Aspects of control theory on infinite-dimensional Lie groups and $G$-manifolds

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Abstract

We develop aspects of geometric control theory on Lie groups $G$ which may be infinite dimensional, and on smooth $G$-manifolds $M$ modelled on locally convex spaces. As a tool, we discuss existence and uniqueness questions for differential equations on $M$ given by time-dependent fundamental vector fields which are $L^1$ in time. We then discuss the closures of reachable sets in $M$ for controls in the Lie algebra $\mathfrak{g}$ of $G$, or within a compact convex subset of $\mathfrak{g}$. Regularity properties of the Lie group $G$ play an important role.

Classification: 22E65 (primary); 28B05, 34A12, 34H05, 46E30, 46E40.

Key words: infinite-dimensional Lie group, Fréchet-Lie group, regular Lie group, measurable regularity, exponential function, Trotter formula, local $\mu$-convexity, geometric control theory, left-invariant vector field, fundamental vector field, $G$-manifold, homogeneous space, reachable set, staircase function, compact set, extreme point, semigroup, bang-bang principle

1 Introduction and statement of main results

We lay some foundations for control theory on smooth $G$-manifolds $M$, for $G$ a Lie group modelled on a locally convex space. As a starting point, we discuss local existence and uniqueness for Carathéodory solutions to time-dependent fundamental vector fields on $M$ with $L^1$-time dependence. Assuming that $G$ is $L^1$-regular, we obtain results concerning reachable points.

Ordinary differential equations in Banach spaces are a classical topic (see, e.g., [5], [8], [30]). Beyond normable spaces, initial value problems on locally convex spaces need not have solutions, and may possess many solutions (see Example 6.1 and Example 6.2 in [32], also [20 §2.4]). But for special classes
of equations, specific results are available, notably when Lie groups come into play. If $G$ is a Lie group modelled on a locally convex space, then $G$ gives rise to a smooth left action $G \times TG \to TG$, $(g, v) \mapsto g.v$ on its tangent bundle $TG$ via left translation, $g.v := T\lambda_g(v)$ with $\lambda_g : G \to G$, $h \mapsto gh$. Let $e \in G$ be the neutral element. If $\gamma : [0, 1] \to g$ is a continuous path in the Lie algebra $g := T_eG$ of $G$, then the initial value problem

$$\dot{y}(t) = y(t).\gamma(t), \quad y(0) = e$$

has at most one $C^1$-solution $\eta : [0, 1] \to G$, by [33, Lemma 7.4]. If $\eta$ exists, it is called the evolution of $\gamma$ and denoted by $\operatorname{Evol} (\gamma) := \eta$. Let $k \in \mathbb{N}_0 \cup \{\infty\}$. If $\operatorname{Evol} (\gamma)$ exists for all $\gamma \in C^k([0, 1], g)$ and the map

$$\operatorname{evol} : C^k([0, 1], g) \to G, \quad \gamma \mapsto \operatorname{Evol} (\gamma)(1)$$

is smooth, then $G$ is called $C^k$-regular (see [17]). The $C^\infty$-regular Lie groups are also called regular; every $C^k$-regular Lie group is regular. Regularity is a central concept in infinite-dimensional Lie theory; see [33, 28, 29, 17, 20], and [23] for further information, notably the survey [34].

If $M$ is a smooth manifold modelled on a locally convex space, endowed with a smooth right $G$-action $\sigma : M \times G \to M$, then each $v \in g$ determines a smooth vector field

$$v_x : M \to TM, \quad x \mapsto (T_e\sigma(x, \cdot))(v)$$

on $M$, the fundamental vector field associated with $v$. If a continuous path $\gamma : [0, 1] \to g$ admits a $C^1$-evolution $\eta : [0, 1] \to G$, it is known that the differential equation

$$\dot{y}(t) = \gamma(t)v_x(y(t))$$

on $M$ satisfies local existence and uniqueness of $C^1$-solutions, since

$$\operatorname{Fl} : [0, 1] \times [0, 1] \times M \to M, \quad (t, t_0, y_0) \mapsto \sigma(y_0, \eta(t_0)^{-1} \eta(t))$$

is a globally defined $C^1$-flow for (1), see [20, Lemma 2.5.13].

As our starting point, we establish analogous existence and uniqueness results for Carathéodory solutions, when fundamental vector fields are $\mathcal{L}^1$ in time.

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1Smoothness of $\gamma$ and sequential completeness of $g$ (which are assumed in loc. cit.) are not used in the proof. The fact also follows from Lemmas 2.5.12 and 2.5.4 in [20].
For simplicity of the formulation, we restrict attention to Lie groups and manifolds modelled on Fréchet spaces in the remainder of the introduction, and also in much of Sections 2 through 6 and 8; in Section 10, we explain the necessary changes allowing us to extend the results to Lie groups and manifolds modelled on more general locally convex spaces. (The Fréchet property is also irrelevant in Sections 7 and 9).

For real numbers \( a < b \) and a Fréchet space \( E \), a function \( \gamma: [a,b] \to E \) is called \( L^1 \) if it is measurable with respect to the Borel \( \sigma \)-algebras on domain and range, the image \( \gamma([a,b]) \) contains a countable dense subset and

\[
\|\gamma\|_{L^1,q} := \int_{[a,b]} q(\gamma(t)) \, d\lambda_1(t) < \infty
\]

for each continuous seminorm \( q \) on \( E \) (where \( \lambda_1 \) denotes Lebesgue-Borel measure). Let \( N \subseteq L^1([a,b],E) \) be the set of \( L^1 \)-functions which vanish almost everywhere (with respect to \( \lambda_1 \)), and write \( [\gamma] := \gamma + N \) for \( \gamma \in L^1([a,b],E) \). Then \( L^1([a,b],E) := L^1([a,b],E)/N \) is a Fréchet space with respect to the locally convex vector topology given by the seminorms \( \| \cdot \|_{L^1,q} \) defined via \( \| [\gamma] \|_{L^1,q} := \| \gamma \|_{L^1,q} \) (see [18, Lemma 1.19]). Following [18, §3], we say that a function \( \eta: [a,b] \to E \) is absolutely continuous \( \footnote{If \( E \) is a Banach space, copying the classical \( \varepsilon-\delta \)-definition of absolute continuity of scalar-valued functions (as in [36, Definition 7.17]), one obtains a more general concept of absolutely continuous functions which need not be absolutely continuous in the above more limited sense (see [4]) unless \( E \) is sufficiently nice (e.g., a Hilbert space), see [7].} \) if there exists \( \gamma \in L^1([a,b],E) \) such that

\[
\eta(t) = \eta(a) + \int_a^t \gamma(s) \, ds \quad \text{for all } t \in [a,b],
\]

writing \( \int_a^t \gamma(s) \, ds \) for the weak integral \( \footnote{The same notation will denote weak integrals with respect to Lebesgue measure.} \int_{[a,t]} \gamma(s) \, d\lambda_1(s) \in E \). Then

\[
\gamma(t) = \eta'(t) \quad \text{for almost all } t \in [a,b],
\]

whence \( [\gamma] \in L^1([a,b],E) \) is uniquely determined by \( \eta \) (see, for instance, [18, Lemma 1.28]). We say that a function \( \eta: I \to E \) on an interval \( I \subseteq \mathbb{R} \) is absolutely continuous if \( \eta|_{[a,b]} \) is so for all real numbers \( a < b \) such that \( [a,b] \subseteq I \). If \( W \subseteq \mathbb{R} \times E \) is a subset and \( f: W \to E \) a function, then a function
\( \gamma : I \to E \) on a non-degenerate interval \( I \subseteq \mathbb{R} \) is called a Carathéodory solution to the differential equation
\[
y'(t) = f(t, y(t))
\]
if \((t, \gamma(t)) \in W\) for all \(t \in I\), the function \(\gamma\) is absolutely continuous, and
\[
\gamma'(t) = f(t, \gamma(t)) \quad \text{for almost all } t \in I.
\] (4)

If \((t_0, y_0) \in W\) is given and, moreover,
\[
t_0 \in I \quad \text{and} \quad \gamma(t_0) = y_0,
\] (5)
then \(\gamma\) is called a Carathéodory solution to the initial value problem
\[
y'(t) = f(t, y(t)), \quad y(t_0) = y_0.
\] (6)

If \(\gamma : I \to E\) is absolutely continuous, \((t, \gamma(t)) \in U\) for all \(t \in I\) and \(t_0 \in I\), then \(\gamma\) is a Carathéodory solution to (4) if and only if
\[
\gamma(t) = y_0 + \int_{t_0}^t f(s, \gamma(s)) \, ds \quad \text{for all } t \in I.
\] (7)

For the special case of Carathéodory solutions to differential equations in Banach spaces, see also [38, Chapter 30].

Since \(C^1\)-functions operate on absolutely continuous functions, we can speak about absolutely continuous functions \(\gamma : I \to M\) to a \(C^1\)-manifold \(M\) modelled on a Fréchet space (a continuous function which is absolutely continuous in local charts). Likewise, we can also speak about Carathéodory solutions to differential equations on \(M\) (see Sections 2 and 4 for details).

Let \(M\) be a \(C^1\)-manifold modelled on a locally convex space, \(J \subseteq \mathbb{R}\) be a non-degenerate interval and
\[
f : J \times M \to TM
\]
be a map such that \(f(t, y) \in T_y M\) for all \((t, y) \in J \times M\) (a time-dependent vector field). Or, more generally, consider a function \(f : W \to TM\) on an open subset \(W \subseteq J \times M\) such that \(f(t, y) \in T_y M\) for all \((t, y) \in W\). It is known that the differential equation
\[
\dot{y}(t) = f(t, y(t))
\]
satisfies local existence and uniqueness of \(C^1\)-solutions if it admits local \(C^1\)-flows (see [20, Lemma 2.5.10]; cf. [10, Appendix A] and [41, Appendix A.4] for special cases). Likewise, we have (with terminology as in Section 4):

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Proposition 1.1 Let $M$ be a $C^1$-manifold modelled on a Fréchet space, $J \subseteq \mathbb{R}$ be a non-degenerate interval and $f : W \to TM$ be a map on an open subset $W \subseteq J \times M$ such that $f(t,y) \in T_yM$ for all $(t,y) \in W$. If
\[
\dot{y}(t) = f(t,y(t))
\]
(8) admits local flows which are pullbacks of $C^1$-maps, then (8) satisfies local existence and uniqueness of Carathéodory solutions.

Let $G$ be a Lie group modelled on a Fréchet space with neutral element $e$ and Lie algebra $\mathfrak{g} = T_eG$. If $\gamma : [a,b] \to \mathfrak{g}$ is an $L^1$-map, then the initial value problem
\[
\dot{y}(t) = y(t).\gamma(t), \quad y(a) = e
\]
has at most one Carathéodory solution $\eta : [a,b] \to G$; as above, $\text{Evol}([\gamma]) := \text{Evol}(\gamma) := \eta \in AC([a,b], G)$ is called the evolution of $\gamma$ (cf. [18]). Write
\[
\text{evol}([\gamma]) := \text{evol}(\gamma) := \text{Evol}(\gamma)(b) \in G.
\]

If each $\gamma \in L^1([0,1], \mathfrak{g})$ has an evolution and
\[
\text{Evol} : L^1([0,1], \mathfrak{g}) \to C([0,1], G)
\]
is smooth as a map to the Lie group $C([0,1], G)$ of continuous $G$-valued maps on $[0,1]$ (or, equivalently, to the Lie group $AC([0,1], G)$ of absolutely continuous $G$-valued maps), then $G$ is called $L^1$-regular (see [18]).

Every Lie group $G$ modelled on a Banach space is $L^1$-regular, as well as $C^\ell(M, G)$ for each compact smooth manifold $M$ and $\ell \in \mathbb{N}_0 \cup \{\infty\}$ (as in [13]) and the Lie group $\text{Diff}(M)$ of all smooth diffeomorphisms of $M$ (as in [31], [21] and [33]), see Theorem C, Proposition 7.11, and Theorem C in [18], respectively.

More generally, given $p \in [1, \infty]$ and a Fréchet space $E$, we can define $L^p$-functions $\gamma : [a,b] \to E$ and a corresponding Fréchet space $L^p([a,b], E)$, replacing the requirement (2) with
\[
\|\gamma\|_{L^p,q} := \left(\int_a^b q(\gamma(t))^p \, dt\right)^{1/p} < \infty
\]
for each continuous seminorm $q$ on $E$; if $p = \infty$, we require that the essential suprema
\[
\|\gamma\|_{L^\infty,q} := \|q \circ \gamma\|_{L^\infty}
\]
be finite. In either case, we write \([\gamma]\) for the equivalence class in \(L^p([a, b], E)\) and \(\|\gamma\|_{L^p,q} := \|\gamma\|_{L^p,q}\). The inclusion \(L^p([a, b], E) \subseteq L^1([a, b], E)\) induces an injective linear map \(L^p([a, b], E) \to L^1([a, b], E)\), which we use to identify \(L^p([a, b], E)\) with a vector subspace of \(L^1([a, b], E)\). Unless the contrary is stated, \(L^p([a, b], E)\) shall be endowed with the \(L^p\)-topology (the locally convex vector topology given by the seminorms \(\| \cdot \|_{L^p,q}\)).

If \(p\) is as before and \(G\) is a Lie group modelled on a Fréchet space such that \(\text{Evol}(\gamma)\) exists for each \(\gamma \in L^p([0, 1], \mathfrak{g})\) and \(\text{Evol}: L^p([0, 1], \mathfrak{g}) \to C([0, 1], G)\) is smooth, then \(G\) is called \(L^p\)-regular (see [18, §5]).

We obtain the following existence and uniqueness result for differential equations on \(G\)-manifolds given by time-dependent fundamental vector fields.

**Theorem 1.2** Let \(G\) be a Lie group modelled on a Fréchet space, \(M\) be a smooth manifold modelled on a Fréchet space and \(\sigma: M \times G \to M\) be a smooth map which is a right \(G\)-action. If \(\gamma \in L^1([a, b], \mathfrak{g})\) has an evolution \(\eta \in AC([a, b], G)\), then the differential equation

\[
\dot{y}(t) = \gamma(t)_{\sharp}(y(t))
\]

on \(M\) satisfies local existence and uniqueness of Carathéodory solutions. The mapping

\[
\text{Fl}: [a, b] \times [a, b] \times M \to M, \quad (t, t_0, y_0) \mapsto \sigma(y_0, \eta(t_0)^{-1} \eta(t))
\]

is the maximal flow of (9). Moreover, (9) admits local flows which are pullbacks of \(C^\infty\)-maps.

In fact, we shall see that Fl is such a pullback.

**Remark 1.3** Consider a smooth right \(G\)-action \(M \times G \to M\) as before, and \(T > 0\). As a special case of Theorem 1.2 we know that if \(\text{Evol}(\gamma)\) exists for an \(L^1\)-map \(\gamma: [0, T] \to \mathfrak{g}\), then for \(x_0 \in M\) the initial value problem

\[
\dot{y}(t) = \gamma(t)_{\sharp}(y(t)), \quad y(0) = x_0
\]

has a unique solution on \([0, T]\), given by \(t \mapsto x_0. \text{Evol}(\gamma)(t)\). The endpoint of this integral curve, for \(t = T\), is

\(x_0. \text{evol}(\gamma)\).
We shall use the following elementary concepts.

Let $X$ be a set. A function $\gamma: [a, b] \rightarrow X$ is called a **staircase function** if there exist $n \in \mathbb{N}$ and real numbers $a = t_0 < t_1 < \cdots < t_n = b$ such that $\gamma|_{[t_{j-1}, t_j]}$ is constant for all $j \in \{1, \ldots, n\}$. If $X$ is a topological space, then a function $\gamma: [a, b] \rightarrow X$ is called **piecewise continuous** if there exist $n \in \mathbb{N}$ and real numbers $a = t_0 < t_1 < \cdots < t_n = b$ such that $\gamma|_{[t_{j-1}, t_j]}$ has a continuous extension $[t_{j-1}, t_j] \rightarrow X$ for all $j \in \{1, \ldots, n\}$.

We write $\text{im}(f) := f(X)$ for the image of a function $f: X \rightarrow Y$.

See [1], [24], [25], [26], [37], and the references therein for geometric control theory in finite dimensions, and [39] for general aspects of control theory. Our results concerning control theory subsume the following.

**Theorem 1.4** Let $G$ be a Lie group modelled on a Fréchet space, with Lie algebra $\mathfrak{g}$. Let $S \subseteq \mathfrak{g}$ be a non-empty subset, $M$ be a smooth manifold modelled on a Fréchet space, and $\sigma: M \times G \rightarrow M$, $(x, g) \mapsto x.g$ be a smooth map which is a right $G$-action. Let $x_0, y_0 \in M$, $T \in [0, \infty[$, and $U \subseteq M$ be an open neighbourhood of $y_0$. Let $p \in [1, \infty]$ or $p = \infty$. If $G$ is $L^1$-regular, then the following conditions are equivalent.

(a) There exists $\gamma \in \mathcal{L}^1([0, T], \mathfrak{g})$ with $\text{im}(\gamma) \subseteq S$ such that $x_0.\text{evol}(\gamma) \in U$.

(b) There exists $\gamma \in \mathcal{L}^p([0, T], \mathfrak{g})$ with $\text{im}(\gamma) \subseteq S$ such that $x_0.\text{evol}(\gamma) \in U$.

(c) There exists a piecewise continuous function $\gamma: [0, T] \rightarrow \mathfrak{g}$ with $\text{im}(\gamma) \subseteq S$ such that $x_0.\text{evol}(\gamma) \in U$.

(d) There exists a staircase function $\gamma: [0, T] \rightarrow \mathfrak{g}$ with $\text{im}(\gamma) \subseteq S$ such that $x_0.\text{evol}(\gamma) \in U$.

If $S$ is convex, then the following condition (e) is equivalent to (d):

(e) There exists a continuous function $\gamma: [0, T] \rightarrow \mathfrak{g}$ with $\text{im}(\gamma) \subseteq S$ such that $x_0.\text{evol}(\gamma) \in U$. 

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If $S$ is convex and the convex hull of the set $\text{ex}(S)$ of extreme points is dense in $S$ (e.g., if $S$ is compact and convex), then the following condition (f) is equivalent to (d):

(f) There exists a staircase function $\gamma: [0, T] \to g$ with $\text{im}(\gamma) \subseteq \text{ex}(S)$ such that $x_0. \text{evol}(\gamma) \in U$.

**Remark 1.5** Note that (f) is an instance of a bang-bang principle: If we can enter $U$ in time $T$ using controls in $S$, then also using piecewise constant controls in the set $\text{ex}(S)$ of extreme points. To prove (f), we shall use the Trotter formula, which is valid for all $L^1$-regular Lie groups, by [18, Theorem I] (cf. [22] for further generalizations).

**Remark 1.6** With regard to (f), we recall that the convex hull $\text{conv}(\text{ex}(S))$ is dense in $S$ for each weakly compact convex subset $S$ of a locally convex space $E$, i.e., a convex subset which is compact in the weak topology $O_w$ on $E$ (which is initial with respect to the set $E'$ of continuous linear functionals on $E$)\footnote{By the Krein-Milman Theorem, $\text{conv}(\text{ex}(S))$ has closure $S$ in $(E, O_w)$. The closure $C$ of $\text{conv}(\text{ex}(S))$ in $E$ satisfies $C \subseteq S$, as $O_w$ is coarser than the given topology on $E$. Being closed and convex, $C$ is an intersection of closed half-spaces $H \subseteq E$, by the Hahn-Banach Separation Theorem. As each $H$ is weakly closed, $C$ is closed in $(E, O_w)$ and so $C = S$.} For example, $\text{conv}(\text{ex}(S))$ is dense in $S$ for the closed unit ball $S$ in any reflexive Banach space $E$ (e.g., in a Hilbert space); likewise for every closed, convex, bounded subset $S \subseteq E$.

**Remark 1.7** Recall that every $L^1$-regular Lie group $G$ has an exponential function $\exp_G: g \to G$. If $v \in g$ is given and real numbers $\alpha < \beta$, then the constant function $\gamma: [\alpha, \beta] \to g$, $t \mapsto v$ satisfies 

$$\text{evol}(\gamma) = \exp_G((\beta - \alpha)v).$$

If $T > 0$ and $\gamma: [0, T] \to g$ is a staircase function, let $0 = t_0 < \cdots < t_n = T$ such that $\gamma$ has a constant value $v_j$ on $[t_{j-1}, t_j]$ for $j \in \{1, \ldots, n\}$. Then

$$\text{evol}(\gamma) = \exp_G((t_1 - t_0)v_1) \exp_G((t_2 - t_1)v_2) \cdots \exp_G((t_n - t_{n-1})v_n).$$

**Remark 1.8** We can interpret Theorem 1.4 as a result concerning the closures of reachable sets. E.g., given a subset $S \subseteq g$ and $x_0 \in M$, let

$$\text{Reach}_S(x_0)$$
be the set of all \( y_0 \in M \) such that \( y_0 = x_0 \) or \( y_0 = x_0 \). Using Remark 1.7 we deduce from Theorem 1.4(d) that

\[
\text{Reach}_S(x_0) = x_0, \quad \langle \exp G([0, \infty[S]) \rangle_+,
\]

where \( \langle Y \rangle_+ \) denotes the subsemigroup of \( G \) generated by a subset \( Y \subseteq G \).

For some conclusions, weaker regularity properties (like \( L^\infty \)-regularity) are sufficient. To enable these variants, we discuss continuity of the evolution map \( \text{Evol} : L^p([0, 1], g) \to C([0, 1], G) \) with respect to the \( L^1 \)-topology on its domain, given by the seminorms

\[
L^p([0, 1], g) \to [0, \infty[, \quad [\gamma] \mapsto \int_0^1 q(\gamma(t)) \, dt
\]

for continuous seminorms \( q \) on \( g \).

