The connected components of the space of Alexandrov surfaces

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Abstract

Denote by $\mathcal{A}(\kappa)$ the set of all compact Alexandrov surfaces with curvature bounded below by $\kappa$ without boundary, endowed with the topology induced by the Gromov-Hausdorff metric. We determine the connected components of $\mathcal{A}(\kappa)$ and of its closure.

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1 Introduction and results

In this note, by an Alexandrov surface we understand a compact 2-dimensional Alexandrov space with curvature bounded below by $\kappa$, without boundary. Roughly speaking, an Alexandrov surface is a closed topological surface endowed with an intrinsic geodesic distance satisfying Toponogov’s angle comparison condition. See [5] or [11] for definitions and basic facts about such spaces.

Denote by $\mathcal{A}(\kappa)$ the set of all Alexandrov surfaces. Endowed with the Gromov-Hausdorff metric $d_{GH}$, $\mathcal{A}(\kappa)$ becomes a Baire space in which Riemannian surfaces form a dense subset [6].

Perelman’s stability theorem (see [7], [9]) states, in our case, that close Alexandrov surfaces are homeomorphic, so Alexandrov surfaces with different topology are in different connected components of $\mathcal{A}(\kappa)$. Here we show that homeomorphic Alexandrov surfaces are in the same component of $\mathcal{A}(\kappa)$. 

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Let \( \mathcal{A}(\kappa, \chi, o) \) denote the set of all surfaces in \( \mathcal{A}(\kappa) \) of Euler-Poincaré characteristic \( \chi \) and orientability \( o \), where \( o = 1 \) if the surface is orientable and \( o = -1 \) otherwise.

**Theorem 1.** If non-empty, \( \mathcal{A}(\kappa, \chi, o) \) is a connected component of \( \mathcal{A}(\kappa) \), for \( \kappa \in \mathbb{R}, \chi \leq 2 \) and \( o = \pm 1 \).

A special motivation for this result comes from the study of most (in the sense of Baire category) Alexandrov surfaces. For example, we prove in [10] that most Alexandrov surfaces have either infinitely many simple closed geodesics, or no such geodesic, depending on the value of \( \kappa \) and the connected component of \( \mathcal{A}(\kappa) \) to which they belong. Moreover, for descriptions of most Alexandrov surfaces given in [1] and [6], one has to exclude from the whole space \( \mathcal{A}(0) \) its components consisting of flat surfaces.

Denote by \( \bar{\mathcal{A}}(\kappa) \) (respectively \( \bar{\mathcal{A}}(\kappa, \chi, o) \)) the closure with respect to \( d_{GH} \) of \( \mathcal{A}(\kappa) \) (respectively \( \mathcal{A}(\kappa, \chi, o) \)) in the space of all compact metric spaces. Using Theorem 1, we can also give the connected components of \( \bar{\mathcal{A}}(\kappa) \).

**Theorem 2.** If \( \kappa \geq 0 \), \( \bar{\mathcal{A}}(\kappa) \) is connected. If \( \kappa < 0 \), the connected components of \( \bar{\mathcal{A}}(\kappa) \) are \( \bigcup_{\chi \geq 0, o = \pm 1} \bar{\mathcal{A}}(\kappa, \chi, o) \) and \( \mathcal{A}(\kappa, \chi, o) \) (\( \chi = -1, -2, \ldots, o = \pm 1 \)).

### 2 Proofs

Perelman’s stability theorem can be found, for example, in [9] or [7]; we only need a particular form of it.

**Lemma 1.** Each Alexandrov surface \( A \) has a neighbourhood in \( \mathcal{A}(\kappa) \) whose elements are all homeomorphic to \( A \).

Let \( \mathbb{M}_\kappa^d \) stand for the simply-connected and complete Riemannian manifold of dimension \( d \) and constant curvature \( \kappa \).

