A long cycle theorem involving Fan-type degree condition and neighborhood intersection

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Abstract

In this short note we give a new sufficient condition for the existence of long cycles in graphs involving Fan-type degree conditions and neighborhood intersection.

1 Introduction

We use Bondy and Murty [2] for terminology and notation not defined here and consider simple graphs only.

Let $G$ be a graph, $H$ a subgraph of $G$, and $v$ a vertex of $G$. We use $N_H(v)$ to denote the set of neighbors of $v$ in $H$, and call $d_H(v) = |N_H(v)|$ the degree of $v$ in $H$. For $x, y \in V(G)$, an $(x, y)$-path is a path $P$ connecting $x$ and $y$, and vertices $x, y$ will be called the end-vertices of $P$. If $x, y \in V(H)$, the distance between $x$ and $y$ in $H$, denoted by $d_H(x, y)$, is the length of a shortest $(x, y)$-path in $H$. When there is no danger of ambiguity, we will use $N(v), d(v)$ and $d(x, y)$ instead of $N_G(v), d_G(v)$ and $d_G(x, y)$, respectively.

Let $G$ be a graph and $G'$ be a subgraph of $G$. We call $G'$ an induced subgraph of $G$ if $G'$ contains every edge $xy \in E(G)$ with $x, y \in V(G')$. A graph isomorphic to $K_{1,3}$ is called a claw. A modified claw is a graph isomorphic to one obtained by attaching an edge to one vertex of a triangle. We say that $G$ is claw-free if $G$ contains no induced subgraph isomorphic to a claw.

In 1984, Fan [5] gave the following long cycle theorem involving the maximum degree of every pair of vertices at distance two in a 2-connected graph.

THEOREM 1 ([5]). Let $G$ be a 2-connected graph such that $\max \{d(u), d(v)\} \geq c/2$ for each pair of vertices $u$ and $v$ at distance 2. Then $G$ contains either a Hamilton cycle or a cycle of length at least $c$.

In [3], Bedrossian, Chen and Schelp gave an improvement of Fan’s theorem. They further weakened the restriction on the pair of vertices $u$ and $v$ in graphs: they must be vertices of an induced claw or an induced modified claw at distance two.

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THEOREM 2 ([3]) Let $G$ be a 2-connected graph such that $\max\{d(u), d(v)\} \geq c/2$ for each pair of nonadjacent vertices $u$ and $v$ that are vertices of an induced claw or an induced modified claw of $G$. Then $G$ contains either a Hamilton cycle or a cycle of length at least $c$.

On the other hand, Shi [7] gave the following sufficient condition for the existence of Hamilton cycles in claw-free graphs.

THEOREM 3 ([7]) Let $G$ be a 2-connected claw-free graph. If $|N(u) \cap N(v)| \geq 2$ for every pairs of vertices $u, v$ with $d(u, v) = 2$, then $G$ contains a Hamilton cycle.

In this short note we give a new sufficient condition for the existence of long cycles involving Fan-type degree condition and neighborhood intersection. It can be seen as a generalization of Theorem 3.

THEOREM 4 Let $G$ be a 2-connected graph such that $\max\{d(u), d(v)\} \geq c/2$ for each pair of nonadjacent vertices $u$ and $v$ in an induced claw, and $|N(x) \cap N(y)| \geq 2$ for each pair of nonadjacent vertices $x$ and $y$ in an induced modified claw. Then $G$ contains either a Hamilton cycle or a cycle of length at least $c$.

2 Proof of Theorem 4

Before our proof, we give some additional useful terminology and notation. A $v_m$-path is a path which has $v_m$ as one end-vertex. If a $v_m$-path is a longest path among all paths, then we call it a $v_m$-longest path. Let $P = v_1v_2\ldots v_m$ be a path and denote by $t = t(P) = \max\{j : v_1v_j \in E(G)\}$.

The proof of Theorem 4 is motivated by [3]. It is mainly based on two lemmas below.

LEMMA 1 ([1]) Let $G$ be a 2-connected graph and $P$ be a longest path with two end-vertices $x$ and $y$. Then $G$ contains a Hamilton cycle or a cycle of length at least $d(x) + d(y)$.

LEMMA 2 Let $G$ be a non-Hamiltonian 2-connected graph satisfying the condition of Theorem 4. Let $P = v_1v_2\ldots v_m(v_m = v)$ be a longest path of $G$. Then there exists a $v$-longest path such that the other end-vertex of the path has degree at least $c/2$.

