A MORITA THEOREM FOR ALGEBRAS OF OPERATORS ON HILBERT SPACE

DAVID P. BLECHER

Abstract. We show that two operator algebras are strongly Morita equivalent (in the sense of Blecher, Muhly and Paulsen) if and only if their categories of operator modules are equivalent via completely contractive functors. Moreover, any such functor is completely isometrically isomorphic to the Haagerup tensor product (= interior tensor product) with a strong Morita equivalence bimodule.

Date: September, 1998.
* Supported by a grant from the NSF.
1. Notation, background and statement of the theorem

Around 1960, in pure algebra, arose the notion of Morita equivalence of rings. Two rings $A$ and $B$ were defined to be Morita equivalent if the two categories $A_{MOD}$ and $B_{MOD}$ of modules are equivalent. The fundamental theorem from those early days of that subject (see [19, 22, 4]) is that these categories are equivalent if and only if there exists a pair of bimodules $X$ and $Y$ such that $X \otimes_B Y \cong A$ and $Y \otimes_A X \cong B$ as bimodules. The theorem goes on to describe these so-called 'equivalence bimodules' and how they arise, and the implications for $A$ and $B$.

In the early 70's M. Rieffel introduced and developed the notion of strong Morita equivalence of $C^*$-algebras (see [27] for a good discussion and survey). It has become a fundamental tool in modern operator algebra and noncommutative geometry (see [16] for example). Rieffel defined strong Morita equivalence in terms of the existence of a certain type of bimodule, possessing certain $C^*$-algebra valued positive definite inner products. Until recently there was no description in terms of a categorical equivalence. Except for the absence of such a theorem, the basic features from pure algebra were shown to carry over quite beautifully. Of course one expects, and obtains, stronger (functional analytic) variants of these basic features. As just one example: in pure algebra, one finds that $B \cong p M_n(A)p$, for a projection $p$ in the $n \times n$ matrices $M_n(A)$. The same thing is true in the case of unital strongly Morita equivalent $C^*$-algebras $A$ and $B$, except that $p$ is an orthogonal projection, and the $\cong$ means 'as $C^*$-algebras', i.e. (isometrically) $*$-isomorphically.

In [9] we showed that two $C^*$-algebras are strongly Morita equivalent if and only if their categories of (left) operator modules (defined below) are equivalent via completely contractive functors. Moreover, any such functor is completely isometrically isomorphic to the Haagerup tensor product (=$\text{interior tensor product [21, 7]}) with an equivalence bimodule.

Here we generalize this result to possibly nonselfadjoint operator algebras, that is, to general norm closed algebras of operators on Hilbert space. Thus we answer the main remaining theoretical question from our study of Morita equivalence of possibly nonselfadjoint operator algebras, begun in [12]. The various ingredients of our proof shows how the algebra and functional analytic structures, in particular, the geometry of the associated Hilbert spaces, are intricately connected. Some major tools, such as von Neumann’s double commutant theorem, do not exist for nonselfadjoint operator algebras; to overcome this we use the theory of $C^*$—dilations of operator modules developed in [10], to transfer the problem to the $C^*$—algebra scenario, where we may more or less use our earlier proof of [9], and the lowersemmontinuity argument on the quasistate space which we used there. Of necessity some of our argument consists of instructions on how to follow along and adapt steps in the proof in [9]. In order to not try the readers patience more than needs be, we attempted to keep these instructions minimal, yet sufficient.

Let us begin by establishing the common symbols and notations in this paper. We shall use operator spaces and completely bounded maps quite extensively, and their connections to operator algebras, operator modules and $C^*$—modules. We refer the reader to [8, 23, 12, 7, 9, 10] for missing background. It is perhaps worth saying to the general reader that it has been clear for some time that to understand a general operator algebra $A$ or operator module, it is necessary not only to take into account the norm, but also the natural norm on $M_n(A)$. That is one of the key perspectives of operator space theory. Hence we are not interested in bounded linear transformations, rather we look for the completely bounded, completely isometric, or completely contractive maps - where the

---

1We note that the work of Morita on purely algebraic equivalence, and many related consequences, was summarized and popularized by Bass as a collection of theorems known as Morita I, II and III (see [4, 19]). Most of the appropriate version of ‘Morita I’ was proved for $C^*$—algebras by Rieffel, and for general operator algebras in [12]. Our main theorem here is a ‘Morita II’ theorem for (possibly nonselfadjoint) operator algebras. The appropriate version of ‘Morita III’ follows easily from what we have done, as in pure algebra, and is omitted.
adjective ‘completely’ means that we are applying our maps to matrices too. This is explained at length in the references mentioned above. The algebraic background needed may be found in any account of Morita theory for rings, such as [14] or [19].

We will use the symbols $\mathcal{A}, \mathcal{B}$ for operator algebras. We shall assume that our operator algebras have contractive approximate identities (c.a.i.’s). It is well known that every $C^*$-algebra is an operator algebra in this sense. We write $\mathcal{C}$ and $\mathcal{D}$ for the universal or maximal $C^*$-algebras generated by $\mathcal{A}$ and $\mathcal{B}$ respectively, see §2 of [10]. The symbol $r_v$ will always mean the ‘right multiplication by $v$’ operator, namely $x \mapsto xv$, whose domain is usually the algebra $\mathcal{A}$ or $\mathcal{C}$. We will use the letters $\mathcal{H}$ and $\mathcal{K}$ for Hilbert spaces, $\zeta, \eta$ are typical elements in $\mathcal{H}$ and $\mathcal{K}$ respectively, and $B(H)$ (resp. $B(H, K)$) is the space of bounded linear operators on $H$ (resp. from $H$ to $K$).

