Quasipolynomial Computation of Nested Fixpoints

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Why Nested Fixpoints?

- Applications of parity games:
  - Model checking for the modal $\mu$-calculus
  - Satisfiability checking for the modal $\mu$-calculus
  - Synthesis for linear-time logics (e.g. LTL)

- Recent breakthrough result: solving parity games is in $\text{QP}$
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We show:

- Nested fixpoints stabilize after quasipolynomially many iterations.
Motivation: Parity Games

Parity games: \((V = V\lozenge \cup V\Box, E \subseteq V \times V, \Omega : V \rightarrow [k]), [k] = \{0, \ldots, k\}\)

- \(\lozenge\)-strategy: \(s : V^* V\lozenge \rightarrow V\) such that \(s(\bar{v}v) \in E(v)\)
- \(\lozenge\) wins \(v \in V\) iff there is \(\lozenge\)-strategy with which all \(v\)-plays are even
Motivation: Parity Games

Parity games: \((V = V_\Diamond \cup V_\Box, E \subseteq V \times V, \Omega : V \rightarrow [k]), [k] = \{0, \ldots, k\}\)

- history-free \(\Diamond\)-strategy: \(s : V_\Diamond \rightarrow V\) such that \(s(v) \in E(v)\)
- \(\Diamond\) wins \(v \in V\) iff there is \(\Diamond\)-strategy with which all \(v\)-plays are even

Central result: parity games are history-free determined.

Observation: \(\text{win}_\Diamond\) (and \(\text{win}_\Box\)) can be specified by \(\mu\)-calculus formula.
Motivation: Reducing Parity Games to Safety Games

Idea: Use deterministic Büchi automaton \( A = (Q, [k], \delta, F) \) accepting exactly the even priority sequences in \( G = (V, E, \Omega : V \to [k]) \).

Parity game \( G \) is equivalent to safety game \( G \otimes A = (V \times Q, E \otimes \delta, F \circ \pi_2) \),

\[
(E \otimes \delta)(v, q) = \{ (w, \delta(q, \Omega(v))) \mid w \in E(v) \}
\]
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Size of suitable automaton $A$?

- Immediate: $|Q| \in \mathcal{O}(|V|^k)$
- Calude et al., 2017: $|Q| \in \mathcal{O}(|V|^{\log k})$, $|Q| \in \mathcal{O}(|V|^4)$ if $k \leq \log |V|$
Finite lattice: \((L, \subseteq), L \neq \emptyset\) finite set, \(\subseteq\) partial order on \(L\) s.t. join \(\bigcup X\) and meet \(\bigcap X\) exist for all \(X \subseteq L\).

Basis of \(L\): \(B_L \subseteq L\) s.t. \(l = \bigcup \{b \in B_L \mid b \subseteq l\}\) for all \(l \in L\).

Examples

- For finite set \(V\), powerset lattice \((\mathcal{P}(V), \subseteq)\) is finite lattice with join \(\bigcup U\), meet \(\bigcap U\) for \(U \in \mathcal{P}(V)\); \(V\) is a basis.
- For finite set \(V\) and number \(n\), \((n^V, \subseteq)\) is finite lattice, where \(n^V = \{f : V \to [n-1]\}\), \(f \subseteq g\) iff for all \(v \in V\), \(f(v) \leq g(v)\).

Fix a finite lattice \(L\) and basis \(B_L\).
Function $f : L^{k+1} \rightarrow L$ is monotone if for all $U_i \subseteq V_i$, $0 \leq i \leq k$,

$$f(U_0, \ldots, U_k) \subseteq f(V_0, \ldots, V_k)$$

### Extremal fixpoints, systems of fixpoint equations

Let $f : L \rightarrow L$ and $f_i : L^{k+1} \rightarrow L$, $0 \leq i \leq k$ be monotone functions.

- **LFP** $f = \bigcap \{Z \subseteq U \mid f(Z) \subseteq Z\}$
- **GFP** $f = \bigcup \{Z \subseteq U \mid Z \subseteq f(Z)\}$

**System of fixpoint equations:**

$$X_i =_{\eta_i} f_i(X_0, \ldots, X_k) \quad 0 \leq i \leq k, \eta_i \in \{\text{LFP}, \text{GFP}\}$$
Fix equation system $\mathbb{E}$ of $k + 1$ equations $X_i = \eta_i f_i(X_0, \ldots, X_k)$.

