Hessian continuity at degenerate points in nonvariational elliptic problems

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Abstract

Established in the 30’s, Schauder a priori estimates are among the most classical and powerful tools in the analysis of problems ruled by 2nd order elliptic PDEs. Since then, a central problem in regularity theory has been to understand Schauder type estimates fashioning particular borderline scenarios. In such context, it has been a common accepted aphorism that the continuity of the Hessian of a solution could never be superior than the continuity of the medium. Notwithstanding, in this article we show that solutions to uniformly elliptic, linear equations with $C^0,\bar{\varepsilon}$ coefficients are of class $C^{2,\alpha}$, for any $0 < \bar{\varepsilon} \ll \alpha < 1$, at Hessian degenerate points, $\mathcal{H}(u) := \{ X \mid D^2 u(X) = 0 \}$.

In fact we develop a more general regularity result at such Hessian degenerate points, featuring into the theory of fully nonlinear equations. Insofar as the optimal modulus of continuity for the Hessian is concerned, the result of this paper is the first one in the literature to surpass the inborn obstruction from the sharp Schauder a priori regularity theory.

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1 Introduction

Among the finest treasures of the theory of elliptic PDEs, Schauder a priori regularity estimates assure that solutions to a linear, uniformly elliptic equation with $C^{0,\theta}$ data, $0 < \theta < 1$, i.e., functions $u$ satisfying

$$Lu := a_{ij}(X)D_{ij}u = f(X)$$

(1.1)
where
\[ 0 < \lambda \leq a_{ij}(X) \leq \Lambda, \quad a_{ij}, f \in C^{0,\theta}, \]
are locally of class \( C^{2,\theta} \). Furthermore, there exists a constant \( C > 0 \), depending only upon dimension, ellipticity constants \( (\lambda, \Lambda) \), and the \( \theta \)-Hölder continuity of the data, \( \|a_{ij}\|_{C^{0,\theta}} \) and \( \|f\|_{C^{0,\theta}} \), such that
\[ \|u\|_{C^{2,\theta}(B_{1/2})} \leq C \cdot \|u\|_{L^\infty(B_1)}. \]  \hfill (1.2)

For comprehensive reference on such a theory, we cite the classical books [6, Chapter 6] and also [7, Chapter 6]. It is also interesting to read [9] and references therein.

The importance of such an estimate to the theory of PDEs and its vast range of applications would hardly be exaggerated. Proven in the 30’s by the Polish mathematician, Juliusz Schauder, [8], Schauder a priori estimate (1.2) is sharp in several ways. Estimate (1.2) does not hold true in the case \( \theta = 0 \), i.e. solutions to elliptic equations (1.1) with merely continuous data are not necessarily of class \( C^2 \). Not even \( C^{1,1} \) estimates are in general available equations with continuous sources. Similarly, Schauder a priori estimate also breaks down at the upper endpoint, \( \theta = 1 \). That is, solutions to elliptic equations (1.1) with Lipchitz data are not necessarily \( C^{2,1} \). Establishing optimal regularity estimates in borderline cases for particular problems involves, in general, new, deep and robust techniques. A classical example we bring up here is the theory of obstacle-type free boundary problems
\[ \mathcal{L}u \approx \chi_{\{v > 0\}}. \]  \hfill (1.3)

Classical elliptic regularity theory gives that a solution to (1.3) is of class \( C^{1,\alpha} \) for any \( \alpha < 1 \). Proving that a solution is indeed \( C^{1,1} \) involves a deep and much finer analysis, for instance quasi-monotonicity formulae [3, 10].

