Asymptotic symmetries of Maxwell theory in arbitrary dimensions at spatial infinity

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ABSTRACT: The asymptotic symmetry analysis of Maxwell theory at spatial infinity of Minkowski space with $d \geq 3$ is performed. We revisit the action principle in de Sitter slicing and make it well-defined by an asymptotic gauge fixing. In consequence, the conserved charges are inferred directly by manipulating surface terms of the action. Remarkably, the antipodal condition on de Sitter space is imposed by demanding regularity of field strength at light cone for $d \geq 4$. We also show how this condition reproduces and generalizes the parity conditions for inertial observers introduced in 3+1 formulations. The expression of the charge for two limiting cases is discussed: null infinity and inertial Minkowski observers. For the separately-treated 3d theory, the boundary conditions and charges are compared to null infinity results in the literature. We also compute the conserved charges for background isometries for $d > 3$.

KEYWORDS: Gauge Symmetry, Field Theories in Higher Dimensions, Global Symmetries

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1 Introduction

In Lagrangian theories with a Lie group of global symmetries, Noether’s first theorem establishes a conserved current to every generator of the corresponding Lie algebra. Noether’s method, however, fails to assign conserved currents to gauge symmetries [1]. Instead, several methods have been proposed to associate 2-form currents $k^\mu$ to gauge symmetries [2–5], which yield conserved surface charges. In gauge theories (and gravity), the asymptotic symmetry group (ASG) is the group of gauge transformations with finite surface charge, and the elements are called large gauge transformations (or large diffeomorphisms). To obtain the asymptotic symmetry group, one fixes an appropriate gauge, and imposes certain fall-off behavior on the fields. Large gauge transformations are then the residual gauge transformations i.e. those which preserve both the boundary conditions and the gauge.

Interest in ASG of Maxwell theory in flat space emanated from the discovery that soft photon theorem in QED is the Ward identity of the asymptotic symmetry group [6, 7].

1Quotiented by transformations with vanishing charge.
Concerned with that motivation, research on asymptotic symmetries is mostly performed in null slicing (Bondi coordinates) of flat space, where the surface of integration is (almost) light-like-separated from the scattering event [6, 8–13]. The charges “at null infinity” have also been generalized to subleading orders [14–16].

The asymptotic symmetry group of Maxwell theory \textit{at spatial infinity} is, as far as we know, restricted to three and four dimensions [14, 17–20]. Spatial infinity examination allows applying the canonical methods and define the ASG in a standard way. In [14], the multipole moments of a static configuration are exhibited as asymptotic symmetry charges. In [18], the charges are defined in de Sitter slicing of flat space (explained later), and the null infinity charges will be recovered if the integration surface approaches null infinity. We will follow much similar path in this paper, recovering [11, 21] at null infinity.

It was shown in [22] that demanding the action principle to be well-defined determines the asymptotic gauge almost completely. This condition automatically ensures conservation of the charges for residual gauge transformations.

In this work, we study the asymptotic structure of Maxwell field in arbitrary dimensions at spatial infinity, and identify a set of boundary conditions with non-trivial ASG, generalizing previous works in four dimensions. The ASG is local-U(1) on celestial sphere $S^{d-2}$, parametrized by arbitrary functions on $S^{d-2}$. The surface charges are obtained by manipulating surface terms arising from variation of the action, circumventing standard methods. To do this, we made the action principle well-defined, by making the timelike boundary term vanish, as done in [23–26].

Previous works on gauge theories in flat space advocate a matching condition [21] for the fields at spatial infinity $i^0$, when it is approached from future and past null boundaries $I^+, I^-$. On the asymptotic de Sitter space, this condition relates the states at past and future boundaries $I^\pm$. In dS/CFT studies, various antipodal conditions are proposed to make the Hilbert space well-defined [27]. A key result of this paper is that an antipodal condition is necessary to ensure regularity of field strength at light cone for $d \geq 4$.

We will work in de Sitter slicing [20, 28] of Minkowski space which makes the boundary conditions manifestly Lorentz invariant. In the 3+1 Hamiltonian approach of [19], the formalism loses manifest Lorentz symmetry and the ASG is presented as the product of two opposite-parity subgroups. We will show how their results regarding conserved charges and parities are recovered and generalized, by focusing on specific slices of de Sitter space.

Finally, 3-dimensional theory is covered in section 4. Asymptotic symmetries of 3d Einstein-Maxwell theory was studied in [17] at null infinity and in [29] in near-horizon geometries. We will show by taking null infinity limit that the same set of charges (in Maxwell sector) can be obtained in a non-logarithmic expansion. In addition, our hyperbolic setup fits completely with [30] on BMS$_3$ symmetry at spatial infinity. Thus, we expect that the combined hyperbolic analysis will reproduce the results of [17] in its non-radiative sector.

2 Rindler patch, action principle and conserved charges

Given an arbitrary point $\mathcal{O}$ in Minkowski space, one can define null coordinates $u = t - r$ and $v = t + r$. The future light cone $L^+$ of $\mathcal{O}$ is the $u = 0$ hypersurface, while the past
Figure 1. Penrose diagrams of Minkowski flat spacetime $\mathcal{M}_d$. The Rindler patch covers the events outside the light cone. The solid lines are constant $T$ slices, while dotted lines are constant $\rho$ hyperboloids.

light cone $\mathcal{L}^-$ is at $v = 0$. $\mathcal{L}^+$ and $\mathcal{L}^-$ intersect at the origin $\mathcal{O}$. We call the set of points with space-like distance to $\mathcal{O}$, the Rindler patch and denote it by $\text{Rind}_{d-1}$ (see figure 1). The Rindler patch is conveniently covered by coordinates $(\rho, T, x^A)$, $A = 1, \cdots , d-2$, in which the metric is

$$ds^2 = d\rho^2 + \frac{\rho^2}{\sin^2 T} (-dT^2 + q_{AB}dx^A dx^B), \quad 0 \leq T \leq \pi .$$

(2.1)

where

$$\begin{align*}
\rho^2 &= r^2 - t^2 \\
\cos T &= t/r \\
T &= \rho \cot T \\
r &= \rho/\sin T
\end{align*}$$

(2.2)

The origin is at $\rho = 0$ and undefined $T$. Future light cone $\mathcal{L}^+$ is at $(\rho = 0, T = 0)$ and past light cone $\mathcal{L}^-$ is at$(\rho = 0, T = \pi)$.

Spatial infinity $i^0$ defined as the destination of spacelike geodesics is at $(\rho \to \infty, 0 < T < \pi)$, shown as the intersection of future and past null infinities on the Penrose diagram. The limit $(\rho \to \infty, T \to 0, \pi)$ covers the portion of null infinity outside the light cone.

The constant $\rho$ hypersurfaces are $(d - 1)$-dimensional de Sitter spaces with radius $\rho$, invariant under Lorentz transformations about $\mathcal{O}$. We will show de Sitter coordinates by $x^a$, $a = 2, \cdots , d$, and the unit $dS_{d-1}$ metric by $h_{ab}$,

$$h_{ab}dx^a dx^b = \frac{1}{\sin^2 T} \left(-dT^2 + q_{AB}dx^A dx^B\right).$$

(2.3)

The study is restricted to solutions with asymptotic power expansion in $\rho$

$$A(\rho, x^a) = \sum_n A^{(n)}(x^a)\rho^{-n}$$

(2.4)

2A point at radius $r$ on $\mathcal{L}^+$ is at $(\rho = 0, T = 0)$, by taking the limit $T \to 0$ such that $\rho = rt$.

