Abstract. We consider small perturbations of expanding maps induced by skew-product mappings whose base dynamics are not invertible necessarily. Adopting a previously developed perturbative spectral approach, we show stability of the densities of the unique absolutely continuous invariant probability measures for expanding maps under these perturbations, and upper bounds on the rate of exponential decay of fiber correlations associated to the measures as the noise level goes to zero.

1. Introduction

As is well known, several statistical properties of dynamical systems (such as the existence of SRB measures, the exponential decay of correlations, and the central limit theorem) can be obtained by demonstrating the spectral gap of the transfer operator of the dynamical system in a suitable Banach space. In addition, these statistical properties and quantities are expected to be stable if the spectrum of the transfer operator” is also stable. (The precise definition and properties of the transfer operator are provided in Section 2) This perturbative spectral approach was developed by Baladi and Young and their contemporaries, who sought a simple proof that a (piecewise) expanding map is stochastically stable (i.e., the densities of the unique absolutely continuous invariant probability measures for the dynamics are stable) under independent and identically distributed perturbations, and that its related statistical quantities, such as the rate of the exponential decay of correlations, are also stable (see [3] and references therein). This approach was extended by Baladi [2] and independently by Bogenschütz [6], to the case of perturbations induced by skew-product mappings. However, these extensions are restricted to mixing or invertible base dynamics. In this paper, an alternative perturbative spectral approach based on the Baladi-Young perturbation lemmas is presented, in which the base dynamics need not be mixing or invertible. Consequently, stochastic stability and upper bounds of the exponential decay of correlations for expanding maps under perturbations induced by skew-product mappings whose base dynamics are not invertible necessarily are demonstrated. Our result extends the result established by Baladi, Kondah, and Schmitt in [4].

1.1. Definitions and results. Let $C^r(M,M)$ be the space of all $C^r$ endomorphisms on a compact smooth Riemannian manifold $M$, endowed with the usual $C^r$ metric $d_{C^r}(\cdot,\cdot)$ with $r > 1$. (Given that $r = k + \gamma$ for some $k \in \mathbb{N}$, $k \geq 1$ and $0 \leq \gamma \leq 1$, $f \in C^r(M,M)$ denotes the $k$-th derivative of $f$ is $\gamma$-Hölder.) $f$ in

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$C^r(M, M)$ is said to be an expanding map when there exist constants $C > 0$ and $\lambda > 1$ such that
\[
\|Df^n(x)v\| \geq C\lambda^n\|v\|, \quad n \geq 1
\]
for each $x \in M$ and $v \in T_xM$. For the properties of expanding maps, the reader is referred to [11]. The expanding constant $\Lambda_r(f)$ of an expanding map $f : M \to M$ is defined by
\[
\Lambda_r(f) = \limsup_{m \to \infty} \left( \sup_{x \in M} \sum_{y \in f^{-m}(x)} \frac{\|D(f^{-m}_y)(x)\|}{|\det Df^m(y)|} \right)^{1/m},
\]
which is strictly smaller than 1 (see (2.16) in [4]). Here, $f^{-m}_y$ is the corresponding local inverse branch in a neighborhood of $x$ for each $y \in f^{-m}(\{x\})$.

Let $\Omega$ be a separable metric space endowed with the Borel $\sigma$-field $\mathcal{B}(\Omega)$ with complete probability measure $P$. Given an expanding map $f_0 : M \to M$ of class $C^r$, let $\{f_n\}_{n \geq 0}$ be a family of continuous mappings defined on $\Omega$ with values in $C^r(M, M)$ such that
\[
\text{ess sup}_{\omega \in \Omega} d_{C^r}(f_n(\omega), f_0) \to 0 \quad \text{as } \epsilon \to 0.
\]
For each $\epsilon > 0$, adopting the notation $f_\epsilon(\omega, \cdot) = f_\epsilon(\omega)$, the distance between $f_\epsilon(\omega, x)$ and $f_\epsilon(\omega', x)$ is bounded by $d_{C^r}(f_\epsilon(\omega), f_\epsilon(\omega'))$ for each $x \in M$ and each $\omega, \omega' \in \Omega$. Thus, it is straightforward to realize that $f_\epsilon : \Omega \times M \to M$ is a continuous (in particular, measurable) mapping. Note also that if $\epsilon > 0$ is sufficiently small, $f_\epsilon(\omega)$ is $P$-almost surely an expanding map of class $C^r$.

Let $\theta : \Omega \to \Omega$ be a measure-preserving measurable transformation on $(\Omega, P)$. For each $\epsilon > 0$ and $n \geq 1$, let $f_\epsilon^{(n)}(\omega, x)$ be the fiber component in the $n$-th iteration of the skew product mapping
\[
\Theta_\epsilon(\omega, x) = (\theta_\omega, f_\epsilon(\omega, x)), \quad (\omega, x) \in \Omega \times M,
\]
where we simply write $\theta_\omega$ for $\theta(\omega)$. Setting the notation $f_\epsilon(\omega) = f_\epsilon^{(n)}(\omega, \cdot)$, the explicit form of $f_\epsilon^{(n)}(\omega)$ is
\[
f_\epsilon^{(n)}(\omega) = f_\epsilon(\theta^{n-1}\omega) \circ f_\epsilon(\theta^{n-2}\omega) \circ \cdots \circ f_\epsilon(\omega).
\]

In [2] and other articles on fiber dynamics, $\theta$ is required to be a bimeasurable transformation, i.e., an invertible measurable transformation whose inverse mapping is also measurable (see, for example, [2, 4, 10]; a significant exception is described in Baladi [2]). However, some framework accommodates important examples that are not generally invertible, as shown in Example 1.4. Let $L^p_\nu(S)$ be the usual $L^p$ space on a measurable space $(S, \Sigma, \nu)$ endowed with the $L^p$ norm $\| \cdot \|_{L^p}$ where $1 \leq p \leq \infty$. For each $u \in L^\infty_\nu(S)$, a functional $\ell_\theta u : L^1_\nu(\Omega) \to \mathbb{C}$ is defined as by $\ell_\theta u(\varphi) = \int u(\omega) \cdot \varphi(\theta_\omega)dP$ for each $\varphi \in L^1_\nu(\Omega)$. Since $P$ is an invariant measure, $|\ell_\theta u(\varphi)| \leq \|u\|_L^\infty \|\varphi \circ \theta\|_{L^1} = \|u\|_L^\infty \|\varphi\|_{L^1}$, i.e., $\|\ell_\theta u\|_{(L^1_\nu(\Omega))'} \leq \|u\|_{L^\infty}$. Thus, by the Riesz representation theorem, $\ell_\theta u \in L^\infty_\nu(\Omega) \cong (L^1_\nu(\Omega))^*$ and $\ell_\theta : L^\infty_\nu(\Omega) \to L^\infty_\nu(\Omega)$ is a bounded operator on $L^\infty_\nu(\Omega)$ such that
\[
\int \ell_\theta u(\omega) \cdot \varphi(\omega)dP = \int u(\omega) \cdot \varphi(\theta_\omega)dP, \quad \varphi \in L^1_\nu(\Omega).
\]
($\ell_\theta$ is called the transfer operator of $\theta$ with respect to $P$.)
Let $C^{r-1}(M)$ be the space of all complex-valued functions on $M$ of class $C^{r-1}$ endowed with the usual $C^{r-1}$ norm $\|\cdot\|_{C^{r-1}}$, and let $m$ be the normalized Lebesgue measure on $M$. Let $L^\infty_{\mathbb{P}}(\Omega, C^{r-1}(M))$ be the Lebesgue-Bochner space of mappings defined on $\Omega$ taking values in the Banach space $C^{r-1}(M)$ endowed with the $L^\infty$ norm $\|u\|_{L^\infty} := \text{ess sup}_{\omega \in \Omega} \|u(\omega)\|_{C^{r-1}}$. Here the usual abuse of notation is adopted (where an $L^\infty$ mapping is identified by its equivalence class). The definition and properties of this space are provided in [9]. Here it is merely stated that if $u \in L^\infty_{\mathbb{P}}(\Omega, C^{r-1}(M))$, then $u$ is Bochner measurable, i.e., $u = \lim_{n \to \infty} u_n$ $P$-almost surely, where $u_n : \Omega \to C^{r-1}(M)$ is a simple function of each $n \geq 1$. Setting $u(\omega, \cdot) = u(\omega)$, for each $x \in M$ the mapping $\omega \mapsto u(\omega, x)$ is $P$-almost surely the limit of the sequence $\{u_n(\cdot, x)\}_{n \geq 1}$ of simple functions, and is thus measurable because $P$ is a complete probability measure. Furthermore, $\|u(\cdot, x)\|_{L^\infty} \leq \|u\|_{L^\infty}$: that is, $u(\cdot, x) \in L^\infty_{\mathbb{P}}(\Omega)$ for each $x \in M$. It is supposed that for $\ell_\theta$ (and therefore $\theta$), there exists a bounded operator $\tilde{\ell}_\theta$ on $L^\infty_{\mathbb{P}}(\Omega, C^{r-1}(M))$ such that the following holds for each $u \in L^\infty_{\mathbb{P}}(\Omega, C^{r-1}(M))$, each bounded linear functional $A : C^{r-1}(M) \to \mathbb{C}$, each bounded operator $A : C^{r-1}(M) \to C^{r-1}(M)$, each $x \in M$ and $P$-almost every $\omega \in \Omega$:

