Non-anticommutative Supersymmetric Field Theory and Quantum Shift

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Abstract

Non-anticommutative Grassmann coordinates in four-dimensional twist-deformed $\mathcal{N} = 1$ Euclidean superspace are decomposed into geometrical ones and quantum shift operators. This decomposition leads to the mapping from the commutative to the non-anticommutative supersymmetric field theory. We apply this mapping to the Wess-Zumino model in commutative field theory and derive the corresponding non-anticommutative Lagrangian. Based on the theory of twist deformations of Hopf algebras, we comment the preservation of the (initial) $\mathcal{N} = 1$ super-Poincaré algebra and on the consequent super-Poincaré invariant interpretation of the discussed model, but also provide a measure for the violation of the super-Poincaré symmetry.
1 Introduction

The idea of the space-time noncommutativity was proposed by Heisenberg and quantum field theory on it was then formulated by Snyder [1]. This study was motivated by an expected improvement of the renormalizability properties of quantum field theory at short distance and later on by the theory including gravity, in which space-time may change its nature at short distance (Planck scale). However, perhaps one of the strongest motivations is the recent development of string theory. It was shown that the noncommutativity of coordinates appears in string theory in the presence of an NS-NS B-field [2]. Field theories on noncommutative space-time, whose coordinates satisfy

\[ [\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}, \]  

(1.1)

where \( \theta_{\mu\nu} \) is an antisymmetric constant matrix, revealed many interesting aspects such as UV/IR mixing, solitonic solutions, etc. (for reviews, see [3, 4] and references therein). As a recent progress, the space-time symmetry of the noncommutative field theory was understood in the language of Hopf algebras as a twisted Poincaré symmetry [5, 6, 7].

It is natural to formulate an extension of the noncommutative space-time (1.1) into superspace. A general extension of (1.1) into non(anti)commutative superspace would involve superspace coordinates \( x^\mu, \theta^\alpha \) and \( \bar{\theta}^{\dot{\alpha}} \) in the algebra. A possible algebra formed by superspace coordinates was studied in Refs. [8, 9]. Then, it was shown that string theory gives rise to non-zero value for non-anticommutative commutation relation of the fermionic (Grassmann) coordinates \( \theta^\alpha \),

\[ \{ \hat{\theta}^\alpha, \hat{\theta}^{\dot{\beta}} \} = C^{\alpha\beta}, \]  

(1.2)

where \( C^{\alpha\beta} \) is a deformation parameter, provided one turns on a self-dual graviphoton field strength in four dimensions [10, 11, 12, 13] or, more generally, Ramond-Ramond 2-form field strength in ten dimensions [14]. Various aspects of field theories on non-anticommutative superspace have been intensively studied. Recently it was shown that the non-anticommutative field theory has twisted super-Poincaré symmetry for \( \mathcal{N} = 1 \) theory [15] and also for the extended supersymmetric ones [16]. The super-Poincaré algebra in the non-anticommutative superspace was represented with higher derivative operators in this quantum superspace [17].

More recently, in the noncommutative field theory, a simple interpretation of the noncommutative space-time coordinate was proposed [18]. There it was suggested that the source of the noncommutativity of the space-time coordinates is their quantum fluctuation. The coordinate operators \( \hat{x}_\mu \) are decomposed into geometrical ones \( x^\mu \) and universal shift operators \( \hat{o}_\mu \)

\[ \hat{x}_\mu = x^\mu + \hat{o}_\mu, \]  

(1.3)

where \( \hat{o}_\mu \) satisfy

\[ [\hat{o}_\mu, \hat{o}_\nu] = i\theta_{\mu\nu}. \]  

(1.4)
This shift gives a simple mapping from commutative to noncommutative field theory. Indeed, it was shown that the Lagrangian of noncommutative field theory, essentially written in terms of Weyl-Moyal $\star$-products, is constructed from the corresponding commutative theory Lagrangian by applying the quantum shift.

The purpose of this paper is to extend this construction to non-anticommutative supersymmetric field theory and to consider its applications. We consider a supersymmetric field theory in which the only deformation of the superspace is given by (1.2) (the rest of the algebra on the superspace remains the same with the usual one once the chiral coordinate is taken). This deformation is defined on Euclidean space-time and it was shown in [11] to possess $\mathcal{N} = 1/2$ supersymmetry.

The plan of this paper is as follows. In Section 2 we briefly review the four-dimensional Euclidean twist-deformed $\mathcal{N} = 1$ supersymmetric field theory and clarify the source of discrepancy in its order of supersymmetry ($\mathcal{N} = 1/2$ vs. $\mathcal{N} = 1$). In Section 3 we introduce the quantum shift to the Grassmann variable appearing in the anticommutator (1.2), and then derive various properties such as the $\star$-product between superfields and integration rules. We also construct the non-anticommutative Wess-Zumino model with the results of the previous section. In the last section, we discuss the violation of super-Poincaré symmetry. Finally we summarize our results and discuss some future applications.

## 2 Euclidean twist-deformed $\mathcal{N} = 1$ non-anticommutative superspace

We start with the four dimensional Euclidean $\mathcal{N} = 1$ superspace $(x^\mu, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$. A non-anticommutative superspace is introduced by imposing on the superspace coordinate operators $(\hat{x}^\mu, \hat{\theta}^\alpha, \hat{\bar{\theta}}^{\dot{\alpha}})$ the anticommutation relations [11]

\[ \{\hat{\theta}^\alpha, \hat{\theta}^\beta\} = C^{\alpha\beta}, \]
\[ \{\hat{\theta}^\alpha, \hat{\bar{\theta}}^{\dot{\alpha}}\} = 0. \]

The deformation by $C^{\alpha\beta}$ only of the anticommutator between $\theta$s is possible for Euclidean space-time as can be seen from the commutation relations (2.1) and (2.2). We still have free choices for the commutation relations among $\theta$ and space-time coordinate. In order to fix them, we introduce the chiral and antichiral coordinates defined as $\hat{y}^\mu = \hat{x}^\mu + i\sigma^\mu \hat{\theta}$ and $\hat{\bar{y}}^{\dot{\mu}} = \hat{x}^{\dot{\mu}} - i\sigma^{\dot{\mu}} \hat{\bar{\theta}}$, and we then choose [11]

\[ [\hat{y}^\mu, \hat{y}^{\nu}] = [\hat{y}^\mu, \hat{\theta}^\alpha] = [\hat{y}^{\dot{\mu}}, \hat{\bar{\theta}}^{\dot{\alpha}}] = 0. \]

