Quantization of hypersurface orbital varieties in $\mathfrak{sl}_n$

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1. Introduction.

1.1. Amongst his many great contributions to mathematics, Alexander Kirillov was a co-founder of the “orbit method”. This exploits the symplectic structure on coadjoint orbits with respect to a Lie algebra $\mathfrak{g}$. From a Lagrangian subvariety one attempts to construct a representation of $\mathfrak{g}$ associated to the given orbit. If $\mathfrak{g}$ is semisimple, the orbit

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is nilpotent and one wishes to construct a highest weight module, then this Lagrangian subvariety should be a so-called orbital variety. Then “quantization” of the latter leads to the required representation [J3].

In this paper we consider only the case when $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$. By [M4] all orbital varieties can be weakly quantized (though this fails [J3, 1.3] for arbitrary $\mathfrak{g}$). It remains to show that they can be strongly quantized. This is a rather more delicate question. We shall settle this positively for orbital varieties which are of codimension 1 in the nilradical of a parabolic. These are called hypersurface orbital varieties.

1.2. Let $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ be a triangular decomposition. We identify $\mathfrak{n}^-$ with $(\mathfrak{n}^+)^*$ through the Killing form. Let $G$ be adjoint group of $\mathfrak{g}$ and $B$ the Borel subgroup with Lie algebra $\mathfrak{b} := \mathfrak{h} \oplus \mathfrak{n}^+$. For any Lie algebra $\mathfrak{a}$, let $S(\mathfrak{a})$ (resp. $U(\mathfrak{a})$) denote its symmetric (resp. enveloping) algebra. It is well-known that $G\mathfrak{n}^+$ is a finite union (of so-called nilpotent) orbits classified by Jordan normal form. Let $O$ be such an orbit. After Spaltenstein [S] the irreducible components of $O \cap \mathfrak{n}^+$ have dimension $1/2 \dim O$. They are called orbital varieties and after Steinberg [St] are classified by the standard tableaux whose shape is specified by $O$. As noted in [J1, 7.3] an orbital variety $V$ can be characterized as an irreducible subvariety of $\mathfrak{n}^+$ for which the ideal $I(V)$ of definition of its closure is stable under the Poisson bracket on $\mathfrak{g}$ (induced by the Lie bracket).

1.3. Retain the notation of 1.2. Roughly speaking $V$ can be strongly quantized if $S(\mathfrak{n}^-)/I(\mathfrak{V})$ can be given the structure of a highest weight module (not necessarily simple). Here the choice of highest weight is a crucial and delicate point. Take $\lambda \in \mathfrak{h}^*$ and let $V(\lambda)$ be a highest weight module with highest weight vector $v_\lambda$ of weight $\lambda$. Let $F$ be the canonical (degree) filtration on $U(\mathfrak{n}^-)$ (resp. on $S(\mathfrak{n}^-)$). We say that $V(\lambda)$ is a strong (resp. weak) quantization of $V$ if $\text{gr } F \text{Ann } U(\mathfrak{n}^-)v_\lambda = I(V)$ (resp. if its radical equals $I(V)$).

1.4. In order to exhibit a strong quantization we have to compute $I(\mathfrak{V})$ (or at least to calculate the formal character of $S(\mathfrak{n}^-)/I(\mathfrak{V})$). In [J2, Lecture 7] the general form that this should take is suggested. Let $N$ be the generic matrix of $\mathfrak{n}^- + 1d$. Then $I(\mathfrak{V})$ should take the form $\text{gr } F < a_1, a_2, \ldots, a_n >$, where $a_i$ are amongst the minors of $N$. This suggestion derived from an algorithm for $\text{Ann } U(\mathfrak{n}^-)V$ based on the Enright functor [J1, 8.4] together with calculations in the thesis of E. Benlolo.

1.5. Of course the above does not tackle the difficult question as to which minors $a_i$ to choose. Recently Benlolo and Sanderson [BS] made an appealing conjecture for
this choice concerning orbital varieties $V$ having codimension 1 in the nilradical $m^+$ of a parabolic subalgebra $p$. Apart from the obvious $1 \times 1$ minors it follows from Krull’s theorem that $I(V)$ is generated by just one element $f$ which can be assumed to be an $h$ weight vector. A result in [M3] describes in particular all the orbital varieties of codimension 1 in a given $m^+$. Using this, Benlolo-Sanderson [BS] show that one may reduce to the case when $f$ cannot be expressed in terms of the generic matrices of $n^-$ obtained by deleting the first columns and last rows. Up to this reduction they show that $m^+$ is given by blocks of size $c_1, c_2, \cdots, c_l$ with $c_1 = c_l = c$ and $c_i \neq c$, $\forall i : 1 < i < l$.

Now let $M(t)$ be the generic matrix of $m^- + t1d$, where $m^-$ is the opposed algebra of $m^+$ identified with $(m^+)^*$ and $t$ is an indeterminate. Let $M(t)$ be the $(n-c) \times (n-c)$ minor of $M(t)$ lying in the bottom left-hand corner. Then $M(t)$ is a polynomial in $t$. They conjecture that $f$ is the coefficient of its lowest degree term. Let $\beta_1$ be the highest root of $n^+$, $\beta_2$ the highest root of the subalgebra of $n^+$ obtained by omitting the first row and last column, and so on. They show that $f$ has weight $-\sum_{i=1}^{c} \beta_i$.

1.6. One may note that $f$ of 1.5 can also be described as $\text{gr}_F M(1)$. However as we shall see there is a certain computational advantage in retaining $t$. (Furthermore BS compute the lowest power of $t$ in $M(t)$). One may also replace $m^- + 1d$ in the above by $n^- + 1d$ as long as one includes the obvious $1 \times 1$ minors in $I(V)$. In this sense $M(1)$ only depends on the first column size $c$. Likewise $f = \text{gr}_FM(1)$ depends only partially on $m^-$. In particular the weight of $f$ only depends on $c$. Although BS do check their conjecture in a number of cases, it is interesting to know if it holds in general since the above independence is somewhat surprising. Moreover even the relatively simple case of hypersurface varieties has some remarkable structure.

1.7. Our proof of the BS conjecture involves showing that $S(m^-)f$ is semi-prime and its zero variety is $P$ stable, where $P \supset B$ is the parabolic subgroup with Lie algebra $p$. This holds irrespective of whether $c_i \neq c$, for $i \neq 1, l$. However we show that $f$ is irreducible if and only if this condition holds. This is a delicate point. We shall give three proofs. One is based on a careful counting of $t$ powers in multiplying out $M(t)$ and a knowledge [M2, M3] of the orbital varieties of codimension 1 in $m^+$. The second is based on a representation theoretic argument combined with [M4] which asserts that the associated variety of a highest weight module of integral highest weight is irreducible. The third is the least computational and is based on the irreducibility of the associated variety of the annihilator of a simple highest weight module.

1.8. We shall exhibit a strong quantization of $V$ by determining a simple highest weight
module $V(\lambda)$ whose formal character $\text{ch} V(\lambda)$ coincides with that of $S(m^-)/S(m^-)f$ and by using the linear independence of the characteristic polynomials of orbital varieties. To determine $\lambda$ and compute $\text{ch} V(\lambda)$, we use the formula of Jantzen [Ja] describing the Shapovalov determinants defined with respect to $p$. Let $I$ be the subset of the set of simple roots $\Pi$ defining $p$. Assume $(\alpha, \alpha) = 2$ for any root. Then it suffices to take $\lambda \in \mathfrak{h}^*$ such that $(\lambda, \alpha) = 0$, $\forall \alpha \in I$, $(\lambda, \beta_i) = c - (n - 1)$ and $(\lambda, \gamma) \notin \mathbb{Z}$, for any proper sum of roots in $\Pi' := \Pi \setminus I$. However we also give a choice of integral $\lambda$. This is more delicate since the condition $c_i \neq c$, $\forall i : 1 \leq i \leq k - 1$ must be invoked. Combined with [M4] it leads to the second proof of the irreducibility of $f$. The non-integral case gives our third proof.

We further show that there is a choice of $\lambda$ such that the annihilator of $V(\lambda)$ is maximal as well as $V(\lambda)$ being a strong quantization of $\mathcal{V}$.

1.9. The truth of BS conjecture has the following remarkable consequence. One may specify positive integers $s, t$ such that $f$ is exactly the highest common divisor of the $t \times t$ minors of $\mathcal{M}(0)^s$. There seems to be no elementary proof of this purely combinatorial fact.

2. Combinatorial Preliminaries and the Benlolo-Sanderson Conjecture

2.1 The base field is assumed algebraically closed of characteristic zero and can be taken to be the complex field $\mathbb{C}$ without loss of generality.

Recall the notation and the hypotheses of 1.1 and 1.2. Put $n = n^+$. In particular an orbital variety $\mathcal{V}$ associated to a nilpotent orbit $\mathcal{O}$ is an irreducible component of $\mathcal{O} \cap n$. Let $V$ denote the set of all orbital varieties of $\mathfrak{g}$ and $W$ the Weyl group for the pair $(\mathfrak{g}, \mathfrak{h})$. We describe first the Steinberg map of $W$ onto $V$. Let $R \subset \mathfrak{h}^*$ denote the set of non-zero roots, $R^+$ the set of positive roots corresponding to $n$ in the triangular decomposition of $\mathfrak{g}$ and $\Pi \subset R^+$ the resulting set of simple roots. Let $X_\alpha = \mathbb{C}x_\alpha$ denote the root subspace corresponding to $\alpha \in R$. Then $n = \bigoplus_{\alpha \in R^+} X_\alpha$ (resp. $n^- = \bigoplus_{\alpha \in -R^+} X_\alpha$).

For each $w \in W$ set $n \cap^w n := \bigoplus_{\alpha \in R^+ \cap w(R^+)} X_\alpha$. For each closed, irreducible subgroup $H$ of $G$ let $H(n \cap^w n)$ be the set of $H$ conjugates of $n \cap^w n$. It is an irreducible locally closed subvariety. Since there are only finitely many nilpotent orbits in $\mathfrak{g}$ it follows that there exists a unique nilpotent orbit $\mathcal{O}$ such that $\overline{G(n \cap^w n)} = \overline{\mathcal{O}}$. A result of Steinberg [St] asserts that $\mathcal{V}_w := \overline{B(n \cap^w n)} \cap \mathcal{O}$ is an orbital variety and that the map $\phi : w \mapsto \mathcal{V}_w$ is a surjection of $W$ onto $V$.

2.2 It is convenient to replace $\mathfrak{sl}_n$ by $\mathfrak{g} = \mathfrak{gl}_n$. This obviously makes no difference.
Note that the adjoint action is just conjugation by $G = GL_n$. Let $n$ be the subalgebra of strictly upper-triangular matrices and let $n^-$ be the subalgebra of strictly lower-triangular matrices. Let $B$ be the (Borel) subgroup of upper-triangular matrices in $G$. All parabolic subgroups we consider further are standard, that is contain $B$.

Let $e_{i,j}$ be the matrix having 1 in the $ij$-th entry and 0 elsewhere.

Take $i < j$ and let $\alpha_{i,j}$ be the root corresponding to $e_{i,j}$. Set $\alpha_{j,i} = -\alpha_{i,j}$. We write $\alpha_{i,i+1}$ simply as $\alpha_i$. Then $\Pi = \{ \alpha_i \}_{i=1}^{n-1}$. Moreover $\alpha_{i,j} \in R^+$ exactly when $i < j$. For each $\alpha \in \Pi$, let $s_\alpha \in W$ be the corresponding reflection and set $s_i = s_{\alpha_i}$.

**2.3** We represent every element of the symmetric group $S_n$ in word form

$$w = [a_1, a_2, \ldots, a_n], \quad \text{where } a_i = w(i).$$

We identify $W$ with $S_n$ by taking $s_i$ to be the elementary permutation interchanging $i$, $i + 1$.

**Definition.** Given $w = [a_1, \cdots, a_n]$. Set $p_w(i) = j$ if $a_j = i$, that is $p_w(i)$ is the place (index) of $i$ in the word form of $w$.

One has $w(p_w(i)) = w(j) = a_j = i$, that is $p_w(i) = w^{-1}(i)$. On the other hand $w(\alpha_{i,j}) = \alpha_{w(i), w(j)}$. Set $S(w) = \{ \alpha \in R^+ \mid w(\alpha) \in -R^+ \}$.

This gives the following result.

**Lemma.** Take $i < j$. Then $\alpha_{i,j} \in S(w^{-1})$ if and only if $p_w(i) > p_w(j)$.

**Remark.** In particular $\alpha_i \in S(w^{-1})$ exactly when $i + 1$ comes before $i$ in the word form of $w$. This is of course well-known.

**2.4** Let $P(n)$ denote the set of partitions of $n$. Then $\lambda := \{ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0 \} \in P(n)$ defines the sizes $\lambda_i$ of the Jordan blocks of some element $x_\lambda \in n$. Set $O_\lambda = Gx_\lambda$. Let $\lambda^* = \{ \lambda_1^* \geq \lambda_2^* \geq \cdots \geq \lambda_l^* > 0 \}$ be the dual partition. By definition $\lambda_i^* = \sharp \{ j \mid \lambda_j \geq i \}$. Then $k = \lambda_1^*$, $l = \lambda_1$ and this notation will be fixed throughout. We view $\lambda$ as a Young diagram $D_\lambda$ with $k$ rows of length $\lambda_1, \ldots, \lambda_k$.

