ON FLUSHED PARTITIONS AND CONCAVE COMPOSITIONS

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Abstract. In this work, we give combinatorial proofs for generating functions of two problems, i.e., flushed partitions and concave compositions of even length. We also give combinatorial interpretations of one problem posed by Sylvester involving flushed partitions and then prove it. For these purposes, we first describe an involution and use it to prove core identities. Using this involution with modifications, we prove several problems of different nature, including Andrews’ partition identities involving initial repetitions and partition theoretical interpretations of three mock theta functions of third order $f(q)$, $\phi(q)$ and $\psi(q)$. An identity of Ramanujan is proved combinatorially. Several new identities are also established.

Keywords: Integer partition, flushed partition, concave composition, involution, mock theta function.

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1 Introduction

In this paper, we are mainly concerned with two problems in the theory of integer partition, namely, flushed partitions and concave compositions of even length.

The definition of flushed partition is given by Sylvester [16]. A partition is called flushed when the number of the parts with length $k$ is odd, where $k = 1, 2, \cdots, 2i - 1$, and the parts with length $2i$ do not occur an odd number of times. Similarly, define unflushed partitions as those not satisfying the above conditions. Sylvester also posed two problems with respect to flushed partitions. One of them, in Sylvester’s words, is stated as follows,

"1. Required to prove, that if any number be partitioned in every possible way, the number of unflushed partitions containing an odd number of parts is equal to the number of unflushed partitions containing an even number of parts.

"Ex. gr.: The total partitions of 7 are 7; 6, 1; 5, 2; 5, 1, 1; 4, 3; 4, 2, 1; 4, 1, 1, 1; 3, 3, 1; 3, 2, 2; 3, 2, 1, 1; 2, 2, 2, 1; 3, 1, 1, 1, 1; 2, 2, 1, 1, 1; 2, 1, 1, 1, 1, 1; 1, 1, 1, 1, 1, 1, 1. Of these, 6, 1; 4, 1, 1, 1; 3, 3, 1; 2, 1, 1, 1, 1; 1, 1, 1, 1, 1, 1, 1, 1 alone are flushed. Of the remaining unflushed partitions, five contain an odd number of parts, and five an even number.

"Again, the total partitions of 6 are 6; 5, 1; 4, 2; 4, 1, 1; 3, 3; 3, 2, 1; 2, 2, 2; 3, 1, 1, 1; 2, 2, 1, 1; 2, 1, 1, 1, 1; 1, 1, 1, 1, 1, 1, 1; of which 5, 1; 3, 2, 1; 3, 1, 1, 1 alone are flushed. Of the remainder, four contain an odd and four an even number of parts.

"N.B.—This transcendental theorem compares singularly with the well-known algebraical one, that the total number of the permuted partitions of a number with an odd number of parts is equal to the number of the same with an even number.

Solution to this problem is given by Andrews in 1970 in [1] by manipulating generating functions, and the generating function for flushed partitions reads as follows,

\[ [x^n] F(x) = \frac{1}{2} \frac{x^{n+1}}{1 - x^{n+1}}. \]

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\[
\frac{1}{(q)_{\infty}} \sum_{n=1}^{\infty} q^{(3n-1)/2}(1 - q^n). \tag{1.1}
\]

However, Andrews doubted that his proofs are what Sylvester expected in the first place. He writes (in [7]): “It is completely unknown whether this was Sylvester’s approach and how he came upon flushed partitions in the first place.”

It is hardly believed that the core of the combinatorial proof of Sylvester’s problem is only one involution. But it turns out to be the case. We will first prove combinatorially the above generating function (1.1), and then by inserting another variable \(z\) into the generating function, we can map bijectively unflushed partition of \(n\) with \(m\) parts into unrestricted partitions of \(n\) with \(m\) parts with additional restrictions involving Durfee Symbols, a very natural concept yet just introduced recently by Andrews in [8]. In this way, we try to understand flushed partitions in a new combinatorial sense and then provide a new proof of Sylvester’s problem.

*Concave composition of even length* was recently introduced by Andrews in the study of orthogonal polynomials, see [6, 8]. It is a sum of the form \(\sum a_i + \sum b_i\) such that

\[a_1 > a_2 > \cdots > a_m = b_m < b_{m-1} < \cdots < b_1,\]

where \(a_m \geq 0\), and all \(a_i\) and \(b_i\) are integers. Let \(CE(n)\) denote the set of concave compositions of even length of \(n\), and let \(ce(n)\) be the cardinality of \(CE(n)\). By transformation formulas, Andrews derived the generating function of \(ce(n)\) as follows ([7, Theorem 1]),

\[
\sum_{n=0}^{\infty} ce(n)q^n = \frac{1}{(q)_{\infty}} \left(1 - \sum_{n=1}^{\infty} q^{n(3n-1)/2}(1 - q^n)\right). \tag{1.2}
\]

Andrews [6] asked for a combinatorial proof of Theorem 1.2. We will give one such proof in this paper.

Note the above two generating functions (1.1) and (1.2) have connections with one another, which lead to our main theorem of this paper as stated as follows.

**Theorem 1.1** The number of unflushed partitions of \(n\) is equal to the number of concave compositions of even length of \(n\).

This paper is organized as follows. In Section 2, we define an involution, which, with modifications, will be used repeatedly. In Section 3, we give combinatorial proofs of several different \(q\)-series identities. Readers who are not interested in these problems can skip directly to Section 4, where we will prove our main theorem about these two generating functions and prove Sylvester’s first problem.

The applications of the involutions in Section 3 consists of several different partition identities. The nature of these problems varies, showing such involutions are indeed useful tools.

One application is on Andrews’ partition identity involving initial repetitions. In [5], Andrews proved the \(q\)-series identity

\[
\sum_{n=0}^{\infty} z^n q^{1+2+2^2+\cdots+n^2} (q)_n (1 - zq^j) = \prod_{j=n+1}^{\infty} (1 - zq^j) q^{\frac{j(j+1)}{2}}, \tag{1.3}
\]

which, when interpreted combinatorially, means that partitions of \(n\) with \(m\) different parts and an even number of distinct parts in which, if part \(j\) is repeated, then all parts smaller than \(j\) is repeated,
are equinumerous with partitions of $n$ with $m$ different parts and an odd number of distinct parts satisfying the same property, unless $n$ is a triangular number $\frac{j(j+1)}{2}$ and $m = j$, when their difference is $(-1)^j$. Partitions with this property are called \textit{partitions with initial 2-repetitions}.

This result (1.3) looks similar to classical Euler’s pentagonal number theorem. The involution plays a role as the role the Franklin’s well known involution has played in Euler’s pentagonal number theorem. Based on this proof, a formula is given to compute the number of partitions into even number of distinct parts.

