AN EQUIVALENT CHARACTERIZATION OF WEAK BMO MARTINGALE SPACES

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Abstract. In this paper, we give an equivalent characterization of weak BMO martingale spaces due to Ferenc Weisz (1998).

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1. INTRODUCTION

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, and $\{\mathcal{F}_n\}_{n \geq 0}$ be an increasing sequence of sub-$\sigma$-algebras of $\mathcal{F}$ such that $\mathcal{F} = \sigma\left(\bigcup_{n \geq 0} \mathcal{F}_n\right)$. The expectation operator and the conditional expectation operator relative to $\mathcal{F}_n$ are denoted by $E$ and $E_n$, respectively. A sequence $f = (f_n)_{n \geq 0}$ of random variables such that $f_n$ is $\mathcal{F}_n$-measurable is said to be a martingale if $E(|f_n|) < \infty$ and $E_n(f_{n+1}) = f_n$ for every $n \geq 0$.

The study of the space BMO (Bounded Mean Oscillation) began with the establishment of the so-called John–Nirenberg theorem in [11]. Basing mainly on the duality and something else, the space BMO plays a remarkable role both in classical analysis and martingale theory. For example, BMO is a good space in operator actions (see e.g. [14], Chapter 4). And the martingale space $BMO_r(\alpha)$ was first introduced by Herz in [4] as the dual of $H^p_{\alpha} (0 < p \leq 1)$ associated with the dyadic filtration (see Example [13] below). With the help of atomic decomposition, Weisz extended this result in [15] to a general case. Let $T$ be the set of all stopping times with respect to $\{\mathcal{F}_n\}_{n \geq 0}$. The martingale space $BMO_r(\alpha)$ ([16], p. 8; or [15]) for $1 \leq r < \infty$ and $\alpha \geq 0$ is defined as

$$BMO_r(\alpha) = \{f = (f_n)_{n \geq 0} : \|f\|_{BMO_r(\alpha)} < \infty\},$$

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288
D. Zhou et al.

where
\[ \|f\|_{BMO_r(\alpha)} = \sup_{\nu \in T} \mathbb{P}(\nu < \infty)^{-1/r - \alpha} \|f - f^\nu\|_r. \]

We present two well-known results (see [16] or [15]). If \(0 < p \leq 1\) and \(\alpha = \frac{1}{p} - 1\), then \(BMO_2(\alpha)\) is the dual space of the Hardy space \(H_p^s\). If the stochastic basis \(\{F_n\}_{n \geq 1}\) is regular, then \(BMO_r(\alpha) = BMO_1(\alpha)\). And recently, Yi et al. proved in [18] that \(BMO_E(\alpha) = BMO_1(\alpha)\), where \(\alpha = 0\) and \(E\) is a rearrangement invariant Banach function space.

In the present paper, we consider a weak BMO martingale space. To characterize the dual of the weak Hardy martingale space \(H_p^s; \infty\), Weisz in [17] first introduced and studied the weak BMO martingale space. Let us recall the definition. We also refer the reader to [12] and [13] for some new results related to weak BMO martingales spaces.

**Definition 1.1.** Let \(1 \leq r < \infty\), \(\alpha r + 1 > 0\). The space \(wBMO_r(\alpha)\) is defined as the set of all martingales \(f \in L_r\) with the norm
\[ \|f\|_{wBMO_r(\alpha)} = \int_0^\infty \frac{t^r_\alpha(x)}{x} dx < \infty, \]
where \(t^r_\alpha(x) = x^{-1/r - \alpha} \sup_{\nu \in T: \mathbb{P}(\nu < \infty) \leq x} \|f - f^\nu\|_r.\)

In the very recent paper [8], the generalized BMO martingale space is introduced as the dual of Hardy–Lorentz martingale space. Strongly motivated by [8], Definition 1.1, we introduce the following new weak BMO martingale space by stopping time sequences.