Another decisive sharpness aspect of the Schauder regularity theory concerns the best exponent for Hölder continuity of the Hessian of a solution to (1.1). That is, fixed \( 0 < \alpha_0 < 1 \), a solution to a uniformly elliptic equation with \( C^{0,\alpha_0} \) data is of class \( C^{2,\alpha_0} \), but it may fail to belong to \( C^{2,\alpha_0+\delta}, \delta > 0 \). Up to our knowledge, there has been no significant advances on the issue of surpassing the universal \( C^{2,\alpha_0} \) regularity obstruction, at least for certain analytical meaningful points. Of particular interest are the Hessian degenerate points of the solution:
\[ \mathcal{H}(u) := (D^2 u)^{-1} \{0\}. \]
The above discussion brings us to the main result of this present work. We shall establish in this manuscript that solutions to a linear elliptic equation

$$a_{ij}(X)D_{ij}u = 0,$$

with $a_{ij} \in C^{0,\bar{\epsilon}}, 0 < \bar{\epsilon} \ll 1$, is of class $C^{2,1-}$ at any point $Y \in \mathcal{H}(u)$. The symbol $C^{2,1-}$ means $C^{2,\alpha}$ for any $\alpha < 1$. This is an unexpected gain of smoothness at Hessian degenerate points, beyond the continuity of the media. It is furthermore simple to see that if $Y \not\in \mathcal{H}(u)$ such a result cannot hold in general. Thus, our result is, in this perspective, optimal. The very same conclusion is obtained if one assumes only that $a_{ij}$ is Dini continuous. To the best of our knowledge, our result is the first one, in the context of non-divergence elliptic regularity theory, to outmatch the classical Schauder a priori estimate, insofar as the continuity of the media is concerned.

To exemplify the gain of smoothness provided by the result above mentioned, suppose, for the sake of illustration, that $u$ solves a linear equation

$$a_{ij}(X)D_{ij}u = 0,$$

where $a_{ij}$ is uniform elliptic and $a_{ij} \in C^{0,0.1}$. Classical Schauder regularity theory gives that $D^2u \in C^{0,0.1}$ at any interior point. However, at a degenerate Hessian point $Y \in \mathcal{H}(u)$, in fact $D^2u$ is much smoother, for instance, $D^2u \in C^{0,0.999}$, i.e., $D^2u$ is asymptotically Lipschitz continuous.

The solution designed for the proof of such a result is, in its very nature, nonlinear. Thus, for the sake of completeness, we shall state and prove a more general result, in the context of fully nonlinear elliptic equations.

Hereafter $Q_1 \subset \mathbb{R}^d$ denotes the unit cube in the $d$-dimensional euclidean space and $\text{Sym}(d)$ stands for the space of $d \times d$ symmetric matrices. Throughout this paper we shall work under uniform ellipticity condition on the operator $F: Q_1 \times \text{Sym}(d) \to \mathbb{R}$, i.e., we assume there exist two positive constants $0 < \lambda \leq \Lambda$ such that, for any $M \in \text{Sym}(d)$, $X \in Q_1$,

$$\lambda \|P\| \leq F(X, M + P) - F(X, M) \leq \Lambda \|P\|, \quad \forall P \geq 0. \quad (H0)$$

As to access estimates on the Hessian of a solution, in this article we shall assume that the model, constant coefficient equation has a priori $C^{2,\alpha_F}$ local estimates, for some $0 < \alpha_F \leq 1$. More precisely, we assume

$$F(0, D^2h) = 0, \text{ in } Q_1 \quad \text{implies} \quad \|h\|_{C^{2,\alpha_F}(Q_{1/2})} \leq \Theta \cdot \|h\|_{L^\infty(Q_1)}, \quad (H1)$$

for some constant $\Theta > 0$. 
Since we will deal with variable coefficient equations, in accordance to [1] (see also [2]), throughout this article we impose an $L^n$ type of $C^{0,\bar{\varepsilon}}$ continuity of the coefficients. More precisely, measuring the oscillation of the coefficients at $0 \in B_1 \subset \mathbb{R}^d$, by

$$\beta_F(X) := \sup_{N \in \text{Sym}(n)} \frac{|F(X, N) - F(0, N)|}{1 + \|N\|},$$

we shall impose the existence of a constant $C_1 > 0$ such that

$$\int_{Q_r} |\beta_F(X)|^n dX \leq C_1^n \cdot r^{n \bar{\varepsilon}},$$

(H2)

for some $0 < \bar{\varepsilon} \ll 1$.

