The General Spherically Symmetric Static Solutions in the Einstein-Aether Theory

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Abstract. In the present work we have analyzed all the possible spherically symmetric vacuum solutions allowed by the Einstein-Aether (EA) theory. We show that there are solutions only for two specific values of \( c_{14} \), zero and two. This result is completely general. For \( c_{14} = 0 \) we have the Schwarzschild spacetime. On the other hand, for \( c_{14} = 2 \) we have a family of new static solutions with an arbitrary function of the coordinate \( r \). As a consequence of \( c_{14} = 2 \) the EA coupling constant is zero, means that there is no coupling with the matter, which does not represent any problem to vacuum solutions. The problem arises when matching the vacuum exterior with some interior source matter since the EA parameter should be the same in the whole spacetime (interior and exterior). In this case the unique possibility would be \( c_{14} = 0 \).

1. Introduction

The Lorentz invariance is an exact symmetry of special relativity, quantum field theories and the standard model of particle physics, and also a local symmetry in freely falling inertial frames in General Relativity [1]. The Lorentz violation in matter interactions is highly constrained by several precision experiments, see [2] for the latest example, while similar studies in gravity are not as well explored. With this motivation, Jacobson and collaborators introduced and analyzed a general class of vector-tensor theories called the Einstein-Aether (EA) theory [3] [4] [5] [6] [7].

The first paper that investigated spherical static vacuum solutions in the EA theory was presented by Eling and Jacobson in 2006 [8]. In this paper the authors found a family of analytical solutions for the metric functions assuming an aether vector proportional to the timelike Killing field. These solutions do not depend on the parameters \( c_2 \) and \( c_3 \) of the EA theory but only depend on the combination \( c_1 + c_4 \), such that \( c_1 + c_4 < 2 \). For \( c_1 + c_4 > 2 \) the coupling constant \( G \) becomes negative, implying that the gravity is repulsive, and for \( c_1 + c_4 = 0 \) this one is exactly the Newtonian gravitational constant.
The authors have shown when \( c_1 + c_4 = 0 \) the Schwarzschild solution can be obtained and also explored some solutions for different choices of \( 0 \leq c_1 + c_4 < 2 \). In another paper, Eling, Jacobson and Miller \([9]\) studied a perfect fluid, in order to model a neutron star, and considered the vacuum solution given in the previous paper \([8]\) for any \( c_1 + c_4 < 2 \). Almost the complete literature on black holes in EA theory can be found in the papers \([10]-[26]\). In this paper we complete a general survey of spherically symmetric static vacuum solutions.

The paper is organized as follows. The Section 2 briefly outlines the EA theory, whose field equations are solved for a general spherically symmetric metric in Section 3 which are further analyzed in Sections 4 and 5. We end with some remarks in Section 6.

2. Field equations in the EA theory

The general action of the EA theory is given by

\[
S = \int \sqrt{-g} \left( L_{\text{Einstein}} + L_{\text{aether}} + L_{\text{matter}} \right) d^4x, \tag{1}
\]

where, the first term is the usual Einstein-Hilbert Lagrangian, defined by \( R \), the Ricci scalar, and \( G_N \), the Newtonian gravitational constant, as

\[
L_{\text{Einstein}} = \frac{1}{16\pi G} R. \tag{2}
\]

The second term, the aether Lagrangian is given by

\[
L_{\text{aether}} = \frac{1}{16\pi G} \left[ -K_{mn}^{ab} \nabla_a u^m \nabla_b u^n + \lambda (g_{ab} u^a u^b + 1) \right], \tag{3}
\]

where the tensor \( K_{mn}^{ab} \) is defined as

\[
K_{mn}^{ab} = c_1 g_{ab} g_{mn} + c_2 \delta_m^a \delta_n^b + c_3 \delta_m^b \delta_n^a - c_4 u^a u^b g_{mn}, \tag{4}
\]

being the \( c_i \) dimensionless coupling constants, and \( \lambda \) a Lagrange multiplier enforcing the unit timelike constraint on the aether, and

\[
\delta_m^a \delta_n^b = g^{\alpha \gamma} g_{\alpha m} g_{\beta n} g_{\beta n}. \tag{5}
\]

Finally, the last term, \( L_{\text{matter}} \) is the matter Lagrangian, which depends on the metric tensor and the matter field.

