SIMPLIFYING SCHMIDT NUMBER WITNESSES VIA HIGHER-DIMENSIONAL EMBEDDINGS

FLORIAN HULPEKE, DAGMAR BRUSS, MACIEJ LEWENSTEIN and ANNA SANPERA
Institut f"ur Theoretische Physik, Universit"at Hannover
D-30167 Hannover, Germany

We apply the generalised concept of witness operators to arbitrary convex sets, and review the criteria for the optimisation of these general witnesses. We then define an embedding of state vectors and operators into a higher-dimensional Hilbert space. This embedding leads to a connection between any Schmidt number witness in the original Hilbert space and a witness for Schmidt number two (i.e. the most general entanglement witness) in the appropriate enlarged Hilbert space. Using this relation we arrive at a conceptually simple method for the construction of Schmidt number witnesses in bipartite systems.

Keywords: Witness operators, Schmidt number, Classification of entanglement

1 Introduction

In spite of the tremendous effort devoted in the recent years to characterize (i.e. to classify, quantify, detect and measure) entanglement [1], the description of entanglement remains an eluding problem whose complexity grows very fast with the number of subsystems of a given composite quantum system and with the dimension of the Hilbert space. Several operational separability criteria have been introduced to determine if a given state \( \rho \) acting on \( \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes ... \otimes \mathcal{H}_n \) is entangled or not (i.e separable). Among them, the criterion of the Positive Partial Transposition (PPT) [2] and the realignment criterion [3] are particularly powerful. Recently Doherty et al. [4] have introduced a new family of separability criteria which gives a characterization of mixed bipartite entangled states with a finite number of tests.

An apparently different approach to treat the same problem is based on entanglement witness operators [5, 6]. An entanglement witness \( W \) is a hermitian operator which has a positive expectation value on all separable states, but a negative one for at least one entangled state. This state is said to be detected by the witness operator. The existence of these operators is a direct consequence of the nested convex structure of the sets of (mixed) states acting on the Hilbert space \( \mathcal{H} \) of a composite system. Since for any given (finite-dimensional) Hilbert space the subset of separable states is convex and closed, it is always possible to find entanglement witness operators regardless of the dimensions and/or the number of subsystems of the composite system. Equivalently each entangled state can be detected by a witness operator. Notice that by using this approach the problem of determining whether a given state \( \rho \) is entangled or not is transformed into the problem of finding a suitable witness operator that detects it. The most suitable witnesses will be those that detect more states than any other ones, and for that reason they are called optimal witnesses.

Remarkably, entanglement witnesses have become a very powerful tool not only for de-
tecting entanglement, but also in the context of various other tasks in quantum information theory. For instance, establishing a secret key in quantum cryptography requires the existence of quantum correlations, which can be characterized by optimal witnesses \[7\]. The activation and distillation properties of a state \( \rho \) (that is, the possibility to locally distill from an ensemble of mixed states a subset of maximally entangled pure states) can also be recast in terms of witness operators \[8\]. By far the best-known and most famous entanglement witnesses are the so-called Bell inequalities \[9\]. It is easy to see from the definition of entanglement witnesses that each Bell inequality corresponds to an entanglement witness. However, this correspondence does not necessarily hold the other way round, as there exist entangled states that do not violate any Bell inequality but nevertheless are detected by a witness operator \[10\]. Thus, for the detection of entanglement witness operators are, in this respect, stronger than Bell inequalities. Furthermore, witness operators can be generalized to distinguish between different types of entanglement as long as the different entanglement types correspond to nested convex subsets. This is indeed the case for bipartite systems and, at least, for the simplest multipartite system, i.e. \( \mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \) \[11\]. For larger multipartite systems, where the structure of entangled states is much richer and much less-known, witness operators are also a useful tool to explore the structure of the Hilbert space. Finally, let us point out that since entanglement witnesses are observables (although not positive semidefinite) they can be measured. The experimental implementation of a witness operator can be realised by means of local measurements \[12, 13\] and has already been achieved in the laboratory \[14, 15\].

The paper is organized as follows. In section 2 we first review the concept of a general witness as well as its optimization following the arguments given in \[14, 16, 17\]. Most of the lemmas and theorems stated in this part of the paper are a straightforward generalization of the formalism developed previously, but for completeness we have included them here. In section 3 we restrict ourselves to bipartite systems, and review first the concepts of Schmidt number and Schmidt number witnesses. We show then that by embedding the state vectors and operators of the original Hilbert space into a higher-dimensional Hilbert space it is possible to connect any Schmidt number witness in the original Hilbert space to a witness of Schmidt number two (i.e. the most general entanglement witness) in the appropriate enlarged Hilbert space. Using this method one can simplify the construction and optimization of the desired general witness. We close this section with an explicit example to illustrate our method. Finally, we present our conclusions in section 4.

## 2 Optimisation of a general witness operator

We consider quantum systems of arbitrary, finite dimensions. By \( \mathcal{H} \) we denote a Hilbert space over the field \( \mathbb{K} \), by \( B(\mathcal{H}) \) the space of bounded operators acting on this Hilbert space and by \( \mathcal{P} \subset B(\mathcal{H}) \) the subset of positive semidefinite operators with trace one (the set of states on \( \mathcal{H} \)).

Consider the following situation, sketched in figure 1, for two given nested convex, closed subsets \( \mathcal{S}_1 \subset \mathcal{S}_2 \subset \mathcal{P} \), does a given \( \rho \in \mathcal{S}_2 \) belong to \( \mathcal{S}_1 \)? Without loss of generality we will
Fig. 1. The structure of the nested convex sets.

Fig. 2. The witness corresponding to the hyper-plane $W_1$ is $(S_1, S_2)$-finer, but not $(S_1, P)$-finer than $W_2$. Both $W_1$ and $W_2$ are $(S_1, P)$-finer than $W_3$.

assume that the identity belongs to $S_1$ and that $S_1$ is not of measure zero in $S_2$ [18].

**Definition 1:** For any convex set $X$ we denote the *border of $X$* by

$$
\delta X := \left\{ a \in X \mid \exists b \in X, \text{s.t. } \forall \lambda > 0 \text{ one has : } (1 + \lambda)a - \lambda b \notin X \right\}
$$

(1)

Some of the operators on the border, that have special properties, are referred to as edge-operators. Notice that a full characterization of the border-operators of $S_1$ immediately implies a full characterization of the operators that belong to $S_2 \setminus S_1$. In fact a full characterization of edge-operators is sufficient for this aim. Without loss of generality one can always shift the set $S_1$ such that $\mathbf{1} \in S_1 \setminus \delta S_1$.

Before answering the question whether a given $\rho \in S_2$ belongs to $S_1$ we first formally define a general witness operator. To simplify the notation, by $W$ we shall denote an $(S_1, S_2)$-witness defined as follows:

**Definition 2:** A hermitian operator $W$ is called an $(S_1, S_2)$-witness iff:

(i) $\tr(W\sigma) \geq 0 \ \forall \sigma \in S_1$.

