Braneworld holography in Gauss-Bonnet gravity

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Abstract

We investigate holography on an \((n-1)\)-dimensional brane embedded in a background of AdS black holes, in \(n\)-dimensional Gauss-Bonnet gravity. We demonstrate that for a critical brane near the AdS boundary, the Friedmann equation corresponds to that of the standard cosmology driven by a CFT dual to the AdS bulk. We show that there is no holographic description for non-critical branes, or when the brane is further away from the AdS boundary. We then derive a Cardy-Verlinde formula for the dual CFT on the critical brane near the boundary. This gives us insight into the remarkable correspondence between Cardy-Verlinde formulæ and Friedmann equations in Einstein gravity.
1 Introduction

In recent years, there has been an enormous amount of research into two important areas of theoretical physics: braneworld cosmology and the holographic principle. The braneworld scenario gained momentum as a solution to the hierarchy problem [1, 2, 3, 4], although the single brane model of Randall and Sundrum provided us with an interesting alternative to compactification [5]. The holographic principle, meanwhile, was first realised in string theory via the AdS/CFT correspondence [6, 7, 8].

At first glance, braneworld physics and holography are two very distinct subjects. However, it was soon realised that this is not the case [9, 10, 11], and so began the study of braneworld holography (see for example [12, 13, 14], or [15] for a review).

The essence of braneworld holography can be captured in the following claim: Randall-Sundrum braneworld gravity is dual to a CFT with a UV cutoff, coupled to gravity on the brane. Formal evidence for this claim was provided by studying a brane universe in the background of the Schwarzschild-AdS black hole. The introduction of the black hole on the gravity side of the AdS/CFT correspondence corresponds to considering finite temperature states in the dual CFT [16]. In the context of braneworld holography, Savonije and Verlinde demonstrated that their induced braneworld cosmology could alternatively be described as the standard FRW cosmology driven by the energy density of this dual CFT [17, 13].

In this article we develop this notion of braneworld holography to include a broader class of bulk gravitational theories – namely we add the Gauss-Bonnet term to the standard Einstein-Hilbert action giving

$$S = \frac{1}{16\pi G_n} \int_M d^n x \sqrt{-g} \left\{ R - 2\Lambda_n + \alpha \mathcal{L}_{GB} \right\},$$  \hspace{1cm} (1)

where

$$\mathcal{L}_{GB} = R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd}.$$  \hspace{1cm} (2)

In $n = 4$ dimensions, the Gauss-Bonnet term is a topological invariant that does not enter the dynamics, but in $n = 5$ or 6, the equations of motion derived from this action include the Lovelock tensor [18]. With the inclusion of this tensor, these are then the most general equations of motion which satisfy the same principles required for the construction of the Einstein-Hilbert action in $n = 4$.

It is therefore natural to want to consider the Gauss-Bonnet gravity in higher dimensions from a purely classical point of view, for generic values of the Gauss-Bonnet parameter, $\alpha$. However, for small values of $\alpha$, the study of Gauss-Bonnet gravity can be motivated by string theory. Curvature squared terms appear as the leading order stringy correction to Einstein gravity in the $\alpha'$ expansion of the heterotic string action [19, 20]. Furthermore, for this theory of gravity to be ghost-free, the curvature squared terms must appear in the Gauss-Bonnet combination [21, 22, 23]. In the AdS/CFT correspondence, the introduction of such higher order terms corresponds to next to leading order corrections in the $1/N$ expansion of the CFT [24, 25, 26]. The importance of Gauss-Bonnet gravity in the framework of braneworld holography is thus self-evident. Previous studies of branes in higher derivative theories of gravity
suggest that no holographic description can be found \cite{27,28,29,30}. However, we demonstrate that a holographic description does exist in Gauss-Bonnet gravity, at least for critical branes, close to the boundary of AdS. This should come as no surprise. When the brane is close to the AdS boundary the UV cutoff in the dual CFT is not too significant, and one can justifiably appeal to the AdS/CFT correspondence.

In Einstein gravity, we were able to relax the constraint on the position of the brane \cite{31}. By calculating the bulk energy exactly via a Hamiltonian method, a larger equivalence was observed. When one reinterprets the black hole’s contribution to the braneworld cosmology as energy density due to the field theory, nonlinear terms in the energy density and pressure are found in the FRW equations. These exactly reproduce those of the unconventional cosmology \cite{32,33} described by a matterfilled brane in a pure AdS bulk. We coin the phrase “exact holography” to describe this equivalence.

The machinery to investigate exact holography in the Gauss-Bonnet scenario has recently become available \cite{34}. Using this, we are able to show that in contrast to Einstein gravity, a holographic description can only be found for flat branes near the AdS boundary. This has two important implications regarding the existence and behaviour of the Cardy-Verlinde formula for the dual field theory. Firstly, in the limit of a valid holographic description, it turns out that we can indeed cast the thermodynamic properties of the dual CFT into a Cardy-Verlinde like formula, provided we make consistent approximations. This is in contrast to previous studies which suggest that no Cardy-Verlinde formula can be found \cite{35,36}. Having found this formula, we are able to study its behaviour at the point that the brane crosses the black hole horizon. In Einstein gravity, we find that we reproduce the Friedmann equation! This does not happen in Gauss-Bonnet gravity. The difference helps us to understand what is special about the Einstein case. We believe that the remarkable correspondence between the Friedmann equation and the Cardy-Verlinde formula in Einstein gravity is related to the existence of exact holography.

The rest of this paper is organised as follows: In section \ref{sec:2} we describe the Gauss-Bonnet braneworld scenario and review the derivation of the Friedmann equations in this case. In section \ref{sec:3} we consider a brane moving in a pure (Gauss-Bonnet) AdS bulk, with additional matter on the brane. This enables us to establish the fine tuning condition for vanishing braneworld cosmological constant, and to derive the connection between the bulk and braneworld Newton constants. In section \ref{sec:4} we demonstrate that a holographic description exists for a flat brane moving near the boundary of a Gauss-Bonnet AdS black hole bulk. The cosmology is well described as being the standard cosmology driven by a dual CFT. In section \ref{sec:5} we show that there is no exact holography in Gauss-Bonnet gravity. In section \ref{sec:6} we derive a Cardy-Verlinde formula for the case where the holographic description is valid. We comment that this doesn’t make sense at the horizon, and gain insight into the remarkable properties of the Cardy-Verlinde formula in Einstein gravity. Finally, section \ref{sec:7} contains some concluding remarks.
2 Equations of motion

