Graceful labeling for open superstar of complete bipartite graphs $S^2(t \cdot K_{m,n})$

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Abstract. A graceful labeling of graph $G$ with $m$ edges is a one-to-one function $f$ from vertices of $G$ to the set $\{0, 1, 2, \ldots, m\}$ such that the induced function $f^*$ from edges of $G$ to the set $\{1, 2, 3, \ldots, m\}$, which is defined by the difference of its incident vertices labels, is a bijection. An open superstar of complete bipartite graphs is an integration of complete bipartite graphs and superstar graphs. A superstar graph $S^2_t$ is a graph obtained by replacing each edge of star $S_t$ by a path $P_3$ of three vertices. An open superstar of complete bipartite graph, $S^2(t \cdot K_{m,n})$, is a graph obtained by identifying each pendant vertex of superstar $S^2_t$ with exactly one vertex of a complete bipartite graph $K_{m,n}$. In this paper, we have given a graceful labeling for the open superstar of complete bipartite graphs $S^2(t \cdot K_{m,n})$ for all positive integers $m$, $n$, and odd positive integers $t$.

1. Introduction

A graph labeling is a mapping that carries a set of graph elements into a set of integers. There are various types of graph labeling, such as: graceful labeling, harmonious labeling, magic labeling, anti-magic labeling and irregular labeling. More result of graph labeling, can be found in Gallian [1].

The graceful labeling was introduced by Rosa [2] in 1967. A graceful labeling of graph $G$ with $m$ edges is a one-to-one function $f$ from the vertices of $G$ to the set $\{0, 1, 2, \ldots, m\}$ such that the induced function $f^*: E(G) \rightarrow \{1, 2, 3, \ldots, m\}$, which is defined by $f^*(uv) = |f(u) - f(v)|$, is a bijection. If graph $G$ can be labeled with graceful labeling, we say that $G$ is graceful. In 1979, Bermond [3] conjecture that family of lobsters graph is graceful. In A Dynamic Survey of Graph Labeling by Gallian [1], there are a lot of graphs with graceful labeling, such as: trees (if less than 36 vertices), caterpillars, cycles $C_n$ (iff $n \equiv 0, 3 \pmod 4$), complete graphs $K_n$ (iff $n \leq 4$) and complete bipartite graph $K_{m,n}$. Although numerous families of graceful graphs are known, a general necessary or sufficient condition to classify a graph is graceful has not yet been found.

A star $S_t$ is graceful since it is a special case of caterpillar. Kaneria et. al [4] introduced a open star of graphs and proved that open star of complete bipartite graphs is graceful. Inspired by Kaneria et. al [4], we generalized the star graphs into superstar graph. A superstar $S^2_t$ is graceful since it is a special case of lobster graph. Then, we proved that open superstar of complete bipartite graphs is graceful, for the odd number of copies of complete bipartite graph, by finding a graceful labeling for this family of graphs.

In this paper, we denote the set of vertices of $G$ by $V(G)$, the set of edges of $G$ by $E(G)$, and $uv$ is the incident edge of vertices $u$ and $v$. The notation $S_t$ denotes a star graph with $t$ edges and $K_{m,n}$ denotes a complete bipartite graph.
2. Open superstar of complete bipartite graphs

Definition 2.1. A superstar graph $S^2_t$ is a graph obtained by replacing each edge of star $S_t$ by a path $P_3$ of three vertices.

Illustration for Definition 2.1 is given in Figure 1 and Figure 2.

![Figure 1. A star $S_7$.](image1)

![Figure 2. A superstar $S^2_7$.](image2)

It can be seen from the figures that a star $S_t$ (Figure 1) is homeomorphic with a superstar $S^2_t$ (Figure 2). We can generalize the Definition 2.1 into the definition of superstar $S^n_t$. Therefore, analogue with definition $S^2_t$, $S^n_t$ is a graph obtained by replacing each edge of star $S_t$ by a path $P_{n+1}$ of $n+1$ vertices. However in this paper, we just discusses about $S^2_t$.

Definition 2.2. An open superstar of complete bipartite graph $S^2(t \cdot K_{m,n})$ is a graph obtained by identifying each pendant vertex of superstar $S^2_t$ with exactly one vertex of a complete bipartite graph $K_{m,n}$.

For more understanding about open superstar of complete bipartite graphs, Figure 3 and Figure 4 show the illustration about how to construct a open superstar $S^2(7 \cdot K_{3,2})$.

