\( \theta_{13}, \mu \tau \) symmetry breaking and neutrino Yukawa textures

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Within the type-I seesaw and in the basis where charged lepton and heavy neutrino mass matrices are real and diagonal, \( \mu \tau \) symmetric four and three zero neutrino Yukawa textures are perturbed by lowest order \( \mu \tau \) symmetry breaking terms. These perturbations are taken to be the most general ones for those textures. For quite small values of those symmetry breaking parameters, permitting a lowest order analysis, current best-fit ranges of neutrino mass squared differences and mixing angles are shown to be accommodable, including a value of \( \theta_{13} \) in the observed range, provided all the light neutrinos have an inverted mass ordering.

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INTRODUCTION

A major recent development in Particle Physics has been the observation \[1–4\] of a significant mixing between the first and third generations of (anti-) neutrinos with a measured angle \( \theta_{13} = 8.8^\circ \pm 1.1^\circ \). The underlying physical implication is rather serious. Certain flavor symmetries in the neutrino sector, such as that under \( \mu \leftrightarrow \tau \) interchange \[5\] (i.e. \( 2 \leftrightarrow 3 \) in relevant matrix elements), which imply a vanishing \( \theta_{13} \), must be broken. The latter became a highly popular idea on account of its prediction of a maximal mixing (\( \theta_{23} = 45^\circ \)) between the second and third generations of neutrinos—a situation still well-allowed by extant data. But, now that \( \theta_{13} \) is sizably nonzero, there is interest in breaking this \( \mu \tau \) symmetry. Its spontaneous breakdown generally requires \[5, 6\] several additional fields. So explicit and radiative modes of \( \mu \tau \) symmetry breaking are the only viable options with the choice of a minimal set of fields. However, radiative breaking, related to a high scale scenario with \( \theta_{13} = 0 \) through Renormalisation Group Evolution, is characterised by \[7\] a small loop-induced constant proportional to \((m_\tau/v)^2\), \( v \) being the EW VEV. The latter, even in the light neutrino mass ordering case producing the largest \( \theta_{13} \), is unable \[7\] to generate a \( \theta_{13} \) larger than 5° which is somewhat disfavoured by the data. A reasonably minimalist approach then would be to try the explicit breaking of \( \mu \tau \) symmetry as a perturbation.

We have in mind a canonical type-I seesaw \[9–12\] mechanism for the generation of the light neutrino Majorana mass matrix \( M_\nu \) with three heavy (>10^9 GeV) right chiral EW singlet Majorana neutrinos. We prefer to introduce the perturbation in the neutrino Yukawa coupling matrix or equivalently in the neutrino Dirac mass matrix \( D \). The latter, rather than \( M_\nu \), is what appears in the Lagrangian. Hence the present approach reflects a more basic way of handling the above explicit symmetry breaking. Our next step is to study the effects of those parameters on predictive neutrino Yukawa textures, specially on four and three zero \( \mu \tau \) symmetric Yukawa textures of \( D \). Such Yukawa texture zeros may arise in a number of models, e.g. those with \[13\] extra dimensions. It may be recalled here that the study of presumed four zero textures has had a distinguished record in the quark sector \[14–16\]. It is thus natural to extend similar ideas to neutrinos modulo the difference due to the type-I seesaw. One can raise the issue of arbitrariness in our choice of textures, but our preference for the maximal number of zeros in the \( \mu \tau \) symmetric neutrino Yukawa coupling matrix is motivated by predictivity. Textures with fewer zeros have many more parameters and do not have testable predictions. Any texture statement is, of course, dependent on the weak basis chosen. We select one in which the charged lepton and heavy right chiral neutrino mass matrices are real, positive and diagonal.

By an \( n \) zero texture, we mean here is an allowed configuration of \( M_D \) with \( n \) vanishing elements. Three and four zero textures provide a predictive and useful framework \[17\] within which one can discuss neutrino masses and mixing angles. As already mentioned, this utility gets much reduced for textures with a fewer number of zeros. Exact \( \mu \tau \) symmetry automatically yields \( \theta_{23} = \pi/4 \) and \( \theta_{13} = 0 \). Additionally, each of the allowed textures leads to an \( M_\nu \) with three independent parameters (two real ones and one phase) apart from an overall mass factor. Our aim here is to study deviations from these consequences of \( \mu \tau \) symmetry due to the symmetry breaking perturbation.

In deciding which \( \mu \tau \) symmetric textures are allowed and which are not, we are guided by twin criteria. First, the observed fact of none of the three light neutrinos being unmixed in flavor means that a block diagonal form of \( M_D \) is inadmissible. Further, if any row of \( M_D \) is orthogonal, element by element (ruling out unnatural cancellations) to each of the other two, one neutrino family decouples—disallowing that texture. Second, in the absence of any fundamental principle dictating as such, none of the neutrinos is taken to be strictly massless, i.e det \( M_\nu \) ≠ 0, which
requires via the seesaw that det $M_D \neq 0$. The presence of three heavy right chiral singlet Majorana neutrinos is crucial here since, with only two, a massless neutrino is inevitable [18]. This means that no entire row or column of $M_D$ can vanish, nor can there be in it a quartet of zeros at the corners of a rectangular array.

With the above constraints, four was shown [19] to be the maximum number of zeros allowed in a texture of $M_D$. All such four zero textures have been discussed extensively [20–22]. These textures were further restricted [23,24] drastically to four allowed ones by the imposition of $\mu\tau$ symmetry. The four allowed textures, named $A_1, A_2$ and $B_1, B_2$ (each pair yielding the same $M_R$), were of course included in the complete list of Ref. [19]. They have also been shown to be capable of leading to a desirable level of baryogenesis via leptogenesis [25]. Interestingly, allowed three zero textures of $M_D$, when made $\mu\tau$ symmetric, are also found to be drastically reduced in number to only two. These are designated $C_1$ and $C_2$ here, both leading to the same $M_R$.

Each one of the $M_D$ textures $A_1, A_2, B_1, B_2, C_1, C_2$ is then perturbed away from $\mu\tau$ symmetry by deviation factors of the form $1 - \epsilon_i e^{i \phi_i}$ (no sum). We now call the perturbed texture $M_D^\epsilon$. It should be emphasized that, while the perturbations explicitly break $\mu\tau$ symmetry, the maximal zero texture is kept intact. The latter is our basic framework which is retained without change. Here the parameters $\epsilon_i$ are real positive numbers, kept small in order that higher order terms can be neglected. Further, the phases $\phi_i$ are unrestricted except for being between $-\pi$ and $+\pi$. Because of the presence of two independent off-diagonal elements in the $\mu\tau$ symmetric form of any of $M_{DA_1}, M_{DA_2}, M_{DB_1}$ and $M_{DB_2}$, only two such deviation factors are needed per texture in complete generality. In contrast, the $\mu\tau$ symmetric form of either of $M_{DC_1}, M_{DC_2}$ has three independent off-diagonal elements; hence three deviation factors each have to be inserted in general in these cases. Additionally, a deviation factor $1 - \delta, \delta$ being real, is introduced in the third element of the diagonal $\mu\tau$ symmetric form of the right chiral heavy neutrino mass matrix $M_R$.

The corresponding complex symmetric light neutrino Majorana mass matrix $M_{\nu, \delta}^\epsilon$ obtains via the type-I seesaw from the above perturbed textures. Thus we have

$$M_{\nu, \delta}^\epsilon \simeq -M_D^\epsilon (M_{R}^\delta)^{-1} M_D^{\epsilon T}. \tag{1}$$

Of course, one needs to carefully follow the interplay between the number of independent parameters in the emergent $M_\nu$ and the number of separate experimental inputs, as was emphasized [26] some time ago. With the parametric form of $M_{\nu, \delta}^\epsilon$ for each of the six textures, we construct the hermitian product

$$H_{\nu, \delta}^\epsilon = M_{\nu, \delta}^\epsilon M_{\nu, \delta}^{\epsilon \dagger}. \tag{2}$$

Through $H_{\nu, \delta}^\epsilon$ we directly connect with five experimentally measured quantities of phenomenological relevance, to wit \(\Delta_1^2 = m_1^2 - m_2^2, |\Delta_2^2| = |m_3^2 - m_2^2|, \theta_{12}, \theta_{23} \text{ and } \theta_{13} \).

