INHOMOGENEOUS DIOPHANTINE APPROXIMATION ON PLANAR CURVES

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Abstract. The inhomogeneous metric theory for the set of simultaneously \( \psi \)-approximable points lying on a planar curve is developed. Our results naturally incorporate the homogeneous Khintchine-Jarník type theorems recently established in [3] and [10]. The key lies in obtaining essentially the best possible results regarding the distribution of ‘shifted’ rational points near planar curves.

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1. Introduction and Statement of Results

1.1. Inhomogeneous approximation in the plane. Throughout \( \psi : \mathbb{N} \to \mathbb{R}^+ \) is a monotonic function such that \( \psi(t) \to 0 \) as \( t \to \infty \) and will be referred to as an approximating function. Given \( \psi \) and a point \( \theta := (\theta_1, \theta_2) \in \mathbb{R}^2 \), let \( S(\psi, \theta) \) denote the set of points \( x := (x_1, x_2) \in \mathbb{R}^2 \) for which there exists infinitely many positive integers \( q \) such that

\[
\max_{1 \leq i \leq 2} \|qx_i - \theta_i\| < \psi(q).
\]

Here and throughout \( \| \cdot \| \) denotes the distance to the nearest integer. In the case that the inhomogeneous factor \( \theta \) is the origin, the corresponding set \( S(\psi) \) is the usual homogeneous set of simultaneously \( \psi \)-approximable points in the plane. In the case \( \psi : t \to t^{-v} \) with \( v > 0 \), let us write \( S(v, \theta) \) for \( S(\psi, \theta) \). The following statement provides a beautiful and simple criterion for the ‘size’ of \( S(\psi, \theta) \) expressed in terms of \( s \)-dimensional Hausdorff measure \( \mathcal{H}^s \).

Theorem. Let \( s \in (0, 2] \), \( \theta \in \mathbb{R}^2 \) and \( \psi \) be an approximating function. Then

\[
\mathcal{H}^s(S(\psi, \theta)) = \begin{cases} 
0 & \text{when } \sum t^{2-s} \psi(t)^s < \infty \\
\mathcal{H}^s(\mathbb{R}^2) & \text{when } \sum t^{2-s} \psi(t)^s = \infty.
\end{cases}
\]
This result generalizes and unifies the classical theorems of Khintchine (1924) and Jarník (1931) and will be refereed to as the Khintchine-Jarník theorem. When \( s = 2 \), the measure \( H^2 \) is equivalent to two-dimensional Lebesgue measure in the plane and, loosely speaking, the theorem corresponds to Khintchine’s Theorem. Actually, the stronger statement that \( H^2(\mathbb{R}^2 \setminus S(\psi, \theta)) = 0 \) if \( \sum \psi(t)^2 = \infty \) is true and the homogeneous case of this statement is due to Khintchine. When \( s < 2 \), the homogeneous case of the theorem corresponds to Jarník’s Theorem and can be regarded as a Hausdorff measure version of Khintchine’s Theorem. For further details see [2, Section 12.1] and references within.

1.2. Inhomogeneous approximation restricted to curves. Let \( C \) be a planar curve. In short, the goal is to obtain an analogue of the above Khintchine-Jarník theorem for \( C \cap S(\psi, \theta) \). The fact that the points \( x := (x_1, x_2) \in \mathbb{R}^2 \) of interest are restricted to \( C \) and therefore are of dependent variables, introduces major difficulties in attempting to describe the measure theoretic structure of \( C \cap S(\psi, \theta) \).

In 1998, Kleinbock & Margulis [7] established the fundamental Baker-Sprindzuk conjecture concerning homogeneous Diophantine approximation on manifolds. As a consequence, for non-degenerate planar curves\(^1\) the one–dimensional Lebesgue measure \( H^1 \) of the set \( C \cap S(v) \) is zero whenever \( v > 1/2 \) – see also [8]. Subsequently, staying within the homogeneous setup, the significantly stronger Khintchine-Jarník type theorem for \( C \cap S(\psi) \) has been established – see [3] for the convergence part and [10] for the divergence part.

