Stability analysis of black holes by the S-deformation method for coupled systems

Masashi Kimura\textsuperscript{1,4} and Takahiro Tanaka\textsuperscript{2,3}

\textsuperscript{1} CENTRA, Departamento de Física, Instituto Superior Técnico, Universidade de Lisboa, Avenida Rovisco Pais 1, 1049 Lisboa, Portugal
\textsuperscript{2} Department of Physics, Kyoto University, Kyoto 606-8502, Japan
\textsuperscript{3} Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan

E-mail: masashi.kimura@tecnico.ulisboa.pt

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Abstract
We propose a simple method to prove the linear mode stability of a black hole when the perturbed field equations take the form of a system of coupled Schrödinger equations. The linear mode stability of the spacetime is guaranteed by the existence of an appropriate S-deformation. Such an S-deformation is related to the Riccati transformation of a solution to the Schrödinger system with zero energy. We apply this formalism to some examples and numerically study their stability.

Keywords: black hole stability analysis, S-deformation method, perturbation

(Some figures may appear in colour only in the online journal)

1. Introduction
To understand the physical properties of black holes, one typically studies their linearised field equations and the motion of test particles. The background spacetime is required to be stable for the perturbative approximation to remain valid. Stability is also important for black hole formation because unstable solutions are not realised as the final states of gravitational collapse. If the background spacetime is highly symmetric, such as for a spherically symmetric static black hole, the perturbed field equations can be written as an expansion of mode functions, leading to a system of ordinary differential equations. When there is only a single degree of freedom, or each degree of freedom is decoupled, the perturbed field equations usually reduce to a single master equation in the form of the Schrödinger equation [1–13]

\[ \left( -\frac{d^2}{dx^2} + V \right) \Phi = \omega^2 \Phi, \]  

(1)
where we have expressed the time dependence of the perturbed fields in terms of the modes $e^{-i\omega t}$. The existence of $\omega^2 < 0$ mode for equation (1) corresponds to an exponentially growing mode. The $S$-deformation method was used to show the non-existence of such modes in [5–15]. In the $S$-deformation method, the existence of a function $S$ that is continuous everywhere and satisfies $V - S^2 + dS/dx \geq 0$ implies the mode stability of the spacetime [5–7]. Recently, it was shown that we can construct a regular solution of

$$V - S^2 + \frac{dS}{dx} = 0,$$

if the spacetime is stable [16, 17].

With two or more degrees of freedom, the perturbed field equations form a coupled system of ordinary differential equations. In some cases, the perturbed field equations take the form of systems of coupled Schrödinger equations [18–23], where the form of the equation is the same as equation (1) but the potential is a matrix and the wave function has multi components. In this paper, to study the mode stability of such systems, we extend the formalism in the previous works [16, 17] to coupled systems. We also discuss the relation between our formalism and the nodal theorem in the pioneering work [24] on this topic.

This paper is organized as follows. In section 2, we develop a formalism to prove stability of the coupled systems based on the $S$-deformation method. In section 3, we apply our formalism to some examples, and numerically study their stability. Section 4 is devoted to summary and discussion. We also provide several appendices. In this paper, we consider that the effective potential is a Hermitian matrix, but we also show that it can be considered as real symmetric matrix problem in appendix A. In appendix B, we discuss basic properties of the Riccati transformation for a system of coupled Schrödinger equations. In appendix C, we give a proof of the existence of $S$-deformation for the case with a compact support potential if the system is stable. In appendix D, we discuss the robustness of the $S$-deformation method, i.e. the reason why we can find a regular $S$-deformation without fine-tuning. In appendix E, we give a proof of the existence of a regular $S$-deformation for a positive definite effective potential. In appendix F, we give an explicit form of $S$-deformation near the horizon for two degrees of freedom case. In appendix G, we show that the existence of two different $S$-deformations implies the non-existence of the zero mode under some assumption.

### 2. Simple test for stability of black hole

#### 2.1. The $S$-deformation method for coupled systems

Consider the case where the perturbed field equations take the form of a system of coupled Schrödinger equations

$$-\frac{d^2}{dx^2} \Phi + V \Phi = \omega^2 \Phi =: E\Phi,$$

where $V$ is an $n \times n$ Hermitian matrix, and $\Phi$ is an $n$-component vector. We assume that $n$ also corresponds to the number of physical degrees of freedom. In this paper, we assume that the domain of $V$ is $-\infty < x < \infty$, and $V$ is piecewise continuous and bounded. For any $n \times n$ matrix $S$, we can show the relation

5 We consider the case where the coupling term in the perturbed Lagrangian takes $\Phi^\dagger V \Phi$. If $V$ is a Hermitian matrix, this term is real. When we consider real symmetric $V$ and real $\Phi$, the following discussion in this paper holds just by taking real parts.
\[- \frac{d}{dx} \left[ \Phi^\dagger \frac{d\Phi}{dx} + \Phi^\dagger S \Phi \right] + \left( \frac{d\Phi^\dagger}{dx} + \Phi^\dagger S \right) \left( \frac{d\Phi}{dx} + S \Phi \right) + \Phi^\dagger \left[ V + \frac{dS}{dx} - S^2 \right] \Phi = E |\Phi|^2, \]

where \( \dagger \) denotes the Hermitian conjugate. If \( S \) is Hermitian and its components are continuous functions\(^6\), the equation

\[- \left[ \Phi^\dagger \frac{d\Phi}{dx} + \Phi^\dagger S \Phi \right] \biggr|_{-\infty}^{\infty} + \int dx \left[ \left| \frac{d\Phi}{dx} + S \Phi \right|^2 + \Phi^\dagger \left( V + \frac{dS}{dx} - S^2 \right) \Phi \right] = E \int dx |\Phi|^2. \tag{4} \]

holds. We consider the boundary condition such that the boundary term in equation (4) vanishes. We can see that the deformed potential for the coupled system is

\[ \tilde{V} := V + \frac{dS}{dx} - S^2. \tag{5} \]

This is an extension of \( S \)-deformation method [5–7], and we also refer to \( \tilde{V} \) as an \( S \)-deformation of the potential \( V \) in this paper. If there exists a continuous \( S \) which gives \( \tilde{V} = K^\dagger K \), since \( \Phi^\dagger \tilde{V} \Phi = |K\Phi|^2 \geq 0 \), we can say the non-existence of the negative energy bound state, i.e. the non-existence of the exponentially growing mode in time. In [25, 26], this method was used for stability analysis.

### 2.2. Stability of the coupled system

Similarly to the single mode case [16, 17], we consider the condition \( \tilde{V} = 0 \), i.e.

\[ V + \frac{dS}{dx} - S^2 = 0, \tag{6} \]

for the coupled system. The existence of a continuous \( S \) as a solution of equation (6) is a sufficient condition for the stability of the spacetime from equation (4).

According to the nodal theorem in [24]\(^7\), for a set of the solutions \( \{ \Phi_i \} (i = 1, 2, \ldots, n) \) of the Schrödinger equation with \( E = 0 \) with the boundary condition such that \( \Phi_i |_{x=L} = 0 \) and \( \left( \frac{d\Phi_i}{dx} \right) |_{x=L} = \nu_i \), where \( \nu_i \) are linearly independent constant vectors, the necessary and sufficient condition for the non-existence of the negative energy bound state for equation (3) is that \( \det(Y) \) does not have a zero except at \( x = L \) if \( L \) is sufficiently large (or sufficiently large negative), where \( Y \) is defined as

\[ Y := (\Phi_1, \Phi_2, \ldots, \Phi_n). \tag{7} \]

In [18, 19, 27–29], the nodal theorem [24] was used for stability analysis.

If we assume that this statement holds even when we take the limit \( L \to \infty \) (or \( L \to -\infty \)), i.e. when we take \( \{ \Phi_i \} \) as a set of \( n \)-decaying modes at \( x \to -\infty \) (or \( x \to \infty \)), \( \det(Y) \neq 0 \) except at infinity when the spacetime is stable, and \( Y^{-1} \) does not diverge at any finite point. In this case, we can construct a continuous \( S \) as a Riccati transformation by

\[ S := -\frac{dY}{dx} Y^{-1}. \tag{8} \]

\(^6\)We also assume that \( dS/dx \) can be defined piecewise continuously, and \( dS/dx \) is bounded at the discontinuity points.

