Hausdorff dimension of the sets of Li-Yorke pairs for some chaotic dynamical systems including $A$-coupled expanding systems

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Abstract

In this paper we consider Hausdorff dimension of the sets of Li-Yorke pairs for some chaotic dynamical systems including $A$-coupled expanding systems. We prove that Li-Yorke pairs of $A$-coupled-expanding system under some conditions have full hausdorff dimension on the invariant set. We generalize the result of [8] on the Hausdorff dimension of Li-Yorke pairs of dynamical systems topologically conjugate to the full shift and having a self-similar invariant set to the case of dynamical system conjugated to some kind of subshifts. And Hausdorff dimension of ”chaotic invariant set” for some kind of $A$-coupled-expanding maps is shown.

Keywords: Li-Yorke chaos, coupled-expanding map, Hausdorff dimension

1 Introduction

The term “chaos” was introduced firstly into mathematics in the paper of Li-Yorke [1] that is based on the existence of Li-Yorke pairs. Li-Yorke pairs are the pairs of points that approach each other for some sequence of moments in the time evolution and that remain separated for other sequences of moments. In [8] was discussed on the Hausdorff dimension of the set of Li-Yorke pairs for some simple classical chaotic dynamical systems. They have shown that, if the dynamical system is topologically conjugate to the full shift symbolic dynamical system and its invariant set is self-similar or a product of self-similar sets and, then its Li-Yorke pairs have full Hausdorff dimension.

On the other hand, the coupled-expanding and $A$-coupled-expanding with a transitive matrix $A$ has been recognized as one of the important criteria of chaos which can be found in [3, 4, 5, 6, 7, 13]. There have been obtained some results on the chaotic properties of

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$A$-coupled-expanding map including chaos in the sense of Li-Yorke and Devaney and its topological entropy otherwise we couldn’t find yet any result on dimensional-theoretical research for it.

Therefore it seems natural to study on the Hausdorff dimension of the set of Li-Yorke pairs for the $A$-coupled-expanding map and to generalize the result of [8] on the Hausdorff dimension of Li-Yorke pairs concerned with the shift, to the case of concerned with subshift.

In this paper we consider these topics by using the concept of symbolic geometric construction which was defined in [9].

The rest of this paper is organized as follows. In Section 2 some basic concepts which will be used later are introduced. In section 3 we prove that for some $A$-coupled-expanding maps under some conditions, their invariant Cantor sets on which they are topologically conjugate to the subshift $\sigma_A$ become limit sets of a symbolic geometric construction concerning the basic sets of the map (Theorem 3.1). This means that Li-Yorke pairs of $A$-coupled-expanding map under some conditions have full Hausdorff dimension on the invariant set (Remark 3.1). We also prove a theorem on the Hausdorff dimension of the set of Li-Yorke pairs for a strictly coupled-expanding map (Thorem 3.2). In section 4, we generalize the result of [8] on the Hausdorff dimension of Li-Yorke pairs of dynamical systems conjugate to a shift and having a self-similar invariant set, to the case of dynamical system conjugated to some kind of subshifts (Theorem 4.1). And by using Theorem 4.1, we obtain a result on Hausdorff dimension of the set of Li-Yorke pairs of a strictly $A$-coupled-expanding map of some special matrices $A$ under some conditions (Theorem 4.2).

2 Preliminaries

In this section we introduce main concepts which are used in this paper. All the others concerned with topological and symbolic dynamics are refered to the notations in [2].

**Definition 2.1**[9] Let $(X, T)$ be a dynamical system on metric space $X$. A pair of points $(x, y) \in X^2$ is said to be Li-Yorke pair for $T$ if

$$\liminf_{n \to \infty} d(T^n x, T^n y) = 0 \quad \text{and} \quad \limsup_{n \to \infty} d(T^n x, T^n y) > 0.$$ 

Given an invariant set $\Lambda \subset X$, i.e. $f(\Lambda) = \Lambda$, we define the set of Li-Yorke pairs in $\Lambda$ for $T$ by

$$LY_T(\Lambda) = \{(x, y) \in \Lambda^2 | (x, y) \text{ is a Li-Yorke pair}\}.$$ 

We say that Li-Yorke pairs in $\Lambda$ have full Hausdorff dimension for $T$ if the Hausdorff dimension of $LY_T(\Lambda)$ coincides with the Hausdorff dimension of $\Lambda^2$, i.e.

$$\dim_H(LY_T(\Lambda)) = \dim_H(\Lambda^2).$$

**Definition 2.2**[5] Let $(X, d)$ be a metric space and $f : D \subset X \to X$. Let $A = ((A)_{ij})_{m \times m}$ be a $m \times m$ transitive matrix, where $m \geq 2$. If there exist $m$ nonempty subsets $V_i (1 \leq
$i \leq m$) of $D$ with pairwise disjoint interiors such that

$$f(V_i) \supset \bigcup_{j: (A)_{ij}=1} V_j, \quad 1 \leq i \leq m$$

then $f$ is said to be $A$-coupled-expanding map in $V_i$, $1 \leq i \leq m$.

Further, the map $f$ is said to be strictly $A$-coupled-expanding map in $V_i$, $1 \leq i \leq m$ if $d(V_i, V_j) > 0$ for all $1 \leq i \neq j \leq m$. In the special case that all entries of $A$ are 1s, the (strictly) $A$-coupled-expanding map is said to be (strictly) coupled-expanding map.

**Definition 2.3**[9] Let $\Sigma^+_m = \{(i_1 \ldots i_n \ldots) : i_j = 1, \ldots, m\}$. Let $Q \subset \Sigma^+_m$ be an invariant set under the shift $\sigma$ on $\Sigma^+_m$ and $\{\Delta_{i_1 \ldots i_n}\}$, $(i_j = 1, \ldots, m)$ be a family of closed sets in $\mathbb{R}^d$ called as basic sets where $\{i_1 \ldots i_n\}$ is an admissible $n$-tuple with respect to $Q$, i.e., there exists $(j_1 \ldots j_n \ldots) \in Q$ such that $j_1 = i_1, \ldots, j_n = i_n$.

