On the generalized Andrews-Curtis-Problem
– A Disproof of the Relative Case –

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Abstract

If $K$ and $L$ are finite, connected $PLCW$–complexes of dimension $n$, which are simple-homotopy equivalent, then there exists a deformation $K \xrightarrow{\varphi} L$, provided $n \geq 3$, see Wall [Wa66]. In Hog-Angeloni/Metzler, Chapter I of [LMS 197], 1993, pages 45,46, the case $n = 1$ is also listed. But in several publications, I am (co-)author of the case $n = 2$ is called questionable. This is the so called generalized Andrews-Curtis-Problem. A positive expectation is called the Andrews-Curtis Conjecture $(AC')$.

For $n \geq 3$ Wall even proved that a common subcomplex of $K$ and $L$ can be kept fix during the deformation. In the case $n = 2$ this relative version is what we disprove in the present paper. It was mentioned as open in Chapter 2 of [LMS 446]. In addition to my own previous work I strongly use two results of Allan J. Sieradski. Whereas in higher dimensions the relative case needs extra labour, dimension 2 does so in the absolute one. The end of the present paper contains hints towards this goal.

I dedicate this paper to friends and colleagues, who were and are partners of my work on $(AC')$, in particular to Cynthia Hog-Angeloni, and to my wives Ingrid Baumann-Metzler as well as to the memory of Helga Metzler (1942 - 1994), who accompanied the development of [LMS 446] resp. [LMS 197].
§1 Bias

For terminology we refer to earlier publications, in particular [LMS 197] and [LMS 446]. This covers the algebraic counterpart of 3-deformations, namely $Q^-$, $Q^*$ and $Q^{**}$ transformations of finite presentations. $Q^*$ and $Q^{**}$ transformations were first defined in [Me76]. The notion of bias is due to Michael N. Dyer and Allan J. Sieradski and concerns how spherical elements lie in the second homology of complexes. An overview can be found in M. Paul Latiolais Chapter III of [LMS 197].

Let $K^2, L^2$ be 2–complexes with isomorphic abelian $\pi_1$ and let $\alpha : \pi_1(K^2) \to \pi_1(L^2)$ be an isomorphism. If $\langle a_1, \ldots, a_g \mid a_1^m, \ldots, a_g^m, [a_i, a_j], [a_i, a_j], [a_i, a_j] \rangle$ with minimal ($= (g^2)$) number of commutators is a presentation of $\pi_1(K^2) = \mathbb{Z}_m \times \cdots \times \mathbb{Z}_m$, $m \neq 1$, $x_{ij}, y_{ij}$ prime to $m$, and likewise a presentation of $\pi_1(L^2)$ with generators $\alpha(a_i)$ and $x'_{ij}, y'_{ij}$ are given, then

(1) $K^2$ and $L^2$ are at most $Q^{**}$– (or homotopy-) equivalent, if a $k$ with $\prod x_{ij}y_{ij} \equiv \pm k^{g-1} \prod x_{ij}y_{ij} \mod m$ exists.

Definition: $m$ is called the bias modulus and the residual class of $\pm (\prod x_{ij}y_{ij}) \cdot (\prod x_{ij}y_{ij})^{-1}$ the bias in this situation.

(1) is the main result in [Me76]. For topological interpretations and generalizations of the bias invariant see [Dyer86] and [Me00].

(2) For $Q^{(*)}$–equivalence of $K^2$ and $L^2$ the bias even has to be $\pm 1$.

In his paper [Si77] Allan J. Sieradski showed that the criteria (1) and (2) are also sufficient and generalized them to a finite number of free products of finite abelian groups with the same $m$ and $g$. This corresponds to forming one-point unions of standard 2–complexes.