Suppose that $\pi$ is a completely contractive representation of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$, and that $X$ is a closed subspace of $B(H)$ such that $\pi(\mathcal{A})X \subset X$. Then $X$ is a left $\mathcal{A}$-module. We shall assume that the module action is nondegenerate (= essential). We say that such $X$, considered as an abstract operator space and a left $\mathcal{A}$-module, is a left operator module over $\mathcal{A}$. By considering $X$ as an abstract operator space and module, we may forget about the particular $\mathcal{H}, \pi$ used. An obvious modification of a theorem of Christensen, Effros and Sinclair [17] tells us that the operator modules are (up to completely isometric isomorphism) exactly the operator spaces $X$ which are (nondegenerate) left $\mathcal{A}$-modules, such that the module action is a ‘completely contractive’ bilinear map (that is $\|ax\| \leq \|a\|\|x\|$ for matrices $a$ and $x$ with entries in $\mathcal{A}$ and $X$ respectively) or equivalently, the module action linearizes to a complete contraction $\mathcal{A} \otimes_h X \to X$, where $\otimes_h$ is the Haagerup tensor product. Such an $X$ is referred to as an abstract operator module. We will use the facts that submodules and quotient modules of operator modules, are again operator modules.

We write $\mathcal{A}OMOD$ for the category of left $\mathcal{A}$-operator modules. The morphisms are $\mathcal{A}CB(X, W)$, the completely bounded left $\mathcal{A}$-module maps. If $X$ is also a right $\mathcal{B}$-module, then $\mathcal{A}CB(X, W)$ is a left $\mathcal{B}$-module where $(bT)(x) = T(bx)$, or equivalently, $bT = Tr_b$. We will write $\mathcal{A}CB^{ess}(X, W)$ for the subset consisting of such maps $bT$, for $b \in \mathcal{B}$. If $X, W \in \mathcal{A}OMOD$ then $\mathcal{A}CB(X, W)$ is an operator space [13]. In this paper, when $X, W$ are operator modules or bimodules, and when we say ‘$X \cong W$’, or ‘$X \cong W$ as operator modules’, we will mean that the implicit isomorphism is a completely isometric module map.

We will need the following important principle from §3 of [10] which we shall use several times here without comment: an isometric surjective $\mathcal{A}$-module map between two Banach $\mathcal{C}$-modules, is a $\mathcal{C}$-module map. This shows that the ‘forgetful functor’ $\mathcal{C}OMOD \to \mathcal{A}OMOD$, embeds $\mathcal{C}OMOD$ as a (non-full) subcategory of $\mathcal{A}OMOD$. To an algebraist, it may be helpful to remark that it is a reflective subcategory in the sense of [14]. The $C^*$-dilation, or maximal dilation, referred to earlier, is the left adjoint of this forgetful functor; and it can be explicitly described as the functor $\mathcal{C} \otimes_h \mathcal{A} \to \mathcal{C}$. Here $\otimes_h \mathcal{A}$ is the module Haagerup tensor product studied in [14]. We will repeatedly use the fact (3.11 in [10]) that the ‘obvious map’ $V \to \mathcal{C} \otimes_h \mathcal{A} V$, is completely isometric, thus $V$ is an $\mathcal{A}$-submodule of its maximal dilation.

We now turn to the category $\mathcal{A}HMOD$ of Hilbert spaces $\mathcal{H}$ which are left $\mathcal{A}$-modules via a nondegenerate completely contractive representation of $\mathcal{A}$ on $\mathcal{H}$. If $\mathcal{A}$ is a $C^*$-algebra, then this is the same as the category of nondegenerate $*-$representations of $\mathcal{A}$ on Hilbert space. By the universal property of the maximal generated $C^*$-algebra, $\mathcal{A}HMOD = \mathcal{C}HMOD$ as objects. In [12] we showed how $\mathcal{A}HMOD$ may be viewed as a subcategory of $\mathcal{A}OMOD$ (see the discussion at the end of Chapter 2, and after Proposition 3.8, there). Briefly, if $\mathcal{H} \in \mathcal{A}HMOD$ then if $\mathcal{H}$ is equipped with its ‘Hilbert column’ operator space structure $\mathcal{H}^c$, then $\mathcal{H}^c \in \mathcal{A}OMOD$. Conversely, if $V \in \mathcal{A}OMOD$ is also a Hilbert column space, then the associated representation $\mathcal{A} \to B(V)$ is completely contractive and nondegenerate. It is well known that for a linear map
$T : H \to K$ between Hilbert spaces, the usual norm equals the completely bounded norm of $T$ as a map $H^c \to K^c$. Thus we see that the assignment $H \mapsto H^c$ embeds $\mathcal{A} \text{MOD}$ as a full subcategory of $\mathcal{A} \text{OMOD}$. In future, if a Hilbert space is referred to as an operator space, it will be with respect to its column operator space structure, unless specified to the contrary.

We are concerned with functors between categories of operator modules. Such functors $F : \mathcal{A} \text{OMOD} \to \mathcal{B} \text{OMOD}$ are assumed to be linear on spaces of morphisms. Thus $T \mapsto F(T)$, from the space $\mathcal{A} \text{CB}(X,W)$ to $\mathcal{B} \text{CB}(F(X), F(W))$, is linear, for all pairs of objects $X, W \in \mathcal{A} \text{OMOD}$. We say $F$ is completely contractive, if this map $T \mapsto F(T)$ is completely contractive, for all pairs of objects $X, W \in \mathcal{A} \text{OMOD}$. We say two functors $F_1, F_2 : \mathcal{A} \text{OMOD} \to \mathcal{B} \text{OMOD}$ are (naturally) completely isometrically isomorphic, if they are naturally isomorphic in the sense of category theory \cite{1, 12}, with the natural transformations being complete isometries. In this case we write $F_1 \cong F_2$ completely isometrically.