The next applications involve combinatorial interpretations of the following three mock theta functions of order 3, defined by Ramanujan in his last letter to Hardy (see [13]),

$$f(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q)^{n}} \tag{1.4}$$

$$\phi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q^2; q^2)^{n}} \tag{1.5}$$

$$\psi(q) = \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q; q^2)^{n}} \tag{1.6}$$

The combinatorial interpretation of (1.4) and (1.5) are given in the end of Section 2. The following combinatorial interpretation of (1.6) was first given by Fine in [14]:

$$f(q) = 1 + \frac{1}{(-q)^{\infty}} \sum_{k \geq 1} (-1)^{k-1} q^k (-q^{k+1})^{\infty} \tag{1.7}$$

We will restate this result as Theorem 3.7. Our proof is new.

Then all these combinatorial interpretations lead to the following identity of Ramanujan:

$$f(q) = \phi(-q) - 2\psi(-q)$$

We will restate this as Corollary 3.8. The first proof of this identity is given by Watson in [17]. Another combinatorial proof was given by Chen, Ji and Liu in [11].

At the end of Section 3, we use the involution to generate several more identities, which all have partition interpretations. We only write such interpretation for one identity as an example.

## 2 An Involution $\alpha$

All notations in the theory of integer partitions follow the book [3]. A partition of a positive integer $n$ is a finite nonincreasing sequence of positive integers $(\lambda_1, \cdots, \lambda_r)$ such that $\sum_{i=1}^{r} \lambda_i = n$. $\lambda_i$ are called the parts of the partition. We use $P$ to denote the set of partitions and $P_n$ to denote the set of partitions of $n$. We are also interested in the partitions into distinct parts. We use $D$ to denote the set of partitions into distinct parts and $D_n$ to denote the set of partitions of $n$ into distinct parts.

At the same time, define the set of partitions into distinct parts which may contain one copy of empty part as $D'$. Naturally, $D'_n$ would denote the set of such partitions of $n$. Similarly, we define $P'$ as the set of partitions which can contain empty parts (maybe more than one yet a limited number of copies). $P'_n$, of course, would be the set of such partitions of $n$.

We can represent a partition as its Young diagram, that is, a pattern of left-justified boxes with $\lambda_i$ squares in row $i$. The square in the $i$th row and the $j$th column can be written simply as the
square \((i, j)\). The Durfee square in \(\lambda\) is the largest square of boxes contained in the partition \(\lambda\).

When the Durfee square of a partition is \(n \times n\), we say it is a Durfee square of size \(n\). The conjugate of \(\lambda = (\lambda_1, \ldots, \lambda_r)\) is a partition \(\lambda'\) with the \(i\)th part \(\lambda'_i\) as the number of parts of \(\lambda\) that are \(\geq i\).

We adopt the following standard notation:

\[
(a; q^k)_n = (1 - a)(1 - aq^k) \cdots (1 - aq^{(n-1)k}),
\]

\[
(a; q^k)_\infty = \prod_{n=0}^{\infty} (1 - aq^{nk}),
\]

\[
(a)_n = (a; q)_n,
\]

\[
(a)_0 = (a; q)_0 = 1.
\]

Now we are ready to state the involution we mentioned, which will be denoted as \(\alpha\). Given a triple \((\lambda, \mu, \rho_{n+\ell})\) satisfying the following properties with a sign \((-1)^{k+\ell}\):  

1. \(\lambda \in D\) has \(n\) distinct parts. The least part has length \(k\). Note that it may contain one empty part.

2. \(\mu\) is a partition with no more than \(\ell\) parts. Or, we can think \(\mu \in P\) as a partition into exactly \(\ell\) parts, which may contain some empty parts.

3. \(\rho_{n+\ell}\) is Sylvester’s triangle \((n + \ell, n + \ell - 1, \ldots, 1)\).

Let the involution \(\alpha\) act on such a triple by comparing \(\lambda_1\) the largest part of \(\lambda\) and \(\mu_1\) the largest part of \(\mu\), as follows.

If \(\lambda_1 \geq \mu_1\), and the number of parts of \(\lambda\) is at least two, then we move the first part of \(\lambda\) and attach it to \(\mu\), making it \(\mu'\). What has been left there is then \(\lambda'\). Since \(\mu'\) has no more than \(\ell + 1\) parts, yet the smallest part of \(\lambda\) does not change, the sign changes.

On the other hand, if \(\lambda_1 < \mu_1\), then we move the first part of \(\mu\) and attach it to \(\lambda\), making it \(\lambda'\). What has been left there is then \(\mu'\) and \(\mu'\) has no more than \(\ell - 1\) parts, yet the smallest part of \(\lambda\) does not change, the sign changes too.

The triples that the involution does not apply to are the ones \((\lambda, \mu, \rho_d)\) where \(\lambda\) has only one part of length \(t\) and \(t \geq \mu_1\); \(\mu\) is a partition with no more than \(d - 1\) parts.

For example, Let \(\lambda = (9, 6, 5, 2), \mu = (6, 4, 4), k = 2, n = 4, \ell = 3\). It is assigned \((-1)^5 = -1\). Then we have that \(\alpha(\lambda, \mu, \rho_{4+3}) = (\lambda', \mu', \rho'_{4+4})\), where \(\lambda' = (6, 5, 2), \mu' = (9, 6, 4, 4), k = 2, n' = 3, \ell' = 4\). It is assigned with \((-1)^6 = 1\). We illustrate this example in Figure 1.

**Remark 1.** For the triple \((\lambda, \mu, \rho_{n+\ell})\), special attention should be given to \(\mu\), because sometimes the number of nonzero parts of \(\mu\) is strictly less than \(\ell\). This situation makes no exception for our arguments. In fact, \((\lambda', \mu', \rho_{(n+1)+(\ell+1)}) = \alpha((\lambda, \mu, \rho_{n+\ell}))\) where \(\ell\) changes by \(\pm 1\).

**Remark 2.** The nature of this involution has a more general form in [10].

Applying this involution, we can prove the following identity:

**Theorem 2.1**

\[
\sum_{n=1}^{\infty} q^{n^2} \frac{q^{n+1}}{(q)_n(1 + q^n)} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^n}{(1 + q \cdots (1 + q^n)).}
\]

**Proof.**

For each term in the left hand side of (2.1), we interpret \(q^{n^2}, \frac{1}{(q)_n}, \frac{1}{(1 + q^n)}\) and \((q^{n+1})_\infty\) as follows, respectively:
the partition \( \lambda = (9, 6, 5, 2) \)

\[ \rightarrow \]

the partition \( \lambda' = (6, 5, 2) \)

the partition \( \mu = (6, 4, 4) \) with \( \rho_7 \)

the partition \( \mu' = (9, 6, 4, 4) \) with \( \rho_7 \)

Figure 1: The involution \( \alpha \)

1. A \( n \times n \) squares,
2. A partition with the largest part at most \( n - 1 \),
3. \( k \) parts of length exactly \( n \), assigned \((-1)^k\),
4. A partition with distinct parts, the least part being larger than \( n \). (Suppose this partition has \( \ell \) parts. It is assigned \((-1)^\ell\).

