EXISTENCE AND HARDNESS OF CONVEYOR BELTS

MOLLY BAIRD, SARA C. BILLEY, ERIK D. DEMaine, MARTIN L. DEMaine, DAVID EPPSTEIN, SÁNDOR FEKETE, GRAHAM GORDON, SEAN GRIFFIN, JOSEPH S. B. MITCHELL, AND JOSHUA P. SWANSON

Abstract. An open problem of Manuel Abellanas asks whether every set of disjoint closed unit disks in the plane can be connected by a conveyor belt, which means a tight simple closed curve that touches the boundary of each disk, possibly multiple times. We prove three main results:

1. For unit disks whose centers are both $x$-monotone and $y$-monotone, or whose centers have $x$-coordinates that differ by at least two units, a conveyor belt always exists and can be found efficiently.
2. It is NP-complete to determine whether disks of varying radii have a conveyor belt, and it remains NP-complete when we constrain the belt to touch disks exactly once.
3. Any disjoint set of $n$ disks of arbitrary radii can be augmented by $O(n)$ "guide" disks so that the augmented system has a conveyor belt touching each disk exactly once, answering a conjecture of Demaine, Demaine, and Palop.

1. Introduction

In 2001 (later published in [1,2]), Manuel Abellanas asked whether every finite collection of disjoint closed disks in the plane can be spanned by a conveyor belt. A conveyor belt for such a collection of disks is a continuously differentiable simple closed curve that touches the boundary of each disk at least once, is disjoint from the disk interiors, and consists of arcs of the disks and bitangents between them. We may imagine such a curve as made out of an elastic band wrapped tightly around the disks. Consequently, all the disks and the conveyor belt with them can turn without slipping. Two disks have the same orientation if they turn in the same direction when the conveyor belt is pulled, or equivalently if they are both on the same side of the belt. See Figure 1.

![Figure 1](https://example.com/conveyor-belt.png)

**Figure 1.** A conveyor belt on 24 nonoverlapping unit disks. The colors and markings of the disks indicate their orientations.

We review some of the history of Abellanas’ question. While Abellanas’ question remains open for unit disks, Tejel and García found an example of non-unit disks that have no conveyor belt [9].

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O’Rourke [15] relaxed the problem by allowing the curve to cross itself or wrap around some arcs of disks more than once, with specified sets of disks of the same orientation. For his variant of the problem, not every system of disjoint unit disks has a conveyor belt, but for a belt of this type to exist it is sufficient for a certain hull-visibility graph of the disks to be connected or for the disks to remain disjoint when expanded by a sufficiently large factor. Demaine, Demaine, and Palop [8, 9] designed puzzle fonts, specified by a system of disks per character, with a unique conveyor belt in the shape of that character. If the conveyor belt is not shown, decoding the font becomes a puzzle for the viewer.

In this paper, we prove several related results on conveyor belts:

1. For unit disks whose sorted orders by x- and by y-coordinates of their centers are the same (i.e., xy-monotone), there always exists a conveyor belt (Theorem 5), and a solution belt can be constructed in linear time after sorting (Theorem 6). The same method also applies to unit disks whose x-coordinates differ by two or more units and, more generally, to monotonically separated configurations (Definition 3).

2. Strengthening the known result that not every system of non-unit disks has a conveyor belt [9], we show that the decision problem is NP-complete for non-unit disks (Theorem 10).

3. A variation of the conveyor belt problem in which we constrain the belt to touch each disk exactly once (and disks may still have non-unit radii) is also NP-complete (Theorem 7).

4. Both versions can be made to have a conveyor belt solution by adding $O(n)$ (non-unit) “guide” disks, answering a conjecture of Demaine, Demaine, and Palop [9] (Theorem 11). Conversely, $\Omega(n)$ guide disks are sometimes necessary (Theorem 13).

2. Preliminaries

Between any two disks $D_1$ and $D_2$, there are four bitangent line segments, as depicted in Figure 2. In the case when the two disks $D_1$ and $D_2$ have centers $(x_1, y_1)$ and $(x_2, y_2)$ with $x_1 < x_2$, define the lower bitangent to be the bitangent which stays entirely below the line through the centers of $D_1$ and $D_2$. The upper bitangent is defined similarly. Define the inner bitangents as the two bitangents that cross the line between the centers of $D_1$ and $D_2$.

Example 1. Suppose the disks are in “general position” in the sense that no line intersects three disks. Let $P$ be any simple polygon with the disk centers as vertices; one often refers to such a $P$ as a (simple) polygonalization of the center points. Such a polygon $P$ must exist; a traveling salesman...
tour of the center points is one such (simple) polygonalization. Another approach is to choose one of the vertices of the convex hull of the circle centers and connect the remaining circle centers into a path in radial sorted order around the chosen vertex. To determine the orientations of the vertices, travel along \( P \) cyclically and declare all disks where we turn left to be of one orientation and all disks where we turn right to be of the other orientation.

