Research Article

Radius Constants for Analytic Functions with Fixed Second Coefficient

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Let \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) be analytic in the unit disk with the second coefficient \( a_2 \) satisfying \( |a_2| \leq 1 \). Sharp radius of Janowski starlikeness is obtained for functions \( f \) whose \( n \)th coefficient satisfies \( |a_n| \leq n (n \geq 2) \). Other radius constants are also obtained for these functions, and connections with earlier results are made.

1. Introduction

Let \( \mathcal{A} \) denote the class of analytic functions \( f \) defined in the open unit disk \( D := \{ z \in \mathbb{C} : |z| < 1 \} \), normalized by \( f(0) = 0 = f'(0) - 1 \), and let \( \mathcal{S} \) denote its subclass consisting of univalent functions. If \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S} \), de Branges [1] obtained the sharp coefficient bound that \( |a_n| \leq n (n \geq 2) \). However, the inequality \( |a_n| \leq n, n \geq 2 \), is not sufficient for \( f \) to be univalent; for example, \( f(z) = z + 2z^2 \) is clearly not a member of \( \mathcal{S} \).

Several subclasses of \( \mathcal{S} \) possess a similar coefficient bound. For instance, the \( n \)th coefficients of starlike functions, convex functions in the direction of imaginary axis, and close-to-convex functions satisfy \( |a_n| \leq n (n \geq 2) \) [2–4]. Other examples include functions which are convex, starlike of order \( 1/2 \), and starlike with respect to symmetric points. The \( n \)th coefficients of these functions satisfy \( |a_n| \leq 1 (n \geq 2) \) [5–7]. The \( n \)th coefficient of close-to-convex functions with argument \( \beta \) satisfies \( |a_n| \leq 1 + (n-1) \cos \beta \) [8], and the coefficients of uniformly starlike functions are bounded by \( 2/n \) [9], while \( |a_n| \leq 1/n \) [10] for uniformly convex functions. Simple examples show that these bounds are not sufficient to characterize the geometric properties of the classes of functions.

In the sequel, we will assume that \( f \in \mathcal{A} \) has the Taylor expansion of the form \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \). Gavrilov [11] showed that the radius of univalence for functions \( f \in \mathcal{A} \) satisfying \( |a_2| \leq n (n \geq 2) \) is the real root \( r_0 = 0.164 \) of the equation \( 2(1-r)^3 - (1+r) = 0 \), and the result is sharp for \( f(z) = 2z - z/(1-z)^2 \). Gavrilov also proved that the radius of univalence for functions \( f \in \mathcal{A} \) satisfying the coefficient bound \( |a_2| \leq M (n \geq 2) \) is \( 1 - \sqrt{M/(1+M)} \). The condition \( |a_2| \leq M \) clearly holds for functions \( f \in \mathcal{A} \) satisfying \( f(z) \leq M \), and for these functions, Landau [12] proved that the radius of univalence is \( M - \sqrt{M^2-1} \). In fact, Yamashita [13] showed that the radius of univalence obtained by Gavrilov [11] is also the radius of starlikeness for functions \( f \in \mathcal{A} \) satisfying \( |a_2| \leq n \) or \( |a_2| \leq M \). Additionally, Yamashita [13] determined that the radius of convexity for functions \( f \in \mathcal{A} \) satisfying \( |a_2| \leq n \) is the real root \( r_0 = 0.090 \) of the equation \( 2(1-r)^4 - (1+4r+r^2) = 0 \), while the radius of convexity for functions \( f \in \mathcal{A} \) satisfying \( |a_2| \leq M \) is the real root of

\[
(M+1)(1-r)^3 - M(1+r) = 0.
\]

Recently, Kalaj et al. [14] obtained the radii of univalence, starlikeness, and convexity for harmonic mappings satisfying certain coefficient inequalities.
For any analytic functions \( f \) and \( g \), the function \( f \) is subordinate to \( g \), denoted by \( f < g \), if there is an analytic self-map \( w \) of \( \mathbb{D} \) with \( w(0) = 0 \) satisfying \( f(z) = g(w(z)) \). If \( g \) is univalent, then \( f < g \) is equivalent to \( f(0) = g(0) \) and \( f(\mathbb{D}) \subseteq g(\mathbb{D}) \).

