Controlling probability transfer in the discrete-time quantum walk by modulating the symmetries

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Abstract
Quantum walks are a useful platform for studying high-efficiency energy transport. In this paper, we study the probability transfer in the discrete-time quantum walk (DTQW) in detail. With intricate design of the initial state, the probability transfer in the DTQW system can exhibit similar behaviors to those in continuous-time quantum walks (CTQWs). Furthermore, without the assumption of a biased initial state and coupling to the environment in the DTQW system, we explore how to achieve directed probability transfer in the DTQW and find that it is closely connected to the time-reversal symmetry of the effective Hamiltonian of the walk. We find that regardless of whether the number of vertices in the DTQW is even or odd, we can control the probability transfer in the DTQW by modulating the time-reversal symmetry of the system. Such control of directed probability transfer can only work well in a CTQW with odd vertices. Even when temporal or spatial disorder emerges in the DTQW, the connection between the probability transfer and the symmetry of the walk still exists. Experimental proposals for observing the probability distributions in the DTQW are discussed.

1. Introduction
Controlling an energy transfer to flow along a directed path is an important topic at present [1]. It has been verified that in many quantum systems, e.g. spins, nanoparticles and the Fenna–Matthew–Olsen (FMO) complex [2–12], by designing the biased initial state or by connecting part of the system to the environment, high-efficiency unidirectional probability transfer can be realized. Due to the good correspondence between the aforementioned quantum systems and the continuous-time quantum walk (CTQW), the phenomenon of energy transfer in these real physical systems can be revealed through the probability transfer among the vertices in the CTQW [13–17]. Besides external designs such as designing the biased initial state or connecting the system to the environment, many researchers have found that the directional probability transfer can be controlled by adding the chiral term $e^{i\phi}$ into the standard CTQW [18, 19]. For these studies of probability transfer in the CTQW without an external design, the emergence of the directed probability transfer is attributed to a break in the probability time-reversal symmetry (PTRS) of the system when the term $e^{i\phi}$ is added.

Although the CTQW is closely connected to many quantum systems, experiments involving small CTQW systems are difficult and the realization of large-scale CTQWs is far beyond our current technology. Only a few works have examined the CTQW from the experimental point of view [19–22]. Another type of quantum walk involving the coin has been provided, and that is the discrete-time quantum walk (DTQW) [23, 24]. The DTQW is the quantum extension of the classical walk and seen as a useful platform to realize fast quantum information processing [25–30]. Compared with the difficult realization of the CTQW, the DTQW can be implemented easily with current available experimental conditions [31–40]. Experimental realizations of DTQWs have already been presented in different systems, e.g., linear optical components, optical orbital angular momentums, ion traps and time bins. Given that the probability transfer among the vertices of the DTQW is closely related to the quantum information processing in the walk [24–26], it is very important to study how to control the probability transfer in the DTQW. Although implementations of some quantum information processing based
on the DTQW have consulted the CTQW [41–46], the differences and the similarities in the probability transfer between the DTQW and the CTQW have not been addressed.

Inspired by the investigations above, in this paper, we study the probability transfer in the DTQW on a circle. Firstly, with an approximate theoretic calculation and strict numerical simulation, we demonstrate that, in the DTQW, we can reproduce the characteristics of probability transfer of the CTQW on a circle. When the number of vertices in the DTQW is even, the symmetric probability distributions among the vertices can be found; while when the number of the vertices is odd, the directed probability transfer and the asymmetric probability distributions emerge. Then we study how to control the probability transfer in the DTQW. The initial state of the DTQW system is unbiased and there is no leakage at the certain vertex of the system. We find that the emergence of the directed probability transfer in the DTQW is closely connected to the time-reversal symmetry (TRS) of the system’s effective Hamiltonian. Directed probability transfer appears when the TRS of the system Hamiltonian is broken. In our discussion, the broken TRS in the DTQW can be implemented by adding an external electrical field or optical devices. Such a connection between the probability transfer and the TRS can be applied to a DTQW involving both even and odd vertices. The requirement for PTRS makes directed probability transfer appear in CTQWs with odd vertices only. So by modulating the TRS of the system, we can control the probability transfer in the DTQW without the requirement for delicate external design. Moreover, although temporal disorder or spatial disorder is introduced in the DTQW on a circle, the averaged probability distribution still spreads symmetrically with respect to the starting vertex when the TRS is kept. Two experimental proposals for the DTQW on a circle are presented at the end of the paper. These realizations of probability transfer in the DTQW can help us to control highly efficient unidirectional energy and quantum information processing under laboratory conditions.

The organization of our work is as follows: in section 2, we derive the evolution for the DTQW and reproduce the characteristics of probability transfer in the CTQW on a circle. Then, in section 3, we take two DTQWs as examples. We analyse the hidden symmetries in the Hamiltonian of these two DTQWs and establish the connection between the probability transfer and the symmetries of the system. In section 4, we discuss the probability transfer when temporal or spatial disorder is introduced. The possible experimental implementations are addressed in section 5 and we conclude in section 6.

2. Reproducing the characteristics of probability transfer in the CTQW in the DTQW

The description of the CTQW is addressed with the Hamiltonian \( H_c \), which contains the interaction between different vertices in the system [13–15]. The Hilbert space for the CTQW is spanned by the vertices \([n]\) and the dynamics of the CTQW can be presented in the form of the Schrödinger equation,

\[
\frac{\partial}{\partial t} \psi_n = H_c \psi_n,
\]

where \( \psi_n \) represents the wave function at the vertex \( n \) in the CTQW. In comparison, the total system of the DTQW contains the coin and the vertices [23, 24]. The Hilbert space for the coin is spanned by \([L]\) and \([R]\), and the coin operator \( C(\theta) \) is

\[
C(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}.
\]

The coin states \([R]\) and \([L]\) are represented as \([R] = (1, 0)^T\) and \([L] = (0, 1)^T\), where the operator \( T \) stands for the matrix transpose. The one-step evolution operator \( U \) in the DTQW comprises the coin operator \( C(\theta) \) and the conditional shift operator \( S \).

\[
S = \sum_{x=0,1,\ldots,N-1} \left| x + 1 \mod N \right\rangle \left\langle x \otimes [R] \otimes [R] + |x - 1 \mod N \left\rangle \left\langle x \otimes [L] \otimes [L].
\right. \right.
\]

