A family of orthogonal polynomials corresponding to Jacobi matrices with a trace class inverse

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Abstract
Assume that \( \{a_n; n \geq 0\} \) is a sequence of positive numbers and \( \sum a_n^{-1} < \infty \). Let \( \alpha_n = ka_n, \beta_n = a_n + k^2 a_{n-1} \) where \( k \in (0, 1) \) is a parameter, and let \( \{P_n(x)\} \) be an orthonormal polynomial sequence defined by the three-term recurrence
\[
\alpha_0 P_1(x) + (\beta_0 - x) P_0(x) = 0, \quad \alpha_n P_{n+1}(x) + (\beta_n - x) P_n(x) + \alpha_{n-1} P_{n-1}(x) = 0
\]
for \( n \geq 1 \), with \( P_0(x) = 1 \). Let \( J \) be the corresponding Jacobi (tridiagonal) matrix, i.e. \( J_{n,n} = \beta_n, J_{n,n+1} = J_{n+1,n} = \alpha_n \) for \( n \geq 0 \). Then \( J^{-1} \) exists and belongs to the trace class. We derive an explicit formula for \( P_n(x) \) as well as for the characteristic function of \( J \) and describe the orthogonality measure for the polynomial sequence. As a particular case, the modified \( q \)-Laguerre polynomials are introduced and studied.

Keywords: orthogonal polynomials; Jacobi matrix; \( q \)-Laguerre polynomials
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1 Introduction
A semi-infinite Jacobi (tridiagonal) matrix will be written in the form

\[
J = \begin{pmatrix}
\beta_0 & \alpha_0 \\
\alpha_0 & \beta_1 & \alpha_1 \\
& \alpha_1 & \beta_2 & \alpha_2 \\
& & & \ddots & \ddots \\
& & & & \ddots
\end{pmatrix}.
\]
\( J \) is always assumed to be real and non-decomposable, i.e. \( \alpha_n \neq 0 \) for all \( n \geq 0 \). Suppose we are given a sequence of positive numbers \( \{a_n; n \geq 0\} \) such that

\[
\sum_{n=0}^{\infty} \frac{1}{a_n} < \infty,
\]

and a parameter \( k \in (0, 1) \). We will focus on sequences \( \{\alpha_n\} \) and \( \{\beta_n\} \) defined as

\[
\alpha_n := ka_n, \quad \beta_n := a_n + k^2a_{n-1} \quad \text{for } n \in \mathbb{Z}_+
\]

(with \( \mathbb{Z}_+ \) standing for non-negative integers).

We aim to study the orthonormal polynomial sequence \( \{P_n(x); n \geq 0\} \) defined by the three-term recurrence with initial data:

\[
\begin{align*}
\alpha_0P_1(x) + (\beta_0 - x)P_0(x) &= 0, \\
\alpha_nP_{n+1}(x) + (\beta_n - z)P_n(x) + \alpha_{n-1}P_{n-1}(x) &= 0 \quad \text{for } n \geq 1, \\
P_0(x) &= 1.
\end{align*}
\]

Our goal is derivation of a formula for the polynomials \( P_n(x) \) and a description of the respective orthogonality measure. This task is closely related to analysis of spectral properties of an operator in the Hilbert space \( \ell^2(\mathbb{Z}_+) \) represented by the Jacobi matrix \( \mathcal{J} \), \( \mathcal{J} \).

A short remark concerning the matrix operator is worthwhile. Let us denote the canonical basis in \( \ell^2(\mathbb{Z}_+) \) as \( \{e_n; n \geq 0\} \). The matrix \( \mathcal{J} \) clearly represents a symmetric operator in \( \ell^2(\mathbb{Z}_+) \), with the domain being equal to the linear hull of \( \{e_n\} \).

The symmetric operator will be called \( \mathcal{J} \). The deficiency indices of \( \mathcal{J} \) are either \((0, 0)\) or \((1, 1)\). As is well known, this happens if and only if the Hamburger moment problem for \( \{P_n(x)\} \) is or is not determinate, respectively \( \mathcal{J} \). Under our assumptions it will turn out that \( \mathcal{J} \) is essentially self-adjoint. We will denote its self-adjoint closure by the symbol \( J \).

In order to formulate the result we introduce two complex functions

\[
\begin{align*}
\mathcal{F}(z) := & \quad 1 + \sum_{m=1}^{\infty} (-1)^m \\
& \times \left( \sum_{0 \leq j_1 < j_2 < \ldots < j_m < \infty} \frac{(1 - k^{2(j_1+1)})(1 - k^{2(j_2-j_1)}) \cdots (1 - k^{2(j_m-j_{m-1})})}{(1 - k^2)^m a_{j_1} a_{j_2} \cdots a_{j_m}} \right) z^m
\end{align*}
\]

and

\[
\begin{align*}
\mathcal{W}(z) := & \quad \sum_{m=0}^{\infty} (-1)^m \\
& \times \left( \sum_{0 \leq j_0 < j_1 < \ldots < j_m < \infty} \frac{k^{2j_0}(1 - k^{2(j_1-j_0)})(1 - k^{2(j_2-j_1)}) \cdots (1 - k^{2(j_m-j_{m-1})})}{(1 - k^2)^m a_{j_0} a_{j_1} a_{j_2} \cdots a_{j_m}} \right) z^m
\end{align*}
\]
With the above assumptions it is readily seen that the functions are both entire. Moreover, we extend definition (6) by defining a countable family of entire functions, indexed by $n \in \mathbb{Z}_+$,

$$\mathcal{W}_n(z) := \sum_{m=0}^{\infty} (-1)^m \times \left( \sum_{n \leq j_0 < j_1 < \ldots < j_m < \infty} \frac{k^{2j_0} (1 - k^2(j_1 - j_0))(1 - k^2(j_2 - j_1)) \ldots (1 - k^2(j_m - j_{m-1}))}{(1 - k^2)^m a_{j_0} a_j_1 a_{j_2} \ldots a_{j_m}} \right) z^m. \tag{7}$$

Hence $\mathcal{W}_0(z) \equiv \mathcal{W}(z)$. It is straightforward to derive the estimate

$$|\mathcal{W}_n(z)| \leq k^{2n} \min\{a_j; j \geq n\} \exp\left( \frac{|z|}{1 - k^2} \sum_{j=n+1}^{\infty} \frac{1}{a_j} \right). \tag{8}$$

The most essential properties of $J$ and $\{P_n(x)\}$ are described in the following two theorems.

**Theorem 1.** Let $\{a_n; n \geq 0\}$ be a sequence of positive numbers satisfying (2) and $k \in (0, 1)$. Furthermore, let $\{\alpha_n\}$ and $\{\beta_n\}$ be the sequences introduced in (3). Then the symmetric operator $\hat{J}$ in $\ell^2(\mathbb{Z}_+)$ which is defined on the linear hull of $\{e_n; n \geq 0\}$ and represented by the Jacobi matrix (1) is essentially self-adjoint. The spectrum of its closure, a self-adjoint operator $J$, satisfies

$$\text{spec}(J) = \text{spec}_p(J) = \{\lambda_j; j \in \mathbb{Z}_+\}, \tag{9}$$

with all eigenvalues $\lambda_j$ being positive and simple. Moreover, it holds true that

$$\sum_{j=0}^{\infty} \frac{1}{\lambda_j} = \sum_{j=0}^{\infty} \frac{1 - k^{2j+2}}{(1 - k^2)a_j} < \infty$$

so that $J^{-1}$ exists and belongs to the trace class. For every $j \in \mathbb{Z}_+$, an eigenvector corresponding to the eigenvalue $\lambda_j$ can be chosen as the column vector

$$\left( \Phi_0(\lambda_j), \Phi_1(\lambda_j), \Phi_2(\lambda_j), \ldots \right)^T \in \ell^2(\mathbb{Z}_+), \text{ where } \Phi_n(z) := (-1)^n k^{-n} \mathcal{W}_n(z), \tag{10}$$

with $\mathcal{W}_n(z)$ defined in (7). The entire function $\mathfrak{g}(z)$ defined in (5) is a characteristic function of $J$ in the sense that, for all $z \in \mathbb{C}$,

$$\mathfrak{g}(z) = \prod_{j=0}^{\infty} \left( 1 - \frac{z}{\lambda_j} \right). \tag{11}$$

**Theorem 2.** Let $\{a_n; n \geq 0\}$ be a sequence of positive numbers satisfying (2), and $k \in (0, 1)$. Furthermore, let $\{P_n(x); n \geq 0\}$ be the orthonormal polynomial sequence
defined in \( \mathcal{H} \), \( \mathcal{J} \). Then, for all \( n \in \mathbb{Z}_+ \),

\[
(-1)^n k^n P_n(x) = 1 + \sum_{m=1}^{n} (-1)^m 
\times \left( \sum_{0 \leq j_1 < j_2 < \ldots < j_m \leq n-1} \frac{(1 - k^{2(j_1+1)})(1 - k^{2(j_2-j_1)}) \ldots (1 - k^{2(j_m-j_{m-1})})}{(1 - k^2)^m a_{j_1} a_{j_2} \ldots a_{j_m}} \right) x^m.
\]

The Hamburger moment problem for \( \{P_n(x)\} \) is determinate. The unique orthogonality measure is supported on the zero set of the function \( \mathcal{F}(z) \) defined in \( \mathcal{H} \), i.e. the measure support coincides with the spectrum \( \{\lambda_j; j \geq 0\} \) of the operator \( J \), as described in Theorem \( \mathcal{J} \). It holds true that

\[
\forall s, t \in \mathbb{Z}_+, \sum_{j=0}^{\infty} \mu_j P_s(\lambda_j) P_t(\lambda_j) = \delta_{s,t}
\]

where the masses \( \mu_j, j \in \mathbb{Z}_+ \), are given by

\[
\mu_j = -\frac{W(\lambda_j)}{\mathcal{F}'(\lambda_j)},
\]

with \( \mathcal{W}(z) \) being defined in \( \mathcal{G} \).