(ii) $\exists \rho \in S_2$ such that $\tr(W\rho) < 0$.

(iii) $\tr W = 1$.

We note in passing that $W$ is not positive semidefinite and that condition (iii) corresponds to a normalization of the operator $W$. This normalization is useful for the comparison between different witnesses.

To prove that $\rho \in S_2 \setminus S_1$ it is sufficient to find a witness operator $W$ that detects $\rho$. For the cases in which the set $S_1$ is also closed (and therefore compact, due to the boundedness of $P$) the existence of a witness that detects $\rho$ is also necessary for $\rho \in S_2 \setminus S_1$ [19].

Note that without the requirement $\mathbf{1} \in S_1$ there could be cases for which no witnesses exist. For example, assume $\sigma \neq \mathbf{1}$, $S_1 = \{\sigma\}$ and $S_2 = \{\rho(\lambda) = \lambda \mathbf{1} + (1 - \lambda)\sigma | 1 \geq \lambda > 0\}$. Then no linear operator exists such that $\tr(W\sigma) \geq 0$ for all $\sigma \in S_1$ and $\tr(W\rho) < 0$ for a $\rho \in S_2$ and $\tr W = 1$. 

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To proceed further, let us introduce now some basic concepts and notations related to witness operators. We shall adopt here the notation developed in Ref. [6] and [17]. Our notation is as follows:

1. \( D_1^S := \{ \rho \in S_2 | Tr(W\rho) < 0 \} \), i.e. the set of the states in \( S_2 \) that are detected by \( W \).
2. \( Q_{S_2} := \{ Z | Tr(Z\rho) \geq 0 \forall \rho \in S_2 \} \), i.e. the set of operators which do not detect any state in \( S_2 \), and, therefore, are non-negative on \( S_2 \).
3. \( P_{i}^{S_1} := \{ |\psi\rangle\langle \psi | \in S_1 | (\psi|W|\psi) = 0 \} \), i.e. the set of one-dimensional projectors (pure states) in \( S_1 \) for which the expectation value of \( W \) vanishes.
4. Finer witness: \( W_1 \) is \((S_1,S_2)\)-finer than \( W_2 \) iff \( D_{W_2}^{S_2} \subset D_{W_1}^{S_2} \), that is, if any state detected by \( W_2 \) is also detected by \( W_1 \).
5. Optimal witness: \( W \) is an \((S_1,S_2)\)-optimal witness iff there exists no other witness that is \((S_1,S_2)\)-finer than \( W \). So for all \( W_1 \) that are \((S_1,S_2)\)-finer than \( W \) the equality \( W_1 = W \) holds.

The concept of a witness being \((S_1,S_2)\)-finer than another one depends on \( S_2 \). As illustrated in figure 2 it is possible that a witness \( W_1 \) is \((S_1,S_2)\)-finer than \( W_2 \), but that there exists an \( S'_2 \) with \( S_2 \subset S'_2 \), such that \( W_1 \) is not \((S_1,S'_2)\)-finer.

The above definitions provide the necessary tools to optimize a given general witness.

Lemma 1 (Lemma 1 in [6]): Let \( W_1 \) be \((S_1,S_2)\)-finer than \( W_2 \) and

\[
\lambda := \inf_{\rho \in D_{W_2}^{S_2}} \frac{Tr \rho W_1}{Tr \rho W_2} \quad (2)
\]

Then for any positive operator \( \rho \) we have:
(i) If \( Tr \rho W_2 = 0 \) then \( Tr \rho W_1 \leq 0 \).
(ii) If \( Tr \rho W_2 < 0 \) then \( Tr \rho W_1 \leq Tr \rho W_2 \).
(iii) If \( Tr \rho W_2 > 0 \) then \( Tr \rho W_1 \leq \lambda Tr \rho W_2 \).
(iv) \( \lambda \geq 1 \). In particular if \( \lambda = 1 \) \( \iff \) \( W_1 = W_2 \).

In complete analogy with the optimization of entanglement witnesses, one can also construct \((S_1,S_2)\)-finer general witnesses by simply subtracting any operator that is positive definite on \( S_2 \) (i.e. \( Z \in Q_{S_2} \)) from the original witness in such a way that the remaining operator still fulfills the necessary conditions for being a witness operator.

Lemma 2 (Lemma 2 in [6]): \( W_1 \) is \((S_1,S_2)\)-finer than \( W_2 \) \( \iff \) there exists a \( Z \in Q_{S_2} \) (i.e. \( Tr(Z\rho) \geq 0 \) for all \( \rho \in S_2 \)), and there exists an \( \epsilon \) with \( 1 > \epsilon \geq 0 \) such that \( W_2 = (1 - \epsilon) W_1 + \epsilon Z \) \( \iff \) for all \( \rho \in S_1 \) : \( Tr(W_2\rho) - (1 - \epsilon) Tr(W_1\rho) \geq 0 \).

Theorem 1 (Theorem 1 in [6]): A witness \( W_1 \) is optimal \( \iff \) for all \( Z \in Q_{S_2} \) and \( \epsilon > 0 \) the operator \( W' := (1 + \epsilon) W_1 - \epsilon Z \) is not a witness, i.e. it does not fulfill \( Tr(W'\rho) \geq 0 \) for all \( \rho \in S_1 \).

Lemma 3 (Lemma 3 in [6]): If \( ZP_{W}^{S_1} \neq 0 \) (i.e. for all \( P \in P_{W}^{S_1} \) one has \( Z \not\perp P \)), then \( Z \neq 0 \) for all \( W \in D_{W_2}^{S_2} \).
cannot be subtracted from $W$, that is, $(1 + \epsilon)W - \epsilon Z$ is not a witness for any $\epsilon > 0$.

**Corollary 1 (Corollary 2 in [8]):** If $P^S_1$ spans $\mathcal{H}$ then $W$ is optimal.

**Lemma 4:** If, for a given $(S_1, S_2)$-witness $W$, there exists a $\rho \in \delta S_1 \setminus \delta S_2$, such that $\text{Tr} W\rho = 0$, then $W$ is $(S_1, S_2)$-optimal. Geometrically $W$ can be interpreted as an $S_1$-edge witness operator, namely it is tangent to the set $S_1$ at a point which does not belong to the border of $S_2$.