**Theorem 1.9** Let \( p \in [1, \infty[ \) or \( p = \infty \). Then

\[
\text{Evol} : L^p([0, 1], g) \to C([0, 1], G)
\]

is continuous with respect to the \( L^1 \)-topology on \( L^p([0, 1], g) \), for each \( L^p \)-regular Lie group \( G \) modelled on a Fréchet space.

It was already shown in [17, Lemma 14.9] that \( \text{evol} : C([0, 1], g) \to G \) is continuous with respect to the \( L^1 \)-topology for each \( C^0 \)-regular, locally \( \mu \)-convex Lie group \( G \), and we can adapt the proof (the concept of local \( \mu \)-convexity, which goes back to [17], is recalled in Definition 6.3). In the meantime, work by Hanusch showed that every \( C^0 \)-regular Lie group is locally \( \mu \)-convex (cf. Theorem 1 in [23, §5]). Since \( L^p \)-regularity implies \( C^0 \)-regularity (see [18, Corollary 5.21]), we can exploit that \( G \) in Theorem 1.9 is locally \( \mu \)-convex.

We mention that more can be shown: The evolution map in Theorem 1.9 is \( C^\infty \) with respect to the \( L^1 \)-topology (see [19, Remark 4.3]).

Using Theorem 1.9 as a tool, we can generalize Theorem 1.4 as follows:

**Theorem 1.10** Instead of requiring \( L^1 \)-regularity, let \( G \) be a Lie group modelled on a Fréchet space such that \( G \) is \( L^q \)-regular for some \( q \in [1, \infty[ \) or \( q = \infty \). Then all conclusions of Theorem 1.4 remain valid if we assume \( p \geq q \) and replace (a) with
(a)' There exists $\gamma \in L^q([0,T], g)$ with $\text{im}(\gamma) \subseteq S$ such that $x_0, \text{evol}(\gamma) \in U$.

A variant of Theorem 1.4 is also available if a Lie group $G$ is only assumed to be $C^0$-regular (see Theorem 9.1).

**Remark 1.11** Using Lusin measurability instead of Borel measurability, it is possible to define $L^p$-maps to sequentially complete locally convex spaces, corresponding absolutely continuous maps, and $L^p$-regularity (see [35]). Using these, we find that Proposition 1.1 remains valid if $M$ is a $C^2$-manifold modelled on a sequentially complete locally convex space and ([8] admits local flows which are pullbacks of $C^2$-maps (see Remark 10.13(a)); Theorems 1.2, 1.4 and 1.10 remain valid if $G$ is a Lie group modelled on a sequentially complete locally convex space and $M$ a smooth manifold modelled on a sequentially complete locally convex space, with $L^1$-regularity (and $L^q$-regularity) as in [35] (see Section 10, notably Theorem 10.19, Remark 10.20 and Remark 10.21). Moreover, Theorem 1.9 remains valid for Lie groups modelled on sequentially complete locally convex spaces and $L^p$-regularity as in [35] (see Remark 10.21). Generalizations to $E$-regular Lie groups modelled on sequentially complete (FEP)-spaces or integral complete locally convex spaces (as in [18]) are also possible, see Remark 10.22 (for the terminology, cf. also Section 7).

Absolutely continuous functions $\eta: [a, b] \to E$ as in (3) to a sequentially complete locally convex space are more difficult to treat than those to Fréchet spaces, as $\eta'(t)$ may not exist (and recover $\gamma(t)$) almost everywhere in this case. Notably, Carathéodory solutions need to be understood in the sense of (7), while (1) might not hold almost everywhere. At least, for each continuous linear map $q: E \to F$ to a Banach space $F$, we still have that $(q \circ \eta)'(t)$ exists and equals $q(\gamma(t))$ for almost all $t$. This will be good enough for the proofs, exploiting that $q \circ f$ for an $E$-valued $C^2$-function $f$ locally factors over a $C^1$-map on an open subset of a Banach space (see Appendix A).

**Remark 1.12** Examples of $L^1$-regular Lie groups modelled on sequentially complete locally convex spaces are direct limits $G = \bigcup_{n \in \mathbb{N}} G_n$ of finite-dimensional Lie groups $G_1 \subseteq G_2 \subseteq \cdots$ (as in [15]), the Lie group $\text{Diff}^\omega(M)$ of real-analytic diffeomorphisms of a compact real-analytic manifold $M$ (as in [29]) and the Lie group $\text{Diff}_c(M)$ of compactly supported smooth diffeomorphisms of a finite-dimensional paracompact smooth manifold $M$ (cf. [31]); $L^1$-regularity was established in [18 Theorem E], [19 Theorem 1.2], and [18].
2 Preliminaries and notation

In the following, \( \mathbb{N} := \{1, 2, \ldots \} \) and \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \). All topological vector spaces are assumed Hausdorff, with the exception of the spaces \( \mathcal{L}^p([a, b], E) \) for a Fréchet space \( E \), real numbers \( a < b \) and \( p \in [1, \infty) \), which are endowed with the vector topology determined by the seminorms \( \| \cdot \|_{L^p,q} \) for continuous seminorms \( q \) on \( E \). The topology induced by \( \mathcal{L}^1([a, b], E) \) on \( \mathcal{L}^p([a, b], E) \) shall be referred to as the \( \mathcal{L}^1\)-topology thereon. We shall use “locally convex space” as an abbreviation of “locally convex topological vector space.” If \( M \) is a subset of a real vector space \( E \), we write \( \text{conv}(M) \) for its convex hull. If \( p : E \to [0, \infty[ \) is a seminorm, we write \( B^p_\varepsilon(x) := \{ y \in E : p(y - x) < \varepsilon \} \) for the open ball of radius \( \varepsilon > 0 \) around \( x \in E \). If \( (E, \| \cdot \|) \) is a normed space and the norm is understood, we also write \( B^E_\varepsilon(x) \) in place of \( B^\|\cdot\|_\varepsilon(x) \). If \( E \) and \( F \) are locally convex spaces and \( U \subseteq E \) is an open subset, we say that a continuous map \( f : U \to F \) is \( C^1 \) if the directional derivative

\[
\frac{df(x, y)}{dt} \Big|_{t=0} f(x + ty)
\]

exists in \( F \) for all \( (x, y) \in U \times E \), and the map \( df : U \times E \to F \) is continuous. Recursively, given \( k \geq 2 \) we say that \( f \) is \( C^k \) if \( f \) is \( C^1 \) and \( df \) is \( C^{k-1} \). If \( f \) is \( C^k \) for all \( k \in \mathbb{N} \), then \( f \) is called \( C^\infty \) or smooth. This approach to calculus in locally convex spaces, which goes back to [2], is known as Keller’s \( C^k \)-theory [27]. We refer to [12], [20], [21], [33], and [34] for introductions to this approach to calculus, cf. also [3]. For the corresponding concepts of manifolds and Lie groups modelled on a locally convex space, see [12], [20], and [34]. As usual, a Lie group (resp., manifold) modelled on a Fréchet space shall be called a Fréchet-Lie group (resp., a Fréchet manifold). If \( M \) is a \( C^1 \)-manifold modelled on a locally convex space, we let \( TM \) be its tangent bundle and write \( T_xM \) for the tangent space at \( x \in M \). If \( V \) is an open subset of a locally convex space \( E \), we identify \( TV \) with \( V \times E \), as usual. If \( f : M \to N \) is a \( C^1 \)-map between \( C^1 \)-manifolds, we write \( Tf : TM \to TN \) for its tangent map. In the case of a \( C^1 \)-map \( f : M \to V \subseteq E \), we write \( df \) for the second component of the tangent map

\[
Tf : TM \to TV = V \times E.
\]
Many preliminaries were already described in the introduction, and need not be repeated. For more background concerning vector-valued $L^p$-functions, vector-valued absolutely continuous functions, and $L^p$-regularity, see [18].

2.1 If $M$ is a $C^1$-manifold modelled on a Fréchet space and $I \subseteq \mathbb{R}$ an interval, we say that a function $\eta: I \to M$ is absolutely continuous if $\eta$ is continuous and, for each $t_0 \in I$, there exist a chart $\phi: U_\phi \to V_\phi$ of $M$ and real numbers $\alpha < \beta$ with $[\alpha, \beta] \subseteq I$ such that $t_0$ is in the interior of $[\alpha, \beta]$ relative $I$, $\eta([\alpha, \beta]) \subseteq U_\phi$, and $\phi \circ \eta|_{[\alpha, \beta]}$ is absolutely continuous. Equivalently, $\phi \circ \eta|_J$ is absolutely continuous for each chart $\phi: U_\phi \to V_\phi$ of $M$ and each interval $J \subseteq I$ such that $\eta(J) \subseteq U_\phi$ (cf. Definition 3.20, Lemma 3.21, Lemma 3.18(a) and 3.15 in [18]).

Let $I \subseteq \mathbb{R}$ be a non-degenerate interval and $E$ a locally convex space. As usual, we say that a map $\eta: I \to E$ is differentiable at $t_0 \in I$ if the limit

$$\eta'(t_0) := \lim_{t \to t_0} \frac{\eta(t) - \eta(t_0)}{t - t_0}$$

(with $t \neq t_0$) exists in $E$. We shall use a well-known fact (cf. [18] Lemma 1.57):

2.2 Let $E$ and $F$ be locally convex spaces, $U \subseteq E$ be open and $f: U \to F$ be a $C^1$-map. If $I \subseteq \mathbb{R}$ is a non-degenerate interval, $\eta: I \to E$ a function with $\eta(I) \subseteq U$ and $t_0 \in I$ such that $\eta'(t_0)$ exists, then also $(f \circ \eta)'(t_0)$ exists and $(f \circ \eta)'(t_0) = df(\eta(t_0), \eta'(t_0))$.

We deduce from 2.2.

2.3 Let $E$ and $F$ be Fréchet spaces, $U \subseteq E$ be open and $f: U \to F$ be a $C^1$-map. If $I \subseteq \mathbb{R}$ is a non-degenerate interval and $\eta: I \to E$ an absolutely continuous function such that $\eta(I) \subseteq U$, then $f \circ \eta: I \to F$ is absolutely continuous (see [18] Lemma 3.18 (a)]). For each $t_0 \in I$ such that $\eta'(t_0)$ exists (which is the case for $\lambda_1$-almost all $t_0 \in I$), we have that $(f \circ \eta)'(t_0)$ exists and $(f \circ \eta)'(t_0) = df(\eta(t_0), \eta'(t_0))$.

2.4 Let $M$ be a $C^1$-manifold modelled on a locally convex space $E$. Let $I \subseteq \mathbb{R}$ be a non-degenerate interval, $\eta: I \to M$ be a continuous map and $t_0 \in I$. We say that $\eta$ is differentiable at $t_0$ if $\phi \circ \eta: \eta^{-1}(U_\phi) \to V_\phi$ is differentiable at $t_0$ for some chart $\phi: U_\phi \to V_\phi \subseteq E$ of $M$ such that $\eta(t_0) \in U_\phi$. By 2.2 the latter then holds for any such chart, and the tangent vector

$$\dot{\eta}(t_0) := T\phi^{-1}((\phi \circ \eta)(t_0), (\phi \circ \eta)'(t_0)) \in T_{\eta(t_0)}M$$
is well defined, independent of the choice of $\phi$. If no confusion is likely, we also write $\frac{d\sigma}{dt}(t_0) := \dot{\sigma}(t_0)$ (e.g., in Definition 4.8(c)).

2.5 Let $f : M \to N$ be a $C^1$-map between $C^1$-manifolds modelled on locally convex spaces. If $I \subseteq \mathbb{R}$ is a non-degenerate interval, $t_0 \in I$ and $\eta : I \to M$ a continuous map which is differentiable at $t_0$, then also $f \circ \eta : I \to N$ is differentiable at $t_0$ and

$$ (f \circ \eta)'(t_0) = T f(\dot{\eta}(t_0)). \quad (11) $$

[Let $\psi : U_\psi \to V_\psi$ be a chart of $N$ with $f(\eta(t_0)) \in U_\psi$ and $\phi : U_\phi \to V_\phi$ be a chart of $M$ with $\eta(t_0) \in U_\phi$ and $f(U_\phi) \subseteq U_\psi$. Let $J \subseteq I$ be a subinterval which is a neighbourhood of $t_0$ in $I$, such that $\eta(J) \subseteq U_\phi$. Then

$$ \psi \circ f \circ \eta|_J = (\psi \circ f \circ \phi^{-1}) \circ (\phi \circ \eta|_J) $$

is differentiable at $t_0$, by 2.2 whence so is $f \circ \eta$. Moreover, using 2.2 we obtain $(\psi(f(\eta(t_0))), (\psi \circ f \circ \eta)'(t_0)) = T(\psi \circ f \circ \phi^{-1})(\phi(\eta(t_0))), (\phi \circ \eta)'(t_0)) = T\psi T f \dot{\eta}(t_0)$. Applying $T\psi^{-1}$ to both sides, (11) follows.]

2.6 If $E_1$, $E_2$, and $F$ are locally convex spaces, $U_1 \subseteq E_1$ and $U_2 \subseteq E_2$ open subsets and $f : U_1 \times U_2 \to F$ a $C^1$-map, then

$$ df((x_1, x_2), (y_1, y_2)) = d_1 f(x_1, x_2; y_1) + d_2 f(x_1, x_2; y_2) $$

for all $(x_1, x_2) \in U_1 \times U_2$ and $(y_1, y_2) \in E_1 \times E_2$, where

$$ d_1 f(x_1, x_2; y_1) := d(f(\cdot, x_2))(x_1, y_1) $$

and $d_2 f(x_1, x_2; y_2) := d(f(x_1, \cdot))(x_2, y_2)$, see [20, Proposition 1.2.8]. Likewise,

$$ df((x_1, x_2, x_3), (y_1, y_2, y_3)) = d_1 f(x_1, x_2, x_3; y_1) + d_2 f(x_1, x_2, x_3; y_2) + d_3 f(x_1, x_2, x_3; y_3) $$

for $C^1$-maps $f : U_1 \times U_2 \times U_3 \to F$, in terms of partial differentials.

If $X$ is a topological space, we write $\mathcal{B}(X)$ for the $\sigma$-algebra of Borel sets (generated by the set of open subsets of $X$). A map between topological spaces is called Borel measurable if it is measurable with respect to the $\sigma$-algebras of Borel sets on domain and range. As usual, we say that a topological space is separable if it has a dense, countable subset. The following fact is useful.
Lemma 2.7 Let $X$, $X_1$, $X_2$, and $Y$ be topological spaces and $f : X_1 \times X_2 \to Y$ be a continuous map. Let $\gamma : X \to X_1$ and $\eta : X \to X_2$ be Borel measurable mappings. If $X = \bigcup_{n \in \mathbb{N}} A_n$ with Borel sets $A_n$ such that $\eta(A_n)$ is separable and metrizable in the topology induced by $X_2$, then

$$f \circ (\gamma, \eta) : X \to Y, \quad x \mapsto f(\gamma(x), \eta(x))$$

is Borel measurable.

Proof. It suffices to show that $f \circ (\gamma, \eta)|_{A_n}$ is measurable on $A_n$, endowed with the trace $\mathcal{B}(X)|_{A_n} = \mathcal{B}(A_n)$, for each $n \in \mathbb{N}$. Since $\eta(A_n)$ is metrizable and separable, $\mathcal{B}(X_1 \times X_2)|_{X_1 \times \eta(A_n)} = \mathcal{B}(X_1 \times \eta(A_n)) = \mathcal{B}(X_1) \otimes \mathcal{B}(\eta(A_n))$ is the product $\sigma$-algebra (see, e.g., [14, Lemma 2.7]). Hence $(\gamma, \eta)|_{A_n} : X \to X_1 \times X_2$ is measurable as a map to $X_1 \times \eta(A_n)$ with the trace of $\mathcal{B}(X_1 \times X_2)$ and hence Borel measurable to $X_1 \times X_2$. Since $f$ is continuous and thus Borel measurable, also the composition $f \circ (\gamma, \eta)|_{A_n}$ is Borel measurable. \hfill $\Box$

2.8 Let $G$ be a Lie group with Lie algebra $\mathfrak{g} := T_e G$, modelled on a locally convex space. Let $\alpha < \beta$ and $a < b$ be real numbers and $\phi : [\alpha, \beta] \to [a, b]$ be the restriction of the unique affine-linear map $\mathbb{R} \to \mathbb{R}$ taking $\alpha$ to $a$ and $\beta$ to $b$. Thus $\phi'$ is the constant function whose value is the slope $m := (b - a)/(\beta - \alpha)$. If $\gamma : [a, b] \to \mathfrak{g}$ is a continuous function admitting an evolution $\eta := \text{Evol}(\gamma) : [a, b] \to G$, then $\zeta : [\alpha, \beta] \to \mathfrak{g}$, $s \mapsto m \gamma(\phi(s))$ has $\eta \circ \phi$ as its evolution, i.e.,

$$\text{Evol}(\zeta) = \text{Evol}(\gamma) \circ \phi.$$ 

In fact, $(\eta \circ \phi)(\alpha) = \eta(a) = e$ holds and $(\eta \circ \phi)'(s) = \dot{\eta}(\phi(s))\phi'(s) = \phi'(s)\dot{\eta}(\phi(s)).\gamma(\phi(s)) = (\eta \circ \phi)(s).m \gamma(\phi(s)) = (\eta \circ \phi)(s).\zeta(s)$ for all $s \in [a, b]$, by the Chain Rule.

Likewise if $G$ is a Fréchet-Lie group, $\gamma \in \mathcal{L}^1([a, b], \mathfrak{g})$ and $\eta$ its evolution in the sense of Carathéodory solutions (cf. [18]).}

3 Initial value problems in Fréchet spaces

We discuss local existence and uniqueness for Carathéodory solutions to initial value problems in Fréchet spaces. The treatment emulates the earlier discussion of existence and uniqueness of $C^1$-solutions in [20 §2.4].
Definition 3.1 Let $E$ be a Fréchet space and $f : W \to E$ be a function on a subset $W \subseteq \mathbb{R} \times E$. We say that the differential equation
\[ y'(t) = f(t, y(t)) \]  
(12)
satisfies \textit{local uniqueness of Carathéodory solutions} if the following holds: For all Carathéodory solutions $\gamma_1 : I_1 \to E$ and $\gamma_2 : I_2 \to E$ of (12) and $t_0 \in I_1 \cap I_2$ such that $\gamma_1(t_0) = \gamma_2(t_0)$, there exists an interval $K \subseteq \mathbb{R}$ which is an open neighbourhood of $t_0$ in $I_1 \cap I_2$ such that
\[ \gamma_1|_K = \gamma_2|_K. \]

Lemma 3.2 In a Fréchet space $E$, consider a differential equation (12) which satisfies local uniqueness of Carathéodory solutions. Assume that $\gamma_1 : I_1 \to E$ and $\gamma_2 : I_2 \to E$ are Carathéodory solutions to (12) such that $\gamma_1(t_0) = \gamma_2(t_0)$ for some $t_0 \in I_1 \cap I_2$. Then
\[ \gamma_1|_{I_1 \cap I_2} = \gamma_2|_{I_1 \cap I_2}. \]

Proof. (Compare [20, Lemma 2.4.6] for $C^1$-solutions). The set $A := \{ t \in I_1 \cap I_2 : \gamma_1(t) = \gamma_2(t) \}$ is closed in $I_1 \cap I_2$ since $E$ is Hausdorff and the functions $\gamma_1$ and $\gamma_2$ are continuous. Since (12) satisfies local uniqueness of Carathéodory solutions, the set $A$ is also open in $I_1 \cap I_2$. By hypothesis, $A \neq \emptyset$. Since $I_1 \cap I_2$ is an interval and hence connected, it follows that $A = I_1 \cap I_2$ and thus $\gamma_1|_{I_1 \cap I_2} = \gamma_2|_{I_1 \cap I_2}$. \qed

Definition 3.3 Let $E$ be a Fréchet space, $J \subseteq \mathbb{R}$ a non-degenerate interval and $f : W \to E$ be a function on a subset $W \subseteq J \times E$. We say that the differential equation (12) satisfies \textit{local existence of Carathéodory solutions} if for all $(t_0, y_0) \in W$, there exists a Carathéodory solution $\gamma : I \to E$ to the initial value problem
\[ y'(t) = f(t, y(t)), \quad y(t_0) = y_0 \]  
(13)
such that $I$ is a relatively open subinterval of $J$.

\[ \text{More precisely, we should speak about local existence of Carathéodory solutions with respect to $J$, but $J$ will always be clear from the context. Likewise in Definition 4.5.} \]
Definition 3.4 Let \( J \subseteq \mathbb{R} \) be a non-degenerate interval, \( E \) be a Fréchet space, \( U \subseteq E \) be a subset and \( f: W \to E \) be a function on an open subset \( W \subseteq J \times U \). Let \( k \in \mathbb{N} \cup \{\infty\} \). We say that the differential equation

\[
y'(t) = f(t, y(t))
\]

admits local flows which are pullbacks of \( C^k \)-maps if, for all \( (\tilde{t}, \overline{y}) \in W \), there exist a relatively open interval \( I \subseteq J \) with \( \tilde{t} \in I \), an open neighborhood \( V \) of \( \overline{y} \) in \( U \) with \( I \times V \subseteq W \) and function \( \Phi: I \times I \times V \to E \) with the following properties:

(a) For all \( (t_0, y_0) \in I \times V \), the function \( I \to E, t \mapsto \Phi_{t_0}(y_0) := \Phi(t, t_0, y_0) \) is a Carathéodory solution to the initial value problem \((13)\);

(b) There is an open \( \overline{y} \)-neighbourhood \( Y \subseteq V \) such that \( \Phi_{t_1}(Y) \subseteq V \) for all \( t_0, t_1 \in I \) and

\[
\Phi_{t_2}(\Phi_{t_1}(y_0)) = \Phi_{t_2}(y_0)
\]

for all \( t_0, t_1, t_2 \in I \) and \( y_0 \in Y \);

(c) There exist Fréchet spaces \( E_1 \) and \( E_2 \), open subsets \( V_1 \subseteq E_1 \) and \( V_2 \subseteq E_2 \), absolutely continuous functions \( \alpha: I \to V_1 \subseteq E_1 \) and \( \beta: I \to V_2 \subseteq E_2 \), and a \( C^k \)-map \( \Psi: V_1 \times V_2 \times V \to E \) such that

\[
\Phi(t, t_0, y_0) = \Psi(\alpha(t), \beta(t_0), y_0)
\]

for all \( (t, t_0, y_0) \in I \times I \times V \).

