An $O(\log \log m)$-competitive Algorithm for Online Machine Minimization

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Abstract

This paper considers the online machine minimization problem, a basic real time scheduling problem. The setting for this problem consists of $n$ jobs that arrive over time, where each job has a deadline by which it must be completed. The goal is to design an online scheduler that feasibly schedules the jobs on a nearly minimal number of machines. An algorithm is $c$-machine optimal if the algorithm will feasibly schedule a collection of jobs on $c \cdot m$ machines if there exists a feasible schedule on $m$ machines. For over two decades the best known result was a $O(\log P)$-machine optimal algorithm, where $P$ is the ratio of the maximum to minimum job size. In a recent breakthrough, a $O(\log m)$-machine optimal algorithm was given. In this paper, we exponentially improve on this recent result by giving a $O(\log \log m)$-machine optimal algorithm.

1 Introduction

In a typical real time scheduling environment, there is a collection of jobs that arrive over time. This collection of jobs could be generated by a task system. Each job has a processing time and a deadline, and must be processed by its deadline. In such a setting, there are typically two types of results in the literature. One type of result is the design of a scheduler, and an analysis that shows that this scheduler can complete all jobs by their deadlines if the job instance satisfies certain conditions. The other type of result is the design of a feasibility test, and an analysis that shows that this test will either determine whether a particular scheduler will feasibly schedule any job instance that might arise from a particular task system on some collection of machines, or determine that there is some job instance that the scheduler will not feasibly schedule on some collection of machines.

One classic scheduling result, of the first type, is that the Earliest Deadline First (EDF) scheduling algorithm is optimal for deadline scheduling on one machine. That is, given a collection of jobs for which there exists a feasible schedule on one machine, EDF will feasibly schedule that collection of jobs on one machine. This result is one of the main reasons EDF is widely used in the real time scheduling literature [1–6].

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Meeting all deadlines becomes more challenging when the jobs can be scheduled on a set of $m$ identical machines. It is known that no optimal online algorithm exists for more than one machine \[7\]. That is, for every online scheduler, and for every $m > 1$, there is a collection of jobs that is feasibly schedulable on $m$ machines, but that this scheduler will not feasibly schedule on $m$ machines. This impossibility result has naturally led to a line of research that involves seeking online algorithms that are near-optimal.

The type of near-optimal algorithm that this paper is concerned with is a $c$-machine optimal algorithm. We say that an algorithm is $c$-machine optimal if the algorithm will feasibly schedule a collection of jobs on $c \cdot m$ machines if there exists a feasible schedule on $m$ machines. The goal of this line of research is to determine how small a machine augmentation parameter $c$ is attainable by an online algorithm. Determining whether there exists an $O(1)$-machine optimal algorithm is considered to be a big open problem in this line of research \[8,9\].

The concept of a $c$-machine optimal algorithm can be related to scheduling task systems in real-time scheduling as follows. If there exists an algorithm that can feasibly schedule jobs generated by a task system on $m$ machines then a $c$-machine algorithm will feasibly schedule the jobs from the task system on $cm$ machines. In particular, a $c$-machine algorithm can be used to schedule an infinite set of jobs generated by a task system feasibly on $cm$ machines so long as some algorithm can feasibly schedule the jobs on $m$ machines.

An important parameter of a job is its relative laxity, which is the job’s laxity divided by the length of its lifespan. (The length of its lifespan is its deadline minus its release time, and its laxity is the length of its lifespan minus its size.) It is relatively straightforward to observe that if all jobs have relative laxity $\Omega(1^{2})$, then EDF is $O(1)$-machine optimal. See \[10\] for details. Unfortunately, the problem is much more challenging when jobs have smaller relative laxity. For over two decades the best known result, when there is no restriction on the laxity, was an $O(\log P)$-machine optimal algorithm, where $P$ is the ratio of the maximum to minimum job size \[8\]. Essentially, Phillips et al. \[8\] observed that $O(\log P)$-machine augmentation trivially reduces the general problem to the easy special case that all jobs have almost the same size. The bound of $O(\log P)$ also has the disadvantage of being dependent on the input data; bounds that are independent of the input data are much stronger.

In a recent major advance, Chen et al. \[10\] gave a novel online algorithm, which we call CMS after the authors’ initials. Their analysis showed that their algorithm is $O(\log m)$-machine optimal for jobs with relative laxity less than $1/2$. The algorithm and analysis are somewhat complex, but the underlying intuition of the CMS algorithm design is to prioritize jobs that have used the largest fraction of their original laxity. Thus, by running EDF on jobs with relative laxity more than $1/2$ on half the machines, and by running the CMS algorithm on jobs with relative laxity at most $1/2$ on half of the machines, the work \[10\] obtains an $O(\log m)$-machine optimal algorithm for arbitrary instances. The work of \[11\] improved on this slightly by observing that one can combine EDF and the CMS algorithm somewhat more cleverly to obtain a $O(\log \frac{m}{\log \log m})$-machine optimal algorithm.

### 1.1 Our Results

Our main result is an exponential improvement on the machine augmentation parameter achieved in \[10,11\]. We give a new algorithm (called Algorithm A in this paper) and analysis showing that Algorithm A is $O(\log \log m)$-machine optimal.

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1. A function $g(x)$ is $O(f(x))$ if there exists a constant $c$ and any value $x_0$ such that $g(x) \leq cf(x)$ for all $x \geq x_0$.

2. A function $g(x)$ is $\Omega(f(x))$ if there exists a constant $c$ and any value $x_0$ such that $g(x) \geq cf(x)$ for all $x \geq x_0$. 

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Our algorithm is constructed from several building blocks. The initial insight that led to our main result, and the first building block, is the observation that the algorithm Shortest Job First (SJF) is $O(1)$-machine optimal if all jobs have approximately the same relative laxity. More precisely, we show the following. This proof of the following lemma is given later in the paper.

