Spiky strings on $AdS_3 \times S^3$ with NS-NS flux

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Abstract: We study rigidly rotating strings in the background of $AdS_3 \times S^3$ with Neveu-Schwarz (NS) fluxes. We find two interesting limiting cases corresponding to the known giant magnon and the new single spike solution of strings in the above background and write down the dispersion relations among various conserved charges. We use proper regularization to find the correct relations among them. We further study the circular strings and infinite spikes on AdS and study their properties.

Keywords: AdS-CFT correspondence, Bosonic Strings.
1. Introduction

AdS/CFT duality \[1\] \[2\] \[3\] relates spectrum of free strings in the string theory side to the operator dimensions in the gauge theory in planar limit. Though proving the duality has been a quite challenging subject primarily due to that fact there are infinite tower of states in the string theory side, in the last few years the conjectured duality has been tested in various limits on both sides of the correspondence, such as the large charge sectors \[4\] \[5\] \[6\] \[7\]. With the realization that the idea of counting the operators from gauge theory side can be elegantly formulated in terms of an integrable spin chain \[8\] , it has further been established that integrability played an important role on both sides of the duality, since the dual string theory also needs to be integrable in the semiclassical limit \[9\] \[10\] \[11\] \[12\] \[13\]. In this connection, Hofman and Maldacena(HM) put forward a special limit \[14\], using which determining the spectrum on both sides of the duality became much easier. With these ideas, various large classes of rigidly rotating and pulsating string solutions have been studied in AdS and non-AdS backgrounds. These studies have also been successful in providing a realization of the string states which would correspond to some exact operators in the gauge theory side (see, for example \[15\] \[16\] \[17\] \[18\] \[19\] \[20\] \[21\] \[22\] \[23\] \[24\] \[25\] \[26\] \[27\] \[28\] \[29\] \[30\]). In HM construction, the spectrum on the field theory side consists of an elementary excitation, the so called magnon which carries a momentum \(p\) along the finitely or infinitely long spin chain \[31\] \[32\]. The dual string state, derived from the rigidly rotating string in the \(\mathbb{R} \times S^3\) presenting the same dispersion relation in the large \(\sqrt{\lambda}\) 't Hooft limit, is known as the giant magnon. However, a more general class of rotating string solutions were also found out in \[33\] which are dual to a higher twist operators in
the boundary field theory. These kind of solutions are called spiky strings. In addition to the rotating strings, the spinning and folded strings have also been found out to have exact correspondence with the dual operators in the gauge theory[34][35]. Indeed a large class of rotating and pulsating string solution in various backgrounds have been studied and the dual operators have also been examined carefully. More recently string theory in the background of $\text{AdS}_3 \times S^3 \times T^4$ with mixed R-R and NS-NS flux have been shown to be integrable and a S-matrix has also been proposed[36][37][38]. Furthermore the giant magnon solution has been studied solving the principal chiral model and the modified dispersion relation has been presented to be

$$E - J_1 = \sqrt{(J_2 - qTp)^2 + 4T^2(1 - q^2)\sin^2\frac{p}{2}},$$

where $p$ is the worldsheet momentum and $T$ is the tension of the string and $q$ parametrizes the Neveu-Schwarz flux. It has been clear that for the interpolating case of $q \neq 1$ the spectrum cannot be detailed using either the WZW model approach ($q = 1$, pure NS-NS) or the Bethe ansatz approach ($q = 0$, pure R-R). Thus the investigation of string states in the mixed flux backgrounds was initiated expecting to find some relation between these two approaches. However, this giant magnon solution (1.1) was expected to correspond to the appropriate operator in the dual theory. The finite size corrections for this giant magnon has also been proposed in [39]. Further in [40] more folded spinning string solutions in this background have been studied. It was also shown, by using SO (2,2) transformation and re-parametrization, that these spinning and folded strings can be related to light like Wilson loops in $\text{AdS}_3$ with Neveu-Schwarz flux. In this work we would like to generalize these ideas further. First of all we solve the string equations of motion in the background of $\mathbb{R} \times S^3$ with NS-NS flux and find two limiting case solutions, one corresponding to the giant magnon[38] and the other corresponding to a new single spike solution. For these cases, the dispersion relation among the Noether charges and deficit angle gets modified by a ‘shift term’ proportional to the worldsheet momenta due to the presence of the NS-NS flux, as shown in [38]. Further we generalize the magnon and spiky string solutions by turning on NS-NS fields both in the $\text{AdS}$ and sphere part, and study more general class of rotating open string solutions. We also study several circular and infinite spike string configurations and find the energy-spin relation with a discussion about the interplay between them.

The rest of the paper is organized as follows. In section-2 we study the general solution for the rigidly rotating strings in the background $\mathbb{R} \times S^3$ with NS-NS flux and find both giant magnon and single spike solutions as two limiting cases. Section-3 is devoted to the study of rotating, circular and helical open strings in the background of $\text{AdS}_3 \times S^3$ with NS-NS flux both on $\text{AdS}$ and on sphere. Finally in section-4 we present our conclusions.

2. Spiky strings in $\mathbb{R} \times S^3$ with NS-NS flux

We start with the metric and the background field for $\text{AdS}_3 \times S^3 \times T^4$ geometry supported by the NS-NS fluxes. This geometry can be obtained by taking the near horizon limit of the $\text{NS1} – 5\text{S}$ background, see for example [11]. The full $\text{AdS}_3 \times S^3$ metric with NS-NS
B-fields is as follows
\[ ds^2 = -\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\phi^2 + d\theta^2 + \sin^2 \theta d\phi_1^2 + \cos^2 \theta d\phi_2^2, \]
\[ b_{t\phi} = q \sinh^2 \rho, \quad b_{\phi_1\phi_2} = -q \cos^2 \theta. \] (2.1)

In this section we wish to review the giant magnon solution proposed in [38] and find a new single spike solution of string equations of motion in the background of \( \mathbb{R} \times S^3 \) with NS-NS B-field. The relevant metric and background field is given by (putting \( \rho = 0 \))
\[ ds^2 = -dt^2 + d\theta^2 + \sin^2 \theta d\phi_1^2 + \cos^2 \theta d\phi_2^2, \quad b_{\phi_1\phi_2} = -q \cos^2 \theta. \] (2.2)

