Abstract

In this study, the numerical solutions of reaction-diffusion systems are investigated via the trigonometric quintic B-spline finite element collocation method. These equations appear in various disciplines in order to describe certain physical facts, such as pattern formation, autocatalytic chemical reactions and population dynamics. The Schnakenberg, Gray-Scott and Brusselator models are special cases of reaction-diffusion systems considered as numerical examples in this paper. For numerical purposes, Crank-Nicolson formulae are used for the time discretization and the resulting system is linearized by Taylor expansion. In the finite element method, a uniform partition of the solution domain is constructed for the space discretization. Over the mentioned mesh, dirac-delta function and trigonometric quintic B-spline functions are chosen as the weighted function and the bases functions, respectively. Thus, the reaction-diffusion system turns into an algebraic system which can be represented by a matrix equation so that the coefficients are block matrices containing a certain number of non-zero elements in each row. The method is tested on different problems. To illustrate the accuracy, error norms are calculated in the linear problem whereas the relative error is given in other nonlinear problems. Subject to the character of the nonlinear problems, the occurring spatial patterns are formed by the trajectories of the dependent variables. The degree of the base polynomial allows the method to be used in high-order differential equation solutions. The algorithm produces accurate results even when the time increment is larger. Therefore, the proposed Trigonometric Quintic B-spline Collocation method is an effective method which produces acceptable results for the solutions of reaction-diffusion systems.
1 Introduction

The reaction diffusion (RD) system is used to model chemical exchange reactions, the transport of ground water in an aquifer, pattern formation in the study of biology, chemistry and ecology. The RD system exhibits very rich dynamics behavior including periodic and quasi-periodic solutions. Theoretical studies have been developed to describe such dynamic behaviors. Most reaction-diffusion systems includes the nonlinear reaction term making it difficult to solve analytically. Attempts have been made to look for the numerical solutions to reveal more dynamic behaviors of the RD system.

The spline functions of various degrees are accompanied to construct numerical methods to solve differential equations of certain order, since the resulting matrix system is always diagonal and can be solved easily and approximate solutions having the degree accuracy of less than the degree of the spline functions, can be set up. High order continuous differentiable approximate solutions can be produced by way of using high order spline functions as solutions of the differential equations. B-splines are defined as a basis of the spline space [16]. Polynomial B-splines are extensively used for finding numerical solutions of differential equations, function approximation and computer-aided design. The numerical procedure based on the B-spline collocation method has been increasingly applied for nonlinear evolution equations in various fields of science. However, application of trigonometric B-spline collocation methods to nonlinear evolution problems is few in comparison with the collocation method based on polynomial B-spline functions. The numerical methods for solving types of ordinary differential equations with quadratic and cubic trigonometric B-spline are given by A. Nikolis [1, 6]. Linear two point boundary value problems of the order of two are solved using the trigonometric cubic B-spline(TCB) interpolation method [11]. Another numerical method employing the TCB is set up to solve a class of linear two-point singular boundary value problems in the study [12]. Recently, a collocation finite difference scheme based on the TCB has been developed for the numerical solution of a one-dimensional hyperbolic equation (wave equation) with a non-local conservation condition [13]. A new two-time level implicit technique based on the TCB, is proposed for the approximate solution of a nonclassical diffusion problem with a nonlocal boundary condition in the study [14]. A new three-time level implicit approach, based on the TCB is presented for the approximate solution of the Generalized Nonlinear Klein-Gordon equation with Dirichlet boundary conditions [15]. Some research in the literature [10] has established spline-based numerical approaches for solving reaction-diffusion equation systems but without the trigonometric B-spline, to our knowledge. In this paper, trigonometric quintic B-splines(TQB) are used to establish a collocation method with suggested numerical method being applied to find numerical solutions of a reaction-diffusion equation system. As a result, the present method makes it possible to approximate solutions as well as derivatives up to an order of four at each point of the problem domain.

When reaction-diffusion systems are studied, it can be understood that different species interact with each other, and also that in chemical reactions two different chemical substances generate new substances, for example. For modeling these types of events,
which have more than one dependent variable, differential equation systems have been used. One-dimensional time-dependent reaction-diffusion equation systems can be defined as follows:

\[
\begin{align*}
\frac{\partial U}{\partial t} &= D_u \frac{\partial^2 U}{\partial x^2} + F(U, V) \\
\frac{\partial V}{\partial t} &= D_v \frac{\partial^2 V}{\partial x^2} + G(U, V)
\end{align*}
\]

where \( U = U(x, t), V = V(x, t), \Omega \subset R^2 \) is a problem domain, \( D_u \) and \( D_v \) are the diffusion coefficients of \( U \) and \( V \) respectively, \( F \) and \( G \) are the growth and interaction functions that represents the reactions of the system. \( F \) and \( G \) are always nonlinear functions. A general one dimensional reaction-diffusion equation system which includes all models we mentioned in this paper, is expressed as:

\[
\begin{align*}
\frac{\partial U}{\partial t} &= a_1 \frac{\partial^2 U}{\partial x^2} + b_1 U + c_1 V + d_1 U^2 V + e_1 UV + m_1 UV^2 + n_1 \\
\frac{\partial V}{\partial t} &= a_2 \frac{\partial^2 V}{\partial x^2} + b_2 U + c_2 V + d_2 U^2 V + e_2 UV + m_2 UV^2 + n_2
\end{align*}
\]

The solution region of the problem \((-\infty, \infty)\) should be restricted as \((x_0, x_N)\) for computational purpose. In this case, system (2)’s initial conditions are either the homogeny Dirichlet boundary conditions

\[
\begin{align*}
U(x_0, t) = U(x_N, t) &= 0, \\
V(x_0, t) = V(x_N, t) &= 0,
\end{align*}
\]

or homogeny Neumann boundary conditions

\[
\begin{align*}
U_x(x_0, t) = U_x(x_N, t) &= 0, \\
V_x(x_0, t) = V_x(x_N, t) &= 0
\end{align*}
\]

will be used. Appropriate coefficients of the system (2) for each test problem will be selected depending on the characteristics of each model in the following sections and documented in Table 1:

| Test Problem | \( a_1 \) | \( a_2 \) | \( b_1 \) | \( b_2 \) | \( c_1 \) | \( c_2 \) | \( d_1 \) | \( d_2 \) | \( e_1 \) | \( e_2 \) | \( m_1 \) | \( m_2 \) | \( n_1 \) | \( n_2 \) |
|--------------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| Linear       | \( d \) | \( d \) | \(-a\)   | 0       | 1       | \(-b\)  | 0       | 0       | 0       | 0       | 0       | 0       | 0       | 0       |
| Brusselator  | \( \epsilon_1 \) | \( \epsilon_2 \) | \(-B + 1\) | \( B \) | 0       | 0       | 1       | \(-1\)  | 0       | 0       | 0       | 0       | A       | 0       |
| Schnakenberg | \( 1 \) | \( d \) | \(-\gamma\) | 0       | 0       | \( \gamma \) | \(-\gamma\) | 0       | 0       | 0       | 0       | 0       | \( \gamma a \) | \( \gamma b \) |
| Gray-Scott   | \( \epsilon_1 \) | \( \epsilon_2 \) | \(-f\) | 0       | 0       | \(-(f + k)\) | 0       | 0       | 0       | \(-1\)  | 1       | \( f \) | 0       | 0       |
2 The Trigonometric Quintic B-spline Collocation Method

Consider the solution space of the differential problem \([a = x_0, b = x_N]\) is partitioned into a mesh of uniform length \(h = x_{m+1} - x_m\) by knots \(x_m\) where \(m = -2, \ldots, N + 2\). On this partition, together with additional knots \(x_{N-2}, x_{N-1}, x_{N+1}, x_{N+2}\) outside the problem domain, the trigonometric quintic B-spline \(T_m^5(x)\) basis functions at knots is given by

\[
T_m^5(x) = \frac{1}{\theta} \begin{cases} 
\frac{p^5(x_{m-3})}{x \in [x_{m-3}, x_{m-2}]}, \\
-\frac{p^4(x_{m-3})p(x_{m-1}) - p^4(x_{m-3})p(x_m)}{x \in [x_{m-2}, x_{m-1}]}, \\
\frac{p^5(x_{m-3})p^4(x_{m-2}) + p^4(x_{m-3})p(x_{m-2})p(x_{m-3})}{x \in [x_{m-1}, x_m]}, \\
-\frac{p^4(x_{m-2})p(x_{m-1}) - p(x_{m-2})p(x_{m-1})p(x_{m})}{x \in [x_m, x_{m+1}]}, \\
\frac{p(x_{m})p(x_{m-1})p(x_{m+2})p(x_{m+3})}{x \in [x_{m+1}, x_{m+2}]}, \\
\end{cases}
\]

where the \(p(x_m), \Theta\) and \(m\) are:

\[
p(x_m) = \sin(\frac{x-x_m}{2}), \quad \Theta = \sin(\frac{5h}{2}) \sin(2h) \sin(\frac{3h}{2}) \sin(h) \sin(\frac{h}{2}),
\]

\(m = O(1) N\)

The \(T_m^5(x)\) functions and its principle derivatives vanish outside the region \([x_{m-3}, x_{m+3}]\). The set of those B-splines \(T_m^5(x)\), \(m = -2, \ldots, N + 2\) are a basis for the trigonometric spline space. An approximate solution \(U_N(x,t)\) and \(V_N(x,t)\) to the unknown solution \(U(x,t)\) and \(V(x,t)\) can be assumed of the forms

\[
U_N(x,t) = \sum_{i=-2}^{N+2} T_i^5(x) \delta_i(t) \quad V_N(x,t) = \sum_{i=-2}^{N+2} T_i^5(x) \gamma_i(t)
\]

where \(\delta_i\) and \(\gamma_i\) are time dependent parameters to be determined from the collocation points \(x_i, i = 0, \ldots, N\) with boundary and initial conditions.
\( T^5_m(x) \) trigonometric quintic B-spline functions are zero behind the interval \([x_{m-3}, x_{m+3}]\) and \( T^5_m(x) \) functions sequentially covers six elements in the interval \([x_{m-3}, x_{m+3}]\) so that, each \([x_m, x_{m+1}]\) finite element is covered by the six \( T^5_{m-2}, T^5_{m-1}, T^5_m, T^5_{m+1}, T^5_{m+2}, \) and \( T^5_{m+3} \) trigonometric quintic B-spline. In this case \((6)\) the approach is given as:

\[
U_N(x, t) = \sum_{i=m-2}^{m+3} T^5_i(x) \delta_i = T^5_{m-2}(x) \delta_{m-2} + T^5_{m-1}(x) \delta_{m-1} + T^5_m(x) \delta_m + T^5_{m+1}(x) \delta_{m+1} + T^5_{m+2}(x) \delta_{m+2} + T^5_{m+3}(x) \delta_{m+3}
\]

\[
V_N(x, t) = \sum_{i=m-2}^{m+3} T^5_i(x) \gamma_i = T^5_{m-2}(x) \gamma_{m-2} + T^5_{m-1}(x) \gamma_{m-1} + T^5_m(x) \gamma_m + T^5_{m+1}(x) \gamma_{m+1} + T^5_{m+2}(x) \gamma_{m+2} + T^5_{m+3}(x) \gamma_{m+3}
\]

(7)

In these numerical approaches, the approximate solutions at the knots can be written in terms of the time parameters using \( T^5_m(x) \) and Eq. \((6)\). After this, by also making necessary calculations, we can write \( T^5_m(x) \) functions for \( U_m \) and \( V_m \) and its first, second, third and fourth derivatives at the knots \( x_m \) are given in terms of parameters by the following relationships.

\[
U_m = \alpha_1 \delta_{m-2} + \alpha_2 \delta_{m-1} + \alpha_3 \delta_m + \alpha_2 \delta_{m+1} + \alpha_1 \delta_{m+2}
\]

\[
U'_m = -\alpha_4 \delta_{m-2} - \alpha_5 \delta_{m-1} + \alpha_5 \delta_{m+1} - \alpha_4 \delta_{m+2}
\]

\[
U''_m = \alpha_6 \delta_{m-2} + \alpha_7 \delta_{m-1} + \alpha_8 \delta_m + \alpha_7 \delta_{m+1} + \alpha_6 \delta_{m+2}
\]

\[
U'''_m = -\alpha_9 \delta_{m-2} + \alpha_10 \delta_{m-1} - \alpha_9 \delta_{m+1} - \alpha_9 \delta_{m+2}
\]

\[
U''''_m = \alpha_11 \delta_{m-2} + \alpha_12 \delta_{m-1} + \alpha_13 \delta_m + \alpha_12 \delta_{m+1} + \alpha_11 \delta_{m+2}
\]