We split the \( n \times n \) squares to a Sylvester's triangle \( \rho_n = (n, n - 1, \ldots, 1) \) and another Sylvester's triangle \( \rho_{n-1} = (n - 1, n - 2, \ldots, 1) \). Then we glue together three objects to form a new partition \( \lambda^* \), which are, the \( k \) parts of length \( n \), \( \rho_{n-1} \) and the partition with the largest part at most \( n - 1 \).

Now we observe its conjugate and denote it as \( \lambda \). It is a partition with distinct parts, \( k \) the length of the least part. Obviously, \( k = 0 \) is allowed, so \( \lambda \) can have one empty part, that is, \( \lambda \in D' \).

Then we attach the Sylvester's triangle \( \rho_n \) under the partition with \( \ell \) parts, the least length of parts larger than \( n \). This is clearly a partition with distinct parts. We divide it into a partition \( \mu \) and Sylvester's triangle \( \rho_{n+\ell} \) in the obvious way. We see that \( \mu \) has no more than \( \ell \) parts. Again, we can regard \( \mu \) as a partition with exactly \( \ell \) parts, where we allow empty parts to exist.

Remember this was assigned \((-1)^{k+\ell}\).

We then invoke the involution \( \alpha \). Since the cases which the involution does not apply to are the triples \( (\lambda, \mu, \rho_0) \) where \( \lambda \) has only one part of length \( t \) and \( t \geq \mu_1 \); \( \mu \) is a partition with no more than \( d - 1 \) parts.

We move this \( t \) squares and attach it into \( \mu \) together, getting a partition with no more than \( d \) parts, which is assigned \((-1)^{t+d-1} \), \( t \) the largest part of \( \mu \), \( d \) is the subscript of the Sylvester's Triangle. This corresponds to the right-hand side of the identity, which completes the proof.

See Figure 2 for a concrete example. The graph in the left is the partition \( (13, 10, 9, 4, 4, 4, 4, 4, 3, 3, 1, 1) \), while the graph in the right is \( \lambda = (9, 6, 5, 2), \mu = (6, 4, 4) \) and \( \rho_4+3 \).

Now we generalize theorem 2.1 to the following two identities:
Corollary 2.2

\[
\sum_{n=1}^{\infty} \frac{z^n q^{n^2} (zq^{n+1})}{(q)_{n-1}(1 + q^n)} = \sum_{n=1}^{\infty} \frac{(-1)^n z^n q^{n(n+1)/2}}{(1 + q) \cdots (1 + q^n)}. \tag{2.2}
\]

\[
\sum_{n=1}^{\infty} \frac{z^n q^{n^2} (zq^{n+1})}{(q)_{n-1}(1 + zq^n)} = \sum_{n=1}^{\infty} \frac{(-1)^n z^n q^{n(n+1)/2}}{(1 + zq) \cdots (1 + zq^n)}. \tag{2.3}
\]

**Proof.** For the first identity, recall the first remark we made after the definition of the involution \(\alpha\). We observe that after the application of the involution \(\alpha\), the size of the Sylvester’s triangle does not change. We use another variable \(z\) to track this size. Following the proof of Theorem 2.1, we write \(z_n + \ell\) to denote the Sylvester’s Triangle \(\rho_{n+\ell}\). In this way, we have proved the identity (2.2).

For the second identity, we insert the variable \(z\) to track the sum of the size of the Sylvester’s triangle and the number of additional parts of length \(n\) apart from the \(n \times n\) squares. In the context of the proof of theorem 2.1, we write \(z_n + \ell + k\) to denote this value. Again, since this value does not change under the involution \(\alpha\), we have proved the identity (2.3).

In (2.2) and in (2.3), let \(q \to q^2\), \(z \to -q^{-1}\), we have the following:

\[
\sum_{n=1}^{\infty} \frac{(-1)^n q^{2n^2-n}(-q^{2n+1}; q^2)_n}{(q^2; q^2)_{n-1}(1 + q^{2n})} = -\sum_{n=1}^{\infty} \frac{q^{n^2}}{(-q^2; q^2)_n} = 1 - \phi(q), \tag{2.4}
\]

\[
\sum_{n=1}^{\infty} \frac{(-1)^n q^{2n^2-n}(-q^{2n+1}; q^2)_n}{(q^2; q^2)_{n-1}(1 - q^{2n+1})} = -\sum_{n=1}^{\infty} \frac{q^{n^2}}{(q; q^2)_n} = -\psi(q), \tag{2.5}
\]

where

\[
\phi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q^2; q^2)_n},
\]

\[
\psi(q) = \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q; q^2)_n}.
\]
are two mock theta functions of third order defined by Ramanujan (see [15]).

By simple calculations with Euler’s identity \((-q)_\infty = \frac{1}{(q;q^2)_\infty}\), we have

\[(-q; q)_\infty (1 - \phi(-q)) = \sum_{n=1}^{\infty} \frac{q^{2n^2-n}}{(q)_{2n-1}(1 + q^{2n})}, \tag{2.6}\]

\[(-q; q)_\infty (-\psi(-q)) = \sum_{n=1}^{\infty} \frac{q^{2n^2-n}}{(q)_{2n-1}(1 + q^{2n})} = \sum_{n=1}^{\infty} \frac{q^{2n^2-n}}{(q)_{2n-2}(1 - q^{2(2n-1)})}. \tag{2.7}\]

We now use \(D_{e,e}(n)\) to denote the number of partitions of \(n\) into an even number of distinct parts in which the smallest part is even. \(D_{e,o}(n)\) denote the number of partitions of \(n\) into an even number of distinct parts in which the smallest part is odd.

Symmetrically, \(D_{o,e}(n)\) (respectively, \(D_{o,o}(n)\)) denote the number of partitions of \(n\) into odd number of distinct parts in which the smallest part is even (respectively, odd).

