Construction of Quantum Target Space from World-Sheet States using Quantum State Tomography

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Abstract

In this paper, we will construct the quantum states of target space coordinates from world-sheet states, using quantum state tomography. To perform quantum state tomography of an open string, we will construct suitable quadrature operators. We do this by first defining the quadrature operators in world-sheet, and then using them to construct the quantum target space quadrature operators for an open string. We will connect the quantum target space to classical geometry using coherent string states. We will be using a novel construction based on a string displacement operator to construct these coherent states. The coherent states of the world-sheet will also be used to construct the coherent states in target space.

1 Introduction

Even though there is a clear connection between target space and world-sheet in classical theory, it is difficult to find a quantum analog of this connection. However, it is also possible to establish a connection between quantum states of world-sheet and the classical target space on which these quantum states propagate. This is done by using the renormalization group flow of world-sheet perturbations. For the quantum theory of world-sheet to remain conformally invariant, it is required that the world-sheet $\beta$-functions vanish. The equations obtained from this procedure are then identified with the equations of motion for the gravitational action in the target space. Higher curvature terms can be obtained from higher loop corrections on the world-sheet. So, with this procedure, a consistent classical target space is constructed on which is consistent with the quantum theory of world-sheet. However, it is important to construct a full quantum theory of target space using quantum theory of world-sheet. We propose that a proper analysis of the quantum information in the world-sheet can be used to obtain the quantum states of target space. To properly analyze the information needed to construct a quantum state of a system, we have to use quantum estimation theory \cite{47, 48}. Now a large set of parameters in quantum estimation can be used to estimate the full quantum state of a system. This estimation of quantum state of the system by large set of parameters is done using quantum state tomography \cite{49, 50}. Thus, quantum state tomography has been used to obtain quantum states of various different quantum systems \cite{45, 46, 51}. It is possible to perform quantum state tomography, even when the system has infinite degrees of freedom, such as a quantum field theory \cite{52, 53}. Several interesting techniques have been developed to perform quantum state tomography of such system \cite{54, 55, 56, 57}. The quantum state tomography has also been used to study the behavior of quantum states in loop quantum gravity \cite{58}. However, the quantum state tomography has not been used to analyze the behavior of quantum states of string theory. So, we will perform quantum state tomography of an open string. We would like to point out that in quantum state tomography for quantum mechanical systems, information obtained
from experiments is used to reconstruct quantum state of the system. Motivated by this observation, we expect that in string theory, information obtained from world-sheet should be used to construct quantum states of target space. It may be noted that the that string correlation functions have been used to analyze the behavior of world-sheet string states [37, 38, 41, 42]. Now as it is known that information about a quantum system can obtain from correlation functions [33, 34, 35, 36], information theory has already been used to analyze the world-sheet string states. We would like to also point out that the correction function of a quantum system can be constructed from its tomogram [43, 44], and this is another motivation to construct the tomography in string theory.

The connection between this quantum state of target space and classical geometry can be made using coherent states. In fact, such a connection between string coherent states and classical geometry has also been used in fuzzball proposal [9, 10, 11, 12]. In this proposal, the entire region of space within event horizon of a black hole is considered to be made up of quantum state of strings. This fuzzball states is expressed as a wave functional in the full string theory, and not its supergravity approximation. The fuzzball quantum state is then connected to classical geometry of a black hole using string coherent states in target space, as these states can be approximated by classical geometric solutions. In fact, it has been demonstrated that fuzzballs are consistent with the gravitational wave observations done on black holes [13, 14]. However, understanding of quantum nature of the string in the fuzzball is important to investigate the black hole information paradox [15, 16]. String coherent states are also important in the analysis of cosmic strings, which are produced at the end of D3-D3 brane inflation [17, 18]. Furthermore, as these cosmic strings can be detected using gravitational wave observation [19, 20, 21, 22, 23], it is important to understand their properties. Thus, it is important to understand the quantum states of target space coordinates, and this can be done by performing quantum states tomography using coherent states. We would like to point out that quantum state tomography has been performed using its coherent states [59, 60]. Here we will generalize these results to string theory.

To analyze such results in string theory, we need to first construct coherent states in string theory. The coherent states of string have been constructed using the DDF formalism [24, 25]. The coherent states in the Neveu-Schwarz sector has been constructed using this formalism [26]. This has been extended to the Ramond sector by supersymmetric transformations in target space [26]. It is known that the DDF operators satisfy the oscillator algebra, and so they can be used to construct coherent string theoretical coherent states, which are analogous to the usual coherent states for quantum mechanical oscillator. However, it is also possible to directly construct the string coherent states using the analogy of the original string algebra with the oscillator algebra [27, 28]. So, in this paper, we will explicitly construct string coherent states. The optical coherent states are important in performed quantum state tomography, using quadrature operators in optical phase space [61, 62, 63, 64]. In fact, optical coherent states can be constructed in quantum optics using quadrature operators in optical phase space [29, 30, 31, 32]. In this paper, we will generalize the quadrature operators in optical phase space to string theory, and use them to perform quantum state tomography of an open string.

