Anomaly and Sign problem in $\mathcal{N} = (2, 2)$ SYM on Polyhedra :
Numerical Analysis

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Abstract

We investigate the two-dimensional $\mathcal{N} = (2, 2)$ supersymmetric Yang-Mills (SYM) theory on the discretized curved space (polyhedra). We first revisit that the number of supersymmetries of the continuum $\mathcal{N} = (2, 2)$ SYM theory on any curved manifold can be enhanced at least to two by introducing an appropriate $U(1)$ gauge background associated with the $U(1)_V$ symmetry. We then show that the generalized Sugino model on the discretized curved space, which was proposed in our previous work, can be identified to the discretization of this SUSY enhanced theory, where one of the supersymmetries remains and the other is broken but restored in the continuum limit. We find that the $U(1)_A$ anomaly exists also in the discretized theory as a result of an unbalance of the number of the fermions proportional to the Euler characteristics of the polyhedra. We then study this model by using the numerical Monte-Carlo simulation. We propose a novel phase-quench method called “anomaly-phase-quenched approximation” with respect to the $U(1)_A$ anomaly. We numerically show that the Ward-Takahashi (WT) identity associated with the remaining supersymmetry is realized by adopting this approximation. We figure out the relation between the sign (phase) problem and pseudo-zero-modes of the Dirac operator. We also show that the divergent behavior of the scalar one-point function gets milder as the genus of the background increases. These are the first numerical observations for the supersymmetric lattice model on the curved space with generic topologies.

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## Contents

1. **Introduction**  
   - 2

2. **$\mathcal{N} = (2, 2)$ Continuum field theory and Generalized Sugino model**  
   - 3
     - 2.1 Rigid supersymmetry on a curved space  
     - 3
     - 2.2 $\mathcal{N} = (2, 2)$ rigid supersymmetry on the Riemann surface  
     - 5
     - 2.3 Generalized Sugino model as a discretization of the rigid SUSY theory  
     - 7

3. **$U(1)_A$ anomaly and a novel phase-quenched method**  
   - 11
     - 3.1 Anomaly-phase-quench method  
     - 11
     - 3.2 Ward-Takahashi identity  
     - 14

4. **Monte Carlo Simulations**  
   - 15
     - 4.1 Algorithm and data analysis  
     - 16
     - 4.2 Simulation parameters  
     - 16
     - 4.3 One-point scalar correlation function  
     - 18
     - 4.4 Ward-Takahashi identity  
     - 18
     - 4.5 Phase of the Pfaffian  
     - 20
     - 4.6 Origin of the anomaly  
     - 22

5. **Summary and Discussion**  
   - 23

A. **Explicit form of the action**  
   - 24
1 Introduction

One of the most important and non-trivial aspects of gauge field theories is the dynamics. Even if the theory is strongly restricted by some symmetry like supersymmetry, it is in general not sufficient to determine the whole dynamics. This leads to a strong motivation to pursue a way to examine the dynamical aspect of the supersymmetric gauge theories nonperturbatively. Among a great deal of approaches in this direction, the application of the lattice technique to supersymmetric gauge theories has been a long-standing theme in theoretical high energy physics \[1, 2, 3, 4\]. Although the lattice regularization breaks the Poincare invariance to its discrete subgroup and the supersymmetry cannot be straightforwardly realized on the lattice, several ways of bypassing the problem have been developed to date. In particular, for low-dimensional gauge theories with extended supersymmetries, lattice models could avoid fine-tuning for the continuum limit due to partially preserved supercharges. In \[5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21\], the lattice supersymmetric gauge models preserving one or two supercharges are proposed based on the discretized topologically twisted gauge theories (for relations among several lattice formulations, see \[22, 23, 24, 25, 26\]). Since the supersymmetries are partially preserved in the models, the numerical simulation can be carried out on the basis similar to lattice QCD \[27, 28, 29, 30, 31, 32, 33, 34\]. The problem of the vacuum degeneracy of lattice gauge fields is also shown to be avoided without using an admissibility condition \[35\]. For discretization of higher dimensional supersymmetric gauge theories without fine-tuning, see \[36, 37\]. Relevance of the lattice supersymmetric models to fermion discretizations is discussed in \[38\].

The lattice formulations of supersymmetric theories are in general discretized on a periodic hypercubic lattice, that is, the topology is restricted only on the torus. However, the topology plays an significant role in supersymmetric gauge theories especially in the context of topological field theory \[39, 40, 41, 42, 43, 44, 45\]. The significance of such theories has recently been re-recognized in relation to the localization technique in supersymmetric gauge theories \[46\].

In \[47\], a discretization of the topologically twisted two-dimensional \(N = (2, 2)\) supersymmetric Yang-Mills (SYM) theory on an arbitrary Riemann surface is proposed (generalized Sugino model). There, the Riemann surface is discretized to a polygon, on which the supersymmetric gauge theory preserving a single supercharge is defined. It is shown that the theory can be defined on any decomposition of the two-dimensional surface and one can take its continuum limit without any fine-tuning. In \[48\], the analytical study based on the localization technique for the model is performed and it was shown that the partition function of the model mainly depends on the Euler characteristics of the background Riemann surface.

In this paper, we investigate the two-dimensional \(N = (2, 2)\) SYM on a curved space-time, where two supersymmetries survive under an appropriate \(U(1)\) gauge backgrounds associated with the gauged \(U(1)_V\) symmetry \[49, 50\], both from the theoretical and numerical viewpoints. 

What we will show is summarized in the followings:

1. We show that the generalized Sugino model is nothing but a discretization of the SUSY enhanced theory, where one of the supersymmetries is preserved and the other is broken but restored in the continuum limit. Both these continuum and discretized supersymmetric theories are not topological but physical in the sense that we can consider any kinds of operators as observables, that is, we do not need to restrict the observables to \(Q\)-cohomology unlike the traditional topological field theories.

2. We show that the \(U(1)_A\) anomaly exists also in the generalized Sugino model as a result of the unbalance of the fermion numbers which is related to the Euler characteristics of
3. We study the generalized Sugino model based on the numerical Monte-Carlo simulation. We show that the flat directions of the scalar fields can be controlled by adding a mass term. By investigating the behavior of the expectation values of the one-point function of scalar fields, we find that the divergent behavior of the expectation value in a small mass region gets milder as the genus of background geometry increases.

4. We divide the phase of the Pfaffian of the Dirac operator into the anomaly-induced phase and the residual phase by introducing a specific operator called the "compensator". We approximate the theory by ignoring the residual phase in the Monte Carlo simulation, which we call the "anomaly-phase-quenched approximation". We show that the Ward-Takahashi (WT) relations expected from the remaining supersymmetry of the model are satisfied based on this approximation.

5. We show that the sign problem in the model originates in the pseudo-zero-modes of the Dirac operators, which causes the $U(1)_A$ anomaly. In other words, the Pfaffian phase except the anomaly-induced phase (the residual phase) does not contribute the path integral regardless of the topology of the background\textsuperscript{1}. This fact guarantees the validity of the anomaly-phase quenched approximation.

This paper is organized as follows: In Sec. 2, we discuss SUSY enhancement in the continuum $\mathcal{N} = (2, 2)$ SYM theory on the curved space with introducing an appropriate $U(1)$ gauge backgrounds. We also discuss its relation to the generalized Sugino model on the discretized curved space. In Sec. 3, we discuss the $U(1)_A$ anomaly in the discretized model and propose the anomaly-phase-quenched approximation. In Sec. 4 we show the results of Monte Carlo simulations for the scalar one-point functions and the WT identity in the model. We also show the origin of the $U(1)_A$ anomaly and the origin of the sign problem based on the numerical methods.

2 $\mathcal{N} = (2, 2)$ Continuum field theory and Generalized Sugino model

2.1 Rigid supersymmetry on a curved space

We first review a general aspect of the rigid supersymmetry on a curved manifold. The rigid supersymmetry on the curved manifold is developed to apply the localization to various field theories, whose general constructions are discussed in [49, 50] and more detailed discussion for two-dimensional case is given in [51].

The rigid supersymmetry on the flat space-time is the usual supersymmetry, that is, it is realized as a fermionic variation of the fields via

$$\delta = \xi^a Q_a,$$

where the index $a$ runs the number of the supercharges which we are considering, $Q_a$ are the supercharges and $\xi^a$ are globally constant fermionic parameters on the whole flat space-time. The supersymmetry of the theory is guaranteed by the invariance of the action and the measure under the variation by (2.1). In particular the Lagrangian of the theory is varied up to the total

\textsuperscript{1} It has been already shown that there is no sign problem in the torus background [33].
derivatives. However, if we consider the same kind of fermionic transformation on a curved space-time, it is in general hard to make an action invariant under the transformation (2.1) since the fixed spinor $\xi$ does not commute with the covariant derivative in general even if it has a position dependence as long as it is a fixed function. Thus we sometimes make the supersymmetry local and build a supergravity by introducing extra fields and symmetries.