We will also need to review the Koebe–Andreev–Thurston circle packing theorem.

**Theorem 2.** \([17]\) For every planar graph \( G \) there exists a system of interior-disjoint disks in the plane, corresponding one-to-one with the vertices of \( G \), such that two vertices are adjacent in \( G \) if and only if the corresponding two disks are tangent.

The centers and radii of these disks are algebraic numbers, but they can be of arbitrarily large degree, so it may not be straightforward to represent them exactly as objects in a symbolic algebra system \([5]\). Nevertheless, there exist algorithms that can compute their coordinates numerically, to arbitrary precision, in time polynomial in the number of vertices of \( G \) and in the number of bits of precision desired \([6,14]\).

A **maximal planar graph** is a graph embedded in the plane so that all of its faces are triangles, including the unbounded outer face. For maximal planar graphs, the corresponding circle packing is unique up to Möbius transformations of the plane, which preserve circles and their tangencies. In this case, the circle packing can be chosen such that the three vertices of the outer face of the graph are three mutually tangent unit disk, with the rest of the disks all fitting into the triangular region bounded by these three unit disks. For this packing, the disks cannot vary more than exponentially in size: there exists a constant \( \varepsilon > 0 \) (not depending on the graph) such that, for any disk \( D \) of radius \( r \) that touches at most \( d \) other disks, the other disks all have radius at least \( \varepsilon^d r \) \([13]\).

We further recall the **power diagram** for a set of disks in the plane \([4]\). The **power distance** from a point \( p \) to a disk \( D \) with center \( c \) and radius \( r \) is \( |p − c|^2 − r^2 \). The power distance is negative if and only if \( p \) is inside \( D \), and otherwise it is the length of a tangent line from \( D \) to \( p \). Associated to \( D \) is a convex polygon comprised of all points in the plane whose power distance to any disk in the configuration is minimized by its power distance to \( D \). The power diagram is the set of these polygons. Since the disks in our configurations are disjoint, \( D \) is contained in the interior of its polygon. Geometrically, the power diagram forms a subdivision of the plane into polygons, one per disk, with each disk interior to its polygon. It has \( O(n) \) vertices and edges \([4, \text{Lemma 1}]\) and can be constructed in time \( O(n \log n) \) \([4, \S 5.1]\).

### 3. Monotonically Separated Disk Configurations

We begin by fixing some notation and vocabulary. Suppose we have a sequence of \( n \) disks with centers \((x_i, y_i)\) for \( 1 \leq i \leq n \). By rotating and relabeling the disks if necessary, we can assume without loss of generality that \( x_1 < x_2 < \cdots < x_n \), so the disks are sorted by their \( x \)-coordinates.

**Definition 3.** We say that a sequence of disks is **monotonically separated** if \( x_1 < x_2 < \cdots < x_n \) and for every \( i < j < k \), the \( k \)th disk is disjoint from the convex hull of the \( i \)th and \( j \)th disks, and the \( i \)th disk is disjoint from the convex hull of the \( j \)th and \( k \)th disks.

**Lemma 4.** Given a sequence of \( n \) unit disks with \( x_1 < \cdots < x_n \), the disks are monotonically separated in either of the following two cases:

1. \((xy\text{-monotone})\) \( y_1 < \cdots < y_n \), so the centers are both \( x \)-monotone and \( y \)-monotone; or
2. \((x\text{-separated})\) for all \( 1 \leq i < n \), we have \( x_{i+1} − x_i \geq 2 \).

**Proof.** In the case where the \( x \)-coordinates differ by at least two units, the vertical line \( x = x_j + 1 \) separates the convex hull of disks \( i \) and \( j \) from disk \( k \). See Figure 3. The other cases are similar. \( \square \)

**Theorem 5.** Every monotonically separated sequence of unit disks has a conveyor belt.
Figure 3. Diagrams of monotonically separated sequences. For unit disks with indices \( i < j < k \) and with centers that are \( x- \) and \( y- \) monotone (left) or whose \( x- \) coordinates differ by at least two units (right), the convex hull of disks \( i \) and \( j \) is confined to a disjoint region from disk \( k \).

Proof. Consider the upper convex hull of the disks, which is the part of the boundary of the convex hull passing clockwise from the leftmost point of the leftmost disk \( L \) to the rightmost point of the rightmost disk \( R \), not including these endpoints. Let \( U_1, U_2, \ldots \) be the disks that contact the upper convex hull in between \( L \) and \( R \), referred to as the upper hull disks and listed with increasing \( x- \) coordinate. Let \( D_0 = L, D_1, D_2, \ldots, D_k = R \) be the disks that are not in the subsequence of upper hull disks, listed with increasing \( x- \) coordinate. We assume \( k \geq 3 \) since otherwise the boundary of the convex hull of the disks is a conveyor belt.

