Holographic dual to charged SYK from 3D Gravity and Chern-Simons

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Abstract

In this paper, we obtain a bulk dual to SYK model, including SYK model with $U(1)$ charge, by Kaluza-Klein (KK) reduction from three dimensions. We show that KK reduction of the 3D Einstein action plus its boundary term gives the Jackiw-Teitelboim (JT) model in 2D with the appropriate 1D boundary term. The dilaton of the JT model turns out to be proportional to the KK radius. Small non-zero value of the dilaton represents a small non-zero KK radius, which naturally corresponds to an effective near-AdS\textsubscript{2} geometry from the 3D scenario. In presence of $U(1)$ charge, the 3D model additionally includes a $U(1)$ Chern-Simons (CS) action. In order to describe a boundary theory with non-zero chemical potential, we have also introduced a coupling between CS gauge field and bulk gravity. The 3D CS action plus the new coupling term with appropriate boundary terms reduce in two dimensions to a BF-type action plus a source term. The bulk dual represents the soft sector of the charged SYK model. The pseudo-Nambu-Goldstone modes of combined Diff/$\mathbb{SL}(2,\mathbb{R})$ and $U(1)_{local}/U(1)$ transformations are represented by combined large diffeomorphisms and large gauge transformations. The bulk action after such transformations reproduce the effective action of the pseudo NG modes of the charged SYK model. We compute chaotic correlators from the bulk and reproduce the result that the contribution from the “boundary photons” corresponds to zero Liapunov exponent.

Dedicated to the memory of Joe Polchinski.

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1 Introduction and Summary

Finding simple field theories dual to gravitational systems has been a long standing goal. In the 1990’s matrix models provided an early impetus to such a search, where a main ingredient of the duality was symmetries. For example, the $c = 1$ matrix model was found to be governed by $W_\infty$ symmetries; consequently the dynamics could be abstracted in terms of coadjoint orbits of $W_\infty$. The latter had a natural realization in terms of two dimensional string theory, thus establishing a connection between $c = 1$ matrix model and two dimensional string theory \cite{1}. A similar approach was taken recently in \cite{2}, based on coadjoint orbits of the Virasoro symmetry group of the SYK model \cite{3–6}, to construct a bulk dual to matrix model and two dimensional string theory \cite{1}. A similar approach was taken recently in \cite{2}, based on coadjoint orbits of the Virasoro symmetry group of the SYK model \cite{3–6}, to construct a bulk dual to matrix model and two dimensional string theory \cite{1}. A similar approach was taken recently in \cite{2}, based on coadjoint orbits of the Virasoro symmetry group of the SYK model \cite{3–6}, to construct a bulk dual to matrix model and two dimensional string theory \cite{1}. A similar approach was taken recently in \cite{2}, based on coadjoint orbits of the Virasoro symmetry group of the SYK model \cite{3–6}, to construct a bulk dual to matrix model and two dimensional string theory \cite{1}.

The SYK model, briefly, is a model of interacting fermions at a single point. Its relevance to the physics of black holes stems from the fact that the model saturates the quantum chaos bound. The main feature of the theory responsible for this is the appearance of time reparametrization symmetry (henceforth called $\text{Diff}^3$ at the strong coupling (IR fixed point) in the large $N$ limit (see the original papers mentioned above as well as \cite{7–11} for further developments). The symmetry is spontaneously broken to $S\text{L}(2, \mathbb{R})$, leading to Nambu Goldstone (NG) modes parameterized by the coset. The IR theory is singular since the NG modes have precisely zero action (there is no analogue of the pion kinetic energy term). To regulate the theory, one has to introduce a small breaking of the reparameterization symmetry by being slightly away from the strong coupling fixed point; the dynamics of the pseudo NG modes is governed by an action described by the Schwarzian of the reparameterization function (see \cite{2} below).

The original SYK model involved Majorana fermions which did not carry any charge. In \cite{12}, a generalized SYK model with Dirac fermions, with a global $U(1)$, was introduced. At the strong coupling fixed point, this $U(1)$ symmetry is enhanced to local $U(1)$ transformations.\footnote{We thank E. Witten for initial suggestions regarding a possible connection of the viewpoint of \cite{2} to three dimensions.} As in the uncharged case, the theory is singular at the strong coupling limit. The dynamics of the combined pseudo NG modes, denoted by $f(\tau) \in \text{Diff}/S\text{L}(2, \mathbb{R})$, and $\exp[i\varphi(\tau)] \in U(1)_{\text{local}}/U(1)$ by the following action \cite{12}:

$$S[f, \varphi] = S_1 + S_2,$$

$$S_1 = -\frac{\gamma}{4\pi^2} \int_0^\beta d\tau \{ \tan(\pi f(\tau)/\beta), \tau \};$$

$$S_2 = \frac{K}{2} \int_0^\beta d\tau [\partial_\tau \varphi - i\mu(\partial_\tau \epsilon(\tau))]^2, \quad \epsilon(\tau) \equiv f(\tau) - \tau.$$  \hspace{1cm} (3)

In the above, $K$ and $\gamma$ are constants depending on the coupling $J$ and chemical potential $\mu$, and are $\sim 1/J$. We have used the following notation for the Schwarzian of a function $f(x)$

$$\{f, x\} \equiv \frac{f'''}{f'} - \frac{3}{2} \left( \frac{f''}{f'} \right)^2$$

The first term in (1) is the familiar Schwarzian from the uncharged case, written at a finite temperature. The second term describes the action for the new set of pseudo Nambu Goldstone modes corresponding to $U(1)$ gauge transformations and their coupling to the $\text{Diff}$ modes. The pseudo NG bosons represent the ‘soft’ sector of the charged SYK model, which satisfies the following condition

$$\omega \ll J, \mu \ll J,$$  \hspace{1cm} (4)

where $\omega$ denotes the frequency corresponding to the relative time $\tau_1 - \tau_2$ of the bilocal variable $G(\tau_1, \tau_2)$. The configurations of the charged SYK model are schematically represented by the left panel of Figure \footnote{\text{Diff} represents $\text{Diff}(S^3)$ or $\text{Diff}(\mathbb{R}^3)$, depending on whether the system is at a finite temperature or zero temperature. For most part of the paper, we will take time to be Euclidean. The group $\text{Diff}$ is also called the Virasoro group.} The pseudo Nambu Goldstone modes of reparameterization invariance of the SYK model are strongly reminiscent of large diffeomorphisms of an asymptotically AdS$_2$ geometry. This idea has been implemented \footnote{Spontaneously broken by the vacuum to $U(1)$.}
in the bulk dual, in the Jackiw-Teitelboim (JT) models \([13,14]\) as well as in the Polyakov gravity model \([2]\), based on the coadjoint orbit point of view mentioned above. A precise form of asymptotic AdS\(_2\) geometries, which implements the large diffeomorphisms, appears in \([2]\) and is given by:\[^8\]

\[
ds^2_{\text{AdS}_2} = \frac{l^2}{z^2} \left( dz^2 + d\tau^2 \left( 1 - \frac{z^2}{2} \{ \tan f(\tau), \tau \} \right)^2 \right),
\]

where \(\tau, z\) are the time and radial coordinates in the gravity and \(l\) denotes AdS length scale. As above, \(f(\tau)\) denotes elements of the Diff group. Note that from the bulk viewpoint, the unbroken subgroup \(\text{SL}(2, \mathbb{R})\) is evident from \((5)\), since the Schwarzian vanishes for \(f \in \text{SL}(2, \mathbb{R})\) and the metric becomes AdS\(_2\). As has been shown in \([2,13,14]\), in the presence of appropriate symmetry-breaking terms, the large diffeomorphisms of the near-AdS\(_2\) geometry well capture the low energy dynamics and thermodynamics of the SYK models.

In this paper, we explore possible origins of the near-AdS\(_2\) geometries in three dimensions. The original motivation came from the observation that the asymptotically AdS\(_2\) geometries \([59]\) of \([2]\) can be obtained as an induced metric on a domain wall from the asymptotically AdS\(_2\) geometries of Brown and Henneaux \([13,15]\). In this paper, we find that a more relevant approach appears to be Kaluza-Klein (KK) reduction from 3D Einstein gravity in a black hole phase, which leads to the JT-type 2D bulk model \([13,14]\). A vanishing limit of the KK radius corresponds to an exact AdS\(_2\) symmetry (although such a limit is singular), while a small non-zero KK radius denotes a near-AdS\(_2\) configuration. Under the KK reduction, the KK radius gets identified with the dilaton of the JT model. The breaking of the AdS\(_2\) symmetry by a non-zero dilaton therefore gets a geometric understanding in the KK scenario.

The main new ingredient in our model is the treatment of the bulk dual of the charged SYK model. We find that in the 3D picture, incorporating the \(U(1)\) charge comes naturally, by including an abelian Chern-Simons term. We add to 3D Einstein gravity a \(U(1)\) Chern-Simons term as well as a term which couples the two theories. The KK reduction to 2D gives a BF theory coupled to the JT model with an appropriate coupling between the two, which reproduces the effective action of the pseudo-Goldstone modes of the charged SYK model. The 2D gauge fields are now acted on by large diffeomorphisms as well as large gauge transformations. The configuration space is represented by the right panel of Figure 1. In the near AdS\(_2\) background mentioned above, these configurations well capture the pseudo NG modes of the charged SYK model and reproduce the effective action \([11]\) (see the detailed summary below).

We should remark here that the bulk duals mentioned above do not purport to be a bulk dual of the full SYK model, rather they represent the ‘soft’ sector \([11]\). It is not clear if there is a local bulk dual for the full SYK model \([17,18]\). For proposals of a bulk dual for the massive modes of SYK model from a 3D viewpoint, see \([19,21]\). Note that the original motivation for considering JT models in \([13,14]\) also comes from higher dimensions, namely from AdS\(_d\) extremal black hole geometries \((d \geq 4)\) which have an AdS\(_2\) near-horizon geometry. The breaking of the AdS\(_2\) isometry in that case is represented here by the physics away from

[^8]: We have chosen the Diff element to be \(\tan f(\tau)\) to represent a large diffeomorphism of the AdS\(_2\) BH solution.

