Accretion dynamics of ideal fluid in the deformed Kerr Spacetime

Subhankar Patra, 1,★ Bibhas Ranjan Majhi, 1 † Santabrata Das 1,‡
1Department of Physics, Indian Institute of Technology Guwahati, Guwahati 781039, Assam, India

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ABSTRACT

We study the properties of a low-angular momentum, inviscid, advective accretion flow in a deformed Kerr spacetime under the framework of general theory of relativity. We solve the governing equations that describe the flow motion in terms of input parameters, namely energy \(E\), angular momentum \(\lambda\), spin \(\alpha_k\) and deformation parameter \(\varepsilon\), respectively. We find that global transonic accretion solutions continue to exist in non-Kerr spacetime. Depending on the input parameters, accretion flow is seen to experience shock transition and we find that shocked induced accretion solutions are available for a wide range of the parameter space in \(\varepsilon\) plane. We examine the modification of the shock parameter space with \(\varepsilon\), and find that as \(\varepsilon\) is increased, the effective region of the parameter space is reduced, and gradually shifted towards the higher \(\lambda\) and lower \(E\) domain. In addition, for the first time in the literature, we notice that accretion flow having zero angular momentum admits shock transition when spacetime deformation is significantly large. Interestingly, beyond a critical limit of \(\varepsilon_{\text{max}}\), the nature of the central object alters from black hole to naked singularity and we identify \(\varepsilon_{\text{max}}\) as function of \(\alpha_k\). Further, we examine the accretion solutions and its properties around the naked singularity as well. Finally, we indicate the implications of the present formalism in the context of astrophysical applications.

Key words: Accretion dynamics – deformed Kerr black hole – naked singularity.

1 INTRODUCTION

Accretion of matter onto compact objects (namely, black holes (BHs) and neutron stars) is the most acceptable and prolific physical process in the context of energy generation in enigmatic objects like active galactic nuclei (AGNs) and X-ray binaries (XRBs) etc (Pringle 1981; Frank et al. 2002; Netzer 2013; Abramowicz & Fragile 2013). It has been noticed from observations that these objects undergo several spectral state transitions (Esin et al., 1998), and these spectral states are classified as “Low/Hard” state (LHS), “High/Soft” state (HSS), and “Intermediate” state (IMS), respectively. In order to understand the aforementioned spectral states, various accretion disc models have been developed. The standard thin disc model, developed considering Keplerian flow, (Shakura & Sunyaev 1973; Novikov & Thorne 1973) was successful to explain the HSS. To interpret the characteristics of LHS and IMS, several other disc models, namely the thick disc model (Paczynsky & Wiita 1980a; Paczynsky & Wiita 1980b; Chakrabarti 1990; Molteni et al. 1994), advective disc model (Liang & Thompson 1980a; Fukue 1987; Chakrabarti 1989), advection-dominated disc model (Narayan & Yi 1994; Chakrabarti 1996; Esin et al. 1997; Narayan et al. 1997; Lu et al. 1999) and truncated disc models (Esin et al. 2001; Done et al. 2007) were studied. Among them, some of the disc models are also potentially viable to comprehend the origin of the relativistic jets and quasi-periodic oscillation (QPOs) phenomena as well. Needless to mention that the underlying scenario of these theories are landed into the fundamental aspects of the general relativity (GR). Since the hosted central object directly impacts on the properties of the accretion disc, such theory can successfully probe the signatures of strong gravity (e.g., event horizon, ergosphere, ISCO and shadow etc.), and eventually, can ascertain the physical parameters (i.e., mass and spin) of the central source.

Meanwhile/Recently, high precision observational measurements of the electromagnetic spectrum reveals some unusual features from the known Kerr signals. Such discriminant is also observed in the gravitational waves spectrum from the BHs or neutron stars binary system (Abbott et al. 2016a; Abbott et al. 2016b). In these circumstances, several research groups have reported the parametric deviations to the Schwarzschild and Kerr black holes (Johannsen & Psaltis 2011; Rezzolla & Zhidenko 2014; Konoplya & Zhidenko 2016). According to the no-hair theorem, such deviation to the original metrics brings alternative gravity theory (i.e., the metrics are no longer the Einstein gravity solutions). Thus, we may anticipate that the non-Kerr spacetime can affect various strong gravity signatures and illustrate the peculiarities in the observations. In the last decade, one of the emerging and smeared alternative gravity is the Johannsen-Psaltis (JP) metric (Johannsen & Psaltis 2011). They first include a deformation function, which contains infinite terms, in the Schwarzschild metric and then apply the Newman-Janis algorithm to convert into a rotational Kerr-like metric. After that, deformation terms are restricted through the observational limitations on the weak-field modification of GR and asymptotic flatness. The finally
obtained metric is characterized by the mass, spin and only one deviation parameter. When the deviation parameter is zero, it is reduced to the original Kerr metric. Their analysis also inflicts one valuable outcome in the calculation of the ISCO and circular photon orbits, and their dependency on the spin and deviation parameters under this proposed background. They show that, depending on the spin parameter, the central singularity of the spacetime becomes naked for outside observers when the deviation parameter crosses some limiting value. Usually, these irregularities in spacetime are described by the negative precession of the closed timelike orbits, which are the observational signature of the naked singular exotic objects.

Meanwhile, various investigations have been performed on the JP metric. For example, Bambi (2011) found the restriction to the spin parameter for non-Kerr BHs through the observational inconsistency in the radiative efficiency for luminous AGNs. In Bambi et al. (2012), the spacial topology of the event horizon for non-Kerr spacetime has been investigated. The properties of the ergosphere and the energy extraction by the Penrose process in a rotating deformed BH are carried out in Liu et al. (2012). Chen & Jing (2012) analyzed the strong gravitational lensing effect in a background of non-Kerr compact objects. Krawczynski (2012) distinguished between the original Kerr BHs hypothesis and non-Kerr BHs, and tested the no-hair theorem through the spectro-polarimetric observational data of the black hole XRBs. A detailed investigation of shadows and restriction to the spacetime parameters have been presented through the observation of polarization angels in Atamurotov et al. (2013). The inclusion of new parametric deviation approach and its challenge to the JP metric are encountered in Rezzolla & Zhidenko (2014). A review on the signatures of alternative gravity by employing the gravitational waves from the BHs merging is presented in Yagi & Stein (2016). The simultaneous existence of closed timelike orbits with negative precession and shadows is reported for the non-Kerr naked singular spacetime in Bambhaniya et al. (2021). Very recently, the properties of the accretion disc around a non-Kerr black hole without reflection symmetry have been revealed in Chen & Yang (2021). All these works evidently indicate that the JP metric attracts huge focus on it and also gets tremendous success for different applications in gravity. However, to the best of our knowledge, no one has conveyed the hydrodynamics of the accreting matter in the background of JP compact objects. Such deficiency in the literature pushes us to serve the present work, where we explore, for the very first time, the accretion dynamics of fluids in a spacetime of alternative gravity. We expect this analysis will lead to a better understanding of the non-Kerr spacetime in the light of accretion dynamics.

In this work, we solve the general relativistic Euler’s equation in the JP spacetime by utilizing the standard definitions of three velocities (Lu 1985) in a co-rotating frame. Even in the strong-field regime, flow equations mimic the Newtonian-like equations and provide the effective potential corresponding to the gravitating object (Dibbingia et al. 2018b). We derive the radial velocity gradient and temperature gradient equations using the relativistic equation of state (REoS) that endure variable adiabatic index ($\Gamma$). After developing the mathematical framework, our primary motivation is to express the influence of the deformed term on the flow properties. We start our analysis accommodating the effect of the deformation parameter ($\varepsilon$) on the nature of critical points and the global transonic solutions around BH. Next, we separate the parameter space (in angular momentum ($\lambda$) - energy ($E$) plane) by means of the nature of the solution topologies and show their modifications with the input parameters. The global shock solutions, including their inherent properties, have been studied in detail. An important result is presented where we depict that even zero angular momentum flow can possesses multiple critical points and consequently suffer the shock transitions. This eventually provides new signatures of accretion dynamics in the non-Kerr BH spacetime. We show that for a given spin parameter ($a_k$), the usual BH accretion solutions continue to present up to a maximum value of the deformation parameter ($\varepsilon_{\text{max}}$). When $\varepsilon > \varepsilon_{\text{max}}$, the nature of the solution changes due to the presence of an extra sonic point very close to the compact object. This possibly happens as the central source seems to appear as naked singularity (Dibbingia et al. 2020) which is examined using numerical as well as analytical means. We further calculate a parameter space spanned by the spin ($a_k$) and the deformation parameter ($\varepsilon$) according to the solution topologies around either BH or naked singularity state of the central objects. A comparison of $\varepsilon_{\text{max}}$ obtained in the pseudo-Newtonian model and analytical approach is presented where good agreement is seen. This evidently indicates that the accretion dynamics provides an alternative window to distinguish the subtle nature of the compact objects.

Finally, we wish to emphasis that the global transonic accretion solutions in the deformed Kerr spacetime continue to exist as in the case of original Kerr spacetime. From our analysis, two new findings are imparted. One of them is the multiple critical point solutions including shock transitions. Another one is the existence of naked singularity for non-Kerr spacetime even if the spin parameter $a_k < 1$. In the alternative gravity theory of GR, these specific findings can be considered observational evidence to distinguish the deformed spacetime from the original Kerr spacetime.

