Improved Approximation Algorithms for Stochastic-Matching Problems*

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Abstract. We consider the Stochastic Matching problem, which is motivated by applications in kidney exchange and online dating. In this problem, we are given an undirected graph. Each edge is assigned a known, independent probability of existence and a positive weight (or profit). We must probe an edge to discover whether or not it exists. Each node is assigned a positive integer called a timeout (or a patience). On this random graph we are executing a process, which probes the edges one-by-one and gradually constructs a matching. The process is constrained in two ways. First, if a probed edge exists, it must be added irrevocably to the matching (the query-commit model). Second, the timeout of a node $v$ upper-bounds the number of edges incident to $v$ that can be probed. The goal is to maximize the expected weight of the constructed matching.

For this problem, Bansal et al. [8] provided a 0.33-approximation algorithm for bipartite graphs and a 0.25-approximation for general graphs. We improve the approximation factors to 0.39 and 0.269, respectively.

The main technical ingredient in our result is a novel way of probing edges according to a not-uniformly-random permutation. Patching this method with an algorithm that works best for large-probability edges (plus additional ideas) leads to our improved approximation factors.

1 Introduction

Maximum-weight matching is a fundamental problem in combinatorial optimization and has applications in a wide-range of areas such as market design [46,1], computer vision [18,54], computational biology [52], and machine learning [50]. The basic version of this problem is well-understood; exact polynomial-time algorithms are available for both bipartite and general graphs due to the celebrated results of [37] and [26], respectively. However, in many applications there are uncertainties associated with the input and typically, the problem of interest is more nuanced. A common approach to model this uncertainty is via randomness; we assume that we have a distribution over a collection of graphs. There are many such models ranging from stochastic edges [20,12] to stochastic vertices [29,14].

In this paper, we study the stochastic-matching model where edges in the graph are uncertain and the exact realization of the graph is obtained by probing the edges. This model has many applications in kidney exchange, online dating, and labor markets (see Subsection 1.3 for details). Further, we study a more general version of the problem, introduced by [19], where the algorithm is constrained by the

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number of probes it can make on the edges incident to any single vertex. This is used to model the notion of timeouts (also called patience) which naturally arises in many applications.

The formulation (described formally in subsection 1.1) is similar to many other well-studied stochastic-optimization problems such as stochastic knapsack [22], stochastic packing [8,16], and stochastic shortest-path problems [44].

1.1 Definitions and Notation

In the stochastic-matching problem, we are given a graph $G = (V,E)$, where $V$ denotes the set of vertices and $E$ denotes the set of potential edges. Additionally, we are given the following functions.

- $p : E \rightarrow [0,1]$ associates every edge $e$ with an independent probability of existence, $p_e$. When an edge is probed, it will exist with probability $p_e$ and must be added to the matching if it exists. Thus, we can only probe edges whose endpoints are currently unmatched.
- $w : E \rightarrow \mathbb{R}^+$ denotes a weight function that assigns a non-negative weight (or profit) $w_e$ to each edge $e$.
- $t : V \rightarrow \mathbb{N}$ is the timeout (or patience) function which sets an upper bound $t_v$ on the number of times a vertex $v$ can have one of its incident edges probed.

An algorithm for this problem probes edges in a possibly adaptive order. When an edge is probed, it is present with probability $p_e$ (independent of all other edges), in which case it must be included in the matching under construction (query-commit model) and provides a weight of $w_e$. We can probe at most $t_v$ edges among the set $\delta(v)$ of edges incident to a node $v$. Furthermore, when an edge $e$ is added to the matching, no edge $f \in \delta(e)$ (i.e., incident on $e$) can be probed in subsequent steps. Each edge may only be probed once. Our goal is to maximize the expected weight of the constructed matching.

A naive approach to solve this problem is to construct an exponential-sized Markov Decision Process (MDP) and solve it optimally using dynamic programming. However, there are no known algorithms to solve this exactly in polynomial time. In fact, the exact complexity of computing the optimal solution is unknown. The naive solution above is in PSPACE; it is unknown if the problem is in either NP or in P. Thus, following prior works [19,33,43], we aim at finding an approximation to the optimal solution in polynomial time. To measure the performance of any algorithm, we use the standard notion of approximation ratio which is defined as follows.

Definition 1 (Approximation ratio). For any instance $I$, let $\mathbb{E}[\text{ALG}(I)]$ denote the expected weight of the matching obtained by the algorithm ALG on $I$. Let $\mathbb{E}[\text{OPT}(I)]$ denote the expected weight of the matching obtained by the optimal probing strategy. Then the approximation ratio of ALG is defined as $\min_I \frac{\mathbb{E}[\text{ALG}(I)]}{\mathbb{E}[\text{OPT}(I)]}$.

We remark that the measure used in [3] is the reciprocal of the ratio defined in Definition 1. Thus, the ratios in [3] are always greater than 1 while the ratios in this paper are at most 1. Bansal et al. [8] provide an LP-based 0.33-approximation when $G$ is bipartite, and via a reduction to the bipartite case, a 0.25-approximation for general graphs (see also [4]).

For the related stochastic-matching problem without patience constraints (or equivalently, all $t_v$ equal infinity), the best-known algorithm achieves an approximation ratio of $1 – 1/e$ [30] and no algorithm can perform better than 0.898 [20]. Since the problem without patience constraints is a special case of the problem studied here, the latter hardness result applies to our setting as well.

Chen et al. [19] formulated and initiated the study of the problem with patience constraints and gave a probing scheme that achieves an approximation ratio of 0.25 in unweighted bipartite graphs. Later, Adamczyk [2] showed that the simple greedy algorithm (probe edges in non-increasing order of probability) achieves a 0.5-approximation for the unweighted case. This model was extended to weighted bipartite graphs by [8] who improved the ratio to 0.33. [8] provided a new algorithm that further improved this ratio to 0.35 which is the current state-of-the-art. For general graphs, the current best approximation is 0.31 [10].
We note that our work and the most-recent prior work [SK] use a natural linear program (LP) – see (LP-BIP) in Section 2—to upper bound the optimal solution. However, it was shown in [13] that no algorithm can achieve an approximation better than 0.544 using this LP even for the unweighted problem.

**Online Stochastic Matching with Timeouts.** We also consider the *Online Stochastic Matching with Timeouts* problem introduced in [3]. Here we are given as input a complete bipartite graph $G = (A \cup B, A \times B)$, where nodes in $B$ are *buyer types* and nodes in $A$ are *items* that we wish to sell. Like in the offline case, edges are labeled with probabilities and profits, and nodes are assigned timeouts. However, in this case timeouts on the item side are assumed to be unbounded. Then a second bipartite graph is constructed in an online fashion. Initially this graph consists of $A \times A$, and nodes are assigned timeouts. However, in this case timeouts on the item side are assumed to be unbounded. Then a second bipartite graph is constructed in an online fashion. Initially this graph consists of $A \times A$, and nodes are assigned timeouts. However, in this case timeouts on the item side are assumed to be unbounded. Then a second bipartite graph is constructed in an online fashion. Initially this graph consists of $A \times A$, and nodes are assigned timeouts.

For this problem, Bansal et al. [8] present a 0.126-approximation algorithm. In his Ph.D. thesis, Li [38] claims an improved 0.249-approximation. However, his analysis contains a mistake [39]. By fixing that, he still achieves a 0.193-approximation ratio improving over [8]. And although a corrected version of this result is not published later anywhere, in essence it would have follow the same lines as the result of Mukherjee [13] who independently obtained exactly the same approximation ratio.

### 1.2 Our Results

Our main result is an approximation algorithm for bipartite Stochastic Matching which improves the 0.33-approximation of Bansal et al. [8] (see Section 2).

**Theorem 1.** There is an expected 0.39-approximation algorithm for Stochastic Matching in bipartite graphs.

A 0.351-approximation, originally presented in [3], can be obtained as follows. We build upon the algorithm in [3], which works as follows. After solving a proper LP and rounding the solution via a rounding technique from [31], Bansal et al. probe edges in uniform random order. Then they show that every edge $e$ is probed with probability at least $x_e \cdot g(p_{max})$, where $x_e$ is the fractional value of $e$ assigned by the LP, $p_{max} := \max_{f \in \delta(e)} \{ p_f \}$ is the largest probability of any edge incident to $e$ ($e$ excluded), and $g(\cdot)$ is a decreasing function with $g(1) = 1/3$.

Our idea is to instead consider edges in a carefully chosen *non-uniform* random order. This way, we are able to show (with a slightly simpler analysis) that each edge $e$ is probed with probability $x_e \cdot g(p_e) \geq \frac{1}{2} x_e$. Observe that we have the same function $g(\cdot)$ as in [3], but depending on $p_e$ rather than $p_{max}$. In particular, according to our analysis, small-probability edges are more likely to be probed than large-probability ones (for a given value of $x_e$), regardless of the probabilities of edges incident to $e$. Though this approach alone does not directly imply an improved approximation factor, we further patch it with a greedy algorithm that behaves best for large-probability edges, and this yields an improved approximation ratio altogether. The greedy algorithm prioritizes large-probability edges for the same value of $x_e$.

We further improve the approximation factor for the bipartite case with the above mentioned updated patching algorithm, hence improving the ratio of 0.351 from [3] up to the current-best-known ratio of 0.39 stated in Theorem 1.

We also improve on the 0.25-approximation for general graphs in [8] (see Section 3).

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7 As in [8], we assume that the probability of each buyer-type $b$ is an integer multiple of $1/n$. 

**Theorem 2.** There is an expected $0.269$-approximation algorithm for Stochastic Matching in general graphs.

This is achieved by reducing the general case to the bipartite one as in prior work, but we also use a refined LP with blossom inequalities in order to fully exploit our large/small probability patching technique.