In the weak-field, slow-motion limit EA theory reduces to Newtonian gravity with a value of Newton’s constant \( G_N \) related to the parameter \( G \) in the action \([11]\) by \([11]\),

\[
G = G_N \left( 1 - \frac{c_1 + c_4}{2} \right). \tag{6}
\]

Note that if \( c_1 = -c_4 \) the EA coupling constant \( G \) becomes the Newtonian coupling constant \( G_N \), without necessarily imposing \( c_1 = c_4 = 0 \).

The field equations are obtained by extremizing the action with respect to independent variables of the system. The variation with respect to the Lagrange multiplier \( \lambda \) imposes the condition that \( u^a \) is a unit timelike vector, thus

\[
g_{ab} u^a u^b = -1, \tag{7}
\]
while the variation of the action with respect $u^a$, leads to 
\[ \nabla_a J^a_b + c_4 a_a \nabla_b u^a + \lambda u_b = 0, \] 
where, 
\[ J^a_m = K^a_{mn} \nabla_b u^n, \] 
and 
\[ a_a = u^b \nabla_b u_a. \] 
The variation of the action with respect to the metric $g_{mn}$ gives the dynamical equations,
\[ G_{ab}^{\text{Einstein}} = T_{ab}^{\text{aether}} + 8\pi G T_{ab}^{\text{matter}}, \] 
where
\[ G_{ab}^{\text{Einstein}} = R_{ab} - \frac{1}{2} g_{ab} R, \]
\[ T_{ab}^{\text{aether}} = \nabla_c [J^c_{(a} u_{b)} + u^c J_{(a} u_{(b)]} - \frac{1}{2} g_{ab} J_d \nabla_c u^d + \lambda u_a u_b \]
\[ + c_1 [\nabla_a u_c \nabla_b u^c - \nabla^c u_a \nabla_c u_b] + c_4 a_a a_b, \]
\[ T_{ab}^{\text{matter}} = - \frac{2}{\sqrt{-g}} \frac{\delta (\sqrt{-g} L_{\text{matter}})}{\delta g_{ab}}. \] 
In a more general situation, the Lagrangian of GR theory is recovered, if and only if, the coupling constants are identically null, e.g., $c_1 = c_2 = c_3 = c_4 = 0$, considering the equations (11) and (7).

3. Solutions of EA field equations

Aiming to know what kind of solutions the EA theory admits we start with the most general spherically symmetric static metric
\[ ds^2 = -e^{2A(r)} dt^2 + e^{2B(r)} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2. \] 
In accordance with equation (7), the aether field is assumed unitarily, timelike and constant, chosen as
\[ u^a = (1, 0, 0, 0). \] 
Assuming (13), we compute the different terms in the field equations (12) for the vacuum, giving
\[ G_{rr}^{\text{aether}} = -\frac{1}{2r^2} (r^2 c_{14} A'^2 + 4r A' - 2e^{2B} + 2) = 0, \] 
\[ G_{\theta\theta}^{\text{aether}} = \frac{r}{2e^{2B}} (2r A'' - r(c_{14} - 2)A'^2 - 2r A'B' + 2A' - 2B') = 0, \] 
\[ G_{\phi\phi}^{\text{aether}} = -\frac{e^{2(A-B)}}{2r^2} [c_{14}(-2r^2 A'B' + r^2 A'^2 + 2r^2 A'' + 4r A') \]
\[-4r B' - 2e^{2B} + 2] = 0, \]
\[ G_{\phi \phi}^{\text{aether}} = G_{\theta \theta}^{\text{aether}} \sin^2 \theta, \]  
(18)

where the symbol prime denotes the derivative in relation to the coordinate \( r \) and \( G_{\mu \nu}^{\text{aether}} = G_{\mu \nu}^{\text{Einstein}} - T_{\mu \nu}^{\text{aether}} \). Here, the constant \( c_{14} \) is defined as

\[ c_{14} = c_1 + c_4. \]  
(19)

