Picard-Fuchs Equations and Prepotentials in $N = 2$ Supersymmetric QCD

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Abstract

The Picard-Fuchs equations for $N = 2$ supersymmetric $SU(N_c)$ Yang-Mills theories with massless hypermultiplets are obtained for $N_c = 2$ and 3. For $SU(2)$ we derive the non-linear differential equations for the prepotentials and calculate full non-perturbative corrections to the effective gauge coupling constant in the weak and strong coupling regions.

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1 Introduction

The work of Seiberg and Witten [1, 2] on the exact solution to the low-energy effective action of $N = 2$ supersymmetric Yang-Mills theory has afforded not only a renewed insight into the charge confinement [2] and the chiral symmetry breaking [2], but also a marvelous insight into topological invariants [3] and conformal field theories in four dimensions [4]. The key ingredients in obtaining these exact results are duality and the appearance of massless monopoles/dyons in the strong coupling regions of the theory. In the weak coupling region, the exact solution enables us to determine full non-perturbative instanton corrections to the effective coupling constant, whose evaluation is otherwise quite cumbersome in the standard framework of quantum field theory.

The low-energy effective action is described in terms of a prepotential which is a single holomorphic function of superfields of $N = 2 U(1)$ vector multiplets. The exact solution for the prepotential may be characterized by the period integrals of the special type of one-form on a hyper-elliptic curve. The curves associated with a variety of $N = 2$ supersymmetric Yang-Mills theories and QCD have been studied extensively [5]−[20]. The moduli space of the curve contains singularities, at which some solitons become massless. In order to investigate the strong coupling physics, one needs to evaluate the period integrals near the singular locus. An efficient approach to study this problem is to use the Picard-Fuchs equation, the differential equation which the periods obey [11, 12, 13, 14].

In the present article, we shall investigate the quantum moduli space of $N = 2$ supersymmetric Yang-Mills theories with massless hypermultiplets and gauge groups $SU(2)$ and $SU(3)$. We shall derive the Picard-Fuchs equation for the scalar part of $N = 2 U(1)$ vector multiplets and their duals. By solving the non-linear differential equation obeyed by the prepotential for $G = SU(2)$, we will explicitly evaluate the non-perturative contributions in the prepotential in both weak and strong coupling regions.
2 \( N = 2 \) Supersymmetric Yang-Mills Theory with Massless Hypermultiplets

We begin with reviewing some basic properties of the low-energy effective action of the \( N = 2 \) supersymmetric \( G = SU(N_c) \) QCD. In \( N = 1 \) superfield formulation, the theory contains chiral multiplets \( \Phi^a \) and chiral field strength \( W^a \) both in the adjoint representation of \( G \), and chiral superfields \( Q^i \) in the \( N_c \) and \( \bar{Q}^i \) in the \( \bar{N}_c \) representation of the gauge group. The superpotential is given by \( W = \bar{Q}T^a\Phi^aQ + M^i_j\bar{Q}^iQ^j \) where \( T^a \) is the generator of \( G \) and \( M^i_j \) is the mass matrix. Along the flat direction the scalar fields \( \phi \) of \( \Phi \) get vacuum expectation values, which break the gauge group to the Cartan subgroup \( U(1)^r \) where \( r = N_c - 1 \) is the rank of \( G = SU(N_c) \). When the squark fields do not have vacuum expectation values, the low-energy effective theory is in the Coulomb branch and contains \( r \) \( U(1) \) vector multiplets \((A^i,W^a_\alpha)\) \((i = 1, \ldots, r)\), where \( A^i \) are \( N = 1 \) chiral superfields and \( W^a_\alpha \) are \( N = 1 \) vector superfields. The quantum moduli space may be characterized by the low-energy effective Lagrangian \( \mathcal{L} \) with the prepotential \( F(A) \)

\[
\mathcal{L} = \frac{1}{4\pi} \text{Im} \left( \int d^2\theta d^2\bar{\theta} A_{Di} \bar{A}^i + \frac{1}{2} \int d^2\theta \tau^{ij} W^i_\alpha W^j_\alpha \right),
\]

