CHAOS IN FRW COSMOLOGY WITH VARIOUS TYPES OF A SCALAR FIELD POTENTIAL

A. TOPORENSKY
Sternberg Astronomical Institute, Universitetsky prospekt,13, Moscow 119899, Russia
E-mail: lesha@sai.msu.ru

The results on chaos in FRW cosmology with a massive scalar field are extended to another scalar field potential. It is shown that for sufficiently steep potentials the chaos disappears. A simple and rather accurate analytical criterion for the chaos to disappear is given. On the contrary, for gently sloping potentials the transition to a strong chaotic regime can occur. Two examples, concerning asymptotically flat and Damour-Mukhanov potentials are given.

1 Introduction

The studies of chaotical dynamics of closed isotropic cosmological model has a long story. They were initiated by papers[1] where the possibility to avoid a singularity at the contraction stage in such a model with a minimally coupled massive scalar field was discovered. Later it was found that this model allows the existence of periodical trajectories[2] and aperiodical infinitely bouncing trajectories having a fractal nature[3]. In[4] the set of periodical trajectories was studied from the viewpoint of dynamical chaos theory. It was proved that the dynamics of a closed universe with a massive scalar field is chaotic and an important invariant of the chaos, the topological entropy, was calculated.

However, most of modern scenarios based on ideas of the string theory and compactification naturally lead to another forms of potential which are exponential or behave as exponential for large $\phi$ (see, for example, the paper of Gunther and Zhuk[5]). This steepness of the potential apparently changes the possibilities of escaping the singularities and alters the structure of infinitely bouncing trajectories. Under some conditions, which are described in Section 2, the chaotic behaviour can completely disappear.

On the other hand, potential, less steep than quadratic one can give rise to chaotic dynamics which differ qualitatively from described in[4]. Two examples are presented in Section 3.

2 Chaotic properties of closed FRW model with a scalar field

We shall consider a cosmological model with an action

$$ S = \int d^4x \sqrt{-g} \left\{ \frac{m_\phi^2}{16\pi} R + \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - V(\varphi) \right\}. $$

For a closed Friedmann model with the metric

$$ ds^2 = dt^2 - a^2(t)d^2\Omega^3, $$

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where \( a(t) \) is a cosmological radius, \( d^2\Omega^{(3)} \) is the metric of a unit 3-sphere and with homogeneous scalar field \( \varphi \) we have the following ODE system:

\[
\frac{m_P^2}{16\pi} \left( \ddot{a} + \frac{\dot{a}^2}{2a} + \frac{1}{2a} \right) + \frac{a\dot{\varphi}^2}{8} - \frac{aV(\varphi)}{4} = 0, \tag{3}
\]

\[
\ddot{\varphi} + \frac{3\dot{\varphi}\dot{a}}{a} + V'(\varphi) = 0. \tag{4}
\]

where the scalar field potential \( V(\phi) \) is a smooth nonnegative function with \( V(0) = 0 \).

This system has one first integral of motion

\[
-\frac{3}{8\pi} m_P^2 (\dot{a}^2 + 1) + \frac{a^2}{2} \left( \dot{\varphi}^2 + 2V(\varphi) \right) = 0. \tag{5}
\]

It is easy to see from Eq. (5) that the points of maximal expansion and those of minimal contraction, i.e. the points, where \( \dot{a} = 0 \) can exist only in the region where

\[
a^2 \leq \frac{3}{8\pi} \frac{m_P^2}{V(\varphi)}, \tag{6}
\]

Sometimes, the region defined by inequalities (6) is called Euclidean, and the opposite region is called Lorentzian. We will use this definition for brevity, though we consider these notations rather meaningless.

It also can be shown that the possible points of maximal expansion are localized inside the region

\[
a^2 \leq \frac{1}{4\pi} \frac{m_P^2}{V(\varphi)}, \tag{7}
\]

while the possible points of minimal contraction lie outside this region (7) being at the same time inside the Euclidean region (6).

In our paper it was shown that the region of possible points of maximal expansion has quite a regular structure. In the left, closest to axis \( a = 0 \) part of this region there are points of maximal expansion after which trajectory goes to singularity. Then one can see the region where after the going through the point of maximal expansion a trajectory undergoes the “bounce” i.e. goes through the point of minimal contraction. Then we have the region where after going through the point of maximal expansion trajectory has a “\( \varphi \)-turn” i.e. has the extremum in the value of the scalar field \( \varphi \) and then falls into a singularity. Then one has the region corresponding to the trajectories having bounce after one oscillation in \( \varphi \) and so on.