\(^7\)Note that this theorem holds when \( V \) is Hermitian while a real symmetric potential is assumed in [24]. This is because the Hermitian case can be considered as a real symmetric problem as shown in appendix A.
We can easily check that $S$ satisfies equation (6) and becomes a Hermitian matrix (see appendix B for basic properties of the Riccati transformation of solutions to systems of coupled Schrödinger equations). This discussion suggests that there exists a continuous solution $S$ for equation (6) if there does not exist a bound state with $E \leq 0$. In appendix C, we show that this is correct for the case with compact support potential (or rapidly decaying potential at $x \to \pm \infty$). Also, in a single mode case, a proof was given under weaker conditions [17].

We introduce $Y_L$ and $Y_R$ to denote $Y$ which are constructed from linearly independent decaying (or constant) modes at $x \to -\infty$ and $x \to \infty$, respectively, and the corresponding $S$ as $S_L = -(dY_L/dx)Y_L$, $S_R = -(dY_R/dx)Y_R$, respectively. As discussed in appendix D, we can show that the general regular $S$ for the compact support potential satisfies the property that all eigenvalues of $S_R - S$ and $S - S_L$ are non-negative. Also, if we solve equation (6) with the boundary conditions that guarantee $S$ to be Hermitian and all eigenvalues of $S_R - S$ and $S - S_L$ to be non-negative, the solution becomes regular for a stable spacetime. We expect that these properties hold even for non-compact potentials with sufficiently rapid convergence at $x \to \pm \infty$.

In practice, if the potential $V$ is positive definite at large $x$, as discussed in appendix E where we show the existence of a regular $S$-deformation for a positive definite potential, we conjecture that $S = 0$ at a large $x$ is an appropriate boundary condition in solving equation (6) to obtain a regular $S$.

2.3. $S$ is bounded if $\text{Tr}(S)$ and $V$ are bounded

When $S$ is divergent at a point, one of its eigenvalues is also divergent there. Let $e$ be a unit eigenvector of $S$ with the eigenvalue $\lambda$. Then, $\lambda$ satisfies the equation

$$\frac{d\lambda}{dx} = \lambda^2 - e^\dagger Ve.$$  \hspace{1cm} (9)

Note that $e$ is not a constant vector. For a bounded $V$, $\lambda$ can be divergent only when $d\lambda/dx \approx \lambda^2$. Since the solution of $d\lambda/dx = \lambda^2 = \lambda = -1/(x - c)$, the eigenvalue can be divergent only at a finite point. Thus, if $V$ are bounded, $S$ can be divergent only at a finite point.

If $\text{Tr}S$ is bounded above and below, $\text{det}(Y)$ does not have zero except at infinity since $d(\ln \text{det}(Y))/dx = \text{Tr}(Y^{-1}dY/dx) = \text{Tr}(S)$. This implies that $S$ is regular if $\text{Tr}S$ is bounded above and below.

3. Application: numerical calculations for coupled systems

In this section, we consider to solve equation (6) numerically and find a regular $S$-deformation of a coupled system. As examples, we apply our formalism to the systems discussed in [21, 22].

3.1. Schwarzschild black hole in dynamical Chern–Simons gravity

The odd parity metric perturbations coupled to a perturbed massless scalar field in dynamical Chern–Simons gravity, the action of this theory is

$$S = \int d^4x \sqrt{-g} [\kappa R - (\partial/4)R^\sigma\rho\tau\nu\rho - (\beta/2)(\partial_i\phi)(\partial_i\phi^\dagger)] = \text{Tr}(S).$$

This implies that $S$ is regular if $\text{Tr}S$ is bounded above and below.

The action of this theory is $S = \int d^4x \sqrt{-g} [\kappa R - (\partial/4)R^\sigma\rho\tau\nu\rho - (\beta/2)(\partial_i\phi)(\partial_i\phi^\dagger)]$, where $\kappa$ is the gravitational constant, and $\beta$ is the coupling constant [21, 26].
\[ \text{d}x^2 = -f \text{d}t^2 + f^{-1} \text{d}r^2 + r^2 (\text{d}\theta^2 + \sin^2 \theta \text{d}\phi^2), \]  
with \( f = 1 - 2M/r \) reduce to the master equations \([21, 26]\)

\[ -\frac{d^2}{dx^2} \Phi_1 + V_{11} \Phi_1 + V_{12} \Phi_2 = E \Phi_1, \]  
\[ -\frac{d^2}{dx^2} \Phi_2 + V_{12} \Phi_1 + V_{22} \Phi_2 = E \Phi_2, \]  
where \( d/dx = f d/d\bar{x} \). The effective potentials are given by

\[ V_{11} = f \left[ \frac{\ell(\ell + 1)}{r^2} - \frac{6M}{r^3} \right], \]  
\[ V_{12} = f \frac{24M \sqrt{\pi} (\ell + 2)(\ell + 1) \ell(\ell - 1)}{\sqrt{3} r^5}, \]  
\[ V_{22} = f \left[ \frac{\ell(\ell + 1)}{r^2} \left( 1 + \frac{576 \pi M^2}{\beta r^6} \right) + \frac{2M}{r^3} \right], \]  
where \( \beta \) is the coupling constant in dynamical Chern–Simons gravity. We note that the \( \beta \to \infty \) limit corresponds to the general relativity case. Recently, the stability of this system was proven analytically for \( \beta \geq 0 \) \([26]\). We apply our formalism to this system as a test problem. Since \( \mathbf{V} \) is real symmetric, we consider a real symmetric \( \mathbf{S} \), then equation (6) becomes

\[ f \frac{d S_{11}}{dr} = S_{11}^2 + S_{12}^2 - V_{11}, \]  
\[ f \frac{d S_{12}}{dr} = S_{12}(S_{11} + S_{22}) - V_{12}, \]  
\[ f \frac{d S_{22}}{dr} = S_{22}^2 + S_{12}^2 - V_{22}. \]  
We solve these equations numerically\(^{10}\) and we plot \( \text{Tr} (\mathbf{S}) = S_{11} + S_{22} \) in figure 1 for \( \ell = 2, \beta M^4 = 1/10 \) case. We adopt the boundary condition such that \( \mathbf{S} = 0 \) at \( r/(2M) = r_{\text{hi}}/(2M) = 10 \), and solve the equations in the domain \( r_L \leq r \leq r_R \), with \( r_L/(2M) = 1 + 10^{-30} \) and \( r_R/(2M) = 500 \). Since \( r \) is not an appropriate coordinate to numerically solve the equations (16)–(18) in the near horizon region \( r_L \leq r \leq r_{\text{hi}} \), we use an alternative coordinate \( \bar{x} = 2M \ln(r/(2M) - 1) \), then \( d/dx = (r/(2M) - 1) d/dr \) and \( r = 2M(1 + e^{\bar{x}/(2M)}) \).

Figure 1 shows that \( \text{Tr} (\mathbf{S}) \) is continuous and bounded, and this implies that \( \mathbf{S} \) is regular (see section 2.3), i.e. the spacetime is stable against the \( \ell = 2 \) mode perturbation in \( \beta M^4 = 1/10 \) case. We also report that we can find a regular \( \mathbf{S} \) even if we change \( r_{\text{hi}} \) to other values, e.g. \( r_{\text{hi}}/(2M) = 50, 100, 200 \), like the single mode case \([16]\).

The left figure in figure 2 shows that the solution asymptotes to the approximate solution in the near horizon region, i.e. the solution of equation (6) with vanishing potential (see appendix F where we show the explicit form of the general \( S \)-deformation for \( n = 2 \) case when the potential vanishes). In figure 3, we plot the right hand side of equations (F.5) and (F.6).

\(^9\)The relation among \( \Phi_1, \Phi_2 \) and perturbed quantities can be seen in \([21, 26]\).

