On the time–modulation of the K–shell electron capture decay of H-like $^{140}$Pr$^{58+}$ ions produced by neutrino mass differences

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According to recent experimental data at GSI, the rate of the number of daughter ions $^{140}$Ce$^{58+}$, produced by the nuclear K–shell electron capture (EC) decay of the H–like $^{140}$Pr$^{58+}$, is modulated in time with a period $T_{EC} = 7.06(8)$ sec and an amplitude $a_{EC} = 0.18(3)$. We show that this phenomenon can be explained by neutrino mass differences and derive a value for the difference of squared masses $\Delta m_2^2 = m_2^2 - m_1^2 = 2.22(3) \times 10^{-4}$ eV$^2$.

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Introduction

The experimental investigation of the K–shell electron capture (EC) and $\beta^+$ decays of the H–like $^{140}$Pr$^{58+}$ and the He–like $^{140}$Pm$^{58+}$ ions, has been recently carried out in the Experimental Storage Ring (ESR) at GSI in Darmstadt [1]. The obtained results showed a dependence of the weak decay rates on the electron structure of the heavy ions. As has been shown in [2], the experimental data on the ratios of the weak $\beta^+$–decays of the H–like $^{140}$Pr$^{58+}$ and the He–like $^{140}$Pm$^{58+}$ ions, obtained in GSI [1], can be described within standard theory of weak interactions of heavy ions and massless Dirac neutrinos [3] with an accuracy better than 3%.

However, a very recent measurement of the time–dependence of the rate of the number of daughter ions $^{140}$Ce$^{58+}$ from the EC–decay of the H–like $^{140}$Pr$^{58+}$ ion, i.e. $^{140}$Pr$^{58+} \rightarrow ^{140}$Ce$^{58+} + \nu_e$, showed a time–modulation of the exponential decay with a period $T_{EC} = 7.06(8)$ s and an amplitude $a_{EC} = 0.18(3)$ [1]. Since the rate of the number of daughter ions is defined by

$$\frac{dN_{EC}^d(t)}{dt} = \lambda_{EC}(t) N_m(t),$$

(1)

where $\lambda_{EC}(t)$ is the EC–decay rate and $N_m(t)$ is the number of mother ions $^{140}$Pr$^{58+}$, the time–modulation of $dN_{EC}^d(t)/dt$ implies a periodic time–dependence of the EC–decay rate $\lambda_{EC}(t)$ [4]

$$\lambda_{EC}(t) = \lambda_{EC} \left\{ 1 + a_{EC} \cos \left( \frac{2\pi t}{T_{EC}} + \phi_{EC} \right) \right\},$$

(2)

where $a_{EC}$, $T_{EC}$ and $\phi_{EC}$ are the amplitude, period and phase of the time–dependent term [4].

Nowadays the existence of massive neutrinos, neutrino–flavour mixing and neutrino oscillations is well established experimentally and elaborated theoretically [5]. However, these phenomena concerned mainly the propagation of solar neutrinos and reactor anti–neutrinos in space and time combined with neutrino oscillations [5]. We show that the observed time–modulation of the rate of the number of daughter ions $dN_{EC}^d(t)/dt$ in the EC–decay of the H–like $^{140}$Pr$^{58+}$ ions is not caused by neutrino oscillations but can be explained by a kind of quantum beats [6] due to mass differences of neutrino mass–eigenstates [5]. This can provide a new method for studying of massive
neutrino mixing, neutrino mass differences and neutrino vacuum polarisation.

**Amplitudes of EC–decays of H–like heavy ions**

The Hamilton operator \( H_W(t) \) of the weak interactions, responsible for the EC and \( \beta^+ \) decays of the H–like heavy ions, can be taken in the standard form \([3]\) but accounting for the neutrino–flavour mixing \([5]\) (see also \([7]–[9]\)). This gives

\[
H_W^{(j)}(t) = \sum_j U_{e j} H_W^{(j)}(t),
\]

where \( U_{e j} \) are matrix elements of the neutrino mixing matrix \( U \). The Hamilton operator of the weak interactions \( H_W^{(j)}(t) \) is defined by

\[
H_W^{(j)}(t) = G_F \sqrt{\frac{\alpha}{2}} V_{ud} \int d^3 x [\bar{\psi}_e(x) \gamma^\mu (1 - g A \gamma^5) \psi_p(x)]
\times [\bar{\psi}_{\nu j}(x) \gamma^\mu (1 - 45^5) \psi_{\nu j}(x)],
\]

with standard notations \([2,3]\).