One exception is when the curved space-time admits covariantly constant Killing spinors,

$$\nabla_\mu \xi = 0. \quad (2.2)$$

In this case, we can make an invariant action under the transformation (2.1) with the parameter $\xi$ satisfying (2.2), since they commute with the covariant derivatives and act as a constant parameter in the variation of the Lagrangian. The existence of the covariantly constant Killing spinor restricts the structure of the manifolds. The Killing spinors exist only on Ricci flat Kähler manifolds for example. This is the reason why the rigid supersymmetry on the curved manifold has not been considered for a long time.

However, there is a route to avoid this obstacle. The point is that many supersymmetric theories have global symmetries called the R-symmetries and the parameter $\xi$ is charged under them. So if we gauge one of the global R-symmetries, the Killing equation (2.2) is modified to

$$\nabla^R_\mu \xi \equiv (\nabla_\mu + i A^R_\mu) \xi = 0, \quad (2.3)$$

where $A^R_\mu$ is a vector field corresponding to the gauged R-symmetry. The solutions of (2.3) include broader possibilities than (2.2) depending on a choice of $A^R_\mu$. In particular, if the field strength from the gauge field $A^R_\mu$ cancels the effect of the curvature, we can obtain the covariantly constant Killing spinor on the curved manifold in the deformed sense. In this way, the rigid supersymmetry on the curved space is constructed by gauging the R-symmetry. This construction of the rigid supersymmetry is also understood from a point of view of fixed background fields in the supergravity theory as discussed in [49, 50].

2.2 $\mathcal{N} = (2, 2)$ rigid supersymmetry on the Riemann surface

Now let us apply the above strategy to the 2D $\mathcal{N} = (2, 2)$ SYM theory. This theory on the two-dimensional flat space-time is obtained by the dimensional reduction from the 4D $\mathcal{N} = 1$ SYM theory, which has four supercharges. We denote the parameters of the supersymmetric variation associated with four supercharges as $\xi_a$ and $\bar{\xi}_{\dot{a}}$ ($a, \dot{a} = 1, 2$). This theory possesses two abelian R-symmetries: one comes from the original R-symmetry of the four-dimensional theory which we call $U(1)_V$ and the other comes from a rotational symmetry of the dimensionally reduced space which is called $U(1)_A$.

We consider the $\mathcal{N} = (2, 2)$ SYM theory on a Riemann surface. It is significant that the topology of the two-dimensional curved manifolds without boundary is completely classified by the number of handles (genus) $h$, and once the topology of the Riemann surface is given, we can always choose the metric of the Riemann surface in a specific coordinate patch $(z, \bar{z})$ as conformally flat:

$$ds^2 = e^{2\sigma(z, \bar{z})} dz \otimes d\bar{z}. \quad (2.4)$$

We thus denote the Riemann surface as $\Sigma_h$ and use the conformal metric (2.4) in the following. All the objects of the differential geometry like covariant derivative, spin connection, curvature, and so on are derived from the function $\sigma(z, \bar{z})$ in this coordinate patch.
Here we gauge the global $U(1)_V$ symmetry. Recalling that the supersymmetry parameters $\xi_a$ and $\bar{\xi}_{\dot{a}}$ have charges $+1$ and $-1$ under $U(1)_V$, respectively, the modified Killing equations \textbf{(2.3)} for the components of $\xi_a$ become

$$
\nabla^R_z \xi_1 = \left( \partial_z - \frac{1}{2} \partial_{\bar{z}} \sigma + i A^R_z \right) \xi_1 = 0,
$$

$$
\nabla^R_{\bar{z}} \xi_2 = \left( \partial_{\bar{z}} + \frac{1}{2} \partial_z \sigma + i A^R_{\bar{z}} \right) \xi_2 = 0,
$$

\textbf{(2.5)}

in the complex coordinates $z$ and $\bar{z}$. These equations do not have a non-vanishing solution for general $A^R_\mu$, but if we choose

$$
A^R_z = -i \frac{1}{2} \partial_z \sigma \quad \text{and} \quad A^R_{\bar{z}} = i \frac{1}{2} \partial_{\bar{z}} \sigma,
$$

\textbf{(2.6)}

by using the conformal factor $\sigma(z, \bar{z})$, the spinor

$$
\xi_\alpha = \begin{pmatrix} \xi_0 \\ 0 \end{pmatrix},
$$

\textbf{(2.7)}

with a constant Grassmann value $\xi_0$ becomes a solution of \textbf{(2.5)} since the spin connection and the gauge potential are canceled with each other for $\xi_1$. Similarly,

$$
\bar{\xi}_{\dot{\alpha}} = \begin{pmatrix} 0 \\ \bar{\xi}_0 \end{pmatrix}
$$

\textbf{(2.8)}

with a constant Grassmann parameter $\bar{\xi}_0$ is a solution of the modified Killing equation. Thus, associated with the Killing spinors \textbf{(2.7)} and \textbf{(2.8)}, we have two conserved supercharges on the curved Riemann surface $\Sigma_h$ by choosing the background gauge field as \textbf{(2.6)}.

For the conserved Killing spinors \textbf{(2.7)} and \textbf{(2.8)}, the supersymmetric transformation of the $N = (2, 2)$ vector multiplets in two-dimensions is written as

$$
\delta A_z = -i \xi_0 e^\sigma \bar{\lambda}_2, \quad \delta A_{\bar{z}} = i \xi_0 e^\sigma \lambda_2,
$$

$$
\delta \Phi = 0, \quad \delta \bar{\Phi} = -2i \xi_0 \bar{\lambda}_1 + 2i \bar{\xi}_0 \lambda_1,
$$

$$
\delta \lambda_1 = 2 \sqrt{2} \xi_0 e^{-\sigma} D_z X_w, \quad \delta \bar{\lambda}_1 = 2 \sqrt{2} \xi_0 e^{-\sigma} D_{\bar{z}} X_w,
$$

$$
\delta \lambda_2 = \sqrt{2} \xi_0 e^{-2\sigma} F_{zz} + i \sqrt{2} \xi_0 [X_w, X_{\bar{w}}] + \frac{i}{\sqrt{2}} \xi_0 D, \quad \delta \bar{\lambda}_2 = -\sqrt{2} \xi_0 e^{-2\sigma} F_{\bar{z}\bar{z}} + i \sqrt{2} \xi_0 [X_w, \bar{X}_{\bar{w}}] - \frac{i}{\sqrt{2}} \bar{\xi}_0 D,
$$

$$
\delta D = \sqrt{2} \xi_0 e^{-\sigma} D^R_z \bar{\lambda}_1 - \sqrt{2} \xi_0 e^{-\sigma} D^R_{\bar{z}} \lambda_1 - i \sqrt{2} \xi_0 [X_w, \bar{\lambda}_2] + i \sqrt{2} \bar{\xi}_0 [X_w, \lambda_2].
$$

\textbf{(2.9)}

In particular, we define a specific linear combination of the supercharges,

$$
Q \equiv Q_1 + \bar{Q}_2.
$$

\textbf{(2.10)}
Defining new fermionic fields by
\[ \lambda_z \equiv -ie^\sigma \bar{\lambda}_2, \quad \lambda_{\bar{z}} \equiv ie^\sigma \lambda_2, \quad \eta \equiv -2i\bar{\lambda}_1 + 2i\lambda_1, \quad \chi \equiv -2i\bar{\lambda}_1 - 2i\lambda_1, \] (2.11)
we see that the supersymmetric transformations under \( Q \) can be simply written as
\[
\begin{align*}
Q\Phi &= 0, \\
QA_\mu &= \lambda_\mu, \\
Q\lambda_\mu &= iD_\mu \Phi, \\
Q\bar{\Phi} &= \eta, \\
Q\eta &= [\Phi, \bar{\Phi}], \\
Q\chi &= \{\Phi, \chi\}, \\
Q\chi &= Y.
\end{align*}
\] (2.12)

It is important that, from the viewpoint of the modified Killing equation, the fermions \( \lambda_\mu \) and \( \{\eta, \chi\} \) can be regarded as 1-form and 0-forms on \( \Sigma_h \), respectively. Using the supercharge \( Q \), the action can be written in the \( Q \)-exact form:
\[
S = Q \int_{\Sigma_h} d^2 x \sqrt{g} \; Tr \left[ \frac{1}{4} \eta[\Phi, \bar{\Phi}] - ig^{\mu\nu} \lambda_\mu D_\nu \bar{\Phi} + \chi(Y - i\Omega) \right],
\] (2.13)
with \( \Omega = 2iF_{z\bar{z}} \). It is easy to see \( Q \) satisfies \( Q^2 = \delta \Phi \) where \( \delta \) is the infinitesimal gauge transformation by the parameter \( \epsilon \). Therefore the invariance of the action under the \( Q \)-transformation is manifest in the expression (2.13).

Note that this choice of the supercharge and the field components is also known as the topological A-twist of \( \mathcal{N} = (2, 2) \) theory on \( \Sigma_h \). Thus the supersymmetric theory on the curved space with the background R-gauge field can be naturally identified to the topologically twisted theory. However, we emphasize that it does not mean that we must restrict the observables only to the \( Q \)-cohomology. We can also regard the theory as a “physical” supersymmetric gauge theory.