In order to identify eventual singularities in the solutions, it is very useful to calculate the Kretschmann scalar invariant \( K \). For the metric (13), it is given by

\[ K = \frac{4}{r^4 e^{4B}} \left( 2B'^2r^2 + e^{4B} - 2e^{2B} + 1 + 2A'^2r^2 + r^4 A''^2 + 2r^4 A''^2 - 2r^4 A'' B' A' + r^4 A'^4 - 2r^4 A'^3 B' + r^4 B'^2 A'^2 \right). \]  
(20)

Substituting the field equation (15) into (17) we can eliminate the term \( e^{2B} \) and find

\[ c_{14} \left( 2rA'' - 2rA'B' + 4A' - 4A' - 4B' = 0. \right. \]  
(21)

With this equation and equation (16) we can eliminate \( A'' \) obtaining

\[ (c_{14} - 2) \left( B' + A' + \frac{1}{2}r c_{14} A'^2 \right) = 0. \]  
(22)

Now using equations (21) and (16) we can eliminate \( B' \) and obtain

\[ (c_{14} - 2) \left( 2rA'' + 4rA'^2 + 4A' + r^2 c_{14} A'^3 \right) = 0. \]  
(23)

From the equations (22) and (23) we can note that there are two possibilities: \( c_{14} \neq 2 \) and \( c_{14} = 2 \). Let us now analyze in details each one.

4. Analysis of the solutions with \( c_{14} \neq 2 \)

In this case we must have from the equations (22) and (23) that

\[ B' + A' + \frac{1}{2}r c_{14} A'^2 = 0, \]  
(24)

and

\[ 2rA'' + 4rA'^2 + 4A' + r^2 c_{14} A'^3 = 0. \]  
(25)

Substituting \( B' \) and \( A'' \) from these equations into the field equations (15)-(17) we get

\[ r^2 c_{14} A'^2 + 4rA' - 2e^{2B} + 2 = 0, \]  
(26)

\[ r^2 A'^2 c_{14} (A' - 1) = 0, \]  
(27)

\[ r^2 c_{14} A'^2 + r^3 c_{14} A'^3 - r^3 c_{14} A'^2 + 4rA' - 2e^{2B} + 2 = 0. \]  
(28)

Solving equation (27) we have three solutions:

(i) \( A = r + \delta \)

(ii) \( A = \gamma \)

(iii) \( c_{14} = 0 \)

where \( \delta \) and \( \gamma \) are integration constants.
4.1. Case (i): $A = r + \delta$

In this case we have

$$B = \frac{1}{2} \ln \left( \frac{1}{2} r^2 c_{14} + 2r + 1 \right)$$

(29)

Substituting $A$ and $B$ into the field equations (15)-(17) we can see that $c_{14} = 2$. This case will be studied in details in the Section 4.

4.2. Case (ii): $A = \gamma$

Substituting $A$ into the field equations (15)-(17) we can see that $B = 0$. Thus, we obtain the analogous of the Minkowski of GR theory.

4.3. Case (iii): $c_{14} = 0$

Substituting $c_{14} = 0$ into the field equations (15)-(17), we have the solution

$$A = \lambda + \ln \sqrt{\left( \frac{re^{\sigma} - 1}{r} \right)}$$

(30)

and

$$B = \frac{\sigma}{2} + \ln \sqrt{\left( \frac{r}{re^{\sigma} - 1} \right)}$$

(31)

With this solution for $A$ and $B$ we can put (13) in the Schwarzschild metric form, that is,

$$ds^2 = -\left( 1 - \frac{e^{-\sigma}}{r} \right) dt^2 + \left( 1 - \frac{e^{-\sigma}}{r} \right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$$

(32)

where $e^{2\lambda+\sigma} dt^2 = dt^2$ with $\lambda$ and $\sigma$ being integration constants.

5. Analysis of the solutions with $c_{14} = 2$

Substituting this condition into field equations (15)-(17), we get

$$2e^{2B} - 2 - 4r A' - 2A'^2 r^2 = 0,$$

(33)

$$2B' - 2A' - 2A'' r + 2B' A' r = 0,$$

(34)

$$4r B' + 2e^{2B} - 2 - 2A'^2 r^2 + 4r^2 A'B' - 4r^2 A'' - 8r A' = 0.$$  

(35)

Solving simultaneously equations (33)-(35) we get the solution

$$B = \ln \left( \pm (1 + r A') \right),$$

(36)

where $A$ is an arbitrary function of the $r$.

