PROJECTIVE CLASSIFICATION OF JETS OF SURFACES IN $\mathbb{P}^4$

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ABSTRACT. We are interested in the local extrinsic geometry of smooth surfaces in 4-space, and classify jets of Monge forms by projective transformations according to $A^3$-types of their central projections.

1. INTRODUCTION

We are concerned with the geometry of the contact of smooth surfaces in the projective space $\mathbb{R}P^4$ with their tangent lines. The contact is measured by types of map germs of central projections of the surfaces. In the present paper we classify jets of generic surfaces by projective transformations which preserve the geometry of the contacts of surfaces with their tangent lines.

For surfaces in $\mathbb{R}P^3$, Platonova [20, 21] completed the classification of generic surfaces by projective transformations. The classification of generic two parameter families of surfaces is done in [8] (see also [1, 10, 11, 12]). The study of surfaces in $\mathbb{R}P^4$ was proposed in [2] with relation to the classification of singularities which appear in central projections of generic surfaces in $\mathbb{R}^4$ by D. Mond [15, 16, 17]. However there have been no results about the classification of surfaces so far. This paper gives an answer to the proposal in [2] with the complete classification list of jets of generic surfaces in $\mathbb{R}P^4$ by projective transformations.

On the other hand, the study of geometric aspects of surfaces in 4-space is a relatively new subject and has a lot of analogy to the study of surfaces in 3-space which have been investigated in [5, 6, 13, 14, 18].

Let $M$ be a smooth surface embedded in $\mathbb{R}^4 \subset \mathbb{R}P^4$ containing the origin of $\mathbb{R}^4$ where $\mathbb{R}^4$ is identified with the open subset $\{[x; y; z; w; 1]\} \subset \mathbb{R}P^4$. We write $(z, w) = f(x, y) = (f_1(x, y), f_2(x, y))$ as the Monge form of $M$ at the origin where $f_i(0, 0) = d f_i(0, 0) = 0$ for $i = 1, 2$. Two jets of surfaces at some points are said to be projective equivalent if there is a projective transformation on $\mathbb{R}P^4$ sending one to the other. Our result is the following.

Theorem 1. There is an open everywhere dense subset $O$ of the space of compact smooth surfaces $M$ in $\mathbb{R}P^4$ such that the germ at each point on $M$ in $O$ is projectively equivalent to a germ with the $k$-jet of the Monge form of one of the cases in Table 1.

Observe that the last column of Table 1 means the types of central projections of corresponding surfaces from view points on asymptotic lines. In section 2, we briefly explain the stratification of the space of jets of Monge forms induced by the stratification of jets of germs of central projections, and review the stratification of the 3-jet space of Monge forms induced from the $A^3$-stratification of central projection germs which was originally done in Ph. D thesis of Mond [15]. In Section 3 we obtain simple normal forms of jets of Monge forms that represent each...
Remark 1.1. Our normal forms in Table 1 contain a lot of moduli parameters including coefficients of higher order terms of degree greater than 4. They must be interpreted as some projective differential invariants. For example, when we look at the $A$-types of the central projection of the $\Pi_5$-type surfaces germs, it is observed that the central projection from a view point on the asymptotic line is $A$-equivalent to $P_3(c): (x, xy^2 + cy^2, xy + y^3)$ where $c$ is a moduli parameter, and $c = \frac{3}{\gamma}$. The first author [1] found also that $\gamma$ and $\lambda$ are expressed by combinations of some cross-ratio invariants and they determines the topological type of BDE (binary differential equations) of asymptotic curves.

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2. The stratification of the 3-jet space of Monge forms

Mond, in Chapter II of his Ph.D thesis [15], stratified the jet space of Monge forms according to $A$-types of germs of central projections. In this section we explain the way to stratify the jet space of Monge forms according to types of central projections, and review Mond’s stratification for the 3-jet space of Monge forms.

