The method of Bregman projections in deterministic and stochastic convex feasibility problems

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Abstract

In this work we study the method of Bregman projections for deterministic and stochastic convex feasibility problems with three types of control sequences for the selection of sets during the algorithmic procedure: greedy, random, and adaptive random. We analyze in depth the case of affine feasibility problems showing that the iterates generated by the proposed methods converge Q-linearly and providing also explicit global and local rates of convergence. This work generalizes from one hand recent developments in randomized methods for the solution of linear systems based on orthogonal projection methods. On the other hand, our results yield global and local Q-linear rates of convergence for the Sinkhorn and Greenhorn algorithms in discrete entropic-regularized optimal transport, for the first time, even in the multimarginal setting.

Keywords. Convex feasibility problem, stochastic convex feasibility problem, Bregman projection method, KL projection, multimarginal regularized optimal trasport, linear convergence.

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1 Introduction

The convex feasibility problem consists in finding a point in the intersection of a finite family of closed convex sets. Such problem arises in several areas of mathematics and applied sciences, such as best approximation theory [23] and image reconstruction [20]. A classical approach to solve that problem is the method of cyclic orthogonal projections [29], which generates a sequence of points by projecting onto each constraint set cyclically. In 1967 Bregman, in [12], generalized this method by allowing non-orthogonal projections which are constructed as follows. Given a well-behaved convex function \( \phi \), construct the so called Bregman distance\(^1\) (or divergence) between two points \( x \) and \( y \) as

\[
D_\phi(x, y) = \phi(x) - \phi(y) - \langle x - y, \nabla \phi(x) \rangle
\]

and then define the Bregman projection of a point \( x \) onto a given closed convex set as the point in the set which minimizes the Bregman distance from \( x \). He studied two versions of the algorithm depending on the strategy (control) for calling the various projections during the algorithmic procedure: cyclic and the most remote set control. Since then, a number of studies followed, extending

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\(^1\)Note that is not a distance in the sense of metric topology and even when \( \phi(x) = (1/2)\|x\|^2 \) it is one half of the square of the distance between \( x \) and \( y \).
the theory to more general set controls: almost cyclic and repetitive [16, 17, 20]. The Bregman projection method, with respect to the Kullback-Leibler divergence, has found numerous applications in optimal transport [11] and probability and statistics, where it is known as the iterative proportional fitting procedure (IPFP) [43].

In this work, we focus on the convex feasibility problem in an Euclidean space \( X \), addressing both the deterministic and the stochastic variants (possibly with an infinite uncountable number of sets) as described in the following problems.

**Problem 1.1** (Deterministic). Let \( C = (C_i)_{i \in I} \) be a family of nonempty closed convex sets in \( X \). Let \( C = \bigcap_{i \in I} C_i \) and suppose that \( C \neq \emptyset \). Find \( x \in C \).

**Problem 1.2** (Stochastic). Let \((I, I)\) be a measurable space, \( C = (C_i)_{i \in I} \) a family of nonempty closed convex sets in \( X \), such that \( i \mapsto C_i \) is a measurable set-valued mapping, and let \( \xi \) be an \( I \)-valued random variable. Let \( C = \{ x \in X \mid x \in C_\xi, \mathbb{P}\text{-a.s.} \} \) and suppose that \( C \neq \emptyset \). Find \( x \in C \).

In dealing with the two problems above we consider the method of Bregman projections. The algorithm is detailed below, where we denote by \( P_{C_i} \) the Bregman projection onto the set \( C_i \) w.r.t. \( \phi \).

**Algorithm 1.3** (The Bregman projection method). Let \( x_0 \in \text{int(dom } \phi) \). Iterate

\[
\text{for } k = 0, 1, \ldots \nabla \begin{align*}
\quad \text{choose } \xi_k & \in I \\
\quad x_{k+1} & = P_{C_{\xi_k}}(x_k).
\end{align*}
\]

(1.2)

The sequence \((\xi_k)_{k \in \mathbb{N}}\) is called the set control sequence and, depending on which of the two problems above we consider, it can be deterministic or stochastic.

### 1.1 Contribution

In the following we summarize the main contribution of this paper. We denote by \( D_C(x) = \inf_{z \in C} D_\phi(z, x) \) the Bregman distance from \( x \) to the set \( C \). The following holds.

- As for Problem 1.1, if the sets \( C_i \)'s are affine and the \( \xi_k \)'s are chosen according to the greedy strategy \( \xi_k \in \arg \max_{i \in I} D_{C_i}(x_k) \) (the most remote set control), we prove that \( D_\phi(P_C(x_0), x_k) = D_C(x_k) \to 0 \) with Q-linear rate and we provide both global and local rates of convergence. See Theorem 4.5.

- As for Problem 1.2, if the \( \xi_k \)'s are random variables which are independent copies of \( \xi \), then we prove that the iterates converge almost surely and in mean square to a random variable taking values in the set \( C \). See Theorem 3.7. Moreover, if the sets \( C_i \)'s are affine and the \( \xi_k \)'s are random variables which are either independent copies of \( \xi \) or with distribution adaptively depending on \( x_k \), we prove that \( \mathbb{E}[D_\phi(P_C(x_0), x_k)] = \mathbb{E}[D_C(x_k)] \to 0 \) with Q-linear rate and we provide both global and local rates of convergence. See Theorem 4.9, Remark 4.10, and Theorem 4.13.

Below we comment on the results. 1) To the best of our knowledge the Q-linear convergence of the Bregman projection method in (deterministic and stochastic) affine feasibility problems is new. This fully generalizes the well-known linear rate of convergence of the orthogonal projection method [31, 37, 44, 40] to Bregman projections. We stress that this extension is truly nontrivial. Indeed, while in the classical setting one works with orthogonal projections w.r.t. an Euclidean norm, in the
Bregman setting this can be done only locally by approximating the Bregman distance through the semi-norm induced by the Hessian (which indeed we allow to be possibly rank deficient) of the Fenchel conjugate of $\phi$. 2) Concerning the stochastic feasibility problem in general, the Bregman projection approach and its converge is also new. 3) The results cover a number of interesting (Legendre) functions $\phi$: among others, Burg entropy, Boltzmann-Shannon entropy, Fermi-Dirac entropy, and $p$-norms with $1 < p \leq 2$. 4) Novel applications include sketch & Bregman project methods for solving linear systems of equations and generalization of Sinkhorn and Greenkhorn algorithms for regularized multimarginal optimal transport, to name a few. In particular, this analysis establishes global and local $Q$-linear convergence of the Greenkhorn algorithm and of the (stochastic and greedy) iterative KL projection algorithm for multimarginal entropic-regularized optimal transport.

1.2 Related works

Although the method of Bregman projections for convex feasibility problems has a long history and has proved to be at the basis of several important algorithms in science [24, 39], unlike its Euclidean version (the method of orthogonal projections), it remained in the domain of deterministic intersection of sets, i.e., Problem 1.1. Indeed, even if quite general deterministic set control sequences, called repetitive, are allowed [16, 17], they are meaningful and can possibly handle the stochastic setting only when the number of sets is finite (see Remark 3.8). On the other hand Problem 1.2, even with an uncountable number of sets, was first considered in [13] and tackled via an expected orthogonal projection method, which was later extended to Bregman projections in [14]. However, these types of methods are different from Algorithm 1.3: indeed they are defined as $x_{k+1} = \mathbb{E}[P_{C_{i_k}}(x_k)]$, so that they generate a non-stochastic sequence. Instead, the method of stochastic orthogonal projections for Problem 1.2 was introduced in [38] and in recent years has received a renewed attention also thanks to the fast development of machine learning and data science [33, 37, 35].

The linear convergence of the method is known only in the deterministic case and for some special affine feasibility problems. A prominent example is that of the Sinkhorn algorithm for entropic optimal transport, which can be viewed as alternating Bregman projections (w.r.t. the Kullback-Leibler divergence) between two suitable affine sets: global and local convergence rates are discussed in [32, 39]. We also mention the work [34], by Iusem, which shows local linear convergence for a row-action method for general linear inequality constraints with an almost cyclic control sequence. The method is not purely alternating projections (as Algorithm 1.3), but is in fact a primal-dual algorithm (originally introduced in [12]) specifically designed for the case that the $C_i$’s are halfspaces. Besides, for this method no global linear rate is known.

Since affine feasibility problems are essentially linear systems of equations, in the following we also discuss connections with the field of numerical linear algebra. In the last decades, mainly triggered by problems in machine learning, randomized linear solvers have emerged as a way to approach large linear systems. One of the main achievements in this direction is the work by Strohmer and Vershynin [44] who studied a randomized version of the Kaczmarz method [31]. More recently, in [28, 27, 37, 40], this method has been extended to the more general framework of sketch & project, which is designed to compute the minimal norm solution of a feasible linear system by making a sequence of orthogonal projections onto sketched systems of smaller dimensions. In [27] different strategies of choosing sketched systems at each iteration were considered: greedy, random, and adaptive. For these strategies linear convergence rates were obtained, which in the special case of the randomized Kaczmarz methods reduce to those in [44]. Our work extends [27, 40, 44], since we allow Bregman projections onto the sketched systems, proving linear convergence for all the
types of sketching strategies described above. We call our method sketch & Bregman project. We stress that when our results specialize to orthogonal projections, they completely recover the ones in [27, 40, 44], revealing that our analysis is indeed tight (see Section 5.1).

1.3 Outline of the paper

In the next section we provide notation and basic facts about the Bregman projections. Section 3 analyzes Algorithm 1.3 for the general case of convex sets. In Section 4 we present the main results of this work, which concerns the linear convergence of the deterministic and stochastic Bregman projection method for the affine feasibility problem. Finally, Section 5 discusses in more details some relevant applications, meaning, sketch & Bregman project methods for solving linear systems and regularized optimal transport problems.

2 Preliminaries

In this section we provide notation and basic concepts and results related to the Bregman projection method.

2.1 Notation and basic background

In this work $X$ is an Euclidean space with scalar product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. The interior of a set $C \subset X$ is denoted by $\text{int}(C)$ and its boundary by $\text{bdry}(C)$. We set $\mathbb{R}_+ = [0, +\infty]$ and $\mathbb{R}_{++} = [0, +\infty[.$

Let $\phi : X \to ]-\infty, +\infty]$ be an extended-real valued function. The set of minimizers of the function $\phi$ is denoted by $\text{arg}\min_{x \in X} \phi(x)$, the domain of $\phi$ is $\text{dom} \phi := \{ x \in X \mid \phi(x) < +\infty \}$ and $\phi$ is proper when $\text{dom} \phi \neq \emptyset$. The function $\phi$ is convex if $\phi(tx + (1-t)y) \leq t\phi(x) + (1-t)\phi(y)$ for all $x, y \in \text{dom} \phi$ and $t \in [0,1]$. If the above inequality is strict when $0 < t < 1$ and $x \neq y$, the function is strictly convex. The function $\phi$ is closed if the sublevel sets $\{ x \in X \mid \phi(x) \leq t \}$ are closed in $X$ for any $t \in \mathbb{R}$. For a convex function $\phi : X \to ]-\infty, +\infty]$, we denote by $\phi^*$ its Fenchel conjugate, that is, $\phi^* : X \to ]-\infty, +\infty]$, $\phi^*(y) := \sup_{x \in X} \{ \langle x, y \rangle - \phi(x) \}$. The conjugate of a convex function is always closed and convex, and if $\phi$ is proper closed and convex, then $(\phi^*)^* = \phi$.

A proper closed and convex function $\phi$ is essentially smooth if it is differentiable on $\text{int}(\text{dom} \phi) \neq \emptyset$, and $\| \nabla \phi(x_n) \| \to +\infty$ whenever $x_n \in \text{int}(\text{dom} \phi)$ and $x_n \to x \in \text{bdry}(\text{dom} \phi)$. The function $\phi$ is essentially strictly convex if $\text{int}(\text{dom} \phi^*) \neq \emptyset$ and is strictly convex on every convex subset of $\text{dom} \phi$. A Legendre function is a proper closed and convex function which is also essentially smooth and essentially strictly convex. A function is Legendre if and only if its conjugate is so. Moreover, if $\phi$ is a Legendre function, then $\nabla \phi : \text{int}(\text{dom} \phi) \to \text{int}(\text{dom} \phi^*)$ and $\nabla \phi^* : \text{int}(\text{dom} \phi^*) \to \text{int}(\text{dom} \phi)$ are bijective, inverses of each other, and continuous. See [42, Sec. 26]. Given a Legendre function $\phi$, the Bregman distance associated to $\phi$ is the function $D_\phi : X \times X \to [0, +\infty]$ such that

$$D_\phi(x, y) = \begin{cases} \phi(x) - \phi(y) - \langle x - y, \nabla \phi(y) \rangle & \text{if } y \in \text{int}(\text{dom} \phi) \\ +\infty & \text{otherwise}. \end{cases}$$

(2.1)

In the following we will use some important properties of Bregman distances generated by Legendre functions. See [6, 8, 9]. Note that item (xi) follows from Taylor’s formula for $\phi$. 

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Fact 2.1. Let $\phi$ be a Legendre function. Then the following properties hold.

(i) $(\forall x \in \text{dom} \phi)(\forall y \in \text{int}(\text{dom} \phi)) D_\phi(x, y) = \phi(x) + \phi^*(\nabla \phi(y)) - \langle x, \nabla \phi(y) \rangle$.

(ii) $(\forall y \in \text{int}(\text{dom} \phi)) D_\phi(\cdot, y)$ is a strictly convex on $\text{int}(\text{dom} \phi)$ and coercive.

(iii) $(\forall x, y \in \text{int}(\text{dom} \phi)) D_\phi(x, y) = 0 \iff x = y$.

(iv) $(\forall x, y \in \text{int}(\text{dom} \phi)) D_\phi(x, y) + D_\phi(y, x) = \langle x - y, \nabla \phi(x) - \nabla \phi(y) \rangle \geq 0$.

(v) (Three-Point Identity [19]) For every $x \in X$ and $y, z \in \text{int}(\text{dom} \phi)$, we have

$$D_\phi(x, z) = D_\phi(x, y) + D_\phi(y, z) + \langle x - y, \nabla \phi(y) - \nabla \phi(z) \rangle.$$  

(vi) $(\forall x, y \in \text{int}(\text{dom} \phi)) D_\phi(x, y) = D_\phi^*(\nabla \phi(y), \nabla \phi(x))$.

(vii) $D_\phi$ is continuous on $\text{int}(\text{dom} \phi) \times \text{int}(\text{dom} \phi)$.

(viii) Suppose that $\phi$ is twice differentiable on $\text{int}(\text{dom} \phi)$. Then

$$\left(\forall x \in \text{int}(\text{dom} \phi), \nabla^2 \phi(x) \text{ is invertible} \right) \Leftrightarrow (\phi^* \text{ is twice differentiable}).$$  

(ix) Suppose that $\text{dom} \phi^*$ is open. Then, for every $x \in \text{int}(\text{dom} \phi)$, the sublevel sets of $D_\phi(x, \cdot)$ are compact, and hence $D_\phi(x, \cdot)$ is lower semicontinuous.

(x) Suppose that $\text{dom} \phi^*$ is open. Then, for every $x \in \text{int}(\text{dom} \phi)$, and every sequence $(y_k)_{k \in \mathbb{N}}$ in $\text{int}(\text{dom} \phi)$

$$D_\phi(x, y_k) \to 0 \Rightarrow y_k \to x.$$  

Consequently, for every $x \in \text{int}(\text{dom} \phi)$ and $\varepsilon > 0$, there exists $\delta > 0$ such that for every $y \in \text{int}(\text{dom} \phi)$, $D_\phi(x, y) < \delta \implies \|x - y\| < \varepsilon$.

(xi) If $\phi$ is twice differentiable on $\text{int}(\text{dom} \phi)$, then for every $x, y \in \text{int}(\text{dom} \phi)$ there exists $\xi \in [x, y]$ such that

$$D_\phi(x, y) = \frac{1}{2} \langle \nabla^2 \phi(\xi)(x - y), x - y \rangle.$$  

Moreover, for every $y \in \text{int}(\text{dom} \phi)$ and every $\varepsilon > 0$ there exists $\delta > 0$ such that, for every $x \in \text{int}(\text{dom} \phi)$ such that $x - y \notin \text{Ker}(\nabla^2 \phi(y))$,

$$\|x - y\| \leq \delta \implies \left| \frac{1}{2} \langle \nabla^2 \phi(y)(x - y), x - y \rangle \right| \leq \varepsilon.$$  

In addition to the above facts, we will use the following ones, too.

