Super-statistical description of thermo-magnetic properties of a system of 2D GaAs quantum dots with gaussian confinement and Rashba spin-orbit interaction

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We examine the effect of non-equilibrium processes modeled by the introduction of a generalized Boltzmann factor on the thermal and magnetic properties of an array of two-dimensional GaAs quantum dots in the presence of an external uniform and constant magnetic field. The model consists of a single-electron subject to a confining Gaussian potential with a spin-orbit interaction in the Rashba approach. We compute the specific heat and the magnetic susceptibility within the formalism of $\chi^2$-superstatistics from the exact solution of the Schrödinger equation. Furthermore, an analytic solution for the partition function allows a study of the impact of the number of subsystems on the superstatistical corrections and confirms that the ordinary thermo-magnetic properties are recovered whenever the thermal distribution can be approximated by a Dirac delta. Also, we found a progressive disappearance of the Schottky anomaly with decreasing number of subsystems, while the specific heat ceases to be a monotonically increasing function with respect to the average temperature when the $\chi^2$-distribution is spread over a large range of temperatures. Remarkably, the introduction of fluctuations in the temperature is found to suppress the paramagnetic phase transition that would otherwise appear at low temperatures. Finally, we emphasize that an appropriate construction of the definition of physical observables is crucial for obtaining a correct description of the physics derived from a non-extensive construction of the entropy.

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I. INTRODUCTION

Semiconductor quantum dots (QDs), also called artificial atoms, consist of charge carriers confined in all directions resulting in a discrete energy spectrum similar to atoms in nature. Such systems are very attractive both from an experimental and a theoretical point of view as they possess a wide range of technological applications in quantum information and photoelectronics [1–4]. This is due to the possibility of controlling several characteristics of such systems. For example, it is nowadays possible to vary with high precision the number of electrons composing a QD [5,8], the size [9,10], the shape [11] and their composition [12]. This leads to direct light-emission technologies such as diodes [13,14], displays [15], luminescent solar concentrators [16,17], lasers [18,19], solar cells [20], among others [21].

In particular, spin-related phenomena in QDs has been studied in extension in the last decades as they are crucial in the semiconductor technology called spintronics [22–24]. For instance, the spin-orbit (SO) coupling mechanisms in semiconductors [25–30] provide a basis for device applications and a source of interesting physics such as the spin transistor [31,32]. The Rashba effect is of particular interest as it provides a SO coupling whose tunability allows SO effects to occur in QDs with few electrons [33]. In fact, some theoretical studies of the impact of the Rashba-SOI on the optical properties of a disk-like QD in the presence of an external magnetic field have been carried out within the framework of the density matrix approach [34]. Notably, a spin dependence of the spatial wave functions is typical in systems involving a Rashba QD in the presence of an external magnetic field, which can affect the thermo-magnetic and optical properties of a QD [35,36].

From a theoretical point of view, a correct description of the confining potential is the key to find new phenomena in the QDs dynamics. By choosing a suitable function of the confining potential and interactions, it is possible in some cases to analytically find the energy spectrum of the system, which allows the theoretical exploration of the thermal, magnetic and optical properties of QD's. It is well established that a harmonic potential is a good approximation which reproduces the main characteristics of such systems [37,38], but it has been demonstrated that the confinement is rather anharmonic and has a finite depth which has been simulated by several authors using a Gaussian potential model [39,40,43]. These works are developed in the context of the Boltzmann-Gibbs statistics, i.e., the QD under study represents the whole system which is in thermal equilibrium with an external bath. Therefore, it is possible to apply the canonical partition function formalism for computing the thermal properties from the spectrum of the confined electrons.