In our work, we focus on the dynamics of the DTQW \( N \)-vertices on a circle (see figure 1). As addressed by conditional shift operator \( S \), the coin state \([R]\) \([L]\) makes the walker travel from position \( x \) to \( x + 1 \) (position \( x \) to \( x - 1 \)). When the walker stands at the position \( N - 1 \), the coin state \([R]\) makes the walker travel to position \( 0 \); when the walker stands at position \( 0 \), the coin state \([L]\) makes the walker travel to position \( N - 1 \) [47, 48]. Considering that the DTQW contains the coin and the position, we use the wave functions \( \psi_L(n,t) \) and \( \psi_R(n,t) \) to represent the complex amplitudes at time \( t \) and the \( n \)th vertex with the coin state \([L]\) and \([R]\), respectively. To obtain the dynamics of the wave functions \( \psi_L(n,t) \) and \( \psi_R(n,t) \), we work in the momentum space by introducing the Fourier transform

\[
\psi_R(n,t) = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \phi_R(k,t)e^{ikt}, \quad \psi_L(n,t) = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \phi_L(k,t)e^{ikt},
\]

where the possible values of \( k \) are restricted to \( k = 2\pi m/N (m \in \mathbb{Z}) \) since the periodic boundary condition is chosen [41, 47]. When expressing the evolution operator \( U \) of the DTQW in the momentum space as \( \hat{U}_k \), we can obtain the Hamiltonian \( \hat{H}_k \) of such a walk satisfying \( \hat{U}_k = e^{-ik\hat{\eta}} \). In our derivation, the rotation angle \( \theta \) of the coin is assumed to be close to \( \pi/2 \). The derivation details are
presented in appendix A. The evolutions of $\phi_R(k, t)$ and $\phi_L(k, t)$ are governed by the Hamiltonian $H_k$. After tracing back to the real space of the walk, we present the time evolution of $\psi_R(n, t)$ and $\psi_L(n, t)$ as

$$i\frac{\partial}{\partial t}\psi_R(n, t) = -i\gamma[\psi_L(n, t) - \psi_L(n - 2, t)], \quad (4a)$$

$$i\frac{\partial}{\partial t}\psi_L(n, t) = -i\gamma[\psi_R(n + 2, t) - \psi_R(n, t)]. \quad (4b)$$

Here, the coefficient $\gamma$ can be addressed as $\gamma = \frac{1}{2}(\frac{\pi}{2} - \theta)$. When we define new wave functions as $\Psi_+(n, t) = \psi_R(n + 1, t) + \psi_L(n, t)$ and $\Psi_-(n, t) = \psi_R(n + 1, t) - \psi_L(n, t)$, equations (4a) and (4b) can be expressed as

$$i\frac{\partial}{\partial t}\Psi_+(n, t) = -i\gamma[\Psi_+(n + 1, t) - \Psi_-(n - 1, t)], \quad (5a)$$

$$i\frac{\partial}{\partial t}\Psi_-(n, t) = i\gamma[\Psi_+(n + 1, t) - \Psi_-(n - 1, t)]. \quad (5b)$$

The dynamics of the DTQW can be described with $\Psi_+(n, t)$ and $\Psi_-(n, t)$. We want to find the connection between the dynamics of these wave functions $\Psi_+(n, t)$, $\Psi_-(n, t)$ and the CTQW next. The Hamiltonian $H_c$ of the $N$-vertices CTQW on a circle is

$$H_c = \sum_{n=0,1,...,N-1} (\gamma e^{i\xi n}|(n - 1) \text{mod } N\rangle \langle n| + \gamma e^{-i\xi n}|(n + 1) \text{mod } N\rangle \langle n|), \quad (6)$$

we can obtain the dynamics of wave function $\psi_n$ at position $n$ in the CTQW as

$$i\frac{\partial}{\partial t}\psi_n = \gamma e^{i\xi n}\psi_{n+1} + \gamma e^{-i\xi n}\psi_{n-1}. \quad (7)$$

When the phase $\alpha_c = -\pi/2$, the dynamics of $\psi_n$ in the CTQW behaves the same as that of $\Psi_+$ in the DTQW (see equation (5a)); and when the phase $\alpha_c = \pi/2$, the dynamics of $\psi_n$ in the CTQW behaves the same as that of $\Psi_-$ in the DTQW (see equation (5b)). The coupling strength $\gamma$ connects the rotation angle of the coin operator in the DTQW and the hopping coefficient in the CTQW. In figure 2, the time evolutions of probabilities for the DTQW and CTQW are presented. Here, the probability in the DTQW $P_n(t)$ is defined as $P_n(t) = |\psi_R(n + 1, t)|^2 + |\psi_L(n, t)|^2$, and the initial state for the DTQW is assumed to be $|\Phi_{ini}(t)\rangle = \frac{1}{\sqrt{2}}[|1\rangle + |2\rangle]$. For the CTQW, we assume that only vertex 1 is occupied in the beginning and the probability of vertex $n$ is represented by $P_n(t) = |\psi_n(t)|^2$. From the figure, the good correspondence between the DTQW and CTQW can be seen. The time evolution of the probability distribution in the DTQW is similar to that in the CTQW. For the DTQW with even vertices, the probability distribution is symmetric with respect to the initial ‘vertex’ 1; while, when the DTQW contains the odd vertices, the directed probability transfer can be found. This symmetric and asymmetric probability transfers have been addressed in the CTQW on a circle involving the chiral term $e^{i\xi n}$ with the even or odd vertices, respectively [19]. Our results reveal that when the rotation angle of the coin $\theta$ in the DTQW is close to $\pi/2$, the dynamics of the DTQW can be described by the two coupled equations involving the wave functions $\psi_R(n, t)$ and $\psi_L(n, t)$. When rearranging these two wave functions into $\Psi_+(n, t)$ and $\Psi_-(n, t)$, we find that the dynamics of $\Psi_+(n, t)$ and $\Psi_-(n, t)$ have similar evolution dynamics to those in the standard CTQW with even or odd vertices. Based on the discussion above, with the precise parameters chosen in the easily implemented DTQW, we can reproduce the characteristic behaviors of probability transfer in the CTQW.
3. Connection of hidden symmetries and probability transfer in the DTQW

In the section above, we have discussed how to express the characteristics of probability transfer of the CTQW in the DTQW. When the rotation angle of the coin operator in the DTQW is close to \(2\pi\), the dynamics of the wave function in the DTQW can be elaborated in the same form as in the CTQW. In the following, we study the probability transfer in the DTQW without the assumption of the rotation angle of the coin being close to \(2\pi\).