\begin{align*}
\text{(1.4)} & \quad \tilde{\ell}_\theta u(\omega, x) = \ell_\theta[u(\cdot, x)](\omega) \\
\text{(1.5)} & \quad \ell_\theta[Au(\cdot)](\omega) = A\tilde{\ell}_\theta u(\omega), \quad \ell_\theta[Au(\cdot)](\omega) = A\tilde{\ell}_\theta u(\omega), \\
\text{and} & \quad \|\tilde{\ell}_\theta u\|_{L^\infty} \leq \|u\|_{L^\infty}.
\end{align*}

Now, some definitions are provided on measure-preserving skew-product transformations. Let $\mathcal{B}(M)$ be the Borel $\sigma$-field of $M$. It is known that for each probability measure $\mu$ on $\Omega \times M$ with marginal $P$ on $\Omega$, there exists a function $\mu(\cdot) : \Omega \times \mathcal{B}(M) \to [0, 1]$, $(\omega, B) \mapsto \mu_\omega(B)$ that satisfies the following three conditions: $\omega \mapsto \mu_\omega(B)$ is measurable for each $B \in \mathcal{B}(M)$; $\mu_\omega$ is $P$-almost surely a probability measure on $M$; $\int \varphi d\mu = \int \varphi d\mu_\omega dP$ for each $\varphi \in L^1_B(\Omega \times M)$. This function, which is $P$-almost surely unique, is called the disintegration of $\mu$ [11, Chapter 1]. Let $f : \Omega \times M \to M$ be a measurable mapping. A measure $\mu$ on $\Omega \times M$ is called invariant under $f$ when $\mu$ is invariant under the skew-product mapping $\Theta(\omega, x) = (\theta\omega, f(\omega, x))$ and the marginal measure of $\mu$ coincides with $P$. [4] Given an absolutely continuous invariant probability measure $\mu$ of a measurable mapping $f : \Omega \times M \to M$, the (operational) forward fiber correlation function $C_{\varphi, u}(\omega, n)$ of $\varphi \in L^1_B(M)$ and $u \in L^\infty_{\mathbb{P}}(M)$ at $\omega \in \Omega$ is defined by

$$
C_{\varphi, u}(\omega, n) = \int \varphi \circ f^{(n)}(\omega) \cdot ud\mu - \int \varphi d\mu_{\theta^n \omega} \int ud\mu,
$$

and we call $\ell_\theta^\infty C_{\varphi, u}(\omega, n)$ the (operational) backward fiber correlation function of $\varphi$ and $u$ at $\omega \in \Omega$. (Since $\mu_\omega$ is $P$-almost surely absolutely continuous, $C_{\varphi, u}(\cdot, n)$ is in $L^\infty_\mathbb{P}(\Omega)$ and $\ell_\theta^\infty C_{\varphi, u}(\cdot, n)$ is well defined.) The backward fiber correlation functions of $(f, \mu)$ are said to decay exponentially fast in a Banach space $E \subset L^\infty_{\mathbb{P}}(M)$ when $\theta$ is a bimeasurable transformation, it follows from Theorem 1.4.5 in [1] which the pushforward measure of $\mu_\omega$ by $f(\omega)$ coincides with $\mu_{\theta^n \omega}$ $P$-almost surely if and only if $\mu$ is invariant under $f$. Such measures $\mu_\omega$ where $\omega \in \Omega$ are called stationary measures in [4].
there exist constants $C > 0$ and $0 < \tau < 1$ (independent of $\omega$) such that for any \( \varphi \in L^1(M) \) and \( u \in E \),
\[
(1.7) \quad |\ell^0_t \varphi(\omega, n)| \leq C\tau^n \|\varphi\|_{L^1} \|u\|_E \quad \text{P-a.s.,}
\]
where \( \|\cdot\|_E \) is the norm of \( E \). Similarly, the (operational) integrated correlation functions of \( (f, \mu) \) decay exponentially fast in a Banach space \( E \subset L^p_{\rho \times \nu}(\Omega \times M) \) when there exist constants \( C > 0 \) and \( 0 < \tau < 1 \) (independent of $\omega$) such that for any \( \varphi \in L^1_{\rho \times \nu}(\Omega \times M) \) and \( u \in E \), the mapping \( \Omega \ni \omega \mapsto C_{\varphi(\omega), u(\omega)}(\omega, n) \) is integrable for each \( n \geq 1 \). Setting $\varphi(\omega) = \varphi(\omega, \cdot)$,
\[
(1.8) \quad \left| \int C_{\varphi(\omega), u(\cdot)}(\cdot, n) dP \right| \leq C\tau^n \|\varphi\|_{L^1} \|u\|_E.
\]
The smallest number \( \tau \) such that \((1.7) \) (or \((1.8) \)) holds for any \( \tau > \tau \) is called the rate of exponential decay of backward fiber correlation functions (resp. integrated correlation functions) in \( E \). When \( \theta \) is bimeasurable, since \( \ell_\theta u = u \circ \theta^{-1} \) (see Example 1.4), then \( \ell^0_\theta |\varphi(\omega, u)\|= \ell^0_\theta |\varphi(u(\omega), \omega)\| \) \( P \)-almost surely. Thus, the exponential decay of backward fiber correlations in \( C^{r-1}(M) \) yields the exponential decay of forward fiber correlations in \( C^{r-1}(M) \) (i.e., \((1.7) \) holds, where \( \ell^0_\theta C_{\varphi, u} \) is replaced by \( C_{\varphi, u} \)) and also the exponential decay of integrated correlations in \( L^\infty(\Omega, C^{r-1}(M)) \). Under these conditions, the mixing of the skew-product mapping is equivalent to the mixing of the base dynamics (see comments in [7] Subsection 0.2)). As is well known, any expanding map \( f : M \to M \) admits a unique absolutely continuous invariant ergodic probability measure (abbreviated to aceip) on \( M \) with a density function of class \( C^{r-1} \). In addition, the correlations decay exponentially fast in \( C^{r-1}(M) \) (see e.g. [12]). The aceip of the expanding map \( f_0 : M \to M \) is denoted by \( \mu^0 \). Let \( \rho : M \to C \) be the density function of \( \mu^0 \). The rate of exponential decay of correlations of \((f_0, \mu^0) \) is denoted by \( \tau_0 \).

