Enumeration formulas for standard young tableaux of approximate C shape

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Abstract. This paper focuses on the combinatorial enumeration of the truncated hollow standard Young tableaux of approximate C shape. The enumeration formula of approximately C-shaped SYT, namely the convolution of Catalan numbers, is derived by using the multi-integral method based on the sequence statistical model. Meanwhile, it is found that the first and last rows differ in the number of boxes. In the case where the number of boxes in the last row $n$ is large enough, the number of SYT does not change, especially if $m=1$, whether it is closely related to the Catalan number. In addition, we present two main closed formulas for SYTs of angular and “T” shape.

1. Introduction
The Enumeration for the standard Young tableaux (SYT) is an important research object in enumerative combinatorics. The number of truncated hollow SYT has been studied recently. G. D. James and M. H. Peel introduced the truncated SYT in [1]. In [2], a new explanation was given for truncated SYTs. R.M. Adin and Y. Roichman discussed about the number of SYTs deleted from some cells in the northeast corner in the literature. The work of Adin et al. uses pivoting theory method in [3] to obtain the number of rectangles and moving staircase truncated by a square or approximate square. In [4], G. Panova presented the formula of a rectangle truncated by staircase or by approximate square according to Schur function. P. Sun pointed out in [5,6] that the number of SYTs of regular, shifted and truncated shapes can be derived by evaluating the distribution of nested sequence statistics. A partition $\lambda$ of a positive integer $n$ is a non-increasing sequence of nonnegative integers $\lambda=(\lambda_1,\cdots,\lambda_y)$, namely $n=\lambda_1+\cdots+\lambda_y$. A Young diagram of shape $\lambda$ has a left-justified array of $n$ cells, with row $i$ (from top to bottom) containing $\lambda_i$ cells where $\lambda$ represents its shape, and $(i,j)$ denotes the number of cells in row $i$ and column $j$. In [7], Ron. M. Adin and Yuval Roichman obtained a standard Young tableau (SYT) of shape $\lambda$ which is a labeling by $\{1,2,\cdots,n\}$ of the cells in the Young diagram, with each row and column increasing respectively from left to right and from top to bottom. It is well known that the number of SYT was given earlier by the hook-length formula. A SYT of rectangular shape is denoted by $n^m=(n,n,\cdots,n)$ and It is clear that the SYT of truncated shapes can be represented by $\lambda\setminus\mu$ or $\lambda/\nu$, namely $\lambda,\mu,\nu$ are partitions of positive integers. Instead of deleting $\mu$ cells from the end of row $i$, a truncated shape $\hat{\lambda}\setminus\mu$ is a left-justified array of
cells which belong to $\lambda$ rather than $\nu$ and the shape $\lambda/\nu$ is the cells which belong to $\lambda$ rather than $\nu$ when cells in the upper left corners are kicked out [8].

The SYT of a hollow truncated shape is denoted by $\lambda \setminus \mu \{ (i_0, j_0) \}$ that the diagram of shape $\lambda$ deletes the cells belonging to shape $\mu$ from $(i_0, j_0)$, while the cells of $\mu$ in the upper left corners are deleted in row $i_0$ and column $j_0$ of $\lambda$. The SYT of truncated shape $\lambda \setminus \mu \{ (i_0, j_0) \}$ is filled by a corresponding truncated diagram with the integers from 1 to $|\lambda| - |\mu|$ so that each row and column is increasing in [4].

For example, $(k+1)^3 \setminus (k) \{(2,2)\}$ is denoted as Figure 1.

![Figure 1. SYT of shape $(k+1)^3 \setminus (k) \{(2,2)\}$](image)

Many scholars have studied enumeration of hollow truncated SYTs. And in [9], P.Sun obtained the formulas for SYT of nearly hollow rectangular shapes and gave the definition of the hollow SYT. In addition, the SYT of shapes like number, letter or other shapes are attracting more and more attention. Lu Chen and Chuanjuan Sun gave the number of 4- and H-shaped SYTs in [10]. As a continuation of the work in this paper, we have obtained the enumeration formulas of the following three near-C-shaped hollow truncated SYTs in Figure 2, especially (b), which is related to the Catalan number.