**Definition 1.2.** Let \(1 \leq r < \infty\) and \(\alpha \geq 0\). The weak BMO martingale space \(wBMO_r(\alpha)\) is defined by
\[ wBMO_r(\alpha) = \{ f \in L_r : \|f\|_{wBMO_r(\alpha)} < \infty \}, \]
where
\[ \|f\|_{wBMO_r(\alpha)} = \sup_{\nu \in T: \mathbb{P}(\nu < \infty) \leq 1} \sum_{k \in \mathbb{Z}} 2^k \mathbb{P}(\nu_k < \infty)^{1-1/r} \|f - f^{\nu_k}\|_r \]
and the supremum is taken over all stopping time sequences \(\{\nu_k\}_{k \in \mathbb{Z}}\) such that \(2^k \mathbb{P}(\nu_k < \infty)^{1+\alpha} \in \ell_\infty\).

It is a very natural question: what is the relationship between \(wBMO_r(\alpha)\) and \(wBMO_r(\alpha)\)? The paper fully answers this question. Our main result can be described as follows. We simply put \(wBMO = wBMO(0)\) and \(wBMO = wBMO(0)\).
Weaker BMO martingale spaces

**Theorem 1.1.** Let $1 \leq r < \infty$ and $\alpha \geq 0$. If the stochastic basis $\{F_n\}_{n \geq 0}$ is regular, then

$$wBMO_r(\alpha) = wBMO_r(\alpha)$$

with equivalent norms. In particular,

$$wBMO_r = wBMO_r$$

with equivalent norms.

In this paper, the set of integers and the set of nonnegative integers are always denoted by $\mathbb{Z}$ and $\mathbb{N}$, respectively. We use $C$ to denote a positive constant which may vary from line to line. The symbol $\subset$ means the continuous embedding.

**2. Preliminaries**

Firstly, we give the definition of Lorentz spaces. We denote by $L_{0}(\Omega, \mathcal{F}, \mathbb{P})$, or simply $L_{0}(\Omega)$, the space of all measurable functions on $(\Omega, \mathcal{F}, \mathbb{P})$. For any $f \in L_{0}(\Omega)$, we define the distribution function of $f$ by

$$\lambda_s(f) = \mathbb{P}\{\omega \in \Omega : |f(\omega)| > s\}, \quad s \geq 0.$$ 

Moreover, denote by $\mu_t(f)$ the decreasing rearrangement of $f$ defined by

$$\mu_t(f) = \inf\{s \geq 0 : \lambda_s(f) \leq t\}, \quad t \geq 0,$$

with the convention that $\inf\emptyset = \infty$.

**Definition 2.1.** Let $0 < p < \infty$ and $0 < q \leq \infty$. Then, the Lorentz space $L_{p,q}(\Omega)$ consists of measurable functions such that $\|f\|_{p,q} < \infty$, where

$$\|f\|_{p,q} = \left[\int_0^\infty (t^{1/p} \mu_t(f))^q \frac{dt}{t}\right]^{1/q}, \quad 0 < q < \infty,$$

and

$$\|f\|_{p,\infty} = \sup_{0 \leq t < \infty} t^{1/p} \mu_t(f), \quad q = \infty.$$ 

**Remark 2.1.** We refer the reader to [2] for the following basic properties.

1. If $p = q$, then $L_{p,q}(\Omega)$ becomes $L_p(\Omega)$.

2. If $0 < p_1 \leq p_2 < \infty$ and $0 < q \leq \infty$, then $\|f\|_{p_1,q} \leq C \|f\|_{p_2,q}$, where $C$ depends on $p_1, p_2$ and $q$. This is due to $\mathbb{P}(\Omega) = 1$.

3. If $0 < p < \infty$ and $0 < q_1 \leq q_2 < \infty$, then $\|f\|_{p,q_2} \leq C \|f\|_{p,q_1}$, where $C$ depends on $q_1, q_2$ and $p$. 