Until the recent proof of the inhomogeneous Baker-Sprindzuk conjecture [4, 5], the theory of inhomogeneous Diophantine approximation on planar curves (let alone manifolds) had remained essentially non-existent and ad-hoc. As a consequence of the measure results in [4, 5] or alternatively the even more recent dimension results in [1], we now know that for any non-degenerate planar curve \( C \) and \( \theta \in \mathbb{R}^2 \),

\[ H^1(C \cap S(v, \theta)) = 0 \quad \text{when} \quad v > 1/2 . \]

Clearly, this statement is far from the desirable Khintchine-Jarník type theorem for \( C \cap S(\psi, \theta) \). As mentioned above, such a statement exists within the homogeneous setup. This paper constitutes part of a programme to develop a coherent inhomogeneous theory for curves, and indeed manifolds in line with the homogeneous theory.

Without loss of generality, we will assume that \( C = C_f := \{(x, f(x)) : x \in I\} \) is given as the graph of a function \( f : I \to \mathbb{R} \), where \( I \) is some interval of \( \mathbb{R} \). As usual, \( C^{(n)}(I) \) will denote the set of \( n \)-times continuously differentiable functions defined on some interval \( I \) of \( \mathbb{R} \). In this paper we establish the inhomogeneous analogues of the main theorems in [3] and [10]; that is, we obtain the following complete Khintchine-Jarník type theorem for planar curves.

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\(^1\)A planar curve \( C \) is non-degenerate if the set of points on \( C \) at which the curvature vanishes is a set of one–dimensional Lebesgue measure zero. Moreover, it is not difficult to show that the set of points on a planar curve at which the curvature vanishes but the curve is non-degenerate is at most countable. In view of this, the curvature completely describes the non-degeneracy of planar curves.
Theorem 1. Let $s \in (1/2, 1]$, $\theta \in \mathbb{R}^2$ and $\psi$ be an approximating function. Let $f \in C^{3}(I)$ and assume that $H^s(\{x \in I : f''(x) = 0\}) = 0$. Then

$$H^s(C_f \cap S(\psi, \theta)) = \begin{cases} 0 & \text{when } \sum t^{1-s} \psi(t)^{s+1} < \infty \\ H^s(C_f) & \text{when } \sum t^{1-s} \psi(t)^{s+1} = \infty \end{cases}.$$ 

Note that a planar curve is one dimensional and so $H^s(C_f \cap S(\psi, \theta)) \leq H^s(C_f) = 0$ for any $s > 1$ irrespective of the approximating function $\psi$. Thus the hypothesis that $s \leq 1$ is essential and obvious. In the case $s = 1$, the theorem is a statement concerning the one-dimensional Lebesgue measure of $C_f \cap S(\psi, \theta)$ and the convergence part actually only requires that $f \in C^2(I)$. Also, as one would expect, the measure zero assumption on the set $\{x \in I : f''(x) = 0\}$ coincides with the definition of non-degeneracy. In the case $s < 1$, we have that $H^s(C_f) = \infty$ and the theorem provides an elegant zero-infinity law for the Hausdorff measure of $C_f \cap S(\psi, \theta)$. In particular, this law implies the following corollary on the Hausdorff dimension of $C_f \cap S(\psi, \theta)$ expressed in terms of the lower order $\lambda_{\psi}$ of $1/\psi$. Recall,

$$\lambda_{\psi} := \liminf_{t \to \infty} \frac{-\log \psi(t)}{\log t}$$

and indicates the growth of the function $1/\psi$ ‘near’ infinity. Note that $\lambda_{\psi}$ is non-negative since $\psi(t) \to 0$ as $t \to \infty$.

