CHARACTERIZATION OF SEGMENT AND CONVEXITY PRESERVING MAPS

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ABSTRACT. In this note functions that transform open segments of a linear space into open segments of another linear space are studied and characterized. Assuming that the range is non-collinear, it is proved that such a map can always be expressed as the ratio of two affine functions.

1. Introduction

Throughout this paper assume that $X$ and $Y$ are (real) linear spaces. For $a, b \in X$ (or $a, b \in Y$) the closed segment $[a, b]$ and the open segment $]a, b[$ connecting the points $a$ and $b$ are defined by

$$[a, b] := \{ta + (1 - t)b \mid 0 \leq t \leq 1\}, \quad ]a, b[ := \{ta + (1 - t)b \mid 0 < t < 1\}.$$  

(Observe that, if the endpoints $a$ and $b$ coincide then $[a, b] = ]a, b[ = \{a\}$.)

Given a convex subset $D \subset X$ and a map $f : D \to Y$, we can consider two convexity preserving properties for $f$. We say that $f$ preserves convexity if $f(K)$ is convex for all convex subset $K \subseteq D$. Analogously, we say that $f^{-1}$ preserves convexity or $f$ is inversely convexity preserving if $f^{-1}(K)$ is convex whenever $K$ is a convex subset of $f(D)$. It is immediate to see that $f$ is convexity preserving if and only if

$$f([x, y]) \subseteq f([x, y]) \quad (x, y \in D).$$

On the other hand, $f$ is inversely convexity preserving if and only if

$$f([x, y]) \supseteq f([x, y]) \quad (x, y \in D).$$

Functions enjoying both of the above properties, i.e., satisfying

$$f([x, y]) = f([x, y]) \quad (x, y \in D),$$

are called segment preserving maps. Therefore, $f$ is segment preserving if and only if it is convexity and also inversely convexity preserving.

If, in (1), (2), and (3) the closed segments are replaced by open segments then we speak about strict convexity and segment preserving properties for $f$. Clearly, strict convexity or
segment preserving maps are always convexity or segment preserving (in the same sense), the converse, however, may not be valid (see the examples below).

The obvious candidates for (strict) convexity and segment preserving maps are affine maps, i.e., functions of the form \( f(x) = A(x) + a \), where \( A : X \to Y \) is linear and \( a \) is a constant vector. It is a natural question if there exist other types of segment preserving functions.

One can trivially see that if \( X = Y = \mathbb{R} \), then \( f : X \to Y \) is segment preserving if and only if it is continuous and either increasing or decreasing; furthermore, \( f \) is strictly segment preserving if and only if it is continuous and either strictly increasing, or strictly decreasing, or constant. Therefore, for the first sight, the class of such maps seems to be even more complicated in the higher-dimensional setting. However, as we shall see, if the range of \( f \) is at least two dimensional, then the description is easier: strict segment preserving maps, moreover strict inversely convexity preserving maps can be expressed as the ratio of an \( Y \)-valued and a real-valued affine map.

In order to prove our results, we shall apply Ceva’s classical theorem known from the elementary geometry (see [Cox73]):

**Ceva’s Theorem.** Let \( x, y, z \) be non-collinear points of a real linear space. Let \( t_x, t_y, t_z, s_x, s_y, s_z \) be positive numbers and

\[
(4) \quad p := \frac{t_y}{t_y + s_z} y + \frac{s_z}{t_y + s_z} z, \quad q := \frac{t_z}{t_z + s_x} z + \frac{s_x}{t_z + s_x} x, \quad r := \frac{t_x}{t_x + s_y} x + \frac{s_y}{t_x + s_y} y.
\]

Then the intersection of the segments \([x, p], [y, q], \) and \([z, r] \) is nonempty if and only if

\[
(5) \quad t_x \cdot t_y \cdot t_z = s_x \cdot s_y \cdot s_z.
\]

