THE GONALITY OF COMPLETE INTERSECTION CURVES

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1. Introduction

The purpose of this paper is to show that for a complete intersection curve $C$ in projective space (other than a few exceptions stated below), any branched covering $C \rightarrow \mathbb{P}^1$ of minimum degree is obtained by projection from a linear space. We also prove a special case of one of the well-known Cayley-Bacharach conjectures due to Eisenbud, Green, and Harris.

Let $C$ be a projective curve. Recall that the gonality of $C$, $\text{gon}(C)$, is the minimum degree of a surjective morphism

$$\tilde{C} \longrightarrow \mathbb{P}^1,$$

where $\tilde{C}$ is the normalization of $C$. Thus, $C$ is rational precisely when $\text{gon}(C) = 1$, and, more generally, gonality measures how far the curve is from being rational. Gonality is a classical invariant, and there has been significant interest in bounding the gonality of various classes of curves and characterizing the corresponding maps to $\mathbb{P}^1$. Specifically, if $C$ is embedded in projective space, it is natural to ask whether the gonality is related to the embedding of the curve.

For example, the gonality of plane curves (i.e. complete intersection curves of codimension one) is well understood. If $C \subset \mathbb{P}^2$ is a smooth curve of degree $d$, then the map $C \rightarrow \mathbb{P}^1$ obtained by projecting from a point in $C$ has degree $d - 1$, giving an upper bound on the gonality. In fact, a classical theorem of Noether states that if $d \geq 3$, then $\text{gon}(C) = d - 1$,

and any covering of $\mathbb{P}^1$ of degree $d - 1$ is obtained by projecting from a point.

Moreover, the gonality of codimension two complete intersection curves was studied first by Ciliberto and Lazarsfeld [CL84] and later by Basili [Bas96]. The former authors proved the uniqueness of the linear series $|O_C(1)|$ for both complete intersection curves and more general classes of curves, such as projectively normal curves. Basili showed that if $C \subset \mathbb{P}^3$ is a smooth complete intersection curve, then the gonality is indeed computed by projection from a line and every minimal covering arises in this way. Recently, Hartshorne and Schlesinger generalized Basili’s results to smooth ACM curves in $\mathbb{P}^3$ with the exception of a few cases [HS11]. The same result holds for many other specific classes of curves in $\mathbb{P}^3$ (e.g. [Bal97], [EF01], [Far01], [Har02], [Mar96], ). See [HS11] for a detailed review on curves in $\mathbb{P}^3$ whose gonality is computed by projection from a line.

At the other extreme, i.e. higher dimensional hypersurfaces, the recent paper [BDE+16] calculated the so-called degree of irrationality of very general hypersurfaces in projective space. If $X$ is a smooth variety of dimension $n$, then the degree of irrationality of $X$, $\text{irr}(X)$,
is the minimum degree of a dominant rational map $X \dashrightarrow \mathbb{P}^n$. Clearly, this definition agrees with the definition of gonality in the $n = 1$ case, and $\text{irr}(X) = 1$ if and only if $X$ is rational. The main result of [BDE+16] states that if $X \subset \mathbb{P}^{n+1}$ is a very general hypersurface of degree $d \geq 2n+1$, then $\text{irr}(X) = d - 1$, and if $d \geq 2n+2$, then any dominant map $X \dashrightarrow \mathbb{P}^n$ is obtained by projecting from a point on $X$.

Returning to curves, the main result about gonality of complete intersection curves in higher dimensional projective spaces to date is a lower bound due to Lazarsfeld:

**Theorem 1.1** (cf. [Laz97], Exercise 4.12). Let $C \subset \mathbb{P}^n$ be a smooth complete intersection curve of type $(a_1, a_2, \ldots, a_{n-1})$, where $2 \leq a_1 \leq \cdots \leq a_{n-1}$. Then

$$\text{gon}(C) \geq (a_1 - 1)a_2 \cdots a_{n-1}.$$ 

In light of the previous examples, one may ask whether every such map is given by projection. Our main result answers this question as long as $C$ satisfies mild degree restrictions:

**Theorem A.** Let $C \subset \mathbb{P}^n$ be a complete intersection curve of type $(a_1, \ldots, a_{n-1})$, with

$$4 \leq a_1 < a_2 \leq \cdots \leq a_{n-1}.$$ 

Then any morphism $C \to \mathbb{P}^1$ of degree equal to the gonality is obtained by projecting from an $(n-2)$-plane. Thus $\text{gon}(C) = a_1a_2 \cdots a_{n-1} - \gamma$, where $\gamma$ is the maximum number of points on $C$ contained in an $(n-2)$-plane.

**Remark 1.2.** In fact, we can weaken the hypotheses of Theorem A slightly. As long as $C$ lies on a smooth complete intersection threefold of type $(w_3, \ldots, w_{n-1})$ and the remaining two degrees $a$ and $b$ cutting out $C$ satisfy $4 \leq a < b$, then the conclusion of the theorem holds.

