Aspects of Chaitin’s Omega *

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Abstract. The halting probability of a Turing machine, also known as Chaitin’s Omega, is an algorithmically random number with many interesting properties. Since Chaitin’s seminal work, many popular expositions have appeared, mainly focusing on the metamathematical or philosophical significance of Omega (or debating against it). At the same time, a rich mathematical theory exploring the properties of Chaitin’s Omega has been brewing in various technical papers, which quietly reveals the significance of this number to many aspects of contemporary algorithmic information theory. The purpose of this survey is to expose these developments and tell a story about Omega, which outlines its multifaceted mathematical properties and roles in algorithmic randomness.

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## Contents

1 **Introduction** ................................................................. 3
   1.1 About this survey ......................................................... 3
   1.2 What is a halting probability? .......................................... 4
   1.3 The number of wisdom .................................................. 5

2 **Omega in algorithmic information theory and metamathematics** .............................. 5
   2.1 Randomness, Incompressibility and Unpredictability of Omega .................... 6
   2.2 Omega in the left-c.e. reals ............................................ 6
   2.3 Undecidability in formal systems and Omega .................................... 7
   2.4 Algorithmic probability and Omega ...................................... 7
   2.5 Algorithmic probability in formal systems and Omega .......................... 8
   2.6 A weakly random version of Omega ..................................... 9
   2.7 Analogues of Omega in the computably enumerable sets ........................ 9

3 **Computable approximations to Omega** .................................................................. 10
   3.1 Rates of convergence amongst left-c.e. reals ..................................... 10
   3.2 Initial segment complexity of Omega numbers .................................... 11
   3.3 Comparing the rates of convergence amongst Omega numbers .................. 13
   3.4 Speeding-up the approximation to Omega ....................................... 13

4 **Omega and computable enumerability** .................................................................. 14
   4.1 Omega and halting problems ................................................ 14
   4.2 Omega and left-c.e. reals or c.e. sets ....................................... 15
   4.3 How similar or different are two Omega numbers? .............................. 16
   4.4 Computational power versus randomness of Omega ............................. 17

5 **Halting probability relative to a set** .................................................................. 18
   5.1 Restricting the output of a universal prefix-free machine ....................... 18
   5.2 Halting probability in an oracle prefix-free machine ............................. 19
   5.3 Omega operators .............................................................. 20

6 **Machine probabilities beyond halting** ................................................................ 21
   6.1 Natural machine properties and their universal probabilities .................. 21
   6.2 Complexity of a property versus algorithmic randomness of its probability ... 22
   6.3 Invariance with respect to different universal machines .......................... 22

7 **Conclusion** ........................................................................... 23
1 Introduction

The two most influential contributions of Gregory Chaitin to the theory of algorithmic information theory are (a) the information-theoretic extensions of Gödel’s incompleteness theorem\(^1\) and (b) the discovery of the halting probability \(\Omega\), as a concrete algorithmically random real number. Chaitin’s numerous popular expositions of these discoveries have attracted a certain amount of criticism, which concerns his philosophical interpretations of the incompleteness results \([97, 89]\) as well as his limited and subjective view of the field of algorithmic information theory and its contributors \([66]\).

The purpose of the present article is to expose a mathematical theory of halting probabilities which was developed in the last 40 years by numerous researchers (much of it without Chaitin’s active participation) and which reveals interesting and deep properties of the number \(\Omega\). Most expositions on \(\Omega\) in the literature such as \([48, 47, 45, 29]\) do very little\(^2\) with regard to its mathematical properties, either because they aim at a very general audience or because they focus on its philosophical significance. In contrast, we take the view that a better argument for the importance of Chaitin’s \(\Omega\) – one that is immune to attacks concerning its philosophical interpretations – is a mathematical theory that reveals its complexities and its relevance to many important topics in contemporary algorithmic information theory.

1.1 About this survey

We adopt a rather informal style of presentation, often omitting technical definitions of established notions, either on the assumption that the reader is familiar with them or on the basis that any ambiguity can be easily resolved by consulting a given technical reference. At the cost of (lack of) self-containment, this approach will allow us to tell a coherent and concise mathematical story of \(\Omega\), based on a number of technical and relatively recent contributions that are absent from popular expositions. The rich reference list and our multiple citations may be regarded as a compensation for our approach to this survey. Due to the large number or theorems about \(\Omega\) that are discussed in the text, and in order to increase the readability of this article and keep it concise, we have (a) avoided theorem displays and proofs, opting for a conversational mention of the statements in the right context; and (b) suppressed the names of the contributors of most results from the main text, merely citing the number of the relevant bibliographic entry.

The protagonist in this survey is \(\Omega\) itself and the style of this presentation aims at showing the impact of Chaitin’s \(\Omega\) in contemporary research in algorithmic randomness in the most straightforward way. Our assumption about the reader is basic familiarity with \(\Omega\) and the part of algorithmic information theory which is sufficient to define it. For technical definitions of notions or exact statements of results that we mention but did not include, the reader is referred to the encyclopedic monographs by Downey and Hirschfeldt \([62]\) and Li and Vitanyi \([81]\), or the specific research articles that are cited during the various discussions. Inevitably, we do not aim or claim to be exhaustive; however we do strive to include or at least mention most technical contributions about \(\Omega\) that fit our narrative. The reader will also find open problems, research suggestions and loose ends, in the context of many discussions about \(\Omega\).

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\(^1\)Here we mean Gödel’s first incompleteness theorem which asserts that every sufficiently powerful formal system is incomplete, in the sense that there are undecidable sentences with respect to it. Chaitin’s extensions of Gödel’s incompleteness theorem were developed in \([35, 36, 37, 42]\) and popularized in many articles and books including \([49, 39, 43]\). We note that Kikuchi \([72]\) and Kritchman and Raz \([75]\) have given proofs of the second incompleteness theorem in the spirit of Chaitin’s arguments, using Kolmogorov complexity.

\(^2\)virtually nothing beyond the basic fact that it is algorithmically random and effectively approximable from below.
1.2 What is a halting probability?

Suppose that we run a universal\(^3\) Turing machine on a random (in the probabilistic sense) binary program. More precisely, whenever the next bit of the program is required during the program execution, we flip a coin and feed the binary output to the machine. On the basis of this thought experiment, which we are often going to refer to as Chaitin’s thought experiment, we can then consider the probability that the universal Turing machine \(U\) will halt. If we consider this problem in the context of self-delimiting machines\(^4\) then the halting probability \(\Omega_U\) of \(U\) takes the following simple expression:

\[
\Omega_U = \sum_{U(\sigma) \downarrow} 2^{-|\sigma|}
\]

where \(\sigma\) represents the random finite binary programs and \(U(\sigma) \downarrow\) denotes the fact that \(U\) halts on input \(\sigma\). Note that what we call Omega or halting probability (or alternatively Omega number) is not a single real number, but a family of real numbers indexed by the corresponding universal Turing machine. The relationships and differences of different Omega numbers will be explored in Section 4.3. Also note that \(\Omega_U\) is a left-c.e. real, i.e. it is the limit of an increasing computable sequence of rationals.

One can view Omega as a compressed version of Turing’s halting set \(H = \{\sigma \mid U(\sigma) \downarrow\}\). Indeed, if we coded the halting set into a binary stream \(\eta\) in a canonical way, ordering the strings first by length and then lexicographically, and representing the outcome \(U(\sigma) \downarrow\) by 1 and the outcome \(U(\sigma) \uparrow\) by 0, then the first \(n\) bits of \(\Omega_U\) can give answers to the first \(m\) bits of \(\eta\), where \(m\) is exponentially larger than \(n\). The precise relationship between these numbers \(n, m\) will be explored in Section 4.

Any prefix-free machine \(M\) has a halting probability \(\Omega_M\) – not just the universal ones. All halting probabilities are left-c.e. reals and vice-versa, as a consequence of the Kraft-Chaitin theorem, every left-c.e. real is the halting probability of some prefix-free machine. In [63] it was shown that there are non-computable left-c.e. reals \(\alpha\) such that any prefix-free machine with halting probability \(\alpha\) has a computable domain.

Returning to universal prefix-free machines, a fundamental result is

**The Omega characterization:** Martin-Löf random left-c.e. reals are exactly the halting probabilities of universal prefix-free machines

and moreover the same holds if we replace ‘universal’ with ‘optimal’ in the sense of Kolmogorov complexity. This was proved in the cumulative work in [91, 24, 76] and will be discussed in more depth in the

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\(^3\)By universal machine (in any standard model of Turing machines) we mean that it can effectively simulate any other Turing machine (in the class that it belongs to). In particular, given an effective numbering of all machines in the class \((M_e)\) there exists a computable function \(e \mapsto \sigma_e\) from indices to strings such that \(U(\sigma_e, e) = M_e(\sigma)\) for all \(\sigma\). This notion of universality also applies to prefix-free machines, and contrasts the notion of optimal prefix-free machines in the context of Kolmogorov complexity (see [81, Definition 2.1]). Optimal machines are the ones with respect to which the Kolmogorov complexity of any string is minimal within an additive constant compared to any other machine.