For \( \beta \in \mathbb{R} \setminus \{1\} \), \( \alpha \geq 0 \), the class \( \mathcal{L}(\alpha, \beta) \) consists of functions \( f \in \mathcal{A} \) satisfying
\[
\frac{z^2 f''(z)}{f(z)} + \frac{z f'(z)}{f(z)} < 1 + (1 - 2\beta) \frac{1 + \beta}{1 - z} \, .
\] (2)

Denote by \( \mathcal{L}_0(\alpha, \beta) \) its subclass consisting of functions \( f \in \mathcal{A} \) satisfying
\[
\frac{z^2 f''(z)}{f(z)} + \frac{z f'(z)}{f(z)} - 1 \leq |1 - \beta| \quad (\beta \in \mathbb{R} \setminus \{1\}, \alpha \geq 0) \, .
\] (3)

These classes were investigated in [15–24].

For \( \beta < 1 \), the class \( \mathcal{L}(0, \beta) \), the class of starlike functions of order \( \beta \), while, for the case \( \beta > 1 \), the class was studied in [25–28].

The class \( \mathcal{S}_T[A, B] \) of Janowski starlike functions [29] consists of functions \( f \in \mathcal{A} \), satisfying the subordination
\[
\frac{zf'(z)}{f(z)} < \frac{1 + Az}{1 + Bz} \quad (-1 \leq B < A \leq 1) \, .
\] (4)

Certain well-known subclasses of starlike functions are special cases of \( \mathcal{S}_T[A, B] \) for appropriate choices of the parameters \( A \) and \( B \). For example, for \( 0 \leq \beta < 1 \), \( \mathcal{S}_T(\beta) \) is the class of starlike functions of order \( \beta \). Denote by \( \mathcal{S}_T[A, B] \) the class \( \mathcal{S}_T[A, B] \) consisting of functions \( f \) in \( \mathcal{A} \) for which \( 0 \leq \beta < 1 \).

This paper studies the class \( \mathcal{A}_b \) consisting of functions \( f(z) = z + \sum_{n=2}^\infty a_n z^n \), \( |a_2| = 2b, 0 \leq b \leq 1 \), in the disk \( \mathbb{D} \). The subclass of univalent functions in \( \mathcal{A}_b \) has been studied in [30–33]. In [33], Ravichandran obtained sharp radii of starlikeness and convexity of order \( \alpha \) for functions \( f \in \mathcal{A}_b \) satisfying \( |a_n| \leq n \) or \( |a_n| \leq M, n \geq 3 \). The author also obtained the radius of uniform convexity and parabolic starlikeness for functions \( f \in \mathcal{A}_b \) satisfying \( |a_n| \leq n, n \geq 3 \).

This paper finds radius constants for functions \( f(z) = z + \sum_{n=2}^\infty a_n z^n \in \mathcal{A}_b \) satisfying either \( |a_n| \leq cn + d \) \( (c, d \geq 0) \) or \( |a_n| \leq c/n \) \( (c > 0, n \geq 3) \). In the next section, sharp \( \mathcal{L}(\alpha, \beta) \)-radius and \( \mathcal{S}_T[A, B] \)-radius are derived for these classes. Several known radius constants are shown to be special cases of the results obtained.

### 2. Radius Constants

A sufficient condition for functions \( f \in \mathcal{A} \) to belong to the class \( \mathcal{L}(\alpha, \beta) \) is given in the following lemma.

**Lemma 1** (see [24, 34]). Let \( \beta \in \mathbb{R} \setminus \{1\} \) and \( \alpha \geq 0 \). If \( f(z) = z + \sum_{n=2}^\infty a_n z^n \in \mathcal{A} \) satisfies the inequality
\[
\sum_{n=2}^\infty (an^2 + (1 - \alpha)n - \beta) |a_n| \leq |1 - \beta| \, .
\] (5)

then \( f \in \mathcal{L}(\alpha, \beta) \).

Making use of this lemma, the sharp \( \mathcal{L}(\alpha, \beta) \)-radius is obtained for \( f \in \mathcal{A}_b \) satisfying the coefficient inequality \( |a_n| \leq cn + d \).