Take a surface $M$ in $\mathbb{R}^3 \subset \mathbb{R}P^4$ with the Monge form $(z, w) = f(x, y) = (f_1(x, y), f_2(x, y))$. Let $V_l$ denote the space of polynomials in $x, y$ of degree greater than $l$ and less than or equal to $l$. Our aim here is to obtain a stratification of the $\ell$-jet space of Monge forms $V_l \times V_l$ which is induced from the $A^\ell$-stratification of $J^\ell(2, 3)$ obtained in [15] [16] as follows.

| Type | Normal form | Condition | cod | Proj. |
|------|-------------|-----------|-----|-------|
| $\Pi_E$ | $(x^2 - y^2 + y^3, xy + \psi_1)$ | $\alpha \neq 0$ | 0 | $-$ |
| $\Pi_3$ | $(x^2 + y^3 + y\phi_3, y^2 + \alpha x^3 + x\psi_3)$ | $\alpha \neq 0$ | 0 | $S$ |
| $\Pi_B$ | $(x^2 + y^3 + y\phi_3, y^2 + x\psi_3)$ | $-\gamma$, $\Lambda \neq 0$ | 2 | $B$ |
| $\Pi_H$ | $(x^2 + \beta xy^2 + y^2 + y\phi_3, xy + x\psi_3)$ | $-\gamma$, $\Lambda \neq 0$ | 1 | $H$ |
| $\Pi_P$ | $(x^2 + \beta xy^2 + \gamma y^3 + \psi_4)$ | $\beta, \gamma, \Lambda \neq 0$ | 2 | $P$ |
| $\Pi_I$ | $(x^2 + \beta xy^2 + \gamma y^3 + \psi_4)$ | $\beta, \gamma, \Lambda \neq 0$ | 2 | $S, B$ |
| $\Pi_I'$ | $(x^2 + y^2 + k_1 x^2 y + y\phi_3, \psi_3 + \psi_4)$ | $b_{30}, b_{10} \neq 0$ | 2 | $S, B$ |
| $\Pi_I''$ | $(x^2 + y^2 + k_2 x^3 + \phi_4, \psi_3 + \psi_4)$ | $b_{30}, b_{20} \neq 0, a_{22} = 0$ | 2 | $S, B, H$ |

Table 1. Strata of codimension $\leq 2$ in the space of 4-jets of Monge forms corresponding to $A^3$-types of central projections on asymptotic lines. Here $\phi_s = \sum_{i+j=s} a_{ij} x^i y^j, \psi_s = \sum_{i+j=s} b_{ij} x^i y^j$, $\alpha, \beta, \gamma, \lambda, k_1, k_2, a_{ij}, b_{ij} \in \mathbb{R}$ are moduli parameters and $\Lambda = 6 \gamma^2 + 4\lambda - 15\gamma + 5$. Surface germs of $\Pi_E$-type do not have asymptotic lines.
Consider a point \( p \in \mathbb{R}P^4 \), which is sometimes called a view point, not lying on \( M \) and define \( \pi_p : \mathbb{R}P^4 - \{p\} \rightarrow \mathbb{R}P^3 \) as the canonical projection which associates \( x \in \mathbb{R}P^4 - \{p\} \) to the line generated by \( x - p \). The central projection of the surface \( M \) from \( p \) is given by the composite map

\[
\varphi_{p,M} := \pi_p \circ \iota : M \rightarrow \mathbb{R}P^3
\]

(see also \[3\]).

We denote the central projection of the surface germ expressed in Monge form \( f \) from a view point \( p \) by \( \varphi_{p,f} \). We stratify \( V_\ell \times V_\ell \) by the difference of \( A_\ell \)-types of \( j^\ell\varphi_{p,f} \) for a view point \( p \in \mathbb{R}P^4 - M \) and the \( \ell \)-jet of a Monge form \( j^\ell f \in V_\ell \times V_\ell \). \( A_\ell \) means the equivalence of jets of map germs, i.e., two jets \( j^h g, j^h h \in J^\ell(2,3) \) are equivalent if and only if there exist jets of diffeomorphism germs \( \sigma, \tau \) of the source and the target at the origins such that \( j^h h = j^\ell(\tau \circ g \circ \sigma^{-1}) \). Remark that even for the same \( \ell \)-jet of the surface the central projection gives different \( A_\ell \)-types depending on the view point. For example, \( j^2\varphi_{p,f} \) is always regular type (i.e. equivalent to \((x,y,0)\)) if and only if \( p \) is outside the tangent plane of the surface germ; and gives a singularity if and only if \( p \) is on the tangent plane to the surface.

