Fundamental tradeoffs between memorization and robustness in random features and neural tangent regimes

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Abstract

This work studies the (non)robustness of two-layer neural networks in various high-dimensional linearized regimes. We establish fundamental trade-offs between memorization and robustness, as measured by the Sobolev-seminorm of the model w.r.t. the data distribution, i.e., the square root of the average squared $L_2$-norm of the gradients of the model w.r.t. its input. More precisely, if $n$ is the number of training examples, $d$ is the input dimension, and $k$ is the number of hidden neurons in a two-layer neural network, we prove for a large class of activation functions that, if the model memorizes even a fraction of the training, then its Sobolev-seminorm is lower-bounded by (i) $\sqrt{n}$ in case of infinite-width random features (RF) or neural tangent kernel (NTK) with $d \gtrsim n$; (ii) $\sqrt{n}$ in case of finite-width RF with proportionate scaling of $d$ and $k$; and (iii) $\sqrt{n/k}$ in case of finite-width NTK with proportionate scaling of $d$ and $k$. Moreover, all of these lower-bounds are tight: they are attained by the min-norm least-squares interpolator when $n$, $d$, and $k$ are in the appropriate interpolating regime. All our results hold as soon as data is log-concave isotropic, and there is label-noise, i.e., the target variable is not a deterministic function of the data. We empirically validate our theoretical results with experiments. Accidentally, these experiments also reveal for the first time a multiple-descent phenomenon in the robustness of the min-norm interpolator.

1 Introduction

Consider a random dataset $D_n = \{(x_1, y_1), \ldots, (x_n, y_n)\}$ consisting of $n$ labeled iid datapoints from a distribution on $\mathbb{R}^d \times \{\pm 1\}$. It is now well-known (e.g., see Bubeck et al. (2020a); Vershynin (2020) and references therein) that a two-layer neural network (NN) $f_{W,v}: \mathbb{R}^d \rightarrow \mathbb{R}$ with appropriate choice of activation function $\sigma: \mathbb{R} \rightarrow \mathbb{R}$ and sufficiently many hidden neurons $k$ (the network’s width), defined for input $x \in \mathbb{R}^d$ by

$$f_{W,v}(x) := \sum_{j=1}^k v_j \sigma(x^\top w_j), \quad \text{with } W = (w_1, \ldots, w_k) \in \mathbb{R}^{k \times d}, \quad v = (v_1, \ldots, v_k) \in \mathbb{R}^k,$$

(1)

can perfectly fit the dataset $D_n$ in the sense that $y_i = f_{W,v}(x_i)$ for all $i \in [n]$. For example, this can be done with $k \gtrsim n/d$ and the thresholding activation Baum (1988) or ReLU Bubeck et al. (2020a). In the model (1), each $w_j \in \mathbb{R}^d$ are the weights or parameter vector of the $j$th in the hidden layer and $v_j$ is the corresponding output weight for that neuron. If the network is over-parametrized in the sense that $k \geq n$, a very smooth/robust interpolation is realizable, in the sense that the input-to-output Lipschitz constant is bounded. This is indeed achievable by appropriately tuning each neuron to only handle one datapoint. Robustness is important in many situations, e.g., in machine learning applications where the goal is to learn a prediction function (aka model) which will perform well on unseen data from the same distribution, and so it is reasonable to ask that the predictions of the model be stable w.r.t. small perturbations in its input $x$, including adversarial perturbations. Recent work Bubeck et al. (2020b) hints that over-parametrization might not just be sufficient, but also necessary for robustness.

In this work, we study the robustness of (in)finite NNs in the random features (RF) Rahimi and Recht (2008, 2009) and neural tangent kernel (NTK) regimes Jacot et al. (2018). We establish quantitative trade-offs
between memorization and robustness in these regimes, as a function of complexity parameters $n$, $d$, and $k$. We also observe for the first time, a multiple-descent phenomenon Belkin et al. (2018); Loog et al. (2020) in the robustness of models in these regimes.

**Notation.** We will use the notation $a_n \gtrsim b_n$ (also written $a_n = \Omega(b_n)$ or equivalently, $b_n = O(n)$) to mean that $a_n \geq c b_n$ for some $c > 0$ and for sufficiently large $n$, while $a_n \lesssim b_n$ means $a_n \leq c b_n$. We will use $\Omega(\cdot)$ to mean $\Omega(\cdot)$ modulo log-factors. The notation $o(1)$ will be used to denote a quantity which goes to zero with $n$. Probabilistic versions of these notations are written with a subscript $\mathbb{P}$, for example $\mathbb{P}_S(\cdot)$, $o_p(\cdot)$, etc. The acronym $a.s.$ means almost-surely, $a.e.$ means almost-everywhere, $w.p.$ means with probability, and $w.h.p.$ means with high probability. The $L_p$-norm of a finite-dimensional vector $w$ is denoted $\|w\|_p$. We will write $\|w\|$ to mean $\|w\|_2$.

1.1 Problem setup

**Generic dataset.** Suppose the distribution $P$ of the dataset $\mathcal{D}_n$ is supported on $\mathcal{S}_{d-1} \times \mathbb{R}$, where $\mathcal{S}_{d-1} := \{ x \in \mathbb{R}^d \mid \|x\| = 1 \}$ is the the unit-sphere in $\mathbb{R}^d$, and the marginal distribution of each $x_i$ is $\pi_d$, the uniform distribution on $\mathcal{S}_{d-1}$. Given a function $f : \mathcal{S}_{d-1} \to \mathbb{R}$ (e.g. a neural network), its training error is defined by $\varepsilon_n(f) := \left( \frac{1}{n} \right) \sum_{i=1}^n (f(x_i) - y_i)^2$ and its generalization error is $\varepsilon_{\text{test}}(f) := \mathbb{E}_{(x,t) \sim \mathcal{P}}((f(x) - t)^2)$.

**Definition 1.1 (Bayes-optimal error).** Let $\varepsilon_{\text{test}}^* \geq 0$ denote the Bayes-optimal error for the problem, that is $\varepsilon_{\text{test}}^* := \inf_f \varepsilon_{\text{test}}(f)$, where the infimum is taken over all measurable functions $f : \mathcal{S}_{d-1} \to \mathbb{R}$.

For concreteness, consider the linear data-generating process with

$$y_i = w_0^T x_i + z_i, \text{ for } i \in \llbracket n \rrbracket$$

(2)

where $w_0 \in \mathbb{R}^d$ with $\|w_0\| \leq 1$ and $z_1, \ldots, z_n$ is an iid sequence of label-noise from $\mathcal{N}(0, \zeta^2)$, independent of the $x_i$’s. A simple calculation reveals then reveals that the Bayes-optimal error for the prediction problem is $\varepsilon_{\text{test}}^* = \zeta^2$. Thus, the variance $\zeta^2$ of the label noise completely controls the difficulty of the learning problem. To avoid being corner cases, we shall assume the following condition non-degeneracy condition.

**Condition 1.1 (Labels are not a deterministic function of inputs).** All through this manuscript, we will assume that $\varepsilon_{\text{test}}^* \geq \varepsilon_0$, for some absolute constant $\varepsilon_0 \in (0, 1/2]$.

A dataset $\mathcal{D}_n$ verifying the above condition will be referred to as a generic dataset. In Bubeck et al. (2020b), the authors considered the noise-only scenario where the $y_i$’s are uniformly distributed in $\{ \pm 1 \}$ and are completely independent of the $x_i$’s, which in our notations, corresponds to taking $w_0 = 0$ and $\zeta = 1$. For later use, let $X := (x_1, \ldots, x_n) \in \mathbb{R}^{n \times d}$ be the corresponding design matrix and let $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ be the corresponding sequence of training targets / labels.

**Definition 1.2 (Memorization).** Given $\varepsilon \geq 0$, $f$ is said to $\varepsilon$-memorize the dataset $\mathcal{D}_n$ if $\varepsilon_n(f) \leq \varepsilon$; by convention, if $\varepsilon \leq \varepsilon_{\text{test}}^*/2$ (or any other absolute fraction), we simply say $f$ memorizes $\mathcal{D}_n$.

