Three perspectives on complexity: entropy, compression, subsymmetry

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Abstract. There is no single universally accepted definition of ‘Complexity’. There are several perspectives on complexity and what constitutes complex behaviour or complex systems, as opposed to regular, predictable behaviour and simple systems. In this paper, we explore the following perspectives on complexity: effort-to-describe (Shannon entropy $H$, Lempel-Ziv complexity $LZ$), effort-to-compress (ETC complexity) and degree-of-order (Subsymmetry or $SubSym$). While Shannon entropy and $LZ$ are very popular and widely used, ETC is relatively a new complexity measure. In this paper, we also propose a novel normalized complexity measure $SubSym$ based on the existing idea of counting the number of subsymmetries or palindromes within a sequence. We compare the performance of these complexity measures on the following tasks: (A) characterizing complexity of short binary sequences of lengths 4 to 16, (B) distinguishing periodic and chaotic time series from 1D logistic map and 2D Hénon map, (C) analyzing the complexity of stochastic time series generated from 2-state Markov chains, and (D) distinguishing between tonic and irregular spiking patterns generated from the ‘Adaptive exponential integrate-and-fire’ neuron model. Our study reveals that each perspective has its own advantages and uniqueness while also having an overlap with each other.

1 Introduction

The seemingly naive procedure of finding decimal expansion of a real number, synchronized beating of heart cells, irregular firing of neurons in the brain, global economy and society, motion of planets and galaxies, the everyday weather and the Internet – are but a few examples of complex systems. While there is little agreement on the precise mathematical definition of complexity, there is no doubt on the ubiquity of complex systems in nature and society.

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The ability to model, analyze and understand complex systems is invaluable since it allows us to control, predict and exploit complex phenomena to our advantage. For example, by characterizing the complexity of the chaotic nature of cardiovascular dynamics, we may one day understand the effects of aging [1]. Understanding complexity may be the key to unlocking the mysteries of the brain since chaotic behaviour is abound in the central nervous system [2,3]. Complex behaviour of neural signals may even be necessary for multiplexing a large number of neural signals in the brain for successful communication in the presence of neural noise and interference [4]. Pseudo-random sequences having high linear (and nonlinear) complexity, but easy to generate in hardware, are employed in various cryptographic protocols [5]. Complexity and randomness is increasingly playing an important role in psychology to decipher behaviours and mental processes pertaining to memory (see [6] and references therein), cognitive effects of aging and human judgments of visual complexity [7]. Such a list is by no means exhaustive, but only serves to indicate the widespread prevalence of complex systems and use of the notion of complexity to characterize such systems.

There are broadly two categories of complex systems – deterministic and non-deterministic (stochastic). Chaotic systems are examples of deterministic complex systems which are capable of producing rich and complex behaviour ranging from periodic to quasi-periodic to completely random-looking and unpredictable outputs. Stochastic systems are inherently non-deterministic (‘random’) and capable of exhibiting complex dynamics which can be predicted only probabilistically, at best. In either case, measuring complexity (of time series measurements from such systems) is an important step in characterizing or modeling the system/phenomenon under study. In this paper, we concern ourselves with the all-important question: What is complexity?

1.1 Complexity: many facets

There is no single universally accepted definition of ‘Complexity’. There are several perspectives on complexity and what constitutes complex behaviour or complex systems, as opposed to regular, predictable behaviour and simple systems. With the recent availability of high speed and low cost computational tools at our disposal, analysis of complex systems has become data-intensive. Lloyd [8] lists three questions which researchers ask in order to quantify the complexity of the thing (object) under study:

- How hard is it to describe? (effort-to-describe)
- How hard is it to create? (effort-to-create)
- What is its degree of organization? (degree-of-order).

We have indicated these questions with the corresponding italicized phrases in parenthesis (not present in Lloyd’s original formulation). In this paper, we study three perspectives of complexity namely: effort-to-describe, effort-to-compress and degree-of-order. We have chosen a new perspective ‘effort-to-compress’ instead of ‘effort-to-create’ and our interpretation of degree-of-order is slightly different from Lloyd’s (as will be evident in this paper). Our reasons for choosing these categories for characterizing complexity are as follows:

- Lloyd [8] rightly observes that though there is a multiplication of complexity measures, several of these measures represent variations of only a handful of

\textsuperscript{1}Most naturally occurring systems are a hybrid since stochastic noise is inevitable.
underlying themes. Hence, Lloyd makes a first attempt to distill the common questions that are asked by researchers in diverse disciplines in order to quantify complexity of the object under study. However, Lloyd makes it clear that his categorization and tabulation of complexity measures is not only incomplete but also not watertight. There could very well be different ways of classification.

- We largely stick to Lloyd’s classification scheme but with a few departures. Lloyd’s ‘difficulty of creation’ is replaced by our ‘effort-to-compress’ category. The measures listed under Lloyd’s ‘difficulty of creation’ are not easy to compute and are not readily applicable to characterize complexity of an arbitrary time series. Whereas, the measure that we propose is very easy to compute, is different from other measures, can be applied to an arbitrary time series and is beginning to be used in several applications.

- Lloyd lists a number of measures to capture the ‘degree of organization’. However, the notion of ‘symmetry’ is intrinsic and a basic measure of the degree of organization and is not directly and simply captured by any of the measures listed by Lloyd. Thus, we have retained the same category of Lloyd, but we propose a new measure which attempts to capture degree of organization by counting the number of ‘subsymmetries’ inherent in the input sequence, in a simple, direct and straightforward manner. This novel measure will be proposed in the next section.

