First-order Logic: Modality and Intensionality

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Abstract. Contemporary use of the term ‘intension’ derives from the traditional logical Frege-Russell’s doctrine that an idea (logic formula) has both an extension and an intension. Although there is divergence in formulation, it is accepted that the extension of an idea consists of the subjects to which the idea applies, and the intension consists of the attributes implied by the idea. From the Montague’s point of view, the meaning of an idea can be considered as particular extensions in different possible worlds.

In this paper we analyze the minimal intensional semantic enrichment of the syntax of the FOL language, by unification of different views: Tarskian extensional semantics of the FOL, modal interpretation of quantifiers, and a derivation of the Tarskian theory of truth from unified semantic theory based on a single meaning relation. We show that not all modal predicate logics are intensional, and that an equivalent modal Kripke’s interpretation of logic quantifiers in FOL results in a particular pure extensional modal predicate logic (as is the standard Tarskian semantics of the FOL). This minimal intensional enrichment is obtained by adopting the theory of properties, relations and propositions (PRP) as the universe or domain of the FOL, composed by particulars and universals (or concepts), with the two-step interpretation of the FOL that eliminates the weak points of the Montague’s intensional semantics. Differently from the Bealer’s intensional FOL, we show that it is not necessary the introduction of the intensional abstraction in order to obtain the full intensional properties of the FOL.

Final result of this paper is represented by the commutative homomorphic diagram that holds in each given possible world of this new intensional FOL, from the free algebra of the FOL syntax, toward its intensional algebra of concepts, and, successively, to the new extensional relational algebra (different from cylindric algebras), and we show that it corresponds to the Tarski’s interpretation of the standard extensional FOL in this possible world.

1 Introduction

The simplest aspect of an expression’s meaning is its extension. We can stipulate that the extension of a sentence is its truth-value, and that the extension of a singular term is its referent. The extension of other expressions can be seen as associated entities that contribute to the truth-value of a sentence in a manner broadly analogous to the way in which the referent of a singular term contributes to the truth-value of a sentence. In many cases, the extension of an expression will be what we intuitively think of as its referent, although this need not hold in all cases, as the case of sentences illustrates. While
Frege himself is often interpreted as holding that a sentence’s referent is its truth-value, this claim is counterintuitive and widely disputed. We can avoid that issue in the present framework by using the technical term ‘extension’. In this context, the claim that the extension of a sentence is its truth-value is a stipulation.

'Extensional' is most definitely a technical term. Say that the extension of a name is its denotation, the extension of a predicate is the set of things it applies to, and the extension of a sentence is its truth value. A logic is extensional if coextensional expressions can be substituted one for another in any sentence of the logic "salva veritate", that is, without a change in truth value. The intuitive idea behind this principle is that, in an extensional logic, the only logically significant notion of meaning that attaches to an expression is its extension. An intensional logics is exactly one in which substitutivity fails for some of the sentences of the logic.

In "Über Sinn und Bedeutung", Frege concentrated mostly on the senses of names, holding that all names have a sense. It is natural to hold that the same considerations apply to any expression that has an extension. Two general terms can have the same extension and different cognitive significance; two predicates can have the same extension and different cognitive significance; two sentences can have the same extension and different cognitive significance. So general terms, predicates, and sentences all have senses as well as extensions. The same goes for any expression that has an extension, or is a candidate for extension.

The distinction between intensions and extensions is important, considering that extensions can be notoriously difficult to handle in an efficient manner. The extensional equality theory of predicates and functions under higher-order semantics (for example, for two predicates with the same set of attributes \( p = q \) is true iff these symbols are interpreted by the same relation), that is, the strong equational theory of intensions, is not decidable, in general. For example, in the second-order predicate calculus and Church’s simple theory of types, both under the standard semantics, is not even semi-decidable. Thus, separating intensions from extensions makes it possible to have an equational theory over predicate and function names (intensions) that is separate from the extensional equality of relations and functions.

The first conception of intensional entities (or concepts) is built into the possible-worlds treatment of Properties, Relations and Propositions (PRP).s. This conception is commonly attributed to Leibniz, and underlies Alonzo Church’s alternative formulation of Frege’s theory of senses ("A formulation of the Logic of Sense and Denotation" in Henle, Kallen, and Langer, 3-24, and "Outline of a Revised Formulation of the Logic of Sense and Denotation" in two parts, Nous, VII (1973), 24-33, and VIII, (1974), 135-156). This conception of PRPs is ideally suited for treating the modalities (necessity, possibility, etc.) and to Montague’s definition of intension of a given virtual predicate \( \phi(x_1, \ldots, x_k) \) (a FOL open-sentence with the tuple of free variables \( (x_1, \ldots x_k) \)) as a mapping from possible worlds into extensions of this virtual predicate. Among the possible worlds we distinguish the actual possible world. For example if we consider a set of predicates of a given Database and their extensions in different time-instances, the actual possible world is identified by the current instance of the time.

The second conception of intensional entities is to be found in in Russell’s doctrine of logical atomism. On this doctrine it is required that all complete definitions of inten-
sional entities be finite as well as unique and non-circular: it offers an algebraic way for definition of complex intensional entities from simple (atomic) entities (i.e., algebra of concepts), conception also evident in Leibniz’s remarks. In a predicate logics, predicates and open-sentences (with free variables) expresses classes (properties and relations), and sentences express propositions. Note that classes (intensional entities) are reified, i.e., they belong to the same domain as individual objects (particulars). This endows the intensional logics with a great deal of uniformity, making it possible to manipulate classes and individual objects in the same language. In particular, when viewed as an individual object, a class can be a member of another class.

The standard semantics of First-order Logic (FOL) are Tarski style models, which are extensional. In this respect, FOL is extensional. But the open question is if it is possible to obtain also an intensional semantics of FOL such that the Tarski’s extensions of its expressions are equal to extensions of concepts (intensional entities) of the same FOL expressions in the actual possible world.

In what follows we denote by $B^A$ the set of all functions from $A$ to $B$, and by $A^n$ a n-folded cartesion product $A \times ... \times A$ for $n \geq 1$. By $f, t$ we denote empty set $\emptyset$ and singleton set $\{<\}$ respectively (with the empty tuple $<$ i.e. the unique tuple of 0-ary relation), which may be thought of as falsity $f$ and truth $t$, as those used in the relational algebra. For a given domain $D$ we define that $D^0$ is a singleton set $\{<\}$, so that $\{f, t\} = \mathcal{P}(D^0)$, where $\mathcal{P}$ is the powerset operator.

**First-order Logic (FOL):** We will shortly introduce the syntax of the FOL language $L$, and its extensional semantics based on Tarski’s interpretations, as follows:

**Definition 1.** The syntax of the First-order Logic language $L$ is as follows:

Logic operators ($\land, \neg, \exists$) over bounded lattice of truth values $2 = \{f, t\}$, $f$ for falsity and $t$ for truth; Predicate letters $p_1^{k_1}, p_2^{k_2}, ...$ with a given arity $k_i \geq 1$, $i = 1, 2, ...$ in $P$; Functional letters $f_1^{k_1}, f_2^{k_2}, ...$ with a given arity $k_i \geq 1$ in $F$ (language constants $c, d, ...$ are considered as particular case of nullary functional letters); Variables $x, y, z, ...$ in $V$, and punctuation symbols (comma, parenthesis).

With the following simultaneous inductive definition of term and formula:

1. All variables and constants are terms.
2. If $t_1, ..., t_k$ are terms and $f_k^i \in F$ is a k-ary functional symbol then $f_k^i(t_1, ..., t_k)$ is a term, while $p_k^i(t_1, ..., t_k)$ is a formula for a k-ary predicate letter $p_k^i \in P$.
3. If $\phi$ and $\psi$ are formulae, then $(\phi \land \psi), \neg \phi$, and $(\exists x_i) \phi$ for $x_i \in V$ are formulae.

An interpretation (Tarski) $I_T$ consists in a non empty domain $D$ and a mapping that assigns to any predicate letter $p_k^i \in P$ a relation $R = I_T(p_k^i) \subseteq D^k$, to any functional letter $f_k^i \in F$ a function $I_T(f_k^i) : D^k \rightarrow D$, or, equivalently, its graph relation $R = I_T(f_k^i) \subseteq D^{k+1}$ where the $k+1$-th column is the resulting function’s value, and to each individual constant $c \in F$ one given element $I_T(c) \in D$.

A Predicate Logic $L_P$ is a subset of the FOL without the quantifier $\exists$.

**Remark:** The propositional logic can be considered as a particular case of Predicate logic when all symbols in $P$ are nullary, that is a set of propositional symbols, while $F$, $D$, and $V$ are empty sets. By considering that $D^0 = \{<\}$ is a singleton set, then for any $p_i \in P$, $I_T(p_i) \subseteq D^0$, that is, $I_T(p_i) = f$ (empty set) or $I_T(p_i) = t$ (singleton set $\{<\}$). That is, $I_T$ becomes an interpretation $I_T : P \rightarrow 2$ of this logic, which can be
homomorphically extended to all formulae in the unique standard way.

In a formula $(\exists x)\phi$, the formula $\phi$ is called "action field" for the quantifier $(\exists x)$. A variable $y$ in a formula $\psi$ is called bounded variable iff it is the variable of a quantifier $(\exists y)$ in $\psi$, or it is in the action field of a quantifier $(\exists y)$ in the formula $\psi$. A variable $x$ is free in $\psi$ if it is not bounded.

The universal quantifier is defined by $\forall = \neg \exists \neg$. Disjunction and implication are expressed by $\phi \lor \psi = \neg (\neg \phi \land \neg \psi)$, and $\phi \rightarrow \psi = \neg \phi \lor \psi$. In FOL with the identity $\equiv$, the formula $(\exists x)\phi(x)$ denotes the formula $(\exists x)\phi(x) \land (\forall x)(\forall y)(\phi(x) \land \phi(y) \Rightarrow (x \equiv y))$.

We can introduce the sorts in order to be able to assign each variable $x_i$ to a sort $S_i \subseteq D$ where $D$ is a given domain for the FOL (for example, for natural numbers, for reals, for dates, etc., as used for some attributes in database relations). An assignment $g : V \rightarrow D$ for variables in $V$ is applied only to free variables in terms and formulae. If we use sorts for variables, then for each sorted variable $x_i \in V$ an assignment $g$ must satisfy the auxiliary condition $g(x_i) \in S_i$. Such an assignment $g \in D^V$ can be recursively uniquely extended into the assignment $g^x : T \rightarrow D$, where $T$ denotes the set of all terms, by:

1. $g^x(t) = g(x) \in D$ if the term $t$ is a variable $x \in V$.
2. $g^x(t) = I_T(c) \in D$ if the term $t$ is a constant $c \in F$.
3. if a term $t$ is $f^k_i(t_1, \ldots, t_k)$, where $f^k_i \in F$ is a k-ary functional symbol and $t_1, \ldots, t_k$ are terms, then $g^x(f^k_i(t_1, \ldots, t_k)) = I_T(f^k_i(g^x(t_1), \ldots, g^x(t_k)))$ or, equivalently, in the graph-interpretation of the function, $g^x(f^k_i(t_1, \ldots, t_k)) = u$ such that $(g^x(t_1), \ldots, g^x(t_k), u) \in I_T(f^k_i) \subseteq D^{k+1}$.

In what follows we will use the graph-interpretation for functions in FOL like its interpretation in intensional logics. We denote by $t/g$ (or $\phi/g$) the ground term (or formula) without free variables, obtained by assignment $g$ from a term $t$ (or a formula $\phi$), and by $\phi[x/t]$ the formula obtained by uniformly replacing $x$ by a term $t$ in $\phi$. A sentence is a formula having no free variables.

A Herbrand base of a logic $\mathcal{L}$ is defined by $H = \{ p^k_1(t_1, \ldots, t_k) \mid p^k_1 \in P \text{ and } t_1, \ldots, t_k \text{ are ground terms} \}$. We define the satisfaction for the logic formulae in $\mathcal{L}$ and a given assignment $g : V \rightarrow D$ inductively, as follows:

If a formula $\phi$ is an atomic formula $p^k_1(t_1, \ldots, t_k)$, then this assignment $g$ satisfies $\phi$ iff $(g^x(t_1), \ldots, g^x(t_k)) \in I_T(p^k_1)$; $g$ satisfies $\neg \phi$ iff it does not satisfy $\phi$; $g$ satisfies $\phi \land \psi$ iff $g$ satisfies $\phi$ and $g$ satisfies $\psi$; $g$ satisfies $\exists x_i \phi$ iff exists an assignment $g' \in D^V$ that may differ from $g$ only for the variable $x_i \in V$, and $g'$ satisfies $\phi$. A formula $\phi$ is true for every assignment $g \in D^V$. A formula $\phi$ is valid (i.e., tautology) iff $\phi$ is true for every Tarski’s interpretation $I_T \in \mathcal{I}_T$. An interpretation $I_T$ is a model of a set of formulae $\Gamma$ iff every formula $\phi \in \Gamma$ is true in this interpretation. We denote by $\text{FOL}(\Gamma)$ the FOL with a set of assumptions $\Gamma$, and by $\mathcal{I}_T(\Gamma)$ the subset of Tarski’s interpretations that are models of $\Gamma$, with $\mathcal{I}_T(\emptyset) = \mathcal{I}_T$. A formula $\phi$ is said to be a logical consequence of $\Gamma$, denoted by $\Gamma \models \phi$, iff $\phi$ is true in all interpretations in $\mathcal{I}_T(\Gamma)$. Thus, $\models \phi$ iff $\phi$ is a tautology.

The basic set of axioms of the FOL are that of the propositional logic with two additional axioms: (A1) $(\forall x)(\phi \Rightarrow \psi) \Rightarrow (\phi \Rightarrow (\forall x)\psi)$, (x does not occur in $\phi$ and it is not bound in $\psi$), and (A2) $(\forall x)\phi \Rightarrow \phi[x/t]$, (neither $x$ nor any variable in $t$ occurs bound in $\phi$). For the FOL with identity, we need the proper axiom (A3) $x_1 \equiv x_2 \Rightarrow (x_1 \equiv x_2)$.


A predicate/propositional multi-modal logic is a standard Propositional Logic extended by a number of existential extensional identity will be denoted in the standard way by relations standard Kripke semantics each modal operator

\( \Diamond \) will be the set of all \( k \)-ary relations over a domain \( D \), where \( k \in \mathbb{N} = \{0, 1, 2, \ldots \} \). Then, this extensional equivalence between two relations \( R_1, R_2 \in \mathcal{R} \) with the same arity will be denoted by \( R_1 \equiv R_2 \), while the extensional identity will be denoted in the standard way by \( R_1 = R_2 \).