**Definition 1.1.** We say that two operator algebras $\mathcal{A}$ and $\mathcal{B}$ are (left) operator Morita equivalent if there exist completely contractive functors $F : \mathcal{A} \text{OMOD} \to \mathcal{B} \text{OMOD}$ and $G : \mathcal{B} \text{OMOD} \to \mathcal{A} \text{OMOD}$, such that $FG \cong \text{Id}$ and $GF \cong \text{Id}$ completely isometrically. Such $F$ and $G$ will be called operator equivalence functors.

There is an obvious adaption to ‘right operator Morita equivalence’, where we are concerned with right operator modules. We remark that for $\mathcal{C}^*$-algebras it is easy to show that left operator Morita equivalence implies right operator Morita equivalence, but this seems much harder for nonselfadjoint operator algebras, although we shall see that it is true.

In \cite{12} we generalized strong Morita equivalence of $\mathcal{C}^*$-algebras to possibly nonselfadjoint operator algebras:

**Definition 1.2.** Two operator algebras $\mathcal{A}$ and $\mathcal{B}$ are strongly Morita equivalent if there exists an $\mathcal{A} - \mathcal{B}$-operator bimodule $X$, and a $\mathcal{B} - \mathcal{A}$-operator bimodule $Y$, such that $X \otimes_h \mathcal{B} Y \cong \mathcal{A}$ completely isometrically and as $\mathcal{A}$-bimodules, and such that $Y \otimes_h \mathcal{A} X \cong \mathcal{B}$ completely isometrically and as $\mathcal{B}$-bimodules. We say that $X$ is an $\mathcal{A} - \mathcal{B}$-strong Morita equivalence bimodule.

The above is not quite the definition given in \cite{12}, although we remarked, without giving a proof, that it is an equivalent definition. Essentially it is the same proof of the corresponding result in pure algebra (see \cite{13} or \cite{14} 12.12.3 and 12.13). In our scenario there is really only one new point, that is the element $u$ described in these texts is not in $\mathcal{A}$, but in $\mathcal{C} \mathcal{B} \mathcal{A}(\mathcal{A}, \mathcal{A}) = \mathcal{R} \mathcal{M}(\mathcal{A})$, the right multipliers of $\mathcal{A}$. However it commutes with $\mathcal{A}$ in $\mathcal{R} \mathcal{M}(\mathcal{A})$, so it falls in the center of the multiplier algebra $\mathcal{M}(\mathcal{A})$ of $\mathcal{A}$, and in fact it is a unitary there. The rest of the proof carries through quite obviously.

We can now state our main theorem.

**Theorem 1.3.** Two operator algebras $\mathcal{A}$ and $\mathcal{B}$ with contractive approximate identities are strongly Morita equivalent if and only if they are left operator Morita equivalent, and if and only if they are right operator Morita equivalent. Suppose that $F, G$ are the left operator equivalence functors, and set $Y = F(\mathcal{A})$ and $X = G(\mathcal{B})$. Then $X$ is an $\mathcal{A} - \mathcal{B}$-strong Morita equivalence bimodule, and $Y$ is a $\mathcal{B} - \mathcal{A}$-strong Morita equivalence bimodule, which is unitarily equivalent to the dual operator module $\tilde{X}$ of $X$. Moreover, $F(V) \cong Y \otimes_h \mathcal{A} V \cong \mathcal{A} \mathcal{K}(X,V)$ completely isometrically isomorphically (as $\mathcal{B}$-operator modules), for all $V \in \mathcal{A} \text{OMOD}$. Thus $F \cong Y \otimes_h \mathcal{A} \cong \mathcal{A} \mathcal{K}(X,-)$ completely isometrically. Similarly $G \cong X \otimes_h \mathcal{B} \cong \mathcal{B} \mathcal{K}(Y,-)$ completely isometrically. Also $F$ and $G$ restrict to equivalences of the subcategory $\mathcal{A} \text{HMOD}$ with $\mathcal{B} \text{HMOD}$, the subcategory $\mathcal{C} \text{HMOD}$ with $\mathcal{D} \text{HMOD}$, and the subcategory $\mathcal{C} \text{OMOD}$ with $\mathcal{D} \text{OMOD}$. 
Lemma 2.2. If \( V \) is completely isometrically isomorphic, where \( R \) is an isometric isomorphism of algebras. Moreover if \( T \) is a completely isometric isomorphism \( \hat{X} \) mentioned in the theorem, we prove the analogue of the results in pure algebra known as ‘Morita I’ (see [9, 4]) That \( \hat{X} \cong Y \) will follow from Theorem 4.1 (see also 4.17 and 4.21) in [12], and so we will not mention \( \hat{X} \) here again.

That strong Morita equivalence implies operator Morita equivalence is the easy direction of the theorem. This follows just as in pure algebra - see [12] §3 for details.

One may adapt the statement of our main theorem above, to allow the operator equivalence functors to be defined on not all of \( OMOD \), but only on a subcategory \( D \) of \( OMOD \) which contains \( HMOD \) and the operator algebra itself, and the maximal C*-algebra it generates. Our proof goes through verbatim (see comments in [1]).

We remark that a number of functional analytic versions of the ‘Morita theorem’ of equivalence of module categories, have been established in various contexts, although the categories and methods used bear little relation to ours (with the exception of [27], which we will use in our proof).

We refer the reader to [27], [5], and [20], for such results in the settings of W*-algebras, unital C*-algebras and Banach algebras, respectively. Recently in [2, 3], Ara gave such a Morita theorem for C*-algebras, which again is completely different to ours.