We begin by describing a key subroutine and some of its properties. We will shortly use this subroutine to iteratively construct a conveyor belt on the given disks. Given two consecutive disks \( D_i \) and \( D_{i+1} \), the “winding process” constructs two partial conveyor belts from \( D_i \) to \( D_{i+1} \) as follows. First, place a ray along the “lower” bitangent between \( D_i \) and \( D_{i+1} \) with the ray’s apex on \( D_i \) and pointing the ray initially towards \( D_{i+1} \). Continuously rotate the ray counterclockwise so that it stays tangent to \( D_i \) at its apex. As we rotate, we may shift the apex of the ray to disks other than \( D_i \) according to the following possible events considered in order.

1. The rotating ray may align with a tangent to some upper hull disk \( U \) that lies between \( D_i \) and \( D_{i+1} \) in the \( x- \) sorted order of disks. If there is more than one such \( U \), choose the one with the smallest \( x- \) coordinate. In this case, add to the current partial conveyor belt the bitangent along the ray between the disk the ray is currently rotating around and \( U \). Shift the apex of the rotating ray to \( U \) and continue rotating the apex of the ray counterclockwise around \( U \) and inscribing an arc along the boundary of \( U \).

2. The rotating ray may align with a tangent to \( D_{i+1} \) for the first time. In this case, we report the partial conveyor belt from \( D_i \) through the apex of the ray to this point of tangency as the “lower” of the two partial conveyor belts between \( D_i \) and \( D_{i+1} \). Continue rotating the ray.

3. The rotating ray may align with a tangent to \( D_{i+1} \) for the second time. In this case, report a partial conveyor belt from \( D_i \) to \( D_{i+1} \) as before, this time calling it the “upper” partial conveyor belt. Stop the winding process.

We claim the winding process has the following properties.

(a) Both partial conveyor belts lie entirely within the convex hull of \( D_i \) and \( D_{i+1} \). More precisely, they lie within the wedge-shaped region bounded by the two tangent lines to \( D_{i+1} \) from the initial starting point on \( D_i \).
(b) Both belts are disjoint from the upper convex hull, even though some upper hull disks may be touched by either belt.
(c) Both belts are valid partial conveyor belts touching $D_i$ and $D_{i+1}$ in the sense that they consist of arcs of disks and bitangents between them whose union is a continuously differentiable curve without self-intersection that is disjoint from all disk interiors.

Property (a) is easy to see, and (b) follows from (a). For (c), every property is clear except possibly that the partial belts may not be disjoint from all disk interiors. The ray initially points along the “lower” bitangent between $D_i$ and $D_{i+1}$, which cannot intersect the interior of an upper hull disk by definition of the upper hull so it is reported as part of the lower belt. As the winding process continues, the upper belt remains disjoint from all upper hull disk interiors with center $x$-coordinate between $D_i$ and $D_{i+1}$ by construction. The remaining disks $D_j$ are disjoint from the convex hull of $D_i$ and $D_{i+1}$ since the disks are monotonically separated by hypothesis, so both belts are disjoint from these remaining disks by (a).

The “unwinding process” is a slight variation on the winding process. Initially, the ray is placed exactly as before along the “lower” bitangent from $D_i$ to $D_{i+1}$ except its direction is reversed, with the apex still at $D_i$. The apex rotates counterclockwise, continuing exactly as before. The belts created by the unwinding process also satisfy (a)-(c) above by virtually the same reasoning.

Now consider any triple of disks $D_{i-1}, D_i, D_{i+1}$. The interior of the convex hull of $D_{i-1}$ and $D_i$ intersects the boundary of $D_i$ in an open semi-circle, and the interior of the convex hull of $D_i$ and $D_{i+1}$ intersects the boundary of $D_i$ in another open semi-circle. Suppose for simplicity that the centers of $D_{i-1}, D_i, D_{i+1}$ are not colinear. The union of these two semicircles is then a single arc on the boundary of $D_i$. Let $\alpha$ denote the complement of this open arc in the boundary of $D_i$. We may now paste together two of the four partial belts from $D_{i-1}$ to $D_i$, an arc containing $\alpha$, and two of the four partial belts from $D_i$ to $D_{i+1}$ as in Figure 4 resulting in four partial belts from $D_{i-1}$ to $D_{i+1}$. Since the disks are monotonically separated, one may check that property (a) ensures each resulting partial belt from $D_{i-1}$ to $D_{i+1}$ has no self-intersections and continues to satisfy (b) and (c).