If for any admissible tuple $(i_1 \ldots i_n i_{n+1})$ with respect to $Q$ it is satisfied that

(i) $\Delta_{i_1 \ldots i_n i_{n+1}} \subset \Delta_{i_1 \ldots i_n}$

(ii) $\Delta_{i_1 \ldots i_n} \cap \Delta_{j_1 \ldots j_n} = \emptyset$, $(i_1 \ldots i_n) \neq (j_1 \ldots j_n)$

and

$$\lim_{n \to \infty} \max_{\text{admissible}} \text{diam}(\Delta_{i_1 \ldots i_n}) = 0$$

then we call a pair $(Q, \{\Delta_{i_1 \ldots i_n}\})$ symbolic geometric construction. And the set

$$F = \bigcap_{n=1}^\infty \bigcup_{\text{admissible}} \Delta_{i_1 \ldots i_n}$$

is said to be limit set of it.

This limit set is Cantor-like set, i.e., it is perfect, nowhere dense and totally disconnected set. The geometric construction is said to be a simple geometric construction if $Q = \Sigma^+_m$ and is said to be Markov geometric construction if $Q = \Sigma^+_m (A)$ for transitive matrix $A$(see [5] for $\Sigma^+_m (A)$).

**Definition 2.4**[11] Let $S_i : \mathbb{R}^d \to \mathbb{R}^d (1 \leq i \leq m)$ be a contraction map with contract ratio coefficient $c_i$, i.e., $|S_i(x) - S_i(y)| = c_i|x - y|$, $0 < c_i < 1$.

If $K = \bigcup_{i=1}^N S_i(K)$, then $K$ is said to be invariant set with respect to $S = \{S_1, \ldots, S_N\}$.

If $K$ is invariant set with respect to $S = \{S_1, \ldots, S_N\}$ and for any $\alpha$ with $\sum_{i=1}^N c_i^\alpha = 1$, satisfies that

$$H^\alpha(K) > 0, \quad H^\alpha(K_i \cap K_j) = 0, \quad (i \neq j)$$

3
then $K$ is said to be self-similar set where $H^\alpha$ is $\alpha$ dimension Hausdorff measure and $K_i = S_i(K)$.

3 $A$-coupled-expanding map with symbolic geometric construction and Hausdorff dimension of the set of Li-Yorke pairs for it.

We now prove that for some $A$-coupled-expanding maps under some conditions, their invariant Cantor sets on which they are topologically conjugate to the subshift $\sigma_A$ refer to the limit sets of symbolic geometric construction concerning the basic sets of the maps, so that their Li-Yorke pairs have full Hausdorff dimension. And we consider Hausdorff dimension of the set of Li-Yorke pairs for a strictly coupled-expanding map.

**Lemma 3.1.** Let $(X,d)$ be a metric space, $f : X \subset D \rightarrow X$ be a map and $A$ be an $m \times m$ irreducible transitive matrix such that

$$\exists i_0 (1 \leq i_0 \leq m), \quad \sum_{j=1}^{m} (A)_{i_0 j} \geq 2.$$ 

Assume that there are $m$ compact subsets $V_i (1 \leq i \leq m)$ of $X$ with pairwise disjoint interiors such that $f$ is continuous and satisfies followings:

i) $f$ is a strictly $A$-coupled-expanding map on the $V_i (1 \leq i \leq m)$,

ii) there exist some constants $\lambda_1, \ldots, \lambda_m (\lambda_i > 1)$ such that

$$d(f(x), f(y)) = \lambda_i d(x, y), \quad x, y \in V_i (1 \leq i \leq m)$$

Then $f$ has an invariant Cantor set $V \subset \bigcup_{i=1}^{m} V_i$ such that $f$ in $V$ is topologically conjugated to the subshift $\sigma_A$.

**Proof.** Put $\lambda_0 = \min_{1 \leq i \leq m} \lambda_i$, then for any $x, y \in V_i$,

$$d(f(x), f(y)) \geq \lambda_0 d(x, y).$$

Thus the desired result follows immediately from the theorem 5.2 of [5].

Next theorem shows that for some $A$-coupled maps, their invariant sets on which they are topologically conjugate to the subshift $\sigma_A$ have "symbolic geometric construction".

**Theorem 3.1.** Let $D \subset \mathbb{R}^d$ be a bounded and closed set and suppose that $f : D \rightarrow \mathbb{R}^d$ satisfies all assumptions of above Lemma. Put $Q = \Sigma_m^+ (A)$ and for admissible sequence $(a_0 a_1 \ldots a_n)$ put

$$\Delta_{a_0 a_1 \ldots a_n} = \bigcap_{j=0}^{n} f^{-j}(V_{a_j}).$$
Then a symbolic geometric construction with the family of basic sets \( \{ \Delta_{a_0 a_1 \ldots a_n} \} \) is constructed and the limit set of this construction is coincide with the set \( V \) in the Lemma, i.e.,
\[
\bigcap_{n=0}^{\infty} \bigcup_{(a_0 a_1 \ldots a_n) \text{ admissible}} \Delta_{a_0 a_1 \ldots a_n} = V.
\]

In other words \( f \)

**Proof.** It is easy to see that for any \( n \in \mathbb{N} \), \( \Delta_{a_0 a_1 \ldots a_{n+1}} \subset \Delta_{a_0 a_1 \ldots a_n} \) and
\[
f(\Delta_{a_0 a_1 \ldots a_n}) \subset \bigcap_{j=0}^{n} f^{1-j}(V_{a_j}) \subset \bigcap_{j=1}^{n} f^{1-j}(V_{a_j}) = \Delta_{a_1 \ldots a_n}.
\]

