Nonlinear differential inequality

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Abstract

A nonlinear inequality is formulated in the paper. An estimate of the rate of growth/decay of solutions to this inequality is obtained. This inequality is of interest in a study of dynamical systems and nonlinear evolution equations. It can be applied to a study of global existence of solutions to nonlinear PDE.

Keywords. Nonlinear inequality; Dynamical Systems Method (DSM); stability.
MSC: 26D10, 37L05, 47J35, 65J15.

1 Introduction

In this paper the following nonlinear differential inequality

\[ \dot{g}(t) \leq -\gamma(t)g(t) + \alpha(t, g(t)) + \beta(t), \quad t \geq t_0, \quad \dot{g} = \frac{dg}{dt}, \quad g \geq 0, \quad (1) \]

is studied. In equation (1), \( \beta(t) \) and \( \gamma(t) \) are continuous functions, defined on \([t_0, \infty)\), where \( t_0 \geq 0 \) is a fixed number.

Inequality (1) was studied in [8] with \( \alpha(t, y) = \tilde{\alpha}(t)y^2 \), where \( 0 \leq \tilde{\alpha}(t) \) is a continuous function on \([t_0, \infty)\). This inequality arises in the study of the Dynamical Systems Method (DSM) for solving nonlinear operator equations. Sufficient conditions on \( \beta, \alpha \) and \( \gamma \) which yields an estimate for the rate of growth/decay of \( g(t) \) were given in [8]. A discrete analog of (1) was studied in [4]. An application to the study of a discrete version of the DSM for solving nonlinear equation was demonstrated in [4].

In [5] inequality (1) is studied in the case \( \alpha(t, y) = \tilde{\alpha}(t)y^p \), where \( p > 1 \) and \( 0 \leq \tilde{\alpha}(t) \) is a continuous function on \([t_0, \infty)\). This equality allows one to study the DSM under weaker smoothness assumption on \( F \) than in the cited works. It allows one to study the convergence of the DSM under the assumption that \( F' \) is locally Hölder continuous. An application to the study of large time behavior of solutions to some partial differential equations was outlined in [5].

In this paper we assume that \( 0 \leq \alpha(t, y) \) is a nondecreasing function of \( y \) on \([0, \infty]\) and is continuous with respect to \( t \) on \([t_0, \infty)\). Under this weak assumption on \( \alpha \) and some assumptions on \( \beta \) and \( \gamma \), we give an estimate for the rate of growth/decay of \( g(t) \) as \( t \to \infty \) in Theorem 1.

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A discrete version of (1) is studied and the result is stated in Theorem 3. In Section 3 an application of inequality (1) to the study of large time behavior of solutions to some partial equations is sketched.

2 Main results

Throughout the paper let us assume that the function \(0 \leq \alpha(t, y)\) is locally Lipchitz-continuous, nondecreasing with respect to \(y\), and is continuous with respect to \(t\) on \([t_0, \infty)\).

**Theorem 1** Let \(\beta(t)\) and \(\gamma(t)\) be continuous functions on \([t_0, \infty)\). Suppose there exists a function \(\mu(t) > 0\), \(\mu \in C^1[t_0, \infty)\), such that

\[
\alpha\left(t, \frac{1}{\mu(t)} \right) + \beta(t) \leq \frac{1}{\mu(t)} \left[ \gamma - \frac{\dot{\mu}(t)}{\mu(t)} \right], \quad t \geq t_0.
\]

Let \(g(t) \geq 0\) be a solution to inequality (1) such that

\[
\mu(t_0)g(t_0) < 1.
\]

Then \(g(t)\) exists globally and the following estimate holds:

\[
0 \leq g(t) < \frac{1}{\mu(t)}, \quad \forall t \geq t_0.
\]

Consequently, if \(\lim_{t \to \infty} \mu(t) = \infty\), then

\[
\lim_{t \to \infty} g(t) = 0.
\]

**Proof.** Denote

\[
v(t) := g(t)e^{\int_{t_0}^{t} \gamma(s)ds}.
\]

Then inequality (1) takes the form

\[
\dot{v}(t) \leq a(t)v(t) + b(t), \quad v(t_0) = g(t_0) := g_0,
\]

where

\[
a(t) = e^{\int_{t_0}^{t} \gamma(s)ds}, \quad b(t) := \beta(t)e^{\int_{t_0}^{t} \gamma(s)ds}.
\]

Denote

\[
\eta(t) = \frac{e^{\int_{t_0}^{t} \gamma(s)ds}}{\mu(t)}.
\]