Then by interpreting (2.6) and (2.7) we get the partition theoretical interpretations of mock theta function \(\phi(-q)\) and \(\psi(-q)\):

**Theorem 2.3**

\[(-q; q)_\infty (1 - \phi(-q)) = \sum_{n=1}^{\infty} (D_{o,e}(n) + D_{o,o}(n) + D_{e,c}(n) - D_{e,o}(n)) q^n, \tag{2.8}\]

\[(-q; q)_\infty (-\psi(-q)) = \sum_{n=1}^{\infty} D_{o,o}(n) q^n. \tag{2.9}\]

**Proof.** For identity (2.8), note that \(q^{2n^2-n} = q^{1+2+\cdots+2n-1}\), and that \(\frac{1}{(1+q^n)} = \sum_{k=0}^{\infty} (-q^{2n})^k\).

Then the right-hand side of the identity (2.8) is the sum of the following three parts,

\[\sum_{n=1}^{\infty} \frac{q^{2n^2-n}}{(q)_{2n-1}}, \quad \sum_{n=1}^{\infty} \frac{q^{2n^2-n}}{(q)_{2n-1}}(q^{4n} + q^{6n} + \cdots) \quad \text{and} \quad -\sum_{n=1}^{\infty} \frac{q^{2n^2-n}}{(q)_{2n-1}}(q^{2n} + q^{6n} + \cdots),\]

which corresponds to

\[\sum_{n=1}^{\infty} (D_{o,e}(n) + D_{o,o}(n)) q^n, \quad \sum_{n=1}^{\infty} D_{e,c}(n) q^n \quad \text{and} \quad -\sum_{n=1}^{\infty} D_{e,o}(n)) q^n,\]

respectively. This proves the identity (2.8). The proof of identity (2.7) is similar.

By multiplying (2.8) with 2 and then subtracting (2.7) we get that

**Corollary 2.4**

\[(-q; q)_\infty (\phi(-q) - 2\psi(-q) - 1) = \sum_{n=1}^{\infty} (D_{o,o}(n) + D_{e,o}(n) - D_{e,e}(n) - D_{o,e}(n)) q^n = \sum_{n=1}^{\infty} (-1)^{n-1} q^n(-q^{n+1})_\infty.\]

**Proof.** To get the last identity, observe that in both sides, the coefficient of \(q^n\) is the number of partitions of \(n\) with the smallest part odd minus the number of partitions of \(n\) with the smallest part even.
3 Several Applications of The Involution

The involution we use in this section is essentially the same with previous $\alpha$, but strictly, we will denote it as $\alpha'$ to indicate that there are minor differences.

We first prove two theorems in [5] and [6]. They serve as responses to Andrews’ questions on finding combinatorial interpretations of these problems.

Recall the definition of Frobenius symbol (see [2]). The Frobenius symbol is two rows of decreasing, non-negative integers of equal length, which is often used as one way of representation of a partition. For example, a partition $\lambda = (7, 7, 6, 4, 4, 2, 2)$ is given, with the following Young diagram representation. (see Figure 3)

![Young diagram of the partition (7, 7, 6, 4, 4, 2, 2)](image)

One counts the number of squares in the 4 rows to the right and up of the diagonal, and the number of squares in the 4 columns to the left and below the diagonal, getting the Frobenius symbol as

$$\begin{pmatrix} 6 & 5 & 3 & 0 \\ 6 & 5 & 2 & 1 \end{pmatrix}$$

Theorem 3.1 ([7, Theorem 4]) The number of partitions of $n$ whose Frobenius symbol has no 0 on the top row equals the number of partitions of $n$ in which the smallest number that is not a summand is odd.

In order to prove Theorem 3.1, we first reduce it to proving the following identity:

Lemma 3.2 ([7, Lemma 12])

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q)_n^2} = \frac{1}{(q)_{\infty}} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2}.$$ 

The left hand side is the generating function for partitions of $n$ whose Frobenius symbol has no 0 in the top row. To see this, we draw a $n \times (n+1)$ rectangle, attach a partition with the largest part at most $n$ under the rectangle and then attach a partition with the largest part at most $n$ to the right of the rectangle.
At the same time, the right hand side is actually generating function for the number of partitions of $n$ in which the smallest number that is not a summand is odd, since:

$$
\frac{1}{(q)_\infty} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2} = \frac{1}{(q)_\infty} \sum_{n=0}^{\infty} q^n (1 - q^{2n+1}) = \sum_{n=0}^{\infty} q^{1+2+\cdots+2n} \prod_{j\neq 2n+1} (1 - q^j).
$$

**Proof of Theorem 3.2.** Rewrite lemma 3.2 by multiplying $(q)_\infty$ in both sides, we get

$$
\sum_{n=0}^{\infty} q^{n^2+n+1} \frac{(q)_\infty}{(q)_n} = \sum_{n=0}^{\infty} (-1)^n q^n(n+1)/2.
$$

Like in the proof of Theorem 2.1 we split and rearrange every term in the left-hand side. This time the $n^2 + n$ squares can be split into two copies of Sylvester’s triangle $\rho_n$. The first $\rho_n$ is glued with the partition whose largest part is at most $n$, constructing the partition $\lambda$. The second $\rho_n$ is glued with partition whose smallest part is at least $n+1$, and get a larger Sylvester’s triangle $\rho_n+\ell$ and the partition $\mu$. Then we have a triple ($\lambda, \mu, \rho_n+\ell$), assigned $(-1)^{\ell}$, such that

1. $\lambda \in D$ is a partition with $n$ distinct parts,
2. $\mu \in P'$ is a partition with $\ell$ parts, while the empty parts are allowed, and
3. $\rho_{n+\ell}$ is a Sylvester’s triangle.

Similar to the previous $\alpha$, we define an involution $\alpha'$ as follows. Compare $\lambda_1$ the first part of $\lambda$ and $\mu_1$ the first part of $\mu$. If $\lambda_1 \geq \mu_1$, we remove the first part of $\lambda$ and attach it to $\mu$. This move changes the sign since it add 1 to $\ell$. If $\lambda_1 < \mu_1$, we remove the first part of $\mu$ and attach it to $\lambda$. This move also changes the sign since it subtract 1 from $\ell$.

All are canceled except those that with $\lambda$ empty partition $\emptyset$, where we have $n = 0$. What have been left are $(-1)^{\ell} \rho_\ell$, thus the conclusion.

**Definition.** ([6]) A partition with initial $k$-repetitions is a partition in which if any $j$ appears at least $k$ times as a part then each positive integer less than $j$ appears at least $k$ times as a part.

In a partition, one part is called a distinct part if it only appears once. Let $D_e(m,n)$ (resp. $D_o(m,n)$) denote the number of partitions of $n$ with initial 2-repetitions, with $m$ different parts and an even (resp. odd) number of distinct parts.