Consider an \( (n-1) \)-dimensional brane moving in an \( n \)-dimensional bulk, where \( n \geq 5 \). The bulk is a solution to Gauss-Bonnet gravity with a negative (bare) cosmological constant, \( \Lambda_n \). It is given by two spacetimes, \( M_1 \) and \( M_2 \), with boundaries \( \partial M_1 \) and \( \partial M_2 \) respectively. The brane can be thought of as a domain wall between the two spacetimes, so that it coincides with \( \partial M_1 \) and \( \partial M_2 \). For simplicity, we will assume that we have \( \mathbb{Z}_2 \) symmetry across the brane. This scenario is described by the following action,

\[
S = S_{\text{grav}} + S_{\text{brane}},
\]

where

\[
S_{\text{grav}} = \frac{1}{16\pi G_n} \int_{M_1 + M_2} d^n x \sqrt{-g} \left\{ R - 2\Lambda_n + \alpha L_{\text{GB}} \right\} + \int_{\partial M_1 + \partial M_2} \text{boundary terms},
\]

\[
S_{\text{brane}} = \int_{\text{brane}} d^{n-1} x \sqrt{-h} L_{\text{brane}}.
\]

The boundary integrals in \( S_{\text{grav}} \) are required for a well defined action principle [37] (see also [38]). We denote the bulk metric and the brane metric by \( g_{ab} \) and \( h_{ab} \) respectively. \( L_{\text{brane}} \) describes the matter content on the brane.

2.1 The bulk

For the action [39], the bulk equations of motion are given by

\[
R_{ab} - \frac{1}{2} R g_{ab} = -\Lambda_n g_{ab} + \alpha \left\{ \frac{1}{2} L_{\text{GB}} g_{ab} - 2RR_{ab} + 4R_{ac} R_{b}^{\ c} + 4R_{acbd} R^{cde} - 2R_{acde} R^{cde} \right\}
\]

(6)

Given the complexity of these equations, it is surprising that they admit the following family of simple static black hole solutions [39, 40] (see also [41, 42]):

\[
ds_n^2 = -h_{\text{BH}}(a) dt^2 + \frac{da^2}{h_{\text{BH}}(a)} + a^2 d\Omega_{n-2}^2,
\]

(7)

where \( d\Omega_{n-2}^2 \) is the metric on a unit \( (n-2) \)-sphere, and

\[
h_{\text{BH}}(a) = 1 + \frac{a^2}{2\tilde{\alpha}} (1 - \xi(a)) \quad \text{for} \quad \xi(a) = \sqrt{1 - 4\tilde{\alpha} k_n^2 + \frac{4\tilde{\alpha} \mu}{a^{n-1}}}.\]

(8)

By \( \mathbb{Z}_2 \) symmetry across the brane we have two identical black holes, one each living on either side of the brane. \( \mu \geq 0 \) is the constant of integration which determines the mass of each of these black holes [13, 14, 15, 34],

\[
M = \frac{(n-2)\Omega_{n-2}\mu}{16\pi G_n}.
\]

(9)
where $\Omega_{n-2}$ is the volume of a unit $(n-2)$-sphere. $k_n$ and $\tilde{\alpha}$ are related to the bulk cosmological constant and the Gauss-Bonnet parameter as follows,

$$\Lambda_n = -\frac{1}{2}(n-1)(n-2)k_n^2, \quad \tilde{\alpha} = (n-3)(n-4)\alpha$$  \hspace{1cm} (10)

Since we could consider associating $\alpha$ with the slope parameter \textit{i.e.} $\alpha'$ of heterotic string theory, from now on we will assume that $\alpha \geq 0$. Furthermore, for the metric to be real, we also have the condition

$$4\tilde{\alpha}k_n^2 \leq 1.$$  \hspace{1cm} (11)

These Gauss-Bonnet black hole solutions are asymptotically maximally symmetric and, in the limit $\alpha \to 0$, they reduce to the standard AdS black hole metric of Einstein gravity.

### 2.2 The brane

We now consider the dynamics of the brane, moving in the static black hole bulk. The brane is given by the section $(t(\tau), a(\tau), x^\mu)$ of the bulk metric, where the parameter $\tau$ corresponds to the proper time of an observer comoving with the brane. This gives the condition

$$-h_{\text{BH}}(a)\dot{t}^2 + \frac{\dot{a}^2}{h_{\text{BH}}(a)} = -1,$$  \hspace{1cm} (12)

where overdot corresponds to differentiation with respect to $\tau$. The induced metric is that of a FRW universe,

$$ds^2_{n-1} = -d\tau^2 + a(\tau)^2d\Omega^2_{n-2},$$  \hspace{1cm} (13)

with Hubble parameter $H = \dot{a}/a$. The equations of motion for the brane are determined by the junction conditions for a braneworld in Gauss-Bonnet gravity [46, 38]. Given that we have $\mathbb{Z}_2$ symmetry across the brane, these take the form

$$2(K_{ab} - Kh_{ab}) + 4\alpha(Q_{ab} - \frac{1}{3}Qh_{ab}) = -8\pi G_n S_{ab},$$  \hspace{1cm} (14)

where the energy momentum tensor on the brane is

$$S_{ab} = -\frac{2}{\sqrt{h}}\frac{\delta S_{\text{brane}}}{\delta h^{ab}},$$  \hspace{1cm} (15)

and

$$Q_{ab} = 2KK_{ac}K_b^c - 2K_{ac}K^{cd}K_{db} + K_{ab}(K_{cd}K^{cd} - K^2) + 2K\mathcal{R}_{ab} + \mathcal{R}K_{ab} - 2K^{cd}\mathcal{R}_{cdef} - 4\mathcal{R}_{ac}K_{b}^c.$$  \hspace{1cm} (16)

For a brane with unit normal, $n_a$, the extrinsic curvature of the brane is given by $K_{ab} = h\delta_{a}^{d}h\delta_{b}^{c}\nabla_{(c}n_{d)}$. $\mathcal{R}_{abcd}$ is the Riemann tensor on the brane, constructed from the induced metric $h_{ab}$.
Since the brane is homogeneous and isotropic, its energy momentum tensor is given in terms of its energy density, $\rho_{\text{brane}}$, and pressure, $p_{\text{brane}}$, as follows,

$$S_{ab} = (\rho_{\text{brane}} + p_{\text{brane}})\tau_a \tau_b + p_{\text{brane}} h_{ab},$$

(17)

where $\tau^a = (\dot{t}(\tau), \dot{a}(\tau), 0)$ is the velocity of a comoving observer. Given that the unit normal is $n_a = (-\dot{a}(\tau), \dot{t}(\tau), 0)$, we can evaluate the $\tau\tau$ component of (14) to give

$$\left(1 + \frac{4\dot{\alpha}^2}{3a^2} + \frac{2\dot{\alpha}}{a^2}\right) \frac{h_{BH}}{a} - 2\dot{\alpha} \left[\frac{h_{BH}}{a}\right]^2 \dot{t} = \frac{4\pi G_n}{(n-2)\rho_{\text{brane}}}.$$  

