THE GAUGING PROCEDURE AND CARROLLIAN GRAVITY

JOSÉ FIGUEROA-O'FARRILL, EMIL HAVE, STEFAN PROHAZKA, AND JAKOB SALZER

Abstract. We discuss a gauging procedure that allows us to construct lagrangians that dictate the dynamics of an underlying Cartan geometry. In a sense to be made precise in the paper, the starting datum in the gauging procedure is a Klein pair corresponding to a homogeneous space. What the gauging procedure amounts to is the construction of a Cartan geometry modelled on that Klein geometry, with the gauge field defining a Cartan connection. The lagrangian itself consists of all gauge-invariant top-forms constructed from the Cartan connection and its curvature. After demonstrating that this procedure produces four-dimensional General Relativity upon gauging Minkowski spacetime, we proceed to gauge all four-dimensional maximally symmetric carrollian spaces: Carroll, (anti-)de Sitter–Carroll and the lightcone. For the first three of these spaces, our lagrangians generalise earlier first-order lagrangians. The resulting theories of carrollian gravity all take the same form, which seems to be a manifestation of model mutation at the level of the lagrangians. The odd one out, the lightcone, is not reductive and this means that although the equations of motion take the same form as in the other cases, the geometric interpretation is different. For all carrollian theories of gravity we obtain analogues of the Gauss–Bonnet, Pontryagin and Nieh–Yan topological terms, as well as two additional terms that are intrinsically carrollian and seem to have no lorentzian counterpart. Since we gauge the theories from scratch this work also provides a no-go result for the electric carrollian theory in a first-order formulation.

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1. INTRODUCTION AND SUMMARY

1.1. Introduction. The scope of this work is two-fold: we first introduce a systematic gauging procedure rooted in Cartan geometry, which we then use to construct carrollian theories of gravity for all maximally symmetric carrollian spacetimes.

It is a frequent claim in the literature that one obtains General Relativity by “gauging the Poincaré algebra”. Taking this at face value one would be forgiven for thinking that the Einstein equations are somehow derivable from the Poincaré algebra without any further choices. As we will discuss in detail in this paper, one needs in fact as an additional ingredient a preferred subalgebra, in this case the Lorentz algebra. These two ingredients form a so-called Klein pair which specifies a particular homogeneous space of the Poincaré group, namely Minkowski spacetime. What is usually known as “gauging the Poincaré algebra” is then just the construction of a Cartan geometry modelled on Minkowski spacetime.\(^1\) Of course, Minkowski spacetime is not the only homogeneous spacetime of the Poincaré group. For example, the carrollian limit of AdS, known variously as anti-de Sitter–Carroll (AdSC) [3] or, as explained in [4], also as the blow-up (Ti\(^\pm\)) of either future or past timelike infinity in Minkowski spacetime, is also a homogenous space of the Poincaré group. As we shall see in this paper, “gauging the Poincaré algebra” in this case does not lead to General Relativity, but in fact to a version of carrollian gravity.

An essential ingredient of a Cartan geometry on a manifold \(M\) modelled on a homogeneous space \(G/H\) is a Cartan connection \(A\), a Lie algebra valued 1-form on a principal \(H\)-bundle over \(M\). If the Klein pair \((g,h) = (\text{Poincaré}, \text{Lorentz})\) form a so-called Klein pair which specifies a particular homogeneous space of the Poincaré group, namely Minkowski spacetime. What is usually known as “gauging the Poincaré algebra” is then just the construction of a Cartan geometry modelled on Minkowski spacetime.\(^1\) Of course, Minkowski spacetime is not the only homogeneous spacetime of the Poincaré group. For example, the carrollian limit of AdS, known variously as anti-de Sitter–Carroll (AdSC) [3] or, as explained in [4], also as the blow-up (Ti\(^\pm\)) of either future or past timelike infinity in Minkowski spacetime, is also a homogenous space of the Poincaré group. As we shall see in this paper, “gauging the Poincaré algebra” in this case does not lead to General Relativity, but in fact to a version of carrollian gravity.

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where \( \Omega^{mn} = d\omega^{mn} + \omega^{m \gamma} \wedge \omega^{\gamma n} \). The first term is the Hilbert–Palatini lagrangian, the second term is the modification due to Holst [5] and the third term is a cosmological constant term, both with undetermined relative coefficients. This procedure also produces the Pontryagin, Gauss–Bonnet and Nieh–Yan boundary terms (see Table 7 for the explicit expressions). Setting the Holst term to zero, this is of course nothing but the first-order formulation of General Relativity.

As mentioned above, one clue as to why we must stress the importance of choosing the Klein pair \((g, h) = (\text{Poincaré}, \text{Lorentz})\) of the Poincaré algebra to reproduce (1.2) is the existence of four-dimensional homogeneous spaces of the Poincaré group other than Minkowski spacetime. One such space, \(\text{AdS}_C\), specified by the Klein pair \((g, h) = (\text{Poincaré}, \text{iso}(3))\) is not lorentzian, but rather Carrollian.\(^2\)

Carroll symmetry [6,7] arises from Lorentz symmetry in the limit where the speed of light goes to zero, as depicted in Figure 1 in terms of what this limit does to the lightcone. Carrollian physics can thus be regarded as describing an ultrarelativistic (or, perhaps more appropriately, an ultra-local) limit.

\[\Gamma(c = 1)\]

\[\Gamma(c \ll 1)\]

\[\Gamma(c = 0)\]

**Figure 1.** In the Carroll limit, the lightcones close up. In the second scenario, where \(c \ll 1\), the dynamics can be described by an expansion of General Relativity around \(c = 0\) [8]. When the lightcone closes completely, all motion ceases to exist (although the limiting behaviour of tachyons allows for motion [9]), and the spacetime becomes ultra-local.

Reasons to be interested in carrollian physics are plentiful, the most obvious one being its connection to null hypersurfaces in lorentzian spacetimes. Thus it plays a prominent rôle in the discussion of null infinity [10] (which in turn is connected to the celestial sphere [4,11,12]), black hole horizons [13] or spatial and timelike infinity [4,14]. Consequently, it has been argued to provide a natural starting point for the discussion of flat space holography, e.g., as discussed in [15] or using carrollian fluids in [16]. Carroll symmetry furthermore appears in the context of gravitational waves [17], cosmology [9], and fractons [18]. Moreover, the carrollian limit of gravity was considered in [19] as a way to describe the strong coupling limit of gravity in the vicinity of a space-like singularity; for other works on carrollian theories of gravity see [8,20,21,22,23].

\(^2\)Indeed, if the action of the Poincaré group is effective, no such spacetime could be lorentzian, since purely on dimensional grounds any Poincaré-invariant metric on a four-dimensional lorentzian manifold must be locally isometric to the Minkowski metric.
Similar to the gauging procedure for the Klein pair of Minkowski spacetime outlined above, we will construct in this work carrollian gravitational theories by gauging all four spatially isotropic maximally symmetric carrollian spacetimes. These have been classified\(^3\) in [3] and are given by the following spacetimes: Carroll C, de Sitter–Carroll dSC, anti-de Sitter–Carroll AdSC and the lightcone LC, whose Klein pairs are described in Section 2 and listed in Table 3.

Some of the novel features of our approach are:

1. The way we arrive at the lagrangians is algorithmic;
2. We cover all maximally symmetric carrollian spacetimes;
3. We construct lagrangians from scratch, rather than obtaining them from a limit; and
4. In particular, this means we provide all possible invariant terms (under the conditions we provide) for the lagrangian, including topological terms.

We will conclude this introduction by a brief overview of our results.

1.2. Summary. The first lagrangian we obtain is similar in structure to the one of Minkowski spacetime (1.2). It is given by a three-parameter family with carrollian Hilbert–Palatini, carrollian Holst and carrollian cosmological constant terms

\[ \mathcal{L}_{\text{carrollian}} = \mathcal{L}_{\text{HP}} + \beta \mathcal{L}_{\text{Holst}} + \lambda \mathcal{L}_{\text{cA}}, \]

which are given explicitly in Table 1. Note that this lagrangian depends on a parameter \( \sigma \) that can be tuned to give either Carroll, AdS Carroll or dS Carroll. That the equations of motion are the same in all three cases is a manifestation of the fact that these spaces are model mutants: Klein geometries giving rise to the same Cartan geometry. We will discuss this in Section 3 and at the end of Section 4.

| 4-form | \( \delta \) | carrollian analogue of |
|--------|--------|----------------------|
| \( \mathcal{L}_{\text{HP}} = \frac{1}{2} \epsilon_{abc} (\theta^a \wedge \theta^b \wedge \Psi^c + \xi \wedge \theta^a \wedge \Omega^{bc}) \) | | Hilbert–Palatini |
| \( \mathcal{L}_{\text{Holst}} = \frac{1}{2} \theta^a \wedge \theta^b \wedge \Omega_{ab} \) | | Holst |
| \( \mathcal{L}_{\text{cA}} = \frac{1}{6} e_{abc} \theta^a \wedge \theta^b \wedge \theta^c \wedge \xi \) | | Cosmological |
| \( \frac{1}{2} \epsilon_{abc} \psi^a \wedge \Omega^{bc} - \sigma (\mathcal{L}_{\text{HP}} + \frac{1}{3} \mathcal{L}_{\text{cA}}) \) | | Pontryagin |
| \( \frac{1}{2} \Omega^{ab} \wedge \Omega_{ab} - \frac{1}{6} \theta^a \wedge \Theta_a \) | | Gauss–Bonnet |
| \( \frac{1}{2} \epsilon_{abc} \psi^a \wedge \Omega^{bc} - \frac{1}{2} \epsilon_{abc} \theta^a \wedge \theta^b \wedge \theta^c \) | | Nich–Yan |
| \( \frac{1}{2} \epsilon_{abc} \theta^a \wedge \theta^b \wedge \Theta^c = \frac{1}{2} d (\epsilon_{abc} \theta^a \wedge \theta^b \wedge \theta^c) \) | | intrinsic |
| \( \frac{1}{2} \epsilon_{abc} \theta^a \wedge \Omega^{bc} - \frac{1}{2} d (\epsilon_{abc} \theta^a \wedge \Omega^{bc}) - \sigma \epsilon_{abc} \theta^a \wedge \theta^b \wedge \theta^c \) | | intrinsic |

**Table 1.** Summary of carrollian gauge-invariant 4-forms and boundary terms (\( \delta \)) for AdSC (\( \sigma = +1 \)), dSC (\( \sigma = -1 \)) and C (\( \sigma = 0 \)). The carrollian curvatures are given by \( \Omega^{ab} = d \omega^{ab} + \omega^a_c \wedge \omega^{cb} + \sigma \theta^a \wedge \theta^b \) and \( \Psi^a = d \phi^a + \omega^a_b \wedge \psi^b + \sigma \xi \wedge \theta^a \) and part of the carrollian torsion is given by \( \Theta^a = d \theta^a + \omega^a_b \wedge \theta^b \).

This lagrangian generalises earlier first-order lagrangian [21,22] in the following way:

1. This gauging generalises to nonzero cosmological constant. For AdSC we set \( \sigma = +1 \), while for dSC we take \( \sigma = -1 \). This lagrangian also has a well-defined flat limit \( \sigma = 0 \) upon which we arrive at the lagrangian for flat carrollian space. In each case we have

\(^3\)This generalises the seminal work of Bacry and Lévy-Leblond [24] by the addition of the lightcone, which was made possible by dropping one of the restrictions of [24].
the choice to set \( \mathcal{L}_\Lambda = 0 \). In all cases this provides the most general lagrangian with nontrivial dynamics, under the conditions we provide.

(2) We have introduced a carrollian Holst term \( \mathcal{L}_{c\text{Holst}} \) which is valid for any cosmological constant. It shares the characteristic feature of its lorentzian counterpart in that it has a nontrivial effect on the equations of motion, but in such a way that the solution space is untouched.

(3) We have derived novel topological carrollian boundary terms. They are analogous to the Pontryagin, Gauss–Bonnet and Nieh–Yan terms, with two additional terms that appear to be intrinsically carrollian, as listed in Table 1.

We remark yet again that AdS Carroll, like Minkowski, is a homogeneous space of the Poincaré group. This explicitly shows that the sentence “gauging the Poincaré algebra” is ambiguous and can lead to inequivalent lagrangians. In fact, it is not just ambiguous but indeed misleading. The phenomenon of model mutation (to be discussed below) shows that it is not the Lie algebra structure of \( \mathfrak{g} \) that is important, but the structure of \( \mathfrak{g} \) as a representation of \( \mathfrak{h} \), at least in the reductive case.

The remaining maximally symmetric carrollian space is the lightcone, for which we obtain the lagrangian

\[
\mathcal{L}_{c\Lambda} = \mathcal{L}_{c\text{HP}} + \beta \mathcal{L}_{c\text{Holst}} + \Lambda \mathcal{L}_{c\Lambda},
\]

with the explicit expressions for the terms given in Table 2. Again the Holst term has the interesting

\[
\mathcal{L}_{c\text{Holst}} = \frac{1}{2} \epsilon_{abc} \Theta^a \wedge \Theta^b - \frac{1}{2} \mathcal{L}_{c\Lambda}
\]

\[
\mathcal{L}_{c\Lambda} = \frac{1}{6} \epsilon_{abc} \theta^a \wedge \theta^b \wedge \theta^c
\]

\[
\mathcal{L}_{c\text{HP}} = \frac{1}{2} \Omega_{ab} \wedge \theta^a \wedge \theta^b + \Theta_a \wedge \theta^a \wedge \xi
\]

characteristic of leading to nontrivial equations of motion that do not restrict the solution space. The boundary terms share some similarities with their minkowskian counterparts and show that we can exchange the Holst and cosmological terms with \( \frac{1}{2} \Theta_a \wedge \Theta^a \) and \( \frac{1}{2} \epsilon_{abc} \Theta^a \wedge \theta^b \wedge \theta^c \), respectively.

While there is some freedom in the carrollian lagrangians we also show the none of them describe the electric carrollian theory [19]. Since we have gauged all maximally symmetric carrollian spacetimes from scratch this provides a novel no-go result for the construction of electric theory in a first-order formulation. See Section 9 for more details and a discussion.

This paper is organised as follows. In Section 2 we review the maximally symmetric carrollian spacetimes. In Section 3 we connect the gauging procedure to Cartan geometry and describe how one can construct gauge invariant lagrangians. We then show in Section 4 that the gauging procedure indeed recovers General Relativity when applied to Minkowski spacetime. In Section 5 we gauge (anti-)de Sitter–Carroll. In Section 6 we gauge Carroll and also look at how the action can be recovered as a limit from the (A)dS counterparts. We also provide a geometric

| 4-form | \( \delta \) | lightcone analogue of |
|--------|--------|---------------------|
| \( \mathcal{L}_{c\text{Holst}} = \frac{1}{2} \Omega_{ab} \wedge \theta^a \wedge \theta^b + \Theta_a \wedge \theta^a \wedge \xi \) | Holst |
| \( \mathcal{L}_{c\Lambda} = \frac{1}{6} \epsilon_{abc} \theta^a \wedge \theta^b \wedge \theta^c \) | Cosmological |
| \( \mathcal{L}_{c\text{HP}} = \frac{1}{2} \Omega_{ab} \wedge \theta^a \wedge \theta^b + \Theta_a \wedge \theta^a \wedge \xi \) | Pontryagin |
| \( \mathcal{L}_{c\text{Holst}} - \frac{1}{2} \Theta_a \wedge \Theta^a = - \frac{1}{2} d(\theta^a \wedge \Theta_a) \) | Gauss–Bonnet |
| \( \mathcal{L}_{c\Lambda} = \frac{1}{6} \epsilon_{abc} \theta^a \wedge \theta^b \wedge \theta^c \) | Nieh–Yan |

Table 2. Summary of lightcone gauge-invariant 4-forms and boundary terms (\( \delta \)) where \( \Omega_{ab} = d\omega_{ab} + \omega_{ac} \wedge \omega^{cb} + \psi^a \wedge \theta^b - \psi^b \wedge \theta^a \), \( \psi^a = d\psi^a + \omega_{ab} \wedge \psi^b + \xi \wedge \psi^a \), \( \Theta^a = d\theta^a + \omega^a \wedge \theta^b - \xi \wedge \theta^a \) and \( \Xi = d\xi + \psi^a \wedge \theta^a \).
interpretation (Section 7). In Section 8 we gauge the lightcone. Section 9 provides a conclusion with further remarks and an outlook.

2. The Klein geometries

Let $M$ be a smooth manifold on which a Lie group $G$ acts smoothly and transitively; that is, $M$ is a homogeneous $G$-space. From the optics of $G$, every point looks just like any other point, so we are free to choose a point $o \in M$ and declare it to be the origin. Let $H$ denote the subgroup of $G$ which fixes the origin. Then $M$ is $G$-equivariantly diffeomorphic to the space $G/H$ of left cosets $gH$ for $g \in G$, where the action of $G$ on $G/H$ is induced by left multiplication on $G$. If we let $g$ and $h$ denote, respectively, the Lie algebras of $G$ and $H$, we see that that we may associate a Klein pair $(g, h)$ to $M$. This turns out to capture the geometry of $M$ as a homogeneous space of $G$ up to coverings. More precisely, as reviewed in [3, Appendix B], there is a bijective correspondence between (effective, geometrically realisable) Klein pairs $(g, h)$ for a fixed $g$ and simply-connected homogeneous spaces of the simply-connected group $G$ with Lie algebra $g$. In this paper we will be applying the gauging procedure to the Klein pairs corresponding to spatially isotropic carrollian spacetimes. These were classified in [3] and studied further in [25] and consist of four spacetimes: Carroll (C), de Sitter Carroll (dSC), anti-de Sitter–Carroll (AdSC) — which are the carrollian limits of the maximally symmetric lorentzian spacetimes Minkowski, de Sitter and anti-de Sitter, respectively — and the lightcone (LC). They are summarised in Table 3, which lists the nonzero Lie brackets of $g = (L_{ab}, B_a, P_a, H)$ where $h = (L_{ab}, B_a)$ and where we omit those brackets involving the rotation generators which are common to all; namely,

$$
\begin{align*}
[L_{ab}, L_{cd}] &= \delta_{bc} L_{ad} - \delta_{ac} L_{bd} - \delta_{bd} L_{ac} + \delta_{ad} L_{bc} \\
[L_{ab}, B_c] &= \delta_{bc} B_a - \delta_{ac} B_b \\
[L_{ab}, P_c] &= \delta_{bc} P_a - \delta_{ac} P_b \\
[L_{ab}, H] &= 0.
\end{align*}
$$

(2.1)

The table uses by now standard abbreviated notation where we drop the vector indices, so that, e.g., $[B, P] = H$ stands for $[B_a, P_a] = \delta_{ab} H$; $[H, P] = B$ stands for $[H_a, P_a] = B_a$; and $[P, P] = L$ stands for $[P_a, P_a] = L_{ab}$.

| Name | Klein pair $(g, h)$ | Nonzero Lie brackets in addition to $[L, L] = L, [L, B] = B, [L, P] = P$ |
|------|---------------------|---------------------------------------------------------------------|
| Carroll $(c, \mathfrak{iso}(3))$ | $(B, P) = H$ | |
| de Sitter–Carroll $(\mathfrak{iso}(4), \mathfrak{iso}(3))$ | $(B, P) = H$ | $[H, P] = -B, [P, P] = -L$ |
| anti-de Sitter–Carroll $(\mathfrak{iso}(3, 1), \mathfrak{iso}(3))$ | $(B, P) = H$ | $[H, P] = B, [P, P] = L$ |
| lightcone $(\mathfrak{so}(4, 1), \mathfrak{iso}(3))$ | $(B, P) = H + L$ | $[H, P] = -P$ |

The Lie algebra $g$ is different in each case and spanned by rotations $L$, (carrollian) boosts $B$, temporal translations $H$ and spatial translations $P$. In all cases the Lie subalgebra of the Klein pair $h = \mathfrak{iso}(3)$ is spanned $L$ and $B$.

