REASONABLE TRIANGULATED CATEGORIES HAVE FILTERED ENHANCEMENTS

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Abstract. We prove that a triangulated category which is the underlying category of a stable derivator has a filtered enhancement, providing an affirmative answer to a conjecture in [3].

INTRODUCTION

Filtered enhancements of triangulated categories were defined in order to allow us to perform various constructions in these categories: In [2, Appendix], where filtered enhancement were first considered, they are used for defining the realization functor. This is a functor $D^b_A \to T$, where $T$ is the enhanced triangulated category and $D_A$ is the bounded derived category over the abelian category $A$, which occurs as the heart of an appropriate t-structure in $T$. In [10] the realization functor is used in the study of tilting and silting objects in triangulated categories (see for example [10, Proposition 5.1]). In [9] and [11] filtered enhancement are used for studying the so-called the weight complex functor of [9, Theorem 3.3.1]. For example in [3, Conjecture 3.3.3 and Remark 3.3.4] it is stated that the weight complex functor can be lifted to a strong version, provided that a filtered enhancement exists. In [3, Remark 3.3.4] it is conjectured that every reasonable triangulated category has a filtered enhancement; in a private communication, M. Bondarko credited Beĭlinson with the formulation of this conjecture. As an evidence for the conjecture, remark that in [2, Example A 2] it is stated that the filtered derived category in the sense of [4, V.1] provides a filtered enhancement for the usual derived category $D_A$ of an abelian category $A$. The whole argument for this fact can be found in [11, Proposition 6.3]. The present paper originates in the observation that the same argument works for showing that the derived category of modules over a dg algebra has a filtered enhancement. Together with [10, Proposition 3.8], which says that filtered enhancements are inherited from a triangulated category to an arbitrary localization, and with Keller and Porta characterization of algebraic triangulated categories as localizations of derived categories of modules over dg algebras (see [7] and [9]) the above conjecture is settled for the case of algebraic triangulated categories. Further, the idea is to generalize this approach by mimicking the argument above in order to construct filtered enhancement for the underlying category $D(e)$ of a stable derivator $D$ inside...
the category \( \text{D}(\mathbb{Z}) \), where \( \mathbb{Z} \) is the poset of integers. This is what we are doing in this work.

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The construction of \( f \)-enhancements

We begin with recalling the definitions of the main notions we will use in the sequel. Consider a triangulated category \( \mathcal{T} \), whose suspension functor is denoted by \( X \mapsto \Sigma X \), for all \( X \in \mathcal{T} \). Then, according to [1, Definition 1.3.1], a \( t \)-structure in \( \mathcal{T} \) is a pair of full subcategories \((\mathcal{T}_{\leq 0}, \mathcal{T}_{\geq 0})\), such that the following conditions are satisfied, where we use the notations \( \mathcal{T}_{\geq n} = \mathcal{T}_{\geq 0}[n] \) and \( \mathcal{T}_{\leq n} = \mathcal{T}_{\leq 0}[n] \):

1. \( \mathcal{T}(X, Y) = 0 \) for all \( X \in \mathcal{T}_{\leq 0} \) and all \( Y \in \mathcal{T}_{\geq 1} \).
2. \( \mathcal{T}_{\leq 0} \subseteq \mathcal{T}_{\leq 1} \) and \( \mathcal{T}_{\geq 1} \subseteq \mathcal{T}_{\geq 0} \).
3. For all \( X \in \mathcal{T} \) there is a triangle

\[
X_{\leq 0} \to X \to X_{\geq 1} \oplus \to
\]

in \( \mathcal{T} \), with \( X_{\leq 0} \in \mathcal{T}_{\leq 0} \) and \( X_{\geq 1} \in \mathcal{T}_{\geq 1} \).

Note that if \((\mathcal{T}_{\leq 0}, \mathcal{T}_{\geq 0})\) is a \( t \)-structure on \( \mathcal{T} \), then \( \mathcal{T}_{\leq 0} \) is called the aisle and \( \mathcal{T}_{\geq 0} \) is called the coaisle associated with this \( t \)-structure. We know that the inclusion functor \( \mathcal{T}_{\leq 0} \to \mathcal{T} \) has a right adjoint and the inclusion functor \( \mathcal{T}_{\geq 0} \to \mathcal{T} \) has a left adjoint. Note also that the \( t \)-structure is called stable if its aisle (or equivalently coaisle) is a triangulated subcategory.