Thus, memorization essentially refers to a model which minimizes the training error way beyond the (Bayes) optimal test-error. It turns out that price of doing this is a degradation in robustness, as measured a sense that will become clear in a bit. Moreover, and as one would expect, this price grows with the sample size.

1.2 Prior works

**Is over-parametrization necessary for robustness?** Recently, it has been conjectured in Bubeck et al. (2020b) that over-parametrization is not just sufficient for robustness, but also necessary. More precisely, suppose the activation function $\sigma$ is 1-Lipschitz and adopting the notation of Bubeck et al. (2020b), let $\mathcal{F}_{d,k}(\sigma) := \{ f_{W,v} \mid W \in \mathbb{R}^{k \times d}, v \in \mathbb{R}^k \}$ be the set of all two-layer neural networks of width $k$, input dimension $d$, and activation function $\sigma$.

**Conjecture 1** (Bubeck et al. (2020b)). It holds with high probability over the dataset $\mathcal{D}_n$ that any $f \in \mathcal{F}_{d,k}(\sigma)$ which memorizes $\mathcal{D}_n$ must satisfy $\text{Lip}(f) \geq \Omega(\sqrt{n/k})$. Thus, in order for $\mathcal{F}_{d,k}(\sigma)$ to contain a neural network which smoothly interpolates $\mathcal{D}_n$, it must be over-parametrized, i.e $k \gtrsim n$ hidden neurons are required.
Recall that the Lipschitz constant of a function \( f : \mathcal{S}_{d-1} \to \mathbb{R} \) is defined by

\[
\text{Lip}(f) := \sup_{x, x' \in \mathcal{S}_{d-1}, \ x' \neq x} \frac{|f(x) - f(x')|}{\|x - x'\|},
\]

and measures the maximum absolute change in the output of \( f \) as a fraction of the change in its input.

**Progress on Conjecture 1** A number of particular cases of Conjecture 1 were proven in Bubeck et al. (2020b). Most notably, the conjecture was proved in the following regimes:

- **Low-dimensional under-complete regime where \( n \geq d \geq k \).** In this regime, a weaker form of the conjecture was proved with \( n \) replaced by \( k \) in the lower-bound. More precisely, it was proved in Theorem 4 of the aforementioned paper that in this case, \( \text{Lip}(f) \geq \tilde{\Omega}(\sqrt{d/k}) \) w.p. \( 1 - e^{-\Omega(d)} \). The condition \( n \geq d \) is crucial for the arguments in that theorem to hold.

- **Lower-bounding a proxy for \( \text{Lip}(f) \).** The authors also proved a weaker form of the conjecture, in which \( \text{Lip}(f) \) of the neural network \( f = f_{W,x} \in F_{k,d}(\sigma) \), is replaced with an upper-bound \( \eta(f) \) defined by \( \eta(f) := \sum_{j=1}^k |v_j| ||w_j||_2 \), which is well-known to be a reasonable measure of complexity for neural networks Bartlett (1998). Bubeck et al. (2018) then proved that
  - With positive probability, any \( f \in F_{k,d}(\sigma) \) which memorizes generic data must verify \( \eta(f) \geq \Omega(\sqrt{n/k}) \). We note that such a result does not say anything useful about Conjecture 1 itself, since \( \eta(f) \) is only an upper-bound for \( \text{Lip}(f) \), the object the conjecture is ultimately about.

- **Converse of the conjecture.** Upper-bounds for the Lipschitz-constant were established in that paper (see Conjecture 2 therein), under different regimes.

- **The case of bounded network parameters.** Very recently, Husain and Balle (2021) studied Conjecture 1 in the restrictive scenario where the parameters of the network are constrained to be bounded.

### 1.3 A new measure of robustness: Sobolev-seminorm

**Limitations of Lipschitz constants to study (non)robustness.** Although a small Lipschitz constant for a model \( f \) immediately implies robustness in the sense that small changes in the input \( x \) can only cause small changes in the output \( f(x) \) (by norm duality), a large Lipschitz constant is uninformative. Indeed, one can imagine an otherwise very smooth \( f \), the norm of whose input-to-output gradient explodes on a subset of the sphere of arbitrarily small measure w.r.t the true distribution of the data. However, such a model could be perfectly robust (for example if the function is constant outside this "bad" set), in any practical sense. Thus unlike Bubeck et al. (2020b) which studies the Lipschitz constants, we propose to instead study the models **Sobolev-seminorm**. To simplify (with abuse of language), we study the average norm of the gradient rather than the maximum (i.e the worst-case).

**Sobolev-seminorm as a measure of robustness.** Let \( f : \mathbb{R}^d \to \mathbb{R} \) be a function which is continuously-differentiable in the usual sense, almost-everywhere (a.e). The spherical gradient of \( f \) is the map \( \nabla_{\mathcal{S}_{d-1}} f : \mathcal{S}_{d-1} \to T\mathcal{S}_{d-1} \) defined for each \( x \in \mathcal{S}_{d-1} \) by

\[
\nabla_{\mathcal{S}_{d-1}} f(x) := P_{\tau_x S_{d-1}}(\nabla f(x)) = P_{x^+}(\nabla f(x)) = (I_d - xx^\top)\nabla f(x),
\]

where \( \nabla : \mathbb{R}^d \to \mathbb{R}^d \) is the usual euclidean gradient of \( f \). Here \( T_x \mathcal{S}_{d-1} \) is the tangent space of \( \mathcal{S}_{d-1} \) at the point \( x \); \( P_{x^+} \in \mathbb{R}^{d \times d} \) is the projector onto the orthogonal complement of \( T_x \mathcal{S}_{d-1} \); and \( T\mathcal{S}_{d-1} := \{ (x, z) \mid x \in \mathcal{S}_{d-1}, \ z \in T_x \mathcal{S}_{d-1} \} \) is the tangent bundle.

**Definition 1.3.** Define the Sobolev-seminorm of \( f \) w.r.t the uniform \( \tau_d \) on \( \mathcal{S}_{d-1} \), denoted \( \mathcal{G}(f) \), by

\[
\mathcal{G}(f) := \|\nabla_{\mathcal{S}_{d-1}} f\|_{L^2(\tau_d)} = \left( \int_{\mathcal{S}_{d-1}} \|\nabla_{\mathcal{S}_{d-1}} f(x)\|^2 d\tau_d(x) \right)^{1/2}.
\]

Thus, \( \mathcal{G}(f)^2 \) is the average squared \( L_2 \)-norm of the input-to-output gradient of \( f \). From the above definition, is clear that
Lip(f) ≥ \Theta(f). Thus, lower-bounds on \Theta(f) immediately translate to lower-bounds on Lip(f)
and upper-bounds on Lip(f) translate to upper-bounds on \Theta(f).

Remark 1.1 (The case of non-differentiable functions). Our restriction to differentiable \( f : \mathbb{R}^d \to \mathbb{R} \) is
only artificial. In case \( f \) is non-differentiable, we may replace pointwise gradient-norm \( \| \nabla_{S_{d-1}} f(x) \|_2 \)
in the above definitions with the strong slope \( \text{De Giorgi et al. (1980); Corvellec and Motreanu (2007);}
Aze and Corvellec (2017) } |\nabla f|(x), defined by \( |\nabla f|(x) := \limsup_{x \to x'} \frac{|f(x) - f(x')|_+}{\| x - x' \|}. \) In
particular, if \( f \) is differentiable at \( x \), then \( |\nabla f|(x) = \| \nabla_{S_{d-1}} f(x) \| \).

1.4 Summary of main contributions

We consider two-layer neural networks (1) in the random features (RF) Rahimi and Recht (2008, 2009) and neural tangent kernel (NTK) Jacot et al. (2018) regimes, in both the finite-width and
infinite-width settings and establish a number of theorems which highlight a fundamental tradeo;
between memorization and robustness.

