Qiao Liu
School of Mathematics and Statistics, Central South University
Changsha, Hunan 410083, China

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Abstract. We study the partial regularity problem for a three dimensional simplified Ericksen–Leslie system, which consists of the Navier–Stokes equations for the fluid velocity coupled with a convective Ginzburg-Landau type equations for the molecule orientation, modelling the incompressible nematic liquid crystal flows. Base on the recent studies on the Navier–Stokes equations, we first prove some new local energy bounds and an \( \varepsilon \)-regularity criterion for suitable weak solutions to the simplified Ericksen-Leslie system, i.e., for \( \sigma \in [0, 1] \), there exists a \( \varepsilon > 0 \) such that if \((u, d, P)\) is a suitable weak solution in \( Q_r(z_0) \) with \( 0 < r \leq 1 \) and \( z_0 = (x_0, t_0) \), and satisfies

\[
r^{-\frac{3}{2}} \int_{t_0 - r^2}^{t_0} \left( \|u\|^2_{H^{-\sigma}(B_r(x_0))} + \|
abla d\|^2_{H^{-\sigma}(B_r(x_0))} + \|P\|^2_{H^{-\sigma}(B_r(x_0))} \right) \, dt \leq \varepsilon,
\]

then \((u, d)\) is regular at \( z_0 \). Here, \( H^{-\sigma}(B_r(x)) \) is the dual space of \( H^\sigma(B_r(x)) \), the space of functions \( f \) in the homogeneous Sobolev space \( H^\sigma(\mathbb{R}^3) \) such that \( \text{supp} \, f \subset B_r(x) \). Inspired by this \( \varepsilon \)-regularity criterion, we then improve the known upper Minkowski dimension of the possible interior singular points for suitable weak solutions from \( \frac{25}{63} (\approx 1.50794) \) given by [24] (Nonlinear Anal. RWA, 44 (2018), 246–259.) to \( \frac{22}{61} (\approx 1.36215) \).

1. Introduction. Liquid crystal, which is a state of matter capable of flow, but its molecules may be oriented in a crystal-like way. Hence, it is commonly considered as the fourth state of matter, different from gases, liquid, and solid. There are numerous attempts to formulate continuum theories describing the behaviour of liquid crystals flows, see Stewart’s monograph [31]. Nematic, termed smectic and cholesteric are three main types of liquid crystals. The nematic phase appears to be the most common one, where the molecules do not exhibit any positional order, but they have long-range orientational order. The hydrodynamic theory of the nematic liquid crystal was first derived by Ericksen and Leslie in 1960s (see [4, 15]).

In Ericksen-Leslie theory, the nematic liquid crystal flows in \( \mathbb{R}^3 \times (0, +\infty) \), is characterized by the velocity field \( u : \mathbb{R}^3 \times (0, +\infty) \to \mathbb{R}^3 \), the pressure \( P : \mathbb{R}^3 \times (0, +\infty) \to \mathbb{R} \) and the (averaged) macroscopic/continuum molecule orientation \( d \):

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\( \mathbb{R}^3 \times (0, +\infty) \rightarrow \mathbb{R}^3 \). Let \( u_0 \) be a given initial velocity with \( \nabla \cdot u_0 = 0 \) in distribution sense, and \( d_0 : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) be a given initial liquid crystal orientation field with \( |d_0| \leq 1 \). Then the simplified version of the Ericksen-Leslie system, modeling from nematic sense, and

\[
\begin{align*}
    \rho_t u - \Delta u + (u \cdot \nabla) u + \nabla P &= -\nabla \cdot (\nabla d \odot \nabla d), & \text{in } \mathbb{R}^3 \times (0, +\infty), \\
    \partial_t d + (u \cdot \nabla) d &= \Delta d - f(d), & \text{in } \mathbb{R}^3 \times (0, +\infty), \\
    \nabla \cdot u &= 0, & \text{in } \mathbb{R}^3 \times (0, +\infty), \\
    (u, d)|_{t=0} &= (u_0, d_0) & \text{in } \mathbb{R}^3.
\end{align*}
\]

(1)

Here, \( \nabla d \odot \nabla d \) denotes the \( 3 \times 3 \) matrix and

\[
(\nabla d \odot \nabla d)_{ij} = \sum_{k=1}^{3} \partial_{x_i} d_k \partial_{x_j} d_k \quad \text{for } i, j = 1, 2, 3.
\]

\( f(d) = \nabla F(d) \) is the Ginzburg-Landau approximation function, where \( F(d) \) is a scale function of \( d \) and is given by

\[
F(d) = \frac{1}{\epsilon}((|d|^2 - 1)^2, \epsilon > 0.
\]

(2)

We notice that when \( \epsilon \) goes to zero, then \( d \) becomes an unit vector field. Throughout the paper, due to the concrete values of the viscosity constants do not play a special role in our discussion, we assume that they are all equal to one for simplicity.

System (1), which retains most of the interesting mathematical properties (without destroying the basic nonlinear structure) of the original Ericksen-Leslie model, is a dissipative system with strong nonlinearities, and thus, its mathematical analysis is full of challenge. In the absence of the orientation, i.e., \( d \equiv (0, 0, 1) \), (1) becomes the 3D Navier–Stokes (NS) equations. The problem of the global regularity of solutions to the NS equations in three and higher space dimensions is a fundamental question in fluid dynamics and is still widely open. Meanwhile, many researchers studied the partial regularity of suitable weak solutions to the NS equations, here we only mention a few works mostly related to ours. In a series of paper [28, 29], Scheffer introduced the idea of suitable weak solutions and partial regularity theory. Caffarelli-Kohn-Nirenberg [1] further strengthened Scheffer’s results, established a \( \epsilon \)-regularity theorem to the suitable weak solutions and provided the nullity of one dimensional parabolic Hausdorff measure (see Definition 2.2 below) of the singular set. The Caffarelli-Kohn-Nirenberg’s \( \epsilon \)-regularity theorem gives that there exists a \( \epsilon > 0 \) such that \( z = (x, t) \in \mathbb{R}^3 \times (0, +\infty) \) is a regular point for the suitable weak solution \( u \) to the 3D NS equations if

\[
\limsup_{r \to 0} r^{-1} \int_{Q_r(z)} |\nabla u|^2 \, dy \, d\tau < \epsilon,
\]

(3)

where \( Q_r(z) \) denotes the parabolic cylinder, and

\[
Q_r(z) = B_r(z) \times (t-r^2, t) \subset \mathbb{R}^3 \times \mathbb{R}_+ \quad \text{and} \quad B_r(z) = \{ y \in \mathbb{R}^3 : |x - y| < r \}.
\]

The proof of (3) is based on the following \( \epsilon \)-regularity criterion that involves both \( u \) and \( P \): there exists a \( \epsilon > 0 \) such that if \( u \) is a suitable weak solution in \( Q_r(z) \) and satisfies

\[
r^{-2} \int_{Q_r(z)} (|u|^3 + |P|^{\frac{3}{2}}) \, dy \, d\tau < \epsilon,
\]

(4)
then \(u \in L^\infty(Q_z(z))\). Later, a new short proof of Caffarelli-Kohn-Nirenberg theorem by an indirect argument was given by Lin [18]. Choe-Lewis [2] improved the parabolic Hausdorff dimension by logarithmic factor. Escauriaza-Seregin-˘Sver´ ak [5] proved the marginal case of the so-called Ladyzhenskaya-Prodi-Serrin condition based on the unique continuation theory for parabolic equations. In 2007, some different type \(\varepsilon\)-regularity criteria were obtained by Gustafson-Kang-Tsai [9]. Recently, by viewing the total pressure \(P + \frac{|u|^2}{2}\) as a signed distribution belonging to a certain negative order Sobolev space in local energy inequality, Guevara-Phuc [8] proved that there exists a \(\varepsilon > 0\) such that if \(u\) is a suitable weak solution in \(Q_r(z)\) and satisfies

\[
r^{-\frac{3}{2}} \int_{t-r^2}^t \left( \|u\|^2 H^{-\sigma}(B_r(x)) + \|P\|^2 H^{-\sigma}(B_r(x)) \right) d\tau < \varepsilon \quad \text{for some } \sigma \in [0, 1],
\]

then \(z = (x,t)\) is a regular point. Here, \(H^{-\sigma}(B_r(x))\) is the dual space of \(H_0^\sigma(B_r(x))\), the space of functions \(f\) in the homogeneous Sobolev space \(H^\sigma(\mathbb{R}^3)\) such that \(\text{supp } f \subset B_r(x)\). By letting \(\sigma = \frac{3}{2}\), then the Sobolev embedding theorem together with (4) yields that the condition

\[
r^{-\frac{3}{2}} \int_{Q_r(z)} (\|u\| H^{-\frac{3}{2}} + |P| H^{-\frac{3}{2}}) dy \, d\tau < \varepsilon,
\]

still get that \(z = (x,t)\) is a regular point. Base on the partial regularity results above, many researchers studied the upper Minkowski dimension (see Definition 2.2) estimate of singular points. The relationship between the Hausdorff dimension and the upper Minkowski dimension comprises the first being less than the second (e.g., see [6]). Using (3), Robinson-Sadowski [27] proved that the upper Minkowski dimension of \(\Sigma\) is \(\frac{5}{3}\) at most. Subsequently, Kukavica [13] showed that the upper Minkowski dimension of the singular points is less than or equal to \(\frac{135}{82}\) \((\approx 1.646)\) and asked whether the dimension of the singular points is 1 at most. Later, Koh-Yang [12] proved that the upper Minkowski dimension of \(\Sigma\) is bounded by \(\frac{95}{53}\) \((\approx 1.508)\). Based on the arguments given by [12] and some careful estimates, Wang-Wu [32] refined the upper Minkowski dimension to \(\frac{395}{360}\) \((\approx 1.300)\). Base on regularity criterion (5), Ren-Wang-Wu [26] improved the upper Minkowski dimension to \(\frac{275}{252}\) \((\approx 1.286)\). We also refer the readers to paper [33], Wang-Yang refined the bound to \(\frac{7}{6}\) \((\approx 1.167)\).

As for system (1), in [20, 21], Lin-Liu gave a systematic mathematical analysis and obtained the following existence and partial regularity results:

- for \(u_0 \in L^2(\mathbb{R}^3)\) with \(\text{div } u_0 = 0\), and \(d_0 - \vec{d} \in H^1(\mathbb{R}^3)\), there exists a global weak solution \((u, d)\) to system (1) such that

\[
\begin{align*}
&u \in L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3)) \quad \text{and } d - \vec{d} \in L^\infty(0, T; H^1(\mathbb{R}^3)) \cap L^2(0, T; H^2(\mathbb{R}^3))
\end{align*}
\]

for any \(T \in (0, +\infty)\). Moreover, the following energy inequality holds

\[
\int_{\mathbb{R}^3} (|u|^2 + |\nabla d|^2 + F(d))(x, t) \, dx + 2 \int_0^t \int_{\mathbb{R}^3} \left( |\nabla u|^2 + |\nabla d|^2 + 2|\nabla f(d)|\nabla d + |f(d)|^2 \right) \, dx \, d\tau \leq \int_{\mathbb{R}^3} (|u_0|^2 + |\nabla d_0|^2)(x) \, dx \quad \text{for all } t \geq 0.
\]

The one-dimensional parabolic Hausdorff measure of the singular points of the suitable weak solution \((u, d)\) is zero. Here by suitable weak solutions we mean solutions that solve (1.1) in the sense of distribution and satisfy the local energy inequality (see Definition 2.1 below).
In [20], the authors also discuss some stability properties and time asymptotic of (1). But, like the NS equations, the uniqueness and regularity of the weak solutions to system (1) are open problems. Hu-Wang [11] established global existence of strong solutions and weak-strong uniqueness to system (1) with suitable initial conditions. Asymptotic behavior of solutions was studied by Wu [34] and Dai-Schonbek [3]. Recently, using the method by [9] on the NS equations, Lan-Ma [14] established some new $\varepsilon$-regularity criteria for suitable weak solutions to system (1) in terms of the velocity field. However, there is still a large gap between the case of existence and the case of the solutions being regular. The loss of the regularity may be due to the formation of singular point. Motivated by Seregin [30] on the NS equations, Liu [25] studied the singular points of weak solutions, and prove that if a weak solution $(u,d)$ to system (1) on $\mathbb{R}^3 \times (0,T)$ with $0 < T < \infty$, and satisfies
\[ \|u(\cdot,t)\|_{L^p(\mathbb{R}^3)} + \|\nabla d(\cdot,t)\|_{L^p(\mathbb{R}^3)} \leq \frac{c}{(T-t)^{\frac{p-3}{2p}}} \quad \text{for some } 3 < p < \infty \text{ and } c > 0, \]
then the number of singular points is finite. Inspired by papers [12,13,27,32] on the NS equations, Liu [24] proved that the upper Minkowski dimension of the singular points $\Sigma$ of the simplified Ericksen-Leslie system (1) is the set of points for suitable weak solution $(u,d)$. We refer the readers to see, e.g., [16,22,35] and other related references cited therein for this system.