The chiral term \(e^{i\alpha_a}\) is added into the walk. The DTQW involving this chiral term can be realized easily in experiments, e.g., electric field, optical devices, and so on \([49–51]\). In our discussion below, two DTQWs have been put forward, one is DTQW I with \(U_{1c} = \) and the other is DTQW II with \(U_{2c} = S_2 C(\theta)\). The conditional shift operator \(S_1\) in the one-step evolution operator of DTQW I \(U_1\) is

\[
S_1 = \sum_{x=0,1,\ldots,N-1} |(x + 1) \mod N \rangle \langle x| \otimes |R\rangle \langle R| e^{i\alpha_a} + |(x - 1) \mod N \rangle \langle x| \otimes |L\rangle \langle L| e^{-i\alpha_a},
\]

(8)

and the conditional shift operator \(S_2\) in the one-step evolution operator of DTQW II \(U_2\) is

\[
S_2 = \sum_{x=0,1,\ldots,N-1} |(x + 1) \mod N \rangle \langle x| \otimes |L\rangle \langle L| e^{i\alpha_a} + |(x - 1) \mod N \rangle \langle x| \otimes |R\rangle \langle R| e^{-i\alpha_a}.
\]

(9)

We only consider the \(N\)-vertices DTQW on a circle. As revealed by the conditional shift operators \(S_1\) and \(S_2\), the coin state \(|R\rangle\) makes the walker travel from position \(x\) to \(x + 1\), and the coin state \(|L\rangle\) makes the walker travel from position \(x\) to \(x - 1\). The difference between DTQW I and DTQW II is that the coin state will change to its orthogonal state after applying the conditional shift operator \(S_2\), but the coin state will remain unchanged when applying the conditional shift operator \(S_1\). When the walker stands at position \(N - 1\), the coin state \(|R\rangle\) makes the walker travel to the position 0; when the walker stands at the position 0, the coin state \(|L\rangle\) makes the walker travel to position \(N - 1\). These two types of DTQWs have been widely studied for quantum algorithms, quantum state transfer and quantum gates \([44]\). The experimental proposals for these two DTQWs involving the chiral term \(e^{i\alpha_a}\) are presented in section 5. In our discussion, to get rid of the effects of the initial state and environment, we assume that the initial state of the DTQW is unbiased and there is no leakage at the certain vertex. The phase \(\alpha_a\) is set as \(N\alpha_a = m\pi (m \in \mathbb{Z})\). If the phase meets \(N\alpha_a = m\pi\), when the wave functions undergoing different cycles arrive at the certain vertex, the phase differences between these wave functions are multiples of \(\pi\). For two symmetric vertices with respect to the starting vertex, the chiral term only introduces different global phases into the states at these two vertices, which cannot appear in the probabilities. The emergence of the symmetric or asymmetric probability transfer in the DTQW only depends on the intrinsic property of the walk. In the description of the CTQW, the emergence of the directed probability transfer is attributed to the broken PTRS \(\{U_{ij}\}^0 \neq \{U_{ji}\}^0\). The presence or not of the PTRS in the CTQW can be found by the elements of step-evolution operator \(U = e^{-iH}t\), where the operator \(H\), describes the Hamiltonian of the

\[\text{Figure 2.}\] The time evolutions of probabilities for different vertices in the DTQW and CTQW on a circle. The probability in the DTQW is \(P_{1r} = \langle \psi_1(x + 1, t) | R \rangle^2 + \langle \psi_1(x, t) | L \rangle^2\), and the probability at the vertex \(n\) in the CTQW is \(P_{1n} = \langle \psi_n(t) | L \rangle^2\). Left, the number of vertices is six; Right, the number of vertices is seven. (a) and (b) The rotation angle in the coin \(\theta = 15\pi/32\). (c) and (d) The rotation angle in the coin \(\theta = -\pi/2\).
CTQW. While, due to the additional coin, the one-step evolution operator \( U \) for the DTQW contains the contribution from the coin, and the observation of the PTRS of \( U \) cannot provide direct evidence of probability transfer among the vertices in the walk. Though the study of PTRS of \( U \) does not work well in the analysis of the probability transfer in the DTQW, it provides a hint so that we can analyse the hidden symmetries in the effective Hamiltonian of the DTQW and explore the relation between the symmetries and the probability transfer in the walk. The one-step evolution operators in the DTQWs are equivalent to those generated by the effective Hamiltonians \( U_1 = e^{-iH_{01}(\theta)} \) and \( U_2 = e^{-iH_{02}(\theta)} \). The Hamiltonians \( H_{01}(\theta) \) and \( H_{02}(\theta) \) for DTQW I and DTQW II are

\[
H_{01}(\theta) = \frac{1}{\pi} \int dk \left[ \xi_k (\theta, k) \tilde{n}_1^2 (\theta, k) \cdot \partial \right] \otimes |k\rangle \langle k|, \\
H_{02}(\theta) = \frac{1}{\pi} \int dk \left[ \xi_k (\theta, k) \tilde{n}_2^2 (\theta, k) \cdot \partial \right] \otimes |k\rangle \langle k|.
\]

(10a)

(10b)

The Fourier transform \(|x\rangle |\sigma\rangle = \frac{1}{\sqrt{2N}} \sum_k e^{ikx} |k\rangle |\sigma\rangle \) has been employed in our derivation. The details of derivation can be found in appendix B. We have set \( \tilde{\sigma} = (\sigma_x, \sigma_y, \sigma_z) \), and sin \( \xi_k = \cos \theta \sin (k + \alpha_d) \), cos \( \xi_k = \sin \theta \cos (k + \alpha_d) \). The vector \( \vec{n}_1 (\theta, k) \) and vector \( \vec{n}_2 (\theta, k) \) are

\[
\vec{n}_1 (\theta, k) = \begin{bmatrix} \sin \theta \cos (k + \alpha_d) \\ -\sin \theta \sin (k + \alpha_d) \\ -\cos \theta \cos (k + \alpha_d) \end{bmatrix}, \\
\vec{n}_2 (\theta, k) = \begin{bmatrix} \cos \theta \sin (k + \alpha_d) \\ -\cos \theta \cos (k + \alpha_d) \\ -\sin \theta \sin (k + \alpha_d) \end{bmatrix}.
\]

(11a)

(11b)

As we have obtained the Hamiltonian \( H_{01} \) and \( H_{02} \) for DTQW I and DTQW II, we can study the symmetries in these two DTQWs and explore the relation between the symmetries and the probability transfer in the DTQW.

**Chiral symmetry.** In our discussion, we assume that the rotation angles at different vertices in the DTQW are the same, and we do not focus on the edge between two bulks with different rotation angles [52]. In this case the chiral symmetry \( \Gamma_\theta \) can be obtained as

\[
\Gamma_\theta \mathcal{H}_\theta \Gamma_\theta^{-1} = -\mathcal{H}.
\]

(12)

For DTQW I, by setting \( \Gamma_{1,0} = e^{-i\xi\tilde{B}_1(0,k)\beta/2} \) with the vector \( \vec{B}_1 (\theta, k) = [\cos \theta, 0, \sin \theta] \), we can obtain \( \Gamma_{1,0} \mathcal{H}_1 \Gamma_{1,0}^{-1} = -\mathcal{H} \). For DTQW II, when we have \( \Gamma_{2,0} = e^{-i\xi\tilde{B}_2(0,k)\beta/2} \) with the vector \( \vec{B}_2 (\theta, k) = [\sin \theta, 0, \cos \theta] \), the relation \( \Gamma_{2,0} \mathcal{H}_2 \Gamma_{2,0}^{-1} = -\mathcal{H} \) is satisfied. The emergence of chiral symmetry in the DTQW makes the eigenvalue spectrum of effective Hamiltonian \( \mathcal{H} \) distribute symmetrically with respect to zero energy [52–54].