We say that a line on the tangent plane which goes through the origin in \( \mathbb{R}^4 \) is an asymptotic line of the surface given in Monge form \( f \) if \( \varphi_{p,f} \) is equivalent to one of singularities worse than a crosscap (\( S_0 \)-type) for all view points \( p \) on the line. In Section 3 we show that the classification of 2-jets of Monge forms by projective transformations coincides with the classification of \( J^2(2,2) \) by \( GL(2) \times GL(2) \)-actions in Table 2, and each orbit is characterized by the number of asymptotic lines \[5, 16\]. For elliptic type with the form \( j^2 f = (x^2 + y^2, xy) \), there are no asymptotic lines; for hyperbolic type with the form \( j^2 f = (x^2, y^2) \), \( x \) and \( y \)-axis are two unique asymptotic lines; for parabolic type with the form \( j^2 f = (x^2, xy) \), \( y \)-axis is a unique asymptotic line; for inflection type with the form \( j^2 f = (x^2 + y^2, 0) \) or \((xy, 0)\), all lines on the tangent plane which go through the origin are asymptotic lines. Thus we study types of central projections from view points on the asymptotic lines, and divide strata in Table 2 into finer ones.

Define a smooth map, the Monge-Taylor map \( \Theta : M \rightarrow V_\ell \times V_\ell \), which associates to each point in \( M \) the \( \ell \)-jet of Monge form \( f = (f_1, f_2) \) at the point. The following is a natural extension of Bruce’s Theorem in \[3\].
Theorem 2. Let \( Z \subset V_1 \times V_1 \) be an \( GL(2) \times GL(2) \)-invariant submanifold. For generic surface \( M \) in \( \mathbb{R}^4 \), the Monge-Taylor map \( \Theta : M \to V_1 \times V_1 \) is transverse to \( Z \).

Proof: The proof follows the same arguments in the proof of Theorem 1 in [3] (see also [6]). \( \square \)

Since the strata are induced from \( A^f \)-types of central projections, they are \( GL(2) \times GL(2) \)-invariant. By Theorem 2 we consider only strata with codimension at most 2. In this section, we study the stratification of 3-jets of Monge forms which is induced by the \( A^3 \)-orbits in [15, 16, 17] or their unions given in Table 3.

Write 
\[
\varphi_{p,f} = \sum_{i+j \geq 2} a_{ij} x^i y^j, \quad f_2(x, y) = \sum_{i+j \geq 2} b_{ij} x^i y^j.
\]

Then \( \varphi_{p,f} \) can be regarded as a map germ \( \mathbb{R}^2, 0 \to \mathbb{R}^3 \). Indeed, for \( p = [a; b; c; d; 1] \in \mathbb{R}^4 - M \), we choose \( a \neq 0 \), then \( \varphi_{p,f} \) is given by 
\[
\varphi_{p,f}(x, y) = \left( \frac{y - b}{x - a} \cdot f_1(x, y) - \frac{c}{x - a} \cdot f_2(x, y) - \frac{d}{x - a} \right).
\]

On the other hand, if \( p \) is taken at infinity and written as \( p = [a; b; c; d; 0] \), \( \varphi_{p,f} \) is given by 
\[
\varphi_{p,f}(x, y) = (y - ux, f_1(x, y) - vx, f_2(x, y) - wx)
\]
with \((u, v, w) = (\frac{b}{a}, \frac{c}{a}, \frac{d}{a})\) (see also [3]).

The following sums up the Propositions III. 2:2, 2:8, 2:14, 2:16 and 2:17 in [15].

Proposition 2.1. (Mond [15]) For a surface germ in Monge form \((z, w) = f(x, y)\) we have the following.

(i) Suppose that \( j^2 f = (x^2, y^2) \). Then:
\[

depth 3 \varphi_{p,f} \sim S \iff a_{03} \neq 0 \quad \text{(resp. } b_{30} \neq 0) \]
\[

depth 3 \varphi_{p,f} \sim B \iff a_{03} = 0 \quad \text{(resp. } b_{30} = 0) \]

for \( p \) on the asymptotic line \( x = 0 \) (resp. \( y = 0 \)).
Statement (iv) for $p(x, y)$. Then:

\[ j^3\varphi_{p,f} \sim H \iff a_{03} \neq 0 \]
\[ j^3\varphi_{p,f} \sim P \iff a_{03} = 0 \text{ and } a_{12}, b_{03} \neq 0 \]
\[ j^3\varphi_{p,f} \sim R \iff a_{03} = b_{03} = 0 \text{ and } a_{12} \neq 0 \]
\[ j^3\varphi_{p,f} \sim T \iff a_{03} = a_{12} = 0 \text{ and } b_{03} \neq 0 \]
\[ j^3\varphi_{p,f} \sim U \iff a_{03} = a_{12} = b_{03} = 0 \]

for $p$ on the unique asymptotic line $x = 0$.

