CONVERGENCE OF GINZBURG-LANDAU FUNCTIONALS IN
3-D SUPERCONDUCTIVITY

S. BALDO, R.L. JERRARD, G. ORLANDI, AND H.M. SONER

Abstract. In this paper we consider the asymptotic behavior of the Ginzburg-Landau model for superconductivity in 3-d, in various energy regimes. We rigorously derive, through an analysis via Γ-convergence, a reduced model for the vortex density, and deduce a curvature equation for the vortex lines. In the companion paper [2] we describe further applications to superconductivity and superfluidity, such as general expressions for the first critical magnetic field $H_{c1}$, and the critical angular velocity of rotating Bose-Einstein condensates.

1. INTRODUCTION

In this paper we investigate the asymptotic behavior as $\epsilon \to 0$ of the functionals

$$E_\epsilon(u) \equiv E_\epsilon(u; \Omega) = \int_\Omega e_\epsilon(u) \, dx = \int_\Omega \frac{1}{2} |Du|^2 + \frac{1}{\epsilon^2} W(u) \, dx,$$

where $\epsilon > 0$, $\Omega$ is a bounded Lipschitz domain in $\mathbb{R}^3$, $u = u^1 + iu^2 \in H^1(\Omega; \mathbb{C})$, $W : \mathbb{R}^2 \simeq \mathbb{C} \to \mathbb{R}$ is nonnegative and continuous, $W(u) = 0 \iff |u| = 1$, and is assumed to satisfy some growth condition at infinity and around its zero set (see hypothesis ($H_q$) below).

In the case $W(u) = \frac{(1-|u|^2)^2}{4}$, one usually refers to $E_\epsilon$ as the Ginzburg-Landau functional. This model is relevant to a variety of phenomena in quantum physics and in fact, as corollaries of its asymptotic analysis we will derive, here and in the companion paper [2], reduced models for density of vortex lines (or curves) in 3-d superconductivity and Bose-Einstein condensation. In these physical application, $\epsilon$ represents a (small) characteristic length, $u$ corresponds to a wavefunction, $|u|^2$ to the density of superconducting or superfluid material contained in $\Omega$. Moreover, the momentum, defined as the 1-form

$$ju \equiv (iu, du) \equiv u^1 du^2 - u^2 du^1,$$

represents the superconducting (resp. superfluid) current, and hence it is natural to interpret the Jacobian $Ju \equiv du^1 \wedge du^2$ as the vorticity, since $2Ju = d(ju)$. We refer the reader to the Appendix for notation used throughout this paper and background on differential forms and related material.

In the 2-d case it has been recognized since [5] that for minimizers $u_\epsilon$ of $E_\epsilon$ (subject to appropriate boundary conditions), as $\epsilon \to 0$, typically the energy scales like $|\log \epsilon|$ and there are a finite number of singular points, called vortices, where the energy density $e_\epsilon(u_\epsilon) dx$ and the vorticity $Ju_\epsilon$ concentrate. Moreover, the rescaled energy $\frac{E_\epsilon(u_\epsilon)}{|\log \epsilon|} dx$ controls the total vorticity. These phenomena are robust, in the sense that analogous results hold in higher dimensions (see [24, 6], where the limiting vorticity
is supported in a codimension 2 minimal surface) and under weaker assumptions
on \( u_\epsilon \), as stated in the following \( \Gamma \)-convergence result:

**Theorem 1 (22, 13).** Let \( K > 0, n \geq 2, \Omega \subset \mathbb{R}^n \) be a bounded Lipschitz domain,
and the potential \( W \) satisfy the growth condition\(^1\)

\[
\liminf_{|u| \to \infty} \frac{W(u)}{|u|^q} > 0, \quad \liminf_{|u| \to 0} \frac{W(u)}{(1 - |u|)^2} > 0,
\]

for some \( q \geq 2 \). Then the following statements hold:

(i) Compactness and lower bound inequality. For any sequence \( u_\epsilon \in H^1(\Omega, \mathbb{C}) \) such that

\[
E_\epsilon(u_\epsilon) \leq K|\log \epsilon|,
\]

we have, up to a subsequence, \( J u_\epsilon \to J \) in \( W^{-1,p} \) for every \( p < \frac{n}{n-1} \), where \( J \) is
an exact measure-valued 2-form in \( \Omega \) with finite mass \( ||J|| = ||J||_{(\Omega)} \), and \( J \) has the
structure of an \( (n-2) \)-rectifiable boundary with multiplicities in \( \pi \cdot \mathbb{Z} \). Moreover,

\[
\liminf_{\epsilon \to 0} \frac{E_\epsilon(u_\epsilon)}{|\log \epsilon|} \geq ||J||.
\]

(ii) Upper bound (in)equality. For any exact measure-valued 2-form \( J \) having the
structure of an \( (n-2) \)-rectifiable boundary in \( \Omega \) with multiplicities in \( \pi \cdot \mathbb{Z} \), there
exist \( u_\epsilon \in H^1(\Omega, \mathbb{C}) \) s.t. \( J u_\epsilon \to J \) in \( W^{-1,p} \) for every \( p < \frac{n}{n-1} \), and

\[
\lim_{\epsilon \to 0} \frac{E_\epsilon(u_\epsilon)}{|\log \epsilon|} = ||J||.
\]

Other energy regimes arise naturally for \( E_\epsilon \) and are interesting for applications.
In particular the energy regime \( E_\epsilon(u_\epsilon) \approx |\log \epsilon|^2 \) corresponds to the onset of the
mixed phase in type-II superconductors, and to the appearance of vortices in Bose-
Einstein condensates. These situations have been extensively studied in the 2-d case,
especially by Sandier and Serfaty in the case of superconductivity (see \cite{30} and
references therein). In this energy regime, the number of vortices is of order \( |\log \epsilon| \),
therefore unbounded as \( \epsilon \to 0 \). Another feature is that the contribution of the vortices
to the energy is of the same order as the contribution of the momentum, so that the
limiting behavior can be described in terms of this last quantity, suitably normalized.
A \( \Gamma \)-convergence result for \( \frac{1}{p} E_\epsilon \) for general energy regimes \( E_\epsilon(u_\epsilon) \lesssim g_\epsilon \ll \epsilon^{-2} \) has
been proved, in the 2-d case, in \cite{23}, see also \cite{30}.

1.1. **Main results.** A first result of this paper extends the asymptotic analysis of
\cite{23} to the 3-d case. We write \( f_\epsilon \ll h_\epsilon \) (or \( h_\epsilon \gg f_\epsilon \)) to express \( f_\epsilon = o(h_\epsilon) \) as \( \epsilon \to 0 \).
We will use the notation

\[
\mathcal{A}_0 := \{(J, v) : J \text{ is an exact measure-valued 2-form in } \Omega, v \in L^2(\Lambda^1 \Omega)\}
\]

Measure-valued \( k \)-forms are discussed in the Appendix, see in particular Sections
5.1.1 and 5.1.2. Our conventions imply that a measure-value form \( J \) has finite
mass, so that \( ||J|| := ||J||_{(\Omega)} < \infty \), where \( ||J|| \) denotes the total variation measure
associated with \( J \). We say that a measure-valued \( k \)-form \( J \) is exact if \( J = dw \) in the
sense of distributions for some measure-valued \( k-1 \)-form \( w \). We show in Lemma
14 that a measure-valued \( (n-1) \)-form \( J \) on a smooth bounded open \( \Omega \subset \mathbb{R}^n \) is
exact if and only if \( dJ = 0 \) and the associated flux through each component of the

\[^1\]cf. condition (2.2) in [1].
boundary \( \partial \Omega \) vanishes. The latter condition follows automatically from the former if \( \partial \Omega \) is connected.

**Theorem 2.** Let \( \Omega \) be a bounded Lipschitz domain in \( \mathbb{R}^3 \), \( W(u) \) satisfy (\( H_q \)) for some \( q \geq 2 \), and \( |\log \epsilon| \ll g_c \ll \epsilon^{-2} \). Then the following statements hold:

(i) Compactness and lower bound inequality. For any sequence \( u_\epsilon \in H^1(\Omega, \mathbb{C}) \) such that

\[
(H_\delta) \quad \text{for some } K > 0, \quad E_\epsilon(u_\epsilon) \leq Kg_c,
\]

there exist \( (J, v) \in \mathcal{A}_0 \) such that after passing to a subsequence if necessary,

\[
|u_\epsilon| \to 1 \quad \text{in } L^q(\Omega), \quad \frac{j u_\epsilon}{|u_\epsilon|} \sqrt{g_c} \to v \quad \text{weakly in } L^2(\Lambda^1 \Omega),
\]

\[
\frac{j u_\epsilon}{\sqrt{g_c}} \to v \quad \text{weakly in } L^{\frac{2q}{q+2}}(\Lambda^1 \Omega).
\]

If \( g_c \leq |\log \epsilon|^2 \), then in addition

\[
\frac{|\log \epsilon|}{g_c} Ju_\epsilon = \frac{|\log \epsilon|}{2g_c} d(j u_\epsilon) \to J \quad \text{in } W^{-1,p}(\Lambda^2 \Omega) \quad \forall \, p < 3/2.
\]

The convergences in (1.5) and (1.6) yield, in different scaling regimes,

\[
(S_1) \quad \text{if } |\log \epsilon| \ll g_c \ll |\log \epsilon|^2 \text{ then } (J, v) \in \mathcal{A}_3 := \{(J, v) \in \mathcal{A}_0 : dv = 0\},
\]

\[
(S_2) \quad \text{if } g_c = |\log \epsilon|^2 \text{ then } (J, v) \in \mathcal{A}_2 := \{(J, v) \in \mathcal{A}_0 : J = \frac{1}{2} dv \in H^{-1}(\Lambda^2 \Omega)\},
\]

\[
(S_3) \quad \text{if } |\log \epsilon|^2 \ll g_c \ll \epsilon^{-2} \text{ then } (J, v) \in \mathcal{A}_3 := \{(J, v) \in \mathcal{A}_0 : J = 0\}.
\]

and in every case,

\[
\liminf_{\epsilon \to 0} \frac{E_\epsilon(u_\epsilon)}{g_c} \geq ||J|| + \frac{1}{2} ||v||^2_{L^2(\Lambda^1 \Omega)}.
\]

(ii) Upper bound (in)equality. Assume that \((g_c)_{c>0}\) satisfies one of the scaling conditions (\( S_k \)), \( k \in \{1, 2, 3\} \), identified above, and that \((J, v) \in \mathcal{A}_k \). Then \( \exists U_\epsilon \in H^1(\Omega, \mathbb{C}) \) such that (1.4), (1.5), (1.6), (1.7) hold, and

\[
\lim_{\epsilon \to 0} \frac{E_\epsilon(U_\epsilon)}{g_c} = ||J|| + \frac{1}{2} ||v||^2_{L^2(\Lambda^1 \Omega)}.
\]

The compactness and lower bound assertions are either very easy, already known, see for example [31], or are proved almost exactly as in the 2d case. The upper bound (1.8) is the main new part of the theorem, and constitutes the most difficult part of the theorem.

**Remark 1.** Assume that \((g_c)_{c>0}\) satisfies one of the scaling conditions (\( S_k \)), \( k \in \{1, 2, 3\} \), identified above, and for \((J, v) \in \mathcal{A}_0\), set

\[
E(J, v) := ||J|| + \frac{1}{2} ||v||^2_{L^2(\Lambda^1 \Omega)} \quad \text{if } (J, v) \in \mathcal{A}_k,
\]

and \(E(J, v) := +\infty\) if \((J, v) \notin \mathcal{A}_k\). We express the \( \Gamma \)-convergence result of Theorem 2 using the notation

\[
\frac{E_\epsilon(u_\epsilon)}{g_c} \Gamma \to E(J, v),
\]
where the $\Gamma$-limit is intended with respect to the convergences \((1.4), (1.5), (1.6)\). Notice that the contributions of vorticity and momentum are decoupled in the $\Gamma$-limit, due to the different scaling factors in \((1.5), (1.6)\), except for the critical regime $g_\varepsilon = |\log \varepsilon|^2$, where the scalings of $Ju_\varepsilon$ and $ju_\varepsilon$ coincide, and the limits satisfy $2J = dv$ (see section 1.2 below). In particular, Theorem 2 expresses the fact that for regimes $g_\varepsilon \ll |\log \varepsilon|^2$, the contribution to the energy is given by the vorticity and the curl-free part of the momentum, while for $g_\varepsilon \gg |\log \varepsilon|^2$ the contribution of the vorticity vanishes asymptotically.

Remark 2. As observed in [22, 1], replacing $W(u)$ by $\sigma \cdot W(u)$, $\sigma > 0$, and letting $\sigma \to 0$, the lower bound \((1.7)\) can be sharpened to
\[
\liminf_{\varepsilon \to 0} \int_\Omega \frac{|\nabla u_\varepsilon|^2}{2g_\varepsilon} \geq \|J\| + \frac{1}{2} \|v\|^2_{L^2(\Lambda')}.
\]
Moreover, for a sequence $u_\varepsilon$ satisfying \((1.8)\), the potential part of the energy is a lower order term, i.e.
\[
\int_\Omega \frac{W(u_\varepsilon)}{\varepsilon^2} = o(g_\varepsilon) \quad \text{as} \quad \varepsilon \to 0.
\]
Inequality \((1.11)\) is also proved in [31].

Remark 3. In the 2-d case the $\Gamma$-convergence result of [23] is formulated exactly as Theorem 2 above, except for the convergence of the normalized Jacobians $\frac{\log \varepsilon}{g_\varepsilon} Ju_\varepsilon$, that takes place there in $W^{1,p}$ for any $p < 2$.

Remark 4. By localization, Theorem 2 implies the following: for any $u_\varepsilon$ satisfying $(H_\varepsilon)$, the rescaled energy densities $\frac{e_{i,(u_\varepsilon)/g_\varepsilon}}{g_\varepsilon}$ converge weakly as measures in $\Omega$, upon passing to a subsequence, to a limiting measure $\mu$, with $|J| + \frac{\varepsilon^2}{2} dx \leq \mu$. It then follows that $\mu = |J| + \frac{\varepsilon^2}{2} dx$ for any sequence $(u_\varepsilon)$ such that the convergences \((1.4), (1.5), (1.6)\) and the upper bound equality \((1.8)\) hold.

Remark 5. The final compactness assertion \((1.6)\) is proved by establishing convergence in $W^{-1,1}$, and then interpolating, using the easy estimate $\|Ju_\varepsilon\|_{L^1} \leq \|Du_\varepsilon\|_{L^1}$. For $|\log \varepsilon| \ll g_\varepsilon \ll \varepsilon^{-2}$, \((1.5)\) already implies that $\frac{\log \varepsilon}{g_\varepsilon} Ju_\varepsilon \to 0$ in $W^{-1,\frac{2}{1-2\gamma}}$. This can also be improved by interpolating with $L^1$ estimates (which imply $W^{-1,3/2}$ estimates) if $\frac{2\gamma}{7} < \frac{3}{2}$.

Remark 6. The convergences \((1.4), (1.5), (1.6)\) have been already established in the analysis of [22, 1, 23]. In particular, for a domain $\Omega \subset \mathbb{R}^n$ with $n \geq 4$, \((1.4)\) and \((1.5)\) still hold true, while the normalized Jacobians converge to $J$ in $W^{-1,p}$ for any $p < \frac{n}{n-1}$. Moreover, assuming $g_\varepsilon \leq \varepsilon^{-\gamma}$ for some $0 < \gamma < 2$, the convergence in \((1.6)\) can be improved according to $\gamma$, see [23]. In [8], following [10], the convergence in \((1.6)\) has been proved also to hold in $W^{1,\frac{2n}{n-2}}$ (as well as in fractional spaces $W^{s,p}$ with $sp = n/(n-1)$) for $n \geq 4$, and even in the case $n = 3$, assuming the condition $u \in L^q(\Omega)$ for $q > 6$ (see [8], Theorem 1.3 and Remark 1.6).

Remark 7. In the scaling $g_\varepsilon = |\log \varepsilon|$ studied in Theorem 1, arguments in the proof of Theorem 2 can easily be adapted to show that $\frac{E_i(u_\varepsilon)}{g_\varepsilon} \rightharpoonup E(J, v)$, where the $\Gamma$-limit is again intended with respect to the convergences \((1.4), (1.5), (1.6)\) and where $E(J, v)$ is defined exactly as in \((1.9)\), except that $E(J, v)$ is set equal to $+\infty$ unless
\[ dv = 0 \] and \( J \) has the structure of a rectifiable boundary. This is an improvement over Theorem 1 (cf. analogous results in [7] for critical points of \( E_\epsilon \), and in [4] for minimizers with local energy bounds), and in fact is valid in \( \mathbb{R}^n \) for any \( n \geq 3 \).

**Remark 8.** The validity of (1.7), (1.8) in dimension \( n \geq 4 \) remains an open issue for energy regimes \( g_\epsilon \gg |\log \epsilon| \). A major difficulty is to determine the correct generalization of the total variation term \( \| J \| \) in (1.9). Different candidates include the total variation with respect to the comass norm, the Euclidean norm, and the mass norm, see [16]. For measure-valued 2-forms in \( \mathbb{R}^3 \), all of these coincide. The most reasonable conjecture is that the mass norm is the suitable one for the higher-dimensional generalization of Theorem 2, but this seems difficult to prove. The arguments we give to prove (1.7) are in fact presented in \( \mathbb{R}^n \), and for \( n \geq 4 \) prove that (1.7) holds with \( \| J \| \) replaced by the comass of \( J \), which in general is strictly less than the mass of \( J \). Lower bounds involving the comass norm in \( \mathbb{R}^n, n \geq 4 \), are also proved in [31].

By way of illustration, for the (constant) measure-valued 2-form \( J = dx^1 \wedge dx^2 + dx^3 \wedge dx^4 \) on an open set \( \Omega \subset \mathbb{R}^4 \), one has comass(\( J \)) = \( |\Omega| \), the Euclidean total variation of \( J \) is \( \sqrt{2} \cdot |\Omega| \), and mass(\( J \)) = \( 2 |\Omega| \).

For \( |\log \epsilon|^2 \ll g_\epsilon \ll \epsilon^{-2} \), the total variation term does not appear in the limiting functional, so the issue of mass versus comass does not arise, and the proof of the lower bound (1.7) is straightforward; in fact it follows from arguments we give here. The upper bound (1.8) is probably also easier in this case than for \( |\log \epsilon| \ll g_\epsilon \leq |\log \epsilon|^2 \).

Replacing assumption \((H_\|)\) for \( W(u) \) with the following one (verified in particular for sequences of minimizers)

\((H_\infty) \quad \exists C > 1 \text{ such that } \| u_\epsilon \| \leq C \quad \forall \epsilon < 1 , \)

and taking into account Remark 6 a variant of Theorem 2 can be formulated as follows:

**Theorem 3.** In the hypotheses of Theorem 2 we have

(i) Compactness. For any sequence \( u_\epsilon \in H^1(\Omega, \mathbb{C}) \) verifying \((H_\|)\) and \((H_\infty)\) we have, up to a subsequence,

\[ \frac{j u_\epsilon}{\sqrt{g_\epsilon}} \to v \text{ weakly in } L^2(\Lambda^1 \Omega) , \quad \frac{|\log \epsilon|}{g_\epsilon} J u_\epsilon \to J \text{ in } W^{-1,3/2}(\Lambda^2 \Omega) , \]

where \( J \) is an exact measure-valued 2-form in \( \Omega \), with finite \( \| J \| \equiv |J|(|\Omega|) \).

(ii) \( \Gamma \)-convergence. Assuming that \( g_\epsilon \) respects one of the scaling conditions \( S_k \) from Theorem 2 we have

\[ \frac{E_\epsilon(u_\epsilon)}{g_\epsilon} \rightharpoonup_{\Gamma} E(J,v) , \]

with respect to the convergence (1.13), where \( E(J,v) \) is defined in (1.9), taking into account the relevant scaling regime.

1.2. The critical regime \( g_\epsilon = |\log \epsilon|^2 \). Let us specialize the statements of Theorems 2 and 3 to the critical regime \( g_\epsilon = |\log \epsilon|^2 \), where the scaling factors in (1.4), (1.5), (1.6) are equal, and hence the normalized vorticity is related to the momentum
by the formula $2J = dv$. We then have

$$\frac{E_v(u_v)}{|\log \varepsilon|^2} \rightharpoonup E(v),$$

where, for $v \in L^2(\Lambda^1 \Omega)$, we define

$$E(v) := E(\frac{dv}{2}, v) = \frac{1}{2} ||dv|| + \frac{1}{2} ||v||^2_{L^2(\Lambda^1 \Omega)}$$

if the mass $||dv|| \equiv |dv|(|\Omega|$ is finite, $E(v) = +\infty$ otherwise. The $\Gamma$-limit is intended with respect to the convergences (1.4), (1.5), (1.6).

Clearly Theorem 3 yields the same conclusion (1.15), this time with respect to the convergence (1.13), which in this case reads

$$\frac{j_{ue}}{|\log \varepsilon|} \rightharpoonup v \text{ weakly in } L^2(\Lambda^1 \Omega), \quad \frac{2j_{ue}}{|\log \varepsilon|} \to dv \text{ in } W^{-1.3/2}(\Lambda^2 \Omega).$$

1.3. Applications to superconductivity. As a first application of the above results in the energy regime $g_\varepsilon = |\log \varepsilon|^2$, we describe the asymptotic behavior of the Ginzburg-Landau functional for superconductivity

$$\mathcal{F}_\varepsilon(u, A) = \int_{\Omega} \frac{|du - iAu|^2}{2} + \frac{1}{\varepsilon^2} W(u) \, dx + \int_{\mathbb{R}^3} \frac{|dA - h_{ex}|^2}{2} \, dx$$

defined for $\Omega \subset \mathbb{R}^3$, where the 2-form $h_{ex} \in L^2_{loc}(\Lambda^2 \mathbb{R}^3)$ is an external applied magnetic field, the 1-form $A \in H^1(\Lambda^1 \mathbb{R}^3)$ is the induced vector potential (gauge field). It does not change the problem to assume that $h_{ex}$ has the form $h_{ex} = dA_{ex}$ for some $A_{ex} \in H^1_{loc}(\Lambda^1 \mathbb{R}^3)$, and we will always make this assumption.

Let $H^1_+ (\Lambda^1 \mathbb{R}^3) := \{ A \in H^1(\Lambda^1 \mathbb{R}^3) : d^+ A = 0 \}$, and define the inner product $(A, B)_{H^1_+ (\Lambda^1 \mathbb{R}^3)} := (dA, dB)_{L^2(\Lambda^2 \mathbb{R}^3)}$. This makes $H^1_+ (\Lambda^1 \mathbb{R}^3)$ into a Hilbert space, satisfying in addition the Sobolev inequality

$$||A||_{L^6(\Lambda^1 \mathbb{R}^3)} \leq C ||A||_{H^1_+ (\Lambda^1 \mathbb{R}^3)}.$$ 

We will study $\mathcal{F}_\varepsilon(v, A)$ for $(v, A) \in H^1(\Omega; \mathbb{C}) \times [A_{ex} + H^1_+ (\Lambda^1 \mathbb{R}^3)]$; this is reasonable in view of the gauge-invariance of $\mathcal{F}_\varepsilon$, that is, the fact that

$$\mathcal{F}_\varepsilon(u, A) = \mathcal{F}_\varepsilon(u e^{i\phi}, A + d\phi) \quad \forall \phi \in H^1(\mathbb{R}^3).$$

It is useful to decompose $\mathcal{F}_\varepsilon$ as follows (see e.g. [9]):

$$\mathcal{F}_\varepsilon(u, A) = E\varepsilon(u) + \mathcal{I}(u, A) + \mathcal{M}(A, h_{ex}) + \mathcal{R}(u, A),$$

with

$$\mathcal{I}(u, A) := - \int_{\Omega} A \cdot j u \, dx,$$

$$\mathcal{M}(A, h_{ex}) := \int_{\Omega} \frac{|A|^2}{2} \, dx + \int_{\mathbb{R}^3} \frac{|dA - h_{ex}|^2}{2} \, dx = \frac{1}{2} ||A||^2_{L^2(\Lambda^1 \Omega)} + \frac{1}{2} ||A - A_{ex}||^2_{H^1_+ (\Lambda^1 \mathbb{R}^3)}.$$ 

and $\mathcal{R}(u, A) = \frac{1}{2} \int_{\Omega} (|u|^2 - 1) |A|^2 \, dx$ is a remainder term of lower order. Thus $\mathcal{F}_\varepsilon(u, A)$ may be written as a continuous perturbation of $E\varepsilon(u) + \mathcal{M}(A, h_{ex})$, and using the stability properties of $\Gamma$-convergence we deduce, as in [23] for the 2-d case, the $\Gamma$-convergence for the functionals $\mathcal{F}_\varepsilon$ in the critical energy regime $g_\varepsilon = |\log \varepsilon|^2$:
Theorem 4. Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain, $W(u)$ satisfy $(H_q)$ with $q \geq 3$, and assume $h_{ex} = dA_{ex,e}$ and that there exists $A_{ex,o} \in H_{loc}^1(\Lambda^1 \mathbb{R}^3)$ such that $A_{ex,o} - A_{ex,0} \rightharpoonup 0$ in $\dot{H}^1_s(\Lambda^1 \mathbb{R}^3)$. Then the following hold.

