Convergence to the traveling wave solution of a biological-physical model through a differential equation with piecewise constant argument

C Ramirez-Carrasco¹, and F Duque-Marín¹,²
¹ Facultad de Ciencias Básicas, Universidad Católica del Maule, Talca, Chile
² Escuela de Investigación en Biomatemática, Universidad del Quindío, Armenia, Colombia

E-mail: carloshrc1989@gmail.com, fedver@gmail.com

Abstract. Getting to know explicitly or approximately the traveling wave solutions of the diffusive delay logistic equation, commonly known as the delayed Kolmogorov-Petrovsky-Piscounov-Fisher equation, is of major importance for understanding various biological and physical phenomena. In this study, we discretize the delay argument of the equation that satisfies the traveling wave and we obtain a second order delay differential equation with piecewise constant argument. We prove the existence and uniqueness of a solution for the discretized equation, and then prove that this solution converges uniformly along the whole straight towards the traveling wave. The methodology posed is based on the upper and lower solutions technique along with the use of a monotone integral operator. Our results show that the technique we developed is another good method for approaching traveling wave solutions. In addition, we suggest that this method can be applied to other reaction-diffusion equations that model a wide range of biological, physical, and chemical phenomena.

1. Introduction
The study of traveling wave solutions of the diffusive delay logistic equation, commonly known as the delayed Kolmogorov-Petrovsky-Piscounov-Fisher equation or simply the delayed KPP-Fisher equation, began with Gourley [1] and Wu [2] on the early 2000s. This equation emerged as a model in biology [3,4], however the physics community has acknowledged that it is closely related to some important physical phenomena such as neutron action, liquid crystal wave movement and the spread of nervous signals in biophysics [5]. That is why it is very important for the researchers to get to approach this explicitly or at least approximately. The delayed KPP-Fisher equation has been widely studied by various authors [6–9], all of whom examine the existence, uniqueness, asymptotic behavior and the more qualitative properties of traveling waves. On the other hand, Gyori [10] provides a method to approximate solutions of delayed differential equations, given an initial function, using differential equations with piecewise constant arguments or commonly known as DEPCA, since then its method has been studied extensively by several authors [11–14]. However, the literature on approximating traveling waves by this method is scarce.

In that order of ideas, the objective of this research is to study convergence to the traveling wave of the delayed KPP-Fisher equation through the solution of a DEPCA. This DEPCA arises from discretizing the delay argument of the equation that satisfies the traveling wave of the delayed KPP-Fisher equation.
The methodology posed in this study is based on the lower-upper solutions technique. We propose an upper and lower solutions for DEPCA and together with the use of a monotone integral operator we show that the unique heterocline solution of DEPCA can be found via an iteration procedure. Finally, we proved that this solution converges to the traveling wave of delayed KPP-Fisher equation as the step of discretization becomes smaller. Our results suggest that this type of discretization is a good approximation to traveling waves in a delayed reaction-diffusion equation, which model a wide array of biological [15], physical [5] and chemical [16] phenomena.

2. Notations and preliminaries

In this section, we think back to some definitions and present the necessary notations for understanding our results. The diffusive logistic equation or KPP-Fisher equation [3, 4] is denoted by Equation (1).

$$\frac{\partial U(z,x)}{\partial z} = \frac{\partial^2 U(z,x)}{\partial x^2} + U(z,x)(1 - U(z,x)) \quad U \geq 0, \; z \in \mathbb{R}, \; x \in \mathbb{R}. \tag{1}$$

Equation (1) can be viewed as a natural extension of the ordinary logistic equation $u'(z) = u(z)(1 - u(z))$. In 1948, Hutchinson [17] posed an important improvement to this model, incorporating a $\tau > 0$ delay as follows: $u'(z) = u(z)(1 - u(z - \tau))$. Since then Equation (2) or delayed KPP-Fisher equation.