(iii) Suppose that $j^2f = (x^2, y^2)$. Then:

\[ j^3\varphi_{p,f} \sim S \text{ or } B \]

for any $p$ on the $xy$-plane.

(iv) Suppose that $j^2f = (xy, 0)$. Then:

\[ j^3\varphi_{p,f} \sim S, B \text{ or } H \iff b_{03} \neq 0 \]
\[ j^3\varphi_{p,f} \sim S, B \text{ or } P \iff b_{03} = 0 \text{ and } a_{30}, b_{21} \neq 0 \]
\[ j^3\varphi_{p,f} \sim S, B \text{ or } R \iff a_{30} = b_{03} = 0 \text{ and } b_{21} \neq 0 \]
\[ j^3\varphi_{p,f} \sim S, B \text{ or } T \iff a_{30} = b_{21} = 0 \text{ and } b_{03} \neq 0 \]
\[ j^3\varphi_{p,f} \sim S, B \text{ or } U \iff a_{30} = b_{03} = b_{21} = 0 \]

for any $p$ on the $xy$-plane.

Proof: Statement (i). It is easy to check that the $x$ and $y$-axes are asymptotic lines at the origin for $M$ in Monge-form with $j^2f = (x^2, y^2)$. Suppose $p$ is at the $y$-axis and written as $p = (0, a, 0, 0)$, then, by coordinate changes, we get

\[ j^3\varphi_{p,f} \sim A^3 (x, a(a_2 + 1)x^2 y + a_{03} y^3, y^2). \]

If $a_{03} \neq 0$, $j^3\varphi_{p,f}(0)$ is of $S$-type, otherwise it is of $B$-type. If $p$ is at infinity on the $y$-axis, we obtain

\[ j^3\varphi_{p,f} \sim A^3 (x, xy, a_{21} x^2 y + a_{03} y^3). \]

Again, if $a_{03} \neq 0$, $j^3\varphi_{p,f}(0)$ is of $S$-type, otherwise it is of $B$-type. By exchanging $x$ and $y$ (also $a_{ij}$ and $b_{ij}$), the case of $p$ on the $x$-axis follows similarly.

Statement (ii). Remark that $y$-axis is a unique asymptotic line at the origin when $j^2f(0) = (x^2, xy)$, hence we put $p = (0, a, 0, 0) \in \mathbb{R}^4 \subset \mathbb{R}P^4 (a \neq 0)$ and we get

\[ j^3\varphi_{p,f} \sim A^3 (x, a_{12} ax y^2 + a_{03} y^3, xy + b_{03} y^3). \]

Statement (ii) for $p \in \mathbb{R}^4$ naturally follows, and the case $p$ at infinity is similar.

Statement (iii). Let view point $p = (a, b, 0, 0) \in \mathbb{R}^4$ and $a \neq 0$, then we get

\[ j^3\varphi_{p,f} \sim A^3 (x, y^2, \xi_1 x^2 y + \xi_2 y^3) \]

where $\xi_1$ and $\xi_2$ are homogeneous polynomials of degree 3 with variables $a$ and $b$ whose coefficients consist of $a_{ij}$ and $b_{ij}$. Statement (iii) for $p = (a, b, 0, 0) \in \mathbb{R}^4$ where $a \neq 0$ naturally follows, and it is easily seen that the projection gives just $S_1$-type for view points $p = (0, b, 0, 0) \in \mathbb{R}^4$ where $b \neq 0$. The case $p$ at infinity is similar.

Statement (iv). For view point $p = (a, b, 0, 0) \in \mathbb{R}^4$ where $a, b \neq 0$, it is easily seen that $j^3\varphi_{p,f} \sim S$ or $B$ in the similar way to the above. Put $p = (a, 0, 0, 0) \in \mathbb{R}^4$ where $a \neq 0$ and we get

\[ j^3\varphi_{p,f} \sim A^3 (x, xy - \frac{a_{30}}{a} y^3, b_{21} x y^2 - \frac{b_{30}}{a} y^3). \]