(8)

\[
V_m = \alpha_1 \gamma_{m-2} + \alpha_2 \gamma_{m-1} + \alpha_3 \gamma_m + \alpha_2 \gamma_{m+1} + \alpha_1 \gamma_{m+2}
\]

\[
V'_m = -\alpha_4 \gamma_{m-2} - \alpha_5 \gamma_{m-1} + \alpha_5 \gamma_{m+1} + \alpha_4 \gamma_{m+2}
\]

\[
V''_m = \alpha_6 \gamma_{m-2} + \alpha_7 \gamma_{m-1} + \alpha_8 \gamma_m + \alpha_7 \gamma_{m+1} + \alpha_6 \gamma_{m+2}
\]

\[
V'''_m = -\alpha_9 \gamma_{m-2} + \alpha_10 \gamma_{m-1} - \alpha_9 \gamma_{m+1} + \alpha_9 \gamma_{m+2}
\]

\[
V''''_m = \alpha_11 \gamma_{m-2} + \alpha_12 \gamma_{m-1} + \alpha_13 \gamma_m + \alpha_12 \gamma_{m+1} + \alpha_11 \gamma_{m+2}
\]

where the coefficients are:
\[
\alpha_1 = \frac{\sin^5\left(\frac{h}{2}\right)}{\Theta}
\]
\[
\alpha_2 = \frac{2\sin^5\left(\frac{h}{2}\right)\cos\left(\frac{h}{2}\right)(16\cos^2\left(\frac{h}{2}\right) - 3)}{\Theta}
\]
\[
\alpha_3 = \frac{2(1 + 48\cos^4\left(\frac{h}{2}\right) - 16\cos^2\left(\frac{h}{2}\right)\sin^5\left(\frac{h}{2}\right))}{\Theta}
\]
\[
\alpha_4 = \frac{\frac{5}{2}\sin^4\left(\frac{h}{2}\right)\cos\left(\frac{h}{2}\right)}{\Theta}
\]
\[
\alpha_5 = \frac{5\sin^4\left(\frac{h}{2}\right)\cos^2\left(\frac{h}{2}\right)(8\cos^2\left(\frac{h}{2}\right) - 3)}{\Theta}
\]
\[
\alpha_6 = \frac{\frac{5}{4}\sin^3\left(\frac{h}{2}\right)(5\cos^2\left(\frac{h}{2}\right) - 1)}{\Theta}
\]
\[
\alpha_7 = \frac{\frac{5}{2}\sin^3\left(\frac{h}{2}\right)(\cos\left(\frac{h}{2}\right)(-15\cos^2\left(\frac{h}{2}\right) + 3 + 16\cos^4\left(\frac{h}{2}\right)))}{\Theta}
\]
\[
\alpha_8 = \frac{-\frac{5}{2}\sin^3\left(\frac{h}{2}\right)(16\cos^6\left(\frac{h}{2}\right) - 5\cos^6\left(\frac{h}{2}\right) + 1)}{\Theta}
\]
\[
\alpha_9 = \frac{\frac{5}{8}\sin^2\left(\frac{h}{2}\right)\cos\left(\frac{h}{2}\right)(25\cos^2\left(\frac{h}{2}\right) - 13)}{\Theta}
\]
\[
\alpha_{10} = \frac{-\frac{5}{4}\sin^2\left(\frac{h}{2}\right)(\cos\left(\frac{h}{2}\right)(8\cos^4\left(\frac{h}{2}\right) - 35\cos^2\left(\frac{h}{2}\right) + 15))}{\Theta}
\]
\[
\alpha_{11} = \frac{\frac{5}{16}(125\cos^4\left(\frac{h}{2}\right) - 114\cos^2\left(\frac{h}{2}\right) + 13)\sin\left(\frac{h}{2}\right))}{\Theta}
\]
\[
\alpha_{12} = \frac{-\frac{5}{8}\sin\left(\frac{h}{2}\right)\cos\left(\frac{h}{2}\right)(176\cos^6\left(\frac{h}{2}\right) - 137\cos^4\left(\frac{h}{2}\right) - 6\cos^2\left(\frac{h}{2}\right) + 15)}{\Theta}
\]
\[
\alpha_{13} = \frac{\frac{5}{8}(92\cos^6\left(\frac{h}{2}\right) - 117\cos^4\left(\frac{h}{2}\right) + 62\cos^2\left(\frac{h}{2}\right) - 13)(-1 + 4\cos^2\left(\frac{h}{2}\right)\sin\left(\frac{h}{2}\right))}{\Theta}
\]

The Crank–Nicholson scheme
\[
U_t = \frac{U^{n+1} - U^n}{\Delta t}, \quad U = \frac{U^{n+1} + U^n}{2}
\]
\[
V_t = \frac{V^{n+1} - V^n}{\Delta t}, \quad V = \frac{V^{n+1} + V^n}{2}
\]
is used to discretize time variables of the unknown $U$ and $V$ and their derivatives, to have the time integrated reaction-diffusion equation system:

$$
\begin{align*}
\frac{U^{n+1} - U^n}{\Delta t} &= -a_1 \frac{U_{xx}^{n+1} + U_{xx}^n}{2} - b_1 \frac{U_{xx}^{n+1} + U_{xx}^n}{2} - c_1 \frac{V_{xx}^{n+1} + V_{xx}^n}{2} - d_1 \frac{(U^2V)^{n+1} + (U^2V)^n}{2} \\
\frac{V^{n+1} - V^n}{\Delta t} &= -a_2 \frac{V_{xx}^{n+1} + V_{xx}^n}{2} - b_2 \frac{U_{xx}^{n+1} + U_{xx}^n}{2} - c_2 \frac{V_{xx}^{n+1} + V_{xx}^n}{2} - d_2 \frac{(U^2V)^{n+1} + (U^2V)^n}{2} \\
\end{align*}
$$

where $U^{n+1} = U(x,t)$ and $V^{n+1} = V(x,t)$ are the solutions of the equations at the $(n+1)$th time level. Here $t_{n+1} = t_n + \Delta t$ and $\Delta t$ is the time step, superscripts denote the $n$th level $t_n = n\Delta t$.