Moreover, we require the existence of a Borel set \( I_0 \subseteq I \) with \( \lambda_1(I \setminus I_0) = 0 \) such that \( \frac{d}{dt}\Phi_{t_0}(y_0) \) exists and

\[
\frac{d}{dt}\Phi_{t_0}(y_0) = f(t, \Phi_{t_0}(y_0))
\]

for all \( y_0 \in V, t_0 \in I, \) and \( t \in I_0 \).

Remark 3.5 If \( y'(t) = f(t, y(t)) \) (as in Definition 3.4) admits local flows which are pullbacks of \( C^k \)-maps, then \( y'(t) = f(t, y(t)) \) satisfies local existence of Carathéodory solutions.

In fact, for any \( (\tilde{t}, \overline{y}) \in W \), the map \( I \to E, t \mapsto \Phi(t, \tilde{t}, \overline{y}) \) is a solution to the initial value problem \( y'(t) = f(t, y(t)), y(\tilde{t}) = \overline{y} \) on the relatively open subinterval \( I \subseteq J \) with \( \tilde{t} \in I \) (with notation as in Definition 3.4).

\(^6\)In particular, \( \Phi \) is continuous.
Lemma 3.6 Let $J \subseteq \mathbb{R}$ be a non-degenerate interval, $E$ be a Fréchet space, $U \subseteq E$ be a subset and $f : W \to E$ be a function on an open subset $W \subseteq J \times U$. If the differential equation $y'(t) = f(t, y(t))$ admits local flows which are pullbacks of $C^1$-maps, then it satisfies local uniqueness of Carathéodory solutions.

Proof. (Compare [20, Proposition 2.4.20] for local $C^1$-flows). Let $\gamma_j : I_j \to E$ be solutions to $y'(t) = f(t, y(t))$ for $j \in \{1, 2\}$ and $\bar{t} \in I_1 \cap I_2$ such that $\gamma := \gamma_1(\bar{t}) = \gamma_2(\bar{t})$. To see that $\gamma_1$ and $\gamma_2$ coincide on a neighborhood of $\bar{t}$ in $I_1 \cap I_2$, we may assume $I_1 \cap I_2 \neq \{\bar{t}\}$ (excluding a trivial case). Thus $I_1 \cap I_2$ is a non-degenerate interval. For $j \in \{1, 2\}$, there exists a Borel set $I_{j,0} \subseteq I_j$ with $\lambda_1(I_j \setminus I_{j,0}) = 0$ such that $\gamma_j$ is differentiable at each $t \in I_{j,0}$ and

$$\gamma'_j(t) = f(t, \gamma_j(t)) \text{ for all } t \in I_{j,0}.$$ 

Let $I, V, \Phi, Y, E_1, E_2, V_1, V_2, \Psi, \alpha, \beta,$ and $I_0$ be as in Definition 3.4. After shrinking $I_0$, we may assume that, moreover, $\alpha'(t)$ and $\beta'(t)$ exist at each $t \in I_0$. For $(t, y_0) \in I \times V$, the partial derivative of $\Phi$ with respect to the second variable,

$$(\partial_2 \Phi)(t, t_0, y_0) := \frac{\partial \Phi}{\partial t_0}(t, t_0, y_0) = d_2 \Psi(\alpha(t), \beta(t_0), y_0; \beta'(t_0)),$$

exists for all $t_0 \in I$ such that $\beta'(t_0)$ exists, and hence for all $t_0 \in I_0$ (where we used [23] and notation as in [24]).

There exists a relatively open interval $K \subseteq I_1 \cap I_2 \cap I$ with $\bar{t} \in K$ such that $\gamma_1(K) \subseteq Y$, $\gamma_2(K) \subseteq Y$ and $\Phi_{t, \bar{t}}(\gamma)$ $\in Y$ for all $t \in K$. After shrinking $K$ if necessary, we can also assume that

$$\theta_j(t) := \Phi_{t, \bar{t}}(\gamma_j(t)) \in Y \text{ for all } t \in K \text{ and } j \in \{1, 2\}.$$ 

Note that $\theta_j$ is absolutely continuous by [23] as

$$\theta_j(t) = \Psi(\alpha(\bar{t}), \beta(t), \gamma_j(t));$$

and

$$\theta'_j(t) = d_2 \Psi(\alpha(\bar{t}), \beta(t), \gamma_j(t); \beta'(t)) + d_3 \Psi(\alpha(\bar{t}), \beta(t), \gamma_j(t); \gamma'_j(t))$$

$$= \partial_2 \Phi(\bar{t}, t, \gamma_j(t)) + d \Phi_{t, \bar{t}}(\gamma_j(t), \gamma'_j(t)) \quad (14)$$

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for all $t \in K_0 := K \cap I_0 \cap I_{1,0} \cap I_{2,0}$. It suffices to show that

$$\gamma_j(t) = \Phi_{t,\bar{t}}(\overline{y}) \quad \text{for all } t \in K \text{ and } j \in \{1, 2\}.$$  

Since $\Phi_{t,\bar{t}} \circ \Phi_{\bar{t},t}|_Y = \text{id}_Y$ for all $t \in I$ (by (a) and (b) in Definition 3.4), the map $\Phi_{t,\bar{t}}|_Y$ is injective. Hence $\gamma_1|_K = \gamma_2|_K$ will hold if we can show that both $\theta_1$ and $\theta_2$ coincide with

$$\theta: K \to E, \quad t \mapsto \Phi_{t,\bar{t}}(\Phi_{\bar{t},t}(\overline{y})) = \overline{y}.$$  

Since $\theta_j(\bar{t}) = \overline{y} = \theta(\bar{t})$ for $j \in \{1, 2\}$, the latter will hold if we can show that

$$\theta_j'(t) = \theta'(t) = 0 \quad \text{for all } t \in K_0$$

(and thus for $\lambda_1$-almost all $t \in K$). Given $z \in Y$, we have

$$z = \Phi_{t,\bar{t}}(\Phi_{\bar{t},t}(z)) = \Psi(\alpha(\bar{t}), \beta(t), \Phi_{t,\bar{t}}(z))$$

for all $t \in I$ (by (a), (b), and (c) in Definition 3.4) and hence, differentiating with respect to $t$,

$$0 = d_2 \Psi(\alpha(\bar{t}), \beta(t), \Phi_{t,\bar{t}}(z); \beta'(t)) + d_3 \Psi(\alpha(\bar{t}), \beta(t), \Phi_{t,\bar{t}}(z), \frac{d}{dt} \Phi_{t,\bar{t}}(z))$$

$$= \partial_2 \Phi(\bar{t}, t, \Phi_{t,\bar{t}}(z)) + d\Phi_{t,\bar{t}}(\Phi_{t,\bar{t}}(z), f(t, \Phi_{t,\bar{t}}(z)))$$

(15)

for all $t \in I_0$ (using that $\frac{d}{dt} \Phi_{t,\bar{t}}(z) = f(t, \Phi_{t,\bar{t}}(z))$ for all $t \in I_0$). For $t \in K_0$ and $j \in \{1, 2\}$, we can consider $z := \theta_j(t) \in Y$; then $\gamma_j(t) = \Phi_{t,\bar{t}}(z)$ . Since $\gamma_j'(t) = f(t, \gamma_j(t)) = f(t, \Phi_{t,\bar{t}}(z))$, using (14) we get

$$\theta_j'(t) = \partial_2 \Phi(\bar{t}, t, \gamma_j(t)) + d\Phi_{t,\bar{t}}(\gamma_j(t), \gamma_j'(t))$$

$$= \partial_2 \Phi(\bar{t}, t, \Phi_{t,\bar{t}}(z)) + d\Phi_{t,\bar{t}}(\Phi_{t,\bar{t}}(z), f(t, \Phi_{t,\bar{t}}(z))) = 0$$

as a special case of (15). \hfill \Box

4 Initial value problems in Fréchet manifolds

We now extend the local theory of Section 3 to the case of Fréchet manifolds. Also, we prove Proposition 1.1 and Theorem 1.2. The treatment emulates the discussion of existence and uniqueness of $C^1$-solutions in [20, §2.5].
Definition 4.1 Let $M$ be a $C^1$-manifold modelled on a Fréchet space $E$ and $f: W \rightarrow TM$ be a function on a subset $W \subseteq \mathbb{R} \times M$ such that $f(t,y) \in T_yM$ for all $(t,y) \in W$. We say that a function $\gamma: I \rightarrow M$ on a non-degenerate interval $I \subseteq \mathbb{R}$ is a Carathéodory solution to the differential equation

\begin{equation}
\dot{y}(t) = f(t,y(t))
\end{equation}

if $\gamma$ is absolutely continuous, $(t,\gamma(t)) \in W$ for all $t \in I$ and

\begin{equation}
\dot{\gamma}(t) = f(t,\gamma(t))
\end{equation}

for $\lambda_1$-almost all $t \in I$ (using notation as in 2.4). If $(t_0,y_0) \in W$ is given and $\gamma$ satisfies, moreover, the condition $\gamma(t_0) = y_0$, then $\gamma$ is called a Carathéodory solution to the initial value problem

\begin{equation}
\dot{y}(t) = f(t,y(t)), \quad y(t_0) = y_0.
\end{equation}

Remark 4.2 If $\phi: U_\phi \rightarrow V_\phi \subseteq E$ is a chart for $M$ in the situation of Definition 4.1, we define a function

$f_\phi: W_\phi \rightarrow E, \quad (t,y) \mapsto d\phi(f(t,\phi^{-1}(y)))$

on $W_\phi := \{(t,y) \in \mathbb{R} \times V_\phi: (t,\phi^{-1}(y)) \in W\} \subseteq \mathbb{R} \times E$. Let $\gamma: I \rightarrow M$ be a continuous function on a non-degenerate interval $I \subseteq \mathbb{R}$ such that $(t,\gamma(t)) \in W$ for all $t \in I$. Then $\gamma$ is a Carathéodory solution to (16) if and only if $\phi \circ \gamma|_K$ is a Carathéodory solution to

\begin{equation}
y'(t) = f_\phi(t,\phi^{-1}(y(t))),
\end{equation}

for each chart $\phi: U_\phi \rightarrow V_\phi \subseteq E$ and each non-degenerate subinterval $K \subseteq I$ such that $\gamma(K) \subseteq U_\phi$. The latter holds if and only if, for each $t_0 \in I$, there are a chart $\phi$ and a subinterval $K \subseteq I$ which is a neighbourhood of $t_0$ in $I$ such that $\gamma(K) \subseteq U_\phi$ and $\phi \circ \gamma|_K$ solves (18), due to 2.3. If $I = [a,b]$, equivalently we may take $K$ in finite set of subintervals which cover $I$.

Definition 4.3 In the situation of Definition 4.1 we say that the differential equation (16) satisfies local uniqueness of Carathéodory solutions if the following holds: For all Carathéodory solutions $\gamma_1: I_1 \rightarrow M$ and $\gamma_2: I_2 \rightarrow M$ of (16) and $t_0 \in I_1 \cap I_2$ such that $\gamma_1(t_0) = \gamma_2(t_0)$, there exists an interval $K \subseteq \mathbb{R}$ which is an open neighbourhood of $t_0$ in $I_1 \cap I_2$ such that $\gamma_1|_K = \gamma_2|_K$. 19
Lemma 4.4 Consider a differential equation (16) as in Definition 4.1, which satisfies local uniqueness of Carathéodory solutions. Assume that \( \gamma_1: I_1 \to M \) and \( \gamma_2: I_2 \to M \) are Carathéodory solutions to (16) such that \( \gamma_1(t_0) = \gamma_2(t_0) \) for some \( t_0 \in I_1 \cap I_2 \). Then \( \gamma_1|_{I_1 \cap I_2} = \gamma_2|_{I_1 \cap I_2} \).

Proof. We can repeat the proof of Lemma 3.2 with \( M \) in place of \( E \). \( \square \)

Definition 4.5 Let \( M \) be a \( C^1 \)-manifold modelled on a Fréchet space, \( J \subseteq \mathbb{R} \) be a non-degenerate interval and \( f: W \to TM \) be a function on a subset \( W \subseteq J \times M \) such that \( f(t,y) \in T_yM \) for all \((t,y) \in W\). We say that the differential equation (16) satisfies local existence of Carathéodory solutions if for all \((t_0,y_0) \in W\), there exists a Carathéodory solution \( \gamma: I \to M \) to the initial value problem (17) such that \( I \) is a relatively open subinterval of \( J \).

Lemma 4.6 Let \( M \) be a \( C^1 \)-manifold modelled on a Fréchet space, \( J \subseteq \mathbb{R} \) be a non-degenerate interval and \( f: W \to TM \) be a function on a subset \( W \subseteq J \times M \) such that \( f(t,y) \in T_yM \) for all \((t,y) \in W\). Assume that the differential equation (16) satisfies both local existence of Carathéodory solutions and local uniqueness. Then, for all \((t_0,y_0) \in W\), there exists a Carathéodory solution \( \gamma: I \to M \) to the initial value problem (17) such that \( I_\eta \subseteq I \) and \( \eta = \gamma|_{I_\eta} \) for each Carathéodory solution \( \eta: I_\eta \to M \) to (17). Moreover, \( I \) is relatively open in \( J \).

Proof. Case 1: Let us first assume that \( t_0 \) is the minimum of \( J \) (Case 2, that \( t_0 \) is the maximum of \( J \), is analogous). The set \( L \) of all \( \tau \in ]t_0, \infty[ \cap J \) such that (17) has a solution \( \eta_\tau: [t_0, \tau] \to M \) is a subinterval of \( ]t_0, \infty[ \cap J \), and also \( I := L \cup \{t_0\} \) is an interval. If \( t \in I \), there exists \( \tau \in L \) such that \( t \in [t_0, \tau] \); we define \( \gamma(t) := \gamma_\tau(t) \).

If \( \sigma, \tau \in L \) and \( \sigma \leq \tau \), then \( \gamma_{\sigma} = \gamma_{\tau}|_{[t_0, \sigma]} \) by Lemma 4.4, entailing that \( \gamma: I \to M \) is well defined. By construction, we have \( \gamma|_{[t_0, \tau]} = \gamma_\tau \) for each \( \tau \in L \); notably, \( \gamma(t_0) = \gamma_\tau(t_0) = y_0 \). If \([a, b] \subseteq I \) with \( a < b \), then \( \gamma|_{[a, b]} = \gamma_b|_{[a, b]} \) is absolutely continuous, whence \( \gamma \) is absolutely continuous.

If \( I \) does not have a maximum, then \( I \) is open in \( J = [t_0, \infty[ \cap J \). We can take an ascending sequence \( t_0 < \tau_1 < \tau_2 < \cdots \) tending to the supremum of \( I \).
Since $\gamma|_{[t_0, \tau]}$ is differentiable at almost all $t$ in its domain, with derivative $f(t, \gamma(t))$, the same is true of $\gamma$. Thus $\gamma$ is a Carathéodory solution to (17).

If $I$ has a maximum $\tau$, then $\tau \in L$ and $\gamma = \gamma_\tau$ is a Carathéodory solution to (17). We show that $\tau$ is also the maximum of $J$, whence $[t_0, \tau] = J$ is open in $J$. If not, using local existence we find a Carathéodory solution $\eta$ of (16) on an interval $K \subseteq J$ with $\tau$ in the interior of $K$ relative to $J$, such that $\eta(\tau) = \gamma(\tau)$. Then $[\tau, \theta] \subseteq K$ for some $\theta > \tau$ and we can extend $\gamma$ to a solution of (17) defined on $[t_0, \theta]$ by taking $t \in [\tau, \theta]$ to $\eta(t)$. Thus $\theta \in L$; since $\theta > \tau$, this contradicts $\tau = \max L$.

Case 3: If $t_0$ is in the interior of $J$ relative to $\mathbb{R}$, then Case 1 provides a solution $\gamma_+$ to the initial value problem on a largest subinterval $I_+ \subseteq J \cap [t_0, \infty[$.

Likewise, Case 2 provides a solution $\gamma_-$ on a largest subinterval $I_-$ of $J \cap ]-\infty, t_0]$. Then $I := I_+ \cup I_-$ and the function $\gamma : I \to M$ which is defined piecewise via $\gamma(t) := \gamma_{\pm}(t)$ for $t \in I_{\pm}$ are as required.

\textbf{Definition 4.7} The solution $\gamma_{t_0, y_0} := \gamma$ described in Lemma 4.6 is called the \textit{maximal solution} to the initial value problem (17); we write $I_{t_0, y_0} := I$ for its domain. We abbreviate

$$\Omega := \bigcup_{(t_0, y_0) \in W} I_{t_0, y_0} \times \{(t_0, y_0)\} \subseteq \mathbb{R} \times \mathbb{R} \times M$$

and call

$$\text{Fl}: \Omega \to M, \quad \text{Fl}(t, t_0, y_0) := \gamma_{t_0, y_0}(t)$$

the \textit{maximal flow} associated with (16). We also write $\text{Fl}_{t, t_0}(y_0) := \text{Fl}(t, t_0, y_0)$.

\textbf{Definition 4.8} Let $J \subseteq \mathbb{R}$ be a non-degenerate interval, $k \in \mathbb{N} \cup \{\infty\}$, $M$ be a $C^k$-manifold modelled on a Fréchet space and $f : W \to TM$ be a function on an open subset $W \subseteq J \times M$ such that $f(t, y) \in T_yM$ for all $(t, y) \in W$. We say that the differential equation $\dot{y}(t) = f(t, y(t))$ admits local flows which are pullbacks of $C^k$-maps if, for all $(\overline{t}, \overline{y}) \in W$, there exist a relatively open interval $I \subseteq J$ with $\overline{t} \in I$, an open neighborhood $V$ of $\overline{y}$ in $M$ with $I \times V \subseteq W$ and function

$$\Phi : I \times I \times V \to M$$

with the following properties:
(a) For all \((t_0, y_0) \in I \times V\), the function \(I \rightarrow E, t \mapsto \Phi_{t,t_0}(y_0) := \Phi(t, t_0, y_0)\) is a Carathéodory solution to the initial value problem (17);

(b) There is an open \(\mathcal{U}\)-neighbourhood \(Y \subseteq V\) such that \(\Phi_{t_1,t_0}(Y) \subseteq V\) for all \(t_0, t_1 \in I\) and

\[
\Phi_{t_2,t_1}(\Phi_{t_1,t_0}(y_0)) = \Phi_{t_2,t_0}(y_0) \quad \text{for all} \quad t_0, t_1, t_2 \in I \quad \text{and} \quad y_0 \in Y;
\]

(c) There exist \(C^k\)-manifolds \(N_1\) and \(N_2\) modelled on Fréchet spaces \(E_1\) and \(E_2\), respectively, absolutely continuous functions \(\zeta_j : I \rightarrow N_j\) for \(j \in \{1, 2\}\) and a \(C^k\)-map \(\Psi : N_1 \times N_2 \times V \rightarrow M\) such that

\[
\Phi(t, t_0, y_0) = \Psi(\zeta_1(t), \zeta_2(t_0), y_0) \quad \text{for all} \quad (t, t_0, y_0) \in I \times I \times V.
\]

Moreover, we require the existence of a Borel set \(I_0 \subseteq I\) with \(\lambda_1(I \setminus I_0) = 0\) such that \(\frac{d}{dt}\Phi_{t,t_0}(y_0)\) exists and

\[
\frac{d}{dt}\Phi_{t,t_0}(y_0) = f(t, \Phi_{t,t_0}(y_0))
\]

for all \(y_0 \in V\), \(t_0 \in I\) and \(t \in I_0\).

Remark 4.9 If \(\dot{y}(t) = f(t, y(t))\) (as in Definition 4.8) admits local flows which are pullbacks of \(C^k\)-maps, then \(\dot{y}(t) = f(t, y(t))\) satisfies local existence of Carathéodory solutions (arguing as in Remark 3.5).

Remark 4.10 We might speak about local flows which are pullbacks of \(C^k\)-maps on Fréchet manifolds in the situation of Definition 4.8 and speak about local flows which are pullbacks of \(C^k\)-maps on open subsets of Fréchet spaces in the situation of Definition 3.4 to distinguish clearly between the concepts (likewise, we should use separate terminology in Definition 10.14). But it will always be clear from the context what is intended.

