ON A CHEEGER TYPE INEQUALITY IN CAYLEY GRAPHS OF
FINITE GROUPS

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ABSTRACT. Let $G$ be a finite group. It was remarked in [BGGT15] that if the cayley graph $C(G, S)$ is an expander graph and is non-bipartite then the spectrum of the adjacency operator $T$ is bounded away from $-1$. In this article we are interested in explicit bounds for the spectrum of these graphs. Specifically we show that the non-trivial spectrum of the adjacency operator lies in the interval $[-1 + \frac{h(G)^4}{\gamma}, 1 - \frac{h(G)^2}{2d^2}]$ where $h(G)$ denotes the (vertex) cheeger constant of the $d$ regular graph $C(G, S)$ with respect to a symmetric set $S$ of generators and $\gamma = 2^d d^d (d + 3)^2$.

1. Introduction

Throughout this article we will consider a finite group $G$ with $|G| = n$. We will denote by $C(G, S)$ for a symmetric subset $S \subset G$ of size $|S| = d$, to be the Cayley graph of $G$ with respect to $S$. Then $C(G, S)$ is $d$ regular. Given a finite $d$ regular Cayley graph $C(G, S)$ we have the normalised adjacency matrix $T$ of size $n \times n$ whose eigenvalues lie in the interval $[-1, 1]$. The normalised Laplacian matrix of $C(G, S)$ denoted by $L$ is defined as

$$L := I_n - T$$

where $I_n$ denotes the identity matrix. The eigenvalues of $L$ lie in the interval $[0, 2]$. It is easy to see that $1$ is always an eigenvalue of $T$ and $0$ that of $L$. We denote the eigenvalues of $T$ as $-1 \leq t_n \leq ... \leq t_2 \leq t_1 = 1$ and that of $L$ as $\lambda_i = 1 - t_i, i = 1, 2, ..., n$. The graph $C(G, S)$ is connected if and only if $\lambda_2 > 0$ (equivalently $t_2 < 1$). The graph is bipartite if and only if $\lambda_n = 2$.

We recall the notion of Cheeger constant.

Definition 1.1 (Vertex boundary of a set). Let $\mathbb{G} = (V, E)$ be a graph with vertex set $V$ and edge set $E$. For a subset $V_1 \subset V$ let $N(V_1)$ denoting the neighbourhood of $V_1$ be

$$N(V_1) := \{v \in V : vv_1 \in E \text{ for some } v_1 \in V_1\}$$

Then the boundary of $V_1$ is defined as $\delta(V_1) := N(V_1) \setminus V_1$

Definition 1.2 (Cheeger constant). The Cheeger constant of the graph $\mathbb{G} = (V, E)$ denoted by $h(\mathbb{G})$ is defined as

$$h(\mathbb{G}) := \inf\{\frac{|\delta(V_1)|}{|V_1|} : V_1 \subset V, |V_1| \leq \frac{|V|}{2}\}$$

This is also called the vertex cheeger constant of a graph.

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Definition 1.3 \((n,d,\epsilon)\) expander. Let \(\epsilon > 0\). An \((n,d,\epsilon)\) expander is a graph \((V,E)\) on \(|V| = n\) vertices having maximal degree \(d\) such that for every set \(X \subseteq V\) satisfying \(|V_1| \leq \frac{n}{2}\), \(|\delta(V_1)| \geq \epsilon|V_1|\) holds, equivalently \(h(G) \geq \epsilon\).

In this article we are interested in the spectrum of the Laplace operator \(L\) for the Cayley graph \(C(G,S)\). The Cayley graph is bipartite if and only if there exists an index two subgroup \(H\) of \(G\) which is disjoint from \(X\). See Prop 2.6. It was observed in [BGGT15] (Appendix E) that if \(C(G,S)\) is an expander graph and is non-bipartite then the spectrum of \(T\) is not only bounded away from 1 but also from \(-1\). Here we show that

Theorem 1.4. Let the Cayley graph \(C(G,S)\) be an expander with \(|S| = d\) and \(h(G)\) denote its Cheeger constant. Then if \(C(G,S)\) is non-bipartite we have

\[ \lambda_n \leq 2 - \frac{h(G)^4}{\alpha d^6(d+3)^2} \]

where \(\lambda_n\) the largest eigenvalue of the normalised laplacian matrix and \(\alpha\) is an absolute constant (we can take \(\alpha = 2^8\))

The strategy of the proof closely follows the combinatorial arguments of Breuillard–Green–Guralnick–Tao in [BGGT15].