A filtered triangulated category or an \( f \)-category for short, is a quintuple \((\mathcal{X}, \mathcal{X}(\geq 0), \mathcal{X}(\leq 0), s, \alpha)\), where \( \mathcal{X} \) is a triangulated category, \( \mathcal{X}(\geq 0) \) and \( \mathcal{X}(\leq 0) \) are full triangulated subcategories, \( s : \mathcal{X} \to \mathcal{X} \) is a triangle autoequivalence and \( \alpha : \text{id}_\mathcal{X} \Rightarrow s \) is a natural transformation; we put \( \mathcal{X}(\geq n) = s^n \mathcal{X}(\geq 0) \) and \( \mathcal{X}(\leq n) = s^n \mathcal{X}(\leq 0) \). These data have to satisfy the following axioms:

1. \( \mathcal{X}(\geq 1) \subseteq \mathcal{X}(\geq 0) \) and \( \mathcal{X}(\leq 0) \subseteq \mathcal{X}(\leq 1) \).
2. \( \mathcal{X} = \bigcup_{n \in \mathbb{Z}} \mathcal{X}(\geq n) = \bigcup_{n \in \mathbb{Z}} \mathcal{X}(\leq n) \).
3. \( \mathcal{X}(X, Y) = 0 \) for all \( X \in \mathcal{X}(\geq 1) \) and all \( Y \in \mathcal{X}(\leq 0) \).
4. For all \( X \in \mathcal{X} \) there is a triangle

\[
X(\geq 1) \to X \to X(\leq 0) \oplus \to
\]

in \( \mathcal{X} \), with \( X(\geq 1) \in \mathcal{X}(\geq 1) \) and \( X(\leq 0) \in \mathcal{X}(\leq 0) \).

5. One has \( \alpha \cdot s = s \cdot \alpha \) as natural transformations \( s \Rightarrow s^2 \).
6. For all \( X \in \mathcal{X}(\geq 1) \) and all \( Y \in \mathcal{X}(\leq 0) \), the map

\[
\mathcal{X}(s(Y), X) \to \mathcal{X}(Y, X) \text{ given by } g \mapsto g \cdot \alpha(Y)
\]

is bijective.

In [11, §7.2] it is introduced a new axiom for \( f \)-categories, which seems to be necessary in order to show that the realization functor is a triangle functor. This new axiom is formulated as follows:
(F7) The morphism of triangles constructed in the solid part of the following diagram fits in the $3 \times 3$ diagram whose rows and columns are triangles:

\[
\begin{array}{ccc}
X(\geq 1) & \rightarrow & X \rightarrow X(\leq 0) \rightarrow \\
\downarrow a(Y(\geq 1)) \cdot f(\geq 1) & \downarrow a(Y) \cdot f & \downarrow a(Y(\leq 0)) \cdot f(\leq 0) \\
s(Y(\geq 1)) & \rightarrow & s(Y) \rightarrow s(Y(\leq 0)) \\
\downarrow & \downarrow & \downarrow \\
Z' & \rightarrow & Z \rightarrow Z'' \\
\downarrow & \downarrow & \downarrow \\
\downarrow & \downarrow & \downarrow \\
\uparrow & \uparrow & \uparrow \\
& & \\
\end{array}
\]

If $T$ is a triangulated category then a \textit{filtered enhancement} for $T$, called also an \textit{f-enhancement} or an \textit{f-category} over $T$, is an $f$-category $\mathcal{X}$ together with a triangle equivalence

\[ T \xrightarrow{\sim} \mathcal{X}(\geq 0) \cap \mathcal{X}(\leq 0). \]

