ANOTHER PROBABILISTIC CONSTRUCTION OF $\phi^{2n}$ IN DIMENSION 2

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1. Introduction

The main input of this note is to provide an alternative probabilistic approach to the $\phi^{2n}$ theory in dimension 2, based on concentration phenomenon of martingales associated to polynomials of Gaussian variables. This is based on an adaptation of the work [LRV18] of Lacoin-Rhodes-Vargas, in which exponential potentials associated to quantum Mabuchi $K$-energy are studied.

We give an alternative proof of the following classical result.

**Theorem 1.1** (Negative exponential moments). Let $n \geq 2$ be an integer and let $R$ be a real, unitary polynomial of even degree $2n$. Let $X$ be the (Dirichlet) Gaussian Free Field on a bounded simply connected domain $\Lambda \subset \mathbb{R}^2$.

Consider the (non-necessary positive) Wick-ordered random measure

$$V_R(\Lambda) = \int_{\Lambda} :R(X)(x) : d^2x$$

with integer $n \geq 2$. Then we have the following estimate

$$\mathbb{E} \left[ e^{-\alpha V_R(\Lambda)} \right] < \infty$$

for some $\alpha > 0$.

This key estimate for the construction of the $\phi^{2n}$ theory (where $R(X) = X^{2n}$) in dimension 2 follows originally from a hypercontractivity argument due to Nelson [Nel66]. Given this estimate the rest of the argument is standard: the book [Ree12] is a good reference for details and developments of the hypercontractivity argument.

The idea of the martingale method is originally used to study more involved models such as the quantum Mabuchi $K$-energy [LRV18] or the Sine-Gordon model [LRV19]. This note shows in particular that this idea can also be successfully implemented to the Euclidean quantum $\phi^{2n}$ theory in dimension 2.

We stress that the purpose of this note is to introduce a new and arguably convenient construction of a classical theory in an elementary fashion. Readers unfamiliar with the classical model can consult [Sim15] for an overview on this subject.

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2. Preliminaries

**Notations.** In the following we denote by $\Lambda \subset \mathbb{R}^2$ some bounded simply connected open subset of the Euclidean plane $\mathbb{R}^2$. We consider $n \geq 2$ be an integer and let $R$ be a real, unitary polynomial of even degree $2n$. We use $X$ to denote the (Dirichlet) Gaussian Free Field (GFF in short) supported on $\Lambda$. The object of interest would be the Wick-ordered polynomial $:R(X) :$ for the GFF. More precisely, we are interested in integrals of type $\int_{\Lambda} :R(X)(x) : d^2x$. 
2.1. **Gaussian Free Field.** We review some of the aspects of the probabilistic construction of the Gaussian Free Field (or GFF after) that will be useful later. We refer to [Dub09] for more information.

Recall that the Green function $K(x, y)$ on the domain $\Lambda$ is defined as $K = (-\Delta_{\Lambda})^{-1}$, where $-\Delta_{\Lambda}$ is the differential operator with Dirichlet boundary condition $g = 0$ on $\partial \Lambda$. In the following we will stick to the Dirichlet boundary condition although the argument works for general boundary conditions.

A (Dirichlet) GFF $X$ on $\Lambda$ is a random distribution taking value in the negative Sobolev space $H^{-s}(\Lambda)$ with $s > 0$. It is characterized by its mean and covariance kernel $K$ on $\Lambda$: for test functions $f, g \in H^{s}(\Lambda)$,

$$
E[\langle X, f \rangle] = 0, \quad E[\langle X, f \rangle \langle X, g \rangle] = \int_{\Lambda} f(x)g(y)K(x, y)d^2xd^2y
$$

where $\langle \cdot, \cdot \rangle$ denotes the dual bracket between $H^{-s}(\Lambda)$ and $H^{s}(\Lambda)$. Recall that the Green function $K$ displays logarithmic divergence on the diagonal, that is

$$
K(x, y) = -\ln|x - y| + F(x, y)
$$

with $F(x, y)$ smooth.

2.2. **Wick ordering and Hermite polynomials.** Let $(B_t)_{t \in \mathbb{R}^+}$ be the standard 1d Brownian motion. We consider the Wick ordering of $(B_t)^{2n}$, defined by

$$
: (B_t)^{2n} := t^n P_{2n}^H \left( \frac{B_t}{\sqrt{t}} \right)
$$

where $P_{2n}^H$ denotes the Hermite polynomial (normalized to have unitary leading coefficient) of degree $2n$. The Wick ordering procedure requires that the expectation vanishes, i.e.

$$
E\left[ P_{2n}^H \left( \frac{B_t}{\sqrt{t}} \right) \right] = 0, \quad \forall t \geq 0.
$$

It follows that the Itō derivative of $P_{2n}^H(B_t)$ with respect to the Brownian filtration has no drift term. The Wick ordering procedure provides a natural martingale parametrized by the time $t$.

