An economic game with stochastic dynamics

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\textbf{Abstract}

In this paper we investigate a stochastic model for an economic game. To describe this model we have used a Wiener process, as the noise has a stabilization effect. The dynamics are studied in terms of stochastic stability in the stationary state, by constructing the Lyapunov exponent, depending on the parameters that describe the model. The numerical simulation that we did justifies the theoretical results.

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\section{Introduction.}

Stochastic modeling plays an important role in many branches of science. In many practical situations, perturbations are expressed in terms of
white noise, modeled by brownian motion. The behavior of a deterministic dynamical system which is disturbed by noise may be modeled by a stochastic differential equation (SDE). The stochastic stability has been introduced by Bertram and Sarachik and is characterized by the negativeness of Lyapunov exponents. In general, it is not possible to determine this exponents explicitly. Many numerical approaches have been proposed, which generally used the simulation of the stochastic trajectories. In the present paper, we study a stochastic dynamical system that are used in economy, in describing a Cournot duopoly game.

In 1838, Cournot introduced the first formal theory of oligopoly, which treated the case of naive expectations, where each player assumes the last values taken by the competitors without estimation of their future reactions. Recently, a lot of articles have shown that the Cournot model may lead to a cyclic or chaotic behavior. Also, in Rosser reviews the development of the theory of complex oligopoly dynamics.

In the present paper we have studied a stochastic Cournot economic game. In Section 2 we present the Lyapunov exponent and stability in stochastic 2d dynamical structures. Section 3 studies the Lyapunov exponent for an economic game with stochastic dynamics. Some numerical simulations are given in Section 4. Finally, Section 5 draws some conclusions.

2 The Lyapunov exponent and stability in stochastic 2d dynamical structures.

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space. It is assumed that the \(\sigma\)-algebra \(\mathcal{F}\) is a filtration that is, \(\mathcal{F}\) is generated by a family of \(\sigma\)-algebra \(\mathcal{F}_t(t \geq 0)\) such that

\[ \mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}, \quad \forall s \leq t, s, t \in I, \]

where \(I = [0, T], T \in (0, \infty)\).

Let \(\{x(t) = (x_1(t), x_2(t))\}_{t \geq 0}\) be a stochastic process. The system of Ito equations:

\[ dx_i(t, \omega) = f_i(t, x(t, \omega))dt + g_i(x(t, \omega))dw(t, \omega), i = 1, 2, \quad (1) \]
with the initial condition \( x(0) = x_0 \) is written as:

\[
x_i(t, \omega) = x_{i0}(\omega) + \int_0^t f_i(x(s, \omega))ds + \int_0^t g_i(x(s, \omega))dw(s, \omega), \quad i = 1, 2, \tag{2}
\]

for almost all \( \omega \in \Omega \) and for each \( t > 0 \), where \( f_i(x) \) are drift functions, \( g_i(x) \) are diffusion functions, \( \int_0^t f_i(x(s))ds, \quad i = 1, 2 \) are Riemann integrals and \( \int_0^t g_i(x(s))dw(s) \) are Itô integrals. It is assumed that \( f_i \) and \( g_i, \quad i = 1, 2 \) satisfy the conditions of existence of solution for this SDE with initial condition \( x(0) = a_0 \in \mathbb{R}^n \).

Let \( x_0 = (x_{10}, x_{20}) \in \mathbb{R}^2 \) be a solution of the system:

\[
f_i(x_0) = 0, \quad i = 1, 2. \tag{3}
\]

The functions \( g_i, \quad i = 1, 2 \) are chosen so that:

\[
g_i(x_0) = 0, \quad i = 1, 2.
\]

In what follows, we consider:

\[
g_i(x) = \sum_{j=1}^2 b_{ij}(x_j - x_{0j}), \quad i = 1, 2,
\]

where \( b_{ij} \in \mathbb{R}, i, j = 1, 2 \).