**Sublattice symmetry.** In the DTQW, the walker hops left or right positions on the lattice according to the outcome from tossing the coin. For one hopping process on the lattice, the positions at which the walker start and end belong to different sublattices. The sublattice symmetry \( \Lambda \) in the DTQW can be defined as

\[
\Lambda = \sum_{x \in \text{even}} |x\rangle \langle x| - \sum_{x \in \text{odd}} |x\rangle \langle x|.
\]

When the DTQW contains sublattice symmetry, the one-step evolution operator \( \mathcal{L} \) satisfies \( \Lambda \mathcal{L} \Lambda^{-1} = -U \), and the effective Hamiltonian \( H \) satisfies \( \Lambda H \Lambda^{-1} = H + \pi \). It means if \( E \) is the eigenvalue of the Hamiltonian \( H \), the value \( E + \pi \) is also the eigenvalue of \( \mathcal{L} \).

**Particle–hole symmetry.** When the system possesses particle–hole symmetry (PHS), the effective Hamiltonian \( H \) of the DTQW satisfies \( \mathcal{P} \mathcal{H} \mathcal{P}^{-1} = -\mathcal{H} \). This operator \( \mathcal{P} \) is the PHS operator and often seen as the complex conjugate operator [52]. It means that when the one-step evolution operator for the DTQW \( U \) has only real elements, the particle–hole symmetry is satisfied. The existence of particle–hole symmetry guarantees that the eigenvalues of \( H \) are symmetric with respect to zero energy.

**Time-reversal symmetry.** The time-reversal symmetry \( T \) requires the system Hamiltonian \( H \) to meet \( \mathcal{T} H \mathcal{T}^{-1} = -H \). It has been verified that when the system possesses chiral symmetry and particle–hole symmetry, the \( T \) is also satisfied for the system, \( \mathcal{T} = \Gamma_\theta \mathcal{L} \mathcal{P} \) [52]. Though the explicit expression of \( T \) is not easy to obtain, from the expression \( \mathcal{T} = \Gamma_\theta \mathcal{L} \mathcal{P} \), we can test whether the system possesses the chiral symmetry and particle–hole symmetry first, and then we can judge the existence of \( T \).

In figure 5, we present the eigenvalues of the Hamiltonians for DTQW I and DTQW II. As addressed in the description of the sublattice symmetry, such symmetry requires the DTQW to contain even vertices. In our study, the number of vertices in the DTQW is six. Our calculation associated with the chiral symmetry \( \Gamma_\theta \) can work well with the rotation angle \( \theta \in (0, \pi/2) \). Because the rotation angle takes 0 or \( \pi/2 \), the eigenstates of the Hamiltonian \( H_1 \left[ \vec{n}_1^2 (\theta, k) \right] \) and \( H_2 \left[ \vec{n}_2^2 (\theta, k) \right] \) are ill-defined. In DTQW I and DTQW II with the rotation angle \( \theta \in (0, \pi/2) \), when the \( \alpha_d \) in the chiral term is \( \alpha_d \in (0, \pi/2) \), the elements of one-step evolution operators \( U_1 \) and \( U_2 \) are not all real numbers. So the PHS for DTQW I and DTQW II are broken with \( \alpha_d \in (0, \pi) \). From the calculation above, no matter what value of \( \alpha_d \) is taken, the chiral term \( e^{i\alpha_d} \) does not affect the chiral symmetry of the effective Hamiltonian. The existence of chiral symmetry guarantees that the energy spectrum is symmetric with respect to zero energy [53, 54]. Considering that the chiral symmetry, PHS and \( T \) have a direct connection, we can obtain that the system does not possess the \( T \) with \( \alpha_d \) of the chiral term satisfying \( \alpha_d \in (0, \pi) \). In contrast, when \( \alpha_d \) of the chiral term takes the value 0 or \( \pi \), the emergence of chiral symmetry and PHS guarantees the existence of \( T \) for the effective Hamiltonians of DTQW I and DTQW II. For sublattice symmetry, we need to verify whether the dynamic evolution \( U \) satisfies \( \Lambda U \Lambda^{-1} = -U \). By putting the forms of \( U_1 \)
and $U_z$ into the expression, we find that when the system contains even vertices, the sublattice symmetry is satisfied (see the eigenvalue spectrum of $H$ in figure 3); in comparison, when the system contains odd vertices, the sublattice symmetry is not satisfied.

Here, we present the probability distributions among the vertices in the DTQW at time $t = 11$ and $t = 12$ in figure 4. When the rotation angle $\theta = \pi/5$ and chiral term $\alpha_d = 0$, the symmetric probability distribution with respect to the starting vertex can be found. When the rotation angle $\theta = \pi/5$ and the chiral term $\alpha_d = \pi/7$, the directed probability transfer has been presented. In our discussion, the number of vertices in the DTQW is five and the system does not contain sublattice symmetry. When chiral term $\alpha_d = 0$, from the analysis above, the chiral symmetry, PHS and TRS are satisfied. When chiral term $\alpha_d = \pi/7$, the chiral symmetry is satisfied (equation (12) and below). Due to the appearance of the imaginary component in the $U_{1,2}$, the PHS for DTQW I and II is broken, so the TRS does not exist in these two walks. Based on the discussion above, the appearance of the directed probability transfer has a close relation to the TRS of the Hamiltonian. When the rotation angle $\theta \in (0, \pi/2)$ and $\alpha_d = 0$, the existence of TRS leads to symmetric distributions among the vertices in the DTQW. The directed probability transfer and the asymmetric distributions among the vertices in the DTQW connect closely with the broken TRS. Furthermore, when checking the elements of step operator $U_{1,2}$, we find that the asymmetric or symmetric probability transfer has no direct connection with the PTRS.