Using (2.3), one has

\[ [\hat{y}^\mu, \hat{y}^{\nu}] = 4\hat{\theta}\hat{\bar{\theta}}C^{\mu\nu}, \]
where $C^{\mu\nu} = C^{\alpha\beta} \epsilon_{\beta\gamma}(\sigma^{\mu\nu})^\gamma_{\alpha}$ and $\sigma_{\mu\nu} = \frac{1}{4}(\sigma_\mu \bar{\sigma}_\nu - \sigma_\nu \bar{\sigma}_\mu)$. The non-trivial anticommutator (2.1) implies that, when functions of $\hat{\theta}$ are multiplied, the result should be Weyl ordered (symmetrized with respect to the Grassmann variables). According to the Weyl-Moyal correspondence,

$$f(\hat{\theta})g(\hat{\theta}) \to f(\theta) * g(\theta) = f(\theta) \exp\left(-\frac{C^{\alpha\beta}}{2} \vec{Q}_\alpha \vec{Q}_\beta\right) g(\theta),$$

(2.5)

where $Q_\alpha$ is a supercharge in the chiral coordinate defined by

$$Q_\alpha = \frac{\partial}{\partial \theta^\alpha}.$$ (2.6)

The other supercharges and covariant derivatives are then

$$\bar{Q}_{\dot{\alpha}} = -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} + 2i \theta^{\alpha} \sigma^{\mu}_{\dot{\alpha} \beta} \frac{\partial}{\partial y^\mu},$$ (2.7)

$$D_\alpha = \frac{\partial}{\partial \theta^\alpha} + 2i \sigma^{\mu}_{\alpha \dot{\beta}} \bar{\theta}^{\dot{\beta}} \frac{\partial}{\partial y^\mu}, \quad \bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}}.$$ (2.8)

It was argued in [11] that the supercharges and covariant derivatives satisfy the following algebra:

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2i \sigma^{\mu}_{\alpha \dot{\beta}} \frac{\partial}{\partial y^\mu},$$ (2.9)

$$\{Q_\alpha, Q_\beta\} = 0,$$ (2.10)

$$\{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = -4C^{\alpha\beta} \sigma^{\mu}_{\dot{\alpha} \dot{\beta}} \bar{\sigma}^{\nu}_{\dot{\beta} \beta} \frac{\partial}{\partial y^\mu} \frac{\partial}{\partial y^\nu},$$ (2.11)

and

$$\{D_\alpha, \bar{D}_{\dot{\beta}}\} = -2i \sigma^{\mu}_{\alpha \dot{\beta}} \frac{\partial}{\partial y^\mu},$$ (2.12)

$$\{D_\alpha, D_\beta\} = \{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} = 0,$$ (2.13)

$$\{D_\alpha, Q_\beta\} = \{\bar{D}_{\dot{\alpha}}, Q_\beta\} = \{D_\alpha, \bar{Q}_{\dot{\beta}}\} = \{\bar{D}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0.$$ (2.14)

One sees that this algebra represents $\mathcal{N} = 1/2$ supersymmetry, i.e. $Q$-supersymmetry is preserved while $\bar{Q}$-supersymmetry is broken.

However, the deformation of the four-dimensional $\mathcal{N} = 1$ superspace into a non-anticommutative superspace with the anticommutation relations (2.1), (2.2) can be achieved also by a twist [15]. We shall not repeat here the construction of the twist deformation, but just use the concept for clarifying the order of supersymmetry in this case. The essence of the twist deformation is that the original algebra of the generators of a certain symmetry (in this case, the $\mathcal{N} = 1$ super-Poincaré algebra) remains the same as in the undeformed case

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2i \sigma^{\mu}_{\alpha \dot{\beta}} \frac{\partial}{\partial y^\mu},$$ (2.15)

$$\{Q_\alpha, Q_\beta\} = 0,$$ (2.16)

$$\{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0.$$ (2.17)
i.e., the action of the generators on functions of the algebra of representation $\mathcal{A}$ is the same as in the usual case. However the twist element
\[
\mathcal{F} = \exp \left[ -\frac{1}{2} C^{\alpha\beta} Q_\alpha \otimes Q_\beta \right]
\]
changes the so-called coproduct of the generators, i.e. the action of the super-Poincaré algebra generators in the tensor product of representations. Moreover, the product in the algebra of representation $\mathcal{A}$ (the algebra of fields, in the case of QFT) has to be changed, in order to be compatible with the twisted coproduct. This deformed product of fields is nothing else but the $\star$-product, analogous to the one induced through the Weyl-Moyal correspondence. We emphasize that, according to the twist-deformation procedure, the $\star$-product is used only between fields and never in the action of the generators of the Hopf algebra on the fields, even if the generators are realized such that the use of ”$\star$-action” seems appropriate (as is the case with the realization \eqref{2.7}).

The origin of the discrepancy between the order of supersymmetry, for the same non-anticommutative theory, obtained in \cite{11} as $\mathcal{N} = 1/2$ and in \cite{15} as $\mathcal{N} = 1$ is elucidated by the use of the concept of twist: though the representations of the twisted super-Poincaré algebra are the same as the ones of the usual super-Poincaré algebra, thus justifying the supersymmetrically invariant interpretation (see \cite{7} for the Lorentz-invariant interpretation of NC QFT) and $\mathcal{N} = 1$, nevertheless the super-Poincaré algebra is broken, which justifies the $\mathcal{N} = 1/2$ result. In this paper we shall refer to this non-anticommutative theory as Euclidean twist-deformed $\mathcal{N} = 1$ supersymmetric theory. However, for good measure, in the last section we shall discuss in qualitative terms its super-Poincaré violation.

Since the anticommutators between supercharge and covariant derivative remain zero, we can still define supersymmetric covariant constraints on superfields as in the commutative case. Chiral and antichiral superfields are defined by
\[
\bar{D}_\dot{\alpha} \Phi = 0, \quad D_\alpha \bar{\Phi} = 0,
\]
respectively. From \eqref{2.19}, the chiral and the antichiral superfields are written as
\[
\Phi(y, \theta) = \phi(y) + \sqrt{2} \theta \psi(y) + \theta^2 F(y), \quad (2.20)
\]
\[
\bar{\Phi}(\bar{y}, \bar{\theta}) = \bar{\phi}(\bar{y}) + \sqrt{2} \bar{\theta} \bar{\psi}(\bar{y}) + \bar{\theta}^2 \bar{F}(\bar{y}). \quad (2.21)
\]
Using these superfields, one can write the non-anticommutative Wess-Zumino Lagrangian as
\[
\mathcal{L} = \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{sp}}, \quad (2.22)
\]
\[
\mathcal{L}_{\text{kin}} = \int d^4 \theta \Phi \star \bar{\Phi}, \quad (2.23)
\]
\[
\mathcal{L}_{\text{sp}} = \int d^2 \theta \left( \frac{m}{2} \Phi \star \Phi + \frac{\lambda}{3} \Phi \star \Phi \star \Phi \right) + \int d^2 \bar{\theta} \left( \frac{\bar{m}}{2} \bar{\Phi} \star \bar{\Phi} + \frac{\bar{\lambda}}{3} \bar{\Phi} \star \bar{\Phi} \star \bar{\Phi} \right). \quad (2.24)
\]
The resultant component Lagrangian is described by
\[
\mathcal{L} = \mathcal{L}(C = 0) + \frac{\lambda}{3} \det CF^3. \quad (2.25)
\]
5
Note that the deformation appears in the holomorphic part of the superpotential, while the other parts remain the same as in the commutative case.