[^9]: It would be interesting to explore possible connections between this approach and the KK reduction studied in our present work.
the near-horizon region. As remarked above, in our case, of KK reduction from 3D, it is rather different; the breaking of the AdS$_2$ symmetry is measured by a non-zero KK radius.

Summary and organization of results

In section 2 we recapitulate the charged SYK model. We closely follow Ref. [12] in this review. In Section 3 we consider the KK reduction of Einstein-U(1) Chern Simons theory in 3D. The salient points are

1. KK reduction of Einstein theory in 3D (with negative cosmological constant) reduces to a generalized Jackiw-Teitelboim model [22, 23]. The original references for this are [24–26]; see also the recent papers [27, 28] which use these ideas in contexts similar to that of the present paper.

2. The dilaton comes from the radius of the KK direction; as emphasized above, a small non-zero KK radius corresponds to a small symmetry breaking parameter of AdS$_2$ symmetry, giving a geometric meaning to such a role played by the dilaton in the JT model.

3. The KK reduction of the BTZ black hole is a 2D black hole with a dilaton solution. The 2D theory admits the asymptotically AdS$_2$ solutions (39) which are obtained by applying large diffeomorphism to the 2D black holes. [10] We discuss why the 3D AdS soliton is not relevant for our discussion.

4. In Section 3.2 we discuss the validity of the KK reduction. We show that the energy regime in which KK reduction is valid coincides with the IR regime of SYK.

5. In Section 3.3 we introduce a U(1) Chern-Simons theory with a term that couples it to the Einstein gravity.

Section 4 contains the main results of the paper.

1. The bulk action for the holographic dual of low energy sector of charged SYK is presented in Section 4. We discuss the equation of motion and class of solutions in this section.

2. The generalized JT model has contribution from the KK gauge fields. We show that a consistent truncation exists under which the KK gauge fields can be set to zero, after which the 2D theory becomes precisely the JT model. In the context of the BTZ solution, this truncation amounts to setting the angular momentum to zero.

3. To account for thermal effects in gravity we take a black hole geometry as the reference geometry. This solution is the BTZ spacetime in 3D gravity. Large diffeomorphisms and large gauge transformations of the fields are performed on this thermal background.

4. In Section 4.3 the effective action of the large diffeomorphisms and large gauge transformations is given by a Schwarzian action plus a sigma model term. The Schwarzian action is already known from [13]. The sigma model in charged SYK is proposed in [12].

5. We conclude this section with a comparison between our bulk results with that of field theory.

In section 5 we compute chaotic correlators from the bulk. We reproduce the result that there is no contribution from “boundary photons” to the Liaponov exponent.

These are KK reductions of BTZ black holes with large diffeomorphisms in 3D in the presence of non-normalizable deformations and hence, are not a subset of Brown-Henneaux geometries.
2 The charged SYK model

The original SYK model involved Majorana fermions. In [12], a generalized SYK model with Dirac fermions with a global $U(1)$ was introduced. The model is given by a hamiltonian

$$H = \sum_{i_1, \ldots, i_q} j_{i_1, \ldots, i_q} \psi_{i_1}^\dagger \cdots \psi_{i_q}^\dagger \psi_{i_q/2}^\dagger \psi_{i_q/2+1} \cdots \psi_{i_q},$$

where $q$ is an even number and $j_{i_1, \ldots, i_q}$ are complex Gaussian random variables with $\langle |j_{i_1, \ldots, i_q}|^2 \rangle = \frac{J^2((q/2)!)^2}{N^{q-1}}$.

Effective action and Schwinger-Dyson equations

The Euclidean action for this model is

$$S = \int d\tau \left[ \frac{1}{2} \psi_i^\dagger (\partial_\tau - \mu) \psi_i - H \right],$$

where $\mu$ denotes the chemical potential. The large $N$ limit is characterized by Schwinger-Dyson equations, which can be derived from the following effective action in terms of the bilocal field $\tilde{G}(\tau_1, \tau_2) = \frac{1}{N} \sum \psi^\dagger(\tau_1)\psi(\tau_2)$ and an auxiliary field $\tilde{\Sigma}(\tau_1, \tau_2)$

$$S[\tilde{G}, \tilde{\Sigma}] = -\frac{N}{2} \text{Tr} \ln \left[ \delta(\tau_1 - \tau_2) \left( -\frac{\partial}{\partial \tau_2} + \mu \right) - \tilde{\Sigma}(\tau_1, \tau_2) \right] - \frac{N}{2} \int d\tau_1 d\tau_2 \left[ \tilde{\Sigma}(\tau_1, \tau_2) \tilde{G}(\tau_1, \tau_2) + J^2 (-1)^{q/2} \left( \tilde{G}(\tau_1, \tau_2) \right)^{q/2} \left( \tilde{G}(\tau_2, \tau_1) \right)^{q/2} \right].$$

In fact, the original fermion integral with a quenched average over the disorder reduces, in the large $N$ limit, to a path integral over the variables $\tilde{G}(\tau_1, \tau_2)$ and $\tilde{\Sigma}(\tau_1, \tau_2)$ with the above action [13]. The large $N$ limit of the above action is characterized by the following Schwinger-Dyson equations,

$$\tilde{\Sigma}(\tau) = -(-1)^{q/2} J^2 G(\tau)^{q/2} G(-\tau)^{q/2},$$

$$G(iw_n) = \frac{[\mu - \Sigma(iw_n)]^{-1}}{iJ^2},$$

here $w_n$ denotes the Matsubara frequency.

2.1 Symmetry transformations

In the IR regime defined by $w_n, \mu \ll J$ [12] one can drop the $(i\mu + \mu)$ term from [10], leading to an emergent time reparameterization and $U(1)$ gauge symmetry. The symmetry transformations of the charged SYK model, at strong coupling (the IR regime), consist of (a) diffeomorphisms parameterized by $f: \tilde{\tau} = f(\tau) \in \text{Diff}$ [12] as well as (b) gauge transformations parameterized by $\exp[i\varphi(\tau)] \in U(1)_{\text{gauge}}$. We will denote the combined symmetry group as $\mathcal{G}$ [13] which acts on the bilocal meson variable $G(\tau_1, \tau_2) = (1/N) \psi_i^\dagger(\tau_1) \psi_i(\tau_2)$ as follows:

$$\tilde{G}(\tilde{\tau_1}, \tilde{\tau_2}) d\tilde{\tau}_1^\Delta d\tilde{\tau}_2^\Delta = G(\tau_1, \tau_2) d\tau_1^\Delta d\tau_2^\Delta \exp[i\varphi(\tau_2) - i\varphi(\tau_1)].$$

[13]Our expression for the effective action differs slightly from that of [12] because of different conventions for the bilocal fields and the conserved charge.

[12]Diff represents $\text{Diff}(S^1)$ or $\text{Diff}(R)$, depending on whether the system is at a finite temperature or zero temperature. For most part of the paper, we will take time to be Euclidean.

[13]The action of $\text{Diff}$ and $U(1)_{\text{gauge}}$ do not commute. The parameterization of the combined symmetry transformation follows the convention that the $\text{Diff}$ transformation is performed first, followed by a gauge transformation, as in [13].
In order to make sense of the theory, one needs to turn on a small value of an irrelevant coupling \( \frac{1}{\bar{J}} \), see [12].

Infinitesimally, for small \( \varphi(\tau) \) and small \( \epsilon(\tau) \equiv f(\tau) - \tau \),

\[
\delta G = \delta \epsilon G + \delta \varphi G,
\]

\[
\delta \epsilon G(\tau_1, \tau_2) = [\epsilon(\tau_1) \partial_{\tau_1} + \Delta \epsilon'(\tau_1) + \epsilon(\tau_2) \partial_{\tau_2} + \Delta \epsilon'(\tau_2)] G(\tau_1, \tau_2),
\]

\[
\delta \varphi G(\tau_1, \tau_2) = i \left( \varphi(\tau_2) - \varphi(\tau_1) \right) G(\tau_1, \tau_2).
\]

The vacuum solution \( G_0(\tau_1, \tau_2) \) at strong coupling \([14]\) breaks the symmetry from \( G \) to \( H = SL(2, \mathbb{R}) \times U(1)_{\text{global}} \). The Nambu-Goldstone (NG) modes are therefore parameterized by the coset space \( (f(\tau), \varphi(\tau)) \) \( \in G/H \). As in the uncharged case, the strict limit \( J = \infty \) is singular since the NG modes have zero action. In order to make sense of the theory, one needs to turn on a small value of an irrelevant coupling \( 1/J \), thereby explicitly breaking the symmetry \( G \). The NG modes now become pseudo Nambu-Goldstone modes, with an action which is a generalization of the Schwarzian action of the uncharged model \([12]\). This action is given by \([11], [2], [4]\).

## 3 Kaluza-Klein reduction of 3D gravity and Chern-Simons

The bulk dual to the SYK model with majorana fermions have been previously discussed in \([2, 14]\). We want to investigate the possible holographic dual to charged SYK using a Kaluza-Klein (KK) reduction of 3D gravity. Below, we propose the bulk dual to be a KK reduced action derived from 3D Einstein gravity with negative cosmological constant and Chern-Simons with a source term. To this end, we perform KK reduction of one of the coordinates on a circle \( S^1 \). It is convenient to look at the reduction of AdS3 and Chern-Simons sector separately.