The linearized system of (2) in \( x_0 \), is given by:

\[
X(t) = \int_0^t AX(s)ds + \int_0^t BX(s)dw(s),
\]

where

\[
X(t) = \begin{pmatrix} x(t, \omega) \\ y(t, \omega) \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix},
\]

\[
a_{ij} = \frac{\partial f_i}{\partial x_j}|_{x_0}, \quad b_{ij} = \frac{\partial g_i}{\partial x_j}|_{x_0}.
\]

The Oseledec multiplicative ergodic theorem [9] asserts the existence of 2 non-random Lyapunov exponents \( \lambda_2 \leq \lambda_1 = \lambda \). The top Lyapunov exponent is given by:

\[
\lambda = \lim_{t \to \infty} \sup \log \sqrt{x(t)^2 + y(t)^2}.
\]
Applying the change to polar coordinates:

\[ x(t) = r(t) \cos(\theta(t)), \quad y(t) = r(t) \sin(\theta(t)) \]

by writing the Itô formula for

\[ h_1(x, y) = \frac{1}{2} \log(x^2 + y^2) = \log(r), \quad h_2(x, y) = \arctan \left( \frac{y}{x} \right) = \theta. \]

we get:

**Proposition 1** [5]. The formulas

\[
\log \left( \frac{r(t)}{r(0)} \right) = \int_0^t q_1(\theta(s)) + \frac{1}{2} (q_4(\theta(s))^2 - q_2(\theta(s))^2) \, ds + \int_0^t q_2(\theta(s)) \, dw(s), \tag{4}
\]

\[
\theta(t) = \theta(0) + \int_0^t q_3(\theta(s)) - q_2(\theta(s)) q_4(\theta(s)) \, ds + \int_0^t q_4(\theta(s)) \, dw(s), \tag{5}
\]

hold, where

\[
q_1(\theta) = a_{11} \cos^2(\theta) + (a_{12} + a_{21}) \cos \theta \sin \theta + a_{22} \sin^2 \theta,
\]

\[
q_2(\theta) = b_{11} \cos^2(\theta) + (b_{12} + b_{21}) \cos \theta \sin \theta + b_{22} \sin^2 \theta,
\]

\[
q_3(\theta) = a_{21} \cos^2(\theta) + (a_{22} - a_{11}) \cos \theta \sin \theta - a_{12} \sin^2 \theta,
\]

\[
q_4(\theta) = b_{21} \cos^2(\theta) + (b_{22} - b_{11}) \cos \theta \sin \theta - b_{12} \sin^2 \theta. \tag{6}
\]

As the expectation of the Itô stochastic integral is null

\[
E \int_0^t q_2(\theta(s)) \, dw(s) = 0,
\]

the Lyapunov exponent is given by:

\[
\lambda = \lim_{t \to \infty} \frac{1}{t} \log \left( \frac{r(t)}{r(0)} \right) = \lim_{t \to \infty} \frac{1}{t} E \int_0^t (q_1(\theta(s)) + \frac{1}{2} (q_4(\theta(s))^2 - q_2(\theta(s))) \, ds.
\]

Applying the Oseledec theorem, if \( r(t) \) is ergodic, we get:

\[
\lambda = \int_0^t (q_1(\theta) + \frac{1}{2} (q_3(\theta)^2 - q_2(\theta))) p(\theta) \, d\theta,
\]

where \( p(\theta) \) is the probability distribution of the process \( \theta \).
An approximation of this distribution is calculated by solving the Fokker-Planck equation.