Finally, a Banach space \( K_P(\Omega, C^{r-1}(M)) \) of random observables as the Kolmogorov quotient (by equality \( P \)-almost everywhere) of the space is introduced
\[
(1.9) \quad K_P(\Omega, C^{r-1}(M)) = \left\{ u \in L^\infty_P(\Omega, C^{r-1}(M)) : \omega \mapsto \int u(\omega) dm \text{ is constant } P\text{-a.s.} \right\}
\]
endowed with the \( L^\infty \) norm. \( L^\infty_P(\Omega, C^{r-1}(M)) \) is the space of all Bochner measurable mappings \( u : \Omega \to C^{r-1}(M) \) with finite \( L^\infty \) norm. (In Proposition 2.2 it shall be proved that \( K_P(\Omega, C^{r-1}(M)) \) is a Banach space and that \( \int u(\cdot) dm \) is measurable.) As before, a mapping in \( K_P(\Omega, C^{r-1}(M)) \) by its equivalence class in \( K_P(\Omega, C^{r-1}(M)) \) is identified.

The following theorem extends Theorems A, B and C in [4] to perturbations induced by skew-product mappings whose base dynamics satisfy \((1.4), (1.5)\) and \((1.6) \).

**Theorem 1.1.** Let \( f_0 : M \to M \) be an expanding map, and \( \{f_\varepsilon\}_{\varepsilon>0} \) be a family of continuous mappings on \( (\Omega, \mathcal{P}) \) with values in \( C^r(M, M) \) satisfying \((1.1) \). Suppose that \( \theta : \Omega \to \Omega \) is a measure-preserving transformation satisfying \((1.1) \), \((1.5) \) and \((1.6) \). Then, for any sufficiently small \( \varepsilon > 0 \), there exists a unique absolutely continuous invariant probability measure \( \mu^\varepsilon \) on \( \Omega \times M \) whose density function \( \rho_{\varepsilon} =$
\[
\frac{d\rho_\epsilon}{ dt(P\times m)}\] is in \(K_\Gamma(\Omega, C^{r-1}(M))\) in the notation \(\rho_\epsilon(\omega) = \rho_\epsilon(\omega, \cdot)\), and we have
\[
\text{ess sup}_{\omega \in \Omega} \| \rho_\epsilon(\omega) - \rho_0 \|_{C^{r-1}} \to 0 \quad \text{as } \epsilon \to 0.
\]

Moreover, for each sufficiently small \(\epsilon > 0\), the backward fiber correlation functions and the integrated correlation functions of \((f_\epsilon, \mu_\epsilon)\) decay exponentially fast with rate \(0 < \tau_\epsilon < 1\) in \(C^{r-1}(M)\) and in \(K_\Gamma(\Omega, C^{r-1}(M))\), respectively, and we have
\[
\lim_{\epsilon \to 0} \tau_\epsilon \leq \max\{\tau_0, A_\tau(f_0)\}.
\]

**Remark 1.2.** Bogenschütz [6] and Baladi [2] also investigated stability problems of expanding maps using perturbative spectral approaches. Apart from the invertibility of the base dynamics, Theorem 1.1 differs from Bogenschütz’s result in which he postulated a perturbation lemma for linear cocycles. Therefore, in his result, the “coefficient” \(C\) in (1.7) may depend on \(\omega\), and the integrated correlations may not decay exponentially fast, as demonstrated by Buzzi in [7, Appendix A]. Within the setting of mixing base dynamics, Baladi obtained a sharper spectral stability, which yields a more satisfactory result for the decay rate stability; compare [2, Theorem 5 and Proposition 3.1] and her Banach space \(B(\alpha)\) with Theorem 1.1 Proposition 2.3 and \(K_\Gamma(\Omega, C^{r-1}(M))\). However, the quasi-compactness of the transfer operator of the skew-product mapping in the Banach space \(B(\alpha)\) implies the mixing in the skew-product mapping (in particular, the mixing in the base dynamics). Thus, Baladi’s Banach space \(B(\alpha)\) is not applicable to setting used in this study, in which the base dynamics are not necessarily mixing.

**Remark 1.3.** It follows from Theorem 1.1 that if \((\theta, P)\) is ergodic, then \((\Theta_\epsilon, \mu_\epsilon)\) is ergodic for any sufficiently small \(\epsilon > 0\). Indeed, let \(A \in B(\Omega) \times B(M)\) be invariant under \(\Theta_\epsilon\), and suppose that \(0 < \mu_\epsilon(A) < 1\). Then, it follows from Theorem 1.1 and the invariance of \(A\) that for each \(B \in B(\Omega) \times B(M)\), if the length of \(B^\omega = \{x \in M : (x, \omega) \in B\}\) is \(P\)-almost surely constant (where the constant is denoted as \(\ell(B)\)), then
\[
(P \times m)(A \cap B) = \mu_\epsilon(A) \cdot (P \times m)(B).
\]

Let \(\Gamma_1 = \{\omega \in \Omega : m(A^\omega) = 0\}\). Then, noting that \(A^\omega = (f(\omega))^{-1} A^\theta\omega\) by the invariance of \(A\) and that \(f(\omega)\) is non-singular with respect to \(m\) for each \(\omega \in \Omega\), \(\theta^{-1}\Gamma_1 = \Gamma_1\). Since \((\theta, P)\) is ergodic and \(P(\Gamma_1) \neq 1\) (otherwise, \(\mu_\epsilon(A) = 0\) by the absolute continuity of \(\mu_\epsilon\), \(P(\Gamma_1) = 0\). On the other hand, \(\Gamma_2 = \{\omega \in \Omega : m(A^\omega) = 1\}\) is not a full measure set since \(\mu_\epsilon(A) < 1\). Thus, the set \(\Gamma_3 = \{\omega \in \Omega : 0 < m(A^\omega) < 1\}\) is a positive measure set, and we can find a positive measure set \(\Gamma \subset \Gamma_3\) and \(B_1, B_2 \in B(\Omega) \times B(M)\) such that \(m(B_1^\omega) = m(B_2^\omega)\) are \(P\)-almost surely constant, \(\ell(B_1) = \ell(B_2) \neq 0\), \(B_1^\omega \cap A^\omega = \emptyset\) and \(B_2^\omega \subset A^\omega\) for each \(\omega \in \Gamma\), and \(B_1^\omega = B_2^\omega\) for each \(\omega \in \Omega \setminus \Gamma\). Since these results contradict (1.10), \((\Theta_\epsilon, \mu_\epsilon)\) is ergodic.

**Example 1.4.** We consider examples of measure-preserving transformations satisfying conditions (1.4), (1.5), and (1.6). The most trivial example is a bimeasurable transformation. When \(\theta : \Omega \to \Omega\) is bimeasurable, \(\ell_\theta u(\omega) = u(\theta^{-1}\omega)\) for each \(u \in L_\theta^2(\Omega)\) and \(P\)-almost every \(\omega \in \Omega\) since \(u(\omega) = u(\theta(\theta^{-1}\omega))\). For each \(u \in L_\theta^2(\Omega, C^{r-1}(M))\), let us define \(\ell_\theta u : \Omega \to C^{r-1}(M)\) by \(\ell_\theta u = u \circ \theta^{-1}\). Then, \(\ell_\theta u\) is Bochner measurable since \(\ell_\theta u\) is the composition of the Bochner measurable mapping \(u : \Omega \to C^{r-1}(M)\) and the measurable mapping \(\theta^{-1} : \Omega \to \Omega\). It is
exists a Bochner measurable mapping \( \tilde{\ell} \in L^\infty_p(\Omega, C^{r-1}(M)) \) and that \( \tilde{\ell} \) is a bounded operator on \( L^\infty_p(\Omega, C^{r-1}(M)) \) satisfying (1.4), (1.5) and (1.6).