**Predicate/Propositional Multi-modal Logics:**

A predicate/propositional multi-modal logic is a standard Predicate/Propositional Logic (see Definition 1) extended by a number of existential modal operators \( \Diamond_i \), \( i \geq 1 \). In the standard Kripke semantics each modal operator \( \Diamond_i \) is defined by an accessibility binary relation \( \mathcal{R}_i \subseteq \mathcal{W} \times \mathcal{W} \), for a given set of possible worlds \( \mathcal{W} \). A more exhaustive and formal introduction to modal logics and their Kripke’s interpretations can easily be found in the literature, for example in [1]. Here only a short version will be given, in order to clarify the definitions used in the next paragraphs.

We define \( \mathcal{N} = \{0, 1, 2, \ldots, n\} \subseteq \mathbb{N} \) where \( n \) is a maximal arity of symbols in the finite set \( P \cup F \) of predicate and functional symbols respectively. In the case of the propositional logics we have that \( n = 0 \), so that \( P \) is a set of propositional symbols (that are the nullary predicate symbols) and \( F = \emptyset \) is the empty set. Here we will present two definitions for modal logics, one for the propositional and other for predicate logics, as is used in current literature:
Definition 2. PROPOSITIONAL MULTI-MODAL LOGIC:
We denote by $\mathcal{M} = (\mathcal{W}, \{R_i | 1 \leq i \leq k\}, \mathcal{D}, I_K)$ a multi-modal Kripke’s interpretation with a set of possible worlds $\mathcal{W}$, the accessibility relations $R_i \subseteq \mathcal{W} \times \mathcal{W}$, $i = 1, 2, ..., n$, and a mapping $I_K: \mathcal{P} \rightarrow 2^\mathcal{W}$, such that for any propositional letter $p_i \in \mathcal{P}$, the function $I_K(p_i): \mathcal{W} \rightarrow 2$ defines the truth of $p_i$ in a world $w \in \mathcal{W}$.

For any formula $\varphi$ we define $\mathcal{M} \models_w \varphi$ iff $\varphi$ is satisfied in a world $w \in \mathcal{W}$. For example, a given letter $p_i$ is true in $w$, i.e., $\mathcal{M} \models_w p_i$, iff $I_K(p_i)(w) = t$.

The Kripke semantics is extended to all formulae as follows:
- $\mathcal{M} \models_w \varphi \land \phi$ iff $\mathcal{M} \models_w \varphi$ and $\mathcal{M} \models_w \phi$.
- $\mathcal{M} \models_w \neg \varphi$ iff not $\mathcal{M} \models_w \varphi$.
- $\mathcal{M} \models_w \Box_i \varphi$ iff exists $w' \in \mathcal{W}$ such that $(w, w') \in R_i$ and $\mathcal{M} \models_{w'} \varphi$.

The universal modal operator $\Box_i$ is equal to $\neg \Box_i \neg$.

A formula $\varphi$ is said to be true in a Kripke’s interpretation $\mathcal{M}$ if for each possible world $w$, $\mathcal{M} \models_w \varphi$. A formula is said to be valid if it is true in each interpretation.

Definition 3. PREDICATE MULTI-MODAL LOGIC:
We denote by $\mathcal{M} = (\mathcal{W}, \{R_i | 1 \leq i \leq k\}, \mathcal{D}, I_K)$ a multi-modal Kripke model with finite $k \geq 1$ modal operators with a set of possible worlds $\mathcal{W}$, the accessibility relations $R_i \subseteq \mathcal{W} \times \mathcal{W}$, non empty domain $\mathcal{D}$, and a mapping $I_K : \mathcal{W} \times (\mathcal{P} \cup \mathcal{F}) \rightarrow \bigcup_{n \in \mathbb{N}} (2 \mathcal{D})^P$, such that for any world $w \in \mathcal{W}$,

1. For any functional letter $f_i^k \in \mathcal{F}$, $I_K(w, f_i^k) : \mathcal{D}^k \rightarrow \mathcal{D}$ is a function (interpretation of $f_i^k$ in $w$).
2. For any predicate letter $p_i^k \in \mathcal{P}$, the function $I_K(w, p_i^k) : \mathcal{D}^k \rightarrow 2$ defines the extension of $p_i^k$ in a world $w$, $\|p_i^k(x_1, ..., x_k)\|_{\mathcal{M}, w} =_{def} \{(d_1, ..., d_k) \in \mathcal{D}^k | I_K(w, p_i^k)(d_1, ..., d_k) = t\}$.

For any formula $\varphi$ we define $\mathcal{M} \models_{w, g} \varphi$ iff $\varphi$ is satisfied in a world $w \in \mathcal{W}$ for a given assignment $g : \mathcal{V} \rightarrow \mathcal{D}$. For example, a given atom $p_i^k(x_1, ..., x_k)$ is satisfied in $w$ by assignment $g$, i.e., $\mathcal{M} \models_{w, g} p_i^k(x_1, ..., x_k)$, iff $I_K(w, p_i^k)(g(x_1), ..., g(x_k)) = t$.

The Kripke semantics is extended to all formulae as follows:
- $\mathcal{M} \models_{w, g} \varphi \land \phi$ iff $\mathcal{M} \models_{w, g} \varphi$ and $\mathcal{M} \models_{w, g} \phi$.
- $\mathcal{M} \models_{w, g} \neg \varphi$ iff not $\mathcal{M} \models_{w, g} \varphi$.
- $\mathcal{M} \models_{w, g} \Box_i \varphi$ iff exists $w' \in \mathcal{W}$ such that $(w, w') \in R_i$ and $\mathcal{M} \models_{w', g} \varphi$.

A formula $\varphi$ is said to be true in a Kripke’s interpretation $\mathcal{M}$ if for each possible world $w$, $\mathcal{M} \models_{w, g} \varphi$. A formula is said to be valid if it is true in each interpretation.

Any virtual predicate $\phi(x_1, ..., x_k)$ has different extensions $\|\phi(x_1, ..., x_k)\|_{\mathcal{M}, w} =_{def} \{(g(x_1), ..., g(x_k)) | g \in \mathcal{D}^k \text{ and } \mathcal{M} \models_{w, g} \phi\}$ for different possible worlds $w \in \mathcal{W}$.

Thus we can not establish the simple extensional identity for two concepts as in FOL.

Apparently it seems that Tarski’s interpretation for the FOL and the Kripke’s interpretation for modal predicate logics are inconceivable. Currently, each modal logic is considered as a kind of intensional logic. The open question is what about the modality in the FOL, if it is intrinsic also in FOL, that is, if there is an equivalent multi-modal transformation of the FOL where the Kripke’s interpretation is an equivalent correspondent to the original Tarski’s interpretation for the FOL. The positive answer to these questions is one of the main contributions of this paper.
The Plan of this work is the following: in Section 2 will be presented the PRP theory and the two step intensional semantics for modal predicate logics, with the unique intensional interpretation \( I \) which maps the logic formulae into the concepts (intensional entities), and the set of extensionalization functions which determine the extension of any given concept in different possible worlds. After that we will define an extensional algebra of relations for the FOL, different from standard Cylindric algebras. In Section 3 we will consider the FOL syntax with the modal Kripke’s semantics for each particular application of quantifiers \( (\exists x) \), and we will obtain a multi-modal predicate logic \( \text{FOL}_K \), equivalent to the standard FOL with Tarski’s interpretation. Moreover, we will define the generalized Kripke semantics for modal predicate logics, and we will show their diagram of fundamental reductions, based on the restrictions over possible worlds. In Section 4 we will consider the intensionality of modal logics, and we will show that not all modal logics are intensional as supposed: in fact the modal translation of the FOL syntax results in a multi-modal predicate logic \( \text{FOL}_K \) that is pure extensional as it is the standard Tarskian FOL. Then we will define the full intensional enrichment for multi-modal predicate logics. Finally, in Section 5 we will consider the minimal intensional enrichment of the FOL (which does not change the syntax of the FOL), by defining \( \text{FOL}_{\mathcal{I}} \) intensional logic with the set of explicit possible worlds equal to the set of Tarski’s interpretations of the standard extensional FOL. We will show that its intensionality corresponds to the Montague’s point of view. Then we will define the intensional algebra of concepts for this intensional \( \text{FOL}_{\mathcal{I}} \), and the homomorphic correspondence of the two-step intensional semantics with the Tarskian semantics of the FOL, valid in every possible world of \( \text{FOL}_{\mathcal{I}} \).

2 Intensionality and intensional/extensional semantics

Contemporary use of the term ‘intension’ derives from the traditional logical doctrine that an idea has both an extension and an intension. Although there is divergence in formulation, it is accepted that the extension of an idea consists of the subjects to which the idea applies, and the intension consists of the attributes implied by the idea. In contemporary philosophy, it is linguistic expressions (here it is a logic formula), rather than concepts, that are said to have intensions and extensions. The intension is the concept expressed by the expression, and the extension is the set of items to which the expression applies. This usage resembles use of Frege’s use of ‘Bedeutung’ and ‘Sinn’ \([2]\). It is evident that two ideas could have the same extension but different intensions. The systematic study of intensional entities has been pursued largely in the context of intensional logic; that part of logic in which the principle of (extensional) substitutivity of equivalent expressions fails. Intensional entities (or concepts) are such things as propositions, relations and properties. What make them ‘intensional’ is that they violate the principle of extensionality; the principle that extensional equivalence implies identity. All (or most) of these intensional entities have been classified at one time or another as kinds of Universals \([3]\). Accordingly, standard traditional views about the ontological status of universals carry
over to intensional entities. Nominalists hold that they do not really exist. Conceptualist accept their existence but deem it to be mind-dependent. Realists hold that they are mind-independent. *Ante rem* realists hold that they exist independently of being true of anything; *in re* realists require that they be true of something. In what follows we adopt the *Ante rem* realism.

In a predicate logics, (virtual) predicates expresses classes (properties and relations), and sentences express propositions. Note that classes (intensional entities) are *reified*, i.e., they belong to the same domain as individual objects (particulars). This endows the intensional logics with a great deal of uniformity, making it possible to manipulate classes and individual objects in the same language. In particular, when viewed as an individual object, a class can be a member of another class.

The extensional reductions, such as, propositional complexes and propositional functions, to intensional entities are inadequate, there are several technical difficulties, so that we adopt the non-reductionist approaches and we will show how it corresponds to the possible world semantics. We begin with the informal theory that universals (properties (unary relations), relations, and propositions in PRP theory) are genuine entities that bear fundamental logical relations to one another. To study properties, relations and propositions, one defines a family of set-theoretical structures, one define the intensional algebra, a family of set-theoretical structures most of which are built up from arbitrary objects and fundamental logical operations (conjunction, negation, existential generalization, etc.) on them.

The value of both traditional conceptions of PRPs (the 'possible worlds' and 'algebraic' Russel's approaches) is evident, and in Bealer's work both conceptions are developed side by side. But Bealer's approach to intensional logic locates the origin of intensionality a single underlying *intensional abstraction* operation which transforms the logic formulae into terms, so that we are able to make reification of logic formulae without the necessity of the second-order logics. In fact, the *intensional abstracts* are so called 'that'-clauses. We assume that they are singular terms; Intensional expressions like 'believe', mean', 'assert', 'know', are standard two-place predicates that take 'that'-clauses as arguments. Expressions like 'is necessary', 'is true', and 'is possible' are one-place predicates that take 'that'-clauses as arguments. For example, in the intensional sentence "it is necessary that A", where A is a proposition, the 'that A' is denoted by the ⟨A⟩, where ⟨⟩ is the intensional abstraction operator which transforms a logic formula A into the term ⟨A⟩. So that the sentence "it is necessary that A" is expressed by the logic atom N⟨A⟩), where N is the unary predicate 'is necessary'. In this way we are able to have the higher-order syntax for our *intensional* logic language (predicates appear in variable places of other predicates), as, for example *HiLog* where the same symbol may denote a predicate, a function, or an atomic formula. In the First-order logic (FOL) with intensional abstraction we have more fine distinction between an atom A and its use as a term 'that A', denoted by ⟨A⟩ and considered as intensional 'name', inside some other predicate, and, for example, to have the first-order formula ¬A ∧ P(t, ⟨A⟩) instead of the second-order *HiLog* formula ¬A ∧ P(t, A).

In this work I will not accept this Baler's approach, and I will consider the *minimal* intensionality in FOL without necessity of intensional abstraction operation. Thus I will consider only basic conceptions of intensional entities: open-sentences (transformed
into virtual predicates with non empty tuple of free variables) express properties and relations, and sentences express propositions. But the concepts (properties, relations and propositions) are denotations for open and closed logic sentences, thus elements of the structured domain \( D = D_{-1} + D_f \), (here + is a disjoint union) where a subdomain \( D_{-1} \) is made of particulars (individuals), and the rest \( D_f = D_0 + D_1 + \ldots + D_n \ldots \) is made of universals (concepts): \( D_0 \) for propositions with a distinct element \( Truth \in D_0 \), \( D_1 \) for properties (unary concepts) and \( D_n, n \geq 2 \), for n-ary concept. The concepts in \( D_f \) are denoted by \( u, v, \ldots \), while the values (individuals) in \( D_{-1} \) by \( a, b, \ldots \) (the empty tuple \(< >\) of the nullary relation is an individual in \( D_{-1} \), with \( D^0 = \{ < > \} \).

Sort \( S \) is a subset of a domain \( D \). For example \( [0, 1] \) is closed-interval of reals sort, \( \{0, 1, 2, 3, \ldots \} \subset D_{-1} \) is the sort of integers, etc. These sorts are used for sorted variables in many-sorted predicate logics so that the assigned values for each sorted variable must belong to its sort. The unsorted variables can be considered as variables with a top sort equal to \( D \).

The intensional interpretation is a mapping between the set \( \mathcal{L} \) of formulae of the logic language and intensional entities in \( D \), \( I : \mathcal{L} \rightarrow D \), is of a kind of "conceptualization", such that an open-sentence (virtual predicate) \( \phi(x_1, \ldots, x_k) \) with a tuple of all free variables \( (x_1, \ldots, x_k) \) is mapped into a \( k \)-ary concept, that is, an intensional entity \( u = I(\phi(x_1, \ldots, x_k)) \in D_0 \), and (closed) sentence \( \psi \) into a proposition (i.e., \( \psi \) is mapped into a proposition \( \phi \)) \( v = I(\psi) \in D_0 \).

