Decomposition numbers of $SO_7(q)$ and $Sp_6(q)$

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Abstract

We complete the $\ell$-modular decomposition numbers of the unipotent characters in the principal block of the special orthogonal groups $SO_7(q)$ and the symplectic groups $Sp_6(q)$ for all prime powers $q$ and all odd primes $\ell$ different from the defining characteristic.

1 Introduction

In the representation theory of finite groups, the $\ell$-modular decomposition numbers link the ordinary representations of a group to its representations over fields of prime characteristic $\ell$. The decomposition numbers of finite groups of Lie type of small rank in the case that $\ell$ differs from the defining characteristic were computed by several authors; see for example [Dud13, Gec90, Gec91, HH13, Him11, His89a, LM80, OW98, OW02, Wak04, Whi90a, Whi90b, Whi95, Whi00].

Finite groups of Lie type which are dual to each other have many properties in common. For example, they have isomorphic Weyl groups. It is a natural question to what extent these similarities carry over to modular representations. The smallest example (in terms of the rank) of two untwisted groups in duality with non-isomorphic root systems are the special orthogonal groups $SO_7(q)$ and the symplectic groups $Sp_6(q)$, having root systems of type $B_3$ and $C_3$, respectively. This note is a contribution to the modular representation theory in non-defining characteristic of these groups. Since one expects, roughly speaking, that the representation theory of a given block of a finite group of Lie type may be reduced to the representation theory of unipotent blocks (see [BR03] and [Eng08] for important results in this direction) the unipotent blocks are of particular relevance.

Let $q$ be a power of a prime $p$ and $\ell$ an odd prime different from $p$. We give the $\ell$-modular decomposition matrices of the unipotent blocks of the groups $SO_7(q)$ and $Sp_6(q)$. By the theory of basic sets [GH91] the decomposition matrices can be reconstructed from the decomposition numbers of the ordinary unipotent
irreducible characters via so-called basic relations. Only the principal block for \( \ell \mid q + 1 \) leads to new results: all other blocks and other primes \( \ell \) yield cyclic blocks which have already been treated by Fong and Srinivasan [FS90] and White [Whi00], or in the case of \( \ell \mid q - 1 \) results by Puig [Pui90] and Gruber and Hiss [GH97b] readily give the decomposition numbers.

The \( \ell \)-modular decomposition matrices of the unipotent blocks of \( \text{SO}_7(q) \) and \( \text{Sp}_6(q) \) are completed by Theorem 4.3, the main result of this paper. It gives all \( \ell \)-modular decomposition numbers of the ordinary unipotent irreducible characters in the principal block of \( \text{SO}_7(q) \) and \( \text{Sp}_6(q) \) for \( \ell \mid q + 1 \). It turns out that, with respect to a suitable ordering of rows and columns, the decomposition matrices of the unipotent blocks of \( \text{SO}_7(q) \) and \( \text{Sp}_6(q) \) coincide. Furthermore, Theorem 4.3 confirms [GH97a, Conjecture 3.4] in the special case of \( \text{SO}_7(q) \) and \( \text{Sp}_6(q) \).

It should be noted that for the symplectic groups \( \text{Sp}_6(q) \) most of the decomposition numbers were previously computed by Köhler [Köl96], An, Hiss [AH06] and White [Whi00] except for two parameters in the Steinberg character. Theorem 4.3 gives the values of these parameters.

The paper is structured as follows: We begin by detailing in Section 2 the groups considered with particular emphasis on the Lie theoretic setting and define certain maximal parabolic subgroups. Section 3 focuses on the ordinary characters of the groups introduced. We give constructions for the characters of the parabolic subgroups, fix notation for the ordinary unipotent characters and use Deligne-Lusztig theory to determine several non-unipotent characters of \( \text{SO}_5(q) \), \( \text{SO}_7(q) \) and \( \text{Sp}_6(q) \). Section 4 treats modular representations and contains the computation of the decomposition numbers, mainly dealing with the case \( \ell \mid q + 1 \).

The tools we use are: Brauer character relations coming from the \( \ell \)-regular restrictions of the non-unipotent characters constructed in Section 3, the decomposition numbers of \( \text{Sp}_4(q) \) [Whi90c, Whi93, OW98] and the approximation of the decomposition matrix of \( \text{Sp}_6(q) \) [Köl96, AH06, Whi00], Harish-Chandra induction of projective indecomposable modules (PIMs) of Levi subgroups, induction of PIMs of cyclic blocks of the parabolic subgroups introduced in Section 2 and a recent method by Dudas [Dud13] based on Deligne-Lusztig varieties and their associated complexes of virtual projective modules. As an intermediate result we obtain the decomposition numbers of the unipotent characters of \( \text{SO}_5(q) \).

The prime \( \ell = 2 \) is bad for the orthogonal and the symplectic groups in the sense of [Car85, p. 28]. As a consequence the 2-modular representation theory of the groups involved seems to be fundamentally different and our approach does not lend itself to an easy adaptation. For example, by [Whi90a] the group \( \text{Sp}_4(q) \) has seven unipotent irreducible 2-modular Brauer characters, but only six ordinary unipotent characters. Also, the blocks of the parabolic subgroups we consider are no longer cyclic for \( \ell = 2 \).
2 Symplectic and orthogonal groups

In this section we give a description of the symplectic and orthogonal groups as a pair of dual groups. For the symplectic groups we largely follow [AH06, Section 2].

2.1 Lie theoretic setting

Let \( q \) be a power of a prime \( p \) and \( \mathbb{F} \) an algebraic closure of the finite field \( \mathbb{F}_q \) with \( q \) elements. We fix a positive integer \( m \) and set \( n := 2m + 1 \) and \( n^* := 2m \).

Let \( I_m \in \mathbb{F}^{m \times m} \) be the identity matrix, \( J_m \in \mathbb{F}^{m \times m} \) the matrix with ones on the anti-diagonal and zeros elsewhere, \( J_n := \begin{bmatrix} 0 & 0 & J_m \\ 0 & 2 & 0 \\ J_m & 0 & 0 \end{bmatrix} \) and \( \tilde{J}_n := \begin{bmatrix} 0 & J_m \\ -J_m & 0 \end{bmatrix} \).

We write vectors \( v \in \mathbb{F}^n \) as \( v = [v_m, \ldots, v_1, v_0, v_1', \ldots, v_m']^t \) and define a quadratic form \( Q_n \) on \( \mathbb{F}^n \) by

\[
Q_n(v) := v_0^2 + \sum_{j=1}^m v_j v'_j.
\]

Let \( \{e_m, \ldots, e_1, e_0, e_1', \ldots, e_m'\} \) be the standard basis of the vector space \( \mathbb{F}^n \). We define

\[
G := \text{SO}_n(\mathbb{F}) := \{x \in \text{SL}_n(\mathbb{F}) \mid Q_n(xv) = Q_n(v) \text{ for all } v \in \mathbb{F}^n\},
\]

\[
G^* := \text{Sp}_{n^*}(\mathbb{F}) := \{x \in \text{GL}_{n^*}(\mathbb{F}) \mid x^t \tilde{J}_n x = \tilde{J}_n\}.
\]

Let \( F : G \to G \) and \( F^* : G^* \to G^* \) be the Frobenius maps raising each matrix entry to its \( q \)-th power. For \( t_1, t_2, \ldots, t_m \in \mathbb{F}^x \) we set

\[
h(t_1, \ldots, t_m) := \text{diag}(t_m, \ldots, t_1, t_1^{-1}, \ldots, t_m^{-1}) \in G,
\]

so that \( T := \{h(t_1, \ldots, t_m) \mid t_1, \ldots, t_m \in \mathbb{F}^x\} \) is an \( F \)-stable maximally split torus of \( G \). The root system \( \Phi \) of the simple algebraic group \( G \) of adjoint type with respect to \( T \) is of type \( B_m \). We write \( \Phi^+ \) for the set of positive roots corresponding to the set \( \{\alpha_1, \ldots, \alpha_m\} \) of simple roots where

\[
\alpha_1 : h(t_1, \ldots, t_m) \mapsto t_1,
\]

\[
\alpha_j : h(t_1, \ldots, t_m) \mapsto t_j t_{j-1}^{-1} \text{ for } j > 1.
\]
Note that $\alpha_1$ is a short root, while the other $\alpha_j$ are long roots. The corresponding reflections in the Weyl group $W$ of $G$ are $w_j := s_j T$ for $j = 1, 2, \ldots, m$ where

$$s_1 := \begin{bmatrix} I_{m-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$s_j := \begin{bmatrix} I_{m-j} & 0 & 0 & 0 \\ 0 & J_2 & 0 & 0 \\ 0 & 0 & I_{2j-3} & 0 \\ 0 & 0 & 0 & J_2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

for $j \in \{2, 3, \ldots, m\}$.

In particular, $W$ is generated by $\{w_1, \ldots, w_m\}$ and we have

$$w_1 h(t_1, \ldots, t_m) = s_1 h(t_1, \ldots, t_m) = h(t_1^{-1}, t_2, \ldots, t_m),$$

$$w_j h(t_1, \ldots, t_m) = s_j h(t_1, \ldots, t_m) = h(\ldots, t_j, t_{j-1}, \ldots)$$

for $j > 1.$ (1)

Similarly, for $t_1, t_2, \ldots, t_m \in \mathbb{F}^\times$ we set

$$h^*(t_1, \ldots, t_m) := \text{diag}(t_m, \ldots, t_1^{-1}, \ldots, t_m^{-1}) \in G^\ast,$$

so that $T^\ast := \{h^*(t_1, \ldots, t_m) \mid t_1, \ldots, t_m \in \mathbb{F}^\times\}$ is an $F^\ast$-stable maximally split torus of $G^\ast$. The root system $\Phi^\ast$ of the simple simply-connected algebraic group $G^\ast$ with respect to $T^\ast$ is of type $C_m$. We write $\Phi^{++}$ for the set of positive roots corresponding to the set $\{\alpha_1^\ast, \ldots, \alpha_m^\ast\}$ of simple roots where

$$\alpha_1^\ast : h^*(t_1, \ldots, t_m) \mapsto t_1^2, \quad \alpha_j^\ast : h^*(t_1, \ldots, t_m) \mapsto t_j t_{j-1}^{-1} \text{ for } j > 1.$$

Note that $\alpha_1^\ast$ is a long root, while the other $\alpha_j^\ast$ are short roots. The corresponding reflections in the Weyl group $W^\ast$ of $G^\ast$ are $w_j^\ast := s_j^\ast T^\ast$ for $j = 1, 2, \ldots, m$ where

$$s_1^\ast := \begin{bmatrix} I_{m-1} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & I_{m-1} \end{bmatrix}$$

and

$$s_j^\ast := \begin{bmatrix} I_{m-j} & 0 & 0 & 0 \\ 0 & J_2 & 0 & 0 \\ 0 & 0 & I_{2j-4} & 0 \\ 0 & 0 & 0 & J_2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

for $j \in \{2, 3, \ldots, m\}$.
In particular, $W^\ast$ is generated by $\{w_1^\ast, \ldots, w_m^\ast\}$ and we have
\begin{align}
  w_1^\ast h^\ast(t_1, \ldots, t_m) &= s_1^\ast h^\ast(t_1, \ldots, t_m) = h^\ast(t_1^{-1}, t_2, \ldots, t_m), \\
  w_j^\ast h^\ast(t_1, \ldots, t_m) &= s_j^\ast h^\ast(t_1, \ldots, t_m) = h^\ast(\ldots, t_j, t_{j-1}, \ldots) & \text{for } j > 1.
\end{align}
(2)

The pairs $(G, F)$ and $(G^\ast, F^\ast)$ are in duality in the sense of [Car85, Chapter 4] and there is an isomorphism $\delta : W \to W^\ast$ mapping $w_j \mapsto w_j^\ast$ for $j = 1, 2, \ldots, m$.

We are interested in the finite groups $G_n := \SO_n(q) := G^F$ and $G_n^\ast := \Sp_n(q) := G^F^\ast$ with $|G| = |G^\ast| = q^{n^2}(q^{2m} - 1)(q^{2m-2} - 1) \cdots (q^2 - 1)$. Additionally, we define $\GO_n^\ast(q)$, $\GO_n(q)$ as in [HN] and set $G_0^\ast := \Sp_0(q) := \GO_0^\ast(q) := \{1\}$. To apply Deligne-Lusztig theory we consider $G^\ast$ as a subgroup of
\[ \tilde{G}^\ast := \CSp_n(q)(\F) := \{x \in \GL_n(q) \mid \exists \lambda \in \F^\times : x^{t^r} J_n x = \lambda J_n\} \]
whose center is connected. We denote the Frobenius map $F^\ast : \tilde{G}^\ast \to \tilde{G}^\ast$ raising each matrix entry to its $q$-th power also by $F^\ast$ and define $G^\ast := G_n^\ast := \tilde{G}^\ast F^\ast$. In particular, we have $|\tilde{G}^\ast| = (q - 1)|G^\ast|$.

**Remark 2.1.** For even $q$ there is a natural isomorphism $\SO_n(q) \to \Sp_n(q)$ mapping each matrix $A$ to the matrix which is obtained from $A$ by removing the middle row and the middle column.