The Polyakov action of a fundamental string in the conformal gauge for the said background can be written as
\[ S = \frac{T}{2} \int d\tau d\sigma \left[ -(\dot{t}^2 - \dot{\tau}^2) + \dot{\theta}^2 - \theta'^2 + \sin^2 \theta (\dot{\phi}_1^2 - \phi_1'^2) + \cos^2 \theta (\dot{\phi}_2^2 - \phi_2'^2) - 2q \cos^2 \theta (\dot{\phi}_1 \phi'_2 - \phi_1' \dot{\phi}_2) \right], \] (2.3)
where the ‘dot’ and ‘prime’ denote the derivatives with respect to \( \tau \) and \( \sigma \) respectively and \( T = \frac{\sqrt{\lambda}}{2\pi} \), where \( \lambda \) is the ‘t Hooft coupling constant. We write the ansatz for the rigidly rotating string as
\[ t = \kappa \tau, \quad \theta = \theta(y), \quad \phi_1 = \omega_1[\tau + g_1(y)], \quad \phi_2 = \omega_2[\tau + g_2(y)], \] (2.4)
where \( y = a\sigma - b\tau \). The equations of motion for \( \phi_1 \) and \( \phi_2 \) give the following
\[ g_{1y} = \frac{1}{a^2 - b^2} \left[ \frac{A_1 + a \omega_2 q}{\omega_1 \sin^2 \theta} - aq \frac{\omega_2}{\omega_1} - b \right], \]
\[ g_{2y} = \frac{1}{a^2 - b^2} \left[ \frac{A_2}{\omega_2 \cos^2 \theta} - aq \frac{\omega_1}{\omega_2} - b \right], \] (2.5)
where \( g_y = \frac{\partial g}{\partial y} \). Solving the equation of motion for \( \theta \), we get
\[ (a^2 - b^2)^2 \theta_y^2 = -a^2(1 - q^2)(\omega_1^2 - \omega_2^2) \sin^2 \theta - \frac{(A_1 + a \omega_2 q)^2}{\sin^2 \theta} - \frac{A_2^2}{\cos^2 \theta} + C, \] (2.6)
where \( C \) is an integration constant. Invoking the boundary condition \( \frac{\partial \theta}{\partial y} \rightarrow 0 \) at \( \theta = \frac{\pi}{2} \), we get
\[ A_2 = 0, \quad C = a^2(1 - q^2)(\omega_1^2 - \omega_2^2) + (A_1^2 + a q \omega_2)^2. \] (2.7)

Putting the equation (2.6) in the Virasoro constraint \( T_{\tau \sigma} = 0 \) we get
\[ C = \frac{\omega_1}{b} (a^2 + b^2)(A_1 + aq \omega_2) + 2aq \omega_2(A_1 + aq \omega_2) - \left[ \frac{1}{b} aq \omega_1 \omega_2 (a^2 + b^2) + a^2(q^2 \omega_1^2 + \omega_2^2) \right]. \] (2.8)

Equating (2.7) and (2.8), we get a quadratic equation for \( A_1 \) and solving that we get the following limiting values of \( A_1 \)
\[ A_1 = b \omega_1 \quad \text{Giant Magnon solution} \] (2.9)
\[ = \frac{a^2 \omega_1}{b} \quad \text{Single Spike solution} \] (2.10)
2.1 First limiting case: Giant Magnon solution

Using $A_1 = b\omega_1$ in the equation (2.6) and (2.7), we get

$$\theta_y = \frac{\Omega}{a^2 - b^2} \cot \theta \sqrt{\sin^2 \theta - \sin^2 \theta_0},$$  \hspace{1cm} (2.11)

where, $\Omega^2 = a^2(1 - q^2)(\omega_1^2 - \omega_2^2)$ and $\sin \theta_0 = \frac{b\omega_1 + aq\omega_2}{\pi}$. Now the conserved charges and deficit angle become

$$E = \kappa T \int d\sigma,$$
$$J_1 = \frac{T}{a^2 - b^2} \int d\sigma \left[ \omega_1 a^2(1 - q^2) \sin^2 \theta + a^2 q^2 \omega_1 - \omega_1 b^2 \right],$$
$$J_2 = \frac{T}{a^2 - b^2} \int d\sigma \left[ \omega_2 a^2(1 - q^2) + (a^2 q^2 \omega_2 + abq\omega_1) \frac{1}{\sin^2 \theta} \right] \cos^2 \theta,$$
$$\Delta \phi_1 = \frac{b\omega_1 + aq\omega_2}{a^2 - b^2} \int dy \frac{\cos^2 \theta}{\sin^2 \theta}.$$  \hspace{1cm} (2.12)

Here, we note that integrals corresponding to both $E$ and $J_1$ diverge, however their difference is finite. The above charges can be shown to satisfy the following relation

$$\frac{E}{\kappa} - J_1 = \frac{J_2 - qT\Delta \phi_1}{\omega_2}.$$  \hspace{1cm} (2.13)

Evaluating the right side of the above equation we get

$$\frac{J_2 - qT\Delta \phi_1}{\omega_2} = \frac{1}{\omega_1} \sqrt{(J_2 - qT\Delta \phi_1)^2 + 4T^2(1 - q^2)(1 - z_0^2)},$$  \hspace{1cm} (2.14)

where $z_0 = \sin \theta_0$ and $\Delta \phi_1 = \pi - 2\theta_0$. Now we can rewrite the equation (2.13) as

$$\tilde{E} - J_1 = \sqrt{(J_2 - qT\Delta \phi_1)^2 + 4T^2(1 - q^2) \sin^2 \frac{\Delta \phi_1}{2}},$$  \hspace{1cm} (2.15)

where $\tilde{E} = \frac{\kappa}{\kappa} E$. This dispersion relation (2.15) was obtained in [38] by solving the principal chiral model along with a Wess-Zumino term in this background. It is evident that the presence of the term linear in $\Delta \phi_1$ (which is identified to the worldsheet momentum $p$) spoils the periodicity of the dispersion relation, which is a completely new feature of these solutions. Although, the implication of this in the dual picture is not clear.