Proof of Proposition 1.1. For each chart \(\phi : U_\phi \rightarrow V_\phi \subseteq E\) of \(M\), let \(f_\phi : W_\phi \rightarrow E\) be as in Remark 4.2. We claim: Each differential equation

\[
y'(t) = f_\phi(t, y(t))
\]

admits local flows which are pullbacks of \(C^1\)-maps, in the sense of Definition 3.4. As a consequence, each of the differential equations (19) satisfies

\footnote{In particular, \(\Phi\) is continuous.}
local uniqueness of Carathéodory solutions, by Lemma 3.6. This implies that
(16) satisfies local uniqueness of Carathéodory solutions (cf. Remark 4.2),
which completes the proof. To establish the claim, let \((\bar{t}, \bar{z}) \in W_\phi\) and
\(\bar{y} := \phi^{-1}(\bar{z}) \in U_\phi\). Then
\[(\bar{t}, \bar{y}) \in W \subseteq J \times M.\]
Let \(I, V, \Phi, Y, N_1, N_2, E_1, E_2, \Psi, \zeta_1, \zeta_2,\) and \(I_0\) be as in Definition 4.8. After
shrinking \(I_0\) if necessary, we may assume that, moreover, \(\dot{\zeta}_1(t)\) and \(\dot{\zeta}_2(t)\) exist
for all \(t \in I_0\).
There exist charts \(\phi_j : U_j \to V_j \subseteq E_j\) of \(N_j\) with \(\zeta_j(\bar{t}) \in U_j\) for \(j \in \{1, 2\}\).
Since \(\Psi(\zeta_1(\bar{t}), \zeta_2(\bar{t}), \bar{y}) = \Phi_{\bar{t}, \bar{y}} = \bar{y} \in U_\phi\) and \(\Psi\) is continuous, after shrink-
ing \(U_1\) and \(U_2\) we find an open \(\bar{y}\)-neighbourhood \(U \subseteq U_\phi\) such that
\[\Psi(U_1 \times U_2 \times U) \subseteq U_\phi.\]
Then \(P := \phi(U)\) is an open \(\bar{z}\)-neighbourhood in \(V_\phi\). After shrinking \(U_1\) and \(U_2\) further, we may assume that
\[\Psi(U_1 \times U_2 \times Z) \subseteq U\]
for some open \(\bar{y}\)-neighbourhood \(Z \subseteq U\). Then \(Q := \phi(Z)\) is an open \(\bar{z}\)-neighbourhood in \(P\). After shrinking \(I\), we may assume that \(\zeta_j(I) \subseteq U_j\) for
\(j \in \{1, 2\}\). Then
\[\Psi_\phi : V_1 \times V_2 \times P \to E, \quad (u, v, w) \mapsto \phi(\phi^{-1}_1(u), \phi^{-1}_2(v), \phi^{-1}(w))\]
is a \(C^1\)-map. Moreover, \(\alpha := \phi_1 \circ \zeta_1 : I \to V_1 \subseteq E_1\) and \(\beta := \phi_2 \circ \zeta_2 : I \to V_2 \subseteq E_2\) are absolutely continuous functions which are differentiable at each
\(t \in I_0\). Define
\[\Phi_\phi : I \times I \times P \to E, \quad (t, t_0, z_0) \mapsto \Psi_\phi(\alpha(t), \beta(t_0), z_0) = \phi(\Phi_{t, t_0}(\phi^{-1}(z_0))).\]
Using 2.5 it is now easy to check that \(\Phi_\phi\) and \(\Psi_\phi\) in place of \(\Phi\) and \(\Psi\), with
\(P\) and \(Q\) in place of \(V\) and \(Y\), satisfy the conditions (a)-(c) of Definition 3.4
for \(f_\phi\) in place of \(f\) and \(\bar{z}\) in place of \(\bar{y}\). This establishes the claim. □

5 Initial value problems on \(G\)-manifolds

We now prove Theorem 1.2 which provides a criterion for local uniqueness of Carathéodory solutions to differential equations on smooth \(G\)-manifolds.
which are given by time-dependent fundamental vector fields.

**Proof of Theorem 1.2.** Abbreviate $I := J := [a, b]$. Define

$$f : I \times M \to TM, \quad f(t, y) := \gamma(t)_y(g).$$

(20)

Since $\eta : I \to G$ is absolutely continuous and the evolution of $\gamma$, there exists a Borel set $I_0 \subseteq I$ with $\lambda_1(I \setminus I_0) = 0$ such that $\eta(t)$ exists for all $t \in I_0$ and

$$\dot{\eta}(t) = \eta(t).\gamma(t) \quad \text{for all} \ t \in I_0.$$

Abbreviate $\sigma_y(g) := y.g := \sigma(y, g)$ for $y \in M$ and $g \in G$. We define

$$\Phi : I \times I \times M \to M, \quad (t, t_0, y_0) \mapsto \sigma(y_0, \eta(t_0)^{-1}\eta(t)).$$

and write $\Phi(t, t_0, y_0) := \Phi(t, t_0, y_0)$. The map

$$\Psi : G \times G \times M \to M, \quad (g, h, y) \mapsto \sigma(y, h^{-1}g)$$

is smooth and

$$\Phi(t, t_0, y_0) = \Psi(\zeta_1(t), \zeta_2(t_0), y_0)$$

for all $(t, t_0, y_0) \in I \times I \times M$ with $\zeta_1 := \zeta_2 := \eta$.

Given $y_0 \in M$ and $t_0 \in I$, we obtain an absolutely continuous function $\theta : I \to G$ via

$$\theta(t) := \eta(t_0)^{-1}\eta(t).$$

Also $\phi : I \to M, \phi(t) := y_0.\theta(t) = \Phi(t, t_0, y_0)$ is absolutely continuous as $\phi(t) = \sigma_{y_0} \circ \theta$. By definition, $\phi(t_0) = y_0$. For each $t \in I_0$, we have

$$\dot{\theta}(t) = \eta(t_0)^{-1}.\dot{\eta}(t) = \eta(t_0)^{-1} \eta(t).\gamma(t),$$

whence $\theta(t)^{-1}.\dot{\theta}(t) = \eta(t)^{-1} \eta(t_0).\eta(t_0)^{-1} \eta(t).\gamma(t) = \gamma(t)$ and

$$\dot{\phi}(t) = \frac{d}{ds} \bigg|_{s=0} \phi(t + s) = \frac{d}{ds} \bigg|_{s=0} y_0.\theta(t + s)$$

$$= \frac{d}{ds} \bigg|_{s=0} (y_0.\theta(t)).\theta(t)^{-1} \theta(t + s) = T(\sigma_{y_0, \theta(t)})(\theta(t)^{-1}.\dot{\theta}(t))$$

$$= T(\sigma_{y_0, \theta(t)}) \gamma(t) = \gamma(t)\frac{d}{dt}(\phi(t)) = f(t, \phi(t)).$$

Hence $\phi$ solves (17). Thus conditions (a) and (c) of Definition 1.8 are satisfied (with $I := J, N_1 := N_2 := G, Y := V := M, \Phi, \Psi, \zeta_1, \zeta_2,$ and $I_0$ independent
of \((\bar{t}, \bar{y}) \in [a, b] \times M\). For all \(t_0, t_1, t_2 \in I\) and \(y_0 \in M\), setting \(y_1 := \Phi_{t_1, t_0}(y_0) = y_0 \cdot \eta(t_0)^{-1} \eta(t_1)\), we have
\[
\Phi_{t_2, t_1}(\Phi_{t_1, t_0}(y_0)) = \Phi_{t_2, t_1}(y_1) = y_1 \cdot \eta(t_1)^{-1} \eta(t_2) = (y_0 \cdot \eta(t_0)^{-1} \eta(t_1)) \cdot \eta(t_1)^{-1} \eta(t_2) = y_0 \cdot \eta(t_0)^{-1} \eta(t_2) = \Phi_{t_2, t_0}(y_0).
\]

Thus condition (b) of Definition 4.8 is satisfied by \(\Phi\). We have shown that \(16\), applied to \(f\) as in \(20\), admits local flows which are pullbacks of \(C^\infty\)-maps. Thus \(16\) satisfies local uniqueness of Carathéodory solutions. As \(t \mapsto \Phi_{t, t_0}(y)\) is a solution to \(17\) defined on all of \(I\), we see that \(I_{t_0, y_0} = I\) for all \((t_0, y_0) \in I \times M\) and \(\gamma_{t_0, y_0}(t) = \Phi_{t, t_0}(y_0)\). The domain of the maximal flow \(\text{Fl}\) of \(16\) is therefore given by \(\Omega = I \times I \times M\) here, and \(\text{Fl}(t, t_0, y_0) = \gamma_{t_0, y_0}(t) = \Phi_{t, t_0}(y_0)\), which completes the proof. □

6 Reachable neighbourhoods in \(G\)-manifolds

In this section, we prove Theorem 1.4. We begin with preparatory results. First, we discuss the approximation of vector-valued \(L^1\)-functions by staircase functions. Using the Trotter product formula, we then provide preparations enabling us to replace controls in the convex hull \(\text{conv}\{v_1, \ldots, v_m\}\) of finitely many vectors with controls in \(\{v_1, \ldots, v_m\}\).

For each Fréchet space \(E\), the space of \(E\)-valued staircase functions is dense in \(L^1([a, b], E)\). Moreover, we have:

**Lemma 6.1** Let \(a < b\) be real numbers, \(E\) be a Fréchet space, and \(\gamma \in L^1([a, b], E)\). Let \(q\) be a continuous seminorm on \(E\) and \(\varepsilon > 0\). Then we have:

(a) There exists a staircase function \(\eta: [a, b] \to E\) such that \(\eta([a, b]) \subseteq \gamma([a, b])\) and \(\|\gamma - \eta\|_{L^1, q} \leq \varepsilon\).

(b) There exists a continuous function \(\theta: [a, b] \to E\) such that \(\theta([a, b]) \subseteq \text{conv}(\gamma([a, b]))\) and \(\|\gamma - \theta\|_{L^1, q} \leq \varepsilon\).

**Proof.** (a) Let \((y_n)_{n \in \mathbb{N}}\) be a sequence in \(\gamma([a, b])\) such that \(\{y_n: n \in \mathbb{N}\}\) is dense in \(\gamma([a, b])\). Abbreviate \(T := b - a\). Define
\[
A_1 := \left\{ t \in [a, b]: q(\gamma(t) - y_1) < \frac{\varepsilon}{6T} \right\}
\]

...
and, recursively,

\[ A_n := \left\{ t \in [a, b] \setminus (A_1 \cup \cdots \cup A_{n-1}) : q(\gamma(t) - y_n) < \frac{\varepsilon}{6T} \right\} \]

for integers \( n \geq 2 \). Then \( (A_n)_{n \in \mathbb{N}} \) is a sequence of pairwise disjoint Borel sets with union \([a, b] \). There exists \( N \in \mathbb{N} \) such that \( R := \bigcup_{n>N} A_n \) satisfies

\[ \int_R q(\gamma(t)) \, dt < \varepsilon/6 \]

and

\[ q(y_1)\lambda_1(R) < \varepsilon/12. \]  

Using characteristic functions of the sets \( A_n \), we define

\[ \gamma_1 : [a, b] \to E, \quad t \mapsto \sum_{n=1}^N 1_{A_n}(t)y_n. \]

If \( n \in \{1, \ldots, N\} \) and \( t \in A_n \), then \( q(\gamma(t) - \gamma_1(t)) = q(\gamma(t) - y_n) < \varepsilon/(6T) \). If \( t \in R \), then \( q(\gamma(t) - \gamma_1(t)) = q(\gamma(t)) \). Hence, abbreviating \( A := A_1 \cup \cdots \cup A_N \),

\[ \|\gamma - \gamma_1\|_{L^1,q} = \int_A q(\gamma(t) - \gamma_1(t)) \, dt + \int_R q(\gamma(t)) \, dt \leq \frac{\varepsilon}{6T} \lambda_1(A) + \frac{\varepsilon}{6} \leq \frac{\varepsilon}{3}. \]

Pick a real number \( C > 0 \) such that

\[ q(y_n) \leq C \text{ for all } n \in \{1, \ldots, N\}. \]

By inner regularity of \( \lambda_1 \), for \( n \in \{1, \ldots, N\} \) we find a compact subset \( K_n \subseteq A_n \) such that

\[ \lambda_1(A_n \setminus K_n) \leq \frac{\varepsilon}{12NC}. \]  

Hence \( \gamma_2 := \sum_{n=1}^N 1_{K_n}y_n \) satisfies

\[ \|\gamma_1 - \gamma_2\|_{L^1,q} = \sum_{n=1}^N q(y_n)\lambda_1(A_n \setminus K_n) \leq \varepsilon/12 \leq \varepsilon/3. \]

The compact sets \( K_1, \ldots, K_N \) are pairwise disjoint, whence we find pairwise disjoint open subsets \( U_1, \ldots, U_N \) of \([a, b] \) such that \( K_n \subseteq U_n \) for all \( n \in \{1, \ldots, N\} \).
\{1, \ldots, N\}. By outer regularity of \(\lambda_1\), after shrinking the sets if necessary we may assume that 

\[
\lambda_1(U_n \setminus K_n) < \frac{\varepsilon}{6NC} \text{ for all } n \in \{1, \ldots, N\}.
\]

After replacing \(U_n\) with a finite number of its connected components, we may assume that each \(U_n\) is a union of finitely many pairwise disjoint intervals which are open in \([a, b]\). Let

\[
U := \bigcup_{n=1}^{N} U_n \text{ and } B := [a, b] \setminus U.
\]

Then \(\eta := y_1 1_B + \sum_{n=1}^{N} y_n 1_{U_n}\) is a staircase function with \(\eta([a, b]) \subseteq \gamma([a, b])\). Since \(\eta(t) = y_n \in \gamma([a, b])\) for each \(n \in \{1, \ldots, N\}\) and \(t \in U_n\), while \(\eta(t) = y_1 \in \gamma([a, b])\) for all \(t \in B\). Since

\[
B \subseteq [a, b] \setminus (K_1 \cup \cdots \cup K_N) = R \cup \bigcup_{n=1}^{N} (A_n \setminus K_n),
\]

using (21) and (22) we estimate 

\[
q(y_1)\lambda_1(B) \leq \varepsilon/12 + \varepsilon/12 = \varepsilon/6.
\]

Thus 

\[
\|\gamma - \eta\|_{L^1,q} = \sum_{n=1}^{N} \lambda_1(U_n \setminus K_n)q(y_n) + \lambda_1(B)q(y_1) \leq \varepsilon/6 + \varepsilon/6 = \varepsilon/3,
\]

whence \(\|\gamma - \eta\|_{L^1,q} \leq \varepsilon/2\). There exist \(n \in \mathbb{N}\) and numbers \(a = t_0 < t_1 < \cdots < t_n = b\) such that \(\eta|_{[t_{j-1}, t_j]}\) is constant, with value \(y_j \in \gamma([a, b])\), for all \(j \in \{1, \ldots, n\}\). Choose \(\delta > 0\) so small that \(2\delta < t_j - t_{j-1}\) for all \(j \in \{1, \ldots, n\}\). For \(k \in \mathbb{N}\), we define \(\theta_k \in C([a, b], E)\) piecewise, as follows: We let \(\theta_k(t) := y_1\) for \(t \in [a, t_1 - \delta/k]\) and \(\theta_k(t) := y_n\) for \(t \in [t_{n-1} + \delta/k, b]\). We let \(\theta_k(t) := y_j\) for all \(j \in \{2, \ldots, n-1\}\) and \(t \in [t_{j-1} + \delta/k, t_j - \delta/k]\). Finally, for \(j \in \{1, \ldots, n-1\}\) and \(t \in [t_j - \delta/k, t_j + \delta/k]\), we define 

\[
\theta_k(t) := y_j + \frac{t - t_j + \delta/k}{(2\delta/k)}(y_{j+1} - y_j); \quad (27)
\]
thus $\theta_k|_{t_j-\delta/k,t_j+\delta/k}$ is a restriction of the unique affine-linear map taking $t_j - \delta/k$ to $y_j$ and $t_j + \delta/k$ to $y_{j+1}$. Note that the image of $\theta_k$ is contained in the convex hull $C$ of $\eta([a,b])$ and hence in the convex hull of $\gamma([a,b])$. As $k \to \infty$, we have $\theta_k(t) \to \eta(t)$ for all $t \in [a,b] \setminus \{t_0, \ldots , t_n\}$ and thus for almost all $t \in [a,b]$. Since $C$ is bounded, also $q(C)$ is bounded. The constant function $g: [a,b] \to [0,\infty]$, $t \mapsto 2\sup q(C)$ is $\lambda_1$-integrable and $q(\eta(t) - \theta_k(t)) \leq g(t)$ for all $t \in [a,b]$. Thus
\[
\|\eta - \theta_k\|_{L^1,q} = \int_a^b q(\eta(t) - \theta_k(t)) \, dt \to 0
\]
as $k \to \infty$ by dominated convergence. Notably, we find $k \in \mathbb{N}$ such that $\theta := \theta_k$ satisfies $\|\eta - \theta\|_{L^1,q} \leq \epsilon/2$. Then $\|\gamma - \theta\|_{L^1,q} \leq \epsilon$.

\[\square\]

**Remark 6.2** If $E$ may not be a Fréchet space, but is an arbitrary locally convex space, and $\gamma: [a,b] \to E$ is a piecewise continuous function, then all conclusions of Lemma 6.1 remain valid, with identical proof.\footnote{Note that $\zeta([a,\beta])$ is compact and metrizable (using \cite{9}, Theorem 4.4.17), whence $\zeta([a,\beta])$ and its subsets are metrizable and separable, for all real numbers $\alpha < \beta$ and each continuous function $\zeta: [a,\beta] \to E$. Hence $\gamma - \eta$ and $\gamma - \theta$ are Borel measurable also in the current variant of Lemma 6.1, and so are the functions $\gamma - \gamma_1$, $\gamma_1 - \gamma_2$, $\gamma_2 - \eta$, and $\eta - \theta_k$ in its proof (by Lemma 2.4).}

We shall use the following concept (cf. \cite{17}, Definition 14.3).\footnote{If $R \geq 1$ there, we may replace $R$ with 1; if $R \leq 1$, replace $p$ with $p/R$ and $R$ with 1.}

**Definition 6.3** A Lie group $G$ modelled on a locally convex space $E$ is called **locally $\mu$-convex** if there exists a chart $\phi: U_\phi \to V_\phi \subseteq E$ of $G$ with $e \in U_\phi$ and $\phi(e) = 0$ with the following property: For each continuous seminorm $q$ on $E$, there exist a continuous seminorm $p$ on $E$ such that
\[
g_1g_2 \cdots g_n \in U_\phi \quad \text{and} \quad q(\phi(g_1 \cdots g_n)) \leq \sum_{j=1}^n p(\phi(g_j))
\]
for all $n \in \mathbb{N}$ and all $g_1, \ldots , g_n \in U_\phi$ such that $\sum_{j=1}^n p(\phi(g_j)) < 1$.

Then every chart taking $e$ to 0 has this property (see \cite{17}, Remark 14.4).

Let $X$ be a set. If $\gamma: [0,a] \to X$ and $\eta: [0,b] \to X$ are staircase functions, we write $\gamma \ast \eta: [0,a+b] \to X$ for the concatenation defined via $t \mapsto \gamma(t)$ for $t \in [0,a]$, $t \mapsto \eta(t-a)$ for $t \in ]a,a+b]$.
Lemma 6.4 Let $G$ be a $C^0$-regular Lie group modelled on a locally convex space, with Lie algebra $\mathfrak{g}$. Let $m \in \mathbb{N}$, $t_1, \ldots, t_m \in [0, \infty]$, $T := t_1 + \cdots + t_m$, $v_1, \ldots, v_m \in \mathfrak{g}$, and $W$ be a neighbourhood of $g := \exp G(t_1v_1 + \cdots + t_mv_m)$ in $G$. Then there exists a staircase function $\gamma: [0, T] \to \mathfrak{g}$ with $\gamma([0, T]) \subseteq \{v_1, \ldots, v_m\}$ such that $\text{evol}(\gamma) \in W$.

**Proof.** The proof is by induction on $m \in \mathbb{N}$. The case $m = 1$ is trivial, as $\exp G(t_1v_1) = \text{evol}(\gamma)$ for the constant function $\gamma: [0, t_1] \to \mathfrak{g}$, $t \mapsto v_1$, which is a staircase function. Now let $m > 1$ and assume the assertion holds for $m-1$ in place of $m$. Abbreviate $v := t_1v_1 + \cdots + t_{m-1}v_{m-1}$. Since $G$ is $C^0$-regular, it is locally $\mu$-convex (see [23]), whence $G$ satisfies the strong Trotter property formulated in [18, p. 7] (see [22]). In particular, the Trotter product formula holds for each pair of Lie algebra elements, and thus

$$g = \exp G(v + t_mv_m) = \lim_{n \to \infty} \left( \exp G(v/n) \exp G(t_mv_m/n) \right)^n.$$ 

We therefore find $n \in \mathbb{N}$ such that

$$\left( \exp G(v/n) \exp G(t_mv_m/n) \right)^n \in W.$$ 

Now $\exp G(t_mv_m/n) = \text{evol}(\theta)$ for the constant function $\theta: [0, t_m/n] \to \mathfrak{g}$, $t \mapsto v_m$. There exists an open neighbourhood $U$ of $\exp G(v/n)$ in $G$ such that

$$(u \text{ evol}(\eta))^n \in W \quad \text{for all } u \in U.$$ 

Since $\exp G(v/n) = \exp G(t_1v_1/n + \cdots + t_{m-1}v_{m-1}/n) \in U$, we have

$$\text{evol}(\eta) \in U$$

for a staircase function $\eta: [0, (T-t_m)/n] \to \mathfrak{g}$ with image in $\{v_1, \ldots, v_{m-1}\}$, by induction. If we define

$$\gamma := (\eta * \theta) * \cdots * (\eta * \theta)$$

10If $\gamma(t) = (\gamma_1 * \cdots \gamma_k)(t)$ except for finitely many $t$, with constant functions $\gamma_1, \ldots, \gamma_k$, then $\text{evol}(\gamma) := \text{evol}(\gamma_1) \cdots \text{evol}(\gamma_k)$. See Remark [7.7] for more details.

11For the case of $L^1$-regular Fréchet-Lie groups considered in Theorem [1.4], the strong Trotter property was already established in [18, Theorem I].
as the concatenation of \( n \) copies of \( \eta \ast \theta \), then

\[
\text{evol}(\gamma) = (\text{evol}(\eta) \text{evol}(\theta))^n \in W.
\]

Moreover, \( \gamma \) is a staircase function with image in \( \{v_1, \ldots, v_m\} \). \( \square \)

**Proof of Theorem 1.4.** (b) implies (a) since \( L^p([0, T], \mathfrak{g}) \subseteq L^1([0, T], \mathfrak{g}) \). Likewise, (d) implies (c) as staircase functions are piecewise continuous, and (c) implies (b) as piecewise continuous functions \([0, T] \rightarrow \mathfrak{g}\) are in \( L^p([0, T], \mathfrak{g}) \).

