SECOND ORDER ESTIMATES FOR COMPLEX HESSIAN EQUATIONS ON HERMITIAN MANIFOLDS

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ABSTRACT. We derive second order estimates for \( \chi \)-plurisubharmonic solutions of complex Hessian equations with right hand side depending on the gradient on compact Hermitian manifolds.

1. Introduction. Let \((M, \omega)\) be a compact complex manifold of complex dimension \( n \geq 2 \). For any smooth function \( u \in C^\infty(M) \), let \( \chi(z, u) \) be a smooth real (1,1) form on \( M \) and \( \psi(z, v, u) \in C^\infty((T^{1,0}(M))^* \times \mathbb{R}) \) be a positive function. The following equation which we shall call it the complex Hessian equation:

\[
(\chi(z, u) + \sqrt{-1} \partial \bar{\partial} u)^k \wedge \omega^{n-k} = \psi(z, Du, u) \omega^n,
\]

(1)

for \( 1 \leq k \leq n \), can be viewed as an intermediate equation between the Laplace equation \((k = 1)\) and the complex Monge-Ampère equation \((k = n)\), where \( D \) is the covariant derivative with respect to the given metric \( \omega \).

A function \( u \in C^2(M) \) is called admissible if \( g = \chi(z, u) + \sqrt{-1} \partial \bar{\partial} u \in \Gamma_k(M) \). We formally define \( \Gamma_k(M) \) in (8). In particular, \( u \) is called \( \chi \)-plurisubharmonic if \( g = \chi(z, u) + \sqrt{-1} \partial \bar{\partial} u > 0 \) (i.e. \( g \in \Gamma_n(M) \)). When \( k = 1 \), (1) is just a quasilinear equation which is well understood. Otherwise, it is fully nonlinear and a natural approach to solve (1) is the continuity method which reduces the solvability to a priori estimates of solutions up to the second order. Higher order estimates follow from Evans-Krylov theory and Schauder estimate. Note that Tian [19] presented a new proof of the \( C^{2, \alpha} \) estimates (for real and complex Monge-Ampère equations), which not only weakens the regularity assumptions on \( \psi(z) \) but also can be applied to more general nonlinear elliptic systems. In this paper, we are mainly concerned with the second order estimate of (1).

On compact Kähler manifolds, the existence and regularity theory for complex Monge-Ampère equations has been studied for a long time since Yau proved the Calabi conjecture in [24], where he also studied the case when the right-hand side may degenerate or have poles. Later after that, Cheng-Yau [1], Kobayashi [12]...
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and Tian-Yau [20, 21] generalized the conjecture to certain complete non-compact Kähler manifolds. The complex Hessian equation also appears in many geometric problems, such as the Calabi-Yau equation studied by Fino-Li-Salamon-Vezzoni [7] and the J-flow studied by Song-Weinkove [17]. The solvability of (1) when $\chi = \omega$ and $\psi = \psi(z)$ was proved in the work of Hou-Ma-Wu [11] and Dinew-Ko lodziej [6].

There has been growing interest in extending the above results to non-Kähler settings. For the complex Monge-Ampère equation, it was solved by Tosatti-Weinkove [22] on Hermitian manifolds, as well as by Chu-Tosatti-Weinkove [5] on almost Hermitian manifolds. Székelyhidi [18] and Zhang [26] independently solved the equation (1) on compact Hermitian manifolds when $\chi = \omega$ and $\psi = \psi(z)$. Chu-Huang-Zhu [4] obtained the second order estimate with $k = 2$ on an almost Hermitian manifold.

When $\psi = \psi(z, Du, u)$, (1) has been much less studied so far. Actually, it is still an open problem even for the real counterparts of (1) to derive a second order estimate. Recently, Guan-Ren-Wang [10] solved this problem for convex solutions. Borrowing the idea in [10] and adapting the techniques for real Hessian equations, Phong-Picard-Zhang [15] obtained the second order estimate for $\chi$-plurisubharmonic solutions of (1) on Kähler manifolds. In this paper, motivated by [15], we study (1) on compact Hermitian manifolds and derive the second order estimate for $\chi$-plurisubharmonic solutions. Our main result is stated as follows.

**Theorem 1.1.** Let $(M, \omega)$ be a compact Hermitian manifold of complex dimension $n$. Suppose $u \in C^4(M)$ is a $\chi$-plurisubharmonic solution of (1) and $\chi(z, u) \geq \varepsilon \omega$. Then we have the uniform second order derivative estimate

$$|D^2 u|_\omega \leq C,$$

where $C$ is a uniform constant depending only on $(M, \omega)$, $\varepsilon$, $n$, $k$, $\chi$, $\psi$, sup$_M |u|$, sup$_M |Du|$.

**Remark 1.** The above estimate can be stated for $g \in \Gamma_{k+1}$, see the remark after Theorem 1 and Remark 2 in [15]. An interesting question is to derive the estimate with the more natural elliptic condition $g \in \Gamma_k$. If $k = n$, $\chi$-plurisubharmonic is already the natural assumption for ellipticity of (1). So our result generalizes the estimate for Monge-Ampère equations on compact Hermitian manifolds. If $k = 2$, the estimate was derived without the plurisubharmonicity assumption, see [4].

An important case of (1) with $\psi$ depending on $Du$ is the Fu-Yau equation. As a reduced version of generalized Hull-Strominger system in higher dimensions, Fu-Yau in [8] introduced a fully nonlinear equation which can be rewritten as a $\sigma_2$-type equation with specific right hand side $\psi(z, Du, u)$. When $n = 2$, the Fu-Yau equation is equivalent to the Strominger system on a toric fibration over a $K3$ surface constructed by Goldstein-Prokushkin, which was solved by Fu-Yau in [8]. For higher dimension $n$, the corresponding problem on compact Kähler or non-Kähler manifolds has been well studied, see [2, 16, 3].

The dependence on the gradient of $u$ in (1) creates substantial new difficulties, due to the appearance of terms such as $|DDu|^2$ and $|D^2 u|^2$ when one differentiates the equation twice. A consequence of this is that we cannot control the bad third order terms straightly as in Li [14] and Guan-Jiao [9]. Furthermore, it is made more difficult by the differences between the real case and the complex case to control the negative third order terms due to complex conjugacy. The differences can be seen right through the definition of third order terms in Section 3 that the good terms “$B$” and “$D$” compared with those in [10] get doubled in the real case. These are
the main difficulties that have been overcome in [15] on Kähler manifolds. In the Hermitian case, there are more bad terms from the appearance of the form $T + D^3 u$, where $T$ is the torsion of $\omega$ and $D^3 u$ represents the third derivatives of the solution $u$. To control these terms, we modify the auxiliary function. This gives us a little more good third order terms which are sufficient to push the argument through.