(i) Compactness. For any sequence $(u_\epsilon, A_\epsilon) \in H^1(\Omega; \mathbb{C}) \times [A_{ex,0} + \dot{H}^1_s(\Lambda^1 \mathbb{R}^3)]$ such that $\mathcal{F}(u_\epsilon, A_\epsilon) \leq K |\log \epsilon|^2$, we have, up to a subsequence,

$$\frac{A_\epsilon}{|\log \epsilon|} - A \rightharpoonup 0 \quad \text{weakly in } \dot{H}^1_s(\Lambda^1 \mathbb{R}^3)$$

for some $A \in A_{ex,0} + \dot{H}^1_s(\Lambda^1 \mathbb{R}^3)$ as well as the convergences (1.4), (1.5), (1.6) of Theorem 2 in the case $g_\epsilon = |\log \epsilon|^2$.

(ii) $\Gamma$-convergence. For $v \in L^2(\Lambda^1 \Omega)$ and $A \in A_{ex,0} + \dot{H}^1_s(\Lambda^1 \mathbb{R}^3)$, define

$$\mathcal{F}(v, A) = \frac{1}{2} ||dv|| + \frac{1}{2} ||v - A||^2_{L^2(\Lambda^1 \Omega)} + \frac{1}{2} ||dA - dA_{ex,0}||^2_{L^2(\Lambda^2 \mathbb{R}^3)}$$

if $||dv|| = ||dv||(\Omega)$ is finite, $\mathcal{F}(v, A) = +\infty$ otherwise.

Then under the convergences (1.22), (1.4), (1.5), (1.6), we have

$$\lim_{\epsilon \to 0} \frac{\mathcal{F}(u_\epsilon, A_\epsilon)}{|\log \epsilon|^2} \Gamma \to \mathcal{F}(v, A).$$

Remark 9. Assuming $(H_\infty)$, the $\Gamma$-limit (1.24) is obtained with respect to the convergences (1.22), (1.17).

Remark 10. The statement of Theorem 4 is not gauge-invariant, as the condition that $A_\epsilon \in A_{ex,e} + \dot{H}^1_s(\Lambda^1 \mathbb{R}^3)$ uniquely determines the function $\phi$ in (1.18). Fixing this degree of freedom is clearly necessary for compactness. Note however that the limiting functional $\mathcal{F}$ has a gauge-invariance property: $\mathcal{F}(v, A) = \mathcal{F}(v + \gamma|\Omega, A + \gamma)$ whenever $d\gamma = 0$.

The Euler-Lagrange equations of the functional $\mathcal{F}$ consist in the Ampère law $d^*H = j$ for the resulting magnetic field $H = dA - h$, generated by the (gauge-invariant) super-current $j = v - A$ in $\Omega$ (see (1.5)), and a curvature equation for the vortex filaments, i.e. the streamlines of the limiting vortex distribution (see (1.7)), which reads, in the regular case,

$$\begin{cases}
\kappa = 2\bar{\tau} \times \bar{\tau} & \text{in } \Omega, \\
\bar{\tau} = 0 & \text{on } \partial \Omega.
\end{cases}$$

whith $\kappa$ and $\bar{\tau}$ denoting respectively the curvature vector and the unit tangent to the vortex filament, $\bar{\tau}$ the vector field corresponding to the super-current $j = v - A$, and $\times$ the exterior product in $\mathbb{R}^3$. Formula (1.25) generalizes the corresponding law in the case of a finite number of vortices (see [7], Theorem 3 (iv), and [13]).

Remark 11. In [2] we analyze in more detail the properties of minimizers of the limiting functional $\mathcal{F}$ through the introduction of a dual variational problem. We use this description to characterize to leading order the first critical field $H_{cr}$. These results extend to 3 dimensions facts about 2-d models of superconductivity first established by Sandier and Serfaty [29], see also [30] and other references cited therein. Following the initial work of Sandier and Serfaty, it was shown in [23] that their results can be recovered via the 2-d analog of the procedure we follow here and in [2].
As far as we know, the relevance of convex duality in these settings was first pointed out by Brezis and Serfaty [12].

Remark 12. In [2] we also apply Theorem 2 to study the Γ-limit of the Gross-Pitaevskii functional for superfluidity, and derive in particular a reduced vortex density model for rotating Bose-Einstein condensates, deducing the corresponding curvature equations and an expression for the critical angular velocity.

Remark 13. Theorem 4 is concerned with the description of the behavior of global minimizers. The convergence of local minimizers with bounded vorticity has been studied, under various assumptions, in [21, 20, 25], relying on techniques related to Theorem 1.

1.4. Plan of the paper. This paper is organized as follows: in Section 2 we prove the lower bound and compactness statement (i) of Theorem 2, while Section 3 is devoted to the proof of the upper bound statement (ii). In Section 4 we prove Theorem 4 and derive the Euler-Lagrange equations of the Γ-limit, obtaining in particular formula (1.25). Section 5 is an Appendix that collects some notation and the proofs of some auxiliary results.

Acknowledgements. This research has been partially funded by G.N.A.M.P.A. of Istituto Nazionale di Alta Matematica (visiting professor program), Università di Verona (funding program Cooperint), and the National Science and Engineering Research Council of Canada under operating Grant 261955. We warmly thank these institutions for support and kind hospitality. We are also grateful to Giovanni Alberti for numerous helpful discussions.

2. LOWER BOUND AND COMPACTNESS

In this section we prove statement (i) of Theorem 2, relying largely on our previous works [23, 1]. We prove everything in Ω ⊂ \( \mathbb{R}^n \), for arbitrary \( n \geq 3 \). We note however that the lower bound inequality (1.7) is not expected to be sharp when \( n \geq 4 \), see Remark 8.

We first derive (1.4) and (1.5). Then, assuming (1.6), we derive the characterization of the limiting spaces \( \mathcal{A}_k \) corresponding to the scaling regimes \( S_k \) identified in the statement of the Theorem. We next turn to the proof of the lower bound (1.7). The compactness statement (1.6) in the case \( p = 1 \) will be obtained during the proof of (1.7), and the case \( 1 < p < \frac{n}{n-1} \), (see Remark 9) will form the case \( p = 1 \) by a short interpolation argument.

Proof of (1.4), (1.5). Observe first that \( |u_\varepsilon| \to 1 \) in \( L^q(\Omega) \) by assumptions (\( H_q \)) on \( W(u) \) and (\( H_g \)) on \( E_\varepsilon \), since

\[
\int_\Omega |1 - |u_\varepsilon||^q \leq C \int_\Omega W(u_\varepsilon) \leq C \varepsilon^2 E_\varepsilon(u_\varepsilon) \leq C \varepsilon^2 g_\varepsilon \to 0.
\]

From the identity \( |u|^2 |\nabla u|^2 = |u|^2 |\nabla |u||^2 + |ju|^2 \) we deduce that

\[
\int_\Omega \frac{|ju_\varepsilon|^2}{|u_\varepsilon|^2 g_\varepsilon} \leq 2 \cdot \frac{E_\varepsilon(u_\varepsilon)}{g_\varepsilon} \leq 2K,
\]

(2.1)
which yields, up to a subsequence, \( \frac{ju_e}{\sqrt{g_e}} \to v \) weakly in \( L^2(\Omega) \), completing the proof of (1.4). Now write
\[
\frac{ju_e}{\sqrt{g_e}} = \frac{ju_e}{|u_e| \sqrt{g_e}} + \left( |u_e| - 1 \right) \cdot \frac{ju_e}{|u_e| \sqrt{g_e}}.
\]
Using (1.4) we deduce that \( \left( |u_e| - 1 \right) \cdot \frac{ju_e}{|u_e| \sqrt{g_e}} \to 0 \) weakly in \( L^\frac{2n}{q+2}(\Omega) \). This yields \( \frac{ju_e}{\sqrt{g_e}} \to v \) weakly in \( L^\frac{2n}{q+2}(\Omega) \), i.e. (1.5).

Next, the characterization of the limiting spaces \( A_k \) follows from (1.4), (1.3) and (1.0), since by (1.3) we deduce that \( d(\frac{ju_e}{g_e}) \to dv \) weakly in \( W^{-1, \frac{2n}{q+2}}(\Omega) \), hence, in the case \( g_e \gg |\log \epsilon|^2 \),
\[
(2.2) \quad \frac{|\log \epsilon|}{g_e} J u_e = \left( \frac{|\log \epsilon|}{\sqrt{g_e}} \right) d \left( \frac{ju_e}{\sqrt{g_e}} \right) \to 0 \cdot dv = 0 \quad \text{in} \quad W^{-1, \frac{2n}{q+2}}(\Omega).
\]

In view of (1.0), this implies \( J = 0 \) by uniqueness of the weak limit. On the other hand, in the case \( g_e \ll |\log \epsilon|^2 \),
\[
d \left( \frac{ju_e}{\sqrt{g_e}} \right) = 2\left( \frac{\sqrt{g_e}}{|\log \epsilon|} \right) \cdot \left( \frac{|\log \epsilon|}{g_e} J u_e \right) \to 0 \cdot J = 0 \quad \text{in} \quad W^{-1, p}(\Omega), \quad p < \frac{n}{n-1},
\]
which implies \( dv = 0 \), again by uniqueness of the weak limit. The above formulas, in the case \( g_e = |\log \epsilon|^2 \), imply that \( dv = 2J \).

We turn to the proof of (1.7) distinguishing two cases, namely \( |\log \epsilon| \ll g_e \ll |\log \epsilon|^2 \), and \( |\log \epsilon|^2 \ll g_e \ll \epsilon^2 \). We begin with the latter case.

**Proof of (1.7) in the case \( g_e \gg |\log \epsilon|^2 \).** In this energy regime, we have just shown that \( J = 0 \), and (1.4) and (2.1) immediately imply
\[
(2.3) \quad \liminf_{\epsilon \to 0} \frac{E_\epsilon(u_e)}{g_e} \geq \frac{1}{2} \int_\Omega |v|^2,
\]
yielding conclusion (1.7).

If it is not true that \( g_e \gg |\log \epsilon|^2 \), then by passing to a subsequence we may suppose that \( g_e \leq C|\log \epsilon|^2 \). By renaming the constant \( K \) in \( (H_g) \) we may also assume that \( C = 1 \). Thus the proof of (1.7) will be completed by the following.

**Proof of (1.7) in the case \( |\log \epsilon| \ll g_e \ll |\log \epsilon|^2 \).** The main step in the proof is the following improvement of [1], Proposition 3.1. We establish it in greater generality than is needed for the proof of (1.7).

We remark that (1.7) in the scaling \( |\log \epsilon| \ll g_e \ll |\log \epsilon|^2 \) is already established in [31], and moreover that a key point in the proof there is a result similar to the following proposition.

**Proposition 1.** Let \( u_e \) be a sequence of smooth maps on \( \Omega \subset \mathbb{R}^n, n \geq 2 \), such that \( (H_g) \) holds, with \( |\log \epsilon| \leq g_e \leq |\log \epsilon|^2 \). Then we have, up to a subsequence,
\[
(2.4) \quad \frac{|\log \epsilon|}{g_e} J u_e \to J \quad \text{in} \quad W^{-1,1}(\Lambda^2 \Omega),
\]
where $J$ is an exact measure-valued 2-form with finite mass in $\Omega$. Moreover, there exists a closed set $C_\varepsilon \subset \Omega$ such that $|C_\varepsilon| \to 0$, and such that for every simple 2-covector $\eta$ such that $|\eta| = 1$ and for every open set $U \Subset \Omega$, it holds

$$
\lim_{\varepsilon \to 0} \inf J(u_\varepsilon; C_\varepsilon) = \inf \frac{E_\varepsilon(u_\varepsilon; C_\varepsilon)}{g_\varepsilon} \geq |(J, \eta)|(U),
$$

where $(J, \eta)$ is the signed measure defined according to \ref{eq:5.4}.

Our proof of Proposition \ref{prop:1} differs from that of the corresponding point (Proposition IV.3) in \cite{31}. One feature of our proof is that the set $C_\varepsilon$ that we construct is manifestly a closed set, whereas in the construction of \cite{31}, a certain amount of work is required even to see that the corresponding set is measurable.

Taking for granted Proposition \ref{prop:1}, we complete the proof of \ref{eq:1.7}. First, a standard localization argument (see \cite{1}, p. 1436) gives, for any finite collection of pairwise disjoint open sets $U_j \Subset \Omega$ and simple unit 2-covectors $\eta_j$,

$$
\sum_j |(J, \eta_j)|(U_j) \leq \liminf_{\varepsilon \to 0} \inf \frac{E_\varepsilon(u_\varepsilon; C_\varepsilon)}{g_\varepsilon}.
$$

Taking the supremum over all choices of pairwise disjoint open sets $U_j$ and unit simple 2-covectors $\eta_j$ on the l.h.s. of \ref{eq:2.6} yields the total comass norm of $J$ in the sense of \cite{10}, section 1.8.1. In the 3-dimensional case this coincides with the total variation (or $L^1$, accordingly) norm of $J$, since all 2-covectors in $\mathbb{R}^3$ are necessarily simple. Hence we may write, for $n = 3$,

$$
|J|(\Omega) \leq \liminf_{\varepsilon \to 0} \frac{E_\varepsilon(u_\varepsilon; C_\varepsilon)}{g_\varepsilon}.
$$

Let now $\Omega_\varepsilon \equiv \Omega \setminus C_\varepsilon$, and $\chi_\varepsilon(x)$ be the characteristic function of $\Omega_\varepsilon$. We may assume after passing to a subsequence that $\chi_\varepsilon(x) \to 1$ as $\varepsilon \to 0$ for a.e. $x \in \Omega$, since $|C_\varepsilon| \to 0$. Then for any $h \in L^2$, $\chi_\varepsilon \cdot h \to h$ in $L^2$ by the dominated convergence theorem, and so it follows from \ref{eq:1.4} that

$$
\int_{\Omega} h \cdot \chi_\varepsilon \cdot \frac{ju_\varepsilon}{|u_\varepsilon| \sqrt[2]{g_\varepsilon}} \to \int h \cdot v \quad \text{as } \varepsilon \to 0.
$$

That is, $\chi_\varepsilon \cdot \frac{ju_\varepsilon}{|u_\varepsilon| \sqrt[2]{g_\varepsilon}} \to v$ weakly in $L^2$. Since

$$
\int_{\Omega} \varepsilon_\varepsilon(u) \geq \frac{1}{2} \int_{\Omega} \varepsilon_\varepsilon \frac{|ju_\varepsilon|^2}{|u_\varepsilon|^2}
$$

we deduce that

$$
\liminf_{\varepsilon \to 0} \frac{E_\varepsilon(u_\varepsilon; \Omega_\varepsilon)}{g_\varepsilon} \geq \liminf_{\varepsilon \to 0} \frac{1}{2} \int_{\Omega} \varepsilon_\varepsilon \frac{|ju_\varepsilon|^2}{|u_\varepsilon|^2} g_\varepsilon \geq \frac{1}{2} \int_{\Omega} v^2.
$$

To conclude observe that $E_\varepsilon(u_\varepsilon; \Omega) = E_\varepsilon(u_\varepsilon; C_\varepsilon) + E_\varepsilon(u_\varepsilon; \Omega_\varepsilon)$, so that

$$
\liminf_{\varepsilon \to 0} \frac{E_\varepsilon(u_\varepsilon; \Omega)}{g_\varepsilon} \geq \liminf_{\varepsilon \to 0} \frac{E_\varepsilon(u_\varepsilon; C_\varepsilon)}{g_\varepsilon} + \liminf_{\varepsilon \to 0} \frac{E_\varepsilon(u_\varepsilon; \Omega_\varepsilon)}{g_\varepsilon}.
$$

Combining \ref{eq:2.9} with \ref{eq:2.8} and \ref{eq:2.7} we obtain \ref{eq:1.7}.

\[ \square \]
We now supply the

**Proof of Proposition 1.** We will proceed in two steps: first, we apply the discretization procedure of [1], Section 3 at a suitable scale \( \ell \) to deduce (2.4) and to obtain a identify a small set \( C_\tau \subset \Omega \) where the Jacobian \( J u_\epsilon \) is essentially confined. Second, we apply the cited procedure again, this time imposing an additional condition that yields good control of the resulting 2-form \( \nu'_\epsilon \) (a discretization of the Jacobian) in a small neighborhood \( C_\tau \) of \( C_\tau \) by the Ginzburg-Landau energy in the same small neighborhood \( C_\tau \). Then we argue that the restriction of \( \nu'_\epsilon \) to a suitable subset of \( C_\tau \) converges to the same limit as \( J u_\epsilon \), so that from lower semicontinuity, bounds on \( \langle \nu'_\epsilon, \eta \rangle \subset C_\tau \) yield estimates on \( (J, \eta) \), thereby proving (2.5).

We carry out these arguments in detail in the case \( n = 3 \) and then we discuss the general case.

**Step 1.** We follow [1], Section 3. Fix a unit simple 2-covector \( \eta \), and an orthonormal basis \((\tilde{e}_i)\) of \( \mathbb{R}^3 \) satisfying \( \eta(\tilde{e}_2 \wedge \tilde{e}_3) = 1 \). Consider a grid \( \Gamma = \Gamma(a, \tilde{e}_i, \ell) \), given by the collection of cubes with edges of size \( \ell \), and vertices having coordinates (with respect to a reference system with origin in \( a \in \mathbb{R}^3 \) and orthonormal directions \((\tilde{e}_i)_{i=1,2,3})\) which are integer multiples of \( \ell \). For \( h = 1, 2 \) denote by \( R_h \) the \( h \)-skeleton of \( \Gamma \), i.e. the union of all \( h \)-dimensional faces of the cubes of \( \Gamma \). Consider also the dual grid having vertices in the centers of the cubes of \( \Gamma \), and denote by \( R'_h \) for \( h = 1, 2 \), its \( h \)-skeleton. From (\( H_\beta \)) and the assumption that \( g_\beta \leq |\log \epsilon|^2 \) we have

\[
(2.10) \quad E_\beta(u_\epsilon; \Omega) \leq K|\log \epsilon|^2, \quad \text{and we set } \ell := |\log \epsilon|^{-10}.
\]

Observe that (2.10) replaces (3.22) and (3.23) in [1]. Choose \( a \equiv a_\epsilon \) by a mean-value argument in such a way that Lemma 3.11 of [1] holds, so that in particular, the restriction of the energy on the 2-d and 1-d skeleton of \( \Gamma \) is controlled by

\[
(2.11) \quad \int_{R_h \cap \Omega} c_\tau(u_\epsilon) dH^h \leq C_0 \ell^{h-3} E_\tau(u_\epsilon; \Omega), \quad h = 1, 2,
\]

for a suitable constant \( C_0 > 1 \), and moreover

\[
(2.12) \quad \ell \int_{\Omega} \frac{c_\tau(u_\epsilon)}{\text{dist}(x, R_1)} dx \leq C_0 E_\tau(u_\epsilon; \Omega).
\]

In view of (2.10), Lemma 3.4 in [1] is satisfied, hence \( |u_\epsilon| \to 1 \) uniformly on \( R_1 \cap \Omega \). In particular, for any face \( Q \in R_2 \), the topological degree \( d_Q := \deg (\frac{u_\epsilon}{|u_\epsilon|}, \partial Q, S^1) \in \mathbb{Z} \) is well-defined (modulo the choice of an orientation of \( Q \) in \( \mathbb{R}^3 \)).

The discretization procedure of [1], Lemmas 3.7 to 3.10, may then take place on any fixed open set \( U \subset \Omega \), yielding an oriented polyhedral 1-cycle (actually, a relative boundary in \( \overline{U} \)) \( M_\tau = \sum (-1)^{\sigma} dQ_i \cdot Q'_i \), where \( Q'_i \subset R'_1 \) is the unique edge of the cubes of the dual grid intersecting the face \( Q_i \subset R_2 \), the sign \( (-1)^{\sigma} \) depends on the orientations of both \( Q_i \) and \( Q'_i \), and the sum is extended to any \( Q_i \subset R_2 \) such that \( Q_i \cap U \neq \emptyset \). Notice that \( M_\tau \) is supported in \( R'_1 \cap U \sqrt{3} \ell \), where \( U \sqrt{3} \ell \) denotes the tubular neighborhood of \( U \) of thickness \( \sqrt{3} \ell \). The cycle \( M_\tau \) gives rise to a (measure-valued) 2-form \( \nu_\tau \), whose action on 2-forms in \( C_\tau^\infty(\Lambda^2 \Omega) \) is defined by

\[
(2.13) \quad \langle \nu_\tau, \varphi \rangle := \pi \cdot \sum_{\substack{Q_i \subset R_2 \\\ Q_i \cap U \neq \emptyset}} (-1)^{\sigma} dQ_i \int_{Q'_i} * \varphi.
\]
The 2-form $\nu_\epsilon$ is exact in $U$, since $M_\epsilon$ is a relative boundary in $\bar{U}$, and enjoys the following properties: it is a measure-valued 2-form supported in $R_1' \cap U \sqrt{3}\ell$, such that its total variation $|\nu_\epsilon|$ is bounded on $U$ by\footnote{cf. [1], (3.29)}
\begin{equation}
|\nu_\epsilon|(U) = \sum_{Q_i \subset R_2} \pi \ell \cdot |dQ_i| \leq C \frac{E_\epsilon(u_\epsilon; \Omega)}{\log \epsilon},
\end{equation}
with $C > 0$ independent of $U \in \Omega$, and such that $\nu_\epsilon$ is close to $Ju_\epsilon$ in the $W^{-1,1}$ norm, namely\footnote{combine (2.10) and (2.12) with (3.7) and (3.14) of [1].}
\begin{equation}
||Ju_\epsilon - \nu_\epsilon||_{W^{-1,1}(A^2U)} \leq C \ell \cdot E_\epsilon(u_\epsilon; \Omega).
\end{equation}
Moreover, the support of $\nu_\epsilon$ is contained in the interior of a set $C'_\epsilon \subset U \sqrt{3}\ell$ given by the union of those cubes of the grid $\mathcal{G}$ having at least one face $Q \subset R_2$, $Q \cap U \neq \emptyset$, such that $dQ \neq 0$. Denote by $I$ the set of indices $i$ in (2.14) for which $dQ_i \neq 0$, or equivalently, $|dQ_i| \geq 1$. By (2.15) we have
\begin{equation}
|C'_\epsilon| \leq \ell^3 \cdot |I| \leq \sum_{i \in I} \ell^3 \cdot |dQ_i| \leq C\ell^2 \frac{E_\epsilon(u_\epsilon; \Omega)}{\log \epsilon},
\end{equation}
so that by (2.10), $|C'_\epsilon| \to 0$ as $\epsilon \to 0$.

Notice moreover that (2.14) and (H$g$) imply that $\frac{\log \epsilon}{g_{\epsilon}} \cdot \nu_\epsilon \rightharpoonup J$ weakly as measures, where $J$ is a measure-valued 2-form in $\Omega$, which is exact and has total variation $|J|(\Omega) \leq C \liminf_{\epsilon \to 0} \frac{E_\epsilon(u_\epsilon; \Omega)}{g_{\epsilon}}$. By (2.15) we finally deduce that $Ju_\epsilon \to J$ in $W^{-1,1}(\Lambda^2U)$ for any $U \in \Omega$, which yields (2.1).

**Step 2.** For $N > 0$ to be chosen below, define $C_\epsilon \equiv C_{N,\epsilon} \equiv \{x \in \Omega, \ \text{dist}(x, C'_\epsilon) \leq 2N\ell\}$ to be the tubular neighborhood of $C'_\epsilon$ of thickness $2N\ell$ intersected with $\Omega$. By (2.16) we have
\begin{equation}
|C_\epsilon| \leq 8N^3|C'_\epsilon| \leq C N^3 \ell^2 \frac{g_{\epsilon}}{\log \epsilon} \to 0 \quad \text{as } \epsilon \to 0,
\end{equation}
as long as $N^3 \leq \ell^{-1}$. In view of (2.10), (2.17) is verified for instance by fixing
\begin{equation}
N \equiv N_\epsilon \equiv |\log \epsilon|^3.
\end{equation}
Observe moreover that
\begin{equation}
E_\epsilon(u_\epsilon; C_\epsilon) \leq E_\epsilon(u_\epsilon; \Omega) \leq K g_{\epsilon} \leq |\log \epsilon|^2.
\end{equation}
Consider the grid $\mathcal{G}^*_\epsilon = \mathcal{G}(b_\epsilon, \bar{e}_1, \ell)$, where $\ell = \ell_\epsilon = |\log \epsilon|^{-10}$ as above and $b_\epsilon$ is chosen such that for an arbitrarily fixed $\delta > 0$, (3.18), (3.19) and (3.20) in Lemma 3.11 of [1] hold true, and moreover (3.17) holds true with $\Omega$ replaced by $C_\epsilon$. In other words, denoting by $R_1^h$ the $h$-skeleton of $\mathcal{G}^*_\epsilon$, $h = 1, 2$, and $R_2$ the union of the faces of the 2-skeleton of $\mathcal{G}^*_\epsilon$ orthogonal to $\bar{e}_1$ we have,
\begin{equation}
\int_{R_2^\epsilon \cap C_\epsilon} e_\epsilon(u_\epsilon)dh^2 \leq (1 + \delta)\ell^{-1}E_\epsilon(u_\epsilon; C_\epsilon),
\end{equation}
\begin{equation}
\int_{R_1^h \cap C_\epsilon} e_\epsilon(u_\epsilon)dh^h \leq C_0 \delta^{-1} \ell^{h-3}E_\epsilon(u_\epsilon; \Omega), \quad h = 1, 2,
\end{equation}

12
S. BALDO, R.L. JERRARD, G. ORLANDI, AND H.M. SONER
Fix an open subset $U \Subset \Omega$. As in Step 1, the procedure of [1] yields a polyhedral cycle
\begin{equation}
M'_\varepsilon = \sum_{Q_i \in R^*_2 \setminus \partial U \neq \emptyset} (-1)^n dQ_i \cdot Q'_i,
\end{equation}
which is a relative boundary in $\bar{U}$ and is supported in $R^*_1 \cap U^\varepsilon$, where $R^*_1$ is the 1-d skeleton of the dual grid to $\mathcal{G}^*$. The corresponding measure-valued 2-form $\nu'_\varepsilon$, defined as in (2.13) by
\begin{equation}
\langle \nu'_\varepsilon, \varphi \rangle := \pi \cdot \sum_{Q_i \in R^*_2 \setminus \partial U \neq \emptyset} (-1)^n dQ_i \cdot \int_{Q'_i} *\varphi, \quad \forall \varphi \in C^\infty_c(\Lambda^2(\Omega)),
\end{equation}
is exact on $U$ and verifies $|\nu'_\varepsilon|(U) \leq C \frac{E^\varepsilon(u(\cdot, \cdot); \Omega)}{\|\log \varepsilon\|}$ with $C > 0$ independent of $U$.

