Canonical Entropy of charged black hole

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Abstract

Recently, Hawking radiation of the black hole has been studied by using the tunnel effect method. It is found that the radiation spectrum of the black hole is not a strictly pure thermal spectrum. How does the departure from pure thermal spectrum affect the entropy? This is a very interesting problem. In this paper, we calculate the partition function through energy spectrum obtained by using the tunnel effect. From the relation between the partition function and canonical entropy, we can derive the entropy of charged black hole. In our calculation, we consider not only the correction to the black hole entropy due to fluctuation of energy, but also the effect of the change in the black hole charges on entropy. There is not any assumption. This makes our result more reliable.
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1. Introduction

Hawking [1] interpreted the quantum effect of the black hole as emission of thermal radiant spectrum from event horizon, which sets a milestone in black hole physics. The discovery of this effect not only solved the problem in black hole thermodynamics, but also revealed the relation among quantum
mechanics, thermodynamics and gravitation. Studying the thermal prop-
erties of various black holes is one of the important subjects of black hole
physics. Hawking pointed out that vacuum fluctuation near the surface of
the black hole would produce virtual particle pair. When the virtual particles
with negative energy come into black hole via tunnel effect, the energy of the
black hole will decrease. At the same time, the particle with positive energy
may thread out the gravitational region outside the black hole. This process
is equivalent to the emission of a particle from the black hole. However, in
Hawking’s proof there is not any potential barrier in the tunnel.
Parikh and Wilczek [2] discussed Hawking radiation by tunnel effect.
They thought that tunnels in the process of the particle radiation had no
potential barrier before particles radiated. Potential barrier is produced by
radiation particles itself. That is, during the process of tunnel effect creation,
the energy of the black hole decreases and the radius of the black hole hori-
zon reduces. The horizon radius gets a new value that is smaller than the
original value. The decrease of radius is determined by the value of energy
of radiation particles. There is a classical forbidden band– potential barrier
between original radius and the one after the black hole radiates. Parikh
and Wilczek skillfully obtained the radiation spectrum of Schwarzschild and
Reissner-Nordstrom black holes. Refs.[3-14] developed the method proposed
by Parikh and Wilczek. They derived the radiation spectrum of the black
hole in all kinds of space-time. Refs.[13-16] obtained radiation spectrum
of Hawking radiation after considering the generalized uncertainty relation.
And Angheben, Nadalini, Vanzo and Zerbini have computed the radiation
spectrum of the arbitrary dimensional black hole, but haven’t obtained uni-
versal expression for radiation spectrum of static mass equal to zero and
charged particle.
In this paper, firstly we obtain the radiation spectrum of the black hole
which is independent of whether static mass of radiation particle equal to zero
by quantum statistical method, and then using this radiation spectrum, we
calculate the black hole canonical entropy and derive the canonical entropy
of the charged black hole.

2. Canonical partition function of charged black hole
For static or stable space-times, since the metric is independent of time
variable, in the spatial region we can constitute a contemporaneous plane
that surrounds the black hole. We put the black hole in contact with a
thermal radiant field with temperature $T$ ($T$ is the radiation temperature of
the black hole). We require that $R \gg r_H$, where $R$ is the radius of the contemporaneous plane and $r_H$ is the horizon radius of the black hole. Since the radius of the contemporaneous plane is much bigger than the horizon radius of the black hole, we can consider the region surrounded by this contemporaneous plane as an isolated thermodynamic system with conserved energy. This region can be divided into three parts: the naked black hole, horizon surface and radiation field. Suppose that the total energy is $E^0$, the total charges is $Q^0$, the initial energy of the naked black hole is $E$, which is Arnowitt-Deser-Misner (ADM) mass, and the charge is $Q$. Initial energy of the horizon surface of the black hole and the charge are zero. The energy of the radiation field is $E_r$, and the charge is $Q_r$. We know that the location of the black hole horizon $r_H$ is a function of energy and charge. It is denoted as $r_H(E, Q)$. When the black hole has Hawking radiation, the horizon location changes. Suppose that the energy of Hawking radiation particles is $E_s$ and the electric charge is $Q_n$, then the horizon location will change from $r_H(E, Q)$ to $r_H(E - E_s, Q - Q_n)$. At this time, a quantum energy layer with energy $E_s$ and the electric charge $Q_n$ is formed between the two horizons. Because the black hole and the horizon are put in contact with a thermal radiant field with temperature $T$, the temperature is invariant during the creation process of radiation. Therefore, we can assume that in this process the temperature of the black hole is a constant. This hypothesis is consistent with the one used when the Hawking radiation is discussed via tunnel effect. At this time, there is no energy exchange between energy layer and the radiation field. As a result, the energy of energy layer and the energy of naked black hole are conservative, and the charge of energy layer and the charge of naked black hole are conservative.