(18)

If we square this equation, and use the condition (12), we obtain the following equation for the Hubble parameter\(^1\), $H$,

$$\left[H^2 + \frac{h_{BH}}{a^2}\right] \left[\frac{4\dot{\alpha}}{3} H^2 + 1 + \frac{2\dot{\alpha}}{a^2} \left(1 - \frac{1}{3} h_{BH}\right)^2\right]^2 = \left(\frac{4\pi G_n}{n-2}\right)^2 \rho_{\text{brane}}.$$  

(19)

This is a cubic equation for $H^2$, with one real solution. We can extract this solution to write down the Friedmann equation for our braneworld, in its standard explicit form. To simplify the appearance of the equation we make the following definitions,

$$\lambda = 3\sqrt{\dot{\alpha}} \left(\frac{4\pi G_n}{n-2}\right) \rho_{\text{brane}},$$  

(20)

$$\zeta_{\pm}(a) = \left(\sqrt{\lambda^2 + \xi(a)^3} \pm \lambda\right)^{\frac{1}{3}}.$$  

(21)

The Friedmann equation now reads

$$H^2 = -\frac{1}{a^2} + \frac{\zeta_+^2(a) + \zeta_-^2(a) - 2}{4\dot{\alpha}}.$$  

(22)

If we expand this equation in $\dot{\alpha}$, then to lowest (zeroth) order, we recover the Friedmann equation for a brane in Einstein gravity \cite{32,33}, as indeed we should,

$$H^2 = \left(\frac{4\pi G_n}{n-2}\right)^2 \rho_{\text{brane}}^2 - k_n^2 - \frac{1}{a^2} + \frac{\mu}{a^{n-1}}.$$  

(23)

3 Brane moving in an AdS background

We now consider the case where the bulk spacetime is pure AdS space. This corresponds to the case $\mu = 0$ in our bulk metric (7). In other words, the bulk metric is now given by

$$ds_n^2 = -h_{\text{AdS}}(a) dt^2 + \frac{da^2}{h_{\text{AdS}}(a)} + a^2 d\Omega_{n-2}^2,$$

(24)

\(^1\)For the original derivation of this equation, see \cite{42}. Alternative forms of the Friedmann equation for a braneworld in the Gauss-Bonnet bulk were derived in \cite{47,71,48,49}.
where
\[ h_{\text{AdS}}(a) = 1 + k_{\text{eff}}^2 a^2. \] (25)

The effective cosmological constant is given by \( \Lambda_{\text{eff}} = -\frac{1}{2} (n-1)(n-2) k_{\text{eff}}^2 \). This differs from the bare cosmological constant, \( \Lambda_n \), because of the Gauss-Bonnet correction,
\[ k_{\text{eff}}^2 = \frac{1 - \beta}{2\bar{\alpha}} \quad \text{where} \quad \beta = \sqrt{1 - 4\bar{\alpha}k_n^2}. \] (26)

We now assume that the energy-momentum of the brane splits into a contribution from the brane tension, \( \sigma \), and a contribution from additional matter fields. We write
\[ \rho_{\text{brane}} = \rho + \sigma, \quad p_{\text{brane}} = p - \sigma, \] (27)
where \( \rho \) and \( p \) are the energy density and pressure respectively, of the additional matter fields. If we define
\[ \bar{\sigma} = 3\sqrt{\bar{\alpha}} \left( \frac{4\pi G_n}{n-2} \right) \sigma, \] (28)
and
\[ \zeta^\pm = \left( \sqrt{\bar{\sigma}^2 + \beta^2} \pm \bar{\sigma} \right)^{\frac{1}{2}}, \] (29)
then we can expand the Friedmann equation (22) about \( \rho = 0 \),
\[ H^2 = A - \frac{1}{a^2} + \frac{2\pi G_n (\zeta^+_2 - \zeta^-_2)}{(n-2)\sqrt{\bar{\alpha}\sqrt{\bar{\sigma}^2 + \beta^2}}} \rho + \mathcal{O}(\rho^2), \] (30)
where
\[ A = \frac{\zeta^+_2 + \zeta^-_2 - 2}{4\bar{\alpha}}. \] (31)

For \( \rho \ll \sigma \), this looks like the Friedmann equation for the standard cosmology of an \((n-1)\)-dimensional \( \kappa = 1 \) universe,
\[ H^2 = A - \frac{1}{a^2} + \frac{16\pi G_{n-1}}{(n-2)(n-3)} \rho, \] (32)
where the cosmological constant \( \Lambda_{n-1} = \frac{1}{2} (n-2)(n-3)A \).

Let us restrict our attention to critical branes with vanishing cosmological constant, \( A = 0 \). This requires us to fine tune the brane tension in the following way,
\[ \bar{\sigma} = (2 + \beta) \sqrt{1 - \beta}. \] (33)

In this case, the coefficient multiplying \( \rho \) in the Friedmann equation can be simplified dramatically, so that we now have
\[ H^2 = -\frac{1}{a^2} + \frac{8\pi G_n k_{\text{eff}}}{(n-2)(2 - \beta)} \rho + \mathcal{O}(\rho^2). \] (34)
For $\rho \ll \sigma$, we can compare this with equation (32) to find an expression for the Newton’s constant on the brane,

$$G_{n-1} = \frac{(n - 3)G_n k_{\text{eff}}}{2(2 - \beta)}.$$  \hspace{1cm} (35)

For the five dimensional bulk, this agrees with the expression derived in [50] (see also [51]). Furthermore, for the case of general $n$, it agrees with the standard relation for critical branes in the $\alpha \to 0$ limit [52].

4 Braneworld holography

We will now consider the case where $\mu > 0$, that is, when the brane is moving in a Gauss-Bonnet black hole bulk. We will assume that there is no additional matter on the brane, so that its energy-momentum only contains brane tension,

$$\rho_{\text{brane}} = \sigma, \quad p_{\text{brane}} = -\sigma. \hspace{1cm} (36)$$

For Einstein gravity ($\alpha = 0$), it has been shown that when the brane is near the AdS boundary, we can think of its dynamics as being described by a radiation dominated FRW universe. This radiation is given by a strongly coupled CFT with an AdS dual description [13].

It is natural to ask if these ideas can be extended to branes moving in a Gauss-Bonnet bulk, with $\alpha \sim k_n^{-2}$. In this section, we will demonstrate that, for critical branes ($A = 0$) near the AdS boundary, they can.