3 Non-anticommutativity and quantum shift

In this section we interpret the non-anticommutativity in (1.2) by the quantum shift with respect to the Grassmann variable $\theta^\alpha$ similarly to the space-time coordinate noncommutativity in Ref. [18].

First we decompose the Grassmann operator $\hat{\theta}^\alpha$ into the geometrical coordinate $\theta^\alpha$ and the quantum fluctuation $\hat{\vartheta}^\alpha$ which is referred to as the quantum shift operator:

$$\hat{\theta}^\alpha = \theta^\alpha + \hat{\vartheta}^\alpha.$$  \hfill (3.1)

This means that the deformation in the anticommutation relation (2.1) is caused by the anticommutator among $\hat{\vartheta}$

$$\{ \hat{\vartheta}^\alpha, \hat{\vartheta}^\beta \} = C^{\alpha\beta}.$$  \hfill (3.2)

As for the operators $\hat{y}^\mu$ and $\hat{\bar{\theta}}^{\dot{\alpha}}$, they are not quantized since their commutation relations (2.2) and (2.3) are not deformed. Thus, we may write

$$\hat{y}^\mu = y^\mu, \quad \hat{\bar{\theta}}^{\dot{\alpha}} = \bar{\theta}^{\dot{\alpha}}.$$  \hfill (3.3)

The quantum shift (3.1) causes a shift of the antichiral operator $\hat{y}$ as

$$\hat{y}^\mu = \bar{y}^\mu - 2i\beta^{\dot{\alpha}} \sigma^\mu_{\alpha\beta} \bar{\theta}^{\dot{\alpha}} \bar{\theta}^{\dot{\beta}} \partial^\mu,$$  \hfill (3.4)

By using this and (3.2) one obtains the commutation relation (2.4).

In Ref. [18], a new Hilbert space $H_s$ corresponding to the quantum shift operator $\hat{\vartheta}_\mu$ in Eq. (1.3) is introduced in addition to a physical Hilbert space $H_{phys}$. The full Hilbert space is then given by the direct product

$$H = H_{phys} \otimes H_s.$$  \hfill (3.5)

In the present case, since we introduce a new quantum shift $\hat{\vartheta}$ as in (3.1), we have to define a new Hilbert space $H_G$ corresponding to this shift. The Hilbert space is then given by

$$H = H_{phys} \otimes H_G.$$  \hfill (3.6)

Let us consider the chiral superfield $\Phi(\hat{y}, \hat{\theta})$, which is a function of the chiral coordinate operators $\hat{y}$ and $\hat{\theta}$. The quantum shift (3.1) leads to the translation in Grassmann coordinate by $\hat{\vartheta}$

$$\Phi(\hat{y}, \hat{\theta}) = \Phi(y, \theta + \hat{\vartheta}) = e^{\hat{\vartheta} \cdot \hat{Q}} \Phi(y, \theta),$$  \hfill (3.7)
where \( \hat{\theta} \cdot Q \equiv \hat{\theta}^\alpha Q_\alpha \). Here the phase factor \( e^{\hat{\theta} \cdot Q} \) is an operator defined in the Hilbert space \( H_G \) and acting on the wave function. Note that the hermitian conjugate \( (e^{\hat{\theta} \cdot Q})^\dagger = (e^{\hat{\theta} \cdot Q})^\dagger \) is not equal to \( e^{\hat{\theta} \cdot Q} \) since now the Grassmann variables \( \theta \) and \( \bar{\theta} \) are independent ones.

Since applying \( e^{\hat{\theta} \cdot Q} \) to the wave function leads to a phase transformation representing a parallel transformation, it cannot be recognized for a single chiral superfield. Let us consider a product of two chiral superfields. In this case, the quantum shift leads to the non-trivial factor in a product of two chiral superfields at different points:

\[
\Phi_1(\hat{y}_1, \hat{\theta}_1)\Phi_2(\hat{y}_2, \hat{\theta}_2) = e^{\hat{\theta} \cdot Q_1} e^{\hat{\theta} \cdot Q_2} \Phi_1(y_1, \theta_1)\Phi_2(y_2, \theta_2), \quad Q_i = \frac{\partial}{\partial \theta_i}. \tag{3.8}
\]

Using the Baker-Campbell-Hausdorff formula in this case, we have

\[
\Phi_1(\hat{y}_1, \hat{\theta}_1)\Phi_2(\hat{y}_2, \hat{\theta}_2) = e^{\hat{\theta} \cdot (Q_1+Q_2)} e^{-\frac{1}{2} C^\alpha_\beta Q_1\alpha Q_2\beta} (\Phi_1(y_1, \theta_1)\Phi_2(y_2, \theta_2))
= e^{\hat{\theta} \cdot (Q_1+Q_2)} (\Phi_1(y_1, \theta_1) * \Phi_2(y_2, \theta_2)) , \tag{3.9}
\]

where

\[
\Phi_1(y_1, \theta_1) * \Phi_2(y_2, \theta_2) = \exp \left( -\frac{1}{2} C^\alpha_\beta Q_1\alpha Q_2\beta \right) \Phi_1(y_1, \theta_1)\Phi_2(y_2, \theta_2). \tag{3.10}
\]

This generalizes the Weyl-Moyal star product, which is obtained for \( y = y_1 = y_2, \ \theta = \theta_1 = \theta_2 \).

\[
\Phi_1(y, \theta) * \Phi_2(y, \theta) = \exp \left( -\frac{1}{2} C^\alpha_\beta Q_1\alpha Q_2\beta \right) \Phi_1(y_1, \theta_1)\Phi_2(y_2, \theta_2) \bigg|_{y=y_1, \theta=\theta_1} \tag{3.11}
\]