Now we consider a piecewise smooth mapping \( \theta : \Omega \to \Omega \) of class \( C^1 \) on a compact region \( \Omega \subset \mathbb{R}^d \), i.e., \( \Omega \) is the disjoint union of connected and open subsets \( \Gamma_1, \ldots, \Gamma_k \) up to a set of Lebesgue measures 0 such that \( \theta|_{\Gamma_j} \) agrees with a \( C^1 \) map \( \theta_j \) defined on a neighborhood of \( \Gamma_j \) and \( \theta_j \) is a diffeomorphism on the mapped image for each \( 1 \leq j \leq k \). For a detailed study of these mappings, the reader is referred to [10]. Let \( V \) be the normalized Lebesgue measure on \( \Omega \) and define the transfer operator \( \ell_{\theta,V} : L^1_V(\Omega) \to L^1_V(\Omega) \) of \( \theta \) with respect to \( V \) as

\[
\ell_{\theta,V} u = \sum_{j=1}^k \frac{1_{\Gamma_j} \cdot u}{|\det D\theta_j|} \circ \theta_j^{-1}, \quad u \in L^1_V(\Omega).
\]

From the change of variables formula, it follows that \( \int \ell_{\theta,V} u \cdot \varphi dV = \int u \cdot \varphi \circ \theta dV \) for each \( u, \varphi \in L^1_V(\Omega) \) satisfying \( u \cdot \varphi \circ \theta \in L^1_V(\Omega) \) (in particular, \( \varphi \in L^\infty_p(\Omega) \)). Thus, if \( P \) is an absolutely continuous invariant measure of \( \theta \), then the density function \( p \in L^1_V(\Omega) \) of \( P \) is a fixed point of \( \ell_{\theta,V} \). It is assumed that \( P \) is an absolutely continuous invariant probability measure whose density function \( p \) is strictly positive \( V \)-almost everywhere. Extensive examples of such measure-preserving transformations \((\theta, P)\) are given in [3]. Then, for each \( u \in L^\infty_p(\Omega) \) and \( \varphi \in L^1_p(\Omega) \), we have

\[
\int u \cdot \varphi dP = \int \ell_{\theta,V}(u \cdot p) \cdot \varphi dV = \int \frac{\ell_{\theta,V}(u \cdot p)}{p} \cdot \varphi dP.
\]

Thus, for each \( u \in L^\infty_p(\Omega) \), \( \ell_{\theta} u = \ell_{\theta,V}(u \cdot p)/p \) \( P \)-almost surely. For each \( u \in L^\infty_p(\Omega, C^{r-1}(M)) \), a mapping \( \tilde{\ell}_{\theta} : \Omega \to C^{r-1}(M) \) is defined as \( \tilde{\ell}_{\theta} u = \sum_{j=1}^k (1_{\Gamma_j} \cdot u \cdot p \cdot |\det D\theta_j|^{-1}) \circ \theta_j^{-1}/p \). Since every subspace of a separable metric space \( C^{r-1}(M) \) is itself a separable space (see e.g. [13] Theorem 16.2.b and 16.11), the (weakly) measurable mappings \( 1_{\Gamma_j}, p, |\det D\theta_j|^{-1} \) \((1 \leq j \leq k)\), and therefore \( \tilde{\ell}_{\theta} u \), are Bochner measurable by the Pettis measurability theorem. Note that \( ||\tilde{\ell}_{\theta} u(\omega)||_{C^{r-1}} \leq ||u||_{L^\infty_p} ||\ell_{\theta} 1_{\Omega} \omega|| \) \( P \)-almost surely, since all of \( 1_{\Gamma_j}, p, |\det D\theta_j|^{-1} \) \((1 \leq j \leq k)\) are independent of \( x \). It follows from this and the fact \( \ell_{\theta} 1_{\Omega} = 1_{\Omega} \) (note that \( \ell_{\theta,V} p = p \)) that \( \tilde{\ell}_{\theta} \) is a bounded operator on \( L^\infty_p(\Omega, C^{r-1}(M)) \) satisfying (1.6). It is straightforward to check by construction that \( \tilde{\ell}_{\theta} \) satisfies (1.4) and (1.5).

Finally, the one-sided shift \( \theta : \Omega \to \Omega \) is considered: \((\Omega, P) = (\Omega^N, P^N)\) is the product space of a probability separable metric space \((\Omega, P)\), in which \((\theta \omega)_j = \omega_{j+1}\) for each \( j \in \mathbb{N} = \{0, 1, \ldots\} \) and each \( \omega = (\omega_0, \omega_1, \ldots) \in \Omega \). We note that for each \( u \in L^\infty_p(\Omega) \) and \( \varphi \in L^1_p(\Omega) \),

\[
\int \left( \int u(\tilde{\omega}) d\tilde{P}(\tilde{\omega}) \right) \cdot \varphi(\omega) dP = \int u(\tilde{\omega}) \varphi(\theta(\omega_0, \omega_1, \ldots)) d\tilde{P}(\tilde{\omega}) dP(\omega).
\]

Thus, \( \ell_{\theta}(\omega) = \int u(\tilde{\omega}) d\tilde{P}(\tilde{\omega}) \) \( P \)-almost every \( \omega \in \Omega \). By Fubini’s theorem (consider the equivalence between the weak measurability and the Bochner measurability of a mapping \( u : \Omega \to C^{r-1}(M) \), for any \( u \in L^\infty_p(\Omega, C^{r-1}(M)) \), there exists a Bochner measurable mapping \( \tilde{\ell}_{\theta} u : \Omega \to C^{r-1}(M) \) given by

\[
\tilde{\ell}_{\theta} u(\omega) = \int u(\tilde{\omega}) d\tilde{P}(\tilde{\omega}), \quad \omega \in \Omega.
\]
Furthermore, (1.6) for this bounded operator $\hat{\theta}$ on $L^p_\mathcal{P}(\Omega, C^{r-1}(M))$ follows from the Bochner integrability of $\hat{\theta} \ni \omega \mapsto u(\hat{\omega})$ for $P$-almost every $\omega \in \Omega$ (by Fubini’s theorem) and the triangle inequality. (1.4) and (1.5) are immediately obtained by construction.

2. The proof

The proof is started by analyzing the spectrum of ”the graph transformation” induced by the transfer operators of the fiber dynamics $f_\omega(\omega)$, which is exactly the transfer operator of the skew-product mapping $\Theta$ with respect to $P \times m$. Given a $C^r$ expanding mapping $f : M \to M$, the transfer operator $L(f) : C^{r-1}(M) \to C^{r-1}(M)$ is defined as

$$L(f)u(x) = \sum_{f(y) = x} \frac{u(y)}{\det Df(y)}, \quad x \in M$$

for each $u \in C^{r-1}(M)$. As is well known, for each $u \in C^{r-1}(M)$ and $\varphi \in L^m_n(M)$, the change of variables formula yields

$$(2.1) \quad \int \varphi \cdot L(f)udm = \int \varphi \circ f \cdot udm.$$ 

It is remarked that $C^r(M, M) \ni f \mapsto L(f)$ is generally not continuous in the norm topology. However, this quantity is continuous in the strong operator topology, as shown below.

