ON THE CONJUGACY PROBLEM FOR CARTER SUBGROUPS

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Abstract

It is proven in the paper, that Carter subgroups of a finite group are conjugate if Carter subgroups in the group of induced automorphisms for every non-Abelian composition factor are conjugate.

Key words. Carter subgroup, almost simple group, group of induced automorphisms.

§ 1. Introduction

Recall that a nilpotent self-normalizing subgroup is called a Carter subgroup. In the paper we consider the following Problem. Are any two Carter subgroups of a finite group conjugate?

In [1] it is proven that the minimal counter example to this problem should be almost simple. We intend to improve results obtained in [1] (see the theorem below). Actually, we shall use ideas of [1] in order to prove stronger theorem.

Our notations is standard. For a finite group $G$ we denote by $\text{Aut}(G)$ the group of automorphisms of $G$. If $Z(G)$ is trivial, then $G$ is isomorphic to the group of its inner automorphisms and we may suppose that $G \leq \text{Aut}(G)$. A finite group $G$ is said to be almost simple if there is a simple group $S$ with $S \leq G \leq \text{Aut}(S)$, i.e., $F^*(G)$ is a simple group. We denote by $F(G)$ the Fitting subgroup of $G$ and by $F^*(G)$ the generalized Fitting subgroup of $G$.

If $G$ is a group, $A, B, H$ are subgroups of $G$ and $B$ is normal in $A$ ($B \triangleleft A$), then $N_H(A/B) = N_H(A) \cap N_H(B)$. If $x \in N_H(A/B)$, then $x$ induces an automorphism $Ba \mapsto Bx^{-1}ax$ of $A/B$. Thus, there is a homomorphism of $N_H(A/B)$ into $\text{Aut}(A/B)$. The image of this homomorphism is denoted by $\text{Aut}_H(A/B)$ while its kernel is denoted by $C_H(A/B)$. In particular, if $S$ is a composition factor of $G$, then for any $H \leq G$ the group $\text{Aut}_H(S)$ is defined.

Definition. A finite group $G$ is said to satisfy condition (*) if, for every its non-abelian composition factor $S$ and for every its nilpotent subgroup $N$, Carter subgroups of $\langle \text{Aut}_N(S), S \rangle$ are conjugate.

Clearly, if a finite group $G$ satisfies (*), then for every normal subgroup $H$ and soluble subgroup $N$ of $G$, groups $G/H$ and $NH$ satisfy (*). Our goal here is to prove the following theorem.

Theorem. If a finite group $G$ satisfies (*), then Carter subgroups of $G$ are conjugate.

Note that a finite group may not contain Carter subgroups. In this case we also say that its Carter subgroups are conjugate. In sections 2 and 3 we are assuming that $X$ is a counter example to the theorem of minimal order, i.e., that $X$ is a finite group satisfying condition (*), $X$ contains nonconjugate Carter subgroups, but Carter subgroups in every group $M$ of order less, that $|X|$, satisfying condition (*), are conjugate.

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§ 2. Preliminary results

Recall that $X$ is a counterexample to the theorem of minimal order.

**Lemma 1.** Let $G$ be a finite group satisfying $(\ast)$ and $|G| \leq |X|$. Let $H$ be a Carter subgroup of $G$. If $N$ is a normal subgroup of $G$, then $HN/N$ is a Carter subgroup of $G/N$.

**Proof.** Since $HN/N$ is nilpotent, we have just to prove that it is self-normalizing in $G/N$. Clearly, this is true if $G = HN$. So, assume $M = HN < G$. By the minimality of $X$, $M^x = M$, $x \in G$, implies $H^x = H^m$ for some $m \in M$. It follows $xm^{-1} \in N_G(H) = H$ and $x \in M$. This proves that $HN/N$ is nilpotent and self-normalizing in $G/N$. □

**Lemma 2.** Let $B$ be a minimal normal subgroup of $X$ and $H, K$ be non-conjugate Carter subgroups of $X$.

(i) $B$ is non-soluble;

(ii) $X = BH = BK$;

(iii) $B$ is the unique minimal normal subgroup of $X$.

**Proof.** (i) We give a proof by contradiction. Assume that $B$ is soluble and let $\pi : X \to X/B$ be the canonical homomorphism. Then $H^\pi$ and $K^\pi$ are Carter subgroups of $X/B$, by Lemma 1. By the minimality of $X$, there exists $x = Bx$ such that $(K^\pi)^x = H^\pi$. It follows $K^x \leq BH$. Since $BH$ is soluble, $K^x$ is conjugate to $H$ in $BH$, hence $K$ is conjugate to $H$ in $X$, a contradiction.