The rest of the paper is organized as follows. In Section 2 we outline our basic theoretical framework. Section 3 contains an enumeration of the forms of the light neutrino mass matrix $M_\nu$ from $\mu\tau$ symmetric allowed four zero and three zero textures. The corresponding perturbed and broken $\mu\tau$ symmetric expression for $M_{\nu, \delta}^\epsilon$ in parametric form is constructed for each case in Section 4 and the corresponding $H_{\nu, \delta}^\epsilon$ is displayed. Section 5 is devoted to a phenomenological discussion of what is allowed/disallowed in the space of parameters from the five experimental inputs and which textures are in/out, given the $3\sigma$ ranges of those measured quantities. Finally, Section 6 contains a summary of our conclusions. In the Appendix we provide some details of the diagonalization procedure.

**THREE AND FOUR ZERO TEXTURES WITH BROKEN $\mu\tau$ SYMMETRY**

We give a compact outline of our theoretical framework here since a major part of it was detailed earlier [17,23–24]. We have already referred to the type-I seesaw and the $H$ matrix in eqns. (1) and (2) in the Introduction. These can be considered without the $\epsilon$ superscript in the unperturbed limit. We can write

$$U^\dagger H_{\nu, \delta}^\epsilon U = \text{diag}(|m_1|^2, |m_2|^2, |m_3|^2), \tag{3}$$

where we have followed the PDG convention [27] in defining $m_1, m_2, m_3$. In [3], $U$ is the unitary PMNS matrix. Under $\mu\tau$ symmetry, $M_3 = M_2$, but - as mentioned earlier - the broken $\mu\tau$ symmetric extension can be given as

$$M_R = \text{diag} \begin{pmatrix} M_1, & M_2, & M_2(1 - \delta) \end{pmatrix}, \tag{4}$$

where $\delta$ is a real parameter which can have either sign. In the $\mu\tau$ symmetric limit, $\delta \rightarrow 0$ and moreover $M_{12} = M_{13}$, $M_{21} = M_{31}, M_{22} = M_{32}$ and $M_{23} = M_{32}$ where $M$ can be either $M_D$ or $M_\nu$. 
Four zero textures

On applying the twin criteria explained in the Introduction and $\mu \tau$ symmetry, only four textures of $M_D$ are found to survive. They are divided pairwise into two categories $A$ and $B$ and are individually called $A_1$, $A_2$ and $B_1$, $B_2$. In terms of arbitrary complex quantities $a$, $b$, $c$, they can be written as

$$M_{DA1}^{(4)} = \begin{pmatrix} a & b & 0 \\ 0 & 0 & c \\ 0 & c & 0 \end{pmatrix}, \quad M_{DA2}^{(4)} = \begin{pmatrix} a & b & 0 \\ 0 & 0 & c \\ 0 & 0 & c \end{pmatrix},$$

$$M_{DB1}^{(4)} = \begin{pmatrix} a & 0 & 0 \\ b & 0 & c \\ b & c & 0 \end{pmatrix}, \quad M_{DB2}^{(4)} = \begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ b & c & 0 \end{pmatrix}.$$  \hfill (5)

The last two are respective transposes of the first two and the superscripts '4' signifies a four zero texture. Each pair in a category yields the same $M_\nu^{(4)}$. With a reparametrisation in terms of an overall complex mass factor $m_{A/B}$ and two real positive quantities $k_{1,2}/l_{1,2}$ and a phase $\alpha/\beta$ (cf. Table I), they appear in Category $A/B$ as

$$M_{\nu A}^{(4)} = m_A \begin{pmatrix} k_1^2 e^{2i\alpha} & 2 k_2 & k_2 \\ k_2 & 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$M_{\nu B}^{(4)} = m_B \begin{pmatrix} l_1^2 & l_1 t_2 e^{i\beta} & l_1 t_2 e^{i\beta} \\ l_1 t_2 e^{i\beta} & l_1^2 e^{2i\beta} & 1 + l_1^2 e^{2i\beta} \\ l_1^2 e^{2i\beta} & 1 & l_1^2 e^{2i\beta} \end{pmatrix}.$$  \hfill (6)

The most general $\mu \tau$ symmetry breaking perturbation on $M_D$ consists of two independent complex terms containing $\epsilon_1 e^{i\phi_1}$, $\epsilon_2 e^{i\phi_2}$. The four textures of (5) are then extended to

$$M_{DA1}^{(4)}(\nu) = \begin{pmatrix} a & b (1 - \epsilon_1 e^{i\phi_1}) & 0 \\ 0 & c (1 - \epsilon_2 e^{i\phi_2}) & 0 \\ 0 & 0 & c \end{pmatrix}, \quad M_{DA2}^{(4)}(\nu) = \begin{pmatrix} a & b & (1 - \epsilon_1 e^{i\phi_1}) \\ 0 & c & 0 \\ 0 & 0 & c \end{pmatrix},$$

$$M_{DB1}^{(4)}(\nu) = \begin{pmatrix} a & 0 & 0 \\ b (1 - \epsilon_1 e^{i\phi_1}) & 0 & c (1 - \epsilon_2 e^{i\phi_2}) \\ b & c & 0 \end{pmatrix}, \quad M_{DB2}^{(4)}(\nu) = \begin{pmatrix} a & 0 & 0 \\ b (1 - \epsilon_1 e^{i\phi_1}) & c (1 - \epsilon_2 e^{i\phi_2}) & 0 \\ b & 0 & c \end{pmatrix}.$$  \hfill (7)

We always work to the lowest order in the epsilons and $\delta$. The seesaw enables the reparametrisation of $M_\nu^{(4)}(\nu)$ in terms of $m_{A/B}$, $k_{1,2}/l_{1,2}$, $\alpha/\beta$ while including $\mu \tau$ symmetry breaking terms involving $\epsilon_1 e^{i\phi_1}$, $\epsilon_2 e^{i\phi_2}$ and $\delta$. Whereas,
in the $\mu \tau$ symmetric limit, there are only two $M^{(4)}_{\nu}$, now there are four $M^{2,4(4)}$:

$$M^{2,4(4)}_{\nu A_1} = m_A \begin{pmatrix} k_2^2 e^{2i\alpha} + 2k_2^2(1 - e^{i\phi_1} - \frac{1}{2}) & k_2(1 - e^{i\phi_1} - e^{i\phi_2} + \delta) & k_2(1 - e^{i\phi_1} + \delta) \\ k_2(1 - e^{i\phi_1} - e^{i\phi_2} + \delta) & 1 - 2e^{i\phi_2} + \delta & 0 \\ k_2(1 - e^{i\phi_1} + \delta) & 0 & 1 \end{pmatrix},$$

$$M^{2,4(4)}_{\nu A_2} = m_A \begin{pmatrix} k_2^2 e^{2i\alpha} + 2k_2^2(1 - e^{i\phi_1} - \frac{1}{2}) & k_2(1 - e^{i\phi_2}) & k_2(1 - e^{i\phi_1} + \delta) \\ k_2(1 - e^{i\phi_2}) & 1 - 2e^{i\phi_2} + 0 & 0 \\ k_2(1 - e^{i\phi_1} + \delta) & 0 & 1 + \delta \end{pmatrix},$$

$$M^{2,4(4)}_{\nu B_1} = m_B \begin{pmatrix} l_1 l_2 e^{i\beta} (1 - e^{i\phi_1}) & l_1 l_2 e^{i\beta} (1 - e^{i\phi_1} + 1 + \delta) & l_1 l_2 e^{i\beta} (1 - e^{i\phi_1}) \\ l_1 l_2 e^{i\beta} (1 - e^{i\phi_1}) & l_2 e^{2i\beta} (1 - e^{i\phi_1}) + 1 + \delta & l_2 e^{2i\beta} (1 - e^{i\phi_1}) \\ l_1 l_2 e^{i\beta} (1 - e^{i\phi_1}) & l_2 e^{2i\beta} (1 - e^{i\phi_1}) + 1 + \delta & l_2 e^{2i\beta} (1 - e^{i\phi_1}) + 1 \end{pmatrix},$$

$$M^{2,4(4)}_{\nu B_1} = m_B \begin{pmatrix} l_1 l_2 e^{i\beta} (1 - e^{i\phi_1}) & l_2 e^{2i\beta} (1 - e^{i\phi_1} - 2e^{i\phi_2} + 1 + \delta) & l_2 e^{2i\beta} (1 - e^{i\phi_1}) \\ l_1 l_2 e^{i\beta} (1 - e^{i\phi_1} - 2e^{i\phi_2} + 1 + \delta) & l_2 e^{2i\beta} (1 - e^{i\phi_1}) + 1 + \delta & l_2 e^{2i\beta} (1 - e^{i\phi_1}) + 1 + \delta \end{pmatrix}. \quad (8)$$