### 3.1 3D Einstein gravity

We begin with the three dimensional Einstein action with a negative cosmological constant,

\[
S_{3\text{Dgrav}} = -\frac{1}{16\pi G_3} \int_{M_3} d\tau dy dz \sqrt{g^{(3)}} \left( R^{(3)} + \frac{2}{l^2} \right) - \frac{1}{8\pi G_3} \int_{\partial M_3} d\tau dy \sqrt{h} \left( K^{(3)} - \frac{1}{l^2} \right).
\]

The 3D manifold \( M_3 \) is parameterized by \( x^M = \{ \tau, y, z \} \), with a boundary at \( z = 0 \), which we will regulate by a cut-off \( z = \delta \). The coordinates on the boundary \( \partial M_3 \) are given by \( x^a = \{ \tau, y \} \) with the induced metric \( h_{\alpha\beta} \). We have taken the cosmological constant to be \( \Lambda = -\frac{1}{l^2} \), where \( l \) denotes AdS length scale. The boundary term above are the usual Gibbons-Hawking contribution and a counter term to have finite boundary current. We will work with Euclidean signature throughout this paper.

Varying the above action with a vanishing boundary condition on \( \delta g_{MN} \) yields Einstein equations,

\[
R_{MN}^{(3)} - \frac{1}{2} \left( R^{(3)} + \frac{2}{l^2} \right) g_{MN}^{(3)} = 0.
\]

### 3.1.1 Kaluza-Klein reduction to 2D

To perform a KK reduction of the 3D gravity we begin with the ansatz,

\[
ds_3^2 = g_{\mu\nu} dx^\mu dx^\nu + l^2 \Phi^2 (d\theta + B_\mu dx^\mu)^2, \quad dy = l d\theta, \quad \theta \equiv \theta + 2\pi,
\]

\[\text{For example, at zero temperature,}\]

\[
G_0(\tau_1, \tau_2) = G_0(\tau) = \begin{cases} 
- C|\tau|^{-2\Delta}, & \tau > 0 \\
- C e^{-2\bar{\epsilon}|\tau|^{-2\Delta}}, & \tau < 0
\end{cases}
\]

where \( \bar{\epsilon} \) is the ‘spectral asymmetry parameters’ satisfying \( \bar{\epsilon} = \lim_{r\to 0} \partial \mu/\partial T|_{\mu}, \tau = \tau_2 - \tau_2, Q \) and \( \mu \) are the U(1) charge and the corresponding chemical potential respectively and \( C \) is a constant. For more details and for the finite temperature two-point function, see [12].
where all the fields in (14) are functions of $x^\mu = \{\tau, z\}$ which parameterize the 2D base manifold $M_2$. $g_{\mu\nu}$ represents the metric on $M_2$ whereas $B_\mu$ and $\Phi$ are a vector and a scalar field on $M_2$ respectively. Using (16), we can derive following equations

\begin{align}
R^{(3)} &= R^{(2)} - 2 \frac{1}{\Phi} \nabla^2 \Phi - \frac{l^2}{4} \Phi^2 \bar{F}^2 , \quad \mathcal{K}^{(3)} = \mathcal{K}^{(2)} + n^\mu \frac{1}{\Phi} D_\mu \Phi , \\
\sqrt{g^{(3)}} &= l \Phi \sqrt{g^{(2)}} , \quad \sqrt{h} = l \Phi \sqrt{7} .
\end{align}

(17)

Here $\bar{F}^2 = \bar{F}_{\mu\nu} \bar{F}^{\mu\nu}$ is the field strength for KK gauge field $B_\mu$ such that $\bar{F}_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$. The induced metric on the 1D boundary $\partial M_3$ is given in terms of $\gamma_{\tau\tau}$ and $n^\nu$ is a unit normal to the boundary manifold. Using these, the 3D action (14) reduces to a Jackiw-Teitelboim (JT) [22, 23] action plus a term due to KK field strength:

\[ S_{3DBTZ} = -\frac{1}{16\pi G_2} \int_{M_3} d\tau dz \sqrt{g} \Phi \left( R + \frac{2}{l^2} - \frac{l^2}{4} \Phi^2 \bar{F}^2 \right) - \frac{1}{8\pi G_2} \int_{\partial M_3} d\tau d\theta \sqrt{\gamma} \Phi \left( \mathcal{K} - \frac{1}{l} \right) . \]

(18)

We have dropped the superscripts for now as we will only be working in 2 dimensions, we will bring them back whenever necessary. Note that $G_3 = 2\pi l G_2$. The details of the above reduction can be found in Appendix A.

### 3.1.2 Example of KK reduction

**BTZ black holes**

The equation of motion of 2D gravity can either be derived from the variational principle or they can equivalently be derived from the equation (15) and (16). We discuss the 2D equations and their solution in detail in section [Ref section 5]. However, we already know one famous black hole solution in 3D, namely BTZ black hole [29], given by the metric

\[ ds^2_{BTZ} = \left( \frac{l^2}{z^2} - Ml + \frac{J^2 z^2}{4l^2} \right) d\tau^2 + \left( \frac{l^2}{z^2} - Ml + \frac{J^2 z^2}{4l^2} \right)^{-1} \frac{l^4}{z^4} dz^2 + \frac{l^2 a^2}{z^4} \left( d\theta - \frac{J}{2} \frac{z^2}{l^2} d\tau \right)^2 , \]

(19)

where the ranges of the coordinates are $z \in (0, \infty) ; \tau = [0, \beta]$, and $\theta = [0, 2\pi]$ represent the conformal boundary $S^1 \times S^1$. The mass $M$ and angular momentum $J$ are the black hole parameters. Equation (19) is a solution to the equations (15).

We can readily identify the solution to the 2D equations, for the action (15), from the above BTZ spacetime. They are,

\begin{align}
\Phi &= \frac{a}{z} , \\
B_\tau &= -i \frac{J z^2}{2l^2} , \quad B_z = 0 .
\end{align}

(20)

Note that, only when the 3D fields are independent of the compactified direction $y$, the mapping of solutions under KK reduction is possible. This means that the 3D solutions of the form (16), where the metric components depend on the coordinates $\{\tau, z\}$ only, provide a solution to the 2D equation of motion directly.

The non-rotating BTZ black hole in the Fefferman-Graham gauge is given as,

\[ ds^2_{3DBTZ} = \left( \frac{l^2}{4z^2} \left( \frac{M^2 z^2}{l} - 1 \right) \right)^2 d\tau^2 + \frac{l^2}{z^2} dz^2 + \frac{l^2 a^2}{4z^2} \left( \frac{M^2 z^2}{l} + 1 \right)^2 d\theta^2 \]

(21)

with $\tau \in [0, \beta], \theta \in [0, 2\pi]$ and $\beta \sim 1/\sqrt{M}$.
From this 3D metric, we can write the solutions in 2D gravity to be,

\[ ds_{2D}^2 = \frac{l^2}{4z^2} \left( \frac{Mz^2}{l} - 1 \right)^2 \, d\tau^2 + \frac{l^2}{z^2} \, dz^2, \]

\[ \Phi = \frac{la}{2z} \left( \frac{Mz^2}{l} + 1 \right). \]  

(22)

The solution above represents a black hole in AdS_2 spacetime with a dilaton \( \Phi \).

3D AdS-soliton

The equations (15) (and hence equations of motion for the action 18) admit another well-known solution, the AdS soliton [30],

\[ ds^2_{\text{soliton}} = \frac{l^2}{4z^2} \left( \frac{Mz^2}{l} + 1 \right)^2 \, d\tau^2 + \frac{l^2}{z^2} \, dz^2 + \frac{l^2a^2}{4z^2} \left( \frac{Mz^2}{l} - 1 \right)^2 \, d\theta^2 \]  

(23)

with \( \tau \in [0, \beta] \), \( \theta \in [0, 2\pi] \) and \( a \sim 1/\sqrt{M} \). The radial coordinate is \( z \in (0, \infty) \) and the other two coordinates have the period as, \( y \in (0, 2\pi l) \) and \( \tau \in (0, \beta) \). It is easy to show that for low temperatures the free energy of the soliton solution is lesser compared to that of BTZ and thus the soliton phase is preferred for low temperatures.

Let us now look at the 2D solutions derived from 3D solutions. The solution from the BTZ is given in equation (22) and the solution from soliton will be,

\[ ds_{2D}^2 = \frac{l^2}{4z^2} \left( \frac{Mz^2}{l} - 1 \right)^2 \, d\tau^2 + \frac{l^2}{z^2} \, dz^2, \quad \Phi = \frac{al}{2z} \left( \frac{Mz^2}{l} - 1 \right). \]

(24)

An apparent puzzle

When one considers KK reduction along the \( \theta \)-circle, then the KK radius, measured by the proper radius of the \( \theta \)-circle at the boundary, must be smaller than various length scales of relevance of physics in the lower dimension. In particular, this proper radius must be smaller than that of the \( \tau \)-circle, which measures the inverse temperature. Now, it is well-known that under this circumstance, the thermodynamically favourable [30] solution is the AdS soliton which has a contractible \( \theta \)-circle and not the BTZ black hole which has a contractible \( \tau \)-circle. In that case, to arrive at the 2D black hole (21) by KK reduction from 3D, one has to start from the sub-dominant saddle point of the Euclidean path integral, namely the BTZ solution. At large \( N \), there is a sense in which one can explore the perturbative neighbourhood of a sub-dominant saddle point since the tunneling amplitude to the dominant saddle point (the AdS soliton) is of order \( e^{-O(N)} \). In doing so we are aware that there is a loss of unitarity up to non-perturbative effects of order \( e^{-O(N)} \). There is a complementary argument which says that KK reduction from the 3D soliton solution along the \( \theta \)-circle does not in any case make sense. The reason is that for the AdS soliton, the KK circle contracts in the bulk; this leads to a vanishing dilaton in 2D (see (24)) and consequently uncontrolled quantum fluctuations.

In this paper, we will take the viewpoint that we will consider the classical saddle point solution (21) and its 2D reduction (22) and explore the orbits of this solution under large diffeomorphism, which represents the correct physics of the SYK model. The relevant of the AdS soliton is unclear from the point of view of the SYK model.

