Abstract

We point out that nontrivial quantum statistics of vortices in planar superfluid systems has its origin in the Schrodinger equation rather than the Chern-Simons term. Vortices in superfluid helium films are not anyons because helium films are compressible.

Quantum Hall effect layers admit vortices with nontrivial statistics thanks to their incompressibility. We consider type II planar superconductors described by time-dependent nonlinear Schrodinger equation. Inside of a vortex core superconductor is in the normal state. Because of that vortices in superconducting films are anyons. We consider also vortex loops in dimensions but a phase factor due to linking of vortex loops’ worldsheets is not a topological invariant.

Vortices in superfluid helium films. In the local limit the Lagrangian density of the model for interacting hard-core bosons is

$$L = i\hbar \Psi^* \partial_t \Psi - \frac{\hbar^2}{2m} \hat{\nabla} \Psi^* \hat{\nabla} \Psi + \mu \Psi^* \Psi - \frac{1}{2} \lambda (\Psi^* \Psi)^2,$$  \hspace{1cm} (1)

where $\Psi$ is the condensate wave-function defined on the plane. We assume quantisation of the model by path integrals does make sense. It is convenient to rescale the fields and coordinates in the above formula

$$\Psi = \sqrt{\frac{\mu}{\lambda}} \psi, \quad \hat{\nabla} = \frac{\hbar}{m} r, \quad \hat{\nabla}^2 = \frac{\hbar^2}{\sqrt{m\mu}} \tilde{\nabla}^2,$$  \hspace{1cm} (2)

and subsequently multiply the whole Lagrangian density by $\frac{m\lambda}{\hbar^2}$ to obtain the dimensionless Lagrangian

$$L = i\psi^* \partial_t \psi - \frac{1}{2} \nabla^2 (\psi^* \psi) - \frac{1}{2} (1 - \psi^* \psi)^2.$$  \hspace{1cm} (3)

A field equation is

$$i\partial_t \psi = -\nabla^2 \psi + (\psi^* \psi - 1) \psi.$$  \hspace{1cm} (4)

The model admits vortex solutions of the form $\psi = h_n(r) \exp \im \theta, \; n$ is the winding number. The profile function satisfies the equation

$$h_n'' + \frac{h_n'}{r} - \frac{n^2}{r^2} h_n + (1 - h_n^2) h_n = 0.$$  \hspace{1cm} (5)

Asymptotics close to the origin and at infinity are respectively

$$h_n(r) \sim h_n^0 r^n + O(r^{n+2}), \quad h_n(r) \sim 1 - \frac{n^2}{2r^2} + O(r^{-4}).$$  \hspace{1cm} (6)

Energy of the vortex is regular at the origin but logarithmically divergent at infinity. The divergence can be regularised by putting the system into a finite-size box. Energy of a vortex-antivortex pair is also finite since the topology of such a configuration is trivial - there is no slowly falling down gradient of the phase of the scalar field at infinity.

Effective Lagrangian for vortices. Let me first describe derivation of an effective Lagrangian for widely separated vortices. The field configuration can be well approximated by the product Ansatz

$$\psi = \prod_v h[\tilde{x} - \tilde{X}_v(t)] \exp[\im \alpha_v \Theta(\tilde{x} - \tilde{X}_v(t))],$$  \hspace{1cm} (7)
where the product runs over particular vortices. $\alpha_v$ is a sign of circulation of the $v$-th vortex and $\theta(\vec{x})$ is the azimuthal angle around zero of its argument.