For $x \in \Omega$ define $f(x) := \text{dist}(x, M_\varepsilon)$, so that $f$ is 1-Lipschitz. Denoting by $C^t = \{x : f(x) \leq t\} \cap \Omega$, we have that $C^{2N\ell} \subset C_\varepsilon$.

**Lemma 1.** There exists $t := t_\varepsilon < N \ell$ such that
\begin{equation}
||\nu'_\varepsilon \wedge C^t - \nu_\varepsilon||_{W^{-1,1}(U)} \leq C(\ell + N^{-1}) g_\varepsilon,
\end{equation}
with $C > 0$ independent of $\varepsilon$ and $U$. In particular, the choices of $\ell$ and $N$ (see (2.10) and (2.18)) imply that
\begin{equation}
\left|\frac{\|\log \varepsilon\|}{g_\varepsilon} \cdot \nu'_\varepsilon \wedge C^t \to J \right| \text{ in } W^{-1,1}(\Lambda^2 U)
\end{equation}
and, for any 2-covector $\eta$,
\begin{equation}
\left|\frac{\|\log \varepsilon\|}{g_\varepsilon} \cdot \nu'_\varepsilon \wedge C^t, \eta \right| \to (J, \eta) \text{ in } W^{-1,1}(U).
\end{equation}

We postpone the proof of Lemma 1 to Section 5.6 of the Appendix. By (2.27) and lower semicontinuity of total variation we deduce
\begin{equation}
\left|(J, \eta)\right|(U) \leq \liminf_{\varepsilon \to 0} \left|\left|\frac{\|\log \varepsilon\|}{g_\varepsilon} \cdot \nu'_\varepsilon \wedge C^t, \eta \right|\right|(U)
\end{equation}
\begin{equation}
\leq \liminf_{\varepsilon \to 0} \left|\left|\frac{\|\log \varepsilon\|}{g_\varepsilon} \cdot \nu'_\varepsilon \wedge C^{N\ell}, \eta \right|\right|(U).
\end{equation}

Observe that specializing (2.22) to the case $\varphi = \psi \eta$, with $\psi \in C^\infty_c(\Omega)$, and letting $\psi$ approach the characteristic function of $C^{N\ell} \cap U$, we have
\begin{equation}
\left|(\nu'_\varepsilon \wedge C^{N\ell}, \eta)\right|(U) = \left|(\nu'_\varepsilon, \eta)\right|(C^{N\ell} \cap U) = \pi \cdot \sum_{Q_i \in R^*_2 \setminus \partial U \neq \emptyset} dQ_i \int_{Q_i \cap C^{N\ell} \cap U} *\eta.
\end{equation}

Notice that for any $Q' \subset R^*_1$ such that $Q' \cap C^{N\ell} \neq \emptyset$, its dual element $Q$ is contained in the tubular neighborhood of thickness $\sqrt{3} \ell$ of $C^{N\ell}$, which is a subset of $C^{2N\ell}$, so that in particular $Q \subset C_\varepsilon$. Recalling from the definitions that $*\eta = dx^1$, which is the oriented arclength element along $Q'_i$ for $Q_i \in R^*_2$, we obtain from (2.29) that
\begin{equation}
\left|(\nu'_\varepsilon \wedge C^{N\ell}, \eta)\right|(U) \leq \sum_{Q_i \in R^*_2 \cap C_\varepsilon} \pi \ell \cdot |dQ|.
\end{equation}
One readily verifies, following [1], p. 1435, that (2.10) and (2.19) allow to apply Lemma 3.10 there (which relied in turn on a fundamental estimate in [20, 28]), to efficiently estimate the sum of the degrees $|d_Q|$ in terms of $E_\epsilon(u_\epsilon; C_\epsilon)$. Namely, for any $r > 0$, and any $Q \subset R^n_2 \cap \Omega$ we have

\[(2.31) \quad (1 - c_r(\epsilon))\pi \cdot |d_Q| \leq \frac{1}{|\log \epsilon|} \int_{Q} e_\epsilon(u_\epsilon) dH^2 + \frac{Kr\ell}{|\log \epsilon|} \int_{\partial Q} e_\epsilon(u_\epsilon) dH^1,\]

where $c_r(\epsilon)$ is independent of $Q$, and $c_r(\epsilon) \to 0$ as $\epsilon \to 0$ (see [1], p. 1435). We may thus write

\[(2.32) \quad (1 - c_r(\epsilon)) \sum_{Q \subset R^n_2 \cap C_\epsilon} \pi \cdot |d_Q| \leq \frac{1}{|\log \epsilon|} \int_{R^n_2 \cap C_\epsilon} e_\epsilon(u_\epsilon) dH^2 + \frac{Kr\ell}{|\log \epsilon|} \int_{\partial R^n_2 \cap C_\epsilon} e_\epsilon(u_\epsilon) dH^1.

Combining (2.30) with (2.32), and taking into account (2.20), (2.21), we are led to

\[(2.33) \quad (1 - c_r(\epsilon)) \left(\frac{|\log \epsilon|}{g_\epsilon} \cdot \nu_\epsilon \cdot C^{N_\ell}, \eta\right)(U) \leq (1 + \delta + \frac{Kr}{\delta}) \frac{E_\epsilon(u_\epsilon; C_\epsilon)}{g_\epsilon}.

Passing to the limit as $\epsilon \to 0$, we have, in view of (2.28),

\[(2.34) \quad |(J, \eta)(U)| \leq (1 + \delta + \frac{Kr}{\delta}) \liminf_{\epsilon \to 0} \frac{E_\epsilon(u_\epsilon; C_\epsilon)}{g_\epsilon}.

Taking $r < \delta^2$ and $\delta$ arbitrarily small yields (2.35).

\[\square\]

**Proof in the general case $n \geq 3$.** The main tool used above is the algorithm from [1] for constructing a polyhedral approximation of the Jacobian $Ju_\epsilon$ and hence a measure-valued 2-form $\nu_\epsilon$, with good estimates of $\|Ju_\epsilon - \nu_\epsilon\|_{W^{-1,1}}$ and of $|(\nu_\epsilon, \eta)|(W)$ for suitable subsets $W \subset \Omega$. The procedure in [1] in fact is presented in $\mathbb{R}^n$, $n \geq 3$, and so can be employed in the general case as for $n = 3$, with purely cosmetic differences. For example, in $\mathbb{R}^n$, the analog of $Q'_n$ in (2.13) and elsewhere is now the unique $n-2$ face of the dual grid that intersects $Q_\ell$. Also, different scalings make it convenient to choose $\ell = |\log \epsilon|^{-(3n+1)}$, say, while we still take $N = |\log \epsilon|^3$. Then it remains true that $g_\epsilon \ll N$, which is needed for the proof of Lemma [1] and that $|C'_\epsilon| \to 0$, which follows from the fact that $N^2 \ell^2 |\log \epsilon|^2 \to 0$ as $\epsilon \to 0$, compare (2.17). Modulo changes of this sort, the argument is identical in the general case.

\[\square\]

**Proof of (1.0)**. Recall that we have assumed that $g_\epsilon \leq |\log \epsilon|^2$. Since

\[(2.35) \quad \|Ju_\epsilon - \nu_\epsilon\|_{L^1(\Lambda^2 U)} \leq \|Ju_\epsilon\|_{L^1(\Lambda^2 U)} + \|\nu_\epsilon\|_{L^1(\Lambda^2 U)} \leq CE_\epsilon(u_\epsilon; \Omega) \leq Cg_\epsilon

for any $U \subset \Omega$, we deduce, by interpolation with (2.15),

\[(2.36) \quad \|Ju_\epsilon - \nu_\epsilon\|_{W^{-1,\nu}(\Lambda^2 U)} \leq C(\ell_\epsilon \cdot g_\epsilon)^{1 - \frac{n(p-1)}{p}} \frac{a(p-1)}{g_\epsilon^{\frac{n(p-1)}{p}}} \leq C\ell_\epsilon^{1 - \frac{n(p-1)}{p}} |\log \epsilon|^2.

The conclusion (1.0) follows by choosing $\ell_\epsilon = \ell_{\epsilon,p} = |\log \epsilon|^{-\frac{3n}{n-p(n-1)}}$, so that the r.h.s of (2.36) vanishes.

\[\square\]

### 3. UPPER BOUND

In this section we prove statement (ii) of Theorem 2.
3.1. **Strategy of proof.** The proof is subdivided in various steps. First of all, we reduce in Section 3.2 to considering an appropriate dense class of the domain of the $\Gamma$-limit, using a suitable finite elements approximation. The construction of the recovery sequence will be based on a Hodge decomposition of the limiting momentum $p$, described in Section 3.3 and a discretization of the limiting vorticity $dp$ in terms of a system of lines where the vorticity is concentrated and quantized; this, and associated estimates of the discretized vorticity and related quantities, are the main points in the proof. An argument à la Biot-Savart then allows us to construct $S^1$-valued maps whose Jacobian is concentrated precisely on the discretized vorticity lines, and we obtain our maps $u_\varepsilon$ by adjusting the modulus around the vortex cores. The proof is completed by the verification of the upper bound inequality, which relies crucially on good properties of the discretized vortex lines and estimates satisfied by associated auxiliary functions.

3.2. **Nice dense class.** We say that a 1-form $p$ on a domain $\Omega \subset \mathbb{R}^3$ is rational piecewise linear if $p$ is continuous, and there exist a family of closed polygons $\{P_i\}$ with pairwise disjoint interiors such that $\Omega \subset \bigcup P_i$ with $p$ linear on each $P_i \cap \Omega$, and if the flux $\int_{T_j} dp$ is a rational number for every face $T_j$ of every polyhedron $P_i$.

**Lemma 2.** Suppose that $\Omega \subset \mathbb{R}^3$ is a bounded open subset and that $\partial \Omega$ is of class $C^1$. Given $p \in L^2(\Lambda^1(\Omega))$ such that $dp$ is a measure, and given $\delta > 0$ small, there exists a polygonal set $\Omega_\delta$ with $\Omega \subset \bigcup \Omega_\delta \subset \Omega^p = \{\text{dist}(x, \Omega) < \delta\}$, and such that $\Omega \simeq \Omega_\delta \simeq \Omega^p$, and a rational, piecewise linear 1-form $p_\delta \in L^2(\Lambda^1 \Omega^p_\delta)$, such that $dp_\delta \in L^1(\Lambda^2 \Omega^p_\delta)$ and

\[
\|p - p_\delta\|_{L^2(\Omega)} \leq \delta
\]

(3.1)

\[
\|p_\delta\|_{L^2(\Omega^p_\delta)}^2 \leq \|p\|_{L^2(\Omega)}^2 + \delta
\]

(3.2)

\[
\int_{\Omega^p_\delta} |dp_\delta| \leq |dp|(\Omega) + \delta.
\]

(3.3)

**Proof.** Step 1. We say that a simplex $P$ is rational if, whenever $p = \sum (a^{ij} x_j + b^i) dx_i$ is a linear 1-form on $P$ with $a^{ij}, b^i$ rational for all $i, j$, the flux of $p$ through every face of $P$ is rational. We claim that the unit cube in $\mathbb{R}^n$ can be covered by closed rational simplices with pairwise disjoint interiors. Repeating the same construction in every integer translate of the unit cube, we can cover $\mathbb{R}^3$ by closed rational simplices with pairwise disjoint interiors. Note also that if we dilate the simplices by any rational factor, the resulting simplices are still rational.

Let $S_0$ denote the standard simplex $\text{co} \{0, e_1, e_2, e_3\}$ in $\mathbb{R}^3$, where $\text{co} \{\cdot \}$ denotes the convex hull. If $p$ is linear on $P$ with rational coefficients, then the flux $\int_P dp$ is a rational number when $T = \text{co} \{0, e_i, e_j\}$ (with either orientation), for any choice of $i, j \in \{1, 2, 3\}$ with $i \neq j$. Since $\int_{P_T} dp = 0$, it follows that the flux through the fourth face is rational as well. Thus $S_0$ is rational.

Similarly, for any $i, j \in \{1, 2, 3\}$ with $i \neq j$, let $S_{ij} = \text{co} \{e_i + e_j, e_i, e_j, e_1 + e_2 + e_3\}$. The same argument as above shows that $S_{ij}$ is rational. We next claim that

\[
[0, 1]^3 \setminus (S_0 \cup S_{12} \cup S_{13} \cup S_{23}) = \text{co} \{e_1, e_2, e_3, e_1 + e_2 + e_3\}.
\]

This follows by noting that $[0, 1]^3 \setminus (S_0 \cup (\cup_{i,j} S_{ij}))$ is convex, and that its extreme points are exactly $\{e_1, e_2, e_3, e_1 + e_2 + e_3\}$. Every face of $\text{co} \{e_1, e_2, e_3, e_1 + e_2 + e_3\}$
is also a face of either $S_0$ or of $S_{ij}$ for some $i,j$, so it follows from what we have already said that $\operatorname{co} \{e_1, e_2, e_3, e_1 + e_2 + e_3\}$ is rational.

**Step 2.** By adapting standard approximation techniques for $BV$ functions as in [13], we can find a set $\Omega'$ such that $\Omega \subset \Omega'$, and a 1-form $p' \in C^\infty(\Lambda_1(\Omega'))$, such that $\|p - p'\|_{L^2(\Omega)} \leq \delta/2$, $\|p''\|_{L^2(\Omega')} \leq \|p\|_{L^2(\Omega)} + \delta/2$ and $|\delta p'|(\Omega') \leq |\delta p|(\Omega) + \delta/2$. Choose now a domain $\Omega_\delta$ such that $\Omega \subset \Omega_\delta \subset \Omega'$, and $\Omega_\delta$ is the union of a finite number of cubes with pairwise disjoint interiors and rational edges.

By the discussion in Step 1 above, we can triangulate $\Omega_\delta$ with rational simplices. Performing dyadic subdivisions of each cube, we may also obtain rational triangulations with arbitrarily small mesh size (and with fixed geometry, since the angles appearing in the triangulation will be precisely those in our original decomposition of the unit cube).

By standard interpolation theory from the finite elements method (see for instance [14], Chapter 3), we can find piecewise linear 1-forms which are arbitrarily close to $p'$ in $W^{1,2}(\Omega_\delta)$: it suffices to choose a sufficiently fine triangulation constructed as above, and to take the (unique) piecewise linear form $p_\delta$ which interpolates $p'$ in the vertices of the triangulation. Moreover, an arbitrarily small change of $p_\delta$ in the vertices makes it rational. \hfill $\square$

We will also need the following variant of the above.

**Lemma 2'.** Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded open subset and that $\partial \Omega$ is of class $C^1$. Given an exact measure-valued 2-form $J$, and given $\delta > 0$ small, there exists a polygonal set $\Omega_\delta'$ such that $\Omega \subset \Omega_\delta' \subset \Omega^\delta = \{ \text{dist}(x, \Omega) < \delta \}$, and such that $\Omega \simeq \Omega_\delta' \simeq \Omega^\delta$, and a rational, piecewise linear 1-form $p_\delta' \in L^2(\Lambda_1 \Omega_\delta')$, such that $d p_\delta' \in L^1(\Lambda^2 \Omega_\delta')$ and such that

\begin{equation}
\|p - p_\delta\|_{W^{-1,1}(\Omega)} \leq \delta, \quad \int_{\Omega_\delta'} |d p_\delta'| \leq |J|_{\Omega} + \delta.
\end{equation}

The proof is a straightforward modification of the proof of Lemma 2 once we note from Corollary 1 in the Appendix that any exact measure-valued 2-form $J$ in $\Omega$ can be written in the form $J = \delta p'$ for some $p' \in \cap_{1 \leq q < \frac{n}{n-1}} L^q(\Lambda^2 \Omega)$

### 3.3. Hodge decomposition of $p_\delta$.

Here we refer for notations and basic theory to section 5.2 of the Appendix. We henceforth write $p$ instead of $p_\delta$.

Since basic results on Hodge theory to which we appeal require some smoothness of the domain, we fix an open set $\Omega_\delta$ with smooth boundary, such that $\Omega \subset \Omega_\delta \subset \Omega^\delta$, and such that $\Omega \simeq \Omega_\delta \simeq \Omega^\delta$. In particular we assume that if $\partial \Omega^P_\delta$ has connected components $(\partial \Omega^P_\delta)_i$, $i = 1, \ldots, b$, then there exist disjoint connected open sets $W_1, \ldots, W_b$ such that

\begin{equation}
\Omega^P_\delta \setminus \bar{\Omega}_\delta = \bigcup_{i=1}^b W_i, \quad \partial W_i = (\partial \Omega^P_\delta)_i \cup (\partial \Omega)_i \quad \forall i.
\end{equation}

Consider the Hodge decomposition $p = \gamma + d \alpha + d^* \beta$ on $\Omega_\delta$ satisfying the boundary conditions (5.11). Thanks to Corollary 1 in the Appendix, we know that $\beta = -\Delta^{-1}_N(dp)$, so that in particular $|\beta|_q \leq C_q dp_{1,1} \forall q < 3/2$. Recall that by $L^2$-orthogonality of the Hodge decomposition we have

\begin{equation}
\int_{\Omega_\delta} |p|^2 = \int_{\Omega_\delta} |\gamma|^2 + |d^* \beta|^2 + |d \alpha|^2.
\end{equation}
We emphasize that in what follows, we will carry out most geometric arguments on the polygonal set $\Omega^p$, but the Hodge decomposition always refers to the smooth set $\Omega \subset \Omega^p$.

### 3.4. Discretization of $dp = dd^* \beta$.

We will use different arguments to approximate the different terms in the Hodge decomposition of $p$. Most of our effort will be devoted to $d^* \beta$. As noted above, the first step in our construction is to discretize $dp = dd^* \beta$, which one can think of as the vorticity.

**Proposition 2.** Let $p$ be a rational 1-form supported on $\Omega^p \subset \mathbb{R}^3$, and fix $\eta \in (0,1)$. For any $h \leq \eta^2$ there exists an exact measure-valued 2-form $q_h$ in $\Omega^p$ such that:

(i) $q_h = dd^* \beta_h$, where $\beta_h = -\Delta_h^{-1} q_h$ in $\Omega$.

(ii) $||q_h - dp||_{W^{-1,1}(\Omega^p)} \leq C\eta$.

(iii) $||q_h||_{(\Omega^p)} \leq ||dp||_{L^1(\Omega^p)} + C\eta$.

(iv) $||d^* \beta_h||_{L^p(\Omega)} \leq C_p ||q_h||_{(\Omega)}$, $d^* \beta_h \to d^* \beta^0$ in $L^p(\Omega)$ for $p < 3/2$,

(v) $||d^* \beta^0 - d^* \beta||_{L^2(\Omega)} \leq C\eta$,

where $C > 0$ is independent of $h, \eta, U$. For any $\varphi \in C^0(\Lambda^2 \Omega^p)$ we have the integral representation

(vi) $\langle q_h, \varphi \rangle = h \int_{\Gamma_h} *\varphi = h \sum_{\ell=1}^{m(h)} \int_{\Gamma_h^\ell} *\varphi$,

where $\Gamma_h = \bigcup_{s=1}^{n(h)} L^s_h \subset \Omega^p$, $L^s_h$ an oriented line segment $\forall s, h$, $m(h) < n(h) \leq Kh^{-1}$, and for any $\ell, h$, $\Gamma^\ell_h$ is an oriented simple piecewise linear curve in $\Omega^p$ such that $\partial \Gamma^\ell_h \cap U = \emptyset$ $\forall U \subset \Omega^p$. In particular, we have $|q_h|(U) = h|\Gamma_h \cap U|$ for any $U \subset \Omega^p$. Moreover

(vi) $\text{dist}(L_1, L_2) > c_0 \eta h^{1/2}$ if $L_1, L_2$ are disjoint closed line segments of $\Gamma_h$, with $c_0 > 0$ independent of $h, \eta$.

Finally, if $L_1, L_2$ be two line segments of $\Gamma^\ell_h$ with exactly one endpoint in common, and $\tau_1, \tau_2$ are their respective unit tangents, then

(vii) $\tau_1 \cdot \tau_2 > -1 + C\eta^2$,

for some $C > 0$ independent of $h, \eta$.

**Remark 14.** The discretized vorticity $q_h$ has a 1-dimensional character, in that it is supported on a union of line segments, so that in realizing it as a (measure-valued) 2-form, rather than a 1-form or vector field, we are departing both from the convention discussed in (5.6) and from standard practice in geometric measure theory. However, this departure is natural that $q_h$ is an approximation of the 2-form $dp$, and it is very useful when we want to appeal to Hodge Theory to solve elliptic equations with $q_h$ on the right-hand side, as in conclusion (i) above.

**Remark 15.** The role of the parameter $\eta$ is to guarantee that $q_h$ enjoys certain properties such as a good lower bound on distance between distinct piecewise linear curves in the support of $q_h$, see conclusion (vi) above. These are essential for the verification of the upper bound inequality.
Remark 16. Our arguments (in particular the proof of (iv)) show that there exists 2-form $q^0$ such that $q_h \rightharpoonup q^0$ weakly as measures, as $h \to 0$. In fact our construction is designed to yield an explicit description of $q^0$, see (3.18). This complicates the construction of $q_h$ but immediately yields uniform estimates of $q^0$, needed for (iv), that would otherwise require some work to obtain.

Proof. The proof of Proposition 2 will be divided in several steps.

Proof of (v). We start by constructing $q_h$, which amounts to constructing a collection $\Gamma_h$ of line segments, see (v). Let $\eta \in (0,1)$ be fixed, and let $p$ be a piecewise linear rational 1-form with respect to the triangulation $\{S_i\}$ of $\Omega_h^p$ as fixed in the proof of Lemma 2. In particular, for each $i$ there exists a vector $v_i = (v_i^1, v_i^2, v_i^3)$ such that $dp\big|_{S_i} = \sum_j v_i^j \cdot dx_j$. For any simplex $S_i$, let $b_i$ its barycenter, and let

$$\tilde{S}_i = (1 - \eta) \cdot S_i + \eta \cdot b_i \subset S_i$$

be a homothetic copy of $S_i$, and let $T_{ij}, \tilde{T}_{ij}, j = 1, \ldots, 4$ be the 2-faces of $S_i, \tilde{S}_i$ respectively, with the induced orientations.

We will arrange that within each $\tilde{S}_i$, our discretization of $dp$ is supported on a finite union of line segments exactly parallel to $v_i$. In order to do this and to match fluxes across the faces of each $S_i$, we discretize the flux through the faces of each $S_i$ and each $\tilde{S}_i$ in related though different ways.

For every $i$ and for $j \neq k \in \{1, \ldots, 4\}$, define $T_{ijk} = \pi^{-1}(\pi(T_{ij}) \cap \pi(T_{ik})) \cap T_{ij}$ (with the orientation of $T_{ij}$), where $\pi \equiv \pi_i$ is the projection on the 2-plane $(v_i)^\perp$. One may think of $T_{ijk}$ as the portion of $T_{ij}$ connected to $T_{ik}$ by flux lines of $dp$. Further define

$$\phi_{ij} = \int_{T_{ij}} dp \in \mathbb{Q}, \quad \phi_{ijk} = \int_{T_{ijk}} dp = \frac{|T_{ij}|}{|T_{ijk}|} \phi_{ij}.$$ 

Clearly $\phi_{ij} = \sum_{j \neq k} \phi_{ijk}$, and hence $\phi_{ijk} \in \mathbb{Q}$, as solutions of a linear systems with rational data. Let $\phi^{-1}$ be the least common denominator of $\{\phi_{ijk}\} \in \mathbb{N}$, so that $\phi_{ijk} \phi^{-1} \in \mathbb{Z}$.

For $N \in \mathbb{N}$, we define $h_N := \frac{\phi}{nN}$, so that $\frac{\phi_{ijk}}{h_N} \in \mathbb{Z}$ for all $i, j, k$, and similarly $\frac{\phi_{ij}}{h_N} \in \mathbb{Z}$ for every $i, j$. We will prove the proposition for every $h_N$ such that $h_N < \eta^2$; for arbitrary $h < \eta^2$, the conclusions of the proposition then hold if we define $q_h := q_{h_N}, \beta_h := \beta_{h_N}$, for $N$ such that $h_N \leq h < h_{N-1}$.

We henceforth fix an arbitrary $N$ such that $h_N < \eta^2$, and we drop the subscript and write simply $h$.