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Denote by $\mathcal{M}$ the set of all martingales $f = (f_n)_{n \geq 0}$ relative to $\{\mathcal{F}_n\}_{n \geq 0}$ such that $f_0 = 0$. For $f \in \mathcal{M}$, denote its martingale difference by $d_nf = f_n - f_{n-1}$ ($n \geq 0$, with the convention $f_{-1} = 0$). Then the maximal function and the conditional quadratic variation of a martingale $f$ are respectively defined by

$$f_n^* = \sup_{0 \leq i \leq n} |f_i|, \quad f^* = \sup_{n \geq 0} |f_n|, \quad s_n(f) = \left( \sum_{i=1}^n \mathbb{E}_{i-1}|d_i f|^2 \right)^{1/2}, \quad s(f) = \left( \sum_{i=1}^{\infty} \mathbb{E}_{i-1}|d_i f|^2 \right)^{1/2}.$$ 

Then we define martingale Hardy–Lorentz spaces as follows.

**Definition 2.2.** Let $0 < p < \infty$ and $0 < q \leq \infty$. Define

$$H_{p,q}^s = \{ f \in \mathcal{M} : \|f\|_{H_{p,q}^s} = \|f^*\|_{p,q} < \infty \},$$

$$H_{p,q}^s = \{ f \in \mathcal{M} : \|f\|_{H_{p,q}^s} = \|s(f)\|_{p,q} < \infty \}.$$

If $p = q$, then the martingale Hardy–Lorentz spaces recover the martingale Hardy spaces $H_p^s$ and $H_p^s$ (see [16]).

Recall that the stochastic basis $\{\mathcal{F}_n\}_{n \geq 0}$ is said to be regular if there exists a positive constant $R > 0$ such that

$$(2.1) \quad f_n \leq R f_{n-1}, \quad \forall n > 0,$$

holds for all nonnegative martingales $f = (f_n)_{n \geq 0}$. Condition (2.1) can be replaced by several other equivalent conditions (see [14], Chapter 7). We refer the reader to [14], p. 265, for examples for regular stochastic basis. Here, we give a special case.

**Example 2.1.** Let $((0,1], \mathcal{F}, \mu)$ be a probability space such that $\mu$ is the Lebesgue measure and subalgebras $\{\mathcal{F}_n\}_{n \geq 0}$ are generated as follows:

$$\mathcal{F}_n = \text{a } \sigma\text{-algebra generated by atoms } \left( \frac{j}{2^n}, \frac{j+1}{2^n} \right], \quad j = 0, \ldots, 2^n - 1.$$ 

Then $\{\mathcal{F}_n\}_{n \geq 0}$ is regular. And all martingales with respect to such $\{\mathcal{F}_n\}_{n \geq 0}$ are called dyadic martingales.

The method of atomic decompositions plays an important role in martingale theory (see, for example, [3]–[5], [15], [17]). The atomic decompositions of Hardy–Lorentz martingale spaces $H_{p,q}^s$ and martingale inequalities are given in [6] and [8]. We also mention that Hardy–Lorentz spaces with variable exponents were investigated very recently in [9] and [10]. Let us first introduce the concept of an atom (see [16], p. 14).

**Definition 2.3.** Let $0 < p < \infty$ and $p < r \leq \infty$. A measurable function $a$ is called a $(1,p,r)$-atom (or $(3,p,r)$-atom) if there exists a stopping time $\nu \in T$ such that $a_\nu = \mathbb{E}_\nu(a) = 0$ if $\nu \geq n$, and

$$\|s(a)\|_r \leq \mathbb{P}(\nu < \infty)^{1/r-1/p}.$$
Remark 2.2. Let $0 < p < r \leq \infty$ and $0 < q < r$. If $a$ is a $(1, p, r)$-atom, then $\|a\|_{H_{p, q}^*} \leq C$. Choose $p_1, p_2$ such that $\frac{1}{p} = \frac{1}{r} + \frac{1}{p_1}, \frac{1}{q} = \frac{1}{r} + \frac{1}{q_1}$. By Hölder's inequality, we have ($\nu$ is the stopping time corresponding to the atom $a$)

$$\|a\|_{H_{p, q}^*} = \|s(\nu)\|_{p, q} \leq C\|s(\nu)\|_{r, r} \|\chi_{\{\nu < \infty\}}\|_{p_1, q_1} \leq CT(\nu < \infty)^{1/r-1/p} \left( \int_0^\infty t^{q_1/p_1-1} \chi_{(0, \infty)}(\nu < \infty) \, dt \right)^{1/q_1} \leq C.$$

Similarly, we have $\|a\|_{H_{p, q}^{**}} \leq C$ for a $(3, p, r)$-atom $a$. If $p = q$, then $C = 1$.