**Lemma 1.** For a collection of jobs with relative laxities in the range $[\lambda_1, \lambda_2] \subseteq (0, 1/2]$, Shortest Job First is $O(\log_2 \lambda_2 / \lambda_1)$-machine optimal for any $\lambda_1, \lambda_2$.

The second building block is that an implication of Lemma 1 is that SJF is $O(1)$-machine optimal if the relative laxities of the jobs lie in the range $[1/2^{2^i}, 1/2^{2^i+1}]$, for some $i \geq 1$. This leads to an algorithm with machine augmentation doubly logarithmic in the inverse of the minimum relative laxity of any job. More precisely, we show that:

**Lemma 2.** Consider some instance where the relative laxities lie in the range $[1/R, 1/2]$. There is a $O(\log \log R)$-machine optimal algorithm.

**Proof.** Partition the jobs into $\lg \lg R$ different groups, where the jobs in group $i$ have relative laxities in the range $[1/2^{2^{i+1}}, 1/2^{2^i}]$. Use SJF to run the jobs in each group on $O(m)$ machines dedicated to that group. By Lemma 1 the total number of machines per group required is $O(m \log_2 (2^{2^{i+1}} / 2^{2^i})) = O(m \log_2 2^{2^i}) = O(m)$. Thus, at most $O(m \log \lg R)$ machines are needed to ensure all job are completed by their deadlines. \qed

The final building block is the observation that by tweaking the analysis in [10], one can show that the CMS algorithm is $O(1)$-machine optimal if the relative laxities of the jobs are all at most $1/m$. The jobs with relative laxities at least $1/2$ are scheduled on separate $O(m)$ machines using EDF as in the work by Chen et al. [10]. Putting all these building blocks together, we obtain our main result, stated in Theorem 3, that the following algorithm $A$ is $O(\log \log m)$-machine optimal. See Section 4 for full the description of the algorithm $A$.

**Theorem 3.** There is a $O(\log \log m)$-machine optimal algorithm.

The results in this paper have further implications regarding another type of resource augmentation, namely speed augmentation, which is commonly used, either instead of, or in conjunction with, machine augmentation. In our context, an $s$-speed $c$-machine optimal algorithm would feasibly schedule a job instance on $cm$ machines of speed $s$ if this job instance is feasibly schedulable on $m$ machines of speed 1. Speed augmentation is widely used for designing near optimal algorithms, and in corresponding feasibility tests [11–6]. The best combined speed and machine augmentation result comes from the paper [12]. This paper showed the existence of a $(1 + \epsilon)$-speed $O(1/\epsilon)$-machine optimal algorithm. As a corollary to our main result, we can give a doubly-exponential improvement on the trade-off of speed and machine augmentation, stated in Corollary 4:

**Corollary 4.** There is a $(1 + \epsilon)$-speed $O(\log \log \frac{1}{\epsilon})$-machine optimal algorithm for any $\epsilon > 0$.

The proof of Corollary 4 follows from the observation that on a $1 + \epsilon$ speed machine, every job (that is feasibly schedulable on a speed 1 machine) has relative laxity at least $\frac{1}{1 + \epsilon}$.

**Application to Task Systems:** We now comment briefly on the application of these results to periodic/real-time scheduling. On the positive side, as these results apply to all jobs instances, they apply to job instances that arise from periodic task systems. A corollary to our results is that if every job instance arising from a particular task system can be scheduled on $m$ machines, then algorithm $A$ will schedule every job instance arising from this task system on $O(m \log \log m)$ machines. On the negative side, in the context of real-time scheduling, one generally also wants
a corresponding efficient feasibility test that matches the optimality result. In our context, such a test would take as input a task system and a number of machines \( m \), and would determine whether the algorithm \( A \) will feasibly schedule any job instance that might arise from this task system on \( O(m \log \log m) \) machines, or determine that there is some job instance that might arise from this task system that is not feasibly schedulable on \( m \) machines. As an example using speed augmentation, the paper \cite{8} showed that EDF is 2-speed optimal, and the paper \cite{13} extended this speed-augmentation optimality result to an efficient feasibility test for EDF for speed 2 processors. That is, this test determines whether every job instance arising from a task system will be feasibly scheduled by EDF on \( m \) processors of speed 2, or determines that some job instance arising from the task system is not feasibly schedulable on \( m \) processors of speed 1. Unfortunately, machine augmentation is more combinatorially complicated than speed augmentation, and we do not yet know how to extend our machine-augmentation optimality result to an efficient feasibility test. The best we can say is that an optimality result is the first step toward achieving a feasibility test.

## 2 Further Related Work

### Non-Periodic Scheduling

In addition to solving the general deadline problem, the paper \cite{10} handled the special cases of laminar and agreeable deadlines\(^4\), showing that their algorithm is \( O(1) \)-machine optimal for these job instances. Following up on this result, Chen et al. \cite{14} shows that there is no non-migratory \( O(1) \)-machine optimal algorithm. A non-migratory algorithm schedules each job on a unique machine. For the definition of laminar and agreeable deadlines, see \cite{10}.

The paper \cite{12} gave an online algorithm that is \( \left(2 - \frac{2(m-1)+mp}{(m-1)(m+1)+mp}\right) \)-speed \( (m+p) \)-machine optimal and they also give a slightly weaker trade-off analysis for the EDF algorithm. The works \cite{8,12} gave lower bounds showing that there is no \( (1+o(1)) \)-machine optimal algorithm.

### Real-time/Periodic Scheduling

For a survey of standard terminology and notable results for real-time scheduling see \cite{15}. Most of the related results in the real-time literature are about partitioned scheduling, where all jobs emanating from the same task have to run on the same machine. The paper \cite{16} shows that the problem of deciding whether an implicit deadline task set is feasible on a certain number of machines is NP-hard. The paper \cite{17} shows that it is NP-hard to differentiate between implicit deadline task sets that are feasible on 2 machines from those that require 3 machines. The paper \cite{2} gives a partition algorithm that guarantees feasibility on speed 3 machines for a constrained deadline task system if there is a partition that is feasible on speed 1 machine. The paper \cite{17,18} provide polynomial-time approximation schemes for some special cases when speed augmentation is adopted. The papers \cite{17,19} rule out the existence of asymptotic approximation schemes for certain types of task systems. Finally, the paper \cite{19} provides polynomial time partitioning algorithms whose approximation ratios are a function of the maximum ratio of the period to the deadline.