Corollary 1. Let $\theta \in \mathbb{R}^2$ and $\psi$ be an approximating function such that $\lambda_{\psi} \in [1/2, 1)$. Let $f \in C^{3}(I)$ such that $f''(x)$ is not identically zero and assume that

$$\dim \{x \in I : f''(x) = 0\} \leq \frac{2 - \lambda_{\psi}}{1 + \lambda_{\psi}}.$$ 

Then,

$$\dim C_f \cap S(\psi, \theta) = \frac{2 - \lambda_{\psi}}{1 + \lambda_{\psi}}.$$ 

This generalizes Theorem 4 of [3] to the inhomogeneous setting. Note that when $\lambda_{\psi} < 1/2$, the condition that $f''(x)$ is not identically zero follows from the assumption that $\dim \{x \in I : f''(x) = 0\} < 1$. We take this opportunity to mention that this necessary condition should also be present in [3, Theorem 4], where it is missing.

1.3. The inhomogeneous counting results. The proof of Theorem 1 rests on understanding the distribution of ‘shifted’ rational points ‘near’ planar curves. In view of the metrical nature of Theorem 1, there is no harm is assuming that the function $f : I \to \mathbb{R}$ is defined on a closed interval $I$ and that $f''$ is continuous and non-vanishing on $I$. By the compactness of $I$, there exist positive and finite constants $c_1, c_2$ such that

$$c_1 \leq |f''(x)| \leq c_2 \quad \forall x \in I.$$ 

(2)
Let $I$ and $f$ be as above. Furthermore, given $\theta = (\theta_1, \theta_2) \in \mathbb{R}^2$, $\delta > 0$ and $Q \geq 1$, consider the counting function

$$N_\theta(Q, \delta) := \text{card} \left\{ (p_1, q) \in \mathbb{Z} \times \mathbb{N} : Q < q \leq 2Q, \frac{(p_1 + \theta_1)}{q} \in I \right\}. \quad (3)$$

In short, the function $N_\theta(Q, \delta)$ counts the number of rational points $(p_1/q, p_2/q)$ with bounded denominator $q$ such that the shifted points $((p_1 + \theta_1)/q, (p_2 + \theta_2)/q)$ lie within the $\delta/Q$-neighborhood of the curve $C_f$. The following result generalizes Theorem 1 of [10] to the inhomogeneous setting.

**Theorem 2.** Let $f \in C^{(2)}(I)$. Suppose that $Q \geq 1$ and $0 < \delta \leq \frac{1}{2}$. Then

$$N_\theta(Q, \delta) \ll \delta Q^2 + \delta^{-\frac{1}{2}} Q.$$

With a mild additional condition on $f$ we are able to extend the validity of the bound in Theorem 2. The following statement is the inhomogeneous analogue of Theorem 3 in [10].

**Theorem 3.** Let $f'' \in \text{Lip}_\phi(I)$, where $0 < \phi < 1$. Suppose that $Q \geq 1$ and $0 < \delta \leq \frac{1}{2}$. Then, for any $\varepsilon > 0$

$$N_\theta(Q, \delta) \ll \delta Q^2 + \delta^{-\frac{1}{2}} Q^{\frac{1}{2} + \varepsilon} + \delta^{\frac{2-\phi}{2}} Q^{\frac{3-\phi}{2}}.$$

**Remark.** When $\phi = 1$ the proof gives the above theorem with the term $\delta^{\frac{2-\phi}{2}} Q^{\frac{3-\phi}{2}}$ replaced by $Q \log(Q/\delta)$, and this is then always bounded by one of the other two terms.

Armed with Theorems 2 and 3, the convergent part of Theorem 1 is established on following the arguments set out in Sections 6 and 7 of [10]. The modifications are essentially obvious and the details are omitted. It is worth mentioning that when $s = 1$, we only need to appeal to Theorem 2 and thus we only require that $f \in C^{(2)}(I)$ when proving the convergent part of Theorem 1.