Since the theory preserves both of the supercharges \( Q^1 \) and \( \bar{Q}^2 \), the action (2.13) is invariant under not only \( Q \) but also another linear combination \( Q' \equiv Q^1 - \bar{Q}^2 \). The supersymmetry algebra of \( Q' \) is obtained roughly by exchanging the role of \( \eta \) and \( \chi \) from that of \( Q \) reflecting the \( U(1)_V \) symmetry, which rotates \( \eta \) and \( \chi \), of the theory.

On the other hand, the \( U(1)_A \) symmetry acts on the fields as
\[
\begin{align*}
A_\mu &\to A_\mu, \quad \Phi \to e^{2i\theta} \Phi, \quad \bar{\Phi} \to e^{-2i\theta} \bar{\Phi}, \quad Y \to Y, \\
\lambda_\mu &\to e^{i\theta} \lambda_\mu, \quad \eta \to e^{-i\theta} \eta, \quad \chi \to e^{-i\theta} \chi,
\end{align*}
\] (2.14)
and the action (2.13) is manifestly invariant under this symmetry. However \( U(1)_A \) symmetry is broken quantum-mechanically by the anomaly. In fact there is a mismatch in the number of the zero modes of the fermions on \( \Sigma_h \) unless the Euler characteristic is not equal to zero. This anomaly plays an important role even in the discretized theory discussed in the next subsection.

### 2.3 Generalized Sugino model as a discretization of the rigid SUSY theory

Let us next consider a discretization of the continuum theory (2.13). Actually, it is already achieved in [47] where the authors gave a theory on an arbitrary discretized Riemann surface (the generalized Sugino model) whose continuum limit is nothing but the theory given by (2.13). Although the original derivation in [47] is based on the topological twisting, we can obtain the same formulation by taking into account the background vector field \( A_\mu^R \) as discussed in the previous subsections. In fact, we can derive the same formulation by simply replacing all derivatives in the supersymmetric algebra (2.12) and the action (2.13) with difference operators.
since they do not contain the spin connection in the spin representation as the consequence of
the background vector field. Therefore, as we have emphasized, we can regard the discretized
Sugino model as a discretized model of the “physical” theory, that is, we do not need to restrict
the observables to the $Q$-cohomology.

Let us briefly review the generalized Sugino model to fix the notation. This model is con-
structed on a discretized Riemann surface which consists of a set of sites, links with directions
and faces. We denote the number of the sites, the links and the faces as $N_S$, $N_L$ and $N_F$, re-
spectively. The bosonic variables are written as $\{\Phi_s, \Phi_s, U_l, Y_f\}$ and the fermionic variables are
written as $\{\eta_s, \lambda_l, \chi_f\}$ where the indices $s \in \{1, \cdots, N_S\}$, $l \in \{1, \cdots, N_L\}$ and $f \in \{1, \cdots, N_F\}$
are the labels of the sites, links and faces, respectively, which stand for constituents of the poly-
gon with which the variable is associated. We assume that there is an $SU(N_c)$ gauge symmetry
and the observables to the $\Phi$-cohomology.

\begin{equation}
\Phi_s \rightarrow g_s \Phi_s g_s^{-1}, \quad \bar{\Phi}_s \rightarrow g_s \bar{\Phi}_s g_s^{-1}, \quad \eta_s \rightarrow g_s \eta_s g_s^{-1},
\end{equation}
\begin{equation}
U_l \rightarrow g_{\text{org}(l)} U_l g_{\text{tip}(l)}^{-1}, \quad \lambda_l \rightarrow g_{\text{org}(l)} \lambda_l g_{\text{org}(l)}^{-1}, \quad Y_f \rightarrow g_f Y_f g_f^{-1}, \quad \chi_f \rightarrow g_f \chi_f g_f^{-1},
\end{equation}
where $g_s \in SU(N_c)$ is a unitary matrix on the site $s$, $\text{org}(l)$ and $\text{tip}(l)$ stand for the site labels of
the origin and the tip of the link $l$, respectively, and the index $f$ in $g_f$ stands for a representa-
tive site of the face $f$. We sometimes use the same character $f$ to express the representative site
of the face $f$. The way of the gauge transformation (2.15) shows that only $U_l$ lives on the link
and the other variables live on the sites. We should note that this simply means that the link and
face variables except for $U_l$ live on the representative sites of the links and faces.

The supersymmetric transformations of the variables are
\begin{equation}
\begin{align*}
Q \Phi_s &= 0, \\
Q \bar{\Phi}_s &= \eta_s, \\
Q U_l &= i \lambda_l U_l, \\
Q Y_f &= [\Phi_f, \chi_f], \\
Q \lambda_l &= i(U_l \Phi_{\text{tip}(l)} U_l^{-1} - \Phi_{\text{org}(l)} + \lambda_l \lambda_l), \\
Q \chi_f &= Y_f,
\end{align*}
\end{equation}
which are the discrete versions of (2.12). Note that $Q$ satisfies $Q^2 = \delta_{\Phi_s}$ in a parallel way to
the continuum $Q$-transformation. The action of the generalized Sugino model is written in the $Q$-exact form,
\begin{equation}
S_0 = Q \Xi \equiv Q \left[ \sum_{s=1}^{N_S} \alpha_s \Xi_s + \sum_{l=1}^{N_L} \alpha_l \Xi_l + \sum_{f=1}^{N_F} \alpha_f \Xi_f \right],
\end{equation}
where
\begin{equation}
\begin{align*}
\Xi_s &= \frac{1}{2g^2} \text{Tr} \left[ \frac{1}{4} \eta_s [\Phi_s, \bar{\Phi}_s] \right], \\
\Xi_l &= \frac{1}{2g^2} \text{Tr} \left[ -i \lambda_l (U_l \Phi_{\text{tip}(l)} U_l^{-1} - \Phi_{\text{org}(l)}) \right], \\
\Xi_f &= \frac{1}{2g^2} \text{Tr} \left[ \chi_f (Y_f - i \beta_f \Omega(U_f)) \right],
\end{align*}
\end{equation}

\footnote{For the possibility of regarding $\Phi_s$ and $\bar{\Phi}_s$ as independent hermitian matrices, see [47].}
where $\Omega(U_f)$ is set as

$$
\Omega(U_f) = \frac{1}{m} \left[ S^{-1}(U_f^m) C(U_f^m) + C(U_f^m) S^{-1}(U_f^m) \right], \quad \left( m \geq \frac{N_c}{4} \right),
$$

(2.19)

with

$$
S(U_f) = -i(U_f - U_f^{-1}), \quad C(U_f) = U_f + U_f^{-1},
$$

(2.20)

in order to single out only the physical vacuum of the gauge fields [35]. Here $U_f$ is the face variable corresponding to the face $f$ defined as

$$
U_f = \prod_{i=1}^{n_f} U_{l_i}^{\epsilon_i},
$$

(2.21)

where $n_f$ is the number of the links which surround the face $f$, $l_1, \ldots, l_{n_f}$ are the labels of those links which construct the face $f$ in this order, and $\epsilon_i$ ($i = 1, \ldots, l_{n_f}$) express the directions of the links in the face, which is determined recursively as follows (see Fig. 1): We set $s_0$ to be the representative site of the face $f$. If $\text{org}(l_1) = s_0$ then $\epsilon_1 = 1$ and $s_1 = \text{tip}(l_1)$, otherwise $\epsilon_1 = -1$ and $s_1 = \text{org}(l_1)$. For $i = 2, \cdots, n_f$, if $\text{org}(l_i) = s_{i-1}$ then $\epsilon_i = 1$ and $s_i = \text{tip}(l_i)$ otherwise $\epsilon_i = -1$ and $s_i = \text{org}(l_i)$.

![Figure 1](image-url)  

Figure 1: An example of the relation between a face $f$ and the links which constructing the face $f$. The directions of the link $l_i$ is expressed by $\epsilon_i = \pm 1$.

It is easy to see that the (tree-level) continuum limit of the discretized action (2.17) becomes (2.13) by choosing the parameter $\alpha_s, \alpha_l, \alpha_f$ and $\beta_f$ appropriately. We can also see that any relevant operator which breaks the $Q$-symmetry does not appear quantum mechanically by the power counting argument. Therefore we do not need any fine-tuning to take the continuum limit. For more detail, see [47, 48].

In the actual Monte Carlo simulation, we also have to control the flat directions of the scalar fields $\Phi_s$ and $\bar{\Phi}_s$. For this purpose, we add a mass term,

$$
S_\mu \equiv \frac{\mu^2}{2} \sum_s \text{Tr}(\Phi_s \bar{\Phi}_s),
$$

(2.22)
to the discretized action (2.17). This term explicitly breaks the $Q$-symmetry, but it is a soft-breaking term. We can explicitly evaluate the WT identity associated with the $Q$-symmetry with $\mu$ dependence as shown in Section 3.2. The explicit form of the action is summarized in the appendix A.

We note that, although the $U(1)_A$ transformation (2.14) is still a symmetry of the discretized action (2.17), the $Q'$-symmetry and the $U(1)_V$ symmetries are broken by the discretization. This is because the discretized version of $\Omega$ in (2.19) does not admit the exchange of the role of $\eta$ and $\chi$. The $U(1)_V$ symmetry and the $Q'$-symmetry will be recovered only in the continuum limit.