The massive neutrinos \( \nu_j \) in the final state of the EC–decay \( m \to d + \nu_e \), where \( m \) and \( d \) are the mother and daughter ions, are indistinguishable in principle, since the electron is entangled with the electron neutrino \( \nu_e \) only \([8]\), which is the superposition of the neutrino mass–eigenstates \( |\nu_e\rangle = \sum_j U_{e j} |\nu_j\rangle \). Such an indistinguishability of massive neutrinos \( \nu_j \) requires to take the amplitude \( A(m \to d + \nu_e)(t) \) of the EC–decay as a coherent sum of the amplitudes \( A(m \to d + \nu_j)(t) \) of the \( m \to d + \nu_j \) decays, describing three alternative ways of the evolution of the initial system, i.e. the mother ion \( m \) into the daughter ion \( d \). According to Feynman \([10]\), the probability amplitude of the evolution of the quantum system should be taken in the form of a coherent sum of the probability amplitudes of every alternative way of the evolution (see also \([11]\)) in contrast to the recent assertion by Giunti \([12]\) and Kienert et al. \([13]\). As a result we get

\[
A(m \to d + \nu_j)(t) = \sum_j U_{e j} A(m \to d + \nu_j)(t),
\]

where the coefficients \( U_{e j} \) take into account that the electron couples to the electron neutrino only.

In the standard time–dependent perturbation theory \([14]\) the amplitudes \( A(m \to d + \nu_j)(t) \), defined in the rest frame of the mother ion, are given by

\[
A(m \to d + \nu_j)(t) =
= -i \int_{-\infty}^t d\tau e^{\varepsilon \tau} \langle d(q) | H_W^{(j)}(\tau) | m(\bar{0}) \rangle,
\]

where \( \bar{q} \) and \( \bar{k}_j \) are 3–momenta of the daughter ion and the neutrino mass–eigenstate \( \nu_j \), respectively. For the regularization of the integral over time we use the \( \varepsilon \to 0 \) regularization procedure.

For the description of the time modulated interference term in the EC–decay rate of the H–like heavy ion Eq.\([4]\) we assume a non–conservation of 3–momenta of neutrino mass–eigenstates in order to deal with differences of 3–momenta of massive neutrinos playing important role in our analysis of the interference term. Non–conservation of 3–momenta of massive neutrinos can be caused by the uncertainties of momenta of the bound electron, proton and neutron in the elementary \( e^- + p \to n + \nu_j \) transition of the EC–decay of the H–like heavy ion (see also \([15]\)).

A quantum mechanical description of the EC–decays of the H–like heavy ions with different 3–momenta of neutrino mass–eigenstates can be carried out using the wave functions of massive neutrinos in the form of the Gaussian wave packets \([16]\)

\[
\psi_{\nu j}(\vec{r}, t) = (2\pi \delta^2)^{3/2} \int d^3 \vec{k} e^{\frac{i}{\hbar} \delta^2(\vec{k} - \vec{E}_j)^2}\times e^{i \vec{k} \cdot \vec{r} - i E_j(\vec{k}) t} u_{\nu j}(\vec{k}, \sigma_{\nu j}),
\]

where \( \delta \) is a spatial spread of the massive neutrino \( \nu_j \), \( \vec{k}_j \) is the neutrino momentum and \( E_j(\vec{k}) = \sqrt{\vec{k}^2 + m_j^2} \) is the energy of a plane wave with the momentum \( \vec{k} \), \( u_{\nu j}(\vec{k}, \sigma_{\nu j}) \) is the Dirac bispinor of the massive neutrino \( \nu_j \). In the limit \( \delta \to \infty \) the wave function \([6]\) reduces to a plane wave.

Following \([2]\), we obtain the amplitude of the EC–decay as a function of time \( t \)

\[
A(m \to d + \nu_j)(t) = -\sqrt{3} 2 M_m^2 E_d(\bar{q}) \times M_{GT} \langle \psi_1^{(Z)} | \langle 2\pi \delta^2)^{3/2} \sum_j U_{e j} \sqrt{E_j(\bar{q})}
\times e^{-\frac{i}{\hbar} \delta^2(\bar{q} + \bar{k}_j)^2} \frac{i (\Delta E_j(\bar{q}) - i \varepsilon) t}{\Delta E_j(\bar{q}) - i \varepsilon} \delta_{M_F, -\frac{1}{2}},
\]

where
where $\Delta E_j(\bar{q}) = E_d(\bar{q}) + E_j(\bar{q}) - M_m$ is the energy difference of the final and initial state, $\bar{q}$ is the 3–momentum of the daughter ion $d$, $E_d(\bar{q}) = \sqrt{\bar{q}^2 + M_d^2}$ and $M_m$ are the energies of the daughter and mother ions, respectively, and $E_j(\bar{q})$ is the energy of the neutrino mass–eigenstate $\nu_j$ with momentum $\bar{q}$. $\mathcal{M}_{GT}$ is the nuclear matrix element of the Gamow–Teller transition $m \to d$ and $|\psi^{(Z)}_{1s}\rangle$ is the wave function of the bound electron in the H–like heavy ion, averaged over the nuclear density $[2]$.