When the rotation angles $\theta = 0$ or $\pi/2$, we have found that no matter what the value $\alpha_d$ is, the symmetric distribution among the vertices in the DTQW can be found. We have stated that when the rotation angle is $\theta = 0$ or $\pi/2$, the effective Hamiltonian of the DTQW and the associated eigenstates are ill-defined. The analysis of symmetries in the walk cannot be applied with $\theta = 0$ or $\pi/2$. Here, we explore the reason for the symmetric distributions among different vertices in the DTQW from the step operator $U_{1,2}$. For DTQW I, when the rotation angle is $\theta = 0$, the step operator $U_1 = \sum_{m=0,1,\ldots,N-1} e^{i\alpha_d}|R\rangle \langle R| \otimes |(m+1)\mod N\rangle \langle m| - e^{-i\omega |L\rangle \langle L| \otimes |(m-1)\mod N\rangle \langle m|}$ when the rotation angle is $\theta = \pi/2$, the step operator $U_1 = \sum_{m=0,1,\ldots,N-1} e^{i\alpha_d}|R\rangle \langle R| \otimes |(m+1)\mod N\rangle \langle m| + e^{-i\omega |L\rangle \langle L| \otimes |(m-1)\mod N\rangle \langle m|}. From the expression of the step operator $U_1$, we find that when the rotation angle is $\theta = 0$, the dynamics of DTQW I can be divided into two parts, one is related to the wave function with the coin state $|R\rangle$, the other describes the evolution of probability with the coin state $|L\rangle$. These two parts have no communication with each other, and the evolution of the probability distribution is symmetric if the initial state at the starting vertex is half probability in $|R\rangle$ and half in $|L\rangle$. For DTQW I with the rotation angle $\theta = \pi/2$, it is apparent that the walker can only jump between the two nearest neighboring vertices of the starting vertex. So the probability distribution of the DTQW I with $\theta = \pi/2$ is confined to the three vertices only. Considering that there are five vertices in the walk on a circle, with the rotation angle $\theta = \pi/2$, the dynamics of DTQW I on a circle is same as that of DTQW I on a linear
chain, and the symmetric probability distribution will emerge with the unbiased initial state. The details of the DTQW I on a linear chain are addressed at the end of this section. For DTQW II, when the rotation angle is \( \theta = 0 \), the step operator \( U^{m} = \sum_{m = 0}^{1} \exp(i \phi |L \rangle \langle R| \otimes (m + 1) \mod N) (m - \exp(-i \phi) |L \rangle \langle R| \otimes (m - 1) \mod N) (m) \); when the rotation angle is \( \theta = \pi/2 \), the step operator \( U_{2} = \sum_{m = 0}^{1} \exp(i \phi |L \rangle \langle R| \otimes (m + 1) \mod N) (m) + \exp(-i \phi) |R \rangle \langle L| \otimes (m - 1) \mod N) (m) \). From the expression of the step operator \( U_{2} \), when the rotation angle is \( \theta = \pi/2 \), the dynamics of DTQW II can be divided into the evolution of wave function associated with the coin state \( |R \rangle \) and that with the coin state \( |L \rangle \) individually. The appearance of the symmetric distribution in DTQW II requires the unbiased initial state of the walk only. While, when the rotation angle \( \theta = 0 \), the walker in the DTQW II jumps between the two nearest neighboring vertices of the starting vertex. The probability distribution in the DTQW II on a circle with \( \theta = 0 \) is same as that in the DTQW II on a linear chain with the same rotation angle. There is no directed probability transfer when the initial state of the walk is unbiased. Though we present the evolutions of the probability distributions on the odd vertices DTQW I and II (figure 4), we have checked that the dynamics of probability distributions on the even vertices DTQW I and II have the same behaviors. We can conclude that the emergence of the symmetric or asymmetric probability transfer depends on the TRS of the effective Hamiltonian of the walk.

In the discussion above, we present the probability distributions in DTQW I and DTQW II with five vertices. Here, we extend the system size and study the probability distribution in the walk with a large number of vertices, see figures 5 and 6. The only difference between figures 5 and 6 is the phase \( \omega_{i} \) in the chiral term. The probability distributions in the DTQW I systems with four different numbers of vertices are addressed. For figures 5(a) and (b) (figures 6(a) and (b)), the initial states of the walks are the same, \( \{\frac{1}{\sqrt{2}} (|R \rangle + i |L \rangle) \}|6 \rangle \). For figures 5 (c) and (d) (figures 6(c) and (d)), the initial states of the walks are the same, \( \{\frac{1}{\sqrt{2}} (|R \rangle + i |L \rangle) \}|11 \rangle \).
The rotation angle in the coin is \( \theta = \pi/5 \). The phase \( \alpha_d \) in the chiral term is \( \alpha_d = \pi/\sqrt{7} \) in figure 5 and \( \alpha_d = 0 \) in figure 6. For these four different systems, when \( \alpha_d = \pi/\sqrt{7} \), the asymmetric probability distributions in the walk can be found (see figure 5). In this case, the TRS of the effective Hamiltonian is broken. While, when \( \alpha_d = 0 \), the TRS of the effective Hamiltonian exists, the symmetric probability distributions in the walk emerge (see figure 6). Besides DTQW I, the probability distributions in DTQW II with a large number of vertices have similar behaviors to those in DTQW I, which are not presented in the text. Our results here verify that the connection between the TRS of the effective Hamiltonian and the probability transfer is not limited to DTQW systems with only a small number of vertices.

In addition, we present the probability distributions in the DTQW on a chain, see figure 7. We can find that though the additional phase \( \alpha_d \) in the chiral term emerges, the asymmetric probability distributions in the walk do not appear. In the figure, we show that the probability distributions in the walk remain symmetric at different times. For one vertex on the chain at a certain time, the wave packets at a previous step from different positions come to this vertex and interfere with each other. The accumulated phases from the chiral term \( e^{-i\alpha_d} \) for these wave packets are same, so the chiral term has no effect on the probability distribution in the walk on the chain. We also obtain the results with \( \alpha_d = 0 \), which is same as shown in figure 7. While, for one vertex on a circle, the wave packets at a previous step from different positions can undergo different cycles in the walk, these wave packets come to this vertex in the next step and interfere with each other. The accumulated phases from the chiral term \( e^{-i\alpha_d} \) for these different wave packets are different, and the probability distributions in the walk on a circle are affected by the chiral term.

Figure 5. The probability distributions for the DTQW I with time are shown. The probability at the vertex \( n \) is \( P(n) = |\psi_n(t)|^2 + |\psi'_n(t)|^2 \). Four different numbers of vertices are chosen. (a) \( N = 11 \), (b) \( N = 12 \), (c) \( N = 21 \), (d) \( N = 22 \). The rotation angle in the coin is chosen \( \theta = \pi/5 \), and the additional phase \( \alpha_d \) in the walk is chosen \( \alpha_d = \pi/\sqrt{7} \). The initial states for the DTQWs are, (a) and (b) \( \frac{1}{\sqrt{2}} (|R\rangle + |L\rangle) \); (c) and (d) \( \frac{1}{\sqrt{2}} (|R\rangle + |L\rangle)|11\rangle \).
Figure 6. The probability distributions for the DTQW I with time are shown. The probability at the vertex $n$ is $P(n) = |\psi_R(n)|^2 + |\psi_L(n)|^2$. Four different numbers of vertices are chosen. (a) $N = 11$, (b) $N = 12$, (c) $N = 21$, (d) $N = 22$. The rotation angle in the coin is chosen $\theta = \pi/5$, and the additional phase $\alpha_d$ in the walk is chosen $\alpha_d = 0$. The initial states for the DTQWs are, (a) and (b) $\frac{1}{\sqrt{2}}(|R\rangle + i|L\rangle)|0\rangle$; (c) and (d) $\frac{1}{\sqrt{2}}(|R\rangle + i|L\rangle)|1\rangle$.