**Figure 4.** The first picture shows the four partial conveyor belts between three adjacent lower disks. The red arc labeled $\alpha$ is required to be part of the final belt. The second picture shows the next iteration of the algorithm after extending the 4 possible belts to include $D_{i+2}$. Again the red part of the belt is required in any further growth of the belt, and the blue forks indicate the 4 optional extensions.

The full algorithm proceeds as follows. Begin with four partial conveyor belts between two adjacent lower disks. Iteratively extend these belts as above to produce four partial conveyor belts between ever larger sets of adjacent lower disks satisfying (b) and (c). For example, see Figure 4. Finally, choose the belt tangent to the lower sides of $L$ and $R$ and glue it to the upper hull by way of arcs along $L$ and $R$ to create a valid conveyor belt touching each disk at least once.\qed
Figure 5. A sequence of $x$- and $y$-monotone unit disks and the conveyor belt (red) produced for it by our algorithm. The blue line segments indicate the alternative bitangents found as preliminary results by our algorithm, connecting to the opposite side of each non-upper-hull disk from the bitangent that is eventually used in the conveyor belt.

Figure 5 depicts the conveyor belt resulting from the construction in the proof in red. The blue segments are part of an upper or lower belt that was unused in the final conveyor belt. Figure 6 shows another example of the final belt constructed in the algorithm where the disks are monotonically separated but not $x$- and $y$-monotone.

Figure 6. Larger example of the conveyor belt produced by the proof of Theorem 5.

**Theorem 6.** The conveyor belt constructed by the algorithm in the proof of Theorem 5 can be constructed in linear time from an input that lists the coordinates of the disk centers in the order of their monotonically separated sequence.

**Proof.** The upper convex hull can be constructed from this sorted order by a standard Graham scan algorithm, in Andrew’s variation using the left-to-right sorted order rather than radial order.
In this algorithm, we use a stack data structure to store the sequence of disks in the upper hull of ever-longer prefixes of the input. To extend the prefix, we check if the next disk in the sequence together with the top two disks on the stack form an upper hull that is unchanged if the middle of these three disks is omitted. While this is true we pop the stack, effectively removing the middle disk from consideration. After no more such pops can be performed, we push the new disk onto the stack. At the end of this process, the disks that remain on the stack form the upper hull. Each disk is pushed and popped at most once, so the total time for this part of the algorithm is linear.

To simulate the continuous ray-rotation process by which we generated the partial conveyor belts through the remaining disks, we replace it by a discrete process consisting only of the events at which the continuous process undergoes a discrete change, when the rotating ray reaches a tangency with $D_{i+1}$ or with one of the disks $U_j$. There may be many disks $U_j$ between $D_i$ and $D_{i+1}$; however, by convexity of the sequence of disks $U_j$, only two of these disks can cause tangency events: the one that is closest in the sequence to $D_i$, and the one that is closest in the sequence to $D_{i+1}$. Therefore, constructing the sequence of discrete events entails only comparing $O(1)$ possible tangencies at each step to determine which of them happens first, and can be performed in constant time per step. Each disk is visited at most once during this process, so the total time for this part of the algorithm is again linear.

A finite sequence of distinct numbers is called bitonic if it has a unique local maximum and a unique local minimum when viewed as a cyclic sequence. Equivalently, it is unimodal up to a cyclic rotation. Since our method constructs conveyor belts from left-to-right, the result is bitonic with respect to the sequence of $x$-coordinates of disk centers. When a set of disks has a bitonic conveyor belt, it is possible to find such a belt in polynomial time by a straightforward adaptation of the well-known dynamic programming algorithm for bitonic traveling salesperson tours. However, for disks with $x$- and $y$-monotone centers but non-uniform radii, it is not always possible to find conveyor belts that are bitonic. Figure 7 (left) provides a counterexample. For unit disks that are not monotonically separated, it is also not generally possible to have a tour that is bitonic. In Figure 7 (right) there are three disks that are consecutive in the sorted order by $x$-coordinates,
but that have no unblocked bitangent between any two of them. All three disks must be separated from each other in the cyclic sequence of contacts with the conveyor belt, and between any two of them there must be a local minimum or local maximum of the $x$-coordinates.

4. One-Touch NP-Completeness

In this section we consider the following one-touch conveyor belt problem. We are given as input a collection of disjoint disks in the plane specified by integer center coordinates and radii. The goal is to determine whether there exists a conveyor belt that contacts each disk exactly once.

Our proof that this problem is NP-complete will serve as a warm up for the proof that the decision problem for conveyor belts that are not restricted to one touch is also NP-complete.

Theorem 7. The one-touch conveyor belt problem is NP-complete.

Proof. We follow the standard outline for a proof that a problem $X$ is NP-complete, by a polynomial-time reduction from a known NP-complete problem $Y$. To do so $X$ and $Y$ must both be decision problems (problems with a yes or no answer). We must show that $X$ is in NP, and that there exists a polynomial time algorithm for translating all inputs of $Y$ into inputs of $X$ that preserves the answer of each translated input.