To sum up, the action of the generalized Sugino model is given by the summation of the $Q$-exact part $S_0$ given by (2.17) and the mass term $S_\mu$ given by (2.22). We divide the $Q$-exact part $S_0$ into the bosonic part and the fermionic part as

$$S_0 = Q\Xi = S_{0,b} + S_{0,f}. \quad (2.23)$$

We further separate the terms including the auxiliary field $Y_f$ from $S_{0,b}$ as

$$S_{0,b} = \tilde{S}_{0,b} + S_Y, \quad (2.24)$$

with

$$S_Y = \frac{1}{2g^2} \sum_{f=1}^{N_F} \alpha_f \text{Tr} \left( Y_f - \frac{i}{2} \beta_f \Omega(U_f) \right)^2. \quad (2.25)$$

Therefore the bosonic part of the whole generalized Sugino model is given by

$$S_b \equiv \tilde{S}_{0,b} + S_Y + S_\mu. \quad (2.26)$$

Since we use the action after integrating out the auxiliary field in the Monte Carlo simulation, it is convenient to define

$$\tilde{S}_b \equiv \tilde{S}_{0,b} + S_\mu. \quad (2.27)$$

On the other hand, the fermionic part of the action can be written in the form of a bilinear of the fermionic variables:

$$S_f \equiv S_{0,f} = \frac{1}{2} \vec{F} \cdot D\vec{F}, \quad (2.28)$$

where $\vec{F}$ is a vector with the size of $(N_2^c-1)(N_S + N_L + N_F)$ which consists of all the elements of the fermionic variables $\{\eta_s, \lambda_l, \chi_f\}$ and $D$ is an anti-symmetric matrix (Dirac operator) with the same size of $\vec{F}$. By integrating out the fermionic variables $\vec{F}$, there appears the Pfaffian of $D$;

$$\int D\vec{F} e^{-S_f} = \text{Pf}(D), \quad (2.29)$$

which in general takes a complex value since we are considering a space-time with the Euclidean signature. We thus write the phase of the Pfaffian as $\theta_{\text{Pf}}$;

$$\text{Pf}(D) = |\text{Pf}(D)| e^{i\theta_{\text{Pf}}}. \quad (2.30)$$

Since the strategy of the Monte Carlo simulation is to regard the Boltzmann factor of the theory as a probability density, it must be real positive. We thus often approximate the Pfaffian to its absolute value in producing a probability density in the Monte Carlo simulation. Therefore the corresponding partition function in the Monte Carlo simulation of the generalized Sugino model with the standard phase-quenched approximation is given by

$$Z_q = \int D\vec{F} |\text{Pf}(D)| e^{-\tilde{S}_b}, \quad (2.31)$$
where $\mathcal{D}\vec{B}'$ is the measure of the bosonic variables except for the auxiliary field $Y_f$:

$$\mathcal{D}\vec{B}' \equiv \left( \prod_{s=1}^{N_S} \mathcal{D}\Phi_s \mathcal{D}\bar{\Phi}_s \right) \left( \prod_{l=1}^{N_L} \mathcal{D}U_l \right) \left( \prod_{f=1}^{N_F} \mathcal{D}Y_f \right). \quad (2.32)$$

3 $U(1)_A$ anomaly and a novel phase-quenched method

In the Monte Carlo computation for the present model, that is, the generalized Sugino model on a discretized Riemann surface, we need to take care of the two deeply-related properties: Pfaffian phase and $U(1)_A$ anomaly. In this section, we examine the $U(1)_A$ anomaly in detail and introduce a novel phase-quenched method with respect to the anomaly.

3.1 Anomaly-phase-quench method

Let us consider the measure of the generalized Sugino model

$$\mathcal{D}\vec{X} = \mathcal{D}\vec{B} \mathcal{D}\vec{F}, \quad (3.1)$$

where

$$\mathcal{D}\vec{B} \equiv \left( \prod_{s=1}^{N_S} \mathcal{D}\Phi_s \mathcal{D}\bar{\Phi}_s \right) \left( \prod_{l=1}^{N_L} \mathcal{D}U_l \right) \left( \prod_{f=1}^{N_F} \mathcal{D}Y_f \right), \quad (3.2)$$

$$\mathcal{D}\vec{F} \equiv \left( \prod_{s=1}^{N_S} \mathcal{D}\eta_s \right) \left( \prod_{l=1}^{N_L} \mathcal{D}\lambda_l \right) \left( \prod_{f=1}^{N_F} \mathcal{D}\chi_f \right). \quad (3.3)$$

Recalling the $U(1)_A$ transformation of the variables (2.14), we see that, even if the action is invariant under the $U(1)_A$ rotation, the measure of the functional integral can have a net $U(1)_A$ charge:

$$[\mathcal{D}\vec{X}]_A = (N_2^c - 1)\chi_h, \quad (3.4)$$

where $[\mathcal{O}]_A$ denotes the $U(1)_A$ charge of an operator $\mathcal{O}$ and $\chi_h = N_S - N_L + N_F$ is the Euler characteristics of the discretized Riemann surface. This nonzero charge of the integral measure corresponds to the lattice counterpart of the $U(1)_A$ anomaly. In the continuum limit, the Euler characteristics is of course intact and the $U(1)_A$ charge of the measure results in the anomaly term in the continuum theory. Since the Pfaffian appears by integrating out the fermionic variables, the Pfaffian has exactly the same $U(1)_A$ charge with that of the integration measure (3.4). Therefore the Pfaffian phase $\theta_{Pf}$ always includes a phase of the $U(1)_A$ origin unless $\chi_h = 0$. This is not an artificial phase coming from the Wick rotation but from the topology of the background. This motivates us to decompose the Pfaffian phase into two parts; the $U(1)_A$-anomaly phase and the residual phase,

$$\theta_{Pf} = \theta_A + \theta, \quad (3.5)$$

where $\theta_A$ stands for the phase originated in $U(1)_A$ anomaly and $\theta$ is the residual phase. Note that there is always an ambiguity in defining $\theta_A$, which we will fix later in (3.11).

An immediate consequence of this observation is that, when the Euler characteristics is not equal to zero, the partition function of the generalized Sugino model without any quenched approximation trivially vanishes:

$$Z \equiv \int \mathcal{D}\vec{B} \mathcal{D}\vec{F} e^{-S_b - S_f} = \int \mathcal{D}\vec{B} \text{Pf}(D)e^{-S_b} = 0, \quad (3.6)$$
because of the unbalance of the number of fermions between the measure and the Boltzmann factor or the non-vanishing background $U(1)_A$ charge. This means that the ordinary definition of the expectation value of an operator $\mathcal{O}$,

$$
\left< \mathcal{O} \right> = \frac{1}{Z} \int \mathcal{D}\mathbf{\bar{F}} \mathcal{D}\mathbf{\bar{F}}' \mathcal{O} e^{-S_b - S_f},
$$

is ill-defined.

Although the denominator of the expectation value is usually set so that the total probability becomes unity, it is more important that (the absolute value of) the Boltzmann factor is proportional to the probability density. In this sense the denominator is just a normalization. We thus use (2.31) instead of (3.7) as the denominator of the definition of the expectation value of our model in this paper:

$$
\langle \mathcal{O} \rangle \equiv \frac{1}{Z_q} \int \mathcal{D}\mathbf{B} \mathcal{D}\mathbf{O} e^{-S_b} ,
$$

(3.8)

which is suitable to the Monte Carlo simulation. Note that this change of the definition does not affect the WT identity we encountered in Section 3.2.

Incidentally, in the Monte Carlo simulation, we often “approximate” the expectation value (3.8) by replacing the Pfaffian of the numerator to its absolute value:

$$
\langle \mathcal{O} \rangle_q \equiv \frac{1}{Z_q} \int \mathcal{D}\mathbf{B} \mathcal{D}\mathbf{O} |\text{Pf}(D)| e^{-S_b} ,
$$

(3.9)

which we call the naive phase-quenched approximation in this paper. However, this is not an appropriate approximation of the expectation value (3.8). In fact, from the same reason why the partition function vanishes, the expectation value (3.8) vanishes unless the $U(1)_A$ charge of the operator $\mathcal{O}$ is equal to $- (N_c^2 - 1) \chi_h$. In particular, if we compute the expectation value of a $U(1)_A$ neutral operator, it must vanish unless the Euler characteristics of the background is zero ($\chi_h = 0$). However the expectation value of such a neutral operator in the naive phase-quenched approximation (3.9) is apparently non-zero even if $\chi_h \neq 0$. Therefore we cannot use (3.9) as an approximation of the expectation value (3.8) of the generalized Sugino model.