### 2.2 Parabolic subgroups

The group $G_n = \SO_n(q)$ acts naturally on the vector space $\mathbb{F}_q^n$ by multiplication from the left. Let $P_n$ be the stabilizer in $G_n$ of the subspace generated by the basis vector $e_m = [1, 0, \ldots, 0]^tr$. Then $P_n$ is a maximal parabolic subgroup of $G_n$ with Levi decomposition $P_n = L_n \times U_n$, where
\[ L_n = \{s_n(x, a) \mid x \in G_{n-2}, a \in \mathbb{F}_q^2\}, \quad U_n = \{u_n(v) \mid v \in \mathbb{F}_q^{n-2}\} \]
and
\[ s_n(x, a) := \begin{bmatrix} a & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & a^{-1} \end{bmatrix}, \quad u_n(v) := \begin{bmatrix} 1 & -v^{t^r} J_{n-2} & -Q_{n-2}(v) \\ 0 & 1 & v \\ 0 & 0 & 1 \end{bmatrix}. \quad (3) \]

The Levi subgroup $L_n$ decomposes as $L_n = L'_n \times A_n \cong \SO_{n-2}(q) \times \mathbb{F}_q^\times$ where
\[ L'_n := \{s_n(x, 1) \mid x \in \SO_{n-2}(q)\} \quad \text{and} \quad A_n := \{s_n(I_{n-2}, a) \mid a \in \mathbb{F}_q^\times\}. \]
The unipotent radical $U_n$ of $P_n$ is elementary abelian of order $q^{n-2}$. For simplicity, we often write $P$, $L$, $U$, $A$, $L'$ instead of $P_n$, $L_n$, $U_n$, $A_n$, $L'_n$, respectively.

Analogously, the group $G^*_n = Sp_n(q)$ acts naturally on the vector space $\mathbb{F}_q^n$ by multiplication from the left and we write $P^*_n$, for the stabilizer in $G^*_n$ of the subspace generated by $e_m$. Hence, $P^*_n$ is a maximal parabolic subgroup with Levi decomposition $P^*_n = L^*_n \ltimes U^*_n$. The Levi subgroup $L^*_n$ is a direct product of a cyclic group $A^*_n \cong \mathbb{F}_q^\times$ and a subgroup $L^*_n' \cong Sp_{n-2}(q)$. Details on the groups $A^*_n$, $L^*_n$, and on the unipotent radical $U^*_n$, are given in [AH06, 2.2]. We often write $P^*$, $L^*$, $U^*$, $A^*$, $L^*$' instead of $P^*_n$, $L^*_n$, $U^*_n$, $A^*_n$, $L^*_n'$, respectively. The orders of the parabolic subgroups are

$$|P_n| = |P^*_n| = q^{m^2}(q - 1)(q^{2m-2} - 1)(q^{2m-4} - 1) \cdots (q^2 - 1).$$

3 Characters

In this section we collect some information on the ordinary characters of the groups $G$, $G^*$, $\tilde{G}^*$ and some of their parabolic subgroups. For the symplectic groups we largely follow [AH06] and [AH11].

3.1 General character theoretic notation

Let $K$ be a subgroup of a finite group $H$. We write $\text{Irr}(H)$ for the set of complex irreducible characters of $H$ and $1_H$ for the trivial character. Let $\langle \cdot, \cdot \rangle_H$ be the usual scalar product on the space of class functions of $H$. If $\chi$ is a character of $H$ we write $\chi|_K^H$ for the restriction of $\chi$ to $K$. If $\varphi$ is a character of $K$ we write $\varphi^K_H$ for the character of $H$ which is induced by $\varphi$. If $K \unlhd H$ and $\psi$ is a character of the factor group $H/K$ then we denote its inflation to $H$ by $\text{Infl}^H_{H/K}(\psi)$.

3.2 Characters of parabolic subgroups

We give a brief summary of the construction of the ordinary irreducible characters of the parabolic subgroup $P$ of $G$ in [HN]. For $n \geq 5$ the conjugation action of $P$ on $\text{Irr}(U)$ has four orbits and we choose a set of representatives $\{1_U, \lambda^0, \lambda^+, \lambda^-\}$ as in [HN]. The characters $\lambda^0$, $\lambda^+$, $\lambda^-$ are non-trivial linear characters and we say that $\chi \in \text{Irr}(P)$ is of Type 1 if it covers $1_U$ and it is of Type $\varepsilon$ if it covers $\lambda^\varepsilon$ for $\varepsilon \in \{0, +, -\}$. The irreducible characters of Type 1 are obtained from $\text{Irr}(L)$ via inflation, we set $\psi_{\sigma} := \text{Infl}^P_{L}(\sigma)$ for $\sigma \in \text{Irr}(L)$. The inertia subgroup in $P$ of the linear character $\lambda^0$ is $I^0 = P_{n-2} \ltimes U$ where $P_{n-2} \cong P_{n-2}$. We identify $\text{Irr}(P_{n-2})$ with $\text{Irr}(P_{n-2})$ via the bijection in [HN, 3.3] and extend $\lambda^0$ trivially to $\lambda^0 \in \text{Irr}(I^0)$. 

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The irreducible characters of Type 0 are parameterized by \( \text{Irr}(P_{n-2}) \) via

\[
0\psi_\mu := (\hat{\lambda}^0 \cdot \text{Infl}^0_{P_{n-2}}(\mu)) \uparrow_{P_{n-2}}^P \quad \text{for} \quad \mu \in \text{Irr}(P_{n-2}) = \text{Irr}(\tilde{P}_{n-2}).
\]

We say that \( \chi \in \text{Irr}(P) \) is of Type 0,\( \varepsilon \) if \( \chi = 0\psi_\mu \) for some \( \mu \in \text{Irr}(P_{n-2}) \) of Type \( \varepsilon \).

The inertia subgroup of \( \lambda^\pm \) in \( P \) is \( I^\pm = L_n^\pm \ltimes U \) where \( L_n^\pm \cong \text{GO}^\pm_{n-3}(q) \) and we extend \( \lambda^\pm \) trivially to \( \hat{\lambda}^\pm \in \text{Irr}(I^\pm) \). The irreducible characters of Type \( \pm \) are parameterized by \( \text{Irr}(\text{GO}^\pm_{n-3}(q)) = \text{Irr}(L_n^\pm) \) via

\[
\pm\psi_\vartheta := (\hat{\lambda}^\pm \cdot \text{Infl}^\pm_{L_n^\pm}(\vartheta)) \uparrow_{L_n^\pm}^P \quad \text{for} \quad \vartheta \in \text{Irr}(\text{GO}^\pm_{n-3}(q)).
\]

For \( n = 3 \) the orbits of \( \lambda^0 \) and \( \lambda^- \) do not exist and there is just one irreducible character of Type + of \( P_3 \), namely the one corresponding to the trivial character of \( \text{GO}^+_3(q) := \{1\} \).

Suppose that \( q \) is odd. The ordinary irreducible characters of the parabolic subgroup \( P^* \) of \( G^* \) were determined by An and Hiss in [AH06]. We choose a set of representatives \( \{1_U, \lambda, \rho_1, \rho_2\} \) for the four orbits of \( P^* \) on \( \text{Irr}(U^*) \) as in [AH06]; see also [AH11] Sections 4, 5) for the particular choice of \( \rho_1, \rho_2 \). The character \( \lambda \) is a non-trivial linear character while \( \rho_1(1) = \rho_2(1) = q^{m-1} \). Following [AH06], we say that \( \chi \in \text{Irr}(P^*) \) is of Type 1 if it covers \( 1_U \), it is of Type 2 if it covers \( \lambda \), and it is of Type 3 if it covers \( \rho_1 \) or \( \rho_2 \). The irreducible characters of Type 1 are obtained from \( \text{Irr}(L^*) \) via inflation, and we set \( 1\psi_\sigma := \text{Infl}^\star_{L^*}(\sigma) \) for \( \sigma \in \text{Irr}(L^*) \). The inertia subgroup of \( \lambda \) in \( P^* \) is \( I^2 = P^*_{n-2} \ltimes U^* \) where \( P^*_{n-2} \cong P^*_{n-2} \). We identify \( \text{Irr}(\tilde{P}^*_{n-2}) \) with \( \text{Irr}(P^*_{n-2}) \) via the bijection in [AH11] Section 2] and extend the linear character \( \lambda \) trivially to \( \hat{\lambda} \in \text{Irr}(I^2) \). The irreducible characters of Type 2 are parameterized by \( \text{Irr}(P^*_{n-2}) \) via

\[
2\psi_\mu := (\hat{\lambda} \cdot \text{Infl}^2_{P^*_{n-2}}(\mu)) \uparrow_{P^*_{n-2}}^P \quad \text{for} \quad \mu \in \text{Irr}(P^*_{n-2}) = \text{Irr}(\tilde{P}^*_{n-2}).
\]

We say that \( \chi \in \text{Irr}(P^*) \) is of Type 2,\( \varepsilon \) if \( \chi = 2\psi_\mu \) for some \( \mu \in \text{Irr}(P^*_{n-2}) \) of Type \( \varepsilon \).

The characters \( \rho_1 \) and \( \rho_2 \) have the same inertia subgroup \( I^3 = (L^* \ltimes Z^*) \ltimes U^* \) where \( Z^* := \langle -I_{n^*} \rangle \). For \( i = 1, 2 \) we choose an extension \( \hat{\rho}_i \) of the character \( \rho_i \) to \( I^3 \) as in [AH06] 2.3.3. For \( \vartheta \in \text{Irr}(L^*) = \text{Irr}(\text{Sp}_{n-2}(q)) \) we write \( \vartheta \cdot 1_{Z^*} \) for the trivial extension of \( \vartheta \) to \( L^* \ltimes Z^* \) and \( \vartheta \cdot 1_{Z^*} \) for the non-trivial extension. The irreducible characters of Type 3 are parameterized by \( \text{Irr}(\text{Sp}_{n-2}(q)) \) via

\[
3\psi_{\vartheta}^{i,\varepsilon} := (\hat{\rho}_i \cdot \text{Infl}^3_{L^* \ltimes Z^*}(\vartheta \cdot 1_{Z^*})) \uparrow_{I^3}^P \quad \text{where} \quad \vartheta \in \text{Irr}(\text{Sp}_{n-2}(q)), \ i = 1, 2, \ \varepsilon \in \{+, -\}.
\]

It is a slight abuse of notation to denote some of the inertia subgroups and irreducible characters of \( P \) and \( P^* \) by the same symbols (the characters of Type 1, for example). However, we will always make clear whether we are working with the orthogonal or the symplectic groups, so that there should be no confusion.
3.3 Characters of symplectic and orthogonal groups

In this section we fix notation for the ordinary unipotent irreducible characters of the groups $SO_{2m+1}(q)$, $Sp_{2m}(q)$ and $CSp_{2m}(q)$ for $m = 2, 3$. We also construct some non-unipotent ordinary characters of these groups as linear combinations of Deligne-Lusztig characters. When we speak of unipotent characters we always mean irreducible characters.

Let $q$ be an arbitrary prime power. Each unipotent character of the groups $SO_{2m+1}(q)$, $Sp_{2m}(q)$ and $CSp_{2m}(q)$ is labeled by a symbol $\Lambda$ or a triple $[\alpha, \beta, d]$ where $(\alpha, \beta)$ is a bipartition and $d$ is the defect of $\Lambda$; see [Car85, Section 13.8] and [HN, Section 7]. Each of the groups $SO_5(q)$, $Sp_4(q)$ and $CSp_4(q)$ has six and each of the groups $SO_7(q)$, $Sp_6(q)$ and $CSp_6(q)$ has twelve unipotent characters. Their degrees and the symbols and bipartitions labeling these characters are given in Tables 1 and 2. In these tables we use the abbreviations $\phi_1 := q - 1$, $\phi_2 := q + 1$, $\phi_3 := q^2 + q + 1$, $\phi_4 := q^2 + 1$ and $\phi_6 := q^2 - q + 1$. Often, we identify a unipotent character with its label $[\alpha, \beta, d]$.

The character tables of $Sp_4(q)$ and $CSp_4(q)$ were computed in [Eno72], [Shi82], [Sri68] and [Yam]. The character table of $CSp_6(q)$ for odd $q$ and $Sp_6(q)$ for even $q$ were determined in [Lüb93]; see also the CHEVIE library [GHL+96].

| Bipartition | Symbol | Degree | Bipartition | Symbol | Degree |
|-------------|--------|--------|-------------|--------|--------|
| [2, −, 1]   | (2)    | 1      | [1, 1, 1]   | (0 2)  | $\frac{1}{2}q\phi_2$ |
| [−, −, 3]   | (0 1 2) | $\frac{1}{2}q\phi_1^2$ | [−, 2, 1]  | (0 1 2) | $\frac{1}{2}q\phi_4$ |
| [1^2, −, 1] | (1 2 0) | $\frac{1}{2}q\phi_4$  | [−, 1^2, 1]| (0 1 2) | $q^4$    |

Table 1: Labels and degrees of the unipotent characters of $SO_5(q)$, $Sp_4(q)$, $CSp_4(q)$.

In the following we construct some non-unipotent irreducible characters of the groups $SO_5(q)$ and $SO_7(q)$ for odd $q$ as linear combinations of Deligne-Lusztig characters. These linear combinations will be used in Section 4 to derive relations between Brauer characters leading to lower bounds for decomposition numbers. In the remaining part of this section we always assume that $q$ is odd.