Denote by \( \mathcal{R}(\kappa) \) the set of all closed Riemannian surfaces with Gauss curvature at least \( \kappa \), and by \( \mathcal{P}(\kappa) \) the set of all \( \kappa \)-polyhedra. Recall that a \( \kappa \)-polyhedron is an Alexandrov surface obtained by naturally gluing finitely many geodesic polygons from \( \mathbb{M}_\kappa^2 \).

A formal proof for the following result can be found in [6].

**Lemma 2.** The sets \( \mathcal{R}(\kappa) \) and \( \mathcal{P}(\kappa) \) are dense in \( \mathcal{A}(\kappa) \).
A convex surface in $\mathbb{M}_3^3$ is the boundary of a compact convex subset of $\mathbb{M}_3^3$ with non-empty interior. Such a surface is endowed with the so-called intrinsic metric: the distance between two points is the length (measured with the metric of $\mathbb{M}_3^3$) of a shortest curve joining them and lying on the surface.

**Lemma 3.** [2] Every convex surface in $\mathbb{M}_3^3$ belongs to $\mathcal{A}(\kappa, 2, 1)$. Conversely, every surface $A \in \mathcal{A}(\kappa, 2, 1)$ is isometric to some convex surface in $\mathbb{M}_3^3$.

In order to settled the case of $\mathcal{A}(0, 0, o)$, we need the following lemma.

**Lemma 4.** $\mathcal{A}(0, 0, 1)$ contains only flat tori, and $\mathcal{A}(0, 0, -1)$ contains only flat Klein bottles.

**Proof.** Recall that geodesic triangulations with arbitrarily small triangle exist for any Alexandrov surface [2].

Consider $A \in \mathcal{A}(0, 0, 1)$ and a geodesic triangulation $T = \{\Delta_i\}$ of $A$. For each $\Delta_i$, consider a comparison triangle $\tilde{\Delta}_i$ (i.e., a triangle with the same edge lengths) in $\mathbb{M}_0^2$. Glue together the triangles $\tilde{\Delta}_i$ to obtain a surface $P$, in the same way the triangles $\Delta_i$ are glued together to compose $A$. By the definition of Alexandrov surfaces, the angles of $\tilde{\Delta}_i$ are lower than or equal to the angles of $\Delta_i$. It follows that the total angles $\theta_1, \ldots, \theta_n$ of $P$ around its (combinatorial) vertices are at most $2\pi$, hence $P$ is a 0-polyhedron. By the Gauss-Bonnet formula for polyhedra,

$$0 = 2\pi \chi = \sum_{i=1}^{n} (2\pi - \theta_i),$$

whence $\theta_i = 2\pi$ and $P$ is indeed a flat torus.

Now consider a sequence of finer and finer triangulations $T_m$ of $A$ and denote by $P_m$ the corresponding flat tori ($m \in \mathbb{N}$). A result of Alexandrov and Zalgaller (Theorem 10 in [3, p. 90]) assures that $P_m$ converges to $A$, which is therefore flat.

The same argument holds for $\mathcal{A}(0, 0, -1)$. □

Now we are in a position to prove Theorem 1.

Notice that, for $\kappa' > \kappa$, $\mathcal{A}(\kappa')$ is a nowhere dense subset of $\mathcal{A}(\kappa)$; indeed, $\mathcal{A}(\kappa')$ is closed and its complement contains the $\kappa$-polyhedra, which are dense in $\mathcal{A}(\kappa)$. Therefore, there is no direct relationship between the connected components of $\mathcal{A}(\kappa)$ and those of $\mathcal{A}(\kappa')$. 

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Proof of Theorem [1]. By Lemma [1] each set $A(\kappa, \chi, o)$ is open in $A(\kappa)$, so we just need to prove that it is connected.

Each Alexandrov surface $A$ is in particular a metric space. Multiplying all distances in $A \in A(\kappa)$ with the same constant $\delta > 0$ provides another Alexandrov surface, denoted by $\delta A$, which belongs to $A(\frac{\kappa}{\delta^2})$. Moreover, it is easy to see that for any metric spaces $M, N$ we have $d_{GH}(\delta M, \delta N) = \delta d(M, N)$. So there is a natural homothety between $A(\kappa)$ and $A(\kappa \delta^2)$, and therefore we may assume that

$$\kappa \in \{-1, 0, 1\}.$$ 

We consider several cases.

