Calculating the Isgur-Wise Function on the Lattice
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We calculate the Isgur-Wise function by measuring the heavy-heavy meson transition matrix element on the lattice. The standard Wilson action is used for both the heavy and light quarks. Our first numerical results are presented.

1. Introduction

Because of the new flavor and spin symmetries in the heavy quark effective field theory (HQEFT), the heavy-heavy meson decay matrix element can be simplified and different decay processes can be related to each other. For example, in the case of $B \to D$ decay the number of unknown form factors can be reduced from two to one, and we have

$$< D_{v'}|\bar{c}\gamma_\mu b|B_v> = \sqrt{m_B m_D} C_{cb} \xi(v' \cdot v)(v + v')_\nu, \quad (1)$$

where $v$, $v'$ are the four-velocity and $m_B, m_D$ are the B and D meson mass, respectively. The constant $C_{cb}$ comes from integrating the full QCD contribution from the heavy quark mass scale down to a renormalization scale $\mu \ll m_D$

$$C_{cb} = \left[ \frac{\alpha_s(m_D)}{\alpha_s(m_B)} \right]^{6/(33-2N)} \left[ \frac{\alpha_s(m_B)}{\alpha_s(\mu)} \right] a(v' \cdot v'), \quad (2)$$

where $a(v' \cdot v')$ is a slowly varying function of $v' \cdot v'$ and vanishes at $v = v'$.

The Isgur-Wise function $\xi(v' \cdot v)$ represents the interactions of the light degrees of freedom in the heavy meson system and can thus be calculated only by nonperturbative methods.

On the lattice the heavy meson system can be studied in two different approaches. One is to keep the heavy quark dynamical by using the standard Wilson action. This may require extrapolation to the physical heavy meson mass of interest. The alternative is to integrate out the heavy quark first and derive an effective action including only the light degrees of freedom and then perform numerical simulation using this effective action. Here we stay with the first approach.

Using the flavor symmetry of HQEFT the Isgur-Wise function relevant to the $B \to D$ decay of Eq. (1) can be obtained also from the $D \to D$ elastic scattering matrix element

$$< D_{v'}|\bar{c}\gamma_\mu c|D_v> = m_D C_{cc}(\mu) \xi(v' \cdot v)(v + v')_\nu, \quad (3)$$

where

$$C_{cc}(\mu) = \left[ \frac{\alpha_s(m_c)}{\alpha_s(\mu)} \right] a(v' \cdot v). \quad (4)$$

This of course requires that the D meson be sufficiently heavy for the onset of the heavy quark limit (HQL). Conventionally a $B \to D$ (here we use B and D as generic names for heavy pseudoscalar mesons, they do not necessarily represent the physical B and D mesons) transition matrix can be parametrized as

$$< D_{p'}|V_\nu|B_p> = f_+(q^2)(p' + p)_\nu + f_-(q^2)(p - p')_\nu, \quad (5)$$

where $q^2 = (p' - p)^2$ is the momentum transfer between the initial and final states and $V_\nu$ is a vector current. It is easy to show that in the elastic scattering case one has $f_-(q^2) = 0$. Using the relation $v_\nu = p_\nu/m$, Eq. (5) becomes

$$< D_{v'}|V_\nu|D_v> = m_D f_+(q^2)(v' + v)_\nu. \quad (6)$$
Comparing this with Eq. (3) one finds the simple relation between \( f_+ \) and \( \xi \)
\[
f_+ = C_{cc} \xi .
\]
The lattice calculation method for \( f_+ \) has been well established [3,4], and thus the result can be easily used to obtain \( \xi \).

2. Considerations in the Lattice Calculation

At \( \beta = 6.0 \) the inverse lattice spacing \( a^{-1} \approx 2.0\,\text{GeV} \). The HQL becomes valid when the heavy quark \( Q \) in a heavy meson has a mass \( m_Q \gg \Lambda_{QCD} \approx 0.2\,\text{GeV} \). The heaviest mass we can take is order of one or less in lattice units, beyond which one would expect large lattice-spacing artifacts. At \( \beta = 6.0 \) this corresponds to a physical mass in the range of D meson. This can be obtained by setting the hopping parameter for the heavy quark \( Q \) to \( \kappa_Q = 0.118 \). For the light quark we take the hopping parameter \( \kappa_q = 0.152 - 0.155 \) and extrapolate to \( \kappa_{q,cr} = 0.157 \) [3].

How far can \( v \cdot v' \) change on the lattice? In our calculations we always have either the initial or the final particle at rest. Thus
\[
v \cdot v' = \begin{cases} E_D/m_D & \text{if } v' = (0, 0, 0, 1) \\ E_D'/m_D & \text{if } v = (0, 0, 0, 1) \end{cases} ,
\]
where \( E_D = \sqrt{m_D^2 + \vec{p}^2} \).