**Proof:** Assuming that $W$ is not $(S_1, S_2)$-optimal, there would exist an $(S_1, S_2)$-finer general witness $W' \neq W$. Following the assumption there has to exist a state $\sigma \in S_2$ that is detected by $W'$ (i.e. $\text{Tr} W'\sigma < 0$) but not by $W$ (i.e. $\text{Tr} W\sigma \geq 0$). Since $\rho \in S_1$ and $W'$ is $(S_1, S_2)$-finer than $W$, we know according to Lemma 1 (i) that $\text{Tr} W'\rho = 0$. Furthermore $\rho \notin \delta S_2$. This leads by definition to the fact that there is a $\lambda > 0$, with the property that the state $(1 + \lambda)\rho - \lambda \sigma \in S_2$. Evaluating both witnesses $W$ and $W'$ on this state we see that $\text{Tr} W((1 + \lambda)\rho - \lambda \sigma) = -\lambda \text{Tr} W\sigma \leq 0$, but $\text{Tr} (W'(1 + \lambda)\rho - \lambda \sigma) = -\lambda \text{Tr} W'\sigma > 0$. This is in contradiction to Lemma 1 (ii). Therefore, we conclude that there exists no witness $W' \neq W$ which is $(S_1, S_2)$-finer than $W$, and since all witnesses which are $(S_1, S_2)$-finer than $W$ have to be equal to $W$, we conclude that $W$ is optimal. 

Witness operators are often used to detect generic entanglement in bipartite systems. In this case the two convex sets $S_1, S_2$ can be identified as $S_1 = S = \{\text{set of all separable states}\}$ and $S_2 = \mathcal{P} = \{\text{set of all positive (semidefinite) operators with trace one}\}$. It is well known that for systems acting on a Hilbert space of dimension $\dim \mathcal{H} > 6$ (where $\dim \mathcal{H}_A, \dim \mathcal{H}_B \geq 2$) there exists entanglement which cannot be detected by means of the partial transposition. The set of states that remain positive under partial transposition (PPT-set) also form a convex set. We can distinguish PPT-entangled states from separable ones if we search for witnesses associated to $S_1 = S$ and $S_2 = \{\text{PPT}\}$ with $S_1 \subset S_2$. Those witnesses $W$ which are capable to detect $\rho \in S_2 \setminus S_1$ are necessarily non-decomposable, since they cannot be written as $W = P + Q^T$, where $P$ and $Q$ are positive semidefinite operators [3]. First examples of non-decomposable entanglement witnesses were provided in [3], and the characterisation of such witnesses was presented in [16]. Notice that the concept of witness operators only relies on a nested subset structure, and thus is not restricted to bipartite systems. For multipartite systems, there exist typically various classes of distinct multipartite entanglement. For instance, for $2 \times 2 \times 2$ systems, there are classes of separable $\mathcal{S}$, biseparable $\mathcal{B}$, $W$ [20] and $\mathcal{G}\mathcal{H}\mathcal{Z}$ mixed states, which are ordered in the nested structure $\mathcal{S} \subset \mathcal{B} \subset \mathcal{W} \subset \mathcal{G}\mathcal{H}\mathcal{Z}$ [11]. For this case and similar ones one can tailor the appropriate witness $W$ to discriminate between the different nested convex sets.

In the next section we will study so-called Schmidt (number) witnesses [21], which detect how many degrees of freedom of the subsystems of a bipartite system are entangled with each other. Thus Schmidt witnesses can distinguish between entangled states from different Schmidt classes. In this sense Schmidt number witnesses provide a refinement of general entanglement witnesses.
3 From Schmidt number witnesses to entanglement witnesses

For bipartite systems it is possible to extend the useful concept of the Schmidt rank for pure states [22] to the Schmidt number for mixed states [23]. The Schmidt number of a mixed state $\rho$ characterises the maximal Schmidt rank of the pure states which is at least needed to construct $\rho$. It is defined as:

$$ SN(\rho) := \min_{\rho = \sum_{i=1}^r p_i |\psi_i \rangle \langle \psi_i |} \max_i \{SR(\psi_i)\}, $$

where $SR(\psi_i)$ is the Schmidt rank of $|\psi_i\rangle$. By $(SN)_k$ we denote the Schmidt class $k$, i.e. the set of states that have Schmidt number $k$ or less. This set is a convex, compact subset of $\mathcal{P}$, and there is a nested structure of the form $(SN)_1 \subset (SN)_2 \subset \ldots \subset (SN)_k \subset \ldots \subset \mathcal{P}$. Clearly, $(SN)_1 = \mathcal{S}$ corresponds to the set of separable states. As explained in [21] it is clear that one can construct a Schmidt number witness operator $S_k$ which is non-negative on all states in $(SN)_{k-1}$, but detects at least one state belonging to $(SN)_k$. Using our previous notation this would correspond to a $((SN)_{k-1}, (SN)_k)$-witness, but as a shorter notation and to be consistent with the notation of [21] we denote $S_k$ as a $k$-Schmidt-witness ($k$-SW). In the same way we define the terms $k$-finer and $k$-optimal as abbreviations.

In the previous part of this paper we have established tools to answer (a) under which conditions a given $k$-SW is $k$-finer than another one, and (b) how to optimise a given $k$-SW (namely one has to subtract operators $Z$ which are positive definite on the set $(SN)_k$, such that the operator $W = S_k - Z$ has the property $\text{Tr} (W\sigma) \geq 0$ for all $\sigma \in (SN)_{k-1}$). To verify positivity on all states from the set $(SN)_{k-1}$ it suffices to restrict oneself to the set of all pure states $|\psi_{k-1}\rangle$, since those are the extremal points of $(SN)_{k-1}$. But nonetheless such verification remains laborious, since these states do not exhibit a particularly useful structure that allows to check easily whether $\text{Tr} (W|\psi_{k-1}\rangle \langle \psi_{k-1}|) \geq 0$ for all $|\psi_{k-1}\rangle$. Notice, however, that for the case of a general entanglement witnesses (or more precisely, of 2-SW’s) this problem becomes much easier, since semi-positivity of an operator $W$ on all product states $|e, f\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ is equivalent to the semi-positivity of the $(\dim \mathcal{H}_A - 1)$-parameter family of operators $\langle e W|e\rangle \in \mathcal{B}(\mathcal{H}_B)$. Thus, rather than checking a $(\dim \mathcal{H}_A - 1)(\dim \mathcal{H}_B - 1)$-dimensional parameter space corresponding to all product vectors, the task is greatly simplified. Furthermore, positivity on a whole space is mathematically a simpler concept than positivity on a given subset.

Let us point out for clarity that every Schmidt number witness is an entanglement witness since it detects some kind of entanglement. A witness for Schmidt number two (2-SW) corresponds to a witness that detects all kinds of entanglement without discriminating between the different types. Therefore we sometimes use the name “entanglement witness” synonymously with “2-SW”.