\(^4\)Self-delimiting machines, are Turing machines such that their halting only depends on the length of the initial segment of the input that they read, and not on the length of the given program. This restriction is only required for the present expression of the halting probability in terms of finite programs. Alternatively one could consider a universal oracle Turing machine \(M\) and define the halting probability as the uniform Lebesgue measure of the binary streams\(^5\) \(X\) such that \(M(X)\) halts on an empty program. Chaitin [38] showed that self-delimiting machines are equivalent to Turing machines with prefix-free domain and vice-versa, in the sense that computations in one model can be simulated by computation in the other. However Juedes and Lutz [71] showed that but this simulation incurs an exponential blow-up on the running time, unless \(P = NP\).

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### Notes

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main part of this survey. In [33] it was shown that there are non-universal and even non-optimal prefix-free machines whose halting probability is 1-random, i.e. the same as the halting probability of a universal prefix-free machine. Moreover it was shown that if the domain of a prefix-free machine $M$ contains the domain of a universal prefix-free machine, then the halting probability of $M$ is 1-random (though $M$ may not be universal).

1.3 The number of wisdom

Just as the halting set holds the answers to many interesting mathematical problems (Goldbach’s Conjecture, Riemann’s hypothesis or any one-quantifier definable problem in formal arithmetic) so does Omega. The difference is that the first $n$ bits of Omega hold more answers than the first $n$ bits of (the characteristic sequence of) $H$. Some authors [67, 52] have noted that the first 10,000 bits of Omega (with respect to a canonical universal Turing machine) contain the answers to many open problems in mathematics, although extracting such answers would take unrealistically long computations. Calude et.al. [28] compute that the answer for Riemann’s hypothesis is contained within the first 7780 many bits of $\Omega_U$, for a canonical $U$ of their own design. Hence, in the words of Bennett [19], this number embodies an enormous amount of wisdom in a very small space and this property has been the main premise in popular articles such as Gardner [67].

Chaitin [41] showed how to produce an exponential diophantine equation whose solution set defines Omega. Variations of this expression of Omega were later obtained in [87, 82, 88]. In [50] Chaitin also gives an expression of Omega in terms of tilings of the half-plane and in [46] he discusses an interpretation of the approximation to Omega by cellular automata. Mathematically, given the classic results which allow to represent computably enumerable in terms of solutions of diophantine equations or tilings of the plane, the above expressions of Omega are not very surprising. In fact, as Gács [66] points out, some of them are straightforward consequences of known results and the fact that Omega is a left-c.e. real. However for Chaitin these examples serve a deeper purpose, namely a demonstration that a certain type of randomness exists in mathematics. Such arguments about the importance of Omega can be found in his popular books such as [47].

2 Omega in algorithmic information theory and metamathematics

In this section we discuss some of the more basic properties of Chaitin’s Omega. In Section 2.1 we briefly expose its properties from the point of view of algorithmic information theory. In Section 2.3 we discuss the other popular aspect of Chaitin’s Omega, which is its role in demonstrating undecidability in formal systems. In Section 2.4 we discuss how the halting probability relates to the concept of algorithmic probability of Solomonoff, and how Chaitin’s randomized machine thought experiment can be used in order to express a number of algorithmic complexity properties in terms of probabilities. This discussion is a step towards

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6There are a number of self-contained presentations of this result including [27, 23]. In our view the best presentation is in [62, Chapter 9].

7The same canonical universal machine $U$ is used in the papers [30, 31].

8an equation involving only addition, multiplication, and exponentiation of non-negative integer constants and variables

9If we consider one variable as parameter, we obtain an infinite series of equations. Then the $n$th bit of Omega is 0 or 1 according to whether the $n$th equation has finitely or infinitely many non-negative integer solutions.
more advanced results that will be discussed in the later sections of this survey. In Section 2.5 we show how algorithmic probability can be combined with a recent result about differences of Omega numbers in order to derive a surprising fact about the probability of undecidable sentences in arithmetic, in the context of Chain’s thought experiment. In Section 2.6 we discuss Tadaki’s ‘watered-down’ versions of Omega and in Section 2.7 we present an analogue of Omega for the class of computably enumerable sets.

2.1 Randomness, Incompressibility and Unpredictability of Omega

The halting probability of a universal Turing machine is an interesting concept, but what is more interesting is its mathematical and algorithmic properties. Since the halting set is computably enumerable, it follows that $\Omega_U$ is the limit of an increasing computable sequence of rationals – it is a left-c.e. real. Chaitin [38] also showed that it is algorithmically random in the most standard sense – Martin-Löf random. Algorithmic randomness is a negative concept, characterized by avoidance of statistical tests, unpredictability with respect to effective betting strategies, or incompressibility with respect to effective compression.

Intuitively, algorithmic randomness means lack of algorithmically identifiable properties. In contrast, the fact that Omega is left-c.e. is a positive property, suggesting that this number is constructible in some algorithmic sense. It is this contrast that makes Omega special. Moreover, the halting probability has a specific mathematical meaning, which contrasts the intuition that algorithmically random objects are unidentifiable. The Omega characterization (1) showed that these two opposing properties – algorithmic randomness and computable enumerability of the left Dedekind cut – characterize the Omega numbers.

2.2 Omega in the left-c.e. reals

Solovay [92] initiated the study of Omega as a member of the class of left-c.e. reals. In order to compare left-c.e. reals according to how fast they can be approximated by monotone rational sequences, he defined the Solovay reducibility on c.e. reals, where $\alpha$ is Solovay reducible to $\beta$ if from any good rational approximation $q < \beta$ to $\beta$ we can effectively obtain a good approximation $f(q) < \alpha$ to $\alpha$:

**Solovay reducibility** $\alpha \lesssim_S \beta$: there exists a partial computable function $f$ and a constant $c$ such that for each $q < \beta$ we have $f(q) \downarrow < \alpha$ and $\alpha - f(q) < c \cdot (\beta - q)$.

Moreover he showed that Chaitin’s Omega is of complete Solovay degree i.e., roughly speaking, any good approximation to Omega encodes a good approximation to any given left-c.e. real. One of the by-products of the proof of the Omega characterization (1) was the converse of the latter statement, giving a characterization of Omega numbers as the complete left-c.e. reals with respect to Solovay reducibility, i.e. the maximum elements with respect to this preorder.

Coarser reducibilities measuring randomness were introduced in [59], giving further characterizations of Omega numbers as the complete left-c.e. reals in certain degree structures induced by the reducibilities. For example,

\[\text{a left-c.e. real } \beta \text{ is an Omega number if and only if } K(\alpha \uparrow_n \mid \beta \uparrow_n) = O(1) \text{ for all left-c.e. } \alpha \] (2)

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10 Before Chaitin’s discovery, the most concrete Martin-Löf random real known was a 2-quantifier definable number exhibited in Zvonkin and Levin [98].
i.e. if each initial segment of any real $\alpha$ is basically coded in the corresponding initial segment of $\beta$, modulo a fixed-length program. Finally,

a left-c.e. real $\beta$ is an Omega number if and only if its prefix-free initial segment complexity dominates the initial segment prefix-free complexity of any other left-c.e. real $\alpha$ modulo an additive constant, i.e. if $K(\alpha \uparrow_n) = K(\beta \uparrow_n) + O(1)$ for all left-c.e. reals $\alpha$. Additional properties that characterize the Omega numbers inside the class of left-c.e. reals will be explored in Section 3.

2.3 Undecidability in formal systems and Omega

Chaitin [39, 42] expressed his incompleteness theorem in terms of Omega: any formal system can determine only finitely many bits of Omega.\(^{11}\) He also combined this fact with the mathematical expressions of Omega (e.g. in terms of exponential diophantine equations) in order to argue again for the existence of randomness in mathematics (see [43] for many discussions on this). Solovay [92] used a fixed-point construction in order to produce a version of Omega for which for which $ZFC$ cannot predict a single bit. Calude [26] combined Solovay’s construction with the Omega characterization in order to show that any Omega number, is the halting probability of a certain universal prefix-free machine (provably in Peano Arithmetic) such that $ZFC$ cannot prove any statement of the form ‘the $i$th bit of the binary expansion of Omega is $k$’ for any position $i$ in the binary expansion of Omega after the maximal prefix of 1s.