3.2 Validity of the Kaluza Klein reduction

Note that in (21), near the boundary, \( z = \delta, \ g_{\theta\theta} \sim (\frac{\delta}{\pi a})^2 \) up to numerical factors. Hence the KK radius is \( R_{KK} \sim \frac{1}{\delta a} \). For the KK reduction to be valid, the typical energy \( E \) of particles in a bulk correlator should
satisfy the relation

\[ E \ll \frac{1}{R_{KK}} \sim \frac{\delta}{l a} \]

Note that the energy \( E \) measured in the bulk is related to the energy \( \omega \) in the boundary theory by the standard AdS/CFT equation: \( E = \frac{\delta}{l} \omega \).\(^{(15)}\)

Later on in this note we will identify the coefficient \( a \) (which is a non-normalizable deformation of the AdS\(_2\) background) with \( 1/J \) (see (76)). Hence the above condition for the validity of the KK reduction reduces to

\[ \omega \ll J \]

which is the same as the condition (see (4)) that we are working in the IR regime of the SYK theory!

### 3.3 3D Chern-Simons

In order to obtain a dual to charged SYK we must include gauge fields in 3 dimensions besides gravity. The most natural candidate in the long wavelength limit is a U(1) Chern-Simons (CS) action in an asymptotically AdS\(_3\) spacetime (recall that the Maxwell term is irrelevant in the IR). The action must have a boundary term for a well-defined variational principle. The CS action, with appropriate boundary conditions and boundary terms, following [31], is given by

\[
S_{CS} = \frac{ik}{8\pi} \int_{AdS_3} dA + \frac{k}{16\pi} \int_{\partial AdS_3} d\tau dy \sqrt{h} \alpha^\alpha A_\alpha A_\beta .
\]

(25)

Here, \( k \) is a real positive parameter. Although the boundary term introduces a coupling of the boundary U(1) field to 3D gravity, the U(1) field in the bulk is not coupled to gravity in the above, and hence the field strength vanishes as usual. This forces the Wilson loop around the time circle to vanish in geometries where this circle is contractible. As we will see below, this implies a vanishing chemical potential. Hence, in order to have the correct bulk dual to a theory with a non-zero chemical potential, we must introduce a coupling between the bulk metric and the bulk CS U(1) field. The simplest such coupling is

\[
S_{coupling} = \frac{ik}{4\pi} \int_{AdS_3} d\tau dz dy \sqrt{g} A_M J^M
\]

(26)

where the \( J^M \) is a given external source, which we will specify in detail shortly (see (28)).

Next, we do a KK reduction along \( y \) direction, as before (further details of the reduction are presented in Appendix A).

#### 3.3.1 KK reduction to 2D

Under KK compactification along the \( y \) direction, the gauge field is taken to be of the form

\[
A_M dx^M = A_\mu dx^\mu + \chi d\theta .
\]

(27)

where \( \chi \) denotes a scalar in 2 dimensions.

Unlike the dynamical gauge field, the external current \( J^M \) can only be a scalar constant in the two dimensional base manifold under KK reduction. That is,

\[
J^\tau = J^z = 0, \quad J^\nu = J_0 = \text{constant}
\]

(28)

Then from equation (16) for the metric, the KK reduction of the action (25) is,

\[
S_{2Dgauge} = \frac{ik}{2} \int_M \chi F_{\tau z} d\tau dz + \frac{ikl}{2} \int_M \sqrt{g} \Phi \chi d\tau dz - \frac{ik}{4} \int_{\partial M} \chi A_\tau d\tau
\]

This is related to the fact that the AdS metric induced on the boundary \( z = \delta \) is \( ds_\delta^2 = \frac{\delta^2}{l^2} (dx^\mu dx_\mu) \).
\[
\frac{kl}{8} \int_{\partial M} d\tau \sqrt{\gamma} \left( \gamma^{\tau \tau} \Phi A^2_\tau + \frac{\chi^2}{l^2 \Phi} + B_\tau \Phi (\chi B^\tau \Phi^2 - 2A^\tau) \right),
\]
(29)
here \( M \) denotes asymptotically AdS\(_2\) spacetime, represented by the geometries (39) and \( \partial M \) is the boundary given by \( z = \delta \). We have used the definition \( F_{\alpha \beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \).

### 4 Combined action in 2D: bulk dual to charged SYK

As indicated in the introduction, a conserved \( U(1) \) current in the boundary theory implies, by usual AdS/CFT correspondence, that the bulk theory should have a \( U(1) \) gauge field. We propose the KK reduced CS plus the JT action discussed above as the bulk dual for the low energy sector of the charged SYK model. The reduced 2 dimensional action consists of two terms,

\[
S_{2D} = S_{2D\text{grav}} + S_{2D\text{gauge}}
\]
(30)
where \( S_{2D\text{grav}} \) and \( S_{2D\text{gauge}} \) are given by equations (18) and (29) respectively.

\[
\begin{align*}
S_{2D} &= -\frac{1}{16\pi G_2} \int_M d\tau dz \sqrt{g} \Phi \left( R + \frac{2}{l^2} \frac{l^2}{4} \phi^2 \tilde{F}^2 \right) - \frac{1}{8\pi G_2} \int_{\partial M} d\tau \sqrt{\gamma} \Phi \left( K - \frac{l}{4} \right) \\
&\quad - \frac{i k}{4} \int_M \chi F - \frac{i k}{4} \int_M \chi A_\tau d\tau + \frac{kl}{8} \int_{\partial M} d\tau \sqrt{\gamma} \left( \gamma^{\tau \tau} \Phi A^2_\tau + \frac{\chi^2}{l^2 \Phi} + B_\tau \Phi (\chi B^\tau \Phi^2 - 2A^\tau) \right) \\
&\quad + J_0 \frac{i k l}{2} \int_M d\tau \sqrt{g} \Phi \chi d\tau dz,
\end{align*}
\]
(31)
In the above, \( F = e^{\mu \nu} F_{\mu \nu} \).

#### 4.1 Equations of motion

The classical equations of motion from the above action are,

\[
R + \frac{2}{l^2} \frac{l^2}{4} \phi^2 \tilde{F}^2 - 8\pi G_2 ik l \chi J_0 = 0,
\]
(32)
\[
\nabla_\mu (\sqrt{g} \Phi^3 \tilde{F}^{\mu \nu}) = 0,
\]
(33)
\[
\left( \nabla_\mu \nabla_\nu \phi - g_{\mu \nu} \nabla^2 \phi + \frac{g_{\mu \nu}}{2} \left( \frac{2}{l^2} - 8\pi G_2 ik \chi J_0 \right) \phi \right) + \frac{\Phi^3 l^2}{4} \left( -\frac{1}{2} g_{\mu \nu} \tilde{F}^2 + 2 g^{\alpha \beta} \tilde{F}_{\alpha \mu} \tilde{F}_{\beta \nu} \right) = 0,
\]
(34)
\[
\partial_\tau \chi = 0, \quad \partial_z \chi = 0,
\]
(35)
\[
F = 2\Phi l \sqrt{g} J_0.
\]
(36)

Note that it is consistent, in equations (33) and (34), to put

\[
B_\mu = 0, \quad \chi = 0
\]
(37)
since all source terms for these fields involve themselves \(17\). With this, the equations reduce to,

\[
R + \frac{2}{l^2} = 0, \quad \nabla_\mu \nabla_\nu \phi - g_{\mu \nu} \nabla^2 \phi + \frac{g_{\mu \nu}}{2} \left( \frac{2}{l^2} - 8\pi G_2 ik \chi J_0 \right) \phi = 0, \quad F = 2\Phi l \sqrt{g} J_0.
\]
(38)

\(^{16}\)Note that (39) continues to be a solution for the metric even after adding to the action the new part (29), since the bulk part of the latter is topological and does not contain the metric.

\(^{17}\)It is important to mention here that any large gauge transformations of the field \( B_\mu \) will not affect our results due to the consistent truncation \( \chi = 0 \).
It is easy to show that these equations possess a family of solutions for the metric which is given by,

\[ ds_{\text{ads}}^2 = \frac{l^2}{z^2} \left( dz^2 + d\tau^2 \left( 1 - \frac{z^2}{2} f(\tau, \tau) \right)^2 \right). \] (39)

All the geometries of the above form are asymptotically AdS and are generated by starting from the AdS metric

\[ ds_{\text{ads}}^2 = \frac{l^2}{\tilde{z}^2} \left( d\tilde{z}^2 + d\tilde{\tau}^2 \right) \] (40)

by performing the large diffeomorphisms,

\[ \tilde{\tau} = f(\tau) - \frac{2z^2 f''(\tau) f'(\tau)^2}{4f'(\tau)^2 + z^2 f''(\tau)^2}, \quad \tilde{z} = \frac{4zf'(\tau)^3}{4f'(\tau)^2 + z^2 f''(\tau)^2}, \] (41)

which are the 2D counterpart of Brown-Henneaux transformations presented in [2]. In this family a general solution for the dilaton is

\[ \Phi(\tau, \zeta) = \frac{a + b \tilde{\tau} + c (\tilde{\tau}^2 + \tilde{z}^2)}{\tilde{z}} \] (42)

where \((\tilde{\tau}, \tilde{z})\) are to be substituted from (41).

**Large diffeomorphisms**

Large diffeomorphisms given in (41) are not bijective maps everywhere inside the bulk but only near the boundary hence they exactly map AdS boundary into boundary of asymptotically AdS. This map is good for our purpose as we will only need the near boundary behaviour of fields in this work. They are also in one to one correspondence with boundary time reparameterization functions \(f(\tau)\) of the \(\text{Diff}\) group. We would like to point out that the BTZ black hole metric can be generated by taking \(f(\tau) = \tan(\pi \tau / \beta)\). Coordinate transformations changing the topology from AdS to black hole seems surprising. However, on a careful look we find that this is possible because these maps become singular at the horizon. Keeping this in mind we make a distinction between geometries produced by large diffeomorphism of AdS and large diffeomorphisms of BTZ. As we are interested in finite temperature systems we will always consider BTZ black hole to be our reference geometry and work with this class of geometries.