Many authors are attracted by some other fully nonlinear equations involving gradient terms. In contrast to the equation (1), those gradient terms appear in $\chi$. For example, see a recent work by Tosatti-Weinkove [23] where they solved the complex Monge-Ampère-type equation with gradient terms when $n > 2$. This study was initiated and studied by Yuan for $n = 2$ in [25]. Very recently, Yuan informed the authors that he solved the same problem as in [23]. Moreover, he studied the Dirichlet problem of more general fully nonlinear equations on Hermitian manifolds. Our method here can also deal with such extra gradient terms. Suppose $a$ is a smooth $(1,0)$-form on $M$. Let

$$\tilde{g}(z, u, Du) = \tilde{\chi}(z, u, Du) + \sqrt{-1} \partial \bar{\partial} u,$$

where $\chi(z, u, Du) = \chi(z, u) + \sqrt{-1} a \wedge \partial u - \sqrt{-1} a \wedge \partial u$, and consider the equation

$$\tilde{g}^k \wedge \omega^{n-k} = \psi(z, Du, u) \omega^n,$$

for $1 \leq k \leq n$. In analogy to Theorem 1.1, we have the following result.

**Theorem 1.2.** Let $(M, \omega)$ be a compact Hermitian manifold of complex dimension $n$. Suppose $u \in C^4(M)$ is a $\tilde{\chi}$-plurisubharmonic solution of (4) and $\chi(z, u) \geq \varepsilon \omega$. Then we have the uniform second order derivative estimate

$$|D^2 u|_\omega \leq C,$$

where $C$ is a uniform constant depending only on $(M, \omega)$, $\varepsilon$, $n$, $\chi$, $\psi$, $a$, $\sup_M |u|$, $\sup_M |Du|$. 

**Remark 2.** If $\psi$ does not depend on $Du$, one can get the above estimate for admissible solutions by estimating the largest eigenvalue of $\tilde{g}$ as in [18]. We only give an outline of the proof of Theorem 1.2. For the complete details of the proof, see the follow-up work [13] by Li-Shen after the current paper.

The rest of the paper is organized as follows. In Section 2, we introduce some notations and calculations. We prove Theorem 1.1 in Section 3 and give an outline of the proof of Theorem 1.2 in Section 4.

### 2. Preliminaries

To be more clear, we shall rewrite the equation (1) in local coordinates. First, we introduce some notations. For $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n$, the $k$-th elementary symmetric function is defined by $\sigma_k(\lambda) = \sum \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}$, where the sum is over $\{1 \leq i_1 < \cdots < i_k \leq n\}$. Define the cone $\Gamma_k$ by

$$\Gamma_k := \{ \lambda \in \mathbb{R}^n : \sigma_j(\lambda) > 0, j = 1, \cdots, k \}.$$  

(6)

Let $\lambda(\alpha)$ denote the eigenvalues of a Hermitian symmetric matrix $(\alpha)$. Define $\sigma_k(\alpha) = \sigma_k(\lambda(\alpha))$. This definition can be naturally extended to complex manifolds. Let $A^{1,1}(M)$ be the space of smooth real $(1,1)$-forms on $(M, \omega)$. For any $h \in A^{1,1}(M)$, define

$$\sigma_k(h) = \binom{n}{k} h^k \wedge \omega^{n-k} \omega^n.$$ 

(7)
Now, $\Gamma_k$ can be defined on $M$ as follows
\[ \Gamma_k(M) := \{ h \in A^{1,1}(M) : \sigma_j(h) > 0, j = 1, \ldots, k \}. \] (8)

For a function $u \in C^\infty(M)$, we denote
\[ g = \chi(z, u) + \sqrt{-1} \partial \bar{\partial} u. \] (9)

With the above notation, in local coordinates, (1) can be rewritten as follows:
\[ \sigma_k(g) = \sigma_k \left( \chi_{\bar{\partial}} + u_{\bar{\partial}} \right) = \psi(z, Du, u). \] (10)

Recall that if $F(A) = f(\lambda_1, \ldots, \lambda_n)$ is a symmetric function of the eigenvalues of a Hermitian matrix $A = (a_{ij})$, then at a diagonal matrix $A$ with distinct eigenvalues, we have
\[ F_{ij} = \delta_{ij} f_i, \]
\[ F_{i}^{\bar{j} \sigma} u_{\bar{j}k} w_{\bar{i} \rho} = \sum_{p=1}^n f_p u_{\bar{i}} u_{\bar{j}k} + \sum_{p \neq q} f_p \frac{f_q}{\lambda_p - \lambda_q} |w_{\rho k}|^2, \]
where $F_{ij} = \frac{\partial F}{\partial a_{ij}}$, $F_{i}^{\bar{j} \sigma} = \frac{\partial F}{\partial a_{i} \bar{\partial} \sigma}$, and $w_{\bar{j}k}$ is an arbitrary tensor. Note that these formulas make sense even when the eigenvalues are not distinct, since it can be interpreted as a limit.

In local complex coordinates $(z_1, \ldots, z_n)$, the subscripts of a function $u$ always denote the covariant derivatives of $u$ with respect to the Chern connection of $\omega$ in the directions of the local frame $(\partial/\partial z^1, \ldots, \partial/\partial z^n)$. Namely,
\[ u_i = D_i u = D_{\partial/\partial z^i} u, \quad u_{\bar{i}} = D_{\bar{\partial}/\bar{\partial} z^i} D_{\partial/\partial z^i} u, \quad u_{\bar{i}l} = D_{\partial/\partial z^l} D_{\bar{\partial}/\bar{\partial} z^i} D_{\partial/\partial z^i} u. \]

