THE LARGE $k$-TERM PROGRESSION-FREE SETS IN $\mathbb{Z}_q^n$

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In memory of Professor Chengdong Pan

Abstract. Let $k$ and $n$ be fixed positive integers. For each prime power $q \geq k \geq 3$, we show that any subset $A \subseteq \mathbb{Z}_q^n$ free of $k$-term arithmetic progressions has size $|A| \leq c_k(q)^n$ with a constant $c_k(q)$ that can be expressed explicitly in terms of $k$ and $q$. As a consequence, we can take $c_k(q) = 0.8415q$ for sufficiently large $q$ and arbitrarily fixed $k \geq 3$.

1. Introduction

In his famous papers [9],[10], Roth first considered the problem of finding upper bounds for the size of large subset of $\{1, 2, ..., N\}$ with no three-term arithmetic progression, and gave the first nontrivial upper bound. Since then, this problem has received considerable attentions by number theorists. Let $r_3(N)$ denote the maximal size of a subset of $\{1, 2, ..., N\}$ with no three-term arithmetic progression. Roth indeed proved $r_3(N) = O(N/\log \log N)$. This was subsequently improved and enhanced by Heath-Brown [7], Szemerédi [14], Bourgain [3], Sanders [12],[13], and Bloom [2]. The best result so far is $r_3(N) = O(N(\log \log N)^4/\log N)$, due to Bloom.

For an (additively written) abelian group $G$, we say that a subset $A$ of $G$ is $k$-term progression-free if there do not exist $a_1, a_2, \ldots, a_k \in A$ such that $a_k - a_{k-1} = a_{k-1} - a_{k-2} = \ldots = a_2 - a_1 \neq 0$, and denote by $r_k(G)$ the maximal size of $k$-term progression-free subsets of $G$.

In [4], Brown and Buhler first proved that $r_3(\mathbb{Z}_3^2) = o(3^n)$, and this was quantified by Meshulam [8] to $r_3(\mathbb{Z}_3^n) = O(3^n/n)$. In their ground-breaking paper, Bateman and Katz [1] proved that $r_3(\mathbb{Z}_3^n) = O(3^n/n^{1+\eta})$ with some positive constant $\eta > 0$. The best known upper bound, $o(2.756^n)$, is due to Ellenberg and Gijswijt [6]. Especially, they proved that, for any prime $p \geq 3$, there exists a positive constant $c = c(p) < p$ such that $r_3(\mathbb{Z}_p^n) = o(c^n)$. For the upper bound of $r_3(\mathbb{Z}_4^n)$, Sanders [11] proved that $r_3(\mathbb{Z}_4^n) = O(4^n/n(\log n)^n)$ with an absolute constant $\eta > 0$. Quite recently, Croot, Lev and Pach [5] developed the polynomial method and drastically improved the above upper bound to $r_3(\mathbb{Z}_4^n) \leq 4^{0.926n}$ in their breakthrough paper.

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For each positive integer \( m \), define
\[
\mathcal{A}(m) = \min_{x \in (0, 1)} \frac{(1 - x^m)}{m(1 - x)x^{m-1}}.
\]

In this paper, we introduce a formal polynomial method and establish the following upper bound of \( r_k(\mathbb{Z}_q^n) \) for \( p \geq 2 \) and \( k \geq 3 \).

**Theorem 1.1.** For any prime powers \( q = p^\alpha \geq k \geq 3 \) and \( n \geq 1 \), we have
\[
r_k(\mathbb{Z}_q^n) \leq \left( q \cdot \mathcal{A}\left(\frac{q}{(L_k, q)}\right) \right)^n,
\]
where \( \mathcal{A}(\cdot) \) is given by (1.1) and \( L_k \) denotes the l.c.m. of 2, 3, \ldots, \( k-1 \).

**Corollary 1.1.**

1. For \( k \geq 3 \) and large \( q \), \( r_k(\mathbb{Z}_q^n) \leq (0.8415q)^n \).
2. For \( k \geq 3 \) and each \( q > (L_k, q) \), \( r_k(\mathbb{Z}_q^n) \leq (0.945q)^n \).