Suppose that $m = 2$ so that $G = SO_5(F)$ and $G^* = Sp_4(F)$. Since $(G, F)$ and $(G^*, F^*)$ are in duality and the center of $G$ is connected, the Lusztig series of the irreducible characters of $G = GF = SO_5(q)$ are parameterized by the $F^*$-stable semisimple conjugacy classes of $G^*$ or equivalently by the semisimple conjugacy classes of $G^* = G^{*F^*} = Sp_4(q)$. To construct some of these irreducible characters we proceed along the lines of [Lüb93, Section 4.1 and Section 7]. Recall that the action of the Frobenius map $F^*$ on the root system $\Phi^*$ of $G^*$ is trivial.

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We apply [Lüb93, Lemma 7.1] to construct the irreducible characters of \( SO \) coincide with the conjugacy classes of \( C \) trivial and the Dynkin type of \( G \) with respect to the maximal torus \( T \) and properties (ii) and (iii) imply that the root system \( \Psi \) induces an action of \( C \) that

\begin{align*}
\text{It follows from (i) that the element } & g \in G \text{ is } F^* \text{-stable. This implies that } x \in G \text{ such that } x^{-1}F^*(x) \in w_{212}^* \text{ and for this } x \text{ property (i) implies that } xg(t_1,t_2) \in G^*. The centralizer } C_{G^*}(g(t_1,t_2)) \text{ is a connected reductive group and properties (ii) and (iii) imply that the root system } \Psi^* \text{ of } C_{G^*}(g(t_1,t_2)) \text{ with respect to the maximal torus } T^* \text{ has type } A_1 \text{ and that the Weyl group } W_{C^*} \text{ of } C_{G^*}(g(t_1,t_2)) \text{ is generated by } w_1^*.
\end{align*}

Since \( xg(t_1,t_2) \) is \( F^* \)-stable its centralizer \( C_{G^*}(xg(t_1,t_2)) \) is also \( F^* \)-stable. This induces an action of \( F^* \) on \( C_{G^*}(g(t_1,t_2)) \), on \( \Psi^* \) and on \( W_{C^*} \) via the isomorphism \( C_{G^*}(g(t_1,t_2)) \to C_{G^*}(xg(t_1,t_2)) \), \( y \mapsto x y \). The action of \( F^* \) on \( \Psi^* \) and on \( W_{C^*} \) is trivial and the Dynkin type of \( C_{G^*}(xg(t_1,t_2))F^* \) is \( A_1 \). The \( F^* \)-conjugacy classes coincide with the conjugacy classes of \( W_{C^*} \) and are given by: \( C_1 = \{1\} \), \( C_2 = \{w_1^*\} \). We apply [Lüb93, Lemma 7.1] to construct the irreducible characters of \( SO_5(q) \)

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
\text{Bipartition} & \text{Symbol} & \text{Degree} & \text{Bipartition} & \text{Symbol} & \text{Degree} \\
\hline
[3, -1] & \left( \begin{array}{c} 3 \\ - \end{array} \right) & 1 & [1^2, 1, 1] & \left( \begin{array}{c} 12 \\ - \end{array} \right) & q^3 \phi_3 \phi_6 \\
\hline
[2, 1, 1] & \left( \begin{array}{c} 0 \ 3 \\ 1 \end{array} \right) & \frac{1}{2}q \phi_3 \phi_4 & [1, 1^2, 1] & \left( \begin{array}{c} 0 \ 1 \ 3 \\ 1 \end{array} \right) & \frac{1}{2}q^4 \phi_3 \phi_4 \\
\hline
[-, 3, 1] & \left( \begin{array}{c} 0 \ 1 \ 3 \\ 3 \end{array} \right) & \frac{1}{2}q \phi_4 \phi_6 & [-, 2, 1, 1] & \left( \begin{array}{c} 0 \ 1 \ 2 \\ - \end{array} \right) & \frac{1}{2}q^2 \phi_2 \phi_6 \\
\hline
[2, -1, 1] & \left( \begin{array}{c} 1 \ 3 \\ 0 \end{array} \right) & \frac{1}{2}q \phi_2 \phi_6 & [1, 3, -1] & \left( \begin{array}{c} 1 \ 2 \ 3 \\ 1 \end{array} \right) & q^2 \phi_2 \phi_6 \\
\hline
[1, -3] & \left( \begin{array}{c} 0 \ 1 \ 3 \\ - \end{array} \right) & \frac{1}{2}q \phi_1 \phi_3 & [-, 1, 3] & \left( \begin{array}{c} 0 \ 1 \ 2 \ 3 \\ 1 \end{array} \right) & \frac{1}{2}q^2 \phi_1 \phi_3 \\
\hline
[1, 2, 1] & \left( \begin{array}{c} 0 \ 2 \\ 2 \end{array} \right) & q^2 \phi_3 \phi_6 & [-, 1^2, 1] & \left( \begin{array}{c} 0 \ 1 \ 2 \ 3 \\ 1 \ 2 \ 3 \end{array} \right) & q^2 \\
\hline
\end{array}
\]

Table 2: Labels and degrees of the unipotent characters of \( SO_7(q), \text{Sp}_6(q), \text{CSp}_6(q) \).
in the Lusztig series labeled by \( g(t_1, t_2) \). The group \( W_C^* \) is isomorphic to the symmetric group \( S_2 \) and its character table is

|   | \( C_1 \) | \( C_2 \) |
|---|---|---|
| \( \phi_1 \) | 1 | 1 |
| \( \phi_2 \) | 1 | -1 |

We use the notation from [Lüb93, Lemma 7.1]. Since the action of the Frobenius map \( F^* \) on \( W_C^* \) is trivial we have \( W_C^{*F^*} = \text{Irr}(W_C^*) \) and \( \alpha_\phi = 1 \) for all \( \phi \in \text{Irr}(W_C^*) \). For \( w^* \in W_C^* \) we write \( R_{w^*} \) for the Deligne-Lusztig character of \( C_{G^*}(g(t_1, t_2))^{F^*} \) corresponding to the trivial character of a maximal torus obtained from \( T^* \) by twisting with \( w^* \). By [Lüb93, Lemma 7.1] the unipotent irreducible characters of \( C_{G^*}(g(t_1, t_2))^{F^*} \) are given by \( R_{\phi_1} = \varepsilon_1 \frac{1}{2}(R_1 + R_{w^*_1}) \) and \( R_{\phi_2} = \varepsilon_2 \frac{1}{2}(R_1 - R_{w^*_1}) \), where \( \varepsilon_1, \varepsilon_2 \) are complex roots of unity. Applying the Jordan decomposition of characters we obtain the irreducible characters

\[
\begin{align*}
\chi_1(t_1, t_2) &= \frac{\varepsilon_1'}{2}(R_{w^*_21}\cdot(t_1, t_2) + R_{w^*_21\cdot w_1}(t_1, t_2)), \\
\chi_2(t_1, t_2) &= \frac{\varepsilon_2'}{2}(R_{w^*_21}\cdot(t_1, t_2) - R_{w^*_21\cdot w_1}(t_1, t_2)),
\end{align*}
\]  

(4)

of \( G = \text{SO}_5(q) \), where \( \varepsilon_1', \varepsilon_2' \) are complex roots of unity. In (4) we write \( R_{w^*}(t_1, t_2) \) for the Deligne-Lusztig character of \( G \) which is labeled by the \( F^* \)-stable maximal torus \( T' \) of \( G \) which is obtained from \( T \) by twisting with \( w \in W \) and the linear character of \( T'^{F^*} \) corresponding to a fixed conjugate of \( g(t_1, t_2) \) in \( T'^{F^*}_{w^{-1}} \). Since \( R_{w^*}(t_1, t_2)(1) = R_{w^*}(1)(1) \) and since we can easily evaluate the Deligne-Lusztig characters \( R_{w^*}(1) \) at the identity element 1 with CHEVIE we get \( \varepsilon_1' = \varepsilon_2' = -1 \) and the degrees are \( \chi_1(t_1, t_2)(1) = 1 \cdot (q - 1)(q^2 + 1) \), \( \chi_2(t_1, t_2)(1) = q \cdot (q - 1)(q^2 + 1) \).

For \( t_1, t_2 \in F^x \) with \( t_1^{q+1} = t_2^{q+1} = 1 \), \( t_1, t_2 \neq \pm 1 \) and \( t_2 \neq t_1^{-1} \), we set

\[
g_{\text{reg}}(t_1, t_2) := h^*(t_1, t_2), \quad w_0 := w_1 w_2 w_1 w_2, \quad w_0^* := \delta(w_0)^{-1} = w_2^* w_1^* w_2^* w_1^*,
\]

so that \( w_0 \) and \( w_0^* \) are the longest elements of \( W, W^* \), respectively. Using (2) it is straightforward to see that \( w_0^{F^*} g_{\text{reg}}(t_1, t_2) = g_{\text{reg}}(t_1, t_2) \) and \( \alpha^*(g_{\text{reg}}(t_1, t_2)) \neq 1 \) for all \( \alpha^* \in \Phi^{*+} \). Thus, \( g_{\text{reg}}(t_1, t_2) \) is conjugate in \( G^* \) to some regular semisimple element in \( T_{w_0^{F^*}}^* \) where \( T_{w_0^{F^*}}^* \) is an \( F^* \)-stable maximal torus of \( G^* \) which is obtained from \( T^* \) by twisting with \( w_0^* \). As described in [Lüb93, Section 3], the element \( g_{\text{reg}}(t_1, t_2) \) corresponds to a linear character of \( T_{w_0^{F^*}}^* \) in general position. Hence, the corresponding Deligne-Lusztig character \( R_{w_0}(t_1, t_2) \) of \( G = G_{w_0}^{F^*} \) is irreducible up to sign. From CHEVIE we get that \( R_{w_0}(t_1, t_2)(1) = (q - 1)^2(q^2 + 1) \), thus

\[
\chi_3(t_1, t_2) := R_{w_0}(t_1, t_2) \in \text{Irr}(G).
\]  

(5)
now suppose that \( m = 3 \) so that \( G = \text{SO}_7(\mathbb{F}) \) and \( G^* = \text{Sp}_6(\mathbb{F}) \). we still assume that \( q \) is odd. similar to the case \( m = 2 \) we use the methods from \([\text{L"ub}93]\) section 4.1 and section 7 to construct irreducible characters of \( G = G^F = \text{SO}_7(q) \). we define elements of the Weyl group \( W \) by:

\[
 w_9 := w_1w_2w_1w_3w_2w_1, \quad w_{13} := w_2w_1w_3w_2w_1w_3w_2, \quad w_{23} := w_2w_1w_3w_2w_1w_3w_2w_3.
\]

Furthermore, let \( w_{32} \) be the longest element of \( W \) and set \( w_j^* := \delta(w_j)^{-1} \in W^* \) for \( j = 9, 13, 23, 32 \). additionally, we define semisimple elements of \( G^* \) by

1. \( g_0(t) := h^*(t, t, t) \) for \( t \in \mathbb{F}^* \) with \( t^{q^2+1} = 1 \), \( t \neq \pm 1 \),
2. \( g_{13}(t_1, t_2) := h^*(t_1, t_2, t_2) \) for \( t_1, t_2 \in \mathbb{F}^* \) with \( t_1 = \pm 1 \), \( t_2^{q^2+1} = 1 \), \( t_2 \neq \pm 1 \),
3. \( g_{23}(t_1, t_2, t_3) := h^*(t_1, t_2, t_3) \) for \( t_1, t_2, t_3 \in \mathbb{F}^* \) with \( t_1 = \pm 1 \), \( t_2^{q^2+1} = t_3^{q^2+1} = 1 \), \( t_2 \neq \pm 1 \), \( t_3 \neq \pm 1 \),
4. \( g_{32}(t_1, t_2, t_3) := h^*(t_1, t_2, t_3) \) for \( t_1, t_2, t_3 \in \mathbb{F}^* \) with \( t_1^{q^2+1} = t_2^{q^2+1} = t_3^{q^2+1} = 1 \), \( t_i \neq t_j^{q^2+1} \) for all \( i, j = 1, 2, 3 \),

where the notation is motivated by the analogy with \([\text{L"ub}93] \) table 17. for each element \( g_0(t) \), where \( t \) satisfies the conditions in 1), we construct characters \( \chi_{9,1}(t), \chi_{9,2}(t), \chi_{9,3}(t) \in \text{Irr}(G) \) which can be written as rational linear combinations of Deligne-Lusztig characters as follows:

\[
\chi_{9,1}(t) = \frac{1}{6}(R_{w_9w_2w_3w_2}(t) + 3R_{w_9}(t) + 2R_{w_9w_2}(t)),
\]

\[
\chi_{9,2}(t) = \frac{1}{6}(2R_{w_9w_2w_3w_2}(t) - 2R_{w_9w_2}(t)),
\]

\[
\chi_{9,3}(t) = -\frac{1}{6}(R_{w_9w_2w_3w_2}(t) - 3R_{w_9}(t) + 2R_{w_9w_2}(t)).
\]

By \( R_w(t) \) we mean the Deligne-Lusztig character of the group \( G = \text{SO}_7(q) \) which is labeled by the \( F \)-stable maximal torus \( T_w \) of \( G \) which is obtained from \( T \) by twisting with \( w \in W \) and the linear character of \( T_w^F \) corresponding to a fixed conjugate of \( g_0(t) \) in \( T_w^{F^*} \). the degrees of \( \chi_{9,1}(t), \chi_{9,2}(t), \chi_{9,3}(t) \) are \( q^2\phi_3^2\phi_4, q^3\phi_3^2\phi_4, q^3\phi_3^2\phi_4 \), respectively.