We have the following commutation formula on Hermitian manifolds
\[ u_{i\bar{j}l} = u_{i\bar{j}l} - u_{p} R_{i\bar{j}l}^{p}, \]
\[ u_{p\bar{j}m} = u_{p\bar{m}j} - T_{mj}^{p} u_{p \bar{n}}, \]
\[ u_{p\bar{i}l} = u_{p\bar{i}l} - T_{\bar{t}l}^{p} u_{p \bar{n}}, \]
\[ u_{i\bar{j}m} = u_{i\bar{m}j} + u_{p\bar{m}j} R_{mij}^{p} - u_{p\bar{m}j} R_{ijp}^{m} - T_{\bar{t}l}^{m} u_{p \bar{n}} - T_{\bar{t}l}^{m} T_{\bar{n}j}^{m} u_{p \bar{n}}. \]

We use the notation
\[ \sigma_k^{\bar{\partial}} = \frac{\partial \sigma_k(g)}{\partial g_{\bar{n}m}}, \quad \sigma_k^{\bar{\partial} \partial} = \frac{\partial^2 \sigma_k(g)}{\partial g_{\bar{n}m} \partial g_{\bar{n}m}}, \quad F = \sum_{p} \sigma_k^{\bar{\partial} \partial}. \]
\[ (15) \]

We also use the following notation as in \[ 15]\]
\[ |D^2u|^2_{\sigma \omega} = \sigma_k^{\bar{\partial} \partial} \omega^{mn} u_{mp} u_{m \bar{n}}, \quad |D \bar{D} u|^2_{\sigma \omega} = \sigma_k^{\bar{\partial} \partial} \omega^{mn} u_{m \bar{n}} u_{m \bar{n}}. \]
\[ (16) \]

and
\[ |\eta|^2_{\sigma} = \sigma_k^{\bar{\partial} \partial} \eta_{m} \eta_{m}, \]
\[ (17) \]
for any 1-form $\eta$.

Now we do some basic calculations which are used in next Section. In the following, $C$ will be a uniform constant depending on the known data as in Theorem \[ 1.1], but may change from line to line.

Our calculations are carried out at a point $z$ on the manifold $M$, and we use coordinates such that at this point $\omega = \sqrt{-1} \sum \delta_{k \ell} dz^k \wedge d\bar{z}^\ell$ and $g_{\bar{i}j}$ is diagonal. Differentiating (10) yields
\[ \sigma_k^{\bar{\partial} \partial} D_{\bar{i}} g_{\bar{n}m} = D_{\bar{i}} \psi. \]
\[ (18) \]
Differentiating the equation a second time gives

\[
\sigma_k^{p,q} D_7 D_7 g_{ij} \leq C(1 + |DDu|^2 + |D\overline{Du}|^2) + \sum_{\ell} \psi_{ij} u_{ij\ell} + \sum_{\ell} \psi_{ij} u_{ij\ell} - C(1 + |DDu|^2 + |D\overline{Du}|^2) + \sum_{\ell} \psi_{ij} u_{ij\ell} + \sum_{\ell} \psi_{ij} D_7 g_{ij} - C\lambda_1.
\]  

(19)

Direct calculation gives the estimate

\[
\sigma_k^{p,q} D_7 D_7 g_{ij} \geq \sigma_k^{p,q} D_7 D_7 D_7 g_{ij} - C(1 + \lambda_1)F.
\]

(20)

Commuting derivatives yields that

\[
D_7 D_7 g_{ij} = D_7 D_7 D_7 g_{ij} - R_{ijp} a u_{a\bar{a}} + R_{\bar{a}p\bar{a}} a u_{a\bar{a}} - T_{p} u_{a\bar{a}} - T_{p} u_{a\bar{a}} - T_{p} u_{a\bar{a}} - T_{p} u_{a\bar{a}} + \sum_{\ell} \psi_{ij} g_{ij\ell} + \sum_{\ell} \psi_{ij} g_{ij\ell} - \sigma_k^{p,q} (T_{p} u_{a\bar{a}})
\]

(21)

Therefore, by (19), (20), (21), we see

\[
\sigma_k^{p,q} D_7 D_7 g_{ij} \geq \sigma_k^{p,q} D_7 g_{ij} D_7 g_{ij} + \sum_{\ell} \psi_{ij} g_{ij\ell} + \sum_{\ell} \psi_{ij} g_{ij\ell} - \sigma_k^{p,q} (T_{p} u_{a\bar{a}})
\]

(22)

\[
\geq - \sigma_k^{p,q} D_7 g_{ij} D_7 g_{ij} + \sum_{\ell} \psi_{ij} g_{ij\ell} + \sum_{\ell} \psi_{ij} g_{ij\ell} - \sigma_k^{p,q} (T_{p} u_{a\bar{a}})
\]

Then,

\[
\sigma_k^{p,q} |Du|^2_{p,q} = \sigma_k^{p,q} \left( u_{mpq} D^m u + u_{m} u_{pq} u_{m} \right) + |DDu|^2_{\sigma\omega} + |D\overline{Du}|^2_{\sigma\omega}
\]

(23)

Using the Cauchy inequality, we have

\[
|\sigma_k^{p,q} T_{mp} u_{pq} D^m u| \leq \sum_{p,\ell} \sigma_k^{p,q} \left( \frac{1}{4} |u_{p\ell}|^2 + (C_p^t)^2 \right) \leq \frac{1}{4} |D\overline{Du}|^2_{\sigma\omega} + CF,
\]

where $C_p^t = T_{mp} D^m u$. Similarly, we have

\[
| - \sigma_k^{p,q} u_{m} u_{mp} u_{mq} + \sum_{\ell} \psi_{ij} u_{ij\ell} + \sum_{\ell} \psi_{ij} u_{ij\ell} - \sigma_k^{p,q} (T_{p} u_{a\bar{a}})| \leq \frac{1}{4} |D\overline{Du}|^2_{\sigma\omega} + CF.
\]
Substituting the above two inequalities into (23), we get
\[
\sigma_k^m |Du|^2_{\bar{D}} \\
\geq \sigma_k^m D_m g_{\bar{\rho} \bar{\sigma}} D^m u + \sigma_k^m u_m D^m g_{\bar{\rho} \bar{\sigma}} + |Du|^2_{\bar{\sigma} \omega} + \frac{1}{2} |D\bar{D}u|^2_{\bar{\sigma} \omega} - CF
\]
(24)
\[
= D_m (\sigma_k) u_{\bar{\sigma} \omega}^m \bar{\tau} + D_{\bar{\tau}}(\sigma_k) u_{\bar{\rho} \bar{\sigma}}^m \bar{\tau} + |Du|^2_{\bar{\sigma} \omega} + \frac{1}{2} |D\bar{D}u|^2_{\bar{\sigma} \omega} - CF.
\]
Using the differential equation (18), we obtain
\[
\sigma_k^m |Du|^2_{\bar{D}} \geq 2 \text{Re} \left\{ \sum_{p,m} (D_p D_m u D_{\bar{\tau}} u + D_p u D_{\bar{\tau}} D_m u) \psi_{\nu_m} \right\}
\]
(25)
\[- C - CF + |Du|^2_{\bar{\sigma} \omega} + \frac{1}{2} |D\bar{D}u|^2_{\bar{\sigma} \omega}.
\]
We also compute that
\[- \sigma_k^m \nu_{\bar{\rho} \bar{\sigma}} = \sigma_k^m (\chi_{\bar{\rho} \bar{\sigma}} - g_{\bar{\rho} \bar{\sigma}}) \geq \varepsilon F - k \psi.
\]
(26)

