The cosmological gravitating $\sigma$ model: 
solitons and black holes

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Abstract

We derive and analyze exact static solutions to the gravitating $O(3)$ $\sigma$ model with cosmological constant in (2+1) dimensions. Both signs of the gravitational and cosmological constants are considered. Our solutions include geodesically complete spacetimes, and two classes of black holes.
1 Introduction

In a recent paper [1], we have constructed and analyzed exact black hole solutions to the gravitating $O(3)$ $\sigma$ model in (2+1) dimensions. Besides Schwarzschild–like black holes, we also found two numerable families of cold black holes with multiple horizons of infinite area and vanishing Hawking temperature. It was then natural to extend this investigation and inquire whether these remarkable cold black hole solutions survive in the case of a non–vanishing cosmological constant. As we shall see below, they do not, but are replaced by equally interesting horizonless, geodesically complete solutions, while the Schwarzschild–like black holes give way to two distinct families of black hole solutions.

Static, rotationally symmetric solutions to the cosmological gravitating $\sigma$ model were previously investigated by Kim and Moon [2]. They integrated numerically the Einstein–$\sigma$ equations with boundary conditions appropriate to solitonic configurations, and obtained topological solitons with integral winding number, as well as non–topological solitons with half–integral winding number. They also found numerically extreme black hole solutions as limits of non–topological solitons. The present investigation is complementary to that of reference [2]. We shall consider only solutions following a geodesic in target space. This approach will enable us to obtain analytically all such geodesic solutions, without prescribed boundary conditions. These exact geodesic solutions were actually all excluded from the analysis of Kim and Moon by their choice of ansätze and/or boundary conditions.

In the next section we introduce the cosmological gravitating $\sigma$ model, which is reduced by the geodesic ansatz to cosmological gravity coupled to a massless scalar field $\sigma$, or to the dual cosmological Einstein–Maxwell theory (to which it is equivalent outside sources). In the case of static rotationally symmetric solutions, the scalar field $\sigma$ depends only on one coordinate, either the radial coordinate $\rho$ or the angular coordinate $\theta$. All the solutions such that $\sigma = \sigma(\rho)$ are derived and analyzed in Sect. 3, for both signs of the cosmological constant and of the gravitational constant. Depending on the value of the integration constant, these solutions either have naked singularities, which however are, for a certain parameter range, at infinite affine distance on timelike or spacelike geodesics. Or they are geodesically complete, with the wormhole spatial topology; these regular solutions qualify as non–topological solitons. Sect. 4 is devoted to the discussion of the solutions with $\sigma = \sigma(\theta)$. These fall into two classes. The first class contains the charged BTZ black holes [3] and related solutions of the cosmological Einstein-Maxwell theory, while the black holes of the second class are of the form $AdS_2 \times S^1$, and may be obtained as the near–horizon limits of extreme black holes of the first class.

2 Model equations

The three–dimensional $O(3)$ non–linear $\sigma$ model coupled to cosmological gravity is defined by the action

$$S = \frac{1}{2} \int d^3x \sqrt{|g|} \left[ -\frac{1}{\kappa} (g^{\mu\nu} R_{\mu\nu} + 2\Lambda) + g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right], \quad (2.1)$$
where the isovector field $\phi$ is valued on the two-sphere
\[ \phi^2 = \nu^2. \] (2.2)

We will allow for both signs of the three–dimensional Einstein constant $\kappa$ (this sign is not fixed a priori in three–dimensional gravity [4]) and of the cosmological constant $\Lambda$. The Euler equations for the $\sigma$–model field may be written as
\[ D_\mu D^\mu \phi - \lambda \phi = 0, \] (2.3)
with $D_\mu$ the spacetime covariant derivative, and $\lambda$ a Lagrange multiplier to be determined from the constraint (2.2).