We first discretize $dp$ on every $T_{ij}$. In order to avoid discretizing any 2-face twice in inconsistent ways, we define

$$\mathcal{T} := \{T_{ij} : \phi_{ij} > 0 \text{ or } T_{ij} \subset \partial \Omega_h^p\}.$$ 

For $T_{ij} \in \mathcal{T}$, let $m = m_{ij} := \frac{\phi_{ij}}{h} \in \mathbb{Z}$, and let $\ell = \ell_{ij}$ verify $(\ell_{ij} - 1)^2 < m \leq \ell_{ij}^2$. Now partition $T_{ij}$ into $\ell_{ij}^2$ closed triangular pieces $\{T_{ij}^a\}_{a=1}^{\ell_{ij}^2}$ with pairwise disjoint interiors, each one isometric to $\ell_{ij}^{-1} T_{ij}$. Select $m$ of these triangles, and let $\{s_{ij}^a\}_{a=1}^{m}$ be the barycentres of the chosen triangles.

If $T_{ij} \notin \mathcal{T}$, then $T_{ij} = -T_{ij'}$ for some $T_{ij'} \in \mathcal{T}$, we set $m = m_{ij} := m_{ij'},$ and $s_{ij}^a = s_{ij'}^a$ for $a = 1 \ldots m_{ij}.$
Next we consider \( \{ \tilde{T}_{ij} \} \). For \( i, j, k \), let \( \tilde{T}_{ijk} \equiv (1 - \eta) \cdot T_{ijk} + \eta \cdot b \) with the orientation of \( T_{ijk} \) and define

\[
\tilde{T} := \{ \tilde{T}_{ijk} : \phi_{ijk} > 0 \}.
\]

Now proceed as above: for each \( \tilde{T}_{ijk} \in \tilde{T} \), let \( m = m_{ijk} := \frac{\tilde{\phi}_{ijk}}{\eta} \in \mathbb{Z} \) and \( \ell_{ijk} := [\sqrt{m}] \), and partition \( \tilde{T}_{ijk} \) into \( \ell^2 \) closed triangular pieces \( \{ \tilde{T}^a_{ijk} \}_{a=1}^{\ell^2} \) with pairwise disjoint interiors, each one isometric to \( \ell^{-1}\tilde{T}_{ijk} \). Select \( m \) of these triangles, and let \( \{ \tilde{s}^a_{ijk} \}_{a=1}^{m} \) be the barycentres of the chosen triangles.

If \( T_{ijk} \notin \tilde{T} \), then \( \phi_{ijk} \leq 0 \). If \( \phi_{ijk} = 0 \) (which in particular happens if \( T_{ijk} = \emptyset \)) we do nothing. If \( \phi_{ijk} < 0 \), then noting that our orientation conventions imply that \( \phi_{ijk} = -\phi_{ikj} \), we see that \( \tilde{T}_{ikj} \in \tilde{T} \), and we define \( \tilde{s}_{ijk} = \pi^{-1}_i \pi^{-1}_j (\tilde{s}^a_{ikj}) \cap T_{ijk} \).

We now define piecewise linear curves as follows. First, for every \( T_{ijk} \in \tilde{T} \), we define

\[
\tilde{\Gamma}^a_{ijk} := [\pi^{-1}_i (\pi^{-1}_j (\tilde{s}^a_{ikj}))] \cap \tilde{S}_i,
\]

oriented so that \( \partial \tilde{\Gamma}^a_{ijk} = \tilde{s}^a_{ijk} - \tilde{s}^a_{ikj} \).

Here and below, if \( c \) is an oriented piecewise smooth curve, we write \( \partial c = p - q \) to mean that \( \int_c df = f(p) - f(q) \) whenever \( f \) is a smooth function. We define

\[
\Gamma_i = \sum_{j,k,a} \tilde{\Gamma}^a_{ijk}, \quad \text{so that } \Gamma_i \subset \tilde{S}_i, \quad \text{and}
\]

\[
\partial \Gamma_i = \sum_{j,k,a} \text{sign}(\phi_{ijk}) \tilde{s}^a_{ijk}.
\]

Moreover that \( \Gamma_i \) is the collection of segments with the smallest total arclength satisfying this condition (as the segments of \( \Gamma_i \) are all parallel to each other.)

Now for each \( i, j \), let \( P_{ij} := \{(1 - \lambda)x + \lambda b_i : x \in T_{ij}, 0 < \lambda < \eta \} \) be the pyramidal frustum having bases \( T_{ij} \) and \( \tilde{T}_{ij} \), and let \( \Gamma_{ij} \) be a collection of (oriented) line segments such that

\[
\partial \Gamma_{ij} = \sum_a \text{sign}(\phi_{ij}) \tilde{s}^a_{ij} - \sum_{k,a} \text{sign}(\phi_{ijk}) \tilde{s}^a_{ijk},
\]

and that minimizes the total arclength among the set of all collections of line segments satisfying the constraint (3.9). Such collections exist, since \( \text{sign}(\phi_{ij}) = \text{sign}(\phi_{ijk}) \) and \( m_{ij} = \sum_{k \neq j} m_{ijk} \), so that \( \sum_{j,a} \text{sign}(\phi_{ij}) - \sum_{j,k,a} \text{sign}(\phi_{ijk}) = 0 \). Hence \( \Gamma_{ij} \) is well-defined, and clearly \( \Gamma_{ij} \subset P_{ij} \).

We define \( \Gamma_h \) to be the union \( \bigcup \Gamma_i \cup \Gamma_{ij} \) of the families of segments constructed above, and \( n(h) \) to be the total number of segments comprising \( \Gamma_h \). We also define \( \Gamma^\ell_h \) for \( \ell = 1, \ldots, m(h) \), where \( m(h) \leq N(h) \), to be the polyhedral curves realizing the connected components of \( \Gamma_h \). It follows from (3.10), proved below, that \( \partial \Gamma^\ell_h = 0 \) in \( \Omega^P_h \).

Finally, we define the measure-valued 2-form \( q_h \) to satisfy statement (v).

In the following we will write “a region” to refer either to one of the \( \tilde{S}_i \) or one of the \( P_{ij} \). We remark that the definition of \( \Gamma_h \) states that, in the language of Brezis, Coron, and Lieb \cite{Brezis}, its restriction to any region is a minimal connection, subject to the condition (3.8) in \( \tilde{S}_i \) and (3.9) in \( P_{ij} \).

**Proof of (i).** By Lemma 11 and Corollary 11 in the Appendix, it suffices to check that \( dq_h = 0 \) in \( \Omega_h \) and that \( \int_{\partial \Omega_h} (q_h)_{\gamma} = 0 \) for every connected component \( (\partial \Omega_h)_i \) of \( \partial \Omega_h \).
To do this, fix any \( f \in C^\infty_c(\mathbb{R}^3) \), and note that (v), (6.8), (6.9) imply that
\[
\langle dq_h, * f \rangle = \langle q_h, d^* f \rangle = \langle q_h, * df \rangle = h \sum_i \int_{\Gamma_i} df + \sum_{i,j} \int_{\Gamma_{ij}} df = h \sum_{i,j,a} (\text{sign} \phi_{ij}) f(s_{ij}^a).
\]
Here all terms of the form \( f(s_{ij}^a) \) have cancelled, since they occur twice, with opposite signs, in (6.8) and (6.9). If \( s_{ij}^a \in \Omega_k^{ij} \), then our construction implies that there exists exactly one \((i', j', a') \neq (i, j, a)\) such that \( s_{ij}^a = s_{i'j'}^{a'} \), and moreover that \( \text{sign} \phi_{ij} = -\text{sign} \phi_{i'j'} \). Thus all contributions from \( \Omega_k^{ij} \) vanish, and the above reduces to
\[
\langle dq_h, * f \rangle = h \sum_{\{i,j,a; s_{ij}^a \in \partial \Omega_k^{ij}\}} (\text{sign} \phi_{ij}) f(s_{ij}^a).
\]
In particular, by considering \( f \in C^\infty(\Omega_\delta) \) we see that \( dq_h = 0 \) in \( \Omega_\delta \).

Now fix some component \( (\partial \Omega_\delta)_k \) of \( \partial \Omega_\delta \). Then (6.8) implies that
\[
0 = \int_{W_k} dg = \int_{\partial W_k} 1 = \int_{(\partial \Omega_\delta)_k} (q_h)_\tau - \int_{(\partial \Omega_h)_k} (q_h)_\tau.
\]
Moreover, it follows from (6.9), (6.10), and the definition of \( (q_h)_\tau \) (see (5.8) in the Appendix) that
\[
\int_{(\partial \Omega_\delta)_k} (q_h)_\tau = \sum_{(i,j): T_{ij} \subset (\partial \Omega_\delta)_k} h(\text{sign} \phi_{ij}) m_{ij}.
\]
However, the definitions of \( m_{ij} \) and \( \phi_{ij} \) imply that the above quantity equals
\[
\sum_{(i,j): T_{ij} \subset (\partial \Omega_\delta)_k} \phi_{ij} = \sum_{(i,j): T_{ij} \subset (\partial \Omega_\delta)_k} \int_{T_{ij}} dp = \int_{(\partial \Omega_\delta)_k} dp = 0.
\]
Then, as remarked above, (i) follows from Lemma 11 and Corollary 1.

**Proof of (iii).** We next estimate the mass of \( q_h \). We will bound the mass on each region \( R \), and then sum up the estimates. We begin by comparing the fluxes of \( q_h \) and \( dp \) across \( \partial R \).

**Lemma 3.** Let \( R \) be a region, and let \( (dp)_\tau \) and \( (q_h)_\tau \) be the tangential parts of \( dp \) and \( q_h \), respectively, on \( \partial R \), i.e., the measures in \( \mathbb{R}^3 \), supported in \( \partial R \), defined as discussed in the Appendix, see (5.8). Then there exists a constant \( C = C(dp, \Omega_\delta) \), independent of \( \eta \) and \( h \), such that
\[
\|(q_h - dp)_\tau\|_{W^{-1,1}(\mathbb{R}^3)} \leq C(\eta + h^{1/2}) \leq C \eta,
\]
**Proof.** First consider the case of a pyramidal frustum \( P_{ij} \).

Arguing as in the proof of (i), we find from (6.9) that \( (q_h)_\tau = h \sum_a \text{sign} \phi_{ij} s_{ij}^a - h \sum_k a \text{sign} \phi_{ijk} s_{ijk}^a \). Similarly, the definition of \( \phi_{ij} \) and the fact that \( T_{ij} \) and \( T_{ij} \) are parallel implies that
\[
\int_{\partial P_{ij}} f(dp)_\tau = \frac{\phi_{ij}}{|T_{ij}|} \int_{T_{ij}} f d\mathcal{H}^2 - \frac{\phi_{ij}}{|T_{ij}|} \int_{T_{ij}} f d\mathcal{H}^2 + O(||f||_{\infty})
\]
where the error term comes from neglecting \( \partial P_{ij} \setminus (T_{ij} \cup T_{ij}) \), which has area bounded by \( C \eta \).
Thus for any continuous $f$,
\[
\int_{\partial P_{ij}} f(dp - q_h) = \left[ \frac{\phi_{ij}}{|T_{ij}|} \int_{T_{ij}} f \, dH^2 - h \sum_a \text{sign}(\phi_{ij}) f(s^{\alpha}_{ij}) \right] - \left[ \frac{\phi_{ij}}{|T_{ij}|} \int_{T_{ij}} f \, dH^2 - h \sum_{a,k} \text{sign}(\phi_{ij}) f(s^{\alpha}_{ijk}) \right] + O(\|f\|_{\infty} \eta).
\]

We will consider only the second term on the right-hand side (which is slightly harder). We assume for simplicity that $\phi_{ij} > 0$; the case $\phi_{ij} < 0$ is essentially identical. Noting that $\frac{\phi_{ij}}{|T_{ij}|} = \frac{\phi_{ij}}{|T_{ijk}|}$ and that $|T_{ijk}| = \ell_{ijk}^2 |T_{ij}|$, and using notation from the first step above, we have
\[
\int_{T_{ij}} f(dp - q_h) = \left( \frac{\phi_{ij}}{|T_{ij}|} \int_{T_{ij}} f \, dH^2 - h \sum_{a,k} f(s^{\alpha}_{ijk}) \right)
\]
\[
= \left( \frac{\phi_{ij}}{|T_{ij}|} - \frac{\phi_{ijk}}{|T_{ijk}|} \right) \int_{T_{ij}} f \, dH^2 + \sum_{k,a} \frac{\phi_{ijk}}{|T_{ijk}|} \int_{T_{ijk}} f - f(s^{\alpha}_{ijk}) \, dH^2
\]
\[
+ \sum_{a,k} \left( \frac{\phi_{ijk}}{\ell_{ijk}} - h \right) f(s^{\alpha}_{ijk}) + \sum_{k} \frac{\phi_{ijk}}{|T_{ijk}|} \int_{T_{ijk} \setminus \cup_a T_{ijk}} f \, H^2.
\]

It is clear from the definition of $\phi_{ij}$ that $|\phi_{ij}| \leq \|dp\|_{\infty} |T_{ij}| \leq C$, and since by definition $(\ell_{ijk} - 1)^2 < m_{ijk} = h^{-1} \phi_{ijk} \leq \ell_{ijk}^2,$
\[
|\phi_{ijk} - \phi_{ij} - h| \leq \frac{2}{m_{ijk}} \phi_{ijk} \leq \frac{C}{m_{ijk}} (h \phi_{ijk})^{1/2} \leq \frac{C}{m_{ijk}} \sqrt{h}.
\]

Similarly one checks that $|T_{ijk} \setminus \cup_a T_{ijk}| = |T_{ijk}| |1 - \frac{m_{ijk}}{\ell_{ijk}}| \leq C |T_{ijk}| \sqrt{h}$. Note also that $|f(x) - f(s^{\alpha}_{ijk})| \leq \|df\|_{\infty} \text{diam}(T_{ijk}) \leq C \|df\|_{\infty} \sqrt{h}$ for $x \in T_{ijk}$. Taking these into account, elementary calculations yield
\[
\left| \int_{T_{ij}} f(dp - q_h) \right| \leq C(\eta + \sqrt{h}) \|f\|_{W^{1,\infty}}.
\]

Since similar computations apply to $T_{ij}$, we deduce that $|\int_{\partial P_{ij}} f(dp - q_h)| \leq C \eta \|f\|_{W^{1,\infty}}$ for every $P_{ij}$. If the region $R$ is a simplex $\bar{S}_i$, then $\int_{\partial S_i} f(dp - q_h)$ is a sum of terms of exactly the form $\int_{T_{ij}} f(dp - q_h)$ already estimated (now with the opposite orientation) and so the conclusion follows in this case as well. \hfill \Box

For future reference, we remark that the above proof shows that that (3.13)
\[
\int_{T_{ij}} f(dp - q_h) \leq C \sqrt{h} \|f\|_{W^{1,\infty}}, \quad \int_{P_{ij}} f(dp - q_h(1 - \eta)^2 - q_h) \leq C \sqrt{h} \|f\|_{W^{1,\infty}}.
\]

Indeed, every term on the right-hand side of (3.12) can be bounded by $Ch^{1/2}$ except for the term $(\frac{\phi_{ij}}{|T_{ij}|} - \frac{\phi_{ijk}}{|T_{ijk}|}) \int_{T_{ij}} f \, dH^2$. This term is not present when one considers $T_{ij}$ rather than $\bar{T}_{ij}$, and it is also not present if one considers $\bar{T}_{ij}$, but weighting the integrand as shown, since $(1 - \eta)^2 = |\bar{T}_{ij}|/|T_{ij}|$, so that (3.13) follows from our earlier arguments.
We will need the following result about continuous dependence of the minimal connection upon its boundary datum.

**Lemma 4.** Let $K$ be a compact convex domain in $\mathbb{R}^3$, $\zeta$ a measure supported on $\partial K$ such that $\int_{\partial K} \zeta = 0$. Then we have

$$\min \{ ||\alpha|| \equiv |\alpha|(K), \ d\alpha = 0 \ in \ K, \ \alpha_\tau = \zeta \ on \ \partial K \} \leq C ||\zeta||_{W^{-1,1}(\mathbb{R}^3)}.$$

The proof of this lemma is postponed to Section 5.3 in the Appendix. Let us apply Lemma 4 first with $K = P_{ij}$, $\zeta = (q_h - dp)\tau$ and let $\alpha_h$ be the measure 2-form that realizes the minimum. By (3.11) and Lemma 3 we deduce $|\alpha_h|(P_{ij}) \leq C\eta$.

As remarked above, the restriction of $\Gamma_h$ to any region $R$ is a minimal connection, and as a consequence, it follows from results proved in Brezis, Coron and Lieb [11] that $q_h \ll R$ has minimal mass among all 2-form-valued measures $q'$ in $R$ such that $(q')\tau = (q_h)\tau$ on $\partial R$ (not merely those corresponding to a union of oriented line segments). We thus have

$$\int_{\partial K} \zeta \leq \int_{\partial K} \|q_h\|.$$

Next, applying Lemma 4 with $K = \tilde{S}_i$, $\zeta = (q_h - dp)\tau$ and arguing exactly as above, we obtain

$$|q_h|([\tilde{S}_i]) \leq \int_{\tilde{S}_i} |dp| + C\eta.$$

Statement (iii) follows by summing over all regions.

**Proof of (ii).** It suffices to show that for every region $R$,

$$\langle \varphi, (dp - q_h) \ll R \rangle = \int_R \langle \varphi, dp \rangle - \langle \varphi, q_h \ll R \rangle \leq C\eta ||\varphi||_{W^{1,\infty}}$$

for every $\varphi \in C_c^\infty(\Lambda^2\mathbb{R}^3)$. This is clear if $R = P_{ij}$, since $|P_{ij}| \leq C\eta$ for all $i, j$, so that $\|dp\|_{L^1(P_{ij})} \leq C\eta$, and hence $|q_h|(P_{ij}) \leq C\eta$ by (3.16).

If $R = S_i$, then we assume, after changing coordinates, that $dp = \lambda dx^2 \wedge dx^3$ on $\tilde{S}_i$ for some $\lambda \in \mathbb{R}$. Now fix $\varphi \in C_c^\infty(\Lambda^2\mathbb{R}^3)$ and let $\Phi \in C_c^\infty(\mathbb{R}^3)$ be a function such that $(\ast d\Phi, dx^2 \wedge dx^3) = (\varphi, dx^2 \wedge dx^3)$ in $S_i$, and such that $||\Phi||_{W^{1,\infty}} \leq C||\varphi||_{W^{1,\infty}}$. Indeed, $(\ast d\Phi, dx^2 \wedge dx^3) = \Phi_{x_1}$, so we can take

$$\Phi(x) := \chi(x) \int_{-\infty}^{x_1} \chi(x)(s, x_2, x_3), dx^2 \wedge dx^3) \ ds$$

where $\chi \in C_c^\infty(\mathbb{R}^3)$ satisfies $\chi \equiv 1$ on $S_i$. Then clearly $\langle dp \ll \tilde{S}_i, \varphi \rangle = \langle dp \ll \tilde{S}_i, \ast d\Phi \rangle$ and it follows from the form of $dp$ and the definition (ie statement (v)) of $q_h$ that $\langle q_h \ll \tilde{S}_i, \varphi \rangle = \langle q_h \ll \tilde{S}_i, \ast d\Phi \rangle$. Thus Lemma 3 implies that

$$\langle \varphi, (dp - q_h) \ll \tilde{S}_i \rangle = \langle \ast d\Phi, (dp - q_h) \ll \tilde{S}_i \rangle = \int_{\partial S_i} \Phi(dp - q_h)\tau \leq C\eta ||\phi||_{W^{1,\infty}}.$$

Thus $\|dp - q_h \ll S_i\|_{W^{-1,1}(\mathbb{R}^3)} \leq C\eta$.

**Proof of (iv).** The estimate $\|d^*\beta_h\|_{L^p(\Omega_3)} \leq C_p|q_h|(\Omega_3) \leq C$, $1 \leq p < 3/2$, follows immediately from Corollary 1 in the Appendix. Thus $d^*\beta_h$ is weakly precompact in these $L^p$ spaces, and we only need to identify the limit, prove that it is unique, and estimate its $L^2$ distance from $d^*\beta$. 
To do this we will show that $q_h \to q^n$ in $W^{-1,1}(\Omega_h)$, where $q^n = (1 - \eta)^{-2} dp$ on $\tilde{S}_i$, while on $\partial P_{ij}$, $q^n$ is defined to be the unique minimizer of the problem

$$
\min \{ \alpha(P_{ij}), \, d\alpha = 0 \text{ in } P_{ij}, \, \alpha_T = \zeta \text{ on } \partial P_{ij} \},
$$

where $\zeta = (dp)_\Gamma$ on $T_{ij}$, $\zeta = (1 - \eta)^{-2}(dp)_\Gamma$ on $\tilde{T}_{ij}$ and $\zeta = 0$ on the remaining faces of $\partial P_{ij}$. Since then $\beta^n = -\Delta^{-1} q^n$, the uniqueness of $\beta^n$ will follow, and we will deduce the estimates of $\beta^n$ from the explicit form of $q^n$, which we find below.

We consider first a truncated pyramidal region $P_{ij}$, which is the harder case. The uniform mass bounds (3.15) imply that $q_h \leq q^n$, such that we have proved that $q_h \to q^n$ in $W^{-1,1}(\Omega_h)$. We consider first a truncated pyramidal region $P_{ij}$, which is the harder case. The uniform mass bounds (3.15) imply that $\lambda(P_{ij}) \leq (3.18)$ and standard elliptic estimates. This concludes the proof of statement (iv).

We now argue that the unique minimizer (3.17) is given by

$$q^*(x) = a \frac{(x - b_i)}{||x - b_i||^2} \cdot \delta x$$

where $b_i$ denotes the barycenter of $S_i$, $\nu_{ij}$ is the unit normal to $T_{ij}$, and $a \in \mathbb{R}$ is adjusted so that $q^* = \zeta$. (A calculation shows that such a number $a$ exists and also that $d\beta = 0$.) The (unique) minimality of $q^*$ now follows from a calibration argument. We briefly recall the idea: Let $f(x) = |x - b_i|$, so that $df = \sum \frac{(x - b_i)}{|x - b_i|} \cdot \delta x$, and $(\alpha, \gamma) = \langle q^*(\cdot), \alpha \rangle$ on $P_{ij}$. For any other 2-form valued measure $\gamma^*$ supported in $P_{ij}$ such that $d\gamma^* = 0$ in $P_{ij}$ and $\alpha_T = \zeta$ on $\partial P_{ij}$, we have

$$|\gamma^*| = \langle q^* \bigwedge P_{ij}, \alpha \rangle \leq \int_{\partial P_{ij}} \gamma^* = \langle q^*, \alpha \rangle \leq |q^*| \leq |q^*|\langle P_{ij} \rangle,$$

since $|\alpha| \leq 1$ everywhere. Hence $q^*$ is a minimizer. Furthermore, if equality holds then, heuristically, $q^*$ is parallel to $\alpha_T$, or more precisely, $q^*$ has the form

$$\langle q^*, \gamma \rangle = \int_{P_{ij}} \gamma^* \| x - b_i \|^{-2} \cdot \gamma \gamma \cdot \delta x \mu$$

for some measure $\mu$. Then one can check that $q^*$ is the only measure-valued 2-form of this form such that $d\gamma = 0$ in $P_{ij}$, $\alpha_T = \zeta$ on $\partial P_{ij}$. Hence $q^* = \zeta$ as asserted.

The proof that $q_h \to q^n$ in $W^{-1,1}(\Omega_h)$ can be carried out on exactly the same lines, except that the limit has a simpler form. It can also be proved by arguing as in the proof of (ii), but using (3.13) instead of (iii). Thus we have proved that $q_h \to q^n$ in $W^{-1,1}(\Omega_h)$.

From the explicit form of $q^n$, noting that $\int_{\partial P_{ij}} |P_{ij}| \leq C \eta$, we see that

$$||q^n - dp||^2_{L^2(\Omega_h)} \leq C \eta.$$

Thus $\|d^* \beta^n - d^* \beta\|_2^2 = \|d^* \Delta^{-1} (q^n - dp)\|_2^2 \leq C \eta$, by (3.19) and standard elliptic estimates. This concludes the proof of statement (iv).

**Proof of (vi).** We prove now the separation properties of the polyhedral curves $\Gamma_h$. Let $L_1$ and $L_2$ be closed line segments of $\Gamma_h$, with endpoints $s_1^\pm$ and $s_2^\pm$, and assume that $L_1$ and $L_2$ are disjoint, so that in particular $\{s_1^+ \} \cap \{s_2^-\} = \emptyset$. We consider first a truncated pyramidal region $P_{ij}$, which is the harder case. The uniform mass bounds (3.15) imply that $q_h \leq q^n$, such that we have proved that $q_h \to q^n$ in $W^{-1,1}(\Omega_h)$.
If \( L_1, L_2 \) belongs to non-adjacent regions of the family \( \{ \tilde{S}_i, P_{ij} \} \) then the conclusion is obvious, so we assume that this is not the case, and we claim that
\[
(3.20) \quad \text{dist}(s_m^\pm, L_n) \geq c_2n^{1/2} \quad \text{for } m \neq n, n \in \{1, 2\}.
\]
To see this, let \( F \) denote the face (some \( T_{ij} \) or \( \tilde{T}_{ij} \)) containing \( s_1^+ \) say. If \( F \) also contains an endpoint of \( L_2 \) (for example \( s_2^+ \)) then by construction \( L_2 \) forms an angle of at least \( cn \) with \( F \), and \( |s_1^+-s_2^+| \geq ch^{1/2} \), and so \( (3.20) \) follows from elementary geometry. The claim is still clearer if neither endpoint of \( L_2 \) is contained in \( F \).