![Figure 2. Three near-C-shaped SYTs.](image)

This paper is structured as follows. In Section 1, we introduce the previous work of the enumeration of SYTs and illustrate the main definitions about SYTs of different shapes. Meanwhile, we present the main structure and contribution of this paper. In Section 2, we present the sum of enumeration formula of hollow truncated SYT of approximate C shape: $(k,1^n, n),(1 \leq n \leq k, m \geq 1)$, and our method is multiple integration based on the sequence statistical model of SYT in [5]. In Section 3, we obtain the main inferences: (i) In counting the number of SYT $(k,1^n, n),(1 \leq n \leq k, m \geq 1)$, if $n = k$, we easily know the enumeration formula of the truncated hollow SYT of approximate C shape $(k,1^n, k),(k, m \geq 1)$, which is the convolution of the Catalan number. (ii) If $n > k$, we get the number of the SYT $(k,1^n, n),(1 \leq k \leq n, m \geq 1)$. (iii) If $m = 1$ and $n > k$, it is easy to obtain the SYT diagram $(k,1,n),(1 \leq k \leq n)$ is a special number in [11, A000245], which is $a_{k+1} - a_k = \frac{1}{k+1} \binom{2k+2}{k+1} - \frac{1}{k+1} \binom{2k}{k}$, especially $a_k = \frac{1}{k+1} \binom{2k}{k}$ is Catalan number in [11, A000108]. (iv) If $n = 1$, we get the number of the SYT of angular shape, it is a simple combination. (v) As the special one of (iv), we also easily get the number of T-shaped SYT.

2. Sum of a combination
This paper focuses on the representation formula of SYT of approximate C shape by means of multiple integration in the nested sequence statistical model combined with the probability distribution.
The function of uniform distribution in the probability theory proposed by P Sun in [5]. And we also use combinatorial identities to prove the formula.

**Proposition 1.** [5] Let \((\xi_{1,j}^{(1)}, \ldots, \xi_{d,j}^{(1)})(1 \leq j \leq d)\) be the independent sequence statistics of group \(d\) subject to \(U(0,1)\), if \(r_j > j\), namely \((\xi_{1,j}^{(r_j)}, \ldots, \xi_{d,j}^{(r_j)})(1 \leq j \leq d - 1)\), \(\bigcap_{j=1}^{d-1} (\xi_{1,j}^{(i)} < \xi_{1,j}^{(r_j)} < \xi_{2,j}^{(i)} < \cdots < \xi_{d,j}^{(i)} < c_j)\) is satisfied, and these \(d\) groups of statistics are called nested sequence statistics, where \(c_j = \xi_{i,j}^{(r_j)}(s_j \geq r_j)\) or 1.

Its distribution is:

\[
P_d = P\left\{\bigcap_{j=1}^{d-1} (\xi_{1,j}^{(i)} < \xi_{1,j}^{(r_j)} < \xi_{2,j}^{(i)} < \cdots < \xi_{d,j}^{(i)} < c_j)\right\},
\]

Such that:

\[
P = \begin{pmatrix}
\xi_{1,j}^{(1)} < \cdots < \xi_{d,j}^{(1)} \\
\wedge \cdots \wedge \cdots \wedge
\xi_{1,j}^{(2)} < \cdots < \xi_{d,j}^{(2)} \\
\wedge \cdots \wedge \cdots \wedge
\cdots \cdots \cdots
\xi_{1,j}^{(d)} < \cdots < \xi_{d,j}^{(d)}
\end{pmatrix}
\]

And the joint probability distribution function of the nested sequence statistics composed of \((\xi_{1,j}^{(1)}, \ldots, \xi_{d,j}^{(d)})\) is

\[
P_d = \int_{S_\lambda} \prod_{j=1}^{d} \lambda_j dx_{i,j} \cdots dz_{j,j}.
\]

where \(S_\lambda\) is its corresponding nested simplex, which can be expressed as:

\[
S_\lambda = \begin{pmatrix}
0 < x_{i,j} < \cdots < x_{d,j} < 1 \\
\wedge \cdots \wedge \cdots \wedge
0 < y_{i,j} < \cdots < y_{d,j} < 1 \\
\wedge \cdots \wedge \cdots \wedge
0 < z_{i,j} < \cdots < z_{d,j} < 1
\end{pmatrix}
\]

And for each sample point \(\omega\) in the following event \(A\),

\[
A = \left\{\omega : \bigcap_{j=1}^{d-1} (\xi_{1,j}^{(i)}(\omega) < \xi_{1,j}^{(r_j)}(\omega), \xi_{2,j}^{(i)}(\omega) < \cdots < \xi_{d,j}^{(i)}(\omega) < c_j(\omega))\right\},
\]

And each event corresponds to a SYT diagram of shape \(\lambda\). Since the samples of the event are independent and identically distributed, all the sample points of the event \(A\) are possible. From a discrete point of view, the distribution of \(U(0,1)\) nested sequence statistics is calculated as Proposition 2.