where \( A_{Di} = \frac{\partial F}{\partial A^i} \) is a field dual to \( A^i \) and \( \tau^{ij} = \frac{\partial^2 F}{\partial A^i \partial A^j} \) the effective coupling constants. We denote the scalar component of \( A^i \), \( A_{Di} \) by \( a^i, a_{Di} \). The pairs \((a_{Di}, a^i)\) are the \( Sp(2r,\mathbb{Z}) \) section over the space of gauge invariant parameters \( s_i \) \((i = 2, \ldots, N_c)\) defined by \( \det(x - \phi) = x^{N_c} - \sum_{i=2}^{N_c} s_i x^{N_c-i} \). The quantum moduli space of the Coulomb branch is parametrized by the gauge invariants \( s_i \) and the eigenvalues \( m_1, \ldots, m_{N_f} \) of the mass matrix.

The sections \((a_{Di}, a^i)\) are obtained as the period integrals of a meromorphic differential \( \lambda \) over the hyper-elliptic curve \( \mathcal{C} \):

\[
y^2 = C(x)^2 - G(x),
\]

\[
C(x) = x^{N_c} - \sum_{i=2}^{N_c} s_i x^{N_c-i} + \frac{\Lambda^{2N_c-N_f}}{4} \sum_{i=0}^{N_f-N_c} t_i(m) x^{N_f-N_c-i},
\]

\[
G(x) = \Lambda^{2N_c-N_f} \prod_{i=1}^{N_f} (x + m_i),
\]

(2)
where $\Lambda$ stands for the QCD scale parameter and $t_i(m)$ is defined by $\Pi_{i=1}^{N_f}(x + m_i) = \sum_{i=0}^{N_f} t_i(m)x^{N_f-i}$ with $t_0(m) = 1$. The terms proportional to $\Lambda^{2N_c-N_f}$ in $C(x)$ are absent in the case of $N_f < N_c$. Notice that for $N_c = 2, N_f = 2$ theory $C(x)$ should read

$$C(x) = x^2 - s_2 + \frac{\Lambda^2}{8}. \quad (3)$$

The meromorphic differential $\lambda$ is given by

$$\lambda = \frac{x}{2\pi i} d\log \frac{C - y}{C + y}. \quad (4)$$

This differential satisfies the relations

$$\frac{\partial \lambda}{\partial s_i} = -\omega_{N_c+1-i} + d(*), \quad (5)$$

where $\omega_i = \frac{x^{i-1}dx}{y}$ is the basis of the holomorphic one-forms of the hyper-elliptic curve $C$. Choosing the basis of homology cycles $(A_i, B_i)$ on $C$ with the canonical intersection form $A_i \cdot B_j = \delta_{ij}$, $A_i \cdot A_j = B_i \cdot B_j = 0$, one can express the pairs $(a_{Di}, a_i)$ as

$$\alpha_i \cdot a_D = \int_{B_i} \lambda, \quad (\lambda_i - \lambda_{i-1}) \cdot a = \int_{A_i} \lambda, \quad i = 1, \cdots, r, \quad (6)$$

where $\alpha_i$ are the simple roots of $G$ and $\lambda_i$ the fundamental weights of $G$ ($\lambda_0 \equiv 0$).

The BPS mass formula $M \geq |q_i a_i + h_i a_{Di}|$ shows that the massless soliton may be characterized by the vanishing cycle $q_i A_i + h_i B_i \sim 0$. Therefore the singularities of the quantum moduli space are determined by the discriminant $\Delta$ of the curve. Let us enumerate explicit forms of the discriminants $\Delta_{N_f}^{N_c}$ for the $SU(N_c)$ theories ($N_c = 2, 3$) with $N_f(< 2N_c)$ matter hypermultiplets. In the following $\Lambda_{N_f}$ denotes the scale parameter corresponding to the $N_f$-flavor theory.