A more general thermal description of the system can take into account the fluctuations in the temperature or any of its intensive observables and thus, the usual Boltzmann-Gibbs formulation cannot be used. In such scenarios, it is necessary to achieve the correct formalism.
II. THEORETICAL MODEL

A. Energy spectrum of the single quantum subsystem

In the present model, we study a system made of $N$ single-electron 2D-QDs in the presence of an external uniform and constant magnetic field. Each QD can be regarded as a different subsystem with local thermal equilibrium but obeying a $\chi^2$-distribution of the temperatures amongst them. This is a good approximation if the QDs are embedded in a low thermal conductivity material. The single-particle Hamiltonian of the QDs in the presence of an external magnetic field with both Zeeman and Rashba SO terms is given by

$$\hat{H} = \frac{1}{2m^*} \left( \vec{p} - \frac{q}{c} A \right)^2 + \hat{H}_G + \hat{H}_R + \frac{1}{2} \mu_B g^* \vec{S} \cdot \vec{B}. \quad (1)$$

The first term in Eq. (1) refers to the minimal coupling between a particle with charge $q$ and a vector field propagating with the speed of light $c$. By ignoring the effects produced by the conduction band electrons, the effective electron mass $m^*$ is assumed to be constant and thus the non-parabolicity of the conduction band is neglected [94–96]. Furthermore, in the coordinate representation, the vector potential $\vec{A}$ is expressed in the symmetric gauge $\vec{A} = \frac{q}{2c} (-y, x, 0)$ which in a polar-coordinate system has the form

$$\vec{A}(r) = \frac{Br}{2} \vec{e}_0. \quad (2)$$

The second term in Eq. (1) is the confining Gaussian potential $\hat{H}_G$ [36, 40, 41] with:

$$\hat{H}_G = -V_0 e^{-r^2/2R^2},$$

and

$$\approx \frac{1}{2} \frac{m^*}{\omega_c} \omega^2 r^2 - V_0. \quad (3)$$

Here $V_0$ and $R$ define the depth and the range of the potential respectively, while the effective confining frequency $\omega$ is given by

$$\omega = 2V_0 \left( \frac{\sqrt{\omega_c^2 + \omega_k^2}}{\hbar + 2m^* \sqrt{\omega_c^2 + \omega_k^2} R} \right), \quad (4)$$

where $\omega_c = qB/m^*$, $\omega_k^2 = V_0/m^* R^2$ and $\hbar$ is the Planck constant.

The third term in Eq. (1) corresponds to the Rashba spin-orbit coupling $\hat{H}_R$ and is given by the general expression

$$\hat{H}_R = \frac{\gamma}{\hbar} r \hat{\sigma} \cdot \left[ \nabla V \times \left( \vec{p} - \frac{q}{c} A \right) \right]_z, \quad (5)$$

where $\hat{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the Pauli matrices vector, so that

$$\hat{H}_R = \frac{1}{2} m^* \omega_c^2 \left( \hat{r} \cdot \hat{\sigma} \right). \quad (6)$$

In this paper, we focus our attention on the effects of the size of a system consisting of $N$ two dimensional single-electron GaAs QDs with Gaussian confinement and Rashba Spin-Orbit coupling. Each subsystem is assumed to be in local thermal equilibrium so that the usual statistical framework well describes its individual thermodynamic properties while thermal fluctuations are allowed between the composing subsystems. Such out-of-equilibrium scenario is described within the $\chi^2$-SE formalism. All the system is in the presence of an external and constant magnetic field. The paper is organized as follows: In Section IIa we give a description of the model by first providing a solution to the single particle Schrödinger equation (Subsection II. A). Afterwards we provide a review of the SE construction of the partition function in Subsection II. B, followed by the application of such formalism for computing the thermomagnetic properties of the $N$ subsystems of single-electron 2D-QDs in Subsection II. C. In Section III we provide the numerical results along with the physical interpretation of the data. The results are summarized in Sec. IV.
with \( s = \pm 1 \) referring to the spin projection. The last term in Eq. (1) is the Zeeman coupling of the external magnetic field \( \mathbf{B} \) with the electron spin \( \hat{\mathbf{S}} \), where \( \mu_B \) is the Bohr magneton and \( g^* \) is the effective Landé factor of the electron.

With all of the above, the Hamiltonian of the system becomes

\[
\hat{H} = -\frac{\hbar^2}{2m^*} \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right] + \frac{\omega_c}{2} \hat{L}_z \\
+ \frac{1}{2} m^* \Omega_s^2 r^2 - i \sigma_z \gamma m^* \omega^2 \frac{\partial}{\partial \theta} + \frac{1}{2} \mu_B g^* \hat{\mathbf{S}} \cdot \mathbf{B} - V_0,
\]

where \( \hat{L}_z \) is the orbital angular momentum operator in 2D and we have defined

\[
\Omega_s^2 = \left( 1 + s\gamma \frac{m^* \omega_c}{\hbar} \right) \omega^2 + \left( \frac{\omega_c}{2} \right)^2.
\]

The eigenvalues of the Hamiltonian of Eq. (7) have been studied in a recent work \[38\] and are given by

\[
E_{nls} = \hbar \Omega_s (2n + |l| + 1) - \frac{1}{2} \hbar \omega_c l \\
+ \left( \gamma m^* \omega^2 l + \frac{1}{4} g^* \hbar \omega_c \right) s - V_0.
\]

The last equation allows us to compute the canonical partition function \( Z_0 \) in an analytical form.