Figure 7. The probability distributions for DTQW I on a chain are shown. The probability at the vertex $n$ is $P(n) = |\psi_R(n)|^2 + |\psi_L(n)|^2$. The step for the walk (a) $t = 4$, (b) $t = 5$. The rotation angle in the coin is chosen $\theta = \pi/5$, and the additional phase $\alpha_d$ in the walk is chosen $\alpha_d = \pi/\sqrt{2}$. The initial states for the DTQWs are $\frac{1}{\sqrt{2}}(|R\rangle + i|L\rangle)|0\rangle$. 
4. Effect of disorder on the probability transfer in the DTQW

In this section, we will study probability transfer in the DTQW with consideration of the disorder. Due to the inevitable interaction with the environment, disorder emerges in the DTQW system and affects the characteristics of the walk. Here, we assume that the disorder is introduced into the rotation angle of the coin, that is \( \theta \in [0 - \delta \theta, \theta + \delta \theta] \). Here, \( \bar{\theta} \) stands for the mean value of distributed \( \theta \). Firstly, we discuss the probability distribution in the DTQW involving the temporal disorder, the rotation angle \( \theta \) of the coin is uniformly randomly distributed over time \( t \), and the strength of temporal disorder is \( \delta \theta = \pi / 16 \). All the figures have been plotted by averaging with 10000 realizations. Error bars indicate the standard deviation. In all figures, the blue bar stands for the probability at the vertex with the chiral term \( e^{i \alpha_d} (\alpha_d = 0) \), and the red bar represents the probability at the vertex with the chiral term \( e^{i \alpha_d} (\alpha_d = \pi / 7) \). The initial states of the DTQWs are \( |R \rangle + |L \rangle \).
above, when the rotation angle $\theta \in (0, \pi/2)$, the existence of the TRS of the effective Hamiltonian of the walk depends on the chiral term $e^{i\alpha_d}$. Here, considering that the mean value of distributed $\theta$ is $\bar{\theta} = \pi/16$, and the strength of spatial disorder is $\bar{\theta}_s = \pi/16$. All the figures have been plotted by averaging with 10000 realizations. Error bars indicate the standard deviation. In all figures, the blue bar stands for the probability at the vertex with the chiral term $e^{i\alpha_d}$ ($\alpha_d = 0$), and the red bar represents the probability at the vertex with the chiral term $e^{i\alpha_d}$ ($\alpha_d = \pi/7$). The initial states for the DTQWs are $R \otimes (|R \rangle + i|L \rangle)|3\rangle$.

Spatial disorder. Here, we study the effect of spatial disorder in the walk on the probability transfer. The rotation angle of the coin $\theta$ is uniformly randomly distributed over position $n$. In our discussion, the mean value of distributed $\theta \in (0, \pi/2)$, and the strength of spatial disorder is $\bar{\theta}_s = \pi/16$. As can be seen from figure 9, the existence of the spatial disorder makes the probability distribution in the walk fluctuate in each realization. By averaging with 10000 realizations, the standard deviation of the probability distribution is indicated by the error bars in the figure. We find that when the chiral term satisfies $\alpha_d = 0$, the symmetric averaged probability transfer with the time evolution in the walk is present; in contrast, the asymmetric averaged probability transfer among the vertices in the walk emerges with the chiral term $e^{i\alpha_d}$. Though the spatial disorder makes rotation angles $\theta$ for different vertices different in one realization, the coin operators at the ensemble level for different vertices are the same. When the chiral term $\alpha_d$ is $\pi/7$, the TRS of the effective Hamiltonian is broken and the directed averaged probability transfer emerges. Our results here verify that such TRS protected symmetric probability transfer is robust to the spatial disorder in the walk.

Figure 9. The time evolution of the probability distribution for DTQW I and DTQW II are shown. The probability at the vertex $n$ is $P(n)$. The left two figures are depicted with DTQW I, and the right two figures are DTQW II. The number of vertices is chosen as five. The spatial disorder is introduced into the coin of the walk, where the mean value of distributed $\theta$ is $\bar{\theta} = \pi/5$, and the strength of spatial disorder is $\bar{\theta}_s = \pi/16$. All the figures have been plotted by averaging with 10000 realizations. Error bars indicate the standard deviation. In all figures, the blue bar stands for the probability at the vertex with the chiral term $e^{i\alpha_d}$ ($\alpha_d = 0$), and the red bar represents the probability at the vertex with the chiral term $e^{i\alpha_d}$ ($\alpha_d = \pi/7$). The initial states for the DTQWs are $R \otimes (|R \rangle + i|L \rangle)|3\rangle$. New J. Phys. 19 (2017) 113049 T Chen et al.
5. Experimental proposals

In this section, we will study how to experimentally observe the probability distribution in the DTQW on a circle. Two different experimental proposals are addressed. The first proposal is presented below, see figure 10.

In this proposal, we use the orbital angular momentum (OAM) modes of light as positions in the DTQW, and the polarization of light as the coin in the walk. We set the horizontal polarization state $H\langle 0 \rangle$ (the vertical polarization state $V\langle 0 \rangle$) as the state $R\langle 0 \rangle$ ($L\langle 0 \rangle$) in the walk. The OAM-BS scheme $[58, 59]$ is addressed in figure 10(a).

The OAM-BS changes the propagation direction of the incident light with odd OAM modes, but does not change the direction of the incident light with even OAM modes. In our proposal, the action of polarization beam splitter (PBS) transmits light with horizontal polarization and reflects light with vertical polarization. The combination of one half-wave plate, one quarter-wave plate and one half-wave plate ($HWP-QWP-HWP$) realizes the addition of the phase on the light $[60, 61]$. In our proposal, the light with the horizontal polarization travels from right to left, and obtains the phase $\alpha_H$ when passing the $HWP-QWP-HWP$; and the light with the vertical polarization travels from left to right, and obtains the phase $-\alpha_V$. For the spiral phase plate (SPP) in our scheme, the light with the horizontal polarization passes the SPP from right to left, the OAM modes of light is shifted by $+1$; while, the light with the vertical polarization passes the SPP from the left to right, the OAM modes of light is shifted by $-1$. The SPP and $HWP-QWP-HWP$ realize the conditional shift operator in the DTQW on a circle $x \mod 4 \left( x + 1 \right)$ $H\langle x \rangle \otimes \left( H\langle x \rangle + V\langle x - 1 \rangle \right) \otimes \langle x \rangle \otimes \langle H\rangle \otimes \left( V\langle x \rangle - e^{-i\alpha_H} \right)$. The cyclic transformation for OAM modes of light can be realized with the OAM-BS $[58]$. In our proposal, we take four OAM modes ($-2, -1, 0$ and $1$) of light to represent the four-vertices (0, 1, 2 and 3) DTQW system on a circle. The light comes from the source $S$, and the $HWP$ serves as the coin operation on the polarization of light. After passing the PBS, the light with the horizontal and vertical polarization travel along the horizontal and vertical direction, respectively. The combination of $HWP-QWP-HWP$, $SPP$, and two OAM-BS realizations the conditional shift operation in the DTQW on a circle $x \mod 4 \left( x + 1 \right)$ $H\langle x \rangle \otimes \left( H\langle x \rangle + V\langle x - 1 \rangle \right) \otimes \langle x \rangle \otimes \langle H\rangle \otimes \left( V\langle x \rangle - e^{-i\alpha_H} \right)$. The one-step evolution in the DTQW on a circle is completed. The light with the horizontal and vertical polarization combine together when reentering the PBS. The two mirrors reflect the light into the beam splitter (BS). A part of light continues to pass the HWP, and travels as the same as mentioned above. So the evolution of DTQW on a circle containing many steps can be realized. The other part of light coming from the output of BS enters into the detector $D$, which is used to measure the outcomes of each step evolution in the walk.