2.4 Algorithmic probability and Omega

Solomonoff [90] defined algorithmic probability in the context of inductive inference (see [81, Chapter 4] for an up-to-date presentation). The context here is that a universal machine gives a natural distribution of weight to the various finite strings (which can be seen as codes for finite objects) which is often called the a priori distribution. This is only a semi-measure (the total weight is less than 1) and although it can be normalized into a probability distribution, as a semi-measure it enjoys appealing effective properties: it is lower semi-computable or, in our terminology, left-c.e. as a function (it can be effectively approximated from the left). Given any string $\sigma$, the a priori probability $P(\sigma)$ of $\sigma$ is the weight of all the strings in the domain of the universal prefix-free machine which output $\sigma$, i.e.

$$P(\sigma) = \sum_{U(\rho) \downarrow = \sigma} 2^{-|\rho|}$$

Equivalently, $P(\sigma)$ is the probability that the universal machine will output $\sigma$, if supplied with random bits as a program. Intuitively, the more mass $\sigma$ accumulates, the less complex it is expected to be. There is a simple but important result in algorithmic information theory which exemplifies this intuition: the prefix-free complexity $K(\sigma)$ of $\sigma$ equals (modulo an additive constant) the negative logarithm of its a priori probability $P(\sigma)$. Then the algorithmic probability $\Omega(A)$ of a set $A$ – the probability that the universal machine outputs a string in $A$ – is simply the sum of all $P(\sigma)$ for $\sigma \in A$ or equivalently:

$$\Omega(A) = \sum_{U(\rho) \downarrow \in A} 2^{-|\rho|}$$

\(^{11}\)Gács [66] correctly points out that this is a consequence of Levin’s classic work on randomness and left-c.e. semi-measures. A stronger version of incompleteness in terms of Kolmogorov complexity is discussed in [80, Section 2.7.1].
The properties of this number will be discussed in Section 5.1.

In this context, many properties of strings or streams involving prefix-free complexity can be expressed in terms of probabilities of events in the above thought experiment which involves running the universal prefix-free machine on a random program. For example, $K(n)$ (meaning the prefix-free complexity of the string consisting of $n$ many 0s) is, modulo an additive constant, the negative logarithm of the weight of all strings in the domain of $U$ such that $|U(\sigma)| = n$. Hence

$$K(n) = \text{negative logarithm of the probability that the randomized universal prefix-free machine outputs a string of length } n.$$  

One can also express conditional probability in terms of conditional prefix-free complexity. For example, consider the property of a real $X$ that $K(X \upharpoonright n | n^*) = O(1)$, where $n^*$ denotes the shortest program for $n$, or equivalently the pair $(n, K(n))$. Then for each $X$ we have

$$K(X \uparrow n | n^*) = O(1) \text{ if and only if the probability of obtaining output } X \upharpoonright n, \text{ provided that the output has length } n, \text{ has a positive lower bound.}$$

Here is why: by symmetry of information we have $K(X \uparrow n | n^*) = K(X \uparrow n) - K(n)$, while the probability of the property in (4) is the probability of output $X \uparrow n$ over the probability of an $n$-bit output, i.e. $2^{-K(X \uparrow n)}/2^{-K(n)}$. Hence $K(X \uparrow n | n^*) = O(1)$ means that $K(X \uparrow n) = K(n) + O(1)$, which in turn is another way to say that $2^{-K(X \uparrow n)}/2^{-K(n)}$ has a positive lower bound. Incidentally, the property in (4) is no other than the well-known and studied $K$-triviality (see [62, Chapter 11] or [86, Chapter 5]). Hence the $K$-trivial streams are those whose initial segments are very likely to be produced (in the above precise sense) when the universal prefix-free machine is run on a random input.

The study of $K$-triviality has been a significant part of research in algorithmic randomness in the last 15 years, and it is an area where Chaitin’s Omega often plays an important role – see [5] or the results discussed in Sections 5.2 and 4.4.

### 2.5 Algorithmic probability in formal systems and Omega

Let us now look at algorithmic probability in the context of formal systems. The strings $\sigma$ which are described by the universal prefix-free machine may now be viewed as sentences in the language of arithmetic. Then the algorithmic probability of the provable sentences is some version of Omega, i.e. the halting probability of some other universal prefix-free machine. This is simply because the set of provable sentences is a c.e. set $A$ and, as observed by Chaitin [44], $\Omega(A)$ is left-c.e. and 1-random when $A$ is computably enumerable and nonempty. A more surprising fact is the following:

The algorithmic probability of the set of undecidable sentences in formal arithmetic is left-c.e. and 1-random. (5)

This is curious: the undecidable sentences are not effectively verifiable or falsifiable, i.e. neither they nor their negations can be obtained by generating all sufficiently long proofs in arithmetic. Yet their algorithmic probability behaves as the algorithmic probability of the theorems of formal arithmetic, being effectively approximable from the left as if they were effectively enumerable.

This curious result is equivalent to saying that $\Omega_U(B)$ is left-c.e. and 1-random when $B$ is any non-empty effectively closed set. Note that $\Omega_U(B)$, just as the algorithmic probability of the undecidable sentences in
formal arithmetic, is a difference of left-c.e. reals. The proof of (5) appeared in [12] and relies heavily on the understanding of differences of halting probabilities and their approximations as these are discussed in Section 3.3. On the top of these facts, the proof requires what is known as a decanter argument, which is a sophisticated method mostly employed for the study of computationally weak sets, such as the $K$-trivial sets, and was originally introduced in [61] for the establishment of the incompleteness of the $K$-trivial sets.

2.6 A weakly random version of Omega

Tadaki [94] defined ‘watered-down’ versions of Omega, in the sense that they are less random and have faster approximations, as

$$\Omega^s = \sum_{U(\sigma)} 2^{-|\sigma|} \quad \text{for } s \in (0, 1]$$

and he showed that $\Omega^s$ is weakly $s$-random in the sense that

$$\exists c \forall n K(\Omega^s \upharpoonright n) \geq s \cdot n - c$$

Equivalent conditions of (6) in terms of Martin-Löf tests or effective betting strategies, just as in the usual notion of Martin-Löf randomness, can be found in [62, Section 13.5], where a direct relation to effective Hausdorff dimension is also established.

Note that when $s < 1$, the number $\Omega^s$ is less compressed than Omega because each convergent computation adds a smaller amount to the probability. In [96] the analogue of the Omega characterization (1) was carried out for these weaker versions of Omega, including the characterization in terms of speed of convergence (mainly the results in [91, 24, 76] which are discussed in Section 3 of the present article). Further work on weak versions of Omega can be found in [32] while a different version of halting probability is introduced in [34].

2.7 Analogues of Omega in the computably enumerable sets

Is there an analogue of Omega in the c.e. sets? Here we are not necessarily looking for a probability, but a real which is computably enumerable as a set, and whose initial segments are universal or maximally complex in some sense akin to the properties of Omega explored in Section 2.2. We can either look at the initial segment complexity of the c.e. sets and ask that it is sufficiently high, or look at a reducibility amongst c.e. sets that measures complexity, and consider the complete sets with respect to this reducibility (if they exist). Our answer will satisfy both of these heuristics, and this fact will support our bid for an analogue of Omega in the c.e. sets.

Considering the reducibilities of Section 2.2, we start with Solovay reducibility on the c.e. sets, which measures hardness of approximation. Hence we could look for the c.e. set analogue of Omega in the class of c.e. sets which are the hardest to approximate in this context. Unfortunately, it was discovered in [1] that there is no complete c.e. set in the Solovay degrees and, even worse, for each c.e. set $A$ there exists a c.e. set $B$ of strictly larger Solovay degree than $A$. Hence we need to consider a coarser reducibility.

Consider the reducibility implicit in (2), which was introduced in [59] by the name of relative $K$ reducibility: $X \leq_{rK} Y$ when $K(X \upharpoonright n \mid Y \upharpoonright n) = O(1)$; moreover this is equivalent to $C(X \upharpoonright n \mid Y \upharpoonright n) = O(1)$. We have seen that amongst the left-c.e. reals, Omega numbers are characterized as the $\leq_{rK}$-complete left-c.e. reals.

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Is there an $rK$-complete c.e. set, i.e. a c.e. set $A$ such that $K(W \upharpoonright_n | A \upharpoonright_n) = O(1)$ for all c.e. sets $W$? Surprisingly, the answer is yes by [8], and moreover the following are equivalent for any c.e. set $A$:

(a) $A$ is Turing complete with respect to a linear oracle-use function;$^{12}$

(b) $K(W \upharpoonright_n | A \upharpoonright_n) = O(1)$ or equivalently $C(W \upharpoonright_n | A \upharpoonright_n) = O(1)$ for all c.e. sets $W$;

(c) there exists $c$ such that $C(A \upharpoonright_n) \geq \log n - c$ for all $n$.

Given that every c.e. set $W$ has a constant $c$ and infinitely many $n$ with $C(W \upharpoonright_n) \leq \log n + c$ this is arguably the best possible candidate as an analogue of Omega in the c.e. sets in terms of initial segment complexity.$^{13}$ Moreover there are natural sets in this class, namely the halting sets of universal machines with respect to Kolmogorov numberings.$^{14}$ Hence the halting sets with respect to Kolmogorov numberings are analogues of Omega in the c.e. sets.