It is evident that \( (3.20) \) implies (vi) if \( L_1 \) and \( L_2 \) belong to distinct but adjacent regions. If \( L_1 \) and \( L_2 \) belong to the same region, then in view of the minimality property of \( q_n \), we obtain statement (vi) from \( (3.20) \) and the following Lemma:

**Lemma 5.** Let \( \{ s_m^\pm \}_{m=1,2} \) satisfy \( |s_1^+-s_1^-| + |s_2^+-s_2^-| \leq |s_1^+-s_2^-| + |s_2^+-s_1^-| \). Also, let \( L_m \) be the segment joining \( s_m^\pm \) and \( s_m^\pm \), for \( m = 1, 2 \). Then
\[
(3.21) \quad \text{dist}(L_1, L_2) \geq \frac{1}{\sqrt{2}} \min_{m \neq n} \text{dist}(s_m^\pm, L_n).
\]

**Proof.** Let \( Q_m \in L_m, m = 1, 2 \) be such that \( \text{dist}(L_1, L_2) = |Q_1 - Q_2| = d \). If either \( Q_m \) is an endpoint then the conclusion is clear, so we assume that both are interior points, in which case the segment from \( Q_1 \) to \( Q_2 \) is orthogonal to both \( L_1, L_2 \). We may then assume without loss of generality that the midpoint \( \frac{Q_1+Q_2}{2} \) is the origin, and that \( Q_1 = (0, 0, \frac{d}{2}), Q_2 = (0, 0, -\frac{d}{2}) \), and moreover that \( L_1 \) and \( L_2 \) are parallel to the directions \( (\cos \theta, \sin \theta, 0), (\cos \theta, -\sin \theta, 0) \) respectively, for some \( \theta \). Define \( \tilde{s}_1^+ = (\pm \lambda \cos \theta, \pm \lambda \sin \theta, \frac{d}{2}), \tilde{s}_2^+ = (\pm \lambda \cos \theta, \mp \lambda \sin \theta, -\frac{d}{2}) \), for \( \lambda > 0 \) chosen so that one of the \( s_m^\pm \) coincides with the closest point to 0 among the original endpoints.

Our hypothesis and the triangle inequality imply that \( |\tilde{s}_1^+-\tilde{s}_2^-| + |\tilde{s}_2^+-\tilde{s}_1^-| \geq |\tilde{s}_1^+-\tilde{s}_1^-| + |\tilde{s}_2^+-\tilde{s}_2^-| \), which reduces to
\[
2\sqrt{4\lambda^2 \cos^2 \theta + d^2} \geq 4\lambda = 2\sqrt{4\lambda^2 (\cos^2 \theta + \sin^2 \theta)}, \quad \text{so that } d^2 \geq 4\lambda \sin^2 \theta.
\]

On the other hand, assuming for concreteness that \( \tilde{s}_1^+ \) agrees with the original endpoint \( s_1^+ \), then since \( \tilde{s}_2^+ \in L_2 \), we use the above inequality to find that we
\[
\text{dist}(s_1^+, L_2) \leq |\tilde{s}_1^+-\tilde{s}_1^-| = \sqrt{4\lambda^2 \sin^2 \theta + d^2} \leq \sqrt{2}d.
\]

**Proof of (vii).** Finally, suppose that \( L_1 \) and \( L_2 \) are adjacent, and that \( L_1 \) precedes \( L_2 \) in the ordering induced by their respective orienting unit tangents \( \tau_1, \tau_2 \). Decompose \( \tau_i \) as \( \tau_i^\perp + \tau_i^\parallel \), where for \( i = 1, 2 \), \( \tau_i^\perp \) is orthogonal to the face \( T_{ij} \) that contains the common endpoint of \( L_1 \) and \( L_2 \). The orientation conventions imply that \( \tau_1^\perp, \tau_2^\perp > 0 \), and, as noted above, each segment forms an angle of at least \( cn \) with \( T_{ij} \), which implies that \( |\tau_i^\perp| \geq cn \) for \( i = 1, 2 \). Statement (vii) follows directly.

The proof of Proposition 2 is now complete.

**3.5. Pointwise estimates for \( d^*\beta_h \).** Let \( G(x) = (4\pi)^{-1} |x|^{-1} \) be the Poisson kernel in \( \mathbb{R}^3 \). We may write
\[
(3.22) \quad d^*\beta_h = d^*(G * q_h) + \Psi_h \quad \Psi_h = d^*(-\Delta_N^{-1} q_h - G * q_h).
\]
In view of statement (i), we deduce that \( d\Psi_h = d^*\Psi_h = 0 \) in \( \Omega_3 \), i.e. \(-\Delta \Psi = 0 \) in \( \Omega \) and \( \Psi_N = -d^*(G * q_h)_N \) on \( \partial \Omega_3 \). From the decomposition \( (3.22) \) we will deduce pointwise and integral estimates for \( d^*\beta_h \).
We begin with the term $d^*(G * q_h) = G * d^*q_h$. The integral representation of $d^*(G * q_h)$ through the Biot-Savart law takes the form
\begin{equation}
(3.23) \quad d^*(G * q_h)(x) = \frac{1}{4\pi} \sum_{i,j=1}^3 \frac{1}{|x-y|^3} \int_{\Gamma_h} (x_j - y_j) dy_k,
\end{equation}
where $\epsilon_{ijk}$ is the usual totally antisymmetric tensor. This can be justified for example by noting that $\langle d^*(G * q_h), \phi \rangle = \langle q_h, G * d\phi \rangle$, since $G$ is even, and then using statement (v) of Proposition 2 to explicitly write out the right-hand side. From (3.23) we readily deduce

**Lemma 6.** Let $l_1, l_2 > 0$, $L = \{(0,0,z)\, |\, -l_1 \leq z \leq l_2\} \subset \mathbb{R}^3$, $q$ the associated measure 2-form, i.e. $\langle q, \varphi \rangle = \int_L \star \varphi$ for $\varphi \in C^0(\Lambda^2 \mathbb{R}^3)$. Then
\begin{equation}
(3.24) \quad d^*(G * q) = \frac{xy dx - ydx}{4\pi (x^2 + y^2)} \left( \frac{l_2 - z}{\sqrt{x^2 + y^2 + (l_2 - z)^2}} + \frac{l_1 + z}{\sqrt{x^2 + y^2 + (l_1 + z)^2}} \right).
\end{equation}
As a result,\n\begin{equation}
(3.25) \quad |d^*(G * q)(p_0)| \leq \frac{1}{2\pi \cdot \text{dist}(p_0, L)} \quad \text{for every } p_0 \in \mathbb{R}^3.
\end{equation}

**Proof.** We obtain (3.24) by particularizing (3.23) to the case $\Gamma_h = L$. We easily deduce (3.25) from (3.24) if $p_0 = (x_0, y_0, z_0)$ with $-l_1 \leq z_0 \leq l_2$, in which case dist$(p_0, L) = \sqrt{x_0^2 + y_0^2}$. If $z_0 > l_2$ then, writing $r_0 = (x_0^2 + y_0^2)^{1/2}$, since $\lambda \rightarrow \sqrt{\frac{\lambda}{\sqrt{\alpha^2 + \lambda^2}}}$ is an increasing function and $0 < z_0 - l_2 < z_0 + l_1$, we find from (3.24) that
\begin{align*}
|d^*(G * q)(p_0)| &\leq \frac{1}{4\pi r_0} \left( 1 - \frac{z_0 - l_2}{\sqrt{r_0^2 + (l_2 - z_0)^2}} \right) \\
&= \left( \frac{\sqrt{r_0^2 + (l_2 - z_0)^2} - (z_0 - l_2)}{r_0} \right) \left( \frac{1}{4\pi \text{dist}(p_0, L)} \right),
\end{align*}
and (3.25) follows, since $\sqrt{a^2 + b^2} \leq a + b$ for $a, b \geq 0$. The same reasoning of course holds if $z_0 < -l_1$.

**Lemma 7.** Let $x \in \Omega_3$ be such that dist$(x, \Gamma_h) \leq \frac{4}{\eta} h^{1/2}$, where $c_0 > 0$ is defined in statement (vi) of Proposition 4. Then there exists a constant $K > 0$ independent of $\eta, h$ such that if $\eta \leq 1$, then
\begin{equation}
(3.26) \quad |d^* \beta_h(x)| \leq \frac{h}{2\pi \cdot \text{dist}(x, \Gamma_h)} + \frac{K}{\eta^2} \quad \text{if dist}(x, \cup_{i,j} \partial S_i \cup \partial P_j) \geq \frac{c_0}{2} \eta h^{1/2},
\end{equation}
\begin{equation}
(3.27) \quad |d^* \beta_h(x)| \leq \frac{h}{\pi \cdot \text{dist}(x, \Gamma_h)} + \frac{K}{\eta^2} \quad \text{if dist}(x, \cup_{i,j} \partial S_i \cup \partial P_j) < \frac{c_0}{2} \eta h^{1/2}.
\end{equation}

**Proof.** The definition (3.22) of $\Psi_h$ implies that for any measure-valued 2-form $q$,
\begin{equation}
(3.28) \quad |d^* \beta_h| \leq |d^*(G * q)| + \|d^*(G * q_h - G * q)\| + |\Psi_h|.
\end{equation}
Fix $x \in \Omega_3 \setminus \Gamma_h$ and let $r = \frac{4}{\eta} h^{1/2}$. Define a measure-valued 2-form by $\langle q, \varphi \rangle = h \sum_{x \in B(x), \cup L \neq \emptyset} \int_{L_h} \star \varphi$, where $\{L_h\}$ is the collection of line segments whose union gives $\Gamma_h$, see Proposition 2 (vi). By Proposition 2 (vi), there is at most 1 term in the sum that defines $q$ if dist$(x, \cup_{i,j} \partial P_{ij} \cup \partial S_i) \geq r$, and otherwise at most 2 terms.
Then $|d^*(G * q)|$ is estimated via Lemma 6 to give the first term on the right-hand sides of (3.26) and (3.27) respectively, and we must show that the other two terms in (3.28) can be bounded by $K/\eta^2$.

Interior regularity for harmonic functions, together with Proposition 2 statements (iii), (iv) allow us to fix some $q \in (1, 3/2)$ and argue as follows:

$$
\|\Psi_h\|_{L^\infty(\Omega)} \leq C \|\Psi_h\|_{W^{2,2}(\Omega)}
\leq C \|\Psi_h\|_{L^2(\Omega)}
= C \|d^*\beta_h - d^*(G * q_h)\|_{L^2(\Omega)}
\leq C(1 + \eta) \|dp\|_{L^2(\Omega)} \leq C.
$$

(3.29)

To estimate the remaining term in (3.28), observe that

$$
|d^*(G * q_h - G * q)(x)| \leq \frac{6}{4\pi} h \sum_{k=1}^{3} \sum_{\ell=1}^{m(h)} \int_{r^k_\ell \cap B_r(x)} \frac{dy_k}{|x - y|^2}
\leq C \sum_{k=1}^{3} \int_{-M}^{M} \left( \sum_{\ell=1}^{m(h)} \frac{h}{|x - y_k^{\ell}|^2} \right) dt
$$

(3.30)

where $M > 0$ is such that $\Omega_h \subset B_M(0)$ and $\{y_k^{\ell}\}_{\ell'} = \cup \Gamma_h \cap \{y_k = t, |y - x| > r\}$, for $|t| \leq M$. For every $k$ and $t$,

$$
\sum_{\ell=1}^{m(h)} \frac{h}{|x - y_k^{\ell}|^2} \leq \sum_{j=1}^{M/r} \frac{h}{r^2} \# \{\ell' : jr \leq |x - y_k^{\ell'}| < (j + 1)r\}.
$$

Consider the collection of (2 dimensional) balls

$$
\{z : z^k = t, |z - y_k^{\ell'}| < r\}, \quad \text{for } y_k^{\ell'} \text{ such that } jr \leq |x - y_k^{\ell'}| < (j + 1)r.
$$

These balls are pairwise disjoint by Proposition 2 (vi), and are contained in the annulus $\{z : z^k = t, (j - 1)r \leq |x - z| < (j + 2)r\}$, which has area $(6j + 3)\pi r^2$. Thus $\# \{\ell' : jr \leq |x - y_k^{\ell'}| < (j + 1)r\} \leq 6j + 3$ for all $j$. In addition, if we write $x^t$ for the projection of $x$ onto the plane $\{z^k = t\}$, then $\# \{\ell' : jr \leq |x - y_k^{\ell'}| < (j + 1)r\} = 0$ if $(j + 1)r < |x - x^t|$. Then elementary estimates lead to the conclusion

$$
\sum_{\ell=1}^{m(h)} \frac{h}{|x - y_k^{\ell'}|^2} \leq C \frac{h}{r^2} \log\left( \frac{M}{|x - x^t|} \right).
$$

Substituting this into (3.30), we see that $|d^*(G * q_h - G * q)(x)| \leq C h r^2 = C(c_0)\eta^{-2}$, completing the proof of the lemma.

The next lemma shows that we get uniform estimates of certain quantities if we mollify on a scale comparable to the minimum distance between the discretized vortex lines.

**Lemma 8.** Let $0 < \mu < 1$ and $r = \mu c_0 h^{1/2}$, for $c_0$ as in statement (vi) of Proposition 2. Then there exists a nonnegative radial function $\phi$ supported in the unit ball, with $\int \phi = 1$, and such that in addition $\phi_r(x) := r^{-3}\phi(x/r)$ satisfies

$$
\|\phi_r * d^*\beta_h\|_{W^{1,p}(\Omega)} \leq K
$$

(3.31)

for any $p < \infty$, where $K = K(\mu, \eta, \|\phi\|_{L^\infty, \Omega})$ is independent of $h$. 

\[ \square \]
Recall that Proposition 2, statement (iv), provides uniform estimates of proving the first part of (3.34) for \( \zeta \). First, let \( B \in B_1 \) and we conclude (3.34) much as in the case twice more and recalling (3.33), we find that (3.31) holds with \( |\cdot| \). Now fix open sets \( \Omega = \Omega_3 \supset \Omega_2 \supset \Omega_1 \subset \Omega_0 = \Omega_3 \) and functions \( \chi_m \) for \( m = 1, 2, 3 \) such that \( \chi_m \in C^\infty(\Omega_{m+1}) \) and \( \chi_m \equiv 1 \) on an open neighborhood of \( \Omega_m \). Fix a mollifier \( \psi^1 \) as above, but such that \( \text{spt}(\psi^1) \subset B_{1/\beta} \), and define \( \psi^2 = \psi^1 * \psi^1 \) and \( \psi^3 = \psi^1 * \psi^2 \). Thus \( \psi^m \) is radial with support in \( B_1 \) for \( m = 1, 2, 3 \), so that \( (3.32) \) applies to \( \psi_m^3 \). Now write \( \zeta_0 = d^* \beta \), and for \( m = 1, 2, 3 \) define \( \zeta_m = \psi_1^m * (\chi_m \zeta_m) \).

If \( h \), and thus \( r \), is small enough (which we will henceforth take to be the case), then

\[
(3.33) \quad \zeta_m = \psi_1^m * \zeta_m = \psi_1^m * d^* \beta \quad \text{on} \ \Omega_m, \quad \text{and} \ z_m \text{ has support in } \Omega_{m-1}.
\]

We claim that

\[
(3.34) \quad \|d\zeta_m\|_{L^p(\Omega_{m-1})} \leq C_m \|\zeta_m - 1\|_{L^p(\Omega_{m-1})} + C(p, \mu, \psi^1, \Omega_h),
\]

\[
\|d^* \zeta_m\|_{L^p(\Omega_{m-1})} \leq C_m \|\zeta_m - 1\|_{L^p(\Omega_{m-1})}.
\]

To see these, note first that \( d\zeta_m = \psi_1^m * (d\chi_m \wedge \zeta_m - 1) + \psi_1^m * (\chi_m d\zeta_m - 1) \). Then Jensen's inequality implies that

\[
\|\psi_1^m * (d\chi_m \wedge \zeta_m - 1)\|_{L^p(\Omega_{m-1})} \leq \|d\chi_m \wedge \zeta_m - 1\|_{L^p(\Omega_{m-1})} \leq C_m \|\zeta_m - 1\|_{L^p(\Omega_{m-1})}.
\]

We estimate \( \psi_1^m * (\chi_m d\zeta_m - 1) \) first in the case \( m = 1 \), when it follows from statement (i) of Proposition 2 that \( \psi_1^m * (\chi_1 d\zeta_0) = \psi_1^m * (\chi_1 \zeta_h) \). Then arguing as in \( (3.32) \) we find that for any \( p < \infty \),

\[
\|\psi_1^m * (\chi_1 \zeta_h)\|_{L^p(\Omega)} \leq C(p, \Omega_h) \|\psi_1^m * (\chi_1 \zeta_h)\|_{L^\infty(\Omega)} \leq C(p, \psi^1, \Omega_h)(c_0 \mu)^2.
\]

proving the first part of \( (3.34) \) for \( m = 1 \). For \( m = 2, 3 \),

\[
\|\psi_1^m * (\chi_m d\zeta_m - 1)\|_{L^p(\Omega_{m-1})} \leq \|d\zeta_m - 1\|_{L^p(\Omega_{m-1})} \quad \text{and} \quad \|d^* \zeta_m\|_p \leq \|d\chi_m \|_{L^p(\Omega_{m-1})} \leq C_m \|\zeta_m - 1\|_{L^p(\Omega_{m-1})}.
\]

Now recall the Gaffney-Garding inequality

\[
(3.35) \quad \|\zeta\|_{W^{1,p}(U)} \leq C(p, U) \left( \|\zeta\|_{L^p(U)} + \|d\zeta\|_{L^p(U)} + \|d^* \zeta\|_{L^p(U)} \right), \quad 1 < p < \infty,
\]

valid for a differential form \( \zeta \) with compact support in \( U \subset \mathbb{R}^n \). Applying this to \( \zeta_m \), taking into account \( (3.34) \) and noting that \( \|\zeta_m\|_{L^p} \leq \|\zeta_m - 1\|_{L^p} \), we find that

\[
(3.36) \quad \|\zeta_m\|_{W^{1,p}(\Omega_{m-1})} \leq C\|\zeta_m - 1\|_{L^p(\Omega_{m-1})} + C.
\]

Recall that Proposition 2, statement (iv), provides uniform estimates of \( \zeta_0 = d^* \beta \) in \( L^p(\Omega_0) \) for every \( p < 3/2 \), so \( (3.36) \) implies uniform estimates of \( \|\zeta_1\|_{W^{1,p}(\Omega_0)} \) for every \( p < 3/2 \), and hence of \( \|\zeta_1\|_{L^p(\Omega_0)} \) for every \( p < 3 \). Iterating this argument twice more and recalling \( (3.33) \), we find that \( (3.31) \) holds with \( \phi = \psi^3 \). \( \square \)
3.6. Construction of the sequence \( u_\varepsilon \) in case \( g_\varepsilon \geq |\log \varepsilon|^2 \). Assume that the sequence \( g_\varepsilon \) satisfies either \( g_\varepsilon = |\log \varepsilon|^2 \) or \( |\log \varepsilon|^2 \ll g_\varepsilon \ll \varepsilon^{-2} \). Suppose that we are given \((J,v) \in A_0\) as defined in \((3.39)\), and moreover that \( J = \frac{1}{2} dv \) if \( g_\varepsilon = |\log \varepsilon|^2 \), and that \( J = 0 \) if \( |\log \varepsilon|^2 \ll g_\varepsilon \ll \varepsilon^{-2} \).

Set \( p = \frac{1}{2} v \). Fix \( \delta > 0 \) and let \( p_\delta \) be the piecewise linear approximation provided by Lemma 2, and recall the Hodge decomposition in Proposition 2.

Moreover, according to Lemma 10, we may write

\[ \phi_\varepsilon \equiv \phi_{\varepsilon,h} = -\Delta_N q_\varepsilon \delta_\varepsilon \]

is an integer for every \( h \). Thus, if we fix \( \bar{x} \in \Omega \) and let \( c_{\bar{x},x} \) denote a path in \( \Omega_\delta \setminus \Gamma_h \) from \( \bar{x} \) to \( x \), it follows that

\[ \phi_h(x) := \frac{1}{h} \int_{c_{\bar{x},x}} d^* \beta_h \] is well-defined function \( \Omega_\delta \setminus \Gamma_h \to \mathbb{R}/\mathbb{Z} \), independent of the choice of \( c_{\bar{x},x} \), and is hence well-defined a.e. in \( \Omega \).

Moreover, according to Lemma 10, we may write \( \gamma = \sum_{j=1}^\kappa a_j \cdot \phi_j \), where \( \phi_j \) is well-defined in \( \mathbb{R}/\mathbb{Z} \) for \( j = 1, \ldots, \kappa \). For any \( j \) let \( n_j = [h^{-1} a_j] \in \mathbb{Z} \) be the integer part of \( h^{-1} a_j \), and consider \( h^{-1} \gamma_h \equiv d\psi_h = \sum_{j=1}^\kappa n_j d\phi_j \), so that \( \psi_h \) is well-defined in \( \mathbb{R}/\mathbb{Z} \). Let finally \( \alpha_h = h^{-1} \alpha \). The map

\[ v_h = \exp(i 2\pi (\phi_h + \psi_h + \alpha_h)) \]
is thus a well-defined map \( \Omega_\delta \to S^1 \), with

\[ j v_h = 2\pi (d\phi_h + d\psi_h + d\alpha_h) = \frac{2\pi}{h} (d^* \beta_h + \gamma_h + d\alpha) \]

and \( jv_h = \frac{2\pi}{h} d^* \beta_h = \frac{2\pi}{h} \cdot q_\varepsilon \cdot h \). Let now

\[ \rho_\varepsilon(x) \equiv \rho_{\varepsilon,h}(x) = \min\left\{ \frac{\text{dist}(x,\Gamma_h)}{\epsilon}, 1 \right\}, \]

for \( \Gamma_h \) as in Proposition 2 statement (v) and set finally

\[ u_\varepsilon \equiv u_{\varepsilon,h} = \rho_\varepsilon \cdot v_h. \]

3.7. Completion of proof of \((1.8)\) in case \( g_\varepsilon \geq |\log \varepsilon|^2 \). We first claim that

\[ \frac{j u_\varepsilon}{\sqrt{g_\varepsilon}} \to 2\pi (d\alpha + d^* \beta_h + \gamma) \] weakly in \( L^q \) for every \( q \in (1,3/2) \).

for \( \beta_h \) as in statement (iv) of Proposition 2. To see this we write

\[ \frac{j u_\varepsilon}{\sqrt{g_\varepsilon}} = 2\pi (d^* \beta_h + \gamma_h + \alpha) + 2\pi (\rho_\varepsilon^2 - 1) (d^* \beta_h + \gamma_h + d\alpha) \]

It is clear from the definition of \( \gamma_h \) that \( \gamma_h \to \gamma \) uniformly as \( \epsilon \) (and thus \( h \)) tend to 0, and we know from Proposition 2 that \( d^* \beta_h \to d^* \beta_0 \) in the relevant \( L^q \) spaces. So we only need to show that the last term in \((3.43)\) vanishes. For this, we use statements (vi), (v), and (iii) of Proposition 2 to see that

\[ \left| \{ \text{dist}(x,\Gamma_h) \leq \epsilon \} \right| \leq C e^{2|\Gamma_h|} = C \frac{e^{2}}{h} |\eta_\varepsilon|(\Omega_\delta) \leq C \frac{e^{2}}{h}. \]

It easily follows from this and from the definition of \( \rho_\varepsilon \) that \((\rho_\varepsilon^2 - 1) \to 0 \) in \( L^r \) for every \( r < \infty \). Thus, fixing \( q \in (1,3/2) \) and \( r \) such that \( \frac{1}{q} + \frac{1}{r} = 1 \), in view of
uniform estimates of \( \|d^* \beta_h\| \) in Proposition 2 (iv), we find from H"older’s inequality that \( (\rho^2 - 1)(d^* \beta_h + \gamma_h + d\alpha) \to 0 \) in \( L^1 \) as \( \epsilon \to 0 \), proving (3.42).