2 String Quadrature Operators

2.1 Bosonic String Theory

In this section, we will review bosonic string theory, and express it in a form where we can use the techniques from quantum optics [29, 30, 31, 32]. The Polyakov action describes the world-sheet of bosonic string theory, and it can be written as

\[
S = -\frac{T}{2} \int d\sigma^2 \eta^{\alpha\beta} \partial_{\alpha} X_{\mu} \partial_{\beta} X^{\mu}
\]

(1)

where the tension of the string \(T = 1/\pi l_s^2\) is related to string length \(l_s\). We can solve the above equation by applying Neumann boundary conditions for open strings.

\[
X^{\mu}(\tau, \sigma) = x^{\mu} + l_{s} \tau \eta^{\mu} + i l_{s} \sum_{m \neq 0} \frac{1}{m} \alpha_{m}^{\mu} e^{-i m \tau} \cos(m \sigma)
\]

(2)

As string theory has a gauge degrees of freedom, we need to fix a gauge before quantization it. In this paper, we will use the light-cone gauge, where the target space coordinates are defined as \(\{X^+, X^-, X^i\}_{i=1}^{24}\), with \(X^+ = (X^0 + X^{25})\), and \(X^- = (X^0 - X^{25})\). In the light-cone gauge, there are no oscillations in the \(X^+\) direction, and the oscillation in \(X^-\) direction can be expressed in terms of other string oscillations. Thus, we need to only consider string oscillations in the \(\{X^i\}_{i=1}^{24}\) direction to obtain information about the behavior of bosonic string theory in light-cone gauge. Now in the light-cone gauge, the string algebra in these directions can be expressed as \([\hat{a}_m, \hat{a}_n^\dagger] = m \eta^{ij} \delta_{m+n, 0}\), which can also be written as

\[
[\hat{a}_m, \hat{a}_{-m}^\dagger] = m \quad (\text{for } n = -m, \text{ also, } (\hat{a}_m^\dagger)^\dagger = \hat{a}_{-m})
\]

(3)
with \( \{i, j\} \), taking values in \( \{1, 2, 3, \ldots, 24\} \). We will suppress the index \( i \) for simplicity, and express the annihilation and creation operators as \( \hat{\alpha}_m, \hat{\alpha}^{-m} \), and put them back at the end of the calculations. We can now define the number operator \( \hat{N}_m \) as the product of the annihilation and creation operators

\[
\hat{N}_m = (\hat{\alpha}_m \cdot \hat{\alpha}^{-m}) , \quad \text{where} \quad m \geq 1
\]

We define \( |k\rangle \) as \( \alpha_k|0\rangle \), and so the eigenstates of the number operator satisfy

\[
\hat{N}_m |k\rangle = k_n |k\rangle
\]

where \( k_m \) is the eigenvalue of \( \hat{N}_m \). As a result, we have \( \hat{N}_m(\hat{\alpha}_m |k\rangle) = (k - m) |\alpha_k|k\rangle \) and \( \hat{N}_m(\hat{\alpha}^{-m}|k\rangle) = (k + m) |\alpha_k|k\rangle \). Now, because \( (\alpha_m |k\rangle) \) and \( (\hat{\alpha}^{-m}|k\rangle) \) are eigenstates of \( \hat{N}_m \) with eigenvalues \( (k - m) \) and \( (k + m) \), it follows that when \( \hat{\alpha}_m \) and \( \hat{\alpha}^{-m} \) operate on \( |k\rangle \), they decrease and increase \( |k\rangle \) by \( m \) units. This is an important point and a distinguishing factor of the world-sheet algebra. This feature will be important in the construction of the quantum target space quadrature operators. We can now write \( \hat{\alpha}_m \) and \( \hat{\alpha}^{-m} \) as

\[
\hat{\alpha}_m \quad \text{and} \quad \hat{\alpha}^{-m}
\]

where

\[
\langle \alpha_k | \hat{\alpha}_m | \alpha_{k'} \rangle = \delta_{k, k'} \delta_{m, 0}
\]

and \( \langle \alpha_k | \hat{\alpha}^{-m} | \alpha_{k'} \rangle = \delta_{k, k'} \delta_{m, 0} \). This is a consequence of the canonical commutation relation satisfied if we define world-sheet mode quadrature operators in string world-sheet phase space. We observe that these world-sheet mode quadrature operators exist in a string analog of optical phase space. We will now use techniques from quantum optics to construct and analyze the properties of string quadrature operators in string world-sheet phase space (which is the string analog of optical phase space) such that they satisfy a commutation relation which is similar to the commutation relation satisfied by the position and momentum operators of a quantum mechanical oscillator. Thus, the string would-sheet mode quadrature operators would be expected to satisfy the canonical commutation relation

\[
[\hat{s}_{\theta, m}, \hat{p}_{\theta, m}] = i
\]