**Notation.** In the following we absorb the renormalization in $t$ for $P_{2n}^H$ and write

$$
P_{2n}(B_t) := t^n P_{2n}^H \left( \frac{B_t}{\sqrt{t}} \right).
$$

**Example.** For $n = 2$, the Wick ordering yields

$$
P_4(B_t) = B_t^4 - 6tB_t^2 + 3t^2
$$

which can be equally written as

$$
P_4(B_t) = (B_t^2 - 3t)^2 - 6t^2
$$

and $P_4$ is bounded from below by $-6t^2$. We also deduce that the envelope of the zero-graph

$$
\{(t, B_t) \in \mathbb{R}^+ \times \mathbb{R}, P_{2n}(B_t) = 0\}
$$

is given by two symmetric branches

$$
\bigcup_{t \in \mathbb{R}^+} \{(t, \sqrt{(3 + \sqrt{6})t}) \} \cup \{(t, -\sqrt{(3 + \sqrt{6})t}) \} \subset \mathbb{R}^+ \times \mathbb{R}.
$$

**General case.** In general, by linear combination, we define the Wick ordered polynomial of $B_t$ for any real, unitary polynomial $R$ of even degree $2n \geq 4$:

$$
: R(B_t) := P_R(B_t).
$$
More precisely, if

\[ R(X) = \sum_{i=0}^{2n} a_i X^i \]

with \(a_{2n} = 1\), then we define the associated Wick ordered polynomial \( P_R(X) \) by

\[ P_R(X) = \sum_{i=0}^{2n} a_i t^i P_i \left( \frac{X}{\sqrt{t}} \right). \]

The martingale property of \( P_R(B_t) \) with respect to the Brownian filtration is preserved by linear combination. The envelope of the graph of the zeros of \( P_R \) is given explicitly by

\[ \bigcup_{t \in \mathbb{R}^+} \{(t, f_R(t))\} \cup \{(t, -f_R(t))\} \subset \mathbb{R}^+ \times \mathbb{R} \]

where the positive branch \( f_R \geq 0 \) can be explicitly calculated. The example above shows that when \( R(X) = X^4 \),

\[ f_{X^4}(t) = \sqrt{(3 + \sqrt{6})t}. \]

The following facts are elementary.

**Proposition 2.1** (Envelope of zeros). Let \( R \) be a real, unitary polynomial of even degree \( 2n \geq 4 \). The function \( f_R \) satisfies the following:

1. There exists some constant \( A > 0 \) only depending on \( n \) such that \( f_R(t) \leq t + A \) for all \( t \in \mathbb{R}^+ \);
2. For every \( \epsilon > 0 \), there exists some constant \( A' = A'(n, \epsilon) \) such that \( f_R(t) \leq \epsilon t + A' \) for all \( t \in \mathbb{R}^+ \).

We will also consider the value of \( P_R \) on the line \( \{ t + A \}_{t \geq 0} \) for some constant \( A \).

**Proposition 2.2** (Values on cones). For large enough \( A \), the function

\[ t \mapsto P_R(t + A) \]

satisfies the following properties:

1. It is positive for \( t \in \mathbb{R}^+ \);
2. It is strictly increasing in \( t \) for \( t \in \mathbb{R}^+ \).

**2.3. Cut-off regularization.** Since the Gaussian Free Field \( X \) only makes sense as a distribution, it is suitable to define the measure

\[ V_R(\Lambda) = \int_{\Lambda} : R(X)(x) : d^2x \]

using a cut-off procedure. We need the following assumption:

**Proposition 2.3** (Smooth white noise decomposition). We choose a cut-off regularization \( (X_u)_{u \in \mathbb{R}^+} \) satisfying the following properties:

1. The covariance kernel \( K \) can be written in the form

\[ K(x, y) = \int_0^\infty Q_u(x, y) du \]

where for all \( x \neq y \), the above integral is convergent; \( Q_u \) is a bounded symmetric positive definite kernel for any \( u \).
2. Setting \( K_t = \int_0^t Q_u du \), there exists a positive constant \( C \) such that

\[ \left| K_t(x, y) - \left( t \wedge \ln \frac{1}{|x - y|} \right) \right| \leq C. \]
(3) We have \( \lim_{x \to \infty} Q_u(x, x) = 1 \) with uniform convergence in \( x \in \Lambda \).

(4) For all \( 0 < \beta < 2 \),

\[
\int_{\Lambda^2} \int_0^\infty e^{\beta u|Q_u(x, y)|} d^2x d^2y < \infty.
\]

It is proven in [LRV18, Section 4.2] that the GFF \( X \) on \( \Lambda \) can be fitted into this assumption. We will thus work under this assumption in the following.

We define \((X_t(x))_{x \in \Lambda, t \geq 0}\) to be the jointly continuous process in \( x \) and \( t \) with covariance kernel

\[
E[X_s(x)X_t(y)] = \int_0^{t \wedge s} Q_u(x, y) \, du.
\]

According to the above assumption, given \( x \in \Lambda \), the process \((X_t(x))_{t \geq 0}\) is very similar to a standard Brownian motion. We assume for readability in the following that

\[
K_t(x, x) = t
\]

so that \((X_t(x))_{t \geq 0}\) is a standard Brownian motion.

2.4. Quadratic variation of martingales. We have the following lemma in probability concerning the exponential martingale:

**Lemma 2.4 (Exponential martingale).** For any continuous local martingale \( M \) and any \( \lambda \in \mathbb{C} \), the process

\[
\exp \left( \lambda M_t - \frac{\lambda^2}{2} \langle M, M \rangle_t \right)
\]

is a local martingale. We also write \( \langle M \rangle_t \) for the quadratic variation \( \langle M, M \rangle_t \).