Similarly, for each element \( g_{13}(t_1, t_2) \), where \( t_1, t_2 \) satisfy the conditions in 2), there are characters \( \chi_{13,1}(t_1, t_2), \chi_{13,2}(t_1, t_2), \chi_{13,3}(t_1, t_2), \chi_{13,4}(t_1, t_2) \in \text{Irr}(G) \) which
can be written as linear combinations of Deligne-Lusztig characters as follows:

\[
\begin{align*}
\chi_{13,1}(t_1, t_2) &= -\frac{1}{4}(R_{w_{13}}(t_1, t_2) + R_{w_{13}w_1}(t_1, t_2) + R_{w_{13}w_3}(t_1, t_2) + R_{w_{13}w_1w_3}(t_1, t_2)), \\
\chi_{13,2}(t_1, t_2) &= -\frac{1}{4}(R_{w_{13}}(t_1, t_2) + R_{w_{13}w_1}(t_1, t_2) - R_{w_{13}w_3}(t_1, t_2) - R_{w_{13}w_1w_3}(t_1, t_2)), \\
\chi_{13,3}(t_1, t_2) &= -\frac{1}{4}(R_{w_{13}}(t_1, t_2) - R_{w_{13}w_1}(t_1, t_2) + R_{w_{13}w_3}(t_1, t_2) - R_{w_{13}w_1w_3}(t_1, t_2)), \\
\chi_{13,4}(t_1, t_2) &= -\frac{1}{4}(R_{w_{13}}(t_1, t_2) - R_{w_{13}w_1}(t_1, t_2) - R_{w_{13}w_3}(t_1, t_2) + R_{w_{13}w_1w_3}(t_1, t_2)).
\end{align*}
\]

The degrees of the characters \(\chi_{13,1}(t_1, t_2), \chi_{13,2}(t_1, t_2), \chi_{13,3}(t_1, t_2), \chi_{13,4}(t_1, t_2)\) are \(\phi_1\phi_5\phi_4\phi_6, q\phi_2\phi_3\phi_4\phi_6, q\phi_1\phi_3\phi_4\phi_6, q^2\phi_1\phi_3\phi_4\phi_6\), respectively.

Furthermore, for each element \(g_{23}(t_1, t_2, t_3)\), where \(t_1, t_2, t_3\) satisfy the conditions in 3), there are characters \(\chi_{23,1}(t_1, t_2, t_3), \chi_{23,2}(t_1, t_2, t_3) \in \text{Irr}(G)\) which can be written as rational linear combinations of Deligne-Lusztig characters as follows:

\[
\begin{align*}
\chi_{23,1}(t_1, t_2, t_3) &= \frac{1}{2}(R_{w_{23}}(t_1, t_2, t_3) + R_{w_{23}w_1}(t_1, t_2, t_3)), \\
\chi_{23,2}(t_1, t_2, t_3) &= \frac{1}{2}(R_{w_{23}}(t_1, t_2, t_3) - R_{w_{23}w_1}(t_1, t_2, t_3)).
\end{align*}
\]

The degrees of \(\chi_{23,1}(t_1, t_2, t_3), \chi_{23,2}(t_1, t_2, t_3)\) are \(\phi_1^2\phi_3\phi_4\phi_6, q^2\phi_2\phi_3\phi_4\phi_6\), respectively.

Finally, for each element \(g_{32}(t_1, t_2, t_3)\), where \(t_1, t_2, t_3\) satisfy the conditions in 4), the character

\[
\chi_{32}(t_1, t_2, t_3) := -R_{w_{32}}(t_1, t_2, t_3)
\]

is an irreducible character of \(G\) of degree \(\phi_1^2\phi_3\phi_4\phi_6\), where \(R_{w_{32}}(t_1, t_2, t_3)\) is the Deligne-Lusztig character corresponding to the maximal torus obtained from \(T\) by twisting with the longest element \(w_{32}\) and a linear character in general position.

We only sketch the construction of \(\chi_{9,1}(t), \chi_{9,2}(t)\) and \(\chi_{9,3}(t)\). The construction of the remaining characters is similar and easier. Let \(t \in \mathbb{F}^\times\) with \(t^{q+1} = 1, t \neq \pm 1\). We set \(\Pi_9^* := \{\alpha_2^*, \alpha_3^*\}\) and \(w_9^* := \delta(w_9)^{-1} = w_1^*w_3^*w_2^*w_1^*w_3^*w_2^*w_1^*\), and we write \(\Psi_9^*\) for the closed subset of \(\Phi^*\) generated by \(\Pi_9^*\). Using (2) it is easy to see that

(i) \(w_9^*F^*(g_9(t)) = g_9(t)\),

(ii) \(\alpha^*(g_9(t)) = 1\) for all \(\alpha^* \in \Pi_9^*\), and

(iii) \(\alpha^*(g_9(t)) \neq 1\) for all \(\alpha^* \in \Phi^* \setminus \Psi_9^*\).

As for \(m = 2\), it follows from (i) that the element \(g_9(t)\) is conjugate in \(G^*\) to some element of \(G^* = F^*F^* \subseteq G^*\). More specifically: By the surjectivity of the Lang map there is \(x \in G^*\) such that \(x^{-1}F^*(x) \subseteq w_9^*\) and for this \(x\) property (i) implies that \(xg_9(t) \subseteq G^*\). The centralizer \(C_G.(g_9(t))\) is a connected reductive group and
(ii), (iii) imply that the root system $\Psi_9^*$ of $C_{G^*}(g_9(t))$ with respect to $T^*$ has type $A_2$ and that the Weyl group $W_9^*$ of $C_{G^*}(g_9(t))$ is generated by $\{w_2^*, w_3^*\}$. The action of $F^*$ on $\Psi_9^*$ is given by $\alpha_2^* \mapsto \alpha_3^*$ and $\alpha_3^* \mapsto \alpha_5^*$ and the action of $F^*$ on $W_9^*$ is given by $w_2^* \mapsto w_3^*$ and $w_3^* \mapsto w_2^*$. The Dynkin type of $C_{G^*}(g_9(t))^{F^*}$ is $2A_2$ and the $F^*$-conjugacy classes of $W_9^*$ are:

$$C_1 := \{1, w_2^*w_3^*, w_3^*w_2^*\}, \quad C_2 := \{w_2^*w_3^*w_2^*\} \quad \text{and} \quad C_3 := \{w_2^*, w_3^*\}.$$ 

The group $W_9^*$ is isomorphic to the symmetric group $S_3$ and its conjugacy classes are $C_1 = \{1\}$, $C_2 = \{w_2^*, w_3^*, w_2^*w_3^*w_2^*\}$ and $C_3 = \{w_2^*w_3^*, w_3^*w_2^*\}$. The character table of $W_9^*$ is

|     | $C_1$ | $C_2$ | $C_3$ |
|-----|------|------|------|
| $\phi_1$ | 1 | 1 | 1 |
| $\phi_2$ | 2 | 0 | -1 |
| $\phi_3$ | 1 | -1 | 1 |

The Frobenius map $F^*$ acts on $W_9^*$ in the same way as the inner automorphism induced by the element $w_0^* = w_2^*w_3^*w_2^*$. Thus, in the notation of $[Lüb93$ Lemma 7.1] we have $W_9^*F^* = \text{Irr}(W_9^*)$ and $\alpha_{\phi} = w_0^*$ for all $\phi \in \text{Irr}(W_9^*)$. For $w^* \in W_9^*$ we write $R_{w^*}$ for the Deligne-Lusztig character of $C_{G^*}(g_9(t))^{F^*}$ corresponding to the trivial character of a maximal torus obtained from $T^*$ by twisting with $w^*$. By $[Lüb93$ Lemma 7.1] the unipotent irreducible characters of $C_{G^*}(g_9(t))^{F^*}$ are

$$R_{\phi_1} = \varepsilon_1 \frac{1}{6}(R_{w_0^*} + 3R_1 + 2R_{w_2^*}), \quad R_{\phi_2} = \varepsilon_2 \frac{1}{6}(2R_{w_0^*} - 2R_{w_2^*}),$$

$$R_{\phi_3} = \varepsilon_3 \frac{1}{6}(R_{w_0^*} - 3R_1 + 2R_{w_2^*}),$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \mathbb{C}$ are roots of unity. Applying the Jordan decomposition of characters we get:

$$\chi_{9,1}(t) = \frac{\varepsilon_1'}{6}(R_{w_0w_2w_3w_2}(t) + 3R_{w_0}(t) + 2R_{w_2w_2}(t)),$$

$$\chi_{9,2}(t) = \frac{\varepsilon_2'}{6}(2R_{w_0w_2w_3w_2}(t) - 2R_{w_2w_2}(t)),$$

$$\chi_{9,3}(t) = \frac{\varepsilon_3'}{6}(R_{w_0w_2w_3w_2}(t) - 3R_{w_0}(t) + 2R_{w_2w_2}(t)),$$

where $\varepsilon_1', \varepsilon_2', \varepsilon_3' \in \mathbb{C}$ are roots of unity. Since $R_w(t)(1) = R_w(1)(1)$ and since we can easily evaluate the Deligne-Lusztig characters $R_w(1)$ at the identity element 1 with CHEVIE we get $\varepsilon_1' = \varepsilon_2' = 1, \varepsilon_3' = -1$ and also $\chi_{9,j}(t)(1)$ for $j = 1, 2, 3$. 

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4 Decomposition numbers

Let $q$ be an arbitrary prime power and $\ell$ an odd prime not dividing $q$, but dividing the group orders $|G| = |G^*| = q^9(q^6 - 1)(q^4 - 1)(q^2 - 1)$. This section is devoted to the proof of the $\ell$-modular decomposition numbers of the unipotent characters for both $G = \text{SO}_7(q)$ and $G^* = \text{Sp}_6(q)$. Let $(K, O, k)$ be an $\ell$-modular splitting system for $G$, $G^*$ and all their subgroups.

If $\ell \nmid q \pm 1$ then all unipotent blocks of $G$ and $G^*$ are cyclic, and we may refer to [FS90] for odd $q$, and to [Whi00] for even $q$ to obtain their decomposition numbers. Note that Remark 2.1 gives an explicit isomorphism $\text{SO}_7(q) \to \text{Sp}_6(q)$ for even $q$. Therefore we only need to consider odd primes $\ell$ dividing $q + 1$ or $q - 1$. The case of $\ell | q - 1$ is readily solved with the help of theorems by Puig, Gruber and Hiss, see Remark 4.1. Hence, we get new results only in the case of $\ell | q + 1$. This case is the topic of this entire section.

Remark 4.1. The decomposition numbers for $\ell | q - 1$ may be computed as follows: By [FS90] for odd $q$ (see also [HK00, Sections 2 and 6]) and [Whi00] for even $q$ the distribution of the ordinary unipotent characters of $G$ and $G^*$ into blocks coincides with the distribution into Harish-Chandra series. More precisely: There are two unipotent blocks: the principal block containing the unipotent characters in the principal series, i.e. those whose symbol has defect 1, and one block containing the unipotent characters $[1, -, 3]$ and $[-, 1, 3]$ whose symbols have defect 3.

If $\ell > 3$ then $\ell$ does not divide the order of the Weyl group and the decomposition numbers of $G$ and $G^*$ follow from a result by Puig [Pui90]. If $\ell = 3$ then we may employ [GH97a, Theorem 4.13] (see also [GH97b]) to immediately infer the decomposition matrices of both blocks with the help of the decomposition matrices of the general linear groups, see [Jam90]. In fact, the decomposition numbers of the unipotent characters of $G$ and $G^*$ for $\ell | q - 1$ coincide and they do not depend on whether $q$ is odd or even; hence they can be read off from the data given in [Whi00, Theorem 2.1].

Hypothesis 4.2. From now on until the end of this paper, we assume that $\ell$ is an odd prime dividing $q + 1$.

The following theorem is the main result of this paper. We write $(q + 1)\ell$ for the largest power of $\ell$ dividing $q + 1$.

Theorem 4.3. For all prime powers $q$ the $\ell$-modular decomposition numbers of the unipotent characters in the principal block of $G = \text{SO}_7(q)$ and $G^* = \text{Sp}_6(q)$ are given in Table 3, where the decomposition numbers $\alpha, \beta, \gamma$ are as follows:

(a) If $(q + 1)\ell = 3$ then $\alpha = \beta = \gamma = 1$.

(b) If $(q + 1)\ell = 5$ then $\alpha = \beta = \gamma = 2$. 

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(c) If \((q + 1)_\ell > 5\) then \(\alpha = \gamma = 2\) and \(\beta = 3\).

\[
\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}
\hline
& \varphi_1 & \varphi_2 & \varphi_3 & \varphi_4 & \varphi_5 & \varphi_6 & \varphi_7 & \varphi_8 & \varphi_9 & \varphi_{10} \\
\hline
ps & ps & A_1 & [B_2, \eta] & A_1 \times A'_1 & A'_1 & [B_2, \text{St}] & c & c & c \\
\hline
[3, -1] & 1 & . & . & . & . & . & . & . & . & . \\
[2, 1, 1] & 1 & 1 & . & . & . & . & . & . & . & . \\
[-3, 1] & 1 & . & 1 & . & . & . & . & . & . & . \\
[1, -3] & . & . & 1 & . & . & . & . & . & . & . \\
[1, 2, 1] & 1 & 1 & 1 & . & . & . & . & . & . & . \\
[1, 2, 1] & 1 & 1 & . & 1 & 1 & 1 & . & . & . & . \\
[1, 2, 1] & 1 & 1 & 1 & \alpha & 1 & 1 & 1 & . & . & . \\
[1, -3] & 1 & . & . & 1 & . & 1 & . & . & . & . \\
[-1, 3] & . & . & 1 & . & . & . & . & . & . & . \\
[-1, 1] & 1 & 1 & 1 & \alpha & 1 & 1 & 1 & \beta & \gamma & 1 \\
\hline
\end{array}
\]

Table 3: The \(\ell\)-modular decomposition numbers of the unipotent characters in the principal blocks of \(\text{SO}_7(q)\) and \(\text{Sp}_6(q)\).