In this paper we will consider only solutions depending on a single scalar potential $\sigma$. Eq. (2.3) then reduces to
\[ D_\mu \sigma D^\mu \sigma \frac{d^2 \phi}{d\sigma^2} - \lambda \phi = -D_\mu D^\mu \sigma \frac{d\phi}{d\sigma}. \] (2.4)
Without loss of generality [5], the potential $\sigma$ may be chosen to be harmonic,
\[ D_\mu D^\mu \sigma = 0, \] (2.5)
so that the $\sigma$–model field follows a geodesic on the sphere $\phi^2 = \nu^2$, i.e. a large circle parametrized by the angle $\sigma$. The action (2.1) then reduces to that of a massless scalar field coupled to cosmological gravity
\[ S = \frac{1}{2} \int d^3x \sqrt{|g|} \left[ -\frac{1}{\kappa} (g^{\mu\nu} R_{\mu\nu} + 2\Lambda) + \nu^2 g^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma \right]. \] (2.6)
As discussed in [1], this effective Einstein–scalar theory is locally equivalent to sourceless Einstein–Maxwell theory in (2+1) dimensions with a cosmological constant,
\[ S = \frac{1}{2} \int d^3x \sqrt{|g|} \left[ -\frac{1}{\kappa} (g^{\mu\nu} R_{\mu\nu} + 2\Lambda) - \frac{1}{2} g^{\mu\nu} g^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma} \right], \] (2.7)
the second group of Maxwell equations $D_\nu F^{\mu\nu} = 0$ being the integrability condition for the duality relation
\[ F^{\mu\nu} = \frac{\nu}{\sqrt{|g|}} \epsilon^{\mu\nu\lambda} \partial_\lambda \sigma, \] (2.8)
while the first group of Maxwell equations leads to the harmonicity condition (2.5).

Now we specialize to static rotationally symmetric solutions. A convenient parametrization for the spacetime metric is then [6, 7]
\[ ds^2 = U dt^2 + V d\theta^2 + \zeta^{-2} \frac{d\rho^2}{UV}, \] (2.9)
where $\theta$ is an angle and the metric fields $U > 0$, $V < 0$ and $\zeta$ depend only on the radial coordinate $\rho$. The correspondence with the parametrization of Kim and Moon [4],
\[ ds^2 = e^{2N} B dt^2 - \frac{d\tau^2}{B} - r^2 d\theta^2, \] (2.10)
\[ r^2 = -V, \quad e^{-2N} = \frac{\zeta^2}{4} \dot{V}^2, \quad B = \frac{\zeta^2}{4} U \dot{V}^2, \] (2.11)

where \( \dot{\equiv} \frac{d}{d\rho} \). In (2.9), the scaling function \( \zeta(\rho) \) may and will be taken equal to 1 after variation. The constraint \( g_{\rho\theta} = 0 \) is consistent with the Euler equations for the action (2.6) iff

\[ T_{\rho\theta} \equiv \nu^2 \partial_\rho \sigma \partial_\theta \sigma = 0, \] (2.12)

leading to the two possibilities \( \sigma = \sigma(\rho) \), or \( \sigma = \sigma(\theta) \).

The first possibility \( \sigma = \sigma(\rho) \) corresponds, in the parametrization of [2], to \( F(r) = \sigma(\rho), n = 0 \). This case was discussed very briefly (in the more general stationary case) at the end of [3]. The variational problem (2.6) reduces to the one–dimensional problem for the effective Lagrangian

\[ L = \frac{\zeta}{2} \left( -\frac{1}{2\kappa} \dot{U} \dot{V} + \nu^2 UV \dot{\sigma}^2 \right) - \frac{\Lambda}{\kappa \zeta}. \] (2.13)

The elimination of the cyclic variable \( \sigma \) in terms of its constant conjugate momentum \( p \) by

\[ \dot{\sigma} = \frac{p}{\nu^2 \zeta UV} \] (2.14)

then leads to the reduced Lagrangian (or, more precisely, the Routhian [8])

\[ \hat{L} \equiv L - p \dot{\sigma} = -\frac{\zeta}{4\kappa} \ddot{U} \dot{V} - \frac{1}{2\zeta} \left( \frac{p^2}{\nu^2 UV} + \frac{2\Lambda}{\kappa} \right). \] (2.15)

Note that the electromagnetic field dual to \( \sigma(\rho) \),

\[ F_{\theta \rho} = \nu p, \] (2.16)

is globally defined only if the spatial topology is such that the loops \( \rho = \text{constant} \) are non–contractible. The solutions of the equations of motion derived from (2.13) are discussed in Sect. 3.

