ON TOPOLOGICAL GROUPS
(LOCALLY) HOMEOMORPHIC TO LF-SPACES

T. BANAKH, K. MINE, D. REPOVŠ, K. SAKAI, AND T. YAGASAKI

Abstract. We prove that a non-metrizable topological group $G$ is homeomorphic to (open subset of) the LF-space $\mathbb{R}^\infty$ or $\mathbb{R}^\infty \times l^2$ if $G = \bigcup_{n \in \omega} G_n$ for some increasing sequence of subgroups $(G_n)_{n \in \omega}$ such that

1) each group $G_n$ is a Polish absolute (neighborhood) retract;
2) for every $n \in \mathbb{N}$ the quotient map $G_n \to G_n/G_{n-1}$ is a locally trivial bundle;
3) for any open sets $U_n \subset G_n$, $n \in \omega$, the set $\bigcup_{n=1}^\infty U_0U_1 \cdots U_n$ is open in $G$.

The problem of recognizing topological structure of topological groups traces its history back to the fifth problem of Hilbert who asked if Lie groups can be characterized as topological groups whose underlying topological spaces are manifolds. This problem was resolved by combined efforts of Gleason [14], Montgomery, Zippin [22], Hoffman [16] and Iwasawa [13] who proved the following

Theorem 1 (Classics). A topological group $G$ is (locally) homeomorphic to an Euclidean space $\mathbb{R}^n$ if and only if $G$ is locally compact and (locally) contractible.

We say that a topological space $X$ is locally homeomorphic to a space $E$ if each point $x \in X$ has an open neighborhood homeomorphic to an open subset of $E$.

Topological groups (locally) homeomorphic to separable Hilbert spaces were characterized by Dobrowolski and Toruńczyk [10]:

Theorem 2 (Dobrowolski-Toruńczyk). A topological group $G$ is (locally) homeomorphic to a separable Hilbert space if and only if $G$ is a (locally) Polish absolute (neighborhood) retract.

In this theorem a Hilbert space can be finite- or infinite-dimensional. A topological space is called locally Polish if each point has a Polish (=separable completely-metrizable) neighborhood.

Topological groups which are (locally) homeomorphic to non-separable Hilbert spaces were characterized by Banakh and Zarichnyi [6]:

Theorem 3 (Banakh-Zarichnyi). A topological group $G$ is (locally) homeomorphic to an infinite-dimensional Hilbert space if and only if $G$ is a completely-metrizable $A(N)R$ which satisfies LFAP.

We say that a topological space $X$ satisfies LFAP (the Locally Finite Approximation Property) if for any open cover $U$ of $X$ there is a sequence of maps $f_n : X \to X$, $n \in \omega$, such that each map $f_n$ is $U$-near to the identity and the family $(f_n(X))_{n \in \omega}$ is locally finite in $X$.

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In this paper we address the problem of recognizing topological groups that are (locally)
homeomorphic to LF-spaces.

We recall that an LF-space is the direct limit lc-$\lim X_n$ of a tower
\[ X_0 \subset X_1 \subset X_2 \subset \cdots \]
of Fréchet (= locally convex linear completely-metrizable) spaces in the category of locally
convex spaces. More precisely, lc-$\lim X_n$ is the union $X = \bigcup_{n \in \omega} X_n$ endowed with the strongest
topology that turns $X$ into a locally convex space and makes the identity inclusions $X_n \to X$, $n \in \omega$, continuous.

The simplest non-trivial example of an LF-space is $\mathbb{R}^\infty$, the direct limit of the tower
\[ \mathbb{R}^1 \subset \mathbb{R}^2 \subset \mathbb{R}^3 \subset \cdots , \]
where each space $\mathbb{R}^n$ is identified with the hyperplane $\mathbb{R}^n \times \{0\}$ in $\mathbb{R}^{n+1}$. The space $\mathbb{R}^\infty$ can be identified with the direct sum $\bigoplus_{n \in \omega} \mathbb{R}$ of one-dimensional Fréchet spaces in the category of locally convex spaces.