The following result is from [8]. And the result about the Hardy space $H_{p, q}^*$ follows from the combining of Theorem 3.3 and Lemma 5.1 in [8].

Theorem 2.1. If $f = (f_n)_{n \geq 0} \in H_{p, q}^*$ for $0 < p < \infty$, $0 < q < \infty$, then there exist a sequence $(a^k)_{k \in \mathbb{Z}}$ of $(1, p, \infty)$-atoms and a positive number $A$ satisfying $\mu_k = A \cdot 2^k \mathbb{P}(\nu_k < \infty)^{1/p}$ (where $\nu_k$ is the stopping time corresponding to $a^k$) such that

$$(2.2) \quad f_n = \sum_{k \in \mathbb{Z}} \mu_k a_n^k \text{ a.e., } n \in \mathbb{N},$$

and

$$\|\{\mu_k\}\|_{l_q} \leq C \|f\|_{H_{p, q}^*}.$$

Conversely, if the martingale $f$ has the above decomposition, then $f \in H_{p, q}^*$ and $\|f\|_{H_{p, q}^*} \approx \inf \|\{\mu_k\}\|_{l_q}$, where the infimum is taken over all the above decompositions.

Moreover, if the stochastic basis $\{\mathcal{F}_n\}_{n \geq 0}$ is regular and if we replace $H_{p, q}^*$ by $H_{p, q}^*$, $(1, p, \infty)$-atoms by $H_{p, q}^*$, $(3, p, \infty)$-atoms, then the conclusions above still hold.

Lemma 2.1 ([II], Lemma 1.2). Let $0 < p < \infty$ and let the nonnegative sequence $\{\mu_k\}$ be such that $\{2^k \mu_k\} \in l^q$, $0 < q \leq \infty$. Further, suppose the nonnegative function $\varphi$ satisfies the following property: there exists $0 < \varepsilon < \min(1, q/p)$ such that, given an arbitrary integer $k_0$, we have $\varphi \leq \psi_{k_0} + \eta_{k_0}$, where $\psi_{k_0}$ and $\eta_{k_0}$ satisfy

$$2^{k_0 p} \mathbb{P}(\psi_{k_0} > 2^{k_0})^\varepsilon \leq C \sum_{k=-\infty}^{k_0-1} (2^k \mu_k)^p,$$

$$2^{k_0 p} \mathbb{P}(\eta_{k_0} > 2^{k_0}) \leq C \sum_{k=k_0}^{\infty} (2^k \mu_k)^p.$$

Then $\varphi \in l_{p, q}$ and $\|\varphi\|_{p, q} \leq C \|\{2^k \mu_k\}\|_{l_q}$.
3. A JOHN–NIRENBERG THEOREM

In this section, we prove a John–Nirenberg theorem when the stochastic basis \( \{F_n\}_{n \geq 0} \) is regular. The main idea and method are similar to those of [8]. The following lemma can be found in [5], [16]. In fact, it follows from Theorem 7.14 in [5] and Corollary 5.13 in [16].

**Lemma 3.1.** Suppose that \( 0 < q \leq \infty \) and the stochastic basis \( \{F_n\}_{n \geq 0} \) is regular.

If \( 0 < p < \infty \), then \( \mathcal{H}^s_{p,q} \) and \( \mathcal{H}^s_{p,q} \) are equivalent.

If \( 1 < p < \infty \), then \( \mathcal{H}^s_{p,q} \), \( \mathcal{H}^s_{p,q} \) and \( L_{p,q} \) are all equivalent.

**Remark 3.1.** (1) According to [7], Remark 2.2, we can conclude that \( \mathcal{H}^s_{p,\infty} = L_2 \) is dense in \( \mathcal{H}^s_{p,\infty} \).

(2) If the stochastic basis \( \{F_n\}_{n \geq 0} \) is regular, then, by the same argument of Remark 2.2 in [7], \( L_\infty \) is dense in \( \mathcal{H}^s_{p,\infty} \).