Moreover, although the bound in Theorem 1.1 is sharp in every degree, we show that a stronger bound holds for very general complete intersection curves:

**Corollary B.** Let $C$ be a very general complete intersection curve with degrees as described in Theorem A. Then

$$a_1 \cdots a_{n-2}(a_{n-1} - 1) \leq \text{gon}(C) \leq a_1a_2 \cdots a_{n-1} - 2n + 3.$$ 

The proof of Theorem A relies on a generalization of the classical Noether-Lefschetz Theorem [Lef21], which we prove in Section 3. It states that, under certain degree restrictions, a complete intersection curve in projective space lies on a complete intersection surface with Picard group generated by the hyperplane class. However, Theorem A also holds in the more general setting of arbitrary curves lying on surfaces with Picard group $\mathbb{Z}$:

**Theorem C.** Let $C \subset \mathbb{P}^n$ be a smooth non-degenerate curve lying on a smooth surface $S$ with $\text{Pic}(S) = \mathbb{Z} \cdot [\mathcal{O}_S(1)]$, and $C \in |\mathcal{O}_S(\alpha)|$, where $\alpha \geq 4$. Then $\text{gon}(C) = \alpha \cdot \deg(S) - \gamma$, where $\gamma$ is the maximum number of points on $C$ contained in an $(n-2)$-plane, and any morphism $C \to \mathbb{P}^1$ of degree equal to the gonality is obtained by projecting from an $(n-2)$-plane.

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[1] Asher Auel and Dave Jensen showed us how to improve the upper bound from an earlier version of this paper by applying a result of Coppens and Martens [CM91].
Remark 1.3. In [Ras15], Rasmussen independently showed that, under stronger hypotheses, the fibers of a morphism \( C \to \mathbb{P}^1 \) of degree less than the degree of \( C \) must lie in hyperplanes, which follows from our Lemma 4.4.

Remark 1.4. A special case of one of the Cayley-Bacharach conjectures posed by Eisenbud, Green, and Harris (Conjecture CB12 in [EGH96]) follows easily from the proof of Theorem C. We discuss this in Section 5.

A statement analogous to Corollary B also holds in this more general setting:

**Corollary D.** With \( C \) and \( S \) as in Theorem C,
\[
\deg(C) - \deg(S) \leq \text{gon}(C).
\]
If, in addition, \( C \) is linearly normal, then
\[
\text{gon}(C) \leq \deg(C) - 2n + 3.
\]

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Regarding conventions, we will often abuse notation by using additive divisor notation when dealing with line bundles. Throughout, we work over the complex numbers.

## 2. The Cayley-Bacharach condition

Suppose \( Z \) is a set of distinct points on a smooth variety \( X \) of dimension \( n \). Let \( L \) be a line bundle on \( X \). The set of points \( Z \) satisfies the Cayley-Bacharach condition with respect to the complete linear system \( |L| \) if every section of \( L \) vanishing at all but one of the points of \( Z \) also vanishes at the remaining point.

**Remark 2.1.** If \( Z \) satisfies the Cayley-Bacharach condition non-vacuously, i.e. \( h^0(L \otimes I_Z - P) \neq 0 \) for some \( P \in Z \), then \( Z \) doesn’t impose independent conditions on \( L \). However, the converse is not true. For instance, if \( P_1, P_2, P_3 \in \mathbb{P}^2 \) lie on a line \( \ell \), and \( P_4 \notin \ell \), then the set of points \( \{P_1, P_2, P_3, P_4\} \) does not impose independent conditions on \( |\mathcal{O}_{\mathbb{P}^2}(1)| \), but it doesn’t satisfy the Cayley-Bacharach condition with respect to \( |\mathcal{O}_{\mathbb{P}^2}(1)| \).

In the case where \( Z \) is a general fiber of a generically finite rational map \( X \to \mathbb{P}^n \), by analyzing the trace map, one obtains the following, a special case of Proposition 4.2 from [LP95]:

**Theorem 2.2.** [LP95] Let \( X \) be a smooth variety of dimension \( n \), and \( X \to \mathbb{P}^n \) a generically finite dominant rational map. Let \( Z \subset X \) be a finite reduced fiber. Then \( Z \) satisfies the Cayley-Bacharach condition with respect to the canonical linear system \( |K_X| \).

If the canonical bundle of \( X \) is sufficiently positive, this forces various geometric constraints on the fibers. For instance, if \( X \) is a hypersurface, a simple geometric argument shows that under certain degree hypotheses the above theorem implies that the general fiber must be
collinear (cf. [BCD14]). The authors of [BDE+16] exploited this fact to compute the degree of irrationality of very general hypersurfaces.

For our purposes, we will only be dealing with the case in which $X$ is a curve. However, in this case, every such map is actually a morphism, and the result is classical and elementary:

**Theorem 2.3.** Let $C$ be a smooth curve and $f : C \to \mathbb{P}^1$ a branched covering. Then any reduced fiber $Z \subset C$ of $f$ satisfies the Cayley-Bacharach condition with respect to $|K_C|$.