3. Cartan geometry and the gauging procedure

One approach to extracting dynamics from a Klein pair, analogous to the way that one obtains Einstein gravity departing from the Klein pair associated to Minkowski spacetime, is via the so-called gauging procedure [26, 27] (see also [20, 28, 29, 30] for the application of this method to non-lorentizan geometry). Mathematically, this is precisely the construction of a Cartan geometry, as we will now explain.
Let us fix a Klein pair \((g, \mathfrak{h})\) with \(\dim g - \dim \mathfrak{h} = n\) and let us fix a connected Lie group \(H\) with Lie algebra \(\mathfrak{h}\). If the Klein pair is locally effective, then the representation map \(\mathfrak{h} \rightarrow \mathfrak{gl}(g/\mathfrak{h})\) is injective, then we can take \(H\) to be the connected subgroup of \(\mathfrak{gl}(g/\mathfrak{h})\) generated by \(\mathfrak{h}\). More concretely, if we pick a basis \((X_1, \ldots, X_n)\) for \(g/\mathfrak{h}\), where \(X_i \in g\) and \(X_i = X_i \mod \mathfrak{h}\), then we may identify \(\mathfrak{gl}(g/\mathfrak{h})\) with \(\mathfrak{gl}(n, \mathbb{R})\), which is simply the Lie algebra of \(n \times n\) real matrices. Under this identification, every \(Y \in \mathfrak{h}\) defines an \(n \times n\) real matrix which we may exponentiate to obtain an invertible matrix \(\exp Y \in \mathfrak{gl}(n, \mathbb{R})\). We define \(H\) to be the subgroup of \(\mathfrak{gl}(n, \mathbb{R})\) consisting of finite products of such exponentials. It is clear from the definition that \(H\) is closed under products and inversion and also that it is connected, since if \(\gamma = \exp(Y_1) \exp(Y_2) \cdots \exp(Y_k) \in H\), then \(c(t) = \exp(tY_1) \exp(tY_2) \cdots \exp(tY_k)\) is a continuous curve in \(H\) with \(c(0) = 1\), the identity matrix, and \(c(1) = \gamma\).

If, in addition, the Klein pair is reductive, so that \(g = \mathfrak{h} \oplus m\) with \([\mathfrak{h}, m] \subset m\) in the obvious notation, then we may replace \(g/\mathfrak{h}\) with \(m\) in the preceding paragraph and pick the \(X_i\) to be in \(m\). We shall not assume that \((\mathfrak{g}, \mathfrak{h})\) is reductive, since one of our Carrollian examples (the lightcone LC) is not reductive.

Let \(M\) be a smooth manifold of dimension \(n = \dim g - \dim \mathfrak{h}\). The gauging procedure typically starts with a Cartan connection \(A\), which is a one-form \(A \in \Omega^1(M, g)\) in \(M\) with values in \(g\), i.e., \(A\) is a Lie algebra \(g\)-valued one-form. For \(X \in g\), we let \(\lambda \in \mathfrak{g}/\mathfrak{h}\) denote its image under the canonical surjection \(g \rightarrow \mathfrak{g}/\mathfrak{h}\). Let \(\theta := \lambda \in \Omega^1(M, \mathfrak{g}/\mathfrak{h})\) denote the projection of \(A\) to \(\mathfrak{g}/\mathfrak{h}\). The main assumption in the gauging procedure is that \(\theta\) is a coframe field; that is, what is often termed an “inverse vielbein”. The existence of such a coframe implies that the cotangent (and hence the tangent) bundle is trivialisable and this is only the case if \(M\) is parallelisable. Since most manifolds are not parallelisable, we need to restrict ourselves to an open subset \(U \subset M\) on which the cotangent bundle admits a trivialisation. We can do this about every point, so let us choose an open cover \([U_\alpha]\) of \(M\) such that the cotangent bundle is trivialisable over each \(U_\alpha\). The gauging procedure then really starts with a collection \(A_\alpha \in \Omega^1(U_\alpha, g)\) of one-forms over each \(U_\alpha\), with the assumption that the projection \(\theta_\alpha \in \Omega^1(U_\alpha, \mathfrak{g}/\mathfrak{h})\) of \(A_\alpha\) is a local coframe. In other words, if we choose a vector space complement \(m\) of \(\mathfrak{h}\) in \(g\) and a basis \(X_i\) for \(m\), then \(X_i \in g/\mathfrak{h}\) are a basis for \(g/\mathfrak{h}\). Expanding \(\theta_\alpha = \theta_\alpha^i X_i\), the one-forms \(\theta_\alpha^i \in \Omega^1(U_\alpha)\) are pointwise linearly independent everywhere on \(U_\alpha\). Each such pair \((U_\alpha, A_\alpha)\) is called a Cartan gauge in [2, §5.1].

This then prompts the question of how the one-forms \(A_\alpha\) and \(A_\beta\) are related in a non-empty overlap \(U_\alpha \cap U_\beta\). This is typically not discussed in the literature on the gauging procedure, but based on the examples at our disposal, it is reasonable to demand that on a non-empty overlap \(U_\alpha \cap U_\beta\), the one-forms \(A_\alpha\) and \(A_\beta\) should be related by an \(H\)-gauge transformation; namely,

\[
A_\beta = Ad(h_{\alpha\beta}^{-1}) A_\alpha + h_{\alpha\beta}^* \theta_1, \tag{3.1}
\]

for some smooth \(h_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow H\) and where \(\theta_1\) is the left-invariant Maurer–Cartan form on the group \(H\). In the case of matrix groups, which we may assume since in our examples \(H\) is a subgroup of \(\mathfrak{gl}(g/\mathfrak{h})\), the above relation says, explicitly, that for all \(p \in U_\alpha \cap U_\beta\),

\[
A_\beta(p) = h_{\alpha\beta}(p)^{-1} A_\alpha(p) h_{\alpha\beta}(p) + h_{\alpha\beta}(p)^{-1} dh_{\alpha\beta}(p). \tag{3.2}
\]

In the reductive case, this equation breaks up into \(\mathfrak{h}\)- and \(m\)-components. Writing \(A_\alpha = \omega_\alpha + \theta_\alpha\), with \(\omega_\alpha \in \Omega^1(U_\alpha, \mathfrak{h})\) and \(\theta_\alpha \in \Omega^1(U_\alpha, m)\), we have that equation (3.1) also splits into two:

\[
\omega_\beta = Ad(h_{\alpha\beta}^{-1}) \omega_\alpha + h_{\alpha\beta}^* \theta_1 \tag{3.3a}
\]

\[
\theta_\beta = Ad(h_{\alpha\beta}) \theta_\alpha, \tag{3.3b}
\]

from where we see that the one-forms \(\omega_\alpha \in \Omega^1(U_\alpha, \mathfrak{h})\) transform as the gauge fields corresponding to an \(H\)-connection. In the general case, where \(g = \mathfrak{h} \oplus m\) is only a vector space decomposition, the transformation law of \(\omega_\alpha\) is different and in particular it does not transform like an \(H\)-connection.
Back to the general case, we cannot split equation (3.1) into components, but we can project it to \( g/h \). The second term in the RHS is \( h \)-valued and hence only the first term survives the projection, resulting in

\[
\theta_\beta = \overline{\text{Ad}}(h_{\alpha \beta}^{-1})\theta_\alpha,
\]

where \( \overline{\text{Ad}} : H \to \text{GL}(g/h) \) is defined by

\[
\overline{\text{Ad}}(h)X = \overline{\text{Ad}}(h)X,
\]

for every \( h \in H \) and \( X \in g \).

As shown in [2, §5.2], and under the assumption that the Klein geometry is effective, condition (3.1) implies that the \( \{h_{\alpha \beta}\} \) satisfy the cocycle condition on non-empty triple overlaps; namely, for all \( p \in U_\alpha \cap U_\beta \cap U_\gamma \),

\[
h_{\alpha \beta}(p)h_{\beta \gamma}(p) = h_{\alpha \gamma}(p).
\]

It is then standard that the \( \{h_{\alpha \beta}\} \) are the transition functions of a principal (right) \( H \)-bundle \( \pi : \mathcal{P} \to \mathcal{M} \) and, furthermore, that the \( \{A_\alpha\} \) then assemble into a global one-form \( A \in \Omega^1(\mathcal{P}, g) \) on \( \mathcal{P} \) with values in \( g \) satisfying the following three-conditions:

1. (non-degeneracy) for all \( p \in \mathcal{P} \), the linear map \( A_p : T_p \mathcal{P} \rightarrow g \) is an isomorphism;
2. (equivariance) for all \( h \in H \), \( R_h^*A = \overline{\text{Ad}}(h^{-1}) \circ A \), where \( R_h : P \rightarrow P \) is the right action of \( h \in H \) on \( P \); and
3. (normalisation) for all \( X \in h \), \( A(\xi_X) = X \), where \( \xi_X \in \mathfrak{X}(\mathcal{P}) \) is the fundamental vector field associated to \( X \), whose value at \( p \in \mathcal{P} \) is given by

\[
\xi_X(p) = \frac{d}{dt}\exp(\xi_X)(p)\bigg|_{t=0}.
\]

The datum \( (\pi : \mathcal{P} \rightarrow \mathcal{M}, A) \) is a **Cartan geometry on \( \mathcal{M} \) modelled on \( (g, h) \) with group \( H \) and \( A \) is said to be a **Cartan connection**.

If the Klein geometry is not effective, so that there is a nontrivial normal subgroup of \( G \) which acts trivially on \( G/H \), then the description of a Cartan geometry via Cartan gauges is not necessarily equivalent to the bundle description. For ease of exposition we will assume that the Klein geometry is effective and in any case this holds for the examples we shall discuss in this paper.

A Cartan geometry is said to be of the **first order** if \( H \) acts faithfully on \( g/h \). This holds by construction in our set-up, since \( H \) has been defined as a subgroup of \( \text{GL}(g/h) \). As shown in [2, §5.3], a first-order Cartan geometry is an \( H \)-structure; that is, \( P \) is a sub-bundle of the frame bundle of \( \mathcal{M} \) where frames transform under local \( H \)-transformations on overlaps. If in addition, the Cartan geometry is reductive, the Cartan connection

\[
A = 0 + \omega
\]

splits into a **soldering form** \( \theta \) taking values in \( m \) and an **Ehresmann \( h \)-connection** \( \omega \) on \( \mathcal{P} \).

The Klein model of a Cartan geometry modelled on \( (g, h) \) with group \( H \) is precisely the case where \( P = G \), so that \( M = G/H \) and \( A \) is the left-invariant Maurer–Cartan form, so that it obeys the structure equation \( dA + \frac{1}{2}[A, A] = 0 \). For a general Cartan geometry, the **curvature** \( F := dA + \frac{1}{2}[A, A] \in \Omega^2(\mathcal{P}, g) \) is not necessarily zero. Indeed, flat Cartan geometries can be shown to be either (open subsets in) \( G/H \) or discrete quotients thereof; in other words, they are locally isomorphic to \( G/H \). Thus we may understand the curvature of a Cartan connection as the failure of the Cartan geometry to be locally isomorphic to the Klein model.

Two different Klein models may give rise to the same Cartan geometry. For example, Minkowski, de Sitter and anti-de Sitter spacetimes are Klein geometries whose corresponding Cartan geometry is Lorentzian geometry. The Klein pairs corresponding to these three spacetimes are of the form \( (g, h) \) where \( h \cong \mathfrak{so}(3, 1) \) in all cases: \( \{\mathfrak{so}(3, 1), \mathfrak{so}(3, 1)\} \) for Minkowski, \( \{\mathfrak{so}(4, 1), \mathfrak{so}(3, 1)\} \) for
de Sitter and \((\mathfrak{so}(3, 2), \mathfrak{so}(3, 1))\) for anti-de Sitter. Although the Lie algebras \(\mathfrak{so}(3, 1), \mathfrak{so}(4, 1)\) and \(\mathfrak{so}(3, 2)\) are certainly not isomorphic, they are isomorphic as representations of the \(\mathfrak{so}(3, 1)\) subalgebras in their respective Klein pairs. In Cartan geometry one says that these Klein geometries are related by mutation and the Klein geometries are said to be model mutants of each other. Restricting to reductive Klein geometries, model mutation gives rise to isomorphic Cartan geometries and there is a unique model mutant \(\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}\) which is isomorphic (as a Lie algebra) to a semidirect product \(\mathfrak{h} \ltimes \mathfrak{m}\) with \(\mathfrak{m}\) an abelian ideal. In the above examples of lorentzian Klein geometries, this distinguished mutant corresponds to Minkowski spacetime.

Cartan connections of model mutants are of course different, since the curvature measures the deviation of the Cartan geometry from the homogeneous model. For the three maximally lorentzian spacetimes above, the distinction is essentially the cosmological constant, which is the scalar component of the curvature in the Ricci decomposition. Something similar will happen with the carrollian limits of these lorentzian geometries: \(C\), \(dSC\) and \(AdSC\), which are all model mutants with Carroll spacetime \(C\) being the distinguished mutant.

In the reductive case, this splits as \(F = \Omega + \Theta\), where \(\Omega \in \Omega^2(\mathfrak{p}, \mathfrak{h})\) and \(\Theta \in \Omega^2(\mathfrak{p}, \mathfrak{m})\) are given by

\[
\Theta = d_\omega \theta + \frac{1}{2} [\theta, \theta]_m \quad \text{and} \quad \Omega = F_\omega + \frac{1}{2} [\theta, \theta]_h, \tag{3.9}
\]

where we used \(\mathfrak{h}\)-invariance of \(\mathfrak{m}\), and where \(F_\omega := d\omega + \frac{1}{2} [\omega, \omega]\) and \(d_\omega \theta := d\theta + [\omega, \theta]\). By \([\theta, \theta]_m\) and \([\theta, \theta]_h\) we mean the projection of \([\theta, \theta]\) onto \(\mathfrak{m}\) and \(\mathfrak{h}\), respectively. It should be remarked that even when the Cartan geometry is a reductive \(H\)-structure, \(\Omega\) need not coincide with the curvature \(F_\omega\) of the Ehresmann connection \(\omega\). In the general case, there is no split of \(F\) and we must work with the whole curvature.

The curvature satisfies the Bianchi identity \(d_A F := dF + [A, F] = 0\). This is an immediate consequence of \(d^2 = 0\) and the Jacobi identity of \(\mathfrak{g}\). In the reductive case, the Bianchi identity splits into two identities:

\[
d_\omega \Omega + [\theta, \Theta]_h = 0 \quad \text{and} \quad d_\omega \Theta + [\theta, \Omega] + [\theta, \Theta]_m = 0, \tag{3.10}
\]

where we in general define the action of \(d_\omega\) as \(d_\omega \alpha := d\alpha + [\omega, \alpha]\) and where the Lie bracket hides a wedge.

Locally on each open subset \(U_\alpha\) we have \(F_\alpha = dA_\alpha + \frac{1}{2} [A_\alpha, A_\alpha] \in \Omega^2(U_\alpha, \mathfrak{g})\), which again satisfies the Bianchi identity \(dF_\alpha + [A_\alpha, F_\alpha] = 0\). It follows from equation (3.1) that on non-empty \(U_\alpha \cap U_\beta\),

\[
F_\beta = \text{Ad}(h^{-1}_{\alpha\beta})F_\alpha = h^{-1}_{\alpha\beta}F_\alpha h_{\alpha\beta}, \tag{3.11}
\]

where the second expression holds for matrix groups.

Since both \([\theta_\alpha]\) and \([F_\alpha]\) transform linearly under \(H\), any \(n\)-form constructed out of wedge products of \(\theta_\alpha\) and \(F_\alpha\) contracted with \(H\)-invariant tensors will agree on overlaps and hence will glue to a global \(n\)-form on \(M\). This is how we will construct gauge-invariant lagrangians \(\mathcal{L} \in \Omega^n(M)\) associated to a Cartan geometry. In odd dimensions, one could also add Chern–Simons terms which are invariant under infinitesimal gauge transformations and built out of the \(\omega_\alpha\). In this paper we are interested in four-dimensional theories and hence will not need to consider Chern–Simons terms.

Varying the lagrangian \(\mathcal{L}\) with respect to \(A\) and using the Bianchi identities to eliminate derivatives on the curvatures, results in algebraic equations for the curvatures and \(\theta\) in the free differential graded commutative algebra (dgrca) generated by \(\theta\). We record here for later use the variation of the curvature

\[
\delta F = d_A \delta A = d\delta A + [A, \delta A]. \tag{3.12}
\]

In the reductive case, this splits as

\[
\delta \Omega = d_\omega \delta \omega + [\theta, \delta \theta]_h \quad \text{and} \quad \delta \Theta = d_\omega \delta \theta + [\theta, \delta \omega] + [\theta, \delta \theta]_m. \tag{3.13}
\]
We now specialise to the cases of interest, which are all in dimension $n = 4$. We seek $H$-gauge-invariant 4-forms made out of $F$ and $\theta$. Schematically we have three kinds of 4-forms, illustrated in Table 4 along with the $H$-representations they belong to. To arrive at $H$-gauge invariant 4-forms, we need to contract them with $H$-invariant tensors in the duals of the representations in the table. This will require determining the $H$-invariant tensors in those representations. Since we assume that $H$ is connected, the Lie correspondence guarantees that it is sufficient to determine the $h$-invariant tensors, which is a much easier (linear-algebraic) problem.

| 4-form | $h$-representation |
|--------|---------------------|
| $F \wedge F$ | $\odot^2 g$ |
| $F \wedge \theta \wedge \theta$ | $g \otimes \wedge^2 (g/h)$ |
| $\theta \wedge \theta \wedge \theta \wedge \theta$ | $\wedge^4 (g/h)$ |

Table 4. Four-forms (schematically) and their representations

In the reductive case, we may refine the above discussion. We now construct the lagrangian 4-form out of the one-form $\theta$ and the two-forms $\Omega$ and $\Theta$ into which the curvature splits. Schematically, the possible 4-forms we can make up out of these ingredients are given in Table 5 together with the $h$-representation they belong to. As before, the $H$-gauge invariant four-forms are constructed by contracting the terms in the table with $H$-invariant tensors in the duals of the representations in the table.

| 4-form | $h$-representation |
|--------|---------------------|
| $\Omega \wedge \Omega$ | $\odot^2 h$ |
| $\Omega \wedge \Theta$ | $h \otimes m$ |
| $\Theta \wedge \Theta$ | $\odot^2 m$ |
| $\theta \wedge \theta \wedge \Omega$ | $\wedge^2 m \otimes h$ |
| $\theta \wedge \theta \wedge \Theta$ | $\wedge^2 m \otimes m$ |
| $\theta \wedge \theta \wedge \theta \wedge \theta$ | $\wedge^4 m$ |

Table 5. Four-forms (schematically) and their representations (reductive)

The construction of the gauge-invariant lagrangian only sees how the various objects (soldering form, curvature,...) transform under $H$. Therefore provided we construct the most general gauge-invariant lagrangian, the resulting lagrangian will be formally invariant under model mutation, even though the explicit expression of the curvatures, the Bianchi identities and the variations will not be, as they depend on the explicit Lie brackets of $g$. Therefore it is perhaps not clear a priori that the resulting lagrangians are physically equivalent.

The three Klein models corresponding to Carroll ($C$), de Sitter–Carroll (dSC) and anti-de Sitter–Carroll (AdSC) are related by mutation, with $C$ being the distinguished mutant with the abelian ideal. As a check of our contention that the gauging procedure is really just constructing a Cartan geometry, we will show that the resulting carrollian gravity theories modelled on $C$, dSC and AdSC, are equivalent.