The topological classification of LF-spaces was obtained by Mankiewicz [18] who proved
that each LF-space is homeomorphic to the direct sum $\bigoplus_{n \in \omega} l^2(\kappa_i)$ of Hilbert spaces for some
sequence of cardinals $(\kappa_i)_{i \in \omega}$. Here $l^2(\kappa)$ denotes the Hilbert space with orthonormal base of
cardinality $\kappa$. A more precise version of Mankiewicz’s classification says that the spaces
\begin{itemize}
  \item $l^2(\kappa)$ for some cardinal $\kappa \geq 0$,
  \item $\mathbb{R}^\infty$,
  \item $l^2(\kappa) \times \mathbb{R}^\infty$ for some $\kappa \geq \omega$, and
  \item $\bigoplus_{n \in \omega} l^2(\kappa_i)$ for a strictly increasing sequence of infinite cardinals $(\kappa_i)_{i \in \omega}$
\end{itemize}
are pairwise non-homeomorphic and represent all possible topological types of LF-spaces. In
particular, each infinite-dimensional separable LF-space is homeomorphic to one of the following
spaces: $l^2$, $\mathbb{R}^\infty$ or $l^2 \times \mathbb{R}^\infty$.

The topology of infinite-dimensional Hilbert spaces $l^2(\kappa)$ was characterized by Toruńczyk
[27], [28]:

**Theorem 4** (Toruńczyk). A topological space $X$ is homeomorphic to (an open subspace of)
an infinite-dimensional Hilbert space $l^2(\kappa)$ if and only if $X$ is a completely-metrizable absolute
(neighborhood) retract of density $\leq \kappa$ and each map $f : C \to X$ from a completely-metrizable
space $C$ of density $\leq \kappa$ can be uniformly approximated by closed topological embeddings.

The topological characterization of the LF-space $\mathbb{R}^\infty$ is due to Sakai [23]:

**Theorem 5** (Sakai). A topological space $X$ is homeomorphic to (an open subspace of) the
LF-space $\mathbb{R}^\infty$ if and only if

1. $X$ is the topological direct limit $\text{t-lim} X_n$ of a tower $(X_n)_{n \in \omega}$ of finite-dimensional
metrizable compacta, and
2. each embedding $f : B \to X$ of a closed subset $B \subset C$ of a finite-dimensional metrizable
compact space $C$ extends to an embedding of (a neighborhood of $B$ in) the space $C$ into
$X$.

By the topological direct limit $\text{t-lim} X_n$ of a tower
\[ X_0 \subset X_1 \subset X_2 \subset \cdots \]
of topological spaces we understand the union $X = \bigcup_{n \in \omega} X_n$ endowed with the strongest topology making the identity inclusions $X_n \to X$, $n \in \omega$, continuous. A tower $(X_n)_{n \in \omega}$ of topological spaces will be called \textit{closed} if each space $X_n$ is closed in $X_{n+1}$.

It follows from the definitions that for a tower of Fréchet spaces $(X_n)_{n \in \omega}$ the identity map $t\lim X_n \to lc\lim X_n$ is continuous. If all the Fréchet spaces $X_n$ are finite-dimensional (which happens in the case of the LF-space $\mathbb{R}^\infty$), then the identity map $t\lim X_n \to lc\lim X_n$ is a homeomorphism. However, for any strictly increasing tower $(X_n)_{n \in \omega}$ of infinite-dimensional Fréchet spaces the topological direct limit $t\lim X_n$ is not homeomorphic to the direct limit $lc\lim X_n$ of this tower in the category of locally convex spaces, see [7]. This means that topological direct limits cannot be used for topological characterization of non-metrizable LF-spaces, distinct from $\mathbb{R}^\infty$.

It was discovered in [2] that the topology of the direct limit $lc\lim X_n$ of a tower $(X_n)_{n \in \omega}$ of locally convex spaces in the category of locally convex spaces coincides with the topology of the direct limit of this tower in the category of uniform spaces and this observation resulted in the topological characterization of LF-spaces given in [4].

By the \textit{uniform direct limit} $u\lim X_n$ of a tower of uniform spaces

$$X_0 \subset X_1 \subset X_2 \subset \cdots$$

we understand the union $X = \bigcup_{n \in \omega} X_n$ endowed with the strongest uniformity making the identity maps $X_n \to X$, $n \in \omega$, uniformly continuous.

