Abstract

$U(n)$ Yang-Mills theory on the fuzzy sphere $S^2_N$ is quantized using random matrix methods. The gauge theory is formulated as a matrix model for a single Hermitian matrix subject to a constraint, and a potential with two degenerate minima. This allows to reduce the path integral over the gauge fields to an integral over eigenvalues, which can be evaluated for large $N$. The partition function of $U(n)$ Yang-Mills theory on the classical sphere is recovered in the large $N$ limit, as a sum over instanton contributions. The monopole solutions are found explicitly.
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## 1 Introduction

Gauge theories provide the best known description of the fundamental forces in nature. At very short distances however, physics is not known, and it is plausible that spacetime is quantized below some scale. This idea has been contemplated for quite some time, and received a boost recently due to the discovery that string theory naturally leads to
noncommutative gauge theories under suitable conditions, as explained in [1]. Gauge theory on noncommutative spaces has been the subject of much research activity in recent years, see e.g. [2] for a review.

There is one major problem with most models of noncommutative gauge theories: their quantization is very difficult. A direct quantization leads to difficulties related to the notorious UV/IR mixing [3], see also [4] for some recent developments. On the other hand, the use of the Seiberg-Witten map [1, 5], which allows a formulation in terms of commutative quantities, yields Lagrangians which become increasingly complicated at each order in the deformation parameter. This seems to rule out perturbative quantization [6].

The motivation behind this paper is to try to develop new tools for the quantization of gauge theories, taking advantage of noncommutativity. The idea is to make use of one very fascinating feature of gauge theory on (some) noncommutative spaces: It is possible to formulate the gauge theories in terms of Lagrangians which have no derivatives. Rather, the dynamical variables are essentially matrices $B_i$, and the action is the trace of products of these matrices. Hence these gauge theories are matrix models. The gauge transformations have the form $B_i \rightarrow U^{-1}B_iU$ for unitary matrices $U$. The kinetic term is generated upon a shift $B_i = X_i + A_i$, and the $A_i$ become the usual gauge fields in the commutative limit. This is very interesting for the quantization, because the path integral can now be defined simply as the integral over the matrices $B_i$, as in a random matrix model. A promising strategy is then to first do the quantization in terms of the $B_i$ fields, and then go to the “classical” variables $A_i$. This is a bit in the spirit of [7].

In general of course, things are still complicated: The actions are multi-matrix models with nontrivial interactions, and the integration over the $B_i$ is highly nontrivial. It must be so, since they describe a nontrivial quantum field theory. Moreover, the matrices are infinite-dimensional for most spaces (such as for $\mathbb{R}^n$). This latter problem does not occur on the so-called fuzzy spaces, in particular the fuzzy sphere $S^2_N$ [8]. This quantum space is characterized by a deformation parameter $\frac{1}{N}$ which measures the size of “Planck cells”, and reduces to the classical sphere for $N \rightarrow \infty$. Moreover the rotation invariance under $SU(2)$ is maintained, hence $S^2_N$ seems particularly well suited to explore this idea.

In this paper, we will show that for pure $U(n)$ Yang-Mills theory on the fuzzy sphere, the quantization can be carried out completely by integrating over the matrices $B_i$. This is achieved by collecting the $B_i$ into a single hermitian matrix, subject to a constraint. Of course, Yang-Mills theory on $S^2$ is a rather simple field theory with no propagating degrees of freedom; however it does have nontrivial monopole sectors, and its quantization is not entirely trivial. We will calculate the partition function for $U(n)$ Yang-Mills theory on $S^2_N$ in the large $N$ limit, and recover the known result [9, 10] for the partition function on the classical sphere. Corrections of order $\frac{1}{N}$ could be calculated in principle, but we do not attempt this here. The main message is the applicability of completely new methods to noncommutative gauge theory, and hence to their commuta-
tive limit. Moreover, our result strongly suggests that the “commutative limit” of pure
gauge theory on the fuzzy sphere is smooth, which is not obvious in view of the UV/IR
mixing effects in noncommutative field theories [3]. This was also found recently on the
quantum torus [11], however with very different methods.

Another important message is that in the approach developed here, the construction
of gauge theory on (some) noncommutative spaces can be simpler than on a classical
space. In particular, there is no need to introduce nontrivial fiber bundles, connections
and other mathematical structures in our approach: the monopole sectors arise auto-
matically in a very simple way, and reproduce the correct classical limit. We explicitly
calculate the gauge fields for all monopole configurations.

This paper is organized as follows. After a brief review of the fuzzy sphere and the
most basic facts about matrix models, we present in Section 3.2 the particular potential
to be used in this paper. We then show that its minima define a fuzzy sphere, and the fluctuations become gauge fields on this fuzzy sphere after imposing a suitable
constraint. In Section 3.5 the monopole sectors of the $U(1)$ case are identified, and the
gauge field for the monopoles is calculated explicitly. We then generalize the construc-
tion to the $U(n)$ case in Section 4 which amounts simply to taking larger matrices.
Section 5 contains the calculation of the path integral, which is the main application of
our construction. Finally we make some simple observations on symmetries, correlation
functions, and show how a small modification leads to gauge theory on the $q$-deformed
fuzzy sphere. The technical part of the path integral calculation is postponed to the
appendix. In general, the focus is on explicit calculations, keeping the formal mathe-
matics to a minimum. The hope is that noncommutative field theory in general and at
least some of the techniques developed here will eventually be useful for physics.

2 The basic fuzzy sphere

We start by recalling the definition of fuzzy sphere [8,12] in order to fix our conventions.
The algebra of functions on the fuzzy sphere is the finite algebra $S^2_N$ generated by
Hermitian operators $x_i = (x_1, x_2, x_3)$ satisfying the defining relations
\[
[x_i, x_j] = i\Lambda_N \epsilon_{ijk} x_k, \quad (1)
\]
\[
x_1^2 + x_2^2 + x_3^2 = R^2. \quad (2)
\]
The noncommutativity parameter $\Lambda_N$ is of dimension length, and can be taken positive.
The radius $R$ is quantized in units of $\Lambda_N$ by
\[
\frac{R}{\Lambda_N} = \sqrt{\frac{N^2 - 1}{4}}, \quad N = 1, 2, \cdots \quad (3)
\]
This quantization can be easily understood. Indeed [11] is simply the Lie algebra $su(2)$,
whose irreducible representation have dimension $N$. The Casimir of the $N$-dimensional
representation is quantized, and related to $R^2$ by (2) and (3). Thus the fuzzy sphere is characterized by its radius $R$ and the “noncommutativity parameters” $N$ or $\Lambda_N$. The algebra of “functions” $S^2_N$ is simply the algebra $\text{Mat}(N)$ of $N \times N$ matrices. It is covariant under the adjoint action of $SU(2)$, under which it decomposes into the irreducible representations with dimensions $(1) \oplus (3) \oplus (5) \oplus \ldots \oplus (2N - 1)$. The integral of a function $f \in S^2_N$ over the fuzzy sphere is given by

$$R^2 \int f(x) = \frac{4\pi R^2}{N} \text{Tr}[f(x)],$$

where we have introduced $\int$, the integral over the fuzzy sphere with unit radius. It agrees with the integral $\int d\Omega$ on $S^2$ in the large $N$ limit. Invariance of the integral under the rotations $SU(2)$ amounts to invariance of the trace under adjoint action. It is convenient to introduce the dimensionless coordinates

$$\lambda_i = x_i/\Lambda_N$$

which satisfy

$$\varepsilon^{ij}_k \lambda_i \lambda_j = i\lambda_k, \quad \lambda_i \lambda^i = \frac{N^2 - 1}{4}.$$  

(6)

The $\lambda_i$ form a $N$-dimensional representation of $SU(2)$, which is given explicitly in Appendix A for convenience. Noting that $[\lambda_i, x_j] = i\varepsilon_{ijk}x^k$, it follows that the rotation operators $J_i$ act on functions $f \in S^2_N$ as

$$J_i f = [\lambda_i, f].$$

(7)

One can now write down actions for scalar fields, such as

$$S_0 = \int \frac{1}{2} \Phi(\Delta + \mu^2)\Phi + \frac{g}{4!} \Phi^4$$

(8)

where $\Phi$ is a Hermitian matrix, and $\Delta = \sum J_i^2$ is the Laplace operator. For gauge fields, the “correct” action is less obvious because the gauge fields have a priori 3 components (because there are 3 independent one-forms on $S^2_N$, [8]), and it is not obvious how to get rid of the normally unwanted 3rd component. Several slightly different approaches have been pursued in the literature [8, 13, 14, 15, 16]. Alternatively, if one keeps the 3rd component which is essentially a scalar field, one finds actions which turn out to describe D2-branes on $SU(2)$ [18].

In this paper, we will develop a particularly simple formulation of gauge theory on $S^2_N$, which makes a clear choice of the preferred actions even in the nonabelian $U(n)$ case, and includes the topologically nontrivial sectors in an extremely simple way. The starting point is the following observation: we can combine the generators $\lambda_i$ which are $N \times N$ matrices into a single $2N \times 2N$ matrix by

$$C = \frac{1}{2} + \sum_i \lambda_i \sigma^i.$$  

(9)
We then observe the following property
\[ C^2 = \left( \frac{N}{2} \right)^2, \] (10)
which follows from (139). Hence the eigenvalues of \( C \) are \( \pm \frac{N}{2} \). To get the multiplicities, note that \( \lambda_i \sigma^i \) is an intertwiner of \( (2) \otimes (N) = (N-1) \oplus (N+1) \) (i.e. it is invariant under \( SU(2) \)), hence the multiplicities are \( N+1 \) resp. \( N-1 \).

This simple observation leads to the idea that one should consider a matrix model for a hermitian matrix \( C = \frac{1}{2} + B_i \sigma^i \), and a potential which has \( \pm \frac{N}{2} \) as degenerate minima. The fluctuations \( B_i = \lambda_i + A_i \) around the above solution should correspond to the gauge fields \( A_i \), and the invariance under \( C \to U^{-1}CU \) for a unitary matrix \( U \) should correspond to gauge transformations (and other symmetries). Indeed this idea works. Before working it out, let us briefly recall some basic facts about matrix models.

\section{Matrix Models and the Fuzzy Sphere}

\subsection{A brief review of single-matrix models}

We briefly recall some basic facts about matrix models which have found many applications in physics. We refer the reader to \cite{18,19} for excellent reviews and more references. Consider the matrix model of a single \( N \times N \) hermitian matrix \( C \) with potential \( V(C) \). The partition function of the model is defined by
\[ Z = \int dCe^{-TV(C)} = \int \prod_{i=1}^{N} dc_i \Delta^2(c) e^{-\sum V(c_i)} \] (11)
where \( c_i \) are the \( N \) eigenvalues of the Hermitian matrix \( C \). Here
\[ \Delta(c) = \prod_{i<j} (c_i - c_j) \] (12)
is the Vandermonde determinant, which is the Jacobian of the transformation \( dC = \prod dc_i dU \Delta^2(c) \), and the integral over the unitary matrices \( U \) is trivial. In the literature on matrix models, the potential \( V \) is usually chosen to be of the form
\[ V(C) = \frac{N}{g^2} v(C), \quad v(C) = \sum_{k \geq 2} g_k C^k \] (13)
where \( g_2 = 1 \) and the couplings \( g_k \) are kept fixed in the large \( N \) limit.