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2. Proofs

There are two notions of expansions in graphs - the vertex expansion as in Definition 1.3 and the edge expansion.

Definition 2.1 (Edge expansion). Let \(G = (V,E)\) be a \(d\)-regular graph with vertex set \(V\) and edge set \(E\). For a subset \(V_1 \subset V\) let \(E(V_1,V\setminus V_1)\) be the edge boundary of \(V_1\) defined as

\[ E(V_1,V\setminus V_1) := \{(v_1,s) \in E : v_1 \in V, v_1 s \in V\setminus V_1\} \]

Then the edge expansion ratio \(\phi(V_1)\) is defined as

\[ \phi(V_1) := \frac{|E(V_1,V\setminus V_1)|}{d|V_1|} \]

Definition 2.2 (Edge-cheeger constant). The edge-cheeger denoted by \(h(G)\) is

\[ h(G) := \inf_{V_1 \subset V, |V_1| \leq |V|/2} \phi(V_1) \]

In a \(d\) regular graph the two cheeger constants are related by the following

Lemma 2.3. Let \(G = (V,E)\) be a \(d\)-regular graph

\[ \frac{h(G)}{d} \leq h(G) \leq h(G) \]
Proof. Let \( V_1 \subset V \) and we consider the map
\[
\psi : E(V_1, V \setminus V_1) \to \delta(V_1) \text{ given by } (v_1, s) \to v_1s.
\]
The map is surjective hence we have the LHS and at the worst case \( d \) to \( 1 \) wherein we get the RHS. \( \square \)

We have the following inequalities, called the discrete Cheeger-Buser inequality. It is the version for graphs of the corresponding inequalities for the Laplace-Beltrami operator on closed Riemannian manifolds first proven by Cheeger \[Che69\] (lower bound) and by Buser \[Bus82\] (upper bound). The discrete version was shown by Alon and Milman \[AM85\] (Proposition 2.4).

**Proposition 2.4** (discrete Cheeger-Buser inequality). Let \( G = (V, E) \) be a finite \( d \)-regular graph. Let \( \lambda_2 \) denote the second smallest eigenvalue of its normalised laplacian matrix and \( h(G) \) be the (edge) Cheeger constant. Then
\[
\frac{h(G)^2}{2} \leq \lambda_2 \leq 2h(G).
\]

**Proof.** See \[Lub94\] prop. 4.2.4 and prop. 4.2.5 or \[Chu96\] sec. 3. \( \square \)

Before proceeding further let us precise the notion of Cayley graph.

**Definition 2.5.** (Cayley graph) Let \( G \) be a finite group and \( S \) be a symmetric generating set of \( G \). Then the Cayley graph \( C(G, S) \) is the graph having the elements of \( G \) as vertices and \( \forall x, y \in G \) there is an edge between \( x \) and \( y \) if and only if \( \exists s \in S \) such that \( sx = y \). If \( 1 \in S \) then the graph has a loop (which we treat as an edge) going from \( x \) to itself \( \forall x \in G \).

By the regularity of the graph we mean the number of half edges connected to each vertex. If \( |S| = d \) it is clear that \( C(G, S) \) will be \( d \)-regular (where \( |S| \) denotes the cardinality of the set \( S \)).

We also note the adjacency matrix associated to any finite undirected graph. For any finite undirected graph \( G \) having vertex set \( V = \{v_1, ..., v|G|\} \) and edge set \( E \), the adjacency matrix \( T \) is the \( |V| \times |V| \) matrix having \( T_{ij} = \) number of edges connecting \( v_i \) with \( v_j \). The discrete Cheeger inequality applies to all finite regular graphs (the inequality also holds for finite non-regular graphs where we need to consider the maximum of the degrees of the all the vertices - see \[Lub94\] prop. 4.2.4 but for our purposes we shall restrict to regular graphs).

We show the following proposition

**Proposition 2.6** (Criteria for non-bipartite property). A finite Cayley graph \( C(G, S) \) is non-bipartite if and only if there does not exist an index two subgroup \( H \) of \( G \) which is disjoint from \( S \).

**Proof.** Let \( C(G, S) \) be bipartite. Then we can partition the vertex set \( G \) into two disjoint sets \( A \) and \( B \) such that \( G = A \sqcup B \). Let \( 1 \in B \). Let \( s \in S \cap B \). Then \( ss^{-1} \in S \) and so \( 1 = ss^{-1} \in A \). This is a contradiction. So \( S \cap B = \emptyset \).