Similar arguments can also be successfully applied to the online case.

**Theorem 3.** There is an expected $0.245$-approximation algorithm for Online Stochastic Matching with Timeouts.

By applying our idea of non-uniform permutation of edges we would get a $0.193$-approximation (the same as in [38], after correcting the mentioned mistake). However, due to the way edges have to be probed in the online case, we are able to finely control the probability that an edge is probed via *damping factors*. This allows us to improve the approximation from $0.193$ to $0.24$. Our idea is similar in spirit to the one used by Ma [40] in his elegant $2$-approximation algorithm for correlated non-preemptive stochastic knapsack. Further application of the large/small probability trick gives an extra improvement up to $0.245$ (see Section 4). We remark that since the publication of the conference version of this work, this has been improved to $0.46$ ([15]) and more recently to $0.51$ ([27]).

### 1.3 Applications

As previously mentioned, the stochastic matching problem is motivated from various applications such as kidney exchange and online dating. In this sub-section, we briefly consider these applications and show how we can use stochastic matching as a tool to solve these problems.

**Kidney exchange in the United States.** Kidney transplantation usually occurs from deceased donors; however, unlike other organs, another possibility is to obtain a kidney from a compatible living donor since people need only one kidney to survive. There is a large waiting list of patients within the US who need a kidney for their survival. As of July 2019, the United Network for Organ Sharing (UNOS) estimates that the current number of patients who need a transplant is 113,265 with only 7,743 donors in the pool [8]. One possibility is to enter the waitlist as a pair, where one person needs a kidney while the other person is willing to donate a kidney. Viewed as a stochastic-matching problem, the vertices are donor-patient pairs. An edge between two vertices $(u,v)$ exists, if the donor in $u$ is compatible with the patient in $v$ and the donor in $v$ is compatible with the patient in $u$. The probability on the edge is the probability that the exchange will take place. Before every transplant takes place, elaborate medical tests are usually performed which is very expensive. More specifically, as described in [19], a test called the crossmatching is performed, that combines the recipient’s blood serum with some of donor’s red blood cells and checks if the antibodies in the serum kill the cells. This test is both expensive and time-consuming. Moreover, each exchange requires the transplant to happen *simultaneously* since organ donation within the United States is *at will*; donors are legally allowed to withdraw at any time including after agreeing for a donation. These constraints impose that for each donor-patient pair, the number of exchange initiations that can happen has to be small, which are modeled by the patience values at each vertex. Given the long wait-lists and the number of lives that depend on these exchanges, even small improvements to the accuracy of the algorithm have drastic effects on the well-being of the population. Prior works (e.g., [12] and references therein) have empirically applied the variants of the stochastic-matching problem to real-world datasets; in fact, the current model that runs the US-wide kidney exchange is based on a stochastic-matching algorithm (e.g., [24] and references therein).

**Online dating.** Online dating is quickly becoming the most popular form for couples to meet each other [45]. Platforms such as Tinder, eHarmony and Coffee meets Bagel generated about 1.7 billion USD

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8 Data obtained from [https://unos.org/data/transplant-trends/](https://unos.org/data/transplant-trends/)
in revenue in the year 2019. Suppose we have the case of a pool of heterosexual people, represented as the two vertex sets $U$ and $V$ in the bipartite graph. For every pair $u \in U$ and $v \in V$, the system learns their compatibility based on the questions they answer. The goal of the online-dating platform is to suggest couples that maximizes the social welfare (i.e., the total number of matched couples). Each individual in a platform has a limited patience and thus, the system wants to ensure that the number of suggestions provided is small and limited. The stochastic-matching problem models this application where the probability on the edges represent the compatibility and the time-out function represents the individual patience.

**Online labor markets.** In online labor markets such as Mechanical Turk, the goal is to match workers to tasks [48]. Each worker-task pair has a probability of completing the task based on the worker’s skills and the complexity of the task. The goal of the platform is to match the pool of tasks to workers such that the (weighted) number of completed tasks is maximized. To keep workers in continued participation, the system needs to ensure that the worker is not matched with many tasks that they are incapable of handling. This once again fits in the model of the stochastic-matching problem where the workers and tasks represent the two sides of the bipartite graph, the edge-probability represents the probability that the task will be completed, and the time-out function represents the patience level of each worker.

### 1.4 Other Related Work

Stochastic-matching problems come in many flavors and there is a long line of research for each of these models. The literature on the broader stochastic combinatorial optimization is (even more) vast (see [51] for a survey) and here we only mention some representative works. When the graph is unweighted and the time-out at every vertex is infinite, the classic RANKING algorithm of [35] gives an approximation ratio of $1 - 1/e$ for bipartite graphs; this in fact works even if the graph is unknown a priori. For general graphs, the work of [20] gives an algorithm that achieves a ratio of 0.573; moreover, it shows that no algorithm can get a ratio better than 0.898 for general graphs. The work of [12] gives an optimal algorithm in the special case of sparse graphs in this model. The paper [30] considers the weighted version of this problem in bipartite graphs and designs algorithms that achieve an approximation ratio of $1 - \frac{1}{e}$. (Recall that the timeouts are infinite in all of these works.)

The other line of research deals with the stochastic-matching problem where instead of a time-out constraint, the algorithm has to minimize the total number of queried edges. The work of [12] first proposed this model which was later considered and improved (by reducing the number of required queries) in many subsequent follow-up works including [50,115].

Online variants of the stochastic matching problems have been extensively studied due to their applications in Internet advertising and other Internet based markets. The paper [29] introduced the problem of Online Matching with Stochastic Inputs. In this model, the vertices are drawn repeatedly and i.i.d. from a known distribution. The algorithm needs to find a match to a vertex each time one is presented, immediately and irrevocably. The goal is to maximize the expected weight of the matching compared to an algorithm that knows the sequence of realizations a priori. The work of [29] gave an algorithm that achieves a ratio of 0.67, which was subsequently improved by [113,114]. Later work extended further to fully-online models of matching where both partitions of the vertex set in the bipartite graph are sampled i.i.d. from a known distribution [25,53]. The matching problem has also been studied in the two-stage stochastic-optimization model [36].

The stochastic-matching problem is also related to the broader stochastic-packing literature, where the algorithm only knows a probability distribution over the item costs, and once it commits to include the item sees a realization of the actual costs [21,22,130,101,16]. Stochastic packing has also been studied in the online [28,231,17] and bandit [327,33] settings.

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[5] https://www.statista.com/outlook/372/100/online-dating/worldwide
2 Stochastic Matching in Bipartite Graphs

In this section we present our improved approximation algorithm for Stochastic Matching in bipartite graphs. We start by presenting a simpler 0.351-approximation in Section 2.1, and then refine it in Section 2.2.

2.1 An Improved Approximation

In this section we prove the following result.

Theorem 4. There is an expected 0.351-approximation algorithm for Stochastic Matching in bipartite graphs.

Let \( \text{OPT} \) denote an optimal probing strategy and let \( E[\text{OPT}] \) denote its expected value. Consider the following LP:

\[
\begin{align*}
\text{max} & \sum_{e \in E} w_e p_e x_e & \quad \text{(LP-BIP)} \\
\text{s.t.} & \sum_{e \in \delta(v)} p_e x_e \leq 1, & \forall v \in V; \\
& \sum_{e \in \delta(v)} x_e \leq t_v, & \forall v \in V; \\
& 0 \leq x_e \leq 1, & \forall e \in E.
\end{align*}
\]

The proof of the following Lemma is already quite standard [4,8,22] — just note that \( x_e = \Pr[\text{OPT probes } e] \) is a feasible solution of (LP-BIP).

Lemma 1. [8] Let \( \text{LP}_{\text{bip}} \) be the optimal value of (LP-BIP). It holds that \( \text{LP}_{\text{bip}} \geq E[\text{OPT}] \).

Our approach is similar to the one of Bansal et al. [8] (see also Algorithm 1 in the figure). We solve (LP-BIP): let \( x = (x_e)_{e \in E} \) be the optimal fractional solution. Then we apply to \( x \) the rounding procedure by Gandhi et al. [31], which we shall call just GKPS. Let \( \hat{E} \) be the set of rounded edges, and let \( \hat{x}_e = 1 \) if \( e \in \hat{E} \) and \( \hat{x}_e = 0 \) otherwise. GKPS guarantees the following properties of the rounded solution:

1. (Marginal distribution) For any \( e \in E \), \( \Pr[\hat{x}_e = 1] = x_e \).
2. (Degree preservation) For any \( v \in V \), \( \sum_{e \in \delta(v)} \hat{x}_e \leq \lceil \sum_{e \in \delta(v)} x_e \rceil \leq t_v \).
3. (Negative correlation) For any \( v \in V \), any subset \( S \subseteq \delta(v) \) of edges incident to \( v \), and any \( b \in \{0, 1\} \), it holds that \( \Pr[\bigwedge_{e \in S}(\hat{x}_e = b)] \leq \prod_{e \in S} \Pr[\hat{x}_e = b] \).

Our algorithm sorts the edges in \( \hat{E} \) according to a certain random permutation and probes each edge \( e \in E \) according to this order, but provided that the endpoints of \( e \) are not matched already. It is important to notice that, by the degree-preservation property, \( \hat{E} \) has at most \( t_v \) edges incident to each node \( v \). Hence, the timeout constraint of \( v \) is respected even if the algorithm probes all the edges in \( \delta(v) \cap \hat{E} \).