When the quantum energy layer is at state $s$ with the charge $Q_n$ and energy $E_s$, the naked black hole can be at any microscopic state with the charge $Q - Q_n$ and energy $E - E_s$. Let $\Omega(E - E_s, Q - Q_n)$ denote the number of microscopic state of the naked black hole with the charge $q = Q - Q_n$ and energy $E_b = E - E_s$. When the quantum energy layer is at state $s$, the number of microscopic states of the naked black hole and quantum energy layer is $\Omega(E - E_s, Q - Q_n)$. According to the equiprobable principle, every microscopic state of the compound system that consists of the naked black hole and quantum energy layer appears with the same probability. The probability that quantum energy layer is at state $s$ is proportional to $\Omega(E - E_s, Q - Q_n)$. That is
Because the number of microscopic state of the system is very big, in convenience, we discuss $\ln \Omega$. We expand $\ln \Omega$ as a power series with respect to $E_s$ and $Q_n$. We have

$$\ln \Omega(E - E_s, Q - Q_n) = \ln \Omega(E, Q) + \left( \frac{\partial \ln \Omega}{\partial E_b} \right)_{E_s=0, Q_n=0} (-E_s) + \frac{1}{2} \left( \frac{\partial^2 \ln \Omega}{\partial E_b^2} \right)_{E_s=0, Q_n=0} E_s^2 + \cdots$$

$$+ \left( \frac{\partial \ln \Omega}{\partial q} \right)_{E_s=0, Q_n=0} (-Q_n) + \frac{1}{2} \left( \frac{\partial^2 \ln \Omega}{\partial q^2} \right)_{E_s=0, Q_n=0} Q_n^2 + \cdots. \quad (2)$$

The first term in the right hand side of Eq. (2) is a constant. So (1) can be rewritten as

$$\rho_{s,n} \propto e^{-\beta E_s + \beta_2 E_s^2 - \alpha Q_n + \alpha_2 Q_n^2 + \cdots}. \quad (3)$$

where $\alpha_k = \frac{1}{k!} \left( \frac{\partial^k \ln \Omega}{\partial q^k} \right)_{E_s=0, Q_n=0}$, $\beta_k = \frac{1}{k!} \left( \frac{\partial^k \ln \Omega}{\partial E_b^k} \right)_{E_s=0, Q_n=0}$. In statistical physics, logarithm of the number of microscopic state of the system should be the entropy of the system. That is

$$S = \ln \Omega. \quad (4)$$

Eq. (2) can be rewritten as

$$S(E - E_s, Q - Q_n) - S(E, Q) = -\alpha Q_n + \alpha_2 Q_n^2 + \cdots - \beta E_s + \beta_2 E_s^2 + \cdots. \quad (5)$$

In (5), $S(E - E_s, Q - Q_n) - S(E, Q)$ is the difference between the entropy before the naked black hole radiates and the entropy after the naked black hole radiates. That is

$$\Delta S = S(E - E_s, Q - Q_n) - S(E, Q). \quad (6)$$

From Eq. (3), the energy spectrum of the black hole radiation is as follows:

$$\rho_{s,n} \propto e^{\Delta S}. \quad (7)$$
According to thermodynamics, $\beta$ should be the inverse of the temperature. Normalizing the distribution function, we obtain

$$\rho_{s,n} = \frac{1}{Z_C} e^{\Delta S} = \frac{1}{Z_C} e^{-\beta (E_s + \Phi Q_n) + \alpha_2 Q_n^2 + \cdots + \beta_2 E_s^2 + \cdots},$$  

(8)

where $\Phi = \left( \frac{\partial E_s}{\partial q} \right)_{E_s=0, Q_n=0}$ is a static electric potential at the black hole horizon. $Z_C$ is named as a canonical partition function. And it is defined as

$$Z_C = \sum_{E_s, Q_n} \rho(E - E_s, Q - Q_n) e^{\Delta S}.$$  

(9)