The key to establishing the divergence part of Theorem 1 is the following covering result that also yields a sharp lower bound for the counting function $N_\theta(Q, \delta)$. Throughout, $|X|$ will denote the one-dimensional Lebesgue measure of a set $X$ in $\mathbb{R}$.

**Theorem 4.** Let $f \in C^{(3)}(I)$. Then for any interval $J \subseteq I$ there are constants $k_1, k_2, C_1, Q_0 > 0$ such that for any choice of $\delta$ and $Q > Q_0$ subject to

$$\frac{k_1}{Q} \leq \delta \leq k_2 \quad (4)$$

one has

$$\left| \bigcup_{(p_1, q) \in A_\theta(Q, \delta, J)} \left( B \left( \frac{p_1 + \theta_1}{q}, \frac{C_1}{Q^2 \delta} \right) \cap J \right) \right| \geq \frac{1}{2} |J| \quad \forall \theta \in \mathbb{R}^2, \quad (5)$$
where
\[ A_\theta(Q, \delta, J) := \left\{ (p_1, q) \in \mathbb{Z} \times \mathbb{N} : Q < q \leq 2Q, (p_1 + \theta_1)/q \in J, \|qf((p_1 + \theta_1)/q) - \theta_2\| < \delta \right\}. \]

Theorem 4 is the inhomogeneous generalization of Theorem 7 in [3]. Armed with Theorem 4, the arguments set out in Section 7 of [3] are easily adapted to prove the divergence part of Theorem 1. The minor modifications are essentially obvious and the details are omitted.

Note that \( N_\theta(Q, \delta) \) is by definition the cardinality of \( A_\theta(Q, \delta, I) \). With this in mind, it trivially follows that
\[ N_\theta(Q, \delta) \cdot \frac{2C_1}{Q^2 \delta} \geq \sum_{(p_1, q) \in A_\theta(Q, \delta, I)} \left| B \left( \frac{p_1 + \theta_1}{q}, \frac{C_1}{Q^2 \delta} \right) \right| \geq \left| \bigcup_{(p_1, q) \in A_\theta(Q, \delta, I)} \left( B \left( \frac{p_1 + \theta_1}{q}, \frac{C_1}{Q^2 \delta} \right) \cap I \right) \right| \geq \frac{1}{2} |I|. \]

In other words, Theorem 4 implies the following statement which is a generalisation of Theorem 6 in [3].

**Theorem 5.** Let \( f \in C^{(3)}(I) \). There are constants \( k_1, k_2, c, Q_0 > 0 \) such that for any choice of \( \delta \) and \( Q > Q_0 \) satisfying (4) we have
\[ N_\theta(Q, \delta) \geq c \delta Q^2 \quad \forall \theta \in \mathbb{R}^2. \]

2. The proof of Theorems 2 and 3
Without loss of generality, assume that \( \theta = (\theta_1, \theta_2) \) satisfies \( 0 \leq \theta_1, \theta_2 < 1 \). Let
\[ J := \left\lfloor \frac{1}{2\delta} \right\rfloor \]
and consider the Fejér kernel
\[ K_J(x) := J^{-2} \left| \sum_{h=1}^J e(hx) \right|^2 = \left( \frac{\sin \pi Jx}{J \sin \pi x} \right)^2. \]

When \( \|x\| \leq \delta \) we have \( |\sin \pi Jx| = \sin \pi \|Jx\| \geq 2\|Jx\| = 2\|J\| \|x\| = 2J\|x\| \), since \( J\|x\| \leq \delta \left\lfloor \frac{1}{2\delta} \right\rfloor \leq \frac{1}{2} \). Hence, when \( \|x\| \leq \delta \), we have
\[ K_J(x) \geq \frac{2\|x\|J}{J\pi \|x\|} = \frac{2}{\pi}. \]