Now suppose \( x, y \in B \) but \( xy \notin B \). So \( xy \in A \). Thus there exists \( s_1, s_2, ..., s_{2r+1} \in S, r \in \mathbb{N} \) such that \( s_1s_2...s_{2r+1}(xy) = y \). This implies that \( s_1s_2...s_{2r+1}x = 1 \in B \). But this is impossible because \( x \in B \) so \( s_1s_2...s_{2r+1}x \in A \). Thus we have a contradiction and \( xy \in B \). So \( B \) is an index \( 2 \) subgroup disjoint from \( S \).

The other direction is clear. \( \square \)
Lemma 2.7. Let $G$ be a finite group and $C(G, S)$ denote its Cayley graph with respect to a symmetric set $S$ of size $d$. Let $S$ be such that 

$$|SA \setminus A| \geq \epsilon' |A|$$

for every set $A \subseteq G$ with $|A| \leq \frac{|G|}{2}$ and some $\epsilon' > 0$. Then we have the estimate 

$$|SA \setminus A| \geq \frac{\epsilon'}{d} |G \setminus A|$$

for all sets $A \subseteq G$ with $|A| \geq \frac{|G|}{2}$.

Proof. Let $A^c = G \setminus A$. The proof is based on the fact that $|SA \setminus A| \geq \frac{1}{d} |SA^c \setminus A^c|$ for all subsets $A \subseteq G$ and $S = S^{-1} \subseteq G$.

Let $s \in S$,

$$|sA^c \cap A| = |s^{-1}(sA^c \cap A)| = |A^c \cap s^{-1}A| \leq |A^c \cap SA|$$

$$\Rightarrow |SA \setminus A^c| = |SA^c \cap A| = \left| \cup_{s \in S} sA^c \cap A \right| \leq \Sigma_{s \in S} |SA \cap A^c| = d |SA \setminus A|$$

Hence we have

$$|SA \setminus A| \geq \frac{1}{d} |SA^c \setminus A^c| \geq \frac{\epsilon'}{d} |A^c| = \frac{\epsilon'}{d} |G \setminus A|$$

(using the property of combinatorial expansion of $S$ and noting $|A| \geq \frac{|G|}{2}$).

To prove Theorem 1.4 we have to show that under the given assumptions we have $t_2 \leq 1 - \frac{h(G)^2}{2d}$ and $t_1 \geq -1 + \frac{h(G)\alpha}{ad^2(4\alpha + 3)^2}$ for some absolute constant $\alpha$.

We divide it into two parts. The first one (i.e., the upper bound for $t_2$) is a well-known result (independent of the assumption on $S$) and is actually the discrete Cheeger inequality (2.4).

We now address the second portion of Theorem 1.4. First we show the following lemma.

Lemma 2.8. Let $G$ be a finite group, $k \geq 1$ and $S = S^{-1} = \{s_1, ..., s_d\}$ be a symmetric generating set of $G$. Let $s_1, s_2, ..., s_d$ be $\epsilon$-expanding in the sense that 

$$|SX \setminus X| \geq \epsilon |X|$$

for every set $X \subseteq G$ with $|X| \leq \frac{|G|}{2}$ and some $\epsilon > 0$. Suppose there exists $\zeta > 0$ such that adjacency matrix $T$ of $(G, S)$ has an eigenvalue in $[-1, -1 + \zeta]$. Fix $\beta = d^2 \sqrt{2\zeta(2 - \zeta)}$.

Then there exists a set $A$ with the following properties

$(1)$ $(\frac{1}{2 + \beta^2}) |G| \leq |A| \leq \frac{1}{2} |G|

(2)$ $|SA \cap A| \leq \frac{1}{\epsilon} \beta |A|

(3)$ $\forall s \in S, g \in G, |sAg\Delta(Ag)| \leq \beta \left( 1 + \frac{d}{\epsilon} + \frac{2}{\epsilon} \right) |A|.$

Proof. We have

(2.1) $\epsilon |X| \leq |SX \setminus X|$ whenever $X \subseteq G$ with $|X| \leq \frac{|G|}{2}$, and using Lemma 2.7 with $|S| = d$

(2.2) $\frac{\epsilon}{d} |G \setminus X| \leq |SX \setminus X|$
whenever $|X| \geq \frac{|G|}{2}$.