There are further connections between such canonical halting sets and Chaitin’s Omega. Consider a computable order $g$ (i.e. nondecreasing and unbounded) such that $\sum_i 2^{-g(i)}$ is an Omega number. Such functions were studied in [20, 21, 22] by the name of Solovay functions and, given that Omega numbers have the slowest approximations amongst the left-c.e. reals, they are very slow-growing. In [7] it was shown that a left-c.e. real is an Omega number if and only if it computes a halting set with respect to a Kolmogorov numbering with oracle-use a non-decreasing Solovay function.

In the same paper it was shown that every halting set with respect to a Kolmogorov numbering computes Omega with use $O(2^n)$. The reader may compare these results with Tadaki [95] who considered halting sets $W$ of universal prefix-free machines, instead of the more compact halting sets of universal plain Turing machines. It was shown that in the prefix-free case, about $2^{n+2\log n}$ bits of $W$ are needed for the computation of $\Omega \upharpoonright_n$, and that $2^{n+\log n}$ do not always suffice. On the other direction, the first $n$ bits of Omega compute the first $2^n$ bits of $W$, and this is optimal up to a multiplicative constant on $2^n$.

3 Computable approximations to Omega

The Omega characterization (1) relied on the study of the monotone computable approximations properties of Omega, which was initiated by Solovay in [91] and was briefly discussed in Section 2.2. In the present section we discuss the approximation to Omega more thoroughly, including some recent results.

3.1 Rates of convergence amongst left-c.e. reals

Recall the Solovay reducibility from Section 2.2 which was used in order to measure the speed of left-c.e. approximations to left-c.e. reals. Solovay showed that the induced degree structure, known as the Solovay

---

$^{12}$One can define $X \leq_{lin} Y$ if $X$ is Turing computable from $Y$ with linear oracle-use $n \mapsto an + b$ (or equivalently $n \mapsto an$) for some positive constants $a, b$. Then $A$ is linearly complete in the c.e. sets if $W \leq_{lin} A$ for all c.e. sets $W$.

$^{13}$Here we also note that the plain initial segment complexity of any c.e. set is bounded above by $2 \log n + O(1)$ and this is optimal up to an additive constant $[14, 77]$. With respect to prefix-free complexity the upper bound is $(2 + \epsilon) \cdot \log n$ for any $\epsilon > 0$ and it sometimes fails for $\epsilon = 0$ by [6].

$^{14}$Kolmogorov numbering is a Gödel numbering to which every other computable numbering can be reduced via a linearly bounded function.
degrees, has a maximum element and he called the members of the maximum degree Ω-like reals. He also showed that any universal halting probability is Ω-like. Then the work in [24, 76] showed that the Ω-like reals are exactly the halting probabilities of universal prefix-free machines.\(^\text{15}\)

The proof that the omega numbers are exactly the halting probabilities of universal prefix-free machines is, in a sense, non-uniform. Given a left-c.e. index of a 1-random left-c.e. real α in \((0, 1)\) the known argument obtains an index of a universal prefix-free machine \(U\) such that \(\Omega_U = \alpha\) by non-effective means.\(^\text{16}\)

The question whether this non-uniformity is necessary, is open. In fact, many results that touch on this characterization of omega numbers are proved by non-uniform arguments, though the necessity of this non-uniformity has not been established. For example, in [13] it was shown that for each universal prefix-free machine \(U\) there exists another universal prefix-free machine \(V\) such that \(\Omega_U \neq \Omega_V + \beta\) for every left-c.e. real – the proof was non-uniform and the necessity of the non-uniformity was left open.

The Solovay reducibility \(\beta \leq_S \alpha\) between left-c.e. reals \(\alpha, \beta\) can be defined equivalently by any of the following clauses:

(a) there exists a rational \(q\) such that \(q \alpha - \beta\) is left-c.e.

(b) there exist a rational \(q\) and \((\alpha_s) \to \alpha, (\beta_s) \to \beta\) such that \(\beta - \beta_s < q \cdot (\alpha - \alpha_s)\) for all \(s\);

(c) there exist a rational \(q\) and \((\alpha_s) \to \alpha, (\beta_s) \to \beta\) such that \(\beta_{s+1} - \beta_s < q \cdot (\alpha_{s+1} - \alpha_s)\) for all \(s\).

Note that the set of rationals \(q\) for which one of the above clauses holds is upward closed - if the clause holds for the rational \(q\) then it also holds for all rationals \(q' > q\). Although it is not explicitly stated in [60], it follows from the proofs that when \(\beta \leq_S \alpha\), the infimums of the rationals \(q\) for which the clauses (a), (b) and (c) hold are equal. A thorough study of the algebraic aspects of the structure of Solovay degrees of left-c.e. reals was undertaken in [60, 58].

There are many ways in that any approximation to Omega is very much slower than any approximation to any left-c.e. real which is not an Omega number. For example, it was shown in [60] (also see [13]) that if \((\alpha_s), (\Omega_s)\) are any left-c.e. approximations to \(\alpha, \Omega\) respectively then

\[
\alpha \text{ is not an Omega number } \iff \lim_{s} \frac{\alpha - \alpha_s}{\Omega - \Omega_s} = 0. \tag{7}
\]

This shows that Omega is much more intractable than any left-c.e. real which is not an Omega number.

### 3.2 Initial segment complexity of Omega numbers

The intuition that Omega is considerably more intractable than any left-c.e. real which is not an Omega number is also reflected in a result of [53] (also see [60] and [76, Remark 3.5]) which says that

\[
\text{Omega cannot be written as the sum of two left-c.e. reals which are not Omega numbers.} \tag{8}
\]

\(^\text{15}\)The same argument shows that this equivalence is not sensitive to many features of the universal machine. For example, the equivalence holds if we consider a universal oracle Turing machine and define its halting probability as the measure of oracles \(X\) which make the machine halt with input the empty string. The same can be said of other models such as the monotone machines of Levin [78, 79] and even the optimal machines [81, Definition 2.1] in the sense of Kolmogorov complexity. However when considering the halting probability restricted to a \(\Pi^0_1\) set of outputs, the randomness of this real is robust for universal machines by [12] but may not be robust for mere optimal machines by [64].

\(^\text{16}\)One constructs a certain Martin-Löf test \((V_i)\) and uses a number \(n\) such that \(\alpha \notin V_i\) for all \(i > n\) in order to define a universal prefix-free machine \(U\) such that \(\Omega_U = \alpha\). The main source of non-uniformity is the choice of this number \(n\).
We note that by [61], for any left-c.e. reals $\alpha, \beta$, the prefix-free complexity of $\alpha + \beta$ is (within a constant) the maximum of the prefix-free complexities of $\alpha$ and $\beta$. Hence, using the definition of randomness in terms of prefix-free complexity, another way to write (8) is:

$$\text{if } \alpha, \beta \text{ are left-c.e. and not random then } \limsup_n \left( n - \max\{K(\alpha \upharpoonright_n), K(\beta \upharpoonright_n)\} \right) = \infty.$$ 

In other words, if $\alpha, \beta$ are left-c.e. and not Omega numbers, their complexity at position $n$ simultaneously drops well below $n$ for infinitely many $n$. It is interesting that by [5], a similar property occurs with regard to the left-c.e. reals which are not $K$-trivial:

$$\text{if } \alpha, \beta \text{ are left-c.e. and not } K\text{-trivial then } \limsup_n \left( \min\{K(\alpha \upharpoonright_n), K(\beta \upharpoonright_n)\} - K(n) \right) = \infty.$$ 

In other words, if two left-c.e. reals have non-trivial prefix-free complexity, then there are infinitely many lengths $n$ at which their complexity simultaneously rises well above $K(n)$.

It is tempting to seek a strengthening of (8) in the statement that initial segment prefix-free complexity of any non-random left-c.e. real diverges from the initial segment prefix-free complexity of Omega.\footnote{This question was asked by Kenshi Miyabe during the conference Aspects of Computation organized by the Institute of Mathematical Sciences in Singapore in August and September 2017, where Yu Liang pointed to the negative answer we describe below.} Such a result would reinforce the intuition that Omega is much more complex than any non-random left-c.e. real. Unfortunately, results from [59, 9] show that this is not true:

$$\text{There exists a left-c.e. real } \alpha \text{ which is not 1-random but } \liminf_n \left( K(\Omega \upharpoonright_n) - K(\alpha \upharpoonright_n) \right) < \infty. \quad (9)$$

Indeed, in [9] it was shown that there are left-c.e. reals which are not computed by any Omega number with oracle-use $n \mapsto n + O(1)$. Then (9) follows from the above result combined with the theorem in [59] that if $\lim_n (K(\Omega \upharpoonright_n) - K(\alpha \upharpoonright_n))$ is infinite and $\alpha$ is a left-c.e. real then $\alpha$ is computable from $\Omega$ with oracle-use $n \mapsto n + O(1)$.