### 4.1.1 BTZ black hole solutions

We can check that equations (38) are satisfied by BTZ black hole solution,

\[ ds_{\text{BTZ}}^2 = \frac{l^2}{z^2} \left( dz^2 + d\tau^2 \left( 1 - \frac{z^2}{\beta^2} \right)^2 \right) \] (43)

with horizon at \(z = \beta / \pi\).

**General solutions in black hole background for dilaton and gauge field**

A general solution for \(\Phi(\tau, z)\) in the black hole background is,

\[ \Phi(\tau, z)_{\text{BTZ}} = (a + c) \frac{\beta^2 + \pi^2 z^2}{2\pi \beta z} + \left( (a - c) \cos \left( \frac{2\pi \tau}{\beta} \right) + b \sin \left( \frac{2\pi \tau}{\beta} \right) \right) \frac{\beta^2 - \pi^2 z^2}{2\pi \beta z}. \] (44)

Additionally, we choose the gauge \(A_z(\tau, z) = 0\). Plugging (41) and (43) into (36), we can find general solution for \(A_\tau(\tau, z)\). The solution has the form

\[ A_\tau(\tau, z) = a_1^{a,b,c,J_0}(\tau) \frac{1}{z^2} + a_2^{a,b,c,J_0}(\tau) \log(z) + a_3^{a,b,c,J_0}(\tau) z^2 + c_1(\tau), \] (45)
here \(a_i^{a,b,c,J_0}\) are time dependent functions parameterized by \(a, b, c\) and the source \(J_0\). \(c_1(\tau)\) is an integration constant. Detailed expressions for these coefficients can be found in Appendix C. Note that \(A_r(\tau, z) = 0\) gauge will restrict large gauge transformations to be only functions of time. Along with this we also have a condition that \(A_r(\tau, z)\) should vanish at the horizon of the black hole (which is required by Stokes’ theorem for non-singular field strengths included at the horizon). This can always be arranged by choosing the integration constant \(c_1(\tau)\) appropriately. Once this is ensured, we get

\[
A_r(\tau, z) = \tilde{A}_r(\tau, z) + a_r(\tau, z)
\]

(46)

where \(\tilde{A}_r(\tau, z)\) contains \(z\) dependent terms which diverge on the boundary. The boundary value of \(a_r(\tau, z)\) in a black hole background is a constant which is \(J_0/\pi l^3(a + c)/\beta\). This boundary value will be identified with \(-i\tilde{\mu}\) where \(\tilde{\mu}\) is the chemical potential of the boundary theory. Look at Appendix C for detailed expression of \(\tilde{A}_r(\tau, z)\). Solutions (37), (44) and (45) is a consistent set of solutions in BTZ black hole background (43).

4.1.2 Large diffeomorphism of BTZ solutions

We will call the set of solutions (37), (43), (44) and (45) as \(G_0\) (cf., Fig.1) and a set containing diffeomorphisms of them as \(G_f\). Large diffeomorphisms of a BTZ black hole has the following near boundary form

\[
ds_f^2 = \frac{l^2}{z^2} (d\tau^2 + dz^2) - d\tau^2 \left( \tan(\pi f(\tau)/\beta), \tau \right) + O(z).
\]

(47)

Above form of the metric can be obtained by performing coordinate transformations given in (41) on (43) near the boundary. Similarly we can find large diffeomorphism of the gauge field, which is of the form

\[
A^f(\tau, z) = \tilde{A}_f^r(\tau, z) + a_f^r(\tau, z),
\]

(48)

where near the boundary,

\[
a_f^r = a_r f'(\tau) \quad \text{and} \quad \tilde{A}_f^r(\tau, z) = \tilde{a}_1^{a,b,c}(\tau) \frac{1}{z^2} + \tilde{a}_2^{a,b,c}(\tau) \log(z) + z^{a,b,c}(\tau) + O(z^2)
\]

(49)

Look at Appendix C for the values of the coefficients above. The near boundary expression for dilaton is always of the form,

\[
\Phi_f(\tau, z) = \frac{\phi_r(\tau)}{z} + O(z),
\]

(50)

where \(\phi_r(\tau)\) is an arbitrary function of time. The set \(G_f\) consists of the solutions (37), (17), (45) and (49).

4.1.3 Counter terms and boundary conditions

In order to have well defined conserved stress tensor and conserved currents we need to add counter terms to our action. We have gravity boundary term

\[
-\frac{1}{8\pi G_2} \int d\tau \sqrt{\gamma} \Phi \, K,
\]

(51)

which will give a quadratic divergence in the stress tensor. We take care of these by adding

\[
\frac{1}{8\pi G_2} \int_{\partial M} d\tau \sqrt{\gamma} \frac{\Phi}{l}
\]

(52)

to our action (33) (this effectively replaces \(K\) by \((K - 1/l))\). We also have gauge boundary term

\[
\frac{lk}{8} \int d\tau \sqrt{\gamma} \gamma^{\tau\tau} \Phi \, \tilde{A}_r^2 + \frac{lk}{4} \int d\tau \sqrt{\gamma} \gamma^{\tau\tau} \Phi \, \tilde{A}_r \, a_r
\]

(53)
giving us quadratic and logarithmic divergence. To take care of this we add
\[-\frac{kl}{8} \int_{\partial M} d\tau (\Phi \sqrt{-g} \gamma^{\tau\tau} \bar{A}_\tau^2 + 2\Phi \sqrt{-g} \gamma^{\tau\tau} A_\tau a_\tau) .\] (54)

With above boundary terms plus counter terms, we can impose Dirichlet boundary condition on all the fields.

We can choose the boundary values of the fields and the path integral will be functional of these boundary values. We choose following boundary conditions in the black hole background,
\[\chi|_b = 0, \quad B_\mu|_b = 0 \quad a_\tau|_b = \frac{J_0 T^\mu (a + c)}{\beta} .\] (55)

These are consistent with the equation (33) and (35). The boundary condition on the dilaton is,
\[\Phi|_b = \frac{\phi_r(\tau)}{\delta} ,\] (56)
where \(\phi_r(\tau)\) is an arbitrary function of mass dimension \(-1\). Such a boundary condition on the dilaton has been discussed previously in (14). If \(J_0 = 0\), which implies that there is no electric flux in the system, then boundary value of \(a_\tau\) can only be pure gauge in the black hole background, as \(\tau\) is a contractible circle.

### 4.2 Deriving an effective action for low energy sector

In this section, we take a detour to set a general understanding to derive effective action for pseudo Nambu Goldstone modes.

Consider a scenario where we have an action \(S_0\) with exact zero modes \(f\). Let us call classical solution of this action to be \(G_0\), then we have \(S_0(G_0) = S_0(G_0')\). Now if we explicitly add a small symmetry breaking term \(\Delta S\) such that total action becomes \(S_0 + \Delta S\). Now the \(f\) modes are not exact zero modes but they become pseudo Nambu Goldstone (pNG) modes. This also means \(S_0(G_0) + \Delta S(G_0) \neq S_0(G_0') + \Delta S(G_0')\). The action for these pNG modes \(S_{pNG}(f)\) can be calculated by taking a difference
\[S_{pNG}(f) = [S_0(G_0') + \Delta S(G_0')] - [S_0(G_0) + \Delta S(G_0)] .\] (57)

For the purpose of calculations we can choose a solution for which \(\Delta S(G_0)\) does not contain any dynamical fields. In that case,
\[S_{pNG}(f) = \Delta S(G_0') + h_0 ,\] (58)

The kinematic term \((h_0)\) can be dropped from the path integral. Note that even though \(G_0'\) is a solution of \(S_0\), it is not the solution of \(\Delta S\) hence \(G_0'\) is really an off shell configuration for \(S_{pNG}(f)\). Therefore this process is not an on shell evaluation of an action \(S_0 + \Delta S\) but a way to construct off shell action for pNG modes.

Let us analyze above discussion in the context of field theory of a complex scalar field,
\[S_0 = \int d^D x [\partial_\mu \phi^* \partial^\mu \phi + g_1 (v^2 - |\phi(x)|^2)^2] .\] (59)

This model has global U(1) symmetry and we can parameterize space of low energy field configurations as \(\phi_0(x) = \rho \exp[i\theta(x)]\) where \(\rho = |\phi(x)|^2\) is constant. Then the action for these low energy configuration becomes,
\[S_0 = \int d^D x [\rho^2 \partial_\mu \theta \partial^\mu \theta(x) + g_1 (v^2 - \rho^2)^2] .\] (60)

If we do a transformation \(\theta(x) \rightarrow \theta(x) + C_0\) where \(C_0\) is constant, We can see that \(C_0\) is an exact zero mode of this model. let us add a term to this model \(\Delta S = g_2 \int d^D x \rho_0 \cos[\theta(x)]\),
\[S_0 + \Delta S = \int d^D x [\rho^2 \partial_\mu \theta \partial^\mu \theta(x) + g_1 (v^2 - \rho^2)^2 + g_2 \rho \cos[\theta(x)]] .\] (61)
addition of this term explicitly breaks U(1) symmetry and now \( C_0 \) is and pseudo NG mode. Now let us choose a solution \( \theta(x) = \theta_0 \) of \( S_0 \) then from above prescription

\[
S_{pNG} = g_2 \int d^Dx \rho(\cos(\theta_0 + C_0) + \cos(\theta_0))
\]

(62)

We could have chosen \( \theta_0 = \pi/2 \) in which the case the second term above would not have appeared. \( \theta_0 = \pi/2 \) is the solution for which \( \Delta S(G_0) \) is zero.

It is also possible that we have a subspace of \( f \) whose elements are denoted by \( g \) for which \( \Delta S(G_0) = \Delta S(G_0^g) \) (this will trivially happen if \( G_0^g = G_0 \)) then we can conclude that \( g \) is an exact zero mode of \( S_{pNG} \).