It is quite evident that the dispersion relation reduces to the usual dyonic giant magnon relation [32] in the $q = 0$ case, which is interpreted as a bound state of $J_2$ number of elementary magnons. In the other limit $q = 1$, it was shown in [38] that with $J_2 = 1$, the magnon dispersion relation simply corresponds to the energy state $\epsilon = 1 - Tp$, which occurs in the perturbative S-matrix as discussed in [37].

2.2 Second limiting case: Single spike solution

Now we take $A_1 = a^2 b\omega_1$, which makes the $\theta$ equation

$$\theta_y = \frac{\Omega}{a^2 - b^2} \cot \theta \sqrt{\sin^2 \theta - \sin^2 \theta_1},$$  \hspace{1cm} (2.16)
where, \( \sin \theta_1 = \frac{a^2 \omega_1 + abq \omega_2}{b_1} \). The conserved charges and deficit angles are

\[
E = \kappa T \int d\sigma,
\]

\[
J_1 = -\frac{T}{a^2 - b^2} \int d\sigma \, \omega_1 a^2 (1 - q^2) \cos^2 \theta,
\]

\[
J_2 = \frac{T}{a^2 - b^2} \int d\sigma \left[ \omega_2 a^2 (1 - q^2) + (a^2 q^2 \omega_2 + a^3 q \omega_1) \frac{1}{\sin^2 \theta} \right] \cos^2 \theta,
\]

\[
\Delta \phi_1 = \frac{a}{a^2 - b^2} \int dy \left[ \frac{a^2 \omega_1 + abq \omega_2}{b \sin^2 \theta} - (aq \omega_2 + b \omega_1) \right].
\]

(2.17)

From above charges we can see that while both \( E \) and \( \Delta \phi_1 \) diverge, the combination \( E - T \Delta \phi_1 \) remains finite, and can be written as

\[
E - T \Delta \phi_1 = 2T(\frac{\pi}{2} - \theta_1).
\]

(2.18)

The dispersion relation between the angular momenta can be written as follows,

\[
J_1 = \sqrt{[J_2 - qT(\Delta \phi_1)_{reg}]^2 + 4T^2(1 - q^2) \sin^2 \frac{(\Delta \phi_1)_{reg}}{2}},
\]

(2.19)

where

\[
J_1 = 2T \omega_1 \sqrt{\frac{1 - q^2}{\omega_1^2 - \omega_2^2}} \cos \theta_1,
\]

\[
J_2 = -2T \omega_2 \sqrt{\frac{1 - q^2}{\omega_1^2 - \omega_2^2}} \cos \theta_1 - 2qT(\pi - 2\theta_1),
\]

\[
(\Delta \phi_1)_{reg} = -2 \cos^{-1} \sin \theta_1,
\]

(2.20)

And the expression of \( \Delta \phi_1 \) has been regularized by the energy in the equation (2.17). These relations (2.18) and (2.19) can be thought of as the generalized spike solution in the presence of background flux. Note that by putting \( q = 0 \) one arrives at the single spike solution obtained in [32]. Of course the \( q = 1 \) limit of the above solution remains to be understood in a better way as before.

3. Strings in \( AdS_3 \times S^3 \) with Neveu-Schwarz flux

In this section we will generalize the solutions presented in the last section by turning on one spin \( S \) along AdS and two angular momenta \( J_1, J_2 \) along the \( S^3 \) of the full geometry \( AdS_3 \times S^3 \) with Neveu-Schwarz fluxes turned on both in the AdS and the sphere part as written in (2.1). The Polyakov action of the fundamental string in conformal gauge for this background is written as

\[
S = \frac{T}{2} \int d\tau d\sigma \left[ -\cosh^2 \rho(t^2 - t') + \rho^2 - \rho'^2 + \sinh^2 \rho (\dot{\phi}^2 - \phi'^2) + \dot{\theta}^2 \right]
\]
\[-\theta'^2 + \sin^2 \theta (\phi_1'^2 - \phi_1'^2) + \cos^2 \theta (\phi_2'^2 - \phi_2'^2) + 2q \sinh^2 \rho (\dot{t} \phi' - t' \dot{\phi}) \]
\[-2q \cos^2 \theta (\dot{\phi}_1 \phi_2' - \dot{\phi}_1' \phi_2) \]. \hspace{1cm} (3.1)

We take the following ansatz to parameterize the three spin rigidly rotating open strings which have one spin $S$ in the AdS and two angular momenta in the $S^3$ part as,

\[ t = \tau + h_1(y), \quad \rho = \rho(y), \quad \phi = \omega_1[\tau + h_2(y)], \]
\[ \phi_1 = \tau + h_3(y), \quad \theta = \theta(y), \quad \phi_2 = \omega_2[\tau + h_4(y)]. \hspace{1cm} (3.2) \]

Solving the equations of motion for $t, \phi, \phi_1$ and $\phi_2$, we have the following differential equations for $h_1, h_2, h_3$ and $h_4$

\[ h_{1y} = \frac{1}{a^2 - b^2} \left[ \frac{A_1 - a\omega_1 q}{\cosh^2 \rho} + a\omega_1 q - b \right], \]
\[ h_{2y} = \frac{1}{a^2 - b^2} \left[ \frac{A_2}{\sinh^2 \rho} + \frac{aq}{\omega_1} - b \right], \]
\[ h_{3y} = \frac{1}{a^2 - b^2} \left[ \frac{A_3 + a\omega_2 q}{\sin^2 \theta} - a\omega_2 q - b \right], \]
\[ h_{4y} = \frac{1}{a^2 - b^2} \left[ \frac{A_4}{\cos^2 \theta} - \frac{aq}{\omega_2} - b \right], \hspace{1cm} (3.3) \]

where $h_y = \frac{\partial h}{\partial y}$. Also, The substraction of Virasoro constraints $T_{\tau \sigma} = 0$ and $T_{\tau \tau} + T_{\sigma \sigma} = 0$ gives the following relation among various constants appearing in the differential equations

\[ -A_1 + A_2 \omega_1^2 + A_3 + A_4 \omega_2^2 = 0. \hspace{1cm} (3.4) \]