These would-sheet mode quadrature operators exist in a string analog of optical phase space. We observe that this commutation relation is satisfied if we define would-sheet mode quadrature operators in string world-sheet phase space (which is the string analog of optical phase space)

\[
\hat{s}_{\theta, m} = \frac{i \rho_\theta m \sigma}{m} \left( \hat{\alpha}_m e^{-i(\theta + m \tau)} - \hat{\alpha}^{-m} e^{i(\theta + m \tau)} \right)
\]

\[
\hat{p}_{\theta, m} = \frac{1}{2 \rho_\theta m \sigma} \left( \hat{\alpha}_m e^{-i(\theta + m \tau)} + \hat{\alpha}^{-m} e^{i(\theta + m \tau)} \right)
\]

Here the string analog of the homodyne quadrature operators \[65, 66, 67, 68\] has been used to obtain information encoded in the phase of the string modes. Thus, using the formalism of standard homodyne quadrature operators \[65, 66, 67, 68\], we have defined \( \theta \) to be the phase associated with world-sheet homodyne quadrature operators, and so \( 0 \leq \theta \leq 2\pi \). Here in this string world-sheet phase space, \( s_{\theta, m}, p_{\theta, m} \) act as coordinates and momentum, and \( \sigma, \tau \) act as parameters.

Now from this perspective we can define fields on this world-sheet phase space, such that they would represent target space coordinates. Thus, using these string would-sheet mode quadrature operators, we can define the larger space quadrature operators in quantum target space \( \hat{X}_\theta, \hat{P}_\theta \), in the center of mass frame as

\[
\hat{X}_\theta = \rho_\theta \sum_{m=1}^{\infty} \cos m \sigma \left( \hat{\alpha}_m e^{i(\theta + m \tau)} - \hat{\alpha}^{-m} e^{-i(\theta + m \tau)} \right) = \sum_{m=1}^{\infty} \hat{s}_{\theta, m}
\]

\[
\hat{P}_\theta = \rho_\theta \sum_{m=1}^{\infty} \cos m \sigma \left( \hat{\alpha}_m e^{i(\theta + m \tau)} + \hat{\alpha}^{-m} e^{-i(\theta + m \tau)} \right) = 2 \sum_{m=1}^{\infty} \hat{p}_{\theta, m} \cos^2 m \sigma
\]

It may be observed that \( \hat{P}_\theta \) can be viewed as the momentum conjugate to \( \hat{X}_\theta \). Here we had suppressed the index \( i = \{1, 2, 3, \ldots, 24\} \) in light-cone gauge, and if write it explicitly, we obtain the quadrature operators for each of the target space coordinates \( X^i_\theta \), and its momentum conjugate \( P^i_\theta \) in light-cone gauge. In general, we can perform the analysis in a general gauge and obtain quadrature operators \( X^i_\theta, P^i_\theta \). Here different values of the world-sheet quadrature operators parameterized by \( \theta \) to construct the quantum state of the target space coordinates. This result generalized the classical relation between the target space coordinates and the world-sheet of a string. In classical geometry, the world-sheet of strings can be used as a probe for the target space, and information about the target space coordinates can be obtained using world-sheet of strings. Here we have constructed the quantum target space using world-sheet phase space.
2.3 Eigenstate of String Quadrature Operator

In this section, we will derive an explicit expression for the eigenstates of the string quadrature operators $\hat{X}_\theta$. To do this, we define eigenstates of the string would-sheet mode quadrature operators as $|s_{\theta,m}\rangle$ with the eigenvalue $s_{\theta,m}$

$$s_{\theta,m}|s_{\theta,m}\rangle = s_{\theta,m}|s_{\theta,m}\rangle$$  \hspace{1cm} (11)

where $s_{\theta,m}$ is the eigenvalue. Now the eigenstate of the string quadrature operator $\hat{X}_\theta$ can be written in terms of the eigenstates of the string would-sheet mode quadrature operators as $|X_\theta\rangle = \prod_m |s_{\theta,m}\rangle$. Thus, this quantum target space quadrature operator satisfies the following eigenvalue equation

$$\hat{X}_\theta|X_\theta\rangle = X_\theta|X_\theta\rangle$$  \hspace{1cm} (12)

where an explicit form for $X_\theta$ can be obtained from the eigenvalue equation of string would-sheet mode quadrature operators as

$$\langle k|\hat{X}_\theta|s_{\theta,m}\rangle = s_{\theta,m}\tilde{\psi}_k(s_{\theta,m})$$

$$= \langle k|\hat{\alpha}_m e^{-i(\theta+mr)} \frac{\cos(m\sigma)}{m} - \hat{\alpha}_m e^{i(\theta+mr)} \frac{\cos(m\sigma)}{m}|s_{\theta,m}\rangle$$

$$= \sqrt{k + 1} e^{-i(\theta+mr)} \tilde{\psi}_{k+1}(s_{\theta,m}) - \sqrt{k} e^{i(\theta+mr)} \tilde{\psi}_{k-1}(s_{\theta,m})$$  \hspace{1cm} (13)

where $\langle k|s_{\theta,m}\rangle = \tilde{\psi}_k(s_{\theta,m})$, $\langle k + m|s_{\theta,m}\rangle = \tilde{\psi}_{k+m}(s_{\theta,m})$ and $\langle k - m|s_{\theta,m}\rangle = \tilde{\psi}_{k-m}(s_{\theta,m})$ Now similarly, for $\psi_{k+m}(s_{\theta,m}) = (s_{\theta,m}|k + m\rangle$, we can write