2. Some properties of equivalence functors

Throughout this section \( A, B, C, D \) are as before, and \( F: AOMOD \to BOMOD \) is an operator equivalence functor, with ‘inverse’ \( G \) (see Definition 1.1). We set \( Y = F(A) \), \( X = G(B) \), \( Z = F(C) \), \( W = F(D) \). In this paper we will silently be making much use of the following two principles which are of great assistance with operator algebras with c.a.i. but no identity. Firstly, Cohen’s factorization theorem, which asserts that a nondegenerate (left) Banach \( A \)-module \( X \) has the property that \( AX = X \), and indeed any \( x \in X \) may be written as \( ax' \) for \( a \in A, x' \in X \). Secondly, if \( E \) is any C*-algebra generated by an operator algebra with c.a.i., then \( E \) is a nondegenerate \( A \)-module, or equivalently, any c.a.i. for \( A \) is one for \( E \). The latter fact is proved in [5]. The following sequence of lemmas will also be used extensively. Their proofs are mostly identical to the analogous results in [4] and are omitted. The first three are comparitively trivial.

Lemma 2.1. Let \( V \in AOMOD \). Then \( v \mapsto r_v \) is a complete isometry of \( V \) into \( ACB(A,V) \). The range of this map is the set \( ACB^{ess}(A,V) \). If \( V \) is also a Hilbert space, then the map above is a completely isometric isomorphism \( V \cong ACB(A,V) \).

Lemma 2.2. If \( V, V' \in AOMOD \) then the map \( T \mapsto F(T) \) gives a completely isometric surjective linear isomorphism \( ACB(V,V') \cong BCB(F(V),F(V')) \). If \( V = V' \), then this map is a completely isometric isomorphism of algebras. Moreover if \( T \in ACB(V,V') \) is a complete isometry, then so is \( F(T) \).

The last assertion of the previous lemma is discussed in the proof of Theorem 8 in [1].

Lemma 2.3. For any \( V \in AOMOD \), we have \( F(R_m(V)) \cong R_m(F(V)) \) and \( F(C_m(V)) \cong C_m(F(V)) \) completely isometrically isomorphically, where \( R_m(V) \) (resp. \( C_m(V) \)) is the operator module of rows (resp. columns) with \( m \) elements from \( V \).

Lemma 2.4. The functors \( F \) and \( G \) restrict to a completely isometric functorial equivalence of the subcategories \( A\text{HMOD} \) and \( B\text{HMOD} \).
Corollary 2.5. The functors $F$ and $G$ restrict to a completely isometric equivalence of $CHMOD$ and $pHMOD$. This restricted equivalence is a normal $*$-equivalence in the sense of Rieffel [25], and $C$ and $D$ are Morita equivalent in the sense of [23] Definition 8.17.

Proof. This is essentially Proposition 5.1 in [10], together with some general observations in [25] (see Definition 8.17 there).

Lemma 2.6. For any operator $A$-module $V$, the canonical map $\tau_V : Y \otimes V \rightarrow F(V)$ given by $y \otimes v \mapsto F(r_v)(y)$, is completely contractive with respect to the Haagerup tensor norm, and has dense range.

Proof. To show $\tau_V$ has dense range, we suppose the contrary, and let $Q$ be the nonzero quotient map $F(V) \rightarrow \frac{F(V)}{N}$, where $N = (\text{Range } \tau_V)$. Then $G(Q) \neq 0$, so that there exists $v \in V$ with $G(Q)wV^1r_v \neq 0$ as a map on $A$, where $wV$ is the natural transformation $GF(V) \rightarrow V$. Hence $FG(Q)F(wV^{-1})F(r_v) \neq 0$, and thus $QTF(r_v) \neq 0$ for some $T : F(V) \rightarrow F(V)$. By Lemma 2.2, $T = F(S)$ for some $S : V \rightarrow V$, so that $QF(r_{v'}) \neq 0$ for $v' = S(v) \in V$. Hence $Q \circ \tau_V \neq 0$, which is a contradiction.

To show $\tau_V$ is contractive it is sufficient to show that if $\|y_1, \ldots, y_n\| < 1$ and $\|v_1, \ldots, v_n\| < 1$, then $\sum_{k=1}^n F(r_{v_k})(y_k) < 1$. Let us rewrite the last expression. Let $w = [v_1, \ldots, v_n]^t$ be regarded as a map in $CB_A(R_n(A), V)$ via right multiplication $r_w$; then clearly $\|r_w\|_{cb} < 1$. By Lemma 2.3, $F(R_n(A)) \cong R_n(F(A))$, so that we may regard $[y_1, \ldots, y_n]$ as an element $u$ of $F(R_n(A))$ of norm $< 1$. We claim that $F(r_w)(u) = \sum_{k=1}^n F(r_{v_k})(y_k)$. This follows because $u = \sum_{k=1}^n F(i_k)(y_k)$, where $i_k$ is the inclusion of $A$ as the $k$-th entry in $R_n(A)$, so that

$$F(r_w)(u) = \sum_{k=1}^n F(r_{v_k})(y_k) = \sum_{k=1}^n F(r_{v_k})(y_k) = \sum_{k=1}^n F(r_{v_k})(y_k).$$

Thus $\|\sum_{k=1}^n r_{v_k}(y_k)\| = \|F(r_w)(u)\| \leq \|F(r_w)\|_{cb} < 1$. The complete contraction is similar.

3. $C^*$-Restrictable Equivalences.