From inequality (3) and relation (9) one gets

\[
v(t_0) = g(t_0) < \frac{1}{\mu(t_0)} = \eta(t_0).
\]

It follows from the inequalities (2), (7) and (10) that

\[
\dot{v}(t) \leq \alpha(t_0, \frac{1}{\mu(t_0)}) + \beta(t_0) \leq \frac{1}{\mu(t_0)} \left[ \gamma - \frac{\dot{\mu}(t_0)}{\mu(t_0)} \right] = \frac{d}{dt} \left[ e^{\int_{t_0}^{t} \gamma(s)ds} \right] \bigg|_{t=t_0} = \dot{\eta}(t_0).
\]
From the inequalities (10) and (11) it follows that there exists \( \delta > 0 \) such that
\[
v(t) < \eta(t), \quad t_0 \leq t \leq t_0 + \delta.
\] (12)

To continue the proof we need two Claims.

Claim 1. If
\[
v(t) \leq \eta(t), \quad \forall t \in [t_0, T], \quad T > t_0,
\] (13)
then
\[
\dot{v}(t) \leq \dot{\eta}(t), \quad \forall t \in [t_0, T].
\] (14)

Proof of Claim 1.

It follows from inequalities (2), (7) and the inequality \( v(T) \leq \eta(T) \), that
\[
\dot{v}(t) \leq e^{\int_{t_0}^{t} \gamma(s) ds} \alpha(t, \frac{1}{\mu(t)}) + \beta(t)e^{\int_{t_0}^{t} \gamma(s) ds}
\leq \frac{e^{\int_{t_0}^{t} \gamma(s) ds}}{\mu(t)} \left[ \gamma - \frac{\dot{\mu}(t)}{\mu(t)} \right]
= \frac{d}{dt} \frac{e^{\int_{t_0}^{t} \gamma(s) ds}}{\mu(t)} \bigg|_{t=t} = \dot{\eta}(t), \quad \forall t \in [t_0, T].
\] (15)

Claim 1 is proved. \( \square \)

Denote
\[
T := \sup \{ \delta \in \mathbb{R}^+ : v(t) < \eta(t), \forall t \in [t_0, t_0 + \delta] \}.
\] (16)

Claim 2. One has \( T = \infty \).

Claim 2 says that every nonnegative solution \( g(t) \) to inequality (1), satisfying assumption (3), is defined globally.

Proof of Claim 2.

Assume the contrary, i.e., \( T < \infty \). The solution \( v(t) \) to (17) is continuous at every point \( t \) at which it is bounded. From the definition of \( T \) and the continuity of \( v \) and \( \eta \) one gets
\[
v(T) \leq \eta(T).
\] (17)

It follows from inequalities (16), (17), and Claim 1 that
\[
\dot{v}(t) \leq \dot{\eta}(t), \quad \forall t \in [t_0, T].
\] (18)

This implies
\[
v(T) - v(t_0) = \int_{t_0}^{T} \dot{v}(s) ds \leq \int_{t_0}^{T} \dot{\eta}(s) ds = \eta(T) - \eta(t_0).
\] (19)

Since \( v(t_0) < \eta(t_0) \) by assumption (3), it follows from inequality (19) that
\[
v(T) < \eta(T).
\] (20)

Inequality (20) and inequality (18) with \( t = T \) imply that there exists a \( \delta > 0 \) such that
\[
v(t) < \eta(t), \quad T \leq t \leq T + \delta.
\] (21)
This contradicts the definition of $T$ in (16), and the contradiction proves the desired conclusion $T = \infty$.

Claim 2 is proved. □

It follows from the definitions of $\eta(t)$, $T$, $v(t)$, and from the relation $T = \infty$, that

$$g(t) = e^{-\int_{t_0}^t \gamma(s)ds} v(t) < e^{-\int_{t_0}^t \gamma(s)ds} \eta(t) = \frac{1}{\mu(t)}, \quad \forall t > t_0.$$  \hspace{1cm} (22)

Theorem 1 is proved. □

**Theorem 2** Let $\beta(t)$ and $\gamma(t)$ be as in Theorem 1. Assume that $0 < \alpha(t, y)$ is continuous with respect to $t$ on $[t_0, \infty)$, is locally Lipschitz-continuous and nondecreasing with respect to $y$ on $[0, \infty)$. Let $0 \leq g(t)$ satisfy (1) and $0 < \mu(t)$ satisfy (2) and (3). Then

$$g(t) \leq \frac{1}{\mu(t)}, \quad \forall t \geq t_0.$$  \hspace{1cm} (23)