Let us now turn to the gauging method applied to the Klein pairs of (anti-) de Sitter–Carroll, flat Carroll and the lightcone, but to ease ourselves into the calculations, we will first review the classical case of Minkowski spacetime.
4. “Gauging the Poincaré algebra”

In this section we review the gauging procedure starting from the Klein pair \((g, h)\) of (four-dimensional) Minkowski spacetime, where \(g\) is the Poincaré algebra and \(h\) is a Lorentz subalgebra. This is often described as “gauging the Poincaré algebra”, but this is clearly imprecise. The Klein pair \((g, h)\) of AdSC also has \(g\) being isomorphic to the Poincaré algebra, but crucially it is \(h\) which differs. A more precise description would be that we are gauging a Klein pair \((g, h)\); although it should be clear from the description above that, if anything is being gauged, it is actually the subalgebra \(h\).

We shall choose a basis \(L_{mn} = -L_{nm}\) and \(P_m\), where \(m = 0, \ldots, 3\), for \(g\) adapted to the reductive split \(g = h \oplus m\), with \(h = \langle L_{mn} \rangle\) and \(m = \langle P_m \rangle\). In this basis, \(g\) comes with the following brackets

\[
\begin{align*}
[L_{mn}, L_{pq}] &= \eta_{np} L_{mq} - \eta_{mp} L_{nq} - \eta_{mq} L_{np} + \eta_{mp} L_{nq} \\
[L_{mn}, P_q] &= \eta_{nq} P_m - \eta_{mq} P_n \\
[P_m, P_n] &= 0,
\end{align*}
\]

(4.1)

where we take \(\eta_{mn}\) to be mostly plus. The split \(g = h \oplus m\) is not just reductive, but also symmetric \((\{m, m\} \subset h)\). We also observe that the Poincaré Lie algebra is \(Z\)-graded, with \(L_{mn}\) in degree 0 and \(P_m\) in degree -1, say, as is typical in geometric applications.

The restriction to \(h\) of the adjoint representation defines an injective map \(h \rightarrow gl(m)\) and we let \(H < GL(m)\) denote the connected Lie group the image of \(h\) generates. By a classic result of Weyl’s [31, Theorem 2.11.A], all \(H\)-invariant tensors are made out of \(\eta_{mn}\) and \(\epsilon_{mnpq}\).

Let \(\mathcal{M}\) be a four-dimensional smooth manifold. We shall work locally in an open subset \(U \subset \mathcal{M}\) with the tacit assumption that we are repeating these calculations on each member of an open cover for \(\mathcal{M}\) with gluing conditions as explained in Section 3, particularly equation (3.1). We shall let \(A = \frac{1}{4} \omega^{mn} L_{mn} + \theta^m P_m \in \Omega^1(U, g)\) and \(F = \frac{1}{4} \Omega^{mn} L_{mn} + \Theta^m P_m \in \Omega^2(U, g)\). Explicitly, we have

\[
\begin{align*}
\Omega^{mn} &= d\omega^{mn} + \omega^m \wedge \omega^n \\
\Theta^m &= d\theta^m + \omega^m \wedge \theta^n,
\end{align*}
\]

(4.2)

where indices are lowered with \(\eta_{mn}\). The Bianchi identities take the form

\[
\begin{align*}
dd \Omega^{mn} &= 0 \quad \text{and} \quad d\Theta^m = \Omega^{mn} \wedge \theta^n.
\end{align*}
\]

(4.3)

where the action of \(d\dd\) is defined by

\[
\begin{align*}
dd \alpha^{mn} := d\alpha^{mn} + \omega^m \wedge \alpha^n + \alpha^m \wedge \omega^n \wedge \alpha^m.
\end{align*}
\]

(4.4)

The space of \(h\)-invariant four-forms constructed out of \(\Omega^{mn}, \Theta^m, \theta^n\) is six-dimensional and spanned by the four-forms in Table 6, which also records their \(h\)-representation and their Lie algebraic degree relative to the grading by which \(L_{mn}\) has degree 0 and \(P_m\) has degree -1.

Under the variations of \(\omega^{mn}\) and \(\theta^m\), the curvatures vary as follows:

\[
\begin{align*}
\delta \Omega^{mn} &= d\dd \delta \omega^{mn} \quad \text{and} \quad \delta \Theta^m = d\dd \delta \theta^m + \delta \omega^{mn} \wedge \theta^n.
\end{align*}
\]

(4.5)

Under such variations, the top two terms in the table vary into exact forms upon using the Bianchi identity (4.3):

\[
\begin{align*}
\delta(\frac{1}{3} \epsilon_{mnpq} \Omega^{mn} \wedge \Omega^{pq}) &= d(\frac{1}{3} \epsilon_{mnpq} \Omega^{mn} \wedge \delta \omega^{pq}) \\
\delta(\frac{1}{4} \Omega_{mn} \wedge \Omega^{mn}) &= d(\frac{1}{2} \Omega_{mn} \wedge \delta \omega^{mn}),
\end{align*}
\]

(4.6)
This expression is $H$-gauge invariant by construction and hence gives rise to a global 4-form $\mathcal{L} \in \Omega^4(M)$, whose first term is the standard Hilbert–Palatini lagrangian, the second term is the modification due to Holst [5] and the third term is a cosmological constant term. It is interesting to note that one could replace the Holst term with $\Theta^m \wedge \Theta_m$. While this does not alter the equations of motion, it is a change of the boundary structure which might be relevant when there are boundaries at finite distance or corner symmetries.

In summary, introducing real parameters $\mu$ and $\Lambda$, the lagrangian can be taken to be

$$\mathcal{L} = \frac{1}{2} \epsilon_{mnpq} \Theta^m \wedge \Theta_n \wedge \Omega^{pq} + \frac{\mu}{2} \Theta^m \wedge \Theta_n \wedge \Omega_{mn} - \frac{\Lambda}{4} \epsilon_{mnpq} \Theta^m \wedge \Theta_n \wedge \Theta^p \wedge \Theta^q. \tag{4.8}$$

and hence neither of these two terms contributes to the Euler–Lagrange equations. Similarly, from the variations of the middle two terms:

$$\delta(\frac{1}{2} \Theta^m \wedge \Theta_m) = d(\Theta_m \wedge \delta \Theta^m) + \Omega_{mn} \wedge \Theta^m \wedge \delta \Theta^n - \Theta^m \wedge \Theta_n \wedge \delta \omega_{mn}$$

and hence neither of these two terms contributes to the Euler–Lagrange equations. Similarly, from the variations of the middle two terms:

$$\delta(\frac{1}{2} \Omega_{mn} \wedge \Theta^m \wedge \Theta^n) = d(\frac{1}{2} \delta \omega_{mn} \wedge \Theta^m \wedge \Theta^n) - \Theta^m \wedge \Theta_n \wedge \delta \omega_{mn} + \Omega_{mn} \wedge \Theta^m \wedge \delta \Theta^n, \tag{4.7}$$

we see that their difference varies into an exact form and hence we need only keep one of them in the lagrangian. Explicitly, if two 4-forms $X, Y \in \Omega^4(M)$ are such that their difference varies into an exact form, $\delta(X - Y) = dZ$ for some $Z \in \Omega^3(M)$, then the equations of motion obtained from the lagrangian $\mathcal{L} = aX + bY$ are the same as those obtained from $\mathcal{L}' = cX$, where $a, b$ and $c$ are real parameters. To see this, simply write $\mathcal{L} = (a + b)X + b(Y - X)$, so that after identifying $c = a + b$, we get $\delta(\mathcal{L} - \mathcal{L}') = dZ$. We have summarised the terms with nontrivial and trivial variation in Table 7.

| 4-form                                      | $\delta$         | $\mathcal{L}$   |
|--------------------------------------------|------------------|-----------------|
| $\frac{1}{2} \epsilon_{mnpq} \Theta^m \wedge \Theta_n \wedge \Theta^p \wedge \Theta^q$ | $\checkmark$ Pontryagin | Hilbert–Palatini |
| $\frac{1}{2} \epsilon_{mnpq} \Theta^m \wedge \Theta_n \wedge \Theta^p \wedge \Theta^q$ | $\checkmark$ Gauss–Bonnet | Cosmological    |
| $\frac{1}{2} \epsilon_{mnpq} \Theta^m \wedge \Theta_n \wedge \Omega^{pq}$     |                 | Holst           |
| $\frac{1}{2} \Omega_{mn} \wedge \Theta^m \wedge \Theta_n$ or $\frac{1}{2} \Theta_m \wedge \Theta^m$ |                 |                 |
| $\frac{1}{2} \epsilon_{mnpq} \Omega^{mn} \wedge \Omega^{pq}$                 |                 |                 |
| $\frac{1}{2} (\Theta_m \wedge \Theta^m - \Omega_{mn} \wedge \Theta^m \wedge \Theta_n) = \frac{1}{2} d(\Theta^m \wedge \Theta_m)$ | $\checkmark$ |                 |

Table 7. Summary of gauge-invariant 4-forms and boundary terms ($\delta$). The Nieh–Yan term shows that we could replace the Holst term by $\Theta^m \wedge \Theta_m$. 

In summary, introducing real parameters $\mu$ and $\Lambda$, the lagrangian can be taken to be

$$\mathcal{L} = \frac{1}{2} \epsilon_{mnpq} \Theta^m \wedge \Theta_n \wedge \Omega^{pq} + \frac{\mu}{2} \Theta^m \wedge \Theta_n \wedge \Omega_{mn} - \frac{\Lambda}{4} \epsilon_{mnpq} \Theta^m \wedge \Theta_n \wedge \Theta^p \wedge \Theta^q. \tag{4.8}$$

This expression is $H$-gauge invariant by construction and hence gives rise to a global 4-form $\mathcal{L} \in \Omega^4(M)$, whose first term is the standard Hilbert–Palatini lagrangian, the second term is the modification due to Holst [5] and the third term is a cosmological constant term. It is interesting to note that one could replace the Holst term with $\Theta^m \wedge \Theta_m$. While this does not alter the equations of motion, it is a change of the boundary structure which might be relevant when there are boundaries at finite distance or corner symmetries.

| 4-form                                      | degree | $\mathfrak{h}$-representation |
|--------------------------------------------|--------|-------------------------------|
| $\frac{1}{2} \epsilon_{mnpq} \Omega^{mn} \wedge \Omega^{pq}$ | 0      | $\mathfrak{h}^2 \mathfrak{h}$ |
| $\frac{1}{2} \Omega_{mn} \wedge \Omega^{mn}$   | 0      | $\mathfrak{h}^2 \mathfrak{h}$ |
| $\frac{1}{2} \Theta_m \wedge \Theta^m$     | 2      | $\mathfrak{h}^2 \mathfrak{m}$ |
| $\frac{1}{2} \Omega_{mn} \wedge \Theta^m \wedge \Theta^n$ | 2      | $\mathfrak{h} \otimes \mathfrak{m}^2$ |
| $\frac{1}{2} \epsilon_{mnpq} \Omega^{mn} \wedge \Theta^p \wedge \Theta^q$ | 2      | $\mathfrak{h} \otimes \mathfrak{m}^2$ |
| $\frac{1}{2} \epsilon_{mnpq} \Theta^m \wedge \Theta^n \wedge \Theta^p \wedge \Theta^q$ | 4      | $\mathfrak{m}^4$            |

Table 6. Gauge-invariant 4-forms
The variation of this lagrangian gives rise to two equations:
\[ \frac{1}{2} \epsilon_{mnpq} \Theta^p \wedge \Theta^q = -\frac{\beta}{4} (\Theta_m \wedge \Theta_n - \Theta_n \wedge \Theta_m) \] (4.9a)
\[ \frac{1}{2} \epsilon_{mnpq} \Omega^{mp} = \beta \Omega_{qp} \wedge \Theta^p + \frac{\Lambda}{8} \epsilon_{mnpq} \Theta^m \wedge \Theta^n \wedge \Theta^p. \] (4.9b)

**Lemma 1.** Equation (4.9a) implies that \( \Theta^m = 0 \) for any (real) value of \( \beta \).

**Proof.** Consider \( \Psi := \frac{1}{2} (\Theta^m \wedge \Theta^n - \Theta^n \wedge \Theta^m) P_m \wedge P_n \in \Omega^2 (U, \wedge^2 \mathfrak{m}) \). Let \( \ast : \wedge^2 \mathfrak{m} \to \wedge^2 \mathfrak{m} \) be the internal Hodge star, defined by
\[ (\ast \Phi)^{mn} = \frac{1}{2} \epsilon^{mnpq} \Phi_{pq}, \] (4.10)
for any \( \wedge^2 \mathfrak{m} \)-valued object \( \Phi \). It is evident that \( \ast \) operator involves the inner product on \( \mathfrak{m} \) and since this is lorentzian, we have that \( \ast^2 = -\text{id}_{\wedge^2 \mathfrak{m}} \). Now equation (4.9a) says that \( \ast \Psi = \beta \Psi \) and applying \( \ast \) again we see that \( (\beta^2 + 1) \Psi = 0 \). Hence if \( \beta \) is real, the only solution is \( \Psi = 0 \), which is equivalent to
\[ \epsilon_{mnpq} \Theta^p \wedge \Theta^q = 0. \] (4.11)
We may expand \( \Theta^p = \frac{1}{2} \Theta^p \wedge \Theta^q \wedge \Theta^s \) and hence this becomes
\[ \epsilon_{mnpq} \Theta^p_{rs} \wedge \Theta^q \wedge \Theta^s = 0, \] (4.12)
or equivalently
\[ \epsilon_{mnpq} \Theta^p_{rs} = 0. \]
Contracting with \( \epsilon^{rsq} \) and using the standard identity
\[ \epsilon^{rsq} \epsilon_{mnpq} = 6 \delta^{[r}_{[m} \delta^s_{p]} \delta^q_{n]}, \] (4.13)
we arrive at
\[ \delta^s_{n} \Theta^p_{nq} + \delta^q_{m} \Theta^p_{qm} + \delta^q_{n} \Theta^p_{mn} = 0. \]
Tracing with \( \delta^s_{n} \), we see that \( \Theta^p_{nq} = 0 \), proving the lemma.

Since \( \Theta = 0 \), the second of the Bianchi identities in (4.3) becomes \( \Omega_{mn} \wedge \Theta^m = 0 \), which kills the \( \beta \)-dependent term in equation (4.9b). Let us write
\[ \Omega_{mn} = -\frac{1}{2} \mathcal{R}_{mnpq} \Theta^p \wedge \Theta^q \] (4.14)
and insert this into equation (4.9b) to obtain
\[ \frac{1}{2} \mathcal{R}_{mnpq} m_{pq} + \mathcal{R}_{mnpq} n_{pq} + \Lambda n_{pq} = 0. \] (4.15)
Let us define the Ricci tensor \( R_{pq} := \eta^{mn} \mathcal{R}_{mnpq} \) and the Ricci scalar \( R = \eta^{pq} R_{pq} \), in terms of which the above equation becomes
\[ \frac{1}{2} R_{pq} - R_{qp} - \Lambda n_{pq} = 0. \] (4.16)
Tracing with \( \eta^{pq} \), we see that \( R = 4 \Lambda \) and re-inserting this into equation (4.16) we arrive at
\[ R_{pq} = \Lambda n_{pq}. \] (4.17)

By construction, the Cartan geometry under discussion, being of the first order, is an H-structure. Since \( H \) leaves invariant a lorentzian inner product on \( \mathfrak{m} \), we get a lorentzian metric on \( M \) which has local expression \( g = \eta_{mn} \Theta^m \Theta^n \) which allows us to identify the bundle \( \mathcal{P} \to M \) as the bundle of oriented (and time-oriented) orthonormal frames. Being reductive, the connection \( \omega \) takes values in \( \mathfrak{h} \) and hence it is metric-compatible and Lemma 1 says that it is torsion-free, hence it is essentially the Levi-Civita connection and \( \mathcal{R}_{mnpq} \) are the coefficients of the Riemann tensor relative to the local orthonormal frame \( \Theta^m \), with \( \mathcal{R}_{mn} \) and \( \mathcal{R} \) the standard Ricci tensor and Ricci scalar, so that equation (4.17) is indeed the vacuum Einstein equation with a cosmological constant.

Let us point out that by tuning the cosmological constant, we can arrange for Minkowski, de Sitter or anti-de Sitter spacetimes to solve the Euler–Lagrange equations. If we didn’t have the
cosmological constant term, only Minkowski spacetime would solve them. In order to find de Sitter or anti-de Sitter spacetimes as solutions, we would have to build a Cartan geometry modelled on these spacetimes. In [1] it is shown that doing so one recovers MacDowell–Mansouri gravity. It is not uncommon to impose on the gauge-invariant lagrangians the condition that the Klein model should be a solution. In the case of the models being the maximally symmetric lorentzian manifolds, the difference between their curvature tensors is just the scalar component in the Ricci decomposition and this precisely corresponds to the cosmological constant term. Therefore by tuning the cosmological constant parameter in the lagrangian we can obtain Minkowski, de Sitter or anti-de Sitter spacetimes (with any radius of curvature) as solutions. It is not clear to us whether this holds for more general model mutations; that is, whether there is a modification of the lagrangian which can exhibit any model mutant as a solution of its Euler–Lagrange equation.

5. Gauging the Klein pairs of dSC and AdSC

Let us apply the gauging procedure to dSC and AdSC. We will treat both cases simultaneously since, as we shall see, mutatis mutandis, the Euler–Lagrange equations take the same form. The Klein pairs $so(3,1)$ and $so(4)$ are described in Table 3: $so(3,1)$, the euclidean Lie algebra, whereas $so(4)$, the Poincaré Lie algebra. We can introduce a sign $\sigma = \pm 1$ and write the nonzero Lie brackets of both Lie algebras as follows:

$$
\begin{align*}
[L_{ab}, L_{cd}] &= \delta_{bc}L_{ad} - \delta_{ac}L_{bd} - \delta_{bd}L_{ac} + \delta_{ad}L_{bc} \\
[L_{ab}, B_c] &= \delta_{bc}B_a - \delta_{ac}B_b \\
[L_{ab}, P_c] &= \delta_{bc}P_a - \delta_{ac}P_b \\
[B_a, P_b] &= \delta_{ab}H \\
[H, P_a] &= \sigma B_a \\
[P_a, P_b] &= \sigma L_{ab},
\end{align*}
$$

(5.1)

where $\sigma = +1$ for AdSC and $\sigma = -1$ for dSC. Taking $\sigma = 0$ we arrive at the Carroll algebra for $C$, but in this section we will assume that $\sigma \neq 0$.

In both of these cases, the connection $A$ will turn out to be a Cartan connection in a first-order, reductive Cartan geometry. Therefore we could arrive at equivalent equations departing from any other Klein model for the carrollian Cartan geometry. There is a unique such model in which $g = h \times m$, with $m$ abelian. The resulting Klein pair $(h, m)$ describes the Carroll spacetime, since $h \times m \cong \epsilon$, the Carroll algebra. The resulting calculations are technically simpler and, formally, they can be obtained from ours by setting $\sigma = 0$. We will discuss this case separately in Section 6.

