ORTHOFIBRATIONS AND MONOIDAL ADJUNCTIONS

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Abstract. We study various types of fibrations over a product of two ∞-categories, and show how they can be dualised over one of the two factors via an explicit construction in terms of spans. Among other things, we use this to prove that given an adjunction between monoidal ∞-categories, there is an equivalence between lax monoidal structures on the right adjoint and oplax monoidal structures on the left adjoint functor.

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1. Introduction

One of the main goals of the present paper is to provide a proof of the following statement:

**Theorem A.** Given two symmetric monoidal ∞-categories C and D, then the extraction of adjoints gives an equivalence between the ∞-category of lax monoidal right adjoint functors C → D and the opposite of the ∞-category of oplax monoidal left adjoint functors D → C.

As far as we are aware this basic result, while most certainly expected, and in all likelihood well-known to experts, has not appeared in the literature (but see the remark below). In [Lu14] Lurie showed that the right adjoint of a strong symmetric monoidal functor is lax symmetric monoidal, which suffices for a great many applications, but the non-existence of the general statement has been a nuisance.

The difficulty lies in the inherent asymmetry in the definition of a symmetric monoidal ∞-category as a certain type of cocartesian fibration over Γ^op: A lax symmetric monoidal functor C → D is simply a diagram

$$
\begin{array}{ccc}
\mathcal{C}^\otimes & \rightarrow & \mathcal{D}^\otimes \\
\downarrow \Gamma^\text{op} & & \\
1
\end{array}
$$

1
of the underlying ∞-operads, whereas to the best of our knowledge there is no direct way to define an oplax symmetric monoidal functor in terms of similar diagrams. Instead, such an oplax functor is defined as a map of the opposite operads \((\Phi^o)^\otimes \to (\mathcal{D}^o)^\otimes\). These are, however, not given by simply taking opposites (this would result in a cartesian fibration over \(\Gamma\) after all), but rather indirectly by straightening of the map \(\Phi^o \to \mathcal{D}^o\), to a functor \(\mathcal{D}^o \to \Gamma\), postcomposing with \((-)^o\): \(\mathcal{D}^o \to \mathcal{D}\), and then unstraightening the result; here and throughout we use \(\mathcal{D}\) to denote the \(\infty\)-category of \(\infty\)-categories. We hope this does not lead to confusion, as the \((2-)\)category of ordinary categories will not appear in the text.

The trouble is that this procedure does not interact too well with the extraction of adjoints: Even if \(L: \mathcal{C} \to \mathcal{D}\) is assumed strong symmetric monoidal, its right adjoint over \(\Gamma^o\) does not even preserve cocartesian lifts, so that it cannot be unstraightened to a natural transformation and analysed under the process above. To circumvent this issue, we systematically study the type of fibration to which a lax monoidal functor can be unstraightened. It should be of interest in its own right, as we hopefully demonstrate:

**Definition.** We call a functor \(X \to A \times B\) a local orthocartesian fibration if it admits a sufficient supply of cocartesian lifts over \(\iota(A) \times B\) and of cartesian lifts over \(A \times \iota(B)\). Here \(\iota(A)\) denotes the core of \(A\). Together with functors over \(A \times B\) preserving such lifts these form a category \(\text{Ortho}_{\text{loc}}(A,B)\).

We show that Lurie’s unstraightening functors give rise to fully faithful embeddings

\[
\text{Un}^{\text{ct}}: \text{Fun}(A^o, \text{Cart}(B)) \to \text{Ortho}_{\text{loc}}(A,B)
\]
\[
\text{Un}^{\text{sc}}: \text{Fun}(B, \text{Cart}(A)) \to \text{Ortho}_{\text{loc}}(A,B)
\]

whose images agree. A local orthocartesian fibration is an orthocartesian fibration or orthofibration for short if it lies in the common image, and the full subcategory they span we will denote \(\text{Ortho}(A,B) \subseteq \text{Ortho}_{\text{loc}}(A,B)\). We also give a direct characterisation of orthofibrations in terms of a certain compatibility condition between the cartesian and cocartesian lifts.

Applying the (un)straightening procedure further, there result equivalences

\[
\text{Ortho}(A,B) \simeq \text{Fun}(A^o \times B, \text{Cat}) \simeq \text{Cocart}(A^o \times B)
\]

and similarly for \(\text{Cart}(A \times B^o)\) in place of the right hand term. Let us refer to the first of these functors as the orthocartesian unstraightening equivalence, denoted \(\text{Un}^{\text{ct}}/\text{St}^{\text{ct}}\).

Now, it turns out that maps of \(\infty\)-operads can be encoded as local orthofibrations over \(\Gamma^o \times [1]\), and Theorem A boils down to extending the correspondence between ortho- and cocartesian fibrations above to local orthofibrations on the left and what we term Gray fibrations on the right: Functors \(p: X \to A \times B\) that admit sufficiently many cocartesian lifts over \(\iota(A) \times B\) and whose restriction to \(A \times \iota(B)\) is a cocartesian fibration as well. Note that such a Gray fibration \(p\) is locally cocartesian, but not necessarily cocartesian.

To achieve this extension we take inspiration from work of Barwick, Glasman and Nardin [BGN18], who produced a simple direct procedure using span categories in order to construct from a cartesian fibration \(Y \to A\) a cocartesian \(X \to A^o\) classified by the same functor \(A^o \to \text{Cat}\) (recall that this is not simply given by opposing!). We extend their method to cover local orthofibrations and obtain:
**Theorem B.** For any two ∞-categories $A$ and $B$, there is an equivalence

$$\text{Dual}^{cc}: \text{Ortho}^{loc}(A, B) \rightarrow \text{Gray}(A, B^{\text{op}})$$

natural under pullback in both $A$ and $B$, which restricts to the identity for $B = *$ and to the equivalence between cartesian and cocartesian fibrations constructed by Barwick, Glasman and Nardin for $A = *$. It restricts to an equivalence between orthofibrations and cocartesian fibrations such that the triangle

$$\begin{array}{ccc}
\text{Ortho}(A, B) & \longrightarrow & \text{Cocart}(A \times B^{\text{op}}) \\
\downarrow^{ \text{Str}^{cc} } & & \downarrow^{ \text{Str}^{cc} } \\
\text{Fun}(A \times B^{\text{op}}, \text{Cat}) & \longleftarrow & \text{Fun}(A \times B^{\text{op}}, \text{Cat})
\end{array}$$

commutes.

In particular, this equivalence restricts further to give an equivalence

$$\begin{array}{ccc}
\text{Bifib}(A, B) & \longrightarrow & \text{LFib}(A \times B^{\text{op}}) \\
\downarrow^{ \text{Str}^{cc} } & & \downarrow^{ \text{Str}^{cc} } \\
\text{Fun}(A \times B^{\text{op}}, \text{An}) & \longleftarrow & \text{Fun}(A \times B^{\text{op}}, \text{An})
\end{array}$$

where An is the ∞-category of animae, or spaces, recovering results of Stevenson [St18]. In fact, bifibrations are precisely those local orthofibrations whose fibres are animae.

With this result in hand it is then not difficult to deduce Theorem A, by considering the local orthofibration over $\Gamma^{\text{op}} \times [1]$ arising from a lax monoidal symmetric monoidal functor, recognising it as a Gray fibration if there is a left adjoint, using the duality equivalence from Theorem B to obtain another local orthofibration over $\Gamma \times [1]$, and finally taking opposites and identifying $[1]$ and $[1]^{\text{op}}$. In fact, nothing is special about the category $\Gamma^{\text{op}}$ here, and we deduce Theorem A in the generality of an arbitrary base operad. In particular, this covers $\mathbb{E}_{n}$-monoidal structures also for finite $n$. In fact, we ultimately show:

**Theorem C.** For any ∞-operad $O$, there is a canonical equivalence of $(\infty, 2)$-categories

$$O\text{MonCat}^{\text{lax},R} \longrightarrow \left( O\text{MonCat}^{\text{op},L} \right)^{(1,2)-\text{op}}$$

extracting adjoints, where the left hand side denotes the $(\infty, 2)$-category of $O$-monoidal categories, lax $O$-monoidal functors that admit (colourwise) left adjoints, and $O$-monoidal transformations, and the right hand side is defined dually using oplax $O$-monoidal functors that admit right adjoints.

Our construction of this equivalence is, furthermore, natural in the base operad, but let us refrain from spelling this out here. Note, however, that Theorem A is contained in the above result by considering the special case $O^{\otimes} = \mathbb{E}_{\infty}$, and taking morphism ∞-categories between the given symmetric monoidal ∞-categories.

As another application of the technology we develop, we settle a recent question of Clausen, and show:

**Corollary D.** The Yoneda embedding $\mathcal{C} \rightarrow P(\mathcal{C})$ canonically extends to a natural transformation of functors $\text{Cat} \rightarrow \text{CAT}$ from the inclusion to the composite

$$\begin{array}{ccc}
\text{Cat} & \xrightarrow{\text{Fun}(-, \text{An})} & (\text{CAT}^{R})^{\text{op}} \\
\downarrow & & \downarrow \\
& \simeq \text{CAT}^{L} & \subseteq \text{CAT}.
\end{array}$$
Again, this result is certainly expected, and we were surprised to learn that it is apparently not contained in the literature yet; essentially by definition the analogous statement about the naturality of the Yoneda embedding is true if one regards $C \mapsto \mathcal{P}(C)$ as a functor $\text{Cat} \to \text{CAT}$ using the fact that $\mathcal{P}(C)$ is the free cocompletion of $C$. However, it is not a priori clear that this functor agrees with the one appearing in the corollary in a fashion compatible with the Yoneda embedding (but this follows from the result above as well).

**Remark.** During the completion of this work (in fact, during our final week of preparation) Haugseng, working entirely independently, posted the paper [Ha20] to the arXiv, which aside from Corollary D proves many of the same results as the present paper, stopping just short of assembling the $(\infty, 2)$-categorical statement of Theorem C, and pursuing other applications instead. The main technical difference in our approaches is that he reduces Theorem B to the case $A = *$ a point, where it can be obtained from Lurie’s unstraightening equivalence, leading in total to a shorter, but less explicit construction, see Remark 4.11.

We also expect that it should be possible to derive the result above from Lurie’s scaled unstraightening equivalence for locally (co)cartesian fibrations in [Lu09b], which is $(\infty, 2)$-categorical in nature. In contrast, aside from the formulation of Theorem C (which is really just a convenient way of packaging naturality statements) our proofs stay within the realm of $(\infty, 1)$-categories.

**Organisation.** Section 2 introduces (local) orthofibrations and Gray fibrations in more detail and establishes some of their basic properties. In Section 3 we revisit Barwick’s theory of span categories for adequate triples, as it features in the dualisation procedure which we describe in Section 4, in particular establishing the first half of Theorem B as 4.10. In Section 5 dualisation is shown to be compatible with various straightening equivalences, in particular finishing the proof of Theorem B in 5.2. The short Section 6 is devoted to the deduction of Corollary D and in the final section 7 we explain the connection to operad maps and lax natural transformations, and establish Theorem C, and thus Theorem A, as 7.21.

**Conventions.** As mentioned above, we will write $\text{Cat}$ for the $\infty$-category of $\infty$-categories to declutter notation and use $\text{An}$ for the $\infty$-category of animae, or spaces, following a recent suggestion of Clausen and Scholze. The letter $\iota$ will denote the core an $\infty$-category, i.e. the anima spanned by its equivalences. By a (full) subcategory of an $\infty$-category we shall mean a functor such that the induced maps on mapping animae are inclusions of path components (resp. equivalences). We will also call such functors faithful. Likewise, a (1-full) sub-2-category of an $(\infty, 2)$-category is a functor inducing subcategory inclusions (resp. full inclusions) on mapping $\infty$-categories.

Throughout, we shall use full capitalisation to indicate the large variants of $\infty$-categories such as $\text{CAT}$ and in the final section boldface such as $\textbf{Cat}$ to indicate the $(\infty, 2)$-categorical variants.

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Sections 2 and 4, the latter of which forms the combinatorial heart of this paper, largely form the content of the 2020 Master thesis of SL written at the University of Bonn.

2. Gray fibrations and orthofibrations

In the present section we introduce both (local) orthocartesian fibrations and Gray fibrations, study their basic properties and provide several useful recognition criteria for them. We start out with the simpler case of Gray fibrations.

2.1. Gray fibrations. To fix conventions, we recall some basic definitions:

2.1. Definition. Let $p : X \to S$ be a functor of $\infty$-categories. Then an edge $f : y \to z$ of $X$ is $p$-cartesian if the square

$$
\begin{array}{ccc}
\Hom_X(x, y) & \xrightarrow{f^*} & \Hom_X(x, z) \\
\downarrow & & \downarrow \\
\Hom_S(p(x), p(y)) & \xrightarrow{p(f)^*} & \Hom_S(p(x), p(z))
\end{array}
$$

is a pullback square in $\mathrm{An}$. We define $X^{p - \text{cart}}$ to be the subcategory of $\text{Fun}(\Delta^1, X)$ spanned by the $p$-cartesian edges of $X$. An edge $f$ is called locally $p$-cartesian if it is cartesian for the restriction $X \times_S [1] \to [1]$.

Let $E \subseteq S$ be a subcategory of $S$. Then we say that $p$ has a sufficient supply of $p$-cartesian lifts over $E$ if the essential image of the map $(p, t) : X^{p - \text{cart}} \to \text{Fun}(\Delta^1, S) \times_S X$ contains $E \times_S X$. In the special case $E = S$ the functor $p$ is a cartesian fibration, and if all edges in $X$ are $p$-cartesian then $p$ is called a right fibration.

Dually, an edge in $X$ is $p$-cocartesian if it is $p^{op}$-cartesian, $p$ has a sufficient supply of $p$-cocartesian lifts over $E$ if $p^{op}$ has sufficient $p^{op}$-cartesian lifts over $E$, and $p$ is a cocartesian/left fibration if $p^{op}$ is a cartesian/right fibration.

We will write Cocart($S$) and Cart($S$), respectively, for the subcategories of Cat/$S$ spanned by the cocartesian and cartesian fibrations, and functors preserving (co)cartesian edges between them. The full subcategories therein spanned by the left and right fibrations are denoted LFib($S$) and RFib($S$).

Note that the inclusions LFib($S$) $\subseteq$ Cat/$S$ and RFib($S$) $\subseteq$ Cat/$S$ are automatically fully faithful.

2.2. Remark. We note that the above definition does not quite agree with [Lu09a, Definition 2.4.2.1]. The relation is as follows: Any cartesian fibration in Lurie’s sense over an $\infty$-category is one in ours, and conversely, for any factorisation of a cartesian fibration $p$ in our sense into a categorical equivalence and followed by an isofibration, the latter is a cartesian fibration in Lurie’s sense, see [Lu09a, Section 2.4.4]. In particular, our definition is simply the closure under equivalence in Cat/$S$ of Lurie’s.

The change will be convenient when we deal with span categories in the next section, but otherwise has no effect on the basic statements.
2.3. **Notation.** For a functor \( p : X \to A \times B \) define \( p_l \) and \( p_r \) by requiring the two squares

\[
\begin{array}{ccc}
X_l & \to & X \\
p_l & & p \\
A \times \iota B & \to & A \times B
\end{array}
\quad \begin{array}{ccc}
X_r & \to & X \\
p_r & & p \\
\iota A \times B & \to & A \times B
\end{array}
\]

to be cartesian. Also define \( p_1 = \text{pr}_A \circ p \) and \( p_2 = \text{pr}_B \circ p \). To ease presentation, we will furthermore indicate

- a \( p \)-cocartesian edge of \( X \) by \( x \mapsto y \),
- a \( p_l \)-cocartesian edge of \( X \) by \( x \mapsto y \),
- a \( p \)-cartesian edges of \( X \) by \( x \mapsto y \), and
- a \( p_r \)-cartesian edge of \( X \) by \( x \mapsto y \).

Furthermore, we will say that \( p : X \to A \times B \) admits cocartesian lifts over \( B \), if it admits a sufficient supply of \( p \)-cartesian edges over \( \iota(A) \times B \). Note that this is generally stronger than merely requiring \( p_l \) or \( p_1 \) to be a cocartesian fibration. In fact, we have:

2.4. **Proposition.** A functor \( p : X \to A \times B \) admits (co)cartesian lifts over \( B \) if and only if \( p_2 \) is a (co)cartesian fibration and \( p(f) \in \iota A \times B \) for any \( p_2 \)-(co)cartesian edge \( f \).

**Proof.** An edge of \( A \times B \) is easily checked to be cocartesian for \( \text{pr}_2 : A \times B \to B \) if and only if it lies in \( \iota A \times B \). Thus by [Lu09a, Proposition 2.4.1.3 (3)] an edge of \( X \) lying over \( \iota A \times B \) is \( p \)-cocartesian if and only if it is \( p_2 \)-cocartesian.

Thus if \( p \) admits cocartesian lifts over \( B \), and we are given an edge \( f : b \to b' \) in \( B \), and a lift \( x \in X \) of \( b \), a \( p \)-cocartesian lift of the edge \( (\text{id}_{p_1(x)}, f) \) in \( \iota A \times B \) is also a \( p_2 \)-cocartesian lift of \( f \), and since cocartesian lifts are essentially unique it follows that they are all taken to \( \iota A \times B \).

Conversely, if \( p_2 \) is a cocartesian fibration, and we are given an edge \( (f, g) \in \iota A \times B \), together with a lift of its source to \( X \), then a \( p_2 \)-cocartesian lift \( l \) of \( g \) starting at \( x \) is \( p \)-cocartesian, and in fact a lift (up to equivalence!) of \( (f, g) \) if \( p_1(l) \) is an equivalence as assumed.

The proof for cartesian lifts is dual.

2.5. **Definition.** We say \( p : X \to A \times B \) is a Gray fibration over \( (A, B) \) if \( p \) admits cocartesian lifts over \( B \), and \( p_l \) is a cocartesian fibration. Denote the subcategory of \( \text{Cat}(A \times B) \) spanned by the Gray fibrations and functors preserving both types of cocartesian lifts by \( \text{Gray}(A, B) \). Dually, we say \( p : X \to A \times B \) is a Gray opfibration if \( p^{\text{op}} \) is a Gray fibration over \( (A^{\text{op}}, B^{\text{op}}) \), and denote the category they span by \( \text{Gray}^{\text{op}}(A, B) \).

Every cocartesian fibration \( p : X \to A \times B \) is in particular a Gray fibration. In this case an edge is \( p_l \)-cocartesian if and only if it is a \( p \)-cocartesian edge over \( A \times \iota(B) \) so we obtain a fully faithful inclusion

\[
\text{Cocart}(A \times B) \subseteq \text{Gray}(A, B)
\]

and one similarly finds

\[
\text{Gray}(A, B) \subseteq \text{Cocart}^{\text{loc}}(A \times B),
\]
the $\infty$-category of locally cocartesian fibrations over $A \times B$, since by [Lu09a, Lemma 2.4.2.7] locally cocartesian lifts of $(f, g)$ can be obtained by composing cocartesian lifts of $(\text{id}, g)$ with $p_l$-cocartesian lifts of $(f, \text{id})$.

To understand the difference between the first two categories, consider an edge $(f, g) \in A \times B$. Given a lift $x$ of the source of this edge we can choose cocartesian lifts as in the solid part of

\[
\begin{array}{c}
z' \\
\downarrow \\
z \\
\downarrow \\
x \\
\end{array} 
\quad \begin{array}{c}
y' \\
\downarrow \\
y \\
\downarrow \\
x \\
\end{array}
\quad \begin{array}{c}
(a', b') \\
\downarrow \\
(a, b) \\
\downarrow \\
(a, b) \\
\end{array}
\quad \begin{array}{c}
(f, \text{id}) \\
\downarrow \\
(f, \text{id}) \\
\downarrow \\
(id, g) \\
\end{array}
\quad \begin{array}{c}
(a', b') \\
\downarrow \\
(a, b) \\
\downarrow \\
(id, g) \\
\end{array}
\quad \begin{array}{c}
(a, b) \\
\downarrow \\
(a', b) \\
\downarrow \\
(id, g) \\
\end{array}
\]

lying over

\[
\begin{array}{c}
(a', b') \\
\downarrow \\
(a, b) \\
\downarrow \\
(a, b) \\
\end{array}
\quad \begin{array}{c}
(id, g) \\
\downarrow \\
(id, g) \\
\downarrow \\
(id, g) \\
\end{array}
\quad \begin{array}{c}
(f, \text{id}) \\
\downarrow \\
(f, \text{id}) \\
\downarrow \\
(id, g) \\
\end{array}
\]

Recall that tailed arrows denote $p$-cocartesian edges and tailed arrows marked by a circle denote $p_l$-cocartesian edges. Now by [Lu09a, Lemma 2.4.2.7] the composition $x \to z \to z'$ in the top diagram is still locally $p$-cocartesian, whence there exists an essentially unique dashed arrow as indicated making the diagram commute.