Now we briefly describe the idea how to use this theorem for segment preserving maps. Assume that \( f : D \to Y \) is segment preserving and \( f(D) \) is at least two dimensional, i.e., it is non-collinear. Then take \( x, y, z \in D \) such that \( f(x), f(y), f(z) \) form a non-degenerate triangle. If \( p \in [y, z], \) \( q \in [z, x], \) and \( r \in [x, y]\) are of the form \((4)\) such that \((5)\) holds, then (by Ceva’s theorem) the intersection of the segments \([x, p], [y, q], \) and \([z, r] \) is nonempty. Denote by \( s \) its single element. Then, the segments \([f(x), f(p)], [f(y), f(q)], \) and \([f(z), f(r)] \) have the point \( f(s) \) in common. We can find positive numbers \( t'_x, t'_y, t'_z, s'_x, s'_y, s'_z \) such that

\[
\begin{align*}
\frac{t'_y}{t'_y + s'_z} f(y) + \frac{s'_z}{t'_y + s'_z} f(z) = &\ f(p), \\
\frac{t'_z}{t'_z + s'_x} f(z) + \frac{s'_x}{t'_z + s'_x} f(x) = &\ f(q), \\
\frac{t'_x}{t'_x + s'_y} f(x) + \frac{s'_y}{t'_x + s'_y} f(y) = &\ f(r).
\end{align*}
\]

Therefore (again by Ceva’s theorem), we get that

\[
t'_x \cdot t'_y \cdot t'_z = s'_x \cdot s'_y \cdot s'_z.
\]

It will turn out that the above property yields a functional equation for \( f \) which will be solved explicitly in this paper. Thus, we will obtain a complete characterization of strict
segment preserving and strict inversely convexity preserving functions with non-collinear range. showing that they are rational maps, i.e., they can be expressed as the ratio of a vector valued and a positive real valued affine map.

The use of the theory of functional equations in characterizing various geometrical transformations has a rich literature. In the papers [Acz66a, AczBen69, AczMck67, Acz71] rational maps of the projective and complex plane are studied by means of the theory of functional equations.

The motivation for this research came from a recent paper [Mol01] of Molnár where the characterizations of the automorphisms of the Hilbert space effect algebras were studied.

An effect algebra \( E(H) \) is simply the operator interval \([0, I]\) of all positive operators on a Hilbert space \( H \) bounded by the identity. They play a fundamental role in the so-called quantum measurement theory. Molnár has arrived at the following problem: Is every mixture preserving bijective map \( \varphi : E(H) \to E(H) \) a mixture-automorphism? In other words, is it true that any bijective map \( \varphi : E(H) \to E(H) \) with the property “\( A \) is a convex combination of \( B \) and \( C \) if and only if \( \varphi(A) \) is also a convex combination of \( \varphi(B) \) and \( \varphi(C) \)” is necessarily affine? The results of this paper have significantly helped Molnár to obtain an affirmative answer to the problem described above.

The paper

2. Main Results

In our first result, we describe a large class of strict segment preserving maps.

**Theorem 1.** Let \( A : X \to Y \) be a linear operator, \( B : X \to \mathbb{R} \) be a linear function, \( a \in Y \) be a vector, and \( b \in \mathbb{R} \) be a scalar. Let \( D = \{ x \mid B(x) + b > 0 \} \) and assume that \( f : D \to Y \) is given by

\[
(6) \quad f(x) = \frac{A(x) + a}{B(x) + b} \quad (x \in D).
\]

Then \( f \) is a strict segment preserving function.

**Proof.** Let \( x, y \in D \) be fixed and let \( 0 < t < 1 \). Then

\[
f(tx + (1-t)y) = \frac{A(tx + (1-t)y) + a}{B(tx + (1-t)y) + b} = \frac{t(A(x) + a) + (1-t)(A(y) + a)}{t(B(x) + b) + (1-t)(B(y) + b)} f(x) + \frac{(1-t)(B(y) + b)}{t(B(x) + b) + (1-t)(B(y) + b)} f(y)
\]

Therefore, we have that \( f([x, y]) \subseteq [f(x), f(y)] \) for all \( x, y \in D \).

To obtain the reversed implication, observe that the function

\[
t \mapsto \lambda(t) := \frac{t(B(x) + b)}{t(B(x) + b) + (1-t)(B(y) + b)}
\]
maps the open unit interval $]0, 1[$ onto itself, therefore, for each $s \in ]0, 1[$ there exists $t \in ]0, 1[$ such that $\lambda(t) = s$. Then, due to the previous relations

$$sf(x) + (1 - s)f(y) = f(tx + (1 - t)y),$$

hence, $sf(x) + (1 - s)f(y) \in f([x, y[)$ for all $s \in ]0, 1[$. Thus, we get that $f(x), f(y) \subseteq f([x, y[)$ for all $x, y \in D$.