Thus
\[ N_\theta(Q, \delta) \leq \frac{\pi}{2} \sum_{Q < q \leq 2Q} \sum_{p_1 + \theta_1 \in qI} K_J(qf((p_1 + \theta_1)/q) - \theta_2). \]
Since
\[ K_J(x) = \sum_{j=-J}^{J} \frac{J-|j|}{J^2} e(jx) \]
we have
\[ N_\theta(Q, \delta) \leq \pi \delta |I|Q^2 + N_1 + O(\delta Q) = N_1 + O(\delta Q^2) \]
where
\[ N_1 := \frac{\pi}{2} \sum_{0<|j| \leq J} \frac{J-|j|}{J^2} \sum_{Q<q \leq 2Q} \sum_{p_1+\theta_1 \in qI} e(jqf((p_1+\theta_1)/q) - j\theta_2). \]

We observe that the function \( F(x) := jqf(x/q) \) has derivative \( jf'(x/q) \). Given \( j \) with \( 0<|j| \leq J \) we define
\[ H_- := [\inf jf'(x)] - 1, \quad H_+ := [\sup jf'(x)] + 1, \]
\[ h_- := [\inf jf'(x)] + 1, \quad h_+ := [\sup jf'(x)] - 1 \]
where the extrema are over \( x \) in the interval \( I \). Then, by Lemma 4.2 of [9],
\[ \sum_{p_1+\theta_1 \in qI} e(jqf((p_1+\theta_1)/q) - j\theta_2) = \sum_{H_- \leq h \leq H_+} \int_{qI-\theta_1} e(jqf((x+\theta_1)/q) - j\theta_2 - hx)dx \]
\[ + O\left( \log(2 + H) \right) \]
where \( H = \max(|H_-|,|H_+|) \). Clearly \( H \ll |j| \leq J \) and so
\[ N_1 = N_2 + O(Q \log \frac{1}{\delta}) \]
where
\[ N_2 := \frac{\pi}{2} \sum_{0<|j| \leq J} \frac{J-|j|}{J^2} \sum_{Q<q \leq 2Q} \sum_{h_-<h<h_+} \int_{qI-\theta_1} e(jqf((x+\theta_1)/q) - j\theta_2 - hx)dx. \]
The integral here is
\[ qe(h\theta_1 - j\theta_2) \int_I e(q(jf(y) - h\theta_2))dy. \]
As in Section 2 of [10], we obtain
\[ N_2 = N_3 + O\left( \delta^{\frac{1}{2}} Q^\frac{3}{2} \right) \]
where
\[ N_3 := \frac{\pi}{2} \sum_{0<|j| \leq J} \frac{J-|j|}{J^2} \sum_{Q<q \leq 2Q} q \sum_{h_0 < h < h_1} e(h\theta_1 - j\theta_2) \int_I e(q(jf(\theta_2) - h\theta_2))d\theta_2 \]
and the sum over \( h \) is taken to be empty when \( h_+ \leq h_- + 1 \). Apart from the twisting factor \( e(h\theta_1 - j\theta_2) \) this expression is identical to (2.3) of [10], with identical properties of \( f \). The analysis of Sections 2 and 4 of [10] can be applied without further change to obtain the concomitant conclusions.
3. The proof of Theorem 4

We will make use of the following result which appears as Lemma 6 in [3].

**Lemma BDV.** Let \( g := (g_1, g_2) : I \rightarrow \mathbb{R}^2 \) be a \( C^{(2)} \) map defined on a compact interval \( I \) such that \( (g'_1 g''_2 - g'_2 g''_1)(x) \neq 0 \) for all \( x \in I \). Given positive real numbers \( \lambda, K, T \) and an interval \( J \subseteq I \), let \( B(J, \lambda, K, T) \) denote the set of \( x \in J \) for which there exists \((q, p_1, p_2) \in \mathbb{Z}^3 \setminus \{0\}\) satisfying the following system of inequalities:

\[
\begin{align*}
|q g_1(x) + p_1 g_2(x) + p_2| & \leq \lambda \\
|q g'_1(x) + p_1 g'_2(x)| & \leq K \\
|q| & \leq T
\end{align*}
\]