More formally, let $Q$ denote the (ordinary) category of shape

\[
\begin{array}{c}
2 \\
\downarrow \\
1 \\
\downarrow \\
0 \\
\end{array} 
\quad \begin{array}{c}
3 \\
\downarrow \\
4 \\
\downarrow \\
0 \\
\end{array}
\]

and consider the induced functor

\[p_* : \text{Fun}(Q, X) \to \text{Fun}(Q, A \times B)\]

and restrict it to the following kind of diagrams: First in the base only allow diagrams that take the maps $0 \to 1$ and $3 \to 4$ to equivalences in $A$, the maps $1 \to 2$ and $0 \to 3$ to equivalences in $B$ and the map $2 \to 4$ to an equivalence in $A \times B$. Then, furthermore, restrict the source to those diagrams that in addition take the first two edges to $p$-cocartesian edges and the second two to $p_l$-cocartesian edges. Let us call the resulting full subcategory of $\text{Fun}(Q, X)$, the category of $p$-interpolating diagrams in $X$, and an edge $f$ in $X$ we call $p$-interpolating it arises as the evaluation of an interpolating diagram at $2 \to 4$.

2.6. Proposition. Let $p : X \to A \times B$ be a Gray fibration. Then $p$ is a cocartesian fibration if and only if every $p$-interpolating edge in $X$ is an equivalence.

Proof. The forwards direction follows from [Lu09a, Propositions 2.4.1.5, 2.4.1.7 & 2.4.2.8]: Since every locally $p$-cocartesian edge of $X$ is $p$-cocartesian if $p$ is a
cocartesian fibration, so are the evaluations of any interpolating diagram at 0 → 2 and 0 → 4, as compositions of such. But then also the evaluation at 2 → 4 is \(p\)-cocartesian and since it covers an equivalence, it is itself an equivalence.

The reverse direction is similar: We will first show that every \(p_l\)-cocartesian edge \(l: w \to x\) of \(X\) is in fact \(p\)-cocartesian. By \([Lu09a, Lemma 2.4.2.7]\) it suffices to check that postcomposing \(l\) with any locally \(p\)-cocartesian edge results in another locally cocartesian edge. So pick one such \(h\), say, covering some \((f, g): (a, b) \to (a', b')\).

Choosing an interpolating diagram over \((f, g)\) as in the diagrams above one finds \(h\) equivalent to the evaluation of this diagram at 0 → 2 and thus per assumption also to the evaluation at 0 → 4. But factoring this edge as 0 → 3 → 4 gives \(hl\) as the composition of two \(p_l\)-cocartesian edges and a \(p\)-cocartesian edge, so it is in particular locally cocartesian.

To obtain that \(p\) is a cocartesian fibration simply invoke \([Lu09a, Lemma 2.4.1.7]\) once more, to find cocartesian lifts of arbitrary edges \((f, g)\) in \(A \times B\) by composing cocartesian lifts for \((f, \text{id})\) and \((\text{id}, g)\). □

The interpolating edges of a Gray fibration \(p: X \to A \times B\) map to \(\iota(A \times B)\) per construction. From the analogous assertion for cocartesian fibrations we therefore immediately obtain:

2.7. **Corollary.** A Gray fibration \(p: X \to A \times B\) is a left fibration if and only if it is conservative, or equivalently if its fibres are animae.

2.2. **Orthofibrations.** We now turn to the mixed case.

2.8. **Definition.** Let \(p: X \to A \times B\) be a functor of \(\infty\)-categories. Then \(p\) is a local orthofibration if \(p\) admits cocartesian lifts over \(A\) and cartesian lifts over \(B\). We shall denote by \(\text{Ortho}^{loc}(A, B)\) the subcategory of \(\text{Cat} / (A \times B)\) spanned by the local orthofibrations and the functors preserving the (co)cartesian lifts required in their definition.

We note that if \(p\) is a local orthofibration, \(p_l\) is a cocartesian fibration and \(p_r\) is a cartesian fibration. While the converse is generally false, we have the following criterion. It also explains the (apparent) asymmetry between the definitions of local orthofibrations and Gray fibrations, which are supposed to correspond under duality.

2.9. **Proposition.** Given a functor \(p: X \to A \times B\) such that either

1. \(p\) admits cocartesian lifts over \(A\) and \(p_r\) is a cartesian fibration, or
2. \(p\) admits cartesian lifts over \(B\) and \(p_l\) is a cocartesian fibration

Then \(p\) is a local orthofibration.

Said differently, if \(p_l\) is a cocartesian fibration and \(p_r\) a cartesian fibration, then the \(p_l\)-cocartesian edges being \(p\)-cocartesian is equivalent to the \(p_r\)-cartesian edges being \(p\)-cartesian.

**Proof.** We will deal with the first case, and thus need to show that a \(p_r\)-cartesian lift \(l: x \to y\) in \(X\) of an arrow \(g: (a, b) \to (c, d)\), for which \(a \to c\) is an equivalence, is automatically \(p\)-cartesian.
Consider thus the black part of the diagram

\[
\begin{array}{c}
\text{\(z\)} \\
\downarrow \\
\text{\(w\)} & \text{\(x\)} & \text{\(y\)} \\
\Downarrow & \Downarrow & \Downarrow \\
\text{\(e,f\)} & \text{\(a,f\)} & \text{\((c,d)\)} \\
\end{array}
\]

which is a lifting problem, in which one has to find a black dashed arrow in an essentially unique manner. First take an (essentially unique) \(p\)-cocartesian lift \(z \to w\) of \((e, f) \to (a, f)\). Since this arrow is cocartesian in all of \(X\), there is an essentially unique dotted red arrow lifting the outer triangle on the right. Since the lower horizontal part of the diagram lives over \(\iota A \times B\) there now exists an essentially unique map \(w \to x\) (not drawn) lifting the lower triangle. The composition with \(z \to x\) is the desired black dotted map, and using that \(z \to w\) is \(p\)-cocartesian one can then complete the diagram in an essentially unique way. The essential uniqueness of the map \(z \to x\) is seen by reading the argument in reverse. \(\square\)

Again local orthofibrations permit interpolating squares: Starting with \(x \in X\) we can pick (co)cartesian edges in the solid diagram

\[
\begin{array}{c}
\text{\(z'\)} \\
\Downarrow \\
\text{\(z\)} \\
\Downarrow \\
\text{\(x\)} & \text{\(y\)} \\
\Downarrow & \Downarrow \\
\text{\((f,\text{id})\)} & \text{\((a, \text{id})\)} & \text{\((f, \text{id})\)} \\
\Downarrow & \Downarrow \\
\text{\((a,b)\)} & \text{\((a', b')\)} & \text{\((a', b')\)} \\
\Downarrow & \Downarrow \\
\text{\((\text{id}, g)\)} & \text{\((\text{id}, g)\)} & \text{\((\text{id}, g)\)} \\
\end{array}
\]

Again, there is an essentially unique dashed arrow making the diagram commute, obtained by first composing \(z \to x \to y\) and then lifting along the cartesian edge \(y' \to y\) and factoring through the cocartesian edge \(z \to z'\) in either order. More formally, considering the ordinary category \(Q'\) given by

\[
\begin{array}{c}
\text{\(2\)} \\
\Downarrow \\
\text{\(1\)} & \text{\(3\)} & \text{\(4\)} \\
\Downarrow & \Downarrow \\
\text{\(1\)} & \text{\(3\)} & \text{\(4\)} \\
\Downarrow & \Downarrow \\
\end{array}
\]

one can again consider the full subcategory of \(\text{Fun}(Q', X)\) spanned by the \(p\)-interpolating diagrams in \(X\), i.e. those diagrams whose edges are \(p\)-(co)cartesian as indicated and whose image in \(A \times B\) takes the maps \(1 \to 0\) and \(4 \to 3\) to equivalences in \(A\), the
maps 1 \to 2 and 0 \to 3 to equivalences in B and the map 2 \to 4 to an equivalence in A \times B. Again we will refer to an edge in X as interpolating if it arises as the evaluation at 2 \to 4 of a p-interpolating diagram Q' \to X.

2.10. **Remark.** Note that we are not including in the notation for interpolating edges whether p is to be considered as a Gray fibration or a local orthofibration, and as we will see in the last chapter, it may well happen that both interpretations are possible. We will be more precise when the need arises, and hope that for now no confusion can arise.

2.11. **Definition.** We call a local orthofibration p: X \to A \times B an orthocartesian fibration or orthofibration for short, if all its interpolating edges are equivalences. We denote the full subcategory they span inside Ortho^{loc}(A, B) by Ortho(A, B).

Since interpolating edges again lie over equivalences in A \times B we find:

2.12. **Corollary.** For a functor p: X \to A \times B the following are equivalent:

1. p is a conservative local orthofibration.
2. p is a local orthofibration whose fibres are animae.
3. p is a local orthofibration and p_1 is a left fibration.
4. p is a local orthofibration and p_r is a right fibration.

If these conditions are satisfied, then p is in particular an orthofibration.

Such functors are called bifibrations, and we will denote the category they span inside Ortho(A, B) by Bifib(A, B). By a discussion entirely similar to Remark 2.2, such functors form the closure under equivalences in Cat/(A \times B) of the notion considered by Lurie in [Lu09a, Section 2.4.7] and studied in detail by Stevenson [St18].

Finally, let us briefly discuss functoriality. Consider the functor t: Ar(Cat) \to Cat extracting the target of a morphism. Its cartesian edges are easily checked to be precisely the pullback squares, so since Cat is complete, we obtain a functor

\[ \text{Cat}^{\op} \to \text{CAT}, \quad S \mapsto \text{Cat}/S \]

by cartesian unstraightening; here and throughout we denote the large variants of appropriate categories by capital letters.

Essentially by [Lu09a, Proposition 2.4.1.3] (co)cartesian fibrations are stable under pullback and the induced map preserves (co)cartesian edges. Therefore one obtains subfunctors

\[ \text{LFib}, \text{RFib}, \text{Cocart}, \text{Cart}: \text{Cat}^{\op} \to \text{CAT} \]

via the construction above and by pulling back along Cat \times Cat \to Cat also

\[ \text{Gray}, \text{Ortho}^{loc}, \text{Ortho}: \text{Cat}^{\op} \to \text{CAT}. \]

For the final statement we implicitly claim that interpolating edges are also stable under pullback, but this too follows immediately from [Lu09a, Proposition 2.4.1.3].

Finally, we note that by combining Lurie’s unstraightening equivalence with [GHN17, Appendix A] one finds that the functors Cocart, Cart, LFib and RFib just described are equivalent to Fun(−, Cat), Fun(−^{op}, Cat), Fun(−, An) and Fun(−^{op}, An), respectively.

We shall establish an analogue for Ortho in Section 5, below.
3. Adequate triples and categories of spans

In the present section we review the theory of adequate triples and their associated span categories developed by Barwick in [Ba17] under the name effective Burnside categories, as the dualisation construction we introduce in the next section will crucially rely on it.

We will use the opportunity to present an alternate viewpoint on the material by translating the assertions along the equivalence between $\infty$-categories and complete Segal spaces. This will allow for simpler proofs with far less explicit combinatorics than in [Ba17].

As input one uses:

3.1. Definition. An adequate triple $(X, X_{in}, X^{eg})$ consists of an $\infty$-category $X$ together with two subcategories $X_{in}$ and $X^{eg}$ containing all equivalences, whose morphisms are called ingressive and egressive, respectively, such that

(1) For any ingressive morphism $f: y \to x$ and any egressive morphism $g: x' \to x$, there exists a pullback

\[
\begin{array}{ccc}
y' & \xrightarrow{f'} & x' \\
g' \downarrow & & \downarrow g \\
y & \xrightarrow{f} & x
\end{array}
\]

(2) and in any such pullback $f'$ is again ingressive and $g'$ egressive.

Squares whose horizontal arrows are ingressive and whose vertical arrows are egressive are called ambigressive, and ambigressive cartesian if they are pullback diagrams.

We define AdTrip, the category of adequate triples, to be the subcategory of $\text{Fun}(\Lambda^2_2, \text{Cat})$ spanned by the adequate triples and those natural transformations whose evaluation at 2 preserves ambigressive pullback squares.

Note that being a natural transformation boils down to preserving ingressive and egressive maps, since these form subcategories. We shall usually drop them from notation to avoid cluttering.

3.2. Example. Of course, any $\infty$-category $S$ can be made part of an adequate triple by declaring the (say) ingressives to be equivalences and egressives to be any set of maps closed under equivalences. Similarly, if $S$ admits all pullbacks, one can simply take the ingressives and egressives to be the entirety of $S$.

Let us also note that the category of adequate triples admits all limits, and that one simply computes

\[
\lim_{\alpha} (X_{in,\alpha}, X_{eg}^{\alpha}) \simeq \left( \lim_{\alpha} X_{in,\alpha}, \lim_{\alpha} X_{in,\alpha}, \lim_{\alpha} X_{eg}^{\alpha} \right).
\]

We shall generate more interesting examples of adequate triples by means of the following criterion:

3.3. Proposition. Let $p: Y \to X$ be a functor and let $(X, X_{in}, X^{eg})$ such that $X^{eg}$ has $p$-cartesian lifts. Then $Y$ is part of an adequate triple $(Y, p^{-1}(X_{in}), Y^1)$, where a map is ingressive if its image under $p$ is, and egressive if it is $p$-cartesian and its image is egressive in $X$.

Proof. Suppose that $y_1 \to y_0 \leftarrow y_2$ is a span in $Y$ consisting of an ingressive and egressive map, with image $x_1 \to x_0 \leftarrow x_2$ in $S$. Since $y_2 \to y_0$ is egressive (in
particular, $p$-cartesian), for any other object $z \in Y$ with image $t \in X$, there is a natural equivalence

$$\text{Hom}_Y(z, y_1) \times \text{Hom}_Y(z, y_2) \xrightarrow{\sim} \text{Hom}_Y(z, y_1) \times \text{Hom}_X(t, x_0) \text{Hom}_X(t, x_2).$$

It follows that for any $p$-cartesian lift $y_3 \rightarrow y_1$ lift of the egressive map $x_1 \times x_0, x_1 \rightarrow x_1$ (which exists by assumption), the unique map $y_3 \rightarrow y_2$ making the square commute exhibits $y_3 \simeq y_1 \times y_0 y_2$ as the fibre product in $Y$. $\square$

We next set out to construct Barwick’s functor

$$\text{Span} : \text{AdTrip} \rightarrow \text{Cat}.$$  

Loosely speaking the objects of $\text{Span}(X)$ are given by those of $X$, a morphism from $x$ to $y$ in $\text{Span}(X)$ is a diagram

$$x \leftarrow w \rightarrow y$$

whose left pointing arrow is egressive and whose right pointing arrow is ingressive, and composition is by forming pullbacks, which is possible precisely by the assumption on the in- and egressives.

We will in fact define $\text{Span}$ as a functor into complete Segal spaces. A general method for the construction of such functors is as follows: Given a cosimplicial object $C : \Delta^{\text{op}} \rightarrow \mathcal{C}$, we obtain a functor $\Sigma(C) : \mathcal{C} \rightarrow \text{sAn}$ by taking the adjoint of the composition

$$\Delta^{\text{op}} \times \mathcal{C} \xrightarrow{Q^{\text{op}} \times \text{id}} \mathcal{C}^{\text{op}} \times \mathcal{C} \xrightarrow{\text{Hom}_{\mathcal{C}}} \text{An}.$$  

If $\mathcal{C}$ is cocomplete, then $\Sigma(C)$ is right adjoint to the colimit extension of $O : \text{sAn} \rightarrow \mathcal{C}$, see [Lu09a, Proposition 5.2.6.3]. For the cosimplicial object $\Delta^{\text{op}} \rightarrow \text{Cat}$, $n \mapsto [n]$ we obtain in this fashion an adjunction

$$\text{asscat} : \text{sAn} \xleftrightarrow{\text{Cat}} \text{N}.$$  

By results of Joyal, Lurie, Rezk and Tierney [JT07, Lu09b, Re01] $\text{N}$ is fully faithful with essential image the full subcategory $\text{cSS}$ of complete Segal spaces inside $\mathcal{S}$, i.e. those simplicial anima $T$ that satisfy the Segal condition and for which the diagram

$$\begin{array}{ccc}
T_0 & \xrightarrow{\Delta} & T_0 \times T_0 \\
\downarrow^s & & \downarrow^{(s,s)} \\
T_3 & \xrightarrow{(d_{0,2}, d_{1,3})} & T_1 \times T_1
\end{array}$$

is cartesian. We can therefore use the functor $\text{asscat}$ to extract a functor $\mathcal{C} \rightarrow \text{Cat}$ from $\Sigma(O)$, about which we have good control if $\Sigma(O)$ takes values in complete Segal spaces, as will be the case below.

To see this in action, consider for example the cosimplicial object

$$\epsilon : \Delta \rightarrow \text{Cat}, \quad [n] \mapsto [n] \ast [n]^{\text{op}}.$$  

Its associated functor $\text{Cat} \rightarrow \text{Cat}$ is well-known to be the twisted arrow construction, where our convention for the twisted arrow category is that source and target define a map

$$(s, t) : \text{Tw}(\mathcal{C}) \rightarrow \mathcal{C} \times \mathcal{C}^{\text{op}}.$$  

Indeed, it is not difficult to check that $\Sigma(\epsilon)(\mathcal{C}) \simeq N \text{Tw}(\mathcal{C})$ so that $\Sigma(\epsilon)$ does indeed take values in complete Segal spaces, using that $\text{Tw}$ arises from a right Quillen functor between the Joyal model structures whose left adjoint sends $[n] \ast [n]^{\text{op}}$ (see [Lu14, Section 5.2.1] or [HNP17, Proposition 4.13]).
3.4. Example. For example the category Tw([2]) is the poset

\[
\begin{array}{ccc}
(0 \leq 2) & \longrightarrow & (1 \leq 2) \\
\downarrow & & \downarrow \\
(0 \leq 1) & \longrightarrow & (1 \leq 1) \\
\downarrow & & \downarrow \\
(0 \leq 0) & & (2 \leq 2)
\end{array}
\]

Now, the categories Tw([n]) again form a cosimplicial object in Cat, and we enhance it to a cosimplicial object Tw: Δ → AdTrip as follows: We declare the left pointing arrows in Tw([n]) egressive and the right pointing ones ingressive. More formally an arrow in Tw([n]) is egressive if its source component is an isomorphism (and thus automatically an identity in [n]), and ingressive if the target component is. We leave the simple check that this really defines an adequate triple, functorial in \( n \in \Delta \), to the reader; the ambigressive pullbacks are precisely given by the diagrams

\[
\begin{array}{ccc}
(i \leq l) & \longrightarrow & (i \leq j) \\
\downarrow & & \downarrow \\
(k \leq l) & \longrightarrow & (k \leq j)
\end{array}
\]

for \( i \leq k \leq j \leq l \).

3.5. Definition. We define Span: AdTrip → Cat as the composition

\[
\text{AdTrip} \xrightarrow{\Sigma(Tw)} \text{sAn} \xrightarrow{\text{asscat}} \text{Cat}.
\]

To obtain control we are left to show:

3.6. Theorem. The essential image of \( \Sigma(Tw) \) is contained in the complete Segal spaces.

The Segal condition gives us the following description of Span(\( X, X_{in}, X^{eg} \)): objects are objects of \( X \), morphisms from \( x \) to \( y \) are spans \( x \leftarrow z \rightarrow y \) with the left arrow egressive and the right arrow ingressive, and composition proceeds by pullback of spans. Completeness furthermore implies that \( \iota \text{Span}(X, X_{in}, X^{eg}) \simeq \iota(X) \) via the degenerate spans consisting of identities.

Instead of following Barwick’s strategy of explicitly filling simplices in a point-set implementation of the above construction, we simply observe that the proof given in [CDH+20, Section 2.1] for the case of stable categories with triple structure given by taking both ingresses and egressives to simply be all maps carries over to the general setting.

In fact, we will prove the following slightly stronger statement:

3.7. Lemma. Let \( X \) be an adequate triple and denote by \( Q_n(X) \) the full subcategory of \( \text{Fun}(\text{Tw}([n]), X) \) spanned by those functors that define elements in \( \text{Hom}_{\text{AdTrip}}(\text{Tw}([n]), X) \), so that \( \iota Q_n(X) \simeq \text{Hom}_{\text{AdTrip}}(\text{Tw}([n]), X) \). Then the cosimplicial category \( Q(X) \) satisfies the Segal and completeness conditions, that
is the Segal maps
\[ Q_n(X) \to Q_1(X) \times_{Q_0(X)} Q_1(X) \times \cdots \times_{Q_0(X)} Q_1(X) \]
are equivalences and
\[
\begin{array}{ccc}
Q_0(X) & \to & Q_0(X) \\
\downarrow{s} & & \downarrow{(s,s)} \\
Q_3(X) & \to & Q_1(X) \times Q_1(X)
\end{array}
\]
is cartesian.