The main result of this paper is contained in the next theorem. It shows that under the non-collinear range assumption, all strict inversely convexity preserving functions can be expressed as the ratio of two affine functions, and as a consequence of the previous theorem, they are also strictly segment preserving.

**Theorem 2.** Let $D \subseteq X$ be a nonempty convex subset and $f : D \to Y$ be a strict inversely convexity preserving function such that $f(D)$ is non-collinear, i.e., it contains a nondegenerate triangle. Then there exist a linear operator $A : X \to Y$, a linear function $B : X \to \mathbb{R}$, a vector $a \in Y$, and a scalar $b \in \mathbb{R}$ such that

$$B(x) + b > 0 \quad \text{for} \quad x \in D$$

and $f$ admits the representation (6).

**Proof.** By the assumption of the theorem, $f : D \to Y$ satisfies

$$f([x, y[) \subseteq f(x), f(y)[ \quad (x, y \in D).$$

In other words, to each point of the segment $]x, y[,$ there corresponds a unique element of $]f(x), f(y)[$. Thus, for $x, y \in D$ with $f(x) \neq f(y)$, we can uniquely define a function $\alpha_{x,y} : \mathbb{R}_+ \to \mathbb{R}_+$ by the formula

$$f\left(\frac{1}{1 + t}x + \frac{t}{1 + t}y\right) = \frac{1}{1 + \alpha_{x,y}(t)}f(x) + \frac{\alpha_{x,y}(t)}{1 + \alpha_{x,y}(t)}f(y) \quad (t > 0).$$

We split the rest of the proof of Theorem 2 into a sequence of lemmas. First we derive elementary properties of the function $\alpha_{x,y}$.

**Lemma 1.** For all $x, y \in D$ such that $f(x) \neq f(y)$, the function $\alpha_{x,y}$ is strictly increasing and the following identity holds

$$\alpha_{x,y}(t) = \frac{1}{\alpha_{y,x}(1/t)} \quad (t > 0).$$

**Proof of Lemma 1.** Let $0 < s < t$ be fixed. Then

$$\frac{1}{1 + t}x + \frac{t}{1 + t}y = \frac{1 + s}{1 + t}x + \frac{s}{1 + s}y + \frac{t - s}{1 + t}y \in ]\frac{1}{1 + s}x + \frac{s}{1 + s}, y[.$$
Therefore, by the property (8) and the definition of $\alpha_{x,y}$,
\[
\frac{1}{1 + \alpha_{x,y}(t)} f(x) + \frac{\alpha_{x,y}(t)}{1 + \alpha_{x,y}(t)} f(y) = f\left(\frac{1}{1 + t} x + \frac{t}{1 + t} y\right)
\subseteq f\left(\frac{1}{1 + s} x + \frac{s}{1 + s} y\right).
\]
Thus, there exist $0 < r < 1$ such that
\[
\frac{1}{1 + \alpha_{x,y}(t)} f(x) + \frac{\alpha_{x,y}(t)}{1 + \alpha_{x,y}(t)} f(y) = r\left(\frac{1}{1 + \alpha_{x,y}(s)} f(x) + \frac{\alpha_{x,y}(s)}{1 + \alpha_{x,y}(s)} f(y)\right) + (1 - r) f(y).
\]
The points $f(x)$ and $f(y)$ being different, the coefficient of $f(x)$ on both sides has to be the same. Hence
\[
\frac{1 + \alpha_{x,y}(s)}{1 + \alpha_{x,y}(t)} = r < 1,
\]
which yields $\alpha_{x,y}(s) < \alpha_{x,y}(t)$ proving that $\alpha_{x,y}$ is strictly increasing.

To prove (9), let $t > 0$. Then we have
\[
\frac{1}{1 + \alpha_{x,y}(t)} f(x) + \frac{\alpha_{x,y}(t)}{1 + \alpha_{x,y}(t)} f(y) = f\left(\frac{1}{1 + t} x + \frac{t}{1 + t} y\right)
= f\left(\frac{1}{1 + 1/t} y + \frac{1/t}{1 + 1/t} x\right)
= \frac{1}{1 + \alpha_{y,x}(1/t)} f(y) + \frac{\alpha_{y,x}(1/t)}{1 + \alpha_{y,x}(1/t)} f(x).
\]
The coefficients of $f(x)$ are the same on the left and right hand sides, therefore,
\[
\frac{1}{1 + \alpha_{x,y}(t)} = \frac{\alpha_{y,x}(1/t)}{1 + \alpha_{y,x}(1/t)},
\]
which results (9).