$$\psi_{k+m}(s_{\theta,m}) = e^{-i(\theta+mr)} \frac{\sqrt{k + 1} e^{-i(\theta+mr)} \tilde{\psi}_{k+1}(s_{\theta,m}) - \sqrt{k} e^{i(\theta+mr)} \tilde{\psi}_{k-1}(s_{\theta,m})}{\sqrt{k + 1} e^{-i(\theta+mr)} \tilde{\psi}_{k+1}(s_{\theta,m}) + \sqrt{k} e^{i(\theta+mr)} \tilde{\psi}_{k-1}(s_{\theta,m})}$$  \hspace{1cm} (14)

Now we would like to find recurrence relations for this $\psi_m$, for different cases. We first observe that for $k = 0$, we can write

$$\psi_m(s_{\theta,m}) = \frac{e^{-i(\theta+mr)}}{\sqrt{m e^{i(\theta+mr)}}} [s_{\theta,m}\psi_0(s_{\theta,m})]$$  \hspace{1cm} (15)

Similarly, for any $k = n$, such that $n < m$, the second term of recurrence will vanish (as $\psi_{n-m} = 0$ for this case), and we can write

$$\psi_{n+m}(s_{\theta,m}) = \frac{e^{-i(\theta+mr)}}{\sqrt{n + m e^{i(\theta+mr)}}} [s_{\theta,m}\psi_n(s_{\theta,m})]$$  \hspace{1cm} (16)

The second term does not vanish $k \geq m$. So, for $m = n$, we have

$$\psi_{2m}(s_{\theta,m}) = e^{-i(\theta+mr)} \frac{\sqrt{2m e^{i(\theta+mr)}} [s_{\theta,m}\psi_m(s_{\theta,m}) + \sqrt{m e^{i(\theta+mr)}} \tilde{\psi}_0(s_{\theta,m})]}{\sqrt{2m e^{i(\theta+mr)}} [s_{\theta,m}\psi_m(s_{\theta,m}) + \sqrt{m e^{i(\theta+mr)}} \tilde{\psi}_0(s_{\theta,m})]}$$  \hspace{1cm} (17)

and for $k = l > m$, we have

$$\psi_{l+m}(s_{\theta,m}) = e^{-i(\theta+mr)} \frac{\sqrt{l + m e^{i(\theta+mr)}} [s_{\theta,m}\psi_l(s_{\theta,m}) + \sqrt{l e^{i(\theta+mr)}} \tilde{\psi}_{l-m}(s_{\theta,m})]}{\sqrt{l + m e^{i(\theta+mr)}} [s_{\theta,m}\psi_l(s_{\theta,m}) + \sqrt{l e^{i(\theta+mr)}} \tilde{\psi}_{l-m}(s_{\theta,m})]}$$  \hspace{1cm} (18)

Here there are various different recursion relations, each with their own modified Hermite polynomials. We have obtained the first three recursion relations to exemplify the procedure in Appendix, and the any other recursion relations can be obtained using the same algorithm. These cases have been listed in Table 1. A consequence of this observation is that in string theory different states can be annihilated by a single annihilation operator.

| Table 1: Values of $m + k'$ for different values of $t'$ |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| $k + m =$ | for $t = 1$ | for $t = 2$ | $\cdots$ | for $t = l$ |
| $tm$ | $m$ | $2m$ | $\cdots$ | $tm$ |
| $tm + 1$ | $m + 1$ | $2m + 1$ | $\cdots$ | $tm + 1$ |
| $tm + 2$ | $m + 2$ | $2m + 2$ | $\cdots$ | $tm + 2$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $tm + (m - 1)$ | $2m - 1$ | $3m - 1$ | $\cdots$ | $tm + (m - 1)$ |
Using these recurrence relation, we can calculate expressions of $\psi_0, \psi_1, ..., \psi_{m-1}$, which can then be used to obtain $m^{th}$ would-sheet mode wave function $\psi_m$. In order to do this, we start by writing the annihilation operator $\hat{\alpha}_m$ in terms of the quadrature operators $(\hat{s}_{\theta,m}, \hat{p}_{\theta,m})$ as

$$\hat{\alpha}_m = \frac{1}{2} e^{-i(m\tau + \theta)} \left( \hat{s}_{\theta,m} + \frac{2I^2 \cos^2(m\theta)}{m} \hat{p}_{\theta,m} \right)$$

(19)

We can use the annihilation operator to annihilate $\psi_0$ as $\hat{\alpha}_m \psi_0 = \langle s_{\theta,m} | \hat{\alpha}_m | 0 \rangle = 0$. As would-sheet mode quadrature operators satisfy the commutation algebra of position and momentum operators, we can express $\hat{p}_{\theta,m}$ as $-id/d\hat{s}_{\theta,m}$, and write

$$\left( \hat{s}_{\theta,m} + \frac{2I^2 \cos^2(m\sigma)}{m} \frac{d}{d\hat{s}_{\theta,m}} \right) \psi_0 = 0$$