Let us note that each locally convex space $X$ (more generally, each abelian topological group) carries the canonical uniformity generated by the entourages $\{(x, y) \in X^2 : x - y \in U\}$, where $U = -U$ runs over open symmetric neighborhoods of the origin in $X$.

According to [2], for any tower of locally convex spaces $(X_n)_{n \in \omega}$, the identity map $u\lim X_n \to lc\lim X_n$ is a homeomorphism, which allows us to identify the direct limit $lc\lim X_n$ with the uniform direct limit $u\lim X_n$.

The topological characterization of LF-spaces given in [4] detects LF-spaces among uniform direct limits of towers of uniform spaces which are (locally) homeomorphic to Hilbert spaces. So, we first recall some definitions related to uniform spaces. For more detailed information we refer the reader to the Chapter 8 of Engelking’s book [12].

The uniformity of a uniform space $X$ will be denoted by $U_X$. A uniform space is \textit{metrizable} if its uniformity is generated by a metric. For a point $a \in X$, a subset $A \subset X$, and an entourage $U \in U$, let $B(a, U) = \{x \in X : (x, a) \in U\}$ and $B(A, U) = \bigcup_{a \in A} B(a, U)$ be the $U$-balls around $a$ and $A$, respectively. By $I = [0, 1]$ we denote the unit interval.

\textbf{Definition 1.} A uniform space $X$ is defined to be \textit{uniformly locally equiconnected} if there is an entourage $U_0 \in U$ and a continuous map $\lambda : U_0 \times I \to X$ such that

- $\lambda(x, y, 0) = x$ and $\lambda(x, y, 1) = y$ for all $x, y \in U_0$, and
- for every entourage $U \in U_X$ there is an entourage $V \in U_X$ such that for each pair $(x, y) \in V \cap U_0$ we get $\lambda(x, y, t) \in B(x, U)$ for all $t \in I$.

\textbf{Definition 2.} A subset $A$ of a uniform space $X$ is defined to be a \textit{uniform neighborhood retract} in $X$ if there is an entourage $V_0 \in U_X$ and a retraction $r : B(A, V_0) \to A$ such that for every entourage $U \in U_X$ there is an entourage $V \in U_X$ such that for any $a \in A$ and $x \in B(a, V \cap V_0)$ we get $r(x) \in B(a, U)$.

\textbf{Definition 3.} We say that a subset $A$ of a uniform space $X$ has a \textit{uniform frill in} $X$ if there is a continuous map $\gamma : A \times [0, 1] \to X$ such that
• for any entourage $U \in \mathcal{U}_X$ there is a number $\delta > 0$ such that $\gamma(a, t) \in B(a, U)$ for any $(a, t) \in A \times [0, \delta]$, and
• for every $\delta \in (0, 1]$ there is an entourage $U \in \mathcal{U}_X$ such that $\gamma(a, t) \notin B(A, U)$ for any $(a, t) \in A \times [\delta, 1]$.

Now we are able to formulate the characterization of non-metrizable LF-spaces which is due to Banakh and Repovš [4].

**Theorem 6 (Banakh-Repovš).** A topological space $X$ is homeomorphic to (an open subspace of) a non-metrizable LF-space if and only if $X$ is homeomorphic to the uniform direct limit $\varprojlim \ X_n$ of a tower $(X_n)_{n \in \omega}$ of metrizable uniform spaces such that each space $X_n$

1. is uniformly locally equiconnected,
2. is a uniform neighborhood retract in $X_{n+1}$,
3. has a uniform frill in $X_{n+1}$, and
4. is homeomorphic to (an open subspace of) a Hilbert space.

Open subspaces of LF-spaces were studied by Mine and Sakai [19], [20] who proved the following Triangulation Theorem.

**Theorem 7 (Mine-Sakai).** Each open subspace $X$ of an infinite-dimensional LF-space $L$ is homeomorphic to the product $K \times L$ for a suitable locally finite-dimensional simplicial complex $K$ endowed with the metric topology.

Combining this result with Classification Theorem for Hilbert manifolds [8, IX.7.3] we obtain the following classification theorem [19]:

**Theorem 8 (Mine-Sakai).** Two open subspaces of an LF-space $l_2(\kappa) \times \mathbb{R}^\infty$ are homeomorphic if and only if they are homotopically equivalent.