Therefore it follows inductively that
\[
f^n(\Delta_{a_0 a_1 \ldots a_n}) \subset \Delta_{a_n} = V_{a_n}.
\]

Otherwise for any \( x, y \in \Delta_{a_0 a_1 \ldots a_n} \subset V_{a_0} \), it follows that
\[
d(x, y) = \frac{1}{\lambda_{a_0}} d(f(x), f(y))
\]
and
\[
f(x), f(y) \in \Delta_{a_1 \ldots a_n} \subset V_{a_1},
\]
this means that
\[
d(f(x), f(y)) = \frac{1}{\lambda_{a_1}} d(f^2(x), f^2(y)).
\]

Proceeding with these processes, it holds that
\[
d(x, y) = \frac{1}{\lambda_{a_0} \lambda_{a_1} \ldots \lambda_{a_n}} d(f^n(x), f^n(y)) \leq \frac{1}{(\min \lambda_i)^n} \text{diam} V_{a_n} \leq \frac{1}{(\min \lambda_i)^n} \text{diam} D.
\]

It means that
\[
\max \text{diam} \Delta_{a_0 a_1 \ldots a_n} \leq \frac{1}{(\min \lambda_i)^n} \text{diam} D,
\]
therefore
\[
\lim_{n \to \infty} \max \text{diam} \Delta_{a_0 a_1 \ldots a_n} = 0.
\]

Otherwise since there is an invariant subset \( V \subset \bigcup_{i=1}^{m} V_i \) such that \( f \) in \( V \) is topologically conjugated to the subshift \( \sigma_A \) by Lemma 3.1, from Theory 4.1 in [5], \( \bigcap_{n=0}^{\infty} f^{-n}(V_{a_n}) \) is singleton for any \( \alpha = (a_0 a_1 \ldots a_n \ldots) \in \Sigma^+_m(A) \) and
\[ V = \bigcup_{\alpha \in \Sigma^+_m(A)} \bigcap_{n=0}^{\infty} f^{-n}(V_{a_i}). \]

Obviously
\[ \bigcup_{\alpha \in \Sigma^+_m(A)} \bigcap_{n=0}^{\infty} f^{-n}(V_{a_i}) = \bigcap_{n=0}^{\infty} \bigcup_{(i_1, \ldots, i_n) \text{ admissible}} \Delta_{a_{i_1}a_{i_2} \ldots a_{i_n}}. \]

**Remark 3.1** From Theorem 5.1 in [8] and above Theorem 3.1, it follows that Li-Yorks pairs of coupled-expanding map satisfying the conditions of Lemma 3.1 have full Hausdorff dimension in the invariant set \( V \).

Next Theorem is concerned with the Hausdorff dimension of this invariant set \( V \), that is, "chaotic set" for some kind of coupled-expanding dynamical systems.

**Theorem 3.2.** Let \( D \subset \mathbb{R}^n \) be a closed bounded set and \( f : D \to \mathbb{R}^n \) be a strictly coupled-expanding map in \( m \) disjointed compact subsets \( V_i \subset D^d(1 \leq i \leq m) \) and continuous in \( \bigcup_{i=1}^{m} V_i \). If there are some constants \( \lambda_1, \ldots, \lambda_m (\lambda_i > 1) \) such that
\[ d(f(x), f(y)) = \lambda_id(x, y), \quad x, y \in V_i (1 \leq i \leq m), \]
then the Hausdorff dimension of the limit set \( V \) for the symbolic geometric construction with \( (Q = \sum_{m}^{+}, \quad \Delta_{a_{i_1}a_{i_2} \ldots a_{i_n}} = \bigcap_{j=0}^{n} f^{-j}(V_{a_j})) \) is the solution of the equation
\[ \left( \frac{1}{\lambda_1} \right)^p + \ldots + \left( \frac{1}{\lambda_m} \right)^p = 1. \]

And if \( p_0 \) is the solution of this equation, then
\[ \dim_H LY_f(V) = 2p_0. \]

**Proof.** For any \( i \in 1, 2, \ldots, m \), put \( U_i = \{ \alpha = (a_0a_1 \ldots) \in \Sigma^+_m : a_0 = i \}. \) From the assumption for \( f \), for any \( \alpha = (a_0a_1 \ldots) \in \Sigma^+_m \) the set \( \bigcap_{n=0}^{\infty} f^{-n}(V_{a_i}) \) is a singleton. Now define a map \( g : \Sigma^+_m \to V \) as following:
\[ \alpha = (a_0a_1 \ldots) \mapsto \bigcap_{n=0}^{\infty} f^{-n}(V_{a_n}). \]

Then from Theorem 4.1 in [5] \( g \) is homeomorphism and holds that \( f \circ g = g \circ \sigma \). Obviously \( g(U_i) \subset V_i \) and it follows that
\[ f(g(U_i)) = g(\sigma(U_i)) = g(\Sigma^+_m) = V. \]
This means that $V$ can be formed by expanding of $\lambda_i$ times of $g(U_i)$. Therefore for any $i(1 \leq i \leq m)$ putting $S_i = (f|_{V_i})^{-1}|_V$, then $S_i$ is a contraction map with contract ratio coefficient $\frac{1}{\lambda_i}$. In fact, for any $x, y \in V$ there are $t, s \in V_i$ such that $f(t) = x, f(s) = y$ since $f$ is expanding in $V_i(1 \leq i \leq m)$. Therefore

$$S_i(x) = (f|_{V_i})^{-1} \circ f(t) = t,$$

$$S_i(y) = (f|_{V_i})^{-1} \circ f(s) = s.$$

and thus

$$d(S_i(x), S_i(y)) = d(t, s) = \frac{1}{\lambda_i}d(x, y).$$

Otherwise if $i \neq j$, then $S_i(V) \cap S_j(V) = \emptyset$ since $S_i(V) = g(U_i) \subset V_i$, therefore $V$ is a self-similar set. And it is clear that

$$V = \bigcup_{i=1}^{m} g(U_i) = \bigcup_{i=1}^{m} S_i(V).$$

Therefore by the Theorem 2 in [10] $\dim_H(V)$ is equal to the solution of the equation

$$\left(\frac{1}{\lambda_1}\right)^p + \ldots + \left(\frac{1}{\lambda_m}\right)^p = 1.$$