5.1. The gauge fields. We consider an open subset $U \subset M$ of a four-dimensional smooth manifold $M$ and introduce a one-form $A \in \Omega^1(U, g)$ with values in $g$ which, relative to the above basis, can be expanded as

$$
A = \frac{1}{2} \omega^{ab} L_{ab} + \psi^a B_a + \Theta^a P_a + \xi H.
$$

(5.2)

Its curvature $\mathcal{F} \in \Omega^2(U, g)$ is given by

$$
\mathcal{F} = dA + \frac{1}{2} [A, A] = \frac{1}{2} \Omega^{ab} L_{ab} + \psi^a B_a + \Theta^a P_a + \Xi H,
$$

(5.3)

where

$$
\begin{align*}
\Omega^{ab} &= d\omega^{ab} + \omega^a \wedge \omega^b + \sigma \theta^a \wedge \theta^b = F^{ab} + \sigma \theta^a \wedge \theta^b \\
\psi^a &= d\psi^a + \omega^a \wedge \psi^b + \sigma \xi \wedge \theta^a = d\mathcal{V}^{a} + \sigma \xi \wedge \theta^a \\
\Theta^a &= d\Theta^a + \omega^a \wedge \theta^b = d\mathcal{V}^{a} \theta^b \\
\Xi &= d\xi + \psi^a \wedge \theta_a,
\end{align*}
$$

(5.4)
where \( dV \) is the \( \tau \)-covariant exterior derivative and \( F \) its curvature two-form. Indices have been lowered with \( \delta_{ab} \). The Bianchi identity \( df + [A, f] = 0 \) also splits as follows:

\[
\begin{align*}
  dV \Omega^{ab} &= \sigma(\Theta^a \land \Theta^b - \Theta^b \land \Theta^a) \\
  dV \psi^a &= \Omega^a_b \land \psi^b + \sigma\theta^a \land \Xi - \sigma\xi \land \Theta^a \\
  dV \Theta^a &= \Omega^a_b \land \Theta^b \\
  d\Xi &= \Theta^a \land \psi^a \land \Theta_a,
\end{align*}
\]

where \( dV \Omega^{ab} = d\Omega^{ab} + \omega^a c \land \Omega^{cb} - \omega^b c \land \Omega^{ca} \).

We now turn to their determination.

5.2. Invariant tensors. The duals of the \( h \)-representations appearing in Table 5 are subspaces of the tensor algebra of \( \mathfrak{g}^* \). The basis \( (\mathfrak{L}_{ab}, B_a, P_a, H) \) for \( \mathfrak{g} \) induces a canonically dual basis \( \{\lambda^{ab}, \beta^a, \pi^a, \eta\} \) for \( \mathfrak{g}^* \) satisfying

\[
\langle \mathfrak{L}_{ab}, \lambda^{cd} \rangle = \delta^d_b \delta^c_a - \delta^c_b \delta^d_a, \quad \langle B_a, \beta^b \rangle = \delta^b_a, \quad \langle P_a, \pi^b \rangle = \delta^b_a, \quad \langle H, \eta \rangle = 1. \quad (5.6)
\]

It is relatively easy to determine the \( h \)-invariant tensors in \( \otimes^2 h^*, \otimes^2 m^*, h^* \otimes m^*, \land m^* \otimes h^*, \land m^* \otimes m^* \) and \( \land^2 m^* \). We first write down the \( \tau \)-invariant tensors, which are made out of \( \delta_{ab} \) and \( \epsilon_{abc} \), and then we check which linear combinations of \( \tau \)-invariant tensors are also invariant under the coadjoint action of the \( B_a \). This action is given by \( B_a \cdot \alpha = -\alpha \circ ad_{B_a} \) for any \( \alpha \in \mathfrak{g}^* \). We find

\[
B_a \cdot \lambda^{bc} = 0, \quad B_a \cdot \beta^b = \lambda^b_a, \quad B_a \cdot \pi^b = 0 \quad \text{and} \quad B_a \cdot \eta = -\pi_a. \quad (5.7)
\]

Let us go through each of the representations in Table 5 in turn. Firstly, \( \land m^* \) is one-dimensional and spanned by \( 12 \epsilon_{abc} \Theta^a \land \Theta^b \land \Theta^c \land \eta \), which is manifestly \( h \)-invariant. This tensor contributes a term \( 12 \epsilon_{abc} \Theta^a \land \Theta^b \land \Theta^c \land \xi \) to the lagrangian.

There are two \( \tau \)-invariant tensors in \( \land^2 m^* \otimes m^* \):

\[
\frac{1}{2} \epsilon_{abc} \Theta^a \land \Theta^b \land \Theta^c \quad \text{and} \quad \delta_{ab} \eta \land \pi^a \land \pi^b. \quad (5.8)
\]

The first term is invariant under the coadjoint action of \( B_a \), whereas the second term is not. The corresponding term in the lagrangian is \( 12 \epsilon_{abc} \Theta^a \land \Theta^b \land \Theta^c \). There are four \( \tau \)-invariant tensors in \( \land^2 m^* \otimes h^* \):

\[
\frac{1}{2} \pi^a \land \pi^b \otimes \lambda_{ab}, \quad \frac{1}{2} \epsilon_{abc} \pi^a \land \pi^b \otimes \beta^c, \quad \eta \land \pi^a \land \beta_a \quad \text{and} \quad \frac{1}{2} \epsilon_{abc} \eta \land \pi^a \land \lambda^{bc}. \quad (5.9)
\]

where we have lowered some indices using \( \delta_{ab} \). The first term is clearly invariant under \( B_a \) and there is a combination of two of the remaining terms which is also invariant, namely \( \frac{1}{2} \epsilon_{abc} (\pi^a \land \pi^b \land \beta^c + \eta \land \pi^a \land \lambda^{bc}) \), while the penultimate term in the list above is not invariant. These invariant tensors contribute the following terms to the lagrangian \( 12 \epsilon_{abc} \Theta^a \land \Theta^b \land \Omega_{ab} \) and \( \frac{1}{2} \epsilon_{abc} (\Theta^a \land \Theta^b \land \Psi + \lambda^{a} \land \Omega^{bc}) \), respectively.

There are two \( \tau \)-invariant tensors in \( \otimes^2 m^* \): \( \pi_a \pi^b \) and \( \eta^2 \). The former is also invariant under \( B_a \), whereas the latter is not. This representation contributes one term to the lagrangian, namely \( \frac{1}{2} \Theta^a \land \Theta_a \).

There are two \( \tau \)-invariant tensors in \( m^* \otimes h^* \), namely \( \frac{1}{2} \epsilon_{abc} \pi^a \land \lambda^{bc} \) and \( \pi^a \land \beta_a \). The former is invariant under \( B_a \), whereas the latter is not. In summary, we have one possible new term in the lagrangian, namely \( \frac{1}{2} \epsilon_{abc} \Theta^a \land \Omega_{abc} \).

Finally, there are three \( \tau \)-invariant tensors in \( \otimes^2 h^* \):

\[
\frac{1}{2} \lambda^{ab} \lambda_{ab}, \quad \beta^a \beta_a \quad \text{and} \quad \frac{1}{2} \epsilon_{ab} b^{b} \lambda^{bc}. \quad (5.10)
\]
Of these, only the first and last are also invariant under $B_a$, contributing two terms to the lagrangian: $\frac{1}{2} \Omega_{ab} \wedge \Omega_{ab}$ and $\frac{1}{2} \epsilon_{abc} \psi^a \wedge \Omega^{bc}$.

Table 8 summarises the above discussion, where we also record the degree relative to the grading described above of $g$. Recall that $L_{ab}$ and $P_a$ have zero degree, whereas $B_a$ and $H$ have degree $\alpha$. Dually, $\lambda^{ab}$, $\pi^a$ have degree 0 whereas $\beta^a$ and $\eta$ have degree $-\alpha$.

| Invariant tensor | 4-form | Degree |
|------------------|--------|--------|
| $\frac{1}{2} \epsilon_{abc} \pi^a \wedge \pi^b \wedge \pi^c \wedge \eta$ | $\frac{1}{6} \epsilon_{abc} \theta^a \wedge \theta^b \wedge \theta^c \wedge \xi$ | $-\alpha$ |
| $\frac{1}{2} \epsilon_{abc} \pi^a \wedge \pi^b \wedge \pi^c$ | $\frac{1}{2} \epsilon_{abc} \theta^a \wedge \theta^b \wedge \theta^c$ | 0 |
| $\pi^a \wedge \pi^b \wedge \lambda_{ab}$ | $\frac{1}{2} \theta^a \wedge \theta^b \wedge \Omega_{ab}$ | 0 |
| $\frac{1}{2} \epsilon_{abc} (\pi^a \wedge \pi^b \wedge \beta^c + \eta \wedge \pi^a \wedge \lambda^{bc})$ | $\frac{1}{2} \epsilon_{abc} (\theta^a \wedge \theta^b \wedge \psi^c + \xi \wedge \theta^a \wedge \Omega^{bc})$ | $-\alpha$ |
| $\pi^a \pi_a$ | $\frac{1}{2} \Theta^a \wedge \Theta_a$ | 0 |
| $\frac{1}{2} \epsilon_{abc} \pi^a \wedge \lambda^{bc}$ | $\frac{1}{2} \epsilon_{abc} \Theta^a \wedge \Omega^{bc}$ | 0 |
| $\frac{1}{2} \lambda^{ab} \lambda_{ab}$ | $\frac{1}{2} \Omega^{ab} \wedge \Omega_{ab}$ | 0 |
| $\frac{1}{2} \epsilon_{abc} \beta^a \lambda^{bc}$ | $\frac{1}{2} \epsilon_{abc} \psi^a \wedge \Omega^{bc}$ | $-\alpha$ |

**Table 8.** $h$-invariant tensors and the corresponding gauge-invariant 4-forms

### 5.3. Variations

We now vary the gauge-invariant 4-forms in Table 8 with respect to $\omega^{ab}$, $\psi^a$, $\Theta^a$, and $\xi$. We shall find that upon application of the Bianchi identities (5.5) that several of the terms vary into exact forms and hence do not contribute to the Euler–Lagrange equations in the absence of a boundary. The variations of the curvatures can be found from the expressions in (3.13) (or by direct variation of the curvatures in (5.4)). One finds the following

$$
\delta \Omega^{ab} = d^v \delta \omega^{ab} + \sigma (\theta^a \wedge \delta \theta^b - \theta^b \wedge \delta \theta^a)
$$

$$
\delta \psi^a = d^v \delta \psi^a + \delta \omega^{ab} \wedge \psi^b + \sigma (\xi \wedge \delta \theta^a - \theta^a \wedge \delta \xi)
$$

$$
\delta \Theta^a = d^v \delta \theta^a + \delta \omega^{ab} \wedge \theta^b
$$

$$
\delta \Xi = d \delta \xi + \psi_a \wedge \delta \theta^a - \theta^a \wedge \delta \psi_a,
$$

where the last line is for completeness, since $\Xi$ does not appear in the lagrangian. Since $\xi$ must therefore appear linearly, we may think of it as a Lagrange multiplier.

We now vary the gauge-invariant 4-forms in turn, starting with those of degree 0 relative to the Lie algebra grading. We do one such calculation in detail to illustrate the method and list the results of the other calculations. Let us consider the variation of $\frac{1}{4} \Omega^{ab} \wedge \Omega_{ab}$:

$$
\delta \left( \frac{1}{4} \Omega^{ab} \wedge \Omega_{ab} \right) = \frac{1}{2} \Omega_{ab} \wedge \delta \Omega^{ab}
$$

$$
= \frac{1}{2} \Omega_{ab} \wedge (d^v \delta \omega^{ab} + 2 \sigma \theta^a \wedge \delta \theta^b)
$$

$$
= d\left( \frac{1}{2} \Omega_{ab} \wedge \delta \omega^{ab} \right) - \frac{1}{2} d^v \Omega_{ab} \wedge \delta \omega^{ab} + \sigma \Omega_{ab} \wedge \theta^a \wedge \delta \theta^b
$$

$$
= d\left( \frac{1}{2} \Omega_{ab} \wedge \delta \omega^{ab} \right) + \sigma \theta_a \wedge \Theta_b \wedge \delta \omega^{ab} + \sigma \Omega_{ab} \wedge \theta^a \wedge \delta \theta^b,
$$

where in the second line we have used

$$
d(\Omega_{ab} \wedge \delta \omega^{ab}) = d^v \omega_{ab} \wedge \Omega_{ab} \wedge d^v \delta \omega^{ab}
$$

and in the third line we have used the first Bianchi identity of (5.5).
In the same way we find the variations of all the gauge-invariant 4-forms in Table 8 of zero Lie algebra degree:

\[ \delta(\frac{1}{2} \Omega^{ab} \wedge \Omega_{ab}) = d(\frac{1}{2} \Omega_{ab} \wedge \delta \omega^{ab}) + \sigma(\Theta_a \wedge \Theta_b \wedge \delta \omega^{ab} + \Omega_{ab} \wedge \Theta^a \wedge \Theta^b) \]

\[ \delta(\frac{1}{2} \Theta^a \wedge \Theta_a) = d(\Theta_a \wedge \delta \Theta^a) + \Omega_{ab} \wedge \Theta^a \wedge \delta \omega^{ab} - \Theta_a \wedge \Theta_b \wedge \delta \omega^{ab} \]

\[ \delta(\frac{1}{2} \epsilon_{abc} \Theta^a \wedge \Omega^{bc}) = d(\frac{1}{2} \epsilon_{abc} (\delta \Theta^a \wedge \Omega^{bc} + \Theta^a \wedge \delta \omega^{bc}) + \Omega_{ab} \wedge \Theta^a \wedge \delta \omega^{bc} - \Theta_a \wedge \Theta_b \wedge \delta \omega^{ab}) \]

\[ \delta(\frac{1}{2} \epsilon_{abc} \Theta^a \wedge \Theta^b \wedge \Theta^c) = d(\frac{1}{2} \epsilon_{abc} (\delta \Theta^a \wedge \Theta^b \wedge \Theta^c) + \Omega_{ab} \wedge \Theta^a \wedge \delta \omega^{bc} - \Theta_a \wedge \Theta_b \wedge \delta \omega^{ab}) \]

In the middle variation, we have used that

\[ \frac{1}{2} \omega^{ad} \wedge (\epsilon_{abcd} \epsilon_c - \epsilon_{abc} \epsilon_d) \wedge \delta \omega^{bc} = 0 \]

and elsewhere we have used that any totally antisymmetric quantity \( A^{abc} \) in three dimensions can be written as \( A^{abc} = \epsilon^{abc} A_{123} \). We see from this that the following 4-forms vary into exact forms:

\[ \frac{1}{2} \epsilon_{abc} \Theta^a \wedge \Omega^{bc} \wedge \Omega_{ab} - \frac{1}{2} \epsilon_{abc} \Theta^b \wedge \Theta^c \wedge \Theta_a - \frac{1}{2} \epsilon_{abc} \Theta^c \wedge \Theta^a \wedge \Theta_b \]

In the same way we find the variations of all the gauge-invariant 4-forms in Table 8 of zero Lie algebra degree in the following:

\[ \delta(\frac{1}{2} \epsilon_{abc} \Theta^a \wedge \Theta^b \wedge \Theta^c \wedge \xi) = -\frac{1}{2} \epsilon_{abc} \xi \wedge \Theta^a \wedge \Theta^b \wedge \delta \Theta^c + \frac{1}{2} \epsilon_{abcd} \Theta^a \wedge \Theta^b \wedge \Theta^c \wedge \delta \xi, \quad (5.15) \]

\[ \delta(\frac{1}{2} \epsilon_{abc} \psi^{a} \wedge \Omega^{bc}) = d \left( \frac{1}{2} \epsilon_{abc} \left( \delta \psi^a \wedge \Omega^{bc} + \psi^a \wedge \delta \omega^{bc} \right) + \frac{1}{2} \epsilon_{abc} \left( \xi \wedge \Omega^{ab} + 2 \psi^a \wedge \Theta^b \right) \wedge \delta \Theta^c \right) - \frac{1}{2} \epsilon_{abc} \left( \Theta^a \wedge \Xi \wedge \Theta^c \wedge \delta \Theta^a \right) \wedge \delta \xi \quad (5.16) \]

and

\[ \delta(\frac{1}{2} \epsilon_{abc} \left( \xi \wedge \Theta^a \wedge \Xi \wedge \Theta^c \wedge \delta \omega^{bc} \right) + \epsilon_{abc} \left( \xi \wedge \Theta^a \wedge \Xi \wedge \Theta^c \wedge \delta \omega^{bc} \right) + \frac{1}{2} \epsilon_{abc} \left( \xi \wedge \Theta^a \wedge \Xi \wedge \Theta^c \wedge \delta \omega^{bc} \right) + \frac{1}{2} \epsilon_{abc} \left( \xi \wedge \Theta^a \wedge \Xi \wedge \Theta^c \wedge \delta \omega^{bc} \right) \]

We notice that the variation of the combination

\[ \frac{1}{2} \epsilon_{abc} \psi^a \wedge \Omega^{bc} - \frac{1}{2} \epsilon_{abc} \left( \Theta^a \wedge \Theta^b \wedge \psi^c + \xi \wedge \Theta^a \wedge \Omega^{bc} \right) - \frac{1}{2} \epsilon_{abc} \left( \Theta^a \wedge \psi^b \wedge \psi^c \wedge \xi \right) \]

is exact form and hence it is not necessary to include all three terms in the lagrangian.

We have summarised these findings in Table 1.

5.4 Euler–Lagrange equations. Let us introduce real parameters \( \beta, \mu, \Lambda \) and let us consider the following gauge-invariant lagrangian 4-form:

\[ \mathcal{L} = \frac{1}{2} \epsilon_{abc} \left( \Theta^a \wedge \Theta^b \wedge \psi^c + \xi \wedge \Theta^a \wedge \Omega^{bc} \right) + \frac{1}{2} \Theta^a \wedge \Theta^b \wedge \Theta^c \wedge \Theta_a + \frac{1}{2} \epsilon_{abc} \Theta^a \wedge \Theta^b \wedge \Theta^c \wedge \xi \]

The first term is the carrollian analog of the Hilbert–Palatini term, the second can be understood as a carrollian Holst term, while the third is a carrollian cosmological constant term, as listed in (4.8).

Its variation results in an expression of the form

\[ \delta \mathcal{L} = \mathcal{E} \wedge \delta \xi + \mathcal{F}_c \wedge \delta \Theta^c + \mathcal{G}_c \wedge \delta \psi^c + \frac{1}{2} \mathcal{H}_{bc} \wedge \delta \omega^{bc} + d \mathcal{Y}, \quad (5.20) \]

where

\[ \mathcal{Y} = \frac{1}{2} \Theta^a \wedge \Theta^b \wedge \delta \omega_{ab} + \frac{1}{2} \epsilon_{abc} \left( \Theta^a \wedge \Theta^b \wedge \delta \psi^c + \xi \wedge \Theta^a \wedge \delta \omega^{bc} \right) \]
and the Euler–Lagrange equations are given by
\[ E = \frac{\Delta}{2} \varepsilon_{abc} \theta^a \wedge \theta^b \wedge \theta^c - \frac{\mu}{2} \varepsilon_{abc} \theta^a \wedge \Omega_{bc} = 0 \]
\[ F_c = \beta \theta^a \wedge \Omega_{ad} - \frac{\Delta}{2} \varepsilon_{abc} \xi^a \wedge \theta^b \wedge \theta^c + \frac{\mu}{2} \varepsilon_{abc} \xi^a \wedge \Omega_{ab} - \mu \varepsilon_{abc} \theta^a \wedge \Psi_b = 0 \]
\[ G_c = \mu \varepsilon_{abc} \theta^a \wedge \Omega^b = 0 \]
\[ H_{bc} = \beta (\theta_b \wedge \Theta_c - \theta_c \wedge \Theta_b) - \mu \varepsilon_{abc} (\theta^a \wedge \Xi - \xi \wedge \Theta^a) = 0. \]

Notice that the lagrangian does not depend explicitly on \( \sigma \).

Notice also that if \( \mu = 0 \), then the equation \( E = 0 \) says that \( \Lambda = 0 \) since the \( \theta^a \) generate a free graded commutative algebra. This then requires \( \beta \neq 0 \) for a nontrivial lagrangian. The equations in this case reduce to only two:
\[ \theta^a \wedge \Omega_{ac} = 0 \quad \text{and} \quad \theta_b \wedge \Theta_c - \theta_c \wedge \Theta_b = 0, \]
which involve neither \( \Xi \) nor \( \Psi^a \), which remain unconstrained. The first equation is solved by \( \Omega_{ab} = \frac{1}{2} R_{abcd} \theta^c \wedge \theta^d \) for \( R_{abcd} \) the components of an algebraic curvature tensor which is otherwise unconstrained. The second equation is solved by \( \Theta^a = \frac{1}{2} \Theta^a_{bc} \theta^b \wedge \theta^c \) where \( \Theta^a_{cb} = 0 \). These are kinematical constraints and do not give rise to interesting dynamics. We will therefore assume from now on that \( \mu \neq 0 \).