Tradeoffs between memorization and robustness. We establish explicit tradeoffs between mem-
orization and robustness in the following form, valid for a large class of models including but not
limited to models obtained via ridge(less) regression (the so-called representer subspace\(^1\)).

\[
\Theta(f) \geq (\hat{\varepsilon}_{\text{test}} - \hat{\varepsilon}_{\text{mem}}(f)) \cdot \bar{\Omega}(\sqrt{n'})
\]

(6)

where \( n' = n \) for infinite-width RF / NTK and finite-width RF, and \( n' = n/k \) for finite-width NTK.
Numerous experiments confirm our theoretical results. Moreover, the bounds (6) are tight: they are attained by the min-norm interpolator with certain choices of activation function \( \sigma \) (including the
ReLU), and setting scaling regimes for \( n, d, \) and \( k \).

To give a sense of these results, let us restrict to models that memorize the dataset \( D_n \). Recall from
Definition 1.2 that a model \( f \) memorizes \( D_n \) if its training error \( \hat{\varepsilon}_{\text{mem}}(f) \) is a constant short of the
Bayes-optimal test error \( \varepsilon^*_{\text{test}} \). For such models, we may breakdown (6) as follows.

- **Infinite-width RF and NTK.** In the high-dimensional setting \( d > n \), we consider infinite-width
neural networks in the RF or NTK regimes and prove that for a variety of activation functions
including the ReLU, and prove in Theorem 3.2 that with probability tending to 1, any model in the induced "representer subspace" which memorizes dataset \( D_n \) must verify
\( \Theta(f) \geq \bar{\Omega}(\sqrt{n}) \) (ignoring log-factors). Moreover, we establish in Theorem 3.3 the tightness of
this lower-bound: it is attained by the min-norm / least squares model with ReLU activation
function.

- **Finite-width RF.** Consider finite-width neural networks (1) in the RF, where the parameter
vector \( w_j \) of each neuron is sampled iid from the uniform-distribution on the unit-sphere
\( S_{d-1} \), and only the output weights \( v \in \mathbb{R}^k \) learned. For \( d > k \), we prove in Theorem 5.1 that
for a large class of activation functions including the ReLU, that with probability tending to
1, any model in the induced "representer space" which memorizes dataset \( D_n \) must verify
\( \Theta(f) \geq \bar{\Omega}(\sqrt{n}) \) (ignoring log-factors). Moreover, we show in Theorem 5.2: it is attained by
the min-norm / least squares interpolator with ReLU activation function.

- **Given hidden weights.** For any choice of hidden weights matrix \( W = (w_1, \ldots, w_k) \in \mathbb{R}^{k \times d} \),
we prove in Theorem 4.2 that w.p tending to 1: any two-layer model in the induced "representer
subspace" which memorizes \( D_n \) satisfies \( \Theta(f) \geq \bar{\Omega}(\sqrt{1/\alpha_\sigma(W)\sqrt{nd}/k}) \), where \( \alpha_\sigma(W) \)
is a kind of condition number of \( W \) w.r.t the activation function \( \sigma \). For example, for the
identity activation function, we have \( \alpha_{\text{id}}(W) = \text{cond}(W)^2 \), where \( \text{cond}(W) \) is the usual / linear-algebraic condition number of \( W \).

\(^1\)The concept of "representer subspace" is formally defined in (7). As we shall see, different regimes (finite /
infinite-width RF; finite / infinite-width NTK, etc.) of neural networks will give rise to different kernels and
different representer subspaces.
Finite-width NTK. Consider finite-width two-layer neural network (1) in the NTK regime, where the parameter vector \( w_j \) of each neuron is as previously. In the setting where \( d \approx k \), we prove in Theorem 6.1 that with probability tending to 1, any model in the induced "representer subspace" which memorizes the dataset \( D_n \) must verify \( \mathbb{E}(f) \geq \Omega(\sqrt{n/k}) \).

These results, stated more formally in the following sections, are empirically confirmed by experiments in section 7.

**Multiple-descent in robustness.** We empirically observe multiple-descent Belkin et al. (2019); Loog et al. (2020); d’Ascoli et al. (2020); Mei and Montanari (2019); Adlam and Pennington (2020) in robustness of two-layer NNs in the above linearized regimes. Refer to Figure 2. To the best of our knowledge, this is the first time such a phenomenon has been observed. We speculate that the multiple-descent phenomenon occurs for a variety of statistical functionals (here robustness) of NNs, other than their generalization error, which is currently under intensive research in the theoretical machine learning community.

## 2 Preliminaries

### 2.1 Warmup: ordinary linear models

Consider the linear data generating process in Example 1.1, with label noise variance \( \zeta^2 > 0 \). Note that the Bayes-optimal error for the problem is \( \epsilon_{test} = \zeta^2 \). We start with a result on linear models, which already illustrates that the price of memorization is robustness.

**Theorem 2.1** (Law of robustness for ordinary linear models). For sufficiently large \( n \) and \( d \), the following holds \( w.p. 1 - n^{-\Omega(1)} \) over \( D_n \): every linear model \( g_w \) which \( \varepsilon \)-memorizes \( D_n \) verifies \( \text{Lip}(g_w) \geq \mathbb{E}(g_w) \geq (\epsilon_{test}^* - \varepsilon)\sqrt{n} \). In particular, for the high-dimensional regime \( d > n \), the min-norm / least squares interpolator \( \hat{g} = g_w \), defined by setting \( \hat{w} = X^\top (XX^\top)^{-1} y \in \mathbb{R}^d \), satisfies 

\[
\Omega(\epsilon_{test}^* \sqrt{n}) \leq \mathcal{E}(\hat{g}) \leq \mathcal{O}(\epsilon_{test}^* \sqrt{n}) \quad w.p. 1 - n^{-\Omega(1)}.
\]

**Remark 2.1.** We note that the second part of the above result was established in Bubeck et al. (2020b), at least for noise-only data where \( w_0 = 0 \).

The proof of Theorem 2.1 (provided in the appendix) is based on standard Rademacher complexity-based generalization bounds for squared loss. The theorem highlights a clear tradeoff between the memorization error \( \epsilon_{test}^* - \mathbb{E}_n(g_w) \) of a linear model \( g_w : x \mapsto x^\top w \), and its robustness as measured by its Sobolev-seminorm \( \mathbb{E}(g_w) \). In the sequel, we will obtain results of this sort, for linearized neural nets like random features and neural tangent models, in both finite and infinite-width regimes.

### 2.2 Kernelization

The rest of the manuscript will be concerned with the complicated case of neural networks in various linearized regimes. We will employ the language and toolbox of kernel methods in order to give a unified treatment.

**Reproducing Kernel Hilbert Spaces (RKHS).** Consider a continuous positive-definite definite kernel \( K : \mathcal{S}_{d-1} \times \mathcal{S}_{d-1} \rightarrow \mathbb{R} \), where positive-definiteness means that \( \sum_{i=1}^N \sum_{j=1}^N b_i b_j K(x'_i, x'_j) \geq 0 \) for every finite sequence \( x'_1, \ldots, x'_N \in \mathcal{S}_{d-1} \) and every \( b_1, \ldots, b_N \in \mathbb{R} \). Let \( \mathcal{H}_K \) be the Reproducing Kernel Hilbert Space (RKHS) induced by \( K \). Note that \( \mathcal{H}_K \subseteq L^2(\mathcal{U}) \) since \( K \) is a Mercer kernel.\(^1\)

Let \( K(X, X) \in \mathbb{R}^{n \times n} \) be the kernel matrix with entries \( K(x_i, x_j) \), where \( X = (x_1, \ldots, x_n) \in \mathbb{R}^{n \times d} \) is the design matrix associated for the generic dataset \( D_n := \{(x_1, y_1), \ldots, (x_n, y_n)\} \) in (2).