Otherwise, since $f$ satisfies in $V$ the condition of the Theorem 5.1 in [8], it follows that

$$\dim_H(LY_f(V)) = \dim_H(V \times V) = 2\dim_H V = 2p_0$$

where $p_0$ is a solution of above mentioned equation. \hfill \Box

4 Li-Yorke pairs of full dimension for systems conjugated to a subshift

In this section we generalize the result of [8] on the Hausdorff dimension of Li-Yorke pairs of dynamical systems topologically conjugate to a shift and having a self-similar invariant set, to the case of dynamical system topologically conjugate to some kind of subshifts, and consider Hausdorff dimension of "chaotic invariant set" for the systems.

We consider a kind of matrices as following:

$$A = \begin{pmatrix} 1 & & & & \\
& 1 & & & \\
& & \ldots & & \\
& & & 1 & \\
& & & & 1 \\
\end{pmatrix}$$

which has $i$ th row and $i$ th column consisted of 1s while other entries are arbitrary.
We are going to prove that the result of [8] above mentioned stands for the map conjugated to the subshift $\sigma_A$ for this kind of matrices $A$.

First, we consider the matrix $A$ which has $i$ th row and $i$ th column consist of 1s while other entries are all 0. Especially, to simplify our consideration we are going to fix the matrix $A$ as following while other cases can be treated similar to this:

$$
A = \begin{pmatrix}
1 & 1 & \cdots & 1 \\
1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 0 & \cdots & 0
\end{pmatrix}
$$

We can see that $\Sigma^+_m(A)$ and $\Sigma^+_m$ are homeomorphic. In fact, for any $s \in \Sigma^+_m(A)$, assume $\bar{s}$ is the sequence obtained from $s$ by eliminating one digit 1 which lies behind of elements different from 1 in $s$, and define a map $\Phi : \Sigma^+_m(A) \to \Sigma^+_m$, by $\Phi(s) = \bar{s}$, then we can see easily that $\Phi$ is homeomorphism.

Now let $\Lambda \subset \mathbb{R}^d$ be a self-similar set constructed by a family of contracting maps $S = \{S_1, \ldots, S_m\}$ satisfying:

1) the constant ratio coefficient of $S_i$ is $c_i$,

2) there exists a compact set $K \subset \mathbb{R}^d$ such that $S_i(K) \subset K$ for any $i(1 \leq i \leq m)$ and $S_i(K) \cap S_j(K) = \emptyset$ if $i \neq j$,

3) $\Lambda = \bigcup_{i=1}^{m} S_i(\Lambda)$.

And define a map $\pi : \Sigma^+_m \to \Lambda$ by

$$
\pi(\alpha) = \lim_{n \to \infty} S_{a_n} \circ \cdots \circ S_{a_0}(K), \quad \alpha = (a_0 a_1 \ldots) \in \Sigma^+_m.
$$

It is easy to see that $\pi$ is homeomorphism and a map $\pi_A : \Sigma^+_m(A) \to \Lambda$ defined by $\pi_A = \pi \circ \Phi$ is obviously homeomorphism.

Then we can get following Lemma.

**Lemma 4.1.** Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a dynamical system with a compact invariant set $\Lambda$. If $(\Lambda, f)$ homeomorphic conjugated to one-sided subshift $(\Sigma^+_m(A), \sigma_A)$, $f \circ \pi_A = \pi_A \circ \sigma_A$, and $\Lambda$ is self-similar then the Li-Yorke pairs have full Hausdorff dimension for $f$, i.e.,

$$
\dim_H(LY_f(\Lambda)) = \dim_H \Lambda \times \Lambda
$$

where the first row and the first column of $A$ consist of 1s and other entries are all 0.

**Proof.** Let $s \in \Sigma^+_m(A)$ and $N = (N_n)$ be a sequence of natural numbers. Consider a set as following:

$$
\Sigma^+_m^N(S) = \{t \in \Sigma^+_m(A) \mid t_k = s_k, k \in \{u_i, u_i + i\};
\text{ } t_{u_i+i+1} = 1, t_{u_i+i+2} = (1 + s_{u_i+i+2}) \text{ (mod } m), t_{u_i+i+3} = 1, t_{u_i+i+1-1} = 1, i = 0, 1, \ldots \}
$$
where \( u_0 = 0, u_1 = N_0 + 5 \) and \( u_i \) is given by the recursion \( u_{i+1} = u_i + N_i + i + 6 \).