![Figure 3. A superstar $S^2_7$ and seven copies of $K_{3,2}$.](image3)

![Figure 4. An open superstar $S^2(7 \cdot K_{3,2})$.](image4)

Basically, this definition can be generalized into the definition of open superstar of some graphs, by replacing the complete bipartite graph with another graph.
3. Graceful labeling for open superstar of complete bipartite graphs

A graph is graceful if it can be labeled with graceful labeling. To find a graceful labeling for an open superstar of complete bipartite graph $S^2(t \cdot K_{m,n})$, firstly we tried to give a graceful labeling to the easier case, that is for the small positive integers $m$, $n$, and $t$. Then, we looking for a pattern of the labels and generalized it to all positive integers $m$ and $n$. In this process, we only found a graceful labeling for the odd positive integers $t$. This result is given in the following Theorem 3.1.

**Theorem 3.1.** An open superstar of complete bipartite graphs $S^2(t \cdot K_{m,n})$ is graceful for odd positive integers $t$ and all positive integers $m$, $n$.

**Proof.** Let $G$ be an open superstar of complete bipartite graph with $t$ copy of complete bipartite graphs $K_{m,n}$. Clearly, $G$ is a graph with $t(m+n+1)+1$ vertices and $t(mn+2)$ edges as shown in the following Figure 5.

Let $s$ be the number of edges of $G$. So, $s = t(mn+2)$. The labeling function $f$ on the vertices of $G$ is defined as follows:

$$
\begin{align*}
f(u_0) &= 0 \\
f(u_i) &= \begin{cases} s - \frac{1}{2}(i - 1), & i \equiv 1 \pmod{2} \\ \frac{1}{2}i, & i \equiv 0 \pmod{2} \end{cases} \\
f(u_{i,j}) &= \begin{cases} \frac{1}{2}(t + i) + (j - 1)t, & i \equiv 1 \pmod{2} \\ s - \frac{1}{2}(t + i - 1) - (j - 1)t, & i \equiv 0 \pmod{2} \end{cases} \text{ for } j = 1, 2, 3, \cdots, m \\
f(v_{i,j}) &= \begin{cases} s - \left( t + \frac{1}{2}(i - 1) - (j - 1)mt \right), & i \equiv 1 \pmod{2} \\ t + \frac{1}{2}i + (j - 1)mt, & i \equiv 0 \pmod{2} \end{cases} \text{ for } j = 1, 2, 3, \cdots, n
\end{align*}
$$

To prove that the function $f$ is a graceful labeling, it must be shown that $f$ is one-to-one and the induced function $f^*$ is bijective. First, we will show that $f$ is an injective function from $V(G)$, the set of all vertices of $G$, to the set $\{0, 1, 2, 3, \cdots, s\}$.
For \( i = 1, 2, 3, \ldots, t \), the labels of \( u_0, u_i, u_{i+1}, \) and \( v_{i+1} \) are the consecutive integers 0, 1, 2, \ldots, \( \frac{1}{2}(t-1) \) and \( s, s-1, s-2, \ldots, s - \frac{1}{2}(t-1) \). So, there are \( s - (t - \frac{1}{2}(t-1)) - t - \frac{1}{2}(t-1) - 1 \) integers between \( t - \frac{1}{2}(t-1) \) and \( s - (t - \frac{1}{2}(t-1)) \) which can be used as labels of \( t(m+n-2) \) remaining vertices. Consider that, \( s - (t - \frac{1}{2}(t-1)) - t - \frac{1}{2}(t-1) - 1 = t(mn+2) - 3t = t(mn-1) \geq (m+n-2) \). It indicates that \( f \) may be an injective function.

Now, it must be shown that each label for remained vertices is different. From the function \( f \), we can see that \( f(u_{ij}) \equiv f(u_{i+1}) \pmod{t} \) and \( f(v_{ij}) \equiv f(v_{i+1}) \pmod{t} \). Furthermore, for \( i \) is odd, \( f(u_{ij}) = \frac{1}{2}(t+1)(\mod{t}) \) and \( f(v_{ij}) = (t - \frac{1}{2}(i-1))(\mod{t}) \). For \( i \) is even, \( f(u_{ij}) \equiv \frac{1}{2}(t-i+1)(\mod{t}) \) and \( f(v_{ij}) \equiv \frac{1}{2}i(\mod{t}) \). Besides that, function \( f \) also has these following properties:

(i) for \( i \) is odd, \( f(u_{ij}) = f(u_{i+1}) + (j-1)t \) and \( f(v_{ij}) = f(v_{i+1}) - (j-1)mt \).