The "representer subspace". We will denote by \( \text{span}_K(X) \subseteq \mathcal{H}_K \) the subspace of functions in \( \mathcal{H}_K \), formed by linear combinations of the \( n \) functions \( x \mapsto K(x_i, x) \), that is

\[
\text{span}_K(X) := \text{span}(\{K(x_1, \cdot), \ldots, K(x_n, \cdot)\}) = \{f_c := \sum_{i=1}^n c_i K(x_i, \cdot) \mid c \in \mathbb{R}^n\}.
\]

\(^1\)By continuity of \( K \) and compactness of the unit-sphere \( \mathcal{S}_{d-1} \)
As we shall see, different regimes (finite / infinite-width RF, finite / infinite-width NTK, etc.) of neural networks will give rise to different kernels and different representer subspaces.

It is a classical result that the RKHS norm of any \( f_c \in \text{span}_K(X) \) is given by the simple formula

\[
\| f_c \|_{H_K} = \sqrt{c^\top K(X, X)c}.
\]

Also, from the so-called Generalized Representer Theorem (GRT) Schölkopf et al. (2001) (also see appendix for a statement), \( \text{span}_K(X) \) contains all models which can be constructed by doing certain kinds of penalized kernel regression in \( H_K \).

**Proposition 2.1** (Schölkopf et al. (2001)) Generalized representer theorem for the sphere. If \( g : \mathbb{R}^+ \to \mathbb{R} \) is a strictly increasing function and \( \ell_n : (S_{d-1} \times \mathbb{R})^n \to \mathbb{R} \) is an arbitrary “cost function”, then every minimizer of the functional

\[
H_K \to \mathbb{R}, \ f \mapsto \ell_n((x_1, y_1, f(x_1)), \ldots, (x_n, y_n, f(x_n))) + R(\| f \|_{H_K})
\]

is an element of \( \text{span}_K(X) \).

Some notable functions that lie in \( \text{span}_K(X) \) include

- The least-squares model \( \hat{f}_n \), defined by \( \hat{f}_n(x) = \hat{c}_n^\top K(X, x) \) with \( \hat{c}_n := K(X, X)^{-1}y \) (provided the kernel gram matrix \( K(X, X) \) is invertible). This precisely the element of \( \text{span}_K(X) \) with minimal RKHS norm, and corresponds to taking \( \ell_n((x_1, y_1, f(x_1)), \ldots, (x_n, y_n, f(x_n))) := \sum_{i=1}^n (f(x_i) - y_i)^2 \) and \( R = 0 \).
- Any function in the so-called *version space* \( \{ f \in H_K \mid f(x_i) = y_i \forall i \in [n] \} \). This corresponds to the same choice of \( \ell_n \) and \( g \) as above. Note that the least-squares estimator above (when it exists) is itself an element of the version space.
- Any ridge interpolator \( \hat{f}_{n, \lambda} \), namely any function of the form \( \hat{f}_{n, \lambda}(x) = \hat{c}_{n, \lambda}^\top K(X, x) \) for all \( x \in S_{d-1} \), where \( \hat{c}_{n, \lambda} := (K(X, X) + n\lambda I_n)^{-1}y \in \mathbb{R}^n \), with \( \lambda \geq 0 \). This corresponds to taking the cost function \( \ell_n \) as in the previous example, and \( R(\| f \|_{H_K}) = \lambda\| f \|^2_{H_K} \).
- etc.

We will conveniently exploit this universal property of \( \text{span}_K(X) \) to give a unified treatment for robustness in dot-product kernels, finite / infinite-width NNs in RF and NTK regimes, etc. (see Table 1), by reducing to questions about the extreme eigenvalues of certain random matrices (including those of the kernel gram matrix \( K(X, X) \)).

| Model class          | Equivalent kernel on unit-sphere |
|----------------------|----------------------------------|
| General dot-product RKHS | \( K(x, x') := \phi(x^\top x'), t := x^\top x', \phi \in C^0([-1, 1] \to \mathbb{R}) \) |
| Laplace RKHS         | \( K_{\text{Lap}}(x, x') = \phi_{\text{Lap}}(t) := e^{-\sqrt{2\pi/s}/t}, s > 0 \) |
| Gaussian RKHS        | \( K_{\text{Gauss}}(x, x') = \phi_{\text{Gauss}}(t) := e^{-(2-2t)/s^2} \) |
| Infinite-width RF    | \( K_{\text{RF}}(x, x') = \mathbb{E}_w[\sigma(x^\top w)\sigma(w^\top x')] \), with \( w \sim \tau_d \) |
| Infinite-width NTK   | \( K_{\text{NTK}}(x, x') = (x^\top x')\mathbb{E}_w[\sigma'(x^\top w)\sigma'(w^\top x')] \) |
| General feature-based | \( K_\Phi(x, x') = \langle \Phi(x), \Phi(x') \rangle, \) with \( \Phi : S_{d-1} \to H_0 \) |
| Finite-width RF      | \( K_{\text{RF}}(x, x') = \Phi_{\text{RF}}(x)^\top \Phi_{\text{RF}}(x') \) |
| Finite-width NTK     | \( K_{\text{NTK}}(x, x') = \Phi_{\text{NTK}}(x)^\top \Phi_{\text{NTK}}(x') \) |

Table 1: Kernel reformulation of different model classes. This allows us to give a unified treatment of otherwise very disparate situations, by considering the "representer" subspace of induced by the appropriate kernel function \( K \).
3 Law of robustness for general dot-product kernels

Consider the case of a dot-product kernel $K : \mathcal{S}_{d-1} \times \mathcal{S}_{d-1} \to \mathbb{R}$ given by $K(x, x') \equiv \phi(x^\top x')$ for some continuous $\phi : [-1, 1] \to \mathbb{R}$. Further, we impose the following regularity condition

**Condition 3.1.** $\phi$ is thrice continuously-differentiable at 0 and verifies $\phi'(0) \neq 0$.

For example, this is the case for exponential-type kernels like the Gaussian kernel on the unit-sphere $\mathcal{S}_{d-1}$, which is known to be the infinite-width version of the Fourier random features Rahimi and Recht (2008, 2009); infinite-width RF and NTK kernels corresponding to the ReLU activation function Bietti and Mairal (2019b); etc.

Let $\varepsilon_{\text{test}}$ be the Bayes-optimal error for the underlying squared-loss regression problem (see Definition 1.1) and let $\varepsilon$ be any error threshold in the interval $[0, \varepsilon_{\text{test}}]$. The following is our first main result.

**Theorem 3.1 (Law of robustness for dot-product kernels).** Under Condition 3.1, in the limit $n, d \to \infty$ such that $n/d \leq \gamma_1 < 1$, it holds w.p. tending to 1 that: every $f \in \text{span}_K(X)$ which $\varepsilon$-memorizes $D_n$ satisfies $\mathcal{S}(f) \geq \Omega((\varepsilon_{\text{test}} - \varepsilon)\sqrt{n})$. In particular, if the kernel gram matrix $K(X, X)$ is nonsingular, then the min-norm interpolator $f_n(x) := K(x, x)^\top K(X, X)^{-1}y$ satisfies $\mathcal{S}(f_n) \geq \Omega(\varepsilon_{\text{test}}\sqrt{n})$.

Like all our other results, the proof is deferred to the appendix. It uses tools from probability theory like the spherical Poincaré inequality Ledoux (1999); Gozlan et al. (2015); Villani (2008) and random matrix theory (RMT) Vershynin (2012); El Karoui (2010).