**Proof.** Let $Z$ be a reduced fiber of $f$. Then the complete linear system $|Z|$ is base-point free, so for any $P \in Z$,

$$\dim |Z - P| = \dim |Z| - 1.$$ 

Applying Riemann-Roch, we get

$$\dim |K_C - Z| = \dim |K_C - (Z - P)|,$$

which is equivalent to the Cayley-Bacharach property. $\square$

### 3. A generalization of the Neother-Lefschetz theorem

In order to prove our main theorem, we need to show that a complete intersection curve lies on a complete intersection surface whose Picard group is generated by the hyperplane class. More precisely, the goal of this section is to prove the following theorem.

**Theorem 3.1.** Let $W$ be a smooth, complete intersection threefold in $\mathbb{P}^n$ with $n \geq 4$ of type $(w_4, \ldots, w_n)$, and let $C$ be a smooth complete intersection curve in $W$ of type $(d - e, d, w_4, \ldots, w_n)$. If $4 \leq d - e < d$, then the very general complete intersection surface $S$ containing $C$ of type $(d, w_4, \ldots, w_n)$ is smooth and satisfies $\text{Pic} S = \mathbb{Z} \cdot [O_S(1)]$.

If the surface of type $(d - e, w_4, \ldots, w_n)$ containing $C$ is smooth, then we recover a special case of [Lop91, Theorem III.2.1]. Here we will deal with the case when it is possibly singular. The assumption that $n \geq 4$ is a convenience—the statement can be extended to $\mathbb{P}^3$ by augmenting Proposition 3.2 below with Lemmas II.3.3’ and II.3.3” of [Lop91]. Following Lopez’s approach, we will restrict our attention to the main technical ingredient (Corollary 3.4) of Theorem 3.1 and refer to [GH78] for the remainder of the argument, which carries through without incident.

Let $T$ be the unique surface of type $(d - e, w_4, \ldots, w_n)$ containing $C$, and let $P$ be the very general surface of type $(e, w_4, \ldots, w_n)$ containing $C$. Let $X$ be the total space of the pencil interpolating $P \cup T$ and $S$; we will regard $X_0 = P \cup T$ as the central fiber, which is singular at the singularities of $T$ and along the double curve $D = P \cap T$.

We may choose $P$ to meet $S$ and $T$ transversely, which leaves $X$ with ordinary double point singularities at the finite intersection

$$P \cap T \cap S = D \cap C = \{p_1, \ldots, p_N\}.$$

Blowing up each of the $p_i$ produces a smooth model of $X$ whose central fiber has a quadric surface $Q_i$ over each singular point. The strict transform of $T$ specifies a ruling of each $Q_i$, and blowing down along each of the specified rulings yields a family $\tilde{X}$ with smooth total
space. Away from the central fiber, $X$ and $\tilde{X}$ are isomorphic, but the central fiber of $\tilde{X}$ is a reducible surface $\tilde{X}_0 = \tilde{P} \cup \tilde{T}$, where $\tilde{P}$ is the blowup of $P$ at each of the $p_i$, and $\tilde{T} \cong T$. $\tilde{T}$ and $\tilde{P}$ meet in a double curve $\tilde{D} \cong D$.

Since $\tilde{T}$ and $\tilde{P}$ meet transversely,

$$\text{Pic } \tilde{X}_0 = \text{Pic } \tilde{T} \times_{\text{Pic } \tilde{P}} \text{Pic } \tilde{P}. \quad (\ast)$$

and our goal is to compute $\text{Pic } \tilde{X}_0$. Consider the restrictions of $\text{Pic } \tilde{T}$ and $\text{Pic } \tilde{P}$:

\[
\begin{array}{ccc}
\text{Pic } \tilde{T} & \xrightarrow{r_1} & \text{Pic } \tilde{D} \\
\downarrow & & \downarrow \\
\text{Pic } \tilde{P} & \xrightarrow{r_2} & \text{Pic } \tilde{D}
\end{array}
\]

The main ingredient is the following proposition:

**Proposition 3.2.** With notation as above,

\begin{itemize}
  \item[(i)] $\text{Pic } P = \mathbb{Z} \cdot [\mathcal{O}_P(1)]$
  \item[(ii)] $\text{Ker } r_2 = \mathbb{Z} \cdot [\mathcal{O}_P(\sum p_i)(-dH)]$
  \item[(iii)] $\text{Im } r_1 \cap \text{Im } r_2 = \mathbb{Z} \cdot [\mathcal{O}_D(1)]$
  \item[(iv)] $r_1$ is injective.
\end{itemize}

**Proof.** By our assumptions on the degrees, (i) follows from the classical Noether-Lefschetz theorem, and (ii) follows from (i). Moreover, (iii) follows from a standard monodromy argument which is given in [Lop91, Lemma II.3.3], and it remains to show (iv).