- All the measures that we have chosen in this work have intuitive appeal and easy to apply to an arbitrary time series, thus readily useful in real-world applications.

By no means is this exhaustive and there are several other perspectives of complexity which escape us for the moment. Table 1 gives the definitions, examples of these three perspectives and their contrasting approaches to characterize complexity of an object. In the rest of the paper, our goal is to compare and evaluate specific measures that are representative of these three perspectives.

2 The three perspectives

In this section, we give in-depth definitions and descriptions of the four measures that together make up the three perspectives. Among the four measures, two of the measures are well known and widely used in several applications (Shannon entropy $H$ and Lempel-Ziv complexity $LZ$). The third complexity measure known as Effort-To-Compress ($ETC$) is a relatively new measure (proposed by the authors) to characterize complexity of time series and is beginning to be used in several applications. The fourth measure (subsymmetries) is a new measure that we propose in this paper for characterizing complexity of time series, though the underlying idea of counting subsymmetries (palindromes) is not new.

2.1 Effort-to-describe: $H$ and $LZ$

Claude Shannon’s notion of entropy as a quantitative measure of information content [10] is very popular and widely used to characterize the complexity of a given time series. The origins of the idea can be traced back to thermodynamics (Clausius, 1965)
Table 1. Three perspectives on ‘Complexity’.

| Perspective       | Definition                                                                 | Examples                                                                 | Remarks                                                                 |
|-------------------|---------------------------------------------------------------------------|--------------------------------------------------------------------------|-------------------------------------------------------------------------|
| Effort-to-describe| The number of bits needed to describe the object, the length of the shortest computer program to describe the object, the size of the dictionary that captures the minimum exhaustive history of the object. | Shannon Entropy ($H$), Kolmogorov-Chaitin complexity, Lempel-Ziv complexity ($LZ$) | Shannon entropy is measured in bits. $LZ$ is popularly used in place of Kolmogorov-Chaitin complexity which is difficult to compute and is used extensively in several applications. Both $H$ and $LZ$ are listed under ‘difficulty of description’ in Lloyd’s original categories [8]. $ETC$ could be defined with other lossless compression algorithms as well. The effort-to-describe of two objects may be the same, and yet admit different effort-to-compress complexities and vice versa. |
| Effort-to-compress| The effort to losslessly compress the object by a pre-determined lossless compression algorithm. | Effort-To-Compress ($ETC$), a recently proposed measure, is based on the lossless compression algorithm known as Non-Sequential Recursive Pair Substitution (NSRPS). | SubSym as a normalized complexity measure (between 0 and 1) for time series is defined for the first time in this paper. SubSym is based on Alexander and Carey’s idea of counting number of subsymmetries (palindromes) as a measure of visual complexity of binary strings [9]. |
| Degree-of-order   | The amount of order or structure that is intrinsic to the object. Symmetry is a good candidate to capture this notion. | Subsymmetries (SubSym) | |

and in statistical physics (Boltzmann and Gibbs, 1900s). Shannon entropy\(^2\) of a discrete random variable $X$ is defined as:

$$H(X) = - \sum_{i=1}^{M} p_i \log_2(p_i) \text{ bits/symbol},$$  \hspace{1cm} (1)

where $X$ takes $M$ possible events with the probability of occurrence of the $i$th event given by $p_i > 0$. The maximum value of the concave function $H(\cdot)$ is achieved for a uniform random variable with all events equally likely and is equal to $H_{\text{max}} = \log_2(M)$ bits.

One simple yet illuminating interpretation of Shannon entropy is by means of Yes/No questions [11]. Entropy can be defined as the minimum expected number of Yes/No questions needed to determine the state of a system.\(^2\)

\(^2\)We shall always refer to first-order empirically estimated Shannon entropy throughout the paper. Here ‘first-order’ refers to the fact that the entropy is computed on the first-order probability distribution, treating symbols as independent of each other.
of Yes/No questions needed to remove the uncertainty in the random experiment. Thus, entropy serves as measure of the effort-to-describe the outcome of a random experiment.

Shannon entropy and other information theoretic measures are widely used in communication, data compression, error control coding, cryptography, biomedical applications, financial time series analysis, non-linear dynamics and as various entropic forms in physics (please refer to [12] and references there in).

2.2 Kolmogorov-Chaitin complexity and Lempel-Ziv complexity

Another very important measure of complexity goes by the name Kolmogorov complexity (1963) which is defined as the length of the shortest computer program (in a pre-determined chosen programming language) that produces the given string/sequence as output. It is also known as descriptive complexity, Kolmogorov-Chaitin complexity, algorithmic entropy, or program-size complexity. Please refer to Li and Vitanyi [13] for further details.

