t) = (1 + t^2)I - tB = \begin{pmatrix} 1 + t^2 & 2tD \\ 2tF & 1 + t^2 \end{pmatrix}. \tag{2.11}$$

Assuming $t \neq 0$ we deduce from (2.11) that

$$M(t) \begin{pmatrix} x \\ y \end{pmatrix} = 0 \iff \begin{cases} (1 + t^2)x = -2tDy, \\ 2tFx = -(1 + t^2)y. \end{cases} \tag{2.12}$$

Here we use the Jordan form theory of the Coxeter transformations constructed in \([St05, Section 3]\). According to \([St05, 3.14]\), \([St05, Propositions 3.3 and 3.9]\) and we see that $t^2$ is an eigenvalue of the affine Coxeter transformation $C_a$, i.e., (2.8) together with (2.6) are proved. $\Box$

For the results of calculations using W. Ebeling’s theorem, see Table 2.1.

**Remark 2.0.2.** 1) The characteristic polynomials $X$ for the Coxeter transformation and $\tilde{X}$ for the affine Coxeter transformation in Table 2.1 are taken from \([St05, Tables 1.1 and 1.2]\). Pay attention to the fact that the affine Dynkin diagram for $B_n$ is $CD_n$, \([Bo, Tab.2]\), and the affine Dynkin diagram for $C_n$ is $\tilde{C}_n$, \([Bo, Tab.3]\), see \([St05, Fig. 2.2]\).

2) The characteristic polynomial $X$ for the affine Coxeter transformation of $A_n$ depends on the index of the conjugacy class $k$ of the Coxeter transformation, see \([St05, 4.5]\). In the case of $A_n$ (for every $k = 1, 2, ..., n$) the quotient $p(\lambda) = \frac{X}{\tilde{X}}$ contains three factors in the denominator, and its form is different from (1.19), see Table 2.1.

For the case $A_{2n-1}$ and $k = n$, we have

$$p(\lambda) = \frac{\lambda^{2n} - 1}{(\lambda - 1)(\lambda^{2n-k} - 1)(\lambda^k - 1)} = \frac{\lambda^{2n} - 1}{(\lambda - 1)(\lambda^n - 1)(\lambda^n - 1)} = \frac{\lambda^n + 1}{(\lambda^n - 1)(\lambda - 1)} \tag{2.13}$$

and $p(\lambda)$ again is of the form (1.19), see Table 2.1.

3) The quotients $p(\lambda)$ coincide for the following pairs:

- $D_4$ and $G_2$, $E_6$ and $F_4$,
- $D_{n+1}$ and $B_n(n \geq 4)$, $A_{2n-1}$ and $C_n$. \tag{2.14}
Note that the second elements of the pairs are obtained by folding operation from the first ones, see \[\text{St05}, \text{Remark 5.4}\].

3. THE ORBIT STRUCTURE OF THE COXETER TRANSFORMATION

Let $\mathfrak{g}$ be the simple complex Lie algebra of type $A, D$ or $E$, $\tilde{\mathfrak{g}}$ be the affine Kac-Moody Lie algebra associated to $\mathfrak{g}$, and $\mathfrak{h} \subseteq \mathfrak{g}$ be, respectively, Cartan subalgebras of $\mathfrak{g} \subseteq \tilde{\mathfrak{g}}$. Let $\mathfrak{h}^\vee$ (resp. $\tilde{\mathfrak{h}}^\vee$) be the dual space to $\mathfrak{h}$ (resp. $\tilde{\mathfrak{h}}^\vee$) and $\alpha_i \in \mathfrak{h}^\vee, i = 1, \ldots, l$ be an ordered set of simple positive roots. Here, we follow B. Kostant’s description \[\text{Kos84}\] of the orbit structure of the Coxeter transformation $C$ on the highest root in the root system of $\mathfrak{g}$. We consider a bipartite graph and a bicolored Coxeter transformation from \[\text{St05}, \text{Section 3}\]. Let $\beta$ be the highest root of $(\mathfrak{h}, \mathfrak{g})$, see \[\text{St05}, \text{Section 4.1}\]. Then

$$w_2 \beta = \beta \text{ or } w_1 \beta = \beta.$$  

In the second case we just swap $w_1$ and $w_2$, i.e., we always have

$$w_2 \beta = \beta. \quad (3.1)$$

Between two bicolored Coxeter transformations \[\text{St05} (3.1)\] we select such one that

$$C = w_2 w_1.$$  

Consider, for example, the Dynkin diagram $E_6$. Here,

$$w_1 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & -1 & -1 \\ 1 & 0 & 1 & 1 & -1 & -1 \\ 1 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}, \quad w_2 = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix}, \quad w_3 = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