Case 1. The sets $A(-1, \chi, o)$ are connected in $A(1)$.

Choose $A_0, A_1 \in A(-1, \chi, o) \cap R(-1)$. There exist a differentiable surface $S$ of Euler-Poincaré characteristic $\chi$ and orientability $o$, and Riemannian metrics $g_0, g_1$ on $S$ such that $A_i$ is isometric to $(S, g_i) (i = 0,1)$. For $\lambda \in [0, 1]$ we set

$$\tilde{g}_{\lambda} = \lambda g_1 + (1 - \lambda) g_0.$$ 

Denote by $\kappa_{\lambda}$ the minimal value of the Gauss curvature of $\tilde{g}_{\lambda}$, and define the Riemannian metric $g_{\lambda}$ on $S$ by

$$g_{\lambda} = \left\{ \begin{array}{ll} \tilde{g}_{\lambda} & \text{if } \kappa_{\lambda} \geq -1, \\ \frac{\tilde{g}_{\lambda}}{\sqrt{-\kappa_{\lambda}}} & \text{if } \kappa_{\lambda} < -1. \end{array} \right.$$ 

A straightforward computation shows that the Gauss curvature $K_{\lambda}$ of $g_{\lambda}$ verifies $K_{\lambda} \geq -1$.

Denote by $\gamma$ the (obviously continuous) canonical map from the set of Riemannian structures on $S$ to $A(-1, \chi, o)$, which maps $g$ to $(S, g)$. Then $A_{\lambda} \overset{\text{def}}{=} \gamma(g_{\lambda})$ defines a path from $A_0$ to $A_1$. Hence $A(-1, \chi, o) \cap R(-1)$ is connected and, by the density of $R(-1)$, so is $A(-1, \chi, o)$.

Next we treat the connected components of $A(0)$.

Case 2. The sets $A(0, 0, 1)$ and $A(0, 0, -1)$ are connected in $A(0)$.

By Lemma[4] the set $A(0, 0, 1)$ contains only flat tori, hence it is continuously parametrized by the parameters describing the fundamental domains. Similarly for $A(0, 0, -1)$, which consists of flat Klein bottles.

Case 3. The set $A(0, 2, 1)$ is connected in $A(0)$.
Denote by $S$ the space of all convex surfaces in $\mathbb{R}^3$, endowed with the Pompeiu-Hausdorff metric. Lemma 3 shows that any surface $A \in A(0, 2, 1)$ can be realized as a convex surface in $\mathbb{R}^3$.

Given two convex surfaces $S_0, S_1$, define for $\lambda \in [0, 1]$

$$S_\lambda = \partial(\lambda \text{conv}(S_1) + (1 - \lambda)\text{conv}(S_0)),$$

where $\partial C$ stands for the boundary of $C$, $\text{conv}(S)$ for the convex hull of $S$, and $+$ for the Minkowski sum. Then $S_\lambda \in S$ and we have a path in $S$ joining $S_0$ to $S_1$. Since the canonical map $\sigma$ from $S$ to $A(0, 2, 1)$ is continuous [2, Theorem 1 in Chapter 4], we obtain a path in $A(0, 2, 1)$.

Case 4. The set $A(0, 1, -1)$ is connected in $A(0)$.

Consider surfaces $A_0, A_1$ in $A(0, 1, -1)$ as quotients of centrally-symmetric convex surfaces $S_0, S_1$ via antipodal identification, $A_i = \sigma(S_i)/\mathbb{Z}_2$ ($i = 0, 1$). Then the surface $S_\lambda$ defined by (1) is also centrally-symmetric, and therefore $A_\lambda = \sigma(S_\lambda)/\mathbb{Z}_2$ defines a path in $A(0, 1, -1)$ from $A_0$ to $A_1$.

We finally treat the two connected components of $A(1)$.

Case 5. The set $A(1, 2, 1)$ is connected in $A(1)$.