Three zero textures

Our twin criteria here leave only two $\mu \tau$ symmetric textures as survivors, each with a vanishing $(1,1)$ element. We categorise them under the general designation Category $C$, calling them $C1$ and $C2$. Thus

$$M^{(3)}_{DC1} = \begin{pmatrix} 0 & b & b \\ c & 0 & d \\ c & d & 0 \end{pmatrix},$$

$$M^{(3)}_{DC2} = \begin{pmatrix} 0 & b & b \\ c & 0 & d \\ c & d & 0 \end{pmatrix}, \quad (9)$$

with $b$, $c$, $d$ being arbitrary complex quantities in general. In analogy with four zero textures the single seesaw induced mass matrix $M^{3}_{\nu C}$ in this case can be reparametrised (cf Table I) in terms of real positive quantities $r_{1,2}$, a phase $\gamma$ and an overall complex mass factor $m_C$ as

$$M^{(3)}_{\nu C} = m_C \begin{pmatrix} 2r_1^2 & r_1 & r_1 \\ r_1 & r_2 e^{2i\gamma} + 1 & r_2 e^{2i\gamma} \\ r_1 & r_2 e^{2i\gamma} & r_2 e^{2i\gamma} + 1 \end{pmatrix}. \quad (10)$$

Here too the superscript $(3)$ refers to the three zero texture origin.

For these textures there can be three independent $\mu \tau$ symmetry breaking perturbing terms in general. We can therefore extend the textures of $[9]$ to

$$M^{(3)}_{DC1} = \begin{pmatrix} 0 & b & b(1 - e^{i\phi_1}) \\ c(1 - e^{i\phi_2}) & 0 & d(1 - e^{i\phi_2}) \\ c & d & 0 \end{pmatrix}, \quad M^{(3)}_{DC2} = \begin{pmatrix} 0 & b & b(1 - e^{i\phi_1}) \\ c(1 - e^{i\phi_2}) & 0 & d(1 - e^{i\phi_2}) \\ c & 0 & d \end{pmatrix}. \quad (11)$$

Working to the lowest order in $\epsilon_{1,2,3}$, the neutrino mass matrix of $[10]$ splits into two in terms of the reparametrised
quantities \( r_{1,2}, \gamma \) and \( m_C \) as follows:

\[
M_{\nu C}^{\epsilon,\delta(3)} = m_C \begin{pmatrix}
2r_1^2(1 - \epsilon_1 e^{i\phi_1} + \frac{\delta}{2}) & r_1(1 - \epsilon_3 e^{i\phi_3}) & r_1(1 + \delta) \\
r_1(1 - \epsilon_3 e^{i\phi_3}) & r_2^2 e^{2i\gamma} (1 + 2\epsilon_2 e^{i\phi_2}) + 1 - 2\epsilon_3 e^{i\phi_3} & r_2^2 e^{2i\gamma} (1 - \epsilon_2 e^{i\phi_2}) \\
r_1(1 + \delta) & r_2^2 e^{2i\gamma} (1 - \epsilon_2 e^{i\phi_2}) & r_2^2 e^{2i\gamma} (1 + \delta)
\end{pmatrix},
\]

\[
M_{\nu C}^{\epsilon,\delta(3)} = m_C \begin{pmatrix}
2r_1^2(1 - \epsilon_1 e^{i\phi_1} + \frac{\delta}{2}) & r_1(1 - \epsilon_1 e^{i\phi_1} - \epsilon_3 e^{i\phi_3} + \delta) & r_1 \\
r_1(1 - \epsilon_1 e^{i\phi_1} - \epsilon_3 e^{i\phi_3} + \delta) & r_2^2 e^{2i\gamma} (1 - \epsilon_2 e^{i\phi_2}) + 1 - 2\epsilon_3 e^{i\phi_3} + \delta & r_2^2 e^{2i\gamma} (1 - \epsilon_2 e^{i\phi_2}) \\
r_1 & r_2^2 e^{2i\gamma} (1 - \epsilon_2 e^{i\phi_2}) & r_2^2 e^{2i\gamma} (1 + \delta)
\end{pmatrix}.
\]

(12)

**CONNECTION TO OBSERVABLES**

The broken \( \mu \tau \) symmetric forms of \( M_{\nu}^{\epsilon,\delta} \), cf. [1], for the allowed three and four zero textures, viz. [8] and [12], can be written in a unified compact form:

\[
M_{\nu}^{\epsilon,\delta} = m \begin{pmatrix}
P & Q & Q \\
Q & R & S \\
Q & S & R
\end{pmatrix} - \epsilon_1 e^{i\phi_1} \begin{pmatrix} x_1 & x_2 & x_3 \\
x_2 & x_4 & x_5 \\
x_3 & x_5 & x_6 \end{pmatrix} - \epsilon_2 e^{i\phi_2} \begin{pmatrix} y_1 & y_2 & y_3 \\
y_2 & y_4 & y_5 \\
y_3 & y_5 & y_6 \end{pmatrix} - \epsilon_3 e^{i\phi_3} \begin{pmatrix} z_1 & z_2 & z_3 \\
z_2 & z_4 & z_5 \\
z_3 & z_5 & z_6 \end{pmatrix} - \delta \begin{pmatrix} t_1 & t_2 & t_3 \\
t_2 & t_4 & t_5 \\
t_3 & t_5 & t_6 \end{pmatrix}.
\]

(13)

The above equation contains all the six allowed \( M_{\nu}^{\epsilon,\delta} \)'s: four (two) from four (three) zero textures. Here \( m \) can be \( m_A, m_B \) or \( m_C \) depending on the category and as given in Table I. The real symmetry breaking parameters \( \delta, \epsilon_i, \phi_i \), which were introduced in [7], [8], [11] and [12], are universal quantities. Both \( \delta \) and \( \epsilon_i \) are kept small in magnitude (\( \leq 0.15 \)) to ensure the validity of keeping only the lowest order perturbation. By construction, \( \epsilon_3 \) vanishes for four zero textures. In contrast, the (generally complex) quantities \( P, Q, R, S \) as well as \( x_1, ..., x_6, y_1, ..., y_6 \) and \( z_1, ..., z_6 \) of (13) vary from category to category, though remaining unchanged within each category. On the other hand, the quantities \( t_1, ..., t_6 \) are real and vary from texture to texture within each category. The expressions for all these in terms of the real positive parameters \( k_{1,2}/l_{1,2}/r_{1,2} \) and phase \( \alpha/\beta/\gamma \) are listed in Table II. In the limit of vanishing \( \delta \) and \( \epsilon_{1,2,3} \), the \( \mu \tau \) symmetric forms of \( M_{\nu A}, M_{\nu B}, M_{\nu C} \) are restored in terms of \( P, Q, R \) and \( S \).

In order to connect to observable quantities, we need to construct \( H_{\nu}^{\epsilon,\delta} \), cf. [2]. Starting from (13), we obtain

\[
H_{\nu}^{\epsilon,\delta} = m^2 \begin{pmatrix}
|P|^2 + 2|Q|^2 & PQ^* + Q(R^* + S^*) & PQ^* + Q(R^* + S^*) \\
|P|^2 + 2|Q|^2 & |P|^2 + |R|^2 + |S|^2 & |Q|^2 + R^*S + R^*S \\
|P|^2 + 2|Q|^2 & |P|^2 + |R|^2 + |S|^2 & |Q|^2 + |R|^2 + |S|^2
\end{pmatrix} - \epsilon_1 \begin{pmatrix} u_1 & u_2^* & u_3^* \\
u_2 & u_4 & u_5 \\
u_3 & u_5 & u_6 \end{pmatrix} - \epsilon_2 \begin{pmatrix} v_1 & v_2^* & v_3^* \\
v_2 & v_4 & v_5^* \\
v_3 & v_5 & v_6 \end{pmatrix} - \epsilon_3 \begin{pmatrix} w_1 & w_2^* & w_3^* \\
w_2 & w_4 & w_5^* \\
w_3 & w_5 & w_6 \end{pmatrix} - \delta \begin{pmatrix} s_1 & s_2 & s_3^* \\
s_2 & s_4 & s_5^* \\
s_3 & s_5 & s_6 \end{pmatrix}.
\]

(14)

We have introduced in [14] the quantities \( u_i, v_i, w_i \) and \( s_i \) \((i = 1, ..., 6)\) which are algebraic functions of \( P, Q, R, S \) and \( \phi_1, \phi_2, \phi_3 \) as well as \( x_i, y_i, z_i, t_i \) \((i = 1, ..., 6)\). They appear explicitly in Table II. Evidently, \( u_k, v_k, w_k, s_k \) are real for \( k = 1, 4, 6 \) and are generally complex otherwise.