The nonlinear terms $(U^2V)^{n+1}$, $(UV^2)^{n+1}$ and $(UV)^{n+1}$ in equation (10) is linearized by using the following forms (11):

$$
\begin{align*}
(U^2V)^{n+1} &= U^{n+1}U^nV^n + U^nU^{n+1}V^n + U^nU^nV^{n+1} - 2U^nU^nV^u \\
(UV^2)^{n+1} &= U^{n+1}V^nV^n + U^nV^{n+1}V^n + U^nV^nV^{n+1} - 2U^nV^nV^u \\
(UV)^{n+1} &= U^{n+1}V^n + U^nV^{n+1} - U^nV^n \\
\end{align*}
$$

When we substitute (11) in (10), the linearized general model equation system takes the form as shown below,

$$
\begin{align*}
-a_1 \frac{U_{xx}^{n+1} + \beta_1 U^{n+1} + \beta_2 V^{n+1}}{2} &= a_1 \frac{U_{xx}^n + \beta_3 U^n + \beta_4 V^n}{2} + n_1 \\
-a_2 \frac{V_{xx}^{n+1} + \beta_5 U^{n+1} + \beta_6 V^{n+1}}{2} &= a_2 \frac{V_{xx}^n + \beta_7 U^n + \beta_8 V^n}{2} + n_2 \\
\end{align*}
$$

where

$$
\begin{align*}
\beta_1 &= \frac{1}{\Delta t} - \frac{b_1}{2} - \frac{d_1}{2}U^nV^n - \frac{c_1}{2}V^n - \frac{m_1}{2}(V^n)^2 \\
\beta_2 &= \frac{1}{\Delta t} + \frac{b_1}{2} - \frac{d_1}{2}U^nV^n - \frac{e_1}{2}U^n - \frac{m_1}{2}U^nV^n \\
\beta_3 &= \frac{1}{\Delta t} + \frac{c_1}{2} - \frac{d_1}{2}(U^n)^2 \\
\beta_4 &= \frac{1}{\Delta t} - \frac{e_1}{2}(U^n)^2 \\
\beta_5 &= \frac{-b_2}{2} - \frac{d_2}{2}U^nV^n - \frac{e_2}{2}V^n - \frac{m_2}{2}(V^n)^2 \\
\beta_6 &= \frac{-b_2}{2} - \frac{e_2}{2}U^n - \frac{m_2}{2}U^nV^n \\
\beta_7 &= \frac{-b_2}{2} - \frac{e_2}{2}U^n - \frac{m_2}{2}U^nV^n \\
\beta_8 &= \frac{-b_2}{2} - \frac{e_2}{2}U^n - \frac{m_2}{2}U^nV^n \\
\end{align*}
$$
To discrete the model system (2) fully by space respectively, we substitute the approximate solution (8) into (12) yielding the fully-discretized equations.

\[
\begin{align*}
    \nu_{m1}\delta_{m+1}^{n+1} + \nu_{m2}\gamma_{m-2}^{n+1} + \nu_{m3}\delta_{m-1}^{n+1} + \nu_{m4}\gamma_{m-1}^{n+1} + \nu_{m5}\delta_{m}^{n+1} + \nu_{m6}\gamma_{m}^{n+1} + \\
    \nu_{m7}\delta_{m+1}^{n+1} + \nu_{m8}\gamma_{m-1}^{n+1} + \nu_{m9}\delta_{m+2}^{n+1} + \nu_{m10}\gamma_{m+2}^{n+1} = \\
    \nu_{m11}\delta_{m-2}^{n} + \nu_{m12}\gamma_{m-2}^{n} + \nu_{m13}\delta_{m-1}^{n} + \nu_{m14}\gamma_{m-1}^{n} + \nu_{m15}\delta_{m}^{n} + \nu_{m16}\gamma_{m}^{n} + \\
    \nu_{m17}\delta_{m+1}^{n} + \nu_{m18}\gamma_{m+1}^{n} + \nu_{m19}\delta_{m+2}^{n} + \nu_{m20}\gamma_{m+2}^{n} + n_1 \\
    \nu_{m21}\delta_{m-2}^{n+1} + \nu_{m22}\gamma_{m-2}^{n+1} + \nu_{m23}\delta_{m-1}^{n+1} + \nu_{m24}\gamma_{m-1}^{n+1} + \nu_{m25}\delta_{m}^{n+1} + \nu_{m26}\gamma_{m}^{n+1} + \\
    \nu_{m27}\delta_{m+1}^{n+1} + \nu_{m28}\gamma_{m+1}^{n+1} + \nu_{m29}\delta_{m+2}^{n+1} + \nu_{m30}\gamma_{m+2}^{n+1} = \\
    \nu_{m31}\delta_{m-2}^{n} + \nu_{m32}\gamma_{m-2}^{n} + \nu_{m33}\delta_{m-1}^{n} + \nu_{m34}\gamma_{m-1}^{n} + \nu_{m35}\delta_{m}^{n} + \nu_{m36}\gamma_{m}^{n} + \\
    \nu_{m37}\delta_{m+1}^{n} + \nu_{m38}\gamma_{m+1}^{n} + \nu_{m39}\delta_{m+2}^{n} + \nu_{m40}\gamma_{m+2}^{n} + n_2
\end{align*}
\]

where the \( \nu_m \) coefficients are:

\[
\begin{align*}
    \nu_{m1} &= \beta_{m1}a_1 - \frac{a_1}{2}\alpha_6 \\
    \nu_{m2} &= \beta_{m2}a_1 \\
    \nu_{m3} &= \beta_{m1}a_2 - \frac{a_1}{2}\alpha_7 \\
    \nu_{m4} &= \beta_{m2}a_2 \\
    \nu_{m5} &= \beta_{m1}a_3 - \frac{a_1}{2}\alpha_8 \\
    \nu_{m6} &= \beta_{m2}a_3 \\
    \nu_{m7} &= \beta_{m1}a_4 - \frac{a_1}{2}\alpha_7 \\
    \nu_{m8} &= \beta_{m2}a_4 \\
    \nu_{m9} &= \beta_{m1}a_5 - \frac{a_1}{2}\alpha_6 \\
    \nu_{m10} &= \beta_{m2}a_5 \\
    \nu_{m11} &= \beta_{m3}a_1 + \frac{a_1}{2}\alpha_6 \\
    \nu_{m12} &= \beta_{m4}a_1 \\
    \nu_{m13} &= \beta_{m3}a_2 + \frac{a_1}{2}\alpha_7 \\
    \nu_{m14} &= \beta_{m4}a_2 \\
    \nu_{m15} &= \beta_{m3}a_3 + \frac{a_1}{2}\alpha_8 \\
    \nu_{m16} &= \beta_{m4}a_3 \\
    \nu_{m17} &= \beta_{m3}a_4 + \frac{a_1}{2}\alpha_7 \\
    \nu_{m18} &= \beta_{m4}a_4 \\
    \nu_{m19} &= \beta_{m3}a_5 + \frac{a_1}{2}\alpha_6 \\
    \nu_{m20} &= \beta_{m4}a_5 \\
    \nu_{m31} &= \beta_{m7}a_1 \\
    \nu_{m32} &= \beta_{m8}a_1 - \frac{a_2}{2}\alpha_6 \\
    \nu_{m33} &= \beta_{m7}a_2 \\
    \nu_{m34} &= \beta_{m8}a_2 - \frac{a_2}{2}\alpha_7 \\
    \nu_{m35} &= \beta_{m7}a_3 \\
    \nu_{m36} &= \beta_{m8}a_3 - \frac{a_2}{2}\alpha_8 \\
    \nu_{m37} &= \beta_{m7}a_4 \\
    \nu_{m38} &= \beta_{m8}a_4 - \frac{a_2}{2}\alpha_7 \\
    \nu_{m39} &= \beta_{m7}a_5 \\
    \nu_{m40} &= \beta_{m8}a_5 - \frac{a_2}{2}\alpha_6
\end{align*}
\]

The system (13) can be converted into the following matrix system:

\[
A\mathbf{x}^{n+1} = B\mathbf{x}^n + F
\]
The system (16) is consists of a $2N + 2$ linear equation in $2N + 10$ unknown parameters with $x^{n+1}, x^n$ and $F$ being the vectors as shown below:

$$
\begin{align*}
\mathbf{x}^{n+1} &= [\delta_{-2}^{n+1}, \gamma_{-2}^{n+1}, \delta_{-1}^{n+1}, \gamma_{-1}^{n+1}, \delta_{0}^{n+1}, \gamma_{0}^{n+1}, \ldots, \delta_{N+1}^{n+1}, \gamma_{N+1}^{n+1}, \delta_{N+2}^{n+1}, \gamma_{N+2}^{n+1}]^T \\
\mathbf{x}^n &= [\delta_{-2}^{n}, \gamma_{-2}^{n}, \delta_{-1}^{n}, \gamma_{-1}^{n}, \delta_{0}^{n}, \gamma_{0}^{n}, \ldots, \delta_{N+1}^{n}, \gamma_{N+1}^{n}, \delta_{N+2}^{n}, \gamma_{N+2}^{n}]^T \\
F &= [n_1, n_2, n_1, n_2, \ldots, n_1, n_2]^T
\end{align*}
$$

To obtain a unique solution an additional eight constraints are needed. While $m = 0$ and $m = N$ by imposing the Dirichlet boundary conditions or the Neumann boundary conditions this will lead us to new relationships to eliminate parameters

$$
\delta_{-2}, \delta_{-1}, \delta_{N+1}, \delta_{N+2}, \gamma_{-1}, \gamma_{N+1}, \gamma_{N+2}
$$

from the system (15). When we eliminate these parameters the resulting $(2N + 2) \times (2N + 2)$ matrix system can be solved by the Gauss elimination algorithm.

The initial parameters of $\mathbf{x}^0 = (\delta_{-2}^0, \gamma_{-2}^0, \delta_{-1}^0, \gamma_{-1}^0, \delta_{0}^0, \gamma_{0}^0, \ldots, \delta_{N+1}^0, \gamma_{N+1}^0, \delta_{N+2}^0, \gamma_{N+2}^0)$ must be found to start the iteration process by using both initial and boundary conditions. The recurrence relationship (15) gives the time evolution of vector $\mathbf{x}^n$. Thus, the nodal values $U_N(x, t)$ and $V_N(x, t)$ can be computed via the equations (8) at the knots.

### 2.1 Results of The Numerical Solutions

In this section, we will compare the efficiency and accuracy of the suggested method on the given reaction-diffusion equation system models. The obtained results for each model will compare with [10] and [3]. The accuracy of the schemes is measured in terms of the following discrete error norm
\[ L_2 = |U - U_N|_2 = \sqrt{h \sum_{j=0}^{N} (U_j - (U_N)_j^2)} \text{ and } L_\infty = |U - U_N|_\infty = \max_j |U_j - (U_N)_j^n|. \]

The relative error \( \sqrt{\sum_{j=0}^{N} |U_j^{n+1} - U_j^n|^2 / \sum_{j=0}^{N} |U_j^{n+1}|} \) is used to measure errors of solutions of the reaction-diffusion systems that do not have an analytic solution.

### 2.1.1 Linear Problem

It is stated that the terms \( F(U, V) \) and \( G(U, V) \) are always nonlinear in the system (1). However, it is not possible to calculate error norms because of the limitations of the analytical solutions of the nonlinear system. The linear problem has been solved to examine error norms for testing this method:

\[
\begin{align*}
\frac{\partial U}{\partial t} &= d \frac{\partial^2 U}{\partial x^2} - aU + V \\
\frac{\partial V}{\partial t} &= d \frac{\partial^2 V}{\partial x^2} - bV. \tag{17}
\end{align*}
\]

The given equation system described above is a linear reaction-diffusion system, which has analytical solutions given as:

\[
\begin{align*}
U(x,t) &= (e^{-(a+d)t} + e^{-(b+d)t}) \cos(x), \\
V(x,t) &= (a - b)(e^{-(b+d)t}) \cos(x). \tag{18}
\end{align*}
\]

Solutions were obtained by solving the reaction-diffusion system (17) in this section. Three different cases were considered in numerical computation of coefficients in the system (17). This system’s initial conditions can be obtained, when \( t = 0 \) in (18) the solutions. When a solution region is selected as \((0, \pi/2)\) interval, the boundary conditions are described as:

\[
\begin{align*}
U_x(0,t) &= 0 & U(\pi/2,t) &= 0, \\
V_x(0,t) &= 0 & V(\pi/2,t) &= 0. \tag{19}
\end{align*}
\]

In numerical calculations, the programme is going to run up to time \( t = 1 \) for various \( N \) and \( \Delta t \) and the reaction and diffusion mechanism is examined for different selections of constants \( a, b, \) and \( d \). The error values \( L_2 \) and \( L_\infty \) that have emerged in the solution, are presented in the tables.