**Proof.** Let $v(t)$ be defined in (6). Then inequality (7) holds. Let $w_n(t)$ solve the following differential equation

$$\dot{w}_n(t) = a(t)\alpha(t, w_n(t)e^{-\int_{t_0}^t \gamma(s)ds}) + b(t), \quad w_n(t_0) = g(t_0) - \frac{1}{n}, \quad n \geq n_0,$$  \hspace{1cm} (24)

where $n_0$ is sufficiently large and $g(t_0) > \frac{1}{n_0}$. Since $\alpha(t, y)$ is continuous with respect to $t$ and locally Lipschitz-continuous with respect to $y$, there exists a unique local solution to (24).

From the proof of Theorem 1 one gets

$$w_n(t) < \frac{e^{\int_{t_0}^t \gamma(s)ds}}{\mu(t)}, \quad \forall t \geq t_0, \forall n \geq n_0.$$  \hspace{1cm} (25)

Let $t_0 < \tau < \infty$ be an arbitrary constant and

$$w(t) = \lim_{n \to \infty} w_n(t), \quad \forall t \in [t_0, \tau].$$  \hspace{1cm} (26)

This and the fact that $w_n(t)$ is uniformly continuous on $[0, \tau]$ imply that $w(t)$ solves the following equation:

$$\dot{w}(t) = a(t)\alpha(t, w(t)e^{-\int_{t_0}^t \gamma(s)ds}) + b(t), \quad w(t_0) = g(t_0), \quad \forall t \in [0, \tau].$$  \hspace{1cm} (27)

Note that the solution $w(t)$ to (27) is unique since $\alpha(t, y)$ is continuous with respect to $t$ and locally Lipschitz-continuous with respect to $y$. From (7), (27), a comparison lemma (see, e.g., [8], p.99), the continuity of $w_n(t)$ with respect to $w_0(t_0)$ on $[0, \tau]$, and (25), one gets

$$v(t) \leq w(t) \leq \frac{e^{\int_{t_0}^t \gamma(s)ds}}{\mu(t)}; \quad \forall t \in [t_0, \tau], \forall n \geq n_0.$$  \hspace{1cm} (28)

Since $\tau > t_0$ is arbitrary, inequality (23) follows from (28).

Theorem 2 is proved. □
Let us consider a discrete analog of Theorem 1. Let
\[
\frac{g_{n+1} - g_n}{h_n} \leq -\gamma_n g_n + \alpha(n, g_n) + \beta_n, \quad h_n > 0, \quad 0 < h_n \gamma_n < 1, \quad p > 1,
\]
and the inequality:
\[
g_{n+1} \leq (1 - \gamma_n) g_n + \alpha(n, g_n) + \beta_n, \quad n \geq 0, \quad 0 < \gamma_n < 1, \quad p > 1,
\]
where \(g_n, \beta_n\) and \(\gamma_n\) are positive sequences of real numbers.

Under suitable assumptions on \(\beta_n\) and \(\gamma_n\), we obtain an upper bound for \(g_n\) as \(n \to \infty\). In particular, we give sufficient conditions for the validity of the relation \(\lim_{n \to \infty} g_n = 0\), and estimate the rate of growth/decay of \(g_n\) as \(n \to \infty\). This result can be used in a study of evolution problems, in a study of iterative processes, and in a study of nonlinear PDE.

**Theorem 3** Let \(\beta_n\) and \(g_n\) be nonnegative sequences of numbers. Assume that
\[
\frac{g_{n+1} - g_n}{h_n} \leq -\gamma_n g_n + \alpha(n, g_n) + \beta_n, \quad h_n > 0, \quad 0 < h_n \gamma_n < 1,
\]
or, equivalently,
\[
g_{n+1} \leq g_n (1 - h_n \gamma_n) + h_n \alpha(n, g_n) + h_n \beta_n, \quad h_n > 0, \quad 0 < h_n \gamma_n < 1.
\]
If there is a sequence of positive numbers \((\mu_n)_{n=1}^\infty\), such that the following conditions hold:
\[
\alpha(n, \frac{1}{\mu_n}) + \beta_n \leq \frac{1}{\mu_n} \left( \gamma_n - \frac{\mu_{n+1} - \mu_n}{\mu_n h_n} \right); \quad (31)
\]
\[
g_0 \leq \frac{1}{\mu_0}; \quad (32)
\]
then
\[
0 \leq g_n \leq \frac{1}{\mu_n} \quad \forall n \geq 0. \quad (33)
\]
Therefore, if \(\lim_{n \to \infty} \mu_n = \infty\), then \(\lim_{n \to \infty} g_n = 0\).