The next point to note is this. With \( i \) and \( j \) running from 1 to 6, \( u_i \) do not involve \( y_j, z_j, t_j, \phi_2 \) and \( \phi_3 \); similarly, \( v_i \) do not involve \( x_j, z_j, t_j, \phi_1 \) and \( \phi_3 \); \( w_i \) do not involve \( x_j, y_j, t_j, \phi_1 \) and \( \phi_2 \); \( s_i \) do not involve \( x_j, y_j, z_j, \phi_1, \phi_2 \) and \( \phi_3 \). The explicit expressions for \( u_i, v_i, w_i \) and \( s_i \) are given in Table III. It is interesting that they are related by certain substitution relations. Thus if \( u_i \) are written as functions \( f_i \) of set of appropriate variables, \( v_i \) as well as \( w_i \)
TABLE II: Expressions for quantities appearing in $M_{\nu}^{\epsilon,\delta,\nu}$ of (13). Parameters $z_{1-6}$, not needed for four zero textures since $\epsilon_3 = 0$, have been kept blank for the latter.

| Quantity | Four zero | | | Three zero | | |
|----------|----------|----------|----------|----------|----------|
| $P$      | $k_1^2 e^{2i\omega} + 2k_2^2$ | $k_1^2 e^{2i\alpha} + 2k_2^2$ | $t_1^2$ | $t_2^2$ | $r_1^2$ | $r_1^2$ |
| $Q$      | $k_2$    | $k_2$    | $l_1 l_2 e^{i\beta}$ | $l_1 l_2 e^{i\beta}$ | $r_1$   | $r_1$   |
| $R$      | 1        | 1        | $l_2^2 e^{2i\beta} + 1$ | $l_2^2 e^{2i\beta} + 1$ | $r_2^2 e^{2i\gamma} + 1$ | $r_2^2 e^{2i\gamma} + 1$ |
| $S$      | 0        | 0        | $l_2^2 e^{2i\beta}$ | $l_2^2 e^{2i\beta}$ | $r_2^2 e^{2i\gamma}$ | $r_2^2 e^{2i\gamma}$ |
| $x_1$    | $2k_2^2$ | $2k_2^2$ | 0        | 0        | $2r_1^2$ | $2r_1^2$ |
| $x_2$    | $k_2$    | 0        | $l_1 l_2 e^{i\beta}$ | $l_1 l_2 e^{i\beta}$ | 0        | $r_1$   |
| $x_3$    | 0        | $k_2$    | 0        | 0        | $r_1$   | 0        |
| $x_4$    | 0        | 0        | $2l_2^2 e^{2i\beta}$ | $2l_2^2 e^{2i\beta}$ | 0        | 0        |
| $x_5$    | 0        | 0        | $l_2^2 e^{2i\beta}$ | $l_2^2 e^{2i\beta}$ | 0        | 0        |
| $x_6$    | 0        | 0        | 0        | 0        | 0        | 0        |
| $y_1$    | 0        | 0        | 0        | 0        | 0        | 0        |
| $y_2$    | $k_2$    | $k_2$    | 0        | 0        | 0        | 0        |
| $y_3$    | 0        | 0        | 0        | 0        | 0        | 0        |
| $y_4$    | 2        | 2        | 2        | 2        | $2r_2^2 e^{2i\gamma}$ | $2r_2^2 e^{2i\gamma}$ |
| $y_5$    | 0        | 0        | 0        | 0        | $r_2^2 e^{2i\gamma}$ | $r_2^2 e^{2i\gamma}$ |
| $y_6$    | 0        | 0        | 0        | 0        | 0        | 0        |
| $z_1$    | -        | -        | -        | -        | 0        | 0        |
| $z_2$    | -        | -        | -        | -        | $r_1$   | $r_1$   |
| $z_3$    | -        | -        | -        | -        | 0        | 0        |
| $z_4$    | -        | -        | -        | -        | 2        | 2        |
| $z_5$    | -        | -        | -        | -        | 0        | 0        |
| $z_6$    | -        | -        | -        | -        | 0        | 0        |
| $t_1$    | $-k_2^2$ | $-k_2^2$ | 0        | 0        | $-r_1^2$ | $-r_1^2$ |
| $t_2$    | $-k_2$  | 0        | 0        | 0        | 0        | $-r_1$   |
| $t_3$    | 0        | $-k_2$  | 0        | 0        | $-r_1$   | 0        |
| $t_4$    | -1       | 0        | -1       | 0        | 0        | $-1$     |
| $t_5$    | 0        | 0        | 0        | 0        | 0        | 0        |
| $t_6$    | 0        | -1       | 0        | -1       | $-1$     | 0        |

and $s_i$ are the same functions $f_i$ of different sets of relevant variables.

$$u_i = f_i(x_1, x_2, ..., x_6, \phi_1),$$
$$v_i = f_i(y_1, y_2, ..., y_6, \phi_2),$$
$$w_i = f_i(z_1, z_2, ..., z_6, \phi_3),$$
$$s_i = f_i(t_1, t_2, ..., t_6, 0).$$

(15)

Before moving on to diagonalise $H_{\nu}^{\epsilon,\delta}$ of (14), let us recall what happens in the $\mu\tau$ symmetric limit. In this case we can obtain \cite{23} the neutrino mass squared differences and mixing angles in terms of three real functions $X_{1,2,3}$ of $P$, \cite{14}.
TABLE III: Expressions for $\mu\tau$ symmetry breaking quantities.