**Semantics of fixpoint equation systems**

For valuation $\sigma : [k] \rightarrow L$, put $\llbracket X_i \rrbracket^\sigma = \eta_i f_i^\sigma$ where, for $A \in L$,

$$f_i^\sigma(A) = f_i(\llbracket X_0 \rrbracket^{\sigma[A/i]}, \ldots, \llbracket X_{i-1} \rrbracket^{\sigma[A/i]}, A, \sigma(i + 1), \ldots, \sigma(k))$$

**Solution** for variable $X_k$ in $\mathbb{E}$: $\llbracket X_k \rrbracket_\mathbb{E} = \llbracket X_k \rrbracket^\epsilon$, where $\text{dom}(\epsilon) = \emptyset$. 
For parity game \((V, E, \Omega : V \rightarrow [k])\), use lattice \(L = \mathcal{P}(V)\) and define

\[
\text{force}(U) = \{v \in V^\Diamond \mid E(v) \cap U \neq \emptyset\} \cup \{v \in V^\square \mid E(v) \subseteq U\}
\]

\[
f_{\text{PG}}(X_0, \ldots, X_k) = \bigcup_{0 \leq i \leq k} (\{v \in V \mid \Omega(v) = i\} \cap \text{force}(X_i))
\]

Define equation system: \(\eta_i = \text{LFP}\) if \(i\) odd, \(\eta_i = \text{GFP}\) otherwise and

\[
X_0 = \text{GFP}\, f_{\text{PG}}(X_0, \ldots, X_k) \quad X_i = \eta_i X_{i-1} \text{ for } i > 0,
\]

**Theorem (e.g. [Dawar, Grädel, 2008])**

\[
\text{win}^\Diamond = \llbracket X_k \rrbracket_{f_{\text{PG}}}
\]
**Even k-graph:** \( G = (W, \delta \subseteq W \times [k] \times W) \) s.t. all \( \delta \)-paths are even

**Definition: History-free witnesses**

Even \( k \)-graph \( (V, S) \) s.t. \( V = B_L \times [k] \) and for all \( (u, j) \in V \),

\[
u \sqsubseteq f_j(S_0(u, j), \ldots, S_k(u, j))
\]

where \( S_i(u, j) = \bigsqcup \{(w, i) \mid ((u, j), i, (w, i)) \in S\} \)

**Note:** \( |V| \in \mathcal{O}(|B_L| \cdot (k + 1)) \)

**Lemma**

There is history-free witness s.t. \( (u, j) \in V \) if and only if \( u \sqsubseteq [X_j]_E \).
Definition - Universal Graphs [Colcombet, Fijalkow, 2019]

Homomorphism from $G = (W, \delta)$ to $G' = (W', \delta')$: $h : W \rightarrow W'$ s.t.

for all $(v, p, w) \in \delta$, we have $(h(v), p, h(w)) \in \delta'$.

$(n, k)$-universal graph $S$: even $k$-graph s.t. for all even $k$-graphs $G$ with $|G| \leq n$, there is homomorphism from $G$ to $S$.

Theorem [Czerwiński et al., 2019]

- There is a deterministic $(n, k)$-universal graph of size $n^{\log k + O(1)}$, and of size $O(n^4)$ if $k \leq \log n$.
- Every $(n, k)$-universal graph has size at least $n^{\log \frac{k}{\log n} - 1}$. 
Fix deterministic \(((|B_L|)(k + 1), k + 1)\)-universal graph \(S = (W, \delta)\).

**Definition - Product fixpoint**

Define \(\mathcal{E} \otimes S : \mathcal{P}(B_L \times [k] \times W) \rightarrow \mathcal{P}(B_L \times [k] \times W)\) by

\[
(\mathcal{E} \otimes S)(Z) = \{(v, p, q) \in B_L \times [k] \times W \mid v \sqsubseteq f_p(Z_0^q, \ldots, Z_k^q)\}
\]

where

\[
Z_i^q = \bigsqcup \{u \in B_L \mid (u, i, \delta(q, i)) \in Z\}.
\]

\(Y =_{\text{GFP}} (\mathcal{E} \otimes S)(Y)\) is **chained product fixpoint** of \(\mathcal{E}\) and \(S\).

**Theorem**

We have \(u \sqsubseteq [X_i]_{\mathcal{E}}\) if and only if there is \(q \in W\) s.t. \((u, i, q) \in [Y]_{\mathcal{E} \otimes S}\).
A Progress Measure Algorithm

Fix total simulation order \( \leq \) on \( W \), least node w.r.t. \( \leq : q_{\min} \)

**Measure:** \( \mu : B_L \times [k] \to W \cup \{ \star \} \); define function Lift on measures:

\[
(Lift(\mu))(v, p) = \min\{q \in W \mid v \sqsubseteq f_p(U_{0}^{\mu, q}, \ldots, U_{k}^{\mu, q})\}
\]

where \( \min(\emptyset) = \star \) and

\[
U_{i}^{\mu, q} = \bigsqcup\{u \in B_L \mid \mu(u, i) \leq \delta(q, i)\},
\]

**Lifting algorithm**

1. Initialize \( \mu(v, p) = q_{\min} \) for all \((v, p) \in B_L \times [k]\).
2. If \( \text{Lift}(\mu) \neq \mu \), then put \( \mu := \text{Lift}(\mu) \) and go to 2. Otherwise go to 3.
3. Return \( \mathcal{B} = \{(v, p) \in B_L \times [k] \mid \mu(v, p) \neq \star\} \).