To show that the one-touch conveyor belt problem is in NP, we describe a nondeterministic polynomial-time algorithm for it: an algorithm that guesses a solution that can be described in a polynomial number of bits, verifies in deterministic polynomial time that the solution is valid, and if so answers yes. It must be the case that every solvable instance has a solution that causes this algorithm to answer yes, and that no non-solvable instance has such a solution. In our case, the solutions can specify the cyclic order of contacts along the belts and the orientations of each disk, information that can be specified in only $O(n \log n)$ bits for $n$ disks, a polynomial number. If there are $n$ disks, since the belt is one-touch, the resulting belt will have $2n$ arcs or bitangent segments. The verification algorithm must then test that only consecutive arcs or bitangents intersect, and then only in a single point with compatible directions. It is straightforward to see that these verifications may be performed with $O(n^2)$ tests, so the verification algorithm takes polynomial time, as required.

To prove NP-hardness, we reduce from a known NP-complete problem, determining the existence of a Hamiltonian cycle in a maximal planar graph. This problem was proven NP-complete by Wigderson in 1982 [16]. It is straightforward to modify Wigderson’s construction to ensure that the maximal planar graphs that it produces always have an even number of vertices; see Remark 8.

The reduction begins by representing a given maximal planar graph as a system of touching disks using the circle packing theorem. The only unblocked bitangent segments in the circle packing are outer tangents for the three outermost disks. Shrinking each disk by a factor of $1 - \delta$ for $\delta$ sufficiently small will create additional unblocked bitangents only between adjacent disks, see Figure 8. When finding the largest possible $\delta$ for a given triple of disks, we will perform a constant number of polynomial operations and root extractions using the disk radii, so $\delta$ is at least a polynomial in the smallest possible ratio $\varepsilon^n$ between tangent disks.

The numerical precision needed to represent the disk centers and their radii accurately enough to perform this shrinkage, and to avoid creating additional bitangencies through numerical inaccuracies, is therefore also at most exponential in the number of disks. Numbers with this level of precision may be represented using a linear number of bits, allowing an approximate numerical representation of the circle packing and its shrunk system of disjoint disks to be constructed numerically in polynomial time. By scaling this numerical representation appropriately, we can cause all the disk centers and radii to become integers, as needed for an input to the one-touch conveyor belt problem.

It remains to verify that this transformation from graphs to systems of disks preserves the yes-or-no answers to every input. That is, we must show that a given maximal planar graph $G$ with
Figure 8. A one-touch conveyor belt for a system of disks constructed by slightly shrinking a circle packing for the graph of the edges and vertices of an octahedron. The belt touches the disks in the order given by a Hamiltonian cycle on the octahedron, and crosses between disks using only inner tangents. Circular arcs on the belt are shown in blue, and bitangents are shown in red.

An even number of vertices has a Hamiltonian cycle if and only if the resulting system of disks has a one-touch conveyor belt. We begin by arguing that, when a one-touch conveyor belt exists, a Hamiltonian cycle also exists. Thus, suppose that we have a one-touch conveyor belt $B$. Since every face of $G$ is a triangle, the only unobstructed bitangents that exist for the disks of the construction are between shrunken disks that, before shrinking, were tangent. Therefore, $B$ can only use these segments to move from one disk to another, and each two consecutive disks along $B$ must be adjacent in $G$. Thus, the cyclic ordering of disks in $B$ corresponds to a cyclic ordering of vertices in $G$ such that each two consecutive vertices in the ordering are adjacent; that is, to a Hamiltonian cycle.

Conversely, suppose we have a Hamiltonian cycle $C$ in $G$; we must show that there exists a one-touch conveyor belt in the corresponding system of shrunken disks. To do so, we use the cyclic ordering of vertices in $C$ as the cyclic order of disks in a belt $B$, and we assign the disks signs that alternate between positive and negative in this cyclic order. This assignment is well-defined since the number of vertices in $G$ is assumed to be even. It corresponds to a curve composed of circular arcs and inner bitangents of pairs of circles that, before shrinking, were tangent. The shrinking process that we perform causes all such inner bitangents to be unobstructed, and disjoint from all the other inner bitangents of other formerly-tangent pairs, so no two such segments can cross, nor can a bitangent segment cross any circular arc. Therefore, the resulting curve is a simple closed curve, composed of arcs and bitangents, with one arc per disk, so it is a one-touch conveyor belt as required. □
Remark 8. We may modify Wigderson’s construction \[16\] to produce graphs with evenly many nodes as follows. From Wigderson’s graph \(K\), one may create a graph \(K'\) by adding an extra vertex at the midpoint of the edge of \(K\) that starts at the central vertex and travels northwest, followed by triangulating the two resulting rectangles by adding two extra edges emanating from the new vertex. In forming Wigderson’s graph \(M\), use one copy of \(K\) and one copy of \(K'\). The resulting variant of Wigderson’s graph \(N\) will have 58 instead of 55 nodes, so the graphs \(G'\) will have a multiple of 58 nodes, an even number.