We now turn to the proof of the upper bound. Since \( h = g^{-1/2} \), we have

\[
(3.45) \quad \frac{E(u_\epsilon; \Omega)}{g_\epsilon} = \frac{h^2}{2} \int_{\Omega} \left| \nabla \rho_\epsilon \right|^2 + \rho_\epsilon^2 |jv_h|^2 + \frac{W(\rho_\epsilon)}{\epsilon^2}.
\]

Let us estimate the various terms contributing to \( g^{-1}E(u_\epsilon; \Omega) \). First note that

\[
\frac{h^2}{2} \int_{\Omega} \left| \nabla \rho_\epsilon \right|^2 + \frac{W(\rho_\epsilon)}{\epsilon^2} \leq \frac{C_h}{\epsilon^2} - \{ \text{dist} (x, \Gamma_h) \leq \epsilon \}\]

for \( C = \frac{1}{2}(1 + \|W\|_{L^\infty(B_1)}) \). It follows from this and (3.34) that

\[
(3.46) \quad \frac{h^2}{2} \int_{\Omega} \left| \nabla \rho_\epsilon \right|^2 + \frac{W(\rho_\epsilon)}{\epsilon^2} \leq C h.
\]

Moreover,

\[
(3.47) \quad \frac{h^2}{2} \int_{\Omega} \rho_\epsilon^2 |jv_h|^2 = 2\pi \int_{\Omega} \rho_\epsilon^2 (|d^* \beta_h|^2 + |d\alpha + \gamma_h|^2 + 2 d^* \beta_h \cdot (d\alpha + \gamma_h)).
\]

We have just shown in the proof of (3.42) that \( \rho^2(\alpha + \gamma) \to \alpha + \gamma \) in \( L^p \) \( \forall p < +\infty \) and that \( d^* \beta_h \to d^* \beta^0 \) weakly in \( L^q \) \( \forall q < 3/2 \). Thus, recalling the estimate \( \|d^* \beta^0 - d^* \beta\|_2^2 \leq C \eta \) from statement (iv) in Proposition 2, we obtain

\[
(3.48) \quad \lim_{\epsilon \to 0} \int_{\Omega} \rho_\epsilon^2 (\alpha + \gamma_h) \cdot d^* \beta_h = \int_{\Omega} d^* \beta^0 \cdot (\alpha + \gamma) = C \sqrt{\pi} + \int_{\Omega} d^* \beta \cdot (\alpha + \gamma),
\]

\[
(3.49) \quad \lim_{\epsilon \to 0} \int_{\Omega} \rho_\epsilon^2 |\alpha + \gamma_h|^2 \leq \int_{\Omega} |\alpha + \gamma|^2.
\]

For the remaining term, fix \( 0 < \mu < 1 \) and set \( r = c_0 \mu \eta h^{1/2} \). Denote \( G^\lambda_h = \{ \text{dist} (x, \Gamma_h) \leq \lambda \} \cap \Omega \). We have

\[
(3.50) \quad 2\pi \int_{\Omega} \rho_\epsilon^2 |d^* \beta_h|^2 = A_\epsilon + B_\epsilon + C_\epsilon,
\]

where

\[
(3.51) \quad A_\epsilon = 2\pi \int_{\Omega} \rho_\epsilon^2 |d^* \beta_h|^2, \quad B_\epsilon = 2\pi \int_{G^\mu_h \setminus G^\lambda_h} |d^* \beta_h|^2, \quad C_\epsilon = 2\pi \int_{\Omega \setminus G^\lambda_h} |d^* \beta_h|^2.
\]

Let us estimate \( A_\epsilon \). By (3.26), (3.27), and (3.40), \( \rho_\epsilon^2 |d^* \beta_h|^2 \leq \frac{h^2}{\epsilon^2} + \frac{2K^2}{\eta^2} \) in \( G^\lambda_h \), so (3.44) implies that

\[
(3.52) \quad A_\epsilon \leq |G^\lambda_h| \left( \frac{h^2}{\epsilon^2} + \frac{2K^2}{\eta^2} \right) \leq C (h + K \frac{\epsilon^2}{\eta^2 h})
\]

so that, since \( h = g^{-1/2} \) and \( |\log \epsilon|^2 \leq g_\epsilon \ll \epsilon^{-2} \), we have

\[
(3.53) \quad \lim_{\epsilon \to 0} \sup A_\epsilon = 0.
\]

Let us turn to \( C_\epsilon \). Let \( \phi_\epsilon \) be the radial mollifier found in Lemma 8. Observe that \( d^* \beta_h \) is harmonic on \( \Omega \setminus G^\mu_h \), and hence coincides there with \( \phi_\epsilon \ast d^* \beta_h \), by the mean-value property of harmonic functions. By (3.31) and Rellich’s Theorem we
deduce that $\phi_\epsilon \ast \delta^* \beta h$ is strongly compact in $L^2(\Omega)$, and hence by Proposition \[2\] statement (iv) that $\phi_\epsilon \ast \delta^* \beta h \to \delta^* \beta^0$ in $L^2(\Omega)$ as $\epsilon \to 0$. We deduce that

$$
\limsup_{\epsilon \to 0} C_\epsilon = \limsup_{\epsilon \to 0} 2\pi^2 \int_{\Omega \setminus G_h^\epsilon} |\phi_\epsilon \ast \delta^* \beta h| \leq \limsup_{\epsilon \to 0} 2\pi^2 \int_{\Omega} |\phi_\epsilon \ast \delta^* \beta h|^2 = 2\pi^2 \int_{\Omega} |\delta^* \beta^0|^2 \\
\leq 2\pi^2 \int_{\Omega} |\delta^* \beta^0|^2 + C\eta.
\tag{3.54}
$$

To estimate $B_\epsilon$ we proceed as follows: let $V_1 = (G_h^r \setminus G_h^\epsilon) \setminus U_{r_0}$, where $U_{r_0} = \{ \text{dist} \left(x, \cup_{i,j} \partial S_i \cup \partial P_j\right) < r_0\} \cap \Omega$ and $r_0 = \frac{\eta}{\sqrt{2}} h^{1/2}$, and set $V_2 = (G_h^r \setminus G_h^\epsilon) \cap U_{r_0}$. For any $\sigma > 0$ we have, using for $\delta^* \beta h$ the bound \(3.26\) on $V_1$ and \(3.27\) on $V_2$,

$$
2\pi^2 \int_{V_1} |\delta^* \beta h|^2 \leq (1 + \sigma) \frac{h^2}{2} \int_{V_1} \frac{dx}{\text{dist}(x, \Gamma_h)^2} + \left(1 + \sigma\right) \frac{2\pi^2 K^2}{\eta^4} |V_1|,
\tag{3.55}
$$

and then statements (iii), (v) of Proposition \[2\] and \(3.57\) give

$$
2\pi^2 \int_{V_2} |\delta^* \beta h|^2 \leq 4(1 + \sigma) \frac{h^2}{2} \int_{V_2} \frac{dx}{\text{dist}(x, \Gamma_h)^2} + \left(1 + \sigma\right) \frac{2\pi^2 K^2}{\eta^4} |V_2|,
\tag{3.56}
$$

so that

$$
B_\epsilon \leq (1 + \sigma) h^2 \pi \log \left(\frac{r}{\epsilon}\right) \left(|\Gamma_h| + 3|\Gamma_h \cap U_{r_0}|\right) + \left(1 + \frac{1}{\sigma}\right) \frac{C\mu^2}{\eta^2} h |\Gamma_h|.
\tag{3.57}
$$

If $g_\epsilon = h^{-2} = |\log \epsilon|^2$ then statements (iii), (v) of Proposition \[2\] and \(3.57\) give

$$
\limsup_{\epsilon \to 0} B_\epsilon \leq \left(1 + \sigma\right) \pi + \left(1 + \frac{1}{\sigma}\right) \frac{C\mu^2}{\eta^2} \cdot (C\eta + ||dp_\delta||_{L^1(\Omega)}) \tag{3.58},
$$

while if $|\log \epsilon|^2 \ll g_\epsilon \ll \epsilon^{-2}$ (i.e., $\epsilon \ll h \ll |\log \epsilon|^{-1}$), we have

$$
\limsup_{\epsilon \to 0} B_\epsilon \leq \left(1 + \frac{1}{\sigma}\right) \frac{C\mu^2}{\eta^2} \cdot (C\eta + ||dp_\delta||_{L^1(\Omega)}) \tag{3.59},
$$

We sum up all the contributions \(3.46\), \(3.48\), \(3.49\), \(3.53\), \(3.54\), \(3.55\) and \(3.59\), noting that the terms estimated in \(3.48\), \(3.49\), and \(3.54\) add up to $2\pi^2 \int_{\Omega_{\delta}} |d\alpha + \gamma + \delta^* \beta|^2 + C\sqrt{\eta} = 2\pi^2 \int_{\Omega_{\delta}} |p_\delta|^2 + C\sqrt{\eta}$. Thus, letting first $\mu \to 0$, then $\sigma \to 0$, in \(3.58\) and \(3.59\), we obtain

$$
\limsup_{\epsilon \to 0} \frac{E_\epsilon(u_\epsilon, \Omega)}{g_\epsilon} \leq \pi \int_{\Omega_{\epsilon}} |dp_\delta| + 2\pi^2 \int_{\Omega_{\epsilon}} |p_\delta|^2 + C\sqrt{\eta} \tag{3.60},
$$

if $g_\epsilon = |\log \epsilon|^2$, and

$$
\limsup_{\epsilon \to 0} \frac{E_\epsilon(u_\epsilon, \Omega)}{g_\epsilon} \leq 2\pi^2 \int_{\Omega_{\epsilon}} |p_\delta|^2 + C\sqrt{\eta} \tag{3.61},
$$

if $|\log \epsilon|^2 \ll g_\epsilon \ll \epsilon^{-2}$. In these estimates $C$ is independent of $\eta$. Thus, since $p = 2\pi v_\epsilon$, and recalling \(3.50\), \(3.52\), \(3.53\), and statement (iv) of Proposition \[2\] we see that as first $\eta$ and then $\delta$ tend to 0, the right-hand sides above converge to
\[ \frac{1}{2} |dv| (\Omega) + \frac{1}{2} \|v\|_{L^2(\Omega)}^2 \] in the case \( g_\epsilon = |\log \epsilon|^2 \), and \( \frac{1}{2} \|v\|_{L^2(\Omega)}^2 \) in the case \( |\log \epsilon|^2 \ll g_\epsilon \ll \epsilon^{-2} \). Thus, we can find sequences \( \eta = \eta_\epsilon \) and \( \delta = \delta_\epsilon \) tending to zero slowly enough that, if we define \( U_\epsilon := u_\epsilon \) with parameters \( \delta_\epsilon \) in the piecewise linear approximation (Lemma 2) and \( \eta_\epsilon \) in the discretization of the vorticity (Proposition 2), then

\[
\limsup_{\epsilon \to 0} \frac{E_\epsilon (U_\epsilon; \Omega)}{g_\epsilon} \leq \frac{1}{2} |dv| (\Omega) + \frac{1}{2} \|v\|_{L^2(\Omega)}^2 \quad \text{if } g_\epsilon = |\log \epsilon|^2
\]

\[
\limsup_{\epsilon \to 0} \frac{E_\epsilon (U_\epsilon; \Omega)}{g_\epsilon} \leq \frac{1}{2} \|v\|_{L^2(\Omega)}^2 \quad \text{if } |\log \epsilon|^2 \ll g_\epsilon \ll \epsilon^{-2}.
\]

This finally proves the upper bound (1.8), recalling that \( J = \frac{1}{2} dv \) for \( g_\epsilon = |\log \epsilon|^2 \) and \( J = 0 \) when \( |\log \epsilon|^2 \ll g_\epsilon \ll \epsilon^{-2} \).

Finally, having established the energy upper bound for \( U_\epsilon \), the compactness assertions (1.4), (1.5), (1.6) imply that \( u \) converges to \( u \) in the required spaces, so it suffices only to identify the limits. In fact, it suffices to show for example that \( \frac{1}{\sqrt{g_\epsilon}} jU_\epsilon \to v \) in the sense of distributions, and this follows (after taking \( \eta_\epsilon \) in the definition of \( U_\epsilon \) converge to zero more slowly, if necessary) from (3.4).

3.8. Construction of the sequence \( u_\epsilon \) in case \( g_\epsilon \ll |\log \epsilon|^2 \). Let \( J \) be an exact measure-value 2-form in \( \Omega \) and \( v \in L^2(\Lambda^1 \Omega) \) such that \( dv = 0 \). Fix \( \delta > 0 \), and let \( p_\delta \) be the rational piecewise linear approximation of \( p := \frac{1}{\sqrt{g_\epsilon}} \) from Lemma 2. Furthermore, let \( p'_\delta \) be the rational piecewise linear function from Lemma 2', so that \( dp' \) approximates \( J \). Our Hodge decomposition gives respectively \( p_\delta = \gamma + da + d^* \beta' \), and \( p'_\delta = \gamma' + da' + d^* \beta' \). Let \( h = \frac{1}{\sqrt{g_\epsilon}} \) and \( h' = \frac{|\log \epsilon|}{g_\epsilon} \), so that \( h = h' \ll \epsilon^{-1} \ll |\log \epsilon| = h' \).

Fix \( \eta > 0 \), and for \( h' < \eta \) let \( d^* \beta_{\epsilon'} \) be the discretization of \( d^* \beta' \) via Proposition 2. Let \( \phi_{\epsilon'} \) be defined as in (3.37), so that \( d\phi_{\epsilon'} = \frac{1}{h'} d^* \beta_{\epsilon'} \), let \( h^{-1} \gamma_h = d\psi_h \) be as in section 3.6 and set \( \omega_h = h^{-1} \alpha_h \).

Finally, let \( u_\epsilon \) be as in (3.40) and define

\[
(3.64) \quad u_\epsilon = \rho_\epsilon \exp (2\pi \cdot (\phi_{\epsilon'} + \psi_h + \omega_h)).
\]

3.9. Completion of proof of (1.8) in case \( g_\epsilon \ll |\log \epsilon|^2 \). We have to estimate

\[
(3.65) \quad E_\epsilon (u_\epsilon; \Omega) = \frac{h^2}{2} \int_{\Omega} |\nabla \rho_\epsilon|^2 + \frac{W(\rho_\epsilon)}{\epsilon^2} + 4\pi^2 \rho_\epsilon^2 \left( \frac{1}{h'} d^* \beta_{\epsilon'} + \frac{1}{h} (g_h + da) \right)^2.
\]

Then \( |\text{dist}(x, \Gamma_h) \leq \eta| \leq \frac{\eta^2}{\pi} \) as in (3.44), so we find as in (3.46) that

\[
(3.66) \quad \frac{h^2}{2} \int_{\Omega} |\nabla \rho_\epsilon|^2 + \frac{W(\rho_\epsilon)}{\epsilon^2} \leq C \frac{h^2}{h'} \to 0.
\]

For the remaining terms we have

\[
(3.67) \quad 2\pi^2 \int_{\Omega} \rho_\epsilon |d\alpha + g_h|^2 \to 2\pi^2 \int_{\Omega} |d\alpha + g_h|^2 \leq 2\pi^2 \int_{\Omega} |p_\delta|^2,
\]

\[
(3.68) \quad 2\pi^2 \frac{h}{h'} \int_{\Omega} \rho_\epsilon^2 d^* \beta_{\epsilon'} \cdot (d\alpha + g_h) \to 0,
\]

\[
(3.69) \quad 2\pi^2 \frac{h^2}{h'} \int_{\Omega} \rho_\epsilon^2 |d^* \beta_{\epsilon'}|^2 = A_\epsilon' + B_\epsilon' + C_\epsilon',
\]
where, in the notation corresponding to (3.69),
\[
A'_\epsilon = 2\pi^2 \frac{h^2}{\eta^2} \int_{G_{\epsilon}'} \rho_{\epsilon}^2 |d^* \beta_{h'}|^2 ,
\]
(3.70)
\[
B'_\epsilon = 2\pi^2 \frac{h^2}{\eta^2} \int_{G_{\epsilon}'' \setminus G_{\epsilon}'} |d^* \beta_{h'}|^2 ,
\]
\[
C'_\epsilon = 2\pi^2 \frac{h^2}{\eta^2} \int_{\Omega \setminus G_{\epsilon}''} |d^* \beta_{h'}|^2
\]
for \( r = c_0 \eta(h')^{1/2} \). Reasoning as in (3.52) and (3.54) we deduce a fortiori that \( \limsup A_\epsilon = \limsup_{\epsilon \to 0} C_\epsilon = 0 \), while following (3.55) and (3.56) we deduce
\[
B'_\epsilon \leq (1 + \sigma) \pi \log(\frac{r}{\epsilon}) (|\Gamma_{h'}| + C|\Gamma_{h'} \cap U_r|) + (1 + \frac{1}{\sigma}) \frac{h^2}{\eta^2} |\Gamma_{h'}| ,
\]
so that \( \limsup B'_\epsilon \leq (1 + \sigma) \pi \int_{\Omega} |dp'_\beta| + C \eta \) by Proposition 2 (iii). Summing up the various contributions and then letting \( \sigma \to 0 \), we obtain
\[
\limsup_{\epsilon \to 0} \frac{E(u_\epsilon)}{\eta(h')} \leq \pi \int_{\Omega} |dp'_\beta| + 2\pi^2 \int_{\Omega} |p_\beta|^2 + C \eta .
\]
We conclude the proof as in the previous cases, by defining \( U_\epsilon := u(\epsilon, \eta, \delta_\epsilon) \) (that is, defining \( u_\epsilon \) as above, but with parameters \( \delta_\epsilon \) in the piecewise linear approximation of Lemma 2 and \( \eta_\epsilon \) in the discretization of the vorticity of Proposition 2) for \( \eta_\epsilon \) and \( \delta_\epsilon \) converging to zero sufficiently slowly, so that \( U_\epsilon \) satisfies the Gamma-limsup inequality (1.3), and then verifying the convergence as before. \( \square \)

4. APPLICATIONS TO SUPERCONDUCTIVITY

In this section we prove Theorem 4 and begin the analysis of the limiting functional \( \mathcal{F} \), deriving the curvature equation for the vortex filaments. We use a good deal of notation that was introduced in Section 1.3.

In the companion paper [2] we analyze in more detail the properties of \( \mathcal{F} \) and derive further applications such as a general expression for the first critical field \( H_{c_1} \).

4.1. Proof of Theorem 4

First, recalling that \( h_{ex} = \varepsilon \Delta x_{ex, c} \), we see immediately from the definition of \( \mathcal{F} \) and of the \( \dot{H}^1_\sigma (\Lambda^1 \mathbb{R}^3) \) norm that
\[
\|A_\epsilon - A_{ex, \epsilon}\|_{\dot{H}^1_\sigma}^2 \leq 2 \mathcal{F}(u_\epsilon, A_\epsilon) \leq K |\log \varepsilon|^2 .
\]
It immediately follows that \( \frac{1}{|\log \varepsilon|}(A_\epsilon - A_{ex, \epsilon}) \) is weakly precompact in \( \dot{H}^1_\sigma (\Lambda^1 \mathbb{R}^3) \), and since \( |\log \varepsilon|^{-1} A_{ex, \epsilon} \to A_{ex, 0} \) in \( \dot{H}^1_\sigma (\Lambda^1 \mathbb{R}^3) \), we deduce (1.22).

The above bounds on \( A_\epsilon \) and the Sobolev embedding \( \dot{H}^1_\sigma \hookrightarrow L^6 \) implies that
\[
\| |\log \varepsilon|^{-1} A_\epsilon \|_{L^6(\Lambda^1 \mathbb{R}^3)} \leq K .
\]
In order to establish the remaining compactness assertions, we use the decomposition (1.19), which implies that
\[
E(\varepsilon, u_\epsilon) \leq \mathcal{F}(u_\epsilon, A_\epsilon) + \int_{\Omega} A_\epsilon \cdot j u_\epsilon \| \leq K |\log \varepsilon|^2 + \int_{\Omega} A_\epsilon \cdot j u_\epsilon \| .
\]
using the fact that $\mathcal{M}(A; dA_{\epsilon, x}) + R(u_{\epsilon}, A_{\epsilon}) \geq 0$. To estimate the right-hand side, note that in general
\[
|ju \cdot A| \leq |u| |Du| |A| \leq \frac{1}{4} |Du|^2 + |u|^2 |A|^2 \leq \frac{1}{4} |Du|^2 + 2|A|^2 + 2(|u| - 1)^2 |A|^2
\]
\[
\leq \frac{1}{4} |Du|^2 + 2|A|^2 + \frac{c}{\epsilon^2} |u| - 1|^3 + Cc^2 |A|^6.
\]
And hypothesis $(H_q)$ with $q \geq 3$ implies that $c |u| - 1|^3 \leq \frac{1}{q} W(u)$ if $c$ is small enough, so that
\[
|\int_\Omega A_{\epsilon} \cdot ju_{\epsilon}| \leq \frac{1}{2} E_{\epsilon}(u_{\epsilon}) + \epsilon C \int_\Omega |A_{\epsilon}|^2 + \epsilon^2 |A_{\epsilon}|^6 dx.
\]
By combining the above inequalities and using (4.1), we find that $E_{\epsilon}(u_{\epsilon}) \leq K |\log \epsilon|^2$, which in view of Theorem 2 implies that (1.13), (1.5), (1.6) hold with $g_\epsilon = |\log \epsilon|$. To prove statement (ii), consider the decomposition of $F_{\epsilon}$ given by (1.19), (1.20), which may be rewritten
\[
\frac{F_{\epsilon}(u_{\epsilon}, A_{\epsilon})}{|\log \epsilon|^2} = E_{\epsilon}(u_{\epsilon}) + M(\frac{A_{\epsilon}}{|\log \epsilon|}, \frac{h_{ex}}{|\log \epsilon|}) + I(\frac{ju_{\epsilon}}{|\log \epsilon|}, \frac{A_{\epsilon}}{|\log \epsilon|}) + \frac{R(u_{\epsilon}, A_{\epsilon})}{|\log \epsilon|^2}.
\]
Recall that (1.15) asserts
\[
\frac{1}{|\log \epsilon|^2} E_{\epsilon}(u_{\epsilon}) \Gamma \rightarrow E(v),
\]
with $E(v)$ defined in (1.10). Note further that $\mathcal{M}$ is lower semicontinuous with respect to the weak $\hat{H}^1_x$ convergence of $\frac{A_{\epsilon}}{|\log \epsilon|}$, and hence, taking into account (1.22), we readily deduce
\[
\mathcal{M}(\frac{A_{\epsilon}}{|\log \epsilon|}, \frac{h_{ex}}{|\log \epsilon|}) \Gamma \rightarrow \mathcal{M}(A, h).
\]
Moreover, by Sobolev embedding, (1.22) implies $\frac{A_{\epsilon}}{|\log \epsilon|} \rightharpoonup A$ strongly in $L^p(\Omega)$, for any $1 \leq p < 6$, whereas (1.5) gives $\frac{u_{\epsilon}}{|\log \epsilon|} \rightharpoonup v$ weakly in $L^{2q/(q+2)}(\Omega)$. For $q \geq 3$ we have $2q/(q+2) \geq 6/5$, so that for any admissible sequence $(u_{\epsilon}, A_{\epsilon})$ we have
\[
\frac{I(\frac{ju_{\epsilon}}{|\log \epsilon|}, \frac{A_{\epsilon}}{|\log \epsilon|})}{|\log \epsilon|^2} \rightarrow I(v, A).
\]
Note finally that for the remainder term $R(u_{\epsilon}, A_{\epsilon})$, since $|1 - |u|^2|^{3/2} \leq CW(u)$,
\[
|R(u_{\epsilon}, A_{\epsilon})| \leq \int_\Omega |1 - |u|^2| |A_{\epsilon}|^2 dx
\]
\[
\leq C \epsilon^{4/3} \left( \int_\Omega \frac{W(u_{\epsilon})}{\epsilon^2} dx \right)^{2/3} \left( \int_\Omega |A_{\epsilon}|^6 dx \right)^{1/3}
\]
\[
\leq C \epsilon^{4/3} E_{\epsilon}(u_{\epsilon})^{2/3} \|A_{\epsilon}\|_{L^6(\Omega)}^2
\]
\[
\leq C \epsilon^{4/3} |\log \epsilon|^{10/3},
\]
so that $\frac{1}{|\log \epsilon|^2} R(u_{\epsilon}, A_{\epsilon}) \leq C(\epsilon|\log \epsilon|)^{4/3}$ converges uniformly to 0.

From the above considerations it follows immediately that
\[
\frac{F_{\epsilon}(u_{\epsilon}, A_{\epsilon})}{|\log \epsilon|^2} \Gamma \rightarrow E(v) + I(v, A) + \mathcal{M}(A, h),
\]
which is formula (1.23).

4.2. Some properties of the $\Gamma$-limit $\mathcal{F}$. In this section we derive the Euler-Lagrange equations for the functional $\mathcal{F}$ and deduce a curvature equation for the limiting vortex filaments. First of all notice that $\mathcal{F}$ is strictly convex and hence admits a unique minimizer $(v,\Lambda)$. We first make variations of $\mathcal{F}$ with respect to $\Lambda$.

Standard computations yield

\begin{equation}
\begin{cases}
\frac{1}{2}(d^*(dA - h) = \mathbf{1}_\Omega \cdot (v - A) & \text{in } \mathbb{R}^3, \\
[(\star (dA - h))_\tau] = [(dA - h)_N] = 0 & \text{on } \partial \Omega,
\end{cases}
\end{equation}

where $\mathbf{1}_\Omega$ denotes the characteristic function of $\Omega$ and $[(dA - h)_N]$ denotes the jump across $\partial \Omega$ of the normal component of $(dA - h)$. Denoting $j = \mathbf{1}_\Omega \cdot (v - A)$ the gauge-invariant supercurrent in $\Omega$ and $H = dA - h$, we recover from (4.6) Ampère law $d^*H = j$ in $\mathbb{R}^3$ for the magnetic field $H$, which has to be coupled with Gauss law $dH = 0$ in $\mathbb{R}^3$, and with the continuity condition $[H] = 0$ on $\partial \Omega$, which is a consequence of $[H_N] = 0$ (by (4.6)) and $[H_\tau] = 0$ on $\partial \Omega$ (by Gauss law $dH = 0$).