- $N_c = 2$ ($u \equiv s_2$)

$$\Delta_0^2 = 256\Lambda_0^8(u - \Lambda_0^2)(u + \Lambda_0^2)$$
\[ \Delta_1^2 = -\Lambda_1^6(27\Lambda_1^6 + 256u^3) \]
\[ \Delta_2^2 = \Lambda_2^4(8u - \Lambda_2^3)^2(8u + \Lambda_2^3)^2/16 \]
\[ \Delta_3^2 = \Lambda_3^2(\Lambda_3^2 - 256u)u^4 \]

\[ N_c = 3 \ (u \equiv s_2, \ v \equiv s_3) \]
\[ \Delta_0^8 = 64\Lambda_0^8(-27(v - \Lambda_0^3)^2 + 4u^3)(-4u^3 + 27(v + \Lambda_0^3)^2) \]
\[ \Delta_1^1 = \Lambda_1^{15}(-3125\Lambda_1^{15} + 256\Lambda_1^5u^5 + 22500\Lambda_1^{10}uv - 1024u^6v \]
\[ -43200\Lambda_1^5u^2v^2 + 13824u^3v^3 - 46656v^5) \]
\[ \Delta_2^2 = 64\Lambda_2^{12}v^2(-4(\Lambda_2^3 - u)^3 - 27v^3)(-4(\Lambda_2^3 + u)^3 + 27v^3) \]
\[ \Delta_3^3 = \Lambda_3^9(-\Lambda_3^3 + 4v^3)(729\Lambda_3^6u^3 - 256u^6 - 3888\Lambda_3^3u^3v \]
\[ +3456u^3v^2 + 2916\Lambda_3^3v^3 - 11664v^4))/16 \]
\[ \Delta_4^4 = 4\Lambda_4^6v^4(\Lambda_4^4u - 8\Lambda_4^2u^2 + 16u^3 + 2\Lambda_4^2v - 72\Lambda_4uv - 108v^2) \]
\[ (-\Lambda_4^4u + 8\Lambda_4^2u^2 - 16u^3 + 2\Lambda_4^2v - 72\Lambda_4uv + 108v^2) \]
\[ \Delta_5^5 = 4\Lambda_5^6v^5(\Lambda_5^4u^3 - 256u^6 + 2\Lambda_5^2u^3v - 528\Lambda_5^4u^3v + \Lambda_5^4uv^2 \]
\[ -300\Lambda_5^2u^2v^2 + 3456u^3v^2 + 4\Lambda_5^3v^3 - 4212\Lambda_5uv^3 - 11664v^4) \]

Note that for the gauge group $SU(2)$ the zeros of the discriminant realize the discrete $\mathbb{Z}_4$ symmetry which arises from the $U(1)_R$ anomaly and the parity transformation in the flavor symmetry $O(2N_f)$. Order of the zeros corresponds to the number of massless hypermultiplets appearing at the singularities. Massless monopoles obey the spinor representation of the flavor symmetry $SO(2N_f)$. For $SU(3)$ symmetries are less obvious particularly for odd flavors.

### 3 The Picard-Fuchs equations

Let us study the Picard-Fuchs equations for $(a_{Di}, a_i)$ and examine their solutions. We first discuss the $SU(2)$ case in detail for simplicity. One can change the curve of fourth order to the third one by using the Möbius transformation. The curves are given by