The second possibility \( \sigma = \sigma(\theta) \) reduces, after using the harmonicity condition (2.5) and the single–valuedness of the \( \sigma \)--model field \( \phi \), to

\[ \sigma = n \theta \] (2.17)

(\( n \) integer), corresponding in the parametrization of [2] to \( F(r) = 0 \). In this case the constant of motion is the dual electromagnetic field density

\[ \Pi^{t\rho} \equiv -\zeta^{-1} F^{t\rho} = n\nu, \] (2.18)

corresponding to the quantized electric charge \( 2\pi n\nu \). Accordingly, the Einstein–Maxwell action (2.7) reduces to the reduced Lagrangian

\[ \hat{L} \equiv L - \Pi^{t\rho} F_{t\rho} = -\frac{\zeta}{4\kappa} \ddot{U} \dot{V} - \frac{1}{\zeta} \left( \frac{\Lambda}{\kappa} - \frac{n^2 \nu^2}{2V} \right). \] (2.19)

The solutions of the corresponding equations of motion, first derived in [4], will be revisited in Sect. 4.
3 Non–topological solitons (σ = σ(ρ))

The Lagrangian (2.13) leads (after setting the scale ζ(ρ) = 1) to the equations of motion

\[ V\dddot{U} = U\dddot{V} = -\frac{2\kappa p^2}{\nu^2 UV}, \quad (3.1) \]

\[ \dot{U} \dot{V} - \frac{2\kappa p^2}{\nu^2 UV} = 4\Lambda. \quad (3.2) \]

The linear combination of (3.1) and (3.2) leads to the equations

\[ (V \dot{U})\dot{V} = (U \dot{V}) = 4\Lambda, \quad (3.3) \]

\[ V\dot{U} = 4\Lambda(\rho - \alpha), \quad (3.4) \]

where α and β are two integration constants. The sum of these two equations may be further integrated (the integration constant being fixed by the Hamiltonian constraint (3.2)) to

\[ UV = 4\Lambda(\rho - \alpha)(\rho - \beta), \quad (3.5) \]

with

\[ \rho_{\pm} = \frac{\alpha + \beta \pm \delta}{2}, \quad \delta = \sqrt{(\alpha - \beta)^2 + \kappa p^2/2\Lambda^2\nu^2}. \quad (3.6) \]

Finally equations (3.3) and (3.4) may be combined with (3.5) to yield equations for \( \dot{U}/U \) and \( \dot{V}/V \) which are readily integrated. The resulting metric depends on the sign of \( \delta^2 \):

1) \( \delta^2 > 0 \). The metric and scalar field are given by

\[
\begin{align*}
\text{ds}^2 &= A|\rho - \rho_+|^{1/2-a} |\rho - \rho_-|^{1/2-a} \, dt^2 - \frac{4|\Lambda|}{A} |\rho - \rho_+|^{1/2-a} |\rho - \rho_-|^{1/2-a} \, d\theta^2 \\
&\quad + \frac{dp^2}{4\Lambda(\rho - \rho_+)(\rho - \rho_-)}, \quad \sigma = \frac{p}{4\Lambda\nu^2\delta} \ln\left(\frac{\rho - \rho_+}{\rho - \rho_-}\right),
\end{align*}
\]

where \( A \) is an integration constant (which we will assume to be positive), and \( a = (\beta - \alpha)/2\delta \), with \( a^2 > 1/4 \) for \( \kappa < 0 \), and \( a^2 < 1/4 \) for \( \kappa > 0 \). The metric (3.7) is Lorentzian and static, with \( \partial_t \) as timelike Killing vector, in the intervals where the product \( UV \) is negative, i.e. in the ranges \( \rho > \rho_+ \) or \( \rho < \rho_- \) for \( \Lambda < 0 \), and \( \rho_- < \rho < \rho_+ \) for \( \Lambda > 0 \).

In order to study the spacetime structure we first compute the scalar curvature

\[ R = -6\Lambda + \frac{\kappa p^2}{4\nu^2(\rho - \rho_+)(\rho - \rho_-)}, \quad (3.8) \]

which diverges at \( \rho = \rho_{\pm} \) and goes to a constant as \( \rho \to \infty \). The nature of the singularities \( \rho = \rho_{\pm} \) can be understood by considering the behavior of the geodesics in their vicinity.
The general expression for the proper time along radial (i.e. $\theta = \text{const.}$) timelike curves is

$$\tau = \int d\rho \sqrt{-g_{tt}g_{\rho\rho}} \sqrt{E^2 - g_{tt}}. \quad (3.9)$$