We begin by clarifying what we mean by “near the AdS boundary”. We mean that the brane position is given by $a(\tau) \gg a_H$, where $a_H$ is the radius of the black hole horizon ($h_{\text{BH}}(a_H) = 0$). However, for critical branes ($A = 0$) and anti-de Sitter branes ($A < 0$), the trajectory will have a maximum value of $a$. In order to have $a \gg a_H$, we require that $a_H \gg \sqrt{\alpha}$. In fact, a discussion of anti-de Sitter branes cannot be included as the large $a$ limit also requires $|A| \ll \alpha^{-1}$ when $A < 0$ (see appendix A).

For $a \gg a_H$, the Friedmann equation (22) can be approximated\(^2\) by

$$H^2 = A - \frac{1}{a^2} + \frac{\zeta^* + \xi^*}{2\sqrt{\sigma^2 + \beta^2}} \frac{\mu}{a^{n-1}}.$$ \hspace{1cm} (37)

As in section 3, we restrict attention to critical branes. The fine-tuning of the brane tension (33) simplifies the coefficient of $\mu$, so that our Friedmann equation becomes

$$H^2 = -\frac{1}{a^2} + \frac{\mu}{(2 - \beta)a^{n-1}}.$$ \hspace{1cm} (38)

\(^2\)We are assuming $1 > 4\tilde{\alpha}k_n^2$, although the argument presented in this section can be modified to include $1 = 4\tilde{\alpha}k_n^2$. 

8
Our main interest lies in the contribution from the black hole masses. As in Einstein gravity, we will show that this contribution can be thought of as coming from the energy density of a dual CFT – we will now calculate this energy density.

The energy of the bulk is given by the sum of the black hole masses, \( E_{\text{bulk}} = 2M \). This energy is measured with respect to the bulk time coordinate, \( t \), whereas an observer on the brane measures energy with respect to the brane time coordinate, \( \tau \). To arrive at the energy of the CFT, we therefore need to scale the bulk energy by \( \dot{t} \), \( E_{\text{CFT}} = E_{\text{bulk}} \dot{t} \).

This redshift factor can be found using equation (18). Given that we are near the AdS boundary we find that

\[
\dot{t} \approx \frac{2\sqrt{\tilde{\alpha} \tilde{\sigma}}}{(1 - \beta)(2 + \beta)a}.
\]  

(39)

We now impose the fine tuning condition (33), to give

\[
\dot{t} \approx \frac{1}{k_{\text{eff}} a}.
\]  

(40)

The CFT energy is therefore given by

\[
E_{\text{CFT}} = 2M \dot{t} \approx \frac{(n - 2)\Omega_{n-2}\mu}{8\pi G_n k_{\text{eff}} a}.
\]  

(41)

To calculate the energy density, we need to divide by the spatial volume of the CFT,

\[
V_{\text{CFT}} = \Omega_{n-2}a^{n-2}.
\]  

(42)

Finally we arrive at the following expression for the energy density of the dual CFT,

\[
\rho_{\text{CFT}} = \frac{(n - 2)\mu}{8\pi G_n k_{\text{eff}} a^{n-1}}.
\]  

(43)

We now rewrite the Friedmann equation (38) in terms of this energy density.

\[
H^2 = -\frac{1}{a^2} + \frac{8\pi G_n k_{\text{eff}}}{(n - 2)(2 - \beta)} \rho_{\text{CFT}}.
\]  

(44)

Using the relation between the braneworld and bulk Newton’s constants (35), the Friedmann equation becomes,

\[
H^2 = -\frac{1}{a^2} + \frac{16\pi G_{n-1} a^{n-1}}{(n - 2)(n - 3)} \rho_{\text{CFT}}.
\]  

(45)

This is just the Friedmann equation for the standard cosmology in \((n - 1)\) dimensions. The cosmology is driven by a strongly coupled CFT, which is dual to the AdS black hole bulk. For critical branes near the AdS boundary, we conclude that there is a holographic description even when the bulk gravity includes a Gauss-Bonnet correction.
We should also note that we can use the thermodynamic relation, \( p = -\partial E/\partial V \), to derive the CFT pressure from the energy density. The equation of state corresponds to that of radiation,

\[
p_{\text{CFT}} = \frac{\rho_{\text{CFT}}}{n-2}.
\]  

(46)

If we differentiate the Friedmann equation (38) with respect to \( \tau \), then the resulting equation,

\[
\dot{H} = \frac{1}{a^2} - \frac{(n-1)\mu}{2(2-\beta)a^{n-1}},
\]  

(47)

can be written as the second of the FRW equations of the standard cosmology,

\[
\dot{H} = \frac{1}{a^2} - \frac{8\pi G n^{-1}}{(n-3)} (\rho_{\text{CFT}} + p_{\text{CFT}}).
\]  

(48)

5 Exact holography?

The AdS/CFT correspondence relates gravity on \( n \)-dimensional AdS space to a CFT on \((n-1)\)-dimensional Minkowski space. In braneworld holography, the field theory on the brane is cutoff in the UV. This cutoff vanishes as the brane approaches the boundary of AdS so that the field theory becomes conformal. Only at this point can we confidently appeal to the AdS/CFT correspondence. Although it is natural to expect a holographic description for critical branes near the AdS boundary there is no reason to expect more. However, in Einstein gravity, it has been shown that a holographic description exists for non-critical branes [14]. Perhaps even more surprisingly, there is a form of exact holography in Einstein gravity [31]. This is where the condition that the brane should be near the AdS boundary is relaxed. In this section we ask whether or not the same generalisations can be made in Gauss-Bonnet gravity.

5.1 Exact holography in Einstein gravity

We start by reviewing precisely what we mean by exact holography in Einstein gravity. Consider a brane moving in pure AdS space, with

\[
h_{\text{AdS}}(a) = k_n^2 a^2 + 1.
\]  

(49)

As in section 3, we assume that the energy-momentum of the brane is made up of tension, \( \sigma \), and additional matter with energy density, \( \rho \), and pressure, \( p \). The Friedmann equation is [32][33]

\[
H^2 = \frac{A}{a^2} + \frac{16\pi G n_{-1}(n-3)\rho}{(n-2)(n-3)} \left[ 1 + \frac{\rho}{2\sigma} \right].
\]  

(50)

This takes the form of the \((n-1)\)-dimensional standard cosmology when \( \rho \ll \sigma \).