One of our main objectives is to improve the bound of the upper Minkowski dimension of the singular points to the simplified Ericksen-Leslie system (1). The singular points $\Sigma$ of the simplified Ericksen-Leslie system (1) is the set of points $z = (x,t)$ such that in any neighborhood of each $z$, $|u| + |\nabla d|$ is unbounded. To this end, we shall first derive a version of local energy estimates using certain type of test function and give an new $\varepsilon$-regularity criterion for suitable weak solutions by modifying the arguments in [8,9]. Since the $\varepsilon$-regularity criteria are local, we suppose our reference balls are in the domain of the equations. Let $H^{-\sigma}(B_r(x))$ ($\sigma \in \mathbb{R}$) be the dual space of
\[ H^\sigma(B_r(x)) \overset{\text{def}}{=} \{ f \in \dot{H}^\sigma(\mathbb{R}^3) : \text{supp } f \subseteq B_r(x) \}, \]
where $\dot{H}^\sigma(\mathbb{R}^3)$ is the homogeneous Sobolev space on $\mathbb{R}^3$, and is defined as
\[ \dot{H}^\sigma(\mathbb{R}^3) \overset{\text{def}}{=} \{ f : f \text{ is the tempered distribution over } \mathbb{R}^3, \text{the Fourier transform of } f \text{ belongs to } L^1_{\text{loc}}(\mathbb{R}^3), \text{ and } \|f\|_{\dot{H}^\sigma(\mathbb{R}^3)} := \left( \int_{\mathbb{R}^3} |\xi|^{2\sigma} |\hat{f}(\xi)|^2 d\xi \right)^{\frac{1}{2}} \text{< } +\infty \}. \]
Our main $\varepsilon$-regularity criterion on suitable weak solution $(u, d)$ to system (1) is as follows:

**Theorem 1.1.** Let $\sigma \in [0, 1]$. There exists a $\varepsilon > 0$ such that if $(u, d, P)$ is a suitable weak solution to the simplified Ericksen-Leslie system (1) in $Q_r(z_0)$ with $0 < r \leq 1$, and satisfies

$$r^{-\frac{5\sigma}{4}} \int_{t_0}^t \left( \|u\|^2 \frac{\partial^2}{\partial t^2} H^{-\sigma}(B_r(x_0)) + \|\nabla d\|^2 \frac{\partial^2}{\partial t^2} H^{-\sigma}(B_r(x_0)) + \|P\|^2 \frac{\partial^2}{\partial t^2} H^{-\sigma}(B_r(x_0)) \right) dt \leq \varepsilon,$$

then $(u, \nabla d)$ is bounded in $Q_{\frac{r}{2}}(z_0)$, i.e., $(u, d)$ is regular at $z_0$.

For any ball $B \subset \mathbb{R}^3$ and $\sigma \in [0, \frac{3}{2})$, notice that the standard Sobolev embedding theory yields that $L^{\frac{6}{5-4\sigma}}(B) \subseteq H^{-\sigma}(B)$ and $\|f\|_{H^{-\sigma}(B)} \leq C\|f\|_{L^{\frac{6}{5-4\sigma}}(B)}$. Thus we have the following corollary.

**Corollary 1.** Let $\alpha \in [\frac{6}{5}, 2]$. There exists a $\varepsilon > 0$ such that if $(u, d, P)$ is a suitable weak solution to the simplified Ericksen-Leslie system (1) in $Q_r(z_0)$ with $0 < r \leq 1$, and satisfies

$$r^{-\frac{\alpha}{2}} \int_{t_0}^t \left( \|u\|^2 \frac{\alpha}{2} \frac{\partial^2}{\partial t^2} L^{\frac{\alpha}{2}}(B_r(x_0)) + \|\nabla d\|^2 \frac{\alpha}{2} \frac{\partial^2}{\partial t^2} L^{\frac{\alpha}{2}}(B_r(x_0)) + \|P\|^2 \frac{\alpha}{2} \frac{\partial^2}{\partial t^2} L^{\frac{\alpha}{2}}(B_r(x_0)) \right) dt \leq \varepsilon,$$  \hspace{1cm} (8)

then $(u, d)$ is regular at $z_0$.

**Remark 1.** By letting $\alpha = \frac{3}{2}$, condition (8) together with the H"older’s inequality implies that

$$r^{-2} \int_{Q_r(z_0)} (|u|^3 + |\nabla d|^3 + |P|^\frac{3}{2}) \, dx \, dt \leq \varepsilon^\frac{5}{2}.$$  \hspace{1cm} (9)

The $\varepsilon$-regularity criterion: condition (9) ensures that $z_0$ is a regular point of $(u, d)$ to system (1) is obtained by [21].

Using our new regularity results above, and modifying the arguments in papers [12, 13, 24, 26, 27, 32, 33], we improve the bound of the upper Minkowski dimension of singular points to system (1) by the following theorem:

**Theorem 1.2.** Let $0 < \gamma < \frac{560}{1839}$, $0 < T \leq \infty$ and $(u, d, P)$ be a suitable weak solution to the simplified Ericksen-Leslie system (1) on $\mathbb{R}^3 \times (0, T)$. Let $\Sigma$ be the possible singular points of $(u, d, P)$. Then for any compact set $D \subseteq \mathbb{R}^3 \times (0, T)$, the possible upper Minkowski dimension of $\Sigma \cap D$ has a bound

$$\overline{\dim}_M(\Sigma \cap D) \leq \frac{835}{613}.$$  

**Remark 2.** Theorem 1.2 gives a better bound of the upper Minkowski dimension of the suitable weak solutions to system (1) than that of [24]. Compare the result on Theorem 1.2 to that of [26, 32, 33], we notice that we can not get better results than that of the NS equation due to system (1) is more complex. Moreover, it should be point out that the bound $\frac{835}{613}$ is not an optimal value, we hope that we can improve it in the future studies.

This paper is organized as follows. In Section 2, we recall the definition of the suitable weak solutions to system (1), the concepts of the Hausdorff dimension and the upper Minkowski dimension on parabolic versions, and two useful lemmas. Section 3 is devoted to proving Theorem 1.1. In Section 4, we shall give the proof of Theorem 1.2.
2. Preliminaries. Let us first recall the definition of the suitable weak solution to the simplified Ericksen-Leslie system (1).

**Definition 2.1.** (suitable weak solution [21]) We say \((u, d, P)\) is a suitable weak solution of the 3d simplified Ericksen-Leslie system (1) if the following condition holds:

(a). \((u, d - \overline{d}) : \mathbb{R}^3 \times [0, T] \mapsto \mathbb{R}^3\) and \(P : \mathbb{R}^3 \times [0, T] \mapsto \mathbb{R}\) satisfy

\[ u \in L^\infty([0, T]; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3)), d - \overline{d} \in L^\infty([0, T]; H^1(\mathbb{R}^3)) \cap L^2(0, T; H^2(\mathbb{R}^3)) \]

and

\[ P \in L^{\frac{3}{2}}(0, T; L^{\frac{3}{2}}(\mathbb{R}^3)); \]

(b). \((u, d, P)\) satisfies (1) in \(\mathbb{R}^3 \times [0, T]\) in the sense of distribution;

(c). for a.e. \(t \in (0, T)\), \((u, d, P)\) satisfies the following local energy inequality:

\[
\int_{\mathbb{R}^3 \times \{t\}} (|u|^2 + |\nabla d|^2) \phi dx + 2 \int_0^t \int_{\mathbb{R}^3} (|\nabla u|^2 + |\nabla^2 d|^2) \phi dx d\tau \\
\leq \int_0^t \int_{\mathbb{R}^3} \{(|u|^2 + |\nabla d|^2)(\phi_t + \Delta \phi) + (|u|^2 + |\nabla d|^2 + 2P)u \cdot \nabla \phi \\
+ ((u \cdot \nabla) d) \cdot \nabla \phi - \nabla f(d) \nabla d \phi\} dx d\tau,
\]

for any nonnegative \(\phi \in C^\infty_0(\mathbb{R}^3 \times [0, T])\).

Next, we shall introduce the concepts of the parabolic Hausdorff dimension and the upper Minkowski dimension on parabolic versions (see e.g., [6]).

**Definition 2.2.** 1. **(The parabolic Hausdorff dimension)** For fixed \(\rho > 0\) and set \(\Sigma \subset \mathbb{R}^3 \times \mathbb{R}\), let \(\{Q_{r_j}(z_j)\}_{j \in \mathcal{A}}\) be the family of all coverings of parabolic cylinders \(Q_{r_j}(z_j)\) that covers \(\Sigma\) with \(0 < r_j \leq \rho\). The parabolic Hausdorff dimension of the set \(\Sigma\) is defined as

\[ \dim_H(\Sigma) = \inf \{\alpha : \mathcal{H}^\alpha(\Sigma) = 0\}, \]

where

\[ \mathcal{H}^\alpha (\Sigma) = \lim_{\varepsilon \to 0} \inf \left\{ \sum_{j \in \mathcal{A}} r_j^{\alpha} : \Sigma \subseteq \bigcup_{j \in \mathcal{A}} Q_{r_j}(z_j), 1 < r_j \leq \varrho \right\}. \]

2. **(The upper parabolic Minkowski dimension)** Let \(N(\Sigma, r)\) denote the minimum number of parabolic cylinders \(Q_r(z)\) required to cover the set \(\Sigma\). Then the parabolic upper Minkowski dimension of the set \(\Sigma\) is defined as

\[ \overline{\dim}_M(\Sigma) = \limsup_{r \to 0} \frac{\log N(\Sigma, r)}{-\log r}. \]

The following two lemmas, which can be found in [7] and [8], respectively, are needed in the proofs of our main results.