Remark 3. Let us mention an interpretation of the function \( \mathcal{W}(z) \) which is related to the notion of the associated Jacobi matrix. By definition, the associated Jacobi matrix \( \mathcal{J}^{(1)} \) is obtained from \( \mathcal{J} \) given in \( \mathcal{J} \) by deleting the first row and column. Under the same assumption as in Theorem \( \mathcal{J} \), there exists exactly one self-adjoint operator \( J^{(1)} \) whose matrix in the canonical basis equals \( \mathcal{J}^{(1)} \), \( J^{(1)} \) is positive definite and its inverse belongs to the trace class, see Proposition \( \mathcal{J} \) below. \( \mathcal{W}(z) \) is a characteristic function of \( J^{(1)} \) in the sense that, for all \( z \in \mathbb{C} \),

\[
\mathcal{W}(z) = \mathcal{W}(0) \prod_{j=0}^{\infty} \left( 1 - \frac{z}{\lambda_j^{(1)}} \right)
\]

where \( \text{spec}(J^{(1)}) = \text{spec}_{p}(J^{(1)}) = \{\lambda_j^{(1)}; j \in \mathbb{Z}_+\} \). The proof of this fact is omitted, however, since it is not substantial for the rest of the paper.

The remainder of the paper is devoted to a proof of Theorems \( \mathcal{J} \) \( \mathcal{J} \) and to an application. The paper is organized as follows. In Section \( \mathcal{J} \) we provide a brief summary of some basic notions and preparatory results. Sections \( \mathcal{J} \) contains a proof of Theorems \( \mathcal{J} \) \( \mathcal{J} \). This task is accomplished by proving a series of separate propositions and lemmas. In Section \( \mathcal{J} \) we derive some summation formulas which are needed in the following section, Section \( \mathcal{J} \) in which a modification of \( q \)-Laguerre polynomials is proposed. It turns out that the orthogonality measure for the modified \( q \)-Laguerre polynomials is supported on the roots of a Jackson \( q \)-Bessel function of the second kind.
2 Preliminaries

It can be shown that if there exists exactly one self-adjoint operator $J$ whose matrix in the canonical basis equals $J$ then a similar assertion is true for the associated Jacobi matrix $J^{(1)}$ as well \cite{[9]}. The self-adjoint operator corresponding to $J^{(1)}$ will be denoted $J^{(1)}$.

Proposition 4. Assume that there exists exactly one self-adjoint operator $J$ in $\ell^2(\mathbb{Z}^+)$ whose matrix in the canonical basis equals a Jacobi matrix $J$. Assume further that $J$ is bounded below by a positive constant and $J^{-1}$ belongs to the trace class. Then the same is true for the self-adjoint operator $J^{(1)}$ corresponding to the associated Jacobi matrix $J^{(1)}$.

Proof. The proof is computational and is outlined below. From the fact that the matrix of $J$ in the canonical basis is tridiagonal it follows that

$$
\beta_0 \langle e_0, J^{-1} e_0 \rangle + \alpha_0 \langle e_1, J^{-1} e_0 \rangle = 1,
\beta_0^2 \langle e_0, J^{-1} e_0 \rangle - \alpha_0^2 \langle e_1, J^{-1} e_1 \rangle = \beta_0.
$$

Using these relations it is straightforward to show that for

$$
M := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \alpha_0 \begin{pmatrix} \langle e_1, J^{-1} e_0 \rangle & \langle e_1, J^{-1} e_1 \rangle \\ \langle e_0, J^{-1} e_0 \rangle & \langle e_1, J^{-1} e_0 \rangle \end{pmatrix}
$$

we have

$$
\det(M) = \beta_0 \langle e_0, J^{-1} e_0 \rangle.
$$

Denote as $\beta_0 \oplus J^{(1)}$ the block diagonal matrix with blocks $(\beta_0)$ and $J^{(1)}$. Equation

$$
(\beta_0 \oplus J^{(1)}) \left( J^{-1} + J^{-1} \left( \sum_{s=0}^{1} \sum_{t=0}^{1} x_{s,t} e_s e_t^T \right) J^{-1} \right) = I
$$

results in four equations for four unknowns $x_{s,t}$, $0 \leq s, t \leq 1$, written in the matrix form

$$
\alpha_0 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \alpha_0 \begin{pmatrix} \langle e_1, J^{-1} e_0 \rangle & \langle e_1, J^{-1} e_1 \rangle \\ \langle e_0, J^{-1} e_0 \rangle & \langle e_1, J^{-1} e_0 \rangle \end{pmatrix} \right) \begin{pmatrix} x_{0,0} & x_{0,1} \\ x_{1,0} & x_{1,1} \end{pmatrix}.
$$

A unique solution is readily obtained. Since the inverse is unique we have

$$
(\beta_0 \oplus J^{(1)})^{-1} = J^{-1} + J^{-1} \left( \sum_{s=0}^{1} \sum_{t=0}^{1} x_{s,t} e_s e_t^T \right) J^{-1}
$$

where

$$
x_{0,0} = \frac{\alpha_0^2 \langle e_1, J^{-1} e_1 \rangle}{\beta_0 \langle e_0, J^{-1} e_0 \rangle}, \quad x_{0,1} = x_{1,0} = \alpha_0, \quad x_{1,1} = \frac{\alpha_0^2}{\beta_0}.
$$

\]
Whence
\[
\text{tr} \left( (J^{(1)})^{-1} \right) = \text{tr}(J^{-1}) + \sum_{s=0}^{1} \sum_{t=0}^{1} x_{s,t}(e_s, J^{-2} e_t) - \frac{1}{\beta_0}.
\]
This shows the proposition.

We still assume that $J$ is a unique self-adjoint operator in $\ell^2(\mathbb{Z}_+)$ whose matrix in the canonical basis equals $\mathcal{J}$, as given in (11). Let $\mu$ be the unique Borel probability measure on $\mathbb{R}$ which is an orthogonality measure for the orthonormal polynomial sequence $\{P_n(x)\}$ defined in (11). Furthermore, let $w_n(z), n \geq 0,$ be the so called functions of the second kind [11],
\[
\forall z \in \varrho(J), \forall n \geq 0, \ w_n(z) := \int \frac{P_n(\lambda)}{\lambda - z} \, d\mu(\lambda).
\]
Here $\varrho(J)$ denotes the resolvent set of $J$. In particular,
\[
w(z) \equiv w_0(z) = \int \frac{d\mu(\lambda)}{\lambda - z}
\]
is the Weyl function of $J$. Thus the Weyl function is the Stieltjes transformation of the measure $\mu$.