(20)

As a result, we get an explicit expression for $\psi_0$ (with $A$ as the normalization constant),

$$\psi_0 = Ae^{-\frac{m^2 \sigma^2}{2l^2 \cos^2(m\sigma)}}, \quad \text{with} \quad A = \left[ \frac{m}{2\pi I^2 \cos^2(m\sigma)} \right]^\frac{1}{4}$$

(21)

Similarly, because $\hat{\alpha}_m | z \rangle = 0$, where $z = (0, 1, ..., m - 1)$, we have $\psi_z = \psi_0$. This behavior of $\psi_m$ can be clearly seen in Fig. 1. For instance, for $m = 1$ case, all the wave functions will be different, since $m = 1$ replicates the usual quantum mechanical behavior. However, for $m = 2$, $\psi_0 = \psi_1$. We have plotted $\psi_4$ to illustrate this behavior.

![Plots showing the behavior of $\psi_m$](image)

Figure 1: The plots show the behavior of $\psi_m$ for the special case of $m = 1, \sigma = 0, \theta = 0$ as a function of $s_{\theta,m}$ and time $\tau$. (a) Ground state wavefunction $\psi_0$ which preserves its Gaussian shape through time, whereas (b) is the first excited state $\psi_1$ which oscillates in time sinusoidally. (c) $\psi_0$ for $m = 2$ and (d) $\psi_4$ for $m=2$.

3 String State Tomography

3.1 String Coherent States

The DDF formalism has been used to construct string coherent states [24, 25]. The DDF formalism has been used in both Neveu-Schwarz sector [26] and [26]. The string coherent states have also been constructed using...
the analogy of the original string algebra with the oscillator algebra [27, 28]. It may be noted that the optical quadrature operators have been used to obtain coherent states in quantum optics [29, 30, 31, 32]. Motivated by this construction, we will construct the string coherent states in target space using string quadrature operators obtained in the previous section. Now to construct string coherent states, we will first construct the coherent states for a single would-sheet mode. Thus, we can define the string would-sheet mode coherent state as states which most closely resembling behavior of a classical string oscillatory modes. So, the string would-sheet mode coherent state $|\varphi_m\rangle$ can be expressed as an eigenstate of $\hat{\alpha}_m$, with eigenvalue $\varphi_m$

$$\hat{\alpha}_m|\varphi_m\rangle = \varphi_m|\varphi_m\rangle$$

(22)

where $|\varphi_m\rangle$ satisfying $\langle \varphi_m|\varphi_m\rangle = 1$. In order to derive an explicit expression for string coherent state $|\varphi_m\rangle$, we expand in terms of $|k\rangle$ string states as

$$|\varphi_m\rangle = \sum_{k=0}^{\infty} |k\rangle \langle k| \varphi_m\rangle$$

(23)

Here we can use $\langle k|\hat{\alpha}_m|\varphi_m\rangle = \varphi_m\langle k|\varphi_m\rangle$ to obtain $\langle k + m|\varphi_m\rangle = \varphi_m\langle k|\varphi_m\rangle/\sqrt{k + m}$. Replace $k$ with $k - m$, we obtain $\langle k|\varphi_m\rangle = \varphi_m\langle k - m|\varphi_m\rangle/\sqrt{k}$, and by repeating this process, we also obtain $\langle k - m|\varphi_m\rangle = \varphi_m\langle k - 2m|\varphi_m\rangle/\sqrt{k - m}$. Using these expression, we can write $\langle k|\varphi_m\rangle = \varphi_m^2 \langle k - 2m|\varphi_m\rangle/\sqrt{k(k - m)}$. Now repeating this procedure $n$ times, we obtain a general expression for $\langle k|\varphi_m\rangle$ as

$$\langle k|\varphi_m\rangle = \frac{\varphi_m^n}{\sqrt{k(k - m)(k - 2m)...(k - (n - 1)m)}}$$

(24)

Now for $k \geq m$, and $k - nm = b$, with $b = (0, 1, 2...m - 1)$, $n = (0, 1, 2, 3...)$, we can write the expression for $\langle k|\varphi_m\rangle$ as

$$\langle k|\varphi_m\rangle = \frac{\varphi_m^n}{\sqrt{k(k - m)(k - 2m)...m}}$$

(25)

We can obtain the expression for $|b\rangle\varphi_m\rangle$ using the string displacement operator $D(\varphi_m)$ and the normalization condition. We define the string would-sheet mode displacement operator as the operator which generates a string would-sheet mode coherent states from the vacuum state, $|\varphi_m\rangle = D(\varphi_m)|0\rangle$. Thus, we can write the explicit expression for string would-sheet mode displacement operator as

$$\hat{D}(\varphi_m) = e^{-\frac{\Delta_m^2}{2}} e^{\varphi_m \hat{\alpha}_m - \frac{1}{2} \hat{\alpha}_m^* \hat{\alpha}_m}$$

(26)