Owing to the uncomputability of Kolmogorov complexity, lossless compression algorithms have been used to approximate as an upper bound. Lempel-Ziv complexity is one such measure [14] that is closely related to Lempel-Ziv universal lossless compression algorithm [15]. It is defined as the size of the dictionary which captures the minimum exhaustive history of the input sequence, from which the entire sequence can be uniquely expressed (or compressed). For example, if the input string is ‘aacgacga’, the dictionary is built up of words that form the minimum exhaustive history as the input sequence is parsed from left to right. In this example, the dictionary is \{a, ac, g, acga\}. Thus the value of Lempel-Ziv complexity (LZ) is the size of the dictionary which is \(c(n) = 4\). In order to allow us to compare LZ complexities of strings of different length, a normalized measure is proposed [16] as \(C_{LZ} = \left(\frac{c(n)}{n}\right) \log_\alpha(n)\), where \(n\) is the length of the input sequence, \(\alpha\) is the number of unique symbols in the input sequence.

Lempel-Ziv complexity (LZ) is widely used in a variety of applications such as estimation of entropy of spike trains, biomedical applications, financial time series analysis, bioinformatics applications, analysis of cardiovascular dynamics and several others (please refer to [12] and references there in).

2.3 Effort-to-compress: ETC

Recently [17], we have proposed a new complexity measure which has a different perspective on complexity. Known as the Effort-To-Compress or ETC, it measures the effort needed to compress the input sequence by a lossless compression algorithm. Specifically, we employ the Non-Sequential Recursive Pair Substitution (NSRPS) algorithm [18]. The algorithm is simple to describe. ETC proceeds by identifying that pair of consecutive symbols in the input sequence that is most frequently occurring and replaces it with a new symbol. In the next pass, this is repeated. In this fashion, at every iteration, the length of the sequence reduces and the algorithm stops when the sequence turns into a constant sequence (this is guaranteed as the length decreases in successive iteration). The ETC complexity measure is defined as the number of iterations needed to reduce the input sequence into a constant sequence. As an example, consider the input sequence aacgacga. In the first iteration, ETC transforms aacgacga \(\mapsto a1g1ga\) (since ac is the most frequently pair, it is replaced by the new symbol ‘1’). In the next iteration, ETC transforms a1g1ga \(\mapsto a22a\) (1g is the most frequently occurring pair). Subsequently, a22a \(\mapsto 32a\mapsto 4a\mapsto 5\). Thus, it took 5 iterations to transform aacgacga to a constant sequence (5). The value of the
Table 2. SubSym: A normalized complexity measure defined using number of subsymmetries (palindromes) [9]. As an example we consider three binary strings of length 6. The normalized measure is defined with respect to the all-zero sequence $Z$ for which the value of the measure is 0.

| String    | Length of substring | Number of palindromes | SubSym |
|-----------|---------------------|-----------------------|--------|
| $Z = 000000'$ | 2                   | 5                     |        |
| '010010'   | 3                   | 4                     |        |
|           | 4                   | 3                     |        |
|           | 5                   | 2                     |        |
|           | 6                   | 1                     |        |
| Total = 15|                     |                       | $1 - \frac{15}{15} = 0$ |
| '011010'   | 2                   | 1                     |        |
|           | 3                   | 2                     |        |
|           | 4                   | 1                     |        |
|           | 5                   | 0                     |        |
|           | 6                   | 1                     |        |
| Total = 5 |                     |                       | $1 - \frac{5}{15} = 0.6667$ |
| '011010'   | 2                   | 1                     |        |
|           | 3                   | 2                     |        |
|           | 4                   | 1                     |        |
|           | 5                   | 0                     |        |
|           | 6                   | 0                     |        |
| Total = 4 |                     |                       | $1 - \frac{4}{15} = 0.7333$ |

complexity measure in this instance is $N = 5$. The normalized value of the measure is given by $\frac{N}{L-1}$, where $L$ is the length of the input sequence (0.8571 for this example). As another example, $aacaacaacac \rightarrow 1c1c1c1c \rightarrow 2222$, implying that the ETC value for $aacaacaacac$ is $N = 2$ and $\frac{N}{L-1} = \frac{2}{11} = 0.1818$. Note that $0 \leq \frac{N}{L-1} \leq 1$ always. Lower the value of the measure, lower is the complexity. Thus, in the above two examples, $aacaacaacac$ has a much lower ETC complexity (0.1818) than $aaacgacga$ (0.8571).

ETC was originally proposed for automatic identification and classification of short noisy sequences from chaotic dynamical systems [17]. It has recently been applied for characterizing dynamical complexity of time series from chaotic flows and maps, as well as two-state Markov chains [12,19], analyzing short RR tachograms from healthy young and old subjects [20], and for developing a measure of integrated information [21].

2.4 Degree-of-order: SubSym

In the original taxonomy of complexities by Lloyd [8], degree of organization was divided into two aspects – the difficulty of describing organizational structure, and the amount of information that is shared between the parts of a system as a result of this organizational structure. We interpret this perspective in a slightly different way. We are interested in the degree-of-order that is intrinsic to an input sequence. The best candidate for such a notion is symmetry.