In this paper we shall apply the Characterization Theorem 6 to detect topological groups that are homeomorphic to (open subspaces) of LF-spaces.

It is well-known that each topological group $G$ carries four natural uniformities:
• the left uniformity $\mathcal{U}^L$ generated by the entourages $\mathcal{U}^L = \{(x, y) \in G^2 : x \in yU\}$,
• the right uniformity $\mathcal{U}^R$ generated by the entourages $\mathcal{U}^R = \{(x, y) \in G^2 : x \in Uy\}$,
• the two-sided uniformity $\mathcal{U}^{LR}$ generated by the entourages $\mathcal{U}^{LR} = \{(x, y) \in G^2 : x \in yU \cap Uy\}$, and
• the Roelcke uniformity $\mathcal{U}^{RL}$ generated by the entourages $\mathcal{U}^{RL} = \{(x, y) \in G^2 : x \in Uy\}$,

where $U = U^{-1}$ runs over open symmetric neighborhoods of the neutral element $e$ of $G$.

The group $G$ endowed with one of the uniformities $\mathcal{U}^L, \mathcal{U}^R, \mathcal{U}^{LR}, \mathcal{U}^{RL}$ is denoted by $G^L, G^R, G^{LR}, G^{RL}$, respectively. These four uniformities on $G$ coincide if and only if the group $G$ is balanced, which means that $G$ has a neighborhood base at $e$ consisting of open sets $U \subset G$ that are invariant in the sense that $U^G = U$ where $U^G = \{huh^{-1} : h \in H, u \in U\}$.

Let $G$ be a topological group and

$$G_0 \subset G_1 \subset \cdots$$

be a tower of closed subgroups of $G$ such that $G = \bigcup_{n \in \omega} G_n$. Endowing the subgroups $G_n, n \in \omega$, with one of four canonical uniformities, we obtain four uniform direct limits $\varprojlim G^L_n, \varprojlim G^R_n, \varprojlim G^{LR}_n, \varprojlim G^{RL}_n$ of the towers of uniform spaces $(G^L_n)_{n \in \omega}, (G^R_n)_{n \in \omega}, (G^{LR}_n)_{n \in \omega}, (G^{RL}_n)_{n \in \omega}$, respectively.
Besides those direct limits, the group $G$ also carries the topology of the group direct limit $\lim_{\to} G_n$ of the tower $(G_n)_{n \in \omega}$. This is the strongest topology that turns $G = \bigcup_{n \in \omega} G_n$ into a topological group and makes the identity maps $G_n \to G$, $n \in \omega$, continuous.

For these direct limits we get the following diagram in which each arrow indicates that the corresponding identity map is continuous:

$$
\begin{array}{ccc}
    u \text{-lim} G^L_n & \longrightarrow & u \text{-lim} G^{RL}_n \\
    t \text{-lim} G_n & \longrightarrow & u \text{-lim} G^R_n \\
    & \uparrow & \uparrow \\
    & G & G \\
\end{array}
$$

We are interested in cases when the identity map $u \text{-lim} G^L_n \to G$ is a homeomorphism.

**Definition 4.** A topological group $G$ carries the strong topology with respect to a tower of subgroups

$$
G_0 \subset G_1 \subset G_2 \subset \cdots
$$

if $G = \bigcup_{n \in \omega} G_n$ and for any neighborhoods $U_n \subset G_n$, $n \in \omega$, of the neutral element $e$ the group product

$$
\prod_{n \in \omega} U_n = \bigcup_{n=1}^{\infty} U_0 U_1 \cdots U_n
$$

is a neighborhood of $e$ in the group $G$.

**Proposition 1.** For a topological group $G$ and a tower of subgroups $(G_n)_{n \in \omega}$ with $G = \bigcup_{n \in \omega} G_n$ the following conditions are equivalent:

1. $G$ carries the strong topology with respect to the tower $(G_n)_{n \in \omega}$;
2. the identity map $u \text{-lim} G^L_n \to G$ is a homeomorphism;
3. the identity map $u \text{-lim} G^R_n \to G$ is a homeomorphism.