PROOF. Suppose not. Now we choose a path $P_1$ such that $t' = t(P_1)$ is as large as possible among all $v_m$-longest paths of $G$. Without loss of generality, we still denote $P_1 = v_1v_2\ldots v_m$.

CLAIM 1 $t' \leq m - 1$.

PROOF. If $v_1v_m \in E(G)$, then $G$ is Hamiltonian or $G$ has a non-Hamilton cycle including all vertices of $P_1$. Hence $G$ has a Hamilton cycle or a path longer than $P_1$ since $G$ is 2-connected, a contradiction.

CLAIM 2. $\{v_1, v_{t' - 1}, v', v_{t' + 1}\}$ induces a modified claw.

PROOF. By Claim 1 and the choice of $t'$, $v_{t' + 1}$ exists and $v_1v_{t' + 1} \notin E(G)$. By the connectedness of $G$ and the choice of $P_1$, $t' \geq 3$. Assume that $v_1v_{t' - 1}, v_{t' - 1}v_{t' + 1} \notin E(G)$. Then $\{v_1, v_{t' - 1}, v', v_{t' + 1}\}$ induces a claw. By the condition of Theorem 4 and the hypothesis that $d(v_{t' - 1}) < c/2$, we have $d(v_{t' - 1}) \geq c/2$ and $d(v_{t' + 1}) \geq c/2$. Let $P'_1 = v_{t' - 1}P_1v'v_{t' + 1}$. Then $P'_1$ is a $v_m$-longest path such that the other end-vertex $v_{t' - 1}$ with the degree at least $c/2$, a contradiction. If $v_{t' - 1}v_{t' + 1} \in E(G)$, then $P'_1 = v_{t' - 1}P_1v'v_{t' + 1}v_m$ is a $v_m$-longest path with $t(P') \geq t' + 1$, a contradiction.
By the assumption of Theorem 4, we have $|N(v_1) \cap N(v_{t'+1})| \geq 2$. By the definition of $t'$, there is a vertex $v_i \in N(v_1) \cap N(v_{t'+1})$, where $2 \leq i \leq t'-2$.

**CLAIM 3.** \( \{v_1, v_i, v_{i+1}, v_{t'+1}\} \) induces a modified claw.

**PROOF** We have \( v_1v_{i+1} \notin E(G) \), since otherwise \( P'_{t} = v_i \overrightarrow{P_1} v_{i+1} \overrightarrow{P_1} v_m \) is a \( v_m \)-longest path such that \( t(P') \geq t' + 1 \), a contradiction. If \( v_{i+1}v_{t'+1} \notin E(G) \), then \( \{v_1, v_i, v_{i+1}, v_{t'+1}\} \) induces an induced claw. By the condition of Theorem 4 and our hypothesis that \( d(v_1) < c/2 \), we have \( d(v_{i+1}) \geq c/2 \). Let \( P'_1 = v_{i+1} \overrightarrow{P_1} v_i \overrightarrow{P_1} v_{i+1} \overrightarrow{P_1} v_m \). Then \( P'_1 \) is a \( v_m \)-longest path with the other end-vertex of degree at least \( c/2 \), a contradiction. Thus, we have \( v_{i+1}v_{t'+1} \in E(G) \) and the proof of this claim is complete.

Now, we consider the same path \( P'_1 = v_{i+1} \overrightarrow{P_1} v_i \overrightarrow{P_1} v_{i+1} \overrightarrow{P_1} v_m \). Then \( P'_1 \) is a \( v_m \)-longest path such that \( t(P'_1) \geq t' + 1 \), a contradiction. It is thus completed the proof of Lemma 2.

**PROOF OF THEOREM 4.** Suppose that \( G \) contains no Hamilton cycles. By using lemma 2 twice, we obtain a longest path with both end-vertices having the degree at least \( c/2 \). Then by Lemma 2, we can find a cycle of length at least \( c \).

### 3 Concluding remarks

In 1989, Zhu, Li and Deng [8] proposed the definition of implicit degree. Based on this definition, many theorems on long cycles including Theorems 1 and 2, are largely improved. For details, see [4,6,8]. Maybe it is interesting to find a version of Theorem 4 under the condition of implicit degree.

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