In order to overcome this problem, we introduce a gauge invariant operator $\mathcal{A}$ with a specific $U(1)_A$ charge $- (N_c^2 - 1) \chi_h$. We assume that the operator $\mathcal{A}$ is invariant not only under any (bosonic) global symmetry transformation of the theory other than $U(1)_A$ but also under the $Q$-transformation:

$$
Q \mathcal{A} = 0, \quad (3.10)
$$

for the later purpose. A notable fact is that if we insert the operator $\mathcal{A}$ into the path integral, it exactly cancels the $U(1)_A$ charge of the integration measure or the Pfaffian. In this sense, we call the operator $\mathcal{A}$ the “compensator”. We can define the $U(1)_A$ part of the Pfaffian phase in (3.5) through the compensator as

$$
\mathcal{A} = |\mathcal{A}| e^{-i\theta_A}. \quad (3.11)
$$

Once we define the anomaly phase $\theta_A$, we can introduce a phase-quench method of a novel type by ignoring only the residual phase of the Pfaffian phase $\theta$ in (3.5):

$$
\left< \mathcal{O} \right>^\hat{q} \equiv \left< \mathcal{O} e^{i\theta_A} \right>^q = \left< \frac{\mathcal{A}^*}{|\mathcal{A}|} \right>^q, \quad (3.12)
$$

\[3\] Of course, instead of using the quenched Pfaffian in the denominator, one can insert the compensator as $\int d\mathbf{\bar{F}} \text{Pf}(D)e^{-S_b}$ with respect to the $Q$-symmetry.

\[4\] This is the reason why the naive-phase-quench approximation works in the Sugino model on the torus.
which we call the \textit{anomaly-phase-quench} method. We see that \( \langle O \rangle^q \) vanishes unless the operator \( O \) has the \( U(1)_A \) charge \( -(N_c^2 - 1)\chi_h \), which is exactly the property of the expectation value of the generalized Sugino model.

Furthermore, we expect that there is no sign problem in the present discretized theory as long as the anomaly-induced phase is cancelled by the compensator. In other words, the residual phase \( e^{i\theta} \) of the Pfaffian in (3.5) will not contribute to the results of the expectation values. As shown in [33], this is actually the case in the torus background where it was shown that the Pfaffian becomes real positive in the continuum limit. The point is that, for the continuum 2D \( \mathcal{N} = (2, 2) \) SYM in the flat background, the eigenvalues of the Dirac operator always appear as complex conjugate pairs [33]. We can apply the same logic to the continuum theory on a curved background discussed in Section 2, and the product of the non-zero eigenvalues of the Dirac operator is expected be real positive for the continuum theory in a curved background as well.

Recalling that the \( U(1)_A \) anomaly of the continuum theory comes from the fermionic zero-modes, this strongly suggests that there is no sign problem in the present theory as long as we eliminate the anomaly-induced phase \( \theta_A \). Therefore we expect that the anomaly-phase-quench method, where we ignore the residual phase (the Pfaffian phase except the anomaly-induced phase), will provide a proper approximation of the expectation value in the Monte Carlo simulation.

We close this subsection by making three comments. First, let us consider the expectation value

\[
\langle O_0 A \rangle
\]

for an arbitrary operator \( O_0 \) with zero \( U(1)_A \) charge \([O_0]_A = 0\). Since the combination \( O_0 A \) has the \( U(1)_A \) charge \( -(N_c^2 - 1)\chi_h \), it has a nontrivial value in general. We see that the anomaly-phase-quenched approximation of (3.13) can be expressed in the naive-phase-quenched approximation by

\[
\langle O_0 A \rangle^q = \langle O_0 A e^{i\theta_A} \rangle^q = \langle O_0 |A| \rangle^q,
\]

which is useful when we evaluate (3.13) in the Monte Carlo simulation.

Second, the compensator is not unique but has many possible choices. We propose the following three different compensators among them; the trace-type compensator,

\[
A_{\text{tr}} = \frac{1}{N_S} \sum_{s=1}^{N_S} \left( \frac{1}{N_c} \text{Tr} (\Phi_s)^2 \right)^{-\frac{N_c^2 - 1}{4}\chi_h},
\]

the Izykson-Zuber (IZ) type compensator,

\[
A_{\text{IZ}} = \frac{1}{N_l} \sum_{l=1}^{N_l} \left( \frac{1}{N_c} \text{Tr} \left( 2\Phi_{\text{org}(l)} U_l \Phi_{\text{tip}(l)} U_l^\dagger + \lambda_l \lambda_l (U_l \Phi_{\text{tip}(l)} U_l^\dagger + \Phi_{\text{org}(l)}) \right) \right)^{-\frac{N_c^2 - 1}{4}\chi_h},
\]

and the determinant-type compensator,

\[
A_{\text{det}} = \frac{1}{N_S} \sum_{s=1}^{N_S} (\text{Det} \Phi_s)^{-\frac{N_c^2 - 1}{2N_c}\chi_h}.
\]

Here we should note that there is an ambiguity in the branch of the exponents when they are fractional numbers. We choose the branch \( 0 \leq \text{arg}(z^{1/n}) \leq \frac{2\pi}{n} \) for \( z \in \mathbb{C} \) and \( n \geq 1 \) in this paper. It is notable that an appropriate type of compensators depends on topology of the background space, which we will discuss in details in the next section.

\footnote{In the present discretized theory, the off-diagonal non-zero modes also contribute to the Pfaffian phase.}
Third, there is no natural principle to fix the overall ± sign of the Pfaffian. In fact, we can easily see that the sign of the Pfaffian can flip by changing the order of the fermionic variables \( \vec{F} \) in defining the fermionic action. (Note that this is not a sign problem, but just ambiguity of a common sign of the Pfaffian.)

### 3.2 Ward-Takahashi identity

As a candidate of the measurements in the Monte Carlo simulation, we derive a WT identity corresponding to the \( Q \)-symmetry \(^6\).

Let us consider the integral,

\[
I \equiv \int D\vec{X} \zeta \Xi(\vec{X}) A(\vec{X}) e^{-S(\vec{X})},
\]

where \( \Xi \) in given in (2.23), \( A \) is a compensator and \( \zeta \) is a constant Grassmann number with the \( U(1)_A \) charge 1. Since the combination \( \zeta \Xi \) is neutral under the \( U(1)_A \) transformation, the integral \( I \) does not have the \( U(1)_A \) charge. The integral \( I \) itself is not altered even if we change all the variables \( X \in \vec{X} \) in the expression (3.18) to \( X' = X + \epsilon QX \). The expansions of \( S(\vec{X}') \) and \( \Xi(\vec{X}') \) by \( \epsilon \) are

\[
S(\vec{X}') = S(\vec{X}) + \frac{\mu^2}{2} \epsilon \sum_s \text{Tr}(\Phi_s \eta_s),
\]

\[
\Xi(\vec{X}') = \Xi + \epsilon Q \Xi = \Xi + \epsilon S_0,
\]

while the measure and the compensator are invariant. We then obtain the relation among the expectation values:

\[
\langle (\tilde{S}_b + S_f)A \rangle + \frac{N_F^2 - 1}{2} N_F \langle A \rangle + \frac{\mu^2}{2} \sum_s \langle \Xi \text{Tr}(\Phi_s \eta_s) A \rangle = 0,
\]

where we have integrated out the auxiliary field \( Y_f \) and divided the integral by \( Z_q \). Since the expectation value of the fermion action is related to the degrees of freedom of the fermions of the system. In fact, we can show in general that

\[
\langle S_f f(\Psi M \Psi) \rangle = -\frac{1}{2} \left( (N_c^2 - 1)(N_S + N_L + N_F) \right) \langle f(\Psi M \Psi) \rangle + \langle (\Psi M \Psi f' (\Psi M \Psi) \rangle,
\]

where \( f(x) \) is an arbitrary function, \( \Psi = (\eta, \lambda, \chi) \) is a vector of all the fermion degrees of freedom and \( M \) is an anti-symmetric matrix with the size of the fermion degrees of freedom. Therefore, we can further estimate \( \langle S_f A \rangle \) as

\[
\langle S_f A_{\text{tr/det}} \rangle = -\frac{1}{2} (N_c^2 - 1)(N_S + N_L + N_F) \langle A_{\text{tr/det}} \rangle,
\]

\(^6\) The identity we give here is not the WT identity in a usual sense but may be similar to the so-called PCAC relation or the PCSC relation considered in [30], since we have added a symmetry breaking term \( 2.22 \) to the action. In this paper, however, we will call it just the WT identity for simplicity.
for the trace-type and determinant-type compensators and

$$\langle S_f A_{IZ} \rangle = -\frac{1}{2} (N_c^2 - 1)(N_S + N_L + N_F) \langle A_{IZ} \rangle$$

$$- \left( \frac{N_c^2 - 1}{4} - \chi_h \right) \frac{1}{N_c} \sum_{l=1}^{N_L} \left\{ \frac{1}{N_c} \text{Tr} \left( \lambda_l \lambda_l \left( U_l \Phi_{\text{tip}(l)} U_l^\dagger + \Phi_{\text{org}(l)} \right) \right) \right\}$$

$$\times \frac{1}{N_c} \left\{ \text{Tr} \left( 2 \Phi_{\text{org}(l)} U_l \Phi_{\text{tip}(l)} U_l^\dagger + \lambda_l \lambda_l \left( U_l \Phi_{\text{tip}(l)} U_l^\dagger + \Phi_{\text{org}(l)} \right) \right) \right\}^{1/N_c^2 - 1/4} \chi_h^{-1},$$