The idea of counting the total number of subsymmetries that is intrinsic in a binary string goes back to Alexander and Carey [9] who were interested in characterizing the cognitive simplicity of binary patterns. However, it has largely been forgotten, only recently revived by Toussaint et al. [7], again in the context of human judgment of visual complexity. The definition is very simple – just count the total number of all substrings of lengths $\geq 2$ which are palindromes.
With the already existing notion of subsymmetries (palindromes), we define a new normalized complexity measure for sequences as follows:

\[
SubSym(x) = 1 - \frac{\text{subsymmetries}(x)}{\text{subsymmetries}(Z)},
\]

\[
= 1 - \frac{\text{subsymmetries}(x)}{\binom{n(n-1)}{2}},
\]

where \(x\) is the input sequence of length \(n\), \(Z\) is the all-zero sequence of length \(n\) and the function \(\text{subsymmetries}(.)\) computes the total number of subsymmetries or palindromes. By a matter of simple counting, it is easy to see that an all-zero sequence of length \(n\) has total number of subsymmetries or palindromes = \(\frac{n(n-1)}{2}\). The idea behind this definition is that the all-zero sequence has the highest order in that it has the most number of subsymmetries. Hence, the number of subsymmetries of the input sequence \(x\) is divided by that of the all-zero sequence. This gives the proportion of order or simplicity in the input sequence. In order to give a measure of complexity we subtract this quantity from 1. The resulting normalized measure always satisfies 0 ≤ SubSym(x) ≤ 1.

As an example, we consider three binary strings of length 6 (refer to Tab. 2) and compute the normalized measure: SubSym. As per the measure, the string ‘000000’ has the highest order (15 palindromes), followed by the string ‘010010’ (5 palindromes) and ‘011010’ (4 palindromes). The normalized complexity measure SubSym for the three strings are 0, 0.6667 and 0.7333 respectively.

### 3 Comparing the three perspectives

In this section, we rigorously evaluate the four complexity measures (just described) arising from the three perspectives. To this end, we consider the following four tasks:

- (A) Characterizing complexity of short binary sequences.
- (B) Distinguishing periodic and chaotic time series.
- (C) Characterizing the complexity of stochastic time series generated from 2-state Markov chains.
- (D) Distinguishing tonic and irregular spiking from a spiking neuron model.

One important point to note is that all these measures work with symbols (such as ‘0’ and ‘1’, \(\{a,c,g,t\}\), etc.). So, we need a procedure to convert the real-valued time series into a symbolic sequence (a sequence of symbols). This is done by first defining a partition and assigning unique symbols to non-overlapping bins (or intervals) which form the partition. For example, consider a real-valued time series that takes values in the range \([\text{Min}, \text{Max}]\). Let us define a partition consisting of 2 bins: \([\text{Min}, \text{Mid}]\) and \([\text{Mid}, \text{Max}]\) where \(\text{Mid} = \frac{\text{Min} + \text{Max}}{2}\). Thus every value of the time series is checked to see in which bin it belongs. If it belonged to the bin \([\text{Min}, \text{Mid}]\), it is coded as ‘0’ and if it belonged to \([\text{Mid}, \text{Max}]\) it is coded as ‘1’. Thus, the resulting symbolic sequence will contain symbols made up of 0s and 1s and is now ready for analysis using the complexity measures. This procedure can be suitably modified for creating a symbolic sequence with more number of symbols, though for all experiments in this work we have used only 2 bins.

\(^3\)The length of the sequence we consider is always \(n > 1\).
### 3.1 Complexity of short binary sequences

Binary sequences or strings arise in several applications. For example, in analyzing membrane potential waveforms (also known as spike trains) which are obtained from neurons, the signal is first converted into a symbolic sequence of zeros and ones indicating ‘no-spiking’ and ‘spiking’ in a moving window of a pre-determined length [22]. The resulting binary sequence is then analyzed in order to estimate its entropy by means of Lempel-Ziv complexity [22]. In several cases, the neuron under study may fire only a few times (10–20 or even lesser) in the window chosen for the study. Thus, it is vital to consider the performance of complexity measures on short binary sequences.

As an example, consider the four binary strings in Table 3 and the corresponding values of the four complexity measures. Intuitively, we expect that binary strings with more structure or order have a lower value of complexity (e.g. ‘000000’ and ‘010010’) than those with less structure (e.g. ‘011010’ and ‘011100’). From the table, we can see that different measures capture different aspects of order and structure. Shannon entropy ($H$) is blind to the arrangement of 0s and 1s and is only sensitive to their relative counts. LZ performs very poorly as it has issues with short data-lengths which has been previously noted [23]. ETC and SubSym seem to yield reasonable values for this example.

SubSym rates ‘011010’ as the string with highest complexity since it has the least number of palindromes (subsymmetries). The number of subsymmetries in ‘010010’ and ‘011100’ are the same and hence have identical values of SubSym. For ETC, ‘011100’ has slightly higher complexity than ‘011010’ since the latter has two patterns of same frequency (‘01’ and ‘10’) whereas the former has only one pattern with highest frequency (‘11’).

The first test that we perform is to characterize the complexity of all possible binary strings (or sequences) of lengths ranging from 4 to 16. One of the metrics that we use to evaluate these measures is the number of distinct complexity values the complexity measure takes across the entire ensemble of binary strings of a particular length. A larger number of distinct values is desirable since it allows to distinguish complexities of individual binary strings more finely.

Figure 1 shows the graph of the number of distinct complexity values (for the four measures) for all binary strings of lengths 4 to 16. As it can be seen, SubSym has the most number of distinct values followed by ETC, H and LZ. It appears that among the measures considered, SubSym is ideally suited for characterizing the complexity of short binary strings. However, this alone is insufficient, and the next test will determine whether these measures can successfully distinguish between periodic and chaotic time series.