## 3 Formal Problem Definition and Notations

A (finite) set of \( n \) jobs arrive over time to be scheduled on \( m \) identical machines/processors. These jobs could be generated by a task system. Each job \( j \) has size \( p_j \) and deadline \( d_j \). The online scheduler only learns of job \( j \) when \( j \) arrives at its release time \( r_j \). This paper assumes that all

\(^4\)In the laminar case, for any pair of jobs \( i \) and \( j \), either the two jobs lifespans are disjoint or one job’s lifespan fully contains the other job’s. In the agreeable deadline case, if job \( i \) is released earlier than job \( j \), then \( i \) has a deadline no later than \( j \).
job characteristics, $p_j$, $d_j$, and $r_j$, are integers. Each machine can process at most one job at a time, and no job can be processed at the same time on two different machines. The paper considers preemptive migratory scheduling, which means that there is no further restrictions on when and where a job is processed. A job $j$ completes if it gets processed for $p_j$ units of time. We say that a job is alive at time $t$ if the job has arrived and hasn’t completed at $t$. Every job $j$ must be processed and completed within their lifespan (processing interval) $I(j) := (r_j, d_j)$. We say that a schedule is feasible if all jobs complete within their lifespans.

A job $j$’s laxity $\ell_j$ is defined as $d_j - r_j - p_j$. In words, job $j$ may spend no more than $\ell_j$ time steps during its lifespan not being processed in a feasible schedule. Intuitively, one can think of $\ell_j$ as $j$’s budget, and the job has to pay a unit cost out of its budget when it is not processed. A job $j$’s relative laxity, denoted as $p_j$, is $\ell_j/I(j)$, the ratio of the job’s laxity to its lifespan length. We say that $j$ covers time $t$ if $t$ is within $j$’s lifespan, i.e. $t \in I(j)$. For a set of jobs $S$, define $I(S) := \bigcup_{j \in S} I(j)$. For a finite collection $I$ of disjoint intervals, let $|I|$ denote the total length of intervals in $I$.

Let $m^*$ denote the minimum number of machines that admits a feasible offline schedule for a given instance. It can be assumed without loss of generality that $m^*$ is known to the algorithm up to a constant factor using a standard doubling trick – if our scheduling fails due to underestimating $m^*$, we simply double our estimate. See [10] for more details. The algorithm used in this paper will be parameterized by $m^*$.

Let $\alpha$ be a scalar. We say that job $j$ is $\alpha$-loose if $p_j \leq \alpha |I(j)|$, otherwise it is $\alpha$-tight. We say a job is simply loose if it is $\frac{1}{2}$-loose. We say that a job is very tight if its relative laxity is at most $1/m$. Note that relative laxity and $\alpha$ are defined differently, so a job that is $\alpha$-loose for a large $\alpha$ actually has a small relative laxity.

Whenever a machine becomes available, the algorithm Shortest-Job-First (SJF) chooses to schedule the uncompleted job $j$ with the smallest original work, $p_j$. Whenever a machine becomes available, the algorithm Earliest-Deadline-First (EDF) chooses to schedule the uncompleted job $j$ with the smallest deadline, $d_j$.

4 Algorithm Description

This section describes our $O(\log \log m)$-machine optimal algorithm, which will be denoted as $A$. The algorithm is hybrid and runs several different procedures depending on the relative laxity of jobs. The algorithm $A$ is parameterized by $m^*$, the minimum number of machines required to feasibly schedule the jobs by any (offline) algorithm. To present our algorithm more transparently, we describe our algorithm assuming that the parameter $m^*$ is known to the algorithm a priori—we will show in Section 4.2 how we can easily remove the assumption by using at most four times more machines. Likewise, we will show in Section 4.2 $A$ can be implemented without knowing parameters $m_{edf}$, $m_{sjf}$ and $m_{cms}$ that appear in the following; all parameters will be shown to be $O(m^*)$.

- Earliest Deadline First (EDF): Jobs with relative laxity at least $1/4$ are scheduled using EDF on $m_{edf}$ dedicated machines.
- Shortest Job First (SJF): Let $L_i$ denote the set of jobs with relative laxity in the range of $(1/2^{2i+1}, 1/2^{2i})$ where $i$ is an integer in the range of $[1, \lceil \log \log m^* \rceil]$; here, $\log$ has a base of 2. For each $i$, a set of $m_{sjf}$ machines, $M_i$, are dedicated to processing jobs in $L_i$ using SJF. At any point in time SJF schedules up to $m_{sjf}$ jobs with the smallest sizes.
• Chen-Megow-Schewoir (CMS): The remaining jobs, which have relative laxity no greater than $1/m^*$, are scheduled using the CMS algorithm [10] on $m_{cms}$ dedicated machines. The description of the CMS algorithm is given in the next section.

Note that EDF, SJF and CMS algorithms are used to process jobs with relative laxities that are high, intermediate, and low, respectively. It is important to note that the three algorithms use disjoint sets of machines. Further, the algorithm separately uses SJF for jobs in each set $L_i$ using a distinct set $M_i$ of machines. It is easy to see that $O(m^* \log \log m^*)$ machines are used by our hybrid algorithm A if $m_{edf}$, $m_{sjf}$ and $m_{cms}$ are all $O(m^*)$.