\(^{10}\)We used the function NDSolve in Mathematica for the numerical calculation and set the parameter WorkingPrecision to 30.
This also shows that the numerical solution can be matched with the approximate solution in $x \lessapprox -10$, and we can see that $x < c_{\pm}$ are satisfied there.

In far region, since the potential is not rapidly decaying as $r \to \infty$, the solutions do not approximately satisfy equation (6) with vanishing potential. However, since $|V_{12}| \ll |V_{11}|,|V_{22}|$ and $|S_{12}| \ll |S_{11}|,|S_{22}|$ in far region, we can expect that the system is approximately decoupled there. The right figure in figure 2 show that the solution asymptotes to the decoupled solution\textsuperscript{11}.

\textsuperscript{11} In far region, $S_{11}$ and $S_{22}$ approximately satisfies the equations $dS_i/dr = \frac{S_i^2 - \ell(\ell + 1)/r^2}{(r + c_i r^{2(1+\ell)})} \to -(\ell + 1)/r$. 

\textbf{Figure 1.} The numerical solution $\text{Tr}(S)$ for $\ell = 2, \beta M^4 = 1/10$ with the boundary condition $r_{\text{ini}}/(2M) = 10$ and $S|_{r_{\text{ini}}} = 0$. The figures are the profile of $\text{Tr}(S)$ (upper), near horizon region (lower left), far region (lower right).

\textbf{Figure 2.} Normalized $\text{Tr}(S)$ in near horizon region (left) and far region (right) by their asymptotic behaviors $\text{Tr}(S) \simeq -2/\ell = -1/(M \ln(r/(2M) - 1))$ and $\text{Tr}(S) \simeq -2(\ell + 1)/r$, respectively. The parameters and the boundary condition are the same as in figure 1.
3.2. Charged squashed Kaluza–Klein black hole

The charged squashed Kaluza–Klein black hole [30] is a solution of the five-dimensional Einstein–Maxwell system. The metric and the gauge 1-form of this spacetime are given by

\[
\begin{align*}
    ds^2 &= -F dt^2 + K^2 d\rho^2 + \rho^2 K^2 (d\theta^2 + \sin^2 \theta d\phi^2) + \left(\rho_0 + \rho_+ + \rho_0 - \rho_+\right) (d\psi + \cos \theta d\phi)^2, \\
    A_\mu dx^\mu &= \sqrt{\frac{3}{2}} \frac{\rho_+ + \rho_-}{\rho} dt,
\end{align*}
\]

with the functions \( F = \frac{(\rho_+ + \rho_-)(\rho_0 + \rho_-)}{K^2} \) and \( K^2 = \frac{(\rho_0 + \rho_0 - \rho_+)}{\rho} \). The ranges of the angular coordinates are \( 0 \leq \theta \leq \pi, 0 \leq \phi \leq 2\pi, 0 \leq \psi \leq 4\pi \). The three parameters \( \rho_\pm, \rho_0 \) satisfy \( \rho_+ \geq \rho_- \geq 0 \) and \( \rho_+ + \rho_0 \geq 0 \), and the black hole horizon locates at \( \rho = \rho_+ \). The relations among these parameters and the Komar mass \( M \), the electric charge \( Q \) and the size of the extra dimension at infinity \( r_\infty \) are

\[
M = \frac{2\pi r_\infty}{G} (\rho_+ + \rho_-), \quad Q = \sqrt{\frac{3\pi r_\infty}{G}} \frac{\sqrt{\rho_+ + \rho_-}}{\sqrt{\rho_0 + \rho_-}}, \quad r_\infty = 2\sqrt{(\rho_0 + \rho_+)/(\rho_0 + \rho_-)},
\]

where \( G \) is the five-dimensional gravitational constant [30, 31]. In the asymptotic region \( \rho \to \infty \), the metric behaves

\[
\begin{align*}
    ds^2 &= -dt^2 + d\rho^2 + \rho^2 (d\theta^2 + \sin^2 \theta d\phi^2) + (r_\infty/2)^2 (d\psi + \cos \theta d\phi)^2 + O(1/\rho).
\end{align*}
\]

Thus, the metric in the asymptotic region behaves the four-dimensional Minkowski spacetime with twisted \( S^1 \).

The gravitational and electro-magnetic perturbation around the charged squashed Kaluza–Klein black hole was discussed in [22], and the master equations for \( K = \pm 1 \) mode...
perturbations take the same form as equations (11) and (12), with \( \frac{d}{dx} = \frac{F/K^2}{d/d\rho} \). The explicit form of the effective potential is given in [22]12.

Since \( V \) is real symmetric, we consider a real symmetric \( S \). Then, equation (6) also takes the same form as equations (16)–(18). At \( \rho \to \infty \), the potential takes \( V_{11} \to 4/r_{\infty}^2, V_{22} \to 4/r_{\infty}^2 \) and \( V_{12} \to 0 \), then the \( S \)-deformation takes \( S_{11} \to -2/r_{\infty}, S_{22} \to -2/r_{\infty}, S_{12} \to 0 \) and \( \text{tr}(S) \to -4/r_{\infty} \). We plot the numerical solutions \( \text{Tr}(S) \) in figure 4 for the cases \( r_{\infty}/\sqrt{GM} = 0.01, 0.1, 1, 10, 100 \). In each case, we plot \( Q/Q_{\text{extremal}} = 0, 0.01, 0.5, 0.99, 1 \), respectively, where \( Q_{\text{extremal}} = \sqrt{3M}/2 \) is the electric charge for the extremal case. We adopt the boundary condition such that \( S = 0 \) at \( \rho = \rho_{\text{ini}} \) and the values of \( \rho_{\text{ini}} \) can be read from figure 4, which correspond to \( Q/Q_{\text{extremal}} = 0, 0.01, 0.5, 0.99, 1 \), from left to right, respectively. Figure 4 shows that \( \text{Tr}(S) \) is bounded, and this implies that the spacetime is stable13. We should note that we can see \( \text{tr}(S) \rightarrow -4/r_{\infty} \) in the cases \( r_{\infty}/\sqrt{M} = 10, 100 \), if we extend the plot range of the \( \rho \) axis. We just report that we can find a regular \( S \) if \( \rho_{\text{ini}} \) is smaller than that for larger values.

Similarly to section 3.1, we use a coordinate \( \bar{x} = \rho_+ \ln(\rho/\rho_+ - 1), \) to solve equations (16)–(18) in the near horizon region. We checked that the right hand sides of equations (F.5) and (F.6) take constants near the horizon, and \( \bar{x} < c_\pm \) are satisfied there. Note that for large \( r_{\infty}, c_\pm \) becomes very large, e.g. \( c_+/\rho_+ \sim 10^9 \), and then \( (-x/2)\text{Tr}(S) \) becomes almost unity at a point extremely close to the horizon, e.g. \( \rho/\rho_+ - 1 \sim 10^{-9} \). In the extremal case, we use a coordinate \( \bar{x} = -\rho_0^2/(\rho - \rho_+) \) to solve equations (16)–(18) in the near horizon region. We note that the stability was already proven analytically in the extremal case [22].

4. Summary and discussion

In this paper, we extended the formalism in [16, 17] to coupled systems where the perturbative field equations take the form of systems of coupled Schrödinger equations. Similar to [16, 17], the existence of \( S \)-deformation implies the linear mode stability of the system. Also, we applied our formalism to the case of the Schwarzschild black hole in dynamical Chern–Simons gravity and a charged Kaluza–Klein black hole [22, 30], and we showed the stability of the spacetime by finding a regular \( S \)-deformation function numerically. While the dynamical Chern–Simons gravity case is a test problem because the stability was already shown in [26], our study of charged Kaluza–Klein black hole case gives a first numerical proof of the stability of the coupled mode for non-extremal cases.