Solovay [91] produced a number of technical results regarding the initial segment complexity of Omega, which are reproduced in [62, Section 10.2]. A consequence of these results is that the initial segment prefix-free complexity of Omega does not bound above the initial segment prefix-free complexity of any 2-random real, i.e. any real which is random relative to the halting problem. In fact, we can elaborate on this statement. Since $\Omega$ is 1-random, we have that $K(\Omega \upharpoonright_n)$ is in the interval $(n, n + K(n))$ (modulo an additive constant). By Chaitin [40] we have

$$\lim_n \left( K(\Omega \upharpoonright_n) - n \right) = \infty$$

and this, in fact, is a characterizing property of every 1-random real number. On the other hand, Miller [83] showed that $\liminf_n (n + K(n) - K(X \upharpoonright_n)) < \infty$ is equivalent to the property that $X$ is 2-random, hence

$$\liminf_n \left( n + K(n) - K(\Omega \upharpoonright_n) \right) = \infty.$$ 

These results highlight the fact that Omega is not as random as a typical algorithmically random number, a theme that is explored further in Section 4.4.
3.3 Comparing the rates of convergence amongst Omega numbers

There is one aspect of approximations to Omega that remained unresolved until very recently, where it was settled in [12]. We have seen in (10) that Omega numbers are in their own league as far as the rate of their monotone effective approximation is concerned. But how do the approximations to two different Omega numbers related? We know by (7) that they are both very slow compared to any approximation to any real which is not an Omega number, but can we compare the two rates with each other? From the Omega characterization, and in particular from [76], we know that given any left-c.e. approximations \((\omega_s) \to \omega, (\Omega_s \to \Omega)\) to two Omega numbers, we have \(\liminf \left[ (\Omega - \Omega_s)/(\omega - \omega_s) \right] > 0\).18 But does this limit exist?

Quite remarkably, in [12] it was shown that given any left-c.e. approximations \((\alpha_s) \to \alpha, (\beta_s) \to \beta\), \(\alpha, \beta\) are Omega numbers \(\Rightarrow \mathcal{D}(\alpha, \beta) := \lim_{s} \frac{\alpha - \alpha_s}{\beta - \beta_s}\) exists and is positive and independent of the chosen approximations \((\alpha_s) \to \alpha, (\beta_s) \to \beta\). (10)

Moreover this limit \(\mathcal{D}(\alpha, \beta)\) has a rather special property – it is the
- infimum of all rationals \(q\) with the property that \(q \cdot \beta - \alpha\) is a left-c.e. real;
- supremum of all rationals \(p\) such that \(p \cdot \beta - \alpha\) is a right-c.e. real.

In fact, given two Omega numbers \(\alpha < \beta\), the value of \(\mathcal{D}(\alpha, \beta)\) determines whether \(\beta - \alpha\) is an Omega number:
- if \(\mathcal{D}(\alpha, \beta) < 1\) then \(\beta - \alpha\) is an Omega number;
- if \(\mathcal{D}(\alpha, \beta) \geq 1\) then \(\beta - \alpha\) is not an Omega number;

More specifically, we have the following trichotomy:
- if \(\mathcal{D}(\alpha, \beta) > 1\) then \(\beta - \alpha\) is right-c.e. and 1-random;
- if \(\mathcal{D}(\alpha, \beta) < 1\) then \(\beta - \alpha\) is left-c.e. and 1-random;
- if \(\mathcal{D}(\alpha, \beta) = 1\) then \(\beta - \alpha\) is not right-c.e. or left-c.e. or 1-random;

Based on (10), a theory of derivation for the field of differences of left-c.e. reals was developed in [84].

3.4 Speeding-up the approximation to Omega

We are interested in functions \(f\) which grow sufficiently fast so that we can use them to approximate Omega faster than the universal effective approximation. This means that

\[
\text{for each } c \text{ there exists } n \text{ such that } \Omega - \Omega_{f(n)} < 2^{-c} \cdot (\Omega - \Omega_n). \tag{11}
\]

It is not hard to see that no computable function \(f\) has this property. In fact, it was shown in [5] that if \(f\) is low for \(\Omega\) (i.e. it is computable from a set relative to which \(\Omega\) is random) then it does not have the above property. It was also shown that every c.e. degree which is not \(K\)-trivial contains a function \(f\) such that

\footnote{This holds even if \(\omega\) is not an Omega number.}
(11) holds. Since the low for $\Omega$ c.e. degrees are exactly the $K$-trivial c.e. degrees, it follows that in the c.e. degrees exactly the ones that are not $K$-trivial compute (or even contain) functions $f$ with the property (11).

A c.e. degree can speed-up the approximation to Omega if and only if it is not $K$-trivial. (12)

Moreover, if a c.e. set $A$ is not $K$-trivial then there exists a canonical function in the degree of $A$ with the property (11), namely the settling time in any computable enumeration of $A$.

This is interesting! Combined with known characterizations of the low for $\Omega$ sets (see Section 5.2), we have that the following are equivalent for c.e. sets:

(a) $A$ can speed-up the approximation to Omega;
(b) $\Omega^A_U$ is not a left-c.e. real;

where $U$ is any universal prefix-free machine.

Can the characterization (12) be generalized outside the class of c.e. degrees? A reasonable guess, given the facts from [5], would be to conjecture that a degree computes $f$ with the property (11) if and only if it is not low for $\Omega$. Although one direction of this equivalence is true, surprisingly the other is not. Miller and Nies, see [86, Section 8.1], showed that non-computable low for $\Omega$ sets are necessarily hyperimmune, i.e. they compute functions which are not dominated by any computable function. This means that non-computable hyperimmune-free degrees are not low for $\Omega$, and these clearly do not compute functions with the property (11), since every function they compute is dominated by a computable function.

On the other hand, given that there exists a function $f \leq_I \theta'$ such that $\Omega - \Omega_{f(n)} < 2^{-n} \cdot (\Omega - \Omega_n)$, it is not hard to see that every array non-computable degree can speed-up the approximation to Omega. The array computable degrees contain the low for Omega degrees, but also contain other degrees which can speed-up the approximation to Omega (such as c.e. degrees which are not $K$-trivial). The question of exactly which degrees can speed up the approximation to Omega remains open.

4 Omega and computable enumerability

We have discussed that Omega can be seen as a compressed version of the halting problem, and moreover it enjoys several completeness properties with respect to the left-c.e. reals. In this section we present a number of results that make the connection between Omega and c.e. sets or other left-c.e. reals precise. In Section 4.1 we discuss the problem of how many bits of Omega are needed for the computation of $n$ bits of a c.e. set like the halting problem, or a left-c.e. real, and vice-versa how many bits of a c.e. set are needed in order to compute $n$ bits of Omega. In Section 4.3 we give a thorough examination to the question of how similar and how different two universal halting probabilities can be.

4.1 Omega and halting problems

We have seen that many open problems in mathematics have their solutions coded in the first few thousands bits of a canonical version of Omega, while some authors have worked-out more precise bounds of this type. Technically speaking, given that such solutions are merely answers to certain halting problems, the issue here is which initial segment of Omega is sufficient to answer a given halting problem. It is possible to
obtain asymptotics that show the lengths of the initial segments of Omega in relation to the amount of halting problems that they encode. The first results of this type were obtained by Solovay [91] and first published in [62, Section 3.13]. If we define

\[ p(n) = |\{ \sigma \in 2^n \mid U(\sigma) \downarrow \}| \]
\[ \mathcal{D}_n = \{ \sigma \in 2^{\leq n} \mid U(\sigma) \downarrow \} \quad \text{and} \quad P(n) = |\mathcal{D}_n| \]

then

\[ p(n) \sim P(n) \sim 2^{n-K(n)} \]
\[ \mathcal{D}_n \]

is uniformly computable from \( \Omega \upharpoonright n \). Moreover

\[ K(\Omega \upharpoonright n \mid \mathcal{D}_n + K(n)) = O(1) \]

which means that the first \( n + K(n) \) bits of the halting problem can be used for the computation of the first \( n \) bits of Omega (with an additional program of fixed length which might not be uniformly given in \( n \)).

Tadaki [95] improved on the latter result by showing the following characterization.\footnote{Formally, there exists an oracle Turing machine \( M \) such that for each \( n \) we have \( M(\mathcal{D}_n + f(n) + O(1))(W)(n) = \Omega \upharpoonright n \).}

Given optimal machines \( W, V \) and a computable function \( f \), we have \( \sum_i 2^{-f(i)} < \infty \) if and only if \( \mathcal{D}_n(W) \) uniformly computes the first \( n - f(n) - c \) bits of \( \Omega_V \) for some constant \( c \).