The rank of a Jordan block of size $r$ is just $r - 1$ so $\text{rk} x_\lambda = \sum_{i=1}^k (\lambda_i - 1) = n - \lambda_1^*$. For all integer $i \geq 1$, the Jordan blocks of $x_\lambda^i$ are obtained by deleting the first $i$ columns of $\lambda$. Then

1. $\text{rk} x_\lambda^i = n - \sum_{j=1}^i \lambda_j^*$.
2. $\text{Again (cf. [H, §3.8]) dim } O_\lambda = n^2 - \sum_{i=1}^k (\lambda_i^*)^2$.

**2.5** Define a partial order on $P(n)$ as follows. Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \lambda_k)$ and $\mu = (\mu_1 \geq \mu_2 \geq \cdots \mu_k)$ be partitions of $n$ which we can assume to correspond to
diagrams of the same height by adding empty rows. Set \( \lambda \leq \mu \) if
\[
\sum_{j=1}^{i} \lambda_j \geq \sum_{j=1}^{i} \mu_j , \quad \text{for all} \quad i = 1, 2, \ldots, k.
\]
The following result of M. Gerstenhaber (cf. [H, §3.10]) shows that this order corresponds to inclusion of nilpotent orbit closures:

**Theorem.** Given two partitions \( \lambda \) and \( \mu \) of \( n \) one has \( \lambda \leq \mu \) if and only if \( \mathcal{O}_\mu \subset \mathcal{O}_\lambda \).

In our convention \( \lambda = (n) \) is the minimal partition of \( n \) and \( \lambda = (1, 1, \ldots, 1) \) is the maximal one.

2.6 For a partition \( \lambda \) of \( n \) one can fill the boxes of \( D_\lambda \) with \( n \) distinct positive integers. If the entries increase in rows from left to right and in columns from top to bottom, we call such an array a Young tableau. If the numbers in Young tableau form a set of integers from 1 to \( n \) we call it standard. Let \( T_n \) be the set of standard Young tableaux of size \( n \).

We shall not distinguish between Young and standard tableaux. Indeed if a Young tableau has entries \( t_1 < t_2 < \cdots < t_n \) then it can be identified with the standard tableau obtained by replacing \( t_i \) by \( i \). Similarly the sequence \( [t_{w(1)}, \ldots, t_{w(n)}] : w \in S_n \) can be viewed as a word form of \( w \) by replacing \( t_i \) by \( i \). We call this process standardization. Occasionally it can cause confusion; but in such cases adequate warning will be given.

We may also concatenate word forms so then \( [a_1, a_2, \ldots, a_n] \) can be written as \( [a', a''] \) where \( a' = [a_1, a_2, \ldots, a_i], \ a'' = [a_{i+1}, \ldots, a_n] \).

The shape of a Young tableau \( T \) is defined to be the Young diagram from which \( T \) was built. It defines a partition of \( n \) which we denote by \( \text{sh} \, T \).

Given \( T \in T_n \), let \( \sigma(T) \) denote the conjugate standard tableau obtained by rotation about the main diagonal. We remark that \( \text{sh} \, \sigma(T) = (\text{sh} \, T)^* \). Again \( \sigma \) takes a row tableau \( R \) into a column tableau \( C \) and vise-versa.

Given any sequence \( S \) of strictly increasing integers (for example a row \( R \) or column \( C \) above) let \( \hat{S} \) denote the reversed sequence of strictly decreasing integers. For any tableau or word form \( S \), let \( \langle S \rangle \) denote the set of entries of \( S \) and \( |S| \) the cardinality of \( \langle S \rangle \).

The Robinson - Schensted correspondence \( w \mapsto (Q(w), R(w)) \) gives a bijection (see, for example [Kn] or [F]) from the symmetric group \( S_n \) onto the pairs of standard Young tableaux of the same shape. By R. Steinberg [St] for all \( w, y \in S_n \) one has \( \mathcal{V}_w = \mathcal{V}_y \).
iff $Q(w) = Q(y)$. This parameterizes the set of orbital varieties $V$ by $T_n$. Moreover $sh Q(w) = \lambda$ if and only if $V_w$ is contained in $O_\lambda$.

If $Q(w) = T$ we set $V_T := V_w$ and $T_{V_w} := Q(w)$.

We define a partial order on Young tableaux and a partial order on orbital varieties analogous to the order on partitions given in 2.5. It is called the geometric order on $V$ or on $T_n$.

**Definition.** (i) Given distinct $V_1, V_2 \in V$. Set $V_2 \succ V_1$ if $V_2 \subset V_1$.

(ii) Given distinct $T_1, T_2 \in T_n$. Set $T_2 \succ T_1$ if $V_{T_2} \succ V_{T_1}$.

In general the problem of the combinatorial description of geometric order is extremely difficult. Partial results are described in [M5], which for the case studied below gives a complete answer. All that is needed here in 2.17; though a more general result is given in [M2].

2.7 Fix $T \in T_n$ with $sh T = \lambda$. For all $i, j \in \{1, \ldots, n\}$ let $T^i_j$ denote the $ij$-th entry of $T$ when it is defined.

**Definition.** For any tableau $T$

(i) Given $s = T^i_j$, set $r_T(s) = i$ and $c_T(s) = j$.

(ii) For all $1 \leq i \leq \lambda^*_i = k$, set $T^i = (T^i_1, \ldots, T^i_k)$. It is the $i$-th row of $T$.

(iii) For all $j : 1 \leq j \leq \lambda_1 = l$, set $T_j = (T^1_j, \ldots, T^\lambda_j)$. It is the $j$-th column of $T$.

(iv) The hook number of the $ij$-th entry of $T$ is defined by

$$h(T^i_j) = 1 + (\lambda^*_j - j) + (\lambda_i - i).$$

(v) $T^i_j$ is called a corner entry of $T$ if $h(T^i_j) = 1$.

(vi) If $i \leq j$, then $T^{i,j}$ denotes the subtableau with rows $T^i, T^i+1, \ldots, T^j$. Otherwise it denotes the empty tableau.

A row (resp. column) of $T$ is determined by its entries since these must increase from left to right (resp. from top to bottom).

Let $T = (T_1, T_2, \ldots, T_l)$, $S = (S_1, S_2, \ldots, S_{l'})$ be Young tableaux given by their columns. Assume that $T, S$ have no common entries. Then we define $(T, S)$ to be the array whose rows are the same as the rows of $(T_1, T_2, \ldots, T_l, S_1, S_2, \ldots, S_{l'})$, that is $r_{(T,S)}(T^i_j) = i$ and $r_{(T,S)}(S^i_j) = i$, and ordered in the increasing order. Of course this involves the shuffling of numbered boxes within a row.
Lemma. \((T, S)\) is a Young tableau.

Proof.

The proof is by induction on rows. Suppose that the integers in the first \(r\) rows have been placed in increasing order and that the resulting array of \(r\) rows obtained from \((T, S)\) is a Young tableau. Let \(i_1 < i_2 < \cdots < i_n\) be the integers of the \(r\)-th row and \(j_1 < j_2 < \cdots < j_m\) the ordered set of entries of \(< T^{r+1}, S^{r+1} >\). Since \(T, S\) are Young tableaux and their entries are distinct from one another, there exists an injective map \(\phi : \{1, 2, \ldots, m\} \rightarrow \{1, 2, \ldots, n\}\) such that \(j_t > i_{\phi(t)}\). Then \(j_s > j_t > i_{\phi(t)}\) for all \(t = 1, 2, \ldots, s\). Thus \(j_s\) exceeds \(s\) elements of the \(r\)-th row. This forces \(j_s > i_s\), as required.

If the entries of \(S\) all exceed those of \(T\) then one only needs to shift numbered boxes (to the left). In a similar fashion \((T^r)\) is defined. One has \(\sigma(T^r) = (\sigma(T)\sigma(S))\).

2.8 Take \(j \in \mathbb{N}\) such that \(j \notin < R >\). Let \(a_i\) be the smallest entry of \(R\) greater than \(j\) and set:

\[(R + j) := (a_1, \ldots, a_{i-1}, j, a_i, \ldots)\]

Define \(w_r, w_c \in W\) by word forms

\[w_r(T) := [T^\lambda, \ldots, T^1], \quad w_c(T) := [\hat{T}_1, \ldots, \hat{\lambda}_1]\]

Lemma. [M5 or M1, 3.2.2] One has

\[Q(w_r(T)) = Q(w_c(T)) = T.\]

Proof.

We give a proof for completion and to clarify the nature of Robinson-Schensted correspondence.

Let \(T\) be a Young tableau and \(b_1\) an integer \(> 0\) not belonging to the entries of \(T\). The Robinson-Schensted (RS) insertion of \(b_1\) into \(T\) gives a new Young tableau \((T \downarrow b_1)\). This is an inductive procedure in which \(b_1, b_2, \ldots\), are defined and \(b_i\) is inserted into the \(i\)-th row of \(T\). In detail, write \(T^1 = (a_1, a_2, \ldots, a_l)\). If \(b_1 > a_l\) then \(b_1\) is inserted into \(T^1\) and the process stops, that is

\[(T \downarrow b_1) = \begin{pmatrix} T^1 + b_1 \\ T^2 \end{pmatrix}.\]
Otherwise let \( i \) be the smallest integer such that \( b_1 < a_i \). Then \( (T \downarrow b_1)^1 \) is obtained by replacing \( a_i \) by \( b_1 \). Set \( b_2 = a_i \). Then

\[
(T \downarrow b_1) = \left( \begin{array}{c}
(T_1 \downarrow b_1)^1 \\
(T_{2,k} \downarrow b_2)
\end{array} \right).
\]

Let \( w = [a_1, a_2, \ldots, a_n] \) be a word form. Then \( Q(w) \) is obtained inductively as follows. Take \( Q_0(w) \) to be the empty tableau and set \( Q_i(w) = (Q_{i-1}(w) \downarrow a_i), \forall i : 1 \leq i \leq n \). Then \( Q(w) = Q_n(w) \).

We prove lemma by induction on rows. Let \( T \) be a Young tableau and set \( T^1 = (a_1, a_2, \ldots, a_l) \). Then \( a_i < a_{i+1}, a_i < T^2_i, \forall i : 1 \leq i \leq l \) (using the convention that the undefined entries \( T^j_i : j > 2 \), of \( T \) are set equal to \( \infty \)). Conversely starting from \( T_{2,k} \) and the sequence \( R = (a_1, a_2, \ldots, a_l) \) with the above properties we obtain a Young tableau \( S \) with \( S^1 = R, S^i = T^i, \forall i > 1 \). One checks that \( Q([w, (T_{2,k}, R)]) = S \) and this proves the first part of the lemma. The second obtains via \( \sigma \).

\[ \text{2.9} \quad \text{Let us describe the jeu de taquin (cf. [Sch]) which removes } T^i_j \text{ from } T. \text{ The resulting tableau is denoted by } T - T^i_j \text{ and is obtained as follows.}
\]

Consider a row \( R = (a_1, \cdots) \). Take \( j \in \mathbb{N} \) such that \( j > a_1 \) and \( j \not\in \langle R \rangle \). Let \( a_i \) be the greatest entry of \( R \) smaller than \( j \) and set:

\[
(R \uparrow j) := (a_1, \cdots, a_{i-1}, j, a_{i+1}, \cdots).
\]

One may remark that \( ((R \uparrow j) \downarrow a_i) = R \).

Take some element \( a_i \) of \( R \) and set:

\[
(R - a_i) := (a_1, \cdots, a_{i-1}, a_{i+1}, \cdots)
\]

Similar operations are defined for a column \( C \) except that we write \( (C \leftarrow j) \) for \( \sigma(\sigma(C) \uparrow j) \).

With these preliminaries we write

1. If \( h(T^i_j) = 1 \), then

\[
(T - T^i_j) = \left( \begin{array}{c}
T_{1,i-1}^{1} \\
(T^i_{i} - T^i_{j}) \\
T^{i+1,m}_{i}
\end{array} \right).
\]

2. If \( h(T^i_j) > 1 \), then
(i) If $T_{j+1}^{i} > T_{j}^{i+1}$ or $\lambda_{i} = j$, set

$$(T - T_{j}^{i}) = \left(\begin{array}{ccc}
T_{1,i}^{1,i} & (T_{i} \uparrow T_{j}^{i+1}) \\
(T_{j+1,m}^{i+1} - T_{j}^{i+1})
\end{array}\right)$$

(ii) If $T_{j+1}^{i} < T_{j}^{i+1}$ or $\lambda_{j}^{*} = i$, set

$$(T - T_{j}^{i}) = (T_{1,j-1}, (T_{j} \leftarrow T_{j+1}^{i}), (T_{j+1}^{i+1}, \lambda_{j} - T_{j+1}^{i}))$$

**Theorem.** [Sch] If $T$ is a Young tableau then so is $(T - T_{j}^{i})$.

2.10 Given $w \in W$, $T \in T_{n}$, $\mathcal{V} \in \mathcal{V}$, a standard parabolic subgroup $P$ and a standard parabolic subalgebra $p$. Let $p_{\alpha_{i}}$ be the minimal standard parabolic subalgebra including $X_{-\alpha_{i}}$ and let $P_{\alpha_{i}}$ be the corresponding standard parabolic subgroup. Define their $\tau$-invariants to be

$$\tau(w) := \Pi \cap S(w^{-1}),$$
$$\tau(T) := \{\alpha_{i} : r_{T}(i) > r_{T}(i)\},$$
$$\tau(P) := \{\alpha_{i} : P_{\alpha_{i}} \subset P\},$$
$$\tau(p) := \{\alpha_{i} : p_{\alpha_{i}} \subset p\},$$
$$\tau(\mathcal{V}) := \{\alpha_{i} : P_{\alpha_{i}}(\mathcal{V}) = \mathcal{V}\}.$$ 

Note that $P$ (resp. $p$) is uniquely determined by its $\tau$-invariant. Let $P_{T} = P_{\mathcal{V}_{T}}$ be the stabilizer of $\mathcal{V}_{T}$. It is a standard parabolic subgroup of $G$ and we set $p_{T} = p_{\mathcal{V}_{T}} = \text{Lie } P_{T}$. By say [J, §9] one has

**Lemma.** $\tau(w) = \tau(\mathcal{V}_{w}) = \tau(Q(w)) = \tau(P_{\mathcal{V}_{w}}) = \tau(p_{\mathcal{V}_{w}})$.

2.11 Given $\mathcal{I} \subset \Pi$, let $P_{\mathcal{I}}$ denote the unique standard parabolic subgroup of $G$ such that $\tau(P_{\mathcal{I}}) = \mathcal{I}$. Let $M_{\mathcal{I}}$ be the unipotent radical of $P_{\mathcal{I}}$ and $L_{\mathcal{I}}$ a Levi factor. Let $p_{\mathcal{I}}$, $m_{\mathcal{I}}$, $l_{\mathcal{I}}$ denote the corresponding Lie algebras. These notation will be conserved throughout though the subscripts may sometimes be dropped.