**Notation.** Throughout this paper, \( p \) with or without subscript, is always reserved for primes. Denote by \((a, b)\) the greatest common divisor of \( a \) and \( b \), \( L_t := [2, 3, \ldots, t-1] \) the least common multiple of 2, 3, \ldots, \( t-1 \). For a set \( S \), denote by \(|S|\) the cardinality of \( S \), and define \( mS := \{ms : s \in S\} \).

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### 2. Some Lemmas

Throughout this section, we fix \( n \geq 1 \) and \( q = p^\alpha \geq k \geq 3 \).

Given a positive integer \( m \), the unknown \( Y \) is said to be a generator of order \( m \), if \( Y^0 = Y^m = 1 \) and \( Y^j \neq 1 \) for \( 1 \leq j \leq m-1 \). For \( 1 \leq i \leq n \), let \( Y_i \) be generators of order \( q \), then \( Y_i^q = 1 \), and
\[
\prod_{i=1}^{n} Y_i^{\lambda_i} = 1 \text{ if and only if } \lambda_i \equiv 0 \pmod{q} \text{ for each } 1 \leq i \leq n.
\]

For \( 1 \leq i \leq n \), put \( X_i = Y_i - 1 \). Let \( \mathbb{Z}_p[X_1, \ldots, X_n] \) denote the linear space spanned by monomials \( \{X_1^{\lambda_1} \cdots X_n^{\lambda_n} : 0 \leq \lambda_i \in \mathbb{Z}\} \) with coefficients over \( \mathbb{Z}_p \). \( F[X_1, \ldots, X_n] = 0 \) means all coefficients of \( F[X_1, \ldots, X_n] \) is 0 over \( \mathbb{Z}_p \). We thus have \( Y_i^q = (X_i + 1)^q = X_i^q + 1 \), which gives \( X_i^q = 0 \) since \( Y_i \) is of order \( q \). Hence it is reasonable to assume that the terms of \( X_1^{\lambda_1} \cdots X_n^{\lambda_n} \) vanish if \( \lambda_i \geq q \) for some \( 1 \leq i \leq n \).

For \( 0 < \alpha < 1/2 \), define
\[
\mathcal{M}_{\alpha, q} := \left\{ (\lambda_1, \lambda_2, \ldots, \lambda_n) \in [0, q)^n \cap \mathbb{Z}^n : \sum_{i=1}^{n} \frac{\lambda_i}{q-1} \leq \alpha n \right\}
\]
and $\overline{M}_{\alpha,q} = ([0,q)^n \cap \mathbb{Z}^n) \setminus M_{\alpha,q}$ denotes the complementary set. It is clear that $|\overline{M}_{\alpha,q}| = |M_{1-\alpha,q}|$.

For $c = (c_1, \ldots, c_n) \in \mathbb{Z}_q^n$, and $X = (X_1, \ldots, X_n)$, define

$$X^c := \prod_{i=1}^{n} X_i^{c_i}.$$ 

For $a, b \in \mathbb{Z}_q^n$, one can define $a_i, b_i, X^a, X^b$ accordingly.

For a set $B \subseteq \mathbb{Z}_q^n$, let $V_B$ denote the sub-space spanned by $\{X^a : a \in B\}$ over $\mathbb{Z}_p$. Then $\dim V_B = |B|$. When $B = \mathbb{Z}_q^n$ is the whole space, we write $V_{\mathbb{Z}_q^n} = V$ and thus $\dim V = q^n$. For each $f \in V$, we may write

$$f = \sum_{a \in \mathbb{Z}_q^n} f(a)X^a$$

with coefficients $f(a), a \in \mathbb{Z}_q^n$.