Our algorithm differs from [8] and subsequent work in the way edges are randomly ordered. Prior work exploits a uniformly-random order on \( \hat{E} \). We rather use the following, more complex strategy. For each \( e \in \hat{E} \) we draw a random variable \( Y_e \) distributed on the interval \( \left[ 0, \frac{1}{p_e \ln \frac{1}{1-p_e}} \right] \) according to the following cumulative distribution: \( \Pr[Y_e \leq y] = \frac{1}{p_e} (1 - e^{-p_e y}) \). Observe that the density function of \( Y_e \) in this interval is \( e^{-p_e y} \) (and zero otherwise). Edges of \( \hat{E} \) are sorted in increasing order of the \( Y_e \)'s, and they are probed according to that order. We let \( Y \) denote the vector \( (Y_e)_{e \in \hat{E}} \), wherein the elements of \( \hat{E} \) are ordered in some fixed manner.
**Algorithm 1:** An algorithm for Bipartite Stochastic Matching (ALG1).

1. Let \((x_e)_{e \in E}\) be the solution to (LP-BIP).
2. Round the solution \((x_e)_{e \in E}\) with GKPS; let \((\hat{x}_e)_{e \in E}\) be the rounded 0-1 solution, and \(E = \{ e \in E \mid \hat{x}_e = 1 \}\).
3. For every \(e \in \hat{E}\), sample a random variable \(Y_e\) distributed as \(\Pr [Y_e \leq y] = \frac{1 - e^{-yp_e}}{p_e}\).
4. For every \(e \in \hat{E}\) in increasing order of \(Y_e\):
   - (a) If no edge \(f \in \delta(e) := \delta(e) \cap \hat{E}\) is yet taken, then probe edge \(e\).

Define \(\hat{\delta}(v) := \delta(v) \cap \hat{E}\). We say that an edge \(e \in \hat{E}\) is safe if, at the time we consider \(e\) for probing, no other edge \(f \in \hat{\delta}(e)\) has already been taken into the matching. Note that the algorithm can probe \(e\) only in this case, and that if we do probe \(e\), it gets added to the matching with probability \(p_e\) independent of all other events.

The main ingredient of our analysis is the following lower-bound on the probability that an arbitrary edge \(e\) is safe.

**Lemma 2.** For every edge \(e\) it holds that \(\Pr [e \text{ is safe} | e \in \hat{E}] \geq g(p_e)\), where

\[
g(p) := \frac{1}{2 + p} \left( 1 - \exp \left( - (2 + p) \frac{1}{p} \ln \frac{1}{1 - p} \right) \right).
\]

**Proof.** In the worst case every edge \(f \in \delta(e)\) that is before \(e\) in the ordering can be probed, and each of these probes has to fail for \(e\) to be safe. Thus

\[
\Pr [e \text{ is safe} | e \in \hat{E}] \geq \mathbb{E}_{\hat{E},Y} \left[ \prod_{f \in \delta(e): Y_f < Y_e} (1 - p_f) | e \in \hat{E} \right].
\]

Now we take expectation on \(Y\) only, and using the fact that the variables \(Y_f\) are independent, we can write the latter expectation as

\[
\mathbb{E}_{\hat{E},Y} \left[ \int_0^{p_e} \ln \frac{1}{p_e} \left( \prod_{f \in \delta(e)} (\Pr [Y_f \leq y] (1 - p_f) + \Pr [Y_f > y]) \right) e^{-p_e y} \mathrm{d}y \mid e \in \hat{E} \right].
\]

Observe that \(\Pr [Y_f \leq y] (1 - p_f) + \Pr [Y_f > y] = 1 - p_f \Pr [Y_f \leq y]\). When \(y > \frac{1}{p_f} \ln \frac{1}{1 - p_f}\), then \(\Pr [Y_f \leq y] = 1\), and moreover, \(\frac{1}{p_f} (1 - e^{-py})\) is an increasing function of \(y\). Thus we can upper-bound \(\Pr [Y_f \leq y]\) by \(\frac{1}{p_f} (1 - e^{-py})\) for any \(y \in [0, \infty]\), and obtain that \(1 - p_f \Pr [Y_f \leq y] \geq 1 - p_f \frac{1}{p_f} (1 - e^{-py}) = e^{-py}\). Thus (5) can be lower-bounded by

\[
\mathbb{E}_{\hat{E},Y} \left[ \int_0^{p_e} \ln \frac{1}{p_e} e^{-\sum_{f \in \delta(e)} p_f y} \mathrm{d}y \mid e \in \hat{E} \right] = \mathbb{E}_{\hat{E},Y} \left[ \frac{1}{\sum_{f \in \delta(e)} p_f + p_e} \left( 1 - e^{-\left( \sum_{f \in \delta(e)} p_f \right) \frac{1}{p_e} \ln \frac{1}{p_e}} \right) \mid e \in \hat{E} \right].
\]

We know from the negative-correlation and marginal-distribution properties that \(\mathbb{E}_{\hat{E},Y} \left[ \hat{x}_f | e \in \hat{E} \right] \leq \mathbb{E}_{\hat{E},Y} [\hat{x}_f] = x_f\) for every \(f \in \delta(e)\), and therefore \(\mathbb{E}_{\hat{E},Y} \left[ \sum_{f \in \delta(e)} p_f \mid e \in \hat{E} \right] \leq \sum_{f \in \delta(e)} p_f x_f \leq 2\), where the last inequality follows from the LP constraints. Consider function \(f(x) := \frac{1}{x + p_e} \left( 1 - e^{-x + p_e} \frac{1}{p_e} \ln \frac{1}{p_e} \right)\).
This function is decreasing and convex. From Jensen’s inequality we know that $E[f(x)] \geq f(E[x])$. Thus

$$E_{\hat{E}\setminus e} \left[ f \left( \sum_{f \in \delta(e)} p_f \right) \right] \geq f \left( E_{\hat{E}\setminus e} \left[ \sum_{f \in \delta(e)} p_f \right] \right) \geq f(2) = \frac{1}{2 + p_e} \left( 1 - e^{-2(1+p_e)/p_e} \right) \ln \frac{1}{1-p_e} = g(p_e).$$

From Lemma 2 and the marginal distribution property, the expected contribution of edge $e$ to the profit of the solution is

$$w_e p_e \cdot \Pr \left[ e \in \hat{E} \right] \cdot \Pr \left[ e \text{ is safe} | e \in \hat{E} \right] \geq w_e p_e x_e \cdot g(p_e) \geq \frac{1}{3} w_e p_e x_e.$$ 

Therefore, our analysis implies a $1/3$ approximation, matching the result in [3]. However, by working with the probabilities appropriately, we can do better as described next.

**Patching with Greedy.** We next describe an improved approximation algorithm, based on the patching of the above algorithm with a simple greedy one. Let $\delta \in (0,1)$ be a parameter to be fixed later. We define $E_{\text{large}}$ as the set of (large) edges $e$ with $p_e \geq \delta$, and let $E_{\text{small}}$ be the remaining (small) edges. Recall that $\text{LP}_{\text{bip}}$ denotes the optimal value of (LP-BIP). Let also $\text{LP}_{\text{large}}$ and $\text{LP}_{\text{small}}$ be the fraction of $\text{LP}_{\text{bip}}$ due to large and small edges, respectively; i.e., $\text{LP}_{\text{large}} = \sum_{e \in E_{\text{large}}} w_e p_e x_e$ and $\text{LP}_{\text{small}} = \text{LP}_{\text{bip}} - \text{LP}_{\text{large}}$. Define $\gamma \in [0,1]$ such that $\gamma \text{LP}_{\text{bip}} = \text{LP}_{\text{large}}$. By refining the above analysis, we obtain the following result.

**Lemma 3.** Algorithm [4] has an expected approximation ratio $\frac{1}{3} \gamma + g(\delta) (1 - \gamma)$.

**Proof.** The expected profit of the algorithm is at least:

$$\sum_{e \in E} w_e p_e x_e \cdot g(p_e) \geq \sum_{e \in E_{\text{large}}} w_e p_e x_e \cdot g(1) + \sum_{e \in E_{\text{small}}} w_e p_e x_e \cdot g(\delta)$$

$$= \frac{1}{3} \text{LP}_{\text{large}} + g(\delta) \text{LP}_{\text{small}} = \left( \frac{1}{3} \gamma + g(\delta) (1 - \gamma) \right) \text{LP}_{\text{bip}}. \quad \square$$

**A greedy algorithm** (Greedy). Consider the following greedy algorithm Greedy. Compute a maximum weight matching $M_{\text{grad}}$ in $G$ with respect to edge weights $w_e p_e$ and probe the edges of $M_{\text{grad}}$ in any order. Note that the timeout constraints are satisfied since we probe at most one edge incident to each node (and timeouts are strictly positive by definition and w.l.o.g.).