Now, the equation of motion for $\rho$ and $\theta$ become

\[ (a^2 - b^2)\rho_{yy} + \frac{1}{a^2 - b^2} \sinh \rho \cosh \rho \left[ \frac{(A_1 - a\omega_1 q)^2}{\cosh^4 \rho} - \frac{\omega_1^2 A_2^2}{\sinh^4 \rho} - a^2 (1 - q^2)(1 - \omega_1^2) \right] = 0, \hspace{1cm} (3.5) \]
\[ (a^2 - b^2)\theta_{yy} + \frac{1}{a^2 - b^2} \sin \theta \cos \theta \left[ a^2 (1 - q^2)(1 - \omega_2^2) - \frac{(A_3 + a\omega_2 q)^2}{\sin^4 \theta} + \frac{\omega_2^2 A_4^2}{\cos^4 \theta} \right] = 0, \hspace{1cm} (3.6) \]

where $\rho_{yy} = \frac{\partial^2 \rho}{\partial y^2}$ and $\theta_{yy} = \frac{\partial^2 \theta}{\partial y^2}$. We can then write the conserved charges as

\[ E = T \int d\sigma \left[ (1 - bh_{1y}) \cosh^2 \rho - a\omega_1 q h_{2y} \sinh^2 \rho \right], \]
\[ S = T \int d\sigma \left[ \omega_1 (1 - bh_{2y}) \sinh^2 \rho - aq h_{1y} \sinh^2 \rho \right], \]
\[ J_1 = T \int d\sigma \left[ (1 - bh_{3y}) \sin^2 \theta - a\omega_2 q h_{4y} \cos^2 \theta \right], \]
\[ J_2 = T \int d\sigma \left[ \omega_2 (1 - bh_{4y}) \cos^2 \theta + aq h_{3y} \cos^2 \theta \right]. \hspace{1cm} (3.7) \]

Next, by choosing particular value of the constants as in the previous section, we would like to look for giant magnon and single spike solutions for the string and further look for the circular and helical string configurations.
3.1 Giant Magnon Solution

For finding out the giant magnon solution, we choose the integration constants as $A_1 = b = A_3$ and $A_2 = 0 = A_4$. The solution of equations (3.5) and (3.6) become

$$\rho_y^2 = \frac{1}{(a^2 - b^2)^2} \left[ a^2 (1 - q^2) (1 - \omega_1^2) - \frac{(b - a \omega_1 q)^2}{\cosh^2 \rho} \right] \sinh^2 \rho, \quad (3.8)$$

and

$$\theta_y^2 = \frac{1}{(a^2 - b^2)^2} \left[ a^2 (1 - q^2) (1 - \omega_2^2) - \frac{(b + a \omega_2 q)^2}{\sin^2 \theta} \right] \cos^2 \theta. \quad (3.9)$$

Note that the above equations can be rewritten as

$$\rho_y = \frac{\Omega_1}{a^2 - b^2} \tanh \rho \sqrt{\cosh^2 \rho - \cosh^2 \rho_0}, \quad (3.10)$$

and

$$\theta_y = \frac{\Omega_2}{a^2 - b^2} \cot \theta \sqrt{\sin^2 \theta - \sin^2 \theta_0}, \quad (3.11)$$

where, $\Omega_1^2 = a^2 (1 - q^2) (1 - \omega_1^2)$, $\Omega_2^2 = a^2 (1 - q^2) (1 - \omega_2^2)$, $\cosh \rho_0 = \frac{b - a \omega_1 q}{\Omega_1}$, and $\sin \theta_0 = \frac{b + a \omega_2 q}{\Omega_2}$. Now the conserved charges (3.7) and deficit angles become

$$E = \frac{T}{a^2 - b^2} \int d\sigma \left[ a^2 (1 - q^2) \cosh^2 \rho + a^2 q^2 - b^2 \right],$$

$$S = \frac{T}{a^2 - b^2} \int d\sigma \left[ a^2 (1 - q^2) + (a^2 q^2 - \frac{abq}{\omega_1}) \right] \frac{1}{\cosh^2 \rho} \sinh^2 \rho,$$

$$J_1 = \frac{T}{a^2 - b^2} \int d\sigma \left[ a^2 (1 - q^2) \sin^2 \theta + a^2 q^2 - b^2 \right],$$

$$J_2 = \frac{T}{a^2 - b^2} \int d\sigma \left[ a^2 (1 - q^2) + (a^2 q^2 + \frac{abq}{\omega_2}) \right] \frac{1}{\sin^2 \theta} \cos^2 \theta,$$

$$\Delta \phi_1 = \frac{1}{a^2 - b^2} \int dy \left( b + a \omega_2 q \right) \frac{\cos^2 \theta}{\sin^2 \theta},$$

$$\Delta t = \frac{1}{a^2 - b^2} \int dy \left( a \omega_1 q - b \right) \frac{\sinh^2 \rho}{\cosh^2 \rho}. \quad (3.12)$$

Here $\Delta t$ is the time difference between two endpoints of the string. It is clear from the above expressions that we have the following relation among various conserved charges

$$E - J_1 = \frac{S - qT \Delta t}{\omega_1} + \frac{J_2 - qT \Delta \phi_1}{\omega_2}. \quad (3.13)$$

As the conserved charges are divergent, we regularize them to remove the divergent part. Let us write

$$\frac{S - qT \Delta t}{\omega_1} = \frac{2Ta}{a^2 - b^2} \int_0^\infty \frac{dp}{\rho_y} (1 - q^2) \sinh^2 \rho = 2T \sqrt{\frac{1 - q^2}{1 - \omega_1^2}} \int_1^\infty dz \frac{z}{\sqrt{z^2 - z_0^2}}, \quad (3.14)$$

where $z = \cosh \rho$ and $z_0 = \cosh \rho_0 = \frac{b - a \omega_1 q}{\Omega_1}$. From the equations (3.12) and (3.13), we can see that $E$ and $J_1$ have divergences in the IR whereas $E - J_1$ has a divergence in UV.
which corresponds to $\rho = \infty$ i.e the AdS boundary. So the divergence can be canceled by generating a counter term by deforming the integration contour away from $\rho = \infty$. This can be visualized in the sense that these strings never reach out to the boundary of $AdS_3$.