**Proposition 2.** [5] For the \(\lambda\)-type nested sequence statistic that obeys the \(U(0,1)\) distribution, its distribution is:
where \( N_\lambda \) is the number of SYT-type chart of shape \( \lambda \), and \( |\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_d \).

**Proposition 3.** [5] Let \( N_\lambda \) be the number of SYT diagram of shape \( \lambda \), then

\[
N_\lambda = \int_{S_\lambda} \cdots \int_{S_\lambda} dx_1 \cdots dx_d.
\]

where \( S_\lambda \) is the nested simplex corresponding to the SYT diagram of \( \lambda \) shape.

Furthermore, in the SYT diagram, for \( 1 \leq n \leq k \), the number \( H_{(k,1^n,:)\lambda} \) for SYT of truncated shape \( (k,1^n,n),(1 \leq n \leq k,m \geq 1) \) is

\[
H_{(k,1^n,:)\lambda} = (n+k+m)! \int_{D_{(k,1^n,:)\lambda}} \cdots dx dy dz
\]

(3)

where \( D_{(k,1^n,:)\lambda} \) (abbreviated as \( D_{(k,1^n,:)\lambda} \)) is the following SYT-type integral scope in Figure 3 (the variables are increasing from left to right and from top to bottom):

\[
\begin{array}{ccccccc}
  x_1 & \cdots & x_{n-1} & x_n & x_{n+1} & \cdots & x_k \\
  y_1 \\
  \vdots \\
  y_m \\
  z_1 & \cdots & z_{n-1} & z_n \\
\end{array}
\]

**Figure 3.** The sequence statistics model same with that of the SYT of shape \( (k,1^n,n),(1 \leq n \leq k,m \geq 1) \).

Considering \( z_{i-1} < x_i < z_i \), \( 2 \leq i \leq n+1 \) and \( y_{j-1} < x_j < y_j \), \( 1 \leq j \leq m+1 \), and there is \( y_0 = x_1, y_{m+1} = z_1, z_{n+1} = 1 \). where \( D_{(k,1^n,:)\lambda} \) is separated into \( D_i(i) \) and \( D_j(j) \) that are illustrated as:

\[
0 < x_i < x_{i-1} < \cdots < x_{i+1} < x_{i+2} < \cdots < x_{n+1}
\]

and

\[
0 < x_i < x_{i-1} < \cdots < x_{i+1} < x_{i+2} < \cdots < x_{n+1}
\]

Therefore, we have **Theorem 1.** For \( m \geq 1,1 \leq n \leq k \), the number \( H_{(k,1^n,:)\lambda} \) of SYT of truncated shape \( (k,1^n,n),(1 \leq n \leq k,m \geq 1) \) is
\[
H_{(k^*,n)} = \binom{n+m+k-1}{k-1} - \binom{n+m+k-1}{n-2}.
\]  

(5)

**Proof.** Write \( I_{i^*,j^*} = \sum_{j=2}^{n+1} I_i(j) + \sum_{j=1}^{n+1} I_j(j) \), where

\[
I_i(j) = \int \cdots \int_{D_i(j)} dx_i dy_i dz_i, I_j(j) = \int \cdots \int_{D_j(j)} dx_j dy_j dz_j.
\]

For \( 3 \leq i \leq n+1 \), \( I_i(i) \) can be simplified as

\[
I_i(i) = \int \cdots \int_{\Omega_i} \frac{(x_i - x_{i-1})^{k-i}}{(k-i)!} \frac{(z_i - x_i)^{m+1}}{(m+1)!} \frac{(1-x_i)^{n+1}}{(n-i+1)!} \ dx_i \cdots dx_i dx_i dz_i \cdots dz_{i-1},
\]

where \( \Omega_i \) is

\[
\Omega_i = \left\{ \begin{array}{l}
0 < x_1 < x_2 < x_3 < \cdots < x_{i-1} \\
\quad \wedge \wedge \wedge \wedge \\
2 < z_2 < z_3 < \cdots < z_{i-1} < x_k < 1
\end{array} \right\}
\]

Moreover, by changing the variables

\[
\begin{align*}
x_r &= x_1 + (x_r - x_1)u_{r-1}, \\
z_r &= x_1 + (z_r - x_1)v_{r-1},
\end{align*}
\]

where \( 2 \leq r \leq i-1 \).