(3.23)

for the IZ-type compensator. We then finally obtain the WT identity,

$$\langle \tilde{S}_b A_{\text{tr/det}} \rangle + \frac{\mu^2}{2} \sum_s \langle \Xi \text{Tr}(\Phi_s \eta_s) A_{\text{tr/det}} \rangle - \frac{N_c^2 - 1}{2} (N_S + N_L) \langle A_{\text{tr/det}} \rangle = 0$$

(3.24)

for the trace-type and determinant-type compensators and

$$\langle \tilde{S}_b A_{IZ} \rangle + \frac{\mu^2}{2} \sum_s \langle \Xi \text{Tr}(\Phi_s \eta_s) A_{IZ} \rangle - \frac{N_c^2 - 1}{2} (N_S + N_L) \langle A_{IZ} \rangle$$

$$- \frac{N_c^2 - 1}{4} - \chi_h \left\{ \frac{1}{N_c} \sum_{l=1}^{N_L} \left\{ \frac{1}{N_c} \text{Tr} \left( \lambda_l \lambda_l \left( U_l \Phi_{\text{tip}(l)} U_l^\dagger + \Phi_{\text{org}(l)} \right) \right) \right\} \right\}$$

$$\times \left\{ \frac{1}{N_c} \text{Tr} \left( 2 \Phi_{\text{org}(l)} U_l \Phi_{\text{tip}(l)} U_l^\dagger + \lambda_l \lambda_l \left( U_l \Phi_{\text{tip}(l)} U_l^\dagger + \Phi_{\text{org}(l)} \right) \right) \right\}^{1/N_c^2 - 1/4} \chi_h^{-1},$$

(3.25)

for the IZ-type compensator. We emphasize that this relation is exact even if it includes the parameter $\mu$ of the SUSY breaking term. In the Monte Carlo simulation, we check if these identities are satisfied in the anomaly-phase-quenched approximation (3.14).

In the next section, we evaluate the left hand side of the equation (3.24) for $\Sigma_0 (S^2)$ and $\Sigma_1 (T^2)$ and the equation (3.25) for $\Sigma_2$ in the anomaly-phase-quenched approximation. It gives a non-trivial check for the following claims:

1. The generalized Sugino model on the curved space is a valid extension of the Sugino model on the flat space $T^2$ and the model correctly reproduces the $U(1)_A$ anomaly through a unbalance of the fermion number due to the topology of the background space-time.

2. The quench of the phase $\theta$ among the whole Pfaffian phase $\theta_{\text{ Pf}} = \theta + \theta_A$ gives a correct result in the lattice simulation on the generic background space. It means that $e^{i\theta}$ is negligible and the sign problem due to the phase is absent in the generalized Sugino model on the generic background space-time.

4 Monte Carlo Simulations

In this section, we show the results of the Monte Carlo simulation of the generalized Sugino model on several polyhedra with $h = 0, 1, \text{ and } 2$. We first check if the flat directions of the scalar fields are properly controlled by the mass term (2.22). We next show that the anomaly-phase-quenched approximation properly works by evaluating the left-hand sides of (3.24) and
We investigate the behavior of the Pfaffian phase in details and show that we have no further sign problem in this model as long as the $U(1)_A$ anomaly-induced phase is cancelled by the compensator. We also check that the origin of the $U(1)_A$ anomaly originates from the light modes (pseudo-zero-modes) of the Dirac operator in the discretized model.

4.1 Algorithm and data analysis

We have used the rational hybrid Monte Carlo algorithm [52], where we ignore the phase of the Pfaffian $\theta_{Pf}$ and estimate the absolute value of the Pfaffian using the pseudo-fermion method by approximating the matrix $(D^\dagger D)^{-1/4}$ by rational functions,

$$(D^\dagger D)^{-1/4} \sim \alpha_0 + \sum_{r=1}^{N_r} \frac{\alpha_r}{\beta_r + D^\dagger D}. \tag{4.1}$$

We have set $N_r = 24$ and the coefficients $\alpha_r$ and $\beta_r$ are determined by the Remez algorithm. For each set of parameters, we have generated 15000 – 20000 configurations in general, while we have generated 80000 – 100000 configurations for some parameters to obtain a sufficient statistics. The simulation code is written in FORTRAN 90 and is not yet parallelized, which had run on personal computers with Intel Core i7. After measuring the autocorrelations, we have estimated the standard error by using the Jack Knife method.

4.2 Simulation parameters

In this paper, we consider only polyhedra with the same shape of the face as discretization of Riemann surfaces. For $h = 0$, we consider the five regular polyhedra, namely, the regular tetrahedron, the regular octahedron, the regular cube, the regular icosahedron and the regular dodecahedron. For $h = 1$, we consider the regular square lattice with the size of 3, 4 and 5. For $h = 2$, we consider such a double torus which is realized by gluing two regular square lattice with the size 3 by one surface (see Fig. 2). We fix the physical 'tHooft coupling as $\lambda_{\text{phys}} \equiv g_{\text{phys}}^2 N_c = 1$, thus the dimensionless coupling can be expressed as $\lambda = g^2 N_c = a^2$ with the lattice spacing $a$ (we note $\lambda_{\text{phys}} \equiv \lambda/a^2$ and $g_{\text{phys}}^2 \equiv g^2/a^2$). We identify the lattice spacing with length of a link and fix the surface area of background as $S_{\text{Area}} = 1$. Hence, the lattice spacing is given by

$$S_{\text{Area}} = a^2 \cdot \sigma \cdot \# \text{face} = 1, \tag{4.2}$$

where $\sigma$ is area of a face with unit sides,

$$\sigma = \begin{cases} \sqrt{3}/4 & \text{for triangle (T)} \\ 1 & \text{for square (S)} \\ 5/4 \cdot \tan(3\pi/10) & \text{for pentagon (P)} \end{cases}. \tag{4.3}$$

The continuum limit can be realized by $a \to 0$. We summarize geometries used for our simulations in Table 1.

The gauge group is restricted to $SU(2)$ and thus the parameter $m$ appeared in (2.19) to single out the physical vacuum was set to $m = 1$. For the scalar mass to stabilize the flat directions of $\Phi_s$, we have chosen the dimensionless mass parameter $\mu$ as appropriate values for each topology.

Recall that the role of the compensator is to cancel the non-vanishing $U(1)_A$ charge of the path-integral measure. Since the unbalance of the $U(1)_A$ charge is caused by the fermions, we
Figure 2: The development of the polyhedron with \( h = 2 \) we adopt in this paper. The links with the same symbol of arrows are identified. The faces are painted by gray. As a result, there are 14 sites, 32 links and 16 faces.

| genus | Euler ch. | geometry  | \( N_S \) | \( N_L \) | \( N_F \) | shape of face | lattice spacing |
|-------|-----------|-----------|----------|----------|----------|---------------|-----------------|
| 0     | 2         | tetra     | 4        | 6        | 4        | T             | 0.7598          |
|       |           | octa      | 6        | 12       | 8        | T             | 0.5373          |
|       |           | cube      | 8        | 12       | 6        | S             | 0.4082          |
|       |           | icosa     | 12       | 30       | 20       | T             | 0.3398          |
|       |           | dodeca    | 20       | 30       | 12       | P             | 0.2201          |
| 1     | 0         | 3 \times 3 \text{reg.lat.} | 9 | 18 | 9 | S | 0.3333 |
|       |           | 4 \times 4 \text{reg.lat.} | 16 | 32 | 16 | S | 0.2500 |
|       |           | 5 \times 5 \text{reg.lat.} | 25 | 50 | 25 | S | 0.2000 |
| 2     | -2        | Fig 2     | 14       | 32       | 16       | S             | 0.2500          |

Table 1: Topology and geometry for lattice simulations which we performed. The symbols in the fifth column T, S, P express “triangle”, “square” and “pentagon”, respectively, as shown in eq. (4.3).
can effectively regard the insertion of the compensator as the insertion of the fermions. On the $h = 0$ background, the number of the fermion modes of $\eta$ and $\chi$ in the measure exceeds that of $\lambda$. Since the $U(1)_A$ phase of $\eta$ and $\chi$ is related with that of the bosonic field $\Phi$ by the $Q$-symmetry, it is a natural and good choice to consider the compensator composed only of $\Phi$ in terms of numerical efficiency. We then adopt $A_{tr}$ for $h = 0$ (sphere). On the $h = 2$ background, the situation is opposite. In this case, we need to choose a compensator containing $\lambda$ itself, so we adopt $A_{IZ}$ in order to cancel the $U(1)_A$ phase efficiently for $h = 2$ (double torus). Since $A_{det}$ and $A_{tr}$ are identical up to sign for the gauge group $SU(2)$, we do not use $A_{det}$ in this paper.

### 4.3 One-point scalar correlation function

First of all, we should check if the flat directions of the scalar fields $\Phi_s$ and $\bar{\Phi}_s$ are properly controlled in the numerical simulation by adding the soft mass term. To this end, we examine the behavior of the one-point function,

$$
\frac{1}{N_s} \sum_s \left\langle \frac{1}{a^2 N_s} \text{Tr} (\Phi_s \bar{\Phi}_s) \right\rangle,
$$

(4.4)

where we have rescaled it by $a^2$ to identify the scalar fields as those in the continuum theory.

