Is topological Skyrme Model consistent with the Standard Model?

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Abstract

The topological Skyrme model is known to give a successful description of baryons. As a consistency check, here it is shown that in view of the recent discovery of charge quantization as an intrinsic and basic property of the Standard Model and the color dependence arising therein, the Skyrme Model is indeed completely consistent with the Standard Model.
It is well known that in SU($N_c$) Quantum Chromodynamics in the limit of $N_c$ going to infinity the baryons behave as solitons in an effective meson field theory [1]. A popular candidate for such an effective field theory is the topological Skyrme Model [2]. It has been extensively studied for two or more flavours [3] and it has been shown that the resemblance of the topological soliton to the baryon in the quark model in the large $N_c$ limit is very strong [4,5]. It’s baryon number and the fermionic character is also well understood [3,6].

Theoretically the most well studied and experimentally the best established model of particle physics is the Standard Model (SM) based on the group $SU(3_c) \otimes SU(2)_L \otimes U(1)_Y$ [3]. The model consists of a priori several disparate concepts which are brought together to give the SM its structure as a whole. The successes of the SM are many however, it is believed to have a few shortcomings. It has been a folklore in particle physics that the electric charge is not quantized in the SM. It was felt that one has to go to the Grand Unified Theories to obtain quantization of the electric charge. It turned out to be a false accusation against the SM. It was clearly and convincingly demo- 

\[ Q(u) = Q(c) = Q(t) = \frac{1}{2}(1 + \frac{1}{N_c}) \quad (1) \]

\[ Q(d) = Q(s) = Q(b) = \frac{1}{2}(-1 + \frac{1}{N_c}) \quad (2) \]

For $N_c = 3$ this gives the correct charges. A short derivation of the result is given in the Appendix. It was also demonstrated by the author [7] that these were the correct charges to use in studies for QCD for arbitrary $N_c$. 

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This was contrary to many who had been using static (i.e. independent of color) charges $2/3$ and $-1/3$ [1,4,5,6].

Hence in addition to the other well known properties of the SM, I would like to stress that the quantization of the electric charge and the structure of the electric charge arising therein, especially its color dependence, should be treated as an intrinsic property of the SM. A consistency with the SM should be an essential requirement for phenomenological models which are supposed to work at low energies and for any extensions of the SM which should be relevant at high temperatures especially in the context of the early universe.

The color dependence of the electric charge shown above should be viewed in two independent but complementary ways. Firstly for $N_c \neq 3$ it is different from the static charges $Q(u)=2/3$ and $Q(d)=-1/3$. Secondly even for $N_c = 3$ it should be viewed as providing an anatomic view of the internal structure of the electric charge, meaning as to how are $2/3$ and $-1/3$ built up and in what way the three colors contribute to it. For example the SM is making the statement that in $1/3$ the $3$ is not entirely due to the $3$ of the QCD group $SU(3_c)$. However this is what the $SU(5)$ Grand Unified Theory says [9,10,11], wherein $Q(d)=-1/3 = \frac{-1}{N_c=3}$. This is conflict with the SM expression where $Q(d)= -1/3 = \frac{1}{2}(-1 + \frac{1}{N_c=3})$. Hence the expression for the electric charge can be a very discriminating and restrictive tool for extensions beyond the SM. This has been used to check consistency of various models in a fruitful manner [9,10,11].

Quite clearly low energy phenomenological models of hadrons should be consistent with the SM in all respects. Is it true for the topological Skyrme Model? It shall be demonstrated below that the answer to the question in the title of the paper is in the affirmative.

To do so let us start with the Skyrme Lagrangian [6]

$$L_S = \frac{f_\pi^2}{4} Tr(L_\mu L'^\mu) + \frac{1}{32e^2} Tr[L_\mu, L_\nu]^2 \quad (3)$$

where $L_\mu = U^\dagger \partial_\mu U$. The $U$ field for the three flavour case for example is

$$U(x) = exp\left(\frac{i\lambda^a \phi^a(x)}{f_\pi}\right)$$

with $\phi^a$ the pseudoscalar octet of $\pi$, $K$ and $\eta$ mesons. In the full topological Skyrme this is supplemented with a Wess-Zumino effective action
\[ \Gamma_{WZ} = \frac{-i}{240\pi^2} \int_{\Sigma} d^5x \epsilon^{\mu\nu\alpha\beta\gamma} Tr[L_\mu L_\nu L_\alpha L_\beta L_\gamma] \] (4)

on surface $\Sigma$. Let the field $U$ be transformed by the charge operator $Q$ as
\[ U(x) \to e^{i\Lambda Q} U(x) e^{-i\Lambda Q}. \]

where all the charges are counted in units of the absolute value of the electronic charge.

Making $\Lambda = \Lambda(x)$ a local transformation the Noether current is [6]
\[ J_{\mu}^{em}(x) = j_{\mu}^{em}(x) + j_{\mu}^{WZ}(x) \] (5)

where the first one is the standard Skyrme term and the second is the Wess-Zumino term
\[ j_{\mu}^{WZ}(x) = \frac{N_c}{48\pi^2} \epsilon_{\mu\nu\lambda\sigma} Tr L_\nu L_\lambda L_\sigma (Q + U U^\dagger Q) \] (6)

In the standard way [6] we take the $U(1)$ of electromagnetism as a subgroup of the three flavour $SU(3)$. Its generators can be found by the canonical methods. As the charge operator can be simultaneously diagonalized along with the third component of isospin and hypercharge we write it as
\[ Q = \begin{pmatrix} q_1 & 0 & 0 \\ 0 & q_2 & 0 \\ 0 & 0 & q_3 \end{pmatrix} \]