**Definition 4. Extensions and extensionalization functions:**

Let \( \mathcal{R} = \bigcup_{k \in \mathbb{N}} \mathcal{P}(D^k) = \sum_{k \in \mathbb{N}} \mathcal{P}(D^k) \) be the set of all \( k \)-ary relations, where \( k \in \mathbb{N} = \{0, 1, 2, \ldots \} \). Notice that \( \{f, t\} = \mathcal{P}(D^0) \in \mathcal{R} \), that is, the truth values are extensions in \( \mathcal{R} \). The extensions of the intensional entities (concepts) are given by the set \( \mathcal{E} \) of extensionalization functions \( h : D \rightarrow \mathcal{R} \), such that

\[
h = h_{-1} + h_0 + \sum_{i \geq 1} h_i : \bigoplus_{i \geq 1} D_i \rightarrow \mathcal{P}(D_{-1}) + \{f, t\} + \sum_{i \geq 1} \mathcal{P}(D^i)
\]

where \( h_{-1} = id : D_{-1} \rightarrow D_{-1} \) is an identity, \( h_0 : D_0 \rightarrow \{f, t\} = \mathcal{P}(D^0) \) assigns the truth values in \( \{f, t\} \) to all propositions with the constant assignment \( h_0(Truth) = t \), and \( h_i : D_i \rightarrow \mathcal{P}(D^i), i \geq 1 \), assigns an extension to each concept.

Consequently, intensions can be seen as names of abstract or concrete concepts, while extensions correspond to various rules that these concepts play in different worlds.

Thus, for any open-sentence \( \phi(x_1, \ldots, x_k) \) we have that its extension, in a given world \( w \in \mathcal{W} \) of the Kripke’s interpretation \( \mathcal{M} = (\mathcal{W}, \{R_i \mid 1 \leq i \leq k\}, D, V) \) for modal (intensional) logics in Definition[3] is equal to:

\[
h(I(\phi(x_1, \ldots, x_k))) = \models (g(x_1), \ldots, g(x_k)) \in D^V \land \mathcal{M} \models w, g, \phi.
\]

From a logic point of view, two possible worlds \( w \) and \( w' \) are indistinguishable if all sentences have the same extensions in them, so that we can consider an extensionalization function \( h \) as a “possible world”, similarly to the semantics of a probabilistic logic, where possible worlds are Herbrand interpretations for given set of predicate letters \( P \) in a given logic. Thus, for a given modal logic we will have that there is a mapping \( is : \mathcal{W} \rightarrow \mathcal{E} \) from the set of possible worlds to the set of extensionalization functions.
Definition 5. Two-step Intensional Semantics: The intensional semantics of the logic language with the set of formulae $L$ can be represented by the mapping

$$L \rightarrow_I D \Rightarrow_{w \in W} \mathcal{R},$$

where $\rightarrow_I$ is a fixed intensional interpretation $I : L \rightarrow D$ and $\Rightarrow_{w \in W}$ is the set of all extensionalization functions $h = is(w) : D \rightarrow \mathcal{R}$ in $E$, where $is : W \rightarrow E$ is the mapping from the set of possible worlds to the set of extensionalization functions. We define the mapping $I_n : L_{op} \rightarrow \mathcal{R}^W$, where $L_{op}$ is the subset of formulae with free variables (virtual predicates), such that for any virtual predicate $\phi(x_1, \ldots, x_k) \in L_{op}$ the mapping $I_n(\phi(x_1, \ldots, x_k)) : W \rightarrow \mathcal{R}$ is the Montague’s meaning (i.e., intension) of this virtual predicate [8,9,10,11,12]. That is, the mapping which returns with the extension of this (virtual) predicate in every possible world in $W$.

We adopted this two-step intensional semantics, instead of well known Montague’s semantics (which lies in the construction of a compositional and recursive semantics that covers both intension and extension) because of a number of its weakness. Let us consider the following two past participles: ‘bought’ and ‘sold’ (with unary predicates $p_1(x)$, ‘$x$ has been bought’, and $p_2(x)$, ‘$x$ has been sold’). These two different concepts in the Montague’s semantics would have not only the same extension but also their intension, from the fact that their extensions are identical in every possible world. Within the two-steps formalism we can avoid this problem by assigning two different concepts (meanings) $u = I(p_1(x))$ and $v = I(p_2(x))$ in $D_1$. Notice that the same problem we have in the Montague’s semantics for two sentences with different meanings, which bear the same truth value across all possible worlds: in the Montague’s semantics they will be forced to the same meaning.

But there is also another advantage of this two-step intensional semantics in Definition 5 here we are able to define an intensional algebra $A_{int}$ over intensional entities in $D$, which is autosufficient, differently from Montague’s semantics where the compositional and recursive semantics of intensions can be defined only by their extensional properties. As we will see in the last Section, this intensional algebra is defined in the way that each extensional mapping $h = is(w) : D \rightarrow \mathcal{R}$ is a homomorphism between this intensional algebra $A_{int}$ and the extensional relational algebra $A_{R}$ that represents the compositional and recursive semantics of the extensions, given by Corollary 1 later in this Section. In this way the compositional and recursive semantics of the intensions in $A_{int}$ coincide with the Montague’s semantics, where, for example, the mapping $I_n(\phi \wedge \psi) : W \rightarrow \mathcal{R}$ (i.e., the Montague’s intension of the composite formula $\phi \wedge \psi$) is functionally dependent on the mappings $I_n(\phi) : W \rightarrow \mathcal{R}$ and $I_n(\psi) : W \rightarrow \mathcal{R}$ (i.e., dependent on the Montague’s intensions of $\phi$ and $\psi$).

Remark: the mapping $I_n$ can be extended also to all sentences (the formulae without free variables), such that for any sentence $\phi$, $I_n(\phi) : W \rightarrow \{t, f\} = P(D^0) \subseteq \mathcal{R}$ is a mapping that defines the truth value (i.e., an extension in $\mathcal{R}$ in Definition 4) of this sentence in each possible world $W$. Equivalently to this Montague’s semantics for intensions of logic formulae, we can use the Carnap’s semantics [13] of concepts in $D$, that is $I_{n,c} : D \rightarrow \mathcal{R}^W$ such hat the intension of a concept $u \in D$ is a mapping $I_{n,c}(u) : W \rightarrow \mathcal{R}$ from possible worlds to extensions. This Carnap’s semantics of concepts is represented by the second mapping of the diagram $D \Rightarrow_{w \in W} \mathcal{R}$ above.
Tarski’s interpretation of the FOL is instead given by a single mapping $I^*_T : ℓ \rightarrow ℙ$, as explained in the introduction dedicated to FOL. Thus, if there is a modal Kripke semantics with a set of possible worlds $\mathcal{W}$ (thus, an intensional semantics) for FOL, equivalent to the standard FOL semantics given by the Tarski’s interpretation $I_T$, then we have to obtain for every possible world $w \in \mathcal{W}$ of such a semantics that $h = is(w)$ is invariant (i.e., the set $\{h = is(w) \mid w \in \mathcal{W}\}$ is a singleton set), and consequently $I^*_T = h \circ I$, where $\circ$ is a composition of functions, such that for any formula $\phi \in ℓ$, $h(I(\phi)) = I^*_T(\phi)$. For any constant $c$ of the FOL language we assume that $I(c) = I^*_T(c) \in D$.

We consider that the domain $D$ is equal in each possible world $w \in \mathcal{W}$. It is demonstrated that also in the case of different domains $D_w$ in different possible worlds, we can always obtain the constant domain model (as in Definition 2.1 in [14]) $D = \bigcup_{w \in \mathcal{W}} D_w$ and by introducing a new built-in binary predicate $e(x, y)$ where $x$ has as domain the set of possible worlds, so that $e(w, d)$ is true if $d \in D_w$. It is important that the set of particulars $D_{-1}$ is the set of rigid objects like “Eiffel tower” or “George Washington”, that have equal extension (denotation) in each possible world: it holds from the fact that for every rigid object $c$, a possible world $w \in \mathcal{W}$, and a given intensional interpretation $I$ we have that $d = I(c) \in D_{-1}$ and its extension is $h(d) = is(w)(d) = d$ constant independently from $w$.

The problem of non-rigid objects and relative complications considered by Fitting in [15],[4],[6], as “the gross domestic product of Denmark” or “the Secretary-General of the United Nations”, here are considered not as constants of the language but as unary predicates, denoted by $p^1_1, p^2_1 \in P$. The intension $u = I(p^1_1) \in D_1$ denotes the property (unary concept) whose extension $is(w)(I(p^1_1))$ is a singleton set (by introducing an axiom $(\exists x)p^1_1(x)$), possibly different in each possible world (for example, the instance of time) $w \in \mathcal{W}$. If we need to use these “non-rigid objects” as arguments inside other predicates, in order to avoid the second-order syntax we can use Bealer’s intensional abstraction [1] which can transform the unary predicates used for non-rigid objects into terms, so that can be used as arguments inside other predicates.

It explains why in these two-step interpretations, intensional and extensional, we can work in an unified general rigid framework, and overcome the major difficulties for modal first-order logics, considered by Fitting in the number of his papers, by introducing new operations like ‘extension of’ operators $↓$ and ‘predicate abstracts’, $\lambda \lambda x_1, ..., x_n, \phi >$ that transforms the logic formula with a tuple of free variables $\phi(x_1, ..., x_n)$ into new atomic formula $\lambda \lambda x_1, ..., x_n, \phi > (t_1, ..., t_n)$ for any given set of terms $t_i, i = 1, ..., n$ (Definition 2.3 in [4]). Notice that differently from this Fitting’s approach we do not consider a virtual predicate $\phi(x_1, ..., x_n)$, as a new atom, but as a standard logic formula.

Another relevant question w.r.t. this two-step interpretations of an intensional semantics is how in it is managed the extensional identity relation $\doteq$ (binary predicate of the identity) of the FOL. Here this extensional identity relation is mapped into the binary concept $Id = I(\doteq (x, y)) \in D_2$, such that $(\forall w \in \mathcal{W})(is(w)(Id) = R_w)$, where $\doteq (x, y)$ denotes an atom of the FOL of the binary predicate for identity in FOL, usually written by FOL formula $x \doteq y$ (here we prefer to distinguish this formal symbol $\doteq \in P$ of the
Formula (the contradiction formula is denoted by
\( \neg \top \)) is used in all mathematical definitions in this paper). That is, for every possible world \( w \) and its correspondent extensionalization function \( h = is(w) \), the extensional identity relation in \( \mathcal{D} \) is the extension of the binary concept \( \mathsf{Id} \in \mathcal{D}_2 \), as defined by Bealer’s approach to intensional FOL with intensional abstraction in [6].

Let \( \mathcal{A}_{\mathsf{FOL}} = (\mathcal{L}, \equiv, \top, \land, \neg, \exists) \) be a free syntax algebra for “First-order logic with identity \( \equiv \)”, with the set \( \mathcal{L} \) of first-order logic formulae, with \( \top \) denoting the tautology formula (the contradiction formula is denoted by \( \neg \top \)), with the set of variables in \( \forall \) and the domain of values in \( \mathcal{D} \). It is well known that we are able to make the extensional algebraization of the FOL by using the cylindric algebras [17] that are the extension of Boolean algebras with a set of binary operators for the FOL identity relations and a set of unary algebraic operators (“projections”) for each case of FOL quantification (\( \exists x \)). In what follows we will make an analog extensional algebraization over \( \mathfrak{R} \) but by interpretation of the logic conjunction \( \land \) by a set of natural join operators over relations introduced by Codd’s relational algebra [18,19] as a kind of a predicate calculus whose interpretations are tied to the database.

In what follows we will use the function \( f_{\leftrightarrow} : \mathfrak{R} \to \mathfrak{R} \), such that for any \( R \in \mathfrak{R} \), \( f_{\leftrightarrow}(R) = \{ \langle > \} \) if \( R \neq \emptyset \); \( \emptyset \) otherwise. Let us define the following set of algebraic operators for relations in \( \mathfrak{R} \):

1. binary operator \( \bowtie : \mathfrak{R} \times \mathfrak{R} \to \mathfrak{R} \), such that for any two relations \( R_1, R_2 \in \mathfrak{R} \), the \( R_1 \bowtie R_2 \) is equal to the relation obtained by natural join of these two relations \( I \in S \) is an empty set of pairs of joined columns of respective relations (where the first argument is the column index of the relation \( R_1 \) while the second argument is the column index of the joined column of the relation \( R_2 \)); otherwise it is equal to the cartesian product \( R_1 \times R_2 \).

For example, the logic formula \( \phi(x_i, x_j, x_k, x_l, x_m) \land \psi(y_i, y_j, y_l, y_m) \) will be translated by the algebraic expression \( R_1 \bowtie R_2 \) where \( R_1 \in \mathcal{P}(\mathcal{D}^k) \), \( R_2 \in \mathcal{P}(\mathcal{D}^l) \) are the extensions for a given Tarski’s interpretation \( I_T \) of the virtual predicate \( \phi, \psi \) relatively, so that \( S = \{(4,1), (2,3)\} \) and the resulting relation will have the following ordering of attributes: \( (x_i, x_j, x_k, x_l, x_m, y_i, y_j) \). Consequently, we have that for any two formulae \( \phi, \psi \in \mathcal{L} \) and a particular join operator \( \bowtie \) uniquely determined by tuples of free variables in these two formulae,

\[ I_T(\phi \land \psi) = I_T(\phi) \bowtie S I_T(\psi). \]

2. unary operator \( \sim : \mathfrak{R} \to \mathfrak{R} \), such that for any \( k \)-ary (with \( k \geq 0 \)) relation \( R \in \mathcal{P}(\mathcal{D}^k) \subseteq \mathfrak{R} \) we have that \( \sim (R) = \mathcal{D}^k \setminus R \in \mathcal{D}^k \), where \( \backslash \) is the substraction of relations.

For example, the logic formula \( \neg \phi(x_i, x_j, x_k, x_l, x_m) \) will be translated by the algebraic expression \( \mathcal{D}^k \setminus R \) where \( R \) is the extensions for a given Tarski’s interpretation \( I_T \) of the virtual predicate \( \phi \). Consequently, we have that for any formula \( \phi \in \mathcal{L} \),

\[ I_T(\neg \phi) = \neg (I_T(\phi)). \]

3. unary operator \( \pi_m : \mathfrak{R} \to \mathfrak{R} \), such that for any \( k \)-ary (with \( k \geq 0 \)) relation \( R \in \mathcal{P}(\mathcal{D}^k) \subseteq \mathfrak{R} \) we have that \( \pi_m(R) \) is equal to the relation obtained by elimination of the \( m \)-th column of the relation \( R \) if \( 1 \leq m \leq k \) and \( k \geq 2 \); equal to \( f_{\leftrightarrow}(R) \) if \( m = k = 1 \); otherwise it is equal to \( R \).

For example, the logic formula \( \exists x_k \phi(x_i, x_j, x_k, x_l, x_m) \) will be translated by the

\[ I_T(\exists x_k \phi(x_i, x_j, x_k, x_l, x_m)) \]
algebraic expression \( \pi_{-3}(R) \) where \( R \) is the extensions for a given Tarski’s interpretation \( I_T \) of the virtual predicate \( \phi \) and the resulting relation will have the following ordering of attributes: \( \{ x_1, x_j, x_l, x_m \} \). Consequently, we have that for any formula \( \phi \in \mathcal{L} \) with a free variable \( x \), where \( m \) is equal to the position of this variable \( x \) in the tuple of free variables in \( \phi \) (or \( m = 0 \) otherwise, where \( \pi_{-0} \) is the identity function), \( I_T((\exists x)\phi) = \pi_{-m}(I_T^{*}(\phi)) \).