Now consider a brane with no additional matter, moving in an AdS black hole bulk, with

\[
h_{\text{BH}}(a) = k_n^2 a^2 + 1 - \frac{\mu}{a^{n-3}}.
\]  

(51)
In this case the Friedmann equation is given by

$$H^2 = A - \frac{1}{a^2} + \frac{\mu}{a^{n-1}}. \tag{52}$$

In [31], we showed how we can calculate exactly the energy density, $\rho_{FT}$, measured by an observer on the brane – this can be done without assuming that the brane is near the AdS boundary. $\rho_{FT}$ is given in terms of $\mu$, so we can rewrite the Friedmann equation (52) to give

$$H^2 = A - \frac{1}{a^2} + \frac{16\pi G_{n-1}}{(n-2)(n-3)}\rho_{FT} \left[ 1 + \frac{\rho_{FT}}{2\sigma} \right]. \tag{53}$$

This takes exactly the same form as the Friedmann equation (50) for the brane moving in pure AdS space with additional matter on the brane. We can therefore think of $\rho_{FT}$ as being the energy density of a field theory living on the brane. This field theory is dual to the AdS black hole bulk, although it is no longer conformal. We think of the dual field theory on the brane as being cut off in the ultra violet – this cutoff disappears as we go closer and closer to the AdS boundary, and we approach a conformal field theory. In this case, we are not assuming that the brane is near the boundary, so the cutoff can be significant.

### 5.2 Exact holography in Gauss-Bonnet gravity?

We shall now investigate whether or not exact holography exists when the bulk is a Gauss-Bonnet black hole. Our result relies on the expression for the Gauss-Bonnet Hamiltonian found in [34]. We will find it convenient to rederive the Friedmann equation from the action given in that paper.

Consider the timelike vector field defined on the brane,

$$\tau^a = (\dot{t}, \dot{a}, 0). \tag{54}$$

This maps the brane onto itself and satisfies $\tau^a \nabla_a \tau = 1$. In principle we can extend the definition of $\tau$ into the bulk, stating only that it approaches the form given by equation (54) as it nears the brane. Now introduce a family of spacelike surfaces, $\{\Sigma_\tau\}$, each labelled by the parameter $\tau$, so that we have a foliation of the bulk spacetime. These surfaces should meet the boundary/brane orthogonally. We decompose $\tau^a$ into the lapse function and shift vector,

$$\tau^a = N r^a + N^a, \tag{55}$$

where $r^a$ is the unit normal to $\Sigma_\tau$.

We can now write the bulk metric in ADM form

$$ds_n^2 = g_{ab}dx^a dx^b = -N^2 d\tau^2 + \gamma_{ab}(dx^a + N^a d\tau)(dx^b + N^b d\tau), \tag{56}$$

where $\gamma_{ab}$ is the induced metric on $\Sigma_\tau$. 

11
If $S_\tau$ is the intersection of the brane and $\Sigma_\tau$, then the family of surfaces $\{S_\tau\}$ is a foliation of the brane. In the same way as for the bulk, we can write the brane metric in ADM form,

$$ds_{n-1}^2 = h_{ab}dx^a dx^b = -N^2 d\tau^2 + \lambda_{ab}(dx^a + N^a d\tau)(dx^b + N^b d\tau), \quad (57)$$

where $\lambda_{ab}$ is the induced metric on $S_\tau$.

Now, from [34], we can write the action as

$$S = S_{\text{grav}} + S_{\text{brane}}, \quad (58)$$

in which

$$S_{\text{grav}} = \frac{1}{8\pi G_n} \int dt \left\{ \int_{\Sigma_t} d^{n-1}x \left( \pi^{ab}_{\gamma_b} - N\mathcal{H} - N^a\mathcal{H}_a \right) - \int_{S_t} d^{n-2}x \sqrt{\lambda}(N\mathcal{J} + N^a\mathcal{J}_a) \right\}, \quad (59)$$

where $\pi^{ab}$ is the momentum conjugate to $\gamma_{ab}$, and $\mathcal{H}$ and $\mathcal{H}_a$ are the Hamiltonian and momentum constraints respectively. We have an overall factor of 2 by $\mathbb{Z}_2$ symmetry across the brane.

The $\tau\tau$ component of the junction conditions at the brane gives rise to the Friedmann equation. This is obtained by varying the brane part of the action with respect to the brane metric. For the $\tau\tau$ component this amounts to variation with respect to $N$, which gives

$$J + N \left( \frac{\delta \mathcal{J}}{\delta N} \right) + N^a \left( \frac{\delta \mathcal{J}_a}{\delta N} \right) = \frac{8\pi G_n}{\sqrt{\lambda}} \frac{\delta S_{\text{brane}}}{\delta N}. \quad (60)$$

The energy-momentum tensor on the brane is given by

$$S^{ab} = \frac{2}{\sqrt{h}} \frac{\delta S_{\text{brane}}}{\delta h_{ab}}. \quad (61)$$

Furthermore, since $h_{\tau\tau} = -N^2$ and $\sqrt{h} = N\sqrt{\lambda}$, we find that

$$\frac{\delta S_{\text{brane}}}{\delta N} = -N^2 \sqrt{\lambda} S^{\tau\tau}, \quad (62)$$

so that the Friedmann equation now reads

$$J + N \left( \frac{\delta \mathcal{J}}{\delta N} \right) + N^a \left( \frac{\delta \mathcal{J}_a}{\delta N} \right) = -8\pi G_n N^2 S^{\tau\tau}. \quad (63)$$

Note that we have a homogeneous isotropic brane,

$$ds_{n-1}^2 = -d\tau^2 + a^2 d\Omega_{n-2}^2, \quad (64)$$

moving in a static bulk,

$$ds_n^2 = -h(a)dt^2 + \frac{da^2}{h(a)} + a^2 d\Omega_{n-2}^2. \quad (65)$$
Now, on the brane, $\tau^a$ is in fact the unit normal to $S_t$. This means that $N = 1$ and $N^a = 0$ on the brane, although this need not be the case in the bulk. We also have $S^{\tau\tau} = \rho_{\text{brane}}$. Putting all this information back into the Friedmann equation, we now have

$$\mathcal{J} + \mathcal{J}' = -8\pi G_n \rho_{\text{brane}}, \quad \text{where} \quad \mathcal{J}' = \frac{\delta \mathcal{J}}{\delta N}. \quad (66)$$

Now consider two different scenarios: (i) a bulk AdS black hole, with the brane matter made up of tension only, and (ii) a pure AdS bulk, with the brane matter made up of tension, and some additional matter such as radiation. For case (i), we have $h(a) = h_{\text{BH}}(a)$ and $\rho_{\text{brane}} = \sigma$, where $\sigma$ is the brane tension. For case (ii), we have $h(a) = h_{\text{AdS}}(a)$ with $\rho_{\text{brane}} = \sigma + \rho$, where $\rho$ is the energy density of the additional matter. From a holographic point of view, we would expect these two cases to be equivalent if the energy density, $\rho$, corresponds to that of a field theory dual to the bulk black hole gravity.