3. **Proof of Theorem 1.1.** We denote by \(\lambda_1, \lambda_2, \ldots, \lambda_n\) the eigenvalues of \(g_{\bar{\tau} \bar{\sigma}} = \chi_{\bar{\tau} \bar{\sigma}} + u_{\bar{\tau} \bar{\sigma}}\) with respect to \(\omega\). When \(k = 1\), the equation (1) becomes
\[
\Delta_{\omega} u + \text{Tr}_{\omega} \chi(z, u) = n\psi(z, Du, u).
\]
(27)
It follows that \(\Delta_{\omega} u\) is bounded, and the desired estimate follows in turn from the positivity of \(g\). Henceforth, we assume that \(k \geq 2\).

We apply the maximum principle to the following test function:
\[
G = \log P_m + \varphi(|Du|^2) + \phi(u),
\]
(28)
where \(P_m = \sum_j \lambda_j^m\). Here \(m\) is a large positive integer to be determined later. Also, \(\varphi\) and \(\phi\) are positive functions to be determined later, which satisfy the following assumptions
\[
\varphi'' - \phi'' \left(\frac{\varphi'}{\phi'}\right)^2 \geq 0, \quad \varphi' > 0, \quad \phi' < 0, \quad \phi'' > 0.
\]
(29)
We assume that the maximum of \(G\) is achieved at some point \(p \in M\). We choose the coordinate system centered at \(p\) such that \(\omega = \sqrt{-1} \sum \delta_k dz_k \wedge d\bar{z}^k\) and \(y_{\bar{\tau} \bar{\sigma}}\) is diagonal with the ordering \(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0\).

Differentiating \(G\), we first obtain the critical equation
\[
\frac{DP_m}{P_m} + \varphi' |Du|^2 + \phi' Du = 0.
\]
(30)
Differentiating \(G\) a second time, using (12) and contracting with \(\sigma_k^{\bar{\rho} \bar{\sigma}}\) yields
\[
0 \geq \frac{m}{P_m} \sum_j \lambda_j^{m-1} \sigma_k^{\bar{\rho} \bar{\sigma}} D_{\bar{\tau}} D_p g_{\bar{\tau} j} + \frac{m\sigma_k^{\bar{\rho} \bar{\sigma}}}{P_m} (m-1) \sum_j \lambda_j^{m-2} |D_p g_{\bar{\tau} j}|^2
\]
\[+ \frac{m\sigma_k^{\bar{\rho} \bar{\sigma}}}{P_m} \sum_{i \neq j} \frac{\lambda_i^{m-1} - \lambda_j^{m-1}}{\lambda_i - \lambda_j} |D_p g_{\bar{\tau} j}|^2 + \sigma_k^{\bar{\rho} \bar{\sigma}} (\phi'' D_p u D_{\bar{\tau}} u + \phi' u_{\bar{\rho} \bar{\sigma}})
\]
\[+ \sigma_k^{\bar{\rho} \bar{\sigma}} (\varphi'' D_p |Du|^2 D_{\bar{\tau}} |Du|^2 + \varphi' |Du|^2_{\bar{\rho} \bar{\sigma}}) - \frac{|DP_m|^2_{\bar{\sigma} \omega}}{P_m^2}.
\]
(31)
Here we used the notation introduced in Section 2.
Using the critical equation (30), we obtain

\[ D_p u D_p u \geq \frac{1}{2} \frac{|D_p P_m|^2}{P_m^2 (\phi')^2} - \left( \frac{\phi'}{\phi} \right)^2 |D_p |D u|_2^2. \]  

(32)

Substituting (22), (25), (26), (32) into (31),

\[ 0 \geq - \frac{C \sum_j \lambda_j^{m-1}}{P_m} (1 + |D Du|^2 + |D D u|^2 + (1 + \lambda_1) F + \lambda_1) \]
\[ + \sum_j \frac{\lambda_j^{m-1}}{P_m} \left( - \sigma_k^2 \int D_j g \sigma D_j g + \sum_\ell \psi_{j\ell} g_{j\ell} + \sum_\ell \psi_{\ell j} \right) \]
\[ - \sigma_k^2 \left( (T_p^a u_{p\sigma}) + \frac{m-1}{P_m} \sum_j \lambda_j^{m-2} |D_p g_{j\ell}|^2 \right) \]
\[ + \frac{1}{P_m} \sum_j \frac{\lambda_j^{m-1} - \lambda_j^{m-1}}{\lambda_1 - \lambda_j} |D_p g_{j\ell}|^2 - \left( 1 - \frac{\phi''}{2(\phi')^2} \right) \frac{|D P_m|^2}{P_m^2} \]  

(33)

It follows that

\[ 0 = \frac{m}{P_m} \sum_j \lambda_j^{m-1} D_\ell g_{j\ell} + \phi' D_\ell (u_p u_{p\sigma}^\sigma) + \phi' D_\ell u. \]

(34)

Similarly, we have

\[ 0 = \frac{m}{P_m} \sum \lambda_j^{m-1} \sum_\ell \psi_{\ell j} D_\ell g_{j\ell} \]
\[ + \frac{\phi'}{m} \sum_\ell \psi_{\ell j} \sum_p (u_{p\ell} u_{p\sigma}^\sigma) + \frac{\phi'}{m} \sum_\ell u_\ell \psi_{\ell j}. \]  

(35)
Then we have

\[
\frac{1}{P_m} \sum_j \lambda_j^{m-1} \sum_{\ell} (\psi_{\ell j} D_{\ell j} g_j + \psi_{\ell j} D_{\ell j} g_j) + 2 \frac{\varphi'}{m} \Re \left\{ \sum_{p, m} (D_p D_m u D_{\pi m} + D_p u D_{\pi m} D_m u) \psi_{\ell m} \right\} \geq - \frac{\varphi'}{m} \sum_{p, k, m} \psi_{\ell m} T_{km} u_k u_{\bar{\pi}}
\]

Using (12), one can obtain the well-known identity

\[
- \sigma_k^{p, q} D_j g_{\pi j} D_j g_{\pi j} = - \sigma_k^{p, q} D_j g_{\pi j} D_j g_{\pi j} + \sigma_k^{p, q} |D_j g_{\pi j}|^2, \tag{38}
\]

where \( \sigma_k^{p, q} = \frac{\partial}{\partial \lambda_p} \frac{\partial}{\partial \lambda_q} \sigma_k(\lambda) \).