$$\frac{\partial U(z,x)}{\partial z} = \frac{\partial^2 U(z,x)}{\partial x^2} + U(z,x)(1 - U(z - \tau, x)) \quad U \geq 0, \; z \in \mathbb{R}, \; x \in \mathbb{R}, \tag{2}$$

is considered a natural prototype of delayed reaction-diffusion equations. We must recall that classic solution $U(z,x) = \phi(x + cz)$ is a traveling wave for Equation (2), where $c > 0$ is the wave velocity, if the profile function $\phi$ is positive and satisfies $\phi(-\infty) = 0$, $\phi(+\infty) = 1$ (See [10]). If we replace the profile function $\phi$ in the Equation (2) we get the ordinary Equation (3), then the existence of traveling waves for Equation (2) is equivalent to the presence of positive heteroclinic connections in the second order nonlinear differential Equation (3).

$$\phi''(t) - cc'\phi'(t) + \phi(t)(1 - \phi(t - r)) = 0 \quad t \in \mathbb{R}, \; r > 0, \tag{3}$$

where $t = x + cz$ and $r = c\tau$. In [8], the uniqueness of an increasing monotone traveling wave was proved for Equation (2), if $(\tau, c) \in \text{int}(\mathcal{D})$, where $\mathcal{D} := \{(\tau, c) : 0 \leq \tau \leq \tau_1 \sim 0.50771166... \text{ and } 2 \leq c \leq c^\ast(\tau)\}$ with $c^\ast(\tau) = +\infty$ for $0 \leq \tau < 1/e$ and $c^\ast(\tau)$ the parametric equation $2 + \sqrt{4 + c^4\tau^2} = c^2\tau^2 \exp((2 - c^2\tau + \sqrt{4 + c^4\tau^2})/2)$ for $1/e \leq \tau \leq \tau_1$.

On the other hand, according to Cooke and Wiener [18], a typical DEPCA is of the form: $x'(t) = f(t, x(t), x(\gamma(t)))$ where the argument $\gamma(t)$ is piecewise constant, for example $\gamma(t) = [t]$ or $\gamma(t) = [t - r]$, where $r$ is a positive integer and $[\cdot]$ denotes the greatest integer function. Then for each $k \in \mathbb{Z}$, we establish $h = r/k > 0$ and define the DEPCA associated with Equation (3) by Equation (4).

$$u''(t) - cu'(t) + u(t)(1 - u(\gamma_h(t - r))) = 0 \quad u \geq 0, \; t \in \mathbb{R}, \; r > 0, \tag{4}$$

with $u(-\infty) = 0$, $u(+\infty) = 1$ and where $\gamma_h(\cdot) = [\cdot/h] h$. Equation (4) is constructed by discretizing the delay argument of Equation (3) through the function $\gamma_h$ which is constant in the intervals $(kh, (k + 1)h], \; k \in \mathbb{Z}$. 


Definition 1. Let $C^2(\mathbb{R})$ be the set of twice continuously differentiable functions. The function $u_h \in C^2(\mathbb{R})$ is called heteroclinic solution of Equation (4) if it satisfies said equation in each interval $[kh, (k + 1)h)$, $k \in \mathbb{Z}$ and $u_h(-\infty) = 0$, $u_h(+\infty) = 1$.

In order to prove the existence of the heteroclinic solution of Equation (4), we present the definitions of upper and lower solutions adopted in [8].

Definition 2. A continuous function $\rho \in C(\mathbb{R})$ is called a lower solution of Equation (4) if $\rho'$ and $\rho''$ exist almost everywhere in $\mathbb{R}$, and they are essentially bounded on $\mathbb{R}$ and if the inequality, Equation (5), holds.

$$
\rho''(t) - c \rho'(t) + f(\rho(t), \rho(t - r)) \leq 0, \quad t \in \mathbb{R}, \ r > 0.
$$

An upper solution of Equation (4) is defined in a similar way by reversing the inequality, Equation (5).