Since \( \mu \neq 0 \), the equations \( G_c = 0 \) and \( H_{bc} = 0 \) reduce to
\[ 0^{|a} \wedge \theta^{|b} = 0 \quad \text{(5.24a)} \]
\[ \theta^a \wedge \Xi - \xi \wedge \theta^a = 0, \quad \text{(5.24b)} \]
where the first follows from \( 0 = \varepsilon^{abc} G_c \).

**Lemma 2.** Equations (5.24a) and (5.24b) imply that \( \Xi = 0 \) and \( \Theta^a = 0 \).

**Proof.** We expand \( \Xi \) and \( \Theta^a \) in terms of the coframe \( \theta^a, \xi \), as follows:
\[ \Theta^a = \frac{1}{2} \Theta^a_{bc} \theta^b \wedge \theta^c + \Theta^a_b \theta^b \wedge \xi, \]
\[ \Xi = \frac{1}{2} \Xi_{ab} \theta^a \wedge \theta^b + \Xi_a \theta^a \wedge \xi. \]

Equation (5.24a) is equivalent to
\[ 0 = \varepsilon_{abc} \theta^b \wedge \Theta^c = \frac{1}{2} \varepsilon_{abc} \Theta_{de} \theta^b \wedge \theta^d \wedge \theta^e + \varepsilon_{abc} \Theta^e_{d} \theta^b \wedge \theta^d \wedge \xi. \]
Both terms are independent and hence must vanish separately. The first term gives the equation
\[ \varepsilon^{bde} \varepsilon_{abc} \Theta^e_{de} = 0 \]
which implies \( \Theta^e_{de} = 0 \) after using the standard identity
\[ \varepsilon^{abc} \varepsilon_{def} = 6 \delta^{|a}_{|d} \delta^{|b}_{|e} \delta^{|c}_{|f}. \]

The second term gives the equation
\[ \varepsilon^{bde} \varepsilon_{abc} \Theta^e_{d} = 0 \implies \Theta^e_{d} = \Theta^e_{b} \delta^e_d, \]
again using equation (5.25). Tracing we see that \( \Theta^e_{b} = 0 \) and hence that \( \Theta^e_{d} = 0 \).

Equation (5.24b) is equivalent to
\[ 0 = \frac{1}{2} \Xi_{cd} \theta^a \wedge \theta^c \wedge \theta^d + \Xi_a \theta^a \wedge \theta^c \wedge \xi - \frac{1}{2} \Theta^a_{cd} \xi \wedge \theta^c \wedge \theta^d. \]

---

4One might wonder whether the theory with \( \mu = 0 \) is the “electric theory” of [8, 19, 23], which only involves the carrollian intrinsic torsion. However, a more careful analysis reveals that this is not, in fact, the electric theory. We comment on this in Section 9.
The $\theta^3$ term is equivalent to $\epsilon^{acd}\Xi_{cd} = 0$, whose only solution is $\Xi_{cd} = 0$. The $\theta^2\xi$ terms become

$$\Xi_d\delta^a_c - \Xi_c\delta^a_d = \frac{1}{2}\Theta^a_{cd}. $$

Tracing and using that $\Theta^d_{de} = 0$, we see that $\Xi_d = 0$ and hence that $\Theta^a_{cd} = 0$, thus proving the lemma.

This lemma is analogous to the well-known fact that in the Palatini formalism of General Relativity, the Euler–Lagrange equation of the connection sets the torsion to zero. Whereas this allows us to solve for the connection in terms of the metric and essentially work in a second-order formalism, this is not the case here. Indeed, the Cartan geometry in question is a carrollian $G$-structure and as shown in [32, Lemma 7], the torsion-free condition does not determine the connection uniquely, but only up to the kernel of the Spencer map, which for carrollian structures corresponds to symmetric tensors of the form $S_{ab}\omega^a\omega^b$.

Since $\Xi$ and $\Theta^a$ vanish, their Bianchi identities then imply that $\Omega^{ab}\wedge\theta^b = 0$ and $\Theta^a \wedge \Psi_a = 0$. Let us expand $\Omega_{ab}$ and $\Psi_a$ in terms of the coframe $\theta^a, \xi$, as follows

$$\Omega_{ab} = -\frac{1}{2}R_{abcd}\theta^c \wedge \theta^d + R_{abc}\theta^c \wedge \xi^d. $$

(5.26)

The Bianchi identity $\Psi^a \wedge \theta_a = 0$ becomes

$$\psi^{[abc]} = 0 \quad \text{and} \quad \psi^{[ab]} = 0. $$

(5.27)

The Bianchi identity $\Omega^{ab} \wedge \theta^b = 0$ becomes

$$R_{a[bcd]} = 0 \quad \text{and} \quad R^{a[bc]} = 0. $$

(5.28)

The latter equation says that $R^{abc} = 0$. Indeed, that equation together with $R^{[ab)c} = 0$ say that

$$R^{abc} = -R^{bac} = -R^{cba} = R^{c[a} = R_{abc} = -R_{abc} = -R^{abc}. $$

(5.29)

The former equation says that $R_{abcd}$ is an algebraic curvature tensor. Let us introduce the algebraic Ricci tensor $R_{ab} = \eta^{cd}R_{acdb}$ and the Ricci scalar $R = \eta^{ab}R_{ab}$. In three dimensions (due to the absence of a Weyl tensor) an algebraic curvature tensor $R_{abcd}$ has the following Ricci decomposition

$$R_{abcd} = R_{ad}\delta_{bc} - R_{ac}\delta_{bd} - R_{bd}\delta_{ac} + R_{bc}\delta_{ad} + \frac{1}{2}R(\delta_{ac}\delta_{bd} - \delta_{bd}\delta_{ac}). $$

(5.30)

so that it is completely determined by its Ricci tensor.

The remaining Euler–Lagrange equations are

$$0 = \frac{1}{6}\epsilon_{abc}\theta^a \wedge \theta^b \wedge \theta^c + \frac{1}{3}\epsilon_{abc}\Omega^{bc} $$

(5.31a)

and

$$0 = -\frac{1}{2}\epsilon_{abc}\theta^a \wedge \theta^b \wedge \xi + \mu\epsilon_{abc}\psi^a \wedge \theta^b + \frac{1}{2}\epsilon_{abc}\Psi^d \wedge \Omega_{ab}. $$

(5.31b)

**Lemma 3.** Equation (5.31a) is equivalent to $R = \frac{2\Lambda}{\mu}$, and equation (5.31b) is equivalent to

$$\psi^a_{\ ab} = 0, \quad \text{and} \quad \Psi_{ab} = R_{ab} - \frac{\Lambda}{\mu}\delta_{ab}. $$

(5.32)

**Proof.** Substitute $\Omega_{ab} = -\frac{1}{2}R_{abcd}\theta^c \wedge \theta^d$ into equation (5.31a) and simplify to obtain

$$-\frac{1}{2}\epsilon_{abc}R_{de}^{bc}\theta^a \wedge \theta^d \wedge \theta^e = \frac{1}{6}\epsilon_{abc}\theta^a \wedge \theta^b \wedge \theta^c \quad \Rightarrow \quad -\frac{1}{2}\epsilon_{abc}R_{de}^{bc}\theta^a = \frac{1}{6}\epsilon_{ade}\epsilon_{ade} \wedge \delta_{cd}, $$

which is seen to be $R = \frac{2\Lambda}{\mu}$ after using the identity (5.25).

Substituting $\Omega_{ab} = -\frac{1}{2}R_{abcd}\theta^c \wedge \theta^d$ and $\Psi_a = \frac{1}{2}\psi_{abc}\theta^b \wedge \theta^c + \Psi_{ab}\theta^b \wedge \xi$ into equation (5.31b), it decomposes into two equations. The $\theta^3$ term says

$$\frac{1}{2}\epsilon_{abc}\psi_{de} \wedge \theta^b \wedge \theta^c \wedge \theta^e \Rightarrow \epsilon_{de} \wedge \psi_{ab} = 0. $$
Using the identity (5.25), this becomes simply the vanishing trace condition $\Psi_{ab}^a = 0$. The $\theta^2 \xi$, terms result in the equation

$$0 = -\frac{\Lambda}{2} \epsilon_{abc} \theta^a \wedge \theta^b \wedge \xi + \mu \epsilon_{abc} \Psi_{ab}^a \theta^b \wedge \theta^d \wedge \xi - \frac{\mu}{\Phi} \epsilon_{abc} R^a_{ab} \theta^d \wedge \theta^e \wedge \xi,$$

or equivalently,

$$0 = -\frac{\Lambda}{2} \epsilon_{abc} \theta^a \wedge \theta^b \wedge \xi + \mu \epsilon_{abc} \Psi_{ab}^a \theta^b \wedge \theta^d \wedge \xi - \frac{\mu}{\Phi} \epsilon_{abc} R^a_{ab} \theta^d \wedge \theta^e \wedge \xi.$$

Repeated use of the identity (5.25) and the fact that $R = \frac{2\Lambda}{\mu}$, and simplifying results in

$$\Psi_a \delta^f_c - \Psi_c^f + R \epsilon_{fde} = 0.$$

Tracing and using again that $R = \frac{2\Lambda}{\mu}$, we see that $\Psi_a^a = -\frac{\Lambda}{\mu}$, and inserting back into the equation we find (after some relabelling) that

$$\Psi_{ab} = R_{ab} - \frac{\Lambda}{\mu} \delta_{ab}.$$

The condition $\Psi_{ab}^a = 0$ together with $\Psi^{abc} = 0$ say that we can write

$$\Psi_{abc} = S_{abc} = S_{debc},$$

for some traceless symmetric tensor $S_{ab}$.

We may summarise these results as follows, where we have set $\mu = 1$ without loss of generality.

**Proposition 4.** The solution of the Euler–Lagrange equations for the lagrangian

$$\mathcal{L} = \frac{1}{2} \epsilon_{abc} \left( \theta^a \wedge \theta^b \wedge \Psi^c + \xi \wedge \theta^a \wedge \Omega^{bc} \right) + \frac{\Phi}{2} \theta^a \wedge \theta^b \wedge \Omega_{ab} + \frac{\Lambda}{2} \epsilon_{abc} \theta^a \wedge \theta^b \wedge \theta^c \wedge \xi$$

are such that $\Xi = \Theta^a = 0$ and

$$\Omega_{ab} = R_{bc} \theta^a \wedge \theta^c - R_{ac} \theta^b \wedge \theta^c - 2 \Lambda \theta_a \wedge \theta_b,$$

$$\Psi_a = \frac{1}{2} \epsilon_{bde} S_{ab} \theta_a \wedge \theta_b + R_{ab} \theta^a \wedge \xi - \Lambda \theta_a \wedge \xi,$$

(5.34)
where $S_{ab}$ is symmetric and traceless.

6. The case in-between: Carroll

As we remarked above, setting $\sigma = 0$ in (5.1) gives rise to the Carroll algebra. Since the homogeneous spaces of dSC, AdSC and Carroll are model mutants, that is to say, the stabiliser $h$ is the same in both cases and the kinematical Lie algebras are isomorphic as $h$-modules, the gauge-invariant 4-forms are still given by those listed in Table 8. Only quantities that depend on the details of $g$ itself are modified. This means that the explicit expressions for the curvatures change, and consequently their variations, and the Bianchi identities change as well. For Carroll, we thus have

$$\Omega^{ab} = d\omega^{ab} + \omega^a \wedge \omega^b$$

$$\psi_a = d\psi^a + \omega^a \Lambda \psi^b$$

$$\Theta^a = d\theta^a + \omega^a \wedge \theta^b,$$

(6.1)

obtained by setting $\sigma = 0$ in (5.4). Setting $\sigma = 0$ in the derivation of (5.19) we arrive at the same lagrangian

$$\mathcal{L} = \frac{1}{2} \epsilon_{abc} \left( \theta^a \wedge \theta^b \wedge \Psi^c + \xi \wedge \theta^a \wedge \Omega^{bc} \right) + \frac{\Phi}{2} \theta^a \wedge \theta^b \wedge \Omega_{ab} + \frac{\Lambda}{2} \epsilon_{abc} \theta^a \wedge \theta^b \wedge \theta^c \wedge \xi.$$

(6.2)
and the same Euler–Lagrange equations where the curvature components are now given by (6.1). Note, however, that boundary terms varying into total derivatives slightly change. We have summarised these terms in Table 1.
We can understand the Carroll lagrangian (6.2) as the “ultra-relativistic” limit of the Einstein–Palatini–Holst lagrangian (4.8). To this end, consider the Poincaré algebra (4.1) in the kinematical basis \((L_{ab}, B_a, P_a, H)\), where \(a = 1, 2, 3\) and
\[
B_a = L_{0a} \quad \text{and} \quad H = P_0.
\]
(6.3)

We can then write the Cartan connection and its associated curvature as
\[
A = \frac{1}{4} \omega^{ab} L_{ab} + \psi^a B_a + \theta^a P_a + \xi H
\]
\[
F = \frac{1}{2} \Omega^{ab} L_{ab} + \psi^a B_a + \Theta^a P_a + \Sigma H,
\]
(6.4)

where \(\theta^m = (\xi, \theta^a)\), \(\omega^{mn} = (\psi^a, \omega^{ab})\), \(\Theta^m = (\Sigma, \Theta^a)\) and \(\Omega^{mn} = (\psi^a, \Omega^{ab})\). Note that these expressions are formally identical to (5.2) and (5.3), although at this stage all we have done is recast the Cartan connection and curvature of Minkowski. Now, the ultra-relativistic limit corresponds to setting the speed of light to zero, so to be able to take this limit we must introduce appropriate factors of \(c\) in the Minkowski metric, which consequently becomes
\[
\eta_{mn} = \text{diag}(-c^2, 1, 1, 1).
\]
(6.5)

In the new basis and with factors of \(c\) restored, the Poincaré algebra (4.1) becomes
\[
[L_{ab}, L_{cd}] = \delta_{bc} L_{ad} - \delta_{ac} L_{bd} + \delta_{ad} L_{bc}
\]
\[
[L_{ab}, B_c] = \delta_{bc} B_a - \delta_{ac} B_b
\]
\[
[L_{ab}, P_c] = \delta_{bc} P_a - \delta_{ac} P_b
\]
\[
[B_a, P_b] = \delta_{ab} H
\]
\[
[B_a, B_b] = c^2 L_{ab}
\]
\[
[B_a, H] = c^2 P_a.
\]
(6.6)

Since \(F = dA + \frac{1}{2}[A, A]\), the curvature components pick up factors of \(c\)
\[
\Omega^{ab} = d^V \omega^{ab} + c^2 \psi^a \wedge \psi^b
\]
\[
\Psi^a = d^V \psi^a
\]
\[
\Theta^a = d^V \theta^a + c^2 \psi^a \wedge \xi
\]
\[
\Sigma = d\xi + \psi^a \wedge \theta^a
\]
(6.7)

which agree with (6.1) when \(c = 0\). The gauge invariant 4-forms are those listed in Table 6. The ones that do not vary into exact forms may now be written as
\[
\frac{1}{4} \epsilon_{mnpq} \theta^m \wedge \theta^n \wedge \theta^p \wedge \theta^q = \frac{1}{4} \epsilon_{abc} \xi \wedge \theta^a \wedge \Omega^{bc} + \frac{1}{2} \epsilon_{abc} \theta^a \wedge \theta^b \wedge \psi^c
\]
\[
\frac{1}{2} \theta^m \wedge \theta^n \wedge \Omega_{mn} = \frac{1}{2} \epsilon_{mnpq} \theta^m \wedge \theta^n \wedge \Omega^{pq} = \frac{1}{2} \theta^a \wedge \theta^b \wedge \Omega_{ab} = c^2 \xi \wedge \theta^a \wedge \psi^a
\]
\[
\frac{1}{3} \epsilon_{mnpq} \theta^m \wedge \theta^n \wedge \theta^p \wedge \theta^q = -\frac{1}{3} \epsilon_{abc} \theta^a \wedge \theta^b \wedge \theta^c \wedge \xi,
\]
(6.8)

where we used that \(\epsilon_{0abc} = \epsilon_{abc}\). This means that we can write the Minkowski lagrangian as
\[
\mathcal{L} = \frac{1}{4} \epsilon_{mnpq} \theta^m \wedge \theta^n \wedge \Omega^{pq} + \frac{1}{2} \theta^m \wedge \theta^n \wedge \Omega_{mn} - \frac{1}{4} \epsilon_{mnpq} \theta^m \wedge \theta^n \wedge \theta^p \wedge \theta^q
\]
\[
= \frac{1}{2} \epsilon_{abc} (\xi \wedge \theta^a \wedge \Omega^{bc} + \theta^a \wedge \theta^b \wedge \psi^c) + \frac{1}{2} \theta^a \wedge \theta^b \wedge \Omega_{ab} + c^2 \beta \xi \wedge \theta^a \wedge \psi^a + \frac{1}{3} \epsilon_{abc} \theta^a \wedge \theta^b \wedge \theta^c \wedge \xi
\]
(6.9)

Setting \(c = 0\) in this expression produces the lagrangian
\[
\mathcal{L}_{c=0} = \frac{1}{2} \epsilon_{abc} (\xi \wedge \theta^a \wedge \Omega^{bc} + \theta^a \wedge \theta^b \wedge \psi^c) + \frac{1}{2} \theta^a \wedge \theta^b \wedge \Omega_{ab} + \frac{1}{3} \epsilon_{abc} \theta^a \wedge \theta^b \wedge \theta^c \wedge \xi,
\]
(6.10)

which is identical to the Carroll lagrangian (6.2).
To match this with the result of [21], we take (6.2) and set \( \mu = 1 \) and \( \beta = \Lambda = 0 \), leaving only the term that descends from the Einstein–Hilbert–Palatini term
\[
\mathcal{L} = \frac{1}{2} \epsilon_{abc} (\theta^a \wedge \theta^b \wedge \Psi^c + \xi \wedge \theta^a \wedge \Omega^{bc}).
\] (6.11)

Then, write for, say, the first term the following
\[
\frac{1}{2} \epsilon_{abc} \theta_\mu^a \theta_\nu^b \Psi_\rho^c \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\sigma = \frac{1}{2} \epsilon_{abc} \epsilon^{\mu \nu \rho \sigma} \theta_\mu^a \theta_\nu^b \Psi_\rho^c \, \mathrm{d}vol,
\] (6.12)

where we introduced local coordinates \((x^\mu)\) on the four-dimensional manifold, and where \(e = \det(\xi_\mu, \theta_\mu^a)\) and we defined the top form
\[
dvol := \frac{1}{2} \epsilon_{abc} \theta^a \wedge \theta^b \wedge \theta^c = \theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4 \in \Omega^4(\mathcal{U}).
\] (6.13)

More specifically, the combination \((\xi_\mu, \theta_\mu^a)\) forms an invertible matrix, and we have that
\[
e = \frac{1}{2} \epsilon^{\mu \nu \rho \sigma} \epsilon_{abc} \xi_\mu \theta_\nu^a \theta_\rho^b \theta_\sigma^c.
\] (6.14)

As above, we denote the inverse of \((\xi_\mu, \theta_\mu^a)\) by \((\kappa^\mu, e_\mu^a)\), and inserting the completeness relation (7.3) twice on the right-hand side of (6.12), we get
\[
\frac{1}{2} \epsilon_{abc} \epsilon^{\mu \nu \rho \sigma} \theta_\mu^a \theta_\nu^b \Psi_\rho^c \, \mathrm{d}vol = \frac{1}{2} \epsilon_{abc} \epsilon^{\mu \nu \rho \sigma} \xi_\mu \theta_\nu^a \theta_\rho^b \theta_\sigma^c \kappa^\lambda \epsilon_a^e \Psi_\lambda^e \, \mathrm{d}vol = \frac{1}{2} \epsilon_{abc} \epsilon^{\mu \nu \rho \sigma} \xi_\mu \theta_\nu^a \theta_\rho^b \theta_\sigma^c \kappa^\lambda \epsilon_a^e \Psi_\lambda^e \, \mathrm{d}vol
\] (6.15)

where in going from the first to the second line, we used that the Levi–Civita symbol forces the d-index on \(\theta_\sigma^d\) to be equal to \(c\). A similar calculation shows that
\[
\frac{1}{2} \epsilon_{abc} \xi \wedge \theta^a \wedge \Omega^{bc} = \frac{1}{2} \epsilon^{ab} e_a^c \theta_\mu^a \Omega_{\mu \nu} \wedge \mathrm{d}vol,
\] (6.16)

and so we recover the lagrangian of [21] in the first-order formulation:
\[
\mathcal{L} = \frac{1}{2} \left( e^{ab} e_a^e \theta_\mu^a \Omega_{\mu \nu} \wedge \mathrm{d}vol + 2 \kappa^a \epsilon_a^e \Psi_{\mu \nu} \wedge \mathrm{d}vol \right).
\] (6.17)

To get the lagrangian in a second order formulation, we make use of the results we obtain in the next section below. There, we will find that
\[
\Omega_{\mu \nu} \wedge \wedge ^{ab} = \epsilon^\lambda_\mu \theta_\rho^a \theta_\sigma^b \Psi_{\mu \lambda} \wedge \wedge ^\rho,
\] (6.18)

where \( R_{\mu \lambda \wedge \wedge \wedge}^\rho \) are the components of the Riemann tensor of a certain affine connection \(\nabla\). Plugging this into the first-order lagrangian (6.16), we get
\[
\mathcal{L} = \frac{1}{2} \gamma^{\mu \nu} \left( R_{\mu \nu} \wedge + \kappa^\rho \xi_\lambda R_{\mu \nu} \wedge ^\lambda \right) \, \mathrm{d}vol,
\] (6.19)

in agreement with [21].