### 3.2 Distinguishing periodic and chaotic time series

In several applications, there is a need to distinguish between periodic behaviour and irregular or chaotic behaviour. For example, it is well known that heart rate in healthy human hearts is not constant or periodic, but shows irregular patterns
that are typically associated with a chaotic dynamical system [24]. Several studies suggest that aging and other pathological physiological conditions result in reduced complexity of heart dynamics, leading to a more regular heart rate variability [1,25].

We are interested in determining whether the four complexity measures ($H$, $LZ$, $ETC$ and $SubSym$) are able to distinguish between chaotic (irregular) and periodic time series. In order to test this, we first consider the 1D chaotic logistic map [26] given by the following equation:

\[ x_{n+1} = ax_n(1 - x_n), \]

where $n$ is the discrete time sample, $\{x_i\}_{i=1}^{LEN}$ is the time series of length $LEN$ and $a$ is the bifurcation parameter. The values of $a$ are chosen such that the resulting time series is periodic ($a = 3.83$), weakly chaotic ($a = 3.75$), strongly chaotic ($a = 3.9$) and fully chaotic $^4$ ($a = 4.0$). The length of the time series generated is varied as $LEN = 20, 40, 80, 100, 150$ and $200$. For each of these lengths, 50 different sets of time series are generated from distinct initial conditions chosen randomly from the interval $(0, 1)$. The resulting ensemble of time series is converted into a symbolic sequence with two symbols: ‘0’ corresponding to the interval $(0, 0.5]$ and ‘1’ corresponding to interval $[0.5, 1)$. These symbolic sequences are subsequently analyzed by the four complexity measures.

Figure 2 shows the mean complexity values vs. length of time series. We make the following observations. Mean Shannon entropy values are fairly constant as the length of the time series is varied. Though entropy is able to separate between the chaotic and periodic time series, it does poorly to distinguish between strong chaos ($a = 3.9$) and full chaos ($a = 4.0$). Both mean $LZ$ and mean $ETC$ values show good separation for

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$^4$Fully chaotic refers to the absence of attracting periodic orbits and the Lyapunov exponent is positive and reaches maximum at $a = 4.0$ indicating the highest degree of chaos.
the four cases, especially with increasing lengths. SubSym is able to clearly separate the periodic ($a = 3.83$) from the chaotic, though it is unable to distinguish between weak chaos ($a = 3.75$), strong chaos ($a = 3.9$) and full chaos ($a = 4.0$).

For the next experiment, we are interested in determining the minimum length of time series needed by these measures to classify periodic time series from chaotic time series. We consider two non-linear chaotic maps – the 1D logistic map (defined previously) and the 2D Hénon map [26] given by:

\[
\text{Hénon map: } x_{n+1} = 1 - ax_n^2 + y_n, \quad y_{n+1} = bx_n.
\]

For this experiment, we consider only two choices for the parameter $a$ for the 1D logistic map: $a = 3.83$ and $a = 4.0$ which correspond to periodic and chaotic behaviour respectively. Similarly, for the 2D Hénon map, with the parameter $b$ fixed at 0.3, the two choices of parameter $a$ are $a = 1.3$ and $a = 1.4$ corresponding to periodic and chaotic behaviour respectively. As before, we take 2 equal sized bins for creating the symbolic sequence, and 50 trials. The aim of this exercise is to determine that length of the time series, below which the measure fails to distinguish between periodic and chaotic cases.

Table 4 shows the minimum length of time series required to statistically distinguish between periodic and chaotic time series for the four measures. We can infer that – for the 1D logistic map, all the measures succeed in discriminating periodic from chaotic time series, though they need different minimum lengths to do the job. For the 2D Hénon map, both Shannon entropy and SubSym fail to discriminate even for lengths as large as 2000. ETC has the best (lowest minimum length) performance for both maps.
Table 4. Minimum length of time series required for statistically distinguishing between periodic and chaotic time series (obtained from randomly chosen initial conditions over 50 trials) from the chosen chaotic maps (logistic and Hénon). ETC requires the least length of time series.

| Chaotic Map       | Min. Len. | Min. Len. | Min. Len. | Min. Len. |
|-------------------|-----------|-----------|-----------|-----------|
| H                 | 15        | 8         | 6         | 10        |
| Hénon map         | > 2000    | 30        | 15        | > 2000    |

Fig. 3. A 2-state discrete Markov chain with transition probabilities $0 < \alpha, \beta < 1$.

Fig. 4. The entropy rate $H_{\text{theor}}$ of discrete 2-state Markov chains for different values of $\alpha$ and $\beta$. The entropy rate is a good measure of complexity of the 2-state Markov chain.

3.3 Markov chains

So far we have only considered deterministic signals. Signals in the real world are almost always stochastic in nature. Markov processes are examples of stochastic processes which has the property that the current state of the system depends only on the immediate past state, but not on the sequence of past states prior to that. Markov chains play an important role in modeling and analysis of large class of stochastic signals in the real world. For example, Markov chains are employed for analyzing cardiovascular dynamics [27–30], base compositions of DNA sequences [31–34], speech recognition [35,36,36,37], modeling of language scripts [38–40], information sources [41–43], financial time series analysis [44,45] and several others.
Fig. 5. Performance of the complexity measures on time series generated from 2-state Markov chains with fixed $\beta = 0.3$ and varying $\alpha$ (between 0.001 to 0.99999 in steps of 0.01). (a) Top: length = 25, (b) middle: length = 40, (c) bottom: length = 65. For easy of visual comparison, we have normalized the graphs by dividing the entire plot by the maximum value of that particular measure. This will only scale the entire graph and not alter the CC in any way.
Fig. 6. Performance of the complexity measures on time series generated from 2-state Markov chains with fixed $\beta = 0.3$ and varying $\alpha$ (between 0.001 to 0.99999 in steps of 0.01). (a) Top: length = 100, (b) middle: length = 200, (c) bottom: length = 500. For easy of visual comparison, we have normalized the graphs by dividing the entire plot by the maximum value of that particular measure. This will only scale the entire graph and not alter the CC in any way.
We consider 2-state discrete Markov chains as depicted in Figure 3 with the following transition probability matrix:

\[
TPM = \begin{bmatrix}
1 - \alpha & \alpha \\
\beta & 1 - \beta
\end{bmatrix}
\]

(2)

where \(0 < \alpha, \beta < 1\). Such a discrete Markov chain is ergodic, admits a stationary distribution, and thus we can explicitly compute the entropy rate (in bits) \([11]\) of the Markov chain as:

\[
\text{Entropy rate} = H_{\text{theor}} = \frac{\beta}{\alpha + \beta} H_{\text{BEF}}(\alpha) + \frac{\alpha}{\alpha + \beta} H_{\text{BEF}}(\beta),
\]

(3)

where \(H_{\text{BEF}}(.)\) is the binary entropy function (in bits) given by:

\[
H_{\text{BEF}}(p) = -p \log_2(p) - (1 - p) \log_2(1 - p).
\]

(4)

3.3.1 Complexity of 2-state Markov chains

The entropy rate of the 2-state Markov chain as given in equation (3) is a very good measure of the complexity of the chain which we can compute analytically given the TPM. We depict the nature of the entropy rate for a few choices of \(\beta\) (with \(\alpha\) varying) in Figure 4. An intuitive understanding of the entropy rate is the following – it is the number of bits needed to represent or encode the current state of the Markov chain, given the immediate past state.

We shall analyze time series generated from 2-state Markov chains and characterize its complexity using the four measures (\(H\), \(ETC\), \(LZ\) and \(SubSym\)). To this end, we consider 2-state Markov chains with a fixed representative value for \(\beta = 0.3\), but varying \(\alpha\) (between 0.001 to 0.99999 in steps of 0.01). The choice of \(\beta\) is arbitrary and in what follows, we expect our inferences and analysis to largely hold for other values of \(\beta\) as well.

We generate several time series from a 2-state Markov chain \((\beta = 0.3, 0.001 \leq \alpha \leq 0.99999)\) with varying lengths. The lengths of time series are 25, 40, 65, 100, 200 and 500. The first three lengths (25, 40, 65) serve as very short length time series which we may typically encounter in neuroscience and other biomedical applications, while the last three lengths (100, 200, 500) are short to medium length time series encountered in other applications. We omit very large length time series for our study (for large length time series several complexity measures are efficient, it is the short and medium length time series that are very hard to characterize and that is the focus of our study).

In Figures 5 and 6, we show the plots of the performance of the three measures namely \(ETC\), \(LZ\), \(SubSym\) for very short length time series (lengths = 25, 40, 65) and medium length time series (lengths = 100, 200, 500) respectively. We also compute the ‘Entropy rate’ (theoretical or \(H_{\text{theor}}\)) using equation (3) which ideally characterizes the complexity of the Markov chain (given knowledge of TPM). Additionally, we also estimate the Entropy rate from the empirically generated time series (\(H_{\text{empr}}\)). This quantity, termed as ‘Entropy rate (Empirical)’ or \(H_{\text{empr}}\), is estimated using the following equation:

\[
H_{\text{empr}} = \lim_{n \to \infty} H(X_n | X_{n-1}) = \lim_{n \to \infty} H(X_n, X_{n-1}) - H(X_{n-1}),
\]

(5)
Table 5. Complexity of 2-state Markov chains. The correlation coefficient (CC) of the three complexity measures \((ETC, LZ\) and \(SubSym\)) with \(H_{theor}\) is computed for varying length of time series from the chain. We have taken \(\beta = 0.3\) and \(\alpha\) varying between 0.001 to 0.99999 in steps of 0.01. We also indicate the correlation coefficient of the empirically determined entropy rate \(H_{empr}\) of the time series with \(H_{theor}\) for purposes of comparison.

| Length | \(CC(H_{theor}, H_{empr})\) | \(CC(H_{theor}, ETC)\) | \(CC(H_{theor}, LZ)\) | \(CC(H_{theor}, SubSym)\) |
|--------|------------------|------------------|------------------|------------------|
| 500    | 0.9877           | 0.9777           | 0.9731           | 0.7222           |
| 200    | 0.9709           | 0.9519           | 0.9516           | 0.8213           |
| 100    | 0.9599           | 0.9209           | 0.9043           | 0.8056           |
| 65     | 0.8960           | 0.8690           | 0.8577           | 0.8609           |
| 40     | 0.8974           | 0.8652           | 0.8068           | 0.8642           |
| 25     | 0.7679           | 0.7352           | 0.6828           | 0.7657           |

where the random variable \(X_n\) and \(X_{n-1}\) correspond to discrete time instants \(n\) and \(n-1\) respectively (for implementation, we drop the limit and take the entire available length of the time series).