Introducing a hard cut-off to regulate the UV divergence in the integral and following the prescription in \cite{42}, we have the regularized value given as

$$\frac{(S - qT\Delta t)}{\omega_1}^{\text{reg}} = -2T \sqrt{\frac{1 - q^2}{1 - \omega_1^2}} \sqrt{1 - z_0^2}. \quad (3.15)$$

From the above expression, we find the following relation

$$\frac{(S - qT\Delta t)}{\omega_1}^{\text{reg}} = -\sqrt{(S - qT\Delta t)^2 + 4T^2(1 - q^2)(1 - z_0^2)}. \quad (3.16)$$

Similarly it is evident that

$$\frac{J_2 - qT\Delta \phi_1}{\omega_2} = \frac{2T a}{a^2 - b^2} \int_{\theta_0}^{\pi/2} \frac{d\theta}{\omega_2} \left(1 - q^2\right) \cos^2 \theta = 2T \sqrt{\frac{1 - q^2}{1 - \omega_2^2}} \int_{x_0}^{1} \frac{dx}{\sqrt{x^2 - x_0^2}}, \quad (3.17)$$

where $x = \sin \theta$ and $x_0 = \sin \theta_0 = \frac{b + a\omega_2 \Omega_2}{\omega_2}$. Which gives

$$\frac{J_2 - qT\Delta \phi_1}{\omega_2} = 2T \sqrt{\frac{1 - q^2}{1 - \omega_2^2}} \sqrt{1 - x_0^2}. \quad (3.18)$$

Further, the above expression can be written as

$$\frac{J_2 - qT\Delta \phi_1}{\omega_2} = \sqrt{(J_2 - qT\Delta \phi_1)^2 + 4T^2(1 - q^2)(1 - x_0^2)}. \quad (3.19)$$

Now the time difference is given by

$$\Delta t = 2 \frac{a\omega_1 q - b}{\Omega_1} \int_{0}^{\infty} d\rho \frac{\tanh \rho}{\cosh^2 \rho - \cosh^2 \rho_0} = -2 \tan^{-1} \frac{z_0}{\sqrt{1 - z_0^2}}. \quad (3.20)$$

Similarly the deficit angle for $\phi_1$ is

$$\Delta \phi_1 = 2 \frac{a\omega_2 q + b}{\Omega_2} \int_{\theta_0}^{\pi/2} d\theta \frac{\cot \theta}{\sqrt{\sin^2 \theta - \sin^2 \theta_0}} = \pi - \frac{\theta_0}{2}. \quad (3.21)$$

Now using equations \cite{42},\cite{19},\cite{20} and \cite{21}, we can rewrite the equation \cite{13} as magnon dispersion relation

$$(E - J_1)^{\text{reg}} = -\sqrt{(S - qT\Delta t)^2 + 4T^2(1 - q^2)\cos^2 \frac{\Delta t}{2}} + \sqrt{(J_2 - qT\Delta \phi_1)^2 + 4T^2(1 - q^2)\sin^2 \frac{\Delta \phi_1}{2}}. \quad (3.22)$$

Note that by putting $q = 0$ we get back the dispersion relation for the three spin giant magnon solution presented in \cite{13}. The above dispersion relation seemingly gives us a
collection of magnon bound states with momentum $\Delta \phi_1$ with another collection of magnon bound states with momentum $\Delta t + \pi$. However here both the number of magnons in the bound states are somehow shifted by the ‘kink’ charges proportional to the ‘momenta’. The total dispersion relation is completely non-periodic in these momenta owing to the presence of the terms as explained above. For the WZW limit it can be expected that naively putting $q = 1$ into the dispersion relation might not be useful. The fate of these open string solutions in that limit has to be investigated in a more rigorous way.

### 3.2 Spike Solution

To obtain the single spike-like solution, we chose the integration constants as: $A_1 = \frac{a_2}{b} = A_3$ and $A_2 = 0 = A_4$. The solution of equations (3.5) and (3.6) now becomes

$$\rho_y = \frac{\Omega_1}{a_2^2 - b^2} \tanh \rho \sqrt{\cosh^2 \rho - \cosh^2 \rho_1}, \quad (3.23)$$

and

$$\theta_y = \frac{\Omega_2}{a_2^2 - b^2} \cot \theta \sqrt{\sin^2 \theta - \sin^2 \theta_1}, \quad (3.24)$$

where, $\cosh \rho_1 = \frac{a_2^2 - ab \omega_1 q}{b a_2}$, and $\sin \theta_1 = \frac{a_2^2 + ab \omega_2 q}{b a_2}$. Now the conserved charges (3.7) and deficit angles become

$$E = \frac{T}{a_2^2 - b^2} \int d\sigma \left[ a_2^2 (1 - q^2) \sinh^2 \rho \right],$$

$$S \omega_1 = \frac{T}{a_2^2 - b^2} \int d\sigma \left[ a_2^2 (1 - q^2) + \left( a_2^2 q^2 - \frac{a_3}{b \omega_1} \right) \frac{1}{\cosh^2 \rho} \right] \sinh^2 \rho,$$

$$J_1 = -\frac{T}{a_2^2 - b^2} \int d\sigma \left[ a_2^2 (1 - q^2) \cos^2 \theta \right],$$

$$J_2 \omega_2 = \frac{T}{a_2^2 - b^2} \int d\sigma \left[ a_2^2 (1 - q^2) + \left( a_2^2 q^2 + \frac{a_3}{b \omega_2} \right) \frac{1}{\sin^2 \theta} \right] \cos^2 \theta,$$

$$\Delta \phi_1 = \int dy \left[ \frac{a_2^2 + ab \omega_2 q}{a_2^2 b - b^3} \frac{\cos^2 \theta}{\sin^2 \theta} + \frac{1}{b} \right],$$

$$\Delta t = \int dy \left[ \frac{ab \omega_1 q - a_2^2}{a_2^2 b - b^3} \frac{\sin^2 \rho}{\cosh^2 \rho} + \frac{1}{b} \right]. \quad (3.25)$$