The reason why these matrix models are so useful is that the models really only depend on the \( N \) eigenvalues \( c_i \), while the matrices have \( N^2 \) degrees of freedom. This
lead to the development of powerful methods (e.g. steepest descent method, orthogonal polynomials, etc [21, 22]) which can be used to analyze the models, and basically solve them explicitly in the large $N$ limit. For example, the saddle-point equation is given by

$$\frac{1}{g^2} v'(c_i) = \frac{2}{N} \sum_{j \neq i} \frac{1}{c_i - c_j}. \quad (14)$$

The sum in the r.h.s. is due to the Vandermonde determinant in the measure and represents a repulsive potential among the eigenvalues. The $N$-dependence of the potential ensures that the repulsive effect is well balanced in the large $N$ limit. Due to this repulsive force, the eigenvalues spread evenly around the classical solution of the equation of motion $c_i = 0$. The distribution of the eigenvalues

$$\rho(c) := \frac{1}{N} \sum_i \delta(c - c_i) \quad (15)$$

becomes continuous in the large $N$ limit and can be solved easily from the equations [21]

$$\frac{1}{g^2} v'(c) = 2 \int dc' \frac{\rho(c')}{c - c'}, \quad \int dc \rho(c) = 1. \quad (16)$$

There is much more to be said about these matrix models. In particular, the distribution of eigenvalues (and correlation functions) can be derived using e.g. the method of orthogonal polynomials, without relying on the saddle-point approximation. However we will not need these techniques due to the simplicity of the model considered here.

### 3.2 A matrix model with degenerate minima

In this paper, we will show that the fuzzy sphere arises as a vacuum solution of another Hermitian matrix model, given by a potential with a different scaling dependence in $N$. Consider the matrix model with action

$$S = Tr V(C) = \frac{1}{g^2 N} Tr \left( C^2 - \left( \frac{N}{2} \right)^2 \right) = \frac{1}{g^2 N} Tr \left( \frac{N^4}{16} - \frac{N^2}{2} C^2 + C^4 \right) \quad (17)$$

where $C$ is a $2N \times 2N$ Hermitian matrix, and $g^2 > 0$ is kept fixed independent of $N$. The shape of the potential is sketched in Figure [11] The following features distinguish it from the matrix models considered before:

- The coefficient of the quadratic term is negative. This implies that the distribution of eigenvalues are peaked at the minima $\pm \frac{N}{2}$ of the polynomial $V(C)$, rather than around the origin.
The specific form and the particular $N$ dependence of $V(C)$ is chosen such that the accumulation of eigenvalues at its minima will lead to the emergence of a (fuzzy) sphere $S^2_N$. The fluctuations will describe a gauge theory on $S^2_N$, as we will show below. Apart from the $N$ dependence, the relative coefficients between the $C^2$ and $C^4$ terms in $V$ can be adjusted to any (negative) number by a rescaling of $C$ and $g$. In general, different spaces may be generated for different potentials $V(C)$, and the properties of this space such as symmetries are related to the details of this eigenvalue distribution.

If we expand $V(C)$ around one of its minima (consider $\frac{N}{2}$ to be specific, setting $C = \frac{N}{2} + \mu_+$), then

$$V(\frac{N}{2} + \mu_+) = \frac{N}{g^2}(\mu^2_+ + \frac{2}{N}\mu_+^3 + \frac{1}{N^2}\mu_+^4)$$

(18)

Now the coefficient of the quadratic term is positive, and comparing with the usual matrix models we expect that the higher-order terms will become irrelevant for large $N$, leading to a simple Gaussian distribution near each minimum.

Let us turn to the eigenvalue distribution in the large $N$ limit. The stationary points of our potential are given by the eigenvalues

$$c = 0, \pm \frac{N}{2}$$

(19)

Since the action for $c = 0$ is of order $N^3$, it is quite obvious that this stationary point does not contribute for large $N$. Hence consider the eigenvalues distribution near the minima $c = \pm \frac{N}{2}$ in the large $N$ limit. We introduce $c_i = \frac{N}{2} + \mu_i$ to facilitate the analysis; the other case $c_i = -\frac{N}{2} + \mu_i$ is similar. Performing a similar analysis as [21], the saddle
point equation becomes

\[ \frac{1}{g^2}(2\mu_i + \frac{6}{N}\mu_i^2 + \frac{4}{N^2}\mu_i^3) = \frac{1}{N} \sum_{j \neq i} \frac{2}{\mu_i - \mu_j}. \]  

(20)

In the large \( N \) limit this becomes again

\[ \frac{1}{g^2}\mu = \int_{-a}^{a} d\mu' \frac{\rho(\mu')}{\mu - \mu'}, \quad \int_{-a}^{a} d\mu \rho(\mu) = 1 \]  

(21)

which gives the standard distribution for Gaussian matrices,

\[ a = \sqrt{2g^2}, \quad \rho(\mu) = \frac{1}{g\pi} \sqrt{2g^2 - \mu^2}. \]  

(22)

In particular, the distribution is only nonzero for \( |\mu_i| < \sqrt{2}g \), and we find a finite spread of eigenvalues. The reason is the explicit \( N \) in front of the quadratic term in the action. This means that only eigenvalues near \( \pm \frac{N}{2} \) will contribute,

\[ \delta\mu_i < \sqrt{2}g. \]  

(23)

Hence near each minimum \( \pm \frac{N}{2} \), the action can be replaced by a Gaussian \( \frac{N}{g^2} \sum_i (\mu_i^\pm)^2 \) for large \( N \), since the higher-order terms are suppressed by \( \frac{1}{N} \).

### 3.3 Matrix model vacua and emergence of the fuzzy sphere

Let us resume the analysis of the particular model given by the polynomial \([17]\). The solutions to the classical equation of motion

\[ C(-N^2/4 + C^2) = 0 \]  

(24)

are characterized by the multiplicities \( n_+, n_-, n_0 \) of the eigenvalues \( \pm \frac{N}{2} \) resp. 0, which satisfy \( n_+ + n_- + n_0 = 2N \). We can assume that \( n_0 = 0 \) as discussed above, since each zero eigenvalue gives a contribution \( \frac{N^3}{16g^2} \) to the action and is highly suppressed. Then the saddle points are characterized by the trace \( \text{Tr}(C) \), given by

\[ \text{Tr}(C) = N/2 \ (n_+ - n_-). \]  

(25)

Consider the vacuum with \( n_+ - n_- = 2 \), i.e. \( n_+ = N + 1, n_- = N - 1 \). Using the \( U(2N) \) invariance, one can put the vacuum in the form

\[ C = \frac{1}{2} + \lambda_i \sigma^i \]  

(26)

where \( \lambda_i \) are precisely the \( N \times N \) Hermitian matrices in \([5], [6]\) which describe the fuzzy sphere, and \( \sigma^i, i = 1, 2, 3 \) are the Pauli matrices. In other words, we can find a unitary matrix \( U \) such that

\[ \text{diag}(\frac{N}{2}, ..., -\frac{N}{2}) = U(\frac{1}{2} + \lambda_i \sigma^i)U^{-1} \]  

(27)
provided \( n_+ - n_- = 2 \), as explained in Section 2. The equations of motion then take the form
\[
\varepsilon^{ij}_k \lambda_i \lambda_j = i \lambda_k, \quad \lambda_i \lambda^i = \frac{N^2 - 1}{4}.
\]
(28)
These are precisely the defining relation (3) of the fuzzy sphere \( S^2_N \) in terms of the dimensionless coordinates. The radius has been set to unit here, since it can easily be reintroduced.

### 3.4 Matrix fluctuations and gauge fields

Now consider a general \( 2N \times 2N \) Hermitian matrix \( C \),
\[
C = C_\alpha \sigma^\alpha = (\frac{1}{2} + \rho) \sigma^0 + B_i \sigma^i
\]
(29)
where \( \sigma^0 = \mathbb{1} \). Plugging this into (17), we obtain
\[
S = \text{Tr} V(C) = \frac{2}{g^2 N} \text{Tr} \left((B_i B^i - \lambda_i \lambda^i)^2 + (B_i + i \varepsilon_{ijk} B^j B^k)(B^i + i \varepsilon^{irs} B_r B_s)\right.
\]
\[+ D_i \rho D^i \rho + N^2 \rho^2 + 2 \rho^3 + \rho^4
\]
\[+ 6(B_i B^i - \lambda_i \lambda^i) \rho (\rho + 1) + 4i \rho \varepsilon_{ijk} (B^i B^j B^k - \lambda^i \lambda^j \lambda^k)\right).
\]
(30)
where
\[
D_i \rho := [B_i, \rho].
\]
(31)
This starts to look like a field theoretic action on the fuzzy sphere \( S^2_N \). Its interpretation is however obscured by the presence of \( \rho \). Comparing with (9), we shall therefore impose the constraint
\[
\rho = 0, \quad \text{i.e.} \quad C_0 = \frac{1}{2}.
\]
(32)
This implies
\[
\text{Tr}(C) = N.
\]
(33)
Then the above action becomes
\[
S = \frac{2}{g^2 N} \text{Tr} \left((B_i B^i - \frac{N^2 - 1}{4})^2 + (B_i + i \varepsilon_{ijk} B^j B^k)(B^i + i \varepsilon^{irs} B_r B_s)\right).
\]
(34)
This is one possible action for a gauge theory on the fuzzy sphere, cp. [8,13,14,15,16,17]. The constraint breaks the original \( SU(2N) \) symmetry down to a smaller subgroup, which contains a \( SU(N) \) gauge symmetry acting as
\[
B_i \rightarrow U^{-1} B_i U.
\]
(35)
We will see that this corresponds to the usual \( U(1) \) local gauge symmetry in the classical limit. Of course, breaking the full \( SU(2N) \) symmetry by the constraint (32) is somewhat
against the spirit of the matrix model, in particular the integration in the partition function cannot be carried out as easily as in (11) any more. This is to be expected, since (34) is a matrix model with 3 interacting matrices. Nevertheless, this approach will allow us to carry out the path integral with some more effort.

It is easy to understand the dominant configurations for the action (34). The term \((B_i + i\varepsilon_{ijk}B_jB_k)^2\) implies that the \(B_i\) approximately generate a representation of \(su(2)\), and the other term implies that \(B_iB^i \approx (N^2 - 1)/4\), which corresponds to the Casimir of the approximate \(su(2)\) representation. Hence one could interpret (34) as a theory of “fluctuating representations” of \(su(2)\), and the dominant configurations will be approximately \(N\)-dimensional irreps of \(su(2)\). This is an important difference to other possible actions without the term \((B_iB^i - \frac{N^2-1}{4})^2\), such as in [38]: there, reducible “block”-solutions with blocks of arbitrary size are allowed, while in (34) they are suppressed. As we will see, this is crucial for the physical interpretation, and only (34) reduces to an ordinary Yang-Mills theory on \(S^2\) in the large \(N\) limit.