The vector $z \in \mathfrak{h}^\vee$ and the highest root $\beta$ are:

$$z = x_1 y_1 x_0 y_2 x_2 y_3,$$  

$$\beta = \begin{pmatrix} 3 \\ 1 \\ 2 \\ 2 \\ 1 \end{pmatrix}, \quad \text{or} \quad \beta = \begin{pmatrix} 3 \\ 1 \\ 1 \\ 2 \\ 2 \\ 1 \end{pmatrix} \quad (3.3)$$

Further, following B.Kostant \[\text{Kos84} \text{Th.1.5}\] consider the alternating products $\tau^{(n)}$:

$$\tau^{(1)} = w_1,$$  

$$\tau^{(2)} = C = w_2 w_1,$$  

$$\tau^{(3)} = w_1 C = w_1 w_2 w_1,$$  

$$\ldots,$$  

$$\tau^{(n)} = \begin{cases} C^k = w_2 w_1 \ldots w_2 w_1 & \text{for } n = 2k, \\ w_1 C^k = w_1 w_2 w_1 \ldots w_2 w_1 & \text{for } n = 2k + 1, \end{cases} \quad (3.4)$$

and the orbit of the highest root $\beta$ under the action of $\tau^{(n)}$:

$$\beta_n = \tau^{(n)} \beta, \quad \text{where } n = 1, \ldots, h - 1 \quad (3.5)$$

($h$ is the Coxeter number, see \[\text{St05} (4.1)\]).
Theorem 3.0.1. (B. Kostant, [Kos84, Theorems 1.3, 1.4, 1.5, 1.8])

1) There exist \( z_j \in \hat{h}^\vee \), where \( j = 0, \ldots, h \), and even integers \( 2 \leq a \leq b \leq h \) (see (1.20) and Table 1.3) such that the generating functions \( P_G(t) \) (see (1.14), (1.15)) are obtained as follows:

\[
[P_G(t)]_i = \begin{cases} 
  \frac{1 + t^h}{(1 - t^a)(1 - t^b)} & \text{for } i = 0, \\
  \sum_{j=0}^{h} z_j t^j & \text{for } i = 1, \ldots, r.
\end{cases}
\] (3.6)

For \( n = 1, \ldots, h - 1 \), one has \( z_n \in h^\vee \) (not just \( \hat{h}^\vee \)). The indices \( i = 1, \ldots, r \) enumerate the vertices of the Dynkin diagram and the coordinates of the vectors \( z_n \); \( i = 0 \) corresponds to the additional (affine) vertex, the one that extends the Dynkin diagram to the extended Dynkin diagram. One has \( z_0 = z_h = \alpha_0 \), where \( \alpha_0 \in \hat{h}^\vee \) is the added simple root corresponding to the affine vertex.

2) The vectors \( z_n \) (we call these vectors the assembling vectors) are obtained as follows:

\[ z_n = \tau^{(n-1)} \beta - \tau^{(n)} \beta. \] (3.7)

3) We have

\[ z_g = 2\alpha_\ast, \quad \text{where } g = \frac{h}{2}, \] (3.8)

and \( \alpha_\ast \) is the simple root corresponding to the branch point for diagrams \( D_n, E_n \) and to the midpoint for the diagram \( A_{2m-1} \). In all these cases \( h \) is even, and \( g \) is integer. The diagram \( A_{2m} \) has been excluded.