To avoid a misunderstanding, let us indicate once more what initial condition space we use. When we start from maximal expansion point, we fix one time derivative (\( \dot{a} = 0 \)). So, for fix the initial condition completely, we need only to specify initial values of \( a \) and \( \varphi \). The initial \( \dot{\varphi} \) is determined from the constraint equation (5), and our initial condition space becomes \( (a, \varphi) \). We will study the structure of this space, investigating the location of points of maximal expansion,
starting from which a trajectory has a bounce, but not the location of bounce itself. The \( \varphi = 0 \) cross-section of regions, leading to bounce, are called as "intervals".

Remembering that a trajectory describing an expanding universe must have a point of maximal expansion, we can further apply this analysis to bouncing trajectories. Their second point of maximal expansion may lie either inside "bounce" regions, defined at the first step or between them. This fact generates a more tiny substructure of the region under consideration. This substructure of regions having two bounces repeats the general structure of regions having at least one bounce and so on and so forth. Continuing this process \textit{ad infinitum} we can get the fractal zero-measure set of infinitely bouncing trajectories escaping the singularity.

Numerical investigations show also that all simple periodical trajectories (i.e. having only one bounce per period) have a full stop point on the Euclidean boundary (the curve defined by the equality in (6)). Moreover, trajectories, going from the boundary inside the Euclidean region has a point of maximal expansion almost immediately and then go towards a singularity. So, periodical trajectory approaches their bounce point on the boundary from the Lorentzian side. Hence, we have a necessary condition for given region of the Euclidean boundary to contain a full stop points of periodical trajectories (it is the condition that a trajectory starting with zero velocities from the Euclidean boundary goes into the Lorentzian region):

\[
\frac{\ddot{\varphi}}{\ddot{a}} < \frac{d\varphi}{da},
\]

where function \( \varphi(a) \) in the right-hand side is the equation of the Euclidean boundary (6).

The point on the Euclidean boundary where a trajectory starting with zero velocities have a direction tangent to the Euclidean boundary can be obtained substituting the equality in (6) into the equations of motion. It was first introduced by Page for massive scalar field potential \( V(\phi) = \frac{m^2 \phi^2}{2} \). In this case

\[
\varphi_{\mathrm{page}} = \sqrt{\frac{3}{4\pi}} m p;
\]
\[
a_{\mathrm{page}} = \frac{1}{m},
\]

except for the trivial solution \( \varphi = 0, a = \infty \).

The first periodical trajectory have full stop point at \( \varphi = 1.19 \varphi_{\mathrm{page}} \) and \( a = 0.83 a_{\mathrm{page}} \). These obtained numerically values, rather close to the Page point, represent a right-down boundary of a set of full stop points of periodical trajectories on the Euclidean boundary.

Using the equation of motion (3)-(4) the criterion (8) gives for an arbitrary scalar field potential

\[
V(\varphi) > \sqrt{\frac{3m^2}{16\pi}} V'(\varphi)
\]

The condition (10) may be treated as restricting a local steepness of the function \( V(\phi) \). In can be easily seen that for power-law function the Page value \( \phi_{\mathrm{page}} \) such that for all \( \phi > \phi_{\mathrm{page}} \) the condition (8) is satisfied, always exists. For steeper potentials the situation changes. For example, the potential \( V(\varphi) = M_0^4 (\cosh(\varphi/\varphi_0) - 1) \),
Figure 1. Example of trajectories with the initial conditions close to the boundary separating trajectories falling into $\varphi = +\infty$ (trajectory 1) and $\varphi = -\infty$ (trajectories 2 – 5) singularities for the case $\varphi_0 < \frac{1}{4\sqrt{\pi}} m_P$. This boundary is sharp, no fractal structure is present. Trajectories 2 – 5 have a zigzag-like form, no periodical trajectories are present. The long-dashed line is the Euclidean boundary, the short-dashed line is the separating curve.

studied in has a Page point only if $\varphi_0 > \frac{\sqrt{3}}{4\sqrt{\pi}} m_P$. In the opposite case the condition (8) is never satisfied and all the trajectories starting from the Euclidean boundary go into the Euclidean region and soon experience the point of maximal expansion. It was confirmed numerically that the chaos is absent in this case (see Fig.1.).

For steeper potentials like $V(\varphi) = M_0^2(\exp(\varphi^2/\varphi_0^2) + \exp(-\varphi^2/\varphi_0^2) - 2)$ inequality (8) is definitely violated for large $\phi$ but, depending on $\phi_0$ it can be satisfied for intermediate $\phi$. For different $\phi_0$ there are two or zero Page points in this case. The value of $\phi_0$ corresponding to the Page points disappearance (0.905$m_P$) differ from the $\phi_0$ of the chaos disappearance obtained numerically (0.96$m_P$) for $\sim 6\%$. It is interesting that when the chaos exists, the number of bounce intervals is finite and the whole picture of the chaos is similar to the picture obtained for a system with a massive scalar field and a hydrodynamical matter.