It will be convenient to separate an ‘easy version’ of our main theorem. We will say that an operator equivalence functor $F$ is $C^*$-restrictable, if $F$ restricts to a functor from $COMOD$ into $pOMOD$. In this section we prove our main theorem under the extra assumption that all functors concerned are $C^*$-restrictable. First we attend to the easy direction of the theorem, which now requires a little extra proof, namely that the canonical equivalence functors which come from a strong Morita equivalence, are $C^*$-restrictable. So suppose that $A$ and $B$ are strongly Morita equivalent, and that $X$ and $Y$ are the strong Morita equivalence bimodules. Then we know from [1] that $C$ and $D$ are strongly Morita $C^*$-algebras, with $D \rightarrow C$-strong Morita equivalence bimodule $Z \cong Y \otimes_{hAC} C$. Set $F(V) = Y \otimes_{hAC} V$, for $V$ a $C$-operator module. However, $Y \otimes_{hAC} V \cong Y \otimes_{hAC} C \otimes_{hC} V \cong Z \otimes_{hC} V$. Hence $F$ restricted to $COMOD$ is equivalent to $Z \otimes_{hC} \rightarrow$, and is thus $C^*$-restrictable.

Conversely, suppose that $F$ and $G$ are $C^*$-restrictable operator equivalence functors. Clearly $F$ and $G$ give an operator Morita equivalence of $COMOD$ and $pOMOD$, when restricted to these subcategories, and in [3] we completely characterized such equivalences. Set $Y = F(A), Z = F(C), X = G(B)$ and $W = G(D)$ as before. From Lemma 2.4, with $V = A$, it follows that $Y$ is a right $A$-operator module. Similarly $X$ is a right $B$-module. From [4] we have that $Z, W$ are strong Morita equivalence bimodules for $C$ and $D$. From [22], the inclusions $A \subset C$ and $B \subset D$ give completely isometric inclusions $Y \rightarrow Z$ and $X \rightarrow W$. 
In [3] it was shown that $F$ takes Hilbert $C$-modules to Hilbert $D$-modules. For any Hilbert $C$-module $K$, we have the following sequence of canonical complete isometries

$$A\cdot CB(X,K) \cong B\cdot CB(B,F(K)) \cong F(K) \cong D\cdot CB(D,F(K)) \cong C\cdot CB(W,K),$$

using Lemmas 2.1 and 2.2. If $R$ is the composition of this sequence of maps, then $R$ is an inverse to the restriction map $C\cdot CB(W,K) \to A\cdot CB(X,K)$. Hence by 3.8 in [10], we have $W \cong C \otimes_{hA} X$ completely isometrically and as $C$-modules, and it is easily checked that this isometry is a right $B$-module map. Similarly, $Z \cong D \otimes_{hB} Y$.

For any $A$-operator module $V$, using the last fact we see that:

$$Y \otimes_{hA} V \subset D \otimes_{hB} (Y \otimes_{hA} V) \cong Z \otimes_{hA} V,$$

completely isometrically. On the other hand we have the following sequence of canonical completely contractive $B$-module maps:

$$Y \otimes_{hA} V \to F(V) \to F(C \otimes_{hA} V) \cong Z \otimes_{hC} (C \otimes_{hA} V) \cong Z \otimes_{hC} V.$$  

The first map in this sequence comes from [2.6], the second map comes from Lemma 2.2, and the third map comes from the main theorem in [10]. The composition of the maps in this sequence coincides with the composition of complete isometries in the last sequence. Hence the canonical map $Y \otimes_{hA} V \to F(V)$ is a complete isometry, and is thus a completely isometric isomorphism since it has dense range.

Finally, $A \cong GF(A) \cong X \otimes_{hB} Y$, and similarly $B \cong Y \otimes_{hA} X$. The remaining assertions of the theorem we leave to the reader, namely some algebraic details such as checking that the transformations are natural).

**Remark.** There is a natural equivalence $A^*\cdot OMOD \cong OMOD_A$, via taking the ‘conjugate operator module’. In view of this, it is reasonable to define a ‘two-sided’ operator Morita equivalence of operator algebras, in which we adjust the definition of left operator Morita equivalence by replacing $F$ with two functors $F_L : A^*\cdot OMOD \to B\cdot OMOD$ and $F_R : OMOD_A \to OMOD_B$, and similarly for $B$. Since $A^*\cdot OMOD \cong OMOD_A$, we get a functor $F_R : A^*\cdot OMOD \to B^*\cdot OMOD$. Since $C\cdot OMOD$ is a subcategory of both $A^*\cdot OMOD$ and $OMOD_A$, it is reasonable to assume that $F_L = F_R$ on $C\cdot OMOD$, and that $F_L$ is $C^*$-restrictable. Indeed, $F_L = F_R$ for the canonical functors $F_L = Y \otimes_{hA} -$ and $F_R = - \otimes_{hA} X$ coming from a strong Morita equivalence. This last interesting fact we leave as an exercise. Thus ‘$C^*$-restrictability’ is a natural condition to impose.

4. **Completion of the proof of the main theorem**

Again $A, B, F, G, X, Y, W, Z$ are as in the previous section, but now we fix $H \in A\cdot HMOD$ to be the Hilbert space of the universal representation of $C$, and fix $K = F(H)$. Then $e(C) \subset B(H)$, where $e(C)$ is the enveloping von Neumann algebra of $C$. By 2.4 and 2.5, $F$ and $G$ restrict to an equivalence of $A\cdot HMOD$ with $B\cdot HMOD$, and restricts further to a normal *-equivalence of $C\cdot HMOD$ with $D\cdot HMOD$. By [21] Propositions 1.1, 1.3 and 1.6, $D$ acts faithfully on $K$, and if we regard $D$ as a subset of $B(K)$ then the weak operator closure $D''$ of $D$ in $B(K)$, is $W^*$-isomorphic to $e(D)$. We shall indeed regard $D$ henceforth as a subalgebra of $B(K)$.