The Fokker-Planck (FPE) equation associated with equation (5) for \( p(t, \theta) \) is

\[
\frac{\partial p}{\partial t} + \frac{\partial}{\partial \theta} \left( (q_3(\theta) - q_2(\theta)q_4(\theta))p \right) - \frac{1}{2} \frac{\partial^2}{\partial \theta^2} (q_4(\theta)^2 p) = 0. \quad (7)
\]

From (7), it results that the solution \( p(\theta) \) of the FPE is solution of the following first order equation:

\[
(-q_3(\theta) + q_1(\theta)q_4(\theta) + q_2(\theta)q_5(\theta))p(\theta) + \frac{1}{2} q_4(\theta)^2 p'(\theta) = p_0, \quad (8)
\]

where \( p' = \frac{dp}{d\theta} \) and

\[
q_5(\theta) = -(b_{12} + b_{21}) \sin 2\theta - (b_{22} - b_{11}) \cos 2\theta.
\]

**Proposition 2** \([5]\). If \( q_4(\theta) \neq 0 \), the solution of the equation (8) is given by:

\[
p(\theta) = \frac{k}{D(\theta)q_4(\theta)^2} \left( 1 + \eta \int_0^\theta D(u) \, du \right)
\]

where \( k \) is determined by the normality condition

\[
\int_0^{2\pi} p(\theta) \, d\theta = 1
\]

and

\[
\eta = \frac{D(2\pi) - 1}{\int_0^{2\pi} D(u) \, du}.
\]

The function \( D \) is given by:

\[
D(\theta) = \exp \left( -2 \int_0^\theta \frac{q_3(u) - q_2(u)q_4(u) - q_4(u)q_5(u)}{q_4(u)^2} \, du \right)
\]

A numerical solution of the phase distribution could be performed by a simple backward difference scheme.
We consider \( N \in \mathbb{R}_+, h = \frac{\pi}{N} \) and

\[
q_1(i) = a_{11} \cos^2(ih) + (a_{12} + a_{21}) \cos(ih) \sin(ih) + a_{22} \sin^2(ih),
q_2(i) = b_{11} \cos^2(ih) + (b_{12} + b_{21}) \cos(ih) \sin(ih) + b_{22} \sin^2(ih),
q_3(i) = a_{21} \cos^2(ih) + (a_{22} - a_{11}) \cos(ih) \sin(ih) - a_{12} \sin^2(ih),
q_4(i) = b_{21} \cos^2(ih) + (b_{22} - b_{11}) \cos(ih) \sin(ih) - b_{12} \sin^2(ih),
q_5(i) = -(b_{12} + b_{21}) \sin(2ih) - (b_{22} - b_{11}) \cos(2ih), \quad i = 0, \ldots, N
\]

The function \( p(i), i = 0, \ldots, N \) is given by the following relations:

\[
p(i) = (p(0) + \frac{q_4(i)^2 p(i - 1)}{2h}) F(i)
\]

where

\[
F(i) = \frac{2h}{2h - q_3(i) + q_2(i)q_4(i) + q_4(i)q_5(i) + q_4(i)^2}.
\]

The Lyapunov exponent is \( \lambda = \lambda(N) \), where

\[
\lambda(N) = \sum_{i=0}^{N} (q_1(i) + \frac{1}{2}(q_4(i)^2 - q_2(i)^2)) p(i)h.
\]

**Proposition 3** If the matrix \( B \) is given by:

\[
b_{11} = \alpha, b_{12} = -\beta, b_{21} = \beta, b_{22} = \alpha
\]

then

\[
p(\theta) = \frac{k}{\beta^2} \exp\left\{ \frac{1}{\beta^2}((a_{21} - a_{12} - \alpha\beta)\theta + \frac{1}{2}(a_{11} - a_{22}) \cos 2\theta + \frac{1}{2}(a_{21} - a_{12}) \sin 2\theta) \right\}
\]

\[
k = \frac{\beta^2}{\int_{0}^{2\pi} \exp\left\{ \frac{1}{\beta^2}((a_{21} - a_{12} - \alpha\beta)\theta + \frac{1}{2}(a_{11} - a_{22}) \cos 2\theta + \frac{1}{2}(a_{21} - a_{12}) \sin 2\theta) d\theta \right\}}
\]

\[
\lambda = \frac{1}{2}(a_{11} + a_{22} + \beta^2 - \alpha^2) + \frac{1}{2}(a_{11} - a_{22})c_2 + \frac{1}{2}(a_{21} + a_{12})s_2,
\]

where

\[
c_2 = \int_{0}^{2\pi} \cos(2\theta)p(\theta) d\theta, \quad s_2 = \int_{0}^{2\pi} \sin(2\theta)p(\theta) d\theta.
\]
3 The Lyapunov exponent for an economic game with stochastic dynamics.