In particular, if \( M_0 = 1 \) and \( \langle M \rangle_\infty < \infty \), then \( M \) is a \( L^2 \)-bounded continuous martingale and we have for \( \alpha > 0 \) the following inequality

\[
\limsup_{t \to \infty} E[\exp(-\alpha M_t)] < \infty
\]

in such a way that the limit of \( M_t \) displays Gaussian concentration.

We refer to the classical text book [RY99] for these results.

3. Proof of the main theorem

We now prove Theorem 1.1 using martingale methods.

3.1. Preliminary notations. Fix a real, unitary polynomial \( R \) of even degree \( 2n \geq 4 \). Hereafter we sometimes drop the dependence on \( R \) where there is no ambiguity.

Borrowing notations from Proposition 2.1, we consider the two-branched envelope

\[
E := \{(t, u) \in \mathbb{R}_+ \times \mathbb{R}; |u| = f_R(t)\}.
\]

The envelop \( E \) depends on \( R \): we write simply \( E \) for readability.

We introduce a cut-off at level \( \pm g(t) \) where

\[
g(t) = t + A
\]

and \( A \geq 0 \) is a large constant chosen later: we require that \( f_R(t) \leq g(t) \) for all \( t \in \mathbb{R}_+ \) and that Proposition 2.2 holds.

We consider also the cone \( C \) with two symmetric branches:

\[
C := \{(t, u) \in \mathbb{R}_+ \times \mathbb{R}; |u| = g(t)\}.
\]

Geometrically, the envelope \( E \) is in between the two branches of \( C \).
Let us rewrite the main theorem with the notations in the preliminary. We define a regularization of the measure $V_R(\Lambda)$ using the smooth white noise decomposition Proposition 2.3

$$D_t = \int_\Lambda P_R(X_t(x))d^2x.$$  

We prove in the following that uniformly in $t$, there exists some $C(A)$ such that for all $\alpha \in \mathbb{R}$,

$$\mathbb{E}[e^{\alpha D_t}] \leq e^{C(A)\alpha^2}.$$  

3.2. Strategy of the proof. One first calculates the quadratic variation of the martingale $D_t$ in view of Lemma 2.4. We have

$$\langle D \rangle_t \leq \int_{\Lambda^2 \times [0,t]} |P'_R(X_u(x))P'_R(X_u(y))|Q_u(x,y)d^2xd^2ydu.$$  

If the graph $(t, X_t(x))$ stays (uniformly in $t$ and in $x$) inside the cone $C$, then $|P'_R(X_u(x))|$ cannot take exceptionally high values and the quadratic variation $\langle D \rangle_t$ is uniformly bounded in $t$ (see Lemma 3.2 below) and the $L^2$-theory of martingales applies. By Lemma 2.4, the limiting measure would display Gaussian concentration bound.

Almost surely this is not the case: the process $X_t(x)$ goes out of the cone $C$ and takes high values. We consider for every $x \in \Lambda$ the stopping time

$$H^x := \inf\{s \geq 0; (s, X_s(x)) \in C\}.$$  

As the zero-value envelope $E$ is inside the cone $C$, after time $H^x$ the process $P_R(X_u(x))$ at point $x$ stays positive until the next time it returns to $E$.

Introduce a sequence of stopping times (always with respect to a fixed $x \in \Lambda$):

$$H^x_k := \inf\{s \geq L^x_{k-1}; (x, X_s(x)) \in C\},$$  

$$L^x_k := \inf\{s \geq H^x_k; (x, X_s(x)) \in E\}.$$  

By convention, $L^x_0 \equiv 0$. We can write as a decomposition of times depending on whether $X_t(x)$ takes low or high values,

$$[L^x, H^x] := \bigcup_{k \in \mathbb{N}} [L^x_k, H^x_{k+1}],$$  

$$[H^x, L^x] := \bigcup_{k \in \mathbb{N}^*} [H^x_k, L^x_k].$$  

It follows that for all $x \in \Lambda$, $[L^x, H^x] \cup [H^x, L^x] = \mathbb{R}_+$ almost surely.

We will take advantage of the positivity between the stopping times $[H^x, L^x]$. More precisely, on one hand the total contribution of $P_R(X_t(x))$ from intervals of the form $[L^x, H^x]$ is bounded in $L^2$ (since it takes values inside the cone $C$), on the other hand the contribution of $P_R(X_t(x))$ from intervals $[H^x, L^x]$ has constant positive sign.

We quantify this observation in the following way:

**Proposition 3.1** (High value cut-off). We consider the following decomposition. Let

$$D_L(t) = \int_\Lambda \left( \int_0^t P'_R(X_s(x))1_{\{s \in [L^x, H^x]\}}ds \right) d^2x$$  

and

$$D_H(t) = \int_\Lambda \left( \int_0^t P'_R(X_s(x))1_{\{s \in [H^x, L^x]\}}ds \right) d^2x$$  

in such a way that

$$D_t = D_L(t) + D_H(t).$$  

Then we have the following inequality

$$D_t \geq D_L(t) - Q$$  

(5)
where \( Q \) denotes the positive quantity

\[
Q := \int_{\Lambda} \left( \sum_{i=1}^{\infty} 1_{\{H_i^e < \infty\}} P_R(g(H_i^e)) \right) \, d^2x.
\]

Note that \( D_L, D_H, Q \) depend on \( R \) but we drop this dependence in the notation.