Lemma 2. For all $x, y, z \in D$ with $f(x) \neq f(y) \neq f(z) \neq f(x)$, the following functional equation holds
\[
\alpha_{x,y}(st) = \alpha_{x,z}(s) \cdot \alpha_{z,y}(t) \quad (s, t > 0).
\]

Proof of Lemma 2. First assume that $x, y, z$ are such that $f(x), f(y), f(z)$ form a nondegenerate triangle. Then $x, y, z$ also form a nondegenerate triangle. (Indeed, if $x, y, z$ were collinear, then one of them would be between the two others, say $z \in [x, y]$. Then, by (8), $f(z) \in [f(x), f(y)]$ which is an obvious contradiction.) Let $s, t > 0$ be fixed and consider the points
\[
p(t) := \frac{t}{1 + t} y + \frac{1}{1 + t} z, \quad q(s) := \frac{s}{1 + s} z + \frac{1}{1 + s} x, \quad r(st) := \frac{1}{1 + st} x + \frac{st}{1 + st} y.
\]
Then, by Ceva’s Theorem, the segments 

\[ |x, p(t)|, \quad |y, q(s)|, \quad \text{and} \quad |z, r(st)| \]

have a nonempty intersection. Therefore, by the inverse convexity preserving property \([\text{S}]\) of \(f\), the intersection of their images

\[(11) \quad |f(x), f(p(t))|, \quad |f(y), f(q(s))|, \quad \text{and} \quad |f(z), f(r(st))| \]

is also nonempty. On the other hand, due to the definition of the function \(\alpha\), we get

\[ f(p(t)) = \frac{\alpha_{x,y}(t)}{1 + \alpha_{x,y}(t)} f(y) + \frac{1}{1 + \alpha_{x,y}(t)} f(z), \quad f(q(s)) = \frac{\alpha_{x,z}(s)}{1 + \alpha_{x,z}(s)} f(z) + \frac{1}{1 + \alpha_{x,z}(s)} f(x), \]

and

\[ f(r(st)) = \frac{1}{1 + \alpha_{x,y}(st)} f(x) + \frac{\alpha_{x,y}(st)}{1 + \alpha_{x,y}(st)} f(y). \]

Thus, again by Ceva’s Theorem, the non-emptiness of the intersection of the segments \([\text{II}]\) yields that \((10)\) is satisfied.

Finally we discuss the case when \(f(x), f(y), f(z)\) are pairwise distinct collinear points. Then there exists a point \(u \in D\) such that \(f(x), f(y), f(z)\), and \(f(u)\) are non-collinear (otherwise the range of \(f\) were covered by a line). Then, using repeatedly what we have proved for the non-collinear case,

\[ \alpha_{x,y}(st) = \alpha_{x,u}(s) \cdot \alpha_{u,y}(t) = \alpha_{x,z}(s) \cdot \alpha_{z,u}(1) \cdot \alpha_{u,y}(t) = \alpha_{x,z}(s) \cdot \alpha_{z,y}(t). \]

Thus the proof is also complete for this case. \(\square\)

In the next lemma we solve the functional equation \((10)\).

**Lemma 3.** There exists a positive constant \(c > 0\) and a positive-valued function \(\varphi : D \to \mathbb{R}_+\) such that

\[(12) \quad \alpha_{x,y}(t) = \frac{\varphi(y)}{\varphi(x)} t^c \quad \text{for} \quad t > 0, x, y \in D \text{ with } f(x) \neq f(y).\]

**Proof of Lemma 3.** Let \(x_0, y_0, z_0 \in D\) be fixed points such that \(f(x_0), f(y_0), f(z_0)\) is non-collinear. Then, by Lemma 3, we have that the functions strictly increasing functions \(\alpha_{x_0,y_0}, \alpha_{x_0,z_0}, \text{and} \alpha_{z_0,y_0}\) satisfy a Pexider-type functional equation. Thus, by the theory of functional equations (see e.g. [Acz66], [Kuc85]), there exists a constant \(c \in \mathbb{R}\) such that

\[(13) \quad \alpha_{x_0,y_0}(t) = \alpha_{x_0,y_0}(1) t^c, \quad \alpha_{x_0,z_0}(t) = \alpha_{x_0,z_0}(1) t^c, \quad \alpha_{z_0,y_0}(t) = \alpha_{z_0,y_0}(1) t^c \quad \text{for} \quad t > 0.\]

Due to the monotonicity properties of these functions, we have that \(c > 0\).