**B. Superstatistical description of the system**

The superstatistics was well described by Beck and Cohen \[54\] as an extension of the usual statistical description of a systems that has not yet reached the full equilibrium. The fluctuations are encoded in the intensity parameter \( \tilde{\beta} \) such that the whole system can be divided in subsystems where \( \tilde{\beta} \) is approximately constant. It is worth mentioning that each subsystem must have a low particle density in order for the Boltzmann statistics to hold for each one of them regardless the temperature.

Thus the system is analyzed as a space-time average of Boltzmann factors \( e^{-\tilde{\beta} \hat{H}} \), where \( \hat{H} \) is the Hamiltonian of a single subsystem, and the fluctuations are taken into account by a probability distribution \( f(\tilde{\beta}, \beta) \). In this sense, the formalism can be regarded as the superposition of two statistics: one referring to the Boltzmann factors \( e^{-\tilde{\beta} \hat{H}} \) and other to \( \beta \).

Mathematically, an averaged Boltzmann factor can be defined as

\[
\mathcal{B}(\hat{H}) = \int_0^\infty \tilde{\beta} f(\tilde{\beta}, \beta) e^{-\tilde{\beta} \hat{H}} \, d\tilde{\beta},
\]

which leads to the identification of the SE partition function as

\[
Z = \text{Tr} \left\{ \mathcal{B}(\hat{H}) \right\}.
\]

Several choices are possible for the distribution obeyed by the parameter \( \tilde{\beta} \) over the ensemble of \( N \)-subsystems, each one of them leading to their corresponding Boltzmann factors \[84\]. In the present document, we chose a \( \chi^2 \) distribution

\[
f(\tilde{\beta}, \beta) = \frac{1}{\Gamma(\overline{N}/2)} \left( \frac{N}{2\beta} \right)^{\overline{N}/2} \tilde{\beta}^{N/2-1} e^{-N\tilde{\beta}/2\beta},
\]

where \( \Gamma(x) \) is the gamma function. This distribution has the property that the parameter \( \beta \) from Eq. (12) corresponds to the average inverse temperature of the whole system, given that

\[
\langle \tilde{\beta} \rangle = \int \tilde{\beta} f(\tilde{\beta}, \beta) \, d\tilde{\beta} = \beta.
\]

Because of this, any further thermodynamic observable will refer to the average temperature of the whole system \( \beta^{-1} \) and not the individual temperatures \( \tilde{\beta}^{-1} \) of the composing subsystems.

By replacing Eq. (12) into Eq. (10), the averaged Boltzmann factor corresponding to a \( \chi^2 \)-distribution is given by

\[
\mathcal{B}(\hat{H}) = \left( 1 + \frac{2}{N} \hat{\beta} \hat{H} \right)^{-N/2}.
\]

The \( \chi^2 \)-distribution is a typical distribution for positive-valued random variables. Additionally, if \( \chi^2 \)-like fluctuations evolve on a long timescale, one ends up with Tsallis statistics in a natural way. Tsallis distributions can easily be related to the fact that there are spatio-temporal fluctuations of an intensive parameter such as the inverse temperature. For other distributions of the intensive parameter, one ends up with more general superstatistics \[54\] which contain Tsallis statistics.
as a special case. For $\hat{H} \ll \beta$, all superstatistics have been shown to have the same first-order corrections to the Boltzmann factor of ordinary statistical mechanics as Tsallis statistics \cite{tsallis2009}. For moderately large $\hat{H}$, the behaviour of the system is often observed to remain similar to that given by Tsallis statistics \cite{tsallis2003}. For very large values of $\hat{H}$ the correction to the Boltzmann factor is strongly dependent on the chosen distribution for the fluctuations \cite{tsallis2003,tsallis2009}. In Subsection II C we obtain an exact expression for the SE-partition function independent of the energy scale and the number of subsystems by summing over all the energy spectrum. Because of this, our construction will necessarily lead to Tsallis entropy regardless of the involved parameters and therefore, a proper prescription for computing the thermodynamic observables is strongly required.