Now to obtain the expression for $\langle b|\varphi_m\rangle$, we observe that

$$\langle b|\varphi_m\rangle = \langle b|D(\varphi_m)|0\rangle = e^{-\frac{\Delta_m^2}{2}} \langle b|e^{\varphi_m \hat{\alpha}_m - \frac{1}{2} \hat{\alpha}_m^* \hat{\alpha}_m}|0\rangle$$

$$= e^{-\frac{\Delta_m^2}{2}} \langle b|(1 + \varphi_m \hat{\alpha}_m + ...)(1 + \varphi_m^* \hat{\alpha}_m + ...)|0\rangle$$

(27)

Thus, using $\hat{\alpha}_m|0\rangle = 0$, we obtain $\langle b|\varphi_m\rangle = e^{-\frac{\Delta_m^2}{2}} \delta_{0,b}$, which vanishes for $b \neq 0$, and so, we can write

$$\langle 0|\varphi_m\rangle = e^{-\frac{\Delta_m^2}{2}}$$

(28)

In the general expression for $\langle k|\varphi_m\rangle$, we observe that $\langle b|\varphi_m\rangle = 0$ for $b \neq 0$, and we obtain a non-vanishing expression only for $k = nm$. So, using the expression for $\langle 0|\varphi_m\rangle$ in the general expression for $\langle k|\varphi_m\rangle$, we obtain

$$\langle k|\varphi_m\rangle = \frac{\varphi_m^n e^{-\frac{\Delta_m^2}{2}}}{\sqrt{k(k - m)(k - 2m)...(k - (n - 1)m)}}$$

(29)

Using this expression for $\langle k|\varphi_m\rangle$ (with $k = nm$) in the general expression for string would-sheet mode coherent states, we can write an explicit expression for a string would-sheet mode coherent state as

$$|\varphi_m\rangle = \sum_{n=0}^{\infty} \frac{\varphi_m^n e^{-\frac{\Delta_m^2}{2}}}{\sqrt{k(k - m)(k - 2m)...(k - (n - 1)m)}}|k\rangle$$

(30)

By repeating this procedure for different string modes, we can obtain string would-sheet mode coherent states for different string modes. Using these string would-sheet mode coherent states, a coherent state for strings in target space can be expressed as

$$|\Phi\rangle = \prod_{m} |\varphi_m\rangle$$

(31)
Thus, we have generalized the construction of coherent states to string theory using coherent states for each of the string modes. It may be noted that even though string coherent states have been obtained before, this is the first time that they have been obtained using string analogs of the quadrature operators. We can construct a coherent state for each of the target space coordinates in the light-cone gauge \( \{ \Phi \}_i \). These can be constructed from string world-sheet mode coherent states, \( | \Phi \rangle = \prod_m | \varphi_m \rangle \), where we can define the string world-sheet mode coherent states for various target space coordinates in light-cone gauge using \( \hat{\alpha}_m | \varphi_m \rangle = \varphi_m | \varphi_m \rangle \). Thus, the world-sheet coherent states can be used to construct coherent states for target space, which can be approximated by classical geometric solutions.

### 3.2 Quantum State Tomography

In the previous section, we obtained string coherent states in target space. As coherent states can be used to perform quantum state tomography for a system [59, 60], we will use the string coherent states obtained in the previous section to perform quantum state tomography of bosonic strings. To do this we will first expand the eigenstates of a string quadrature operator in quantum target space using the previous section to perform quantum state tomography of bosonic strings. To do this we will first expand the string would-sheet mode quadrature operator as

\[
| \theta, m \rangle = \sum_{t=0}^{\infty} | t \rangle | \theta, m \rangle
\]

Using this expression for a string would-sheet mode quadrature operator, we can write the eigenstates of a string quadrature operator as

\[
| X_\theta \rangle = \prod_m | \theta, m \rangle.
\]

Now from this expression for the eigenstates of a string quadrature operator in quantum target space \( | X_\theta \rangle \), we can define the quantum state tomogram for a string as \( \Omega(\theta, \theta) \), where

\[
\Omega(\theta, \theta) = \langle X_\theta | \hat{\rho} | X_\theta \rangle
\]

with \( \hat{\rho} \) as the string density matrix for the given system. Here we have also used the normalization condition

\[
\int \Omega(\theta, \theta) dX_\theta = 1
\]

It may be noted that in analogy with quantum optics [29, 30, 31, 32], the tomogram of the pure open string state in target space is represented by the density matrix \( \hat{\rho} = | \Phi \rangle \langle \Phi | \) can be written as \( \Omega(\theta, \theta) = | \langle X_\theta | \Phi \rangle |^2 \).