The Lagrangian density (3) can be rewritten as

$$L = -\rho \partial_t \chi - \rho(\nabla \chi)^2 - (\nabla \rho \chi)_t^2 - \frac{1}{2} (1 - \rho)^2,$$

(8)

where we have replaced $\psi = \rho \chi \exp i \chi$. With the use of the product Ansatz (6) the Lagrangian takes the form

$$L = - E_0 - \int d^2 x [\rho \partial_t \chi + \sum_{v \neq w} [\chi_v \chi_w \nabla \Theta(\vec{x} - \vec{x}_v) \nabla \Theta(\vec{x} - \vec{x}_w)]] .$$

(9)

The first term is minus the net energy due to nonzero winding number and to core contributions. The second term contains information about vortex interactions. This term is regularised by $\rho$ - the modulus of the scalar field. The modulus is close to 1 almost everywhere except the cores of vortices. Loosely speaking the phase gradients and the time derivative of the phase in the second term have to be integrated over the whole plane except vortex cores. The first term to be integrated can be rewritten as

$$- \sum_v \alpha_v \int d^2 x \rho \partial_t \Theta(\vec{x} - \vec{x}_v(t)) .$$

(10)

If there were only one vortex a careful treatment of the integration of the multivalued phase would lead to

$$L^{eff}_{1} = - \pi \alpha_v \vec{X}_v(t) \times \dot{\vec{X}}_v(t)$$

(11)

Upon quantisation by path integrals this factor would lead to a phase picked up by the wave-function. If a vortex trajectory were closed then

$$S^{eff}_{1} = \int_{t_1}^{t_2} L^{eff}_{1} = 2 \pi \alpha_v \times "area\ enclosed\ by\ trajectory" .$$

(12)

The area should be taken with a positive sign for a clockwise motion and with a negative sign for an anticlockwise motion. For trajectories like, say, the numeral "8" one part of the contour gives a positive contribution and the other one negative. An equivalent interpretation is that the phase picked up by the wave function is proportional to the total mass of the superfluid enclosed by the contour. This is the whole story in the one-vortex case.

If there were more than one vortex an additional term would arise from (11) in addition to (12)

$$S^{eff}_{2} = M_0 \int_{t_1}^{t_2} \sum_{v > w} \alpha_v \alpha_w \frac{d}{dt} \Theta[\vec{x}_v(t) - \vec{x}_w(t)] ,$$

(13)

where

$$M_0 = 2 \pi \int r dr [1 - h^2_r(r)]$$

(14)

is the defect of the superfluid mass with respect to the uniform background. With the asymptotics (3) it becomes clear that the integral is logarithmically divergent. Thus there is no dilute vortex gas limit in superfluid helium in which vortices might be anyons.

Discussion. There is no long-range statistical interaction between vortices in the nonlinear Schrodinger equation. Compressibility of superfluid manifests itself in divergence of the integral in Eq. (14). In the incompressible case this integral would be convergent but, as it was shown in (2), two-dimensional superfluid films are compressible. It is so because of the contribution to the energy density from the term $|\nabla \chi|^2$ which is nontrivial in the case of nonzero net superfluidity. In other words, the term $\frac{\kappa}{\hbar} \hbar_\alpha$ in Eq. (4) gives rise to the long-range term in the second of Eqs. (9). This term might be matched if the local interaction in the model (1) were replaced by a two-body repulsive potential $V(\vec{R}) = \frac{\hbar^2}{2m}$ with some critical value of the constant $g$. Such a model would have nothing to do with superfluid helium as long-range interactions are attractive there. However it shows that one can have anyons in a model just because of the Schrodinger term and without any explicit parity-breaking terms like Chern-Simons interaction. The Schrodinger term has a remarkable form very similar to the Berry phase in Quantum Mechanics. If we consider evolution of the condensate wave-function due to slow motion of a vortex the contribution of the Schrodinger term to the effective action is

$$i \int_{t_1}^{t_2} dt \int \psi(\vec{x}, \vec{X}) \frac{d}{dX} \psi(\vec{x}, \vec{X}) > ,$$

(15)

where $\psi$ is the condensate wave-function dependent on a vortex position $\vec{X}$, $<>$ is the standard scalar product.