### 4.1 Algorithm Chen-Megow-Schewoir (CMS)

Since the algorithm CMS is not as well known as EDF or SJF, we give a full description of CMS including its pseudocode. The algorithm CMS takes as input a parameter $m_{cms}$. The algorithm processes jobs using $m_{cms} + 1$ machines, and either outputs a feasible schedule using $m_{cms}$ machines or declares failure. The $m_{cms} + 1$ machines are indexed by $1, 2, \cdots, m_{cms} + 1$ in an arbitrary but fixed order. The last machine $m_{cms} + 1$ is forbidden, meaning that the algorithm declares failure if it ever processes a job on the machine. Each job $j$ is initially given a budget equal to its laxity, $\ell_j = d_j - r_j - p_j$, and its budget is equally distributed to the $m_{cms} + 1$ machines. We emphasize that the budget is never shared between machines. Let $b_{ji}(t)$ denote $j$'s budget for machine $i$ at time $t$. Note that $b_{ji}(r_j) = \ell_j/(m_{cms} + 1)$ for all $i \in [m_{cms} + 1]$.

We now describe how the algorithm CMS decides which jobs to schedule and which to delay at $t$ at each fixed time $t$. Consider the incomplete jobs in decreasing order of their arrival times, breaking ties in an arbitrary but fixed order. When considering job $j$, let $i$ be the least indexed machine a job is not currently being scheduled on at the fixed time $t$. We assign $j$ to machine $i$, which doesn’t necessarily mean that $i$ processes job $j$ at the moment. If job $j$ has any budget left for machine $i$, i.e. $b_{ij}(t) > 0$, do not process $j$, i.e. delay it, decreasing $b_{ij}(t)$ at a rate of 1 at the instantaneous time $t$. If the budget is empty, i.e., $b_{ij}(t) = 0$, schedule job $j$ on machine $i$. After either delaying or processing $j$ on machine $i$, we consider the next incomplete job. As mentioned before, the algorithm CMS declares failure if it ever has the forbidden machine $m_{cms} + 1$ process a job.

The algorithm CMS keeps the same schedule, i.e. schedule exactly the same job on the same machine, until time $t''$ when a new job arrives or a job assigned to machine $i$ completely uses its budget for machine $i$ while getting delayed. Since the above procedure is invoked only for such events, it is easy to see that CMS runs in polynomial time. For more details, see the pseudo-codes, Algorithms [1] and [2].

It is worth noting that it could happen that a job uses its budget for machine $i$ before depleting it budgets for lower-indexed machines, $1, 2, \cdots, i - 1$. Thus, when the algorithm declares failure, that is, processes a certain job $j$ on machine $m_{cms} + 1$ at time $t$, it must be the case that $b_{j,m_{cms}+1}(t) = 0$, but not necessarily $b_{ji}(t) = 0$ for all $i \in [m_{cms}]$.

We now take a close look at CMS taking into account issues arising in its implementation. Algorithm [1] CMS, is described assuming that the first job arrives at time 0 and no two jobs arrive at the same time. These assumption can be made w.l.o.g. by shifting the time horizon and breaking ties between jobs with the same arrival time in an arbitrary but fixed order. Algorithm [1] uses Algorithm [2] Sub-CMS, as a sub-procedure. When CMS calls Sub-CMS, it passes to the sub-procedure the set of alive jobs at the moment, $A_t$, and the number of the given machines, $m_{cms}$, along with jobs’ remaining budgets $\{b_{ji}(t)\}$ and remaining sizes $\{p_j(t)\}$; here $p_j(t)$ denotes $j$’s remaining size at time $t$. Then, Sub-CMS finds an assignment of each job to a machine at
Algorithm 1: Algorithm Chen-Megow-Schewior (CMS)

**Input:** A sequence of jobs arriving online; $m_{\text{cms}}$ machines indexed by $1, 2, \cdots, m_{\text{cms}}$

**Output:** Either yields a feasible schedule or declares failure; a feasible schedule is always output if $m_{\text{cms}} \geq c_{\text{cms}} \cdot m^*$

1. $t' = 0$ // the latest time when the Sub-CMS was called;
2. $t = 0$ // the current time;
3. while $A_t := \{ j \mid r_j \leq t, p_j(t) > 0 \} \neq \emptyset$
   do
    // $A_t$: jobs alive at time $t$
    4. $\psi \leftarrow \text{Sub-CMS}(A_t, m_{\text{cms}}, \{ b_{ji}(t) \}_{j \in A_t, i \in [m_{\text{cms}}+1]}, \{ p_j(t) \}_{j \in A_t});$
    5. if $\exists j \in A_t$ such that $\psi(j, t) = m_{\text{cms}} + 1$ and $b_{j,\psi(j,t)} = 0$ then
       declare failure and terminate;
    6. $\Delta_1 = \min \{ b_{ji,\psi(j,t)}(t) \mid b_{ji,\psi(j,t)} > 0, j \in A_t \};$
    7. $\Delta_2 = \min \{ p_j(t) \mid b_{ji,\psi(j,t)} = 0, j \in A_t \};$
    8. $\Delta = \min \{ \Delta_1, \Delta_2 \};$
    9. $t' = t;$
    10. $t = t' + \Delta;$
    11. if a new job $j$ arrives before time $t$ then
        for all $i \in [m_{\text{cms}} + 1]$ do
          12. $b_{ji}(r_j) = \ell_j / (m_{\text{cms}} + 1);$  
        end
        13. $t = r_j;$
        14. for all $j \in A_{t'}$ such that $b_{ji,\psi(j,t')} > 0$ do
            15. $b_{ji,\psi(j,t')}(t) = b_{j,\psi(j,t')}(t') - (t - t');$
        end
        16. for all $j \in A_{t'}$ such that $b_{ji,\psi(j,t')} = 0$ do
            17. $p_j(t) = p_j(t') - (t - t');$
        end
    end
  end

Algorithm 2: Algorithm Sub-CMS

**Input:** $A_t$: machines $1, 2, \cdots, m_{\text{cms}} + 1$; $b_{ji}(t)$ for all $j \in A_t$ and $i \in [m_{\text{cms}} + 1]$; $p_j(t)$ for all $j \in A_t$.