Statement (iv) for $p \in \mathbb{R}^4$ naturally follows, and the case $p$ at infinity is similar. \(\square\)
Based on Proposition 2.1, we stratify the 3-jet space of Monge forms into strata with codimension at most 2 as in Table 4. In the next section we give simple normal forms of 4-jets of Monge forms which represent each stratum in Table 4 by projective transformations.

| Name | Type of 2-jet | Condition | cod | Proj. |
|------|--------------|-----------|-----|-------|
| $\Pi_E$ | $(x^2 - y^2, xy)$ | $a_{03} \cdot b_{30} \neq 0$ | 0 | $S$ |
| $\Pi_S$ | $(x^2, y^2)$ | $b_{30} = 0, a_{03} \neq 0$ | 1 | $S, B$ |
| $\Pi_B$ | $(x^2, y^2)$ | $a_{03} = b_{30} = 0$ | 2 | $B$ |
| $\Pi_H$ | $(x^3, xy)$ | $a_{03} \neq 0$ | 1 | $H$ |
| $\Pi_P$ | $(x^3, xy)$ | $a_{03} = 0, a_{12} \cdot b_{03} \neq 0$ | 2 | $P$ |
| $\Pi_7^I$ | $(x^3 + y^2, 0)$ | $b_{30} \neq 0$ | 2 | $S, B$ |
| $\Pi_7^S$ | $(xy, 0)$ | $b_{03} \neq 0$ | 2 | $S, B, H$ |

Table 4. Strata of codimension $\leq 2$ in the space of 3-jets of Monge forms corresponding to $\mathcal{A}^3$-types of central projections from view points on asymptotic lines. Surface germs of $\Pi_E$-type do not have asymptotic lines.

Remark 2.1. By taking higher order terms of the simple normal forms in the Table 1 we can consider a finer stratification of the space of Monge forms which corresponds to $\mathcal{A}$-types of central projections as Mond did in [15]. For instance, we take the Monge form of the $\Pi_S$-type in Table 1 and write $f = (x^2 + y^3 + \sum_{i+j\geq 4} a_{ij} x^i y^j, y^2 + \alpha x^3 + \sum_{i+j\geq 4} b_{ij} x^i y^j)$ with $\alpha, a_{ij}, b_{ij} \in \mathbb{R}, \alpha \neq 0$ and $a_{04} = b_{04} = 0$. Then the condition $a_{03} \neq 0, b_{13} \neq 0$ determines the proper stratum of the $\Pi_S$-stratum where the $\mathcal{A}$-types of the central projection can be determined as the regular, crosscap, $S_1$ or $S_2$-type depending on the position of the view points (See [15]).

3. The classification of Monge forms by projective transformations and proof of Theorem 1

In this section we consider a classification of jets of Monge-forms of generic surfaces by projective transformations based on the stratification in Table 4. The projective linear group $PGL(5)$ is defined as the quotient space $GL(5)/\sim$, where $A \sim A'$ if $\exists \lambda \in \mathbb{R}$ such that $A = \lambda A'$. To consider the action on $V_5 \times V_5$ ($5$-jet-spaces of Monge-forms), we define the following subgroup

$$G(5) := \{ \Psi \in PGL(5) \mid \Psi(0) = 0, \ \Psi(W) = W \}$$

of $PGL(5)$, where $0 = [0; 0; 0; 0; 1]$ is the origin and $W$ is the $xy$-plane in $\mathbb{R}^4$. Thus $G(5)$ form a 16-dimensional subgroup of $PGL(5)$ and acts on $V_5 \times V_5$.

Let $f = (f_1, f_2)$ and $g = (g_1, g_2)$ be Monge forms of surface-germs at the origin. We say that the $k$-jets of these Monge forms are projectively equivalent and write $j^k f \sim j^k g$ if there exists $\Psi \in G(5)$ which transforms one to the other. Remark that $\mathcal{A}^k$-types of central projections of jets of smooth surfaces are invariant under projective transformations of surfaces, that is, $j^k \varphi_{p,f} \sim_{\mathcal{A}^k} j^k \varphi_{\Phi(p),\Phi(f)}$ from view point $p$ and $\Phi \in G(5)$.
In this paper we check the equivalence of jets of Monge forms in the following way. With the coordinate \((x, y, z, w)\) of \(\mathbb{R}^4\), a projective transformation \(\Psi \in G(5)\) is regarded locally as a diffeomorphism germ \(\mathbb{R}^4, 0 \to \mathbb{R}^4, 0\) given by