Figure 8 depicts this construction for the maximal planar graph given by the vertices and edges of a regular octahedron. The one-touch belt shown in the figure connects the six disks shown in the order corresponding to a Hamiltonian cycle through the six vertices of the octahedron.

Remark 9. Whether counting the number of solutions is \#P-complete remains open. The NP-completeness reduction above is close to parsimonious, but it counts cycles through edges of the outer triangle differently than cycles that do not use those edges. More precisely, one may check that the number of conveyor belts corresponding to a given Hamiltonian cycle is 1 if the cycle involves 0 or 1 edges of the outer triangle, and 2 if the cycle involves 2 edges of the outer triangle.

5. Multi-Touch NP-Completeness

Theorem 10. It is NP-complete to determine whether a given system of disks has a conveyor belt, even allowing the belt to touch a single disk multiple times along disjoint arcs.

Proof. As in the one-touch case, we reduce from the Hamiltonian circuit problem, but in a different class of graphs, the cubic 3-connected planar graphs. These graphs are the dual graphs of the
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Figure 10. Adding three small disks to the outer triangle of a circle packing restricts a conveyor belt to enter and exit the interior of the packing as shown, up to rotation and reflection.

maximal planar graphs used for the one-touch problem. Again, testing Hamiltonicity of these graphs is known to be NP-complete \cite{11}; indeed, Wigderson’s reduction in \cite{16} is to this older result.

Our reduction is similar to the one-touch reduction. Let $G$ be a cubic 3-connected planar graph with $n > 3$ vertices. Let $G'$ be the maximal planar graph dual to $G$. Represent $G'$ by a circle packing, and then slightly shrink the disks. This time, we modify the construction by adding another small disk at the radical center of each triple of disks that represent an interior triangle of $G'$ (Figure 9), and by adding three more disks tangent to pairs of the outer three disks, small enough that they lie within the convex hull of the outer three disks (Figure 10). The intent of these changes is to force the conveyor belt to visit every face of $G'$ and thereby represent a Hamiltonian cycle for the originally given cubic graph $G$, whose vertices correspond to these faces. Since $G$ has $n > 3$ vertices, there is at least one interior triple of disks.

For each pair of disks that were initially tangent, a conveyor belt may pass between them at most twice. Call such a crossing single-ply if the conveyor belt passes between those disks once and double-ply if the conveyor belt passes between those disks twice. All of the ways that a belt can enter and exit one of the inner triangles of the circle packing are depicted in Figure 9. By inspecting each case, we find that a belt enters through a single-ply crossing if and only if it exits through a single-ply crossing. It follows that any segment of the conveyor belt passing through the interior of the outer triangle of the circle packing must consist entirely of single-ply or entirely of double-ply crossings.

The disks added for the outside triangle can only be touched by a conveyor belt as shown in Figure 10. We consider each depicted case in turn.

- In Figure 10 (left), the conveyor belt enters and exits the interior of the disk construction via two single-ply connections in regions which correspond to two neighboring triangles of the outer triangular face of $G'$. Consequently, if a one-touch conveyor belt exists for the disk construction, it must be single-ply everywhere in the interior as shown in the unique single-ply connection pattern from Figure 9 (lower right). Furthermore, the conveyor belt must enter and exit each region corresponding with a triangle in $G'$ exactly once, so it must correspond to a Hamiltonian circuit of $G$. Conversely, it is not hard to see that every Hamiltonian circuit of $G$ can be transformed to a conveyor belt on the constructed disk configuration. Therefore, the disk configuration derived from $G$ has a conveyor belt if and only if $G$ has a Hamiltonian circuit.

- In Figure 10 (middle), the conveyor belt enters and exits the interior of the constructed disk configuration via three double-ply connections. The interior connections must then all be double-ply as well, as depicted in Figure 9 (upper left) and (lower left). Furthermore, we can observe from the pictures that both strands enter together from one gap between
disks and exit together from another gap between disks. In this sense, double-ply segments form paths. Starting at one of the double-ply connections entering the interior from the outside triangle, the belt must form a path through a sequence of regions corresponding with triangles in $G'$. If the path ended at Figure 9 (lower left), there would be a disconnected segment of the belt, so it must end at another connection to the outside triangle. However, the third connection to the outside triangle would then be disconnected from the other two. So, there are no belts of this form.