(ii) for \( i \) is even, \( f(u_{ij}) = f(u_{i+1}) - (j-1)t \) and \( f(v_{ij}) = f(v_{i+1}) + (j-1)mt \).

It is clear that for \( i \neq k \), \( f(u_{ij}) \) and \( f(u_{ik}) \) are in different classes of modulo \( t \) and it also stand for \( f(v_{ij}) \) and \( f(v_{ik}) \), but not for \( f(u_{ij}) \) and \( f(v_{ik}) \). For example, in case \( i = 2 \) and \( k = t - 1 \), we have

\[
f(u_{2j}) \equiv \frac{1}{2}(t-1)(\mod{t}) \equiv f(v_{t-1,l}).
\]

Since \( f(u_{2j+1}) > f(u_{2j+2}) \) for \( j_1 < j_2 \) and \( f(v_{t-1,l+1}) < f(v_{t-1,l+2}) \) for \( l_1 < l_2 \), we just need to compare the smallest value of \( f(u_{2j}) \) and the biggest value of \( f(v_{t-1,l}) \). The smallest value of \( f(u_{2j}) \) is \( f(u_{2m}) = mnt - mt + \frac{1}{2}(t-1) \). While the biggest value of \( f(v_{t-1,l}) \) is \( f(v_{t-1,n}) = t + \frac{1}{2}(t-1) + mnt - mt \). Therefore, we get

\[
f(v_{t-1,l}) < f(v_{t-1,l+1}) < \cdots < f(v_{t-1,n}) < f(u_{2m}) < f(u_{2m-1}) < \cdots < f(u_{21}).
\]

By generalized this case, we obtained that although \( f(u_{ij}) \) and \( f(v_{kj}) \) are in the same class of modulo \( t \), the values of \( f(u_{ij}) \) and \( f(v_{kj}) \), for all \( j \) and \( l \), are distinct.

Second, it must be proved that the induced function \( f^* \) is bijective. We know that for \( uv \in E(G) \), \( f^*(uv) = |f(u) - f(v)| \). It is easy to show that for the superstar part of \( G \), we have

\[
f^*(u_0u_i) = \begin{cases} s - \frac{1}{2}(i-1), & i \equiv 1 \pmod{2} \\ \frac{1}{2}i, & i \equiv 0 \pmod{2} \end{cases}, \quad f^*(u_iu_{i+1}) = s - i - \frac{1}{2}(t-1), \quad i = 1, 2, \ldots, t.
\]

So, the edges labels of superstar part of \( G \) are the consecutive positive integers \( 1, 2, 3, \ldots, \frac{1}{2}(t-1) \) and \( s - t - \frac{1}{2}(t-1), s - (t-1) - \frac{1}{2}(t-1), \ldots, s \). There are \( tmn \) integers between \( \frac{1}{2}(t-1) \) and \( s - t - \frac{1}{2}(t-1) \) that can be used as the labels of edges in each copy of complete bipartite graph. There are \( t \) copies of complete bipartite graphs \( K_{m,n} \). Every complete bipartite graphs \( K_{m,n} \) has \( mn \) edges. So, there are \( tmn \) edges and \( tmn \) consecutive integers for labeling it. To prove \( f^* \) is bijective, it must be proved that for every \( i, j, \) and \( k \), the value of \( f^*(u_{ij}v_{ik}) \) is all distinct and \( \frac{1}{2}(t-1) < f^*(u_{ij}v_{ik}) < s - t - \frac{1}{2}(t-1) \).

Since \( f(u_{ij}) \) is in the same class of modulo \( t \) for all \( j \) and \( f(v_{ik}) \) is in the same class of modulo \( t \) for all \( k \), \( f^*(u_{ij}v_{ik}) \) is in the same class of modulo \( t \) for all \( j \) and \( k \). So, if two edges are located in the same copy of complete bipartite graph, its labels are in the same class of modulo \( t \). From the definition and properties of \( f \), it can be shown that \( f^*(u_{ij}v_{ik}) = (\frac{1}{2}(t+1) - i)(\mod{t}) \), for all \( i, j, \) and \( k \).