To evaluate this tomogram for an open string, we need to express it in terms of tomogram for world-sheet string modes, which can be written as \( \Omega(\theta, m, \theta_m) = | \langle \theta, m | \varphi_m \rangle |^2 \). Thus, to perform quantum state tomography of an open string, we evaluate \( \langle \theta, m | \varphi_m \rangle \), and write a tomogram for world-sheet string modes as

\[
| \langle \theta, m | \varphi_m \rangle |^2 = \left| \sum_{t=0}^{\infty} \sum_{n=0}^{\infty} J_{tm}(s_{\theta, m}) e^{-i(\theta + m\tau)} \psi_0(s_{\theta, m}) \varphi_m^n e^{-i\omega_m |^2 \sqrt{nm((nm) - m)((nm) - 2m)...m}} \right|^2
\]

Now using this expression for tomogram for world-sheet string modes, we can write the tomogram for the target space as

\[
| \langle X_\theta | \Phi \rangle |^2 = \left| \prod_m \langle \theta, m | \varphi_m \rangle \right|^2
\]

Thus, we can perform the quantum state tomography in a quantum target space, using this expression in terms of world-sheet modes. Here if we do not suppress the index, and repeat this procedure we can express all
the information needed to construct a quantum target space using \( \{ \Omega_i(X_i, \theta) \} \), where \( \Omega_i(X_i, \theta) = |\langle X_i | \Phi \rangle|^2 \), with \( i = \{1, 2, 3, \ldots, 24\} \) in light-cone coordinates. We plot the tomogram for target space in Fig. 2, Fig. 3 and Fig. 4 with various parameter choices. In Fig. 2, we see the variation of the tomogram as we add modes to the system. It is clear that the addition of modes centralizes the tomogram. This is evident from Eq. 37, when we take the product of multiple modes, there is constructive interference in the central region but destructive interference elsewhere. Fig. 3 is the variation of the tomogram with the time-like coordinate \( \tau \). Changing \( \tau \) simply translates the tomogram and this is clear from Eq. 37. Lastly, in Fig. 4 we investigate the variation of the tomogram with different values of \( \sigma \), the space-like coordinate on the world sheet as can be seen in Eq. 2. Due to the structure of the modified Hermite polynomials \( J_m(t) \), we expect that the tomogram is undefined when \( m \sigma = \pi \) and thus we vary \( \sigma \) through odd divisions of \( \pi \). From this it is clear why the tomogram squishes for certain values of \( \sigma \).

Figure 2: Contour plot of the tomogram \( \Omega(X_{\theta, m}, \theta) \), where \( N = 20, \tau = 0, \sigma = 0 \) and \( M = \{3, 5, 8, 10\} \). Figures are enumerated from left to right.

Figure 3: Contour plot of the tomogram \( \Omega(X_{\theta, m}, \theta) \), where \( M = 10, N = 20 \). For \( \sigma = 0 \), it shows the time evolution \( \tau = \{0, 0.1\} \). It simply translates as time progresses and this is clear from Eq. 37.
Figure 4: Contour plot of the tomogram $\Omega(X_\theta, \mu, \theta)$, where $M = 10$, $N = 20$. For $\tau = 0$ and $\sigma = \{\pi, \pi/3, \pi/7, \pi/11, \pi/25, \pi/37, \pi/49, 0\}$. Figures are enumerated from left to right.

4 Conclusion

It is known that the quantum state tomography contains all the information needed to construct a given quantum state. This motivated the construction of quantum state tomography in string theory. To perform quantum state tomography of an open string, we constructed quadrature operators for an open string in quantum target space. These string quadrature operators were constructed using quadrature operators for different world-sheet modes of an open string. We also defined a suitable string displacement operator which would convert a vacuum state into a string world-sheet mode coherent state. These string world-sheet mode coherent states were then used to construct the coherent state for an open string. We used these coherent states along with the string quadrature operators to perform quantum state tomography of an open string. We would like to point out that we performed quantum state tomography on a flat target space. However, it would be interesting to generalize these results to open string coupled to gravity. Furthermore, it would be interesting to perform the quantum state tomography for string states representing a fuzzball and a cosmic string. We expect to obtain first order corrections to the classical behavior of such states using quantum tomography. These results can then be used to analyze black hole information paradox [15, 16]. To perform this analysis it is important to analyze the quantum tomogram for strings in curved space-time.

It would be interesting to generalize these results to closed strings. We expect that the quadrature operators for an closed string could also be constructed using the quadrature operators for different modes of a closed string. We expect that we will need to construct too the quadrature operators, which would correspond to right and left movers. We can also obtain coherent states for a closed string using the same algorithm. These closed string coherent states along with the corresponding quadrature operators can be used to perform quantum state tomography for closed strings. It would also be interesting to generalize these results to thermal string states. We can construct thermal coherent states for those thermal states [8]. This can again be done by defining thermal coherent string world-sheet mode states, and then using them to construct the thermal coherent states for strings. These thermal coherent states can then be used to perform quantum tomography for such thermal string states. This can be done for both open and closed string states.