(13)

Here we may use \( 2 \log n \) as a simple representative of the functions \( f \) with the property \( \sum_i 2^{-f(i)} < \infty \).

Then (13) says that given any optimal machines \( W, V \), in order to compute \( n - 2 \log n \) bits of the halting probability with respect to \( V \), we need to know the halting problem with respect to \( W \), for all strings of length at most \( n \).

Tadaki also gave an analogous result concerning computations of halting sets from halting probabilities.

Given optimal machines \( W, V \) and a computable function \( f \), we have \( f = O(1) \) if and only if the first \( n \) bits of \( \Omega_V \) uniformly compute \( \mathcal{D}_{n+f(n)-c}(W) \), for some constant \( c \).

In other words, the first \( n \) bits of the halting probability of \( V \) can only solve the halting problem of \( W \) for the inputs of length at most \( n \) (plus or minus a constant). We stress Tadaki’s results apply to arbitrary optimal machines – not necessarily universal, and characterize the computational relation between universal halting problems and universal halting probabilities for prefix-free machines.

### 4.2 Omega and left-c.e. reals or c.e. sets

Since Omega is Turing-complete, it computes all left-c.e. reals and all c.e. sets. We can then obtain asymptotics regarding these computations. How many bits of Omega are needed in order to compute \( n \) bits of an arbitrary left-c.e. real or a arbitrary c.e. set?

In [7] it was shown that given any computable \( h : \mathbb{N} \to \mathbb{N} \),

\[ \text{if } \sum_n 2^{n-h(n)} < \infty \text{ converges, then Omega computes every left-c.e. real } \alpha \text{ with oracle-use } h. \quad (14) \]

Moreover the converse of this implication was shown under the additional assumption that \( h(n) - n \) is non-decreasing: if the sum in (14) is unbounded, then there are left-c.e. reals which are not computable from Omega with oracle-use \( h \). In fact, the following stronger statement was obtained:

\[ \text{If } \sum_n 2^{n-h(n)} = \infty \text{ then there exist two c.e. reals such that no left-c.e. real can compute both of them with oracle-use } h + O(1). \]
Hence we see that Omega computes all left-c.e. reals with oracle-use \( n + 2 \log n \) but oracle-use \( n + \log n \) is not always sufficient for this purpose. There is also a uniform (hence stronger) version of the latter fact, which was obtained recently by Fang Nan:\(^{20}\)

There exists a left-c.e. real which are not computed by *any* Omega number with oracle-use 
\[ n \mapsto n + g(n), \]
for any computable non-decreasing function \( g \) such that \( \sum_n 2^{-g(n)} = \infty \).

Analogous results were obtained in [7, 73] with respect to c.e. sets, although now the oracle-uses are appropriately tighter. Given any computable function \( g \),

\[
\text{if } \sum_n 2^{-g(n)} < \infty, \text{ then every c.e. set is computable from Omega with oracle-use } g.
\]

Here, just as in (14), the statements hold for all versions of Omega. A strong converse is also given, under the additional assumption that \( g \) is nondecreasing:

\[
\text{if } \sum_n 2^{-g(n)} = \infty \text{ then Omega cannot compute all c.e. sets with oracle-use } g. \text{ In fact, in this case, no linearly complete c.e. set can be computed by any c.e. real with oracle-use } g.
\]

One can draw interesting conclusions about the computational relation between halting probabilities and halting problems from the last two results. Note that the halting set \( H \) with respect to plain Turing machines and a Kolmogorov numbering of all programs, is linearly complete. Hence any such canonical halting problem \( H \) is not computable by Omega with oracle-use \( \log n \) but it is computable by Omega with oracle-use \( 2 \log n \). We may also contrast these results with Tadaki’s results that we discussed in Section 4.1, which referred to halting problems with respect to prefix-free machines.

These results give worse-case bounds on the number of bits of Omega needed for the computations. One would guess that the number of bits of Omega that are needed for the computation of \( n \) bits of a left-c.e. real or a c.e. set \( X \) should depend on how much information is encoded in \( X \upharpoonright_n \). The latter, in turn, would be reflected in the Kolmogorov complexity of the first \( n \) bits of \( X \). An upper bound on the oracle-use along these lines was obtained in [6]:

\[
\text{Every left-c.e. real } X \text{ can be computed from } \Omega \text{ with oracle-use } g(n) = \min_{i \geq n} K(X \upharpoonright_i). \tag{15}
\]

This is remarkable! The prefix-free complexity \( K(X \upharpoonright_n) \) is supposed to measure the amount of information – in bits – encoded in the first \( n \) bits of \( X \). Then (15) says that the number of bits of \( \Omega \) containing the information \( X \upharpoonright_n \) is \( K(X \upharpoonright_n) \), i.e. precisely the amount of information in \( X \upharpoonright_n \); and \( \Omega \) has this property with respect to every left-c.e. real \( X \). This is another testimony of the compactness of retrievable information in the initial segments of Omega.

### 4.3 How similar or different are two Omega numbers?

In Kolmogorov complexity we know that changing universal machines is like changing coordinate system in geometry: the theory remains the same, and the complexity function only changes by a constant. How is the change of the universal machine reflected in the halting probability? How similar or different are the halting probabilities with respect to two different universal machines?

\(^{20}\)A special case of this result for oracle-use \( n \mapsto n + O(1) \) was obtained earlier in [9].
We know that all (universal) halting probabilities are in the same Turing degree, the degree of the halting problem. So they can compute each other. Moreover the oracle-use functions in these reductions are computably bounded, so all Omega numbers are in the same weak truth-table degree as the halting problem \([25]\). However these facts no longer hold for truth-table reductions. For example, the halting problem is not truth-table reducible to any Omega number \([25]\).\(^{21}\)

In \([64]\) it was shown that there are universal \(U, V\) such that the corresponding halting probabilities have incomparable truth-table degrees. In \([13]\) it was shown that for each universal prefix-free machine \(U\) there exists another universal prefix-free machine \(V\) such that \(\Omega_U \neq \Omega_V + \beta\) for every left-c.e. real. This shows that given two Omega numbers, one is not necessarily the translation of the other by a left-c.e. real.

But how many bits of \(\Omega_U\) are needed to compute \(\Omega_V\) given any two universal prefix-free machines \(U, V\)? In \([11]\) it was shown that for each \(\epsilon > 1\) the oracle-use function \(n \mapsto n + \epsilon \cdot \log n\) suffices for this computation while, in general, \(n \mapsto n + \log n\) does not.\(^{22}\)

### 4.4 Computational power versus randomness of Omega

We have discussed the computational power of Omega in Sections 1.3 and 4. From these discussions it is clear that Omega is a rather special algorithmically random sequence: it is Turing-complete\(^{23}\) and a left-c.e. real. On the other hand, intuitively one would expect that algorithmically random numbers do not solve interesting problems such as the halting problem. Denis Hirschfeldt in many of his expository talks distinguishes two kinds of algorithmically random numbers: the ones that are ‘truly’ random and the ones, such as Omega, that pass the Martin-Löf tests because they have enough information so that they can ‘fake’ randomness. In fact, this distinction is quantifiable and reflected in many theorems. If one slightly increases the strength of randomness under consideration, then all random numbers are necessarily incomplete. In other words, Omega has just enough randomness to qualify as Martin-Löf random, but no more. Difference randomness \([65]\) is exactly this slight strengthening of Martin-Löf randomness which characterizes the Martin-Löf random reals which do not compute the halting problem. Furthermore, incomplete Martin-Löf random reals not only fail to compute the halting problem, but are computationally weak in many other ways\(^{24}\) in line with our intuition about algorithmically random numbers.

All these facts show that Omega is a computationally powerful random real, which is necessarily not very random (in the sense that it fails slightly stronger notions of algorithmic randomness). In all of the examples from Sections 1.3 and 4 demonstrating the computational power of Omega, computations use entire initial segments of Omega rather than individual bits. Is this necessary? In other words, can we still solve interesting problems if we only have access to a certain, possibly non-adjacent, bits of Omega? This question

\(^{21}\)This is a consequence of a more general theorem of Demuth \([54]\) which says that any non-computable set which is truth-table reducible to a Martin-Löf random set is Turing equivalent to a Martin-Löf random set.

\(^{22}\)The actual results are more general. Given any Omega number \(\Omega\) there exists another Omega number which is not computable from \(\Omega\) with oracle-use \(n+\log n\). Moreover, given any non-decreasing computable function \(g\) such that \(\sum_i 2^{-g(i)}\) is finite, any Omega number is computable from any other Omega number with oracle-use \(n + n \mapsto g(n)\). An exact characterization of the required worse-case redundancy is still lacking. For example, is oracle-use \(n + \log n + 2\log\log n\) sufficient for the computation of any omega number from any other Omega number?

\(^{23}\)In fact, the standard notion of Martin-Löf randomness that we use allows the existence of random numbers that compute any given sequence. In particular, given any sequence \(A\), by the Kučera-Gács theorem, there exists a Martin-Löf random \(X\) which computes \(A\).