**Lemma 4.** Greedy has an expected approximation ratio of at least $\delta \gamma$.

**Proof.** It is sufficient to show that the expected profit of the obtained solution is at least $\delta \cdot \text{LP}_{\text{large}}$. Let $x = (x_e)_{e \in E}$ be the optimal solution to (LP-BIP). Consider the solution $x' = (x'_e)_{e \in E}$ that is obtained from $x$ by setting to zero all the variables corresponding to edges in $E_{\text{small}}$, and by multiplying all the remaining variables by $\delta$. Since $p_e \geq \delta$ for all $e \in E_{\text{large}}$, $x'$ is a feasible fractional solution to the following matching LP:

$$\max \sum_{e \in E} w_e p_e z_e \quad \text{(LP-MATCH)} \quad (6)$$

$$\text{s.t.} \quad \sum_{e \in \delta(u)} z_e \leq 1, \quad \forall u \in V;$$

$$0 \leq z_e \leq 1, \quad \forall e \in E. \quad (7)$$
The value of \( x' \) in the above LP is \( \delta \cdot \text{LP}_{\text{large}} \) by construction. Let \( \text{LP}_{\text{match}} \) be the optimal profit of (LP-MATCH). Then \( \text{LP}_{\text{match}} \geq \delta \cdot \text{LP}_{\text{large}} \). Given that the graph is bipartite, (LP-MATCH) defines the matching polyhedron, and we can find an integral optimal solution to it. But such a solution is exactly a maximum weight matching according to weights \( w_e p_e \), i.e. \( \sum_{e \in M_{\text{opt}}} w_e p_e = \text{LP}_{\text{match}} \). The claim follows since the expected profit of the greedy algorithm is precisely the weight of \( M_{\text{opt}} \). \( \square \)

A hybrid algorithm of ALG\(_1\) and Greedy. The overall algorithm, denoted by Hybrid(\( \delta \)) is stated as follows. For a given \( \delta \), we simply compute the value of \( \gamma \), and run Greedy if \( \gamma \delta \geq \left( \frac{1}{3} \gamma + g(\delta) (1 - \gamma) \right) \), and ALG\(_1\) otherwise.\(^{10}\)

The approximation factor of Hybrid(\( \delta \)) is given by \( \max \{ \frac{\gamma}{3} + (1 - \gamma)g(\delta), \gamma \delta \} \), and the worst case is achieved when the two quantities are equal, i.e., \( \gamma = \frac{g(\delta)}{\delta + g(\delta) - \frac{1}{4}} \), yielding an approximation ratio of \( \frac{\delta + g(\delta)}{\delta + g(\delta) - \frac{1}{4}} \). Now we just need to maximize this ratio. Since this function is quite complicated and so finding algebraically its maximum seems impossible, we need to compute it numerically. To avoid issues of numerical error, let us just notice that for \( \delta = 0.6 \), the ratio is approximately 0.351563 – which allows us to claim the ratio of 0.351 from Theorem 3.

### 2.2 A Refined Approximation

We now describe the approach to achieve the 0.39-approximation ratio stated in Theorem 1. The main algorithm, denoted by MAIN, consists of two sub-routines. One is ALG\(_1\) as described in Algorithm 1 in Section 2.1. The other is a new patching algorithm, denoted by ALG\(_2\), which is described in Algorithm 2.

---

**Algorithm 2: Patching algorithm (ALG\(_2\))**

1. Construct and solve (LP-BIP) for the input instance. Let \( \vec{x} \) be an optimal solution.
2. Consider the vector \( \vec{y} \) such that \( y_e := p_e x_e \) for every edge \( e \in E \).
3. Run GKPS with \( \vec{y} \) as the input to obtain an integral vector \( \vec{Y} \).
4. Probe all edges \( e \in E \) with \( Y_e = 1 \).

---

Let \( \vec{x} \) be the optimal solutions to (LP-BIP). Recall the definition of function \( g : [0, 1] \rightarrow [0, 1] \) from Lemma 2:

\[
g(p) := \frac{1}{2 + p} \left( 1 - \exp \left( -\frac{2 + p}{p} \ln \left( \frac{1}{1 - p} \right) \right) \right).
\]

From the same Lemma 2 we know that the expected total weight achieved by ALG\(_1\) is

\[
\mathbb{E} \left[ \text{ALG}_1 \right] \geq \sum_{e \in E} w_e p_e x_e \cdot g(p_e).
\]

Consider now vector \( \vec{y} \) defined as \( y_e := p_e x_e \). Notice that vector \( \vec{y} \) is a feasible solution to (LP-MATCH). And in Algorithm 2 we can see that it is \( \vec{y} \) that guides ALG\(_2\).

We shall prove in a moment that the expected outcome of ALG\(_2\) is

\[
\mathbb{E} \left[ \text{ALG}_2 \right] = \sum_{e \in E} w_e p_e x_e.
\]

Our main algorithm, MAIN, is formally stated in Algorithm 3.

To prove Theorem 1 it remains to bound the expected outcome of ALG\(_2\). We then use lowerbounds on \( \mathbb{E} \left[ \text{ALG}_1 \right] \) and \( \mathbb{E} \left[ \text{ALG}_2 \right] \) to bound the approximation achieved by MAIN. We now lower bound the profit of ALG\(_2\) in the following lemma.

\(^{10}\) Note that we cannot run both algorithms and take the better solution, due to the probe-commit constraint.
Algorithm 3: The main algorithm (MAIN).

1. Construct and solve (LP-BIP) for the input instance; let \( \bar{x} \) be its optimal solution.
2. Run ALG 1 if \( \sum_{e \in E} w_e p_e x_e \cdot g(p_e) \geq \sum_{e \in E} w_e p_e^2 x_e \); otherwise run ALG 2.

Lemma 5. The total expected weight of the matching obtained by ALG 2 is \( \sum_{e \in E} w_e p_e^2 x_e \) where \( \bar{x} \) denotes the optimal solution to (LP-BIP).

Proof. Recall that \( \bar{y} := \bar{p} \cdot \bar{x} \) is a feasible solution to (LP-MATCH): so, the use of the GKPS dependent-rounding procedure on the polytope from (LP-MATCH) is allowed. First, from property (P1) of GKPS, the probability that an edge \( e \in E \) has \( Y_e = 1 \) is \( y_e = p_e x_e \). Second, from property (P2) and the fact that \( \sum_{e \in \delta(v)} y_e = \sum_{e \in \delta(v)} p_e x_e \leq 1 \), we have that the subgraph induced by the edges with \( Y_e = 1 \) has at most one edge incident to any vertex \( v \in V \). Thus, given that an edge \( e \) has \( Y_e = 1 \), it is guaranteed to be probed and the probability that it is eventually chosen into the matching is its probability of existing \( p_e \). Putting these two facts together, the probability that any edge \( e \in E \) is included in the final matching is \( p_e^2 x_e \). Using the linearity of expectation, we obtain that the total expected weight of the matching is \( \sum_{e \in E} w_e p_e^2 x_e \).

We now have all the ingredients to prove Theorem 4.

Proof (of Theorem 4). Algorithm MAIN chooses either ALG 1 or ALG 2 depending on the maximum of the lowerbounds of ALG 1 and ALG 2. Hence to find its worst case behaviour we have to characterize an instance that minimizes

\[
\max \left\{ \sum_{e \in E} w_e p_e x_e \cdot g(p_e), \sum_{e \in E} w_e p_e^2 x_e \right\}.
\]

Let \( E(q) \) denote the set of edges such that \( p_e = q \). Let \( h_q := \sum_{e \in E(q)} w_e x_e \). Notice that the quantity \( \int_{q=0}^{1} g(q) dq \) is just the value of (LP-BIP), which is our upperbound on \( OPT \). The outcome of ALG 1 is thus \( \int_{q=0}^{1} g(q) \cdot q^2 h_q dq \). And the value of (LP-MATCH) induced by \( \bar{y} \) is \( \int_{q=0}^{1} q h_q dq \), which is at the same time the outcome of ALG 2. Thus, the adversary wants to minimize the following mathematical program.

\[
\min \left\{ \max \left\{ \int_{q=0}^{1} g(q) dq, \int_{q=0}^{1} q^2 h_q dq \right\} \right\} \text{ such that } \int_{q=0}^{1} q h_q dq = 1
\]

The normalization constraint \( \int_{q=0}^{1} q h_q dq = 1 \) ensures that the optimal value to the adversarial program (5) is the approximation ratio for the algorithm MAIN.

Such a mathematical program may be hard to solve in full generality, due to the fact that the variable over which we optimize is in fact a (not necessarily continuous) probability distribution. However, in our case we can characterize the optimal solution, i.e., function \( q h_q \) such that \( \int_{q=0}^{1} q h_q dq = 1 \), algebraically. Let us next formulate our problem in a more compact way where we define \( f := q h_q \):

\[
\min \left\{ \max \left\{ \int_{q=0}^{1} g(q) \cdot f(q) dq, \int_{q=0}^{1} q \cdot f(q) dq \right\} \text{ s.t. } \int_{q=0}^{1} f(q) dq = 1 \right\}.
\]
Since \( g \) is a concave function, it is point-wise at least as large as a linear function \((1 - q) g(0) + q \cdot g(1)\) for all \( q \in (0, 1) \). Hence, the minimum of \([9]\) is at least as large as the minimum of the following program:

\[
\begin{align*}
\min_f \quad & \max_q \left( \int_{q=0}^{1} ((1 - q) g(0) + q \cdot g(1)) \cdot f(q) dq \right) \\
\text{s.t.} \quad & \int_{q=0}^{1} f(q) dq = 1.
\end{align*}
\]

(10)

Here we have two linear functions, i.e., \((1 - q) g(0) + q \cdot g(1)\) and \(q\). This allows us to simplify it further:

\[
\int_{q=0}^{1} ((1 - q) g(0) + q \cdot g(1)) \cdot f(q) dq = g(0) + (g(1) - g(0)) \int_{q=0}^{1} q \cdot f(q) dq.
\]

Even though the variable \( f \) in program \([10]\) is a density function, we can consider the whole integral \(\int_{q=0}^{1} g \cdot f(q) dq\) as a single real variable from \([0, 1]\), and the program simplifies to

\[
\begin{align*}
\min_{\alpha} \quad & \max \left( g(0) + (g(1) - g(0)) \cdot \alpha, \alpha \right) \\
\text{s.t.} \quad & \alpha \in [0, 1].
\end{align*}
\]

Since function \( \alpha \) is increasing and function \( g(0) + (g(1) - g(0)) \cdot \alpha \) is decreasing, the minimum is obtained for \( \alpha \) for which \( g(0) + (g(1) - g(0)) \cdot \alpha = \alpha \). This yields a value of \( \alpha \) such that

\[
\alpha = \frac{g(0)}{1 + g(0) - g(1)} = \frac{1}{2} \left( 1 - e^{-2} \right) - \frac{1}{3} = \frac{3e^{-2} - 3}{7e^{-2} - 3} \approx 0.39338739.
\]

Solution \( \alpha \) which is a number has to be translated into solution \( f \) of program \([10]\) which is a density function. This is however straightforward: \( f \) is a density function which places \( \alpha \) mass on point 1, and \( 1 - \alpha \) mass on point 0. At the same time \( f \) is a solution to the initial program \([9]\). And since the value of program \([9]\) for such an \( f \) is also equal \( \frac{3e^{-2} - 3}{7e^{-2} - 3} \), we conclude that it is the actual minimal value of it.

