What makes it possible to learn probability distributions in the natural world?

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In statistical mechanics we routinely analyze the joint probability distribution for very large numbers of variables; in field theory this number is infinite, at least formally \[ N \approx 2 \text{.} \] There is considerable interest in giving a similar probabilistic description outside the traditional domains of physics, spurred in part by the availability of “big data” from a wider variety of complex systems. But the models we consider in most physics problems are highly constrained; without these constraints we must learn the underlying distribution from the data. If what we observe are discrete events, then the probability distribution is a list of numbers, one for each possible outcome, and this number is beyond astronomical: in an image with just \( N = 100 \) pixels, where each pixel can be black or white, the number of possible images (\( 2^N \sim 10^{30} \)) is larger than the age of the universe in seconds. Under these conditions it is physically impossible to “measure” the underlying probability distribution from data alone, and it will continue to be impossible no matter how our technology evolves \[ \sim N \times 2^N \times \text{bits} \text{.} \]

The conventional approach is to hypothesize a family of simplified models. A classic example is the Ising model with pairwise interactions among spins (binary variables); with \( N \) spins these models have \( \sim N^2 \) parameters, many fewer than the \( \sim 2^N \) required to describe an arbitrary distribution. The availability of larger data sets, along with the computational power to analyze these data, has led to exploration of much more complex models, such as deep neural networks \[ \sim N \times 2^N \text{ bits} \text{.} \] The number of parameters in these models is much larger than once thought practical, but still vastly smaller than would be needed to describe an arbitrary probability distribution. While one can sometimes make useful predictions without knowing the full distribution, learning this distribution is almost by definition a crucial step in constructing a statistical mechanics of complex systems.

We would like to go beyond the exploration of particular models to have more general criteria for the learnability of distributions, in the spirit of theories for the learnability of functions or rules \[ \sim \text{bits} \text{.} \] To be concrete, suppose that what we observe is a collection of \( N \) binary variables, \( \sigma \equiv \{ \sigma_1, \sigma_2, \ldots, \sigma_N \} \), so that there are \( 2^N \) possible states, and in general learning the distribution would require many more than \( 2^N \) observations. Can we state conditions on the distribution that are sufficient to guarantee effective learning from a much smaller number of examples, perhaps linear or polynomial in \( N \)? Importantly, are these conditions satisfied in natural data?

Here we suggest that two separate conditions are sufficient for learnability: the consistent sub–extensivity of the mutual information between parts of a system, and the compressibility of interactions into an efficient representation. Both of these connect to our intuitions about the equilibrium statistical physics of systems with local interactions, as well as related results regarding quantum systems \[ \sim \text{bits} \text{.} \] We check for sub–extensivity of mutual information in natural data \[ \sim \text{bits} \text{.} \] and compressibility of interactions in models from statistical physics.

If we knew that the \( N \) binary variables could be broken into two independent halves, then the full probability distribution could be written in terms of \( 2 \times 2^N \) parameters, vastly less than \( 2^N \). More generally, imagine that we can place a bound on the mutual information between the two halves, \( I_{1/2}(N) \). If this information is unconstrained, then we need \( \sim 2^N \) parameters to describe the system, while if \( I_{1/2}(N) \rightarrow 0 \) then we can use only \( 2 \times 2^N \) and still give an exact description. It seems plausible that if \( I_{1/2}(N) \) is sufficiently small, there should be a good approximation that has roughly \( 2 \times 2^N \) parameters.

Let’s call the two halves of our system right and left,

\[
\sigma_R \equiv \{ \sigma_1, \sigma_2, \ldots, \sigma_{N/2} \} \quad (1)
\]

\[
\sigma_L \equiv \{ \sigma_{N/2+1}, \sigma_{N/2+2}, \ldots, \sigma_N \} \quad (2)
\]

We recall that the shortest possible code which represents...
the states $\sigma$ is based on exact knowledge of the probability distribution, where each state $\sigma$ is represented by a code word of length $L(\sigma) \sim -\ln P(\sigma)$, so that the mean code length is the entropy of the distribution $H(\sigma)$.

Codes built from approximate models of the distribution will be longer, on average, by an amount $\langle \Delta L \rangle$ equal to the Kullback–Leibler divergence between the model and the true distribution,

$$\langle \Delta L \rangle = \sum_{\sigma} P(\sigma) \left( -\log P_{\text{approx}}(\sigma) - -\log P(\sigma) \right)$$

$$= \sum_{\sigma} P(\sigma) \log \left[ \frac{P(\sigma)}{P_{\text{approx}}(\sigma)} \right], \quad (3)$$

and this provides a measure of model quality. If our approximate model is the one in which the two halves of the system are independent,

$$P_{\text{approx}}(\sigma) = P_R(\sigma_R)P_L(\sigma_L) \quad (4)$$

then this coding cost becomes

$$\langle \Delta L \rangle = \sum_{\sigma} P(\sigma_R, \sigma_L) \log \left[ \frac{P(\sigma_R, \sigma_L)}{P_R(\sigma_R)P_L(\sigma_L)} \right] \quad (5)$$

$$= I_{1/2}(N), \quad (6)$$

the mutual information between the two halves.