**Theorem 3.3** [6, Theorem 2]

$$
D_e(m,n) - D_o(m,n) = \begin{cases} (-1)^j, & \text{if } m = j, n = j(j+1)/2; \\ 0, & \text{otherwise.} \end{cases}
$$

**Proof.** We first write down the bivariate generating function as follows:

$$
\sum_{n,m \geq 0} (D_e(m,n) - D_o(m,n)) z^m q^n = \sum_{n=0}^{\infty} \frac{z^n q^{1+2+\cdots+n+2}}{(q)_n} \prod_{j=n+1}^{\infty} (1 - zq^j)
$$

$$
= \sum_{n=0}^{\infty} \frac{z^n q^{n^2+n+(q^{n+1})}}{(q)_n}.
$$
Then we reduce the proof to the following identity:

\[
\sum_{n=0}^{\infty} \frac{z^n q^{n^2 + n} (zq^{n+1})}{(q)_n} = \sum_{j=0}^{\infty} (-1)^j z^j q^{j(j+1)/2}.
\] (3.1)

Now we apply the involution \(\alpha'\), while inserting a new variable \(z\). In the context of the proof of Theorem 3.2 we use \(z^{n+\ell}\) to denote the Sylvester’s triangle \(\rho_{n+\ell}\). Since the application does not change the value of \(n + \ell\), the proof is finished.

In (3.1), let \(q \to q^2\) and \(z \to q^{-1}\), and we have

\[
(q; q^2) \sum_{n=0}^{\infty} \frac{q^{2n^2 + n}}{(q)_{2n}} = \sum_{j=0}^{\infty} (-1)^j q^{j^2}.
\]

or, by the classical identity of Euler, \((q; q^2) = 1/(-q; q)_{\infty}\), we have:

\[
\sum_{n=0}^{\infty} \frac{q^{1+2+\cdots+2n}}{(q)_{2n}} = (-q) \sum_{j=0}^{\infty} (-1)^j q^{j^2}.
\]

Let \(Q_E(n)\) denote the number of partitions of \(n\) into even number of distinct parts; \(Q(n)\) denote the number of partitions of \(n\) into distinct parts. So we have proved the following identity combinatorially.

**Theorem 3.4**

\[Q_E(n) = Q(n) - Q(n-1^2) + Q(n-2^2) - Q(n-3^2) + \cdots.\]

**Remark.** Interestingly enough, for unrestricted partitions, the same conclusion holds. Compare the above proof with the combinatorial proof of the following identity in [18]:

\[p_E(n) = p(n) - p(n-1^2) + p(n-2^2) - p(n-3^2) + \cdots,\]

where \(p_E(n)\) and \(p(n)\) denote the number of partitions into an even number of parts and the number of partitions, respectively.

Using the same involution of Theorem 3.1 we can also prove the following two identities. We omit the proofs.

**Corollary 3.5**

\[
\sum_{n=1}^{\infty} \frac{q^{n^2}}{(q)_{n-1}(q)_n} = \frac{1}{(q)_{\infty}} \sum_{n=1}^{\infty} (-1)^{n-1} q^{n(n+1)/2},
\]

or, more generally,

\[
\sum_{n=1}^{\infty} \frac{z^n q^{n^2} (zq^{n+1})}{(q)_{n-1}} = \sum_{n=1}^{\infty} (-1)^{n-1} z^n q^{n(n+1)/2}.
\]

**Corollary 3.6**

\[
\sum_{n=0}^{\infty} \frac{z^n q^{n^2}}{(zq)_n(q)_n} = \frac{1}{(zq)_{\infty}}.
\] (3.2)
**Remark.** Corollary 3.6 can be proved using Durfee square (see [3]). Here we have a new combinatorial proof, though a little more complicated than the standard one.

The next application of the involution is an identity involving another Ramanujan’s third order mock theta function: $f(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q^n)^{12}}$. Fine [14] derived the following identity applying some transformation formulas.

**Theorem 3.7** ([15, pp. 56])

$$f(q) = 1 + \frac{1}{(q; q)_\infty} \sum_{n \geq 1} (-1)^{n-1} q^n (-q^{n+1})_\infty.$$  

**Remark.** Combinatorially, $f(q) - 1 = \sum_{n=1}^{\infty} (N_e(n) - N_o(n)) q^n$, where $N_e(n)$ (respectively, $N_o(n)$) is the number of partitions of $n$ with even (respectively, odd) rank.

On the other hand, $\sum_{k \geq 1} (-1)^{k-1} q^k (-q^{k+1})_\infty = \sum_{n=1}^{\infty} (L_o(n) - L_e(n)) q^n$, where $L_e(n)$ (respectively, $L_o(n)$) is the number of partitions of $n$ into distinct parts with the smallest part even (respectively, odd). So the above identity relates these two enumeration problems.

Differently with previously involutions, we will compare the smallest parts of $\lambda$ and $\mu$ along the way. All other procedures are similar.

**Proof.** As before, we rewrite the identity as

$$\sum_{n=1}^{\infty} \frac{q^{n^2} (-q^{n+1})_\infty}{(q)_n} = \sum_{n \geq 1} (-1)^{n-1} q^n (-q^{n+1})_\infty. \quad (3.3)$$

Again, we split the $n^2$ squares into a Sylvester’s triangle $\rho_n$ and another Sylvester’s triangle $\rho_n'$. We glue $\rho_n$ with partition whose largest part is at most $n$, and transpose to its conjugate to get a partition $\lambda$. Then we glue $\rho_n$ with the partition into distinct parts whose smallest part is at least $n + 1$, split it to get a larger Sylvester’s triangle $\rho_{n+\ell}$ and a partition $\mu$. Thus, we have a triple $(\lambda, \mu, \rho_{n+\ell})$ assigned $(-1)^{k-n+1}$, such that

1. $\lambda \in \mathcal{D}'$ is a partition with $n$ distinct parts (empty part may be included). Set the largest part of $\lambda$ as $k$.
2. $\mu \in \mathcal{P}'$ is a partition with $\ell$ parts (empty parts may be included).
3. $\rho_{n+\ell}$ is a Sylvester’s triangle.

Similar to the previous involution, we define an involution $\sigma''$ as follows. Compare $\lambda_{\text{small}}$ the smallest part of $\lambda$ and $\mu_{\text{small}}$ the smallest part of $\mu$ (both parts could be empty parts). If $\lambda_{\text{small}} \leq \mu_{\text{small}}$, we remove the smallest part of $\lambda$ and attach it to $\mu$. This move changes the sign since it subtracts 1 from $n$. If $\lambda_{\text{small}} > \mu_{\text{small}}$, we remove the least part of $\mu$ and attach it to $\lambda$. This move also changes the sign since it add 1 from $n$.