Eq. (3.9) can be generalized to \( n \)-points function:

\[
\Phi_1(\hat{y}_1, \hat{\theta}_1)\Phi_2(\hat{y}_2, \hat{\theta}_2) \cdots \Phi_n(\hat{y}_n, \hat{\theta}_n) = \exp(\sum_{i=1}^{n} \hat{\theta} \cdot Q_i) e^{D} (\Phi_1(y_1, \theta_1)\Phi_2(y_2, \theta_2) \cdots \Phi_n(y_n, \theta_n)) , \tag{3.12}
\]

where

\[
D = -\frac{1}{2} C^\alpha_\beta \sum_{a<b} Q_{a\alpha} Q_{b\beta} , \quad (a, b = 1, \cdots n) . \tag{3.13}
\]

As for a product at the same points, one similarly finds

\[
\Phi_1(\hat{y}, \hat{\theta})\Phi_2(\hat{y}, \hat{\theta}) \cdots \Phi_n(\hat{y}, \hat{\theta}) = e^{\hat{\theta} \cdot Q} (\Phi_1(y, \theta) * \Phi_2(y, \theta) * \cdots * \Phi_n(y, \theta)) . \tag{3.14}
\]

Next we consider the antichiral superfield. Taking Eq. (3.14) into account, the quantum shift produces the following exponential factor in a single antichiral superfield

\[
\Phi(\hat{\bar{y}}, \hat{\bar{\theta}}) = \exp (-2i \sigma^\mu \bar{\theta} \partial_\mu) \Phi(\bar{y}, \bar{\theta}) . \tag{3.15}
\]
Using the Baker-Campbell-Hausdorff’s formula, one finds the product of two antichiral superfields to be
\[
Φ_1(\hat{y}_1, \hat{θ}_1)Φ_2(\hat{y}_2, \hat{θ}_2) = \exp \left( -2i\hat{θ}^μ \sum_{i=1}^{2} (\hat{θ}_i \partial_μ) \right) \exp(2\bar{θ}^a C^{aμ} \partial_{\bar{a}ν} Φ_1(\hat{y}_1, \hat{θ}_1)Φ_2(\hat{y}_2, \hat{θ}_2)) .
\] (3.16)

The generalization to a product of \( n \) antichiral superfields is given by
\[
Φ_1(\hat{y}_1, \hat{θ}_1)Φ_2(\hat{y}_2, \hat{θ}_2) \cdots Φ_n(\hat{y}_n, \hat{θ}_n)
= \exp \left( -2i\hat{θ}^μ \hat{θ}^ν \sum_{i=1}^{n} \partial_{iμ} \right) e^G(Φ_1(\hat{y}_1, \hat{θ}_1)Φ_2(\hat{y}_2, \hat{θ}_2) \cdots Φ_n(\hat{y}_n, \hat{θ}_n)) ,
\] (3.17)

where
\[
G = 2\bar{θ}^a C^{aμ} \sum_{a < b} \partial_{αμ} \partial_{βν} .
\] (3.18)

The product of antichiral superfields at the same points is described by
\[
Φ_1(\hat{y}, \hat{θ})Φ_2(\hat{y}, \hat{θ}) \cdots Φ_n(\hat{y}, \hat{θ})
= \exp \left( -2i\hat{θ}^μ \hat{θ}^ν \partial_{μ} \right) 2\bar{θ}^a C^{aμ} \partial_{aν} Φ_1(\hat{y}, \hat{θ}) * Φ_2(\hat{y}, \hat{θ}) * \cdots * Φ_n(\hat{y}, \hat{θ}) ,
\] (3.19)

where the \(*\)-product is expressed as
\[
* = \exp \left( 2C^{aμ} \bar{θ}^a \partial_{μ} \right) .
\] (3.20)

Now we define the integration rule and formulas for chiral and antichiral superfields. The integration of a product of chiral superfields over Grassmann variable is translational invariant in \( θ \) :
\[
\int d^2 θ Φ_1(\hat{y}, \hat{θ})Φ_2(\hat{y}, \hat{θ}) \cdots Φ_n(\hat{y}, \hat{θ}) = \int d^2 θ e^{Q} (Φ_1(y, θ) * Φ_2(y, θ) * \cdots * Φ_n(y, θ))
= \int d^2 θ Φ_1(y, θ) * Φ_2(y, θ) * \cdots * Φ_n(y, θ) .
\] (3.21)

Note that the exponential phase factor does not contribute to the integral, since a product of chiral superfields is a chiral superfield and the action of \( Q \) on it does not give a term proportional to \( θ^2 \), which \( θ \)-integral only picks up.

As for a product of antichiral superfields, the integration over the space-time coordinate is invariant under the quantum shift
\[
\int d^4 x Φ_1(\hat{y}, \hat{θ})Φ_2(\hat{y}, \hat{θ}) \cdots Φ_n(\hat{y}, \hat{θ}) = \int d^4 x e^{-2iθ^a \partial_α\bar{θ}_ν} (Φ_1(\bar{y}, \bar{θ}) * Φ_2(\bar{y}, \bar{θ}) * \cdots * Φ_n(\bar{y}, \bar{θ}))
= \int d^4 x Φ_1(\bar{y}, \bar{θ}) * Φ_2(\bar{y}, \bar{θ}) * \cdots * Φ_n(\bar{y}, \bar{θ}) .
\] (3.22)

Once again the exponential factor does not contribute since the surface term vanishes at the boundary of the \( x \)-integral.
One can easily show that the integrals are invariant under the cyclic permutations
\[
\int d^2\theta \Phi_1(y, \theta) \ast \Phi_2(y, \theta) \ast \cdots \ast \Phi_n(y, \theta) = \int d^2\theta \Phi_n(y, \theta) \ast \Phi_1(y, \theta) \ast \cdots \ast \Phi_{n-1}(y, \theta),
\]
\[
(3.23)
\]
\[
\int d^2\bar{\theta} \Phi_1(\bar{y}, \bar{\theta}) \ast \Phi_2(\bar{y}, \bar{\theta}) \ast \cdots \ast \Phi_n(\bar{y}, \bar{\theta}) = \int d^2\bar{\theta} \Phi_n(\bar{y}, \bar{\theta}) \ast \Phi_1(\bar{y}, \bar{\theta}) \ast \cdots \ast \Phi_{n-1}(\bar{y}, \bar{\theta}).
\]
\[
(3.24)
\]
It is also seen that in the star product we can replace one of the star-products by a usual dot product under the integral:
\[
\int d^2\theta \Phi_1(y, \theta) \ast \cdots \ast \Phi_n(y, \theta) = \int d^2\theta \Phi_1(y, \theta) \cdot (\Phi_2(y, \theta) \ast \cdots \ast \Phi_n(y, \theta))
\]
\[
\cdots
\]
\[
(3.25)
\]
\[
\int d^4x \Phi_1(\hat{y}, \hat{\theta}) \ast \cdots \ast \Phi_n(\hat{y}, \hat{\theta}) = \int d^4x \Phi_1(\hat{y}, \hat{\theta}) \cdot (\Phi_2(\hat{y}, \hat{\theta}) \ast \cdots \ast \Phi_n(\hat{y}, \hat{\theta}))
\]
\[
\cdots
\]
\[
(3.26)
\]
Now we are ready to go to applications of the quantum shift.