First, note that the singularities of $T$ are isolated, since $C$ is smooth and moves in $T$. Furthermore, $T$ is a complete intersection and hence has, for instance, Cohen-Macaulay singularities, so $T$ is normal. Notationally, since $\tilde{T} \cong T$ and $\tilde{D} \cong D$, we will work with $T$ and $D$ for simplicity.

Lemma II.2.4 of [Lop91] shows that unless $e = 1$ and $T$ is either ruled by lines, the Veronese surface, or its general projection to $\mathbb{P}^4$ or $\mathbb{P}^3$, then there exists a pencil $|\mathcal{V}|$ of irreducible curves within $|\mathcal{O}_T(e)|$. If $T$ is the Veronese surface or its general projection to $\mathbb{P}^4$, $r_1$ is clearly injective. The general projection of the Veronese surface to $\mathbb{P}^3$ (the Steiner surface) is not normal (and we have excluded $n = 3$).

Therefore, if we assume that $T$ is not ruled, then we may assume that $T$ possesses a pencil $|\mathcal{V}|$ of irreducible curves within $|\mathcal{O}_T(e)|$. Let $f : T' \to \mathbb{P}^1$ be a resolution of $|\mathcal{V}|$:

\[
\begin{array}{ccc}
T' & \xrightarrow{f} & \mathbb{P}^1 \\
\downarrow & \nearrow \quad \nearrow \\
T & \xrightarrow{\epsilon} & \mathbb{P}^1
\end{array}
\]

Let $L$ be an element of $\text{Pic } T$, and assume $L$ that restricts trivially to a general member of $|\mathcal{V}|$. Then $\epsilon^*L$ restricts trivially to every fiber of $f$ over an open subscheme $U$ of $\mathbb{P}^1$, and cohomology and base change implies that $\epsilon^*L|_{f^{-1}(U)}$ is actually the pullback of an invertible sheaf on $\mathbb{P}^1|U$. Since there are finitely many fibers in the complement of $f^{-1}(U)$, all of which are irreducible, $\epsilon^*L$ is globally the pullback of a line bundle $\mathcal{O}_{\mathbb{P}^1}(\ell)$ on $\mathbb{P}^1$. 


But \( f^*\mathcal{O}_{\mathbb{P}^1}(\ell) \cong \ell(eH - E) \), where \( E \) is supported on the exceptional divisor \( \text{Exc}(\epsilon) \) of \( \epsilon \). On the complement of \( \text{Exc}(\epsilon) \), \( \epsilon^*L \cong L \cong \ell eH \), and since \( T \) is normal, the isomorphism \( L \cong \ell eH \) extends across all of \( T \). It follows that \( L \) is trivial, and the restriction map \( \text{Pic} \, T \to \text{Pic} \, C \) is injective for the very general curve \( C \) in \( |\mathcal{V}| \).

Next, assume that \( T \) is ruled by lines and \( \epsilon = 1 \). If \( T \) is smooth, then it cannot be ruled—it would be of general type. So \( T \) possesses some singularities, and the upshot is that \( T \) must be a cone:

**Lemma 3.3.** Let \( T \) be a singular, normal, irreducible surface in \( \mathbb{P}^n \) which is ruled by lines. Then \( T \) is a cone over a smooth, degenerate curve.

**Proof.** Let \( Z \) be a connected component of the curve in \( \mathbb{G}(1, n) \) which sweeps out \( T \), and let \( Z' \) be its normalization. Note that \( Z \) is irreducible, as otherwise \( T \) would be singular along a line, and likewise the normalization \( Z' \to Z \) is bijective. The universal line \( \Phi \to Z \) pulls back to a family of lines \( \Phi' \to Z' \), and there is a natural map \( \pi : \Phi' \to T \).

First, we claim that \( \pi \) is birational. Assume, for the sake of contradiction, that \( \deg \pi \geq 2 \), and let \( \Lambda \) be the line corresponding to an arbitrary point on \( Z \). For every point \( p \) along \( \Lambda \), there is an additional line on \( T \) meeting \( \Lambda \) at \( \pi \), and since \( Z \) is irreducible, it follows that every line which comes from \( Z \) meets \( \Lambda \). Applying the same argument to three distinct \( \Lambda \) yields that \( T \) is a plane, which contradicts various of the hypotheses on \( T \).

Second, we claim that \( \pi \) contracts a curve. This follows from the birationality of \( \pi \), the normality of \( T \), and Zariski’s Main Theorem, as well as the fact that \( \pi \) cannot be an isomorphism. Let \( E \) denote a curve contracted by \( \pi \). By definition of \( \Phi', E \) cannot be supported on the fibers of \( \Phi' \to Z' \), so \( E \) must meet every fiber. Then \( \pi(E) \) lies on every line which comes from \( Z \), and by taking a general hyperplane \( H \cap T \) section of \( T \), we see that \( T \) may be regarded as the cone over \( H \cap T \) with vertex \( \pi(E) \).