We may assume that \( \lambda_1 \geq 1 \). By the assumption \( \varphi'' - \varphi''(\frac{\varphi'}{\varphi})^2 \geq 0 \) in (29), the main inequality (33) becomes

\[
0 \geq \frac{1}{P_m} \sum_j \lambda_j^{m-1} \left( - \sigma_k^{p, q} D_j g_{\pi j} D_j g_{\pi j} + \sigma_k^{p, q} |D_j g_{\pi j}|^2 \right) + \frac{2\sigma_k^{p, q}}{P_m} \sum_j \lambda_j^{m-1} \Re \left( T_{p j} u_{\pi j} \right) + \frac{\sigma_k^{p, q}}{P_m} (m - 1) \sum_j \lambda_j^{m-2} |D_p g_j|^2
\]

\[
+ \frac{\sigma_k^{p, q}}{P_m} \sum_{i \neq j} \lambda_i^{m-1} - \lambda_j^{m-1} |D_p g_j|^2 - \left( 1 - \frac{\varphi''}{2(\varphi')^2} \right) \frac{|D_{P_m} g_{\pi j}|^2}{m \Pi_{P_m}}
\]

\[
+ \frac{\varphi'}{m} \left( |D u|^2 + \frac{1}{2} |D \sqrt{u}|^2 \right) + \left( - \frac{\varphi'}{m} \varepsilon - \frac{\varphi'}{m} - C \right) F
\]

\[
+ C \frac{\varphi'}{m} \frac{m}{C} - C \frac{\varphi'}{\lambda_1} (1 + |D u|^2 + |D \sqrt{u}|^2).
\]

Let

\[
\tilde{A}_j = \frac{1}{P_m} \lambda_j^{m-1} \sum_{p, q} \sigma_k^{p, q} D_j g_{\pi j} D_j g_{\pi j}, \quad \tilde{B}_q = \frac{1}{P_m} \sum_{j, p} \lambda_j^{m-1} \sigma_k^{p, q} |D_j g_{\pi j}|^2,
\]

\[
C_p = \frac{m - 1}{P_m} \sigma_k^{p, q} \sum_j \lambda_j^{m-2} |D_j g_{\pi j}|^2, \quad \tilde{D}_p = \sigma_k^{p, q} \sum_{j \neq i} \lambda_j^{m-1} - \lambda_i^{m-1} |D_p g_j|^2,
\]

\[
E_i = \frac{m}{P_m} \sigma_k^{p, q} \sum_p \lambda_j^{m-1} |D_i g_{\pi j}|^2, \quad H_p = \frac{2\sigma_k^{p, q}}{P_m} \sum_{j, a} \lambda_j^{m-1} \Re(T_{p j} u_{\pi j}).
\]
Then (39) becomes
\[
0 \geq - \sum_j \hat{A}_j + \sum_q \hat{B}_q + \sum_p C_p + \sum_p \hat{D}_p - \sum_p H_p \\
- \left(1 - \frac{\phi''}{2(\phi')} \right) \sum_i E_i + \frac{\phi'}{m} \left( |DDu|_{\sigma_\omega}^2 + \frac{1}{2} |D\bar{D}u|_{\sigma_\omega}^2 \right) \\
+ \left(- \frac{\phi'}{m} \phi' - C \phi' - C \right) \mathcal{F} - C \left(- \frac{\phi'}{m} \phi' + 1 \right) \\
- \frac{C}{\lambda_1} \left(1 + |DDu|^2 + |D\bar{D}u|^2 \right) .
\]

(40)

We first deal with the torsion term \(H_p\). By (13), for any \(0 < \beta < 1\), we have
\[
H_p \leq \frac{2\sigma^p \Sigma}{P_m} \sum_{j, a} \lambda_j^{m-1} |T_{pj} D_p g_{pj}| + C \sigma^p \Sigma \\
\leq \frac{\sigma^p \Sigma}{P_m} \sum_{j, a} \left( \beta \lambda_j^{m-2} |D_p g_{pj}|^2 + \frac{1}{\beta} \lambda_j^m |T_{pj}|^2 \right) + C \sigma^p \Sigma \\
\leq \frac{\sigma^p \Sigma}{P_m} \beta \sum_{j \neq p} \lambda_j^{-2} |D_p g_{pj}|^2 + \frac{\sigma^p \Sigma}{P_m} \beta \sum_{j} \lambda_j^{m-2} |D_p g_{pj}|^2 + C \sigma^p \Sigma .
\]

(41)

By direct computation, we have
\[
\hat{D}_p = \frac{\sigma^p \Sigma}{P_m} \sum_{j \neq i} \sum_{s=0}^{m-2} \lambda_i^{m-2-s} \lambda_j^{s} |D_p g_{pj}|^2 \geq \frac{\sigma^p \Sigma}{P_m} \sum_{j \neq i} \lambda_i^{m-2} |D_p g_{pj}|^2 .
\]

(42)

Now we see that
\[
- \sum_p H_p + \sum_p C_p + \sum_p \hat{D}_p \\
\geq (1 - \beta) \sum_p \hat{D}_p + (1 - \frac{\beta}{m-1}) \sum_p C_p - \frac{C}{\beta} \mathcal{F} .
\]

(43)

Substituting (43) into (40) yields
\[
0 \geq - \sum_j \hat{A}_j + \sum_q \hat{B}_q + (1 - \beta) \left( \sum_p C_p + \sum_p \hat{D}_p \right) \\
- \left(1 - \frac{\phi''}{2(\phi')} \right) \sum_i E_i + \frac{\phi'}{m} \left( |DDu|_{\sigma_\omega}^2 + \frac{1}{2} |D\bar{D}u|_{\sigma_\omega}^2 \right) \\
+ \left(- \frac{\phi'}{m} \phi' - C \phi' - C \right) \mathcal{F} - C \left(- \frac{\phi'}{m} \phi' + 1 \right) \\
- \frac{C}{\lambda_1} \left(1 + |DDu|^2 + |D\bar{D}u|^2 \right) ,
\]

as \(m > 2\) will be chosen large enough.