(ii) Assume that $BH < X$. By Lemma 1 and the minimality of $X$, $BH/B$ and $BK/B$ are conjugate in $X/B$: so there exists $x \in X$ such that $K^x \leq BH$. It follows that $K^x$ is conjugate to $H$ in $BH$, hence $K$ is conjugate to $H$ in $X$, a contradiction.

(iii) Suppose that $M$ is a minimal normal subgroup of $X$ different from $B$. By (i), $M$ is non-
soluble. On the other hand, $MB/B \simeq M$ is a subgroup of the nilpotent group $X/B \simeq H/H \cap B$, a contradiction. □

The following lemma is useful in many applications, so we prove it here, though we need only a part of its proof in our later arguments.

**Lemma 3.** Let $G$ be a finite group. Let $H$ be a Carter subgroup of $G$. Assume that there exists a normal subgroup $B = T_1 \times \ldots \times T_k$ of $G$ such that $T_i \simeq \ldots \simeq T_k \simeq T$, $Z(T_i) = \{1\}$ for all $i$, and $G = H(T_1 \times \ldots \times T_k)$. Then $\text{Aut}_H(T_i)$ is a Carter subgroup of $\langle \text{Aut}_H(T_i), T_i \rangle$.

**Proof.** Assume that our statement is false and $G$ is a counterexample with $k$ minimal, then $k > 1$. Clearly, $G$ acts transitively, by conjugation, on the set $\Omega := \{T_1, \ldots, T_k\}$. We may assume that the $T_j$’s are indexed so that $G$ acts primitively on the set $\{\Delta_1, \ldots, \Delta_p\}$, $p > 1$, where for each $i$:

$$\Delta_i := \{T_{1+(i-1)\ell}, \ldots, T_{i\ell}\}, \quad k = pl.$$  

Denote by $\varphi : G \to \text{Sym}_p$ the induced permutation representation. Clearly, $B \leq \ker \varphi$, so that $G^\varphi = (BH)^\varphi = H^\varphi$ is a primitive nilpotent subgroup of $\text{Sym}_p$. Hence $p$ is prime and $G^\varphi$ is a cyclic group of order $p$. In particular, $Y := \ker \varphi$ coincides with the stabilizer of any $\Delta_i$, so that $\varphi$ is permutationally equivalent to the representation of $G$ on the right cosets of $Y$. For each $i = 1, \ldots, p$, let $S_i = T_{1+(i-1)\ell} \times \ldots \times T_{i\ell}$: then $Y = N_G(S_i)$ and $B = S_1 \times \ldots \times S_p$. Consider $\xi : Y \to \text{Aut}_Y(S_1)$, let $A = Y^\xi$, $S = S_1^\xi$. Clearly $S$ is a normal subgroup of $A$; moreover, $S$ is
isomorphic to $S_1$, since $S_1$ has trivial center. On the other hand, for each $i \neq 1$, $S_i \leq \ker \xi$, since $S_i$ centralizes $S_1$.

Denote by $A \wr C_p$ the wreath product of $A$ and a cyclic group $C_p$ and let \{\(x_1 = e, \ldots, x_p\)\} be a right transversal of $Y$. Then the map $\eta : G \to A \wr C_p$ such that, for each $x \in G$:

\[
x \mapsto \left(\left(x_1 x_1^{-1}\right)^\xi, \ldots, \left(x_p x_p^{-1}\right)^\xi\right) x^\varphi
\]

is a homomorphism. Clearly $Y^n$ is a subdirect product of the base subgroup $A^p$ and

\[
S^n_1 = \{(s, 1, \ldots, 1)|s \in S\}, B^n = \{(s_1, \ldots, s_p)|s_i \in S\} \leq Y^n.
\]

Moreover, $\ker \eta = C_G(B) = \{e\}$, so we may identify $G$ with $G^n$. We choose $h \in H \setminus Y$. Then $G = \langle Y, h \rangle$, $h^p \in Y$, $H = \langle Y \cap H \rangle \langle h \rangle$ and we may assume

\[
h = (a_1, a_2, \ldots, a_p)\pi, a_i \in A, \pi = (1, 2, \ldots, p) \in C_p.
\]

For each $i$, $1 \leq i \leq p$, let $\psi_i : A^p \to A$ be the canonical projection and let $H_i := (H \cap Y)^{\psi_i}$. Clearly, $Y^{\psi_i} = A$. Moreover, for each $i \geq 2$, $H_i = H_i^{h^{-1}} = H_i^{a_1 \cdots a_i - 1}$ since $h$ normalizes $Y \cap H$.