This paper is arranged as follows. In Section 2, we develop the mathematical framework of the accretion disc theory and set up the critical point conditions. In Section 3, we present the effect of $\varepsilon$ on the sonic point analysis, global flow solutions and modification of the parameter space for the non-Kerr BH. Section 4 analyzes the shock-induced global accretion solutions and their parameter spaces. The dependence of shock properties on $\varepsilon$ is also established in this section. In Section 5, we show the flow solutions associated with zero angular momentum flows. In Section 6, we depict how $\varepsilon$ incorporates the naked singularity in the system through sonic point analysis and their corresponding transonic solutions. In addition, the change of parameter space of multiple critical points for non-Kerr naked singularity has been presented in this section. In Section 7, we represent the properties of the spacetime parameters in the JP metric. Finally, in Section 8, we present conclusions.

2 ASSUMPTIONS AND GOVERNING EQUATIONS

We present the basic equations governing the accretion flow using general relativistic hydrodynamics. To avoid mathematical complexity, the accretion disc is assumed to remain confined along the equatorial plane of the central object. We further consider the flow to be steady, inviscid and advective, where energy dissipations due to viscosity, thermal conduction, magnetic fields and radiative cooling are neglected.

2.1 Governing equations

In the standard Boyer-Lindquist coordinates ($t, r, \theta, \phi$), the deformed Kerr metric (known as JP metric) is expressed as (Johannsen & Psaltis...
The above relation is explicitly written for our metric (1). Since the motion is considered around the disk equatorial plane, and there are time translation as well as azimuthal symmetries, the equations (2) and (3) for $i = r$ are simplified as (Dihingia et al. 2018b, 2020),

$$\frac{e + p}{\rho} \frac{dp}{dr} - \frac{de}{dr} = 0$$  \hspace{1cm} (7)

and

$$\gamma_r^2 \frac{dv_r}{dr} + \frac{1}{h_1 \rho} \frac{dp}{dr} = 0.$$  \hspace{1cm} (8)

For the same reason, $i = \theta$ component of equation (3) is trivially satisfied and therefore does not lead to any new equation. However, for $i = \phi$, equation (3) provides the conservation of specific angular momentum (Dihingia et al. 2018b), which we already encountered through the Killing symmetries in the metric. Following (Chakrabarti 1989; Dihingia et al. 2018b, and references therein), we identify $\Phi_{\text{eff}}$ in equation (8) as the effective pseudo-potential and is given by

$$\Phi_{\text{eff}} = 1 + 0.5 \ln(\Phi).$$  \hspace{1cm} (9)

For the metric given in equation (1), $\Phi$ is given by

$$\Phi = \frac{(\Delta + a_h^2)(1 + h)r}{a_h^2(r + 2)(1 + h) - 4a_k \lambda (1 + h) + r^2 + \lambda^2 (r - 2)(1 + h)}.$$  \hspace{1cm} (10)

Integrating the mass conservation equation, we obtain the mass accretion rate ($M$) which is given by (Kumar & Chattopadhyay 2017; Dihingia et al. 2018b),

$$M = -4\pi \rho v_c \gamma_c H \sqrt{(\Delta + a_h^2)(1 + h)},$$  \hspace{1cm} (11)

where $H$ is the local half thickness of the disk. Following the work of Chattopadhyay & Ryu (2009), we adopt the relativistic equation of state (R EOS) and pressure ($p$) as,

$$e = \frac{p f}{\tau} \quad \text{and} \quad p = \frac{2 \rho \Theta}{\tau},$$  \hspace{1cm} (12)

where $\tau = 2 - \xi (1 - 1/\chi)$, the composition ratio $\xi = n_e/n_p$ and the mass ratio $\chi = m_e/m_p$. The number density and the mass of the $i$th species (electron, proton) are denoted by $n_i \in \{n_e, n_p\}$ and $m_i \in \{m_e, m_p\}$, respectively. Moreover, we consider $\xi = 1$ throughout our analysis. Here the quantity $f$ is obtained in terms of dimensionless temperature ($\Theta = k_B T/m_e c^2, k_B$ is the Boltzmann constant and $T$ is the flow temperature in Kelvin) as

$$f = (2 - \xi) \left[ 1 + \Theta \left( \frac{9 \Theta + 3}{3 \Theta + 2} \right) \right] + \xi \left[ \frac{1}{\chi} + \Theta \left( \frac{9 \Theta + 3/\chi}{3 \Theta + 2/\chi} \right) \right].$$  \hspace{1cm} (13)

For R EOS, polytropic index ($N$), adiabatic index ($\Gamma$) and sound speed ($C_s$) are defined as

$$N = \frac{1}{2} \frac{df}{d\Theta}, \quad \Gamma = 1 + \frac{1}{N}, \quad \text{and} \quad C_s^2 = \frac{\Gamma \rho}{e + p} = \frac{2 \Gamma \Theta}{f + 2 \Theta}.$$  \hspace{1cm} (14)

Considering the hydrodynamic equilibrium in the vertical direction, the local half thickness of the disk ($H$) is calculated as (Lasota 1994; Riffert & Herold 1995; Peitz & Appl 1997),

$$H = \sqrt{\frac{pr^3}{\rho F}} = \sqrt{\frac{2r^3 \Theta}{\tau F}},$$  \hspace{1cm} (15)

where

$$F = \frac{\gamma_r^2 (r^2 + a_h^2)^2 + 2 \Delta a_h^2}{(r^2 + a_h^2)^2} - 2 \Delta a_h^2.$$  \hspace{1cm} (16)

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Integrating equation (7) and using equation (12), the mass density is obtained as (Kumar et al. 2013; Chattopadhyay & Kumar 2013; Dihingia et al. 2020),

$$p = \mathcal{K} \exp \left( k_4 \Theta^{3/2} (3 \Theta + 2)^{k_1} (3 \Theta + 2/\chi)^{k_2} \right),$$

where \( k_1 = 3(2-\xi)/4, k_2 = 3\xi/4, k_3 = (f-\tau)/(2\Theta), \) and \( \mathcal{K} \) refers the entropy constant. Using equations (11) and (17), we compute the entropy accretion rate as (Chattopadhyay & Kumar 2016; Kumar & Chattopadhyay 2017),

$$\mathcal{M} = \frac{\dot{M}}{4\pi \mathcal{K}} = \nu \gamma_2 H \sqrt{(\Delta + a_k^2 h)(1 + h)}$$

$$\times \exp \left( k_4 \Theta^{3/2} (3 \Theta + 2)^{k_1} (3 \Theta + 2/\chi)^{k_2} \right).$$

Considering logarithmic derivative of equation (11) and setting the condition of constant mass accretion rate (i.e., \( d\mathcal{M}/dr = 0 \)), the temperature gradient is expressed as,

$$\frac{d\Theta}{dr} = -\frac{2\Theta}{2N + 1}$$

$$\times \left[ \frac{a_k^2}{v} \frac{d\nu}{dr} + N_{11} + N_{12} - 3\varepsilon \left( \frac{a_k^2}{\Delta + a_k^2 h} + \frac{1}{1 + h} \right) \right],$$

where

$$N_{11} = \frac{5}{2} + \frac{r - a_k^2(1 + h)}{r(\Delta + a_k^2 h)} \text{ and } N_{12} = -\frac{1}{2F} \frac{dF}{dr}.$$

The explicit form of the quantity \( \frac{1}{F} \frac{dF}{dr} \) is obtained by taking the logarithmic derivative of equation (16) and is given by,

$$\frac{1}{F} \frac{dF}{dr} = \gamma_v^2 \Delta \frac{d\nu}{dr} + 4a_k^2 (r^2 + a_k^2) \left[ \frac{(r^2 + a_k^2)\Delta}{(r^2 + a_k^2)^2} - 4\Delta r \right]$$

$$\times \frac{1}{(r^2 + a_k^2 + \Delta + a_k^2 h)^2},$$

where

$$\Delta = \frac{d\Delta}{dr} = 2(r - 1);$$

$$\Omega = \frac{d\Omega}{dr} = -2(1 + h)$$

$$\times \left[ \frac{a_k^2(1 + h) - 2a_k^2 \lambda(1 + h) + a_k[\lambda^2(1 + h) + 3(r^2)] + r^2 \lambda(r - 3)}{[a_k^2(r + 2)(1 + h) - 2a_k \lambda(1 + h) + r^3]^2} \right]$$

$$- \frac{3\varepsilon \lambda(r - 2 - 2a_k)}{r[a_k^2(r + 2)(1 + h) - 2a_k \lambda(1 + h) + r^3]^2}.$$  

Finally, we capitalize equation (14) and obtain the radial velocity gradient from equation (8) as,

$$\frac{dv}{dr} = \frac{N}{D},$$

where the explicit form of the denominator \( D \) and that of the numerator \( N \) are represented by,

$$D = \gamma_v^2 \left[ v - \frac{2C_s^2}{(\Gamma + 1)v} \right]$$

and

$$N = \frac{2C_s^2}{\Gamma + 1}$$

$$\times \left[ N_{11} + N_{12} - \frac{3\varepsilon}{2r^4} \left( \frac{a_k^2}{\Delta + a_k^2 h} + \frac{1}{1 + h} \right) \right] - \frac{d\Phi_{\text{ext}}}{dr}.$$