Then an element \( t \in \Sigma_A^N(S) \) has a form as following:

\[
t = s_0 1\tilde{t}_2 1 \cdots 1 s_{u_1} s_{u_1+1} 1 \tilde{t}_{u_1+3} 1 \cdots 1 s_{u_2} s_{u_2+1} s_{u_2+2} 1 \tilde{t}_{u_2+4} 1 \cdots 1 \cdots
\]

where
\[
\tilde{t}_2 = (1 + s_2) \mod m \\
\tilde{t}_{u_1+3} = (1 + s_{u_1+3}) \mod m \\
\tilde{t}_{u_2+4} = (1 + s_{u_2+4}) \mod m \\

\ldots
\]

Then a pair \((s, t) \in \Sigma_+ m \times \Sigma_+ m(A), t \in \Sigma_A^N(s)\), is a Li-Yorke pair for \( \sigma_A \). In fact it holds that

\[
\lim_{i \to \infty} d(\sigma_A^{u_i}(s), \sigma_A^{u_i}(t)) \leq \lim_{i \to \infty} 2^{-i} = 0.
\]

\[
\lim_{i \to \infty} d(\sigma_A^{u_i+i+1}(s), \sigma_A^{u_i+i+1}(t)) \geq \frac{1}{2} > 0.
\]

Now define a map \( pr : \Sigma_+ m(A) \to \Sigma_+ N(A) \) by

\[
pr(t) = s_0 1\tilde{t}_2 1 t_0 \cdots t_{N_0-1} 1 s_{u_1} s_{u_1+1} 1 \tilde{t}_{u_1+3} t_{N_0} \cdots t_{N_0+N_1-1} \cdots \in \Sigma_+ N(A)
\]

for \( t = (t_0 t_1 \ldots) \in \Sigma_+ m(A) \). We can see that the map \( pr \) is continuous injection. Let

\[
\bar{\Sigma} = pr(\Sigma_+ m(A)) \subset \Sigma_+ N(s) \\
\Lambda_N(s) = \pi_A(\Sigma_A^N(s)) \\
\tilde{\Lambda} = \pi_A(\bar{\Sigma}).
\]

Then \( \tilde{\Lambda} \subset \Lambda_N(s) \).

Now we prove that

\[
\dim_H \Lambda_N(s) = \dim_H \Lambda = D
\]

where \( D \) is a solution of the equation

\[
c_1^D + \ldots + c_m^D = 1
\]

for a sequence \((N_n)\) satisfying

\[
\lim_{M \to \infty} \frac{(M + 6)^2}{M-1 \sum_{n=0}^M N_n} = 0
\]

(for example \( N_n = n^2 \)).

It is clear that \( \dim_H \Lambda = D \) from the Theorem 2 in [10]. Since \( \Lambda_N(s) \subset \Lambda \), this yields \( \dim_H \Lambda_N(s) \leq D \).
For the opposite inequality it is sufficient to prove that \( \dim H(\tilde{\Lambda}) \geq D \) because \( \Lambda_N(s) \supset \tilde{\Lambda} \).

Now let \( \nu \) be a Bernoulli measure on \( \Sigma_m^+ \) corresponding the probability vector \( (c_1^D, \ldots, c_m^D) \).

And define

\[
\mu = \nu \circ \Phi \circ pr^{-1} \circ \pi_A^{-1}
\]

then \( \mu \) becomes a probability measure on the \( \tilde{\Lambda} \).

If we prove that

\[
\liminf_{\rho \to 0} \frac{\log \mu(B_\rho(x))}{\log \rho} \geq D
\]

for any \( x \in \tilde{\Lambda} \), then we will have \( \dim H(\tilde{\Lambda}) \geq D \) using the Theorem 6.6.3 in [12].

For any \( x \in \tilde{\Lambda} \) there is an unique sequence \( \alpha = (a_0a_1\ldots) \in \Sigma_m^+ \) such that

\[
\pi_A \circ pr \circ \Phi^{-1}(\alpha) = x
\]

by bijectivity of the map \( pr \) on \( \tilde{\Sigma} \). Since \( \{\pi_A \circ pr \circ \Phi^{-1}([a_0, \ldots, a_k])\}_{k=0}^\infty \) is a contracting family of compact subsets containing \( x \), of which diameter goes to 0, we have

\[
\{x\} = \bigcap_{k=0}^\infty \pi_A \circ pr \circ \Phi^{-1}([a_0, \ldots, a_k])
\]

where

\[
[a_0, \ldots, a_k] = \{t = (t_i) \in \Sigma_m^+ | t_i = a_i \ \text{for} \ i = 1, \ldots, k\}.
\]

And it is clear that \( pr \circ \Phi^{-1}([a_0, \ldots, a_k]) \) is also cylinder set in \( \tilde{\Sigma} \).

For a cylinder \( [b_0, \ldots, b_k] \), denote \( c([b_0, \ldots, b_k]) \) as follows:

\[
c([b_0, \ldots, b_k]) = c_{b_1} \ldots c_{b_k},
\]

\[
c_i' = \begin{cases} 
c_i & i \neq 1 \\
c_1 & i = 1 \end{cases}.
\]

Then actually \( c([b_0, \ldots, b_k]) \) is a product of \( c_i \)'s because of the property of \( A \). Put

\[
d = \min_{i \neq j} \text{dist}(S_i(K), S_j(K))
\]

where

\[
\text{dist}(A, B) = \inf \{\text{dist}(x, y) | x \in A, y \in B\}.
\]

Then we can see easily that the sequence \( \{d \cdot c(pr \circ \Phi^{-1}([a_0, \ldots, a_k]))\}_{k=1}^\infty \) converges to 0 as \( k \to \infty \). Therefore for any \( \rho > 0 \) there is a \( k = k(\rho) \) such that

\[
d \cdot c(pr \circ \Phi^{-1}([a_0, \ldots, a_k])) \leq \rho < d \cdot c(pr \circ \Phi^{-1}([a_0, \ldots, a_{k-1}])).
\]

Assume that \( (\tilde{a}_0, \ldots, \tilde{a}_k) \neq (a_0, \ldots, a_k) \), i.e., there is an \( l \in \{0, \ldots, k\} \) such that \( \tilde{a}_l = a_l(i = 0, \ldots, l - 1), a_l \neq a_l \). Then we have

\[
\text{dist} \left( \pi_A \circ pr \circ \Phi^{-1}([a_0, \ldots, a_k]), \pi_A \circ pr \circ \Phi^{-1}([\tilde{a}_0, \ldots, \tilde{a}_k]) \right) \geq
\]