**Theorem 2.** Let \( \beta \in \mathbb{R} \setminus \{1\} \), \( 6\alpha + 3 - \beta \geq 0 \), and \( \alpha \geq 0 \). The \( \mathcal{L}(\alpha, \beta) \)-radius for \( f(z) = z + \sum_{n=2}^\infty a_n z^n \in \mathcal{A}_b \), satisfying the coefficient inequality \( |a_n| \leq cn + d \), \( c, d \geq 0, n \geq 3 \), is the real root in \( (0, 1) \) of the equation
\[
(\alpha \beta + c + d)(1 - \beta) + 2(\alpha \beta + c + d)(1 - r) + (1 - r)^2
\] (6)

For \( \beta < 1 \), this number is also the \( \mathcal{L}(\alpha, \beta) \)-radius of \( f \in \mathcal{A}_b \). The results are sharp.

**Proof.** The number \( r_0 \) is the \( \mathcal{L}(\alpha, \beta) \)-radius for \( f \in \mathcal{A}_b \) if and only if \( f(r_0 z) = f(z) \) in \( \mathcal{L}(\alpha, \beta) \). Therefore, by Lemma 1, it is sufficient to verify the inequality
\[
\sum_{n=2}^\infty (an^2 + (1 - \alpha)n - \beta) |a_n| r_0^n \leq |1 - \beta| \, .
\] (7)

where \( r_0 \) is the real root in \( (0, 1) \) of (6). Using the known expansions
\[
\sum_{n=3}^\infty r_0^{n-1} = \frac{1}{1 - r_0} - 1 - r_0, \quad n \geq 3, (8)
\]

\[
\sum_{n=3}^\infty n^2 r_0^{n-1} = \frac{1}{(1 - r_0)^3} - 1 - 4r_0, \quad n \geq 3, (10)
\]

leads to
\[
\sum_{n=2}^\infty (an^2 + (1 - \alpha)n - \beta) |a_n| r_0^n \leq 2(\alpha \beta + c + d)br_0
\]

\[
= 2(\alpha \beta + c + d)br_0 + ca\left(\frac{1 + 4r_0 + r_0^2}{(1 - r_0)^3} - 1 - 8r_0\right)
\]

\[
+ ((1 - \alpha)c + ad)\left(\frac{1 + r_0}{(1 - r_0)^3} - 1 - 4r_0\right)
\]

\[
+ ((1 - \alpha)d - \beta c)\left(\frac{1}{(1 - r_0)^2} - 1 - 2r_0\right)
\]

\[
- \beta d\left(\frac{1}{1 - r_0} - 1 - r_0\right)
\]
\[ (c + d) (\beta - 1) - (2x + 2 - \beta) (2 (c - b) + d) r_0 \]
\[ + \left( c a (1 + 4 r_0 + r_0^2) + ((1 - \alpha) c + a d) (1 - r_0^2) \right) \]
\[ + ((1 - \alpha) d - \beta c) (1 - r_0) \]
\[ - \beta d (1 - r_0) \]
\[ = |1 - \beta| \cdot \tag{12} \]

For \( \beta < 1 \), consider the function
\[ f_0 (z) = z - 2 b z^2 - \sum_{n=3}^{\infty} (c n + d) z^n \]
\[ = (c + 1) z + 2 (c - b) z^2 - \frac{c z}{1 - z} - \frac{d z^3}{1 - z} \cdot \tag{13} \]

At the root \( z = r_0 \) in \((0,1)\) of \((6)\), \( f_0 \) satisfies
\[ \text{Re} \left( \frac{z^2 f''_0 (z)}{f_0 (z)} + \frac{z f'_0 (z)}{f_0 (z)} \right) = 1 - \frac{N (r_0)}{D (r_0)} = \beta \cdot \tag{14} \]

where
\[ N (r_0) = -2 (c - b) (2x + 1) r_0 + \frac{2 c r_0 (2x + 1)}{1 - r_0} \]
\[ + \frac{6 c a r_0^2}{(1 - r_0)^3} + \frac{2 d r_0^2 (3x + 1)}{1 - r_0} \]
\[ + \frac{d r_0^3 (6x + 1)}{(1 - r_0)^2} + \frac{2 d r_0^3 a}{(1 - r_0)^2} - \frac{d r_0^3}{1 - r_0} \cdot \tag{15} \]
\[ D (r_0) = c + 1 + 2 (c - b) r_0 - \frac{c}{1 - r_0} - \frac{d r_0}{1 - r_0} \cdot \]