Derivators were first introduced by Grothendieck in an initially unpublished manuscript which is nowadays available online, see [6], thanks to the efforts of M. Künnzer, J. Malgoire and G. Maltsiniotis. In our consideration about this subject we will follow closely the exposition in [5]. According to [5] Definition 1.1], a \textit{prederivator} is a strict 2-functor $D : \text{Cat}^{op} \rightarrow \text{CAT}$ where $\text{Cat}$ is the 2-category of all small categories and $\text{CAT}$ is the 2-category of not necessarily small categories. We neglect set theoretical issues coming from the fact that CAT has no small hom–sets, since they play no role in our considerations. For two small categories $J$ and $K$ and a functor $u : J \rightarrow K$ we will denote $u^* = D(u) : D(K) \rightarrow D(J)$. We entirely stick on the convention in [5], in the sense that $D$ reverses the sense of functors, but preserves the sense of natural transformation. That is, if $\alpha : u \Rightarrow v$ is a natural transformation in Cat, then the induced natural transformation is defined as $\alpha^* : u^* \Rightarrow v^*$. With the notations above we call \textit{homotopy left (right) Kan extension} along $u$ a left (respectively right) adjoint for $u^*$. We will denote the left adjoint by $u_l : D(J) \rightarrow D(K)$ and the right adjoint by $u_s : D(J) \rightarrow D(K)$.

Denote by $e = [0]$ the singleton category. Then the category $D(e)$ is called the \textit{underlying category} of the prederivator $D$. Remark that for every small category $K$ and every $k \in K$, there is a functor $k : e \rightarrow K$ sending $0$ to $k \in K$. The induced functor $k^* : D(K) \rightarrow D(e)$ is called the \textit{evaluation functor}. For all objects $X, Y$ and all morphisms $f : X \rightarrow Y$ in $D(K)$ we will denote $X_k = k^*(X)$ and $f_k = k^*(f)$. The right (left) Kan extension along the unique functor $K \rightarrow e$ is called the \textit{homotopy (co)limit} of shape $K$ and is denoted $\text{holim}_K$, respectively $\text{hocolim}_K$. Let $u : K \rightarrow J$ be a functor between two small categories. For every $k \in K$ we consider the categories $J/k$ and $J\backslash k$ whose objects are pairs of the form $(j, u(j) \rightarrow k)$, respectively $(j, k \rightarrow u(j))$, where $j \in J$ and $u(j) \rightarrow k, k \rightarrow u(j)$ are maps in $K$, and whose morphisms are those of $K$ which make commutative the obvious triangles. We call $J/k$ and $J\backslash k$ the \textit{categories of u-objects over}, respectively \textit{under}, $k$. In both cases, there is a functor $\pi : J/k \rightarrow J$, respectively $\pi : J\backslash k \rightarrow J$. 
which forgets the morphism component. With these data, one can construct two natural maps (see [5, Section 1.1] for details):
\[
\hocolim_{J/k} \pi^*(X) \to u_!(X)_k \quad \text{and} \quad u_*(X)_k \to \holim_{J \setminus k} \pi^*(X).
\]

A prederivator \(D : \text{Cat}^{op} \to \text{CAT}\) is called derivator if it satisfies the following axioms:

(D1) \(D\) sends coproducts to products; in particular, \(D(\emptyset)\) is trivial.

(D2) A morphism \(f : X \to Y\) in \(D(K)\) is an isomorphism if and only if \(f_k : X_k \to Y_k\) is an isomorphism in \(D(e)\) for every \(k \in K\).

(D3) For every functor \(u : J \to K\), there are homotopy left and right Kan extensions along \(u\) (here and below, the left adjoint is depicted up and the right one down):
\[
\begin{array}{ccc}
D(J) & \xrightarrow{u_!} & D(K) \\
\downarrow{u_*} & & \downarrow{u_*}
\end{array}
\]

(D4) For every functor \(u : J \to K\) and every \(k \in K\), the canonical morphisms \(\hocolim_{J/k} \pi^*(X) \to u_!(X)_k\) and \(u_*(X)_k \to \holim_{J \setminus k} \pi^*(X)\) are isomorphisms for all \(X \in D(J)\).