C. Partition function and thermodynamic quantities

By taking the trace prescription from Eq. (11) of the generalized Boltzmann factor in Eq. (14) for a system with the quantum numbers from Subsection II A, the SE-partition function turns to be

$$Z_N(\beta) = \sum_{s=\pm1} \sum_{n=0}^{\infty} \sum_{l=-\infty}^{\infty} \left( 1 + \frac{2}{N}\beta E_{nl} \right)^{-N/2} .$$

(15)

The sum over the energy levels from Eq. (9) can be analytically found to be

$$Z_N(\beta) = \sum_{s=\pm1} \left\{ \lambda_s^-(\beta, N) \sum_{n=0}^{\infty} \zeta \left[ \frac{N}{2}, \Delta_n^-(\beta, N) \right] 
+ \lambda_s^+(\beta, N) \sum_{n=0}^{\infty} \zeta \left[ \frac{N}{2}, 1 + \Delta_n^+(\beta, N) \right] \right\} ,$$

(16)

where $\zeta(a, x)$ is the Hurwitz zeta function and the auxiliary functions $\lambda_s^\pm$ and $\Delta_n^\pm$ are defined as

$$\lambda_s^\pm(\beta, N) = \left[ \frac{N}{2\beta \hbar (\Omega_s \pm \omega_c/2 - \gamma m^* \omega^2 s/\hbar)} \right]^{N/2}$$

(17a)

and

$$\Delta_n^\pm(\beta, N) = \frac{N/2 \beta \hbar + (2n + 1)\Omega_s + g^* \omega_c s/4 - V_0/\hbar}{\Omega_s \pm \omega_c/2 - \gamma m^* \omega^2 s/\hbar} .$$

(17b)

Certainly, the partition function contains the full thermodynamic of the system which can be reached from the derivatives of its natural logarithm. To achieve an accurate description of the system’s response functions, note that the Eq. (19) corresponds precisely with the partition $Z_q$ of the Tsallis probability distribution \cite{tsallis2009}

$$p_n(\beta) = \left[ 1 - (1 - q)\beta E_n \right]^{1/(1-q)} Z_q(\beta) ,$$

(18a)

with

$$Z_q(\beta) = \sum_n [1 - (1 - q)\beta E_n]^{1/(1-q)} .$$

(18b)

The entropic index $q = 1 + 2/N$ characterizes the degree of subextensivity of the entropy addition rule as the usual Boltzmann entropy is recovered in the limit $q \rightarrow 1$ or $N \rightarrow \infty$.

Tsallis et al. \cite{tsallis2009} showed that in order to preserve the Legendre structure of the statistical thermodynamics, the $q$-logarithm has to be introduced \cite{tsallis2009}:

$$\ln_q x = \frac{x^{1-q} - 1}{1-q} ,$$

(19)

so that $\ln_1 x = \ln x$, and therefore, the Free Energy, the average energy and the specific heat are given by

$$F_q(\beta) = U_q(\beta) - TS_q = -\frac{1}{\beta} \ln_q Z_q(\beta) ,$$

(20a)

$$U_q(\beta) = -\frac{\partial}{\partial \beta} \ln_q Z_q(\beta)$$

(20b)

and

$$C_q(\beta) = \frac{\partial U_q(\beta)}{\partial T} .$$

(20c)

In the expression above $S_q$ is the well-known non-extensive Tsallis entropy \cite{tsallis2009}:

$$S_q = k_B \left( 1 - \sum_n p_n^q \right) \forall q \in \mathbb{R} ,$$

(21)

with the constriction

$$\sum_n p_n = 1 .$$

(22)

