ANALYZING BENARDETE’S COMMENT ON
DECIMAL NOTATION

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Abstract. Philosopher Benardete challenged both the conventional wisdom and the received mathematical treatment of zero, dot, nine recurring. An initially puzzling passage in Benardete on the intelligibility of the continuum reveals challenging insights into number systems, the foundations of modern analysis, and mathematics education. A key concept here is, in Terry Tao’s terminology, that of an ultralimit.

Keywords: real analysis; infinitesimals; decimal notation; procedures vs ontology

1. Introduction

Philosopher José Benardete in his book Infinity: An essay in metaphysics argues that some natural pre-mathematical intuitions cannot be properly expressed if one is limited to an overly restrictive number system:

The intelligibility of the continuum has been found—many times over—to require that the domain of real numbers be enlarged to include infinitesimals. This enlarged domain may be styled the domain of continuum numbers. It will now be evident that .9999... does not equal 1 but falls infinitesimally short of it. I think that .9999... should indeed be admitted as a number ... though not as a real number. [Benardete 1964, p. 279] (emphasis in the original)

To a professional mathematician, Benardete’s remarks may seem naive. The equality between 1 and 0.999... is not in the realm of speculation, but rather an established fact. This follows from the very definition of 0.999... as the limit of the sequence 0.9, 0.99, 0.999, ... where the limit of a sequence \( (u_n) \) is defined, following any calculus textbook, as the real number \( L \) such that for every \( \epsilon > 0 \) there exists
an $N > 0$ such that if $n > N$ then $|u_n - L| < \epsilon$ (and even in the hyperreal number system they are still equal, as Bryan Dawson recently noted in [Dawson 2016]).

It patently follows from the definition that $0.999\ldots$ equals precisely 1 because the value $L = 1$ satisfies the definition stated, and there is nothing more to discuss. Or is there? Earlier studies in this direction include [Ely 2010], [Katz & Katz 2010a], [Katz & Katz 2010b].

2. Vicious circle

As is patently evident from the definition given above, the procedure of taking the limit is real-valued by definition. If one is to treat Benardete’s comment charitably, one cannot presuppose the answer to his query, namely that the number in question is necessarily a real number. The equality should, if possible, be conceived independently from a conception of $0.999\ldots$ in terms of limits that presupposes that limits are, by definition, real-valued.

While the equality $0.999\ldots = 1$ (whether it results from a definition or from a mathematical demonstration) is absolutely necessary in order to ensure the coherence of the algebra with the real numbers, in some approaches to the real numbers the identification of $0.999\ldots$ with 1 is itself part of the definition of the real number line rather than a theorem. Thus, the real numbers can be defined in terms of unending strings usually referred to as their decimal expansions. In this approach to the real numbers, the strings 1.000\ldots and 0.999\ldots are postulated to be the same number, i.e., they represent the same real number; more generally, the equivalence relation on such formal strings is defined in such a way that each terminating string is equivalent to the related one with an infinite tail of 9s.

In this approach to the real numbers, it is indeed correct to assert that the equality (in reality masking an equivalence) is not a theorem but rather a definition. It can be debated whether practically speaking this approach is a good approach to the real numbers; arguably it is for some purposes, but not for others.

Historically the first scholar to recognize that it is useful to represent numbers by unending decimals was Simon Stevin already in the 16th century, before the golden age of the calculus, and even before the symbolic notation of Vieta; see [Katz & Katz 2012] as well as [Blaszczyk, Katz & Sherry 2013, Section 2]. The identification of the two strings is first found in the second half of the 18th century in the work of Lambert (1758) and Euler (1765).
3. Other meanings

Can one assign any meaning to the symbolic string “0.999...” other than defining it to be 1? That question cannot be answered without analyzing what informal meaning is assigned to 0.999..., prior to interpreting it in a formal mathematical sense.

Beginning calculus students often informally describe this as “nought, dot, nine recurring” or alternatively “zero, point, followed by infinitely many 9s.” The second description may not necessarily refer to any sophisticated number system like the real number system (e.g., equivalence classes of Cauchy sequences of rational numbers expressed in Zermelo–Fraenkel set theory), since at this level the students will typically not have been exposed to such mathematical abstractions, involving as they do equivalence classes of Cauchy sequences, Dedekind cuts, or other techniques from analysis.

It is also known that at this level, about 80 percent of the students feel that such an object necessarily falls a little bit short of 1. The question is whether such intuitions are necessarily erroneous, or whether they could lend themselves to a mathematically rigorous interpretation in the context of a suitable number system.

A possible mathematical interpretation, in the context of the sequence 0.9, 0.99, 0.999, ..., mentioned in the introduction, is as follows. Instead of talking the limit of the sequence, one takes, in the terminology of Terry Tao, an ultralimit, resulting in a hyperreal number that falls infinitesimally short of 1.