Two firms enter the market with a homogenous consumption product. The elements which describe the model are: the quantities which enter the market from the two firms \( x_i \geq 0, \ i = 1, 2 \); the inverse demand function \( p : \mathbb{R}_+ \to \mathbb{R}_+ \) \( p \) is a derivable function with \( p'(x) < 0, \lim_{x \to a_1} p(x) = 0, \)
\( \lim_{x \to 0} p(x) = b_1, \ a_1 \in \mathbb{R}, \ b_1 \in \mathbb{R} \); the cost functions \( C_i : \mathbb{R}_+ \to \mathbb{R}_+ \) \( C_i \) are derivable functions with \( C_i'(x_i) > 0, \ C_i'' \geq 0, \ i = 1, 2 \).

In our study we consider \( p(x) = \frac{1}{x}, x > 0 \) and \( C_i(x_i) = c_i x_i + d_i, i = 1, 2 \).

The mathematical model of the stochastic dynamic economic game is described by the stochastic system of equations:

\[
\begin{align*}
    x_1(t) &= x_1(0) + k_1 \int_0^t \left( \frac{x_2(s)}{(x_1(s) + x_2(s))^2} - c_1 \right) ds + \int_0^t (b_{11} x_1(s) + b_{12} x_2(s) + \gamma_1) dw(s) \\
    x_2(t) &= x_2(0) + k_2 \int_0^t \left( \frac{x_1(s)}{(x_1(s) + x_2(s))^2} - c_2 \right) ds + \int_0^t (b_{21} x_1(s) + b_{22} x_2(s) + \gamma_2) dw(s)
\end{align*}
\]

where \( b_{ij} \in \mathbb{R}, i, j = 1, 2, k_1 > 0, k_2 > 0, x_i(t) = x_i(t, \omega), i = 1, 2 \).

\[
\gamma_1 = -\frac{b_{11} c_2 + b_{12} c_1}{(c_1 + c_2)^2}, \quad \gamma_2 = -\frac{b_{21} c_2 + b_{22} c_1}{(c_1 + c_2)^2}.
\]

For \( b_{ij} = 0, i, j = 1, 2 \) model (9) is reduced to the classical model of the economic game [2], [8].

The system of stochastic equations (9), (SDE), has the form (2) from section 2, where:

\[
\begin{align*}
    f_1(x_1, x_2) &= \frac{x_2}{(x_1 + x_2)^2} - c_1, \ g_1(x_1, x_2) = b_{11} x_1 + b_{12} x_2 + \gamma_1, \\
    f_2(x_1, x_2) &= \frac{x_1}{(x_1 + x_2)^2} - c_2, \ g_2(x_1, x_2) = b_{21} x_1 + b_{22} x_2 + \gamma_2.
\end{align*}
\]

Applying the results from section 2, we have:

**Proposition 4** (i) The stationary state of (SDE) (9) is given by:

\[
\begin{align*}
    x_{10} &= \frac{c_2}{(c_1 + c_2)^2}, \quad x_{20} = \frac{c_1}{(c_1 + c_2)^2}.
\end{align*}
\]
(ii) The elements of the matrix $A$, which characterize linearized equation (9) in $(x_{10}, x_{20})$ are:

$$a_{11} = -2k_1c_1(c_1 + c_2), \quad a_{12} = -k_1(c_1^2 - c_2^2)$$

$$a_{21} = k_2(c_1^2 - c_2^2), \quad a_{22} = -2k_2c_2(c_1 + c_2);$$

(iii) The roots of the characteristic equation:

$$\mu^2 - (a_{11} + a_{22})\mu + a_{11}a_{22} - a_{12}a_{21} = 0 \quad (10)$$

have the real part:

$$\text{Re}(\mu_{1,2}) = -(k_1c_1 + k_2c_2)(c_1 + c_2);$$

(iv) If $b_{11} = \alpha$, $b_{12} = -\beta$, $b_{21} = \beta$, $b_{22} = \alpha$, $\beta \neq 0$, then the Lyapunov coefficient of (SDE) (6) is:

$$\lambda = -(k_1c_1 + k_2c_2)(c_1 + c_2) + \frac{1}{2}(\beta^2 - \alpha^2) - (k_1c_1 - k_2c_2)(c_1 + c_2)D_2 +$$

$$+ \frac{1}{2}(k_2 - k_1)(c_1^2 - c_2^2)E_2 \quad (11)$$

where

$$D_2 = \int_0^{2\pi} \cos(2\theta)p(\theta)d\theta, \quad E_2 = \int_0^{2\pi} \sin(2\theta)p(\theta)d\theta$$

and

$$p(\theta) = kg(\theta), \quad k = \frac{1}{\int_0^{2\pi} g(\theta)d\theta},$$

$$g(\theta) = \frac{1}{\beta^2} \exp\left\{ \frac{1}{\beta^2}((k_1 + k_2)(c_1^2 - c_2^2) + \alpha\beta)\theta - (k_1c_1 - k_2c_2)(c_1 + c_2)\cos(2\theta) + \right.$$  

$$+ \frac{1}{2}(k_1 + k_2)(c_1^2 - c_2^2)\sin(2\theta)\}.$$

4 Numerical Simulations.

We have done the numerical simulations using a program in Maple 12. For $c_1 = 0.2$, $c_2 = 2$, $k_1 = 0.2$, $k_2 = 0.4$, $\beta = 2$, in figure 1 is displayed $(\alpha, \lambda(\alpha))$, where $\lambda(\alpha)$ is given by (11). For $\alpha \in (-\infty, -1.2) \cup (1.1, \infty)$, the Lyapunov exponent is negative, then (SDE) has an asymptotically stable stationary state. For $\alpha \in (-1.2, 1.1)$, the Lyapunov exponent is positive and (SDE) has an asymptotically unstable stationary state.

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If $\beta$ is a real parameter and $\alpha = 2$, the figure 2 shows the behavior of the top Lyapunov exponent as a function of $\beta$: $(\beta, \lambda(\beta))$.

For $\beta \in (-\infty, -2.6) \cup (2.6, \infty)$ the Lyapunov exponent is positive and (SDE) has an asymptotically unstable stationary state. For $\beta(-2.6, 2.6)$ the Lyapunov exponent is negative and (SDE) has an asymptotically stable stationary state.
The Euler second order scheme for (SDE) (2) is given by:

\[
x_1(n+1) = x_1(n) + h \left( \frac{x_2(n)}{(x_1(n) + x_2(n))^2} - c_1 \right) + (b_{11} x_1(n) + b_{12} x_2(n) + \gamma_1) G(n) + \frac{(b_{11} x_1(n) + b_{12} x_2(n) + \gamma_1) G(n)^2 - h}{2} + \frac{(b_{11} x_1(n) + b_{12} x_2(n) + \gamma_1) G(n)^2 - h}{2} + \left( b_{11} - \frac{2x_2(n)}{(x_1(n) + x_2(n))^3} \right) (b_{11} x_1(n) + b_{12} x_2(n) + \gamma_1) h G(n) \frac{hG(n)}{2},
\]

\[
x_2(n+1) = x_2(n) + h \left( \frac{x_1(n)}{(x_1(n) + x_2(n))^2} - c_2 \right) + (b_{21} x_1(n) + b_{22} x_2(n) + \gamma_2) G(n) + \frac{(b_{21} x_1(n) + b_{22} x_2(n) + \gamma_2) G(n)^2 - h}{2} + \frac{(b_{21} x_1(n) + b_{22} x_2(n) + \gamma_2) G(n)^2 - h}{2} + \left( b_{21} - \frac{2x_1(n)}{(x_1(n) + x_2(n))^3} \right) (b_{21} x_1(n) + b_{22} x_2(n) + \gamma_2) h G(n) \frac{hG(n)}{2},
\]

where \( G(n) = w((n+1)h) - w(nh), \ n = 1, 2, \ldots, \) and \( x_i(n) = x_i(nh, \omega), \ i = 1, 2. \)