With the proposal above, we can realize the experimental observation of probability distribution in the DTQW on a circle containing many steps. This proposal is suitable for the four-vertices DTQW system on a circle. The odd-vertices DTQW system on a circle can be realized experimentally by referring to $[48]$. The light

![Figure 10. Experimental proposal for observing the probability distribution in the DTQW on a circle. The coin operator in the walk is realized by one half-wave plate with a certain angle. The conditional shift operator in the DTQW on a circle is realized with the combination of one half-wave plate, one quarter-wave plate, one half-wave plate (HWP–QWP–HWP), one spiral phase plate (SPP) and two OAM-BSs. The function of the OAM-BS is addressed in (a). The loop corresponds to one step of the walk.](image-url)
with the horizontal or vertical polarization can be separated by beam displacers (BDs). The light at different spatial positions are registered as different positions in the DTQW. In this proposal, at each step of evolution in the DTQW on a circle, we need to relabel the spatial positions of light with the positions of the DTQW. Given that the additional phase $\alpha_q$ is required in our DTQW on a circle, by referring to figure 5 in [48], we need to add the HWP–QWP–HWP at the end of each step in the walk [60, 61]. In this way we can realize to observe the probability distributions in the odd-vertices DTQW on a circle. Furthermore, with different schemes in this proposal involving BDs, we can realize the even-vertices DTQW on a circle [48]. Compared to the proposal with the OAM modes of light above, the advantage of this proposal with BDs is suitable for both even and odd-vertices DTQWs on a circle, but the weakness of this proposal is that many optical apparatuses are required at each step evolution of the walk, and it is difficult to realize many evolution steps of the DTQW on a circle. Moreover, though these two experimental proposals are for DTQW I in our study, DTQW II can also be experimentally realized by adding one HWP with the angle $\pi /4$ at the end of each step evolution in these two proposals.

6. Summary

In this paper, we study how to control the probability transfer among the vertices in the DTQW on a circle. In our discussion, by using the approximate analytic method and strict numerical simulation, we have presented the non-directed probability transfer in the DTQW involving even vertices and directed probability transfer in the DTQW containing odd vertices. Such probability transfer behaviors related to odd or even vertices have been addressed in the description of the CTQW. Here, we have established the connection between the DTQW and the CTQW. We have presented that, in the DTQW, we can reproduce the characteristic of the probability transfer in the CTQW. Next, without the assumption of the biased initial state and the leakage at the certain vertex, we have explored the reason for the directed probability transfer in the DTQW and revealed that the emergence of the directed probability transfer connects closely to the TRS of the effective Hamiltonian of the system. Considering that the rotation angle belongs to $\theta \in (0, \pi /2)$, when the additional chiral term does not affect the TRS of DTQW, the probability among the vertices of the walk distributes symmetrically with respect to the starting position, and there the directed probability transfer does not exist. The directed probability transfer appears with the broken TRS in the walk. For the CTQW, it has been verified before that the emergence of the symmetric probability transfer relates to the TRS. As addressed in the description of the CTQW, the satisfaction of PTRS requires that the number of vertices in the walk is even. For our studied DTQW, the TRS can be verified in walks with both odd and even vertices, so the appearance of the symmetric or asymmetric probability transfer has nothing to do with the number of vertices in the walk. When the rotation angle is 0 or $\pi /2$, the time evolution of the DTQW is either divided into two individual dynamics associated with one of the two coin states $|R\rangle$ and $|L\rangle$, or confined into the nearest neighboring vertices of the starting position in the DTQW. So the chiral term does not affect the symmetric probability distributions among the vertices in the walk. Our results revealed that though the temporal or spatial disorder is introduced into the walk, the symmetric averaged probability transfer still exists with the appearance of the TRS in the effective Hamiltonian of the DTQW.

Controlling the probability or energy flow along a directed path is an important topic, especially without the requirement for a well-designed initial state or leakage at the certain vertex. Our results here elucidate the relation between the probability transfer and the hidden symmetries in the quantum walk system. Besides our theoretic analysis, we have presented two experimental proposals to observe the probability distributions in the DTQWs on a circle. With these proposals, we can verify our results about the connection between the TRS of the system and the probability transfer in the walk, which will be our next investigation.

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Appendix A. Evolution dynamics for $\psi_L(n, t)$ and $\psi_R(n, t)$ of the DTQW on a circle

We present the derivation of the dynamics for the wave functions $\psi_L(n, t)$ and $\psi_R(n, t)$ in detail. As $\psi_L(n, t)$ and $\psi_R(n, t)$ represent the complex amplitudes at time $t$ and the $n$th vertex with the coin state $|L\rangle$ and $|R\rangle$, respectively. By applying the Fourier transform, the state of the DTQW in the momentum space is
\[ \psi(n, t) = \begin{pmatrix} \psi_R(n, t) \\ \psi_L(n, t) \end{pmatrix} = \frac{1}{2\pi} \int_{-\pi}^{\pi} dk \begin{pmatrix} \phi_R(k, t) \\ \phi_L(k, t) \end{pmatrix} e^{i k t}. \]  

(A1)

Given that all vertices in our DTQW form a circle, the possible values of \( k \) are restricted to \( k = 2\pi m/N (m \in \mathbb{Z}) \) [47]. The evolutions of \( \phi_R(k, t) \) and \( \phi_L(k, t) \) in the momentum space are

\[
\begin{align*}
\phi_R(k, t) &= U_t^R \phi_R(k, 0), \\
\phi_L(k, t) &= U_t^L \phi_L(k, 0),
\end{align*}
\]

(A2)

where the evolution operator \( U_k \) of the DTQW is addressed in the \( k \) space as

\[
U_k = e^{-ik_0} e^{-ik_0} \sigma_x = e^{-ik_0} e^{-i\theta/2-\delta} \sigma_x = e^{-ik_0} e^{i\theta/2} \sigma_x.
\]

(A3)

Here, we assume that the parameter of the coin \( \theta \) is close to \( \pi/2 \), and \( \theta = \pi/2 - \delta \). We have

\[
\begin{align*}
U_k^R &= e^{-ik_0} e^{i\theta_0} \sigma_x e^{-ik_0} e^{i\theta_0} \sigma_x = e^{-ik_0} e^{i\theta_0} e^{i\theta_0}, \\
&= e^{i\theta(\cos 2k - \delta \sin 2k)} e^{-i\theta} = e^{ik(\cos 2k - \delta \sin 2k)} + \mathcal{O}(\delta^2) + \mathcal{O}(\delta^2).
\end{align*}
\]

(A4)