From (3.25), we get the following spike-like relation between the conserved charges

$$\omega_1 E - S - q T (\Delta \phi_1 - \Delta t) = 2q T (\frac{\pi}{2} - \theta_1). \quad (3.26)$$

For the sake of completeness we wish to compute $J_1$ and $J_2$ as

$$J_1 = 2T \sqrt{\frac{1 - q^2}{1 - \omega_1^2}} \cos \theta_1,$$

$$J_2 = -2T \omega_2 \sqrt{\frac{1 - q^2}{1 - \omega_2^2}} \cos \theta_1 - q T (\pi - 2\theta_1). \quad (3.27)$$

From (3.27), we have the following relation between $J_1$ and $J_2$

$$\omega_2 J_1 + J_2 = q T (2\theta_1 - \pi). \quad (3.28)$$
3.3 Circular Strings

In this section, we describe the rotating circular string which is rotating at a fixed $\rho$ value. For this we choose the integration constants as $A_4 = 0$ and $A_3 = b$. Now form the equation $\surd \rho$ we can get $A_1 = A_2 \omega_1^2 + b$. Using above constant values in the equations (3.3) and (3.6), we get

$$A_1 = A_2 \omega_1^2 + b.$$  

where $\alpha = \frac{b + aq\omega_1}{\sqrt{a^2(1-q^2)(1-\omega_1^2)}}$ and $\theta$ value runs from $\frac{\pi}{2}$ to $\theta_{max} = \sin^{-1} \alpha$. The solution for the above equation is

$$\cos \theta = \sqrt{1 - \alpha^2} \cosh \beta y,$$  

where $\beta = \frac{1}{a^2 - b^2} \sqrt{a^2(1-q^2)(1-\omega_1^2) - (b + aq\omega_2)^2}$. From the solution, we can see that at $\tau = 0$, $\theta = \frac{\pi}{2}$ corresponds to $\sigma = \pm \infty$ and $\theta_{max}$ corresponds to $\sigma = 0$, which shows the range of sigma to be $-\infty < \sigma < \infty$. Now the solution to equation (3.5) can be written as

$$(a^2 - b^2)\rho_y^2 = a^2(1-q^2)(1-\omega_1^2) \sinh^2 \rho + \frac{(A_1 - aq\omega_1)}{\cosh^2 \rho} - \frac{A_2^2 \omega_1^2}{\sinh^2 \rho} + C,$$  

where $C$ is the integration constant can be chosen from the $T_{\tau\sigma} = 0$ Virasoro constraint which is

$$C = 2aq\omega_1(A_1 - A_2) - a^2q^2\omega_1^2 - b^2.$$  

Now, we can write the equation (3.35) as

$$\rho_y = \pm \sqrt{\frac{A}{(a^2 - b^2) \cosh \rho \sinh \rho}},$$  

where

$$A = \sqrt{M \sinh^6 \rho + N \sinh^4 \rho + P \sinh^2 \rho + R},$$  

$$M = a^2(1-q^2)(1-\omega_1^2),$$  

$$N = a^2(1-q^2 - \omega_1^2) - b^2 + 2aq\omega_1 [b + A_2(\omega_1^2 - 1)],$$  

$$P = 2bA_2\omega_1^2 - 2aq\omega_1 A_2 + \omega_1^2 A_2^2(\omega_1^2 - 1),$$  

$$R = -A^2_2 \omega_1^2.$$  

The above equation can be written as

$$A = \sqrt{(x - r_1)(x - r_2)(x - r_3)},$$  

where $x = \sinh^2 \rho$ and $r_{i=1,2,3} = r_i(a, b, q, \omega_1, A_2)$ are the roots of the above polynomial. The equation (3.35) gives the minimum value for $\rho_{min} = \sinh^{-1} \sqrt{r_1}$, where $r_1$ is the
smallest root of the polynomial which corresponds to the minimum value of $AdS$ radius i.e. $\rho_{\text{min}}$. Using equation (3.3) and (3.33), we can write

$$\frac{\partial \rho}{\partial \phi} = \frac{A \sinh \rho}{[A_2 \omega_1 + (a q - b \omega_1) \sinh^2 \rho] \cosh \rho}. \quad (3.36)$$

The above differential equation describes the shape of the string on $AdS$ part. The simple solution of the above equation (3.36) is given by the string located at $\rho = \rho_{\text{min}}$ where $A$ is zero. From the equation (3.3) at a fixed time $\tau = 0$ where $y = a \sigma$, the string configuration in $\phi$-direction is given by

$$\phi = \frac{a \omega_1}{a^2 - b^2} \left[ \frac{A_2}{\omega_1} + \frac{b A_2}{r_1} \right] \sigma. \quad (3.37)$$