If \(D : \text{Cat}^{op} \to \text{Cat}\) is a prederivator and \(K\) is a small category, one defined \(D^K : \text{Cat}^{op} \to \text{CAT}\), by \(D^K(J) = D(K \times J)\). From [5, Theorem 1.25], we learn that if \(D\) is a derivator, then so is \(D^K\) too. As we already noticed, an object \(k \in K\) gives rise to a functor \(k^* : D(K) \to D(e)\). Under the categorical exponential law we obtain a functor
\[
dia_K : D(K) \to D(e)^K
\]
which sends every \(X \in D(K)\) to its underlying diagram in \(D(e)^K\). Replacing \(D\) with \(D^J\) we get a functor
\[
dia_{K,J} : D(K \times J) \cong D(J \times K) \to D(J)^K.
\]
The derivator \(D\) is called strong if the functor \(\text{dia}_{[1],J}\) is full and essentially surjective for each small category \(J\), where \([1]\) is the category associated to the poset \([0 \leq 1]\) (see [5, Definition 1.8]). A derivator is called pointed if its underlying category has a zero object (that is, it is pointed). Because it is quite technical, we do not recall here the definition of a stable derivator; we refer the interested readers to [5, Definition 4.1]. Actually we only need the facts stated in [5, Theorem 4.16 and Corollary 4.19]: If \(D\) is a stable derivator, then for every small category \(K\) the category \(D(K)\) has a canonical triangulated category structure, and for every functor \(u : J \to K\) the induced functors \(u^*, u_1\) and \(u_*\) are triangle functors.

The following result is known, but, for an easier reference, we state it here exactly in the form we need it:

**Lemma 1.** Consider the diagram of triangulated categories and functors:
\[
\begin{array}{ccc}
S & \xrightarrow{i} & T \\
\downarrow{i_*} & & \downarrow{j_*}
\end{array}
\]
\[
\begin{array}{ccc}
T & \xrightarrow{j} & U
\end{array}
\]

The following are equivalent:

(i) The functors \(i\) and \(j_\ast\) are fully faithful and \((\text{Im}i_\ast, \text{Im}j_\ast)\) is a stable \(t\)-structure in \(T\).
(ii) The functor $i_!$ is fully faithful, $j^*$ has a fully faithful right adjoint $j_*$ and $\text{Im} \ i_! = \text{Ker} \ j^*$.

(iii) The functor $j_*$ is fully faithful, $i^*$ has a fully faithful left adjoint $i_!$ and $\text{Im} \ j_* = \text{Ker} \ i^*$.

**Proof.** The equivalence between (i) and (ii) follows by [8, Proposition 4.9.1].

By the same [8, Proposition 4.9.1], the solid part of the diagram above satisfies (ii) if and only if there is a localization functor $L : T \to T$, such that $S = \text{Ker} \ L$. Dually the dotted part of the same diagram satisfies (iii) if and only if there is a colocalization functor $\Gamma : T \to T$, such that $\text{Ker} \ \Gamma = U$. The equivalence between conditions concerning localization and colocalization functors follows by [8, Proposition 4.12.1].

□

From now on let $D : \text{Cat}^{op} \to \text{CAT}$ be a stable derivator. Consider the poset $(\mathbb{Z}, \geq)$ viewed as a category, that is

$$Z = [\ldots \leftarrow -2 \leftarrow -1 \leftarrow 0 \leftarrow 1 \leftarrow 2 \leftarrow \ldots].$$

For an object $X \in D(Z)$, let

$$(\dagger) \ldots \leftarrow X_{-2} \leftarrow X_{-1} \leftarrow X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \ldots$$

be the underlying diagram of $X$.

**Lemma 2.** There exists an autoequivalence $s : D(Z) \to D(Z)$ and natural transformation $\alpha : \text{id}_{D(Z)} \Rightarrow s$ such that $s(X)_k = X_{k-1}$, $\alpha(X)_k$ is the map $X_k \to X_{k-1}$ in the diagram $(\dagger)$, for all $X \in D(Z)$ and all $k \in \mathbb{Z}$, and $\alpha \cdot s = s \cdot \alpha$ as natural transformations $s \Rightarrow s^2$.