Thus an obstacle is identified. The core must be well-defined and for the core to be well-defined we have to make the fluid less compressible. The contribution from the phase-gradient energy can be matched by gauging the model $\partial_\mu \chi \rightarrow \partial_\mu \chi + i A_\mu$. An example are the models with Chern-Simons interaction being a phenomenological description of the quantum Hall effect. In these nonrelativistic models (1) (3) (4) (5) the scalar field of the nonlinear Schrodinger equation is minimally coupled to the C-S field

$$\partial_\mu \psi \rightarrow \partial_\mu \psi - ia_\mu \psi .$$

(16)

The kinetics of the gauge field is governed by the Chern-Simons term

$$\kappa \epsilon^{\mu \nu \alpha} a_\mu \partial_\nu a_\alpha .$$

(17)

All the cited models possess the Bogomolny limit (10) and in this limit there are static multivortex solutions.

The dynamics of CS vortices in the Bogomolny limit has been intensively studied in relativistic models (1) (11) and also in the nonrelativistic context (12) (13). One starts from a static multivortex solution in the Coulomb gauge. In this gauge the phase of the scalar field is a sum of azimuthal angles exactly the same as in (1). Since the gauge field is transverse the CS term does not contribute to the effective Lagrangian. The only contribution comes from the Schrodinger term $i \psi^* \partial_\mu \psi$. This is exactly the same as in the nonlinear Schrodinger equation. A different choice of the gauge would not change the topology of the phase of the scalar field - the nontrivial phase factor for a closed trajectory would still come solely from the Schrodinger term. Thus the nontrivial statistics of vortices comes from the Schrodinger term and not from the CS term. The CS term transmutes the quantumstatistics of the elementary field quanta only. The statistical interaction between vortices arises because there is a definite mass defect inside the closed trajectory of a vortex (or a
mass excess for Jackiw-Pi solitons). The Gauss’ law establishes connection between matter density and magnetic field

\[ \kappa B = \psi^* \psi. \]  

(18)

The mass defect is quantised because the magnetic flux of the vortex is quantised. The interaction term at long distances is up to a numerical factor proportional to

\[ \kappa \sum_{\psi > w} \alpha_v \alpha_w \frac{d}{dt} \Theta [\vec{x}_v(t) - \vec{x}_w(t)]. \]  

(19)

The prefactor \( \kappa \) is the same as in the CS term. This coincidence might suggest that the CS term is responsible for the statistical interaction between vortices. In fact it is rather a dynamical effect of the CS term - in the limit of very small \( \kappa \) vortices become very thin - the mass defect goes to zero. On the other hand for \( \kappa = 0 \), in the nonlinear Schroedinger equation, the core width is infinite. The limit \( \kappa \to 0 \) is singular - one can not analytically continue Eq. 19 to vanishing \( \kappa \). The crucial effect of the Chern-Simons gauge field is to make the mass defect finite - all the fields around a vortex are well localised.

Vortices appear to be anyons in the gauged nonlinear Schroedinger equation with Chern-Simons or Chern-Simons-Maxwell term. One might wonder if it is also possible to have anyons with just the Maxwell term. The answer is negative. The phase factor is proportional the mass missing in the vortex core (\( \rho \) is less than 1 there). However in the Maxwell-Schroedinger case \( \rho \) is also charge density and because of screening the net charge (mass defect) of a vortex must be zero. Addition of the Chern-Simons term modifies the Gauss’ law and nonzero net charge becomes possible.

**Vortices in type II superconductors.** The fact that Maxwell-Schroedinger dynamics does not admit anyons does not rule out anyons in real superconductors. The evolution of the condensate wavefunction is described by such a model but there are normal charges and currents in addition to those of the condensate. Because of screening the total net vortex charge has to vanish. Such a weakened restriction does not rule out the possibility of nonzero net superconducting charge and that is what really matters. Below we will derive statistical interaction between vortices with an assumption of local charge neutrality but this assumption can be released in the dilute vortex gas limit.