We can notice that $\phi$ ranges as $\sigma$, which is $-\infty < \phi < \infty$. So the solution corresponds to the circular string having infinite number of windings. The conserved charges and deficit angles for this configuration are given by

$$E = \frac{T}{a^2 - b^2} \int d\sigma \left[ a^2 (1 - q^2) \sinh^2 \rho_{\text{min}} + a^2 - b^2 - A_2 b \omega_1^2 - A_2 a q \omega_1 \right],$$

$$S = \frac{T}{a^2 - b^2} \int d\sigma \left[ a^2 (1 - q^2) \sinh^2 \rho_{\text{min}} - b A_2 - (A_2 a q \omega_1 + \frac{ab q}{\omega_1} - a^2 q^2) \frac{\sinh^2 \rho_{\text{min}}}{\cosh^2 \rho_{\text{min}}} \right],$$

$$J_1 = \frac{T}{a^2 - b^2} \int d\sigma \left[ a^2 - b^2 - a^2 (1 - q^2) \cos^2 \theta \right],$$

$$J_2 = \frac{T}{a^2 - b^2} \int d\sigma \left[ a^2 - b^2 - a^2 q^2 + \frac{ab q}{\omega_2} \cos^2 \theta \right],$$

$$\Delta \phi_1 = \int d\sigma \frac{b + a q \omega_2}{a^2 - b^2} a \cot^2 \theta,$$

$$\Delta t = \int d\sigma \frac{a}{a^2 - b^2} \left[ \frac{A_2 \omega_1^2}{\cosh^2 \rho_{\text{min}}} - (b - a q \omega_1) \frac{\sinh^2 \rho_{\text{min}}}{\cosh^2 \rho_{\text{min}}} \right]. \quad (3.38)$$

Now choosing $\omega_1^2 = 1 - b^2$ and $A_2 = \frac{2}{T}$, from the above equations we can see the relation among the conserved quantities as

$$E - \frac{S - q T \Delta t}{\omega_1} = \frac{a^2 + b^2}{a^2 - b^2} \left[ \frac{J_1}{\omega_1} + \frac{J_2 - q T \Delta \phi_1}{\omega_2} \right]. \quad (3.39)$$

Note that the above relation reduces to the one mentioned in [24] in the limit $q = 0$.

### 3.4 Infinite spikes on $AdS$

The equation (3.36) also describes another kind of string solution where the string extends from $\rho_{\text{min}}$ to $\infty$ with the infinite winding number and the infinite angular momentum in the $\phi$ direction. As the string is now extended in the radial direction of $AdS$, $\rho$ is not fixed anymore and becomes the function of $\sigma$ (as we are considering the case at $\tau = 0$). In the asymptotic region, the solution to the equation (3.33) is in the form $\sigma - C \sim e^{-\rho}$, where $C$ is an integration constant. So when $\rho \to \infty$, $\sigma$ goes to $C$, which tells us $\rho_{\text{min}} < \rho < \infty$ covers the finite range of $\sigma$ which corresponds to that of one $AdS$ spike solution. To cover
all $\sigma$, we must include an array of infinite spikes on the AdS. As the integral range of $\rho$ covers only one spike, we can write conserved charges and deficit angles for a single AdS spike as

$$E = 2T \int_{\rho_{\min}}^{\infty} d\rho \left[ a(1 - q^2) \sinh^2 \rho + a - \frac{b^2}{a} - A_2 b \frac{\omega_1}{a} - A_2 q \omega_1 \right] \cosh \rho \sinh \rho \frac{A}{A},$$

$$S_{/\omega_1} = 2T \int_{\rho_{\min}}^{\infty} d\rho \left[ a(1 - q^2) \sinh^2 \rho - \frac{b}{a} A_2 - (A_2 q \omega_1 + \frac{bq}{\omega_1} - a q^2) \tanh^2 \rho \right] \cosh \rho \sinh \rho \frac{A}{A},$$

$$\Delta t = 2 \int_{\rho_{\min}}^{\infty} d\rho \left[ \frac{A_2 \omega_1^2}{\cosh^2 \rho} - (b - aq \omega_1) \tanh^2 \rho \right] \cosh \rho \sinh \rho \frac{A}{A},$$

$$\Delta \phi = 2 \int_{\rho_{\min}}^{\infty} d\rho \left[ A_2 \omega_1 + (aq - b \omega_1) \sinh^2 \rho \right] \coth \rho \frac{A}{A}. \quad (3.40)$$

We use the same value of $A_2 = 2/b$ and $\omega_1^2 = 1 - b^2$. From the above conserved charges we can find the relation as

$$E - \frac{S - q T \Delta t}{\omega_1} = 2T \frac{a^2 + b^2}{a^2 q - ab \omega_1} \left[ \frac{\Delta \phi}{2} - I \right], \quad (3.41)$$

where

$$I = A_2 \omega_1 \int_{\rho_{\min}}^{\infty} \frac{d\rho \coth \rho}{A} = \frac{\omega_1}{b} \int_{r_1}^{\infty} \frac{dx}{x \sqrt{(x - r_1)(x - r_2)(x - r_3)}} \left[ \Pi \left( \frac{r_3}{r_3 - r_1}, \frac{r_3 - r_2}{r_3 - r_1} \right) - \mathbb{K} \left( \frac{r_3 - r_2}{r_3 - r_1} \right) \right], \quad (3.42)$$

where $\Pi$ and $\mathbb{K}$ are complete elliptic integrals of third and first kind and $\Pi(n, k) = \int_0^1 \frac{dx}{(1 + nx^2)\sqrt{(1 - x^2)(1 - k^2 x^2)}}$. Now as $\theta$ covers all the ranges of $\sigma$, we can write the conserved charges and deficit angles with the help of $\theta$ equation where $\rho$ is a complicated function of $\theta$,

$$\frac{d\rho}{d\theta} = \frac{A \tan \theta}{\sqrt{a^2(1 - q^2)(1 - \omega_1^2)} \cosh \rho \sinh \rho \sqrt{\sin^2 \theta - \alpha^2}}. \quad (3.43)$$