Given $w \in W$, let $\ell(w)$ denote its reduced length. Let $W_{\mathcal{I}}$ be the subgroup of $W$ generated by the $s_{\alpha} : \alpha \in \mathcal{I}$ and $w_{\mathcal{I}}$ its unique longest element. It is also defined by condition $S(w_{\mathcal{I}}) = R^{+} \cap N_{\mathcal{I}}$. The following is well known and easy to check.

**Lemma.** $\mathcal{V}_{\mathcal{I}} := \mathcal{V}_{w_{\mathcal{I}}}$ is the unique orbital variety with closure $m_{\mathcal{I}}$. Moreover if $\tau(\mathcal{V}) \supset \mathcal{I}$, then $\mathcal{V} \subset \overline{\mathcal{V}_{\mathcal{I}}} = m_{\mathcal{I}}$.

**Remark.** Set $\mathcal{O}_{\mathcal{I}} = G\mathcal{V}_{\mathcal{I}}$. One calls $\mathcal{V}_{\mathcal{I}}$ the Richardson component (of $\mathcal{O}_{\mathcal{I}} \cap n$) defined by $\mathcal{I}$. With respect to the order relation defined in 2.6, $\mathcal{V}_{\mathcal{I}}$ is the unique minimal orbital
variety with $\tau$–invariant $I$. In general if $n \cap w \subset n \cap w$ then trivially $V_y \subset B(n \cap w) = \overline{V}_w$. By the lemma the converse holds if $w = w_x$, for some $I \subset \Pi$; but fails in general [M5, 4.1.1].

2.12 Given $I \subset \Pi$. Write $T_{\forall x}$ simply as $T_I$. It is obtained by the following rules

(i) $r_T(1) = 1$.
(ii) Given $s : 1 < s \leq n$

\[
    r_T(s) = \begin{cases} 
    1, & \text{if } \alpha_{s-1} \notin I, \\
    r_T(s-1) + 1, & \text{otherwise}.
    \end{cases}
\]

A useful way to present $T_I$ is as follows. Partition $\{1, 2, \ldots, n\}$ into connected subsets $C_j := \{b_j, b_j + 1, \ldots, b_{j+1} - 1\}$ by choosing a strictly increasing sequence $1 = b_1 < b_2 < \cdots < b_{l+1} = n + 1$. Setting $I = \{\alpha_i \mid i, i + 1 \text{ belong to some } C_j\}$ defines a bijection between the set of all such partitions and the set of subsets of $\Pi$. Given $I \subset \Pi$, let $\{C_i^I : i = 1, 2, \ldots, l\}$ be the corresponding connected subsets which we view as columns. (Sometimes we may omit the $I$ superscript.) Then in the notation of 2.7 we have $T_{\forall x} = (C_1^I, C_2^I, \ldots, C_l^I)$. Of course this involves some sliding of boxes to the left. However there are some advantages in this presentation. For example $T_{\forall x} - 1$ is obtained by simply replacing $C_1$ by $C_1 - 1$ and $T_{\forall x} - n$ is obtained by simply replacing $C_l$ by $C_l - n$. Again one easily checks the

Proposition. For all $I \subset \Pi$, the word form of $w_x$ is given by

\[
    w_x = [\hat{C}_1^I, \hat{C}_2^I, \ldots, \hat{C}_l^I].
\]

We call $C_i^I$ the $i$–th chain of $T_{\forall x}$ and $T_I = (C_1^I, C_2^I, \ldots, C_l^I)$ the chain form of $T_I$.

Example

Consider $I = \{\alpha_1, \alpha_4, \alpha_5, \alpha_6, \alpha_9\}$ in $\mathfrak{sl}_{10}$. Then

\[
    T_I = \begin{array}{cccc}
    1 & 3 & 4 & 8 & 9 \\
    2 & 5 & 10 \\
    6 \\
    7
    \end{array}
\]

or in chain form $T_{\forall x} = (C_1, C_2, C_3, C_4, C_5)$ where $C_1 = \{1, 2\}$, $C_2 = \{3\}$, $C_3 = \{4, 5, 6, 7\}$, $C_4 = \{8\}$ and $C_5 = \{9, 10\}$.
2.13 Given an orbital variety \( V \), set \( O = G V \). It is a nilpotent orbit. Given a nilpotent orbit \( O' \subset \overline{O} \) one may try to describe \( V \cap O' \). In general it is not even known if this intersection is equidimensional. However if \( V = V_{\mathcal{I}} \) for some \( \mathcal{I} \subset \Pi \) a complete answer given by [S1], [M5 or M1] and 2.11. Set \( O_{\mathcal{I}} = G V_{\mathcal{I}} \).

**Theorem.** Let \( O \) be a nilpotent orbit in \( \overline{O_{\mathcal{I}}} = G m_{\mathcal{I}} \). Then \( m_{\mathcal{I}} \cap O \) is the union of orbital varieties \( V \) in \( O \cap n \) satisfying \( \tau(V) \supset \mathcal{I} \).

**Proof.**

By [S1, p. 456, last corollary] \( m_{\mathcal{I}} \cap O \) is equidimensional.

By [M1, 4.1.8 or M5, 3.5.6] it contains at least one orbital variety and so \( \dim V = \frac{1}{2} \dim O \), for every irreducible component \( V \) of \( O \cap m_{\mathcal{I}} \). Yet \( V \subset O \cap n \), which also has dimension \( \frac{1}{2} \dim O \). Hence \( V \) is an orbital variety associated to \( O \). Then by 2.11 every orbital variety \( V \) associated to \( O \) lies in \( m_{\mathcal{I}} \) if and only if \( \tau(V) \supset \mathcal{I} \).

**Remark.** By 2.10, \( \alpha_i \in \tau(V_{\mathcal{I}}) \) if and only if \( r_T(i + 1) > r_T(i) \). In particular if \( V > V_{\mathcal{I}} \) then for any \( i \) one has \( r_{V_{\mathcal{I}}}(i) \geq r_T(i) \).

2.14 Take \( \mathcal{I} \subset \Pi \) and recall the notation of 2.11. Set \( B_{\mathcal{I}} = B \cap L_{\mathcal{I}} \), \( n_{\mathcal{I}} = n \cap I_{\mathcal{I}} \) and \( W_{\mathcal{I}} = \{ w \in W \mid wI \subset R^+ \} \). We have decompositions \( B = M_{\mathcal{I}} \times B_{\mathcal{I}} \), \( n = n_{\mathcal{I}} \oplus m_{\mathcal{I}} \), \( W_{\mathcal{I}} \times W_{\mathcal{I}} \cong W \). They define projections \( B \to B_{\mathcal{I}} \), \( n \to n_{\mathcal{I}} \) and \( W \to W_{\mathcal{I}} \) which we will denote by \( \pi_{\mathcal{I}} \), or simply by \( \pi \). Let \( V_{\pi(w)}^\mathcal{I} \) be the orbital variety of \( I_{\mathcal{I}} \) with closure \( \overline{B_{\mathcal{I}}(n_{\mathcal{I}} \cap \pi(w) n_{\mathcal{I}})} \).

**Proposition.** [M1, 4.1.2 or M5, 4.2.2] For all \( w \in W \), \( \mathcal{I} \subset \Pi \) one has \( \pi_{\mathcal{I}}(V_{\mathcal{I}}) = \overline{V_{\mathcal{I}}(\pi(w))} = \overline{V_{\pi(w)}} \).

**Remark 1.** It is clear that \( \pi_{\mathcal{I}} \) is inclusion preserving. Thus if \( V_w \), \( V_y \) are orbital varieties of \( g \) with \( V_w \subset V_y \), then \( \overline{V_{\mathcal{I}}(\pi(w))} \subset \overline{V_{\pi(y)}} \).

**Remark 2.** Suppose \( a \subset n \) is \( h \) stable. Since root subspaces are one-dimensional we obtain \( \pi_{\mathcal{I}}(a) = a \cap n_{\mathcal{I}} \). Again \( M_{\mathcal{I}} \) acts by 1 on \( n/m_{\mathcal{I}} \). Then since \( \pi_{\mathcal{I}} \) is continuous

\[
\pi_{\mathcal{I}}(B a) \subset \pi_{\mathcal{I}}(B a) \subset \overline{B a \cap n_{\mathcal{I}}} \subset \pi_{\mathcal{I}}(B a)
\]

and so \( \overline{V_{\pi(w)}} = \overline{V_w \cap n_{\mathcal{I}}} \).

For \( 1 \leq i < j \leq n \) set \( [i, j] = \{ r \in \mathbb{N} \mid i \leq r \leq j \} \), \( \Pi_{i,j} = \{ \alpha_r \mid r, r + 1 \in [i, j] \} \) and \( \pi_{i,j} = \pi_{\Pi_{i,j}} \). Define \( T_n \to T_{j-i+1} \), through the jeu de taquin applied to the entries (taken in any order) of \( T \) not lying in \( [i, j] \). By [M1, 2.4.16, 4.1.1 or M5 4.3.3] one has

\[
\pi_{i,j}(\overline{V_T}) = \overline{V_{\pi_{i,j}(T)}}. \tag{*}
\]
2.15 Definition. Let $\geq$ be an order relation on $V$. Given $V_{T_2} > V_{T_1}$ we call $V_{T_2}$ a descendant of $V_{T_1}$ (respectively $T_2$ a descendant of $T_1$) if for any orbital variety $W$ such that $V_{T_2} \geq W \geq V_{T_1}$ one has $W = V_{T_2}$ or $W = V_{T_1}$. One calls $V_{T_2}$ a geometric descendant of $V_{T_1}$ when $\geq$ is the geometric order.

If unqualified descendant will mean geometric descendant.

The set of descendants of any $T_I : I \subset \Pi$ is described explicitly in [M2, 2.6]. The simplification that results in our case can be understood as follows.

Define the Duflo (or weak Bruhat) order on $W$ by $y \supseteq w$ if $n \cap y \subset n \cap w$. Through the map $w \mapsto Q(w)$ it induces an order relation $\supset$ on $T_n$ and hence on $V$ called the (induced) Duflo order. Remark 2.11 states just that $V_y \subset V_w$ if $y \supseteq w$, but the converse can fail unless $w = w_I$. The description of the induced Duflo order on $T_n$ is itself a non-trivial problem; but the complete solution was given in [M1, 3.3.3, 3.4.5 or M5, 3.3.3, 3.4.4]. By the above remark the set of geometric descendants of a Richardson component is contained in the set of Duflo descendants in turn described by [M2, 2.6].

Present $T_I$ as in 2.12, that is we write $T_I = (C_I^1, C_I^2, \ldots, C_I^l)$. Set $c_i = |C_i|$. Then $\varsigma_i = C_i^{c_i}$ is the largest entry of $C_i$ and moreover belongs to $T_{c_i}^i$. For any $1 \leq i \leq l$, suppose that there exists $j : 1 \leq j < i$ such that $c_j \geq c_i$. Take the maximal integer $i' : 1 \leq i' < i$ such that $c_{i'}$ is minimal with the property that $c_{i'} \geq c_i$. Define

$$T_I(i) := \begin{pmatrix} T^{1,c_i-1} \\ T^{c_i-\varsigma_i} \\ T^{c_{i}+1,c_{i'}} \\ T^{c_{i'}+1+\varsigma_i} \\ T^{c_{i'}+2,k} \end{pmatrix}$$

In other words we move $\varsigma_i$ from the $c_i$-th to the $(c_{i'}+1)$-th row according to the rules of 2.9. In example 2.12 one has $c_2 = 1 > c_1 = 2$; $c_3 = 4$ is maximal, $c_4 = c_2 = 1$ and $c_5 = c_1 = 2$. Thus our procedure defines 3 tableaux:

$$T_I(2) = \begin{array}{cccc} 1 & 4 & 8 & 9 \\ 2 & 5 & 10 & \end{array}, \quad T_I(4) = \begin{array}{cccc} 1 & 3 & 4 & 9 \\ 2 & 5 & 8 & 10 \\ 6 & 7 & \end{array}, \quad T_I(5) = \begin{array}{cccc} 1 & 3 & 4 & 8 & 9 \\ 2 & 5 & \end{array}.$$

Lemma. $T_I(i)$ is a Young tableau.