Consider next the equations of motion,

\[
[B^i, B_jB^j - \frac{N^2-1}{4}]_+ + (B + i\varepsilon BB)_i + i\varepsilon^{ijk}[B_j, (B + i\varepsilon BB)_k] = 0. \tag{36}
\]

The “vacuum” solution is

\[
B_i = \lambda_i, \tag{37}
\]

up to gauge transformation. In fact then \(S = 0\), and this is the unique solution with \(S = 0\) up to \(SU(N)\) gauge invariance, because both \(B + i\varepsilon BB = 0\) and \(B_iB^i = \frac{N^2-1}{4}\) must hold. This means that \(B_i\) is a representation of \(su(2)\) with fixed Casimir. If we now expand

\[
B_i = \lambda_i + A_i, \tag{38}
\]

then

\[
B_iB^i - \lambda_i\lambda^i = \lambda_iA_i^i + A_i\lambda^i + A_iA^i \tag{39}
\]

and

\[
B^i + i\varepsilon^{ikl}B_kB_l = \frac{1}{2}\varepsilon^{ikl}F_{kl}, \quad F_{kl} := i[\lambda_k, A_l] - i[\lambda_l, A_k] + i[A_k, A_l] + \varepsilon_{klm}A^m. \tag{40}
\]

Notice that the kinetic terms in \(F_{kl}\) arise automatically due to the shift (38). The \(SU(N)\) gauge symmetry acts on \(A_i\) as

\[
A_i \rightarrow U^{-1}A_iU + U^{-1}[\lambda_i, U] \tag{41}
\]

which for \(U = \exp(ih(x))\) and \(N \rightarrow \infty\) becomes the usual (abelian) gauge transformation for a gauge field.

However, the gauge field \(A_i\) has 3 components, which does not seem to match with the degrees of freedom in a 2-dimensional gauge theory. To understand this, we should
translate the action into conventional field theory language, which can be done for large $N$. One can then decompose the field $A_i$ into a tangential component $A_i$ and a radial component $\varphi$ as follows\(^1\):

$$A_i = \frac{4}{N^2 - 1} \lambda_i \varphi + A_i \approx \frac{2}{N} x_i \varphi + A_i$$

(42)

where $\varphi$ and $A_i$ are defined such that

$$\lambda^i A_i = 0,$$

(43)

$$\varphi := \lambda^i A_i.$$  (44)

Then

$$B_i B^i - \lambda_i \lambda^i = 2\varphi - [\lambda_i, A_i] + A_i^2 + \frac{1}{N} T(\varphi, A).$$

(45)

Here $T(\varphi, A)$ stands for functions of $\varphi$ and $A_i$ which are suppressed by $\frac{1}{N}$. Similarly, all terms involving $\varphi$ in the “field strength” $F_{kl}$ (40) are suppressed\(^2\) by $\frac{1}{N}$. Therefore the only term involving $\varphi$ which contributes for large $N$ is the square of (45). We can now simply integrate out $\varphi$ (i.e. consider it an auxiliary variable), replacing it by

$$\varphi = \frac{1}{2} (|\lambda^i A_i| - A_i^2)$$

(46)

for large $N$, which is smooth (i.e. high angular momenta are suppressed, assuming the $A_i$ are smooth). Hence all terms in $F_{kl}$ containing $\varphi$ can be omitted for large $N$, being suppressed by $\frac{1}{N}$. Another way of arriving at this conclusion is to use the field $\phi := \frac{1}{N} \varphi$, which has a large mass of order $N$. At any rate, we can now write

$$F_{kl} = i[\lambda_k, A_l] - i[\lambda_l, A_k] + i[A_k, A_l] + \varepsilon_{klm} A^m$$

(47)

for large $N$, involving the tangential gauge field only. As a consequence of (43) and (6), $F_{kl}$ is “tangential”

$$x^k F_{kl} = o(1/N)$$

(48)

for large $N$, and it becomes the field strength of an abelian gauge theory on $S^2$.

To summarize, we found that the radial fluctuations $\varphi$ decouple in the large $N$ limit, and (34) reduces to a $U(1)$ Yang-Mills theory on a unit sphere with action

$$S = \frac{1}{g^2} \int F_{mn} F^{mn}.$$  (49)

Here

$$F_{kl} = iJ_k A_l - iJ_l A_k + \varepsilon_{klm} A^m$$

(50)

\(^1\)this decomposition as defined here is gauge-invariant only in the large $N$ limit. From that point of view $\tilde{\phi} := B_i B^i - \lambda_i \lambda^i$ would be a nicer radial field, however this leads to a nontrivial Jacobian in the path integral, which makes the decoupling argument below more subtle.

\(^2\)assuming that $\varphi$ and $A_i$ is “smooth”, so that $[\varphi, \lambda_i]$ is finite. This is justified by (46) and the kinetic terms in the action.
is the field strength for the $U(1)$ gauge potential $A_i$. The fields are tangential in the sense\[24\]

$$x^i A_i = 0, \quad x^i F_{ij} = 0. \tag{51}$$

This is a description of the 2d gauge theory in terms of a 3-component gauge field $A_i$ subject to the tangential constraint $\Box$. This formulation is manifestly invariant under $SO(3)$ rotations. To put it in a more familiar form, let us assume that we are sitting on the north pole. Then only $A_1, A_2$ and $F_{12}$ survive the constraints, and

$$iJ_1 = -\partial_2, \quad iJ_2 = \partial_1. \tag{52}$$

Our gauge theory can then be identified with a gauge theory with only tangential gauge fields $A_{(cl)}^i, i = 1, 2$, whose field strength takes the usual form

$$F_{12}^{(cl)} = \partial_1 A_{2(\text{cl})} - \partial_2 A_{1(\text{cl})} \tag{53}$$

if we identify

$$A_{1(\text{cl})} = -A_2, \quad A_{2(\text{cl})} = A_1. \tag{54}$$

In coordinate independent form, this is

$$\vec{A}^{(cl)} = \vec{r} \times \vec{A} \tag{55}$$

where $\vec{r}$ is the radial unit vector. This identification will be useful in the next subsection.

Since the volume of the gauge group is finite here, we do not have to fix the $SU(N)$ gauge using e.g. the Faddeev-Popov method. Instead we can keep the integral over all configurations. Indeed, working with $B_i$ or even $C$ seems much easier, and makes all the symmetries manifest. The beauty of our formulation is that it allows to apply the powerful methods of random matrix theory, after suitable modifications. One can hope that this will lead to new methods for studying gauge theories.

**Alternative version of constraint.** It is important to realize that the matrix model $\text{Tr} V(C)$ describes a Yang-Mills theory only if we impose the constraint $\Box$, so that the last terms in $\Box$ vanishes. Without that constraint, $4i\rho \varepsilon_{ijk}(B^i B^j B^k - \lambda^i \lambda^j \lambda^k)$ contains a term $\propto NF \rho$, which after integrating out $\rho$ cancels the YM term $\text{Tr} FF$. However, there is another possibility, namely to consider

$$S' = \text{Tr}(V(C)) - N\text{Tr}(C_0 - \frac{1}{2})^2. \tag{56}$$

The last terms of course implies the $C_0 = \frac{1}{2}$ constraint in the large $N$ limit. This leads to an additional term $-\text{Tr} N \rho^2$ in the action $\Box$, which allows to integrate out $\rho$ for large $N$ leaving a rescaled YM term $\propto \text{Tr} FF$. The path integral can then be carried out in the same way as we will do below. Basically this seems to be a matter of taste, and we chose to impose $\rho = 0$ directly.
3.5 Monopole sectors

If we claim to have a fuzzy version of Yang-Mills theory on the 2-sphere, we should be able to recover the monopole sectors as well. They are not hard to find here: as discussed above, the dominant contributions for the action (34) are “approximate” representations of \( su(2) \) with Casimir \( B_i B^i \approx \frac{N^2 - 1}{4} \). This suggests to consider irreps of dimension \( M \) slightly different from \( N \). They indeed turn out to describe monopoles\(^3\).

Hence consider the same action (17), but for matrices of different size. Let

\[
C^{(M)} = \frac{1}{2} + B^{(M)}_i \sigma^i
\]

be a \( 2M \times 2M \) matrix with

\[
M = N - m, \quad m \in \mathbb{Z}.
\]

This implies in particular

\[
Tr(C) = M,
\]

which again picks out the sector \( n_+ - n_- = 2 \). One can easily see that the equation of motion (36) resp. \( Tr(V'(C)\delta C) = 0 \) has solutions of the form \( B^{(M)}_i = \alpha_m \lambda_i^{(M)} \) if

\[
3(\alpha_m - 1)^2 + \alpha_m^2 M^2 - N^2 = 0,
\]

which gives

\[
\alpha_m = 1 + \frac{m}{N} \quad \text{for } N \gg m.
\]

Hence we found the new solution

\[
C^{(M)} = \frac{1}{2} + \alpha_m \lambda_i^{(M)} \sigma^i.
\]

Then

\[
(F^{(M)})_i = i \varepsilon_{ijk}(B^{(M)})_j(B^{(M)})_k + (B^{(M)})_i \rightarrow \frac{m}{2} x_i
\]

and

\[
B_{(M)} \cdot B_{(M)} - \frac{N^2 - 1}{4} \rightarrow 0
\]

for \( N \rightarrow \infty \). In particular, the field strength is tangential in the sense (48), with \( |F| = \frac{m}{2} \). This is just like the field strength of a monopole of charge \( m \), suggesting that \( m \) is the monopole charge. The action is

\[
S(C^{(M)}) = \frac{m^2}{2g^2}
\]

\(^3\)The basic idea that fuzzy spheres of different size correspond to monopoles was also proposed in [25], without calculating the gauge field.
for large $N$. We will show that this interpretation as a monopole is correct by writing $(B(M))_i$ as an excitation over the fuzzy sphere solution $B_i = \lambda_i^{(N)}$. The corresponding gauge field will take the usual form of Dirac monopole of charge $m$ in the large $N$ limit. To see this, write this solution as a block matrix

$$C = \frac{1}{2} + \begin{pmatrix} \alpha_m \lambda_i^{(M)} & \sigma^j \\ 0 & 0 \end{pmatrix} = \frac{1}{2} + \lambda_i^{(N)} \sigma^j + A_i \sigma^i,$$  \hspace{1cm} (66)

where $A_i$ is interpreted as gauge field. Hence

$$A_i = \alpha_m \lambda_i^{(M)} - \lambda_i^{(N)}. \hspace{1cm} (67)$$

Using the representation (136) for the $\lambda_i = \lambda_i^{(N)}$, one obtains the following non-vanishing matrix elements

$$(A_3)_{kk} = \begin{cases} -(\lambda_3)_{kk} \left( 1 - \alpha_m \left( 1 - \frac{m}{N+1-2k} \right) \right) & 1 \leq k \leq N - m, \\ -(\lambda_3)_{kk}, & k > N - m, \end{cases} \hspace{1cm} (68)$$

$$(A_+)_{k,k+1} = \begin{cases} -(\lambda_+)_{k,k+1} \left( 1 - \alpha_m \sqrt{1 - \frac{m}{N-k}} \right), & 1 \leq k \leq N - m, \\ -(\lambda_+)_{k,k+1}, & k > N - m \end{cases} \hspace{1cm} (69)$$

and $A_-$ is obtained by transposition. This can be translated into functions on the fuzzy sphere via (5), which takes the form

$$x_i = \frac{2}{N} \lambda_i \hspace{1cm} (70)$$

in the large $N$ limit. We also note that $k$ is related to the “height” $x_3$ through

$$k = \frac{N + 1}{2} - (\lambda_3)_{kk}. \hspace{1cm} (71)$$

The quantities in (68)-(69) have a smooth limit for large $N$, except at the finite set of “points” $N - m + 1 < k < N$ (located at the south pole) where $A_i$ develops a singularity. This singularity corresponds to the Dirac string. In the patch covered by $1 \leq k \leq N - m$, which represents the sphere without the south pole, we obtain

$$A_3 = -\frac{m}{2} (1 - x_3), \hspace{1cm} (72)$$

$$A_+ = -\frac{m x_+}{2(1 + x_3)} + \frac{m x_+}{2}, \hspace{1cm} (73)$$

$$A_- = -\frac{m x_-}{2(1 + x_3)} + \frac{m x_-}{2}, \hspace{1cm} (74)$$

in the large $N$ limit (recall that $R = 1$ throughout). It is easy to check that $A_i$ satisfy the constraint (51). This looks almost but not quite right; however, recall that we must
use the identification (55) to find the corresponding classical gauge field $A_{i}^{(cl)}$. This comes indeed out as

$$\vec{A}^{(cl)} = \vec{r} \times \vec{A} = \frac{1}{2} \begin{pmatrix} x_2 & -x_1 \\ -x_1 & 0 \end{pmatrix}$$