3The point at retarded time $u$ on future null infinity is reached by taking the limit $\rho \to \infty$ such that $u = -\rho \tau/2$. Similarly, taking the limit with fixed $v = \rho(\pi - \tau)/2$, one arrives at the advanced time $v$ on past null infinity.
Figure 2. The region where we define the action problem. It is confined by initial and final cones $I_1$ and $I_2$ (e.g. at constant $T$), intersecting at $O$. The region is not bounded in $\rho$ direction, so $I_{1,2}$ are Cauchy surfaces where initial and final data are fixed. The boundary terms are computed at constant-$\rho$ hyperboloids ($B$). Dashed lines show the light-cone.

In some cases, we drop the superscript ($n$) for the leading term (the least $n$) in each component to reduce clutter.

2.1 The action principle

In the Lagrangian formulation of physical theories, the classical trajectories of the dynamical variables $\Phi^i$ are stationary points of an action functional

$$\frac{\delta S}{\delta \Phi^i} \bigg|_{\Phi^i_{cl}} = 0 \quad (2.5)$$

for fixed initial and final values. In field theories, the functional derivative of the action is well-defined, if variation of dynamical fields leaves no boundary terms. In our setup, there are two spacelike boundaries $I_{1,2}$ and one timelike boundary $B$ lying on asymptotic de Sitter space (see figure 2). Data on spacelike boundaries is fixed, so we must ensure that the boundary term on $B$ either vanishes or itself is a total derivative.

For Maxwell theory with action

$$S = \int_{M_d} \sqrt{g} \left( -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} + A_\mu J^\mu \right), \quad (2.6)$$

the timelike boundary term is

$$\int_B \rho^{d-1} \sqrt{h} \delta A_\alpha F^{\alpha \rho} \quad (2.7)$$

We will show that for specific boundary conditions and an asymptotic gauge fixing, the boundary term does vanish.

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4Notation: $g = |\text{det} g_{\mu \nu}|$ for all metrics involved.
2.2 Conserved charges

For the specific example of Maxwell theory, we show that with a well-defined action principle at hand, one can define conserved charges for gauge transformations of the theory, and identify the asymptotic symmetry group as the group of gauge transformations having finite charge.

Consider variation of the action around a solution to equations of motion\(^5\)

\[
\delta S[\Phi] \approx \int_{I_2} \mathcal{I}(\Phi, \delta \Phi) - \int_{I_1} \mathcal{I}(\Phi, \delta \Phi) + \int_B \mathcal{B}(\Phi, \delta \Phi)
\]  

(2.8)

If the field variation is a gauge transformation (or a diffeomorphism in gravity theories), then, the \(I\) integrands in (2.8) become total derivatives, so the first two terms becomes codimension-2 integrals on \(\partial I_1\) and \(\partial I_2\). This can be checked in specific examples, and a proof is given in [31].

If the action principle is well-defined, the \(B\)-integral on the timelike boundary is either vanishing, or a total derivative on the hyperboloid (so that it becomes a surface integral on boundaries of \(B\)). As a result, the gauge transformation of the action becomes the difference of two codimension-2 integrals on shell

\[
\delta_{\lambda} S \approx \int_{\partial I_2} \mathcal{C}(\Phi, \delta_{\lambda} \Phi) - \int_{\partial I_1} \mathcal{C}(\Phi, \delta_{\lambda} \Phi).
\]  

(2.9)

The left-hand-side depends on the explicit form of the action. If the action is gauge invariant \((\delta_{\lambda} S = 0)\), (2.9) shows that the integral \(\int_{\partial I} \mathcal{C}\) is independent of the surface of integration; thus we can identify the codimension-2 integrals as the conserved charges corresponding to the gauge transformation \(\delta_{\lambda}\).

**Covariant phase space method.** Let us compare the procedure above with covariant phase space method. The symplectic form of the theory is nothing but variation of the action surface terms

\[
\Omega = \int_{I} \mathcal{I}(\delta \Phi, \delta' \Phi)
\]  

(2.10)

for two field variations \(\delta, \delta'\), and \(\mathcal{I}\) is defined in (2.8). Taking a second variation of (2.8) shows that in general \(\Omega\) is not conserved since its flux at timelike boundary \(B\) is non-vanishing and given by

\[
\int_B \mathcal{B}(\delta \Phi, \delta' \Phi)
\]  

(2.11)

where the integrand \(\mathcal{B}\) is again defined in (2.8). Therefore, eliminating the symplectic flux is equivalent to making the action principle well-defined. For the conservation of the symplectic form, the flux (2.11) need not be strictly vanishing. It is enough, if possible, to make it a total divergence reducing the expression to codimension-2 integrals on \(\partial B\):

\[
\int_B \mathcal{B}(\delta \Phi, \delta' \Phi) \equiv -\Omega^{\nu \cdot \delta \nu} / \partial I_2
\]  

(2.12)

\(^5\)Notation: \(\approx\) is equality when equations of motion hold.
Finally, \( \Omega^{\text{dry}} \) can be added to \( \Omega \) as a surface term, leading to conserved charges. This subtraction is a **Y ambiguity** in covariant phase space terminology [3]. This procedure was done in [18] for 4d Maxwell theory. It can be readily generalized to arbitrary dimensions by appropriate choice of boundary conditions. However, we decide to bypass the symplectic form by working directly with the action.

### 2.3 Field equations in de Sitter slicing

Written in coordinates \((\rho, x^a)\), the field equations and Bianchi identities are

\[
D_a F^{a \rho} = J^\rho 
\]  
(2.13a)

\[
\frac{1}{\rho^{d-1}} \partial_\rho (\rho^{d-1} F^{\rho a}) + D_a F^{ba} = J^a 
\]  
(2.13b)

\[
\partial_\rho F_{ab} + 2 \partial_{[a} F_{b] \rho} = 0 
\]  
(2.13c)

\[
\partial_{[a} F_{bc]} = 0 
\]  
(2.13d)

where \( D \) is the covariant derivative on \( dS_{d-1} \). Analyzing the solutions suggests appropriate boundary conditions for the theory. Note that \( F^a \) and \( F_{ab} \) are distinct Lorentz invariant components. First we ask if there are solutions to equations of motion once either of them is set to zero.

1. If we set \( F_{ab} = 0 \), by (2.13b) we have \( F^{a \rho} \propto \rho^{1-d} \). Furthermore, by (2.13c) and (2.13a),

\[
F_{a \rho} = \partial_a \psi = \rho^{3-d} \partial_a \psi (x^b), \quad D^a D_a \psi = J^\rho 
\]  
(2.14)

In this case, the solution consists of a scalar degree of freedom \( \psi \).

2. In general, \( F_{ab} \) is closed on de Sitter space by (2.13d), thus it is locally exact \( F_{ab} = (dA)^{ab}_{(0)} \). Switching \( F_{a \rho} \) off, fixes the \( \rho \)-dependence by (2.13c) to \( F_{ab} \propto \rho^0 \). Finally, the field equation (2.13b) reduces to

\[
D^a D_{[a} A^{(0)}_{b]} = 0 
\]  
(2.15)

(Notation is explained in (2.4)).

Any other solution involves both \( F_{ab} \) and \( F_{a \rho} \). The solutions with power-law fall-off in \( \rho \) correspond to multipoles of electric and magnetic branes. Electric monopoles generate the independent solution (2.14) for \( F_{a \rho} \), while magnetic monopoles(-branes) generate the independent solution (2.15) for \( F_{ab} \). Their multi-poles generate fields of lower fall-off which mix \( F_{a \rho} \) and \( F_{ab} \). On the contrary, arranging monopoles to build lines of charge will generate stronger fields at infinity, but in any case mix \( F_{a \rho} \) and \( F_{ab} \).

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\(^6\)Notation: \( \sim O(\rho^n) \) means all powers not exceeding \( n \), while \( \propto \rho^n \) means the \( n \)-th power of \( \rho \) exclusively.