### 2.2 Critical point conditions

In an accretion process around a gravitating object, infalling matter starts accreting with negligible radial velocity from the outer edge of the disk (usually far away from the horizon) and remain sub-sonic. On the other hand, accretion flow enters into the black hole supersonically in order to satisfy the inner boundary conditions imposed by the event horizon. Since the motion of the flow generally remains smooth everywhere, accreting matter experiences sonic state transition at some point to become transonic (Liang & Thompson 1980b; Abramowicz & Zurek 1981) and such a point is referred as critical point (\( r_c \)). At \( r_c \), the radial velocity gradient takes \( (dv/dr)_{r_c} = 0/0 \) (equation 23) form as it must be real and finite, and hence, we obtain the critical point conditions by setting \( D = N = 0 \) simultaneously which are given by,

$$v_c^2 = \frac{2C_s^2}{(\Gamma + 1)}$$

and

$$C_s^2 = \frac{\Gamma_c + 1}{2} \left( \frac{d\Phi_{\text{ext}}}{dr} \right)_{r_c} \times$$

$$\left[ (N_{11})_{r_c} + (N_{12})_{r_c} - \frac{3\varepsilon}{2r_c^4} \left( \frac{a_k^2}{\Delta_c + a_k^2 h_c} + \frac{1}{1 + h_c} \right) \right]^{-1},$$

where the subscript “\( c \)” refers quantities measured at the critical point. We evaluate \( (dv/dr)_{r_c} \) by applying l’Hôpital’s rule which is obtained as

$$\frac{dv}{dr}_{r_c} = -\frac{B \pm \sqrt{B^2 - 4AC}}{2A},$$

where the resulted form of quantities \( A, B \) and \( C \) are presented in Appendix A. In general, critical points are classified in three different categories. For saddle type critical points, both values of \( (dv/dr)_{r_c} \) are real with opposite sign. For nodal type critical points, both values of \( (dv/dr)_{r_c} \) are real and same sign, whereas for O-type critical point, \( (dv/dr)_{r_c} \) becomes imaginary. It is noteworthy that any physically acceptable accretion solution only passes through the saddle type critical point (Das 2007; Chakrabarti & Das 2004, and references therein), and hence, in this work, we focus only those accretion solutions that possess saddle type critical point (hereafter critical point). We further mention that accretion flow may contain multiple critical points depending of the flow parameters and flow of this kind are potentially favourable to contain shock wave (see §4).

### 3 Hydrodynamics with deformation

In this section, we explore the role of the deformation parameter (\( \epsilon \)) in deciding the nature of the critical points as well as the accretion solutions in the non-Kerr (deformed) spacetime. While doing this, we identify the range of parameters that allows accretion solutions around black holes. We also put efforts in examining the nature of the accretion solutions beyond black hole environment as well.

#### 3.1 Critical points analysis

As the accretion solutions embrace the critical points, we start our analysis by understanding the nature of critical points. For that we calculate the specific energy \( (E_c) \) at a critical point \( (r_c) \) by solving...
equations (4), (26) and (27) using the global parameters, namely $\lambda$, $\varepsilon$ and $a_\text{k}$, respectively. In Fig. 1, the variation of $E_c$ as a function of $r_c$ is presented for different $\lambda$ with $\varepsilon = 3$ (see panel (a)) and for different $\varepsilon$ with $\lambda = 2.8$ (see panel (b)). Presently, we choose the Kerr parameter $a_\text{k} = 0$. However, we mention that there are no qualitative differences between the characteristics of critical points for the non-spinning and spinning black holes. Hence, in this work, most of the analyses have been carried out considering $a_\text{k} = 0$, although there are instances where results for $a_\text{k} \neq 0$ are presented according to the necessity. In the figure, different $\lambda$ and $\varepsilon$ values are marked in the respective panels. We use black, blue, red and orange curves for $\lambda = 2.5, 2.7, 2.9$ and 3.1 respectively. The same color curves are used to represent results for $\varepsilon = 0, 3, 6$ and 9, respectively. In each panel, a given curve is generally comprised with saddle, nodal and O-type critical points and they are demonstrated by the solid, dotted and dashed curves respectively. Moreover, these critical points appear in sequence as saddle-nodal-spiral-nodal-saddle as $r_c$ is increased. In addition, we observe that all curves have an asymptotic behaviour towards $E_c \approx 1$ (dash-dotted horizontal line) for larger values of $r_c$ irrespective of $\lambda$ and $\varepsilon$ values. Depending on $E_c$, $\lambda$ and $\varepsilon$, flow may contain either single or multiple critical points. Usually, critical points formed near and far away from the horizon are called as inner ($r_\text{in}$) and outer critical points ($r_\text{out}$), respectively. It is evident from the figure that there exists a range of $E_c$ that yields multiple critical points and such energy range is strictly depends on $\lambda$ and $\varepsilon$ values. Following this, in §3.3, we put effort to identify the effective region of the parameter space based on the nature of the accretion solutions. Overall, it is now evident that $\varepsilon$, $\lambda$ and $E_c$ play pivotal role in determining the nature of the critical points and its associated properties.

### 3.2 Effect of $\varepsilon$ on global accretion solutions

Here, we examine the impact of $\varepsilon$ on the accretion solutions. While doing so, we calculate the location of the critical point ($r_c$), and the corresponding radial velocity ($v_c$) and dimensionless temperature ($\Theta_c$) at $r_c$ by simultaneously solving equations (4), (14), (26) and (27) for a given set of input parameters ($\lambda$, $E$, $a_\text{k}$ and $\varepsilon$). We employ $\Theta_c$ and $v_c$ as the initial values at $r_c$ to simultaneously solve equations (19) and (26) once inward up to the horizon ($r_\text{H}$) and then outward up to the outer edge of the disk ($r_\text{edge}$). Finally, we join this two parts of the solution to obtain the complete radial profiles of velocity ($v$) and temperature ($\Theta$). In Fig. 2, we depict the accretion solutions ($M$ vs. $r$) for different $\varepsilon$, where $E = 1.001$, $\lambda = 2.8$ and $a_\text{k} = 0$ are chosen. In panels (a-d), the variation of Mach number ($M$) as function of radial distances ($r$) is presented for $\varepsilon = 0, 3, 6$ and 9, respectively. Here, the solid curve represents the accretion solution whereas dashed curve denotes the corresponding wind solution. In the figure, filled circles refer the critical points, where inner ($r_\text{in}$) and outer ($r_\text{out}$) critical points are marked. We observe that for $\varepsilon = 0$, the flow passes through the outer critical point at $r_\text{out} = 198.6334$ and connects the outer edge of the accretion disc ($r_\text{edge} = 1000$) to the black hole horizon ($r_\text{H}$) (see panel (a)). As the deformation parameter is increased (say $\varepsilon = 3$) keeping other input parameter unchanged, inner critical point is appeared at $r_\text{in} = 5.7136$ along with the outer critical point at $r_\text{out} = 198.5498$. Interestingly, the solution passing through the outer critical point continues to connect $r_\text{H}$ and $r_\text{edge}$, however the inner critical point solutions fail to do so at it terminates at a radius ($r_\text{t} = 20.3336$) in between inner and outer critical points as $r_\text{in} < r_\text{t} < r_\text{out}$ as shown in panel (b). For $\varepsilon = 6$, the nature of the flow solution remains qualitatively similar to panel (b) although $r_\text{t} = 77.7947$ is increased (see panel (c)). We wish to emphasize that solutions presented in panel (b-c) may experience shock transition and we plan to discuss it elaborately in §4. With the further increase of deformation parameter $\varepsilon = 9$, the solution characteristics are changed completely as shown in panel (d). We find that the solution passing through $r_\text{in} = 3.4628$ smoothly connects $r_\text{edge}$ to $r_\text{H}$, but the possesses $r_\text{out} = 198.3757$ fails to do so. Hence, it is evident that $\varepsilon$ plays a decisive role in determining the nature of the accretion solutions around the central object under consideration.
3.3 Parameter space based on nature of accretion solutions

In this section, we separate the effective region of the parameter space (in $\lambda - E$ plane) according to the nature of the accretion solutions. The obtained results are plotted in Fig. 3(A), where four regions, namely (a), (b), (c) and (d) are identified according to the nature of the solution topologies. We present their respective solutions in inset panels. Here, the Mach number ($M = v/C_s$) is plotted with the radial distances ($r$) in each panel. The accretion and wind solutions are denoted by black (solid) and blue (dashed) lines respectively, and the filled circles are used for the saddle type critical points. Note that all the solutions which have been drawn in the inset panels contain the saddle type critical points only. The area with in red (dashed) line represents the region (b) and the remaining area with in the grey (solid) line represents the region (c). The notion behind the finding of the region (b) is that the entropy accretion rate at the inner critical point is more compared to the outer critical point; however, for region (c), the adopted logic is just the reverse one. For panel (a), we choose the flow parameters as $\lambda = 2.5$ and $E = 1.001$. In this case, the critical point is found at $r_{cut} = 203.562$, which is a saddle one. The solutions corresponding to panel (b) are obtained for input parameters $(\lambda, E) = (2.80, 1.001)$, where the critical points are obtained at $r_{in} = 4.5132$ (saddle-type), $r_{i} = 13.0295$ (O-type) and $r_{out} = 198.4749$ (saddle-type). In panel (c), we plot the solutions while choosing the flow parameters as $(\lambda, E) = (3.1, 1.0001)$. Here, we calculate the critical points as $r_{in} = 4.0795$ (saddle-type), $r_{i} = 17.0861$ (O-type) and $r_{out} = 194.5943$ (saddle-type). Finally, we find the solution in panel (d) for injected flow parameters $(\lambda, E) = (3.1, 0.0045)$ and evaluate the critical point as $r_{in} = 4.0445$ (saddle-type). Therefore, if we consider any flow parameter $(\lambda, E)$ from the regions (a) and (d), the solution possesses one critical point only. However, any flow parameters $(\lambda, E)$ from the regions (b) and (c) provide the solutions containing both the critical points.