Let us begin by considering the case with the bulk black hole. The Friedmann equation reads

$$\mathcal{J}_{\text{BH}} + \mathcal{J}'_{\text{BH}} = -8\pi G_n \sigma. \quad (67)$$

In order to see if we have a holographic description in the way we have just described, we need to calculate the energy density of the bulk, as measured by an observer on the brane—in other words, we want to calculate the energy density of the bulk using $\tau$ as our time coordinate. This is done by evaluating the Hamiltonian with a suitable choice of background. We choose the background, $\tilde{\mathcal{M}}$, to be pure AdS space (with the same effective cosmological constant as the black hole spacetime),

$$ds_{\text{AdS}}^2 = -h_{\text{AdS}}(a)dt^2 + \frac{da^2}{h_{\text{AdS}}(a)} + a^2 d\Omega_{n-2}^2, \quad (68)$$
cut off at a surface, $\partial \tilde{\mathcal{M}}$, given by

$$T = T(\tau), \quad a = a(\tau) \quad \text{where} \quad -h_{\text{AdS}}(a)\dot{T}^2 + \frac{\dot{a}^2}{h_{\text{AdS}}(a)} = -1. \quad (69)$$

This ensures that the geometry on $\partial \tilde{\mathcal{M}}$ is the same as that on the brane.

Given that $N = 1$ and $N^a = 0$ on the brane, we evaluate the Hamiltonian to derive the energy of the bulk measured with respect to $\tau$,

$$\mathcal{E} = \frac{1}{8\pi G_n} \int_{S_\tau} d^{n-2}x \sqrt{\lambda} (\mathcal{J}_{\text{BH}} - \mathcal{J}_{\text{AdS}}), \quad (70)$$

where we have a factor of two because there are two copies of the bulk. To get the energy density we need to divide by the spatial volume of the brane,

$$V = \int_{S_\tau} d^{n-2}x \sqrt{\lambda}. \quad (71)$$

We thus see that

$$\rho = \frac{\mathcal{E}}{V} = \frac{1}{8\pi G_n} (\mathcal{J}_{\text{BH}} - \mathcal{J}_{\text{AdS}}). \quad (72)$$
If we substitute this back into the Friedmann equation (67), we obtain
\[ J_{\text{AdS}} + J'_{\text{AdS}} + (J'_{\text{BH}} - J'_{\text{AdS}}) = -8\pi G_n (\sigma + \rho). \] (73)

It is clear that for the holographic description to be valid, we need to ignore the contribution from \((J'_{\text{BH}} - J'_{\text{AdS}})\). This is achieved if we satisfy the condition
\[ |J| \gg |J'|. \] (74)

For Gauss-Bonnet gravity, we have
\[ J = 2K + 12\alpha \delta_{[b}^i \delta_{c]}^m \delta_{c]}^n K_l^a \left[ \hat{R}^{bc}_{\text{mn}} - 2H^b_m H^c_n - \frac{2}{3} K^b_m K^c_n \right], \] (75)

where \(\hat{R}^{bc}_{\text{mn}}\) is the Riemann tensor on \(S_\tau\), \(K^a_b\) is the extrinsic curvature of \(S_\tau\) in \(\Sigma_\tau\) and \(H^a_b\) is the extrinsic curvature of \(S_\tau\) in the brane. To evaluate \(J'\), we need to vary equation (75) with respect to \(N\). However, the only term that depends on \(N\) is \(H^a_b\), which is proportional to \(1/N\). Therefore,
\[ J' = \frac{48}{\tilde{\alpha}} 2(n-2) \sqrt{H^2 + \frac{h(a)}{a^2}} \left\{ 1 + 2\tilde{\alpha} \left[ \frac{1}{a^2} - \frac{4}{3} H^2 - \frac{1}{3} \frac{h(a)}{a^2} \right] \right\}, \] (78)

where \(\tilde{\alpha} = (n-3)(n-4)\alpha\), as adopted in (10). For \(\alpha \sim k_n^{-2}\), the condition (74) amounts to,
\[ \alpha^{-1} \gg H^2. \] (80)

We are now ready to ask if we can have exact holography like we did for Einstein gravity. For \(a \sim \sqrt{\alpha}\), it is clear that both sides of (80) are of order \(\alpha^{-1}\), and the
condition (74) does not hold. There is no exact holography in Gauss-Bonnet gravity. This is not really surprising – the fact that we found exact holography for Einstein gravity was remarkable. As we suggested earlier, via the AdS/CFT motivation for braneworld holography, we would only really expect to find a holographic description for critical branes near the AdS boundary.

Given that we know that the condition (74) doesn’t hold for general values of $a$, we now ask what happens when $a$ is large. Recall that we must immediately eliminate the possibility of anti-de Sitter branes ($\mathcal{A} < 0$). For critical branes and de Sitter branes, $H^2 \sim \mathcal{A}$ for large $a$, so the condition (74) only holds if we have

$$a^{-1} \gg \mathcal{A}. \quad (81)$$

This suggests that we would only find a holographic description for critical branes satisfying $\mathcal{A} = 0$. Unlike in Einstein gravity, there will be no extension to non-critical branes.

To sum up, we have found a condition that determines when a holographic description will be valid for a brane moving in a black hole bulk. This condition is satisfied for Einstein gravity, regardless of the brane’s position, or the value of the braneworld cosmological constant. For Gauss-Bonnet gravity, the condition is far more restrictive. It only holds for critical branes close to the AdS boundary. We conclude that the holographic description shown in the last section is the only one you can find for branes in Gauss-Bonnet gravity.

## 6 Cardy-Verlinde formulæ

The Cardy-Verlinde formula [17] for a CFT is the generalisation to arbitrary dimensions of the well known Cardy formula [53] for 1 + 1-dimensional CFTs. It relates the entropy, $S$, of the CFT, to its energy, $E$, and Casimir energy, $E_c$. The Casimir energy can be thought of as providing the non-extensive part of the formula. In this paper, we will discuss the local version of the Cardy-Verlinde formula for a CFT living on an FRW brane.

We begin with the thermodynamic relation

$$dE = T \, dS - p \, dV, \quad (82)$$

where $T$, $p$ and $V$ are the temperature, pressure and volume of the CFT respectively. If we introduce the following densities,

$$s = \frac{S}{V}, \quad \rho = \frac{E}{V}, \quad (83)$$

we can use the fact that $V \sim a^{n-2}$ to rewrite the thermodynamic relation in the following form,

$$d\rho = T \, ds + \gamma d \left( \frac{1}{a^2} \right), \quad (84)$$

15
where
\[
\gamma = \frac{(n - 2)a^2}{2}(\rho + p - Ts). \tag{85}
\]
We can think of \(\gamma\) as describing the variation of \(\rho\) with respect to the spatial curvature, \(1/a^2\). If the entropy and energy were purely extensive, \(\gamma\) would vanish. \(\gamma\) will therefore give the non-extensive part of the local Cardy-Verlinde formula.