For notation convenience, we shall call

$$[F]_{n, \bar{\varepsilon}} := \inf\{C_1 \in [0, \infty) \mid \text{(H2) holds}\}. \quad (1.4)$$

Similarly, we measure the oscillation of the source function $f: Q_1 \to \mathbb{R}$ around $0$ by

$$[f]_{n, \gamma} := \inf\{K \in [0, \infty) \mid \int_{Q_r} |f(X)|^n dX \leq K^n \cdot r^{n \gamma}\}. \quad (1.5)$$

When $[f]_{n, \gamma} < +\infty$, we say $f$ is $C^{0,\gamma}$ continuous at $0$ in the $L^n$ sense.

We recall that under $C^{0,\bar{\varepsilon}}$ continuity of the coefficients (hypothesis (H2)) and $C^{2,\alpha_F}$ a priori estimates, $0 < \bar{\varepsilon} < \alpha_F$, for $F(0, D^2 h) = 0$ (hypothesis (H1)), Luis Caffarelli in [1] shows that solutions to the variable coefficients equation, $F(X, D^2 \xi) = 0$ is $C^{2,\bar{\varepsilon}}$, with appropriate a priori estimates. In a parallel to the classical, linear Schauder regularity theory, the exponent $\bar{\varepsilon}$ in Caffarelli’s $C^2$ estimates cannot be surpassed, in general.

Before continuing, we set that throughout this paper, any given constant $\kappa$ that depends only upon dimension $d$, ellipticity $(\lambda, \Lambda)$, $[F]_{n, \bar{\varepsilon}}$, $[f]_{n, \gamma}$ and the $C^{2,\alpha_F}$ regularity estimates for the constant coefficient equation, i.e., the constant $\Theta$ in hypothesis (H1) will be called universal. We now state the general regularity Theorem at Hessian degenerate points of this present manuscript.

**Theorem 1.** Let $u \in C(Q_1)$ be a viscosity solution to $F(X, D^2 u) = f(X)$ in $Q_1$. Assume $F: Q_1 \times \text{Sym}(d) \to \mathbb{R}$ satisfy conditions (H0), (H1) and (H2) and that $0 \in \mathcal{K}(u)$. Assume further that $f$ is $C^{0,\gamma}$ continuous at $0$ in the $L^n$ sense, for some $0 < \gamma < 1$. Then $u$ is $C^{2,\min\{\alpha_F, \gamma\}}$ at the origin, i.e., for any $\beta \in (0, \alpha_F) \cap (0, \gamma]$,

$$|u(X) - [u(0) + \nabla u(0) \cdot X]| \leq C_\beta |X|^{2+\beta},$$

for a constant $C_\beta > 0$ that depends only on $\beta$, $|f(0)|$, and universal parameters.
We highlight once more that the key information provided in Theorem 1 is that at Hessian degenerate points, solutions disregard the rough continuity of the media (coefficients), and its regularity theory is asymptotically as strong as the constant coefficient one. Within the classical regularity theory for 2nd order elliptic PDEs, this is at the very least a surprising result.

Notice, furthermore, that since harmonic functions are of class $C^2$, indeed it follows as a Corollary of Theorem 1 that solutions to a linear, uniform elliptic homogeneous equation

$$a_{ij}(X)D_{ij}u = 0,$$

with $C^{0,\bar{\epsilon}}$ coefficients are of class $C^{2,1-}$ at Hessian degenerate points. The very same $C^{2,1-}$ regularity at Hessian degenerate points holds true for fully nonlinear equation governed by an operator $F \in C^{1,\bar{\epsilon}}(Q_1, \times\text{Sym}(d))$ that is convex or concave w.r.t. the matrix variable $M$.

A careful analysis of the proof of Theorem 1, to be delivered in the next section, when projected to the linear setting, reveals that the $C^{2,1-}$ regularity at Hessian degenerate points holds as soon as the coefficients are "continuous enough" as to allow a priori $C^2$ estimates for $L$-harmonic functions. A classical example of such condition is the so called Dini continuity of the coefficients. Recall a function $\phi$ is said to be Dini continuous at $0$ if

$$\int_0^1 \sup_{B_t} \frac{|\phi(X) - \phi(0)|}{t} dt < \infty.$$

In particular any $C^{0,\epsilon}$ Hölder continuous function is Dini continuous. We state the conclusion of the above discussion as a Theorem.