The rate of the neutrino spectrum of the $EC$–decay as a function of time is defined by

$$\frac{dN_{\nu_e}(t)}{dt} = \frac{1}{2M_m} \int \frac{d^3q}{(2\pi)^3 2E_d(\bar{q})} \times \frac{1}{2F + 1} \frac{d}{dt} \sum_{M_F = \pm \frac{1}{2}} |A(m \to d + \nu_e)(t)|^2. \tag{8}$$

The integration over $\bar{q}$ can be carried out if the width $\delta$ of the wave packets of the wave functions of neutrino mass–eigenstates is sufficiently large, so that the Gaussian function $e^{-\delta^2 (\bar{q} - \bar{p})^2}$ is localised in the vicinity of $\bar{q} \simeq -\bar{p}$, where $\bar{p} = \vec{k}_j$ for the diagonal term and $\bar{p} = (\vec{k}_i + \vec{k}_j)/2 = \vec{k}_ij^{\dagger}(+)$ for the interference term. The result of the integration over $\bar{q}$ is given by two terms

$$\frac{dN_{\nu_e}(t)}{dt} = \frac{dN_{\nu_e}^{(1)}(t)}{dt} + \frac{dN_{\nu_e}^{(2)}(t)}{dt}, \tag{9}$$

where we have denoted the diagonal term

$$\frac{dN_{\nu_e}^{(1)}(t)}{dt} = \frac{3}{2F + 1} |\mathcal{M}_{GT}|^2 |\langle \psi^{(Z)}_{1s}\rangle|^2 (\pi \delta^2)^{3/2} \times \sum_j U^*_{ej} U_{cj} E_j(\vec{k}_j) \frac{2\varepsilon}{(\Delta E_j(\vec{k}_j))^2 + \varepsilon^2} e^{\varepsilon t} \tag{10}$$

and the interference term

$$\frac{dN_{\nu_e}^{(2)}(t)}{dt} = \frac{3}{2F + 1} |\mathcal{M}_{GT}|^2 |\langle \psi^{(Z)}_{1s}\rangle|^2 (\pi \delta^2)^{3/2} \times \sum_{i > j} U^*_{ei} U_{ej} e^{-\delta^2 (\vec{k}_ij^{\dagger}(-))^2} \sqrt{E_i(\vec{k}_ij^{\dagger}(+)E_j(\vec{k}_ij^{\dagger}(+))} \times \left[ \frac{2\varepsilon}{(\Delta E_j(\vec{k}_ij^{\dagger}(+))^2 + \varepsilon^2} + \frac{2\varepsilon}{(\Delta E_j(\vec{k}_ij^{\dagger}(+))^2 + \varepsilon^2} \right] e^{\varepsilon t} \times \cos \left[ \left( E_i(\vec{k}_ij^{\dagger}(+) - E_j(\vec{k}_ij^{\dagger}(+)) \right) t \right]. \tag{11}$$

Here $\vec{k}_ij^{\dagger}(-) = (\vec{k}_i - \vec{k}_j)/2$ and $\vec{k}_ij^{\dagger}(+) = (\vec{k}_i + \vec{k}_j)/2$ are the difference and averaged neutrino momenta and $E_j(\vec{k}_ij^{\dagger}(+) = \sqrt{(\vec{k}_ij^{\dagger}(+))^2 + m_j^2}$. The former is an approximate relation between the energy of the massive neutrino $\nu_j$ and a momentum $\vec{k}_ij^{\dagger}(+)$, since 3–momenta of massive neutrinos are not conserved and massive neutrinos are off–shell. As a result such a relation cannot be used for the analysis of energy differences $E_i(\vec{k}_ij^{\dagger}(+) - E_j(\vec{k}_ij^{\dagger}(+))$, which are sensitive to the off–shell and on–shell states of massive neutrinos. We show this below.

**The $EC$–decay rate $\lambda_{EC}$ of the H–like ions**

The diagonal part of the rate of the neutrino spectrum $dN_{\nu_e}^{(1)}(t)/dt$ defines the meanvalue of the $EC$–decay rate $\lambda_{EC} = \lambda_{EC}$. Taking the limit $\varepsilon \to 0$ in Eq. (10) we get

$$\frac{dN_{\nu_e}^{(1)}(t)}{dt} = \frac{3}{2F + 1} |\mathcal{M}_{GT}|^2 |\langle \psi^{(Z)}_{1s}\rangle|^2 (\pi \delta^2)^{3/2} \times \sum_j |U_{ej}|^2 E_j(\vec{k}_j) 2\pi \delta (\Delta E_j(\vec{k}_j)), \tag{12}$$

where the $\delta$–functions $\delta(\Delta E_j(\vec{k}_j))$ with the arguments $\Delta E_j(\vec{k}_j) = E_d(\vec{k}_j) + E_j(\vec{k}_j) - M_m$ describe the conservation of energy in the $EC$–decay channels $m \to d + \nu_j$. The solution of the equations $\Delta E_j(\vec{k}_j) = 0$ gives the energies of neutrino mass–eigenstates $\nu_j$ shown in Fig. 1 with the energy differences $E_i(\vec{k}_i) - E_j(\vec{k}_j) = \omega_{ij} = \Delta m_{ij}^2/2M_m$.