Notice that the ordering of attributes of resulting relations corresponds to the method used for generating the ordering of variables in the tuples of free variables adopted for virtual predicates, as explained in the introduction to FOL.

**Corollary 1**  
**Extensional FOL Semantics:**

Let us define the extensional relational algebra for the FOL by,  
\[
\mathcal{A}_\mathfrak{R} = (\mathfrak{R}, \mathcal{R}_=, \{ < > \}, \{ \triangleright \}_{S \in \mathcal{P}(\mathfrak{R})}, \sim, \{ \pi_{-n} \}_{n \in \mathbb{N}}),
\]

where \( \{ < > \} \in \mathfrak{R} \) is the algebraic value correspondent to the logic truth, and \( \mathcal{R}_= \) is the binary relation for extensionally equal elements. We will use ‘=’ for the extensional identity for relations in \( \mathfrak{R} \).

Then, for any Tarski’s interpretation \( I_T \) its unique extension to all formulae \( I_T^{*} : \mathcal{L} \rightarrow \mathfrak{R} \) is also the homomorphism \( I_T^{*} : \mathcal{A}_{\text{FOL}} \rightarrow \mathcal{A}_{\mathfrak{R}} \) from the free syntax FOL algebra into this extensional relational algebra.

**Proof:** Directly from definition of the semantics of the operators in \( \mathcal{A}_{\mathfrak{R}} \) defined in precedence. Let us take the case of conjunction of logic formulae of the definition above where \( \phi(x_i, x_j, x_k, x_l, x_m, y_i, y_j) \) (it’s tuple of variables is obtained by the method defined in the FOL introduction) is the virtual predicate of the logic formula \( \phi(x_i, x_j, x_k, x_l, x_m) \) and \( \psi(x_i, y_i, x_j, y_j) \):

\[
I_T^{*}(\phi \land \psi) = I_T^{*}(\phi) \cup I_T^{*}(\psi)
\]

Thus, it is enough to show that is valid also \( I_T^{*}(\top) = \{ < > \} \) and \( I_T^{*}(\bot) = \emptyset \). The first property comes from the fact that \( \top \) is a tautology, thus satisfied by every assignment \( g \), that is it is true, i.e. \( I_T^{*}(\top) = t \) (and \( t \) is equal to the empty tuple \( \{ < > \} \)). The second property comes from the fact that \( I_T^{*}(\neg \top) = \sim \) (and \( \sim \) is the built-in binary predicate, that is, with the same extension in every Tarski’s interpretation.

Consequently, the mapping \( I_T^{*} : (\mathcal{L}, \hat{\top}, \top, \land, \neg, \exists) \rightarrow \mathcal{A}_{\mathfrak{R}} \) is a homomorphism that represents the extensional Tarskian semantics of the FOL.

\( \Box \)

Notice that \( \mathfrak{R} \) is a poset with the bottom element \( \emptyset \) and the top element \( \{ < > \} \), and the partial ordering \( \preceq \) defined as follows: for any two relations \( R_1, R_2 \in \mathfrak{R} \),
$R_1 \preceq R_2$ iff "for some operation $\triangleright_S$ it holds that $(R_1 \triangleright_S R_2) = R_1$".

It is easy to verify that for any $R \in \mathcal{R}$ and operation $\triangleright_S$ it holds that $(R \triangleright_S \emptyset) = \emptyset$, and $(R \triangleright_S \{<>\}) = R$. That is, $\emptyset \preceq R \preceq \{<>\}$.

3 First-order logic and modality

In propositional modal logics the possible worlds are entities where a given propositional symbol can be true or false. Thus, from logical point of view the possible worlds [20,21] in Kripke’s relational semantics are characterized by property to determine the truth of logic sentences. The important question relative to the syntax of the FOL is if there is a kind of basic set of possible worlds that have such properties. The answer is affirmative.

In fact, if we consider a k-ary predicate letter $p^k_i$ as a new kind of ‘propositional letter’, then an assignment $g : \mathcal{V} \rightarrow \mathcal{D}$ can be considered as an intrinsic (par excellence) possible world, where the truth of this ‘propositional letter’ $p^k_i$ is equal to the truth of the ground atom $p^k_i(g(x_1),\ldots,g(x_k))$.

Consequently, in what follows we will denote by $\mathcal{W}$ the set of explicit possible worlds (defined explicitly for each particular case of modal logics), while the set $\mathcal{D}^V$ we will be called as the set of intrinsic possible worlds (which is invariant and common for every predicate modal logic). In the case when $\mathcal{V} = \emptyset$ is the empty set we obtain the singleton set of intrinsic possible worlds $\mathcal{D}^V = \{\ast\}$, with the empty function $* : \emptyset \rightarrow \mathcal{D}$.

By $\mathcal{W} \subseteq \{(w,g) | w \in \mathcal{W}, g \in \mathcal{D}^V\}$ we will denote the set of (generalized) possible worlds. In this way, as in the case of propositional modal logic, we will have that a formula $\phi$ is true in a Kripke’s interpretation $\mathcal{M}$ if for each (generalized) possible world $u = (w,g) \in \mathcal{W}$, $\mathcal{M} \models_w \phi$.

We denote by $|\phi| = \{(w,g) \in \mathcal{W} \mid \mathcal{M} \models_w \phi\}$ the set of all worlds where the formula $\phi$ is satisfied by interpretation $\mathcal{M}$. Thus, as in the case of propositional modal logics, also in the case of predicate modal logics we have that a formula $\phi$ is true iff it is satisfied in all (generalized) possible worlds, i.e., iff $|\phi| = \mathcal{W}$.

With this new arrangement we can reformulate the standard semantics for multimodal predicate logic in Definition 3. $\pi_1$ and $\pi_2$ denote the first and the second projections:

**Definition 6.** Generalized Kripke semantics for multi-modal logics:

We denote by $\mathcal{M} = (\mathcal{W}, \{R_i\}, \mathcal{D}, I_K)$ a multi-modal Kripke’s interpretation with a set of (generalized) possible worlds $\mathcal{W}$, a set of explicit possible worlds $\mathcal{W} = \pi_1(\mathcal{W})$ and $\pi_2(\mathcal{W}) = \mathcal{D}^V$, the accessibility relations $R_i \subseteq \mathcal{W} \times \mathcal{W}$, $i = 1, 2, \ldots$, non empty domain $\mathcal{D}$, and a mapping $I_K : \mathcal{W} \times (P \cup F) \rightarrow \bigcup_{n \in \mathbb{N}} (2 \cup \mathcal{D})^{\mathcal{D}^V}$, such that for any world $w \in \mathcal{W}$,

1. For any functional letter $f^k_i \in F$, $I_K(w, f^k_i) : \mathcal{D}^V \rightarrow \mathcal{D}$ is a function (interpretation of $f^k_i$ in $w$).
2. For any predicate letter $p^k_i \in P$, the function $I_K(w, p^k_i) : \mathcal{D}^V \rightarrow \mathcal{D}$ defines the extension of $p^k_i$ in a world $w$, $\|p^k_i(x_1,\ldots,x_k)\|_{\mathcal{M},w} =_{def} \{d_1,\ldots,d_k \in \mathcal{D}^k \mid I_K(w, p^k_i)(d_1,\ldots,d_k) = t\}$.

Now we will see that we have two particular “projections” of the generalized Kripke semantics for multi-modal logics, defined above: the first one is explicit-worlds “projec-
tion” resulting in the Kripke semantics of multi-modal propositional logics; the second one is intrinsic-worlds "projection" resulting in the Kripke semantics of FOL logic.

**Proposition 1**  **explicit-worlds "projection" of generalized semantics:**
The Kripke semantics of multi-modal propositional logic given by Definition 2 is a particular case of the Definition 6 when \(D, F, V\) are empty sets and \(P\) has only nullary symbols, that is, the propositional symbols.

**Proof:** In this case when \(V = \emptyset\) is the empty set we have that \(D^V\) is a singleton set, denoted by \(\{*\}\), with unique element equal to the empty function \(* : V \rightarrow D\) (i.e., the function whose graph is empty). Thus \(W = V \times \{*\}\) is equivalent to the set of explicit worlds \(W\), so that the original satisfaction relation \(M \models_{w,g}\) of predicate modal logic in Definition 3 can be equivalently reduced to the satisfaction relation \(M \models_w\) for only explicit worlds of propositional logic in Definition 2.

While \(I_K : W \times P \rightarrow \bigcup_{n \in N} 2^{D^n}\) where in this case \(N = \{0\}\) we obtain is reduced to \(I_K : W \times P \rightarrow 2^{D^0}\), where \(D^0 = \{<>\}\) is a singleton set, so that \(2^{D^0}\) is equivalent to 2, so that we obtain the reduction into the mapping \(I_K : W \times P \rightarrow 2\), and by currying (the \(\lambda\) abstraction), we obtain the mapping \(I'_K : P \rightarrow 2^W\), such that for any \(p_i \in P\) and \(w \in W\) we obtain that \(I_K(w, p_i) = I'_K(p_i)(w) \in 2\) is the truth value of propositional letter (nullary predicate symbol in \(P\)) in the explicit possible world \(w\). It is easy to verify that this obtained mapping \(I'_K\) is that of the propositional modal logic given in Definition 2.

**Remark:** an interesting consequence of this explicit-world "projection" of the generalized Kripke semantics is the idea of the extension of a propositional nullary predicate symbol \(p_i\) given by Definition 6 by \(\|p_i\|_{M,w} = \|p_i^0\|_{M,w} = \text{def} \{ (d_0) \in D^0 = \{<>\} \mid I_K(w, p_i^0) = t \} = \{ <> \mid I'_K(p_i)(w) = t \} = I'_K(p_i)(w)\). That is, it is equal to \(t = \{<>\}\) if \(p_i\) is true in the explicit world \(w\), or equal to \(f\) (the empty set) if \(p_i\) is false in the explicit world \(w\). It is analogous to the consideration of extensions of sentences defined in the intensional semantics, as defined in Section 2, where the truth is the extension of sentences, distinct from their meaning that is, their intension (from Montague’s point of view, the intension of the propositional letter \(p_i\) defined above would be the function \(I'_K(p_i) : W \rightarrow 2\).

It is well known for a predicate logic and FOL (which is a predicate logic extended by logic quantifiers) that we have not any defined set of explicit possible worlds. Thus, trying to define the Kripke semantics (given by Definition 6) to FOL, we can assume that the generalized possible worlds coincide with the intrinsic possible worlds, that is, \(W = D^V\). In the case of the pure predicate logic (without quantifiers) we do not need the possible worlds: a ground atom in order to be true in such a Kripke’s interpretation has to be true in every possible world (or, alternatively, it has to be false in every possible world). Consequently, in the predicate logics the truth of ground atoms and sentences is invariant w.r.t. the possible worlds, which renders useless the definition of possible worlds and Kripke’s semantics for these logics. But in the case of the modal interpretation of the FOL, the FOL quantifiers has to be interpreted by modal operators and their accessibility relations: thus, the possible worlds are necessary in order to determine the truth of logic formulae with quantifiers.
Differently from the FOL with original Tarski’s interpretation for the unique existential operator ∃, the modal point of view for the FOL with Kripke’s interpretation have a particular existential modal operator ◊x, here denoted by (∃x), for each variable x ∈ V. As usual, the universal modal operators are defined by (∀x) = ¬(∃x)¬.

Consequently the same syntax for the FOL of a formula (∃x)φ can have two equivalent semantics: the original Tarski’s interpretation that interprets the unique existential operator ∃ for a variable x and parenthesis (, ), and Kripke’s relational interpretation where the whole expression (∃x) is interpreted as one particular existential modal operator ◊x. This is valid approach based on the fact that, from the algebraic point of view, the syntax of (∃x) can be interpreted as an unary operation which is additive, that is, it holds that (∃x)(φ ∨ ψ) = (∃x)(φ) ∨ (∃x)(ψ), and normal, i.e., (∃x)(⊥) = ⊥ where ⊥ denotes a contradiction sentence (the negation of the tautology ⊤); this property is common for all existential modal operators of the normal Kripke modal logics. In fact, the generalization inference rule (G) of FOL here becomes the rule of necessitation, and the axiom (A1) a particular case of Kripke axiom of normal modal logics.

Thus, we have the following particular case of Definition 6 (here the symbol "\" is the set subtruction operation):

**Definition 7. INTRINSIC-WORLDS "PROJECTION" OF GENERALIZED SEMANTICS:**

We denote by \( M = (\mathbb{W}, \{R_x \mid x \in V\}, D, I_K) \) a multimodal Kripke’s interpretation of the FOL, with a set of (generalized) possible worlds \( \mathbb{W} = D^V \), equal to the set of intrinsic possible worlds (assignments) \( D^V \), the accessibility relation \( R_x = \{(w_1, w_2) \in \mathbb{W} \mid x \in V \text{ and for all } y \in V \setminus \{x\}\{w_1(y) = w_2(y)\}\} \) for existential modal operator (∃x) for each variable x ∈ V, non empty domain D, and a mapping \( I_K : \mathbb{W} \times (P \cup F) \rightarrow \bigcup_{k \in \mathbb{N}}(2 \cup D)^{D^V} \), such that for any world \( w \in \mathbb{W} \),

1. For any functional letter \( f^k_i \in F \), \( I_K(w, f^k_i) : D^k \rightarrow D \) is a function (interpretation of \( f^k_i \) in \( w \)).
2. For any predicate letter \( p^k_i \in P \), the function \( I_K(w, p^k_i) : D^k \rightarrow 2 \) defines the extension of \( p^k_i \) in a world \( w \).

Such an interpretation is the Kripke model of the FOL if, for any \( (d_1, ..., d_k) \in D^k \), for all \( w' \in \mathbb{W} \), \( I_K(w', p^k_i)(d_1, ..., d_k) = I_K(w, p^k_i)(d_1, ..., d_k) \) and \( I_K(w', f^k_i(d_1, ..., d_k)) = I_K(w, f^k_i(d_1, ..., d_k)) \).

We will denote by "FOL-K" the FOL with these modal Kripke models. We recall that FOL-K has the same syntax as FOL, that is, the same set of formulae, and the same domain D, differently from the standard embedding of the modal predicate logics into the FOL: it introduces a new built-in predicate symbol for each binary accessibility relation \( R_i \), and enlarges the original domain D with the set of possible worlds, and enlarges the set of variables V by the new variables for these new built-in symbols.