**Lemma 2.3.** Let \(I(t)\) be a bounded nonnegative function in the interval \([r_1, r_2]\). Assume that for every \(r_1 \leq s \leq t \leq r_2\) and

\[ I(s) \leq (A(t - s)^{-\alpha} + B(t - s)^{-\beta} + C) + \theta I(t) \]

with \(A, B, C \geq 0, \alpha > \beta > 0\) and \(\theta \in [0, 1]\). Then there holds

\[ I(r_1) \leq c(\alpha, \theta)[A(r_2 - r_1)^{-\alpha} + B(r_2 - r_1)^{-\beta} + C], \]

where \(c(\alpha, \theta)\) is a positive constant depending only on \(\alpha\) and \(\theta\).
Lemma 2.4. Let \( h \) be a harmonic function in \( B_{2r}(x_0) \subset \mathbb{R}^3 \) and \( \sigma \in [0,1] \). Then there exists a positive constant \( C \) such that
\[
\|h\|_{L^2(B_r(x_0))} \leq C \|r^{-\sigma}h\|_{H^{-\sigma}(B_{2r}(x_0))}.
\]

3. Proof of Theorem 1.1. In the light of the natural scaling property of system (1), we first introduce the following scaling invariant quantities. For \( z_0 = (x_0, t_0) \), \( f = u \) or \( \nabla d \), define
\[
A(f, z_0, r) := \sup_{0 \leq t \leq 2} \int_{B_r(x_0)} |f(y, t)|^2 dy, \quad B(f, z_0, r) := r^{-1} \int_{B_r(x_0)} |\nabla f(y, t)|^2 dy dt,
\]
\[
G_\sigma(f, z_0, r) := r^{\frac{3}{2}} \int_{0 \leq t \leq r^2} \|f(\cdot, t)\|_{H^{-\sigma}(B_r(x_0))}^2 dt, \quad H_\sigma(f, z_0, r) := r^{\frac{3}{2}} \int_{0 \leq t \leq r} \|f(\cdot, t)\|_{H^{-\sigma}(B_r(x_0))}^2 dt,
\]
\[
E_{\alpha}(f, z_0, r) := r^{-\frac{d}{\alpha - \sigma}} \|\nabla f\|_{L^\infty(\Omega)}^{\frac{4\alpha}{4\alpha - 2\sigma}} (Q_r(z_0)) := r^{-\frac{d}{\alpha - \sigma}} \int_{0 \leq t \leq r} \|f\|_{L^\infty(\Omega)}^{\frac{4\alpha}{4\alpha - 2\sigma}} (Q_r(z_0)) dt
\]
\[
F_{\alpha}(P, z_0, r) := r^{-\frac{d}{\alpha - \sigma}} \|\nabla d\|_{L^\infty(\Omega)}^{\frac{4\alpha}{4\alpha - 2\sigma}} (Q_r(z_0)) := r^{-\frac{d}{\alpha - \sigma}} \int_{0 \leq t \leq r} \|\nabla d\|_{L^\infty(\Omega)}^{\frac{4\alpha}{4\alpha - 2\sigma}} (Q_r(z_0)) dt,
\]
where \( \sigma \in [0,1] \) and \( \alpha \in \left[\frac{d}{2}, 2\right] \). We also denote \( A(u, \nabla d, z_0, r) \) by \( A(u, z_0, r) \) or \( A(\nabla d, z_0, r) \), \( B(u, \nabla d, z_0, r) \), \( E_{\alpha}(u, \nabla d, z_0, r) \) and \( G_\sigma(u, \nabla d, z_0, r) \) by similar notations. Besides, we use \( C \) to denote a positive absolute constant which will change from line to line.

In the following Lemma, by using the modified argument in Guevara-Phuc [8] and the idea of viewing the total Pressure \( \frac{1}{2}(|u|^2 + |\nabla d|^2) + P \) as a signed distribution in \( H^{-\sigma} \), gives the bound of \( A(u, \nabla d, z_0, r) \) and \( B(u, \nabla d, z_0, r) \).

Lemma 3.1. Suppose that \((u, d, P)\) is a suitable weak solution to the simplified Ericksen-Leslie system (1) in \( Q_r(z_0) \) with \( 0 < r < 1 \). Then it holds that
\[
A(u, \nabla d, z_0, r \frac{r}{2}) + B(u, \nabla d, z_0, r \frac{r}{2}) \leq C \left[ G_\sigma(u, \nabla d, z_0, r)^{\frac{2}{2-\sigma}} (1 + G_\sigma(u, \nabla d, z_0, r)^{\frac{2-\sigma}{2}}) + H_\sigma(P, z_0, r)^{2-\sigma} \right]
\]
(11)
for any \( \sigma \in [0,1] \).

Proof. For \( \frac{s}{2} \leq s < \rho \leq r < 1 \), let \( \eta_1 \in C_0^\infty(\mathbb{R}^3) \) such that \( 0 \leq \eta_1 \leq 1 \) in \( \mathbb{R}^3 \), \( \eta_1(x) \equiv 1 \) on \( B_\rho(x) \) and
\[
|\nabla^\alpha \eta_1| \leq \frac{c}{(\rho - s)^{|\alpha|}}
\]
for all multi-indices \( \alpha \), where \( c \) is a positive constant. Let \( \eta_2 \in C_0^\infty(t_0 - \rho^2, t_0 + \rho^2) \) such that \( 0 \leq \eta_2 \leq 1 \) in \( \mathbb{R} \), \( \eta_2(t) \equiv 1 \) for \( t \in [t_0 - s^2, t_0 + s^2] \) and
\[
|\eta_2'(t)| \leq \frac{c}{\rho^2 - s^2} \leq \frac{c}{r(\rho - s)}
\]
with positive constant \( c \).

Define \( \phi(x, t) := \eta_1(x) \eta_2(t) \), then for all multi-indices \( \alpha \), it holds that
\[
|\nabla^\alpha \phi| \leq \frac{c}{(\rho - s)^{|\alpha|}}, \quad |\nabla^\alpha \phi| \leq \frac{c}{r(\rho - s)^{1 + |\alpha|}}.
\]

For \( \sigma \in [0,1] \), using \( \phi(x, t) \) defined above as the test function in the local energy inequality (10), one obtains after by using the duality relation between \( H^{-\sigma}(B_\rho(x_0)) \)
and $H^s_0(B_\rho(x_0))$ that

$I(s) := A(u, \nabla d, z_0, s) + B(u, \nabla d, z_0, s)$

$$
\begin{align*}
&\leq \frac{1}{s} \int_{t_0}^{t_0} \|u\|^2 + |\nabla d|^2 \|H^{-s}(B_\rho(x_0))\| \phi_1 + \Delta \phi \|H^s(B_\rho(x_0))\| dt \\
&+ \frac{1}{s} \int_{t_0}^{t_0} \|u\|^2 + |\nabla d|^2 + 2|P||H^{-s}(B_\rho(x_0))\| |u| |\nabla \phi|H^s(B_\rho(x_0))| dt + \frac{1}{s} \int_{Q_\rho(x_0)} |\nabla d|^2dxdt \\
&=: I_1 + I_2 + I_3,
\end{align*}
$$

where $I_3$ can be bounded by

$$
I_3 \leq \frac{\rho^3}{s} \sup_{t_0 - \rho^2 \leq t \leq t_0} \frac{1}{\rho} \int_{B_\rho(x_0)} |\nabla d|^2 d\rho \leq \frac{1}{4} A(u, \nabla d, z_0, \rho) < \frac{1}{4} I(\rho),
$$

when $\rho \leq \min\{r, \frac{1}{22^2}\}$. To bound $I_1$ and $I_2$, let us first define

$$
X = \int_{t_0 - \rho^2}^{t_0} \|u\|^2 + |\nabla d|^2 \|H^{-s}(B_\rho(x_0))\| dt \\
Y = \int_{t_0 - \rho^2}^{t_0} |P||H^{-s}(B_\rho(x_0))\| dt.
$$

Notice that it holds the interpolation inequality $\|f\|_{H^s(\mathbb{R})} \leq \|f\|_{1-\theta}^{1-\theta} \|f\|_{H^{1\ell}(\mathbb{R})}$ for all $f \in H^{1\ell}(\mathbb{R}) \cap H^{1\ell}(\mathbb{R})$, $\ell_1, \ell_2 \in \mathbb{R}$, where $0 \leq \theta \leq 1$ and $\ell = (1 - \theta)\ell_1 + \theta\ell_2$. Then, applying properties of the test function $\phi$, and the Hölder's inequality, it follows that

$$
I_1 \leq \frac{1}{s} \int_{t_0 - \rho^2}^{t_0} \|u\|^2 + |\nabla d|^2 \|H^{-s}(B_\rho(x_0))\| |\nabla \phi|H^s(B_\rho(x_0))| dt \\
\leq C\frac{\rho^3}{s} \frac{\rho^2}{(\rho - s)^2 + \sigma} \int_{t_0 - \rho^2}^{t_0} \|u\|^2 + |\nabla d|^2 |H^{-s}(B_\rho(x_0))| dt \\
\leq C\frac{\rho^3}{(\rho - s)^2 + \sigma} \int_{t_0 - \rho^2}^{t_0} \|u\|^2 + |\nabla d|^2 \|H^{-s}(B_\rho(x_0))\| \|\nabla \phi\|_{L^2(B_\rho(x_0))} \|\nabla \phi\|_{L^2(B_\rho(x_0))} \|\nabla \phi\|_{L^2(B_\rho(x_0))} dt \\
\leq C\frac{\rho^3}{(\rho - s)^2 + \sigma} X^{2 - \sigma}.
$$

Similarly,

$$
I_2 \leq \frac{1}{s} \int_{t_0 - \rho^2}^{t_0} \|u\|^2 + |\nabla d|^2 + 2|P||H^{-s}(B_\rho(x_0))\| |u| |\nabla \phi|H^s(B_\rho(x_0))| |\nabla |u|\nabla \phi|H^s(B_\rho(x_0))| dt \\
\leq C\frac{1}{s} \left( \int_{t_0 - \rho^2}^{t_0} \|u\|^2 + |\nabla d|^2 + 2|P||H^{-s}(B_\rho(x_0))\| dt \right)^\frac{2 - \sigma}{2} \\
\times \left( \int_{t_0 - \rho^2}^{t_0} \|u\| |\nabla \phi|H^s(B_\rho(x_0))| |\nabla |u|\nabla \phi|H^s(B_\rho(x_0))| dt \right)^\frac{2}{2} \\
\leq C\frac{1}{s} \left( \int_{t_0 - \rho^2}^{t_0} \frac{1}{(\rho - s)^2 + \sigma} \left( \int_{Q_\rho(x_0)} |u|^2 dxdt + \frac{\rho^2}{t_0 - \rho^2} \sup_{t_0 - \rho^2 \leq t \leq t_0} \int_{B_\rho(x_0)} |u|^2 dx \right) \right)^\frac{1 - \sigma}{2} \\
\times \left( \frac{\rho^3}{(\rho - s)^2 + \sigma} (X + Y)^{2 - \sigma} \left( \sup_{t_0 - \rho^2 \leq t \leq t_0} \int_{B_\rho(x_0)} |u|^2 dx \right) \right)^\frac{1 - \sigma}{2} \\
\leq C\frac{\rho^3}{(\rho - s)^2 + \sigma} (X + Y)^{2 - \sigma} \left( \sup_{t_0 - \rho^2 \leq t \leq t_0} \int_{B_\rho(x_0)} |u|^2 dx \right) \frac{1 - \sigma}{2} \\
\leq C\frac{\rho^3}{(\rho - s)^2 + \sigma} \left( X + Y \right)^{2 - \sigma} \left( \frac{\rho^3}{(\rho - s)^2 + \sigma} A(u, z_0, \rho) + \frac{\rho^3}{(\rho - s)^2} B(u, z_0, \rho) + \frac{\rho^3}{(\rho - s)^2} A(u, z_0, \rho) \right)^\frac{1 - \sigma}{2} \\
\leq C\frac{\rho^3}{(\rho - s)^2 + \sigma} \left( X + Y \right)^{2 - \sigma} \left( \frac{\rho^3}{(\rho - s)^2 + \sigma} I(\rho) \right)^\frac{1 - \sigma}{2} \\
\leq \frac{1}{4} I(\rho) + C \left( \frac{\rho^3}{(\rho - s)^2 + \sigma} + \frac{\rho^3}{(\rho - s)^2} \right) \left( X + Y \right)^{2 - \sigma}.
Combining all estimates of \( I_i \) \((i = 1, 2, 3)\) together, and noticing that \( \frac{r}{2} \leq s < \rho \leq r \), we obtain that

\[
I(s) \leq C \left( \frac{r^{\frac{4}{3} + \sigma}}{(\rho - s)^{2 + \sigma}} X + Y \right)^{\frac{2 - \sigma}{2}} + C \left( \frac{r^{2\sigma - 1}}{(\rho - s)^{2 + 2\sigma}} + \frac{r^{-1}}{(\rho - s)^{\frac{3}{2}}} \right) (X + Y)^{2 - \sigma} + \frac{1}{2} I(\rho).
\]

Hence, by using Lemma 3.2, it follows that

\[
I(\frac{r}{2}) \leq C \left( \frac{1}{r^{\frac{3}{2}}} X + Y \right)^{\frac{2 - \sigma}{2}} + \frac{1}{2} (X + Y)^{2 - \sigma},
\]

from which together with the definitions of \( G_\sigma(u, \nabla d, z_0, r) \) and \( H_\sigma(P, z_0, r) \) yields (11). Thus we complete the proof of Lemma 3.1.

The next lemma provides bounds for the pressure.