\[ y^2 = x^2(x - u) + \frac{1}{4}\Lambda_0^4x \]
for $N_f = 0$ and
\begin{equation}
    y^2 = x^2(x - u) - \frac{1}{64} \Lambda_{N_f}^{2(4-N_f)}(x - u)^{N_f-1}
\end{equation}
for $N_f = 1, 2, 3$. The meromorphic differential $\lambda$ for this curve is given by
\begin{equation}
    \lambda = \sqrt{2} \frac{2u - (4 - N_f)x}{8\pi y} dx
\end{equation}
which satisfies
\begin{equation}
    \frac{d\lambda}{du} = \sqrt{2} \frac{dx}{8\pi y} + d(*).
\end{equation}
It is not difficult to see that the period $\Pi = \oint \lambda$ satisfies the differential equation
\begin{equation}
    p(u) \frac{d^2\Pi}{du^2} + \Pi = 0,
\end{equation}
where $[11, 12, 14]
\begin{equation}
    \begin{cases}
        4(u^2 - \Lambda_0^4) & N_f = 0 \\
        4u^2 + \frac{27\Lambda_1^6}{64u} & N_f = 1 \\
        4\left(u^2 - \frac{\Lambda_2^6}{64u}\right) & N_f = 2 \\
        u\left(4u - \frac{\Lambda_3^2}{64}\right) & N_f = 3.
    \end{cases}
\end{equation}
Note that the regular singularities of $[13]$ correspond to the zeros of the discriminant. Introducing a new variable
\begin{equation}
    z = \begin{cases}
        \left(\frac{u}{\Lambda_0^2}\right)^2 & N_f = 0 \\
        -\frac{256}{27} \left(\frac{u}{\Lambda_1^2}\right)^3 & N_f = 1 \\
        64 \left(\frac{u}{\Lambda_2^2}\right)^3 & N_f = 2 \\
        256 \left(\frac{u}{\Lambda_3^2}\right) & N_f = 3
    \end{cases}
\end{equation}
it is shown that the Picard-Fuchs equation (13) turns out to be the Gauss’s hypergeometric system

\[ z(1 - z) \frac{d^2 \Pi}{dz^2} + (\gamma - (\alpha + \beta + 1)z) \frac{d\Pi}{dz} - \alpha \beta \Pi = 0, \quad (16) \]

where

\[ \alpha = \beta = \frac{-1}{2(4 - N_f)}, \quad \gamma = \frac{3 - N_f}{4 - N_f}. \quad (17) \]

Using the fundamental solutions of (16), we may evaluate the power series solutions for \((a_D, a)\) near the singularities, which has been explicitly done in our previous work [14].

Next we write down the Picard-Fuchs equations for \(SU(3)\) theory with flavors \(N_f \leq 5\). The periods \(\Pi = \oint \lambda\) now depend on two parameters \(u\) and \(v\). The Picard-Fuchs equations take the form

\[ L_1 \Pi = 0, \quad L_2 \Pi = 0, \quad (18) \]

where \(L_1\) and \(L_2\) are given by

- \(N_f = 0\)

\[ L_1 = \frac{1}{3} (-27 \Lambda_0^6 + 4u^3 + 27v^2) \frac{\partial^2 v}{\partial v^2} + 12uv \frac{\partial u}{\partial v} + 3v \frac{\partial v}{\partial v} + 1 \]

\[ L_2 = \frac{1}{u} (-27 \Lambda_0^6 + 4u^3 + 27v^2) \frac{\partial^2 u}{\partial u^2} + 12uv \frac{\partial u}{\partial v} + 3v \frac{\partial v}{\partial v} + 1 \quad (19) \]

- \(N_f = 1\)

\[ L_1 = \frac{-25 \Lambda_1^5 u^2 + 84u^3 v + 405v^3}{45v} \frac{\partial^2 v}{\partial v^2} + \frac{64u^4 - 1125 \Lambda_1^5 v + 2160uv^2}{180v} \frac{\partial u}{\partial v} \]

\[ + \frac{16u^3 + 135v^2}{45v} \frac{\partial v}{\partial v} + 1 \]

\[ L_2 = \frac{4(-25 \Lambda_1^5 u^2 + 84u^3 v + 405v^3)}{-25 \Lambda_1^5 + 84uv} \frac{\partial^2 u}{\partial u^2} \]

\[ + \frac{(-25 \Lambda_1^5 + 36uv)(-25 \Lambda_1^5 + 96uv)}{-25 \Lambda_1^5 + 84uv} \frac{\partial u}{\partial v} \]

\[ + \frac{3v(-25 \Lambda_1^5 + 36uv)}{-25 \Lambda_1^5 + 84uv} \frac{\partial v}{\partial v} + 1 \quad (20) \]
\( \bullet N_f = 2 \)