**Proof.** Let us prove (33) by induction. Inequality (33) holds for \(n = 0\) by assumption (32). Suppose that (33) holds for all \(n \leq m\). From inequalities (29), (31), and from the induction hypothesis \(g_n \leq \frac{1}{\mu_n}, \quad n \leq m\), one gets
\[
g_{m+1} \leq g_m (1 - h_m \gamma_m) + h_m \alpha(m, g_m) + h_m \beta_m
\]
\[
\leq \frac{1}{\mu_m} (1 - h_m \gamma_m) + h_m \alpha(m, \frac{1}{\mu_m}) + h_m \beta_m
\]
\[
\leq \frac{1}{\mu_m} (1 - h_m \gamma_m) + \frac{h_m}{\mu_m} \left( \gamma_m - \frac{\mu_{m+1} - \mu_m}{\mu_m h_m} \right)
\]
\[
= \frac{1}{\mu_{m+1}} - \frac{\mu_{m+1}^2 - 2\mu_{m+1} \mu_m + \mu_m^2}{\mu_n \mu_{m+1}^2} \leq \frac{1}{\mu_{m+1}}. \quad (34)
\]
Therefore, inequality (33) holds for \(n = m + 1\). Thus, inequality (33) holds for all \(n \geq 0\) by induction. Theorem 3 is proved. \(\square\)
Setting $h_n = 1$ in Theorem 3, one obtains the following result:

**Theorem 4** Let $\beta_n, \gamma_n$ and $g_n$ be sequences of nonnegative numbers, and

$$g_{n+1} \leq g_n(1 - \gamma_n) + \alpha(n, g_n) + \beta_n, \quad 0 < \gamma_n < 1. \quad (35)$$

If there is sequence $\left(\mu_n\right)_{n=1}^\infty > 0$ such that the following conditions hold

$$g_0 \leq \frac{1}{\mu_0}, \quad \alpha(n, \frac{1}{\mu_n}) + \beta_n \leq \frac{1}{\mu_n} \left(\gamma_n - \frac{\mu_{n+1} - \mu_n}{\mu_nh_n}\right), \quad \forall n \geq 0, \quad (36)$$

then

$$g_n \leq \frac{1}{\mu_n}, \quad \forall n \geq 0. \quad (37)$$

### 3 Applications

Here we sketch an idea for possible applications of our inequalities in a study of dynamical systems in a Hilbert space $H$.

In this Section we assume without loss of generality that $t_0 = 0$. Let

$$\dot{u} + Au = h(t, u) + f(t), \quad u(0) = u_0, \quad \dot{u} := \frac{du}{dt}, \quad t \geq 0. \quad (38)$$

To explain the ideas, let us make simplifying assumptions: $A > 0$ is a selfadjoint time-independent operator in a real Hilbert space $H$, $h(t, u)$ is a nonlinear operator in $H$, locally Lipschitz with respect to $u$ and continuous with respect to $t \in \mathbb{R}_+ := [0, \infty)$, and $f$ is a continuous function on $\mathbb{R}_+$ with values in $H$, $\sup_{t \geq 0} \|f(t)\| < \infty$. The scalar product in $H$ is denoted $\langle u, v \rangle$. Assume that

$$\langle Au, u \rangle \geq \gamma \langle u, u \rangle, \quad \gamma = \text{const} > 0, \quad \|h(t, u)\| \leq \alpha(t, \|u\|), \quad \forall u \in D(A), \quad (39)$$

where $\alpha(t, y) \leq cy^p$, $p > 1$ and $c > 0$ are constants, and $\alpha(t, y)$ is a non-decreasing $C^1([0, \infty))$ function of $y$. Our approach works when $\gamma = \gamma(t)$ and $c = c(t)$, see Examples 1,2 below. The problem is to estimate the behavior of the solution to (38) as $t \rightarrow \infty$ and to give sufficient conditions for a global existence of the unique solution to (38). Our approach consists of a reduction of this problem to the inequality $1$ and an application of Theorem $1$. A different approach, studied in the literature (see, e.g., [6], [7]), is based on the semigroup theory.