So, studying the possibility to satisfy the condition (8) at the Euclidean boundary, we have an easily calculated and rather accurate criterion for the existence of the chaotic dynamics in the system (3)-(5).
3 Chaotic dynamics for gently sloping potentials

In this section we describe the opposite case - the potential which is less steep than the quadratic one. We will see that in this case the transition to a qualitatively stronger chaos may occur.

3.1 Asymptotically flat potentials and merging of the bounce intervals

Let us start with the potential

\[ V(\varphi) = M_0^4 (1 - \exp(-\varphi^2/\varphi_0^2)), \quad (11) \]

where \( M_0 \) and \( \varphi_0 \) are parameters. \( M_0 \) determines the asymptotical value of the potential for \( \varphi \to \pm \infty \).

It can be easily checked from the equations of motion that multiplying the potential to a constant (i.e. changing the \( M_0 \)) leads only to rescaling \( a \). So, this procedure do not change the chaotic properties of our dynamical system. On the contrary, this system appear to be very sensitive to the value of \( \varphi_0 \). We plotted in Fig.2. the \( \varphi = 0 \) cross-section of bounce intervals depending on \( \varphi_0 \). In our numerical investigations we use the units in which \( m_P/\sqrt{16\pi} = 1 \), and below we present our results in these units because now most of the interesting events occur for the range of parameters of the order of unity. This plot represents a situation, qualitatively different from studied previously for potentials like \( V \sim \varphi^2 \) and steeper. Namely, the bounce intervals can merge.

Let us see more precisely what does it means. For \( \varphi_0 > 0.82 \) the picture is qualitatively as for a massive scalar field - trajectories from 1-st interval have a bounce with no \( \varphi \)-turns before it, trajectories which have initial point of maximal expansion between 1-st and 2-nd intervals fall into a singularity after one \( \varphi \)-turn, those from 2-nd interval have a bounce after one \( \varphi \)-turn and so on. For \( \varphi_0 \) a bit smaller than the first merging value the 2-nd interval contains trajectories with two \( \varphi \)-turns before bounce, the space between 1-st interval (which is now the product of two merged intervals) and the 2-nd one contains trajectories falling into a singularity after two \( \varphi \)-turns. There are no trajectories going to a singularity with exactly one \( \varphi \)-turn. Trajectories from the 1-st interval can experience now a complicated chaotic behavior which can not be described in as similar way as above.

With \( \varphi_0 \) decreasing further, the process of interval merging being to continue leading to growing chaoticisation of trajectories. When \( n \) intervals merged together, only trajectories with at least \( n \) oscillations of the scalar field before falling into a singularity are possible. Those having exactly \( n \) \( \varphi \)-turns have their initial point of maximal expansion between 1-st bounce interval and the 2-nd one (the second interval now contains trajectories having a bounce after \( n \) \( \varphi \)-turns). For initial values of the scale factor larger then those from the 2-nd interval, the regular quasiperiodic structure described above is restored.

Numerical analysis shows also that the fraction of very chaotic trajectories as a function of \( \varphi_0 \) grows rapidly with \( \varphi_0 \) decreasing below the first merging value. To illustrate this point we studied the behavior of trajectories starting from the point of maximal expansion with initial values of \( a \) located in the range of the first two
Figure 2. The $\varphi = 0$ cross-section of the bounce intervals for the potential (11) depending on $\varphi_0$. Consecutive merging of 5 first intervals can be seen in this range of $\varphi_0$. Total number of trajectories for each $\varphi_0$ is equal to 1000. We plotted in Fig.3 the number of trajectories which do not fall into a singularity during first 50 oscillations of the scalar field $\varphi$. We do not include trajectories with the next point of maximal expansion located outside the 2-nd (or the 1-st one, if merging occurred) interval, so all counted trajectories avoid a singularity during this sufficiently long time interval due to their extreme chaoticity, but not due to reaching the slow-roll regime. Before merging, the measure of so chaotic trajectories is extremely low and they are undistinguishable on our grid. When $\varphi_0$ becomes slightly lower than the value of the first merging, this number begin to grow rather rapidly and for $\varphi_0 \sim 0.6$ near 10% of trajectories from the 1-st interval on our grid experience at least 50 oscillation before falling into a singularity.

We point out that for a simple massive scalar field potential only $\sim 10^{-2}$ trajectories in the same range of the initial scale factors have at least one bounce (using the abovementioned quasiperiodicity in the structure of initial condition space ($\alpha, \varphi$) we can estimate this measure as a ratio of interval width and distance between near-by intervals). The width of subintervals containing trajectories not falling into a singularity after only one bounce is about one hundred times less than the width of "main" intervals and so on. The common numerical calculation accuracy is insufficient for distinguishing even the sole trajectory with 50 oscillations and $\alpha$ being in the range of first two intervals.