\[ \mathcal{L}_1 = \frac{-8\Lambda_2^4 u + 8u^3 + 27v^2}{3} \frac{\partial_v^2}{v} + \frac{4(2\Lambda_2^8 - 4\Lambda_2^4 u^2 + 2u^4 + 27uv^2)}{9v} \partial_u \partial_v \]
\[ + \frac{-8\Lambda_2^4 u + 8u^3 + 27v^2}{9v} \partial_v + 1 \]
\[ \mathcal{L}_2 = \frac{-8\Lambda_2^4 u + 8u^3 + 27v^2}{2u} \partial_u^2 + \frac{3v(\Lambda_2^4 + 3u^2)}{u} \partial_v \partial_v + 1 \] \hspace{1cm} (21)

\( \bullet N_f = 3 \)

\[ \mathcal{L}_1 = \frac{(4v - \Lambda_3^2)(4u^3 + 9v^2)}{4v} \partial_v^2 + \frac{u(64u^3 - 81\Lambda_3^2 v + 432v^2)}{36v} \partial_u \partial_v \]
\[ + \frac{16u^3 + 27v^2}{9v} \partial_v + 1 \]
\[ \mathcal{L}_2 = \frac{4u^3 + 9v^2}{u} \partial_u^2 + \frac{u(-9\Lambda_3^2 + 32v)}{4} \partial_u \partial_v - v \partial_v + 1 \] \hspace{1cm} (22)

\( \bullet N_f = 4 \)

\[ \mathcal{L}_1 = \frac{-\Lambda_4^1 u - 56\Lambda_2^4 u^2 + 240u^3 + 324v^2}{36} \frac{\partial_v^2}{v} + \frac{-8\Lambda_2^4 u^2 + 32u^3 + 27v^2}{9v} \partial_v \]
\[ + \frac{2\Lambda_4^1 u^2 - 16\Lambda_4^1 u^2 + 32u^4 - 9\Lambda_4^1 v^2 + 108uv^2}{9v} \partial_u \partial_v + 1 \]
\[ \mathcal{L}_2 = \frac{-\Lambda_4^1 u - 56\Lambda_2^4 u^2 + 240u^3 + 324v^2}{\Lambda_4^1 + 60u} \partial_u^2 \]
\[ + \frac{v(\Lambda_4^1 + 12u)(-\Lambda_4^1 + 36u)}{\Lambda_4^1 + 60u} \partial_u \partial_v + \frac{3v(\Lambda_4^1 - 36u)}{\Lambda_4^1 + 60u} \partial_v + 1 \] \hspace{1cm} (23)

\( \bullet N_f = 5 \)

\[ \mathcal{L}_1 = \frac{v(-2\Lambda_5^2 u^2 + 528u^3 + 57\Lambda_5 uv + 324v^2)}{5\Lambda_5 u + 36v} \partial_v^2 \]
\[ - \frac{-5\Lambda_5^2 u^3 + 1280u^4 - \Lambda_5^3 uv + 332\Lambda_5 u^2 v - 4\Lambda_5^2 v^2 + 1728uv^2}{4(5\Lambda_5 u + 36v)} \partial_u \partial_v \]
The Picard-Fuchs equations for $N_c = 3, N_f = 0$ theory have been investigated in detail by Klemm, Lerche and Theisen [12]. They show that the system reduces to the Appell’s hypergeometric system of type $F_4$. Although other cases are not classified as the Appell’s system, the solutions of these equations may be obtained in the form of the power series expansion in $u$ and $v$.

\section*{4 Calculation of the Prepotentials}

Solving the system of the Picard-Fuchs equations, we evaluate the non-perturbative corrections to the prepotentials, which can be regarded as the multi-instanton contributions in the weak coupling region and the threshold corrections in the strong coupling regions. In the previous work [14], we have calculated some non-trivial corrections by using the solutions of the hypergeometric system. In this article, we shall instead take more direct approach by solving the non-linear differential equation for the prepotential in $SU(2)$ theory. The present approach was proposed originally by Matone for $SU(2)$ pure Yang- Mills theory [13].