\(^7\)For example, for an electric dipole, \( F_{a \rho} \propto \rho^{2-d} \), and by Bianchi identity (2.13c), \( F_{a \rho} \propto \rho^{3-d} \). Define \( \psi_a \equiv F^{a (d-2)}_{\rho a} \). Then,

\[
(d - 3) F^{(d-3)}_{ab} = (d \psi)_{ab}, \quad D^b (d \psi)_{ba} - (d - 3) \psi_a = 0 
\]  
(2.16a)

away from sources.
Denote the set of solutions for electric monopoles given in (2.14) by $E$. This space covers moving electric charges in space, which are passing the origin simultaneously at $t=0$, hence their worldlines cross $O$. The field strength is $F_{\rho a} \propto \rho^{3-d}$ with no subleading terms. For an arbitrary configuration of freely moving charges, the leading component of asymptotic field is an element of $E$, but subleading terms are generally present. In other words, the definition of $E$ is Lorentz invariant, but not Poincaré invariant. $E$ encodes the information of charge values $q_n$ and their velocities $\tilde{\beta}_n$. The space $E$ is isomorphic to the space of boost vectors $\tilde{\beta}$, that is $\mathbb{R}^{d-1}$. The space of conserved electric charges we will construct is also isomorphic to $\mathbb{R}^{d-1}$: each point of this space with coordinate vector $\tilde{\beta}$ is a conserved charge and gives the total electric charge in space, moving with that specific boost.

The set of solutions (2.15) covers magnetic monopoles moving freely in space and crossing the origin at $t = 0$. In dimensions larger than 4, the magnetic monopoles are replaced by extended magnetic branes since the dual field strength $F$ is a $(d-2)$-form in that case. We are considering boundary conditions which exclude magnetic charges in this work.

3 Four and higher dimensions

In this section, we exploit the asymptotic symmetries of Maxwell theory in dimensions higher than three. First, we present a set of well-motivated boundary conditions on field strength tensor. Nonetheless, existence of large gauge transformations demand that the gauge field be finite at infinity. That will necessitate an asymptotic gauge choice to make the action principle well-defined. Finally, we find the conserved charges of the theory at spatial infinity by computing the on-shell action.

3.1 Boundary conditions and the action principle

The electromagnetic field of a static electric charge is $F_{\gamma} \propto \rho^{3-d}$. Applying a boost (which belongs to the isometry group of the hyperboloid) will turn on other de Sitter components of $F_{\gamma}$ with the same fall-off; so one generally has $F_{\gamma} \propto \rho^{3-d}$. Therefore, we propose the following boundary conditions for $d$-dimensional theory

$$F_{\rho a} \sim O\left(\rho^{3-d}\right), \quad F_{ab} \sim O\left(\rho^{3-d}\right).$$

The $F_{ab}$ components arise because of electric multipoles (cf. section 2.3). The leading component of $F_{\gamma}$ is in $E$ of section 2.3 and satisfies

$$F_{\gamma}^{(d-3)} = \partial_\gamma \psi, \quad D_\gamma D^\gamma \psi = j^{(1-d)}$$

*For a spherically symmetric field we have

$$Q = \int_{s^{d-2}} \sqrt{q^{d-2}} F_{\gamma} \rightarrow F_{\gamma} = \frac{Q}{a_{d-2}} \rho^{3-d}$$

where $a_{d-2}$ is the area of a $(d-2)$-sphere. In hyperbolic coordinates we have

$$F_{\gamma} = -\frac{\rho}{\sin \tau} F_{\gamma} = -\frac{Q}{a_{d-2}} \rho^{3-d} \sin^{d-3} \tau.$$
Components of gauge field that saturate (3.1) behave like
\[ A_a \sim \mathcal{O} \left( \rho^{3-d} \right) \quad A_\rho \sim \mathcal{O} \left( \rho^{3-d} \right) \] (3.3)

Plugging into (2.7), the boundary term falls like \( \mathcal{O} \left( \rho^{3-d} \right) \). For \( d > 3 \), the action principle is well-defined. However, this choice will make the charges for all gauge transformations vanish. For instance, the Gauss law
\[ Q = \int_{S_{d-2}} * \mathcal{F} \] (3.4)
is regarded as the charge for gauge transformation with \( \lambda = 1 \), which is excluded if \( A_a \sim \mathcal{O} \left( \rho^{3-d} \right) \) in dimensions higher than three. The theory enjoys non-trivial ASG, only if \( \delta A_a \sim \mathcal{O} \left( \lambda = 1 \right) \). Thus, our prescribed boundary condition is as follows:
\[ A_a = \partial_a \phi + \sum_{n=d-3} A_a^{(n)} \rho^{-n} \quad \partial_\rho \phi = 0 \] (3.5)
\[ A_\rho = \sum_{n=d-3} A_\rho^{(n)} \rho^{-n} \quad A_\rho^{(d-3)} = \psi(x^a) \] (3.6)
As a result, the above boundary conditions respect (3.1), while \( A_a \sim \mathcal{O} \left( \lambda = 1 \right) \) by a pure gauge fluctuation\(^9 \partial_a \phi \). Previous works in four dimensional Maxwell theory allow magnetic monopoles. That would make \( \mathcal{F}_{ab} \sim \mathcal{O} \left( \lambda = 1 \right) \) so the leading term of the gauge field would not be pure gauge. Here we are not taking account of magnetic charges though.

With the aforementioned boundary condition, the boundary term of the action will be finite
\[ \int_B \sqrt{\hbar} \delta A^{(0)}_a F_{ab}^{(d-3)} = \int_B \sqrt{\hbar} \delta A^{(0)}_b (d-3) \] (3.7)
According to (3.1), \( F_{ab}^{(0)} = 0 \) so the leading term is (locally) pure gauge \( A_b^{(0)} = \partial_b \phi \). Consequently, after integration by parts, the boundary term of the action vanishes on shell, by equation of motion \( D^2 \psi = 0 \) (up to a total divergence on \( B \)). However, we request off-shell vanishing of the boundary term, since the variational principle must entail the equations of motion, and they can not be used a priori.

One way out is to fix the asymptotic gauge \( \delta D^a A^{(0)}_a = 0 \), for which the boundary term becomes a total divergence on \( B \) after an integration by parts. There are also other possibilities. The Lorenz gauge at leading order is
\[ D^a A^{(0)}_a + \alpha (d-2) A^{(1)}_\rho = 0 \quad \alpha = 1 \] (3.8)
and by our boundary conditions on field strength, \( A^{(1)}_\rho = 0 \) for \( d > 4 \). Thus, the Lorenz gauge, or its extension to general \( \alpha \) will make the action principle well-defined in dimensions strictly higher than 4. In four spacetime dimensions, \( A^{(1)}_\rho = \psi \) (up to a constant number which drops from derivatives) so it is necessary to add a boundary term
\[ S_b = -\alpha \int_{B_3} \sqrt{\hbar} \psi^2 \quad \text{for } d = 4 \] (3.9)
to make the action well-defined [22].