3 \ Stochastic Matching in General Graphs

In this section, we present our improved approximation algorithm for Stochastic Matching in general graphs as stated in Theorem 2.

We consider the linear program LP-GEN which is obtained from LP-BIP by adding the following blossom inequalities:

\[
\sum_{e \in E(W)} p_e x_e \leq \frac{|W| - 1}{2} \quad \forall W \subseteq V, |W| \text{ odd.} \tag{11}
\]

Here \( E(W) \) is the subset of edges with both endpoints in \( W \). We remark that, using standard tools from matching theory, we can solve LP-GEN in polynomial time despite its exponential number of constraints \([2]\). Also, in this case, \( x_e = \Pr[\text{OPT probes } e] \) is a feasible solution of LP-GEN, hence the analogue of Lemma 1 still holds.

Our stochastic-matching algorithm for the case of a general graph \( G = (V, E) \) works via a reduction to the bipartite case. First we solve LP-GEN; let \( x = (x_e)_{e \in E} \) be the optimal fractional solution. Second we randomly split the nodes \( V \) into two sets \( A \) and \( B \), with \( E_{AB} \) being the set of edges between them. On the bipartite graph \((A \cup B, E_{AB})\) we apply the algorithm for the bipartite case, but using the fractional solution \((x_e)_{e \in E_{AB}}\) induced by LP-GEN rather than solving LP-BIP. Note that \((x_e)_{e \in E_{AB}}\) is a feasible solution to LP-BIP for the bipartite graph \((A \cup B, E_{AB})\).
The analysis differs only in two points w.r.t. the one for the bipartite case. First, with \( \hat{E}_{AB} \) being the subset of edges of \( E_{AB} \) that were rounded to 1, we have now that \( \Pr \left[ e \in \hat{E}_{AB} \right] = \Pr \left[ e \in E_{AB} \right] \).

\[
\Pr \left[ e \in \hat{E}_{AB} \bigg| e \in E_{AB} \right] = \frac{1}{2} x_e.
\]

Second, but for the same reason, using again the negative correlation and marginal distribution properties, we have

\[
E \left[ \sum_{f \in \delta(e)} p_f \bigg| e \in \hat{E}_{AB} \right] \leq \sum_{f \in \delta(e)} p_f \Pr \left[ f \in \hat{E}_{AB} \bigg| f \in E_{AB} \right] = \sum_{f \in \delta(e)} \frac{p_f x_f}{2} \leq \frac{2 - 2p_e x_e}{2} \leq 1.
\]

Repeating the steps of the proof of Lemma 2 and including the above inequality we get the following.

**Lemma 6.** For every edge \( e \) it holds that \( \Pr \left[ e \text{ is safe} \bigg| e \in \hat{E}_{AB} \right] \geq h(p_e) \), where

\[
h(p) := \frac{1}{1 + p} \left( 1 - \exp \left( - (1 + p) \frac{1}{p} \ln \frac{1}{1 - p} \right) \right).
\]

Since \( h(p_e) \geq h(1) = \frac{1}{2} \), we directly obtain a 1/4-approximation which matches the result in [8]. Similarly to the bipartite case, we can patch this result with the simple greedy algorithm (which is exactly the same in the general graph case). For a given parameter \( \delta \in [0, 1] \), let us define \( \gamma \) analogously to the bipartite case. Similarly to the proof of Lemma 3 one obtains that the above algorithm has approximation factor \( \frac{\delta}{2} + \frac{1}{2} \frac{h(\delta)}{h(\delta) - 1/2} \). Similarly to the proof of Lemma 4 the greedy algorithm has approximation ratio \( \gamma \delta \) (here we exploit the blossom inequalities that guarantee the integrality of the matching polyhedron). We can conclude similarly that in the worst case \( \gamma = \frac{h(\delta)}{2\delta + h(\delta) - 1/2} \), yielding an approximation ratio of \( \frac{\delta h(\delta)}{2\delta + h(\delta) - 1/2} \). Maximizing (numerically) this function over \( \delta \) gives, for \( \delta = 0.5580 \), the 0.269 approximation ratio claimed in Theorem 2.

## 4 Online Stochastic Matching with Timeouts

Let \( G = (A \cup B, A \times B) \) be the input graph, with items \( A \) and buyer types \( B \). We use the same notation for edge probabilities, edge profits, and timeouts as in Stochastic Matching. Following [8], we can assume w.l.o.g. that each buyer type is sampled uniformly with probability \( 1/n \). Consider the following linear program:

\[
\text{max} \quad \sum_{a \in A, b \in B} w_{ab} p_{ab} x_{ab} \quad \text{(LP-ONL)}
\]

\[
\text{s.t.} \quad \sum_{b \in B} p_{ab} x_{ab} \leq 1, \quad \forall a \in A
\]

\[
\sum_{a \in A} p_{ab} x_{ab} \leq 1, \quad \forall b \in B
\]

\[
\sum_{a \in A} x_{ab} \leq t_b, \quad \forall b \in B
\]

\[
0 \leq x_{ab} \leq 1, \quad \forall ab \in E.
\]

The above LP models a bipartite stochastic-matching instance where one side of the bipartition contains exactly one buyer per buyer type. In contrast, in the online case, several buyers of the same buyer type (or none at all) can arrive, and the optimal strategy can allow many buyers of the same type to probe edges. This is not a problem though, since the following lemma from [8] allows us just to look at the graph of buyer types and not at the actual realized buyers.
Lemma 7. (S, Lemmas 9 and 11) Let $\mathbb{E}[\text{OPT}]$ be the expected profit of the optimal online algorithm for the problem. Let $LP_{onl}$ be the optimal value of LP-ONL. It holds that $\mathbb{E}[\text{OPT}] \leq LP_{onl}$.

We will devise an algorithm whose expected outcome is at least $0.245 \cdot LP_{onl}$, and then Theorem 3 follows from Lemma 7.

The algorithm. We initially solve LP-ONL and let $(x_{ab})_{a \in A \times B}$ be the optimal fractional solution. Then buyers arrive. When a buyer of type $b$ is sampled, then: (a) if a buyer of the same type $b$ was already sampled before we simply discard her, do nothing, and wait for another buyer to arrive, and (b) if it is the first buyer of type $b$, then we execute the following subroutine for buyers. Since we take action only when the first buyer of type $b$ comes, we shall denote such a buyer simply by $b$, as it will not cause any confusion.

Subroutine for buyers. Let us consider the step of the online algorithm in which the first buyer of type $b$ arrived, if any. Let $A_b$ be the items that are still available when $b$ arrives. Our subroutine will probe a subset of at most $t_b$ edges $ab$, $a \in A_b$. Consider the vector $(x_{ab})_{a \in A_b}$. Observe that it satisfies the constraints $\sum_{a \in A_b} p_{ab} x_{ab} \leq 1$ and $\sum_{a \in A_b} x_{ab} \leq t_b$. Again using GKPS, we round this vector in order to get $(\hat{x}_{ab})_{a \in A_b}$ with $\hat{x}_{ab} \in \{0, 1\}$, and satisfying the marginal distribution, degree preservation, and negative correlation properties.

Let $\hat{A}_b$ be the set of items $a$ such that $\hat{x}_{ab} = 1$. For each $ab$, $a \in \hat{A}_b$, we independently draw a random variable $Y_{ab}$ with distribution: $\Pr[Y_{ab} \leq y] = \frac{1}{p_{ab}} (1 - \exp(-p_{ab} \cdot y))$ for $y \in \left[0, \frac{1}{p_{ab}} \ln \frac{1}{p_{ab}}\right]$. Let $Y = (Y_{ab})_{a \in \hat{A}_b}$.

Next we consider items of $\hat{A}_b$ in increasing order of $Y_{ab}$. Let $\alpha_{ab} \in \left[\frac{1}{2}, 1\right]$ be a dumping factor that we will define later. With probability $\alpha_{ab}$ we probe edge $ab$ and as usual we stop the process (of probing edges incident to $b$) if $ab$ is present. Otherwise (with probability $1 - \alpha_{ab}$) we simulate the probe of $ab$, meaning that with probability $p_{ab}$ we stop the process anyway — like if edge $ab$ was probed and turned out to be present. Note that we do not get any profit from the latter simulation since we do not really probe $ab$.

Dumping factors. It remains to define the dumping factors. For a given edge $ab$, let

$$\beta_{ab} := \mathbb{E}_{\hat{A}_b \setminus a, Y} \left[ \prod_{a' \in \hat{A}_b \setminus a, Y_{a' < Y_{ab}}} (1 - p_{a'b}) \right] a \in \hat{A}_b.$$

Using the inequality $\sum_{a \in \hat{A}_b} p_{ab} x_{ab} \leq 1$, by repeating the analysis from Section 2 we can show that

$$\beta_{ab} \geq h(p_{ab}) = \frac{1}{1 + p_{ab}} \left(1 - \exp\left(-1 + p_{ab} \frac{1}{p_{ab}} \ln \frac{1}{1 - p_{ab}}\right)\right) \geq \frac{1}{2}.$$

Let us assume for the sake of simplicity that we are able to compute $\beta_{ab}$ exactly. We set $\alpha_{ab} = \frac{1}{2\beta_{ab}}$. Note that $\alpha_{ab}$ is well defined since $\beta_{ab} \in [1/2, 1]$.