Application of the metric (3.7) tells us that for instance the singularity $\rho = \rho_+ \equiv \rho_0$ is reached in a finite proper time only if $a \geq -1/2$, whereas for $a < -1/2$ (which is possible only for $\kappa < 0$) there is always a turning point at some $\rho_1 > \rho_+$, and $\rho_+$ is never reached. For null geodesics the affine parameter

$$v = \frac{1}{E} \int d\rho \sqrt{-g_{tt}g_{\rho\rho}} \quad (3.10)$$

goes to infinity for $\rho \to \rho_+$ provided that $a < -3/2$, and is bounded otherwise. Similar considerations apply for the singularity at $\rho = \rho_- \equiv \rho_0$ (note that the solution (3.7) is invariant under the involution $\rho_+ \leftrightarrow \rho_-, \alpha \leftrightarrow -\alpha, p \leftrightarrow -p$). In the case $\Lambda < 0$, the other boundary of the spacetime, $\rho = \pm \infty$, is null and spacelike complete, while all timelike geodesics bounce inward at some finite $\rho_2$. Therefore, despite the behavior of $R$ in eq. (3.8), for $a < -3/2$ (resp. $a > 3/2$) the whole spacetime $\rho > \rho_+$ (resp. $\rho < \rho_-$) is null and timelike complete and so physically regular (only spacelike trajectories ‘feel’ the singularity at the point $\rho = \rho_\pm$). The global behavior of timelike geodesics, always bounded between $\rho_1$ and $\rho_2$, and the existence of two timelike regular boundaries ($\rho = \rho_\pm$ and $\rho = \pm \infty$) are exactly what happens also in pure AdS spaces. As in those cases, the full Penrose diagram is an infinite strip extending in the vertical $t$ direction.

We find it interesting to comment on some limiting cases. The limits $a \to \pm 1/2$, or $\beta - \alpha \to \pm \delta$, correspond to $\kappa p^2 \to 0$, i.e. the $\sigma$-model field decouples. Then $R = -6\Lambda$ and all curvature singularities disappear. In particular, $a = 1/2$ is nothing but the BTZ black hole [3] ($\rho_+$ is the location of the event horizon) and $a = -1/2$ the Deser-Jackiw-'t Hooft solution [4]. The connection with the corresponding solutions of the gravitating $\sigma$–model in the case $\Lambda = 0$ (see [1]) is more involved. The limit $\Lambda \to 0$ can be performed only locally, e.g. in the vicinity of $\rho_+$. Starting from the metric (3.7), let us introduce the coordinates $(x, \tilde{t})$ via ($c$ is an appropriate constant)

$$\rho = \rho_+ + c|\Lambda|x^{4/(1+2a)}, \quad \tilde{t} = |\Lambda|t. \quad (3.11)$$

and take the limit $\Lambda \to 0$ while keeping the quantity $A|\Lambda|^{1/2+a} = \text{const}$ (the relation between $a$ and the parameter $\alpha$ used in [1] is $\alpha = (1 - 2a)/(1 + 2a)$). In particular, for $a < -3/2$ and in the limit $x \to 0$ a horizon appears and, therefore, a black hole by analytic continuation to negative values of $x$. These features were absent in the original metric (3.7).

2) $\delta^2 = 0$. In this case (which can occur only for $\kappa < 0$), the two singularities coincide, $\rho_+ = \rho_- \equiv \rho_0$, so that the spacetime metric can be Lorentzian and static only for $\Lambda < 0$. Invariance under translations in $\rho$ allows us to take $\rho_0 = 0$, leading to the form of the solution

$$ds^2 = A|\rho|e^{b/2\rho}dt^2 + \frac{4\Lambda}{A}|\rho|e^{-b/2\rho}d\theta^2 + \frac{d\rho^2}{4\Lambda\rho^2},$$

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\[
\sigma = -\frac{p}{4\Lambda \nu^2 \rho}, \quad (3.12)
\]
with \(A > 0\) and \(b = (-\kappa p^2/2\Lambda^2 \nu^2)^{1/2}\) (we have taken into account the involution mentioned above to keep only the positive root), and the scalar curvature

\[
R = -6\Lambda + \frac{\Lambda b^2}{2\rho^2}. \quad (3.13)
\]

The nature of the singularity at \(\rho = 0\) depends on the range \((\rho > 0\) or \(\rho < 0\)) considered. For \(\rho > 0\), it is not only null, but also spacelike complete, whereas timelike geodesics always bounce back and never reach it. The global causal structure of this regular spacetime is the same as for the solutions \(a < -3/2\) analyzed in case 1 (the present case corresponds to \(a \to -\infty\)). For \(\rho < 0\) all geodesics terminate at \(\rho = 0\) for a finite affine parameter, i.e. the spacetime is truly singular there. Finally, for \(b = 0\) the geometry becomes regular and corresponds to the BTZ vacuum (the quantity \(\delta^2\) measures the mass of the solution).