Consider in $\mathbb{R}^4$ the subspace $\mathbb{R}^3 = \mathbb{R}^3 \times \{0\}$, and the open half-sphere $H$ of center $c = (0, 0, 0, 1)$ and radius 1 included in $\mathbb{R}^3 \times [0, 1]$.

Let $q : \mathbb{R}^3 \to H$ be the homeomorphism associating to each $x \in \mathbb{R}^3$ the intersection point of the segment $[xc]$ with $H$. Clearly, $q$ maps segments of $\mathbb{R}^3$ to geodesic segments of $H$, and thus it maps bijectively convex sets in $\mathbb{R}^3$ to convex sets in $H$. Denote $S_H$ the set of convex surfaces in $H$. We can define $Q : S \to S_H$ by $Q(S) \overset{\text{def}}{=} q(S)$. Hence $S_H$ is homeomorphic to $S$, which is connected by Case (4).

Consider now two surfaces $A_0, A_1 \in A(1, 2, 1)$ and choose

$$\mu < \min \left\{ \frac{\pi}{2 \text{diam}(A_0)}, \frac{\pi}{2 \text{diam}(A_1)}, 1 \right\}.$$

Obviously, $A_i$ is path-connected to $\mu A_i$ in $A(1, 2, 1)$, and the diameter of $\mu A_i$ is less than $\pi/2$ ($i = 0, 1$). By Lemma 3, $\mu A_i$ is isometric to a surface $S_i$ in $M_1^3$; moreover, the smallness of $\mu$ easily implies that $S_i$ is isometric to a surface in $S_H$, and $S_H$ is connected.

Case 6. The set $A(1, 1, -1)$ is connected in $A(1)$.
This follows directly from the previous argument, because the universal covering of any surface \( \tilde{A} \in \mathcal{A}(1, 1, -1) \) is a surface \( A \in \mathcal{A}(1, 2, 1) \) endowed with an isometric involution without fixed points, \( \tilde{A} = A/\mathbb{Z}_2 \).

The proof of Theorem 1 is complete. \( \square \)

Recall that the 2-dimensional Hausdorff measure \( \mu(A) \) is always finite and positive for \( A \in \mathcal{A}(\kappa) \). The following result is Corollary 10.10.11 in [4, p. 401], stated in our framework.

**Lemma 5.** Let \( A_n \in \mathcal{A}(\kappa) \) converge to a compact space \( X \). Then \( \dim(X) < 2 \) if and only if \( \mu(A_n) \to 0 \).

**Proof of Theorem 2.** We may assume, as in the proof of Theorem 1, that \( \kappa \in \{-1, 0, 1\} \).

To prove that \( \bar{A}(\kappa) \) is connected for \( \kappa \geq 0 \), it suffices to show that the space consisting of a single point belongs to the closure of any connected component of \( \mathcal{A}(\kappa) \). This is indeed the case, because for any \( A \in \mathcal{A}(\kappa, \chi, o) \) and \( 0 < \delta \leq 1 \) we have \( \delta A \in \mathcal{A}(\kappa, \chi, o) \), and \( \lim_{\delta \to 0} \delta A \) is a point.

This also implies that

\[
\bigcup_{\chi = 0, 1, 2} \bar{A}(-1, \chi, o)
\]

is connected.

Consider now \( A \in \mathcal{A}(-1, \chi, o) \) with \( \chi < 0 \). Let \( \omega \) be the curvature measure on \( A \) (see [3]). Y. Machigashira [8] proved that \( \omega \geq \kappa \mu \) holds for any Alexandrov surface of curvature bounded below by \( \kappa \). Therefore, by a variant of the Gauss-Bonnet theorem,

\[
2\pi\chi = \omega(A) \geq \kappa \mu(A) = -\mu(A),
\]

hence \( \mu(A) \geq 2\pi|\chi| \). Lemma 5 shows now that \( \mathcal{A}(-1, \chi, o) \) is closed in the space of all compact metric spaces \( (o = \pm 1, \chi < 0) \). \( \square \)

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