**Lemma 2.1.** Suppose $k \geq 3$ and $0 < \alpha < 1/2$. Let $A$ be a subset of $\mathbb{Z}_q^n$ satisfies $ra \neq rb$ for $a \neq b \in A$ with $1 \leq r \leq k - 1$. Suppose $P \in V_{M_{2,\alpha,q}}$ satisfies $P(2a - b)P(3a - 2b) \cdots P((k - 1)a - (k - 2)b) = 0$ for every pair $a, b$ of distinct elements in $A$. Then there exists an element $c \in A$ such that $P(c) = 0$ when $|A| > 2^{k-2} |\overline{M}_{\alpha,q}|$.

**Proof.** For brevity, we only prove the lemma for $k = 4$, and the method also works in the general case.

For $0 < \alpha < 1/2$, put $m_{\alpha} = \{X^\lambda : \lambda \in \overline{M}_{\alpha,q}\}$, so we can write

$$P(X) = \sum_{f,g \in m_{2\alpha}} c_{f,g} f(X)g(X).$$

In each term of the summand, at least one of $f$ and $g$ is in $m_{\alpha}$. Hence

$$P(X) = \sum_{f \in m_{\alpha}} f(X)F_f(X) + \sum_{g \in m_{\alpha}} g(X)G_g(X).$$

We thus have

$$P(X)P(Y) = \sum_{f,f_1 \in m_{\alpha}} f(X)f_1(Y)F_f(X)F_{f_1}(Y)$$

$$+ \sum_{g,g_1 \in m_{\alpha}} g(X)g_1(Y)G_g(X)G_{g_1}(Y)$$

$$+ \sum_{f,g_1 \in m_{\alpha}} f(X)g_1(Y)F_f(X)G_{g_1}(Y)$$

$$+ \sum_{f_1,g \in m_{\alpha}} f_1(X)g(Y)F_{f_1}(X)G_g(Y)$$
for some families of polynomials $F, G$ indexed by $m_\alpha$.

Write $A = \{a_1, a_2, \ldots, a_t\}$. Now let $B$ be the $t \times t$ matrix whose $i, j$ entry is $P(2a_i - a_j) P(3a_i - 2a_j)$. Then

$$B_{ij} = \sum_{f, f_1 \in m_\alpha} f(2a_i) f_1(3a_i) F_f(-a_j) F_{f_1}(-2a_j)$$

$$+ \sum_{g, g_1 \in m_\alpha} G_g(2a_i) G_{g_1}(3a_i) g(-a_j) g_1(-2a_j)$$

$$+ \sum_{f, g_1 \in m_\alpha} f(2a_i) G_{g_1}(3a_i) F_f(-a_j) g_1(-2a_j)$$

$$+ \sum_{f_1, g \in m_\alpha} G_g(2a_i) f_1(3a_i) g(-a_j) F_{f_1}(-2a_j)$$

$$= B^{(1)}_{ij} + B^{(2)}_{ij} + B^{(3)}_{ij} + B^{(4)}_{ij},$$

say. Hence $(B^{(s)}_{ij})$ is a sum of at most $|\mathcal{M}_{\alpha,q}|^2$ matrices for each $s$. One may see that each matrix in $(B^{(1)}_{ij})$ has the form

$$\begin{pmatrix}
  f(2a_1) f_1(3a_1) \\
  f(2a_2) f_1(3a_2) \\
  \cdots \\
  f(2a_t) f_1(3a_t)
\end{pmatrix}
\begin{pmatrix}
  F_f(-a_1) F_{f_1}(-2a_1), \\
  F_f(-a_2) F_{f_1}(-2a_2), \\
  \cdots, \\
  F_f(-a_t) F_{f_1}(-2a_t)
\end{pmatrix}$$

and of rank 1 or 0; the rank is 0 unless there exists some $a_i$ such that $f(2a_i) f_1(3a_i) = 1$, and the number of such $a_i$ is at most $|\mathcal{M}_{\alpha,q}|$. This yields the rank of $(B^{(1)}_{ij})$ is at most $|\mathcal{M}_{\alpha,q}|$. Similarly, one can show that the rank of $(B^{(2)}_{ij})$ is also at most $|\mathcal{M}_{\alpha,q}|$.