The time-dependent Ginzburg-Landau model is defined by the following Lagrangian density

\[ L = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} + \frac{1}{2} (\psi^* D_0 \psi - c.c.) \]

\[ -\frac{1}{2} D_k \psi^* D_k \psi - \frac{1}{8} \lambda (1 - \psi^* \psi)^2 + A_0 \rho_0 + A_\mu J_n^\mu. \]

(20)

The Lagrangian is already in rescaled dimensionless units. We will consider this model as \( 2 + 1 \) dimensional and later on discuss modifications in for real thin superconducting films. \( \lambda \) is a dimensionless constant which in the considered case of type II superconductors is larger then 1. The covariant derivatives are \( D_\mu \psi = \partial_\mu \psi + i A_\mu \psi \). \( \rho_0 \) is an uniform positive background charge density which in our rescaled model is equal to \(+1\). \( J_n \) is the mentioned normal current minimally coupled to the gauge field.

Variation with respect to \( A_0 \) leads to Gauss’ law

\[ -\nabla^2 \A_0 + \partial_i (\text{div} \A) = \rho_0 - \psi^* \psi + J_n^0. \]

(21)

Now we assume local charge neutrality. In other words we assume the distribution of the normal current is governed by some fast degrees of freedom and it quickly adjusts itself so that the R.H.S. of the above equation vanishes locally. This assumption is far too strong for our needs and it is going to be released. With this assumption we can choose the Coulomb gauge \( \text{div} \A = 0 \) together with \( A_0 = 0 \). We can forget now about the scalar potential and thus the model in a static case reduces to the Ginzburg-Landau theory

\[ L = \frac{1}{2} F_{12} F_{12} + \frac{1}{2} D_k \psi^* D_k \psi + \frac{1}{2} \lambda (1 - \psi^* \psi)^2. \]

(22)

It is well known that such a model admits well localised vortex solutions. Field equations are

\[ \nabla^2 A_1 = \frac{1}{2} i (\psi^* D_1 \psi - c.c.), \]

\[ D_k D_k \psi = \frac{1}{2} \lambda (\psi^* \psi - 1) \psi. \]

(23)

With an Ansatz

\[ \psi(r, \theta) = h_n(r)e^{i n \theta}, \quad A_1(r, \theta) = \frac{n}{r} A_n(r) \]

(24)

a vortex solution with a winding number \( n \) is obtained. Both the scalar field and the magnetic field are exponentially localised. For \( \lambda > 1 \) the penetration length of the magnetic field is larger then the coherence length of the scalar field. The gauge invariant combination \( \partial_1 \psi + i A_1 \psi \) is exponentially falling down with a distance from a vortex core. Multivortex solutions can be approximated by a product Ansatz the same as in Eq. 20, supplemented by an additive Ansatz for the gauge fields.

Once again we can consider Berry phases picked up by the wave-function thanks to the Schroedinger term. This term is still the only term which can contribute to the Berry phase because the only other term with time derivatives in the Lagrangian is already quadratic: \( \frac{1}{2} \partial_t A_0 \partial_t A_k \). Thus the considerations as to the Berry phase in the second section can be directly applied to the present model with a result

\[ S_{2}^{eff} = M_0 \int_{t_1}^{t_2} \sum_{\psi > w} \alpha_v \alpha_w \frac{d}{dt} \Theta [\vec{x}_v(t) - \vec{x}_w(t)], \]

(25)

where this time \( M_0 \) is a well-defined integral

\[ M_0 = 2 \pi \int_{0}^{\infty} rdr [1 - h_1^2(r)] \equiv \pi R_c^2, \]

(26)

being a total mass defect of the superfluid directly related to the missing charge \( Q_0 = -M_0 \). The last equivalence in Eq.(27) is a definition of the radius of the vortex core. In this model vortex is a composite of a nonzero net superfluid charge and a nonzero net magnetic flux. One might think that this is the origin of the Aharonov-Bohm effect. In fact it is just a mere coincidence.

Another effect considered before in a microscopic setting is a Berry phase for a single vortex

\[ S_{1}^{eff} = \int_{t_1}^{t_2} E_{1}^{eff} = 2 \pi a_v \times \text{”area enclosed by the trajectory”}. \]

(27)
This Berry phase is responsible for the Magnus force.