Let $g(t) := \|u(t)\|$. Problem (38) has a unique local solution under our assumptions. This local solution exists globally if $\sup_{t \geq 0} \|u(t)\| < \infty$. Multiply (38) by $u$ and use (39) to get

$$\dot{g} \leq -\gamma(t)g^2 + \alpha(t, g)g + \beta(t)g, \quad \beta(t) := \|f(t)\|. \quad (40)$$

Since $g \geq 0$, one gets

$$\dot{g} \leq -\gamma(t)g + \alpha(t, g(t)) + \beta(t). \quad (41)$$

Now Theorem 1 is applicable and yields sufficient conditions (2) and (3) for the global existence of the solution to (38) and estimate (4) for the behavior of $\|u(t)\|$ as $t \rightarrow \infty$. The choice of $\mu(t)$ in Theorem 1 is often straightforward. For example, if $\alpha(t, g(t)) = \frac{c}{\mu(t)}g^2$,
where \( \lim_{t \to \infty} a(t) = 0, \) \( \dot{a}(t) < 0, \) then one can often choose \( \mu(t) = \frac{\lambda}{a(t)}, \) \( \lambda = \text{const} > 0, \)

The outlined approach is applicable to stability of the solutions to nonlinear differential equations, to semilinear parabolic problems, to hyperbolic problems, and other problems. There is a large literature on the stability of the solutions to differential equations (see, e.g., [1], [2], and references therein). Our approach yields some novel results. If the selfadjoint operator \( A \) depends on \( t, \) \( A = A(t), \) and \( \gamma = \gamma(t) > 0, \) \( \lim_{t \to \infty} \gamma(t) = 0, \) one can treat problems with degenerate, as \( t \to \infty, \) elliptic operators \( A.\)

For instance, if the operator \( A \) is a second-order elliptic operator with matrix \( a_{ij}(x,t), \) and the minimal eigenvalue \( \lambda(x,t) \) of this matrix satisfies the condition \( \min_x \lambda(x,t) := \gamma(t) \to 0 \) as \( t \to \infty, \) then Theorem 1 is applicable under suitable assumptions on \( \gamma(t), h(t,u) \) and \( f(t). \)

**Example 1.** Consider

\[
\dot{u} = -\gamma(t)u + a(t)u(t)|u(t)|^p + \frac{1}{(1+t)^q}, \quad u(0) = 0, \tag{42}
\]

where \( \gamma(t) = \frac{c}{(1+t)^r}, \) \( a(t) = \frac{1}{(1+t)^m}, \) \( p, q, b, c, \) and \( m \) are positive constants. Our goal is to give sufficient conditions for the solution to the above problem to converge to zero as \( t \to \infty. \) Multiply \( (42) \) by \( u, \) denote \( g := u^2, \) and get the following inequality

\[
\dot{g} \leq -2 \frac{c}{(1+t)^r}g + 2 \frac{1}{(1+t)^m}g(t)^{1+0.5p} + 2 \frac{1}{(1+t)^q}g^{0.5}, \quad g = u^2. \tag{43}
\]

Choose \( \mu(t) = \lambda(1+t)^\nu \), where \( \lambda > 0 \) and \( \nu > 0 \) are constants.

Inequality (2) takes the form:

\[
\frac{2}{(1+t)^m} \lambda(1+t)^\nu]^{-1-0.5p} + \frac{2}{(1+t)^q} \lambda(1+t)^\nu]^{-0.5} \leq [\lambda(1+t)^\nu]^{-1} \left( 2 \frac{c}{(1+t)^b} - \frac{\nu}{1+t} \right). \tag{44}
\]

Choose \( p, q, m, c, \lambda \) and \( \nu \) so that inequality (44) be valid and \( \lambda u(0)^2 < 1, \) so that condition (3) with \( t_0 = 0 \) holds. If this is done, then \( u^2(t) \leq \frac{1}{\lambda(1+t)^\nu}, \) so \( \lim_{t \to \infty} u(t) = 0. \) For example, choose \( b = 1, \) \( \nu = 1, \) \( q = 1.5, \) \( m = 1, \) \( \lambda = 4, \) \( c = 4, \) \( p \geq 1. \) Then inequality (44) is valid, and if \( u(0)^2 < 1/4, \) then (3) with \( t_0 = 0 \) holds, so \( \lim_{t \to \infty} u(t) = 0. \) The choice of the parameters can be varied. In particular, the nonlinearity growth, governed by \( p, \) can be arbitrary in power scale. If \( b = 1 \) then three inequalities \( m + 0.5p\nu \geq 1, \) \( q - 0.5\nu \geq 1, \) and \( \lambda^{1/2} + \lambda^{-0.5p} \leq c - 0.5\nu \) together with \( u(0)^2 < \lambda^{-1} \) are sufficient for (3) and (44) to hold, so they imply \( \lim_{t \to \infty} u(t) = 0. \)