For the coupled system, there is already a criteria, called the nodal theorem [24], for the existence and non-existence of the negative energy bound state as an extension of the Sturm–Liouville theorem. From the nodal theorem, we obtained a suggestion such that there also exists a regular \( S \)-deformation for the coupled system if the spacetime is stable. We showed that this is correct when the potential is of a compact support (or rapidly decaying at infinity) in appendix C. We mention some merit of our \( S \)-deformation method compared to the nodal theorem [24]. (i) If we discuss the stability based on the nodal theorem, we need to solve the Schrödinger equation from \( x = L \) (or \( x = -L \)) with sufficiently large \( L \), but if the spacetime

12 Since the effective potential in [22] is not symmetric, we need to consider the transformation of the master variable \( \Phi_1 := r_{\infty}\phi_{IG}, \Phi_2 := \phi_{LE} \) so that the effective potential is real symmetric. \( V_{12} \) is given by multiplying \( r_{\infty} \) to equation (52) in [22]. Note that \( V_{11}, V_{22} \) are same form as equations (51) and (54) in [22], respectively. Since under the transformation, \( (\phi_{IG}, \phi_{LE}) \rightarrow (\Phi_1, \Phi_2) \), the determinant of the effective potential does not change, the discussion based on using the determinant in [22] also does not change.

13 In figure 1 in the erratum for [32], the curves for \( \rho_0/\rho_+ = 30, 50, 100 \) apparently looks regular, but in fact they are divergent near the horizon. If we adopt the boundary condition \( S = 0 \) at \( \rho/\rho_+ = 50 \), then \( S \) becomes regular everywhere.
is almost marginally stable, it is not trivial how large \( L \) we should consider. If we discuss the stability based on the \( S \)-deformation method, we can solve the equation from a finite point, and we do not need to care about the boundary condition at infinity very much. (ii) To understand the proof of the nodal theorem [24] is not easy, but it is obvious that the existence of the regular \( S \)-deformation is a sufficient condition for the stability. (iii) We can easily show the non-existence of the zero mode just by showing two different regular \( S \)-deformation functions (see appendix G). In an usual way, we need to solve the Schrödinger equation for \( E = 0 \) with the decaying boundary condition at \( x \to \infty \) or \( x \to -\infty \), which is not always easy.

Finally, we briefly discuss an extension of the \( S \)-deformation method. Let us consider a system

\[
-\frac{d^2}{dx^2} \Phi + V \Phi = \omega^2 C \Phi, \tag{21}
\]

where \( V \) and \( C \) are Hermitian matrices and \( C \) is positive definite. Since \( \Phi^\dagger C \Phi \geq 0 \), from the similar discussion in section 1, the existence of regular solution of \( dS/dx = S^2 - V \) implies the mode stability of this system.

Figure 4. \( \text{Tr}(S) \) for the numerical solution \( S \) for various parameters. The boundary condition is \( S = 0 \), and hence \( \text{Tr}(S) = 0 \), at \( \rho = \rho_{\text{ini}} \). The values of \( \rho_{\text{ini}} \) can be read from each figure, the corresponding charge is \( Q/Q_{\text{extremal}} = 0, 0.01, 0.5, 0.99, 1 \) from left to right, respectively.
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Appendix A. Hermitian $V$ case as real symmetric problem

We decompose $n \times n$ Hermitian potential $V$ and $\Phi$ into the real part and imaginary part as $V = V_R + iV_I$ and $\Phi = \Phi_R + i\Phi_I$. From $V^\dagger = V$, we obtain $V_R = V_R, V_I = -V_I$, where $T$ denotes a transposition. The Schrödinger equation (3) can be written in the form

$$\frac{d^2 \Phi_R}{dx^2} + V_R \Phi_R - V_I \Phi_I = E \Phi_R,$$

(A.1)

$$\frac{d^2 \Phi_I}{dx^2} + V_I \Phi_R + V_R \Phi_I = E \Phi_I.$$

(A.2)

So, we can consider this problem as $2n \times 2n$ real potential problem with

$$U = \begin{pmatrix} V_R & -V_I \\ V_I & V_R \end{pmatrix}.$$  

(A.3)

We should note that $U$ is real symmetric since

$$U^T = \begin{pmatrix} V_R^T & V_I^T \\ -V_I^T & V_R^T \end{pmatrix} = \begin{pmatrix} V_R & -V_I \\ V_I & V_R \end{pmatrix} = U.$$  

(A.4)

When $\{\Phi_{\mu}\} = \{\Phi_{\mu R} + i\Phi_{\mu I}\}, (\mu = 1, 2, \cdots, m \text{ and } m \leq 2n)$, are linearly independent with complex coefficients,

$$\begin{pmatrix} \Phi_{1R} \\ \Phi_{1I} \end{pmatrix}, \begin{pmatrix} -\Phi_{2I} \\ \Phi_{2R} \end{pmatrix}, \begin{pmatrix} -\Phi_{3I} \\ \Phi_{3R} \end{pmatrix}, \cdots, \begin{pmatrix} \Phi_{nR} \\ \Phi_{nI} \end{pmatrix}, \begin{pmatrix} -\Phi_{1I} \\ -\Phi_{1R} \end{pmatrix}, \begin{pmatrix} -\Phi_{2I} \\ -\Phi_{2R} \end{pmatrix}, \cdots, \begin{pmatrix} -\Phi_{nI} \\ -\Phi_{nR} \end{pmatrix},$$

(A.5)

are linearly independent with real coefficients. For $n$ linearly independent solutions $\{\Phi_i\}$ ($i = 1, 2, \cdots, n$), defining $2n \times 2n$ real matrix

$$Z = \begin{pmatrix} \Phi_{1R} & \Phi_{2R} & \cdots & \Phi_{nR} & -\Phi_{1I} & -\Phi_{2I} & \cdots & -\Phi_{nI} \\ \Phi_{1I} & \Phi_{2I} & \cdots & \Phi_{nI} & \Phi_{1R} & \Phi_{2R} & \cdots & \Phi_{nR} \end{pmatrix},$$

(A.6)

this corresponds to equation (B.5) to the potential problem for $U$. We can show the relation

$$|\det(Y)|^2 = \det(Z),$$

(A.7)

where $n \times n$ matrix $Y$ is

$$Y = (\Phi_1, \Phi_2, \cdots, \Phi_n) = (\Phi_{1R} + i\Phi_{1I}, \Phi_{2R} + i\Phi_{2I}, \cdots, \Phi_{nR} + i\Phi_{nI}).$$

(A.8)

Thus, $\det(Y) = 0$ if and only if $\det(Z) = 0$. 

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Appendix B. Riccati transformation for a system of coupled Schrödinger equations

We introduce the Riccati transformation for a system of coupled Schrödinger equation (3). If there exists an $n \times n$ matrix $S$ which satisfies $\frac{d}{dx} \Phi = -S \Phi$, we can show

$$\frac{d^2 \Phi}{dx^2} = -\frac{dS}{dx} \Phi + S^2 \Phi.$$  \hspace{1cm} (B.1)

We should note that $S$ is not necessarily a Hermitian matrix in this section. From the Schrödinger equation (3), which $\Phi$ is supposed to solve, we obtain

$$\left( V - E + \frac{dS}{dx} - S^2 \right) \Phi = 0,$$  \hspace{1cm} (B.2)

where we omitted to write an unit matrix in front of $E$. If $\Phi$ is a zero vector, the equation

$$\det \left( V - E + \frac{dS}{dx} - S^2 \right) = 0$$  \hspace{1cm} (B.3)

should hold. In fact, instead of solving the Schrödinger equation (3), we can solve the equation

$$V - E + \frac{dS}{dx} - S^2 = 0.$$  \hspace{1cm} (B.4)

Once a solution $S$ is given, we can solve the equation $\frac{d\Phi}{dx} = -S \Phi$ with respect to $\Phi$, then $\Phi$ satisfies the Schrödinger equation (3).