4) The series of assembling vectors \( z_n \) is symmetric:

\[ z_{g+k} = z_{g-k} \text{ for } k = 1, \ldots, g. \] (3.9)

In the case of the Dynkin diagram \( E_6 \), the vectors \( \tau^{(n)} \beta \) are given in Table 3.1, and the assembling vectors \( z_n \) are given in Table 3.2. The vector \( z_6 \) coincides with \( 2\alpha_{x_0} \), where \( \alpha_{x_0} \) is the simple root corresponding to the vertex \( x_0 \), see (3.3). From Table 3.2 we see, that

\[ z_1 = z_{11}, \quad z_2 = z_{10}, \quad z_3 = z_9, \quad z_4 = z_8, \quad z_5 = z_7. \] (3.10)

Denote by \( z(t)_i \) the polynomial \( \sum_{j=0}^{h} z_j t^j \) from (3.6). In the case of \( E_6 \) we have:

\[
\begin{align*}
  z(t)_{x_0} &= t^2 + t^4 + 2t^6 + t^8 + t^{10}, \\
  z(t)_{x_1} &= t^4 + t^8, \\
  z(t)_{x_2} &= t^4 + t^8, \\
  z(t)_{y_1} &= t^3 + t^5 + t^7 + t^9, \\
  z(t)_{y_2} &= t^3 + t^5 + t^7 + t^9, \\
  z(t)_{y_3} &= t + t^5 + t^7 + t^{11}.
\end{align*}
\] (3.11)
Table 3.1. The orbit of the Coxeter transformation on the highest root

| $\beta$ | $\tau^{(1)} \beta = w_1 \beta$ | $\tau^{(2)} \beta = C \beta$ |
|---------|-------------------------------|-------------------------------|
| 1 2 3 2 1 | 1 2 3 2 1 | 1 2 2 2 1 |
| 2       | 1               | 1               |
| $\tau^{(3)} \beta = w_1 C \beta$ | $\tau^{(4)} \beta = C^2 \beta$ | $\tau^{(5)} \beta = w_1 C^2 \beta$ |
| 1 1 2 1 1 | 0 1 1 1 0 | 0 0 1 0 0 |
| 1       | 1               | 0               |
| $\tau^{(6)} \beta = C^3 \beta$ | $\tau^{(7)} \beta = w_1 C^3 \beta$ | $\tau^{(8)} \beta = C^4 \beta$ |
| 0 0 -1 0 0 | 0 -1 -1 -1 0 | -1 -1 -2 -1 -1 |
| 0       | -1             | -1             |
| $\tau^{(9)} \beta = w_1 C^4 \beta$ | $\tau^{(10)} \beta = C^5 \beta$ | $\tau^{(11)} \beta = w_1 C^5 \beta$ |
| -1 -2 -2 -2 -1 | -1 -2 -3 -2 -1 | -1 -2 -3 -2 -1 |
| -1       | -1             | -2             |

Table 3.2. The assembling vectors $z_n = \tau^{(n-1)} \beta - \tau^{(n)} \beta$

| $z_1 = \beta - w_1 \beta$ | $z_2 = w_1 \beta - C \beta$ | $z_3 = C \beta - w_1 C \beta$ |
|-----------------------------|-----------------------------|-----------------------------|
| 0 0 0 0 0 0 1               | 0 0 1 0 0 0 0               | 0 1 0 1 0 0 0               |
| $z_4 = w_1 C \beta - C^2 \beta$ | $z_5 = C^2 \beta - w_1 C^2 \beta$ | $z_6 = w_1 C^2 \beta - C^3 \beta$ |
| 1 0 1 0 1 1 0               | 0 1 0 1 0 1 0               | 0 0 2 0 0 0 0               |
| 0       | 1               | 0               |
| $z_7 = C^3 \beta - w_1 C^3 \beta$ | $z_8 = w_1 C^3 \beta - C^4 \beta$ | $z_9 = C^4 \beta - w_1 C^4 \beta$ |
| 0 1 0 1 0 1 0               | 1 0 1 0 1 0 1               | 0 1 0 1 0 0 0               |
| 1       | 0               | 0               |
| $z_{10} = w_1 C^4 \beta - C^5 \beta$ | $z_{11} = C^5 \beta - w_1 C^5 \beta$ |
| 0 0 1 0 0 0 0               | 0 0 0 0 0 0 0               |
| 0       | 1               |