**Theorem 2.** Let $u \in C(Q_1)$ be a viscosity solution to $a_{ij}(X)D_{ij}u = 0$ in $Q_1$, where $a_{ij}$ is uniform elliptic and Dini continuous matrix. Then $u$ is $C^{2,1-}$ at any Hessian degenerate point, $Y \in \mathcal{H}(u) \cap B_{1/2}$. That is, for any $\alpha < 1$

$$|u(X) - [u(Y) + \nabla u(Y) \cdot (X - Y)]| \leq C_{\alpha}|X - Y|^{2+\alpha},$$

for a constant $C_{\alpha} > 0$ that depends only on $(1 - \alpha)$, dimension, ellipticity constants and Dini continuity of the coefficients.

We conclude this introduction by mentioning that the statement of Theorem 2 as well as the insights for the proof of Theorem 1 have their essential roots in the affluent theory of geometric free boundary problems, see [4] and also [5, Chapter 5]. Of particular interest are the classes of pseudo free boundary problems of the form

$$\max\{\mathcal{L}\phi - \phi^\mu, -\phi\} = 0, \quad \mu > 0,$$
for some nonhomogeneous elliptic operator $\mathcal{L}$. The limiting case $\mu = 0$ represents the classical obstacle problem, already mentioned above. For $\mu > 0$, solutions are of class $C^2$ across the free interface $\partial \{ \phi > 0 \}$. Thus, all free boundary points are Hessian degenerate. In accordance to Theorem 2, $\phi \in C^{2,1-}$ at free boundary points, neglecting the eventual rough continuity of the coefficients of $\mathcal{L}$.

## 2 Proof

### 2.1 Hessian degenerate approximations

In this first part of the proof we establish a primary compactness result. It states that if the coefficients are nearly constant, the source is nearly zero and the Hessian of the solution at 0 is tiny enough, then one can find an $\mathcal{F}$-harmonic function close to $u$, for which 0 is a Hessian degenerate point.

**Lemma 3.** Let $u \in C(Q_1)$ be a viscosity solution to $F(X,D^2u) = f(X)$ in $Q_1$ and $|u| \leq 1$. Assume $F: Q_1 \times \text{Sym}(d) \to \mathbb{R}$ satisfies conditions (H0), (H1), (H2) and $f(0) = 0$. Given $\delta > 0$, there exists an $\varepsilon$, depending only on $\delta$ and universal parameters, such that if

$$ [F]_{n,\varepsilon} + [f]_{n,\gamma} + |D^2u(0)| \leq \varepsilon,$$

then we can find a function $h \in C(Q_{1/2})$, satisfying

$$ \mathcal{F}(D^2h) = 0 \text{ in } Q_{1/2} \quad \text{and} \quad D^2h(0) = 0,$$

where $\mathcal{F}: \text{Sym}(d) \to \mathbb{R}$ is under the conditions (H0) and (H1) and

$$ \sup_{Q_{1/2}} |u - h| \leq \delta.$$

**Proof.** Suppose, for the sake of contradiction, that the thesis of the Lemma fails. This means that there exist a sequence of functions

$$ u_j \in C(Q_1) \quad \text{with} \quad |u_j| \leq 1,$$

a sequence of operators

$$ F_j: Q_1 \times \text{Sym}(d) \to \mathbb{R} \quad \text{satisfying conditions (H0), (H1) and (H2),}$$

and a sequence of sources $f_j$ with $f_j(0) = 0$, all linked by the equation

$$ F_j(X,D^2u_j) = f_j(X), \quad (2.1) $$
Hessian continuity

in the viscosity sense. There also holds

\[ [F_j]_{n,\xi} + [f_j]_{n,\gamma} + |D^2 u_j(0)| = o(1), \quad (2.2) \]
as \( j \to \infty \). However, for a fixed \( \delta_0 > 0 \),

\[ \inf_{h \in \mathbb{H}} \| u_j - h \|_{L^\infty(Q_{1/2})} \geq \delta_0, \quad (2.3) \]

where

\[ \mathbb{H} := \left\{ h \mid J(D^2 h) = 0 \text{ in } Q_{1/2}, \text{ for some } J \text{ verifying (H0) and (H1)} \right\}. \]