Since $T$ has an eigen-value in $[-1, -1 + \zeta]$, $T^2$ has a non-trivial eigenvalue say $t'$ in $((1 - \zeta)^2, 1]$.

Now consider the set $S^2$ (obtained by identifying all equal elements in the multi-set $S.S$) and the multi-set $S' = S.S$ (without identification). $T^2$ is the adjacency matrix associated with $S'$ and $|S^2| \leq |S'| = d^2$. Let $h(G, S')$ denote the Cheeger constant for $G$ with respect to the multi-set $S'$ and let $h(G, S^2)$ denote the Cheeger constant for the weighted Cayley graph of $G$ with respect to $S^2$. It is clear that $h(G, S^2) \leq h(G, S')$.

We have $t' > (1 - \zeta)^2$. Let $L$ denote the Laplacian matrix of the graph of $G$ with respect to $S'$ with the adjacency operator $T^2$ and let its eigenvalues be denoted by $0 = L_1 \leq L_2 \leq ... \leq L_n \leq 2$. We know that $L_2 = 1 - t' < 1 - (1 - \zeta)^2 = \zeta(2 - \zeta)$.

By the discrete Cheeger-Buser inequality (Prop 2.4) for the graph of $G$ with respect to $S'$ we have

$$h^2(G, S') \leq L_2 < \zeta(2 - \zeta) \Rightarrow h(G, S^2) \leq h(G, S') < \sqrt{2\zeta(2 - \zeta)}.$$ 

This implies that $\exists A \subset G$ with $|A| \leq \frac{|G|}{2}$ such that

$$(2.3) \quad \frac{|S^2A \setminus A|}{|A|} < \beta$$

It is clear that $|A \cup SA| \geq \frac{|G|}{2}$ for $\zeta \leq \frac{2}{\beta}$. $A \cup SA$ satisfies the hypothesis of Lemma 2.7 and we can use equation 2.2 and equation 2.3 to get

$$\frac{\epsilon}{d} |G \setminus (A \cup SA)| \leq |S(A \cup SA) \setminus (A \cup SA)| \leq |S^2A \setminus A| \leq \beta |A|.$$ 

Noting the fact that $|G \setminus (A \cup SA)| = |G| - |A \cup SA|$, we have

$$|G| - \frac{\beta}{\epsilon} |A| \leq |A \cup SA|$$

We use the fact that $|SA| \leq |S^2A| \leq |A| + \beta |A|$ to conclude that

$$(2.4) \quad (\frac{1}{2} + \frac{\beta}{\epsilon} + \frac{d\epsilon}{\beta}) |G| \leq |A|.$$ 

For arbitrary sets $X, Y, Z \subset G$ we have $X(Y \cap Z) \subset XY \cap XZ$.

Hence

$$|S(A \cap SA) \setminus (A \cap SA)| \leq |S^2A \setminus A| \leq \beta |A|.$$ 

As $|A| \leq \frac{|G|}{2}$ clearly $|A \cap SA| \leq \frac{|G|}{2}$. Hence the hypothesis of $\epsilon$ expansion applies to $A \cap SA$ and we have

$$(2.5) \quad |A \cap SA| \leq \frac{1}{\epsilon} |A|.$$
Also $sA$ for any $s \in S$ is almost $A^c$. This is because
\[ |sA \Delta A^c| = |G| - |sA\Delta A| \leq |G| - (2 - \frac{2}{\epsilon} \beta)|A| \]
\[ \leq \left( \beta + \frac{d\beta}{\epsilon} + \frac{2\beta}{\epsilon} \right)|A| \]
Thus we have
\[ (2.6) \quad |sA \Delta A^c| \leq \left( \beta + \frac{d\beta}{\epsilon} + \frac{2\beta}{\epsilon} \right)|A| \]

Now let $g \in G$ be arbitrary, then we have
\[ |sAg \Delta Ag| \geq (2 - \frac{2}{\epsilon} \beta)|A| \]
and
\[ (2.7) \quad |sAg \Delta (Ag)^c| \leq \beta \left( 1 + \frac{d}{\epsilon} + \frac{2}{\epsilon} \right)|A| \]

**Theorem 2.9.** Let $G$ be a finite group, $k \geq 1$ and $S = S^{-1} = \{s_1, ..., s_d\}$ be a symmetric generating set of $G$. Suppose that $G$ does not have an index two subgroup $H$ disjoint from $S$. Let $s_1, s_2, ..., s_d$ be $\epsilon-$expanding i.e,
\[ |SA \setminus A| \geq \epsilon|A| \]
for every set $A \subseteq G$ with $|A| \leq \frac{|G|}{2}$ and some $\epsilon > 0$. Then all the eigenvalues of the operator $T$ are $\geq -1 + \frac{\epsilon^d}{\alpha d(d+3)^2}$ where $\alpha$ is an absolute constant (we can take $\alpha = 2^5$).