Now we investigate the black hole solutions for spin parameter $a_k = 0.99$, checking whether it is consistent or not. In Fig. 3(B), we separate the parameter space as (a), (b), (c) and (d) by following different solution topologies for a rapidly rotating black hole ($a_k = 0.99$) and capitalize $\varepsilon = 0.015$. Here, all lines and dots bring identical information as in panel (A). The global accretion solution associated with the inflow parameters $(\lambda, E) = (1.8, 1.015)$ has been presented in inset panel (a), which contain only outer critical point $r_{out} = 357.8267$. In panel (b), we depict the solutions corresponding to the input parameters $(\lambda, E) = (1.97, 1.0005)$. In this case, solutions possess multiple critical points $(r_{in} = 1.3283, r_{i} = 4.433$ and $r_{out} = 355.8871)$. For $(\lambda, E) = (2.1, 0.007)$, we display the accretion solutions in inset panel (c). The sonic points are obtained as $r_{in} = 1.2606, r_{i} = 5.6399$ and $r_{out} = 34.3714$. However, the solutions in panel (d) promote inner critical point only $(r_{in} = 1.2575)$. It is to be noted that even for the highly rotating black holes, the flow solutions are similar to that of the nonrotating black holes.

We additionally figure out various flow parameters (defined in equations (12), (14), (18) and (23)) in Fig. 4, associated with the global accretion solution in inset panel (a) of Fig. 3(A). Moreover, we consider that the flow enters at the outer edge $r_{edge} = 1000$ of the accretion disc. The evolution of the bulk velocity ($v$), dimensionless temperature ($\theta$), mass density ($\rho$), local pressure ($p$), adiabatic index ($\Gamma$) and entropy accretion rate ($M$) as function of radial coordinates ($r$) are presented in panels (a), (b), (c), (d), (e) and (f) respectively. It is depicted that all inspected quantity increases monotonically (see panels (a) - (d)) as we move towards the inner egde until it gently fall into the black hole horizon. However, the adiabatic index decreases (see panel (e)) with the decrease of radial coordinates due to the rise of temperature. We calculate the entropy accretion rate as $M = 2.7202 \times 10^7$ from the panel (f), revealing a constant quantity for a particular solution.

Variability in the parameter space of a set of parameters corresponding to given other parameters is quite often in the context of accretion dynamics. To acknowledge this identity, we represent the alteration of parameter space associated with multiple critical points (saddle types) in the $\lambda - E$ plane for different $\varepsilon$ in Fig. 5. We consider the Kerr parameter $a_k = 0$ and mark the various $\varepsilon$ in this panel. Different potential regions are obtained while using blue (solid), black (dashed), magenta (dotted) and red (dash-dotted) lines for $\varepsilon = 0, 5, 10$ and 15.2 respectively. Here, we observe that the parameter space is shifted towards lower angular momentum and higher energy with the increase of the deformation parameters for a specific value of Kerr parameter. However, it is understandable that more deformations to spacetime endorse the larger areas in the parameter space. Interestingly, a small nozzle-shaped area for multiple critical points is found at zero angular momentum for $\varepsilon = 15.2$. It is essential to highlight that choice $\varepsilon = 15.2$ is not arbitrary; it provides the minimum value of the deformation parameter, which delivers a small energy range corresponding to the multiple sonic points for $\lambda = 0$. When we take any $\varepsilon$ beyond the value mentioned above, we can carry out the multiple sonic points even for $\lambda = 0$; obviously, it depends on the spin parameter. This finding is entirely new, and no one ever has reported it in the literature. But the big question is whether the associated
accretion solutions for zero angular momentum are consistent with the general black hole solutions corresponding to non-zero angular momentum. Section 5 has discussed a detailed analysis on this subject.

In the above case, we discovered the parameter space for the non-rotating black hole. Now it’s time to invent the effect of the non-zero spin on parameter space for a given deformation parameter. Here, we fix $\epsilon = 0.02$, which grants the previously discussed accretion solutions for the black hole of different spins. In Fig. 6, we study the parameter space modification for saddle type multiple sonic points in the $\lambda - E$ plane for different Kerr parameters ($a_k$) with $\epsilon = 0.02$. In this panel, we mark $a_k$ values as well. The functional domains are separated using the black (solid), blue (dashed) and red (dotted) curves corresponding to $a_k = 0, 0.5$ and 0.99 respectively. It is clarified that the segment within a curve moves into the lower angular momentum and higher energy sites. Moreover, the area under an identified province is more in the case of a rapidly rotating object than that of a moderately spinning black hole. Therefore, the above two analyses suggest that the Kerr parameter and the deformation parameter play the equivalent role in controlling the parameter space.

Figure 3. Division of parameter space in $\lambda - E$ plane according to the nature of the flow solution topologies. For panel (A) and (B), we choose $(a_k, \epsilon) = (0, 5)$ and $(0.99, 0.015)$, respectively. In each plot, four regions are marked as (a), (b), (c), (d) and the corresponding solutions are shown in the inset panels. See text for details.

Figure 4. Variation of (a) bulk velocity ($v$), (b) temperature ($\Theta$), (c) density ($\rho$), (d) pressure ($p$), (e) adiabatic index ($\Gamma$), (f) entropy accretion rate ($\dot{M}$) with the radial distances ($r$). Here, we choose $\lambda = 2.5$, $E = 1.001$, $a_k = 0$ and $\epsilon = 5$, respectively. See text for details.

Figure 5. Modification of the parameter space in $\lambda - E$ plane for multiple critical points with deformation parameter ($\epsilon$). Here, we fix Kerr parameter $a_k = 0$. Regions bounded with blue (solid), black (dashed), magenta (dotted) and red (dash-dotted) curves are for $\epsilon = 0, 5, 10$ and 15.2, respectively. See text for details.
4 ACCRETION SOLUTIONS WITH SHOCK TRANSITIONS

In Subsection 3.2, we mentioned about the possibility of existence of shock solution as the necessary (but not sufficient) condition for shock transition – possession of higher entropy content in the subsonic branch compared to the supersonic branch – was shown to be satisfied. In principle, the solution enters subsonically from the outer edge of the disc which becomes supersonic after being passed through the external critical point and continues to accrete towards the horizon. In the meantime, discontinuous shock transitions (Fukue 1987; Chakrabarti 1989; Yang & Kafatos 1995; Chakrabarti & Das 2004; Chattopadhyay & Kumar 2016; Kumar & Chattopadhyay 2017; Dihingia et al. 2018b; Dihingia et al. 2018a; Dihingia et al. 2019a,b) happen in the flow variables when the centrifugal repulsion becomes comparable to the black hole’s gravitational pull. Therefore, at the centrifugally driven shock location, fluid has to simultaneously satisfy the following relativistic shock conditions (Taub 1948):

\[
[pu'] = 0; \quad [(e + p)u' u'] = 0; \quad [(e + p)u' u' + pp'] = 0,
\]

along with the pre-mentioned necessary condition. Here the terms in the square brackets refer to the difference in respective quantities before and after shock. In Fig. 7, we represent the global accretion solutions which comprise the shock transition for fluid parameters \((\lambda, E) = (3, 1.0005)\). Here, we choose \(a_k = 0, \varepsilon = 3, E = 1.0005\) and \(\lambda = 3\). In this figure, the Mach number \((M)\) is plotted with the radial distances \((r)\). It is observed that the solution (solid, grey) accreting through the outer critical point \(r_{\text{out}} = 339.7504\) builds a steady connection between the outer edge accretion disc and the horizon. However, the solution (dashed, magenta) passing through the inner critical point \(r_{\text{in}} = 4.7916\) is closed one and connects the horizon only. The global shock solutions is intimated by the red (solid) curve and the vertical line denotes the shock transition at the shock location \(r_{\text{sh}} = 50.1706\), which is calculated by using (29). The arrow heads indicate the overall flow direction.

Shock has a significant impact on the fluid and disc parameters. Here, we invent how the shock transitions affect the flow variables. In Fig. 8, we illustrate the dynamics of several flow parameters corresponding to the global shock solution of Fig. 7. We depict the variation of radial velocity \((v)\), mass density \((\rho)\), dimensionless temperature \((\Theta)\), local pressure \((p)\), adiabatic index \((\Gamma)\) and entropy accretion rate \((\mathcal{M})\) with the radial coordinate \((r)\) in panels (a), (b), (c), (d), (e) and (f) respectively. In panel (a), it is explicated that the radial velocity experiences a sudden jump at the shock location.
Therefore, the fluid gets compressed and correspondingly a sharp transition in the local density is detected in panel (b). It is interpreted from the panel (c) that the temperature changes substantially when the flow transitions to the subsonic arm from the supersonic one due to the conversion of kinetic energy into thermal energy. Hence, the fluid components (electrons, ions, etc.) collide more and stimulate the high-pressure content after the shock transition (see panel (d)). As the temperature in the post-shock disc is higher than the pre-shock disc, such discontinuity in other temperature-dependent quantities is also exposed (see panels (e) and (f)). However, after shock transition, the electron clouds in the post-shock disc become ultra-relativistic due to high temperature and trigger the inverse Compton effect to soft photons coming from the pre-shock disc; hence, produce the hard and non-thermal flux distribution in the electromagnetic spectrum of the accretion disc. Not only that, several numerical simulations show that extremely thermalized electrons are deflected along both directions of the BH’s rotation axis and thereby address the bipolar relativistic jets (Molteni et al. 1994; Chattopadhay & Das 2007; Das & Chattopadhay 2008; Kumar & Chattopadhay 2013; Das et al. 2014; Kumar et al. 2014; Kumar & Chattopadhay 2017). This analysis intimates that the shock transitions play a crucial role in controlling the flow parameters and the black hole’s spectral properties. So far, we have not confessed the the role of \( \varepsilon \) governing the shock-induced global accretion solutions and flow parameters. The following subsection shape the above mentioned requirement through explicit analysis.