| Functions | Expressions |
|-----------|-------------|
| $u_1$     | $[P^*x_1 + Q^*(x_2 + x_3)] e^{i\phi_1} + c.c.$ |
| $u_2$     | $[P^*x_2 + Q^*(x_4 + x_5)] e^{i\phi_1} + [Qx_4^* + Rx_5^* + Sx_5^*] e^{-i\phi_1}$ |
| $u_3$     | $[P^*x_3 + Q^*(x_5 + x_6)] e^{i\phi_1} + [Qx_5^* + Sx_2^* + Rx_3^*] e^{-i\phi_1}$ |
| $u_4$     | $[Q^*x_4 + R^*x_4 + S^*x_5] e^{i\phi_1} + c.c.$ |
| $u_5$     | $[Q^*x_3 + R^*x_5 + S^*x_6] e^{i\phi_1} + [Qx_5^* + Sx_3^* + Rx_3^*] e^{-i\phi_1}$ |
| $u_6$     | $[Qx_3 + S^*x_5 + R^*x_6] e^{i\phi_1} + c.c.$ |
| $v_1$     | $[P^*y_1 + Q^*(y_2 + y_3)] e^{i\phi_2} + c.c.$ |
| $v_2$     | $[P^*y_2 + Q^*(y_4 + y_5)] e^{i\phi_2} + [Qy_4^* + Ry_5^* + Sy_5^*] e^{-i\phi_2}$ |
| $v_3$     | $[P^*y_3 + Q^*(y_5 + y_6)] e^{i\phi_2} + [Qy_5^* + Sy_2^* + Ry_3^*] e^{-i\phi_2}$ |
| $v_4$     | $[Qy_2 + R^*y_1 + S^*y_3] e^{i\phi_2} + c.c.$ |
| $v_5$     | $[Qy_3 + R^*y_5 + S^*y_6] e^{i\phi_2} + [Qy_5^* + Sy_2^* + Ry_3^*] e^{-i\phi_2}$ |
| $v_6$     | $[Qy_3 + S^*y_5 + R^*y_6] e^{i\phi_2} + c.c.$ |
| $w_1$     | $[P^*z_1 + Q^*(z_2 + z_3)] e^{i\phi_3} + c.c.$ |
| $w_2$     | $[P^*z_2 + Q^*(z_4 + z_5)] e^{i\phi_3} + [Qz_5^* + Rz_2^* + Sz_3^*] e^{-i\phi_3}$ |
| $w_3$     | $[P^*z_3 + Q^*(z_5 + z_6)] e^{i\phi_3} + [Qz_5^* + S_2^* + Rz_3^*] e^{-i\phi_3}$ |
| $w_4$     | $[Q^*z_2 + R^*z_4 + S^*z_5] e^{i\phi_3} + c.c.$ |
| $w_5$     | $[Q^*z_3 + R^*z_5 + S^*z_6] e^{i\phi_3} + [Qz_5^* + S_1^* + Rz_3^*] e^{-i\phi_3}$ |
| $w_6$     | $[Q^*z_3 + S^*z_5 + R^*z_6] e^{i\phi_3} + c.c.$ |
| $s_1$     | $[P^*t_1 + Q^*(t_2 + t_3)] + c.c.$ |
| $s_2$     | $[P^*t_2 + Q^*(t_4 + t_5)] + [Qt_1^* + Rt_2^* + St_3^*]$ |
| $s_3$     | $[P^*t_3 + Q^*(t_5 + t_6)] + [Qt_1^* + St_2^* + Rt_3^*]$ |
| $s_4$     | $[Qt_2 + R^*t_4 + S^*t_5] + c.c.$ |
| $s_5$     | $[Q^*t_3 + R^*t_5 + S^*t_6] + [Qt_2^* + St_1^* + Rt_3^*]$ |
| $s_6$     | $[Q^*t_3 + S^*t_5 + R^*t_6] + c.c.$ |

$Q, R, S$ with $X = (X_1^2 + X_2^2)^{1/2}$:

\[
\Delta_{21}^2 = m_2^2 - m_1^2 = m^2 X,
\]

\[
\Delta_{32}^2 = m_3^2 - m_2^2 = \frac{m^2}{2} (X_3 - X),
\]

\[
\tan 2\theta_{12} = \frac{X_1}{X_2},
\]

\[
m_{1,2} = \sqrt{2} \left( 2 - \frac{X_3 + X}{2X} \right)^{1/2},
\]

\[
m_3 = \Delta_{21}^2 / X^{1/2},
\]

with

\[
X_1 = 2\sqrt{2} |PQ^* + Q(R^* + S^*)|,
\]

\[
X_2 = |R + S|^2 - |P|^2,
\]

\[
X_3 = |R + S|^2 - |P|^2 - 4(|Q|^2 + RS^* + R^* S).
\]

Turning to $H^e_{\nu\mu}$ of \[^{14}\] for broken $\mu\tau$ symmetry, it also can be diagonalized in a similar fashion to yield $(\Delta_{21}^e)^{\epsilon,\delta} = (m_2^e)^2 - (m_1^e)^2$, $(\Delta_{32}^e)^{\epsilon,\delta} = (m_3^e)^2 - (m_2^e)^2$, $\theta_{12}^e$, $\theta_{23}^e$, $\theta_{13}^e$. The superscripts explicitly signify that $\mu\tau$ symmetry breaking has been taken into account. Some details of the diagonalization procedure are given in the Appendix. The complicated algebraic expressions for those quantities can be simplified by defining another set of functions $U_i, V_i, W_i, S_i$ ($i = 1, \ldots, 6$) of the quantities introduced in \[^{15}\]. The detailed expressions are given in Table IV with
\[ c_{12} = \cos \theta_{12}, \quad s_{12} = \sin \theta_{12}, \quad \theta_{12} \text{ being the unperturbed mixing angle between the first two generations and the phase } \psi \text{ being defined as } \]

\[ \psi = \arg [P^*Q + Q^*(R + S)] \]  \hspace{1cm} (18)

Note once again that each of the functions \( U_i, V_i, W_i, S_i \) depends on the subset of the quantities \( \{u_i, v_i, w_i, s_i\} \) and is related to the others by a set of substitution rules analogous to (15):

\[
\begin{align*}
U_1 &= F_1(u_1, u_2, ..., u_6), \\
V_1 &= F_1(v_1, v_2, ..., v_6), \\
W_1 &= F_1(w_1, w_2, ..., w_6), \\
S_1 &= F_1(s_1, s_2, ..., s_6).
\end{align*}
\]  \hspace{1cm} (19)

The final expressions for the observable quantities in terms of their unperturbed values are

**TABLE IV:** Expressions for the functions appearing in (20).

| Functions | Expressions |
|-----------|-------------|
| \( U_1 \) | \[ \frac{1}{2}[-2c_{12}u_1 + \sqrt{2}c_{12}s_{12}[(u_2 + u_3)e^{-iv} + (u_2 + u_3)e^{iv}] - s_{12}^2(u_4 + u_6 + u_5 + u_5^*)] \] |
| \( U_2 \) | \[ \frac{1}{2}[\sqrt{2}c_{12}(u_2 + u_3)e^{-iv} + \sqrt{2}s_{12}^2(u_2 + u_3)e^{iv} + c_{12}s_{12}(u_4 + u_6 - 2u_1 + u_5 + u_5^*)] \] |
| \( U_3 \) | \[ \frac{1}{2}[\sqrt{2}c_{12}(u_2 - u_3)e^{-iv} + s_{12}(u_6 - u_4 + u_5 - u_5^*)] \] |
| \( U_4 \) | \[ \frac{1}{2}[-2s_{12}^2u_1 - \sqrt{2}c_{12}s_{12}[(u_2 + u_3)e^{-iv} + (u_2 + u_3)e^{iv}] - c_{12}^2(u_4 + u_6 + u_5 + u_5^*)] \] |
| \( U_5 \) | \[ \frac{1}{2}[\sqrt{2}s_{12}(u_2 - u_3)e^{-iv} - c_{12}(u_6 - u_4 + u_5 - u_5^*)] \] |
| \( U_6 \) | \[ \frac{1}{2}(u_5 + u_5^* - u_4 - u_6) \] |
| \( V_1 \) | \[ \frac{1}{2}[-2c_{12}v_1 + \sqrt{2}c_{12}s_{12}[(v_2 + v_3)e^{-iv} + (v_2 + v_3)e^{iv}] - s_{12}^2(v_4 + v_6 + v_5 + v_5^*)] \] |
| \( V_2 \) | \[ \frac{1}{2}[-\sqrt{2}c_{12}(v_2 + v_3)e^{-iv} + \sqrt{2}s_{12}^2(v_2 + v_3)e^{iv} + c_{12}s_{12}(v_4 + v_6 - 2v_1 + v_5 + v_5^*)] \] |
| \( V_3 \) | \[ \frac{1}{2}[\sqrt{2}c_{12}(v_2 - v_3)e^{-iv} + s_{12}(v_6 - v_4 + v_5 - v_5^*)] \] |
| \( V_4 \) | \[ \frac{1}{2}[-2s_{12}^2v_1 - \sqrt{2}c_{12}s_{12}[(v_2 + v_3)e^{-iv} + (v_2 + v_3)e^{iv}] - c_{12}^2(v_4 + v_6 + v_5 + v_5^*)] \] |
| \( V_5 \) | \[ \frac{1}{2}[-\sqrt{2}s_{12}(v_2 - v_3)e^{-iv} - c_{12}(v_6 - v_4 + v_5 - v_5^*)] \] |
| \( V_6 \) | \[ \frac{1}{2}(v_5 + v_5^* - v_4 - v_6) \] |
| \( W_1 \) | \[ \frac{1}{2}[-2c_{12}w_1 + \sqrt{2}c_{12}s_{12}[(w_2 + w_3)e^{-iv} + (w_2 + w_3)e^{iv}] - s_{12}^2(w_4 + w_6 + w_5 + w_5^*)] \] |
| \( W_2 \) | \[ \frac{1}{2}[-\sqrt{2}c_{12}(w_2 + w_3)e^{-iv} + \sqrt{2}s_{12}^2(w_2 + w_3)e^{iv} + c_{12}s_{12}(w_4 + w_6 - 2w_1 + w_5 + w_5^*)] \] |
| \( W_3 \) | \[ \frac{1}{2}[\sqrt{2}c_{12}(w_2 - w_3)e^{-iv} + s_{12}(w_6 - w_4 + w_5 - w_5^*)] \] |
| \( W_4 \) | \[ \frac{1}{2}[\sqrt{2}s_{12}(w_2 - w_3)e^{-iv} - c_{12}(w_6 - w_4 + w_5 - w_5^*)] \] |
| \( W_5 \) | \[ \frac{1}{2}(w_5 + w_5^* - w_4 - w_6) \] |
| \( W_6 \) | \[ \frac{1}{2}(w_5 + w_5^* - w_4 - w_6) \] |
| \( S_1 \) | \[ \frac{1}{2}[-2c_{12}s_1 + \sqrt{2}c_{12}s_{12}[(s_2 + s_3)e^{-iv} + (s_2 + s_3)e^{iv}] - s_{12}^2(s_4 + s_6 + s_5 + s_5^*)] \] |
| \( S_2 \) | \[ \frac{1}{2}[-\sqrt{2}c_{12}(s_2 + s_3)e^{-iv} + \sqrt{2}s_{12}^2(s_2 + s_3)e^{iv} + c_{12}s_{12}(s_4 + s_6 - 2s_1 + s_5 + s_5^*)] \] |
| \( S_3 \) | \[ \frac{1}{2}[-\sqrt{2}c_{12}(s_2 - s_3)e^{-iv} + s_{12}(s_6 - s_4 + s_5 - s_5^*)] \] |
| \( S_4 \) | \[ \frac{1}{2}[\sqrt{2}c_{12}(s_2 - s_3)e^{-iv} + s_{12}(s_6 - s_4 + s_5 - s_5^*)] \] |
| \( S_5 \) | \[ \frac{1}{2}[\sqrt{2}s_{12}(s_2 - s_3)e^{-iv} - c_{12}(s_4 - s_6 + s_5 + s_5^*)] \] |
| \( S_6 \) | \[ \frac{1}{2}[-\sqrt{2}s_{12}(s_2 - s_3)e^{iv} - c_{12}(s_6 - s_4 + s_5 - s_5^*)] \] |
\( (m_1^{e,\delta})^2 = m_1^2 + m^2 [U_{1e1} + V_{1e2} + W_{1e3} + S_1 \delta] , \)
\( (m_2^{e,\delta})^2 = m_2^2 + m^2 [U_{2e1} + V_{2e2} + W_{2e3} + S_2 \delta] , \)
\( (m_3^{e,\delta})^2 = m_3^2 + m^2 [U_{3e1} + V_{3e2} + W_{3e3} + S_3 \delta] . \)