In contrast to this, the chaos for the potential (11) is really significant. Detail of intervals merging including the description out of $\varphi = 0$ cross-section require...
Figure 3. Number $N$ of trajectories do not falling into a singularity during 50 oscillating times for the potential (11) depending on the parameter $\varphi_0$. The scale factor of the initial maximal expansion point varies in the range of the 1-st and 2-nd intervals which merge at $\varphi_0 = 0.82$. Total number of trajectories is equal to 1000.

For large initial $a$ the configuration of bounce intervals for potential (11) looks like the configuration for a massive scalar field potential with the effective mass $m_{\text{eff}} = (\sqrt{2} M_0^2)/\varphi_0$. The periods of corresponding structures coincides with a good accuracy ($m\Delta a \sim 2.8$ in the initial condition space) though the widths of the intervals for the potential (11) is bigger then for $V = (m_{\text{eff}}^2 \varphi^2)/2$.

### 3.2 Damour-Mukhanov potentials

The very chaotic regime described above is possible also for potentials, which are not asymptotically flat, if the potential growth is slow enough. We will illustrate this point describing a particular (but rather wide) family of potentials having power-law behavior – Damour-Mukhanov potentials. They was originally introduced to show a possibility to have an inflation behaviour without slow-roll regime. After, various issues on inflationary dynamics and growth of perturbation for this kind of scalar field potential was studied.

The explicit form of Damour-Mukhanov potential is

$$V(\varphi) = \frac{M_0^4}{q} \left[ \left( 1 + \frac{\varphi^2}{\varphi_0^2} \right)^{q/2} - 1 \right],$$

with three parameters $-M_0, q$ and $\varphi_0$.

For $\varphi \ll \varphi_0$ the potential looks like the massive one with the effective mass $m_{\text{eff}} = M_0^2/\varphi_0$. In the opposite case of large $\varphi$ it grows like $\varphi^q$.

As in the previous section, the chaotic behavior does not depend on $M_0$. So, we have a two-parameter family of potentials with different chaotic properties. Numerical studies with respect to possibility of bounce intervals merging shows the following picture (see Fig.4): for a rather wide range of $q$ there exists a corresponding critical value of $\varphi_0$ such that for $\varphi_0$ less than critical, the very chaotic regime exists. Increasing $q$ corresponds to decreasing critical $\varphi_0$. 
Surely, since this regime is absent for quadratic and more steep potentials, \( q \) must at least be less than 2. We can see clearly the very chaotic regime for \( q < 1.24 \). The case \( q = 1.24 \) lead to strong chaos for \( \varphi_0 < 1.4 \times 10^{-5} \) and the critical \( \varphi_0 \) decreases with increasing \( q \) very sharply at this point. We did not investigated further these extremely small values of \( \varphi_0 \), because the physical meaning of such kind of potential is very doubtful.

References

1. L. Parker and S.A. Fulling, Phys. Rev. D 7, 2357 (1973); A.A. Starobinsky, Pisma A.J. 4, 155 (1978) [Sov. Astron. Lett. 4, 82 (1978)].
2. S.W. Hawking, in Relativity, Groups and Topology II, Les Houches, Session XL 1993 Eds. B.S. DeWitt and R. Stora, (North Holland, Amsterdam, 1984).
3. D.N. Page, Class. Quant. Grav. 1, 417 (1984).
4. N. Cornish and E. Shellard, Phys. Rev. Lett. 81, 3571 (1998).
5. U. Gunther and A. Zhuk, Phys. Rev. D56, 6391 (1997).
6. A. Yu. Kamenshchik, I. M. Khalatnikov and A. V. Toporensky, Int. J. Mod. Phys. D6, 673 (1997).
7. A. V. Toporensky, Int. J. Mod. Phys. D8, 739 (1999).
8. A. Yu. Kamenshchik, I. M. Khalatnikov and A. V. Toporensky, Int. J. Mod. Phys. D 6, 649 (1997).
9. T. Damour and V. F. Mukhanov, Phys. Rev. Lett. 80, 3440 (1998).
10. A. R. Liddle and A. Mazumdar, Phys. Rev. D58, 083508 (1998).
11. A. Taruya, Phys. Rev. D59, 103505 (1999).
12. V. C. Cardenas and G. Palma, "Some remarks on oscillationary inflation". astro-ph/9904313.
13. A. Yu. Kamenshchik, I. M. Khalatnikov, S. V. Savchenko and A. V. Toporensky, Phys. Rev. D59, 123516 (1999).