One can check that [9]
\[
\forall z \in \varrho(J), \forall n \geq 0, \ w_n(z)P_n(z) = \langle e_n, (J - z)^{-1} e_n \rangle.
\]
Particularly, if $J$ is bounded below by a positive constant $\gamma$ and so $0 \in \varrho(J)$ then
\[
\sum_{n=0}^{\infty} \langle e_n, J^{-1} e_n \rangle = \sum_{n=0}^{\infty} w_n(0)P_n(0).
\]
Hence $J^{-1}$ belongs to the trace class if and only if this sum converges and, if so, the sum equals $\text{tr} J^{-1}$. Moreover, the support of the unique orthogonality measure is contained in $[\gamma, \infty)$. Necessarily, all roots of the polynomials $P_n(z), n \in \mathbb{N}$, are contained in this interval, too [2]. We have
\[
\forall z \in \mathbb{C} \setminus [\gamma, \infty), \forall n \geq 0, \ w_n(z) = -\left( \sum_{j=n}^{\infty} \frac{1}{\alpha_j P_j(z) P_{j+1}(z)} \right) P_n(z).
\]
In addition, as also proven in [9], if $J$ is bounded below by a positive constant and $J^{-1}$ is a trace class operator then
\[
\exists_{\text{char}}(z) := 1 - z \sum_{n=0}^{\infty} w_n(0)P_n(z)
\]
is a characteristic function of $J$ in the sense that
\[
\exists_{\text{char}}(z) = \prod_{j=0}^{\infty} \left( 1 - \frac{z}{\lambda_j} \right) \text{ where } \text{spec}(J) = \text{spec}_p(J) = \{\lambda_j; j \geq 0\}.
\]
3 Proofs of Theorems 1 and 2

All the claims contained in Theorems 1 and 2 will be proven step by step as separate lemmas or propositions while assuming everywhere that \( \{a_n\} \) is a sequence of positive numbers satisfying (2) and the sequences \( \{\alpha_n\} \) and \( \{\beta_n\} \) are given by equations (3). Furthermore, \( \{P_n(x)\} \) denotes the orthonormal polynomial sequence defined in (4).

**Proposition 5.** The symmetric operator \( \tilde{J} \) in \( \ell^2(\mathbb{Z}_+) \), with the matrix \( J \) given in (1), is essentially self-adjoint, and the Hamburger moment problem for \( \{P_n(x)\} \) is determinate. The closure of \( \tilde{J} \), i.e. the self-adjoint operator \( J \), is positive definite and \( J^{-1} \) belongs to the trace class.

**Proof.** From (3), (4) we infer that the three-term recurrence in case \( x = 0 \) can be reduced to a two-term recurrence, namely we get

\[
ka_n(P_{n+1}(0) + k^{-1}P_n(0)) + k^2a_{n-1}(P_n(0) + k^{-1}P_{n-1}(0)) = 0 \quad \text{for} \quad n \geq 1.
\]

Taking into account the initial data we have

\[
P_n(0) = (-1)^nk^{-n} \quad \text{for} \quad n \geq 0. \tag{19}
\]

Thus the sum \( \sum_n P_n(0)^2 \) is divergent and the Hamburger moment problem is determinate [1]. This in turn means that the symmetric operator \( \tilde{J} \) is essentially self-adjoint [1].

The matrix \( J \) can be decomposed,

\[
J = (I + kE)J_0(I + kE^T),
\]

where \( I \) is the unite matrix, \( J_0 := \text{diag}(a_0, a_1, a_2, \ldots) \) and \( E \) is a strictly lower triangular matrix whose only nonzero elements are \( E_{n+1,n} = 1, n \geq 0 \). Since all factors are band matrices the product is well defined. It follows that

\[
\forall f \in \text{span}\{e_n\}, \quad \langle f, Jf \rangle \geq a_{\min} \| (I + kE^T)f \|^2 \geq a_{\min} (1 - k)^2 \| f \|^2
\]

where

\[
a_{\min} := \min\{a_n; n \geq 0\}.
\]

Since \( \text{span}\{e_n\} \) is a core for \( J \) we have

\[
J \geq \gamma \quad \text{where} \quad \gamma := a_{\min} (1 - k)^2 > 0.
\]

In view of (19) and recalling (17), (16) we get

\[
w_n(0) = (-1)^n \sum_{j=n}^{\infty} \frac{k^{2j-n}}{a_j} = (-1)^n k^n \sum_{j=0}^{\infty} \frac{k^{2j}}{a_{n+j}} \tag{20}
\]
and
\[ \text{tr } J^{-1} = \sum_{n=0}^{\infty} w_n(0) P_n(0) = - \sum_{n=0}^{\infty} P_n(0)^2 \sum_{j=n}^{\infty} \frac{1}{\alpha_j P_j(0) P_{j+1}(0)} = \sum_{n=0}^{\infty} k^{-2n} \sum_{j=n}^{\infty} \frac{k^{2j}}{a_j} = \sum_{j=0}^{\infty} \frac{1}{a_j} \sum_{n=0}^{j} k^{2(j-n)} = \sum_{j=0}^{\infty} \frac{1 - k^{2j+2}}{(1 - k^2)a_j} < \infty. \]

Hence \( J^{-1} \) belongs to the trace class \( \square \).

**Remark 6.** We have derived that
\[ \text{tr } J^{-1} = \sum_{j=0}^{\infty} \frac{1 - k^{2j+2}}{(1 - k^2)a_j}. \]

**Proposition 7.** The polynomial sequence \( \{ P_n(x) \} \) defined in (4) satisfies (12).

**Proof.** Denote by \( \pi_n(x) \) the RHS of (12). We have to verify that the sequence \( \{ (-1)^n k^{-n} \pi_n(x) \} \) satisfies (4). The initial condition is actually fulfilled since \( \pi_0(x) = 1 \). Furthermore, \( \pi_1(x) = 1 - x/a_0 \), and thus the beginning of the recurrence for \( n = 0 \) is immediately seen to be satisfied, too. So we can focus on the recurrence for \( n \geq 1 \).

In view of (3), the recurrence we wish to prove takes the form
\[ a_n(\pi_{n+1}(x) - \pi_n(x)) - k^2 a_{n-1}(\pi_n(x) - \pi_{n-1}(x)) = -x \pi_n(x). \] (21)

Note that in the outer sum on the RHS of (12) we can write \( \sum_{m=1}^{\infty} \) instead of \( \sum_{m=1}^{n} \) owing to the constraint on the indices \( j_i \) in the inner sum. Considering the formula for \( \pi_{n+1}(x) \) and separating the cases \( j_m < n \) and \( j_m = n \) we obtain
\[ \pi_{n+1}(x) = \pi_n(x) + \frac{(1 - k^{2(n+1)})x}{(1 - k^2)a_n} + \sum_{m=2}^{\infty} (-1)^m x^m \times \sum_{0 \leq j_1 < j_2 < \ldots < j_{m-1} < n} \frac{(1 - k^{2(j_1+1)})(1 - k^{2(j_2-j_1)}) \ldots (1 - k^{2(j_{m-1}-j_{m-2})})(1 - k^{2(n-j_{m-1})})}{(1 - k^2)^m a_{j_1} a_{j_2} \cdots a_{j_{m-1}} a_n}. \]

This can be rewritten as
\[ a_n(\pi_{n+1}(x) - \pi_n(x)) = x \left( \frac{1 - k^{2(n+1)}}{1 - k^2} - x \sum_{m=1}^{\infty} (-1)^m \right. \]
\[ \times \left( \sum_{0 \leq j_1 < j_2 < \ldots < j_m \leq n-1} \frac{(1 - k^{2(j_1+1)})(1 - k^{2(j_2-j_1)}) \ldots (1 - k^{2(j_m-j_{m-1})})}{(1 - k^2)^m a_{j_1} a_{j_2} \cdots a_{j_m}} \times \frac{1 - k^{2(n-j_m)}}{1 - k^2} \right) x^m. \]

A similar formula holds for \( a_{n-1}(\pi_n(x) - \pi_{n-1}(x)) \) where the constraint \( \ldots < j_m \leq n-2 \) can be replaced by \( \ldots < j_m \leq n-1 \) owing to the factor \( (1 - k^{2(n-1-j_m)}) \). Now it is straightforward to evaluate the LHS of (21) and consequently to find out that equation (21) is actually valid. \( \square \)
Proposition 8. The function $\mathfrak{F}(z)$ defined in (3) is a characteristic function of $J$ in the sense that equations (9), (17) hold.