### 4.1 Effect of \( \varepsilon \) on global shock solutions and shock properties

This subsection starts our discussion by considering the black hole of spin \( a_b = 0 \). Here, we assume the flows entered at the outer edge \( r_{\text{edge}} = 300 \) with the energy \( E = 1.0013 \) and angular momentum \( \lambda = 3 \). In Fig. 9, we represent the variation of Mach number \( (M) \) as the function of radial distances \( (r) \) for different \( \varepsilon \). The shock locations are moving away from the black hole horizon with the increase of deformation parameter. For this specific case the shock locations are computed as \( r_{\text{sh}} = 22.5278, 36.1334, 61.0066 \) and 86.8639 corresponding to \( \varepsilon = 0, 1, 2 \) and 2.5, respectively. Here, we choose \( \lambda, E = (3, 0.0013) \), and \( a_b = 0 \). In each panel, shock locations are indicated by the vertical lines at \( r_{\text{sh}} = 22.5278, 36.1334, 61.0066 \) and 86.8639, respectively. See text for details.

In Fig. 10(a), we represent the bulk radial velocity \( (v) \) profile with the radial coordinates \( (r) \) corresponding to the above mentioned deformation parameters which embrace the shock fonts. Furthermore, we consider the same set of input parameters as we took in Fig. 9. As the shock fonts move away from the horizon, the vertical jumps in \( v \) decreases with the increase of \( \varepsilon \). After a specific value of \( \varepsilon \) (we call this as \( \varepsilon_c \)) shock transitions cease to exist and smooth accretion through the extreme critical point will continue again. The exact value of \( \varepsilon_c \) is not calculated yet; however, its accurate estimation will be given later when discussing the shock properties. We also characterise other flow parameters for the shock solutions corresponding to the input parameters as in Fig. 10(a). The mass density \( (\rho) \), dimensionless temperature \( (\Theta) \), local pressure \( (p) \), adiabatic index \( (\Gamma) \) and entropy accretion rate \( (\dot{\mathcal{M}}) \) are plotted as a function of \( r \) in Fig. 10(b), (c), (d), (e) and (f) respectively. In all panels, we use black (solid), blue (dotted), red (dashed) and magenta (dash-dotted) and angular momentum \( \lambda = 3 \). In Fig. 9, we represent the variation of Mach number \( (M) \) as the function of radial distances \( (r) \) for different \( \varepsilon \). The black (solid), blue (dotted), red (dashed) and magenta (dash-dotted) curves are used for \( \varepsilon = 0, 1, 2 \) and 2.5 respectively. The sonic points are spotted by the filled circles corresponding to different \( \varepsilon \). The shock transitions cease to exist and smooth accretion through the extreme critical point will continue again. The exact value of \( \varepsilon_c \) is not calculated yet; however, its accurate estimation will be given later when discussing the shock properties. We also characterise other flow parameters for the shock solutions corresponding to the input parameters as in Fig. 10(a). The mass density \( (\rho) \), dimensionless temperature \( (\Theta) \), local pressure \( (p) \), adiabatic index \( (\Gamma) \) and entropy accretion rate \( (\dot{\mathcal{M}}) \) are plotted as a function of \( r \) in Fig. 10(b), (c), (d), (e) and (f) respectively. In all panels, we use black (solid), blue (dotted), red (dashed) and magenta (dash-dotted)
curves curves for $\varepsilon = 0, 1, 2$ and 2.5 respectively. The difference in entities across the shock transition become less when the deformation in the spacetime is increased due to the movement of shock locations towards the outer edge of the accretion disc. This analysis glimpse the consequence of the deformation parameter on the dynamics of global shock solutions and flow variables but is unable to provide complete information. Next, to shape our objective more powerfully, we focused on the shock properties.

A detailed analysis of the shock properties has been done here. We explore various shock properties corresponding to different angular momentum ($\lambda$) flows of energy $E = 1.0012$, injected at the outer edge $r_{\text{edge}} = 1000$ of the accretion disc. The variation of shock location ($r_{\text{sh}}$) with $\varepsilon$ is presented in Fig. 11(a). We apply the black (solid), red (dotted), blue (dashed), orange (dash-dotted) and magenta (long dashed) curves for $\lambda = 3, 3.025, 3.05, 3.075$ and 3.1 respectively. It is found that the shock location is increased with the increase of $\varepsilon$ for a given $\lambda$. When the deformation parameter exceeds a critical value ($\varepsilon_c$), shock does not exist due to the violation of conditional relations at the shock location. A sophisticated calculation leads the values of $\varepsilon_c$ are $2.82, 2.47, 1.8, 1.43$, and 0.81 for the respective $\lambda$ mentioned here. The decrease in $\varepsilon_c$ is noticed for the high angular momentum flows when $(a_k, E)$ are fixed. Hence, $\varepsilon_c$ is not a universal quantity and it strongly depends on the injected input parameters. We also notice that the shock locations advance towards the outer edge with the increase of $\lambda$ for a given $\varepsilon$. This outcome implies that the shock solutions are probably driven by centrifugal repulsion. As we mentioned earlier, the flow density and temperature are substantially increased in the post-shock region due to fall off the radial velocity at the shock location. Therefore, it is significant to investigate the density and temperature distributions across the standing shock. We define the compression ratio ($R$) as the ratio of post-shock surface density ($\Sigma_+$) to pre-shock surface density ($\Sigma_-$). Since the surface density $\Sigma \sim \rho H$ (Chakrabarti & Das 2001), we obtain the expression of $R$ at the shock location ($r_{\text{sh}}$) with the help of (11) as (Das 2007; Das & Chakrabarti 2008; Sarkar et al. 2018; Sarkar & Das 2018).

$$R = \frac{\Sigma_+(r_{\text{sh}})}{\Sigma_-(r_{\text{sh}})} = \frac{p_+(r_{\text{sh}})H_+(r_{\text{sh}})}{p_-(r_{\text{sh}})H_-(r_{\text{sh}})} = \frac{\gamma_+}{\gamma_-} \frac{v_+(\gamma_+)_{+}}{v_-(\gamma_+)_{-}}$$

where $-\text{ and } +$ refer to the quantities before and after shock. In Fig. 11(b), we depict the variation of $R$ as function of $\varepsilon$ for the same set of input parameters as in Fig. 11(a). Here, the compression ratio decreases with the increase of deformation parameter. This result is quite expected because when the shock fonts move towards the outer edge, the compression of flow in the post-shock region becomes less (see Fig. 10b). To understand the temperature jump at the shock transition, we define the shock strength ($S$) as the ratio of the pre-shock Mach number ($M_-$) to the post-shock Mach number ($M_+$)

$$S = \frac{\Sigma_+(r_{\text{sh}})}{\Sigma_-(r_{\text{sh}})} = \frac{p_+(r_{\text{sh}})H_+(r_{\text{sh}})}{p_-(r_{\text{sh}})H_-(r_{\text{sh}})}$$

In Fig. 11(c), we analyze the variation of the shock strength ($S$) with $\varepsilon$. Here, we consider the same input parameters, line styles and line colors as used in Fig. 11(a) and (b). The result explicits that the shock strength decreases with the increase of $\varepsilon$. As the shock locations step away from the horizon, we would expect the shock strength to decrease and hence our analysis is precisely justified here (see Fig. 10c). The next paragraph will stick to shock properties analysis, setting $\lambda$ as a universal parameter instead of $E$.

In Fig. 12, we explore the variation of the shock variables with $\varepsilon$ for flows injected with identical angular momentums $\lambda = 3$ but possessing different energies $(E)$. Here, we consider the flows enter at $r_{\text{edge}} = 1000$ and choose $a_k = 0$. Various shock properties ($r_{\text{sh}}, R$ and $S$) are plotted in panels (a), (b) and (c) respectively. In each panel, we employ the black (solid), red (dotted), blue (dashed), orange (dash-dotted) and magenta (long dashed) curves corresponding to $E = 1.0011, 1.0013, 1.0015, 1.0017$ and 1.0019 respectively. For a fixed $E$, the shock fonts are shifted outward from the central
object with the increase of \( \varepsilon \). However, the standing shock transitions monotonically wipes out as one increase \( \varepsilon \). When \( \varepsilon > \varepsilon_c \), shock conditions are failed to maintain, hence shock solutions disappear. Here, we also explicitly calculate the quantity \( \varepsilon_c \) as 3.04, 2.68, 2.03, 1.61, and 1.07, respectively associated with the energies as mentioned earlier. It is to be noticed that \( \varepsilon_c \) decreases with the increase of energies and analogously verifies the preferential dependence of \( \varepsilon_c \) on the initial parameters. Moreover, the shock location increases with the increase of energy for a fixed deformation parameter. We also observe that the compression ratio and shock strength decreases with the increase of \( \varepsilon \) due to less flow compression towards outer edge. Following the above studies, we should mention that without knowing the parameter space that encompasses the shock solutions, playing with any analysis connected to shock transitions is relatively obscure, especially when dealing with the larger span of the parameter space. Once we finish it, our job becomes more accessible, and also, we can call any parameters during the investigation. We will calculate the shock parameter space in the next subsection.