Since the main objective of this study is to prove that the solution of Equation (4) uniformly converges to the traveling wave of Equation (2) when step $h$ tends towards zero, we begin next section proving the existence and uniqueness of a solution for Equation (4).

3. Results

In this section we present the results of the study. We start by proposing an upper and lower solutions for Equation (4) and together with the use of a monotone integral operator we show that the unique heteroclinic solution of Equation (4) can be found via an iteration procedure. Finally, we proved that this solution converges to the travelling wave of Equation (2) as the step $h$ tends to zero.

3.1. Existence

Remember that if $(\tau, c) \in \text{int}(D)$, then Equation (2) has an unique monotone traveling wave for $r = c \tau$. If we disturb the parameter $\tau$ as $\tau + h/c$ for $h > 0$ small enough that $(\tau + h/c, c) \in \text{int}(D)$ then traveling wave for $R = r + h$, which we will denote as $\phi_-$, is a lower solution to Equation (4); this is proven in the following Proposition 1.

Proposition 1. Let $\phi_-$ be traveling wave solution of Equation (2) for the parameter pair $(\tau + h/c, c)$ then $\phi_-$ satisfies the inequality, Equation (6).

$$
\phi_-''(t) - c \phi_-'(t) + \phi_-(t)(1 - \phi_-(\gamma h(t - r))) \leq 0,
$$

that is $\phi_-$ is a lower solution for Equation (4).

Proof. We know that $\phi_-$ is increasing it is also easy to see that $t - R \leq \gamma h(t - r)$ then $\phi_-(t) \phi_-(t - R) \leq \phi_-(t) \phi_-(\gamma h(t - r))$. Thus $\phi_-''(t) - c \phi_-'(t) + \phi_-(t)(1 - \phi_-(\gamma h(t - r))) \leq \phi_-''(t) - c \phi_-'(t) + \phi_-(t)(1 - \phi_-(t - R))$. Since $\phi_-$ is solution of Equation (3) for the parameters $(\tau + h/c, c)$, the result is obtained.

Remark 1. In [8], it was proven that function, Equation (7), defined by sections is upper solution of Equation (3).

$$
\phi_+(t) = \begin{cases} 
1 - e^{\lambda_2 t} + e^{\beta t} & t > t_0 \\
1 - e^{\lambda_2 t_0} + e^{\beta t_0} & t \leq t_0,
\end{cases}
$$

where, $\beta \in (\lambda_1, \lambda_2)$ with $\lambda_1 < \lambda_2 < 0$ roots of the equation $\chi_1(z) = z^2 - cz - e^{-rz}$ and $t_0 = (\ln(-\beta) - \ln(-\lambda_2))/(\lambda_2 - \beta)$ the point where function, Equation (7), reaches its minimum.
In this study we prove that the function, Equation (7), is also an upper solution for Equation (4).

Proposition 2. The function, Equation (7), satisfies the inequality, Equation (8)

\[ \phi''_+(t) - c\phi'_+(t) + \phi_+(t) - \phi_+(t)\phi_+(\gamma_h(t - \tau)) \geq 0, \quad (8) \]

that is \( \phi_+ \) is an upper solution for Equation (4).

Proof. We note that \( \phi_+ \) is non-decreasing and \( \gamma_h(t) \leq t \) then \( \phi_+(t)\phi_+(\gamma_h(t - \tau)) \leq \phi_+(t)\phi_+(t - \tau) \). Thus \( \phi''_+(t) - c\phi'_+(t) + \phi_+(t)(1 - \phi_+(t - \tau)) \leq \phi''_+(t) - c\phi'_+(t) + \phi_+(t)(1 - \phi_+(\gamma_h(t - \tau))) \). Finally, since \( \phi_+ \) it is an upper solution for Equation (3), the result is obtained.