\(^9\)By “pure gauge” we mean a flat connection; a configuration gauge equivalent to \( A_a = 0 \), although it may involve an improper gauge transformation (that with non-zero charge).
3.2 Conserved charges from action

The action with a solution to equations of motion plugged in, is a functional of initial and final field values (or boundary values in Euclidean versions); that is how classical trajectories are defined. For Maxwell theory,

\[ S \approx - \int_{I_2} \sqrt{g} n_{\mu} A_\mu F^{\mu\nu} + \int_{I_1} \sqrt{g} n_{\mu} A_\mu F^{\mu\nu} + \int_B \sqrt{\hbar} \partial_\alpha \phi \partial^\alpha \psi \]  

(3.10)

where \( n_{\mu} = \partial_{\mu} T \). Varying (3.10) by gauge transformations \( \delta A_\mu = \partial_\mu \Lambda \), and using field equations following an integration by parts gives\(^\text{10}\)

\[ \delta_\Lambda S = - \int_{I_2} \sqrt{g} \Lambda J^T + \int_{I_1} \sqrt{g} \Lambda J^T \approx - \int_{\partial I_2} \sqrt{g} \Lambda F^{\rho T} + \int_{\partial I_1} \sqrt{g} \Lambda F^{\rho T} + \int_B \sqrt{\hbar} \partial_\alpha \lambda \partial^\alpha \psi. \]  

(3.11)

where \( \lambda = \Lambda^{(0)} \). The explicit form of Maxwell action (2.6) shows that the left-hand-side above is the flux through spatial boundary:

\[ \delta_\Lambda S = - \int_{I_2} \sqrt{g} \Lambda J^T + \int_{I_1} \sqrt{g} \Lambda J^T = \int_B \sqrt{\hbar} \rho^{d-1} \lambda J^\rho \]  

(3.12)

We can make this “charge flux” vanish asymptotically by the additional assumption \( J^\rho \sim \mathcal{O} (\rho^{-d}) \). This condition ensures that the system is localized and the charges are conserved. So far we made the left-hand-side in (3.11) vanish; let us look at the other side.

Recall that the action principle necessitated fixing the asymptotic Lorenz gauge (3.8), leaving residual gauge transformations

\[ \delta A_\mu^{(0)} = \partial_\alpha \lambda, \quad D^\alpha D_\alpha \lambda = 0, \]  

(3.13)

with arbitrary subleading terms. The condition on \( \lambda \) allows us to turn the very last term in the right-hand-side of (3.11) into a total divergence on \( B \). As a result we manage to prove that the quantity

\[ Q_\Lambda = \int_{\partial I} \sqrt{g} (\lambda F^{\rho T} - \partial^T \lambda \psi) \]  

(3.14)

is independent of \( I \); i.e. conserved.

3.3 Light cone regularity and antipodal identification

\( \lambda \) and \( \psi \) both satisfy

\[ D^\rho D_\alpha f(x^\lambda) = 0 \]  

(3.15)

and the solution is obtained by spectral decomposition of Laplace operator on \( S^{d-2} \), being \( D^2 Y_\ell (\hat{x}) = -\ell (\ell + d - 3) Y_\ell (\hat{x}) \). Then, (3.15) will simplify to

\[ (1 - y^2) f_\ell''(y) + (d - 4) y f_\ell'(y) + \ell (\ell + d - 3) f_\ell (y) = 0 \quad y = \cos \tau. \]  

(3.16)

\(^{10}\)In equation (3.11), the induced metric on \( \partial I \) yields a determinant factor \( \rho^{d-2} \sin^{2-d} \tau \). On the other hand, \( n_\tau = -\frac{\sqrt{g}}{\sin \tau} (0, 1, 0) \). The combination is equal to \( -\sqrt{g} \).
The general solution is
\[
    f(y,\hat{x}) = (1-y^2)^{\frac{d-2}{4}} \sum_{\ell=1}^{\frac{d-4}{2}} Y_{\ell}(\hat{x}) \left( a_{\ell} P_{(2\ell+d-4)/2}^{(d-2)/2}(y) + b_{\ell} Q_{(2\ell+d-4)/2}^{(d-2)/2}(y) \right),
\]
where \( P_{l}^{m} \) and \( Q_{l}^{m} \) are associated Legendre functions of the first and second kind respectively. For \( \ell = 0 \), the solutions are
\[
    a_0 + b_0 y \begin{pmatrix} \frac{1}{2} \frac{4-d}{2} \frac{3}{2} y^2 \end{pmatrix}. \tag{3.18}
\]

As far as field equations are concerned, the whole set of solutions in (3.17) with two sets of coefficients are admissible both for \( \psi \) and \( \lambda \). In previous works in four dimensional Maxwell theory, a boundary condition, the antipodal matching condition, was imposed such that one of branch of the solutions in (3.17) was allowed for \( \psi \) and the other for \( \lambda \). Here we will provide a rationale for the antipodal matching condition in higher dimensions.

The field strength tensor \( F \) being a physical field must be regular at light cone \( \mathcal{L}^\pm \) (i.e. \( u = 0 \) and \( v = 0 \) surfaces in advanced/retarded Bondi coordinates). Recall that in \( E \) space, \( F_{\alpha\rho} = \rho^{3-d} \partial_\alpha \psi \) in \( d \) dimensions, which diverges at \( \rho = 0 \) in dimensions larger than three. Near \( \mathcal{L}^+ \) (located at \( \rho = 0, T = 0 \)), \( \psi \) must decay at least like \( T^2 \), to make \( F_{\tau\rho} \) finite.

The light cone behavior of solutions (3.17) is
\[
    f_- = T^{d-2} \tilde{\psi}(\hat{x}) + O \left( T^d \right), \quad f_+ = \tilde{\lambda}(\hat{x}) + O \left( T^2 \right). \tag{3.19}
\]
Request for light cone regularity leads us to take \( f_- \) for \( \psi \) as a boundary condition on \( \mathcal{L}^+ \), hence the notation \( \tilde{\psi} \). Similar argument can be made at \( \mathcal{L}^- \) at \( T \to \pi \). Extracting \( f_- \) from (3.17) amounts to setting \( b_0 = 0 \) in even dimensions, and setting \( a_{\ell} = 0 \) in odd dimensions (and keeping \( b_0 \) in all dimensions). These conditions can be summarized as antipodal identification of solutions on \( dS_{d-1} \)
\[
    \psi(T, \hat{x}) = -\psi(\pi - T, -\hat{x}). \tag{3.20}
\]
This is a well-known condition in dS/CFT studies [32]. Gauge parameters with non-vanishing charge (3.14) must reside in \( f_+ \) set. These are even under de Sitter antipodal map
\[
    \lambda(T, \hat{x}) = \lambda(\pi - T, -\hat{x}). \tag{3.21}
\]
Note that the conditions (3.20) and (3.21) hold on the entire de Sitter space and in particular for \( T = 0 \), relating the fields on future and past boundaries of the hyperboloid
\[
    \psi(0, \hat{x}) = -\psi(\pi, -\hat{x}) \tag{3.22}
\]
\[
    \lambda(0, \hat{x}) = \lambda(\pi, -\hat{x}) \tag{3.23}
\]
The fields on left-hand-side live on the past of future null infinity \( T^+ \) while those on right-hand-side live on the future of past null infinity \( T^- \). Therefore \( \lambda \) and \( F_{\tau\rho} = \partial_\alpha \psi \) are both even under antipodal map between future and past null infinity.

\[\text{In four dimensions, the subleading term for } f_+ \text{ is } O \left( T^2 \log T \right).\]
3.4 Charge at null infinity

In the Rindler patch, one can approach the light cone hypersurface $\mathcal{L}^+ \cup \mathcal{L}^-$ from outside. The charge (3.14) takes a simpler form in that limit: the second term in (3.14) vanishes, while the first term becomes

$$Q_\lambda = -(d-2) \int_{S^{d-2}} \sqrt{g} \lambda \bar{\psi}$$

(3.24)

The leading field strength at null infinity becomes

$$F_{ur} = -\frac{1}{r} F_{\tau \rho} = (d-2)r^{2-d} \bar{\psi}(\hat{x}) + O\left(r^{1-d}\right)$$

(3.25)

Hence, the familiar expression for surface charges at future null infinity is recovered

$$Q_\lambda = \int_{S^{d-2}} \sqrt{g} \lambda F_{ur}$$

(3.26)

3.5 Inertial observers

Consider a Minkowski observer with coordinates $(t, r, x^A)$, who advocates a “3+1 formulation” of $d$-dimensional theory. Boundary conditions restrict Cauchy data residing in constant-time hypersurfaces at large $r$. It is implicitly presumed that time interval $\Delta t$ between Cauchy surfaces is much smaller than the radius $r$ beyond which is conceived as “asymptotic region”. This $\Delta t/r \to 0$ condition makes all Cauchy surfaces to converge at $T = \pi/2$ “throat” on the asymptotic de Sitter space. Infinitesimal Lorentz boosts will incline this surface, though, to $T = \beta \cdot \hat{x} + \pi/2$.