Let us take a look at how this works out in the SYK model. At the conformal fixed point, \( \text{Diff} \) group which is represented by \( f(\tau) \) is an exact zero mode of the SYK action (it is easy to see this when SYK action is written in terms of bilocals \( \Sigma \) and \( G \)). When we explicitly break this symmetry by adding a \( 1/J \) proportional term to the action these zero modes become pNG modes. We can evaluate the action for these pNG modes exactly as given above. For the conformal two point function of SYK, \( G_0^g = G \) for \( g \) is being a \( SL(2,R) \) function. Thus from above discussion we can already predict that \( SL(2,R) \) functions are exact zero modes of the action for pNG modes.

**Explicit symmetry breaking terms in \( S_{2D} \)**

We can separate the 2D action as

\[
S_{2D} = S_0 + \Delta S
\]

(63)

where

\[
S_0 = -\frac{1}{16\pi G_2} \int_M d\tau dz \sqrt{g} \Phi \left( R + \frac{2}{l^2} - \frac{l^2}{4} \Phi^2 \right) - \frac{ik}{4} \int_M \chi F + J_0 \frac{ikl}{2} \int_M d\tau \sqrt{g} \Phi \chi d\tau dz
\]

(64)

and

\[
\Delta S = -\frac{1}{8\pi G_2} \int_{\partial M} d\tau \sqrt{\gamma} \Phi \left( \mathcal{K} - \frac{1}{l^2} \right) - \frac{ik}{4} \int_{\partial M} \chi A_\tau d\tau
\]

\[
+ \frac{kl}{8} \int_{\partial M} d\tau \sqrt{\gamma} \left( \gamma^{\tau\tau} \Phi A^2_\tau + \frac{\chi^2}{l^2 \Phi} + B_\tau \chi \Phi \left( \chi B^2 \Phi - 2A^2 \right) \right)
\]

\[
- \frac{kl}{8} \int_{\partial M} d\tau \left( \Phi \sqrt{\gamma} \gamma^{\tau\tau} \tilde{A}^2_\tau + 2\Phi \sqrt{\gamma} \gamma^{\tau\tau} \tilde{A}_\tau a_\tau \right).
\]

(65)

Here \( S_0 \) is a completely gauge invariant and general coordinate invariant term but boundary terms along with counter break these symmetries explicitly.

**4.3 The Schwarzian effective action**

Drawing parallel with Section 4.2 in the gravity sector of the action (31) the analog of \( \Delta S \) is given by the boundary terms whereas the bulk term make up \( S_0 \). To evaluate the effective action we use the field values discussed in Section 4.1.2 and Section 4.1.3. For every point on the \( \text{Diff} \) orbit, bulk part of the JT action vanishes and only contribution comes from the boundary term

\[
\Delta S = -\frac{1}{8\pi G_2} \int d\tau \sqrt{\gamma} \Phi_0 \left( \mathcal{K} - \frac{1}{l^2} \right).
\]

(66)

\( \Delta S(G_0) \) is non-dynamical and \( \Delta S(G_0^f) \) will be given by

\[
S_{pNG} = -\frac{1}{8\pi G_2} \int d\tau \phi_\tau(\tau) \left\{ \tan \left( \pi f(\tau)/\beta \right), \tau \right\}.
\]

(67)
Note that in the gravity path integral which is integral over $\text{Diff}$ orbit, we always have asymptotically $\text{AdS}_2$ geometries (39) and boundary condition (56) hence above effective action should not be considered as an on shell evaluation of JT action but an effective action where $f(\tau)$ is the dynamical variable and $\phi_r$ is an external coupling.

4.4 Sigma model effective action

Under a diffeomorphism followed by a gauge transformation, the gauge fields transform as

$$A_{\mu}^f(x) = \tilde{A}_\mu(\tilde{x})\frac{\partial \tilde{x}^\nu}{\partial x^\mu} + \partial_\mu \phi(x).$$

(68)

Here, the subscript $f$ and $\phi$ denote the large diffeomorphism and large gauge transformations respectively. Recall that because of $A_z = 0$ gauge we have restricted set of gauge transformations to be only function of time. If we perform gauge transformations on (48) we will get

$$A_{\mu}^f(\tau,z) = \bar{A}_\mu^f(\tau) + (a_\tau f'(\tau) + \varphi'(\tau)).$$

(69)

We can think of this as only gauge transformation of $a_\tau^f$. Near the boundary we have

$$a_{\tau}^f f' + \partial_\tau \phi(\tau).$$

(68)

On the set of solutions $G_0$, $S_0$ vanishes and $\Delta S$ is non-dynamical, then $\Delta S(G_0^f)$ will be given by

$$S = \frac{k}{8} \int d\tau \phi_r(\tau) \left[\partial_\tau \phi(\tau)\right]^2.$$

(71)

Above action should also not be thought as an on shell evaluation of the gauge action but as an effective action, where the only inputs were boundary conditions.

Sigma model effective action with $a_\tau = 0$

We can consider the case where $a_\tau$ is 0, the case which is equivalent to putting $J_0 = 0$. Physically, this means that there is no electric flux in the system, this case is equivalent to putting the boundary value of the gauge field as pure gauge. This action is dual to U(1) charged SYK model without chemical potential. Then effective action is

$$S = \frac{k}{8} \int d\tau \phi_r(\tau) \left[\partial_\tau \phi(\tau)\right]^2.$$

(72)

these results have recently appeared in [27] and [28].

4.5 Combined effective action

We combine (71) and (67)

$$S = \int d\tau \phi_r(\tau) \left[\frac{k}{8} \left(a_\tau f'(\tau) + \partial_\tau \phi(\tau)\right)^2 - \frac{1}{8\pi G_2} \left\{\tan \left(\frac{\pi f(\tau)}{\beta}\right), \tau\right\}\right]$$

(73)

here dynamical variables are $\phi(\tau)$ and $g(\tau)$ are dynamical variables and $\phi_r(\tau)$ is an external coupling which can be made constant by choosing a new boundary time as $\phi(\tau)(d\tilde{\tau}/d\tau) = a$ [14]. Rewriting $f(\tilde{\tau}) = \tilde{\tau} + \epsilon(\tilde{\tau})$ and ignoring surface terms plus constants from the action we get

$$S = \int_0^\beta d\tilde{\tau} a \left[\frac{k}{8} \left(a_\tau \epsilon'(\tilde{\tau}) + \partial_\tau \phi(\tilde{\tau})\right)^2 - \frac{1}{8\pi G_2} \left\{\tan \left(\frac{\pi (\tilde{\tau} + \epsilon(\tilde{\tau}))}{\beta}\right), \tilde{\tau}\right\}\right]$$

(74)
4.6 Comparison with field theory

An effective action for the pseudo NG modes of the U(1) charged SYK model was presented in [12] (where we have substituted chemical potential using definitions in Appendix B of [12]),

\[ S = N \int_0^\beta d\tilde{\tau} \left[ \frac{K}{2} (\partial_\tau \varphi(\tilde{\tau}) - i \tilde{\mu} \epsilon'(\tilde{\tau}))^2 - \frac{\gamma}{4\pi^2} \{ \tan (\pi (\tilde{\tau} + \epsilon(\tilde{\tau}))/\beta), \tilde{\tau} \} \right] \]  
(75)

here K and \( \gamma \) are field theory thermodynamic parameters determining compressibility and specific heat respectively. In the large \( J \) small \( \mu \) (near conformal limit) and large \( q \) limit, \( K = q^2/(16J) \) and \( \gamma = 2\pi^2/(J q^2) \). Our results match with (60) with following identifications

\[ a = \frac{1}{J}, \quad a_\tau = -i\tilde{\mu}, \quad \frac{Nq^2}{4a} = k, \quad \frac{4\pi N}{q^2} = \left( \frac{1}{G_\xi} \right). \]  
(76)

5 Quantum chaos

So far our discussion has mostly focussed on the soft modes of the charged SYK model and their bulk dual. As in case of the uncharged model, there are other modes represented by operators of the form \( \psi_i \bar{\partial}_\tau \psi_i \), \( \psi_i \bar{\partial}_\tau \psi_i \), and \( \psi_i \bar{\partial}_\tau \psi_i \). Here the two-sided arrow schematically represents a specific combination of terms with \( n \) derivatives on the first fermion and \( p - n \) derivatives on the second fermion. The first two types of operators are charged while the third set are neutral. The story of correlators of the neutral operators is similar to that in the uncharged SYK model (briefly reviewed below in Sec 5.1). Therefore, in what follows, we will be concerned with charged operators. Let us denote a typical pair of conjugate charged operators by \( O(\tau), O^*(\tau) \), with scaling dimension \( \Delta \).

As in [2,13], to compute correlators of these operators from the bulk viewpoint we consider probe scalar fields \( \eta(\tau, z), \eta^*(\tau, z) \) whose masses are given by the standard AdS/CFT correspondence between mass and operator dimensions. The classical action for the probe scalars will be given by

\[ S_{\text{matter}} = \frac{1}{16\pi G_N} \int_M \sqrt{g} \left[ (g^{\alpha\beta} \partial_\alpha \eta^* \partial_\beta \eta + m^2 |\eta|^2) + \ldots \right] \]  
(77)

where the ... terms are higher order in the fields \( \eta \).