Euler’s beta integral implies that

\[
I_i(i) = \frac{(m+k+i-3)!}{(m+n+k)!} J_i(i), 3 \leq i \leq n+1,
\]

(6)

where

\[
J_i(i) = \int \cdots \int_{\Omega_i} \frac{u_i^{m+1}(1-u_{i-2})^{k-i}}{(m+1)(k-i)!} \ du_i dv_i,
\]

Finally, according to the integral matrix method, by integrating respect to \( u_1, u_2, \ldots, u_{i-2}, v_1, v_2, \ldots, v_{j-2} \), it implies that

\[
J_i(i) = \int \cdots \int_{\Omega_i} \frac{u_i^{m+3}(1-u_{i-2})^{i-1}}{(m+3)(k-i)!} \ du_i dv_i
\]

\[
= \cdots \int_0^{u_{i-2}^{(i-3)!}} \int_0^{u_{i-2}^{(i-1)!}} \frac{1}{(i-3)!} \frac{1}{(m+i-1)!} \ du_{i-2} \ dv_{i-2}
\]

5
\[
\frac{1}{(m+i-1)! (k-2)!} - \frac{1}{(i-3)! (m+k)!}
\]

By similar arguments, we directly have
\[
I(2) = \int_{0 < x_i < x_j < 1} \left( \frac{x_i - x_j}{k-2} \right)^{m+k-1} \left( \frac{1-x_i}{m+k-1} \right)^{n-1} \, dx_i \, dx_j = \frac{1}{(m+n+k)!} \left( \frac{m+k-1}{k-2} \right).
\]

Combining Equation (6)-(8), we get:
\[
\sum_{i=2}^{m} I(i) = \frac{1}{(m+n+k)!} \sum_{i=0}^{m+1} \left[ \binom{m+k+i-1}{k-2} - \binom{m+k+i-1}{m+k} \right].
\]

and for \(2 \leq j \leq m\)
\[
I(j) = \int_{0 < x_i < x_j < 1} \left( \frac{x_j - x_i}{k-2} \right)^{k+j-3} \left( \frac{1-x_j}{k+j-3} \right)^{m-j+n+1} \, dx_i \, dx_j = \frac{1}{(m+n+k)!} \left( \frac{k+j-3}{k-2} \right).
\]

By similar method, we have
\[
I(1) = \frac{1}{(m+n+k)!}
\]
\[
I(m+1) = \frac{1}{(m+n+k)!} \binom{m+k-2}{k-2}
\]

From equations (4) and (9)-(12), by using the combinatorial identity of \(\sum_{k=0}^{m} \binom{x+k}{x} = \binom{x+n+1}{n}\), we complete the proof of lemma 1.

3. Main corollaries
Theorem 1 shows that the number of SYT of approximate C shape is involved in combinatorial summation. Actually, when we derived Lemma 1, we obtained the SYT of similar shapes, as shown in Figure 2. And simple formulas for some special cases are also given.

**Corollary 1.** For \(n = k\), the number of SYT of shape \((k, 1^n, k)\) given in theorem 1, we easily have the number of SYT of shape \((k, 1^n, k)\) is
\[
H_{(k, 1^n, k)} = \binom{2k+m}{k} - \binom{2k+m-1}{k+1}.
\]

It is the convolution of the Catalan number.

**Corollary 2.** For \(n > k\), due to the number of SYT of shape \((k, 1^n, k)\) given in theorem 1, we easily have the number of SYT of shape \((k, 1^n, n)\) given in theorem 1 is
\[
H_{(k, 1^n, n), (1 \leq k \leq n, m \geq 1)} = \binom{2k+m}{k} - \binom{2k+m-1}{k+1}.
\]

The number is independent of \(n\). The last row of the shape \((k, 1^n, k)\) add some boxes after \(z_i\) in Figure 2 (b), and the number of SYT is not variable.

**Corollary 3.** For \(m = 1\), the number of SYT of shape \((k, 1, n)\) is
\[
H_{(k, 1, n)} = \frac{2k+2}{k+1} - \frac{2k}{k+1} = C(k+1) - C(k).
\]

The result is closely related to Catalan number, and actually \(C(k)\) is the general term of Catalan number [11, A000108].
Corollary 4. For $n=1$, the number of SYT of shape $(k,1^{n+1})$ denoted by Figure 4 is

$$H_{(k,1^{n+1})} = \binom{k+m}{k-1}.\tag{16}$$

Corollary 5. The number of SYT of shape $(l+k)^{n+2}/1^{n+1}\setminus((k-1)^{n+1})\setminus\{(l+2,2)\}$ such as T shape in Figure 5, is

$$H_{((l+k)^{n+2}/1^{n+1}\setminus((k-1)^{n+1})\setminus\{(l+2,2)\})} = \binom{k+m}{k-1}.\tag{17}$$

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