The solutions to (second order) equations of motion on $dS_{d-1}$ are specified by initial/final data on past/future boundaries of de Sitter space $\mathcal{I}^-/\mathcal{I}^+$. When an additional antipodal condition is imposed, only one set of data on either boundary suffices (and the other one is determined by e.o.m.). When the spacetime is restricted to a cylinder around $T = \pi/2$, the solution can be specified by a couple of independent data $\Phi$ and $\partial_T \Phi$ (and higher time derivatives determined by e.o.m.). The antipodal condition then halves the possibilities in each one by a restriction on angular dependence, as explained below.

Here, we would like to focus around $T = \pi/2$ surface and translate previous results to a canonical language. First of all, the coordinates are related as

$$t = \rho \cot T \equiv \rho \left(\frac{\pi}{2} - T\right), \quad r = \frac{\rho}{\sin T} \equiv \rho.$$

(3.27)

Next, recall that $A^{(0)}_a = \partial_a \phi$, which implies that

$$A_t(\hat{x}) = \frac{1}{r} \partial_r \phi \left(\frac{\pi}{2}, \hat{x}\right), \quad A_B(\hat{x}) = \partial_B \phi \left(\frac{\pi}{2}, \hat{x}\right)$$

(3.28)

In four dimensions, $A_t$ receives an additional contribution $-\psi(\frac{\pi}{2}, \hat{x})/r$. The radial components may be written as

$$A_r = \partial_r \Lambda + r^{3-d} \tilde{A}_r(\hat{x}) + O\left(r^{2-d}\right), \quad \Lambda \sim O\left(r^0\right).$$

(3.29)
where
\[ \vec{A}_r(\hat{x}) = \psi \left( \frac{\pi}{2}, \hat{x} \right) \] (3.30)

The field strength is given by
\[ \pi^r \equiv \sqrt{g} F^{rt} = -\sqrt{g} \partial_r \psi \left( \frac{\pi}{2}, \hat{x} \right), \quad \pi^B \equiv \sqrt{g} F^{Bt} \sim {\mathcal O}(r^{-2}) \] (3.31)

The “momenta” \( \pi^i \) are symbolic in this discussion, but they are equal to momenta in a true Hamiltonian formulation. Finally, the gauge parameter divides into
\[ \lambda(\hat{x}) \equiv \lambda \left( \frac{\pi}{2}, \hat{x} \right), \quad \mu(\hat{x}) \equiv \partial_r \lambda \left( \frac{\pi}{2}, \hat{x} \right). \] (3.32)

The antipodal conditions (3.20) and (3.21) imply
\[ \vec{A}_r(\hat{x}) = -\vec{A}_r(-\hat{x}) \quad \pi^r(\hat{x}) = +\pi^r(-\hat{x}) \quad A_B(\hat{x}) = -A_B(-\hat{x}) \] (3.33)
and\(^\text{12}\)
\[ \mu(\hat{x}) = -\mu(-\hat{x}) \quad \lambda(\hat{x}) = +\lambda(-\hat{x}) \] (3.34)

In even spacetime dimensions, these are *parity conditions*, cause the antipodal map \( \hat{x} \rightarrow -\hat{x} \) reverses the orientation of \( S^{d-2} \) (the volume form shifts sign). In odd dimensions, however, the map is a rotation about the origin, preserving the orientation. These conditions are preserved under boosts. The connected part of Lorentz group \( SL(d - 1, 1) \), commutes with parity and time-reversal, thus the antipodal conditions (3.20) and (3.21) hold in any Lorentz frame. Explicitly, for an infinitesimally boosted frame and keeping the terms at zeroth order of \( r \) we have
\[ \psi' \left( t' = \frac{\pi}{2}, -\hat{x}' \right) = \psi \left( \frac{\pi}{2} - \hat{\beta} \cdot \hat{x}, -\hat{x} \right) = -\psi \left( \frac{\pi}{2} - \hat{\beta} \cdot \hat{x}, \hat{x} \right) = -\psi' \left( t' = \frac{\pi}{2}, \hat{x}' \right) \] (3.35)

In the second equality we have used the antipodal conditions and the temporal argument is found by \( \pi - (\frac{\pi}{2} - \hat{\beta} \cdot (-\hat{x})) = \frac{\pi}{2} - \hat{\beta} \cdot \hat{x} \).

The conserved charge (3.14) is rewritten as
\[ Q_\lambda = - \int_{S^{d-2}} \sqrt{g} (\lambda \pi^r - \mu \vec{A}_r) \] (3.36)

One must note that \( \mu \) transforms like a vector under boosts, for it is the \( T \)-derivative of a scalar.

### 3.6 Finite action and symplectic form

Here we will show that the symplectic form is finite in dimensions higher than 4. In analogy with mechanical systems, the symplectic 2-form \( \Omega \) in field theories is defined from the boundary term of the Lagrangian. For Maxwell theory in Rindler patch, it is
\[ \Omega = - \int \sqrt{g} \delta A_\nu \delta F^{\mu T} + \Omega^{\text{v.dary}}, \] (3.37)

\(^{12}\)Parity of \( \pi^B \) can not be inferred from leading fields. For electric dipoles, \( \pi^B(-\hat{x}) = +\pi^B(\hat{x}) \).
with $\Omega^{\text{bdary}}$ being a surface term introduced in [18] for $d = 4^{13}$ as

$$\Omega^{\text{bdary}} = - \int_{S^2} \sqrt{h} \delta A^0_{(1)} \delta A^1_{(0)}$$

(3.38)

where $h_{ab}$ is the boundary de Sitter metric (2.3). Using (2.12), the surface term in higher dimensions can be shown to be

$$\Omega^{\text{bdary}} = - \int_{S^{d-1}} \sqrt{h} \delta A^0_{(d-3)} \partial^T \delta \phi .$$

(3.39)

In four dimensions, the bulk symplectic form is logarithmically divergent, since

$$F_{ab} = \int \sqrt{\rho} d^3 x d\rho \left( \delta A^0_{(1)} \delta F_{RT}^{(1)} + \delta A^0_{(0)} \delta F_{RT}^{(0)} \right) + O (\rho^0)$$

(3.40)

The second term in brackets which corresponds to magnetic monopoles is excluded in our boundary condition (3.1). The first term, however has the form $R \partial^T$. If the integration surface is $T = 2$, this term vanishes by antipodal condition (3.20). This remains true for boosted frames too. Nevertheless, it is not clear if the divergence cancels for arbitrary spacelike surfaces $I$, and we are not aware of any resolution. Similar divergence occurs in computing the on-shell action, where the cancellation around $T = \pi/2$ surface is again ensured by antipodal conditions.

In higher dimensions, $\delta A^0_{(1)} = 0$, and no large $\rho$ divergence appears.