Regarding $(B^{(3)}_{ij})$, each of the $|\mathcal{M}_{\alpha,q}|^2$ matrices has the form

$$\begin{pmatrix}
  f(2a_1) G_{g_1}(3a_1) \\
  f(2a_2) G_{g_1}(3a_2) \\
  \cdots \\
  f(2a_t) G_{g_1}(3a_t)
\end{pmatrix}
\begin{pmatrix}
  F_f(-a_1) g_1(-2a_1), \\
  F_f(-a_2) g_1(-2a_2), \\
  \cdots, \\
  F_f(-a_t) g_1(-2a_t)
\end{pmatrix}$$

and of rank 1 or 0; the rank is 0 unless there exists some $a_i$ and $a_j$ such that $f(2a_i) = 1$ and $g_1(-2a_j) = 1$, then this matrix has only one non-zero element. The number of such $a_i$ is at most $|\mathcal{M}_{\alpha,q}|$, hence the row rank of $(B^{(3)}_{ij})$ is also at most $|\mathcal{M}_{\alpha,q}|$, which also applies similarly to $(B^{(4)}_{ij})$. Thus the rank of $B$ is at most $4|\mathcal{M}_{\alpha,q}|$.

On the other hand, by the hypothesis on $P$, $B$ must be a diagonal matrix. This completes the proof. \qed
Lemma 2.2. Let $q \geq k \geq 3$, $A$ a subset of $\mathbb{Z}_q^n$ which doesn’t contain $k$-term arithmetic progressions. Then we have

$$|A| \leq (2^{k-2} + 1)d^n|\mathfrak{M}_{1/3,q/d}|,$$

where $d = (L_k, q)$.

Proof. Suppose $t \geq 3$ and $b_1, b_2, \ldots, b_t$ is a non-trivial $t$-term arithmetic progression, then

$$b_t - b_{t-1} = b_{t-1} - b_{t-2} = \cdots = b_2 - b_1 \neq 0,$$

and

$$b_j = b_1 + (j-1)(b_2 - b_1) = (j-1)b_2 - (j-2)b_1, \quad 3 \leq j \leq t.$$ 

Hence each non-trivial $t$-term arithmetic progression $b_1, b_2, \ldots, b_t$ is determined by $b_2$ and $b_1$ only, taking the order into account.

Let $F$ be the kernel of the homomorphism of $\mathbb{Z}_q^n$ defined by $g \mapsto q^{(L_k,q)}g$ ($g \in \mathbb{Z}_q^n$), then

$$F = (L_k, q)\mathbb{Z}_q^n \cong \mathbb{Z}_q^n_{(L_k,q)},$$

and

$$\mathbb{Z}_q^n / F \cong \mathbb{Z}_{(L_k,q)}.$$ 

Let $\mathfrak{R}$ be the set of all $F$-cosets, we write $\mathfrak{R} = \{R_1, R_2, \ldots, R_{(L_k,q)^n}\}$. For $1 \leq j \leq (L_k, q)^n$, let $A_j := A \cap R_j$, we choose one element $r_j \in A_j$, and then we have

$$R_j = r_j + F.$$ 

Without loss of generality, we consider $A_1$. First, we have

$$(A_1 - r_1) \cap R_1 \subseteq F = (L_k, q)\mathbb{Z}_q^n \cong \mathbb{Z}_q^n_{(L_k,q)},$$

Therefore $A_1$ doesn’t contain any $k$-term arithmetic progression. Define $B$ by

$$A_1 - r_1 = (L_k, q)B, \quad B \subseteq \mathbb{Z}_q^n.$$ 

Hence $B$ doesn’t contain any $k$-term arithmetic progression and satisfies $ra \neq rb$ for $a \neq b \in B$ with $1 \leq r \leq k - 1$.