Proof.
Indeed the largest element $\varsigma_i$ of the $i$–th chain is relocated on the $i'$–th chain where it becomes the largest element since $i' < i$. Then the assertion follows from lemma 2.7.

\[\]
By Remark 2.13 we have

\[ r_T(s) \geq r_{T_2}(s), \quad \text{for all } s. \]

On the other hand by 2.4(2) and 2.5, \( \text{sh} T \) is obtained from \( \text{sh} T_2 \) by lowering exactly one box by row. Combined with the previous result this means that there is exactly one entry, say \( m \) of \( T_2 \) which is displaced on passing to \( T \) and moreover it appears exactly one row below. Suppose \( \alpha_m \in \tau(T^*_I) \). Since \( \tau(T) \supset \tau(T_2) \) by 2.11, it follows by Remark 2.13, that

\[ r_T(m + 1) \geq r_T(m) + 1 = r_{T_2}(m) + 2, \quad \text{whilst } r_{T_2}(m + 1) = r_{T_2}(m) + 1. \]

This contradicts \( r_T(m + 1) = r_{T_2}(m + 1) \). Thus, in the previous notation \( m = \zeta_i \), the latter being the longest element of some chain \( C_i \) of \( T_2 \) of length \( c_i \). By the above we also have \( T^{c_i} = T^{c_i}_I - m \) and \( T^{c_i+1} = T^{c_i+1}_I + m \) whilst the remaining rows of \( T, T_2 \) must coincide. Finally we must have some \( i' < i \) with \( c_{i'} = c_i \), for otherwise \( m \) would be strictly less than an entry in \( T \) above it. We conclude that \( T = T^*_I(i) \), as required.

2.18 By 2.17 we conclude that the orbital varieties of codimension 1 in \( \mathcal{V}_I \) are obtained by moving \( \zeta_i \) from row \( c_i \) to row \( (c_i + 1) \) wherever \( |T^{c_i}_I| \geq |T^{c_i+1}_I| + 2 \). Indeed if \( \zeta_i \) belongs to \( T_j \) (and here \( j \leq i \) because some sliding of boxes from the chain form of \( T_2 \) may be necessary) then the condition \( c_{i'} = c_i \) implies that \( |T_{j-1}| = |T_j| \).

More precisely there are exactly \( t + 1 := |\{j \mid 1 \leq j \leq i \text{ with } c_j = c_i \}| \) columns of \( T \), namely \( T_j, T_{j-1}, \ldots, T_{j-t} \), having length \( |T_j| \). In addition suppose that \( i \) is maximal for a given value \( c_i \) that is \( c_j \neq c_i \) for any \( j > i \). Then there are exactly \( t \) choices of \( s \) such that \( T^*_I(s) \) is defined and has the same shape as \( T^*_I(i) \).

Observe that the partition \( \lambda_I = \text{sh} T_2 \) is defined by chain lengths namely we have \( \lambda^*_I = \{c_i : i = 1, 2, \ldots, l\} \) appropriately ordered. Fix \( c \in \lambda^*_I \) and set \( t+1 = |\{i \mid c_i = c\}| \). Suppose \( t \geq 1 \) and let \( \mathcal{O}_I(c) \) be defined through the partition \( \lambda_I(c) \) obtained from \( \lambda^*_I \) by replacing the appropriate pair \( (c, c) \) in \( \lambda^*_I \) by \( (c+1, c-1) \). Then we may summarize the above by

**Proposition.** Fix \( I \subset \Pi \) and \( c \in \mathbb{N}^+ \). Set \( t = |\{i \mid c_i = c\} - 1 \). Assume \( t \geq 1 \). Then \( m_i \cap \mathcal{O}_I(c) \) is a union of \( t \) orbital varieties of codimension 1 in \( m_2 \). Moreover these orbital varieties correspond to the standard tableaux \( T^*_I(i) : c_i = c \), except for the minimal \( i \) that is \( i : c_i = c \) and \( c_j \neq c \) for all \( j < i \).
Note that orbital varieties of codimension 1 in the nilpotent radical of a parabolic appear exactly when there are repetitions in the dual partition associated to this parabolic.

2.19 Let $m^-$ be the opposed algebra of $m = m^+$ identified with $m^*$ through the Killing form. By Krull’s theorem the ideal $I(\mathcal{V})$ of definition of an orbital variety closure $\mathcal{V}$ in the conclusion of 2.18 is principal in $S(m^-)$ and we denote by $f_z(i)$, or simply $f$, the corresponding irreducible generator in $S(m^-)$.

The form of $f_z(i)$ was conjectured by Benlolo and Sanderson in [BS]. We refer to this and their conjecture simply as BS. We prove this conjecture in Sect. 3.

As explained in 1.4, 1.5 the general form of $f_z(i)$ is $\text{gr } a$ for some minor $a$ of a generic matrix in $m_I + \text{Id}$. They took a suitable minor lying in a bottom left-hand corner. In this they needed results similar to [M5] to reduce to the case when $I, \ c$ are chosen so that $c = c_1 = c_l$ and $c_i \neq c$ for $1 < i < l$. Then their conjecture is equivalent to saying that $a$ is just the $(n - c) \times (n - c)$ minor lying in the bottom left-hand corner.

The BS reduction can be read off from 2.16 and 2.18 which makes precise the result of [M5] which one needs and we believe clarifies their analysis. A key point is the knowledge of the orbital varieties of codimension 1 in $m_I$ given by 2.17 developed from the results of [M5]. We remark that the codimension 1 case is relatively easy and barely uses the full power of [M5].

For all $1 \leq i, j \leq n$, set $X_{i,j} = \mathbb{C}e_{i,j}$. Given $w \in W$, we give the affine $|R^+| - \ell(w)$ dimensional space $X(w)$ a matrix presentation through $X(w)_{i,j} = X_{i,j}$, wherever $i < j$ and $w(i) < w(j)$.

Let $x_{j,i}$ denote the coordinate function on $\mathfrak{g}$ defined by

$$x_{j,i}(e_{r,s}) = \begin{cases} -1, & \text{if } (r, s) = (i, j), \\ 0, & \text{otherwise}. \end{cases}$$

Then the Poisson bracket $\{,\}$ defined on $S(\mathfrak{g}^*)$ through the Lie bracket on $\mathfrak{g}$ satisfies

$$\{x_{i,j}, x_{r,s}\} = \delta_{j,r}x_{i,s} - \delta_{s,i}x_{r,j} \tag{*}$$

where $\delta_{i,j}$ is the Kronecker delta. Setting $x_{i,j} = e_{i,j}$ for $i > j$ identifies $m^-$ with $m^*$.

Given $w \in W$, let $\mathcal{M}(w)$ be the matrix with entries

$$\mathcal{M}(w)_{i,j} = \begin{cases} x_{i,j}, & \text{if } X_{j,i} \subset X(w), \\ 0, & \text{otherwise}. \end{cases}$$

When $w = w_I$, we set $\mathcal{M}(w) = \mathcal{M}_I$.  

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Assume \( c_1 = c_1 =: c \) and let \( M_{I}^{c}(t) \) denote the \((n-c) \times (n-c)\) minor in the bottom left-hand corner of \( M_{I} + t\text{Id} \), that is

\[
M_{I}^{c}(t) := \begin{vmatrix}
x_{c+1,1} & \cdots & x_{c+1,c} & t & 0 & \cdots & 0 \\
x_{c+2,1} & \cdots & \cdots & \cdots & t & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
x_{n-c,1} & \cdots & \cdots & \cdots & \cdots & \cdots & t \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
x_{n,1} & \cdots & x_{n,c} & x_{n,c+1} & \cdots & x_{n,n-c}
\end{vmatrix}
\]

Note that there are some zeros in the dot places of the determinant in correspondence with the definition of matrix \( M_{I} \).

**Example**

Consider \( \mathcal{I} = \{ \alpha_1, \alpha_4 \} \) in \( \mathfrak{sl}_5 \). In that case \( c_1 = c_3 = 2 \) and

\[
M_{I} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
x_{4,1} & x_{4,2} & 0 & 0 & 0 \\
x_{4,1} & x_{4,2} & x_{4,3} & 0 & 0 \\
x_{5,1} & x_{5,2} & x_{5,3} & 0 & 0
\end{bmatrix}
\text{ and } M_{I}^{2}(t) := \begin{bmatrix}
x_{3,1} & x_{3,2} & t \\
x_{4,1} & x_{4,2} & x_{4,3} \\
x_{5,1} & x_{5,2} & x_{5,3}
\end{bmatrix}
\]

Developing \( M_{I}^{c}(t) \) in powers of \( t \) one obtains

\[
M_{I}^{c}(t) = m_{n-c} + m_{n-c-1}t + \cdots + m_{c}t^{n-2c}.
\]

Set

\[
d_i = \begin{cases} 
    c_i - c & \text{if } c_i > c \\
    0 & \text{otherwise}
\end{cases}, \quad d_{I} = \sum_{i=0}^{k} d_i, \quad l_{I} = n - c - d_{I}
\]

By [BS, lemma 2] the lowest non-zero coefficient is \( m_{l_{I}} \) that is the coefficient of \( t^{d_{I}} \).

The BS conjecture states that \( f_{I}(t) = m_{l_{I}} \).

In our example \( d_{I} = 0 \) and \( l_{I} = 3 \) so that

\[
m_{l_{I}} = x_{3,1}x_{4,2}x_{5,3} + x_{3,2}x_{4,3}x_{5,1} - x_{3,1}x_{4,3}x_{5,2} - x_{3,2}x_{4,1}x_{5,3}.
\]

It is clear that \( m_{l_{I}} = \text{gr } M_{I}^{c}(1) \). We do not need to know the explicit power of \( t \) which divides \( M_{I}^{c}(t) \), though this is used in one of our three proofs of irreducibility. The BS conjecture can be expressed as the following theorem which we prove in the next section.

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Theorem. Take $\mathcal{I} \subset \Pi$ and let $T_\mathcal{I} = (C_1, C_2, \ldots, C_l)$ be its chain form. Set $c_i = |C_i|$, for all $i$.

(i) Assume $c_1 = c_l =: c$ and $c_j \neq c$, $\forall j : 1 < j < l$. Then $I(\mathcal{V}_{T_\mathcal{I}(l)})$ is generated by
\[
\{ x_{ij} : X_{ij} \in p_\mathcal{I} \cap \mathfrak{n}^- \} \quad \text{and} \quad m_\mathcal{I}.
\]

(ii) Suppose there exist $i < j$ such that $c_i = c_j$ and $c_s \neq c_i$, $\forall s : i < s < j$. Set $u = C_i^1$, $v = C_j^c$ and $\mathcal{I}' = \mathcal{I} \cap \Pi_{u,v}$. Then $I(\mathcal{V}_{T_\mathcal{I}(j)})$ is generated by
\[
\{ x_{ij} : X_{ij} \in p_\mathcal{I} \cap \mathfrak{n}^- \} \quad \text{and} \quad m_{\mathcal{I}'} \quad \text{defined with respect to} \quad \mathcal{V}_{T_{\mathcal{I}'}} \quad \text{viewed as an orbital variety in } \pi_{u,v}(\mathfrak{n}).
\]

Note that the part (i) of the theorem is a special case of part (ii) and part (ii) will be deduced from part (i).

2.20 Let $i$ be a positive integer $< \frac{n}{2}$ and set
\[
\beta_i = \sum_{j=i}^{n-i} \alpha_j
\]
Take $c$ as in 2.19 and set
\[
\gamma_c = \sum_{i=1}^{c} \beta_i.
\]

As noted in [BS, VIII] one may check that $M_c(t)$, and hence $m_\mathcal{I}$ has a weight $-\gamma_c$. Moreover with respect to the Cartan inner product $(,)$ one has $(\gamma_c, \alpha) = 0$, $\forall \alpha \in \Pi \setminus \{\alpha_c, \alpha_{n-c}\}$. In particular
\[
(\gamma_c, \alpha) = 0, \quad \forall \alpha \in \mathcal{I}. \quad (*)
\]

3. Proof of the Benlolo-Sanderson conjecture.

3.1 Let us recall some general facts about ideals of definition of orbital variety closures.

Retain the notation of 2.11 and 2.18. Let $P$ be some standard parabolic subgroup of $G$, $\mathfrak{p} = \text{Lie}(P)$ and $\mathfrak{m}$ its nilradical. Let $V$ be a closed subvariety of $\mathfrak{m}$ and $I$ (resp. $J$) its ideal of definition in $S(\mathfrak{m}^-)$ (resp. $S(\mathfrak{g})$). One has $J = I + S(\mathfrak{g})\mathfrak{p}$ and $J \cap S(\mathfrak{m}^-) = I$ from which one easily checks the

Lemma. The following are equivalent

(i) $\{I, I\} \subset I$ and $\{\mathfrak{p}, I\} \subset J$.
(ii) $\{I, I\} \subset I$ and $V$ is $P$ invariant.
(iii) \( \{J, J\} \subset J \).

3.2 One calls \( \mathcal{V} \) involutive if any one of the above holds. If \( \mathcal{V} \) is involutive then so are its irreducible components.

Suppose \( \mathcal{V} \subset m \) is irreducible. Then \( \mathcal{O} \mathcal{V} \) contains a unique dense orbit \( \mathcal{O} \). If in addition \( \mathcal{V} \) is involutive then \( \dim(\mathcal{V} \cap \mathcal{O}) \geq \frac{1}{2} \dim \mathcal{O} \) since \( \mathcal{O} \) is a symplectic variety. Yet \( \dim(n \cap \mathcal{O}) = \frac{1}{2} \dim \mathcal{O} \) and so \( \mathcal{V} \cap \mathcal{O} \) must be an irreducible component of \( n \cap \mathcal{O} \) hence an orbital variety associated to \( \mathcal{O} \) with closure \( \mathcal{V} \).