Since \(H_{theor}\) is the ideal characterization of the complexity of the 2-state Markov chain (for a fixed \(\beta\), and varying \(\alpha\)), we compute the correlation coefficient\(^5\) (CC) between the four measures \((H_{empr}, ETC, LZ, SubSym)\) and \(H_{theor}\). This is shown in Table 5.

We draw the following conclusions:

- In all instances, the empirically estimated entropy rate \((H_{empr})\) comes closest to the theoretically computed entropy rate \((H_{theor})\), as indicated by the best (highest) values of correlation coefficient (CC).

- For very low length time series, \(SubSym\) outperforms both \(LZ\) and \(ETC\). This is indicated by a higher value of correlation coefficient (refer to Tab. 5).

- For medium length time series, \(ETC\) outperforms both \(LZ\) and \(SubSym\) complexity measures (refer to Tab. 5).

- As expected, for smaller lengths, there is a huge variation in the values of the complexity measures (as can be seen from the plots).

We now consider the final experiment of distinguishing between tonic (regular) spiking patterns and irregular (chaotic) spiking patterns from a spiking neuron model.

### 3.4 Spiking neuron model

There are \(\approx 86\) billion neurons in the human central nervous system. These cells are the basic information processing units and they form a highly complex network. These neurons respond to a stimulus (either an external or an internal stimulus) by firing different patterns of short duration cell membrane potential spikes (known as ‘action potentials’). These carry important information about the stimuli [46].

In order to investigate the dynamics of neuronal behaviour, different models of their firing have been proposed. One such single spiking neuron model is the ‘Adaptive Exponential Integrate-and-Fire’ (AdEx) model, initially proposed by Brette and Gerstner [47]. This is a simple and realistic model used commonly and has only two

\(^5\)The standard Pearson’s linear correlation coefficient defined as \(CC(A, B) = \frac{\text{covariance}(A, B)}{\sqrt{\text{variance}(A)} \sqrt{\text{variance}(B)}}\) is used.
Fig. 7. Membrane potential $V(t)$ (mV) vs. time (ms) from the ‘Adaptive Exponential Integrate-and-Fire’ (AdEx) neuron firing model used for the study. Top: tonic spiking waveform showing periodic spikes. Bottom: irregular spiking waveform showing chaotic behaviour.

The equations and a reset condition. The different firing patterns produced by this model are described in Naud et al. [48]. The equations governing the model are:

$$C \frac{dV}{dt} = -g_L(V - E_L) + g_L \Delta T \exp \left( \frac{V - V_T}{\Delta T} \right) + I - w,$$

$$\tau_w \frac{dw}{dt} = a(V - E_L) - w.$$

In the above equations, $V(t)$ is the membrane potential, $I(t)$ is the injection current and $w$ is the adaptation current. The model has 9 parameters: total capacitance
Fig. 8. Shannon entropy of tonic and irregular spiking using a moving window of 50 ms with 90% overlap. Mean entropy is higher for tonic than that of irregular spiking which is not intuitive.

Fig. 9. LZ complexity measures of tonic and irregular spiking using a moving window of 50 ms with 90% overlap. Mean LZ for tonic and irregular spiking are similar indicating the inability of LZ complexity measure to distinguish between the two spike trains.

\(C\), total leak conductance \(g_L\), effective rest potential \(E_L\), threshold slope factor \(\Delta T\), effective threshold potential \(V_T\), conductance \(a\), time constant \(\tau_w\), spike triggered adaptation \(b\) and reset potential \(V_r\). When the current drives the voltage beyond the threshold value of \(V_T\), the exponential term causes a sharp increase (upswing) in the action potential. The rise is stopped at reset threshold that has been fixed at 0 mV. The decrease (downswing) in the action potential is represented by the reset condition: if \(V > 0\) mV then \(V \rightarrow V_r, w \rightarrow w_r\) where \(w_r = w + b\).

By varying the 9 parameters, various firing patterns can be obtained [48], such as – tonic spiking, adaptation, initial burst, regular bursting, delayed accelerating,
Fig. 10. ETC complexity measures of tonic and irregular spiking using a moving window of 50 ms with 90% overlap. There is a clear difference for the two spiking patterns. The mean ETC for irregular spiking is greater than mean ETC for tonic spiking indicating that ETC is able to distinguish between the two spike trains.

Fig. 11. SubSym complexity measure of tonic and irregular spiking using a moving window of 50 ms with 90% overlap. Mean SubSym is higher for tonic than that of irregular spiking which is not intuitive.

delayed regular bursting, transient spiking and irregular spiking. We are interested in two firing patterns namely: tonic spiking and irregular spiking. These are shown in Figure 7. In tonic spiking, spikes are produced at regular intervals, whereas the inter-spike intervals in irregular spiking are aperiodic and is also highly sensitive to initial conditions (a hallmark of chaos). Thus, discriminating between these two patterns is akin to distinguishing periodic signals from chaotic ones.

We follow the procedure of [22] for converting our waveforms into symbolic sequences of 0s and 1s. The entire waveform is partitioned into non-overlapping time
windows of length 1 ms. In each window, if one or more spikes are present, it is coded as ‘1’, else coded as ‘0’. The four complexity measures (\(H\), \(LZ\), \(ETC\) and \(SubSym\)) are estimated for the symbolic sequence in a moving time window of 50 ms (with an overlap of 90% between successive windows) and the results are shown in Figures 8–11.