**Lemma 3.2.** Suppose that \((u, d, P)\) is a suitable weak solution to the simplified Ericksen-Leslie system (1) in \( Q_r(z_0) \) with \( 0 < r < 1 \). Then for any \( \rho \in (0, \frac{r}{2}) \), it holds that

\[
\rho^{-3} \int_{t_0 - r^2}^{t_0} \| [P(\cdot, t) - [P(\cdot, t)]_{x_0, \rho}]_{L^2(B_\rho(x_0))} \| dt \\
\leq C \left( \frac{\rho}{r} \right)^3 \int_{t_0 - r^2}^{t_0} \| (P(\cdot, t) - [P(\cdot, t)]_{x_0, \rho}) \|_{L^1(B_\rho(x_0))} dt + \left( \frac{r}{\rho} \right)^3 A(u, \nabla d, z_0, r)^{\frac{1 + 2\sigma}{2}} B(u, \nabla d, z_0, r)^{\frac{3 - 2\sigma}{2}}
\]

and

\[
\rho^{-3} \int_{t_0 - r^2}^{t_0} \| P(\cdot, t) - [P(\cdot, t)]_{x_0, \rho} \|_{L^1(B_\rho(x_0))} dt \\
\leq C \left[ H_\sigma(P, z_0, r) + \left( \frac{r}{\rho} \right)^3 A(u, \nabla d, z_0, r)^{\frac{1 + 2\sigma}{2}} B(u, \nabla d, z_0, r)^{\frac{3 - 2\sigma}{2}} \right],
\]

where \( \sigma \in [0, 1] \), and

\[
[f]_{x_0, \rho} := \int_{B_\rho(x_0)} f(x) dx = \frac{1}{|B_\rho(x_0)|} \int_{B_\rho(x_0)} f(x) dx
\]
denotes the spatial average of a function \( f \) over the ball \( B_\rho(x_0) \).

**Proof.** For a.e. \( t \in (t_0 - r^2, t_0) \), define \( h \) such that

\[
h = P - \tilde{P} \quad \text{in} \ B_r(x_0),
\]

where \( \tilde{P} \) is defined by

\[
\tilde{P} := R_i R_j [(U_{i, j} + D_{i, j}) \chi_{B_r(x_0)}].
\]

Here \( R_i = \partial_i (\Delta)^{- \frac{1}{2}} \), \( i = 1, 2, 3 \), is the \( i \)-th Riesz transform, \( U_{i, j} = (u_i - [u_i]_{x_0, r})(u_j - [u_j]_{x_0, r}) \), \( D_{i, j} = (\partial_i d - [\partial_i d]_{x_0, r}) \cdot (\partial_j d - [\partial_j d]_{x_0, r}) \), and \( \chi_{B_r(x_0)} \) is the characteristic function of \( B_r(x_0) \). Notice that from (1)_1 and (1)_3, \( P \) satisfying

\[
-\Delta P = \text{div} \text{div}(u \otimes u + \nabla d \otimes \nabla d)
\]
in the distributional sense, hence, we see that \( h \) is harmonic in \( B_r(x_0) \) for a.e. \( t \in (t_0 - r^2, t_0) \). Then for any \( \rho \in (0, \frac{r}{2}) \), one has (see, e.g., [10])

\[
\left( \int_{B_\rho(x_0)} \| h - [h]_{x_0, \rho} \|_{L^2} dx \right)^{\frac{1}{2}} \leq C \frac{\rho}{r} \int_{B_r(x_0)} \| h - [h]_{x_0, r} \| dx.
\]
Hence, using $P = \tilde{P} + h$, one has
\[
\int_{B_\rho(x_0)} |P(x,t) - [P(\cdot,t)]_{x_0,\rho}|^2 dx
\leq \int_{B_\rho(x_0)} |\tilde{P}(x,t) - [\tilde{P}(\cdot,t)]_{x_0,\rho}|^2 dx + \int_{B_\rho(x_0)} |h(x,t) - [h(\cdot,t)]_{x_0,\rho}|^2 dx
\leq 2\|\tilde{P}(\cdot,t)\|_{L^2(B_\rho(x_0))}^2 + C\frac{\rho_4}{\rho^8} \left( \int_{B_\rho(x_0)} |h(x,t) - [h(\cdot,t)]_{x_0,\rho}|^2 dx \right)^2
\leq 2\|\tilde{P}(\cdot,t)\|_{L^2(B_\rho(x_0))}^2 + C\frac{\rho_4}{\rho^8} \left( \int_{B_\rho(x_0)} |P(x,t) - [P(\cdot,t)]_{x_0,\rho}|^2 dx + 2\|\tilde{P}(\cdot,t)\|_{L^1(B_\rho(x_0))} \right)^2
\leq C\|\tilde{P}(\cdot,t)\|^2_{L^2(B_\rho(x_0))} + C\frac{\rho_4}{\rho^8} \left( \int_{B_\rho(x_0)} |P(x,t) - [P(\cdot,t)]_{x_0,\rho}|^2 dx \right),
\]  
(14)
where we have used the fact that $\rho \leq \frac{\rho_2}{5}$. By using the Calderón-Zygmund estimate and the interpolation inequality\(^1\), one gets
\[
\|\tilde{P}(\cdot,t)\|^2_{L^2(B_\rho(x_0))} \leq C \int_{B_\rho(x_0)} (|u - [u]_{x_0,\rho}|^2 + |\nabla d - [\nabla d]_{x_0,\rho}|^2) dx
\leq C \left[ \left( \int_{B_\rho(x_0)} |\nabla u|^2 dx \right)^2 \left( \int_{B_\rho(x_0)} |u|^2 dx \right)^\frac{1}{2} + \left( \int_{B_\rho(x_0)} |\nabla^2 d|^2 dx \right)^2 \left( \int_{B_\rho(x_0)} |\nabla^2 d|^2 dx \right)^\frac{1}{2} \right]^\frac{1}{2}
\leq C r^2 A(u, \nabla d, x_0, \rho)^\frac{1}{2} \left( \int_{B_\rho(x_0)} (|\nabla u|^2 + |\nabla^2 d|^2) dx \right)^\frac{3}{2},
\]  
(15)
Combining (14) and (15) together, one gets
\[
\rho^{-\frac{3}{2}} \|P(x,t) - [P(\cdot,t)]_{x_0,\rho}\|_{L^2(B_\rho(x_0))} \leq C \left( \frac{1}{r^4} \int_{B_{r}(x_0)} |P(x,t) - [P(\cdot,t)]_{x_0,\rho}| dx + C^{\frac{3}{4}} \frac{1}{\rho^2} A(u, \nabla d, x_0, \rho)^\frac{1}{2} \left( \int_{B_\rho(x_0)} (|\nabla u|^2 + |\nabla^2 d|^2) dx \right)^\frac{3}{2} \right).
\]
Integrating the above inequality respect to $t$ over the interval $(t_0 - \rho^2, t_0)$, we get (12).

To prove (12), notice that $h$ is harmonic in $B_r(x_0)$ for a.e. $t \in (t_0 - r^2, t_0)$. Then for any $\rho \in (0, \frac{\rho_2}{5}]$, one has (see, e.g., [10])
\[
\left( \int_{B_\rho(x_0)} |h|^2 dx \right)^\frac{1}{2} \leq C \int_{B_\rho(x_0)} |h| dx.
\]

\(^1\)Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain. For any function $u \in W^{1,2}(\Omega)$ such that $\int_{\Omega} u(x) dx = 0$, then for all $q \in [2, 6]$, it holds that
\[
\int_{\Omega} |u|^q dx \leq C_q \left( \int_{\Omega} |u|^2 dx \right)^\frac{q}{2} \left( \int_{\Omega} |\nabla u|^2 dx \right)^{-\frac{q-2}{2}} + \frac{2q}{q-2}.
\]
Then using $P = \hat{P} + h$ again, it follows that
\[
\int_{B_\rho(x_0)} |P(x,t)| \, dx \leq \int_{B_\rho(x_0)} |\hat{P}(x,t)| \, dx + \int_{B_\rho(x_0)} |h(x,t)| \, dx
\]
\[
\leq \int_{B_\rho(x_0)} |\hat{P}(x,t)| \, dx + \rho^3 \left( \int_{B_\rho(x_0)} |h(x,t)|^2 \, dx \right)^{\frac{1}{2}}
\]
\[
\leq \int_{B_\rho(x_0)} |\hat{P}(x,t)| \, dx + \left( \frac{\rho}{\rho^3} \right)^{\frac{3}{2}} \left( \int_{B_\rho(x_0)} |h(x,t)| \, dx \right)^{\frac{3}{2}}
\]
\[
\leq \int_{B_\rho(x_0)} |\hat{P}(x,t)| \, dx + \frac{\rho^3}{\rho^{\frac{3}{2} + \sigma}} \left( \int_{B_\rho(x_0)} |h(x,t)| \, dx \right)^{\frac{3}{2}}
\]
\[
\leq \|\hat{P}(\cdot,t)\|_{L^1(B_\rho(x_0))} + \frac{\rho^3}{\rho^{\frac{3}{2} + \sigma}} \|h(\cdot,t)\|_{H^{-\sigma}(B_\rho(x_0))}
\]
\[
\leq \|\hat{P}(\cdot,t)\|_{L^1(B_\rho(x_0))} + \frac{\rho^3}{\rho^{\frac{3}{2} + \sigma}} (\|\hat{P}(\cdot,t)\|_{H^{-\sigma}(B_\rho(x_0))} + \|P(\cdot,t)\|_{H^{-\sigma}(B_\rho(x_0))}),
\]
where we have used the Hölder’s inequality and Lemma 2.4 in the above estimates. On the other hand, by using the Sobolev embedding $L^{\frac{6}{\sigma}}(B_r(x_0)) \hookrightarrow H^{-\sigma}(B_r(x_0))$ with $\sigma \in [0, \frac{3}{2})$, it holds that
\[
\|\hat{P}(\cdot,t)\|_{H^{-\sigma}(B_r(x_0))} \leq C \|\hat{P}(\cdot,t)\|_{L^{\frac{6}{\sigma}}(B_r(x_0))},
\]
and using the Hölder’s inequality, there holds
\[
\|\hat{P}(\cdot,t)\|_{L^1(B_r(x_0))} \leq C r^{\frac{1}{2} - \sigma} \|\hat{P}(\cdot,t)\|_{L^{\frac{6}{\sigma}}(B_r(x_0))}.
\]
Thus,
\[
\rho^{-3} \|P(\cdot,t)\|_{L^1(B_\rho(x_0))} \leq C \frac{1}{r^{\frac{3}{2} + \sigma}} \|\hat{P}(\cdot,t)\|_{L^{\frac{6}{\sigma}}(B_r(x_0))} + C \frac{1}{r^{\frac{3}{2} + \sigma}} \|P(\cdot,t)\|_{H^{-\sigma}(B_r(x_0))}
\]
\[
\leq C \frac{1}{r^{\frac{3}{2} + \sigma}} \left( \int_{B_r(x_0)} |u - [u]_{x_0,r}| \frac{12}{1 + 2\sigma} + |\nabla d - [\nabla d]_{x_0,r}| \frac{12}{1 + 2\sigma} \right) \, dx + C \frac{1}{r^{\frac{3}{2} + \sigma}} \|P(\cdot,t)\|_{H^{-\sigma}(B_r(x_0))}
\]
\[
\leq C \frac{1}{r^{\frac{3}{2} + \sigma}} \left( \int_{B_r(x_0)} |\nabla u|^2 \, dx \right)^{\frac{3 - 2\sigma}{2}} \left( \int_{B_r(x_0)} |u|^2 \, dx \right)^{\frac{1 + 2\sigma}{2}} + C \frac{1}{r^{\frac{3}{2} + \sigma}} \|P(\cdot,t)\|_{H^{-\sigma}(B_r(x_0))}
\]
\[
\leq C \frac{1}{r^{\frac{3}{2} + \sigma}} A(u, \nabla d, x_0, r)^{\frac{1 + 2\sigma}{2}} \left( \int_{B_r(x_0)} |\nabla u|^2 + |\nabla^2 d|^2 \right) \, dx + C \frac{1}{r^{\frac{3}{2} + \sigma}} \|P(\cdot,t)\|_{H^{-\sigma}(B_r(x_0))}
\]
Integrating the above inequality with respect to $t$ over the interval $(t_0 - \rho^2, t_0)$, and using the Hölder’s inequality, we obtain (13). Thus we complete the proof of Lemma 3.2.