**Output:** $\psi(j, t)$ for all all jobs $j \in A_t$.

1. Order jobs in $A_t$ in non-increasing order of their arrival times;
2. $i = 1;$
3. for each $j \in A_t$ do
   4. $\psi(j, t) = i;$
   5. if $b_{ij} = 0$ then
      6. $i = i + 1;$
   end
4. end
Lemma 5. Given a c-machine-optimal online algorithm that takes $m^*$ as a parameter, we can convert it into one that is 4c-optimal without using the parameter.

To convey the main idea more transparently and illustrate how it is used, we show this theorem for a specific algorithm EDF—however, the proof is completely oblivious to EDF, and therefore, we will have the theorem immediately. Recall that we use EDF to process jobs with relative laxity at least $1/4$, whose set is denoted as $\mathcal{L}_{\text{high}}$. The following theorem shown in [10] (Theorem 2.3),

Theorem 6. If all jobs have relative laxity at least $\rho$, EDF is $1/\rho^2$-machine optimal.

implies that if we run EDF using $cm^*$ machines, where $c = 16$, we can feasibly schedule all jobs in $\mathcal{L}_{\text{high}}$. If we had known $m^*$ from the beginning, we could have dedicated $cm^*$ or more machines from time 0 and we would have successfully scheduled all jobs in $\mathcal{L}_{\text{high}}$. To avoid using the knowledge of $m^*$, at a high-level, we will partition the time horizon $[0, \infty)$ online into disjoint intervals, $I_1 := [t_0 := 0, t_1), I_2 := [t_1, t_2), \ldots, I_\kappa := [t_{\kappa-1}, t_\kappa = \infty)$. Each interval $I_k$ is associated with $2^{k-1}$
machines that are exclusively dedicated to processing jobs arriving during \( I_k \), which we denote as \( J(I_k) \). We now describe how we define the times \( t_1, t_2, \cdots \) online. Initially, we use only one machine to run EDF. Whenever a new job arrives at time \( t \), we simulate EDF’s schedule until we complete all jobs that have arrived pretending that no more jobs arrive. If EDF can feasibly complete all jobs alive at the time using the single machine, then we do nothing. Otherwise, we set \( t_1 = t \). As mentioned before, jobs arriving by time \( t_1 \) are scheduled by the initial single machine. We repeat this recursively: Say the current time \( t \) is such that \( t \geq t_{k-1} \) but we haven’t set \( t_k \) yet. When a new job \( j \) arrives at time \( t \), we simulate EDF’s schedule pretending that no more jobs arrive and set \( t_k = t \) if it fails to yield a feasible schedule for jobs that have arrived after \( t_{k-1} \) using \( 2^{k-1} \) machines; otherwise we do nothing.

It is clear that this algorithm always gives a feasible schedule as we use more machines whenever we need more. Therefore, it only remains to show that the number of machines we will have used at the end is not far from \( m^* \). We show that \( 2^{k-1} \leq 2cm^* \), meaning that we use at most \( \sum_{k=1}^{\infty} 2^{k-1} < 2^k \leq 4cm^* \). To see this, for the sake of contradiction suppose \( 2^{k-1} > 2cm^* \). Thus, we have \( 2^{k-2} > cm^* \). This means that even if we dedicated more than \( cm^* \) machines, we couldn’t feasibly schedule all jobs arriving during \( I_{k-1} \) and had to use more machines from time \( t_k \). This is a contradiction to the precondition that all jobs, including those jobs arriving during \( I_{k-1} \), are schedulable on \( cm^* \) machines. Recalling \( c = 16 \), we can feasibly schedule all jobs in \( L_{\text{high}} \) using 64\( m^* \) machines. As mentioned, this proof is oblivious to the algorithm, hence we have Lemma 5.

Thus, we can use this doubling trick for each run of EDF for jobs in \( L_{\text{high}} \), SJF for each \( L_i, i \in [1, \lfloor \log \log m^* \rfloor] \), and CMS for the other jobs. Each run is guaranteed to find a feasible schedule for the jobs it is assigned when using \( O(m^*) \) machines. Since we use at most four times more machines for each run by doubling and there are \( 2 + \lfloor \log m^* \rfloor \) runs, we use at most \( O(m^* \log \log m^*) \) machines, as desired.

5 Algorithm Analysis

In this section, the theoretical guarantees of algorithm \( A \) are shown.

The main challenge in analyzing the optimality of an online algorithm is discovering strong lower bounds on \( m^* \), the minimum number of machines needed to feasibly schedule a particular job instance. In subsection 5.1 and subsection 5.2 we strengthen two lower bounds found in [10].

With new lower bounds on \( m^* \) in place, Subsection 5.3 proves Lemma [1].

Subsection 5.4 explains why CMS is \( O(1) \)-machine optimal for very tight jobs, and EDF is \( O(1) \)-machine optimal for loose jobs.

Our main result, that algorithm \( A \) is \( O(\log \log m) \)-machine optimal, follows by combining these results.

5.1 First Lower Bound

This section gives a new lower bound on \( m^* \). To do so, the following important definition originating from [10] is needed.

**Definition 1 ([10])** Let \( G \) be a set of \( \alpha \)-tight jobs and let \( T \) be a non-empty finite union of time intervals. For some \( \mu \in \mathbb{N} \) and \( \beta \in (0, 1) \), a pair \((G, T)\) is called \((\mu, \beta)\)-critical if

1. each time \( t \in \mathbb{N} \) and \( \beta \in (0, 1) \), a pair \((G, T)\) is called \((\mu, \beta)\)-critical if
2. \( |T \cap I(j)| \geq \beta \ell_j \) for all \( j \) in \( G \).
Based on this definition, Chen et al. \cite{10} gave the following novel lower bound on \( m^* \).

**Theorem 7 (\cite{10}).** If there exists a \((\mu, \beta)\)-critical pair, then \( m^* \) is \( \Omega(\frac{\mu}{\log 1/\beta}) \).