We make the following observations based on our results. Across both the spike trains, it can be observed that LZ shows similar variations while ETC shows greater variation for the irregular spike train as compared to the tonic case. Further, statistical evaluation shows that the mean LZ values of both tonic and irregular spike trains are same while there is an appreciable difference between the means of the ETC measure. This is a clear indicator that the periodic and chaotic nature of the tonic and irregular spike trains are well discriminated by ETC measure while LZ fails to do so. In the cases of Shannon entropy and \(SubSym\) measures, mean values of the tonic is higher than that of irregular spiking. This is undesirable (as well as non-intuitive)\(^6\) and hence limits the use of these measures for the purpose of discrimination.

To further validate the results, the complexity values from each of the windows (50 ms duration) were used as sample values and statistical difference was analyzed using confidence interval plots. Using the same sample values, 2 sample student’s \(t\)-test was also done to complement the graphical analysis with the more formal hypothesis test.

The \(t\)-test results may be summarized as follows:

- The mean LZ complexity of irregular spiking pattern \((0.62 \pm 0.12)\) is not significantly greater \((t_{36} = 0, p = 0.50)\) than that of the tonic spiking pattern \((0.62 \pm 0.10)\).
- The mean ETC complexity of irregular spiking pattern \((0.28 \pm 0.04)\) is significantly greater \((t_{36} = 5.84, p = 0.00)\) than that of the tonic spiking pattern \((0.21 \pm 0.03)\).
- The mean Shannon entropy \(H\) of irregular spiking pattern \((0.37 \pm 0.09)\) is not significantly greater \((t_{36} = -4.93, p = 1.00)\) than that of the tonic spiking pattern \((0.47 \pm 0.03)\).
- The mean \(SubSym\) complexity of irregular spiking pattern \((0.65 \pm 0.10)\) is not significantly greater \((t_{36} = -6.51, p = 1.00)\) than that of the tonic spiking pattern \((0.80 \pm 0.01)\).

Based on the sample data, at a 5% significance level (overall error rate) for the statistical test, there is sufficient evidence to conclude that the mean LZ, mean Shannon entropy \(H\) and mean \(SubSym\) complexity measures of irregular spiking patterns are not significantly higher than those of the corresponding tonic patterns, while the mean ETC complexity measure of irregular spiking pattern is significantly higher than that of the tonic pattern. Thus we assert that ETC complexity measure is able to characteristically (statistically) differentiate between the two neuron firing patterns, while the other three complexity measures fail to do so.

4 Discussion and conclusions

Complexity has many facets and perspectives. Inspired by the original categorization of Lloyd, we have chosen three specific viewpoints – effort-to-describe (Shannon Entropy and Lempel-Ziv Complexity), effort-to-compress (ETC) and degree-of-order

\(^6\)This strange behaviour of Shannon entropy and \(SubSym\) measures needs further investigation.
Each of these perspectives is distinct, unique and attempts to capture a particular notion of complexity. They are not strictly independent of each other. For example, entropy and lossless compression are deeply related (through Shannon’s theorems [10,11]). Compressible patterns are typically symmetrical (with a higher degree of intrinsic organization) whereas random incompressible patterns are mostly asymmetrical (lower degree of intrinsic organization). The effort-to-compress is implicitly dependent upon how redundant or compressible the data is. However, two sequences (e.g. ‘011010’ and ‘011100’ in Tab. 3) could have identical values of Entropy and Lempel-Ziv complexity, but differ in the effort-to-compress and the degree-of-organization complexities.

‘Subsymmetries’ relates to intrinsic order of arrangements of the symbols that make up a string. It relates to the perspective of visual complexity which is rooted in the cognitive ability of humans to perceive structure/order and distinguish it from the lack of it. A normalized measure based on subsymmetries (SubSym) was proposed for the first time in this paper. SubSym takes a large number of distinct values for binary strings which is desirable since it allows to distinguish different binary patterns more finely. This new measure deserves to be tested further in various applications.

ETC, a recently proposed measure, has intuitive appeal, is easy to compute, and also demonstrates superior performance in discriminating between periodic and chaotic time series (both for nonlinear chaotic dynamical systems as well as for a spiking neuron model). From a modeling perspective, Shannon entropy is a statistical approach to modeling, Lempel-Ziv complexity is a learning based approach, ETC is a hybrid of statistical and learning based approach whereas SubSym is purely structural or pattern based approach.

We analyzed the complexities of 2-state Markov chains and stochastic time series generated by them. The entropy rate is an ideal measure to characterize the complexity of a Markov chain, provided we know the transition probability matrix (TPM). We generated several stochastic time series from 2-state Markov chains by varying $\alpha$ (the probability of transitioning from state ‘0’ to state ‘1’). Interestingly, for very short length time series, SubSym outperforms both ETC and LZ measures. For medium length time series, ETC does better than SubSym and LZ. But, in all cases, the empirically estimated entropy rate was the closest to the theoretical value of entropy rate. However, as the number of states and the order of the Markov chain increases, we need to study how these measures perform. This would have serious implications on real applications as well.

We conclude by noting that complexity remains a multi-dimensional notion and no single measure can capture it in its entirety. Several perspectives and measures need to work in unison to help us arrive at a better understanding of complexity.

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