For the calculation of the $EC$–decay rate $\lambda_{EC}$ we can set neutrino masses zero and use the unitarity of the $U$–matrix $\sum_j |U_{ej}|^2 = 1$. This gives

$$\lambda_{EC} = \int \frac{d^3k}{(2\pi)^3 2E_{\nu}} \frac{1}{(\pi \delta^2)^{3/2}} \frac{dN_{\nu_e}^{(1)}(t)}{dt}, \tag{13}$$

where $E_{\nu} = |\vec{k}|$ and $(\pi \delta^2)^{3/2}$ is related to the normalisation of the neutrino wave function $[3]$. Hav-
ing integrated over the neutrino phase volume we obtain \( \lambda_{EC} \):

\[
\lambda_{EC} = \frac{1}{2F + 1} \frac{3}{2} \left| M_{\chi} \right|^2 |(\psi_{1s})|^2 Q_H^2 \pi, \quad (14)
\]

where \( Q_H = 3.348(6) \text{MeV} \) for the \( EC \)-decay of \( ^{140}\text{Pr}^{58+} \to ^{140}\text{Ce}^{58+} + \nu_e \) \( [2] \). The \( EC \)-decay rate \( \lambda_{EC} \) has been calculated in \( [2] \) (see also \( [11] \)) within standard theory of weak interactions of heavy ions \( [3] \). Since experimental value of the matrix element \( U_{13} = \sin \theta_{13} e^{i\delta_{CP}} \), where \( \delta_{CP} \) is a CP–violating phase \( [3] \), is very close to zero, we set \( \theta_{13} = 0 \) and below deal with two neutrino mass–eigenstates only.

The elements of the mixing matrix are set equal to \( U_{e1} = \cos \theta_{12} \) and \( U_{e2} = \sin \theta_{12} \) \( [5] \).

**EC–decays of H–like heavy ions as analog of quantum beats of atomic transitions**

The time–modulation of the \( EC \)-decays of the H–like heavy ions bears similarity with quantum beats of atomic transitions \( [6] \), since in the \( EC \)-decays one deals with the transitions from the initial state \( |m\) to the final state \( |d\nu_c\), where the electron neutrino is the coherent state of two neutrino mass–eigenstates with the energy difference equal to \( \omega_{21} = \Delta m^2_{21}/2M_m \). The time differential detection of the daughter ions from the \( EC \)-decays in the GSI experiments with a time resolution \( \tau_d \approx 1 \text{s} \) introduces an energy uncer-

tainty \( \delta E_d \sim 2\pi\hbar/\tau_d = 4.14 \times 10^{-15} \text{eV} \). Thus, if \( \delta E_d > \omega_{21} \), following the analogy with quantum beats of atomic transitions \( [6] \) one should expect a periodic time–dependence of the \( EC \)-decay rate with a frequency \( \omega_{21} \) and a period \( T_{EC} \sim M_m \).

**Analysis of the interference term of \( EC \)-decay rates of H–like heavy ions**

The dependence of the interference term on the average momentum \( \vec{k}_{21} = (\vec{k}_2 + \vec{k}_1)/2 \) can produce an impression that the frequency of the periodic time–dependence of the interference term is of order of \( \Omega_{21} = \Delta m^2_{21}/2Q_H \) only \( [17] \):

\[
\cos \left( (E_{21}(\vec{k}_{21}^{(+)}) - E_{11}(\vec{k}_{21}^{(+)})/t = \cos \left( \frac{m^2_2 - m^2_1}{E_{21}(\vec{k}_{21}^{(+)}) + E_{11}(\vec{k}_{21}^{(+)})} \right) \right) = \cos \left( \Omega_{21}t \right), \quad (15)
\]

where \( E_{11}(\vec{k}_{21}^{(+)}) \approx E_{21}(\vec{k}_{21}^{(+)}) \approx Q_H \) and on–shell relations between the energies and momenta have been used \( [17] \). However, this result, being only partly correct, leads to the missing of the frequency \( \omega_{21} \) due to the use of on–shell relations between energies and momenta for the energy difference. The existence of the frequency \( \omega_{21} \) can be shown using non–conservation of neutrino 3–momenta, their differences and noticing that the behaviour of the interference term is governed by the Gaussian function \( e^{-\frac{1}{2}(E_{21}^{(+)})^2} \)

The region of variation of 3–momenta of massive neutrinos can be divided into two parts with the absolute values \( |\vec{k}_2| > |\vec{k}_1| \) and \( |\vec{k}_2| < |\vec{k}_1| \), respectively, and both cases lead to totally different modulation frequencies of the interference term.

For \( |\vec{k}_2| > |\vec{k}_1| \) the energies \( E_{11}(\vec{k}_1), E_{11}(\vec{k}_{21}), E_{21}(\vec{k}_{21}) \) and \( E_{21}(\vec{k}_2) \) satisfy the inequality:

\( E_{11}(\vec{k}_1) < E_{11}(\vec{k}_{21}) < E_{21}(\vec{k}_{21}) < E_{21}(\vec{k}_2) \) as it shown in Fig.2. The deviations as indicated in Fig.2 are of order of \( O(1/\delta) \). Indeed, for sufficiently large \( \delta \), that has been already assumed above for the calculation the rate of neutrino spectrum Eq.(10) and Eq.(11), due to the function \( e^{-\frac{1}{2}(E_{21}^{(+)})^2} \) the region of the 3–momenta of massive neutrinos is constrained by \( \delta \kappa \sim \kappa \).