4 Three dimensions

This section is devoted to three dimensional Maxwell theory. The asymptotic symmetry at null infinity was discussed in [17]. The reason for separate consideration of three dimensional case is the simple form of solutions: $dS_2$ is conformally flat and the solution is a whole set of left- and right-moving scalar modes. For this simplest case, we will also translate the boundary conditions to Bondi coordinates ($u, r, \varphi$). The unit $dS_2$ metric is simpler in coordinates $x^\pm = \varphi \pm T$, which is

$$ds^2 = \frac{-dT^2 + d\varphi^2}{\sin^2 T} = \frac{dx^+ dx^-}{\sin^2 T} .$$

(4.1)

4.1 Boundary conditions and solution space

The boundary conditions (3.1) for $d = 3$ become$^{14}$

$$F_{\alpha \rho} \sim O (\rho^0) , \quad F_{bc} \sim O (\rho^0) .$$

(4.2)

This boundary condition is realized by following power-law asymptotic expansion on the gauge field

$$A_\rho = \sum_{n=0} A^{(n)}_\rho \rho^{-n} , \quad A_a = \sum_{n=0} A^{(n)}_a \rho^{-n} .$$

(4.3)

$^{13}$It exists also in higher dimensions. We did not need to introduce it for the charges were derived from the action.

$^{14}$The static Coulomb solution is $F_{tr} = q/r$. The electric field in hyperbolic coordinates becomes $F_{\tau r} = -q$. 

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The asymptotic behavior adopted here allows moving electric charges in 2 + 1 dimensions. One can fix a radial gauge $A_{a}(\rho, x^a) = 0$, by an appropriate gauge transformation on (4.3). The price is the introduction of divergent terms in the new $\tilde{A}_a$ expansion:

$$
\tilde{A}_a = \partial_a \psi \rho + \partial_a \chi \log \rho + \sum_{n=0}^{\infty} \tilde{A}_a^{(n)} \rho^{-n} \tag{4.4}
$$

In order to determine a solution for $A_{a}(\rho, x^a)$, fixing initial data at (say) future boundary of de Sitter space at $(\rho, T \to 0)$ is demanded. Hence, (4.4) would be our prescribed asymptotic expansion in radial gauge. However, in the present discussion, we will not fix any gauge and work with (4.3), except at the leading order $A^0$.

Analyzing the free solutions is simpler for the Hodge dualized degrees of freedom. In three dimensions, Maxwell theory is dual to a scalar field:

$$
d\Phi = *F. \tag{4.5}
$$

In hyperbolic coordinates the relation is

$$
\partial_a \Phi = \epsilon^{ab} a F_{\rho b} \quad \partial_{\rho} \Phi = \epsilon^{ab} \rho F_{\rho ab}. \tag{4.6}
$$

so that $\Phi \sim O(1)$. Maxwell free field equations of motion are trivially satisfied $d^2 \Phi = 0$, while the Bianchi Identity $dF = 0$ becomes $d * d\Phi = 0$:

$$
D_a D^a \Phi + \partial_\rho (\rho^2 \partial_\rho \Phi) = 0. \tag{4.7}
$$

The general solution for the electromagnetic field strength is found by (4.6) and (4.5).

At leading and subleading order, $F_{a \rho}^{(0)} = \partial_a A_{\rho}^{(0)}$ and $F_{a \rho}^{(1)} = \partial_a A_{\rho}^{(1)}$. The relation with the dual scalar field is

$$
\partial_a A_{\rho}^{(0)} = \tilde{\epsilon}_{ab} \partial_b \Phi^{(0)} \quad F_{ab}^{(0)} = -\frac{1}{2} \tilde{\epsilon}_{ab} \Phi^{(1)} \quad \partial_a A_{\rho}^{(1)} = \tilde{\epsilon}_{ab} \partial_b \Phi^{(1)} \tag{4.8}
$$

where $\tilde{\epsilon}$ is the volume-form on unit $dS_2$. The equation of motion for $A_{\rho}^{(0)}$ and leading Bianchi identity give

$$
D_a D^a A_{\rho}^{(0)} = 0, \quad D_a D^a \Phi^{(1)} = 0. \tag{4.9}
$$

The differential operator is the Laplacian on $dS_2$, which takes a nicer form

$$
\partial_+ \partial_- \psi = 0. \tag{4.10}
$$

The general solution with periodic boundary condition $\psi(T, \varphi) = \psi(T, \varphi + 2\pi)$ is the following.

$$
\psi(T, \varphi) = a_0 + b_0 T + \sum_{n \neq 0} \left( a_n e^{in\varphi} + b_n e^{-in\varphi} \right) \tag{4.11}
$$
4.2 Action principle and charges

The boundary term with fall-off (4.3) is finite

$$\int_B \sqrt{h} \delta A_{\alpha}^{(0)} \partial^\alpha A_{\rho}^{(0)} = \int_B \sqrt{h} \left[ D^a \left( \delta A_{\alpha}^{(0)} A_{\rho}^{(0)} \right) - D^a \delta A_{\alpha}^{(0)} A_{\rho}^{(0)} \right]$$  \hspace{1cm} (4.11)$$

Integration by parts and fixing the asymptotic gauge $D^a A_{\alpha}^{(0)} = 0$ makes the integrand a total divergence. In contrast to higher dimensions, fixing the Lorenz gauge $\nabla_{\mu} A^{\mu}$ is not consistent with our power-law asymptotic expansion: the Lorentz gauge in hyperbolic coordinates is

$$D^a A_{\alpha} + \partial_\rho (\rho^2 A_{\rho}) = 0,$$  \hspace{1cm} (4.12)$$

hence at leading order it implies either $A_{\rho}^{(0)} = 0$ or $A_{\alpha} \sim O(\rho)$. The latter violates the boundary conditions (4.3), while the former eliminates physical solutions and charges (note: the index on $D_a$ is raised by unit $dS_2$ metric). In the asymptotic gauge $D^a A_{\alpha}^{(0)} = 0$, the boundary term (4.11) becomes an integral on boundaries of $B$, which give no contribution when the initial and final data are fixed.

The asymptotic gauge fixing leaves residual gauge transformations satisfying $D_aD^a \lambda = 0$. The conserved charges are obtained by the same method explained before.

$$Q_\lambda = \int_{S^1} d\varphi \left( \partial_\tau \lambda A_{\rho}^{(0)} - \lambda \partial_\tau A_{\rho}^{(0)} \right) = \int_{S^1} d\varphi \left( \partial_+ \lambda A_{\rho}^{(0)} - \lambda \partial_+ A_{\rho}^{(0)} \right) - (+ \leftrightarrow -)$$  \hspace{1cm} (4.13)$$

With solutions

$$A_{\rho}^{(0)}(T, \varphi) = \psi_0^+ + \psi_0^- T + \sum_{n \neq 0} \left( \psi_n^+ e^{inx^+} + \psi_n^- e^{inx^-} \right)$$  \hspace{1cm} (4.14)$$

$$\lambda(T, \varphi) = \lambda_0^+ + \lambda_0^- T + \sum_{n \neq 0} \left( \lambda_n^+ e^{inx^+} + \lambda_n^- e^{inx^-} \right)$$  \hspace{1cm} (4.15)$$

the charge becomes

$$Q_\lambda[A] = \psi_0^+ \lambda_0^- - \psi_0^- \lambda_0^+ + 2i \sum_{n > 0} n(\psi_n^+ \lambda_n^- - \psi_n^- \lambda_n^+)$$  \hspace{1cm} (4.16)$$

The whole set of charges form two copies of angle-dependent U(1) symmetries, labeled by two functions

$$\lambda^\pm = \sum_n \lambda_n^\pm e^{in\varphi}.$$  \hspace{1cm} (4.17)$$

We will cut off one copy of the charges by an antipodal condition, explained below.