**Proof.** Fix \( x \in \Lambda \) and one can check the following claims:

- If \( k \in \mathbb{N} \) is such that \( t \in [L_k^e, H_{k+1}^e] \), then
  \[
  \int_0^t P'_R(X_s(x)) 1_{\{s \in [L^e, H^e]\}} \, ds = (P_R(X_t(x)) - P_R(X_{L_k^e}(x))) + \sum_{i=0}^{k-1} (P_R(X_{H_{i+1}^e}(x)) - P_R(X_{L_i^e}(x))).
  \]

This is because for every \( l < k \), the increment of the process \( X_t(x) \) on the interval \([L_l^e, H_{l+1}^e]\) contributes exactly to one term in the above summation.

- If now \( k \in \mathbb{N} \) is such that \( t \in [H_k^e, L_k^e] \), then we have \( P_R(X_t(x)) \geq 0 \) and
  \[
  \int_0^t P'_R(X_s(x)) 1_{\{s \in [L^e, H^e]\}} \, ds = \sum_{i=0}^{k-1} (P_R(X_{H_{i+1}^e}(x)) - P_R(X_{L_i^e}(x))).
  \]

Notice now that for all \( i \in \mathbb{N} \), \( P_R(X_{L_i^e}(x)) = 0 \) by definition of the zero envelope \( E \) and hitting times \( L_i^e \). A similar argument as above shows that \( D_H(t) \leq 0 \) for all \( t \in \mathbb{R}_+ \), so that

\[ D_t \geq D_L(t). \]

To prove Equation (5), write the above in the following form:

\[
P_R(X_t(x)) \geq \int_0^t P'_R(X_s(x)) 1_{\{s \in [L^e, H^e]\}} \, ds - \sum_{i=0}^{\infty} 1_{\{H_i^e < \infty\}} P_R(X_{H_{i+1}^e}(x)).
\]

Equation (5) follows by integrating over \( x \in \Lambda \).

Now the proof of the main theorem boils down to two estimates, of which the first one corresponds to the \( L^2 \) part, and the second one corresponds to the high-value part.

**Lemma 3.2** (Low value contribution). \( D_L(t) \) is an honest martingale that has bounded quadratic variation: it converges in \( L^2 \) and satisfies the Gaussian concentration bound

\[
\exists C(A), \forall \alpha \in \mathbb{R}, \mathbb{E} \left[ e^{\alpha D_L(\infty)} \right] \leq e^{C(A)\alpha^2}.
\]

**Lemma 3.3** (High value contribution). The other quantity \( Q \) in the decomposition also satisfies a Gaussian concentration bound:

\[
\exists C(A), \forall \alpha \in \mathbb{R}, \mathbb{E} \left[ e^{\alpha Q} \right] \leq e^{C(A)\alpha^2}.
\]

Combining these two lemmas, Theorem 1.1 follows.

### 3.3. Proofs of technical estimates

We start by proving Lemma 3.2.

**Proof of Lemma 3.2.** The fact that \( D_L(t) \) is a martingale follows from construction. It suffices to show that \( \langle D_L \rangle_\infty \) is bounded from above by a constant: Gaussian concentration then follows by Lemma 2.1. The calculation goes as follows:

\[
\langle D_L \rangle_t \leq \int_{\Lambda^2 \times [0,t]} |P'_R(X_s(x))P'_R(X_s(y))| \, 1_{\{s \in [L^e, H^e] \cap \{t \}} P_R(x,y) d^2xd^2ydu.
\]
Since \( P^*_R(X_s(x)) \) is polynomial of degree \( 2n - 1 \), it has subexponential growth at infinity and the conditioning on \( s \) implies that \( |X_s(x)| \leq g(s) \). We bound the above by

\[
\langle DL \rangle_t \leq C \int_{\Lambda^2 \times [0,t]} e^{\frac{1}{2} g(s)} e^{\frac{1}{2} g(s)} 1_{s \in [L^x, H^x] \cap [L^y, H^y]} Q_s(x, y) d^2x d^2y ds \\
\leq C \int_{\Lambda^2 \times [0,t]} e^s Q_s(x, y) d^2x d^2y ds
\]

for some constant \( C = C(R) \). The last integral is finite by the last item of Proposition 2.3. □

Proof of Lemma 3.3. Recall some preliminaries on Doob martingales. Define the positive quantity

\[
Q^x = \sum_{i=1}^{\infty} 1_{H^*_i < \infty} P_R(g(H^*_i))
\]

(\( Q^x \) depends on \( R \) but we alleviate the notation) so that

\[
Q = \int_{\Lambda} Q^x d^2x.
\]

Lemma 3.4 (\( L^1 \)-boundedness). We have \( \mathbb{E}[Q] < \infty \).

Proof. We bound \( \mathbb{E}[Q^x] \) uniformly in \( x \in \Lambda \): the claim follows from integrating over \( \Lambda \).