Note that this immediately implies the previous theorem, as \( \iota : \text{Cat} \to \text{An} \) preserves limits.

**Proof.** Let \( \mathcal{J}_n \subseteq \text{Tw}([n]) \) denote the subposet consisting of those \((i \leq j)\) with \(j \leq i + 1\), i.e. the zig-zag along the bottom. Since \( \mathcal{J}_n \) evidently decomposes as an iterated pushout
\[
\mathcal{J}_n \simeq \text{Tw}([1]) \cup_{\text{Tw}([0])} \text{Tw}([1]) \cup \cdots \cup_{\text{Tw}([0])} \text{Tw}([1])
\]
along the Segal maps, the iterated pullback appearing in the Segal condition is equivalent to the full subcategory \( \mathcal{J}_n(X) \) of \( \text{Fun}(\mathcal{J}_n, X) \) spanned by those functors taking left pointing edges in \( \mathcal{J}_n \), i.e. those of the form \((i \leq i + 1) \to (i \leq i)\), to ingressives and right pointing arrows, namely \((i \leq i + 1) \to (i + 1 \leq i + 1)\), to egressives. Furthermore, this translates the Segal map to the map \( Q_n(X) \to \mathcal{J}_n(X) \) induced by the restriction
\[
\text{Fun}(\text{Tw}([n]), X) \to \text{Fun}(\mathcal{J}_n, X).
\]

But from the pointwise formulae for Kan extensions one finds that a diagram \( F : \text{Tw}([n]) \to X \) lies in \( Q_n(X) \) if and only if it is right Kan extended from its restriction to \( \mathcal{J}_n \), which has to lie in \( \mathcal{J}_n \). The claim now follows from [Lu09a, Proposition 4.3.2.15].

Similarly, for completeness we first note that the map \( P : Q_3(X) \) from the pullback in question to the lower left corner is fully faithful since the degeneracy \( Q_0(X)^2 \to Q_1(X)^2 \) is (since \( |\text{Tw}([1])| \simeq * \)). But we claim that its essential image consists exactly of those diagrams all of whose edges are equivalences; since also \( |\text{Tw}([3])| \simeq * \) these are precisely the constant ones which gives the result. So consider a diagram
\[
\begin{array}{ccc}
F(0 \leq 3) & & \leftarrow F(0 \leq 2) \\
\downarrow{F(0 \leq 1)} & & \downarrow{F(1 \leq 2)} \\
F(0 \leq 0) & & \leftarrow F(1 \leq 1) \\
\downarrow{F(2 \leq 2)} & & \downarrow{F(2 \leq 2)} \\
F(3 \leq 3) & & \leftarrow F(3 \leq 3)
\end{array}
\]
all of whose squares are (ambigressive) cartesian and such that the four compositions
\[
\begin{align*}
F(0 \leq 2) & \to F(0 \leq 0), \quad F(0 \leq 2) \to F(2 \leq 2) \\
F(1 \leq 3) & \to F(1 \leq 1), \quad F(1 \leq 3) \to F(3 \leq 3)
\end{align*}
\]
are equivalences. Then it first follows that, as pullbacks of equivalences, also \(F(0 \leq 3) \to F(0 \leq 1)\) and \(F(0 \leq 3) \to F(3 \leq 3)\) are equivalences and then by two-out-of-six the entire outer slopes are. But then by commutativity of the larger rectangles also \(F(0 \leq 1) \to F(1 \leq 1)\) and \(F(2 \leq 3) \to F(2 \leq 2)\) are equivalences, and then finally as pullbacks thereof also \(F(0 \leq 2) \to F(1 \leq 2)\) and \(F(1 \leq 3) \to F(1 \leq 2)\).

We shall also need to control functors into span categories. To this end we need a few more definitions. Note that for any \(\infty\)-category \(K\), \(\text{Tw}(K)\) is part of an adequate triple with egressives inducing equivalences on source components and ingressives inducing equivalences on targets (this follows from Proposition 3.3, applied to the cartesian fibration \(t : \text{Tw}(K) \to K^{\text{op}}\)). For an adequate triple \((X, X_{\text{in}}, X_{\text{ex}})\), then define \(Q_K(X)\) as the full subcategory of functors \(\text{Tw}(K) \to X\) that are maps of adequate triples. Since any ambigressive pullback square in \(\text{Tw}(K)\) is contained (up to equivalence) in \(\text{Tw}([3])\) for some map \([3] \to K\), this is equivalent to

\[
\text{Tw}([3]) \to \text{Tw}(K) \to X
\]

being a morphism of adequate triples for every map \([3] \to K\). The condition above is obviously preserved under restriction, so we obtain a functor \(Q_- (X) : \text{Cat}^{\text{op}} \to \text{Cat}\).

3.8. Proposition. The functor \(Q_- : \text{Cat}^{\text{op}} \to \text{Cat}\) preserves limits.

Proof. First consider the case of a diagram \(F : I \to \text{Cat}\) such that the natural map

(*) \[ \colim_{i \in I} N(F_i) \to N(\colim_{i \in I} F_i). \]

is an equivalence. Note that there are natural equivalences

\[ N(\text{Tw}([i])) \cong [i \mapsto \text{Hom}_{\text{Cat}}([i] \to [i]^{\text{op}}, C)] \cong [i \mapsto \text{Hom}_{\text{sAn}}(\Delta^i \to (\Delta^i)^{\text{op}}, N(C))]. \]

In particular, since \(\Delta^i \to (\Delta^i)^{\text{op}} \cong \Delta^{2i+1}\) is small in \(\text{sAn}\), we find

\[ \colim_{i \in I} \text{Tw}(F_i) \to \text{Tw}(\colim_{i \in I} F_i) \]

because of (*); let us also warn the reader that this statement is not generally true without any assumption on \(F\) as the diagram \([1] \xleftarrow{0} [0] \to [1]\) with pushout \([2]\) shows. In the same vein we find that the map

\[ \colim_{i \in I} \text{Hom}_{\text{Cat}}([3], F_i) \to \text{Hom}_{\text{Cat}}([3], \colim_{i \in I} F_i) \]

is an equivalence, whenever \(F\) satisfies (*).

The first statement then gives that the map

\[ \text{Fun}(\text{Tw}(\colim_{i \in I} F_i), X) \to \lim_{i \in I} \text{Fun}(\text{Tw}(F_i, X)) \]

is an equivalence, and the second shows that the full subcategory \(Q_{\colim_{i \in I} F_i}(X)\) of the left corresponds to the full subcategory \(\lim_{i \in I} Q_{F_i}(X)\) on the right.

Now it formally follows that

\[ Q_K(X) \cong \lim_{\Delta \text{N}(K)} Q_k(X) \]

in \(\text{Cat}\) since the functor \(\Delta : \text{N}(K) \to \text{sAn} \xrightarrow{\text{ascat}} \text{Cat}\) tautologically satisfies (*).

To finally see that \(Q_- (X) : \text{Cat}^{\text{op}} \to \text{Cat}\) takes arbitrary colimits to limits we note that this can be tested after precomposition with \(\text{ascat}^{\text{op}} : \text{sAn}^{\text{op}} \to \text{Cat}^{\text{op}}\).

We claim that \(Q_{\text{ascat}(-)}(X) : \text{sAn}^{\text{op}} \to \text{Cat}\) indeed agrees with the limit preserving
extension $F$ of $Q(X) : \Delta^{op} \to \text{Cat}$. By Lemma 3.7, $Q(X)$ is a complete Segal object of $\text{Cat}$, so (essentially by definition of the latter) the limit extension $F$ is invariant under completion, and the same statement evidently holds for $Q_{\text{asscat}(-)}(X)$. But for a complete Segal space $T$ we compute

$$F(T) \simeq \lim_{\Delta^k \to T} Q_k(X) \simeq \lim_{\Delta^k \to N(\text{asscat}(T))} Q_k(X) \simeq Q_{\text{asscat}(T)}(X)$$

using the equivalence above. \qed

3.9. **Corollary.** The functor $\iota Q_{\text{asscat}(-)} : s\text{An}^{op} \to \text{An}$ is canonically equivalent to $\text{Hom}_{s\text{An}}(-, \iota Q(X))$ for any adequate triple $(X, X_{\text{in}}, X_{\text{eg}})$. In particular, we find an equivalence

$$\text{Hom}_{\text{AdTrip}}(\text{Tw}(K), X) \simeq \text{Hom}_{\text{Cat}}(K, \text{Span}(X)),$$

natural in both $K$ and $X$.

Here $\text{Tw}(K)$ is equipped with the adequate structure explained right before Lemma 3.8.

*Proof.* For the first statement note only that both functors are limit preserving extensions of $Q(X)$, and for the second simply insert the nerve of $K$ into the first. \qed

Before we move on to analyse (co)cartesian fibrations among span categories, let us quickly record:

3.10. **Lemma.** There is a canonical equivalence

$$\text{Span}(X)^{op} \simeq \text{Span}(X^{\text{rev}})$$

natural in the adequate triple $(X, X_{\text{in}}, X_{\text{eg}})$, where $X^{\text{rev}}$ denotes the new adequate triple $(X, X^{\text{eg}}, X_{\text{in}})$.

*Proof.* Generally $\text{asscat}(T)^{op} \simeq \text{asscat}(T^{\text{rev}})$, where the superscript denotes the reversal of a simplicial object. The reversal of $\text{Hom}_{\text{AdTrip}}(\text{Tw}([-], X))$ is obtained by reversing the cosimplicial object $\text{Tw}([\cdot])$. But $\text{Tw}([\cdot])^{\text{rev}} \simeq \text{Tw}([\cdot])$ induced by the unique equivalence $[n]^{op} \simeq [n]$, and this swaps ingressive and egressive morphisms. \qed

To understand (co)cartesian edges in span categories, as we must, we again take our cue from [CDH+20], more precisely from the proof of the additivity theorem in Grothendieck-Witt and K-theory, which essentially boils down to the same analysis in the context of stable categories. To state the result we need:

3.11. **Notation.** Given a morphism $p : X \to Y$ of adequate triples we write $p_{\text{in}}$ for $p$ restricted to $X_{\text{in}} \to Y_{\text{in}}$ and $p^{\text{eg}}$ for $p$ restricted to $X^{\text{eg}} \to Y^{\text{eg}}$.

3.12. **Theorem** (Barwick). Let $(X, X_{\text{in}}, X_{\text{eg}})$ and $(Y, Y_{\text{in}}, Y_{\text{eg}})$ be two adequate triples and $p : X \to Y$ be a morphism in $\text{AdTrip}$. Let, furthermore, $f$ be an edge in $Y_{\text{in}}$ such that the following conditions hold:

1. Every pullback of $f$ along an egressive edge has a lift in $X_{\text{in}}$ which is simultaneously $p$-cocartesian and $p_{\text{in}}$-cocartesian with arbitrarily given source.

2. Consider any commutative square $\sigma$ in $X$.
such that $p(\sigma)$ is an ambigressive pullback in $Y$, the edge $g'$ is ingressive, the morphism $\phi$ is egressive, and the morphism $g$ is a $p$-cocartesian and ingressive lift of $f$. Then $g'$ is $p$-cocartesian if and only if the square is an ambigressive pullback.

Then an edge $\sigma: x \to y$ of $\text{Span}(X)$ represented by the span

\[
x \leftarrow w \rightarrow z
\]

with $\psi$ covering $f$ is $\text{Span}(p)$-cocartesian if $\phi$ is $p^{\text{eg}}$-cartesian and $\psi$ is $p$-cocartesian as indicated.

Here we have mildly abused the scheme of Notation 2.3 by marking the $p^{\text{eg}}$-cartesian edge with a circle. This will, however, fit well with the examples we consider in the next section.

Proof. Unwinding definitions (this is where the Segal space perspective pays off) we have to show that for any span $x \leftarrow w \to z$ the solid diagram (ignore the numbers for a moment)

\[
\begin{array}{ccc}
w & \xrightarrow{w} & \bullet \\
\downarrow & & \downarrow \\
x & \xleftarrow{\phi} & y \\
\end{array}
\]

admits an essentially unique dashed filling lying over a given entirely solid diagram in $\text{Span}(Y)$, such that all left pointing arrows are egressive and all right pointing arrows ingressive, and the top square is cartesian. We then first observe that the second condition on $f$ (the image of $\psi$ in $Y$) implies that the assertion that the square is cartesian and $\bullet \to y$ is egressive is equivalent to the assertion that the map $w \to \bullet$ is $p$-cocartesian.

We can then fill the diagram step by step, as indicated by the numbers in the above diagram, essentially uniquely each time:

(1) There exists a unique egressive filler because $\phi$ is $p^{\text{eg}}$-cartesian.

(2) The first assumption on $f$ provides a $p$-cocartesian edge, that is also $p_{\text{m}}$-cocartesian, which is automatically unique.

(3) There exists a unique filler making the middle square commute because $w \to \bullet$ is $p$-cocartesian.

(4) There is a unique ingressive filler because $w \to \bullet$ is $p_{\text{m}}$-cocartesian.

\[\square\]

3.13. Remark. We feel obliged to point out two errors in the statement of the above result in [Ba17, 12.2 Theorem]:

\[\square\]
Barwick requires that the edge $\phi$ be $p$-cartesian and not $p^\pm$-cartesian. Our proof below hopefully makes it transparent why this is not enough. For an explicit counterexample consider $X = Y = [1]^2$ with $p$ the identity, where we equip the source with the triple structure with the horizontal maps (and the identities) ingressive and everything but the horizontal maps egressive and the target with the same ingressive but all maps egressive.

As he is working at the point-set level, Barwick has to show that $\text{Span}(p)$ is an inner fibration. To this end he assumes that $p$ is an inner fibration, but in fact requires the stronger assumption that $p$ is an isofibration, as can be seen by lifting a 2-simplex in $\text{Span}(Y)$ of the form

$$
\begin{aligned}
f &\quad \downarrow \quad \downarrow w \\
x &\quad \downarrow \quad \downarrow g \\
x &\quad \downarrow \quad \downarrow x
\end{aligned}
$$

where $f$ and $g$ are inverse equivalences and all other maps identities.

In order to carry out iterated span constructions in $\text{BGN18}$ one therefore needs to check that if $p$ is an isofibration, then so is $\text{Span}(p)$. This is indeed the case, but requires the characterisation of equivalences in span categories as spans consisting of equivalences (which we obtain as part of the completeness assertion for $\text{ιQ}(X)$). Note also that these latter problems are entirely circumvented by our more invariant approach.

### 4. Dualisation of orthofibrations

In this section we will apply the span functor to construct dualisations of the various fibrations considered in Section 2. The basic observation that makes this possible is:

**4.1. Proposition.** Let $A$ and $B$ be two $\infty$-categories. Then the triple

$$
A \times B^\perp = (A \times B, A \times \text{ι}B, \text{ι}A \times B)
$$

is adequate and

$$
\text{Span}(A \times B^\perp) \cong A \times B^{\text{op}}.
$$

naturally in $A, B \in \text{Cat}$.

**4.2. Remark.** Via Corollary 3.9, this result (for $B = \ast$, but this implies the general case) is in fact equivalent to the assertion that

$$
(\text{Tw}(I))[w^{-1}] \simeq I
$$

via the evaluation at the source for general $I \in \text{Cat}$, where $w$ denotes those maps whose source component is an equivalence. The proof above reduces this to the case $I = [n]$, where it is even a Bousfield localisation (which it is not in general).

**Proof of Proposition 4.1.** For the first claim apply Proposition 3.3 to the cartesian fibration $A \times B \rightarrow B$, where $B$ is endowed with the trivial structure of an adequate triple $(B, \text{ι}B, B)$.

For the second part note that $A \times B^\perp$ decomposes as a product of the triples $(A, A, \text{ι}A)$ and $(B, \text{ι}B, B)$ in the $\infty$-category $\text{AdTrip}$. Furthermore, as a right adjoint the span construction clearly preserves limits, so by Lemma 3.10 it is enough to show
that \( \text{Span}(A, A, \iota A) \simeq A \). Now note that the space of functors from an adequate triple \((X, X_{in}, X^{es})\) into \((A, A, \iota A)\) is equivalent to the space of functors \(X \to A\) which invert the edges in \(X^{es}\).

So we obtain a natural equivalence

\[
N_n \text{Span}(A, A, \iota A) \simeq \text{Hom}_{\text{Cat}}((\text{Tw}([n])[[\text{Tw}([n])^{es}])^{-1}], A).
\]

Recall that the egressives in \(\text{Tw}([n])\) are given by maps \((i \leq j) \to (i \leq j')\) and note that the inclusion of \([n]\) into \(\text{Tw}([n])\) as the \(n\)-simplex \((0 \leq n) \to (0 \leq n - 1) \to \cdots \to (0 \leq 0)\)

witnesses \([n]\) as a right Bousfield localisation of \(\text{Tw}([n])\) at \(\text{Tw}([n])^{es}\). Consequently, \(N \text{Span}(A, A, \iota(A))\) is equivalent to the Rezk nerve of \(A\), finishing the proof. \(\square\)

4.3. Definition. Let us write \(R\text{Cart}(A, B) \subseteq \text{Cat}/A \times B\) for subcategory of functors \(p: X \to A \times B\) which admit \(p\)-cartesian lifts over \(B\), and maps between them that preserve such \(p\)-cartesian lifts over \(B\). Likewise, we will write \(R\text{Cocart}(A, B)\) for the subcategory of functors admitting \(p\)-cocartesian lifts over \(B\).

We can now build a functor

\[
F: R\text{Cart}/(A, B) \to \text{AdTrip}/(A \times B^\perp).
\]

This functor sends a map \(p: X \to A \times B\) which admit \(p\)-cartesian lifts over \(B\) to the functor of adequate triples

\[
p: (X, p^{-1}(A \times \iota B), X^\dagger) \to A \times B^\perp
\]

from Proposition 3.3. Let \(\text{Dual}_c\) denote the composition

\[
R\text{Cart}(A, B) \xrightarrow{F} \text{AdTrip}/A \times B^\perp \xrightarrow{\text{Span}} \text{Cat}/\text{Span}(A \times B^\perp) \xrightarrow{\sim} \text{Cat}/(A \times B^{op}).
\]

4.4. Observation. The triples \(F(X) = (X, p^{-1}(A \times \iota B), X^\dagger)\) and \(A \times B^\perp\) all have the special property that every ambigressive square in them is automatically cartesian. In particular this implies that mapping spaces into these adequate triples in \(\text{AdTrip}\) agrees with the mapping spaces in \(\text{Fun}(\Lambda^2_2, \text{Cat})\).

4.5. Lemma. The functors constructed above canonically assemble into a natural transformation

\[
\text{Dual}_c: R\text{Cart} \to \text{Cat}/(- \times -^{op})
\]

of functors \(\text{Cat}^{op} \times \text{Cat}^{op} \to \text{CAT}\).

Proof. This is clear for the first and last parts of the defining composition. On the other hand it is clear that \(\text{Span}: \text{AdTrip}/X \to \text{Cat}/\text{Span}(X)\) is compatible with the covariant functoriality in \(X\) via postcomposition. To obtain a natural transformation for the contravariant functoriality it thus remains to check that the Beck-Chevalley transformation associated to the square

\[
\begin{array}{ccc}
\text{AdTrip}/X & \xrightarrow{f_*} & \text{AdTrip}/Y \\
\downarrow & & \downarrow \\
\text{Cat}/\text{Span}(X) & \xrightarrow{f_*} & \text{Cat}/\text{Span}(Y)
\end{array}
\]

is an equivalence for every \(f: X \to Y\). But this follows from Corollary 3.9 which implies that \(\text{Span}: \text{AdTrip} \to \text{Cat}\) preserves pullbacks. \(\square\)
4.6. **Lemma.** Let \( p : X \to A \times B \) be a functor admitting \( p \)-cartesian lifts over \( B \). Then for every \((a, b) \in A \times B\),

\[
p^{-1}(a, b) \cong \text{Dual}^{cc}(p)^{-1}(a, b).
\]

**Proof.** Note that the span construction commutes with pullbacks and \( p \)-cartesian edges that happen to lie a fibre of \( p \) are equivalences. Therefore, writing \( X_{a,b} \) for \( p^{-1}(a, b) \), we find that \( \text{Dual}^{cc}(p)^{-1}(a, b) \) is naturally equivalent to \( \text{Span}(X_{a,b}, X_{a,b}, \iota_{X_{a,b}}) \). The claim is now a special case of Proposition 4.1. \( \square \)

4.7. **Theorem.** The functor \( \text{Dual}^{cc} : \text{RCart}(A, B) \to \text{Cat}/(A \times B^{op}) \) has the following properties:

1. \( \text{Dual}^{cc} \) takes values in \( \text{RCocart}(A, B^{op}) \).
2. \( \text{Dual}^{cc} \) restricted to \( \text{Ortho}^{loc}(A, B) \) takes values in \( \text{Gray}(A, B^{op}) \).
3. \( \text{Dual}^{cc} \) restricted to \( \text{Ortho}(A, B) \) takes values in \( \text{Cocart}(A, B^{op}) \).
4. \( \text{Dual}^{cc} \) restricted to \( \text{Gray}^{op}(A, B) \) takes value in \( \text{Ortho}^{loc}(B^{op}, A) \).
5. \( \text{Dual}^{cc} \) restricted to \( \text{Cart}(A \times B) \) takes values in \( \text{Ortho}(B^{op}, A) \).