The Kostant numbers $a, b$ (see Table 1.3) for $E_6$ are $a = 6$, $b = 8$. From (3.8) and (3.11), we have

$$[P_G(t)]_{x_0} = \frac{t^2 + t^4 + 2t^6 + t^8 + t^{10}}{(1 - t^6)(1 - t^8)},$$

$$[P_G(t)]_{x_1} = [P_G(t)]_{x_2} = \frac{t^4 + t^8}{(1 - t^6)(1 - t^8)},$$

$$[P_G(t)]_{y_1} = [P_G(t)]_{y_2} = \frac{t^3 + t^5 + t^7 + t^9}{(1 - t^6)(1 - t^8)},$$

$$[P_G(t)]_{y_3} = \frac{t + t^5 + t^7 + t^{11}}{(1 - t^6)(1 - t^8)}.$$
Since
\[1 - t^6 = \sum_{n=0}^{\infty} t^{6n}, \quad 1 - t^8 = \sum_{n=0}^{\infty} t^{8n},\]
we have
\[[P_G(t)]_{x_1} = [P_G(t)]_{x_2} = \sum_{i,j=0}^{\infty} (t^{6i+8j+4} + t^{6i+8j+8}),\]
\[[P_G(t)]_{y_1} = [P_G(t)]_{y_2} = \sum_{i,j=0}^{\infty} (t^{6i+8j+3} + t^{6i+8j+5} + t^{6i+8j+7} + t^{6i+8j+9}),\]
\[[P_G(t)]_{y_3} = \sum_{i,j=0}^{\infty} (t^{6i+8j+1} + t^{6i+8j+5} + t^{6i+8j+7} + t^{6i+8j+11}),\]
\[(3.13)\]

Recall that \(m_\alpha(n)\), where \(\alpha = x_1, x_2, y_1, y_2, y_3\), are the multiplicities of the indecomposable representations \(\rho_\alpha\) of \(G\) (considered in the context of the McKay correspondence, \[Kos84\]) in the decomposition of \(\pi_n|G\) \[1.17\]. These multiplicities are the coefficients of the Poincaré series \[3.14\], see \[1.12, 1.14, 1.15\]. For example,
\[[P_G(t)]_{x_1} = [P_G(t)]_{x_2} = t^4 + t^8 + t^{10} + t^{12} + t^{14} + 2t^{16} + t^{18} + 2t^{20} + \ldots\]
\[(3.15)\]
\[m_{x_1} = m_{x_2} = \begin{cases} 0 & \text{for } n = 1, 2, 3, 5, 6 \text{ and } n = 2k + 1, k \geq 3, \\ 1 & \text{for } n = 4, 8, 10, 12, 14, 18, \ldots, \\ 2 & \text{for } n = 16, 20, \ldots \\ \ldots \end{cases}\]
\[(3.16)\]

In particular, the representations \(\rho_{x_1}(n)\) and \(\rho_{x_2}(n)\) do not enter in the decomposition of \(\pi_n\) of \(SU(2)\) (see \[1.3\]) for all odd \(n\).

In \[Kos04\], concerning the importance of the polynomials \(z(t)_i\), B. Kostant points out: “Unrelated to the Coxeter element, the polynomials \(z(t)_i\) are also determined in Springer, \[Sp87\]. They also appear in another context in Lusztig, \[Lus83\] and \[Lus99\]. Recently, in a beautiful result, Rossmann, \[Ros04\], relates the character of \(\gamma_i\) to the polynomial \(z(t)_i\).”

4. McKay’s observation relating the Poincaré series

In this section we prove McKay’s observation \[McK99\] relating the Poincaré series, or rather Molien-Poincaré series. In our context these series are the Kostant generating functions \(P_G(t)\) corresponding the indecomposable representations of group \(G\):
\[[P_G(t)]_i = \frac{z(t)_i}{(1 - t^a)(1 - t^b)},\]
\[(4.1)\]
see \[3.6, 3.11\].