3.1 Law of robustness for infinite-width random features and neural tangent kernel

Now, consider an infinite-width (i.e. having $k = \infty$ hidden neurons) neural network in the RF or NTK regime. The learning problem is reduced to RKHS regression with kernels $K_{\text{RF/NTK}}^\infty : (\mathcal{S}_{d-1})^2 \to \mathbb{R}$,

$$K_{\text{NTK}}^\infty(x, x') := (x^\top x')\mathbb{E}_w[\sigma'(x^\top w)\sigma(w^\top x')], \quad (\text{1st layer-only kernel}),$$

$$K_{\text{RF}}^\infty(x, x') := \mathbb{E}_w[\sigma'(x^\top w)\sigma(w^\top x')], \quad (\text{2nd layer-only kernel})$$

(9)

for $w \sim \tau_d$ (see Bietti and Mairal (2019b), e.g). Different choices for the activation function $\sigma$ give rise to different kernels. For example, in the case of the ReLU activation, the corresponding kernels are given by

$$K_{\text{RF/NTK}}^\infty(x, x') = \begin{cases} (x^\top x')\phi_0(x^\top x'), & \text{if NTK (i.e 1st layer-only kernel)}, \\ \phi_1(x^\top x'), & \text{if RF (i.e 1st layer-only kernel)}, \end{cases}$$

(10)

where $\phi_0(t) = \pi^{-1}\arccos(-t)$ and $\phi_1(t) = \pi^{-1}(t \arccos(-t) + \sqrt{1 - t^2})$ are the arc-cosine kernels of order 0 and 1 respectively Bietti and Mairal (2019b). In this particular case, both $K_{\text{RF}}^\infty$ and $K_{\text{NTK}}^\infty$ are dot-product kernels.

**Condition 3.2.** $K_{\text{RF/NTK}}^\infty$ is a dot-product kernel by means of a continuous function $\phi_{\text{RF/NTK}}^\infty : [-1, 1] \to \mathbb{R}$ which is thrice continuously-differentiable at 0 w.p. $(\phi_{\text{RF/NTK}}^\infty)'(0) > 0$.

For example, the absolute-value activation function fails to satisfy this condition. On the other hand, the ReLU, tanh, and the gaussian error-function (erf) satisfy the condition. Table 1 of Louart et al. (2018) provides explicit formula for $\Phi_{\text{RF/NTK}}^\infty$ for a variety of activation functions, including: ReLU, absolute-value, sign, sin, cos, gaussian erf, etc.

When the kernel gram matrix $K_{\text{RF/NTK}}^\infty(X, X) \in \mathbb{R}^{n \times n}$ is invertible (which happens for example in the high-dimensional regime $d > n$ with ReLU activation function), let $\tilde{f}_{\text{RF/NTK}}^\infty$ be the min-norm interpolator defined by (see Arora et al. (2019); Liang and Rakhlin (2020); Hastie et al. (2019))

$$\tilde{f}_{\text{RF/NTK}}^\infty(x) := K_{\text{RF/NTK}}^\infty(X, x)^\top K_{\text{RF/NTK}}^\infty(X, X)^{-1}y, \ x \in \mathcal{S}_{d-1}.$$  

(11)

The following theorem is an important corollary to Theorem 3.1, and establishes a quantitative tradeoff between memorization and robustness for two-layer neural networks (1) in the infinite-width RF and NTK regimes. The proof of the theorem (given in the appendix) makes use of the spherical Poincaré inequality Ledoux (1999); Villani (2008, 2003); Gozlan et al. (2015), together with the classical generalization theory for kernel methods Boucheron, Stéphane et al. (2005).
Theorem 3.2 (Law of robustness for infinite-width RF / NTK). Assume Condition 3.2. In the limit
\( n, d \to \infty \) such that \( n/d \to \gamma_1 \) for some \( \gamma_1 \in (0, 1) \), the following holds w.p. \( 1 - n^{-\Omega(1)} \) over
\( \mathcal{D}_n \): every \( f \in \text{span}_{\text{RF/NTK}}(\mathcal{X}) \) which \( \varepsilon \)-memorizes \( \mathcal{D}_n \) satisfies \( \mathcal{G}(f) \geq \tilde{\Omega}((\varepsilon_{\text{test}} - \varepsilon)\sqrt{n}) \). In
particular, the min-norm interpolator \( \hat{f}_{\text{RF/NTK}}^\infty \) satisfies \( \mathcal{G}(\hat{f}_{\text{RF/NTK}}^\infty) \geq \tilde{\Omega}(\varepsilon_{\text{test}}^\star \sqrt{n}) \) w.p. \( 1 - o(1) \).

In light of the above theorem, in the high-dimensional \( d > n \) regime, it might thus be dangerous to do
min-norm / unpenalized interpolation with infinite-width NNs as advocated in Liang and Rakhlin (2020).
Regularization should be used to select a good tradeoff between fit and robustness. This is
also empirically confirmed in section 7.

Proof of Theorem 3.2. Follows directly from Theorem 3.1 with the kernel \( \mathcal{K} \) taken to be the dot-
product kernel \( \mathcal{K}_{\text{RF/NTK}}^\infty : (x, x') \mapsto \phi_{\text{RF/NTK}}^\infty(x^\top x') \).

A matching upper-bound: the min-norm interpolator. Still in the high-dimensional setting
where both \( n \) and \( d \) are large with \( d \geq n \), we now establish the tightness of the lower-bounds in
Theorem 3.2 for RF approximation. To this end, we will prove that in the case of the ReLU activation
function, the reverse bound is satisfied by the min-norm estimator \( \hat{f}_{\text{RF}}^\infty \). First, we must ensure that
the kernel gram matrix \( \mathcal{K}_{\text{RF}}^\infty(X, X) \) is invertible w.h.p. when the activation function \( \sigma \) is the ReLU.

Lemma 3.1 (Invertibility of RF kernel gram matrix). For the ReLU activation function and for
sufficiently large \( n \) and \( d \) with \( n/d \leq \gamma_1 < 1 \), it holds w.p. \( 1 - d^{-1+o(1)} \) that the eigenvalues
\( \lambda_1 \geq \ldots \geq \lambda_n \) of the RF kernel gram matrix \( \mathcal{K}_{\text{RF}}^\infty(X, X) \) satisfy
\( c \leq \lambda_n \leq \ldots \leq \lambda_1 \leq C \), for constants \( c, C > 0 \) which only depend on \( \gamma_1 \). In particular, \( \mathcal{K}_{\text{RF}}^\infty(X, X) \) is invertible w.p. \( 1 - d^{-1+o(1)} \).

Equipped with this lemma, following result establishes tightness of the lower-bound in Theorem 3.2.

Theorem 3.3 (Tightness of lower-bound in Theorem 3.2). For sufficiently large \( n \) and \( d \) such that
\( n/d \leq \gamma_1 < 1 \), it holds w.p. \( 1 - n^{-\Omega(1)} \) over \( \mathcal{D}_n \) that the RF min-norm interpolator \( \hat{f}_{\text{RF}}^\infty \) defined in
(11) with ReLU activation function verifies \( \mathcal{G}(\hat{f}_{\text{RF}}^\infty) \leq \text{Lip}(\hat{f}_{\text{RF}}) \leq O(\varepsilon_{\text{test}}^\star \sqrt{n}) \).

The proof of the theorem is given in the appendix.

4 Laws of robustness for feature-based kernels

Let \( \mathcal{H}_0 \) is a separable Hilbert space. For concreteness, take \( \mathcal{H}_0 = \mathbb{R}^m \). A continuous embedding
mapping \( \Phi : \mathcal{S}_{d-1} \to \mathbb{R}^m \) induces a kernel \( \mathcal{K}_\Phi : \mathcal{S}_{d-1} \times \mathcal{S}_{d-1} \to \mathbb{R} \) defined by
\[
\mathcal{K}_\Phi(x, x') := \langle \Phi(x), \Phi(x') \rangle_{\mathcal{H}_0},
\] (12)
which in turn induces an RKHS \( \mathcal{H}_{\mathcal{K}_\Phi} \). The embedding map \( \Phi \) may also be referred to as a
dictionary, with atoms \( \Phi_j : \mathcal{S}_{d-1} \to \mathbb{R} \) given by \( \Phi_j(x) := \langle \Phi(x) \rangle_j \).