Proof. As explained in Section 2 we have to show that the function $\mathfrak{F}_{\text{char}}(z)$ defined in (18) equals the entire function $\mathfrak{F}(z)$. To this end we can use the already proven formulas (12) and (20). Thus we can evaluate the coefficient at $z^{m+1}$, $m \in \mathbb{N}$, in the RHS of (18). For an $m$-tuple of indices $j_1, \ldots, j_m \in \mathbb{Z}_+$ let us denote

$$s(j_1, \ldots, j_m) := \frac{(1 - k^{2(j_1+1)}) (1 - k^{2(j_2-j_1)}) \cdots (1 - k^{2(j_m-j_{m-1})})}{(1 - k^2)^m a_{j_1} a_{j_2} \cdots a_{j_m}}.$$

Then the coefficient at $z^{m+1}$ equals, up to the sign $(-1)^{m+1}$,

$$\sum_{n=m}^{\infty} \sum_{j=0}^{\infty} \frac{k^{2j}}{a_{n+j}} \sum_{0 \leq j_1 < j_2 < \ldots < j_m \leq n-1} s(j_1, \ldots, j_m)$$

$$= \sum_{0 \leq j_1 < j_2 < \ldots < j_m < \infty} \sum_{n=m+1}^{\infty} \sum_{j=0}^{\infty} \frac{k^{2j}}{a_{n+j}} s(j_1, \ldots, j_m)$$

$$= \sum_{0 \leq j_1 < j_2 < \ldots < j_m < \infty} \sum_{j=0}^{m+1-j_m-1} \frac{k^{2j} s(j_1, \ldots, j_m)}{a_{j+m+1}} (1 - k^{2(j_m+1-j_m)})$$

and this is in agreement with (5). One can proceed similarly for the coefficient standing at $z$. This coefficient equals, again up to a sign,

$$\sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \frac{k^{2j}}{a_{n+j}} = \sum_{j_1=0}^{j_1} \sum_{j=0}^{j_2} \frac{k^{2j}}{a_{j_1}} = \sum_{j_1=0}^{\infty} \frac{1 - k^{2(j_1+1)}}{(1 - k^2)a_{j_1}}.$$

This shows that $\mathfrak{F}(z) = \mathfrak{F}_{\text{char}}(z)$. \hfill \Box

Lemma 9. The sequence of functions $\{\Phi_n(z)\}$ defined in (10) and (7) obeys the recurrence equation

$$\alpha_n \Phi_{n+1}(z) + (\beta_n - z) \Phi_n(z) + \alpha_{n-1} \Phi_{n-1}(z) = 0 \quad \text{for } n \geq 1,$$

and also

$$\alpha_0 \Phi_1(z) + (\beta_0 - z) \Phi_0(z) - \mathfrak{F}(z) = 0.$$

Proof. From (7) it is seen that

$$\mathcal{M}_n(z) - \mathcal{M}_{n+1}(z) = \frac{k^{2n}}{a_n} \sum_{m=0}^{\infty} (-1)^m$$

$$\times \left( \sum_{n+1 \leq j_1 < j_2 < \ldots < j_m < \infty} \frac{(1 - k^{2(j_1+1)})(1 - k^{2(j_2-j_1)}) \cdots (1 - k^{2(j_m-j_{m-1})})}{(1 - k^2)^m a_{j_1} a_{j_2} \cdots a_{j_m}} \right) z^m.$$
Here again, the constraint in the inner sum, \( n + 1 \leq j_1 < \ldots \), can be replaced by \( n \leq j_1 < \ldots \) because of the factor \((1 - k^2(j_1 - n))\). We have a similar expression for \( \mathcal{W}_{n-1}(z) - \mathcal{W}_n(z) \). Now one can readily verify that

\[
a_n (\mathcal{W}_n(z) - \mathcal{W}_{n+1}(z)) - k^2 a_{n-1} (\mathcal{W}_{n-1}(z) - \mathcal{W}_n(z)) = z \mathcal{W}_n(z).
\]

In regard of (3) and (10), this equation is equivalent to (22).

Let us extend the sequence \( \{a_n; n \geq 0\} \) by prepending to it an additional member \( a_{-1} \), \( 0 < a_{-1} < a_{\min} \). Then one can define functions \( \mathcal{W}_{-1}(z) \) and \( \Phi_{-1}(z) \) correspondingly, just by extending definitions (7), (10) to \( n = -1 \). With this extended definition, equation (22) holds also for \( n = 0 \). It is immediate to see that

\[
\lim_{a_{-1} \to 0^+} a_{-1} \mathcal{W}_{-1}(z) = k^{-2} \mathfrak{F}(z) \quad \text{whence} \quad \lim_{a_{-1} \to 0^+} ka_{-1} \Phi_{-1}(z) = -\mathfrak{F}(z). \tag{24}
\]

Letting \( n = 0 \) in (22) and applying this limit we obtain (23).

**Lemma 10.** Let \( \{\Phi_n(z)\} \) be the sequence defined in (10), (7). The Wronskian of the sequences \( \{P_n(z)\} \) and \( \{\Phi_n(z)\} \) is a constant and fulfills

\[
\alpha_n (P_n(z)\Phi_{n+1}(z) - P_{n+1}(z)\Phi_n(z)) = \mathfrak{F}(z) \text{ for all } n \geq 0. \tag{25}
\]

**Proof.** The sequences \( \{P_n(z)\} \) and \( \{\Phi_n(z)\} \) satisfy both the same three-term recurrence and therefore their Wronskian is a constant for \( n \geq 0 \). To find the constant one can use the same reasoning as in the proof of Lemma 9 and consider an extended sequence of positive numbers \( \{a_n; n \geq -1\} \). Then (22) is valid also for \( n = 0 \). The second equation in (11) is valid for \( n = 0 \) as well provided we let \( P_{-1}(z) = 0 \). Thus the constant equals

\[
\alpha_{-1} (P_{-1}(z)\Phi_0(z) - P_0(z)\Phi_{-1}(z)) = -k a_{-1} \Phi_{-1}(z).
\]

Applying the limit (24) in this expression one obtains (25). \( \square \)

**Proposition 11.** If \( \lambda \) is an eigenvalue of \( J \) then the column vector

\[
\Phi(\lambda) := (\Phi_0(\lambda), \Phi_1(\lambda), \Phi_2(\lambda), \ldots)^T \in l^2(\mathbb{Z}_+)
\]

is an eigenvector of \( J \) corresponding to \( \lambda \). Here \( \{\Phi_n(z)\} \) is again the sequences defined in (10), (7). The norm of the eigenvector fulfills

\[
\|\Phi(\lambda)\|^2 = -\mathfrak{F}(\lambda) \mathcal{W}(\lambda).
\]

**Proof.** From (8) and (10) it follows that

\[
|\Phi_n(z)| \leq \frac{k^n}{\min\{a_j; j \geq n\}} \exp \left( \frac{|z|}{1 - k^2} \sum_{j=n+1}^{\infty} \frac{1}{a_j} \right) \text{ for } n \geq 0, \tag{26}
\]

and so the sequence \( \{\Phi_n(z)\} \) is square summable for every \( z \in \mathbb{C} \). If \( \lambda \) is an eigenvalue of \( J \) then, by Proposition 8, \( \mathfrak{F}(\lambda) = 0 \) and, according to (25), the sequences \( \{P_n(\lambda)\} \)
and \(\{\Phi_n(\lambda)\}\) are linearly dependent. From the recurrence (11) one infers that \(\Phi(\lambda)\) is an eigenvector of \(J\) corresponding to the eigenvalue \(\lambda\).

Using again the extended sequence of positive numbers \(\{a_n; n \geq -1\}\), as in the proof of Lemma 9 and referring to the Cristoffel-Darboux formula [1] we have, for \(N \in \mathbb{N}\) and every couple \(\lambda, \theta \in \mathbb{C}\),

\[
(\lambda - \theta) \sum_{n=0}^{N} \Phi_n(\lambda)\Phi_n(\theta) = \alpha_{-1}(\Phi_{-1}(\lambda)\Phi_0(\theta) - \Phi_0(\lambda)\Phi_{-1}(\theta)) - \alpha_N(\Phi_N(\lambda)\Phi_{N+1}(\theta) - \Phi_{N+1}(\lambda)\Phi_N(\theta)).
\]

Owing to the estimate (26) one can send \(N \to \infty\). In addition, one can again apply the limit (24) to this expression. This way we get

\[
(\lambda - \theta) \sum_{n=0}^{\infty} \Phi_n(\lambda)\Phi_n(\theta) = -\mathcal{G}(\lambda)\Phi_0(\theta) + \Phi_0(\lambda)\mathcal{G}(\theta).
\]

In the case considered \(\lambda\) is an eigenvalue of \(J\) and \(\mathcal{G}(\lambda) = 0\). Then in the limit \(\theta \to \lambda\) we obtain

\[
\sum_{n=0}^{\infty} \Phi_n(\lambda)^2 = -\mathcal{G}'(\lambda)\Phi_0(\lambda).
\] (27)

But note that \(\Phi_0(z) = \mathfrak{M}(z)\). □

**Corollary 12.** Let \(\mu\) be the unique Borel probability measure on \(\mathbb{R}\) which is an orthogonality measure for \(\{P_n(x)\}\). Then for all eigenvalues \(\lambda\) of \(J\),

\[
\mu(\{\lambda\}) = -\frac{\mathfrak{M}(\lambda)}{\mathcal{G}(\lambda)}.
\]

**Proof.** According to the general theory [1, Subsec. 2.5] and by Proposition 8,

\[
supp \mu = \text{spec } J = \mathcal{G}^{-1}(\{0\})
\]

and

\[
\forall \lambda \in \text{spec } J, \quad \mu(\{\lambda\}) = \left(\sum_{n=0}^{\infty} P_n(\lambda)^2\right)^{-1}.
\]

