Quasi-Classical Dynamics

(Joint work with M. Correggi, M. Olivieri)
Quantum←Classical Physical Systems

We are interested in studying systems, that we call Quantum←Classical, satisfying the following properties:

- The system is composed of two parts, one quantum and the other classical;
- The quantum part feels the action of the classical one, e.g. the latter exerts a force on the former;
- The action of the quantum part on the classical part is negligible.

Possible examples are given by:

- An electron subjected to the electromagnetic field of a nucleus;
- An atom in an optical lattice, or in another trap;
- A quantum dot in an electrically driven device;
- A subatomic particle in a background gravitational field.
Aim

- Derive the Quantum←Classical description as an effective theory, starting from a more fundamental theory.

In order to do that, we use the following scheme:

1. We start considering a fully microscopic theory where the to-be-classical part has quantum effects, and feels the action of the to-remain-quantum part.

2. We introduce a quasi-classical parameter, that measures the “quantumness” of the to-be-classical part only, and we perform the quasi-classical limit in which such quantumness vanishes.

3. Taking this limit is sufficient, as we will see, to both cancel the quantum effects on the now-classical part, and the action of the remaining-quantum part on the now-classical part.
Mathematical Structure of Quantum-Classical Systems

- **Quantum Part:**
  - Observables are bounded operators on a Hilbert space $\mathcal{H}$.
  - States are positive trace (=1) class operators on $\mathcal{H}$.

- **Classical Part:**
  - Phase space $\mathfrak{h}$.
  - Observables are (real- or complex-valued) functions on $\mathfrak{h}$.
  - States are probability measures on $\mathfrak{h}$.
Action of the Classical Part on the Quantum one:

- Quantum Observables would in general depend on the configuration $z \in \mathfrak{h}$ of the Classical Part:
  
  $\mathfrak{h} \ni z \mapsto \mathcal{F}(z) \in \mathcal{B}(\mathcal{H})$.

- Quantum States also would depend on the configuration of the Classical Part, as well as their evolution:
  
  $\mathbb{R} \times \mathfrak{h} \ni (t, z) \mapsto \mathcal{U}_t(z) \gamma(z) \mathcal{U}_t^*(z) \in \mathcal{L}^{1,+}(\mathcal{H})$

Action of the Quantum Part on the Classical one:

- None.
**Mixed Probabilistic Description:**

For the quantum part it is possible to define the noncommutative probability (state), at time $t$, $\gamma\big|_{t,E}$, conditioned to an observed value $\lambda$ of the classical observable $f$:

$$\gamma\big|_{t,f=\lambda} = \int_{\mathfrak{h}} \mathbb{1}_{\{f=\lambda\}}(z) \mathcal{U}_t(z) \gamma(z) \mathcal{U}_t^*(z) d\mu_{t,\lambda}(z),$$

where $\mu_{t,\lambda}$ is the disintegration of $\mu_t$ w.r.t. $f$.

The complete state of the system is described by a state-valued measure:

$$m \in \mathcal{M}(\mathfrak{h}, L^1_+ (\mathcal{H})).$$

In fact, if $\mathfrak{h}$ is separable, it is possible to decompose any state-valued measure $m$ in a scalar (probability) measure $\mu_m$, and a Radon-Nikodým derivative $\gamma_m(z) = \frac{d m(z)}{d \mu_m(z)} \in L^1_+ (\mathcal{H})$, with $\text{tr}_{\mathcal{H}}(\gamma_m(z)) = 1$ for $\mu_m$-a.a. $z \in \mathfrak{h}$. 
The Microscopic Model

- **Coupled Quantum System:**
  - The Hilbert space consists of two parts, $\mathcal{H}$ for the to-remain-quantum part, $\mathcal{K}_\varepsilon$ for the to-be-classical part.
  - On the to-be-classical part there is a semiclassical structure, that depends on the quasi-classical parameter $\varepsilon \to 0$:
    \[
    [a_\varepsilon(k), a_\varepsilon^*(k')] = \varepsilon \delta(k - k'),
    [a_\varepsilon(k), a_\varepsilon(k')] = [a_\varepsilon^*(k), a_\varepsilon^*(k')] = 0.
    \]
  - The Hamiltonian of the system is of the form
    \[
    H_\varepsilon = H_{\text{trq}}(x, -i\nabla) + H_{\text{tbc}}(a_\varepsilon, a_\varepsilon^*) + H_I(x, -i\nabla, a_\varepsilon, a_\varepsilon^*).
    \]
  - A state is a coupled density matrix $\Gamma_\varepsilon \in \mathcal{L}_+^1(\mathcal{H} \otimes \mathcal{K}_\varepsilon)$, with quantum evolution
    \[
    \Gamma_\varepsilon(t) = e^{itH_\varepsilon} \Gamma_\varepsilon e^{-itH_\varepsilon}.
    \]
The Microscopic Model