**Proof.** We will prove in each case that dualisation sends the objects of each category to objects of the other. In the process of proving this we will obtain explicitly what the (co)cartesian edges are in each case, in terms of spans of (co)cartesian edges. Therefore the dualisation functor will trivially send morphisms of each subcategory to morphisms in the other.

To prove (1) we have to show that edges of \( \iota A \times B^{op} \) have \( \text{Span}(p) \)-cocartesian lifts. Note that under the equivalence of Proposition 4.1, \( \iota A \times B^{op} \) corresponds to the span category \( \text{Span}(A \times B, \iota(A \times B), \iota A \times B) \) within \( \text{Span}(A \times B, A \times B^{op}, \iota A \times B) \). Let us therefore pick an arrow in the former span category of the form

\[
\begin{array}{ccc}
(a_2', b') & \sim & (a''_2, b'') \\
(a_1, b) & \leftarrow & \rightarrow
\end{array}
\]

where the left morphism lies in \( \iota A \times B \) and the right morphism is an equivalence, together with an element \( x \in X \) lying over \((a, b)\). Then a lift of the above span to a morphism in \( \text{Span}(F(X)) \) is given by \( x \leftarrow x' \xrightarrow{\sim} x'' \) where the left arrow is \( p \)-cartesian and the right arrow is an equivalence. To see that this is \( \text{Span}(p) \)-cocartesian, we want to invoke Theorem 3.12 (note that the left arrow is also \( p^{\#} \)-cartesian, as \( p^{\#} \) is a pullback of \( p \) in the present situation, and the right is \( p \)-cocartesian).

So we are left to show that every edge \( f \in \iota(A \times B) \) satisfies the conditions of Theorem 3.12. The fact that equivalences trivially have (co)cartesian lifts gives condition (1).

Next consider a square \( \sigma \) of the form

\[
\begin{array}{ccc}
x' & \xrightarrow{g'} & z' \\
\phi \downarrow & & \downarrow \psi \\
x & \xrightarrow{g} & z
\end{array}
\]

in \( X \) such that \( p(\sigma) \) is an ambigressive pullback in \( A \times B^\perp \), the edge \( g' \) is ingressive, the morphism \( \phi \) is egressive, and the morphism \( g \) is a \( p \)-cocartesian lift of the
equivalence \( f \). Then \( g \) is an equivalence and \( g' \) is an edge over an equivalence. So condition (2) reduces to the claim that the square is an ambigressive pullback if and only if \( g' \) is an equivalence, which is obvious.

Assertion (2) is now trivial: Simply note that \( \text{Dual}^{cc}(p)_l = \text{Dual}^{cc}(p)|_{A \times B} \) is naturally equivalent to \( p_l \), so that \( \text{Dual}^{cc}(p)_l \) is a cocartesian fibration as soon as \( p_l \) is. Assertion (4) follows similarly upon investing Proposition 2.9.

The remaining two cases require unwinding the definition of interpolating edges \( \text{Dual}^{cc}(p) \). To this end, we provide the following guide of what we proved above (using the scheme from Notation 2.3):

| In Span\((F(X))\) | In \(X\) |
|-------------------|-----------------|
| (1) \(\text{Dual}^{cc}(p)\)-cocartesian over \(A \times B\) | \(\text{•} \rightarrow \text{•} \leftarrow \sim \rightarrow \text{•}\) |
| (2) \(\text{Dual}^{cc}(p)_l\)-cocartesian | \(\text{•} \rightarrow \text{•} \leftarrow \sim \rightarrow \text{•}\) |
| (4) \(\text{Dual}^{cc}(p)_l\)-cartesian | \(\text{•} \rightarrow \text{•} \leftarrow \sim \rightarrow \text{•}\) |

With this at our disposal, let us consider assertion (3). Let \( p \) be an orthofibration, and consider an interpolating diagram

in \(\text{Span}(F(X))\), the source of \(\text{Dual}^{cc}(p)\), where we use the colours to indicate compositions correspondingly here and in the table below. The process of forming the interpolating edge unwinds to the following:

| In \(\text{Span}(F(X))\) | In \(X\) |
|-------------------|-----------------|
| | |
All left pointing arrows are $p$-cartesian, while right pointing arrows are $p_l$-cocartesian. In the first step, one simply computes composition by forming a pullback of equivalences. In the second step, when computing the pullback square in the middle we use the assumption that the interpolating edges of $p$ are equivalences: Recall first that a pullback in $X$ with one leg cartesian can be computed by simply picking the opposite edge cartesian as well (i.e. the left pointing grey edge above). The interpolating edges of $p$ being equivalences then implies that the right-pointing gray arrow, and thus also the right pointing blue arrow, is $p_l$-cocartesian. Finally, the uniqueness of cocartesian edges implies that the dashed arrows in the third picture are equivalences. This shows that, given an orthocartesian fibration $p$, the interpolating edges of $\text{Dual}^{cc}(p)$ are equivalences and therefore by Proposition 2.6 that $\text{Dual}^{cc}(p)$ is a cocartesian fibration.

Finally for (5), let $p$ be a cartesian fibration, and let

be an interpolating diagram of $\text{Dual}^{cc}(p)$. Then the process of forming the interpolating edge proceeds as follows:

Here, the uniqueness of cartesian lifts suffices to imply all calculations, so we conclude that the interpolating edges of $\text{Dual}^{cc}(p)$ are equivalences as desired. □

4.8. Example. As a simple example consider the dual of the bifibration $(t, s) : \text{Ar}(X) \to X \times X$ from [Lu09a, Corollary 2.4.7.11]. Unraveling the definitions, we find that $\mathbb{N}_q \text{Dual}^{cc}(\text{Ar}(X))$ is given by the anima of diagrams $\phi : \text{Tw}([n]) \times [1] \to X$ that take edges in $\text{Tw}([n])^{e\times \{0\}}$ and $\text{Tw}([n])^{e\times \{1\}}$ to equivalences. We now claim
that the localisation of $\text{Tw}([n]) \times [1]$ at these subcategories is naturally equivalent to $[n] \star [n]^{\text{op}}$ via the map

$$\text{Tw}([n]) \times [1] \longrightarrow [n] \star [n]^{\text{op}}, \quad ((i \leq j), \epsilon) \mapsto \begin{cases} i & \epsilon = 0 \\ j & \epsilon = 1 \end{cases},$$

where we have used subscripts to indicate join factors. Restriction along this map then yields a natural equivalence

$$(\text{Tw}(X) \longrightarrow X \times X^{\text{op}}) \longrightarrow \text{Dual}^{\text{cc}}(\text{Ar}(X) \longrightarrow X^2).$$

To see the claim we note that both restrictions $\text{Tw}([n]) \times \{\epsilon\} \rightarrow X$ of a diagram $\phi$ as considered above take all squares in $\text{Tw}([n])$ to pushout squares in $X$ (namely ones in which two opposite edges are equivalences). Using the pointwise formula for left Kan extensions, one then readily checks that $\phi: \text{Tw}([n]) \times [1] \rightarrow X$ is then left Kan extended from the subposet $J_n \times [1]$, where $J_n$ is the arch along the top of $\text{Tw}([n])$ consisting of all $(i \leq j)$ with $i = 0$ or $j = n$. It follows that the inclusion $J_n \times [1] \rightarrow \text{Tw}([n]) \times [1]$ induces an equivalence upon localisation.

Now note that $J_n$ consists of two copies of $[n]$ glued along the initial vertex. The claim then follows from the fact that the localisation of $[n] \times [1]$ at $[n] \times \{1\}$ is given by $[n + 1]$ and likewise for the localisation at $[n] \times \{0\}$; this realises the localisation of $J_n$ as the pushout of $[n + 1]$ and $[1 + n]$ along $[1]$, embedded into the former as the terminal segment, and into the latter as the initial one. We leave it to the reader to check that the localisation map $J_n \rightarrow [n + 1] \cup [1 + n] \cong [n] \star [n]^{\text{op}}$ just described is indeed the restriction of $\phi$.

Our next goal will be to prove that $\text{Dual}^{\text{cc}}$ is an equivalence. To see this, note that there is an obvious analogue of the dualisation functor, which we will denote $\text{Dual}^{\text{ct}}$, from objects that are cocartesian over $B$ to objects that are cartesian over $B$, defined by

$$\text{Dual}^{\text{ct}} = \text{op} \circ \text{Dual}^{\text{cc}} \circ \text{op}.$$

4.9. **Theorem.** The functor $\text{Dual}^{\text{cc}}: \text{RCart}(A, B) \rightarrow \text{RCocart}(A, B^{\text{op}})$ is an equivalence of categories with inverse $\text{Dual}^{\text{ct}}$.

4.10. **Corollary.** The functor $\text{Dual}^{\text{cc}}$ restricts to give an equivalence between the following categories:

- Ortho$^{\text{loc}}(A, B)$ and Gray$(A, B^{\text{op}})$
- Ortho$(A, B)$ and Cocart$(A \times B^{\text{op}})$

4.11. **Remark.** It follows from Proposition 2.4 that $\text{RCart}(A, B)$ is equivalent to $\text{Cart}(B)/\text{pr}_B$, where $\text{pr}_B: A \times B \rightarrow B$. This can be used to reduce Theorem 4.9 to the case where $A = \ast$, which is then handled by [BGN18, Theorem 1.4] (or without an explicit description of the equivalence by simply applying cartesian straightening and then cocartesian unstraightening, which is the method employed by Haugseng in [Ha20]).

We have instead opted to rerun the proof of [BGN18] in the generality of 4.9, in particular correcting a few oversights, see e.g. Remarks 3.13 and 5.6 (2). Furthermore, our use of Segal spaces simplifies several of the more technical steps in [BGN18, Section 4] considerably.
To begin, we note that it is enough to construct a natural equivalence from $\text{Dual}^\text{ct} \circ \text{Dual}^\text{cc}$ to the identity. Having done this, we can construct a natural equivalence from $\text{Dual}^\text{cc} \circ \text{Dual}^\text{ct}$ to the identity as follows:

\[
\text{Dual}^\text{cc} \circ \text{Dual}^\text{ct} \simeq \text{op} \circ \text{op} \circ \text{Dual}^\text{cc} \circ \text{Dual}^\text{ct} \cong \text{op} \circ \text{Dual}^\text{ct} \circ \text{op} \cong \text{op} \circ \text{op} \cong \text{id}.
\]

Let us therefore construct the natural equivalence from $\text{Dual}^\text{ct} \circ \text{Dual}^\text{cc}$ to the identity. The first thing to do is to describe the composite $\text{op} \circ \text{Dual}^\text{cc} \circ \text{op} \circ \text{Dual}^\text{cc}$ more explicitly.

4.12. Observation. Let us write $\iota^{\bullet}_{Q^\text{c}} : \text{sAn} \to \text{sAn}$ for the functor sending a simplicial anima $S$ to the simplicial anima given by

\[
\iota^{\bullet}_{Q^\text{c}}(S) = \text{Hom}_{\text{sAn}}(\text{N}(\text{Tw}([n])), S).
\]

For any $p : X \to A \times B$ in $R\text{Cart}(A, B)$, it follows from the construction of the map $\text{Dual}^\text{cc}(p) : \text{Dual}^\text{cc}(X) \to A \times B^{\text{op}}$ that there is a natural transformation of functors $R\text{Cart}(A, B) \to \text{sAn}$

\[
\text{N}(\text{Dual}^\text{cc}(X)) \to \iota^{\bullet}_{Q^\text{c}}(\text{N}(X)).
\]

In every simplicial degree, this is given by an inclusion of path components of the space of functors $\text{Tw}([n]) \to X$ (on those functors $\text{Tw}([n]) \to X$ that are maps of adequate triples). Applying this reasoning twice, one sees that for $p : X \to A \times B$ in $R\text{Cart}(A, B)$, there is a natural map of simplicial anima

\[
\text{N}(\text{Dual}^\text{cc}(\text{Dual}^\text{cc}(X)^{\text{op}})^{\text{op}}) \to \iota^{\bullet}_{Q^\text{c}}(\text{N}(\text{Dual}^\text{cc}(X)^{\text{op}}))^{\text{op}} \to \iota^{\bullet}_{Q^\text{c}}(\iota^{\bullet}_{Q^\text{c}}(\text{N}(X)^{\text{op}}))^{\text{op}}
\]

which is an inclusion of path components in each degree. Here we have used $(-)^{\text{op}}$ to denote the opposite simplicial anima, obtained by composing with $(-)^{\text{op}} : \Delta \to \Delta$.

To unravel the target of the above map, let us denote by

\[
\text{Tw}^{(2)}(K) = \text{Tw}(\text{Tw}(K))
\]

the twofold iterated twisted arrow category of an $\infty$-category $K$. We then obtain a natural equivalence

\[
\iota^{\bullet}_{Q^\text{c}}(\iota^{\bullet}_{Q^\text{c}}(\text{N}(X)^{\text{op}}))^{\text{op}} \simeq \text{Hom}_{\text{sAn}}(\text{N}(\text{Tw}([n])^{\text{op}}), \iota^{\bullet}_{Q^\text{c}}(\text{N}(X)^{\text{op}}))^{\text{op}} \\
\simeq \lim_{[m] \in \Delta/\text{N}(\text{Tw}([n])^{\text{op}})} \text{Hom}_{\text{sAn}}(\text{N}(\text{Tw}([m])), \text{N}(X)) \\
\simeq \text{Hom}_{\text{Cat}}(\text{Tw}^{(2)}([n]), \text{N}(X)).
\]

Here the third line uses that $\text{Tw}(\mathcal{C}^{\text{op}}) \simeq \text{Tw}(\mathcal{C})$ twice together with the fact that $\text{N}(\text{Tw}(-))$ preserves those colimits of $\infty$-categories that are preserved by the nerve functor (see the proof of Proposition 3.8). Summarising, we see that there is a map of simplicial anima, depending functorially on $p \in R\text{Cart}(A, B)$, which is a degreewise inclusion of path components

\[
\text{N}(\text{Dual}^\text{cc}(\text{Dual}^\text{cc}(X)^{\text{op}})^{\text{op}}) \to \text{Hom}_{\text{Cat}}(\text{Tw}^{(2)}(-), X).
\]

To identify the essential image of (4.13), let us make the following construction:
4.14. **Construction.** Note that $\text{Tw}^{(2)}([n])$ is equivalent to the poset whose objects are tuples $abcd$ with $0 \leq a \leq b \leq c \leq d \leq n$, corresponding to a map $(a \leq d) \rightarrow (b \leq c)$ in $\text{Tw}([n])$. The partial order is then given by $abcd \leq ab'b'c'd'$ when $a \leq a' \leq b' \leq b \leq c \leq c' \leq d' \leq d$. We define the following four wide subcategories of $\text{Tw}^{(2)}([n])$:

1. $\text{Tw}^{(2)}([n])_1$ is the subcategory spanned by the edges $abcd \rightarrow ab'cd$.
2. $\text{Tw}^{(2)}([n])_2$ is the subcategory spanned by the edges $abcd \rightarrow ab'cd$.
3. $\text{Tw}^{(2)}([n])_3$ is the subcategory spanned by the edges $abcd \rightarrow abe'cd$.
4. $\text{Tw}^{(2)}([n])_4$ is the subcategory spanned by the edges $abcd \rightarrow abcd'$.

4.15. **Proposition.** Let $p: X \rightarrow A \times B$ be in $\text{RCart}(A, B)$ and let $[n] \in \Delta$. Then the natural transformation (4.13) identifies the domain with those path components in $\text{Hom}_{\text{Cat}}(\text{Tw}^{(2)}([n]), X)$ consisting of maps $f: \text{Tw}^{(2)}([n]) \rightarrow X$ such that

$$
\begin{align*}
\text{Tw}^{(2)}([n])_1 & \xrightarrow{f} X^\top & \text{Tw}^{(2)}([n])_2 & \xleftarrow{f} \iota X \\
\text{Tw}^{(2)}([n])_3 & \xrightarrow{f} p^{-1}(A \times \iota B) & \text{Tw}^{(2)}([n])_4 & \xleftarrow{f} \iota X
\end{align*}
$$

where $X^\top$ is spanned by the $p$-cartesian lifts of arrows in $\iota A \times B$.

**Proof.** Recall from Lemma 3.10 that $\text{Dual}^{cc}(X)^{op} = \text{Dual}^{cc}(X^{rev})$, where we abusively write $\text{Dual}^{cc}(X^{rev}) = \text{Span}(F(X)^{rev})$ for the span category associated to the adequate triple with ingressives and egressives reversed. A functor

$$
f: [n] \rightarrow \text{Dual}^{cc}(\text{Dual}^{cc}(X^{rev})^{rev})
$$

then corresponds by adjunction to a map $\text{Tw}([n]) \rightarrow \text{Dual}^{cc}(X^{rev})^{rev}$ of adequate triples, i.e. a map of $\infty$-categories $f': \text{Tw}([n]) \rightarrow \text{Dual}^{cc}(X^{rev})$ such that

$$
f'(\text{Tw}([n])_{in}) \subseteq \text{Dual}^{cc}(X^{rev})^{eg} \quad \text{and} \quad f'(\text{Tw}([n])^{eg}) \subseteq \text{Dual}^{cc}(X^{rev})_{in}.
$$

By Corollary 3.9, the map of $\infty$-categories underlying $f'$ corresponds itself to a map $\text{Tw}(\text{Tw}([n])) \rightarrow X^{rev}$ of adequate triples, i.e. a map $f''$: $\text{Tw}(\text{Tw}([n])) \rightarrow X$ such that

$$
f''(\text{Tw}(\text{Tw}([n]))_{in}) \subseteq X^{eg} \quad \text{and} \quad f''(\text{Tw}(\text{Tw}([n]))^{eg}) \subseteq X_{in}.
$$

Let us now unravel these conditions using the description of $\text{Tw}^{(2)}([n])$ from Construction 4.14. Note that $\text{Tw}(\text{Tw}([n])_{in})$ consists of maps of tuples of the form $abcd \leq ab'cd'$, while $\text{Tw}(\text{Tw}([n]))^{eg}$ consists of maps $abcd \leq ab'e'cd$. On the other hand, $\text{Tw}(\text{Tw}([n])_{in})$ consists of maps of the form $abcc \leq ab'bc$ and $\text{Tw}(\text{Tw}([n]))^{eg}$ consists of maps $aacd \leq aa'cd'$.

Furthermore, $\text{Dual}^{cc}(X^{rev})_{in}$ is the subcategory consisting of spans of the form

$$
\begin{array}{ccc}
\bigtriangleup & \xrightarrow{\sim} & \bigtriangledown \\
\bigtriangleup & \xrightarrow{\sim} & \bigtriangledown
\end{array}
$$

where the right map lives over $A \times \iota B$. Similarly, $\text{Dual}^{cc}(X^{rev})^{eg}$ is the subcategory whose morphisms are spans of the form

$$
\begin{array}{ccc}
\bigtriangleup & \xleftarrow{\sim} & \bigtriangledown \\
\bigtriangleup & \xleftarrow{\sim} & \bigtriangledown
\end{array}
$$

(see the table in the proof of Theorem 4.7). Combining all this, the above conditions translate as follows:

1. Every $abcd \leq ab'cd'$ is mapped to $X^\top$.
2. Every $abcd \leq ab'e'cd$ is mapped to $p^{-1}(A \times \iota B)$.
3. Every $abcc \leq ab'bc$ is sent to $X^\top$ and every $abcc \leq ab'bc$ is sent to an equivalence.
(4) Every $aacd \leq aac'd$ is sent to $p^{-1}(A \times \iota B)$ and every $aacd \leq aac'd'$ is sent to an equivalence.