This shows that \( r_0 \) is the sharp \( \mathcal{L}(\alpha, \beta) \)-radius for \( f \in \mathcal{A}_b \). For \( \beta < 1 \), \((14)\) shows that the rational expression \( N(r_0)/D(r_0) \) is positive, and therefore the equality
\[ \left| \frac{z^2 f''_0 (z)}{f_0 (z)} + \frac{z f'_0 (z)}{f_0 (z)} - 1 \right| = 1 - \beta \cdot \tag{16} \]

holds. Thus, \( r_0 \) is the sharp \( \mathcal{L}(\alpha, \beta) \)-radius for \( f \in \mathcal{A}_b \) when \( \beta < 1 \).

For \( \beta > 1 \), the function
\[ f_0 (z) = z + 2 b z^2 + \sum_{n=3}^{\infty} (c n + d) z^n \]
\[ = (1 - c) z + 2 (b - c) z^2 + \frac{c z}{1 - z} + \frac{d z^3}{1 - z} \cdot \tag{17} \]

demonstrates sharpness of the result. The derivation is similar to the case \( \beta < 1 \) and is omitted. \qed

Theorem 3. Let \( \beta \in \mathbb{R} \setminus \{1\} \) and \( \alpha \geq 0 \). The \( \mathcal{L}(\alpha, \beta) \)-radius of \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}_b \) satisfying the coefficient inequality \( |a_n| \leq c/n \) for \( n \geq 3 \) and \( c > 0 \) is the real root in \((0,1)\) of the equation
\[ \left[ c (1 - \beta) + |1 - \beta| + (2x + 2 - \beta) r \left( \frac{c}{2} - 2b \right) \right] (1 - r)^2 \]
\[ = c \alpha + (1 - \alpha) c (1 - r) + \beta c (1 - r)^2 \log(1 - r) \cdot \tag{18} \]

For \( \beta < 1 \), this number is also the \( \mathcal{L}_0(\alpha, \beta) \)-radius of \( f \in \mathcal{A}_b \). The results are sharp.

Proof. By Lemma 1, \( r_0 \) is the \( \mathcal{L}(\alpha, \beta) \)-radius of functions \( f \in \mathcal{A}_b \) when inequality \((7)\) holds for the real root \( r_0 \) of \((18)\) in \((0,1)\). Using \((8)\) and \((9)\) together with
\[ \sum_{n=3}^{\infty} r_0^{n-1} = -\frac{\log(1 - r_0)}{r_0} - 1 - \frac{r_0}{2} \cdot \tag{19} \]
leads to
\[ \sum_{n=3}^{\infty} \left( a_n^2 + (1 - \alpha) n - \beta \right) |a_n| r_0^{n-1} \]
\[ \leq 2 (2x + 2 - \beta) |br_0| \]
\[ + \sum_{n=3}^{\infty} \left( a_n^2 + (1 - \alpha) n - \beta \right) \left( \frac{c}{n} \right) r_0^{n-1} \]
\[ = 2 (2x + 2 - \beta) |br_0| + c a \left( \frac{1}{1 - r_0} - 1 - 2r_0 \right) \]
\[ + (1 - \alpha) c \left( \frac{1}{1 - r_0} - 1 - r_0 \right) \]
\[ - \beta c \left( \frac{- \log(1 - r_0) - 1 - r_0}{r_0} \right) \]
\[ = c (\beta - 1) + (2x + 2 - \beta) r_0 \left( 2b - \frac{c}{2} \right) \]
\[ + c a r_0 + (1 - \alpha) c (1 - r_0) r_0 + \beta c (1 - r_0)^2 \log(1 - r_0) \]
\[ = \left|1 - \beta\right| \cdot \tag{20} \]

To verify sharpness for \( \beta < 1 \), consider the function
\[ f_0 (z) = z - 2 b z^2 - \sum_{n=3}^{\infty} c_n z^n \]
\[ = (1 + c) z + \left( \frac{c}{2} - 2b \right) z^2 + c \log(1 - z) \cdot \tag{21} \]