To conclude the proof of Proposition 3.2 note that \( \text{Cl} \, T \) is isomorphic to \( \text{Pic} \, D \) by [Har77, Exercise II.6.3]. Since \( T \) is normal, \( \text{Pic} \, T \) embeds into \( \text{Cl} \, T \), and thus the restriction map \( r_1 : \text{Pic} \, T \to \text{Pic} \, D \) is injective.

**Corollary 3.4.** \( \text{Pic} \, \tilde{X}_0 = \mathbb{Z} \cdot [\mathcal{O}_{\tilde{X}_0}(1)] \oplus \mathbb{Z} \cdot [M] \), where

\[
M|_{\tilde{T}} = \mathcal{O}_{\tilde{T}}
\]

\[
M|_{\tilde{P}} = \mathcal{O}_{\tilde{P}}(\epsilon) \otimes \mathcal{O}_{\tilde{P}}(\tilde{D})
\]

Note that by our description of the Picard group in (3), it suffice to specify \( M \) by specifying its restrictions to the components of \( \tilde{X}_0 \). The remainder of the proof of Theorem 3.1 follows the argument given in [GH78, pg. 37-39], which carries through in this setting without revision.
4. Proofs of the main theorems

The structure of the proof of Theorem C is as follows: we take a surface of Picard rank one containing the curve, and apply the following theorem of Griffiths and Harris to construct a vector bundle on the surface.

**Theorem 4.1.** [GH78] Let $S$ be a smooth projective surface, $L$ a line bundle on $S$, and $Z \subset S$ a reduced set of points. Then there exists a rank two vector bundle $\mathcal{E}$ with $\det \mathcal{E} = L$ along with a section $s \in H^0(\mathcal{E})$ with $Z(s) = Z$ if and only if $Z$ satisfies the Cayley-Bacharach property with respect to $|K_S + L|$.

We then determine that the vector bundle is Bogomolov unstable, which gives a lower bound on the degree of the fiber and forces the fiber to be contained in a hyperplane. Analyzing the geometry, we conclude that each hyperplane must contain a single fiber, and that the hyperplanes lie in a linear pencil.

For the remainder of the paper, let $C$ be a smooth curve satisfying the assumptions of Theorem C. Combining the Griffiths-Harris theorem with Theorem 2.3, we easily obtain the following.

**Lemma 4.2.** Let $C \to \mathbb{P}^1$ be a surjective morphism, and let $\Gamma \subset C$ be a general fiber. Then there is a rank two vector bundle $\mathcal{E}$ on $S$ sitting in the short exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{E} \to \mathcal{I}_{\Gamma,S}(\alpha) \to 0. \quad (4.1)$$

**Proof.** By Theorem 2.3, $\Gamma$ satisfies the Cayley-Bacharach condition with respect to the canonical linear series $|K_C|$. In particular, by adjunction, it satisfies the Cayley-Bacharach condition with respect to $|K_S + \alpha H|$ where $H$ is a hyperplane section on $S$. The statement then follows from Theorem 4.1. \hfill $\Box$

Using (4.1), we can check that $\mathcal{E}$ is Bogomolov unstable, producing a second representation of $\mathcal{E}$ as extension. The plan is to compare the two. We first recall Bogomolov’s Instability Theorem.

**Theorem 4.3.** [Bog78] Let $\mathcal{F}$ be a rank two vector bundle on a smooth projective surface $X$. If

$$c_1(\mathcal{F})^2 - 4c_2(\mathcal{F}) > 0, \quad (4.2)$$

then $\mathcal{F}$ is Bogomolov unstable. That is, there exists a finite subscheme $Z \subset X$ (possibly empty), plus line bundles $L$ and $M$ on $X$ sitting in an exact sequence

$$0 \to L \to \mathcal{F} \to M \otimes \mathcal{I}_Z \to 0 \quad (4.3)$$

where $(L - M)^2 > 0$ and $(L - M)A > 0$ for all ample divisors $A$.

We can now use this along with (4.1) to prove our key lemma. The technique of the proof is similar to that of Reider’s Theorem (cf. [Laz97] Theorem 2.1, or [Rei88] for Reider’s original proof).

**Lemma 4.4.** Let $C \to \mathbb{P}^1$ be a map of degree less than $d_C := \deg(C)$, and let $\Gamma \subset C$ be a general fiber. Then
(1) $\Gamma$ lies in a hyperplane, and
(2) $d_\Gamma := \deg(\Gamma) \geq \deg(S) \cdot (\alpha - 1)$.

