A note on a conjecture of Gromov about non-free isometric immersions

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Abstract

We extend the results obtained in [DL07] towards the proof of a conjecture of M. Gromov on isometric immersions via non-free maps.

1 Introduction

The problem of isometric immersions was solved first by J. Nash with his celebrated theorem [Nas56] and then, in a much wider setting, by M. Gromov [GR70, Gro86]. In this context it was shown that, given a smooth $m$-dimensional manifold $M$, the operator $D(f) = f^*g_{\text{can}}$, which associates to every smooth map $f : M \to \mathbb{R}^q$ its pull-back of the Euclidean metric $g_{\text{can}}$ on $\mathbb{R}^q$, is an open map for $q \geq m(m+5)/2$ when restricted to the set of free maps (throughout the paper we endow functional spaces with the Whitney topology). We recall that free maps are maps which are injective on the osculating space $J^2_0(\mathbb{R}, M)$, i.e. such that their first and second derivatives are linearly independent, and that the set of all free maps $\text{Free}(M, \mathbb{R}^q)$ is an open subset of $C^\infty(M, \mathbb{R}^q)$ which is empty for $q < m(m+3)/2$ and dense for $q \geq m(m+5)/2$.

We focus here on the particular case $M = \mathbb{R}^m$ and use $D_{m,q}$ for the operator $D$ acting on $C^\infty(\mathbb{R}^m, \mathbb{R}^q)$. In this case it is known that $\text{Free}(\mathbb{R}^m, \mathbb{R}^q)$ is non-empty for $q \geq m(m+3)/2$ (see Example 2 below) so that, in particular, it turns out that, for every $q \geq m(m+3)/2$, there is a non-empty open set $A$ on which the restriction of $D_{m,q}$ is an open map (we say that such a $D$ is infinitesimally invertible over the open set $A$): very little instead is known about $D_{m,q}$ for smaller values of $q$, when no free map can arise for dimensional reasons. A conjecture by Gromov (see [Gro86], p.162) claims that every $D_{m,q}$ is infinitesimally invertible over an open dense set for every $q \geq m(m+3)/2 - \sqrt{m/2}$, namely that there exist non-empty dense open sets $A_{m,q} \subset C^\infty(\mathbb{R}^m, \mathbb{R}^q)$ such that $D_{m,q}|_{A_{m,q}}$ are open maps.

In a recent paper [DL07] steps were taken towards the full proof of the conjecture by showing, through an explicit construction that made use of the Lie equations after Gromov’s idea in [Gro86] (p. 152), that $D_{2,4}$ is infinitesimally invertible over an open set $A_{2,4}^{DL}$. In the next section we improve this result
by finding a larger set $A_{2,4} \supset A_{2,4}^{DL}$ on which $D_{2,4}$ is open and defining open sets $A_{m,q}$, for every $q \geq m(m+3)/2 - 1$, over which the operators $D_{m,q}$ are infinitesimally invertible. Our result is a direct consequence of the following well-known theorem by Duistermaat and Hormander \[DH72\].

**Theorem DH.** Let $M$ be an open manifold. Then the first order differential operator $X(f) = L_\xi f + \lambda f$, where $\xi$ is a vector field on $M$ and $\lambda$ a smooth function, is surjective on $C^\infty(M)$ iff $\xi$ admits a global transversal, i.e. an embedded hypersurface that cuts in a single point each of its integral trajectories.

## 2 Proof and Examples

Our aim is finding an open set of functions $A$ such that, if $f_0 \in A$ and $g_0 = D(f_0)$, the equation

$$D(f) = g$$

has solutions for every $g$ close enough to $g_0$. We recall that, by a general theorem \[Gro86\], the existence of solutions of \[1\] is granted by the existence of solutions of its linearized version, which in turn is equivalent (e.g. see \[GR70\]) to the following algebraic system:

$$\begin{cases}
\delta_{ij} \partial_\alpha f^i \delta f^j = h_\alpha \\
\delta_{ij} \partial_{\alpha \beta} f^i \delta f^j = (\partial_\alpha h_\beta + \partial_\beta h_\alpha - \delta g_{\alpha \beta}) / 2
\end{cases} \tag{2}$$

where the $\delta f^i$ are the $q$ unknowns, the $\delta g_{\alpha \beta}$ are $m(m+1)/2$ given functions and the $h_\alpha$ are $m$ arbitrary functions. Therefore it is enough for our purposes to show that, for some open set of smooth functions, we can always choose the $h_\alpha$ so that system \[2\] has a solution.

**Theorem.** If $q \geq \frac{m(m+3)}{2} - 1$ the operators $D_{m,q}$ are infinitesimally invertible over non-empty open sets $A_{m,q}$.

**Proof.** When $q \geq m(m+3)/2$ the statement is trivially true because it is enough to choose $A_{m,q} = Free(\mathbb{R}^m, \mathbb{R}^q)$. We will assume therefore in the remainder of the proof that $q = m(m+3)/2 - 1$, i.e. that the number of equations is exactly one more than the number of unknowns $\delta f^i$.