If the variables $\{\sigma_l\}$ are arranged in real space such that there is a finite correlation length $\xi$, then the division into right and left halves can be taken literally, and the mutual information between the halves arises from correlations among spins within $\xi$ of the boundary. As a result the mutual information must be related to the area of the boundary, not the volume of the system, and hence is sub–extensive: if the system is of linear dimension $\ell$ in $d$ dimensions, we have $N \sim \ell^d$ and $I_{1/2} \sim \ell^{d-1}$, hence $I_{1/2}(N) = c N^\alpha$ with $\alpha = 1 - 1/d$.

We can ask more generally about systems which, when divided in half, exhibit a mutual information between the halves that behaves as $I_{1/2}(N) = c N^\alpha$ with $\alpha < 1$. Then the approximation of the system as two independent halves has a cost that per degree of freedom

$$\frac{\langle \Delta L \rangle}{N} = c N^{\alpha - 1}, \quad (7)$$

which vanishes as $N$ becomes large. Thus sub–extensive behavior of the mutual information is sufficient to insure that, for large systems, the reduction in number of parameters from $2^N$ down to $2 \times 2^{N/2}$ will result in a model that makes only small errors per degree of freedom.

We can now think about systems in which the mutual information is consistently sub–extensive, that is when we look at properly chosen pieces of the system with $n$ variables, and cut these pieces in half, we always find a mutual information between the halves $I_{1/2}(n) \leq c n^\alpha$. This means that we can keep cutting the variables in half, approximating the distribution as being composed of independent halves, and in the process we make errors that are small when measured as the cost of coding per degree of freedom.

If we make $b$ cuts, we have

$$\langle \Delta L \rangle = c N^\alpha + 2c \left( \frac{N}{2} \right)^\alpha + 4c \left( \frac{N}{4} \right)^\alpha + \cdots + 2^{b-1}c \left( \frac{N}{2^{b-1}} \right)^\alpha \quad (8)$$

$$= c N^\alpha \frac{2^b(1-\alpha) - 1}{2^{1-\alpha} - 1} \leq \hat{c} N^\alpha \left( \frac{N}{n_0} \right)^{1-\alpha}, \quad (9)$$

where $\hat{c} = c/(2^{1-\alpha} - 1)$ and $n_0 = 2^{-\hat{b}} N$, so that

$$\frac{\langle \Delta L \rangle}{N} \leq \frac{\hat{c}}{n_0^{1-\alpha}}. \quad (10)$$

This means that we can guarantee a cost $\langle \Delta L \rangle/N \leq \ell$ if we stop cutting once the pieces are of size

$$n_0 = (\hat{c}/\ell)^{1/(1-\alpha)}.$$

The distribution of $n_0$ binary variables requires at most $2^{n_0}$ parameters, and this is independent of $N$. We need one such model for each of the $N/n_0$ pieces.

Thus, when the mutual information is consistently sub–extensive we can make an approximate model that has $P \sim (N/n_0) 2^{n_0}$ parameters, and the error that we make corresponds to an excess coding cost of $\ell$ bits per degree of freedom, with $n_0$ and $\ell$ connected through Eq (11). This number of parameters is linear in the number of degrees of freedom, and hence we expect that the model can be learned from a number of examples which is also linear in the system size, rather than being exponentially larger.

To make a meaningful connection to the idea of learnability, we need to show two things. First, it must be that typical probability distributions do not have consistently sub–extensive mutual information. Second, we need to show that data in interesting problems really do exhibit this special property.

Here, we take a typical probability distribution to be one in which the probabilities $P(\sigma_R, \sigma_L)$ are nearly independent random numbers, constrained only by normalization. But then the probability of each state in one half of the system

$$P_R(\sigma_R) = \sum_{\sigma_L} P(\sigma_R, \sigma_L), \quad (12)$$

is the sum of a large number of nearly independent random variables, and from the central limit theorem this should approach its expectation value. But the average distribution is uniform, and has the maximal entropy of $N/2$ bits, which predicts $I_{1/2}(N) = N - S(N)$, where $S(N)$ is the entropy of the full $N$–variable system; this is
both our expectation for the typical system, and an upper bound for any system. We also know that the mutual information cannot be larger than the entropy of either half system, and these entropies cannot be larger than $S(N)$ itself. These two bounds require any distribution to lie within the triangle in Fig 1, see also Ref. [11] for analogous bounds in quantum systems.