\[
\Psi(x, y, z, w) = \left( \frac{q_1(x, y, z, w)}{p(x, y, z, w)}, \frac{q_2(x, y, z, w)}{p(x, y, z, w)}, \frac{q_3(x, y, z, w)}{p(x, y, z, w)}, \frac{q_4(x, y, z, w)}{p(x, y, z, w)} \right),
\]

where \(q_i = q_{i1}x + q_{i2}y + q_{i3}z + q_{i4}w\), for \(i = 1, 2\), \(q_j = q_{j3}z + q_{j4}w\), for \(j = 3, 4\) and \(p = 1 + p_1x + p_2y + p_3z + p_4w\). Define

\[
F_1(x, y, z, w) = \frac{q}{p} - f_1\left(\frac{q'}{p'}, \frac{q''}{p''}\right),
\]

\[
F_2(x, y, z, w) = \frac{q}{p} - f_2\left(\frac{q'}{p'}, \frac{q''}{p''}\right).
\]

Then

\[
F_1(x, y, g_1, g_2) = F_2(x, y, g_1, g_2) = o(k)
\]

where \(o\) is Landau’s symbol implies \(j^k f \sim j^k g\).

Hence, to check the equivalence, we have to solve algebraic equations \(F_1 = F_2 = o(2)\) for any \(j^2f, j^2g \in V_2 \times V_2\) gives equations of just \(q_{i1}, q_{i2}, q_{j3}, q_{j4}\) with \(i = 1, 2\) and \(j = 3, 4\), and the classification by projective transformations is reduced to the classification of \(V_2 \times V_2 \subset J^2(2, 2)\) by the natural action of \(G = GL(2, \mathbb{R}) \times GL(2, \mathbb{R})\). The \(G\)-orbits are classified in Table 2, described as Table 2. We classify now the higher jets of germs with a 2-jet as in Table 2.

3.2. Elliptic case. Suppose that \(j^2f = (x^2 - y^2, xy)\) and write

\[
j^3(f_1, f_2) = (x^2 - y^2 + \sum_{i+j=3} a_{ij}x^iy^j, xy + \sum_{i+j=3} b_{ij}x^iy^j)
\]

where \(a_{ij}, b_{ij} \in \mathbb{R}\). The following equivalence

\[
j^3(f_1, f_2) \sim (x^2 - y^2 + y^2\phi_1, xy),
\]

is given by projective transformation \(\Phi\) with

\[
q_1 = x + b_{03}z + (-a_{31} + b_{12} - b_{30})w, \quad q_2 = y - b_{30}z + (-b_{21} + b_{03} + a_{30})w, \\
q_3 = z, \quad q_4 = w, \quad p = 1 + (a_{30} + 2b_{03})x + (2b_{12} - a_{21})y.
\]

Here \(\phi_k\) means homogeneous polynomials of degree \(k\). Consider

\[
j^4(f_1, f_2) = (x^2 - y^2 + y^2\phi_1 + \sum_{i+j=4} c_{ij}x^iy^j, xy + \sum_{i+j=4} d_{ij}x^iy^j)
\]

where \(c_{ij}, d_{ij} \in \mathbb{R}\), then

\[
j^4(f_1, f_2) \sim (x^2 - y^2 + y^2(\phi_1 + \phi_2), xy + \phi_4),
\]

by \(\Phi\) with \(q_1 = x, \quad q_2 = y, \quad q_3 = z, \quad q_4 = w, \quad p = 1 + c_{40}z + c_{31}w\).
3.3. **Hyperbolic case.** Suppose that \( j^2f = (x^2, y^2) \) and write
\[
j^3(f_1, f_2) = (x^2 + \sum_{i+j=3} a_{ij} x^i y^j, y^2 + \sum_{i+j=3} b_{ij} x^i y^j)
\]
where \( a_{ij}, b_{ij} \in \mathbb{R} \). The following equivalence
\[
j^3(f_1, f_2) \sim (x^2 + a_{03} y^3, y^2 + b_{30} x^3)
\]
is given by projective transformation \( \Phi \) with
\[
q_1 = x + \frac{1}{b}(-a_{03} + b_{12})z - \frac{1}{b}a_{12}w, \quad q_2 = y - \frac{1}{b}b_{21}z + \frac{1}{b}(a_{21} - b_{03})w, \quad q_3 = z, \quad q_4 = w, \quad p = 1 + b_{12}x + a_{21}y.
\]
We can eliminate more two coefficients in 4-jet. Put
\[
j^4(f_1, f_2) = (x^2 + a_{03} y^3, y^2 + b_{30} x^3 + \sum_{i+j=4} c_{ij} x^i y^j)
\]
where \( c_{ij}, d_{ij} \in \mathbb{R} \), then
\[
j^4(f_1, f_2) \sim (x^2 + a_{03} y^3 + y \phi_3, y^2 + b_{30} x^3 + x \psi_3),
\]
by \( \Phi \) with \( q_1 = x, \ q_2 = y, \ q_3 = z, \ q_4 = w, \ p = 1 - c_{04}z - d_{04}w \). Here \( \phi_3 \) and \( \psi_3 \) means homogeneous polynomials of degree 3.