Note that these conditions are certainly implied by the ones from the proposition. Conversely, given these conditions, the ones from the statement follow: for example, the map $abcd \leq abcd'$ fits into a square

$$
\begin{array}{ccc}
abcd & \rightarrow & abcd' \\
\downarrow & & \downarrow \\
abcd & \rightarrow & abcd'
\end{array}
$$

The top horizontal arrow is mapped to an equivalence in $X$ by (4), the vertical arrows are sent to maps over $A \times \iota B$ by (2). In particular, the bottom arrow maps to $A \times \iota B$, but it is also a $p$-cartesian lift of an arrow in $\iota A \times B$ by (1), and hence an equivalence. □

Using this, the functor $\text{Dual}^\text{ct} \circ \text{Dual}^\text{cc}$ can be equivalently described as follows. Let $T_4$ be the $\infty$-category given by gluing 4 edges at their targets. We will call objects of $\text{Fun}(T_4, \text{Cat})$ quintuples of $\infty$-categories. Construction 4.14 defines a cosimplicial object $T_{w(2)} : \Delta \rightarrow \text{Fun}(T_4, \text{Cat})$. This cosimplicial object $T_{w(2)}$ induces a functor

$$
\text{Span}(2) : \text{Fun}(T_4, \text{Cat}) \rightarrow \text{Fun}(T_4, \text{Cat})/A \times B^\top.
$$

We also define a functor

$$
F(2) : \text{RCart}(A, B) \rightarrow \text{Fun}(T_4, \text{Cat})/(A \times B)^\top,
$$

where $(A \times B)^\top$ is the quintuple consisting of $A \times B$ with the subcategories

$(A \times B)_1 = \iota A \times B \quad (A \times B)_3 = A \times \iota B \quad (A \times B)_2 = (A \times B)_4 = \iota(A \times B)$.

The functor $F(2)$ sends $p : X \rightarrow A \times B$ to the map $F(2)(p) : F(2)(X) \rightarrow A \times B^\top$ where $F(2)(X)$ has subcategories

$F(2)(X)_1 = X^\top \quad F(2)(X)_3 = p^{-1}(A \times \iota B) \quad F(2)(X)_2 = F(2)(X)_4 = \iota(X)$.

Proposition 4.15 can then be reformulated as follows:

4.16. Corollary. For every $p : X \rightarrow A \times B$ in $\text{RCart}(A, B)$, there are natural degreewise inclusions of path components between simplicial anima

$$
\text{N}(\text{Dual}^\text{ct}(\text{Dual}^\text{cc}(X))) \xrightarrow{(4.13)} \text{Hom}_{\text{Cat}}(\text{Tw}(2)(-) , X) \xrightarrow{\Sigma(Tw(2))} \Sigma(F(2)(X))
$$

whose essential images coincide. In particular, $\Sigma(Tw(2))(F(2)X)$ is a complete Segal space and there is a natural equivalence of $\infty$-categories

$$
\text{Span}(2)(F(2)(X)) \simeq \text{Dual}^\text{ct}(\text{Dual}^\text{cc}(X)).
$$

Applying this to $\text{id}_{A \times B}$, we obtain that $\text{Span}(2)(F(2)(A \times B)) \simeq A \times B$. Therefore we can identify the functor $\text{Dual}^\text{ct} \circ \text{Dual}^\text{cc}$ with the composition:

$$
D(2) : \text{RCart}/(A, B) \xrightarrow{F(2)} \text{Fun}(T_4, \text{Cat})/(A \times B)^\top \xrightarrow{\text{Span}(2)} \text{Cat}/(A \times B).
$$
To prove Theorem 4.9, it will therefore suffice to construct a natural equivalence from $D^{(2)}$ to the identity. This equivalence will arise naturally as a zig-zag of natural equivalences, where the intermediate object is constructed as follows. Consider the cosimplicial object $\text{Ar}(-) : \Delta^{op} \to \text{Fun}(\Lambda^2_2, \text{Cat})$, which is given on objects by $[n] \mapsto (\text{Ar}([n]), \text{Ar}([n])_{in}, \text{Ar}([n])^{eg})$.

The $\infty$-category $\text{Ar}([n]) = \text{Fun}([1], [n])$ is equivalent to the poset

$$
\begin{array}{cccccccc}
00 & \to & 01 & \to & \cdots & \to & 0m & \to & 0n \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
11 & \to & \cdots & \to & 1m & \to & 1n & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \to & \cdots & \to & \cdots & & & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
mm & \to & mm & & \cdots & & & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
nm & & nn & & & & & & \\
\end{array}
$$

where $m = n - 1$. Under this identification, we define $\text{Ar}([n])^{eg}$ to be generated by the edges $ij \to ik$ and $\text{Ar}([n])_{in}$ to be generated by the edges $ik \to jk$. As usual, this cosimplicial object induces a functor

$$
\Sigma(\text{Ar}) : \text{AdTrip} \to \text{sAn}.
$$

The precomposition of $\Sigma(\text{Ar})$ with $F : \text{RCart}(A,B) \to \text{AdTrip}$ comes equipped with a natural transformation of simplicial animae

$$
\eta : \Sigma(\text{Ar}) \circ F \to N
$$

where $N$ sends each $p : X \to A \times B$ to the Rezk nerve of $X$. To see this, notice that the degeneracy map $\Delta^1 \to \Delta^0$ induces a natural transformation $s : [n] \to \text{Ar}([n])$, which simply includes the diagonal in the above picture. In turn, this induces a natural transformation $\Sigma(\text{Ar}) \to \Sigma(\Delta)$ as cosimplicial objects of $\text{Fun}(\Lambda^2_2, \text{Cat})$, where we view each simplex as an object in $\text{Fun}(\Lambda^2_2, \text{Cat})$ by taking the subcategory $\iota[n]$ twice. Pre-composing with $F$ we obtain a natural transformation from $\Sigma(\Delta) \circ F$ to $\Sigma(\Delta) \circ F$. Since $\Sigma(\Delta) \circ F$ is equivalent to the Rezk nerve, we then obtain the desired natural transformation $\eta$.

4.17. **Lemma.** The natural transformation $\eta : \Sigma(\text{Ar})(A \times B^\perp) \to N(A \times B)$ is an equivalence.

**Proof.** Note that $A \times B^\perp$ is a product of adequate triples $(A, A, \iota A)$ and $(B, \iota B, B)$, and that $\Sigma(\text{Ar})(X^{rev}) = \Sigma(\text{Ar}^{rev})(X) = \Sigma(\text{Ar})(X)$ (where $X^{rev}$ has its ingressives and egressives interchanged). It is therefore enough to show that $\Sigma(\text{Ar})(A, A, \iota A)$ is equivalent to the Rezk nerve of $A$. The $n$-th space of $\Sigma(\text{Ar})(A, A, \iota A)$ is given by the space of functors $\text{Ar}([n]) \to A$ that invert the edges in $\text{Ar}([n])^{eg}$. This is equivalent to the space of functors from the localisation

$$
\text{Ar}([n])[\text{Ar}([n])^{eg}^{-1}] \to A.
$$

But the inclusion $s : [n] \to \text{Ar}([n])$ given by $i \mapsto ii$ exhibits $[n]$ as the right Bousfield localisation of $\text{Ar}([n])$ at $\text{Ar}([n])^{eg}$. This implies that restriction along $s$, i.e. the
4.18. Proposition. The map $\eta$ is a natural equivalence. In particular, $\Sigma(Ar)(F(p))$ is a complete Segal space.

Proof. We have to show that the induced map

$$\eta_n : \text{Hom}_{\text{Fun}(\Lambda_2^n, \text{Cat})}(\text{Ar}([n]), F(X)) \to \text{Hom}_{\text{Cat}}([n], X)$$

is an equivalence for every $n$ and $p : X \to A \times B$ in $\text{RCart}(A, B)$. We have already seen in the previous lemma that this map is an equivalence for $A \times B \perp$. It therefore suffices to pick a map of adequate triples $\sigma : \text{Ar}([n]) \rightarrow A \times B \perp$ and verify that $\eta_n$ induces an equivalence between the fibres over $\sigma$, resp. $\eta_n(\sigma)$.

The case $n = 0$ is completely trivial. For $n = 1$ the claim amounts to the following: $\sigma$ gives (up to equivalence) a 2-simplex in $A \times B$ of the form $(c, \text{id}) \circ (\text{id}, d)$ and one has to prove that for any arrow $h : x \to y$ covering the composite, there is a contractible space of factorisations $h \simeq g \circ f$ over $\sigma$ such that $g$ is cartesian. This space is contractible or empty, by the requirement that $g$ must be cartesian. It is furthermore non-empty, because $p$ admits cartesian lifts over $B$.

Let us next treat the case $n = 2$, the generalisation to higher $n$ will then be clear. The domain of the map $\eta_2$ consists of diagrams of the shape

$$x_{00} \rightarrow x_{01} \rightarrow x_{02}$$

covering (up to equivalence) a diagram $\sigma$ in $A \times B$ whose vertical arrows lie in $\iota A \times B$ and horizontal arrows lie in $A \times \iota B$, such that all vertical maps are $p$-cartesian in $X$. The map $\eta_2$ sends this to the long diagonal 2-simplex $x_{00} \rightarrow x_{11} \rightarrow x_{22}$.

Applying the result for $n = 1$ twice, we find that the projection from the diagram of the shape

$$x_{00} \rightarrow x_{01}$$

covering (up to equivalence) a diagram $\sigma$ in $A \times B$ whose vertical arrows lie in $\iota A \times B$ and horizontal arrows lie in $A \times \iota B$, such that all vertical maps are $p$-cartesian and (2) the restriction to the diagonal copy of $[n]$ is equivalent to $\sigma$. Notice that this is precisely the space of $p$-right Kan extensions of $\tau$ over $\sigma$ [Lu09a, Definition 4.3.2.2], hence it is either empty or contractible [Lu09a, Definition 4.3.2.2]. It is
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nonempty since all morphisms in $\iota A \times B$ have $p$-cartesian lifts by assumption, using (the opposite of) \[Lu09a, Example 4.3.1.4, Lemma 4.3.2.13\] and the fact that for each $ij \in \text{Ar}([n])$, there is a coinitial functor $\{jj\} \to \{ij\}/\Delta$. □

To relate the complete Segal space $\Sigma(\text{Ar})(F(p))$ to $D(2)(p)$, consider the composition

$$
[k] \times [1] \to [k] \star [k] \to [k] \star [k]^{\text{op}} \star [k] \star [k]^{\text{op}},
$$

where the first map is the unique natural transformation between the two inclusions $[k] \to [k] \star [k]$ of the summands into the join. This induces a natural transformation

$$z: Tw(2)(-) \to \text{Ar}(-)
$$

sending a tuple $abcd$ in $Tw(2)([n])$ to $ac$ in $\text{Ar}([n])$. Note that this sends $Tw(2)([n])_1 \mapsto \text{Ar}([n])^{\text{cs}}$ and $Tw(2)([n])_4 \mapsto \text{Ar}([n])_{\text{in}}$, while $Tw(2)([n])_2$ and $Tw(2)([n])_4$ are both sent to equivalences. Consequently, restriction along $z$ defines a natural map of simplicial animae

$$(4.19) \quad \zeta: \Sigma(\text{Ar})(F(p)) \to \Sigma(Tw(2))(F(2)(p)) = ND(2)(p).$$

for every $p: X \to A \times B$ in $\text{RCart}(A, B)$.

4.20. Lemma. For every $p: X \to A \times B$ in $\text{RCart}(A, B)$, the natural transformation $\zeta$ (4.19) is an equivalence.

Proof. By Corollary 4.16 and Proposition 4.18, both

$$
\Sigma(\text{Ar})(F(p)) \to \Sigma(\text{Ar})(A \times B^\perp) \quad \text{and} \quad ND(2)(p) \to ND(2)(A \times B^\top)
$$

are the Rezk nerves of maps of $\infty$-categories. Both $D(2)(p)$ and the $\infty$-category associated to $\Sigma(\text{Ar})(F(p))$ come with a natural map to (an $\infty$-category equivalent to) $B$, which is a cartesian fibration. We now observe that $\zeta$ preserves the corresponding cartesian arrows. Indeed, fixing some arrow in $B$, a cartesian lift of this arrow in the $\infty$-category associated to $\Sigma(\text{Ar})(F(p))$ is given by a 2-simplex

$$
\begin{array}{ccc}
  x_{00} & \sim & x_{01} \\
  \downarrow^f & \swarrow \\
  x_{11}
\end{array}
$$

with downwards pointing maps that are $p_2$-cartesian (with $p_2: X \to B$ the second component of $p$). This arrow is sent to an arrow in the $\infty$-category associated to $\Sigma(Tw(2))(F(2)(p))$ of the form

$$
\begin{array}{ccc}
  x_{0000} & \sim & x_{001} & \sim & x_{011} \\
  \downarrow^f & \swarrow & \downarrow^f & \swarrow \\
  x_{1111}
\end{array}
$$

where $x_{0000} = x_{00}, x_{001} = x_{01}$ and $x_{1111} = x_{11}$. This is indeed a cartesian arrow of the projection $\text{Span}(2)(F(2)(p)) \to \text{Span}(2)(A \times B^\top) \to B$.

By $\text{Lu09a, Corollary 2.4.4.4}$, it therefore remains to verify that the $\zeta$ induces an equivalence between fibres over $b \in B$. Writing $X_b$ for the fibre of $p_2$ over $b$, this map is given by

$$
\zeta_b: \Sigma(\text{Ar})(X_b, X_b, \iota X_b) \to \text{NSpan}(2)(X_b, \iota X_b, \iota X_b).
$$
The $n$-th space of the target is equal to space of natural transformations in $\text{Fun}(T_4, \text{Cat})$ from $\text{Tw}^{(2)}([n])$ to $(X_b, X_b, \iota X_b, \iota X_b)$. This is equivalent to the space of functors $\text{Tw}^{(2)}([n]) \to X_b$ that invert the edges in $\text{Tw}^{(2)}([n])$, for $i = 1, 2, 4$ (see Construction 4.14).

Likewise, the $n$-th space of $\Sigma(\text{Ar})(X_b, X_b, \iota X_b)$ consists of maps $\text{Ar}([n]) \to X_b$ inverting $\text{Ar}([n])$ for $i = 1, 2, 4$ (see Construction 4.14).

To see this, let $t: \text{Ar}([n]) \to [n]$ be the codomain projection. Note that $t$ admits a fully faithful left adjoint, so that $t$ is a localisation functor which inverts the ingressive maps in $\text{Ar}([n])$ by construction. On the other hand, $t \circ z$ also realises the localisation of $\text{Tw}^{(2)}([n])$ at the maps of type 1, 2 and 4. Indeed, let

$$[n] \simeq C_3 \xrightarrow{i_3} C_2 \xrightarrow{i_2} C_1 \xrightarrow{i_1} \text{Tw}^{(2)}([n])$$

be the full subcategories of tuples $00cn$, $00cd$ and $0bcd$, respectively. Then $i_1$ admits a right adjoint, $i_2$ a left adjoint and $i_3$ a right adjoint. Composing all these right adjoints gives a localisation functor equivalent to $t \circ z$, which precisely inverts the classes 1, 2 and 4.

Proof of Theorem 4.9. The desired natural equivalence is now given by

$$\text{Dual}^{cc} \circ \text{Dual}^{ct} \xrightarrow{\zeta} D^{(2)} \leftarrow \text{asscat} \circ \Sigma(\text{Ar}) \circ F \xrightarrow{\eta} \text{id}.$$ 

Here we use that $\zeta$ and $\eta$, which a priori only define natural equivalences of $\infty$-categories, also induce natural equivalences in $\text{Cat}/(A \times B)$: indeed, they are natural transformations of diagrams in $\text{Cat}_\infty$ where the indexing category has a terminal object, which is sent to $A \times B$. 

5. Straightening of orthofibrations

In the present section we shall explain how to straighten an orthofibration over $A \times B$ into a functor $A \times B^{op} \to \text{Cat}$. We first do so by iterating Lurie’s cartesian and cocartesian straightening constructions, which can be done in two ways, depending on the order in which they are applied. We then show that these constructions agree both with one another and also with the (co)cartesian straightening obtained through the dualisation procedure of the previous section.

To get started recall from 2.4 that for a local orthofibration $p: X \to A \times B$ the composition $p_1: X \to A$ is a cocartesian fibration and $p_2: X \to B$ is a cartesian fibration. Restricting attention to the former we can then consider

$$\xymatrix{ X \ar[rr]^p \ar[rd]_{p_1} & & A \times B \ar[ld]^{p_2} \ar[ld]^{pr_A} \\
& A \times B }$$

as a diagram of cocartesian fibrations, since 2.4 can be interpreted as the statement that $p$ preserves cocartesian edges. Straightening the two fibrations results in a natural transformation $\text{Str}^{cc}(p_1) \Rightarrow \text{const}_B$, which we can interpret as an object of $\text{Fun}(A, \text{Cat}/B)$. If we let $\text{Cart}^{lax}(B)$ denote the full subcategory of $\text{Cat}/B$ spanned by the cartesian fibrations, we in fact find:
5.1. Proposition. The construction just described gives rise to an equivalence

\[ \text{Str}^{cc} : \text{Ortho}^{loc}(A, B) \leftrightarrow \text{Fun}(A, \text{Cart}^{\text{lax}}(B))^{\text{cart}} : \text{Un}^{cc} \]

natural in \( A, B \in \text{Cat} \), where the right hand side is the wide subcategory of \( \text{Fun}(A, \text{Cart}^{\text{lax}}(B)) \) spanned by those natural transformations that pointwise preserve cartesian edges. The functors restrict to inverse equivalences

\[ \text{Ortho}(A, B) \leftrightarrow \text{Fun}(A, \text{Cart}(B)) \quad \text{and} \quad \text{Bifib}(A, B) \leftrightarrow \text{Fun}(A, \text{RFib}(B)) \]

and there is a dual statement for

\[ \text{Str}^{ct} : \text{Ortho}^{loc}(A, B) \leftrightarrow \text{Fun}(B^{\text{op}}, \text{Cocart}^{\text{lax}}(A))^{\text{cocart}} : \text{Un}^{ct} \]

Before diving into the proof observe that we can now compose \( \text{Str}^{cc} \) with the cartesian unstraightening functor \( \text{Str}^{ct} \) to obtain an equivalence

\[ \text{Ortho}(A, B) \simeq \text{Fun}(A, \text{Cart}(B)) \simeq \text{Fun}(A, \text{Fun}(B^{\text{op}}, \text{Cat})) \simeq \text{Fun}(A \times B^{\text{op}}, \text{Cat}) \]

as desired. Evidently considering \( p_2 \) instead of \( p_1 \) and reversing the order of the two straightening functors produces another such equivalence.

Proof. Letting \( \text{pr}_A : A \times B \to A \) be the projection, we start out with the observation that the forgetful functor

\[ (\text{Cat}/A)/\text{pr}_A \to \text{Cat}/(A \times B) \]

is an equivalence, natural in both \( A \) and \( B \). Let us denote by \( C \) the subcategory of the left hand side corresponding to that of local orthofibrations on the right. Lemma 2.4 tells us that \( C \subseteq \text{Cocart}(A)/\text{pr}_A \). Applying cocartesian straightening, which is natural by the results of [GHN17, Appendix A], we find an equivalence

\[ \text{Str}^{ct} : \text{Cocart}(A)/\text{pr}_A \simeq \text{Fun}(A, \text{Cocart}(A))^{\text{const}_{B} : \text{Un}^{ct}} \]

and it remains to identify the image of \( C \) on the right. Straightening takes a map \( p : X \to A \times B \) to a functorial extension of \( a \to (X(a) \times B, \cdot) \). If \( p \) is a local orthofibration then all of these maps are cartesian fibrations per definition. Conversely, given a functor \( A \to \text{Cart}^{\text{lax}}(B) \) we claim that its unstraightening is a local orthofibration. We first observe that the unstraightening \( p : X \to A \times B \) of any functor \( A \to \text{Cat}/B \) admits cocartesian lifts over \( A \times \iota(B) \) by 2.4.

Thus 2.9 implies that a functor \( A \to \text{Cart}^{\text{lax}}(B) \) indeed unstraightens to a local orthofibration.

By 2.4 we also find that a map between two local orthofibrations \( f : p \to p' \) over \( A \times B \) preserves cocartesian lifts of arrows in \( A \times B \) if and only if it preserves cocartesian lifts when considered as a map \( p_1 \to p'_1 \), so to complete the first assertion, we are only left to check that such a map \( f \) preserves cartesian lifts of morphisms in \( \iota(A) \times B \) if and only if the associated natural transformation between the straightenings preserves cartesian lifts pointwise. But one easily checks that the latter condition is equivalent to \( f \) preserving cartesian lifts of the restrictions of \( p \) to \( \iota(A) \times B \), which gives the claim.