These are not all the terms in the effective action. Another term is a potential interaction derived in [4] which is exponentially suppressed for widely separated vortices. Finally, as we already have mentioned, there are terms in the Lagrangian [20] quadratic in time derivatives $\dot{\psi} \partial_\kappa \partial_\kappa \psi$. These terms contribute to a term in the effective Lagrangian quadratic in velocities

$$S^{\text{eff}}_t = \int_{t_1}^{t_2} dt \sum_l \frac{1}{2} m_{eff} \dot{X}_l \dot{X}_l' .$$

One could naively think that $m_{eff}$ can be found by substituting to $\frac{1}{2} \partial_\kappa \partial_\kappa \partial_\kappa \partial_\kappa$ multivortex fields with time-dependent positions of vortices. In fact one would have to take into account deformations of the multivortex fields for a given trajectory up to terms linear in velocities. The calculations are involved but they have been performed for relativistic Chern-Simons vortices [10]. We are going to pursue this topic in a separate publication. There is an intriguing possibility that $m_{eff}$ can be determined by the parameters of the model [20].

Thus we have shown that unlike vortices in superfluid helium films those in superconducting films are anyons. In fact we have considered a strictly 2 + 1 dimensional model. In reality superconducting films are immersed in 3-space. Vortices in such quasi-planar systems are not so well localised. The combination $\partial_\kappa \psi + i A_\kappa \psi$ outside of the core does not fall down exponentially but rather like $\frac{1}{|\vec{r}|}$ [20]. In superfluid helium $\partial_\kappa \psi$ falls like $\frac{1}{R}$. The decay rate in a superconducting film is strong enough for the integral in Eq.(28) to remain convergent. Thus our conclusions are qualitatively unchanged for thin superconducting films.

Finally let me stress that the local charge neutrality assumption is not essential. The essential point is that there is a nonzero net superconducting charge associated with a vortex. This net charge is enclosed by the trajectory of another vortex. It does not matter if there are any local charge fluctuations provided they are localised close to the core of the vortex.

**Extension to 3+1 dimensions.** So far we have considered only planar systems. Our goal was to make our discussion somewhat parallel to the literature around the quantum Hall effect. However because our anyons arise without any help from any Chern-Simons term one is free to think about generalisations to 3+1 dimensions. Of course there is a trivial embedding of planar solutions in 3-space. There should be statistical interaction between parallel vortex lines. From a theoretical point of view statistical interaction between vortex loops would be more interesting. One could consider closed worldsheets of the loops. If it were not relevant one could similarly as on a plane define a reference loop and close any open worldsheet by continuing its boundaries to the chosen reference loop.

Similarly as in the planar case one can work out the Berry phase [4] for vortex loops parametrised by time $t$ and length-like parameters $\sigma_l$

$$S^{\text{eff}} = \frac{2\pi}{3} \sum_l \int dt d\sigma_l \hat{X}_l (\hat{X}_l \times \hat{X}_l') ,$$

where $l$ runs over the loops. "-'" means differentiation with respect to the appropriate parameter $\sigma$. The parametrisation by $\sigma$'s is so chosen that the circulation of the phase is clockwise when one looks along the tangent to the loop. This is the form of the Berry phase in the limit of vanishing core thickness. The modulus of the phase for a closed loop worldsheet is $2\pi$ times the volume enclosed by the worldsheet. Similarly as on the plane one has to be careful with its sign.

Mutual statistical interactions arise when one takes into account the finite core thickness. The general formula for the Berry phase in a compressible case (finite core thickness) reads [4]

$$\frac{1}{2} \sum_l \int dt d^3x \int d\hat{X}_l (\hat{X}_l \times \hat{x} - \hat{X}_l) \rho(t, \vec{x}) .$$

If the core thickness were neglected one could put $\rho = 1$ everywhere to obtain Eq.(29). Thus the modification to Eq.(29) comes only from vortex cores where $\rho$ deviates from the vacuum value. The Berry phase in 3 dimensions is not a topological index as can be seen in the following example.