Since the coefficients of the system \[2\] are exactly the $m(m+3)/2$ vector fields $\{\partial_\alpha f^i, \partial_{\alpha \beta} f^i\}$, then clearly there exist non-identically zero functions $\lambda^\alpha$ and $\lambda^{\alpha \beta} = \lambda^{\beta \alpha}$, such that identically

$$\lambda^\alpha \partial_\alpha f^i + \lambda^{\alpha \beta} \partial_{\alpha \beta} f^i = 0.$$

This reflects in the following compatibility condition for system \[2\]:

$$2\lambda^\alpha h_\alpha + \lambda^{\alpha \beta}(\partial_\alpha h_\beta + \partial_\beta h_\alpha - \delta g_{\alpha \beta}) = 0$$

which is convenient to rewrite as

$$X^\alpha h_\alpha = \phi \tag{3}$$
where $\phi = \lambda^{\alpha\beta} \delta g_{\alpha\beta}$, $X^\alpha$ is the first-order non-homogeneous differential operator

$$X^\alpha = L_{\xi^\alpha} + 2\lambda^\alpha,$$

$L_{\xi^\alpha}$ is the Lie derivative with respect to the vector field

$$\xi^\alpha = \lambda^{\alpha\beta} \partial_\beta$$

and the functions $\lambda^\alpha$ must be thought as the corresponding zero-order multiplication operators.

Now, let $\mathcal{A}_{m,q} \subset C^\infty(\mathbb{R}^m, \mathbb{R}^q)$ be the open set of immersions $f$ such that their second-order jet $J^2_0(\mathbb{R}, \mathcal{A}) : J^2_0(\mathbb{R}, \mathbb{R}^m) \to J^2_0(\mathbb{R}, \mathbb{R}^q)$ has full rank and, in some coordinate system, there is a coordinate $\alpha_0$ of $\mathbb{R}^m$ such that the functions $\lambda^{\alpha_0\beta}$ are never zero at the same time. Then, after setting $h_\beta = \lambda^{\alpha_0\beta} h$, $\beta = 1, \ldots, m$, for some unknown function $h$, the equation (3) becomes

$$Y h = \psi$$

where $Y = L_\zeta + \lambda'$ for some vector field $\zeta$ and function $\lambda'$. A short computation shows that the component $\alpha_0$ of $\zeta$ is equal to $(\lambda^{\alpha_01})^2 + \cdots + (\lambda^{\alpha_0m})^2$ and therefore it is never zero by hypothesis. In particular this means that every surface $x^{\alpha_0} = \text{const}$ is a global transversal for $\zeta$ and therefore, by Theorem DH, $Y$ is a surjective first-order partial differential operator. Hence for every function belonging to $\mathcal{A}_{m,q}$ it is always possible to choose the $h_\alpha$ in function of the $\delta g_{\alpha\beta}$ so that the compatibility condition (2) is satisfied.

**Example 1.** Consider any pair $(g, h)$ of free maps from $\mathbb{R}$ to $\mathbb{R}^2$. Then the function $F_{gh} : \mathbb{R}^2 \to \mathbb{R}^4$ defined by $F_{gh}(x, y) = (g(x), h(y))$ belongs to $\mathcal{A}_{2,4} \subset C^\infty(\mathbb{R}^2, \mathbb{R}^4)$. Indeed in this case $\partial_{xy} F_{gh} = 0$, so that we can choose

$$\lambda^x = \lambda^y = \lambda^{xx} = \lambda^{yy} = 0, \quad \lambda^{xy} = \lambda^{yx} = 1$$

and therefore the compatibility condition becomes simply

$$\partial_x h_y + \partial_y h_x = \delta g_{xy}$$

which is trivially solvable. E.g. in concrete the function $F(x, y) = (x, e^x, y, e^y)$ belongs to $\mathcal{A}_{2,4}$. Note that, while $\mathcal{A}^{DL}_{2,4} \subset \mathcal{A}_{2,4}$, $F$ does not belong to $\mathcal{A}^{DL}_{2,4}$, i.e. $\mathcal{A}_{2,4}$ is strictly bigger of the set introduced in [DL07].

**Example 2.** Let $F \in \text{Free}(\mathbb{R}^m, \mathbb{R}^{m(m+3)/2})$ be the canonical free map given by

$$F(x^1, \ldots, x^m) = (x^1, \ldots, x^m, (x^1)^2, x^1 x^2, \ldots, (x^m)^2)$$

and $\pi$ any projection $\pi : \mathbb{R}^{m(m+3)/2} \to \mathbb{R}^{m(m+3)/2-1}$ which “forgets” any one of the last $m(m+1)/2$ components. Then their composition $F_\pi = \pi \circ F$ belongs to $\mathcal{A}_{m,m(m+3)/2-1}$ because clearly the second-order jet of $F_\pi$ has full rank and one of its double derivatives, say $\partial_{x_1 x_2} F_\pi$, is identically zero, so we can choose the corresponding factor $\lambda^{x_1 x_2}$ identically equal to 1. In the $(m, q) = (2, 4)$ case for example we get the functions $F_1(x, y) = (x, y, xy, y^2)$, $F_2(x, y) = (x, y, x^2, y^2)$ and $F_3(x, y) = (x, y, x^2, xy)$.  

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Note that, exactly like in [DL07], for $q = m(m + 3)/2 − 1$ the set of $[m(m + 3)/2] \times q$ matrices not satisfying the conditions introduced in the proof to define the open sets $A_{m,q}$ has just codimension 1 in the fibers of the bundle $J^2(\mathbb{R}^m, \mathbb{R}^q) \to J^0(\mathbb{R}^m, \mathbb{R}^q)$ while we would need at least codimension 3 in order to apply the transversality theorems. In particular the sets $A_{m,q}$ are not dense.

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