**Lemma 2.1.** There exists a $C^r$ neighborhood $\mathcal{N}(f_0)$ of $f_0$ such that for each $u \in C^{r-1}(M)$, the map $f \mapsto L(f)u$ is a continuous map from $\mathcal{N}(f_0)$ to $C^{r-1}(M)$.

**Proof.** To prove this lemma, the argument in [4, Lemma A.1] is adopted. Let $\mathcal{N}(f_0)$ be a small $C^r$ neighborhood of $f_0$ so that any $f \in \mathcal{N}(f_0)$ is an expanding map. We recall that all orbits of $f \in \mathcal{N}(f_0)$ are strongly shadowable: if $\hat{f}$ is in a $\epsilon$-neighborhood of $f$ where $\epsilon > 0$ is sufficiently small, then for a fixed $x \in M$, there is a natural bijection between the sets $\{y | f(y) = x\}$ and $\{\hat{y} | \hat{f}(\hat{y}) = x\}$ such that the distance between paired points is at most $O(\epsilon)$. Given an integer $0 \leq j \leq k$, a straightforward calculation shows that the $j$-th derivative of $L(f)u$ takes the following form:

$$(2.2) \quad (L(f)u)^{(j)}(x) = \sum_{n=1}^{N(j,f)} a_n(u, f; y),$$

where $N(j, f) \in \mathbb{N}$ and constant for all $f \in \mathcal{N}(f_0)$ (as can be seen by shrinking the neighborhood), and the terms of $a_n(u, f; \cdot) : M \to \mathbb{C}$ for each $1 \leq n \leq N(j, f)$ involve only the $m$-th derivative of $u$, $|\det Df|^{-1}$ and $Df^{-1} \circ f$ with $m \leq j$, abusing the notation of the inverse branch of $f$ by $f^{-1}$. Hence, for each $f \in \mathcal{N}(f_0)$, $1 \leq j \leq k$ (particularly for $j = k$) and $1 \leq n \leq N(j, f)$, the $j$-th derivative of $a_n(u, f; \cdot)$ is $\gamma$-Hölder, and the $\gamma$-Höder coefficient of the $j$-th derivative of $a_n(u, f; \cdot) - a_n(u, \hat{f}; \cdot)$ is bounded by $\delta_{j,f}(u, \|u\|_{C^{r-1}})$, where $\delta_{j,f}(u, \hat{f})$ is a positive number that tends to zero as $f$ converges to $\hat{f}$ in the $C^r$-topology. The conclusion immediately follows. \hfill \Box

For simplicity, it is written as $L(\epsilon; \omega)$ and $L_n(\epsilon; \omega)$ for $L(f_\omega(\omega))$ and $L(f_\omega^{(n)}(\omega))$, respectively, where $n \geq 1$, $\epsilon > 0$ is sufficiently small and $\omega \in \Omega$. For each $u \in \mathcal{N}$,
for each \( u \in C^{r-1}(M) \). Hence, it follows from (2.2) and the estimate of \( a \in (1.4) \) above), and that for each \( \omega \in \Omega \), the mapping \( \Omega \ni \omega \mapsto \alpha(\omega, \omega') \) is measurable since \( L(\epsilon; \omega) \) is continuous. Hence, by [8, Lemma 3.14], \( \alpha : \Omega \times \Omega \to C^{r-1}(M) \) and \( \tilde{L}_u \) are both measurable. Moreover, reiterating the argument in Example 1.4 on Bochner measurability, it is deduced that \( \tilde{L}_u : \Omega \to C^{r-1}(M) \) is a Bochner measurable mapping.

Now, the weak Lasota-Yorke inequality for expanding maps (see e.g. [14, Lemma 4.2]) is adopted: that is, for each \( C^r \) expanding map \( f : M \to M \), there exists a constant \( C_f > 0 \) such that

\[
\|L(f)u\|_{C^{r-1}} \leq C_f \|u\|_{C^{r-1}}
\]

for each \( u \in C^{r-1}(M) \). Hence, it follows from (2.2) and the estimate of \( a_n(u, f, \cdot) - a_n(u, f, \cdot) \) in Lemma 2.1 that if \( \epsilon > 0 \) is sufficiently small, then we have

\[
\|\tilde{L}_u(\omega)\|_{C^{r-1}} \leq (C_{f_0} + \delta(\epsilon(\omega), f_0))\|u(\omega)\|_{C^{r-1}}, \quad \text{P.a.s.,}
\]

for each \( u \in L^\infty_p(\Omega, C^{r-1}(M)) \), where the notation \( \delta(\cdot, \cdot) \) adopted in the proof of Proposition 2.1 is used, i.e., \( \tilde{L}_u \) is a bounded operator on \( L^\infty_p(\Omega, C^{r-1}(M)) \).

Recalling that \( \tilde{L}_0 \) is a bounded operator on \( L^\infty_p(\Omega, C^{r-1}(M)) \), we can define a bounded operator \( L_\epsilon : L^\infty_p(\Omega, C^{r-1}(M)) \to L^\infty_p(\Omega, C^{r-1}(M)) \) by

\[
L_\epsilon = \tilde{L}_0 \tilde{L}_\epsilon.
\]

\( \tilde{L}_\epsilon \) is "the transfer operator" of the skew-product mapping \( \Theta_\epsilon \) with respect to \( P \times m \). Indeed, if \( u \in L^\infty_p(\Omega, C^{r-1}(M)) \), then \( \Omega \times M \ni (\omega, x) \mapsto u(\omega, x) \) is measurable using the notation \( u(\omega, \cdot) = u(\omega) \) by virtue of [8, Lemma 3.14] together with the fact that \( \omega \mapsto (\omega, x) \) is measurable for each \( x \in M \) (recall the argument in 1.4 above), and that \( x \mapsto u(\omega, x) \) is continuous for each \( \omega \in \Omega \). Hence, for each \( u \in L^\infty_p(\Omega, C^{r-1}(M)) \) and \( \varphi \in L^1_p(\Omega \times M) \), applying (1.3), (1.4) and (2.1) together with Fubini’s theorem, we have

\[
\int \varphi \cdot L_\epsilon u dm dP = \int \left( \int \varphi(\omega, x) \cdot \tilde{L}_\epsilon u(\cdot, x)(\omega) dm \right) dP
\]

\[
= \int \left( \int \varphi(\theta\omega, x) \cdot \tilde{L}_\epsilon u(\omega, x) dm \right) dP = \int \varphi(\theta\omega, f(\omega, x)) \cdot u(\omega, x) dm dP.
\]

Moreover, for each \( n \geq 1 \) and \( u \in L^\infty_p(\Omega, C^{r-1}(M)) \), applying \( n \) iterations of (2.4) together with (1.2), (1.3), (1.4), (2.1), and Fubini’s theorem, we have

\[
L^n_\epsilon u(\omega) = \tilde{L}_0^n [L_n(\epsilon, \cdot)u(\cdot)](\omega) \quad \text{P.a.s.}
\]

The following proposition is not difficult to prove but is important.