**Proof.** Let $u : Z \to Z$ given by $u(k) = k-1$. Then $u$ is an order isomorphism, therefore $u^* : D(Z) \to D(Z)$ is an autoequivalence. Denote $s = u^*$. By construction $s(X)_k = X_{k-1}$ for all $k \in \mathbb{Z}$. Moreover $k \geq u(k)$ for all $k \in \mathbb{Z}$ providing a (unique) natural transformation $\phi : \text{id}_Z \Rightarrow u$. If we denote by $\alpha = \phi^* : \text{id}_{D(Z)} \Rightarrow s$ the induced natural transformation, then clearly $\alpha(X)_k$ is the respective map from $(\dagger)$. Since $\phi \cdot u = u \cdot \phi$ is the unique natural transformation $u \Rightarrow u^2$ we deduce $\alpha \cdot s = s \cdot \alpha$.

□

For all $n \in \mathbb{Z}$ consider $Z_{\geq n} = \{ k \in \mathbb{Z} \mid k \geq n \}$ and $Z_{\leq n} = \{ k \in \mathbb{Z} \mid k \leq n \}$.

The increasing inclusion maps $[\geq n] : Z_{\geq n} \to Z$ and $[\leq n] : Z_{\leq n} \to Z$ can be viewed as fully faithful functors between the respective categories. Therefore they induce functors depicted in the diagram:

$$D(Z_{\geq n}) \xrightarrow{[\geq n]^*} D(Z) \xleftarrow{[\leq n]^*} D(Z_{\leq n}).$$

Note that $[\geq n]^*(X)_k = X_{[\geq n](k)} = X_k$ for all $k \geq n$ and $[\leq n]^*(X)_k = X_k$, for all $k \leq n$. Moreover, the left and right homotopy Kan extensions $[\geq n]^!$ and $[\geq n]^*$, respectively $[\leq n]^!$ and $[\leq n]^*$, are fully faithful by [5, Proposition 1.20].

By $\epsilon_{\geq n} : [\geq n] \cdot [\geq n]^* \to \text{id}_{D(Z)}$ and $\eta_{\leq n} : \text{id}_{D(Z)} \to [\leq n]^! \cdot [\leq n]^*$ we will denote the respective morphisms of adjunction. We also need to consider
The increasing maps \( \geq n \), \( \leq n \) : \( Z \to Z \) given by
\[
[\geq n](k) = \begin{cases} 
  k & \text{for } k \geq n \\
  n & \text{for } k < n 
\end{cases} \quad [\leq n](k) = \begin{cases} 
  k & \text{for } k \leq n \\
  n & \text{for } k > n 
\end{cases}
\]
and the induced functors \( [\geq n]^*, [\leq n]^* : D(Z) \to D(Z) \).

It will be useful to characterize pointwise \( \text{Im}[\leq n] \), as in the next:

**Lemma 3.** For every \( n \in \mathbb{Z} \), it holds
\[
\text{Im}[\leq n] = \{ X \in D(\mathbb{Z}) \mid X_k = 0 \text{ for } k > n \}.
\]

*Proof.* Observe that if \( [\leq n](j) \to k \) is a map in \( \mathbb{Z} \), then \( [\leq n](j) \geq k \), so \( n \geq k \) showing that \( k \in \text{Im}[\leq n] \). This proves that \( [\leq n] \) is a cosieve. The conclusion follows by [5, Proposition 3.6]. \( \square \)

**Lemma 4.** For all \( X \in D(\mathbb{Z}) \) and all \( k \in \mathbb{Z} \), we have:
\[
(a) \ [\geq n]_! \cdot [\geq n]^*(X)_k = X_{[\geq n](k)}.
(b) \ [\leq n]^* \cdot [\leq n]*_!(X)_k = X_{[\leq n](k)}.
\]

*Proof.\ (a).* For an object \( X \in D(\mathbb{Z}) \) we have \( [\geq n]^*(X)_k = X_k \) for all \( k \in \mathbb{Z}_{\geq n} \). Further the left homotopy Kan extension is computed by
\[
[\geq n]_! \cdot [\geq n]^*(X)_k = \text{hocolim}_{\geq n/k} \pi^*(([\geq n]^*(X))_
\]
for all \( k \in \mathbb{Z} \), where
\[
\mathbb{Z}_{\geq n/k} = \{(i, i \to k) \mid i \in \mathbb{Z}_{\geq n}\} \cong \begin{cases} 
  \mathbb{Z}_{\geq k} & \text{for } k \geq n \\
  \mathbb{Z}_{\geq n} & \text{for } k < n 
\end{cases}
\]
is the category of objects \( \geq n \)-over \( k \) and \( \pi : \mathbb{Z}_{\geq n/k} \to \mathbb{Z}_{\geq n} \) is the functor which forgets the morphism component. Thus
\[
[\geq n]_! \cdot [\geq n]^*(X)_k = \begin{cases} 
  \text{hocolim}(X_k \leftarrow X_{k+1} \leftarrow \ldots) = X_k & \text{for } k \geq n \\
  \text{hocolim}(X_n \leftarrow X_{n+1} \leftarrow \ldots) = X_n & \text{for } k < n 
\end{cases}
\]
Note that in both cases above, the diagram whose homotopy colimit is to be computed has an terminal object, hence, according to [5, Lemma 1.19], the homotopy colimit is the evaluation at this object.