**Theorem 4.0.1.** (McKay’s observation \[McK99\] \((*)\)) For diagrams \(G = D_n, E_n\) and \(A_{2m-1}\), the Kostant generating functions \([P_G(t)]_i\) are related as follows:
\[ (t + t^{-1})[P_G(t)]_i = \sum_{i\leftrightarrow j} [P_G(t)]_j, \]
\[(4.2)\]
where \( j \) runs over all vertices adjacent to \( i \), and \( [P_G(t)]_0 \) related to the affine vertex \( \alpha_0 \) occurs in the right side only: \( i = 1, 2, \ldots, r \).

By (4.1), McKay’s observation (4.2) is equivalent to the following one:

\[
(t + t^{-1}) z(t)_i = \sum_{j \sim i} z(t)_j, \quad \text{where } i = 1, \ldots, r. \tag{4.3}
\]

So, we will prove (4.3).

The adjacency matrix \( A \) for types \( ADE \) is the matrix containing non-diagonal entries \( a_{ij} \) if and only if the vertices \( i \) and \( j \) are connected by an edge, and then \( a_{ij} = 1 \), and all diagonal entries \( a_{ii} \) vanish:

\[
A = \begin{pmatrix}
0 & -2D \\
-2D^t & 0
\end{pmatrix}. \tag{4.4}
\]

Let \( \alpha_0 \) be the affine vertex of the graph \( \Gamma \), and \( u_0 \) be a vertex adjacent to \( \alpha_0 \). Extend the adjacency matrix \( A \) to the semi-affine adjacency matrix \( A^\gamma \) (in the style to the McKay definition of the semi-affine graph in \cite{McK99}) by adding a row and a column corresponding to the affine vertex \( \alpha_0 \) as follows: 0 is set in the \((u_0, \alpha_0)\)th slot and 1 is set in the \((\alpha_0, u_0)\)th slot, all remaining places in the \(\alpha_0\)th row and the \(\alpha_0\)th column are 0, see Fig. 4.1. Note, that for the \( A_n \) case, we set 1 in two places: \((\alpha_0, u_0)\) and \((\alpha_0, u'_0)\) corresponding to vertices \( u_0 \) and \( u'_0 \) adjacent to \( \alpha \).

**Figure 4.1.** The semi-affine adjacent matrix \( A^\gamma \)

Using the semi-affine adjacency matrix \( A^\gamma \) we write McKay’s observation (4.3) in the matrix form:

\[
(t + t^{-1}) z(t)_i = (A^\gamma z(t))_i, \quad \text{where } z(t) = \{ z(t)_0, \ldots, z(t)_r \} \text{ and } i = 1, \ldots, r. \tag{4.5}
\]

To prove (4.3), we consider the action of the adjacency matrix \( A \) and the semi-affine adjacency matrix \( A^\gamma \) related to the extended Dynkin diagram of types \( ADE \) on assembling vectors \( z \) (3.7).

**Proposition 4.0.2.** 1) For the vectors \( z_i \in \h^\vee \) from (3.7), we have

\[
\begin{align*}
(a) & \quad A z_i = z_{i-1} + z_{i+1} \quad \text{for } 1 < i < h - 1, \\
(b) & \quad A z_1 = z_2 \quad \text{and} \quad A z_{h-1} = z_{h-2}. \tag{4.6}
\end{align*}
\]
2) Consider the same vectors \( z_i \) as vectors from \( \tilde{h}^\gamma \), so we just add a zero coordinate to the affine vertex \( \alpha_0 \). We have

\[
\begin{align*}
(a) & \quad \mathcal{A}^\gamma z_i = z_{i-1} + z_{i+1} \quad \text{for} \quad 1 < i < h - 1, \\
(b) & \quad \mathcal{A}^\gamma z_1 = z_2 \quad \text{and} \quad \mathcal{A}^\gamma z_{h-1} = z_{h-2}, \\
(c) & \quad \mathcal{A}^\gamma z_0 = z_1 \quad \text{and} \quad \mathcal{A}^\gamma z_h = z_{h-1}.
\end{align*}
\]

\((4.7)\)

Proof.