\( (\Delta_{21}^2)^{e,\delta} = \Delta_{21}^2 + m^2 \{ (U_{1e4} - U_{1e1}) + (V_{1e4} - V_{1e2}) + (W_{1e4} - W_{1e3}) + (S_1 - S_4 - S_1) \delta \} , \)
\( (\Delta_{32}^2)^{e,\delta} = \Delta_{32}^2 + m^2 \{ (S_{2e1} - S_{4e1}) + (U_{3e4} - U_{3e1}) + (V_{3e4} - V_{3e2}) + (W_{3e4} - W_{3e3}) \} , \)
\( (\sin \theta_{12})^{e,\delta} = \left| s_{12} + c_{12} m^2 \left( \frac{S_2^2 \delta + U_2^e \epsilon_1 + V_2^e \epsilon_2 + W_2^e \epsilon_3}{\Delta_{21}^2} \right) \right| , \)
\( (\sin \theta_{23})^{e,\delta} = \frac{1}{\sqrt{2}} \left| s_{12} m^2 \left( \frac{S_3^2 \delta + U_3^e \epsilon_1 + V_3^e \epsilon_2 + W_3^e \epsilon_3}{\Delta_{21}^2 + \Delta_{32}^2} \right) + c_{12} m^2 \left( \frac{S_2^2 \delta + U_2^e \epsilon_1 + V_2^e \epsilon_2 + W_2^e \epsilon_3}{\Delta_{32}^2} \right) \right| , \)
\( (\sin \theta_{13})^{e,\delta} = c_{12} m^2 \left( \frac{S_3^2 \delta + U_3^e \epsilon_1 + V_3^e \epsilon_2 + W_3^e \epsilon_3}{\Delta_{21}^2 + \Delta_{32}^2} \right) + s_{12} m^2 \left( \frac{S_2^2 \delta + U_2^e \epsilon_1 + V_2^e \epsilon_2 + W_2^e \epsilon_3}{\Delta_{32}^2} \right) . \)

(20)

The CP-violating Jarlskog invariant, which is nonvanishing here because of \( \mu \tau \) symmetry breaking, can be obtained from

\[
J_{CP} = \text{Im} \left( \frac{(H^e_{\nu e})_{12} (H^e_{\nu e})_{23} (H^e_{\nu e})_{31}}{(\Delta_{21}^2)^{e,\delta} (\Delta_{21}^2)^{e,\delta} (\Delta_{32}^2)^{e,\delta}} \right)
\]

\[
= \frac{1}{8} \sin 2 \theta_{12}^e \cos \theta_{13}^e \sin \theta_{23}^e \sin \delta_D^e .
\]

(21)

where \( \delta_D^e \) refers to Dirac phase in the PMNS matrix \( U^{e,\delta} \).

**NUMERICAL RESULTS AND PHENOMENOLOGICAL DISCUSSION**

The breaking of \( \mu \tau \) symmetry generates a nonzero \( \theta_{13} \) as well as a deviation in \( \theta_{23} \) from its maximal value of \( \pi/4 \). In the present work, the five experimentally measured inputs [28,30], viz. the two mass squared differences and the three mixing angles, are varied within their 3\( \sigma \) experimental ranges (Table V). We also constrain the sum of the

**TABLE V: Input experimental values [30]**

| Quantity | 3\( \sigma \) ranges/other constraint |
|----------|--------------------------------------|
| \( \Delta_{21}^2 \) | \( 7.00 < \Delta_{21}^2 (10^2 \text{ eV}^{-2}) < 8.09 \) |
| \( \Delta_{12}^2 < 0 \) | \( -2.649 < \Delta_{12}^2 (10^3 \text{ eV}^{-2}) < -2.242 \) |
| \( \Delta_{13}^2 > 0 \) | \( 2.195 < \Delta_{13}^2 (10^3 \text{ eV}^{-2}) < 2.625 \) |
| \( \theta_{12} \) | \( 31.09^\circ < \theta_{12} < 35.89^\circ \) |
| \( \theta_{23} \) | \( 35.80^\circ < \theta_{23} < 54.80^\circ \) |
| \( \theta_{13} \) | \( 7.19^\circ < \theta_{13} < 9.96^\circ \) |
| \( \sum m_i \) | \( < 0.5 \text{ eV} \) |
| \( \delta_D \) | Unconstrained |

neutrino masses to be less than 0.5 eV from cosmological considerations and leave the Dirac phase \( \delta_D \) of the PMNS matrix \( U \) to be unconstrained. On feeding the inputs, one can check which of the six textures can accommodate them. One universal remark can be made about the \( \mu \tau \) symmetry breaking parameters \( \epsilon_i, \delta \). We restrict them to be less than 0.15 in magnitude so that our neglect of \( \delta^2, \delta \epsilon_i, \epsilon_i \epsilon_j \) and higher order terms is numerically justified. Thus \( 0 < |\epsilon_i|, |\delta| < 0.15 \) in general. There is nothing special about the number 0.15 which could have been 0.1. However, we want to maximize the ranges of \( |\epsilon_i|, \delta \) though we neglect their second order contributions; this led to our choice of 0.15 as the upper limit. Among the six textures, we find only two of them, viz. A1 and C1, to survive the imposition of experimental data. The remaining four textures A2, B1, B2 and C2 are ruled out due to the following reasons.

- **Texture A2**: ruled out due to a value of \( \Delta_{32}^2 \) outside the admissible range.
- **Texture B1**: ruled out due to its \( \theta_{13} \) value being outside the allowed interval.
• Texture B2: excluded due to its $\theta_{12}$ value being outside the observed range.