The use of the natural logarithm instead of the $q$-logarithm given in Eq. (19) will therefore break the Legendre structure of the potentials. Even though it could be argued that there is no reason to believe that in a non-extensive scenario the Legendre structure has to remain valid, it must be stressed that in such scenario the parameter $\beta$ in Eqs. (20) is not the Lagrange multiplier associated to the internal energy constraint. Furthermore, as studied in the field of Geometrothermodynamics (GTD), Legendre transformations can be regarded as diffeomorphisms that leave the space of equilibrium states unchanged. In such manner, the curvature of the induced metric in the space of equilibrium states is truly independent of the thermodynamic potential used to describe a given system \cite{tsallis2009}. Because of this, a stability criterion within the Tsallis formalism is well-defined via the positivity of the specific heat \cite{tsallis2009}.
FIG. 2: Thermodynamic functions by following the natural logarithm prescription of Eqs. (25) for the SE-partition function. Panel (a) displays the numerical calculations of the specific heat as a function of temperature using a Rashba spin-orbit coupling of $\gamma = 0.15 \text{ nm}^2$ for $B = 5 \text{ T}$ and $V_0 = 20 \text{ meV}$. The panel (b) shows the magnetic phase diagram for $V_0 = 8 \text{ meV}$, $N = 100$, and $\gamma = 0.15 \text{ nm}^2$. The gray region corresponds to the diamagnetic phase ($\chi < 0$), whereas the white region to the paramagnetic phase ($\chi > 0$).

In particular, from the $q$-expectation value for the internal energy

$$ U = \sum_n p_n^q(\beta)E_n, \quad (24) $$

the parameter $\beta$ can be associated with the Lagrange parameter of the average energy and the Legendre structure is naturally recovered.

In [110], a prescription to construct a renormalized temperature from the parameter $\beta$ is given in order to avoid some undesirable features of the theory such as non-additivity of the internal energy and loss of norm $\langle 1 \rangle_q \neq 1$. In the present work we are not evaluating the effects of such a renormalization but we emphasize that the use of the $q$-logarithm already reproduces most of the features that a thermodynamic theory requires.

III. RESULTS AND DISCUSSION

In order to compute the thermal and magnetic properties of our system, we set the parameters as follows: $m^* = 0.067m_0$ is the effective electron mass for a GaAs QD where $m_0$ is the free electron mass; $g^* = -0.44$ is the effective Landé constant, $\gamma = 0.15 \text{ nm}^2$, $R = 10 \text{ nm}$ is the radio of the QD, and $V_0 = 10 \text{ meV}$ as the value of the confining potential.

In Fig. 2 we show (a) the specific heat $C_v$ and (b) the magnetic susceptibility $\chi$ obtained by taking the natural logarithm of the SE-partition function, i.e., by using the relations

$$ C_v = k_B \frac{\partial}{\partial T} \left( T^2 \frac{\partial}{\partial T} \ln Z_N \right), $$

$$ \chi = k_B T \frac{\partial^2}{\partial B^2} \ln Z_N, \quad (25) $$

This will lead to nonphysical results such as a negative specific heat in the low temperature region and the emergence of a spurious paramagnetic region at high external magnetic fields. Both phenomena are persistent even for a large number of subsystems. This is an effect of the truncation scheme of the Taylor series of the natural logarithm in comparison with the $q$-logarithm, namely

$$ \ln_q Z_N = \ln Z_N - \frac{1}{N} \ln^2 Z_N + \mathcal{O}(N^{-2}) $$

$$ \approx \ln Z_0 + \frac{1}{N} \left( \frac{\beta^2}{Z_0} \frac{\partial^2 Z_0}{\partial \beta^2} - \ln^2 Z_0 \right), \quad (26) $$

where $Z_0$ is the partition function coming from the standard formalism of physical statistics. This means that for large but finite number of subsystems $N$, the out-of-equilibrium effects will be enhanced in the low-temperature regime. Moreover, the introduction of the $q$-logarithm will lead to corrections to the response functions of $\mathcal{O} \left( \frac{1}{N} \ln Z_0 \right)$ that will only be negligible when the natural logarithm of the partition function itself is small with respect to the number of subsystems $N$. This will in general depend in a non-trivial way on the thermodynamic variables of $Z_0$.