Proposition 3. There are translations of \( \phi_-(t) \) y \( \phi_+(t) \) such that \( \phi_-(t) < \phi_+(t) \) for all \( t \in \mathbb{R} \).

Proof. We know that \( \phi_-(\infty) = 0 \) y \( \phi_+(t) = 1 - e^{\lambda_2 t_0} + e^{\beta t_0}, \forall t \leq t_0 \). So, for all \( t \) small enough \( \phi_-(t) < \phi_+(t) \). On the other hand, according to [8] an asymptotic representation of \( \phi_- \) in \( +\infty \) is \( 1 - e^{\lambda} + O(e^{(2\lambda_2 + \sigma)}) \). Then, for all \( t \) large enough \( \phi_-(t) < \phi_+(t) \), whenever \( \beta > 2\lambda_2 + \sigma \) with \( \sigma > 0 \) is small enough. Therefore, there is \( t_s > 0 \) large enough so that \( \phi_-(t - t_s) < \phi_+(t) \) for all \( t \in \mathbb{R} \). Finally, denoting \( t - t_s \) for \( \phi_-(t) < \phi_+(t) \) for all \( t \in \mathbb{R} \) is obtained.

In order to prove the existence and uniqueness of a heterocline solution of Equation (4), we defined the operator, Equation (9), using the method of parameter variation.

\[(A_h\psi)(t) = \int_t^{+\infty} \frac{e^{\lambda(t-s)} - e^{\mu(t-s)}}{\mu - \lambda} \psi(s)\psi(\gamma_h(t - r))ds, \quad (9)\]

where \( 0 < \lambda \leq \mu \) are roots of the equation \( \chi_0(z) = z^2 - cz + 1 \).

Proposition 4. The operator, Equation (9), satisfies \( \phi_-(t) \leq (A_h\phi_-)(t) \) and \( (A_h\phi_+)(t) \leq \phi_+(t) \).

Proof. Let us denote \( \omega^*(t) = \phi''_+(t) - c\phi'_+(t) + \phi_+(t)(1 - \phi_+(\gamma_h(t - \tau))) \geq 0 \). Thus \( \phi''_+(t) - c\phi'_+(t) + \phi_+(t) = \omega^*(t) + \phi_+(t)\phi_+(\gamma_h(t - \tau)) > 0 \). Applying parameter variation in this last equation, we obtain Equation (10).

\[ \phi_+(t) = \int_t^{+\infty} \frac{e^{\lambda(t-s)} - e^{\mu(t-s)}}{\mu - \lambda} [\omega^*(s) + \phi_+(s)\phi_+(\gamma_h(t - s))] ds. \quad (10) \]

Thus

\[ \phi_+(t) = \int_t^{+\infty} \frac{e^{\lambda(t-s)} - e^{\mu(t-s)}}{\mu - \lambda} \omega^*(s) ds + (A_h\phi_+)(t) \geq (A_h\phi_+)(t). \quad (11) \]

On the other hand, denoting \( \omega_-(t) = \phi''_-(t) - c\phi'_-(t) + \phi_-(t)(1 - \phi_-(\gamma_h(t - \tau))) \leq 0 \) similarly we obtain \( \phi_-(t) \leq (A_h\phi_-)(t) \).

Next, through an iterative process on the two inequalities of Proposition 4, we build two monotonic successions and show that they converge towards two functions that satisfy Equation (4).

Lemma 1. Let \( \phi_j(t) = (A_{h_j}\phi_-)(t) \) and \( \psi_j(t) = (A_{h_j}\phi_+)(t) \) with \( j \in \mathbb{N} \). Si \( \Phi = \lim_{j \to +\infty} \phi_j \) and \( \Psi = \lim_{j \to +\infty} \psi_j \) then \( \Phi(t) \leq \Psi(t) \) and both satisfy Equation (4) for all \( t \in \mathbb{R} \).
Proof. Note that if \(0 < \chi \leq f(t) \leq f_2\) then \((A_h f_1)(t) \leq (A_h f_2)(t)\). Also by Proposition 4 we get \(\phi_-(t) \leq \phi_1(t) \leq \phi_2(t) \leq ... \leq \Phi(t) \leq \Psi(t) \leq ... \leq \psi_2(t) \leq \psi_1(t) \leq \phi_+(t)\). Thus the succession of integrable functions are bounded, that is \(|\phi_2| < \phi_+\) and \(|\psi_2| < \phi_+\), then by Lebesgue’s dominated convergence theorem we have Equation (12).