**Example 2.** Consider problem (38) with \( A, h \) and \( f \) satisfying (39) with \( \gamma \equiv 0. \) So one gets inequality (41) with \( \gamma(t) \equiv 0. \) Choose

\[
\mu(t) := c + \lambda(1+t)^{-b}, \quad c > 0, b > 0, \lambda > 0, \tag{45}
\]

where \( c, \lambda, \) and \( b \) are constants. Inequality (2) takes the form:

\[
\alpha(t, \frac{1}{\mu(t)}) + \beta(t) \leq \frac{b\lambda}{\mu(t)(1+t)[\lambda + c(1+t)^b]} \tag{46}
\]
Let $\theta \in (0, 1)$, $p > 0$, and $C > 0$ be constants. Assume that
\[
\alpha(t, |y|) \leq \theta C|y|^p \frac{b\lambda}{(\lambda + c)(1 + t)^{1+p}}, \quad \beta(t) \leq (1 - \theta) \frac{b\lambda}{(c + \lambda)^2(1 + t)^{1+b}}, \quad (47)
\]
for all $t \geq 0$, and
\[
C = \begin{cases} 
   e^{p-1} & \text{if } p > 1, \\
   \frac{(\lambda + c)^{p-1}}{(c + \lambda)^{p-1}} & \text{if } p \leq 1.
\end{cases} \quad (48)
\]
Let us verify that inequality (46) holds given that (47) and (48) hold.

It follows from (45) that $c < \mu(t) \leq c + \lambda$, $\forall t \geq 0$. This and (47) imply
\[
\beta(t) \leq (1 - \theta) \frac{1}{(c + \lambda)^2(1 + t)^{1+b}} \leq (1 - \theta) \frac{1}{\mu(t)} \frac{1}{(1 + t)(c + \lambda(1 + t)^{b})}. \quad (49)
\]
From (48) and (45) one gets
\[
\frac{C}{\mu^{p-1}(t)} \leq C \max(c^{1-p}, (c + \lambda)^{1-p}) \leq 1, \quad \forall t \geq 0. \quad (50)
\]
From (47) and (50) one obtains
\[
\alpha(t, \frac{1}{\mu(t)}) \leq \theta C \frac{1}{\mu(t)} \frac{1}{\mu^{p-1}(t)} \frac{1}{(1 + t)(\lambda(1 + t)^b + c(1 + t)^b)} \frac{b\lambda}{b\lambda} \leq \theta \frac{1}{\mu(t)} \frac{1}{(1 + t)[\lambda + c(1 + t)^{b}]}. \quad (51)
\]
Inequality (46) follows from (49) and (51). From (46) and Theorem 1 one obtains
\[
g(t) \leq \frac{1}{\mu(t)} < \frac{1}{c}, \quad \forall t > 0, \quad (52)
\]
provided that $g(0) < (c + \lambda)^{-1}$. From (11) with $\gamma(t) = 0$ and (47)–(52), one gets $\dot{g}(t) = O(\frac{1}{(1+t)^{1+b}})$. Thus, there exists finite limit $\lim_{t \to \infty} g(t) = g(\infty) \leq c^{-1}$.

References

[1] Yu. L. Daleckii, M. G. Krein, Stability of solutions of differential equations in Banach spaces, Amer. Math. Soc., Providence, RI, 1974.

[2] B. P. Demidovich, Lectures on stability theory, Nauka, Moscow, 1967 (in Russian)

[3] N. S. Hoang, A. G. Ramm, Dynamical Systems Gradient Method for solving nonlinear operator equations with monotone operators, Acta Appl. Math., 106, (2009), 473-499.

[4] N.S. Hoang and A. G. Ramm, A nonlinear inequality, Jour. Math. Ineq., 2, N4, (2008), 459-464.

[5] N.S. Hoang and A. G. Ramm, A nonlinear inequality and applications, Nonlinear Analysis: Theory, Methods & Applications, 71, (2009), 2744- 2752.

[6] S. G. Krein, Linear differential equations in Banach spaces, Amer. Math. Soc., Providence, RI, 1971.
[7] A. Pazy, Semigroups of linear operators and applications to partial differential equations, Springer-Verlag, New York, 1983.

[8] A. G. Ramm, Dynamical systems method for solving operator equations, Elsevier, Amsterdam, 2007.