It is easy to verify that each accessibility binary relation \( R_x, x \in V \), is reflexive, transitive and symmetric relation. Thus, each pair of modal operators (∃x) and (∀x) is an example of existential and universal modal operators of the S5 modal logics, so that (∀x) is an "it is known for all values assigned to x that" modal operator, whose semantics is equivalent to standard FOL "for all values assigned to x" semantics.

It is analogous to monadic algebras of Halmos and his algebraic study of quantifiers, where S5 modal logic is characterized by the class of all closure algebras in which
each closed element is also open. In fact, the complex algebra (over the set of possible worlds) of this S5 multi-modal logic $FOL_K$ uses several S5 algebraic modal operators to provide a Boolean model features of FOL as indicated by Davis [23] in his doctoral thesis supervised by Garret Birkhoff. The Definition 7 as supposed by me, is the first attempt to give a relational Kripke semantics to the FOL, analogous to such an algebraic approach.

As we can see from the definition of Kripke models of the FOL, every function and predicate are rigid in it, that is, they have the same extension in every possible world $w \in W$.

**Theorem 1** The modal Kripke semantics in Definition 7 is an adequate semantics for the FOL. For each Tarski’s interpretation $I_T$ there is an unique Kripke’s interpretation $\mathcal{M}$ (exactly a Kripke model of FOL), and vice versa, such that for any $p_i^k \in P$, $f_i^k \in F$, $(d_1, ..., d_k) \in D^k$ and any intrinsic world (assignment) $g : Var \rightarrow D$ it holds that:

$I_K(g, p_i^k)(d_1, ..., d_k) = t$ iff $(d_1, ..., d_k) \in I_T(p_i^k)$ and

$I_K(g, f_i^k)(d_1, ..., d_k) = u$ iff $u = I_T(f_i^k)(d_1, ..., d_k)$.

We define $\mathcal{K}_F(\Gamma) = \{I_K \text{ defined above from } I_T \mid I_T \in \mathcal{K}_T(\Gamma)\}$ with the bijection $b : \mathcal{K}_T(\Gamma) \simeq \mathcal{K}_F(\Gamma)$, so that for any Tarski’s interpretation we have its equivalent Kripke’s interpretation $I_K = b(I_T)$, where $\Gamma$ is a set of assumptions in this FOL.

Moreover, the following commutative diagram of reductions is valid

![Diagram](image)

**Proof:** Let $(d_1, ..., d_k) \in I_T(p_i^k)$, then from definition in this theorem $I_K(g, p_i^k)(d_1, ..., d_k) = t$, and from Definition 7 we obtain that $|p_i^k(d_1, ..., d_k)| = W$, that is, the ground atom $p_i^k(d_1, ..., d_k)$ is true also in correspondent Kripke’s modal semantics. Viceversa, if $(d_1, ..., d_k) \notin I_T(p_i^k)$, then $|p_i^k(d_1, ..., d_k)| = \emptyset$ is a empty set, that is, the ground atom $p_i^k(d_1, ..., d_k)$ is false also in correspondent Kripke’s modal semantics.

Let us suppose that for any formula $\phi/g$ with $n$ logic connectives, true w.r.t. Tarski’s interpretation $I_T$, it holds that $|\phi/g| = W$, that is, it is true in the correspondent Kripke’s interpretation $I_K = b(I_T)$. Let us show that it holds for any formula $\psi$ with $n + 1$ logic connectives, true w.r.t. Tarski’s interpretation $I_T$; there are the following three cases:

1. $\psi = \phi_1 \land \phi_2$. Then $|\psi| = |\phi_1| \cap |\phi_2| = W$, from the fact that both formulae $\phi_1, \phi_2$ must be true in Tarski’s interpretation $I_T$ and that have less than or equal to $n$ logic connectives, and, consequently (by inductive assumption), $|\phi_i| = W$ for $i = 1, 2$. That is, $\psi$ is true in the correspondent Kripke’s interpretation $I_K = b(I_T)$.

2. $\psi = \neg \phi$. Then $|\psi| = W \setminus |\phi| = W$, from the fact that the formula $\phi$ must be false in Tarski’s interpretation $I_T$ and that has less than or equal to $n$ logic connectives, and, consequently (by inductive assumption), $|\phi| = \emptyset$. That is, $\psi$ is true in the correspondent Kripke’s interpretation $I_K = b(I_T)$.

3. $\psi = (\exists x)\phi(x)$ where $\phi(x)$ denotes the formula $\phi$ with the unique free variable $x$. [Revised beginning of sentence: “Then” instead of “Then”]
From the fact that \((\exists x)\phi(x)\) is true in Tarski’s interpretation we have that there is a value \(u \in D\) such that a sentence \(\phi[x/u]\), that is a formula \(\phi\) where the variable \(x\) is substituted by the value \(u \in D\), is true sentence. Then we obtain:
\[|((\exists x)\phi(x))| = \{ w \mid \text{exists } w' \text{ such that } (w, w') \in R_x \text{ and } M \models_{w'} \phi(x) \} = W,\]
because for any \(w \in D^V\) there is \(w' \in D^V\) such that \(w'(x) = u\) and for all \(y \in (V \backslash \{x\}) \) \(w'(y) = w(y)\), and consequently \((w, w') \in R_x\). It holds that \(M \models_{w'} \phi(x)\), i.e., \(\phi(x)\) is satisfied for the assignment \(w'\), because \(w'(x) = u\) and \(\phi(w'(x))\) is equivalent to \(\phi[x/u]\) which is true sentence with (from inductive hypothesis) \(|\phi[x/u]| = W\).
Consequently, any sentence which is true in Tarski’s interpretation \(I_T\) is also true in the Kripke’s interpretation \(I_K = \delta(I_T)\). Vice versa, for any sentence true in Kripke’s interpretation \(I_K \in \mathcal{JK}(\Gamma)\) it can be analogously shown that it is also true in the Tarski’s interpretation \(I_T = \delta^{-1}(I_K)\), where \(\delta^{-1}\) is inverse of the bijection \(\delta\). Thus, the Kripke’s semantics given in Definition \[\delta\] is an adequate semantics for the FOL.
Consequently, both “projections” in diagram above, where \(\text{FOL}_K\) denotes this adequate Kripke’s version of the FOL (i.e., FOL where the quantifiers \((\exists x)\) are interpreted as modal existential operators), are valid.
Let us show that also other two reductions into Propositional logics are valid and render commutative the diagram above:
1. Actual world reduction, when \(\mathcal{W} = \{h\}\): then, for this unique explicit actual world \(h\) we have that we can have only one (non empty) accessibility relation \(R_i = \{(h, h)\}\), so that unique possible existential modal operator \(\diamond_i\), of this modal propositional logic obtained by this reduction, is an identity operation: that is, obtained reduction is a propositional logic without modal operators, i.e., it is a pure propositional logic. In fact, we have that for any propositional formula \(\phi\) and the unique explicit world \(h \in \mathcal{W}\),
\[M \models_h \diamond_i \phi \iff \text{exists } w' \in \mathcal{W} \text{ such that } (h, w') \in R_i \text{ and } M \models_{w'} \phi,\]
iff \(M \models_h \phi\). The Kripke’s mapping \(I_K : P \rightarrow 2^\mathcal{W}\), for the singleton set \(\mathcal{W} = \{h\}\) and the bijection \(2^{\{h\}} \simeq 2\), becomes the propositional interpretation \(I'_K : P \rightarrow 2\). Thus, we obtained a pure propositional logic in the actual world.
2. Reduction \(\mathcal{V} = D = F = \emptyset\) from \(\text{FOL}_K\): from the fact that \(\mathcal{V}\) is the empty set of variables, we have that in \(P\) all symbols become nullary symbols, that is propositional symbols, so that the obtained logic is without modal operators (that is without existential \(\text{FOL}\) quantifier \(\exists\)). Consequently, the obtained logic is a propositional logic with a unique generalized world equal to the empty function \(* : \emptyset \rightarrow D\), from the fact that \(\mathcal{W} = D^\emptyset = \{\emptyset\}\). The Kripke’s mapping \(I_K : \mathcal{W} \times P \rightarrow \bigcup_{n \in \mathbb{N}} 2^{D^n}\) for \(\text{FOL}_K\) in this reduction becomes the mapping \(I'_K : \{\emptyset\} \times P \rightarrow 2^{D^0}\) where \(D^0\) is the singleton set \(\{<>\}\), so that, from bijections \(\{\emptyset\} \times P \simeq P\) and \(2^{\{<\}} \simeq 2\), this mapping becomes the propositional interpretation \(I'_K : P \rightarrow 2\). We can consider the unique generalized world \(*\) equivalent to the unique actual world, so that we obtain exactly the same propositional logics in the actual world, as in the case above.

\[\square\]
There is a surprising result from this theorem and its commutative diagram: we obtained that a FOL (more precise its modal interpretation of quantifiers in \(\text{FOL}_K\)) is a particular reduction from the predicate modal logics. But it is well known that the propositional modal logics can be, based on modal correspondence theory \[24,25\], embedded into the FOL by transforming each propositional letter \(p_i\) into an unary predicate \(p_i^1(x)\) (where
Proposition 2 The intension (sense) of any virtual predicate \( \phi(x_1, \ldots, x_k) \) in the \( \text{FOL}_K \), with a set of only intrinsic possible worlds \( \mathbb{W} = \mathbb{D}^V \), is equivalent to its extension in a given Tarski’s interpretation of the FOL. That is, it is impossible to support the intensions in the standard FOL with Tarski’s semantics.

Proof: We have to show that the intension \( I^*_{\phi}(\phi(x_1, \ldots, x_k)) : \mathbb{W} \to \mathfrak{R} \) of any FOL formula \( \phi(x_1, \ldots, x_k) \) in a given Tarski’s interpretation \( I^*_{\phi} : \mathcal{L} \to \mathfrak{R} \) is a constant function.
from the set of possible worlds $\mathcal{W} = \mathcal{D}^\mathcal{V}$ in FOL$_\mathcal{K}$, such that for all $w \in \mathcal{W}$ we have that $I_n(\phi(x_1, ..., x_k))(w) = R$, where $R = I_T^i(\phi(x_1, ..., x_k))$ is the extension of this formula in this Tarski’s interpretation. We can show it by the structural recursion:

1. Case when $\phi(x_1, ..., x_k)$ is a predicate letter $p^k \in P$. Then,

$$I_n(p^k_i(x_1, ..., x_k))(w) = \|p^k_i(x_1, ..., x_k)\|_{\mathcal{M}, w}$$

$$= \{ (d_1, ..., d_k) \in \mathcal{D}^k \mid \mathcal{M} \models_w p^k_i(d_1, ..., d_k) \}$$

$$= \{ (d_1, ..., d_k) \mid I_K(w, p^k_i)(d_1, ..., d_k) = t \}$$

$$= \{ (d_1, ..., d_k) \mid (d_1, ..., d_k) \in I_T(p^k_i) \} = I_T(p^k_i).$$

2. Case when $\phi(x_1, ..., x_k)$ is a virtual predicate. From Theorem 1, it holds that if for a given assignment $g \in \mathcal{W}$ a ground formula $\phi(x_1, ..., x_k)/g$ is true in a given Tarski’s interpretation $I_T$ (i.e., when $I_T(\phi(x_1, ..., x_k)/g) = t$), then $|\phi(x_1, ..., x_k)/g| = \mathcal{W}$ (it is true in the correspondent Kripke’s interpretation), that is, $\mathcal{M} \models_w \phi(x_1, ..., x_k)/g$ for every possible world $w \in \mathcal{W}$. Thus, we have that the intension of this virtual predicate is,

$$I_n(\phi(x_1, ..., x_k))(w) = |\phi(x_1, ..., x_k)|_{\mathcal{M}, w}$$

$$= \{ (g(x_1), ..., g(x_k)) \in \mathcal{D}^k \mid g \in \mathcal{W} \text{ and } \mathcal{M} \models_w \phi(x_1, ..., x_k)/g \}$$

$$= \{ (g(x_1), ..., g(x_k)) \mid g \in \mathcal{W} \text{ and } I_T^i(\phi(x_1, ..., x_k)/g) = t \}$$

$$= \{ (g(x_1), ..., g(x_k)) \mid (g(x_1), ..., g(x_k)) \in I_T^i(\phi(x_1, ..., x_k)) \}$$

$$= I_T^i(\phi(x_1, ..., x_k)).$$

Thus, the function $I_n$ is invariant w.r.t. the possible worlds $w \in \mathcal{W}$, and returns with the extension, of a considered (virtual) predicate, determined by a given Tarski’s interpretation.

What does it mean? First of all it means that not every modal logic with a given set of possible worlds $\mathcal{W}$ is an intensional logic, and that the quality of the intensionality which can be expressed by a given modal logic depends on the set of possible worlds $\mathcal{W}$ and their capacity to model the possible extensions of logic formulae. For example, if $\mathcal{W}$ is a finite set with very small cardinality, it often wold not be able to express the all possible extensions for logic formulae, and, consequently, its intensional capability will be very limited. But also if $\mathcal{W}$ is infinite, as in the case above when $\mathcal{D}$ is an infinite domain, we demonstrated that they are not able to express the intensionality. Consequently, in order to be able to express the full intensionality in a given modal logic, it is very important to chose the new appropriate set of possible worlds, independently from the original set of possible worlds of the particular given modal logic.

In fact, from this point of view, the left arrow in the diagram in Theorem 1 represents the logics with (partial) intensionalities, while the right arrow of the same diagram represents two extremal reductions of the intensionality, by identifying it with the pure extensionality (the propositional logic can be seen as a modal logic with the unique actual possible worlds, so that the intensionality corresponds to the extensionality, as in the case of the FOL$_\mathcal{C}$).

**Remark:** The natural choice for the set of explicit possible worlds for the fully intensional logic is the set of interpretations of its original logic (modal or not, determined by its set of axioms, inference relations, and a predefined set $\Gamma$, possibly empty, for which these interpretations are models), because such a set of interpretations is able to express the all logically possible extensions of the formulae of the original (not fully intensional) logic. In what follows we will do this intensional upgrade for the standard (not modal) FOL($\Gamma$), but generally it can be done to every kind of logics, thus to
any kind of modal logics, consequently also to the modal logic $\text{FOL}_K(\Gamma)$: in that case we obtain the two-levels modal logic (as in \cite{26,27}). At the lower-level we will have original modal logics with their original set of possible worlds (the set $D^V$ in the case of $\text{FOL}_K(\Gamma)$), while at the new upper-level each new explicit possible world would correspond to the particular Kripke’s interpretations of the original modal logics. The obtained upper-level intensional logic has a kind of rigid semantics, where the domains and the extensions of built-in predicates/propositions of the "lower-level" modal logics are identical in every upper-level possible world.

In intensional logics a $k$-ary functional symbol $f^k \in F$ is considered as the new $k + 1$-ary "functional" predicate symbol $f^{k+1} \in P$ whose extension is the graph of this function, such that cannot exists two tuples $(d_1, \ldots, d_k, u_1), (d_1, \ldots, d_k, u_2)$ in its extension with $u_1 \neq u_2$ (by introducing new axiom $(\exists_1 x_{k+1}) f^{k+1}(x_1, \ldots, x_{k+1})$). Thus, in what follows we will have only the set of predicate symbols.