**Proof.** By Theorem 9.1 of [3], the topology of the uniform direct limit $u \text{-lim} G^L_n$ is generated by the base $\mathcal{T}$ consisting of the products $\prod_{n \in \omega} U_n$ of open subsets $U_n \subset G_n$, $n \in \omega$. This description of the topology of $u \text{-lim} G^L_n$ implies the equivalence of the statements (1) and (2).

The equivalence of (2) and (3) follows from the fact that the inversion

$$
(\cdot)^{-1} : u \text{-lim} G^L_n \to u \text{-lim} G^R_n, \quad (\cdot)^{-1} : x \mapsto x^{-1},
$$

is a (uniform) homeomorphism.

Now we shall apply Theorem [for recognizing towers of topological groups whose uniform direct limits are (locally) homeomorphic to LF-spaces.

**Theorem 9.** For a tower $(G_n)_{n \in \omega}$ of topological groups the uniform direct limit $u \text{-lim} G^L_n$ is homeomorphic to (an open subspace of) an LF-space if for every $n \in \omega$ the uniform space $G^L_n$ a uniform neighborhood retract in $G^L_{n+1}$ and $G_n$ is homeomorphic to (an open subspace of) a Hilbert space.
Proof. Let \( G = \bigcup_{n \in \omega} G_n \). By the hypothesis, each group \( G_n \) is homeomorphic to (an open subspace) of a Hilbert space \( l_2(\kappa_n) \) having an orthonormal base of cardinality \( \kappa_n \). Let \( \kappa = \sup_{i \in \omega} \kappa_i \) and consider the following three cases.

**Case 1.** The cardinal \( \kappa \) is finite. Then there is \( m \in \omega \) such that \( \kappa_i = \kappa \) for all \( i \geq m \). For every \( i \geq m \), the groups \( G_i \subset G_{i+1} \) are Lie groups of the same finite dimension. Consequently, \( G_i \) is a closed-and-open subgroup of \( G_{i+1} \). In this case the group \( G = \nu-lim G_n^l = \tau-lim G_n \) is a Lie group. The connected component of the group \( G_m \) coincides with the connected component of each group \( G_n \), \( n \geq m \), and hence coincides with the connected component of the group \( G \). Since each group \( G_n \), \( n \geq m \), is homeomorphic to an open subspace of the Euclidean space \( l_2(\kappa) \), its connected component (being open in \( G_n \)) has at most countable index in \( G_n \). Consequently, the connected component of the group \( G \) also has at most countable index in \( G \). The connected component of the group \( G_m \) admits an open embedding into the Euclidean space \( l_2(\kappa) \) and so does the connected component of the group \( G \). Since this component has at most countable index in \( G \), the whole group \( G \) is homeomorphic to an open subset of \( l_2(\kappa) \) (because \( l_2(\kappa) \times \mathbb{N} \) admits an open embedding into \( l_2(\kappa) \)).

**Case 2.** The cardinal \( \kappa \) is infinite but there is \( m \in \omega \) such that for every \( n \in \omega \) the group \( G_n \) is open in \( G_{n+1} \). Repeating the argument from the previous case, we can show that \( G = \nu-lim G_n^l \) is locally homeomorphic to the infinite-dimensional Hilbert space \( l_2(\kappa) \). Since \( G \) has density \( \sup_{n \in \omega} \text{dens}(G_n) \leq \text{dens}(l_2(\kappa)) = \kappa \), the space \( G \), being an \( l_2(\kappa) \)-manifold of density \( \leq \kappa \), admits an open embedding into \( l_2(\kappa) \) by the Open Embedding Theorem for Hilbert manifolds, see [8, IX.7.1].

**Case 3.** The group \( G_n \) is not open in \( G_{n+1} \) for infinitely many numbers \( n \). Passing to a suitable subsequence \( (G_{n_k})_{k \in \omega} \), we may assume that each group \( G_n \) is not open in \( G_{n+1} \). Then also each group \( G_n \) is nowhere dense in \( G_{n+1} \). In this case we can apply Theorem [6] and show that the uniform direct limit \( \nu-lim G_n^l \) is homeomorphic to (an open subset of) a non-metrizable LF-space. The conditions (2) and (4) of that theorem hold by our hypothesis. The remaining conditions (1) and (3) are established in the following two lemmas.