The allowed region of deviations is of order of
\[ \left| \mathbf{k}_{21}^{(-)} \right| \sim 1/\delta. \]

For these momenta the energies of neutrino mass–eigenstates obey the obvious relation
\[ E_2(\mathbf{k}_{22}) \approx E_1(\mathbf{k}_1) \gg \left| \mathbf{k}_{21}^{(-)} \right| \sim 1/\delta. \]

Making the expansions of the energies of massive neutrinos in powers of \( \mathbf{k}_{21}^{(-)} \) and keeping only the first order contributions we get
\[ E_2(\mathbf{k}_{21}^{(+)})(E_2(\mathbf{k}_{21}^{(-)})) = E_2(\mathbf{k}_2) - \Delta E_2 \]
and
\[ E_1(\mathbf{k}_{21}^{(+)})(E_1(\mathbf{k}_{21}^{(-)})) = E_1(\mathbf{k}_1) + \Delta E_1, \]
where we have denoted \( \Delta E_2 = \mathbf{k}_2 \cdot \mathbf{k}_{21}^{(-)}/E_2(\mathbf{k}_2) \) and \( \Delta E_1 = \mathbf{k}_1 \cdot \mathbf{k}_{21}^{(-)}/E_1(\mathbf{k}_1) \).

Without loss of generality we can set \( \Delta E_2 \approx \Delta E_1 \approx \Delta E \), where \( \Delta E = O(1/\delta) \) and
\[ E_2(\mathbf{k}_2) \approx E_1(\mathbf{k}_1) \gg |\Delta E| = O(1/\delta). \]
This shows that the deviations of \( E_2(\mathbf{k}_{21}^{(+)}) \) and \( E_1(\mathbf{k}_{21}^{(+)}) \) from \( E_2(\mathbf{k}_2) \) and \( E_1(\mathbf{k}_1) \), respectively, are of order of \( O(1/\delta) \) and the interference term should have a periodic time dependence with a period \( \omega_{21} \).

In the region \( |\mathbf{k}_2| > |\mathbf{k}_1| \) of 3–momenta of massive neutrinos the calculation of the periodic term runs as follows
\[
\cos((E_2(\mathbf{k}_{21}^{(+)}) - E_1(\mathbf{k}_{21}^{(+)})\hat{t})) \xrightarrow{1/\delta} \cos((E_2(\mathbf{k}_2) - E_1(\mathbf{k}_1))\hat{t}) = \cos(\omega_{21}\hat{t}), \]
where we have used the on–shell relations between neutrino energies and momenta only for the expansions
\[ E_2(\mathbf{k}_{21}^{(+)}) = E_2(\mathbf{k}_2) - \Delta E = E_2(\mathbf{k}_2) - O(1/\delta) \]
and
\[ E_1(\mathbf{k}_{21}^{(+)}) = E_1(\mathbf{k}_1) + \Delta E = E_1(\mathbf{k}_1) + O(1/\delta) \]
but not for the energy difference. This gives
\[ E_2(\mathbf{k}_{21}^{(+)}) - E_1(\mathbf{k}_{21}^{(+)}) = E_2(\mathbf{k}_2) - E_1(\mathbf{k}_1) - 2\Delta E = \omega_{21} - O(2/\delta). \]

As a result the validity of the interference term to have a frequency \( \omega_{21} \) should be confirmed by the inequality \( \omega_{21} \gg 2/\delta \), which can be always satisfied for the sufficiently large \( \delta \).

In turn, for \( |\mathbf{k}_2| < |\mathbf{k}_1| \) the energies \( E_1(\mathbf{k}_{21}^{(+)}) \), \( E_1(\mathbf{k}_1) \), \( E_2(\mathbf{k}_2) \) and \( E_2(\mathbf{k}_{21}^{(+)}) \) satisfy the inequality:
\[ E_1(\mathbf{k}_{21}^{(+)}) < E_1(\mathbf{k}_1) < E_2(\mathbf{k}_2) < E_2(\mathbf{k}_{21}^{(+)}) \]
as it shown in Fig. 3. In this case a periodic time–dependence of the interference term can be defined by the frequency, which is greater than \( \omega_{21} \).

In order to take into account both frequencies of the periodic time–dependence of the interference term we introduce the function
\[
\rho(\tilde{\mathbf{k}}_2, \tilde{\mathbf{k}}_1) = \theta(|\mathbf{k}_2| - |\mathbf{k}_1|) \rho'(\tilde{\mathbf{k}}_2, \tilde{\mathbf{k}}_1) + \theta(|\mathbf{k}_1| - |\mathbf{k}_2|) \rho''(\tilde{\mathbf{k}}_2, \tilde{\mathbf{k}}_1), \tag{17}
\]
having a meaning of the probability density for neutrino mass–eigenstates \( \nu_2 \) and \( \nu_1 \) to have 3–momenta \( \mathbf{k}_2 \) and \( \mathbf{k}_1 \), respectively. The Heaviside step functions \( \theta(|\mathbf{k}_2| - |\mathbf{k}_1|) \) and \( \theta(|\mathbf{k}_1| - |\mathbf{k}_2|) \) take into account two possibilities \( |\mathbf{k}_2| > |\mathbf{k}_1| \) and \( |\mathbf{k}_2| < |\mathbf{k}_1| \) for the 3–momenta of neutrino mass–eigenstates shown in Fig. 2 and Fig. 3.