From Fig. 2(a) it can be noticed that for some values of $N$ and $T$, the specific heat becomes negative and
FIG. 3: Specific heat as a function of temperature using a Rashba spin-orbit coupling of $\gamma = 0.15 \text{ nm}^2$ for increasing values of the external magnetic field $B$ and different system sizes: $N = 10$ for panels (a) and (b); $N = 50$ for panels (c) and (d); $N = 100$ for panels (e) and (f) and $N = 10^3$ for panels (g) and (h). In each panel, the solid lines represent the super-statistical behaviour obtained from Eq. (20c), while the Boltzmann statistics are depicted as dotted lines and can regarded as the asymptotic behaviour for $N \to \infty$. 
always tends to $-2k_B$ as $T \to 0$. Adding the correction from Eq. (25) will recover the $N \to \infty$ behaviour depicted in the dotted line and positivity is restored. It is worth mentioning that if either the interactions are long-range or if the system is small concerning the interaction range, then negative specific heats can be, and indeed have been, observed [109]. Such systems include among others gravitational systems (long-range interactions) [109,105] and atomic clusters (small systems) [106,108]. Nevertheless, we attribute such behaviour in the system under study in the present work to a poor choice of the definition of the prescription for obtaining physical observables.

On the other hand, by comparison with Refs. 36 and 87, the magnetic phase diagram from Fig. 2(b) shows the emergence of a paramagnetic phase (represented by the white region at high magnetic fields). Those results are non-intuitive and contradict the stability condition for the subsystems conforming the array of QD’s [107]. By this means, the necessity for introducing $q$-expectation values becomes more evident.

It is worth mentioning that several authors have used the standard definition of an expectation value for their calculations in order to explore the thermodynamic consequences of a superstatistical treatment in different scenarios: the thermodynamical properties of the anharmonic canonical ensemble within the cosmic-string framework [40], the impact of the non-commutativity of the space for systems with thermal fluctuations [40], the effective Quantum Chromodynamics phase diagram [67] and the anharmonic oscillator for non-relativistic and relativistic Klein-Gordon equations [104], among others [34,90]. We suggest the restoration of the Legendre invariance in Refs. 46, 49, 53, 67, 90, 101 to regain a closed thermodynamic treatment.

In the following, only $q$-expectation values obtained from the full SE-partition function are evaluated and discussed.

The specific heat $C_v$ as a function of temperature for several values of the external magnetic field is shown in Fig. 3. To appreciate the effect of SE, we compare the results for a varying number of subsystems $N$ with respect to the extensive Boltzmann statistics represented by the respective dotted lines. As can be noticed from Figs. 3(a), (e), (e) and (g), the Schottky anomaly slowly disappears as the full system decreases in size. On the other hand, Figs. 3(b), (d), (f) and (h) show that for a small number of subsystems the specific heat ceases to be a monotonically increasing function with respect to the average temperature.

This behavior can be understood in terms of the probability distribution function of Eq. (12) which is plotted if Fig. 4 for $\beta = 1/2$ ($T = 2$ when $k_B = 1$) and for different values of $N$. For a small number of subsystems and a small average temperature, a wide range of fluctuations can contribute to the weighted Boltzmann factor. In contrast, the distribution function for a very large number of composing systems $N \to \infty$ resembles a

Dirac delta and the temperature of the whole system can be approximated as a unique value and thermodynamic equilibrium is established. Nevertheless, when the number of subsystems is large but finite, the $\chi^2$-distribution will get narrower for increasing values of the average temperature, asymptotically approaching to a Dirac delta for the infinite temperature limit. Therefore, one can expect high deviations from the Boltzmann statistics for systems near room average temperature whenever they are far from the thermodynamic equilibrium.

The low-temperature peak of the Schottky anomaly is closely related to the energy required for a thermal transition between the ground state and the first excited state of the system (with energy $\Delta E$) as it can be interpreted as a resonance in $k_B T_s \sim \Delta E$. In the case of a broad distribution of temperatures (small values of $N$) the resonance also becomes broader and eventually smears out.

The specific heat in the high-temperature regime asymptotically approaches a constant as would be expected from a Dulong-Petit-like behaviour. Surprisingly, this is truth even for relatively small number of subsystems, where the value for such asymptotic constant seems to be lower for decreasing values of $N$. In other words, out-of-equilibrium effects will introduce corrections that effectively lower the heat capacity of the system. This seems to be a consequence of the $\chi^2$-distribution with low number of subsystems, where contributions from $\beta$ lower than the average $\beta$ (i.e., at high temperatures) are dominant, meaning that most of the subsystems already have higher local temperatures than the average temperature of the system and therefore less heat transfer is necessary to increase it.