**Antipodal condition.** The whole set of solutions (4.10) are regular at light cone. Nevertheless, we opt to impose conditions (3.20) which include physical solutions.\footnote{\scriptsize $A_{\rho}^{(0)}$ is defined up to a constant function; so is the antipodal condition on $A_{\rho}^{(0)}$.}

$$A_{\rho}^{(0)}(T, \varphi) = -A_{\rho}^{(0)}(\pi - T, \varphi + \pi)$$  \hspace{1cm} (4.18)$$
The antipodal map \((T, \varphi) \to (\pi - T, \varphi + \pi)\) is equivalent to \(x^+ \leftrightarrow x^-\). As a result, (4.10) is divided into even and odd parts

\[
A_{(0)}^0(T, \varphi) = c_0 T + \sum_{n \neq 0} \frac{c_n}{n} e^{in\varphi} \sin nT, \quad c_n = c^*_n \quad \text{odd (4.19a)}
\]

\[
\lambda(T, \varphi) = d_0 + \sum_{n \neq 0} d_n e^{in\varphi} \cos nT, \quad d_n = d^*_n \quad \text{even (4.19b)}
\]

By this boundary condition, the field strength is obtained by taking a derivative of \(A_{(0)}^0\). One can explicitly check that for a boosted electric charge, the gauge field lies in (4.19a).

The set of charges consists of one copy of angle-dependent \(\text{U}(1)\) transformations on circle

\[
Q\lambda[A] = -\sum_n d^*_n c_n.
\]

**Boundary conditions and charges at null infinity limit.** In this section we compare our results with those in [17] at null infinity \((\rho \to \infty, T \to 0)\) in Bondi coordinates \((u, r, x^4)\). In that work, the Einstein-Maxwell theory was considered. We will switch the gravity sector off to compare the Maxwell sector with this work. The boundary conditions are

\[
A_r = 0 \quad A_u = \mathcal{O} \left( \ln \frac{r}{r_0} \right) \quad A_\varphi = \mathcal{O} \left( \ln \frac{r}{r_0} \right)
\]

(4.21)

The leading terms in the solution space were parametrized as

\[
A_\varphi = \alpha(u, \varphi) \ln \frac{r}{r_0} + A^0_\varphi(u, \varphi) + \mathcal{O} \left( r^{-1} \right)
\]

(4.22)

\[
A_u = -\lambda \ln \frac{r}{r_0} + A^0_u(u, \varphi) + \mathcal{O} \left( r^{-1} \right)
\]

(4.23)

where \(\alpha(u, \varphi) = -\omega(\varphi) - u\lambda'(\varphi)\) and \(A^0_\varphi = -\lambda'(\varphi) + (A^0_u)'\). Here, the function \(\omega(\varphi)\) is an “integration constant”, and \(A^0_u(u, \varphi)\) is the electromagnetic news.

The field strength tensor at leading order is

\[
F_{u\varphi} = -\lambda'(\varphi) \quad F_{\varphi r} = \frac{\omega(\varphi) + u\lambda'(\varphi)}{r} \quad F_{ur} = \frac{\lambda(\varphi)}{r}
\]

(4.24)

(\(\lambda\) in notation of [17] is a physical component of the gauge field while in our notation it is the gauge parameter not appearing in the field strength) The surface charges in [17] consist of gravitational and gauge parts. By setting the super-rotation and supertranslation generators to zero, and in absence of electromagnetic news function \(A^0_u = 0\), the charge is integrable and conserved and is given by

\[
Q_E = \frac{1}{4\pi G} \int d\varphi \lambda(\varphi) \tilde{E}(\varphi) \quad \text{(Maxwell sector only)}
\]

(4.25)

where \(\tilde{E}(\varphi)\) is the leading term of the gauge parameter in this specific case. Clearly, this non-gravitational subgroup of the asymptotic symmetry group is \(\text{U}(1)\) at every angle.
In our analysis, the dual scalar field behaves as $\Phi \sim O(1)$ in $\rho$-expansion. According to (4.7), the leading order $\Phi^{(0)}$ is even under antipodal condition, and the general solution is given in (4.19b), which has the following behaviour near future null infinity at $T = 0$

$$\Phi^{(0)}(T, \varphi) = p_0 + \sum_{n \neq 0} p_n e^{i n \varphi} \cos nT = \Phi^{(0)} - \Phi^{(0)\nu} \frac{u}{r} + \cdots \quad (4.26)$$

where we have used $\tau^2 \approx -2u/r$, and defined

$$\Phi^{(0)} = \sum_n p_n e^{i n \varphi} \quad (4.27)$$

(note that the superscript $(n)$ shows the $\rho$-expansion at spatial infinity).

For the subleading order $\Phi^{(1)}$, we take it to be odd (like $A^{(0)}_\rho$) under antipodal map. In this case, the solution is given in (4.19a), and at $T \to 0$ it behaves as

$$\Phi^{(1)}(T, \varphi) = q_0 T + \sum_{n \neq 0} q_n e^{i n \varphi} \sin nT = \Phi^{(1)} - \frac{\Phi^{(1)}_r}{r} + \cdots \quad (4.28)$$

where we have defined

$$\Phi^{(1)} = \sum_n q_n e^{i n \varphi} \quad (4.29)$$

The dual field strength at leading order will be

$$F_{\rho r} = \epsilon^{\rho r} \partial_r \Phi = \frac{\Phi^{(0)\nu}}{r} + \cdots \quad (4.30a)$$

$$F_{\rho \varphi} = \epsilon_{\rho \varphi} \left( \partial_r \Phi - \partial_\varphi \Phi \right) = -\Phi^{(0)\nu} + \cdots \quad (4.30b)$$

$$F_{\varphi r} = \epsilon_{\varphi r} \partial_r \Phi = \frac{-\Phi^{(1)} + \omega \Phi^{(0)\nu}}{r} + \cdots \quad (4.30c)$$

These are exactly the leading null infinity behaviour (4.24) of ref. [17], for $\lambda(\varphi) = \Phi^{(0)\nu}$ and $\omega(\varphi) = -\Phi^{(1)}$.

There only remains to check the agreement of conserved charges. Close to the future null infinity at $T = 0$, the fields behave as follows

$$A^{(0)}_\rho = \tilde{A}^{(0)}_\rho (\varphi) T + O(T^3) \quad \text{gauge field} \quad (4.31a)$$

$$\lambda = \tilde{\lambda}(\varphi) + O(T^2) \quad \text{gauge parameter} \quad (4.31b)$$

At null infinity, only the second term of the charge (4.13) remains non-vanishing,

$$Q_\lambda = \int_{S^1} d\varphi \tilde{\lambda} \tilde{A}^{(0)}_\rho = \int_{S^1} d\varphi \tilde{\lambda} \Phi^{(0)\nu} \quad (4.32)$$

where we have used $\partial_r \tilde{A}^{(0)}_\rho = \Phi^{(0)\nu}$. This expression is in agreement with (4.25) by substitution $\tilde{\lambda} \to \tilde{E}$ and $\tilde{\Phi}^{(0)\nu} \to \lambda$. In conclusion, the spatial infinity power-law boundary condition (4.3) advocated here reproduces the leading order field strength behavior at null infinity, prescribed in [17]. Finally, we mention that [17] takes a larger set of initial data (and hence solution space) than considered here, by including logarithmic terms in the asymptotic expansion. This difference is however irrelevant to the asymptotic symmetry group.
5 Minkowski isometry charges

In previous sections, we computed the conserved charges for large gauge transformations. We can also apply the same method to obtain the charges for the background ISO$(d - 1, 1)$ isometries. The Minkowski Killing vectors include Lorentz transformations $L$ and Translations $P$. Denoting the Minkowski coordinates by $X^\mu$, we have $X^\mu X_\mu = \rho^2$ and we define

$$
\vec{X}^\mu = \frac{X^\mu}{\rho}, \quad \vec{M}^\mu_a = \frac{\partial \vec{X}^\mu}{\partial x^a}.
$$

The isometries in hyperbolic coordinates are the following.