(75)
or $A_{i}^{(cl)} = \frac{m}{2} \begin{pmatrix} 1 + x_3 \\ ydx - xdy \end{pmatrix}$, which is precisely the (tangential) gauge field of a Dirac monopole of charge $m$ on the sphere. The field strength was already calculated in (63), and is constant with the correct quantization

$$\int F \equiv \int F_{i}x^{i} = 4\pi \frac{m}{2}$$

(76)

Notice that $F$ is constant in spite of (or rather because of) the “non-classical” term $[A_{i}, A_{j}]$ in the definition of $F$. The same calculation applies for $m < 0$, hence we get both negative and positive monopole charge as it should be. The singularity at the south pole can of course be moved around using suitable $SU(M)$ gauge transformations. At finite $N$ resp. $M$, this configuration should therefore be interpreted as a fuzzy monopole.

This point of view considering the monopole sectors as matrices of different size is quite compatible with the treatment of nontrivial topological sectors in [20], where sections in nontrivial bundles are represented by $N \times M$ matrices. Clearly our gauge fields $B_{i}$ of the appropriate size can act on these from the left resp. right, and one can define covariant derivatives in this way. This will be elaborated elsewhere.

**More careful embedding of monopole sectors.** We should address a somewhat unsatisfactory aspect of the above treatment of the monopole sectors: Formally we have been considering distinct matrices $C^{(M)}$ for different $M$, but more properly they should all be considered as block-matrices embedded

$$... \leftrightarrow \text{Mat}(N - 1) \leftrightarrow \text{Mat}(N) \leftrightarrow \text{Mat}(N + 1) \leftrightarrow ...$$

(77)
as in (66), cp. also [27]. Then there is a small problem with (66): the $\frac{1}{2}$ in the lower-right block in (66) must be there in order to satisfy the constraint $\rho = 0$. However the eigenvalues of this small block are far from $\pm \frac{N}{2}$. Therefore this type of block-matrix configuration would strictly speaking be highly suppressed by the action, because the Dirac-string contributes $o(N^3)$ to the action. One way to cure this problem is to replace the action by

$$S' = \text{Tr}(\frac{C - \frac{1}{2}}{N^2} V(C))$$

(78)

which now has $\pm \frac{N}{2}, \frac{1}{2}$ as degenerate minima, and all fluctuations are Gaussian as in (18). Here $C \in \text{Mat}(M)$ should be a matrix of fixed size $M$ which is large enough to accommodate all relevant solutions, i.e. $N \ll M \ll 2N$. Now the block-matrix (66) is really a solution of the equation of motion, with action $\frac{m^2}{2g^2}$. In terms of the $B_{i}$ fields,
we note that \((C - \frac{1}{2})^2 = B_i B^i + i \varepsilon^{ikl} B_k B^l \sigma^k\), so that the action differs from (34) by terms of the form \(\text{Tr}((\frac{\lambda_i}{N} + \frac{1}{2} i \varepsilon^i \sigma^j) V(C))\) which are suppressed for large \(N\). The crucial difference is that the Dirac-string now has vanishing action. Therefore all monopole configurations can be obtained as distinct solutions in one single configuration space \(\text{Mat}(M)\). This is conceptually very appealing, because it shows that the nontrivial topological sectors arise here automatically as different solutions for the same action. This is even simpler than in the classical case: there is no need to introduce nontrivial principal bundles, they just come out. However the calculation of the partition function below would be somewhat more complicated for the action (78). Since the path integral is the main focus of this paper, we shall not pursue this point of view here, and consider the monopole configuration as truly “distinct” sectors for simplicity, as classically.

### 4 Nonabelian case: \(U(n)\) Yang-Mills theory

Now consider the same matrix model \(S = \text{Tr} V(C)\) as in (17), but for larger matrices of size \(2M \times 2M\) with

\[ M = nN - m, \quad \text{Tr}(C) = M. \tag{79} \]

The last constraint implies that the multiplicities of the (dominant) eigenvalue distributions of \(C\) are now \(n_+ - n_- = 2n\). We will see that this leads to a non-abelian \(U(n)\) Yang-Mills theory.

First we should find the ground state. For \(M = nN\), the absolute minima of the action \(S = \text{Tr} V\) are now given by any matrix \(C\) with \(n_+ = M + n\) eigenvalues \(\frac{N}{2}\) and \(n_- = M - n\) eigenvalues \(-\frac{N}{2}\). In a suitable basis, \(C\) takes the form

\[ C = \left( \frac{1}{2} + \lambda_i^{(N)} \sigma^i \right) \mathbf{1}_{n \times n}, \tag{80} \]

which is a block matrix consisting of \(k\) blocks of the solutions \((\frac{1}{2} + \lambda_i^{(N)} \sigma^i)\) of Section 3.3. The action is then zero, and clearly all other saddle points have a positive action. In general, we can write again any \(2M \times 2M\) matrix \(C\) in the form

\[ C = \left( \frac{1}{2} + \rho \right) + B_i \sigma^i \tag{81} \]

where the \(B_i\) and \(\rho\) are now \(M \times M\) matrices. In order to obtain a Yang-Mills gauge theory, we shall impose again the constraint

\[ \rho = 0, \tag{82} \]

so that the action (17) reduces to

\[ S = \frac{2}{g^2 N} \text{Tr} \left( \left( B_i B^i - \frac{N^2 - 1}{4} \right)^2 + (B_i + i \varepsilon_{ijk} B^j B^k)(B_i + i \varepsilon^{irs} B_r B_s) \right), \tag{83} \]
which has the same form as (34) but for different size \( M \) of the matrices. It is invariant under the gauge transformations \( B_i \to U^{-1} B_i U \) for \( U \in U(M) \). To understand its meaning, we write the fluctuations of \( B \) resp. \( C \) in the form

\[
B_i = B_{i,a} t^a = \lambda_i^{(N)} t^0 + A_i
\]  

(84)

where \( B_i \) and \( A_i \) carry a \( u(n) \) index,

\[
A_i = A_{i,0} t^0 + A_{i,a} t^a.
\]  

(85)

Here \( t_a \) denote the Gell-Mann matrices of \( su(n) \), which satisfy

\[
t_a t_b = \frac{1}{n} g_{ab} + \frac{1}{2} (d_{ab}^c + i f_{ab}^c) t_c
\]  

(86)

and \( t_0 = 1 \) is the \( n \times n \) unit matrix. The rest of the analysis of Section 3.4 goes essentially through. In particular, we can split the gauge fields again into tangential and radial components

\[
A_i = \frac{4\lambda_i}{N^2 - 1} \varphi + A_i.
\]  

(87)

We suppress here the \( u(n) \) labels of \( \varphi \) and \( A_i \), which are defined by

\[
\varphi := \lambda^i A_i, \quad \lambda^i A_i = 0.
\]  

(88)

Then (45) implies as before that all components of \( \varphi \) decouple and can be integrated out. It remains

\[
B_i = \lambda_i + A_i = \lambda_i + A_{i,0} t^0 + A_{i,a} t^a
\]  

(89)

involving only the tangential components of \( A_i \), and in the large \( N \) limit we obtain a theory with action

\[
S = \operatorname{Tr} V = \frac{1}{g^2} \int F_{mn} F^{mn}
\]  

(90)

where again

\[
B^2 + i \varepsilon^{ikl} B_k B_l = \frac{1}{2} \varepsilon^{ikl} F_{kl}, \quad F_{kl} = i [\lambda_k, A_l] - i [\lambda_l, A_k] + i [A_k, A_l] + \varepsilon_{k0m} A^m
\]  

(91)

is tangential

\[
x^k F_{kl} = o(1/N).
\]  

(92)

Spelling out the \( u(n) \) structure explicitly and omitting terms which vanish for large \( N \), this action becomes

\[
S = \operatorname{Tr} V = \frac{1}{g^2} \int (F_{mn,0} F^{mn,0} + F_{mn,a} F^{mn,a}),
\]  

(93)

where

\[
F_{kl,0} = i J_k A_{l,0} - i J_l A_{k,0} + \varepsilon_{k0m} A_{m,0},
\]

\[
F_{kl,a} = i J_k A_{l,a} - i J_l A_{k,a} + i A_{k,b} A_{l,c} f_{a}^{bc} + \varepsilon_{k0m} A_{m,a}.
\]  

(94)

This is the action of a \( U(n) \) Yang-Mills theory on the sphere. Recall that the only difference to the abelian case in Section 3.4 is the size \( M \approx nN \) of the matrices.
Saddle points. The remaining saddle-points can now be found from the equation of motion \((56)\) as in the previous section. Clearly any (reducible, in general) representation of \(su(2)\) with a suitable normalization as in \((62)\) will give a solution. Therefore the (dominant) saddle-points are given by the block-matrices

\[
C^{(m_1,\ldots,m_n)} = \begin{pmatrix}
C^{(M_1)} & 0 & \cdots & 0 \\
0 & C^{(M_2)} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & C^{(M_n)}
\end{pmatrix},
\]

(95)

where each block has the form \((62)\) with size

\[
M_i = N - m_i, \quad m_i \in \mathbb{Z},
\]

(96)

such that \(m_1 + \ldots + m_n = m\). Their action is

\[
S(C^{(m_1,\ldots,m_n)}) = \frac{1}{2g^2} \sum_i m_i^2
\]

(97)

for large \(N\), hence configurations with large \(|m_i|\) are suppressed.

We can now write the saddle-points \((95)\) as fluctuations around the ground state, as in Section 3.5. After arranging the blocks appropriately (by a gauge transformation), they take the form

\[
A_i = \begin{pmatrix}
m_1A_i & 0 & \cdots & 0 \\
0 & m_2A_i & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & m_nA_i
\end{pmatrix}
\]

(98)

(for large \(N\)) with \(m_1 + \ldots + m_n = m\), where \(A_i\) is the basic abelian monopole field found in Section 3.5. Hence the sectors with \(m_i \neq 0\) correspond to nontrivial \(U(n)\) gauge field configurations, which are precisely the “instantons”\(^4\) of the \(U(n)\) YM theory found in [28,29].