Fact 2.2. Let $A : X \to Y$ be a linear operator and let $A^\dagger$ be its Moore-Penrose pseudoinverse. Then $AA^\dagger = A(A^\dagger A)^\dagger A^\dagger$ is the orthogonal projector onto $\text{Im}(A)$, and $\|A^\dagger\|^{-1} = \inf_{z \in \text{Ker}(A)^\perp \setminus \{0\}} \|Az\|/\|z\|$ is the smallest positive singular value of $A$.

Fact 2.3 ([25, Example 5.1.5]). Let $\zeta_1$ and $\zeta_2$ be independent random variables with values in the measurable spaces $Z_1$ and $Z_2$ respectively. Let $\varphi : Z_1 \times Z_2 \to \mathbb{R}$ be measurable and suppose that $\mathbb{E}[\varphi(\zeta_1, \zeta_2)] < +\infty$. Then $\mathbb{E}[\varphi(\zeta_1, \zeta_2) | \zeta_1] = \psi(\zeta_1)$, where for all $z_1 \in Z_1$, $\psi(z_1) = \mathbb{E}[\varphi(z_1, \zeta_2)]$. 

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Fact 2.4 ([25, Theorem 3.2.4]). Let \((x_k)_{k \in \mathbb{N}}\) be a sequence of \(X\)-valued random variable and let \(x\) be an \(X\)-valued random variable. Then the following hold.

(i) Suppose that \(x_k\) are uniformly essentially bounded, i.e., \(\sup_{k \in \mathbb{N}} \text{ess sup} \|x_k\| < +\infty\). Then \(x_k \to x\) \(\mathbb{P}\text{-a.s.} \Rightarrow \mathbb{E}[\|x_k - x\|^2] \to 0\).

(ii) Suppose that \(x_k \in U \subset X\) \(\mathbb{P}\text{-a.s.}\) and \(T: U \to Y\) is continuous. Then \(x_k \to x\) in distribution \(\Rightarrow T(x_k) \to T(x)\) in distribution.

2.2 The Bregman projection onto a convex set

In this section we recall the definition and the main properties of the Bregman projection operator. We first address the general case of a convex set and then the special case of an affine set.

Let \(C \subset X\) be a nonempty closed and convex set. Let \(\phi\) be a Legendre function on \(X\) such that \(C \cap \text{int}(\text{dom } \phi) \neq \emptyset\) and let \(D_\phi\) be its associated Bregman distance. Let \(x \in \text{int}(\text{dom } \phi)\). Then, the following optimization problem

\[
\min_{z \in C} D_\phi(z, x) \tag{2.7}
\]

has a unique solution which is in \(C \cap \text{int}(\text{dom } \phi)\) [6, Corollary 7.9]. In other words, the operator \(P_C: \text{int}(\text{dom } \phi) \to C \cap \text{int}(\text{dom } \phi)\) such that, for all \(x \in \text{int}(\text{dom } \phi)\),

\[
P_C(x) := \arg \min_{z \in C} D_\phi(z, x), \tag{2.8}
\]

is well defined. It is called the Bregman projector onto \(C\) with respect to \(\phi\). The point \(P_C(x)\) is the Bregman projection of \(x\) onto \(C\) with respect to \(\phi\) and is characterized by the following variational inequality [8, Proposition 3.16]

\[
(\forall z \in C) \quad \langle z - P_C(x), \nabla \phi(x) - \nabla \phi(P_C(x)) \rangle \leq 0, \tag{2.9}
\]

or equivalently, using (2.2), by the condition

\[
(\forall z \in C) \quad D_\phi(z, x) \geq D_\phi(z, P_C(x)) + D_\phi(P_C(x), x). \tag{2.10}
\]

In the special case that \(C\) is an affine set, in (2.10) and (2.9) equalities hold. We also define the Bregman distance to \(C\) with respect to \(\phi\) as

\[
D_C: \text{int}(\text{dom } \phi) \to [0, +\infty]: x \mapsto \inf_{z \in C} D_\phi(z, x). \tag{2.11}
\]

As regards the Bregman projection, the following holds [10].

Fact 2.5. Let \(C \subset X\) be a nonempty closed convex set and let \(P_C\) be the Bregman projection onto \(C\) as defined in (2.8). Then the following hold.

(i) \((\forall x \in \text{int}(\text{dom } \phi)) P_C(x) = x \iff x \in C \iff D_C(x) = 0\).

(ii) \((\forall x \in C)(\forall y \in \text{int}(\text{dom } \phi)) D_\phi(x, P_C(y)) \leq D_\phi(x, y) - D_\phi(P_C(y), y)\).

(iii) Suppose that \(\text{dom } \phi^*\) is open. Then the operator \(P_C: \text{int}(\text{dom } \phi) \to \text{int}(\text{dom } \phi) \cap C\) and the function \(D_C: \text{int}(\text{dom } \phi) \to \mathbb{R}\) are continuous.
We now consider Bregman projections onto affine sets. Hence, in the rest of the section we assume that

$$C := \{ z \in X \mid Az = b \},$$

where $A : X \to Y$ is a linear operator between Euclidean spaces and $b \in Y$. We wish to characterize the Bregman projector onto $C$. Let $x \in \text{int}(\text{dom } \phi)$. Then, problem (2.7) turns into

$$\min_{z \in X \atop Az = b} D_\phi(z, x).$$

(2.13)

The dual problem of (2.13) (in the sense of Fenchel-Rockafellar) is

$$\min_{\lambda \in Y} \phi^*(\nabla \phi(x) + A^* \lambda) - \phi^*(\nabla \phi(x)) - \langle \lambda, b \rangle =: \Psi_C^*(\lambda).$$

(2.14)

We denote by $x_* = P_C(x)$ the unique solution of (2.13) and by $\lambda_*$ a solution (not necessarily unique) of the dual problem (2.14). They are characterized by the following KKT conditions

$$x_* \in \text{int}(\text{dom } \phi), \quad Ax_* = b, \quad \text{and} \quad \nabla \phi(x) + A^* \lambda_* = \nabla \phi(x_*).$$

(2.15)

A direct consequence of the KKT conditions are the following useful properties.

**Fact 2.6.** Let $x \in \text{int}(\text{dom } \phi)$ and $C = \{ z \in X \mid Az = b \}$. Let $x_* = P_C(x)$ and let $\lambda_*$ be a minimizer of $\Psi_C^*$. Then the following holds.

(i) $(\forall x, y \in \text{int}(\text{dom } \phi)) \ P_C(x) = P_C(y) \Leftrightarrow \nabla \phi(x) - \nabla \phi(y) \in \text{Im}(A^*)$.

(ii) $x_* = \nabla \phi^*(\nabla \phi(x) + A^* \lambda_*)$ and $A \nabla \phi^*(\nabla \phi(x) + A^* \lambda_*) = b$.

(iii) (Pythagoras’s theorem) $(\forall z \in C \cap \text{dom } \phi) \ D_\phi(z, x) = D_\phi(z, x_*) + D_\phi(x_*, x)$.

**Remark 2.7.** Suppose that $C$ is a hyperplane, that is, $C = \{ x \in X \mid \langle a, x \rangle = b \}$. Then, by Fact 2.6(ii), any dual solution $\lambda_*$ satisfies $\langle a, \nabla \phi^*(\nabla \phi(x) + \lambda_*) \rangle = b$, which is an equation in $\mathbb{R}$ and hence can be easily solved via a number of iterative methods (e.g., bisection, gradient descent, Newton). Moreover, in this case the dual solution is unique since $\lambda_* = \langle a, \nabla \phi(x_*) - \nabla \phi(x) \rangle / \| a \|^2$. In [24] several examples in which such equation can be solved explicitly are provided.

**Lemma 2.8.** Let $C_1$ and $C_2$ be two closed affine sets such that $C_2 \subset C_1$ and let $x \in \text{int}(\text{dom } \phi)$. Then, $P_{C_2}(x) = P_{C_2}(P_{C_1}(x))$ and $D_{C_2}(P_{C_1}(x)) + D_{C_1}(x) = D_{C_2}(x)$.

**Proof.** Let $x_i = P_{C_i}(x), \ i = 1, 2$ and $z \in C_2$. Then using Fact 2.6 (iii), $D_\phi(x_2, x_1) + D_\phi(x_1, x) = D_\phi(x_2, x) \leq D_\phi(z, x) = D_\phi(z, x_1) + D_\phi(x_1, x)$, which yields $D_\phi(x_2, x_1) \leq D_\phi(z, x_1)$. Hence $x_2 = P_{C_2}(x_1)$ and $D_{C_2}(x_1) + D_{C_1}(x) = D_{C_2}(x)$. \[\square\]

**Lemma 2.9.** Let $(x, \lambda) \in \text{int}(\text{dom } \phi) \times Y$ be such that $\nabla \phi(x) + A^* \lambda \in \text{int}(\text{dom } \phi^*)$. Then, the following hold.

(i) $(\forall z \in C \cap \text{dom } \phi) \quad \Psi_C^*(\lambda) = D_\phi(z, \nabla \phi^*(\nabla \phi(x) + A^* \lambda)) - D_\phi(z, x)$;

(ii) $(\forall z \in C \cap \text{dom } \phi) \quad D_\phi(z, P_{C_1}(x)) \leq D_\phi(z, \nabla \phi^*(\nabla \phi(x) + A^* \lambda))$. 

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Proof. Since $Az = b$, it follows from (2.14) and Fact 2.1(vi) that

$$
\Psi_C^\ast(\lambda) = \phi^*(\nabla \phi(x) + A^*\lambda) - \phi^*(\nabla \phi(x)) - \langle z, A^*\lambda \rangle \\
= D_{\phi^*}(\nabla \phi(x) + A^*\lambda, \nabla \phi(x)) + \langle x - z, A^*\lambda \rangle \\
= D_{\phi^*}(x, \nabla \phi^*(\nabla \phi(x) + A^*\lambda)) + \langle x - z, A^*\lambda \rangle.
$$

(2.16)

Moreover, it follows from Fact 2.1(v) that

$$
D_{\phi}(z, \nabla \phi^*(\nabla \phi(x) + A^*\lambda)) = D_{\phi}(z, x) + D_{\phi}(x, \nabla \phi^*(\nabla \phi(x) + A^*\lambda)) + \langle x - z, A^*\lambda \rangle,
$$

which together with (2.16) yields (i). Next, since $P_C(x) \in C$, weak duality yields $D_{\phi}(P_C(x), x) \geq -\Psi_C^\ast(\lambda)$. Then, by (i), $D_{\phi}(P_C(x), x) \geq D_{\phi}(z, x) - D_{\phi}(z, \nabla \phi^*(\nabla \phi(x) + A^*\lambda))$. Statement (ii) follows by Pythagora’s theorem given in Proposition 2.6(iii).

\]

2.3 D-Fejér monotone sequences [7, 18]

Let $C \subset X$ be a nonempty closed convex set. Let $\phi$ be a Legendre function such that $C \cap \text{int}(\text{dom } \phi) \neq \emptyset$. A sequence $(x_k)_{k \in \mathbb{N}}$ in $\text{int}(\text{dom } \phi)$ is Bregman monotone or D-Fejér monotone w.r.t. $C$ if

$$
(\forall x \in C)(\forall k \in \mathbb{N}) \quad D_{\phi}(x, x_{k+1}) \leq D_{\phi}(x, x_k).
$$

(2.17)

For D-Fejér monotone sequences, the following properties are known [7, Proposition 4.1, Example 4.7, and Theorem 4.1(i)].

Proposition 2.10. Let $(x_k)_{k \in \mathbb{N}}$ be a D-Fejer monotone sequence with respect to $C$. Then the following hold.

(i) $(\forall x \in C \cap \text{dom } \phi) \quad (D_{\phi}(x, x_k))_{k \in \mathbb{N}}$ is decreasing.

(ii) $(D_{C}(x_k))_{k \in \mathbb{N}}$ is decreasing.

(iii) $(\forall k \in \mathbb{N})(\forall p \in \mathbb{N}) \quad D_{C}(x_{k+p}) \leq D_{C}(x_k) - D_{\phi}(P_C(x_k), P_C(x_{k+p})).$

(iv) $(\forall x \in C \cap \text{dom } \phi)(\forall x' \in C \cap \text{dom } \phi) \quad \langle x - x', \nabla \phi(x_k) \rangle$ is convergent.

(v) Suppose that dom $\phi^*$ is open. Then $(x_k)_{k \in \mathbb{N}}$ is bounded.

(vi) If all cluster points of $(x_k)_{k \in \mathbb{N}}$ lie in $C$, then $(x_k)_{k \in \mathbb{N}}$ converges to some point in $C \cap \text{int}(\text{dom } \phi)$.

Concerning Proposition 2.10(vi), we now give a result ensuring that the cluster points of $(x_k)_{k \in \mathbb{N}}$ lie in $C$. In the sequel we will consider the following sequential consistency assumption [7].

SC For all bounded sequences $(z_k)_{k \in \mathbb{N}}$ and $(y_k)_{k \in \mathbb{N}}$ in $\text{int}(\text{dom } \phi)$

$$
D_{\phi}(z_k, y_k) \rightarrow 0 \implies z_k - y_k \rightarrow 0.
$$

Proposition 2.11. Suppose that $D_{C}(x_k) \rightarrow 0$ and that SC holds. Then $(x_k)_{k \in \mathbb{N}}$ converges to some point in $C \cap \text{int}(\text{dom } \phi)$. 

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**Proof.** Let \( x \in C \cap \text{int}(\text{dom} \, \phi) \). It follows from Proposition 2.10(i) and Fact 2.1(ix) that \( (x_k)_{k \in \mathbb{N}} \) is contained in the compact set \( \{ D_\phi(x, \cdot) \leq D_\phi(x, x_0) \} \subset \text{int}(\text{dom} \, \phi) \). Hence the set of cluster points of \( (x_k)_{k \in \mathbb{N}} \) is nonempty and contained in \( \text{int}(\text{dom} \, \phi) \). Moreover, it follows from Proposition 2.10(iii) (with \( k = 0 \)) and Fact 2.1(ix) that \( (P_C(x_p))_{p \in \mathbb{N}} \) is contained in the compact set \( \{ D_\phi(P_C(x_0), \cdot) \leq D_\phi(x_0) \} \) and hence it is bounded. Let \( x \) be a cluster point of \( (x_k)_{k \in \mathbb{N}} \) and let \( (x_{n_k})_{k \in \mathbb{N}} \) be a subsequence such that \( x_{n_k} \to x \). Then, we saw that \( x \in \text{int}(\text{dom} \, \phi) \). Moreover, \( D_\phi(P_C(x_{n_k}), x_{n_k}) = D_\phi(x_{n_k}) \to 0 \) and hence in virtue of \( \text{SC} \), we have that \( P_C(x_{n_k}) - x_{n_k} \to 0 \). Therefore, \( P_C(x_{n_k}) \to x \), which implies that \( x \in C \), since \( C \) is closed. Thus, we proved that all cluster points of \( (x_k)_{k \in \mathbb{N}} \) lie in \( C \cap \text{int}(\text{dom} \, \phi) \) and therefore, by Proposition 2.10(vi) we derive that \( (x_k)_{k \in \mathbb{N}} \) converges to some point in \( C \). \( \square \)

### 3 The method of Bregman projections

In this section we study the main properties of the method of Bregman projections in general convex feasibility problems. We first address the well known deterministic case for Problem 1.1 in which the various projections are performed in a greedy manner. Then, we introduce the stochastic version of the algorithm which is designed for Problem 1.2. In either case we make the following basic assumption.

**H0** \( C \neq X \), \( \phi \) is a Legendre function, \( \text{dom} \, \phi^* \) is open, and \( \text{int}(\text{dom} \, \phi) \cap C \neq \emptyset \).

Moreover, in this section we will also consider the condition \( \text{SC} \) above.

We note that, referring to Problem 1.1, when \( I \) is finite, say \( I = \{1, \ldots, n\} \), a standard implementation of Algorithm 1.3 is that of cyclic projections, in which we have \( \xi_k = (k \mod n) + 1 \) [1]. In this work depending on the problem at hand we consider instead the following set control schemes.

**C1** For every \( k \in \mathbb{N} \), \( \xi_k \in \text{arg max}_{i \in I} D_\phi(P_C(x_k), x_k) \).

**C2** The \( \xi_k \)'s are \( I \)-valued random variables which are independent copies of \( \xi \).

We call them greedy and random, respectively. The first one was considered in the pioneering work by Bregman [12] and was called the most remote set control in [20]. Instead, random set controls appear in [38] in the context of the orthogonal projection method and in [33] in the study of stochastic fixed point equations. In Section 4.3 we will consider another type of random set control scheme which we call adaptive random which is inspired by the work [27].