In the following, it is shown that Theorem 7 can be strengthened.

**Theorem 8.** If all jobs are \( \alpha \)-tight, and there exists a \((\mu, \beta)\)-critical pair, then \( m^* \) is \( \Omega(\frac{\mu}{\log_2(1-\alpha)/\beta}) \).

The rest of this section is devoted to proving Theorem 8. The proof builds on the analysis given in \cite{10}. The proof of Theorem 7 in \cite{10} establishes the following.

**Lemma 9 (\cite{10}).** If there exists a pair \((G, T)\) that is \((\mu, \beta)\)-critical then there exists a collection \( S_1, S_2, \ldots S_{\lceil 2m^*/\alpha \rceil} \) of pairwise disjoint sets of \( \alpha \)-tight jobs where \( I(S_1) \subseteq I(S_2) \subseteq \cdots \subseteq I(S_{\lceil 2m^*/\alpha \rceil}) \). Further if \( |I(S_{\lceil 2m^*/\alpha \rceil})| \geq \gamma |I(S_1)| \) then \( m^* \geq \Omega(\frac{\mu}{\log_2 \beta}) \) for any scalar \( \gamma \).

After proving this lemma, the proof in \cite{10} is completed by showing \( \gamma \geq 2 \).

Given the previous lemma, to prove Theorem 8 it is sufficient to establish a stronger lower bound on \( \gamma \), namely that \( \gamma \geq \frac{1}{32(1-\alpha)} \). This is done in Lemma 10. This and Lemma 13 gives a contradiction to the definition of \( m^* \).

**Lemma 10.** Let \( S_1, S_2, \ldots S_{\lceil 2m^*/\alpha \rceil} \) be pairwise disjoint sets of \( \alpha \)-tight jobs such that \( I(S_1) \subseteq I(S_2) \subseteq \cdots \subseteq I(S_{\lceil 2m^*/\alpha \rceil}) \). It is the case that \( 32(1-\alpha)|I(S_{\lceil 2m^*/\alpha \rceil})| \geq |I(S_1)| \).

**Proof.** Suppose that the lemma is not true and \( 32(1-\alpha)|I(S_{\lceil 2m^*/\alpha \rceil})| < |I(S_1)| \). To begin, we construct sets \( S'_1 \subseteq S_i \) such that \( I(S'_1) = I(S_i) \) and for each \( t \in I(S'_1) \) there are at most two jobs \( j \in S'_t \) where \( t \in I(j) \). The construction of such a set is standard in the scheduling community.

A full proof can be found in \cite{10}. Such a set can be constructed using a simple greedy procedure where jobs are chosen greedily such that you always choose to add the job to \( S'_t \) from \( S_i \) with the latest deadline that covers the smallest uncovered time \( t \in I(S_i) \).

Fix a set \( S'_t \). Partition the jobs in \( S'_t \) into two sets \( J_{i, 1} \) and \( J_{i, 2} \). Let \( J_{i, 1} \) contain a job \( j \in S'_t \) if \( |I(j) \cap I(S_1)| \geq 4(1-\alpha)|I(j)| \) and otherwise job \( j \) is in \( J_{i, 2} \).

First say that there exists an \( i \) such that \( |I(J_{i, 2}) \cap I(S_1)| \geq \frac{1}{4}|I(S_1)| \). Then we have the following.

\[
|I(S_{\lceil 2m^*/\alpha \rceil})| \\
\geq |I(S_1)| \quad \text{[Since } I(S_1) \subseteq S_{\lceil 2m^*/\alpha \rceil}] \\
\geq |I(S'_i)| \quad \text{[By definition of } I(S'_i)] \\
\geq |I(J_{i, 2})| \quad \text{[Since } J_{i, 2} \subseteq S'_t] \\
\geq \frac{1}{2} \sum_{j \in J_{i, 2}} |I(j)| \quad \text{[Definition of } S'_t] \\
\geq \frac{1}{8(1-\alpha)} \sum_{j \in J_{i, 2}} |I(j) \cap I(S_1)| \quad \text{[Definition of } J_{i, 2}] \\
\geq \frac{1}{32(1-\alpha)} |I(S_1)| \quad \text{[since } |I(J_{i, 2}) \cap I(S_1)| \geq \frac{1}{4}|I(S_1)|]\]

We note that the fourth inequality uses the fact that no time is covered by more than two jobs in \( S'_t \).
This contradicts the assumption that $32(1 - \alpha)|I(S|^{2m*/\alpha})| < |I(S_1)|$. Thus, we may assume that there is no $i$ where $|I(J_{i,2}) \cap I(S_1)| \geq \frac{1}{4}|I(S_1)|$. In particular, since $I(S_1) \subseteq I(S_i) = I(J_{i,1} \cup J_{i,2})$, it is the case that $|I(J_{i,1}) \cap I(S_1)| \geq \frac{3}{4}|I(S_1)|$.

We will draw a contradiction by showing that the amount of work that must be done during $I(S_1)$ is greater than any feasible schedule can complete using $m^*$ machines.

Consider any job $j \in \cup_i J_{i,1}$. Consider the amount of work of job $j$ that must be done during $I(S_1)$ by any feasible schedule. This is at least $q_j := p_j - (|I(j)| - |I(S_1) \cap I(j)|)$. Knowing that $j$ is $\alpha$-tight, we have that $q_j \geq \alpha(|I(j)|) - (|I(j)| - |I(S_1) \cap I(j)|) = |I(S_1) \cap I(j)| - (1 - \alpha)|I(j)|$.

Let $\lambda$ be such that $\lambda|I(j)| = |I(S_1) \cap I(j)|$. Then, we have $|I(S_1) \cap I(j)| - (1 - \alpha)|I(j)| = (1 - \frac{\lambda}{\alpha})|I(S_1) \cap I(j)|$. By definition of $J_{i,1}$ it is the case that $|I(j) \cap I(S_1)| \geq 4(1 - \alpha)|I(j)|$ and so $\lambda \geq 4(1 - \alpha)$. Therefore we have that $q_j \geq (1 - \frac{\lambda}{\alpha})|I(S_1) \cap I(j)| \geq \frac{3}{4}|I(S_1) \cap I(j)|$.