If we adopt the semi-classical method, the partition function in Eq. (9) can be rewritten as

$$Z_C = \int dE_s dQ_n \rho(E - E_s, Q - Q_n) e^{-\beta (E_s - \Phi Q_n) + \alpha_2 Q_n^2 + \cdots + \beta_2 E_s^2 + \cdots},$$  

(10)

where $\rho(E - E_s, Q - Q_n)$ is the state density when the energy and the charge are given by $E - E_s$ and $Q - Q_n$, respectively. In our calculation, when we neglect the higher-order terms and only keep the first order term, we have

$$Z_G = \int dE_s dQ_n \rho(E - E_s, Q - Q_n) e^{-\beta (E_s - \Phi Q_n)}.$$  

(11)

When the energy of the radiation particles of the black hole is $E_s$ and the charge is $Q_n$, the energy of the black hole is $E_b = E - E_s$, and the charge is $q = Q - Q_n$. For the black hole, when the energy is $E_b$ and the charge is $q$, the corresponding state density is $\rho(E - E_s, Q - Q_n)$. Then

$$\rho(E - E_s, Q - Q_n) = \exp \left[ S_{MC}(E - E_s, Q - Q_n) \right].$$  

(12)

The integral in Eq. (11) can be performed in general by the saddle point approximation, provided the microcanonical entropy $S_{MC}(E - E_s, Q - Q_n)$ can be Taylor-expanded around the average equilibrium energy $E$ and $Q$

$$S_{MC}(E - E_s, Q - Q_n) = S_{MC}(E, Q) - \alpha Q_n + \alpha_2 Q_n^2 + \cdots - \beta E_s + \beta_2 E_s^2 + \cdots.$$  

(13)

Neglecting the higher-order terms, we get the partition function given by

$$Z_C = \int dE_s dQ_n e^{S_{MC}(E, Q)} e^{2[\beta E_s - \alpha Q_n + \alpha_2 Q_n^2 + \beta_2 E_s^2]}.$$  

(14)
When $\alpha = -\beta \Phi > 0$, the main contribution to the integral in Eq. (14) comes from the case when $E_s$ and $Q_n$ are very small. Letting the integral upper limit be infinity, we have

$$Z_C(\beta, \alpha) = e^{S_{MC}(E, Q)} \left[ \frac{1}{2} \sqrt{\frac{\pi}{-2\beta^2}} \exp \left( \frac{\beta^2}{-2\beta^2} \right) \left( 1 - erf \left( \frac{\beta}{\sqrt{-2\beta^2}} \right) \right) \right] \times$$

$$\left[ \frac{1}{2} \sqrt{\frac{\pi}{-2\alpha^2}} \exp \left( \frac{\alpha^2}{-2\alpha^2} \right) \left( 1 - erf \left( \frac{\alpha}{\sqrt{-2\alpha^2}} \right) \right) \right].$$

(15)

where

$$erf(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt$$

is the error function.

3. Canonical entropy of charged black hole

Using the standard formula from equilibrium statistical mechanics

$$S_C = \ln Z_C - \beta \frac{\partial \ln Z_C}{\partial \beta} - \alpha \frac{\partial \ln Z_C}{\partial \alpha},$$

it is easy to deduce that the canonical entropy is given in terms of the microcanonical entropy by

$$S_C(E, Q) = S_{MC}(E, Q) + \Delta S,$$

(17)

where

$$\Delta S = \ln f(\beta, \beta_2) - \beta \frac{\partial \ln f(\beta, \beta_2)}{\partial \beta} + \ln f(\alpha, \alpha_2) - \alpha \frac{\partial \ln f(\alpha, \alpha_2)}{\partial \alpha},$$

(18)

$$f(\beta, \beta_2) = \frac{1}{2} \sqrt{\frac{\pi}{-2\beta^2}} \exp \left( \frac{\beta^2}{-2\beta^2} \right) \left[ 1 - erf \left( \frac{\beta}{\sqrt{-2\beta^2}} \right) \right].$$

Making use of the asymptotic expression of the error function

$$erf(z) = 1 - \frac{e^{-z^2}}{\sqrt{\pi} z} \left[ 1 + \sum_{k=1}^{\infty} (-1)^k \frac{(2k-1)!!}{(2z^2)^k} \right], \quad |z| \to \infty,$$