**Proposition 2.2.** For any \( \epsilon > 0 \), \( L_\epsilon \) preserves a Banach space \( K_P(\Omega, C^{r-1}(M)) \) given in (1.9).
Proof: It is first shown that $K_P(\Omega, C^{-1}(M))$ is a Banach space. For each $u \in L_P^\infty(\Omega, C^{-1}(M))$, $I(u; \cdot) : \Omega \to \mathbb{C}$ is defined as $I(u; \omega) = \int u(\omega)dm$. As discussed above \[2.4\], $\Omega \times M \ni (\omega, x) \mapsto u(\omega, x)$ is measurable, and $I(u; \cdot) : \Omega \to \mathbb{C}$ is measurable by Fubini’s theorem. If $u \in K_P(\Omega, C^{-1}(M))$, then $I(u; \cdot)$ is $P$-almost surely constant. The constant is denoted by $\tilde{I}(u)$. Since the space $L_P^\infty(\Omega, C^{-1}(M))$ is complete, a Cauchy sequence $\{\tilde{u}_n\}_{n \geq 1} \subset K_P(\Omega, C^{-1}(M))$ has a limit $\tilde{u}$ of $\{\tilde{u}_n\}$ in $L_P^\infty(\Omega, C^{-1}(M))$ with respect to the norm $\| \cdot \|_{L^\infty}$. Hence, it suffices to show that $I(\tilde{u}; \cdot)$ is $P$-almost surely constant. We define $\Gamma = \cup_{n \geq 1} \Gamma_n$ with zero measure sets $\Gamma_n = \{ \omega : I(\tilde{u}_n; \omega) \neq \tilde{I}(u_n) \}$. Then, it is easily seen that $P(\Gamma) = 0$, and $I(\tilde{u}_n; \omega) = \tilde{I}(u_n)$ for all $\omega \in \Omega \setminus \Gamma$ and $n \geq 1$. We also note that for each $u, v \in L_P^\infty(\Omega, C^{-1}(M))$, $|I(u; \omega) - I(v; \omega)| \leq \|u(\omega) - v(\omega)\|_{C^{r-1}(M)} \leq \|u - v\|_{L^\infty}$ for $P$-almost every $\omega \in \Omega$. Thus, $I(\tilde{u}_n; \cdot)$ $P$-almost surely converges to $I(\tilde{u}; \cdot)$, and $I(\tilde{u}_n)$ converges to a number $\tilde{I}$, and therefore $I(\tilde{u}; \omega) = \tilde{I}$ for $P$-almost every $\omega$ in the full measure set $\Omega \setminus \Gamma$.

Next, it is shown that $L_\epsilon$ preserves $K_P(\Omega, C^{-1}(M))$. By \[2.4\], for each $u \in L_P^\infty(\Omega, C^{-1}(M))$,

\[2.6\]
$I(\tilde{L}_\epsilon u; \omega) = \int L(\epsilon; \omega)u(\omega)dm = \int u(\omega) \cdot 1_M \circ f(\omega)dm, \quad \omega \in \Omega,
$

which coincides with $I(u; \omega)$ since $1_M \circ f = 1_M$ for any mapping $f : M \to M$ on $M$. That is, $L_\epsilon u \in K_P(\Omega, C^{-1}(M))$ for each $u \in K_P(\Omega, C^{-1}(M))$. If $u \in L_P^\infty(\Omega, C^{-1}(M))$, then $\Omega \times M \ni (\omega, x) \mapsto u(\omega, x)$ is measurable as discussed above. Hence, it follows from \[1.3\], \[1.4\] and Fubini’s theorem that for each $\varphi \in L_P^\infty(\Omega)$

\[
\int \varphi(\omega) \cdot I(\tilde{\theta}_\epsilon u; \omega)dP = \int \varphi(\omega) \cdot \tilde{\theta}_\epsilon u dm dP = \int \varphi(\theta(\omega)) \cdot u(\omega, x)dP,\]

which, again by Fubini’s theorem, coincides with $\int \varphi(\theta(\omega)) \cdot I(u; \omega)dP$. Specifying $u \in K_P(\Omega, C^{-1}(M))$, it is written as $\int \varphi(\theta(\omega)) \cdot I(u; \omega)dP = \int \varphi(\omega) \cdot I(u)dP$ since $P$ is an invariant measure. Thus, $\tilde{\theta}_\epsilon u$ is also in $K_P(\Omega, C^{-1}(M))$, and

\[2.7\]
$I(\tilde{\theta}_\epsilon u) = I(u)$.

It immediately follows from this demonstration and \[2.4\] that $L_\epsilon u = \tilde{\theta}_\epsilon \tilde{L}_\epsilon u$ is in $K_P(\Omega, C^{-1}(M))$, and the conclusion is obtained. \[\Box\]

The spectrum of $L_\epsilon$ is now analyzed by showing that the operator $L_\epsilon$ closely matches the operator $L(f_0)$. However, $L_\epsilon$ and $L(f_0)$ are not directly relatable because the two operators act on different spaces. To obtain a meaningful comparison, the transfer operator of the skew-product mapping $\Theta_0 : (\omega, x) \mapsto (\theta(\omega), f_0(x))$ is considered. A bounded operator $L_0 : L_P^\infty(\Omega, C^{-1}(M)) \to L_P^\infty(\Omega, C^{-1}(M))$ is defined as $L_0 = \tilde{\theta}_0 \tilde{L}_0$, where\[L_0 u(\omega) = L(f_0)u(\omega), \quad \omega \in \Omega\]

for each $u \in L_P^\infty(\Omega, C^{-1}(M))$. Hereafter, $L(f_0)$ is expressed in the simplified form $L_0$. Note that Proposition \[2.2, 2.4, 2.6\] and \[2.2\] with $\Theta_0$, $\tilde{L}_0$ and $L_\epsilon$ replaced by $\Theta_0$, $\tilde{L}_0$ and $L_0$ hold by the arguments used to develop the proof of Proposition \[2.2\] and the respective equations.
The following proposition is essential for proving Theorem 1.1. Let $\sigma(A)$ be the spectrum of a bounded operator $A : E \to E$ on a Banach space $E$. In particular, the spectrum of $L_0 : K_P(\Omega, C^{r-1}(M)) \to K_P(\Omega, C^{r-1}(M))$ and $L_0 : C^{r-1}(M) \to C^{r-1}(M)$ is denoted as $\sigma(L_0)$ and $\sigma(L_0)$, respectively.

**Proposition 2.3.** $L_0$ is quasicompact on $K_P(\Omega, C^{r-1}(M))$ with spectral radius 1, and its spectrum with absolute value 1 consists only of a simple eigenvalue 1. Moreover, $\Pi_0$ is the projection into the eigenspace of $L_0$ belonging to the eigenvalue 1.

Let $\rho_0 := \Pi_01_M$. Then $\rho_0 \neq 0$. Indeed, by (2.1),

$$\int \rho_0 dm = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \int 1_M \circ f_0^k dm = 1.$$  

As is well known, 1 is the simple eigenvalue of the transfer operator $L(f)$ on $C^{r-1}(M)$ for each $C^r$ expanding map $f : M \to M$ (see [4, Section 2]). It therefore follows from (2.9) that $\rho_0$ is the unique eigenfunction of $L_0$ up to a constant belonging to the eigenvalue 1.