To see the rough behavior of the one-point function (4.4), it is useful to consider a real free scalar field $\phi(x)$ with mass $m$ in a two-dimensional curved background. The one-point function of $\phi(x)$ can be evaluated as

$$
\langle \phi(x)\phi(x) \rangle = \log \Lambda \frac{\Lambda^2}{m^2} + \frac{R}{24\pi m^2} + \mathcal{O}(\Lambda^{-2}),
$$

(4.5)

where $\Lambda$ is a UV cutoff and $R$ is the scalar curvature. This shows that the behavior of the one-point function against $m^2$ depends on the geometry of the background. Recalling $R > 0$ for the sphere, $R = 0$ for the torus and $R < 0$ for the double-torus, this suggests that (4.4) diverges by power for $h = 0$, diverges logarithmically for $h = 1$ and diverges milder than logarithm or converges to a finite value for $h \geq 2$ in taking the limit of $m \to 0$ (recall that the left hand side of (4.5) is positive).

We show the results of the corresponding numerical simulation of the discretized theory in Fig. 3 where we have plotted (4.4) against the square of the physical scalar mass $\mu^2/a^2$ for $h = 0, 1, 2$. We have used the logarithmic scale both in the $x$-axis and $y$-axis for $h = 0$ and only in the $x$-axis for $h = 1$ and 2. The fitting function is $f(x) = \alpha x^{\beta}$ for $h = 0$, $f(x) = \alpha \log(x) + \beta$ for $h = 1$ and $f(x) = \alpha x^{\beta} + \gamma$ for $h = 2$ with $x = \mu^2/a^2$ and the fitting is carried out by using the minimum number of the data for fitting from the smallest value of $\mu^2$ for each polyhedron. The fitting results are shown in Table. 2. These results are consistent with the observations from (4.5). We can thus conclude that we properly control the flat directions by adding the mass term. In particular, this result shows that the scalar fields on a sphere are unstabler than on a torus while the flat directions are expected to be effectively lifted up for $h \geq 2$.

### 4.4 Ward-Takahashi identity

In Fig 4, we show the WT identities using the anomaly-phase-quenched approximation for $h = 0$ (sphere), $h = 1$ (torus) and $h = 2$ (double torus), respectively. We plot the left-hand sides of the expressions (3.24) and (3.25) evaluated in the anomaly-phase-quenched approximation normalized by $\frac{1}{2} (N_c^2 - 1)(N_S + N_L) \langle \mathcal{A} \rangle^8$. We see that the WT identity is in good agreement with the theoretical predictions for the three cases of spacetime backgrounds. These results
Figure 3: Plot and fit of expectation values of the scalar bilinear \(\langle \phi \phi \rangle\). The fitting functions and results are summarized in Table 2.

Table 2: The fitting results of Fig. 3. The fitting function is chosen as

\[ f(\mu^2/a^2) = \alpha (\mu^2/a^2)^\beta \] for \( h = 0 \),

\[ f(\mu^2/a^2) = \alpha \log(\mu^2/a^2) + \beta \] for \( h = 1 \),

and

\[ f(\mu^2/a^2) = \alpha (\mu^2/a^2)^\beta + \gamma \] for \( h = 2 \).
indicate the three significant facts: Firstly, the anomaly-phase-quenched approximation works well. Secondly, the $U(1)_A$ anomaly is correctly reproduced in the present model since, if it is not the case, the anomaly-phase-quenched approximation does not work. Thirdly, the WT identity predicted from the analytical investigation is realized in the present model.

![Graphs showing WT identities for different values of $h$ and $\mu^2$.]

Figure 4: The left hand side of the WT identities (3.24) (the panels (1) and (2)) and (3.25) (the panel (3)) in the anomaly-phase-quenched approximation normalized by $\frac{1}{N^2(c-1)(N_S+N_L)}\langle A \rangle^R$ against to $\mu^2$ for $h = 0$ (left), $h = 1$ (middle) and $h = 2$ (right). We have used the compensator $A_{tr}$ for $h = 0$ and $A_{1Z}$ for $h = 2$ while we have set $A = 1$ for $h = 1$ since we do not need the compensator when $h = 1$.

4.5 Phase of the Pfaffian

In this subsection, we investigate the behavior of Pfaffian phase and show that the insertion of compensators settles the sign problem due to the anomaly-induced phase.

Let us first show the histogram of the phase of the Pfaffian $\theta_{PF}$ for $h = 1$ (torus) with $\mu^2 = 0.01$ in the left panel of Fig.5. We see that the phase is localized around $\theta_{PF} = 0$. This reproduces the previous result on the absence of the sign problem in 2D $\mathcal{N} = (2,2)$ SYM on the flat space-time shown in [33].

In the right panel of Fig.5, we show the histogram of the Pfaffian for $h = 0$ and $h = 2$, where we only show the result for the background with the smallest lattice spacing (dodecahedron) for $h = 0$ since the results for the others are the same. As seen from this figure, the phase
(1) The phase of the Pfaffian for $h = 1$

(2) The phase of the Pfaffian for $h = 0$ and $h = 2$

Figure 5: The histogram of the phase of the pfaffian for the $h = 1$ (left) and $h = 0, 2$ (right). For $h = 1$, we have shown the results for all the background whereas we have shown only the result for the dodecahedron for $h = 0$. The mass parameter is $\mu^2 = 0.01$.

is uniformly distributed to the whole region in contrast with that for torus ($h = 1$). This is not surprising because the integral measure or the Pfaffian is not $U(1)_A$-neutral except for the background with $\chi_h = 0$, and this property makes an expectation value of any $U(1)_A$-neutral operators exactly zero in the naive simulation.

Figure 6: The histogram of the phase of $\text{Pf}(D) A_{tr}$ for $h = 0$ (left) and $\text{Pf}(D) A_{IZ}$ for $h = 2$ (right). We have shown only the result for the dodecahedron for $h = 0$. The mass parameters are $\mu^2 = 0.01, 0.1$ and 0.03.

In order to manifest that the freely rotating phase of the Pfaffian for $h = 0, 2$ originates from the $U(1)_A$ charge of the Pfaffian, we show the histogram of the phase of the combination $\text{Pf}(D) A_{tr}$ for $h = 0$ and $\text{Pf}(D) A_{IZ}$ for $h = 2$ in Figs. 6. To look into the mass dependence of the residual phase, we plot the results with $\mu^2 = 0.001, 0.01$ and 0.03. As expected, there appear peaks in the small $\mu$ region in both of $h = 0$ and $h = 2$ as expected. The appearance of two peaks is inevitable because there is an ambiguity of signs in defining $A_{tr}$ and $A_{IZ}$ for $SU(2)$ for each configuration. The location of the peaks (around the $\pm \pi/2$) does not matter since it just
depends on the notation at the beginning. This result strongly suggests that the anomaly phase of the Pfaffian and the compensator cancel with each other and there is no sign problem as long as we introduce the appropriate compensator.

4.6 Origin of the anomaly

As well-known for the continuum gauge theory, if zero modes of Dirac operator carry $U(1)_A$ charges, the difference of numbers of the left- and right-handed zero modes, called the “index”, leads to the $U(1)_A$ anomaly. In a continuum version of the present theory on the curved space, the number of zero modes responsible for $U(1)_A$ anomaly, or equivalently the index, is proportional to the absolute value of the Euler characteristics $|\chi_h|$, which we expect to be equal to or larger than $\dim(G) \cdot |\chi_h|$ based on the index theorem. (These modes also contain “accidental zero modes”, which can get zero depending on values of the scalar field.) The counterparts of the zero modes relevant to $U(1)_A$ anomaly in the discretized theory are not exactly zero in general, since zero eigenvalues of the Dirac operator are lifted as lattice artifacts in general. Thus we call the modes relevant to $U(1)_A$ anomaly “pseudo-zero-modes” in the discretized theory. The number of pseudo-zero-modes is also expected to be equal to or larger than $\dim(G) \cdot |\chi_h|$.

![Figure 7: Six ($(N_c^2-1)\chi_h$) pseudo-zero-modes are plotted as orange points. Among non-zero modes (blue circle and green cross), the modes plotted as green crosses on the real axis are identified as Fourier modes.](image_url)

Now, let us manifest an algorithm to pick up the pseudo-zero-modes from the eigenvalues of the Dirac matrix of the discretized theory. We show a typical distribution plot of the eigenvalues for dodecahedron ($h = 0$) for a certain gauge configuration in Fig 7. Note that the distribution has a point symmetry because the Dirac matrix is anti-symmetric. As we mentioned, the number of pseudo-zero-modes is at least $(N_c^2-1)\chi_h = 6$ for $N_c = 2$, $\chi_h = 2$. Looking at Fig. 7 we see that there are always two modes close to the origin and four modes around them (orange points). There are also almost fixed modes on the real axis (green crosses), which are identified as Fourier modes. We thus regard the nearest 6 modes to the origin except for those on the real axis as (a part of) the pseudo-zero-modes.