Definition 4.1 (Condition number of embedding \( \Phi \)). The condition number of the embedding map
\( \Phi : \mathcal{S}_{d-1} \to \mathbb{R}^m \), denoted \( \alpha_\Phi \), is defined by
\[
\alpha_\Phi = \frac{\|\Phi\|^2_{L^2(\mathcal{H}_0)}}{\lambda_{\min}(C_{\Phi})}.
\] (13)

We have the following theorem, a law of robustness for general feature-based models.

Theorem 4.1. The following holds w.p. \( 1 - n^{-\Omega(1)} \) over the generic dataset \( \mathcal{D}_n \); every \( f \in \text{span}_{\mathcal{K}_\Phi}(\mathcal{X}) \)
which \( \varepsilon \)-memorizes the generic dataset \( \mathcal{D}_n \) satisfies \( \mathcal{G}(f) \geq \tilde{\Omega}((\varepsilon_{\text{test}} - \varepsilon)\sqrt{n}) \). In
particular, if the gram matrix \( \mathcal{K}_\Phi(X, X) := \Phi(X)\Phi(X)^\top \) is invertible (which necessarily
implies \( n \leq d \)), then w.p. \( 1 - n^{-\Omega(1)} \) over \( \mathcal{D}_n \) it holds that the least-squares model \( \hat{f}_\Phi(x) := \mathcal{K}_\Phi(x, X)^\top \mathcal{K}_\Phi(X, X)^{-1} y \) satisfies
\( \mathcal{G}(\hat{f}_\Phi) \geq \tilde{\Omega}((\varepsilon_{\text{test}} - \varepsilon)\sqrt{n}) \).
4.1 Ordinary linear models (again)

A remarkable property of the bound in 4.1 is that its is completely free of the design matrix $X$. To illustrate the potential benefit of this, reconsider the ordinary linear model from section B.4. This is equivalent to taking $m = d$ and $\tilde{\Phi}(x) = \Phi_{id}(x) := x$ for all $x \sim S_{d-1}$. One easily computes $\mathbb{E}_{x \sim \mathcal{N}(0, I_d)}[\|\Phi_{id}(x)\|^2] = \mathbb{E}[\|x\|^2] = 1$ and $C_{\Phi_{id}} = \text{Cov}_{x \sim \mathcal{N}(0, I_d)}(\sqrt{d}x) = I_d$, so that $\lambda_{\text{max}}(C_{\Phi_{id}}) = 1$. We deduce that

$$\alpha_{\Phi_{id}} = 1 \leq \frac{\lambda_{\text{max}}(XX^\top)}{\lambda_{\text{min}}(XX^\top)} = \text{cond}(X)^2,$$

where $\text{cond}(X)$ is the condition number of the design matrix $X$. We thus have the following improved version of Corollary B.1.

**Corollary 4.1** (Law of robustness for linear model (improved bound)). For every $\varepsilon \in (0, \varepsilon_{\text{test}}^*)$, the following holds with probability $1 - n^{-\Omega(1)}$ over the generic dataset $\mathcal{D}_n$: every $f \in \text{span}_{\Phi_{id}}(X)$ which $\varepsilon$-memorizes $\mathcal{D}_n$ satisfies $\mathcal{G}(f) \geq \Omega((\varepsilon_{\text{test}}^* - \varepsilon)\sqrt{n})$. In particular, if $n/d \leq \gamma_1 < 1$, then for the min-norm interpolator $f_n(x) := x^\top X^\top (XX^\top)^{-1}y$, it holds w.p. $1 - n^{-\Omega(1)}$ over $\mathcal{D}_n$ that $\mathcal{G}(f_n) \geq \Omega(\varepsilon_{\text{test}}^*\sqrt{n})$.

Just as in the case of Theorem 2.1, the above result is tight because a $\sqrt{n}$ upper-bound for the Lipschitz constant of the min-norm interpolator was obtained in Bubeck et al. (2020b).

4.2 Finite-width networks with prescribed hidden weights

Let $f_{W,v} \in \mathcal{F}_{d,k}(\sigma)$ be two-layer neural network on the unit sphere $S_{d-1}$, with $k$ hidden neurons, activation function $\sigma : \mathbb{R} \to \mathbb{R}$, as defined in (1). Fix the hidden weights matrix $W \in \mathbb{R}^{k \times d}$ (for example, consider a random matrix or a pretrained matrix), and consider the subset $\mathcal{F}_W(\sigma) \subseteq \mathcal{F}_{k,d}(\sigma)$ of two-layer neural networks (1) with hidden weights matrix fixed at $W$. The embedding function

$\Phi_W : S_{d-1} \to \mathbb{R}^k$, $\Phi_W(x) = \frac{1}{\sqrt{k}}\sigma(Wx) := \frac{1}{\sqrt{k}}(\sigma(x^\top w_1), \ldots, \sigma(x^\top w_k))$

induces a kernel $K_W(x, x') = \Phi_W(x)^\top \Phi_W(x') = \frac{1}{k} \sum_{j=1}^k \sigma(x^\top w_j)\sigma(x^\top w_j)^\top$. This is an instance of (12) with $\mathcal{H}_0 = \mathbb{R}^k$ and $\Phi$ given by (15). Note that both $\Phi_W$ and $K_W$ depend on frozen value of $W$. Later in this section, we will consider the scenario where the hidden weights matrix $W$ is random. Let $C_{\sigma}(W) \in \mathbb{R}^{k \times k}$ be the covariance matrix of $\sqrt{d}\sigma(Wx) \in \mathbb{R}^k$ for $x \sim \mathcal{N}(0, I_d)$. The following result, one of our main contributions, can be used used to evaluate the robustness of NNs with given hidden weights, e.g. trained neural networks.

**Theorem 4.2** (Law of robustness with given hidden weights). The following holds with probability $1 - n^{-\Omega(1)}$ over the generic dataset $\mathcal{D}_n$: for every $W \in \mathbb{R}^{k \times d}$, any $f \in \text{span}_{\Phi_{id}}(X)$ which $\varepsilon$-memorizes $\mathcal{D}_n$ satisfies $\mathcal{G}(f) \geq (\varepsilon_{\text{test}}^* - \varepsilon)\sqrt{\frac{\lambda_{\min}(C_{\sigma}(W))}{\|W\|_F} \Omega(\sqrt{nd}) \geq (\varepsilon_{\text{test}}^* - \varepsilon)\sqrt{\frac{\lambda_{\min}(C_{\sigma}(W))}{\|W\|_F}} \Omega(\sqrt{nd})}$.

Note that the quantities $\lambda_{\min}(C_{\sigma}(W))$, $\|W\|_F$, and $\|W\|_F$, appearing in the theorem can be computed on unlabeled data. The ratio $\kappa_\sigma(W) := \sqrt{\lambda_{\min}(C_{\sigma}(W))/\|W\|_F}$ can is a kind of (inverse) condition number for $W$. Indeed, in the special case where activation function $\sigma$ is the identity, one easily computes $\kappa_{\sigma}(W) = \sqrt{\lambda_{\min}(WW^\top)/\lambda_{\max}(WW^\top)} = \text{cond}(W)^{-1}$, where $\text{cond}(W)$ is the usual / linear-algebraic condition number of $W$. In the case of random features models where $W$ is frozen at its random value at initialization, random matrix theory (RMT) can used to bound $\kappa_\sigma(W)$ away from zero.

**Proof of Theorem 4.2.** The result is Corollary Theorem 4.1. We need to compute the following quantities

- $\|\sigma \circ W\|_{L_2(\mathcal{N})} = \mathbb{E}_{x \sim \mathcal{N}}[\|\sigma(Wx)\|^2]$,
- $\lambda_{\min}(C_{\sigma}(W))$, where $C_{\sigma}(W) \in \mathbb{R}^{k \times k}$ is the covariance matrix of the random vector $\sqrt{d}\Phi_W(x) = \sqrt{d}\sigma(Wx)$ for $x \sim \mathcal{N}(0, I_d)$. 