In this section we will show that it is always possible to reformulate the problem of finding a witness of Schmidt class $k$ into finding a 2-SW denoted by $\mathcal{S}$ in a higher-dimensional Hilbert space. Our method relies on embedding the original Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ into an enlarged Hilbert space, such that each pure state with Schmidt rank $k$ or less in the original space becomes a product state in the enlarged space. The enlarged Hilbert space consists of the original one with two added local ancillas of dimension $k$, i.e. $\mathcal{H}_{\text{enlarged}} = (\mathcal{H}_A \otimes K^k) \otimes (\mathcal{H}_B \otimes K^k)$. The embedding is performed by means of a map $I_k$. We shall study...
the effect of this map on the set of operators acting on $\mathcal{H}_A \otimes \mathcal{H}_B$ and show that this map also connects the expectation values of operators in the original Hilbert space with the expectation values of operators in the enlarged Hilbert space.

3.1 Mapping from the original Hilbert space to the enlarged one

In this subsection we first define a map which transforms states in the original Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ into states in the enlarged one. We then study the effect of this map on the operators acting on $\mathcal{H}_A \otimes \mathcal{H}_B$ and define for each operator $S$ an operator $S$ acting in the enlarged space.

Definition 3: We denote by $I_k: \mathcal{H}_A \otimes \mathcal{H}_B \rightarrow (\mathcal{H}_A \otimes K^k) \otimes (\mathcal{H}_B \otimes K^k)$ a map that transforms every pure state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ into a pure state $|I_k(\psi)\rangle \in (\mathcal{H}_A \otimes K^k) \otimes (\mathcal{H}_B \otimes K^k)$. This map is defined by:

$$|\psi\rangle = \sum_{i=1}^{n} \lambda_i |a_i b_i\rangle \mapsto |I_k(\psi)\rangle = \left( \sum_{i=1}^{k} |a_i\rangle \otimes |i\rangle \right) \otimes \left( \sum_{j=1}^{k} \lambda_j |b_j\rangle \otimes |j\rangle \right)$$

$$+ \left( \sum_{i=k+1}^{2k} |a_i\rangle \otimes |i-k\rangle \right) \otimes \left( \sum_{j=k+1}^{2k} \lambda_j |b_j\rangle \otimes |j-k\rangle \right) + \ldots$$

$$+ \left( \sum_{i=u+1}^{n} |a_i\rangle \otimes |i-u\rangle \right) \otimes \left( \sum_{j=u+1}^{n} \lambda_j |b_j\rangle \otimes |j-u\rangle \right). \quad (4)$$

where $n \leq \min(\dim \mathcal{H}_A, \dim \mathcal{H}_B)$. Here we have fixed a basis $\{|1\rangle, \ldots, |k\rangle\}$ for both ancilla spaces. The ancilla states are orthogonal, i.e. $\langle i|j\rangle = \delta_{ij}$. The vectors $|a_i\rangle, |b_i\rangle$ denote the Schmidt bases of $|\psi\rangle$ and $u := \text{floor } (n/k)k$ where floor indicates the integer part $\lfloor \cdot \rfloor$. We denote the action of this map as “lifting up”. In the following we describe the properties of this map and its action on states and operators.

Remark 2: For each pure state $|\psi\rangle = \sum_{i=1}^{n} \lambda_i |a_i b_i\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ it holds that:

$$I_k(|\psi\rangle) = I_k(\sum_{i=1}^{k} \lambda_i |a_i b_i\rangle) + I_k(\sum_{i=k+1}^{2k} \lambda_i |a_i b_i\rangle) + \ldots + I_k(\sum_{i=u+1}^{n} \lambda_i |a_i b_i\rangle). \quad (5)$$

In particular, $I_k$ maps by definition pure states in $\mathcal{H}_A \otimes \mathcal{H}_B$ with Schmidt rank $k$ or less into pure product states in $(\mathcal{H}_A \otimes K^k) \otimes (\mathcal{H}_B \otimes K^k)$.

We now define for each operator acting on $\mathcal{H}_A \otimes \mathcal{H}_B$ an operator acting on $(\mathcal{H}_A \otimes K^k) \otimes (\mathcal{H}_B \otimes K^k)$.

Definition 4: For a given $S = \sum_{i,j,l,m} \sigma_{i,j,l,m} |i\rangle \langle l| \otimes |j\rangle \langle m|_B$ acting on $\mathcal{H}_A \otimes \mathcal{H}_B$, and $k \in \mathbb{N}$ we define an operator $S_k$ on $(\mathcal{H}_A \otimes K^k) \otimes (\mathcal{H}_B \otimes K^k)$ by:

$$S_k := \sum_{s,t=1}^{k} \sum_{i,j,l,m} \sigma_{i,j,l,m} |i,s\rangle \langle l,t| \otimes |j,s\rangle \langle m,t|. \quad (6)$$
If Tr (S) = 1, then Tr (S_k) = k.

**Remark 3:** Reordering the tensor-product structure of the enlarged Hilbert space as \( H_A \otimes H_B \otimes K^n \otimes K^n \) leads to the following expression for \( S_k \):

\[
S_k = \sum_{s,t=1}^{k} S \otimes |ss\rangle \langle tt|.
\] (7)

In the following the notation \( S \) (i.e. blackboard font) will be used only for those operators acting on \( (H_A \otimes K^n) \otimes (H_B \otimes K^n) \), which can be written like in eqn. (7), with a corresponding operator \( S \in B(H_A \otimes H_B) \).

**Remark 4:** For every pure state \(|\psi\rangle \in H_A \otimes H_B \) with Schmidt rank less or equal to \( k \), i.e. \(|\psi\rangle = \sum_{i=1}^{u} \lambda_i |a_i b_i\rangle \) with \( \lambda_i \geq 0 \) and every operator \( S \) it holds that:

\[
\langle \psi | S | \psi \rangle = \sum_{s,t=1}^{k} \lambda_s^* \lambda_t \langle a_s b_s | S | a_t b_t \rangle = \sum_{i,j,m,s,t=1}^{k} \lambda_i^* \lambda_m \langle a_i | \langle b_j | S | a_m \rangle \langle ij|ss\rangle \langle tt|lm\rangle
\]

\[
= \langle I_k(\psi) | S_k | I_k(\psi) \rangle.
\] (8)

**Remark 5:** Notice that the same construction holds for a pure state of Schmidt rank larger than \( k \), namely \(|\psi\rangle = \sum_{i=1}^{u} \lambda_i \sum_{j=1}^{n-k} |a_i b_i\rangle \) with \( \lambda_i \geq 0 \) and every operator \( S \) it holds that:

\[
\langle \psi | S | \psi \rangle = \sum_{i,j=1}^{u} \langle \psi_i | S | \psi_j \rangle = \sum_{i,j=1}^{u} \langle I_k(\psi_i) | S_k | I_k(\psi_j) \rangle
\]

\[
= \langle I_k(\psi) | S_k | I_k(\psi) \rangle.
\] (9)

**Remark 6:** Given a mixed state \( \rho \) and an arbitrary decomposition \( \rho = \sum_{i=1}^{l} p_i |\phi_i\rangle \langle \phi_i| \), it follows that

\[
\text{Tr} (S \rho) = \sum_{i=1}^{l} p_i \langle \phi_i | S | \phi_i \rangle = \sum_{i=1}^{l} p_i \langle I_k(\phi_i) | S_k | I_k(\phi_i) \rangle.
\] (10)

Thus, by defining a (non-normalized) mixed state \( \Gamma_k := \sum_{i=1}^{l} p_i |I_k(\phi_i)\rangle \langle I_k(\phi_i)| \in B((H_A \otimes K^n) \otimes (H_B \otimes K^n)) \) one arrives at

\[
\text{Tr} (S \rho) = \text{Tr} (S_k \Gamma_k).
\] (11)

### 3.2 Mapping from the enlarged Hilbert space to the original one

In this part we now define a map which transforms states in the enlarged Hilbert space into states in the original one.