We start with the Picard-Fuchs equation (13). Notice that the equation carries no first derivative term. This implies that the Wronskian of the system

\begin{equation}
W = \begin{vmatrix}
a & a_D \\
\partial_u a & \partial_u a_D
\end{vmatrix}
\end{equation}

(25)
is independent of $u$; $\partial_u W = 0$. When we integrate this with respect to $u$ there appears a constant, the explicit value of which is evaluated at any singularity in the moduli space. One finds that

\begin{equation}
W = \frac{ib}{4\pi},
\end{equation}

(26)
where \( b = 4 - N_f \) is the coefficient of the beta-function of \( SU(2) \) \( N = 2 \) QCD. In the present case, we can further simplify the equation (26) since the Wronskian is written in the form of the total derivative

\[
W = \partial_a \left( a \frac{\partial \mathcal{F}}{\partial a} - 2 \mathcal{F} \right). \tag{27}
\]

Thus we get the following relation between the prepotential and the gauge invariant parameter \( u \)

\[
a \frac{\partial \mathcal{F}}{\partial a} - 2 \mathcal{F} = \frac{ib}{4\pi} u + \text{const.} \tag{28}
\]

The constant term may be absorbed in the definition of \( \mathcal{F} \) by shifts, which does not affect the form of the low-energy effective action. This identity has been generalized for any \( N = 2 \) Yang-Mills theories with or without matter hypermultiplets [21, 22] and is important in view of its relation to the soliton theory [23].

The relation (28) allows us to express \( u \) in terms of \( a \)

\[
u = \mathcal{G}(a) \equiv \frac{4\pi}{ib} \left( a \frac{\partial \mathcal{F}}{\partial a} - 2 \mathcal{F} \right). \tag{29}
\]

The Picard-Fuchs equation becomes

\[
- p(\mathcal{G}(a)) \frac{d^2 \mathcal{G}}{da^2} + a \left( \frac{d\mathcal{G}}{da} \right)^3 = 0. \tag{30}
\]

This non-linear equation is utilized to determine the instanton coefficients recursively. Near the singularity at \( u = \infty \), the prepotential takes the form

\[
\mathcal{F}(a) = \begin{cases} 
\frac{ia^2}{2\pi} \left\{ 2 \log \frac{a^2}{\Lambda_0^2} + \sum_{k=0}^{\infty} F_k(0) \left( \frac{\Lambda_0}{\alpha} \right)^{4k} \right\} & N_f = 0 \\
\frac{ia^2}{4\pi} \left\{ (4 - N_f) \log \frac{a^2}{\Lambda_{N_f}^2} + \sum_{k=0}^{\infty} F_k(N_f) \left( \frac{\Lambda_{N_f}^2}{a^2} \right)^{k(4-N_f)} \right\} & N_f = 1, 2, 3.
\end{cases} \tag{31}
\]

The first four coefficients are listed in the table
and summarized by the formulas \((N_f = 1, 2, 3)\):

\[
\begin{array}{|c|c|c|c|c|}
\hline
N_f & F_1 & F_2 & F_3 & F_4 \\
\hline
0 & -\frac{1}{2} & -\frac{5}{211} & -\frac{3}{218} & -\frac{1469}{218} \\
1 & \frac{3}{212} & -\frac{3.17}{2^{238}} & \frac{5.7.11}{2^{38}} & \frac{3^2.5.7.9679}{2^{29}} \\
2 & -\frac{1}{211} & -\frac{5}{230} & -\frac{3}{236} & -\frac{13.113}{2^{26}} \\
3 & -\frac{1}{211} & -\frac{1}{212} & 0 & -\frac{5}{211} \\
\hline
\end{array}
\]

\[
F_1(N_f) = -\frac{1}{2^{b}c(N_f)} \frac{1}{2 b^2}, \\
F_2(N_f) = -\frac{1}{2^{2b}c(N_f)^2} \frac{5 - 8b + 4b^2}{8^2 b^4}, \\
F_3(N_f) = \frac{1}{2^{3b}c(N_f)^3} \frac{(1 - b)(1 - 2b)(-23 + 30b - 16b^2)}{1728 b^6}, \\
F_4(N_f) = \frac{1}{2^{4b}c(N_f)^4} \frac{(1 - 2b)(-677 + 3910b - 8124b^2 + 8456b^3 - 4672b^4 + 1152b^5)}{294912 b^8}, \\
\]