We next turn to the claim that orthofibrations correspond precisely to those functors, whose induced maps preserve cartesian lifts. So consider a local orthofibration \( p : X \to A \times B \), and a morphism \( f : a \to a' \) in \( A \). We wish to understand
when the functor $f_*$ in
\[
X_{\{a\} \times B} \xrightarrow{f_*} X_{\{a'\} \times B}
\]

preserves cartesian lifts.

The image $f_*(\phi)$ of an edge $\phi: x \to x'$ in $X_{\{a\} \times B}$ per construction of the straightening participates in a commutative diagram
\[
\begin{array}{ccc}
x & \xrightarrow{y} & y \\
\downarrow{\phi} & & \downarrow{f_* (\phi)} \\
x' & \xrightarrow{y'} & y',
\end{array}
\]
whose horizontal maps are $p_1$-cocartesian lifts of $f$, so in particular project to $A \times \iota(B)$. Choosing a cartesian lift $\psi$ of $p(f_*(\phi))$ with target $y'$ we can enlarge this diagram to
\[
\begin{array}{ccc}
x & \xrightarrow{y} & y'' \\
\downarrow{\phi} & \xrightarrow{f_* (\phi)} & \downarrow{\psi} \\
x' & \xrightarrow{y'} & y',
\end{array}
\]
where the dashed arrow is uniquely determined by $\psi$ being cartesian. If now $\phi$ is cartesian, then this dashed arrow is (per construction) an interpolating edge for $p$. Thus if all interpolating edges for $p$ are equivalences it follows that $f_*(\phi)$ is cartesian, and conversely, if $f_*(\phi)$ is cartesian again, then the uniqueness of cartesian lifts forces the interpolating edge to be an equivalence. Since $\phi$ was arbitrary this gives the claim.

The final statement about bifibrations is now obvious by the characterisation of these as orthofibrations with groupoid fibres, and similar for right fibrations among cartesian fibrations. □

Denote by $F$ the functor
\[
\text{Cat}^{\text{op}} \times \text{Cat}^{\text{op}} \to \text{CAT}, \quad (A,B) \mapsto \text{Fun}(A \times B^{\text{op}}, \text{Cat}).
\]

Our next goal is to show:

5.2. Theorem. The four natural equivalences Ortho $\to$ $F$ given by
\[
\begin{align*}
\text{Ortho}(A,B) &\xrightarrow{\text{Dual}^{cc}} \text{Cocart}(A \times B^{\text{op}}) \xrightarrow{\text{Str}^{cc}} \text{Fun}(A \times B^{\text{op}}, \text{Cat}) \\
\text{Ortho}(A,B) &\xrightarrow{\text{Dual}^{ct}} \text{Cart}(A^{\text{op}} \times B) \xrightarrow{\text{Str}^{ct}} \text{Fun}(A \times B^{\text{op}}, \text{Cat}) \\
\text{Ortho}(A,B) &\xrightarrow{\text{Str}^{cc}} \text{Fun}(A, \text{Cart}(B)) \xrightarrow{\text{Str}^{ct}} \text{Fun}(A \times B^{\text{op}}, \text{Cat})
\end{align*}
\]
and
\[
\begin{align*}
\text{Ortho}(A,B) &\xrightarrow{\text{Str}^{ct}} \text{Fun}(B^{\text{op}}, \text{Cocart}(A)) \xrightarrow{\text{Str}^{cc}} \text{Fun}(A \times B^{\text{op}}, \text{Cat})
\end{align*}
\]
are pairwise equivalent in an essentially unique fashion.
5.3. **Definition.** We shall refer to any of the functors above as the *orthocartesian (un)straightening equivalence*, in formulae
\[
\text{Str}^\text{oc} : \text{Ortho}(A, B) \leftrightarrow \text{Fun}(A \times B^{\text{op}}, \text{Cat}) : \text{Un}^\text{oc}.
\]

By naturality in the input categories this equivalence commutes with restriction to fibres. Consequently, by 2.12 it restricts in particular to an equivalence
\[
\text{Str}^\text{bi} : \text{Bifib}(A, B) \leftrightarrow \text{Fun}(A \times B^{\text{op}}, \text{An}) : \text{Un}^\text{bi},
\]
recovering Stevenson’s result from \[St18\]. Note also that the case \(A = [0]\) is \[BGN18, \text{Theorem 1.4}\], since in this case the first and third, and similarly the second and fourth equivalences agree per definition (and similarly in case \(B = [0]\)).

5.4. **Example.** From 4.8 we find \(\text{Str}^\text{bi}(\text{Ar}(\mathcal{C}) \xleftarrow{(t,s)} \mathcal{C} \times \mathcal{C}) \simeq \text{Hom}_{\mathcal{C}^{\text{op}}}\) as functors \(\mathcal{C} \times \mathcal{C}^{\text{op}} \to \text{An}\).

For the proof of 5.2 we shall follow the strategy of proof from \[BGN18\] in simply computing the automorphism space of the functor \(\mathcal{F}\). The result is:

5.5. **Proposition.** *Evaluation at \([(0), (0)] \in \text{Cat}^2* determines an equivalence
\[
\text{Aut}(\mathcal{F}) \rightarrow \text{Aut}(	ext{Cat}).
\]

In particular, this space is discrete, and \(\pi_0 \text{Aut}(\mathcal{F}) = \mathbb{Z}/2\), the nontrivial element induced by \((-)^{\text{op}} : \text{Cat} \to \text{Cat}\).

It follows that also the space of equivalences \(\text{Ortho} \to \mathcal{F}\) has precisely two components, detected after evaluation at \([(0), (0)]\), and both are discrete. By 4.6 all candidates for the orthocartesian straightening functor lie in the same component, so 5.2 follows.

5.6. **Remark.** (1) A result analogous to 5.5 holds with \text{Cat} replaced by \text{An}, with the conclusion that the automorphism space of \((A, B) \mapsto \text{Fun}(A \times B, \text{An})\) is contractible (in fact our proof of 5.5 below applies mutatis mutandis). Subject to checking the naturality of Stevenson’s equivalence, which can be achieved using the techniques of \[GHN17, \text{Appendix A}\], this implies that our equivalence \(\text{Str}^\text{bi}\), in fact agrees with Stevenson’s from \[St18\]. We shall refrain from working out further details, as we shall not need the statement.

(2) The proof in \[BGN18\] of the analogue of 5.5 for the functor \(A \mapsto \text{Fun}(A, \text{Cat})\), which is one step in the proof of \[BGN18, \text{Theorem 1.4}\], seems incomplete: In loc.cit. the authors use Yoneda’s lemma to deduce that the automorphisms of the functor
\[
\text{Cat}^{\text{op}} \to \text{AN}, \quad A \mapsto \text{Hom}_{\text{CAT}}(A, \text{Cat})
\]
are given by \(\text{Aut}(\text{Cat})\), which is the discrete space \(\mathbb{Z}/2\) by a theorem of Toën, see \[To05\]. This suffices to establish the analogue of 5.2 after taking groupoid cores, but it is unclear to us how to obtain the general statement from this information.

By contrast, our more elaborate proof of 5.5 below also invests intermediary steps of Toën’s results. The strategy applies equally well in the situation of \[BGN18\], but at any rate 5.2 and 5.5 subsume \[BGN18, \text{Theorem 1.4}\].

**Proof of 5.5.** We shall spare ourselves one opposite throughout the proof, and consider
\[
\mathcal{F} : \text{Cat}^{\text{op}} \times \text{Cat}^{\text{op}} \to \text{CAT}, \quad (\mathcal{C}, \mathcal{D}) \mapsto \text{Fun}(\mathcal{C} \times \mathcal{D}, \text{Cat}).
\]
This clearly has no effect on the statement to be proved.

Recall that precomposition with the localisation \( \text{asscat} : \text{sAn} \to \text{Cat} \) gives a fully faithful embedding

\[
\text{Fun}(\text{Cat}^{\text{op}} \times \text{Cat}^{\text{op}}, \mathcal{X}) \longrightarrow \text{Fun}(\text{sAn}^{\text{op}} \times \text{sAn}^{\text{op}}, \mathcal{X})
\]

for any category \( \mathcal{X} \). Since \( \text{asscat} \) preserves colimits, this inclusion furthermore preserves the property of commuting with limits in each variable separately, and on the right the full subcategory spanned by such functors is equivalent to \( \text{Fun}(\Delta^{\text{op}} \times \Delta^{\text{op}}, \mathcal{X}) \). Since \( \mathcal{F} : \text{Cat}^{\text{op}} \times \text{Cat}^{\text{op}} \to \text{CAT} \) does indeed preserve limits in each variable, we find that \( \text{Aut}(\mathcal{F}) \) agrees with the automorphisms of the (large) simplicial category \( \mathcal{F}|_{\Delta^2} \) given by \( (n, m) \mapsto \text{Fun}([n] \times [m], \text{Cat}) \), with functoriality arising from restriction.

Now note that \( \mathcal{F}|_{\Delta^2} \) takes values in the subcategory \( \text{CAT}^R \) of \( \text{CAT} \) of functors admitting left adjoints (in fact the induced maps admit both adjoints via Kan extension, but we shall not need to consider the right adjoints). While the inclusion \( \text{CAT}^R \subset \text{CAT} \) is not fully faithful, it is so after restricting to groupoid cores, and by 5.8 below this feature persists to simplicial objects. We may therefore compute the automorphisms of \( \mathcal{F}|_{\Delta^2} \) as a simplicial object in \( \text{CAT}^R \) without effecting change.

Along with another application of 5.8, this allows us to exchange automorphisms of \( \mathcal{F}|_{\Delta^2} \) for the automorphisms of the (large) bicosimplicial category \( \mathcal{G} \)

\[
(n, m) \mapsto \text{Fun}([n] \times [m], \text{Cat})
\]

with functoriality arising by left Kan extension. Now let us denote for each \( (n, m) \) by \( \mathcal{H}(n, m) \subseteq \mathcal{G}(n, m) \) the full subcategory of functors \( [n] \times [m] \to \text{Cat} \) that arise as the left Kan extension of a functor \( [k] : [0] \to \text{Cat} \) with value a simplex along any vertex inclusion \( (i, j) : [0] \to [n] \times [m] \). Up to equivalence, such a functor is given by

\[
(5.7) \quad [n] \times [m] \to \text{Cat}, \quad (a, b) \mapsto \begin{cases} [k] & a \geq i, b \geq j \\ \emptyset & \text{else} \end{cases}
\]

By transitivity of left Kan extensions, every induced map on homotopy categories \( \pi \mathcal{G}(n, m) \to \pi \mathcal{G}(n', m') \) carries \( \pi \mathcal{H}(n, m) \) into \( \pi \mathcal{H}(n', m') \) (of course these latter two categories are equivalent to \( \mathcal{H}(n, m) \) and \( \mathcal{H}(n', m') \), respectively, but never mind that). Since there are pullbacks

\[
\begin{array}{ccc}
\mathcal{H}(n, m) & \longrightarrow & \mathcal{G}(n, m) \\
\downarrow & & \downarrow \\
\pi \mathcal{H}(n, m) & \longrightarrow & \pi \mathcal{G}(n, m)
\end{array}
\]

it follows from the functoriality of pullbacks that the \( \mathcal{H}(n, m) \) assemble into a functor \( \Delta^2 \to \text{Cat} \) equipped with a natural transformation \( \mathcal{H} \to \mathcal{G} \). Furthermore, using the above formula for the left Kan extension and the fact that left Kan extension is adjoint to restriction, one sees that there are equivalences

\[
\mathcal{H}(n, m) \simeq [n]^{\text{op}} \times [m]^{\text{op}} \times \Delta
\]
where \((i, j, [k])\) corresponds to the functor \((5.7)\). In particular, the \(\infty\)-categories \(\mathcal{H}(n, m)\) are 0-truncated, i.e. equivalent to ordinary categories with discrete core; one can use this to conclude that the above equivalence is natural in \((n, m)\) (which is now a property, rather than a structure), for the obvious functoriality on the right leaving \(\Delta\) fixed.

Note that \(\mathcal{H}(n, m)\) generates \(\mathcal{G}(n, m)\) under colimits, or equivalently, mapping out of all objects in \(\mathcal{H}(n, m)\) detects equivalences (using that by adjunction, mapping the left Kan extension \((5.7)\) into \(F\) is equivalent to mapping \([k]\) into \(F(i, j)\)).

The natural fully faithful inclusion \(\mathcal{H}(n, m) \rightarrow \mathcal{G}(n, m)\) therefore exhibits \(\mathcal{G}(n, m)\) as a natural Bousfield localisation of \(\mathcal{P}(\mathcal{H}(n, m))\) for each \((n, m) \in \Delta^2\).

Consider now the induced maps

\[
\operatorname{Hom}_{\text{Cat}^{2}}(\mathcal{G}, \mathcal{G}) \rightarrow \operatorname{Hom}_{\text{Cat}^{2}}(\mathcal{H}, \mathcal{G}) \leftarrow \operatorname{Hom}_{\text{Cat}^{2}}(\mathcal{H}, \mathcal{H})
\]

which we claim are inclusions of path components. Writing out the terms as spaces of natural transformations and applying \([\text{GHN17, Proposition 5.1}]\), we find

\[
\operatorname{Hom}_{\text{Cat}^{2}}(\mathcal{G}, \mathcal{G}) \simeq \lim \operatorname{Hom}_{\text{Cat}}(\mathcal{G}(n, m), \mathcal{G}(n', m'))
\]

\[
\subseteq \lim \operatorname{Hom}_{\text{Cat}}(\mathcal{P}(\mathcal{H}(n, m)), \mathcal{G}(n', m'))
\]

\[
\simeq \lim \operatorname{Hom}_{\text{Cat}}(\mathcal{H}(n, m), \mathcal{G}(n', m'))
\]

\[
\simeq \operatorname{Hom}_{\text{Cat}^{2}}(\mathcal{H}, \mathcal{G})
\]

where the limits run over \([f: (n, m) \rightarrow (n', m')]\) \(\in \text{Tw}(\Delta \times \Delta)\). The second term is a set of path components in the third; indeed, this is so before taking limits, so that the fibre over a point in the target is the limit of a diagram only taking values \(\emptyset\) and \(\ast\), and thus also either empty or contractible itself. Similarly, we find

\[
\operatorname{Hom}_{\text{Cat}^{2}}(\mathcal{H}, \mathcal{H}) \simeq \lim \operatorname{Hom}_{\text{Cat}}(\mathcal{H}(n, m), \mathcal{H}(n', m'))
\]

\[
\subseteq \lim \operatorname{Hom}_{\text{Cat}}(\mathcal{H}(n, m), \mathcal{G}(n', m'))
\]

\[
\simeq \operatorname{Hom}_{\text{Cat}^{2}}(\mathcal{H}, \mathcal{G})
\]

Now any automorphism \(\varphi\) of \(\mathcal{G}\) induces an automorphism on \(\mathcal{G}(0, 0) = \text{Cat}\), which preserves the full subcategory \(\Delta \subset \text{Cat}\) by \([\text{Lu09b, Corollary 4.4.11 \& Proposition 4.4.13}]\). It follows immediately from the description of the image of \(\mathcal{H}(n, m)\) in \(\mathcal{G}(n, m)\) as a Kan extension, that \(\varphi\) preserves the full subcategory \(\mathcal{H}(n, m) \subset \mathcal{G}(n, m)\), and thus restricts to an automorphism of \(\mathcal{H}\). This implies that the inclusions of path components above refine to inclusions of path components

\[
\operatorname{Aut}(\mathcal{G}, \mathcal{G}) \subseteq \operatorname{Aut}(\mathcal{H}, \mathcal{H}) \subseteq \operatorname{Hom}_{\text{Cat}^{2}}(\mathcal{H}, \mathcal{G})
\]

In particular, the claim that \(\operatorname{Aut}(\mathcal{G}, \mathcal{G})\) is discrete with two components will follow from the analogous statement for \(\mathcal{H}\). The discreteness here is immediate since all isomorphisms in the categories \([n]^{\text{op}} \times [m]^{\text{op}} \times \Delta\) are identities, so \(\operatorname{Hom}_{\text{Cat}}(\mathcal{H}(n, m), \mathcal{H}(n', m'))\) is always discrete. But any automorphism of \(\mathcal{H}\) induces one on \(\mathcal{H}(0, 0) = \Delta\), and this restriction determines the entire transformation: The composite

\[
[n]^{\text{op}} \times [m]^{\text{op}} \times \Delta \xrightarrow{\varphi_{n,m}} [n]^{\text{op}} \times [m]^{\text{op}} \times \Delta \rightarrow \Delta
\]

is determined by naturality for the degeneracy map \((n, m) \rightarrow (0, 0)\) and the composite

\[
[n]^{\text{op}} \times [m]^{\text{op}} \times \Delta \xrightarrow{\varphi_{n,m}} [n]^{\text{op}} \times [m]^{\text{op}} \times \Delta \rightarrow [n]^{\text{op}} \times [m]^{\text{op}}
\]
by naturality with respect to the boundary maps \((0, 0) \to (n, m)\). Thus
\[
\text{Aut}(\mathcal{H}) \simeq \text{Aut}(\Delta) = \mathbb{Z}/2
\]
as desired. \(\square\)

We used the following observation in the proof above:

5.8. Lemma. If \(F : \mathcal{C} \to \mathcal{D}\) is faithful then the diagram

\[
\begin{array}{ccc}
\text{Fun}(\mathcal{E}, \mathcal{C}) & \longrightarrow & \text{Fun}(\mathcal{E}, \mathcal{D}) \\
\downarrow & & \downarrow \\
\text{Fun}(\pi \mathcal{E}, \pi \mathcal{C}) & \longrightarrow & \text{Fun}(\pi \mathcal{E}, \pi \mathcal{D})
\end{array}
\]

is cartesian and, in particular, \(F_* : \text{Fun}(\mathcal{E}, \mathcal{C}) \to \text{Fun}(\mathcal{E}, \mathcal{D})\) is again faithful for any \(\mathcal{E} \in \text{Cat}\). If, furthermore, the restriction \(\iota(\mathcal{C}) \to \iota(\mathcal{D})\) is an inclusion of path components, then so is
\[
\iota(\text{Fun}(\mathcal{E}, \mathcal{C})) \to \iota(\text{Fun}(\mathcal{E}, \mathcal{D})).
\]

5.9. Remark. Note that the functor
\[
\pi \text{Fun}(\mathcal{E}, \mathcal{C}) \to \text{Fun}(\pi \mathcal{E}, \pi \mathcal{C})
\]
is not usually an equivalence, so the diagram of the lemma, is not the canonical one displaying \(F_*\) as faithful.

Proof. The first assertion is immediate from \(\text{Fun}(\pi \mathcal{E}, \pi \mathcal{D}) \simeq \text{Fun}(\mathcal{E}, \mathcal{D})\) and the analogous assertion for \(\mathcal{C}\) in place of \(\mathcal{D}\). The second assertion then follows, since the lower horizontal functor is clearly faithful and faithful functors are closed under pullback: This is most easily seen from the characterisation that all induced maps on morphism complexes have empty or contractible fibres, which is evidently stable under pullback. The third statement similarly follows from the analogue for ordinary categories by applying cores to the diagram of the lemma. \(\square\)

6. NATURALITY OF THE YONEDA EMBEDDING

In this short section we deduce, as an application of 5.5, the following result:

6.1. Theorem. The Yoneda embedding \(\mathcal{C} \to \mathcal{P}(\mathcal{C})\) canonically extends to a natural transformation of functors \(\text{Cat} \to \text{CAT}\) from the inclusion to the composite

\[
\text{Cat} \xrightarrow{\text{Fun}(\mathcal{C}, \text{An})} (\text{CAT}^R)^{\text{op}} \simeq \text{CAT}^L \subseteq \text{CAT}.
\]

As mentioned in the introduction this question was recently posed to the first author (among others) by D. Clausen during a visit to the University of Copenhagen, as it appears missing from the literature so far. It is a lucky accident that our methods answer it.