**Lemma 3.2.** Let \( 0 < p \leq 1 \). If the stochastic basis \( \{F_n\}_{n \geq 0} \) is regular, then

\[
(\mathcal{H}^s_{p,\infty})^* = wBMO_1(\alpha), \quad \alpha = \frac{1}{p} - 1.
\]

**Proof.** Let \( g \in wBMO_1(\alpha) \). Define

\[
\phi_g(f) = \mathbb{E}(fg), \quad f \in L_\infty.
\]

Then, by Theorem [2,1], we find that \( (\nu_k) \) is the stopping time corresponding to the
atom $a_k$ for every $k \in \mathbb{Z}$)
\[ |\phi_g(f)| \leq \sum_{k \in \mathbb{Z}} |\mu_k| E(a_k^*(g - g^{\nu_k})) \leq \sum_{k \in \mathbb{Z}} |\mu_k| ||a_k||\infty ||g - g^{\nu_k}||_1 \]
\[ \leq C \sum_{k \in \mathbb{Z}} |\mu_k| ||a_k||\infty ||g - g^{\nu_k}||_1 \]
\[ \leq C \sum_{k \in \mathbb{Z}} |\mu_k| \mathbb{P}(\nu_k < \infty)^{-1/p} ||g - g^{\nu_k}||_1 \]
\[ = C \cdot A \sum_{k \in \mathbb{Z}} 2^k ||g - g^{\nu_k}||_1. \]

By the definition of $|| \cdot ||_{wBMO_r(x)}$, we obtain
\[ |\phi_g(f)| \leq C \cdot A \sup_k 2^k \mathbb{P}(\nu_k < \infty)^{1/p} ||g||_{wBMO_1(x)} \]
\[ \leq C ||f||_{H_{p,\infty}} ||g||_{wBMO_1(x)}. \]

Since the stochastic basis $\{\mathcal{F}_n\}_{n \geq 0}$ is regular, $L_\infty$ is dense in $\mathcal{H}_{p,\infty}^*$ (see Remark 3.1(2)). Then $\phi_g$ can be uniquely extended to be a continuous linear functional on $\mathcal{H}_{p,\infty}^*$.

Conversely, let $\phi \in (\mathcal{H}_{p,\infty}^*)^*$. Since $L_2$ is dense in $\mathcal{H}_{p,\infty}^*$ (see Remark 3.1(2)), there exists $g \in L_2 \subset L_1$ such that
\[ \phi(f) = E(fg), \quad f \in L_\infty. \]

Let $\{\nu_k\}_{k \in \mathbb{Z}}$ be a stopping time sequence satisfying $\{2^k \mathbb{P}(\nu_k < \infty)^{1/p}\}_{k \in \mathbb{Z}} \in l_\infty$ and let
\[ h_k = \text{sign}(g - g^{\nu_k}), \quad a_k = \frac{1}{2}(h_k - h_k^{\nu_k}) \mathbb{P}(\nu_k < \infty)^{-1/p}. \]

Then $a_k$ is a $(3, p, \infty)$-atom. Let $f^N = \sum_{k=-N}^N 2^{k+1} \mathbb{P}(\nu_k < \infty)^{1/p} a_k^{\nu_k}$, where $N$ is an arbitrary nonnegative integer. By Theorem 2.4.1, we have $f^N \in H_{p,\infty}^*$ and
\[ ||f^N||_{H_{p,\infty}} \leq C \sup_k 2^k \mathbb{P}(\nu_k < \infty)^{1/p}. \]

Consequently,
\[ \sum_{k=-N}^N 2^k ||g - g^{\nu_k}||_1 = \sum_{k=-N}^N 2^k E(h_k(g - g^{\nu_k})) = \sum_{k=-N}^N 2^k E((h_k - h_k^{\nu_k})g) \]
\[ = E(f^N g) = \phi(f^N) \leq ||f^N||_{H_{p,\infty}} ||\phi|| \]
\[ \leq C \sup_k 2^k \mathbb{P}(\nu_k < \infty)^{1/p} ||\phi||. \]
Thus we have
\[ \sum_{k=-N}^{N} 2^k \| g - g'^k \|_1 \sup_k 2^k \mathbb{P}(\nu_k < \infty)^{1/p} \leq C\|\phi\|. \]
This implies \( \|g\|_{wBMO_{1}(\alpha)} \leq C\|\phi\| \). The proof is complete. \( \blacksquare \)