3.3 Let \( J_o \) be an ideal of \( S(g) \) with radical \( J \). Suppose \( \{J_o, J_o\} \subset J_o \). It is generally false that this implies \( \{J, J\} \subset J \), so one cannot conclude that the zero variety of \( J_o \) is involutive. This is a delicate point in general. In the present situation, this difficulty can be avoided. Call \( f \in S(m^-) \) multilinear if its degree in any one of the variables \( x_{i,j} \) is at most 1. This property passes to the irreducible factors \( f_1, f_2, \ldots, f_t \) and we let \( \mathcal{V}(f_1), \ldots, \mathcal{V}(f_t) \) denote their sets of zeros in \( m \). Consider the condition

\[
\{p, f\} \subset S(m^-)f + S(g)p. \tag{*}
\]

**Proposition.** Suppose \( f \in S(m^-) \) is multilinear. Then

(i) \( S(m^-)f \) is semiprime.
(ii) \( S(m^-)f + S(g)p \) is semiprime.
(iii) If (*) holds then the \( \mathcal{V}(f_i) \) are orbital varieties of codimension 1 in \( m \).

**Proof.**

(i) is immediate. (ii) is easily deduced from (i). Finally (iii) follows from (ii) and 3.1, 3.2.

3.4 Our proof of the BS conjecture involves three steps. The first is to show that \( f = m_{l_\zeta} \) satisfies 3.3 (*). This is rather obvious especially from the quantum viewpoint of [J2, Lecture 7]. Nevertheless a delicate point is that \( M_{\zeta}(t) \) itself does not satisfy (*). We may view its zero set as a deformation of \( \mathcal{V}(f) \); but which is not itself orbital nor \( P \) invariant. (Our original motivation for such deformation came from trying to define an Enright functor on orbital varieties itself inspired by the algorithm for \( \text{Ann}_{U(n^-)^v} \) in [J1, 8.4] based on the Enright functor.) This result is given in 3.11.

The second step is to show that \( f \) is irreducible. This is a delicate point. However we can give three different proofs. The first, given in 3.15, is a fairly explicit but includes some computations involving the precise knowledge of \( M_{\zeta} \). The second, outlined in 4.3, uses representation theory. We construct a simple highest weight module \( L \) with integral
highest weight which is a strong quantization of $\mathcal{V}(f)$. This is of interest in its own right. Then we use the linear independence of the characteristic polynomials of orbital varieties and the irreducibility of the associated variety of $L$ which holds [M4] in type $A$. The third method, given in 4.7, is the least computational; but the most sophisticated. Here we construct a strong quantization; but not necessary having integral highest weight. Then we use the difficult fact that $V(\text{Ann } L)$ is the closure of a nilpotent orbit. In type $A$ this has the relatively easy proof using mainly that orbit closures are normal [BK]. Finally we apply 2.18. In all these methods it is crucial to use that $c_i \neq c, \forall 1 < i < l$. Otherwise $V(f)$ has exactly $t := \#\{i \mid c_i = c\} - 1$ components. A computational proof in some special cases when $m_{lx} = M^x_l(0)$ was also given in [BS]. Even that is not trivial though this case can be viewed as an easy consequence of 2.4 (1) combined with 2.18.

Step three is to show that $M^x_l$ vanishes on $X(w)$ for some $w : Q(w) = T^x_l(\ell)$. Then $P^x_l$ invariance implies $\mathcal{V}(f) \supset B(n \cap w n)$. This is shown in 3.8. Since both have codimension 1 in $m_x$, irreducibility finishes the proof. Notice this does not use involutivity; but by 3.1 the latter is equivalent to $P^x_l$ invariance. One may also avoid this last step by combining 3.3. and 2.18; but this is less satisfying.

3.5 Given $w \in S_n$ with word form $[a_1, a_2, \ldots, a_n]$, let $w - s \in S_{n-1}$ be defined by deleting $s$ and standardizing the word form. For any matrix $M$ let $M^{i,j}$ be the matrix obtained from $M$ by deleting the $i$–th row and $j$–th column. Recall the definition of $X(w)$ given in 2.19.

Lemma. For all $w \in W, s \in \{1, 2, \ldots, n\}$ one has $X(w - s) = X(w)^{s,s}$.

Proof. Take $1 \leq i < j \leq n$. Then $w(\alpha_{i,j}) \in R^+$ if and only if $w(i) < w(j)$ with a similar assertion for $y = w - s$. Yet recalling that we are standardizing $w - s$ we have up to standardization

$$y(r) = \begin{cases} a_r, & \text{if } r < w^{-1}(s), \\ a_{r+1}, & \text{if } r \geq w^{-1}(s). \end{cases}$$

From this the assertion readily follows.

3.6 Suppose $T = Q(w)$. It is generally false that $Q(w - s) = T - s$, though it is true for the canonical elements $w_r(T)$ and $w_c(T)$ defined in 2.8. Here we shall only need to describe $Q(w_r(T) - s)$, when $s \in T^k$, that is to say when $s$ lies in the last row of $T$. Recall that $w_r(T) = [T^k, T^{k-1}, \ldots, T^1]$. 

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Lemma. One has $w_r(T) - T^k_j = w_r(T - T^k_j)$ for every entry $T^k_j$ of the last row $T^k$ of $T$.

Proof. Indeed

$$(T - T^k_j) = \left( \frac{T^{1,k-1}}{(T^k - T^k_j)} \right)$$

which gives the required assertion.

3.7 Fix $\mathcal{I} \subset \Pi$ and $s \in \{1, 2, \ldots, n\}$. Define $\mathcal{I}' \subset \Pi \setminus \{\alpha_{s-1}\}$ by $\mathcal{I}' = \{\alpha_i \mid i < s - 1, \alpha_i \in \mathcal{I}\} \cup \{\alpha_{i-1} \mid i \geq s, \alpha_i \in \mathcal{I}\}$. Write $T_\mathcal{I} = (C_1^\mathcal{I}, C_2^\mathcal{I}, \ldots, C_l^\mathcal{I})$ as in 2.12.

Lemma.

(i) $w_{\mathcal{I}} - s = w_{\mathcal{I}'}$.

(ii) $Q(w_{\mathcal{I}} - s) = T_{\mathcal{I}'}$.

(iii) Suppose $s \in C_j^\mathcal{I}$. Then up to standardization $T_{\mathcal{I}'} = (C_1^\mathcal{I}', C_2^\mathcal{I}', \ldots, C_l^\mathcal{I}')$ where

$$C_i^{\mathcal{I}'} = \begin{cases} C_i^\mathcal{I}, & \text{if } i \neq j, \\ C_j^\mathcal{I} - s, & \text{if } i = j. \end{cases}$$

(iv) If $s$ lies in the last row of $T_{\mathcal{I}}$ then $T_{\mathcal{I}'} = (T_{\mathcal{I}} - s)$ after standardization.

Proof. Recall from 2.15 the notation $c_i := |C_i|$ and $c_i := C_i^{c_i}$. One has

$$\mathcal{I} = \Pi \setminus \bigcup_{i=1}^{l-1} \alpha_{c_i}$$

Then (i) is clear if $s \neq c_i$ for any $i$. Otherwise $\alpha_s \notin \mathcal{I}$ but also $\alpha_{s-1} \notin \mathcal{I}'$. Hence (i). (ii) follows from (i) by 2.12. (iii) is obtained from 2.12 just by definition of chains. Finally (iii) implies (iv) since in that case $T_{\mathcal{I}'}$ is obtained by eliminating $s$ and sliding the rest to the left.

3.8 Retain the notation of 2.19 and assume $c = c_1 = c_l$. Then $T_{\mathcal{I}}(l)$ is defined. Set $M = M_{\mathcal{I}}^c(t)$.

Proposition. Set $w = w_r(T_{\mathcal{I}}(l))$. Then $M_{\mathcal{I}}^c(t)$ vanishes on $n \cap w n$, that is $M(X(w)) = 0$.

Proof.
The proof is by induction on \( d_x \) defined in 2.19. If \( d_x = 0 \), then \( c_i \leq c, \forall 1 \leq i \leq l \) and so \( n \) occurs in the last row of \( T_\mathcal{I} \). Moreover \( T_\mathcal{I}(l) \) has \( c + 1 \) rows with \( n \) the unique entry in row \( c + 1 \). Consequently the word form of \( w \) starts with \( n \). By 2.3 this forces \( X_{i,n} \not\in n \cap^w n, \forall i < n \). Thus the entries \( x_{n,i} : i = 1, 2, \ldots, n - c \), of the last row of \( M \) vanish on \( X(w) \) and consequently so does \( M \).

Let us assume the assertion holds for \( d_x = d \) and take \( d_x = d + 1 \). In particular there is some \( c_i > c \). Let \( k \) be the number of rows of \( T := T_\mathcal{I} \). Since \( c_i > c \) this is also the number of rows of \( T' := T_\mathcal{I}(l) \) and so the word form of \( w \) starts with \( s = T_1^k \), where \( s \not\subset C_1 > \). By 2.3 we conclude that \( X_{i,s} \not\subset n \cap^w n, \forall i < s \). Thus the entries \( x_{s,i} : i < s \) in \( M'_x(t) \) vanish on \( X(w) \). This leaves \( t \) as the only non-zero entry of row \( s \). Consequently

\[
M(X(w)) = tM(X(w)^{s,s})
\]

it being understood that \( n \) is reduced by 1 in defining the right hand side.

By 3.5 one has \( X(w)^{s,s} = X(w - s) \) whilst by 3.6 one has \( X(w - s) = X(w_r(T' - s)) \). Recall the definition of \( T' \) from 3.7. By 3.7 (iv), \( T - s = T_\mathcal{I} \). Then by 2.17 \( T' - s \) is a descendant of \( T_\mathcal{I} \). Moreover \( d_\mathcal{I} = d_x - 1 = d \) and \( C_1^{T'} = C_1^T, C_i^{T'} = C_i^T \) up to standardization. In particular \( |C_i^{T'}| = |C_i^T| = c \). Set \( w' = w_r(T' - s), M' = M'_x(t) \). By the above and the induction hypothesis \( M'(X(w')) = 0 \). Then by (*) and the above \( M(X(w)) = 0 \).

3.9 Retain the notation and hypotheses of 3.8. For \( 1 \leq i, j \leq n \), let \( M^{i,j} \) denote \( ij \)-th cofactor of \( M \). Set \( d = d_x \) and recall that \( t^d \) divides \( M \). The following is clear by row expansion of the determinant.

**Lemma.** Choose \( i, j \) such that \( M_{i,j} = x_{c+i,j} \in m_\mathcal{I} \). Then \( M^{i,j} \) is divisible by \( t^d \).

3.10 Take \( f \in S(m^-) \). In computing \( \{x_{i,j}, f\} \) via 2.19 (*) we obtain two types of terms. The first (resp. second) is obtained from the first (resp. second factor on the right-hand side. We call them the terms obtained from \( j \) (resp. \( i \)).

**Lemma.** Suppose \( i \in \{1, 2, \ldots, n - 1\} \setminus \{c, n - c\} \). Then

\[
\{x_{i,i+1}, M\} \subset S(m^-_\mathcal{I})M + S(g)p_\mathcal{I}.
\]

**Proof.**

Let \( \mathcal{X} \) be matrix with entries \( \mathcal{X}_{i,j} = x_{i,j} \) and \( X \) the \((n - c) \times (n - c) \) minor in the bottom left-hand corner of \( \mathcal{X} + t\text{Id} \). Then \( M - X \in S(g)p_\mathcal{I} \). Since \( p_\mathcal{I} \) is a subalgebra it is enough to prove the corresponding assertion for \( X \).
Consider \( \{x_{i,i+1}, X\} \).

(i) Suppose \( i > c \). Then the sum of terms coming from \((i+1)\) forms a determinant \( X^{i+1} \) with the same rows as \( X \) except that the entries \( x_{i+1,s} + \delta_{i+1,s}t \) on the \((i+1-c)\)-th row of \( X \) are replaced by \( x_{i,s} \). Hence this term equals \(-tX^{i+1-c,i} \).

(ii) Suppose \( i < n - c \). Then the sum of terms coming from \( i \) forms a determinant \( X_i \) with the same columns as \( X \) except that the entries \( x_{s,i} + \delta_{s,i}t \) on the \( i \)-th column of \( X \) are replaced by \(-x_{s,i+1} \). Hence this term equals \( tX^{i+1-c,i} \).

If \( c < i < n - c \) then terms cancel. If \( i < c \) or \( i > n - c \), both terms are zero. This proves the lemma.

3.11 Let \( f = m_{IZ} \).

**Lemma.** One has \( \{p_z, f\} \subset S(m_z^+)f + S(g)p_z \).

**Proof.**

By 2.20 (*) \( f \) has weight zero with respect to the Cartan subalgebra \([i_z, i_z] \cap h\) of the semisimple Lie algebra \([i_z, i_z] \). Since \( S(g) \) and \( S(g)i_z \) are locally finite \( i_z \) modules it follows from the theory of finite dimensional \( i_z \) modules that it suffices to show \( \{x_{i,i+1}, f\} \subset S(m_z^+)f + S(g)p_z \), for all \( 1 \leq i \leq n - 1 \) and \( \{x_{i+1,i}, f\} \subset S(m_z^+)f + S(g)p_z \), for all \( \alpha_i \in \tau(p_z) \). By 3.10 among all \( \{x_{i,i+1}, f\} \) it remains to consider the cases \( i = c, n - c \). Both are similar and we consider only the first. Consider \( \{x_{c,c+1}, M\} \). The sum of terms coming from \( c + 1 \) all lie in \( S(g)p \). The sum of terms coming from \( c \) equal to \( tM^{1,c} \mod S(g)p \). Yet \( M_{1,c} = x_{c+1,c} \in m_z^- \) so by 3.9 this expression is divisible by \( t^{d+1} \mod S(g)p_z \). Thus \( \{x_{c,c+1}, f\} = 0 \mod S(g)p_z \).