Remark 4.4. (a) For \((q + 1)_\ell > 5\) the decomposition numbers in Table 3 and the statement in part (c) of Theorem 4.3 were obtained independently by Olivier Dudas and Gunter Malle (private communication).

(b) For \(G^* = \text{Sp}_6(q)\) the decomposition numbers in Table 3 were already computed by An, Hiss [AH06] and Köhler [K06] for odd \(q\) and White [Whi00] for even \(q\) except for the entries \(\beta\) and \(\gamma\), for which they proved lower and upper bounds.

(c) By [FS89] and [Whi00] the characters \([21, -1, 1]\) and \([-21, 1]\) are contained in a cyclic block and do not belong to the principal block; see [FS90], [Whi00] for the Brauer tree of this cyclic block.

(d) For each of the groups \(G = \text{SO}_7(q)\) and \(G^* = \text{Sp}_6(q)\) there are ten irreducible Brauer characters \(\varphi_1, \varphi_2, \ldots, \varphi_{10}\) in the principal block. The second row of Table 3 lists for each \(\varphi_j\) the modular Harish-Chandra series containing \(\varphi_j\); see [GH97a, Section 2] for a definition of modular Harish-Chandra series.

We write \(ps\) for the principal series and \(c\) for cuspidal Brauer characters. The standard Levi subgroups corresponding to the sets \(\{\alpha_1^{(s)}\}, \{\alpha_2^{(s)}\}\) and \(\{\alpha_1^{(s)}, \alpha_3^{(s)}\}\) of simple roots each have a unique cuspidal unipotent Brauer character, namely the modular Steinberg character. We denote the corresponding modular Harish-Chandra series of \(G\) and \(G^*\) by \(A_1\), \(A'_1\) and \(A_1 \times A'_1\).
Furthermore, we will see in Section 4.3 that the Levi subgroup $L^*$ has exactly two cuspidal unipotent Brauer characters: the restriction of the cuspidal ordinary unipotent character to the $\ell$-regular elements and the modular Steinberg character. We write $[B_2, \eta]$ and $[B_2, \text{St}]$ for the corresponding modular Harish-Chandra series of $G$ and $G^*$.

(e) For each of the groups $G = \text{SO}_7(q)$ and $G^* = \text{Sp}_6(q)$ the set of ordinary unipotent characters is partitioned into six families; see the information available in the GAP-part of CHEVIE [GAP13], [GHL+96]. The distribution into families is indicated by the dotted lines in the leftmost column of Table 3.

(f) Theorem 4.3 confirms [GH97a, Conjecture 3.4] in the special case of $\text{SO}_7(q)$ and $\text{Sp}_6(q)$ for all prime powers $q$. We will see later in Theorem 4.9 that the conjecture also holds for the groups $\text{SO}_5(q)$ and all prime powers $q$.

In Section 4.1 we derive some relations for the Brauer characters of $G$ and $G^*$. Section 4.2 gives some cyclic blocks of the parabolic subgroups $P$ and $P^*$. In order to find a good initial approximation for the decomposition matrices of $G$ and $G^*$, we compute as a first step the decomposition matrix of $\text{SO}_5(q)$ in Section 4.3. It turns out that it coincides with the decomposition matrix of $\text{Sp}_4(q)$ in [OW98]. The approximation is given in Section 4.4, in which we also show that almost all entries are decomposition numbers. Finally, in Section 4.5 we dispel the last ambiguities in the decomposition matrices and finish the proof of Theorem 4.3.

To eliminate these ambiguities we apply two approaches: for $(q+1)\ell > 5$ we make use of a recent result by Dudas (see [Dud13]) exploiting some deep results of Deligne-Lusztig theory to obtain information on projective modules for $G$ and $G^*$ which give upper bounds for the decomposition numbers. By providing suitable relations of Brauer characters, we can show that these bounds are tight. For small $(q + 1)\ell$ however, we need to find better upper bounds than the ones Deligne-Lusztig theory provides. Hence for $(q + 1)\ell \leq 5$ we consider the above-mentioned cyclic blocks of the parabolic subgroups $P$ and $P^*$. Again, we show that the upper bounds thus obtained are tight by providing suitable character relations.

4.1 Relations

In this section we always assume that $q$ is an odd prime power. Let $H$ be one of the groups $G = \text{SO}_7(q)$, $G^* = \text{Sp}_6(q)$, $G^* = \text{CSp}_6(q)$, $G_5 = \text{SO}_5(q)$ or $G_5^* = \text{Sp}_4(q)$. For a class function $\psi$ of $H$ we write $\tilde{\psi}$ for the restriction of $\psi$ to the $\ell$-regular elements of $H$. Since $\ell$ is odd the prime $\ell$ is good for the groups $G$, $G^*$ and $G^*$ in the sense of [Car85, p. 28]. A general result of Geck and Hiss [GH91] implies that $\{\tilde{\chi} \mid \chi \in \text{Irr}(H) \text{ unipotent}\}$ is a basic set for the unipotent blocks of $H$. In this section we write certain Brauer characters as $\mathbb{Z}$-linear combinations of the above
basic sets. The relations obtained in this way will be used in Sections 4.3-4.5 to determine lower bounds for certain decomposition numbers of $H$. We denote a unipotent character of $H$ by the bipartition labeling it; see Section 3.3.

**Lemma 4.5.** Let $q$ be odd.

(a) There are ordinary characters $\chi_1, \chi_2$ of $SO_5(q)$ such that

$$
\begin{align*}
\tilde{\chi}_1 &= -[2, -1]^\gamma - [-, -3]^\gamma + [-, 2, 1]^\gamma; \\
\tilde{\chi}_2 &= -[-, -3]^\gamma - [1^2, -1]^\gamma + [-, 1^2, 1]^\gamma;
\end{align*}
$$

(b) If $(q + 1)\ell > 3$ then there is an ordinary character $\chi_3$ of $SO_5(q)$ such that

$$
\tilde{\chi}_3 = [2, -1]^\gamma - 2 \cdot [-, -3]^\gamma - [1^2, -1]^\gamma + [-, 1^2, 1]^\gamma.
$$

(c) Let $H \in \{SO_7(q), Sp_6(q)\}$. There are ordinary characters $\chi_{9,i}$ and $\chi_{13,j}$ of $H$ for $i = 1, 2, 3$ and $j = 1, 2, 3, 4$ such that

$$
\begin{align*}
\tilde{\chi}_{9,1} &= [3, -1]^\gamma - [2, 1, 1]^\gamma - [-, 3, 1]^\gamma + [1, 2, 1]^\gamma; \\
\tilde{\chi}_{9,2} &= -[-, 3, 1]^\gamma - [1, -3]^\gamma + [1, 2, 1]^\gamma - [1^2, 1, 1]^\gamma + [1^3, -1]^\gamma + [-, 1, 3]^\gamma, \\
\tilde{\chi}_{9,3} &= [1^2, 1, 1]^\gamma - [1^2, 1, 1]^\gamma - [1^3, -1]^\gamma + [-, 1^3, 1]^\gamma; \\
\tilde{\chi}_{13,1} &= [-3, -1]^\gamma + [-, 3, 1]^\gamma + [1, -3]^\gamma - [1, 2, 1]^\gamma + [1^2, 1, 1]^\gamma; \\
\tilde{\chi}_{13,2} &= -[-, 3, 1]^\gamma - [1, -3]^\gamma - [1^2, 1, 1]^\gamma + [1^2, 1, 1]^\gamma + [1^3, -1]^\gamma; \\
\tilde{\chi}_{13,3} &= [-2, 1, 1]^\gamma - [-, 3, 1]^\gamma + [1, 2, 1]^\gamma + [1^3, -1]^\gamma + [-, 1, 3]^\gamma; \\
\tilde{\chi}_{13,4} &= -[1, 2, 1]^\gamma + [1^2, 1, 1]^\gamma - [1^3, -1]^\gamma + [-, 1, 3]^\gamma + [-, 1^3, 1]^\gamma.
\end{align*}
$$

(d) Let $H \in \{SO_7(q), Sp_6(q)\}$. If $(q + 1)\ell > 3$ then there are ordinary characters $\chi_{23,1}, \chi_{23,2}$ of $H$ such that

$$
\begin{align*}
\tilde{\chi}_{23,1} &= [3, -1]^\gamma - 2 \cdot [-, 3, 1]^\gamma - 2 \cdot [1, -3]^\gamma + [1, 2, 1]^\gamma - 2 \cdot [1^2, 1, 1]^\gamma \\
&\quad + [1, 1^2, 1]^\gamma + [1^3, -1]^\gamma; \\
\tilde{\chi}_{23,2} &= [2, 1, 1]^\gamma + [-, 3, 1]^\gamma - 2 \cdot [1, 2, 1]^\gamma + [1^2, 1, 1]^\gamma - 2 \cdot [1^3, -1]^\gamma \\
&\quad - 2 \cdot [-, 1, 3]^\gamma + [-, 1^3, 1]^\gamma.
\end{align*}
$$

(e) Let $H \in \{SO_7(q), Sp_6(q)\}$. If $(q + 1)\ell > 5$ then there is an ordinary character $\chi_{32}$ of $H$ such that

$$
\begin{align*}
\tilde{\chi}_{32} &= [-3, -1]^\gamma + [2, 1, 1]^\gamma + 3 \cdot [-, 3, 1]^\gamma + 2 \cdot [1, -3]^\gamma - 3 \cdot [1, 2, 1]^\gamma \\
&\quad + 3 \cdot [1^2, 1, 1]^\gamma - [1, 1^2, 1]^\gamma - 3 \cdot [1^3, -1]^\gamma - 2 \cdot [-, 1, 3]^\gamma + [-, 1^3, 1]^\gamma.
\end{align*}
$$
Proof. (a): Let $H := \text{SO}_5(q)$. In the notation of Section 5.3 there is $t_2 \in \mathbb{F}^\times$ with $t_2^{p+1} = 1$, $t_2 \neq \pm 1$ such that $g(1, t_2)$ is an $\ell$-element. Now we invoke [DM91, Proposition 12.6]: Let $f$ denote the characteristic function on $\ell$-regular classes, i.e. for $x \in H$ we set $f(x) := 1$ if and only if $\ell \nmid |x|$, and $f(x) := 0$ otherwise. Note that $f \in C(G)_\ell$, i.e. is $p$-constant as $\ell \neq p$, so $f(x) = f(x')$. We therefore obtain that $R_w(1, t_2)^\sim = R_{\tilde{w}}(1)^\sim$ for $w \in \{w_{212}, w_{212}w_1\}$. Using CHEVIE we decompose $R_w(1)^\sim, R_{w_{212}w_1}(1)^\sim$ into a $\mathbb{Z}$-linear combination of $\{\tilde{\chi} \mid \chi \in \text{Irr}(H)\}$ unipotent characters. It follows from (i) and from $\varepsilon'_1 = \varepsilon'_2 = -1$ that $\chi_1 := \chi_1(1, t_2)$ and $\chi_2 := \chi_2(1, t_2)$ satisfy the equations in (a).

(b): If $(q + 1)/\ell > 3$ then there are $t_1, t_2 \in \mathbb{F}^\times$ with $t_1^{p+1} = t_2^{p+1} = 1$, $t_1, t_2 \neq \pm 1$ and $t_2 \neq t_1^{-1}$ such that $g_{\text{reg}}(t_1, t_2)$ is an $\ell$-element. In the same way as in (a) we can derive from (5) that $\chi_3 := \chi_3(t_1, t_2)$ satisfies the equation in (b).

(c),(d),(e): Suppose first that $H = \text{SO}_7(q)$. As in the proof of (a), (b) we can use equations (6)-(9), [DM91, Proposition 12.6] and CHEVIE to see that $\chi_{9,j} := \chi_{9,j}(t)$ for $j = 1, 2, 3$, $\chi_{13,j} := \chi_{13,j}(t_1, t_2)$ for $j = 1, 2, 3, 4$, $\chi_{23,j} := \chi_{23,j}(t_1, t_2, t_3)$ for $j = 1, 2$ and $\chi_{32} := \chi_{32}(t_1, t_2, t_3)$ for suitable choices of the field elements $t, t_1, t_2, t_3$ satisfy the equations in (c),(d),(e).