In section \(\text{[3]}\) we showed that for a critical brane near the AdS boundary, the dynamics is driven by a CFT that is dual to the Gauss-Bonnet AdS black hole bulk. Eventually we will state the Cardy-Verlinde formula for this CFT. However, first we will review the form of the Cardy-Verlinde formula for a CFT that is dual to an AdS black hole bulk in Einstein gravity.

### 6.1 Cardy-Verlinde formulæ for CFTs with AdS duals in Einstein gravity

Now consider a critical brane moving in a black hole bulk, in Einstein gravity\(\text{[13]}\). When it is near the AdS boundary, the CFT on the brane obeys the following Cardy-Verlinde formula,
\[
s^2 = \left(\frac{4\pi}{n - 2}\right)^2 \gamma \left(\rho - \frac{\gamma}{a^2}\right). \tag{86}
\]
A remarkable connection between this formula, and the Friedmann equation was noted in \(\text{[13]}\). At the point that the brane crosses the horizon, the entropy density is given by the Hubble entropy,
\[
s = \frac{(n - 3)H}{4G_{n-1}}, \tag{87}
\]
and \(\gamma = (n - 2)(n - 3)/16\pi G_{n-1}\). The Cardy-Verlinde formula now reads
\[
H^2 = -\frac{1}{a^2} + \frac{16\pi G_{n-1}}{(n - 2)(n - 3)} \rho. \tag{88}
\]
This is precisely the Friedmann equation for the standard cosmology!

However, we should be cautious. Contrary to the claim made in \(\text{[13]}\), the formula \((86)\) is only valid when \(k^{-1}_n \ll a_H \ll a\). This must be the case because the dual field theory ceases to be conformal for smaller values of \(a\). We would therefore expect the structure of the formula to change, and an extra scale to be introduced, reflecting the fact that conformal invariance has been broken.

Since the holographic description exists in Einstein gravity for all values of \(a\), it was possible to find a more exact version of this formula \(\text{[31]}\).

\[
s^2 = \left(\frac{4\pi}{n - 2}\right)^2 \gamma \left(1 + \frac{\rho}{\sigma}\right) \left[\rho \left(1 + \frac{\rho}{2\sigma}\right) - \frac{\gamma}{a^2} \left(1 + \frac{\rho}{\sigma}\right)\right]. \tag{89}
\]
The brane tension, \(\sigma\) now appears as the new scale in the formula. For \(\rho \ll \sigma\), this formula reduces to its conformal version \((86)\).
It is interesting to evaluate this formula at the point that the brane crosses the horizon. Once again, the entropy density is given by the Hubble entropy, but this time we have a more precise formula for $\gamma$. It is given by

$$\gamma \left(1 + \frac{\rho}{\sigma}\right) = \frac{(n-2)(n-3)}{16\pi G_{n-1}}. \quad (90)$$

The generalised Cardy-Verlinde formula now coincides with the exact braneworld Friedmann equation

6.2 Cardy-Verlinde formulæ for CFTs with AdS duals in Gauss-Bonnet gravity

Now consider the case of the critical brane moving in a Gauss-Bonnet black hole bulk. For the brane near the AdS boundary we have a holographic description, but not otherwise. We shall now derive the Cardy-Verlinde formula for the CFT on the brane, when the holographic description holds.

The entropy of the CFT is just given by the total entropy of the two Gauss-Bonnet black holes \cite{40} (see also \cite{52, 53}),

$$S = 2 \frac{\Omega_{n-2} a_H^{n-2}}{4 G_n} \left[1 + 2 \left(n-2 \frac{a_H}{\tilde{\alpha}^2}ight) \frac{\tilde{\alpha}}{a_H^2}\right]. \quad (91)$$

Note that the Gauss-Bonnet black hole entropy does not obey the area law which exists in Einstein gravity. The temperature of the CFT is given by the temperature of the black hole, $T_{\text{BH}}$, with the appropriate redshift factor, $\dot{t} \approx 1/k_{\text{eff}} a$,

$$T = \frac{T_{\text{BH}}}{k_{\text{eff}} a}, \quad \text{where} \quad T_{\text{BH}} = \frac{h_{\text{BH}}'(a_H)}{4\pi}. \quad (92)$$

In section 4 we found the energy density and pressure of the CFT (equations (43) and (46) respectively). In principle we can now calculate $\gamma$, and attempt to construct a Cardy-Verlinde formula in the form of equation (86). However, as noted in \cite{36} this will be impossible if one attempts to include all the Gauss-Bonnet corrections.

Before we lose hope, it is important to take stock of what we are actually trying to do. We are trying to find a Cardy-Verlinde formula for a CFT that is dual to a Gauss-Bonnet AdS bulk. It only makes sense to think of this CFT when the holographic description is valid, that is when $\sqrt{\tilde{\alpha}} \ll a_H \ll a$. It is therefore inappropriate to include the $\tilde{\alpha}/a_H^2$ correction in the entropy formula (91). In the holographic limit we can make the following consistent approximations:

$$s = \frac{1}{2G_n} \left(\frac{a_H}{a}\right)^{n-2} \left[1 + \mathcal{O}\left(\frac{\tilde{\alpha}}{a_H^2}\right)\right] \quad (93)$$

$$\rho = \frac{n-2}{8\pi G_n \tilde{\alpha} k_{\text{eff}}} \left(\frac{a_H}{a}\right)^{n-1} \left[\tilde{\alpha} k_n^2 + \mathcal{O}\left(\frac{\tilde{\alpha}}{a_H^2}\right)\right] \quad (94)$$

$$\gamma = \frac{n-2}{8\pi G_n \tilde{\alpha} k_{\text{eff}}} \left(\frac{a_H}{a}\right)^{n-3} \left[\chi + \mathcal{O}\left(\frac{\tilde{\alpha}}{a_H^2}\right)\right] \quad (95)$$
where \( \chi = 1 - 2 \left( \frac{\alpha^2}{\pi^2} \right) \tilde{a} k_n^2 \). We can now cast these quantities into a Cardy-Verlinde formula:

\[
 s^2 = \frac{1}{\chi} \left( \frac{k_{\text{eff}}}{k_n} \right)^2 \left( \frac{4\pi}{n-2} \right)^2 \gamma \left[ \rho - \frac{\gamma}{\tilde{a}} \right] \left( 1 + O \left( \frac{\tilde{a}}{a_H^2} \right) \right),
\]

Note that this formula agrees with (86) in the limit \( \alpha \to 0 \).

We can now ask whether this formula bears any resemblance to the Friedmann equation at the point that the brane crosses the black hole horizon – the answer is no. However, it doesn’t make sense to evaluate this formula at \( a = a_H \), as it is only valid for \( a \gg a_H \). In Einstein gravity, this was also the case, but the Cardy-Verlinde formula (86) evaluated at the horizon, still gave the Friedmann equation of the standard cosmology. We believe we now understand why this was the case.