A simple example with bias modulus $m = 5$ is contained in chapter XII of [LMS 197] written by Cynthia Hog-Angeloni and myself on pages 377 and 378 together with an explicit list of (semisplit) $Q$–transformations from $P_1 \cup P_2$ to $Q_1 \cup Q_2$. It reads: Let $P_1, Q_1, P_2$ and $Q_2$ be presentations of $\mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_5$ given by

$P_1 = \langle a_1, a_2, a_3 \mid a_1^5, a_2^5, a_3^5, [a_1, a_2], [a_1, a_3], [a_2, a_3] \rangle$,

$Q_1 = \langle a_1, a_2, a_3 \mid a_1^5, a_2^5, a_3^5, [a_1, a_2], [a_1, a_3], [a_2, a_3] \rangle$,
\[ P_2 = \langle b_1, b_2, b_3 \mid b_1^5, b_2^5, b_3^5, [b_1, b_2], [b_1, b_3], [b_2, b_3] \rangle \text{ and } Q_2 = \langle b_1, b_2, b_3 \mid b_1^5, b_2^5, b_3^5, [b_1, b_2], [b_1, b_3], [b_2, b_3] \rangle, \quad g = 3. \]

In [LMS 197] the \(Q\)–transformations \(P_1 \lor P_2 \rightarrow Q_1 \lor Q_2\) were used to show that a factorization of a presentation class with freely indecomposable factors, if possible, is not unique in general.

We now use this example for the

**Theorem** There is no \(Q^{**}\)–transformation \(P_1 \lor P_2 \rightarrow Q_1 \lor Q_2\) rel. the joint \(1\)–skeleton of the standard \(2\)–complexes \(K^2(P_1 \lor P_2)\) and \(L^2(Q_1 \lor Q_2)\), the map of which is homotopic to the one given by the initial \(Q\)–transformation from \(P_1 \lor P_2\) to \(Q_1 \lor Q_2\).

§2 Proof of the Theorem

The bias is a homotopy invariant of maps (see M. Paul Latiolais, Chapter III in [1]). Because of being induced by the \(Q\)–transformation \(P_1 \lor P_2 \rightarrow Q_1 \lor Q_2\), its fundamental group map is (homotopic to) the identity, and by (2) the bias has a value \(\pm 1\).

For the proof we need in addition from [Si85] that for finite abelian \(\pi_1\) an automorphism can be decomposed into row transformations and diagonal ones. Such a decomposition is possible even if the automorphism is the identity but the commutators of the presentations contain nontrivial exponents.

Keeping fix the \(1\)–skeleton of \(P_1 \lor P_2\) and \(Q_1 \lor Q_2\) (up to homotopy) during a \(Q^{**}\)–transformation would mean that the free factors of \(\pi_1\) would be fixed. There would hold an equation

\[ (3) \quad \pm 1 \equiv k^{g-1} \cdot 2 \cdot 2 \mod 5, \quad \text{one} \ 2 \text{ belonging to } P_1 \lor P_2, \quad \text{the other one to } Q_1 \lor Q_2. \]

But fixing the free factors, by (1) above and [Si85] the two factors 2 in (3) aren’t quadratic residues \(\mod 5\), although their product is. This is a behaviour similar to the fact that a “product” of two Möbius bands results in a torus. Hence a \(Q^{**}\)–transformation with the properties of the theorem doesn’t exist. \(\square\)

**Remark:**

The example of our theorem and similar ones, which are based on bias give rise to the two special cases for \((AC'):\)

\[ 1^{\text{I don’t know a generalization of Sieradski’s result for more than one free factor.}} \]
A) In general is is impossible to fix a subcomplex during a deformation.

B) In general it is likewise impossible to choose the final map being homotopic to the initial (simple-)homotopy equivalence.

As Cynthia Hog-Angeloni has mentioned, in non-bias situations the cases may disagree.

§3  An outlook to the absolute case

Our theorem stimulates the idea to show the necessity of 4–expansions in the absolute case, which – astonishingly enough – could be avoided in the relative one. This idea may be made concrete by thickening the above example at those subcomplexes that were fixed so far. And the Möbius bands may give assistance of algebraic topology. Of course, other strategies may be useful in addition.

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