10
\[
\geq \text{dist} \left( \pi_A \circ pr \circ \Phi^{-1}([a_0, \ldots, a_l]), \{ \pi_A \circ pr \circ \Phi^{-1}([\bar{a}_0, \ldots, \bar{a}_l]) \} \right).
\]

We can denote \( pr \circ \Phi^{-1}([a_0, \ldots, a_{l-1}]) \) by \([u_0, \ldots, u_t]\) since it is a cylinder in \( \tilde{\Sigma} \). And put

\[
S_{[i_1, \ldots, i_k]} = S_{i_k} \circ \cdots \circ S_{i_1}.
\]

From the definition of \( \pi_A \) we have

\[
\pi_A \circ pr \circ \Phi^{-1}([a_0, \ldots, a_{l-1}]) \subset S_{\Phi([u_0, \ldots, u_t])}(K)
\]

\[
\pi_A \circ pr \circ \Phi^{-1}([a_0, \ldots, a_l]) \subset S_{\Phi([u_0, \ldots, u_t, a_l])}(K) \subset S_{\Phi([u_0, \ldots, u_t])}(K)
\]

\[
\pi_A \circ pr \circ \Phi^{-1}([\bar{a}_0, \ldots, \bar{a}_l]) \subset S_{\Phi([u_0, \ldots, u_t, \bar{a}_l])}(K) \subset S_{\Phi([u_0, \ldots, u_t])}(K).
\]

Therefore

\[
\text{dist} \left( \pi_A \circ pr \circ \Phi^{-1}([a_0, \ldots, a_l]), \{ \pi_A \circ pr \circ \Phi^{-1}([\bar{a}_0, \ldots, \bar{a}_l]) \} \right) \geq
\]

\[
\geq \text{dist} \left( S_{\Phi([u_0, \ldots, u_t, a_l])}(K), S_{\Phi([u_0, \ldots, u_t, \bar{a}_l])}(K) \right)
\]

\[
\geq d \cdot c([u_0, \ldots, u_t]) = d \cdot c(pr \circ \Phi^{-1}([a_0, \ldots, a_{l-1}])) > \rho.
\]

This means that

\[
\text{dist} \left( \pi_A \circ pr \circ \Phi^{-1}([a_0, \ldots, a_k]), \pi_A \circ pr \circ \Phi^{-1}([\bar{a}_0, \ldots, \bar{a}_k]) \right) > \rho,
\]

thus

\[
\Lambda \cap B_\rho(x) \subset \pi_A \circ pr \circ \Phi^{-1}([a_0, \ldots, a_k])
\]

and

\[
\mu(B_\rho(x)) \leq \mu(\pi_A \circ pr \circ \Phi^{-1}([a_0, \ldots, a_k])) = \nu([a_0, \ldots, a_k]) = (c_{a_0} \cdot \ldots \cdot c_{a_k})^D.
\]

Otherwise

\[
c(\Phi^{-1}([a_0, \ldots, a_k])) = c_{a_0} \cdot \ldots \cdot c_{a_k}.
\]

In fact, \( \Phi^{-1}(a_0a_1 \ldots) \) is the sequence obtained by setting 1 behind each digit of \((a_0a_1 \ldots)\) not being 1, and denoting the digits not being 1 by \(a_{n_1}, \ldots, a_{n_p}\) in order, we have

\[
c(\Phi^{-1}([a_0, \ldots, a_k])) = c_{a_0} \cdot \ldots \cdot c_{a_{n_1}-1} \cdot \frac{c_{a_{n_1}}}{c_1} \cdot c_{1} \cdot \ldots \cdot c_{a_{n_2}-1} \cdot \frac{c_{a_{n_2}}}{c_1} \cdot c_{1} \cdot \ldots \cdot c_{a_{n_{p-1}}-1} \cdot \frac{c_{a_{n_{p-1}}}}{c_1} \cdot c_{1} \cdot \ldots \cdot c_{a_k}
\]

\[
= c_{a_0} \cdot \ldots \cdot c_{a_k}.
\]
Therefore
\[ \mu(B_\rho(x)) \leq (c(\Phi^{-1}([a_0, \ldots, a_k])))^D = \]
\[ = \frac{(c(\Phi^{-1}([a_0, \ldots, a_k])))^D}{(c(pr \circ \Phi^{-1}([a_0, \ldots, a_k])))^D} \cdot (c(pr \circ \Phi^{-1}([a_0, \ldots, a_k])))^D \]
\[ \leq \frac{(c(\Phi^{-1}([a_0, \ldots, a_k])))^D}{(c(pr \circ \Phi^{-1}([a_0, \ldots, a_k])))^D} \cdot d^{-D} \cdot \rho^D. \]

It follows that
\[ \log \mu(B_\rho(x)) \leq D \left( \log \rho - \log d - \log (c(pr \circ \Phi^{-1}([a_0, \ldots, a_k])))^D \right) \]
\[ \leq D \left( \log \rho - \log d - \delta(k) \cdot \log \mathcal{C} \right) \]
where \( \mathcal{C} = \min_i c_i \) and
\[ \delta(k) = z(pr \circ \Phi^{-1}([a_0, \ldots, a_k])) - \sharp(\Phi^{-1}([a_0, \ldots, a_k])), \]
here \( z \) denotes the length of the cylinder. Then dividing by \( \log \rho \) we have
\[ \frac{\log \mu(B_\rho(x))}{\log \rho} \geq D + D \left( - \frac{\log d}{\log \rho} - \delta(k) \cdot \frac{\log \mathcal{C}}{\log \rho} \right) \geq \]
\[ \geq D + D \left( - \frac{\log d}{\log \rho} - \frac{\delta(k) \cdot \log \mathcal{C}}{\log d + \log c(pr \circ \Phi^{-1}([a_0, \ldots, a_{k-1}]))} \right) \]
\[ \geq D + D \left( - \frac{\log d}{\log \rho} - \frac{\delta(k) \cdot \log \mathcal{C}}{\log d + (k + \delta(k - 1)) \cdot \log \mathcal{C}} \right) \]
where \( \mathcal{C} = \max_i c_i \).