- **Coupled Quantum System:**
  - The Hilbert space consists of two parts, $\mathcal{H}$ for the to-remain-quantum part, $\mathcal{K}_\varepsilon$ for the to-be-classical part.
  - On the to-be-classical part there is a semiclassical structure, that depends on the quasi-classical parameter $\varepsilon \to 0$:
    \[
    [a_\varepsilon(k), a^*_\varepsilon(k')] = \varepsilon \delta(k - k') , [a_\varepsilon(k), a_\varepsilon(k')] = [a^*_\varepsilon(k), a^*_\varepsilon(k')] = 0 .
    \]
  - The Hamiltonian of the system is of the form
    \[
    H_\varepsilon = H_{\text{trq}}(x, -i\nabla) + ( )H_{\text{tbc}}(a_\varepsilon, a^*_\varepsilon) + H_\text{I}(x, -i\nabla, a_\varepsilon, a^*_\varepsilon) .
    \]
  - A state is a coupled density matrix $\Gamma_\varepsilon \in L^1_+(\mathcal{H} \otimes \mathcal{K}_\varepsilon)$, with quantum evolution
    \[
    \Gamma_\varepsilon(t) = e^{itH_\varepsilon} \Gamma_\varepsilon e^{-itH_\varepsilon} .
    \]
The Microscopic Model

- **Coupled Quantum System:**
  - The Hilbert space consists of two parts, $\mathcal{H}$ for the to-remain-quantum part, $\mathcal{K}_\varepsilon$ for the to-be-classical part.
  - On the to-be-classical part there is a semiclassical structure, that depends on the quasi-classical parameter $\varepsilon \to 0$:
    \[
    [a_\varepsilon(k), a_\varepsilon^*(k')] = \varepsilon \delta(k - k'), [a_\varepsilon(k), a_\varepsilon(k')] = [a_\varepsilon^*(k), a_\varepsilon^*(k')] = 0. 
    \]
  - The Hamiltonian of the system is of the form
    \[
    H_\varepsilon = H_{\text{trq}}(x, -i\nabla) + \left(\frac{1}{\varepsilon}\right) H_{\text{tbc}}(a_\varepsilon, a_\varepsilon^*) + H_{\text{I}}(x, -i\nabla, a_\varepsilon, a_\varepsilon^*). 
    \]
  - A state is a coupled density matrix $\Gamma_\varepsilon \in \mathcal{L}^1_+(\mathcal{H} \otimes \mathcal{K}_\varepsilon)$, with quantum evolution
    \[
    \Gamma_\varepsilon(t) = e^{itH_\varepsilon} \Gamma_\varepsilon e^{-itH_\varepsilon}. 
    \]
Quasi-Classical Limit of States

Theorem ([Fa 2018, CoFaOl 2019] – Extending [AmNi 2008–])

\[ \Gamma_\varepsilon \xrightarrow[\varepsilon \to 0]{} m \]
Quasi- Classical Limit of States

Theorem ([Fa 2018, CoFaOl 2019] – Extending [AmNi 2008–])

\[ \exists \delta > 0 \exists C > 0 \quad \text{Tr}(\Gamma_{\epsilon}(\int a_{\epsilon}^*(k)a_{\epsilon}(k)dk)^{\delta}) \leq C \implies \exists \epsilon_n \to 0 \exists m \in \mathcal{M}(\mathfrak{h}, L^1_+(\mathcal{H})) : \]

\[ \Gamma_{\epsilon_n} \to m \]
Quasi-Classical Limit of States

Theorem ([Fa 2018, CoFaOl 2019] – Extending [AmNi 2008–])