In the next section we show an example where we consider the metric function $g_{tt}$ as the same of the correspondent to the Schwarzschild solution.
5.1. Example of an asymptotically flat similar to Schwarzschild spacetime in EA theory

In the equation (36) we consider

\[ A = \frac{1}{2} \ln \left( 1 - \frac{2M}{r} \right), \]

obtaining from (36)

\[ B = \frac{1}{2} \ln \left( \frac{(M - r)^2}{(2M - r)^2} \right). \]

Then, the metric is given by

\[ ds^2 = -\left( 1 - \frac{2M}{r} \right) dt^2 + \frac{(M - r)^2}{(2M - r)^2} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2, \]

where \( M \) is a constant.

The Riemann scalar is given by

\[ K = \frac{4M^2}{r^4(M - r)^6} \left( 26M^4 - 90Mr^3 + 119M^2r^2 - 64r^3M + 12r^4 \right), \]

and its limits are

\[ \lim_{r \to M} K = \infty, \]

\[ \lim_{r \to 0} K = \infty. \]

This spacetime is Minkowski at the infinity

\[ \lim_{r \to \infty} ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2. \]

6. Conclusions

In the present work we have analyzed all the possible solutions for the vacuum with spherical symmetry allowed by the EA theory. We show that there are solutions only for two specific values of \( c_{14} \), zero and two. This result is completely general. For \( c_{14} = 0 \) we have the Schwarzschild spacetime. On the other hand, for \( c_{14} = 2 \) we have a family of new static solutions with an arbitrary function of the coordinate \( r \). As a consequence of \( c_{14} = 2 \) the EA coupling constant is zero, means that there is none coupling with the matter, which does not represent any problem to vacuum solutions. The problem arises when matching the vacuum exterior with some interior source matter since the EA parameter should be the same in the whole spacetime (interior and exterior). In this case the unique possibility would be \( c_{14} = 0 \).
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References