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Since the activation function $\sigma$ is 1-Lipschitz, one easily upper-bound the first quantity as
\[
\|\sigma \circ W\|_{L^2(\tau_d)}^2 = \mathbb{E}_{x \sim \tau_d}[\|\sigma(W x)\|^2] \leq \mathbb{E}_{x \sim \tau_d}[\|W x\|^2] = \text{Tr}(W W^T \text{Cov}_{x \sim \tau_d}(x))
\]
\[
= \frac{\|W\|_F^2}{d} \leq \frac{k \|W\|_{op}^2}{d}.
\]

The result then follows directly from Theorem 4.1. \qed

5 Law of robustness for finite-width / empirical RF

Consider the finite-width RF regime Rahimi and Recht (2008, 2009); Bach (2017); Bietti and Mairal (2019b); Ji et al. (2020), where the rows of the hidden weights matrix $W = (w_1, \ldots, w_k) \in \mathbb{R}^{k \times d}$ are chosen iid according to $\tau_d$ (the uniform distribution on the unit-sphere $S_{d-1}$) or equivalently, from $\mathcal{N}(0, (1/d) I_d)$, and only the output weights vector $v \in \mathbb{R}^k$ is optimized. With this choice of $W$, let $\Phi_{RF} : S_{d-1} \to \mathbb{R}^k$ be the feature map resulting from (15) with corresponding kernel $K_{RF}(x,x') = \Phi_{RF}(x)^T \Phi_{RF}(x')$, which can be seen as an empirical version of the kernel given in (9). However, $K_{RF}$ is not a dot-product kernel; this leads to technical difficulties.

**Definition 5.1** (Curvature constants of activation function). For $z \sim \mathcal{N}(0,1)$, define scalars

\[
\beta_0(\sigma) := \mathbb{E}_z[\sigma(z)^2] \geq 0, \quad \beta_1(\sigma) := \mathbb{E}_z[z \sigma(z)^2] \geq 0, \quad \beta_\star(\sigma) := \mathbb{E}_z[\sigma(z)^2] - \beta_0(\sigma) - \beta_1(\sigma) \in \mathbb{R}. \tag{16}
\]

These quantities, which measure the degree of nonlinearity and curvature of the activation function $\sigma$, appear naturally in our analysis of the eigenvalues of the random matrix $C_\sigma(W) := \text{Cov}_{x \sim \tau_d}(\sqrt{d} \sigma(W x))$, an essential step in our analysis of the Sobolev-semi-norm of functions in representer subspace $\text{span}_{K_{RF}}(X)$. They have also appeared in the analysis of the generalization error of neural networks in RF and NTK regimes Mei and Montanari (2019); Montanari and Zhong (2020); Gerace et al. (2020) and also in the analysis of the multiple-descent phenomenon Adlam and Pennington (2020); d’Ascoli et al. (2020).

Observe that one may write $C_\sigma(W) = \bar{C}_\sigma(W) - \mu_\sigma(W) \mu_\sigma(W)^T$, where $\mu_\sigma(W) := \mathbb{E}_{x \sim \tau_d}[\sigma(W x)]$ and the entries of $\bar{C}_\sigma(W) \in \mathbb{R}^{k \times k}$ are given by

\[
\bar{C}_\sigma(W)_{j,\ell} := \mathbb{E}_{x \sim \tau_d}[\sqrt{d} \sigma(x^T w_j) \sqrt{d} \sigma(x^T w_\ell)]. \tag{17}
\]

We will need the following technical condition.

**Condition 5.1.** (1) There exists a continuous function $\phi_\sigma : [-1,1] \to \mathbb{R}$ which is thrice continuously-differentiable at 0 such that $\bar{C}_\sigma(W)_{j,\ell} = \phi_\sigma(w_j^T w_\ell)$ for all $j, \ell \in [k]$ and at least one of the following two-conditions holds:

\[\begin{align*}
(2A) & \quad \beta_\star(\sigma) > 0. \\
(2B) & \quad \beta_\star(\sigma) \geq 0, \beta_1(\sigma) > 0 \text{ and } k/d \leq \gamma_1 \text{ for some absolute constant } \gamma_1 \in [0,1).
\end{align*}\]

The above condition is satisfied when the underlying activation function $\sigma$ is the ReLU or absolute-value, or the gaussian error-function. Part (1) implies $C_\sigma(W) = \bar{C}_\sigma(W) - \phi_\sigma(0) 1_k 1_k^T$ is a dot-product kernel matrix. Part (2A) was introduced in Mei and Montanari (2019); Pennington and Worah (2017); Montanari and Zhong (2020); Hastie et al. (2019) in the analysis of the generalization error for finite-width neural networks in various linearized regimes (RF, NTK, etc.).

**Theorem 5.1** (Law of robustness for finite-width RF). Assume $k \asymp d$ and Condition 5.1. Then it holds w.p. $1 - \left(n \wedge d\right)^{-\Omega(1)}$ over $W$ and the generic dataset $D_n$ that every $f \in \text{span}_{K_{RF}}(X)$ which $\varepsilon$-memorizes $D_n$ verifies $\mathbb{S}(f) \geq \hat{\Omega}(\varepsilon_{\text{test}} - \varepsilon) \sqrt{n}$.

Note that the above lower-bound matches the infinite-width bound established in Theorem 3.2. Intuitively, this was to be expected as using fewer than infinitely many hidden neurons can only make the resulting model less smooth / robust (it is easier to smoothly draw when given more options).
5.1 Matching upper-bound for min-norm interpolator

We now establish a matching $\sqrt{n}$ an upper-bound which proves that the lower-bound in Theorem 5.1 is tight: it is achieved by the min-norm interpolator. We consider the following so-called proportionately scaling regime, where $n$, $d$, and $k$ are allowed to simultaneously go to infinity at the same rate, i.e. according to

**Condition 5.2 (Proportionate scaling).** $n, d, k \to \infty$ in such a way that

$$n/d \to \gamma_1 \in (0, \infty), \quad k/d \to \gamma_2 \in (0, \infty), \quad n/k \to \gamma = \gamma_1/\gamma_2 \in (0, \infty).$$

For a ridge penalty parameter $\lambda \geq 0$, consider the ridged random features interpolator

$$\tilde{f}_{RF, \lambda}(x) := K_{RF}(X, x)\top (K_{RF}(X, X) + (k\lambda/d)I_k)^{-1}y.$$  

(19)

In Mei and Montanari (2019), a fine analysis was done and explicit analytic formulae for the test error, the training error, and the norm of the optimal output weights vector $\tilde{v}_{RF, \lambda}$ were obtained. Most importantly, it was shown that the training error is close to zero for $\lambda$ close to zero; the norm of $\tilde{v}_{RF, \lambda}$ increases interpolation threshold ($k = n$) where it diverges to infinity; then beyond this threshold, it converges to a constant as $\gamma \to \infty$. This behavior was proposed as an explanation of the origins of the double-descent phenomenon, and then decreases.

**The upper-bound.** For stating and proving the upper-bound we promised, will need the following technical restriction from Mei and Montanari (2019).

**Condition 5.3 (Mei and Montanari (2019)).** The activation function $\sigma : \mathbb{R} \to \mathbb{R}$ is weakly differentiable and satisfies the growth condition $|\sigma(t)|, |\sigma'(t)| \leq c_0 e^{c_1|t|}$, for some $c_0, c_1 \in [0, \infty)$. Recall the definition of the coefficients $\beta_0(\sigma) \geq 0$, $\beta_1(\sigma) \geq 0$, and $\beta_\ast(\sigma) \in \mathbb{R}$ from (16). Assume that $\beta_\ast(\sigma) > 0$ and define the coefficient $\theta = \theta(\sigma) := \sqrt{\beta_1(\sigma)/\beta_\ast(\sigma)}$.