It is important in what follows to keep in mind the canonical right module action of $B$ on $X$. $xb = F(r_b)(x)$, for $x \in X, b \in B$, where as in section 2, $r_b : B \to B$ is the map $c \mapsto cb$. By 2.6, $X$ is an operator $A-B$-bimodule. Similarly, $Y$ is canonically an operator $B-A$-bimodule, and $Z$ and $W$ are, respectively, operator $B-C^*$- and $A-D^*$-modules. Using the last assertion in 2.2 the inclusion $i$ of $A$ in $C$ induces a completely isometric inclusion $F(i)$ of $Y$ in $Z$. It is easy to see that
$F(i)$ is a $B - A$-bimodule map. We will regard $Y$ as a $B - A$-submodule of $Z$, and, similarly, $X$ as an $A - B$-submodule of $W$.

As we saw in Lemma 2.4, there is a left $B$-module map $Y \otimes X \to F(X)$ defined by $y \otimes x \mapsto F(r_x)(y)$. Since $F(X) = F G(B) \cong B$, we get a left $B$-module map $Y \otimes X \to B$, which we shall write as $[\cdot]$. In a similar way we get a module map $(\cdot) : X \otimes Y \to A$. In what follows we may use the same notations for the 'unlinearized' bilinear maps, so for example we may use the symbols $[y, x]$ for $[y \otimes x]$. These maps $[\cdot]$ and $(\cdot)$ have natural extensions, which are denoted by the same symbols, to maps from $Y \otimes W \to D$ and $X \otimes Z \to C$ respectively. Namely, $[y, w]$ is defined via $\tau_W$. These maps $[\cdot]$ and $(\cdot)$ all have dense range, by Lemma 2.4.

**Lemma 4.1.** The canonical maps $X \to CB(X, Y)$ and $Y \to CB(X, A)$ induced by $[\cdot]$ and $(\cdot)$ respectively, are complete isometries. Similarly, the extended maps $W \to CB(Y, D)$, and $Z \to CB(X, C)$ are complete isometries.

The proof of this is identical to the proof of the analogous result in [9].

The following maps $\Phi : Z \to B(H, K)$, and $\Psi : W \to B(K, H)$ will play a central role in the remainder of the proof. Namely, $\Phi(z)(\zeta) = F(r_z)(\zeta)$, and $\Psi(w)(\eta) = \omega_H G(r_\eta)(w)$, where $\omega_H : GF(H) \to H$ is the $A$-module map coming from the natural transformation $GF \cong Id$. Here $r_z : C \to H$ and $r_\eta : D \to K$. Since $\omega_H$ is an isometric surjection between Hilbert space it is unitary, and hence is also a $C$-module map. It is straightforward algebra to check that:

$$\Psi(x)\Phi(z) = (x, z) \quad \& \quad \Phi(y)\Psi(w) = [y, w]V \quad (\dagger)$$

for all $x \in X, y \in Y, z \in Z, w \in W$, and where $V \in B(K)$ is a unitary operator in $D'$ composed of two natural transformations.

**Lemma 4.2.** The map $\Phi : Z \to B(H, K)$ (resp. $\Psi : W \to B(K, H)$) is a completely isometric $B - C$-module map (resp. $A - D$-module map). Moreover, $\Phi(z_1)^*\Phi(z_2) \in C'' = e(C)$ for all $z_1, z_2 \in Z$, and $\Psi(w_1)^*\Psi(w_2) \in D''$ for $w_1, w_2 \in W$.

**Proof.** This is also almost identical to the analogous result in [9]. One first establishes, for example, that for $T \in C'$, we have $\Phi(y)T = F(T)\Phi(y)$, and this gives the 2nd commutant assertions as in [9].

We shall simply give a few steps in the calculation showing that $\Phi$ is a complete isometry; the missing steps may be found by comparison with [9]:

$$\|\Phi(z_{ij})\| = \sup\{\|\Phi(z_{ij})(\zeta_{kl})\| : [\zeta_{kl}] \in Ball(M_m(H^c)), m \in \mathbb{N}\}$$

where we used the last part of Lemma 1.1 in the last line. Thus $\Phi$ is a complete isometry. 

**Lemma 4.3.** The unitary $V$ is in the center of the multiplier algebra of $D$; and $\Phi(y)\Psi(w) \in D$ for all $y \in Y, w \in W$.

**Proof.** We will use the facts stated in the first part of the proof of the previous lemma. By $(\dagger)$ we know that $A = [\Phi(X)\Phi(Y)]$. Hence, using the second equation in $(\dagger)$, we see that

$$A\Psi(X)V^{-1}\Phi(Y) = [\Psi(X)\Phi(Y)]\Psi(X)V^{-1}\Phi(Y) \subset [\Phi(X)\Psi(X)V^{-1}\Phi(Y)] \subset [\Psi(X)\Phi(Y)] = A$$

If $T \in C'$ is such that $F(T) = V^{-1}$, then $\Phi(X)\Psi(X)V^{-1}\Phi(Y) = \Psi(X)\Phi(Y)T$. Thus $AAT \subset A$, so that $AT \subset A$. Since $Y = YA$ we have $\Phi(Y)T \subset \Phi(Y)$. Thus

$$\Phi(y)\Psi(w) = V[y, w] = VV^{-1}\Phi(y)\Psi(w) = V\Phi(y)T\Psi(w) \in V\Phi(Y)\Psi(W) \subset D.$$
for \( y \in Y, w \in W \). Since \([ \cdot ]\) has dense range in \( \mathcal{D} \), we see the multiplier assertion. \(\square\)

**Theorem 4.4.** The quantity \( \Psi(w)^*\Psi(w) \), which is in \( \mathcal{D}' \) by Lemma 4.2, is actually in \( \mathcal{D} \) for all \( w \in W \); and similarly \( \Phi(z)^*\Phi(z) \in \mathcal{C} \) for all \( z \in Z \).