The argument above gives that each job $j \in \cup_i J_{i,1}$ must be processed for $\frac{3}{4}|I(S_1) \cap I(j)|$ time units during $I(S_1)$. The total amount of work that must be done during $I(S_1)$ for jobs in $J_{i,1}$ is at least the following for any $i$.

$$\sum_{j \in J_{i,1}} \frac{3}{4}|I(S_1) \cap I(j)| \geq \frac{3}{4}|I(J_{i,1}) \cap I(S_1)| \geq \frac{9}{16}|I(S_1)| \text{ since } |I(J_{i,1}) \cap I(S_1)| \geq \frac{3}{4}|I(S_1)|.$$

There are $[2m^*/\alpha]$ sets $S_i$ and unique jobs in each set. Thus, the total volume that must be processed during $I(S_1)$ is greater than $[2m^*/\alpha] \frac{q_m}{m}|I(S_1)| > m^*|I(S_1)|$. This is more work than any algorithm with $m^*$ machines can do during $I(S_1)$, contradicting the definition of $m^*$.

\[\square\]

5.2 The Second Lower Bound

The authors in [10] give another lower bound based on a variant of the definition of a critical pair.

**Definition 2.** Let $G$ be a set of $\alpha$-tight jobs and let $T$ be a non-empty finite union of time intervals. For some $\mu \in \mathbb{N}$ and $\beta \in (0, 1)$, a pair $(G,T)$ is called weakly $(\mu,\beta)$-critical if

1. each time $t$ belonging to an interval in $T$ is covered by at least $\mu$ distinct jobs in $G$.
2. $|T| \geq \beta/\mu \cdot \sum_{j \in G} \ell_j$.

**Theorem 11.** If there exists a weakly $(\mu,\beta)$-critical pair, then $m^* = \Omega(\frac{\mu}{\log 1/\beta})$.

This theorem can be strengthened as was done for the first lower bound.

**Theorem 12.** If there exists a weakly $(\mu,\beta)$-critical pair, then $m^* = \Omega(\frac{\mu}{\log_{1/(1-\alpha)} 1/\beta})$.

The proof of Theorem 12 extends the proof of Theorem 11 exactly as the proof of Theorem 8 extends the proof of Theorem 7.

The proof in [10] shows the following lemma.

**Lemma 13.** If there exists a pair $(G,T)$ that is weakly $(\mu,\beta)$-critical then there exists a collection $S_1, S_2, \ldots, S_{[2m^*/\alpha]}$ of pairwise disjoint sets of $\alpha$-tight jobs where $I(S_1) \subseteq I(S_2) \subseteq \ldots \subseteq I(S_{[2m^*/\alpha]})$. Further if $|I(S_{[2m^*/\alpha]})| \geq \gamma |I(S_1)|$ then $m^* \geq \Omega(\frac{\mu}{\log_{1/(1-\alpha)} 1/\beta})$.

Combining this lemma with Lemma 10 proves Theorem 12.
5.3 Analysis of SJF on Jobs with Similar Relative Laxities

This section is devoted to proving Lemma 1. Fix $\lambda_1$ and $\lambda_2$ such that all jobs have relative laxity in $[\lambda_1, \lambda_2]$. For the lemma it is assumed that $[\lambda_1, \lambda_2] \subseteq (0, 1/2]$. Consider such a job instance where all jobs are scheduled using SJF on $m$ machines and $m^*$ is the minimum number of machines required for any algorithm to feasibly schedule this problem instance. That is, at each time jobs are sorted by their original processing times and the $m$ jobs with smallest processing times are processed on the $m$ machines. Let $j$ be the first job SJF couldn’t complete before its deadline. Let $T$ be the set of times when job $j$ was not being processed during its lifespan. Let $G$ denote the set of jobs SJF schedules at times in $T$. Note that there are at least $m$ jobs processed by SJF at each time in $T$, meaning that each time in $T$ is covered by at least $m$ distinct jobs in $G$. Thus, $(G, T)$ satisfies the first property in Definition 2 for $\mu = m$.

It now remains to show the second property in Definition 2. To see this we first upper bound the total size of jobs in $G$. We know that every job $i$ in $G$ is 1/2-tight and no larger than job $j$. Thus, $i$’s lifespan length, $|I(i)| \leq 2p_i \leq 2p_j \leq 2|I(j)|$. Since $i$’s lifespan intersects $j$’s lifespan, $i$’s lifespan must be contained in $(r_j - 2|I(j)|, d_j + 2|I(j)|)$, implying that the total length of jobs in $G$ is at most $5m^*|I(j)|$. This is because the total amount of work any algorithm with $m^*$ machines can do during $(r_j - 2|I(j)|, d_j + 2|I(j)|)$ is upper bounded by $5m^*|I(j)|$.

This implies that $\sum_{i \in G} \lambda_i \leq 5m^*|I(j)|$ since a job’s laxity is at most $\lambda_2$ times its lifespan. Finally, we know that $|T| \geq \lambda_j \geq \lambda_1|I(j)|$ as $j$’s relative laxity is at least $\lambda_1$. Thus we have, $|T| \geq \frac{\lambda_1}{5\lambda_2} \cdot \frac{1}{m^*} \sum_{i \in G} \ell_i \geq \frac{\lambda_1}{5\lambda_2} \cdot \frac{1}{m^*} \sum_{i \in G} \ell_i$; the last inequality follows since our algorithm uses as many machines as the optimal scheduler. This implies that the second property is satisfied for $\beta = \frac{\lambda_1}{5\lambda_2}$ and $\mu = m$. Hence the pair $(G, T)$ is weakly $(m, \frac{\lambda_1}{5\lambda_2})$-critical. Since all jobs are $1 - \lambda_2$ tight because the relative laxities are at most $\lambda_2$, by Theorem 12 we have $m^* \geq c \cdot \frac{\mu}{\log_{1/(1-\alpha)} 1/\beta} = c \cdot \frac{\mu m}{\log_{1/(1-\alpha)} \frac{5\lambda_2}{\lambda_1}}$ for a certain constant $c > 0$. Thus, we have $m \leq \frac{1}{c} \cdot (\log_{1/(1-\alpha)} 5\lambda_2/\lambda_1) \cdot m^*$. Therefore, we had run SJF on more than $\frac{1}{c} \cdot (\log_{1/(1-\alpha)} 5\lambda_2/\lambda_1) \cdot m^*$ machines, we would get a contradiction. This completes the proof of Lemma 1.