(b). This statement is the dual of (a). \( \square \)

**Corollary 5.** The following natural transformations:
\[
(a) \ \epsilon_{\geq n}(X) : [\geq n]_! \cdot [\geq n]^*(X) \to X, \text{ with } X \in \text{Im}[\geq n]^*
\]
\[
(b) \ \eta_{\leq n}(X) : X \to [\leq n]^* \cdot [\leq n]^*(X), \text{ with } X \in \text{Im}[\leq n - 1]^*.
\]
are isomorphisms.

*Proof.\ (a).* Observe that an object \( X \in \text{Im}[\geq n]^* \) is characterized by the condition \( X_k = X_n \) for all \( k < n \). From Lemma 4 we learn that for such an \( X \), the morphisms
\[
\epsilon_{\geq n}(X)_k : [\geq n]_! \cdot [\geq n]^*(X)_k \to X_{[\geq n](k)} = X_k
\]
are actually isomorphisms for all \( k \in \mathbb{Z} \), hence \( \epsilon_{\geq n}(X) \) is also an isomorphism.
(b). By the argument dual to the one used in (a), we can show that 
\[ \eta_{\leq n}(X) \] is an isomorphism for all
\[ X \in \text{Im[} \leq n] = \{ X \in D(\mathbb{Z}) \mid X_k = X_n \text{ for } k \geq n \}. \]
For concluding we only have to note that by Lemma 3
\[ \text{Im[} \leq n-1] = \{ X \in D(\mathbb{Z}) \mid X_k = 0 \text{ for } k \geq n \} \subseteq \text{Im[} \leq n]. \]

\[ \square \]

**Lemma 6.** The following equalities hold:

(a) \( s \text{Im[} \leq n] = \text{Im[} \leq n+1] \)

(b) \( s \text{Im[} \leq n] = \text{Im[} \leq n+1] \).

**Proof.** The conclusion follows at once, by the pointwise characterizations of \( \text{Im[} \leq n] \) and \( \text{Im[} \leq n] \).

\[ \square \]

**Theorem 7.** For every stable derivator \( D \), the underlying triangulated category \( D(e) \) has an \( f \)-enhancement, satisfying the supplementary axiom (F7).

**Proof.** Let
\[ \mathcal{X} = \bigcup_{m \leq n} (\text{Im[} \leq m] \cap \text{Im[} \leq n-1]). \]
Because \( \leq m \) and \( \leq n-1 \) are triangle functors, their images are triangulated subcategories, hence the same is true for the intersection. Further \( \mathcal{X} \) is triangulated as a directed union of triangulated subcategories. Note that an object \( X \in \mathcal{X} \) has the underlying diagram of the form:
\[ (\dagger) \ldots = X_{m-1} = X_m \leftarrow X_{m-1} \leftarrow \ldots \leftarrow X_{n-2} \leftarrow X_{n-1} \leftarrow 0 \leftarrow 0 \leftarrow \ldots \]
We denote \( \mathcal{X}(\geq n) = \mathcal{X} \cap \text{Im[} \leq n] \) and \( \mathcal{X}(\leq n) = \mathcal{X} \cap \text{Im[} \leq n-1] \), for all \( n \in \mathbb{Z} \). Since for all \( k \in \mathbb{Z} \) we have \( s(X)_k = X_{k-1} \), it follows \( s(X) \in \mathcal{X} \) for every \( X \in \mathcal{X} \), hence \( s \) induced a well-defined functor (denoted with the same symbol) \( s : \mathcal{X} \to \mathcal{X} \). The restriction the natural morphism \( \alpha = id_{\mathcal{X}} \Rightarrow s \). Moreover \( s\mathcal{X}(\geq n) = \mathcal{X}(\geq n+1) \) and \( s\mathcal{X}(\leq n) = \mathcal{X}(\leq n+1) \).