Now suppose that $H = \text{Sp}_6(q)$ so that $H = G^*$ is a subgroup of $\tilde{G}^* = \text{CSp}_6(q)$. The character table of $\tilde{G}^*$ was computed by Lübeck [Lüb93] and is contained in the CHEVIE library. It follows from [DM91, p. 140], [Bon06, Theorem 11.12] and the degrees of the unipotent characters of $G^*$ and $\tilde{G}^*$ that restriction induces a bijection between the set of unipotent characters of $G^*$ and the set of unipotent characters of $\tilde{G}^*$. If $\chi_{\Lambda}$ is a unipotent character of $G^*$ with the label $\Lambda$ we denote its unipotent extension to $\tilde{G}^*$ also by $\chi_{\Lambda}$ or just $\Lambda$. Using the notation from [Lüb93] and the explicit knowledge of the character table of $G^*$ we get

$$(\chi_{9,1}(k_1, k_2)|_{\tilde{G}^*})^\sim = ([3, -1]|_{\tilde{G}^*})^\sim - ([2, 1, 1]|_{\tilde{G}^*})^\sim = ([3, -1] - [2, 1, 1] - [3, 1] + [1, 2, 1])^\sim$$

for a suitable choice of the parameters $k_1, k_2$. Thus, with this choice of $k_1$ and $k_2$, the character $\chi_{9,1} := \chi_{9,1}(k_1, k_2)$ satisfies the first equation in (c). Similarly, $\chi_{9,j} := \chi_{9,j}(k_1, k_2)|_{\tilde{G}^*}$ for $j = 2, 3$, $\chi_{13,j} := \chi_{13,j}(k_1, k_2)|_{\tilde{G}^*}$ for $j = 1, 2, 3, 4$, $\chi_{23,j} := \chi_{23,j}(k_1, k_2, k_3)|_{\tilde{G}^*}$ for $j = 1, 2$, and finally $\chi_{32} := \chi_{32}(k_1, k_2, k_3, k_4)|_{\tilde{G}^*}$ for suitable parameters $k_1, k_2, k_3, k_4$ satisfy the remaining equations in (c),(d),(e).

Remark 4.6. The proof of Lemma 4.5 is constructive in the sense that it gives explicit descriptions of the ordinary characters $\chi_i$ and $\chi_{i,j}$ on the left hand side of the equations in (a)-(e).
4.2 Cyclic blocks of parabolic subgroups

In this section we determine certain cyclic blocks of the maximal parabolic subgroups $P = P_7$ of $G = \text{SO}_7(q)$ and $P^* = P_6^*$ of $G^* = \text{Sp}_6(q)$. Owing to the simple structure of their projective indecomposable modules, these will serve our purposes twofold: via induction they provide a source of projective modules for an approximation of the decomposition matrices of their overgroups, while also simultaneously giving small bounds for the decomposition numbers.

In Section 4.3 we calculate the decomposition numbers of the principal block of $G_5 = \text{SO}_5(q)$ to later use this information for $G = \text{SO}_7(q)$. Therefore, we begin this section by determining cyclic blocks of the parabolic subgroup $P_5$.

To enable a concise statement of the following assertions, we need to introduce some notation. Recall from Section 3.2 that the inertia subgroup $I_\lambda$ of $\lambda$ in $P_n$ decomposes as $L_\lambda \rtimes U_n$ where $L_\lambda \cong \text{GO}_{n-3}(q)$. If $n = 5$ we have $L_5 \cong \text{GO}_2(q)$, which is a dihedral group of order $2(q + 1)$; see [Tay92, Theorem 11.4]. For odd $q$, let $\nu_\lambda$ be the non-trivial linear character of $L_5$ with $\text{SO}_2(q) \leq \ker(\nu_\lambda)$; see [HN, Remark 7.1] for additional information. If $q$ is even then $L_5$ only possesses two linear characters, and in this case we write $\nu_1$ for the unique non-trivial linear character of $L_5$.

Lemma 4.7. Let $P_5$ be the parabolic subgroup of $G_5 = \text{SO}_5(q)$ defined in Section 2.2. For all prime powers $q$ there is a cyclic block $b_5$ of $P_5$ whose Brauer tree is given in Table 4 with $\xi_1 := -\psi_1$, $\xi_2 := -\psi_{\nu_1}$, and $\xi_{\text{exc}} := -\psi_{\Xi}$, where $\Xi$ is the sum of all irreducible characters of degree 2 in the principal block of $L_5$.

$$\xi_1 \quad \xi_{\text{exc}} \quad \xi_2$$

Table 4: The Brauer tree of cyclic blocks of $P_5$, $P_7$, and $P_6^*$.

Proof. This is an application of Fong-Reynolds correspondence. Let $\xi \in \text{Irr}(U_5)$, then $\xi$ is the only ordinary character in its block $b$, since $U_5$ is an $\ell'$-group. By [Fon61, 2B] there is a bijection of blocks, preserving defect groups and decomposition matrices, between the blocks $\text{Bl}(P_5 | b)$ covering $b$, and the blocks of the inertia subgroup of $b$ covering $b$. We take $\xi := \lambda$ as in Section 3.2 and write $b^-$ for the block of $U_5$ containing $\lambda^-$. Thus we have a bijection between the sets $\text{Bl}(P | b^-)$ and $\text{Bl}(I^- | b^-)$.
As $I^- \cong L^- \ltimes U_5$ is a split extension and $\lambda^-$ extends to $I^-$, [Fon61, 2D] gives a bijection between $\text{Bl}(I^-|b^-)$ and the blocks of $L^- \cong D_{2(q+1)}$ preserving defect groups and decomposition matrices. As this correspondence is realized via the character constructions detailed in Section 3.2, the claim follows from the well-known blocks of the dihedral group $L^- \cong D_{2(q+1)}$.

For the symplectic group $G^*$ and odd $q$, we follow [AH11] notationwise: we define the ordinary irreducible characters $\nu_7$ and $\nu_8$ of the group $\text{Sp}_2(q) = \text{SL}_2(q)$ of degree $(q - 1)/2$ in the same way as [AH11, Lemma 5.1].

**Lemma 4.8.** Let $P = P_7$ be the parabolic subgroup of $G = \text{SO}_7(q)$ and $P^* = P_6^*$ the parabolic subgroup of $G^* = \text{Sp}_6(q)$ defined in Section 2.2.

(a) For all prime powers $q$ there is a cyclic block $b$ of $P$ whose Brauer tree is given in Table 4 with

$$\xi_1 := 0^\psi(-\psi_{L^-_5}), \quad \xi_2 := 0^\psi(-\psi_{\xi}), \quad \text{and} \quad \xi_{\text{exc}} := 0^\psi(-\psi_{\Xi}),$$

where $\Xi$ is the sum of all irreducible characters of degree 2 in the principal block of $L^-_5$.

(b) If $q$ is odd then there is a cyclic block $b^*$ of $P^*$ whose Brauer tree is given in Table 4 with

$$\xi_1 := 2^\psi(\psi_{\xi}), \quad \xi_2 := 2^\psi(\psi_{\xi}^{1+}), \quad \text{and} \quad \xi_{\text{exc}} := 2^\psi(\psi_{\Xi}^{1+}),$$

where $\Xi$ is the sum of all irreducible characters of degree $q - 1$ lying in the quasi-isolated block of $\text{Sp}_2(q) = \text{SL}_2(q)$ containing $\nu_7$ and $\nu_8$.

**Proof.** We give the proof for the block $b$ in part (a). Let $\lambda^0$ be the non-trivial linear character of $U$ defined in Section 3.2. Analogously to the proof of Lemma 4.1 Fong-Reynolds correspondence gives a bijection between the blocks of $P$ covering the block $b^0 := \{\lambda^0\}$ of $U$ and the blocks of $P_5$, also preserving defect groups and decomposition matrices. We may repeat the same arguments for $P_5$ and $\lambda^- \in \text{Irr}(U_5)$ to extend this correspondence to $L^-_5$. Again, this bijection is realized via the character constructions of Section 3.2 so the claim follows.

The existence of the block $b^*$ can be proved by the same arguments using the character constructions in [AH06, 2.3].

### 4.3 The decomposition numbers of $\text{SO}_5(q)$

In this section we apply our overall approach to the group $G_5 = \text{SO}_5(q)$ to determine the $\ell$-modular decomposition numbers of the unipotent characters of $G_5$. 

20
Theorem 4.9. For all prime powers $q$ the $\ell$-modular decomposition numbers of the unipotent characters in the principal block of $\text{SO}_5(q)$ are given in Table 5, where the decomposition number $\alpha = 1$ if $(q + 1)\ell = 3$, and $\alpha = 2$ if $(q + 1)\ell > 3$.

| $\ell$ | $\text{ps}$ | $\text{c}$ | $A_1$ | $A_1$ | $c$ |
|-------|-------------|-------------|-------|-------|-----|
| $[2, -1]$ | 1           | 1           | $\text{A}_1$ | $\text{A}_1$ | $c$ |
| $[-, -3]$ | 1           | 1           | $\text{A}_1$ | $\text{A}_1$ | $c$ |
| $[1, -1]$ | 1           | 1           | $\text{A}_1$ | $\text{A}_1$ | $c$ |
| $[-, 2, 1]$ | 1           | $\alpha$ | 1 | 1 | 1 |

Table 5: The $\ell$-modular decomposition numbers of the unipotent characters in the principal block of $\text{SO}_5(q)$.

Remark 4.10. (a) The character $[1, 1, 1]$ has defect 0 and is not contained in the principal block.

(b) The set of ordinary unipotent characters of $\text{SO}_5(q)$ is partitioned into three families; see the information available in the GAP part of CHEVIE. The distribution into these families is indicated by the dotted lines in the leftmost column of Table 5.

(c) There are five irreducible Brauer characters $\psi_1, \ldots, \psi_5$ in the principal block of $\text{SO}_5(q)$ and each belongs to a different Harish-Chandra series. We write $\text{ps}$ for the principal series and $c$ for cuspidal Brauer characters. A Levi subgroup corresponding to the short simple root $\alpha_1$ has a unique cuspidal unipotent Brauer character, the modular Steinberg character, and we denote the corresponding Harish-Chandra series of $\text{SO}_5(q)$ by $\text{A}_1$. The same is true for the Levi subgroup corresponding to the long simple root $\alpha_2$ and we write $\tilde{\text{A}}_1$ for the corresponding Harish-Chandra series of $\text{SO}_5(q)$.

(d) Theorem 4.9 confirms [GH97a, Conjecture 3.4] in the special case of $\text{SO}_5(q)$ for all prime powers $q$.

Proof. Suppose that $q$ is odd. We begin by constructing several projective characters of $G_5$ in the sense that these characters are sums of the ordinary characters of projective indecomposable $kG_5$-modules (PIMs).

Let $1_B$ be the trivial $kB$-module of a Borel subgroup $B$ of the group $G_5$ and $\Phi_{1_B}$ the ordinary character of a projective cover of $1_B$. We write $\psi_{\text{St}}$ for the modular Steinberg module of a Levi subgroup $L_{\text{short}}$ corresponding to the short simple root
\[\begin{array}{ccccc}
[2, -1] & 1 & . & . & . \\
[-3, 1] & . & 1 & . & . \\
[1^2, -1] & 1 & . & 1 & . \\
[-2, 1] & 1 & . & . & 1 \\
[-1^2, 1] & 1 & \frac{(q+1)\ell - 1}{2} & 1 & 1 & 1 \\
\end{array}\]

Table 6: Scalar products of the unipotent characters of \(G_5\) with the projective characters \(\Psi_1, \Psi_2, \ldots, \Psi_5\).

\(\alpha_1\) and let \(\Phi_{\varphi}\) be the ordinary character of a projective cover of \(\varphi\). Similarly, we write \(\tilde{\varphi}\) for the modular Steinberg module of a Levi subgroup \(L_{\text{long}}\) corresponding to the long simple root \(\alpha_2\) and let \(\Phi_{\tilde{\varphi}}\) be the ordinary character of a projective cover of \(\tilde{\varphi}\). Furthermore, let \(\Phi_{\nu}\) be the ordinary character of a projective cover of the simple \(kP_5\)-module with Brauer character \((-1)\psi_{\nu}\) in the cyclic block \(b_5\) of the parabolic subgroup \(P_5\) described in Lemma 4.7. Hence, in the notation of Lemma 4.7 we have \(\Phi_{\nu} = -\psi_{\nu} + \psi_{\Xi}\). We define characters of \(G_5\) as follows:

\(\Psi_1 := R_{G_5}^{G_5}(\Phi_{\nu}), \quad \Psi_2 := \Phi_{\nu}^{G_5}_{P_5}, \quad \Psi_3 := R_{L_{\text{long}}}^{G_5}(\Phi_{\tilde{\varphi}}), \quad \Psi_4 := R_{L_{\text{short}}}^{G_5}(\Phi_{\varphi}),\)

and let \(\Psi_5\) denote the Gelfand-Graev character of \(G_5\). The characters \(\Psi_1, \ldots, \Psi_4\) are projective since induction and Harish-Chandra induction preserve projectives; see [His89, Lemma 4.4.3]. The character \(\Psi_5\) is projective because it is induced from an \(\ell\)-subgroup.

The scalar products of \(\Psi_1, \ldots, \Psi_5\) with the ordinary unipotent characters of \(G_5\) are given in Table 6 and can be determined as follows: Since Harish-Chandra induction commutes with taking unipotent quotients (see [His93, Lemma 6.1]) it is straightforward to compute the scalar products of \(\Psi_1, \Psi_3, \Psi_4\) with the ordinary unipotent characters of \(G_5\). The scalar products of \(\Psi_2\) follow from [HN, Theorem 7.2] and Frobenius reciprocity. By [Car85, Section 12.1] the Steinberg character \(\text{St}_{G_5}\) is the only unipotent constituent of \(\Psi_5\) and it has multiplicity one.