Firstly, the equation system (17) coefficients are chosen as \( a = 0.1, b = 0.01 \) and \( d = 1 \) which is a diffusion dominated case. The boundary and initial conditions are chosen to coincide with the polynomial quintic B-spline collocation method (PQBCM) [10]. The programme is run up to \( t = 1 \) and the obtained results for \( U \), in terms of \( L_2 \) and \( L_\infty \) norms are given in Table 3.

In Table 3, \( L_2 \) and \( L_\infty \) error norms are calculated for both \( U \) and \( V \), for \( N = 512 \) and various \( \Delta t \) with results of [10] and [3] is also given in the same table. When Table 3 is examined, it seems that, the accuracy of the obtained results for function \( V \) are more
efficient than obtained results for function $U$. When we compare the results, the proposed method has better accuracy against the other references under the same conditions.

Table 3: Error norms $L_2$ and $L_\infty$ for diffusion dominant case for $a=0.1$, $b=0.01$, $d=1$

| $N$ | $\Delta t$ | $U$ | $V$ | $U$ | $V$ |
|-----|------------|-----|-----|-----|-----|
|     | $L_2 \times 10^4$ | $L_\infty \times 10^4$ | $L_2 \times 10^6$ | $L_\infty \times 10^6$ | $L_2 \times 10^4$ | $L_\infty \times 10^6$ |
| 512 | 0.005 | 0.008090 | 0.009120 | 0.029344 | 0.033079 | 0.015123 | 0.017048 | 0.062416 | 0.070361 |
|     | 0.01 | 0.053460 | 0.060265 | 0.216594 | 0.241162 | 0.060493 | 0.068193 | 0.249667 | 0.281444 |
|     | 0.02 | 0.234949 | 0.264853 | 0.965627 | 1.085530 | 0.241983 | 0.272782 | 0.998702 | 1.125815 |
|     | 0.04 | 0.961033 | 1.083353 | 3.962253 | 4.466566 | 0.968068 | 1.091283 | 3.995334 | 4.503855 |
| CN-MG method [3] | | | | | | | | |

Secondly, the constants of system equation (17) are selected as $a=2$, $b=1$, $d=0.001$ which is a reaction dominated case. The programme is run up to $t=1$, and the obtained results in terms of $L_2$ and $L_\infty$ norms are given in Table 4.

In Table 4, $L_2$ and $L_\infty$ error norms are calculated both for $U$ and $V$, for $N=512$ and various $\Delta t$ and the results of [10] and [3] are given in the same table.

Table 4: Error norms $L_2$ and $L_\infty$ for reaction dominated case for $a=2$, $b=1$, $d=0.001$

| $N$ | $\Delta t$ | $U$ | $V$ | $U$ | $V$ |
|-----|------------|-----|-----|-----|-----|
|     | $L_2 \times 10^4$ | $L_\infty \times 10^4$ | $L_2 \times 10^6$ | $L_\infty \times 10^6$ | $L_2 \times 10^4$ | $L_\infty \times 10^6$ |
| 512 | 0.005 | 0.026827 | 0.030241 | 0.068087 | 0.076573 | 0.026832 | 0.030247 | 0.068124 | 0.076795 |
|     | 0.01 | 0.107324 | 0.120984 | 0.272462 | 0.307141 | 0.107329 | 0.120989 | 0.272499 | 0.307183 |
|     | 0.02 | 0.429339 | 0.483984 | 1.089996 | 1.228729 | 0.429344 | 0.483990 | 1.090033 | 1.228771 |
|     | 0.04 | 1.717837 | 1.936481 | 4.360663 | 4.915683 | 1.717842 | 1.936487 | 4.360700 | 4.915725 |
| CN-MG method [3] | | | | | | | | |

Last, we will obtain a numerical solution of the reaction-diffusion equation for $a=100$, $b=1$, $d=0.001$ which is a reaction dominated case with stiff reaction.

In Table 5, $L_2$ and $L_\infty$ error norms are calculated both for $U$ and $V$, for $N=512$ and various $\Delta t$.
Table 5: Error norms $L_2$ and $L_\infty$ for diffusion dominated case with stiff reaction for $a = 100$, $b = 1$, $d = 0.001$

| TQB                  | Polynomial quintic B-spline,\[10\] |
|----------------------|-------------------------------------|
| $N$                  | $\Delta t$  | $U$ | $V$ | $U$ | $V$ |
| $L_2 \times 10^5$    | $L_\infty \times 10^5$ | $L_2 \times 10^4$ | $L_\infty \times 10^4$ | $L_2 \times 10^3$ | $L_\infty \times 10^3$ |
| 512                  | 0.005       | 0.068087 | 0.076753 | 0.067406 | 0.075986 | 0.067443 | 0.076027 |
|                      | 0.01        | 0.272462 | 0.307141 | 0.269738 | 0.30407 | 0.272499 | 0.307183 | 0.269774 | 0.304111 |
|                      | 0.02        | 1.089996 | 1.228729 | 1.079096 | 1.216442 | 1.090033 | 1.228771 | 1.079133 | 1.216484 |
|                      | 0.04        | 4.360663 | 4.915684 | 4.317057 | 4.866527 | 4.360700 | 4.915725 | 4.317093 | 4.866568 |
| CN-MG method \[3\]   |                         | 0.0760 |                             | 0.304 |                             | 1.22 |                             | 4.87 |

2.1.2 Nonlinear Problem (Brusselator Model)

The Brusselator model is a general nonlinear reaction-diffusion system that models predicting oscillations in chemical reactions. The system was first presented by Prigogine and Lefever \[5\] showing two variable autocatalytic reactions. This is one of the simplest reaction-diffusion equations exhibiting Turing instability, and that large-scale studies have been conducted on this model with the system being investigated both analytically and numerically. The general reaction-diffusion equation for this model given as:

\[
\begin{align*}
\frac{\partial U}{\partial t} &= \epsilon_1 \frac{\partial^2 U}{\partial x^2} + Au^2V - (B + 1)U \\
\frac{\partial V}{\partial t} &= \epsilon_2 \frac{\partial^2 V}{\partial x^2} + BU - U^2V
\end{align*}
\]  \quad (20)

where $\epsilon_i$, $i = 1, 2$ are diffusion constants, $x$ is the spatial coordinate and $U, V$ are functions of $x$ and $t$ representing concentrations. The initial conditions are selected similar to the reference \[7\].