4.2 Shock parameter space

We now estimate the shock parameter space to enlarge the shock identity. Here, we first characterise the effective regions of the parameter space corresponding to different \( \varepsilon \) while keeping \( a_k \) fixed. In Fig. 13, we find the shock parameter space in \( \lambda - E \) plane for four different \( \varepsilon = 0, 5, 10 \) and 15 respectively with \( a_k = 0 \). Identified provinces are bounded with the blue (solid), black (dashed), magenta (dotted) and red (dash-dotted) curves corresponding to respective \( \varepsilon \). It is observed that the area under the curve increases with the increase of deformation parameters. Additionally, the shock parameter space is shifted towards the lower angular momentums and to the higher energies as deformation in the system increases for a given spin of the black hole.

Next we study the modification of the shock parameter space for different \( a_k \), but this time \( \varepsilon \) is taken as constant. Here, we pick up \( \varepsilon = 0.02 \), which holds a similar cause as in Fig. 6. We represent the shock parameter space in \( \lambda - E \) plane for distinct \( a_k \) in Fig. 14. Black (solid), blue (dashed) and red (dotted) lines have been sponsored to characterize the valid regions for shock transitions corresponding to \( a_k = 0, 0.5 \) and 0.99 respectively, and Kerr parameters are appropriately annotated. We observe that the parameter space moves into the lower angular momentums and higher energies zone as we increase black hole spin for a given deformation parameter. As the shock parameter space is the shrink version of the whole multiple critical point parameter space, its alteration process is not surprising because the entire parameter space for multiple sonic points, as discussed in Subsection 3.3, is shifted accordingly.

5 FLOW SOLUTIONS CORRESPONDING TO \( \lambda = 0 \)

There are few unique interesting features are being observed here in accretion flow due to the deformation in the spacetime which were absent in Kerr spacetime. Presence of such properties clearly isolates the central object from the usual Kerr. In this section we will discuss one of them.

We have already encountered the multiple sonic points for flow with zero angular momentum in Subsection 3.3. In this analysis, we examine the presence of accretion solutions associated with these sonic points and remark on them. We assume that the flows enter at the extreme edge \( r_{\text{edge}} = 1000 \) with the angular momentum \( \lambda = 0 \). However, the choice of deformation parameters must be \( \varepsilon \geq 15.2 \) for \( a_k = 0 \), as mentioned before. Let us initiate our survey with the global accretion solutions for \( (\varepsilon, E) = (16, 1.017) \) in Fig. 15(a), where the Mach number (\( M \)) is sketched with the radial locations (\( r \)). In this case, solid and dashed curves have been applied to signify the accretion and wind solutions respectively. We use the filled dots corresponding to the saddle type critical points calculated here. It is clarified that the solutions occupy only the outer sonic point (\( r_{\text{out}} = 19.8723 \)) and join the exterior part of the accretion disc to the black hole horizon. The entropy accretion rate at the sonic point is calculated as \( M_{\text{out}} = 20.5276 \times 10^7 \). In Fig. 15(b), we plot the accretion solutions associated with \( (\varepsilon, E) = (17, 1.017) \) and interpret that the solutions won both inner and outer critical points. The critical points are computed as \( r_{\text{in}} = 4.0993 \) (saddle-type), \( r_{\text{s}} = 5.3533 \)
(O-type) and $r_{\text{out}} = 19.6517$ (saddle-type). In this case, the solutions flowing down the external sonic point are connected with the central singular point. But the solutions passing through the inner sonic point seem to form a closed loop near the horizon. The entropy accretion rates in subsonic and supersonic branches are $M_{\text{in}} = 24.7699 \times 10^7$ and $M_{\text{out}} = 20.4652 \times 10^7$ respectively. Since the entropy content for the inner one is higher than the outer one, the solution may experience a shock transition and attains both critical points. However, we do not compute the global shock solutions in this regard. For shock-induced global accretion solution, we set $(\varepsilon, E) = (18, 1.017)$. In Fig. 15(c), the red (solid) curve illustrates shock solution in association with the background general accretion solutions. The satisfied shock conditions at $r_{\text{sh}} = 8.7829$ and the result $M_{\text{in}}(22.6956 \times 10^7) > M_{\text{out}}(20.4013 \times 10^7)$ lead the sharp jump into the subsonic branch from the supersonic branch, which has been presented by the vertical line in this panel. Location of two saddle-type sonic points are given by $r_{\text{in}} = 3.8059$ and $r_{\text{out}} = 19.4199$ and the spiral one is obtained as $r_s = 6.0624$. An exhibition of accretion solutions for $(\varepsilon, E) = (18.95, 1.019)$ has been done in Fig. 15(d). In this case, the solutions possess multiple critical points ($r_{\text{in}} = 3.636$, $r_s = 6.8765$ and $r_{\text{out}} = 17.0132$) and deliver the higher entropy content for external component ($M_{\text{out}} = 22.1992 \times 10^7$) compared to the internal one ($M_{\text{in}} = 21.3056 \times 10^7$). The solutions passing through the inner critical points behave differently due to their extension through the outer edge towards the horizon. However, the solution going down the outer sonic point is closed and disconnects the black hole’s horizon with the outer edge. We notice that the outer sonic points and shock locations are constructed efficiently close to the horizon for $\lambda = 0$ case (Bondi flows). Therefore, all the prerequisites of the advection-dominated accretion flows (ADAF) are maintained here. At last, we illustrate another solution corresponding to $(\varepsilon, E) = (18, 1.03)$, which provides only inner ($r_{\text{in}} = 3.6695$) sonic point (see Fig. 15e). Here, the entropy accretion rate at the inner critical point is evaluated as $M_{\text{in}} = 27.1305 \times 10^7$ and also observed that the subsonic solutions at $r_{\text{edge}}$ are turned into the supersonic flows when it crosses $r_{\text{in}}$. This analysis depicts that for $\lambda = 0$ flows, we get the different accretion solutions at $r_{\text{edge}}$ in addition to the global shock solutions. Most importantly, our new identifications are the existence of the multiple sonic points solutions for “Bondi flows”, and these solutions can advertise the considerably deformed Kerr black holes due to the higher values of $\varepsilon$. When we consider any non-zero spin parameters, we anticipate that the above solutions to appear for the lower values of $\varepsilon$. Overall, this analysis has a direct influence on identifying the non-Kerr BHs.

6 DEFORMATION PROVIDES NAKED SINGULARITY

In deformed Kerr spacetime, the flow solutions associated with the central object can behave differently depending on $\varepsilon$ for a particular $a_k$. One of such instance has been explored in the last section. An elaborate discussion on the dependency between $a_k$ and $\varepsilon$, which separates those unfamiliar solutions from the usual black hole solutions, will be presented now. Here, we impose the contribution of the deformation parameter on the overhead objective.

6.1 Impact on the sonic points

Whenever we have investigated for a rapidly rotating black hole ($a_k = 0.99$), we have taken $\varepsilon = 0.02$, and the justification behind it has been mentioned several times. In this section, we consider $\varepsilon = 0.03$ (one of the higher values compared to taken in last discussions). To inflect our analysis, we display the variation of the specific energy ($E_c$) with the critical point coordinates ($r_c$) for different specific angular momentum ($\lambda$) in Fig. 16. We use the black, blue, red and orange
lines for $\lambda = 1.84, 1.86, 1.88$ and $1.9$ respectively. Additionally, solid, dotted and dashed curves highlight the saddle, nodal and spiral type sonic points in each curve. Here, we notice that in the region $E_c \geq 1$, there could be maximum of four critical points irrespective of $\lambda$. However, for $E_c < 1$, we find that the maximum critical points are three and the minimum is one, which depends on $\lambda$. Interestingly, in both the cases, one critical point revealed as O-type and is developed very close to the central singular point. As we have already seen, this type of sonic point is not present in previous parameter ranges; even if it appears, still the visibility is lost due to its formation inside the black hole horizon. However, the innermost critical point can be disclosed when the horizon is opened up along the equatorial plane of the central object (Bambhaniya et al. 2021) and may promote the naked singular central object. It is observed that for motion through the outer edge up to the singular point, critical points change as saddle-nodal-spiral-nodal-saddle-spiral. This sequence is maintained for those $\lambda$ who deliver the four sonic points. Moreover, the maximum energy associated with these multiple critical points is shifted towards the lower values with the increase of angular momentum. After a specific value of $\lambda$, we can’t get the energies that deliver the global accretion solutions corresponding to four sonic points. In Subsection 6.3, we will discuss the parameter space in detail. At the moment, we intend to discover the global accretion solutions corresponding to these sonic points in the following subsections and try to build their glance.