**Remark 7:** Every pure product state \(|A\rangle \otimes |B\rangle = |A, B\rangle \in (H_A \otimes K^n) \otimes (H_B \otimes K^n) \) can be
expressed by using the Schmidt decomposition of each pure state $|A\rangle, |B\rangle$ in the bipartite splitting $\mathcal{H}_{A,B} \otimes \mathcal{K}^k$ as:

$$|A\rangle = \sum_{i=1}^{k} \lambda_i |a_i\rangle \otimes |c_i\rangle \quad ; \quad |B\rangle = \sum_{j=1}^{k} \mu_j |b_j\rangle \otimes |d_j\rangle,$$

(12)

where the Schmidt coefficients $\lambda_i$ and $\mu_j$ are positive and $\sum_i \lambda_i^2 = 1, \sum_j \mu_j^2 = 1$.

**Definition 5:** By $\tilde{J}_k : (\mathcal{H}_A \otimes \mathcal{K}^k) \otimes (\mathcal{H}_B \otimes \mathcal{K}^k) \rightarrow \mathcal{H}_A \otimes \mathcal{H}_B$ we denote the map that transforms any pure product state $|A\rangle \otimes |B\rangle \in (\mathcal{H}_A \otimes \mathcal{K}^k) \otimes (\mathcal{H}_B \otimes \mathcal{K}^k)$ into a pure state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ with Schmidt rank less or equal to $k$. This map is defined by:

$$\tilde{J}_k : |A\rangle \otimes |B\rangle \mapsto |\tilde{J}_k(A \otimes B)\rangle = \sum_{i=1}^{k} (ii) \left( \sum_{l=1}^{k} \lambda_l |a_l\rangle \otimes |c_l\rangle \right) \otimes \left( \sum_{m=1}^{k} \mu_m |b_m\rangle \otimes |d_m\rangle \right)$$

$$= \sum_{i,m=1}^{k} F_{im} \lambda_i \mu_m |a_i b_m\rangle$$

(13)

where $F_{im} := \sum_{s=1}^{k} (ii) c_l d_m$.

Recalling that the Schmidt rank of a pure state is equal to the rank of its reduced density matrices we find:

$$\text{Tr}_B |\tilde{J}_k(A \otimes B)\rangle \langle \tilde{J}_k(A \otimes B)| = \sum_{s=1}^{n} \langle b_s | \sum_{i,j,l,m=1}^{k} F_{ij}^* F_{lm} \lambda_i^* \lambda_j \mu_j^* \mu_m |a_i\rangle \otimes |b_j\rangle \langle a_l b_m| b_s\rangle$$

$$= \sum_{i,j=1}^{k} \lambda_i^* \lambda_j \sum_{s=1}^{k} F_{is}^* F_{js} |\mu_s|^2 |a_i\rangle \langle a_j|.$$

(14)

The rank of this operator cannot exceed $k$, and therefore, the Schmidt rank of $|\tilde{J}_k(A \otimes B)\rangle$ does not exceed $k$.

This map can be now straightforwardly extended to map entangled pure states of the enlarged space into the original one by using the Schmidt decompositions according to the split $(\mathcal{H}_A \otimes \mathcal{K}^k) \otimes (\mathcal{H}_B \otimes \mathcal{K}^k)$ (Remark 7) and applying the map to each Schmidt term separately.

**Definition 6:** By $J_k$ we define the extension of $\tilde{J}_k$ on all pure entangled states $|\Psi\rangle$ in $(\mathcal{H}_A \otimes \mathcal{K}^k) \otimes (\mathcal{H}_B \otimes \mathcal{K}^k)$:

$$J_k : |\Psi\rangle = \sum_{i=1}^{l} \lambda_i |A_i\rangle \otimes |B_i\rangle \mapsto |J_k(\Psi)\rangle := \sum_{i=1}^{l} \lambda_i |\tilde{J}_k(A_i \otimes B_i)\rangle.$$

(15)

We call the action of the map $J_k$ "lifting down".

**Remark 8:** Notice that for all $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ with Schmidt rank $k$ or less, it holds that: $|J_k(I_k(\psi))\rangle = |\psi\rangle$. Thus, when restricted to these states, the map $J_k$ is the inverse map of $I_k$. 

9
3.3 Connection between the expectation values of operators

After the definition of the two maps (the “lifting up” map $I_k$ and the “lifting down” map $J_k$) one observes that there is a close relation between the expectation value of an operator in the original space and the expectation value of the corresponding operator in the enlarged space.

**Lemma 5:** Given two arbitrary pure product states $|A_1B_1\rangle,|A_2B_2\rangle \in (\mathcal{H}_A \otimes \mathcal{K}^k) \otimes (\mathcal{H}_B \otimes \mathcal{K}^k)$ the following equation:

$$\langle A_1| \otimes \langle B_1| S_k |A_2\rangle \otimes |B_2\rangle = \langle J_k(A_1 \otimes B_1)|S|J_k(A_2 \otimes B_2)\rangle \quad (16)$$

holds.

**Proof:** We use Remark 3 to express $S_k$ as a function of $S$, and express each pure state $|A_i\rangle$ (respectively $|B_i\rangle$) in its Schmidt decomposition according to Remark 7. The expectation value of $S_k$ is thus given as:

$$\langle A_1B_1|S_k|A_2B_2\rangle = \left( \sum_{i,j=1}^{k} \lambda_i^* \mu_j^* \langle a_i|b_j|S( \sum_{l,m=1}^{k} \nu_l \xi_m |e_l f_m\rangle) \right) \sum_{s,t=1}^{k} \langle c_is|ss\rangle \langle tt|g_i h_m\rangle$$

$$= \left( \sum_{i,j=1}^{k} F_{ij}^* \lambda_i^* \mu_j^* \langle a_i|b_j\rangle \right) S \left( \sum_{l,m=1}^{k} G_{lm} \nu_l \xi_m |e_l f_m\rangle \right)$$

$$= \langle J_k(A_1 \otimes B_1)|S|J_k(A_2 \otimes B_2)\rangle, \quad (17)$$

where $F_{ij} := \sum_{s=1}^{k} \langle ss|c_is\rangle$ and $G_{ij} := \sum_{l=1}^{k} \langle tt|g_i h_j\rangle$.