Let now $J(v)$ denote the convex and positively 1-homogeneous function $J(v) := ||dv||$, and let $\partial J$ be its subdifferential. Making variations of $\mathcal{F}$ with respect to $v$ yields the differential inclusion

\begin{equation}
\frac{1}{2} \partial J(v) + v - A.
\end{equation}

Assume the minimizer $v$ is regular and $\text{spt } |dv| = U$, with $U$ an open subset of $\Omega$. In particular, if $U$ is a proper subset of $\Omega$, then one may view $\Omega \cap \partial U$ as a kind of free boundary. This situation has a counterpart in the 2-d case (see [30], [23]). Then (4.7) corresponds to

\begin{equation}
\frac{1}{2} \int_U \frac{dv}{|dv|} \wedge \star d\phi + \int_{\Omega} (v - A) \wedge \star \phi = 0
\end{equation}

for any $\phi \in C^\infty(\Lambda^1 \Omega)$ such that $\text{spt } \phi \subseteq \Omega \setminus \partial U$. Testing (4.8) with $\phi \in C^\infty(\Lambda^1(\Omega \setminus \bar{U}))$ we deduce $v = A$ in $\Omega \setminus \bar{U}$. Testing now with those $\phi \in C^\infty(\Lambda^1(\Omega))$ such that $\text{spt } \phi \subseteq \bar{U} \setminus (\Omega \cap \partial U)$ and integrating by parts (4.8) we further deduce

\begin{equation}
\frac{1}{2} d^* \left( \frac{dv}{|dv|} \right) + v - A \wedge \star \phi + \int_{\partial U \cap \bar{U}} (\phi \wedge \frac{dv}{|dv|})_\tau = 0,
\end{equation}

whence

\begin{equation}
\begin{cases}
\frac{1}{2} d^* \left( \frac{dv}{|dv|} \right) = 2(A - v) & \text{in } U, \\
(\star \frac{dv}{|dv|})_\tau = 0 & \text{on } \bar{U} \cap \partial \Omega.
\end{cases}
\end{equation}

Notice that $\tau = \star \frac{dv}{|dv|}$ is the unit tangent covector field to the streamlines of the covector distribution $\star dv$, which correspond to the limiting vorticity. From (4.10) we obtain in particular

\begin{equation}
\begin{cases}
\tau \wedge d\tau = 2\tau \wedge (v - A) = 2\tau \wedge j & \text{in } U, \\
\tau_\tau = 0 & \text{on } \bar{U} \cap \partial \Omega.
\end{cases}
\end{equation}

Denoting respectively by $\vec{\tau}$ and $\vec{j}$ the vector fields corresponding to $\tau$ and $j$, we notice that $\star (\tau \wedge j)$ corresponds to $\vec{\tau} \times \vec{j}$, and $d\tau$ corresponds to the vector field
\[ \nabla \times \vec{\tau}, \text{ so that } \star (\tau \wedge \star d\tau) \text{ corresponds to the curvature vector } \vec{k} = \nabla \times (\nabla \times \vec{\tau}). \text{ We thus deduce the curvature equation (1.25).} \]

**Remark 17.** Notice that \( d^* \tau = \star d(\frac{d\tau}{\lvert \lvert d\tau \rvert \rvert}) = 0 \) (or equivalently \( \nabla \cdot \vec{\tau} = 0 \)) in \( \Omega \). From (4.10) we deduce that \( \tau \) satisfies the Hodge system

\[
\begin{align*}
\frac{d\tau}{\lvert \lvert d\tau \rvert \rvert} &= 2j & \text{in } \Omega \\
\frac{d^* \tau}{\lvert \lvert d\tau \rvert \rvert} &= 0 & \text{in } \Omega \\
\tau_T &= 0 & \text{on } \partial \Omega,
\end{align*}
\]

or respectively

\[
\begin{align*}
\nabla \times \vec{\tau} &= 2j & \text{in } \Omega \\
\nabla \cdot \vec{\tau} &= 0 & \text{in } \Omega \\
\vec{\tau}_T &= 0 & \text{on } \partial \Omega,
\end{align*}
\]

under the pointwise constraint \( \lvert \lvert \tau \rvert \rvert = 1 \) (resp. \( \lvert \lvert \vec{\tau} \rvert \rvert = 1 \)) in \( \text{spt } j \).

**Remark 18.** From (4.9), (4.10) we recover in particular the continuity equation \( d^* j = d^* (v - A) = 0 \) (or equivalently, \( \nabla \cdot \vec{j} = 0 \)). If \( A \) is in the Coulomb gauge \( d^* A = 0 \) (which happens in particular if \( A_{ex} = cx^2dx^3 - x^2dx_1 \)) and \( A \in H^1 \), so that \( d^* (A - A_{ex}) = 0 \), then it follows that \( v \) satisfies

\[
\begin{align*}
\frac{d^* v}{\lvert \lvert d\tau \rvert \rvert} &= 0 & \text{in } \Omega \\
v_N &= 0 & \text{on } \partial \Omega.
\end{align*}
\]

5. **APPENDIX**

In this Appendix we recollect basic facts and notation that we use throughout the paper, as well as background on differential forms, Hodge decompositions, minimal connections. We also provide the proofs of Lemma 11 and Lemma 12.

5.1. **Differential forms.** For \( 0 \leq k \leq n \), let \( \Lambda^k \mathbb{R}^n \) be the space of \( k \)-covectors in \( \mathbb{R}^n \), i.e. \( \theta \in \Lambda^k \mathbb{R}^n \) if \( \theta = \sum \theta_I dx^I \), where \( dx^I := dx^{i_1} \wedge \ldots \wedge dx^{i_k} \), \( 1 \leq i_1 < \ldots < i_k \leq n \). For \( \theta, \beta \in \Lambda^k \mathbb{R}^n \), their inner product is given by \( \langle \theta, \beta \rangle := \sum \theta_I \cdot \beta_I \).

Let \( \Omega \subset \mathbb{R}^n \) be a smooth bounded open set. We will denote by \( C^\infty(\Lambda^k \Omega) := C^\infty(\Omega; \Lambda^k \mathbb{R}^n) \) the space of smooth \( k \)-forms on \( \Omega \). Similarly we denote by \( L^p(\Lambda^k \Omega) \), \( W^{1,p}(\Lambda^k \Omega) \) the spaces of \( k \)-forms of class \( L^p \) and \( W^{1,p} \) respectively. For \( \omega \in C^\infty(\Lambda^k \Omega) \), denote by \( \omega_T \in C^\infty(\Lambda^k \partial \Omega) \) its tangential component\(^{10}\) on \( \partial \Omega \), and by \( \omega_N := \omega \circ \partial \Omega - \omega_T \) its normal component on \( \partial \Omega \). The operators \( \omega \mapsto \omega_T \) and \( \omega \mapsto \omega_N \) extend to bounded linear operators \( W^{1,p}(\Lambda^k \Omega) \to L^p(\partial \Omega; \Lambda^k \mathbb{R}^n) \).

The Hodge star operator \( \star : \Lambda^k \mathbb{R}^n \to \Lambda^{n-k} \mathbb{R}^n \) is defined in such a way that \( \theta \wedge \star \varphi = \langle \theta, \varphi \rangle dx^{i_1} \wedge \ldots \wedge dx^{i_k} \). The \( L^2 \) inner product of \( \omega, \eta \in C^\infty(\Lambda^k \Omega) \) is defined by

\[
\langle \omega, \eta \rangle := \int_{\Omega} \langle \omega, \eta \rangle dL^n = \int_{\Omega} \omega \wedge \star \eta.
\]

Let \( T \subset \Omega \) be a piecewise smooth \( m \)-dimensional submanifold with boundary. Integration of (the tangential component of) a smooth \( m \)-form \( \omega \) on \( T \) will be denoted by \( \int_T \omega \equiv \int_T \omega_T = \int_T i^* \omega \), with \( i : T \to \Omega \) the inclusion map.

The adjoint with respect to \( \langle \cdot, \cdot \rangle \) of the \( \star \) operator on \( k \)-forms is \( (-1)^{k(n-k)} \star \).

\(^{10}\)i.e. \( \omega_T := i^* \omega \), where \( i : \partial \Omega \to \Omega \) is the inclusion map
5.1.1. measure-valued forms. A distribution-valued $k$-form $\mu$ is an element of the dual space\(^7\) of $C^\infty(\Lambda^k\Omega)$, and we express the duality pairing through the notation $\langle \cdot, \cdot \rangle$. In particular, we will say that $\mu$ is a measure-valued $k$-form (cf. [3], Definition 2.1) if

$$\langle \mu, \varphi \rangle \leq C||\varphi||_{\infty} \quad \forall \varphi \in C^\infty_c(\Lambda^k\Omega).$$

A measure-valued $k$-form $\mu$ can be represented by integration (cf. [3], Proposition 2.2) as follows:

$$\langle \mu, \varphi \rangle = \int_\Omega (\nu, \varphi) \, d|\mu|,$$

where $|\mu|$ is the total variation measure of (the vector measure) $\mu$ and $\nu$ is a $|\mu|$-measurable $k$-form such that $(\nu, \nu)^{1/2} =: |\nu| = 1$ $|\mu|$-a.e. in $\Omega$. We denote by $||\mu|| := |\mu|(\Omega)$ the total variation norm of $|\mu|$. It coincides with the $L^1$ norm $||\mu||_1 = \int_\Omega |\mu|$ if $\mu \in L^1(\Lambda^k\Omega)$. We denote by $\mu\llcorner U$ the restriction of $\mu$ to $U \subset \Omega$, defined by

$$\langle \mu\llcorner U, \varphi \rangle = \int_U (\nu, \varphi) \, d|\mu|.$$

Moreover, for $\eta$ a unit $k$-covector and $\mu$ a measure $k$-form in $\Omega$, the component along $\eta$ of $\mu$ is a signed measure denoted $\langle (\mu, \eta) \rangle$ defined by

$$\langle (\mu, \eta)(U) \rangle := \langle \mu(U), \eta \rangle = \int_U (\nu, \eta) \, d|\mu| \quad \forall U \in \Omega,$$

with variation measure $|(\mu, \eta)|$ given by

$$|(\mu, \eta)|(U) = \int_U |(\nu, \eta)| \, d|\mu| \quad \forall U \in \Omega.$$

Notice that an oriented piecewise smooth $k$-dimensional submanifold $T \subset \Omega$ can be identified with a measure $k$-form $\hat{T}$, whose action on smooth $k$-forms $\varphi$ is given by

$$\langle \hat{T}, \varphi \rangle = \int_T \varphi.$$

Let $d$ be the exterior differentiation operator, and $d^* = (-1)^{n(k+1)+1} \star d\star$ its adjoint with respect to $\langle \cdot, \cdot \rangle$, i.e. $\langle d\omega, \eta \rangle = \langle \omega, d^*\eta \rangle$ for $\omega$ a $k$-form, and $\eta$ an $(n-k-1)$-form. We define the action of $d$ and $d^*$ on a measure-valued distribution $\mu$ by duality, so that $\langle d\mu, \eta \rangle := \langle \mu, d^*\eta \rangle$ and $\langle d^*\mu, \eta \rangle := \langle \mu, d\eta \rangle$ for $\eta$ with compact support.

Stokes’ Theorem reads $\int_T d\varphi = \int_{\partial T} \varphi\tau$, for $\varphi$ a smooth $(k-1)$-form and $T$ as above. Notice that by (5.6) we have

$$\langle \hat{T}, d\varphi \rangle = \langle d^*\hat{T}, \varphi \rangle = \langle \partial T, \varphi \rangle,$$

so that $\partial T = d^*\hat{T}$.

A measure-valued $k$-form $\mu$ is said to be closed if $d\mu = 0$, and it is exact if there exists a measure-valued $k-1$-form $\psi$ such that $\mu = d\psi$.

\[^7\]One can thus identify a distribution-valued $k$-form with a $k$-current, see [10], although we generally choose not to do so.
5.1.2. the tangential part of measure-valued forms. Suppose that \( \omega \) is a closed measure-valued \( n - 1 \)-form defined on an open subset \( \Omega \subset \mathbb{R}^n \). If we fix an open \( U \subset \Omega \) with piecewise smooth boundary \( \partial U \), we will use the notation \( \omega_T \) to denote the distribution defined by

\[
(5.8) \quad \int f \omega_T := \int_U df \wedge \omega \quad \text{for all } f \in C^\infty(U) \cap C(\bar{U}).
\]

Thus our definition states that \( \omega_T := \ast d(\chi_U \omega) \) in the sense of distributions, where \( \chi_U \) is the characteristic function of \( U \). Although the notation \( \omega_T \) does not explicitly indicate the set \( U \), it will normally be clear from the context, and when it is not, we will write for example “\( \omega_T \) on \( \partial U \)”.

In general \( \omega_T \) is a distribution supported on \( \partial U \). We claim that

\[
(5.9) \quad \int f \omega_T \ \text{depends only on } f|_{\partial U}, \ \text{for smooth } f.
\]

To verify this, it suffices to check that \( \int_U df \wedge \omega = 0 \) for \( \omega \) as above, whenever \( f = 0 \) on \( \partial U \). Toward this end, let \( \chi_\epsilon \) denote a smooth function with compact support in \( U \), such that \( 0 \leq \chi_\epsilon \leq 1 \), \( |\nabla \chi_\epsilon| \leq C/\epsilon \), \( \chi_\epsilon(x) = 1 \) if \( \text{dist}(x, \partial U) \geq \epsilon \), and \( \chi_\epsilon = 0 \) if \( \text{dist}(x, \partial U) \leq \epsilon/2 \). Then

\[
\int_U df \wedge \omega = \lim_{\epsilon \to 0} \int_U \chi_\epsilon df \wedge \omega = \lim_{\epsilon \to 0} \int_U f d\chi_\epsilon \wedge \omega
\]

since \( \omega \) is closed. Since \( f \) is smooth and \( f = 0 \) on \( \partial U \), \( |f d\chi_\epsilon| \leq (C\epsilon)(C/\epsilon) \leq C \)

when \( \text{dist}(x, \partial U) < \epsilon \), so the right-hand side is bounded by \( \|\omega\|(\text{supp} d\chi_\epsilon) \). Since \( |\omega| \)

has finite total mass by assumption, we easily conclude that there exists a sequence \( \epsilon_k \downarrow 0 \) such that \( \lim_{k \to \infty} \int_U \chi_\epsilon_k df \wedge \omega = 0 \), proving (5.9).

It follows from (5.9) that expressions such as \( \int_{\partial U} \omega_T \) are well-defined.

In this paper it will often be the case that \( \omega_T \) is a measure supported on \( \partial U \), and when this holds, we may also think of \( \omega_T \) as a measure-valued \( (n - 1) \)-form on \( \partial U \). In particular, if \( \omega \) is smooth enough, then \( \int f \omega_T \) agrees with the classical expression discussed above, \( \int_{\partial U} f(x) \ast \omega(x) \), where \( i : \partial U \to \Omega \) is the inclusion map.

5.1.3. harmonic forms. If \( d\omega = d^*\omega = 0 \) then \( \omega \) is said to be harmonic. Denote by

\[
\mathcal{H}^k = \mathcal{H}^k(\Omega) := \{ \omega \in L^2 \cap C^\infty(\Lambda^k \Omega) \mid d\omega = 0, \ d^*\omega = 0 \}
\]

the space of harmonic \( k \)-forms on \( \Omega \), and by

\[
\mathcal{H}_N^k = \{ \omega \in \mathcal{H}^k \mid \omega_T = 0 \}, \quad \mathcal{H}_N^k = \{ \omega \in \mathcal{H}^k \mid \omega_N = 0 \},
\]

the spaces of harmonic forms with vanishing tangential and normal components on \( \partial \Omega \). Since \( \ast \omega_N = (\ast \omega)_T \) and \( \ast \ast = (-1)^{(n-k)} \), we have the bijections

\[
\ast : \mathcal{H}_N^k \to \mathcal{H}_N^{n-k}, \quad \ast : \mathcal{H}_N^k \to \mathcal{H}_N^{n-k}.
\]

Harmonic forms in \( \mathcal{H}_N^k \cup \mathcal{H}_N^k \) are smooth up to \( \partial \Omega \). Denote by \( H(\omega) \) (resp. \( H_T(\omega) \), \( H_N(\omega) \)) the orthogonal projection of a \( k \)-form \( \omega \) on \( \mathcal{H}^k \) (resp. \( \mathcal{H}_N^k \), \( \mathcal{H}_N^k \)). With respect to an orthonormal basis \( \{\gamma_i\}_{i=1, \ldots, \ell} \) of \( \mathcal{H}^k \) (resp. \( \mathcal{H}_N^k \), \( \mathcal{H}_N^k \)), the orthogonal projection is of course given by \( \sum_{i=1}^\ell \langle \gamma_i, \gamma_i \rangle \gamma_i \).

The Laplace operator \( -\Delta = dd^* + d^*d \) on smooth \( k \)-forms is positive semidefinite, commutes with \( \ast, d, d^* \), and \( h \in \mathcal{H}^k \Rightarrow -\Delta h = 0 \).
In both cases, we have so that
\[ \gamma \in H^k_N, \quad \alpha \in W^{1,p}(\Lambda^{k-1}\Omega), \quad \beta \in W^{1,p}(\Lambda^{k+1}\Omega), \quad \beta_N = 0. \]

Then \( \gamma = H_N(\omega) \). Moreover there exists a unique \( \Psi \in W^{2,p}(\Lambda^k\Omega) \) such that
\[ -\Delta \Psi = \omega - H_N(\omega), \quad \Psi_N = 0, \quad (d\Psi)_N = 0, \]
and
\[ ||d\Psi||_{1,p} + ||d^*\Psi||_{1,p} \leq C_p ||\omega||_p. \]

We will write \( \Psi = -\Delta_N^{-1}(\omega - H_N(\omega)) \).

The operator \( -\Delta_N^{-1} \) is self-adjoint on \( H^k_N \), and similarly \( -\Delta_N^{-1} \) is self-adjoint on \( H_N^\tau \).

**Remark.** In case \( \Omega = \mathbb{R}^n \), basic properties of harmonic functions imply that \( H^k = \{0\} \). For \( \omega \) compactly supported the potential \( \Psi \) is given in particular by \( \Psi = G*\omega \), where \( G(x) = c_n|x|^{n-2} \) is the Poisson kernel on \( \mathbb{R}^n, \ n \geq 3 \). The Hodge decomposition of \( \omega \) reads \( \omega = d\alpha + d^*\beta \) with \( \beta = G*\omega d \) and \( \alpha = G*d\omega \). In this case \( \alpha, \beta \in W^{1,p} \) rather than \( W^{1,p} \).

For \( \omega \in L^1(\Lambda^k\Omega) \) or more generally a measure-valued \( k \)-form, the decomposition \((5.10)\) fails in general, but decompositions of the form \((5.12), (5.15)\) still hold, in view of this variant of \cite{[J].}

**Proposition 3.** Let \( \mu \) be a measure-valued \( k \)-form in \( \Omega \). If \( H_N(\mu) = 0 \), there exists a unique \( \Psi \in W^{1,q}(\Lambda^k\Omega) \forall q < n/(n-1) \), denoted by \( \Psi = -\Delta_N^{-1}(\mu) \), such that
\[ -\Delta \Psi = \mu, \quad \Psi_N = 0, \quad (d\Psi)_N = 0, \]
so that in particular \( H_N(\Psi) = 0 \).

If \( H_T(\mu) = 0 \), then there exists a unique \( \Psi \in W^{1,q}(\Lambda^k\Omega) \forall q < n/(n-1) \), denoted by \( \Psi = -\Delta_T^{-1}(\mu) \), such that
\[ -\Delta \Psi = \mu, \quad \Psi_T = 0, \quad (d^*\Psi)_T = 0, \]
and in particular \( H_T(\Psi) = 0 \).

In both cases, we have
\[ ||d\Psi||_q + ||d^*\Psi||_q \leq C_q||\mu|| \quad \forall q < \frac{n}{n-1}. \]

**Proof.** The proof of Proposition 3 follows exactly the duality argument à la Stampacchia carried out in \cite{[J]}, taking into account the elliptic estimates \((5.13)\) for the operators \( -\Delta_N \) and \( -\Delta_T \), and observing that they are self-adjoint.
Corollary 1. A measure-valued k-form \( \mu \) is exact if and only if \( d\mu = 0 \) and \( H_N(\mu) = 0 \). In addition, if \( \mu \) is exact then \( \mu = d\zeta \), for \( \zeta := d^*(\Delta_N)^{-1}\mu \in \cap_{q \leq n/n-1} L^q(\Lambda^{k-1}\Omega) \), and \( \|\zeta\|_q \leq C_q\|\mu\| \).

Similarly, a measure-valued k form \( \mu \) is co-exact (that is, can be written \( \mu = d^*\psi \) for some measure-valued k+1-form \( \psi \)) if and only if \( d^*\mu = 0 \) and \( H^\tau(\mu) = 0 \), and if these conditions hold, then \( \mu = d^*\zeta \) for \( \zeta := d(-\Delta)^{-1}\mu \in \cap_{1 \leq q < n/n-1} L^q(\Lambda^{k+1}\Omega) \), and \( \|\zeta\|_q \leq C_q\|\mu\| \).

Proof. If \( d\mu = 0 \) and \( H_N(\mu) = 0 \) then we appeal to Proposition 3 and define \( \zeta = d^*(\Delta^{-1}_N\mu) \), and it follows that \( \mu = d\zeta \). Conversely, \( \mu = d\psi \) in \( \Omega \) for some measure-valued k – 1-form \( \psi \), then it is clear that \( d\mu = 0 \) in \( \Omega \), and if \( \varphi \in \mathcal{H}_N^\tau \), then for \( \chi \epsilon \) as in the proof of (5.9),

\[
\int \phi \cdot \mu = \lim_{\epsilon \to 0} \int \chi \epsilon \varphi \cdot d\psi = \lim_{\epsilon \to 0} \int d^*(\chi \epsilon \varphi) \cdot \psi.
\]

Next, the fact that \( \varphi \in \mathcal{H}_N^\tau \), and properties of \( \chi \epsilon \) imply that \( \|d^*(\chi \epsilon \varphi)\| = \|d\chi \epsilon \wedge \rho \varphi\| \leq C \), independent of \( \epsilon \). We then conclude as in the proof of (5.9) that \( \int \phi \cdot \mu = 0 \), and hence that \( H_N(\mu) = 0 \).

The assertions about co-exact forms are proved in exactly the same way. \( \square \)

Remark 20. In case \( \Omega = \mathbb{R}^n \), \( \mu \) compactly supported, we have in particular \( \zeta = d^*(G \ast \mu) \) (resp. \( \zeta = d(G \ast \mu) \)).

Remark 21. If \( \varphi \) is a smooth k-form and \( \varphi_N = 0 \) (resp. \( \varphi_{\tau} = 0 \)), then \( (d^*\varphi)_N = 0 \) (resp. \( (d\varphi)_\tau = 0 \)). The form \( \zeta \) of Corollary 11 is only in \( L^q \), and so does not have a normal (resp. tangential) trace, but can be shown to satisfy \( \zeta_N = 0 \) (resp. \( \zeta_{\tau} = 0 \)) in a sort of distributional sense, as a consequence of the fact that \( \zeta = d^*\Psi \) (resp. \( \beta = d^*\Psi \)) for \( \Psi = -\Delta^{-1}_N \mu \in W^{1,q} \), with \( \Psi_N = 0 \) (resp. \( \Psi = -\Delta^{-1}_\tau \mu \), \( \Psi_{\tau} = 0 \)).

This distributional trace (of which our definition (5.8) of \( q_{\tau} \) for a closed measure-valued \( n-1 \)-form \( q \) is a special case) is strong enough to provide uniqueness assertions in the setting of Corollary 11. For example, if \( d\mu = 0 \), then there is a unique \( \zeta \in L^q(\Lambda^{k-1}\Omega) \) satisfying \( d\xi = \mu, d^*\xi = 0 \), and \( \zeta_N = 0 \) in the distributional sense.

Remark 22. Through the Green operators \( -\Delta^{-1}_N \) (resp. \( -\Delta^{-1}_\tau \)), one obtains an integral expression for the linking number of a k-cycle and a (relative) \( (n-k-1) \)-boundary (resp. a relative k-cycle with a \( (n-k-1) \)-boundary) in \( \Omega \) (see e.g. [15]). Let for instance \( \Gamma \) be a relative \( (n-k-1) \)-boundary in \( \Omega \), i.e. \( \Gamma = \partial R + \Gamma' \) with \( R \subset \Omega \) and \( \Gamma' \subset \partial \Omega \). One immediately verifies that \( H_{\tau}(\Gamma) = 0 \), and hence \( H_{N}(\ast\Gamma) = 0 \). Let \( \beta = -\Delta^{-1}_N(\ast\Gamma) \). Hence we have \( d^*\beta \in L^p(\Lambda^{1}\Omega) \) for \( p < \frac{n}{n-1} \) and \( \beta \) is smooth outside \( \Gamma \). Hence, for a k-cycle \( \gamma \subset \Omega \setminus \Gamma \) we have \( 0 = \partial^\gamma = d^*\gamma \), and moreover

\[
\int_\gamma d^* \beta = (d^*\Delta^{-1}_N(\ast\Gamma), \gamma) = (\Gamma, d(-\Delta^{-1}_N \gamma)) = (\partial R, d(-\Delta^{-1}_N \gamma))
\]

\[
= (\widehat{\partial R}, d(-\Delta^{-1}_N \gamma)) = (\widehat{\partial R}, *\Delta^{-1}_N^{-1}(d \ast \gamma))
\]

\[
= (\widehat{\partial R}, *\gamma) = \langle \gamma \bigwedge R, *1 \rangle = \sum_{a_i \in \gamma \cap R} \ast(\tau_{a_i} \wedge \tau_R(a_i)) \in \mathbb{Z}.
\]

(5.17)

Observe that in case \( \Gamma = \partial R \subset \Omega \) is a \( (n-k-1) \)-boundary in \( \Omega \), we have \( H(\Gamma) = 0 \), hence we may consider \( \beta = -\Delta^{-1}(\ast\Gamma) = G \ast (\ast\Gamma) \) with \( G \) the Poisson kernel in \( \mathbb{R}^n \),
and deduce for $d^*\beta$ the integral representation
\begin{equation}
\label{eq:5.18}
d^*\beta = G \ast (\ast d\Gamma) = (\ast dG) \ast \tilde{\Gamma} = \int_\Gamma \ast dG(x-\cdot),
\end{equation}
which in the case $n = 3$, $k = 1$ reads more familiarly
\begin{equation}
\label{eq:5.19}
d^*\beta = \sum_{i,j,k=1}^3 4\pi dx^i \epsilon_{ijk} \int_{\Gamma^h_k} \frac{(x_j - y_j)dy^k}{|x-y|^3}.
\end{equation}
Following \cite{5.24}, we thus deduce the Biot-Savart formula for the linking number
\begin{equation}
\label{eq:5.20}
\int_{\gamma_x} \int_{\Gamma_y} \ast dG(x-y) = \langle \dot{R}, \ast \gamma \rangle = \sum_{a_i \in \gamma \cap \partial R} \ast (\tau_\gamma \wedge \ast \tau_R(a_i)) \in \mathbb{Z}.
\end{equation}
Notice that the integral formula \eqref{eq:5.20} gives $\text{link}(\Gamma, \gamma)$ also when $\Gamma$ is just a cycle, i.e. $\partial \Gamma = 0$, not necessarily a boundary. In fact, considering $\gamma \times \Gamma = 0 \in \mathbb{R}^n \times \mathbb{R}^n$, and $\ast d\Gamma(x-y) = |S^{n-1}|^{-1} \psi^*(ds)$, where $\psi : \gamma \times \Gamma \to S^{n-1} \subset \mathbb{R}^n$ is given by $\psi(x,y) = \frac{x-y}{|x-y|}$ and $ds$ is the volume form of $S^{n-1}$. Hence
\begin{equation}
\label{eq:5.21}
\int_{\gamma_x} \int_{\Gamma_y} \ast d\Gamma(x-y) = \frac{1}{|S^{n-1}|} \int_{\gamma \times \Gamma} \psi^*(ds) = \text{deg}(\psi) \in \mathbb{Z}.
\end{equation}

5.3. Representation of harmonic 1-forms. We describe next the spaces $H^1_\Omega$, (resp. $H^1_\Gamma$), of harmonic 1-forms on $\Omega \subset \mathbb{R}^n$ with zero normal (resp. tangential) component on $\partial \Omega$. Since $H^{n-1}_\Omega = \ast H^1_\Gamma$ (resp. $H^{n-1}_\Gamma = \ast H^1_\Omega$), this yields also a representation for harmonic $(n-1)$-forms.