To derive the equation above, we use the Baker–Campbell–Hausdorff formula, that is

\[
e^{-ik_0} \sigma_x e^{ik_0} = \sigma_x + (-1)[ik_0 \sigma_x, \sigma_x] + \frac{(-1)^2}{2!} [ik_0, [ik_0, \sigma_x]] + ... = \sigma_x \cos 2k - \sigma_y \sin 2k.
\]

(A5)

Thus we have

\[
U_k^R = (U^2)^{t/2} = \exp[-i\theta t \sin(\cos k \sigma_x + \sin k \sigma_y)] = \exp[-i2\gamma t \sin(\cos k \sigma_x + \sin k \sigma_y)],
\]

(A6)

with \( \delta t \to 2\gamma \). If we set \( \delta t = t \), then \( \delta = 2\gamma \). Due to the time evolution \( U_k = e^{-ik_0} \), we can extract the Hamiltonian for this walk as

\[
H_k = 2\gamma \sin(\cos k \sigma_x + \sin k \sigma_y) = -i\gamma \begin{pmatrix} 0 & 1 - e^{-i2k} \\ e^{i2k} & 0 \end{pmatrix}
\]

(A7)

Considering that the unitary operator \( U_k \) and the Hamiltonian \( H_k \) describe the evolution of the DTQW, we trace back to the real space of the DTQW, and obtain the evolution of \( \psi_R(n, t) \) and \( \psi_L(n, t) \) as

\[
\begin{align*}
\frac{i}{\hbar} \frac{\partial}{\partial t} \psi_R(n, t) &= -i\gamma [\psi_L(n, t) - \psi_L(n - 2, t)], \\
\frac{i}{\hbar} \frac{\partial}{\partial t} \psi_L(n, t) &= -i\gamma [\psi_R(n + 2, t) - \psi_R(n, t)].
\end{align*}
\]

(A8a,b)

**Appendix B. Hamiltonians \( H_1(\theta) \) and \( H_2(\theta) \) for DTQW I and DTQW II**

Here, we provide the derivation of Hamiltonians \( H_1(\theta) \) and \( H_2(\theta) \) for DTQW I and DTQW II, respectively. As the one-step evolution in the DTQW contains one coin operation and one shift operation, the shift operators \( S_1 \) and \( S_2 \) can be expressed in the \( k \) space as

\[
S_1(k) = \sum_k e^{ik} |k\rangle \otimes |R\rangle \langle R| e^{i\alpha} + e^{-ik} |k\rangle \otimes |L\rangle \langle L| e^{-i\alpha}
\]

\[
= \sum_k |k\rangle \langle k| \begin{pmatrix} e^{i(k+\alpha)} & 0 \\ 0 & e^{-i(k+\alpha)} \end{pmatrix}
\]

(B1a)

\[
S_2(k) = \sum_k e^{ik} |k\rangle \otimes |L\rangle \langle R| e^{i\alpha} + e^{-ik} |k\rangle \otimes |R\rangle \langle L| e^{-i\alpha}
\]

\[
= \sum_k |k\rangle \langle k| \begin{pmatrix} 0 & e^{-i(k+\alpha)} \\ e^{i(k+\alpha)} & 0 \end{pmatrix}
\]

(B1b)

The one-step evolution operators in the DTQW I \( U_1 \) and DTQW II \( U_2 \) in the \( k \) space are

\[
U_1(k) = \begin{pmatrix} \cos \theta e^{i(k+\alpha)} & \sin \theta e^{i(k+\alpha)} \\ \sin \theta e^{-i(k+\alpha)} & -\cos \theta e^{-i(k+\alpha)} \end{pmatrix},
\]

(B2a)
\[ U_2(k) = \begin{pmatrix} \cos \theta e^{-i(k + \alpha_d)} & \sin \theta e^{-i(k + \alpha_d)} \\ -\sin \theta e^{i(k + \alpha_d)} & \cos \theta e^{i(k + \alpha_d)} \end{pmatrix}. \]  \tag{B2b}

The eigenvalues of \( U_1(k) \) are \( \lambda_{\pm}^1 = \pm e^{i\xi}, \) with \( \sin \xi = \cos \theta \sin (k + \alpha_d). \) The eigenstate corresponding to the value \( \lambda_{\pm}^1 \) is
\[ |\lambda_{\pm}^1 \rangle = \begin{pmatrix} \frac{\sin \theta}{\sqrt{J^2 - 2 \cos \theta \cos (\frac{\alpha}{2} - k - \alpha_d)}} \\ -\frac{e^{i(\frac{\alpha}{2} - k - \alpha_d)} - \cos \theta}{\sqrt{J^2 - 2 \cos \theta \cos (\frac{\alpha}{2} - k - \alpha_d)}} \end{pmatrix}. \]  \tag{B3}

The eigenvalues of \( U_2(k) \) are \( \lambda_{\pm}^2 = e^{i\xi}, \) with \( \cos \xi = \sin \theta \cos (k + \alpha_d). \) The eigenstate corresponding to the value \( \lambda_{\pm}^2 \) is
\[ |\lambda_{\pm}^2 \rangle = \begin{pmatrix} \frac{\cos \theta}{\sqrt{J^2 - 2 \sin \theta \cos (\frac{\alpha}{2} + k + \alpha_d)}} \\ -\frac{e^{i(\frac{\alpha}{2} + k + \alpha_d)} - \sin \theta}{\sqrt{J^2 - 2 \sin \theta \cos (\frac{\alpha}{2} + k + \alpha_d)}} \end{pmatrix}. \]  \tag{B4}

To study the hidden symmetries in DTQW I and DTQW II, we need to obtain the effective Hamiltonian \( H_1 \) and \( H_2 \) for DTQW I and DTQW II, respectively. When expressing \( U_1 = e^{-iH_1(\theta)} \) and \( U_2 = e^{-iH_2(\theta)} \), the Hamiltonians \( H_1(\theta) \) and \( H_2(\theta) \) are
\[ H_1(\theta) = \int_{-\pi}^{\pi} dk [\xi(k, \theta) n_{1x}(k, \theta) \cdot \sigma] \otimes |k\rangle \langle k|, \]  \tag{B5a}
\[ H_2(\theta) = \int_{-\pi}^{\pi} dk [\xi(k, \theta) n_{2x}(k, \theta) \cdot \sigma] \otimes |k\rangle \langle k|. \]  \tag{B5b}

We set \( \sigma = (\sigma_x, \sigma_y, \sigma_z). \) The vector \( n_{1x}(\theta, k) \) and vector \( n_{2x}(\theta, k) \) are
\[ n_{1x}(\theta, k) = [\sin \theta \cos (k + \alpha_d), -\sin \theta \sin (k + \alpha_d), -\cos \theta \cos (k + \alpha_d)] / \cos \xi, \]  \tag{B6a}
\[ n_{2x}(\theta, k) = [\cos \theta \sin (k + \alpha_d), -\cos \theta \cos (k + \alpha_d), -\sin \theta \sin (k + \alpha_d)] / \sin \xi. \]  \tag{B6b}

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