5 The path integral

The quantization of gauge theory on the 2-sphere has been studied extensively, using a variety of methods including lattice formulations and a generalization of the Duistermaat-Heckmann localization theorem, see e.g. [9,10,28,29,30,31,32,33]. In particular, the partition function and correlation functions of Wilson loops have been

\(^4\)we refer to any critical point of the YM action as instanton, as is customary in the related literature. In general they are unstable.
calculated. It is therefore natural to ask whether such calculations can also be done on the fuzzy sphere. Some of the known methods, in particular the localization theorem, might well be applicable as shown in [11] for the case of a torus. However this method is rather indirect, and we want to calculate the path integral directly taking advantage of the above formulation as matrix model.

One of the nice features of the fuzzy sphere is the fact that all path integrals are finite, simply because there are only finitely many degrees of freedom. However, this does not necessarily make them easy to evaluate: e.g. for scalar fields, one is forced to resort to perturbation theory (see e.g. [34]), which is even more complicated than in the classical case.

The main advantage of our matrix formulation of gauge theory is that it allows to explicitly carry out the path integral. This provides a truly new approach to gauge theory, since this formulation is possible only in the noncommutative case. Without the constraint $\rho = 0$, the integration would even be “trivial” as in Section 3.1, but this does not describe a YM theory on the sphere. We will now show how this constraint can be handled using the known matrix model technology, and calculate the partition function directly by integrating over the gauge fields for large $N$. We will recover the known result [9,10] for the partition function of $U(n)$ YM theory on the sphere for $N \to \infty$. While the explicit calculation in Appendix B may seem a bit involved for our present application, one can hope that the idea will be useful in less “trivial” cases as well.

We want to quantize the gauge theory with action (83) by integrating over the $M \times M$ matrices $B_i$. This will be done by integrating over the $2M \times 2M$ matrices $C$ in the action (17), imposing the constraint (32). We will not attempt here to calculate the full generating functional for the gauge field, only the partition function

\[
Z = \int dB_i \exp(-S(B)) = \int dC \delta(C_0 - \frac{1}{2}) \exp(-\text{Tr}V(C)) = \int \int d\Lambda_i \Delta^2(\Lambda_i) \exp(-\text{Tr}V(\Lambda)) \int dU \delta((U^{-1}\Lambda U)_0 - \frac{1}{2})
\]

where $dU$ is the integral over $2M \times 2M$ unitary matrices, and $C = U^{-1}\Lambda U$. Here $\delta(C_0 - \frac{1}{2})$ is a product over $M^2$ delta functions, which can be calculated as follows: define

\[
J = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix} = K \sigma^0
\]

where $K$ is a $M \times M$ matrix. Then

\[
\delta((U^{-1}CU)_0 - \frac{1}{2}) = \int dK \exp(i\text{Tr}(U^{-1}(C - \frac{1}{2})UJ)).
\]
By gauge invariance, the r.h.s. depends on the eigenvalues $\Lambda_i$ of $C$ only. Hence

$$Z = \int dK \int d\Lambda_i \Delta^2(\Lambda_i) \exp(-\text{Tr} V(\Lambda)) \int dU \exp(i\text{Tr}(U^{-1} \Lambda U J - \frac{1}{2} J))$$

$$= \int dK \ Z[J] \ e^{-\frac{1}{2} \text{Tr} J} \ (102)$$

where

$$Z[J] := \int dC \ \exp(-\text{Tr} V(C) + i \text{Tr}(CJ)) \ (103)$$

depends only on the eigenvalues $J_i$ of $J$. Diagonalizing $K = V^{-1} k V$, we get

$$Z = \int d k_i \Delta^2(k) \int d\Lambda_i \Delta^2(\Lambda_i) \exp(-\text{Tr} V(\Lambda)) \int dU \exp(i\text{Tr}(U^{-1}(\Lambda - \frac{1}{2}) U J))$$

$$= \int dK \ Z[J] \ e^{i \text{Tr} k_i J_i} \ (104)$$

where $\int dV$ was absorbed in $\int dU$. The main step is now to carry out the integral over $\int dU$, which can be done using the Itzykson-Zuber-Harish-Chandra formula [35, 36],

$$\int dU \exp(i\text{Tr}(U^{-1}CUJ)) = \text{const} \ \frac{\det(e^{i\Lambda_i J_j})}{\Delta(\Lambda_i) \Delta(J_i)}.$$  \ (105)

This depends only on the eigenvalues of $J$ and $C$, with Vandermonde-determinants $\Delta(\Lambda_i)$ and $\Delta(J_i)$. Note that the Vandermonde-determinants are totally antisymmetric, and so is $\det(e^{i\Lambda_i J_j})$. Therefore this expression is manifestly symmetric in both $\Lambda$ and $J$.

In this step we have reduced the number of integrals from $M^2$ to $2M$. This means basically that the integral over fields on $S^2_N$ is reduced to the integral over functions in one variable. This is a huge step, just like in the usual matrix models. The constraint however forces us to evaluate in addition the integral over $k_i$, which is quite complicated due to the rapid oscillations in $\det(e^{i\Lambda_i J_j})$; recall that $\Lambda_i \approx \pm N/2$. Nevertheless, it is shown in Appendix B how the integrals over $\Lambda_i$ and $k_i$ can be evaluated for large $N$, with the result

$$Z_m = \sum_{m_1 + \ldots + m_n = m} \int_{-\infty}^{\infty} d\kappa_1 \ldots d\kappa_n \ \Delta^2(\kappa) \ e^{i\kappa_i m_i} \ \exp(-g^2/2 \ \sum \kappa_i^2).$$  \ (106)

for matrices of size $M = nN - m$ (omitting overall constants). This form of $Z$ was found in [31] for a $U(n)$ Yang-Mills theory on the ordinary 2-sphere, apart from the constraint $\sum m_i = m$ which will be removed soon. It can be rewritten in the “localized” form as a weighted sum of saddle-point contributions, as advocated by Witten [28]. This can be seen as follows:

$$Z_m = \sum_{m_1 + \ldots + m_n = m} \Delta^2(-i \frac{\partial}{\partial m_i}) \int d\kappa_i \ e^{i\kappa_i m_i} \ \exp(-g^2/2 \ \sum \kappa_i^2).$$

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\[ \propto \sum_{m_1 + \ldots + m_n = m} \Delta^2 \left( \frac{\partial}{\partial m_i} \right) \exp \left( -\frac{1}{2g^2} \sum m_i^2 \right) \]
\[ \propto \sum_{m_1 + \ldots + m_n = m} \Delta \left( \frac{\partial}{\partial m_i} \right) \Delta (m_i) \exp \left( -\frac{1}{2g^2} \sum m_i^2 \right) \]
\[ = \sum_{m_1 + \ldots + m_n = m} P(m_i, g) \exp \left( -\frac{1}{2g^2} \sum m_i^2 \right). \quad (107) \]

Here \( P(m_i, g) \) is a totally symmetric polynomial in the \( m_i \). The last exponential is precisely the action \( (97) \) for the saddle-point \((m_1, \ldots, m_n)\) as discussed in Section 4, which is weighted by the polynomial \( P(m_i, g) \) (e.g. for \( n = 2 \), one finds \( P(m_i, g) = (m_1 - m_2)^2 - 2g^2 \)). This shows that the “localization” \( (28) \) also holds in the noncommutative case for gauge theory on the fuzzy sphere, at least in the large \( N \) limit. However we did not use any localization theorem here, it comes out by an explicit computation of the purely bosonic path integral, without having to introduce auxiliary fermionic fields as in \( (28) \). Note in particular that we did not do any gauge-fixing, which is not necessary here because the volume of the gauge group \( U(M) \) is finite.

To include all monopole configurations, we should sum over matrices of different sizes \( M = nN - m \) as explained in Section 4, keeping \( V(C) \) constant. Then \( m \) is the \( U(1) \) monopole charge. Hence the full partition function is obtained by summing\(^5\) over all \( Z_m \),

\[ Z = \sum_m Z_m = \sum_{m_1, \ldots, m_n = -\infty}^{\infty} \int d\kappa_i \Delta^2(\kappa) e^{i\kappa_i m_i} \exp \left( -\frac{g^2}{2} \sum \kappa_i^2 \right). \quad (108) \]

One can now perform a Poisson-resummation as in \( (31) \),

\[ \sum_m f(m) = \sum_p \tilde{f}(2\pi p) \quad (109) \]

where \( \tilde{f}(p) = \int \frac{dx}{2\pi} f(x)e^{-ipx} \). This gives

\[ Z = \sum_{p_1, \ldots, p_n \in \mathbb{Z}} \Delta^2(p) \exp \left( -2\pi^2 g^2 \sum p_i^2 \right). \quad (110) \]

This is the partition function of a \( U(n) \) Yang-Mills theory on the ordinary 2-sphere. As shown in \( (31) \), this is equivalent to the form found in \( (9, 10) \)

\[ Z = \sum_{R} (d_R)^2 \exp \left( -4\pi^2 g^2 C_{2R} \right), \quad (111) \]

where the sum is over all representations of \( U(n) \) and \( d_R \) is the dimension of the representation and \( C_{2R} \) the quadratic casimir. Hence the limit \( N \to \infty \) of the partition function

\(^5\)the relative weights of \( Z_m \) for different \( m \) is strictly speaking not determined here. However, it could be calculated using the embedding \( (27) \) as explained in Section 3.5.
for $U(n)$ YM on the fuzzy sphere is well-defined, and reproduces the result for YM on
the classical sphere. This strongly suggests that the same holds for the full YM theory
on the fuzzy sphere, and that there is nothing like UV/IR mixing for pure gauge theory
on $S^2_N$. This is unlike the case of a scalar field, which exhibits a “non-commutative
anomaly” \cite{Rajeev} related to UV/IR mixing.

6 Remarks on symmetries and correlation functions.

Let us try to understand the symmetries of our model in more detail. Recall that
2-dimensional Yang-Mills theory is invariant under area-preserving diffeomorphisms
(APD’s), because the field strength $F = F_{ij}dx^i dx^j = f \omega$ has only one component,
and $\int F \ast F = \int f^2 \omega$ is invariant under APD’s. Here $\omega$ denotes the volume form.

It is easy to understand the quantization of APD’s on the fuzzy sphere. The fuzzy
sphere arises as quantization of the Poisson structure $\{x_i, x_j\} = \epsilon_{ijk} x_k$, which corre-
sponds to the (canonical) symplectic form $\omega = x^i dx^j dx^k \epsilon_{ijk}$ on $S^2$ which coincides with
the volume form. Hence any function $f(x)$ on the sphere defines a Hamiltonian flow,
which preserves $\omega$. It is therefore an area-preserving diffeomorphism. Explicitly, the
vector field $X_f$ generating the APD with “hamiltonian” $f$ is determined by

$$\mathcal{L}_{X_f} g = \{ g, f \}$$

(112)

where $\{,\}$ are the Poisson brackets. Hence the derivation

$$\mathcal{L}_{X_f} (x_i) = \{ x_i, f \}$$

(113)

is an infinitesimal APD determined by $f$ acting on the coordinate function $x_i$. After
quantization, this is replaced by the commutator

$$\delta_f (x_i) = [ x_i, f ]$$

(114)

which can therefore be interpreted as quantized infinitesimal APD on the fuzzy sphere.
In this way, the APD’s on the fuzzy sphere can be identified with $SU(N)$.