It follows from (2.2), (2.1) and standard reasoning that, up to a subsequence, we can assume,

\[ F_j \to \mathcal{F}_\infty, \quad \text{locally uniformly in } Q_1 \times \text{Sym}(d) \]
\[ u_j \to u_\infty, \quad \text{locally uniformly in } Q_1. \]

Since \([F_j]_{n,\xi} = o(1)\), the limiting operator \( \mathcal{F}_\infty \) has constant coefficients. From uniform convergence, \( \mathcal{F}_\infty \) too satisfies (H1) and (H2). Also, by the same reasoning employed in [1], Lemma 13, we deduce

\[ \mathcal{F}_\infty(D^2 u_\infty) = 0 \quad (2.4) \]
in the viscosity sense. Furthermore, by Caffarelli’s \( C^{2,\beta} \) regularity estimate, see [1] Theorem 3, and the fact that \( |D^2 u_j(0)| = o(1) \), we conclude

\[ D^2 u_\infty(0) = 0. \quad (2.5) \]

We have proven that the limiting function \( u_\infty \) lies in the functional space \( \mathbb{H} \). Hence, we reach a contradiction on (2.3) if we take \( h = u_\infty \) and \( j \gg 1 \). The proof of Lemma 3 is concluded. \( \square \)

2.2 Discrete regularity

Our next step is to establish a discrete version of the main Theorem. It states that if the oscillations of the data of the equation are universally small, then a step-one version of the continuous thesis of Theorem [1] holds. More precisely, we have
Lemma 4. Let \( u \in C(Q_1) \) be a viscosity solution to \( F(X, D^2u) = f(X) \) in \( Q_1 \) and \( |u| \leq 1 \). Assume \( F: Q_1 \times \text{Sym}(d) \to \mathbb{R} \) satisfies conditions \( (H_0), (H_1), (H_2) \) and \( f(0) = 0 \). Given \( 0 < \beta < \alpha_F \), there exists an \( \varepsilon > 0 \), that depends only on \( \beta \) and universal parameters such that if

\[
[F]_{n, \varepsilon} + [f]_{n, \gamma} + |D^2u(0)| \leq \varepsilon,
\]

then there exist numbers \( 0 < \theta \ll 1 \ll C < \infty \), depending only on \( \beta \) and universal parameters, and an affine function \( \ell_0(X) = a_0 + \tilde{b}_0 \cdot X \), with universally bounded coefficients,

\[
|a_0|_{\mathbb{R}} + |\tilde{b}_0|_{\mathbb{R}^d} \leq C
\]

such that the following control is granted

\[
\sup_{Q_{\theta}} |u(X) - \ell_0(X)| \leq \theta^{2+\beta}.
\]

Proof. For a small number \( \delta > 0 \) to be chosen later, let \( h \) be the function sponsored by Lemma 3 that is within a \( \delta \) distance from \( u \) in the \( L^\infty \) topology. That is, \( h \) solves

\[
\mathcal{F}(D^2h) = 0 \text{ in } Q_{1/2},
\]

for some constant coefficient operator \( \mathcal{F} \) which satisfies \( (H_0) \) and \( (H_1) \) and also \( D^2h(0) = 0 \) is verified. We notice that within the scope of From Lemma 3 once we adjust \( \delta > 0 \) by a universal decision, the choice of \( \varepsilon > 0 \) in the statement of this present Lemma shall also be universal.