In view of (27), it suffices to observe that for every \(\lambda \in \mathcal{G}^{-1}(\{0\})\) and all \(n \geq 0\), \(P_n(\lambda) = \Phi_n(\lambda)/\Phi_0(\lambda)\). Recall again that \(\Phi_0(z) = \mathfrak{M}(z)\). □

**Proposition 13.** The functions of the second kind defined in (14) can be expressed as

\[
\forall n \geq 0, \quad w_n(z) = \frac{\Phi_n(z)}{\mathfrak{G}(z)}.
\] (28)

Here again, \(\{\Phi_n(z)\}\) is the sequences defined in (10), (7). In particular, for \(n = 0\) we have an expression for the Weyl function (15),

\[
w(z) = \frac{\mathfrak{M}(z)}{\mathfrak{G}(z)}.
\] (29)
Proof. This assertion is a direct consequence of the following well known fact (for instance, it is in principle contained in [1], using somewhat different terminology it can be found in [10, Chap. 2], a detailed discussion is also provided in [9]): in the Hamburger determinate case and for every \( z \in \mathcal{g}(J) \), \( \{w_n(z)\} \) is the unique square summable sequence satisfying

\[
\alpha_0 w_1(z) + (\beta_0 - z) w_0(z) = 1,
\]

\[
\alpha_n w_{n+1}(z) + (\beta_n - z) w_n(z) + \alpha_{n-1} w_{n-1}(z) = 0 \quad \text{for } n \geq 1
\]

The sequence \( \{\Phi_n(z)\} \) is square summable, see (26), and a comparison with (22), (23) leads to (28). \( \square \)

Remark 14. Let us point out that there is an alternative way how to derive the formula for the masses of atoms of the discrete measure \( \mu \). We know that \( \mu \) is supported on the spectrum of \( J \) which consists of simple eigenvalues \( \lambda_j, j \geq 0 \). On the other hand, the Weyl function equals the Stieltjes transformation of \( \mu \). Combining (15) and (29) we have

\[
\mathfrak{W}(z) = \int d\mu(\lambda) = \sum_{j=0}^{\infty} \frac{\mu_j}{\lambda_j - z}
\]

where \( \mu_j := \mu(\{\lambda_j\}) \). From here we immediately obtain (13).

Proof of Theorem 1. Proposition 5, Remark 6, Proposition 8 and Proposition 11 jointly imply Theorem 1. \( \square \)

Proof of Theorem 2. Proposition 7, Proposition 8, Proposition 5 and Corollary 12 (or Remark 14) jointly imply Theorem 2. \( \square \)

4 Auxiliary summation identities

Everywhere in what follows \( q \in (0,1) \). In the sequel we are using standard notations as far as the \( q \)-Pochhammer symbol and the basic hypergeometric functions are concerned, see [3, 7].

Remark 15. We shall need the identity

\[
\sum_{n=0}^{\infty} \frac{q^{rn}}{(q^n w; q)_{r+1}} = \frac{1}{(1 - q^r)(w; q)_r}, \quad \forall r \in \mathbb{N}.
\]

The both sides are regarded as meromorphic functions in \( w \in \mathbb{C} \).

The identity is a straightforward consequence of a well known formula for \( \phi_0(q^n; q, w) \) telling us that [3, 7]

\[
\frac{1}{(w; q)_m} = \sum_{s=0}^{\infty} (q^n; q)_s w^s.
\]

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Here \( w \in \mathbb{C}, |w| < 1, \) and \( m \in \mathbb{Z}_+. \) In fact, it suffices to show (31) for \( |w| < 1. \) Then, in view of (32), the LHS of (31) equals

\[
\sum_{n=0}^{\infty} q^n \sum_{s=0}^{\infty} \frac{(q^{r+1}; q)_s}{(q; q)_s} q^{ns} w^s = \sum_{s=0}^{\infty} \frac{(q^{r+1}; q)_s}{(1 - q^{r+s})(q; q)_s} w^s = \frac{1}{1 - q^r} \sum_{s=0}^{\infty} \frac{(q^r; q)_s}{(q; q)_s} w^s
\]

\[
= \frac{1}{(1 - q^r)(w; q)_r}.
\]

**Lemma 16.** For every \( w \in \mathbb{C}, |w| < 1, \) and \( m \in \mathbb{N}, \)

\[
(1 - q^m)(w; q)_m \sum_{j=0}^{\infty} q^{(m+1)j} w^j \sum_{n=0}^{\infty} q^{(j+m+1)n} = \sum_{j=0}^{\infty} q^{mj} w^j.
\]

**Proof.** We can express the LHS of (33) as

\[
\frac{(w; q)_m}{w} \sum_{j=0}^{\infty} q^{(m+1)j} w^j \sum_{n=0}^{\infty} q^{(j+m)n} \left( \frac{1}{(q^n w; q)_n} - \frac{1}{(q^{n+1} w; q)_n} \right)
\]

\[
= \frac{(w; q)_m}{w} \sum_{j=0}^{\infty} q^{(m+1)j} w^j \left( \frac{1}{(w; q)_m} + \sum_{n=0}^{\infty} \frac{(q^{j+m} - 1)q^{(j+m)n}}{(q^{n+1} w; q)_n} \right)
\]

\[
= \frac{1}{w} \sum_{j=0}^{\infty} q^{(m+1)j} w^j - \frac{(w; q)_m}{w} \sum_{n=0}^{\infty} q^{mn} \frac{1}{(q^{n+1} w; q)_n}.
\]

Refraining to (33), this expression can be further simplified and we obtain

\[
\frac{1}{w} \sum_{j=0}^{\infty} q^{(m+1)j} w^j - \frac{(w; q)_m}{w(1 - q^m)(qw; q)_m} = \frac{1}{w} \sum_{j=0}^{\infty} q^{(m+1)j} w^j - \frac{1 - w}{w(1 - q^m)(1 - q^m w)}
\]

\[
= \frac{1}{1 - q^m w} + \sum_{j=0}^{\infty} q^{(m+1)(j+1) w^j}.
\]

Now one can readily check that the last expression actually equals the RHS of (33). \( \square \)

**Proposition 17.** For every \( a > 0 \) and \( m \in \mathbb{Z}_+, \)

\[
\sum_{0 \leq n_0 \leq n_{m-1} \leq \ldots \leq n_0 < \infty} q^{n_0 + \ldots + n_{m-1} + n_m} (q^{n_0 + a}; q)_2 (q^{n_{m-1} + a + 2}; q)_2 \cdots (q^{n_1 + a + 2m-2}; q)_2 (q^{n_0 + a + 2m}; q)_1
\]

\[
= \frac{1}{(q; q)_m (q^a; q)_m} \sum_{j=0}^{\infty} q^{(m+a)j} \frac{1}{1 - q^{j+m+1}}.
\]

**Remark 18.** Note that the LHS of (34) can be rewritten as

\[
\sum_{n_0=0}^{\infty} \sum_{n_1=0}^{\infty} \cdots \sum_{n_m=0}^{\infty} q^{n_0 + 2n_1 + \ldots + (m+1)n_m} (q^{n_0 + a}; q)_2 (q^{n_{m-1} + a + 2}; q)_2 \cdots (q^{n_1 + a + 2m-2}; q)_2 (q^{n_0 + n_1 + \ldots + n_m + a + 2m}; q)_1.
\]
To see it one can simply apply in this expression the substitution
\[ n_j + n_{j+1} + \ldots + n_m = n_j', \quad 0 \leq j \leq m. \]

**Proof.** The claim is true for \( m = 0 \) since
\[
\sum_{n=0}^{\infty} \frac{q^n}{1 - q^{n+a}} = \sum_{j=0}^{\infty} \frac{q^{aj}}{1 - q^{j+1}}.
\]

Let us denote (in this proof only) the LHS of (34) as \( T_m(a) \). Note that, for \( \ell \in \mathbb{Z}_+ \),
\[
\sum_{\ell \leq n_m \leq n_{m-1} \leq \ldots \leq n_0 < \infty} \frac{q^{n_0+\ldots+n_{m-1}+n_m}}{(q^{n_0+a};q)_2(q^{n_{m-1}+a+2};q)_2 \ldots (q^{n_1+a+2m-2};q)_2(q^{n_0+a+2m};q)_1}
= q^{(m+1)\ell} T_m(\ell + a).
\]