\[
U(x, 0) = 0.5, \quad V(x, 0) = 1 + 5x
\]  \quad (21)

and the additional boundary conditions

\[
\begin{align*}
U_{xx}(x_0, t) &= 0, & U_{xx}(x_N, t) &= 0, \\
V_{xx}(x_0, t) &= 0, & V_{xx}(x_N, t) &= 0.
\end{align*}
\]

In the equation system (20), the coefficients are taken as $\epsilon_1 = \epsilon_2 = 10^{-4}$, $A = 1$, $B = 3.4$. The solutions are obtained in the region $x \in [0, 1]$, and the program is run by the time $t = 15$; for space discretization $N = 200$ split point and for time discretization $\Delta t = 0.01$ time step is used. The solutions under these selections, are given in Fig. 1 and
Fig. 2. which show changes of density of the functions. When wave action is examined, we observe that both $U$ and $V$ exhibit periodic wave motion under these conditions.

Figure 1: Periodic wave motion for $U$ for $N = 200 \Delta t = 0.01$

Figure 2: Periodic wave motion for $V$ for $N = 200 \Delta t = 0.01$

The density values for periodic motion are given in Table 6. We see that this wave is observed as a period of about 7.8; whereas the period 7.7 is found when the polynomial quintic B-spline collocation algorithm is implemented (7)

| Density | $t$ | $x = 0.0$ | $x = 0.2$ | $x = 0.4$ | $x = 0.6$ | $x = 0.8$ | $x = 1.0$ |
|---------|-----|-----------|-----------|-----------|-----------|-----------|-----------|
| $U$     | 3   | 0.284595  | 0.317799  | 0.377380  | 0.604709  | 1.623703  | 0.691906  |
|         | 10.8| 0.344555  | 0.321243  | 0.376194  | 0.605486  | 1.715194  | 0.716792  |
|         | 6   | 0.400865  | 0.687572  | 2.884364  | 0.549937  | 0.323697  | 0.348838  |
|         | 13.8| 0.398971  | 0.680057  | 2.911740  | 0.533798  | 0.322405  | 0.347582  |
| $V$     | 3   | 3.363723  | 4.250910  | 5.066610  | 5.546754  | 1.650507  | 2.507119  |
|         | 10.8| 3.309473  | 4.240150  | 5.062313  | 5.651837  | 1.591938  | 2.473710  |
|         | 6   | 5.258678  | 5.632343  | 1.073700  | 2.739517  | 4.300681  | 4.755329  |
|         | 13.8| 5.241915  | 5.634312  | 1.065232  | 2.769906  | 4.269058  | 4.737755  |
Table 7: Density values for periodic motion for quintic B-spline [10]

| Density | t   | x = 0.0   | x = 0.2   | x = 0.4   | x = 0.6   | x = 0.8   | x = 1.0   |
|---------|-----|-----------|-----------|-----------|-----------|-----------|-----------|
| U       | 3   | 0.284657  | 0.317966  | 0.377959  | 0.612881  | 1.519483  | 0.648434  |
|         | 10.7| 0.347747  | 0.321168  | 0.376204  | 0.611218  | 1.626310  | 0.680742  |
|         | 6   | 0.401741  | 0.706734  | 2.71642   | 0.510302  | 0.326204  | 0.352411  |
|         | 13.7| 0.398904  | 0.691408  | 2.769059  | 0.500480  | 0.324523  | 0.350579  |
| V       | 3   | 3.363896  | 4.251219  | 5.066734  | 5.537413  | 1.732740  | 2.580615  |
|         | 10.7| 3.299664  | 4.233913  | 5.056668  | 5.637796  | 1.659946  | 2.534846  |
|         | 6   | 5.257254  | 5.606791  | 1.137215  | 2.825295  | 4.355469  | 4.798749  |
|         | 13.7| 5.234725  | 5.613815  | 1.119445  | 2.846165  | 4.317357  | 4.774541  |

2.1.3 Nonlinear Problem (Schnakenberg Model)

The Schnakenberg model is a well-known reaction-diffusion model which is a simplified version of the Brusselator model. It is a relatively easy system for modeling the reaction-diffusion mechanism. There are many studies in the literature on this model. Firstly it is modeled by Schakenberg [4] and given as:

\[
\begin{align*}
\frac{\partial U}{\partial t} &= \frac{\partial^2 U}{\partial x^2} + \gamma(a - U + U^2V) \\
\frac{\partial V}{\partial t} &= d \frac{\partial^2 V}{\partial x^2} + \gamma(b - U^2V)
\end{align*}
\] (22)

where \( U \) and \( V \) denote the concentration of activator and inhibitor respectively, \( d \) is diffusion coefficient, \( \gamma \), \( a \) and \( b \) are rate constants of the biochemical reactions. The oscillation problem is taken into account for the Schnakenberg Model. Accordingly, the parameters for system (22) are selected as \( a = 0.126779, b = 0.792366, d = 10 \) and \( \gamma = 10^4 \). The problem’s initial conditions:

\[
\begin{align*}
U(x, 0) &= 0.919145 + 0.001 \sum_{j=1}^{25} \frac{\cos(2\pi jx)}{j} \\
V(x, 0) &= 0.937903 + 0.001 \sum_{j=1}^{25} \frac{\cos(2\pi jx)}{j}
\end{align*}
\] (23)

are on the interval \([-1, 1]\). The boundary conditions left, right and additional boundary conditions are:

\[
\begin{align*}
U_x(x_0, t) &= 0 \quad U_x(x_N, t) = 0, \\
V_x(x_0, t) &= 0 \quad V_x(x_N, t) = 0, \\
U_{xxx}(x_0, t) &= 0 \quad U_{xxx}(x_N, t) = 0, \\
V_{xxx}(x_0, t) &= 0 \quad V_{xxx}(x_N, t) = 0.
\end{align*}
\]

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Computations are performed to the $t = 2.5$ for space/time combinations given in Table 8. Obtained relative error values are given in Table 8 together with the results of quintic B-spline collocation method [10].