**Remark 3.1.** According to Algorithm 1.3, for every \( k \geq 1 \), \( x_k \in C_{\xi_{k-1}} \) and hence \( D_{C_{\xi_{k-1}}}(x_k) = 0 \). So, if the greedy scheme \( \text{C1} \) is adopted, then \( D_{C_{\xi_k}}(x_k) = \max_{i \in I} D_{C_i}(x_k) > 0 \) (otherwise \( x_k \in \bigcap_{i \in I} C_i = C \) and the algorithm would stop) and hence \( \xi_k \neq \xi_{k-1} \). This shows that if \( I = \{1, 2\} \), \( \text{C1} \) reduces to alternating projections.

### 3.1 Convex feasibility problem

We start by approaching Problem 1.1 via Algorithm 1.3 with the greedy set control scheme \( \text{C1} \).

**Proposition 3.2.** Referring to Problem 1.1, suppose that **H0** holds and let \( (x_k)_{k \in \mathbb{N}} \) be generated by Algorithm 1.3. Then, the following hold.
(i) \((\forall k \in \mathbb{N})(\forall x \in C)\) \(D_\phi(x, x_{k+1}) \leq D_\phi(x, x_k) - D_\phi(x_{k+1}, x_k)\). Hence \((x_k)_{k \in \mathbb{N}}\) is D-Fejer monotone with respect to \(C\).

(ii) \(\sum_{k=0}^{+\infty} D_\phi(x_{k+1}, x_k) < +\infty\)

Proof. (i): Let \(k \in \mathbb{N}\). Since \(x_{k+1} = P_{C_{\xi_k}}(x_k)\), it follows from (2.10) that

\[(\forall x \in C_{\xi_k})\quad D_\phi(x, x_k) \geq D_\phi(x, x_{k+1}) + D_\phi(x_{k+1}, x_k).\]

Since \(C \subset C_{\xi_k}\), the statement follows.

(ii): Let \(x \in C \cap \text{dom} \phi\). Then, by (i) we derive that \(D_\phi(x_{k+1}, x_k) \leq D_\phi(x, x_k) - D_\phi(x, x_{k+1})\) and hence \(\sum_{k=0}^{+\infty} D_\phi(x_{k+1}, x_k) < D_\phi(x, x_0) < +\infty\).

The following is essentially Theorem 2 in [12].

Theorem 3.3 (Greedy control scheme). Referring to Problem 1.1, suppose that H0 and SC hold, and that \((x_k)_{k \in \mathbb{N}}\) is generated by Algorithm 1.3 with the greedy set control scheme C1. Then \(D_C(x_k) \to 0\) and \(x_k \to \hat{x} \in C \cap \text{int}(\text{dom} \phi)\).

3.2 Stochastic convex feasibility problems

We now consider Problem 1.2. We denote by \(\mu\) the distribution of the random variable \(\xi\) and by \((\Omega, \mathcal{A}, \mathbb{P})\) the underlying probability space. We recall that the set-valued mapping \(i \in I \to C_i \subset X\) (with closed values) is measurable if for all Borel set \(Z \subset X\), \(\{i \in I \mid C_i \cap Z \neq \emptyset\} \in \mathcal{I}\). Therefore for all \(x \in X\), \(\{i \in I \mid x \in C_i\}\) is measurable and thanks to [15, Lemma III.39] (with \(\varphi(i, z) = -D_\phi(z, x)\) and \(\Sigma(i) = C_i\)) we have that, for every \(x \in \text{int}(\text{dom} \phi)\), the function \(i \in I \to D_{C_i}(x) \in \mathbb{R}\) is also measurable. We will study Algorithm 1.3 adopting the random set control scheme C2, meaning that the \(\xi_k\)'s are independent copies of the random variable \(\xi\). Note that now \(x_k\) is a random variable and more precisely \(x_k = x_k(\xi_0, \ldots, \xi_{k-1})\), so that \(x_k\) and \(\xi_k\) are independent random variables. We denote by \(\mathcal{X}_k\) the \(\sigma\)-algebra generated by \(x_0, \ldots, x_k\). Finally, we set

\[\overline{D}_C : \text{int}(\text{dom} \phi) \to \mathbb{R} : x \mapsto \mathbb{E}[D_{C_{\xi_k}}(x)].\]

(3.1)

Remark 3.4. Since \(\{i \in I \mid x \in C_i\} \in \mathcal{I}\) and \(\xi\) is measurable, we have \(\{x \in C_{\xi}\} = \xi^{-1}(\{i \in I \mid x \in C_i\}) \in \mathcal{A}\). Moreover, for every \(x \in \text{int}(\text{dom} \phi)\), since \(i \in I \to D_{C_i}(x)\) is measurable, \(D_{C_{\xi}}(x)\) is measurable too. Note that

\[\mathbb{P}(\{x \in C_{\xi}\}) = \mathbb{P}(\xi^{-1}(\{i \in I \mid x \in C_i\})) = \mu(\{i \in I \mid x \in C_i\})\]

\[(3.2)\]

\[\mathbb{E}[D_{C_{\xi}}(x)] = \int_{\Omega} D_{C_{\xi}(\omega)}(x)\mathbb{P}(d\omega) = \int_{\mathcal{I}} D_{C_i}(x)\mu(di).\]

(3.3)

Therefore, the definitions of \(C\), in Problem 1.2, and (3.1) depend only on the distribution \(\mu\) of \(\xi\). Finally note that by (3.2) one derives that

\[\mathbb{P}(\{x \in C_{\xi}\}) = 1 \iff \mu(\{i \in I \mid x \in C_i\}) = 1.\]

(3.4)

Proposition 3.5. Referring to Problem 1.2 and assuming H0, the following hold.

(i) The function \(\overline{D}_C : \text{int}(\text{dom} \phi) \to \mathbb{R}\) is lower semicontinuous.

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(iii) There exists a $\mu$-negligible set $J \subset I$ such that, $C = \bigcap_{j \in I \setminus J} C_j$. Moreover, if $(\xi_k)_{k \in \mathbb{N}}$ is a sequence of $I$-valued random variables each one having the same distribution of $\xi$, then there exists a $\mathbb{P}$-negligible set $N$ such that, for every $k \in \mathbb{N}$, $C = \bigcap_{\omega \in \Omega \setminus N} C_{\xi_k(\omega)}$.

**Proof.** (i): Let $(x_k)_{k \in \mathbb{N}}$ be a sequence in $\text{int}(\text{dom} \phi)$ and $x \in \text{int}(\text{dom} \phi)$ be such that $x_k \to x$ and let $z \in C \cap \text{int}(\text{dom} \phi)$. Then, there exists a $\mathbb{P}$-negligible set $N \subset \Omega$ such that, for every $\omega \in \Omega \setminus N$, $z \in C_{\xi(\omega)}$. It follows from Fact 2.5(iii) that, for every $\omega \in \Omega \setminus N$, $D_{C_{\xi(\omega)}}(z) \to D_{C_{\xi(\omega)}}(x)$. Therefore, $D_C(x_k) \to D_C(x)$ almost surely and hence by Fatou’s lemma we have $\mathbb{E}[D_C(x_k)] \leq \liminf_{k \to +\infty} \mathbb{E}[D_C(x_k)]$.

(ii): Let $x \in \text{int}(\text{dom} \phi)$. Since the integrand in $D_C(x)$ is positive, it follows from Fact 2.5(i) that $D_C(x) = 0 \Leftrightarrow D_C(x) = 0 \mathbb{P}$-a.s. $\Leftrightarrow x \in C_{\xi}$ $\mathbb{P}$-a.s.

(iii): Let $Q$ be a countable dense subset of $C$ and let $x \in Q$. Then, since $x \in C$, it follows from (3.4) that there exists a $\mu$-negligible subset $J_x \subset I$ such that $x \in C_i$ for every $i \in I \setminus J_x$. Set $J = \bigcup_{i \in J_x} J_i$. Then $J$ is $\mu$-negligible and, for every $i \in I \setminus J$, $x \in C_i$; hence, $x \in \bigcap_{i \in I \setminus J} C_i$. We then proved that $Q \subset \bigcap_{i \in I \setminus J} C_i$. Since this latter intersection is closed we have $C = \text{cl}(Q) \subset \bigcap_{i \in I \setminus J} C_i$. On the other hand, if $x \in \bigcap_{i \in I \setminus J} C_i$, then, we have $x \in C$, again by (3.4). Suppose now that $(\xi_k)_{k \in \mathbb{N}}$ is a sequence of $I$-valued random variables each one distributed according to $\mu$. Set $N = \bigcup_{k \in \mathbb{N}} \xi_k^{-1}(J)$ and let $k \in \mathbb{N}$. We first prove that $C \subset \bigcap_{\omega \in \Omega \setminus N} C_{\xi_k(\omega)}$. Let $x \in C$. Then, since $\mathbb{P}(\xi_k^{-1}(J)) = \mu(J) = 0$, $N$ is $\mathbb{P}$-negligible and for every $\omega \in \Omega \setminus N$ we have $\xi_k(\omega) \in I \setminus J$ and hence $x \in C_{\xi_k(\omega)}$. Thus, $x \in \bigcap_{\omega \in \Omega \setminus N} C_{\xi_k(\omega)}$. The other inclusion follows from the true definition of $C$ in Problem 1.2, noting that if $x \in \bigcap_{\omega \in \Omega \setminus N} C_{\xi_k(\omega)}$, then $\mathbb{P} \{ x \in C_{\xi_k} \} = \mathbb{P} \{ x \in C_{\xi_k} \} = 1$. \hfill \Box

**Remark 3.6.** Proposition 3.5(iii) is essentially [33, Lemma 2.4 and Corollary 2.6].

**Theorem 3.7 (Random control scheme).** Referring to Problem 1.2, suppose that assumption $H_0$ holds.

Let the function $D_C : \text{int}(\text{dom} \phi) \to \mathbb{R}$ be defined as in (3.1). If $(x_k)_{k \in \mathbb{N}}$ is generated by Algorithm 1.3 with the random set control scheme $C_2$, then the following hold.

(i) The sequence $(x_k)_{k \in \mathbb{N}}$ is $D$-Fejér monotone w.r.t. $C$ $\mathbb{P}$-a.s. and contained in a compact subset of $X$ $\mathbb{P}$-a.s.

(ii) $(D_C(x_k))_{k \in \mathbb{N}}$ is $\mathbb{P}$-a.s. decreasing and $D_C(x_k) \to 0$ $\mathbb{P}$-a.s.

(iii) There exists an $X$-valued random variable $\hat{x}$ and a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ such that $\hat{x} \in C \cap \text{int}(\text{dom} \phi)$ $\mathbb{P}$-a.s. and $x_{n_k} \to \hat{x}$ in distribution.

(iv) $D_C(x_k) \to 0$ $\mathbb{P}$-a.s. and $\mathbb{E}[D_C(x_k)] \to 0$.

(v) If SC holds, then there exists an $X$-valued random variable $\hat{x}$ such that $\hat{x} \in \text{int}(\text{dom} \phi) \cap C$ $\mathbb{P}$-a.s. and $x_k \to \hat{x}$ $\mathbb{P}$-a.s. and $\mathbb{E}[\|x_k - \hat{x}\|^2] \to 0$.

**Proof. (i):** Proposition 3.5(iii) yields that there exists a $\mathbb{P}$-negligible set $N$ such that $C = \bigcap_{\omega \in \Omega \setminus N} C_{\xi(\omega)}$ for every $k \in \mathbb{N}$. Let $\omega \in \Omega \setminus N$ and let $x \in C$. Then, for every $k \in \mathbb{N}$, $x \in C_{\xi_k(\omega)}$, and, hence, since $x_{k+1}(\omega) = P_{C_{\xi(\omega)}}(x_k(\omega))$, it follows from (2.10) that, for every $k \in \mathbb{N}$,

$$D_{\phi}(x, x_{k+1}(\omega)) \leq D_{\phi}(x, x_k(\omega)) - D_{\phi}(x_{k+1}(\omega), x_k(\omega)).$$

(3.5)
This shows that \((x_k)_{k \in \mathbb{N}}\) is \(D\)-Fejér monotone w.r.t \(C\) \(\mathbb{P}\)-a.s. and hence, Proposition 2.10 yields that \((D_\phi(x, x_k))_{k \in \mathbb{N}}\) is \(\mathbb{P}\)-a.s. decreasing. So, if we pick \(x \in C \cap \text{int}(\text{dom } \phi)\), it follows from Fact 2.1(ix) that \((x_k)_{k \in \mathbb{N}}\) is \(\mathbb{P}\)-a.s. contained in the compact sublevel set \(\{D_\phi(x, \cdot) \leq D_\phi(x, x_0)\} \subset \text{int}(\text{dom } \phi)\).

(ii)-(iii): It follows from (i) and Proposition 2.10(ii) that \((D_C(x_k))_{k \in \mathbb{N}}\) is \(\mathbb{P}\)-a.s. decreasing. Also, (i) yields that the sequence \((x_k)_{k \in \mathbb{N}}\) is uniformly essentially bounded, i.e., \(\sup_{k \in \mathbb{N}} \text{ess sup} \|x_k\| < +\infty\). So, Prokhorov’s theorem ensures that there exists a subsequence \((x_{n_k})_{k \in \mathbb{N}}\) converging in distribution to some random vector \(\hat{x}\). Let \(x \in C \cap \text{int}(\text{dom } \phi)\). Since, in virtue of Fact 2.1(ix), \(D_\phi(x, \cdot)\) is positive and lower semicontinuous on \(X\), Portmanteau theorem (see e.g., [3, Theorem 2.8.1]) yields that

\[
\mathbb{E}[D_\phi(x, \hat{x})] \leq \liminf_{k \to \infty} \mathbb{E}[D_\phi(x, x_{n_k})] \leq D_\phi(x, x_0) < +\infty. \tag{3.6}
\]

Thus, \(D_\phi(x, \hat{x}) < +\infty\) \(\mathbb{P}\)-a.s. and hence \(\hat{x} \in \text{int}(\text{dom } \phi)\) \(\mathbb{P}\)-a.s. Now, taking the conditional expectation in (3.5) and using Fact 2.3, we get

\[
\mathbb{E}[D_\phi(x, x_{k+1}) | X_k] \leq D_\phi(x, x_k) - \mathbb{E}[D_\phi(P_{C_{x_k}}(x_k), x_k) | X_k]
\]

\[
= D_\phi(x, x_k) - \mathbb{E}[D_{C_{x_k}}(x_k) | X_k]
\]

\[
= D_\phi(x, x_k) - \overline{D}_C(x_k)
\]

and hence \(\mathbb{E}[D_\phi(x, x_{k+1})] \leq \mathbb{E}[D_\phi(x, x_k)] - \mathbb{E}[\overline{D}_C(x_k)]\). This shows that \((\mathbb{E}[D_\phi(x, x_k)])_{k \in \mathbb{N}}\) is decreasing, hence convergent, and also that \(\mathbb{E}\left[\sum_{k=0}^{+\infty} \overline{D}_C(x_k)\right] = \sum_{k=0}^{+\infty} \mathbb{E}[\overline{D}_C(x_k)] < +\infty\). Then we have \(\overline{D}_C(x_k) \to 0\) and \(\sum_{k=0}^{+\infty} \overline{D}_C(x_k) < +\infty\) \(\mathbb{P}\)-a.s., so that \(\overline{D}_C(x_k) \to 0\) \(\mathbb{P}\)-a.s. Now, since, in virtue of Proposition 3.5(i), the function \(\overline{D}_C : \text{int}(\text{dom } \phi) \to \mathbb{R}\) is lower semicontinuous and \(x_{n_k} \to \hat{x}\) in distribution, another application of Portmanteau theorem gives \(\mathbb{E}[\overline{D}_C(\hat{x})] \leq \liminf_{k \to +\infty} \mathbb{E}[\overline{D}_C(x_{n_k})] = 0\). This, yields that \(\overline{D}_C(\hat{x}) = 0\) \(\mathbb{P}\)-a.s. and hence, by Proposition 3.5(ii), that \(\hat{x} \in C\) \(\mathbb{P}\)-a.s.