On the other hand, for given $n$-linearly independent solutions $\{\Phi_i\}, (i = 1, 2, \ldots, n)$ of equation (3), we can construct $S$ which satisfies equation (B.4) in the following way. For an $n \times n$ matrix

$$Y := (\Phi_1, \Phi_2, \ldots, \Phi_n),$$  \hspace{1cm} (B.5)

which satisfies

$$-\frac{d^2}{dx^2} Y + YY = EY,$$  \hspace{1cm} (B.6)

we define $S$ as

$$S := -\frac{dY}{dx} Y^{-1},$$  \hspace{1cm} (B.7)

where we consider a domain such that $\det(Y) \neq 0$. Since $S$ satisfies $dY/dx = -SY$, the relation

$$\left( V - E + \frac{dS}{dx} - S^2 \right) Y = 0$$  \hspace{1cm} (B.8)

holds. Since we focus on a domain with $\det(Y) \neq 0$, this equation implies that $S$ satisfies equation (B.4). For a solution of equation (B.4), we can solve $dY/dx = -SY$ with respect to $Y$, then it satisfies equation (B.6). This solution $Y$ contains $2n^2$ integral constants which coincides with the number of integral constants of the general solution of equation (B.6).
We also discuss the condition for \( S \) defined by equation (B.7) to be a Hermitian matrix. We can easily show that \( S \) becomes Hermitian if it is chosen to be so at a point in solving equation (B.7). This criterion is useful, but ‘a point’ should be a finite point, cannot be infinity. To discuss the relation between the decaying boundary condition at infinity and the Hermiticity of \( S \), here we introduce another criterion. Defining \( \rho \) as

\[
\rho := Y \frac{dY}{dx} - \frac{dY}{dx} Y, \quad (B.9)
\]

one can show \( d\rho/dx = 0 \) and hence

\[
\rho = \text{const.} \quad (B.10)
\]

Thus, if \( \rho = 0 \) at some point, \( S \) becomes a Hermitian matrix from the relation

\[
S^\dagger - S = (Y^\dagger)^{-1} \rho Y^{-1}. \quad (B.11)
\]

When all \( \{\Phi_i\} \) are decaying modes (or constant modes), \( \rho \) becomes zero since both \( Y \) and \( dY/dx \) (only \( dY/dx \) for constant modes) vanish at infinity, and \( S \) becomes a Hermitian matrix.

**Appendix C. Existence of S-deformation of a compact support potential**

For simplicity, we consider the case with a compact support potential, i.e. \( V = 0 \) for \( |x| \geq L \) with a positive constant \( L \). Note that the following discussion can be extended to the case with a non-compact support potential if it rapidly decays at \( x \to \pm \infty \). Also, if \( V \to \text{diag}[v_1, \ldots, v_n] \) with \( v_i \geq 0, (i = 1, \cdots, n) \) at \( x \to \infty \) in some basis and vanishing components are rapidly decaying, the following discussion remains to be valid by a minor modification\(^{15}\).

We denote \( Y_L^{(E)} \) and \( Y_R^{(E)} \) are solutions of equation (B.6) for an energy \( E(\leq 0) \) which behave as

\[
Y_L^{(E)} |_{x \leq -L} = \text{diag}[e^{\sqrt{-E}x}, e^{\sqrt{-E}x}, \ldots, e^{\sqrt{-E}x}], \quad (C.1)
\]

\[
Y_R^{(E)} |_{x \geq L} = \text{diag}[e^{-\sqrt{-E}x}, e^{-\sqrt{-E}x}, \ldots, e^{-\sqrt{-E}x}]. \quad (C.2)
\]

For \( E < 0 \), \( Y_L^{(E)} \) and \( Y_R^{(E)} \) are decaying mode at \( x \to -\infty \) and \( x \to \infty \), respectively. It is clear that both \( Y_L^{(E)} \) and \( Y_R^{(E)} \) are continuous functions of \( E(\leq 0) \) at a finite point of \( x \). We also define the corresponding \( S \) as

\[
S_L^{(E)} := -\frac{dY_L^{(E)}}{dx} \left(Y_L^{(E)}\right)^{-1}, \quad S_R^{(E)} := -\frac{dY_R^{(E)}}{dx} \left(Y_R^{(E)}\right)^{-1}. \quad (C.3)
\]

In this section, we show that \( S_L^{(0)} \) and \( S_R^{(0)} \) are continuous for \(-\infty < x < \infty \) when there exists no bound state with \( E \leq 0 \). Hereafter, we mainly focus on \( S_L^{(E)} \), but the same kind of discussion also holds for \( S_R^{(E)} \).

**Lemma C.1.** We denote Hermitian solutions of equation (B.4) for \( E_1 \) and \( E_2 \) with \( E_2 > E_1 \) by \( S^{(E_1)} \) and \( S^{(E_2)} \), respectively. We assume that \( S^{(E_1)} \) and \( S^{(E_2)} \) are continuous for \( x_- \leq x \leq x_+ \). If all eigenvalues of \( S^{(E_2)} - S^{(E_1)} \) are positive at \( x = x_- \), those for \( x_- \leq x \leq x_+ \) are also positive.

\(^{14}\)From equation (B.4) and its Hermitian conjugate, an equation \( d(S^\dagger - S)/dx = (S^\dagger - S)S^\dagger + S(S^\dagger - S) \) holds. If \( S = S^\dagger \) is satisfied at some point \( x = a \), \( S \) becomes a Hermitian matrix.

\(^{15}\)Note that \( V \) at \( x \to -\infty \) is zero because it corresponds to the horizon. If \( V \) at \( x \to \infty \) is not diagonalized, we can diagonalize it by a linear transformation of the wave functions.
**Proof.** Suppose that \( S^{(E_1)} - S^{(E_2)} \) has a zero eigenvalue at \( x = a \) with \( x_- < a < x_+ \), and it does not have a zero eigenvalue for \( x_- \leq x < a \). At \( x = a \), the equations

\[
\left( S^{(E_2)} - S^{(E_1)} \right) \hat{e}_a = 0, \tag{C.4}
\]

\[
\hat{e}_a \left( \frac{dS^{(E_2)}}{dx} - \frac{dS^{(E_1)}}{dx} \right) \hat{e}_a \leq 0, \tag{C.5}
\]

should hold, where \( \hat{e}_a \) is an unit eigenvector of \( S^{(E_1)} - S^{(E_2)} \) for the zero eigenvalue at \( x = a \).\(^{16}\) We note that \( \hat{e}_a \) is a constant vector.\(^ {17}\) However, at \( x = a \), we also have the relation

\[
\hat{e}_a \left( \frac{dS^{(E_2)}}{dx} - \frac{S^{(E_1)}}{dx} \right) \hat{e}_a = E_2 - E_1 + \hat{e}_a \left( S^{(E_2)} \right)^2 \hat{e}_a - \hat{e}_a \left( S^{(E_1)} \right)^2 \hat{e}_a = E_2 - E_1 > 0, \tag{C.6}
\]

where we used equation (C.4) in the second equality. This is a contradiction. \( \square \)

**Lemma C.2.** Let us assume that \( S \) is continuous for \( x_- < x < x_+ \) and \( S \) is divergent at \( x = x_- \) (\( x = x_+ \)). Then, the eigenvalue of \( S \) that diverges behaves as \(-1/(x-x_+) \to +\infty \) at \( x \to x_+ - 0 \) \((-1/(x-x_-) \to -\infty \) at \( x \to x_- + 0 \)).

**Proof.** From the assumption, one of the eigenvalue of \( S \) is divergent at \( x = x_+ \). Let \( e \) be the unit eigenvector of \( S \) with the eigenvalue \( \lambda \)

\[
Se = \lambda e, \tag{C.7}
\]

and \( |\lambda| \to \infty \) at \( x = x_+ \). Note that \( e \) is not a constant vector. Taking \( x \) derivative of equation (C.7) and multiplying \( e^\dagger \) from the left side, we obtain

\[
e^\dagger \frac{dS}{dx} e = \frac{d\lambda}{dx}. \tag{C.8}
\]

From equation (B.4), this equation can be written in the form

\[
\frac{d\lambda}{dx} = \lambda^2 - e^\dagger (V - E)e. \tag{C.9}
\]

Thus, when \( |\lambda| \to \infty \) at \( x \to x_+ \), \( \lambda \) approximately satisfies \( d\lambda/dx \simeq \lambda^2 \), and the approximate solution is \( \lambda \simeq -1/(x-x_+) \to +\infty \) at \( x \to x_+ - 0 \).\(^ {18}\) \( \square \)

**Lemma C.3.** Suppose that \( \det(Y) \) is not identically zero. \( S \) is divergent at \( x = x_+ \) if and only if \( \det(Y) = 0 \) at \( x = x_+ \).