1a) Let us prove \((4.6 \text{a})\). According to \((3.7)\) we have

\[
z_{2n} = w_1 \mathbf{C}^{n-1} \beta - \mathbf{C}^n \beta = (1 - w_2) w_1 \mathbf{C}^{n-1} \beta,
\]

\((4.8)\)

and

\[
z_{2n+1} = \mathbf{C}^n \beta - w_1 \mathbf{C}^n \beta = (1 - w_1) \mathbf{C}^n \beta.
\]

\((4.9)\)

Thus, for \( i = 2n \), eq. \((4.8)\) is equivalent to

\[
\mathcal{A}(1 - w_2) w_1 \mathbf{C}^{n-1} \beta = (1 - w_1) \mathbf{C}^{n-1} \beta + (1 - w_1) \mathbf{C}^n \beta,
\]

\((4.10)\)

and for \( i = 2n + 1 \), eq. \((4.8)\) is equivalent to

\[
\mathcal{A}(1 - w_1) \mathbf{C}^n \beta = (1 - w_2) w_1 \mathbf{C}^{n-1} \beta + (1 - w_2) w_1 \mathbf{C}^n \beta.
\]

\((4.11)\)

To prove relations \((4.10)\) and \((4.11)\), it suffices to show that

\[
\mathcal{A}(1 - w_2) w_1 = (1 - w_1) + (1 - w_1) \mathbf{C} = (1 - w_1)(1 + \mathbf{C}),
\]

\((4.12)\)

and

\[
\mathcal{A}(1 - w_1) \mathbf{C} = (1 - w_2) w_1 + (1 - w_2) w_1 \mathbf{C} = (1 - w_2) w_1(1 + \mathbf{C}).
\]

\((4.13)\)

In \((3.1), (3.2), \) \( w_1 \) and \( w_2 \) are chosen as

\[
w_1 = \begin{pmatrix} I \\ -2D^t \\ -I \end{pmatrix}, \quad w_2 = \begin{pmatrix} -I \\ -2D \\ I \end{pmatrix},
\]

\((4.14)\)

So, by \((4.10)\) we have

\[
(1 - w_1) = \begin{pmatrix} 0 & 0 & 2I \\ 2D^t & 2I \end{pmatrix}, \quad 1 - w_2 = \begin{pmatrix} 2I & 2D \\ 0 & 0 \end{pmatrix},
\]

\[
(1 - w_2) w_1 = \begin{pmatrix} 2I & 4DD^t & -2D \\ 0 & -2D & 0 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 4DD^t - I & 2D \\ -2D^t & -I \end{pmatrix},
\]

\((4.15)\)

\[
\mathcal{A}(1 - w_2) w_1 = \begin{pmatrix} 0 \\ -4D^t + 8D^tDD^t \\ 4D^tD \end{pmatrix},
\]

\[
1 + \mathbf{C} = \begin{pmatrix} 4DD^t \\ -2D^t \\ 2D \end{pmatrix}.
\]

By \((4.15)\) we have

\[
(1 - w_1)(1 + \mathbf{C}) = \begin{pmatrix} 0 \\ -4D^t + 8D^tDD^t \\ 4D^tD \end{pmatrix},
\]

\((4.16)\)

and \((4.12)\) is true. Further, we have

\[
(1 - w_1) \mathbf{C} = \begin{pmatrix} 0 \\ 8D^tDD^t - 6D^t \\ 4D^tD - 2I \end{pmatrix},
\]

\[
\mathcal{A}(1 - w_1) \mathbf{C} = \begin{pmatrix} -16DD^tDD^t + 12DD^t \\ -8DD^tD + 4D \\ 0 \end{pmatrix},
\]

\((4.17)\)
By (4.11) we obtain
\[(1 - w_2)w_1(1 + C) = \begin{pmatrix} -16DD^tDD^t + 12DD^t & -8DD^tD + 4D \\ 0 & 0 \end{pmatrix}, \tag{4.18}\]
and (4.11) is also true.