We need a lemma from [10].
Lemma 3.1 ([10]). Suppose $1 \leq \ell < k \leq n$, and let $\alpha = 1/(k - \ell)$. Let $W = (w_{pq})$ be a Hermitian tensor in the $\Gamma_k$ cone. Then for any $\theta > 0$,

$$-\sigma_k^{p,q}(W)w_{p\ell}w_{q\ell} + \left(1 - \alpha + \frac{\alpha}{\theta}\right) \frac{|D_\ell \sigma_k(W)|^2}{\sigma_k(W)} \geq \sigma_k(W)(\alpha + 1 - \alpha \theta) \left| \frac{D_\ell \sigma_k(W)}{\sigma_\ell(W)} \right|^2 - \frac{\sigma_k(W)}{\sigma_\ell(W)} \sigma_k^{p,q}(W)w_{p\ell}w_{q\ell}. \quad (45)$$

Taking $\ell = 1$ and $\theta = 1/2$ in the above lemma, we have

$$-\sigma_k^{p,q}D_1g_{pt}D_\ell g_{q\ell} + K|D_\ell \sigma_k|^2 \geq 0, \quad (46)$$

for $K > (1 - \alpha + \alpha/\theta)(\inf \psi)^{-1}$ if $2 \leq k \leq n$. We shall denote

$$A_j = \frac{1}{P_m} \lambda_j^{m-1} \left(K|D_j \sigma_k|^2 - \sigma_k^{p,q}D_j g_{pt}D_\ell g_{q\ell}\right),$$

$$B_q = \frac{1}{P_m} \sum_p \lambda_p^{m-1} \sigma_k^{p,q} |D_q g_{pp}|^2,$$

$$D_i = \frac{1}{P_m} \sum_{p \neq i} \frac{\sigma_k^{p,q} \lambda_p^{m-1} - \lambda_i^{m-1}}{\lambda_p - \lambda_i} |D_i g_{pp}|^2.$$

Define $H_{p\ell} = D_j \chi_{p\ell} - D_q \chi_{j\ell}$. For any constant $0 < \tau < 1$, we can estimate

$$\sum_q \tilde{B}_q \geq \frac{1}{P_m} \sum_{j,q} \lambda_j^{m-1} \sigma_k^{j\ell q}\left|D_j g_{j\ell}\right|^2 = \frac{1}{P_m} \sum_{j,q} \lambda_j^{m-1} \sigma_k^{j\ell q}\left|D_j u_{j\ell} - T_{j\ell}^u u_{a3} + D_q \chi_{j\ell} + H_{j\ell q}\right|^2 \geq \frac{1}{P_m} \sum_{j,q} \lambda_j^{m-1} \sigma_k^{j\ell q}\left((1 - \tau)|D_q g_{j\ell}|^2 - (\frac{1}{\tau} - 1)|H_{j\ell q} - T_{j\ell}^u u_{a3}|^2\right) = (1 - \tau) \sum_q B_q - \frac{1 - \tau}{P_m} \sum_{q,j} \lambda_j^{m-2} \left(\sigma_k^{j\ell q} \lambda_j\right)|H_{j\ell q} - T_{j\ell}^u u_{a3}|^2.$$

Now we use $\sigma_1(\lambda|i)$ and $\sigma_1(\lambda|ij)$ to denote the $l$-th elementary function of

$$(\lambda|i) = (\lambda_1, \ldots, \hat{\lambda_i}, \ldots, \lambda_n) \in \mathbb{R}^{n-1}$$

and

$$(\lambda|ij) = (\lambda_1, \ldots, \hat{\lambda_i}, \ldots, \hat{\lambda_j}, \ldots, \lambda_n) \in \mathbb{R}^{n-2},$$

where we use the symbol “$\cdot$” to mean “omit”. Then, we have $\sigma_k^{\ell} = \sigma_{k-1}(\lambda|i)$, $\sigma_k^{p,\ell} = \sigma_{k-2}(\lambda|pi)$. By the formula $\sigma_1(\lambda) = \sigma_1(\lambda|p) + \lambda_p \sigma_{1-1}(\lambda|p)$ for any $1 \leq p \leq n$, we obtain

$$\frac{1}{\tau P_m} \sum_{q,j} \lambda_j^{m-2} \left(\sigma_k^{j\ell q} - \sigma_{k-1}(\lambda|j\ell)\right)|H_{j\ell q} - T_{j\ell}^u u_{a3}|^2 \leq \frac{C}{\tau F},$$

which implies

$$\sum_q \tilde{B}_q \geq (1 - \tau) \sum_q B_q - \frac{C}{\tau F}. $$
Similarly, we may estimate
\[
\sum_p D_p \geq \frac{1}{P_m} \sum_{j \neq i} \sigma_k^j \frac{\lambda_{i}^{m-1} - \lambda_{j}^{m-1}}{\lambda_{i} - \lambda_{j}} |D_j g_{ij}|^2
\]
\[
\geq \frac{1}{P_m} \sum_{j \neq i} \sigma_k^j \frac{\lambda_{i}^{m-1} - \lambda_{j}^{m-1}}{\lambda_{i} - \lambda_{j}} \left((1 - \tau)|D_j g_{ij}|^2 - \frac{C}{\tau} \lambda_i^2\right)
\]
\[
\geq (1 - \tau) \sum_i D_i - \frac{C}{\tau} F.
\]
Note that \(\frac{\lambda_{i}^{m-1}}{P_m} |D_j \sigma_k|^2 \leq \frac{C}{\lambda_1} (|DDu|^2 + |D\overline{D}u|^2)\). Then (44) becomes
\[
0 \geq \sum_i A_i + (1 - \tau) \sum_i B_i + (1 - \beta) \sum_i C_i
\]
\[
+ (1 - \beta)(1 - \tau) \sum_i D_i - \left(1 - \frac{\phi''}{2 (\phi')^2}\right) \sum_i E_i
\]
\[
+ \frac{\varphi'}{m} \left(|DDu|^2 + \frac{1}{2} |D\overline{D}u|^2\right) - \frac{C(K)}{\lambda_1} \left(|DDu|^2 + |D\overline{D}u|^2\right)
\]
\[
+ \left(- \frac{\phi'}{m} - C \frac{\varphi'}{\beta} - C \frac{C}{\tau} \right) F - C(\varphi' - \phi' + 1),
\]
when \(\lambda_1\) is sufficiently large. Let \(1 - \delta = (1 - \beta)(1 - \tau)\). We then have
\[
0 \geq (1 - \delta) \sum_i \left(A_i + B_i + C_i + D_i\right) - \left(1 - \frac{\phi''}{2 (\phi')^2}\right) \sum_i E_i
\]
\[
+ \frac{\varphi'}{m} \left(|DDu|^2 + \frac{1}{2} |D\overline{D}u|^2\right) - \frac{C(K)}{\lambda_1} \left(|DDu|^2 + |D\overline{D}u|^2\right)
\]
\[
+ \left(- \frac{\phi'}{m} - C \frac{\varphi'}{\beta} - C \frac{C}{\tau} \right) F - C(\varphi' - \phi' + 1).
\]