**Lemma 3.3.** Let \( 0 < p \leq 1, 1 < r < \infty \). If the stochastic basis \( \{ \mathcal{F}_n \}_{n \geq 0} \) is regular, then
\[ (\mathcal{H}_{p,\infty})^* = wBMO_r(\alpha), \quad \alpha = \frac{1}{p} - 1. \]

**Proof.** By Hölder’s inequality, we have \( \|f\|_{wBMO_{1}(\alpha)} \leq \|f\|_{wBMO_r(\alpha)} \) for any \( f \in wBMO_r(\alpha) \). Let \( g \in wBMO_r(\alpha) \subset L_r \). We define
\[ \phi_g(f) = \mathbb{E}(fg), \quad \forall f \in L_r. \]

Then, by Lemma 5.2, we have
\[ |\phi_g(f)| \leq C\|f\|_{H_{p,\infty}} \|g\|_{wBMO_{1}(\alpha)} \leq C\|f\|_{H_{p,\infty}} \|g\|_{wBMO_r(\alpha)}. \]
It follows from Remark 5.1(2) that \( L_r^r \) is dense in \( \mathcal{H}_{p,\infty}^* \). Thus \( \phi_g \) can be uniquely extended to be a continuous linear functional on \( \mathcal{H}_{p,\infty}^* \).

Conversely, if \( \phi \in (\mathcal{H}_{p,\infty}^*)^* \), by Doob’s maximal inequality, we have \( L_r^r = H_{p,r,r}^r \subset \mathcal{H}_{p,\infty}^* \). Then \( (\mathcal{H}_{p,\infty}^*)^* \subset (L_r^r)^* = L_r \). Thus there exists \( g \in L_r \) such that
\[ \phi(f) = \phi_g(f) = \mathbb{E}(fg), \quad \forall f \in L_r. \]

Let \( \{ \nu_k \}_{k \in \mathbb{Z}} \) be a stopping time sequence such that \( \{ 2^k \mathbb{P}(\nu_k < \infty)^{1/p} \}_{k \in \mathbb{Z}} \subset l_\infty \) and \( N \) be an arbitrary nonnegative integer. Let
\[ h_k = \frac{|g - g'^k|^{r-1}\text{sign}(g - g'^k)}{\|g - g'^k\|_{r}^{r-1}}, \quad f = \sum_{k=-N}^{N} 2^k \mathbb{P}(\nu_k < \infty)^{1/r'}(h_k - h'^k_k). \]

For an arbitrary integer \( k_0 \) which satisfies \( -N \leq k_0 \leq N \) (for \( k_0 \leq -N \), let \( G = 0 \) and \( H = f \); for \( k_0 > N \), let \( H = 0 \) and \( G = f \)), let
\[ f = G + H, \]
where
\[ G = \sum_{k=-N}^{k_0-1} 2^k \mathbb{P}(\nu_k < \infty)^{1/r'}(h_k - h'^k_k), \]
and
\[ H = \sum_{k=k_0}^{N} 2^k \mathbb{P}(\nu_k < \infty)^{1/r'}(h_k - h'^k_k). \]
Thus we obtain
\[ \Pr(G^* > 2^{k_0}) \leq \frac{1}{2^{k_0}} \|G^*\|_{r'} \leq C \frac{1}{2^{k_0}} \|G\|_{r'} \]
\[ \leq C \frac{1}{2^{k_0}} \left( \sum_{k=-N}^{k_0-1} 2^k \Pr(\nu_k < \infty)^{1/r'} \right). \]