So far, we have defined the action of the maps on pure states and operators and we have shown how the maps permit to “jump” from the original space to the enlarged one (and vice versa). We have also shown the relation between the expectation value of an operator in the original space and the corresponding operator in the enlarged space. We proceed now to show that a Schmidt number witness ($k$-SW) acting in $\mathcal{H}_A \otimes \mathcal{H}_B$ corresponds to an entanglement witness ($2$-SW) acting on $(\mathcal{H}_A \otimes \mathcal{K}^k) \otimes (\mathcal{H}_B \otimes \mathcal{K}^k)$. The main results of our paper are stated in the following two theorems.

**Theorem 2:**

i) Given two arbitrary operators $S, \rho \in B(\mathcal{H}_A \otimes \mathcal{H}_B)$ and an arbitrary decomposition $\rho = \sum_{i=1}^{l} p_i |\phi_i\rangle \langle \phi_i|$, it holds that $\text{Tr } (S \rho) = \text{Tr } (S \rho_k)$, where $\rho_k := \sum_{i=1}^{l} \sum_{j=1}^{k} p_i |J_k(\phi_i)\rangle \langle J_k(\phi_i)|$.

ii) Given two arbitrary operators $S_k, \Theta \in B((\mathcal{H}_A \otimes \mathcal{K}^k) \otimes (\mathcal{H}_B \otimes \mathcal{K}^k))$ and an arbitrary decomposition $\Theta = \sum_{i=1}^{l} p_i |\Phi_i\rangle \langle \Phi_i|$, it holds that $\text{Tr } (S_k \Theta) = \text{Tr } (S \theta)$, where $\theta := \sum_{i=1}^{l} \sum_{j=1}^{k} |J_k(\Phi_i)\rangle \langle J_k(\Phi_i)|$.

**Proof:** (i) See Remark (6). (ii) The proof is a concatenation of the previous remarks.

**Theorem 3:**

i) If $S$ is a $k$-SW acting on $\mathcal{H}_A \otimes \mathcal{H}_B$, then $S_{k-1}$ is a $2$-SW acting on $(\mathcal{H}_A \otimes \mathcal{K}^{k-1}) \otimes (\mathcal{H}_B \otimes \mathcal{K}^{k-1})$.

ii) If $S_{k-1}$ is a $2$-SW, then $S$ is an $n$-SW with $k \leq n \leq 2k$.
iii) If $S'_{k-1} = (1 + \epsilon)S_{k-1} - \epsilon Z_{k-1}$ is a 2-SW which is 2-finer than $S_{k-1}$, then the corresponding $S'$ is $k$-finer than $S$.

**Proof:** (i) According to Remark 4 and to the fact that $S$ is a $k$-SW one observes that $S_{k-1}$ has a positive expectation value for all pure product-states in $(\mathcal{H}_A \otimes \mathcal{K}^{k-1}) \otimes (\mathcal{H}_B \otimes \mathcal{K}^{k-1})$. It remains to be shown that there exists a state with Schmidt rank 2 in $(\mathcal{H}_A \otimes \mathcal{K}^{k-1}) \otimes (\mathcal{H}_B \otimes \mathcal{K}^{k-1})$ for which $S_{k-1}$ has a negative expectation value. Since $S$ is a $k$-SW there exists a state $|\psi\rangle$ with Schmidt rank $k$ that is detected by $S$. The Schmidt decomposition of such a state can be written as:

$$|\psi\rangle = \sum_{i=1}^{k} \lambda_i |a_i\rangle |b_i\rangle.$$  \hspace{1cm} (18)

Notice then that $I_{k-1}$ will map this pure state of Schmidt rank $k$ into a pure state of Schmidt rank 2. According to Remarks 2 and 4 the expectation value of $\langle I_{k-1} | S_{k-1} | I_{k-1} \rangle = \langle \psi | S | \psi \rangle < 0$. So a state with Schmidt rank 2 in $(\mathcal{H}_A \otimes \mathcal{K}^{k-1}) \otimes (\mathcal{H}_B \otimes \mathcal{K}^{k-1})$ is detected and $S_{k-1}$ is a (non-normalized) 2-SW.

ii) Let $S_{k-1}$ be a (non-normalized) 2-SW, then there exists a state $|\Psi\rangle := \kappa_1 |A_1\rangle \otimes |B_1\rangle + \kappa_2 |A_2\rangle \otimes |B_2\rangle$ with $\langle \Psi | S_{k-1} | \Psi \rangle < 0$, but since $S_{k-1}$ is a 2-SW it is non-negative on all separable states in $(\mathcal{H}_A \otimes \mathcal{K}^{k-1}) \otimes (\mathcal{H}_B \otimes \mathcal{K}^{k-1})$.

Since all states of Schmidt rank less than $k$ are mapped by $I_{k-1}$ to product states, one obtains that $S$ is non-negative on all states with Schmidt rank less than $k$. Furthermore $S$ cannot be positive, since $S_{k-1}$ is not positive. So it remains to be shown that there exists a state with Schmidt rank $n$ (with $k \leq n \leq 2k$) that is detected. Writing the pure states $|A_i\rangle$, $|B_i\rangle$ in their Schmidt decomposition as in eqn. (17), the expectation value of $S_{k-1}$ is given as:

$$\langle \Psi | S_{k-1} | \Psi \rangle = |\kappa_1|^2 \langle A_1 B_1 | S_{k-1} | A_1 B_1 \rangle + |\kappa_2|^2 \langle A_2 B_2 | S_{k-1} | A_2 B_2 \rangle + \kappa_1^* \kappa_2^* \langle A_1 B_1 | S_{k-1} | A_2 B_2 \rangle + \kappa_2^* \kappa_1^* \langle A_2 B_2 | S_{k-1} | A_1 B_1 \rangle.$$  

By use of Lemma 5 one can relate each of the above terms to a matrix element of $S$ and arrives at:

$$\langle \Psi | S_{k-1} | \Psi \rangle = (\kappa_1^* (J_{k-1}(A_1 \otimes B_1)) + \kappa_2^* (J_{k-1}(A_2 \otimes B_2))) S((\kappa_1 | J_{k-1}(A_1 \otimes B_1) \rangle + \kappa_2 | J_{k-1}(A_2 \otimes B_2) \rangle)).$$