Then
\[
j^4(f_1, f_2) \sim \begin{cases} 
(x^2 + y^3 + y \phi_3, y^2 + \alpha x^3 + x \psi_3), & \text{if } a_{03}, b_{30} \neq 0; \\
(x^2 + y^3 + y \phi_3, y^2 + x \psi_3), & \text{if } a_{03} 
eq 0 \text{ and } b_{30} = 0; \\
(x^2 + y^3, y^2 + x \psi_3), & \text{if } a_{03} = b_{30} = 0.
\end{cases}
\]

3.4. **Parabolic case.** Suppose that \( j^2f = (x^2, xy) \) and write
\[
j^3(f_1, f_2) = (x^2 + \sum_{i+j=3} a_{ij} x^i y^j, xy + \sum_{i+j=3} b_{ij} x^i y^j)
\]
where \( a_{ij}, b_{ij} \in \mathbb{R} \). It is easy to show that
\[
j^3(f_1, f_2) \sim (x^2 + a_{12} xy^2 + a_{03} y^3, xy + b_{12} xy^2 + b_{03} y^3)
\]
where \( b_{12} = b_{12} - \frac{1}{a_{21}} \). If \( a_{03} \neq 0 \), then
\[
j^3(f_1, f_2) \sim (x^2 + (a_{12} + 3b_{03}) xy^2 + a_{03} y^3, xy)
\]
with the equivalence given by \( \Phi \) with
\[
q_1 = x - \frac{-b_{12}a_{03} + 3a_{12}b_{03} + 3b_{21}a_{03}}{a_{03}} w, \\
q_2 = \frac{b_{12}}{a_{03}} x + y + \frac{b_{21}(a_{12}b_{03} - a_{03}b_{12})}{a_{03}} z - \frac{b_{12}(2b_{21}a_{03} + b_{12}a_{03})}{a_{03}} w, \\
q_3 = z, \\
q_4 = \frac{b_{12}}{a_{03}} z + w, \quad p = 1 + \frac{b_{12}}{a_{03}} x - \frac{-b_{12}a_{03} + 4a_{12}b_{03} + 3b_{21}a_{03}}{a_{03}} y.
\]
Then the 4-jet can be written in the form
\[
j^4(f_1, f_2) \sim (x^2 + \beta xy^2 + y^3 + y \phi_3, xy + x \psi_3),
\]
where \( \beta = \frac{(a_{12} + 3b_{03})}{a_{03}} \), \( \phi_3 \) and \( \psi_3 \) mean homogeneous polynomials of degree 3.