∃δ > 0 ∃C > 0 \( \text{Tr}(\Gamma_{ε}(\int a_{ε}^{*}(k)a_{ε}(k)dk)^{δ}) \leq C \) \( ⇒ \) \( \exists ε_n \to 0 \exists m \in M(\hbar, L_{1}^{+}(\mathcal{H})) : \Gamma_{ε_n} \xrightarrow{n→∞} m \)

- The above convergence holds as a weak-* convergence on \( L_{1}^{+}(\mathcal{H}) \) of Fourier transforms: \( ∀ξ \in \hbar, \lim_{n→∞} \hat{Γ}_{ε_n}(ξ) = \lim_{n→∞} \text{tr}_{\mathcal{H}_{ε_n}} (Γ_{ε}e^{i(a_{ε}^{*}(ξ)+a_{ε}(ξ)))} = \int_{\hbar} e^{2i\Re(⟨ξ,z⟩)\hbar} γ_{m}(z) dμ_{m}(z) = \hat{m}(ξ) . \)

- **Warning:**

\[ 0 ≤ μ_{m}(\hbar) ≤ 1 ! \]
Quasi-Classical Dynamics: Egorov-type Theorem

Theorem ([CoFaOl 2019])

\[ \Gamma_{\varepsilon_n} \xrightarrow{n \to \infty} m \iff \forall t \in \mathbb{R}, \Gamma_{\varepsilon_n}(t) \xrightarrow{n \to \infty} m_t. \]
The Quasi-Classical Limit

Quasi-Classical Dynamics: Egorov-type Theorem

Theorem ([CoFaOl 2019])

\[ \Gamma_{\epsilon_n} \xrightarrow{n \to \infty} m \iff \forall t \in \mathbb{R}, \Gamma_{\epsilon_n}(t) \xrightarrow{n \to \infty} m_t. \]

- \( m_t \) is given by the Radon-Nikodým decomposition

\[ d m_t = \gamma_m(z) d \mu_{m_t}, \]

where

\[ \gamma_{m_t} = \mathcal{U}_{t,0}(z) \gamma_m(z) \mathcal{U}_{t,0}^*(z), \]

\[ \mu_{m_t} = \Phi_t \ast \mu_m, \]

with \( \mathcal{U}_{t,0}(z) \) the unitary evolution on \( \mathcal{H} \) generated by

\[ \mathcal{H} = H_{trq}(x, -i\nabla) + H_I(x, -i\nabla, \Phi_t z, \overline{\Phi_t z}), \]
and $\Phi_t$ is

$$
\Phi_t = \begin{cases} 
\text{Ham. flow gen. by } H_{tbc}(z, \bar{z}) & \text{if } \frac{1}{\epsilon} H_{tbc}(a_\epsilon, a_\epsilon^*) \\
\text{id} & \text{if } H_{tbc}(a_\epsilon, a_\epsilon^*) 
\end{cases}.
$$

**Remarks**

- The loss of mass can be only lost at the initial time: $\forall t \in \mathbb{R}$,

  $$
  \mu_{m_t}(\mathfrak{h}) = \mu_m(\mathfrak{h}).
  $$

- This description fits very nicely with the requirements asked for a Quantum$\leftrightarrow$Classical theory.
Outline of the Proof

1. Given $\Gamma_{\varepsilon_n} \to m$, extract a subsequence $\Gamma_{\varepsilon_{n_k}}$ such that $\forall t \in \mathbb{R}$, $\Gamma_{\varepsilon_{n_k}}(t) \to m_t$.

2. By newly developed techniques of semiclassical analysis, study the limit $k \to \infty$ of the equation:

$$\Gamma_{\varepsilon_{n_k}}(t) = \Gamma_{\varepsilon_{n_k}} - i \int_0^t [H_{\varepsilon_{n_k}}, \Gamma_{\varepsilon_{n_k}}(\tau)] \, d\tau.$$ 

3. Study the properties of the transport equation for the quasi-classical measure $m_t$:

$$d m_t(z) = d m(z) - i \int_0^t [\mathcal{H}(x, -i \nabla, \Phi_t z, \Phi_t \bar{z}), \gamma_{m_t}(z)] \, d\mu_{m_t} \, d\tau,$$

in particular the uniqueness of solutions that satisfy suitable regularity properties (that we know a priori to be satisfied by limit measures).
Thank you for the attention.