3) \(\delta^2 < 0\). Again in this case necessarily \(\kappa < 0\) and \(\Lambda < 0\). Putting \(\eta^2 \equiv -\delta^2/4\), and choosing \(\alpha + \beta = 0\), we obtain the solution

\[
\begin{align*}
\sigma &= \frac{p}{4\Lambda \nu^2 \eta} \arctan(\rho/\eta) \\
\end{align*}
\]

\((\tilde{\alpha} = \alpha/\eta)\), and the corresponding scalar curvature

\[
R = -6\Lambda + \frac{\kappa p^2}{4\Lambda \nu^2 (\rho^2 + \eta^2)}. \quad (3.15)
\]

This solution is everywhere regular in the whole range \(-\infty < \rho < +\infty\). It is a non-topological soliton, disproving the claim of nonexistence of such \(n = 0\) regular solutions made by Kim and Moon \[2\]. This claim relied on the occurrence of a logarithmic divergence of \(F(r) \equiv \sigma(\rho)\) at \(r^2 \equiv -g_{\theta \theta} = 0\); however our solution (3.14) has the wormhole topology, with \(-g_{\theta \theta}\) positive everywhere.

This completes the analysis of the regular solutions of the \(\sigma\)-model equations for \(\sigma = \sigma(\rho)\). Let us add that the last two cases \((\Lambda < 0, \kappa < 0\) and \(\delta^2 \leq 0\)), which lead to geodesically complete spacetimes with the topology \(R^2 \times S^1\), also correspond to fully regular electrostatic solutions of the cosmological Einstein–Maxwell theory \[2.7\], with closed lines of force from (2.10).
4 The black hole class ($\sigma = \sigma(\theta)$)

We now turn to the analysis of the Lagrangian (2.19). The equations of motion, after setting $\zeta(\rho) = 1$, are

$$\ddot{V} = 0,$$

$$\ddot{U} = \frac{2\kappa n^2 \nu^2}{V^2},$$

$$\dot{U}\dot{V} = 4\Lambda - \frac{2\kappa n^2 \nu^2}{V}.$$

Eq. (4.1) is integrated by $V = a\rho + b$ ($a, b$ constants), leading to two cases (see also [7]).

1) In the generic case $a \neq 0$, we can always translate the radial coordinate $\rho$ and rescale the time coordinate $t$ so that $V = -2\rho$ ($\rho \geq 0$). Then the metric is given in Schwarzschild form by

$$ds^2 = -\Lambda(r^2 - 2\gamma^2 \ln(r/r_0)) \, dt^2 - r^2 \, d\theta^2 + \frac{dr^2}{\Lambda(r^2 - 2\gamma^2 \ln(r/r_0))},$$

with $r^2 = 2\rho$, and $\gamma^2 = -\kappa n^2 \nu^2 / 2\Lambda$. The Ricci scalar is

$$R = -2\Lambda \left(3 - \frac{\gamma^2}{r^2}\right),$$

indicating the presence of a curvature singularity at $r = 0$.

For $\Lambda < 0$, we recognize in (4.4) the electrically charged BTZ solution [3]. The horizons are defined as the surfaces where $f(r) \equiv -\Lambda(r^2 - 2\gamma^2 \ln(r/r_0)) = 0$. Let us first consider the case $\kappa > 0$ ($\gamma^2 > 0$). Then $f(r)$ is minimum for $r = \gamma$. We may distinguish the following three different cases:

i) $\gamma > \sqrt{er_0}$ ($f(\gamma) < 0$). This means that $f(r)$ vanishes for two values $r_\pm$ of $r$, $r_+$ being the black hole horizon and $r_-$ the inner horizon.

ii) $\gamma = \sqrt{er_0}$ ($f(\gamma) = 0$). This is the extremal solution where the two horizons merge $r_+ = r_- = \gamma$.

iii) $\gamma < \sqrt{er_0}$ ($f(\gamma) > 0$). In this case there are no horizons and the singularity is naked. In these three cases the causal structure of the solution (4.4) is the same as in the corresponding cases of the 4-dimensional Reissner-Nordström-Anti-de Sitter (RN-AdS) solution. Furthermore, it can also be verified that all geodesics behave as in the RN-AdS spacetime.