From a holographic perspective, we need to ask what is the difference between an Einstein bulk and a Gauss-Bonnet bulk. The difference lies in the existence of exact holography for Einstein gravity but not for Gauss-Bonnet gravity. This means we cannot say anything sensible about the CFT at the time the brane crosses the horizon for the Gauss-Bonnet bulk. For the Einstein bulk, we can happily trust the exact holographic description and use the generalised version of the Cardy-Verlinde formula (89) at \( a = a_H \). This formula agrees with the braneworld Friedmann equation at each order of \( \rho \). Since the braneworld Friedmann equation and the standard Friedmann equation agree up to order \( \rho \), it is clear what is happening when we evaluate the approximate formula (86) at the horizon. We are just seeing the agreement of (89) with the braneworld Friedmann equation, up to order \( \rho \). However, we should note that the \( \rho^2 \) corrections are not small at the horizon.

In Gauss-Bonnet gravity there is no exact holography, and therefore no correct way to describe the physics of a dual field theory at the time the brane crosses the black hole horizon. The Cardy-Verlinde formula (86) will only be valid near the boundary of AdS.

7 Discussion

In this paper we have attempted to extend the ideas of braneworld holography in Einstein gravity, to Gauss-Bonnet gravity. We have found that there exists a holographic description of a critical brane moving in a Gauss-Bonnet AdS black hole bulk, but only when it is close to the boundary. This is in contrast to Einstein gravity, when a holographic description can be found even when the brane is not near the AdS boundary. This has important implications when one considers the Cardy-Verlinde formulae for the dual field theories on the brane. It was previously thought that one could not cast the thermodynamic quantities for the CFT dual to a Gauss-Bonnet AdS bulk into a Cardy-Verlinde like formula. However, by making approximations consistent with the limit in which the holographic description is valid, it turns out that one can.

Finding a Cardy-Verlinde formula for the CFT with the Gauss-Bonnet AdS dual enabled us to compare its properties with its analogue in Einstein gravity. In particular, if we evaluate the Cardy-Verlinde formula for Einstein gravity at the point at
which the brane crosses the black hole horizon, it gives us the Friedmann equation. This relationship between the Cardy-Verlinde formula and the Friedmann equation has been somewhat of a mystery, although our study of Gauss-Bonnet braneworld holography has enabled us to shed some light on the problem. We found that the relationship does not exist for Gauss-Bonnet gravity. We believe that this is because there is no exact holography for Gauss-Bonnet gravity, and hence we cannot make sense of the CFT physics at the time the brane crosses the horizon – this is explained in detail in section 6.

We would like to finish off by commenting on earlier studies of Gauss-Bonnet braneworld holography [27, 28, 29, 30]. From these, one draws negative conclusions about the existence of a holographic description. However, these studies all use an alternative Friedmann equation, derived in [56, 27]. The difference occurs because they have different boundary terms in the Gauss-Bonnet action [57, 58]. In this paper, we have used the boundary terms derived by Myers [37]. It is encouraging that we have succeeded in gaining some positive results using this method. The study of braneworld cosmology using the Friedmann equations discussed in this article has recently begun [59, 60]. It would be an interesting avenue of research to investigate the possible connections between the existence of a holographic description of braneworld cosmology, and the nature of the alternative boundary terms employed in the study of braneworld cosmology in a Gauss-Bonnet bulk. In particular, the treatment of the brane as a thin wall in the bulk spacetime is not a trivial matter in the Gauss-Bonnet braneworld model [61]. Before one takes a thin wall limit to approach the standard Randall-Sundrum braneworld model, the thick wall in these models has an internal structure – it remains an open question as to whether the nature of this internal structure affects the existence of a holographic description of braneworld cosmology.

Acknowledgements

We would like to thank Dominic Breacher, Christos Charmousis, Stephen Davis, Shin’ichi Nojiri and Simon Ross for helpful correspondence. JPG would particularly like to thank Ulf Danielsson for stimulating discussions. AP would also like to thank Syksy Räsänen and John March-Russell for helpful conversations. AP was funded by PPARC. JPG and AP acknowledge the invaluable support of B. Bird.

A Requirements for large $a$ limit

In this section, we will justify some of the claims made in section 4 regarding the limit $a \gg a_H$.

Note that the condition $h(a_H) = 0$ implies that

$$\tilde{\alpha}_\mu = A a_H^{n-1}$$

(97)

where

$$A = \tilde{\alpha} k_n^2 + \frac{\tilde{\alpha}}{a_H^2} + \left( \frac{\tilde{\alpha}}{a_H^2} \right)^2$$

(98)
is of order one.

If we assume that $a_H \ll a$, we see that $4\bar{\alpha}\mu/a^n \ll 1$, so that the Friedmann equation takes the form given in equation (97):

$$H^2 = \mathcal{A} - \frac{1}{a^2} + \frac{B}{A} \frac{\mu}{a^n - 1}$$

(99)

where $B$ is also of order one. For the de Sitter brane ($\mathcal{A} > 0$) there are clearly a number of scenarios in which $H$ never vanishes and the brane can reach to arbitrarily large values of $a$ (see section 5.5 in [15]). However, for critical branes ($\mathcal{A} = 0$) and anti-de Sitter branes ($\mathcal{A} < 0$), $a$ will have a maximum value $a_{\text{max}}$, where $H$ vanishes.

Consider the critical brane at $a_{\text{max}}$. It follows from (99) that

$$\mu = \frac{A}{B} a_{\text{max}}^{-3}$$

(100)

Combining this with equation (97), we find

$$\frac{\bar{\alpha}}{a_H^2} = B \left( \frac{a_H}{a_{\text{max}}} \right)^{n-3} \ll 1$$

(101)

where we have used the fact that $B$ is order one.

Now consider the anti-de Sitter brane at $a_{\text{max}}$. This time we get

$$\mu = \frac{A}{B} (|\mathcal{A}| a_{\text{max}}^{n-1} + a_{\text{max}}^{n-3})$$

(102)

Again, we combine this with equation (97) to give

$$B = \bar{\alpha} |\mathcal{A}| \left( \frac{a_{\text{max}}}{a_H} \right)^{n-1} + \bar{\alpha} \left( \frac{a_{\text{max}}}{a_H} \right)^{n-3}.$$  

(103)

Since $B$ is order one, and $a_{\text{max}} \gg a_H$, we must have

$$\bar{\alpha} |\mathcal{A}| \ll 1, \quad \frac{\bar{\alpha}}{a_H^2} \ll 1.$$  

(104)

This means that the anti-de Sitter brane is ruled out in the large $a$ analysis.

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