This two-level intensional modal logic with the orthogonality of old possible worlds of the original modal logic $\mathcal{W} = \mathcal{W} \times D^V$ and the new set of explicit possible worlds $\mathcal{G}_K$ (the set of all Kripke interpretations $I_K \in \mathcal{G}_K(\Gamma)$, of the original (non-intensional) modal logic, in which all assumptions in $\Gamma$ (possibly empty set) are true), means that the obtained intensional modal logic has the set of explicit possible worlds equal to the cartesian product of old explicit worlds $\mathcal{W}$ and new added worlds in $\mathcal{G}_K(\Gamma)$, so that new generalized possible worlds are equal to the set $\mathcal{W} = (\mathcal{G}_K(\Gamma) \times \mathcal{W}) \times D^V$.

Consequently, the Kripke semantics of fully intensional modal logic, obtained as an enrichment of the original modal logic, can be given by the following definition:

**Definition 8. INTENSIONAL ENRICHMENT OF MULTI-MODAL LOGICS:**

Let $\mathcal{M} = (\mathcal{W}, \{\mathcal{R}_i\}, \mathcal{D}, I_K)$ be a Kripke’s interpretation of an original multi-modal logic with the set of (generalized) possible worlds $\mathcal{W} = \mathcal{W} \times D^V$ and the set of existential modal operators $\diamond_i$ with accessibility relations $\mathcal{R}_i$, given by Definition\[9\]. Then we denote by $\hat{\mathcal{M}} = (\hat{\mathcal{W}}, \{\hat{\mathcal{R}}_i\}, \hat{\mathcal{D}}, \hat{I}_K)$ a Kripke’s interpretation of its intensional enrichment with the set of possible worlds $\hat{\mathcal{W}} = \mathcal{G}_K(\Gamma) \times \mathcal{W}$, the optional set of new intensional modal operators $\hat{\diamond}_j$ with the accessibility relations $\hat{\mathcal{R}}_j$ over the worlds in $\mathcal{G}_K(\Gamma)$, and new set of explicit worlds $\hat{\mathcal{W}} = \mathcal{G}_K(\Gamma) \times \mathcal{W}$, such that for any explicit world $(I_K, w, p^k) \in \mathcal{G}_K(\Gamma) \times \mathcal{W}$ and $p^k \in P$ we have that $\hat{I}_K(I_K, w, p^k) =_{\text{def}} I_K(w, p^k) : D^k \rightarrow 2$. The satisfaction relation $\models_{I_K, w, g}$ for a given world $(I_K, w, g) \in \hat{\mathcal{W}}$ is defined as follows:

1. $\hat{\mathcal{M}} \models_{I_K, w, g} p^k(x_1, \ldots, x_k)$ iff $\hat{I}_K(I_K, w, p^k)(g(x_1), \ldots, g(x_k)) = t$.
2. $\hat{\mathcal{M}} \models_{I_K, w, g} \varphi \land \phi$ iff $\hat{\mathcal{M}} \models_{I_K, w, g} \varphi$ and $\hat{\mathcal{M}} \models_{I_K, w, g} \phi$.
3. $\hat{\mathcal{M}} \models_{I_K, w, g} \neg \varphi$ iff not $\hat{\mathcal{M}} \models_{I_K, w, g} \varphi$.
4. $\hat{\mathcal{M}} \models_{I_K, w, g} \diamond_i \varphi$ iff exists $w' \in \mathcal{W}$ such that $(w, w') \in \mathcal{R}_i$ and $\hat{\mathcal{M}} \models_{I_K, w', g} \varphi$.
5. $\hat{\mathcal{M}} \models_{I_K, w, g} \hat{\diamond}_j \varphi$ iff exists $I_K' \in \mathcal{G}_K$ such that $(I_K, I_K') \in \hat{\mathcal{R}}_j$ and $\hat{\mathcal{M}} \models_{I_K', w, g} \varphi$.

Notice that this intensional enrichment is maximal one: in fact we have taken all Kripke’s interpretations of the original modal logics for the possible worlds of this new intensional logic. We can obtain partial intensional enrichments if we take only a strict subset of $S \subset \mathcal{G}_K(\Gamma)$ in order to define generalized possible worlds $\hat{\mathcal{W}} = S \times \mathcal{W}$. In that case
we would introduce the non monotonic property for obtained intensional logic.

**Example 1**: Let us consider the intension enrichment of the multi-modal logic $\text{FOL}_K(\Gamma)$ given by Definition\ref{def:intension_enrich} with the Kripke’s interpretation $\mathcal{M} = (\mathcal{W}, \{\mathcal{R}_i \mid x \in \mathcal{V}\}, D, I_K)$ of the FOL($\Gamma$), with a set of (generalized) possible worlds $\mathcal{W} = D^V$ and the accessibility relation $\mathcal{R}_x = \{(w_1, w_2) \in \mathcal{W} \times \mathcal{W} \mid x \in \mathcal{V}\}$ and for all $y \in \mathcal{V}\setminus\{x\}(w_1(y) = w_2(y))$ for existential modal operator ($\exists x$) for each variable $x \in \mathcal{V}$.

Then $\tilde{\mathcal{M}} = (\tilde{\mathcal{W}}, \{\tilde{\mathcal{R}}_j\}, \tilde{D}, \tilde{I}_K)$ is a Kripke’s interpretation of its intensional enrichment with the set of generalized possible worlds $\tilde{\mathcal{W}} = \mathcal{I}_K(\Gamma) \times \mathcal{W} = \mathcal{I}_K \times D^V$ (here the set of explicit worlds is $\mathcal{I}_K(\Gamma)$), the optional set of new modal operators $\hat{\Box}_j$ with the accessibility relations $\tilde{\mathcal{R}}_j$ over the worlds in $\mathcal{I}_K(\Gamma)$, and new mapping $\tilde{I}_K : (\mathcal{I}_K(\Gamma) \times D^V) \times P \to \bigcup_{n \in \mathbb{N}} 2^{\mathcal{D}^n}$, such that for any explicit world $(I_K, g) \in \mathcal{I}_K(\Gamma) \times D^V$ and $p^k_i \in P$ we have that $\tilde{I}_K(I_K, g, p^k_i) =_{def} I_K(g, p^k_i) : D^k \to 2$, with $\tilde{M} =_{I_K,g} \hat{\Box}_j \phi$ iff there exists $g' \in D^V$ such that $(g, g') \in \mathcal{R}_x$ and $\tilde{M} =_{I_K,g'} \phi$.

Then, from Definition\ref{def:intension semantics} for the intension semantics, the mapping $I_n : \mathcal{L}_{op} \to \mathcal{R}^{\mathcal{I}_K(\Gamma)}$, where $\mathcal{L}_{op}$ is the subset of formulae with free variables (virtual predicates), such that for any virtual predicate $\phi(x_1, \ldots, x_k) \in \mathcal{L}_{op}$ the mapping $I_n(\phi(x_1, \ldots, x_k)) : \mathcal{I}_K(\Gamma) \to \mathcal{R}$ is the Montague’s meaning (intension) of this virtual predicate, i.e. mapping which turns with the extension of this predicate in every explicit possible world (i.e., Kripke’s interpretation of FOL$_K(\Gamma)$) $I_K \in \mathcal{I}_K(\Gamma)$. That is, we have that $I_n(\phi(x_1, \ldots, x_k))(I_K) =_{def} \|\phi(x_1, \ldots, x_k)\|_{\mathcal{M}, I_K} \subseteq \{(g(x_1), \ldots, g(x_k)) \in D^k \mid g \in D^V \text{ and } \tilde{M} =_{I_K,g} \phi(x_1, \ldots, x_k)\}$.

In what follows, the minimal (i.e. without new intensional modal operators $\hat{\Box}_j$) intensional enrichment of the multi-modal logic $\text{FOL}_K$ we will denote by $\text{FOL}_{Kx}$.

This two-level intensional modal logic, described above, has the following correspondence property between the Kripke’s interpretation $\mathcal{M}$ of the original modal logic and the Kripke’s interpretation of $\tilde{\mathcal{M}}$ its intensional enrichment:

**Proposition 3** For any logic formulae $\phi$ of the original multi-modal logic, with the set of (generalized) possible worlds $\mathcal{W} = \mathcal{V} \times D^V$ and the set of existential modal operators $\hat{\Box}_i$ with accessibility relations $\mathcal{R}_i$ given by Definition\ref{def:original semantics} the following property is valid: $\tilde{\mathcal{M}} =_{I_K,w,g} \phi$ iff $\mathcal{M} =_{w,g} \phi$, where $\mathcal{M} = (\mathcal{W}, \{\mathcal{R}_i\}, D, I_K)$ is a Kripke’s interpretation of the original multi-modal logic. Consequently, $\phi$ is true in the intensionally enriched multi-modal logic iff it is valid in the original multi-modal logic.

**Proof**: Let us demonstrate it by structural induction on the length of logic formulae. For any atom $\phi = p^k_i(x_1, \ldots, x_k)$ we have from Definition\ref{def:original semantics} that $\tilde{\mathcal{M}} =_{I_K,w,g} p^k_i(x_1, \ldots, x_k)$ iff $\tilde{I}_K(I_K, w, p^k_i(g(x_1), \ldots, g(x_k))) = I_K(w, p^k_i(g(x_1), \ldots, g(x_k))) = t$ iff $\mathcal{M} =_{w,g} p^k_i(x_1, \ldots, x_k)$. Let us suppose that such a property holds for every formula $\phi$ with less than $n$ logic connectives of the original multi-modal logic (thus without new intensional connectives $\hat{\Box}_i$), and let us show that it holds also for any formula with $n$ logic connectives. There are the following cases:

1. The case when $\phi = \neg \psi$ where $\psi$ has $n - 1$ logic connectives. Then $\tilde{\mathcal{M}} =_{I_K,w,g} \phi$
We extend the satisfaction relation to the set of generalized possible worlds in obtained intensional semantics is equal to the set of Tarski’s interpretations of FOL

We denote by \( M \) with the set of possible worlds \( W \) as standard FOL (thus without other (modal) logic connectives), but enriched of FOL (thus, satisfied in every explicit possible world, that is, in every Tarski’s interpretation are so that the set of generalized possible worlds in obtained intensional semantics is equal to the set of all Tarski’s interpretations of FOL

Two virtual predicates with the same tuple of free variables, \( \phi(x_1, \ldots, x_n) \) and \( \psi(x_1, \ldots, x_n) \), are intensionally equal iff the formula \( \phi(x_1, \ldots, x_n) \equiv \psi(x_1, \ldots, x_n) \) is true in this FOL\(_I\) (thus, satisfied in every explicit possible world, that is, in every Tarski’s interpretation of FOL\(_I\)).

Let us define this minimal intensional first-order logic FOL\(_I\) which has the same syntax as standard FOL (thus without other (modal) logic connectives), but enriched with the set of possible worlds \( W = \mathcal{I}_F \times D^V \).

**Definition 9. Minimal Intensional First-order Logic (FOL\(_I\)):**

We denote by \( M_{\text{FOL}\_I} = (W, D, I_K) \) the Kripke’s interpretation of the Intensional logic FOL\(_I\) with a set of (generalized) possible worlds \( W \), a set of explicit possible worlds equal to the set of Tarski’s interpretation of FOL\(_I\), \( W = \pi_1(W) = \mathcal{I}_F \) and \( \pi_2(W) = D^V \), non empty domain \( D \), and the mapping \( I_K : W \times P \rightarrow \bigcup_{n \in N} D^n \).

We extend the satisfaction relation \( \models_{w,g} \) of Kripke semantics to the first-order quantification \( \exists \) by: \( M_{\text{FOL}\_I} \models_{w,g} (\exists x) \phi \) iff

1. \( M_{\text{FOL}\_I} \models_{w,g} \phi \), if \( x \) is not a free variable in \( \phi \);
2. exists \( u \in D \) such that \( \mathcal{M}_{FOL,\{\Gamma\}} \models_{w,g} \phi[x/u] \), if \( x \) is a free variable in \( \phi \) and \( \phi[x/u] \) the formula obtained by substitution of \( x \) by the value \( u \) in \( \phi \).

Such an interpretation is the Kripke model of Intensional FOL if for any explicit world (Tarski’s interpretation) \( w = I_T \in W, p_k^1 \in P \), and a tuple \((d_1, \ldots, d_k) \in D^k\), we have that:

\[
I_K(p_k^1)(d_1, \ldots, d_k) = t \iff (d_1, \ldots, d_k) \in w(p_k^1).
\]

Notice that the intensional semantics above is given for the ordinary syntax of the First-order logic with the existential quantifier \( \exists \), without modal operators, thus with the empty set of accessibility binary relations over the set of explicit possible worlds \( W = \pi_1(W) = I_T(\Gamma) \): this is the reason to denominate it by “minimal”.

Let us show that this unique intensional Kripke model \( \mathcal{M}_{FOL,\{\Gamma\}} = (W, D, I_K) \) models the Tarskian logical consequence of the First-order logic with a set of assumption in \( \Gamma \), so that the added intensionality preserves the Tarskian semantics of the FOL.

**Proposition 4** Let \( \mathcal{M}_{FOL,\{\Gamma\}} = (W, D, I_K) \) be the unique intensional Kripke model of the First-order logic with a set of assumptions in \( \Gamma \), as defined in Definition[9] Then, a formula \( \phi \) is a logical consequence of \( \Gamma \) in the Tarskian semantics for the FOL, that is, \( \Gamma \models \phi \), iff \( \phi \) is true in this Kripke intensional model \( \mathcal{M}_{FOL,\{\Gamma\}} \).

Let \( I_n : \mathcal{L}_{op} \rightarrow \mathcal{R}^W \) be the mapping given in Definition[5] Then, for any (virtual) predicate \( \phi(x_1, \ldots, x_k) \), the mapping \( I_n(\phi(x_1, \ldots, x_k)) : W \rightarrow \mathcal{R} \) represents the Montague’s meaning (intension) of this logic formula, such that:

\[
\text{for any } w \in W = \pi_1(W), \; I_n(\phi(x_1, \ldots, x_k))(w) = w^*(\phi(x_1, \ldots, x_k)).
\]

**Proof:** Let us show that for any first-order formula \( \phi \) it holds that, \( \mathcal{M}_{FOL,\{\Gamma\}} \models_{w,g} \phi \iff w^*(\phi/g) = t \), where \( w^* \) is the unique extension of Tarski’s interpretation \( w = I_T \in W = I_T(\Gamma) \) to all formulae.