Since \( \lim_{\rho \to 0} k(\rho) = \infty \), now we prove that \( \lim_{k \to \infty} \frac{\delta(k)}{k} = 0. \)
For \( k \in \mathbb{N} \) there is a \( M = M(k) \) such that
\[ \sum_{n=0}^{M-1} N_n < 2(k + 1) \leq \sum_{n=0}^M N_n \]
where we notes \( \sharp(\Phi^{-1}([a_0, \ldots, a_k])) \) is no more than \( 2(k + 1) \). From the definition of \( \delta, \Sigma^A_N(s) \) and \( pr \), we have
\[ \delta(k) < \sum_{n=0}^M (n + 6) < (M + 6)^2 \]
\[ \frac{\delta(k)}{2(k + 1)} < \frac{(M + 6)^2}{\sum_{n=0}^{M-1} N_n} \rightarrow 0 (k \to \infty). \]
Therefore

\[ \frac{\delta(k)}{k} \to 0 \quad (k \to \infty) \]

and thus

\[ \liminf_{\rho \to 0} \frac{\log \mu(B_\rho(x))}{\log \rho} \geq D. \]

Now put

\[ \Pi_N = \{(s, t) | s \in \Sigma^+_m(A), t \in \Sigma^+_N(s)\}. \]

Since \((s, t) \in \Pi_N\) is Li-Yorke pair for \(\sigma_A\), we have

\[ \Pi_N \subset LY_{\sigma_A}(\Sigma^+_m(A)). \]

And putting

\[ S_A = \{(x, y) \in \Lambda \times \Lambda | x \in \Lambda, y \in \Lambda^A(\pi_A^{-1}(x))\} = \pi_A(\Pi_N) \]

we can get

\[ S_A \subset LY_f(\Lambda). \]

In fact, if \((s, t) \in LY_{\sigma_A}(\Sigma^+_m(A))\), then from its definition we have

\[ \liminf_{n \to \infty} d(\sigma^n_A(s), \sigma^n_A(t)) = 0, \quad \limsup_{n \to \infty} d(\sigma^n_A(s), \sigma^n_A(t)) > 0. \]

Since \(\pi_A\) is continuous, it follows

\[ \liminf_{n \to \infty} d(\pi_A(\sigma^n_A(s)), \pi_A(\sigma^n_A(t))) = 0, \quad \limsup_{n \to \infty} d(\pi_A(\sigma^n_A(s)), \pi_A(\sigma^n_A(t))) > 0. \]

and since \(f \circ \pi_A = \pi_A \circ \sigma_A\), we get

\[ \liminf_{n \to \infty} d(f^n(\pi_A(s)), f^n(\pi_A(t))) = 0, \quad \limsup_{n \to \infty} d(f^n(\pi_A(s)), f^n(\pi_A(t))) > 0. \]

Thus

\[ (\pi_A(s), \pi_A(t)) \in LY_f(\Lambda). \]

Now using \(\Pi_N \subset LY_{\sigma_A}(\Sigma^+_m(A))\) we get

\[ S_A = \pi_A(\Pi_N) \subset LY_f(\Lambda). \]

Otherwise the Theorem 4.1 in [8] implies that

\[ \dim_H S^A = \dim_H \Lambda \times \Lambda \]

and using \(S^A \subset LY_f(\Lambda) \subset \Lambda \times \Lambda\) we can get

\[ \dim_H LY_f(\Lambda) = \dim_H \Lambda \times \Lambda. \]
Next, let \( A \) be a matrix which has \( i \)th row and \( i \)th column consisted of 1s while other entries are arbitrary. As above, we assume that \( i = 1 \). Now, we will generalize definition of the map \( \Phi : \Sigma^+_m(A) \to \Sigma^+_m \). For any \( s \in \Sigma^+_m(A) \), assume \( \bar{s} \) is the sequence obtained from \( s \) by eliminating one digit which lies behind of elements different from 1 in \( s \), and define a map \( \Phi : \Sigma^+_m(A) \to \Sigma^+_m \), by \( \Phi(s) = \bar{s} \), then we can prove that \( \Phi \) is continuous surjection. (Note that eliminating digit might not be 1.)

Similarly to above consideration, we define a map \( \pi_A : \Sigma^+_m(A) \to \Lambda \) as \( \pi_A = \pi \circ \Phi \) (the map \( \pi : \Sigma^+_m \to \Lambda \) is already defined as \( \pi(\alpha) = \lim_{n \to \infty} S_{a_n} \circ \ldots \circ S_{a_0} (K), \alpha = (a_0 a_1 \ldots) \in \Sigma^+_m \) and then \( \pi_A \) is obviously continuous surjection.

Now by using Lemma 4.1 we can generalize the theorem 5.1 in [8] to the case of some kind of subshifts.