On the other hand, \( \{H^* > 0\} \subset \bigcup_{k=k_0}^{N} \{\nu_k < \infty\} \). Then, for each 0 < \( \varepsilon \) < 1, we have

\[ 2^{k_0} \varepsilon \Pr(H^* > 2^{k_0}) \leq 2^{k_0} \varepsilon \Pr(H^* > 0) \leq 2^{k_0} \varepsilon \sum_{k=k_0}^{N} \Pr(\nu_k < \infty) \]
\[ \leq \sum_{k=k_0}^{N} 2^{k_0} \Pr(\nu_k < \infty) = \sum_{k=k_0}^{N} (2^{k_0} \Pr(\nu_k < \infty)^{1/p})^p \]
\[ \leq \sum_{k=k_0}^{\infty} (2^{k_0} \Pr(\nu_k < \infty)^{1/p})^p. \]

By Lemma 2.1, we have \( f^* \in L_{p,\infty} \) and \( \|f^*\|_{p,\infty} \leq C \|\{2^k \Pr(\nu_k < \infty)^{1/p}\}_{k \in \mathbb{Z}}\|_{\infty}. \) Thus, \( f \in H^*_{p,\infty} \) and

\[ \|f\|_{H^*_{p,\infty}} \leq C \sup_k 2^k \Pr(\nu_k < \infty)^{1/p}. \]

Consequently,
\[ \sum_{k=-N}^{N} 2^k \Pr(\nu_k < \infty)^{1-1/r} \|g - g^{p_k}\|_r = \sum_{k=-N}^{N} 2^k \Pr(\nu_k < \infty)^{1/r'} \mathbb{E}(h_k(g - g^{p_k})) \]
\[ = \sum_{k=-N}^{N} 2^k \Pr(\nu_k < \infty)^{1/r'} \mathbb{E}((h_k - h_k^{p_k})g) \]
\[ = \mathbb{E}(fg) = \varphi(f) \leq \|f\|_{H^*_{p,\infty}} \|\varphi\| \]
\[ \leq C \sup_k 2^k \Pr(\nu_k < \infty)^{1/p}. \]

Thus we obtain
\[ \sum_{k=-N}^{N} 2^k \Pr(\nu_k < \infty)^{1-1/r} \|g - g^{p_k}\|_r \]
\[ \leq C \|\varphi\|. \]

Taking \( N \to \infty \) and the supremum over all stopping time sequences satisfying \( \{2^k \Pr(\nu_k < \infty)^{1/p}\}_{k \in \mathbb{Z}} \subset \mathcal{L}_\infty \), we get \( \|g\|_{wBMO(\alpha)} \leq C \|\varphi\| \).
Now we formulate the weak version of the John–Nirenberg theorem, which directly follows from Lemmas 3.2 and 3.3.

**Theorem 3.1.** Let $0 < r < \infty$ and $\frac{1}{r} \leq \frac{1}{\alpha} < \frac{1}{\alpha} + 1 < \infty$. If the stochastic basis $\{F_n\}_{n \geq 0}$ is regular, then

$$wBMO_r(\alpha) = wBMO_1(\alpha)$$

with equivalent norms.

According to Lemma 3.1, Lemma 3.3 holds if we replace $H^s_{p, \infty}$ by $H^s_{p, \infty}$. Without regularity of stochastic basis $\{F_n\}_{n \geq 0}$, we also get a duality result.

**Proposition 3.1.** Let $0 < p < 1$. Then $(H^s_{p, \infty})^* = wBMO_2(\alpha)$ with $\alpha = 1/p - 1$.

**Proof.** Note that $H^2_p = L_2$ is dense in $H^s_{p, \infty}$ by Remark 3.1(1). The first part of the proof is similar to that of Lemma 3.2, and the converse part is similar to that of Lemma 3.3 with $r = 2$. We omit the proof. \qed

### 4. Proof of the Main Theorem

In this section we complete the proof of Theorem 1.1.

Let $\overline{H}^s_{p, \infty}$ be the $H^s_{p, \infty}$ closure of $H^s_{\infty}$. Since $H^s_{\infty} \subset H^s_{2} = L_2$, using Remark 3.1(1), we have $\overline{H}^s_{p, \infty} \subset H^s_{p, \infty}$. Then $(H^s_{p, \infty})^* \subset (\overline{H}^s_{p, \infty})^*$.