All are canceled except those $(\lambda, \mu)$ with sign $(-1)^{k-1}$ satisfying that $\lambda$ only contains one part $k-1$ and $\mu$ contains $\ell$ nonempty parts, each $\geq k-1$. Then attaching this $k-1$ under $\mu$, aside with the Sylvester’s triangle $\rho_{\ell+1}$, we get the right-hand side of the identity. 

Combining both the above theorem and Corollary 2.4, we get the following well known relation, which is first derived by Ramanujan:

**Corollary 3.8**

$$\phi(-q) - 2\psi(-q) = f(q).$$
Just like Corollary 3.5 and Corollary 3.6, we get two identities of Theorem 3.7. The proof is also similar to that of Theorem 3.7, except that this time, the partition \( \lambda \) is not allowed to have empty parts, which left more terms in the right-hand side.

**Corollary 3.9**

\[
\sum_{n=1}^{\infty} \frac{q^{n^2+n}}{(-q)_n} = \frac{1}{(-q)_{\infty}} \sum_{k \geq 1} \sum_{n \geq 1} (-1)^{n-1} q^{\frac{k(k+1)}{2}} q^{k+n} (-q^{k+n+1})_{\infty}. \tag{3.4}
\]

\[
\sum_{n=1}^{\infty} \frac{q^{n^2}}{(-q)_{n-1}(-q)_n} = \frac{1}{(-q)_{\infty}} \sum_{k \geq 1} \sum_{n \geq 1} (-1)^{n-1} q^{\frac{k(k+1)}{2}} q^{k+n+1} (-q^{k+n+2})_{\infty}. \tag{3.5}
\]

As in Theorem 3.3, we can generalize (3.3), (3.4) and (3.5) by inserting a new variable \( z \) into the identities to track different values, getting the following:

**Corollary 3.9**

\[
\sum_{n=1}^{\infty} \frac{z^n q^{n^2} (-zq^{n+1})}{(-q)_n} = \sum_{k \geq 1} (-1)^{k-1} z q^k (-zq^{k+1})_{\infty}. \tag{3.6}
\]

\[
\sum_{n=1}^{\infty} \frac{z^n q^{n^2+n} (-zq^{n+1})}{(-q)_n} = \sum_{k \geq 1} \sum_{n \geq 1} (-1)^{n-1} z^k q^{\frac{k(k+1)}{2}} q^{k+n} (-zq^{n+k+1})_{\infty}. \tag{3.7}
\]

\[
\sum_{n=1}^{\infty} \frac{z^n q^{n^2} (-zq^{n+1})}{(-q)_{n-1}} = \sum_{k \geq 1} \sum_{n \geq 1} (-1)^{n-1} z^{k+1} q^{\frac{k(k+1)}{2}} q^{k+n+1} (-zq^{n+k+2})_{\infty}. \tag{3.8}
\]

**Proof.** For all three identities, we use \( z^{n+\ell} \) to denote the Sylvester’s triangle \( \rho_{n+\ell} \), which, under the application of the involution \( \alpha'' \), does not change. All the other procedures follow the proof of Theorem 3.7. \( \blacksquare \)

The above three identities can be translated into the language of partitions. We only do this for identity (3.7), stating it as a theorem, which, as will be seen, is in the spirit of Andrews’ “partition with initial repetitions”. For more details, see [5].

Let \( I_e(m, n) \) (resp. \( I_o(m, n) \)) denote the number of partitions of \( n \) with initial 2-repetitions, with \( m \) different parts and an even (resp. odd) number of repeated parts. We say a partition into distinct parts is with a initial Sylvester’s triangle \( \rho_k, k \geq 1 \) when it contains \( 1, 2, \ldots, k \) and does not contain \( k + 1 \) and it is not \( \rho_k \) itself. Let \( S_e(m, n) \) (resp. \( S_o(m, n) \)) denote the number of partitions of \( n \) with initial Sylvester’s triangle, with \( m \) different parts, and the first gap (i.e. the difference between neighboring parts which is larger than one) of parts is even (resp. odd).

**Theorem 3.10**

\[
I_e(m, n) - I_o(m, n) = S_o(m, n) - S_e(m, n).
\]

**Proof.** Since in the right-hand side of (3.7) the coefficient of \( z^m q^n \) is \( S_o(m, n) - S_e(m, n) \). To finish the proof, we only need to rewrite the left-hand side of (3.7) as follows:
\[
\sum_{n=1}^{\infty} z^n q^{n^2+n(-2q^{n+1})} = \sum_{n=1}^{\infty} z^n q^{1+2+2+\cdots+n-2} \prod_{j=n+1}^{\infty} (1 + zq^j)
\]
\[
= \sum_{n,m \geq 1} (I_e(m,n) - I_o(m,n)) z^m q^n.
\]

4 Flushed Partitions, Concave Compositions and Proper Partitions

We first prove two identities:

Lemma 4.1

\[
\sum_{n=1}^{\infty} q^{(3n-1)/2}(1 - q^n) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{n(n+1)/2}}{(1 + q) \cdots (1 + q^n)}. \tag{4.1}
\]

\[
\sum_{n=1}^{\infty} q^{n(3n-1)/2}(1 - q^n) = \sum_{n=1}^{\infty} \frac{q^{n(2n-1)}}{(-q)2n}. \tag{4.2}
\]

Proof. For the second identity, let \(P_o(D_n)\) (resp. \(P_e(D_n)\)) denote the number of partitions \(\lambda\) of \(n\) with distinct parts and the largest part is odd (resp. even). The following identity is due to Fine (see [13]) which can be reached by Franklin’s well known involution.

\[
\sum_{n=1}^{\infty} q^{n(3n-1)/2}(1 - q^n) = \sum_{n=1}^{\infty} (P_o(D_n) - P_e(D_n)) q^n. \tag{4.3}
\]

To prove the first identity, it is sufficient to prove the following

\[
\sum_{n=1}^{\infty} (P_o(D_n) - P_e(D_n)) q^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{n(n+1)}}{(1 + q) \cdots (1 + q^n)}. \tag{4.4}
\]

Identity (4.4) can be shown as follows. Given a partition with distinct \(n\) parts \(\lambda \in D\), we split the partition into a Sylvester’s triangle \(\rho_n = (n, n-1, \cdots, 1)\) and a partition \(\mu \in P\). The conjugate of \(\mu\) is a partition, \(\mu^*\), with the largest part at most \(n\) and with the number of parts \(r\). Note that \(\lambda_1\), the largest part of \(\lambda\) is \(n + r\). So in the left-hand side of (4.3), the above \(\lambda\) is actually assigned \((-1)^{n-1+r}\).