We shall prove that $|B| \leq (2^{k-2} + 1)|\mathfrak{M}_{1/3,q/(L_k,q)}|$, which would yield

$$|A| \leq (L_k, q)^n|B| \leq (2^{k-2} + 1)(L_k, q)^n|\mathfrak{M}_{1/3,q/(L_k,q)}|.$$
Assuming, contrary to what we want to prove, that $|B| > (2^{k-2} + 1)|\mathcal{M}_{1/3,q/(L_k,q)}|$. Let $W$ denote the linear space spanned by $\{X^\lambda : \lambda \in B \cap \mathcal{M}_{2/3,q/(L_k,q)}\}$, then

$$\dim W \geq |\mathcal{M}_{2/3,q/(L_k,q)}| + |B| - (q/(L_k,q))^n - |\mathcal{M}_{2/3,q/(L_k,q)}|$$

$$= |B| - |\mathcal{M}_{2/3,q/(L_k,q)}|$$

$$= |B| - |\mathcal{M}_{1/3,q/(L_k,q)}|$$

$$> 2^{k-2}|\mathcal{M}_{1/3,q/(L_k,q)}|.$$

We can choose some $b_i \in B$ such that

$$P := X^{b_1} + X^{b_2} + \cdots + X^{b_t} \in W \subseteq V_{\mathcal{M}_{2/3,q/(L_k,q)}}.$$

Let $B_1 := \{b_1, b_2, \ldots, b_t\} \subseteq B$.

By assumption we have $P(2a - b)P(3a - 2b) \cdots P((k - 1)a - (k - 2)b) = 0$ for every pair $a, b$ of distinct elements in $B_1$. Taking $\alpha = 1/3$ in Lemma 2.1, we have $P(b_i) = 0$ for some $b_i \in B_1$, this is a contradiction. Hence

$$|B| \leq (2^{k-2} + 1)|\mathcal{M}_{1/3,q/(L_k,q)}|,$$

and the lemma follows. \hfill \Box

**Lemma 2.3.** We have

$$|\mathcal{M}_{1/3,q}| \leq q^n \mathcal{A}(q)^n.$$

*Proof.* Write $\xi_i = \lambda_i/(q - 1)$, and we regard $\xi_i$ as random variables uniformly distributed in the set

$$\left\{0, \frac{1}{q - 1}, \frac{2}{q - 1}, \ldots, \frac{q - 1}{q - 1}\right\}.$$

Then

$$\left(2.5\right) \quad \frac{|\mathcal{M}_{1/3,q}|}{q^n} = \Pr\left(\sum_{i=1}^{n} \xi_i \leq n/3\right) = \Pr\left(x^{\sum_{i=1}^{n} \xi_i} \geq x^{n/3}\right)$$

for any $x \in (0, 1)$. By Chernoff bound, we have

$$\frac{|\mathcal{M}_{1/3,q}|}{q^n} \leq x^{-n/3} E\left[x^{\sum_{i=1}^{n} \xi_i}\right] = \left(\prod_{i=1}^{n} x^{-1/3} E[x^{\xi_i}]\right).$$

On the other hand, from the uniform distribution of $\xi_i$ ($1 \leq i \leq n$), it follows that

$$x^{-1/3} E[x^{\xi_i}] = \frac{1}{q^{1/3}} \sum_{j=0}^{q-1} x^{\frac{j}{q^{1/3}}} = \frac{1 - y^q}{q(1 - y)y^{q^{1/3}}},$$

with $y = x^{1/q^{1/3}}$. 

Hence we may conclude that
\[
\frac{|\mathcal{M}_{1/3, q}|}{q^n} \leq \left( \frac{1 - y^q}{q(1 - y)^{\frac{q-1}{3}}} \right)^n.
\]
The lemma then follows from the arbitrariness of \(x\) (and thus of \(y\)). \(\square\)

3. Proof of Theorem 1.1

Now we give the proof of Theorem 1.1.