Now, we need to prove that in each copy of complete bipartite graph, there is no edges which has the same label. For the odd \( i \), let \( f(u_{i1}) = a \) and \( f(v_{i1}) = b \). Then, we get \( f(u_{ij}) = a + (j-1)t \), for \( j = 1, 2, 3, \ldots, m \), and \( f(v_{ik}) = b - (k-1)mt \), for \( k = 1, 2, 3, \ldots, n \). It is clear that for all \( j \) and \( k \),

\[
f^*(u_{ij}v_{ik}) = (b-a) - (k-1)mt - (j-1)t.
\]
If there is edge $u_i,p v_{i,q}$ with the same label, it must be shown that $u_i,p v_{i,q} = u_i,j v_{i,k}$. Consider that

$$f^*(u_{i,j} v_{i,k}) = f^*(u_{i,p} v_{i,q}) \Leftrightarrow q - k = \frac{j - p}{m}.$$ 

Since $1 - m \leq j - p \leq m - 1$ and $qk$ is an integer, then $qk$ must be zero. So, we get $q = k$ and $j = p$, i.e. $u_i,p v_{i,q} = u_i,j v_{i,k}$. For the even $i$, the process is analogue.

We already proved that for every $i$, $j$, and $k$, the value of $f^*(u_{i,j} v_{i,k})$ are distinct. Then, we continue to prove that $\frac{1}{2}(t - 1) < f^*(u_{i,j} v_{i,k}) < s - t - \frac{1}{2}(t - 1)$. Consider that the smallest edges label in each copy of $K_{m,n} \text{ is } f^*(u_{i,m} v_{i,n}) = \frac{1}{2}(3t+1) - i$. Then, for the biggest $i$, we have the smallest $f^*(u_{i,j} v_{i,k})$, it is $f^*(u_{t,m} v_{t,n}) = \frac{1}{2}(t + 1)$. The biggest label in each copy of $K_{m,n}$ is $f^*(u_{1,1} v_{1,1}) = s - t - \frac{1}{2}(t - 1) - i$. Then, for the smallest $i$, we have the biggest $f^*(u_{i,j} v_{i,k})$, it is $f^*(u_{1,1} v_{1,1}) = s - t - \frac{1}{2}(t - 1) - 1$.

Since $f$ is an injective from $V(G)$ to the set \{0, 1, 2, 3, \ldots, s\} and the induced function $f^*$ is a bijective from $E(G)$ to the set \{1, 2, 3, \ldots, s\}, we can say that $f$ is a graceful labeling for $G$ and graph $G$ is graceful. So, for positive integers $m$, $n$ and odd positive integers $t$, graph $S^2(t \cdot K_{m,n})$ is graceful. 

**Illustration 3.1.** The open superstar of 5 copy of $K_{2,3}$ and its graceful labeling are shown in Figure 6.

![Figure 6. A graph $S^2(5 \cdot K_{2,3})$ and its graceful labeling.](image)

**Illustration 3.2.** We know that the complete bipartite graphs $K_{m,n}$ and $K_{n,m}$ are isomorphic. However, in the open superstar case, $S^2(t \cdot K_{m,n})$ and $S^2(t \cdot K_{n,m})$ are not isomorphic graphs for $m \neq n$. For example, we can compare $S^2(5 \cdot K_{2,3})$ with $S^2(5 \cdot K_{3,2})$ as can be seen in Figure 7.

4. **Conclusion and future work**

In this paper, we have introduced a definition of open superstar of complete bipartite graphs. We also have given a graceful labeling for $S^2(t \cdot K_{m,n})$ with odd positive integers $t$. Therefore, for positive integers $m$, $n$ and odd positive integers $t$, open superstar of complete bipartite graphs $S^2(t \cdot K_{m,n})$ is graceful. Our future research is construct a graceful labeling for open superstar of complete bipartite graphs $S^2(t \cdot K_{m,n})$ with even positive integers $t$ and another families of open superstar graphs.
Figure 7. A graph $S^2(5 \cdot K_{3,2})$ and its graceful labeling.

References
[1] Gallian J A 2016 The Electronic Journal of Combinatoric #DS6
[2] Rosa A 1967 On certain valuation of the vertices of a graph Theory of Graphs (Int. Symp., Rome, July 1966) pp 329–335
[3] Bermond J C 1979 Graceful graphs, radio antennae and French windmills Graph Theory and Combinatorics (London: Pitman) pp 18–37
[4] Kaneria V J, Meghpara M and Makadia H M 2014 International Journal of Mathematics and Statistics Invention (IJMSI) 2 19 – 23