At the root \( z = r_0 \) in \((0, 1)\) of (18), \( f_0 \) satisfies
\[
\text{Re} \left( \alpha \frac{z^2 f''_0(z) + zf'_0(z)}{f_0(z)} \right) = 1 - \left( -\frac{c}{2} - 2b \right) r_0 (2\alpha + 1) + \frac{c r_0 \alpha}{(1 - r_0)^2}
+ \frac{c}{1 - r_0} + \frac{c \log(1 - r_0)}{r_0}
\times \left( 1 + c + \left( \frac{c}{2} - 2b \right) r_0 + \frac{c \log(1 - r_0)}{r_0} \right)^{-1} = \beta.
\] (22)

Thus, \( r_0 \) is the sharp \( L(\alpha, \beta) \)-radius for \( f \in \mathcal{A}_b \). For \( \beta < 1 \), the rational expression in (22) is positive, and therefore
\[
\left| \alpha \frac{z^2 f''_0(z) + zf'_0(z)}{f_0(z)} - 1 \right| = 1 - \beta,
\] (23)
which shows that \( r_0 \) is the sharp \( L(\alpha, \beta) \)-radius for \( f \in \mathcal{A}_b \).

For \( \beta > 1 \), sharpness of the result is demonstrated by the function \( f_0 \) given by
\[
f_0(z) = z + 2bz^2 + \sum_{n=3}^{\infty} \frac{c}{n} z^n
= (1 - c) z + \left( 2b - \frac{c}{2} \right) z^2 - c \log(1 - z).
\] (24)

**Remark 4.** The results obtained above yield the following special cases.

(1) For \( \alpha = 0 \), \( \beta = 0 \), \( c = 1 \), \( d = 0 \), and \( 0 \leq b \leq 1 \), Theorem 2 yields the radius of starlikeness obtained by Yamashita [13].

(2) For \( \alpha = 0 \), \( c = 1 \), and \( d = 0 \), Theorem 2 reduces to Theorem 2.1 in [33, page 3]. When \( \alpha = 0 \), \( c = 0 \), and \( d = M \), Theorem 2 leads to Theorem 2.5 in [33, page 5].

(3) For \( \alpha = 0 \), Theorem 3 yields the radius of starlikeness of order \( \beta \) for \( f \in \mathcal{A}_b \) obtained by Ravichandran [33, Theorem 2.8].

The following result of Goel and Sohi [35] will be required in our investigation of the class of Janowski starlike functions.

**Lemma 5** (see [35]). Let \(-1 \leq B < A \leq 1\). If \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A} \) satisfies the inequality
\[
\sum_{n=2}^{\infty} ((1 - B)n - (1 - A)) |a_n| \leq A - B,
\] (25)
then \( f \in \mathcal{S}_T[A, B] \).

The next result finds the sharp \( \mathcal{S}_T[A, B] \)-radius for \( f \in \mathcal{A}_b \), satisfying the coefficient inequality \( |a_n| \leq cn + d \).

**Theorem 6.** Let \(-1 \leq B < A \leq 1\). The \( \mathcal{S}_T[A, B] \)-radius for \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}_b \) satisfying the coefficient inequality \( |a_n| \leq cn + d, n \geq 3 \) and \( c, d \geq 0 \), is the real root in \((0, 1)\) of the equation
\[
[(A - B) (c + d + 1)
- (2b - 2c - d) (2(1 - B) - (1 - A)) r] (1 - r)^3
= c (1 - B) (1 + r) + (d (1 - B) - c (1 - A)) (1 - r)
- (1 - A) d (1 - r)^2.
\] (26)

This radius is sharp.