5.1 OTO correlators in the uncharged model

To begin, we will quickly review the calculation in the uncharged case. Let us consider the Euclidean correlator \( \langle O(\tau_1)O(\tau_2)O(\tau_3)O(\tau_4) \rangle \), which is given by

\[ \langle O(\tau_1)O(\tau_2)O(\tau_3)O(\tau_4) \rangle = \left. \frac{1}{Z[J]} \frac{\delta}{\delta J(\tau_1)} \frac{\delta}{\delta J(\tau_2)} \frac{\delta}{\delta J(\tau_3)} \frac{\delta}{\delta J(\tau_4)} Z[J] \right|_{J=0} \]  
(78)

By the usual rules of AdS/CFT, the quantity \( Z[J] \) is given by a bulk path integral with the following non-normalizable (source-type) boundary condition over the dual scalar field \( \eta (\nu = \Delta - \frac{1}{2}) \)

\[ \eta(\tau, z) = z^{\frac{1}{2} - \nu} J(\tau) + \text{(higher order in } z), \]  
(79)

in addition to the non-normalizable dilaton boundary condition [50] we described in the previous sections. We will (a) first compute the matter path integral, for a fixed metric characterized by a particular Diff element \( f \) (e.g. [17]), and then (b) integrate over the metrics (i.e. integrate over the functions \( f \)). This procedure gives us

\[ Z[f, J] = \exp \left[ \int d\tau d\tau' J(\tau) G_{f,\Delta}(\tau, \tau') J(\tau') \right], \quad G_{f,\Delta}(\tau, \tau') \equiv [f'(\tau)f'(\tau')]^{\Delta} G_{0,\Delta}(f(\tau) - f(\tau')) \]  
(80)

\[ ^{18} \text{In this section, we will reserve the notation } \Delta \text{ for the scaling dimension of } O, \text{ rather than for that of the fermion.} \]
\[ Z[J] = \int [d\mu[f]] \exp[S_{\text{eff}}[f[\tau]]] Z[f, J] \]  

(81)

Here \( d\mu[f] \equiv \frac{\partial f(\tau)}{\partial f(\tau')} \) is an SL(2, \mathbb{R})-invariant measure \[2,32\]. The effective action \( S_{\text{eff}}[f] \) is given by \[67\]. To derive the first line, note that if we consider \( f = \tau \) (the identity transformation), corresponding to the 2D BTZ metric \[43\], we will have the familiar result (using tilde coordinates to distinguish the starting point of the orbit of large diffeomorphism)

\[ Z[0, J] = \exp \left[ \int d\tilde{\tau} d\tilde{\tau}' \tilde{J}(\tilde{\tau}) G_{0,\Delta}(\tilde{\tau} - \tilde{\tau}') J(\tilde{\tau}') \right] \]

where \( G_{0,\Delta}(\tilde{\tau} - \tilde{\tau}') \) is the two-point function

\[ G_{0,\Delta}(\tilde{\tau} - \tilde{\tau}') = \langle O(\tilde{\tau}) O(\tilde{\tau}') \rangle = \left( \frac{\pi}{\beta \sin \left( \frac{\pi (\tilde{\tau} - \tilde{\tau}')}{\beta} \right)} \right)^{2\Delta} \]

To compute \( Z[f, J] \) we need to do this computation in the metric \( \[39\] \) and \( \[42\] \). This can be done by applying a large diffeomorphism transformation asymptotically and noting that under such a transformation the source term \( J \) transforms as

\[ \tilde{z}^{\frac{1}{2}} - \nu \tilde{J}(\tilde{\tau}) = \tilde{z}^{\frac{1}{2}} - \nu J(\tau) \]

by virtue of \[70\] and the fact that \( \eta \) is a scalar. This leads to the expression for \( G_{f,\Delta} \) in \( \[80\] \).

Eq. (81) is simply obtained by noting that the integral over the space of metrics, as shown in \[2\], reduces to the integration of the \( f \)-variables with the Schwarzian effective action.

Using \( \[78\] \), \( \[80\] \), \( \[81\] \) we get

\[ \langle O(\tau_1) O(\tau_2) O(\tau_3) O(\tau_4) \rangle = \int [d\mu[f]] \exp[S_{\text{eff}}[f]] \left( G_{f,\Delta}(\tau_1, \tau_2) G_{f,\Delta}(\tau_3, \tau_4) + (\tau_2 \leftrightarrow \tau_3) + (\tau_2 \leftrightarrow \tau_4) \right) \]

(82)

In the \( G_N \to 0 \) limit, it is enough to expand \( S_{\text{eff}}[f] \) up to quadratic order in \( \epsilon \) defined by \( f(\tau) = \tau + \sqrt{16\pi G_N} \epsilon(\tau) \):

\[ S_{\text{eff}}[f] = a \frac{1}{2} \int_0^\beta d\tau \left( (\epsilon')^2 - \left( \frac{2\pi}{\beta} \right)^2 \langle \epsilon' \rangle^2 \right) + ... \]

with propagator

\[ D(\omega) = \frac{1}{a} \frac{1}{\omega^2 - (2\pi/\beta)^2} \]

(83)

Correspondingly, in \( \[82\] \), we need to expand the Green’s functions to leading order in \( \epsilon \); thus:

\[ G_{f,\Delta}(\tau_1, \tau_2) = G_{0,\Delta}(\tau_1, \tau_2) + \epsilon(\tau_1) \delta G_1 + \epsilon(\tau_2) \delta G_2 + O(\epsilon^2), \]

\[ G_{f,\Delta}(\tau_3, \tau_4) = G_{0,\Delta}(\tau_3, \tau_4) + \epsilon(\tau_3) \delta G_3 + \epsilon(\tau_4) \delta G_4 + O(\epsilon^2). \]

(84)

Collecting all, the Euclidean four-point correlator is of the form

\[ \langle O(\tau_1) O(\tau_2) O(\tau_3) O(\tau_4) \rangle = G_{0,\Delta}(\tau_1, \tau_2) G_{0,\Delta}(\tau_3, \tau_4) + (\tau_2 \leftrightarrow \tau_3) + (\tau_2 \leftrightarrow \tau_4) + \frac{1}{16\pi G_N} \left( \delta G_2 \delta G_3 \langle \epsilon(\tau_2) \epsilon(\tau_3) \rangle + \frac{1}{2} \left( \sum_{j=2}^{4} \langle 2, j \rightarrow (1, 3) \rangle + \langle 2, j \rightarrow (1, 4) \rangle + \langle 2, j \rightarrow (2, 4) \rangle \right) \right) \]

(85)

In the same manner as in the boundary theory calculation \[8\], an OTO correlator \( \langle O(0) O(T) O(0) O(T) \rangle \) can be obtained by an appropriate analytic continuation of the above expression. The chaotic growth originates from the \( \epsilon \)-propagators in the second line of the above equation (see the schematic representation in Figure \[2\] of the first term in parenthesis in the above equation). The pole \( \omega = -2\pi/\beta \) of the propagator
under the Lorentzian continuation leads to a growing term proportional to \(\exp[\lambda_L T]\) for large \(T\), with \(\lambda_L = 2\pi/\beta^{19}\).

5.2 OTO correlators in the charged model

Let us now come back to the charged model. We would like to compute, from the bulk dual, the OTO correlator of a charged operator \(O\) (with charge \(q\)): \(\langle O(0)O^*(t)O(0)O^*(t)\rangle\). The computation follows along similar lines as in the uncharged case; we will thus highlight the essential differences.

The bulk path integral is now given by (cf. [50], [51])

\[
Z[f, \varphi, J] = \exp \left[ \int d\tau d\tau' J(\tau)G_{f,\varphi,\Delta}(\tau, \tau') J(\tau') \right],
\]

\[
G_{f,\varphi,\Delta}(\tau, \tau') \equiv [f'(\tau)f'(\tau')]^{-\Delta} G_{0,\Delta}(f(\tau) - f(\tau')) \exp[iq(\varphi(\tau_2) - i\varphi(\tau_2))]
\]

(86)

\[
Z[J] = \int [d\mu[f, \varphi]] \exp[S_{\text{eff}}[f[\tau], \varphi[\tau]]]Z[f, \varphi, J]
\]

(87)

The effective action \(S_{\text{eff}}\) is now given by (1). The additional part in the expression for \(G_{f,\varphi,\Delta}\), involving the large gauge transformation \(\varphi\), comes from the fact from the gauge transformation of \(J\) which is inherited from that \(\eta\) (cf. [29]). The counterpart of (54) is now given by (up to leading term)

\[
G_{f,\varphi,\Delta}(\tau_1, \tau_2) = G_{0,\Delta}(\tau_1, \tau_2) + \epsilon(\tau_1)\delta G_1 + \epsilon(\tau_2)\delta G_2 + \varphi(\tau_1)\delta G'_1 + \varphi(\tau_2)\delta G'_2,
\]

\[
G_{f,\varphi,\Delta}(\tau_3, \tau_4) = G_{0,\Delta}(\tau_3, \tau_4) + \epsilon(\tau_4)\delta G_3 + \epsilon(\tau_4)\delta G_4 + \varphi(\tau_3)\delta G'_3 + \varphi(\tau_4)\delta G'_4.
\]

(88)

Now, \((\varphi)\), \textit{a priori}, leads to mixed propagators (\(c\varphi\)). However, by making a field redefinition \(\varphi \rightarrow \tilde{\varphi} = \varphi - i\mu\epsilon \varphi\), the two fields get decoupled. This leads to an expression similar to (55) which involves terms with \(c\)-propagators as well as \(\tilde{\varphi}\) propagators. The latter propagator is \(\propto 1/\omega^2\), and does not have any other nontrivial pole in the complex plane. The structure of the Euclidean propagator is schematically represented in Figure 4. The Liapunov growth is governed by the graviton propagator, as before; the boundary photon propagator does not show any chaotic growth— its Liapunov exponent is zero.