Now consider “abelian” gauge theory on $S^2_N$ as discussed in Section \cite{NCG}. It is well
known (see e.g. \cite{Stern}) that gauge transformations on noncommutative spaces are closely
related to certain diffeomorphism groups. Indeed according to the above discussion, the
gauge transformations

$$B_i \rightarrow U^{-1} B_i U$$

(115)

with $U = \exp (i f(x)) \in SU(N)$ can be interpreted as APD acting on the $B_i$, viewed as
3 scalar fields. This statement has no classical analog, and has nothing to do with the
action of APD’s on classical gauge fields $A_i$. The induced transformation

$$F_{ij} \rightarrow U^{-1} F_{ij} U$$

(116)

I want to thank S. Rajeev for explaining this to me
of the field strength becomes trivial in the commutative limit, since then $F_{ij}$ commutes with all functions. For finite $N$ however, (116) is nontrivial even in the “abelian” case at hand, and can be interpreted as action of the APD determined by $U = e^{i f(x)}$ on each component of $F_{ij}$ (which is again different from the classical action of APD’s on a tensor!). Hence gauge transformations are necessarily “non-local”, and cannot be disentangled from general coordinate transformations on $S^2_N$. This phenomenon is quite common on noncommutative spaces: e.g. on the canonical quantum plane $R^d_\theta$, the translations are inner and hence part of the gauge transformations of $F_{ij}$. One could interpret this from a fiber-bundle point of view by saying that base space and fiber become unified in some sense.

Apart from this $SU(N)$ group of gauge transformations (or APD’s), the constraint $C_0 = \frac{1}{2}$ is also preserved by the $SU(2)$ group $\exp(i \alpha_i \sigma^i)$ which acts on the indices of the $B_i$ only. The overall symmetry group is therefore $SU(N) \times SU(2)$. Note that the “physical rotations” $\exp(i \alpha_i (\lambda^i + \frac{1}{2} \sigma^i))$ are combinations of these $SU(2)$ rotations and gauge transformations.

Let us extract some information about the correlation functions using these symmetries, without trying to calculate them explicitly. We only consider the abelian case for simplicity. The correlation functions are defined by

$$\langle C_1...C_m \rangle = \frac{1}{Z} \int dC C_1...C_m \delta(C_0 - \frac{1}{2}) \exp(-\text{Tr} V(C))$$

where the indices $1,...,m$ indicate $2N \times 2N$ matrix labels, or in terms of the components

$$\langle (B_i)_1...(B_j)_m \rangle = \frac{1}{Z} \int dB (B_i)_1...(B_j)_m \exp(-\text{Tr} V(B)).$$

They are highly constrained by the $SU(N) \times SU(2)$ symmetry. For example,

$$\langle B_i \rangle = 0$$

using $SU(2)$ invariance. This might seem strange, because we were using an expansion $B_i = \lambda_i + A_i$ with $A_i$ being a “small” fluctuation. However, this is not a contradiction: we do not assume any spontaneous symmetry breaking (or gauge fixing), and the solution $B_i = \lambda_i$ is only one possible gauge choice. To get nontrivial results, we should of course consider quantities which are gauge invariant or contain gauge invariant information. Consider for example

$$\langle F_i(x) F_j(y) \rangle = c \delta_{ij} g_{ab} \nu^a \otimes \nu^b$$

using the $SU(N) \times SU(2)$ symmetry, where $\nu^a$ denotes the $SU(N)$ Gell-Mann matrices. Here $x$ and $y$ stand for the first respectively second tensor slot, interpreted as functions on $S^2_N$. Since $g_{ab} \nu^a \otimes \nu^b$ is the reproducing kernel, it should be interpreted as delta-function $\delta(x,y)$ on the sphere, and we can write

$$\langle F_i(x) F_j(y) \rangle = c \delta_{ij} \delta(x,y).$$
This makes sense: there are no propagating modes, therefore there is no correlation
between fields at different points. The normalization can be calculated e.g. for coinciding
points, \( \int \langle F_i(x) F_j(y) \rangle \) which is certainly nonzero but finite. Similarly, it follows that
\[
\langle F_i(x) F_j(y) F_k(z) \rangle = \varepsilon_{ijk} f_{abc} \nu^a \otimes \nu^b \otimes \nu^c
\]
where \( f_{abc} \) is the structure constant of \( SU(N) \), since the r.h.s. is the only invariant
tensor (\( d_{abc} \) cannot occur since the l.h.s is totally symmetric). Hence it is proportional to
the function
\[
f(x, y, z) := f_{abc} \nu^a \otimes \nu^b \otimes \nu^c
\]
on \( S_N^2 \), which in the classical limit is the unique function \( f(x, y, z) \) on the sphere which
is invariant under APD’s and vanishes for coinciding points.

In general, one would expect that all correlation functions of the field strength have
a well-defined classical limit in this model, just like the partition function. In principle
it should be possible to calculate them explicitly in terms of integrals over eigenvalues,
and we expect no problem related to UV/IR mixing or renormalization. It would also
be interesting to know whether it is possible to relate them to integrable models. These
issues are left for further investigations.

7 The \( q \)-deformed fuzzy sphere revisited

Remarkably, we can repeat the same construction with a slightly different potential, and
obtain a gauge theory on the \( q \)-deformed fuzzy sphere \([21]\). Consider
\[
C^{(q)} = \frac{1}{2} + \lambda_i^{(q)} \sigma_i^{(q)}
\]
where \( \lambda_i^{(q)}, i = 1, 2, 3 \) are \( N \times N \) hermitian matrices and \( \sigma_i^{(q)} \) are the \( q \)-deformed sigma
matrices, both of which are defined to be the Clebsches of the corresponding irreps
of \( U_q(su(2)) \). To simplify the notation, we shall sometimes omit the label \( (q) \), which
is understood throughout this section. We also need the \( q \)-deformed invariant tensors
\( \varepsilon^{ij}_k = (\varepsilon^{(q)})^{ij}_k \) and \( g_{ij} = g^{(q)}_{ij} \) which can be found e.g. in \([21]\). The generators satisfy
\[
\varepsilon^{ij}_k \lambda_i \lambda_j = i \lambda_k, \quad \lambda_i \lambda_j g^{ij} = \frac{[N - 1]_q[N + 1]_q}{[2]_q^2}
\]
where \([n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}\), and
\[
\varepsilon^{ij}_k \sigma^i \sigma^j = i[2]_q \sigma^k, \quad \sigma^i \sigma^j g_{ij} = [3]_q
\]
which implies
\[
\sigma^i \sigma^j = i \varepsilon^{ij}_k \sigma^k + g^{ij}
\]
It follows that
\[
(\lambda_i^{(q)} \sigma_i^{(q)})^2 = -(\lambda_i^{(q)} \sigma_i^{(q)}) + \frac{[N - 1]_q [N + 1]_q}{[2]_q^2}.
\]  
(128)
This means that \( C^{(q)} \) has eigenvalues \( \pm \sqrt{\frac{[N - 1]_q [N + 1]_q}{[2]_q^2} + \frac{1}{4}} \) with multiplicities \( N \pm 1 \). It is therefore a minimum of the matrix potential
\[
V_q(C) = \frac{1}{g^2 N} \left( C^2 - \left( \frac{[N - 1]_q [N + 1]_q}{[2]_q^2} + \frac{1}{4} \right)^2 \right).
\]  
(129)
Expanding now a general matrix \( C \) as
\[
C = \frac{1}{2} + B_i \sigma_i^j, \quad B_i = \lambda_i + A_i
\]  
(130)
imposing again the constraint \( C_0 = \frac{1}{2} \), one obtains
\[
C^2 - \left( \frac{[N - 1]_q [N + 1]_q}{[2]_q^2} + \frac{1}{4} \right) = (B_i B_j g^{ij} - \frac{[N - 1]_q [N + 1]_q}{[2]_q^2}) + \sigma^i F_i
\]  
(131)
where
\[
F_i = B_i + i \varepsilon^{kl} B_k B_l, \quad F_{kl} := i[\lambda_k, A_l] - i[\lambda_l, A_k] + i[A_k, A_l] + \varepsilon_{klm} A^m
\]  
(132)
is indeed the appropriate \( q \)-field strength as used in [23]. Hence we naturally recover gauge models on the \( q \)-deformed fuzzy sphere. However it is not clear how to define the trace: If we take the classical trace
\[
S = \text{Tr} \, V_q(C),
\]  
(133)
a strange term of the form \( \text{Tr}((B_i B^i) F_3) \) appears, because \( \sigma^3_i = \left( \begin{array}{cc} q & 0 \\ 0 & -q^{-1} \end{array} \right) \) has a non-vanishing trace:
\[
\text{tr}(\sigma_i^{(q)} \sigma_j^{(q)}) = 2 g^{ij}_q + i(q - q^{-1}) \varepsilon^{ij}_3.
\]  
(134)
Hence it seems more natural to take the quantum trace over the \( \sigma^i \) space, which has the property that \( \text{tr}_q(\sigma_i^{(q)}) = 0 \), so that \( \text{tr}_q(\sigma_i^{(q)} \sigma_j^{(q)}) = [2]_q g^{ij}_q \). Then taking the classical trace over the \( N \times N \) matrices would lead to the action
\[
S = \frac{2}{g^2 N} \text{Tr} \left( (B_i B^i - \frac{N^2 - 1}{4})^2 + (B_i + i \varepsilon_{ijk} B^i B^j B^k)(B^i + i \varepsilon^{irs} B_i B_s) \right).
\]  
(135)
where \( q \)-deformed tensors are understood. This is again invariant under \( SU(N) \) (hence solvable), but the “physical” rotations defined similar as in Section 6 are violated. On the other hand, taking the quantum trace over the full \( 2N \times 2N \) matrices would break the gauge invariance but make the model formally \( U_q(su(2)) \) invariant as in [23]. This may shed some light on the issues raised in the application of \( q \)-deformed gauge theories on \( D \)-branes, see [37].
8 Discussion and outlook

We presented a new formulation of pure gauge theory on the fuzzy sphere, which allows to carry out the path integral explicitly. The partition function for $U(n)$ Yang-Mills theory on $S^2_N$ is calculated in the large $N$ limit, and the known result $^{[9,10]}$ for the classical sphere is recovered for large $N$.

There are several messages that should be stressed. First and foremost, gauge theories on noncommutative spaces are accessible to new methods and tools which could not be applied on commutative space. Of course, Yang-Mills theory on $S^2$ is a rather simple field theory with no propagating degrees of freedom; however it does have nontrivial monopole sectors, and its quantization is not entirely trivial. If the methods presented here can be generalized, noncommutative gauge theory may become a useful alternative to lattice gauge theory, even from an analytical point of view. Of course it should also be interesting from a numerical point of view.

Another important message is that the “classical limit” of pure gauge theory on the fuzzy sphere is smooth, at least for the partition function. This is not obvious in view of the UV/IR mixing effects in noncommutative field theories. It would be very interesting to know if this generalizes to higher dimensions. An interesting such space is fuzzy $\mathbb{C}P^2$ $^{[39]}$, which is the subject of current investigations.