**Proof.** We first observe that as in (3) the natural transformations \( \Psi \) (which we recall, is \( W \) whenever a property is established for \( W \) statements hold for \( D A \) few paragraphs there, by \( \Psi \)). Therefore from Theorem 4.15 of [12] we conclude that \( L \) we let \( F \) algebra containing \( E \) yields \( M \). This is a subalgebra by (4). As in [9] this implies that \( \Phi(\bar{\psi}(z)) \) is a right \( C \) module over \( W \). It is easily seen, using equation (4), that \( \Phi(w) \in D \). Theorem 4.5.

**Theorem 4.5.** The \( C^* \)-algebras \( \mathcal{C} \) and \( \mathcal{D} \) are strongly Morita equivalent. In fact \( Z \), which we have seen to be a \( B-C \)-operator bimodule, is a \( \mathcal{D}-\mathcal{C} \)-strong Morita equivalence bimodule. Similarly, \( W \) is a \( \mathcal{C}-\mathcal{D} \)-strong Morita equivalence bimodule, and indeed \( W \cong \bar{Z} \) unitarily (and as operator bimodules).

**Proof.** We will use some elementary theory or notation from \( C^* \)-modules as may be found in [21] for example. It follows by the polarization identity, and the previous theorem, that \( W \) is a RIGHT \( C^* \)-module over \( \mathcal{D} \) with inner product \( \langle w_1 \mid w_2 \rangle_{\mathcal{D}} = \Psi(w_1)^*\Psi(w_2) \). The induced norm on \( W \) from the inner product coincides with the usual norm. Similarly \( Z \) (or equivalently \( \Phi(Z) \)) is a right \( C^* \)-module over \( \mathcal{C} \). Also, \( W \) is a LEFT \( \mathcal{C} \)-module over \( \mathcal{E} = [\Psi(W)^*] \), indeed it is clear that \( \mathcal{E} \cong \mathbb{K}C(Z) \), the so-called imprimitivity \( C^* \)-algebra of the right \( C^* \)-module \( Z \). The inner product is obviously \( \mathcal{E} \langle w_1 \mid w_2 \rangle = \Psi(w_1)^*\Psi(w_2)^* \). We will show that \( \mathcal{E} = \mathcal{C} \). Analogous statements hold for \( \mathcal{D} \) and \( \Phi \), and we will assume below, without writing it down explicitly, that whenever a property is established for \( W \), the symmetric matching assertions for \( Z \).

Let \( \mathcal{L} \) be the linking \( C^* \)-algebra for the right \( C^* \)-module \( W \), viewed as a subalgebra of \( B(H \oplus K) \). We let \( \mathcal{F} = [\Psi(W)^*\Phi(Y)] \). It is easily seen, using equation (5) and Lemma 4.3, that \( \mathcal{F} \) is an operator algebra containing \( \mathcal{A} \), and that the c.a.i. of \( \mathcal{A} \) is a c.a.i. for \( \mathcal{F} \). We let \( \mathcal{G} = [\Phi'(Y)] \), and we define \( \mathcal{M} \) to be the following subset of \( B(H \oplus K) \):

\[
\begin{bmatrix}
\mathcal{F} & \Psi(W) \\
\mathcal{G} & \mathcal{D}
\end{bmatrix}
\]

This is a subalgebra by (6) and Lemma 4.3. It is also easy to check that \( \mathcal{L}\mathcal{M} = \mathcal{M} \) and \( \mathcal{M}\mathcal{L} = \mathcal{L} \). Therefore from Theorem 4.15 of [12] we conclude that \( \mathcal{L} = \mathcal{M} \). Comparing corners of these algebras yields \( \mathcal{E} = \mathcal{F} \) and \( G = \Psi(W)^* \). Thus we see that \( \mathcal{A} \subset \mathcal{E} \), from which it follows that \( \mathcal{C} \subset \mathcal{E} \), since \( \mathcal{C} \) is the \( C^* \)-algebra generated by \( \mathcal{A} \) in \( B(H) \). Thus we have finally seen that \( W \) is a left \( \mathcal{C} \)-module,
and that \( \Psi \) is a left \( \mathcal{C} \)-module map. By symmetry, \( Z \) is a left \( \mathcal{D} \)-module and \( \Phi \) is a \( \mathcal{D} \)-module map, so that

\[
\Psi(W)^* = \mathcal{G} = [\mathcal{D}\Phi(Y)] \subset \Phi(Z).
\]

Also

\[
\Psi(X\mathcal{D})\Phi(Y) \subset [\Psi(X)\Phi(Z)] \subset \mathcal{C},
\]

and so \( \mathcal{E} = \mathcal{F} \subset \mathcal{C} \). Thus \( \mathbb{K}\mathcal{C}(Z) \cong \mathcal{E} = \mathcal{C} \). By symmetry note that \( \Psi(W)^* = \Phi(Z) \), and that

\[
\mathcal{D} = [\Phi(Z)\Phi(Z)^*] = [\Psi(W)^*\Psi(W)].
\]

Thus the conclusions of the theorem all hold. \( \square \)

**Corollary 4.6.** Operator equivalence functors are automatically \( C^* \)-restrictable.

**Proof.** We keep to the notation used until now. We will begin by showing that \( W \) is the maximal dilation of \( X \), and \( Z \) is the maximal dilation of \( Y \). We saw above that the set which we called \( \mathcal{G} \), equals \( Z \), so that \( Y \) generates \( Z \) as a left operator \( \mathcal{D} \)-module. We have the following sequence of fairly obvious maps, using Lemmas 2.1 and 2.2 above:

\[
\mathcal{A}CB(X, H) \cong \mathcal{B}CB(B, K) \cong K \cong \mathcal{D}CB(D, K) \to \mathcal{A}CB(W, H).
\]

It is easily checked that \( \eta \in K \) corresponds under the last two maps in the sequence to the map \( w \mapsto \Phi(w)(\eta) \), which lies in \( \mathcal{C}CB(W, H) \) since \( \Phi \) is a left \( \mathcal{C} \)-module map. Thus if \( R \) is the composition of all the maps in this sequence, then the range of \( R \) is contained in \( \mathcal{C}CB(W, K) \). Moreover, \( R \) is an inverse to the restriction map \( \mathcal{C}CB(W, K) \to \mathcal{A}CB(X, K) \). Thus \( \mathcal{C}CB(W, K) = \mathcal{A}CB(X, K) \).