The functor \( 0 : e \to \mathcal{Z} \) is fully faithful, therefore \[ 5 \] Proposition 1.20 implies \( 0 : D(e) \to D(\mathbb{Z}) \) has the same property. Moreover for all \( T \in D(e) \) and \( k \in \mathbb{Z} \) we have \( 0_k(T) = \text{hocolim}_{e/k} \pi^*(T) \), where
\[ e/k = \{(0, 0 \to k)\} \cong \begin{cases} 0 \text{ for } k \leq 0 \\ \emptyset \text{ for } k > 0 \end{cases} \]
is the category of objects \( e \)-over \( k \) and \( \pi : e/k \to e \) is the functor which forgets the morphism component. Therefore
\[ 0_k(T) = \begin{cases} T, \text{ for } k \leq 0 \\ 0, \text{ for } k > 0 \end{cases}, \]
thus \( 0 \) induces an equivalence between \( D(e) \) and \( \text{Im}_0 = \mathcal{X}(\geq 0) \cap \mathcal{X}(\leq 0) \).

We claim that \( (X, \mathcal{X}(\geq 0), \mathcal{X}(\leq 0), s, \alpha) \) is an \( f \)-enhancement for \( D(e) \), hence we have to verify the axioms (F1)-(F6).

The axiom (F1) follows by Lemma 6, the axiom (F5) follows by Lemma 2 and the axiom (F2) is direct a consequence of the construction of \( \mathcal{X} \).
The axioms (F3) and (F4) are equivalent to the condition that
\((\mathcal{X}(\geq n + 1), \mathcal{X}(\leq n))\)
is a stable t-structure on \(\mathcal{X}\). In order to prove this, denote by
\((\geq n) : \mathcal{X}(\geq n) \to \mathcal{X}\) and \((\leq n)_* : \mathcal{X}(\leq n) \to \mathcal{X}\)
the inclusion functors. By Lemma 3 and Corollary 5, the functors
\((\geq n)^* : \mathcal{X} \to \mathcal{X}(\geq n), (\geq n)^*(X) = [\geq n]_! \cdot [\geq n]^*(X)\)
and
\((\leq n)^* : \mathcal{X} \to \mathcal{X}(\leq n), (\leq n)^*(X) = [\leq n]_* \cdot [\leq n]^*(X)\)
provide well defined right, respectively left adjoint for these inclusion functors. Noting that
\(\text{Ker}(\geq n + 1)^* = \{X \in \mathcal{X} \mid X_k = 0 \text{ for all } k \leq n\} = \text{Im}(\leq n)_*\)
our claim follow by Lemma 1.

For proving (F6) let \(X \in \mathcal{X}(\geq 1)\) and \(Y \in \mathcal{X}(\leq 0)\), that is \(X_k = X_1\) for all \(k \leq 1\) and \(Y_k = 0\) for all \(k > 0\). We want to show that the map
\(\varphi : \mathcal{X}(s(Y), X) \to \mathcal{X}(Y, X)\) given by \(\varphi(g) = g \cdot \alpha(Y)\)
is bijective. If \(\varphi(g) = 0\) for some \(g : s(Y) \to X\), then \(g \cdot \alpha(Y) = 0\) so
\((g \cdot \alpha(Y))_k = 0\) for all \(k \in \mathbb{Z}\). Since for \(k \leq 1\) we have \(X_{k-1} = X_k\), the
construction of \(\alpha\) implies \(g_k = (g \cdot \alpha(Y))_{k-1} = 0\). On the other side, for \(k > 1\) we have \(g_k = 0\) because \(s(Y)_k = Y_{k-1} = 0\). Therefore all \(g_k\) vanish,
consequently \(g = 0\). This proves that \(\varphi\) is injective. For showing that \(\varphi\) is
surjective, let \(f \in \mathcal{X}(Y, X)\). Then take
\[g_k = \begin{cases} f_{k-1} : s(Y)_k = Y_{k-1} & \rightarrow X_{k-1} = X_k, \text{ for } k \leq 1 \\ 0 : s(Y)_k = 0 & \rightarrow X_k, \text{ for } k > 1 \end{cases}.\]
Since the derivator \(\mathcal{D}\) is strong, we find a morphism \(g : s(Y) \to X\) in \(\mathcal{D}(\mathbb{Z})\), or equivalently in \(\mathcal{X}\), such that \(g \cdot \alpha(Y)_k = f_k\) for all \(k \in \mathbb{Z}\), that is \(g \cdot \alpha(Y) = f\).