Now we choose \(\phi\) and \(\varphi\) to satisfy (29). Let \(\varphi(t) = e^{N t}\) and \(\phi(s) = e^{M(-s + L)}\) where \(L \geq |u| C^{2} + 1\) is a constant. Then, we see
\[
\varphi'' - \phi'' \frac{\varphi'^2}{\phi'^2} = N^2 e^{N t} - \frac{N^2 e^{2 N t}}{e^{M(-s + L)}} > 0, \quad \varphi' > 0, \quad \phi' < 0, \quad \phi'' > 0,
\]
when \(M \gg N > 1\), which shows the assumption (29) is satisfied. Since, at \(p\),
\[
\frac{\phi''}{2 (\phi')^2} = \frac{1}{2 e^{M(-u(p) + L)}}
\]
by choosing
\[
\beta = \tau = \frac{1}{6 e^{M(-u(p) + L)}},
\]
we obtain that \(1 - \delta \geq 1 - \frac{\phi''}{2 (\phi')^2}\). By the arguments as in [15], for sufficiently large \(m\), we can assume without loss of generality that
\[
A_i + B_i + C_i + D_i - E_i \geq 0, \quad \forall i = 1, \ldots, n.
\]
Note that the arguments in [15] for this step did not make use of the commutation formula, so there will be no more extra torsion terms.
By direct calculation and Cauchy inequality, we have
\[ |DDu|^2_{σ_ω} + \frac{1}{2} |D Dw|^2_{σ_ω} \geq \frac{1}{C λ_1} (|DDu|^2 + |D Dw|^2) \geq \frac{1}{C λ_1} |DDu|^2 + \frac{1}{C}. \]
For the inequality \( σ_k^T \geq \frac{k}{n} λ_1 \), one can find in [11, Lemma 2.2 (1)]. Now (48) becomes
\[ 0 \geq \left( \frac{φ'}{mC} - C(K) \right) λ_1 + \frac{1}{λ_1} \left( \frac{φ'}{mC} - C(K) \right) |DDu|^2 + \left( -\frac{φ'}{m} - C - \frac{C}{m} \right) F - C(φ' - φ' + 1). \]
Taking \( N \) large enough, we can ensure that \( \frac{φ'}{mC} - C(K) > 0 \). For fixed \( N \), it follows that
\[ -\frac{φ'}{m} - C - \frac{C}{m} = \frac{M}{m} φ - C N φ - C φ > 0 \]
when \( M \gg N \). This leads to
\[ 0 \geq \left( \frac{φ'}{mC} - C(K) \right) λ_1 - C, \]
which finally gives an upper bound of \( λ_1 \).

4. **Outline of proof of Theorem 1.2.** We can rewrite (4) as follows:
\[ σ_k(φ) = σ_k \left( φ_{i\overline{j}} + u_{i\overline{j}} + a_iu_{i\overline{j}} + a_{i\overline{j}}u_i \right) = ψ(z, Du, u). \]

By direct calculation and Cauchy inequality, we have
\[ σ_k^{P\overline{P}} D_{P\overline{P}} g_{i\overline{j}} \]
\[ \geq - σ_k^{P\overline{P}} D_{P\overline{P}} g_{i\overline{j}} + \sum_{\ell} ψ_{\ell \overline{\ell}} g_{i\overline{j}} + \sum_{\ell} ψ_{\ell \overline{\ell}} g_{i\overline{j}} - σ_k^{P\overline{P}} (T_{P\overline{P}} u_{i\overline{m}}) \]
\[ + T_{P\overline{P}} (D_{P\overline{P}} u_{i\overline{m}}) - C(1 + |DDu|^2 + |D Dw|^2 + F + λ_1 F + λ_1) + σ_k^{P\overline{P}} (a_iu_{i\overline{j}} + a_{i\overline{j}}u_i) - a_iu_{i\overline{j}} - a_{i\overline{j}}u_i - |DDu|^2_{σ_ω} - |DDu|^2_{σ_ω}. \]
and
\[ σ_k^{P\overline{P}} |Du|_{σ_ω}^2 \geq 2 Re \left\{ \sum_{\ell, m} (D_{\ell\overline{m}} Du_{i\overline{m}} + D_{\ell\overline{m}} Du_{i\overline{m}} u_{i\overline{m}} + D_{\ell\overline{m}} Du_{i\overline{m}} u_{i\overline{m}}) ψ_{\ell\overline{m}} \right\} \]
\[ - C - C F + \frac{1}{2} |DDu|^2_{σ_ω} + \frac{1}{2} |D Dw|^2_{σ_ω}. \]
We also have
\[ \left| \frac{2σ_k}{m} Re \left\{ a_i \left( D_{P\overline{P}} u_{i\overline{m}} + φ' D_{P\overline{P}} u_{i\overline{m}} \right) \right\} \right| \]
\[ \leq \frac{φ''}{4φ'^2} |D_{P\overline{P}} u_{i\overline{m}}|^2_m + \frac{σ_k}{m} φ'^2 |φ''| D_{P\overline{P}} u_{i\overline{m}}|^2 + C φ'^2 F. \]