Let $N := (H_1 \times \cdots \times H_p) \setminus Y$. $N$ is normalized by $H$, since $H = \langle N \cap H \rangle \langle h \rangle$ and $H_i^{h} = H_{i+1} \pmod{p}$. We claim that $H_1$ is a Carter subgroup of $A$. Assume $n_1 \in N_A(H_1) \setminus H_1$. From $Y = \langle Y \cap H \rangle B$, it follows $n_1 = h_1 s$, $h_1 \in H_1$, $s \in N_S(H_1) \setminus H_1$. Let $b := (s, s^{a_1}, \ldots, s^{a_1 \cdots a_p - 1}) \in B$. Then $b$ normalizes $N$, for:

$H_i^b = H_i^{s^{a_1 \cdots a_i - 1}} = H_i^{a_1 \cdots a_i - 1} = H_i^{a_1 \cdots a_i - 1} = H_i^{a_1 \cdots a_i - 1} = H_i$.

Now $[b, h^{-1}] := b^{-1} h b h^{-1} \in Y$ is such that:

$[b, h^{-1}]^{\psi_i} = 1$ if $i \neq p$, $[b, h^{-1}]^{\psi_p} = [s, (a_1 \cdots a_p)^{-1}]^{a_1 \cdots a_p - 1}$,

where $a_1 \cdots a_p = (h^p)^{\psi_i} \in H_1$. Since $s \in N_S(H_1)$, it follows

$[s, (a_1 \cdots a_p)^{-1}] \in H_1, [s, (a_1 \cdots a_p)^{-1}]^{a_1 \cdots a_p - 1} \in H_p$.

So $[b, h^{-1}] \in N$ and $b \in N_G(N \langle h \rangle)$. But $H \leq N \langle h \rangle$, implies $N_G(N \langle h \rangle) = N \langle h \rangle$. Indeed, if $g \in N_G(N \langle h \rangle)$, then $H^g$ is a Carter subgroup of $N \langle h \rangle$. But $N \langle h \rangle$ is soluble, hence there exists $y \in N \langle h \rangle$ with $H^g = H^y$. Now $H$ is a Carter subgroup of $G$, thus $gy^{-1} \in H$ and $g \in N \langle h \rangle$. Therefore $b \in N, s \in H_1$, i.e., $n_1 \in H_1$, a contradiction.

Now $A = H_1(T_1 \times \cdots \times T_l)$ and $l < k$. By induction we have that $\text{Aut}_{H_1}(T_1)$ is a Carter subgroup of $\langle \text{Aut}_{H_1}(T_1), T_1 \rangle$. In view of our construction, $\text{Aut}_H(T_1) = \text{Aut}_{H_1}(T_1)$ and the lemma follows.

\section{Proof of the Theorem}

Write $B = T_1 \times \cdots \times T_k$, $T_i \simeq T$, a non-abelian simple group. What remains to prove is $k = 1$. In the notations of the proof of Lemma 3, we have shown that $H_1$ is a Carter subgroup of $A$. Clearly, since each $H_i$ is conjugate to $H_1$ in $A$, $N_A(H_i) = H_i$, $i = 1, \ldots, p$. It follows easily that $N$ is a Carter subgroup of $Y$. For, let $y := (y_1, \ldots, y_p) \in N_Y(N)$: from $N^{\psi_i} = H_i$, we have $y_i \in N_A(H_i) = H_i$, for each $i$, hence $y \in N$.

We have seen that, to each Carter subgroup $H$ of $X$ we can associate a Carter subgroup $N = N_H$ of $Y$, such that $H$ normalizes $N_H$. Clearly, $N_H \neq \{e\}$, otherwise $X$ would have order $p$. So let $K$ be a Carter subgroup of $X$, not conjugate to $H$, and let $N_K$ be the Carter subgroup of $Y$ corresponding to $K$. By the minimality of $X$, for each $x \in X$ we have $\langle H^x, K \rangle = X$. On the other hand, by the inductive hypothesis, there exists $x \in Y$ such that $N_K = (N_H)^x$. Hence $N_K$ is normal in $\langle K, H^x \rangle = X$, a contradiction as $\{e\} \neq N_K$ is nilpotent.
§ 4. Some properties of Carter subgroups

Here we prove some lemmas that are useful in investigation of Carter subgroups in finite groups, in particular in almost simple groups.

**Lemma 4.** Let $G$ be a finite group satisfying $(\ast)$, $H$ be a normal subgroup of $G$, and $K$ be a Carter subgroup of $G$. Then $KH/H$ is a Carter subgroup of $G/H$.

*Proof.* This fact is proven in Lemma 1 under additional assumption $|G| \leq |X|$. Now we proved the theorem, thus $|X| = \infty$ and this lemma holds true for any finite group $G$. $\square$

**Lemma 5.** Assume that $G$ is a finite group. Let $K$ be a Carter subgroup of $G$, with center $Z(K)$. Assume also that $e \neq z \in Z(K)$ and $C_G(z)$ satisfies $(\ast)$.