Before going to give the proof Theorem 1.1, let us recall the following $\varepsilon$-regularity criterion for suitable weak solutions to the simplified Ericksen-Leslie system (1) obtained by Lan-Ma [14] (see also Liu [25]).

**Lemma 3.3.** Let $(u,d,P)$ be a suitable weak solution to the nematic liquid crystal flow (1) on $Q_t(x_0)$ for some $0 < r \leq 1$. There exists a constant $\varepsilon_* > 0$ such that if
\[
\sup_{0 \leq \rho \leq r} A(u, \nabla d, x_0, \rho) \leq \varepsilon_*,
\]
the $(u,d)$ is regular at $x_0$. 

In what follows, let us turn to prove Theorem 1.1.

**Proof of Theorem 1.1** We first notice that for $0 < r \leq 1$, our assumption is that
\[
G_\sigma(u, \nabla d, z_0, r) + H_\sigma(P, z_0, r) \leq \varepsilon,
\]
where $0 < \varepsilon < 1$ is to be determined, and $\sigma \in [0, 1]$. By using Lemma 3.1, it is enough to prove that
\[
\sup_{0 < \rho \leq \frac{r}{2}} (A(u, \nabla d, z_0, \rho) + B(u, \nabla d, z_0, \rho)) \leq C \varepsilon^{\frac{2m}{n}} \leq \varepsilon^*.
\]
Here $C$ is a positive constant and independent of $r$. By using the inductive argument used by [1, 17], we get that to prove (17), it suffices to show a discrete version, i.e.,
\[
A(u, \nabla d, z_0, \theta^r r) + B(u, \nabla d, z_0, \theta^r r) \leq C \varepsilon^{\frac{2m}{n}},
\]
for some fixed $\theta \in (0, \frac{1}{4})$ and for all $n \in \mathbb{N}$.

Now, let $\theta \in (0, \frac{1}{4})$ be determined later and define
\[
r_n = \theta^n r, \quad n \in \mathbb{N}.
\]
Then we can find that when $\varepsilon$ is sufficiently small enough (depending on $\theta$), the hypothesis (16) together with Lemma 3.1 yields that inequality (18) holds in the case $n = 1$. Suppose now that inequality (18) holds for $n = 1, \ldots, m - 1$ with an integer $m \geq 2$. In what follows, we shall prove that (18) is still hold for $n = m$. Let $0 \leq \chi \leq 1$ be a smooth cutoff function which vanishes outside of $Q_{2m}^*(z_0)$, equals 1 on $Q_{2m}^*(z_0)$, and for all $n \in \mathbb{N}$.

Define $\phi_m = \chi \psi_m$, then it can be seen that $\phi_m \geq 0$, $(\partial_t + \Delta)\phi_m = 0$ in $Q_{2m}^*(z_0)$, and direct calculation shows that
\[
|\partial_\tau + \Delta|\phi_m | \leq C \quad \text{on } Q_{2m}^*(z_0);
\]
\[
c_0 r_m^{-3} \leq \phi_m \leq C r_m^{-3} |\nabla \phi_m| \leq C r_m^{-4} \quad \text{on } Q_{r_m}(z_0), m \geq 2.
\]
\[
\phi_m \leq C r_m^{-3} |\nabla \phi_m| \leq C r_k^{-4} \quad \text{on } Q_{r_k}(z_0)\setminus Q_{r_k-1}(z_0), 1 \leq k \leq m.
\]
Here $c_0 > 0$ is an uniform constant, and the constant $C = C(\theta) > 0$ is independent of $m$ and $r$. Taking the text function $\phi_m$ in the local energy inequality, it follows that
\[
\int_{B_r(z_0)} (|u|^2 + |\nabla d|^2)\phi_m dx + 2 \int_{Q_r(z_0)} (|\nabla u|^2 + |\nabla^2 d|^2)\phi_m dx dt
\]
\[
\leq \int_{Q_r(z_0)} \left\{(|u|^2 + |\nabla d|^2)(|\partial_t + \Delta|\phi_m) + |\nabla \phi_m||u||(|u|^2 + |\nabla d|^2) + 2P(u \cdot \nabla \phi_m) + [u \cdot \nabla d \otimes \nabla d] \cdot \nabla \phi_m + [\nabla f(d) \cdot \nabla d] \phi_m \right\} dx dt,
\]
which yields that
\[
A(u, \nabla d, z_0, r_m) + B(u, \nabla d, z_0, r_m)
\]
\[
\leq C \left( r_m^2 \int_{Q_{2r_m}(z_0)} (|u|^2 + |\nabla d|^2)(|\partial_t + \Delta|\phi_m) dx dt + r_m^2 \int_{Q_{2r_m}(z_0)} (|u|^3 + |\nabla d|^3 + |u||\nabla d|^2) \nabla \phi_m dx dt \right.
\]
\[
+ r_m^2 \int_{Q_{2r_m}(z_0)} P u \cdot \nabla \phi_m dx dt \left. + r_m^2 \int_{Q_{2r_m}(z_0)} |\nabla d|^2 \phi_m dx dt \right)
\]
\[
=: I_1 + I_2 + J_3 + J_4.
\]
In what follows, we shall give the estimate of $J_i$ ($i = 1, 2, 3, 4$) term by term. To do it, let us first write
\[ \phi_m = \chi_1 \phi_m = \sum_{k=1}^{m-1} (\chi_k - \chi_{k+1})\phi_m + \chi_m \phi_m, \]
where $\chi_k$ (with $k = 1, 2, \cdots, m$) is a smooth function such that $0 \leq \chi_k \leq 1$, vanishes outside of $Q_{r_k}(z_0)$, equals 1 on $Q_{r_k}(r_0)$, and
\[ |\nabla \chi_k| \leq \frac{C}{r_k}, \quad |\partial_t \chi_k|, |\Delta \chi_k| \leq \frac{C}{r_k^2}. \]

By the hypothesis that (18) holds for $n = 1, 2, \cdots, m-1$, one has
\[
J_1 \leq C r_m \sum_{k=1}^{m-1} \int_{Q_{r_k}(z_0)} (|u|^2 + |\nabla d|^2)(|\nabla (\chi_k - \chi_{k+1})\phi_m| + |\Delta (\chi_k - \chi_{k+1})\phi_m| + |\nabla (\chi_k - \chi_{k+1})\nabla \phi_m|) dx dt
\]
\[ + C r_m \int_{Q_{r_k}(z_0)} (|u|^2 + |\nabla d|^2)(|\partial_t (\chi_m - \phi_m)| + |\Delta \chi_m - \phi_m| + |\nabla (\chi_m - \phi_m)|) dx dt\]
\[ \leq C r_m \sum_{k=1}^{m-1} r_k^{-5} \int_{Q_{r_k}(z_0)} (|u|^2 + |\nabla d|^2) dx dt + C r_m^{-3} \int_{Q_{r_m}(z_0)} (|u|^2 + |\nabla d|^2) dx dt\]
\[ \leq C r_m \sum_{k=1}^{m-1} r_k^{-2} A(u, \nabla d, z_0, r_k) + C \theta^{-1} A(u, \nabla d, z_0, r_{m-1})\]
\[ \leq C (r_m \sum_{k=1}^{m-1} r_k^{-2} + \theta^{-1}) \varepsilon^{\frac{2-\alpha}{2}} \leq C(\theta) \varepsilon^{\frac{2-\alpha}{2}}. \]

Notice that it holds that (see, e.g., [1, 21])
\[ \rho^{-2} \int_{Q_{\rho}(z_0)} |f|^3 dx dt \leq C \left( \frac{\rho}{r} \right)^3 A(f, \rho, r)^{\frac{3}{2}} B(f, \rho, r)^{\frac{3}{2}} + C \left( \frac{\rho}{r} \right)^3 A(f, \rho, r)^{\frac{3}{2}}, \]
for all $0 \leq \rho \leq r$, and $f$ is a function in $Q_{r}(z_0)$. This together with the properties of $\phi_m$ yields that
\[
J_2 \leq C r_m \sum_{k=1}^{m-1} \int_{Q_{r_k}(z_0)} (|u|^3 + |\nabla d|^3)(|\nabla (\chi_k - \chi_{k+1})\phi_m| + |(\chi_k - \chi_{k+1})\nabla \phi_m|) dx dt
\]
\[ + C r_m \int_{Q_{r_m}(z_0)} (|u|^3 + |\nabla d|^3)(|\nabla \chi_m\phi_m| + |\chi_m\nabla \phi_m|) dx dt\]
\[ \leq C r_m \sum_{k=1}^{m-1} r_k^{-4} \int_{Q_{r_k}(z_0)} (|u|^3 + |\nabla d|^3) dx dt + C r_m^{-2} \int_{Q_{r_m}(z_0)} (|u|^3 + |\nabla d|^3) dx dt\]
\[ \leq C r_m \sum_{k=2}^{m} r_k^{-2} \left[ \left( \frac{r_{k-1}}{r_k} \right)^3 A(u, \nabla d, z_0, r_{k-1}) \right]^{\frac{3}{2}} B(u, \nabla d, z_0, r_{k-1})^{\frac{3}{2}} + \left( \frac{r_k}{r_{k-1}} \right)^3 A(u, \nabla d, z_0, r_{k-1})^{\frac{3}{2}} \]
\[ + C r_m \sum_{k=2}^{m} r_k^{-2} \left[ \left( \frac{r_{k-1}}{r_k} \right)^3 A(u, \nabla d, z_0) \right]^{\frac{3}{2}} B(u, \nabla d, z_0) \]
\[ \leq C (r_m \sum_{k=2}^{m} r_k^{-2} + 1)(\theta^3 + \theta^{-3}) \varepsilon^{\frac{3(2-\alpha)}{4}} \leq C(\theta) \varepsilon^{\frac{2-\alpha}{2}}, \]
where we have used the Lemma 3.1 in the inequality above. The term $J_4$ can be estimated as
\[ J_4 \leq C r_m^2 \int_{Q_{r_1}(z_0)} |\nabla d|^2 dx dt \leq C r_m^2 \theta A(\nabla d, z_0, r_1) \leq C(\theta) \varepsilon^{\frac{2-\alpha}{2}}. \]
For the term $J_3$, it is easy to see that

$$J_3 \leq C r_m \left[ \sum_{k=1}^{m-1} \int_{Q_{r_k}(z_0)} P u \cdot \nabla ([\chi_k - \chi_{k+1}] \phi_m) dx dt \right] + C r_m \left[ \int_{Q_{r_k}(z_0)} P u \cdot \nabla (\chi_m \phi_m) dx dt \right]$$

$$\leq C r_m \left[ \sum_{k=1}^{m-1} \int_{Q_{r_k}(z_0)} (P - [P]_{z_0,r_k}) u \cdot \nabla ([\chi_k - \chi_{k+1}] \phi_m) dx dt \right]$$

$$+ C r_m \left[ \int_{Q_{r_k}(z_0)} (P - [P]_{z_0,r_m}) u \cdot \nabla (\chi_m \phi_m) dx dt \right]$$

$$\leq C r_m \sum_{k=2}^{m} r_k^{-\frac{2}{3}} \int_{Q_{r_k}(z_0)} \|P - [P]_{z_0,r_k}\|_{L^2(B_{r_k}(z_0))} dt$$

$$+ C r_m \sum_{k=2}^{m} r_k^{-\frac{2}{3}} \int_{Q_{r_k}(z_0)} \|P - [P]_{z_0,r_k}\|_{L^2(B_{r_k}(z_0))} dt$$

$$\leq C r_m \sum_{k=2}^{m} r_k^{-\frac{2}{3}} \int_{Q_{r_k}(z_0)} \|P - [P]_{z_0,r_k}\|_{L^2(B_{r_k}(z_0))} dt$$

$$+ C \theta^{-1} \int_{Q_{r_k}(z_0)} \|P - [P]_{z_0,r_k}\|_{L^2(B_{r_k}(z_0))} dt$$

$$\leq \Pi(r_k) = r_k^{-\frac{2}{3}} \int_{t_0 - r_k^2}^{t_0} \|P - [P]_{z_0,r_k}\|_{L^2(B_{r_k}(z_0))} dt \quad \text{for } k = 1, 2, \ldots, m.$$  