As for \( \{x_{i+1,i}, f\} \) we show that \( \{x_{i+1,i}, M\} = 0 \) exactly in the same manner as in 3.10. Let us sketch the proof. The cases \( i < c \) or \( i + 1 > n - c \) are the same so let us show this for \( i + 1 > n - c \). In this case the sum of terms coming from \( i + 1 \) is zero and the sum of terms coming from \( i \) forms a determinant \( M^i \) with the same rows as \( M \) except that the \((i-c)\)-th row is replaced by the \((i-c+1)\)-th row so that \( M^i = 0 \). Now if \( c < i < n - c - 1 \) then for all \( s \neq i, i + 1 \) one has \( M_{i-c,s} = 0 \) iff \( M_{i+1-c,s} = 0 \). As well for all \( s \neq i - c, i + 1 - c \) one has \( M_{s,i} = 0 \) iff \( M_{s,i+1} = 0 \) Using this we get exactly as in 3.10 that the sum of terms coming from \( i \) results in \(-tM^{i-c,i+1} \) and the sum of terms coming from \((i+1)\) results in \( tM^{i-c,i+1} \) so that the terms cancel.

3.12 We may summarize the consequence of 3.3 (ii), 3.8, 3.11 as follows. Adopt the notation and hypotheses of 2.19 (i). In particular set \( T' = T_Z(l) \).
**Proposition.** The zero variety $\mathcal{V}(m_{l^2})$ of $m_{l^2}$ in $m_{l^2}^-$ is a union of orbital variety closures of codimension 1 one of which is $\mathcal{V}_{T'}$.

**Remark.** We have not yet used that $c_j \neq c$ for $1 < j < l$. This is needed for irreducibility of $\mathcal{V}(m_{l^2})$. This would follow from 2.18 and the irreducibility of $G\mathcal{V}(m_{l^2})$; but we only have a direct proof of the latter using representation theory (cf. 4.7).

**3.13** Let us show that (ii) of Theorem 2.19 results from (i). This obtains from the following general remark. Take $I \subset \Pi$ and $T_{I} = (C_1, \ldots, C_l)$ be its chain form. Fix $i, j$: $1 \leq i \leq j \leq l$ and set $T_{I'} = (C_i, \ldots, C_j)$. Let $T' \in T_{|T_{I'}}$ satisfy $T' > T_{I'}$ and set $T = (C_1, \ldots, C_{i-1}, T', C_{j+1}, \ldots, C_l)$. Then $T > T_{I}$ and $\pi(T_{I}) = T_{I'}$, $\pi(T) = T'$, where $\pi = \pi_{i-1, j}$. Again by 2.14 ($\ast$) $\pi(\mathcal{V}_{T}) = \mathcal{V}_{T'}$, and by 2.4 (2) the codimension of $\mathcal{V}_{T}$ in $m_{l^2}$ equals the codimension of $\mathcal{V}_{T'}$ in $m_{l^2'} = \pi(m_{l^2})$. The implication $(i) \implies (ii)$ in 2.19 follows from the

**Lemma.** With the above hypotheses

$$I(\overline{\mathcal{V}}_{T}) = S(m_{l^2}^-)I(\overline{\mathcal{V}}_{T'}).$$

**Proof.**

From the commuting diagram

$$\begin{array}{ccc}
S(\pi(m_{l^2}^-)) & \xrightarrow{\pi^*} & S(m_{l^2}^-) \\
\downarrow & & \downarrow \\
R[\overline{\mathcal{V}}_{T'}] & \xrightarrow{\pi^*} & R[\overline{\mathcal{V}}_{T}]
\end{array}$$

we obtain the inclusion $\supset$. On the other hand

$$S(m_{l^2}^-)/IS(m_{l^2}^-) \cong S(\pi(m_{l^2}^-))/I \otimes S(\ker \pi),$$

so $IS(m_{l^2}^-)$ is prime. Equality of codimensions finishes the proof.

**3.14** Let $V$ be a vector space of dimension $n < \infty$. Let $\{v_i\}_{i=1}^n$ be a basis of $V$ and set $v_0 = v_{n+1} = 0$. Define $e, f \in \text{End} V$, by $ev_i = v_{i-1}$, $fv_i = v_{i+1}$, $\forall i$. Let $r, s$ be integers $> 0$ and set $A_{r,s}^n(t) = \text{Det} (te^s + f^r)$.

**Lemma.** For all $r, s, n > 0$ one has

$$A_{r,s}^n(t) = \begin{cases} ((-1)^{r+s}t^r)^h & \text{if } h(r + s) = n, \\
0 & \text{if } r + s \text{ does not divide } n.\end{cases}$$
Proof.

If \( r + s > n \), then there are only zeros in the \( n - s + 1 \)-th row, so we can suppose \( r + s \leq n \). Then the first \( r \) rows have exactly one entry, namely \( t \). Similarly the first \( s \) columns have exactly one entry, namely 1. Developing these rows and columns gives

\[
A_{r,s}^n(t) = (-1)^{r+s}t^r A_{r,s}^{n-r-s}(t),
\]

and hence the required result.

3.15 The irreducibility of \( m_{I} \) is established by induction on \( n \). In view of 3.13 it is therefore enough to consider only the situation described in 2.19 (i). Precisely we show the

Proposition. Take \( I \in \Pi \) and \( T_I = (C_1, C_2, \ldots, C_l) \) the chain form of \( T_I \). Set \( c_i = |C_i| \). Assume \( c_1 = c_l =: c \) and \( c_j \neq c, \forall j : 1 < j < l \). Then \( m_{I} \) is irreducible.

Proof.

Otherwise by 2.17, 3.12 and 3.13, there is a subset \( I' \) of \( \Pi \) obtained from \( I \) exactly as in 2.19 (ii) such that the corresponding \( f' := m_{I'} \) is an irreducible factor of \( f = m_{I} \). We obtain a contradiction by showing there exists a point \( x \in m \) such that \( f(x) = (-1)^c, \ f'(x) = 0 \).

Set \( a = |T^c_{I}^{+1,k}| \) and

\[
<T^1_{I}^{c}> = \{p_1, p_2, \ldots, p_{n-a}\}, \quad <T^{c+1}_{I}^{+1,k}> = \{q_1, q_2, \ldots, q_a\},
\]

written in increasing order. Define \( x \in m \) by

\[
x_{r,s} = \begin{cases} 1, & \text{if } r = p_i, \text{ for } i : c < i \leq n - a \text{ and } s = p_{c-i} \\ 0, & \text{otherwise.} \end{cases}
\]

Set \( M := M^c_{I}(t) \). Let \( b \) be an integer \( 0 < b \leq n - c \) and let \( \Delta_b \) denote the set of subsets of \( \{c+1, \ldots, n\} \) of cardinality \( b \). Given \( \sigma = \{r_1, r_2, \ldots, r_b\} \in \Delta_b \), let \( M^\sigma \) be obtained from \( M \) by deleting the \( (r_i - c) \)-th rows and \( r_i \)-th columns where \( i = 1, 2, \ldots, b \). Let \( M^\sigma_0 \) be the evaluation of \( M^\sigma \) at \( t = 0 \). Developing \( M \) gives the term \( (-1)^ct^bM^\sigma \) and moreover

\[
m_{n-c-b} = (-1)^c \sum_{\sigma \in \Delta_b} M^\sigma_0.
\]

Recall that \( l_{I} = n - c - a \). For \( \sigma \in \Delta_a \) one has

\[
M^\sigma_0(x) = \begin{cases} 1, & \text{if } \sigma = \{q_1, q_2, \ldots, q_a\}, \\ 0, & \text{otherwise.} \end{cases}
\]
Consequently

(i) \( f(x) = (-1)^c. \)

In the notation of theorem 2.19 (ii) set \( c' := c_i = c_j \) and \( M' = M^c_{T'}(t). \) Then

(ii) \( f'(x) = 0, \) if \( c' > c. \)

Indeed in this case the last row of \( M' \) is indexed by the largest integer in \( C_j. \) By the hypothesis this is an entry of \( T^c_{T+1,k}. \) Hence even \( M'(x) = 0. \)

Define \( u, v \) as in theorem 2.19 (ii) and set \( n' = v - u + 1. \) Set \( M'_{T}(x) = 0 \) unless \( \sigma \supset \sigma', \) so that \( M'(x) = (-1)^c' t^{d'} A_{c-c',c}(t). \) This is obtained through a development of \( M' \) similar to that for \( M \) and noting that \( M'_{\sigma}(x) = 0 \) unless \( \sigma \supset \sigma', \) so that \( M'(x) = (-1)^c' t^{d'} A_{c-c',c}(t). \)

Let us show finally that

(iv) \( f'(x) = 0, \) if \( c' < c. \)

By 3.14 it suffices to show that \( h(c - c') + d' > d_{T'} \) when \( n' = hc. \)

Let \( C_{m_1}, C_{m_2}, \ldots, C_{m_r} \) be the chains of length \( > c \) between \( C_i \) and \( C_j. \) One has

\[
\sum_{s=1}^{r} (c_{m_s} - c') \leq d' + r(c - c')
\]

and so the assertion holds if \( r < h. \) On the other hand \( hc = n' \geq (r + 2)c' + d_{T'}. \) Thus

\[
h(c - c') + d' \geq (r + 2)c' + d_{T'} > d_{T'} \quad \text{if} \quad r \geq h.
\]

This proves (iv).

Finally the hypothesis itself excludes \( c = c' \) and so the proposition follows from (i), (ii) and (iv).

\[\square\]

**Remark.** The proof of theorem 2.19 is now complete.

3.16 Adopt the hypotheses of 2.19 (i). Let \( r \) be the number of chains \( C_i \) of length \( > c \) and \( s + 2c \) the sum of the lengths of the remaining chains.

**Proposition.** The generator \( f_T(l) \) of \( I(\overline{\nu}_{T,l}(l)) \) in \( S(m^-) \) is the highest common divisor of the (non-vanishing) \( (s + c) \times (s + c) \) minors of \( M^{r+1}_{T}. \)

**Proof.**

Under the hypotheses of 2.19 (i), it follows from 2.18 (in notation of 2.18) that \( m_T \cap \mathcal{O}_T(c) = \nu_{T,l}(l). \) By 2.4 (i) the nilpotent orbit to which an element \( x \) belongs is determined by \( \{ \text{rk} x^i : i = 1, 2, \ldots, n \}. \)
Set $\lambda = \text{sh} T_I$. Since $n - \sum_{j>r+1} \lambda^*_j = s + c$, it follows that $\text{rk} x^{r+1} = s + c$, for all $x \in m \cap O_I$.

Now $\text{sh} T_I(l)$ is obtained from $\lambda$ by lowering a box from $c$-th to the $(c+1)$-th row. Under the hypotheses of 2.19 (i) this takes it from the $(r+2)$-th to the $(r+1)$-th column.

From the three paragraphs above we conclude that

$$m_x \cap O_I(c) = \{x \in m \mid \text{rk} x^{r+1} = s + c - 1\}$$

Thus this hypersurface in $m_x$ is the set of common zeros of the $(s + c) \times (s + c)$ minors of $M^{r+1}_{r+1}$. Hence $f_z(l)$ is exactly their largest common divisor.

3.17 It is of course not too easy to explicitly determine $f_z(l)$ through 3.16. Moreover these power rank conditions (i.e. 2.4 (i)) are in general insufficient to obtain ideals of definition of orbital variety closures. This was already observed by van Leeuwen [vanL, §8] and also resulted independently from [M5, §4.3] because otherwise the chain order defined in [M5, 4.3.1] would coincide with the geometric order which fails in $\mathfrak{sl}_7$ as it is shown in [M5, 4.3.6].

Combining 3.16 and 2.19 (i) (which implies that $f_z(l) = m_{l_z}$) gives a remarkable combinatorial fact, namely 3.16, about generic matrices. Here we note that $\deg m_{l_z} = (r+1)c + s$ whilst the minors in the conclusion of 3.16 have degree $(r+1)(s+c)$. Thus these degrees coincide if and only if $rs = 0$. Of course when degrees coincide $f_z(l)$ is proportional to any non-zero $(s + c) \times (s + c)$ minor of $M^{r+1}_{r+1}$. Using this Benlolo and Sanderson were able to prove their conjecture, that is Theorem 2.19 (i), under hypothesis $r = 0$ or the hypothesis $c = 1$ (which forces $s = 0$). These cases are relatively easy up to proving irreducibility for which they developed a special trick. A purely combinatorial proof of the case $s = 0$ already appears to be rather difficult.

4. Strong quantization of hypersurface orbital varieties

4.1 We begin with a combinatorial lemma. Let $\mathbb{R}$ be the real field.

**Lemma.** Let $c, c_2, \ldots, c_l$ be positive integers with $c = c_l$. Suppose that the system of inequalities for $\{b_i\}_{i=2}^l$ defined by

1) $1 + \sum_{i=2}^s b_i \leq \sum_{i=2}^{s-1} c_i + \max(c, c_s) : s = 2, \ldots, l-1$.

2) $1 + \sum_{i=s+1}^l b_i \leq \sum_{i=s+1}^{l-1} c_i + \max(c, c_s) : s = 2, 3, \ldots, l-1$.  

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3) \( 1 + \sum_{i=2}^{l} b_i = \sum_{i=2}^{l} c_i \)

has a solution \( b_i \in \mathbb{R}, \forall i \). Then \( c_s \neq c, \forall s : 2 \leq s < l \). Moreover, if the latter holds one can find a solution with \( b_i \in \mathbb{N}, \forall i \).