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\(^{16}\) Let us denote \( e \) be a unit eigenvector of \( S^{(E_1)} - S^{(E_2)} \) with the eigenvalue \( \Lambda \) which vanishes at \( x = a \). The derivative of \( \Lambda \) becomes \( d\Lambda/dx = e^\dagger (S^{(E_1)} - S^{(E_2)})e \), where we used \((d e^\dagger / dx)(S^{(E_1)} - S^{(E_2)})e + e^\dagger (S^{(E_2)} - S^{(E_1)}) (de/dx) = \Lambda (de^\dagger / dx)e + e^\dagger (de/dx) = 0 \). Since \( e = \hat{e}_a \) at \( x = a \), we obtain equation (C.5) from \( d\Lambda/dx_{\text{eval}} \leq 0 \).

\(^{17}\) In this paper, we use the hat symbol to denote a constant vector.

\(^{18}\) Precisely speaking, this is just a sketch of the proof, but we can also give a rigorous proof.
Proof. It is trivial that \( \det(Y) = 0 \) at \( x = x_+ \) if \( S \) is divergent at \( x = x_+ \) because \( S = -(dY/dx)Y^{-1} \), and \( Y \) and \( dY/dx \) are continuous in \(-\infty < x < \infty\). Thus, we need to show that \( S \) is divergent at \( x = x_+ \) if \( \det(Y) = 0 \) at \( x = x_+ \). Let us assume that \( S \) is not divergent at \( x = x_+ \) when \( \det(Y) = 0 \) at \( x = x_+ \). In that case, from the relation
\[
\frac{d \det(Y)}{dx} = \det(Y) \text{Tr}(Y^{-1}dY/dx) = -\det(Y) \text{Tr}(S),
\]
\( \det(Y) \) vanishes everywhere. This contradicts with the assumption that \( \det(Y) \) is not identically zero. \( \square \)

**Lemma C.4.** If \( \det(Y(E)) \) has a zero at \( x = x_E \), there exists \( E_1( < E_2) \) such that \( \det(Y(E)) \) has a zero at \( x = x_E \) for \( E_1 < E < E_2 \).

**Proof.** From lemma C.3, if \( \det(Y(E)) \) has a zero at \( x = x_E \), \( S(E) \) is divergent at \( x = x_E \). From lemma C.2, one of the eigenvalue of \( S(E) \), which is denoted by \( \lambda(E) \), behaves like \( (\lambda(E))^{-1} \approx -(x - x_E) \) near \( x = x_E \). Since \( (\lambda(E))^{-1} \) is a continuous function of \( E \), the property that \( (\lambda(E))^{-1} \) has a zero at \( x = x_E \) with \( d(\lambda(E))^{-1}/dx|_{x_E} = -1 \) should hold for the range \( E_1 < E < E_2 \) with some \( E_1( < E_2) \).

**Lemma C.5.** Let us denote \( x_E \) as a zero of \( \det(Y(E)) \) if it exists and assume that \( \det(Y(E)) \neq 0 \) for \(-\infty < x < x_E \). Then, \( x_E \) is monotonically non-increasing function of \( E( < 0) \) if it exists.

**Proof.** Since at \( x \leq -L \),
\[
S(E) = \text{diag} [\sqrt{-E}, \sqrt{-E}, \ldots, \sqrt{-E}],
\]
for \( E_1 < E_2 < 0 \), all eigenvalues of \( S(E_1) - S(E_2) \) are positive at \( x \leq -L \). If we assume \( x_{E_1} < x_{E_2} \), \( S(E_1) \) and \( S(E_2) \) are continuous in \(-\infty < x < x_{E_1} \). From lemma C.1, all eigenvalue of \( S(E_1) - S(E_2) \) are positive for \(-\infty < x < x_{E_1} \). However, since \( \lambda(E) \) holds and \( \lambda(E) \to \infty \) at \( x \to x_{E_1} \), this contradicts with the assumption \( x_{E_1} < x_{E_2} \).

**Lemma C.6.** \( \det(Y(E)) \) does not have any zero for sufficiently large negative \( E \).

**Proof.** Let \( e \) be a unit eigenvector of \( S(E) \) with the eigenvalue \( \lambda(E) \):
\[
S(E)e = \lambda(E)e.
\]
For a sufficiently large negative \( E \), the function \( e^\dagger Ve - E \) is positive and bounded everywhere. In this case, from the equation

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19 We used a well known fact: for two Hermitian matrices \( A \) and \( B \), if all eigenvalues of \( A - B \) are positive, the inequality \( \lambda_i > \lambda_i^0 \) holds, where \( \lambda_i \) and \( \lambda_i^0 \) are the \( i \)th largest eigenvalues of \( A \) and \( B \), respectively.
we can see that $\lambda^{(E)}_L$ is bounded from the same discussion as [16]\(^{20}\) Thus, the eigenvalue of $S^{(E)}_L$ is continuous everywhere for a large negative $E$, hence $\det \left( Y^{(E)}_L \right)$ also does not have any zero for a large negative $E$.

\begin{proof}
To show that $S^{(0)}_L$ is continuous for a stable spacetime, we need to show that there exists a negative energy eigenstate if $\det \left( Y^{(0)}_L \right)$ has a zero. From lemmas C.4–C.6, there exists $E_0(<0)$ such that $x_L \to \infty$ when $E \to E_0 + 0$. Thus, there exists $E_1$ with $E_0 < E_1 < 0$ such that for $E_0 < E < E_1$, $x_E$, a zero of $\det \left( Y^{(E)}_L \right)$, locates in $x \geq L$ and $\det \left( Y^{(E)}_L \right) \neq 0$ for $-\infty < x < x_E$. For $x > L$, $Y^{(E)}_L$ can be written in the form

\[ Y^{(E)}_L \big|_{x \geq L} = (e^{2\sqrt{E}x} A^{(E)}_L + 1)e^{-\sqrt{E}x} B^{(E)}_L, \]

where $A^{(E)}_L$ and $B^{(E)}_L$ are constant matrices. Note that $\det \left( B^{(E)}_L \right) \neq 0$ because $\det \left( Y^{(E)}_L \right) \big|_{x > L}$ is not identically zero. If $\det(Y^{(E)}_L) = 0$ at $x = x_E$, the equation

\[ \det \left( e^{2\sqrt{E}x} A^{(E)}_L + 1 \right) = 0, \]

holds. Since $x_E \to \infty$ for $E \to E_0$, $\det \left( A^{(E_0)}_L \right) = 0$ should be satisfied. Let $u$ be an eigenvector of $A^{(E_0)}_L$ with zero eigenvalue, $Y^{(E_0)}_L \left( B^{(E_0)}_L \right)^{-1} u$ behaves like

\[ Y^{(E_0)}_L \left( B^{(E_0)}_L \right)^{-1} u \big|_{x < -L} = e^{2\sqrt{E}x} \left( B^{(E_0)}_L \right)^{-1} u, \]

\[ Y^{(E_0)}_L \left( B^{(E_0)}_L \right)^{-1} u \big|_{x \geq L} = e^{-\sqrt{E}x} u. \]

Thus, $Y^{(E_0)}_L \left( B^{(E_0)}_L \right)^{-1} u$ is a negative energy eigenstate of the Schrödinger equation.
\end{proof}

\section*{Appendix D. Robustness of the S-deformation method}

In a single mode case, the robustness of the S-deformation method was discussed in [17], i.e. the general regular S-deformation has one parameter degree of freedom. In this section, we extend the discussion in [17] to coupled systems when the potential $V$ has a compact support.