**Proof.** First we use lemma 2.8 to conclude that there exists $A \subset G$ with $|A| \geq \left( \frac{1}{2 + \beta + \frac{4\beta}{\epsilon}} \right)|G|$ with $|SA \cap A| \leq \frac{\epsilon}{3}|A|$ and $\forall s \in S, g \in G, |sAg \Delta (Ag)^c| \leq \beta \left( 1 + \frac{4}{\epsilon} + \frac{2}{\epsilon} \right)|A|$.

Taking $A_g := A \cap Ag$ and $A_g' := (A \cup Ag)^c = A^c \cap (Ag)^c$, and $B = A_g \cup A_g'$ we can show that either

1. $|B| \leq \frac{|G|}{2}$ in which case
\[ (2.8) \quad |B| \leq \frac{4d\beta}{\epsilon^2} \left( \epsilon + d + 2 \right)|A| \]

or

2. $|B| > \frac{|G|}{2}$ in which case $|B'| \leq \frac{|G|}{2}$ and then
\[ (2.9) \quad |G \setminus B| \leq \frac{4d\beta}{\epsilon^2} \left( \epsilon + d + 2 \right)|A| \]

This implies that
\[ (2.10) \quad \left( 1 - \frac{2d\beta}{\epsilon^2} \left( \epsilon + d + 2 \right) \right)|A| \leq |A \cap Ag| \]
Thus for any $g \in G$ we have either

(i) $|A \cap Ag| \geq \left(1 - \frac{2d\beta}{\epsilon^2}(\epsilon + d + 2)\right)|A|$ or

(ii) $|A \cap Ag| \leq \frac{|B|}{2} \leq \frac{2d\beta}{\epsilon^2}(\epsilon + d + 2)|A|$ (this follows from $B = A_g \cup A'_g$ and $|A'_g| \geq |A_g|$ when $|A| \leq \frac{|G|}{2}$).

The trick now is to use the method of Freiman in [Fm73] to find a subgroup $H$ of $G$.

**Claim 2.10.** If $H := \{g \in G: |A \cap Ag| \geq r|A|\}$ where $r = 1 - \frac{2d\beta}{\epsilon^2}(\epsilon + d + 2)$. Fix $\beta = \frac{\epsilon^2}{2d(d+3)}$. Then $H$ is a subgroup of $G$ of index 2.

**Proof of claim.** We have $H = H^{-1}$, $1 \in H$ and $r > \frac{1}{2} + \frac{2d\beta}{\epsilon^2}(\epsilon + d + 2)$. Also for $g, h \in H$ we have by the triangle inequality

$$|A \setminus Ag| \leq |A \setminus Ah| + |Ah \setminus Ag| \leq 2(1 - r)|A|$$

$$\Rightarrow |A \cap Ag| \geq (2r - 1)|A|$$

This means that $gh$ cannot belong to (ii) for $\Rightarrow gh$ belongs to (i) $\Rightarrow gh \in H$. So $H$ is a subgroup of $G$.

Now from the estimate

$$|A|^2 = \sum_{g \in G} |A \cap Ag| \leq |H||A| + \frac{2d\beta}{\epsilon^2}(\epsilon + d + 2)|A||G \setminus H|$$

Let $z = \frac{2d\beta}{\epsilon^2}(\epsilon + d + 2)$ we have

$$(1 - z)|H| \geq \left(1 - \frac{1}{2 + \beta + \frac{d\beta}{\epsilon}}\right)|G| - z|G|$$

using the fact that $|A| \geq \left(\frac{1}{2 + \beta + \frac{d\beta}{\epsilon}}\right)|G|$.

The index of $H$ in $G$ is 2 iff $|H| > \frac{|G|}{3}$ and this is equivalent to

$$\left(1 - \frac{1}{2 + \beta + \frac{d\beta}{\epsilon}} - z\right) > \frac{1 - z}{3}$$

which on simplification gives us $\beta \leq \frac{\epsilon^2}{2d(d+3)}$ where we can take $\alpha = 2^3$.