The contribution of the interference term
\( \lambda_{EC}^{(2)}(t) \) to the EC-decay rate we define as follows

\[
\lambda_{EC}^{(2)}(t) = \lim_{\varepsilon \to 0} \frac{1}{\pi \varepsilon^3} \int \frac{dN_{\psi_s}(t)}{dt} \times \rho(\vec{k}_2, \vec{k}_1) \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \cdot (2E_1(\vec{k}_1) - 2E_2(\vec{k}_2)).
\]

(18)

Substituting Eq. (17) into Eq. (18) we get

\[
\lambda_{EC}^{(2)}(t) = \lambda_{EC}^{(2)}(t) + \lambda_{EC}^{(2)''}(t),
\]

(19)

where \( \lambda_{EC}^{(2)}(t) \) and \( \lambda_{EC}^{(2)''}(t) \) are determined by the contributions of the regions \( |\vec{k}_2| > |\vec{k}_1| \) and \( |\vec{k}_2| < |\vec{k}_1| \), respectively. They are given by the following momentum integrals

\[
\lambda_{EC}^{(2)'}(t) = \lim_{\varepsilon \to 0} \frac{3}{2F + 1} |\mathcal{M}_{GT}|^2 |\langle \psi_s(1) \rangle|^2 \frac{1}{2} \sin 2\theta_{12}
\times \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \theta(|\vec{k}_2| - |\vec{k}_1|)
\times \rho(\vec{k}_2, \vec{k}_1) e^{-\delta^2(2\vec{k}_2)^2} \sqrt{E_2(\vec{k}_2)^+ E_1(\vec{k}_1)^+}
\times \left[ \left( \Delta E_2(\vec{k}_2)^+ \right)^2 + \varepsilon^2 \right] + \left( \Delta E_1(\vec{k}_1)^+ \right)^2 + \varepsilon^2
\times \cos \left[ \left( E_2(\vec{k}_2)^+ - E_1(\vec{k}_1)^+ \right) t \right] e^{\varepsilon t}
\]

(20)

and

\[
\lambda_{EC}^{(2)''}(t) = \lim_{\varepsilon \to 0} \frac{3}{2F + 1} |\mathcal{M}_{GT}|^2 |\langle \psi_s(1) \rangle|^2 \frac{1}{2} \sin 2\theta_{12}
\times \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \theta(|\vec{k}_1| - |\vec{k}_2|)
\times \rho''(\vec{k}_2, \vec{k}_1) e^{-\delta^2(2\vec{k}_2)^2} \sqrt{E_2(\vec{k}_2)^+ E_1(\vec{k}_1)^+}
\times \left[ \left( \Delta E_2(\vec{k}_2)^+ \right)^2 + \varepsilon^2 \right] + \left( \Delta E_1(\vec{k}_1)^+ \right)^2 + \varepsilon^2
\times \cos \left[ \left( \frac{m_2^2 - m_1^2}{E_2(\vec{k}_2)^+ + E_1(\vec{k}_1)^+} \right) t \right] e^{\varepsilon t}.
\]

(21)

According to Figs. 2 and 3, the frequencies in two integrals Eqs. (20) and (21) should be equal to \( \omega_{21} \) and \( \Omega_{21} \), respectively.

Making expansions in powers of \( \Delta E(\vec{k}_2)^+ \sim 1/\varepsilon^3 \) and taking the limit \( \varepsilon \to 0 \) we reduce the r.h.s.

\[5\text{Formally this leads to the replacements } \Delta E_2(\vec{k}_2)^+ \to \Delta E_2(\vec{k}_2), \Delta E_1(\vec{k}_1)^+ \to \Delta E_1(\vec{k}_1), E_2(\vec{k}_2)^+ \to E_2(\vec{k}_2) \text{ and } E_1(\vec{k}_1)^+ \to E_1(\vec{k}_1). \]

of Eqs. (20) and (21) to the form

\[
\lambda_{EC}^{(2)'}(t) = \frac{3}{2F + 1} |\mathcal{M}_{GT}|^2 |\langle \psi_s(1) \rangle|^2 \frac{1}{2} \sin 2\theta_{12}
\times \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \theta(|\vec{k}_2| - |\vec{k}_1|)
\times \rho(\vec{k}_2, \vec{k}_1) e^{-\delta^2(2\vec{k}_2)^2} \sqrt{E_2(\vec{k}_2) E_1(\vec{k}_1)}
\times \left[ 2\pi \delta(\Delta E_2(\vec{k}_2)) + 2\pi \delta(\Delta E_1(\vec{k}_1)) \right]
\times \cos \left[ \left( E_2(\vec{k}_2) - E_1(\vec{k}_1) \right) t \right]
\]