4 Non-anticommutative field theory

Let us assume \( \Phi_1(y, \theta) \cdot \Phi_2(y, \theta) = \Phi_3(y, \theta) \). One can find the following inconsistency similarly to the noncommutative case \[18\]
\[
\Phi_1(\hat{y}, \hat{\theta}) \Phi_2(\hat{y}, \hat{\theta}) = e^{\hat{\theta} \cdot Q}(\Phi_1(y, \theta) \ast \Phi_2(y, \theta))
\]
\[
\neq e^{\hat{\theta} \cdot Q}(\Phi_1(y, \theta) \Phi_2(y, \theta)) = \Phi_3(y, \theta).
\]
\[
(4.1)
\]
Thus, before introducing the quantum shift, we have to uniquely factorize a given operator into a primitive one. The rule we apply is the following: we decompose a given operator into primitive factors or their linear combination and replace \( y, \theta \) and \( \bar{\theta} \) with \( \hat{y}, \hat{\theta} \) and \( \hat{\bar{\theta}} \), respectively in the primitive superfields, whose components are incoming fields:
\[
\Phi_{\text{in}} = \phi_{\text{in}} + \sqrt{2} \theta \psi_{\text{in}} + \theta^2 F(\phi_{\text{in}}, \psi_{\text{in}}),
\]
\[
(4.2)
\]
where \( \phi_{\text{in}} \) and \( \psi_{\text{in}} \) are incoming bosonic and fermionic fields, respectively. Applying this rule to the superfield \( \Phi_{\text{in}} \), we may write it as
\[
\Phi_{\text{in}}(\hat{y}, \hat{\theta}) = e^{\hat{\theta} \cdot Q} \Phi_{\text{in}}(y, \theta).
\]
\[
(4.3)
\]
This does not hold for the Heisenberg field $\Phi$ since it involves interaction and should be factorized as
\[
\Phi(\hat{y}, \hat{\theta}) = e^{\hat{\vartheta} \cdot \mu Q_{\alpha \beta} y_{\beta} \delta_{\mu} \rho_{\alpha}} \Phi(y, \theta),
\]
(4.4)
to be consistent with the rule. Here $\Phi_\theta$ will be recognized as a non-anticommutative chiral superfield as shown below. Similarly, the antichiral Heisenberg superfield is factorized as
\[
\bar{\Phi}(\bar{\hat{y}}, \bar{\hat{\theta}}) = e^{-2i \hat{\vartheta} \sigma^\mu \delta_{\mu} \rho_{\alpha}} \bar{\Phi}_\theta(y, \theta).
\]
(4.5)

Let us apply the quantum shift to a simple supersymmetric model, the Wess-Zumino model, with the rule we mentioned above. The starting point is the Lagrangian of the commutative supersymmetric field theory,
\[
\mathcal{L} = \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{sp}},
\]
(4.6)
\[
\mathcal{L}_{\text{kin}} = \int d^4 \theta \Phi \bar{\Phi},
\]
(4.7)
\[
\mathcal{L}_{\text{sp}} = \int d^2 \theta \left( \frac{m_2}{2} \Phi^2 + \frac{\lambda}{3} \Phi^3 \right) + \int d^2 \bar{\theta} \left( \frac{\bar{m}_2}{2} \bar{\Phi}^2 + \frac{\bar{\lambda}}{3} \bar{\Phi}^3 \right).
\]
(4.8)

Applying the shift (3.1) to the kinetic term (4.7), it becomes
\[
S_{\text{kin}} \rightarrow \int d^4 x d^4 \theta e^{\hat{\vartheta} \cdot (Q - 2i \sigma^\mu \delta_{\mu})} \left( \Phi_\theta(y, \theta) \exp \left( iC_{\alpha \beta} Q_{\alpha \beta} \sigma^\mu \delta_{\mu} \rho_{\alpha} \vartheta \right) \right) \Phi_\theta(\bar{y}, \bar{\theta})
\]
\[
= \int d^4 x d^4 \theta \Phi_\theta(y, \theta) \exp \left( iC_{\alpha \beta} Q_{\alpha \beta} \sigma^\mu \delta_{\mu} \rho_{\alpha} \vartheta \right) \Phi_\theta(\bar{y}, \bar{\theta})
\]
\[
= \int d^4 x d^4 \theta \Phi_\theta(y, \theta) \Phi_\theta(\bar{y}, \bar{\theta}).
\]
(4.9)

In the first equality, we used the fact that the phase factor including the quantum shift operator $\hat{\vartheta}$ does not contribute to the integral, since the action of $Q$ does not give any term contributing to the integral over $\theta$, while the $\bar{y}^\mu$-derivative gives only a surface term. In the final step, we used the fact that the exponential operator between chiral and antichiral superfields becomes the usual product.

As for the superpotential part (4.8), with the help of (3.14), (3.19), (3.21) and (3.22), it is easy to see that
\[
S_{\text{sp}} \rightarrow \int d^4 x d^2 \theta \left( \frac{m_2}{2} \Phi_\theta \Phi_\theta + \frac{\lambda}{3} \Phi_\theta \Phi_\theta \Phi_\theta \Phi \right) + \int d^4 x d^2 \bar{\theta} \left( \frac{\bar{m}_2}{2} \bar{\Phi}_\theta \bar{\Phi}_\theta + \frac{\bar{\lambda}}{3} \bar{\Phi}_\theta \bar{\Phi}_\theta \bar{\Phi}_\theta \right).
\]
(4.10)

The equation of motion of this system is derived by using (3.26):
\[
- \frac{1}{4} D^2 \Phi_\theta + \bar{m} \bar{\Phi}_\theta + \bar{\lambda} \bar{\Phi}_\theta \Phi_\theta = 0.
\]
(4.11)

We can see that the total action is a non-anticommutative version of the Wess-Zumino model written in terms of the superfields $\Phi_\theta$ and $\bar{\Phi}_\theta$ after applying the quantum shift.
5 Violation of super-Poincaré invariance

In this section, we consider the violation of super-Poincaré invariance. Both the Poincaré symmetry and the supersymmetry are violated. In the commutative case, the invariance of the S-matrix under the Lorentz generator $M_{\mu\nu}$ of the super-Poincaré group is expressed through the commutator:

$$[S, M_{\mu\nu}] = 0 .$$

(5.1)

For single chiral superfields, this commutation relation is zero since the theory is Lorentz invariant as be seen in Eq. (2.25). However, the Lorentz symmetry violation appears for the general Wess-Zumino model. For instance, for three different chiral superfields, one has

$$[S_\vartheta, M_{\mu\nu}] \neq 0 .$$

(5.2)

Here $S_\vartheta$ is the S-matrix in the non-anticommutative theory, given as

$$S_\vartheta = T \exp \left( i \int d^4x L^\vartheta_{\text{int}} \right)$$

(5.3)

where $L^\vartheta_{\text{int}}$ is the interaction part of the non-anticommutative Lagrangian.