Since $|J_{k-1}(A_1 \otimes B_1)\rangle$ and $|J_{k-1}(A_2 \otimes B_2)\rangle$ both have a Schmidt rank less than $k$, the Schmidt rank of their sum cannot exceed $2(k-1)$. So there exists also a minimal $n$ with $k \leq n \leq 2(k-1)$ and a pure state in $\mathcal{H}_A \otimes \mathcal{H}_B$ with Schmidt rank $n$, that is detected by $S$.

iii) Let $|\phi\rangle = \sum_{i=1}^{k} \lambda_i |a_i b_i\rangle = |\psi\rangle + \lambda_k |a_k b_k\rangle$ be an arbitrary pure state with Schmidt rank less or equal $k$ in $\mathcal{H}_A \otimes \mathcal{H}_B$, i.e. $\lambda_k \geq 0$. Since $S'_{k-1}$ is 2-finer than $S_{k-1}$ the operator $Z_{k-1}$ has to be non-negative on all pure states $|\Psi\rangle = \mu_1 |A_1 B_1\rangle + \mu_2 |A_2 B_2\rangle$ with Schmidt rank two or less in $(\mathcal{H}_A \otimes \mathcal{K}^{k-1}) \otimes (\mathcal{H}_B \otimes \mathcal{K}^{k-1})$, i.e. $\mu_{1,2} \geq 0$. In particular this has to hold for $\mu_1 |A_1 B_1\rangle = I_{k-1}(|\psi\rangle)$ and $\mu_2 |A_2 B_2\rangle = I_k(\lambda_k |a_k\rangle \otimes |b_k\rangle) = |a_k\rangle \otimes |1\rangle \otimes \lambda_k |b_k\rangle \otimes |1\rangle$. By calculating the expectation value of $Z_{k-1}$ on $|\Psi\rangle$ and using Lemma 5 and Remark 8 one
Therefore, \( Z \in \mathcal{Q}(SN)_k \), and due to Lemma 2 \( S' \) is \( k \)-finer than \( S \). \( \square \).

**Remark 9:** An operator \( S \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B) \) is a \( k \)-SW if the operator \( S_l \) is a (non-normalized) entanglement witness for all \( l \leq k \), but no entanglement witness for all \( l > k \).

**Lemma 6:** If there exists some \( Z \in \mathcal{Q}(SN)_k \) such that \( ZP_{S_k \times k} = 0 \) and

\[
\lambda_0 = \inf_{\langle A \rangle \in \mathcal{H}_A \otimes \mathcal{H}_B} \left[ \left( \langle A | Z_{k-1} | A \rangle \right)^{-1/2} \langle A | S_{k-1} | A \rangle \left( \langle A | Z_{k-1} | A \rangle \right)^{-1/2} \right]_{\text{min}} \tag{19}
\]

\[
= \left( \sup_{\langle A \rangle \in \mathcal{H}_A \otimes \mathcal{H}_B} \left[ \left( \langle A | S_{k-1} | A \rangle \right)^{-1/2} \langle A | Z_{k-1} | A \rangle \left( \langle A | S_{k-1} | A \rangle \right)^{-1/2} \right]_{\text{max}} \right)^{-1} > 0 ,
\]

where we denote by \([\cdot]_{\text{min}} / \text{max}\) the minimal/maximal eigenvalue of an operator, then the operator

\[
S'(\lambda) := (S - \lambda Z)/(1 - \lambda)
\]

with \( \lambda \geq 0 \) is a \( k \)-SW if and only if \( \lambda \leq \lambda_0 \).

**Proof:** Let us find out for which values of \( \lambda > 0 \) the operator \( S'(\lambda) \) is a \( k \)-SW and, therefore, \( S'_{k-1}(\lambda) \) is an entanglement witness. To this aim we demand that \( \langle A | S'_{k-1}(\lambda) | A \rangle \geq 0 \), i.e.

\[
\langle A | S_{k-1} | A \rangle - \lambda \langle A | Z_{k-1} | A \rangle \geq 0. \tag{20}
\]

On one hand, multiplying equation (20) from the left and from the right by \( \langle A | Z_{k-1} | A \rangle \)^{-1/2} we obtain \( \langle A | Z_{k-1} | A \rangle^{-1/2} \langle A | S_{k-1} | A \rangle \langle A | Z_{k-1} | A \rangle^{-1/2} \geq \lambda I \), which leads to \( \lambda \leq \lambda_0 \) given in the first part of the eqn. (19). On the other hand, multiplying equation (20) by \( \langle A | S_{k-1} | A \rangle \)^{-1/2} from the right and the left side we obtain \( \langle A | S_{k-1} | A \rangle^{-1/2} \langle A | Z_{k-1} | A \rangle \langle A | S_{k-1} | A \rangle^{-1/2} \leq 1/\lambda \), which immediately leads to \( \lambda \leq \lambda_0 \), given in the second equality of eqn. (19). \( \square \).

### 3.4 Example

The aim of this subsection is to illustrate the previous method and results with an explicit example. Consider the following one parameter family of operators

\[
S(a) := \frac{1}{1 - a} \left( \frac{1}{9} I - a |\psi\rangle \langle \psi| \right) \tag{21}
\]

acting on \( C^3 \otimes C^3 \) where \( |\psi\rangle = \frac{1}{\sqrt{3}} (|00\rangle + |11\rangle + |22\rangle) \) and \( a > 0 \).

Our goal is to determine for which parameters \( a \) the witness operator \( S(a) \) is able to detect Schmidt number 3 only, i.e. for which it is non-negative on states with Schmidt number 2. Note that the partial transpose of \( S(a) \) provides a family of states that for some \( a \) are \( n \)-copy non-distillable \( \underline{25} \). In \( \underline{23} \), where the possibility of the existence of non-distillable states with
non-positive partial transpose was discussed, these states were investigated, and the result that we are going to derive below, was obtained by using a direct method. The aim of the example presented here is thus to illustrate how one can arrive at such result by transforming the problem to the task of checking if in some extended space a corresponding new operator is an entanglement witness.