It is notable that, since the Pfaffian is roughly the product of the half of the eigenvalues of the Dirac operator, the Pfaffian includes the half of the pseudo-zero-modes. Therefore the
Pfaffian except for these pseudo-zero-modes, which we call the subtracted Pfaffian $\text{Pf}'(D)$, is expected to be neutral under $U(1)_A$. Fig. 8 shows the histograms of the phase of the subtracted Pfaffian for the dodecahedron (left) with $\mu^2 = 0.001, 0.01, 0.03$. The phase of the subtracted Pfaffian strongly localizes around $\theta = 0, \pi$ in the both cases. The appearance of the two peaks is the result of the ambiguity of the sign of the subtracted Pfaffian, which appears because there is a choice of the eigenvalue from the pair $\pm \lambda_i$ in constructing $\text{Pf}'(D)$. This strongly supports that the origin of the anomaly in the generalized Sugino model is the pseudo-zero-modes of the Dirac operator. This result also justifies our identification of the six pseudo-zero-modes relevant to $U(1)_A$ anomaly, although full understanding on relation between pseudo-zero-modes and $U(1)_A$ anomaly will be investigated in our future work.

5 Summary and Discussion

In this paper, we have investigated the discretized two-dimensional $\mathcal{N} = (2, 2)$ supersymmetric theory on curved backgrounds both from the theoretical and numerical viewpoints. We made the global $U(1)_V$ symmetry gauged in the continuum $\mathcal{N} = (2, 2)$ supersymmetric theory, and showed that we can preserve two supercharges on any curved background by adding an appropriate $U(1)_V$ gauge field as a background. The model we consider in this paper is a discretization of this theory with keeping one of the two supercharges, where the other supercharge is restored in the continuum limit. We emphasize that the theory is a physical gauge theory, that is, we do not need to restrict the observables to the $Q$-cohomology.

We proposed the numerical calculation based on the novel phase-quenched method, which we call “the anomaly-phase quenched approximation”. This method is in general applicable to the cases that the Pfaffian phase of the Dirac operator includes the anomaly-induced phase in part. In the numerical calculation, we found that a WT identity associated with the $Q$-symmetry is satisfied in our model and the model correctly reproduces the $U(1)_A$ anomaly through the Euler characteristics of the background space. We also figure out the relation between the sign problem and pseudo-zero-modes of the Dirac operator and show how the scalar fields lift the flat direction depending on the topology.
In this paper, we have concentrated on the WT identity corresponding to the preserved supersymmetry and the anomaly of the discretized model, thus we do not take the continuum limit. Since we expect that another supersymmetry, which is explicitly broken by the discretization, is recovered in the continuum limit, we should check if the WT identity corresponding to the broken supersymmetry becomes to be satisfied in the continuum limit.

The construction of the generalized Sugino model is applicable to other two-dimensional gauge theories as well. It is definitely interesting to discretize the theory with the maximal supersymmetry, namely, 2D $\mathcal{N} = (8,8)$ SYM theory. In particular, we can modify this theory so that it allows fuzzy sphere solution with keeping supersymmetries, and it is straightforward to discretize the modified theory on a Riemann surface $\Sigma_h$. By repeating the discussion given in [36], we will be able to realize 4D $\mathcal{N} = 4$ SYM on $\Sigma_h \times \mathbb{R}^2_\theta$ or $\Sigma_h \times S^2$, where $\mathbb{R}^2_\theta$ expresses the Moyal plane with the non-commutative parameter $\theta$. This will give a strong method to investigate 4D $\mathcal{N} = 4$ SYM non-perturbatively.

Another application of the construction is a discretization of 2D SQCD, which has richer structure than SYM. For example, by adding matter multiplets, the partition function becomes sensitive to the topology of the gauge bundle in the continuum theory. It will be interesting to understand how it happens in the discretized theory.

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A Explicit form of the action

In this appendix, we show the explicit form of the generalized Sugino model. As mentioned in the subsection 2.3, the action is given by the summation of the $Q$-exact part (2.17) and the mass term (2.22). The explicit form can be straightforwardly derived by acting $Q$ to each part.

The bosonic action after integrating out the auxiliary field can be written as

$$\tilde{S}_b = S_b^S + S_b^L + S_b^F + S_\mu,$$

(A.1)
where

\[ S_b^S = \frac{1}{2g^2} \sum_{s=1}^{N_S} \alpha_s \text{Tr} \left[ \frac{1}{4} [\Phi_s, \bar{\Phi}_s]^2 \right], \quad (A.2) \]

\[ S_b^L = \frac{1}{2g^2} \sum_{l=1}^{N_L} \alpha_l \text{Tr} \left[ (U_l \Phi_{\text{tip}(l)} U_l^{-1} - \Phi_{\text{org}(l)}) (U_l \bar{\Phi}_{\text{tip}(l)} U_l^{-1} - \bar{\Phi}_{\text{org}(l)}) \right], \quad (A.3) \]

\[ S_b^F = \frac{1}{2g^2} \sum_{f=1}^{N_F} \alpha_f \beta_f^2 \frac{1}{4} \text{Tr} \left[ \Omega(U_f)^2 \right], \quad (A.4) \]

\[ S_\mu = \frac{1}{2g^2} \sum_{s=1}^{N_S} \text{Tr} \left[ \frac{\mu^2}{2} \Phi_s \bar{\Phi}_s \right]. \quad (A.5) \]

Remind that a face can be expressed by oriented links, so the link variable \( U_f \) in the face part is defined by \((2.21)\).

The fermion action is written as

\[ S_f = S_f^S + S_f^L + S_f^F \quad (A.6) \]

with

\[ S_f^S = \frac{1}{2g^2} \sum_{s=1}^{N_S} \alpha_s \text{Tr} \left[ -\frac{1}{4} \eta_s [\Phi_s, \eta_s] \right], \quad (A.7) \]

\[ S_f^L = \frac{1}{2g^2} \left\{ \sum_{l=1}^{N_L} \alpha_l \lambda_l [U_l \Phi_{\text{tip}(l)} U_l^{-1}, \lambda_l] \right. \\
+ \sum_{l=1}^{N_L} \frac{i}{2} \alpha_l \text{Tr} \left[ \lambda_l (U_l \eta_{\text{tip}(l)} U_l^{-1} - \eta_{\text{org}(l)}) \right] \\
+ \sum_{s=1}^{N_S} \left( \frac{-i}{2} \right) \text{Tr} \left[ \eta_s \left( \sum_{l \in <\bullet,s>} (\alpha_l U_l^{-1} \lambda_l U_l) - \sum_{l' \in <s,\bullet>} (\alpha_{l'} \lambda_{l'}) \right) \right] \right\}, \quad (A.8) \]

\[ S_f^F = \frac{1}{2g^2} \sum_{f=1}^{N_F} \alpha_f \text{Tr} \left[ \chi_f [\Phi_f, \chi_f] + \frac{i}{2} \beta_f \left\{ \chi_f Q \Omega(U_f) - (Q \Omega(U_f))^\dagger \chi_f \right\} \right], \quad (A.9) \]

where \( <\bullet,s> \) and \( <s,\bullet> \) mean a set of oriented links with \( s = \text{tip}(l) \) and \( s = \text{org}(l) \), respectively. In addition, we choose \( \Phi_f \) in the face part as a scalar field on a representative site included in the face. \( Q \Omega(U_f) \) in the second term of the face action can be written as

\[ Q \Omega(U_f) = \sum_{l \in f} \lambda_l T_a \frac{\partial \Omega(U_f)}{\partial A_l^a}, \quad \frac{\partial U_l}{\partial A^a} = iT^a U_l, \quad (A.10) \]

where the hermitian matrix \( T^a \) is a basis of the \( SU(N) \) Lie algebra and \( a \) is the gauge index.
running over $1, \cdots, \dim(G)$. More explicitly,

$$Q\Omega(U_f) = \frac{1}{m} \left[ -S^{-1}(U_f) \cdot Q\mathcal{S}(U_f) \cdot S^{-1}(U_f) \mathcal{C}(U_f) + S^{-1}(U_f) \cdot Q\mathcal{C}(U_f) \\
+ Q\mathcal{C}(U_f) \cdot S^{-1}(U_f) - \mathcal{C}(U_f) S^{-1}(U_f) \cdot Q\mathcal{S}(U_f) \cdot S^{-1}(U_f) \right], \quad (A.11)$$

$$QU_f^m = \sum_{n=1}^{m} U_f^{n-1} (QU_f) U_f^{m-n}, \quad (A.12)$$

$$QU_f = \sum_{j=1}^{n_f} U_{ij}^1 \cdots U_{ij}^{j-1} (QU_{ij}^j) U_{ij+1}^{j+1} \cdots U_{in_f}^{n_f}, \quad (A.13)$$

where

$$QU_f = i\lambda_i U_f, \quad QU_f^{-1} = -iU_f^{-1}\lambda_i. \quad (A.14)$$

$\mathcal{S}(U_f)$ and $\mathcal{C}(U_f)$ are defined by (2.20). Here, we have written the fermionic action so that the anti-symmetricity of the Dirac matrix becomes manifest.

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