**Lemma 1.** If a topological group \( G \) is locally contractible, then the uniform space \( G^l \) is uniformly locally equiconnected.

**Proof.** Since \( G \) is locally contractible, there is a neighborhood \( U = U^{-1} \) of the neutral element \( e \in G \) and a continuous map \( \gamma : U \times [0, 1] \to G \) such that \( \gamma(x, 0) = x \) and \( \gamma(x, 1) = e \) for all \( x \in U \). Replacing the map \( \gamma \) by the map

\[
\gamma' : U \times [0, 1] \to G, \quad \gamma'(x, t) = \gamma(e, t)^{-1} \cdot \gamma(x, t),
\]

we may additionally assume that \( \gamma(e, t) = e \) for all \( t \in [0, 1] \).

The neighborhood \( U \) determines the entourage \( U^l = \{ (x, y) \in G^2 : x \in yU \} \) that belongs to the left uniformity on \( G \). Then the function

\[
\lambda : U^l \times [0, 1] \to G, \quad \lambda : (x, y, t) \mapsto y \cdot \gamma(y^{-1}x, t),
\]

witnesses that the group \( G^l \) endowed with the left uniformity is uniformly locally equiconnected.

**Lemma 2.** If \( H \) is a closed nowhere dense subgroup of a locally path-connected topological group \( G \), then \( H^l \) has a uniform frill in the uniform space \( G^l \).
Proof. Since $H$ is nowhere dense in the locally path-connected group $G$ there is a continuous map $\gamma : [0, 1] \to G$ such that $\gamma(0) = e$ and $\gamma(1) \notin H$. We may additionally assume that $\gamma^{-1}(H) = \{0\}$. In the opposite case take the real number $b = \max \gamma^{-1}(H) \in [0, 1)$ and consider the map

$$\gamma' : [0, 1] \to G, \quad \gamma'(t) = \gamma(b) \cdot \gamma(b + (1 - b)t) \quad \text{for} \quad t \in [0, 1].$$

The map $\gamma'$ will have the required property: $\gamma'(0) = \gamma(b)^{-1} \cdot \gamma(b)$ and $\gamma'(t) \notin H$ for $t > 0$. It is easy to check that the map

$$\alpha : H^L \times [0, 1] \to G^L, \quad \alpha : (h, t) \mapsto h \cdot \gamma(t),$$

determines a uniform frill of $H^L$ in $G^L$. □

Combining Theorem 9 with Proposition 1, we get

**Corollary 1.** A topological group $G$ is homeomorphic to (an open subset of) an LF-space if $G$ carries the strong topology with respect to a tower of subgroups $(G_n)_{n \in \omega}$ such that each group $G_n$ is homeomorphic to (an open subset of) a Hilbert space and is a uniform neighborhood retract in the uniform space $G_{n+1}^L$.

In light of this corollary it is important to recognize subgroups which are uniform neighborhood retracts in ambient groups (endowed with their left uniformity).

An obvious condition, which implies that a closed subgroup $H \subset G$ is a uniform neighborhood retract in $G^L$ is that $H^L$ is a uniform absolute neighborhood retract.

Following [21], we define a metric space $X$ to be a uniform absolute (neighborhood) retract if $X$ is a uniform (neighborhood) retract in each metric space $M$ that contains $X$ as a closed isometrically embedded subspace. By Proposition 2.1 and 2.2 of [26], each convex subset of a locally convex linear metric space is a uniform absolute retract. By [21] and [26], each absolute (neighborhood) retract is homeomorphic to a uniform absolute (neighborhood) retract.

By Birkhoff-Kakutani Metrization Theorem [23], the left uniformity of any first countable topological group $G$ is generated by some left-invariant metric. So we can think of topological groups as metric spaces endowed with a left-invariant metric.

**Problem 1.** Recognize topological groups that are uniform absolute neighborhood retracts.

This problem is not trivial because of the following.