\(^{24}\)For example, they do not compute any complete extension of Peano arithmetic \([93]\).
has a strongly negative answer, for the following reasons. If useful information was coded into individual bits of Omega inside an infinite computable set $A$ with infinite complement, one would be able to solve interesting problems by truth-table queries on the sequence of bits $\Omega_A$ of Omega restricted on positions in $A$. However it is known that such a sequence $\Omega_A$ is Turing-incomplete and Martin-Löf random, so by [54] any non-computable set which is truth-table reducible to $\Omega_A$ is Turing equivalent to an incomplete Martin-Löf random set. But we know that such sets do not code any problems of interest, which are mostly located inside c.e. degrees, and in fact the complete c.e. degree.

Using more advanced results, we can obtain further demonstrations that parts of Omega such as its even bits do not compute useful non-trivial problems, or even any considerably non-trivial problem. Let $\Omega_0, \Omega_1$ be the even and odd bit-sequences of Omega respectively. Then $\Omega_0$ is an incomplete 1-random real below $0'$. Such reals are known to have small computational power – for example, any c.e. set that they compute is $K$-trivial. As a result, not much useful information can be recovered by querying the even bits of Omega. A similar statement is true for any nontrivial computable subset of bits of Omega, which means that information is not coded into individual bits of Omega but rather into its initial segments. Having said this, there are non-computable c.e. sets that are computable from both $\Omega_0$ and $\Omega_1$. Moreover, a c.e. set is computable from $\Omega_0$ if and only if it is computable from $\Omega_1$ [69]. One can generalize this by partitioning the bits of Omega into $k$ many disjoint families, and considering the sets that are computable by all of these $k$ many reals. Interestingly, one then gets a strictly decreasing sequence of nontrivial subclasses of the $K$-trivial sets [68].

5 Halting probability relative to a set

There are more than one ways that one can relativize the halting probability – here we consider two. In Section 5.1 we discuss the probability $\Omega_U(X)$ that the universal prefix-free machine $U$ halts with an output inside a given set $X$, focusing on the randomness properties of it. In Section 5.2 we instead add the given set $X$ as an oracle to the prefix-free machine $U$, and consider the halting probability $\Omega^X_U$ of $U$ with oracle $X$. Finally in Section 4.4 we report on some recent work concerning computations from subsequences of the binary expansion of Omega, and connections with lowness classes and $K$-triviality.

5.1 Restricting the output of a universal prefix-free machine

Given a universal prefix-free machine $U$ we may consider the probability $\Omega_U(X)$ that $U$ halts with output in a given set $X$ of strings. Such probabilities were considered in [74, 15, 64, 12], initially as an attempt to produce concrete numbers which are more random than Chaitin’s Omega. In [15] it was shown that if $X$ is $\Sigma^0_n$-complete or $\Pi^0_n$-complete for any $n > 1$ then $\Omega_U(X)$ is 1-random. On the other hand, in the same paper it was shown that if $n > 1$ and $X$ is any $\Sigma^0_n$ or $\Pi^0_n$ set then $\Omega_U(X)$ is not $n$-random.

The case $n = 1$ is particularly interesting. Chaitin [44] had already noticed that if $X$ is any non-empty $\Sigma^0_1$ set, then $\Omega_U(X)$ is 1-random and left-c.e., so it is just another Omega number. The case when $X$ is $\Pi^0_1$ is the basis of [85, Question 8.10] which was also discussed in [16, 15, 64], and remained open until recently. In [64] an optimal (in terms of Kolmogorov complexity) but not universal prefix-free machine $U$ was constructed, and a $\Pi^0_1$ set $X$ such that $\Omega_U(X)$ is not Martin-Löf random. The question whether $\Omega_U(X)$ is always an omega number when $U$ is universal and $X$ is a non-empty $\Pi^0_1$ set, required a deeper understanding.

[25]The same result for an apparently stronger notion of completeness was obtained earlier in [74].
of omega numbers and their differences. This is hardly surprising since, when $X$ is a $\Pi^0_1$ set the real $\Omega_U(X)$ is the difference of two omega numbers. In [12] it was shown that $\Omega_U(X)$ is Martin-Löf random and left-c.e. i.e. just another omega number when $U$ is universal and $X$ is any $\Pi^0_1$ set. The proof uses the results about differences of omega numbers that we discussed in Section 3 and a decanter argument for the construction of a suitable Martin-Löf test.

Many questions which ask for the randomness strength of $\Omega_U(X)$ when $X$ has some type of universality remain open. For example, if $X$ is uniformly $\Sigma^0_n$-complete for all $n$ (i.e. it is in the many-one degree of $\emptyset^{(\omega)}$) then what level of randomness can we expect from $\Omega_U(X)$?

### 5.2 Halting probability in an oracle prefix-free machine

Another way to relativize the halting probability is to consider universal oracle prefix-free machines $U$ and the probability $\Omega_U^X$ that they halt when equipped with oracle $X$. The study of these numbers, as well as the map $X \mapsto \Omega_U^X$ was initiated in [57]. First, we note that the Omega characterization (1) relativizes to any oracle $X$ in a rather straightforward way. For example, the halting probabilities of universal prefix-free machines with oracle $X$ are exactly the $X$-left-c.e. real numbers which are 1-random relative to $X$. In the same way, Solovay reducibility relativizes to $X$ and the numbers $\Omega^X_U$ are exactly the $X$-left-c.e. real numbers with the slowest $X$-left-c.e. approximations. Second, in [57] it was shown that

$$\Omega \text{ is 1-random relative to } X \text{ iff } \Omega_U^X \text{ is left-c.e. for some universal prefix-free machine } U.$$ 

Sets $X$ with this property are called low for $\Omega$. Recall that $X$ is $K$-trivial if there exists some $c$ such that $K(X | n) \leq K(n) + c$ for all $n$, and note that in general $X \not\equiv_T \Omega_U^X$. A remarkable result from [57] is that

$$X \text{ is } K\text{-trivial iff } \Omega_U^X \text{ is left-c.e. for all universal prefix-free machines } U.$$ 

and moreover, $X$ is $K$-trivial if and only if $X' \equiv_T \Omega_U^X$.

Perhaps the leading motivating question behind [57] was the degree invariance of the operator $X \mapsto \Omega_U^X$. A very strong negative answer was given to this question:

given any universal prefix-free machine $U$, there are sets $A, B$ which only differ on finitely many bits, and such that $\Omega_U^A$ is Turing incomparable to $\Omega_U^B$.

Another very important consequence of this work concerns the invariance of the relativized Omega with respect to the choice of the underlying universal prefix-free machine. Recall that Chaitin’s omega with respect to different universal prefix-free machines are very similar – certainly Turing equivalent. Is this true for the relativized Omega? If the oracle $X$ is $K$-trivial, then it follows from the above discussion that $\Omega_U^X$ is always in the Turing degree of the halting problem. A remarkable fact is that

if $X$ is not $K$-trivial, then there are universal Turing machines $U, V$ such that $\Omega_U^X$ and $\Omega_V^X$ have incomparable Turing degree.

In [83, 10] it was shown that the low for $\Omega$ reals are very related to relativized prefix-free complexity. In particular, the following are equivalent for each $X$:

(a) $X$ is low for $\Omega$;

(b) there exists $c$ such that $K(n) \leq K^X(n) + c$ for infinitely many $n$.
Finally low for $\Omega$ sets $X$ have interesting growth-rate properties, as we briefly mentioned in Section 3.4. If $X$ is non-computable and low for $\Omega$ then it computes a function which is not dominated by any computable function (by Miller and Nies, see [86, Section 8.1]). On the other hand, by [5] if $X$ is low for $\Omega$ then any function $f$ which is computable by $X$ does not speed-up the effective approximations to Omega, in the sense that there exists a constant $c$ such that $\Omega - \Omega_n \leq c \cdot (\Omega - \Omega_{f(n)})$ for all $n$.

### 5.3 Omega operators

Omega can be turned into an operator, mapping binary streams or subsets of $\mathbb{N}$ to reals, in a number of different ways. Perhaps the more natural such examples are the following:

(a) $Z \mapsto \sum_{U(\sigma) \downarrow \in Z} 2^{-|\sigma|}$

(b) $X \mapsto \sum_{U^X(\sigma) \downarrow} 2^{-|\sigma|}$

(c) $X \mapsto \sum_{n} 2^{-K(X|_n)}$

(d) $X \mapsto \sum_{U(\sigma) \downarrow \prec X} 2^{-|\sigma|}$

where $Z \subseteq \mathbb{N}$, $X$ is a binary stream and ‘$\prec$’ denotes the prefix relation.