(a) implies (d): Let \( \gamma \in L^1([0, T], \mathfrak{g}) \) such that \( x_0.\text{evol}(\gamma) \in U \). Since \( \text{evol}: L^1([0, T], \mathfrak{g}) \rightarrow G \) and the action \( M \times G \rightarrow M \) are continuous, there exists \( \varepsilon > 0 \) and a continuous seminorm \( P \) on \( \mathfrak{g} \) such that

\[
x_0.\text{evol}(\eta) \in U \text{ for all } \eta \in L^1([0, T], \mathfrak{g}) \text{ such that } \|\gamma - \eta\|_{L^1, P} < \varepsilon.
\]

By Lemma 6.1(a), such an \( \eta \) can be chosen as a staircase function with values in \( \gamma([0, T]) \subseteq S \).

Now assume that \( S \) is convex.

(a) implies (e): For \( \gamma, \varepsilon \) and \( P \) as before, Lemma 6.1(b) provides a continuous function \( \theta: [0, T] \rightarrow \mathfrak{g} \) such that \( \|\gamma - \theta\|_{L^1, P} < \varepsilon \) (whence \( x_0.\text{evol}(\theta) \in U \)) and \( \theta([0, T]) \subseteq \text{conv } \gamma([0, T]) \subseteq S \).

(e) implies (c) as each continuous function \([0, T] \rightarrow \mathfrak{g}\) is piecewise continuous.

Now assume that \( S \) is convex and \( \text{conv}(\text{ex}(S)) \) is dense in \( S \).

(d) implies (f): Let \( \gamma: [0, T] \rightarrow S \) be a staircase function with \( x_0.\text{evol}(\gamma) \in U \). Let \( V \) be an open neighbourhood of \( \text{evol}(\gamma) \) in \( G \) such that \( x_0.V \subseteq U \). There are \( 0 = t_0 < t_1 < \cdots < t_\ell = T \) such that \( \gamma|_{t_{j-1}, t_j} \) is a constant function with value \( w_j \in S \) for all \( j \in \{1, \ldots, \ell\} \). The function

\[
\phi: \mathfrak{g}^\ell \rightarrow L^1([0, T], \mathfrak{g}), \ u = (u_1, \ldots, u_\ell) \mapsto u_\ell 1_{[t_{\ell-1}, t_\ell]} + \sum_{j=1}^{\ell-1} u_j 1_{[t_{j-1}, t_j]}
\]

is linear and continuous as \( \|\phi(u)\|_{L^1, P} \leq T \max\{P(u_1), \ldots, P(u_\ell)\} \) for each continuous seminorm \( P \) on \( \mathfrak{g} \). Thus there exists an open neighbourhood \( W \) of \( (w_1, \ldots, w_\ell) \) in \( \mathfrak{g}^\ell \) such that \( \text{evol}(\phi(W)) \subseteq V \). Since \( \text{conv}(\text{ex}(S)) \)^\ell is dense in \( S^\ell \) and \( (w_1, \ldots, w_\ell) \in S^\ell \), we find \( u = (u_1, \ldots, u_\ell) \in (\text{conv}(\text{ex}(S)))^\ell \cap W \).
After replacing $\gamma$ with $\phi(u)$, we may assume that $\gamma([0,T]) = \{w_1, \ldots, w_\ell\} \subseteq \text{conv}(\text{ex}(S))$. Since

$$\text{evol}(\gamma) = \exp_G((t_1 - t_0)w_1) \cdots \exp_G((t_\ell - t_{\ell-1})w_\ell) \in V,$$

we find open neighbourhoods $W_j$ of $\exp_G((t_j - t_{j-1})w_j)$ in $G$ for $j \in \{1, \ldots, \ell\}$ such that $W_1W_2 \cdots W_\ell \subseteq V$.

For each $j \in \{1, \ldots, \ell\}$, we have

$$w_j = \sum_{i=1}^{m_j} t_{j,i}v_{j,i}$$

for suitable $m_j \in \mathbb{N}$, $v_{j,i} \in \text{ex}(S)$ and $t_{j,i} > 0$ for $i \in \{1, \ldots, m_j\}$ with $\sum_{i=1}^{m_j} t_{j,i} = 1$. By Lemma 6.4, for each $j \in \{1, \ldots, \ell\}$, there is a staircase function $\gamma_j: [0, t_j - t_{j-1}] \to g$ with image in $\{v_{j,1}, \ldots, v_{j,m_j}\} \subseteq \text{ex}(S)$ such that $\text{evol}(\gamma_j) \in W_j$.

Then $\eta := \gamma_1 \circ \cdots \circ \gamma_\ell: [0,T] \to g$ is a staircase function with values in $\text{ex}(S)$ such that $\text{evol}(\eta) = \text{evol}(\gamma_1) \circ \cdots \circ \text{evol}(\gamma_\ell) \in W_1 \cdots W_\ell \subseteq V$ and thus $x_0, \text{evol}(\eta) \in U$.

(f) implies (d): This is trivial. □.

7 Continuity of evolution in the $L^1$-topology

For our next theorem, we need terminology from [18]. We recall:

7.1 Assume that, for each Fréchet space $E$ and real numbers $a < b$, a vector subspace $\mathcal{E}([a,b], E)$ of $L^1([a,b], E)$ has been chosen, together with a locally convex vector topology on $\mathcal{E}([a,b], E)$ such that the inclusion map $\mathcal{E}([a,b], E) \to L^1([a,b], E)$ is continuous. Following [18, Definition 3.1], we call $\mathcal{E}$ a bifunctor on Fréchet spaces if (a) and (b) hold:

(a) For each continuous linear map $\lambda: E_1 \to E_2$ between Fréchet spaces, we have $[\lambda \circ \gamma] \in \mathcal{E}([a,b], E_2)$ for all $a < b$ and $[\gamma] \in \mathcal{E}([a,b], E_1)$, and the linear map $\mathcal{E}([a,b], \lambda): \mathcal{E}([a,b], E_1) \to \mathcal{E}([a,b], E_2)$, $[\gamma] \mapsto [\lambda \circ \gamma]$ is continuous.
(b) For each Fréchet space $E$, real numbers $a < b$ and $c < d$ and each mapping $f : [c, d] \to [a, b]$ which is the restriction of a strictly increasing affine-linear map $\mathbb{R} \to \mathbb{R}$, we have $[\gamma \circ f] \in \mathcal{E}([c, d], E)$ for each $[\gamma] \in \mathcal{E}([a, b], E)$ and the linear map $\mathcal{E}(f, E) : \mathcal{E}([a, b], E) \to \mathcal{E}([c, d], E)$, $[\gamma] \mapsto [\gamma \circ f]$ is continuous.

7.2 Recall that a locally convex space $E$ is called sequentially complete (or also: an scle-space, for short) if every Cauchy sequence in $E$ is convergent in $E$. We say that $E$ is integral complete if the weak integral $\int_0^1 \gamma(t) \, dt$ exists in $E$ for each continuous function $\gamma : [0, 1] \to E$. It is known (cf. [12]) that $E$ is integral complete if and only if $E$ has the metric convex compactness property (metric CCP) discussed in [40], requiring that the closure of conv $K$ in $E$ be compact for each compact, metrizable subset $K \subseteq E$. Sequential completeness implies integral completeness, but not conversely (cf. [40]). The modelling space of every $C^0$-regular Lie group is integral complete (see [17, Theorem C (a)]).

7.3 Following [18, Definition 1.38], we say that a locally convex space $E$ has the Fréchet exhaustion property (FEP) if every separable closed vector subspace $S \subseteq E$ is a union of vector subspaces $F_1 \subseteq F_2 \subseteq \cdots$ of $E$ which are Fréchet spaces in the induced topology. All Fréchet spaces, strict (LF)-spaces, and locally convex direct sums of Fréchet spaces are (FEP)-spaces, as well as their closed vector subspaces. Bifunctors on sequentially complete (FEP)-spaces are defined as in 7.1, replacing Fréchet spaces with sequentially complete (FEP)-spaces.

7.4 If $E$ is a locally convex space, then the vector space $L_{\infty}^\infty([a, b], E)$ of all Borel measurable functions $\gamma : [a, b] \to E$ can be considered such that the closure of $\gamma([a, b])$ in $E$ is compact and metrizable, and the corresponding space $L^\infty_{\infty}([a, b], E)$ of equivalence classes (see [14]). Bifunctors on integral complete locally convex spaces are defined as in 7.1, replacing Fréchet spaces with integral complete locally convex spaces and each symbol $L^1$ with $L^\infty_{\infty}$.

7.5 Let $\mathcal{E}$ be a bifunctor on Fréchet spaces (resp., on sequentially complete (FEP)-spaces, resp., on integral complete locally convex spaces) and $E$ be such a locally convex space. Given real numbers $a < b$, a function $\gamma : [a, b] \to E$ can be considered such that the closure of $\gamma([a, b])$ in $E$ is compact and metrizable, and the corresponding space $L^\infty_{\infty}([a, b], E)$ of equivalence classes (see [14]). Bifunctors on integral complete locally convex spaces are defined as in 7.1, replacing Fréchet spaces with integral complete locally convex spaces and each symbol $L^1$ with $L^\infty_{\infty}$.

\[\text{For example, } \mathcal{E} = L^p \text{ with } p \in [1, \infty] \text{ in cases 1 or 2; or } \mathcal{E} = L^\infty_{\infty} \text{ in all three cases.}\]
$\eta: [a,b] \to E$ is called a $AC_E$ if there exists $[\gamma] \in \mathcal{E}([a,b], E)$ such that

$$\eta(t) = \eta(a) + \int_a^t \gamma(s) \, ds \quad \text{for all } t \in [a,b],$$

(23)

where the integrals exist as weak integrals in $E$, i.e., $\lambda(\eta(t)) = \lambda(\eta(a)) + \int_a^t \lambda(\gamma(s)) \, ds$ for all $\lambda \in E'$. Then $[\gamma] \in \mathcal{E}([a,b], E)$ is uniquely determined by $\eta$ (see [18]), and we write $\eta' := [\gamma]$. 

To avoid clumsy formulations, in the following proof square brackets will frequently be omitted; it will be clear from the context whether $\gamma$ or $[\gamma]$ is intended. Notably, we may write $\eta' = \gamma$ for a representative $\gamma$ of $[\gamma]$.

**Theorem 7.6** Let $\mathcal{E}$ be a bifunctor on Fréchet spaces (resp., on sequentially complete (FEP)-spaces, resp., on integral complete locally convex spaces) which satisfies the locality axiom, the pushforward axioms, and such that smooth functions acts smoothly on $AC_E$. If $\mathcal{E}$ satisfies the subdivision axiom, then the evolution map

$$\text{Evol}: \mathcal{E}([0,1], g) \to C([0,1], G)$$

is continuous with respect to the $L^1$-topology on $\mathcal{E}([0,1], g)$, for each Lie group $G$ modelled on a locally convex space of the preceding form such that $G$ is $\mathcal{E}$-regular.

**Proof of Theorem 7.6** Let $U \subseteq G$ be an open identity neighbourhood. After shrinking $U$, we may assume that there exists a $C^\infty$-diffeomorphism $\phi: U \to V$ onto an open 0-neighbourhood $V \subseteq g$ such that $\phi(e) = 0$ and $d\phi|_g = \text{id}_g$. Let $q$ be a continuous seminorm on $g$. Since $G$ is $\mathcal{E}$-regular, it is $C^0$-regular (see [18] Corollaries 5.21 and 5.22) and hence locally $\mu$-convex (see [23]). Thus, we find a continuous seminorm $p$ on $g$ such that

$$g_1 \cdots g_n \in U \quad \text{and} \quad q(\phi(g_1 \cdots g_n)) \leq \sum_{j=1}^n p(\phi(g_j))$$

(24)

for all $n \in \mathbb{N}$ and all $g_1, \ldots, g_n \in U$ such that $\sum_{j=1}^n p(\phi(g_j)) < 1$. Let $Y \subseteq U$ be an open identity neighbourhood such that $YY \subseteq U$. Then $Z := \phi(Y) \subseteq V$ is an open 0-neighbourhood. The map

$$\mu: Z \times Z \to V, \quad (x,y) \mapsto \phi(\phi^{-1}(x)\phi^{-1}(y))$$

...
is smooth. There exist an open 0-neighbourhood $N \subseteq \mathbb{Z}$ and a continuous seminorm $P$ on $\mathfrak{g}$ such that

$$d_2\mu(N \times \{0\} \times B^P_1(0)) \subseteq B^p_1(0),$$

entailing that

$$p(d_2\mu(x, 0, y)) \leq P(y) \quad \text{for all } x \in N \text{ and } y \in \mathfrak{g}.$$

By continuity of $\text{Evol}: \mathcal{E}([0, 1], \mathfrak{g}) \rightarrow C([0, 1], G)$, there exists an open 0-neighbourhood $W \subseteq \mathcal{E}([0, 1], \mathfrak{g})$ such that

$$\text{Evol}(W) \subseteq C([0, 1], \phi^{-1}(N)).$$

For $\eta \in W$, we deduce that $\theta := \phi \circ \text{Evol}(\eta)$ satisfies

$$\theta(t) = \int_0^t \theta'(s) \, ds = \int_0^t d_2\mu(\theta(s), 0, \eta(s)) \, ds$$

for all $t \in [0, 1]$, whence

$$p(\theta(t)) \leq \int_0^t p(d_2\mu(\theta(s), 0, \eta(s))) \, ds \leq \int_0^t P(\eta(s)) \, ds \leq ||\eta||_{L^1,P}. \quad (25)$$

Now let $\gamma \in \mathcal{E}([0, 1], \mathfrak{g})$ such that $||\gamma||_{L^1,P} < 1$. By the subdivision property, there exists $n \in \mathbb{N}$ such that $\gamma_{n,k} \in W$ for all $k \in \{0, 1, \ldots, n-1\}$, where

$$\gamma_{n,k} : [0, 1] \rightarrow \mathfrak{g}, \quad t \mapsto \frac{1}{n} \gamma((k + t)/n)$$

(see [18, Definition 5.24]). We fix $n$. Note that

$$\sum_{k=0}^{n-1} ||\gamma_{n,k}||_{L^1,P} = ||\gamma||_{L^1,P} < 1.$$

For $k \in \{0, \ldots, n\}$ and $t \in [0, 1]$, we get $\eta_n := \text{Evol}(\gamma_{n,k}) \in C([0, 1], \phi^{-1}(N))$ and

$$p(\phi(\eta_n(t))) \leq ||\gamma_{n,k}||_{L^1,P},$$

by (25). Given $t \in [0, 1]$, there is $k \in \{0, \ldots, n-1\}$ with $t \in [k/n, (k+1)/n]$. By (24), we have

$$\text{Evol}(\gamma)(t) = \text{evol}(\gamma_{n,0}) \cdots \text{evol}(\gamma_{n,k-1}) \text{Evol}(\gamma_{n,k})(nt - k) \in U,$$
since \( g_j := \text{evol}(\gamma_{n,j-1}) \) for \( j \in \{1, \ldots, k\} \) and \( g_{k+1} := \text{Evol}(\gamma_{n,k})(nt - k) \) are elements of \( \phi^{-1}(N) \subseteq U \) which satisfy
\[
\sum_{j=1}^{k+1} p(\phi(g_j)) \leq \sum_{j=0}^{k} \|\gamma_{n,j}\|_{L^1,P} < 1.
\]
Thus \( \text{Evol}(\gamma) \in \mathcal{C}([0,1],U) \) for all \( \gamma \in \mathcal{E}([0,1],\mathfrak{g}) \) such that \( \|\gamma\|_{L^1,P} < 1 \), showing that \( \text{Evol}: \mathcal{E}([0,1],\mathfrak{g}) \to \mathcal{C}([0,1],G) \) is continuous at 0 with respect to the \( L^1 \)-topology.

Since \( G \) is \( \mathcal{E} \)-regular, \( \mathcal{E}([0,1],\mathfrak{g}) \) can be made a group with neutral element 0 and group multiplication given by
\[
[\gamma] \circ [\eta] := [\text{Ad}(\text{Evol}(\eta))^{-1} \cdot \gamma] + [\eta]
\]
(see [18, Definition 5.34]). In view of [18, Lemma 5.10], the right translation
\[
\rho_{[\eta]}: \mathcal{E}([0,1],\mathfrak{g}) \to \mathcal{E}([0,1],\mathfrak{g}), \quad [\gamma] \mapsto [\gamma] \circ [\eta]
\]
is continuous with respect to the \( L^1 \)-topology on both sides and hence a homeomorphism, for each \( [\eta] \in \mathcal{E}([0,1],\mathfrak{g}) \). For \( \beta := \text{Evol}(\eta) \), the right translation
\[
\rho_\beta: \mathcal{C}([0,1],G) \to \mathcal{C}([0,1],G), \quad \zeta \mapsto \zeta \beta
\]
is continuous. Since \( \text{Evol} \) is continuous at 0, we deduce from
\[
\text{Evol} = \rho_\beta \circ \text{Evol} \circ \rho_{[\eta]^{-1}}
\]
that the map \( \text{Evol}: \mathcal{E}([0,1],\mathfrak{g}) \to \mathcal{C}([0,1],G) \) is continuous at \( [\eta] \) with respect to the \( L^1 \)-topology on its domain. □

**Remark 7.7** Let \( G \) be a \( C^0 \)-regular Lie group modelled on a locally convex space and \( \mathcal{P}C([0,1],\mathfrak{g}) \) be the space of piecewise continuous \( \mathfrak{g} \)-valued functions on \([0,1]\). Given \( \gamma \in \mathcal{P}C([0,1],\mathfrak{g}) \), let \( 0 = t_0 < \cdots < t_m = 1 \) be a subdi-


division of \([0,1]\) such that \( \gamma|_{[t_{j-1},t_j]} \) has a continuous extension \( \gamma_j: [t_{j-1},t_j] \to \mathfrak{g} \) for all \( j \in \{1, \ldots, m\} \). Given \( t \in [0,1] \), there is \( j \in \{1, \ldots, m\} \) such that \( t \in [t_{j-1},t_j] \). We define
\[
\text{Evol}(\gamma)(t) := \text{evol}(\gamma_1) \cdots \text{evol}(\gamma_{j-1}) \cdot \text{Evol}(\gamma_j)(t),
\]
using evolution maps \( \text{evol} \) with domain \( \mathcal{C}([t_{i-1},t_i],\mathfrak{g}) \) for \( i \in \{1, \ldots, j-1\} \) and \( \text{Evol} \) with domain \( \mathcal{C}([t_{j-1},t_j],\mathfrak{g}) \) on the right-hand side. Then \( \text{Evol}(\gamma) \in \mathcal{C}([0,1],G) \).
$C([0,1], G)$ is well defined, independent of the choice of subdivision\textsuperscript{13} and we obtain a map

$$\text{Evol}: PC([0,1], g) \to C([0,1], G).$$

(26)

It is known from the work of Hanusch that the evolution map\textsuperscript{26} is continuous with respect to the $L^1$-topology\textsuperscript{14} on its domain $PC([0,1], g)$. This can be shown as in the previous proof, with the modification that the subdivisions used to define the $\gamma_{n,k}$ need not be equidistant but include the points $t_0 < \cdots < t_m$ just considered (and hence the points of discontinuity of $\gamma$), to ensure that the $\gamma_{n,k}$ are continuous functions; moreover, the subdivisions need to be chosen such that the mesh tends to 0 for $n \to \infty$.

Using an affine-linear reparametrization, we can replace $[0,1]$ with $[a,b]$ for any $a < b$ in the preceding results concerning evolution maps (cf. 2.8).

**Proof of Theorem 1.9.** Theorem 1.9 becomes a special case of Theorem 7.6 if we consider $L^p$ as a bifunctor on Fréchet spaces. The required axioms and hypotheses were verified in [18]. □

8 Analogues using only $L^q$-regularity

**Proof of Theorem 1.10.** (b) implies (a)' since $L^p([0,T], g) \subseteq L^q([0,T], g)$.

The implications “(d)$\Rightarrow$(c)”, “(c)$\Rightarrow$(b)”, “(e)$\Rightarrow$(c)” and “(f)$\Rightarrow$(d)” can be shown as in the proof of Theorem 1.4.

(a)' implies (d): Let $\gamma \in L^q([0,T], g)$ such that $x_0.evol(\gamma) \in U$. Since $\text{Evol}: L^q([0,T], g) \to G$ is continuous with respect to the $L^1$-topology (cf. Theorem 1.9) and the action $M \times G \to M$ is continuous, there exists $\varepsilon > 0$ and a continuous seminorm $P$ on $g$ such that

$$x_0.evol(\eta) \in U \text{ for all } \eta \in L^q([0,T], g) \text{ such that } \|\gamma - \eta\|_{L^1,P} < \varepsilon.$$

By Lemma 6.1(a), such an $\eta$ can be chosen as a staircase function with values in $\gamma([0,T]) \subseteq S$.

\textsuperscript{13}It is unchanged if we add one point to a subdivision, and hence under passage to joint refinements of two given subdivisions.

\textsuperscript{14}By definition, this is the (not necessarily Hausdorff) locally convex vector topology defined by the seminorms $PC([0,1], g) \to [0, \infty]$, $\gamma \mapsto \|\gamma\|_{L^1,q}$ for $q$ in the set of all continuous seminorms on $g$. 36
The implication “(a)′⇒(e)” can be shown like “(a)⇒(e)” in the proof of Theorem 1.4.

The implication “(d)⇒(f)” can be shown as in the proof of Theorem 1.4, except that we consider φ as a map from \( g^\ell \) to \( L^q([0,T], g) \), endowed with the \( L^1 \)-topology. □

9 Analogues requiring only \( C^0 \)-regularity

For \( C^0 \)-regular Lie groups, we can say the following.