Then for any interval \( J \subseteq I \) there is \( C > 0 \) such that for any choice of numbers \( \lambda, K, T \) satisfying

\[
0 < \lambda \leq 1, \quad T \geq 1, \quad K > 0 \quad \text{and} \quad \lambda K T \leq 1
\]

one has

\[
|B(J, \lambda, K, T)| \leq C \max \left( \lambda^{1/3}, (\lambda K T)^{1/9} \right) |J|.
\]

To begin the proof of Theorem 4 define \( g(x) := (g_1(x), g_2(x)) \) by setting

\[
g_1(x) := x f'(x) - f(x) \quad \text{and} \quad g_2(x) := -f'(x).
\]

Then \( g \in C^{(2)}(I) \). Also, note that

\[
g'(x) = (xf''(x), -f''(x)) \quad \text{and} \quad g''(x) = (f''(x) + xf'''(x), -f'''(x))
\]

and

\[
(g'_1 g''_2 - g'_2 g''_1)(x) = f''(x)^2.
\]

As \( f''(x) \neq 0 \) everywhere, Lemma BDV is applicable to this \( g \). In view of (2) and the fact that \( g'_2(x) = -f''(x) \), it follows that

\[
c_1 \leq |g'_2(x)| \leq c_2 \quad \forall x \in I.
\]

For a fixed \( x \in I \), consider the following system of inequalities:

\[
\begin{align*}
|q g_1(x) + p_1 g_2(x) + p_2| & \leq c_0^3 \delta \\
|q g'_1(x) + p_1 g'_2(x)| & \leq c_2 (c_0^6 Q \delta)^{-1} \\
|q| & \leq c_0^2 Q
\end{align*}
\]

Here \( c_0 < 1 \) is a real parameter to be determined later. Note that with \( q, p_1, p_2 \) regarded as real variables, the system defines a convex body \( D \) in \( \mathbb{R}^3 \) symmetric about the origin.

Next, fix an interval \( J \subseteq I \). By definition, the set \( B(J, \lambda, K, T) \) with

\[
\lambda := c_0^3 \delta, \quad K := c_2 (c_0^6 Q \delta)^{-1}, \quad T := c_0^2 Q
\]

be a
consists of points \( x \in J \) such that there exists a non-zero integer solution \((q, p_1, p_2)\) to the system (10) with \(|q| \leq c_0^4Q\). By Lemma BDV, for sufficiently large \( Q \) we have that
\[
|B(J, \lambda, K, T)| \leq C|J| \max \left\{ (c_0^3\delta)^{1/3}, (c_2c_0)^{1/9} \right\}
= C(c_2c_0)^{1/9}|J| \leq |J|/4
\]
provided that \( c_0 \leq c_2^{-1}(4C)^{-9} \). Therefore, with \( \lambda, K, T \) given by (11) and \( Q \) sufficiently large
\[
|\frac{3}{4}J \setminus B(J, \lambda, K, T)| \geq |J|/2 ,
\]
where \( \frac{3}{4}J \) is the interval \( J \) scaled by \( \frac{3}{4} \).