This argument is instantiated by the random energy model (REM), in which each of $2^N$ states has an energy drawn at random from a Gaussian distribution with variance $\langle E^2 \rangle = N$, with probabilities given by the Boltzmann distribution at temperature $T$ [15]. The states can be labeled by binary numbers and the digits assigned arbitrarily as left and right halves of a spin system. In Figure 1a, we show $I_{1/2}(N)$ vs $S(N)$ for these models, with varying $T$ and $N$, and compare with bounds derived above. We see that as $N$ increases the information per spin increases to approach the bounds, indicating that $I_{1/2}(N)$ is extensive everywhere above the freezing transition. In contrast, models with sub-extensive mutual information would approach the x-axis in this plot. The peak in the mutual information shows that, while the REM is unlearnable everywhere in the high-temperature regime, it is most un-learnable in an intermediate regime between $T_c$ and $T_\infty$, while the entropy is monotonically increasing as a function of $T$ (Fig 1b).

As an example of interesting real world data, we consider ensembles of images extracted from a large database of natural movies [12]. We downsample the raw data by a factor of two, and then discretize to black/white binary pixels with a threshold such that black and white are equally likely across each ensemble. We then analyze contiguous patches of $N$ pixels, where the cut of the system in half is literally a right/left cut. With 1200 frames and roughly 200,000 image samples from within these frames, we are able to make reliable estimates of entropy and mutual information out to $N \sim 16$ pixels [10]. In Fig 2 (inset) we see that $I_{1/2}(N)$ vs $S(N)$ moves away from the bounds with increasing $N$, and in the main figure we see explicitly that $I_{1/2}(N) \propto N^\alpha$ is strongly sub-extensive, with $\alpha = 0.1 \pm 0.03$, consistently across different natural contexts.

It is perhaps surprising that real world data meet the conditions for being well approximated by a model of independent pieces. Still, this is unsatisfying, and we would like to do better. Can we build a model in which the total cost $\Delta L$ is finite, even as the number of degrees of freedom $N$ becomes large? We will see that this is possible if shared information is compressible.

Let us break the $N$ spins into two groups,

$$\vec{\sigma}_K \equiv \{\sigma_1, \sigma_2, \ldots, \sigma_K\} \quad (13)$$

$$\vec{\sigma}_{N-K} \equiv \{\sigma_{K+1}, \sigma_{K+2}, \ldots, \sigma_N\}, \quad (14)$$

with $K \ll N$. The smaller group of $K$ spins could be one of the blocks of size $n_0$ from above, but this is not essential. Because the mutual information

$$I_0(N,K) \equiv I(\vec{\sigma}_K; \vec{\sigma}_{N-K}) \quad (15)$$

is finite, even as $N \to \infty$, it is plausible that we don’t need to specify all the details of the $N-K$ spins in order to capture their influence on the $K$ spins. The general idea is to compress our description of $\vec{\sigma}_{N-K}$ while maintaining as much information as possible about $\vec{\sigma}_K$, and this is the information bottleneck problem [17]. Concretely, we map $\vec{\sigma}_{N-K} \to X$, maximizing

$$-\mathcal{F} = I(X;\vec{\sigma}_K) - TI(X;\vec{\sigma}_{N-K}). \quad (16)$$

![FIG. 1: Mutual information between halves of the system for the random energy mode. (a) Along each curve at fixed $N$, we vary $T$, and compare with the bounds (dashed lines). (b) For $N = 22$, the mutual information between halves of the system versus $T$. The infinite size system (solid line) has a cusp in the mutual information, while the entropy (inset) is monotonically increasing with $T$.](image1)

![FIG. 2: Mutual information between halves of image patches vs patch size in pixels. Data from snapshots out of the Chicago motion database, with different natural environments analyzed separately [12]; error bars at the largest $N$ are a few percent, smaller than the symbols. Inset shows $I_{1/2}(N)$ vs $S(N)$, moving farther away from the bounds as $N$ increases.](image2)
We can solve this problem with $X$ being a discrete variable of cardinality $||X||$. As $T \to 0$ we recover a deterministic mapping $\bar{\sigma}_{N-K} \to X$, and this mapping captures a fraction of the available information,

$$I_{T=0}(X; \bar{\sigma}_K) = [1 - \epsilon_N(||X||)] I_0(N,K),$$

(17)

where the notation reminds us that the efficiency of capturing information may depend on $N$.

The intuition of compressibility is that with only $I_0$ bits available, we should be able to express the interaction between $\bar{\sigma}_K$ and $\bar{\sigma}_{N-K}$ in roughly $I_0$ bits, or in a compressed variable $X$ with $\log_2 ||X|| \sim I_0$. To be more precise, let’s define a function $F_N(\epsilon)$, such that if we compress to within a factor $F$ we capture information to within a factor $\epsilon$,

$$\log_2 ||X|| = F_N(\epsilon) I_0 \Rightarrow \epsilon_N(||X||) = \epsilon.$$  

(18)

This is illustrated in Fig 3.