We are going to show that

\[(14) \quad \alpha_{x,y}(t) = \alpha_{x,y}(1) t^c \quad \text{for} \quad t > 0, x, y \in D \text{ with } f(x) \neq f(y).\]

By \((13)\), and by Lemma 1, we have that \((14)\) holds if \(x, y \in D_0 := \{x_0, y_0, z_0\} \).
Finally, if $f \not\in D_0$, $y \in D_0$ and $f(x) \neq f(y)$. Assume that $y = x_0$. Then one of the triples $f(x), f(x_0), f(y_0)$ and $f(x), f(x_0), f(z_0)$ is non-collinear, say the first. Then, using (13) and (14),

$$\alpha_{x,x_0}(t) = \alpha_{x,y_0}(1) \cdot \alpha_{y_0,x_0}(t) = \alpha_{x,y_0}(1) \cdot \alpha_{y_0,x_0}(1) t^c = \alpha_{x,x_0}(1) t^c \quad \text{for} \quad t > 0.$$  

We can also get (14) in the cases $y = y_0$, $y = z_0$ similarly.

The proof of (14) for the case when $x \in D_0$, $y \not\in D_0$ follows from the previous case and from the identity (9).

Now we consider the case when $x, y \not\in D_0$. Then for one of the points of $D_0$, say for $x_0$, the point $f(x_0)$ is non-collinear together with $f(x)$ and $f(y)$. Then, using Lemma 2 again and what we have already shown,

$$\alpha_{x,y}(t) = \alpha_{x,x_0}(1) \cdot \alpha_{x_0,y}(t) = \alpha_{x,x_0}(1) \cdot \alpha_{x_0,y}(1) t^c = \alpha_{x,y}(1) t^c \quad \text{for} \quad t > 0.$$  

Thus (14) is proved.

To complete the proof of the lemma, define $\varphi : D \to \mathbb{R}_+$ by

$$\varphi(x) := \begin{cases} \alpha_{x_0,x}(1) & \text{if } f(x) \neq f(x_0), \\ \alpha_{x_0,y}(1) \cdot \alpha_{y,x}(1) & \text{if } f(x) = f(x_0). \end{cases}$$

In view of (14), it is sufficient to prove that

$$\tag{15} \alpha_{x,y}(1) = \frac{\varphi(y)}{\varphi(x)} \quad \text{for} \quad x, y \in D \text{ with } f(x) \neq f(y).$$

We distinguish three different cases.

The first case is when $\{f(x), f(y)\}$ does not contain $f(x_0)$, that is, when $f(x) \neq f(x_0) \neq f(y)$. Then

$$\alpha_{x,y}(1) = \alpha_{x,x_0}(1) \cdot \alpha_{x_0,y}(1) = \frac{\alpha_{x_0,y}(1)}{\alpha_{x_0,x}(1)} = \frac{\varphi(y)}{\varphi(x)}.$$  

The second case is when $\{f(x), f(y)\}$ contains $f(x_0)$, but it does not contain $f(y_0)$. Then there are two possibilities. If $f(x) = f(x_0)$ and $f(y) \neq f(y_0)$, then

$$\alpha_{x,y}(1) = \alpha_{x,y_0}(1) \cdot \alpha_{y_0,y}(1) = \alpha_{x,y_0}(1) \cdot \alpha_{y_0,x_0}(1) \cdot \alpha_{x_0,y}(1) = \frac{\alpha_{x,y_0}(1)}{\alpha_{x_0,y_0}(1) \cdot \alpha_{y_0,x}(1)} = \frac{\varphi(y)}{\varphi(x)}.$$  

If $f(y) = f(x_0)$ and $f(x) \neq f(y_0)$, then the proof is similar to the previous argument.

The third case is when $\{f(x), f(y)\}$ contains $f(x_0)$ and also $f(y_0)$. Then again, there are two possibilities. If $f(x) = f(x_0)$ and $f(y) = f(y_0)$, then

$$\alpha_{x,y}(1) = \alpha_{x,y_0}(1) \cdot \alpha_{y_0,y}(1) = \alpha_{x,y_0}(1) \cdot \alpha_{y_0,x_0}(1) \cdot \alpha_{x_0,y}(1) \cdot \alpha_{x_0,y_0}(1) = \frac{\alpha_{x,y_0}(1)}{\alpha_{x_0,y_0}(1) \cdot \alpha_{y_0,x}(1)} = \frac{\varphi(y)}{\varphi(x)}.$$  

Finally, if $f(y) = f(x_0)$ and $f(x) = f(y_0)$, then we can argue in a similar way.