| $\Delta t$   | Nu. of steps | $U$     | $U_{[10]}$     | $V$     | $V_{[10]}$     |
|--------------|--------------|---------|----------------|---------|----------------|
| $5 \times 10^{-6}$ | 500000       | 0       | $5.7160 \times 10^{-14}$ | $5.4418 \times 10^{-17}$ | $5.4564 \times 10^{-14}$ |
| $5 \times 10^{-5}$ | 50000        | $6.2202 \times 10^{-17}$ | $1.5653 \times 10^{-10}$ | $1.6794 \times 10^{-16}$ | $1.1105 \times 10^{-10}$ |
| $1 \times 10^{-4}$ | 25000        | $1.7593 \times 10^{-16}$ | $9.8744 \times 10^{-10}$ | $2.4423 \times 10^{-16}$ | $8.8599 \times 10^{-10}$ |
| $1.20 \times 10^{-4}$ | 20833        | $1.5668 \times 10^{-16}$ | $1.5055 \times 10^{-09}$ | $2.2996 \times 10^{-16}$ | $1.3790 \times 10^{-09}$ |
| $1.32 \times 10^{-4}$ | 18939        | $1.4610 \times 10^{-16}$ | $1.0564 \times 10^{-11}$ | $2.9664 \times 10^{-16}$ | $1.0301 \times 10^{-11}$ |
| $1 \times 10^{-3}$ | 2500         | $2.5895 \times 10^{-14}$ | -                | $2.0341 \times 10^{-14}$ | -                |
| $2 \times 10^{-3}$ | 1250         | $5.4591 \times 10^{-09}$ | -                | $3.9448 \times 10^{-09}$ | -                |
| $5 \times 10^{-3}$ | 500          | $5.4960 \times 10^{-06}$ | -                | $4.7003 \times 10^{-06}$ | -                |

As can be seen from Table 8, the algorithm produces accurate results even when the time increment is larger. The Figure 3 was drawn to show the oscillation movements for values $\Delta t = 5 \times 10^{-5}$, $N = 100$ and $N = 200$. It is shown in Fig. 3 that the functions $U$ and $V$ make 9 oscillations when $N = 200$ and $N = 100$. This result with the references [8] and [9] shows that a finer mesh is necessary for accurate solutions.

Fig. 3: The oscillation movement for $N = 100$ and $N = 200$ in the moment $t = 2.5$

2.1.4 Nonlinear Problem (Gray-Scott Model)

The Gray-Scott model is a reaction-diffusion system which models the forming of certain spatial patterns by a few chemical species, that exit in the nature. It was put forward by Gray and Scott [2] and the reaction-diffusion system is given as:
\[ \frac{\partial U}{\partial t} = \varepsilon_1 \frac{\partial^2 U}{\partial x^2} - U^2 V + f(1 - U) \]
\[ \frac{\partial V}{\partial t} = \varepsilon_2 \frac{\partial^2 V}{\partial x^2} + U^2 V - (f + k)V \]  \hspace{1cm} (24)

In this section, the numerical method was tested with repetitive spot patterns on Gray-Scott Model. The parameters for system (24) were chosen as the reference [17]

\[ \varepsilon_1 = 1, \quad \varepsilon_2 = 0.01, \quad a = 9, \quad b = 0.4 \]

with these parameters the initial conditions of system (24) were taken as

\[ U(x, 0) = 1 - \frac{1}{2} \sin^{100} \left( \frac{\pi (x-L)}{2L} \right) \]
\[ V(x, 0) = \frac{1}{4} \sin^{100} \left( \frac{\pi (x-L)}{2L} \right) \]  \hspace{1cm} (25)

and solutions were investigated in interval \([-L, L]\) and \(L = 50\). For space discretization \(N = 400\) and for time discretization \(\Delta t = 0.2\) were selected. Dirichlet boundary conditions

\[ U(x_0, t) = U(x_N, t) = 1, \]
\[ V(x_0, t) = V(x_N, t) = 0 \]

together with additional Neuman boundary conditions

\[ U_x(x_0, t) = U_x(x_N, t) = 0, \]
\[ V_x(x_0, t) = V_x(x_N, t) = 0 \]

are used. Numerical computations were made until \(t = 100\) and \(t = 500\) so that repetitive patterns were obtained. Under these initial conditions, primarily two pulses were created and separated from each other, with each pulse then being split into two again to form four pulses, as shown in Fig. 5, until time \(t = 1000\), as time evolved. This self-replicating process goes on to cover the spatial domain. These splitting movements of the functions \(U\) and \(V\) due to time and space are presented in Figs 4-5.
Figure 4: The splitting process of repetitive spot pattern of waves for $t = 100$ and $t = 500$

Figure 5: The splitting process of repetitive spot pattern of waves for $t = 1000$

The intensity changes of functions $U$ and $V$ due to time and space are presented in Fig. 6 and Fig. 7, respectively. These spatial patterns, which known as repetitive spot patterns, initially starting with two waves of splitting movement, seem to cover the whole domain with branching over time.
2.2 Discussion

Proposed algorithm has been used for calculating numerical solutions of reaction-diffusion equation systems. Solutions of linear and nonlinear RD systems are shown on the models of certain chemical problems: the Brusselator model, Schnakenberg model and Gray-Scott models are simulated suitably by use of the suggested algorithm. The proposed TQB algorithm is an alternative method to the more usual polynomial quintic B-spline
collocation methods (PQBCM). The results of the suggested algorithm are documented together with those obtained with PQBCM and Crank-Nicolson Multigrid solver method (CN-MG) for the test problem. It can be seen from the tables (3-5) that the accuracy of the algorithms are almost the same with that for the PQBCM and are better than the CN-MG. Solutions of the nonlinear problems, which have no analytical solutions in general, are given graphically. Model solutions are represented fairly and can be compared with the equivalent graphs given in the studies [7, 8, 9, 10]. Use of the trigonometric B-spline having continuity of the order four allow us to have an approximate functions in the order of four. Therefore, differential equations in the order of four can be solved numerical by using the trigonometric B-spline functions to have solutions of continuity in the order of four. Consequently, the TQB collocation method produces fairly acceptable results for numerical solutions of reaction-diffusion systems. Thus, it is also recommended to finding solutions of the other partial differential equations.

**Competing Interest**

The authors declare that they have no competing interests.

**Authors’ contributions**

AOT carried out the algorithm implementation and conducted test studies, participated in the sequence alignment and drafted the manuscript. NA conceived of the study, participated in developing algorithm for Numeric Solutions of trigonometric quintic B-splines and helped to draft the manuscript. ID participated in its design and coordination and helped to draft the manuscript. All authors read and approved the final manuscript.

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