Compression means that we are approximating

$$P(\bar{\sigma}_K|\bar{\sigma}_{N-K}) \approx P(\bar{\sigma}_K|X).$$

(19)

This approximate model has $2^K ||X||$ states, and hence this many parameters. To describe the whole system we need $N/K$ of these models, so the total number of parameters $P$ is given (somewhat generously) by

$$\log_2 P = K + \log_2 ||X|| + \log_2(N/K).$$

(20)

The cost of coding in this approximate model is the total mutual information we are missing,

$$\Delta L = (N/K)\epsilon_N(||X||) I_0.$$  

(21)

So to achieve a fixed $\Delta L$ at large $N$, we need to have

$$\epsilon = \frac{K\Delta L}{NI_0},$$

(22)

which means

$$\log_2 P = K + F_N \left( \epsilon = \frac{K\Delta L}{NI_0} \right) I_0 + \log_2(N/K).$$

(23)

Thus the number of parameters is set by the behavior of $F_N(\epsilon = K\Delta L/NI_0)$ at large $N$.

The most favorable possibility is that

$$\lim_{N \to \infty} F_N(\epsilon = 0) = f(K).$$

(24)

Then we have

$$\log_2 P = K + f(K) I_0 + \log_2(N/K),$$

(25)

and hence $P \sim N$. This is what happens in physics problems with local interactions: all of the impact of the $N-K$ spins on the small region of $K$ spins can be captured by enumerating a fixed number of variables even as $N \to \infty$.

The next case is where there is a logarithmic divergence at small $\epsilon$, so that

$$\lim_{N \to \infty} F_N \left( \epsilon = \frac{K\Delta L}{NI_0} \right) = g(K) \log_2 \left( \frac{NI_0}{K\Delta L} \right) + \text{constant},$$

(26)

which implies

$$\log_2 P \sim [1 + g(K)I_0] \log_2 N + \text{constant}. $$

(27)

Thus the number of parameters is polynomial in the number of spins, although possibly with a large power.

The logarithmic behavior of $F_N(\epsilon = K\Delta L/NI_0)$ as $N \to \infty$ is realized in certain models with long ranged interactions, including mean field models. This is easiest to see at $K = 1$, where the impact of all $N-1$ spins on the one spin of interest can always be summarized by an effective field $h$. As $N \to \infty$, this field becomes a continuous variable, chosen from a distribution $P(h)$ which could be different at every spin. Compressing the state of the $N-1$ spins is equivalent to representing the continuous $h$ by the discrete $X$; information is lost because there is some range of $h$ values that are assigned to the same $X$. If $||X||$ is large and this information loss is small, we will have $\epsilon \sim (\langle \delta h^2 \rangle_X) X$, the variance of $h$ at fixed $X$. Crudely speaking, compression takes the full dynamic range $H_N$ of the effective field, which may depend on $N$, and divides it into $||X||$ bins, so that

$$\langle (\delta h)^2 \rangle_X \sim \frac{H_N^2}{||X||^2},$$

(28)

and hence $F_N(\epsilon) \sim \log_2 (H_N^2/\epsilon)$, so that

$$\log_2 P \sim \log_2 \left( \frac{NH_N^2}{\Delta L} \right) + \log_2(N),$$

(29)
where we drop $N$–independent constants.

As an example, in the disordered phase of a mean field ferromagnet, we have $H_N \sim 1/\sqrt{N}$, which gives a number of parameters again linear in the number of spins [18]. Even if $H_N \sim 1$ at large $N$, we have $P \propto N^2$ [19]. Notice that these results, perhaps surprisingly, don’t depend on the usual assumption of pairwise interactions.

While a logarithmic divergence in $F_N(\epsilon)$ leaves us with a polynomial number of parameters, a linear divergence implies that the code needed to describe the effect of $N - K$ spins on the small cluster of $K$ spins has $\sim N$ bits, and no compression is possible. In this case we are back to a number of parameters that is exponential in $N$.

To summarize, the consistent sub–extensivity of mutual information makes possible approximate models that have a number of parameters linear in the number of degrees of freedom while suffering a cost per degree of freedom that vanishes in the thermodynamic limit, and compressibility of the mutual information makes it possible to have only finite total cost in this limit. These results suggest, strongly, that complexity can be tamed without making assumptions about the nature of interactions, generalizing our intuition from physics problems that we understand to probabilistic models much more generally. Perhaps this also provides new perspective on why simple models work in the traditional problems of statistical physics.

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