Finally, it is noteworthy from Fig. 4 that the param-
magnetic region in the low-temperature low-external field regime typical from GaAs QDs \[36, 37\] disappears when the out-of-equilibrium corrections are introduced using the \(\chi^2\)-SE formalism for decreasing number of subsystems \(N\). This is physically interpreted as a lack of order in the spins coming from the fact that for small \(N\), a low average temperature will still have important contributions from subsystems with higher local temperature that will break the order of the spins in a global description.

### IV. SUMMARY

We examined the effect of non-equilibrium processes modeled by the \(\chi^2\)-superstatistics on the thermal and magnetic properties of an array of two-dimensional GaAs quantum dots with Rashba spin-orbit interaction in the presence of an external uniform and constant magnetic field.

First, we used the present model to quantitatively emphasize the importance of an appropriate construction of physical observables for obtaining a correct description of the physics derived from a non-extensive construction of the entropy.

Afterwards, we offered an improved calculation obtained from the analytic solution for the partition function. This allowed us to study the impact of an arbitrary number of subsystems on the superstatistical corrections and confirms that the ordinary thermo-magnetic properties are recovered whenever the thermal distribution of the composing subsystems can be approximated by a Dirac delta.

Finally, we found that the most remarkable out-of-equilibrium effects appear for a small number of subsystems or at the low-temperature regime, this is, whenever the \(\chi^2\)-distribution is spread over a large range of temperatures. In terms of the response functions, this means that the introduction of a broad range of fluctuations in the local temperatures of the system is responsible for a progressive disappearance of the Schottky anomaly, while the high-average temperature specific heat gets effectively decreased. Furthermore, a small number of composing subsystems is found to suppress the paramagnetic phase transition that would otherwise appear at low temperatures.

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[1] P. Michler, A. Kiraz, C. Becher, W. V. Schoenfeld, P. M. Petroff, Lidong Zhang, E. Hu, A. Imamoli, Science 290, 5500 (2000).
[2] A. Kiraz, M. Atatüre, and A. Imamoli, Phys. Rev. A 69, 032305 (2004); Erratum Phys. Rev. A 70, 059904 (2004).
[3] Portalupi S.L., Michler P., Resonantly Excited Quantum Dots: Superior Non-classical Light Sources for Quantum Information. In: Michler P. (eds) Quantum Dots for Quantum Information Technologies. Nano-Optics and Nanophotonics. Springer, Cham (2017).
[4] Simone Luca Portalupi, Gaston Hornecker, Valrian Giesz, Thomas Grange, Aristeide Lematre, Justin Demory, Isabelle Sagnes, Norberto D. Lanzillotti-Kimura, Loc Lanco, Alexia Auffves, and Pascale Senellart, Nano Lett. 15 (10), 6290-6294, (2015).
[5] D. Loss and D. P. DiVincenzo, Phys. Rev. A 57, 120 (1998).
[6] G. Burkard, D. Loss, and D. P. DiVincenzo, Phys. Rev. B 59, 2070 (1999).
[7] A. Barenco, C. H. Bennett, R. C., D. P. DiVincenzo, N. Margolus, P. Shor, T. Sleator, J. A. Smolin, and H. Weinfurter, Phys. Rev. A 52, 3457 (1995).
[8] C. F. Ramirez-Gutierrez, J. D. Castano-Yepes, and M. E. Rodriguez-Garcia, J. Appl. Phys. 121, 025103 (2017).
[9] Ankur Khare, Andrew W. Wills, Lauren M. Ammerman, David J. Norrisz and Eray S. Aydil, Chem. Commun. 47, 1172111723 (2011).
[10] Jianbing Zhang, Ryan W. Crisp, Jianbo Gao, Daniel M. Kroupa, Matthew C. Beard, and Joseph M. Luther, J. Phys. Chem. Lett. 6 (10), 18301833 (2015).
[11] X. Peng, L. Manna, W. Yang, J. Wickham, E. Scher, A. Kadavanich, A. P. Alivisatos, Nature, 404 (6773), 59. (2000).
[12] Young-Shin Park, Jaehoon Lim and Victor I. Klimov, Nature Materials 18, 249255 (2019).
[13] V. L. Colvin, M. C. Schlamp and A. P. Alivisatos, Nature 370, 354357 (1994).
[14] J. Kwik et al, Nano Lett. 15, 37933799 (2015).
[15] T. H. Kim, et al. Nat. Photon. 5, 176182 (2011).
[16] K. Wu, H. Li, and V. I. Klimov, Nat. Photon. 12, 105110 (2018).
[17] N. D. Bronstein, et al. ACS Photon. 2, 15761583 (2015).
[18] V. I. Klimov, et al. Science 290, 314317 (2000).
[19] J. Lim, Y. S. Park, and V. I. Klimov, Nat. Mater. 17, 4248 (2018).
[20] Jialong Duan, Huihui Zhang, Qunwei Tang, Benlin He and Liangmin Yu, J. Mater. Chem. A 3 17497-17510, (2015).
[21] Debasis Bera, Lei Qian, Teng-Kuan Tseng, and Paul H. Holloway, Materials 3 2260-2345, (2010).
[22] S. A. Wolf, D. D. Awschalom, R. A. Buhrman, J.
A 286, 156 (2000).