The right hand side of (4.4), at the same time, is the generating function of a pair of partitions \((\rho_n, \mu^*)\), where \(\rho_n\) is Sylvester’s triangle and \(\mu\) has largest part no more than \(n\) and with \(r\) parts. Each term is assigned \((-1)^{n-1+r}\), thus the conclusion.

The proof of the second identity is in the same fashion of (4.4), while the parity of the Sylvester’s triangle should be noted. Again, for a partition of \(n\) with distinct parts \(\lambda \in D_n\) assigned \((-1)^{\lambda_1}\), we split it into a Sylvester’s triangle \(\rho_{2n-1} = (2n-1, 2n-2, \cdots, 1)\) and a partition \(\mu\) with at most \(2n\) parts, assigned \((-1)^{\mu_1}\), where \(\mu_1\) is the largest part of \(\mu\).
Definition: Durfee Symbol was introduced by Andrews in [7], which is defined as follows: using Corollary 3.6, we can represent an unrestricted partition as its Durfee square of size \( n \) and two partitions with the largest part at most \( n \). We then denote the partition with two rows of integers, the top row listing the parts of the conjugate of the partition to the right of the Durfee square and the bottom row listing the parts of the partition under the Durfee square. We also write down a subscript \( n \) in the end to denote the size of the Durfee square. Take the partition \( \lambda = (11, 11, 9, 7, 5, 5, 4, 4, 3) \) for example, whose Young diagram is depicted in Figure 4, The Durfee symbol representation of \( \lambda \) is as follows:

\[
\begin{pmatrix}
5 & 5 & 4 & 4 & 3 & 3 \\
5 & 5 & 4 & 4 & 3 & 0 \\
\end{pmatrix}
\]

Suppose the Durfee square of a partition is of size \( n \), we call a partition proper if its Durfee symbol has the same number of \( n \)'s in both the top and bottom rows. All other partitions are improper partitions. The number of proper partitions of \( n \) is denoted as \( PR(n) \). The number of improper partitions of \( n \) is denoted as \( IMPR(n) \). A typical example of a proper partition, \( \lambda = (11, 11, 9, 7, 5, 5, 4, 4, 3) \), has its Young diagram in Figure 4. In its Durfee symbol, both the top and bottom rows have two 5’s.

![Figure 4: \( \lambda = (11, 11, 11, 9, 7, 5, 5, 4, 4, 3) \)](image)

Obviously, the generating function of proper partitions is

\[
\sum_{n=0}^{\infty} PR(n)q^n = 1 + \sum_{n=1}^{\infty} \frac{q^n}{(q^2; q^2)_n - (1 - q^{2n})}.
\]

As we stated in the introduction, a partition \( \lambda \) is called flushed if the smallest part to appear an even number of times is even. The unflushed partition are, of course, those partitions in which the smallest part to appear an even number of times is odd. We denote the number of flushed partitions of \( n \) as \( F(n) \).

In a flushed partition, suppose \( 1, 2, \cdots, 2i-1 \) all appear an odd number of times, and \( 2i \) appear an even number of times (zero times included). We extract \( 1 + 2 + \cdots + 2i - 1 \) and left \( 1, 2, \cdots, 2i \) all appearing an even number of times. So one easily writes down the generating function of \( F(n) \) as follows:

\[
\sum_{n \geq 1} F(n)q^n = \sum_{n=1}^{\infty} q^{n(2n-1)} \frac{1}{(q^2; q^2)_n (q^{2n+1})_{\infty}} = \sum_{n=1}^{\infty} \frac{q^{n(3n-1)/2} (1 - q^n)}{(q)_n q^{n} (q^2; q^2)_{n} (q^{2n+1})_{\infty}},
\]

where the second equality is by (4.2).

Relating Lemma [4.1] and Theorem [2.1] we get the following:
**Theorem 4.2 (Generating Functions for Flushed Partitions)** The number of flushed partitions of \( n \) is equal to the number of proper partitions of \( n \). Thus, they share the same generating function,

\[
\sum_{n \geq 1} F(n)q^n = \sum_{n = 1}^{\infty} PR(n)q^n = \frac{1}{(q)_\infty} \sum_{n = 1}^{\infty} q^{n(3n-1)/2}(1 - q^n).
\]

**Remark:** In [12] Dyson defined the rank of a partition as the largest part minus the number of parts. We denote the number of partitions of \( n \) with rank \( m \) by \( N(m, n) \). Then in [9] Atkin and Swinnerton-Dyer derived the generating function of \( N(0, n) \) as follows:

\[
\sum_{n = 1}^{\infty} N(0, n)q^n = \frac{1}{(q)_\infty} \sum_{n = 1}^{\infty} (-1)^{n-1} q^{n(3n-1)/2}(1 - q^n).
\]

It would be interesting to compare this with Theorem 4.2. Notice that there is some kind of symmetry between proper partitions and unrestricted partitions with rank 0. Since by the classical pentagonal number theorem, we have \((q)_\infty = \sum_{n = 0}^{\infty} (-1)^n q^{n(3n-1)/2}(1 + q^n)\), so, the above identity and theorem 4.2 reveal the relations of pentagonal number theorem with three different variations of signs.

**Theorem 4.3 (Generating Functions for Concave Compositions of Even Length)** The number of concave compositions of even length of \( n \) is equal to the number of improper partitions of \( n \). So, they share the same generating function,

\[
\sum_{n = 0}^{\infty} cc(n)q^n = \sum_{n = 0}^{\infty} IMPR(n)q^n = \frac{1}{(q)_\infty} \left(1 - \sum_{n = 1}^{\infty} q^{n(3n-1)/2}(1 - q^n)\right).
\]

**Remark:** Theorem 1.1 is the combination of theorem 4.2 and theorem 4.3

**Proof.** We will construct a bijection to finish the proof. Define a map \( \phi \) from concave compositions of even length of \( n \) to unrestricted partitions of \( n \) as follows. First take a concave composition \( C \) :

\[a_1 > a_2 > \cdots > a_m = b_m < b_{m-1} < \cdots < b_1.\]

The partition \( \phi(C) \) will depend on four cases.

1. When \( a_m = b_m = 0, a_{m-1} = b_{m-1} \), let the first row of the Young diagram have \( a_1 \) squares. Then draw \( b_1 \) squares under the square \((1, 1)\); Begin with the position \((2, 2)\), \( a_2 \) squares are put in the second row. Then we put \( b_2 \) squares under the position \((2, 2)\). Continuing in this fashion, we get a diagram. This diagram represent a partition of \( n \), which is denoted as \( \phi(C) \). Now we claim that the resulting partition is an improper partition. In fact, since \( a_m = b_m = 0 \), one knows that the Durfee square of \( \phi(C) \) is with size \( m - 1 \). Thus, in its Durfee symbol, in the top row, the number of \( m - 1 \)'s is \( a_m - 1 \); in the bottom row, the number of \( m - 1 \)'s is \( b_{m-1} \). For example, if the concave composition is \( C : 2 > 1 > 0 = 0 < 1 < 2, \) we have \( \phi(C) = (2, 2, 2) \), as illustrated in Figure 5.