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\(\textbf{19}\) The way this happens in practice is a bit more subtle [5]; the poles of (59), \(\omega = 0, \pm 2\pi/\beta\), correspond to \(SL(2,\mathbb{R})\) zero modes, and hence are excluded from the Matsubara sum involved in the real time propagator \(\langle \varphi(0)\varphi(t)\rangle\). However, the sum over all other frequencies leads to a contour integral which gets deformed to a new contour which includes the only the poles \(\omega = 0, \pm 2\pi/\beta\).
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A KK reduction of 3D action

We start with the general KK ansatz,

\[ ds^2 = g^{(3)} dx^M dx^N = e^{2\alpha\phi} g_{\mu\nu} dx^\mu dx^\nu + \ell^2 e^{2\beta\phi} (d\theta + B_\mu dx^\mu)^2 \]  

(89)

here \( \alpha \) and \( \beta \) are some constants. The labels \( M, N \) run over 3D coordinates \( \{\tau, z, \theta\} \), and \( \mu, \nu \) run over 2D coordinates \( \{\tau, z\} \). The metric components \( g_{\mu\nu} \), scalar and vector fields, \( \phi \) and \( B_\mu \) are independent of the compactified direction \( \theta \). Under this identification we obtain,

\[
R^{(3)} = e^{-2\alpha\phi} \left[ R^{(2)} - 2\beta^2 \partial_\mu \phi \partial^\mu \phi - 2(\alpha + \beta) \nabla^2 \phi - \frac{\ell^2 e^{-2(2\alpha - \beta)\phi}}{4} \tilde{F}^2 \right] ; \\
K^{(3)} = e^{\alpha\phi} \left[ K^{(2)} + \frac{\alpha + \beta}{\beta} e^{-\phi} n^a \nabla_a e^{\beta\phi} \right] \\
\sqrt{g^{(3)}} = 1 e^{(2\alpha + \beta)\phi} \sqrt{g^{(2)}} ; \quad \sqrt{h} = 1 e^{(\alpha + \beta)\phi} \sqrt{\gamma} 
\]  

(90)

\( \tilde{F} \) is the field strength tensor for the KK vector field \( B = B_\mu dx^\mu \). Thus the 3D action (14) reduces to a 2D action,

\[
S_{2\text{Dgrav}} = -\frac{\ell}{8G_3} \int_{M_2} d\tau dz \left[ \sqrt{g^{(2)}} \Phi^{-\beta} \left( R^{(2)} + \frac{2}{\ell^2} - \frac{2}{\Phi^2} (\alpha + \beta - 2\beta^2) \partial_\mu \Phi \partial^\mu \Phi + 2(\alpha + \beta) \frac{1}{\Phi} \nabla^2 \Phi \\
- \frac{\ell^2 \Phi^2 (2\alpha - \beta)}{4} \tilde{F}^2 \right) \right] - \frac{1}{4G_3} \int_{\partial M_2} d\tau \sqrt{\gamma} \Phi^{-1} \left[ K^{(2)} - \frac{\alpha + \beta}{\Phi} n^a \nabla_a \Phi - \frac{1}{\ell} \right] 
\]  

(91)

where we have redefined \( \phi \) to \( -\log \Phi \). Choosing specific values for the constants as \( \alpha = 0 \) and \( \beta = -1 \), we can remove the kinetic terms for the dilaton using divergence theorem. This choice of constants leads to,

\[
S_{2\text{Dgrav}} = -\frac{1}{8G_3} \int_{M_2} d\tau dz \sqrt{g^{(2)}} \Phi \left( R^{(2)} + \frac{2}{\ell^2} - \frac{\ell^2 \Phi^2}{4} \tilde{F}^2 \right) - \frac{1}{4G_3} \int_{\partial M_2} d\tau \sqrt{\gamma} \Phi \left( K^{(2)} - 1 \right) 
\]  

(92)

With the identification, \( G_3 = 2\pi iG_2 \), we obtain the action (18).

Under KK reduction we write the 3D CS gauge field \( A_M dx^M = A_\mu dx^\mu + \chi d\theta \), where \( \chi \) is 2D scalar field. Using metric and gauge field reduction and dropping all the \( y \) derivatives, it is straightforward to obtain action (20).

B Asymptotically AdS\(_2\) uplifted to 3D

As seen in section 4.1 with the solution \( B_\mu = 0 \) and \( \chi = 0 \), the general solutions to the equation (32) are the asymptotically AdS\(_2\) spacetimes,

\[
ds^2_{\text{ads}} = \frac{\ell^2}{z^2} \left( dz^2 + d\tau^2 \left( 1 - \frac{z^2}{2} \{f, \tau\} \right)^2 \right). 
\]  

(93)
In this background the solution for the dilaton is,
\[ \Phi = \frac{a + b\tilde{r} + c(\tilde{r}^2 + \tilde{z}^2)}{\tilde{z}} , \]  
(94)
where the \( \{ \tilde{r}, \tilde{z} \} \) are given by,
\[ \tilde{r} = f(\tau) = \frac{2z^2 f''(\tau)f'(\tau)^2}{4f'(\tau)^2 + z^2 f''(\tau)^2} , \quad \tilde{z} = \frac{4z f'(\tau)^3}{4f'(\tau)^2 + z^2 f''(\tau)^2} , \]  
(95)
The 3D oxidation of this solution will be,
\[ ds_{\text{uplift}}^2 = \frac{l^2}{z^2} dz^2 + \frac{l^2}{z^2} \left( 1 - \frac{z^2}{2} \{ f, \tau \} \right)^2 d\tau^2 + l^2 \Phi^2 d\theta^2 . \]  
(96)
It satisfies the 3D Einstein equations with \( \Lambda = -1/l^2 \), see equations (15). This 3D solution violates the Brown and Henneaux conditions. In the metric (96) the \( g_{\theta\theta} \) component corresponds to a non-normalizable deformation of AdS3 (parameterized by the \( f(\tau) \)).

C General expression for gauge field in black hole background

The BTZ black hole metric is
\[ ds_{\text{BTZ}}^2 = \frac{l^2}{z^2} \left( dz^2 + d\tau^2 \left( 1 - \frac{z^2}{\beta^2} \right)^2 \right) . \]  
(97)
The general solution for dilaton in this background is
\[ \Phi(\tau, z)_{\text{BTZ}} = (a + c) \frac{(\beta^2 + \pi^2 z^2)}{2\pi z} + (a - c) \cos \left( \frac{2\pi \tau}{\beta} \right) + b \sin \left( \frac{2\pi \tau}{\beta} \right) \frac{\beta^2 - \pi^2 z^2}{2\pi z} . \]  
(98)
Using the equation of motion (36) in the gauge \( A_\tau(\tau, z) = 0 \) we get,
\[ \partial_z A_\tau(\tau, z) = 2l J_0 \sqrt{g_{\text{BTZ}}} \Phi(\tau, z)_{\text{BTZ}} . \]  
(99)
General solution for the above equation is
\[ A_\tau(\tau, z) = a_1^{a,b,c, J_0}(\tau) \frac{1}{z^2} + a_2^{a,b,c, J_0}(\tau) \log(z) + a_3^{a,b,c, J_0}(\tau) z^2 + c_1(\tau) \]  
(100)
where
\[ a_1^{a,b,c, J_0}(\tau) = -\frac{\beta J_0 \beta^3}{2\pi} (a + c + (a - c) \cos \Theta + b \sin \Theta) \]  
(101)
\[ a_2^{a,b,c, J_0}(\tau) = -\frac{2\pi J_0 \beta^3}{\beta} ((a - c) \cos \Theta + b \sin \Theta) \]  
(102)
\[ a_3^{a,b,c, J_0}(\tau) = -\frac{\pi^3 J_0 \beta^3}{2\beta^3} (a + c - (a - c) \cos \Theta - b \sin \Theta) \]  
(103)
where \( \Theta = 2\pi \tau / \beta \). We also have to ensure that this \( A_\tau(\tau, z) \) vanishes on the horizon \( z = \beta / \pi \) this can be arranged by choosing integration constant
\[ c_1(\tau) = \frac{\pi J_0 \beta^3 (a + c)}{\beta} + \frac{\pi J_0 \beta^3}{\beta} (2(a - c) \log(\beta / \pi) \cos \Theta + 2b \log(\beta / \pi) \sin \Theta) \]  
(104)
We can see that constant part of the gauge field \( A_\tau \) comes from the constant part of \( c_1(\tau) \) which is \( \pi J_0 \beta^3 (a + c) / \beta \). We rewrite equation (100) as \( A_\tau(\tau, z) = \tilde{A}(\tau, z) + A_\tau(\tau, z) \). Here, \( \tilde{A}(\tau, z) \) contains all the \( z \) dependent terms which diverge at the boundary and \( A_\tau(\tau, z) \) at the boundary is just a constant.
When we perform diffeomorphisms on $A_r(\tau, z) = a_r(\tau, z) + \bar{A}_r(\tau, z)$ we will get $A_r^f(\tau, z) = a_r^f(\tau, z) + \bar{A}_r(\tau, z)$. At the boundary, $a_r^f = a_r f^r(\tau)$ and

$$
\bar{A}_r^f(\tau, z) = \bar{a}_1^{a,b,c,j_0}(\tau) \frac{1}{2 \tau^2} + \bar{a}_2^{a,b,c,j_0}(\tau) \log(\tau) + \bar{c}^{a,b,c,j_0}(\tau) + O(z^2)
$$

$$
\bar{a}_1^{a,b,c,j_0}(\tau) = - \frac{\beta j_0^3}{2 \pi f^r(\tau)} (a + c + (a - c) \cos(\Theta^f) + b \sin(\Theta^f))
$$

$$
\bar{a}_2^{a,b,c,j_0}(\tau) = - \frac{2 \pi j_0^3 f^r(\tau)}{\beta} ((a - c) \cos(\Theta^f) + b \sin(\Theta^f))
$$

$$
\bar{c}^{a,b,c,j_0}(\tau) = \frac{j_0^3}{4 \pi \beta f^r(\tau)^3} \left( - \beta^2 f''(\tau)^2 ((a - c) \cos(\Theta^f) + a + b \sin(\Theta^f) + c)
+ 4 \pi^2 f'(\tau)^4 (-2 \log(\Theta^f/2)) ((a - c) \cos(\Theta^f) + b \sin(\Theta^f))
+ \beta^2 f'(\tau)^2 (a - c) \cos(\Theta^f) + a + b \sin(\Theta^f) + c)
+ 2 \beta f'(\tau)^2 f''(\tau) ((a - c) \sin(\Theta^f) + b \cos(\Theta^f)) \right)
$$

where $\Theta^f = 2 \pi f(\tau)/\beta$.

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