Let us consider a pair of loops. One of them is a circle of radius $R$ in the x-y plane centered at the origin

$$\hat{X}_1(t, \sigma_1) = (R \cos \sigma_1, R \sin \sigma_1, 0) .$$

The second loop is also a circle in a plane parallel to the x-y plane but its radius and center are time-dependent

$$\hat{X}_2(t, \sigma_2) = [(R + a \cos \omega t) \cos \sigma_2, (R + a \cos \omega t) \sin \sigma_2, a \sin \omega t] .$$

The worldsheet of the second loop is a torus with the first loop inside of it. The only contribution to the Berry phase comes from the motion of the second loop and is equal to $-2\pi \times (\text{"volume of the torus"} - 2\pi R R^2)$. The second term is the volume of the thin vortex core of the first loop. This term is an analogue of the statistical interaction of vortices on a plane - it is the mass missing inside the closed worldsheet. This term is not a topological invariant because its value depends continuously on the radius $R$ while the topology is $R$-independent.

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**References**

[1] J.C.Neu, Physica D 43 (1990) 385,
[2] F.Lund, Phys.Lett.A 159 (1991) 245,
[3] K.Lee, preprint CU-TP-652, [cond-mat/9409040]
[4] M.Hatsuda, S.Yahikozawa, P.Ao, D.J.Thouless, Phys.Rev.B 49 (1994) 15870,
[5] J.Dziarmaga, preprint TPJU 6/95, [cond-mat/9503068]
[6] V.T.Ginzburg, P.Pitaevskii, Zh.Eksp.Teor.Fiz. 34 (1958) 1240 [Sov.Phys.JETP 7 (1958) 888]; P.Pitaevskii, ibid. 40 (1961) 646 [13 (1961) 451]; E.P.Gross, Nuovo Cimento 20 (1961) 454,
[7] F.D.M. Haldane, Y.-S. Wu, Phys. Rev. Lett. 55 (1985) 2887,
[8] R. Jackiw and E.J. Weinberg, Phys. Rev. Lett. 64 (1990) 2334;
R. Jackiw, K. Lee and E.J. Weinberg, Phys. Rev. D 42 (1990) 3488,
[9] S.K. Kim and H. Min, Phys. Lett. B 281 (1992) 81; Y. Kim, K. Lee,
Phys. Rev. D 49 (1994) 2041; J. Dziarmaga, Phys. Rev. D 49 (1994) 5469,
[10] J. Dziarmaga, hep-th/9412180, to appear in Phys. Rev. D,
[11] R. Jackiw and S.-Y. Pi, Phys. Rev. Lett. 64 (1990) 2969,
Phys. Rev. D 42 (1990) 3500,
[12] L. Hua, C. Chou, Phys. Lett. B 308 (1993) 286; Q. Liu,
Phys. Lett. B 321 (1994) 219,
[13] N. Read, Phys. Rev. Lett. 65 (1990) 1502; Z. F. Ezawa and
A. Iwazaki, Phys. Rev. B 47 (1993) 7295,
[14] I. V. Barashenkov, A. O. Harin, Phys. Rev. Lett. 72 (1994) 1575,
[15] J. Dziarmaga, Phys. Rev. D 50 (1994) R2376,
[16] E. B. Bogomol'nyi, Sov. J. Nucl. Phys. 24 (1976) 449,
[17] A. A. Abrikosov, J. Exp. Theor. Phys. 5 (1957) 1174;
H. B. Nielsen, P. Olesen, Nucl. Phys. B 61 (1973) 45,
[18] P. Ao, D. J. Thouless, Phys. Rev. Lett. 70 (1993) 2159,
[19] L. Peres, J. Rubinstein, Physica D 64 (1993) 299,
[20] P. G. de Gennes, "Superconductivity in Metals and Alloys"
(Addison-Wesley, New York, 1989), Chapter 3,