The electric charge of pseudoscalar octet mesons are known. these give
\[ q_1 - q_2 = 1, q_2 = q_3 \] (7)

Hence one obtains
\[ Q = (q_2 + \frac{1}{3}) 1_{3\times3} + \frac{1}{2} \lambda_3 + \frac{1}{2\sqrt{3}} \lambda_8 \] (8)

In the standard way we use $U = A(t) U_c(x) A(t)^{-1}$ where $A$ is the collective coordinate. We obtain the $B=1$ electric charge from the Skyrme term in terms of the left-handed generators $L_\alpha$ only as
\[ Q^{em} = \frac{1}{2} (L_3 - (A^\dagger \lambda_3 A)_8 \frac{N_c B(U_c)}{\sqrt{3}}) + \frac{1}{2\sqrt{3}} (L_8 - (A^\dagger \lambda_8 A)_8 \frac{N_c B(U_c)}{\sqrt{3}}) \] (9)
The Wess-Zumino term contributes

\[ Q^{WZ} = N_c B(U_c)(q_2 + \frac{1}{3} + \frac{1}{2\sqrt{3}}(A_8^\dagger \lambda_3 A)_8 + \frac{1}{6}(A_8^\dagger \lambda_3 A)_8) \]  

(10)

Hence the total electric charge is [6]

\[ Q = I_3 + \frac{1}{2} Y + (q_2 + \frac{1}{3}) N_c B(U_c) \]  

(11)

For the hypercharge we take \( Y = \frac{N_c}{3} \) [12] and demanding that the proton charge be unit for any arbitrary value of \( N_c \) we find that \( q_2 \) is equal to \( Q(d) \) as given in eq. (2) and hence all the correct color dependent electric charges as demanded by the Standard Model are reproduced by the Skyrme model. Hence it is heartening to conclude that the Skyrme model is fully consistent with the Standard Model.

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Appendix

To demonstrate charge quantization as an intrinsic property of the SM the complete machinery which makes the SM is required. As required by the SM one has the repetitive structure for each generation of the fermions. Let us start by looking at the first generation of quarks and leptons $(u, d, e, \nu)$ and assign them to $SU(N_c) \otimes SU(2)_L \otimes U(1)_Y$ representation as follows $[7,8]$.

$$q_L = \begin{pmatrix} u \\ d \end{pmatrix}_L, (N_c, 2, Y_q)$$

$$u_R; (N_c, 1, Y_u)$$

$$d_R; (N_c, 1, Y_d)$$

$$l_L = \begin{pmatrix} \nu \\ e \end{pmatrix}; (1, 2, Y_l)$$

$$e_R; (1, 1, Y_e)$$

(12)

$N_c = 3$ corresponds to the Standard Model case. To keep things as general as possible this brings in five unknown hypercharges.

Let us now define the electric charge in the most general way in terms of the diagonal generators of $SU(2)_L \otimes U(1)_Y$ as

$$Q' = a'I_3 + bY$$

(13)

We can always scale the electric charge once as $Q = \frac{Q'}{a'}$ and hence $(b = \frac{b'}{a'})$

$$Q = I_3 + bY$$

(14)

In the SM $SU(N_c) \otimes SU(2)_L \otimes U(1)_Y$ is spontaneously broken through the Higgs mechanism to the group $SU(N_c) \otimes U(1)_{em}$. In this model the Higgs is assumed to be doublet $\phi$ with arbitrary hypercharge $Y_\phi$. The isospin $I_3 = -\frac{1}{2}$ component of the Higgs develops a nonzero vacuum expectation
value $\langle \phi \rangle_\phi$. Since we want the $U(1)_{em}$ generator $Q$ to be unbroken we require $Q < \phi >_\phi = 0$. This right away fixes $b$ in (3) and we get

$$Q = I_3 + \left( \frac{1}{2Y_\phi} \right) Y$$

(15)

Next one requires that the fermion masses arise through Yukawa coupling and also by demanding that the triangular anomaly cancels (to ensure renormaligability of the theory) (see [7,8] for details); one obtains all the unknown hypercharge in terms of the unknown Higgs hypercharge $Y_\phi$. Ultimately $Y_\phi$ is cancelled out and one obtains the correct charge quantization as follows.

$$q_L = \left( \begin{array}{c} u \\ d \end{array} \right)_L, \quad Y_q = \frac{Y_\phi}{N_c},$$

$$Q(u) = \frac{1}{2}(1 + \frac{1}{N_c}), \quad Q(d) = \frac{1}{2}(-1 + \frac{1}{N_c})$$

$$u_R, Y_u = Y_\phi(1 + \frac{1}{N_c}), \quad Q(u_R) = \frac{1}{2}(1 + \frac{1}{N_c})$$

$$d_R, Y_d = Y_\phi(-1 + \frac{1}{N_c}), \quad Q(d_R) = \frac{1}{2}(-1 + \frac{1}{N_c})$$

$$l_L = \left( \begin{array}{c} \nu \\ e \end{array} \right)_L, \quad Y_l = -Y_\phi, \quad Q(\nu) = 0, \quad Q(e) = -1$$

$$e_R, Y_e = -2Y_\phi, \quad Q(e_R) = -1$$

(16)

A repetitive structure gives charges for the other generation of fermions also [7,8].

Note that the Generalized Gell Mann Nishijima expression of the SU(6) (flavour) quark model is consistent with the above SM expression (eqn. 1 and 2). One takes $B = \frac{1}{N_c}$ in the expression $Q = I_3 + (B + S + C + b + t)$ with the standard values of $S, C, b, t$ for the quarks [7].
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