\[
\Phi(t) = \lim_{j \to +\infty} \phi_j(t) = \lim_{j \to +\infty} (A_h \phi_{j-1})(t) = \left(A_h \lim_{j \to +\infty} \phi_{j-1}\right)(t) = (A_h \Phi)(t).
\]

Thus

\[
\Phi(t) = \int_t^{+\infty} e^{\lambda(t-s)} - e^{\mu(t-s)} \Phi(s)\Phi(\gamma_h(s-r))ds,
\]

(13)

\[
c\Phi'(t) = \int_t^{+\infty} e^{\lambda(t-s)} - e^{\mu(t-s)} \Phi(s)\Phi(\gamma_h(s-r))ds,
\]

(14)

\[
\Phi''(t) = \int_t^{+\infty} e^{\lambda(t-s)} - e^{\mu(t-s)} \Phi(s)\Phi(\gamma_h(s-r))ds + \Phi(t)\Phi(\gamma_h(t-r))ds.
\]

(15)

Since \(\lambda^2 - c\lambda + 1 = 0\) and \(\mu^2 - c\mu + 1 = 0\) the Equation (13), Equation (14), and Equation (15) satisfy \(\Phi''(t) - c\Phi'(t) + \Phi(t) = \Phi(t)\Phi(\gamma_h(t-r))\), that is to say \(\Phi\) satisfies Equation (4). Similarly, it is proven that \(\Psi\) also satisfies Equation (4).

3.2. Uniqueness

To prove the uniqueness of \(\Phi\) and \(\Psi\) we will follow the classical path by assuming that \(\Phi\) is different of \(\Psi\) and thus obtain some contradiction.

Lemma 2. The functions \(\Phi\) and \(\Psi\) are such that \(\Phi = \Psi\).

Proof. Suppose that the functions \(\Phi\) and \(\Psi\) that satisfy Equation (4) are such that \(\Phi \neq \Psi\). Consider the Bielecki norm \(||\Phi - \Psi|| = \max_{t \in R} |\Phi(t) - \Psi(t)|e^{-\lambda z t}\), where \(\lambda_2\) is one of the negative roots of the equation \(\chi_1(z) = z^2 - cz - e^{-rz}\). If the maximum of the Bielecki norm is reached in some \(t_0\), denoting \(\omega(t_0) = ||\Phi - \Psi||\) we have Equation (16).

\[
||\Phi - \Psi|| = |\Phi(t_0) - \Psi(t_0)|e^{-\lambda z t_0} < \frac{e^{-\lambda_2 t_0}}{\mu - \lambda} \int_{t_0}^{+\infty} (e^{\lambda(t_0-s)} - e^{\mu(t_0-s)}) |\Phi(s)\Phi(\gamma_h(s-r)) - \Psi(s)\Psi(\gamma_h(s-r))|ds
\]

\[
< \frac{e^{-\lambda_2 t_0}}{\mu - \lambda} \int_{t_0}^{+\infty} (e^{\lambda(t_0-s)} - e^{\mu(t_0-s)}) (|\Phi(s) - \Psi(s)||\Phi(\gamma_h(s-r))| + |\Psi(s)||\Phi(\gamma_h(s-r)) - \Psi(\gamma_h(s-r))|)ds
\]