In this section, since we only consider $E = 0$ case, we omit to write the superscript (0), e.g. $Y_L$ instead of $Y^{(0)}_L$. We assume that there exists no bound state with $E \leq 0$, then $\det(Y_L)$ and $\det(Y_R)$ do not have zero in $-\infty < x < \infty$. The general solution of equation (B.6) with $E = 0$ is

\[^{20}\lambda^{(E)}_L$ behaves $\lambda^{(E)}_L \big|_{x < -L} = -\sqrt{E}$, so we need to check that it is bounded for $x > -L$. For positive $e^1 V e - E$, once $\lambda^{(E)}_L$ takes a value less than $\sqrt{e^1 V e - E}$ it is bounded in $x > -L$, and this condition is satisfied at $x = -L$.\]
\[ Y = (Y_L M + Y_R)N. \]  

(D.1)

Since we are interested in regular $S$-deformations which correspond to the case with $\det(Y) \neq 0$, we only need to consider $\det(N) \neq 0$. In that case, all $N$ gives the same $S = -(dY/dx)Y^{-1}$.

Thus, it is enough to study the case that $N$ is the identity matrix.

D.1. All eigenvalues of $S_R - S_L$ are positive

In $x \leq -L$, from the behavior of $Y_L$ and $Y_R$.

\[ Y_L |_{x \leq -L} = \text{diag}[1, \ldots, 1], \quad (D.2) \]

\[ Y_R |_{x \leq -L} = C_R + xD_R, \quad (D.3) \]

$S_L$ and $S_R$ becomes

\[ S_L |_{x \leq -L} = \text{diag}[0, \ldots, 0], \quad (D.4) \]

\[ S_R |_{x \leq -L} = -(D_R^{-1}C_R + x)^{-1}. \quad (D.5) \]

Note that $\det(D_R) \neq 0$ because we assume that there exists no bound state with $E \leq 0$. Since $S_R$ is bounded, $D_R^{-1}C_R + x$ does not have zero eigenvalue in $x \leq -L$, hence $D_R^{-1}C_R - L$ is negative definite and $S_R |_{x < -L}$ is positive definite in $x \leq -L$. Thus, all eigenvalues of $S_R - S_L$ are positive in $x \leq -L$.

Suppose that $S_R - S_L$ has a zero eigenvalue at $x = a (> -L)$. Also, let $e$ be a unit eigenvector of $S_R - S_L$ with the eigenvalue $\Lambda$,

\[ (S_R - S_L)e = \Lambda e. \quad (D.6) \]

then $\Lambda = 0$ at $x = a$. From this equation, we can derive

\[ \frac{d\Lambda}{dx} = \Lambda \left( \Lambda + 2e^*S_L e \right). \quad (D.7) \]

Thus, if $\Lambda = 0$ at $x = a$, $\Lambda = 0$ everywhere from the uniqueness of the ordinary differential equation. However, this contradicts with the fact that $\Lambda > 0$ in $x \leq -L$ as shown above.

D.2. General regular $S$

Defining $\rho_{LR}$ by

\[ \rho_{LR} = \frac{dY^\dagger_L Y_R - Y^\dagger_L dY_R}{dx}, \quad (D.8) \]

the relation

\[ S_R - S_L = (Y^\dagger_L)^{-1}\rho_{LR}(Y_R)^{-1} =: W^{-1}, \quad (D.9) \]

holds. From $Y = Y_L M + Y_R$,

\[ W_L := Y \rho_{LR}^{-1} Y^\dagger_L = Y_L M \rho_{LR}^{-1} Y^\dagger_L + Y_R \rho_{LR}^{-1} Y^\dagger_L \]

\[ = Y_L M \rho_{LR}^{-1} Y^\dagger_L + W. \quad (D.10) \]
From the relation,
\[
\frac{dY^+_L}{dx} Y - Y^+_L \frac{dY}{dx} = \rho_{LR},
\]  \hspace{1cm} (D.11)
we can show
\[
S - S_L = (Y^+_L)^{-1} \rho_{LR} Y^{-1} = W^{-1}_L.
\]  \hspace{1cm} (D.12)

Since $S$ is Hermitian, $W_L$ should also be Hermitian. From equation (D.10), this condition is that $M \rho_{LR}^{-1}$ is Hermitian. If $M \rho_{LR}^{-1}$ is non-negative definite, from equation (D.10), $Y \rho_{LR}^{-1} Y_L$ becomes positive definite. In that case, $Y$ does not have a zero eigenvalue because if $e_1^T Y = 0$ with a vector $e_1$ at some point, it contradicts with $e_1^T Y \rho_{LR}^{-1} Y_L e_1 > 0$. Thus, $M \rho_{LR}^{-1} \geq 0$ is the sufficient condition so that $S$ is regular everywhere.

From the relations $Y_L M \rho_{LR}^{-1} Y_L^* = O(x^2)$ and $W = O(x)$ at $x \rightarrow \infty$, and $Y_L M \rho_{LR}^{-1} Y_L = O(x^0)$ and $W = O(x)$ at $x \rightarrow -\infty$, if $M \rho_{LR}^{-1}$ has a negative eigenvalue, $W_L$ has a negative eigenvalue near $x \rightarrow \infty$ and $W_L$ is positive definite near $x \rightarrow -\infty$. Thus, $W_L$ should have a zero eigenvalue with an eigenvector $e$ somewhere. However, this implies $Y (\rho_{LR}^{-1} Y_L e) = 0$, and then $Y$ has a zero eigenvalue. This is a contradiction. Thus, $M \rho_{LR}^{-1} \geq 0$ is the necessarily and sufficient condition so that $S$ is regular everywhere.

D.3. Robustness of the S-deformation method

We have already obtained that all eigenvalues of $S - S_L$ are positive in the previous subsection. We repeat the same discussion for $S_R - S$. From the equation $Y^+ = M^T Y^+_L + Y^+_R$, the relation
\[
W_R := Y_R \rho_{LR}^{-1} (M^T)^{-1} Y^+_R = (Y_R \rho_{LR}^{-1} Y^+_L + Y_R \rho_{LR}^{-1} (M^T)^{-1} Y^+_R)
\]  \hspace{1cm} (D.13)
holds. From the relation,
\[
\frac{dY^+_L}{dx} Y_R - Y^+_L \frac{dY}{dx} = M^T \rho_{LR},
\]  \hspace{1cm} (D.14)
we can show
\[
S_R - S = (Y^+_R)^{-1} M^T \rho_{LR} Y_R^{-1} = W_R^{-1}.
\]  \hspace{1cm} (D.15)
From the same discussion above, we impose that $M^T \rho_{LR}$ is Hermitian and $M^T \rho_{LR}$ is positive definite\(^{21}\) so that $S$ is Hermitian and continuous. Thus, $S_R - S$ is positive definite everywhere for regular $S$.

If we solve equation (6) with the boundary condition such that all eigenvalues of $S_R - S$ and $S - S_L$ are positive, this relations hold everywhere. In that case, all eigenvalues satisfy $\lambda_i^L < \lambda_i < \lambda_i^R$, where $\lambda_i^L$, $\lambda_i$, $\lambda_i^R$ are $i$th largest eigenvalue of $S_L, S, S_R$, respectively. This is the reason why we can find the S-deformation without fine-tuning in numerically solving equation (6).

\(^{21}\) This is same as that $M \rho_{LR}^{-1}$ is Hermitian and positive definite.


Appendix E. Regular $S$ for positive definite $V$

In this section, we consider the case with positive definite $V$, where the system is manifestly stable. In this case, the following proposition holds.

**Proposition E.1.** Let us assume that $V$ is positive definite, and let $S$ be a solution of equation (6). If all components of $S$ are zero at a point, $S$ is Hermitian and bounded everywhere.

**Proof.** Suppose that $x = a$ be a point such that all components of $S$ are zero there, then all eigenvalues are also zero at $x = a$. As shown in footnote 14, if $S$ is Hermitian at a point, the solution of equation (6) is Hermitian everywhere. Let $e$ and $\lambda$ be a unit eigenvector and an eigenvalue of $S$, respectively. Then, the relation

$$\frac{d\lambda}{dx} = \lambda^2 - e^\dagger Ve$$

(E.1)

holds and $\lambda = 0$ at $x = a$. From the same discussion as [16], once $\lambda = 0$ at a point, $\lambda$ is bounded above and below everywhere for positive $e^\dagger Ve$. Since this discussion holds for all eigenvalues of $S$, the solution of equation (6) is bounded everywhere.