Rob Ely argues that such intuitions are not necessarily mathematically erroneous because they can find a rigorous implementation in the context of a hyperreal number system, where a number with an infinite tail of 9s can fall infinitesimally short of 1 [Ely 2010]. Namely, if \( H \) is an infinite hypernatural, then the hyperfinite sum \( \sum_{n=1}^{H} \frac{9}{10^n} = 0.999...9 \) contains \( H \) occurrences of the digit 9, and falls short of 1 by the tiny amount \( (0.1)^H \).

Infinite hypernaturals like \( H \) belong to a hyperreal line, say \( \star \mathbb{R} \), which obeys the same rules, namely the rules of real-closed ordered fields, as the usual real line \( \mathbb{R} \). Numbers like \( H \) behave in \( \star \mathbb{R} \) as the usual natural numbers behave in \( \mathbb{R} \), by the transfer principle. This is in sharp

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1See e.g., https://terrytao.wordpress.com/tag/ultralimit-analysis
2The transfer principle is a type of theorem that, depending on the context, asserts that rules, laws or procedures valid for a certain number system, still apply (i.e., are “transfered”) to an extended number system. Thus, the familiar extension \( \mathbb{Q} \hookrightarrow \mathbb{R} \) preserves the properties of an ordered field. To give a negative example, the extension \( \mathbb{R} \hookrightarrow \mathbb{R} \cup \{\pm \infty\} \) of the real numbers to the so-called extended reals does
contrast with Cantorian theories where neither cardinals nor ordinals obey the rules of an ordered field. Thus, there is no such thing as $\omega - 1$, while $\aleph_0 + 1 = \aleph_0$. On the other hand, according to Tirosh and Tsamir, students tend to attribute properties of finite sets to infinite ones:

Research in mathematics education indicates that in the transition from given systems to wider ones learners tend to attribute the properties that hold for the former to the latter. In particular, it has been found that, in the context of Cantorian Set Theory, learners tend to attribute properties of finite sets to infinite ones—using methods which are acceptable for finite sets, to infinite ones. [Tirosh & Tsamir 2013]

The hyperrational number $\sum_{n=1}^{H} \frac{9}{10^n} = 0.999\ldots9$ can be visualized (virtually) on a hyperreal numerical line in the halo of 1 by means of Keisler’s microscope, with the difference being $10^{-H}$. To return to the real numbers, we have the equality $\text{st}(0.999\ldots9) = 1$ where “st” is the operation of taking standard part (shadow).\(^3\)

Thus, there are many hyperreal numbers representable as 0 followed by infinitely many 9s, e.g., 0.999\ldots9000\ldots (where the zeros start from some infinite rank), 0.999\ldots9123123\ldots, 0.999\ldots999\ldots, etc. Among such numbers only one is the natural extension of the usual sequence 0.(9) (unending 9s) and this indeed equals 1. The matter was explored in more detail by [Lightstone 1972].

The hyperrational number $\sum_{n=1}^{H} \frac{9}{10^n} = 0.999\ldots9$ is a terminating infinite string of 9s different from the one usually envisioned in real analysis, but it respects student intuitions. Intuitions like these underlie Leibniz’s and Peirce’s conceptions of the continuum. They can be helpful in learning the calculus, as argued in [Ely 2010]; see also [Vinsonhaler 2016], [Katz & Polev 2017].

The pedagogical issue is a separate one but what could be emphasized here is that the existence of such an interpretation suggests that we indeed do assume that such a string represents a real number when we prove that it necessarily equals 1. The idea that such an assumption can be challenged is in line with Benardete’s comment.

\(^3\)To elaborate further, note that each hyperreal $r$ can be represented by means of its extended decimal expansion $r = \pm n.r_1r_2\ldots r_i\ldots r_H\ldots$ where $n \in {}^*\mathbb{N} \cup \{0\}$ and $r_i \in \{0, 1, 2, \ldots, 9\}$ for $i \in {}^*\mathbb{N}$.
4. Conclusion

It is known that the real number system provides an adequate foundation for mathematics. Yet the real number system does not explain the well-known phenomenon of student unease about the identity $1 = 0.999\ldots$ (nor the philosopher’s unease).

It is similarly known that a set-theoretic justification of a hyperreal number system involves more work than is justified by the need of explaining $0.999\ldots$ to the students. Yet the crucial distinction between set-theoretic justification (ontology), on the one hand, and the procedures involving infinitesimals, on the other, has been emphasized in \cite{Bair et al. 2016} and \cite{Bascelli et al. 2016}.

Working with infinitesimals and exploiting the procedures based on them to great scientific effect was a salient feature of the output of the pioneers of analysis like Fermat (see \cite{Katz et al. 2013}), Leibniz (see \cite{Bascelli et al. 2014}), Euler (see \cite{Bair et al. 2013}), and Cauchy (see \cite{Bascelli et al. 2018}).

What we show in our paper is that such infinitesimal procedures can be used to address both the student unease and the philosopher’s unease. Leibniz did not develop a set-theoretic ontology for infinitesimals for the simple reason that set theory did not exist yet, but he may have been more open to an infinitesimal gap between “zero, dot, followed by an infinite number of 9s” and 1 than many traditionally trained mathematicians today.

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