7. Geometric interpretation

We can recast our results in terms of the carrollian structure on the manifold \( M \). We work in a Cartan gauge \((\mathcal{U}, A)\) where \((\mathcal{U}, x^\mu)\) is also a chart with local coordinates \(x^\mu = (x^0, x^1, x^2, x^3)\), relative to which the Cartan connection \( A \in \Omega^1(\mathcal{U}; \mathfrak{g})\) given in (5.2) has components
\[
A_\mu = \frac{1}{2} \omega_\mu \wedge ^{ab} l_{ab} + \psi_\mu \wedge ^a B_a + \theta_\mu \wedge ^a P_a + \xi_\mu H.
\] (7.1)

Since \((A)\mathrm{d}SC\) and \( C \) are reductive Klein geometries, the \( m \)-component \( \theta^a P_a + \xi H \) of \( A \) defines a coframe: an isomorphism \( T_P M \to \mathfrak{m} \), sending \( \partial_\mu \to \theta^a P_a + \xi_\mu H \). Let \((\kappa, e_\mu)\) denote the canonically dual frame. The carrollian structure is given by \((\kappa, h)\), where \( h = \delta_{ab} \theta^a \theta^b \). Relative to local coordinates,
\[
\kappa = \kappa^a \partial_\mu, \quad \xi = \xi_\mu \mathrm{d}x^\mu, \quad \theta^a = \theta^a_\mu \mathrm{d}x^\mu \quad \text{and} \quad e_\mu = e_\mu^a \partial_\mu.
\] (7.2)
satisfying the following completeness relations:
\[ \xi_\mu k^\mu = 1, \quad \theta^a_\mu k^\mu = 0, \quad \xi_\mu e^a_b = 0, \quad \theta^a_\mu e^b_c = \delta^b_c \quad \text{and} \quad \xi_\mu k^\gamma + \theta^a_\mu e^\gamma_b = \delta^\gamma_b. \quad (7.3) \]

It is convenient to define \( \gamma \in \Gamma(\mathcal{O}^2 TM) \) via \( \gamma = \delta^{ab} e_a e_b \) with components \( \gamma^{\mu\nu} = \delta^{ab} e^a_\mu e^b_\nu \). It follows from the completeness relations that
\[ \gamma^{\mu\nu} h_{\mu\nu} + k^{\mu} \xi_\nu = \delta^\mu_\nu. \quad (7.4) \]

where \( h_{\mu\nu} = \delta_{ab} \theta^a_\mu \theta^b_\nu \) are the components of \( h \).

The \( b \)-component \( A^b \) of \( A \) defines an Ehresmann connection on the principal \( H \)-bundle \( P \to M \) of the Cartan geometry. This connection induces a Koszul connection on the so-called fake tangent bundle \( P \times_M \mathfrak{m} \to M \), which is the vector bundle associated to \( P \) via the \( H \)-representation \( \mathfrak{m} \). Locally, sections of \( P \times_M \mathfrak{m} \) are \( \mathfrak{m} \)-valued functions on \( U \). The coframe \((\xi, \theta^a)\) gives a bundle isomorphism \( TM \to P \times_M \mathfrak{m} \) sending \( \partial_a \mapsto \xi_\mu H + \theta^a_\mu P_a \) and allowing us to transport the Koszul connection to an affine connection \( \nabla \) on \( TM \). This affine connection has connection coefficients \( \Gamma^\rho_{\mu\nu} \) defined by
\[ \nabla_\mu \partial_\nu = \Gamma^\rho_{\mu\nu} \partial_\rho. \quad (7.5) \]

They can be determined in terms of the Cartan connection via the so-called Vierbein postulate, which says that we obtain the same result if we differentiate \( \partial_\mu \) with the affine connection \( \nabla \) and then map to \( P \times_M \mathfrak{m} \) or first map to \( P \times_M \mathfrak{m} \) and differentiate with the \( h \)-part of the Cartan connection. Explicitly, in the former operation we obtain
\[ \partial_\nu \nabla_\mu = \Gamma^\rho_{\mu\nu} \partial_\rho, \quad (7.6) \]

whereas in the latter we obtain
\[ \partial_\nu \nabla_\mu \partial_\rho - \theta^a_\mu \partial_\rho \partial_\mu \xi^a_b H + \xi^a_b \partial_\rho \partial_\mu \xi^a_b = \partial_\mu \xi^a_b \partial_\rho \partial_\nu \xi^a_b = \partial_\mu \xi^a_b \partial_\rho \partial_\nu \xi^a_b = \partial_\mu \xi^a_b \partial_\rho \partial_\nu \xi^a_b. \quad (7.7) \]

where \( A^b \) is the \( b \)-component of the Cartan connection. Equating the two expressions, we obtain
\[ \Gamma^\rho_{\mu\nu} (\xi_\rho H + \theta^a_\rho P_a) = (\partial_\rho \xi^a_b H + \xi^a_b \partial_\rho \xi^a_b) H + (\partial_\rho \theta^a_\nu + \omega^a_\nu \theta^b_\rho) P_a. \quad (7.8) \]

from where we read off the following identities:
\[ \Gamma^\rho_{\mu\nu} \xi_\rho = \partial_\rho \xi^a_b + \theta^a_\rho \theta^b_\nu \]
\[ \Gamma^\rho_{\mu\nu} \theta^a_\rho = \partial_\rho \theta^a_\nu + \omega^a_\nu \theta^b_\rho. \quad (7.9) \]

The completeness relations (7.3) say that
\[ \Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu} \delta^\rho = \Gamma^\rho_{\mu\nu} (\xi_\rho k^\rho + \theta^b_\rho e^a_b)
= \Gamma^\rho_{\mu\nu} \xi_\rho k^\rho + \Gamma^\rho_{\mu\nu} \theta^b_\rho e^a_b
= (\partial_\rho \xi^a_b H + \xi^a_b \partial_\rho \xi^a_b) k^\rho + (\partial_\rho \theta^a_\nu + \theta^a_\nu \theta^b_\rho) e^a_b. \quad (7.10) \]

so that
\[ \Gamma^\rho_{\mu\nu} = k^\rho \partial_\mu \xi_\nu + k^\rho \theta^a_\nu \theta^b_\rho + \xi^a_b \partial_\nu \theta^b_\rho + \theta^a_\nu \theta^b_\rho \omega^a_\mu \theta^b_\nu. \quad (7.11) \]

Similarly, the curvature \( F = dA + \frac{1}{2} [A, A] \) given in (5.3) has components
\[ F_{\mu\nu} = \frac{1}{2} \Omega_{\mu\nu}^{ab} L_{ab} + \Psi_{\mu\nu}^{ab} B_{ab} + \Theta_{\mu\nu}^{a} P_a + \Xi_{\mu\nu} H, \quad (7.12) \]

where \( \Omega_{\mu\nu}^{ab} \), \( \Psi_{\mu\nu}^{ab} \), \( \Theta_{\mu\nu}^{a} \), and \( \Xi_{\mu\nu} \) are the curvature components.
where the local forms of (5.4) read
\[
\begin{align*}
\Omega_{\mu\nu}^{ab} &= 2\partial_{[\mu}\omega_{\nu]}^{ab} + 2\omega_{[\mu}^{a}c\omega_{\nu]}^{b} + 2\sigma\theta_{[\mu}^{a}\theta_{\nu]}^{b} \\
\Psi_{\mu\nu}^{a} &= 2\partial_{[\mu}\psi_{\nu]}^{a} + 2\omega_{[\mu}^{a}b\psi_{\nu]}^{b} + 2\sigma\epsilon_{[\mu}^{a}\theta_{\nu]}^{b} \\
\Theta_{\mu\nu}^{a} &= 2\partial_{[\mu}\theta_{\nu]}^{a} + 2\omega_{[\mu}^{a}b\theta_{\nu]}^{b} \\
\Xi_{\mu\nu} &= 2\omega_{[\mu}^{a}b\theta_{\nu]}^{a}.
\end{align*}
\] (7.13)

The last term in each of the two first lines — which are also the only terms involving the sign \(\sigma\) — constitute the only difference compared to gauging the Carroll algebra; that is, the algebra (5.1) with \(\sigma = 0\) (see [20,21,22]).

In general, only for the Klein model in which \(m\) is abelian, does the curvature of the affine connection derived from the \(h\)-component of the Cartan connection agree with the \(h\)-component of the curvature of the Cartan connection. The Klein model with \(m\) abelian is the one for which \(g\) is the Carroll algebra; that is, the case where \(\sigma = 0\). Adorning with a hat the curvature with \(\sigma = 0\), we may thus write
\[
\begin{align*}
\hat{\Omega}_{\mu\nu}^{ab} &= \hat{\Omega}_{\mu\nu}^{ab} + 2\sigma\theta_{[\mu}^{a}\theta_{\nu]}^{b}, \\
\hat{\Psi}_{\mu\nu}^{a} &= \hat{\Psi}_{\mu\nu}^{a} + 2\sigma\epsilon_{[\mu}^{a}\theta_{\nu]}^{a},
\end{align*}
\] (7.14)

which makes explicit the dependence on \(\sigma\). As we will see explicitly below, the hatted curvature is the curvature of the \(h\)-component of the Cartan connection.

The torsion of the affine connection (7.11) satisfies
\[
\Gamma_{\mu\nu}^{\lambda} = \hat{\Gamma}_{\mu\nu}^{\lambda} - \Gamma_{\nu\lambda}^{\mu} = \nu^{\lambda}\Sigma_{\mu\nu} + c_{\lambda}^{\mu\nu}a,
\] (7.15)

and hence the equations of motion \(\Theta^{a} = \Xi = 0\) imply that the torsion of the affine connection vanishes. In the literature, conditions like \(\Theta^{a} = \Xi = 0\) are sometimes called curvature\(^5\) constraints and, in the absence of an action from which they arise as equations of motion, are imposed by hand in order that the affine connection can be solved for in terms of the soldering form.

Since the affine connection \(\nabla\) comes from an Ehresmann connection on \(P\), it is adapted to the carrollian structure; that is, the tensors \((\kappa, h)\) defining the carrollian structure are parallel:
\[
\nabla_{\mu}\kappa^{\nu} = 0 \quad \text{and} \quad \nabla_{\mu}h_{\nu\lambda} = 0.
\] (7.16)

As shown in [20,33], the most general torsion-free connection adapted to the carrollian structure \((\nu, h)\) has coefficients given by
\[
\Gamma_{\mu\nu}^{\lambda} = \kappa^{\lambda}\delta_{[\mu}\xi_{\nu]} + \frac{1}{2}\nu^{\lambda}\left(\partial_{\mu}h_{\nu\rho} + \partial_{\nu}h_{\rho\mu} - \partial_{\rho}h_{\mu\nu}\right) - \kappa^{\mu}\Sigma_{\mu\nu},
\] (7.17)

where \(\Sigma_{\mu\nu} = \Sigma_{[\mu\nu]}\) is an arbitrary symmetric tensor subject to the condition
\[
\kappa^{a}\Sigma_{\mu\nu} = 0,
\] (7.18)

arising from the fact that \(\kappa\) is parallel. The connection coefficients
\[
\Gamma_{\mu\nu}^{\lambda} = \kappa^{\lambda}\delta_{[\mu}\xi_{\nu]} + \frac{1}{2}\nu^{\lambda}\left(\partial_{\mu}h_{\nu\rho} + \partial_{\nu}h_{\rho\mu} - \partial_{\rho}h_{\mu\nu}\right)
\] (7.19)

define another adapted connection \(\hat{\nabla}\) and the difference between the two adapted connections \(\nabla\) and \(\hat{\nabla}\) is the \((1,2)\)-tensor field with components \(\kappa^{\lambda}\Sigma_{\mu\nu}\). Notice that both \(\Gamma^{\alpha}_{\mu\nu}\) in equation (7.11) and \(\Gamma^{\alpha}_{\mu\nu}\) in equation (7.19) are given in terms of the components of the Cartan connection, and hence so are their difference. This means that we may solve for \(\Sigma_{\mu\nu}\) in terms of the Cartan connection. Since the affine connection \(\nabla\) is torsion free, we can write (7.11) as
\[
\Gamma_{\mu\nu}^{\lambda} = \kappa^{\lambda}\delta_{[\mu}\xi_{\nu]} + \kappa^{\lambda}\delta_{[\mu}\psi_{\nu]}^{a} + \epsilon_{\lambda}^{\alpha}a\delta_{[\mu}\theta_{\nu]}^{a} + \epsilon_{\lambda}^{\alpha}b\omega_{[\mu}^{a}b\theta_{\nu]}^{b}.
\] (7.20)

\(\text{Or, occasionally, conventional constraints.}\)
On the other hand, the connection coefficients $\tilde{\Gamma}^a_{\mu\nu}$ of the adapted connection $\tilde{\nabla}$ can be written in terms of the coframe $\theta^a_\mu$ by writing $h_{\mu\nu} = \delta_{ab}\theta^a_\mu \theta^b_\nu = \theta^a_\mu \theta^a_\nu$, in which case we find that

$$\tilde{\Gamma}^a_{\mu\nu} = \kappa^a \delta_{[\mu} \xi_{\nu]} + e^a \delta_{[\mu \theta^a_\nu]} + \frac{1}{2} \gamma^a \theta^a_\nu (d\theta^a)_{\mu\nu} + \frac{1}{2} \gamma^a \theta^a_\mu (d\theta^a)_{\nu\mu}$$

which we used in (6.19) to write our lagrangian in a second-order formulation. This, together with the relations (7.3), then implies that

$$\Gamma^a_{\mu\nu} - \Gamma^a_{\nu\mu} = \kappa^a \theta^a_\mu \psi_\nu \partial^\alpha.$$  \hspace{1cm} (7.22)

The Riemann tensor of an affine connection $\nabla$ is defined by

$$R^\nabla (X, Y) Z = \nabla_X \nabla_Y Z - \nabla_{\nabla_X Y} Z,$$  \hspace{1cm} (7.23)

whose components $R^\nabla_{\mu\nu\rho} \sigma$ are defined by

$$R^\nabla_{\mu\nu\rho} \sigma \delta_\sigma = R^\nabla (\partial_\mu, \partial_\nu) \partial_\rho = -[\nabla_\mu, \nabla_\nu] \partial_\rho.$$  \hspace{1cm} (7.24)

In terms of the connection coefficients, they are given by

$$R^\nabla_{\mu\nu\rho} \sigma = -\partial_\mu \Gamma^\sigma_{\nu\rho} + \partial_\nu \Gamma^\sigma_{\mu\rho} - \Gamma^\sigma_{\rho\nu} \Gamma^\rho_{\mu\lambda} + \Gamma^\rho_{\sigma\nu} \Gamma^\sigma_{\mu\lambda}.$$  \hspace{1cm} (7.25)

Comparing with the Riemann tensor of $\tilde{\nabla}$, which has a similar expression but with $\tilde{\Gamma}^a_{\mu\nu}$ replacing $\Gamma^a_{\mu\nu}$, we have

$$R^\nabla_{\mu\nu\lambda} \rho = R^\nabla_{\mu\nu\lambda} \rho + 2\kappa^a \tilde{\nabla}_{\mu} \tilde{\nabla}_{\nu} \psi_\lambda.$$  \hspace{1cm} (7.26)

Using the explicit expression (7.11) for the connection coefficients and using the Vierbein postulate (7.9), we may write the Riemann tensor of the affine connection in terms of the curvatures that feature in (7.12):

$$R^\nabla_{\mu\nu\lambda} \rho = -\kappa^a \theta^a_\lambda \psi_{\mu\nu} + \theta^a_\lambda e^a_\rho \psi_{\mu\nu} + 2\sigma \kappa^a \theta^a_\lambda \xi_{[\mu \theta^a_\nu]} - 2\sigma \theta^a_\lambda e^a_\rho \psi_{\mu \theta^a_\nu},$$  \hspace{1cm} (7.27)

which, as advertised earlier, admits a simpler form in terms of the pure (pseudo-)carollian curvatures defined in (7.14):

$$R^\nabla_{\mu\nu\lambda} \rho = -\kappa^a \theta^a_\lambda \tilde{\psi}_{\mu\nu} + \theta^a_\lambda e^a_\rho \tilde{\psi}_{\mu\nu} + \theta^a_\lambda e^a_\rho \tilde{\psi}_{\mu\lambda} a b.$$  \hspace{1cm} (7.28)

This, together with the relations (7.3), then implies that

$$\tilde{\Omega}_{\mu\nu\rho \sigma} = e^a \psi_\rho \tilde{\psi}_{\mu\nu} \psi_{\sigma \lambda}$$

$$\tilde{\psi}_{\mu\nu} a = -\xi_{\rho} \theta^a_\mu \tilde{\psi}_{\nu\lambda} \rho,$$  \hspace{1cm} (7.29)

which we used in (6.19) to write our lagrangian in a second-order formulation.