The discussion in appendix D suggests that all eigenvalues of $S_R - S$ and $S - S_L$ are positive everywhere for regular $S$. Thus, from proposition E.1 we can expect that $S_R$ is positive definite, and $S_L$ is negative definite for positive definite $V$. If the potential is positive definite only at large $x$, the similar discussion holds in the asymptotic region, hence we can expect that the solution of equation (6) with the boundary condition $S = 0$ at large $x$ becomes regular for a stable spacetime. This is just a rough discussion, but it is worth to try this boundary condition if $V$ is positive definite at large $x$. We should note that the numerical studies in section 3 support this.

Appendix F. Approximate solution in $n = 2$

Usually, the potential $V$ is proportional to $r - r_H$ near the horizon of a non-extremal black hole, where $r_H$ is the horizon radius. Since the relation between $x$ and $r$ near the horizon is $r/r_H \simeq 1 + e^{i/n}$, the potential is rapidly decaying to zero at $x \to -\infty$ as $V \propto e^{i/n} \to 0$. In this case, we can find approximate solutions of the equation $V + dS/dx - S^2 = 0$ as

$$S_{11} = \frac{-2x + c_+ + c_- - (c_+ - c_-) \cos \theta_0}{2(x - c_+)(x - c_-)},$$

(F.2)

22 When $V = 0$, the general solution of $d^2Y/dx^2 = 0$ is $Y = R_{\Theta}(x + A)B$ with constant matrices $R_{\Theta}$, $A$, and $B$ ($\det(B) \neq 0$). For the latter convenience, we put the rotation matrix $R_{\Theta}$

$$R_{\Theta} = \begin{pmatrix} \cos \Theta & -\sin \Theta \\ \sin \Theta & \cos \Theta \end{pmatrix}.$$  

(F.1)

We should note that $B$ does not appear in $S = -(dY/dx)Y^{-1}$. Imposing $S$ to be symmetric, $A$ also should be symmetric. Since the non-diagonal part of $A$, can be removed by the degrees of freedom of $R_{\Theta}$ and $B$, i.e. $R_{\Theta} \to R_{\Theta}R_{\Theta}$ and $B \to R_{\Theta}B$ and choose $\Theta_1$ so that the non-diagonal part vanishes, we only need to consider $Y = R_{\Theta} \text{diag}(x - c_+, x - c_-)B$. Defining $\theta_0 = 2\Theta$, we obtain equations (F.2)-(F.4) from $S = -(dY/dx)Y^{-1}$.
$S_{12} = \frac{-(x_+ - x_-) \sin \theta_0}{2(x - c_+)(x - c_-)}, \quad (F.3)$

$S_{22} = \frac{-2x + c_+ + c_- + (c_+ - c_-) \cos \theta_0}{2(x - c_+)(x - c_-)}. \quad (F.4)$

In $x \to -\infty$, we can see $S_{11} \simeq -1/x, S_{12} \simeq (-(x_+ - x_-) \sin \theta_0)/(2x^2)$ and $S_{22} \simeq -1/x$.

These equations can be written in the form

$$c_\pm = x - \frac{S_{11} + S_{22} \pm \sqrt{4S_{12}^2 + (S_{11} - S_{22})^2}}{2(S_{12} - S_{11}S_{22})}, \quad (F.5)$$

$$\cos \theta_0 = \frac{-S_{11} + S_{22}}{\sqrt{4S_{12}^2 + (S_{11} - S_{22})^2}}. \quad (F.6)$$

For a numerical solution $S$, if the right hand sides of the above equations take constants in the asymptotic region, it implies that equations (F.2)–(F.4) become a good approximation there. Also, if $x < c_\pm$ are satisfied in the asymptotic region, it implies that $S$ is bounded in the asymptotic region.

Appendix G. Non-existence of the zero mode

Similarly to the case of the single degree of freedom [17], the following proposition on the zero mode, i.e. the non-trivial solution of equation (3) with zero energy which is decaying (or constant) at both $x \to -\infty$ and $x \to \infty$, holds:

**Proposition G.1.** Suppose that there exist two different regular solutions of equation (6), $S_1$ and $S_2$. If $\det(S_1 - S_2)$ is not identically zero, no zero mode exists.

**Proof.** If there exists a zero mode $\Phi$, it satisfies

$$\frac{d\Phi}{dx} + S_i \Phi = 0, \quad (i = 1, 2) \quad (G.1)$$

from equation (4), and hence, $(S_1 - S_2)\Phi = 0$ holds. However, from the assumption this cannot be satisfied.

If we obtain two different $S$ from the boundary conditions $S = 0$ at two large $x$, usually, the assumption ‘$\det(S_1 - S_2)$ is not identically zero’ is satisfied.

**ORCID iDs**

Masashi Kimura ⊕ https://orcid.org/0000-0001-5427-5593

**References**

[1] Regge T and Wheeler J A 1957 *Phys. Rev.* 108 1063
[2] Vishveshwara C V 1970 *Phys. Rev.* D 1 2870
[3] Zerilli F J 1970 *Phys. Rev. Lett.* 24 737
[4] Zerilli F J 1970 Phys. Rev. D 2 2141
[5] Kodama H and Ishibashi A 2003 Prog. Theor. Phys. 110 701
[6] Ishibashi A and Kodama H 2003 Prog. Theor. Phys. 110 901
[7] Kodama H and Ishibashi A 2004 Prog. Theor. Phys. 111 29
[8] Dotti G and Gleiser R J 2005 Class. Quantum Grav. 22 L1
[9] Dotti G and Gleiser R J 2005 Phys. Rev. D 72 044018
[10] Gleiser R J and Dotti G 2005 Phys. Rev. D 72 124002
[11] Takahashi T and Soda J 2010 Prog. Theor. Phys. 124 911
[12] Takahashi T and Soda J 2009 Phys. Rev. D 79 104025
[13] Takahashi T and Soda J 2009 Phys. Rev. D 80 104021
[14] Beroiz M, Dotti G and Gleiser R J 2007 Phys. Rev. D 76 024012
[15] Takahashi T and Soda J 2010 Prog. Theor. Phys. 124 711
[16] Kimura M 2017 Class. Quantum Grav. 34 235007
[17] Kimura M and Tanaka T 2018 Class. Quantum Grav. 35 195008
[18] Winstanley E and Sarbach O 2002 Class. Quantum Grav. 19 689
[19] Baxter J E and Winstanley E 2016 J. Math. Phys. 57 022506
[20] Sarbach O and Winstanley E 2001 Class. Quantum Grav. 18 2125
[21] Molina C, Pani P, Cardoso V and Gualtieri L 2010 Phys. Rev. D 81 124021
[22] Nishikawa R and Kimura M 2010 Class. Quantum Grav. 27 215020
[23] Cardoso V, Kimura M, Maselli A and Senatore L 2018 Phys. Rev. Lett. 121 251105
[24] Amann H and Quittner P 1995 J. Math. Phys. 36 4553
[25] Aybat S M and George D P 2010 J. High Energy Phys. JHEP09(2010)010
[26] Kimura M 2018 Phys. Rev. D 98 024048
[27] Garaud J and Volkov M S 2008 Nucl. Phys. B 799 430
[28] Chen F W, Gu B M and Liu Y X 2018 Eur. Phys. J. C 78 131
[29] Garaud J and Volkov M S 2010 Nucl. Phys. B 839 310
[30] Ishihara H and Matsumo K 2006 Prog. Theor. Phys. 116 417
[31] Kurita Y and Ishihara H 2007 Class. Quantum Grav. 24 4525
[32] Kimura M, Murata K, Ishihara H and Soda J 2008 Phys. Rev. D 77 064015
Kimura M, Murata K, Ishihara H and Soda J 2017 Phys. Rev. D 96 089902 (erratum)