Consider the following quantity:

\[
Q^{x,m} = \sum_{i=1}^{\infty} 1_{H^*_i \in (m-1, m]} P_R(g(m)).
\]

By Proposition 2.2 choose \( A \) large enough such that \( P_R \) is strictly increasing on \( \mathbb{R}_+ \) and

\[
P_R(g(m)) = \sup_{v \in (m-1, m]} P_R(g(v))
\]

such that

\[
Q^x \leq \sum_{m=1}^{\infty} Q^{x,m}.
\]

We now prove a standard estimate

\[
\mathbb{E}[\# \{ i : H^*_i \in (m-1, m] \}] \leq \frac{8}{\sqrt{2\pi m}} e^{-\frac{m}{2}}.
\]

Given this and that the polynomial \( P_R \) has sub-exponential growth at infinity, i.e.

\[
P_R(g(m)) \leq C(A)e^{-\frac{m}{2}},
\]

the result follows by summing over \( m \) then integrating over \( x \).

Notice that, with \( A \geq 1 \),

\[
\mathbb{P}[\exists i, H^*_i \in (m-1, m]] \leq \mathbb{P}\left[ \sup_{s \leq m} |B_s| \geq m \right] \leq \frac{4}{\sqrt{2\pi m}} e^{-\frac{m}{2}}
\]

by a standard Gaussian tail estimate. Using the Markov property for the Brownian motion,

\[
\mathbb{P}[\# \{ i : H_i \in (m-1, m] \geq k+1 \} | \# \{ i : H_i \in (m-1, m] \geq k \}] \leq \frac{1}{2}
\]

and Equation (8) follows from summing over \( k \). □
The Doob martingale $Q^x_t$ is defined as

$$Q^x_t = \mathbb{E}[Q^x_t | \mathcal{F}_t]$$

(recall that $\mathcal{F}_t = \sigma\{X_s, s \in [0, t]\}$) and since it is a martingale associated to the Brownian filtration \{X_t(x)\}, we can write

$$dQ^x_t = A^x_t dX_t(x).$$

Then the bracket $\langle Q \rangle_\infty$ can be written as

$$\langle Q \rangle_\infty = \int_{\mathbb{R}^2 \times \mathbb{R}^+_t} A^x_s A^y_s Q_u(x, y) d^2x d^2y du. \quad (9)$$

We now control $A^x_t$ uniformly in $x$, according to whether $t \in [H^x, L^x]$ or $t \in [L^x, H^x]$. In the following we drop the dependency on $x$ to alleviate the notations. More precisely, we prove that uniformly over all $t$, with some constant $C(A)$ independent of $x$,

$$|A_t| \leq C(A) e^{t/2}. \quad (10)$$

Lemma 3.3 then follows from Equation (9) and Lemma 2.3, together with Lemma 2.4.

To prove Equation (10), we apply coupling techniques to the Brownian motion $X_t(x)$.

3.3.1. **First case:** $t \in [H^x, L^x]$. Suppose $t \in [H^x, L^x]$ for some $k \in \mathbb{N}$.

Let $\mathbb{P}_z$ be the law of a standard Brownian motion $(B_t)_{t \geq 0}$ starting at point $z$. By the strong Markov property of $X_t(x)$ as a Brownian motion $B_t$ (we drop the index $x$ afterwards), write $Q_t$ as

$$Q_t = \sum_{i=1}^{k-1} P_R(g(H_i)) + \mathbb{E}_{X_t(x)} \left[ \sum_{i=1}^\infty 1_{\{\bar{T}^i < \infty\}} P_R(g(t + \bar{T}^i)) \right]$$

where the stopping time sequence $\bar{T}^i_k = \bar{T}^i_k(B)$ is defined recursively by $\bar{T}^i_0 = 0$ and

$$\begin{align*}
\bar{T}^i_k &= \inf\{s \geq \bar{T}^i_{k-1}; B_s \in E\}; \\
\bar{T}^i_k &= \inf\{s \geq \bar{T}^i_{k}; B_s \in C\}. \quad (11)
\end{align*}$$

We deduce the expression for $A_t$ in this case:

$$A_t = \partial_z \left( \mathbb{E}_{\mathbb{P}_z} \left[ \sum_{i=1}^\infty 1_{\{\bar{T}^i < \infty\}} P_R(g(t + \bar{T}^i)) \right] \right) \bigg|_{z = X_t(x)}.$$ 

We show that the expression in the definition of $A_t$ that we derive is Lipschitz in $z$ with the adequate Lipschitz constant: this will imply Equation (10).

Let $t \in [H, L]$ and consider a coupling between two independent Brownian motions, starting from points $z_1 < z_2$ with $|z_1 - z_2|$ small, denoted respectively by $B^1$ and $B^2$. Suppose that the two Brownian motions evolve independently until the first time they meet

$$\tau = \inf\{s > 0; B^1_s = B^2_s\}$$

and jointly afterwards. Each Brownian motion in this coupling defines its own hitting time $\bar{T}^{(t, j)}_i$ and $\bar{T}^{(t, j)}_i$ for $j \in \{1, 2\}$ similarly as in Equations (11). The hitting times are identical up to a shift in the indices after merging at time $\tau$.