- In Figure 10 (right), the conveyor belt makes two single-ply connections and one double-ply connection to the interior of the construction. The double-ply connection must pass through the interior to the single-ply connections in order for the belt to be connected, though as noted above the interior connections are all single-ply or all double-ply, so this is not possible and there are again no belts of this form.

Thus, the only possible conveyor belts are ones that makes single-ply connections only, each of which correspond to a Hamiltonian cycle.

6. Guide Disks

In this section, we explore the question of how many additional disks must be added in order to guarantee that any disk configuration has a conveyor belt. We show that a linear number of additional disks suffice. Our construction uses the power diagram of a disk configuration described in Section 2.

**Theorem 11.** Any system of $n$ disks can be augmented by $O(n)$ additional disks so that the resulting augmented system of disks has a (one-touch) conveyor belt.

*Proof.* First construct the power diagram of the disks. The dual graph of the power diagram is the graph whose nodes are polygons in the power diagram, where nodes are connected if their polygons share an edge. Pick a spanning tree of this dual graph. Imagine traveling around the “outside” of the spanning tree, which gives a cyclic sequence $S$ of disks and edges of the dual graph. The disks in $S$ may be repeated, consecutive disks have power diagram cells that are adjacent to each other, and each disk is included at least once in $S$. Since the number of edges of the power diagram is linear in $n$, the length of $S$ is linear in $n$.

Let $C$ and $D$ be consecutive disks in $S$ corresponding to an edge in the spanning tree. Take $y$ to be the midpoint of the edge between the polygons of $C$ and $D$ in the power diagram; if the edge is infinite, we may take $y$ to be any point on the interior of the edge. Let $x$ and $z$ be points on the line segments from the centers of $C$ and $D$ to $y$ just outside of $C$ and $D$, respectively. We may represent the edge between $C$ and $D$ in the tree geometrically by a polyline consisting of the two line segments $xy$ and $yz$. The polytopes representing distinct edges do not cross. Taken together, they geometrically represent the spanning tree of the dual graph. By adding small guide disks near $x$, $y$, and $z$ as needed, we may form a conveyor belt that represents traveling around the “outside” of the spanning tree. Since the length of $S$ is linear, $O(n)$ guide disks are needed.

In the one-touch model, we may perform a similar construction, but when it would use more than one arc of an input disk we may instead route the belt near the edges of the corresponding power diagram polygon for all but one of these arcs. This will require at most a constant number of disks per edge in the power diagram, so there are again $O(n)$ guide disks total.

**Lemma 12.** There exists a configuration $C$ of disks with the property that, whenever $C$ appears as part of a larger configuration $C'$, with all disks of $C' \setminus C$ disjoint from the convex hull of $C$, every conveyor belt for $C'$ must include at least one guide disk interior to the convex hull of $C$.

*Proof.* Let $G$ be a cubic 3-connected planar graph in which the longest path has fewer than $|V(G)|/3$ vertices; the existence of $G$ follows from known results on the shortness exponent of cubic 3-connected planar graphs [12]. As in the proof of Theorem 10, represent the dual graph of $G$ by a
Figure 11. A system of \( n \) disks requiring \( \Omega(n) \) guide disks for a one-touch conveyor belt

circle packing (choosing arbitrarily which triple of circles to make exterior), place another smaller tangent circle within each triangle of circles of the packing, and then shrink all of the circles by a small amount to allow a conveyor belt to pass between them without making any additional bitangents available to the belt. Let \( C \) be the resulting system of circles.

Then, as in the proof of Theorem 10, any conveyor belt must enter \( C \) via at most three single-ply or double-ply crossings between the three pairs of outer circles of \( C \). If there were no guide disks within the convex hull of \( C \), the same analysis in Theorem 10 would show that such a belt would necessarily correspond to at most three paths through \( G \) that either enter \( C \) through one of these crossings and exit through another, or enter \( C \) through a double-ply crossing and stop somewhere within \( C \). However, because all paths are short, it is not possible for three or fewer paths to touch all vertices of \( G \); correspondingly, a conveyor belt without a guide disk within \( C \) cannot touch all of the smaller circles within each triangle of circles of \( C \). Therefore, at least one guide disk within the convex hull of \( C \) is needed.  

\[ \square \]

**Theorem 13.** In order for one-touch or unconstrained conveyor belts to include \( n \) given disks, \( \Omega(n) \) guide disks are sometimes necessary.

**Proof.** For the one-touch case, place \( n - 1 \) small disks at the vertices of a regular polygon, and a large disk at the center of the polygon, in such a way that no two small disks can see each other (Figure 11). Then, in the cyclic sequence of disks given by any one-touch conveyor belt, each of the small disks must be separated from each other by a contact with some other disk, but only one of those contacts can be the large disk. Therefore, there must be at least \( n - 2 \) other guide disks to provide these contacts.