**Theorem**

We have \((v, p) \in \mathcal{B}\) if and only if \( v \sqsubseteq \left[ X_p \right]_E \).
Coalgebraic $\mu$-calculus [Cirstea, Kupke, Pattinson 2011]: Generic fixpoint logic framework, subsuming e.g. graded, probabilistic and alternating-time $\mu$-calculi
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- Reduction of model checking [H., Schröder, CONCUR 2019] for the coalgebraic $\mu$-calculus to solving fixpoint equation systems.

Corollary
Model checking for coalgebraic $\mu$-calculi is in $QP$. 
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**Corollary**
Model checking for coalgebraic $\mu$-calculi is in QP.

- Reduction of satisfiability checking [H., Schröder, FoSSaCS 2019] for the coalgebraic $\mu$-calculus to solving fixpoint equation systems.

**Corollary**
Satisfiability checking for coalgebraic $\mu$-calculi can be done in time $O(2^{nk\log n})$ (down from $O(2^{n^2k^2\log n})$).
More Examples, Many-valued Games and Logics

- **Finite latticed $\mu$-calculus** [Bruns, Godefroid, 2004], latticed parity games [Kupferman, Lustig, 2007]

- Games / logics with combined parity and quantitative objective:
  - **Energy** parity games [Chatterjee, Doyen, 2012], energy $\mu$-calculus [Amram, Maoz, Pistiner, Ringert, 2020]
  - **Mean-payoff** parity games; recover [Daviaud, Jurdzinski, Lazic, 2018]
Energy parity game: \((V, E, \Omega, w), w : E \to \mathbb{Z}\); player \(\lozenge\) wins even plays with starting credit \(c\) if energy value always remains non-negative.
Energy parity game: $(V, E, \Omega, w)$, $w : E \to \mathbb{Z}$; player $\diamond$ wins even plays with starting credit $c$ if energy value always remains non-negative.

- **History-dependent** $\diamond$-strategies: $s(1) = 1$, $s(1, 1) = 2$
- [Chatterjee, Doyen, 2012]: bound on history $c = |V| \cdot k \cdot W$
Energy parity game: $(V, E, \Omega, w)$, $w : E \to \mathbb{Z}$; player $\Diamond$ wins even plays with starting credit $c$ if energy value always remains non-negative.

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Equation system over lattice $L = c^V$ with elements $g : V \to \{0, \ldots, c\}$

Function $f_{\text{EPG}} : L^{k+1} \to L$ is formula of energy $\mu$-calculus.

**Theorem [Amram, Maoz, Pistiner, Ringert, 2020]**

Player $\Diamond$ wins $\nu$ with initial credit $c$ if and only if $(X_k f_{\text{EPG}})(\nu) = c$. 

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Hausmann, Schröder – Quasipolynomial Computation of Nested Fixpoints
Unifying progress measure algorithm leads to novel complexity results:

| setting            | game solving | model checking | satisfiability checking |
|--------------------|--------------|----------------|------------------------|
| coalgebraic        | QP           | QP             | $2^\mathcal{O}(nk \log n)$ |
| latticed           | QP           | QP             | ?                      |
| energy             | pseudo-QP    | QP in c        | ?                      |
| mean pay-off       | pseudo-QP    | ?              | ?                      |
## Conclusion

### Results:

- Quasipolynomial solving of fixpoint equations by universal graphs
- Highly general quasipolynomial progress measure algorithm for
  - Energy parity games, model checking energy $\mu$-calculus
  - Latticed parity games, model checking finite latticed $\mu$-calculi
  - Coalgebraic parity games, model checking / satisfiability checking for coalgebraic $\mu$-calculus

### Future work:

- Cover more variants of games (e.g. stochastic setting)
- Does this work for all games with finite-history strategies?
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Fixpoint parity game for equation system $E$

Parity game $(V, E, \Omega)$, nodes: $V = (B_L \times [k]) \cup L^{k+1}$

| node       | priority | owner | moves to                                      |
|------------|----------|-------|-----------------------------------------------|
| $(u, j) \in B_L \times [k]$ | $\text{ad}(j)$ | $\Diamond$ | $\{ U \in L^k \mid u \sqsubseteq f_j(U) \}$ |
| $U$           | 0        | $\Box$ | $\{(v, i) \mid v \in U_i \}$                |

where $U = (U_0, \ldots, U_k) \in L^{k+1}$

Theorem [König et al. 2019]

Eloise wins node $(u, i)$ if and only if $u \in \lfloor X_i \rfloor_E$.

Problem: exponential size
- still useful, e.g. for showing history-freeness for equation systems.