On the other hand, in the case $\kappa < 0$ ($\gamma^2 < 0$), the equation $f(r) = 0$ has always one solution, so that there is a single horizon, and all geodesics terminate at the spacelike singularity $r = 0$ in a finite proper time and affine parameter. This spacetime is the analogue of the Schwarzschild-Anti-de Sitter (or of the static BTZ) spacetime.

Let us now consider the case of a positive cosmological constant, $\Lambda > 0$. Then, for $\kappa > 0$ ($\gamma^2 < 0$), there is always one horizon at $r = r_h$, with $f(r) > 0$ for $0 < r < r_h$. 

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The Penrose diagram is obtained by a \( \pi/2 \) rotation from that of the Schwarzschild-Anti-de Sitter black hole. For \( \kappa < 0 \) (\( \gamma^2 > 0 \)), the spacetime contains sectors where the the Killing vector \( \partial_t \) is timelike only if \( \gamma > \sqrt{|\kappa r_0|} \). These sectors are connected by two horizons with cosmological sectors terminating respectively at the spacelike singularity \( r = 0 \) and at spacelike infinity. The Penrose diagram is similar to that of the Reissner-Nordström-Anti-de Sitter black hole, rotated by \( \pi/2 \) (or of the Schwarzschild-de Sitter black hole, including the case \( \gamma = \sqrt{|\kappa r_0|} \) where the black hole and the de Sitter horizons merge).

2) The second case is given by \( V \) constant, leading from Eq. (4.3) to \( V = -\gamma^2 \). Then eq. (4.2) can be solved and the general form of the metric is

\[
\begin{align*}
 ds^2 &= -\frac{2\Lambda}{\gamma^2} (\rho^2 + c) \, dt^2 - \gamma^2 \, d\theta^2 + \frac{d\rho^2}{2\Lambda(\rho^2 + c)},
\end{align*}
\]

where \( c \) is an integration constant. This metric is Lorentzian for \( \kappa \Lambda < 0 \) if the range of \( \rho \) is such that \( \Lambda(\rho^2 + c) < 0 \). The mixed Ricci tensor components are constant, \( R_t^t = R_\rho^\rho = -2\Lambda, R_\theta^\theta = 0 \), showing that the geometry is regular for all \( \rho \).

The causal structure depends on the signs of \( \Lambda \) and \( c \). For \( \Lambda < 0 \) (\( \kappa > 0 \), the metric (4.6) is Lorentzian for \( \rho^2 + c > 0 \). If \( c > 0 \), the solution is globally \( AdS_2 \times S^1 \). If \( c < 0 \), then the spacetime is the direct product of the \( AdS_2 \) black hole [10] (or extreme black hole if \( c = 0 \)) of mass \( |c| \) with the circle \( S^1 \). This spacetime has two horizons at \( \rho_{\pm} = \pm\sqrt{|c|} \) and two asymptotic regions \( \rho = \pm\infty \). Its boundaries are null and spacelike complete, while timelike geodesics cross both horizons and are bounded between \( \rho_1 \) (\( \rho_- \)) and \( \rho_2 \) (\( \rho_+ \)). Finally, for \( \Lambda > 0 \) (\( \kappa < 0 \)), then the metric (4.6), is Lorentzian for \( \rho^2 + c < 0 \) (implying \( c < 0 \)), and is the 3–dimensional version of the Nariai solution, i.e. the direct product of the 2–dimensional de Sitter spacetime \( dS_2 \) with the circle \( S^1 \).

The enhanced symmetry of these black hole solutions suggests that they may be obtained as near horizon geometries of the generic black holes of case 1) in the limit where two horizons become coincident. To show this (a similar proof for generic 2D dilaton gravity theories is presented in [11] and for \( D \geq 4 \) Einstein-Maxwell theory in [12]), let us start with the generic black hole metric (4.4) and define

\[
\begin{align*}
 r_0 &\equiv \gamma e^{-1/2+c\alpha^2/\gamma^2}, \quad t = \frac{\tilde{t}}{\alpha\gamma}, \quad r = \gamma + \alpha \rho.
\end{align*}
\]

When the near-horizon limit \( \alpha \to 0 \) (\( r \to \gamma \)) is taken (for either \( \Lambda < 0 \), \( \kappa > 0 \) or \( \Lambda > 0 \), \( \kappa < 0 \)), the metric (4.6) is obtained.

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