2. When \( a_m = b_m \neq 0 \), we do symmetrically as in case one. let the first column of the Young diagram have \( b_1 \) squares. Then draw \( a_1 \) squares to the right of the square \((1, 1)\); Begin with the position \((2, 2)\), \( b_2 \) squares are put in the second column. Then we put \( a_2 \) squares to the right of the position \((2, 2)\). Continuing in this fashion, we get a Young diagram. This diagram represent a
partition, which, denote by $\phi(C)$, is improper. To see this, we first observe that the Durfee square of $\phi(C)$ is of size $m$. In the top row of the Durfee symbol, the number of $m$’s is $a_m$, while in the bottom row, the number of $m$’s is $b_m - 1$. For example, if the concave composition is $C : 2 > 1 = 1 < 2$, we have $\phi(C) = (3, 3)$ as illustrated in Figure 6.

3. When $a_m = b_m = 0, a_{m-1} \neq b_{m-1}$, if $a_{m-1} > b_{m-1}$, we follow the procedure of case two. We get a diagram, representing a partition of $n$, whose Durfee square has size $m - 1$. We denote this partition as $\phi(C)$ and observe that, in its Durfee symbol, the number of $m - 1$’s in the top row is $a_{m-1}$, while the number of $m - 1$’s in the bottom row is $b_{m-1} - 1$. Since $a_{m-1} - b_{m-1} + 1 \geq 2$, we have that $\phi(C)$ is improper.

For example, if the concave composition is $C : 3 > 2 > 0 = 0 < 1 < 2$, we have $\phi(C) = (4, 4)$, as illustrated in Figure 7.

4. When $a_m = b_m = 0, a_{m-1} \neq b_{m-1}$, if $a_{m-1} > b_{m-1}$. We follow the procedure of case one. We get a diagram which represents a partition of $n$, whose Durfee square has size $m - 1$. We denote this partition as $\phi(C)$ and observe that, in its Durfee symbol, the number of $m - 1$’s in the top row is $a_{m-1} - 1$, while the number of $m - 1$’s in the bottom row is $b_{m-1}$. Since $b_{m-1} - a_{m-1} + 1 \geq 2$, we have that $\phi(C)$ is improper.

Now we finished the definition of the map $\phi$, which map all concave compositions of even length of $n$ into improper partitions of $n$. We still need to see that $\phi$ is a bijection. Given an improper partition, $C'$, and suppose for accuracy its Durfee square’s size is $m$. We then write down its Durfee
symbol, and observe the difference of the number of $m$'s in its top row and bottom row. We divide it into four cases, namely, when the difference is $1$, $-1$, $\geq 2$ and $\leq -2$. Each of these cases corresponds to one the four cases we analyzed above, then we can map them back to the concave compositions of even length, which shows $\phi$ is indeed a bijective map.

In order to prove Sylvester’s first problem, we have the following Corollary of the above theorems.

**Corollary 4.4** The number of unflushed partitions of $n$ with $m$ parts is equal with the number of unrestricted partitions of $n$ with $m$ parts minus the number of partitions of $n$ with $m$ parts in which when the Durfee square is $k \times k$, the Durfee symbol contains no $k$'s in the top row and contains an even number of $k$'s in the bottom row.

**Proof.** Just like what we have done in the previous two sections, we try to insert another variable $z$ into the identities and start all the procedures of finding generating function of unflushed partitions all over again. Then the generating function for unflushed partitions of $n$ with $m$ parts can be denoted as:

\[
\sum_{n=0}^{\infty} \frac{z^{2n}q^n(2n+1)}{(q^2; q^2)_{2n+1}} \frac{1}{(zq^{2n+2})_\infty}
\]

\[
= \frac{1}{(zq)_\infty} \sum_{n=0}^{\infty} \frac{z^{2n}q^n(2n+1)}{(-zq)_{2n+1}}
\]

\[
= \frac{1}{(zq)_\infty} \sum_{n=0}^{\infty} (-1)^n \frac{z^n q^{n(2n+1)/2}}{(-zq)_n}
\]

\[
= \sum_{n=0}^{\infty} \frac{z^n q^n}{(zq)_n (q)_n} - \sum_{n=1}^{\infty} \frac{z^n q^n}{(zq)_{n-1} (q)_{n-1} (1 - z^2 q^{2n})}
\]

the first identity is by an analysis similar to that of (4.5), the second identity is from the proof of Lemma 4.1, and the third identity is by (2.3) and (3.2).

We interpret the first and the last sums of the above identity and get the conclusion.

We conclude this paper with a combinatorial proof of Sylvester’s problem.

**Theorem 4.5** Unflushed partitions of $n$ with odd number of parts are equinumerous with unflushed partitions of $n$ with even number of parts.

**Proof.** By Corollary 4.4 we only need to analyze the parity of the number of unrestricted partitions of $n$ with $m$ parts, in which, when the Durfee square is $k \times k$, either there are an odd number of $k$'s in the bottom row, or there are an even number of $k$’s in the bottom row and at least one $k$ in the top row.

If the total number of $k$’s in the Durfee symbol is odd, say, $2i - 1$, then we can arrange these $k$’s in top and bottom rows in $2i$ ways, by putting $0, 1, \cdots, 2i - 1$ copies of $k$’s in the bottom row and the rest $k$’s in the top row. Observe that the parity of number of parts changes accordingly, that is, among these $2i$ partitions, $i$ ones have an even number of parts and $i$ ones have an odd number of parts.

If, however, the total number of $k$’s in the Durfee symbol is even, say, $2i$, remember that the top row must contain at least one $k$. Then we arrange these $k$’s in top and bottom rows in $2i$ ways, by putting $0, 1, \cdots, 2i - 1$ copies of $k$’s in the bottom row and the rest in the top row. In this case the parity of number of parts changes accordingly, too, that is, among these $2i$ partitions, $i$ ones have an even number of parts and $i$ ones have an odd number of parts.
5 Possible Further Works

1. In [16], Sylvester also asked another problem, as he put it, “2. Required to prove that the same proposition holds when any odd number is partitioned without repetitions in every possible way.” We don’t know whether or not a similar combinatorial proof can be given for this second problem.

2. Regarding Section 3, it seems the method in the proof of Theorem 2.1 involving the involutions $\alpha, \alpha'$ and $\alpha''$ can be applied to more problems.

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