If \( a_{03} = 0 \) but \( a_{12} \neq 0 \) we obtain
\[
j^3(f_1, f_2) \sim (x^2 + xy^2, xy + y^3)
\]
with the projective transformation \( \Psi \) given by
\[
q_1 = a_{12} x + a_{12} b_{12} w, \quad q_2 = y, \quad q_3 = a_{12}^2 z, \\
q_4 = a_{12} w, \quad p = 1 + 2b_{12} y,
\]
where $\gamma = \frac{b_{03}}{a_{12}}$. If we put
\[ j^4(f_1, f_2) = (x^2 + xy^2 + \sum_{i+j=4} a_{ij} x^i y^j, x y + \beta y^3 + \sum_{i+j=4} b_{ij} x^i y^j), \]
then $\gamma \neq 0$ leads to
\[ j^4(f_1, f_2) \sim (x^2 + xy^2 + \lambda y^4, xy + \gamma y^3 + \phi_4) \]
by a projective transformation $\Psi$ with
\[ q_1 = x + \frac{1}{2}(q_{21}^2 + p_1)z + (3\gamma q_{21} - 3q_{21})w, \]
\[ q_2 = y + q_{21}x + \frac{1}{2}(2\gamma q_{21} + q_{21}^3 + p_1 q_{21})z + \frac{1}{2}(q_{21}^2 + p_1)w, \]
\[ q_3 = z, \quad q_4 = q_{21}z + w, \quad p = 1 + p_1 x - (6\gamma q_{21} - 4q_{21})y + p_3 z + p_4 w, \]
where $p_1 = \frac{1}{\xi_1}, p_3 = \frac{1}{\xi_2}, p_4 = \frac{1}{\xi_3}, q_{21} = -\frac{1}{\xi_4}, \xi_i$ are combinations of the coefficients of 4-jet and $\Lambda = 6\gamma^2 + 4\lambda - 15\gamma + 5 \neq 0$. If $\Lambda = 0$ the terms of $j^4(f_1, f_2)$ of order 4 can not be removed. The $\phi_4$ is a homogeneous polynomials of degree 4.

3.5. **Inflection case.** Suppose that $j^2 f = (x^2 + y^2, 0)$ and write
\[ j^3(f_1, f_2) = (x^2 + y^2 + \sum_{i+j=3} a_{ij} x^i y^j, \sum_{i+j=3} b_{ij} x^i y^j). \]
Let $b_{30} - b_{12} \neq 0$. It follows that
\[ j^3(f_1, f_2) \sim (x^2 + y^2 + k_1 x^2 y, \phi_3) \]
by $\Psi$ with
\[ q_1 = x, \quad q_2 = y, \quad q_3 = z + (a_{30} - a_{12})w, \quad q_4 = (b_{30} - b_{12})w, \]
\[ p = 1 - \frac{(a_{20} b_{12} - a_{12} b_{30})}{(b_{30} - b_{12})} x - \frac{(a_{20} b_{03} - a_{12} b_{03} - a_{03} b_{12} + a_{03} b_{12})}{(b_{30} - b_{12})} y. \]
Here, $k_1$ is scalar constant. Now, we take
\[ j^4(f_1, f_2) = (x^2 + y^2 + k_1 x^2 y + \sum_{i+j=4} c_{ij} x^i y^j, \phi_3 + \sum_{i+j=4} d_{ij} x^i y^j), \]
where $c_{ij}, d_{ij} \in \mathbb{R}$, then it follows that
\[ j^4(f_1, f_2) \sim (x^2 + y^2 + k_1 x^2 y + y \psi_3, \phi_3 + \phi_4) \]
by $\Psi$ with $q_1 = x, \quad q_2 = y, \quad q_3 = z, \quad q_4 = w, p = 1 + c_{40} z$. Here $\phi_k$ and $\psi_k$ means homogeneous polynomials of degree $k$.

Next, suppose
\[ j^3(f_1, f_2) = (xy + \sum_{i+j=3} a_{ij} x^i y^j, \sum_{i+j=3} b_{ij} x^i y^j). \]
If $b_{03} \neq 0$, then
\[ j^3(f_1, f_2) \sim (xy + k_2 x^3, \phi_3) \]
by $\Psi$ with
\[ q_1 = x, \quad q_2 = y, \quad q_3 = z + (a_{21} + a_{03})w, \quad q_4 = b_{03}w, \]
\[ p = 1 - \frac{(a_{21} b_{03} - a_{03} b_{21})}{b_{03}} x - \frac{(a_{21} b_{03} - a_{03} b_{21})}{b_{03}} y. \]
The $k_2$ is a scalar constants. Finally, we consider
\[ j^4(f_1, f_2) = (xy + k_2 x^3 + \sum_{i+j=4} c_{ij} x^i y^j, \phi_3 + \sum_{i+j=4} d_{ij} x^i y^j), \]
where $c_{ij}, d_{ij} \in \mathbb{R}$. Thus, it follows that

$$j^4(f_1, f_2) \sim (xy + k_2x^3 + \xi_4, \phi_3 + \phi_4)$$

by $\Psi$ with $q_1 = x$, $q_2 = y$, $q_3 = z$, $q_4 = w$, $p = 1 + c_{22}z$. The $\phi_k$ means homogeneous polynomials of degree $k$ and $\bar{\xi}_4$ is a homogeneous polynomials of degree 4 without the term $x^2y^2$.

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