(iv): From (ii) we have that \((D_C(x_k))_{k \in \mathbb{N}}\) converges almost surely to some nonnegative and finite random variable \(\zeta\) and that \(D_C(x_k) \leq D_C(x_0)\) \(\mathbb{P}\)-a.s. Thus, it follows from the Lebesgue’s dominated convergence theorem that \(\mathbb{E}[D_C(x_k)] \to \mathbb{E}[\zeta]\). Now, from (iii) we have that \(x_{n_k} \to \hat{x}\) in distribution and \(\hat{x} \in C \cap \text{int}(\text{dom } \phi)\) \(\mathbb{P}\)-a.s. However, \(D_C\) is continuous on \(\text{int}(\text{dom } \phi)\), and hence, in virtue of Fact 2.4, \(D_C(x_{n_k}) \to D_C(\hat{x}) = 0\) in distribution. Therefore, \(\mathbb{E}[D_C(x_{n_k})] \to 0\). This shows that \(\mathbb{E}[\zeta] = 0\) and hence \(\zeta = 0\) \(\mathbb{P}\)-a.s. So, we have that \(D_C(x_k) \to 0\) \(\mathbb{P}\)-a.s.

(v): By item (i), there exists a negligeble set \(\Omega \subset \mathbb{N}\) such that, for all \(\omega \in \mathbb{N} \setminus \Omega\), \((x_k(\omega))_{k \in \mathbb{N}}\) is \(D\)-Fejér monotone w.r.t. \(C\). Then Proposition 2.11 yields that, for every \(\omega \in \mathbb{N} \setminus \Omega\), \((x_k(\omega))_{k \in \mathbb{N}}\) converges to some point in \(C \cap \text{int}(\text{dom } \phi)\). So, convergence \(\mathbb{P}\)-a.s. follows, which, by Fact 2.4, implies convergence in mean square.

\[\square\]

Remark 3.8. In [17], in relation to Algorithm 1.3 and for finite index set \(I\), a repetitive control sequence \((\xi_i)_{i \in \mathbb{N}}\) was used, meaning that for every \(i \in I\), the set \(\{k \in \mathbb{N} | \xi_k = i\}\) is infinite. This type of control was also called random in [8]. We show here that this concept can indeed cover the stochastic setting analyzed in Theorem 3.7, when \(I\) is finite. Indeed, assume that, for every \(i \in I\), \(\mathbb{P}(\xi = i) = p_i > 0\). Let \(i \in I\) and set, for every \(k \in \mathbb{N}\), \(S_k = \{\xi_k = i\}\). Then \(\sum_{k \in \mathbb{N}} \mathbb{P}(S_k) = +\infty\) and, since \((\xi_k)_{k \in \mathbb{N}}\) is an independent sequence of random variables, we have that \((S_k)_{k \in \mathbb{N}}\) is an independent sequence of events. Hence, by the second Borel-Cantelli lemma [25, Theorem 2.3.6], we derive that \(\mathbb{P}(\limsup_{k} S_k) = 1\). Note that \(\omega \in \limsup_{k} S_k \iff \{k \in \mathbb{N} | \omega \in S_k\}\) is infinite. Moreover, since \(\Omega = \limsup_{k} S_k\) is a set of probability one, so is \(\Omega = \bigcap_{i \in I} \Omega_i\). Therefore, if we pick \(\omega \in \Omega\), we have that \((\xi_k(\omega))_{k \in \mathbb{N}}\) is a repetitive control sequence and hence the almost sure convergence in Theorem 3.7(iv)-(v) can be derived from [8, Theorem 8.1] (see also [17, Theorem 3.2]).
4 Convergence of the method of Bregman projections for affine sets

In this section we analyze the convergence of the Bregman projection method for affine feasibility problems. We prove global and local Q-linear convergence of the method, providing also explicit global and local rates. We cover three options for the set control sequence: greedy, random, and adaptive random.

We will make the following assumption.

**H1** The sets $C_i$’s are affine, i.e., for every $i \in I$, $C_i = \{ x \in X \mid A_i x = b_i \}$ for some nonzero linear operator $A_i : X \to Y_i$ and some $b_i \in Y_i$.

In this situation in both Problem 1.1 and Problem 1.2 the set $C$ is affine. We therefore let $A : X \to Y$ be a linear operator and $b \in Y$ such that

$$C = \{ x \in X \mid Ax = b \}.$$  \hfill (4.1)

As before, we will make assumption **H0** and in addition, depending on the fact that we are dealing with Problem 1.1 or Problem 1.2, we will also make one of the following technical assumptions.

**H2** $\phi^*$ is twice differentiable and, for all $x \in \text{int}(\text{dom } \phi) \cap C$, $A \nabla^2 \phi^*(\nabla \phi(x)) \neq 0$ and

$$\sup_{i \in I} \| A_i^* (A_i \nabla^2 \phi^*(\nabla \phi(x)) A_i^*)^\dagger A_i \| < +\infty.$$  

**H2** $\phi^*$ is twice differentiable and, for all $x \in \text{int}(\text{dom } \phi) \cap C$, $A \nabla^2 \phi^*(\nabla \phi(x)) \neq 0$ and

$$\text{ess sup} \| A_i^*(A_i \nabla^2 \phi^*(\nabla \phi(x)) A_i^*)^\dagger A_i \| < +\infty.$$  

The result below shows that **H2** and **H2** hold in a number of significant cases.

**Proposition 4.1.** Suppose that $\phi^*$ is twice differentiable. Then, referring to Problem 1.1 (resp. Problem 1.2), assumption **H2** (resp. assumption **H2**) holds if one of the following conditions is met:

(a) for every $x \in \text{int}(\text{dom } \phi) \cap C$, $\nabla^2 \phi^*(\nabla \phi(x))$ is invertible,

(b) $\phi$ is twice differentiable on $\text{int}(\text{dom } \phi)$,

(c) $I$ is finite and for $x \in \text{int}(\text{dom } \phi) \cap C$, $A \nabla^2 \phi^*(\nabla \phi(x)) \neq 0$,

(d) $I$ is finite, $X = \mathbb{R}^n$, $0 \notin C$, and $\phi(x) = (1/p) \| x \|_p^p$, for $p \in ]1, 2[.$

**Proof.** We will prove only the case **H2**. The other is similar.

(a): Let $x \in \text{int}(\text{dom } \phi) \cap C$ and set $H = [\nabla^2 \phi^*(\nabla \phi(x))]^{1/2}$. Since $H^2$ is invertible, we have $A \neq 0 \Rightarrow AH^2 \neq 0$. Moreover, it follows from Fact 2.2 that $Q_i(x) = HA_i^*(A_i H^2 A_i^*)^\dagger A_i H$ is the orthogonal projector onto $\text{Im}(HA_i^*)$. Since $H$ is invertible, we have $A_i^*(A_i H^2 A_i^*)^\dagger A_i = H^{-1}Q_i(x)H^{-1}$ and hence $\| A_i^*(A_i H^2 A_i^*)^\dagger A_i \| \leq \| H^{-1} \| \| Q_i(x) \| \| H^{-1} \| = \| H^{-1} \| ^2$. Thus, the last condition in **H2** follows.

(b): Since both $\phi$ and $\phi^*$ are twice differentiable, their Hessians are inverse to each other. Hence, (a), and a fortiori **H2** holds.

(c): If $I$ is finite, then the last condition in **H2** trivially holds.
(d): First, note that \( \phi^*(y) = q^{-1}\|y\|^q \), where \( p^{-1} + q^{-1} = 1 \), and that \( \text{dom } \phi = \text{dom } \phi^* = \mathbb{R}^n \).

Next, let \( x \in \text{int(dom } \phi) \cap C \) and \( y = \nabla \phi(x) \). Then, for every \( j \in \{1, \ldots, n\} \), \( y_j = \text{sgn}(x_j) |x_j|^{p-1} \).

Hence, \( \nabla^2 \phi^*(y) \) is a diagonal matrix such that \( [\nabla^2 \phi^*(y)]_{j,j} = (q-1) |y_j|^{q-2} = (q-1) |x_j|^{(p-1)(q-2)} \), for every \( j \in \{1, \ldots, n\} \). Now, since \( x \neq 0 \), if we define the vector \( v \in \mathbb{R}^n \) as

\[
(\forall j \in [n]) \quad v_j = \begin{cases} (q-1)^{-1} \text{sgn}(x_j) |x_j|^{1-(p-1)(q-2)} & \text{if } x_j \neq 0 \\ 0 & \text{otherwise,} \end{cases}
\]

we have \( v \neq 0 \) and \( \nabla^2 \phi^*(y)v = x \), and hence \( A \nabla^2 \phi^*(y)v = Ax = b \neq 0 \) (since \( 0 \notin C \)). So, the first part of \( H_2 \) follows. The last condition in \( H_2 \) follows from (c).

\[ \square \]

### 4.1 Greedy set control scheme

We address Problem 1.1 assuming \( H_0, H_1 \), and \( H_2 \). We first give a general convergence theorem and then we study rate of convergence going through several technical results and finally providing a theorem.

**Theorem 4.2.** Under the assumptions of Problem 1.1 and \( H_0 \), let \( (x_k)_{k \in \mathbb{N}} \) be generated by Algorithm 1.3 using any (deterministic) sequence \( (\xi_k)_{k \in \mathbb{N}} \) in I and let \( x_* = P_C(x_0) \). Then the following hold.

(i) For every \( k \in \mathbb{N} \), \( \nabla \phi(x_*) - \nabla \phi(x_k) \in \text{Im}(A^*) \).

(ii) \( (\forall k \in \mathbb{N}) \quad P_C(x_k) = x_* \) and \( D_C(x_{k+1}) + D_{C_{\xi_k}}(x_k) = D_C(x_k) \).

(iii) Suppose that \( SC \) and \( C_1 \) hold. Then \( x_k \to x_* \).

**Proof.** (i): Let, for every \( i \in I, C_i = \{ x \in X | A_i x = b_i \} \) for some linear operator \( A_i : X \to Y \) and \( b_i \in Y \). Since \( C \subset C_i \), we have \( \text{Ker}(A) \subset \text{Ker}(A_i) \) and hence \( \text{Im}(A_i^*) = \text{Ker}(A_i)^\perp \subset \text{Ker}(A)^\perp = \text{Im}(A^*) \).

Moreover, (2.15) yields that \( \nabla \phi(x_*) = \nabla \phi(x_0) + v_* \) for some \( v_* \in \text{Im}(A^*) \) and, for every \( k \in \mathbb{N} \), \( \nabla \phi(x_{k+1}) = \nabla \phi(x_k) + v_{\xi_k} \) for some \( v_{\xi_k} \in \text{Im}(A_{\xi_k}^*) \subset \text{Im}(A^*) \). Therefore, \( \nabla \phi(x_k) = \nabla \phi(x_0) + \sum_{j=0}^{k-1} v_{\xi_j} \) and hence \( \nabla \phi(x_*) = \nabla \phi(x_k) = v_* - \sum_{j=0}^{k-1} v_{\xi_j} \in \text{Im}(A^*) \).

(ii): Let \( k \in \mathbb{N} \). Since \( C \subset C_{\xi_k} \) and \( x_{k+1} = P_{C_{\xi_k}}(x_k) \), Lemma 2.8 (with \( x = x_k \)) yields that

\[
P_C(x_k) = P_C(x_{k+1}) \quad \text{and} \quad D_C(x_{k+1}) + D_{C_{\xi_k}}(x_k) = D_C(x_k). \tag{4.3}
\]

Thus, the second part of the statement follows immediately, while the first part follows by applying the first equation in (4.3) recursively.

(iii): By Theorem 3.3 we have \( x_k \to \hat{x} \) for some \( \hat{x} \in C \cap \text{int(dom } \phi) \). Since \( P_C \) is continuous on \( \text{int(dom } \phi) \), we have \( P_C(x_k) \to P_C(\hat{x}) = \hat{x} \). However, it follows from (ii) that \( P_C(x_k) = x_* \), for every \( k \in \mathbb{N} \). Therefore, \( \hat{x} = x_* \). \[ \square \]

Next, we provide Lemma 4.3 containing the definition of \( \gamma_C \), which governs the local linear rate. Then, we introduce Lemma 4.4 that defines \( \sigma_C \), which is critical for obtaining the global convergence rate. Finally, we conclude with Theorem 4.5 covering the Q-linear convergence of the greedy Bregman projection method for affine sets.

**Lemma 4.3.** Referring to Problem 1.1, suppose that assumptions \( H_0, H_1 \), and \( H_2 \) hold. Let, for every \( x \in \text{int(dom } \phi) \cap C \), \( Q_i(x) \) be the orthogonal projection onto \( \text{Im}(\nabla^2 \phi^* \nabla \phi(x))^{1/2} A_i^* \), \( V(x) = \text{Im}(\nabla^2 \phi^* \nabla \phi(x))^{1/2} A^* \), and

\[
\gamma_C(x) = \inf_{v \in V(x) \setminus \{0\}} \sup_{i \in I} \frac{\|Q_i(x)v\|^2}{\|v\|^2}. \tag{4.4}
\]
Then, for every \( x \in \text{int}(\text{dom } \phi) \cap C \), \( \gamma_C(x) \in [0,1] \). Moreover, for every \( \varepsilon \in [0,1] \) there exists \( \delta > 0 \), such that for every \( x \in \text{int}(\text{dom } \phi) \),

\[
D_C(x) < \delta \quad \implies \quad \inf_{i \in I} D_C(P_C(x)) \leq \frac{1+\varepsilon}{1-\varepsilon} [1-\gamma_C(P_C(x))] D_C(x).
\] (4.5)

Proof. Let \( x \in \text{int}(\text{dom } \phi) \) and \( y = \nabla \phi(x) \). First of all we note that assumption \( H_2 \) yields \( [\nabla^2 \phi^*(y)]^{1/2} [\nabla^2 \phi^*(y)]^{1/2} A^* = [\nabla^2 \phi^*(y)] A^* \neq 0 \) and hence \( [\nabla^2 \phi^*(y)]^{1/2} A^* \neq 0 \), which in turns implies \( V(x) \neq \emptyset \). Clearly, \( \gamma_C(x) = \min_{v \in V(x), ||v|| = 1} \sup_{i \in I} ||Q_i(x)v||^2 \) and hence \( \gamma_C(x) \in [0,1] \). Now assume, by contradiction, that \( \gamma_C(x) = 0 \). Then, there exists \( v \in V(x) \), with \( ||v|| = 1 \), such that

\[
(\forall i \in I) \quad v \in \text{Ker}(Q_i(x)) = \text{Im}([\nabla^2 \phi^*(y)]^{1/2} A^*)^\perp = \text{Ker}(A_i[\nabla^2 \phi^*(y)]^{1/2}).
\] (4.6)

that is, \( [\nabla^2 \phi^*(y)]^{1/2} v \in \text{Ker}(A_i) \). Then, since \( \cap_{i \in I} \text{Ker}(A_i) = \text{Ker}(A) \), we have that \( [\nabla^2 \phi^*(y)]^{1/2} v \in \text{Ker}(A) \), and hence, \( v \in \text{Ker}(A[A^2 \phi^*(y)]^{1/2}) = V(x)^\perp \). Thus, since \( v \neq 0 \), we obtain a contradiction, and hence necessarily \( \gamma_C(x) > 0 \).

The proof of (4.5) is quite technical. Therefore, we will proceed through 6 steps. First note that if \( x \in \text{int}(\text{dom } \phi) \cap C \), then, recalling Fact 2.5(i), we have \( 0 = D_C(x) = D_C(P_C(x)) \), for every \( i \in I \), hence the inequality in (4.5) trivially holds. Therefore, in the following we let \( x \in \text{int}(\text{dom } \phi) \setminus C \) and set \( x_\tau = P_C(x) \), \( y = \nabla \phi(x) \) and \( y_\tau = \nabla \phi(x_\tau) \). Additionally, for the sake of brevity, we set \( H = [\nabla^2 \phi^*(y_\tau)]^{1/2} \).

**Step 1:** We have

\[
(\forall v_i \in \text{Im}(A_i^*)) \quad y + v_i \in \text{int}(\text{dom } \phi^*) \quad \implies \quad D_C(x_i) \leq D_{\phi^*}(y + v_i). \] (4.7)

Indeed, Lemma 2.8 yields \( D_C(x_i) = D_{\phi^*}(P_C(x_i), x_i) = D_{\phi^*}(P_C(x_i), x_i) = D_{\phi^*}(x_\tau, P_C(x_i)) \), with \( x_\tau \in C_i \). Hence, using Lemma 2.9 and Fact 2.1(vi), (4.7) follows.

**Step 2:** There exists \( \tilde{w} \in \text{Im}(A^*) \) such that \( ||H\tilde{w}|| = 1 \) and, for all \( \tau > 0 \),

\[
u_{\tau} := H(y_\tau - y + \tau \tilde{w}) \in V(x_\tau) \setminus \{0\}.
\] (4.8)

Indeed, first recall that \( \text{Im}(HA^*) = V(x_\tau) \neq \emptyset \). It follows from (2.15) that \( y_\tau - y \in \text{Im}(A^*) \). Now, if \( H(y_\tau - y) \neq 0 \) we define \( \tilde{w} = (y_\tau - y)/||H(y_\tau - y)|| \) and (4.8) follows. Otherwise, since \( \text{Im}(HA^*) \neq \emptyset \), we can pick \( \tilde{w} \in \text{Im}(A^*) \) such that \( ||H\tilde{w}|| = 1 \) and again (4.8) follows.