From this point onwards, \( x \in \frac{3}{4}J \setminus B(J, \lambda, K, T) \) and is fixed. Then,
\[
q > c_0^4Q
\]
for any non-zero integer solution \((q, p_1, p_2)\) to the system (10). In other words, the first consecutive minimum of the body (10) is at least \( c_0 \). Let \( \lambda_1 \leq \lambda_2 \leq \lambda_3 \) be the consecutive minima of the convex body \( D \) given by (10). Thus, \( \lambda_1 \geq c_0 \). By Minkowski’s theorem on consecutive minima [6], we have that
\[
\lambda_1\lambda_2\lambda_3V \leq 2^3 ,
\]
where \( V \) is the volume of \( D \). It is readily verified that
\[
V = 8|g_2'(x)|^{-1}c_2 \geq 8c_2^{-1}c_2 = 8 .
\]
Therefore,
\[
\lambda_3 \leq 8\lambda_1^{-2}V^{-1} \leq c_0^{-2}
\]
and it follows that there are three linearly independent integer vectors
\[
a^{(i)} := (q^{(i)}, p_1^{(i)}, p_2^{(i)}) \quad (1 \leq i \leq 3)
\]
satisfying the system of inequalities
\[
\begin{align*}
|q^{(i)}g_1(x) + p_1^{(i)}g_2(x) + p_2^{(i)}| &\leq c_0\delta \\
|q^{(i)}g_1'(x) + p_1^{(i)}g_2'(x)| &\leq c_2(c_0^6Q\delta)^{-1} \\
0 &\leq q^{(i)} \leq c_0Q .
\end{align*}
\]
For each \( i \), define
\[
G_i(x) := q^{(i)}g_1(x) + p_1^{(i)}g_2(x) + p_2^{(i)} .
\]
Now with \( \theta = (\theta_1, \theta_2) \in \mathbb{R}^2 \) fixed, consider the following system of linear equations with respect to the real variables \( \eta_1, \eta_2, \eta_3 \):
\[
\begin{align*}
\eta_1 G_1(x) + \eta_2 G_2(x) + \eta_3 G_3(x) &= \theta_1 f'(x) - \theta_2 \\
\eta_1 G_1'(x) + \eta_2 G_2'(x) + \eta_3 G_3'(x) &= \theta_1 f''(x) \\
\eta_1 q^{(1)} + \eta_2 q^{(2)} + \eta_3 q^{(3)} &= 2Q.
\end{align*}
\]

(16)

The determinant of this system is equal to
\[
-f''(x) \begin{vmatrix}
p_2^{(1)} & p_2^{(2)} & p_2^{(3)} \\
p_1^{(1)} & p_1^{(2)} & p_1^{(3)} \\
q^{(1)} & q^{(2)} & q^{(3)}
\end{vmatrix}
\]

and so is non-zero. Therefore, there is a unique solution \( \eta_1, \eta_2, \eta_3 \) to the system (16). Let,
\[
t_i := \lfloor \eta_i \rfloor \quad \text{when} \quad q^{(i)} \geq 0 \quad \text{and} \quad t_i := \lceil \eta_i \rceil \quad \text{when} \quad q^{(i)} < 0.
\]

Therefore,
\[
|\eta_i - t_i| < 1 \quad (1 \leq i \leq 3)
\]

(17)

Also, define \( a = (q, p_1, p_2) \in \mathbb{Z}^3 \setminus \{0\} \) by setting
\[
a = (q, p_1, p_2) := \sum_{i=1}^3 t_i a^{(i)}
\]

(18)

In view of the last equation of (16) and the definition of \( t_i \), it follows that \( q \leq 2Q \). Furthermore, using the fact that \( |q^{(i)}| \leq c_0 Q \) for each \( i \) (this follows from the last equation of (15)) we get that \( q \geq 2Q - 3c_0 Q \geq Q \) provided that \( c_0 \leq 1/3 \). Thus, we have that
\[
Q \leq q \leq 2Q.
\]

(19)

Further,
\[
|q g_1'(x) + p_1 g_2'(x) - \theta_1 f''(x)| \leq \sum_{i=1}^3 t_i G_i'(x) - \theta_1 f''(x)
\]

(18)

\[
\leq \sum_{i=1}^3 |t_i - \eta_i| G_i'(x)
\]

(16)

\[
\leq \sum_{i=1}^3 |G_i'(x)|
\]

(17)

\[
\leq 3c_2(c_0^8 Q\delta)^{-1}.
\]

(20)

In view of (18) and (20), we have that
\[
|q x f''(x) - p_1 f''(x) - \theta_1 f''(x)| < 3c_2(c_0^8 Q\delta)^{-1}.
\]