**Example 1.** There is a linear metric space $L$ which is an AR but is not a uniform AR.

Proof. According to a celebrated result of Cauty [9], there exists a $\sigma$-compact linear metric space $E$, which is not an absolute retract. Let $D \subset E$ be a countable dense subset and $L$ be its linear hull in $E$. The space $L$ is an AR, being a countable union of finite-dimensional compacta, see [17]. By [21, 1.4] a metric space is a uniform AR if it contains a dense subspace that is a uniform AR. Since $E$ fails to be a (uniform) AR, its dense AR-subspace $L$ fails to be a uniform AR. □

There is another (less obvious) condition on a subgroup $H$ of a topological group $G$ implying that $H$ is a uniform neighborhood retract in $G^L$.

**Definition 5.** A subgroup $H$ is called locally topologically complemented in $G$ (briefly, LTC-subgroup) if the quotient map $q : G \to G/H = \{Hx : x \in G\}$ is a locally trivial bundle. This happens if and only if the quotient map $\gamma : G \to G/H$ has a local section.

A tower of topological groups $(G_n)_{n \in \omega}$ will be called an LTC-tower if each group $G_n$ is an LTC-subgroup of $G_{n+1}$.
Proposition 2. If $H$ is an LTC-subgroup of a topological group $G$, then $H$ is a uniform neighborhood retract in the uniform space $G^L$.

Proof. By our hypothesis, the quotient map $q : G \to G/H$ is a locally trivial bundle and as such, has a local continuous section $s : U \to G$ defined on an open neighborhood $U \subset G/H$ of the distinguished element $\bar{e} = He \in G/H$ (here $e$ stands for the neutral element of $H$).

Replacing the section $s$ by the section $s' : y \mapsto s(\bar{e})^{-1} \cdot s(y)$, we can additionally assume that $s(\bar{e}) = e$. Now it is easy to check that the formula

$$r(x) = x \cdot (s \circ q(x))^{-1}, \ x \in q^{-1}(U),$$

determines a regular retraction of the uniform neighborhood $q^{-1}(U) \subset G$ of $H$ onto $H$, witnessing that $H$ is a uniform neighborhood retract in $G^L$. □

Combining Theorem 9 with Proposition 2 we get:

Corollary 2. For an LTC-tower of topological groups $(G_n)_{n \in \omega}$ the uniform direct limit $\varinjlim G^L_n$ is homeomorphic to (an open subspace of) an LF-space if each group $G_n$ is homeomorphic to (an open subspace of) a Hilbert space.

Combining Corollary 1 with Proposition 2, we get

Corollary 3. A topological group $G$ is homeomorphic to (an open subset of) an LF-space if $G$ carries the strong topology with respect to an LTC-tower of subgroups $(G_n)_{n \in \omega}$ such that each $G_n$ group is homeomorphic to (an open subset of) a Hilbert space.

Combining this corollary with the Dobrowolski-Toruńczyk Theorem 2, we shall obtain the following theorem that will be applied in [1] and [5] for recognizing homeomorphism and diffeomorphism groups that are homeomorphic to (open subspaces of) the LF-space $\mathbb{R}^\infty \times l_2$.

Theorem 10. A non-metrizable topological group $G$ is homeomorphic to (an open subset of) the LF-space $\mathbb{R}^\infty$ or $\mathbb{R}^\infty \times l_2$ if $G$ carries the strong topology with respect to an LTC-tower of Polish $A(N)R$-subgroups $(G_n)_{n \in \omega}$.

Proof. We consider two cases.

1. Each group $G_n$ is locally compact and hence is a Lie group by the result of Hoffman [16]. In this case we shall show that $G$ is homeomorphic to (an open subset of) of $\mathbb{R}^\infty$.

We claim that infinitely many groups $G_n$ are nowhere dense in $G_{n+1}$. Assuming the converse, we would conclude that there is $m \in \omega$ such that each subgroup $G_n$, $n \geq m$, is open in $G_{n+1}$. As $G$ carries the strong topology with respect to the tower $(G_n)_{n \in \omega}$, the group $G_m = \prod_{n \in \omega} G_{\min\{n,m\}}$ is open in $G$. In this case $G$ is metrizable, which is a contradiction. Therefore infinitely many groups $G_n$ are nowhere dense in $G_{n+1}$. Passing to a suitable subsequence we can assume that each group $G_n$ is nowhere dense in $G_{n+1}$.