(32)

where \(c(N_f) = \eta_N f 2^8 (4 - N_f)^{N_f - 4} (N_f = 1, 2, 3)\) and \(\eta_1 = -1, \eta_2 = \eta_3 = 1\). The \(N_f = 0\) result \([12, 13]\) may be obtained by replacing \(c(N_f)\) with 1 in the \(N_f = 2\) case. The results \([32]\) indeed agree with those obtained previously \([14]\).

Similarly we may solve the non-linear differential equation in the strong coupling region where monopoles become massless. In this case the dual field \(a_D\) is a good coordinate around the singularity \(u = u_0\) where

\[
u_0 = \begin{cases} \\
\Lambda_0^2 & N_f = 0 \\
-3 \cdot 2^{-\frac{8}{3}} \Lambda_1^2 & N_f = 1 \\
\Lambda_2^2/8 & N_f = 2 \\
0 & N_f = 3. \\
\end{cases}
\]

(33)

\footnote{We have corrected an error of \(F_2(N_f)\) in the previous work \([14]\).}
Introduce the dual prepotential $F_D(a_D)$ satisfying $a = \frac{dF_D}{da_D}$. Then, instead of (27), the Wronskian is expressed in terms of $F_D$. Hence around $u = u_0$ we get

$$u = u_0 + \frac{4\pi}{b} \left( 2F_D - a_D \frac{dF_D}{da_D} \right).$$

Similarly to the weak-coupling computation we now obtain the non-linear differential equation which makes it possible to calculate $F_D$ recursively. Using the expansion

$$F_D = \frac{i}{8\pi} a_D^2 \left\{ k \log \left( \frac{a_D^2}{\Lambda^2_{N_f}} \right) + \sum_{n=1}^{\infty} F_{Dn}(N_f) \left( \frac{a_D}{\Lambda_{N_f}} \right)^n \right\}$$

with $k = 1$ ($N_f = 0$) and $k = 2^{N_f-1}$ ($N_f = 1, 2, 3$), we find the coefficients as follows

$$F_D^{N_f=0} = 16kb^2\tilde{c}(N_f),$$

$$F_D^{N_f=1} = \frac{2 \cdot 2k}{3\tilde{c}(N_f)} \frac{(1 - 2b)(-5 + 2b)}{16b^2},$$

$$F_D^{N_f=2} = \frac{-2k}{2\tilde{c}(N_f)^2} \frac{(1 - 2b)(-61 + 190b - 92b^2 + 8b^3)}{1152b^4},$$

$$F_D^{N_f=3} = \frac{2 \cdot 2k}{5\tilde{c}(N_f)^3} \frac{(1 - 2b)(1379 - 8162b + 14596b^2 - 7288b^3 + 960b^4)}{110592b^6},$$

where $\tilde{c}(1) = -3^{-1/2} \cdot 2^{-17/6}, \tilde{c}(2) = -i2^{-9/2}$ and $\tilde{c}(3) = 2^{-13/2}$. The $N_f = 0$ result is obtained by putting $b = 2$ and $\tilde{c}(0) = -i/4$. Here we have corrected some misprints in the previous work [14].

For $SU(3)$ theory the prepotential $F$ obeys the basic identity analogous to (28) [21, 22]. In contrast to the $SU(2)$ case, combining this with the $SU(3)$ Picard-Fuchs equations does not yield a simple recursion relation. It is not clear at present if the existence of such recursion relation is peculiar to $SU(2)$ theory. It will be interesting if one could find an additional constraint equation for $F$ which eventually leads to the direct evaluation of instanton corrections.
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