[90] S. Sargolzaeipor, H. Hassanabadi and W. S. Chung, Mod. Phys. Lett. A , 34, No. 03, 1950023 (2019).

[91] B. Boyacioglu and A. Chatterjee, J. Appl. Phys. 112 (8), 083514 (2012).

[92] M. Abbarchi et al. J. Phys.: Conf. Ser. 210, 012012 (2010).

[93] S. Saravana Kumar and A. John Peter, Journal of Nanoelectronics and Optoelectronics 7, 1 (2012).

[94] A. R. Jeice, Sr. G. Jayam, and K. S. J. Wilson, Int. J. Mod. Phys. B 32, 1850122 (2018).

[95] A. R. Jeice, Sr. G. Jeyam, and K. S. J. Wilson, Indian J. Phys. 90, 805 (2016).

[96] A. Sivakami, A. R. Jeice, and K. Navaneethakrishnan, Int. J. Mod. Phys. B 24, 5561 (2010).

[97] V. Lopes-Oliveira, L. K. Castelano, G. E. Marques, S. E. Ulloa, and V. Lopez-Richard, Phys. Rev. B 92, 035441 (2015).

[98] A. V. Moroz and C. H. W. Barnes, Phys. Rev. B 61, R2464(R) (2000).

[99] C. Beck, Continuum Mech. Thermodyn. 16, 293 (2004).

[100] C. Beck, Braz. J. Phys. 39, 357 (2009).

[101] Bing-Qian Wang, Zheng-Wen Long, Chao-Yun and Long Shu-Rui Wu, Phys. A 517, 163 (2019).

[102] F. Staniscia, A. Turchi, D. Fanelli, P. H. Chavanis, and G. De Ninno, Phys. Rev. Lett. 105, 010601 (2010).

[103] D. Lynden-Bell and R. Wood, Mon. Not. R. Astron. Soc. 138, 495 (1968).

[104] W. Thirring, Z. Phys. 235, 339 (1970).

[105] D. Lynden-Bell and R. M. Lynden-Bell, Mon. Not. R. Astron. Soc. 181, 405 (1977).

[106] P. Labastie and R. L. Whetten, Phys. Rev. Lett. 65, 1567 (1990).

[107] J. A. Reyes-Nava, I. L. Garzón, and K. Michaelian, Phys. Rev. B 67, 165401 (2003).

[108] W. Thirring, H. Narnhofer, and H. A. Posch, Phys. Rev. Lett. 91, 130601, (2003).

[109] A. Ramírez-Hernández, H. Larralde and F. Leyvraz, Phys. Rev. Lett. 100, 120601 (2008).

[110] C. Tsallis et al., Phys. A 261, 534 (1998).

[111] Alessandro Bravetti, Cesar S Lopez-Monsalvo, Francisco Nettel, Hernando Quevedo, Journal of Geometry and Physics 81, 1 (2014).

[112] C. Tsallis, J. Stat. Phys. 52, 479 (1988).

[113] C. Tsallis, Quimica Nova 17, 468, (1994).

[114] T. Wada, Phys. Lett. A 297, (2002).