In figures 3 and 4 are displayed the orbits: \((n, x_1(n, \omega))\) for (SDE) and \((n, x_1(n))\) for (ODE):

![Fig 3. \((n, x_1(n, \omega))\)](image1)

![Fig 4. \((n, x_1(n))\)](image2)
In figures 5 and 6 are displayed the orbits: \((n, x_2(n, \omega))\) for (SDE) and \((n, x_2(n))\) for (ODE):

Fig 5. \((n, x_2(n, \omega))\)  
Fig 6. \((n, x_2(n))\)

In figures 7 and 8 are displayed the orbits: \((x_1(n, \omega), x_2(n, \omega))\) for (SDE) and \((x_1(n), x_2(n))\) for (ODE):

Fig 7. \((x_1(n, \omega), x_2(n, \omega))\)  
Fig 8. \((x_1(n), x_2(n))\)

5 Conclusions.

In the present paper we investigate an economic game with stochastic dynamics. We focus on a particular game and determine the Lyapunov exponent for the stochastic system of equations that describes the mathematical model. The calculation of the top Lyapunov exponent enables us to decide
whether a stochastic system is stable or not. Using a program in Maple 12, we display the Lyapunov exponent and the system orbits.

References

[1] Arnold, L., *Random dynamical systems*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 1998.

[2] Bundău O., Neamțu M., Opriș D., *Rent seeking games with tax evasion*, Pannonian Applied Mathematical Meetings, B.A.M-CX, 2007; 2295-2315: 121-128; arXiv:math.DS. 0706.0664v1, 6 june, 2007.

[3] Bischi G.I., Naimzada A., *Global analysis of a dynamic duopoly game with bounded rationality*. In Advances in Dynamic Games and Applications, Vol. 5., Birkhaur, Boston, 1999.

[4] Cournot A., *Researches into the principles of the theory of wealth*. Engl. trans., Chapter VII, Irwin Paper Back Classics in Economics, 1963.

[5] Jedrzejewski, F., Brochard, D., *Lyapunov exponents and stability in stochastic dynamical structures*.

[6] Hu. B.Y., Mahommed, S.E., Yan, F., *Discrete-time approximation of stochastic delay equations*, The Annals of Probability, vol. 32, Nr 1A (2004), 265-314.

[7] Kloeden, P.E., Platen, E., *Numerical solution of stochastic differential equations*, Springer-Verlag, 1992.

[8] Mircea G., Neamțu M., Opriș D.: *Dynamical systems from economy, mechanics, biology described by differential equations with time delay (in Romanian)*, Mirton Publishing house, (2003).

[9] Oseledec, V.I., *A multiplicative Ergodic theorem, Lyapunov characteristic numbers for dynamical systems*, Trans. Moscow Math. Soc. 1968, no.19, 197-231.

[10] Puu T., *Chaos in duopoly pricing*. Chaos, Solitons and Fractals 1991; 1: 573-581.
[11] Puu T., *The Chaotic duopolists revisited*. J. of Economic Behavior and Organization Vol. 1998; 37: 385-394.

[12] Puu T., *Attractors, bifurcations and chaos: Nonlinear phenomena in economics*, Springer-Verlag, 2000.

[13] Rosser J.B., *The development of complex oligopoly dynamics theory*. In *Text Book Oligopoly Dynamics: Models and Tools* Springer-Verlag, 2002.

[14] Schurz, H., *Moment contractivity and stability exponents of nonlinear stochastic dynamical systems*, IMA Print Series #1656, 1999.