Thus, in all cases, the proof of (15) is complete. \qed
Lemma 4. The functions \(f\) and \(\varphi\) satisfy the functional equation

\[
(16) \quad f\left(\frac{tx + sy}{t + s}\right) = \frac{t^c \varphi f(x) + s^c \varphi f(y)}{t^c \varphi(x) + s^c \varphi(y)} \quad (x, y \in D, \ t, s > 0).
\]

Here and in the subsequent lemmas, we denote by \(\varphi f\) the function defined by \(\varphi f(x) = \varphi(x) f(x) \quad (x \in D)\).

Proof of Lemma 4. If \(f(x) = f(y)\), then \(\frac{tx + sy}{t + s} \in ]x, y[\) and (13) yield that

\[
f\left(\frac{tx + sy}{t + s}\right) \in ]f(x), f(y)[ = \{f(x)\},
\]

that is, the left hand side of (16) equals \(f(x)\). However, the right hand side is also identical to \(f(x)\), hence (16) holds trivially in this case.

Therefore, we can assume that \(f(x) \neq f(y)\). Then, using the definition of \(\alpha_{x,y}\) and the previous lemma, we get

\[
f\left(\frac{tx + sy}{t + s}\right) = f\left(\frac{1}{1 + s/t} x + \frac{s/t}{1 + s/t} y\right)
\]

\[
= \frac{1}{1 + \alpha_{x,y}(s/t)} f(x) + \frac{\alpha_{x,y}(s/t)}{1 + \alpha_{x,y}(s/t)} f(y)
\]

\[
= \frac{1}{1 + (\varphi(x)s^c)/(\varphi(x)t^c)} f(x) + \frac{(\varphi(y)s^c)/(\varphi(x)t^c)}{1 + (\varphi(y)s^c)/(\varphi(x)t^c)} f(y)
\]

\[
= \frac{t^c \varphi f(x) + s^c \varphi f(y)}{t^c \varphi(x) + s^c \varphi(y)}.
\]

Thus (16) is valid in this case as well. \(\square\)

In the next lemma, we extend the functional equation (16) to three variables.

Lemma 5. For \(x, y, z \in D\) such that \(f(x), f(y),\) and \(f(z)\) are non-collinear, the functions \(f\) and \(\varphi\) satisfy the functional equation

\[
(17) \quad f\left(\frac{tx + sy + rz}{t + s + r}\right) = \frac{t^c \varphi f(x) + s^c \varphi f(y) + r^c \varphi f(z)}{t^c \varphi(x) + s^c \varphi(y) + r^c \varphi(z)} \quad (t, s, r > 0).
\]

Proof of Lemma 5. For \(t, s, r > 0\), consider the point

\[p(t, s, r) := \frac{tx + sy + rz}{t + s + r}.
\]

Then

\[p(t, s, r) = \frac{t}{t + s + r} x + \frac{s + r}{t + s + r}, \quad \frac{sy + rz}{s + r} \in ]x, y[ \text{ and }] s, y[ \text{ and } s, z[.
\]

Therefore, by the inverse convexity preserving property (8) and by Lemma 4, we get

\[
f(p(t, s, r)) \in f\left(\left[ x, \frac{sy + rz}{s + r} \right]\right) \subseteq \left[ f(x), f\left(\frac{sy + rz}{s + r}\right) \right] = \left[ f(x), \frac{s^c \varphi f(y) + r^c \varphi f(z)}{s^c \varphi(y) + r^c \varphi(z)} \right].
\]
The functions \( \phi \) satisfy the functional equations
\[
\begin{align*}
\varphi(f(x) + (1 - t)y) &= t \varphi(f(x)) + (1 - t) \varphi(f(y)), \\
\varphi(tx + (1 - t)y) &= t \varphi(x) + (1 - t) \varphi(y)
\end{align*}
\]
for \( x, y \in D, t \in [0, 1] \), and \( c = 1 \).