(19)
we have

\[ f(\beta, \beta_2) = \frac{1}{2\beta} \left[ 1 + \sum_{k=1}^{\infty} (-1)^k \left( \frac{2k - 1}{2^k} \right) \left( \sqrt{-\frac{2\beta_2}{\beta}} \right)^{2k} \right]. \]  

(20)

In \( \Delta_S \), we only consider the logarithm terms. When \( \Phi < 0 \), the logarithm terms in \( \Delta_S \) are given by

\[ \Delta_S = \ln \left[ 1 + \sum_{k=1}^{\infty} (-1)^k \left( \frac{2k - 1}{2^k} \right) \right] 
+ \ln \left[ 1 + \sum_{k=1}^{\infty} (-1)^k \left( \frac{2k - 1}{2^k} \right) y^k \right] + 2 \ln T - \ln |\Phi|. \]  

(21)

where \( C \) is the thermal capacity under the condition that the charge is invariable, \( y = \left( \frac{\partial}{\partial q} \left( \frac{1}{\beta \Phi} \right) \right)_E \).

In the asymptotic expression of error function given by Eq. (19), we take the sum from 1 to \( n \) as the approximate value of the series. When \( z \) is a real number, its error does not exceed the absolute value of the first term that has been neglected in the series. Therefore, when \( C < -1 \) or \( C > 1 \), the first term in \( \Delta_S \) is not divergent. However, when \( -1 \leq C \leq 1 \), it is possible that the entropy becomes a complex number. The fact that the entropy is a complex number implies that the black hole is not stable in thermodynamics. This is still an unsolved problem.

When \( |y| < 1 \), the second term in \( \Delta_S \) is not divergent. Therefore, we can prove that when the charged black hole is in the stable state, the relation between the energy and temperature of the charges satisfies

\[ \Phi \left( \frac{\partial Q}{\partial \Phi} \right)_E < T \left( \frac{\partial Q}{\partial T} \right)_E. \]  

(22)

4. Conclusion and discussion

For Schwarzschild space-time, when we take the first approximation, the logarithm correction term to entropy is

\[ \Delta_S = \ln T = -\frac{1}{2} \ln \frac{A}{4} + \text{const.} \]  

(23)

Ref.[17] discussed the correction to the black hole entropy using the generalized uncertainty relation and derived the following result.
Based on Eq. (24), there is an uncertain factor $\alpha^2$ in the logarithm term in the correction to the black hole entropy. However, there is no uncertain factor in our result.

When the correction to the black hole thermodynamic quantities due to thermal fluctuation is considered, the expression of entropy is given by [18-21]

$$S = \ln \rho = S_{MC} - \frac{1}{2} \ln(C T^2) + \cdots.$$  \hspace{1cm} (25)

There is a limitation in the above result. Since the thermal capacity of Schwarzschild black hole is negative. The entropy given by Eq. (25) is a complex number. So this relation is not valid for Schwarzschild black hole. However, when we take a proper approximation or limit, general four-dimensional curved space-times can return to Schwarzschild space-times. This implies that Eq. (25) is not universal. In our result we only request the thermal capacity satisfies $C < -1$ or $C > 1$. A big Schwarzschild black hole satisfies this condition. According to this condition, when the energy of Schwarzschild black hole satisfies $M^2 > 1/8\pi$, we can obtain the entropy which is not divergent. Then we can derive the lowest energy of Schwarzschild black hole in the universe.

In addition, the research of the black hole entropy is based on the fact that the black hole has thermal radiation and the radiation spectrum is a pure thermal spectrum. However, Hawking showed that the radiation spectrum is a pure thermal spectrum only under the condition that the background of space-time is invariable. During this radiation process, the information may get lost. The information loss of the black hole means that the pure quantum state will disintegrate to a mixed state. This violates the unitarity principle in quantum mechanics. When we discuss the black hole radiation using the tunnel effect method, after considering the conversion of energy and the change of the horizon, we obtain the result that the radiation spectrum is no longer a strictly pure thermal spectrum. This method avoids the limitation of Hawking radiation and shows that it is the self-gravitation which provides the potential barrier of quantum tunnel.

Our discussion is based on the quantum tunnel effect of the black hole radiation which is independent of that whether static mass of radiation particles equal to zero. So our discussion is very reasonable. We provide a way for
studying the quantum correction to Bekenstein-Hawking entropy. Based on our method, we can further check the string theory and single loop quantum gravity and determine which one is perfect.

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