By using (12) in Lemma 3.2, it follows that for all $2 \leq k \leq m$,

$$\Pi(r_k) \leq C_1 \theta \Pi(r_{k-1}) + C_1 \theta^{-\frac{2}{3}} A(u, \nabla d, z_0, r_{k-1})^{-\frac{1}{2}} B(u, \nabla d, z_0, r_{k-1})^{-\frac{1}{2}},$$

where $C_1$ is a positive constant and is independent of $k$ and $\theta$. By choosing $0 < \theta \leq \frac{1}{\sqrt{2}}$, and then iterating the above inequality, one gets that

$$\Pi(r_k) \leq \frac{1}{4k-1} \Pi(r_1) + C_1 \theta^{-\frac{2}{3}} \sum_{i=1}^{k-1} \frac{1}{4^{i-1}} A(u, \nabla d, z_0, r_{k-1})^{-\frac{1}{2}} B(u, \nabla d, z_0, r_{k-1})^{-\frac{1}{2}}$$

$$\leq \frac{1}{4k-1} \Pi(r_1) + C_1 \theta^{-\frac{2}{3}} \sum_{i=1}^{k-1} \frac{1}{4^{i-1}} \varepsilon^{\frac{2(m-k)}{3}}$$

$$\leq \Pi(r_1) + C \theta^{-\frac{2}{3}} \varepsilon^{\frac{2(m-k)}{3}}, \quad 2 \leq k \leq m,$$
where we have used the assumption that (18) holds for \( n = 1, 2, \cdots, m - 1 \), and \( C \) is independent of \( m \) and \( \theta \). By using Lemma 3.2 again, one obtains that

\[
\Pi(r_1) \leq C_2 \theta^{-1} r^{-3} \int_0^{r_0} \| P - [P]_{x_0} \|^2 \| \partial_x \theta \|_{L^1(B_{r_0}(x_0))} \, dx + C_2 \theta^{-1} A(u, \nabla d, z_0, r) \frac{r}{2} B(u, \nabla d, z_0, r) \frac{1}{\theta} \\
\leq C_2 \left[ \theta^{-1} (H_\sigma(P, z_0, r) + \theta^2 A(u, \nabla d, z_0, r) \frac{1}{\theta^2} B(u, \nabla d, z_0, r) \frac{1}{\theta}) \right] \\
+ \theta^2 A(u, \nabla d, z_0, r) \frac{r}{2} B(u, \nabla d, z_0, r) \frac{1}{\theta^2} \\
\leq C_2 \theta^{-1} (\varepsilon + \theta^2 \varepsilon^{\frac{2-\sigma}{2-\sigma}}) \leq C_2 \theta^{-1} \varepsilon^{\frac{2-\sigma}{2-\sigma}},
\]

where \( C_2 \) is independent of \( \theta \). Inserting the above two estimates into (20), it follows that

\[
J_3 \leq C \sum_{k=2}^{m} \frac{r_k^{-2}}{\varepsilon^{\frac{2-\sigma}{2-\sigma}}} (\Pi(r_1) + \theta^{-\frac{1}{2}} \varepsilon^{\frac{2-\sigma}{2-\sigma}}) + C \theta^{-1} \varepsilon^{\frac{2-\sigma}{2-\sigma}} \Pi(r_1) \\
\leq C(\varepsilon) \theta^{\frac{12-\sigma}{4-\sigma}} + C(\varepsilon) \theta^{\frac{2-\sigma}{2-\sigma}} \Pi(r_1) \leq C(\varepsilon) \theta^{\frac{2-\sigma}{2-\sigma}}.
\]

Inserting the estimates of \( J_i \) \( (i = 1, 2, 3, 4) \) into (19), one gets that

\[
A(u, \nabla d, z_0, r_m) + B(u, \nabla d, z_0, r_m) \leq C(\varepsilon) \theta^{\frac{2-\sigma}{2-\sigma}} \leq \varepsilon^{\frac{2-\sigma}{2-\sigma}}
\]

provided \( \varepsilon \) is small enough. This proves that (18) holds for \( n = m \), and then completes the proof of Theorem 1.1. \( \square \)

4. **Proof of Theorem 1.2.** In this section, we shall give the proof of Theorem 1.2. We first notice that Theorem 1.2 is inspired by the new \( \varepsilon \)-regularity criterion (8) in Corollary 1 and is obtained by the iterate method modifying the argument used in [12, 24]. To apply (8), we need to adapt some decay estimates for \( E_\alpha(u, \nabla d, z_0, \rho) \) and \( F_\alpha(P - [P]_{x_0, \rho}, z_0, \rho) \), as given in the following lemma, which plays important role in the proof.

**Lemma 4.1.** Let \( \alpha \in \left[ \frac{\sigma}{\alpha}, \frac{10}{7} \right] \). For \( 0 \leq \rho \leq \frac{r}{2} \), there exists positive constant \( C \) independent of \( \rho \) and \( r \), such that

\[
E_\alpha(u, \nabla d, z_0, \rho) \leq C \left( \frac{\rho}{r} \right)^{\frac{2a}{\alpha}} A(u, \nabla d, z_0, \rho) \left( \frac{\rho^{a-1}}{\alpha} \right) B(u, \nabla d, z_0, r) \left( \frac{\rho}{r} \right)^{\frac{a}{\alpha}} \\
+ C \left( \frac{\rho}{r} \right)^{\frac{2a}{\alpha}} A(u, \nabla d, z_0, \rho) \left( \frac{\rho^{a-1}}{\alpha} \right) B(u, \nabla d, z_0, r) \left( \frac{\rho}{r} \right)^{\frac{a}{\alpha}}.
\]

Let \( \alpha \in \left[ \frac{\sigma}{\alpha}, \frac{2}{\alpha} \right] \). For \( 0 \leq \rho \leq \frac{r}{2} \), there exists positive constant \( C \) independent of \( \rho \) and \( r \), such that

\[
F_\alpha(P - [P]_{x_0, \rho}, z_0, \rho) \leq C \left( \frac{\rho}{r} \right)^{\frac{4a}{\alpha}} E_\alpha(u, \nabla d, z_0, r) + C \left( \frac{\rho}{r} \right)^{\frac{4a}{\alpha}} F_\alpha(P - [P]_{x_0, \rho}, z_0, r).
\]

**Proof.** By applying the Hölder’s inequality and the Poincaré-Sobolev inequality, one has

\[
\int_{B_{r}(x_0)} |u - [u]_{x_0, r}|^{2a} \, dx \leq C \left( \int_{B_{r}(x_0)} |u - [u]_{x_0, r}|^{2} \, dx \right)^{\alpha - 1} \left( \int_{B_{r}(x_0)} |u - [u]_{x_0, r}|^{\frac{2a}{\alpha}} \, dx \right)^{\frac{2-\alpha}{\alpha}} \\
\leq C \rho^{5-3a} \left( \int_{B_{r}(x_0)} |u - [u]_{x_0, r}|^{2} \, dx \right)^{\alpha - 1} \left( \int_{B_{r}(x_0)} |u - [u]_{x_0, r}|^{6} \, dx \right)^{\frac{1}{3}} \\
\leq C \rho^{5-3a} \int_{B_{r}(x_0)} |u|^{2} \, dx \int_{B_{r}(x_0)} \nabla u \, dx.
\]
from which, together with the triangle inequality and the Hölder’s inequality yields that
\[ \|u\|^2_{L^\alpha(B_\rho(x_0))} \leq \|u\|^2_{L^\alpha(B_\rho(x_0))} \leq \int_{B_{\rho}(x_0)} |u - u|_{x_0, r}^2 \, dx + \int_{B_{\rho}(x_0)} \|u|_{x_0, r}^2 \, dx \]
\[ \leq C \rho^{3-3\alpha} \left( \int_{B_{\rho}(x_0)} |u|^2 \, dx \right)^{\alpha-1} \int_{B_{\rho}(x_0)} |\nabla u|^2 \, dx + C \frac{\rho^\beta}{\tau^\alpha} \left( \int_{B_{\rho}(x_0)} |u|^2 \, dx \right)^\alpha. \]
Integrating \( \|u\|^2_{L^\alpha(B_\rho(x_0))} \) in time on \((t_0 - \rho^2, t_0)\), and using the Hölder’s inequality, it follows that
\[ \int_{t_0 - \rho^2}^{t_0} \|u\|^2_{L^\alpha(B_\rho(x_0))} \, dt \leq \rho^\frac{2\alpha}{\tau^\alpha} \left( \sup_{t_0 - \rho^2 \leq \tau \leq t_0} \int_{B_{\rho}(x_0)} |u|^2 \, dx \right) \left( \int_{Qr(x_0)} |\nabla u|^2 \, dx \right)^\frac{\alpha}{\alpha-1} \]
\[ + C \rho^\frac{2\alpha}{\tau^\alpha} \left( \sup_{t_0 - \rho^2 \leq \tau \leq t_0} \int_{B_{\rho}(x_0)} |u|^2 \, dx \right) \left( \frac{\tau}{\rho} \right)^\frac{8\alpha}{\tau^\alpha} A(u, z_0, r) \frac{\tau}{\rho}. \]
where we have used the fact that \( \alpha \leq \frac{10}{7} \). From the inequality above, we get that
\[ E_\alpha(u, z_0, \rho) \leq C \left( \frac{r}{\rho} \right)^\frac{4\alpha}{\tau^\alpha} A(u, z_0, r) \frac{4(\alpha-1)}{\tau^\alpha} B(u, z_0, r) \frac{4}{\tau^\alpha} + C \left( \frac{r}{\rho} \right)^\frac{8\alpha}{\tau^\alpha} A(u, z_0, r) \frac{4}{\tau^\alpha}. \]
In a similar way, one can get the estimate of \( E_\alpha(\nabla d, z_0, \rho) \), thus we complete the proof of (21).