From Table 6 we get the unitriangular shape of the decomposition matrix of the principal block \(B_0(G_5)\) leading to a bijection between the set of ordinary unipotent characters and the set of irreducible unipotent Brauer characters of \(B_0(G_5)\). Let \(5\varphi_j\) be the irreducible Brauer character corresponding to the \(j\)-th column of Table 6 and let \(5\Phi_j\) be the ordinary character of its projective cover. In particular, \(5\varphi_5\) is the modular Steinberg character of \(G_5\).

The cuspidality of \(5\varphi_2\) and \(5\varphi_5\) is a consequence of [His93, Lemma 4.3] and [GHM94, Theorem 4.2], respectively. The Harish-Chandra series of \(5\varphi_1, 5\varphi_3, 5\varphi_4\) follow from the construction of \(\Psi_1, \Psi_3, \Psi_4\). Thus, all entries of Table 6 are decomposition numbers except for \(\langle [-, 1^2, 1], \Psi_2 \rangle_{G_5}\). Let \(\alpha := \langle [-, 1^2, 1], 5\Phi_2 \rangle_{G_5}\). It
follows from Lemma 4.5 (a), (b) that $\alpha \geq 1$ and that $\alpha \geq 2$ if $(q + 1)\ell > 3$. Hence, Table 3 implies that $\alpha = 1$ if $(q + 1)\ell = 3$.

To prove that $\alpha \leq 2$ in general, we use the method of Dudas detailed in [Dud13]. Let $S$ denote the simple $kG_5$-module with Brauer character $\Phi_5$. Let $w$ be a Weyl group element of minimal length with the property that the Deligne-Lusztig variety associated to $w$ gives rise to a virtual module $P_w$ such that with respect to the usual pairing $\langle \, , \rangle$ of simple and projective $kG_5$-modules we have $\langle P_w, S \rangle \neq 0$. Then by [BR03, 8.10,8.12] there is a bounded perfect complex

$$0 \to Q_\ell(w) \to Q_{\ell(w)+1} \to \cdots \to Q_{2\ell(w)} \to 0$$

giving the homology with compact support on the variety, and the projective cover of $S$ appears only in $Q_\ell(w)$. The same holds for the unipotent summand of the complex whose character is given by the Deligne-Lusztig character $\Phi_5^w(1)$, so that we may verify the minimality property of $w$ by taking scalar products of Deligne-Lusztig characters (which are readily available in the GAP-part of CHEVIE) and

$$5\Phi_5 = [2, -,-,1]^\circ - \alpha [-,-,3]^\circ - [1^2, -,1]^\circ - [-,2,1]^\circ + [-,1^2,1]^\circ.$$ 

For $w = w_1w_2$ we obtain the scalar product $2 - \alpha$, while elements of lesser length yield scalar product 0. We assume that $\alpha \neq 2$ and consider the unipotent summand of the associated complex $0 \to Q_2 \to Q_3 \to Q_4 \to 0$ of projective $kG_5$-modules. Owing to the wedge shape of the decomposition matrix, we see that $R^G_{T_w}(1)$ is the unipotent part of $5\Phi_1 + 5\Phi_2 - 5\Phi_3 - 5\Phi_4 + (2 - \alpha)\cdot 5\Phi_5$. By the above we have the information that $2 - \alpha > 0$. Therefore, $\alpha = 2$ or $\alpha < 2$, hence $\alpha \leq 2$.

Now suppose that $q$ is even. The ordinary character table of $Sp_4(q)$ is available in the CHEVIE library. The character values on the unipotent classes imply that the isomorphism $SO_5(q) \to Sp_4(q)$ from Remark 2.1 maps each unipotent character with label $\Lambda$ to the unipotent character with the same label, except for possibly swapping $[1^2, -,1]$ and $[-,2,1]$. Hence, the decomposition numbers of the unipotent characters of $G_5 = SO_5(q)$ can be read off from those of $Sp_4(q)$ in [Whi05, Theorem 2.2] and [OW98, Theorem 2.3]. The modular Harish-Chandra series can be determined by the same arguments as for odd $q$.

4.4 Approximations

In this section we start with the determination of the decomposition numbers of the groups $G = SO_7(q)$ and $G^* = Sp_6(q)$. In [AH06] an approximation of the decomposition matrix of $G^*$ is constructed for odd $q$. Here we mainly follow the same approach and show that similar arguments also provide an approximation of the decomposition matrix of $G$ and $G^*$. 

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At the starting point of this construction lie the decomposition matrices of the groups \( G_5 = \text{SO}_5(q) \) and \( G^*_4 = \text{Sp}_4(q) \); see Theorem 4.9, [Whi90b, Theorem 4.2], [Whi95, Theorems 2.2, 5.1], [OW98, Theorem 2.3]. The PIMs thus provided are a good source of projectives for \( G \) and \( G^* \) via Harish-Chandra induction. Further projective modules for \( G \) and \( G^* \) can be obtained by inducing PIMs of the cyclic blocks of \( P \) and \( P^* \) established in Lemma 4.8.

**Lemma 4.11.** We use the notation of Lemma 4.8 and set \( m_{\text{exp}} := \frac{(q+1)\ell-1}{2} \). Let \( b \) and \( b^* \) be the cyclic blocks of \( P \) and \( P^* \), respectively, described in Lemma 4.8. When dealing with the block \( b^* \) we assume that \( q \) is odd. Let \( \Phi_{\xi_j} \) be the ordinary character of a projective cover of the simple \( kP^* \)-module in the block \( b^* \). Then the unipotent parts of \( \Phi_{\xi_j} \) are given by

\[
\begin{align*}
(\Phi_{\xi_1}|_{P^*})_{\text{uni}} &= [1^3, -1, 1] + m_{\text{exp}} \cdot [-1^3, 1], \\
(\Phi_{\xi_2}|_{P^*})_{\text{uni}} &= [-1, 3] + m_{\text{exp}} \cdot [-1^3, 1].
\end{align*}
\]

**Proof.** For the block \( b \) this follows from [HN, Theorem 8.1]; for \( b^* \) it follows from [AH11, Table 4], [AH06, Corollary 3.3 and Example 3.5].

The following lemma proves all modular Harish-Chandra series and all decomposition numbers in Table 3 except for \( \beta \) and \( \gamma \). The value of \( \beta \) and \( \gamma \) will be determined in Section 4.5.

**Lemma 4.12.** For all prime powers \( q \) the parts of the \( \ell \)-modular decomposition matrices of \( G \) and \( G^* \) corresponding to the unipotent characters in the principal block are given by Table 3. The same is true for the Harish-Chandra series of the irreducible Brauer characters in the principal block of \( G \) and \( G^* \). The decomposition numbers \( \alpha, \beta, \gamma \) satisfy the following conditions:

(a) \( \alpha = 1 \) if \( (q+1)\ell = 3 \), and \( \alpha = 2 \) if \( (q+1)\ell > 3 \).

(b) \( 1 \leq \beta, \gamma \leq \frac{1}{2}((q+1)\ell - 1) \). Furthermore, if \( (q+1)\ell > 3 \) then \( \beta, \gamma \geq 2 \), and if \( (q+1)\ell > 5 \) then \( \beta \geq 3 \).

**Proof.** Suppose that \( q \) is odd. The arguments for \( G \) and \( G^* \) are entirely similar, allowing us to just state them for \( G \). We start by constructing several projective characters of \( G \).

The Levi subgroup \( L \) of \( P \) is isomorphic to \( A \times G_5 \cong \mathbb{F}_q^* \times \text{SO}_5(q) \), hence we obtain PIMs of \( L \) by inflating the PIMs of \( G_5 \). For \( j = 1, \ldots, 5 \), we denote the inflated PIM corresponding to the \( j \)-th column of Table 5 by \( ^5\Phi_j \). Additionally, we set \( ^5\Phi_6 := \text{Infl}^L_{G_5}([1, 1, 1]) \). Let \( L_{1,3} \cong G_3 \times \text{GL}_2(q) \) be the standard Levi subgroup corresponding to the set \( \{\alpha_1, \alpha_3\} \) of simple roots and let \( \Phi_{\varphi_{\alpha_1}^L\varphi_{\alpha_3}^L} \) be the ordinary character of a projective cover of its modular Steinberg module.
Table 7: Scalar products of the unipotent characters of $G$ with the projective characters $\Psi_1, \Psi_2, \ldots, \Psi_{10}$.

For $j = 1, 2$, let $\Phi_{\xi_j}$ be the ordinary character of a projective cover of the simple $kP$-module with Brauer character $\hat{\xi}_j$ in the cyclic block $b$ of $P$ described in Lemma 4.8. Hence, in the notation of Lemma 4.8 (a) we have:

$$\Phi_{\xi_1} = 0 \psi^{(q+1)r-1}_1 + 0 \psi^{(q+1)r-1}_2$$

and $\Phi_{\xi_2} = 0 \psi^{(q+1)r-1}_3 + 0 \psi^{(q+1)r-1}_4$.

We define characters of $G$ as follows:

\[
\begin{align*}
\Psi_1 &:= R^G_L(5\Phi_1), \quad \Psi_2 := R^G_L(5\Phi_6), \quad \Psi_3 := R^G_L(5\Phi_4), \\
\Psi_4 &:= R^G_L(5\Phi_2), \\
\Psi_5 &:= R^G_{L,1,3}(\Phi_{\varphi_{\text{St}}\otimes\varphi'_{\text{St}}}), \quad \Psi_6 := R^G_L(5\Phi_3), \\
\Psi_7 &:= R^G_L(5\Phi_5), \quad \Psi_8 := \Phi_{\xi_1}^{G}, \quad \Psi_9 := \Phi_{\xi_2}^{G}, \\
\Psi_{10} &:= \text{ind}_{L}^{G}(\psi_{\varphi_1}^{G})
\end{align*}
\]

and let $\Psi_{10}$ denote the Gelfand-Graev character of $G$. The characters $\Psi_1, \ldots, \Psi_{10}$ are projective since induction and Harish-Chandra induction preserve projectives; see [His89b, Lemma 4.4.3]. The character $\Psi_{10}$ is projective because it is induced from an $\ell'$-subgroup.

The scalar products of $\Psi_1, \Psi_2, \ldots, \Psi_{10}$ with the ordinary unipotent characters of $G$ are given in Table 7 and can be determined as follows: Since Harish-Chandra induction commutes with taking unipotent quotients (see [His93, Lemma 6.1]) it is straightforward to compute the scalar products of $\Psi_1, \Psi_2, \ldots, \Psi_7$ with the ordinary unipotent characters of $G$ with the help of CHEVIE. The scalar products of $\Psi_8$ and $\Psi_9$ follow from Lemma 4.11. By [Car85, Section 12.1] the Steinberg character $\text{St}_G$ is the only unipotent constituent of $\Psi_{10}$ and it has multiplicity one.

From Table 7 we get the unitriangular shape of the decomposition matrix of the principal block $B_0(G)$ leading to a bijection between the set of ordinary unipotent characters and the set of irreducible unipotent Brauer characters of $B_0(G)$. Let $\varphi_j$ be the irreducible Brauer character corresponding to the $j$-th column of Table 7.
and let \( \Phi_j \) be the ordinary character of its projective cover. In particular, \( \varphi_{10} \) is the modular Steinberg character of \( G \).

The cuspidality of \( \varphi_{10} \) is a consequence of [GHM94 Theorem 4.2]. We get the lower bounds for \( \beta \) and \( \gamma \) from the relations involving \( \tilde{\chi}_{9,3}, \tilde{\chi}_{13,4}, \tilde{\chi}_{23,2}, \tilde{\chi}_{32} \) in Lemma 4.12. Since \( \Psi_7 \) is Harish-Chandra induced from a proper Levi subgroup and \( \varphi_{10} \) is cuspidal we get that \( \langle [\cdot, 1^3, 1], \Phi_7 \rangle_G = 1 \). This proves all statements about \( \Phi_7, \Phi_8, \Phi_9, \Phi_{10} \) in Lemma 4.12. Furthermore, we see that \( \varphi_7 \) is the only irreducible Brauer character in the Harish-Chandra series \( [B_2, St] \). The construction of \( \Psi_6 \) and the fact that \( \varphi_7 \) belongs to the Harish-Chandra series \( [B_2, St] \) imply that \( \langle [1, 1^2, 1], \Phi_6 \rangle_G = 1 \). From the relation involving \( \tilde{\chi}_{9,2} \) in Lemma 4.12 we get that \( \langle [1^3, -1], \Phi_6 \rangle_G = 1 \). Again, the cuspidality of \( \varphi_{10} \) tells us that \( \langle [\cdot, 1^3, 1], \Phi_6 \rangle_G = 1 \), proving all statements about \( \Phi_6 \) and about the Harish-Chandra series of \( \varphi_6 \).

The construction of \( \Psi_5 \) and the Harish-Chandra series of \( \varphi_7 \) and \( \varphi_{10} \) imply all statements about \( \Phi_5 \) in Lemma 4.12. Additionally, we see that \( \varphi_5 \) is the only irreducible Brauer character in the Harish-Chandra series \( A_1 \times A_1' \). From the modular Harish-Chandra series we get \( \langle [1, 1^2, 1], \Phi_4 \rangle_G = \alpha \). Then the relation involving \( \tilde{\chi}_{9,3} \) in Lemma 4.12 implies that \( \langle [\cdot, 1^3, 1], \Phi_4 \rangle_G = \alpha \). The lower bound for \( \gamma \) gives us \( \langle [\cdot, -1, 3], \Phi_4 \rangle_G = 1 \), proving all statements about \( \Phi_4 \). By the construction of \( \Psi_4 \), the character \( \varphi_4 \) is the only irreducible Brauer character in the Harish-Chandra series \( [B_2, \eta] \). Again from our knowledge of the modular Harish-Chandra series we get all statements about \( \Phi_3, \Phi_2 \) and the Harish-Chandra series of \( \varphi_3 \).