Proof. Let $E$ be the vector bundle on $S$ obtained in Lemma 4.2. First, we show that $E$ is Bogomolov unstable. By (4.1), the Chern classes of $E$ are given by
\[ c_1(E) = \alpha H, \]
\[ c_2(E) = d_\Gamma, \]
where $d_\Gamma$ is the length of $\Gamma$. Let $d_S$ be the degree of $S$. Then
\[ c_1(E)^2 - 4c_2(E) = \alpha^2 d_S - 4d_\Gamma, \]
which greater than zero since $d_\Gamma < d_C = d_S \alpha$, and $\alpha \geq 4$. Thus, $E$ sits in the short exact sequence
\[ 0 \to L \to E \to M \otimes I_Z \to 0 \] (4.4)
satisfying the conditions from Theorem 4.3.

Now we show that $M$ is effective. Since $\text{Pic}(S) = \mathbb{Z}$, we can write $L = O_S(\lambda)$. By (4.1) and (4.4),
\[ c_1(E) = c_1(L) + c_1(M) = c_1(\alpha H). \]
Thus, $M \equiv \alpha H - L$. By the instability of $E$, we have
\[ (2L - \alpha H) \cdot H = (2\lambda - \alpha) d_S > 0. \]
So
\[ 2\lambda > \alpha \geq 4. \] (4.5)
In particular, $\lambda$ is positive. Thus, the composite map
\[ L \to E \to I_{\Gamma,S} \otimes O_S(\alpha) \]
is nonzero, as otherwise $L$ would map to the kernel, $O_S$, of the right-hand map. Twisting down by $\lambda$, we obtain a nonzero map
\[ O_S \to I_{\Gamma,S} \otimes O_S(\alpha - \lambda) = I_{\Gamma,S} \otimes M. \]
This implies that
\[ h^0(M) \geq h^0(I_{\Gamma,S} \otimes M) > 0. \]
Therefore, there is an effective curve
\[ C_0 \in |M| \]
which contains $\Gamma$. Also, $\alpha > \lambda$ (since $\Gamma$ is nonempty).

Now we approximate the intersection pairing $L \cdot M$. Let $d_Z$ denote the length of $Z$. Then by (4.1) and (4.4), we obtain
\[ d_\Gamma = c_2(E) = L \cdot M + d_Z. \]
Thus $d_\Gamma \geq L \cdot M$.

Collecting inequalities, we have
\[ \deg C = \alpha d_S > d_\Gamma \geq L \cdot M = \lambda(\alpha - \lambda) d_S. \]
Combining this inequality with (4.5), we get $0 < \alpha - \lambda < 2$. Thus, $\lambda = \alpha - 1$, which proves (2). For (1), notice

$$L = O_S(\alpha - 1), \quad M = O_S(1).$$

In particular, $\Gamma \subset C_0$ lies in a hyperplane, as desired.

Now that we know a general fiber of a map of minimal degree will lie in a hyperplane, it only remains to show that the corresponding pencil of hyperplanes forms a linear pencil and that a member of the pencil contains only one fiber.

**Proof of Theorem C.** Let $f : C \to \mathbb{P}^1$ be a morphism of degree equal to the gonality of $C$. First we show that each fiber of $f$ spans a unique hyperplane. For the sake of contradiction, suppose two distinct fibers lie in $H \subset \mathbb{P}^n$, a hyperplane. Then by Lemma 4.4

$$d_S(\alpha - 1) \leq \text{gon}(C) \leq \deg C = \frac{1}{2} d_S \alpha.$$

But $\alpha \geq 4$, so we get a contradiction.

Let $\Gamma \subset C$ be a general fiber of $f$, and suppose it doesn’t span a hyperplane. That is, assume $\Gamma \subset G$, where $G \subset \mathbb{P}^n$ is an $(n - 2)$-dimensional linear space. Then projection from $G$ determines a morphism $C \to \mathbb{P}^1$ of degree at most

$$d_C - \text{gon}(C) \leq \alpha d_S - (\alpha - 1) d_S = d_S < \text{gon}(C),$$

which is a contradiction.

Define the incidence correspondence

$$\Lambda := \{(Q, P) \mid Q \in \text{span}(f^{-1}(P))\} \subset \mathbb{P}^n \times \mathbb{P}^1,$$

with projections

$$\begin{array}{ccc}
\mathbb{P}^n & \overset{q}{\longrightarrow} & \Lambda \\
\downarrow \pi & & \downarrow \pi \\
\mathbb{P}^1 & & \mathbb{P}^1
\end{array}$$

Consider the (reducible) curve $q^{-1}(C) \in \Lambda$. Define $Y$ to be the union of the components of $q^{-1}(C)$ that dominate $\mathbb{P}^1$. By slight abuse of notation, we can write

$$Y = C \cup B,$$

where $C$ is mapped by $q$ isomorphically onto its image, and $B$ is a (possibly reducible) curve. Note that $Y$ has relative degree $d_C$ over $\mathbb{P}^1$, and $C$ has relative degree $\text{gon}(C)$ over $\mathbb{P}^1$. Thus, $B$ has relative degree $d_C - \text{gon}(C) > 0$ over $\mathbb{P}^1$.