**Theorem 9.1** Instead of requiring \( L^1 \)-regularity, let \( G \) be a Lie group modelled on a locally convex space such that \( G \) is \( C^0 \)-regular in the situation of Theorem 1.4. Then conditions (c) and (d) of Theorem 1.4 are equivalent. If \( S \) is convex, then (d) is equivalent to (e). If \( S \) is convex and \( \text{conv}(\text{ex}(S)) \) is dense in \( S \), then (d) is equivalent to (f).

**Proof.** The implications “(d)⇒(c)”, “(e)⇒(c)”, and “(f)⇒(d)” are trivial.

(c) implies (d): Let \( \gamma \in PC([0,T], g) \) such that \( x_0.\text{evol}(\gamma) \in U \). Since \( \text{evol}: PC([0,T], g) \to G \) is continuous with respect to the \( L^1 \)-topology (cf. Remark 7.7), and the action \( M \times G \to M \) is continuous, there exists \( \varepsilon > 0 \) and a continuous seminorm \( P \) on \( g \) such that

\[ x_0.\text{evol}(\eta) \in U \text{ for all } \eta \in PC([0,T], g) \text{ such that } \|\gamma - \eta\|_{L^1,P} < \varepsilon. \]

By the conclusion of Lemma 6.1(a), such an \( \eta \) can be chosen as a staircase function with values in \( \gamma([0,T]) \subseteq S \) (see Remark 6.2).

(c) implies (e): In view of Remark 6.2, starting with a piecewise continuous function \( \gamma \), this can be shown like “(a)⇒(e)” in the proof of Theorem 1.4.

The implication “(d)⇒(f)” can be shown as in the proof of Theorem 1.4, except that we consider \( \phi \) as a map from \( g^\ell \) to \( PC([0,T], g) \), endowed with the \( L^1 \)-topology. □

10 Analogues beyond Fréchet-manifolds

In this section, we explain how our results can be extended from the case of Fréchet manifolds (and Fréchet-Lie groups) to the case of manifolds (and Lie
groups) modelled on sequentially complete locally convex spaces. The main point is that an analogue of Lemma 3.6 can be established also in the current higher generality (see Lemma 10.15), as well as an analogue of Theorem 1.2 (see Theorem 10.19). Once this foundation is established in the necessary detail, it will be enough to revisit the other results, and describe which minor modifications are necessary in the statements and proofs.

10.1 We recall that a mapping \( \gamma : I \to X \) from an interval \( I \subseteq \mathbb{R} \) to a topological space \( X \) is called Lusin measurable if there exists a sequence \( (K_j)_{j \in \mathbb{N}} \) of compact subsets \( K_j \subseteq I \) such that

(i) The restriction \( \gamma|_{K_j} : K_j \to X \) is continuous for each \( j \in \mathbb{N} \);

(ii) \( \lambda_1(I \setminus \bigcup_{j \in \mathbb{N}} K_j) = 0 \).

See [11] and the references therein for further information, also [35].

**Remark 10.2** (a) Let \( \tilde{\lambda} : \mathcal{B}(\mathbb{R}) \to [0, \infty] \) be Lebesgue measure. If \( X \) is second countable, then a map \( \gamma : I \to X \) is Lusin measurable if and only if \( \gamma \) is measurable as a function from \( (I, \mathcal{B}(I)) \) to \( (X, \mathcal{B}(X)) \), where \( \mathcal{B}(I) := \mathcal{B}(\mathbb{R})|_I \) (see [35, Lemma 4.1.8]).

(b) If a Lusin measurable map \( \gamma : I \to X \) and \( (K_j)_{j \in \mathbb{N}} \) are as in 10.1, after replacing \( \gamma(x) \) with a constant \( c \in X \) for all \( x \in I \setminus \bigcup_{j \in \mathbb{N}} K_j =: K_0 \), we can achieve that \( \gamma \) is measurable from \( (I, \mathcal{B}(I)) \) to \( (X, \mathcal{B}(X)) \). If \( X \) is Hausdorff, moreover \( \gamma(K_j) \) is compact and metrizable and hence separable and metrizable for each \( j \in \mathbb{N} \) (and so is \( \gamma(K_0) \), which is \( \{c\} \) or \( \emptyset \)). Notably, \( \gamma(I) = \bigcup_{j \in \mathbb{N}} \gamma(K_j) \) is separable.

10.3 Henceforth, if \( E \) is a locally convex space, \( I \subseteq \mathbb{R} \) an interval and \( p \in [1, \infty] \), we write \( \mathcal{L}^p(I, E) \) for the vector space of all Lusin measurable mappings \( \gamma : I \to E \) such that \( \|\gamma\|_{\mathcal{L}^p} := \|q \circ \gamma\|_{\mathcal{L}^p} < \infty \) for all continuous seminorms \( q \) on \( E \). We give \( \mathcal{L}^p(I, E) \) the locally convex vector topology defined by the seminorms \( \| \cdot \|_{\mathcal{L}^p} \). In the usual way, one now obtains Hausdorff locally convex spaces \( \mathcal{L}^p(I, E) \) of equivalence classes \([\gamma]\) modulo Lusin measurable functions vanishing almost everywhere (see [35, p. 43]). If \( \gamma : I \to E \) is locally integrable in the sense that \( \gamma|_{[a,b]} \in \mathcal{L}^1([a,b], E) \) for all \( a < b \) with \( [a,b] \subseteq I \), again we write \([\gamma]\) for the equivalence class modulo Lusin measurable functions which vanish almost everywhere.
10.4 If $E$ is an sclc-space, given real numbers $a < b$, following [35] we call a function $\eta: [a, b] \to E$ absolutely continuous if it is a primitive of some $\gamma \in L^1([a, b], E)$, as in (23). Then $\eta' := [\gamma]$ is uniquely determined (see [35, Lemma 4.2.6]). Let $p \in [1, \infty]$. If $\eta' = [\gamma]$ with $\gamma \in L^p([a, b], E)$, we call $\eta$ an $AC_{L^p}$-map. If $I \subseteq \mathbb{R}$ is an interval, we say that a function $\eta: I \to E$ is absolutely continuous (resp., $AC_{L^p}$) if $\eta|_{[a, b]}$ is so for all real numbers $a < b$ such that $[a, b] \subseteq I$. Again, there is a locally integrable function $\gamma: I \to E$ with primitive $\eta$ and $\eta' := [\gamma]$ is uniquely determined.

10.5 (Chain Rule). Let $E$ and $F$ be sclc-spaces, $U \subseteq E$ be an open subset, $f: U \to F$ a $C^1$-map and $\eta: I \to E$ be an absolutely continuous function on a non-degenerate interval $I \subseteq \mathbb{R}$ such that $\eta(I) \subseteq U$. Let $\gamma: I \to E$ be a locally integrable function with $\eta' = [\gamma]$. Then $f \circ \eta: I \to F$ is absolutely continuous and

$$(f \circ \eta)' = [t \mapsto df(\eta(t), \gamma(t))],$$

by [35, Lemma 4.2.16] and its proof.

10.6 If $E$ is an sclc-space, $W \subseteq \mathbb{R} \times E$ a subset, $f: W \to E$ a function and $(t_0, y_0) \in W$, we call a function $\gamma: I \to E$ on an interval $I \subseteq \mathbb{R}$ a Carathéodory solution to the initial value problem (6) if $\gamma$ is absolutely continuous, $t_0 \in I$ holds, $(t, \gamma(t)) \in W$ for all $t \in I$, and the integral equation (7) is satisfied, or equivalently

$$\gamma' = [t \mapsto f(t, \gamma(t))] \quad \text{and} \quad \gamma(t_0) = y_0. \quad (27)$$

10.7 If $q: E \to F$ is a continuous linear map to a Fréchet space in the situation of 10.6 then $q \circ \gamma$ is absolutely continuous and

$$(q \circ \gamma)' = [t \mapsto q(f(t, \gamma(t)))],$$

by 10.5. As $q \circ \gamma$ is an absolutely continuous map to a Fréchet space, we deduce that there is a Borel set $I_0 \subseteq I$ with $\lambda_1(I \setminus I_0) = 0$ such that $q \circ \gamma$ is differentiable at each $t \in I_0$ and

$$(q \circ \gamma)'(t) = q(f(t, \gamma(t))) \quad \text{for all } t \in I_0.$$

10.8 Let $M$ be a $C^1$-manifold modelled on a locally convex space and $TM$ be its tangent bundle, with the bundle projection $\pi_{TM}: TM \to M$. If $\gamma: I \to TM$ is a Lusin measurable function on an interval $I \subseteq \mathbb{R}$, we write $[\gamma]$ for the set of all Lusin measurable functions $\eta: I \to TM$ such that $\pi_{TM} \circ \gamma = \pi_{TM} \circ \eta$ and $\gamma(t) = \eta(t)$ for almost all $t \in I$. 

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10.9 Let $M$ be a $C^1$-manifold modelled on an sclc-space $E$. For real numbers $a < b$, consider a continuous function $\eta: [a, b] \to M$. If $\eta([a, b]) \subseteq U_\phi$ for some chart $\phi: U_\phi \to V_\phi \subseteq E$, we say that $\eta$ is absolutely continuous if $\phi \circ \eta: I \to E$ is so, and let

$$\dot{\eta} := [t \mapsto T\phi^{-1}((\phi \circ \eta)(t), \gamma(t))]$$

with $\gamma \in L^1([a, b], E)$ such that $(\phi \circ \eta)' = [\gamma]$. By 10.5, absolute continuity of $\eta$ is independent of the choice of $\phi$, and so is $\dot{\eta}$. In the general case, we call $\eta$ absolutely continuous if $[a, b]$ can be subdivided into subintervals $[t_{j-1}, t_j]$ such that $\eta([t_{j-1}, t_j])$ is contained in a chart domain and $\eta|_{[t_{j-1}, t_j]}$ is absolutely continuous. If $(\eta|_{[t_{j-1}, t_j]})' = [\gamma_j]$, we let $\dot{\eta} := [\gamma]$ with $\gamma(t) := \gamma_j(t)$ if $t \in [t_{j-1}, t_j]$ or $j$ is maximal and $t \in [t_{j-1}, t_j]$. If $I \subseteq \mathbb{R}$ is an interval, we call a function $\eta: I \to M$ absolutely continuous if $\eta|_{[a, b]}$ is so for all $a < b$ such that $[a, b] \subseteq I$. We define $\dot{\eta} = [\gamma]$ where $\gamma$ is defined piecewise using representatives of $(\eta|_{[a, b]})'$ for $[a, b]$ in a countable cover of $I$.

10.10 Let $f: M \to N$ be a $C^1$-map between $C^1$-manifolds modelled on sclc-spaces. Let $I \subseteq \mathbb{R}$ be a non-degenerate interval and $\eta: I \to M$ be absolutely continuous. Let $\gamma: I \to TM$ be a Lusin measurable function such that $\pi_{TM} \circ \gamma = \eta$ and $\dot{\eta} = [\gamma]$. Then $f \circ \eta: I \to N$ is absolutely continuous and

$$(f \circ \eta)' = [t \mapsto Tf(\gamma(t))],$$

as a consequence of 10.5.

10.11 If $M$ is a $C^1$-manifold modelled on an sclc-space, $W \subseteq \mathbb{R} \times M$ a subset, $f: W \to TM$ a function such that $f(t, y) \in T_y M$ for all $(t, y) \in W$ and $(t_0, y_0) \in W$, we call a function $\gamma: I \to M$ on a non-degenerate interval $I \subseteq \mathbb{R}$ a Carathéodory solution to the initial value problem

$$\dot{y}(t) = f(t, y(t)), \quad y(t_0) = y_0$$

(28)

if $\gamma$ is absolutely continuous, $t_0 \in I$ holds, $(t, \gamma(t)) \in W$ for all $t \in I$, (29)

$$\dot{\gamma} = [t \mapsto f(t, \gamma(t))], \quad \text{and} \quad \gamma(t_0) = y_0.$$

Solutions to the differential equation $\dot{y}(t) = f(t, y(t))$ are defined analogously.

10.12 Using the concepts of absolutely continuous functions, $AC_{L^p}$-functions, and Carathéodory solutions just described, one can define $L^p$-regular Lie groups modelled on sclc-spaces for $p \in [1, \infty]$ (see 35, Definition 4.3.7), as in the case of Fréchet-Lie groups already discussed in this article.
Remark 10.13 Although $\eta'(t) = [\gamma(t)]$ is still uniquely determined for an absolutely continuous function $\eta: [a, b] \to E$ to an sclc-space, in contrast to the Fréchet case $\eta'(t)$ may not exist for almost all $t$, so that we cannot work with genuine derivatives at a point anymore. Generalizing the results and proofs obtained so far in the Fréchet case, this problem can be frequently be avoided by replacing statements like

$$\eta'(t) = \gamma(t)$$

with $\eta' = [t \mapsto \gamma(t)]$. Notably, Carathéodory solutions to initial value problems are now defined via (27) and (29).

For example, using this interpretation, Definitions 3.1 and 3.3 can be extended to the case that $E$ is an sclc-space, and Lemma 3.2 and its proof remain valid in this generality.

There is one exception, however: Derivatives at a point played an essential role in the proof of Lemma 3.6. We therefore discuss the necessary adaptations in detail now, starting with a suitable modification of Definition 3.4.

As we shall see, the property 10.7 will be good enough to achieve our goals.

Definition 10.14 Let $J \subseteq \mathbb{R}$ be a non-degenerate interval, $E$ be an sclc-space, $U \subseteq E$ be a subset and $f: W \to E$ be a function on an open subset $W \subseteq J \times U$. Let $k \in \mathbb{N} \cup \{\infty\}$. We say that the differential equation

$$y'(t) = f(t, y(t))$$

admits local flows which are pullbacks of $C^k$-maps if, for all $(\bar{t}, \bar{y}) \in W$, there exist a compact interval $I \subseteq J$ which is a neighbourhood of $\bar{t}$ in $J$, an open neighborhood $V$ of $\bar{y}$ in $U$ with $I \times V \subseteq W$ and function

$$\Phi: I \times I \times V \to E$$

such that conditions (a) and (b) from Definition 3.4 are satisfied, and (c)’:

(c)’ There exist sclc-spaces $E_1$ and $E_2$, open subsets $V_1 \subseteq E_1$ and $V_2 \subseteq E_2$, absolutely continuous functions $\alpha: I \to V_1 \subseteq E_1$ and $\beta: I \to V_2 \subseteq E_2$, and a $C^k$-map $\Psi: V_1 \times V_2 \times V \to E$ such that

$$\Phi(t, t_0, y_0) = \Psi(\alpha(t), \beta(t_0), y_0) \quad \text{for all } (t, t_0, y_0) \in I \times I \times V.$$
of a Borel set $I_0 \subseteq I$ with $\lambda_1(I \setminus I_0) = 0$ and an open subset $\Omega_q \subseteq V$ with $C \subseteq \Omega_q$ such that $\frac{d}{dt} q(\Phi_{t,t_0}(y_0))$ exists and
\[
\frac{d}{dt} q(\Phi_{t,t_0}(y_0)) = q(f(t, \Phi_{t,t_0}(y_0)))
\]
for all $y_0 \in \Omega_q$, $t_0 \in I$, and $t \in I_q$.

Note that Remark 3.5 remains valid in sclc-spaces if we replace Definition 3.4 with Definition 10.14.

**Lemma 10.15** Let $J \subseteq \mathbb{R}$ be a non-degenerate interval, $E$ be an sclc-space, $U \subseteq E$ be a subset and $f : W \to E$ be a function on an open subset $W \subseteq J \times U$. If the differential equation $y'(t) = f(t, y(t))$ admits local flows which are pullbacks of $C^2$-maps, then it satisfies local uniqueness of Carathéodory solutions.

**Proof.** Let $\gamma_j : I_j \to E$ be solutions to $y'(t) = f(t, y(t))$ for $j \in \{1, 2\}$ and $t \in I_1 \cap I_2$ such that $\bar{y} := \gamma_1(\bar{t}) = \gamma_2(\bar{t})$. To see that $\gamma_1$ and $\gamma_2$ coincide on a neighbourhood of $\bar{t}$ in $I_1 \cap I_2$, we may assume that $I_1 \cap I_2$ is a non-degenerate interval. Let $I$, $V$, $\Phi$, $Y$, $E_1$, $E_2$, $V_1$, $V_2$, $\Psi$, $\alpha$, and $\beta$ be as in Definition 10.14.

There exists a compact interval $K \subseteq I_1 \cap I_2 \cap I$ which is a neighbourhood of $\bar{t}$ in $I_1 \cap I_2$. Let $\Phi_{\bar{t},t} | Y : Y \to Y$ be as in Definition 10.14.

Note that $\theta_j$ is absolutely continuous by 10.3 as $\theta_j(t) = \Psi(\alpha(\bar{t}), \beta(t), \gamma_j(t))$. It suffices to show that $\gamma_j(t) = \Phi_{t,\bar{t}}(\bar{y})$ for all $t \in K$ and $j \in \{1, 2\}$.

It suffices to show that $q \circ \theta = q \circ \theta_j$.
for \( j \in \{1, 2\} \), for each continuous linear map \( q: E \to \mathbb{R} \). Since \( \theta_j(t) = \bar{y} = \theta(t) \) for \( j \in \{1, 2\} \), the latter will hold if we can show that
\[
(q \circ \theta_j)'(t) = (q \circ \theta)'(t) = 0
\]
for almost all \( t \in K \). By Lemma [A.1] there exist Banach spaces \( F_j \) for \( j \in \{1, 2\} \) and \( F \), open subsets \( Q_j \subseteq F_j \) and \( Q \subseteq F \), continuous linear maps \( p_j: E_j \to F_j \) and \( p: E \to F \), and open subsets \( V_j, 0 \subseteq V_j \) and \( V_0 \subseteq V \), such that
\[
\alpha(K) \subseteq V_{1,0}, \quad \beta(K) \subseteq V_{2,0}, \quad \gamma_1(K) \cup \gamma_2(K) \subseteq V_0,
\]
\[
p_1(V_{1,0}) \subseteq Q_1, \quad p_2(V_{2,0}) \subseteq Q_2, \quad p(V_0) \subseteq Q
\]
and a \( C^1 \)-map \( \Psi_q: Q_1 \times Q_2 \times Q \to \mathbb{R} \) such that
\[
q(\Psi(x, y, z)) = \Psi_q(p_1(x), p_2(y), p(z)) \quad \text{for all } (x, y, z) \in V_{1,0} \times V_{2,0} \times V_0.
\]
For \( j \in \{1, 2\} \), there exists a Borel set \( I_{j,0} \subseteq I_j \) with \( \lambda_1(I_j \setminus I_{j,0}) = 0 \) such that \( p \circ \gamma_j \) is differentiable at each \( t \in I_{j,0} \) and
\[
(p \circ \gamma_j)'(t) = p(f(t, \gamma_j(t))) \quad \text{for all } t \in I_{j,0},
\]
see [10.7]. Let \( I_p \subseteq I \) and \( \Omega_p \) be analogous to \( I_q \) and \( \Omega_q \) in Definition [10.14] (c)', applied with \( p \) in place of \( q \) and \( C := \theta_1(K) \cup \theta_2(K) \). After shrinking \( I_p \), we may assume that, moreover, \( (p_2 \circ \beta)'(t) \) exists at each \( t \in I_p \). For each \( t \in K \), we have
\[
q(\theta_j(t)) = q(\Phi(t, \gamma_j(t))) = q(\Psi(\alpha(t), \beta(t), \gamma_j(t))) = \Psi_q((p_1 \circ \alpha)(t), (p_2 \circ \beta)(t), (p \circ \gamma_j)(t)).
\]
Differentiating at \( t \in K_0 := K \cap I_{1,0} \cap I_{2,0} \cap I_p \), we obtain
\[
\frac{d}{dt} q(\theta_j(t)) = \frac{d}{dt} \Psi_q((p_1 \circ \alpha)(t), (p_2 \circ \beta)(t), (p \circ \gamma_j)(t))
\]
\[
= d_2 \Psi_q((p_1 \circ \alpha)(t), (p_2 \circ \beta)(t), (p \circ \gamma_j)(t); (p \circ \gamma_j)'(t))
\]
\[
+ d_3 \Psi_q((p_1 \circ \alpha)(t), (p_2 \circ \beta)(t), (p \circ \gamma_j)(t); (p \circ \gamma_j)'(t)) = p(f(t, \gamma_j(t)))
\]
Using the Chain Rule, the second summand can be rewritten as
\[
d_3(q \circ \Psi)(\alpha(t), \beta(t), \gamma_j(t); f(t, \gamma_j(t))) = q(d_3 \Phi(t, \gamma_j(t), f(t, \gamma_j(t)))).
\]
Assume that \( z \in Y \cap \Omega \) and \( t \in K_0 \) are given such that \( \Phi(t, \bar{t}, z) \in V_0 \). Then \( \Phi(\tau, \bar{t}, z) \in V_0 \) for \( \tau \in K \) close to \( t \), and
\[
q(z) = q(\Phi(\bar{t}, \tau, \Phi_{\tau, \bar{t}}(z))) = \Psi_q((p_1 \circ \alpha)(\bar{t}), (p_2 \circ \beta)(\tau), p(\Phi_{\tau, \bar{t}}(z)))
\]
is independent of \( \tau \). Differentiating (30) with respect to \( \tau \) at \( \tau = t \), we obtain
\[
0 = d^2 \Psi_q((p_1 \circ \alpha)(\bar{t}), (p_2 \circ \beta)(t), p(\Phi_{t, \bar{t}}(z)); (p_2 \circ \beta)'(t)) + q(d\Phi_{t, \bar{t}}(\Phi_{t, \bar{t}}(z), f(t, \Phi_{t, \bar{t}}(z))))
\] repeating the arguments used to calculate \( \frac{d}{dt} q(\theta_j(t)) \); note that
\[
\frac{d}{dt} p(\Phi_{t, \bar{t}}(z)) = p(f(t, \Phi_{t, \bar{t}}(z)))
\] since \( t \in I_p \) and \( z \in \Omega_p \). For \( j \in \{1, 2\} \) and any \( t \in K_0 \), we may choose \( z := \theta_j(t) \) here. In fact, then \( z \in \Omega_p \) by choice of \( C \); also, \( z = \theta_j(t) \in Y \) and
\[
\Phi_{t, \bar{t}}(z) = \Phi_{t, \bar{t}}(\Phi_{t, \bar{t}}(\gamma_j(t))) = \gamma_j(t) \in V_0.
\] Substituting \( \Phi_{t, \bar{t}}(z) = \gamma_j(t) \) into (31), we obtain
\[
0 = d^2 \Psi_q((p_1 \circ \alpha)(\bar{t}), (p_2 \circ \beta)(t), (p \circ \gamma_j)(t); (p_2 \circ \beta)'(t)) + q(d\Phi_{t, \bar{t}}(\gamma_j(t), f(t, \gamma_j(t)))) = \frac{d}{dt} q(\theta_j(t)),
\] by the above calculation. \( \square \)

**Remark 10.16** Remark 4.2 concerning associated differential equations in local charts, Definition 4.3 of local uniqueness, and Lemma 4.4 (and its proof) concerning global uniqueness remain meaningful for manifolds with sequentially complete modelling spaces, if Definition 4.1 is replaced with 10.11. In Definition 4.5 of local existence, Lemma 4.6 concerning maximal solutions, and Definition 4.7 of maximal flows, we can replace Fréchet spaces with sclc-spaces, without further changes.