The non-vanishing commutator (5.2) represents the amount of the violation of super-Poincaré invariance. In the following, we derive this expression in the Wess-Zumino model. However, for single chiral superfield, this commutation relation is zero since the theory is Lorentz invariant as can be seen from Eq. (2.25). Thus, we consider the general Wess-Zumino model in deriving the expression for the commutator, but the expression can be extended to the case of the most general superpotential (with flat Kähler metric).

Explicitly, the commutator (5.2) can be written as

$$[S_\vartheta, M_{\mu\nu}] = T \left[ i \int d^4x [L^\vartheta_{\text{int}}, M_{\mu\nu}] \exp \left( i \int d^4x' L^\vartheta_{\text{int}} \right) \right] .$$

(5.4)

The interaction part of the Wess-Zumino Lagrangian in the non-anticommutative theory is given by

$$L^\vartheta_{\text{int}} = -|F|^2 + \int d^2\theta W_\ast(\Phi_{\vartheta i}) + \int d^2\bar{\theta} W_\ast(\bar{\Phi}_{\vartheta i})$$

$$= -|F|^2 + \int d^2\theta \left( \frac{m_{ij}}{2} \bar{\Phi}_{\vartheta i} \ast \Phi_{\vartheta j} + \frac{\lambda_{ijk}}{3} \bar{\Phi}_{\vartheta i} \ast \Phi_{\vartheta j} \ast \Phi_{\vartheta k} \right)$$

$$+ \int d^2\bar{\theta} \left( \bar{m}_{ij} \bar{\Phi}_{\vartheta i} \ast \bar{\Phi}_{\vartheta j} + \frac{\bar{\lambda}_{ijk}}{3} \bar{\Phi}_{\vartheta i} \ast \bar{\Phi}_{\vartheta j} \ast \bar{\Phi}_{\vartheta k} \right) .$$

(5.5)
Here the masses $m_{ij}$ are symmetric in their indices, however the coupling $\lambda_{ijk}$ is not necessarily symmetric. Recalling that the square terms for the auxiliary field and the antiholomorphic part are not deformed, we can rewrite (3.13) as

$$\mathcal{L}_\text{int}^\sigma = \mathcal{L}_\text{com} + \int d^2\theta e^D \left( \frac{\lambda_{ijk}}{3} \Phi_{\bar{\nu}i}(y_1, \theta_1) \Phi_{\bar{\nu}j}(y_2, \theta_2) \Phi_{\bar{\nu}k}(y_3, \theta_3) \right) \exp \left( i \int d^4x' \mathcal{L}_\text{int}^\sigma \right) \bigg|_{y=y_1=y_2=y_3, \theta=\theta_1=\theta_2=\theta_3},$$

$$\mathcal{L}_\text{com} = -|F|^2 + \int d^2\theta \frac{m_{ij}}{2} \Phi_{\bar{\nu}i} \Phi_{\bar{\nu}j} + \int d^2\theta \left( \frac{m_{ij}}{2} \Phi_{\bar{\nu}i} \Phi_{\bar{\nu}j} + \frac{\lambda_{ijk}}{3} \Phi_{\bar{\nu}i} \Phi_{\bar{\nu}j} \Phi_{\bar{\nu}k} \right),$$

where $D$ is defined by (3.13) with $a, b = 1, 2, 3$:

$$D = -\frac{1}{2} C^{\alpha\beta}(Q_{1\alpha}Q_{2\beta} + Q_{1\alpha}Q_{3\beta} + Q_{2\alpha}Q_{3\beta}).$$

With this expression, the commutator of $\mathcal{L}_\text{int}^\sigma$ and $M_{\mu\nu}$ in Eq. (5.3) has the form

$$[\mathcal{L}_\text{int}^\sigma, M_{\mu\nu}] = [\mathcal{L}_\text{com}, M_{\mu\nu}]$$

$$+ \frac{\lambda_{ijk}}{3} \int d^2\theta e^D \left[ \Phi_{\bar{\nu}i}(y_1, \theta_1) \Phi_{\bar{\nu}j}(y_2, \theta_2) \Phi_{\bar{\nu}k}(y_3, \theta_3), M_{\mu\nu} \right] \bigg|_{y=y_1=y_2=y_3, \theta=\theta_1=\theta_2=\theta_3}$$

In order to calculate Eq. (5.9), we introduce the superspace representation of the angular momentum tensor

$$[\Phi, M_{\mu\nu}] = \mathcal{M}_{\mu\nu}\Phi,$$

$$\mathcal{M}_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) + i\theta^\alpha (\sigma^\mu\nu)_\alpha^\beta \frac{\partial}{\partial \theta^\beta} + i\bar{\theta}_\alpha (\bar{\sigma}^\mu\nu)_\alpha^\beta \frac{\partial}{\partial \bar{\theta}^\beta}. (5.11)$$

The action of the Lorentz generator on the composite object $\Phi_{\bar{\nu}i}(y_1, \theta_1) \Phi_{\bar{\nu}j}(y_2, \theta_2) \Phi_{\bar{\nu}k}(y_3, \theta_3)$ in (5.9) is given as

$$[\Phi_{\bar{\nu}i}(y_1, \theta_1) \Phi_{\bar{\nu}j}(y_2, \theta_2) \Phi_{\bar{\nu}k}(y_3, \theta_3), M_{\mu\nu}] = (\mathcal{M}^1_{\mu\nu} \Phi_{\bar{\nu}i}(y_1, \theta_1)) \Phi_{\bar{\nu}j}(y_2, \theta_2) \Phi_{\bar{\nu}k}(y_3, \theta_3)$$

$$+ \Phi_{\bar{\nu}i}(y_1, \theta_1)(\mathcal{M}^2_{\mu\nu} \Phi_{\bar{\nu}j}(y_2, \theta_2)) \Phi_{\bar{\nu}k}(y_3, \theta_3)$$