6.2 Finding the global accretion solutions

This section prefaces different accretion solutions in the deformed Kerr spacetime with $(a_k, \varepsilon) = (0.99, 0.03)$ and demonstrates them elaborately. For finding the flow solutions, we obtain the temperature ($\Theta_c$) and the radial velocity ($v_r$) at the critical point locations ($r_c$) using the same procedure as used in Subsection 3.2. After that, with the help of equation (28), we calculate the radial velocity gradient $(dv/dr)_c$ at a sonic point. In order to identify the accretion solution, the negative value of $(dv/dr)_c$ has been adopted. Now treating the calculated values of $\Theta_c$ and $v_r$ as the given initial conditions, we perform the RK4 method for the numerical integration to the equation (23), starting from $r_\infty$ to the singular point and then from $r_\infty$ towards the outer edge of the accretion disc. Finally, combining these two outcomes, we obtain the global accretion solution in the central object background. For wind solutions, we pick up the positive value of $(dv/dr)_c$ and apply the same methodology as used for the accretion solutions. Fig. 17(a) displays the global accretion solution for $\lambda = 1.82$ and $E = 1.0137$. The accretion and wind solutions are plotted with black (solid) and blue (dashed) lines respectively. Here, critical points are calculated as $r_{\infty} = 1.1493$ (O-type) and $r_{\infty} = 17.4339$ (saddle-type) and marked in this figure. We clarify that the Mach number increases when flow travels the whole path from the outer to the inner end of the disc, and it becomes supersonic while crossing the external sonic point ($r_{\infty}$). However, an intense decrease in the Mach number is detected close to the innermost sonic point ($r_{\infty}$). Such peculiarity, we have not witnessed in black hole accretion solutions. Moreover, the entropy accretion rate associated with this solution is calculated as $M_{\text{acc}} = 14.6153 \times 10^7$. In Fig. 17(b), we characterize the variation of the Mach number ($M$) with the radial coordinates ($r$) corresponding to the input parameters $(\lambda, E) = (1.85, 1.0137)$. In this case, the sonic points are found as $r_\infty = 1.1832$, $r_{\text{in}} = 1.8155$, $r_{\text{out}} = 3.7319$ and $r_{\text{out}} = 16.9517$. Spiral-type ($r_{\text{in}}$, $r_{\text{out}}$) and saddle-type critical points ($r_{\text{in}}$, $r_{\text{out}}$, $r_{\text{out}}$) are also annotated in this figure by the solid (black) dot and star marks respectively. In addition, we calculate the entropy accretion rate ($M$) for solutions that pass through both critical points. The results show that the solution goes through $r_{\text{in}}$ possesses higher entropy content ($M_{\text{acc}} = 15.9171 \times 10^7$) than the solution passing through $r_{\text{out}}$ ($M_{\text{acc}} = 14.4893 \times 10^7$). Here, it is manifested that the flow passes through the external critical point made a smooth connection with the origin of the singular object to $r_{\text{edge}}$. However, the solutions go down the inner critical point are closed one and disconnect the accretion disc’s outer edge. Interestingly, substantial fall in the Mach numbers are observed for the solutions passing down both $r_{\text{in}}$ and $r_{\text{out}}$. This feature is absent in the usual black hole case. Next, we illustrated the accretion solutions for the flow accompanied by the injected input parameters $(\lambda, E) = (1.87, 1.0137)$. We demonstrate $M$’s trajectory as a function of $r$ concerning the accretion solution in Fig. 17(c). It is clear from the figure that the accretion solutions passing through the $r_{\text{out}} = 16.6046$ and $r_{\text{in}} = 1.5073$ possess similar behavior with the inner and outer sonic point solutions respectively of Fig. 17(b). Here, the calculated values of entropy accretion rates are $M_{\text{acc}} = 14.1243 \times 10^7$ and $M_{\text{acc}} = 14.4022 \times 10^7$. Therefore, the entropy content is more for the outer one compared to the inner one. Furthermore, we compute the remaining two O-type critical points as $r_{\text{in}} = 1.2339$ and $r_{\text{out}} = 4.2651$ and mark them including previous two saddle-type critical points in this plot. Finally, in Fig. 17(d), we represent the evolution of $M$ with respect to $r$ for flow parameters $(\lambda, E) = (1.85, 1.025)$ as the universal parameters. Here, the flow contains only two critical points, which are very close to the origin ($r_{\text{in}} = 1.1856$ and $r_{\text{out}} = 1.6893$). The sonic points are spotted into the panel and calculated the entropy accretion rate at $r_{\text{in}}$ as $M_{\text{acc}} = 17.3916 \times 10^7$. In this case, the solutions that enter at the outer edge of the disc are smoothly accreted towards the central object and made a connection between the inner and outer edge of the accretion disc after being passed via $r_{\text{in}}$. Here, we also notice the decrement in $M$ near $r_{\text{in}}$ and distinguish it from the inner critical point solutions of the black hole. Such solutions were reported earlier in (Dinhingia et al. 2020) in the background of the naked singular object. Therefore we expect that the present value of deformation makes the central object naked singular, at least along the equatorial plane. To ensure it, we study an analytical method, where we can calculate the event horizon locations using the condition $g^{rr} = 1/g_{rr} = 0$ i.e.,

\[
 r^3 - 2r^4 + a_k^2 r^3 + a_k^2 \varepsilon = 0.
\] (31)

Equation (31) does not provide any real roots for the input parameters $(a_k, \varepsilon) = (0.99, 0.03)$. Hence, it assures that the central object is no longer a black hole and provides the naked singular spacetime. A thorough discussion on the parameter space of $a_k$ and $\varepsilon$, which isolates the deformed spacetime into the black hole and naked singularity, will be done later in Section 7.

6.3 Parameter space for maximum four sonic points

Subsection 6.1 flavors the existence of parameter space in the $\lambda - E$ plane that possesses a maximum of four critical points; between them, two must be O-type critical points. However, the other two can be saddle or nodal types or blended by the saddle and nodal type critical points. Our goal is to determine the parameter space for saddle type multiple critical points because any practical justification dispenses the saddle type critical points only. In Fig. 18, we demonstrate the modification of parameter space for different deformation parameter ($\varepsilon$). This figure, considers a highly rotating object with the Kerr parameter $a_k = 0.99$. Red (solid), blue (dashed), black (dotted) and orange (dash-dotted) lines surround the classified domains corresponding to $\varepsilon = 0.03, 0.0325, 0.035$ and 0.0375 respectively. We notice that the identified regions are shifted towards the lower
angular momentums with the increase of deformation parameter. On the other hand, the lower end of the energy range for multiple critical points moves to the higher values, but the upper limit to the energy does not change too much. Moreover, areas under the parameter space curves shrink when the deformation parameter reaches higher values for a given spin parameter. If we deform the spacetime further, we cannot find any energy and angular momentum, which give the accretion solutions that contains two saddle-type sonic points.

7 UNDERSTANDING THE METRIC PARAMETERS \((a_k, \varepsilon)\) OF THE DEFORMED KERR SPACETIME

In the previous sections, we have taken the allowed deformation parameters corresponding to the presented Kerr parameters in accordance with the interest of investigation regarding the global accretion solutions. This section identifies a parameter space of \(a_k\) and \(\varepsilon\) for both the black hole and naked singular case in Fig. 19 by using the solution topologies. The shaded grey region denotes the valid parameter space for naked singular objects and the remaining part includes the black hole objects. We isolate the identified regions into two domains by drawing a black (solid) line that comprises the maximum values of the deformation parameter \((\varepsilon_{\text{max}})\) associated with the familiar black hole solutions. Therefore, for a given \(a_k\), there must exist \(\varepsilon_{\text{max}}\) which promotes the central object into the black hole state. If we cross this limiting value, the central singularity is no longer the black hole. According to Bambhaniya et al. (2021), its horizon disappear along the equitorial plane and becomes naked singular. The investigated \(\varepsilon_{\text{max}}\) values are listed in Table 1 corresponding to several spin parameters \(a_k\). It is revealed that \(\varepsilon_{\text{max}}\) is shifted towards the lower values when the central objects are spinning fast. Even a tiny deformation parameter to the original Kerr metric leads into the naked singular spacetime. Equation (31) also shows that the central object can provide the naked singularity, depending on the input parameters \((a_k, \varepsilon)\). Table 1 characterizes the comparison between \(\varepsilon_{\text{max}}\) values based on the pseudo-Newtonian model and analytical perspective as well. Here, we restrict our analysis to the observational limit of the deformation parameter \(\varepsilon \leq 19\), following Atamurotov et al. (2013). Excitingly, the evaluated results are very similar in
both cases, which justifies our analysis significantly. Therefore, we could firmly state that identifying central singular nature through the accretion solutions may offer an alternative window despite the analytical approach. In addition, we determine the maximum angular momentum ($\lambda_{\text{max}}$) associated with each solution topology within the specified black hole zone, which may deliver the shock-induced global accretion solutions. We present a three-dimensional surface projection of $\lambda_{\text{max}}$ on the two-dimensional $a_k - \varepsilon$ plane in Fig. 19. The right side color bar of the panel represents the identified range of $\lambda_{\text{max}}$ corresponding to different spacetime parameters ($a_k, \varepsilon$). The minimum and maximum values of $\lambda_{\text{max}}$ are found as 1.17 and 3.37 respectively. Uniting this, this analysis inflicts a great understanding between the metric parameters to settle down the nature of central singularity and helps to play with various astrophysical aspects in the future.

8 SUMMARY AND CONCLUSIONS

Several observational verifications have been reported to establish the Einstein gravity in the weak-field regime. However, some fundamental problems, like dark energy, dark matter, quasi-normal modes, quantization of the gravitational interactions, nature of central singularity, etc., emerge as challenges to the Einstein gravity. Theoretical scientists developed the modified gravity theory in GR to solve the above mentioned problems and rectified observational evidences in the strong-field regime. Motivated by this, in our work, we analyzed the accretion dynamics in the JP spacetime (Johannsen & Psaltis 2011). Following Dihingia et al. (2018b), a pseudo-Newtonian approach has been used in the basic hydrodynamical flow equations. Such effective potential has significant advantages in accretion physics. It is valid in the overall general relativistic regime and is suitable for compact objects up to the maximum possible spin parameter. Therefore, it can be used in any astrophysical event related to different components of the accretion disc. In addition to Killing symmetries in the metric, we considered the ideal fluids to move only in the equatorial plane of an advective disc, such that any mathematical complexity does not appear. We have derived the condition for transonic flows in the sub-Keplerian disc and calculated the velocity gradient at the sonic point in Appendix A. As the deviation term $\varepsilon/\nu^3$ (for $\theta = \pi/2$) in the JP metric puts substantial deformation in the strong-field regime, we emphasized the deformation parameter's role in controlling the global accretion solutions including shock. Moreover, we observed that this spacetime can become exotic like the naked singularity depending on the spacetime parameters ($a_k, \varepsilon$). Here, we summarize our findings below.