The main motivation for the study of operator (a) in [16, 15, 17, 18] was the discovery of concrete highly random numbers. Some facts about operator (a) were discussed in Section 5.1. Much about the initial excitement regarding operator (b) had to do with the possibility that it provides a uniform degree-invariant solution to Post’s problem, which is a long-standing question in classical computability theory. As we discussed in Section 5.2, the results in [57] provide a strong negative answer to such a hope.

Several results regarding the analytic behavior of operator (b), focused on its continuity and the complexity of its range, were obtained in [57, Section 9]. It was shown that

- Operator (b) is lower semi-continuous but not continuous;
- Operator (b) is continuous exactly on the 1-generic reals.

Remarkably, the supremum of (b) is achieved, and the argument $X$ that achieves it is always 1-generic. Several questions about the nature of operator (b) remain, and the interested reader is referred to [57].

Recently a study of operator (c) was initiated in [70], where it was shown that

- operator (c) is continuous and almost everywhere differentiable with derivative 0
- operator (c) is co-meagerly non-differentiable and nowhere monotonic.

Note that, strictly speaking, the value of (c) on any real depends on the choice of the underlying universal machine $U$. One can construct a universal machine $U$ with respect to which operator (c) maps any random stream to a non-random real. On the other hand, there exists a stream $X$ which is always mapped to a non-random real via (c), independently of the underlying optimal machine $U$. It was also shown in [70] that the inverse image of (c) on any real is null. Finally we note that operator (d) is very similar to operator (c), and the results we presented about (c) (all from [70]) also hold for (d) by the same or similar proofs.
6 Machine probabilities beyond halting

In turns out that the probability of many properties of a universal randomized Turing machine can be expressed in terms of relativized halting probabilities. This phenomenon was the topic of [3]. In Section 6.1 we give such examples and their characterizations in terms of algorithmic randomness. Then in Section 6.2 we note that although there is usually a rough correspondence between the arithmetical complexity of a property and the strength of the algorithmic randomness of its probability, this is no always the case, even for some very natural properties such as a computable infinite output from a randomized universal machine. Finally in Section 6.3 we note that for properties of higher arithmetical complexity than halting (for example 2-quantifier definable properties) the degree or even the algorithmic randomness or computability of the probability is very sensitive to the choice of underlying universal machine, in contrast with the halting probability.

6.1 Natural machine properties and their universal probabilities

For example, consider an oracle Turing machine $M$ which has one tape for the oracle $X$, one input string $\sigma$ and one output tape for the output string, incase the computation halts. Then we may treat the oracle $X$ as a random variable, so that the machine is viewed as a probabilistic machine. Then we can consider the probability that the partial function $\sigma \mapsto M^X(\sigma)$ has certain properties, when $M$ is universal. The probability that $\sigma \mapsto M^X(\sigma)$ is total (the totality probability of $M$) is the same as the non-halting probability of a universal prefix-free machine with oracle the halting problem $\emptyset'$. In other words, given a universal oracle Turing machine $M$, there exists a universal oracle prefix-free machine $U$ such that the totality probability of $M$ is equal to $1 - \Omega_{U'}^{\emptyset'}$. We depict this fact in Table 1 by saying that totality (of a universal oracle machine) corresponds to $1 - \Omega_{U'}^{\emptyset'}$; in other words, totality probability of a universal oracle machine is a 0'-right-c.e. 2-random real.

Next, consider a universal oracle-machine $M$ which, given an oracle $X$, it enumerates a set of strings $W^X$. Note that this can be formalized as a property of the function $\sigma \mapsto M^X(\sigma)$, for example by letting $W^X$ be the domain (or the range) of this function. Again, if we consider $X$ as a random variable, we can consider the probability that $W^X$ is a computable set. It turns out that this probability is the same as the halting probability of some universal prefix-free machine with oracle $\emptyset''$. We note this fact in Table 1 by saying that the property of probabilistically enumerating a computable set via a universal oracle Turing machine corresponds to $\Omega_{\emptyset''}^{\emptyset'}$. We may continue with many of the properties that one meets in a first course in computability theory such as the property that $W^X$ is co-finite or the property that $W^X$ computes the halting problem. These characterizations are included in Table 1. Similar characterizations may be obtained for different models of Turing machines even when there are no oracles present. For example, given a prefix-free machine $U$, Wallace (see [55, Section 0.2.2] and [56, Section 2.5]) considered the measure of reals $X$ such that the prefix-free machine $\sigma \mapsto U(X \upharpoonright_n *\sigma)$ is universal for all $n$, and called it the universality probability of $U$. In [4] it was shown that the universality probabilities of universal prefix-free machines are exactly the non-halting probabilities relative to $\emptyset^{(3)}$. 
Table 1: Universal probabilities as Omega numbers

| Property                                      | Probability            |
|-----------------------------------------------|------------------------|
| Totality                                      | $1 - \Omega^{(0)}$     |
| Enumeration of a computable set               | $\Omega^{(2)}$         |
| Enumeration of a co-finite set                | $\Omega^{(2)}$         |
| Enumeration of a set which computes $\emptyset'$ | $\Omega^{(2)}$         |
| Universality probability                     | $1 - \Omega^{(0)}$     |

6.2 Complexity of a property versus algorithmic randomness of its probability

The reader may have noticed a pattern in the examples of Table 1. The arithmetical complexity of each of the properties we considered matches the level of randomness that they correspond to. For example, totality is a $\Pi^0_2$ property and it corresponds to 2-random $\emptyset'$-right-c.e. reals, which is the analogue of $\Pi^0_2$ in algorithmically random reals. Indeed, recall that the $\emptyset'$-right-c.e. reals are exactly the reals whose left Dedekind cut is $\Pi^0_2$. Similarly, computability is a $\Sigma^0_3$ property\(^\text{26}\) and the property that $W^X$ is computable corresponds to the 3-random $\emptyset^{(2)}$-left-c.e. reals $\Omega^{(2)}$ which are the analogue of $\Sigma^0_3$ in algorithmically random reals. Finally, universality is a $\Pi^0_4$ property and, sure enough, the universality probabilities are characterized by the numbers $1 - \Omega^{(0)}$ which are the analogues of $\Pi^0_4$ for algorithmically random reals.

Is this a coincidence, or is there a general theorem that characterizes the various universal probabilities in terms of algorithmically random numbers of the same arithmetical complexity? With so many examples and a uniform methodology for their analysis (see [3]) it is tempting to guess a positive answer to this question. Quite surprisingly, there is a simple example of a universal probability which behaves very differently and, in fact, its complexity depends strongly on the choice of the universal machine. Given a universal oracle Turing machine $M$ consider the probability that $n \mapsto M^X(n)$ is computable (and hence, total). This question can also be phrased in terms of monotone machines, where the input is a stream $X$ and the output is either a string or a stream $Y$, and we are looking for the probability that the output is a computable stream. Note that the property in question is $\Sigma^0_3$, so according to the previous discussion, one would expect these probabilities to be characterized as $\Omega^{(2)}$. However it was shown in [2] that this is never a 3-random number, and depending on the underlying universal machine it can be as complex as $\Omega^{(0)}$ or $1 - \Omega^{(0)}$, and as simple as $1/2$.

6.3 Invariance with respect to different universal machines

We have seen that, although certain differences may exist between two universal halting probabilities, their Turing degree remains fixed to the degree of the halting problem. This is no longer true for probabilities of properties of higher complexity, such as the ones we discussed in Section 6.1. In many cases, if one examines the proofs of this failure of invariance, one reason is the fact from [57] that the Turing degree of relativizations of Omega are very sensitive to the choice of the universal prefix-free machine.\(^\text{27}\) For

\(26\)For example, the set $\{e \mid W_e$ is computable\} is $\Sigma^0_3$.

\(27\)Of course the probabilities of Section 6.1 are not with respect to oracle machines, but as Table 1 shows they can be characterized as such.
example, in [4] it was shown that there are two universal prefix-free machines with universality probabilities having incomparable Turing degrees. Similar results were obtained in [3] with regard to the other probabilities that we discussed.

Perhaps the most extreme example with respect to sensitivity over the choice of underlying universal machine $U$ is the probability of producing an infinite computable stream – the example discussed in Section 6.2. In this case it is not only the Turing degree of the probability that depends on $U$, but most computational aspects of this number. Depending on the choice of $U$ it could be as simple as $1/2$ or as complicated as $\Omega^{\emptyset'}$ or even the mirror image $1 - \Omega^{\emptyset'}$ (namely a right-c.e. real relative to the halting problem $\emptyset'$ which is Martin-Löf random relative to $\emptyset'$).

7 Conclusion

We have discussed several mathematical aspects of Chaitin’s halting probability, which go well beyond its simple algorithmic properties that were established Chaitin’s seminal paper. We have also provided a rich bibliography on this topic, where further results and properties may be found, which have not found a place in our discussions. We hope we have shown that Chaitin’s Omega plays an important role in contemporary studies in algorithmic randomness.

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