Suppose first that $\dim(q(B)) = 0$. Then $q(B)$ is a finite set of points contained in the span of every fiber. If $q(B)$ spans an $(n - 2)$-dimensional space, then $\Lambda$ determines the unique pencil of hyperplanes containing $q(B)$, and we’re done. Thus, we can assume $q(B)$ spans a smaller dimensional space. Choose an $(n - 2)$-plane in $\mathbb{P}^n$ containing $q(B) \cup \{P\}$ where $P \in C - q(B)$. Projection from this linear space then yields a map to $\mathbb{P}^1$ of degree smaller than the gonality, a contradiction.
It remains to rule out the possibility that \( \dim(q(B)) = 1 \). In this case, let \( B_1 \subset B \) be an irreducible component which dominates \( C \). Then \( B_1 \) has relative degree

\[
b \leq d_C - \operatorname{gon}(C) < \operatorname{gon}(C)
\]

over \( \mathbb{P}^1 \). In particular, \( \operatorname{gon}(B_1) < \operatorname{gon}(C) \). However, since we have a branched covering \( B_1 \rightarrow C \), this is a contradiction.

\[\square\]

Corollary D follows quickly from Lemma 4.4 (2) and a theorem of Coppens and Martens.

**Proof of Corollary D.** The lower bound is immediate from Lemma 4.4 (2).

For the upper bound, since \( S \) is non-degenerate, \( \deg(S) \geq n - 1 \). Thus, \( \deg(C) \geq 4n - 4 \). Theorem A from \[CM91\] implies that if \( \deg(C) \geq 4n - 7 \), then \( C \) has a \( 2n - 3 \)-secant \((n - 2)\)-plane. Projecting from such a plane yields the upper bound.

\[\square\]

Notice that Corollary B is obtained by applying Corollary D to a complete intersection curve.

Theorem A follows easily from Theorem C and Theorem 3.1.

**Proof of Theorem A.** If \( n = 2 \) or 3, the Theorem follows from Noether’s or Basili’s Theorem, respectively, so we assume \( n \geq 4 \). Consider the linear subsystem \( D \) of \( |O_{\mathbb{P}^n}(a_{n-1})| \) consisting of sections vanishing on \( C \). Since \( C \) is a complete intersection, it is generated in its highest degree \( a_{n-1} \), so the base locus of \( D \) is \( C \). Thus, by the strong Bertini Theorem (see e.g. \[EH16, Proposition 5.6\]), a general member of \( D \) is smooth. Choosing such a hypersurface, and proceeding by induction, we can find a smooth complete intersection cubic of type \((a_3, a_4, \ldots, a_{n-1})\) containing \( C \) that is smooth. By Theorem 3.1, we can then find a smooth complete intersection surface \( S \) of type \((a_2, \ldots, a_{n-1})\) satisfying the hypotheses of Theorem C. The conclusion follows by applying Theorem C.

\[\square\]

5. A Cayley-Bacharach Conjecture

By modifying the proof of Lemma 4.4, we are able to prove the following special case of one of the Cayley-Bacharach conjectures (Conjecture CB12 of \[EGH96\]).

**Theorem 5.1.** Let \( \Gamma \) be any subscheme of a zero-dimensional complete intersection of hypersurfaces of degrees \( d_1 \leq \cdots \leq d_n \) in \( \mathbb{P}^n \), so that \( \Gamma \) is contained in a complete intersection surface \( S \) of type \((d_3, \ldots, d_n)\) with \( \operatorname{Pic}(S) \) generated by the hyperplane class. Set

\[
k := d_3 + \cdots + d_n - n - 1.
\]

If \( \Gamma \) fails to impose independent conditions on hypersurfaces of degree \( k + e + 2 \), where \( 0 \leq e \leq d_2 - 1 \), then

\[
\deg(\Gamma) \geq (e + 1) \cdot d_3 \cdot d_4 \cdot \cdots \cdot d_n.
\]

**Remark 5.2.** In the notation of the original statement of the conjecture, we are considering the case \( s = 3 \), and setting \( m = k + e + 2 \). Notice that in this case, we are able to obtain a stronger bound than the one given in the conjecture.
Before the proof, we need a slightly more general formulation of Theorem 4.1 in order to deal with non-reduced 0-cycles.