Now the conserved charges and deficit angles can be written by the help of the above equation (3.43)

$$E = \frac{2T}{a^2 \Lambda} \int_{\pi/2}^{\theta_{\max}} d\theta \left[ a^2(1 - q^2) \sinh^2 \rho + a^2 - b^2 - A_2 b \omega_1^2 - A_2 a q \omega_1 \right] \tan \theta \frac{\tan \theta}{\sqrt{\sin^2 \theta - \alpha^2}},$$

$$S_{/\omega_1} = \frac{2T}{a^2 \Lambda} \int_{\pi/2}^{\theta_{\max}} d\theta \left[ a^2(1 - q^2) \sinh^2 \rho - b A_2 - (A_2 a q \omega_1 + \frac{abq}{\omega_1} - a q^2) \cosh^2 \rho \right] \sinh^2 \rho \frac{\tan \theta}{\sqrt{\sin^2 \theta - \alpha^2}},$$

$$J_1 = \frac{2T}{a^2 \Lambda} \int_{\pi/2}^{\theta_{\max}} d\theta \left[ a^2 - b^2 - a^2(1 - q^2) \cos^2 \theta \right] \frac{\tan \theta}{\sqrt{\sin^2 \theta - \alpha^2}},$$

$$J_2 = \frac{2T}{a^2 \Lambda} \int_{\pi/2}^{\theta_{\max}} d\theta \left[ a^2(1 - q^2) \cos^2 \theta + (a q^2 + \frac{ab}{\omega_2}) \cos^2 \theta \right] \frac{\tan \theta}{\sqrt{\sin^2 \theta - \alpha^2}},$$

$$\Delta \phi_1 = \frac{b + a q \omega_2}{a \Lambda} \int_{\pi/2}^{\theta_{\max}} d\theta \cot^2 \theta \frac{\tan \theta}{\sqrt{\sin^2 \theta - \alpha^2}},.$$
\[ \Delta t = \frac{2}{a\Lambda} \int_{\theta/2}^{\theta_{\text{max}}} d\theta \left[ \frac{A_2^2 \omega_1^2}{\cosh^2 \rho_{\text{min}}} - (b - aq \omega_1) \frac{\sinh^2 \rho}{\cosh^2 \rho} \right] \tan \theta \sqrt{\sin^2 \theta - \alpha^2}. \] 

(3.44)

where \( \Lambda = \sqrt{(1 - q^2)(1 - \omega_2^2)} \). These charges for the array of infinite spikes also satisfy the same relation as in the equation (3.39), which can be rewritten as

\[ E - S' - J'_1 = 2T \frac{a^2 + b^2}{a^2 - b^2} \sqrt{\frac{1 - q^2}{2}} \sin \frac{\Delta \phi_1}{2}. \] 

(3.45)

where

\[
S' = \frac{S - qT \Delta t}{\omega_1}, \quad J'_1 = \frac{a^2 + b^2}{a^2 - b^2} J_1, \quad \Delta \phi_1 = 2\theta_{\text{max}} - \pi
\]

\[
J_2 - qT \Delta \phi_1 = -2T \sqrt{\frac{1 - q^2}{2}} \cos \theta_{\text{max}}.
\]

This also looks analogous to the relation mentioned in [24] with \( q = 0 \). In the limit of \( b \to 0 \) and \( \sin (\Delta \phi_1/2) \to 1 \), the above relation (3.45) becomes

\[ E - S' - J_1 = 2T \sqrt{\frac{1 - q^2}{2}}. \] 

(3.46)

The above relation (3.46) looks like circular string rotating at \( \rho_{\text{min}} \) with the infinite angular momentum \( S \) which is shifted by the kink charge and has the shape of a magnon on \( S^2 \). Interestingly we also get the same relation (3.46) from the circular string equation (3.39) using the same limit of \( b \) and \( \Delta \phi_1 \). So both the circular string and infinite spikes resemble the shape of a magnon solution in the \( AdS_3 \times S^2 \), when we use the specific limits.

4. Conclusions and Outlook

In this paper, we have analyzed a large variety of semiclassical string solutions in \( AdS_3 \times S^3 \) motivated mainly by recent studies on string theory in \( AdS_3 \times S^3 \times T^4 \) with mixed fluxes. First we have studied the strings in \( \mathbb{R} \times S^3 \) with a background NS-NS flux. In [37] the generalization of the known dyonic giant magnon solution was found using a WZ term describing the NS-NS flux in a principal chiral model. The ‘shift term’ in the angular momentum proportional to magnon momenta \( p \) was described to be arising due to the ambiguity in the conserved charges associated to the boundary terms of the theory. In our case we find that out from classical string solutions, since it is equivalent to the chiral model in the conformal gauge. We also found out the single spike solution is modified due the background field. While the spike relation presented in [32] does not change, the dispersion relation between the angular momenta \( J_1 \) and \( J_2 \) appear to be different. In the last section, we have investigated various classes of rotating open string solutions for the string moving on \( AdS_3 \times S^3 \) with one spin in the \( AdS \) space and two angular momenta on the sphere. Here the NS-NS fluxes parameterized by \( q \) are turned on in both \( AdS \) and the
sphere. The three spin giant magnon solution in this case is also found to be modified by the ‘shift terms’ already presented in the previous case. While the angular momenta part is modified by a term proportional to the dyonic giant magnon momenta as before, the $AdS$ spin part seems to modified by a kink charge proportional to $\Delta t$. We also discuss a spike-like solution in this background. Further following $[24]$ we found the circular string solutions and an array of infinitely many spikes in the $AdS$. We presented the generalized dispersion relations between conserved quantities in these cases for some specific choice of constants and showed that they reduce to the relations presented in $[24]$ when the B-fields are turned off. It would be challenging to identify the dual gauge theory operators corresponding to these string solutions presented in section-2, though we can not expect a spin chain interpretation of the integrable system for $q \neq 0$ due to the non-periodicity of the solution in the momentum $p$. It is to be noted that all our solutions reduce to the ones known in the literature for the $q = 0$ case. However $q = 1$ appears to be a special case as world sheet theory is related to a WZW model, and it will be interesting to investigate the behaviour of our solutions in this limit. Also since $AdS_3 \times S^3$ with NS-NS flux can be represented as a $SL(2,R) \times SU(2)$ WZW model, we can try to find the three spin giant magnon solution following the formulation in $[37]$ and comment on the ambiguity in the $B$ field for the $AdS$ part also. The three spin string solutions presented in section 3 are related to the open strings and are interesting by themselves. However, apriori it is not to us clear whether one can learn about them in the boundary theory by using $AdS/CFT$ duality. Furthermore it will be interesting to investigate the finite size effects on the solutions presented here as it is shown in $[39]$ that the leading finite size effects in $\mathbb{R} \times S^3$ with a $B$-field is quite different from the usual ones. We wish to come back to these issues in near future.

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