It follows from the \( C^{2,\alpha_F} \) regularity theory available for the operator \( \mathcal{F} \) that

\[
|h(X) - [h(0) + \nabla h(0) \cdot X]| \leq \Theta |X|^{2+\alpha_F},
\]

where \( \Theta > 0 \) is the universal constant from \( (H_1) \). In the sequel we elect \( \theta \) and \( \delta \) as

\[
\theta := \frac{\alpha_F - \beta}{2\Theta} \\
\delta := \frac{1}{2} \left( \frac{1}{2\Theta} \right)^{\frac{2+\beta}{\alpha_F - \beta}}.
\]

The selections above depend only on \( \beta \) and universal parameters. We further take

\[
\ell_0(X) := h(0) + \nabla h(0) \cdot X.
\]
Finally we estimate
\[
\sup_{Q_0} |u(X) - [h(0) + \nabla h(0) \cdot X]| \leq \sup_{Q_0} |u(X) - h(X)| \\
+ \sup_{Q_0} |h(X) - [h(0) + \nabla h(0) \cdot X]| \\
\leq \delta + \Theta \cdot \theta^{2+\alpha_F} = \theta^{2+\beta},
\]
(2.12)
and the proof of Lemma 4 is complete. \qed

2.3 Iterative flatness improvement

We have now gathered all the elements we need to conclude the proof of Theorem 1. Let us fix a number \( \beta \in (0, \alpha_F) \cap (0, \gamma] \). By normalization, expanding variables and translating the equation, if necessary, we can start off the proof by assuming, with no loss of generality, that
\[
f(0) = 0, \quad |u| \leq 1, \quad [F]_{n,\bar{\epsilon}} + [f]_{n,\gamma} \leq \epsilon,
\]
(2.13)
where \( \epsilon > 0 \) is the universal number found in Lemma 4 for our fixed choice of \( \beta \). Recall it is part of the key hypothesis of Theorem 1 that
\[
D^2 u(0) = 0.
\]
(2.14)

Our strategy is to show the existence of a sequence of affine functions \( \ell_k = a_k + b_k X \), satisfying
\[
|a_k - a_{k-1}|_{\mathbb{R}} + \theta^k |\vec{b}_k - \vec{b}_{k-1}|_{\mathbb{R}^d} \leq C \cdot \theta^{k(2+\beta)}
\]
(2.15)
and
\[
\sup_{B_{\theta^k}} |u(X) - \ell_k(X)| \leq \theta^{k(2+\beta)}.
\]
(2.16)

We establish (2.15) and (2.16) by an induction argument. The step \( k = 1 \) is precisely the statement of Lemma 4. Suppose we have verified the \( k \)th step of induction. Define the normalized function
\[
v_k(X) := \frac{u(\theta^k X) - \ell_k(\theta^k X)}{\theta^{k(2+\beta)}},
\]
(2.17)
the fully nonlinear operator
\[
F_k(X, M) := \frac{1}{\theta^{k\beta}} F(\theta^k X, \theta^{k\beta} M),
\]
(2.18)
and the source function
\[ f_k(X) := \frac{1}{\theta^{k\beta}} f(\theta^k X). \] (2.19)

Easily one verifies that
\[ F_k(X, D^2 v) = f_k(X) \] (2.20)
in the viscosity sense and that \( v, F_k \) and \( f_k \) satisfy all the hypotheses from Lemma 4. Thus, it follows from the thesis of that Lemma the existence of an affine function \( \ell_*(X) = a_\star + \tilde{b}_\star \cdot X \), with
\[ |a_\star|_R + |b_\star|_{R^d} \leq C, \] (2.21)
such that
\[ \sup_{Q_\theta} |v(X) - \ell_*(X)| \leq \theta^{2+\beta}. \] (2.22)

Finally if we define
\[ \ell_{k+1}(X) := \ell_k(X) + \theta^{k(2+\beta)} \ell_*(\theta^{-k} X), \] (2.23)
and translate back estimate (2.22) to the original function \( u \), we obtain the \((k+1)\)th step of the induction process. Furthermore, estimate (2.15) follows readily from (2.21) and (2.23).

It is now a mere of routine to conclude the proof of Theorem 1, see for instance [11]. We omit the details here. \( \square \)

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