For a spatial lattice size \( L = 24 \), we have injected momenta
\[
\vec{p} = \frac{2\pi}{L} (1, 0, 0), \frac{2\pi}{L} (1, 1, 0) .
\]
Since \( m_D \approx 1.0 \) in lattice unit, one gets \( v \cdot v' \approx 1.034 \) and 1.066 respectively. To get larger values for \( v \cdot v' \) one needs to inject larger lattice momenta which would in turn introduce large statistical noise in the matrix element calculations. Thus in practice \( v \cdot v' \) can not be much more than \( \sim 1.1 \).

However, this apparent restriction in the range of \( v \cdot v' \) in the lattice calculations has little physical consequence as the validity of HQL requires that the momentum transfer between the initial and final light degrees of freedom, \( \sim \Lambda_{QCD}^2 (v \cdot v' - 1) \), be \( \ll \Lambda_{QCD}^2 \). This in turn means that \( v \cdot v' \) should be close to one.

Removing the lattice artifacts. One of the major concerns in this calculation is the size of the lattice artifacts. Since the heavy meson mass is near one in lattice units, the lattice artifacts could be significant. Ultimately the lattice artifacts can be brought under control either by comparing data at different \( \beta \) values or using the lattice improved actions. However, for simulations at a given \( \beta \) value there are several ways to check the size of the lattice artifacts.

Eq. (3) and consequently Eq. (4) require Lorentz invariance to hold. In Euclidean space the Lorentz transformation becomes a four dimensional Euclidean rotation. The lattice theory, however, does not have exact Euclidean rotational invariance for finite lattice spacing \( a \). Thus \( f_+ \) is not exactly zero. The amplitude of \( f_+ \) (or \( f_-/f_+ \)) gives a measure for the violation of the Euclidean invariance on lattice.

We can also estimate the size of the lattice artifacts by checking the simulation results against known continuum matrix element values at some special points. For example, when both the initial and final D mesons are at rest, \( v = v' = (0, 0, 0, 1) \), the continuum matrix element of \( \bar{c}_c c_T \) is known because of the quark flavor current conservation [6]
\[
< D | \bar{c}_c c_T | D > = 2m_D .
\]
At this so-called “recoil-point” we have \( \xi (1) = 1 \). Both Eqs. (11) and (12) will have \( O(a) \) corrections on lattice. These lattice artifacts may come from different origins. Part of the \( O(a) \) effect can be approximately included by using a normalization factor
\[
< \psi(x) \bar{\psi}(0) >_{cont} = 2\kappa u_0 e^m < \psi(x) \bar{\psi}(0) >_{latt} ,
\]
where [3]
\[
e^m = 1 + \frac{1}{u_0} \left( \frac{1}{2\kappa} - \frac{1}{2\kappa_{cr}} \right) ,
\]
with \( u_0 \) the “tadpole improvement” factor.

Another correction comes from the use of (non-conserved) local vector current \( V_\mu = \bar{c}_c c_T \). This effect can be corrected by introducing a
This way both \( \beta \) quarks are treated as dynamical. We use data at \( n = 3 \) in the quenched limit. Both heavy and light \( Z \) configurations on the \( 24 \times 39 \) lattice and the Lorentz index \( \nu \). We will use Eq. (13) as the normalization condition for matrix element \( <D_\nu|V_\mu|D_\nu> \). This way both \( e(m) \) and \( Z^{loc} \) factors are taken out automatically (together with any other multiplicative \( O(a) \) factors).

3. Numerical results

We use the standard Wilson action for quarks in the quenched limit. Both heavy and light quarks are treated as dynamical. We use data at \( \beta = 6.0 \) on \( 16^3 \times 39 \) and \( 24^3 \times 39 \) lattices. There are 19 configurations on the \( 16^3 \times 39 \) lattice and 8 configurations on the \( 24^3 \times 39 \) lattice. The techniques for measuring the two-point function and the three-point matrix elements are standard. The fittings are done for time slices from \( t=10 \) to \( t=15 \). To reduce the statistical error, we have used symmetry properties of the Green functions and averaged over \( \pm t \) and \( \pm \vec{p} \) directions.