Analysis. Let us denote by $A_b$ the event that at least one buyer of type $b$ arrives. The probability that an edge $ab$ is probed can be expressed as:

$$\Pr[A_b] \cdot \Pr[\text{no b' takes a before b}| A_b] \cdot \Pr[b \text{ probes a}| A_b \land a \text{ is not yet taken}].$$

The probability that $b$ arrives is $\Pr[A_b] = 1 - (1 - \frac{1}{n})^n \geq 1 - \frac{1}{e}$. We shall show first that

$$\Pr[b \text{ probes a}| A_b \land a \text{ is not yet taken}]$$

In this case, we have a bipartite graph where one side has only one vertex, and here GKPS reduces to Srinivasan’s rounding procedure for level-sets [19].
is exactly \( \frac{1}{2}x_{ab} \), and later we shall show that \( \Pr \left[ \text{no } b' \text{ takes } a \text{ before } b \mid \mathcal{A}_b \right] \) is at least \( \frac{1}{1 + \frac{3}{2}(1 - \frac{1}{e})} \). This will yield that the probability that \( ab \) is probed is at least

\[
\left( 1 - \frac{1}{e} \right) \frac{1}{1 + \frac{3}{2}(1 - \frac{1}{e})} \cdot \frac{1}{2} x_{ab} = \frac{e - 1}{3e - 1} x_{ab} > 0.24 x_{ab}.
\]

Consider the probability that some edge \( a'b \) appearing before \( ab \) in the random order blocks edge \( ab \), meaning that \( ab \) is not probed because of \( a'b \). Observe that each such \( a'b \) is indeed considered for probing in the online model, and the probability that \( a'b \) blocks \( ab \) is therefore \( \alpha_{a'b} p_{a'b} + (1 - \alpha_{a'b}) p_{a'b} = p_{a'b} \). We can conclude that the probability that \( ab \) is not blocked is exactly \( \beta_{ab} \).

Due to the dumping factor \( \alpha_{ab} \cdot \beta_{ab} \), the probability that we actually probe edge \( ab \in \mathcal{A}_b \) is exactly \( \alpha_{ab} \cdot \beta_{ab} = \frac{1}{2} \). Recall that \( \Pr \left[ a \in \mathcal{A}_b \right] = x_{ab} \) by the marginal distribution property. Altogether

\[
\Pr \left[ b \text{ probes } a \mid \mathcal{A}_b \land a \text{ is not yet taken} \right] = \frac{1}{2} x_{ab}.
\]

Next let us condition on the event that buyer \( b \) arrived and lower-bound the probability that \( ab \) is not blocked on the \( a \)'s side in such a step, i.e., that no other buyer has taken \( a \) already. The buyers, who are first occurrences of their type, arrive uniformly at random. Therefore, we can analyze the process of their arrivals as if it was constructed by the following procedure: every buyer \( b' \) is given an independent random variable \( Y_{b'} \) distributed exponentially on \([0, \infty)\), i.e., \( \Pr \left[ Y_{b'} < y \right] = 1 - e^{-y} \); buyers arrive in increasing order of their variables \( Y_{b'} \). Once buyer \( b' \) arrives, it probes edge \( ab' \) with probability (exactly) \( \alpha_{ab} \beta_{ab} x_{ab'} = \frac{1}{2} x_{ab'} \) — these probabilities are independent among different buyers. Thus, conditioning on the fact that \( b \) arrives, we obtain the following expression for the probability that \( a \) is safe at the moment when \( b \) arrives:

\[
\Pr \left[ \text{no } b' \text{ takes } a \text{ before } b \mid \mathcal{A}_b \right] \\
\geq \mathbb{E} \left[ \prod_{b' \in B \setminus b : Y_{b'} < Y_b} \left( 1 - \Pr \left[ \mathcal{A}_{b'} \mid \mathcal{A}_b \right] \Pr \left[ b' \text{ probes } ab' \mid \mathcal{A}_{b'} \right] \right) \mathcal{A}_b \right] \\
= \int_0^\infty \prod_{b' \in B \setminus b} \left( 1 - \Pr \left[ \mathcal{A}_{b'} \mid \mathcal{A}_b \right] \cdot \Pr \left[ Y_{b'} < y \mid \mathcal{A}_b \right] \cdot \Pr \left[ b' \text{ probes } ab' \mid \mathcal{A}_{b'} \right] p_{ab'} \right) e^{-y} dy.
\]

Now let us upper-bound each of the probability factors in the above product. First of all \( \Pr \left[ \mathcal{A}_{b'} \mid \mathcal{A}_b \right] = 1 - (1 - \frac{1}{n})^{n-1} \leq 1 - \frac{1}{e} \). Second, \( \Pr \left[ Y_{b'} < y \mid \mathcal{A}_b \right] = 1 - e^{-y} \) just by definition. Third, from (12) we have that \( \Pr \left[ b' \text{ probes } ab' \mid \mathcal{A}_{b'} \right] = \frac{x_{ab}}{2e} \).

Thus the above integral can be lower-bounded by

\[
\int_0^\infty \prod_{b' \in B \setminus b} \left( 1 - \left( 1 - \frac{1}{e} \right) (1 - e^{-y}) \cdot \frac{1}{2} x_{ab'} \cdot p_{ab'} \right) e^{-y} dy \\
\geq \int_0^\infty \prod_{b' \in B \setminus b} \exp \left( - \left( 1 - \frac{1}{e} \right) \frac{1}{2} x_{ab'} \cdot p_{ab'} \cdot y \right) e^{-y} dy \\
= \frac{1}{1 + (1 - \frac{1}{e}) \frac{1}{2} \left( \sum_{b' \in B \setminus b} p_{ab'} \cdot x_{ab'} \right)} \\
\geq \frac{1}{1 + \frac{1}{2}(1 - \frac{1}{e})} = \frac{2e}{3e - 1}.
\]

The \( \mathcal{A}_{b'} \) event in the condition simply indicates that \( Y_{b'} \) was drawn.
In the first inequality above, we used the fact that $1 - c(1 - e^{-y}) \geq e^{-cy}$ for $c \in [0, 1]$ and any $y \in \mathbb{R}$; here $c = (1 - \frac{1}{e}) \frac{\ln(1 - \frac{1}{e})}{\ln \frac{1}{e}} \geq 0$, so that the probability is bounded below by $(\frac{1}{e})^2$. In the first equality we used $\int_0^\infty e^{-ax}dx = \frac{1}{a}$. In the last inequality we used the LP constraint $\sum_{b' \in B \setminus b} p_{ab'} \cdot x_{ab'} \leq 1$.

 altogether, as anticipated earlier,

$$
\Pr \{ab \text{ is probed} \} \geq \left(1 - \frac{1}{c}\right) \frac{x_{ab}}{2} \cdot \frac{2c}{3e - 1} = x_{ab} \cdot \frac{e - 1}{3e - 1} > 0.24 \cdot x_{ab}.
$$

**Technical details.** Recall that we assumed that we are able to compute the quantities $\beta_{ab}$, hence the desired dumping factors $\alpha_{ab}$. Indeed, for our goals it is sufficient to estimate them with large enough probability and with sufficiently good accuracy. This can be done by simulating the underlying random process a polynomial number of times. This way the above probability can be lower bounded by $(\frac{1}{e^2} + \varepsilon)x_a$ for an arbitrarily small constant $\varepsilon > 0$. In particular, by choosing a small enough $\varepsilon$ the factor 0.245 is still guaranteed. The details are given in Appendix A.

The approximation factor can be further improved to 0.245 via the technique based on small and big probabilities that we introduced before. This is discussed in the next section. Theorem 3 follows.

### 4.1 Combination with Greedy in the Online Case

Recall that $h(p) = \frac{1}{1 - p} \left(1 - \exp \left(- \frac{1}{p} \ln \left(\frac{1}{1 - p}\right)\right)\right)$. We are again applying the big/small probabilities trick, so let $\delta \in (0, 1)$ be a parameter to be fixed later. Consider again the subroutine for buyers. We previously used dumping factors $\alpha_{ab} = \frac{1}{\gamma_{ab}}$, where we had set $\beta_{ab} \geq h(p_{ab})$.