For the unconstrained case, we form a configuration of \( \lfloor n/|C| \rfloor \) copies of the configuration \( C \) described in Lemma 12 with disjoint convex hulls. Each copy of \( C \) requires a guide disk interior to its convex hull, so all the copies together require at least \( \lfloor n/|C| \rfloor \) guide disks.  

\[ \square \]

7. Future Work

We conclude with a few questions for further exploration:
(1) How many guide disks are necessary to ensure that any traveling salesman tour on the centers of \( n \) disks can be a subsequence of a conveyor belt?

(2) Is the problem of finding the number of conveyor belts for a given disk configuration \#P-complete?

(3) Given \( n \) points in the plane, is finding the number of polygons with vertices at the \( n \) points also \#P-complete? See [10] for related results and further questions on hardness of counting problem in discrete geometry.

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References
[1] M. Abellanas. Conectando puntos: poligonizaciones y otros problemas relacionados. Gaceta de la Real Sociedad Matematica Espanola, 11(3):543–558, 2008.
[2] M. Abellanas. Linking geometric objects. In P. Ramos and V. Sacristán, editors, XIV Spanish Meeting on Computational Geometry, In Honor of Ferran Hurtado’s 60th Birthday, Alcalá de Henares, June 27–30, 2011, pages 31–32. Centre de Recerca Matemática, Bellaterra, 2011. URL: https://ddd.uab.cat/pub/llibres/2011/tdt1_2072_200199/Documents08_web.pdf
[3] A. M. Andrew. Another efficient algorithm for convex hulls in two dimensions. Information Processing Letters, 9(5):216–219, 1979. doi:10.1016/0020-0190(79)90072-3
[4] F. Aurenhammer. Power diagrams: properties, algorithms and applications. SIAM J. Comput., 16(1):78–96, 1987. doi:10.1137/0216006
[5] M. J. Bannister, W. E. Devanny, D. Eppstein, and M. T. Goodrich. The Galois complexity of graph drawing: why numerical solutions are ubiquitous for force-directed, spectral, and circle packing drawings. J. Graph Algorithms & Applications, 19(2):619–656, 2015. arXiv:1408.1422 doi:10.7155/jgaa.00349
[6] C. R. Collins and K. Stephenson. A circle packing algorithm. Comp. Geom., 25(3):233–256, 2003. doi:10.1016/S0925-7721(02)00099-8
[7] T. H. Cormen, C. E. Leiserson, R. Rivest, and C. Stein. Introduction to Algorithms. MIT Press, 3rd edition, 2009.
[8] E. D. Demaine and M. L. Demaine. Fun with fonts: algorithmic typography. Theoret. Comput. Sci., 586:111–119, 2015. doi:10.1016/j.tcs.2015.01.054
[9] E. D. Demaine, M. L. Demaine, and B. Palop. Conveyor-belt alphabet. In H. Aardse and A. van Baalen, editors, Findings in Elasticity, pages 86–89. Pars Foundation, Lars Müller Publishers, April 2010.
[10] D. Eppstein. Counting polygon triangulations is hard. CoRR, abs/1903.04737, 2019. Accepted to SoCG, June 2019. URL: http://arxiv.org/abs/1903.04737
[11] M. R. Garey, D. S. Johnson, and R. E. Tarjan. The planar Hamiltonian circuit problem is NP-complete. SIAM J. Comput., 5(4):704–714, 1976. doi:10.1137/0205049
[12] B. Grünbaum and H. Walther. Shortest exponents of families of graphs. Journal of Combinatorial Theory. Series A, 14:364–385, 1973. doi:10.1016/0097-3165(73)90012-5
[13] S. Malitz and A. Papakostas. On the angular resolution of planar graphs. SIAM J. Discrete Math., 7(2):172–183, 1994. doi:10.1137/0307014
[14] B. Mohar. A polynomial time circle packing algorithm. Discrete Math., 117(1–3):257–263, 1993. doi:10.1016/0012-365X(93)90340-Y
[15] J. O’Rourke. String-wrapped rotating disks. In A. Márquez, P. Ramos, and J. Urrutia, editors, XIV Spanish Meeting on Computational Geometry, EGC 2011, Dedicated to Ferran Hurtado on the Occasion of His 60th Birthday, Alcalá de Henares, Spain, June 27–30, 2011, Revised Selected Papers, volume 7579 of Lecture Notes in Computer Science, pages 65–78, Cham, 2011. Springer. doi:10.1007/978-3-642-34191-5_6
[16] A. Wigderson. The complexity of the Hamiltonian circuit problem for maximal planar graphs. Technical Report 298, Princeton University Department of Computer Science, 1982.

[17] G. M. Ziegler. Steinitz’ theorem for 3-polytopes. In Lectures on Polytopes, Lecture 4, pages 103–126. Springer-Verlag, 1995.