1b) Let us move on to (4.6 b). This is equivalent to
\[A(\beta - w_1\beta) = w_1\beta - C\beta, \quad \text{or} \quad A(1 - w_1)\beta = (1 - w_2)w_1\beta. \tag{4.19}\]
By (4.4), (4.15) eq. (4.19) is equivalent to
\[
\begin{pmatrix} -4DD^t & -4D \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2I - 4DD^t & -2D \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{4.20}\]
where
\[\beta = \begin{pmatrix} x \\ y \end{pmatrix} \tag{4.21}\]
is given in two-component form corresponding to a bipartite graph. Eq. (4.20) is equivalent to
\[-2Dy = 2x. \tag{4.22}\]
Since the matrix \(-2D\) contains a 1 at the \((i, j)\)th slot if and only if the vertices \(i\) and \(j\) are connected, eq. (4.22) follows from the well-known fact formulated in Remark 4.0.3(b).

**Remark 4.0.3** (On the highest root and imaginary roots). (a) The highest root \(\beta\) for types \(ADE\) coincides with the minimal positive imaginary root (of the corresponding extended Dynkin diagram) without affine coordinate \(\alpha_0\), see [St05, Section 2.4]. The coordinates of the imaginary vector \(\delta\) are given on [St05, Fig. 2.2].

(b) For any vertex \(x_i\) of the the highest root \(\beta\) (except the vertex \(u_0\) adjacent to the affine vertex \(\alpha_0\)), the sum of coordinates of the adjacent vertices \(y_j\) coincides with the doubled coordinate of \(\alpha_{x_i}\):
\[
\sum_{y_j \to x_i} \alpha_{y_j} = 2\alpha_{x_i}. \tag{4.23}\]

(c) For any vertex \(x_i\) of the the imaginary root \(\delta\), the sum of coordinates of the adjacent vertices \(y_j\) coincides with the doubled coordinate of \(\alpha_{x_i}\) as above in (4.23).

Since we can choose partition (4.21) such that \(\alpha_0\) belongs to subset \(y\), we obtain (4.22). The second relation of (4.6 b) follows from the symmetry of assembling vectors, see (3.9) of the Kostant theorem (Theorem 3.0.1).

2) Relations (a), (b) of (4.7) follow from the corresponding relations in (4.6) since the addition of a “1” to the \((\alpha_0, u_0)\)th slot of the matrix \(A\) (4.1) is neutralized by the affine coordinate 0 of the vectors \(z_i\), where \(i = 1, \ldots, h - 1\).

Let us prove (c) of (4.7). First,
\[
A^\gamma z_0 = \alpha_{u_0}, \quad \text{for } D_n, E_n, \tag{4.24}
\]
\[
A^\gamma z_0 = \alpha_{u_0} + \alpha_{u'_0}, \quad \text{for } A_{2m-1},
\]
where $\alpha_{u_0}$ (resp. $\alpha_{u'_0}$) is the simple root with a “1” in the $u_0$th (resp. $u'_0$th) position. Thus, by (4.15) (c) is equivalent to

$$\alpha_{u_0} = (1 - w_1)\beta = \begin{pmatrix} 0 \\ 2D^t x + 2y \end{pmatrix}, \quad \text{for } D_n, E_n,$$

$$\alpha_{u_0} + \alpha_{u'_0} = (1 - w_1)\beta = \begin{pmatrix} 0 \\ 2D^t x + 2y \end{pmatrix}, \quad \text{for } A_{2m-1}. \quad (4.25)$$

Again, by Remark 4.0.3(b), we have $2D^t x + 2y = 0$ for all coordinates excepting coordinate $u_0, u'_0$. For coordinate $u_0$ (resp. $u'_0$), by Remark 4.0.3(c), we have $(\alpha_{u_0} - 2D^t x)u_0 = 2y_{u_0}$ (resp. $(\alpha_{u'_0} - 2D^t x)u'_0 = 2y_{u'_0}$ for $A_{2m-1}$. $\square$