Notice that $S(a)$ is positive semidefinite and, therefore, not a witness if $a \leq \frac{1}{3}$. For an arbitrary $|e⟩ ∈ H_A$ with $|e⟩ = λ_0|0⟩ + λ_1|1⟩ + λ_2|2⟩$ the expectation value of $S(a)$ becomes:

$$S(a) = (1 - a)|e⟩⟨e|S(a)|e⟩ = \frac{1}{9}|e⟩|1_A⟩|e⟩|1_B⟩ - a|e⟩⟨e|⟨e|ψ⟩$$

$$= \frac{1}{9}|1_B⟩ - a\left(λ_0^2|0⟩⟨0| + λ_0^2λ_1|0⟩⟨1| + λ_0^2λ_2|0⟩⟨2| + λ_0^2λ_3|1⟩⟨0| + |λ_1^2|1⟩⟨1| + λ_1^2λ_2|1⟩⟨2| + λ_1^2λ_3|0⟩⟨2| + λ_2^2λ_1|2⟩⟨1| + |λ_2^2|2⟩⟨2|\right)$$

$$= \left(\begin{array}{ccc}
\frac{1}{9}λ_0 & -\frac{1}{9}λ_0λ_1 & -\frac{1}{9}λ_0λ_2 \\
-\frac{1}{9}λ_1 & \frac{1}{9} - \frac{1}{9}λ_1^2 & -\frac{1}{9}λ_1λ_2 \\
-\frac{1}{9}λ_2 & -\frac{1}{9}λ_2λ_1 & \frac{1}{9} - \frac{1}{9}λ_2^2 \\
\end{array}\right).$$

The operator $S(a) e$ has the eigenvalues $\{\frac{1}{9}, \frac{1}{9}, \frac{1}{9} - \frac{1}{9}a\}$. Therefore, by definition $S(a)$ is an entanglement witness for $\frac{1}{3} < a < \frac{2}{3}$. Does exist, however, a region of the parameter space for which $S(a)$ is a 3-SW, i.e. it detects a state with Schmidt rank 3 but does not detect any state with Schmidt rank 2? Clearly for all $\frac{1}{3} < a < \frac{2}{3}$ where $S(a)$ is no 3-SW it is a 2-SW.

According to Theorem 3 an operator $S(a)$ on a $3 \times 3$-dimensional Hilbert-space is a 3-SW iff $S_2(a) := S(a) \otimes (|00⟩⟨00| + |11⟩⟨11|)$ is a (non-normalized) 2-SW. This new operator fulfills that for all pure states $|E⟩ ∈ H_A \otimes C^2$ the operator $⟨E|S_2(a)|E⟩ ≥ 0$. Furthermore since any pure state $|E⟩$ can be decomposed in an arbitrary basis $\{|1⟩, |2⟩, |3⟩\} ∈ H_A$ as $|E⟩ = μ_0|0⟩ + μ_1|1⟩ + μ_2|2⟩ + μ_3|3⟩$, the above operator can be expressed as:

$$⟨E|S_2(a)|E⟩ = \frac{1}{1 - a} \begin{pmatrix}
A - \frac{1}{9}a|μ_0|^2 & B - \frac{1}{9}aμ_0μ_1 & C - \frac{1}{9}aμ_0μ_2 \\
B - \frac{1}{9}aμ_0μ_1 & D - \frac{1}{9}a|μ_1|^2 & A - \frac{1}{9}aμ_1μ_2 \\
C - \frac{1}{9}aμ_0μ_2 & A - \frac{1}{9}aμ_1μ_2 & D - \frac{1}{9}a|μ_2|^2
\end{pmatrix}$$

with $A = (\frac{1}{9}|μ_0|^2 + |μ_2|^2 + |μ_4|^2)$, $B = (\frac{1}{9}μ_0^2μ_1 + μ_2^2μ_3 + μ_4^2μ_5)$, $C = (\frac{1}{9}μ_0^2μ_2 + μ_3^2μ_4 + μ_5^2μ_2)$ and $D = \frac{1}{9}(|μ_1|^2 + |μ_3|^2 + |μ_5|^2)$. It is tedious but straightforward to check that this operator is positive definite for $\frac{1}{3} ≥ a ≥ 0$. Thus, we obtain that

$$S(a) \begin{cases}
\text{positive (no witness)} & \frac{1}{3} ≥ a ≥ 0 \\
\text{a 3-SW} & \frac{2}{3} ≥ a > \frac{1}{3} \\
\text{a 2-SW} & \frac{1}{3} ≥ a > \frac{1}{6}
\end{cases}$$

4 Conclusions

In this article we have first reviewed some properties of general witness operators, as well as their optimisation. We have then focussed on Schmidt number witnesses for bipartite systems,
i.e. those witness operators which are capable to detect the minimal number of entangled
degrees of freedom between both parties (their Schmidt number). We have shown that it is
possible to relate any witness operator for Schmidt number \( k \) to a witness of Schmidt number
2 in an enlarged Hilbert space in such a way that the original subset of states with Schmidt
number \((k - 1)\) corresponds to the subset of separable states in the enlarged space. The
fact that one can establish this correspondence between a \(k\)-Schmidt witness in the original
Hilbert space and a Schmidt witness of number 2 (i.e. a general entanglement witness) in
an enlarged Hilbert space substantially simplifies the construction and optimization of the
desired \(k\)-Schmidt witness. The reason for this is the fact that it is, in general, a much easier
task to check whether an operator is positive semidefinite on pure product states, rather than
to check positivity on pure states of a given Schmidt rank larger than one. Nevertheless a
word of caution is needed when using this method for optimization purposes only, as the
concept of “being finer” is not generally preserved under the lifting map. Therefore it is not
always possible to optimize a \(k\)-Schmidt witness by optimizing the corresponding 2-Schmidt
witness in the enlarged space.

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18. If \( S_1 \) is of measure 0 in \( S_2 \) and is convex there exists a lower-dimensional subspace \( S'_2 \) of \( S_2 \) s.t. \( S_1 \subset S'_2 \) but not of measure 0. To check whether an element does not belong to a subspace is fairly easy (by application of the projection onto that subspace). So without loss of generality \( S_1 \) is not of measure 0 in \( S_2 \).

19. When the set \( S_1 \) is closed there exists, according to the Hahn-Banach theorem, for any \( \rho \notin S_1 \) a linear functional \( f \) on the space of all bounded operators on \( B(\mathcal{H}) \), with the property that \( f(\rho) < 0 \), but \( f(\sigma) \geq 0 \) for all \( \sigma \in S_1 \). Furthermore, every linear functional corresponds to a hermitian operator \( W(f) \) defined through \( f(\sigma) = \text{Tr} (W(f)\sigma) \). Since \( 1 \in S_1 \) we know that \( \text{Tr} (W(f)) > 0 \), therefore the operator \( W(f)/( \text{Tr} (W(f))) \) is a general \( (S_1, S_2) \)-witness that detects \( \rho \).

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24. The Schmidt decomposition is unique only up to a permutation of the terms. Thus, strictly speaking, the map \( I_k \) is not defined unambiguously, due to different possible orderings of the Schmidt terms. However, the arguments in the remainder of the paper are independent of this permutation dependence, and hold for any order of the positive Schmidt coefficients. Therefore, in the following we use for every pure state \( |\psi\rangle \) an arbitrary but fixed order \( |\psi\rangle = \sum_{i=1}^{n} \lambda_i |a_i\rangle \otimes |b_i\rangle \), where for all \( i \) larger than the Schmidt rank of \( |\psi\rangle \) we have \( \lambda_i = 0 \).

25. W. Dür, J.I. Cirac, M. Lewenstein, and D. Bruß, Phys. Rev. A 61, 062313 (2000).