(22)

and

\[
\lambda_{EC}^{(2)''}(t) = \frac{3}{2F + 1} |\mathcal{M}_{GT}|^2 |\langle \psi_s(1) \rangle|^2 \frac{1}{2} \sin 2\theta_{12}
\times \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \theta(|\vec{k}_1| - |\vec{k}_2|)
\times \rho''(\vec{k}_2, \vec{k}_1) e^{-\delta^2(2\vec{k}_2)^2} \sqrt{E_2(\vec{k}_2) E_1(\vec{k}_1)}
\times \left[ 2\pi \delta(\Delta E_2(\vec{k}_2)) + 2\pi \delta(\Delta E_1(\vec{k}_1)) \right]
\times \cos \left[ \left( \frac{m_2^2 - m_1^2}{E_2(\vec{k}_2) + E_1(\vec{k}_1)} \right) t \right].
\]

(23)

By virtue of the \( \delta \)-functions \( \delta(\Delta E_1(\vec{k}_1)) \) and \( \delta(\Delta E_2(\vec{k}_2)) \) with arguments \( \Delta E_1(\vec{k}_1) = E_1(\vec{k}_1) - 2M_1 \) and \( \Delta E_2(\vec{k}_2) = E_2(\vec{k}_2) - 2M_2 \), respectively, providing energy conservation in the EC-decay channels \( m \to d + \nu_1 \) and \( m \to d + \nu_2 \), and the function \( e^{-\delta^2(\vec{k}_2)^2} \), the frequencies of the periodic terms can be replaced by \( \omega_{21} \) and \( \Omega_{21} \), respectively. As a result the contribution of the interference term reads

\[
\lambda_{EC}^{(2)}(t) = a_{EC} \cos(\omega_{21} t) + \tilde{a}_{EC} \cos(\Omega_{21} t),
\]

(24)

where \( \lambda_{EC} \) is defined by Eq. (14). The amplitudes \( a_{EC} \) and \( \tilde{a}_{EC} \) are given by the integrals

\[
a_{EC} = 2\pi^2 Q_{HI} \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \theta(|\vec{k}_1| - |\vec{k}_2|)
\times \rho(\vec{k}_2, \vec{k}_1) e^{-\delta^2(2\vec{k}_2)^2} \sqrt{E_2(\vec{k}_2) E_1(\vec{k}_1)}
\times \left[ \delta(\Delta E_2(\vec{k}_2)) + \delta(\Delta E_1(\vec{k}_1)) \right] \theta(|\vec{k}_2| - |\vec{k}_1|)
\]

(25)
and
\[ a_{EC} = \sin 2\theta_{12} \frac{2\pi^2}{Q_H} \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \left( \rho(\vec{k}_2, \vec{k}_1) e^{-\delta^2(\vec{k}_{21}^2)^2/2} \sqrt{E_2(\vec{k}_2)E_1(\vec{k}_1)} \right) \times \left[ \delta(\Delta E_2(\vec{k}_2)) + \delta(\Delta E_1(\vec{k}_1)) \right] \theta(|\vec{k}_1| - |\vec{k}_2|). \] (26)

The amplitudes of the periodic functions, depending on the mixing angle \( \theta_{12} \) and the parameter \( \delta \), are considered as empirical parameters determined by the experiment.

We would like to emphasize that due to non-conservation of 3-momenta the massive neutrinos are off-shell. This means that on-shell relations between energies and momenta of massive neutrinos are only by the low frequency term \( \delta(\Delta E_2(\vec{k}_2)) + \delta(\Delta E_1(\vec{k}_1)) \) leads to the missing of the frequency \( \omega_{21} \) (see \[17\]).

**Time-dependent \( EC \)-decay rates of H-like heavy ions**

Taking into account the contribution of the interference term, the total \( EC \)-decay rate we get in the form
\[ \frac{\lambda_{EC}(t)}{\lambda_{EC}} = 1 + a_{EC} \cos(\omega_{21} t) + a_{EC} \cos(\Omega_{21} t). \] (27)

The \( EC \)-decay rate Eq. (27) contains two periodic terms. In the laboratory frame the periods are equal to
\[ T_{EC} = \frac{2\pi\gamma}{\omega_{21}}, \quad T_\theta = \frac{2\pi\gamma}{\Omega_{21}}. \] (28)

where \( \gamma = 1.43 \) is the Lorentz factor of the motion of mother ions in the laboratory frame.

We identify \( T_{EC} = 2\pi\gamma/\omega_{21} \approx M_m \) with the experimental period of the time-modulation \( T_{EC} = 7.06(8) \) s, because it is shown experimentally that the period of the time-modulation does not depend on the \( Q \)-value of the weak transition. For the \( EC \)-decay of the H-like \( ^{142}\text{Pm}^{60+} \) ion with \( Q_H \approx 4827 \text{keV} \) the period of modulation \( T_{EC} = 7.10(22) \) s is equal to \( T_{EC} = 7.06(8) \) s of the H-like \( ^{140}\text{Pr}^{58+} \) ion within the experimental error bars. Since the mass difference of the H-like ions \( ^{142}\text{Pm}^{60+} \) and \( ^{140}\text{Pr}^{58+} \) is small we expect only small differences in the periods of the time-modulation as shown by the experiment.