Thus we get, for \( m \geq 1 \),
\[
T_m(a) = \sum_{n=0}^{\infty} \frac{q^n}{(q^{n+a};q)_2} \sum_{n \leq n_{m-1} \leq \ldots \leq n_0 < \infty} q^{n_0+\ldots+n_{m-1}} \times ((q^{n_{m-1}+a+2};q)_2 \ldots (q^{n_1+a+2m-2};q)_2(q^{n_0+a+2m};q)_1)^{-1}
= \sum_{n=0}^{\infty} \frac{q^{(m+1)n}}{(q^{n+a};q)_2} T_{m-1}(n + a + 2).
\]

Hence in order to prove the formula by mathematical induction on \( m \) it suffices to show that the RHS of (34) satisfies the same recurrence. Thus we have to verify that, for all \( m \geq 1 \),
\[
\frac{1}{(q; q)_m(q^a; q)_m} \sum_{j=0}^{\infty} \frac{q^{(m+a)j}}{1 - q^{j+m+1}} = \sum_{n=0}^{\infty} \frac{q^{(m+1)n}}{(q^{n+a};q)_2} \frac{1}{(q^{n+a+2};q)_{m-1}} \sum_{j=0}^{\infty} \frac{q^{(m+n+a+1)j}}{1 - q^{j+m}}.
\]

This equation can be simplified so that it takes the form
\[
\sum_{j=0}^{\infty} \frac{q^{(m+a)j}}{1 - q^{j+m+1}} = (1 - q^m)(q^a; q)_m \sum_{j=0}^{\infty} \frac{q^{(m+a+1)j}}{1 - q^{j+m}} \sum_{n=0}^{\infty} \frac{q^{(m+j+1)n}}{(q^{n+a}; q)_{m+1}}.
\]

But this is a consequence of Lemma [16] for \( w = q^a \). \( \square \)

**Proposition 19.** Let \( m \in \mathbb{Z}_+ \) and \( c_0, c_1, \ldots, c_m > 0 \). Then
\[
\frac{1}{(1 - q)^m} \sum_{0 \leq j_0 \leq j_1 \leq \ldots \leq j_m < \infty} q^{c_0(j_0+c_1j_1+\ldots+c_mj_m)}(1 - q^{j_1-j_0})(1 - q^{j_2-j_1}) \ldots (1 - q^{j_m-j_{m-1}})
= \frac{q^{c_1 + 2c_2 + \ldots + mc_m}}{(q^{c_0+c_1+c_2+\ldots+c_m};q)_1(q^{c_1+c_2+\ldots+c_m};q)_2(q^{c_2+\ldots+c_m};q)_2 \ldots (q^{c_m}; q)_2}. \tag{35}
\]
Proof. Let us denote (in this proof only) the LHS of (35) as \(\Psi_m(c_0, c_1, \ldots, c_m)\). We proceed by mathematical induction on \(m\). For \(m = 0\) the equation is just the sum of a geometric series. For \(m \geq 1\) we use the substitution

\[ j_0 = j, \; j_1 = j_1' + j, \ldots, j_m = j_m' + j, \]

and get

\[
\Psi_m(c_0, c_1, \ldots, c_m) = \frac{1}{(1 - q)^m} \sum_{j=0}^{\infty} q^{(c_0 + c_1 + \ldots + c_m)j} \sum_{0 \leq j_1 \leq \cdots \leq j_m < \infty} q^{c_1 j_1 + \ldots + c_m j_m} \times (1 - q^{j_1})(1 - q^{j_2-j_1}) \cdots (1 - q^{j_m-j_{m-1}})
\]

By the induction hypothesis, the last expression equals

\[
\frac{q^{c_2 + \ldots + (m-1)c_m}}{(1 - q)(1 - q^{c_0 + c_1 + \ldots + c_m})(q^{c_2 + \ldots + c_m}; q)_2 \cdots (q^{cm}; q)_2}
\]

and this is readily seen to be equal to the RHS of (35). \(\square\)

**Proposition 20.** Let \(m \in \mathbb{Z}_+\) and \(c_0, c_1, \ldots, c_m > 0\). Then

\[
\frac{1}{(1 - q)^{m+1}} \sum_{0 \leq j_0 < j_1 < \cdots < j_m < \infty} q^{c_0 j_0 + c_1 j_1 + \ldots + c_m j_m} \times (1 - q^{j_0+1})(1 - q^{j_1-j_0})(1 - q^{j_2-j_1}) \cdots (1 - q^{j_m-j_{m-1}})
\]

\[
= \frac{q^{c_1 + 2c_2 + \ldots + mc_m}}{(q^{c_0 + c_1 + \ldots + c_m}; q)_2(q^{c_1 + c_2 + \ldots + c_m}; q)_2 \cdots (q^{cm}; q)_2}.
\]

Proof. The proof is similar to that of Proposition 19. We again denote the LHS of (36) as \(\Psi_m(c_0, c_1, \ldots, c_m)\) and proceed by mathematical induction on \(m\). Verification of the equation for \(m = 0\) is elementary. For \(m \geq 1\) we use the substitution

\[ j_0 = j, \; j_1 = j_1' + j + 1, \ldots, j_m = j_m' + j + 1, \]

and get

\[
\Psi_m(c_0, c_1, \ldots, c_m) = \frac{q^{c_1 + \ldots + c_m}}{(1 - q)^{m+1}} \sum_{j=0}^{\infty} q^{(c_0 + c_1 + \ldots + c_m)j} (1 - q^{j+1})
\]

\[
\times \sum_{0 \leq j_1 \leq \cdots \leq j_m \leq \infty} q^{c_1 j_1 + \ldots + c_m j_m} (1 - q^{j+1})(1 - q^{j_2-j_1}) \cdots (1 - q^{j_m-j_{m-1}})
\]

\[
= \frac{q^{c_1 + \ldots + c_m}}{(q^{c_0 + c_1 + \ldots + c_m}; q)_2} \Psi_{m-1}(c_1, \ldots, c_m).
\]

From here the induction step readily follows. \(\square\)
Proposition 21. Let $m \in \mathbb{N}$, $s_1, \ldots, s_m \in \mathbb{N}$, and $a > 0$. Then

$$
\sum_{0 \leq n_m \leq \ldots \leq n_2 \leq n_1 < \infty} q^{s_1 n_1 + \ldots + s_m n_m} \left( (q^{n_m+a}; q)_{s_m+1} (q^{n_{m-1}+s_m+a+1}; q)_{s_m-1+1} \right. \\
\times \left. (q^{n_{m-2}+s_{m-1}+s_m+a+2}; q)_{s_{m-2}+1} \cdots (q^{n_1+s_2+\ldots+s_m+m-1}; q)_{s_1+1} \right)^{-1} = \\
\frac{1}{(1-q^{s_1})(1-q^{s_1+s_2}) \cdots (1-q^{s_1+\ldots+s_m})(q^a; q)_{s_1+\ldots+s_m}}.
$$

(37)

Remark 22. In particular,

$$
\sum_{0 \leq n_m \leq n_{m-1} \leq \ldots \leq n_1 < \infty} q^{n_1 + \ldots + n_{m-1} + n_m} \frac{(q^{n_m+2}; q)_2(q^{n_{m-1}+1}; q)_2 \cdots (q^{n_1+2m}; q)_2}{(q; q)_m(q^2; q)_m} = \frac{1}{q^{1/2}}.
$$

(38)

Proof. Let us denote the LHS of (37) as $S_m(s_1, \ldots, s_m; a)$. We proceed by mathematical induction on $m$. For $m = 1$ the equation reduces to (31). For $m \geq 2$ we have

$$
S_m(s_1, \ldots, s_m; a) = \sum_{n=0}^{\infty} \frac{q^{s_1 + \ldots + s_m}n}{(q^{n+a}; q)_{s_m+1}} \sum_{0 \leq n_m \leq \ldots \leq n_2 \leq n_1 < \infty} q^{s_1 n_1 + \ldots + s_m n_m} \\
	imes \left( (q^{n_m-1+s_m+n+a+1}; q)_{s_m-1+1} (q^{n_{m-1}+s_m+n+a+2}; q)_{s_{m-2}+1} \cdots (q^{n_1+s_2+\ldots+s_m+n+m-1}; q)_{s_1+1} \right)^{-1} \\
= \sum_{n=0}^{\infty} \frac{q^{s_1 + \ldots + s_m}n}{(q^{n+a}; q)_{s_m+1}} S_{m-1}(s_1, \ldots, s_{m-1}; n+a+s_m+1).
$$

By the induction hypothesis and again by equation (31) the last expression equals

$$
\frac{1}{(1-q^{s_1})(1-q^{s_1+s_2}) \cdots (1-q^{s_1+\ldots+s_{m-1}})} \sum_{n=0}^{\infty} \frac{q^{s_1 + \ldots + s_m}n}{(q^{n+a}; q)_{s_1+\ldots+s_{m-1}+s_m+1}} = \\
\frac{1}{(1-q^{s_1})(1-q^{s_1+s_2}) \cdots (1-q^{s_1+\ldots+s_{m-1}})(1-q^{s_1+\ldots+s_{m-1}+s_m})(q^a; q)_{s_1+\ldots+s_{m-1}+s_m}}.
$$

This concludes the proof. \( \square \)

5 Modified $q$-Laguerre polynomials

We are going to consider a particular case of (34) when

$$
a_n = q^{-2(n+1)}(1-q^{n+1}), \quad k = q^{1/2},
$$

(39)
\[
\alpha_n = q^{-2n-3/2} (1 - q^{n+1}), \\
\beta_n = q^{-2n-2} (1 - q^{n+1}) + q^{-2n+1} (1 - q^n) = q^{-2(n+1)} (1 - q^{n+1} + q^3 (1 - q^n)).
\]

The corresponding orthonormal polynomial sequence is again denoted as \{P_n(x)\}.