Let us demonstrate it by the structural induction on the length of logic formulae. For any atom \( \phi = p_k^1(x_1, \ldots, x_k) \) we have from Definition[8] that \( \mathcal{M}_{FOL,\{\Gamma\}} \models_{I_T,g} p_k^1(x_1, \ldots, x_k) \iff I_K(I_T, p_k^1)(g(x_1), \ldots, g(x_k)) = t \iff (g(x_1), \ldots, g(x_k)) \in I_T(p_k^1) \iff I_T(p_k^1)(x_1, \ldots, x_k)/g = t \). Let us suppose that such a property holds for every formula \( \phi \) with less than \( n \) logic connectives of the FOL, and let us show that it holds also for any formula with \( n \) logic connectives. There are the following cases:

1. The case when \( \phi = \neg \psi \) where \( \psi \) has \( n-1 \) logic connectives. Then \( \mathcal{M}_{FOL,\{\Gamma\}} \models_{I_T,g} \phi \iff \mathcal{M}_{FOL,\{\Gamma\}} \models_{I_T,g} \neg \psi \iff \text{not } \mathcal{M}_{FOL,\{\Gamma\}} \models_{I_T,g} \psi \iff (by \text{ inductive hypothesis}) \iff I_T^*(\neg \psi/g) = t \iff I_T^*(\phi/g) = t \).

2. The case when \( \phi = \psi_1 \land \psi_2 \), where both \( \psi_1, \psi_2 \) have less than \( n \) logic connectives, is analogous to the case 1.

3. The case when \( \phi = (\exists x)\psi \) where \( \psi \) has \( n-1 \) logic connectives. It is enough to consider the case when \( x \) is a free variable in \( \psi \). Then \( \mathcal{M}_{FOL,\{\Gamma\}} \models_{I_T,g} \phi \iff \mathcal{M}_{FOL,\{\Gamma\}} \models_{I_T,g} (\exists x)\psi \iff \exists u \in D \) such that \( \mathcal{M}_{FOL,\{\Gamma\}} \models_{I_T,g} \psi[x/u] \) (by inductive hypothesis) \iff \exists u \in D \) such that \( I_T^*(\psi[x/u]/g) = t \iff I_T^*((\exists x)\psi/g) = t \).

It is easy to verify that the intension of predicates in the FOL\(_T(\Gamma)\) defined above can be expressed by the mapping \( I_n \) such that for any \( p_k^1 \in P \), \( I_n(p_k^1(x_1, \ldots, x_k))(w) = w(p_k^1) \), and, more general, for any virtual predicate \( \phi(x_1, \ldots, x_k) \),
The minimal other modal operator. Because of that we denominated such an intensional FOL as the required by Bealer [6], that is, we do not need intensional abstraction operator or another modal operator in FOL or as modal operators in FOL

\[ \{ (g(x_1), g(x_k)) \in \mathcal{D}^k | g \in \mathcal{D}^V \text{ and } \mathcal{M}_{\text{FOL}_x} \models_{w,g} \phi(x_1, ..., x_k) \} \]

\[ = \{ (g(x_1), g(x_k)) | g \in \mathcal{D}^V \text{ and } w^*(\phi(x_1, ..., x_k)) = t \} \]

\[ = \{ (g(x_1), g(x_k)) | g \in \mathcal{D}^V \text{ and } (g(x_1), ..., g(x_k)) \in w^*(\phi(x_1, ..., x_k)) \} \]

\[ = w^*(\phi(x_1, ..., x_k)) \]

where \( w^* \) is the unique extension of Tarski’s interpretation \( w \in \mathcal{W} = \mathcal{I}_T(\Gamma) \) to all formulae. Consequently, \( I_n(\phi(x_1, ..., x_k)) : \mathcal{W} \rightarrow \mathcal{R} \) is the Montague’s meaning (i.e., the intension) of the (virtual) predicate \( \phi(x_1, ..., x_k) \).

It is clear that in Kripke semantics of this intensional first-order logic, denoted by FOL\(_x(\Gamma)\), if the set of assumptions is empty \( (\Gamma = \emptyset) \), then a formula \( \phi \) is true in the intensional Kripke model \( \mathcal{M}_{\text{FOL}_x(\Gamma)} \) iff it is valid in Tarskian semantics of the FOL, that is, iff \( \models \phi \) in the FOL.

The main difference between Tarskian semantics and this intensional semantics is that this unique intensional Kripke model \( \mathcal{M}_{\text{FOL}_x(\Gamma)} \) encapsulates the set of all Tarski models of the First-order logic with a (possibly empty) set of assumptions \( \Gamma \).

**Corollary 2** The intensionalities of two different minimal intensional enrichments of the first-order syntax, given by intensional logics FOL\(_x(\Gamma)\) and FOL\(_{Kx}(\Gamma)\) (in Example 1), are equivalent and correspond to Montague’s intensionality.

**Proof:** Let us denote by \( I^n_{\text{FOL}_x(\Gamma)} \), \( I^n_{\text{FOL}_{Kx}(\Gamma)} : \mathcal{L}_{op} \rightarrow \mathcal{R} \) the intensional mappings (from Definition 3 of the intensional semantics) for these two intensional enrichments of the FOL(\( \Gamma \)). Notice that the set of explicit possible worlds \( \mathcal{W} \) in FOL\(_x(\Gamma)\) is equal to \( \mathcal{I}_T(\Gamma) \) while in FOL\(_{Kx}(\Gamma)\) is equal to \( \mathcal{I}_K(\Gamma) \), with the bijection (from Theorem 1) \( b : \mathcal{I}_T(\Gamma) \simeq \mathcal{I}_K(\Gamma) \). We have to show that for any formulae \( \phi(x_1, ..., x_n) \in \mathcal{L}_{op} \) its extension, in a given explicit world \( I_T \in \mathcal{I}_T(\Gamma) \) of the intensional logic FOL\(_x(\Gamma)\), is equal to its extension in the correspondent explicit world \( I_K = b(I_T) \in \mathcal{I}_K(\Gamma) \) of the intensional logic FOL\(_{Kx}(\Gamma)\). In fact, we have that:

\[ I^n_{\text{FOL}_x(\Gamma)}(\phi(x_1, ..., x_k))(I_T) = \| \phi(x_1, ..., x_k) \|_{\mathcal{M}_{\text{FOL}_x(\Gamma)}, I_T} \]

\[ = \{ (g(x_1), g(x_k)) \in \mathcal{D}^k | g \in \mathcal{D}^V \text{ and } \mathcal{M}_{\text{FOL}_x(\Gamma)} \models_{I_T,g} \phi(x_1, ..., x_k) \} \]

\[ = \{ (g(x_1), g(x_k)) | g \in \mathcal{D}^V \text{ and } I^n_T(\phi(x_1, ..., x_k)) = t \} \]

\[ = \{ (g(x_1), g(x_k)) | g \in \mathcal{D}^V \text{ and } \mathcal{M} \models_g \phi(x_1, ..., x_k) \} \]

\[ = \| \phi(x_1, ..., x_k) \|_{\mathcal{M}_{\text{FOL}_{Kx}(\Gamma)}, I_T} = I^n_{\text{FOL}_{Kx}(\Gamma)}(\phi(x_1, ..., x_k))(I_K) \]

That is, independently on how we interpret the quantifiers of the FOL, as in standard FOL or as modal operators in FOL\(_K\), the intensionality of the FOL is obtained only by one adequate semantic enrichment, without modifying its syntax. Consequently, we have demonstrated that an intensional FOL does not need the other logic operators as required by Bealer [6], that is, we do not need intensional abstraction operator or another modal operator. Because of that we denominated such an intensional FOL as the minimal intensional logic. Another intensional FOL without the intensional abstraction is given in the following example:
Example 2: In order to be able to recognize the intensional equivalence between (virtual) predicates, that may be used in intensional mapping between P2P databases, we need to extend this minimal intensional FOL also syntactically, by introducing the new modal existential operator $\quad$, so that $\phi(x_1,...,x_n)$ and $\psi(x_1,...,x_n)$ are intensionally equivalent iff the modal First-order formula $\quad\phi(x_1,...,x_n) = \quad\psi(x_1,...,x_n)$ is true in this modal FOL. The Kripke semantics for this extended modal first-order logic is a $\mathcal{S}_5$ modal FOL with the accessibility relation $\mathcal{R} = \mathcal{W} \times \mathcal{W}$.

Two intensional equivalent predicates does need to have equal extensions in each explicit possible world as is required by intensional equality (equal meaning from Montague’s point of view) when $\phi(x_1,...,x_n) = \psi(x_1,...,x_n)$ is true, where ‘$\equiv$’ is the standard logic equivalence connective.

Notice that if they are intensionally equal, it does not mean that they are equal concepts, i.e. that $I(\phi(x_1,...,x_n)) = I(\phi(x_1,...,x_n)) \in D$, but only that they are necessarily equivalent. In fact, the two atoms $p_1^1(x)$, ”$x$ has been bought”, and $p_2^1(x)$, ”$x$ has been sold”, are necessarily equivalent, that is, it holds that $p_1^1(x) \equiv p_2^1(x)$ but they are two different concepts, that is $I(p_1^1(x)) \neq I(p_2^1(x))$ i.e., $(I(p_1^1(x)),I(p_2^1(x))) \not\in h(Id) = R = _C$.

Such an distinction of equal concepts and of the intensional equality (i.e., the necessary equivalence) is not possible in the Montague’s semantics, and explain why we adopted PRP theory and two-step intensional semantics in Definition 5 analogously to Bealer’s approach.

In fact, we can show that two first-order open formulae $\phi(x_1,...,x_n)$ and $\psi(x_1,...,x_n)$ are intensionally equivalent iff $\quad\phi(x_1,...,x_n)$ and $\quad\psi(x_1,...,x_n)$ are intensionally equal. We have that $I_n(\quad\phi(x_1,...,x_k))(\phi(x_1,...,x_k)) = \quad I(\phi(x_1,...,x_k))(\phi(x_1,...,x_k)) = \{ (g(x_1),...,g(x_k)) | g \in D^k \text{ and } M \models w,g \quad \phi(x_1,...,x_k) \}$ = $\{ (g(x_1),...,g(x_k)) | g \in D^k \text{ and } M \models w,g \quad \phi(x_1,...,x_k), \text{ exists } w \text{ such that } (w,w) \in \mathcal{R}, \text{ and } M \models w,g \quad \phi(x_1,...,x_k) \}$ = $\{ (g(x_1),...,g(x_k)) | g \in D^k \text{ and } exists \ w \text{ such that } M \models w,g \quad \phi(x_1,...,x_k) \}$ = $\bigcup_{w \in \mathcal{W}} I_n(\phi(x_1,...,x_k))(\phi(x_1,...,x_k)) = \bigcup_{w \in \mathcal{W}} w^*(\phi(x_1,...,x_k))$, that is, the intension of $\quad\phi(x_1,...,x_n)$ is a constant function.

Thus, $\phi(x_1,...,x_n)$ and $\psi(x_1,...,x_n)$ are intensionally equivalent if $\bigcup_{w \in \mathcal{W}} I_n(\phi(x_1,...,x_k))(\phi(x_1,...,x_k)) = \bigcup_{w \in \mathcal{W}} I_n(\psi(x_1,...,x_k))(\psi(x_1,...,x_k))$, i.e.,

if $I_n(\quad\phi(x_1,...,x_k))(\phi(x_1,...,x_k)) = I_n(\quad\psi(x_1,...,x_k))(\psi(x_1,...,x_k))$ for every world $w \in \mathcal{W}$, i.e., if $\quad\phi(x_1,...,x_n)$ and $\quad\psi(x_1,...,x_n)$ are intensionally equal.

Another extension of this minimal intensional FOL is of course the intensional FOL defined by Bealer in [6], if we define the mapping $is : \mathcal{W} \rightarrow \mathcal{E}$ in Definition 5 as the Montague-Bealer’s isomorphism (bijection) between possible worlds and the set of extensionalization functions.

Notice that both versions of intensional FOL are modal logics, thus we can define two different logic inferences for them: the local inference relation $\vdash_w$ and the global inference relation $\vdash$, as follows:

1. For a given set of logic formulae $\Gamma$ we tell that they locally infer the formula $\phi$ in a possible world $w \in \mathcal{W}$, that is,

$$\Gamma \vdash_w \phi \iff (\forall \text{ models } M)((\forall g)(\forall \psi \in \Gamma).M \models w,g \psi \Rightarrow M \models w,g \quad \phi)).$$
2. For a given set of logic formulae $\Gamma$ we tell that they globally infer the formula $\phi$, that is, $\Gamma \vdash \phi$ iff $\forall$ models $M$ $\forall w \in W$ $\forall$ $w \in W$, $M \models_{w,g} \psi$ implies $M \models_{w,g} \phi$.

The intensional First-order logic $FOL_I(\Gamma)$ in Definition 9 has one unique Kripke model $M = M_{FOL_I(\Gamma)}$. Thus, we obtain that in this modal intensional logic $FOL_I(\Gamma)$:

1. $\Gamma \models_{w} \phi$ iff $\phi$ is true in the Kripke model $M_{FOL_I(\Gamma)}$ in a given possible world $w \in J_T(\Gamma)$, that is, if $\phi$ is true in the Tarski’s model $I_T = w$ of $\Gamma$. Thus, this local inference $\models_{w}$ corresponds to the derivation of true formulae in a given Tarski model $I_T = w$ of $\Gamma$.

2. $\Gamma \models \phi$ iff $\phi$ is true in the Kripke model $M_{FOL_I(\Gamma)}$, that is, iff $\Gamma \models_{w} \phi$. So that the global inference $\models_{w}$ corresponds to the Tarskian logical consequence $\models$ in the standard First-order logic.

In the rest of this section we will consider the full homomorphic (algebraic) extensions of intensional semantics defined in Definition 5. The first step is to define the intensional algebra $A_{int}$ of concepts, analogous to Concept languages as, for example, in the case of the Description Logic (DL).

Concept languages steam from semantic networks [30,31,32] which for a large group of graphical languages used in the 1970s to represent and reason with conceptual knowledge. But they did not have a rigorously defined statement as emphasized by Brachman and Levesque [33,34]. After that, different versions of DL [35] with formal semantics appeared, as a family of knowledge representation formalisms that represent the knowledge of an application domain by first defining the relevant concepts and roles as a terminology (TBox) and then the assertions (ABox) about named individuals in terms of this terminology. The concepts denote sets of individuals, and roles denote binary relationships between individuals.

In our approach we will use not only binary, but also general $k$-ary relationships between individuals, in order to manage not only unary (as in DL) but all $k$-ary concepts. This approach is similar to Bealer’s intensional algebra, with the difference that our algebra is not an extension of intensional Boolean algebra as in the Bealer’s work, where the intensional conjunction is extensionally interpreted by set intersection (here, instead, it is interpreted by the natural join operations, defined in the FOL extensional algebra $A_R$ in Corollary 4). Moreover, we will define only the minimal intensional algebra (with minimal number of operators), able to support the homomorphic extension of the intensional mapping $I : L \rightarrow D$.