The latter combined with (2) gives that
\[
|x - \frac{p_1 + \theta_1}{q}| \leq \frac{3c_2}{q|f''(x)|c_0^8 Q\delta} \leq \frac{3c_2}{c_1 c_0^8 Q^2 \delta} = \frac{C_1}{Q^2 \delta},
\]

(21)
where $C_1 := \frac{3e}{C\epsilon_0}$. For $Q$ sufficiently large, the right hand side of (21) can be made arbitrarily small which together with the fact that $x \in \frac{3}{4}J$ ensures that
\[ \frac{p_1 + \theta_1}{q} \in J. \] (22)

Also,
\[ |qg_1(x) + p_1g_2(x) + p_2 - (\theta_1 f'(x) - \theta_2)| \leq \left| \sum_{i=1}^{3} t_i G_i(x) - (\theta_1 f'(x) - \theta_2) \right| \]
\[ \leq \left| \sum_{i=1}^{3} (t_i - \eta_i) G_i(x) \right| \]
\[ \leq \sum_{i=1}^{3} |G_i(x)| \]
\[ \leq 3c_0 \delta. \] (23)

By Taylor’s formula,
\[ f\left( \frac{p_1 + \theta_1}{q} \right) = f(x) + f'(x) \left( \frac{p_1 + \theta_1}{q} - x \right) + \frac{1}{2} f''(\tilde{x}) \left( \frac{p_1 + \theta_1}{q} - x \right)^2 \] (24)
for some $\tilde{x}$ between $x$ and $(p_1 + \theta_1)/q$. Thus $\tilde{x} \in J$. By (8), the left hand side of (23) equals $|q(x f'(x) - f(x)) - p_1 f'(x) + p_2 - (\theta_1 f'(x) - \theta_2)|$. Hence,
\[ 3c_0 \delta \geq |q(x f'(x) - f(x)) - p_1 f'(x) + p_2 - (\theta_1 f'(x) - \theta_2)| \]
\[ = |(qx - p_1 - \theta_1)f'(x) + p_2 + \theta_2 - qf(x)| \]
\[ \leq |p_2 + \theta_2 - qf\left( \frac{p_1 + \theta_1}{q} \right) + \frac{q}{2} f''(\tilde{x}) (x - \frac{p_1 + \theta_1}{q})^2|. \] (25)

Therefore, for $Q$ sufficiently large
\[ |q f\left( \frac{p_1 + \theta_1}{q} \right) - p_2 - \theta_2| \leq |p_2 + \theta_2 - qf\left( \frac{p_1 + \theta_1}{q} \right) + \frac{q}{2} f''(\tilde{x}) (x - \frac{p_1 + \theta_1}{q})^2| \]
\[ + \left| \frac{q}{2} f''(\tilde{x}) (x - \frac{p_1 + \theta_1}{q})^2 \right| \]
\[ \leq 3c_0 \delta + \frac{q}{2} c_2 \left( \frac{C_1}{\epsilon_0^2} \right)^2 \]
\[ \leq 3c_0 \delta + \frac{q}{2} c_2 \left( \frac{C_1}{\epsilon_0} \right)^2 = 3c_0 \delta + \frac{c_0 C_2^2}{2k_1^2} Q^{-1} \]
\[ \leq 3c_0 \delta + \frac{c_0 C_2^2}{2k_1^2} \delta < \delta \] (26)
provided that $c_0 < 1/6$ and $\frac{c_0 C_2^2}{2k_1^2} < 1/2$. Thus, for any $x \in \frac{3}{4}J \setminus B(J, \lambda, K, T)$ there exists some $(q, p_1, p_2)$ such that (19), (22) and (26) are satisfied. In other words, $(q, p_1) \in \ldots$
$A_\theta(Q, \delta, J)$ and moreover, in view of (12) we have that (5) is satisfied for all $Q$ sufficiently large. This completes the proof of Theorem 4.

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