Let us turn to prove (22). Let \( \phi \in C_0^\infty(B_{\rho}(x_0)) \) such that \( \phi \equiv 1 \) on \( B_{3\rho}(x_0) \), with \( 0 \leq \phi \leq 1 \) and \( |\nabla \phi| \leq C \tau^{-1}, |\nabla^2 \phi| \leq C \tau^{-2} \). Using the divergence free condition (1), one has
\[ \partial_t \partial_\tau (P \phi) = -\phi \partial_t \partial_\tau (U_{i,j} + D_{i,j}) + 2\partial_t \phi \partial_\tau P + P \partial_\tau \partial_t \phi, \]
where \( U_{i,j} \) and \( D_{i,j} \) are defined as before. Thus for any \( x \in B_{3\rho}(x_0) \), one has
\[ P(x) = \mathcal{G} * \{ -\phi \partial_t \partial_\tau (U_{i,j} + D_{i,j}) + 2\partial_t \phi \partial_\tau P + P \partial_\tau \partial_t \phi \} \]
\[ := P_1(x) + P_2(x), \]
with
\[ P_1(x) = - \partial_t \partial_\tau \mathcal{G} * [\phi(U_{i,j} + D_{i,j})]; \]
\[ P_2(x) = 2\partial_\tau \mathcal{G} * [\partial_\tau \phi(U_{i,j} + D_{i,j}) - \mathcal{G} * [\partial_\tau \partial_\tau \phi(U_{i,j} + D_{i,j})] \]
\[ + 2\partial_\tau \mathcal{G} * (\partial_\tau \phi P) - \mathcal{G} * (\partial_\tau \partial_\tau P). \]
Here, \( \mathcal{G} \) is the standard normalized fundamental solution of the Laplace equation. Due to the fact that \( \phi \equiv 1 \) on \( x \in B_{3\rho}(x_0) \), it is easy to see that \( P_2(x) \) is a harmonic function on \( B_{\rho}(x_0) \), i.e.,
\[ \Delta P_2 = 0 \quad \text{in} \; B_{\rho}(x_0). \]
By applying the interior estimate of the harmonic function, it follows that for all \( x' \in B_{\rho}(x_0) \),
\[ |\nabla P(x')| \leq C \frac{r}{\tau^\alpha} \|P\|_{L^1(B_{3\rho}(x'))} \leq C \frac{r}{\tau^\alpha} \|P\|_{L^1(B_{\rho}(x_0))} \]
\[ \leq C \frac{r}{\tau^{1+\frac{\alpha}{\tau^\alpha}}} \|P\|_{L^\alpha(B_{\rho}(x_0))}, \]
where we have used the Hölder’s inequality, from which, we get that
\[ \|P_2\|_{L^\infty(B_{\rho}(x_0))} \leq \frac{C}{r^{1+\frac{\alpha}{\tau^\alpha}}} \|P\|_{L^\alpha(B_{\rho}(x_0))}. \]
Combining the estimate above with the mean value theorem, we have for any $\rho \leq \frac{r}{8}$,
\[
\|P_2 - [P_2]_{x_0,\rho}\|_{L^\infty(B_\rho(x_0))} \leq C\rho^3 \|P_2 - [P_2]_{x_0,\rho}\|_{L^\infty(B_\rho(x_0))} \leq C\rho^{3+\alpha} \|\nabla P_2\|_{L^\infty(B_\rho(x_0))} .
\]
\[
\leq C\rho^{3+\alpha} \|\nabla P_2\|_{L^\infty(B_{\frac{r}{8}}(x_0))} \leq C\frac{\beta}{\alpha} \|P_2\|_{L^\infty(B_{\frac{r}{8}}(x_0))} .
\]
Notice that $P_2 - [P_2]_{x_0,\frac{r}{8}}$ is also a harmonic function on $B_{\frac{r}{8}}(x_0)$, hence it holds that
\[
\|P_2_{x_0,\rho}\|_{L^\infty(B_\rho(x_0))} \leq C\frac{\beta}{\alpha} \|P_2\|_{L^\infty(B_{\frac{r}{8}}(x_0))} .
\]
On the other hand, the Calderón-Zygmund estimate yields that for $\rho \leq \frac{r}{8}$,
\[
\|P_1\|_{L^\infty(B_\rho(x_0))} \leq \|P_1\|_{L^\infty(B_{\frac{r}{8}}(x_0))} \leq C \int_{B_{\frac{r}{8}}(x_0)} (|u - [u]_{x_0,\rho}|^2 + |\nabla d - [\nabla d]_{x_0,\rho}|^2)\,dx = C\|u\|^2L^\infty(B_{\frac{r}{8}}(x_0)) + \|\nabla d\|_{L^\infty(B_{\frac{r}{8}}(x_0))} .
\]
Hence, integrating $\|P - [P]_{x_0,\rho}\|_{L^\infty(B_{\frac{r}{8}}(x_0))}$ in time on $(t_0 - r^2, t_0)$, and using the triangle inequality, (23) and (24), it follows that
\[
\int_{t_0 - r^2}^{t_0} \|P - [P]_{x_0,\rho}\|_{L^\infty(B_{\frac{r}{8}}(x_0))} \,dt \leq C\int_{t_0 - r^2}^{t_0} (\|P_1 - [P_1]_{x_0,\rho}\|_{L^\infty(B_{\frac{r}{8}}(x_0))} + \|P_2 - [P_2]_{x_0,\rho}\|_{L^\infty(B_{\frac{r}{8}}(x_0))}) \,dt \leq C\int_{t_0 - r^2}^{t_0} \|P_1\|_{L^\infty(B_{\frac{r}{8}}(x_0))} \,dt + C\int_{t_0 - r^2}^{t_0} \|P_2\|_{L^\infty(B_{\frac{r}{8}}(x_0))} \,dt \leq C\int_{t_0 - r^2}^{t_0} (\|u\|^2L^\infty(B_{\frac{r}{8}}(x_0)) + \|\nabla d\|^2L^\infty(B_{\frac{r}{8}}(x_0))d\tau + C\int_{t_0 - r^2}^{t_0} \|P - [P]_{x_0,\rho}\|_{L^\infty(B_{\frac{r}{8}}(x_0))} \,dt ,
\]
from which we obtain (22), and the proof of Lemma 4.1 is completed.

After arguing by contradiction, as given by [12,13,24,27], Theorem 1.2 is shown to be a consequence of the following theorem.

**Theorem 4.2.** Let $0 < T \leq \infty$. Suppose that $(u, d, P)$ is a suitable weak solution to the simplified Ericksen-Leslie system (1) on $\mathbb{R}^3 \times (0, T)$. Then, for any $0 < \gamma < \frac{5\alpha}{18\beta}$, $z = (x, t)$ is a regular point provided that a sufficiently small universal positive constant $\epsilon_1$ and $0 < r < 1$ exist such that
\[
\int_{Q_{r}(z)} (|\nabla u|^2 + |\nabla d|^2 + |u|^2 + |\nabla d|^2 + |P - [P]_{x,B_{r}(z)}|^2 + |\nabla P|^2)\,dx\,dt \leq r^{\frac{3}{8} - \gamma} \epsilon_1 .
\]

**Remark 3.** The proof of Theorem 4.2 follows the approach utilized in [12,26]. Compared with the proof of [24], we notice that the appearance of the pressure in (1) is in terms of $\nabla P$ rather than $P$, hence, we can always replace $P$ by $P - [P]_{x,B_{r}(z)}$. Based on this fact, we could make full use of a better decay estimate of pressure $P - [P]_{x,B_{r}(z)}$ than that of [24]. Moreover, we utilize the quantity $\|\nabla P\|_{L^\infty(0,T)}$ bounded by the initial energy as widely as possible.
Proof. We may prove Theorem 4.2 with assuming that \( z = 0 \) (i.e., \( x = 0 \) and \( t = 0 \)). Suppose that condition (25) is true for some \( 0 < 2\rho < r < 1 \) such that \( \rho^2 < \frac{5}{8} \), where \( \xi \) is determined later, and

\[
\int_{Q_{2\rho}(0)} (|\nabla u|^2 + |\nabla^2 d|^2 + |u|^{\frac{15}{14}} + |\nabla d|^{\frac{15}{14}} + |P - [P]_{x,B_r(x)}|^{\frac{5}{2}} + |\nabla P|^{\frac{5}{2}}) dx \leq (2\rho)^{\frac{5}{2} - \gamma} \epsilon_1. \tag{26}
\]

First, we shall assert that

\[
A(u, \nabla d, 0, 0) \leq C\rho^{-\frac{9\gamma}{16}}. \tag{27}
\]

Indeed, let \( \phi(x, t) \) be a smooth positive function supported in \( Q_{2\rho}(0) \) and with value 1 on \( Q_{\rho}(0) \), such that

\[
|\nabla \phi| \leq C\rho^{-1} \quad \text{in } Q_{2\rho}(0) \quad \text{and} \quad |\partial_t \phi|, |\Delta \phi| \leq C\rho^{-2} \quad \text{in } Q_{2\rho}(0).
\]

Then, employing the divergence free condition (1)_3, the Hölder’s inequality and the Gagliardo-Nirenberg inequality, we deduce that

\[
\int_{Q_{2\rho}(0)} (|u|^2 + |\nabla d|^2)(\partial_t \phi + \Delta \phi) dx d\tau \leq C\rho^{\frac{9}{16}} \int_{Q_{2\rho}(0)} (|u|^{\frac{15}{14}} + |\nabla d|^{\frac{15}{14}}) dx d\tau.
\]

and

\[
\int_{Q_{2\rho}(0)} \nabla f(d) \nabla \phi dx d\tau \leq C \int_{Q_{2\rho}(0)} |\nabla d|^2 dx \leq C\rho^2 \left( \int_{Q_{2\rho}(0)} |\nabla d|^{\frac{15}{14}} dx \right)^{\frac{5}{2}}.
\]
The inequalities above combined with the local energy inequality (10) leads to

\[
\sup_{-\rho^2 \leq t \leq 0} \int_{B_t(0)} (|u|^2 + |\nabla d|^2) dx + \int_{Q_t(0)} (|\nabla u|^2 + |\nabla^2 d|^2) dx dt \\
\leq C(1+\rho^2)(\|u\|_{L^\infty(Q_{2\rho}(0))} + \|\nabla d\|_{L^\infty(Q_{2\rho}(0))})^2 + C\rho^{-\frac{1}{2}}(\|u\|_{L^\infty(Q_{2\rho}(0))} + \|\nabla d\|_{L^\infty(Q_{2\rho}(0))})^3 \\
+ C\rho^{-\frac{1}{2}} \left( \|u\|_{L^6(Q_{2\rho}(0))} + \|\nabla d\|_{L^6(Q_{2\rho}(0))} \right) \left( \|\nabla u\|_{L^2(Q_{2\rho}(0))} + \|\nabla^2 d\|_{L^2(Q_{2\rho}(0))} \right) \\
+ C\rho^{-\frac{3}{2}} \|P - [P]_{0,2\rho}\|_{L^6(Q_{2\rho}(0))} \|\nabla P\|_{L^6(Q_{2\rho}(0))} \|u\|_{L^6(Q_{2\rho}(0))} \\
\leq \varepsilon_1^\gamma \rho^{-\frac{1}{2}} + \varepsilon_2^\gamma \rho^{-\frac{1}{2}} + \varepsilon_3^\gamma \rho^{-\gamma} \leq \varepsilon_1^\gamma \rho^{-\frac{1}{2}} + \varepsilon_2^\gamma \rho^{-\frac{1}{2}} + \varepsilon_3^\gamma \rho^{-\gamma},
\]

where we have used (26) and the assumption \( \gamma \leq \frac{2}{3} \). Hence, we get (27).

Second, by iterating (22) in Lemma 4.1, it is easy to see that for any \( \theta \in (0, \frac{1}{2}) \) and \( 0 < \mu \leq \rho \)

\[
F_\alpha(P - [P]_{0,\theta^N \mu}, \theta^N \mu) \\
\leq C \sum_{k=1}^N \theta^{-\frac{40}{7}} - \frac{12-2\alpha(1-k)}{7-6} \mu E_\alpha(u, \nabla d, 0, \theta^{N-k} \mu) + C\theta^{\frac{12}{23}} - \frac{12}{23} \mu F_\alpha(P - [P]_{0,\mu}, \mu).
\]

(28)

By using the Hörder’s inequality and the Poincaré-Sobolev inequality, one obtains

\[
\|P - [P]_{0,\mu}\|_{L^{23-\alpha}(Q_{\mu}(0))} \leq C\|P - [P]_{0,\mu}\|_{L^{23-\alpha}(Q_{\mu}(0))} \|P - [P]_{0,\mu}\|_{L^{30-19\alpha}(Q_{\mu}(0))} \\
\leq C\mu^{\frac{23}{23-\alpha}} \|\nabla P\|_{L^{23-\alpha}(Q_{\mu}(0))} \|P - [P]_{0,\mu}\|_{L^{30-19\alpha}(Q_{\mu}(0))},
\]

where we have used \( \frac{30}{23} \leq \alpha \leq \frac{10}{7} \), which yields that

\[
F_\alpha(P - [P]_{0,\mu}, \theta, \theta^N \mu) \leq C\mu^{-\frac{40}{7}} \|\nabla P\|_{L^{23-\alpha}(Q_{\mu}(0))} \|P - [P]_{0,\mu}\|_{L^{30-19\alpha}(Q_{\mu}(0))}.
\]

(29)

Inserting the inequality above into (28), one gets that

\[
F_\alpha(P - [P]_{0,\theta^N \mu}, \theta^N \mu) \leq C \sum_{k=1}^N \theta^{-\frac{40}{7}} - \frac{12-2\alpha(1-k)}{7-6} \mu E_\alpha(u, \nabla d, 0, \theta^{N-k} \mu) \\
+ C\theta^{\frac{12}{23}} - \frac{12}{23} \mu \|\nabla P\|_{L^{23-\alpha}(Q_{\mu}(0))} \|P - [P]_{0,\mu}\|_{L^{30-19\alpha}(Q_{\mu}(0))}.
\]

(29)