Given $u \in L_P^\infty(\Omega, C^{r-1}(M))$, a measurable mapping $\Pi_0 u : \Omega \to C^{r-1}(M)$ is defined as

$$\Pi_0 u(\omega) = \rho_0 \int u(\omega) dm, \quad \omega \in \Omega.$$  

(The measurability of $\Pi_0 u$ follows from the proof of Proposition 2.3) It follows from $||\Pi_0 u||_{L^\infty} \leq ||\rho_0||_{C^{r-1}} ||u||_{L^\infty}$ that $\Pi_0$ is a bounded operator on $L_P^\infty(\Omega, C^{r-1}(M))$. Moreover, $\Pi_0$ is the projection into the eigenspace of $L_0$ restricted on the Banach space $K_P(\Omega, C^{r-1}(M))$ belonging to the eigenvalue 1: (2.10) yields $\Pi_0 \Pi_0 = \Pi_0$, and it follows from (2.9) that for each $u \in K_P(\Omega, C^{r-1}(M))$ and $P$-almost every $\omega \in \Omega$,

$$L_0 \Pi_0 u(\omega) = \tilde{\ell}_\theta |L_0 \rho_0 \cdot I(u, \cdot)||(\omega) = \rho_0 \tilde{I}(u) = \Pi_0 u(\omega),$$

where the notations $I(u, \cdot)$ and $\tilde{I}(u)$ adopted in the proof of Proposition 2.2 are used. Another projector is now defined as $\Pi_1 := \text{Id} - \Pi_0$, and decompose $L_0$ into $K = L_0 \Pi_0$ and $R = L_0 \Pi_1$. Since $\Pi_0 K_P(\Omega, C^{r-1}(M)) \subseteq \mathbb{C} \rho_0$, $K$ is a compact operator on $K_P(\Omega, C^{r-1}(M))$. Furthermore, by virtue of (2.1), (2.7), and (2.11),

$$L_0 \Pi_0 u(\omega) = \rho_0 \tilde{I}(\tilde{\ell}_\theta u) = \rho_0 \int \tilde{\ell}_\theta u(\omega) dm = \rho_0 \int L_0 \tilde{\ell}_\theta u(\omega) dm, \quad P\text{-a.s.}$$
for each \( u \in K_P(\Omega, C^{r-1}(M)) \) is obtained. Thus, by (1.3), \( L_0 \Pi_j = \Pi_j L_0 \) is obtained, where \( j = 0, 1 \). In particular, we get for each \( n \geq 1 \),
\[
(2.12) \quad \mathcal{R}^n = L_0^n \Pi_1. 
\]

Similarly, let us define bounded operators \( \pi_0, \pi_1 \) on \( C^{r-1}(M) \) as
\[
\pi_0 u = \rho_0 \int u dm, \quad \pi_1 u = u - \pi_0 u, \quad u \in C^{r-1}(M). 
\]

Then, it is straightforward to check that \( \pi_0, \pi_1 \) are projections, and that \( \pi_0 C^{r-1}(M) \)

is the one-dimensional eigenspace of \( L_0 \) belonging to the eigenvalue 1. In other words, \( \pi_0 \) coincides with \( \bar{\pi}_0 \). Now, \( L_0 \) is decomposed into a compact operator \( K = L_0 \pi_0 \) and a bounded operator \( R = L_0 \pi_1 \). By the approach used to demonstrate (2.12), it can be observed that \( L_0 \) preserves \( \pi_1 C^{r-1}(M) \). We recall that the transfer operator \( L(f) : C^{r-1}(M) \to C^{r-1}(M) \) of a \( C^r \) expanding map \( f : M \to M \) is quasi-compact with spectral radius 1, and its spectrum with absolute value 1 solely consists of the simple eigenvalue 1 (see [4, Section 2]). Therefore, \( \bar{\pi}_0 < 1 \), and there exists a constant \( C > 0 \) such that for any \( u \in C^{r-1}(M) \) and \( n \geq 1 \),
\[
\| L_0^n \pi_1 u \|_{C^{r-1}(M)} \leq C \bar{\pi}_0^n \| u \|_{C^{r-1}(M)}.
\]

It follows from (1.1), (2.5), and (2.12) that for any \( u \in K_P(\Omega, C^{r-1}(M)) \), \( n \geq 1 \), and \( P \)-almost every \( \omega \in \Omega \),
\[
\| \mathcal{R}^n u(\omega) \|_{C^{r-1}} = \| \rho_0 (L_0^n [\Pi_1 u(\cdot)])(\omega) \|_{C^{r-1}} \leq \| L_0^n \pi_1 [ u(\omega) ] \|_{C^{r-1}} \leq C \bar{\pi}_0^n \| u(\omega) \|_{C^{r-1}},
\]
i.e., the spectral radius of \( \mathcal{R} \) is bounded by \( \bar{\pi}_0 \). The conclusion follows from a straightforward check that the spectral radius of \( L_0 \) is 1.

Now, the Baladi-Young perturbation lemmas can be applied to families of linear operators. The relevant lemmas are Lemmas 1, 2, 3, and the comment below Lemma 1 in [4]. Let \( X = K_P(\Omega, C^{r-1}(M)) \), \( T_0 = L_0 \), \( T_\epsilon = L_\epsilon \), \( X_0 = \Pi_0 K_P(\Omega, C^{r-1}(M)) \), \( X_1 = \Pi_1 K_P(\Omega, C^{r-1}(M)) \), \( \kappa_0 = 1 \), \( \kappa_1 = \max\{\bar{\pi}_0, \Lambda_\epsilon(f_0)\} \). \( \kappa \) is arbitrarily close to (and slightly bigger than) \( \kappa_1 \), and \( \Pi_0 \) and \( \Pi_1 \) are the projections given in the proof of Proposition (2.3). Indeed, it is straightforward to verify that hypotheses (A.1) and (A.3) in the lemmas are satisfied by Proposition (2.2) and (2.3) and that hypothesis (A.2) follows from [4] Lemma A.1. From the Baladi-Young perturbation lemmas, it follows that there exists a family of decompositions \( K_P(\Omega, C^{r-1}(M)) = X_0 \oplus X_1 \), \( \epsilon > 0 \), in which the projections \( \Pi_0 : X_0 \oplus X_1 \to X_0 \) satisfy
\[
(2.13) \quad \| \Pi_0 - \Pi'_0 \|_{L_\infty} \to 0 \quad \text{as} \ \epsilon \to 0,
\]
and
\[
\sigma(L_\epsilon|_{X_0}) \to \sigma(L_0|_{X_0}) \quad \text{as} \ \epsilon \to 0,
\]
in terms of the Hausdorff distance and using the notation \( \bar{\epsilon}_\epsilon = \sup\{|z| : z \in \sigma(L_\epsilon|_{X_1})\} \), we have
\[
(2.14) \quad \lim_{\epsilon \to \infty} \bar{\epsilon}_\epsilon \leq \kappa_1.
\]

Let \( \lambda_\epsilon \in \sigma(L_\epsilon|_{X_0}) \) be the simple eigenvalue that converges to 1, and let \( \rho_\epsilon := \Pi_0|_{\Omega \times M} \). It will now be shown that \( \lambda_\epsilon = 1 \) for any sufficiently small \( \epsilon > 0 \). For
measure with the density function $\rho(2.13)$, which demonstrates that $\mu$ invariant probability measure on $0$ and $-\text{almost every } P$. Therefore, by (2.11) and (2.7), $P$-almost surely we have

$$\int \rho_\epsilon(\omega)dm = \int \rho_0dm - \int(\rho_0 - \rho_\epsilon(\omega))dm \geq \int \rho_0dm - \|\rho_\epsilon - \rho_0\|_{L^n} is obtained.
$$

From (2.10) and (2.13), it follows that $\int \rho_\epsilon(\omega)dm > 0$ for any sufficiently small $\epsilon > 0$ and $P$-almost every $\omega \in \Omega$.

On the other hand, $\tilde{\lambda}_\epsilon \rho_\epsilon(\omega, x) = \tilde{\ell}_\theta \tilde{\mathcal{L}}_\epsilon \rho_\epsilon(\omega, x)$ for each $x \in M$ and $P$-almost every $\omega \in \Omega$. Therefore, by (2.11) and (2.7), $P$-almost surely we have

$$\int \rho_\epsilon(\omega)dm = \tilde{\lambda}_\epsilon^{-1} \tilde{1}(\tilde{\ell}_\theta \tilde{\mathcal{L}}_\epsilon \rho_\epsilon) = \tilde{\lambda}_\epsilon^{-1} \tilde{1}(\tilde{\mathcal{L}}_\epsilon \rho_\epsilon) = \tilde{\lambda}_\epsilon^{-1} \int \rho_\epsilon(\omega) \cdot 1_M \circ f_\epsilon(\omega)dm,$$

which coincides with $\tilde{\lambda}_\epsilon^{-1} \int \rho_\epsilon(\omega)dm$. This implies that $\tilde{\lambda}_\epsilon = 1$ for any sufficiently small $\epsilon > 0$.