**Step 3:** Suppose that \( HA^*_i \neq 0 \). We prove that, for every \( \tau > 0 \), there exists \( v_{i,\tau} \in \text{Im}(A_i^*) \) such that \( y + v_{i,\tau} - y_\tau \notin \text{Ker}(H) \) and

\[
||y + v_{i,\tau} - y_\tau|| \leq (1 + M ||H||^2) ||y_\tau - y|| + 3\tau M ||H||,
\] (4.9)

where \( \sup_{i \in I} ||A_i^* (A_i H^2 A_i^*)^\dagger A_i || \leq M < +\infty \), due to \( H_2 \). Indeed, since \( HA_i^* \neq 0 \), there exists \( w_i \in \text{Im}(HA_i^*) \) such that \( ||w_i|| = 2 \). Now, note that

\[
Q_i(x_\tau) = HA_i^* [A_i H^2 A_i^*]^\dagger A_i H
\] (4.10)

and let, for every \( \tau > 0 \),

\[
v_{i,\tau} := A_i^* (A_i H^2 A_i^*)^\dagger A_i H(u_{\tau} + \tau w_i) \in \text{Im}(A_i^*).
\] (4.11)

Then, recalling (4.8), (4.10), and the fact that \( w = H\tilde{w} \), we have

\[
H(y_\tau - y - v_{i,\tau}) = H(y_\tau - y) - Q_i(x_\tau)(u_{\tau} + \tau w_i)
\]

\[
= [I - Q_i(x_\tau)]H(y_\tau - y) - \tau Q_i(x_\tau)(w + w_i)
\]
and, since \( Q_i(x_*) \) is the projector onto \( \text{Im}(HA^*_i) \) and \( w_i \in \text{Im}(HA^*_i) \), we have
\[
\|H(y + v_{i,τ} - y_*)\|^2 = \|\left( I - Q_i(x_*) \right) H(y_*) - y\|^2 + τ^2 \|Q_i(x_*)w + w_i\|^2. \tag{4.12}
\]
In the above formula we have \( Q_i(x_*)w \neq w_i \), since \( \|w_i\|^2 = 2 \) while \( \|Q_i(x_*)w\| \leq \|w\| = 1 \). Therefore \( \|H(y + v_{i,τ} - y_*)\|^2 > 0 \) and hence \( y + v_{i,τ} - y_* \notin \text{Ker}(H) \). Finally, inequality (4.9) follows by bounding \( \|v_{i,τ}\| \) using (4.11), (4.8), assumption H21, and the fact that \( \|w_i\| = 2 \) and \( \|H \tilde{w}\| = 1 \).

**Step 4:** Suppose that \( HA^*_i \neq 0 \) and let \( ε \in ]0, 1[ \). We prove that for \( τ > 0 \) sufficiently small
\[
\frac{1 - ε}{2} \|u_τ\|^2 \leq D_C(x) \quad \text{and} \quad D_C(x_i) \leq \frac{1 + ε}{2} \left( \|\left( I - Q_i(x_*)\right)u_τ\|^2 + \sqrt{τ}D_C(x) \right). \tag{4.13}
\]
Indeed, it follows from the second part of Fact 2.1(xi), applied to \( D_{φ^τ} \), that there exists \( \tilde{δ} > 0 \) such that if \( \|\tilde{y} - y_*\| < \tilde{δ} \) and \( \tilde{y} - y_* \notin \text{Ker}(H) \), then
\[
\frac{\sqrt{1 - ε}}{2} (H^2(\tilde{y} - y_*), \tilde{y} - y_* \leq D_{φ^τ}(\tilde{y}, y_*) \leq \frac{1 + ε}{2} (H^2(\tilde{y} - y_*), \tilde{y} - y_*).
\]
Therefore, setting \( β_* = 1 + \max\{3M \|H\|^2 + M \|H\|, \|\tilde{w}\|\} \) \( > 1 \), it follows from the inequality \( \|y_* - y - τ\tilde{w}\| \leq \|y_* - y\| + τ\|\tilde{w}\| \) and (4.9) that if \( τ \leq \|y - y_*\| \) and \( \|y - y_*\| \leq \tilde{δ}/β_* \), we have
\[
D_{φ^τ}(y + v_{i,τ}, y_*) \leq \frac{1 + ε}{2} (H^2(y + v_{i,τ} - y_*), y + v_{i,τ} - y_* \tag{4.14}
\]
and
\[
D_{φ^τ}(y + τ\tilde{w}, y_*) \geq \frac{\sqrt{1 - ε}}{2} (H^2(y + τ\tilde{w} - y_*), y + τ\tilde{w} - y_*). \tag{4.15}
\]
Now, the continuity of \( \nabla φ \) and Fact 2.1(x) yields that there exists \( δ > 0 \) such that if \( D_C(x) < δ \) then, \( \|y - y_*\| < \frac{δ}{β_*} \), and hence, collecting (4.7) and (4.14), we obtain \( D_C(x_i) \leq D_{φ^τ}(y + v_{i,τ}, y_*) \leq ((1 + ε)/2) \|H(y + v_{i,τ} - y_*)\|^2 \). However, it also holds that \( \|H(y + v_{i,τ} - y_*)\| = \|\left( I - Q_i(x_*)\right)u_τ + τ(w - w_i)\| \leq \|\left( I - Q_i(x_*)\right)u_τ\| + 3τ \). Therefore, since \( \|u_τ\| \leq \|H(y - y_*)\| + τ\|H \tilde{w}\| \leq (\|H\| + 1)\|y - y_*\|, \)
\[
D_C(x_i) \leq \frac{1 + ε}{2} \left( \|\left( I - Q_i(x_*)\right)u_τ\|^2 + 9τ^2 + 6τ\|u_τ\| \right)
\]
\[
\leq \frac{1 + ε}{2} \left( \|\left( I - Q_i(x_*)\right)u_τ\|^2 + 3τ(2\|H\| + 5)\|y - y_*\| \right)
\]
which, for \( τ \leq τ_*^{(1)} := \min\{\|y_* - y\|, 9^{-1} D_C(x)^2 \|y_* - y\|^{-2}(2\|H\| + 5)^{-2}\} \), gives
\[
D_C(x_i) \leq \frac{1 + ε}{2} \left( \|\left( I - Q_i(x_*)\right)u_τ\|^2 + \sqrt{τ}D_C(x) \right). \tag{4.16}
\]
On the other hand, \( D_C(x) = D_{φ^τ}(x_*, x) = D_{φ^τ}(y, y_*) = 0 \). Hence, using the continuity of \( D_{φ^τ}(\cdot, y_*), \) we have that there exists \( τ_*^{(2)} > 0 \) such that for every \( τ \leq τ_*^{(2)} \), \( D_C(x) \geq \sqrt{1 - ε} D_{φ^τ}(y + τ\tilde{w}, y_*) \). So, (4.15) yields
\[
D_C(x) \geq \frac{1 - ε}{2} (H^2(y + τ\tilde{w} - y_*), y + τ\tilde{w} - y_*) = \frac{1 - ε}{2} \|u_τ\|^2. \tag{4.17}
\]
**Step 5:** For \( τ \leq \min\{τ_*^{(1)}, τ_*^{(2)}\} \), we have
\[
\frac{D_C(x_i)}{D_C(x)} \leq \frac{1 + ε}{1 - ε} \left( 1 - \frac{\|Q_i(x_*)u_τ\|^2}{\|u_τ\|^2} \right) + \frac{1 + ε}{2}\sqrt{τ}. \tag{4.18}
\]
This follows from (4.13) when $HA^*_r \neq 0$. However, (4.18) holds even when $\text{Im}(HA^*_r) = \{0\}$. Indeed in such case, recalling the definition of $Q_i(x_*)$, we have $Q_i(x_*) \equiv 0$. Hence, since $D_C(x_i) \leq D_C(x)$, we have that (4.18) actually holds for every $\tau > 0$.

**Step 6:** Note that inequality (4.18) holds for all $i \in I$ and every $\tau \leq \min\{\tau^{(1)}_\ast, \tau^{(2)}_\ast\}$. Consequently, taking the infimum on both sides of (4.18) we have
\[
\inf_{i \in I} \frac{D_C(x_i)}{D_C(x)} \leq \frac{1 + \varepsilon}{1 - \varepsilon} \left(1 - \sup_{i \in I} \frac{\|Q_i(x_*)u_\tau\|^2}{\|u_\tau\|^2}\right) + \frac{1 + \varepsilon}{2}\sqrt{\tau}.
\]
Then (4.5) follows by recalling that $u_\tau \in V(x_*) \setminus \{0\}$, and letting $\tau \to 0$.

**Lemma 4.4.** Under the same assumptions of Lemma 4.3, define the function $\sigma_C$: $\text{int}(\text{dom } \phi) \to [0, 1]$ such that, for every $x \in \text{int}(\text{dom } \phi)$,
\[
\sigma_C(x) = \begin{cases} 
\sup_{z \in K(x) \cap C} \left[\inf_{i \in I} \frac{D_C(P_C(z))}{D_C(z)}\right] & \text{if } x \notin C, \\
1 - \gamma_C(x) & \text{if } x \in C,
\end{cases}
\]
where $K(x) := \{z \in \text{int}(\text{dom } \phi) \mid P_C(z) = P_C(x), \ D_C(z) \leq D_C(x)\}$. Then,

(i) $(\forall x \in \text{int}(\text{dom } \phi)) \sigma_C(x) < 1$ and $\inf_{i \in I} D_C(P_C(x)) \leq \sigma_C(x) D_C(x)$;

(ii) $(\forall x, y \in \text{int}(\text{dom } \phi) \setminus C) y \in K(x) \Rightarrow \sigma_C(y) \leq \sigma_C(x)$.

**Proof.** (i): Clearly, by definition of $\sigma_C$, for every $x \notin C$ (since $x \in K(x)$), we have $\inf_{i \in I} D_C(P_C(x)) \leq \sigma_C(x) D_C(x)$, while for $x \in C$, the inequality is satisfied trivially. Next, let $x \in \text{int}(\text{dom } \phi)$ and let $x_\ast = P_C(x)$. We prove that $\sigma_C(x) < 1$. According to Lemma 4.3 and definition (4.19), if $x \in C$, then $\sigma_C(x) < 1$. So, in the rest of the proof we assume $x \notin C$. Let $z \in K(x) \setminus C$. Then $D_C(z) > 0$ and, since $C = \cap_{i \in I} C_i$, there exists $j \in I$, such that $D_{C_j}(z) > 0$. Now, let $\gamma_z \in [0, 1]$ be such that $D_{C_j}(z) \geq \gamma_z D_{C_j}(z)$. Since, in virtue of Fact 2.5(iii), $D_{C_j}$ and $D_C$ are continuous, there exists an open set $U_z \subset \text{int}(\text{dom } \phi)$ such that $z \in U_z$ and for all $z \in U_z$, $D_{C_j}(z) \geq \gamma_z D_{C_j}(z)$. Then, since $C \subset C_j$, by Lemma 2.8, we have $D_C(P_C(z)) = D_C(z) - D_{C_j}(z) \leq (1 - \gamma_z) D_{C_j}(z)$, and hence
\[
(\forall z \in U_z) \inf_{i \in I} D_C(P_C(z)) \leq (1 - \gamma_z) D_C(z), \quad \text{with } 1 - \gamma_z < 1.
\]

Next, let $z \in K(x) \cap C$ and take $\varepsilon > 0$ such that $(1 + \varepsilon)(1 - \varepsilon)^{-1}[1 - \gamma_C(x_\ast)] < 1$. Moreover, in virtue of Lemma 4.3, there exists $\delta > 0$ such that (4.5) holds. So, define $U_z := \{z \in \text{int}(\text{dom } \phi) \mid D_\phi(z, z_\ast) < \delta\}$. Then, for $z \in U_z$, since $z \in C$, we have that $D_C(z) \leq D_\phi(z, z_\ast) < \delta$, and hence, by (4.5),
\[
\inf_{i \in I} D_C(P_C(z)) \leq \frac{1 + \varepsilon}{1 - \varepsilon}[1 - \gamma_C(P_C(z))] D_C(z).
\]

Now, the set $\tilde{K}(x) = \{z \in \text{int}(\text{dom } \phi) \mid D_\phi(P_C(x), z) \leq D_C(x)\}$ is compact by Fact 2.1(ix). Moreover, by Fact 2.6(i), $K(x) = \{z \in \tilde{K}(x) \mid \nabla_\phi(z) - \nabla_\phi(x) \in \text{Im}(A^*_r)\}$. Thus, it follows from the continuity of $\nabla_\phi$ that $K(x)$ is a closed subset of the compact set $\tilde{K}(x)$, and hence it is compact. Thus, since $\bigcup_{z \in K(x)} U_z$ is an open covering of $K(x)$, there exist points $z_1, \ldots, z_m \in K(x)$ so that $K(x) \subset U_{z_1} \cup \cdots \cup U_{z_m}$, and we can define
\[
\sigma = \max \{1 - \gamma_{z_1}, \ldots, 1 - \gamma_{z_m}, (1 + \varepsilon)(1 - \varepsilon)^{-1}[1 - \gamma_C(P_C(x_\ast))]\} < 1.
\]
We let \( z \in K(x) \setminus C \). Then, there exists \( j \in \{1, \ldots, m\} \), such that \( z \in U_j \). If \( z_j \notin C \), then we derive from (4.20) that \( \inf_{i \in I} D_C(P_C(z)) / D_C(z) \leq 1 - \gamma_{z_j} \leq \sigma \), while if \( z_j \in C \), since \( P_C(z) = x_* \), by (4.21) we have \( \inf_{i \in I} D_C(P_C(z)) / D_C(z) \leq (1 + \varepsilon)(1 - \varepsilon)^{-1}[1 - \gamma_C(x_*)] \leq \sigma \). Therefore,

\[
\sigma_C(x) = \sup_{z \in K(x) \setminus C} \left[ \inf_{i \in I} \frac{D_C(P_C(z))}{D_C(z)} \right] \leq \sigma < 1. \tag{4.23}
\]

(ii): Let \( x, y \in \text{int(dom } \phi) \setminus C \) be such that \( P_C(y) = P_C(x) \) and \( D_C(y) \leq D_C(x) \). Then, clearly \( K(y) \subset K(x) \) and hence \( K(y) \setminus C \subset K(x) \setminus C \). Then, since \( x, y \notin C \), the statement follows from the definition of \( \sigma_C \) in (4.19).

\[\square\]

**Theorem 4.5** (Greedy set control scheme). With reference to Problem 1.1, suppose that \( H0 \), \( H1 \), and \( H2_1 \) hold. Let \( (x_k)_{k \in \mathbb{N}} \) be generated by Algorithm 1.3 with the control \( C1 \) and let \( x_* = P_C(x_0) \). Then the following hold.

(i) \( \forall k \in \mathbb{N} \) \( D_C(x_{k+1}) \leq \sigma_C(x_k) D_C(x_k) \).

(ii) \( \forall n \in \mathbb{N} \) \( \forall k \in \mathbb{N} \) \( D_C(x_{k+1+n}) \leq \sigma_C(x_n) D_C(x_{k+n}) \).

(iii) Either \( D_C(x_k) \to 0 \) in a finite number of iterations or the sequence \( (\sigma_C(x_k))_{k \in \mathbb{N}} \) is decreasing and

\[\lim_{k \to \infty} \sup_{i \in I} \frac{D_C(x_{k+1})}{D_C(x_k)} \leq \lim_{k \to \infty} \sigma_C(x_k) \leq \sigma C(x_*) \].

**Proof.** (i): Let \( k \in \mathbb{N} \). By Theorem 4.2(ii), we have that

\[D_C(x_{k+1}) = D_C(x_k) - D_{C_{x_k}}(x_k), \tag{4.24}\]

where the greedy choice ensures that \( D_{C_{x_k}}(x_k) = \sup_{i \in I} D_C(x_k) \). Thus, \( D_C(x_{k+1}) = \inf_{i \in I} (D_C(x_k) - D_C(x_k)) \) and hence, by Lemma 2.8,

\[D_C(x_{k+1}) = \inf_{i \in I} D_C(P_C(x_k)). \tag{4.25}\]

So, the statement follows from Lemma 4.4(i).