To proceed further, we shall set \( r = \rho^{\eta} = \theta^{N} \mu, \theta = \rho^\xi \) and \( r = \mu = \theta^{\eta} r = \rho^{\eta} \) (\( i = 1, 2, \cdots, N \)), where \( \eta \) and \( \xi \) will be determined by \( \gamma \). Their precise selection will be explained at the end. Thus, from (29), one can derive that

\[
E_\alpha(u, \nabla d, 0, r) + F_\alpha(P, 0, r) \leq C \sum_{k=1}^N \theta^{-\frac{40}{7}} - \frac{12-2\alpha(1-k)}{7-6} \mu E_\alpha(u, \nabla d, 0, r) \\
+ C\theta^{\frac{12}{23}} - \frac{12}{23} \mu \|\nabla P\|_{L^{23-\alpha}(Q_{r\mu}(0))} \|P - [P]_{0,\mu}\|_{L^{30-19\alpha}(Q_{r\mu}(0))} \\
: = J_3 + J_6,
\]

(30)

where we have used the fact that

\[
E_\alpha(u, \nabla d, 0, r) \leq C\theta^{-\frac{40}{7}} - \frac{12-2\alpha(1-k)}{7-6} \mu E_\alpha(u, \nabla d, 0, \theta^{-1} r).
\]
Hence, by using (21) in Lemma 4.1, (26) and (27), it follows that
\[
E_\alpha(u, \nabla d, 0, r_k) \leq C \left( \frac{\rho}{r_k} \right)^{\frac{4\alpha}{n-\beta}} A(u, \nabla d, 0, \rho)^{\frac{4\alpha}{n-\beta}} B(u, \nabla d, 0, \rho)^{\frac{4\alpha}{n-\beta}}
\]
\[
+ \left( \frac{r_k}{\rho} \right)^{\frac{4\alpha}{n-\beta}} A(u, \nabla d, 0, \rho)^{\frac{4\alpha}{n-\beta}}
\]
\[
\leq C \varepsilon_1^{\frac{-\alpha}{105(n-\beta)}} \left( \frac{\rho}{r_k} \right)^{\frac{12\alpha}{5(n-\beta)}} A(u, \nabla d, 0, \rho)^{\frac{4\alpha}{n-\beta}} B(u, \nabla d, 0, \rho)^{\frac{4\alpha}{n-\beta}}
\]
which implies that
\[
J_5 \leq C \varepsilon_1^{\frac{-\alpha}{105(n-\beta)}} \sum_{k=1}^N \left( \frac{4n\alpha + (12 - 2\alpha)(k-1)}{n-\beta} \right)^{\frac{4\alpha}{n-\beta}} \left( \frac{4n\alpha + (12 - 2\alpha)(k-1)}{n-\beta} \right)^{\frac{4\alpha}{n-\beta}} + \rho^{\frac{8\alpha(n-\beta)}{n-\beta}} - \frac{8\alpha}{n-\beta}
\]
\[
= C \varepsilon_1^{\frac{-\alpha}{105(n-\beta)}} \rho^{\frac{16\alpha}{n-\beta}} - \frac{8\alpha}{n-\beta} - \frac{4\alpha}{n-\beta} - \frac{8\alpha(n+\gamma)}{n-\beta} - \frac{(12 - 10\alpha)(n)}{n-\beta}
\]
where we have used the fact that \( \frac{20}{23} \leq \alpha \leq \frac{10}{7} \) implies that \( 2\alpha + 12 > 0 \) and \( 12 - 10\alpha < 0 \). To minimize the right-hand side of the inequality above, by choose
\[
\eta = \frac{1}{6\alpha} (6\alpha + 4) + (\alpha + 6) \xi - \frac{\gamma}{5} - (6 - 5\alpha) \xi N
\]
and then we conclude that for a sufficiently large \( N \),
\[
J_5 \leq C \varepsilon_1^{\frac{-\alpha}{105(n-\beta)}} \rho^{\frac{16\alpha}{n-\beta}} - \frac{8\alpha}{n-\beta} - \frac{4\alpha}{n-\beta} - \frac{8\alpha(n+\gamma)}{n-\beta} - \frac{(12 - 10\alpha)(n)}{n-\beta}
\]
Let us turn to bound \( J_6 \). To do it, we temporarily assume that \( r_N \leq \rho \), i.e.,
\[
\rho^{\alpha - \varepsilon N} \leq \rho.
\]
Using this assumption together with (26) and (31), it follows that
\[
J_6 \leq C \rho^{\frac{(12 - 2\alpha)N}{n-\beta} - \frac{4\alpha}{n-\beta}} \left\| \nabla \left[ P - \frac{2\alpha - 30}{L^2(Q_{r_N}(0))} \right] \right\|_{L^2(Q_{r_N}(0))}^{\frac{30 - 10\alpha}{n-\beta}}
\]
\[
\leq C \rho^{\frac{(12 - 2\alpha)N}{n-\beta} - \frac{4\alpha(n - \varepsilon N)}{n-\beta}} + \frac{2\alpha - 30}{L^2(Q_{r_N}(0))} - \frac{2\alpha}{n-\beta} - \frac{2\alpha}{n-\beta} - \gamma
\]
In what follows, we shall use Corollary 1 with \( \alpha = \frac{10}{7} \) to complete the proof, i.e.,
that there exists constant \( r > 0 \) such that \( E_{\frac{10}{7}}(u, \nabla d, 0, r) + F_{\frac{10}{7}}(P, 0, r) < \varepsilon \). When we setting \( \alpha = \frac{10}{7} \), then (30) becomes
\[
\eta = \frac{7}{30} \left( \frac{4\xi N}{7} + \frac{26\xi}{7} + \frac{104}{21} - \frac{\gamma}{10} \right)
\]
and the estimates of \( J_5 \) and \( J_6 \) can be rewritten as
\[
J_5 \leq C \varepsilon_1^{\frac{-\alpha}{105(n-\beta)}} \rho^{\frac{41\alpha}{21}} - \frac{4\alpha}{n-\beta} - \frac{14\alpha^2}{105(n-\beta)}
\]
and
\[
J_6 \leq C \varepsilon_1^{\frac{-\alpha}{105(n-\beta)}} \rho^{\frac{20\alpha}{21}} + \frac{1}{n-\beta} - \frac{26\alpha}{21} - \frac{26\alpha}{21},
\]
respectively. To complete the proof, we shall claim that if \( 0 < \gamma < \frac{560}{1839} \), there exist a small number \( \xi > 0 \) and a sufficiently large \( N > 0 \) such that \( J_5 + J_6 \leq C \varepsilon_1^6 \leq \varepsilon \). To this end, we need

\[
- \frac{41\xi}{21} + \frac{4}{9} - \frac{4N\xi}{21} - \frac{142\gamma}{105} \geq 0
\]

and

\[
\frac{32\xi N}{21} + \frac{1}{63} - \frac{26\xi}{21} - \frac{29\gamma}{30} \geq 0.
\]

In addition, from (32), it follows that \( \eta - \xi N - 1 \geq 0 \), i.e.,

\[
\eta - \xi N - 1 = -\frac{13\xi N}{15} + \frac{13\xi}{15} + \frac{14}{90} - \frac{7\gamma}{300} \geq 0.
\]

Hence, one may sum up all the restrictions on \( \gamma \) in the following

\[
\gamma \leq \min \left\{ \frac{5(28 - 12\xi N - 123\xi)}{426}, \frac{10(1 + 96\xi N - 78\xi)}{619}, \frac{20(7 - 39\xi N + 39\xi)}{21}, \frac{5}{3} \right\}.
\]

(35)

By maximizing the bound above on \( \gamma \) with respect to \( \xi N \), one obtains

\[
\xi N = \frac{105}{613}.
\]

Furthermore, from (35), it follows that for a sufficiently large natural number \( N \),

\[
\xi = \frac{105}{613N} \leq \frac{142}{205}(\frac{560}{1839} - \gamma).
\]

Hence, by selecting a sufficiently small \( \xi \), one can have any \( \gamma < \frac{560}{1839} \). Then, by selecting

\[
\eta = \frac{7}{30} \left( \frac{60}{613} + \frac{26\xi}{7} + \frac{104}{7} - \frac{\gamma}{10} \right).
\]

At this stage, from (30), (33) and (34), one obtains for a sufficiently small \( \varepsilon_1 \),

\[
E_{\bar{\omega}}(u, \nabla d, z_0, r) + F_{\bar{\omega}}(P - [P]_{x_0, r}, z_0, r) \leq C \varepsilon_1^2 \leq \varepsilon,
\]

with \( r = \rho^\xi \) and \( z_0 = 0 \). By using Corollary 1, one gets that \( z_0 \) is a regular point at this stage. This completes the proof of Theorem 4.2.

Using Theorem 4.2, we are now in a position to prove Theorem 1.2.

**Proof of Theorem 1.2** Notice that Theorem 4.2 implies that if \( z_0 \in \Sigma \) is a singular point, then for all sufficiently small \( 0 < \rho < r \), one has

\[
\rho^{\frac{\xi}{2}} - \gamma \varepsilon_1 \leq \int_{Q_{\rho}(z_0)} (|\nabla u|^2 + |\nabla^2 d|^2 + |u|^{\frac{4\xi}{7}} + |\nabla d|^{\frac{4\xi}{7}} + |P - [P]_{x_0, r}|^{\frac{\xi}{2}} + |\nabla P|^{\frac{\xi}{2}}) dx d\tau.
\]

Now, by fixing \( 5\rho < 1 \) small enough and consider the covering \( \{Q_{\rho}(z) : z = (x, t) \in \Sigma \cap D \} \). By the Vitali covering Lemma, there is a disjoint sub-family

\[
\{Q_{\rho}(z_i) : i = 1, 2, \cdots, M\}
\]
such that Σ ∩ D ⊆ ∪_{i=1}^{M} Q_{5\rho}(z_i). Summing the inequality above at z_i for i = 1, 2, ..., M yields
\begin{align*}
M^{1-\gamma/2} & \leq \sum_{i=1}^{M} \int_{Q_{5\rho}(z_i)} (|\nabla u|^2 + |\nabla^2 d|^2 + |u|^{\frac{10}{3}} + |\nabla d|^{\frac{10}{3}} + |P - [P]_{x_i,B\rho(x_i)}|^2 + |\nabla P|^2) dx \, dt \\
& \leq \int_{0}^{T} (|\nabla u|_{L^2(\mathbb{R}^3)}^2 + |\nabla^2 d|_{L^2(\mathbb{R}^3)}^2 + |u|_{L^{\frac{10}{3}}(\mathbb{R}^3)}^{\frac{10}{3}} + |\nabla d|_{L^{\frac{10}{3}}(\mathbb{R}^3)}^{\frac{10}{3}} + \|P\|_{L^2(\mathbb{R}^3)}^2 + |\nabla P|^2) dt \\
& \leq C \int_{0}^{T} (|\nabla u|_{L^2(\mathbb{R}^3)}^2 + |\nabla^2 d|_{L^2(\mathbb{R}^3)}^2 + |u|_{L^{\frac{10}{3}}(\mathbb{R}^3)}^{\frac{10}{3}} + |\nabla d|_{L^{\frac{10}{3}}(\mathbb{R}^3)}^{\frac{10}{3}}) dt \quad \text{(see [24] for more details)} \\
& \leq C \left(1 + \|u\|_{L^2(0,T,L^2(\mathbb{R}^3))}^2 + |\nabla d|_{L^2(0,T,L^2(\mathbb{R}^3))}^2\right) \int_{0}^{T} (|\nabla u|_{L^2(\mathbb{R}^3)}^2 + |\nabla^2 d|_{L^2(\mathbb{R}^3)}^2) dt \\
& =: \tilde{C} < \infty,
\end{align*}

where we have used the Calderón-Zygmund theorem, the interpolation inequality and the energy inequality (6). Let N(Σ ∩ D; ρ) denote the minimum number of parabolic cylinders Q_ρ(z) required to cover the set Σ ∩ D. Then one has
\begin{align*}
N(\Sigma \cap D; \rho) & \leq M \leq \tilde{C} \varepsilon^{-\frac{3}{1-\gamma}} \rho^{-\frac{3}{2}-\gamma},
\end{align*}

which implies that
\begin{align*}
\limsup_{\rho \to 0} \frac{\log N(\Sigma \cap D; \rho)}{-\log \rho} & \leq \frac{5}{3} - \gamma.
\end{align*}

Since from Theorem 4.2, γ can be arbitrarily close to \(\frac{560}{1539}\), this completes the proof of Theorem 1.2.

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E-mail address: liuqiao2005@163.com