Hence by 3.8 in [10], \( W \) is the maximal dilation of \( X \). A similar argument works for \( Z \).

Let \( V \in \mathcal{C}O\text{MOD} \). By Lemmas 2.1 and 2.2 above, and 3.9 in [10], we have

\[
F(V) \cong \mathcal{B}CB^{ess}(B, F(V)) \cong \mathcal{A}CB^{ess}(X, V) \cong \mathcal{C}CB^{ess}(W, V) \cong Z \otimes_{h\mathcal{C}} V,
\]

as left \( \mathcal{B} \)-operator modules, where the last ‘\( \cong \)’ is from [3] Theorem 3.10. Now the latter space is a left \( \mathcal{D} \)-operator module, hence \( F(V) \) is a \( \mathcal{D} \)-operator module, and (by a comment in §1, which is 3.3 in [10]) the identity \( F(V) \cong Z \otimes_{h\mathcal{C}} V \) above, is also valid as \( \mathcal{D} \)-operator modules. One may easily check that this last identity is a natural isomorphism. But \( Z \otimes_{h\mathcal{C}} \) — is clearly a \( \mathcal{D} \)-module functor. Hence \( F \) is \( C^* \)-restrictable. \( \square \)

Hence, by the result in the previous section, our main theorem is proved.

**References**

[1] F.W. Anderson and K.R. Fuller, *Rings and categories of modules (2nd Ed.*)*, Graduate texts in Math. Vol 13, Springer-Verlag, New York, 1992.

[2] P. Ara, *Morita equivalence and Pedersen’s ideals*, Preprint.

[3] P. Ara, *Morita equivalence for rings with involution*, Preprint.

[4] H. Bass, *The Morita Theorems*, Lecture Notes, University of Oregon, Eugene, 1962.

[5] W. Beer, *On Morita equivalence of nuclear \( C^* \)-algebras*, J. Pure Appl. Algebra 26 (1982), 249-267.

[6] D. P. Blecher, *A generalization of Hilbert modules*, J. Funct. Analysis, 136 (1996), 365-421.

[7] D. P. Blecher, *A new approach to Hilbert \( C^* \)-modules*, Math Ann. 307 (1997), 253-290.

[8] D. P. Blecher, *Some general theory of operator algebras and their modules*, in Operator algebras and applications, A. Katavolos (editor), NATO ASI, Vol. 495, Kluwer, Dordrecht, 1997.

[9] D. P. Blecher, *On Morita’s fundamental theorem for \( C^* \)-algebras*, To appear Math. Scand.

[10] D. P. Blecher, *Modules over operator algebras and the maximal \( C^* \)-dilation*, Preprint (1998).

[11] D. P. Blecher, P. S. Muhly and Q. Na, *Morita equivalence of operator algebras and their \( C^* \)-envelopes*, to appear journal of the London Math. Soc.

[12] D. P. Blecher, P. S. Muhly and V. I. Paulsen, *Categories of operator modules - Morita equivalence and projective modules*, (1998 Revision), To appear Memoirs of the A.M.S.

[13] D. P. Blecher, Z-J. Ruan, and A. M. Sinclair, *A characterization of operator algebras*, J. Functional Anal. b89 (1990), 188-201.
MORITA THEOREM FOR OPERATOR ALGEBRAS

[14] S. MacLane, *Categories for the working mathematician*, Springer-Verlag New York-Heidelberg-Berlin (1971).
[15] P. M. Cohn, *Morita equivalence and duality*, Queen Mary College Mathematical Notes, University of London, London, 1966.
[16] A. Connes, *Noncommutative geometry*, Academic Press (1994).
[17] E. Christensen, E. Effros, and A. Sinclair, *Completely bounded multilinear maps and C*-algebraic cohomology*, Inv. Math. 90 (1987), 279-296.
[18] E. Effros and Z-j. Ruan, *Representations of operator bimodules and their applications*, J. Operator Theory 19 (1988), 137-157.
[19] C. Faith, *Algebra I: Rings, Modules, and Categories*, Springer-Verlag Berlin Heidelberg New York (1981).
[20] N. Gronbaek, *Morita equivalence for Banach algebras, a transcription*, Journal of Pure and Applied Algebra 99 (1995), 183-219.
[21] E. C. Lance, *Hilbert C*-modules - A toolkit for operator algebraists*, London Math. Soc. Lecture Notes, Cambridge University Press (1995).
[22] K. Morita, *Duality of modules and its applications to the theory of rings with minimum condition*, Sci. Rep. Tokyo Kyoiku Daigaku, Sect. A 6 (1958), 85-142.
[23] V. Paulsen, *Completely bounded maps and dilations*, Pitman Research Notes in Math., Longman, London, 1986.
[24] G. Pedersen, *C*-algebras and their automorphism groups, Academic Press (1979).
[25] M. Rieffel, *Morita equivalence for C*-algebras and W*-algebras*, J. Pure Appl. Algebra 5 (1974), 51-96.
[26] M. Rieffel, *Induced representations of C*-algebras*, Adv. Math. 13 (1974), 176-257.
[27] M. Rieffel, *Morita equivalence for operator algebras*, Proceedings of Symposia in Pure Mathematics 38 Part 1 (1982), 285-298.

Department of Mathematics, University of Houston, Houston, TX 77204-3476
E-mail address: dblecher@math.uh.edu