We studied the nature of the critical points (saddle, nodal and spiral) corresponding to the global transonic solutions. An energy range for maximum three sonic points was noticed, and its modifications were also observed irrespective of the angular momentum and deformation parameter changes (Fig. 1). We noticed that different solution topologies are incorporated into the non-Kerr BH system due to the effect of $\varepsilon$ (Fig. 2). However, the multiple sonic point solutions are not secluded; instead, a parameter space exists. We found the parameter space for different solution topologies in the $\lambda - E$ plane for a spinless and rapidly rotating deformed BH (Fig. 3). It has been clarified that the solutions are qualitatively the same in both cases but must possess quantitative differences. We also studied the parameter space change for multiple sonic point solutions by considering several input parameters. We observed that, for a fixed $a_k$, they shifted towards the lower angular momentum and higher energy sides with the increment of $\varepsilon$. Consequently, the area of the effective regions increases (Fig. 5). A similar effect was also revealed in another case where $\varepsilon$ is fixed and $a_k$ is increased (Fig. 6). Therefore, $a_k$ and $\varepsilon$ react similarly to own the parameter space of close solution topologies. The spectral states of AGNs and microquasars are characterized by the hard power-law tails, bipolar relativistic jets, and so others. The standing shock transitions (Fukue 1987; Chakrabarti 1989; Yang & Kafatos 1995; Chakrabarti & Das 2004; Chattopadhyay & Kumar 2016; Kumar & Chattopadhyay 2017; Dihingia et al. 2018b,c; Dihingia et al. 2018a; Dihingia et al. 2019a,b) to the flow parameters justify these crucial features. The above circumstances push us to explore the global shock solutions around the non-Kerr BH (Fig. 7). Flows are coming through the outer critical points, becoming supersonic and in the meanwhile taking a sudden jump into the subsonic arm due to the centrifugal barrier. Such solutions are always preferable when the subsonic components have higher entropy content than the supersonic ones. We examined the evolution of the flow parameters for global transonic solutions without and with shock (Figs 4 and 8). A substantial increase in temperature due to flow compression at the shock locations produces hot and ultrarelativistic electrons in the PSD. Accordingly, low-temperature soft photons get inverse Comptonize that supply power to the jets (Molteni et al. 1994; Chattopadhyay & Das 2007; Das & Chattopadhyay 2008; Kumar & Chattopadhyay 2013; Das et al. 2014; Kumar et al. 2014; Kumar & Chattopadhyay 2017) and create hard X-radiation in the BH spectrum. To glimpse the impact of $\varepsilon$ on shock, we found the global shock solutions for different $\varepsilon$ when flows are entered with identical energy and angular momentum. It has been noticed that the shock locations are shifted away from the horizon when $\varepsilon$ is increased. (Fig. 9). As a result, the change in flow variables across the shock transitions decreases (Fig. 10). After exceeding some critical value ($\varepsilon_c$), shock transitions cease to exist. Needless to say, $\varepsilon_c$ dependent on the input parameters. To investigate the change in flow temperature and density during shock, we explored the variation of shock properties (e.g., shock location, compression ratio ($R$), and shock strength ($S$)) as a function of $\varepsilon$ (Figs 11 and 12). The decrease in both $R$ and $S$ with the increase of $\varepsilon$ is quite evident because the shock fronts move towards the outer edge of the accretion disc. However, for a fixed $\varepsilon$, the shock forms at a large radial distance when $\lambda$ increases, which is a direct evidence that the centrifugal repulsion drives the shock transitions. Shock solutions can also occupy an ample parameter space. We obtained the effective regions in the $\lambda - E$ plane, which delivers the shock-induced global accretion solutions. Moreover, their movement
into the lower angular momentums and higher energies expand the effective parameter space of the shock solutions (Figs 13 and 14).

The accretion solutions associated with $\lambda = 0$ flows (Bondi flows) have been presented (Fig. 15) for non-Kerr BH. We noticed that the global transonic solutions can possess multiple sonic points even for Bondi flows, and consequently, it can suffer the shock transitions. Therefore, we set up a new property to the Bondi flows for the first time. One notable thing is that all the sonic points were formed very close to the horizon in this particular case. For global ADAF, the inner boundary conditions are such that the flow became transonic only close to the horizon. As the “Bondi flows” are the subset of the ADAF flows, we could state that the above result does not break any prerequisites of the accretion physics.

In all the above analyses, we considered those values of $\varepsilon$, which holds usual black hole solutions. We observed some unusual flow properties when $\varepsilon$ crosses some limiting values ($\varepsilon_{\text{max}}$) depending on $a_k$. We have explored those in our work. We studied the critical point analysis (Fig. 16) for distinct $\lambda$ with the particular choice of spacetime parameters ($a_k$, $\varepsilon > \varepsilon_{\text{max}}$). In this case, an extra sonic point was noticed close to the compact object, which is absent for the BH accretion dynamics (Fig. 1). When the horizon is opened up along the equatorial plane of the central object (Bambhaniya et al. 2021), the innermost sonic point is visible and may promote the naked singularity. We calculated the flow solutions corresponding to these critical points (Fig. 17). It has been observed that identified solutions are precisely similar to those investigated for the Kerr-Taub-NUT (KTN) naked singular object by Dihingia et al. (2020). As the JP metric possesses an extra parameter $\varepsilon$, we expect such spacetime can deliver the naked singularity depending on ($a_k$, $\varepsilon$). Motivated by this, we reconfirmed the naked singularity by the analytical calculation of horizon locations (equation 31). Therefore, accretion dynamics can be an alternative way to identify the singular objects (see Table 1). The modification of parameter space in $\lambda - E$ plane for four sonic points has been found in Fig. 18. Parameter spaces were shrunk when the $\varepsilon$ is increased for a given $a_k$. This feature is completely reversed from the BH case. We divided the parameter space ($a_k - \varepsilon$) of JP metric into two domains (BH and naked singularity) using various solution topologies, which preface this spacetime very well (Fig. 19). Finally, we found the maximum angular momentum ($J_{\text{max}}$) associated with the closed solutions. This identification comprises the shock transitions in the overall parameter space available for deformed BH and explain the astrophysical events linked with shock.

Our analysis collectively argues that accretion dynamics is a valuable physical process for explaining the electromagnetic spectrum excited from the accretion disc of strong gravitating compact objects. This theory also reveals impressive features like Bondi flow with multiple sonic points and naked singularity, which can be the signatures of deformed gravity theory. It is to be remembered that we consider the inviscid disc in our model. However, this assumption is simplistic regarding natural accretion flows that include heat dissipation due to viscous stress. Further, we omit the effect of the magnetic fields on the disc. It is challenging to develop the mathematical framework if we include all these realistic situations in general relativistic hydrodynamics. As these scopes are beyond our present work, we may bring them in some future works.

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APPENDIX A: CALCULATION OF $\frac{dv}{dr}$ AT THE CRITICAL POINT $r_c$

After applying the L’“Hospital” rule, we get the radial velocity gradient at the critical point as

$$\frac{dv}{dr} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A},$$

where the explicit form of the quantities $A$, $B$ and $C$ are calculated as follows:

$$A = \gamma_0^2 \left[ 1 + \frac{2C_2^2}{(\Gamma + 1)^2} \right] + \frac{4\gamma_0^4}{v^2(2N + 1)} \frac{\partial}{\partial \theta} \left( C_2^2 \frac{\gamma_0^4}{\Gamma + 1} \right),$$

$$B = \frac{8\gamma_0^2}{v^2(2N + 1)} \left[ N_{11} + N_{12} - \frac{3e}{2r^4} N_{13} \right] \frac{\partial}{\partial \theta} \left( C_2^2 \frac{\gamma_0^4}{\Gamma + 1} \right),$$

$$C = \frac{d^2\Phi}{dr^2} = \frac{4\theta}{2N + 1} \left[ N_{11} + N_{12} - \frac{3e}{2r^4} N_{13} \right]^2 \frac{\partial}{\partial \theta} \left( C_2^2 \frac{\gamma_0^4}{\Gamma + 1} \right)$$

$$- \frac{2C_2^2}{\Gamma + 1} \left[ N_{11}' + N_{12}' + N_{14} + \frac{3a_1^2\varepsilon}{2r^4(\Delta + a_1^2h)^2} \left( \Delta' - \frac{3a_1^2\varepsilon}{r^4} \right) \right].$$

$$N_{11} = \frac{5}{2r} \frac{r - a_1^2(1 + h)}{r(\Delta + a_1^2h)}, N_{12} = -\frac{1}{2F} \frac{dF}{dr}, N_{13} = \frac{1}{2r} + \frac{a_1^2}{\Delta + a_1^2h},$$

$$N_{14} = \frac{6\varepsilon}{r(1 + h)} - \frac{9e^2}{2r^6(1 + h)^2} + \frac{6a_1^2\varepsilon}{r^5(\Delta + a_1^2h)}.$$