The Ricci tensor of $\nabla$ is symmetric. This follows from the fact, proved in Appendix A, that the Ricci tensor for a torsion-free affine connection on an orientable manifold is symmetric if and only if around every point there exists a locally defined parallel volume form. The assumption of orientability is harmless by passing, if needed, to the orientation double cover. This is $\mathfrak{h}$-invariant and hence parallel relative to $\nabla$. Indeed,

$$\nabla_\mu \text{vol} = \frac{1}{2} e^{abc}_{\mu} \nabla_\mu \theta^a \wedge \theta^b \wedge \theta^c \wedge \xi + \frac{1}{6} e^{abc}_{\mu} \theta^a \theta^b \wedge \theta^c \wedge \nabla_\mu \xi.$$  \hspace{1cm} (7.30)

From the Vierbein postulate (7.9), we see that

$$\nabla_\mu \theta^a = -\omega^a_{\mu} \psi_\theta^d$$

$$\nabla_\mu \xi = -\psi_{\mu} \psi_\theta^d.$$  \hspace{1cm} (7.31)

and inserting into $\nabla_\mu \text{vol}$ we arrive at

$$\nabla_\mu \text{vol} = \frac{1}{2} e^{abc}_{\mu} \theta^a \wedge \theta^b \wedge \theta^c \wedge \xi + \frac{1}{6} e^{abc}_{\mu} \theta^a \theta^b \wedge \theta^c \wedge \psi_{\mu} \psi_\theta^d.$$  \hspace{1cm} (7.32)
The second term contains the wedge product of four 6s, but only three are available, hence it vanishes. This leaves the first term
\[
\frac{1}{2} \epsilon_{abc} \omega^a d \theta^d \wedge \theta^b \wedge \theta^c \wedge \xi = \frac{1}{2} \epsilon_{abc} \epsilon^{d} \omega^{a} d \text{ dvol} = \delta^{d}_{a} \omega^{a} d \text{ dvol,} \tag{7.33}
\]
which is zero because \( \omega_{ab} = -\omega_{ba} \). In other words, \( \nabla \text{ dvol} = 0 \) and from Proposition A.1, it follows that the Ricci tensor of \( \nabla \) is symmetric.

The Ricci tensor of \( \nabla \) is also symmetric. Indeed, as shown in Proposition A.2, the Ricci tensor of \( \nabla \) is symmetric if and only if the trace of the contorsion is a closed one-form, but from equation (7.17) the contorsion tensor has components \( \kappa^a \Sigma_{\mu \nu} \) and hence its trace is \( \kappa^a \Sigma_{\mu \nu} \) which vanishes since \( \kappa \) is parallel with respect to any adapted connection (see equation (7.18)).

8. Gauging the lightcone

In this section we apply the gauging procedure to the lightcone, by which we mean the future lightcone with deleted apex as depicted in Figure 2. Cartan geometries modelled on the lightcone have previously been considered in [34]. The lightcone is a homogeneous space of the connected component of the identity of the Lorentz group \( \text{SO}(4,1) \). Its Lie algebra \( g = \text{so}(4,1) \) is spanned by \( L_{ab}, B_a, P_a, H \) with \( a,b = 1, 2, 3 \) and subject to the nonzero brackets
\[
[L_{ab}, L_{cd}] = \delta_{bc} L_{ad} - \delta_{ac} L_{bd} - \delta_{bd} L_{ac} + \delta_{ad} L_{bc} \\
[L_{ab}, B_c] = \delta_{bc} B_a - \delta_{ac} B_b \\
[L_{ab}, P_c] = \delta_{bc} P_a - \delta_{ac} P_b \\
[H, B_a] = B_a \\
[H, P_a] = -P_a \\
[B_a, P_b] = \delta_{ab} H + L_{ab}. \tag{8.1}
\]
The Klein pair \( \langle g, h \rangle \) for the lightcone is such that \( h \) is the span of \( L_{ab} \) and \( B_a \), which is isomorphic to the euclidean algebra \( \text{iso}(3) \), in common with the other maximally symmetric carrollian Klein pairs: \( C, dSC \) and \( AdSC \). In contrast with these other carrollian Klein pairs, the Klein pair of the lightcone is not reductive. Letting \( \langle g', h' \rangle \) be any one of these other carrollian Klein pairs, we have that \( h \cong h' \), but \( g' \) is not isomorphic to \( g \) as a representation of \( h \). Whereas \( C, dSC \) and \( AdSC \) are all mutants of each other, the lightcone is not related any of them by mutation. The carrollian Cartan geometry associated to the lightcone is therefore in principle different to the

![Figure 2. Future lightcone without apex](image-url)
one modelled on $C$, $dSC$ or $AdSC$. In fact, the carrollian structure of the lightcone is of totally umbilical type (in the classification of $[32]$), as shown in $[25, \S7.3]$, and not of totally geodesic type as that of $C$, $dSC$ or $AdSC$.

We let $\mathfrak{g} = (L_{ab}, \mathfrak{g}) \cong \mathfrak{so}(3)$ and $\mathfrak{g} = (L_{ab}, B_a, \mathfrak{g}) \cong \mathfrak{so}(4, 1)$, with the brackets given explicitly by equation (8.1).

8.1. The gauge fields. Let $M$ be a four-dimensional manifold and $U \subset M$ an open subset and let $A \in \Omega^1(U, \mathfrak{g})$ be given relative to the basis of $\mathfrak{g}$ by

$$A = \frac{1}{2} \omega^{ab} L_{ab} + \psi^a B_a + \theta^a p_a + \xi H.$$  

Its curvature $F \in \Omega^2(U, \mathfrak{g})$ is given by

$$F = dA + \frac{1}{2}[A, A] = \frac{1}{2} \Omega^{ab} L_{ab} + \Psi^a B_a + \Theta^a p_a + \Xi H, \quad (8.2)$$

where

$$\Omega^{ab} = d\omega^{ab} + \omega^a \wedge \omega^b + \psi^a \wedge \theta^b - \psi^b \wedge \theta^a = F g^{ab} + \psi^a \wedge \theta^b - \psi^b \wedge \theta^a$$

$$\Psi^a = d\psi^a + \omega^a \wedge \psi^b + \xi \wedge \psi^a = d\nabla \psi^a + \xi \wedge \psi^a$$

$$\Theta^a = d\theta^a + \omega^a \wedge \theta^b - \xi \wedge \theta^a = d\nabla \theta^a - \xi \wedge \theta^a$$

$$\Xi = d\xi + \psi^a \wedge \theta_a.$$ 

where again $d\nabla$ and $F_{\nabla}$ are the $\tau$-covariant exterior derivative and curvature, respectively. These gauge fields satisfy the Bianchi identity $dF + [A, F] = 0$, which decomposes into the following relations

$$d\nabla \Omega^{ab} = -\psi^a \wedge \Theta^b + \psi^b \wedge \Theta^a - \theta^a \wedge \psi^b + \theta^b \wedge \psi^a$$

$$d\nabla \Psi^a = \Omega^{ab} \wedge \psi^b + \psi^a \wedge \Xi - \xi \wedge \psi^a$$

$$d\nabla \Theta^a = \Omega^{ab} \wedge \theta^b - \theta^a \wedge \Xi + \xi \wedge \Theta^a$$

$$d\Xi = \theta^a \wedge \Psi_a - \psi^a \wedge \Theta_a. \quad (8.5)$$

We now proceed to construct the lagrangian 4-form out of $F$ and the projection $\mathbf{F}$ to $\mathfrak{g}/\mathfrak{h}$ of $A$. We will let $\mathbf{P}_a$ and $\mathbf{H}$ denote the basis for $\mathfrak{g}/\mathfrak{h}$ obtained by projecting $\mathfrak{P}_a$ and $\mathfrak{H}$, respectively, via the canonical map $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$. Then the projection $\mathbf{F} \in \Omega^1(U, \mathfrak{g}/\mathfrak{h})$ is given by

$$\mathbf{F} = \theta^a \mathbf{P}_a + \xi \mathbf{H}. \quad (8.6)$$

The possible 4-forms we can construct out of these ingredients are $F \wedge F, F \wedge \mathbf{F} \wedge \mathbf{F}$ and $\mathbf{F} \wedge \mathbf{F} \wedge \mathbf{F} \wedge \mathbf{F}$, which we must contract with all possible $\mathfrak{h}$-invariant tensors in the relevant representations. We now determine them.

8.2. Invariant tensors. The $\mathfrak{h}$-representations of interest are $\circ \mathfrak{g}^*, \mathfrak{g}^* \otimes \wedge^2 (\mathfrak{g}/\mathfrak{h})^*$ and $\wedge^4 (\mathfrak{g}/\mathfrak{h})^*$. Let $\lambda^{ab}, \beta^a, \pi^a, \eta$ be the canonical dual basis for $\mathfrak{g}^*$ with $\pi^a, \eta$ the basis for $(\mathfrak{g}/\mathfrak{h})^* = \mathfrak{b}^0 \subset \mathfrak{g}^*$ canonically dual to $\mathfrak{P}_a, \mathfrak{H}$. Since $\mathfrak{h}$-invariant tensors are in particular $\tau$-invariant, we start by determining the possible $\tau$-invariant tensors in these representations and then look for the subspace of $\tau$-invariant tensors which are annihilated by the action of $B_a$. The $\tau$-invariant tensors are determined using the classic result of Weyl’s [31, Theorem 2.11.A], which says that they are built out of $\delta_{ab}$ and $\epsilon_{abc}$. This results in a 14-dimensional space of $\tau$-invariant tensors spanned by the following:

$$\left(\wedge^4 (\mathfrak{g}/\mathfrak{h})^*\right)^\tau = \left(\frac{1}{2} \epsilon_{abc} \pi^a \wedge \pi^b \wedge \pi^c \wedge \eta\right)$$

$$\left(\circ \mathfrak{g}^*\right)^\tau = \left(\frac{1}{2} \epsilon_{ab} \lambda^{ab}, \frac{1}{2} \epsilon_{abc} \lambda^{ab} \beta^c, \frac{1}{2} \epsilon_{abc} \lambda^{ab} \pi^c, \frac{1}{2} \beta^a \beta_b, \frac{1}{2} \pi^a \pi_a, \frac{1}{2} \eta^2\right)$$

$$\left(\mathfrak{g}^* \otimes \wedge^2 (\mathfrak{g}/\mathfrak{h})^*\right)^\tau = \left(\frac{1}{2} \lambda_{ab} \wedge \pi^a \wedge \pi^b, \frac{1}{2} \epsilon_{abc} \lambda^{ab} \otimes \pi^c \wedge \eta, \frac{1}{2} \epsilon_{abc} \beta^a \otimes \pi^b \wedge \pi^c, \beta_a \wedge \pi^a \wedge \pi^b \wedge \pi^c, \beta_a \otimes \pi^a \wedge \eta, \pi_a \otimes \pi^a \wedge \eta\right). \quad (8.7)$$
To check that we have them all, we may argue as follows. Let $V$ denote the three-dimensional vector representation of $\mathfrak{so}(3)$ and we shall let $R$ denote the one-dimensional trivial representation. It follows that as representations of $\mathfrak{r} \cong \mathfrak{so}(3)$, $\mathfrak{g}/h \cong V \oplus R$ and $g \cong 3V \oplus R$, where we have used that $\wedge^2 V \cong V$. The same holds for their duals, so that $(\mathfrak{g}/h)^* \cong V \oplus R$ and $g^* \cong 3V \oplus R$. It then follows that

$$\wedge^4 (\mathfrak{g}/h)^* \cong \wedge^4 (V \oplus R) \cong R \implies \dim (\wedge^4 (\mathfrak{g}/h)^*)^r = 1,$$

$$\circ^2 g^* \cong \circ^2 (V \oplus R^3 \oplus R) \cong \wedge^2 V \wedge \wedge^2 R^3 \wedge \wedge^2 R \wedge \wedge^2 R \cong V \wedge R^3 \wedge \circ_0^2 V \wedge R^6 \oplus R \wedge R^6 \oplus R \cong 3V \wedge 6 \circ_0^2 V \wedge 7R \implies \dim (\circ^2 g^*)^r = 7,$$

and

$$g^* \wedge (\mathfrak{g}/h)^* \cong (3V \oplus R) \wedge (\mathfrak{g}/h)^* \cong (3V \oplus R) \wedge 2V = 6 (V \wedge V) \wedge 2V = 8V \wedge 6 \circ_0^2 V \wedge 6R \implies \dim (g^* \wedge (\mathfrak{g}/h)^*)^r = 6,$$

where we have used that $V \wedge V = \wedge^2 V \wedge \circ^2 V$ and that $\wedge^2 V \cong V$ and $\circ^2 V = \circ_0^2 V \wedge R$, with $\circ_0^2 V$ the symmetric traceless tensors. In summary, we see that there space of $\mathfrak{r}$-invariant tensors is indeed 14-dimensional and hence the list in equation (8.7) is complete.

The action of $B_a$ on $\mathfrak{g}/h$ can be read off from the brackets in (8.1) by simply projecting the result to $\mathfrak{g}/h$:}

\[
B_a \cdot \overline{P}_b = [B_a, P_b] = \delta_{ab} \overline{H}
\]

\[
B_a \cdot \overline{H} = [B_a, H] = 0.
\]

The action of $B_a$ on $\mathfrak{g}^*$ can also be read off from the brackets in (8.1), even if not as readily, to obtain

\[
B_c \cdot \lambda^{ab} = -\lambda^{ab} \circ ad_{B_c} = -\delta^{ac}_e \pi^b + \delta^{bc}_a \pi^e
\]

\[
B_c \cdot \beta^a = -\beta^a \circ ad_{B_c} = \delta^{ea}_b \pi^c + \delta^{bc}_a \lambda^{ed}
\]

\[
B_c \cdot \pi^a = -\pi^a \circ ad_{B_c} = 0
\]

\[
B_c \cdot \eta = -\eta \circ ad_{B_c} = -\delta_{ac} \pi^a,
\]

where we used (5.6). It is now a simple, albeit somewhat tedious, matter to use these formulæ to determine the space of $h$-invariant tensors in the representations of interest:

\[
\begin{aligned}
\left(\wedge^4 (\mathfrak{g}/h)^*\right)^b &= \left(\frac{1}{4} \epsilon_{abc} \pi^a \wedge \pi^b \wedge \pi^c \wedge \eta\right) \\
\left(\circ^2 g^*\right)^b &= \left(\frac{1}{4} \pi^a_\alpha \pi^\alpha \cdot \frac{1}{2} \epsilon_{abc} \lambda^{ab} \pi^c, \frac{1}{2} \lambda^a_b \lambda^{ab} - \beta^a_b \pi^a - \frac{1}{4} \pi^a_\alpha \right) \\
\left(g^* \wedge (\mathfrak{g}/h)^*\right)^b &= \left(\frac{1}{2} \lambda^a_b \pi^a \wedge \pi^b + \pi^a \otimes \pi^a \wedge \eta, \frac{1}{2} \epsilon_{abc} \pi^a \otimes \pi^b \wedge \pi^c \right).
\end{aligned}
\]

These give rise to six gauge-invariant 4-forms listed in Table 9.

8.3. Variations. We now proceed to vary the gauge-invariant forms in Table 9 with respect to $\omega^{ab}, \psi^a, \theta^a$ and $\xi$. We shall discard those terms which upon application of the Bianchi identities (8.5) vary into exact forms. The variation of the curvature $F$ is given as usual by $\delta F = d \delta A + [A, \delta F]$, which unpacks into the following variations

\[
\begin{aligned}
\delta \Omega^{ab} &= d \nabla \delta \omega^{ab} + \psi^a \wedge \delta \omega^b - \psi^b \wedge \delta \omega^a + \theta^a \wedge \delta \psi^b - \theta^b \wedge \delta \psi^a \\
\delta \psi^a &= d \nabla \delta \psi^a + \xi \wedge \delta \psi^a + \delta \omega^{a}_b \wedge \psi^b - \psi^a \wedge \delta \xi, \\
\delta \theta^a &= d \nabla \delta \theta^a - \xi \wedge \delta \theta^a + \delta \omega^{a}_b \wedge \theta^b + \theta^a \wedge \delta \xi, \\
\delta \xi &= d \delta \xi + \psi_a \wedge \delta \theta^a - \theta_a \wedge \delta \psi^a.
\end{aligned}
\]
and the Euler–Lagrange equations are given by the vanishing of the following:

Similarly, the following two linear combinations also vary into exact forms:

If \( \beta \neq 0 \), then the \( F_\alpha \) and \( G \) equations have no solution unless \( \Lambda = 0 \) as well in which case the lagrangian is identically zero. Hence at least one of \( \beta, \mu \) must be nonzero. At first let us assume that \( \mu \neq 0 \). It will turn out that the equations do not depend on \( \beta \).

The \( E_\alpha \) equation says that \( \theta^a \wedge \Theta^a = 0 \). Inserting that into the \( D_{ab} \) equation we have that

\[
\epsilon_{abc} (\Omega^c \wedge \theta^a - \theta^c \wedge \Xi + \xi \wedge \Theta^a) + \frac{1}{2} \Omega^c \wedge (\epsilon_{acd} \theta_b - \epsilon_{bcd} \theta_a) = 0.
\]
We shall also introduce the shorthand where
\[
\frac{1}{2} \epsilon_{abc} (\theta^b \wedge \Psi^c - \frac{1}{2} \Omega^{bc} \wedge \xi) - \frac{1}{2} \epsilon_{abc} \xi \wedge \theta^b \wedge \theta^c = 0,
\]
and in turn the G equation becomes
\[
\frac{1}{2} \epsilon_{abc} \Omega^{ab} \wedge \theta^c + \Delta \epsilon_{abc} \theta^a \wedge \theta^b \wedge \theta^c = 0.
\]
To go further let us expand the curvatures \( \Omega^{ab} \) and \( \Psi^a \) in terms of the vielbeins:
\[
\Omega_{ab} = \frac{1}{2} R_{abcd} \theta^c \wedge \theta^d + S_{abc} \theta^c \wedge \xi,
\]
\[
\Psi_a = \frac{1}{2} T_{abc} \theta^b \wedge \theta^c + U_{ab} \theta^b \wedge \xi.
\]
We shall also introduce the shorthand \( \lambda := \frac{\Delta}{\mu} \). The G equation says that
\[
\frac{1}{2} \epsilon_{abc} (\frac{1}{2} R^{ab}_{\quad de} \theta^d \wedge \theta^c + S^{ab}_{\quad d} \theta^d \wedge \xi + \frac{1}{2} \lambda \theta^a \wedge \theta^b) \wedge \theta^c = 0,
\]
which breaks up into two equations:
\[
\frac{1}{2} \epsilon_{abc} S^{ab}_{\quad d} \theta^c \wedge \theta^d \wedge \xi = 0 \quad \Rightarrow \quad S^{ab}_{\quad b} = 0 \quad \text{and} \quad \frac{1}{2} \epsilon_{abc} R^{ab}_{\quad de} \theta^c \wedge \theta^d \wedge \theta^e = 0 \quad \Rightarrow \quad R = 2 \lambda,
\]
where \( R := R^{ab}_{\quad ba} \). The \( F_a \) equation becomes
\[
\epsilon_{abc} \theta^b \wedge (\frac{1}{2} T^{c}_{\quad de} \theta^d \wedge \theta^c + \bar{U}^c_{\quad d} \theta^d \wedge \xi) - \frac{1}{2} \epsilon_{abc} R^{bc}_{\quad de} \theta^d \wedge \theta^c \wedge \xi - \frac{1}{2} \epsilon_{abc} \lambda \theta^a \wedge \theta^b \wedge \theta^c = 0,
\]
which again breaks up into two equations. The first equation says
\[
\frac{1}{2} \epsilon_{abc} T^{c}_{\quad de} \theta^b \wedge \theta^d \wedge \theta^e = 0 \quad \Rightarrow \quad T^{b}_{\quad ba} = 0,
\]
and the second equation says that
\[
(\epsilon_{ade} U^c_{\quad e} - \frac{1}{2} \epsilon_{abc} R^{bc}_{\quad de} - \frac{1}{2} \epsilon_{ade} \lambda) \theta^d \wedge \theta^e \wedge \xi = 0,
\]
which we will analyse after we take into account the Bianchi identities (8.23). The first of the Bianchi identities becomes
\[
\theta^a \wedge (\frac{1}{2} T_{abc} \theta^b \wedge \theta^c + U_{ab} \wedge \theta^b \wedge \xi) = 0 \quad \Rightarrow \quad T_{[abc]} = 0 \quad \text{and} \quad U_{[ab]} = 0.
\]
The second Bianchi identity becomes
\[
(\frac{1}{2} R_{abcd} \theta^c \wedge \theta^d + S_{abc} \theta^c \wedge \xi) \wedge \theta^b = 0 \quad \Rightarrow \quad R_{[abcd]} = 0 \quad \text{and} \quad S_{[abc]} = 0.
\]
The latter condition, together with the fact that \( S_{abc} = -S_{bac} \) says that \( S_{abc} = 0 \); whereas the former condition says that \( R_{abcd} \) is an algebraic curvature tensor, which in three dimensions is determined by its Ricci tensor \( R_{ad} := \delta^{bc} R_{abcd} \) via the Ricci decomposition:
\[
R_{abcd} = R_{ad} \delta_{bc} - R_{bd} \delta_{ac} + R_{bc} \delta_{ad} - R_{ac} \delta_{bd} + \frac{1}{2} R (\delta_{ac} \delta_{bd} - \delta_{bc} \delta_{ad}).
\]
We now return to equation (8.31), which after relabelling indices and using the definition of the Ricci tensor and the fact that \( \bar{R} = 2 \lambda \), becomes
\[
U_{ab} = -U_{bd} + R_{ab} = 0,
\]
where \( U = \delta^{ab} U_{ab} \). Taking the trace we see that \( \bar{U} = \frac{1}{2} \bar{R} \) and hence that \( -U_{ab} \) is the Einstein tensor:
\[
U_{ab} = -(R_{ab} - \frac{1}{2} \bar{R} \delta_{ab}) = -R_{ab} + \lambda \delta_{ab}.
\]
Finally, let us write \( T^a_{\quad bc} = \epsilon_{bcd} W^{ad} \). Then \( T^a_{\quad ab} = 0 \) and \( T_{[abc]} = 0 \) imply that \( W^{ab} \) is symmetric and traceless.
We may summarise the preceding discussion as follows, where we have put \( \mu = 1 \) without loss of generality.