Comparing the measured \( f_0 \) value at \( q^2 = 0 \) \((f_0(0) = f_+(q^2 = 0)) \) to the known continuum value \( f_0(0) = 1 \), we observed that the lattice artifacts are typically \( 20\% \) - \( 40\% \) at \( \kappa_Q = 0.118 \) and less than \( 10\% \) at \( \kappa_Q = 0.135 \). We also measured the ratio \( f_-/f_+ \). We find the violation of Euclidean invariance is typically \( 5\% \) - \( 15\% \) for \( \kappa_Q = 0.118 \) and \( 3\% \) - \( 10\% \) for \( \kappa_Q = 0.135 \). Note that the reduction of the lattice artifacts when \( \kappa_Q \) is changed from 0.118 to 0.135 agrees with our intuitive expectations. One may try to use the factors \( Z^{loc} \) and \( e(m) \) to remove part of the \( O(a) \) effects. At \( \beta = 6.0 \) we get from perturbative calculation Eq. (13) that \( Z^{loc} \approx 0.7 \) (using the shifted effective gauge coupling \( \tilde{g}^2 \approx 1.7 \) as suggested in ref. [8]). The factor \( e(m) \) is about 2 for \( \kappa_Q = 0.118 \) and 1.5 for \( \kappa_Q = 0.135 \). Including both these factors the corrected \( f_0(0) \) becomes 1 within errors. This is in agreement with other observations that \( Z^{loc} \) and \( e(m) \) factors seem to account for the largest part of the lattice artifacts.

According to the discussions in Section 2, we define \( \xi = v \cdot v'/f_0(0) \) and list the results in Table 1. We emphasize that this definition removes all momentum-independent \( O(a) \) effects simultaneously, including both \( e(m) \) and \( Z^{loc} \) factors. Note that in Eq. (13) there is a factor \( C_{cc} \) in the connection between \( f_+ \) and \( \xi \). This factor comes from integrating out the QCD effects from the heavy quark scale down to a light scale \( \mu \). For the lattice calculation, however, \( \mu \) is taken to be \( O(1/a) \). Thus in our case \( \mu \sim m_D \sim m_c \). Also \( v \cdot v' \) is very close to one. So for practical purposes we can set \( C_{cc} = 1 \) according to Eq. (13).

We plot our results for the Isgur-Wise function in Fig. 1. For comparison we also plotted the theoretical bounds on the Isgur-Wise function. The top and bottom curves are the upper and lower bounds derived in ref. [9]. They are obtained using the dispersion relation for the two-point functions, with the requirements of unitarity and causality and with some assumptions on the analytic properties of the form factors. The curve in the middle is an upper bound on the Isgur-Wise function derived from current-algebraic sum rules [9]. This is a tighter upper bound. Our data obtained on the \( 24^3 \times 39 \) lattice appear to be, within errors, inside of the upper bound of Bjorken [10] and the lower bound of de Rafael and Taron [9]. The data from the \( 16^3 \times 39 \) lattice is consistent.

| \( \kappa_Q \) | .118, \( 16^3 \times 39 \) | .118, \( 24^3 \times 39 \) | .135, \( 24^3 \times 39 \) |
|---|---|---|---|
| \( \xi \) | 1.00(4) | 0.99(11) | 0.974(20) |
| \( v \cdot v' \) | 1.0567(8) | 1.1103(15) | 1.0259(5) | 1.0512(10) | 1.0543(12) | 1.1059(23) |

Table 1: The Isgur-Wise function. The heavy quark hopping parameter \( \kappa_Q \) and the lattice sizes are shown in the table.
Table 2
Comparison of the lattice calculation for the slope at $v \cdot v' = 1$ with various model calculations.

| Lattice   | [11] | [12] | [13] | [14] | [10,9] |
|-----------|------|------|------|------|--------|
|           | 1.0(8) | 1.6(4) | 1.4(6) | 1.05(20) | 0.65(15) |

with the Bjorken upper bound within rather large errors.

Figure 1. The Isgur-Wise function is plotted against $v \cdot v'$. The open circles represent data on the $16^3 \times 39$ lattice. The solid circles and squares represent data on the $24^3 \times 39$ lattice. The heavy quark hopping parameter is set at $\kappa_Q = 0.118$ (open and solid circles) and at $\kappa = 0.135$ (squares) while the light quark hopping parameters are extrapolated to the chiral limit $\kappa = \kappa_c \approx 0.157$.

Close to $v \cdot v' = 1$ the Isgur-Wise function can be parametrized as

$$\xi(v \cdot v') = 1 - \rho^2(v \cdot v' - 1) + O((v \cdot v' - 1)^2).$$

If we calculate the slope using the data point closest to the $v \cdot v' = 1$ axis, we get $\rho^2 = 1.0(8)$. In Table 2 we list our result along with $\rho^2$ values estimated by other authors.

Note that the lattice meson mass we used is in the range of physical D meson. Thus the $O(1/m_Q)$ correction may be quite significant. For a reliable calculation of the Isgur-Wise function we need to repeat this calculation for several different masses and then extrapolate to the infinite mass limit. For a check on the residual $O(a)$ effects we plan to repeat our calculation at $\beta = 6.3$.

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