This time we define $\alpha_{ab} = \frac{1}{\beta_{ab}}h(\delta)$ for $ab$ such that $p_{ab} \leq \delta$, and $\alpha_{ab} = \frac{1}{\beta_{ab}} \frac{1}{2}$ otherwise. We again assume here that we can calculate $\beta_{ab}$ (see Section A). Define $E_{\text{large}} = \{ab \in E | p_{ab} \geq \delta\}$ and $E_{\text{small}} = E \setminus E_{\text{large}}$, and let $LP_{\text{large}} = \gamma \cdot LP_{\text{onl}}$. Therefore, for edge $ab$ the probability that $ab$ is probed when $b$ scans items is exactly $h(\delta)$ for $ab \in E_{\text{small}}$ and $\frac{1}{2}$ for $ab \in E_{\text{large}}$. Now by repeating the steps in the proof of Section A we obtain that the probability that $ab$ is not blocked on $a$’s side is at least

$$
\frac{1}{1 + \left(1 - \frac{1}{e}\right) \left(\sum_{b' \in B \setminus b} p_{ab'} \cdot \alpha_{ab'} \cdot \beta_{ab'} \cdot x_{ab'}\right)} \geq \frac{1}{1 + \left(1 - \frac{1}{e}\right) h(\delta) \left(\sum_{b' \in B \setminus b} p_{ab'} \cdot x_{ab'}\right)} \geq \frac{1}{1 + \left(1 - \frac{1}{e}\right) h(\delta)}.
$$

since $\alpha_{ab'} \cdot \beta_{ab'} = h(\delta)$ for small edges and $\alpha_{ab'} \cdot \beta_{ab'} = \frac{1}{2} \leq h(\delta)$ for large edges. Therefore, the approximation ratio of such an algorithm is at least

$$
\left(1 - \frac{1}{e}\right) \left(\frac{\frac{1}{2}}{1 + h(\delta) \left(1 - \frac{1}{e}\right)} + (1 - \gamma) \frac{h(\delta)}{1 + h(\delta) \left(1 - \frac{1}{e}\right)}\right)
= \left(1 - \frac{1}{e}\right) \frac{1}{1 + h(\delta) \left(1 - \frac{1}{e}\right)} \left(\frac{\gamma}{2} + (1 - \gamma) h(\delta)\right).
$$

An alternative algorithm simply computes a maximum weight matching w.r.t. weights $p_a w_a$ in the graph corresponding to LP-ONL, and upon arrival of the first copy of a buyer type $b$ probes only the edge incident to $b$ in the matching (if any). By the same argument as in the offline case, this matching has weight at least $\gamma \cdot \delta \cdot LP_{\text{onl}}$, and every buyer type is sampled with probability at least $1 - \frac{1}{e}$. So, the approximation ratio of the greedy algorithm is at least $\left(1 - \frac{1}{e}\right) \gamma \delta$.

For a fixed $\delta$, depending on the value of $\gamma$ (that we can compute offline) we can run the algorithm with best approximation ratio according to the above analysis. Thus the overall approximation ratio is

$$
\left(1 - \frac{1}{e}\right) \max \left\{ \frac{1}{1 + h(\delta) \left(1 - \frac{1}{e}\right)} \left(\frac{\gamma}{2} + (1 - \gamma) h(\delta)\right) \right\}.
$$
As in Section 2.1 the worst-case is obtained when the two quantities are equal. This yields

\[ \gamma = \frac{h(\delta)}{\delta \cdot (1 + h(\delta)(1 - \frac{1}{e})) - \frac{1}{2} + h(\delta)}. \]

Hence the actual approximation ratio is

\[ \left(1 - \frac{1}{e}\right) \frac{\delta \cdot h(\delta)}{\delta \cdot (1 + h(\delta)(1 - \frac{1}{e})) - \frac{1}{2} + h(\delta)}, \]

and now we just need to optimize over \( \delta \). When we set \( \delta = 0.74 \), the approximation ratio becomes approximately 0.245712219628 – which allows us to claim the bound of 0.245 from Theorem 3.

References

1. Abdulkadiroğlu, A., Pathak, P. A., and Roth, A. E. The new york city high school match. *American Economic Review* 95, 2 (2005), 364–367.
2. Adamczyk, M. Improved analysis of the greedy algorithm for stochastic matching. *Information Processing Letters* 111, 15 (2011), 731–737.
3. Adamczyk, M., Grandoni, F., and Mukherjee, J. Improved approximation algorithms for stochastic matching. In *Algorithms-ESA 2015*. Springer, 2015, pp. 1–12.
4. Adamczyk, M., Sviridenko, M., and Ward, J. Submodular stochastic probing on matroids. In *STACS 2014*, pp. 29–40.
5. Assadi, S., Khanna, S., and Li, Y. The stochastic matching problem with (very) few queries. In *Proceedings of the 2016 ACM Conference on Economics and Computation* (2016), ACM, pp. 43–60.
6. Assadi, S., Khanna, S., and Li, Y. The stochastic matching problem: Beating half with a non-adaptive algorithm. In *Proceedings of the 2017 ACM Conference on Economics and Computation* (2017), ACM, pp. 99–116.
7. Badanidiyuru, A., Kleinberg, R., and Slivkins, A. Bandits with knapsacks. *Journal of the ACM (JACM)* 65, 3 (2018), 13.
8. Bansal, N., Gupta, A., Li, J., Mestre, J., Nagarajan, V., and Rudra, A. When LP is the cure for your matching woes: Improved bounds for stochastic matchings. *Algorithmica* 63, 4 (2012), 733–762.
9. Bansal, N., Korula, N., Nagarajan, V., and Srinivasan, A. Solving packing integer programs via randomized rounding with alterations. *Theory Comput.* 8 (2012), 533–565.
10. Barveja, A., Chavan, A., Nikiforov, A., Srinivasan, A., and Xu, P. Improved bounds in stochastic matching and optimization. In *Approximation, randomization, and combinatorial optimization. Algorithms and techniques*, vol. 40 of LIPICS. Leibniz Int. Proc. Inform. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2015, pp. 124–134.
11. Behnezhad, S., and Reyhani, N. Almost optimal stochastic weighted matching with few queries. In *Proceedings of the 2018 ACM Conference on Economics and Computation* (2018), ACM, pp. 235–249.
12. Blum, A., Dickerson, J. P., Haghtalab, N., Procaccia, A. D., Sandholm, T., and Sharma, A. Ignorance is almost bliss: Near-optimal stochastic matching with few queries. In *Proceedings of the Sixteenth ACM Conference on Economics and Computation* (2015), ACM, pp. 325–342.
13. Brubach, B., Grimmel, N., and Srinivasan, A. Vertex-weighted online stochastic matching with patience constraints. *CoRR* (2019).
14. Brubach, B., Sankararaman, K. A., Srinivasan, A., and Xu, P. New algorithms, better bounds, and a novel model for online stochastic matching. In *24th Annual European Symposium on Algorithms (ESA 2016)* (2016), Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik.
15. Brubach, B., Sankararaman, K. A., Srinivasan, A., and Xu, P. Attenuate locally, win globally: An attenuation-based framework for online stochastic matching with timeouts. In *Sixteenth International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2017)*, 2017.
16. Brubach, B., Sankararaman, K. A., Srinivasan, A., and Xu, P. Algorithms to approximate column-sparse packing problems. In *Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms* (2018), Society for Industrial and Applied Mathematics, pp. 311–330.
17. Buchbinder, N., Naor, J. S., et al. The design of competitive online algorithms via a primal–dual approach. *Foundations and Trends® in Theoretical Computer Science* 3, 2–3 (2009), 93–263.
18. Caetano, T. S., McAuley, J. J., Cheng, L., Le, Q. V., and Smola, A. J. Learning graph matching. *IEEE transactions on pattern analysis and machine intelligence* 31, 6 (2009), 1048–1058.

19. Chen, N., Immorlica, N., Karlin, A. R., Mahdian, M., and Rudra, A. Approximating matches made in heaven. In *ICALP 2009*, pp. 266–278.

20. Costello, K. P., Tetali, P., and Tripathi, P. Stochastic matching with commitment. In *International Colloquium on Automata, Languages, and Programming* (2012), Springer, pp. 822–833.

21. Dean, B. C., Goemans, M. X., and Vondrák, J. Adaptivity and approximation for stochastic packing problems. In *Proceedings of the Sixteenth Annual ACM-SIAM Symposium on Discrete Algorithms* (2005), ACM, New York, pp. 395–404.

22. Dean, B. C., Goemans, M. X., and Vondrák, J. Approximating the stochastic knapsack problem: the benefit of adaptivity. *Math. Oper. Res.* 33, 4 (2008), 945–964.

23. Devanur, N. R., Jain, K., Sivan, B., and Wilkens, C. A. Near optimal online algorithms and fast approximation algorithms for resource allocation problems. *Journal of the ACM (JACM)* 66, 1 (2019), 7.

24. Dickerson, J. P., and Sandholm, T. Multi-organ exchange. *Journal of Artificial Intelligence Research* 60 (2017), 639–679.

25. Dickerson, J. P., Sankararaman, K. A., Srinivasan, A., and Xu, P. Assigning tasks to workers based on historical data: Online task assignment with two-sided arrivals. In *Proceedings of the 17th International Conference on Autonomous Agents and MultiAgent Systems* (2018), AAMAS ’18, pp. 318–326.

26. Edmonds, J. Paths, trees, and flowers. *Canadian Journal of mathematics* 17 (1965), 449–467.

27. Fata, E., Ma, W., and Simchi-Levi, D. Multi-stage and multi-customer assortment optimization with inventory constraints. Available at SSRN 3443109 (2019).

28. Feldman, J., Henzinger, M., Korula, N., Mirrokni, V. S., and Stein, C. Online stochastic packing applied to display ad allocation. In *European Symposium on Algorithms* (2010), Springer, pp. 182–194.

29. Feldman, J., Mehta, A., Mirrokni, V., and Muthukrishnan, S. Online stochastic matching: Beating 1-1/e. In *2009 50th Annual IEEE Symposium on Foundations of Computer Science* (2009), IEEE, pp. 117–126.

30. Gamlath, B., Kale, S., and Svensson, O. Beating greedy for stochastic bipartite matching. In *Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms* (2019), SODA ’19, pp. 2841–2854.

31. Gandhi, R., Khuller, S., Parthasarathy, S., and Srinivasan, A. Dependent rounding and its applications to approximation algorithms. *Journal of the ACM* 53, 3 (2006), 324–360.