$$+ \Phi_{\bar{\nu}i}(y_1, \theta_1) \Phi_{\bar{\nu}j}(y_2, \theta_2)(\mathcal{M}^3_{\mu\nu} \Phi_{\bar{\nu}k}(y_3, \theta_3))$$

$$= 0 \quad (5.12)$$

where $\mathcal{M}^i_{\mu\nu} = M_{\mu\nu}(y_i, \theta_i, \bar{\theta}_i)$. Substituting (5.12) into (5.9), we have

$$[\mathcal{L}_\text{int}^\sigma, M_{\mu\nu}] = [\mathcal{L}_\text{com}, M_{\mu\nu}]$$

$$+ \frac{\lambda_{ijk}}{3} \int d^2\theta e^D \mathcal{M}_{\mu\nu} \left( \Phi_{\bar{\nu}i}(y_1, \theta_1) \Phi_{\bar{\nu}j}(y_2, \theta_2) \Phi_{\bar{\nu}k}(y_3, \theta_3) \right) \bigg|_{y=y_1=y_2=y_3, \theta=\theta_1=\theta_2=\theta_3}$$

$$= [\mathcal{L}_\text{com}, M_{\mu\nu}]$$
+ \frac{\lambda_{ijk}}{3} \int d^2 \theta e^D \mathcal{M}_{\mu\nu} e^{-D} e^D (\Phi_{\theta_i}(y_1, \theta_1) \Phi_{\theta_j}(y_2, \theta_2) \Phi_{\theta_k}(y_3, \theta_3)) \bigg|_{y=y_1=y_2=y_3, \theta_1=\theta_2=\theta_3} \\
= [\mathcal{L}_{\text{com}}, M_{\mu\nu}] + \frac{\lambda_{ijk}}{3} \int d^2 \theta \mathcal{M}_{\mu\nu} \Phi_{\theta_i}(y, \theta) \Phi_{\theta_j}(y, \theta) \Phi_{\theta_k}(y, \theta) \\
+ \frac{\lambda_{ijk}}{3} \int d^2 \theta[D, \mathcal{M}_{\mu\nu}] e^D (\Phi_{\theta_i}(y_1, \theta_1) \Phi_{\theta_j}(y_2, \theta_2) \Phi_{\theta_k}(y_3, \theta_3)) \bigg|_{y=y_1=y_2=y_3, \theta_1=\theta_2=\theta_3} (5.13)

The commutator in the last line is easily calculated as

$$[D, \mathcal{M}_{\mu\nu}] = -\frac{i}{2} \sum_{a<b} C^{\alpha\beta}(\sigma_{\mu\nu})_{\gamma} \left( \frac{\partial}{\partial \theta_a^\gamma} \frac{\partial}{\partial \theta_b^\alpha} + \frac{\partial}{\partial \theta_a^\alpha} \frac{\partial}{\partial \theta_b^\gamma} \right).$$

(5.14)

Note that the action of $\mathcal{M}_{\mu\nu}$ on $D$ results in changing the deformation parameter $C^{\alpha\beta}$ in $D$ into $-C^{\alpha\gamma}(\sigma_{\mu\nu})_{\beta\gamma}$ (first term in the r.h.s.) and $-C^{\beta\gamma}(\sigma_{\mu\nu})_{\alpha\gamma}$ (second term in the r.h.s.). Thus, we can represent the r.h.s. in (5.14) by an auxiliary Lorentz generator, which transforms $C^{\alpha\beta}$ as a Lorentz symmetric tensor

$$[D, \mathcal{M}_{\mu\nu}] \equiv \mathcal{M}^{\theta}_{\mu\nu} D.$$ (5.15)

To find the representation of $\mathcal{M}^{\theta}_{\mu\nu}$, we decompose $C^{\alpha\beta}$ into two independent tensors $b^{\alpha\beta}$ and $b^{\beta\alpha}$ which are not symmetric

$$C^{\alpha\beta} = b^{\alpha\beta} + b^{\beta\alpha}.$$ (5.16)

With this, (5.14) is rewritten as

$$[D, \mathcal{M}_{\mu\nu}] = -\frac{i}{2} \sum_{a<b} b^{\alpha\beta} \left\{ (\sigma_{\mu\nu})_{\beta} \gamma \left( \frac{\partial}{\partial \theta_a^\gamma} \frac{\partial}{\partial \theta_b^\alpha} - \frac{\partial}{\partial \theta_a^\alpha} \frac{\partial}{\partial \theta_b^\gamma} \right) + (\sigma_{\mu\nu})_{\alpha} \gamma \left( \frac{\partial}{\partial \theta_a^\alpha} \frac{\partial}{\partial \theta_b^\gamma} - \frac{\partial}{\partial \theta_a^\gamma} \frac{\partial}{\partial \theta_b^\alpha} \right) \right\}.$$ (5.17)

From (5.17) a representation of $\mathcal{M}^{\theta}_{\mu\nu}$ can be read off to be

$$\mathcal{M}^{\theta}_{\mu\nu} = i b^{\alpha\beta} \left( (\sigma_{\mu\nu})_{\beta} \gamma \frac{\partial}{\partial \theta^\alpha} + (\sigma_{\mu\nu})_{\alpha} \gamma \frac{\partial}{\partial \theta^\beta} \right).$$ (5.18)

With Eqs. (5.15) and (5.18), the commutator (5.4) becomes

$$[S_{\theta}, M_{\mu\nu}] = T \left[ i \int d^4 x \left( [\mathcal{L}_{\text{com}}, M_{\mu\nu}] + \frac{\lambda_{ijk}}{3} \int d^2 \theta \mathcal{M}_{\mu\nu} \Phi_{\theta_i}(y_1, \theta_1) \Phi_{\theta_j}(y_2, \theta_2) \Phi_{\theta_k}(y_3, \theta_3) \right) \bigg|_{y=y_1=y_2=y_3, \theta_1=\theta_2=\theta_3} \\
+ \frac{\lambda_{ijk}}{3} \int d^2 \theta \mathcal{M}^{\theta}_{\mu\nu} \Phi_{\theta_i}(y_1, \theta_1) \Phi_{\theta_j}(y_2, \theta_2) \Phi_{\theta_k}(y_3, \theta_3) \bigg|_{y=y_1=y_2=y_3, \theta_1=\theta_2=\theta_3} \right] \times \exp \left( i \int d^4 x' \mathcal{L}_{\text{int}}^{\theta} \right)$$ (5.19)

$$\times \exp \left( i \int d^4 x' \mathcal{L}_{\text{int}}^{\theta} \right)$$ (5.20)
Here the first two terms in Eq. (5.20) vanish since they are just surface terms. Then, taking that
\[ M_\vartheta \mu \nu \int = M_\vartheta \mu \nu \left( -|F|^2 + \int d^2 \theta W_\vartheta (\Phi_\vartheta) + \int d^2 \bar{\theta} W_\vartheta (\bar{\Phi}_\vartheta) \right) \]

\[ = \int d^2 \theta M_\vartheta \mu \nu D e^D \left( \frac{\lambda_{ijk}}{3} \Phi_{\vartheta i}(y_1, \theta_1) \Phi_{\vartheta j}(y_2, \theta_2) \Phi_{\vartheta k}(y_3, \theta_3) \right) \bigg|_{y=y_1=y_2=y_3, \theta=\theta_1=\theta_2=\theta_3} \]  