**Lemma 9.** (Description of $H^1_\Gamma$). Let $(\partial \Omega)_i$, $i = 0, \ldots, b$ denote the connected components of $\partial \Omega$. Then $\gamma \in H^1_\Gamma$ if and only there exist constants $c_1, \ldots, c_b$ such that $\gamma = d\phi$, where $\phi$ is the unique harmonic function in $\Omega$ such that $\phi \equiv c_i$ on $(\partial \Omega)_i$, for $i \geq 1$, and $\phi = 0$ on $(\partial \Omega)_0$.

**Proof.** In fact $H^1_\Gamma$ is isomorphic to the first relative de Rham cohomology group of $\Omega$, that is $H^1_{\partial \Omega}(\Omega, \mathbb{R}^n)$, (see for example \cite{17} vol. 1, Corollary 1, section 5.2.6 and $H^1_{\partial \Omega}(\Omega, \mathbb{R}^n) \cong \mathbb{R}^b$, as it is shown in Lemma \ref{lem:12} below. Finally, the family of 1-forms described in the above statement span a $b$-dimensional subspace of $H^1_\Gamma$. \hfill \Box

**Lemma 10.** (Description of $H^1_\Omega$). Let $\kappa$ denote the dimension of $H^1_\Omega$. Then there exists an orthogonal basis $\{H_j\}_{j=1}^\kappa$ for $H^1_\Omega$ normalized so that for each $j$ there exists a $\mathbb{R}/\mathbb{Z}$-valued function $\phi_j$ such that $H_j = d\phi_j$, so that $e^{2\pi i \phi_j}$ is well-defined.

**Proof.** In fact $H^1_\Omega$ is isomorphic to the first de Rham cohomology group $H^1_{\partial \Omega}(\Omega)$, which in turn is isomorphic to $\text{Hom}(H_1(\Omega, \mathbb{Z}), \mathbb{R})$, and these are all finitely generated. (See e.g. \cite{17} vol.1, Corollary 1 in section 5.2.6 and Theorem 3 in Section 5.3.2.) It follows that if $\{\gamma_i\}_{i=1}^\kappa$ are cycles that form a basis for $H_1(\Omega, \mathbb{Z})$, then there exists a (unique) basis $\{H_i\}_{i=1}^\kappa$ for $H^1_\Omega$ such that $\int_{\gamma_i} H_j = \delta_{ij}$ for $i, j = 1, \ldots, \kappa$.

We now fix $x_0 \in \Omega$ and define $\phi_j(x) := \int_{\gamma(x_0,x)} H_j$, $j = 1, \ldots, \kappa$, where $\gamma(x_0,x)$ is any path in $\Omega$ that starts at $x_0$ and ends at $x$. If $\gamma'(x_0,x)$ is another such path, then $\gamma(x_0,x) - \gamma'(x_0,x)$ is homologous to an integer linear combination of the $\gamma_i$’s, so that $\int_{\gamma(x_0,x)} H_j - \int_{\gamma'(x_0,x)} H_j \in \mathbb{Z}$. Thus $\phi_j$ is well-defined as a function $\Omega \to \mathbb{R}/\mathbb{Z}$. It is immediate that $H_j = d\phi_j$. \hfill \Box
Remark 23. Although this fact is not needed in this paper, we remark that if $H \in \mathcal{H}^b_{N}$, and $K = H_{\uparrow} = H \setminus \partial \Omega$ is its tangential component on $\partial \Omega$, then $K$ is a harmonic $k$-form in $\partial \Omega$. (A special case of this fact is used in the proof of Lemma [10] above.) Indeed, since $dH = 0$ and $(dH)_{N} = 0$ we have $dK = (dH)_{\uparrow} = dH - (dH)_{N} = 0$. Moreover, one can check that $d \ast_{\uparrow} K = (d \ast H)_{\uparrow}$ since $H_{N} = 0$, where $\ast_{\uparrow}$ denotes the star operator on the tangent space of $\partial \Omega$. Hence $d \ast_{\uparrow} K = 0$ and the conclusion follows.

We describe next an exactness criterion for closed $(n-1)$-forms in $\Omega \subset \mathbb{R}^n$.

Lemma 11. A measure-valued $(n-1)$ form $q$ on a smooth bounded open set $\Omega \subset \mathbb{R}^n$ is exact if and only if $dq = 0$ and $\int_{(\partial \Omega)_i} q_{\uparrow} = 0$ for every connected component $(\partial \Omega)_i$ of $\partial \Omega$.

Proof. Let $\gamma \in \mathcal{H}^b_{N-1}$, so that $\ast \gamma \in \mathcal{H}^b$ and hence, by Lemma [9], $\ast \gamma = d \varphi$, where $\Delta \varphi = 0$ in $\Omega$ and $\varphi \equiv c_i$ on the $i$-th connected component $(\partial \Omega)_i$. Then

\begin{equation}
\langle q, \gamma \rangle = \int_{\Omega} q \wedge \ast \gamma = \int_{\Omega} q \wedge d \varphi = \sum_{i=1}^{b} c_i \int_{(\partial \Omega)_i} q_{\uparrow}
\end{equation}

We deduce that $H_{N}(q) = 0$ if and only if $\int_{(\partial \Omega)_i} q_{\uparrow} = 0$ for every $i$. The conclusion now follows from Corollary [11].

5.4. Proof of Lemma [9] completed. We need the following easy result, whose proof uses the language of algebraic topology (see e.g. [32]).

Lemma 12. Let $U$ be a connected Lipschitz domain in $\mathbb{R}^n$, such that $\partial U$ has $b+1$ connected components. Then $H^1_{\partial R}(U, \partial U) \simeq \mathbb{R}^b$.

Proof. From the exact sequence in singular homology for the pair $(\bar{U}, \partial U)$ we have

\begin{equation}
H_1(\partial U) \overset{i_*}{\rightarrow} H_1(\bar{U}) \overset{\Phi_*}{\rightarrow} H_1(\bar{U}, \partial U) \overset{\partial_*}{\rightarrow} H_0(\partial U) \overset{i^0_*}{\rightarrow} H_0(\bar{U}) \rightarrow 0
\end{equation}

which gives rise to the short exact sequence

\begin{equation}
0 \rightarrow \text{Im } \Phi_* \rightarrow H_1(\bar{U}, \partial U) \rightarrow \text{Ker } i^0_* \rightarrow 0
\end{equation}

By hypothesis we have $H_0(U) = \mathbb{Z}, H_0(\partial U) = \mathbb{Z}^{b+1}$, and (5.23) implies $\text{Ker } i^0_* = \mathbb{Z}^b$.

By the Mayer-Vietoris exact sequence for $V = \bar{U}, W = \mathbb{R}^n \setminus U$ we have

\begin{equation}
H_2(V \cup W) \rightarrow H_1(V \cap W) \overset{(i_* \circ i^*_*)}{\rightarrow} H_1(V) \oplus H_1(W) \rightarrow H_1(V \cup W)
\end{equation}

which yields, since $V \cup W = \mathbb{R}^n$ is contractible,

\begin{equation}
0 \rightarrow H_1(\partial U) \overset{(i_* \circ i^*_*)}{\rightarrow} H_1(\bar{U}) \oplus H_1(\mathbb{R}^n \setminus \bar{U}) \rightarrow 0,
\end{equation}

so that $(i_*, i^*_*)$ is an isomorphism. In particular $i_* = \pi_1 \circ (i_*, i^*_*)$ is onto, hence $H_1(\bar{U}) = \text{Im } i_* = \text{Ker } \Phi_*$, which yields $\text{Im } \Phi_* = 0$, so that (5.24) implies that $H_1(U, \partial U)$ is isomorphic to $\text{Ker } i^0_* = \mathbb{Z}^b$. From the regularity assumption on $U$ we have in particular $H_1(U, \partial U) \simeq H_1(U, \partial U)$. Finally, from the relation

\begin{equation}
H^1(U, \partial U) ; \mathbb{R} = \text{Hom}(H_1(U, \partial U); \mathbb{R}) = \text{Hom}(\mathbb{Z}^b; \mathbb{R}) \simeq \mathbb{R}^b
\end{equation}

the conclusion follows, since the first singular relative cohomology group with real coefficients $H^1(U, \partial U); \mathbb{R}$ is isomorphic to the first de Rham relative cohomology group $H^1_{\partial R}(U, \partial U)$.
5.5. Proof of Lemma 4

Step 1. We have: \( \inf \{ \| \alpha \|_{L^1(\Lambda^2 K)} , \ d \alpha = 0 \ in \ K , \ \alpha_T = \zeta \ on \ \partial K \} = \| \zeta \|_{W^{-1,1}(K)} \), where

\[
\| \zeta \|_{W^{-1,1}(K)} = \sup \left\{ \int \varphi \zeta : \ varphi \ in \ W_c^{1,\infty}(\mathbb{R}^3) , \ \| d \varphi \|_{L^\infty(K)} \leq 1 \right\}.
\]

This follows by a straightforward modification of an argument in Federer [16]. We provide a sketch: define a linear functional acting on \( C_c(\mathbb{R}^3) \) by

\[
A(\varphi) := \int_K \varphi \zeta, \quad \varphi \ in \ C_c(\mathbb{R}^3).
\]

Given any measure-valued 2-form \( \alpha \), we similarly define a linear functional \( B_\alpha \) acting on \( C_c(\Lambda^1 \mathbb{R}^3) \) by

\[
B_\alpha(\psi) = \int_K \psi \wedge \alpha, \quad \psi \ in \ C_c(\Lambda^1 \mathbb{R}^3).
\]

And generally, for a linear functional \( C \) on \( C_c(\Lambda^1 \mathbb{R}^3) \), we define \( \partial C(\varphi) := C(d \varphi) \) for \( \varphi \ in \ C_c(\mathbb{R}^3) \). Then the definitions (see (5.28) in particular) imply that \( A = \partial C \) and \( \| C \| < \infty \) if and only if \( C = B_\alpha \) for some measure-valued 2-form \( \alpha \) such that \( d \alpha = 0 \) in \( K \) and \( \alpha_T = \zeta \) on \( \partial K \). Next, we note that \( \| \zeta \|_{W^{-1,1}(K)} = F_{\text{hom},K}(A) \), where \( F_{\text{hom},S}(A) \) denotes the homogeneous flat norm of \( A \) in \( K \), see [16]. Then as observed in section 4.1.12 of [16] in a slightly different setting, the Hahn-Banach Theorem implies that

\[
F_{\text{hom},K}(A) = \min \{ \| C \| , \ \text{spt} C \subset K , \ \partial C = A \}
\]

and this translates our claim, in view of our earlier remarks.

Step 2. We claim that \( \| \zeta \|_{W^{-1,1}(K)} \leq C \| \zeta \|_{W^{-1,1}(\mathbb{R}^3)} \), where

\[
\| \zeta \|_{W^{-1,1}(\mathbb{R}^3)} = \sup \left\{ \int_{\mathbb{R}^3} \varphi \zeta : \ varphi \ in \ W_c^{1,\infty}(\mathbb{R}^3) , \ \| \varphi \|_{W^{1,\infty}(\mathbb{R}^3)} \leq 1 \right\}.
\]

It suffices to show that there exists \( C > 0 \) such that, for any \( \varphi \ in \ W_c^{1,\infty}(\mathbb{R}^3) \) with \( \| d \varphi \|_{L^\infty(K)} \leq 1 \), there exists \( \psi \ in \ W_c^{1,\infty}(\mathbb{R}^3) \) such that

\[
(5.28) \quad \int \varphi \zeta = \int \psi \zeta \quad \text{and} \quad \| \psi \|_{W^{1,\infty}(\mathbb{R}^3)} \leq C.
\]

Indeed, given \( \varphi \) such that \( \| d \varphi \|_{L^\infty(K)} < \infty \), we fix \( x_0 \ in \ K \) and we define \( \psi(x) = \varphi(x) - \varphi(x_0) \) for \( x \ in \ K \). Since \( K \) is convex, \( \varphi \) and hence \( \psi \) are 1-Lipschitz on \( K \), so that \( |\psi(x)| \leq |x - x_0| \leq \text{diam}(K) \) in \( K \). Next, we extend \( \psi \) to \( \mathbb{R}^3 \setminus K \), such that the extended function is still 1-Lipschitz and moreover satisfies \( \| \psi \|_{L^\infty(\mathbb{R}^3)} \leq \text{diam}(K) \), and has compact support.

Since \( \zeta \) is a measure supported on \( \partial K \), clearly \( \int \psi \zeta \) depends only on the behavior of \( \psi \) on \( \partial K \), and hence \( \int \psi \zeta = \int (\varphi - \varphi(x_0)) \zeta = \int \varphi \zeta \), since \( \int_{\partial K} \zeta = 0 \), proving (5.28).

5.6. Proof of Lemma 1

Step 1. We will show below that there exists a piecewise smooth oriented 2-manifold with boundary \( S = S_e \) such that

\[
(5.29) \quad \partial S = M_e - M'_e \ in \ U \ and \ \mathcal{H}^2(S \cap U) \leq C \ell \cdot E_e(u_e; \Omega) \leq C \ell g_e,
\]

with \( C > 0 \) independent of \( \epsilon \) and \( U \). (See the proof of Proposition 1 for notation used here and below.) We first complete the proof of the lemma, assuming (5.29).
We may assume that $S$ intersects transversally the level set $f^{-1}(t)$ for a.e. $t$, since if not, we can arrange that this condition is satisfied after an arbitrarily small perturbation of $S$ that leaves $\partial S$ fixed. Noting that $f^{-1}(t)$ coincides with $\partial C^t$ for a.e. $t$, we deduce that $S \cap \partial C^t$ is piecewise smooth for a.e. $t > 0$.

Since $f$ is 1-Lipschitz, the same is true for $f \subseteq S$, so that $|\nabla (f \subseteq S)| \leq 1$ a.e., and

$$\mathcal{H}^2(\partial S \cap 3Nt) \cap U \geq 0 \int_{(S \cap C^Nt) \cap U} |\nabla (f \subseteq S)|d\mathcal{H}^2 = 0 \int_0^N \mathcal{H}^1((S \cap \partial C^t) \cap U)dt,$$

by the coarea formula. We deduce that there exists $t_e$ s.t.

$$(5.30) \quad \mathcal{H}^1((S \cap \partial C^{t_e}) \cap U) \leq (Nt)^{-1}\mathcal{H}^2(S \cap U) \leq CN^{-1}g_e.$$

In $U$ it holds

$$\partial (S \cap C^{t_e}) = (\partial S) \cap C^{t_e} + S \cap (\partial C^{t_e})$$

$$= (M_e - M'_e) \cap C^{t_e} + S \cap (\partial C^{t_e}) = M_e - M'_e \cap C^{t_e} + S \cap (\partial C^{t_e}).$$

In particular, for $\phi \in C_0^\infty(A^1U)$, we have

$$\langle \nu_e - \nu_e \subseteq C^{t_e}, \phi \rangle = \int_{S \cap C^{t_e}} d* \phi - \int_{S \cap (\partial C^{t_e})} * \phi,$$

(5.32)

$$\|\nu_e - \nu_e \subseteq C^{t_e}\|_{W^{-1,1}(U)} \leq \mathcal{H}^2(S \cap C^{t_e} \cap U) + \mathcal{H}^1(S \cap \partial C^{t_e} \cap U)$$

$$\leq (1 + (Nt)^{-1})\mathcal{H}^2(S \cap U) \leq C(\ell + N^{-1})g_e$$

by (5.30) and (5.29). This gives precisely (2.24).

**Step 2.** To conclude, we supply the proof of our earlier claim (5.28).

Let $g(x) = \left|\text{dist}(x, R_1)\right|^{-1} + \left|\text{dist}(x, R'_1)\right|^{-1}$. By the coarea formula, we have

$$\int_0^1 ds \int_{u^{-1}(s)} g(x)d\mathcal{H}^1(x) = \int_{\Omega} g(x)|Ju_e|dx \leq \int_{\Omega} g(x)e_\nu(u_e)dx,$$

so that by a mean-value argument, (2.12), and (2.22), we deduce from (5.33) that there exists a regular value $s$ of $u_e$ such that $|s| < 1/2$ and, denoting $M_s := u_e^{-1}(s)$, we have

$$\int_{M_s} g(x)d\mathcal{H}^1(x) = \int_{M_s} \frac{d\mathcal{H}^1(x)}{|\text{dist}(x, R_1)|} + \int_{M_s} \frac{d\mathcal{H}^1(x)}{|\text{dist}(x, R'_1)|} \leq \frac{K\nu_e(u_e;\Omega)}{\pi \delta \ell}.$$

Define as in [1], Lemma 3.8 (i), the map $\Phi : \mathbb{R}^3 \setminus R_1 \to R'_1$ and, accordingly, the map $\Phi^* : \mathbb{R}^3 \setminus R'_1 \to R'_1$. Set $\Psi(t, x) = (1 - t)x + t\Phi(x)$, $\Psi^*(t, x) = (1 - t)x + t\Phi^*(x)$, and define $S_1 = \Psi([0, 1] \times M_s)$ and $S_2 = \Psi^*([0, 1] \times M_s)$. Note, following [1], Lemma 3.8 (ii), that since $M_s$ has no boundary in $U$, we have $\partial S_1 = \Phi\# M_s - M_s$ and $\partial S_2 = \Phi^\# M_s - M_s$ in $U$. However, from [1], Lemma 3.8 (i), we know that $\Phi\# M_s = M_s$ the point being that the intersection number of $M_s$ with any 2-face $Q_i$ agrees with $(-1)^{n-d}Q_i$, due to orientation conventions and elementary properties of topological degree. Similarly $\Phi^\# M_s = M'_s$, so if we define $S := S_1 - S_2$, then $\partial S = M_s - M'_s$ in $U$, which is the first part of (5.29). Following the proof of [1], Lemma 3.8 (ii), we readily deduce that

$$\mathcal{H}^1(S \cap U) = \mathcal{H}^1(S_1 \cap U) + \mathcal{H}^1(S_2 \cap U) \leq Ct^2 \int_{M_s} g(x)d\mathcal{H}^1(x).$$
Combining (5.35) and (5.34), claim (5.29) follows.

References

[1] G. Alberti, S. Baldo, G. Orlandi, Variational convergence for functionals of Ginzburg-Landau type, Indiana Univ. Math. J. 54 (2005), 1411–1472.
[2] S. Baldo, G. Orlandi, R. Jerrard, M. Soner, Vortex density models for superconductivity and superfluidity, preprint 2010.
[3] S. Baldo, G. Orlandi, A note on the Hodge theory for functionals with linear growth, Manuscripta Math. 97 (1998), 453–467.
[4] S. Baldo, G. Orlandi, S. Weitkamp, Convergence of minimizers with local energy bounds for the Ginzburg-Landau functionals, Indiana Univ. Math. J., to appear
[5] F. Bethuel, H. Brezis, F. Hélein, Ginzburg-Landau vortices Progress in Nonlinear Differential Equations and their Applications, 13 Birkhäuser Boston, 1994.
[6] F. Bethuel, H. Brezis, G. Orlandi, Asymptotics for the Ginzburg-Landau equation in arbitrary dimensions, J. Funct. Anal. 186 (2001), 432–520.
[7] F. Bethuel, G. Orlandi, D. Smets, Vortex rings for the Gross-Pitaevskii equation, J. Eur. Math. Soc. 6 (2004), 17–94.
[8] F. Bethuel, G. Orlandi, D. Smets, Approximations with vorticity bounds for the Ginzburg-Landau functional Commun. Contemp. Math. 6 (2004), 803–832.
[9] F. Bethuel, T. Riviè re, Vorticité dans les modèles de Ginzburg-Landau pour la supraconduction, Séminaire X EDP, 1993–1994, Exp. No. XVI, cole Polytechnique, Palaiseau
[10] J. Bourgain, H. Brezis, P. Mironescu, $H^{1/2}$ maps with values into the circle: minimal connections, lifting, and the Ginzburg-Landau equation, Publ. Math. Inst. Hautes Etudes Sci. 99 (2004), 1–115.
[11] H. Brezis, J.M. Coron, E. Lieb, Harmonic maps with defects, Comm. Math. Phys. 107 (1986), 649–705.
[12] H. Brezis, S. Serfaty, A variational formulation for the two-sided obstacle problem with measure data. Commun. Contemp. Math. 4 (2002), no. 2, 357374.
[13] D. Chiron, Boundary problems for the Ginzburg-Landau equation Commun. Contemp. Math. 7 (2005), 597–648.
[14] P. G. Ciarlet, The finite element method for elliptic problems., SIAM Philadelphia, 2002.
[15] G. De Rham, Sur la théorie des formes différentielles harmoniques, Ann. Univ. Grenoble 22 (1946), 135–152.
[16] H. Federer, Geometric measure theory, Grundlehren der mathematischen Wissenschaften, 153 Springer-Verlag New York,1969.
[17] M. Giaquinta, G. Modica, J. Souček, Cartesian currents and the Calculus of variations, vol 1. and 2., Springer
[18] E. Giusti, $BV$ Functions and set of finite perimeter, Birkhäuser, Berlin (1982).
[19] T. Iwaniec, C. Scott, B. Stroffolini, Nonlinear Hodge theory on manifolds with boundary Ann. Mat. Pura Appl. (4) 177 (1999), 37–115.
[20] R. Jerrard, Lower bounds for generalized Ginzburg-Landau functionals, SIAM J. Math. Anal. 30 (1999), 721–746.
[21] R. Jerrard, A. Montero, P. Sternberg, Local minimizers of the Ginzburg-Landau energy with magnetic field in three dimensions, Comm. Math. Phys. 249 (2004), 549–577.
[22] R. Jerrard, H.M. Soner, The Jacobian and the Ginzburg-Landau energy, Calc. Var. Partial Differential Equations 14 (2002), 151–191.
[23] R. Jerrard, H.M. Soner, Limiting behavior of the Ginzburg-Landau functional, J. Funct. Anal. 192 (2002), 524–561.
[24] F.H. Lin, T. Rivière, Complex Ginzburg-Landau equations in high dimensions and codimension two area minimizing currents, J. Eur. Math. Soc. 1 (1999), 237–311.
[25] A. Montero, Hodge decomposition with degenerate weights and the Gross-Pitaevskii energy, J. Funct. Anal. 254 (2008), 1926–1973.
[26] A. Montero, P. Sternberg, W. Ziemer, Local minimizers with vortices in the Ginzburg-Landau system in three dimensions, Comm. Pure Appl. Math. 57 (2004), 99–125.
[27] C. B. Morrey, A variational method in the theory of harmonic integrals. II, Amer. J. Math. 78 (1956), 137–170.
[28] E. Sandier, *Lower bounds for the energy of unit vector fields and applications*, J. Funct. Anal. 152 (1998), 379–403. Erratum: Ibid. 171 (2000), 233.

[29] E. Sandier, S. Serfaty, *A rigorous derivation of a free-boundary problem arising in superconductivity*, Ann. Sci. cole Norm. Sup. (4) 33 (2000), no. 4, 561-592.

[30] E. Sandier, S. Serfaty, *Vortices in the magnetic Ginzburg-Landau model*, Progress in Nonlinear Differential Equations and their Applications, 70. Birkhäuser Boston, 2007.

[31] E. Sandier, S. Serfaty, *A product-estimate for Ginzburg-Landau and corollaries*, J. Funct. Anal. 211 (2004), 219–244.

[32] E. Spanier, *Algebraic Topology*. Mc Graw-Hill, New York, 1966.