32. Guha, S., and Munagala, K. Approximation algorithms for partial-information based stochastic control with markovian rewards. In *48th Annual IEEE Symposium on Foundations of Computer Science (FOCS’07)* (2007), IEEE, pp. 483–493.

33. Immorlica, N., Sankararaman, K. A., Schapire, R., and Slivkins, A. Adversarial bandits with knapsacks. In *Proceedings of the sixtieth annual IEEE foundations on computer science (FOCS ’19)* (2019), IEEE.

34. Jaillet, P., and Lu, X. Online stochastic matching: New algorithms with better bounds. *Mathematics of Operations Research* 39, 3 (2013), 624–646.

35. Karp, R. M., Vazirani, U. V., and Vazirani, V. V. An optimal algorithm for on-line bipartite matching. In *Proceedings of the twenty-second annual ACM symposium on Theory of computing* (1990), ACM, pp. 352–358.

36. Katrriell, I., Kenyon-Mathieu, C., and Upfal, E. Commitment under uncertainty: Two-stage stochastic matching problems. In *International Colloquium on Automata, Languages, and Programming* (2007), Springer, pp. 171–182.

37. Kuhn, H. W. The hungarian method for the assignment problem. *Naval research logistics quarterly* 2, 1-2 (1955), 83–97.

38. Li, J. *Decision making under uncertainty*. PhD thesis, University of Maryland, 2011.

39. Li, J. Private communication, 2015.

40. Ma, W. Improvements and generalizations of stochastic knapsack and multi-armed bandit approximation algorithms: Extended abstract. In *SODA 2014*, pp. 1154–1163.

41. Manshadi, V. H., Gharam, S. O., and Saberi, A. Online stochastic matching: Online actions based on offline statistics. *Mathematics of Operations Research* 37, 4 (2012), 559–573.

42. Molinaro, M., and Ravi, R. The query-commit problem. *arXiv preprint arXiv:1110.0990* (2011).

43. Mukherjee, J. Approximation Algorithms for Stochastic Matchingsand Independent Sets. PhD thesis, The Institute of Mathematical Sciences, Chennai, 2019.
A Computing Dumping Factors

Recall that we assumed the knowledge of quantities $\beta_{ab}$, which are needed to define the dumping factors $\hat{\beta}_{ab}$. Though we are not able to compute the first quantities exactly in polynomial time, we can efficiently estimate them and this is sufficient for our goals. Let us focus on a given edge $ab$. Recall that

$$\beta_{ab} := \mathbb{E}_{\hat{A}_b \setminus a,Y} \left[ \prod_{a' \in A_b : Y_{a'b} < Y_{ab}} (1 - p_{a'b}) \bigg| a \in \hat{A}_b \right]$$

$$\geq \frac{1}{1 + p_{ab}} \left( 1 - \exp \left( - (1 + p_{ab}) \left( \frac{1}{p_{ab}} \ln \frac{1}{1 - p_{ab}} \right) \right) \right) = h (p_{ab}) .$$

Let us simulate the subroutine for buyers $N$ times without the dumping factors: in a simulation we run GKPS and sample the $Y$ variables, but simulate the probes of the edges without actually probing any edge. We shall set $N$ later. Let $S^1, S^2, \ldots, S^N$ be 0-1 indicator random variables of whether $a$ was safe or not in each simulation. Note that $\mathbb{E} [S^i] = \beta_{ab} x_{ab} \in [h (p_{ab}) x_{ab}, x_{ab}]$.

Suppose that $x_{ab} \geq \frac{\epsilon}{n}$, where $n$ is the number of buyers. The expression $\hat{s}_{ab} = \frac{1}{N} \sum_{i=1}^{N} S^i$ should be a good estimation of $\beta_{ab} \cdot x_{ab}$, i.e.,

$$\hat{s}_{ab} \in [\beta_{ab} x_{ab} (1 - \epsilon), \beta_{ab} x_{ab} (1 + \epsilon)]$$

with probability $1 - \frac{1}{n^2}$. Set $N = \frac{\ln (2np(Z)^2)}{\epsilon^2} \ln (2n^2 Z)$ for $Z = \frac{3}{\epsilon} + 1$.

Applying Chernoff’s bound $\Pr \left[ |X - \mathbb{E} [X]| > \epsilon \mathbb{E} [X] \right] \leq 2e^{-\frac{\epsilon^2}{3} \mathbb{E}[X]}$ with $X = \sum_{i=1}^{N} S_i$ one obtains:

$$\Pr \left[ \sum_{i=1}^{N} S_i \notin \left[ (1 - \epsilon) \beta_{ab} x_{ab} \cdot N, (1 + \epsilon) \beta_{ab} x_{ab} \cdot N \right] \right] \leq 2 \exp \left( -\frac{\epsilon^2}{3} \beta_{ab} x_{ab} \cdot N \right) \leq 2 \exp \left( -\frac{\epsilon^2}{3} \frac{1}{2} \cdot N \right) \leq 2 \exp \left( -\frac{\epsilon^3}{6n} \cdot N \right) = \frac{1}{n^2} \frac{1}{Z}.$$
From the union bound, with probability at least $1 - \frac{1}{2}$, we have that $s_{ab} \in [\beta_{ab} x_{ab} (1 - \epsilon), \beta_{ab} x_{ab} (1 + \epsilon)]$ for every edge $ab$ such that $x_{ab} \geq \frac{x}{n}$. Now let us assume that this happened, i.e., that we have good estimates. We set $\alpha_{ab} = \max\{\frac{1}{2}, \min\{\frac{1}{2} x_{ab}, 1\}\}$ which belongs to $\left[\frac{1}{2} \beta_{ab} x_{ab} (1 + \epsilon), \frac{1}{2} \beta_{ab} x_{ab} (1 - \epsilon)\right]$, but only for edges $ab$ such that $x_{ab} \geq \frac{x}{n}$. For edges $ab$ such that $x_{ab} < \frac{x}{n}$, we just put $\alpha_{ab} = 1$ (so we do not dump such edges actually). Two elements of the proof were depending on the dumping factors. First, now the probability that edge $ab$ is taken is $\alpha_{ab} \beta_{ab} x_{ab} \in \left[\frac{x_{ab}}{2(1 + \epsilon)}, \frac{x_{ab}}{2(1 - \epsilon)}\right]$. Second, recall that the probability of edge $ab$ not to be blocked is:

$$\frac{1}{1 + (1 - \frac{1}{2}) \left(\sum_{b' \in B \setminus b} p_{ab'} \cdot \alpha_{ab'} \beta_{ab'} \cdot x_{ab'}\right)}.$$  

We have that

$$\sum_{b' \in B \setminus b} p_{ab'} \cdot \alpha_{ab'} \beta_{ab'} \cdot x_{ab'} = \sum_{b' \in B \setminus b: x_{ab'} \geq \frac{x}{n}} p_{ab'} \cdot \alpha_{ab'} \beta_{ab'} \cdot x_{ab'} + \sum_{b' \in B \setminus b: x_{ab'} < \frac{x}{n}} p_{ab'} \cdot \alpha_{ab'} \beta_{ab'} \cdot x_{ab'} \leq \frac{1}{2} (1 - \epsilon) x_{ab'} \leq \sum_{b' \in B \setminus b: x_{ab'} \geq \frac{x}{n}} p_{ab'} \cdot \alpha_{ab'} \beta_{ab'} \cdot x_{ab'} + \frac{1}{2} (1 - \epsilon) x_{ab'} \leq \frac{1}{2} (1 - \epsilon) + \epsilon = \frac{1}{2} + O(\epsilon).$$

So the probability that $a$ is not blocked is at least $\frac{1}{1 + (1 - \frac{1}{2}) (\frac{1}{2} + O(\epsilon))}$. The final probability that edge $ab$ is probed is at least

$$\left(1 - \frac{1}{2}\right) \cdot \frac{x_{ab}}{2(1 + \epsilon)} \cdot \frac{1}{1 + (1 - \frac{1}{2}) (\frac{1}{2} + O(\epsilon))} = \frac{x_{ab}}{1 + \epsilon} \cdot \frac{e - 1}{2e + (e - 1)(1 + O(\epsilon))} = \frac{x_{ab}}{3e - 1 + O(\epsilon)} > 0.24 \cdot x_{ab}.$$  

In the last inequality above we assumed $\epsilon$ to be small enough.

With probability at most $\frac{1}{2}$ we did not obtain good estimates of the dumping factors. Still we have that $\alpha_{ab} \in [\frac{1}{2}, 1]$, and therefore $\alpha_{ab} \beta_{ab} \in [\frac{1}{4}, 1]$. In this case quantity (13) can be just lower-bounded by $\frac{1}{1 + (1 - \frac{1}{2})}$, and the probability that edge $ab$ is probed in the subroutine for buyers is at least $\frac{24 \cdot x_{ab}}{4}$. Thus the probability that edge $ab$ is probed during the algorithm is at least $\left(1 - \frac{1}{2}\right) \cdot \frac{x_{ab}}{4} \cdot \frac{1}{1 + (1 - \frac{1}{2})} = \frac{x_{ab}}{4} \cdot \frac{e - 1}{2e - 1} > 0.097x_{ab}$. The total expected outcome of the algorithm is therefore, for sufficiently small $\epsilon$, at least

$$LP_{\text{onl}} \left(\left(1 - \frac{1}{2}\right) \cdot \frac{e - 1}{3e - 1 + O(\epsilon)} + \frac{1}{4} \cdot \frac{e - 1}{2e - 1}\right)^{\frac{e - 1}{2} + 1} \geq 0.24 \cdot LP_{\text{onl}}.$$  

The above approach can be combined with the small/big probability trick from Section 4.1. By choosing $\epsilon$ small enough the approximation ratio is 0.245 as claimed.