(ii): First note that Theorem 4.2(ii) yields that \( (D_C(x_k))_{k \in \mathbb{N}} \) is decreasing and that \( (P_C(x_k))_{k \in \mathbb{N}} \equiv x_* \). Let \( n, k \in \mathbb{N} \). If \( D_C(x_{k+1+n}) = 0 \), then the inequality holds. Suppose that \( D_C(x_{k+1+n}) > 0 \). Then, \( 0 < D_C(x_{k+1+n}) \leq D_C(x_{k+n}) \leq D_C(x_n) \) and hence \( x_n, x_{k+n} \notin C \) and, using the notation of Lemma 4.4, \( x_{k+n} \in K(x_n) \). So, by Lemma 4.4(ii), we have \( \sigma_C(x_{k+n}) \leq \sigma_C(x_n) \) and the statement follows from (i).

(iii): If \( D_C(x_n) = 0 \) for some \( n \in \mathbb{N} \), since \( (D_C(x_k))_{k \in \mathbb{N}} \) is decreasing, then, for every integer \( k \geq n \), \( D_C(x_k) = 0 \) and hence \( D_C(x_k) \to 0 \) in a finite number of iterations. Suppose that for every \( k \in \mathbb{N} \), \( D_C(x_k) > 0 \), that is, \( x_k \notin C \). Then \( x_{k+1} \in K(x_k) \) and hence, by Lemma 4.4(ii), \( \sigma_C(x_{k+1}) \leq \sigma_C(x_k) \). Moreover, according to Lemma 4.3, given \( \varepsilon \in [0, 1] \) there exists \( \delta > 0 \) be such that (4.5) holds. Since, by (ii), \( D_C(x_k) \to 0 \), there exists \( n \in \mathbb{N} \) such that for all integer \( k \geq n \), \( D_C(x_k) < \delta \), and hence for every \( z \in K(x_k) \setminus C \), since \( D_C(z) \leq D_C(x_k) \leq \delta \), (4.5) yields

\[\inf_{i \in I} \frac{D_C(P_C(z))}{D_C(z)} \leq \frac{1 + \varepsilon}{1 - \varepsilon}[1 - \gamma_C(x_*)]\]

and, recalling (4.19), \( \sigma_C(x_k) \leq (1 + \varepsilon)(1 - \varepsilon)^{-1}\sigma_C(x_*) \). Therefore, using also (i),

\[\lim_{k \to \infty} \frac{D_C(x_{k+1})}{D_C(x_k)} \leq \lim_{k \to \infty} \sigma_C(x_k) = \inf_{k \in \mathbb{N}} \sigma_C(x_k) \leq \frac{1 + \varepsilon}{1 - \varepsilon}\sigma C(x_*) \tag{4.26}\]

Since \( \varepsilon \) is arbitrary in \([0, 1]\), the statement follows. \[\square\]
4.2 Random set control scheme

We now address Problem 1.2. Following the same line of presentation as in the greedy case, we first give a general theorem of convergence and then we analyze the rate of convergence.

**Theorem 4.6.** With reference to Problem 1.2, suppose that H0 and H1 hold. Let \((x_k)_{k \in \mathbb{N}}\) be generated by Algorithm 1.3 using the random set control scheme C2 and let \(x_\star = P_C(x_0)\). Then the following hold.

(i) \(\forall k \in \mathbb{N}, \nabla \phi(x_\star) - \nabla \phi(x_k) \in \text{Im}(A^*) \mathbb{P}\text{-a.s.}\)

(ii) \(\forall k \in \mathbb{N}, P_C(x_k) = x_\star \mathbb{P}\text{-a.s. and } D_C(x_{k+1}) + D_C(x_k) = D_C(x_k) \mathbb{P}\text{-a.s.}\)

(iii) Under the assumptions of Theorem 3.7, \(x_k \to x_\star \mathbb{P}\text{-a.s. and } \mathbb{E}[\|x_k - x_\star\|^2] \to 0\).

**Proof.** It follows from Proposition 3.5(iii) that there exists a \(\mathbb{P}\)-negligible set \(N \subset \Omega\) such that, for every \(k \in \mathbb{N}\), \(C = \bigcap_{\omega \in \Omega \setminus N} C_{\xi_k}(\omega)\). Let \(\omega \in \Omega \setminus N\) and \(k \in \mathbb{N}\).

(i) Since \(x_{k+1}(\omega) = P_{C_{\xi_k}(\omega)}(x_k(\omega))\) and \(C \subset C_{\xi_k}(\omega)\), \(\text{Im}(A_{\xi_k}(\omega)^*) \subset \text{Im}(A^*)\) and, proceeding as in the proof of Theorem 4.2(i), one gets \(\nabla \phi(x_\star) - \nabla \phi(x_k(\omega)) \in \text{Im}(A^*)\).

(ii) As in the proof of Theorem 4.2(ii), since \(x_{k+1}(\omega) = P_{C_{\xi_k}(\omega)}(x_k(\omega))\) and \(C \subset C_{\xi_k}(\omega)\), it follows from Lemma 2.8 that

\[P_C(x_k(\omega)) = P_C(x_{k+1}(\omega)) \quad \text{and} \quad D_C(x_{k+1}(\omega)) + D_C(x_k(\omega)) = D_C(x_k(\omega)).\]

The statement follows.

(iii) By Theorem 3.7 we have \(x_k \to \hat{x} \mathbb{P}\text{-a.s. for some } \hat{x} \in C \cap \text{int}(\text{dom } \phi)\). Since \(P_C\) is continuous on \(\text{int}(\text{dom } \phi)\), we have that \(P_C(x_k) \to P_C(\hat{x}) = \hat{x} \mathbb{P}\text{-a.s.}\). However, item (ii) yields that \(P_C(x_k) = x_\star \mathbb{P}\text{-a.s.}\), for every \(k \in \mathbb{N}\). Therefore, \(\hat{x} = x_\star \mathbb{P}\text{-a.s.}\). Finally, convergence in mean square follows by Fact 2.4. \(\square\)

As before, we give two technical lemmas followed by the theorem on linear convergence.

**Lemma 4.7.** Under the assumptions of Problem 1.2, suppose that H0, H1, and H2 hold. Let, for every \(x \in \text{int}(\text{dom } \phi) \cap C\), \(Q(x)\) be the orthogonal projection onto \(\text{Im}(\nabla^2 \phi^*(\nabla \phi(x))^{1/2} A^*)\), \(V(x) = \text{Im}(\nabla^2 \phi^*(\nabla \phi(x))^{1/2} A^*)\), let \(\mathbf{Q}(x) := \mathbb{E}[Q_\xi(x)]\), and let

\[
\gamma_{C,\mu}(x) = \inf_{v \in V(x) \setminus \{0\}} \frac{\langle \mathbf{Q}(x)v, v \rangle}{\|v\|^2}.
\]

Then the following hold.

(i) For every \(x \in \text{int}(\text{dom } \phi) \cap C\), \(V(x) = \text{Ker}(\mathbf{Q}(x)^\perp)\), hence, \(\gamma_{C,\mu}(x) \in ]0,1]\) is the smallest nonzero eigenvalue of \(\mathbf{Q}(x)\).

(ii) For every \(\varepsilon \in ]0,1[\) there exists \(\delta > 0\) such that for every \(x \in \text{int}(\text{dom } \phi)\)

\[D_C(x) < \delta \implies \mathbb{E}[D_C(P_C(x))] \leq \frac{1 + \varepsilon}{1 - \varepsilon} [1 - \gamma_{C,\mu}(P_C(x))] D_C(x).
\]

**Proof.** (i): Let \(x \in \text{int}(\text{dom } \phi) \cap C\) and set \(y = \nabla \phi(x)\) and \(H = [\nabla^2 \phi^*(y)]^{1/2}\). We prove that \(\text{Ker}(\mathbf{Q}(x)) = \text{Im}(HA^*)^\perp\). Since \(\langle \mathbf{Q}(x)v, v \rangle = 0 \iff \langle Q_\xi(x)v, v \rangle = 0 \mathbb{P}\text{-a.s.}\), we have

\[v \in \text{Ker}(\mathbf{Q}(x)) \iff v \in \text{Ker}(Q_\xi(x)) \mathbb{P}\text{-a.s.}\]

(4.29)
Moreover, \( \text{Ker}(Q(x)) = \text{Im}(Q(x))^\perp = \text{Im}(HA^*)^\perp = \text{Ker}(A,H) \). Therefore, \( v \in \text{Ker}(\overline{Q}(x)) \iff Hv \in \text{Ker}(A_\xi) \) \( \mathbb{P}\)-a.s. Now, we observe that, since \( x \in C \), we have \( x \in C_\xi \) \( \mathbb{P}\)-a.s., and, hence \( C = \text{Ker}(A) + x \) and \( C_\xi = \text{Ker}(A_\xi) + x \) \( \mathbb{P}\)-a.s. Thus, recalling the definition of \( C \) in Problem 1.2,

\[
(\forall u \in X) \ u \in \text{Ker}(A) \iff u + x \in C \iff u + x \in C_\xi \mathbb{P}\text{-a.s.} \iff u \in \text{Ker}(A_\xi) \mathbb{P}\text{-a.s.}
\]

Therefore, \( v \in \text{Ker}(\overline{Q}(x)) \iff Hv \in \text{Ker}(A_\xi) \mathbb{P}\text{-a.s.} \iff Hv \in \text{Ker}(A) \iff v \in \text{Ker}(AH) = \text{Im}(HA^*)^\perp \).

The first part of the statement follows. Now, in view of what we have just proved, we note that

\[
\gamma_{C,\mu}(x) = \min_{\|v\| = 1} \overline{Q}(x)v, v.
\]

Since, for every \( i \in I \), \( Q_i(x) \) is self-adjoint and positive, it follows from linearity of expectation that \( \overline{Q}(x) \) is self-adjoint and positive, too. Hence, in virtue of Fact 2.2, \( \gamma_{C,\mu}(x) > 0 \) is smallest positive eigenvalue of \( \overline{Q}(x) \).

(ii): It follows from Proposition 3.5(iii) that there exists a \( \mathbb{P}\)-negligible set \( N \subset \Omega \) such that \( C = \bigcap_{\omega \in N \setminus \{\xi(\omega)\}} C_\xi(\omega) \). Note that, if \( x \in \text{int}(\text{dom} \phi) \cap C \), then \( 0 = D_C(x) = D_C(\text{PC}_\xi(\omega)(x)) \), for every \( \omega \in \Omega \setminus N \), hence (4.28) holds trivially. Therefore, we let \( x \in \text{int}(\text{dom} \phi) \setminus C \) and let \( x_\ast = \text{PC}(x) \), \( y = \nabla \phi(x) \), \( y_\ast = \nabla \phi(x_\ast) \). Now, let \( \omega \in \Omega \setminus N \) and set \( x_\xi(\omega) = \text{PC}_\xi(\omega)(x) \) and \( y_\xi(\omega) = \nabla \phi(x_\xi(\omega)) \). Then, following the same reasoning as in Lemma 4.3, we obtain (similarly to (4.18)) that for every \( \varepsilon \in [0,1] \) there exists \( \delta > 0 \) such that if \( D_C(x) < \delta \), then

\[
\frac{D_C(\text{PC}_\xi(\omega)(x))}{D_C(x)} \leq 1 + \varepsilon \, \frac{1 + \varepsilon}{1 - \varepsilon} \left( 1 - \frac{\|Q_\xi(x_\ast)u_\tau\|^2}{\|u_\tau\|^2} \right) + \frac{1 + \varepsilon}{2} \sqrt{\tau},
\]

where \( u_\tau = [\nabla^2 \phi^\ast(y_\ast)]^{1/2}(y_\ast - y^\tau \bar{w}) \in V(x_\ast) \setminus \{0\} \) and \( \tau > 0 \) is small enough and independent on \( \xi(\omega) \). The above inequality implies that

\[
\frac{D_C(\text{PC}_\xi(x))}{D_C(x)} \leq 1 + \varepsilon \, \frac{1 + \varepsilon}{1 - \varepsilon} \left( 1 - \frac{\|Q_\xi(x_\ast)u_\tau\|^2}{\|u_\tau\|^2} \right) + \frac{1 + \varepsilon}{2} \sqrt{\tau}, \quad \mathbb{P}\text{-a.s.}
\]

Hence, taking the expectation and recalling definition (4.27), we have

\[
\mathbb{E}\left[ \frac{D_C(\text{PC}_\xi(x))}{D_C(x)} \right] \leq 1 + \varepsilon \, \frac{1 + \varepsilon}{1 - \varepsilon} \left( 1 - \frac{\|Q_\xi(x_\ast)u_\tau\|^2}{\|u_\tau\|^2} \right) + \frac{1 + \varepsilon}{2} \sqrt{\tau} \leq 1 + \frac{\varepsilon}{1 - \varepsilon} \left[ 1 - \gamma_C(x_\ast) \right] + \frac{1 + \varepsilon}{2} \sqrt{\tau}.
\]

Finally, letting \( \tau \to 0 \) in the above inequality the statement follows. \( \square \)

**Lemma 4.8.** Under the same assumptions of Lemma 4.7, define the function \( \sigma_{C,\mu} : \text{int}(\text{dom} \phi) \to [0,1] \) such that

\[
\sigma_{C,\mu}(x) = \begin{cases} 
\sup_{z \in K(x) \setminus C} \left[ \frac{\mathbb{E}[D_C(\text{PC}_\xi(z))]}{D_C(z)} \right] & \text{if } x \notin C, \\
1 - \gamma_{C,\mu}(x) & \text{if } x \in C,
\end{cases}
\]

where, \( K(x) \) is defined as in Lemma 4.4. Then, the following hold.

(i) Suppose that \( C = \bigcap_{i \in I} C_i \). Then, for every \( x \in \text{int}(\text{dom} \phi) \), \( \sigma_C(x) \leq \sigma_{C,\mu}(x) \).
(ii) For every \( \varepsilon \in [0, 1[ \) there exists \( \delta > 0 \) such that
\[
(\forall x \in \text{int}(\text{dom } \phi)) D_C(x) \leq \delta \Rightarrow \sigma_{C, \mu}(x) \leq \frac{1 + \varepsilon}{1 - \varepsilon} \left(1 - \gamma_{C, \mu}(P_C(x))\right).
\]

(iii) \( (\forall x \in \text{int}(\text{dom } \phi)) \sigma_{C, \mu}(x) < 1 \) and \( \mathbb{E}[D_C(P_{C_\varepsilon}(x))] \leq \sigma_{C, \mu}(x) D_C(x) \).

(iv) \( (\forall x, y \in \text{int}(\text{dom } \phi) \setminus C) y \in K(x) \Rightarrow \sigma_{C, \mu}(y) \leq \sigma_{C, \mu}(x) \).

**Proof.** (i): Comparing (4.31) with (4.19), the statement is clear if \( x \not\in C \). On the other hand, for \( x \in C \), recalling (4.4) and (4.27), the statement follows from the fact that \( \langle \mathbb{E}[Q_\varepsilon(x)]v, v \rangle = \mathbb{E}[(Q_\varepsilon(x)v, v)] \leq \sup_{i \in I}(Q_i(x)v, v) = \sup_{i \in I}\|Q_i(x)v\|^2 \).

(ii): Let \( \varepsilon \) and \( \delta \) as in Lemma 4.7(ii) and \( x \in \text{int}(\text{dom } \phi) \) such that \( D_C(x) \leq \delta \). If \( x \in C \), then, by definition, \( \sigma_{C, \mu}(x) = 1 - \gamma_{C, \mu}(x) \) and the statement holds. Suppose that \( x \not\in C \). Then, for every \( z \in K(x) \setminus C \), we have \( D_C(z) \leq D_C(x) \leq \delta \) and hence, by (4.28), \( \mathbb{E}[D_C(P_{C_\varepsilon}(z))] / D_C(z) \leq (1 + \varepsilon)(1 - \varepsilon)^{-1}(1 - \gamma_{C, \mu}(P_C(x))) \). The statement follows from the definition of \( \sigma_{C, \mu} \) in (4.31).