If $\tau < \min(\bar{T}^{(t, 1)}_1, \bar{T}^{(t, 2)}_1)$, then each Brownian motion gives rise to the same contribution in the expression of $A_t$. In particular, this also holds for $\tau < \min(\bar{T}^{(t, 1)}_1, \bar{T}^{(t, 2)}_1, 1)$. Hereafter let

$$\mathcal{T} = \min(\bar{T}^{(t, 1)}_1, \bar{T}^{(t, 2)}_1, 1).$$
It suffices to show that the following bound:

\[
|E_{z_1} \left[ \sum_{i=1}^{\infty} 1_{\{t_i < \infty\}} P_R(g(t + t_i)) \right] - E_{z_2} \left[ \sum_{i=1}^{\infty} 1_{\{t_i < \infty\}} P_R(g(t + t_i)) \right] | \leq \mathbb{P} [\tau > \mathcal{T}] \times \mathbb{E} \left[ \sum_{j=1,2} \sum_{i=1}^{\infty} 1_{\{t_i < \infty\}} P_R(g(t + t_i))^j | \tau > \mathcal{T} \right].
\]

(12)

• One first shows by standard coupling estimate on Brownian motions that

\[
\mathbb{P} [\tau > \min\{\mathcal{T}_1^{(t,1)}, \mathcal{T}_1^{(t,2)}\}] \leq C|z_1 - z_2|.
\]

It is a standard Brownian coupling result that \( \mathbb{P} [\tau > 1] \leq C|z_1 - z_2| \). It remains to show

\[
\mathbb{P} [\tau > \mathcal{T}_1^{(t,1)}] \leq C|z_1 - z_2|.
\]

Provided that we choose a large enough \( A \) in the definition of \( g_t \), we have \( \mathcal{T}_1^{(t,1)} \geq \min\{s; |B^1_s - z_1| \geq 1\} \)

and

\[
\mathbb{P} [\tau > \mathcal{T}_1^{(t,1)}] \leq \mathbb{P}(0, z_2 - z_1) \left[ \mathcal{T}_A > \mathcal{T}_{[-1,1] \times \mathbb{R}} \right] \leq C|z_1 - z_2|
\]

where \( \Delta = \{(x, x); x \in \mathbb{R}\} \) and \( \mathcal{T}_A \) denotes the hitting time of a set \( A \) by a two-dimensional Brownian motion. This is a standard estimate (for a detailed proof, [LRV13, Appendix B]).

• We now show that

\[
\mathbb{E} \left[ \sum_{j=1,2} \sum_{i=1}^{\infty} 1_{\{t_i < \infty\}} P_R(g(t + t_i))^j | \tau > \min\{\mathcal{T}_1^{(t,1)}, \mathcal{T}_1^{(t,2)}\} \right] \leq C e^{t/2}.
\]

By linearity it suffices to show it for \( j = 1 \), the calculation for \( j = 2 \) is similar. We apply Markov property at \( \mathcal{T} = \min\{\mathcal{T}_1^{(t,1)}, \mathcal{T}_1^{(t,2)}\} \) and distinguish two subcases:

- If \( \mathcal{T} < \mathcal{T}_1^{(t,1)} \), we apply Markov property for the Brownian motion \( B^1_t \) at \( \mathcal{T}_1^{(t,1)} \). Since at time \( \mathcal{T}_1^{(t,1)} \) the Brownian motion \( B^1 \) takes value \( f_R(t + \mathcal{T}_1^{(t,1)}) \), the above quantity is dominated by

\[
\sup_{r \in [t, t + 1]} \sup_{s \geq r} \mathbb{E}_{f_R(s)} \left[ \sum_{i=1}^{\infty} 1_{\{t_i < \infty\}} P_R(g(s + t_i)) \right].
\]

As in Lemma 3.4 together with Proposition 2.2 for any \( r \in [t, t + 1] \) and \( s \geq r \),

\[
\mathbb{E}_{f_R(s)} \left[ \sum_{i=1}^{\infty} 1_{\{t_i < \infty\}} P_R(g(s + t_i)) \right] \leq C \sum_{n \geq 1} \mathbb{E}_{f_R(s)} \left[ \# \{t_i < \infty\} \in (n - 1, n] \right] P_R(g(s + n))
\]

(13)

\[
\leq C \sum_{n \geq 1} \frac{1}{\sqrt{n}} e^{-\frac{((1-\epsilon)s+n)^2}{2n}} e^{(s+n)/3}
\]

\[
\leq C e^{-s/3}.
\]

We used the Proposition 2.2 that one can assume \( f_R(s) < \epsilon s + A'(\epsilon) \) uniformly for any \( \epsilon > 0 \).