**Theorem 5.3** (cf. [Laz97], Prop 3.9). Let $S$ be a smooth projective surface, $Z \subset S$ a zero-dimensional subscheme, and $L$ a line bundle on $S$. Given an element $\eta \in \text{Ext}^1(L \otimes \mathcal{I}_{Z,S}, \mathcal{O}_S)$, denote by $\mathcal{F}_\eta$ the sheaf arising from the extension:

$$0 \to \mathcal{O}_S \to \mathcal{F}_\eta \to L \otimes \mathcal{I}_{Z,S} \to 0.$$ 

Then $\mathcal{F}_\eta$ fails to be locally free if and only if there exists a proper (possibly empty) subscheme $Z' \subsetneq Z$ such that

$$\eta \in \text{Im} \left\{ \text{Ext}^1(L \otimes \mathcal{I}_{Z',S}, \mathcal{O}_S) \to \text{Ext}^1(L \otimes \mathcal{I}_{Z,S}, \mathcal{O}_S) \right\}.$$ 

**Proof of Theorem 5.3.** Set $m := k + e$. 

First we show that there is some non-empty subscheme $Z \subset \Gamma$ such that $h^1(\mathcal{I}_{Z,\mathbb{P}^n}(m)) \neq 0$ but $h^1(\mathcal{I}_{Z',\mathbb{P}^n}(m)) = 0$ for all proper subschemes $Z' \subsetneq Z$ by induction on the length of $\Gamma$. Notice that since $h^1(\mathcal{O}_{\mathbb{P}^n}(m)) = 0$, the condition $h^1(\mathcal{I}_{Z,\mathbb{P}^n}(m)) \neq 0$ is equivalent to $Z$ failing to impose independent conditions on $\mathcal{O}_{\mathbb{P}^n}(m)$, so if the proper subschemes of $\Gamma$ all satisfy the desired cohomological property, we are done. Otherwise, there is some nonempty $Z \subsetneq \Gamma$ such that $h^1(\mathcal{I}_{Z,\mathbb{P}^n}(m)) \neq 0$, and we are done by the induction hypothesis. For the base case, assume $\Gamma$ consists of a single point. Then the only proper subscheme is the empty set, which trivially imposes independent conditions on $\mathcal{O}_{\mathbb{P}^n}(m)$, as desired. Since $\deg(\Gamma) \geq \deg(Z)$, we can replace $\Gamma$ with $Z$ for the remainder of the proof.

Since $S$ is a complete intersection with embedding line bundle $\mathcal{O}_S(1)$, we know $h^1(\mathcal{O}_S(m)) = 0$. Thus, we obtain the following commutative diagram with exact rows:

\[
\begin{array}{cccccc}
    h^0(\mathcal{O}_{\mathbb{P}^n}(m)) & \longrightarrow & h^0(\mathcal{O}_Z(m)) & \longrightarrow & h^1(\mathcal{I}_{Z,\mathbb{P}^n}(m)) & \longrightarrow & 0 \\
    \downarrow & & \downarrow \neq & & \downarrow & & \\
    h^0(\mathcal{O}_S(m)) & \longrightarrow & h^0(\mathcal{O}_Z(m)) & \longrightarrow & h^1(\mathcal{I}_{Z,S}(m)) & \longrightarrow & 0
\end{array}
\] 

(5.1)

A straightforward diagram chase shows that $h^1(\mathcal{I}_{Z,\mathbb{P}^n}(m)) = 0$ if and only if $h^1(\mathcal{I}_{Z,S}(m)) = 0$. Thus, since $\omega_S = \mathcal{O}_S(k)$, Serre duality yields

$$\text{Ext}^1(\mathcal{I}_{Z,S}(e + 2), \mathcal{O}_S) \cong h^1(\mathcal{I}_{Z,S}(m))^\vee \neq 0$$

and similarly

$$\text{Ext}^1(\mathcal{I}_{Z',S}(e + 2), \mathcal{O}_S) = 0.$$

Therefore, by Theorem 5.3, we can find a nontrivial extension

$$0 \to \mathcal{O}_S \to \mathcal{E} \to \mathcal{I}_{Z,S}(e + 2) \to 0$$

(5.2)

with $\mathcal{E}$ locally free.

For the sake of contradiction, assume

$$\deg(Z) < (e + 1) \cdot d_3 \cdot d_4 \cdots \cdot d_n = e \cdot \deg(S).$$

Then

$$c_1(\mathcal{E})^2 - 4c_2(\mathcal{E}) = (e + 2)^2 \deg(S) - 4 \deg(Z) > ((e + 2)^2 - 4(e + 1)) \deg(S) \geq 0.$$
Thus, by Theorem 4.3, $E$ is Bogomolov unstable, and we can write it as an extension

$$0 \to L \to E \to M \otimes I_W \to 0,$$

where $L$ and $M$ are line bundles satisfying the conditions from Theorem 4.3 and $W \subset S$ is a finite subscheme. Since the Picard group of $S$ is generated by the hyperplane class, we can set $L = \mathcal{O}_S(\lambda)$.

Following the proof of Lemma 4.4 ($e + 2$ taking the place of $\alpha$), we conclude that

$$2\lambda > e + 2 \geq 2,$$

$$e + 2 > \lambda,$$

and

$$\lambda(e + 2 - \lambda) \cdot \deg(S) \leq \deg(Z) < (e + 2) \cdot \deg(S).$$

Combining these inequalities, we obtain

$$e + 2 - \lambda = 1.$$

This implies

$$(e + 1) \cdot \deg(S) \leq \deg(Z),$$

which gives us a contradiction. □

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