Lorentz: $\xi_L$

$$
L^\mu = 2 \vec{X}^\mu \vec{M}^\nu h^{ab} \partial_b
$$

AdS-Translation: $\xi_T$

$$
P^\mu = -\vec{X}^\mu \partial_\rho - \frac{1}{\rho} \vec{M}^\mu h^{ab} \partial_b.
$$

Variation of Maxwell action under Killing vectors is the following

$$
\delta_\xi S = -\frac{1}{4} \int_M \sqrt{g} \xi^\alpha \partial_\alpha \left( F_{\mu\nu} F^{\mu\nu} \right) = -\frac{1}{4} \int_M \sqrt{g} \nabla_\alpha \left( \xi^\alpha F_{\mu\nu} F^{\mu\nu} \right)
$$

$$
= -\frac{1}{4} \int_{\partial M} \sqrt{\gamma} n_\alpha \left( \xi^\alpha F_{\mu\nu} F^{\mu\nu} \right)
$$

where in the second equality we used Killing property $\nabla_\mu \xi^\mu = 0$, and $\gamma$ is the induced metric on the boundary. Therefore, the action is invariant up to boundary terms, as expected. Next, we compute the on-shell surface terms of the action as in (2.8):

$$
\int_{\partial M} \sqrt{\gamma} n_\alpha \delta_\xi A_\mu F^{\mu\alpha} = -\int_{\partial M} \sqrt{\gamma} n_\alpha \left( \xi^\alpha F_{\mu\mu} + \partial_\mu (\xi^\alpha A_\alpha) \right) F^{\mu\alpha}
$$

Equating (5.4) and (5.5) leads to the on-shell identity

$$
\int_{\partial M} \sqrt{\gamma} n_\alpha \left( \xi_\beta T^{\beta\alpha} + \partial_\mu (\xi^\alpha A_\beta) F^{\mu\alpha} \right) = 0
$$

where

$$
T_{\mu\nu} = F_{\mu}^{\phantom{\mu}^\alpha} F_{\alpha\nu} - \frac{1}{4} g_{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}
$$

is the energy-momentum tensor of Maxwell theory. The boundary $\partial M$ consists of two spacelike hypersurfaces $I_{1,2}$ and a timelike hypersurface $B$. If the contribution from timelike boundary $B$ vanished, (5.6) would imply a conservation law, and the charge would be the left-hand-side of (5.6) integrated on an arbitrary spacelike hypersurface $I$. However, it turns out that the boundary contribution is non-vanishing for the prescribed boundary conditions, equal to

$$
\int_B \rho^{d-1} \sqrt{h} \left( \xi_\alpha T^{\alpha\rho} + D_\alpha (\xi^\beta A_\beta) F^{\alpha\rho} \right)
$$

According to boundary conditions (3.1), $T^{\alpha\rho} \sim \mathcal{O} (\rho^{4-2d})$, so the first term diminishes at large $\rho$ for $d > 3$. Restricting the discussion to $d > 3$, we have $A_\alpha = \partial_\alpha \phi + \cdots \sim \mathcal{O} (1)$. The second term above is finite for Lorentz transformations and vanishes for translations.
As a result, there boundary term is completely vanishing for translations in $d > 3$. The second term in (5.8) for Lorentz transformations is

$$
\int_B \sqrt{h} D_a (\xi^b \partial_b \phi) \partial^a \psi = \int_{\partial B} \sqrt{h} m_a D^a (\xi^b \partial_b \phi) \psi + \int_B \sqrt{h} D_a D^a (\xi^b \partial_b \phi) \psi \quad (5.9)
$$

The second term is vanishing for Lorentz transformations which are the Killing vectors of boundary de Sitter space:

$$
D_a D^a (\xi^b \partial_b \phi) = D_a D^a (\mathcal{L}_\xi \phi) = \mathcal{L}_\xi (D_a D^a \phi) = 0 \quad D^2 \phi = 0 \text{ by gauge condition} \quad (5.10)
$$

Thus we have shown that the contribution of the timelike boundary $B$ to (5.6) is on its boundaries $S^{d-2}$, given in (5.9). As a result, (5.6) is equivalent to

$$
\int_{I_2} \sqrt{\gamma} n_\alpha \left( \xi_\beta T^{\beta \alpha} + \partial_\mu (\xi_\beta A_\beta) F^{\mu \alpha} \right) - \int_{I_1} \sqrt{\gamma} n_\alpha \left( \xi_\beta T^{\beta \alpha} + \partial_\mu (\xi_\beta A_\beta) F^{\mu \alpha} \right) = -\int_{\partial B} \sqrt{h} m_a D^a (\xi^b \partial_b \phi) \psi \quad (5.11)
$$

We conclude that the following quantity is equal on $I_2$ and $I_1$:

$$
Q_\xi = \int \sqrt{\gamma} n_\alpha \xi_\beta T^{\beta \alpha} + \int \sqrt{h} m_a \left[ D^a (\xi^b \partial_b \phi) \psi - (\xi^b \partial_b \phi) D^a \psi \right] \quad d > 3 \quad (5.12)
$$

This completes our derivation of conserved charges for Minkowski Killing vector fields. The surface terms in (5.13) are present only for Lorentz transformations and delicate interplay between fall-off conditions and gauge conditions lead to conserved quantity, directly derived from action. We must note that the boundary flux is non-vanishing for $d = 3$, which needs more careful study. Perhaps additional restrictions on boundary conditions are needed to ensure conservation of isometry charges in three dimensional case.

Although we did not use the covariant phase space method here, one can show that Lorentz transformations lack integrable canonical charges via original symplectic form of the theory (see a Hamiltonian analysis in [19]). The modified conserved symplectic form (3.37), (3.39), however, provides integrable charges equal to (5.13).

6 Discussion

In this note, we worked out asymptotic symmetries of Maxwell theory in three and higher dimension at spatial infinity. We tried to bypass standard methods for computing surface charges, by making the action principle well-defined, applying a gauge transformation on it, and interpreting the resulting conserved quantity as the charge. This work excludes magnetic charges to avoid technical difficulties, although they are discussed in various four dimensional treatments.

We showed that regularity of field strength tensor at light cone implies a certain antipodal condition on de Sitter space in four and higher dimensions, which was familiar in dS/CFT context. In addition, the charges depend on the scalar field $\psi$ on de Sitter space in all dimensions. It is interesting if dS/CFT quantum considerations applied to $\psi$ have implications on Maxwell theory.
In three dimension, the solution space is more transparent as the asymptotic de Sitter space is conformally flat. The light cone regularity argument does not work in three dimension, although it is satisfied by the solution for moving electric charges. For this simple model, we could solve the gauge condition and translate the boundary conditions into Bondi coordinates which are better suited for null infinity discussions.

As an interesting generalization, note that in three dimensions, non-trivial vorticity for gauge field is possible. Gauge transformations considered here are regular, so preserve vorticity. Addition of singular gauge transformations which lead to vorticity might lead to an unexpected relation with electric charges considered here; as is the case in four dimensions [12, 33].

We compared our treatment with Hamiltonian formulations of the theory. Symplectic form and on-shell action are finite in $d > 4$ and their divergences $d = 3, 4$ cancel in inertial frames by virtue of parity conditions. Nonetheless, cancellation in arbitrary slice of asymptotic de Sitter space remains elusive.

Finally, we derived the isometry charges from variation of the action. Although the translation charges involve only the energy-momentum tensor of the theory, there are subtle surface terms in case of Lorentz transformations which appear in the final expression of the charges. We commented on similar phenomena in covariant phase space and Hamiltonian methods. The boundary flux is non-vanishing in three-dimensional case, which means that there are no well-defined isometry charges in that case with present boundary conditions.

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