**Proof.**

Assume that the system has a solution \( b_i \in \mathbb{R}, \forall i \). Adding 1) and 2) gives

\[
2 + \sum_{i=2}^{l} b_i \leq \sum_{i=2}^{l} c_i + 2 \max(c, c_s) - c - c_s
\]

Then in view of 3) we obtain \( 1 \leq 2 \max(c, c_s) - c - c_s \), that is \( c \neq c_s \) for any \( s : s < k \).

Conversely equality in 1) and 3) gives

\[
b_i = \begin{cases} 
\max(c, c_2) - 1, & \text{if } i = 2 \\
(c_{i-1} + \max(c, c_i) - \max(c, c_{i-1})), & \text{if } 2 < i \leq l 
\end{cases}
\]

These imply that

\[
\sum_{i=s+1}^{l} b_i = \sum_{i=s}^{l} c_i - \max(c, c_s) = \sum_{i=s+1}^{l-1} c_i + \max(c, c_s) + ((c + c_s) - 2 \max(c, c_s))
\]

Then 2) holds if and only if \( c + c_s \leq 2 \max(c, c_s) - 1 \) for all \( s = 2, \ldots, l - 1 \). Obviously this is equivalent to \( c_s \neq c \) for any \( s = 2, \ldots, l - 1 \). Note that by (*) these inequalities provide a solution with \( b_i \in \mathbb{N} \) for all \( i \).

**4.2** Recall the notation of 2.7. Given \( \mathcal{I} \subset \Pi \) let \( T_\mathcal{I} = (C_1, \ldots, C_l) \) be the corresponding tableau. Assume that \( c_1 = c_l = c \). One has \( \Pi' := \Pi \setminus \mathcal{I} = \{\alpha_{c_i} : i = 1, 2, \ldots, l-1\} \).

For \( i = 1, 2, \ldots, c \) we set

\[
\beta_i = \sum_{j=i}^{n-1-i} \alpha_j.
\]

Set \( \beta = \beta_1 \). Recall that the BS element \( m_{l_\mathcal{I}} \) defined in 2.18 has weight \( \text{wt}(m_{l_\mathcal{I}}) = -\sum_{i=1}^{c} \beta_i \).

Call \( \nu \in \mathfrak{h}^* \), \( \mathcal{I} \)-regular if \( (\alpha, \nu + \rho) \neq 0, \forall \alpha \in R_\mathcal{I} \). Set \( w.\nu := w(\nu + \rho) - \rho \). Set

\[
\mathcal{S}_\nu = \{(m, \gamma) \in \mathbb{N}^+ \times R^+ \setminus R_\mathcal{I}^+ | (\gamma^\vee, \nu + \rho) = m\}
\]

\[
\mathcal{S}_\nu^o = \{(m, \gamma) \in \mathcal{S}_\nu | s_\gamma.\nu + \rho \text{ is } \mathcal{I} \text{-regular}\}
\]

Take \( c_i \neq c \) \( i = 2, 3, \ldots, l - 1 \) in 4.1 and recall our normalization, making \( (\alpha, \alpha) = 2, \forall \alpha \in R \). Define \( \mu \in P(\Pi) \), by \( (\mu + \rho, \alpha) = 1 \), for \( \alpha \in \mathcal{I} \) and \( (\mu + \rho, \alpha_{c_i}) = -(b_{i+1} - 1) \) for \( i = 1, 2, \ldots, l - 1 \) and where \( \{b_i\}_{i=2}^{l} \) are given by the conclusion of 4.1.
Proposition. \( S^o_\mu = \{(c, \beta)\} \).

Proof.

One has

\[
\sum_{\alpha \in \mathcal{I}} (\mu + \rho, \alpha) = \sum_{i=1}^l (c_i - 1) = \sum_{i=2}^l (c_i - 1) + c - 1 = \sum_{i=2}^l (b_i - 1) + c = -\sum_{\alpha \in \Pi'} (\mu + \rho, \alpha) + c
\]

by 4.1 (3) and so

\[
(\mu + \rho, \beta^\lor) = (\mu + \rho, \beta) = \sum_{\alpha \in \mathcal{I}} (\mu + \rho, \alpha) + \sum_{\alpha \in \Pi'} (\mu + \rho, \alpha) = c.
\]

Thus \((c, \beta) \in S_\mu\).

Observe that if \((c', \gamma) \in S_\mu : \gamma \in R^+ \setminus R_\mathcal{I}^+, c' \in \mathbb{N}\), then \((c', \gamma) \in S^o_\mu\) means that

\[
s_\gamma(\mu + \rho) = \mu + \rho - (\gamma^\lor, \mu + \rho) \gamma = \mu + \rho - c' \gamma
\]

does not vanish on any \(\delta \in R_\mathcal{I}^+\). Note that for \(\delta \in R_\mathcal{I}^+\) and \(\gamma \in R^+ \setminus R_\mathcal{I}^+\) one has

\[
(\delta, \gamma) = \begin{cases} 0 & \text{if } \gamma - \delta \notin R^+ \\ 1 & \text{if } \gamma - \delta \in R^+ \end{cases}
\]

Thus vanishing on some \(\delta \in R_\mathcal{I}^+\) means that \(\gamma - \delta\) is a root and \((\delta, \mu + \rho) = c'\) or simply that \(\gamma - \delta\) is a root such that \((\mu + \rho, \gamma - \delta) = (\mu + \rho, \gamma) - (\mu + \rho, \delta) = 0\).

Note that \(\beta - \delta\) is a root for \(\delta \in R_\mathcal{I}^+\) iff \(\delta = \alpha_1 + \cdots + \alpha_i\) or \(\delta = \alpha_{n-1} + \cdots + \alpha_{n-i}\)

where \(i < c\). Thus \((\mu + \rho, \beta - \delta) > 0\) and so \((c, \beta) \in S^o_\mu\).

Consider \(\gamma = \alpha_1 + \alpha_{i+1} + \cdots + \alpha_j\) for \(i < c\) and \(j > n - c\). One has \((\mu + \rho, \gamma) = c - (i - 1) - (n - j - 1) = c'\). Then unless \(\gamma = \beta\) one can find \(\delta \in R_\mathcal{I}^+\) starting at \(\alpha_i\) or ending in \(\alpha_j\) such that \((\mu + \rho, \gamma - \delta) = 0\). We conclude that \((c', \gamma) \notin S^o_\mu\).

Set \(\delta^-_i = \alpha_1 + \alpha_2 + \cdots + \alpha_{i-1}\) for \(2 \leq i \leq l\) and \(\delta^+_j = \alpha_{c_i} + \alpha_{c_i+1} + \cdots + \alpha_{n-1}\) for \(1 \leq j \leq l - 1\). One has

\[
(\mu + \rho, \delta^-_i) = \sum_{i=1}^{s-1} (c_i - 1) + \sum_{i=2}^s -(b_i - 1) = c + \sum_{i=2}^{s-1} c_i - \sum_{i=2}^s b_i \geq c + 1 - \max(c, c_s)
\]

\[
(\mu + \rho, \delta^+_s) = \sum_{i=s+1}^l (c_i - 1) + \sum_{i=s+2}^l -(b_i - 1)
\]

\[
= \sum_{i=s+1}^{l-1} c_i - \sum_{i=s+1}^l b_i + c \geq c - \max(c, c_s) + 1
\]
Now consider $\gamma_{s,l} = \beta - \delta_s^-$ with $2 \leq s \leq l-1$ Then $(\mu + \rho, \gamma_{s,l}) \leq \max(c, c_s) - 1$. Thus it follows that $(c', \gamma_{s,l}) \not\in \mathcal{S}_\mu^\circ$. Indeed if $c_s > c$ we can subtract $\delta = \alpha_{s-1} + \cdots + \alpha_i$ for some $i : \varsigma_{s-1} + 1 \leq i < \varsigma_s$ from $\gamma_{s,l}$ so that $(\mu + \rho, \gamma_{s,l} - \delta) = 0$ and if $c_s < c$ we can subtract $\delta = \alpha_{n-1} + \cdots \alpha_i$ for some $i : \varsigma_l - 1 < i \leq n-1$ from $\gamma_{s,l}$ so that $(\mu + \rho, \gamma_{s,l} - \delta) = 0$. A similar assertion holds if replace $\gamma_{s,l}$ by $\gamma = \gamma_{s,l} - (\alpha_{s-1} + \cdots + \alpha_i) - (\alpha_{n-1} + \cdots \alpha_j)$ for any $i : \varsigma_{s-1} + 1 \leq i < \varsigma_s$ and $j : \varsigma_l - 1 < j \leq n-1$, that is to say when some of these roots have already been subtracted.

A similar conclusion holds for $\gamma_{1,s} = \beta - \delta_s^+$ with $2 \leq s \leq l-1$ and for $\gamma = \gamma_{1,s} - (\alpha_1 + \cdots + \alpha_i) - (\alpha_{s-1} + \cdots \alpha_j)$ for any $i : 1 \leq i < c$ and $j : \varsigma_{s-1} < j \leq \varsigma_s - 1$ (by the obvious symmetry).

Finally consider $\gamma_{i,j} = \beta - \delta_i^- - \delta_j^+$, with $2 \leq i < j \leq l-1$. We have 

$$(\mu + \rho, \gamma_{i,j}) \leq \max(c, c_i) + \max(c, c_j) - c - 2.$$ 

Then a similar conclusion holds in this case also with slightly stronger reason.

**Remark.** One may also check that the inequalities 1) and 2) of 4.1 are necessary for the conclusion of the proposition to hold with $\mu + \rho$ satisfying $(\mu + \rho, \alpha) = 1$, $\forall \alpha \in \mathcal{I}$ and $(\mu + \rho, \beta) = c$. Consequently the proposition fails when $c = c_i$, for some $i : 1 < i < l$.

**4.3** Take $\mu$ as in 4.2 and recall the definition of $\beta_i$ from 2.20. Recall that $\gamma_c = \sum_{i = 1}^s \beta_i$. Then $(\gamma_c, \alpha) = 0$, $\forall \alpha \in \mathcal{I}$. Moreover $\mu + \rho - \gamma_c$ is the unique $\mathcal{I}$ dominant element of $W_\mathcal{I}(\mu + \rho - c\beta) = W_\mathcal{I}(s_\beta(\mu + \rho))$.

Let $\{M_i^j(\mu)\}_{i = 0}^\infty$ be the Jantzen filtration of $M_\mathcal{I}(\mu)$. Given $(m, \gamma) \in \mathcal{S}_\mu^\circ$, let $\omega(s_\gamma, \mu)$ be the unique $\Pi'$ dominant element in $W_\mathcal{I}(s_\gamma, \mu)$ and sign $(s_\gamma, \lambda) := (-1)^{l(w)}$ where $w \in W_\mathcal{I}$ is the unique element satisfying $w.s_\gamma, \mu = \omega(s_\gamma, \mu)$. The Jantzen sum formula combined with 4.2 gives

$$\sum_{i = 1}^\infty \operatorname{ch} M_i^j(\mu) = \sum_{(m, \gamma) \in \mathcal{S}_\mu^\circ} \operatorname{sign}(s_\gamma, \mu) \operatorname{ch} M_\mathcal{I}(\omega(s_\gamma, \mu)) = \operatorname{ch} M_\mathcal{I}(\mu - \gamma_c), \quad (*)$$

by the above noting that $\operatorname{sign}(s_\beta, \mu)$ is necessarily positive because there is only this term on the right hand side. One may recall that the right hand side of the Jantzen sum formula obtains from the zero of the Shapovalov determinants. Then the appearance of just one term on the right hand side means that the Shapovalov determinant is non-zero on $M_\mathcal{I}(\mu)_{\nu} : \nu > \omega(s_\beta, \mu)$ and vanishes on $M_\mathcal{I}(\mu)_{\omega(s_\beta, \mu)}$.

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(Equivalently, but more directly by Jantzen [Ja, Satz 2] the Shapovalov determinant on the \( \nu \) weight subspace of \( M_{\mathcal{I}}(\mu) \) takes form
\[
\prod_{m=1}^{\infty} \prod_{\gamma \in R^+ \setminus R^+_I} ((\gamma, \mu + \rho) - m) \chi'_\nu(\mu - m\gamma)
\]
where
\[
\chi'_\nu(\lambda) = \sum_{w \in W_\mathcal{I}} (-1)^{l(w)} \dim M(w.\lambda)_\nu
\]
This has a zero if and only if some \((m, \gamma) \in S_\mu \) and the sum of the corresponding exponents is strictly positive. Now by (*), \( \chi'_\nu(\lambda) \neq 0 \) for some \( \nu \) if and only if \( \lambda + \rho \) is \( \mathcal{I} \) regular, so we further require \((\mu, \gamma) \in S_\mu^0 \). Moreover in this case \( \chi'_\nu(\lambda) \) equals \( \text{sign}(\lambda) \dim M_\mathcal{I}(\omega(\lambda))_\nu \). Since in addition \( S_\mu^0 = \{(m, \beta)\} \) in the present case we must have \( \chi'_\nu(s_{\beta, \mu}) \geq 0 \) for all \( \nu \) and of course no cancellations occur. Moreover from (*) we obtain
\[
\chi'_\nu(s_{\beta, \mu}) = \begin{cases} 
0, & \text{if } \nu > \omega(s_{\beta, \mu}), \\
1, & \text{if } \nu = \omega(s_{\beta, \mu}).
\end{cases}
\]
which gives the required assertion.)