[1] D. G. Moore and V. H. Satheeshkumar, “The fate of Lorentz frame in the vicinity of black hole singularity,” Int. J. Mod. Phys. D 22, 134026 (2013) [arXiv:1305.7221 [gr-qc]].
[2] H. Pihan-Le Bars et al., “New Test of Lorentz Invariance Using the MICROSCOPE Space Mission,” Phys. Rev. Lett. 123, no. 23, 231102 (2019) [arXiv:1912.03030 [physics.space-ph]].
[3] T. Jacobson and D. Mattingly, “Gravity with a dynamical preferred frame,” Phys. Rev. D 64, 024028 (2001) [gr-qc/0007031].
[4] C. Eling and T. Jacobson, “Static post-Newtonian equivalence of GR and gravity with a dynamical preferred frame,” Phys. Rev. D 69, 064005 (2004) [gr-qc/0310044].
[5] T. Jacobson and D. Mattingly, “Einstein-Aether waves,” Phys. Rev. D 70, 024003 (2004) [gr-qc/0402005].
[6] C. Eling, T. Jacobson and D. Mattingly, “Einstein-Aether theory,” [gr-qc/0411001].
[7] B. Z. Foster and T. Jacobson, “Post-Newtonian parameters and constraints on Einstein-aether theory,” Phys. Rev. D 73, 064015 (2006) [gr-qc/0509083].
[8] C. Eling and T. Jacobson, “Black Holes in Einstein-Aether Theory,” Class. Quant. Grav. 23, 5643 (2006) Erratum: [Class. Quant. Grav. 27, 049802 (2010)] [gr-qc/0604088].
[9] C. Eling, T. Jacobson and M. Coleman Miller, “Neutron stars in Einstein-aether theory,” Phys. Rev. D 76, 042003 (2007) Erratum: [Phys. Rev. D 80, 129906 (2009)] [arXiv:0705.1565 [gr-qc]].
[10] B. Z. Foster, “Noether charges and black hole mechanics in Einstein-aether theory,” Phys. Rev. D 73, 024005 (2006) [gr-qc/0509121].
[11] D. Garfinkle, C. Eling and T. Jacobson, “Numerical simulations of gravitational collapse in Einstein-aether theory,” Phys. Rev. D 76, 024003 (2007) [gr-qc/0703093 [GR-QC]].
[12] R. A. Konoplya and A. Zhidenko, “Perturbations and quasi-normal modes of black holes in Einstein-Aether theory,” Phys. Lett. B 644, 186 (2007) [gr-qc/0605082].
[13] T. Tamaki and U. Miyamoto, “Generic features of Einstein-Aether black holes,” Phys. Rev. D 77, 024026 (2008) [arXiv:0709.1011 [gr-qc]].
[14] E. Barausse, T. Jacobson and T. P. Sotiriou, “Black holes in Einstein-ether and Horava-Lifshitz gravity,” Phys. Rev. D 83, 124043 (2011) [arXiv:1104.2889 [gr-qc]].
[15] C. Gao and Y. G. Shen, “Static Spherically Symmetric Solution of the Einstein-aether Theory,” Phys. Rev. D 88, 103508 (2013) [arXiv:1301.7122 [gr-qc]].
[16] C. Ding, A. Wang and X. Wang, “Charged Einstein-aether black holes and Smarr formula,” Phys. Rev. D 92, no. 8, 084055 (2015) [arXiv:1507.06618 [gr-qc]].
[17] C. Ding, C. Liu, A. Wang and J. Jing, “Three-dimensional charged Einstein-aether black holes and the Smarr formula,” Phys. Rev. D 94, no. 12, 124034 (2016) [arXiv:1608.00290 [gr-qc]].
[18] C. Ding, A. Wang, X. Wang and T. Zhu, “Hawking radiation of charged Einstein-aether black
holes at both Killing and universal horizons,” Nucl. Phys. B 913, 694 (2016) [arXiv:1512.01900 [gr-qc]].

[19] E. Barausse, T. P. Sotiriou and I. Vega, “Slowly rotating black holes in Einstein-ther theory,” Phys. Rev. D 93, no. 4, 044044 (2016) [arXiv:1512.05894 [gr-qc]].

[20] C. Ding, “Quasinormal ringing of black holes in Einstein-aether theory,” Phys. Rev. D 96, no. 10, 104021 (2017) [arXiv:1707.06747 [gr-qc]].

[21] M. Bhattacharjee, S. Mukohyama, M. B. Wan and A. Wang, “Gravitational collapse and formation of universal horizons in Einstein-aether theory,” Phys. Rev. D 98, no. 6, 064010 (2018) [arXiv:1806.00142 [gr-qc]].

[22] K. Lin et al., “Gravitational waveforms, polarizations, response functions, and energy losses of triple systems in Einstein-aether theory,” Phys. Rev. D 99, no. 2, 023010 (2019) [arXiv:1810.07707 [astro-ph.GA]].

[23] T. Zhu, Q. Wu, M. Jamil and K. Jusufi, “Shadows and deflection angle of charged and slowly rotating black holes in Einstein-ther theory,” Phys. Rev. D 100, no. 4, 044055 (2019) [arXiv:1906.05673 [gr-qc]].

[24] C. Ding, “Gravitational quasinormal modes of black holes in Einstein-aether theory,” Nucl. Phys. B 938, 736 (2019) [arXiv:1812.07994 [gr-qc]].

[25] K. Lin, F. H. Ho and W. L. Qian, “Charged Einstein-aether black holes in $n$-dimensional spacetime,” Int. J. Mod. Phys. D 28, no. 03, 1950049 (2018) [arXiv:1704.06728 [gr-qc]].

[26] C. Zhang, X. Zhao, A. Wang, B. Wang, K. Yagi, N. Yunes, W. Zhao and T. Zhu, “Gravitational waves from the quasicircular inspiral of compact binaries in Einstein-aether theory,” Phys. Rev. D 101, no. 4, 044002 (2020) [arXiv:1911.10278 [gr-qc]].