**Proposition 5.** The solution of the Euler–Lagrange equations for the lagrangian

\[
\mathcal{L} = \frac{1}{2} \epsilon_{abc} \Omega^{ab} \wedge \Theta^c + \frac{1}{2} \Theta^a \wedge \Theta_a + \frac{1}{6} \epsilon_{abc} \theta^a \wedge \theta^b \wedge \theta^c \wedge \xi,
\]

are such that \( \Xi = \Theta^a = 0 \) and

\[
\Omega_{ab} = R_{ac} \theta_b \wedge \theta^c - R_{bc} \theta_a \wedge \theta^c + 2 \Lambda \theta_a \wedge \theta_b,
\]

\[
\Psi_a = \frac{1}{2} \epsilon^{bcd} W_{ab} \theta_c \wedge \theta_d - (R_{ab} - \Lambda \delta_{ab}) \theta^b \wedge \xi,
\]

where \( W_{ab} \) is symmetric and traceless.

9. **Conclusion and outlook**

In this work we framed the construction of gravitational lagrangians via the “gauging of spacetime algebras” in terms of Cartan geometry. We emphasized that, in this procedure, it is in fact not a spacetime symmetry algebra that is being gauged but rather a Klein pair \((g, h)\) determining the underlying homogeneous space, or flat model, of the Cartan geometry. In Section 4 we reviewed the classic example of the Klein pair \((\text{iso}(3,1), \text{so}(3,1))\) corresponding to four-dimensional Minkowski spacetime. The resulting lagrangian (4.8) turned out to be the Hilbert-Palatini lagrangian together with the Holst term, a cosmological constant term and topological terms (Nieh–Yan, Pontryagin and Gauss–Bonnet). In Sections 5 and 6 we subsequently applied the gauging procedure to the carrollian homogeneous spaces \(\mathcal{A}_{dSC}\) and \(\mathcal{C}_{spacetime}\), which resulted in the lagrangians (5.19) and (6.2), respectively. The terms in these lagrangians can be identified with carrollian counterparts of Hilbert–Palatini, Holst, and cosmological constant terms. In addition, there are carrollian analogues of the Nieh–Yan, Pontryagin and Gauss–Bonnet topological terms, as well as other topological terms (in the reductive cases) which are intrinsically carrollian and seem to have no relativistic analogues. Furthermore, we showed explicitly in Section 6 that the lagrangian of flat Carroll space can be recovered as a limit of a relativistic lagrangian. Finally, we applied the gauging procedure to the four-dimensional lightcone, the only remaining four-dimensional carrollian space in the classification of [3], and determined its lagrangian to be given by (8.18).

We will close this work with a few comments on the generality of the approach and possible applications of the theories constructed herein.

**Do we get all gauge-invariant lagrangians?** The approach presented in this work is rather general, as it can be applied to the construction of gravitational theories for all kinematical spacetimes considered, e.g., in [3]. However, given a kinematical spacetime one cannot recover all gauge-invariant, gravitational lagrangians with second-order equations of motion using the gauging procedure. Consider, for instance, the lagrangian for “electric Carroll gravity”

\[
\mathcal{L}_e = \frac{1}{4} \text{dvol} (\mathcal{L}_e \mu \nu \gamma_{\mu \nu} (\gamma_{\mu \nu} \gamma_{\rho \sigma} - \gamma_{\mu \nu} \gamma_{\rho \sigma}))
\]

originally found by Henneaux in [19] that recently resurfaced in works such as [8, 23]. This lagrangian leads to second-order equations of motion and can be obtained from a limiting procedure of Einstein gravity in the second-order formulation using either hamiltonian [19, 23] or lagrangian [8] methods. However, as we will discuss below, it is not equivalent to the lagrangian (5.19) for any choice of parameters. It thus seems likely that also for the other carrollian Klein pairs there exist theories that are not described by the lagrangians we have constructed in this paper.
A different route to building invariant lagrangians. With the geometric interpretation of Section 7 comes a different way of building H-gauge invariant lagrangians, namely by using that any gauge-invariant 4-form $X$ is of the form

$$X = F \, d\text{vol}$$

where $F \in \mathcal{C}^\infty(M)$ is a locally H-invariant function built from the carrollian structure and its "inverse". This perspective was taken in [20] and allows for the construction of a much larger class of lagrangians.

Given a carrollian structure $(\kappa^\mu, h_{\mu\nu})$, the simplest invariant object that we can write down is the second fundamental form

$$K_{\mu\nu} = \frac{1}{2} L_\kappa h_{\mu\nu},$$

which captures the intrinsic torsion of the carrollian structure [32]. By construction, this object satisfies $\kappa^\mu K_{\mu\nu}$, and is thus spatial. Using Cartan’s magic formula, we may relate $K_{\mu\nu}$ to the torsion $\Theta^a$, since

$$L_\kappa \theta^a = t_\kappa \Theta^a - t_\kappa (\omega^a_b) \theta^b,$$

which in components becomes

$$L_\kappa \theta^a_{\mu} = \kappa^\rho \Theta^\rho_{\mu a} - \kappa^\rho \omega^\rho_{\mu b} \theta^b_{\mu}.$$  

This means that

$$K_{\mu\nu} = 2 \delta_{ab} \theta^b_{\mu} L_\kappa \theta^a_{\nu} = 2 \kappa^\rho \Theta^\rho_{\mu a} - 2 \kappa^\rho \omega^\rho_{\mu b} \theta^b_{\nu},$$

where we used antisymmetry of $\omega^a_{\mu b}$. Now, using the object $\gamma^\mu_{\nu} = \delta_{ab} e^a_{\mu} e^b_{\nu}$, which transforms as follows under local Carroll boosts

$$\delta \gamma^\mu_{\nu} = 2 \kappa^\mu \lambda^\nu,$$

where $\lambda^\mu = \gamma^\mu_{\nu} \lambda^\nu$ is the boost parameter, we may build the following invariants out of the second fundamental form (9.3)

$$\gamma^\mu_{\nu} \gamma^\rho_{\sigma} K_{\mu\nu} K_{\rho\sigma} := K_{\mu\nu} K_{\mu\nu}$$

Using these objects, the simplest Carroll-invariant lagrangian that is of the second order in derivatives takes the form

$$\mathcal{L} = d\text{vol} \left( K_{\mu\nu} K_{\mu\nu} - K^2 \right),$$

which is the electric theory (9.1).

The preceding discussion also makes clear the fact that the electric theory is not equivalent to any of the theories we constructed for any choice of parameters. The theory (9.9) allows for solutions with non-vanishing $K_{\mu\nu}$, see e.g., [8] for the explicit equations of motion. However, for all theories we construct the component $\kappa \cdot \Theta^a$ of the torsion is set to zero, even though $\Theta^a$ does not need to vanish in general, e.g., for the lagrangian (5.19) with $\mu = 0$. Given equation (9.6) it is then clear that for all of our theories $K_{\mu\nu} = 0$.

It would be interesting to see whether at least some of the more general theories of the form (9.2), in particular the electric theory (9.9), can be obtained by coupling the lagrangians considered in this work to suitably chosen matter fields or by loosening some of the assumptions of our gauging procedure. We leave this for future study.

Matter couplings. In this paper we have restricted attention to pure gravity theories, but as in any gauge theory, gravity theories may be coupled to matter. A Cartan connection on a principal H-bundle $P \to M$ defines a covariant derivative on sections of any associated bundle. For example, if $V$ is a linear representation of $H$, we may form an associated vector bundle $P \times_H V$,

\footnote{In contrast to Section 7, we no longer take the torsion to be zero. This means that the affine connection is no longer given by (7.17), although the expression (7.11), relating the affine connection to the Cartan connection continues to hold.}
whose sections can be interpreted as $H$-equivariant functions $P \to V$. The covariant derivative induced by the Cartan connection on such sections allows us to write down gauge-invariant terms in the lagrangian describing the coupling of such matter to gravity. Similarly, if $N$ is a manifold on which $H$ acts smoothly, we may form an associated fibre bundle $P \times_H N$ whose sections carry a nonlinear realisation of $H$ and which may be differentiated covariantly. It might be interesting to explore Cartan geometries defined via the Euler–Lagrange equations involving matter couplings.

**Topological terms and asymptotic symmetries.** In the case of Einstein gravity in asymptotically flat spacetimes, one finds that the Poincaré symmetries are enhanced asymptotically to infinite-dimensional BMS symmetries [35, 36], which have recently garnered attention in the context of flat space holography. It has been argued that these symmetries need to be further enhanced by so-called dual supertranslations that arise most naturally from the first-order lagrangian (4.8) with the Holst term included [37, 38]. Asymptotic symmetries of carrollian gravity were studied in [39, 40]. It would be interesting to see whether the symmetries found in these works get additional contributions from the analogous terms in our carrollian lagrangians (5.19) and (6.2).

**Other dimensions.** We have considered only $(3 + 1)$-dimensional theories. However, it is clear that our approach is applicable to any spacetime dimension depending on which one could find additional contributions to the lagrangian. For instance, for odd-dimensional spacetimes one could add Chern–Simons terms to the lagrangian. For lower-dimensional models of carrollian gravity see, e.g., [41, 42, 43] for $(2 + 1)$ and [44, 45] for $(1 + 1)$-dimensional toy-models. In a different context, the $(2 + 1)$-dimensional lightcone theory has been discussed in [46].

**Solution spaces.** In this work we have largely restricted ourselves to the construction of gravitational lagrangians for carrollian theories and analysing the equations of motion. We have not attempted to obtain solutions to the corresponding equations of motion in terms of the vielbeins. As is well-known, the equations of motion of the relativistic lagrangian (4.8) lead to a plethora of interesting metrics such as gravitational waves or (colliding) black holes to name but two such classes of metrics. Thinking of the carrollian theories as arising from a limit of General Relativity, one would expect the solution spaces of the carrollian theories to contain fewer interesting solutions, which, however, might be easier to construct. Nevertheless, these solutions can still have interesting physical interpretations, such as describing the dynamics of General Relativity near a spatial singularity; cf. [19], for example. A better understanding of the equations of motion of the carrollian theories would therefore be of interest, cf. [8, 9, 22, 39, 40].

**Gravitational vacua and (pseudo-)carrollian spaces.** The above-mentioned enhancement of Poincaré to BMS at null infinity in asymptotically flat spacetimes implies that the (radiative) vacuum of gravity in asymptotically flat spacetimes is infinitely degenerate [47]. The different gravitational vacua are related by supertranslations and superrotations. At null infinity, these vacua can be understood in terms of Cartan geometry based on a certain homogeneous space of the Poincaré group [48, 49]. In [46], it was shown that the superrotation sector can be derived from a three-dimensional action.

The enhancement of Poincaré to BMS can also be demonstrated at spatial [50, 51, 52] and time-like infinity [53]. In the recent work [4], we showed that both space-like and time-like infinities can be understood as (pseudo-)carrollian, homogeneous spaces of the Poincaré group. In particular, the blow-up of time-like infinity can be described by AdSC. It is thus suggestive that (a constrained version of) the lagrangian (5.19) for AdSC describes the BMS vacuum sector of General Relativity near time-like infinity. Similar comments apply to the case of space-like infinity with associated pseudo-carrollian Klein pair $\mathcal{S}_i$, that is related to the Ashtekar–Hansen structure at spatial infinity [14, 54]. While we have not discussed the gauging of this Klein pair, it can be obtained from (5.19) with only minor adjustments.
General Relativity: The view from timelike infinity. More speculative, but also more rewarding, is the idea that the relation between General Relativity on asymptotically flat spacetimes and the gaugings of AdS/C and Spi extends also away from the vacuum sector. In [4] it was shown how points in Minkowski spacetime correspond to certain higher-dimensional geometries in AdS/C/Sp. Perhaps the lagrangian (5.19) based on the Klein pair of AdS/C and its solutions encode certain aspects of General Relativity on asymptotically flat spacetimes. Since the map between the flat models of the two Cartan geometries, Minkowski and AdS/C/Sp, is non-local, the putative construction would likely have a twistorial flair. We leave this interesting possibility for further study.

ACKNOWLEDGMENTS

We are grateful to Jelle Hartong, Yannick Herfray, Niels Obers and Alfredo Pérez for useful discussions.

The work of EH is supported by the Royal Society Research Grant for Research Fellows 2017 “A Universal Theory for Fluid Dynamics” (grant number RGF\R1\180017).

SP was supported by the Leverhulme Trust Research Project Grant (RPG-2019-218) “What is Non-Relativistic Quantum Gravity and is it Holographic?”.

JS was supported by a Marina Solvay-fellowship and the F.R.S.-FNRS Belgium through the convention IISN 4.4503.

SP and JS acknowledge support of the Erwin Schrödinger Institute (ESI) in Vienna where part of this work was conducted during the thematic programme “Geometry for Higher Spin Gravity: Conformal Structures, PDEs, and Q-manifolds”. EH and SP would like to thank the organisers of the Carroll Workshop in Vienna where part of this work was completed.

Appendix A. The Ricci tensor of a torsion-free affine connection

In this appendix we give a proof of the following well-known result.

Proposition A.1. Let $M$ be an $n$-dimensional orientable manifold with a torsion-free affine connection $\nabla$. Then the Ricci tensor $\nabla^\omega$ is symmetric if and only if around every point of $M$ there exists a locally defined parallel volume form.

Proof. Let $\omega\in\Omega^n(M)$ be an orientation. Then for all vector fields $X\in\mathfrak{X}(M)$, the $n$-form $\nabla_X\omega$ is proportional to $\omega$: $\nabla_X\omega = f_X\omega$ for some function $f_X\in C^\infty(M)$ depending on $X$. Since $\nabla_X$ is tensorial in $X$, there exists a one-form $\alpha\in\Omega^1(M)$ such that $f_X = \alpha(X)$. The action of the curvature operator $R^\nabla(X,Y)$ on $\omega$ is

$$R^\nabla(X,Y)\cdot\omega = \nabla_{[X,Y]}\omega - [\nabla_X,\nabla_Y]\omega$$

$$= \alpha([X,Y])\omega - \nabla_X(\alpha(Y)\omega) + \nabla_Y(\alpha(X)\omega)$$

$$= \alpha([X,Y])\omega - (\nabla_X\alpha)(Y)\omega - \alpha(\nabla_XY)\omega - \alpha(Y)\nabla_X\omega$$

$$+ (\nabla_Y\alpha)(X)\omega + \alpha(\nabla_YX)\omega + \alpha(X)\nabla_Y\omega$$

$$= -[\nabla_X\alpha](Y)\omega + [\nabla_Y\alpha](X)\omega$$

$$= -d\alpha(X,Y)\omega,$$

(A.1)

where we have used that $\nabla$ is torsion-free. Now the curvature operator $R^\nabla(X,Y)$ acts on $n$-forms via scalar multiplication by the negative of the trace $\text{Tr}(Z\mapsto R^\nabla(X,Y)Z)$ of the curvature operator acting on vector fields. Using the algebraic Bianchi identity (in the absence of torsion) and the
linearity of the trace, we have that

\[
- \text{Tr}(Z \mapsto R^\nabla(X,Y)Z) = \text{Tr}(Z \mapsto R^\nabla(Z,X)Y) + \text{Tr}(Z \mapsto R^\nabla(Y,Z))X
\]

\[
= \text{Tr}(Z \mapsto R^\nabla(Z,X)Y) - \text{Tr}(Z \mapsto R^\nabla(Z,Y)X)
\]

\[
= -r^\nabla(X,Y) + r^\nabla(Y,X).
\]

In other words, we have that

\[
r^\nabla(X,Y) - r^\nabla(Y,X) = d\alpha(X,Y),
\]

so that the Ricci tensor is symmetric if and only if \(d\alpha = 0\). By the Poincaré Lemma, \(d\alpha = 0\) if and only if locally \(\alpha = df\) for some locally-defined function \(f\). But in this case,

\[
\nabla_X \omega = df \cdot \omega = X(f) \omega
\]

and we can then modify \(\omega\) locally to \(d\text{vol} := e^{-f} \omega\) so that \(\nabla_X d\text{vol} = 0\).

In Section 7 we showed that the Cartan connection defines a locally-defined volume form which is parallel relative to the affine connection \(\nabla\) it defines. Therefore, we conclude that its Ricci tensor is symmetric.

Let \(\nabla\) be a torsion-free affine connection on an orientable manifold \(M\) whose Ricci tensor is symmetric and let \(\nabla'\) be a second torsion-free affine connection on \(M\). Let \(K_X Y := \nabla_X Y - \nabla'_X Y\) denote the contorsion \((1,2)\)-tensor field and define a one-form \(\beta \in \Omega^1(M)\) by \(\beta(X) := \text{Tr}(Y \mapsto K_X Y)\).

**Proposition A.2.** The Ricci tensor of \(\nabla'\) is symmetric if and only if \(d\beta = 0\).

**Proof.** By Proposition A.1, the Ricci tensor \(r^\nabla\) is symmetric if and only if there exists about every point a locally-defined parallel volume form \(d\text{vol}\) which is parallel with respect to \(\nabla\). Let \(d\text{vol}\) be a locally-defined parallel volume form for \(\nabla\). Then \(d\text{vol} = f d\text{vol}\) for some locally-defined nowhere-vanishing function \(f\). Without loss of generality we can assume that \(d\text{vol}\) and \(d\text{vol}\) define the same orientation so that \(f\) is positive, say \(f = e^g\) for some locally-defined smooth function \(g\). Then

\[
\nabla_X d\text{vol} = \nabla_X (e^g d\text{vol})
\]

\[
= e^g X(g) d\text{vol} + e^g \nabla_X d\text{vol}
\]

\[
= X(g) e^g d\text{vol} + e^g K_X \cdot d\text{vol}
\]

\[
= e^g (X(g) - \beta(X)) d\text{vol},
\]

so that \(d\text{vol}\) is parallel if and only if \(\beta(X) = X(g) = dq(X)\) for all vector fields \(X\) and for some function \(g\). In other words, \(\beta = dq\), so that it is locally exact or, equivalently, it is closed.

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