**Theorem 4.1.** Let \( \Lambda \) be a above mentioned self-similar compact set and \( f : \mathbb{R}^d \to \mathbb{R}^d \) be a dynamical system with the invariant set \( \Lambda \) and \( A \) be a matrix which has first row and first column consisted of 1s while other entries are arbitrary. If \( (\Lambda, f) \) is topologically semi-conjugated to an one-sided subshift \( (\Sigma^+_m(A), \sigma_A) \), \( f \circ \pi_A = \pi_A \circ \sigma_A \), then the Li-Yorke pairs have full Hausdorff dimension for \( f \), i.e.,

\[
\dim_H(LY_f(\Lambda)) = \dim_H \Lambda \times \Lambda
\]

**Proof.** Put

\[
A' = \begin{pmatrix}
1 & 1 & \cdots & 1 \\
1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 0 & \cdots & 0
\end{pmatrix}
\]

Then \( \Sigma^+_m(A') \subset \Sigma^+_m(A) \), and the restriction of \( \Phi \) to \( \Sigma^+_m(A') \) is homeomorphic. Thus

\[
\pi_A|_{\Sigma^+_m(A')} = \pi \circ \Phi|_{\Sigma^+_m(A')} = \pi_{A'}
\]

is also homorphism from \( \Sigma^+_m(A') \) to \( \Lambda \). Also

\[
f \circ \pi_{A'} = f \circ \pi_A|_{\Sigma^+_m(A')} = \pi_A \circ \sigma_A|_{\Sigma^+_m(A')} = \pi_{A'} \circ \sigma_{A'}
\]

and therefore \( (\Lambda, f) \) is homeomorphic conjugated to \( (\Sigma^+_m(A'), \sigma_{A'}) \), i.e,

\[
f \circ \pi_{A'} = \pi_{A'} \circ \sigma_{A'}
\]

From Lemma 4.1,

\[
\dim_H(LY_f(\Lambda)) = \dim_H \Lambda \times \Lambda.
\]

Next theorem is on the Hausdorff dimension of ”chaotic invariant set” for \( A \)-coupled-expanding systems for special matrix \( A \).
Theorem 4.2. Let $A$ be $m \times m$ transitive matrix which first row and first column consisted of 1s while other entries are all 0. Assume that there are $m$ disjoint compact subsets $V_i (1 \leq i \leq m) (m \geq 2)$ of $X$ such that $f$ satisfies the conditions in the Lemma 3.1, i.e., $f$ is continuous and satisfies followings:

i) $f$ is a strictly $A$-coupled-expanding map on the $V_i (1 \leq i \leq m)$,

ii) there exist some constants $\lambda_1, \ldots, \lambda_m (\lambda_i > 1)$ such that

$$d(f(x), f(y)) = \lambda_i d(x, y), x, y \in V_i (1 \leq i \leq m)$$

Then the Hausdorff dimension of the Cantor invariant subset $V \subset \bigcup_{i=1}^{m} V_i$ on which $f$ is topologically conjugated to subshift $\sigma_A$ (see lemma 3.1), is the solution of the equation

$$\left(\frac{1}{\lambda_1}\right)^p + \left(\frac{1}{\lambda_1 \lambda_2}\right)^p + \ldots + \left(\frac{1}{\lambda_1 \lambda_m}\right)^p = 1.$$ 

And if $p_0$ is the solution of this equation, then

$$\dim_H LY_f(V) = 2p_0.$$ 

Proof. Put

$$U_i = \{ \alpha \in \Sigma_m^+ (A) : a_0 = i \}.$$ 

Then we have

$$\sigma_A(U_i) = \Sigma_m^+ (A), \quad \sigma_A(U_1) = U_1 (2 \leq i \leq m).$$

Using the same way as in the proof of Theorem 3.2, there exists a homeomorphism $g : \Sigma_m^+ (A) \rightarrow V$ such that

$$f \circ g = g \circ \sigma_A \quad g(U_i) \subset V_i (i = 1, \ldots, m)$$

and

$$f(g(U_1)) = g(\sigma_A(U_1)) = V.$$ 

Therefore we can see that $V$ is obtained by $\lambda_1$ times expanding of $g(U_1)$. And since

$$f^2(g(U_i)) = f(g(\sigma_A(U_i))) = f(g(U_1)) = V \quad (2 \leq i \leq m),$$

we also can see that $V$ is obtained by $\lambda_1 \lambda_i$ times expanding of $g(U_i)$. This leads that by putting

$$S_1 = (f|_{V_1})^{-1}|_{V_i}, \quad S_i = (f^2|_{V_i})^{-1}|_{V_i},$$

$S_1, S_i$ are contracting maps with the contract ratio coefficients $\frac{1}{\lambda_1}, \frac{1}{\lambda_1 \lambda_i}$ and $V$ is the self-similar set for $\{S_1, \ldots, S_m\}$.

Thus, from the Theorem 2 in [10], $\dim_H V$ is the solution of the equation

$$\left(\frac{1}{\lambda_1}\right)^p + \left(\frac{1}{\lambda_1 \lambda_2}\right)^p + \ldots + \left(\frac{1}{\lambda_1 \lambda_m}\right)^p = 1.$$
And by using the Theorem 4.1, we have
\[
\dim_H \text{LY}_f(V) = \dim_H V \times V = 2\dim_H V = 2p_0.
\]

5 Conclusion

Through this work we have several interesting observations: ”Chaotic invariant set” for some kind of \(A\)-coupled-expanding maps refers to a limit set of symbolic geometric construction concerning the basic sets of them (in this paper, by ”chaotic invariant set” we mean the invariant Cantor set on which the map is topologically conjugate to the shift \(\sigma\) or subshift \(\sigma_A\) since these shift and subshift actually are all chaotic in several senses) and Li-Yorke pairs of these kind of \(A\)-coupled-expanding maps have full Hausdorff dimension in the invariant set. And the result of [8] on the Hausdorff dimension of Li-Yorke pairs of dynamical systems conjugate to a shift and having a self-similar invariant set is generalized to the case on some kind of subshifts. Moreover Hausdorff dimension of ”chaotic invariant set” for some kind of \(A\)-coupled-expanding maps has been counted.

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