The permutation character \( 1_{P}^{G} \) restricts to the principal block as \( [3, -1, 1] + [2, 1, 1] \). As \( \ell \) divides \( [G : P] \), there is a trivial constituent in the \( \ell \)-modular reduction \( [2, 1, 1] \) so that \( \langle [2, 1, 1], \Phi_1 \rangle_G = 1 \). Obviously, the trivial Brauer character \( \varphi_1 \) belongs to the principal series. Now the modular Harish-Chandra series and the relation involving \( \tilde{\chi}_{13,3} \) in Lemma 4.12 imply all statements about \( \Phi_1 \). Considering the character \( R^{G}_{F}(1_{T}) \) where \( T := T^{F} \) we see that the Harish-Chandra series of \( \varphi_2, \varphi_8 \) and \( \varphi_9 \) are as claimed. This completes the proof of Lemma 4.12 for odd \( q \).

Suppose that \( q \) is even. The ordinary character table of \( \text{Sp}_6(q) \) is available in the CHEVIE library. The character values on the unipotent classes imply that the isomorphism \( \text{SO}_7(q) \to \text{Sp}_6(q) \) from Remark 2.1 maps each unipotent character with label \( \Lambda \) to the unipotent character with the same label. Hence, the decomposition numbers of the unipotent characters of \( G = \text{SO}_7(q) \) can be read off from those of \( \text{Sp}_6(q) \) in [Whi00 Theorem 2.2] except for the upper bounds on \( \beta \) and \( \gamma \) in Lemma 4.12 and we only get \( \langle [2, 1, 1], \Phi_1 \rangle_G \leq 1 \). In the same way as for odd \( q \) we get \( \langle [2, 1, 1], \Phi_1 \rangle_G = 1 \). By inducing the two PIMs in the cyclic block \( b \) of \( P \) from Lemma 4.12 (b) we obtain the upper bounds for \( \beta \) and \( \gamma \). The modular Harish-Chandra series then follow by the same arguments as for odd \( q \).  

\( \square \)
4.5 Proof of Theorem 4.3

In this final section we combine the information we have derived previously and complete the proof of the decomposition numbers of $G = \text{SO}_7(q)$ and $G^* = \text{Sp}_6(q)$ given in Theorem 4.3. According to Lemma 4.12 we only have to settle the ambiguities $\beta$ and $\gamma$. For $(q + 1)\ell > 5$ we employ again Dudas’ method of [Dud13] to show that the lower bounds in Theorem 4.3 are tight. If $(q + 1)\ell \in \{3, 5\}$, then Lemma 4.12 already yields sufficient bounds. In this whole section, $q$ is an odd or even prime power.

**Lemma 4.13.** We have $\gamma \leq 2$ and $\beta \leq \gamma + 1$.

**Proof.** Our approach follows the proof of Theorem 4.9. We give the argument for the group $G$; the proof for $G^*$ may be copied verbatim. For a given element $w$ of the Weyl group $W$ of $G$, we consider the Deligne-Lusztig character $R^G_{G^*}(1)$ and compute the scalar product with the approximated Brauer character

$$\varphi_{10} = -[3, -1] + [2, 1, 1] + \beta \cdot [-, 3, 1] + \gamma \cdot [1, -3, 1] - \beta \cdot [1, 2, 1] + \beta \cdot [1^2, 1, 1] - [1, 1^2, 1] - \beta \cdot [1^3, -1, 1] - \gamma \cdot [-, 1, 3] + [-, 1^3, 1].$$

We obtain the scalar products $2\gamma - 4$ for $w = s_1 s_2 s_3 =: w'$, and $2\beta - 2\gamma - 2$ for $w = s_1 s_2 s_1 s_2 s_3 =: w''$; elements of lesser length yield scalar product 0. Note that with the GAP-part of CHEVIE we get:

$$R^G_{G^*}(1) = [3, -1] - [2, 1, 1] + [1, -3, 1] + [1, 1^2, 1] - [-, 1, 3] - [-, 1^3, 1],$$

$$R^G_{G^*}(1) = [3, -1] - [21, -1] - [1, -3, 1] - [1, 2, 1] + [1^2, 1, 1] + [-, 21, 1] + [-, 1, 3] - [-, 1^3, 1].$$

Owing to the wedge shape of the decomposition matrix, we see that the restriction of $R^G_{G^*}(1)$ to the principal block $B_0(G)$ coincides with the unipotent part of

$$\Phi_1 - 2\Phi_2 - \Phi_3 + \Phi_4 + 2\Phi_5 - \Phi_6 + (2 - \alpha)\Phi_7 - 2\Phi_9 + 2(\gamma - 2)\Phi_{10}; \quad (10)$$

and that the restriction of $R^G_{G^*}(1)$ to $B_0(G)$ is the unipotent part of

$$\Phi_1 - \Phi_2 - \Phi_3 - \Phi_4 + \Phi_6 + \alpha\Phi_7 - 2\Phi_8 + 2\Phi_9 + 2(\beta - \gamma - 1)\Phi_{10}. \quad (11)$$

Therefore, either $\gamma = 2$, or (10) gives $(2\gamma - 4) < 0$, hence $\gamma \leq 2$ as claimed. Analogously, we obtain from (11) that either $\beta = \gamma + 1$, or $\beta < \gamma + 1$, giving the bound for $\beta$, too.

**Corollary 4.14.** If $(q + 1)\ell > 5$ then $\beta = 3$ and $\gamma = 2$. 

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Proof. This follows from the lower bounds for $\beta$ and $\gamma$ given in Lemma 4.12 and the upper bounds of Lemma 4.13 above.

We now treat the remaining two cases $(q+1)_\ell = 3$ and $(q+1)_\ell = 5$, thereby completing the proof of Theorem 4.3.

**Corollary 4.15.** If $(q+1)_\ell = 3$ then $\beta = \gamma = 1$, and if $(q+1)_\ell = 5$ then $\beta = \gamma = 2$.

**Proof.** This is an immediate consequence of the bounds in Lemma 4.12 (b).

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**References**

[AH06] J. An and G. Hiss. Restricting the Steinberg character in finite symplectic groups. *J. Group Theory*, 9(2):251–264, 2006.

[AH11] J. An and G. Hiss. Restricting unipotent characters in finite symplectic groups. *Comm. Algebra*, 39(3):1104–1130, 2011.

[Bon06] C. Bonnafé. Sur les caractères des groupes réductifs finis à centre non connexe: applications aux groupes spéciaux linéaires et unitaires. *Astérisque*, 306, 2006.

[BR03] C. Bonnafé and R. Rouquier. Catégories dérivées et variétés de Deligne-Lusztig. *Publ. Math. Inst. Hautes Études Sci.*, (97):1–59, 2003.

[Car85] R. W. Carter. *Finite groups of Lie type*. Pure and Applied Mathematics (New York). John Wiley & Sons Inc., New York, 1985. Conjugacy classes and complex characters, A Wiley-Interscience Publication.

[DM91] F. Digne and J. Michel. *Representations of finite groups of Lie type*, volume 21 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1991.

[Dud13] O. Dudas. A note on decomposition numbers for groups of Lie type of small rank. *To appear in J. Algebra*, 2013.

[Eng08] M. Enguehard. Vers une décomposition de jordan des blocs des groupes réductifs finis. *J. Algebra*, 319(3):1035—-1115, 2008.
[Eno72] H. Enomoto. The characters of the finite symplectic group $\text{Sp}_4(q)$, $q = 2^f$. Osaka J. Math., 9:75–94, 1972.

[Fon61] P. Fong. On the characters of $p$-solvable groups. Trans. Amer. Math. Soc., 98:263–284, 1961.

[FS89] P. Fong and B. Srinivasan. The blocks of finite classical groups. J. Reine Angew. Math., 396:122–191, 1989.

[FS90] P. Fong and B. Srinivasan. Brauer trees in classical groups. J. Algebra, 131:179–225, 1990.

[GAP13] The GAP Group. GAP – Groups, Algorithms, and Programming, Version 4.6.5, 2013.

[Gec90] M. Geck. Irreducible Brauer characters of the 3-dimensional special unitary groups in non-defining characteristic. Comm. Algebra, 18:563–584, 1990.

[Gec91] M. Geck. Generalized Gelfand-Graev characters for Steinberg’s triality groups and their applications. Comm. Algebra, 19:3249–3269, 1991.

[GH91] M. Geck and G. Hiss. Basic sets of Brauer characters of finite groups of Lie type. J. reine angew. Math., 418:173–188, 1991.

[GH97a] Meinolf Geck and Gerhard Hiss. Modular representations of finite groups of Lie type in non-defining characteristic. In Finite reductive groups (Luminy, 1994), volume 141 of Progr. Math., pages 195–249. Birkhäuser Boston, Boston, MA, 1997.

[GH97b] Jochen Gruber and Gerhard Hiss. Decomposition numbers of finite classical groups for linear primes. J. Reine Angew. Math., 485:55–91, 1997.

[GHL+96] M. Geck, G. Hiss, F. Lübeck, G. Malle, and G. Pfeiffer. CHEVIE – A system for computing and processing generic character tables for finite groups of Lie type, Weyl groups and Hecke algebras. Appl. Algebra Engrg. Comm. Comput., 7:175–210, 1996.

[GHM94] M. Geck, G. Hiss, and G. Malle. Cuspidal unipotent Brauer characters. J. Algebra, 168(1):182–220, 1994.

[HH13] F. Himstedt and S.-c. Huang. On the decomposition numbers of Steinberg’s triality groups $^3D_4(2^n)$ in odd characteristics. Comm. Algebra, 41:1484–1498, 2013.
[Him11] F. Himstedt. On the decomposition numbers of the Ree groups $^2F_4(q^2)$ in non-defining characteristic. *J. Algebra*, 325:364–403, 2011.

[His89a] G. Hiss. On the decomposition numbers of $G_2(q)$. *J. Algebra*, 120:339–360, 1989.

[His89b] G. Hiss. *Zerlegungszahlen endlicher Gruppen vom Lie-Typ in nicht-definierender Charakteristik*. Habilitationsschrift, RWTH Aachen, 1989.

[His93] G. Hiss. Harish-Chandra series of Brauer characters in a finite group with a split BN-pair. *J. London Math. Soc.*, 48:219–228, 1993.

[HK00] G. Hiss and R. Kessar. Scopes reduction and Morita equivalence classes of blocks in finite classical groups. *J. Algebra*, 230(2):378–423, 2000.

[HN] F. Himstedt and F. Noeske. Restricting unipotent characters in finite special orthogonal groups. *In preparation.*

[Jam90] Gordon James. The decomposition matrices of $GL_n(q)$ for $n \leq 10$. *Proc. London Math. Soc. (3)*, 60(2):225–265, 1990.

[Kö6] C. Köhler. Unipotente Charaktere und Zerlegungszahlen der endlichen Chevalleygruppen vom Typ $F_4$. Dissertation, RWTH Aachen, Germany, 2006.

[LM80] P. Landrock and G. O. Michler. Principal 2-blocks of the simple groups of Ree type. *Trans. Amer. Math. Soc.*, 260:83–111, 1980.

[Lüb93] F. Lübeck. Charaktertafeln für die Gruppen $CSp_6(q)$ mit ungeradem $q$ und $Sp_6(q)$ mit geradem $q$. Dissertation, Universität Heidelberg, Germany, 1993.

[OW98] T. Okuyama and K. Waki. Decomposition numbers of $Sp(4,q)$. *J. Algebra*, 199(2):544–555, 1998.

[OW02] T. Okuyama and K. Waki. Decomposition numbers of $SU_3(q^2)$. *J. Algebra*, 255:258–270, 2002.

[Pui90] Lluís Puig. Algèbres de source de certains blocs des groupes de Chevalley. *Astérisque*, (181-182):9, 221–236, 1990.

[Shi82] K. Shinoda. The characters of the finite conformal symplectic group $CSp(4,q)$. *Comm. Alg.*, 10(13):1369–1419, 1982.
[Sri68] B. Srinivasan. The characters of the finite symplectic group $\text{Sp}_4(q)$. Trans. Amer. Math. Soc., 131:488–525, 1968.

[Tay92] D. E. Taylor. The geometry of the classical groups, volume 9 of Sigma Series in Pure Mathematics. Heldermann, 1992.

[Wak04] K. Waki. A note on decomposition numbers of $G_2(2^n)$. J. Algebra, 274:602–606, 2004.

[Whi90a] D. L. White. The 2-decomposition numbers of $\text{Sp}(4,q)$, $q$ odd. J. Algebra, 131:703–725, 1990.

[Whi90b] D. L. White. Decomposition numbers of $\text{Sp}(4,q)$ for primes dividing $q \pm 1$. J. Algebra, 132:488–500, 1990.

[Whi95] D. L. White. Decomposition numbers of $\text{Sp}_4(2^a)$ in odd characteristics. J. Algebra, 177:264–276, 1995.

[Whi00] D. L. White. Decomposition numbers of unipotent blocks of $\text{Sp}_6(2^a)$ in odd characteristics. J. Algebra, 227:172–194, 2000.

[Yam] H. Yamada. The characters of $\text{Sp}_4(q)$, $q$ odd. Preprint.