Furthermore, this new formulation of gauge theory on the fuzzy sphere is arguably simpler than on the classical sphere. It is defined by a potential for a hermitian matrix plus a constraint. This leads not only to the correct kinetic terms, but also all the monopole sectors arise “automatically” in a very simple way, with the correct classical limit. In particular, there is no need to introduce nontrivial fiber bundles, connections and other mathematical structures in our approach. What is missing so far is the inclusion of fermions in this formalism; this will be discussed elsewhere.

There are several other aspects which require further work. First, one should be able to calculate the correlation functions or other suitable observables for pure gauge theory. Perhaps there are some connections with integrable models. One may also try to extend this approach to other gauge groups. Of course it would be very desirable to simplify the calculation in Appendix B, and to systematically calculate the corrections for finite $N$. Furthermore, there exist other interesting gauge models with similar action but without the term $(B_iB^i - N^2/4)^2$, as discussed in $^{[18]}$. They do not fit very well into the formalism presented here. However then the “radial” field $\varphi$ $^{[43]}$ becomes dynamical, and these models describe branes on $SU(2)$ near the origin rather than a Yang-Mills gauge theory on a sphere. All these questions certainly deserve further study.
Acknowledgements

First and foremost I want to thank Chong-Sun Chu for many discussions, collaboration on several aspects of this paper and an invitation to Durham; in particular some of the results in Section 3 were obtained together with him. I also wish to thank H. Grosse for his interest, encouragement and extensive discussions as well as initiations to the ESI in Vienna. Furthermore I am indebted to J. Wess, P. Aschieri, B. Jurco and P. Schupp for drawing my attention to noncommutative gauge theories, and I enjoyed useful discussions with E. Langmann, J. Madore, S. Rajeev, and P. Presnajder.

Appendix A. Some useful formulae

The irreducible \( N \)-dimensional representation of the \( su(2) \) algebra \( \lambda_i \) (6) is given by

\[
(\lambda_3)_{kl} = \delta_{kl} \frac{N + 1 - 2k}{2},
\]

\[
(\lambda_+)_{}_{kl} = \delta_{k+1,l} \sqrt{(N - k)k},
\]

where \( k, l = 1, \ldots, N \) and \( \lambda_\pm = \lambda_1 \pm i\lambda_2 \).

Furthermore, recall that

\[
\sigma^i \sigma^j = \delta^{ij} + i\varepsilon^{ijk} \sigma^k.
\]

Together with (6), this implies the following crucial property of the matrix \( \lambda_i \sigma^i \):

\[
(\lambda_i \sigma^i)^2 = \frac{N^2 - 1}{4} - (\lambda_i \sigma^i),
\]

which means that the eigenvalues (in Mat\( (2N) \)) of the matrix \( \lambda_i \sigma^i \) are \( \frac{1 \pm N}{2} \). To get the multiplicities, we note that \( \lambda_i \sigma^i \) is an intertwiner of \( (2) \otimes (N) = (N - 1) \oplus (N + 1) \) (i.e. it is invariant under \( SU(2) \)), hence the multiplicities are \( N + 1 \) resp. \( N - 1 \).

Appendix B: Evaluation of the partition function.

Consider the expression

\[
\frac{\det(e^{i\lambda_i J_i})}{\Delta(J_i)}
\]

in (105). At first sight it may appear ill-defined, because the denominator is singular due to the form (100) of \( J \). However, the fraction is analytic in \( J \), because all poles are
canceled by zeros in the determinant (it must be so, because the lhs of (105) is clearly analytic). To see this explicitly, assume the following “regularization”

$$\mathcal{J} = \begin{pmatrix} K + \epsilon I & 0 \\ 0 & K \end{pmatrix}$$

(141)

for some infinitesimal constant $\epsilon$, and consider $\Delta(\mathcal{J}_j)$ in more detail. By treating the contributions from the 2 blocks as above separately, it is easy to see that

$$\Delta(\mathcal{J}_j) = \Delta^4(k_i) e^M.$$  

(142)

where $k_i$ are the eigenvalues of $K$. It is useful to order the $\Lambda$ as

$$\mathbf{\Lambda} = (\Lambda_1^+, \Lambda_2^+, \ldots, \Lambda_n^+), \Lambda_1^-, \Lambda_2^-, \ldots, \Lambda_n^-)$$

where $\Lambda_i^+ \approx \pm \frac{N}{2}$. We then have to evaluate the determinant

$$\det(e^{i\Lambda_i \mathcal{J}_j}) = \det \begin{pmatrix} e^{i\Lambda_i^+(k_j + \epsilon)} & e^{i\Lambda_i^+ k_j} \\ e^{i\Lambda_i^-(k_j + \epsilon)} & e^{i\Lambda_i^- k_j} \end{pmatrix},$$

(143)

which clearly contains a factor $e^M$ due to the degeneracy in the $\{k_i\}$. To proceed, we expand it by choosing $M + n$ resp. $M - n$ columns in the upper resp. lower block of (143) as follows

$$\det(e^{i\Lambda_i \mathcal{J}_j}) = \sum_{\{\mathcal{J}^+\}, \{\mathcal{J}^–\}} (-1)^{\sigma(\{\mathcal{J}^+\}, \{\mathcal{J}^–\})} \det(e^{i\mu^+_{\mathcal{J}^+} \mathcal{J}^+}) \det(e^{i\mu^-_{\mathcal{J}^–} \mathcal{J}^–})$$

(144)

where $\{\mathcal{J}^+\}, \{\mathcal{J}^–\} \subset \{k_i + \epsilon, k_i\}$ are complementary subsets with $|\mathcal{J}^+| = M + n$, $|\mathcal{J}^–| = M - n$, and the sign is given by this choice of subsets. In the terms $\det(e^{i\mu^+_{\mathcal{J}^+} \mathcal{J}^+})$, the $\mathcal{J}^+$ are assumed to be ordered as in $(k_1 + \epsilon, ..., k_M + \epsilon, k_1, ..., k_M)$, so that only the choice of the subsets $\{\mathcal{J}^\pm\}$ matters for the sign. In terms of the fluctuations

$$\mu_j^- = (\Lambda_j^- + \frac{N}{2}), \quad j = 1, 2, ..., M - n,$$

$$\mu_j^+ = (\Lambda_j^+ - \frac{N}{2}), \quad j = 1, 2, ..., M + n$$

(145)

this becomes

$$\det(e^{i\Lambda_i \mathcal{J}_j}) = \sum_{\{\mathcal{J}^+\}, \{\mathcal{J}^–\}} (-1)^{\sigma(\{\mathcal{J}^+\}, \{\mathcal{J}^–\})} \det(e^{i\mu^+_{\mathcal{J}^+} \mathcal{J}^+}) \det(e^{i\mu^-_{\mathcal{J}^–} \mathcal{J}^–}) e^{\frac{N}{2}(\sum \mathcal{J}^+ - \sum \mathcal{J}^–)}.$$

(146)

Now the rapidly oscillating terms have been isolated in the last exponential, and it turns out that the correct expansion is in the number of these rapidly oscillating variables $k_i$ in $e^{i\frac{N}{2} \mathcal{J}}$. This depends on the choice of $\{\mathcal{J}^+\}$ (which of course fixes $\{\mathcal{J}^–\}$): Let $\{\kappa^+\}$ be the set of $k_i$’s which occur twice in $\{\mathcal{J}^+\}$ (for $\epsilon = 0$). Because $|\mathcal{J}^+| = M + n$, there are at least $n$ such $\kappa^+$’s. Assume that $|\{\kappa^+\}| = n + d$: then there must be in addition $d$ elements $\kappa^-_j$ among the $\{k_i\}$ which occur twice in $\{\mathcal{J}^–\}$. The last term in (146) is then
We have hence recovered most of the $\epsilon$ factors, which means that there are $n + 2d$ rapidly oscillating variables among the $k_i$. We can therefore expect that $d = 0$ will give the dominating contribution, and concentrate on this case first. Then $\{J^+\} = \{J^-\} \cup 2\{\kappa^+\}$ for $\epsilon = 0$, and

$$
\det(e^{i\Lambda_i J_i})_{d=0} = \sum_{\{J^+\},\{J^-\} : d=0} (-1)^{\sigma(\{J^+\},\{J^-\})} \det(e^{i\mu^+_i J^+_i}) \det(e^{i\mu^-_i J^-_i}) e^{iN(\Sigma \kappa^+)}.
$$

(147)

We will omit the superscript of $\kappa$ from now on. Consider $\{J^+\}$ in more detail, for fixed $\{\kappa\}$. It has the form $\{\kappa\} \cup \{\kappa + \epsilon\} \cup \{J^- \pm \epsilon\}$, and each different sign choice in $\{J^- \pm \epsilon\}$ gives a different contribution to the sum which must be added. Different choices are related by an exchange of the elements $k_i$, $k_i + \epsilon \notin \{\kappa\}$ in $J^\pm$. This amounts to an exchange of the corresponding columns in $\{J^+\}$, hence the two contributions come with a relative $-$ sign. Among these, there is one “ordered” choice $(J^+)_0 = (k_1 + \epsilon, ..., k_M + \epsilon, \kappa_1, ..., \kappa_n)$, $(J^-)_0 = (k'_1, \kappa_1, ..., \kappa_n)$, which depends only on $\{\kappa\}$ and serves to fix the sign $(-1)^{\sigma(\kappa)} := (-1)^{\sigma((J^+)_0, (J^-)_0)}$. The following notation is useful:

$$
T^\epsilon f(k_1, ..., k_M) = f(k_1, ..., k_i + \epsilon, ..., k_M)
$$

(148)

so that e.g.

$$
(J^+)_0 = \prod_{i=1}^M T^\epsilon_i (k_1, ..., k_M, \kappa_1, ..., \kappa_n) = \prod_i T^\epsilon_i J^+
$$

(149)

where $J^+ = (k_1, ..., k_M, \kappa_1, ..., \kappa_n)$, $J^- = (k'_1, ..., k'_{M-n})$ from now on. Summing over all these permutations for each set $\{\kappa\}$ leads to

$$
\det(e^{i\Lambda_i J_i})_{d=0} = \sum_{\{\kappa\}} (-1)^{\sigma(\kappa)} e^{iN(\Sigma \kappa)} \prod_{i \notin \{\kappa\}} \left((T^\epsilon_i)^+ - (T^\epsilon_i)^-\right) \det(e^{i\mu^+_i J^+_i}) \det(e^{i\mu^-_i J^-_i}).
$$

(150)

Here $(T^\epsilon_i)^\pm$ indicates that the $T^\epsilon_i$ operator acts only on $\det(e^{i\mu^+_i J^+_i})$. The sign $(-1)^{\sigma(\kappa)}$ now depends only on the choice of $\{\kappa\} \subset \{k_i\}$. Using

$$
(T^\epsilon_i)^+ - (T^\epsilon_i)^- = ((T^\epsilon_i)^+ - 1) - ((T^\epsilon_i)^- - 1) \to \epsilon \left((\partial_{k'}^+ - \partial_{k'}^-\right)
$$

(151)

for $\epsilon \to 0$, we get

$$
\det(e^{i\Lambda_i J_i})_{d=0} = \epsilon^{M-n} \sum_{\{\kappa\}} (-1)^{\sigma(\kappa)} e^{iN(\Sigma \kappa)} \prod_{\{k'\}} \left((\partial_{k'}^+ - \partial_{k'}^-\right) \det(e^{i\mu^+_i J^+_i}) \det(e^{i\mu^-_i J^-_i}).
$$

(152)

We have hence recovered most of the $\epsilon$ factors; the missing $\epsilon^n$ is contained in $\det(e^{i\mu^+_i J^+_i})$.