**Lemma 4.1** ([17], Corollary 6). Let $0 < p < 2$. Then the dual space of $\overline{H}^s_{p, \infty}$ is $wBMO_2(\alpha)$ with $\alpha = 1/p - 1$.

**Lemma 4.2** ([17], Corollary 8). Suppose that the stochastic basis $\{F_n\}_{n \geq 0}$ is regular and $1 \leq r < \infty$. If $\alpha r + 1 > 0$ for a fixed $\alpha$, then

$$wBMO_r(\alpha) = wBMO_2(\alpha)$$

with equivalent norms.

**Theorem 4.1.** Suppose that $\alpha > 0$. Then

$$wBMO_2(\alpha) = wBMO_2(\alpha)$$

with equivalent norms.

**Proof.** Let $p = \frac{1}{1 + \alpha}$. Since $(H^s_{p, \infty})^* \subset (\overline{H}^s_{p, \infty})^*$, it follows from Proposition 3.1 and Lemma 4.1 that

$$wBMO_2(\alpha) \subset wBMO_2(\alpha).$$

To obtain

$$wBMO_2(\alpha) \supset wBMO_2(\alpha),$$
we shall show that
\[ C \|f\|_{\text{wBMO}_2(\alpha)} \geq \|f\|_{\text{wBMO}_2(\alpha)} \]
for any \( f \in \text{wBMO}_2(\alpha) \). Suppose that \( \{\nu_k\}_{k \in \mathbb{Z}} \) is an arbitrary stopping time sequence such that \( \{2^k \mathbb{P}(\nu_k < \infty)^{1/p}\}_{k \in \mathbb{Z}} \in \ell_{\infty} \). Let
\[ B = \sup_k 2^k \mathbb{P}(\nu_k < \infty)^{1/p}. \]

We can claim that
\[ \sum_{k=-\infty}^{\infty} t_2^2(B^p 2^{-kp}) \leq C \|f\|_{\text{wBMO}_2(\alpha)}. \]

To this end, let \( C_k = B^p 2^{-kp} \). Then, for any \( x \in (C_{k+1}, C_k) \), we have
\[ C_{k+1}^{1/2+\alpha} t_2^2(C_{k+1}) \leq x^{1/2+\alpha} t_2^2(x) \leq C_k^{1/2+\alpha} t_2^2(C_k). \]

We refer to [17], p. 144, for a more general case of the inequalities above. Hence,
\[ \int_0^\infty \frac{t_2^2(x)}{x} dx = \sum_{k=-\infty}^{\infty} \int_{C_{k+1}}^{C_k} \frac{t_2^2(x)}{x} dx \geq (1 - 2^{-p}) 2^{-p(1/2+\alpha)} \sum_{k=-\infty}^{\infty} t_2^2(B^p 2^{-kp}). \]

On the other hand, since \( B^p 2^{-kp} \geq \mathbb{P}(\nu_k < \infty) \) for all \( k \), we have
\[ \sum_{k=-\infty}^{\infty} t_2^2(B^p 2^{-kp}) \geq \sum_{k=-\infty}^{\infty} \frac{2^k (B^p 2^{-kp})^{1/2}\|f - f^{\nu_k}\|_2}{B} \geq \sum_{k=-\infty}^{\infty} \frac{2^k \mathbb{P}(\nu_k < \infty)^{1/2}\|f - f^{\nu_k}\|_2}{B}. \]

By the definition of \( \text{wBMO}_2(\alpha) \), we complete the proof. ■

**Remark 4.1.** If one proves the dual space of \( \mathcal{H}_{p,\infty}^s \) is \( \text{wBMO}(\alpha) \), then Theorem 4.1 holds. If one shows \( \mathcal{H}_{p,\infty}^s = \mathcal{H}_{p,\infty}^s \), then Proposition 3.1 implies Theorem 4.1. We leave the proofs to the interested reader.

Now we are ready to prove the main result of the paper.

**Proof of Theorem 1.1.** It directly follows from Theorems 3.1 and 3.1 and Lemma 4.2. ■

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