6.2. Remark. Let us briefly comment on statements we are aware of in the literature: It follows from [Lu09a, Section 5.1.5] that the Yoneda embedding gives a natural transformation between the inclusion \(\text{Cat} \to \text{CAT}\) and the extension of the assignment \(\mathcal{C} \mapsto \mathcal{P}(\mathcal{C})\) to a functor using the characterisation of the latter as the free cocompletion (essentially per construction of this functoriality). But it is not a priori clear that this second functoriality on \(\mathcal{P}\) agrees with the one described in the Proposition above; let us call refer to them as the free cocompletion and
Kan extension functoriality, respectively. Without considering the naturality of the Yoneda embedding, their agreeance can be seen as follows:

Both functorialities on $\mathcal{P}$ are easily checked to factor as

$$
\text{Cat} \xrightarrow{(-)^\sharp} \text{Cat}^\sharp \xrightarrow{\mathcal{P}} \text{CAT} \supset \text{CAT}^\text{tiny},
$$

where the second term denotes the category of small idempotent complete categories (and the first functor idempotent completion), and the third term the category of cocomplete categories admitting a set of tiny (or completely compact in the terminology of $[\text{Lu09a}]$) objects that jointly detect equivalences, and functors among them preserving both colimits and tiny objects. By an argument analogous to $[\text{Lu09a}, \text{Proposition 5.5.7.8}]$, the free cocompletion functoriality makes $\mathcal{P}$ an equivalence between the middle two terms, and since the two functorialities agree morphisimwise by $[\text{Lu09a}, \text{Proposition 5.2.6.3}]$, the same follows for the Kan extension functoriality. But by a minor modification of Toën’s theorem $\text{Cat}^\sharp$ has discrete automorphism space consisting of two objects (the identity and opposing). Thus there is a unique natural equivalence between the two functorialities on $\mathcal{P}$.

The trouble is that it is not clear (to us!) how to conclude that this equivalence is pointwise given by the identity of $\mathcal{P}(\mathcal{C})$ in general, which seems required to use this strategy above to upgrade the Yoneda embedding to a natural transformation as in 6.1.

**Proof of 6.1.** Recall that the Yoneda embedding $\mathcal{C} \to \mathcal{P}(\mathcal{C})$ is defined by adjoining the functor $\text{Hom}_\mathcal{C}: \mathcal{C}^{\text{op}} \times \mathcal{C} \to \text{An}$, which in turn is given by the cocartesian unstraightening of $\text{Tw}(\mathcal{C}) \to \mathcal{C}^{\text{op}} \times \mathcal{C}$.

Now regard $\text{Tw}$ as a functor from $\text{Cat}$ to the full subcategory of $\text{Cat}/((-)^{\text{op}} \times -)$ spanned by the left fibrations, where

$$
\begin{array}{ccc}
\text{Cat}/((-)^{\text{op}} \times -) & \longrightarrow & \text{Ar}(\text{Cat}) \\
\downarrow & & \downarrow \text{t} \\
\text{Cat} \times \text{Cat} & \xrightarrow{(-)^{\text{op}} \times -} & \text{Cat}
\end{array}
$$

is cartesian. This subcategory can also be regarded as the cartesian unstraightening of the functor $(\mathcal{C}, \mathcal{D}) \mapsto \text{LFib}(\mathcal{E}^{\text{op}} \times \mathcal{D})$. There are equivalences

$$
\begin{array}{ccc}
\text{LFib}(\mathcal{E}^{\text{op}} \times \mathcal{D}) & \xrightarrow{\text{Sh}^{\text{st}}} & \text{Fun}(\mathcal{E}^{\text{op}} \times \mathcal{D}, \text{An}) \\
& & \simeq \text{Fun}(\mathcal{D}, \mathcal{P}(\mathcal{C}))
\end{array}
$$

which are natural in the input categories. Calling the composition of the functor above by these equivalences $\mathcal{F}: (\text{Cat}^{\text{op}})^2 \to \text{CAT}$, the resulting functor

$$
\text{Tw}: \text{Cat} \longrightarrow \text{Un}^{\text{ct}}(\mathcal{F})
$$

takes a category $\mathcal{C}$ to its Yoneda embedding and on morphism it witnesses the lax commutativity of the diagrams

$$
\begin{array}{ccc}
\mathcal{C} & \longrightarrow & \mathcal{P}(\mathcal{C}) \\
\downarrow \mathcal{f} & & \downarrow \mathcal{f}' \\
\mathcal{D} & \longrightarrow & \mathcal{P}(\mathcal{D})
\end{array}
$$

and therefore contains all the requisite data for a natural transformation as in the statement.
To extract it, curry \( \mathcal{F} \) into a functor

\[
\text{Cat}^{\text{op}} \to \text{Fun}(\text{Cat}^{\text{op}}, \text{CAT}), \quad \mathcal{D} \mapsto (\mathcal{C} \mapsto \text{Fun}(\mathcal{D}, \mathcal{P}(\mathcal{C})))
\]

Now observe that this functor takes values in the subcategory of \( \text{Fun}(\text{Cat}^{\text{op}}, \text{CAT}) \) spanned by functors to \( \text{CAT}^R \) and left adjointable squares as morphisms. Via cartesian unstraightening this category is equivalent to \( \text{BICART}(\text{Cat}) \), the intersection the two subcategories \( \text{COCART}(\text{Cat}) \) and \( \text{CART}(\text{Cat}) \) in the over category \( \text{CAT}/\text{Cat} \). Via cocartesian unstraightening this is then also equivalent to the subcategory of \( \text{Fun}(\text{Cat}, \text{CAT}) \) spanned by functors into \( \text{CAT}^L \) and right adjointable squares.

Applying these equivalences to \( \mathcal{F} \) results in a functor \( \mathcal{G} \) in

\[
\text{Fun}(\text{Cat}^{\text{op}}, \text{Fun}(\text{Cat}^{\text{op}}, \text{CAT})) \cong \text{Fun}(\text{Cat} \times \text{Cat}^{\text{op}}, \text{CAT}),
\]

which still has \( \mathcal{G}(\mathcal{C}, \mathcal{D}) \cong \text{Fun}(\mathcal{D}, \mathcal{P}(\mathcal{C})) \), but now the functoriality in the \( \mathcal{C} \)-variable is by left Kan extension.

We now claim that there is a canonical equivalence \( \text{Un}^{\text{ct}}(\mathcal{F}) \cong \text{Un}^{\text{oe}}(\mathcal{G}) \) (among the source categories!) via 5.5 above. To see this we first note that generally by 5.5 the functor

\[
\text{Un}^{\text{ct}}: \text{Fun}(\text{Cat}^{\text{op}}, \text{Fun}(\text{Cat}^{\text{op}}, \text{CAT})) \to \text{Cart}(\text{Cat} \times \text{Cat}^{\text{op}})
\]

agrees with the composite

\[
\text{Fun}(\text{Cat}^{\text{op}}, \text{Fun}(\text{Cat}^{\text{op}}, \text{Cat})) \xrightarrow{\text{Un}^{\text{ct}}} \text{Fun}(\text{Cat}^{\text{op}}, \text{Cart}(\text{Cat})) \xrightarrow{\text{Un}^{\text{ct}}} \text{Cart}(\text{Cat} \times \text{Cat}^{\text{op}})
\]

under the currying equivalence, where the second functor is produced just as its orthocartesian counterpart before 5.1. But as noted above the first map takes \( \mathcal{F} \) into \( \text{Fun}(\text{Cat}^{\text{op}}, \text{BICART}(\text{Cat})) \). Using the description of

\[
\text{Un}^{\text{oe}}: \text{Fun}(\text{Cat} \times \text{Cat}^{\text{op}}, \text{Cat}) \to \text{Ortho}(\text{Cat} \times \text{Cat}^{\text{op}})
\]

as

\[
\text{Fun}(\text{Cat}^{\text{op}}, \text{Fun}(\text{Cat}, \text{Cat})) \xrightarrow{\text{Un}^{\text{oe}}} \text{Fun}(\text{Cat}^{\text{op}}, \text{Cocart}(\text{Cat})) \xrightarrow{\text{Un}^{\text{ct}}} \text{Ortho}(\text{Cat} \times \text{Cat}^{\text{op}})
\]

\( \mathcal{G} \) is defined to have the same image as \( \mathcal{F} \) in \( \text{Fun}(\text{Cat}^{\text{op}}, \text{BICART}(\text{Cat})) \), which gives the claim.

The resulting functor

\[
\text{Tw}: \text{Cat} \to \text{Un}^{\text{oe}}(\mathcal{G})
\]

again takes \( \mathcal{C} \) to its Yoneda embedding, but this time morphisms witness the lax commutativity of

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{f} & \mathcal{P}(\mathcal{C}) \\
\downarrow f & & \downarrow f_i \\
\mathcal{D} & \xrightarrow{f_i} & \mathcal{P}(\mathcal{D}).
\end{array}
\]

But by [Lu09a, Proposition 5.2.6.3] (or rather the second step of its proof), the tautological natural transformation between the two functors in this diagram is an equivalence, i.e. the diagram actually commutes.

Armed with this information consider the diagram
Unoc(G) \longrightarrow \text{Unoc}(\text{Fun}(-,-))
\downarrow
\text{Cat} \times \text{Cat} \xrightarrow{(\mathcal{P},\text{incl})} \text{CAT} \times \text{CAT}

whose top right corner is the lax arrow category Ar^lax(CAT) of large categories. Via the inclusion \text{Hom}_{\text{CAT}}(\mathcal{E},\mathcal{D}) \subseteq \text{Fun}(\mathcal{E},\mathcal{D}) it contains the actual arrow category Ar(CAT) as a wide subcategory and by the previous observation the composite

\text{Cat} \xrightarrow{\text{Tw}} \text{Unoc}(G) \longrightarrow \text{Ar}^lax(CAT)

actually takes values in this subcategory. The resulting functor \text{Cat} \to \text{Ar}(\text{CAT}) is then the natural transformation we set out to construct. □

6.3. Remark. It is of course not necessary to talk about orthofibrations in the proof above, this is simply language convenient in the context of this paper. The non-trivial input is rather that cartesian fibrations over a product can be unstraightened in a two step procedure, which is implied by 5.5.

7. LAX NATURAL TRANSFORMATIONS AND MONOIDAL ADJUNCTIONS

In this section we will use the results of Section 4 and 5 to study some properties of the operation of taking adjoint functors. More precisely, we will first show that for any lax natural transformation \rho: \mathcal{C} \xrightarrow{lax} \mathcal{D} between two diagrams of \infty-categories, by which we shall mean a map between the associated cocartesian fibrations (not necessarily preserving cocartesian edges), such that each component of \rho is a right adjoint, the pointwise left adjoints assemble into an oplax natural transformation \lambda: \mathcal{D} \xrightarrow{\text{op}} \mathcal{C}, i.e. a map between the classifying cocartesian fibrations.

In the last part of the paper we then deduce that for any lax monoidal functor \text{G}: \mathcal{C} \xrightarrow{\otimes} \mathcal{D}, whose underlying functor is a right adjoint, the left adjoint inherits a unique oplax monoidal structure, proving both Theorem A and Theorem C from the introduction.

Both parts can be seen as an \infty-categorical incarnation of the ‘calculus of mates’.

7.1. Adjoint\ of\ lax\ natural\ transformations. Recall that for an \infty-category \text{A}, the full subcategory \text{Cocart}^{lax}(A) \subseteq \text{Cat}/A on the cocartesian fibrations can be thought of as the \infty-category of functors \text{A} \to \text{Cat} with lax natural transformations between them. Likewise, the full subcategory \text{Cart}^{\text{op},\text{L}}(A^{\text{op}}) \subseteq \text{Cat}/A on the cartesian fibrations can be thought of as the category of functors \text{A} \to \text{Cat} with oplax natural transformations between them. Our first goal will be to prove the following:

7.1. Proposition. Let \text{A} be an \infty-category and let us write \text{Cocart}^{lax,R}(A), resp. \text{Cart}^{\text{op},\text{L}}(A), for the wide subcategories whose maps induce right (resp. left) adjoint functors between the fibres. Then there is a natural equivalence of \infty-categories

(7.2) \text{Adj}: \text{Cocart}^{lax,R}(A) \xrightarrow{\sim} \text{Cart}^{\text{op},\text{L}}(A^{\text{op}})^{\text{op}}

sending each cocartesian fibration to the cartesian fibration classifying the same functor \text{A} \to \text{Cat}. 

Informally, this functor takes a lax natural transformation consisting of right adjoint functors to the oplax natural transformation consisting of its left adjoints as follows: Let \( G : X \to Y \) be a map between cocartesian fibrations over \( A \) corresponding on the one hand to a lax natural transformation

\[
\begin{array}{ccc}
X(a) & \xrightarrow{G} & Y(a) \\
\downarrow^{f_!} & \searrow_{\rho_f} & \downarrow^{f_!} \\
X(a') & \xrightarrow{G} & Y(a').
\end{array}
\]

(7.3)

Regarding \( G \) as an object of \( \text{Fun}(1^{\text{op}}, \text{Cocart}^{\text{lax}}(A)) \) it corresponds on the other hand to a local orthocartesian fibration over \( p : T \to A \times [1] \) via Proposition 5.1. The constituent maps of the natural transformation \( \rho_f \) can be obtained in terms of the interpolating edges of \( p \) as follows: for every object \( x_a \in X(a) \simeq p^{-1}(a,1) \), take a diagram in \( T \) of the form

\[
\begin{array}{ccc}
y_a & \xrightarrow{\rho_f(x_a)} & y_{a'} \xrightarrow{\lambda_f(y_a)} x_{a'} \\
\downarrow & & \downarrow \\
x_a & \xrightarrow{\lambda_f(y_a)} & x_{a'} \\
\end{array}
\]

(7.4)

where the downwards pointing maps are \( p \)-cocartesian and the top horizontal map is \( p \)-cartesian. The bottom row decomposes the bottom map into a map in the fibre over \((a',0)\), followed by a \( p \)-cartesian arrow. The map \( \rho_f(x_a) : y_{a'} \to y'_{a'} \) is then the component of the natural transformation \( \rho_f \) at \( x_a \in X(a) \).

Now suppose that \( G \) admits fibrewise left adjoints, then Proposition 7.1 says that these assemble into an oplax natural transformation

\[
\begin{array}{ccc}
X(a) & \xleftarrow{F} & Y(a) \\
\downarrow^{f!} & \nearrow_{\lambda_f} & \downarrow^{f!} \\
X(a') & \xleftarrow{F} & Y(a').
\end{array}
\]

(7.5)

Pointwise, this transformation is obtained as follows: Pick \( y_a \) in the fibre \( Y(a) \simeq p^{-1}(a,0) \) and take a diagram in \( T \) of the form

\[
\begin{array}{ccc}
y_a & \xrightarrow{x_a} & x_{a'} \\
\downarrow & & \downarrow \\
y'_{a'} & \xrightarrow{\lambda_f(y_a)} & x'_{a'} \\
\end{array}
\]

(7.6)

where the vertical arrows are \( p \)-cocartesian, the top horizontal arrow is locally \( p \)-cocartesian and the bottom factorisation decomposes the bottom map into a locally \( p \)-cocartesian map, followed by a map in the fibre over \((a',1)\). Then the map \( \lambda_f(y_a) : x'_{a'} \to x_{a'} \) is the component of the natural transformation \( \lambda_f \). Along with Proposition 7.1 we will verify:

7.7. Lemma. Let \( G : X \to Y \) be a map in \( \text{Cocart}^{\text{lax}}(A) \), corresponding to a lax natural transformation (7.3), then the adjoint oplax transformation from Proposition 7.1 really is pointwise given by (7.6). Furthermore, it is morphismwise equivalent to the Beck–Chevalley transformation associated to \( \rho \), i.e. its component at
some \( f : a \to a' \) is equivalent to the composition

\[
Ff_! \xrightarrow{F\eta} Ff_!GF \xrightarrow{F\rho\eta} F\xi F \xrightarrow{Ff} f_!F.
\]

Here, \( \eta \) denotes the adjunction unit, and \( \epsilon \) the counit. Also we refer the reader to [Lu09a, Section 7.3.1] for a brief discussion of Beck-Chevalley transformations.

Proof of Proposition 7.1 and Lemma 7.7. Let \( B \) be another \( \infty \)-category and note that for a functor \( p = (p_1, p_2) : X \to A \times B \), the following two conditions are equivalent by 2.9:

1. \( p \) is a local orthocartesian fibration whose restriction to \( \iota(A) \times B \) is a cocartesian fibration as well.
2. \( (p_2, p_1) : X \to B \times A \) is a Gray fibration whose restriction to \( B \times \iota(A) \) is a cartesian fibration as well.

Let us write \( \mathcal{M}(A, B) \) for the anima of such functors, so that there are natural inclusions of path components

\[
\iota\text{Gray}(B, A) \supseteq \mathcal{M}(A, B) \subseteq \iota\text{Ortho}^{\text{loc}}(A, B).
\]

Likewise, for a functor \( q = (q_1, q_2) : Y \to A \times B \), the following are equivalent:

1. \( q \) is a local orthocartesian fibration whose restriction to \( A \times \iota(B) \) is a cartesian fibration as well.
2. \( (q_2, q_1) : Y \to A \times B \) is a Gray opfibration whose restriction to \( A \times \iota(B) \) is a cocartesian fibration as well.

Let us write \( \mathcal{N}(A, B) \) for the anima of such functors, so that there are natural inclusions of path components

\[
\iota\text{Gray}^{\text{op}}(A, B) \supseteq \mathcal{N}(A, B) \subseteq \iota\text{Ortho}^{\text{loc}}(A, B).
\]

Now recall from Proposition 5.1 that there are natural equivalences of anima

\[
\iota\text{Ortho}^{\text{loc}}(A, B) \simeq \text{Hom}(B^{\text{op}}, \text{Cocart}^{\text{lax}}(A))
\]

\[
\iota\text{Ortho}^{\text{loc}}(B, A) \simeq \text{Hom}(B, \text{Cart}^{\text{opl}}(A)).
\]

When \( A = \ast \), these equivalences are given by the usual straightening of (co)cartesian fibrations over \( B \). By naturality in \( A \), it then follows that these two equivalences restrict to equivalences between path components

\[
\mathcal{M}(A, B) \simeq \text{Hom}_{\text{CAT}}(B^{\text{op}}, \text{Cocart}^{\text{lax}, R}(A))
\]

\[
\mathcal{N}(B, A) \simeq \text{Hom}_{\text{CAT}}(B, \text{Cart}^{\text{opl}, L}(A)).
\]

Indeed, being contained in \( \mathcal{M}(A, B) \) or \( \mathcal{N}(B, A) \) is a property that is detected pointwise in \( A \) (note the variable switch in the second term). Next, consider the dualisation functor with respect to \( A \)

\[
\text{Dual}^{\text{rt}} : \text{Gray}(B, A) \xrightarrow{\sim} \text{Ortho}^{\text{loc}}(B, A^{\text{op}})
\]

When \( A = \ast \), this restricts to a natural self-equivalence of the category of cocartesian fibrations over \( B \) (homotopic to the identity by Proposition 5.5). By naturality in \( A \), one therefore sees that the dualisation preserves those objects that restrict for each \( a \in A \) to a bicartesian fibration over \( \{a\} \times B \), i.e. it restricts to an equivalence on path components

\[
\text{Dual}^{\text{rt}} : \mathcal{M}(A, B) \xrightarrow{\sim} \mathcal{N}(B, A^{\text{op}}).
\]
Combining all these observations, we obtain a diagram of spaces, depending functorially on $B$:

$$
\begin{align*}
\Hom_{\text{Cat}}(B^{\text{op}}, \text{Cocart}^{\text{lax, R}}(A)) & \xleftarrow{\text{Str}^c} \mathcal{M}(A, B) \\
\text{Dual}^c \downarrow & \sim \\
N(B, A^{\text{op}}) & \xrightarrow{\text{Str}^c} \Hom_{\text{Cat}}(B, \text{Cart}^{\text{opl, L}}(A^{\text{op}})).
\end{align*}
$$

By naturality in $B$, this determines the desired equivalence of $\infty$-categories (7.2). Furthermore, when $B = *$ is a point, the above sequence of equivalences sends a cocartesian fibration over $A$ to the corresponding cartesian fibration classifying the same functor $A \to \text{Cat}$. It follows that the equivalence (7.2) indeed has the correct behaviour on objects and unwinding definitions also shows the first part of Lemma 7.7.

Finally, we explain the identification with the Beck-Chevalley transformation. For the sake of exposition we shall give details for a pointwise identification first, and then indicate the necessary modifications to get the full statement afterwards. Let $p : X \to A \times [1]$ be the local orthocartesian fibration classifying (7.3) and the adjoint (7.5). For any $y \in Y(a) \simeq p^{-1}(a, 0)$ and any map $f : a \to a'$ in $A$, there is a commuting diagram in $T$ (constructed from the outside to the inside by picking (co)cartesian lifts and using lifting properties) that we can depict as

$$
\begin{array}{ccc}
y & \xrightarrow{\eta} & GF(y) \\
\downarrow & & \downarrow \\
f \circ \eta & \xrightarrow{f \circ G \eta} & Gf \circ F(y).
\end{array}
$$

Recall that $\rightarrow$ and $\rightarrow$ denote $p$-cocartesian and $p$-cartesian edges, and note that also the top horizontal composite is locally $p$-cocartesian. The outer square corresponds precisely to the outer square of diagram (7.6), so that the bottom horizontal composite factors uniquely into a locally $p$-cocartesian arrow, followed by $\lambda_f$ (by definition of $\lambda$). Now note that the maps $f \circ \eta$ and $\rho_f$ are both contained in the fibre $p^{-1}(a', 0) \simeq Y(a')$, so that choosing locally $p$-cocartesian lifts over $(a', 0) \to (a', 1)$ yields a commuting diagram

$$
\begin{array}{ccc}
Ff(y) & \xrightarrow{F \circ f \circ \eta} & Ff \circ GF(y) \\
\downarrow & & \downarrow \\
Ff \circ f \circ Gf(y) & \xrightarrow{F \circ \rho_f} & GFf \circ F(y).
\end{array}
$$

Here the top lives in $p^{-1}(a', 0)$ and the bottom in $p^{-1}(a', 1)$. In particular, this shows that the bottom horizontal composite is equivalent to the map $\lambda$ in the diagram (7.6), which proves the result pointwise.