(iii): We proceed as in the proof of Lemma 4.4(i). Let \( x \in \text{int}(\text{dom } \phi) \) and set \( x_* = P_C(x) \). According to Lemma 4.7, if \( x \in C \), then \( \sigma_{C, \mu}(x) < 1 \). So, assume \( x \not\in C \). Then, for any \( z \in K(x) \setminus C \), \( D_C(z) > 0 \) and, according to Proposition 3.5, \( \overline{D}_C(z) > 0 \). Thus, there exist \( 0 < \gamma_z \leq 1 \) such that \( D_C(z) - \gamma_z D_C(z) > 0 \), but, since \( D_C \) is continuous and \( \overline{D}_C \) is lower semicontinuous, there exists an open set \( U_z \subset \text{int}(\text{dom } \phi) \) such that \( z \in U_z \) and for all \( \tilde{z} \in U_z \), \( \overline{D}_C(\tilde{z}) \geq \gamma_z D_C(\tilde{z}) \). Using again Lemma 2.8, we have that \( \mathbb{E}[D_C(P_{C_\varepsilon}(\tilde{z}))] \leq (1 - \gamma_z) D_C(\tilde{z}) \). The rest of the proof follows the same lines of argument as in Lemma 4.4, just by replacing \( \inf_{i \in I} D_C(P_{C_\varepsilon}(.)) \), \( \gamma_C \), \( \sigma_C \) and Lemma 4.3 by \( \mathbb{E}[D_C(P_{C_\varepsilon}(.))], \gamma_{C, \mu}, \sigma_{C, \mu} \), and Lemma 4.7, respectively.

(iv): The proof is identical to that of Lemma 4.4(ii), but now uses (4.31). \( \square \)

**Theorem 4.9** (Random set control scheme). With reference to Problem 1.2, suppose that \( \mathbf{H}_0 \), \( \mathbf{H}_1 \), and \( \mathbf{H}_2 \) hold. Let \( (x_k)_{k \in \mathbb{N}} \) be generated by Algorithm 1.3 using the random set control scheme \( \mathbf{C}_2 \) and let \( x_* = P_C(x_0) \). Then, the following hold.

(i) \( (\forall k \in \mathbb{N}) \mathbb{E}[D_C(x_{k+1}) | X_k] \leq \sigma_{C, \mu}(x_k) D_C(x_k) \mathbb{P}\text{-a.s.} \)

(ii) \((\forall n \in \mathbb{N}) (\forall k \in \mathbb{N}) \mathbb{E}[D_C(x_{k+1+n})] \leq (\text{ess sup } \sigma_{C, \mu}(x_n)) \mathbb{E}[D_C(x_{k+n})]. \)

(iii) For every \( \varepsilon \in [0, 1[ \), there exists \( \delta > 0 \), such that if \( \|x_n - x_*\| \leq \delta \) holds \( \mathbb{P}\text{-a.s.} \) for some \( n \in \mathbb{N} \), then for every \( k \geq n \)
\[
\mathbb{E}[D_C(x_{k+1})] \leq \frac{1 + \varepsilon}{1 - \varepsilon} \sigma_{C, \mu}(x_*) \mathbb{E}[D_C(x_k)].
\]

(iv) For every \( \varepsilon, \alpha \in [0, 1[ \), there exists \( \mathcal{A} \in \mathfrak{F} \) and \( n \in \mathbb{N} \), such that \( \mathbb{P}[\Omega \setminus \mathcal{A}] \leq \alpha \) and, for all \( k \geq n \), we have
\[
\mathbb{E}[D_C(x_{k+1}) | \mathcal{A}] \leq \frac{1 + \varepsilon}{1 - \varepsilon} \sigma_{C, \mu}(x_*) \mathbb{E}[D_C(x_k) | \mathcal{A}].
\]

**Proof.** (i): First, observe that from Theorem 4.6(ii) for every \( k \in \mathbb{N} \), \( P_C(x_k) = x_* \) and, therefore, using Lemma 4.8(iii) and Fact 2.3, the inequality follows.

(ii): Again Theorem 4.6(ii) yields that \( (D_C(x_k))_{k \in \mathbb{N}} \) is decreasing \( \mathbb{P}\text{-a.s.} \) and that \( P_C(x_k) = x_* \) \( \mathbb{P}\text{-a.s.} \), for every \( k \in \mathbb{N} \). So, also in virtue of (i), there exists a \( \mathbb{P}\)-negligible set \( N \) such that for
every $\omega \in \Omega \setminus N$, \((D_C(x_k(\omega)))_{k \in \mathbb{N}}\) is decreasing, \((P_C(x_k(\omega)))_{k \in \mathbb{N}} \equiv x_*\), and $\mathbb{E}[D_C(x_{k+1}) | X_k](\omega) \leq \sigma_{C,\mu}(x_k(\omega))D_C(x_k(\omega))$. Let $n, k \in \mathbb{N}$ and $\omega \in \Omega \setminus N$. We prove that
\[
\eta_{k+1+n}(\omega) := \mathbb{E}[D_C(x_{k+1+n}) | X_{k+n}](\omega) \leq \sigma_{C,\mu}(x_n(\omega))D_C(x_{k+n}(\omega)). \tag{4.34}
\]
Then, since the above inequality holds $\mathbb{P}$-a.s., the statement will follow by just majorizing $\sigma_{C,\mu}(x_n)$ with its essential supremum and then taking expectation. Now, if $\eta_{k+1+n}(\omega) = 0$, then (4.34) holds. Otherwise, since $\sigma_{C,\mu}(x_{k+n}(\omega)) \leq 1$, we have $0 < \eta_{k+1+n}(\omega) \leq \sigma_{C,\mu}(x_{k+n}(\omega))D_C(x_{k+n}(\omega)) \leq D_C(x_{k+n}(\omega))$. Hence $x_{k+n}(\omega), x_n(\omega) \notin C$ and, using the notation of Lemma 4.8, $x_{k+n}(\omega) \in K(x_n(\omega))$. Therefore, Lemma 4.8(iv) yields $\sigma_{C,\mu}(x_{k+n}(\omega)) \leq \sigma_{C,\mu}(x_n(\omega))$ and hence $\eta_{k+1+n}(\omega) \leq \sigma_{C,\mu}(x_n(\omega))D_C(x_{k+n}(\omega))$, so that (4.34) holds.

(iii): Let $\varepsilon \in [0, 1]$ and $\delta > 0$ be from Lemma 4.8(ii). Since $D_\phi(x_*, \cdot)$ is continuous, there exists $\delta_1 > 0$ such that if $\|x_n - x_*\| \leq \delta_1$ $\mathbb{P}$-a.s. for some $n \in \mathbb{N}$, then $D_C(x_n) = D_\phi(x_*, x_n) \leq \delta$ $\mathbb{P}$-a.s. Then, for every integer $k \geq n$, we have $D_C(x_k) \leq D_C(x_n) \leq \delta$ $\mathbb{P}$-a.s. and hence, by Lemma 4.8(ii), $\sigma_{C,\mu}(x_k) \leq (1 + \varepsilon)(1 - \varepsilon)^{-1}\sigma_{C,\mu}(x_n)D_C(x_k)$. The statement follows by using (i) and then taking the expectation.

(iv): Let $\varepsilon \in [0, 1]$ and $\delta > 0$ as in Lemma 4.7(ii) and set $A_k := \{D_C(x_k) \leq \delta\}$. Then, denoting by $\chi_{A_k}$ the characteristic function of the set $A_k$ and using, as before, Lemma 4.8 and (i), we have
\[
\mathbb{E}[D_C(x_{k+1}) | X_k]\chi_{A_k} \leq \frac{1 + \varepsilon}{1 - \varepsilon}\sigma_{C,\mu}(x_n)D_C(x_k)\chi_{A_k}. \tag{4.35}
\]
Moreover, Markov inequality yields $\mathbb{P}[\Omega \setminus A_k] \leq \mathbb{E}[D_C(x_k)]/\delta$, so, since, in virtue of (ii) with $n = 0$, $\mathbb{E}[D_C(x_k)] \to 0$, for every $\alpha \in [0, 1]$ there exists $n \in \mathbb{N}$ such that $\mathbb{P}(\Omega \setminus A_n) \leq \alpha$. Let $\mathcal{A} := A_n$. Since $(D_C(x_k))_{k \in \mathbb{N}}$ is decreasing $\mathbb{P}$-a.s., then for all $k \geq n$, $\mathcal{A} \subset A_k$, except for a negligible set. Hence, from (4.35) we obtain
\[
\mathbb{E}[D_C(x_{k+1}) | X_k]\chi_{A} \leq \frac{1 + \varepsilon}{1 - \varepsilon}\sigma_{C,\mu}(x_n)D_C(x_k)\chi_{A}. \tag{4.33}
\]
But, using $\mathbb{E}[D_C(x_{k+1})\chi_{A}] = \mathbb{E}[\mathbb{E}[D_C(x_{k+1})\chi_{A}] | X_k] = \mathbb{E}[\mathbb{E}[D_C(x_{k+1}) | X_k]\chi_{A}]$ and $\mathbb{E}[\chi_{A}] = \mathbb{E}[\chi_{A}]P(\mathcal{A})$ we obtain (4.33).

Remark 4.10. Taking $n = 0$ in Theorem 4.9(ii) we get the global $Q$-linear rate in expectation: $\mathbb{E}[D_C(x_{k+1})] \leq \sigma_{C,\mu}(x_0)\mathbb{E}[D_C(x_k)]$, where $\sigma_{C,\mu}(x_0) < 1$. However, when $n > 0$, even though $\sigma_{C,\mu}(x_n) < 1$ $\mathbb{P}$-a.s., we are not ensured that $\text{ess sup} \sigma_{C,\mu}(x_n) < 1$.

4.3 Adaptive random set control scheme

In this section we address again Problem 1.2, but as for Algorithm 1.3, we do not assume anymore that the set control indexes $\xi_k$’s are independent copies of $\xi$, but, instead, that they are set adaptively during the algorithm. More precisely, we consider the case where the probability distribution of the next step is adapted to the information that we may obtain in the present. We formalize this set control scheme in the following assumption, where $\mu$ is again the distribution of $\xi$ and $\overline{D}_C$ is still defined as in (3.1).

\textbf{C3} $\nu: \mathcal{I} \times \text{int}(\text{dom} \phi) \to \mathbb{R}_+$ is a probability kernel such that\(^2\)
\[
(\forall x \in \text{int}(\text{dom} \phi)) \quad \nu(\cdot, x) = \begin{cases} \frac{D_{C_{\nu}}(x)}{\overline{D}_C(x)} & \text{if } x \notin C \\ \mu & \text{if } x \in C. \end{cases} \tag{4.36}
\]
\(^2\)This means that $\nu(A, x) = \int_A [D_{C_{\nu}}(x) / \overline{D}_C(x)] \mu(\cdot) d\mu$ if $x \notin C$ and $\nu(A, x) = \mu(A)$ if $x \in C$. 22
Moreover, \((\xi_k)_{k \in \mathbb{N}}\) is a sequence of \(I\)-valued random variables and \((x_k)_{k \in \mathbb{N}}\) is a sequence of \(X\)-valued random variables defined recursively according to Algorithm 1.3. For all \(k \in \mathbb{N}\), \(x_k\) is the sigma algebra generated by \(x_0, \ldots, x_k\), and \(P_{\xi_k|X_k} : \mathcal{I} \times \Omega \to \mathbb{R}_+\) denotes the conditional distribution of \(\xi_k\), given \(X_k\). Finally,

\[
(\forall \omega \in \Omega) \quad P_{\xi_k|X_k}(:, \omega) = \nu(:, x_k(\omega)).
\]

(4.37)

**Remark 4.11.** The above set control scheme subsume the knowledge of the Bregman distances to all \(C_i\)'s from the current realization \(x_k(\omega)\) of the \(k\)-th iterate and the possibility to compute the expectation \(\mathbb{E}[D_{C_i}(x_k(\omega))]\). This is feasible and can be efficiently implemented in the orthogonal sketch & project method \[27\]. This is also the case for KL-projections in entropic regularized optimal transport, however, we postpone a proof of this fact in a subsequent work.

**Remark 4.12.** Let \(k \in \mathbb{N}\) and \(x_0, \ldots, x_{k-1}, x \in \text{int}(\text{dom } \phi)\). One can define the conditional probability of \(\xi_k\) given \(x_0 = x_0, \ldots, x_{k-1} = x_{k-1}, x_k = x\)

\[
P_{\xi_k|x_0=x_0,\ldots,x_{k-1}=x_{k-1},x_k=x} : \mathcal{I} \to \mathbb{R}_+
\]

(4.38)

Then (4.37) is equivalent to ask for

\[
(\forall x \in \text{int}(\text{dom } \phi)) \quad P_{\xi_k|x_0=x_0,\ldots,x_{k-1}=x_{k-1},x_k=x} = \nu(:, x),
\]

(4.39)

which also implies \(P_{\xi_k|x_0=x_0,\ldots,x_{k-1}=x_{k-1},x_k=x} = P_{\xi_k|x_k=x}\) and does not depend on \(k\).

**Theorem 4.13** (Adaptive random set control). Under the assumptions of Problem 1.2, suppose in addition that \(H_0, H_1, \text{ and } H_2\) hold. Let \(V : \text{int}(\text{dom } \phi) \to \mathbb{R}_+\) and \(\beta : \text{int}(\text{dom } \phi) \to \mathbb{R}_+\) be such that

\[
V(x) := \begin{cases} \\
\mathbb{V}
\left[D_{C_i}(x) \right]/D_{C_i}(x) & \text{if } x \notin C \\
0 & \text{if } x \in C,
\end{cases}
\]

(4.40)

Let \((x_k)_{k \in \mathbb{N}}\) be generated by Algorithm 1.3 using the adaptive random set control scheme \(C_3\) and let \(x_* = P_C(x_0)\). Set \(\beta_{\infty} := \lim\inf_{k \to \infty} (\text{ess inf } \beta(x_k)) \geq 1\) and, for every \(k \in \mathbb{N}\), \(\tilde{\sigma}_{C,\mu}(x_k) = \beta(x_k)\sigma_{C,\mu}(x_k) + 1 - \beta(x_k)\). Then the following holds.

(i) \((\forall k \in \mathbb{N}) \ 0 \leq \tilde{\sigma}_{C,\mu}(x_k) \leq \sigma_{C,\mu}(x_k) \ \mathbb{P}\text{-a.s.}\)

(ii) \((\forall k \in \mathbb{N}) \ \mathbb{E}[D_{C}(x_{k+1})] | x_k] \leq \tilde{\sigma}_{C,\mu}(x_k)D_{C}(x_k) \ \mathbb{P}\text{-a.s.}\)

(iii) \((\forall n \in \mathbb{N}) (\forall k \in \mathbb{N}) \ \mathbb{E}[D_{C}(x_{k+1+n})] \leq (\text{ess sup } \sigma_{C,\mu}(x_n))\mathbb{E}[D_{C}(x_{n+k})]\)

(iv) For every \(\varepsilon > 0\) such that \((1 + \varepsilon)(1 - \varepsilon)^{-1}\sigma_{C,\mu}(x_*) < 1\) and every \(\alpha \in ]0, 1[\) there exists \(A \in \mathcal{A}\) with \(\mathbb{P}(\Omega \setminus A) \leq \alpha\) and \(n \in \mathbb{N}\), such that, for all \(k \geq n\),

\[
\mathbb{E}[D_{C}(x_{k+1})] | A] \leq \left(\beta_{\infty}\sigma_{C,\mu}(x_*) + 1 - \frac{1 + \varepsilon}{1 - \varepsilon}\beta_{\infty}\right) \mathbb{E}[D_{C}(x_k) | A]
\]

Proof. (i)-(ii): The second inequality in (i) is immediate, taking into account that, since \(\sigma_{C,\mu}(x_k) \leq 1\), \(\tilde{\sigma}_{C,\mu}(x_k)\) is decreasing in \(\beta(x_k) \geq 1\). In the following we prove the first inequality of (i) and the
inequality in (ii) at the same time. Using Lemma 2.8, similarly to the proof of Theorem 4.6(ii), one obtains

$$E[D_{C}(x_{k+1}) \mid \mathcal{X}_k] = D_{C}(x_k) - E[D_{C \xi_k}(x_k) \mid \mathcal{X}_k]. \tag{4.41}$$

Now, using (4.37) and (4.36), we have that, for every $\omega \in \Omega$,

$$E[D_{C \xi_k}(x_k) \mid \mathcal{X}_k](\omega) = \int_I D_{C}(x_k(\omega))P_{\xi_k|x_k}(di, \omega) = \int_I D_{C}(x_k(\omega))\nu(di, x_k(\omega)).$$

If $x_k(\omega) \in C$, then, clearly, $E[D_{C \xi_k}(x_k) \mid \mathcal{X}_k](\omega) = \overline{D}_{C}(x_k(\omega)) = 0$ and (ii) and the first of (i) hold when evaluated at $\omega$. Otherwise, we have that

$$E[D_{C \xi_k}(x_k) \mid \mathcal{X}_k](\omega) = \int_I \left[ \frac{D_{C}(x_k(\omega))}{\overline{D}_{C}(x_k(\omega))} \right] \mu(di) = \overline{D}_{C}(x_k(\omega))(V(x_k(\omega)) + 1), \tag{4.42}$$

where we used the fact that

$$\nabla \left[ \frac{D_{C}(x)}{\overline{D}_{C}(x)} \right] = E \left[ \left( \frac{D_{C}(x)}{\overline{D}_{C}(x)} \right)^2 \right] - \left( E \left[ \frac{D_{C}(x)}{\overline{D}_{C}(x)} \right] \right)^2 = E \left[ \left( \frac{D_{C}(x)}{\overline{D}_{C}(x)} \right)^2 \right] - 1. \tag{4.43}$$

Set, for the sake of brevity, $\beta_k = \beta(x_k(\omega))$. Then, by the definition of $\beta$, (4.42) yields $E[D_{C \xi_k}(x_k) \mid \mathcal{X}_k](\omega) = \beta_k \overline{D}_{C}(x_k(\omega))$. Thus, by (4.41), we have $E[D_{C}(x_{k+1}) \mid \mathcal{X}_k](\omega) = D_{C}(x_k(\omega)) - \beta_k \overline{D}_{C}(x_k(\omega)) = (1 - \beta_k)D_{C}(x_k(\omega)) + \beta_k(D_{C}(x_k(\omega)) - \overline{D}_{C}(x_k(\omega))) = (1 - \beta_k)D_{C}(x_k(\omega)) + \beta_kE[D_{C}(P_{\xi_k|x_k}) \mid \mathcal{X}_k](\omega) \leq (1 - \beta_k) + \beta_k(\sigma_{C,\mu}(x_k(\omega))) \overline{D}_{C}(x_k(\omega))$, where we used the definition of $\sigma_{C,\mu}$ in (4.31) and that $x_k(\omega) \in K(x_k(\omega)) \setminus C$. So, the inequality in (ii) and the first of (i) hold when evaluated at $\omega$.