By Proposition 1 the identity map $\varinjlim G^L_n \to G$ is a homeomorphism. Since each space $G_n$ is locally compact, by Proposition 5.4 of [2], the identity map $\varprojlim G_n \to \varinjlim G^L_n$ is a homeomorphism. Each group $G_n$, being Polish and locally compact, is the topological direct limit of a sequence of finite-dimensional metrizable compacta. Consequently, so is the space $G = \varprojlim G_n.$
By Sakai’s Characterization Theorem \(5\), to prove the topological equivalence of \(G\) to (an open subset of) the space \(\mathbb{R}^\infty\), it suffices to check that each map \(f : B \rightarrow G\) defined on a closed subset \(B\) of a finite-dimensional metrizable compact space \(C\) has a continuous extension \(\tilde{f} : N(B) \rightarrow G\) to a neighborhood of \(B\) in \(C\) (which is equal to \(C\) if each group \(G_n\) is contractible). The subspace \(f(B)\), being compact in the topological direct limit \(G = \lim\downarrow G_n\), lies in some subgroup \(G_n\). Since \(G_n\) is an ANR, the map \(f\) can be extended to a continuous map \(\tilde{f} : N(B) \rightarrow G_n\) defined on some closed neighborhood \(N(B)\) of \(B\) in \(C\). If the group \(G_n\) is an absolute retract, then we can additionally assume that \(N(B) = C\). Since the quotient space \(N(B)/B\) is finite-dimensional, there is an embedding \(g : N(B)/B \rightarrow [0,1]^d\) for some \(d \in \mathbb{N}\) such that \(g\) maps the distinguished point \(\{B\}\) of \(N(B)/B\) to the distinguished point \((0,\ldots,0)\) of the cube \([0,1]^d\).

Since the subgroup \(G_n\) is a nowhere dense submanifold in the Lie group \(G_{n+1}\), we can construct a topological embedding \(h_1 : N(B) \times [0,1] \rightarrow G_{n+1}\) such that \(h_1(x,0) = x\) for all \(x \in N(B)\). Continuing by induction, we construct a topological embedding \(h_d : N(B) \times [0,1]^d \rightarrow G_{n+d}\) such that \(h_d(x,0^d) = x\) for all \(x \in N(B)\).

Now consider the quotient map \(q : N(B) \rightarrow N(B)/B\) and define the embedding \(\tilde{f} : N(B) \rightarrow G_{n+d} \subset G\) that extends \(f\) by the formula

\[
\tilde{f}(x) = h_d(\tilde{f}(x), g \circ q(x)), \quad x \in N(B).
\]

2. There is \(n \in \omega\) such that the group \(G_n\) is not locally compact. In this case all groups \(G_m, m \geq n\), are not locally compact. Being non-locally compact Polish A(N)Rs, the groups \(G_m, m \geq n\), are not homeomorphic to (open subspaces of) the separable Hilbert space \(l_2\) by the Dobrowolski-Torunczyk Theorem \(2\). In this case the direct application of Corollary \(3\) yields that \(G\) is homeomorphic to (an open subset of) an LF-space \(L\). Being separable, not metrizable and not \(\sigma\)-compact, the LF-space \(L\) is homeomorphic to \(\mathbb{R}^\infty \times l_2\) by the Classification Theorem of Mankiewicz \(18\). \(\square\)

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(T. Banakh) **Institute for Mathematics, Physics and Mechanics, University of Ljubljana,** Slovenia

E-mail address: tbanakh@yahoo.com, T.O.Banakh@gmail.com

(K. Mine) **Institute of Mathematics, University of Tsukuba, Tsukuba, 305-8571, Japan**

E-mail address: pen@math.tsukuba.ac.jp

(D. Repovš) **Faculty of Mathematics and Physics, and Faculty of Education, University of Ljubljana, P. O. Box 2964, Ljubljana, Slovenia 1001**

E-mail address: dusan.repovs@guest.arnes.si

(K. Sakai) **Institute of Mathematics, University of Tsukuba, Tsukuba, 305-8571, Japan**

E-mail address: sakaiktr@sakura.cc.tsukuba.ac.jp

(T. Yagasaki) **Division of Mathematics, Graduate School of Science and Technology, Kyoto Institute of Technology, Kyoto, 606-8585, Japan**

E-mail address: yagasaki@kit.ac.jp