To obtain the identification as natural transformations we must check that the diagram above can be produced functorially in $y \in Y(a)$, in the same manner that the unit and counit transformations are constructed. To this end recall that the unit of an adjunction for example is built by functorially choosing cocartesian lifts over the edges $\{p(y)\} \times [1]$ starting at $y$ (which is possible by [Lu09a, Proposition]...
and the fact that for any cocartesian fibration \( q: A \to B \) the induced map \( A^q \to \text{Ar}(B) \times_B A \) is an equivalence), then functorially choosing cartesian lifts over the same edges meeting the previously constructed ones at the end points, and then finally factoring the cocartesian lifts through the cartesian ones. But this very same procedure can be applied multiple times to iteratively construct the two diagrams above, since each of their arrows is given by either a (locally) cocartesian lift, a (locally) cartesian lift or a factorisation through one such. As carrying this out is not difficult but notationally involved, and we shall not need the full statement below, we leave details to the reader. □

Proposition 7.1 is not quite sufficient to deduce Theorems A and C, since it does not take into account non-invertible 2-morphisms between lax natural transformations. To fix this, we introduce the following construction: Let

\[
\text{Cocart}^{\text{lax}}_S(A \times S) \subseteq \text{Cart}^{\text{pl,l}}_S(A \times S)
\]

denote the full subcategory of cocartesian fibrations which are locally constant on \( S \), i.e. whose unstraightening to a functor \( A \times S \to \text{Cat} \) factors over \( A \times |S| \).

7.8. Lemma. Let \( A \) be an \( \infty \)-category. Then the association above can be considered as a functor

\[
\text{Cat}^{\text{op}} \longrightarrow \text{CAT}; \quad S \longmapsto \text{Cocart}^{\text{lax}}_S(A \times S)
\]

which preserves limits.

The same assertion holds if we instead take \( \text{Cocart}^{\text{lax,R}}_S(A \times S), \text{Cart}^{\text{pl}}_S(A \times S) \) or \( \text{Cart}^{\text{pl,l}}_S(A \times S) \), which are defined analogously.

Proof. For any \( \infty \)-category \( B \), note that

\[
\text{Hom}_{\text{CAT}}(B, \text{Cocart}^{\text{lax}}(A \times S)) \cong \text{Hom}_{\text{loc}}(A \times S, B) \cong \text{Hom}_{\text{CAT}}(A \times S, \text{Cart}^{\text{pl}}(B)).
\]

This implies that \( S \mapsto \text{Cocart}^{\text{lax}}(A \times S) \) preserves limits (though this can of course also be checked directly). Since the functor induced on limits by a natural transformation consisting of fully faithful functors is itself fully faithful, each diagram \( S_\alpha \) in \( \text{Cat} \) with colimit \( S \) gives rise to a commuting diagram

\[
\begin{array}{ccc}
\text{Cocart}^{\text{lax}}_S(A \times S) & \longrightarrow & \lim_{\alpha} \text{Cocart}^{\text{lax}}_{S_\alpha}(A \times S_\alpha) \\
\downarrow & & \downarrow \\
\text{Cocart}^{\text{lax}}(A \times S) & \longrightarrow & \lim_{\alpha} \text{Cocart}^{\text{lax}}(A \times S_\alpha)
\end{array}
\]

The result then follows from the fact that a cocartesian fibration over \( A \times S \) is locally constant on \( S \) if and only if it is locally constant when restricted to each \( S_\alpha \) (since \( A \times |\cdot| : \text{Cat} \to \text{Cat} \) preserves colimits). □

Lemma 7.8 implies that the bisimplicial space

\[
([m], [n]) \longmapsto \text{Hom}_{\text{CAT}}([m], \text{Cocart}^{\text{lax}}_{[n]}(A \times [n]))
\]

satisfies the complete Segal conditions in both simplicial directions. Furthermore, when \( [m] = [0] \) the resulting simplicial space is essentially constant by construction, which justifies the following:
7.10. **Definition.** Let $A$ be a small $\infty$-category. We define $\text{Cocart}^{\text{lax}}(A)$ to be the $(\infty, 2)$-category associated to the 2-fold Segal space (7.9). We define the $(\infty, 2)$-categories $\text{Cart}^{\text{opl}}(A)$, $\text{Cocart}^{\text{lax, L}}(A)$ and $\text{Cart}^{\text{opl, R}}(A)$ similarly.

7.11. **Remark.** Unraveling the definitions, one sees that $\text{Cocart}^{\text{lax}}(A)$ has objects given by cocartesian fibrations $X \to A$, i.e. functors $A \to \text{Cat}$ and 1-morphisms given maps $G : X \to Y$, i.e. lax natural transformations between such functors. A 2-morphism $\mu : G \to G'$ between such natural transformations is then given by natural transformations $\mu_a : G_a \to G'_a$ that commute with the lax structure maps, in the sense that for each $a \to a'$, there is a commuting diagram

![Diagram of lax transformations](attachment:Diagram.png)

Depicting this diagram cubically, it can also be viewed as a lax natural transformation between two functors $A \times [1] \to \text{Cat}$ that are constant along the interval (which is exactly how $\text{Cocart}^{\text{lax}}(A)$ was defined). Note that $\text{Cocart}^{\text{lax, L}}(A) \subseteq \text{Cocart}^{\text{lax}}(A)$ is the 1-full sub-2-category whose morphisms are lax natural transformations consisting of right adjoints.

7.12. **Remark.** Note that for any two $\infty$-categories $A$ and $S$, taking opposite $\infty$-categories defines an equivalence $(-)^{\text{op}} : \text{Cocart}^{\text{lax}}(A \times S) \to \text{Cart}^{\text{opl, L}}(A^{\text{op}} \times S^{\text{op}})$.

Using this, one deduces that taking opposite $\infty$-categories defines an equivalence of $(\infty, 2)$-categories, where in the target the 2-morphisms are reversed

$$(-)^{\text{op}} : \text{Cocart}^{\text{lax}}(A) \to \text{Cart}^{\text{opl}}(A^{\text{op}})^{2-\text{op}}.$$  

7.13. **Theorem.** Let $A$ be an $\infty$-category. Then the equivalence (7.2) extends to a natural equivalence of $(\infty, 2)$-categories

$$\text{Adj} : \text{Cocart}^{\text{lax, R}}(A) \xrightarrow{\sim} \text{Cart}^{\text{opl, L}}(A^{\text{op}})^{(1,2)-\text{op}}$$

where in the target the directions of 1- and 2-morphisms is changed.

**Proof.** This follows immediately from the fact that the natural equivalence of $\infty$-categories

$$\text{Adj} : \text{Cocart}^{\text{lax, R}}(A \times [n]) \to \text{Cart}^{\text{opl, L}}(A^{\text{op}} \times [n]^{\text{op}})^{\text{op}}$$

identifies the full subcategory of cocartesian fibrations that are constant along the simplex $[n]$ with that of cartesian fibrations that are constant along $[n]^{\text{op}}$. □
7.2. Lax monoidal adjunctions. Recall that an ∞-operad \( O \) is a map of ∞-categories \( p: O^\otimes \to \Gamma^{\text{op}} \) to the category of finite sets and partial maps, satisfying the following conditions:

1. every partial bijection in \( \Gamma^{\text{op}} \) has \( p \)-cartesian lifts.
2. let \( x \in O^\otimes \) be an object with \( p(x) = \langle n \rangle \) and let \( \rho^i: x \to x_i \) be \( p \)-cartesian lifts of the partial bijections \( \rho^i: \langle n \rangle \to \langle 1 \rangle \) sending \( i \) to 1. For every \( f: \langle m \rangle \to \langle n \rangle \) and \( y \in O^\otimes_{\langle m \rangle} \), postcomposition with the \( \rho^i \) induces an equivalence

\[
\text{Hom}^\rho_{O^\otimes}(y, x) \to \prod_i \text{Hom}^\rho_{O^\otimes}(y, x_i).
\]

3. for every tuple \( (x_1, \ldots, x_n) \) of objects in \( O^\otimes_{\langle 1 \rangle} \), there exists an \( x \in O^\otimes_{\langle m \rangle} \) together with \( p \)-cartesian lifts \( \rho^i: x \to x_i \).

A morphism in \( O^\otimes \) is called inert if it is the cocartesian lift of a partial bijection in \( \Gamma^{\text{op}} \).

7.15. Definition. The \((\infty, 2)\)-category \( \text{OMonCat}^{\text{lax}}(O^\otimes) \) of \( O \)-monoidal ∞-categories and \textit{lax} \( O \)-monoidal functors between them is given by the 1-full sub-2-category of \( \text{Cocart}^{\text{lax}}(O^\otimes) \) whose:

1. objects are cocartesian fibrations \( p: C^\otimes \to O^\otimes \) satisfying the Segal condition:
   - for each \( x \in O^\otimes_{\langle n \rangle} \) with inert maps \( \rho^i: x \to x_i \), the functor between fibres \( (\rho^i)_*: C^\otimes(x) \to \prod_i C^\otimes(x_i) \) is an equivalence.
2. morphisms are functors \( C^\otimes \to D^\otimes \) over \( O^\otimes \) that preserve the cocartesian morphisms lying over inert morphisms in \( O^\otimes \).

By definition, the underlying \((\infty, 1)\)-category of \( \text{OMonCat}^{\text{lax}}(O^\otimes) \) is the full subcategory of \( \infty \)-operads over \( O \) whose objects are the \( O \)-monoidal categories. A \textit{strong} \( O \)-monoidal functor corresponds to a morphism \( C^\otimes \to D^\otimes \) preserving all cartesian edges.

7.16. Example. Let us explicitly mention the special case \( O^\otimes = \Gamma^{\text{op}} \), where \( \text{OMonCat}^{\text{lax}} \) has objects symmetric monoidal ∞-categories, 1-morphisms lax symmetric monoidal functors, and 2-morphisms symmetric monoidal transformations. In particular, Theorem A from the introduction is a statement about morphism categories therein (and in the oplax analogue defined below).

Another noteworthy special case is that of the trivial operad \( O^\otimes = \Gamma_{\text{inert}}^{\text{op}} \), in which case \( \text{OMonCat}^{\text{lax}} \) is simply the \((\infty, 2)\)-category \( \text{Cat} \) of \((\infty, 1)\)-categories, with 1-morphisms functors and 2-morphisms natural transformations.

7.17. Definition. The \((\infty, 2)\)-category \( \text{OMonCat}^{\text{opl}}(O^\otimes) \) of \( O \)-monoidal ∞-categories and \textit{oplax} \( O \)-monoidal functors between them is the sub-2-category of \( \text{Cart}^{\text{opl}}((O^\otimes)^{\text{op}}) \) whose:

1. objects are cartesian fibrations \( p: C_\otimes \to (O^\otimes)^{\text{op}} \) satisfying the Segal condition:
   - for each \( x \in (O^\otimes)^{\text{op}}_{\langle n \rangle} \) with inert maps \( \rho_i: x_i \to x \) in \( (O^\otimes)^{\text{op}} \), the functor between fibres \( (\rho^i)_*: C_\otimes(x) \to \prod_i C_\otimes(x_i) \) is an equivalence.
2. morphisms are functors \( C_\otimes \to D_\otimes \) that preserve cartesian morphisms lying over inert morphisms in \( (O^\otimes)^{\text{op}} \).

Note in particular that the objects in \( \text{OMonCat}^{\text{opl}} \) are a priori not \( O \)-monoidal categories: one has to take the cocartesian fibration over \( O^\otimes \) corresponding to the
cartesian fibration over \((\mathcal{O}\otimes)^{\text{op}}\) to get an \(\mathcal{O}\)-monoidal \(\infty\)-category in the usual sense. The following lemma thus simply asserts that essentially by definition, an oplax \(\mathcal{O}\)-monoidal functor is a lax \(\mathcal{O}\)-monoidal functor between the opposite categories.

**7.18. Lemma.** Taking opposite categories defines an equivalence of \((\infty, 2)\)-categories
\((-)^{\text{op}}: \mathcal{O}\text{MonCat}^{\text{opl}} \to (\mathcal{O}\text{MonCat}^{\text{lax}})^{2\text{-op}}\).

**Proof.** It suffices to verify that the equivalence of Remark 7.12 identifies the relevant sub-2-categories. Given a cartesian fibration \(\mathcal{C}\otimes \to \mathcal{O}\otimes\), let us write \(\mathcal{C}\otimes \to \mathcal{O}\otimes\) for the opposite cocartesian fibration. The Segal map \((\rho^*_i): \mathcal{C}\otimes(x) \to \prod_i \mathcal{C}\otimes(x_i)\)

is then the opposite of the Segal map \((\rho^i): \mathcal{C}\otimes(x) \to \prod_i \mathcal{C}\otimes(x_i)\), so that one is an equivalence if and only if the other is. Finally, a functor preserving cartesian lifts of inert morphisms is sent to the functor between opposite categories, which preserves cocartesian lifts of inert morphisms. \(\square\)

For example, for \(\mathcal{O}\otimes = \Gamma^{\text{op}}_{\text{inert}}\) we also find \(\mathcal{O}\text{MonCat}^{\text{opl}} = \text{Cat}\), and taking opposite categories indeed defines an equivalence \(\text{Cat} \simeq \text{Cat}^{2\text{-op}}\).

**7.19. Lemma.** Let \(G: \mathcal{C}\otimes \to \mathcal{D}\otimes\) be a lax \(\mathcal{O}\)-monoidal functor, i.e. a morphism in \(\mathcal{O}\text{MonCat}^{\text{lax}}\). Then the following two conditions are equivalent:

1. For every \(x \in \mathcal{O}\otimes\), the induced map on fibres \(G\): \(\mathcal{C}\otimes(x) \to \mathcal{D}\otimes(x)\) is a right adjoint.
2. For every \(x \in \mathcal{O}\otimes\), the induced map on fibres \(G\): \(\mathcal{C}\otimes(x) \to \mathcal{D}\otimes(x)\) is a right adjoint.

**Proof.** This follows from the fact that for each \(x \in \mathcal{O}\otimes\), there is a commuting square
\[
\begin{array}{ccc}
\mathcal{C}\otimes(x) & \xrightarrow{G} & \mathcal{D}\otimes(x) \\
(\rho^i)^* | & & | (\rho^i) \\
\prod_i \mathcal{C}\otimes(x_i) & \xrightarrow{(G_{x,i})} & \prod_i \mathcal{D}\otimes(x_i)
\end{array}
\]
where \(\rho^i: x \to x_i\) are the canonical inert maps decomposing \(x\) into its components \(x_i \in \mathcal{O}\otimes_{(1)}\). \(\square\)

**7.20. Definition.** A lax \(\mathcal{O}\)-monoidal functor \(G: \mathcal{C}\otimes \to \mathcal{D}\otimes\) is a lax \(\mathcal{O}\)-monoidal left adjoint if it satisfies the equivalent conditions of Lemma 7.19.

Likewise, an oplax \(\mathcal{O}\)-monoidal functor \(F: \mathcal{C}\otimes \to \mathcal{D}\otimes\) is called an oplax \(\mathcal{O}\)-monoidal right adjoint if it induces right adjoint functors between the fibres over each \(x \in \mathcal{O}\otimes_{\text{op}}\) (equivalently, all \(x \in \mathcal{O}\otimes_{(1)}^{\text{op}}\)).

**7.21. Theorem.** For each \(\infty\)-operad \(\mathcal{O}\), there is a natural equivalence of \((\infty, 2)\)-categories
\[
\text{Adj}: \mathcal{O}\text{MonCat}^{\text{lax}, R} \xrightarrow{\sim} \left(\mathcal{O}\text{MonCat}^{\text{opl}, L}\right)^{(1,2)\text{-op}}
\]
between the 1-full sub-2-categories whose morphisms are lax \(\mathcal{O}\)-monoidal right adjoints and oplax \(\mathcal{O}\)-monoidal left adjoints.
When $\mathcal{O}$ is the trivial operad, then under the identification of the left hand term with $\text{Cat}^R$ and the right hand side with $(\text{Cat}^L)^{(1,2)-\text{op}}$ this equivalence simply sends a right adjoint functor to its left adjoint.

By naturality, the theorem asserts that for any lax $\mathcal{O}$-monoidal right adjoint $G$, the corresponding left adjoint functor has a canonical oplax $\mathcal{O}$-monoidal structure. Theorem A is a direct consequence by considering morphism categories for $\mathcal{O}^{\otimes} = \Gamma^{\text{op}}$.

**Proof.** It suffices to show that the equivalence of Theorem 7.13 identifies the two relevant sub-2-categories

\[
\text{OMonCat}^{\text{lax, R}} \longrightarrow (\text{OMonCat}^{\text{opl, L}})^{(1,2)-\text{op}}
\]

At the level of objects, note that the functor Adj sends a cocartesian fibration over $\mathcal{O}^{\otimes}$ to the cartesian fibration over $\mathcal{O}^{\otimes, \text{op}}$ classifying the same functor $\mathcal{O}^{\otimes} \longrightarrow \text{Cat}$. In particular, a cocartesian fibration over $\mathcal{O}^{\otimes}$ satisfies the Segal conditions if and only if its image under Adj does.

It remains to verify that the functor Adj sends a map $G: \mathcal{C}^{\otimes} \longrightarrow \mathcal{D}^{\otimes}$ that preserves cocartesian lifts of inerts maps to a map $F: \mathcal{D}^{\otimes} \longrightarrow \mathcal{C}^{\otimes}$ of cartesian fibrations over $\mathcal{O}^{\otimes, \text{op}}$ that preserves cartesian lifts of inert maps (for the reverse implication, reverse the roles of $F$ and $G$ in the next argument). By Lemma 7.7, this comes down to the following assertion: for any inert map $\beta: x \longrightarrow x'$ in $\mathcal{O}^{\otimes}$, the lax $\mathcal{O}$-monoidal functor $G$ defines the commuting left square

\[
\begin{array}{ccc}
\mathcal{C}^{\otimes}(x) & \xrightarrow{G_x} & \mathcal{D}^{\otimes}(x) \\
\downarrow_{\beta} & & \downarrow_{\beta} \\
\mathcal{C}^{\otimes}(x') & \xrightarrow{G_{x'}} & \mathcal{D}^{\otimes}(x')
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{C}^{\otimes}(x) & \xleftarrow{F_x} & \mathcal{D}^{\otimes}(x) \\
\downarrow_{\beta} & & \downarrow_{\beta} \\
\mathcal{C}^{\otimes}(x') & \xleftarrow{F_{x'}} & \mathcal{D}^{\otimes}(x')
\end{array}
\]

and we have to verify that the associated Beck-Chevalley transformation on the left is an equivalence. Using the Segal condition, these squares can be identified with

\[
\begin{array}{ccc}
\prod_{i \in I} \mathcal{C}^{\otimes}(x_i) & \xrightarrow{(G_{x_i})} & \prod_{i \in I} \mathcal{D}^{\otimes}(x_i) \\
\downarrow_{pr} & & \downarrow_{pr} \\
\prod_{j \in J} \mathcal{C}^{\otimes}(x_j) & \xrightarrow{(G_{x_j})} & \prod_{j \in J} \mathcal{D}^{\otimes}(x_j)
\end{array}
\]

\[
\begin{array}{ccc}
\prod_{i \in I} \mathcal{C}^{\otimes}(x_i) & \xleftarrow{(F_{x_i})} & \prod_{i \in I} \mathcal{D}^{\otimes}(x_i) \\
\downarrow_{pr} & & \downarrow_{pr} \\
\prod_{j \in J} \mathcal{C}^{\otimes}(x_j) & \xleftarrow{(F_{x_j})} & \prod_{j \in J} \mathcal{D}^{\otimes}(x_j)
\end{array}
\]

where the vertical functors are projections associated to an inclusion of finite sets $J \subseteq I$. But for such projections, the Beck–Chevalley transformation is always an equivalence (since the unit and counit maps can be computed in each factor).

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