Recall that the \(q\)-Laguerre polynomials are defined as follows \[8, 7\]

\[
L^{(a)}_n(x; q) := \frac{(q^{a+1}; q)_n}{(q; q)_n} \phi_1(q^{-n}; q^{a+1}; q, -q^{n+a+1} x),
\]

particularly,

\[
L^{(0)}_n(x; q) := \phi_1(q^{-n}; q, -q^{n+1} x), \quad L^{(1)}_n(x; q) := \frac{1 - q^{n+1}}{1 - q} \phi_1(q^{-n}; q^2; q, -q^{n+2} x).
\]

It is known that

\[
\lim_{q \to 1} L^{(a)}_n(1 - q^n; x; q) = L^{(a)}_n(x).
\]

Recall, too, that the Jackson \(q\)-Bessel functions of the second kind are defined as \[5, 6, 7\]

\[
J^{(2)}_\nu(x; q) := \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} \left( \frac{x}{2} \right)^\nu \phi_1(q^{\nu+1}; q, -q^{\nu+1} x^2/4),
\]

particularly,

\[
J^{(2)}_1(x; q) := \frac{x}{2(1 - q)} \phi_1(q^2; q, -q^2 x^2/4).
\]

It is known that all the roots of \(J^{(2)}_\nu(x; q)\) for \(\nu > -1\) are real and simple with the only cluster point at infinity \[4\].

Here we propose a modification of the \(q\)-Laguerre polynomials though this is done for the parameter \(a = 0\) only.

**Definition 23.** The modified \(q\)-Laguerre polynomials are introduced by the equation

\[
\tilde{L}_n(x; q) := q^n L^{(0)}_n(x; q) + (1 - q) L^{(1)}_n(x; q), \quad n \in \mathbb{Z}_+.
\]

Comparing (41) to (H1) one finds that, for all \(n \geq 0\), \(\tilde{L}_n(0; q) = 1\). Clearly,

\[
\lim_{q \to 1} \tilde{L}_n((1 - q)x; q) = L^{(0)}_n(x).
\]

**Lemma 24.** It holds true that

\[
q^n L^{(0)}_n(x; q) + L^{(1)}_{n-1}(x; q) - L^{(1)}_n(x; q) = 0, \quad \forall n \in \mathbb{Z}_+.
\]

**Proof.** Verification is straightforward. It is sufficient to use definition (H1) and expand the basic hypergeometric functions in the resulting expression into power series in \(x\).
Proposition 26. The modified $q$-Laguerre polynomials obey the three-term recurrence

\[
(1 - q^{n+1})\tilde{L}_{n+1}(x; q) - (1 - q^{n+1} + q^3(1 - q^n))\tilde{L}_n(x; q) + q^3(1 - q^n)\tilde{L}_{n-1}(x; q) = -xq^{2n+2}\tilde{L}_n(x; q), \quad \forall n \in \mathbb{Z}_+
\]

(\tilde{L}_1(x; q) \text{ is undefined}, \text{ and } \tilde{L}_0(x; q) = 1).

Proof. First note that $\tilde{L}_0(x; q) = \tilde{L}_0(0; q) = 1.$

The $q$-Laguerre polynomials are known to obey the three-term recurrence \[7\]

\[
-2q^{n+1}xL_n^{(0)}(x; q) = (1 - q^{n+1})(L_{n+1}^{(0)}(x; q) - L_n^{(0)}(x; q))
- q(1 - q^n)(L_n^{(0)}(x; q) - L_{n-1}^{(0)}(x; q))
\]

and

\[
-2q^{n+2}xL_n^{(1)}(x; q) = (1 - q^{n+1})(L_{n+1}^{(1)}(x; q) - L_n^{(1)}(x; q))
- q(1 - q^n)(L_n^{(1)}(x; q) - L_{n-1}^{(1)}(x; q)).
\]

Using \[45\] one can rewrite \[48\] as

\[
-2q^{n+2}xL_n^{(1)}(x; q) = (1 - q^{n+1})(L_{n+1}^{(1)}(x; q) - L_n^{(1)}(x; q) - q^nL_n^{(0)}(x; q))
- q^3(1 - q^n)(L_n^{(1)}(x; q) - L_{n-1}^{(1)}(x; q) - q^nL_n^{(0)}(x; q)).
\]

Taking an appropriate linear combination of \[47\] and \[49\] and using the defining equation \[46\] one obtains \[46\]. \qed

Proposition 26. Let $\{P_n(x); n \geq 0\}$ be the orthonormal polynomial sequence defined in \[4\], with $\alpha_n, \beta_n$ given in \[40\]. Then

\[
P_n(x) = (-1)^nq^{-n/2}\tilde{L}_n(x; q), \quad \forall n \in \mathbb{Z}_+.
\]

The Hamburger moment problem for $\{P_n(x)\}$ is determinate, the corresponding orthogonality measure $\mu$ (normalized as a probability measure) is supported on the roots of the function

\[
\mathfrak{F}(z) = \frac{1 - q}{\sqrt{z}} J_1^{(2)}(2\sqrt{z}; q),
\]

with all the roots being positive. The masses of the roots satisfy

\[
\forall \lambda \in \mathfrak{F}^{-1}(\{0\}), \quad \mu(\{\lambda\}) = \frac{\mathfrak{M}(\lambda)}{\mathfrak{F}'(\lambda)}
\]

where

\[
\mathfrak{M}(z) = \frac{(1 - q)q}{z} J_2^{(2)}(2\sqrt{qz}; q) + \sum_{m=0}^{\infty} q^{(m+3)(m+1)/2} \varphi_1(q^{m+2}; q; q^{m+2}; q, q^{m+2}) (q; q)_m(q^2; q)_m (z)^m.
\]
Remark. Let us point out once more that the respective Weyl function satisfies
\[ w(z) := \int \frac{d\mu(\lambda)}{\lambda - z} = \frac{\mathfrak{W}(z)}{\mathfrak{F}(z)}, \]
see \((29)\) and \((30)\).

Proof. The choice \((40)\) is covered by Theorems \(1\) and \(2\) as a particular case. This means, among others, that the polynomials \(P_n(x)\) satisfy \((12)\), the orthogonality measure \(\mu\) is supported on the roots of the function \(\mathfrak{F}(z)\), with all the roots being positive, the masses of the roots satisfy \((13)\), and functions \(\mathfrak{F}(z)\) and \(\mathfrak{W}(z)\) are defined in \((5)\) and \((6)\), respectively.

Let us show \((50)\). Comparing \((4)\), where \(\alpha_n, \beta_n\) are given in \((40)\), with \((46)\) in Proposition \(46\) we see that the sequences \(\{P_n(x)\}\) and \(\{(−1)^nq^{−n/2}\tilde{L}_n(x; q)\}\) obey the same three-term recurrence as well as the same initial condition. Hence the sequences are necessarily equal.