• Texture C2: excluded owing to a $\theta_{13}$ value outside the allowed interval.

We now make some phenomenological remarks on the surviving textures A1 and C1. Consider A1 first. The sensitivity of its three $\mu \tau$ symmetry breaking real parameters $\epsilon_1$, $\epsilon_2$ and $\delta$, cf.(8), to the observable quantities is discussed below.

(i) The parameter that is least sensitive to the input data, in particular the value of $\theta_{13}$, is $\delta$. It is possible to accommodate all the data shown in Table I with any $\delta$ within our chosen range. For simplicity, we set $\delta = 0$ in our further analysis.

(ii) Interestingly, it is $\epsilon_2$, which significantly affects the value of the angle $\theta_{23}$. For $\epsilon_2 = 0$ and very near zero but sizable values of $\epsilon_1$ and $\delta$, $\theta_{23}$ is always in the first octant, i.e. lower than $45^\circ$. On the other hand, a value of $\epsilon_2$ greater than 0.01 and in the range 0.01 - 0.15 takes to $\theta_{23}$ to the second octant, i.e. exceeding $45^\circ$. Thus the octant determination of $\theta_{23}$ will help pin down the value of $\epsilon_2$.

(iii) The crucial role in generating a non zero $\theta_{13}$ is played by $\epsilon_1$. The range of $\epsilon_1$, that is needed to accommodate the observed $\theta_{13}$, is $0.05 \leq \epsilon_1 \leq 0.15$. The lower bound can be tinkered with a bit by suitably large values of $\epsilon_2$ and $\delta$, but it is a minor effect.

Let us now turn to phases and masses. While one needs to restrict $\phi_1$ to $85^\circ \leq \phi_1 \leq 100^\circ$ to fit the data, $\phi_2$ is found to be completely unrestricted. The allowed region of the $k_1$-$k_2$ parametric plane for the surviving texture is shown in the top left panel of Fig.1. The permitted ranges of these parameters are found to be $1.06 < k_1 < 1.42$, $0.25 < k_2 < 0.70$. We find $\alpha$ to be constrained to be very close to $\pi/2$: $89^\circ < \alpha < 90^\circ$. This A1 texture allows only an inverted ordering of the neutrino masses with $\Delta_{32}^2 < 0$. The allowed value of the sum $\sum_i m_i$ has been plotted against $m_1$ in the topmost right panel of Fig.1 and so have those of $m_2$ and $m_3$. We can say that $0.067 \leq m_1(eV)^{-1} \leq 0.160$, $0.068 \leq m_2(eV)^{-1} \leq 0.162$, $0.048 \leq m_3(eV)^{-1} \leq 0.150$ and moreover $0.184 \leq (\sum_i m_i)(eV)^{-1} \leq 0.480$. Thus a quasidegenerate neutrino mass spectrum with an inverted ordering is established. Going down, we have successively plotted the allowed magnitude of the Dirac phase $\delta_D$ vs that of the Jarlskog invariant $J_{CP}$ and the Majorana phases $\alpha_{M1}$ vs $\alpha_{M2}$. We see that $0.001^\circ \leq |\delta_D| \leq 90^\circ$ and $7.0 \times 10^{-7} \leq |J_{CP}| \leq 0.039$ whereas $29^\circ \leq \alpha_{M1} \leq 87^\circ$ and $12^\circ \leq \alpha_{M2} \leq 29^\circ$.

Coming to the three zero texture C1, we find in this case that there are four $\mu \tau$ symmetry breaking real parameters $\epsilon_1$, $\epsilon_2$, $\epsilon_3$ and $\delta$ as well as three phase angles $\phi_1$, $\phi_2$ and $\phi_3$. It is seen that with $\epsilon_1 = \epsilon_2 = 0$ and nonzero values of the other parameters, it is possible to accommodate the input experimental results. So, henceforth, we set $\epsilon_{1,2} = 0$. One may also choose to set $\delta = 0$; however, in that case, the range of $\theta_{13}$ comes out as a rather narrow one: $8.3^\circ < \theta_{13} < 8.9^\circ$ and $\theta_{23} \sim 41.4^\circ$, i.e, the latter lies in the first octant. A different solution is found with $\theta_{23} \geq 45^\circ$ (i.e. in the second octant) for $\delta \geq 0.02$ and $\epsilon_3 \geq 0.06$, but the lower bounds on those parameters have to be increased to 0.03 and 0.07 respectively in order to accommodate the full $3\sigma$ range of $\theta_{23}$. It may be noted that $\epsilon_3$ is the crucial parameter here for the generation of a nonzero $\theta_{13}$. On the other hand, the results are not sensitive to the upper bounds on $\delta$ and $\epsilon_3$. Finally, the effects of $\phi_i$ ($i = 1, 2, 3$) are marginal: $\phi_{1,2,3}$ can swing from 0 to $\pi$. This texture C1 also has $\Delta_{32}^2 < 0$. Plots, similar to those for texture A1, are shown in the top left panel of Fig.2 with the variation of parameters in the ranges $0.06 \leq \epsilon_3 \leq 0.15$ and $0.02 \leq \delta \leq 0.15$. From these plots we obtain for the parameters $r_1$, $r_2$ and $\gamma$ of (12) the ranges $0.655 < r_1 < 1.130$, $0.968 < r_2 < 1.350$, $89^\circ < \gamma < 90^\circ$. The top right panel of Fig.2, showing the neutrino mass interrelations, now implies $0.049 \leq m_1(eV)^{-1} \leq 0.077$, $0.050 \leq m_2(eV)^{-1} \leq 0.078$ and $0.015 \leq m_3(eV)^{-1} \leq 0.049$ and also $0.112 \leq (\sum_i m_i)(eV)^{-1} \leq 0.203$. A weak inverted hierarchy is thus established. The bottom left plot of Fig.2 implies $0.003^\circ \leq |\delta_D| \leq 85^\circ$ and $1.8 \times 10^{-6} \leq |J_{CP}| \leq 0.037$, while the bottom right plot leads to $-88^\circ \leq \alpha_{M1} \leq -25^\circ$ and $4^\circ \leq \alpha_{M2} \leq 46^\circ$.

Finally, we should mention that we have performed similar numerical analyses with the ranges of the input parameters obtained by the global analyses of Ref.28 and Ref.29. Though there are very minor variations in the ranges in Figs.1 and 2, and in the consequent bounds given above, our basic conclusions are unaltered. In ending this section, let us comment on the feasibility of measuring the CP-violating parameters $\delta_D$ and $J_{CP}$ whose magnitudes we have plotted. Information on the latter can be extracted from experiments seeking CP violation with neutrino and antineutrino beams by measuring the difference in oscillation probabilities $P(\nu_\mu \rightarrow \nu_e) - P(\bar{\nu}_\mu \rightarrow \bar{\nu}_e)$. This is reviewed in detail in Ref.31. We need to state here that the $1\sigma$ fit to $\delta_D$, reported in the global analysis of Ref.30, implies $\delta_D = (300^{+66}_{-138})^\circ$. The Majorana phases $\alpha_{M1}$, $\alpha_{M2}$ can be probed in neutrinoless double $\beta$ decay experiments but determining them is a challenging task.
SUMMARY AND CONCLUSION

In this paper we have considered $\mu\tau$ symmetric four and three zero neutrino Yukawa textures allowed by our twin criteria of (1) no massless and (2) no unmixed neutrino. We have further introduced the most general $\mu\tau$ symmetry breaking terms into these textures as a perturbation treated to the lowest order, but keeping the textures intact. All these textures have then been subjected to the minimal type-I seesaw in the weak basis defined by mass diagonal (with real and positive values) charged leptons and heavy right chiral neutrinos. The resulting light neutrino Majorana mass matrix $M_\nu$ has been constructed in each case and its consequences compared quantitatively with the $3\sigma$ ranges of five experimental inputs: $\Delta^2_{21}$, $|\Delta^2_{32}|$, $\theta_{12}$, $\theta_{23}$, $\theta_{13}$. It is found that, out of the originally allowed four 4-zero and two 3-zero textures, only one 4-zero and one 3-zero texture survive. The survivor in each case can admit only an inverted mass ordering of the light neutrinos. For the 4-zero case, a quasidegenerate light neutrino mass spectrum is established while the 3-zero case leads to a weak inverted hierarchy. Allowed ranges of the neutrino masses, of their Dirac and Majorana phases as well as their CP-violation strength and of the magnitudes of the $\mu\tau$ symmetry breaking parameters have been shown.
Appendix