5.4 Analysis of the CMS Algorithm on Very Tight Jobs and EDF on Loose Jobs

First consider the analysis of CMS on very tight jobs. Observe that, in Theorem 8, the base of the logarithm is $m$. Further, the analysis of [10] gives a $(\mu, 1/\mu)$-critical pair where $\mu = m' + 1$ if the algorithm cannot feasibly schedule all of the jobs on $m'$ machines. If CMS uses $\Theta(m^*)$ machines then this implies that all jobs can be feasibly scheduled since otherwise the theorem would give a contradiction.

The paper of [10] shows that for any $\alpha \in (0, 1)$, the algorithm Earliest Deadline First (EDF) is a $1/(1 - \alpha)^2$-machine optimal when all jobs are $\alpha$-loose. Hence EDF will feasibly schedule all 1/2-loose jobs on $4m^*$ machines.

6 Conclusion

This paper shows that if a given set of jobs can be feasibly scheduled on $m$ machines by any algorithm then there is an online algorithm that will feasibly schedule the jobs on $O(m \log \log m)$ machines. We point out two exciting open questions remaining in this line of work. One is to reduce the number of machines required to feasibly schedule the jobs to $O(m)$. The other is to give
a feasibility test for the algorithm. That is, given a task system, determine if the algorithm will feasibly schedule the jobs arising from this task system.

References

[1] C. L. Liu and J. W. Layland, “Scheduling algorithms for multiprogramming in a hard-real-time environment,” Journal of the ACM, vol. 20, no. 1, pp. 46–61, 1973.

[2] S. K. Baruah and N. Fisher, “The partitioned multiprocessor scheduling of deadline-constrained sporadic task systems,” IEEE Transactions on Computers, vol. 55, no. 7, pp. 918–923, 2006.

[3] ——, “The partitioned dynamic-priority scheduling of sporadic task systems,” Real-Time Systems, vol. 36, no. 3, pp. 199–226, 2007.

[4] J. Chen and S. Chakraborty, “Resource augmentation bounds for approximate demand bound functions,” in Real-Time Systems Symposium, 2011, pp. 272–281.

[5] S. Ahuja, K. Lu, and B. Moseley, “Partitioned feasibility tests for sporadic tasks on heterogeneous machines,” in 2016 IEEE International Parallel and Distributed Processing Symposium, IPDPS 2016, Chicago, IL, USA, May 23-27, 2016, 2016, pp. 1013–1020.

[6] B. Andersson and E. Tovar, “Competitive analysis of partitioned scheduling on uniform multiprocessors,” in 21th International Parallel and Distributed Processing Symposium (IPDPS 2007), Proceedings, 26-30 March 2007, Long Beach, California, USA, 2007, pp. 1–8.

[7] M. Dertouzos and A. Mok, “Multiprocessor online scheduling of hard-real-time tasks,” IEEE Transactions on software engineering, vol. 15, no. 12, pp. 1497–1506, 1989.

[8] C. A. Phillips, C. Stein, E. Torgn, and J. Wein, “Optimal time-critical scheduling via resource augmentation,” Algorithmica, vol. 32, no. 2, pp. 163–200, 2002.

[9] K. Pruhs, J. Sgall, and E. Torgn, “Online scheduling,” in Handbook of Scheduling - Algorithms, Models, and Performance Analysis., 2004.

[10] L. Chen, N. Megow, and K. Schewior, “An O(log m)-competitive algorithm for online machine minimization,” in ACM-SIAM Symposium on Discrete Algorithms, 2016, pp. 155–163.

[11] Y. Azar and S. Cohen, “A note on online machine minimization,” in Workshop on Models and Algorithms for Planning and Scheduling Problems, 2017.

[12] T. W. Lam and K. K. To, “Trade-offs between speed and processor in hard-deadline scheduling,” in ACM-SIAM Symposium on Discrete Algorithms, 1999, pp. 623–632.

[13] V. Bonifaci, A. Marchetti-Spaccamela, and S. Stiller, “A constant-approximate feasibility test for multiprocessor real-time scheduling,” Algorithmica, vol. 62, no. 3-4, pp. 1034–1049, 2012.

[14] L. Chen, N. Megow, and K. Schewior, “The power of migration in online machine minimization,” in ACM Symposium on Parallelism in Algorithms and Architectures, 2016, pp. 175–184.

[15] R. I. Davis and A. Burns, “A survey of hard real-time scheduling for multiprocessor systems,” ACM Computing Surveys, vol. 43, no. 4, pp. 35:1–35:44, 2011.
[16] A. K. Mok, “Fundamental design problems of distributed systems for the hard-real-time environment,” 1983, mIT Technical report.

[17] J. Chen and S. Chakraborty, “Partitioned packing and scheduling for sporadic real-time tasks in identical multiprocessor systems,” in *Euromicro Conference on Real-Time Systems*, 2012.

[18] S. K. Baruah, “The partitioned EDF scheduling of sporadic task systems,” in *IEEE Real-Time Systems Symposium*, 2011, pp. 116–125.

[19] J.-J. Chen, N. Bansal, and S. Chakraborty, “Packing sporadic real-time tasks on identical multiprocessor systems,” preprint.