A measure $\mu^\epsilon$ on $\Omega \times M$ is defined as $\mu^\epsilon(dw, dx) = \rho_\epsilon(\omega, x)(P \times m)(dw, dx)$. By virtue of (2.4) and noting that $\tilde{\lambda}_\epsilon = 1$, $\mu^\epsilon$ is invariant with respect to $\Theta_\epsilon$. Furthermore, it follows from Proposition 2.3 that $\mathcal{L}_\epsilon$ is quasi-compact on $K_P(\Omega, C^{\epsilon^{-1}}(M))$ with spectral radius 1, and that its spectrum with absolute value 1 solely consists of the simple eigenvalue 1 for each small $\epsilon > 0$. This implies that when the essential spectral radius of $\mathcal{L}_\epsilon$ is denoted by $\hat{\kappa}_\epsilon$, the following inequality holds for any $n \geq 1$ and $u \in K_P(\Omega, C^{\epsilon^{-1}}(M))$

$$\|\mathcal{L}_\epsilon^n u\|_{L^n} \leq \|\Pi_0^n u\|_{L^n} + O(\hat{\kappa}_\epsilon^n)\|\Pi_1 u\|_{L^n}.$$

This inequality is bounded by $C\|u\|_{L^n}$, where the constant $C > 0$ is independent of $u$ and $n$. Hence, we can define a bounded operator $\overline{\Pi}_0$ on $K_P(\Omega, C^{\epsilon^{-1}}(M))$ of the form

$$\overline{\Pi}_0 u = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^n \mathcal{L}_\epsilon^k u, \quad u \in K_P(\Omega, C^{\epsilon^{-1}}(M)).$$

As in the proof of Proposition 2.3 it can be verified that $\overline{\Pi}_0$ coincides with the eigenprojection $\Pi_0 : K_P(\Omega, C^{\epsilon^{-1}}(M)) \rightarrow X_0$. Thus, $\mu^\epsilon$ is a probability measure on $\Omega \times M$ (in particular, the disintegration $\mu_\omega^\epsilon$ of $\mu^\epsilon$ is $P$-almost surely a probability measure on $M$): by (2.4),

$$\mu^\epsilon(\Omega \times M) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^n \int \mathcal{L}_\epsilon^k 1_{\Omega \times M} \cdot 1_{\Omega \times M} dm dP = 1.$$

Hence, recalling that $\pi_0^{-1} \Gamma = \Gamma \times M$ for each $\Gamma \in \mathcal{F}$, we have

$$\mu^\epsilon(\pi_0^{-1} \Gamma) = \int_{\Gamma} \mu_\omega^\epsilon(M) dP = P(\Gamma),$$

which demonstrates that $\mu^\epsilon$ is a unique absolutely continuous invariant probability measure with the density function $\rho_\epsilon$ in $K_P(\Omega, C^{\epsilon^{-1}}(M))$. Furthermore, from (2.13), $\rho_\epsilon$ converges to the density function $\rho_0$ of the absolutely continuous ergodic invariant probability measure $\mu^0$ of $f_0$ with respect to the norm $\|\cdot\|_{L^n}$.

Since the eigenprojection $\Pi_0 : K_P(P, E) \rightarrow X_0 \cong \mathbb{C}\rho_\epsilon$ is unique, it can be easily confirmed that $\overline{\Pi}_0$ coincides with a bounded operator $\overline{\Pi}_0$ on $K_P(\Omega, C^{\epsilon^{-1}}(M))$ given by

$$\overline{\Pi}_0 u(\omega) = \rho_\epsilon(\omega) \int u(\omega) dm, \quad \omega \in \Omega.$$
(which $P$-almost surely coincides with $\rho_0 \tilde{I}(u)$ for each $u \in K_P(\Omega, C^{r-1}(M))$. From the argument used to prove Proposition 2.3 it can also be verified that $\mathcal{L}_\tau$ preserves $\Pi_1^0 K_P(\Omega, C^{r-1}(M)) = (\mathbb{I} - \Pi_0^0) K_P(\Omega, C^{r-1}(M))$. On the other hand, by (1.3), (1.4), (2.1), and (2.3), for each $\varphi \in L^1_m(M)$ and $u \in C^{r-1}(M)$, we have a standard rewriting of the backward correlations:

$$
\ell^n_\theta \varphi(\omega, n) = \int \varphi \cdot \tilde{L}_\theta^n \ell_n(\cdot; \cdot) u(\omega) dm - \int \varphi \cdot \tilde{L}_\theta^n [\mathcal{L}_\tau^n \rho_\tau(\theta^n \cdot)](\omega) dm \int udP
$$

which coincides with $\int \varphi \cdot \mathcal{L}_\tau^n \Pi_1^0 u(\omega) dm$ for $P$-almost every $\omega \in \Omega$ by $\hat{\Pi}_0^0 = \Pi_0^0$.

(3.9) In the second equality, we use relation $\tilde{L}_\theta^n u(\theta^n) = u(\cdot)$, which holds for each $u \in L^\infty_0(\Omega, C^{r-1}(M))$ because $\int \varphi \cdot \tilde{L}_\theta^n u \circ \theta dP = \int (\varphi \cdot u) \circ \theta dP = \int \varphi \cdot udP$ for any $\varphi \in L^1_0(\Omega)$. Thus, $\ell^n_\theta \varphi(\omega, n)$ is bounded by $C \tau^n \|L_1\| \|u\|_{C^{r-1}}$ for $P$-almost every $\omega \in \Omega$, where $C > 0$ is a constant independent of $\omega$ and $n$. Similarly, for each $\varphi \in L^1_m(\Omega \times M)$ and $u \in K_P(\Omega, C^{r-1}(M))$,

$$
\left| \int C_{\varphi(\theta^n \cdot), u(\cdot)}(\cdot, n) dP \right| = \left| \int \varphi \cdot \mathcal{L}_\tau^n udP - \int \varphi \cdot \mathcal{L}_\tau^n \rho_\tau \cdot \tilde{I}(u) dP \right|
$$

$$
= \left| \int \varphi \cdot \mathcal{L}_\tau^n \Pi_1^0 udP \right| \leq C \tau^n \|L_1\| \|u\|_{C^{r-1}}
$$

where the constant $C > 0$ is independent of $\omega$ and $n$. Moreover, it is straightforward to see that $\bar{\tau}_0$ and $\bar{\tau}_r$ equal the rate $\tau_0$ of the exponential decay of correlations of $(f_0, \mu^0)$ and the rate $\tau_r$ of exponential decay of integrated/backward fiber correlations of $(f_r, \mu^r)$, respectively (see e.g. [3, Remark 2.3]). Finally, $\lim_{r \to 0} \bar{\tau}_r \leq \kappa_1 = \max\{\tau_0, \Lambda_r(f_0)\}$ by (2.13), and we complete the proof of Theorem 1.3.

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Graduate School of Human and Environmental Studies, Kyoto University Yoshida Nihonmatsu-cho, Sakyo-ku, Kyoto, 606-8501, Japan

E-mail address: nakano.yushi.88m@st.kyoto-u.ac.jp