**Definition 10. INTENSIONAL FOL ALGEBRA:** Intensional FOL algebra is a structure $A_{int} = (D,I,d,True,\{conjs\}_{S \in P(|S|^2)},neg,\{exists_n\}_{n \in \mathbb{N}})$, with binary operations $\text{conjs} : D_1 \times D_1 \rightarrow D_1$, unary operation $\text{neg} : D_1 \rightarrow D_1$, and unary operations $\exists_{n} : D_1 \rightarrow D_1$, such that for any extensionalization function $h \in \mathcal{E}$, and $u \in D_k, v \in D_j, k,j \geq 0$,

1. $h(Id) = R_m$ and $h(True) = \{<>\}$.
2. $h(\text{conjs}(u,v)) = h(u) \triangleright S h(v)$, where $\triangleright S$ is the natural join operation defined in Corollary 7 and $\text{conjs}(u,v) \in D_m$, where $m = k + j - |S|$ if for every pair $(i_1, i_2) \in S$ it holds that $1 \leq i_1 \leq k, 1 \leq i_2 \leq j$ (otherwise $\text{conjs}(u,v) \in D_{k+j}$).
3. $h(\text{neg}(u)) = \sim (h(u)) = D^k \setminus (h(u))$, where $\sim$ is the operation defined in Corollary 7 and $\text{neg}(u) \in D_k$.
4. $h(\exists x u) = \pi_n(h(u))$, where $\pi_n$ is the operation defined in Corollary 7 and $\exists x u \in D_{k-1}$ if $1 \leq n \leq k$ (otherwise $\exists x u$ is the identity function).

We define the following homomorphic extension of the intensional interpretation $I : \mathcal{L} \rightarrow \mathcal{D}$:

1. The logic formula $\phi(x_1, x_2, x_3, x_4, x_5)$ will be intensionally interpreted by the concept $u_1 \in D_5$, obtained by the algebraic expression $\text{conj}_S(u, v)$ where $u = I(\phi(x_1, x_2, x_3, x_4, x_5)) \in D_5$, $v = I(\varphi(x_1, y_1, x_2, y_2)) \in D_4$ are the concepts of the virtual predicates $\phi, \varphi$, relatively, and $S = \{(4, 1), (2, 3)\}$. Consequently, we have that for any two formulae $\phi, \psi \in \mathcal{L}$ and a particular operator $\text{conj}_S$ uniquely determined by tuples of free variables in these two formulae, $I(\phi \land \psi) = \text{conj}_S(I(\phi), I(\psi))$.
2. The logic formula $\neg \phi(x_1, x_2, x_3, x_4, x_5)$ will be intensionally interpreted by the concept $u_1 \in D_5$, obtained by the algebraic expression $\text{neg}(u)$ where $u = I(\phi(x_1, x_2, x_3, x_4, x_5)) \in D_5$ is the concept of the virtual predicate $\phi$. Consequently, we have that for any formula $\phi \in \mathcal{L}$, $I(\neg \phi) = \text{neg}(I(\phi))$.
3. The logic formula $\exists x \phi(x_1, x_2, x_3, x_4, x_5)$ will be intensionally interpreted by the concept $u_1 \in D_5$, obtained by the algebraic expression $\exists x \phi(u, v)$ where $u = I(\phi(x_1, x_2, x_3, x_4, x_5)) \in D_5$ is the concept of the virtual predicate $\phi$. Consequently, we have that for any formula $\phi \in \mathcal{L}$ and a particular operator $\exists x \phi$ uniquely determined by the position of the existentially quantified variable in the tuple of free variables in $\phi$ (otherwise $n = 0$ if this quantified variable is not a free variable in $\phi$), $I(\exists x \phi) = \exists x \phi (I(\phi))$.

**Corollary 3** INTENSIONAL/EXTENSIONAL FOL SEMANTICS: For any Tarski’s interpretation $I_T$ of the FOL, the following diagram of homomorphisms commutes:

\[
\begin{array}{ccc}
\mathcal{A}_{\text{int}} (\text{concepts/meaning}) & \xrightarrow{h} & \mathcal{A}_{\text{ext}} (\text{extensionaliz.}) \\
\mathcal{I} (\text{intensional interpretation}) & \xrightarrow{\text{Frege/Russell}} & \mathcal{A}_R (\text{denotation}) \\
\mathcal{A}_{\text{FOL}} (\text{syntax}) & & \\
\end{array}
\]

where $h = \text{is}(w)$ where $w = I_T \in \mathcal{W}$ is the explicit possible world of the minimal intensional first-order logic in Definition 9.

**Proof:** The homomorphism of intensional mapping $I$ is defined by intensional interpretation above. Let us show that also the isomorphism is between the extensionalization mappings $h$ and Tarski’s interpretations $I_T$ is uniquely determined in order to make homomorphic and commutative the diagram above. It can be done by inductive structural recursion on the length of FOL formulae in $\mathcal{L}$: for any atom $p_i(x_1, ..., x_k) \in \mathcal{L}$ we define $is : I_T \mapsto h$ by requirement that $h(I(p_i(x_1, ..., x_k))) = I_T(p_i^k)$. Let us suppose that for any formula $\phi$ with $n$ logic connectives it holds that the mapping $is : I_T \mapsto h$...
satisfies requirement that $h(I(\phi)) = I^*_{T}(\phi)$. Let us show that it holds also for any logic formula $\phi$ with $n+1$ logic connectives. It is enough to show it in the case when $\varphi = \phi \land \psi$ (the other two cases are analogous):

$h(I(\varphi)) = h(I(\phi \land \psi)) = h(\text{conj}_{S}(I(\phi), I(\psi)))$ (from the homomorphic property of $I$)  

$=_{\text{def}} h(I(\phi)) \bowtie_{S} h(I(\psi))$ (from Definition 10)  

$= I^*_{T}(\varphi)$ (by inductive hypothesis)  

$= I^*_{T}(\phi)$, from the fact that the same conjunctive formula $\varphi$ is mapped by $I$ into $\text{conj}_{S_{1}}$ and by $I^*_{T}$ into $\bowtie_{S_{2}}$ where $S_{1} = S_{2}$.

This homomorphic diagram formally express the fusion of Frege’s and Russell’s semantics [2,36,37] of meaning and denotation of the FOL language, and renders mathematically correct the definition of what we call an “intuitive notion of intensionality”, in terms of which a language is intensional if denotation is distinguished from sense: that is, if both a denotation and sense is ascribed to its expressions. This notion is simply adopted from Frege’s contribution (without its infinite sense-hierarchy, avoided by Russell’s approach where there is only one meaning relation, one fundamental relation between words and things, here represented by one fixed intensional interpretation $I$), where the sense contains mode of presentation (here described algebraically as an algebra of concepts (intensions) $A_{\text{int}}$, and where sense determines denotation for any given extensionalization function $h$ (correspondent to a given Traski’s interpretation $I^*_{T}$). More about the relationships between Frege’s and Russell’s theories of meaning may be found in the Chapter 7, “Extensionality and Meaning”, in [6].

As noted by Gottlob Frege and Rudolf Carnap (he uses terms Intension/extension in the place of Frege’s terms sense/denotation [13]), the two logic formulae with the same denotation (i.e., the same extension for a given Tarski’s interpretation $I^*_{T}$) need not have the same sense (intension), thus such co-denotational expressions are not substitutable in general.

In fact there is exactly one sense (meaning) of a given logic formula in $\mathcal{L}$, defined by the uniquely fixed intensional interpretation $I$, and a set of possible denotations (extensions) each determined by a given Tarski’s interpretation of the FOL as follows from Definition 5:

$$\mathcal{L} \rightarrow_{I} D \implies h = h_{I}(I^*_{T}) \land I^*_{T} \in W = \mathcal{M}(I^*_{T}) \text{ for all } R.$$  

Often ‘intension’ has been used exclusively in connection with possible worlds semantics, however, here we use (as many others; as Bealer for example) ‘intension’ in a more wide sense, that is as an algebraic expression in the intensional algebra of meanings (concepts) $A_{\text{int}}$ which represents the structural composition of more complex concepts (meanings) from the given set of atomic meanings. Consequently, not only the denotation (extension) is compositional, but also the meaning (intension) is compositional.

Notice that this compositional property holds also for the generation of subconcepts: for example, given a virtual predicate $\phi(x_{1},..x_{n})$ with correspondent concept $I(\phi) \in D_{n}$, its subconcept is defined by $I(\phi[x_{i}/c]) = I(\phi(x_{1},..,x_{i-1},x_{i}/c,x_{i+1},..,x_{n})) \in D_{n-1}$, where the $i$-th free variable of the original virtual predicate is substituted by a language constant $c$.

The following compositional relationship exists between extensions of concepts and their subconcepts:
**Proposition 5** For any extensionalization function $h$ and a virtual predicate $\phi$ with a tuple of free variables $(x_1, ..., x_{i-1}, x_i, x_{i+1}, ..., x_n)$, $n \geq i \geq 1$, it holds that,

\[
    h(I(\phi|x_i/c)) =
\]

\[
= \pi_i(\{(u_1, ..., u_{i-1}, u_i, u_{i+1}, ..., u_n) \in h(I(\phi)) \mid u_i = I(c)\}), \quad \text{if } n \geq 2;
\]

\[
= f_{<\phi>(\{(u) \in h(I(\phi)) \mid u = I(c)\}), \quad \text{if } i = n = 1.
\]

For the sentences we have that for any virtual predicate $\phi(x_1, ..., x_n)$ and an assignment $g$, $h(I(\phi/g) = t$ if and only if $(g(x_1), ..., g(x_n)) \in h(I(\phi))$. 

**Proof:** Directly from the homomorphic diagram of Frege/Russell’s intensional semantics in Corollary 5. Let us consider the first case when $n \geq 2$, then:

\[
    h(I(\phi|x_i/c)) = I_{<\phi>}(\phi|x_i/c)) = \{(g(x_1), ..., g(x_{i-1}), g(x_{i+1}), ..., g(x_n)) \in D_n | g \in D^n \}
\]

\[
= \pi_i(\{(g(x_1), ..., g(x_{i-1}), g(x_i), g(x_{i+1}), ..., g(x_n)) \in D_n \mid g \in D^n \}
\]

\[
= \pi_i(\{(g(x_1), ..., g(x_{i-1}), g(x_i), g(x_{i+1}), ..., g(x_n)) \in I_{<\phi>}(\phi) \mid g \in D^n \}
\]

\[
= \pi_i(\{(u_1, ..., u_{i-1}, u_i, u_{i+1}, ..., u_n) \in I_{<\phi>}(\phi) \mid u_i = I(c)\})
\]

\[
= \pi_i(\{(u_1, ..., u_{i-1}, u_i, u_{i+1}, ..., u_n) \in h(I(\phi)) \mid u_i = I(c)\}).
\]

The other cases are analogous.

From this proposition it is clear the importance of the homomorphic extensions of the two-step intensional semantics in Definition 5. Without this homomorphic commutativity, the Tarski’s interpretations are not able to specify the interdependence of extensions of correlated concepts in $D$. Thus, the homomorphic extension of Frege/Russell’s intensional semantics is not only a meaningful theoretical contribution but also a necessarily issue in order to be able to define the correct intensional semantics for the FOL.

The commutative homomorphic diagram in Corollary 6 explains in which way the Tarskian semantics neglects meaning, as if truth in language where autonomous. This diagram shows that such a Tarskian approach, quite useful in logic, is very approximative. In fact the Tarskian fact “$A$ is a true sentence” (horizontal arrow in the diagram above with $I_{<\phi>}(A) = t$), is equivalent to “$A$ expresses a true proposition” (where the proposition is an intensional entity equal to $I(A)$, and its truth is obtained by extensionalization mapping $h(I(A)) = t$). That is, the diagram above considers also the theory of truth as a particular case of the theory of meaning, which we are dealing with propositions in $D_0 \subset D$.

Because of that, the intensionality is a strict generalization of the Tarskian theory of truth that is useful in mathematical logic but inessential to the semantics for natural language. It explains why the modern intelligent information retrieval in Web P2P database systems requires the intensionality, and the application of the general theory of meaning in the place of the singular Tarskian theory of truth.

6 Conclusion

Semantics is the theory concerning the fundamental relations between words and things. In Tarskian semantics of the FOL one defines what it takes for a sentence in a language
to be true relative to a model. This puts one in a position to define what it takes for a sentence in a language to be valid. Tarskian semantics often proves quite useful in logic. Despite this, Tarskian semantics neglects meaning, as if truth in language were autonomous. Because of that the Tarskian theory of truth becomes inessential to the semantics for more expressive logics, or more ’natural’ languages, and it is the starting point of my investigation about how to provide the necessary, or minimal, intensionality to the syntax of the FOL.

Both, Montague’s and Bealer’s approaches were useful for this investigation, but the first is not adequate and explains why we adopted two-step intensional semantics (intensional interpretation with the set of extensionalization functions), and the second consider that the intensionality is exclusive consequence of ”intensional abstraction”. First, we show that not all modal predicate logics are intensional logics but only a strict subset of them are intensional. Also the set of pure extensional predicate logics is the strict subset of modal predicate logics.

We defined a modal FOL\(_K\) logic where the quantifiers are interpreted as modal operators, and we have shown that such a modal predicate logic (heaving the same syntax as ordinary FOL) with Kripke’s possible world semantics is pure extensional logic as is FOL with standard Tarskian semantics. We show that the transformation of this predicate modal logic FOL\(_K\) into FOL, by using correspondence modal theory, is impossible, from the fact that by transformation of the modal formulae we obtain the second-order formulae (because the possible worlds are the functions of assignments). In the same way, the transformation of the intensional first-order logic FOL\(_I\) into FOL is impossible (the set of possible worlds are the functions of Tarski’s models of the standard FOL with a set of assumptions \(\Gamma\)).

We have shown that minimal intensional enrichment of the FOL (which does not change the syntax of the FOL) is obtained by adopting the PRP theory, that is a theory of properties, relations, and propositions for the domain \(D\) of the FOL, and by adopting the two-step intensional interpretation. The set of possible worlds of this 'minimal' intensional logic FOL\(_I\) is the set of Tarski’s models of the standard FOL with a set of assumptions \(\Gamma\), with the intensionality equal to Montague’s point of view of the meaning. The global logical inference relation of this intensional first-order logic FOL\(_I\) is equal to the standard Tarskian logical consequence relation of the FOL.

At the end of this work we defined an intensional algebra and an extensional algebra (different from standard cylindric algebras for the FOL), and the commutative homomorphic diagram between them, in Corollary 3 that express the generalization of the Tarskian theory of truth for the FOL into the Frege/Russell’s theory of meaning in this minimal intensional enrichment of the FOL.

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