**Proof.** It is evident that \( r_0 \) is the \( \mathcal{S}_T[A, B] \)-radius of \( f \in \mathcal{A}_b \) if and only if \( f(r_0 z)/r_0 \in \mathcal{S}_T[A, B] \). Hence, by Lemma 5, it suffices to show that
\[
\sum_{n=2}^{\infty} ((1 - B)n - (1 - A)) |a_n| r_0^{n-1} \leq A - B
\] (27)
\((-1 \leq B < A \leq 1),
where \( r_0 \) is the root in \((0, 1)\) of (26). From (8), (9), and (10), it follows that
\[
\sum_{n=2}^{\infty} ((1 - B)n - (1 - A)) |a_n| r_0^{n-1}
\leq 2 (2(1 - B) - (1 - A)) b r_0
+ \sum_{n=3}^{\infty} ((1 - B)n - (1 - A)) (cn + d) r_0^{n-1}
= 2 (2(1 - B) - (1 - A)) b r_0
+ c (1 - B) \left( \frac{1 + r_0}{1 - r_0} \right)^3 - 1 - 4r_0
+ (d (1 - B) - c (1 - A)) \left( \frac{1}{1 - r_0} - 1 - 2r_0 \right)
- (1 - A) d \left( \frac{1}{1 - r_0} - 1 - r_0 \right)
= (B - A) (c + d) + (2b - 2c - d)
\times (2(1 - B) - (1 - A)) r_0
+ (c (1 - B) (1 + r_0)
+ (d (1 - B) - c (1 - A)) (1 - r_0)
- (1 - A) d(1 - r_0)^2) \times (1 - r_0)^{-3}
= A - B.
\] (28)
The function \( f_0 \) given by (13) shows that the result is sharp. Indeed, at the point \( z = r_0 \) where \( r_0 \) is the root in \((0,1)\) of (26), the function \( f_0 \) satisfies

\[
\left| \frac{zf_0'(z)}{f_0(z)} - 1 \right| = \left( -2 (c-b) r_0 + \frac{2dr_0^2}{1-r_0} + \frac{dr_0^3}{(1-r_0)^2} + \frac{2cr_0}{1-r_0} \right) \times \left( c + 1 + 2 (c-b) r_0 - \frac{c}{1-r_0} - \frac{dr_0^2}{1-r_0} \right)^{-1},
\]

or equivalently \( f_0 \in \mathcal{S}[A, B] \).

**Theorem 7.** Let \(-1 \leq B < A \leq 1\). The \( \mathcal{S}[A, B] \)-radius for \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) satisfying the coefficient inequality \( |a_n| \leq c/n, n \geq 3 \) and \( c > 0 \), is the real root in \((0,1)\) of the equation

\[
(1-r) \left( c + 1 + 2 (c-b) r_0 - \frac{c}{1-r_0} - \frac{dr_0^2}{1-r_0} \right)^{-1}.
\]

Then, (26) yields

\[
\left| \frac{zf_0'(z)}{f_0(z)} - 1 \right| = \left| A - B \frac{zf_0'(z)}{f_0(z)} \right|, \quad (-1 \leq B < A \leq 1, z = r_0),
\]

or equivalently \( f_0 \in \mathcal{S}[A, B] \).

**Proof.** By Lemma 5, condition (27) assures that \( r_0 \) is the \( \mathcal{S}[A, B] \)-radius of \( f \in \mathcal{A}_b \) where \( r_0 \) is the real root of (31). Therefore, using (8) and (19) for \( f \in \mathcal{A}_b \) yields

\[
\sum_{n=2}^{\infty} ((1-B)n-(1-A)) |a_n| r_0^{n-1} \leq 2 (2(1-B)-(1-A)) B r_0 + \sum_{n=3}^{\infty} ((1-B)n-(1-A)) \left( \frac{c}{n} \right) r_0^{n-1} = 2 (2(1-B)-(1-A)) B r_0 + c (1-B) \left( \frac{1}{1-r_0} - 1 - r_0 \right)
\]

\[
= c (B-A) + (2(1-B)-(1-A)) r_0 \left( 2b - \frac{c}{2} \right)
\]

\[
+ c (1-B) r_0 + c (1-A) \left( 1-r_0 \right) \log \left( \frac{1-r_0}{r_0} \right)
\]

\[
\left( 1+\frac{c}{2-2b} \right) r_0 + c (1-B) \left( \frac{c}{2-2b} \right) r_0 + \frac{c A \log \left( \frac{1-r_0}{r_0} \right)}{r_0}
\]

\[
\left( 1+\frac{c}{2-2b} \right) r_0 + c (1-B) \left( \frac{c}{2-2b} \right) r_0 + \frac{c \log \left( \frac{1-r_0}{r_0} \right)}{r_0}
\]

at the root \( z = r_0 \) in \((0,1)\) of (31). Evidently, the function \( f_0 \) satisfies (30), and hence the result is sharp.

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.
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