(5.21)  

into account, we arrive at the following simple form
\[ [S_\vartheta, M_\mu \nu] = M_\mu \nu S_\vartheta. \]  

(5.22)  

This expression in the r.h.s. of (5.22) represents the amount of Lorentz invariance violation of the S-matrix in the case of non-anticommutative field theory with flat Kähler potential. For single chiral superfield, one can see that Eq. (5.22) gives zero. However, for multi chiral superfields, the Lorentz violation appears. For instance, considering three chiral superfields, one has
\[ \int d^2 \theta \Phi_{\vartheta 1} * \Phi_{\vartheta 2} * \Phi_{\vartheta 3} = \phi_1 \phi_2 F_3 - \phi_1 \psi_2 \psi_3 + \phi_2 \psi_1 \psi_3 - C^{\alpha \beta} (\psi_1 \psi_3 F_2 - \psi_2 \psi_3 F_1) \]

\[ + F_1 \phi_2 \phi_3 + \phi_1 F_2 \phi_3 - \psi_1 \psi_2 \phi_3. \]  

(5.23)  

Indeed, the fourth and the fifth terms break the Lorentz invariance and such a breaking can be measured by \( M_\mu \nu. \)

Similarly, we can calculate the commutation relation between the S-matrix and the anti-supercharge, \( [S, \bar{Q}] \). In the commutative case it is zero, however in the non-anticommutative case, it is nonvanishing. Straightforward calculation leads to
\[ [S, \bar{Q}] = T \left[ i \int d^4 x [L_\text{int}^\vartheta, \bar{Q}] \exp \left( i \int d^4 x' L_\text{int}^\vartheta \right) \right], \]

(5.24)  

where
\[ \int d^4 x [L_\text{int}^\vartheta, \bar{Q}] = \frac{\lambda_{ijk}}{3} \int d^4 x d^2 \theta [D, \bar{Q}] e^D (\Phi_{\vartheta i}(y_1, \theta_1) \Phi_{\vartheta j}(y_2, \theta_2) \Phi_{\vartheta k}(y_3, \theta_3)) \bigg|_{y=y_1=y_2=y_3, \theta=\theta_1=\theta_2=\theta_3} , \]

\[ [D, \bar{Q}] = -i C^{\alpha \beta} \sum_{1 < a < b < c} \left( Q_{a \alpha} \sigma_{\beta \delta}^\mu \frac{\partial}{\partial y_\mu^a} - Q_{b \beta} \sigma_{\alpha \delta}^\mu \frac{\partial}{\partial y_\mu^a} \right). \]  

(5.25)  

Eq. (5.25) corresponds to Eq. (5.14), exhibiting the violation of the super-Poincaré invariance by calculating (5.25).

### 6 Summary and Discussions

Non-anticommutative field theory with nontrivial anticommutation relation of the Grassmann variables was introduced in [11], where it was shown also that the \( \mathcal{N} = 1 \) super-Poincaré algebra is effectively broken to \( \mathcal{N} = 1/2 \). We argue that, since this theory can be obtained through a
twist deformation, the algebra of the super-Poincaré generators survives actually undeformed, as $\mathcal{N} = 1$ supersymmetry, even if the generators are in a realization which might suggest the use of a $*$-action. Just as noncommutative field theory can be interpreted in a Lorentz invariant manner \cite{7}, the non-anticommutative supersymmetric field theory can be interpreted in a super-Poincaré invariant way. Therefore, we call this theory a twist-deformed $\mathcal{N} = 1$ non-anticommutative supersymmetric field theory. We obtain it in the four-dimensional $\mathcal{N} = 1$ Euclidean superspace from its commutative counterpart by using a simple mapping defined by $\hat{\theta}^\alpha \rightarrow \theta^\alpha + \hat{\vartheta}^\alpha$, where $\theta$ and $\hat{\vartheta}$ are the geometrical coordinate and an operator, respectively. This is a generalization of corresponding approach in noncommutative field theory \cite{18}. The quantum shift produces the star-product between (anti)chiral superfields, and also the phase factors involving $\hat{\vartheta}$ which will not appear in the action. Different phase factors appear in Kähler potential, holomorphic and anti-holomorphic parts in the superpotential. However, they are either total derivative with respect to the space-time, or give rise to a term not contributing to the $\theta$-integration. Thus, we finally obtained the non-anticommutative Lagrangian not depending on $\hat{\vartheta}$. The point is that $\hat{\vartheta}$ always appears in a product with the space-time or the Grassmann coordinate derivatives, but upon integration such derivative operators do not give any contribution.

It would be interesting to study quantum shifts corresponding to commutators $[y^\mu, y^\nu] \neq 0$ and $[y^\mu, \theta_\alpha] \neq 0$. These non-zero commutation relations together with (1.2) were derived in the string framework \cite{14} and the Wess-Zumino model on this non-anticommutative superspace was addressed in Ref. \cite{21}. Application of the quantum shift to supersymmetric theory in Minkowski non-anticommutative superspace \cite{22} is also interesting direction. Our arguments on $\mathcal{N} = 1/2$ vs. $\mathcal{N} = 1$ are also valid for $\mathcal{N} = (1, 1)$ supersymmetric field theory in non-anticommutative superspace \cite{23}. We will get twist-deformed $\mathcal{N} = (1, 1)$ non-anticommutative supersymmetric field theory through the twist deformation. We could obtain this theory by extending the map $\hat{\theta}^\alpha \rightarrow \theta^\alpha + \hat{\vartheta}^\alpha$ in deformed $\mathcal{N} = 1$ theory into one of the deformed $\mathcal{N} = (1, 1)$ case. One might consider to promote the Grassmann quantum shift operators to local ones which depend on the space-time coordinates. By considering the local quantum shift operator in noncommutative field theory as $\hat{\vartheta} \rightarrow \delta(x)$, one can expect that a theory of noncommutative gravity may emerge. Similarly, in non-anticommutative supersymmetric field theory, a local Grassmann shift operator $\hat{\vartheta}(x)$ may lead to non-anticommutative supergravity.

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