Hence the index of $H$ in $G$ is 2 when $\sqrt{2\zeta(2 - \zeta)} = \beta \leq \frac{\epsilon^2}{2d(d+3)}$. This gives us $\zeta = \frac{\epsilon^4}{2^6d^6(d+3)^2}$. We conclude that the index of $H$ in $G$ is 2 for small $\zeta$ i.e., the order of $\zeta$ should be $\frac{\epsilon^4}{d^6(d+3)^2}$.

This completes the proof.

□

We have

$$\sum_{g \in H} |A \cap Ag| = |A \cap H|^2 + |A \setminus (A \cap H)|^2$$

so using (i), $|A \cap H|^2 + |A \setminus (A \cap H)|^2$ is of the order of

$$\left(1 - \frac{2d\beta}{\epsilon^2}(\epsilon + d + 2)\right)|A||H|$$
\[ (1 - \frac{2d\beta}{e^2} (\epsilon + d + 2))|A|^2 \]

We can write \( A \) as \( (A \cap H) \cup (A \setminus A \cap H) \) and so

\[
|A \cap H|^2 + |A \setminus (A \cap H)|^2 \geq |(A \cap H) \cup (A \setminus A \cap H)|^2 - \frac{2d\beta}{e^2} (\epsilon + d + 2)|A|^2
\]

Noting the fact that \( A \cap H \) and \( A \setminus A \cap H \) are disjoint we get that

\[
|A \cap H||A \setminus A \cap H| \leq \frac{2d\beta}{e^2} (\epsilon + d + 2)|A|^2
\]

which translates to either \( A \cap H \) is small or \( A \setminus A \cap H \) is small. Let \( g \in G \setminus H \). So either

\[
|A \Delta H g| \leq \frac{2d\beta}{e^2} (\epsilon + d + 2)|A|
\]

But we have seen that \( \forall s \in S \),

\[
|sA \Delta A^c| \leq \beta \left( 1 + \frac{d}{\epsilon} + \frac{2}{\epsilon} \right)|A|
\]

Thus \( sA \) is nearly the complement of \( A \) while \( Hg \) is nearly \( A \), so \( H \) and \( sA \) have a large intersection for all \( s \in S \). This means that \( s^{-1}H \) and \( A \) have a large intersection. It is clear to see in fact

\[
|sH \cap H| \leq \frac{4d\beta}{e^2} (\epsilon + d + 2)|A|
\]

for all \( s \in S \).

This implies that \( S \subset G \setminus H \) (as \( H \) is a subgroup of index 2) contradicting the hypothesis (as long as \( |sH \cap H| \) is small or \( \beta \leq \frac{e^2}{2 \cdot 2^{d(d+3)}} \Leftrightarrow \zeta \leq \frac{e^4}{2^{d(d+3)}} \)) and proving the theorem.

\[ \Box \]

3. Concluding Remarks

The above bound is dependent on the Cayley graph structure and does not hold for general non-bipartite finite, regular graphs. In the setting of arbitrary regular graphs some recent works are worth mentioning. Bauer and Jost in [BJ13] introduced a dual cheeger constant \( \bar{h} \) which encodes the bipartiteness property of finite regular graphs. The dual cheeger constant \( \bar{h} \) of a \( d \) regular graph is defined as

\[
\bar{h} := \max_{V_1, V_2, V_1 \cup V_2 \neq \emptyset} \frac{2|E(V_1, V_2)|}{\text{vol}(V_1) + \text{vol}(V_2)}
\]

for a partition \( V_1, V_2, V_3 \) of the vertex set \( V \), \( \text{vol}(V_k) = d|V_k| \) and \( |E(V_1, V_2)| \) denotes the number of edges going from \( V_1 \) into \( V_2 \). For a general regular graph it was shown by Bauer-Jost (and independently by Trevisan [Tre09]) that

**Theorem 3.1** (Bauer-Jost [BJ13]). Let \( \lambda_n \) be the largest eigen-value of the graph laplace operator. Then \( \lambda_n \) satisfies

\[
\frac{(1 - \bar{h})^2}{2} \leq 2 - \lambda_n \leq 2(1 - \bar{h})
\]

and the graph is bipartite if and only if \( \bar{h} = 1 \).
There is also the concept of higher order Cheeger constants introduced by Miclo in \cite{Mic08}.

Some recent works treating higher order Cheeger inequalities for general finite graphs are those by Lee–Gharan–Trevisan in \cite{LOGT12} and Liu \cite{Liu15} (for the dual case) etc.

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