- If \( \mathcal{T} > \mathcal{T}_1^{(t,1)} \), then since \( \mathcal{T} \leq \mathcal{T}_1^{(t,1)} \) by definition, we know that

\[
|B^1_T| \leq g(t + \mathcal{T}).
\]
By assumption, \( T \leq 1 \) and we can control the contribution by

\[
\sup_{r \in [t+1]} \sup_{|z| \leq g(r)} \mathbb{E}_z \left[ \sum_{i=1}^{\infty} 1_{\{H_i^{(r,1)} < \infty\}} P_R(g(r + H_i^{(r,1)})) \right].
\]

Again, by the same argument as in Lemma 3.4 with \(|z| \leq g(r)\) and Proposition 2.2,

\[
\mathbb{E}_z \left[ \sum_{i=1}^{\infty} 1_{\{H_i^{(r,1)} < \infty\}} P_R(g(r + H_i^{(r,1)})) \right]
\leq C \sum_{n \geq 1} \mathbb{P}_z \left[ \exists i; H_i^{(r,1)} \in (n-1, n] \right] P_R(g(r + n))
\leq C \sum_{n \geq 1} \frac{1}{\sqrt{n}} e^{-n/2} e^{(r+n)/3}
\leq C e^{r/3}
\]

so that the contribution above is control by \( Ce^{r/3} \). This is an appropriate Lipschitz constant for Equation (10).

3.3.2. Second case: \( t \in [L^2, H^2] \). In this case, the above strategy fails for the first term \( i = 1 \). Indeed, if both Brownian motions start near the cone \( C \), the probability that they merge before either of them hitting \( C \) is arbitrarily small and Equation (5.3.1) cannot be reproduced. However, the same argument works for \( i \geq 2 \) (since in this case they both have to travel from the inner envelope \( E \) to the outer cone \( C \)). We thus have to look more carefully into the term \( i = 1 \).

For the \( i = 1 \) case, we use a different “parallel” coupling. Consider two Brownian motions, \( B_1^1 \) starting at \( z_1 \) and \( B_2^2 \) starting at \( z_2 \) (by symmetry, suppose that \( z_2 > z_1 \)) coupled as

\[
B_2^2 = B_1^1 + (z_2 - z_1).
\]

Denote by \( S_1 \) (resp. \( S_2 \)) the hitting time of \( B_1^1 \) (resp. \( B_2^2 \)) at the outer cone \( C \). We show that

\[
(14) \quad \left| \mathbb{E} \left[ 1_{\{S_1 < \infty\}} P_R(g(t + S_1)) \right] - \mathbb{E} \left[ 1_{\{S_2 < \infty\}} P_R(g(t + S_2)) \right] \right| \leq C(A)|z_1 - z_2| e^{t/2}.
\]

By symmetry we can add the indicator of the event that \( S_1 < S_2 \) (otherwise change \((z_1, z_2)\) into \((-z_1, -z_2))\). This is only a geometric data: with the assumption that \( z_1 < z_2 \), the event \( S_1 < S_2 \) is equivalent to the event that the first time any of the Brownian motions \( B_1^1 \) and \( B_2^2 \) hits the the outer cone \( C \), the location is at the lower branch of \( C \).

Since the above inequality is an absolute value, we should separate into two subcases:

- We can choose \( A \) large enough so that \( P_R(g(\cdot)) \) is strictly increasing on \( \mathbb{R}_+ \) by Proposition 2.2

Notice that with the conditioning \( S_1 < S_2 \) and Markov property at \( S_1 \),

\[
\mathbb{E} \left[ 1_{\{S_2 < \infty\}} P_R(g(t + S_2))1_{\{S_1 < S_2\}} \right]
\geq \mathbb{E} \left[ 1_{\{S_2 < \infty\}} P_R(g(t + S_1))1_{\{S_1 < S_2\}} \right]
\geq \mathbb{E} \left[ 1_{\{S_1 < \infty\}} P_R(g(t + S_1))1_{\{S_1 < S_2\}} \right] \inf_{S_1 > 0} \mathbb{P} \left[ S_2 < \infty | S_1 < \infty \right]
\]

and thus we have

\[
\mathbb{E} \left[ 1_{\{S_1 < \infty\}} P_R(g(t + S_1))1_{\{S_1 < S_2\}} \right] - \mathbb{E} \left[ 1_{\{S_2 < \infty\}} P_R(g(t + S_2))1_{\{S_1 < S_3\}} \right]
\leq \left( \sup_{S_1 > 0} \mathbb{P} \left[ S_2 = \infty | S_1 < \infty \right] \right) \mathbb{E} \left[ 1_{\{S_1 < \infty\}} P_R(g(t + S_1))1_{\{S_1 < S_2\}} \right].
\]
Repeating arguments as before, we have

\[ \mathbb{E} \left[ 1_{\{s_1 < \infty\}} P_R(g(t + S_1)) 1_{\{s_1 < s_2\}} \right] \leq \mathbb{E} \left[ 1_{\{s_1 < \infty\}} P_R(g(t + S_1)) 1_{\{s_2 < s_2' < \infty\}} \right] \leq C(A) e^{t/2}. \]

It remains to control the other conditional probability. Using the strong Markov property for \(B^2\) at \(S_1\), for all \(s_1 > 0,\)

\[ \mathbb{P} [S_2 < \infty | S_1 < \infty] \geq \mathbb{P}_{S_1} \{ \exists s; B_s = s \} = e^{-2(z_2 - z_1)} \]

by standard diffusion process identity. Indeed, for \(z \leq 0,\)

\[ u(z) = \mathbb{P}_z \{ \exists s; B_s = s \} \]

solves the differential equation

\[ u''(z) - 2u'(z) = 0 \]

with initial condition \(u(0) = 1\) and \(u(-\infty) = 0\). Together this proves one direction in Equation (14).