Proof of (4.26). Since,

$$z(t) = \sum_{j=0}^h z_j t^j, \quad z(t)_i = (\sum_{j=0}^h z_j t^j)_i, \quad \text{where } i = 1, \ldots, r,$$

by (4.27) we have

$$\mathcal{A}^\gamma z(t) = \sum_{j=0}^h \mathcal{A}^\gamma z_j t^j =
\begin{align*}
z_1 + z_2 t + (z_1 + z_3) t^2 + \cdots \\
(z_{h-3} + z_{h-1}) t^{h-2} + z_{h-2} t^{h-1} + z_{h-1} t^h =
\end{align*}

$$t^{-1}(z_1 t + z_2 t^2 + z_3 t^3 + \cdots + z_{h-1} t^{h-1}) +
\begin{align*}
t(z_1 t + \cdots + z_{h-2} t^{h-2} + z_{h-1} t^{h-1}) =
\end{align*}

$$t^{-1}(z(t) - z_0 - z h^t).

Since $z_0 = z_h$ we have

$$\mathcal{A}^\gamma z(t) = (t + t^{-1})z(t) - (t + t^{-1})(1 + t^h)z_0. \quad (4.26)$$

Coordinates $(z_0)_i = (z_h)_i$ are zeros for $i = 1, \ldots, r$, and

$$(\mathcal{A}^\gamma z(t))_i = (t + t^{-1})z(t)_i, \quad i = 1, \ldots, r. \quad (4.27)$$

For the coordinate $i = 0$, corresponding to affine vertex $\alpha_0$, by definition of $\mathcal{A}^\gamma$ (see Fig. 4.1) we have $(\mathcal{A}^\gamma z(t))_0 = 0$, and $z(t)_0 = (1 + t^h)z_0$. $\square$

Let us check McKay’s observation for the Kostant generating functions for the case of $E_6$. According to (3.11) we should get the following relations:

1) For $x_0$: \quad $$(t + t^{-1})z(t)x_0 = z(t)y_1 + z(t)y_2 + z(t)y_3,$$
2) For $y_1$: \quad $$(t + t^{-1})z(t)y_1 = z(t)x_0 + z(t)x_1,$$
3) For $y_2$: \quad $$(t + t^{-1})z(t)y_2 = z(t)x_0 + z(t)x_2,$$
4) For $x_1$: \quad $$(t + t^{-1})z(t)x_1 = z(t)y_1,$$
5) For $x_2$: \quad $$(t + t^{-1})z(t)x_2 = z(t)y_2,$$
6) For $y_3$: \quad $$(t + t^{-1})z(t)y_3 = z(t)x_0 + z(t)y_0. \quad (4.28)$$
1) For $x_0$, we have
\[
(t + t^{-1})(t^2 + t^4 + 2t^6 + t^8 + t^{10}) = \\
2(t^3 + t^5 + t^7 + t^9) + (t + t^5 + t^7 + t^{11}),
\]
or
\[
(t^3 + t^5 + 2t^7 + t^9 + t^{11}) + (t + t^3 + 2t^5 + t^7 + t^9) = \\
t + 2t^3 + 3t^5 + 3t^7 + 2t^9 + t^{11}.
\]

2) For $y_1$ (the same for $y_2$), we have
\[
(t + t^{-1})(t^3 + t^5 + t^7 + t^9) = \\
(t^2 + t^4 + 2t^6 + t^8 + t^{10}) + (t^4 + t^8),
\]
or
\[
(t^4 + t^6 + t^8 + t^{10}) + (t^2 + t^4 + t^6 + t^8) = \\
(t^2 + 2t^4 + 2t^6 + 2t^8 + t^{10}).
\]

3) For $x_1$ (the same for $x_2$), we have
\[
(t + t^{-1})(t^4 + t^8) = t^3 + t^5 + t^7 + t^9.
\]

4) For $y_3$, we have
\[
(t + t^{-1})(t + t^5 + t^7 + t^{11}) = \\
(t^2 + t^4 + 2t^6 + t^8 + t^{10}) + (1 + t^{12}),
\]
or
\[
(t^2 + t^6 + t^8 + t^{12}) + (1 + t^4 + t^6 + t^{10}) = \\
(t^2 + t^4 + 2t^6 + t^8 + t^{10}) + (1 + t^{12}).
\]
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