1. Every subgroup $Y$ which contains $K$ and satisfies $(\ast)$ is self-normalizing in $G$.
2. No conjugate of $z$ in $G$, except $z$, lies in $Z(G)$.
3. If $H$ is a Carter subgroup of $G$, non-conjugate to $K$, then $z$ is not conjugate to any element in the center of $H$.

In particular the centralizer $C_G(z)$ is self-normalizing in $G$, and $z$ is not conjugate to any power $z^k \neq z$.

*Proof.* This lemma is proven in [2, Lemma 3.1] for a minimal counter example to the problem and therefore its usage for finding Carter subgroups heavily depends on the classification of finite simple groups. We state here stronger version of the lemma in order to avoid such dependence.

1. Take $x \in N_G(Y)$. Then $K^x$ is a Carter subgroup of $Y$. By the theorem, Carter subgroups of $Y$ are conjugate. Therefore there exists $y \in Y$ with $K^x = K^y$. Hence $xy^{-1} \in N_G(K) = K \leq Y$ and $x \in Y$.
2. Assume $z^{x^{-1}} \in Z(K)$ for some $x \in G$. Then $z$ belongs to the center of $\langle G, G^x \rangle \leq C_G(z)$. Since $C_G(z)$ satisfies $(\ast)$, there exists $y \in C_G(z)$ such that $K^x = K^y$. From $xy^{-1} \in C_G(z)$, we get $z^{y^{-1}} = z$ hence $z^x = z^y = z$. We conclude $z^{x^{-1}} = z$.
3. If our claim is false, substituting $H$ with some conjugate $H^x$ (if necessary), we may assume $z \in Z(K) \cap Z(H)$, i.e. $z \in Z((K, H)) \leq C_G(z)$. Again since $C_G(z)$ satisfies $(\ast)$, there exists $y \in C_G(z)$ such that $H = K^y$. A contradiction. $\square$

Note that for every known finite simple group $G$ (and hence almost simple, since the group of outer automorphisms is soluble) and for most elements $z \in G$ of prime order we have that composition factors of $C_G(z)$ are known simple groups. Indeed, for sporadic groups this statement can be checked by using [3]. Composition factors of $C_{A_n}(z)$ are alternating groups. If $G$ is a finite simple group of Lie type over a field of characteristic $p$ and $(|z|, p) = 1$, then $z$ is semisimple and composition factors of $C_G(z)$ are finite groups of Lie type. If $|z| = p$ and $p$ is a good prime for $G$, then [4, Theorems 1.2 and 1.4] implies that all composition factors of $C_G(z)$ are finite groups of Lie type. The only case, where the structure of centralizers of unipotent elements of order $p$ is not completely known: $p$ is a bad prime for $G$.

Therefore if we are classifying Carter subgroups of almost simple finite group $A$ by induction we may assume that $C_A(z)$ satisfies $(\ast)$ for most elements of prime order $z \in A$. In particular, we can improve table from [2], using results of present paper and [5]. In the table below $A$ is an almost simple group with conjugate Carter subgroups.
\[
\begin{array}{|c|c|}
\hline
\text{Soc}(A) = G & \text{Conditions for } A \\
\hline
\text{alternating, sporadic;} & \text{none} \\
A_1(r^t), B_r(r^t), C_t(r^t), t \text{ even if } r = 3; & \\
2B_2(2^{2n+1}), G_2(r^t), F_4(r^t), 2F_4(2^{2n+1}); & \\
E_7(r^t), r \neq 3; E_8(r^t), r \neq 3,5 & \\
D_{2\ell}(r^t), 3D_4(3^t), 2D_{2\ell}(2^{2t}), & A/(A \cap \hat{G}) \text{ a } 2 - \text{group} \\
t \text{ even if } r = 3 \text{ and, if } G = D_4(r^t), & \text{or} \\
|(\text{Field}(G) \cap A) : (\hat{G} \cap A)| \geq 1 & |\hat{G} : (A \cap \hat{G})| \leq 2 \\
B_t(3^t), C_t(3^t), D_{2\ell}(3^t), 4D_4(3^t); 2D_{2\ell}(3^{2t}), & A = G \\
D_{2\ell+1}(r^t), 2D_{2\ell+1}(2^{2t}), G_2(3^{2n+1}), & \\
E_6(r^t), 2E_6(2^{2t}), E_7(3^t), E_8(3^t), E_8(5^t) & \\
A_\ell(r^t), 2A_\ell(r^{2t}), \ell > 1 & G \leq A \leq G, \\
\hline
\end{array}
\]

**LITERATURE**

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