- Now we prove the other direction. Let \(T = \min\{S_2, S_2'\}\) with

\[ S_2' = \inf\{s > S_1; B_s^2 = -f_R(t + s)\}. \]

Only the lower branch of the envelope \(E\) is concerned because the assumptions \(z_1 < z_2\) and \(S_1 < S_2\) imply geometrically that at time \(S_1\), the Brownian motion \(B_{S_1}^2\) is located at the lower part of the cone \(C\) and the other coupled Brownian motion \(B_{S_1}^2\), at time \(S_1\), is between the lower part of \(E\) and the lower part of \(C\). It follows that after time \(S_1\), the Brownian motion \(B^2\) first hits either the lower part of \(E\) (corresponding to \(S_2'\)) or the lower part of \(C\) (corresponding to \(S_2\)).

We give bounds on \(\mathbb{E} \left[ 1_{\{S_2 < \infty\}} P_R(g(t + S_2)) 1_{\{s_1 < s_2\}} \right]\) depending on how \(S_2\) compares to \(S_2'\). We are going to show that in one case

\[ \mathbb{E} \left[ 1_{\{S_2 < \infty\}} P_R(g(t + S_2)) 1_{\{s_1 < s_2\}} 1_{\{s_2 < s_2' < \infty\}} \right] \leq \mathbb{E} \left[ 1_{\{s_1 < \infty\}} P_R(g(t + S_1)) 1_{\{s_1 < s_2\}} \right] \]

and in the other case

\[ \mathbb{E} \left[ 1_{\{S_2 < \infty\}} P_R(g(t + S_2)) 1_{\{s_1 < s_2\}} 1_{\{s_2' < s_2 < \infty\}} \right] \leq \mathbb{C}|z_1 - z_2| \]

The sum of these two equalities yields a constant order Lipschitz coefficient for Equation (10).

- For the first case, consider the Brownian motion

\[ \overline{B}_s = B_{S_1}^2 + s - B_{S_1}^2. \]

Now in this case, \(T - S_1\) is a stopping time for \(\overline{B}\). Since conditioned on the event that \(S_1 < \infty\) and \(S_1 < S_2\), \(\{P_R(B_{S_1}^2 + \overline{B}_u)\}_{u \geq 0}\) is a (positive) martingale and for the filtration

\[ \mathcal{F}_u = \mathcal{F}_{S_1} \cup \sigma(\overline{B}_s; s \leq u), \]

up until time \((T - S_1)\). Fatou’s lemma for the conditional expectation yields (since \(P_R(f_R(\cdot)) = 0)\)

\[ \mathbb{E} \left[ 1_{\{S_2 < \infty\}} P_R(g(t + S_2)) 1_{\{s_1 < s_2\}} 1_{\{s_2 < s_2' < \infty\}} \right] \]

\[ \leq \mathbb{E} \left[ 1_{\{s_1 < \infty\}} P_R(B_{S_1}^2) 1_{\{s_1 < s_2\}} \right] \]

\[ \leq \mathbb{E} \left[ 1_{\{s_1 < \infty\}} P_R(g(t + S_1)) 1_{\{s_1 < s_2\}} \right] \]

provided that \(P_R(B_{S_1}^2) \leq P_R(g(t + S_1))\) by Proposition 2.1.

- For the second case, consider

\[ \mathbb{E} \left[ P_R(g(t + S_2)) 1_{\{s_1 < s_2' < s_2 < \infty\}} \right]. \]

By applying Markov property at time \(S_2',\)

\[ \mathbb{E} \left[ P_R(g(t + S_2)) 1_{\{s_1 < s_2' < s_2 < \infty\}} \right] \leq \mathbb{P} \left[ S_2' < S_2 < \infty \right] \max_{s \geq t} \mathbb{E}_{f_R(s)} \left[ P_R(g(H_1^t)) 1_{\{H_1^t < \infty\}} \right]. \]
The second term on the right hand side is bounded by a constant, see Equation (13). The rest reduces to the estimate (by applying Markov property at time $S'_2$)

$$\mathbb{P}\left[S'_2 < S_2 | S_1 < \infty\right]$$

$$= \mathbb{P}_{z_1 - z_2} \left[ \min\{s; B_s = f_R(s + t) - t - A\} < \min\{s; B_s = s\} \right]$$

$$\leq \mathbb{P}_{z_1 - z_2} \left[ \min\{s; B_s = \epsilon(s + t) + A'(-) - t - A\} < \min\{s; B_s = s\} \right]$$

where we used the fact that $f_R(s) \leq \epsilon s + A'(-)$ for arbitrarily small $\epsilon > 0$. The last probability can be shown to be smaller than $C |z_1 - z_2|$ by diffusion process estimate. Indeed, it can be bounded by

$$u(z_1 - z_2) = \mathbb{P}_{z_1 - z_2} \left[ \min\{s; B_s = s + A' - A\} < \min\{s; B_s = s\} \right]$$

where the last term solves the differential equation

$$u''(z) - 2u'(z) = 0$$

with initial conditions $u(0) = 0, u(A' - A) = 1$. Computation yields

$$u(z) = \frac{1 - e^{-2z}}{1 - e^{-2(A' - A)}} \leq C |z|$$

which completes the proof. $\square$

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