\[
< \frac{e^{-\lambda_2 t_0}}{\mu - \lambda} \int_{t_0}^{+\infty} (e^{\lambda(t_0-s)} - e^{\mu(t_0-s)}) (\omega(s)e^{\lambda z s} + \omega(\gamma(s-r))e^{\lambda_2 \gamma(s-r)})ds
\]

\[
< \frac{\omega(t_0)e^{-\lambda_2 t_0}}{\mu - \lambda} \int_{t_0}^{+\infty} (e^{\lambda(t_0-s)} + \lambda_2 s + e^{\lambda(t_0-s)} + \lambda_2 s - e^{\mu(t_0-s)} + \lambda_2 s)ds
\]

\[
< \frac{\omega(t_0)e^{-\lambda_2 t_0}}{\mu - \lambda} \left(\frac{e^{\lambda_0(\lambda_2 - \mu)}}{\lambda_2 - \mu} + \frac{e^{\mu_0 - \lambda_2}}{\lambda_2 - \lambda}\right)
\]

\[
< \frac{\omega(t_0)}{\mu - \lambda} \left(1 + e^{-\lambda_2 t_0} - \frac{1 + e^{-\lambda_2 t_0}}{\lambda_2 - \lambda}\right) < \omega(t_0)(1 + e^{-\lambda_2 t_0}) (\lambda_2 - \lambda)(\lambda_2 - \mu),
\]

(16)
but $\lambda^2 - c\lambda + 1 = 1 + e^{-\lambda t}$ then from inequality, Equation (16), we obtain $\|\Phi - \Psi\| < \frac{\omega(1+e^{-\lambda t})}{(\lambda - \lambda)(\lambda - \mu)} = \omega(t_0)$, which is a contradiction. Therefore $\Phi = \Psi$.

From now on we will denote $u_h := \Phi = \Psi$. To prove that $u_h$ is a heterocline solution of Equation (4) we would only need to prove that $u_h(-\infty) = 0$ and $u_h(+\infty) = 1$. This we prove below, in addition we prove that this heterocline solution is monotone increasing.

Lemma 3. The function $u_h$ is monotone increasing for every $t \in \mathbb{R}$ and also satisfies $u_h(-\infty) = 0$ and $u_h(+\infty) = 1$.

Proof. First, we will demonstrate that $u_h'(t) > 0$ for all $t \in \mathbb{R}$. Conversely, we suppose there is $t_0 \in \mathbb{R}$ such that $u_h'(t_0) = 0$, then, Equation (17),

$$u_h'(t_0) = cu_h(t_0) - u_h(t_0)(1 - u_h(\gamma h(t_0 - r))) = -u_h(t_0)(1 - u_h(\gamma h(t_0 - r))).$$  (17)

Since Equation (17) is less than zero then $u_h$ reaches its maximum in $t_0$. This is impossible, because $u_h(+\infty) = 1$ since $\phi \leq u \leq \phi_+$. Therefore $u_h'(t) > 0$ for all $t \in \mathbb{R}$, that is to say $u_h$ is monotone increasing. To prove that $u_h(-\infty) = 0$, we integrate from $0$ to $t$ Equation (17) and obtain Equation (18).

$$u_h(t) = u_h'(0) + cu_h(t) - cu_h(0) - \int_0^t u_h(s)(1 - u_h(\gamma h(s - r)))ds.$$  (18)

Equation (18) implies that $u_h(-\infty) \in \{0,1\}$ on the contrary is $u_h'(\infty) = \infty$. but $u_h(+\infty) = 1$ therefore $u_h(-\infty) = 0$.

Finally, we can summarize the previous results in the first main theorem of this study.

Theorem 1. The Equation (4) or DEPCA associated with Equation (3) has unique increasing heterocline solution $u_h$.

Proof. The result is a consequence of Lemma 1, Lemma 2, and Lemma 3.

3.3. Convergence

Here we prove the second main theorem of this study. When the step $h$ tends to zero, the heteroclinic solution $u_h$ of Equation (4) converges uniformly over the whole straight towards the traveling wave of Equation (2).