**Proof of Lemma 6.** First assume that \( x, y \in D \) such that \( f(x) \neq f(y) \). Take \( z \in D \) arbitrarily such that \( f(x), f(y), \) and \( f(z) \) are non-collinear. Then, by Lemma 5, we have that
\[
\varphi(f(x) + (1 - t)y + rz) = \frac{t \varphi(f(x)) + (1 - t) \varphi(f(y)) + r \varphi(f(z))}{t \varphi(x) + (1 - t) \varphi(y) + r \varphi(z)}
\]
for \( t \in [0, 1], r > 0 \).

On the other hand, by Lemma 4, we obtain that
\[
\varphi(f(tx + (1 - t)y + rz)) = \frac{\varphi(f(tx + (1 - t)y) + rz)}{1 + r}.
\]
Thus, for \( t \in [0, 1], r > 0 \),
\[
\frac{t \varphi(f(x)) + (1 - t) \varphi(f(y)) + r \varphi(f(z))}{t \varphi(x) + (1 - t) \varphi(y) + r \varphi(z)} = \frac{\varphi(f(tx + (1 - t)y) + r \varphi(f(z))}{\varphi(tx + (1 - t)y) + r \varphi(z)}.
\]
Rearranging this equation and using the identity
\[
[t \varphi(f(x)) + (1 - t) \varphi(f(y))] \varphi(tx + (1 - t)y) = [t \varphi(x) + (1 - t) \varphi(y)] \varphi(f(tx + (1 - t)y)
\]
(which is a consequence of Lemma 4), then simplifying by \( r^c \varphi(z) > 0 \), we get that
\[
t^c \varphi \cdot f(x) + (1 - t)^c \varphi \cdot f(y) + \varphi(z) \varphi(tx + (1 - t)y) = [t^c \varphi(x) + (1 - t)^c \varphi(y)]f(z) + \varphi f(tx + (1 - t)y) \quad (t \in [0, 1]).
\]

The point \( z \) being arbitrary (such that the non-collinearity condition holds), this equality results
\[
\varphi f(tx + (1 - t)y) = t^c \varphi \cdot f(x) + (1 - t)^c \varphi \cdot f(y) = t^c \varphi(x) + (1 - t)^c \varphi(y) \quad (t \in [0, 1]).
\]

Now we consider the case \( f(x) = f(y) \). We are going to show that \( (19) \) is satisfied in this case, too.

Choose \( z \in D \) such that \( f(z) \neq f(x) \). Then, by \( (19) \), we have that
\[
\varphi \left( \frac{x + z}{2} \right) = \frac{\varphi(x) + \varphi(z)}{2^c},
\]
whence
\[
(20) \quad \varphi(x) = 2^c \varphi \left( \frac{x + z}{2} \right) - \varphi(z).
\]

Observe that
\[
f \left( \frac{x + z}{2} \right) \neq f(y) \quad \text{and} \quad f(z) \neq f(tx + (1 - t)y) = f(x).
\]

Thus, using \( (20) \) and applying \( (19) \) twice (in the cases when it has already been proved), we get
\[
t^c \varphi(x) + (1 - t)^c \varphi(y) = (2t)^c \varphi \left( \frac{x + z}{2} \right) + (1 - t)^c \varphi(y) - t^c \varphi(z)
\]
\[
= (1 + t)^c \left[ \left( \frac{2t}{1 + t} \right)^c \varphi \left( \frac{x + y}{2} \right) \right] - \varphi(z)
\]
\[
= (1 + t)^c \left[ \left( \frac{2t}{1 + t} \cdot \frac{1 + t}{1 + t} \cdot \frac{1}{1 + t} \cdot \frac{1}{1 + t} \cdot \frac{1}{1 + t} \cdot \frac{1}{1 + t} \right) \varphi(z) \right] - \varphi(z)
\]
\[
= (1 + t)^c \left[ \left( \frac{t}{1 + t} \right)^c \varphi(z) \right] - \varphi(z)
\]
\[
= \varphi(tx + (1 - t)y).
\]

Repeating the same argument with the function \( \varphi \cdot f \), we get that the first equation of \( (19) \) is also valid for all \( x, y \in D \).

Putting \( x = y \) into the second equation of \( (19) \), we get that \( t^c + (1 - t)^c = 1 \) for all \( t \in [0, 1] \). Hence, \( c = 1 \), and then \( (19) \) reduces to \( (18) \) which was to be proved.