Proof. For \(q = p^\alpha\), let \(A\) be a subset of \(\mathbb{Z}_q^n\) free of \(k\)-term arithmetic progressions. By Lemmas 2.2 and 2.3, for \(q > (L_k, q)\) we have
\[
|A| \leq (2^{k-2} + 1) \cdot (L_k, q)^n |\mathcal{M}_{1/3, \frac{q}{(L_k, q)}}| \leq (2^{k-2} + 1) \cdot q^n \cdot \mathfrak{A}\left(\frac{q}{(L_k, q)}\right)^n.
\]
For each positive integer \(v\), using the tensor trick, the set \(A \times A \times \cdots \times A \subseteq \mathbb{Z}_{q^v}^n\) is \(k\)-term progression-free, and therefore
\[
|A|^v \leq (2^{k-2} + 1) \cdot q^{vn} \cdot \mathfrak{A}\left(\frac{q}{(L_k, q)}\right)^{vn}.
\]
This implies Theorem 1.1 by letting \(v\) approach to infinity. \(\square\)

4. Proof of Corollary 1.1

Now we give the proof of Corollary 1.1. Here \(q\) is not necessary to be a prime power and we thus suppose \(q = \prod_{i=1}^I p_i^{\alpha_i}\) as the standard factorization of \(q\).

Proof. (1) For \(q \to +\infty\), we must have
\[
M := \max_{1 \leq i \leq I} p_i^{\alpha_i} \to +\infty.
\]
For large \(M\), taking \(x = 1 - \frac{\alpha}{M}\) with \(\alpha = 2.148\), we then have
\[
\lim_{M \to +\infty} \frac{1 - x^M}{M(1 - x)x^{\frac{M-1}{3}}} = \lim_{M \to +\infty} \frac{1 - (1 - \frac{\alpha}{M})^M}{\alpha(1 - \frac{\alpha}{M})^{\frac{M-1}{3}}} = \frac{e^{\alpha/3} - e^{-2\alpha/3}}{\alpha} < 0.8415,
\]
which yields \(\mathfrak{A}(M) \leq 0.8415\) for all sufficiently large \(M\). It follows that
\[
r_k(\mathbb{Z}_q^n) \leq (q/M)^n r_k(\mathbb{Z}_M^n) \leq (q/M)^n M^n \mathfrak{A}(M)^n \leq (0.8415q)^n.
\]
(2) For \(x = 1 - \frac{\beta}{N}\), \(\beta = 1.6\), we have
\[
\frac{1 - x^N}{N(1 - x)x^{\frac{N-1}{3}}} = \frac{1 - (1 - \frac{\beta}{N})^N}{\beta(1 - \frac{\beta}{N})^{\frac{N-1}{3}}} = \frac{(1 - \frac{\beta}{N})^{-\frac{N-1}{3}} - (1 - \frac{\beta}{N})^\frac{2N-1}{3}}{\beta}.
\]
When \(N \geq 13\), the above quantity is at most
\[
e^{-\frac{N-1}{3} - \beta/3} - e^{-\frac{2N+1}{3} - \beta/3} \leq e^{-\frac{4\beta}{13} - \beta} - e^{-\frac{9\beta}{13} - \beta} < 0.92.
\]
On the other hand, for all prime powers \( N < 13 \), we have the following list of explicit bounds for \( A(N) \): 

\[
\begin{align*}
A(2) &< 0.94495, \\
A(3) &< 0.9184, \\
A(4) &< 0.9027, \\
A(5) &< 0.8924, \\
A(7) &< 0.8795, \\
A(8) &< 0.8753, \\
A(9) &< 0.8718, \\
A(11) &< 0.8667.
\end{align*}
\]

Hence we may state, for each prime power \( N \geq 2 \), that

\[
A(N) < 0.945.
\]  

For \( q > (L_k, q) \), there exists some prime power \( p^\alpha \| q \|_{(L_k, q)} \). We may apply (4.1) with \( N = p^\alpha \), getting

\[
\begin{align*}
r_k(\mathbb{Z}_q^n) &\leq (q/N)^n r_k(\mathbb{Z}_N^n) \\
&\leq (q/N)^n N^n A(N^n) \\
&\leq (0.945 q)^n.
\end{align*}
\]

This establishes Corollary 1.1.

\[\square\]

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