**Definition 10.17** Let \( M \) be a \( C^k \)-manifold modelled on an sclc-space, with \( k \in \mathbb{N} \cup \{\infty\} \). Let \( J \subseteq \mathbb{R} \) be a non-degenerate interval, \( W \subseteq J \times M \) be an open subset, and \( f : W \rightarrow TM \) be a function such that \( f(t, y) \in T_y M \) for all \( (t, y) \in W \). We say that the differential equation \( \dot{y}(t) = f(t, y(t)) \) admits local flows which are pullbacks of \( C^k \)-maps if \( \dot{y}(t) = f_\phi(t, y(t)) \) does so, for each chart \( \phi : U_\phi \rightarrow V_\phi \).
Remark 10.18 (a) Using Definition 10.17 we immediately deduce from Lemma 10.15 that Proposition 1.1 remains valid for differential equations admitting local flows which are pullbacks of $C^2$-maps, on a $C^2$-manifold $M$ modelled on an sclc-space (instead of a Fréchet space).

(b) In the case of a Fréchet manifold, the condition in Definition 10.17 (see proof of Proposition 1.1). We avoided a direct analogue of Definition 4.8 in the case of sequentially complete modelling spaces. The proof of Theorem 1.2 now becomes more technical.

Theorem 10.19 Theorem 1.2 remains valid if $G$ and $M$ are modelled on sequentially complete locally convex spaces (which need not be Fréchet spaces).

Proof. Given $g \in G$, write $\lambda_g : G \to G$, $h \mapsto gh$ for left translation by $g$. Write $\sigma_y : G \to M$, $g \mapsto \sigma(y, g)$ for $y \in M$; thus
\[ T\sigma_y(v) = v(y) \quad \text{for all } v \in g. \] (32)

Since $\sigma(y, gh) = \sigma(y, g), h)$ for all $y \in M$ and $g, h \in G$, we have $\sigma_y \circ \lambda_g = \sigma_{\sigma(y, g)}$ and thus
\[ T\sigma_y \circ T\lambda_g = T\sigma_{\sigma(y, g)} \quad \text{for all } y \in M \text{ and } g \in G. \] (33)

If we define Fl as in (10) and abbreviate $\text{Fl}_{t,t_0}(y_0) := \text{Fl}(t, t_0, y_0)$ for $t,t_0 \in [a, b]$ and $y_0 \in M$, then
\[ \text{Fl}_{t_2,t_1}(\text{Fl}_{t_1,t_0}(y_0)) = \text{Fl}_{t_2,t_0}(y_0) \quad \text{for all } t_0, t_1, t_2 \in [a, b] \] (34)
holds as the left-hand side is
\[ \sigma(\sigma(y_0, \eta(t_0)^{-1}\eta(t_1)), \eta(t_1)^{-1}\eta(t_2)) = \sigma(y_0, \eta(t_0)^{-1}\eta(t_1)\eta(t_2)) \]
and thus equals the right-hand side. Recall that
\[ \dot{\eta} = [t \mapsto \eta(t), \gamma(t)] = [t \mapsto T\lambda_{\eta(t)}\gamma(t)]. \] (35)

Given $t_0 \in [a, b]$ and $y_0 \in M$, let $\zeta(t) := \text{Fl}_{t,t_0}(y_0)$ for $t \in [a, b]$. Since $\zeta = \sigma_{y_0} \circ \lambda_{\eta(t_0)^{-1} \circ \eta}$, the map $\zeta$ is absolutely continuous and we have
\[ \dot{\zeta} = [t \mapsto T\sigma_{y_0} T\lambda_{\eta(t_0)^{-1}} T\lambda_{\eta(t)} \gamma(t)] \\
= [t \mapsto T\sigma_{y_0} T\lambda_{\eta(t)}} \gamma(t)]
= [t \mapsto T\sigma_{\sigma(y_0, \eta(t))^{-1}} \gamma(t)] = [t \mapsto T\sigma_{\zeta(t)} \gamma(t)]
= [t \mapsto \gamma(t)]_{\dot{\zeta}(\zeta(t))}, \]
using \ref{10.10} and \eqref{35} for the first equality, then \eqref{33}, and eventually \eqref{32}.

Thus \( \zeta \) is a Carathéodory solution to the initial value problem

\[ \dot{y}(t) = \gamma(t)\zeta(y(t)), \quad y(t_0) = y_0. \]  \tag{36} \]

We define \( f: [a, b] \times M \to TM, (t, y) \mapsto \gamma(t)\zeta(y) \).

If \( \phi: U_\phi \to V_\phi \subseteq E \) is a chart of \( M \), consider the corresponding function

\[ f_\phi: [a, b] \times V_\phi \to E, \quad (t, z) \mapsto d\phi(f(t, \phi^{-1}(z))), \]

as in Remark \ref{4.2}. Given \( \tilde{t} \in [a, b] \) and \( \tilde{z} \in V_\phi \), let \( \tilde{y} := \phi^{-1}(\tilde{z}) \).

Let \( X \) be the modelling space of \( G \) and \( \psi: U_\psi \to V_\psi \subseteq X \) be a chart of \( G \) such that \( \eta(\tilde{t}) \in U_\psi \). There exists an open \( \tilde{y} \)-neighbourhood \( A \subseteq U_\phi \) and an open \( \eta(\tilde{t}) \)-neighbourhood \( B \subseteq U_\psi \) such that

\[ \sigma(A \times B^{-1}B) \subseteq U_\phi. \]

There exists an open \( \tilde{y} \)-neighbourhood \( K \subseteq A \) and an open \( \eta(\tilde{t}) \)-neighbourhood \( L \subseteq B \) such that

\[ \sigma(K \times L^{-1}L) \subseteq A. \]

Then \( V := \phi(A) \) and \( Y := \phi(K) \) are open neighbourhoods of \( \tilde{z} \) in \( V_\phi \) such that \( Y \subseteq V \). There exists a compact interval \( I \subseteq [a, b] \) which is a neighbourhood of \( \tilde{t} \) in \( [a, b] \), such that \( \eta(I) \subseteq L \). We now show that

\[ \Phi: I \times I \times V \to V_\phi, \quad (t, t_0, z_0) \mapsto \Phi_{t, t_0}(z_0) := \phi(Fl_{t, t_0}(\phi^{-1}(z_0))) \]

satisfies the conditions (a), (b), and (c)' of Definition \ref{10.14} for \( k = \infty \), with \( f_\phi \) in place of \( f \) and \( (\tilde{t}, \tilde{y}) \) in place of \( (t, \tilde{y}) \). If this is true, then each \( f_\phi \) (and hence \( f \)) admits local flows which are pullbacks of \( C^\infty \)-maps, whence \( \dot{y}(t) = f(t, y(t)) \) satisfies local uniqueness (see Lemma \ref{10.15}) and existence of Carathéodory solutions, with maximal flow the map \( Fl \) already introduced.

For \( t_0 \in I \) and \( z_0 \in V \), the map \( I \to U_\phi, t \mapsto Fl_{t, t_0}(\phi^{-1}(z_0)) \) is a Carathéodory solution to \eqref{30} with \( y_0 := \phi^{-1}(z_0) \); using \ref{10.10}, we deduce that \( I \to V_\phi, t \mapsto \Phi_{t, t_0}(z_0) \) is a Carathéodory solution to \( \dot{z}(t) = f_\phi(t, z(t)), z(t_0) = z_0 \). Thus condition (a) in Definition \ref{10.14} is satisfied, and (b) follows from \eqref{34}.

To verify condition (c)', we shall use the smooth mappings

\[ \Theta: L \times L \times A \to U_\phi, \quad (g, h, y) \mapsto \sigma(y, h^{-1}g) \]
Then \( \Phi(t, t_0, z) = \Psi(\alpha(t), \beta(t_0), z) \) for all \((t, t_0, z) \in I \times I \times V\) using the absolutely continuous functions \(\alpha, \beta : I \to \psi(L) \subseteq X\),

\[
\alpha := \psi \circ \eta|_I \quad \text{and} \quad \beta := \psi \circ \eta|_I.
\]

Let \(q : E \to F\) be a continuous linear map to a Banach space \(F\) and \(C \subseteq V\) be a compact subset. By Lemma A.1, there exist continuous linear maps \(p_1 : X \to Z_1, p_2 : X \to Z_2\) and \(p : E \to Z\) to Banach spaces \(Z_1, Z_2,\) and \(Z\), open subsets \(Q_1 \subseteq Z_1, Q_2 \subseteq Z_2,\) and \(Q \subseteq Z\), open subsets \(U_1 \subseteq \psi(L), U_2 \subseteq \psi(L),\) and \(V_q \subseteq V\), and a \(C^1\)-function

\[
\Psi_q : Q_1 \times Q_2 \times Q \to F
\]

such that

\[
\alpha(I) \subseteq U_1, \quad \beta(I) \subseteq U_2, \quad C \subseteq V_q,
\]

\[
p_1(U_1) \subseteq Q_1, \quad p_2(U_2) \subseteq Q_2, \quad p(V_q) \subseteq Q,
\]

and

\[
q(\Psi(x, y, z)) = \Psi_q(p_1(x), p_2(y), p(z)) \quad \text{for all } (x, y, z) \in U_1 \times U_2 \times V_q.
\]

Note that \(p_1 \circ \alpha = p_1 \circ \psi \circ \eta|_I\) is an absolutely continuous map to \(Z_1\) and

\[
(p_1 \circ \alpha)'(t) = p_1(d\psi T\lambda_{\eta(t)}(t)),
\]

using [10.10]. Since \(Z_1\) is a Banach space, we therefore find a Borel set \(I_q \subseteq I\) with \(\lambda_1(I \setminus I_q) = 0\) such that \((p_1 \circ \alpha)'(t)\) exists for all \(t \in I_q\), and is given by

\[
(p_1 \circ \alpha)'(t) = p_1(d\psi T\lambda_{\eta(t)}(t)).
\]

Let \((t_0, z_0) \in I \times V_q\) and \(y_0 := \phi^{-1}(z_0)\). Then

\[
\kappa : I \to F, \quad t \mapsto q(\Phi_{t,t_0}(z_0)) = \Psi_q(p_1(\alpha(t)), p_2(\beta(t_0)), p(z_0))
\]

is absolutely continuous and differentiable at each \(t \in I_q\) (see [2.3]), with

\[
\kappa'(t) = d_1\Psi_q(p_1(\alpha(t)), p_2(\beta(t_0)), p(z_0); p_1(d\psi T\lambda_{\eta(t)}(t))
\]

\[
= d_1(q \circ \Psi)(\alpha(t), \beta(t_0), z_0; d\psi T\lambda_{\eta(t)}(t))
\]

\[
= d(q \circ \phi \circ \Theta(\cdot, \eta(t_0), y(t_0))T\lambda_{\eta(t)}(t)
\]

\[
= q \circ \phi T\sigma_{\eta(y_0, \eta(t_0))^{-1}}T\lambda_{\eta(t)}(t)
\]

\[
= q \circ \phi \gamma(t)_z(\phi^{-1}(\Phi_{t,t_0}(z_0))) = q(\phi(t, \Phi_{t,t_0}(z_0))).
\]
at \( t \in I_q \), using that \( \Theta(\cdot, \eta(t_0), y_0) = \sigma(y_0, \eta(t_0)^{-1} \cdot) = \sigma_{\sigma(y_0, \eta(t_0)^{-1})} \) and

\[
T \sigma_{\sigma(y_0, \eta(t_0)^{-1})} T \lambda_\eta(t) \gamma(t) = T \sigma_{\sigma(y_0, \eta(t_0)^{-1}, \eta(t))} \gamma(t) = \gamma(t)_\sharp(\Phi_{t,t_0}(y_0)) = \gamma(t)_\sharp(\phi^{-1}(\Phi_{t,t_0}(z_0))).
\]

Thus condition (c)' is verified. \( \Box \)

**Remark 10.20** Lemma 6.1 remains valid if, more generally, \( E \) is a locally convex space and \( \gamma : [a, b] \to E \) a Lusin measurable \( L^1 \)-function.

[To see this, let \( (K_n)_{n \in \mathbb{N}} \) be a sequence of compact subsets \( K_n \subseteq [a, b] \) such that \( \gamma|_{K_n} \) is continuous and \( A := \bigcup_{n \in \mathbb{N}} K_n \) satisfies \( \lambda_1([a, b] \setminus A) = 0 \). Let \( c \in \gamma([a, b]) \). If we define \( \bar{\gamma}(t) := \gamma(t) \) if \( t \in A \), \( \bar{\gamma}(t) := c \) if \( t \in [a, b] \setminus A \), then \( \bar{\gamma} : [a, b] \to E \) is Lusin measurable and Borel measurable, and \( \gamma(t) = \bar{\gamma}(t) \) almost everywhere. In the proof, we replace \( \gamma \) with \( \bar{\gamma} \); then \( \bar{\gamma} = \gamma_1, \gamma_2 - \eta, \) and \( \eta - \theta_k \) are Borel measurable by Lemma 2.7. Eventually, \( \|\gamma - \eta\|_{L^1,q} \leq \|\gamma - \bar{\gamma}\|_{L^1,q} + \|\gamma - \eta\|_{L^1,q} = \|\gamma - \eta\|_{L^1,q} \leq \varepsilon \) as \( \|\gamma - \bar{\gamma}\|_{L^1,q} = 0 \). Likewise, \( \|\gamma - \theta\|_{L^1,\varepsilon} \leq \varepsilon \) implies \( \|\gamma - \eta\|_{L^1,q} \leq \varepsilon \).

Having generalized Lemma 6.1, the proof of Theorem 1.4 remains valid for \( G \) and \( M \) modelled on sclc-spaces.

**Remark 10.21** Theorem 1.9 remains valid if \( G \) is an \( L^p \)-regular Lie group modelled on an sclc-space; simply replace \( \mathcal{E} \) with \( L^p \) in the proof of Theorem 7.6, the subdivision property for \( L^p \)-spaces based on Lusin measurability is provided in 35, Lemma 4.3.12. Having generalized Theorem 1.9, the proof of Theorem 1.10 is unchanged for \( G \) and \( M \) modelled on sclc-spaces.

**Remark 10.22** (a) In [18, 1.40], \( L^p \)-spaces associated with Borel measurable \( E \)-valued functions have also been considered if an sclc-space \( E \) has the Fréchet exhaustion property (recalled in 7.3), as well as \( L^p \)-regularity for Lie groups modelled on such spaces. As observed in 35, Remark 4.1.16, these \( L^p \)-spaces are isomorphic to those based on Lusin measurable maps (and \( L^p \)-regularity as in [18] is then equivalent to that in [35]). We therefore need not develop further variants of our current results for (FEP)-spaces, they are covered by the Lusin theory.

(b) If \( E \) is a locally convex space, we recalled in 7.4 the vector spaces \( \mathcal{L}_{rc}^{\infty}([a, b], E) \) and \( \mathcal{L}_{rc}^{\infty}([a, b], E) \) introduced in [14]. Any such \( \gamma \) is the uniform limit of a sequence of measurable functions with finite image (see 14).
Proposition 3.18), and it is Lusin measurable; we can interpret $L^\infty_{rc}([a,b],E)$ with a vector subspace of $L^\infty([a,b],E)$ in the Lusin sense. Now the point is that sequential completeness of $E$ is unnecessary to ensure that primitives $\eta: [a,b] \to E,$
\[ \eta(t) := c + \int_a^t \gamma(s) \, ds \]
(which we call $AC_{L^\infty_{rc}}$-functions) exist for $\gamma \in L^\infty_{rc}([a,b],E);$ they exist whenever $E$ is integral complete in the sense recalled in 7.2.

An $L^\infty_{rc}$-function $\gamma: [a,b] \to E$ is called regulated if $\gamma$ is the uniform limit of a sequence of $E$-valued staircase functions on $[a,b]$; following [18, 1.31], we let $R([a,b],E)$ be the vector space of all such functions and $R([a,b],E) \subseteq L^\infty_{rc}([a,b],E)$ be the corresponding vector subspace. Note that $R([a,b],E)$ contains all piecewise continuous functions, continuous functions, and staircase functions. See [18] for corresponding concepts of $L^\infty_{rc}$-regular Lie groups and $R$-regular Lie groups modelled on integral complete locally convex spaces.

All of our results for $L^p$ with $p = \infty$ and corresponding absolutely continuous functions remain valid for integral complete locally convex spaces and modelling spaces in place of sclc-spaces, by straightforward adaptations. Notably, $E = L^\infty_{rc}$ and $E = R$ are possible choices for $E$ in Theorem 7.6 (as it stands).

In Theorems 1.2 and 10.19 which are the basis for our work, the maximal flow is globally defined, on all of $[a,b] \times [a,b] \times M$. We mention two basic general facts concerning maximal flows, which may be of interest elsewhere.

10.23 Let $M$ be a $C^1$-manifold modelled on an sclc-space, $J \subseteq \mathbb{R}$ be a non-degenerate interval and $f: W \to TM$ be a function on a subset $W \subseteq J \times M$ such that $f(t,y) \in T_yM$ for all $(t,y) \in W$. Assume that the differential equation (16) satisfies both local existence of Carathéodory solutions and local uniqueness. Let $\Omega \subseteq J \times J \times M,$ the maximal flow $Fl: \Omega \to M,$ and $\gamma_{t_0,y_0}: I_{t_0,y_0} \to M$ for $(t_0,y_0) \in W$ be as in Definition 4.7 (cf. also Remark 10.16). For $t,t_0 \in J,$ we define
\[ \Omega_{t,t_0} := \{ y_0 \in M : (t,t_0,y_0) \in \Omega \} \]
and abbreviate $Fl_{t,t_0}(y_0) := Fl(t,t_0,y_0)$ for $y_0 \in \Omega_{t,t_0}$.

The following fact can be proved like Lemmas 2.4.9 and 2.5.7 in [20].

Lemma 10.24 In the situation of 10.23, we have:
(a) For \((t_0, y_0) \in W\) and \(t_1 \in I_{t_0,y_0}\), we have \(\gamma_{t_0,y_0} = \gamma_{t_1,y_1}\) with \(y_1 := \gamma_{t_0,y_0}(t_1) = Fl_{t_1,t_0}(y_0)\).

(b) If \(t_2 \in I_{t_1,y_1}\) in (a), then \(t_2 \in I_{t_0,y_0}\) and \(Fl_{t_2,t_0}(y_0) = Fl_{t_2,t_1}(Fl_{t_1,t_0}(y_0))\).

(c) For all \(t_1, t_0 \in J\), the map \(Fl_{t_1,t_0}: \Omega_{t_1,t_0} \to M\) is injective, \(Fl_{t_1,t_0}(\Omega_{t_1,t_0}) = \Omega_{t_0,t_1}\) holds, and \(Fl_{t_0,t_1} = (Fl_{t_1,t_0})^{-1}\). □

**Definition 10.25** Let \(J \subseteq \mathbb{R}\) be a non-degenerate interval, \(M\) be a \(C^1\)-manifold modelled on an sclc-space and \(f: W \to TM\) be a function on an open subset \(W \subseteq J \times M\) such that \(f(t, y) \in T_y M\) for all \((t, y) \in W\). We say that the differential equation \(\dot{y}(t) = f(t, y(t))\) admits local \(C^0\)-flows in the sense of Carathéodory solutions if, for all \((\bar{t}, \bar{y}) \in W\), there exist a relatively open interval \(I \subseteq J\) with \(\bar{t} \in I\), an open \(\bar{y}\)-neighbourhood \(V\) in \(M\) and a continuous function \(\Phi: I \times I \times V \to M\) which satisfies conditions (a) and (b) stated in Definition 4.8.

The following fact can be proved like Theorem 2.5.15 in [20] (taking a singleton set of parameters in [20, Theorem 2.5.18] and reading ‘solution’ as ‘Carathéodory solution’).