Theorem 2. Let $\phi$ be the traveling wave of Equation (2) and $u_h$ heterocline solution of Equation (4), then, Equation (19),

$$\lim_{h \to 0} \max_{t \in \mathbb{R}}|\phi(t) - u_h(t)| = 0.$$  (19)

Proof. Let $(\phi^n)_{n \in \mathbb{N}}$ and $(u_h^n)_{n \in \mathbb{N}}$ two sequences defined by operator, Equation (9), this is $\phi^n = A^n(\phi_+) y u_h^n = A^n(\phi_+)$. Let us define $w(\phi_+, h, t) = \sup_{t_1, t_2} \{ |\phi_+(t_1) - \phi_+(t_2)| : |t_1 - t_2| \leq h \}$ and denote $k(t,s) = (e^{\lambda(t-s)} - e^{\lambda(t-s)})/(\mu - \lambda)$. Thus if $n = 1$, it is easy to see that $|\phi^1(t) - u_h^1(t)| \leq w(\phi_+, h, t)$. For $n = 2$, we obtain Equation (20).
\[ |\phi^2(t) - u_h^2(t)| = |A(\phi(t)) - \mathcal{A}_h(u_h(t))| \]
\[ = \left| \int_t^\infty k(t,s)\phi^1(s)\phi^1(s-r)ds - \int_t^\infty k(t,s)u_h^1(s)u_h^1(\gamma_h(s-r))ds \right| \]
\[ \leq \int_t^\infty k(t,s)|\phi^1(s)\phi^1(s-r) - u_h^1(s)u_h^1(\gamma_h(s-r))|ds \]
\[ = \int_t^\infty k(t,s)|\phi^1(s)\phi^1(s-r) - u_h^1(s)\phi^1(s-r) + u_h^1(s)\phi^1(s-r) - u_h^1(\gamma_h(s-r))|ds \]
\[ \leq \int_t^\infty k(t,s)|\phi^1(s)\phi^1(s-r) - u_h^1(s)|ds + \int_t^\infty k(t,s)|u_h^1(s)\phi^1(s-r) - u_h^1(\gamma_h(s-r))|ds \]
\[ \leq 2w(\phi_+, h, t). \tag{20} \]

Suppose that \( |\phi^{n-1}(t) - u_h^{n-1}(t)| \leq 2^{n-2}w(\phi_+, h, t) \) then, Equation (21),
\[ |\phi^n(t) - u_h^n(t)| = |A(\phi^{n-1})(t) - \mathcal{A}_h(u_h^{n-1})(t)| \]
\[ = \left| \int_t^\infty k(t,s)\phi^{n-1}(s)\phi^{n-1}(s-r)ds - \int_t^\infty k(t,s)u_h^{n-1}(s)u_h^{n-1}(\gamma_h(s-r))ds \right| \]
\[ \leq \int_t^\infty k(t,s)|\phi^{n-1}(s)\phi^{n-1}(s-r) - u_h^{n-1}(s)\phi^{n-1}(s-r) + u_h^{n-1}(s)\phi^{n-1}(s-r) - u_h^{n-1}(\gamma_h(s-r))|ds \]
\[ \leq \int_t^\infty k(t,s)|\phi^{n-1}(s)\phi^{n-1}(s-r) - u_h^{n-1}(\gamma_h(s-r))|ds + \int_t^\infty k(t,s)|u_h^{n-1}(s)\phi^{n-1}(s-r) - u_h^{n-1}(\gamma_h(s-r))|ds \]
\[ \leq 2^{n-2}w(\phi_+, h, t) + 2^{n-2}w(\phi_+, h, t) = 2^{n-1}w(\phi_+, h, t). \tag{21} \]

Notice that \( \lim_{h \to 0} w(\phi_+, h, t) = 0 \), so for every \( n \in \mathbb{N} \), \( \lim_{h \to 0} |\phi^n(t) - u_h^n(t)| = 0 \). Finally, we can write \( |\phi(t) - u_h(t)| \leq |\phi(t) - \phi^n(t)| + |u_h(t) - u_h^n(t)| + |\phi^n(t) - u_h^n(t)|. \) Therefore, for every \( n \in \mathbb{N} \) large enough, Equation (19) is satisfied.