Diagonalization of $H_{\nu}^{\epsilon,\delta}$

We describe here the methodology of diagonalizing the perturbed $H_{\nu}^{\epsilon,\delta}$ to obtain the results given in (20). First of all, the unperturbed matrix $H_{\nu}$ can be diagonalized by the unitary matrix

$$U = \begin{pmatrix} e^{-i\psi} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -\frac{s_{12}}{\sqrt{2}} & \frac{c_{12}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{c_{12}}{\sqrt{2}} & \frac{s_{12}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \tag{22}$$

where $c_{12} = \cos \theta_{12}$, $s_{12} = \sin \theta_{12}$. Expressions for the unperturbed $\theta_{12}$ and neutrino masses have already been given in (16). The phase $\psi$ is given in (18). In the first step of diagonalization, we rotate $H_{\nu}^{\epsilon,\delta}$ by the unperturbed $U$ of (22). In consequence, off-diagonal terms appear only as being linear in $\epsilon, \delta$, thereby vanishing in the unperturbed limit. The rotated form of $H_{\nu}^{\epsilon,\delta}$ is

$$U^\dagger H_{\nu}^{\epsilon,\delta} U = \begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix} + m^2 \epsilon_1 \begin{pmatrix} U_1 & U_2 & U_3 \\ U_2 & U_4 & U_5 \\ U_3 & U_5 & U_6 \end{pmatrix} + m^2 \epsilon_2 \begin{pmatrix} V_1 & V_2 & V_3 \\ V_2 & V_4 & V_5 \\ V_3 & V_5 & V_6 \end{pmatrix}$$

$$+ m^2 \epsilon_3 \begin{pmatrix} W_1 & W_2^* & W_3^* \\ W_2 & W_4 & W_5^* \\ W_3 & W_5 & W_6 \end{pmatrix} + m^2 \delta \begin{pmatrix} S_1 & S_2 & S_3 \\ S_2 & S_4 & S_5^* \\ S_3 & S_5 & S_6 \end{pmatrix}. \tag{23}$$

Looking at (A.2), we see the need to further rotate it by a matrix which deviates from the identity by terms linear in $\epsilon, \delta$. For that purpose, let us consider the following matrix
\[ V^{\epsilon,\delta} = \begin{pmatrix} 1 & \epsilon_1 X_1^* + \epsilon_2 X_2^* + \epsilon_3 X_3^* + \delta X_4^* & \epsilon_1 Y_1^* + \epsilon_2 Y_2^* + \epsilon_3 Y_3^* + \delta Y_4^* \\ -\epsilon_1 X_1 - \epsilon_2 X_2 - \epsilon_3 X_3 - \delta X_4 & 1 & \epsilon_1 Z_1^* + \epsilon_2 Z_2^* + \epsilon_3 Z_3^* + \delta Z_4^* \\ -\epsilon_1 X_1 - \epsilon_2 X_2 - \epsilon_3 X_3 - \delta X_4 & -\epsilon_1 X_1 - \epsilon_2 X_2 - \epsilon_3 X_3 - \delta X_4 & 1 \end{pmatrix} \]  

(24)

\( V^{\epsilon,\delta} \) is unitary up to the neglect of terms beyond linear order in\( \epsilon_i \) and \( \delta \): \( V^{\epsilon,\delta} V^{\epsilon,\delta\dagger} = I + O(\epsilon_i \epsilon_j) + O(\epsilon_i \delta) + O(\delta^2) \).

Thus we consider the final rotation

\[ U^{\epsilon,\delta} = U V^{\epsilon,\delta} \]  

(25)

and demand to have as a result a diagonal matrix with squares of the perturbed masses as the entries. In this process we impose the vanishing condition on each coefficient of \( \epsilon_i \) and \( \delta \) in each off-diagonal element. That gives us the expressions for \( X_i \), \( Y_i \), \( Z_i \) \((i=1-4)\) and hence the complete mixing matrix

\[ U^{\epsilon,\delta\dagger} H^{\nu,\delta} U^{\epsilon,\delta} = \begin{pmatrix} (m_1^{\epsilon,\delta})^2 & 0 & 0 \\ 0 & (m_2^{\epsilon,\delta})^2 & 0 \\ 0 & 0 & (m_3^{\epsilon,\delta})^2 \end{pmatrix} \]

\[ = \begin{pmatrix} m_1^2 + m^2 U_1 \epsilon_1 + m^2 V_1 \epsilon_2 + m^2 W_1 \epsilon_3 + m^2 S_1 \delta & 0 & 0 \\ 0 & m_2^2 + m^2 U_4 \epsilon_1 + m^2 V_4 \epsilon_2 + m^2 W_4 \epsilon_3 + m^2 S_4 \delta & 0 \\ 0 & 0 & m_3^2 + m^2 U_6 \epsilon_1 + m^2 V_6 \epsilon_2 + m^2 W_6 \epsilon_3 + m^2 S_6 \delta \end{pmatrix} \]  

(26)

The vanishing condition for (2, 1) element after final rotation leads to four equalities from the requirement of vanishing coefficients of \( \epsilon_1, \epsilon_2, \epsilon_3, \delta \). Those are

\[ X_1 = \frac{m_2^2 U_2}{m_2^2 - m_1^2}, \]

\[ X_2 = \frac{m_2^2 V_2}{m_2^2 - m_1^2}, \]

\[ X_3 = \frac{m_2^2 W_2}{m_2^2 - m_1^2}, \]

\[ X_4 = \frac{m_2^2 S_2}{m_2^2 - m_1^2}. \]  

(27)

Similarly, from the required vanishing of the (3, 1) element, we have

\[ Y_1 = \frac{m_3^2 U_3}{m_3^2 - m_1^2}, \]

\[ Y_2 = \frac{m_3^2 V_3}{m_3^2 - m_1^2}, \]

\[ Y_3 = \frac{m_3^2 W_3}{m_3^2 - m_1^2}, \]

\[ Y_4 = \frac{m_3^2 S_3}{m_3^2 - m_1^2}. \]  

(28)
The vanishing of the $(3,2)$ element yields
\[
Z_1 = \frac{m^2 U_5}{m_3^2 - m_2^2}, \\
Z_2 = \frac{m^2 V_5}{m_3^2 - m_2^2}, \\
Z_3 = \frac{m^2 W_5}{m_3^2 - m_2^2}, \\
Z_4 = \frac{m^2 S_5}{m_3^2 - m_2^2}.
\]
(29)

Finally, the mixing angles obtain, cf.(20), as
\[
\sin^2 \theta_{12}^\delta = |V_{12}| = |s_{12} + c_{12}\{X_1^1 \epsilon_1 + X_2^2 \epsilon_2 + X_3^3 \epsilon_3 + X_4^4 \epsilon_4\}|, \\
\sin^2 \theta_{13}^\delta = |V_{13}| = |s_{12} \{Z_1^1 \epsilon_1 + Z_2^2 \epsilon_2 + Z_3^3 \epsilon_3 + Z_4^4 \epsilon_4\} + c_{12} \{Y_1^1 \epsilon_1 + Y_2^2 \epsilon_2 + Y_3^3 \epsilon_3 + Y_4^4 \epsilon_4\}|, \\
\sin^2 \theta_{23}^\delta = |V_{23}| = |\frac{1}{\sqrt{2}} + \frac{s_{12}}{\sqrt{2}} \{Y_1^1 \epsilon_1 + Y_2^2 \epsilon_2 + Y_3^3 \epsilon_3 + Y_4^4 \epsilon_4\} - \frac{c_{12}}{\sqrt{2}} \{Z_1^1 \epsilon_1 + Z_2^2 \epsilon_2 + Z_3^3 \epsilon_3 + Z_4^4 \epsilon_4\}|.
\]
(30)

Here we have substituted the expressions obtained for $X_i$, $Y_i$, $Z_i$ ($i=1-4$) from (27), (28) and (29) directly into (30). The perturbed neutrino squared masses in (20) follow from (A.5).

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