**Proposition 10.26** Let \(J \subseteq \mathbb{R}\) be a non-degenerate interval, \(M\) be a \(C^1\)-manifold modelled on an sclc-space and \(f: W \to TM\) be a function on an open subset \(W \subseteq J \times M\) such that \(f(t, y) \in T_y M\) for all \((t, y) \in W\). Assume that \(\dot{y}(t) = f(t, y(t))\) satisfies local uniqueness of Carathéodory solutions and admits local \(C^0\)-flows in the sense of Carathéodory solutions. Let \(Fl: \Omega \to M\) be the maximal flow of \(\dot{y}(t) = f(t, y(t))\). Then \(\Omega\) is open in \(J \times J \times M\) and \(Fl: \Omega \to M\) is continuous. Moreover, \(\Omega_{t,t_0}\) is open in \(M\) for all \(t, t_0 \in J\) and the map \(Fl_{t,t_0}: \Omega_{t,t_0} \to \Omega_{t_0,t}\) is a homeomorphism, with inverse \(Fl_{t_0,t}\). □

## A Local factorization of \(C^{k+1}\)-mappings over Banach spaces

The following lemma is used in the proof of Lemma 10.15. See [6] for a precursor in the case \(k = 1\); cf. also [16, §7] and [18, Lemma 1.62] for related results.

**Lemma A.1** Let \(E\) be a locally convex space, \((Z, \| \cdot \|)\) be a Banach space, \(U \subseteq E\) be open, \(K \subseteq U\) a compact subset, \(k \in \mathbb{N}_0\) and \(f: U \to Z\) be a \(C^{k+1}\)-map. Then the following holds:
We let \( \tilde{\pi} \) instead. And also Lemma A.1 (b) follows: If Lemma A.1 (a) follows immediately from the next lemma, which we prove

\[ g: Q \to Z \]

on an open subset \( Q \subseteq F \), and an open subset \( V \subseteq U \) with \( q(V) \subseteq Q \) such that \( K \subseteq V \) and

\[ g \circ q|_V = f|_V. \]

(b) If \( E = E_1 \times \cdots \times E_m \) with locally convex spaces \( E_1, \ldots, E_m \) and \( U = U_1 \times \cdots \times U_m \) with open subsets \( U_j \subseteq E_j \) for \( j \in \{1, \ldots, m\} \), we can achieve in (a) that \( V = V_1 \times \cdots \times V_m \) with open subsets \( V_j \subseteq U_j \), \( F = F_1 \times \cdots \times F_m \) with Banach spaces \( F_j \), \( Q = Q_1 \times \cdots \times Q_m \) with open subsets \( Q_j \subseteq F_j \) and \( q = q_1 \times \cdots \times q_m \) with continuous linear maps

\[ q_j: E_j \to F_j \text{ such that } q_j(V_j) \subseteq Q_j \text{, for all } j \in \{1, \ldots, m\}. \]

The following concepts are useful for the proof.

**A.2** If \( E \) is a locally convex space and \( p \) a continuous seminorm on \( E \), then \( N := \{ x \in E : p(x) = 0 \} \) is a vector subspace of \( E \) and \( \| x + N \|_p := p(x) \) is well defined for \( x + N \in E_p := E/N \) and provides a norm \( \| \cdot \|_p \) on \( E_p \). If \( \pi_p: E \to E_p \), \( x \mapsto x + N \) is the canonical map, then \( \| \pi_p(x) \|_p = p(x) \) for all \( x \in E \), entailing that

\[ \pi_p(B_{\varepsilon}^p(x)) = B^p_{\varepsilon\|p}(\pi_p(x)) \text{ for all } x \in E \text{ and } \varepsilon > 0. \] (37)

We let \( \tilde{E}_p \) be a completion of \( E_p \) with \( E_p \subseteq \tilde{E}_p \) and use the same notation, \( \| \cdot \|_p \), for the norm on \( \tilde{E}_p \). We write \( \tilde{\pi}_p \) for \( \pi_p \), considered as a map to \( \tilde{E}_p \).

Lemma [A.1](a) follows immediately from the next lemma, which we prove instead. And also Lemma [A.1](b) follows: If \( E = E_1 \times \cdots \times E_m \) and \( U = U_1 \times \cdots \times U_m \), we let \( \text{pr}_j: E \to E_j \) be the projection onto the \( j \)th component for \( j \in \{1, \ldots, m\} \) and set \( K_j := \text{pr}_j(K) \subseteq U_j \). After increasing \( K \), we may assume that \( K = K_1 \times \cdots \times K_m \). For \( P \) as in Lemma [A.3] there exists a continuous seminorm \( p \geq P \) on \( E \) satisfying

\[ p(x_1, \ldots, x_m) = \max\{p_1(x_1), \ldots, p_m(x_m)\} \text{ for all } (x_1, \ldots, x_m) \in E \]

with continuous seminorms \( p_j \) on \( E_j \) (for example, we can take \( p_j(x_j) := mP(0, \ldots, 0, x_j, 0, \ldots, 0) \) with \( x_j \) in the \( j \)th slot). For any such, we have

\[ \tilde{E}_p \cong \tilde{E}_{p_1} \times \cdots \times \tilde{E}_{p_m} \]
in a natural way and \( Q_p := \tilde{\pi}_p(K) + B_1^{\tilde{E}_p}(0) \) corresponds to \( Q_1 \times \cdots \times Q_m \) with \( Q_j := \tilde{\pi}_{p_j}(K_j) + B_1^{E_{p_j}}(0) \).

**Lemma A.3** Let \( E \) be a locally convex space, \( (Z, \| \cdot \|) \) be a Banach space, \( U \subseteq E \) be open, \( K \subseteq U \) a compact subset, \( k \in \mathbb{N}_0 \) and \( f : U \to Z \) be a \( C^{k+1} \)-map. Then there exists a continuous seminorm \( P \) on \( E \) with \( K + B_1^P(0) \subseteq U \) with the following property: For each continuous seminorm \( p \) on \( E \) such that \( p \geq P \) pointwise, there exists a \( C^k \)-map

\[
g : Q_p \to Z
\]
on the open subset \( Q_p := \tilde{\pi}_p(K) + B_1^{\tilde{E}_p}(0) \) of \( \tilde{E}_p \) such that \( g \circ \tilde{\pi}_p |_{V_p} = f |_{V_p} \) holds for the open subset \( V_p := K + B_1^p(0) \subseteq U \) with \( K \subseteq V_p \).

**Proof of Lemma A.3.** The proof is by induction on \( k \). The case \( k = 0 \): We assume that \( f \) is \( C^1 \). There exists \( M \in [0, \infty) \) such that \( \sup \| f(K) \| < M \). Let \( B \subseteq Z \) be the open unit ball. As \( df : U \times E \to Z \) is continuous and \( df(x,0) = 0 \) for all \( x \in U \), the set \( (df)^{-1}(B) \) is an open neighbourhood of \( K \times \{0\} \) in \( U \times E \). By the Wallace Lemma, there exist an open subset \( O \subseteq U \) with \( K \subseteq O \) and an open 0-neighbourhood \( W \subseteq E \) such that \( O \times W \subseteq (df)^{-1}(B) \). After shrinking \( O \), we may assume that

\[
\| f(x) \| < M \text{ for all } x \in O. \tag{38}
\]

After shrinking \( W \), we may assume that \( K + 2W \subseteq O \) and \( W = B_1^p(0) \) for a continuous seminorm \( P \) on \( E \). Then

\[
\| df(x,y) \| \leq P(y) \text{ for all } x \in O \text{ and } y \in E.
\]

We now define \( V := K + W \). Let \( y, z \in V \). If \( z - y \in W \), pick \( x \in K \) such that \( y - x \in W \). Then \( z - x = (z - y) + (y - x) \in 2W \) and thus \( y, z \in x + 2W \), which is a convex set contained in \( O \). By the Mean Value Theorem,

\[
\| f(z) - f(y) \| = \left\| \int_0^1 df(y + t(z - y), z - y) \, dt \right\| \\
\leq \int_0^1 \| df(y + t(z - y), z - y) \| \, dt \leq P(z - y).
\]

If \( z - y \notin W \), then \( P(z - y) \geq 1 \) and thus

\[
\| f(z) - f(y) \| \leq \| f(z) \| + \| f(y) \| \leq 2M \leq 2M P(z - y).
\]

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Thus, after replacing $P$ with $\max\{1, 2M\} P$ (and shrinking $W$ and $V$ accordingly), we may assume $\|f(z) - f(y)\| \leq P(z - y)$ for all $y, z \in V$. Hence

$$\|f(z) - f(y)\| \leq p(z - y) \text{ for all } y, z \in V_p$$  \hspace{1cm} (39)

for each continuous seminorm $p$ on $E$ such that $p \geq P$ pointwise, with $V_p := K + B^p_1(0)$. Note that

$$\pi_p(V_p) = \pi_p(K) + B^p_1(0)$$

is open in $E_p$. If $y, z \in V_p$ such that $\pi_p(y) = \pi_p(z)$ and thus $p(z - y) = 0$, we have $f(y) = f(z)$ by (39), showing that $h_p: \pi_p(V_p) \to Z, \pi_p(y) \mapsto f(y)$ is well defined. By construction, $h_p \circ \pi_p|_{V_p} = f|_{V_p}$. Note that $\pi_p(V_p) = \pi_p(K) + B^p_1(0)$ is dense in

$$Q_p := \pi_p(K) + B^p_1(0).$$

Moreover, $h_p$ is Lipschitz continuous (and thus uniformly continuous), since

$$\|h_p(\pi_p(z)) - h_p(\pi_p(y))\| \leq \|f(z) - f(y)\| \leq p(z - y) \leq \|\pi_p(z) - \pi_p(y)\|_p$$

for all $y, z \in V_p$. As $(Z, \| \cdot \|)$ is complete, the uniformly continuous map $h_p$ has a unique uniformly continuous extension

$$g_p: Q_p \to Z.$$

Then $g_p \circ \pi_p|_{V_p} = h_p \circ \pi_p|_{V_p} = f|_{V_p}$.

To perform the induction step, let $f$ be $C^{k+2}$ and assume that the assertion of the lemma holds for $k$. Let $P, V_p, g_p, Q_p \to Z$ for $p \geq P$ be as in the step $k = 0$. The map $df: U \times E \to Z$ is $C^{k+1}$ and $K \times \{0\}$ a compact subset of $U \times E$. By the inductive hypothesis, there is a continuous seminorm $R$ on $E \times E$ such that $(K \times \{0\}) + B^R_1(0) \subseteq U \times E$ and there exists a $C^k$-map

$$h_r: S_r \to Z$$

on $S_r := \pi_r(K \times \{0\}) + B^{(E \times E)}_{-1}(0)$ such that $h_r \circ \pi_r|_{W_r} = df|_{W_r}$ on $W_r := (K \times \{0\}) + B^R_1(0)$, for each continuous seminorm $r$ on $E \times E$ such that $r \geq R$. Note that

$$p(x) := \max\{P(x), 2R(x, 0), 2R(0, x)\}$$
defines a continuous seminorm on $E$ such that $p \geq P$. Moreover, $r(x, y) := \max\{p(x), p(y)\}$ defines a continuous seminorm on $E \times E$ with $r \geq R$. Then

$$W_r = V_p \times B^p_1(0) \quad \text{and} \quad S_r = Q_p \times B^\tilde{E}_p(0),$$

identifying $(E \times E)^r$ with $\tilde{E}_p \times \tilde{E}_p$.

We now show that the directional derivative $dg_p(x, y)$ exists for all $x \in Q_p \cap E_p$ and $y \in B^\tilde{E}_p(0)$, and is given by

$$dg_p(x, y) = h_r(x, y).$$

There exists $z \in K$ such that

$$x \in E_p \cap (\tilde{\pi}_p(z) + B^\tilde{E}_1(0)) = E_p \cap B^\tilde{E}_1(\pi_p(z)) = B^E_1(\pi_p(z)).$$

Let $\varepsilon > 0$ such that $\|x - \pi_p(z)\|_p + \varepsilon < 1$. Let $a \in B^\tilde{E}_1(z)$ such that $\pi_p(a) = x$ and $b \in B^\tilde{E}_1(0)$ such that $\pi_p(b) = y$. Then $p(a + s\varepsilon b - z) \leq p(a - z) + \varepsilon p(b) = \|x - \pi_p(z)\|_p + \varepsilon\|y\|_p < 1$ for all $s \in [-1, 1]$, whence $a + s\varepsilon b \in B^\tilde{E}_1(z)$. For $0 \neq t \in [-\varepsilon, \varepsilon]$, we get

$$\frac{g_p(x + ty) - g_p(x)}{t} = \frac{1}{\varepsilon} \frac{g_p(x + (t/\varepsilon)(\varepsilon y)) - g_p(x)}{(t/\varepsilon)}$$

$$= \frac{1}{\varepsilon} \frac{f(a + (t/\varepsilon)(\varepsilon b)) - f(a)}{(t/\varepsilon)}$$

$$\rightarrow \frac{1}{\varepsilon} df(a, \varepsilon b) = df(a, b) = h_r(x, y)$$

as $t \to 0$, as we set out to show.

For all $(x, y) \in (Q_p \cap E_p) \times E_p$ and $s > 0$ such that $y \in sB^E_1(0)$, the directional derivative $dg_p(x, s^{-1}y)$ exists by the preceding, whence also $dg_p(x, y) = dg_p(x, s(s^{-1}y))$ exists by a standard argument, and is given by

$$dg_p(x, y) = sdg_p(x, s^{-1}y) = sh_r(x, s^{-1}y). \quad (40)$$

Thus $d(g_p|_{Q_p \cap E_p}) : (Q_p \cap E_p) \times E_p \to Z$ exists and is a $C^k$-map, since the sets $U_s := (Q_p \cap E_p) \times sB^E_1(0)$ form an open cover of $(Q_p \cap E_p) \times E_p$ for $s \in ]0, \infty[$ and $d(g_p|_{Q_p \cap E_p})(x, y)$ is $C^k$ in $(x, y) \in U_s$, being given by (40). Thus $g_p|_{Q_p \cap E_p}$ is $C^{k+1}$ and thus $C^1$. 

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For each $x_0 \in \mathbb{Q}_p$, there exists $\rho > 0$ such that $B_{2\rho}^E(x_0) \subseteq \mathbb{Q}_p$. Let $\Omega := B_{\rho}^E(x_0)$. We show that $g_p|\Omega$ is $C^{k+1}$; as such sets $\Omega$ form an open cover of $\mathbb{Q}_p$, this implies that $g_p$ is $C^{k+1}$. The function

$$\Delta : \Omega \times B_1^E(0) \times [-\rho, \rho] \to Z, \quad (x, y, t) \mapsto \int_0^1 h_r(x + t y, y) \, ds$$

is continuous, being a parameter-dependent integral (see [20, Lemma 1.1.11]). For $(x, y) \in (\Omega \cap E_p) \times B_1^E(0)$ and $0 \neq t \in [-\rho, \rho]$, we have

$$\Delta(x, y, t) = \int_0^1 d(g_p|_{Q_p \cap E_p})(x + t y, y) \, ds = \frac{g_p(x + ty) - g(x)}{t}, \quad (41)$$

by the Mean Value Theorem. Note that the right-hand side of (41) defines a continuous $Z$-valued function on $\Omega \times B_1^E(0) \times [-\rho, \rho] \setminus \{0\}$, in which $(\Omega \cap E_p) \times B_1^E(0) \times [-\rho, \rho] \setminus \{0\}$ is dense. As also the left-hand side defines a continuous function there and $Z$ is Hausdorff, we deduce that

$$\Delta(x, y, t) = \frac{g_p(x + ty) - g(x)}{t} \quad \text{for all} \quad \Omega \times B_1^E(0) \times (-\rho, \rho) \setminus \{0\}.$$  

Letting $t \to 0$, we see that $dg_p(x, y)$ exists for all $x \in \Omega$ and $y \in B_1^E(0)$, and is given by

$$dg_p(x, y) = \Delta(x, y, 0) = h_r(x, y).$$

As above, we deduce from this that $g_p|\Omega$ as $C^{k+1}$. □

**Acknowledgements.** The research was supported by Deutsche Forschungsgemeinschaft (DFG), project GL 357/9-1. The authors thank the anonymous referee, whose comments helped to improve the presentation.

**References**

[1] Agrachev, A. A. and Y. L. Sachkov, “Control Theory from the Geometric Viewpoint,” Spinger, Berlin, 2004.

[2] Bastiani, A., Applications différentiables et variétés différentiables de dimension infinie, J. Anal. Math. 13 (1964), 1–114.

[3] Bertram, W., H. Glöckner, and K.-H. Neeb, Differential calculus over general base fields and rings, Expo. Math. 22 (2004), 213–282.
[4] Bochner, S., *Absolut-additive abstrakte Mengenfunktionen*, Fundam. Math. **21** (1933), 211–213.

[5] Cartan, H., “Calcul différentiel,” Hermann, Paris 1967.

[6] Chasiotis, A.-V., “Vektorwertige absolut stetige Funktionen,” Bachelor’s thesis, University of Paderborn, 2020 (advised by H. Glöckner).

[7] Clarkson, J. A., *Uniformly convex spaces*, Trans. Amer. Math. Soc. **40** (1936), 396–414.

[8] Dieudonné, J., “Foundations of Modern Analysis,” Academic Press, New York, 1969.

[9] Engelking, R., “General Topology”, Heldermann Verlag, Berlin, 1989.

[10] Eyni, J. M., *The Frobenius theorem for Banach distributions on infinite-dimensional manifolds and applications in infinite-dimensional Lie theory*, preprint, [arXiv:1407.3166](http://arxiv.org/abs/1407.3166).

[11] Florencio, M., F. Mayoral, and P. J. Paúl, *Spaces of vector-valued integrable functions and localization of bounded subsets*, Math. Nachr. **174** (1995), 89–111.

[12] Glöckner, H., *Infinite-dimensional Lie groups without completeness restrictions*, pp. 43–59 in: Strasburger, A. et al. (eds.), “Geometry and Analysis on Finite- and Infinite-Dimensional Lie Groups,” Banach Center Publications **55**, Warsaw, 2002.

[13] Glöckner, H., *Lie group structures on quotient groups and universal complexifications for infinite-dimensional Lie groups*, J. Funct. Anal. **194** (2002), 347–409.

[14] Glöckner, H., *Lie groups of measurable mappings*, Can. J. Math. **55** (2003), 969–999.

[15] Glöckner, H., *Fundamentals of direct limit Lie theory*, Compositio Math. **141** (2005), 1551–1577.

[16] Glöckner, H., *Aspects of differential calculus related to infinite-dimensional vector bundles and Poisson vector spaces*, Axioms **2022**, 11, 221.

[17] Glöckner, H., *Regularity properties of infinite-dimensional Lie groups, and semiregularity*, preprint, [arXiv:1208.0715](http://arxiv.org/abs/1208.0715).
[18] Glöckner, H., *Measurable regularity properties of infinite-dimensional Lie groups*, preprint, arXiv:1601.02568.

[19] Glöckner, H., *Lie groups of real analytic diffeomorphisms are $L^1$-regular*, preprint, arXiv:2007.15611.

[20] Glöckner, H. and K.-H. Neeb, “Infinite-Dimensional Lie Groups,” book in preparation.

[21] Hamilton, R. S., *The inverse function theorem of Nash and Moser*, Bull. Amer. Math. Soc. 7 (1982), 65–222.

[22] Hanusch, M., *The strong Trotter property for locally $\mu$-convex Lie groups*, J. Lie Theory 30 (2020), 25–32.

[23] Hanusch, M., *Regularity of Lie groups*, Commun. Anal. Geom. 30 (2022), 53–152.

[24] Hilgert, J., K. H. Hofmann, and J. D. Lawson, “Lie Groups, Convex Cones, and Semigroups,” Clarendon Press, Oxford, 1989.

[25] Jurdjevic, V., “Geometric Control Theory,” Cambridge Univ. Press, 1997.

[26] Jurdjevic, V. and H. J. Sussmann, “Control Systems on Lie Groups,” J. Differ. Equations 12 (1972), 313–329.

[27] Keller, H. H., “Differential Calculus in Locally Convex Spaces”, Springer, Berlin, 1974.

[28] Kriegl, A. and P. W. Michor, *Regular infinite dimensional Lie groups*, J. Lie Theory 7 (1997), 61–99.

[29] Kriegl, A. and P. W. Michor, “The Convenient Setting of Global Analysis,” AMS, Providence, 1997.

[30] Lang, S., “Fundamentals of Differential Geometry,” Springer, New York, 1999.

[31] Michor, P. W., “Manifolds of Differentiable Mappings,” Shiva Publ., Orpington, 1980.

[32] Milnor, J., *On infinite-dimensional Lie groups*, preprint, Institute for Advanced Study, Princeton, 1982.

[33] Milnor, J., *Remarks on infinite-dimensional Lie groups*, pp. 1007–1057 in: B. S. DeWitt and R. Stora (eds.), “Relativité, groupes et topologie II,” North-Holland, Amsterdam, 1984.
[34] Neeb, K.-H., *Towards a Lie theory of locally convex groups*, Jpn. J. Math. 1 (2006), 291–468.

[35] Nikitin, N., “Regularity Properties of Infinite-Dimensional Lie Groups and Exponential Laws,” doctoral dissertation, University of Paderborn, 2021; see nbn-resolving.de/urn:nbn:de:hbz:466:2-39133

[36] Rudin, W., “Real and Complex Analysis,” McGraw-Hill, New York, 1987.

[37] Sachkov, Y. L., *Control theory on Lie groups*, J. Math. Sci., New York 156 (2009), 381–439.

[38] Schechter, E., “Handbook of Analysis and its Foundations,” Academic Press, San Diego, 1997.

[39] Sontag, E. D., “Mathematical Control Theory,” Springer, New York, 1998.

[40] Voigt, J., *On the convex compactness property for the strong operator topology*, Note Mat. 12 (1992), 259–269.

[41] Walter, B., *Weighted diffeomorphism groups of Banach spaces and weighted mapping groups*, Diss. Math. 484 (2012), 126 pp.

[42] Weizsäcker, H. von, *In which spaces is every curve Lebesgue-Pettis integrable?*, preprint, arXiv:1207.6034

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