4. Conclusions
This study establishes a new method for approximating traveling waves from delayed reaction-diffusion equations. We discretize the delay argument of the delayed KPP-Fisher equation and obtained a DEPCA. Our results show that as the length of the constancy interval of the delay argument of the DEPCA gets smaller, its solution converges uniformly to the traveling wave of the delayed KPP-Fisher equation. We suggest that this method of approximation by DEPCA may be replicated in other reaction-diffusion models, even without delay.

Acknowledgments
The authors would like to thank the doctoral scholarship of the Universidad Católica del Maule, Chile, for their support in the development of this work.
References
[1] Gourley S A 2000 Travelling front solutions of a nonlocal Fisher equation J. Math. Biol. 4 272
[2] Wu J H, Zou X F 2001 Traveling wave fronts of reaction-diffusion systems with delay J. Dynam. Diff. Eqs. 13 651
[3] Kolmogorov A N, Petrovsky I G, Piscounov N S 1937 Etude de lequation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique Bull. Univ. dEtat a Moscou, Ser. Intern. 1 1
[4] Fisher R A 1937 The wave of advance of advantageous genes Ann. of Eugenics 7 335
[5] Wang X, Lu Y 1990 Exact solutions of the extended Burgers-Fisher equation Chinese Physics Letters 7(4) 145
[6] Ai S 2007 Traveling wave fronts for generalized Fisher equations with spatio-temporal delays Journal of Differential Equations 232(1) 104
[7] Hadeler K P 2008 Transport, reaction, and delay in mathematical biology, and the inverse problem for traveling fronts Journal of Mathematical Sciences 149(6) 1658
[8] Gomez A, Trofimchuk S 2011 Monotone traveling wavefronts of the KPP-Fisher delayed equation Journal of Differential Equations 250(4) 1767
[9] Ashwin P, Bartuccelli M V, Bridges T J, Gourley S A 2002 Travelling fronts for the KPP equation with spatio-temporal delay Z. Angew. Math. Phys. 53 103
[10] Gyori I 1991 Approximation of the solution of delay differential equations by using piecewise constant arguments Internat. J. Math. Sci. 14 111
[11] Shah S M, Wiener J 1983 Advanced differential equations with piecewise constant argument deviations International Journal of Mathematics and Mathematical Sciences 6 671
[12] Cooke K L, Wiener J 1984 Retarded differential equations with piecewise constant delays Journal of Mathematical Analysis and Applications 99(1) 265
[13] Yuan R 2003 On the second-order differential equation with piecewise constant argument and almost periodic coefficients Nonlinear Analysis 52 1411
[14] Chiu K S, Li T 2019 Oscillatory and periodic solutions of differential equations with piecewise constant generalized mixed arguments Mathematische Nachrichten 292(10) 2153
[15] Arino O, Hbid M L, Dads E A 2007 Delay Differential Equations and Applications. Proceedings of the NATO Advanced Study Institute held in Marrakesh Morocco 9-21 September 2002 vol 205 (Netherlands: Springer)
[16] Fellner K, Tang B Q 2018 Convergence to equilibrium of renormalised solutions to nonlinear chemical reaction–diffusion systems. Zeitschrift für angewandte Mathematik und Physik 69(3) 54
[17] Hutchinson G E 1948 Circular causal systems in ecology Ann. New York Acad. Sci. 50 221
[18] Cooke K L, Wiener J 1984 Retarded differential equations with piecewise constant delays J. Math. Anal. Appl. 99 265