We would like to point out that coherent states of strings are also important in the context of AdS/CFT correspondence. This is because the micro-states of geometric objects like black holes can be analyzed using string states [1, 2]. This is because the micro-states of an AdS black hole can be obtained from the micro-states of the conformal field theory dual to it, using the AdS/CFT correspondence [3, 4]. It is possible to
obtain information about correlation functions of a conformal field theory by using the Skenderis-van-Rees prescription [5, 6]. This prescription is based on AdS/CFT correspondence, and in it the initial and final states are represented by coherent states in AdS spacetime [7]. The Skenderis-van-Rees prescription has been generalized to conformal field theories at finite temperature, which are dual to a black hole in AdS [8]. Thus, it is important to investigate coherent states in AdS to understand the conformal field theories dual to black holes. It would be interesting to investigate this correspondence beyond supergravity approximation. This can be done by analyzing a possible duality between quantum state tomography of target space in AdS with the quantum state tomography of the boundary conformal field theory.

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Appendix

Here we explicitly derive recurrence relation for three different cases. This will be done by calculating the complete set of these modified Hermite polynomials.

- **Case 1: \( k+m=tm \)** We start from

\[
\psi_{k+m}(s_{\theta,m}) = e^{-i(\theta+m\tau)} \sqrt{k+m \cos(m\sigma)} \left[ s_{\theta,m} \psi_{k}(s_{\theta,m}) + \sqrt{k} e^{-i(\theta+m\tau)} \psi_{k-m}(s_{\theta,m}) \right]
\]  

For \( t = 1 \), we obtain

\[
\psi_{m}(s_{\theta,m}) = e^{-i(\theta+m\tau)} \frac{1}{\sqrt{m \cos(m\sigma)}} [s_{\theta,m} \psi_{0}(s_{\theta,m})]
\]  

and for \( t = 2 \), we obtain

\[
\psi_{2m}(s_{\theta,m}) = e^{-2i(\theta+m\tau)} \frac{1}{\sqrt{2m \cos(m\sigma)}} \left[ s_{\theta,m} s_{\theta,m} + \sqrt{m \cos(m\sigma)} \psi_{0}(s_{\theta,m}) \right]
\]  

In general, \( \psi_{tm} \) can be written as follows

\[
\psi_{tm}(s_{\theta,m}) = e^{-ti(\theta+m\tau)} J_{tm}(s_{\theta,m}) \psi_{0}(s_{\theta,m})
\]  

where \( J_{tm} \) is a new polynomial defined by the following recurrence relation;

\[
J_{(t+1)m}(s_{\theta,m}) = \frac{1}{\sqrt{(t+1)m \cos(m\sigma)}} [s_{\theta,m} J_{tm}(s_{\theta,m}) + \sqrt{(t+1)m} \psi_{0}(s_{\theta,m})]
\]  

with \( J_0 = 1 \).

- **Case 2: \( k+m=tm+1 \)** Here we use

\[
\psi_{k+m}(s_{\theta,m}) = \psi_{tm+1}(s_{\theta,m}) = e^{-i(\theta+m\tau)} \frac{1}{\sqrt{tm+1 \cos(m\sigma)}} \left[ (s_{\theta,m}) \psi_{(t-1)m+1} + \sqrt{(t-1)m+1} e^{-i(\theta+m\tau)} \psi_{(t-2)m+1} \right]
\]  

We also use

\[
\psi_{tm+1}(s_{\theta,m}) = e^{-ti(\theta+m\tau)} J_{tm+1}(s_{\theta,m}) \psi_{1}(s_{\theta,m})
\]
Now $J_{(t+1)m+1}(s\theta,m)$ is a new polynomial defined by the following recurrence relation:

$$J_{(t+1)m+1}(s\theta,m) = \frac{e^{-i(\theta + m\tau)}}{\sqrt{(t + 1)m + 1}} \frac{\cos(m\sigma)}{m} \left[ s\theta,m J_{m(t+1)}(s\theta,m) + \sqrt{m(t+1)} \cos(m\sigma) m J_{m(t-1)+1}(s\theta,m) \right]$$

(46)

with $J_{m+1} = \frac{1}{\sqrt{m+1}} (s\theta,m)$ and $J_1 = 1$. Similarly, we can write the polynomial expressions for $k + m = tm + 2$ and other such expressions. Hence, we can directly derive the expression for $k + m = tm + (m-1)$.

**Case $m-1$: $k+m=tm+(m-1)$** Here we use

$$\psi_{k+m}(s\theta,m) = \psi_{(t+1)m-1}(s\theta,m)$$

$$= e^{-i(\theta + m\tau)} \left[ s\theta,m \psi_{(t+1)m+(m-1)}(s\theta,m) + \sqrt{(t+1)m+(m-1)} \cos(m\sigma) e^{-i(\theta + m\tau)} \psi_{(t+2)m+(m-1)}(s\theta,m) \right]$$

(47)

Now $J_{(t+1)m-1}$ is a new polynomial defined by the following recurrence relation:

$$J_{(t+2)m-1}(s\theta,m) = \frac{1}{\sqrt{(t + 2)m - 1}} \frac{\cos(m\sigma)}{m} \left[ s\theta,m J_{(t+1)m-1}(s\theta,m) + \sqrt{(t+1)m - 1} \cos(m\sigma) m J_{(tm)-1}(s\theta,m) \right]$$

(48)

with $J_{2m-1} = \frac{1}{\sqrt{2m-1}} (s\theta,m)$ and $J_{m-1} = 1$. 