To proceed, we can now integrate over $\mu^\pm$ in $\{148\}$, noting that

$$
\Delta(A) = N^x \Delta(\mu^+)\Delta(\mu^-) \exp(n(\Sigma \mu^- - \Sigma \mu^+))
$$

(153)

29
up to corrections of order $1/N$. In fact we can even neglect the last exponential compared to the leading terms $\exp(-N g^2 \sum (\mu^2)^2)$ in the action, expanded as in (13). Furthermore,

$$\int d\mu^+ \Delta(\mu^+) \det(e^{i\mu^+_i J^+_j}) \exp(-N g^2 \sum (\mu^2)^2) = c \Delta(J^+) \exp(-g^2 4N \sum (J^+)^2) \quad (154)$$

for some constant $c$ by antisymmetry in $J^+$, and similarly for $\mu^-$. Noting also that

$$\Delta(J^+) \Delta(J^-) = (-1)^{\sigma(\kappa)} e^{\kappa \Delta(J^+)} \prod (k' - \kappa)^2 + \left( \Delta(k') \right)_+ \quad (155)$$

we get using (142)

$$Z_{d=0} = \int dk_i e^{-i \sum k_i} \frac{\Delta(k)^2}{\Delta(k)} \sum_{\kappa} \Delta^4(\kappa) \exp \left( iN \left( \sum \kappa - \frac{g^2}{2N} \sum \kappa^2 \right) \right) \prod \left( (\partial_{k'_i} + (\partial_{k'_i})^- \right) \prod (k' - \kappa)^2 \exp(-g^2 4N \sum k'^2) \right)_+ \left( \Delta(k') \exp(-g^2 4N \sum k'^2) \right)_-$$

always ignoring numerical constants. Notice that all signs have disappeared. Since

$$\left( \partial_{k_1} - \partial_{l_1} \right) (f(k_1) f(l_1)) |_{k_1 = l_1} = 0, \quad (156)$$

only the term $\partial_{k'_i} \prod (k' - \kappa)^2 = (\sum i \frac{2}{k_1 - \kappa}) \prod (k' - \kappa)^2$ survives the derivatives w.r.t. $k'_i$, and the term $(\sum i \frac{2}{k_1 - \kappa})$ can be moved outside of the remaining derivatives. Repeating this, we find

$$Z_{d=0} = \int dk_i e^{-i \sum k_i} \frac{\Delta(k)^2}{\Delta(k)} \sum_{\kappa} \Delta^2(\kappa) e^{iN(\kappa)} \exp(-g^2 2N \sum k^2) \left( \prod_{\{k'\}} \partial_{k'_i} \prod (k' - \kappa)^2 \right) \quad (157)$$

since $\Delta^2(k) = \Delta^2(\kappa) \Delta^2(k') \prod (k' - \kappa)^2$. Here

$$\sum \prod \frac{1}{k' - \kappa} = \sum \left( \prod \frac{1}{k' - \kappa_1} \right) \left( \prod \frac{1}{k' - \kappa_n} \right) \quad (158)$$

is the sum over all possible splittings of $\{k'\}$ into $n$ partitions $\{k'\}_{N_1}, ..., \{k'\}_{N_n}$ of size $N_1 + ... + N_n = M - n$, and $(\prod_{k'_i} \frac{1}{k' - \kappa_1}) \equiv (\prod_{\{k'\}_{N_1}} \frac{1}{k' - \kappa_1})$. That is, each $k'_i$ is “linked” to one $\kappa_j$ by a factor of the form $\frac{1}{k'-\kappa_j}$, and there are $N_j$ such $k'$ linked to $\kappa_j$. For fixed $(N_1, ..., N_n)$ each of these terms gives the same contribution, therefore we simply get a

\footnote{If we were more careful to include the factor $\exp(n(\Sigma \mu^- - \Sigma \mu^+))$, there would be additional contributions which are suppressed by $\frac{1}{N}$}
multiplicity factor $\frac{(M-n)!}{N_1!...N_n!}$. Furthermore each different choice of $\{\kappa\} \subset \{k\}$ gives the same integral since the form is identical. Therefore

$$Z_{d=0} = \int dk_i e^{-i\sum k_i+iN(\sum \kappa)} \Delta^2(\kappa)$$

$$\sum_{N_1+...+N_n=M-n} \frac{(M-n)!}{N_1!...N_n!} \prod_{k'\neq \kappa_1} \prod_{k'\neq \kappa_n} \exp\left(-\frac{g^2}{2N} \sum k'^2\right). \quad (159)$$

We have now taken just one fixed subset $\{\kappa\} \subset \{k\}$, since all choices give the same result.

The integral over the $k'_i$ has now the same structure for all $i$ and can be carried out. Notice that there are poles in (159), which seems a bit surprising because the original expression $\frac{\text{det}(e^{i\Lambda_i^j})}{\Delta(\Lambda_i)\Delta(J_j)}$ is perfectly regular. However, recall that this corresponds to the sum over all possible terms in (146), in particular over all choices of $\{\kappa\} \subset \{k\}$. In (159) these different contributions are included in the integral over all values $\int_{-\infty}^\infty dk_i$, contributing with the same $|k_i - \kappa_j|$ but opposite sign. Therefore the cancellations in the sum over $\{\kappa\}$ are now reflected in the cancellations in the contributions to the integral from both sides of the poles with the same $|k_i - \kappa_j|$. We should therefore use the principal value of the integral $\int dk_i$ in (159), which is perfectly regular and well-defined because there are only simple poles. To put it differently, we can restrict the range of integration to the space of $k_i$ with $|k_i - k_j| < \epsilon'$, say; this must give the correct result for $\epsilon' \to 0$. But this is just the definition of the principal value of the integral. In fact, notice that if there were no poles in the above formula, each integral $\int dk'_i$ would produce an exponential factor of order $e^{-N/g^2}$, and $Z$ would vanish for large $N$. The contributions from the poles will give the correct, finite result.

Using the identity

$$\int du \frac{1}{u} f(u) = \frac{1}{2} \left( \int du \frac{f(u) - f(-u)}{u} \right) \quad (160)$$

we get

$$\int dk' \frac{1}{k' - \kappa} \exp\left(-ik' - \frac{g^2}{2N} (k')^2 \right) =$$

$$= e^{-\kappa} \frac{2\kappa}{2\pi} \int du \left( -i \frac{\sin(u)}{u} - e^{iu} \frac{\sinh\left(\frac{g^2}{2N} \kappa u\right)}{u} \right) \exp\left(-\frac{g^2}{2N} u^2\right)$$

Now $\frac{1}{u} \sin(u) \approx \pi \delta(u)$, and $\frac{1}{u} \sinh\left(\frac{g^2}{2N} \kappa u\right) \approx 0$ under the integral, hence

$$\int dk' \frac{1}{k' - \kappa} \exp\left(-ik' - \frac{g^2}{2N} (k')^2 \right) \to -i\pi e^{-\kappa} \frac{2\kappa}{2\pi} \quad \text{for large } N. \quad (161)$$
We therefore obtain

\[ Z_{d=0} = \sum_{N_1 + \ldots + N_n = M-n} \frac{(M-n)!}{N_1! \ldots N_n!} \int d\kappa_i e^{i\kappa(N-(N_i+1))} \Delta^2(\kappa) \exp\left(-\frac{g^2}{2} \sum \kappa_i^2 \left(\frac{N_i + 1}{N}\right)\right). \] (162)

In terms of the integers \( m_i = N - (N_i + 1) \) which satisfy

\[ \sum m_i = nN - M = m, \] (163)

the combinatorial factor is

\[ \frac{(M-n)!}{N_1! \ldots N_n!} \approx \frac{(M-n)!}{((\frac{M}{n} - 1))!^n} \] (164)
as long as the \( m_i \) are small (which will be justified below), up to corrections of order \( \frac{1}{N} \). Hence we can drop this factor, and obtain

\[ Z_{d=0} = \sum_{m_1 + \ldots + m_n = m} \int d\kappa_i \Delta^2(\kappa) e^{i\kappa m_i} \exp\left(-\frac{g^2}{2} \sum \kappa_i^2 (1 - \frac{m_i}{N})\right). \] (165)

This is now a perfectly nice integral. The \( \frac{m_i}{N} \) in the exponential can be neglected, and (106) follows.

Several remarks are in order.

- If one would similarly treat the case of more oscillating factors \( d > 0 \) in (146), we would get similar formulas with \( \kappa \) replaced by \( n + 2d \) variables \( \kappa^+ \) and \( \kappa^- \), which come with oscillating terms \( e^{iN\kappa^\pm} \). The remaining analysis would be similar, with only \( N - n - 2d \) variables \( k' \) which contribute an integral as in (161). This leads to an expression similar as in (162), however there are now not enough phase factors \( e^{-i\kappa_i^\pm} \) to cancel the rapid oscillations in \( e^{iN\sum \kappa_i^\pm} \). Therefore this integral will have additional rapidly oscillating terms, which lead to an exponential suppression. Therefore the contributions \( Z_{d>0} \) can be neglected for large \( N \).

- The approximation replacing the combinatorial factor in (164) by a constant is justified, since only small \( |m_i| \) contribute to the final result (107).

- one can understand the above calculations, in particular the dominant contribution from the poles in (159) intuitively as follows: Each contribution from the poles with \( (N_1, \ldots, N_n) \) corresponds to a “clustering” \( \{k'\}_{N_i} \approx \kappa_i \), \( \ldots \), \( \{k'\}_{N_n} \approx \kappa_n \), which gives the dominant contribution to the integral over the \( k_i \). These clusters correspond precisely to the saddle-points discussed in Section 4 which are also clusters of \( N_i + 1 \) eigenvalues of \( C \). Hence the leading contribution comes form the saddle-points.
• This calculation could be generalized to the potential \((\text{78})\), ordering the eigen-\values as \(\vec{\Lambda} = (\Lambda_1^{(+)}, \Lambda_2^{(+)}, \ldots, \Lambda_n^{(+)}, \Lambda_1^{(0)}, \ldots, \Lambda_1^{(-)}, \Lambda_2^{(-)}, \ldots, \Lambda_n^{(-)})\) \nine where \(\Lambda_i^{(\pm)} \approx \pm \frac{N}{2}\) and \(\Lambda_i^{(0)} \approx \frac{1}{2}\). This would allow to explicitly calculate the relative weights of the different topological sectors in \((\text{108})\), which have been put by hand here.

• with some effort, it should also be possible to compute the leading correction terms in \(\frac{1}{N^2}\). The relevant approximations are those in \((\text{164})\) and \((\text{161})\), which can certainly be improved. The other approximations (taking only \(d = 0\), ignoring the exponential in in \((\text{153})\) and the higher terms in \((\text{18})\) apparently give corrections which are exponentially suppressed. An exact calculation is certainly desirable, but would require more sophisticated tools.

• needless to say, it would be nice to simplify this calculation.

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