Thus, for \( T_{EC} = 7.06(8) \) s we get \( (\Delta m_{21}^2)_{\text{GSI}} = 2.22(3) \times 10^{-4} \text{eV}^2 \) (see also \[18\]). The value \( (\Delta m_{21}^2)_{\text{GSI}} = 2.22(3) \times 10^{-4} \text{eV}^2 \) is by a factor of 2.75 larger than \( (\Delta m_{21}^2)_{\text{KamLAND}} = 0.80^{+0.06}_{-0.05} \times 10^{-4} \text{eV}^2 \), obtained as a best-fit of the global analysis of the solar–neutrino and KamLAND experimental data \[39\].

For \( (\Delta m_{21}^2)_{\text{GSI}} = 2.22(3) \times 10^{-4} \text{eV}^2 \) we get \( T_\theta \approx 1.8 \times 10^{-4} \) s. These fast oscillations, averaged over the experimental time resolution \( \Delta T = 0.32 \) s \[4\], are not observable. As a result the time-dependent \( EC \)-decay rate Eq. (27) is given only by the low frequency term
\[ \lambda_{EC}(t) = \lambda_{EC}(1 + a_{EC} \cos(\omega_{EC} t)), \] (29)
where \( \omega_{EC} = \omega_{21} = 2\pi/T_{EC} \) and the amplitude \( a_{EC} \) is defined by Eq. (28).

**Discussion and summary**

We have shown that the experimental data on the time-modulation of the rate of the number of daughter ions \( ^{140}\text{Ce}^{58+} \), observed in the \( EC \)-decay of the H-like ion \( ^{140}\text{Pr}^{58+} \) \[4\], can be explained by the neutrino mass differences. However, the difference of squared neutrino masses \( (\Delta m_{21}^2)_{\text{GSI}} = 2.22(3) \times 10^{-4} \text{eV}^2 \), derived from the period \( T_{EC} = 7.06(8) \) s of the rate of the number of daughter ions \( ^{140}\text{Ce}^{58+} \), is 2.75 times larger than the value measured by KamLAND \[5\]. A solution of this problem in terms of neutrino mass corrections, induced by the interaction of massive neutrinos with a strong Coulomb field of the daughter ion through virtual \( \ell^- W^+ \) pair creation, is proposed in \[15\].

In summary we argue that the mechanism of
the time–dependence of the $EC$–decay rates of the H–like heavy ions bears similarity with quantum beats of atomic transitions [6], as the $EC$–decays are the transitions of the initial state $|m\rangle$ into the final state $|d\nu\rangle$, where the electron neutrino is the coherent state of two massive neutrinos with energy difference $\omega_{21}$. A sophisticated calculation of the interference term shows also the existence of a periodic time–dependence with a frequency $\Omega_{21} \gg \omega_{21}$. In order to take into account the contributions of two possible frequencies of the periodic time–dependence of the interference term we have introduced the probability density $\rho(\vec{k}_2, \vec{k}_1)$ for neutrino mass–eigenstates $\nu_2$ and $\nu_1$ to get 3–momenta $\vec{k}_2$ and $\vec{k}_1$, respectively. The calculation of the function $\rho(\vec{k}_2, \vec{k}_1)$ is rather hard problem. We can only argue that in the massless limit it should be equal to

$$\rho(\vec{k}_2, \vec{k}_1) = 2\sqrt{E_{\nu}(\vec{k}_1)E_{\nu}(\vec{k}_2)}(2\pi)^3\delta(3)(\vec{k}_2 - \vec{k}_1),$$

which is required by the necessity to retain the meanvalue of the $EC$–decay rate $\lambda_{EC}$.

The necessary condition for the appearance of the interference term in the $EC$–decay rate is the overlap of the energy levels of neutrino mass–eigenstates. Non–conservation of 3–momenta of neutrino mass–eigenstates provides a possibility to place the interference term of the $EC$–decay rate in the energy region of the diagonal term, defining the meanvalue of the $EC$–decay rate $\lambda_{EC}$. A validity of such a transformation is supported by the constraint $\omega_{21} = \Delta m_{21}^2/2M_m \gg 2/\delta$, which makes impossible the limit $\Delta m_{21}^2 \to 0$ in the argument of the interference term, where the terms of order $O(1/\delta)$ are dropped with respect to $\Delta m_{21}^2/4M_m$. This means that in the interference term of the $EC$–decay rate Eq. [29] one cannot set $\Delta m_{21}^2 \to 0$ in order to reduce the time–dependent $EC$–decay rate to the $EC$–decay rate $\lambda_{EC}$ given by Eq. [14]. A correct reduction of $\lambda_{EC}(t)$ to $\lambda_{EC}$ can be carried out only by means of the averaging over time $\langle \lambda_{EC}(t) \rangle = \lambda_{EC}$. In spite of momentum non–conservation, energy of the $EC$–decay is conserved in our approach. This is required by Fermi Golden Rule [14].

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