Now we apply the maximum principle to the test function (28) bearing in mind that \( λ_1, λ_2, \ldots, λ_n \) are the eigenvalues of \( g_{i\overline{j}} = φ_{i\overline{j}} + u_{i\overline{j}} + a_iu_{i\overline{j}} + a_{i\overline{j}}u_i \) with respect to \( ω \). Instead of the assumptions in (29), we now require
\[ φ'' - 2φ'' \left( \frac{φ'}{φ'} \right)^2 \geq 0, \ φ' > 0, \ φ'' < 0. \]
With the above calculations, similar to (39), we have

\[
0 \geq \frac{1}{P_m} \sum_j \lambda_j^{m-1} \left( -\sigma_k^{\nu \nu} \frac{\partial^2 \varphi}{\partial \eta^2} D_j \tilde{g}_j \partial_{\eta} \tilde{g}_j + \sigma_k^{\nu \nu} \frac{\partial^2 \varphi}{\partial \eta^2} |D_j \tilde{g}_j|^2 \right)
- \frac{2\sigma_k^{\nu \nu}}{P_m} \sum_j \lambda_j^{m-1} \text{Re} \left( \frac{\partial_{\nu}}{\partial_{\eta}} u_{\nu j} \right) + \frac{(m-1)\sigma_k^{\nu \nu}}{P_m} \sum_j \lambda_j^{m-2} |D_p \tilde{g}_j|^2
+ \frac{\sigma_k^{\nu \nu}}{P_m} \sum_{i,j \neq j} \lambda_i^{m-1} |\lambda_j - \lambda_i| |D_p \tilde{g}_j|^2 - \frac{1}{4(m-1)^2} |D_p|^2 m \sum_j \lambda_j^{m-2} |D_p \tilde{g}_j|^2 \right)
\]

\[
\frac{\sigma_k^{\nu \nu}}{P_m} \sum_{i,j \neq j} \lambda_j^{m-1} \left( a_j u_{\eta j p}^2 + a_j u_{\eta j p}^2 - a_i u_{\eta j p}^2 - a_i u_{\eta j p}^2 \right),
\]

where we used \( \varphi'' - 2\varphi'' \frac{\partial^2 \varphi}{\partial \eta^2} \geq 0 \) in (54) and \( \lambda_1 > 1 \).

We use the same notation for third order terms as in Section 3. Denote

\[
I_p = \frac{\sigma_k^{\nu \nu}}{P_m} \sum_j \lambda_j^{m-1} \left( a_j u_{\eta j p}^2 + a_j u_{\eta j p}^2 - a_i u_{\eta j p}^2 - a_i u_{\eta j p}^2 \right).
\]

For any \( 0 < \beta < 1 \), we can estimate

\[
|I_p| \leq \frac{\beta \sigma_k^{\nu \nu}}{4 P_m} \sum_j \lambda_j^{m-2} \left( |D_p \tilde{g}_j|^2 + |D_p \tilde{g}_j|^2 \right) + \frac{C}{\beta} \sigma_k^{\nu \nu} + \frac{C}{\lambda_1} \sigma_k^{\nu \nu} \sum_j |u_{j p}|^2
\]

\[
\leq \frac{\beta \sigma_k^{\nu \nu}}{2 P_m} \sum_{j,s} \lambda_j^{m-2} |D_p \tilde{g}_j|^2 + \frac{C}{\beta} \sigma_k^{\nu \nu} + \frac{C}{\lambda_1} \sigma_k^{\nu \nu} \sum_j |u_{j p}|^2
\]

and

\[
H_p \leq \frac{2\sigma_k^{\nu \nu}}{P_m} \sum_{j,s} \lambda_j^{m-1} |D_p \tilde{g}_j|^2 + C \sigma_k^{\nu \nu} + \frac{C}{\lambda_1} \sigma_k^{\nu \nu} \sum_j |u_{j p}|^2
\]

\[
\leq \frac{\beta \sigma_k^{\nu \nu}}{2 P_m} \sum_{j,s} \lambda_j^{m-2} |D_p \tilde{g}_j|^2 + \frac{C}{\beta} \sigma_k^{\nu \nu} + \frac{C}{\lambda_1} \sigma_k^{\nu \nu} \sum_j |u_{j p}|^2.
\]

Now we have

\[
- \sum_p H_p + \sum_p I_p + \sum_p C_p + \sum_p \tilde{D}_p
\]

\[
\geq (1 - \beta) \sum_p \tilde{D}_p + (1 - \beta) \sum_p C_p - \frac{C}{\beta} F - \frac{C}{\lambda_1} |DDu|_{\eta}^2
\]

Define \( H'_{j p} = D_j(\chi_{\eta} + a_i u_{\eta} + a_i u_{j}) - D_p(\chi_{\eta} + a_i u_{\eta} + a_i u_{j}) \). Note that \( |H'_{j p}| \leq C + C \lambda_1 \), since \( u_{j p} - u_{j p} = T_{j p} u_{k} \leq C \). For any \( 0 < \tau < 1 \), we can estimate

\[
\sum_q \tilde{B}_q + \sum_p \tilde{D}_p \geq (1 - \tau) \left( \sum_q B_q + \sum_i D_i \right) - \frac{C}{\tau} F.
\]
Let $1 - \delta = (1 - \beta)(1 - \tau)$. Similar to (48), when $\frac{C}{A_1} \leq \frac{\varphi''}{4m}$, we obtain

$$0 \geq (1 - \delta) \sum_i (A_i + B_i + C_i + D_i) - \left(1 - \frac{\varphi''}{4(\phi')^2}\right) \sum_i E_i$$

$$+ \frac{\varphi'}{4m} \left(|DDu|^2_{\omega^m} + |D\overline{D}u|^2_{\omega^m}\right) - \frac{C(K)}{A_1} \left(|DDu|^2 + |D\overline{D}u|^2\right)$$

$$+ \left(\frac{\phi'}{m} \varepsilon - C \frac{\varphi'}{m} - \frac{C}{\tau} - \frac{C\phi'^2}{\phi''}\right) \mathcal{F} - C(\phi' - \phi' + 1).$$

(56)

Still choose $\phi$ and $\varphi$ as before and (54) is guaranteed by $M \gg N$. If we choose $\beta = \tau = \frac{\phi'}{m \phi''}$, we get $1 - \delta \geq 1 - \frac{\varphi''}{4(\phi')^2}$. Now the proof is the same as that of Theorem 1.1.

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