\textbf{Lemma 7.} Then there exist a linear operator \( A : X \to Y \), a linear function \( B : X \to \mathbb{R} \), a vector \( a \in Y \), and a scalar \( b \in \mathbb{R} \) such that \( (7) \) is valid and
\[
(21) \quad \left\{ \begin{array}{l}
\varphi f(x) = A(x) + a \\
\varphi(x) = B(x) + b
\end{array} \right. \quad (x \in D).
\]
Proof of Lemma 7. We prove the statement for the function \( \varphi \). The proof for \( \varphi \cdot f \) is identical. We know from Lemma 6, that \( \varphi \) satisfies the second equation of (18). Let \( 0 < t < 1 \) be fixed. Then the meaning of (18) is that \( \varphi \) is \( t \)-affine on \( D \). By the extension theorem of [Pál02, Theorem 5], there exists a uniquely determined extension \( \varphi_t : \text{aff}(D) \to \mathbb{R} \) of \( \varphi \) which satisfies the functional equation

\[
\varphi_t(tx + (1-t)y) = t\varphi_t(x) + (1-t)\varphi_t(y) \quad (x, y \in \text{aff}(D)),
\]

where

\[
\text{aff}(D) := \{ sx + (1-s)y \mid x, y \in D, s \in \mathbb{R} \},
\]

which is called the affine hull of \( D \). One can check that, if \( D \) is convex, then \( \text{aff}(D) \) is the smallest affine subspace containing \( D \). On the other hand, by the results of [DarPal87], a \( t \)-affine function is always \((1/2)\)-affine, or in other terms, Jensen-affine. Therefore, \( \varphi_t \) also satisfies

\[
\varphi_t\left(\frac{x+y}{2}\right) = \frac{\varphi_t(x) + \varphi_t(y)}{2} \quad (x, y \in \text{aff}(D)),
\]

that is, \( \varphi_t \) is a Jensen-affine extension of \( \varphi \) to \( \text{aff}(D) \). Due to the uniqueness of such extensions, we get that \( \varphi_t = \varphi_{1/2} \) for all \( t \in ]0, 1[ \). Consequently, \( \varphi^* := \varphi_{1/2} \) satisfies

\[
(22) \quad \varphi^*(tx + (1-t)y) = t\varphi^*(x) + (1-t)\varphi^*(y) \quad (x, y \in \text{aff}(D)).
\]

Let \( x_0 \) be an arbitrary point of \( D \). Then \( L := \text{aff}(D-x_0) \) is a linear subspace of \( X \). It follows form the results concerning Jensen-affine functions that there exists an additive function \( B^* : L \to \mathbb{R} \) and a constant \( b^* \in \mathbb{R} \) such that

\[
(23) \quad \varphi^*(x) = B^*(x-x_0) + b^* \quad (x \in \text{aff}(D)).
\]

Putting this form of \( \varphi^* \) into (22), we get (with the substitutions \( u = x-x_0 \) and \( v = y-x_0 \)) that

\[
B^*(tu + (1-t)v) = tB^*(u) + (1-t)B^*(v) \quad (u, v \in L, t \in ]0, 1[).
\]

Using the additivity of \( B^* \) this reduces (with \( w = u-v \)) to

\[
B^*(tw) = kB^*(w) \quad (w \in L, t \in ]0, 1[).
\]

Thus \( B^* \) is also homogeneous, and hence it is linear. Now, using transfinite induction, we can construct a linear extension \( B : X \to \mathbb{R} \) of \( B^* \). Then, it follows from (23) that

\[
\varphi(x) = \varphi^*(x) = B^*(x-x_0) + b^* = B(x-x_0) + b^* = B(x) + b^* - B(x_0) = B(x) + b \quad (x \in D),
\]

i.e., \( \varphi \) is of the desired form (21). \( \square \)

The proof of Theorem 2 is now completed, since, by Lemma 7, we have that

\[
f(x) = \frac{\varphi \cdot f(x)}{\varphi(x)} = \frac{A(x) + a}{B(x) + b} \quad (x \in D).
\]

Taking \( D = X \) in our main theorem, we get the following consequence.

\( \square \)
Corollary. Let $f : X \to Y$ be a strict inversely convexity preserving function such that $f(X)$ is non-collinear. Then there exist a linear operator $A : X \to Y$ and a vector $a \in Y$ such that

$$f(x) = A(x) + a \quad (x \in X).$$

Proof. By Theorem 2, $f$ is of the form (6) and (7) is satisfied on $D = X$. Therefore, $B$ must be identically zero. Thus, $b > 0$. Without loss of generality, we can take $b = 1$. Then (6) reduces to the desired representation (24). □

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