This forces there to be a highest weight vector in \( M_{\mathcal{I}}(\mu) \) of weight \( \omega(s_{\beta, \mu}) = \mu - \gamma_c \). Since \((\alpha, \mu) = 0 \) for all \( \alpha \in \mathcal{I} \) it follows that \( M_{\mathcal{I}}(\mu) \) is a rank 1 free \( U(\mathfrak{m}_\mathcal{I}^-) \) module. Again \((\alpha, \mu - \gamma_c) = 0 \), for all \( \alpha \in \mathcal{I} \). Consequently this highest weight vector must generate a submodule of \( M_{\mathcal{I}}(\mu) \) isomorphic to \( M_{\mathcal{I}}(\mu - \gamma_c) \) necessarily contained in the maximal submodule \( M_{\mathcal{I}}^1(\mu) \) of \( M_{\mathcal{I}}(\mu) \). By (*) we then conclude that \( M_{\mathcal{I}}^1(\mu) = M_{\mathcal{I}}(\mu - \gamma_c) \) and so \( V(\mu) = M_{\mathcal{I}}(\mu) / M_{\mathcal{I}}(\mu - \gamma_c) \) is simple.

By [M4] the associated variety of \( V(\mu) \) is irreducible and hence an orbital variety \( \mathcal{V} \).

Set \( p_\mathcal{I} = \prod_{\alpha \in R^+_\mathcal{I}} \alpha \). Since \( \text{ch} V(\mu) = (\text{ch} M_\mathcal{I}(\mu))(1 - e^{-\gamma_c}) = \text{ch} S(\mathfrak{m}_\mathcal{I}^-) e^\mu (1 - e^{-\gamma_c}) \), it follows that the characteristic polynomial of \( \mathcal{V} \) is just \( \gamma_c p_\mathcal{I} \).

Let \( f_\mathcal{I}(l) \) be the BS element constructed in 2.18 and \( \mathcal{V}_{T_\mathcal{I}(l)} \) the corresponding hypersurface orbital variety. By 3.12 the zero variety \( \mathcal{V}(f_\mathcal{I}(l)) \) of \( f_\mathcal{I}(l) \) in \( \mathfrak{m}_\mathcal{I}^- \) is a union of orbital variety closures of codimension 1, one of which is \( \mathcal{V}_{T_\mathcal{I}(l)} \). Yet \( \text{wt} f_\mathcal{I}(l) = -\gamma_c \) and so the characteristic polynomial of \( \mathcal{V}(f_\mathcal{I}(l)) \) is also \( \gamma_c p_\mathcal{I} \). By the linear independence of characteristic polynomials of orbital varieties [J1] we conclude that \( \mathcal{V} = \mathcal{V}(f_\mathcal{I}(l)) = \mathcal{V}_{T_\mathcal{I}(l)} \).
This proves that $f_{\mathcal{I}}(l)$ is irreducible. Finally $R[\mathcal{V}_{T\mathcal{I}}(l)] = S(\mathcal{m}_{\mathcal{I}})/S(\mathcal{m}_{\mathcal{I}})f_{\mathcal{I}}(l)$, so $\text{ch} R[\mathcal{V}_{T\mathcal{I}}(l)] = \text{ch} S(\mathcal{m}_{\mathcal{I}})(1 - e^{-\gamma_c})$. Comparison with $\text{ch} V(\mu)$ which is above expression up to a shift, shows that $V(\mu)$ is a strong quantization of $\mathcal{V}_{T\mathcal{I}}(l)$. We have proved the

**Theorem.** Every hypersurface orbital variety in $\mathfrak{sl}_n$ admits a strong quantization being a simple highest weight module with integral highest weight given by the above procedure.

4.4 Let us consider the case where we take equality in 1) of 4.1. Using 3) we obtain

$$\sum_{i=2}^{s} b_i = \sum_{i=2}^{s-1} c_i + \max(c, c_s) - 1, \quad \sum_{i=s+1}^{l} b_i = \sum_{i=s}^{l} c_i - \max(c, c_s)$$

Then in the notation of 4.2 for $2 \leq s \leq l - 1$ we obtain equalities

$$(\mu + \rho, \delta_{s}^{-}) = \sum_{i=1}^{s-1} (c_i - 1) + \sum_{i=2}^{s} -(b_i - 1) = c - \max(c, c_s) + 1$$

$$(\mu + \rho, \delta_{s}^{+}) = \sum_{i=s+1}^{l} (c_i - 1) + \sum_{i=s+1}^{l} -(b_i - 1) = \max(c, c_s) - c_s$$

Consequently $(\mu + \rho, \gamma_{s,t}) = \max(c, c_s) - 1$ for $2 \leq s \leq l - 1$ and $(\mu + \rho, \gamma_{1,s}) = c + c_s - \max(c, c_s)$ for $2 \leq s < l$. Finally for any $\gamma_{s,t}$ with $1 < s < t < l$ one has

$$(\mu + \rho, \gamma_{s,t}) = (\mu + \rho, \gamma_{1,t}) - (\mu + \rho, \delta_{s}^{-}) = c + c_t - \max(c, c_t) - (c - \max(c, c_s) + 1)$$

$$= (c_t - 1) - \max(c, c_t) + \max(c, c_s). \quad (*)$$

Now let $\gamma_{i,j}^{-}$ denote $\gamma_{i,j}$ with the $(c_i - 1)$ roots of the $i$-th column and $(c_j - 1)$ of the $j$-th column removed that is

$$\gamma_{i,j}^{-} = \alpha_{\varsigma_{i+1}} + \cdots + \alpha_{\varsigma_{j-1}}.$$  

Then

$$(\mu + \rho, \gamma_{i,j}^{-}) = c + c_t - \max(c, c_t) - (c - 1) - (c_t - 1) = 2 - \max(c, c_t).$$

Yet to $\gamma_{i,j}^{-}$ we can always add either the roots from the first column or from $t$-th column to obtain some $\gamma := \delta + \gamma_{i,j}^{-} : \delta \in R_{T}^{+}$. Such $\gamma$ for which $(\mu + \rho, \gamma)$ is maximum satisfies

$$(\mu + \rho, \gamma) = 2 - \max(c, c_t) + \max(c, c_t) - 1 = 1 > 0. \quad (**)$$  

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Similarly for $2 \leq s < t \leq l$ by ($\ast$) one has

$$(\mu + \rho, \gamma_{s,t}) = \max(c, c_s) - \max(c, c_t) - (c_s - 1).$$

To $\gamma_{s,t}$ we can add either the roots from the $s$-th column or from $t$-th column. Again in the best case for such $\gamma$ one has

$$(\mu + \rho, \gamma) = \max(c, c_s) + \max(c, c_t) - \max(c, c_t) - c_s \geq 0.$$  \hfill (***)

Let us define

$$\hat{\mathcal{S}}_\mu = \{(m, \beta) \in \mathbb{N}^+ \times R^+ \setminus R^+_I \mid (\beta \vee \mu + \rho) = m\}$$

and

$$\hat{\mathcal{S}}^o_\mu = \{(m, \beta) \in \hat{\mathcal{S}}_\mu \mid s_\beta(\mu + \rho) \text{ is } I \text{ regular}\}.$$

**Lemma.** With the above choice of $\mu$ one has $\hat{\mathcal{S}}^o_\mu = \emptyset$.

**Proof.**

As before given $\gamma \in \hat{\mathcal{S}}_\mu$ we must find $\delta \in R^+_I$ such that $0 = (\delta, s_\gamma(\mu + \rho)) = (\delta, \mu + \rho + (\mu + \rho, \gamma)\gamma)$. It is enough to show that $\gamma + \delta$ is a root vanishing on $\mu + \rho$.

For this notice for $\gamma \in \hat{\mathcal{S}}_\mu$ we must first have $r := (\gamma, \mu + \rho) \in -\mathbb{N}^+$. If for example $\gamma = \gamma_{s,t} : 1 \leq s < t \leq l$, then (**) and (***) show that there exists $\delta \in R^+_I$ such that $\gamma + \delta$ is a root and $(\gamma + \delta, \mu + \rho)$ takes all possible integer values from $r$ to an integer $\geq 0$. The general case is similar (and easier).

4.5 Let $O_I$ be the subcategory of $O$ in which the Levi factor defined by $I$ acts finitely. By [JLT, 9.6]

**Proposition.** Given $\hat{\mathcal{S}}^o_\mu = \emptyset$, then $M_I(\mu)$ is projective in $O_I$.

4.6 **Theorem.** Define $\mu$ as in 4.4. Then $\text{Ann} V(\mu) \in \text{Max} U(\mathfrak{g})$.

**Proof.**

Since $\mathfrak{g}$ is of type $A$ the natural map $U(\mathfrak{g}) \to F(M_I(\mu), M_I(\mu))$ is surjective. Set $F^\mu = F(M_I(\mu), M_I(\mu)) = U(\mathfrak{g})/\text{Ann} M_I(\mu)$. Let $P$ be a maximal ideal of $F^\mu$. By [JLT, 10.9], 4.4 and 4.5 we have

$$P = \text{Ann}_{F^\mu} (M_I(\mu)/PM_I(\mu)).$$

Hence $PM_I(\mu) \subset M_I(\mu)$. Yet $M_I(\mu - \gamma_c)$ is the maximal submodule of $M_I(\mu)$ as a $U(\mathfrak{g})$ module. This forces $P \subset \text{Ann} V(\mu)$ and hence equality.
4.7 Take \( c = c_1, c_2, \ldots, c_l = c \). Proceeding as in 4.2 one may rather easily show that there exists \( \mu \in \mathfrak{h}^* \) satisfying \( (\mu + \rho, \alpha) = 1 \), for all \( \alpha \in \mathcal{I} \) and such that \( \mathcal{S}_\mu^0 = \{(c, \beta)\} \). Then as in 4.3, one checks that \( V(\mu) \) is a strong quantization of \( \mathcal{V}(f_\mathcal{I}(l)) \). Here we do not need \( c_i \neq c \), for \( i : 1 < i < l \), however the resulting \( \mu \) will not be integrable, nor will the associated variety of \( V(\mu) \) be irreducible. However it will be contained in the associated variety of \( \text{Ann} V(\mu) \) which by Borho-Kraft [BK] is just the closure of a nilpotent orbit specifically \( \mathcal{O}_{T_\mathcal{I}(l)} := \mathbf{G} \mathcal{V}_{T_\mathcal{I}(l)} \) in this case. Consequently the associated variety of \( V(\mu) \) is contained in \( \mathfrak{m} \cap \mathcal{O}_{T_\mathcal{I}(l)} \) and we recall that the latter is irreducible if and only if \( c_i \neq c \), for all \( i : 1 < i < l \). This concludes the third proof of the irreducibility of the BS elements.

Appendix: Index of Notation

Symbols appearing frequently are given below in order of appearance.

1.1 \( g \)
1.2 \( n^-, n^+, \mathfrak{h}, b, \mathbf{G}, \mathbf{B}, S(\cdot), U(\cdot), \mathcal{O}, \mathcal{V}, I(\mathcal{V}) \)
1.3 \( \lambda, V(\lambda), \mathcal{F} \)
1.5 \( m^+, m^-, p, \mathcal{M}(t), M(t) \)
1.7 \( P \)
2.1 \( n, V, W, R, R^+, \Pi, X_\alpha, n \cap w n, \mathcal{V}_w \)
2.2 \( g, \mathbf{G}, n, n^-, \mathbf{B}, e_{i,j}, \alpha_{i,j}, \alpha_i, s_\alpha, s_i \)
2.3 \( S_n, [a_1, \ldots, a_n], p_w(i), S(w) \)
2.4 \( P(n), \lambda, \lambda^*, k, l, D_{\lambda} \)
2.6 \( T_n, [a', a''], \text{sh} T, \sigma(T), \tilde{\mathfrak{s}}, < \cdot, \cdot, | \cdot |, Q(w), \mathcal{V}_w, T_V, \mathcal{V}_1 > \mathcal{V}_2, T_1 > T_2 \)
2.7 \( T_j, r_r(\cdot), e_r(\cdot), T^i, T_j, h(T_j^i), T^{i,j}, (T, S), (T_S) \)
2.8 \( (R + j), w_r(T), w_c(T), (T \downarrow b) \)
2.9 \( (R \uparrow j), (R - a), (C \leftarrow j), (T - T_j^i) \)
2.10 \( \tau(w), \tau(T), \tau(P), \tau(p), \tau(\mathcal{V}), P_T, P_V, p_T, p_V \)
2.11 \( \mathcal{I}, P_\mathcal{I}, M_\mathcal{I}, L_\mathcal{I}, p_\mathcal{I}, m_\mathcal{I}, I_\mathcal{I}, \ell(w), W_\mathcal{I}, w_\mathcal{I}, V_\mathcal{I}, \mathcal{O}_\mathcal{I} \)
2.12 \( T_\mathcal{I}, C_1^\mathcal{I}, C_2^\mathcal{I}, \ldots, C_l^\mathcal{I} \)
2.14 \( B_\mathcal{I}, n_\mathcal{I}, W_\mathcal{I}, \pi_\mathcal{I}, \pi, \mathcal{V}_\mathcal{I}(w), \pi_{i,j} \)
2.15 \( c_i, s_i, T_\mathcal{I}(i) \)
2.16 \( \mathcal{I}_n, I_1 \)
2.18 \( \lambda_\mathcal{I}, \mathcal{O}_\mathcal{I}(c) \)
2.19 \( f_\mathcal{I}(i), X(w), x_{i,j}, \{, \}, M(w), c, M_\mathcal{I}(t), d_\mathcal{I}, l_\mathcal{I}, m_{l_\mathcal{I}} \)
2.20 $\beta_i$, $\gamma_c$

3.4 $f$

3.5 $w - s$, $M^{i,j}$

3.8 $M$

3.9 $M^{i,j}$

4.2 $\beta$, $S_\nu$, $S'_\nu$, $\mu$

4.4 $\tilde{S}_\mu$, $\tilde{S}'_\mu$

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