(iii): It is sufficient to prove that $(D_{C}(x_k))_{k\in\mathbb{N}}$ is $\mathbb{P}$-a.s. decreasing and that $P_{C}(x_k) = x_\ast \mathbb{P}$-a.s., for every $k \in \mathbb{N}$. Indeed, once we have proved this, we can follow the same line of argument as in the proof of items (ii)-(iv) of Theorem 4.9. To that purpose, by Proposition 3.5(iii) we have that there exists $\mu$-negligible set $J$ such that $C = \bigcap_{l \in \mathbb{N} \setminus J} C_{\xi_l}$. Let $k \in \mathbb{N}$, set $N_k = \xi_k^{-1}(J)$, and let $\omega \in \Omega$. Then $\mathbb{P}(\xi_k^{-1}(J) \mid \mathcal{X}_k) = P_{\xi_k|x_k}(J, x_k(\omega))$. Hence, if $x_k(\omega) \in C$, then $\mathbb{P}(\xi_k^{-1}(J) \mid \mathcal{X}_k) = \mu(J) = 0$, otherwise

$$\mathbb{P}(\{x \in C_{\xi_k} \mid \mathcal{X}_k\}(\omega) = \int_J \frac{D_{C}(x_k(\omega))}{\overline{D}_{C}(x_k(\omega))} \mu(di) = 0.$$

Therefore in any case $\mathbb{P}(N_k \mid \mathcal{X}_k) = 0$ and hence $\mathbb{P}(N_k) = E[\mathbb{P}(N_k \mid \mathcal{X}_k)] = 0$. Now, as in the proof of Proposition 3.5(iii), we prove that $C \subset \bigcap_{\omega \in \Omega \setminus N_k} C_{\xi_k}(\omega)$. Indeed, let $x \in C$. Then, for every $\omega \in \Omega \setminus N_k$, we have $\xi_k(\omega) \in I \setminus J$ and hence, since $x \in C = \bigcap_{l \in \mathbb{N} \setminus J} C_l$, we have $x \in C_{\xi_k}(\omega)$. Then, for every $\omega \in \Omega \setminus N_k$, $x_{k+1}(\omega) = P_{\xi_{k+1}|x_k}(x_k(\omega))$ and $C \subset C_{\xi_k}(\omega)$. Hence, using Lemma 2.8, we have $P_{C}(x_{k+1}(\omega)) = P_{C}(x_k(\omega))$ and $D_{C}(x_{k+1}(\omega)) = D_{C}(x_k(\omega)) - D_{\phi}(x_{k+1}(\omega), x_k(\omega)) \leq D_{C}(x_k(\omega))$. This proves that $P_{C}(x_{k+1}) = P_{C}(x_k) \mathbb{P}$-a.s. and $D_{C}(x_{k+1}) \leq D_{C}(x_k) \mathbb{P}$-a.s.

(iv): Let $\varepsilon > 0$ with $(1 + \varepsilon)(1 - \varepsilon)^{-1} \sigma_{C,\mu}(x_\ast) < 1$ and $\delta > 0$ be as in Lemma 4.8(ii). The proof follows that of Theorem 4.9(iv). Set $A_k := \{D_{C}(x_k) \leq \delta\}$ and let $n \in \mathbb{N}$ be sufficiently large so that $A := A_n$ is such that $\mathbb{P}(\Omega \setminus A) \leq \alpha$. Let $k \in \mathbb{N}$, with $k \geq n$. Then $A \subset A_k$ and hence, denoting by
Table 1: Legendre functions \( \phi(x) = \sum_{i=1}^{n} \varphi(x_i) \) with \( \varphi^* \) twice differentiable and \( \text{dom } \varphi^* \) open.

| Legendre function | \( \varphi(t) \) | \( \text{dom } \varphi \) | \( \text{dom } \varphi^* \) | Bregman distance/App. |
|-------------------|----------------|----------------|----------------|-----------------|
| Burg entropy      | \(- \log t\) | \(\mathbb{R}_{++}\) | \(\mathbb{R}_{--}\) | Itakura-Saito divergence [22] |
| Boltzmann-Shannon entropy | \(t \log t - t\) | \(\mathbb{R}_{+}\) | \(\mathbb{R}\) | Kullback-Leibler divergence [11, 39] |
| Fermi-Dirac entropy | \(t \log t + (1-t) \log(1-t)\) | \([0, 1]\) | \(\mathbb{R}\) | Logistic loss [30, 22] |
| Hellinger entropy | \(-\sqrt{1-t^2}\) | \([-1, 1]\) | \(\mathbb{R}\) | Hellinger distance [22] |
| Positive power 0 < \(\beta < 1\) | \(t^\beta - \beta t + \beta - 1 \over \beta(\beta - 1)\) | \(\mathbb{R}_{+}\) | \(-\infty, \frac{1}{1-\beta}\) | \(\beta\)-divergence [22, 26] |
| Tsallis entropy 0 < \(q < 1\) | \(\frac{1}{q-1}(t^q - t)\) | \(\mathbb{R}_{+}\) | \(-\infty, \frac{1}{1-q}\) | Tsallis relative \(q\)-entropy [36] |
| \(p\)-norm 1 < \(p \leq 2\) | \(\frac{1}{p}|t|^p\) | \(\mathbb{R}\) | \(\mathbb{R}\) | Compressed sensing [4] |

Now let \(n \in \mathbb{N}\) be sufficiently large so that \(\inf_{h \geq n}(\text{ess inf } \beta(x_h)) > (1 - \varepsilon)(1 + \varepsilon)^{-1}\beta_{\infty}\). Then the statement follows after taking expectation.

Remark 4.14. Theorem 4.13 shows the advantage, in terms of convergence rate, of the adaptive random set control strategy \(C_3\) against \(C_2\). In particular, in the inequality in Theorem 4.13(iv), if \(\beta_{\infty} > 1\) one can choose \(\varepsilon > 0\) sufficiently small and get a better rate than that of Theorem 4.9(iv).

5 Applications

In this section we present two applications, where our analysis provides novel results and/or possibilities. We provide only general insights, leaving a deeper treatment of the subjects for future work. In Table 1 we provide several examples of Legendre functions satisfying the assumptions of our convergence theorems that can be used in such applications.

5.1 Sketch & Bregman project methods

In recent years, a new class of iterative solvers for linear systems, called sketch & project [28], has emerged, taking its origin in the work by Strohmer and Vershynin [44], which, in turn, generalized
the Kaczmarz method [31]. Below we describe the method. Given the problem

\[ x_* = \arg \min \frac{1}{2} \langle Bx, x \rangle \quad \text{subject to} \quad Ax = b, \]

where \( A \in \mathbb{R}^{m \times n}, \ b \in \mathbb{R}^m, \) and \( B \in \mathbb{R}^{n \times n} \) is a symmetric positive definite matrix, one introduces a family of sketch matrices \((S_i)_{i \in I}\) such that \( S_i \in \mathbb{R}^{m \times n_i}, \ n_i \geq 1, \ i \in I, \) and then replaces the task of solving \( Ax = b \) by that of projecting \( B \)-orthogonally onto the solution set of the sketched systems \( S_i^* Ax = S_i^* b, \ i \in I, \) according to different types of sketching control schemes (cyclic, greedy, random, adaptive) [27]. This method indeed leads to the computation of the minimal \( B \)-norm solution of \( Ax = b \) under the assumption that \( x_0 = 0 \) and that the sketches are chosen in a way that the whole solution space is covered by the sketching process. Denoting the solution sets by

\[ C = \{ x \in \mathbb{R}^n | Ax = b \} \quad \text{and} \quad C_i = \{ x \in \mathbb{R}^n | S_i^* Ax = S_i^* b \} (i \in I), \]

the last assumption means that \( C = \bigcap_{i \in I} C_i, \) for a deterministic sketching strategy, or \( C = \{ x \in \mathbb{R}^n | x \in C_\mathbb{P}\ \text{a.s.} \}, \) when the sketches are chosen randomly. This last condition was referred to as the exactness assumption, and can be equivalently expressed as \( \text{Im}(A) \cap \text{Ker}(\mathbb{E}[S_\xi(S_\xi^* AA^* S_\xi)\{S_\xi^*\}]) = \{ 0 \} \) [40, Theorem 3.5]. Global linear convergence rates were obtained in [27, 44, 40], which extend those related to the Kaczmarz-type methods.

A direct application of our results is that the existing sketch & project methodology can be successfully extended to Bregman projections, leading to the novel archetypal method of sketch & Bregman project. In such a way, instead of finding the minimal norm solution of the linear system as in the classical setting, we compute the solution that minimizes the Legendre function \( \phi \) over the solutions of \( Ax = b, \) provided that \( \nabla \phi(x_0) = 0. \) In Table 1 we give some typical separable Legendre functions that can be used within our framework together with references.

The most studied family of sketches is the one associated to the standard basis of \( \mathbb{R}^m, \) e.g., \( S_i = e_i, \) which produces the popular Kaczmarz (or randomized Kaczmarz) method. In our setting this leads to Bregman-Kaczmas methods (greedy, random, adaptive random). In this specific case, the expression of the rates \( \sigma_C \) and \( \sigma_{C,\mu} \) can be simplified. For instance, let \( \mu = 1/m \) be the probability vector of the uniform distribution on \( I = \{ 1, \ldots, m \}. \) Then the constants in (4.19) and (4.31) become

\[ \sigma_C(x_*) = 1 - \min_{v \in \text{Im}(A^*)} \frac{\| A^* v \|_\infty}{\| v \|_2} \quad \text{and} \quad \sigma_{C,\mu}(x_*) = 1 - \frac{1}{m} \lambda_{\mu \min}^+(A^* A), \quad (5.1) \]

respectively, where, setting \( D = \text{diag}(\| \nabla^2 \phi(x_*) \|^{-1}/2 A_1, \ldots, \| \nabla^2 \phi(x_*) \|^{-1}/2 A_m, \|_2), \) we have \( \bar{A} = D^{-1} A. \) These results, when specialized to \( \phi = (1/2)\| \cdot \|_B^2, \) recover the ones of the Kaczmarz method [31, 44].

We finish the section by addressing a comparison with [27]. When \( \phi = (1/2)\| \cdot \|_B^2 \) (so that \( D_\phi(x, y) = (1/2)\| x - y \|_B^2 \)), Fact 2.1(xi) holds for an arbitrarily large \( \delta \) and hence, the local linear rates given in Theorems 4.5(iii), 4.9(iii) and 4.13(iv), are indeed global and match the ones in [27]. Moreover we proved \( \mathbb{P}\text{-a.s.} \) convergence of the iterates, which is, up to our best knowledge, a new result even in the classical setting of stochastic sketch & project methods.

### 5.2 Multimarginal regularized optimal transport

The role of the Bregman projection method in optimal transport (OT) is well-known and deeply studied. In this section we describe the more general setting of multimarginal OT. In particular, we consider the discrete multimarginal regularized optimal transport problem as described in [11]. Let
Let, for every $i = 1, \ldots, m$, $\rho_i \in \Delta_{n_i}$, where $\Delta_\ell = \{ x \in \mathbb{R}_+^\ell \mid \| x \|_1 = 1 \}$ is the unit simplex of $\mathbb{R}^\ell$. Set $X = \mathbb{R}^{n_1 \times \cdots \times n_m}$ and define, for every $i = 1, \ldots, m$, the push-forward (linear) operator $A_i : X \to \mathbb{R}^{n_i}$

$A_i \pi = \left( \sum_{h_1=1}^{n_1} \cdots \sum_{h_i-1=1}^{n_i-1} \sum_{h_{i+1}=1}^{n_{i+1}} \cdots \sum_{h_m=1}^{n_m} \pi_{h_1,\ldots,h_{i-1},h_{i+1},\ldots,h_m} \right)_{1 \leq h_i \leq n_i}. \tag{5.2}$

The objective of multimarginal (entropic) regularized OT can be formulated as

$\pi_* = \arg \min_{\pi \in C_1 \cap \cdots \cap C_m} \text{KL}(\pi, \kappa), \quad C_i = \{ \pi \in X \mid A_i \pi = \rho_i \}, \quad i = 1, \ldots, m, \tag{5.3}$

where $\kappa = e^{-\alpha/\eta} \in X$ is the Gibbs kernel, $\alpha \in X$ the cost multidimensional matrix, and $\text{KL}(\pi, \kappa)$ is the Kullback-Leibler divergence, defined as

$\text{KL}(\pi, \kappa) = \sum_{h_1,\ldots,h_m} \pi_{h_1,\ldots,h_m} \log \left( \frac{\pi_{h_1,\ldots,h_m}}{\kappa_{h_1,\ldots,h_m}} \right) - \pi_{h_1,\ldots,h_m} + \kappa_{h_1,\ldots,h_m}, \tag{5.4}$

which is the Bregman distance associated to the Boltzmann-Shannon entropy

$\phi(\pi) = \sum_{h_1,\ldots,h_m} \pi_{h_1,\ldots,h_m} (\log \pi_{h_1,\ldots,h_m} - 1).$

See Table 1.

Since the KL projection onto the affine sets $C_i$’s can be computed explicitly and efficiently [11, Proposition 4], our results show that the Bregman projection algorithm with greedy or random set control sequence applied to problem (5.3), and initialized with $\pi_0 = \kappa$, converges Q-linearly, in the KL distance, to $\pi_*$. In this setting, when $m = 2$, the Bregman projection method with greedy set control scheme reduces to the Sinkhorn algorithm [39] (see Remark 3.1), for which both global and local linear rates of convergence are known [32, 39]. However, for $m > 2$ the above results are new. Finally, note that the linear operator

$A : X \to \mathbb{R}^{n_1 \times \cdots \times n_m}, \quad A \pi = (A_i \pi)_{1 \leq i \leq m}, \tag{5.5}$

can be identified with a (possibly huge) matrix of dimensions $(n_1 + \cdots + n_m) \times (n_1 \cdots n_m)$. So, we could even consider to perform Bregman projections onto each single row of the linear system

$A \pi = \left[ \begin{array}{c} \rho_1 \\ \vdots \\ \rho_m \end{array} \right]. \tag{5.6}$

This approach, if implemented with a greedy set control sequence, leads to the Greenkhorn algorithm which was studied in [2] only for the standard bimarginal OT problem. We stress that, to the best of our knowledge, the Q-linear rate of convergence is new for this algorithm too (even when $m = 2$).

6 Conclusion and future work

We studied the Bregman projection method with greedy and random set control sequences in deterministic and stochastic convex feasibility problems. We focused in particular on affine feasibility problems.
problems showing the Q-linear convergence of the method. In our future work on this subject we will devote special attention to applications in regularized (multimarginal) OT, which here we just touched. Among other issues, we will address the explicit computation of the local Q-linear rate in terms of the problem data, the computation of the rate for Wasserstein barycenters formulated as a multimarginal OT, and the effects of different sketching control schemes on the convergence rate.

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