Let us show \((51)\). Expressing the \(q\)-Bessel function in terms of a basic hypergeometric function, see \((43)\), one finds that equation \((51)\) means that
\[ \mathfrak{F}(z) = 0 \phi_1( ; q^2; q, -q^2z). \] (53)

Recalling \((5)\) and making the choice \((39)\) we have
\[ \mathfrak{F}(z) = 1 + \sum_{n=1}^{\infty} (-1)^n \]
\[ \times \left( \sum_{0 \leq j_1 < j_2 < \ldots < j_n < \infty} \frac{(1 - q^{j_1+1})(1 - q^{j_2+j_1}) \ldots (1 - q^{j_n-j_{n-1}})}{(1 - q^n)(1 - q^{j_1+1})(1 - q^{j_2+1}) \ldots (1 - q^{j_n+1})} q^{2(j_1+j_2+\ldots+j_n)+2n} \right) z^n. \] (54)

Thus, comparing the coefficients at respective powers of \(z\) on the right-hand sides of \((53)\) and \((54)\), one can see that \((53)\) is equivalent to the countably many equations, numbered by \(m \in \mathbb{N}\),
\[ \sum_{0 \leq j_1 < j_2 < \ldots < j_m < \infty} \frac{(1 - q^{j_1+1})(1 - q^{j_2+j_1}) \ldots (1 - q^{j_m-j_{m-1}})}{(1 - q^m)(1 - q^{j_1+1})(1 - q^{j_2+1}) \ldots (1 - q^{j_m+1})} q^{2(j_1+j_2+\ldots+j_m)} = \frac{q^{m(m-1)}}{(q; q)_m(q^2; q)_m}. \]

For a given \(m \in \mathbb{N}\), the LHS here can be actually simplified with the aid of \((36)\) and
also \( (38) \), and we obtain
\[
\frac{1}{(1 - q)^m} \sum_{n_1=0}^{\infty} \cdots \sum_{n_m=0}^{\infty} q^{n_1+n_2+\ldots+n_m} \\
\times \sum_{0 \leq j_1 < \ldots < j_m < \infty} q^{(n_1+2)j_1+\ldots+(n_m+2)j_m} (1 - q^{j_1+1})(1 - q^{j_2-j_1}) \cdots (1 - q^{j_m-j_{m-1}}) \\
= q^{m(m-1)} \sum_{n_1=0}^{\infty} \cdots \sum_{n_m=0}^{\infty} q^{n_1+2n_2+\ldots+mn_m} \\
= q^{m(m-1)} \sum_{0 \leq n_m \leq n_{m-1} \leq \ldots \leq n_1 < \infty} (q^{n_1+2n_2+\ldots+n_m}; q^2) \cdots (q^{n_m+2}; q^2) \\
= \frac{q^{m(m-1)}}{(q; q)_m(q^2; q)_m}.
\]

Let us show \((52)\). Recalling \((50)\) and making the choice \((39)\) we have
\[
\mathcal{W}(z) = q^2 \sum_{m=0}^{\infty} (-1)^m \left( \sum_{0 \leq j_0 < j_1 < j_2 < \ldots < j_m} q^{3j_0+2j_1+2j_2+\ldots+2j_m} \right. \\
\times \left. \frac{(1 - q^{j_1-j_0})(1 - q^{j_2-j_1}) \cdots (1 - q^{j_m-j_{m-1}})}{(1 - q^{j_0+1})(1 - q^{j_1+1}) \cdots (1 - q^{j_m+1})} \right) \left( \frac{q^2 z}{1 - q} \right)^m.
\]

For a given \( m \in \mathbb{Z}_m \) let
\[
X_m := \frac{1}{(1 - q)^m} \sum_{0 \leq j_0 < j_1 < j_2 < \ldots < j_m} q^{3j_0+2j_1+2j_2+\ldots+2j_m} \\
\times \left. \frac{(1 - q^{j_1-j_0})(1 - q^{j_2-j_1}) \cdots (1 - q^{j_m-j_{m-1}})}{(1 - q^{j_0+1})(1 - q^{j_1+1}) \cdots (1 - q^{j_m+1})} \right)
\]
\[
= \frac{1}{(1 - q)^m} \sum_{n_0=0}^{\infty} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_m=0}^{\infty} q^{n_0+n_1+n_2+\ldots+n_m} \\
\times \sum_{0 \leq j_0 < j_1 < j_2 < \ldots < j_m} q^{(n_0+3)j_0+(n_1+2)j_1+(n_2+2)j_2+\ldots+(n_m+2)j_m} (1 - q^{j_1-j_0})(1 - q^{j_2-j_1}) \cdots (1 - q^{j_m-j_{m-1}}).
\]

Using formula \((35)\) we can compute
\[
X_m = \sum_{n_0=0}^{\infty} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_m=0}^{\infty} q^{n_0+2n_1+3n_2+\ldots+(m+1)n_m+(m+1)m} \\
\times \left. \frac{(q^{n_0+n_1+n_2+\ldots+n_m+2m+3}; q)_1(q^{n_1+n_2+\ldots+n_m+2m}; q)_2(q^{n_2+\ldots+n_m+2m-2}; q)_2 \cdots (q^{n_m+2}; q)_2}{(q^{n_0+2m+3}; q)_1(q^{n_1+2m}; q)_2(q^{n_2+2m-2}; q)_2 \cdots (q^{n_m+2}; q)_2} \right)
\]
\[
= q^{(m+1)m} \sum_{0 \leq n_m \leq \ldots \leq n_2 \leq n_1 \leq n_0 < \infty} (q^{n_0+2m+3}; q)_1(q^{n_1+2m}; q)_2(q^{n_2+2m-2}; q)_2 \cdots (q^{n_m+2}; q)_2.
\]
Writing

\[
\frac{q}{1 - q^{n_0 + 2m + 3}} = \frac{1}{1 - q^{n_0 + 2m + 2}} = \frac{1 - q}{(q^{n_0 + 2m + 3}; q)_2}
\]

we get

\[
X_m = q^{(m+1)m-1} \sum_{0 \leq n_m \leq \ldots \leq n_2 \leq n_1 \leq n_0 < \infty} \frac{q^{n_0 + n_1 + n_2 + \ldots + n_m}}{(q^{n_0 + 2m + 2}; q)_1(q^{n_1 + 2m}; q)_2(q^{n_2 + 2m - 2}; q)_2 \ldots (q^{n_m + 2}; q)_2}
\]

\[
- (1 - q)q^{(m+1)m-1} \sum_{0 \leq n_m \leq \ldots \leq n_2 \leq n_1 \leq n_0 < \infty} \frac{q^{n_0 + n_1 + n_2 + \ldots + n_m}}{(q^{n_0 + 2m + 2}; q)_2(q^{n_1 + 2m}; q)_2(q^{n_2 + 2m - 2}; q)_2 \ldots (q^{n_m + 2}; q)_2}
\]

Next we use identity (34) and again (38) thus obtaining

\[
X_m = \frac{q^{(m+1)m-1}}{(q; q)_m(q^2; q)_m} \sum_{j=0}^{\infty} \frac{q^{(m+2)j}}{1 - q^{j+m+1}} - \frac{(1 - q)q^{(m+1)m-1}}{(q; q)_m(q^2; q)_m(q^3; q)_m} \sum_{j=0}^{\infty} \frac{q^{(m+2)j}}{1 - q^{j+m+2}} + \frac{q^{(m+1)m}}{(q; q)_m(q^2; q)_m(q^3; q)_m}.
\]

From the last expression one can deduce that

\[
\mathfrak{W}(z) = \sum_{m=0}^{\infty} \left( \frac{q^{(m+3)(m+1)}}{(q; q)_m(q^2; q)_m} \sum_{j=0}^{\infty} \frac{q^{(m+2)j}}{1 - q^{j+m+2}} + \frac{q^{(m+2)(m+1)}}{(q; q)_m(q^2; q)_m(q^3; q)_m} \right) (-z)^m.
\]

Let us write \(\mathfrak{W}(z) = \mathfrak{W}_1(z) + \mathfrak{W}_2(z)\) where

\[
\mathfrak{W}_1(z) = \frac{q^2}{1 - q^2} \sum_{m=0}^{\infty} \frac{q^m(m-1)}{(q; q)_m(q^2; q)_m} (-q^4 z)^m = \frac{q^2}{1 - q^2} \phi_1(1; q^3; q, -q^4 z)
\]

and

\[
\mathfrak{W}_2(z) = \sum_{m=0}^{\infty} \frac{q^{m+3}(m+1)}{(q; q)_m(q^2; q)_m} \sum_{j=0}^{\infty} \frac{q^{(m+2)j}}{1 - q^{j+m+2}} (-z)^m.
\]

Recalling (32) we have

\[
\mathfrak{W}_1(z) = \frac{(1 - q)q}{z} J_2^{(2)}(2 \sqrt{q^2}; q).
\]

Furthermore,

\[
\sum_{j=0}^{\infty} \frac{q^{aj}}{1 - q^{j+a}} = \frac{1}{1 - q^a} \sum_{j=0}^{\infty} \frac{(q^a; q)_j}{(q^{a+1}; q)_j} q^{aj} = \frac{2 \phi_1(q^a; q, q^{a+1}; q, q^a)}{1 - q^a},
\]

and therefore

\[
\mathfrak{W}_2(z) = \sum_{m=0}^{\infty} \frac{q^{(m+3)(m+1)}}{(q; q)_m(q^2; q)_m(q^3; q)_m} \frac{2 \phi_1(q^{m+2}; q, q^{m+3}; q, q^{m+2})}{(q; q)_m(q^2; q)_m(q^3; q)_m} (-z)^m.
\]

This concludes the proof. \(\square\)
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