Finally we have to verify (F7). For an object \(X \in \mathcal{X}\), we denote
\(X(\geq n) = (\geq n)_! \cdot (\geq n)^*(X)\) and \(X(\leq n) = (\leq n)_* \cdot (\leq n)^*(X)\).
If \(f : X \to Y\) is a map in \(\mathcal{X}\), or equivalently in \(\mathcal{D}(\mathbb{Z})\), consider the diagram
\[
\begin{array}{ccc}
X(\geq 1) & \longrightarrow & X \\
\downarrow \alpha(Y(\geq 1)) \cdot f(\geq 1) & & \downarrow \alpha(Y) \cdot f \\
X(\leq 0) & \longrightarrow & X(\leq 0)
\end{array}
\]
whose rows are canonical triangle decompositions with respecting the t-structure \((\mathcal{X}(\geq 1), \mathcal{X}(\leq 0))\). The derivator \(\mathcal{D}^Z : \text{Cat}^{op} \to \text{CAT}\) is stable by [5] Proposition 4.3. As in [5] Definition 3.18 there exists a cone functor
\(\mathcal{D}(\mathbb{Z} \times [1]) = \mathcal{D}^Z([1]) \to \mathcal{D}^Z([1]) = \mathcal{D}(\mathbb{Z} \times [1])\)
corresponding to the derivator \(\mathcal{D}^Z\). Applying the cone functor to all vertical morphisms above, we obtain a diagram whose rows and columns are triangles in \(\mathcal{D}(\mathbb{Z})\) exactly as it is required in (F7). \(\square\)
REASONABLE TRIANGULATED CATEGORIES HAVE FILTERED ENHANCEMENTS

References

[1] A. Beilinson, J. Bernstein, P. Deligne, Faisceaux Pervers, in Analysis and topology on singular spaces, Soc. Math. France, Paris, Asterisque, 100 (1982), 5–171.

[2] A. A. Be˘ılinson, On the derived category of perverse sheaves, in K- theory, arithmetic and geometry, Lecture Notes in Math., 1289 (1987) Springer-Verlag, Berlin, 27–41.

[3] M. Bondarko, Weight structures vs. t-structures; weight filtrations, spectral sequences, and complexes (for motives and in general), J. K-Theory 6 (2010), 387–504.

[4] L. Illusie, Complexes cotangent et d eformations I, Lecture Notes in Mathematics, 239 (1971), Springer-Verlag, Berlin, 1971.

[5] M. Groth, Derivators, pointed derivators and stable derivators, Algebraic & Geometric Topology 13 (2013), 313–374.

[6] A. Grothendieck, Les Dérivateurs, manuscript available at: https://webusers.imj-prg.fr/ georges.maltsiniotis/groth/Derivateurs.html.

[7] B. Keller, Deriving DG categories, Ann. Sci. École Norm. Sup. 27 (1994), 63–102.

[8] H. Krause, Localization theory for triangulated categories, in Triangulated categories, London Math. Soc. Lecture Note Ser., 375 (2010), Cambridge Univ. Press, Cambridge, 161–235.

[9] M. Porta, The Popescu–Gabriel theorem for triangulated categories, Adv. Math. 225 (2010), 1669–1715.

[10] C. Psaroudakis, J. Vitória, Realization functors in tilting theory, Math. Z., 2017, DOI 10.1007/s00209-017-1923-y.

[11] O. Schnürer, Homotopy categories and idempotent completeness, weight structures and weight complex functors, preprint, arXiv:1107.1227 [math.CT].

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