For example, the ReLU and the tanh activation functions satisfy the above condition. This condition was introduced in Pennington and Worah (2017); Mei and Montanari (2019); Hastie et al. (2019) to help compute the traces of random matrices involving the nonlinear gram matrix $\Phi_{RF}(X)^\top \Phi_{RF}(X) \in \mathbb{R}^{n \times n}$, which eventually yield analytic formula for train error (MSE), test error, and squared norm of output weights $\tilde{v}_{RF, \lambda}$. This is akin to the use of so-called Gaussian Equivalence Conjecture in the analysis of shallow neural networks, whereby random nonlinear features can be replaced by noisy linear ones (noise), to obtain an equivalent model which has the same training error, test error, etc. asymptotics.

**Theorem 5.2 (Upper-bound for nonrobustness in finite-width RF regime).** For a large class of activation functions including the ReLU, tanh, gaussian error-function (erf), and the absolute-value, we have the following. In the limit when $n, d, k \to \infty$ in the sense of (18) and fixed ridge parameter $\lambda \geq 0$, it holds w.p tending to 1 that $\mathbb{E}(\tilde{f}_{RF, \lambda}) \approx \varepsilon_{\text{test}}^* \sqrt{n}$ if $(\gamma, \lambda) \neq (1, 0)$ and $\mathbb{E}(\tilde{f}_{RF, \lambda})/(\varepsilon_{\text{test}}^* \sqrt{n}) \to \infty$ otherwise.

6 Law of robustness for finite-width / empirical NTK

Now consider the finite-width NTK regime where the rows of the hidden weights matrix $W \in \mathbb{R}^{k \times d}$ are drawn iid from $\tau_d$, producing a feature map $\Phi_{NTK} : \mathcal{S}_{d-1} \to \mathbb{R}^{kd}$ given by

$$\Phi_{NTK}(x) := \frac{1}{\sqrt{k}}\sigma'(Wx) \otimes x = \frac{1}{\sqrt{k}}(\sigma'(x^\top w_1)x, \ldots, \sigma'(x^\top w_k)x),$$

(20)

with associated kernel $K_{NTK}(x, x') := \Phi_{NTK}(x)^\top \Phi_{NTK}(x')$, an empirical version of the infinite-width NTK kernel $K_{\infty}$ given in (9). The generalization properties for regression with this kernel have studied extensively in the literature (see Mei and Montanari (2019); Adlam and Pennington (2020) for example). Our contribution here focuses on robustness, and its interplay with memorization. We will need the function condition on the activation function $\sigma$.

**Condition 6.1.** Condition 5.1 holds with the activation function $\sigma$ replaced by its first derivative $\sigma'$. 

This condition ensures that the smallest eigenvalue of the covariance matrix of $C_{\Phi_{\text{NTK}}(x)} \in \mathbb{R}^{kd \times kd}$ of $\Phi_{\text{NTK}}(x)$ for $x \sim \tau_d$, is lower-bounded i.e lower-bounded, by $\Omega(1/k)$ w.h.p. On the other hand, a simple calculation reveals that $\|\Phi_{\text{NTK}}\|_{L^2(\tau_n)} = O(1)$. Thus, the ratio $\alpha_{\Phi_{\text{NTK}}}$ defined in (13) is $O(k)$ w.h.p. This gives the following corollary to Theorem 4.1 (proved in the appendix).

**Theorem 6.1** (Law of robustness for finite-width NTK). Assume Condition 6.1 holds. For sufficiently large $n$, $d$, and $k$ such that $d \approx k \lesssim O(n)$, the following holds w.p $1 - d^{-\Omega(1)}$ over $W$ and the generic dataset $D_n$: every $f \in \text{span}_{\Phi_{\text{NTK}}}(X)$ which $\epsilon$-memorizes $D_n$ satisfies $\Theta(f) \geq \Omega((\epsilon_{\text{test}}^* - \epsilon) \sqrt{\frac{n}{k}})$.

**Remark 6.1** (Implications for Conjecture 1). Since it is believed that neural networks trained via gradient-descent (GD) behave like NTK approximations Jacot et al. (2018), the above theorem suggests that Conjecture 1 might be true for models trained via GD.

### 6.1 Consequences for min-norm and ridged finite-width NTK models

In Montanari and Zhong (2020), the following scaling for finite-width NTK was considered

$$n, d, k \to \infty \text{ such that } kd \gtrsim n(\log d)^C, \text{ and } n \gtrsim d \gtrsim k \geq d^\delta,$$  

(21)

for constants $C, \delta > 0$. For any $\lambda \geq 0$, consider the ridged interpolator $\hat{f}_{\text{NTK},\lambda}$ defined by

$$\hat{f}_{\text{NTK},\lambda}(x) := K_{\text{NTK}}(X, x)^\top(K_{\text{NTK}}(X, X) + \lambda I_n)^{-1} y.$$  

(22)

Note that $\hat{f}_{\text{NTK}} = \hat{f}_{\text{NTK},0}$ is the min-norm interpolator, on the event that the kernel gram matrix $K_{\text{NTK}}(X, X)$ is invertible. Under the scaling limit (21), it was established in Montanari and Zhong (2020) that

$$\lambda_{\min}(K_{\text{NTK}}(X, X)) \geq \text{var}_{z \sim \mathcal{N}(0, 1)}(\sigma'(z)) - o_p(1),$$  

(23)

which immediately implies $\text{rank}(K_{\text{NTK}}(X, X)) = n$ w.p tending to 1, and so the min-norm / least squares interpolator corresponding to $\lambda = 0$ in (22), perfectly memorizes generic the generic dataset $D_n$. In the following result, we establish lower-bound on the robustness this interpolator.

**Theorem 6.2** (Nonrobust memorization in finite-width NTK). For a large class of activation functions including the ReLU, tanh, and the absolute-value, in the scaling (21), it holds w.p tending to 1 that

$$\Theta(\hat{f}_{\text{NTK}}) \gtrsim \Omega(\epsilon_{\text{test}}^* \sqrt{n/k}).$$

### 7 Experiments

#### 7.1 Experimental setup

**Experiment 1: Finite-width NTK (only first-layer kernel).** For this experiment, we fix the input dimension $d = 50$ and the width of the neural network to $k = 40$. The number of samples $n$ sweeps the range of integers from 20 through 3020, in increments of 100. We consider a variety of activation functions: ReLU, tanh, absolute-value, and the gaussian error-function (erf). For each value of $n$, we sample 10 generic datasets with $n$ samples on the unit-sphere in $\mathcal{S}_{d-1}$, more precisely, random data points drawn iid from $\tau_d$ and given labels according to (2), with the $w_0 \in \mathcal{S}_{d-1}$ and noise level $\zeta$ sweeping from 0 through 1 in steps of 0.2. For each such dataset $D_n$, we also sample 15 iid realizations of $k$ rows of hidden weights matrix $W$, iid from $\tau_d$. Finally, for each $\lambda \in \{0, 10^{-5}, 10^{-4}, 10^{-3}\}$, we do ridge-regression to get an instance $f_{\text{NTK},\lambda}$ of the model (22).

**Experiment 2: Finite-width RF (NTK with only second-layer kernel).** The experimental setting is as in Experiment 1, except that now: $d = 300$, $k$ sweeps from 100 through 1000 in steps of 50, while $n$ sweeps the random of integers from 200 through 1000 in steps of 100. For each such dataset and each value of $\lambda$ as in Experiment 1, we do ridge-regression to get $\hat{f}_{\text{RF}}$ as in (19).

**Experiment 3: RF and NTK with infinite-width.** We run a similar experiment as in Experiment 1 and 2, but with $d = 500$ and $n$ sweeps from 100 through 1000 in steps of 100, $k = \infty$, and $\lambda = 0$. For each dataset, we compute the min-norm interpolator in RF and NTK regimes $\hat{f}_{\text{RF/NTK}}$ via (11).
Metrics. For each fitted model \( \hat{f} \) in each experiment, we estimate its Sobolev-seminorm (our measure of robustness) \( \mathcal{S}(\hat{f}) \) by drawing 500 random points i.i.d. from \( \tau_d \) (the uniform distribution on the unit-sphere \( S_{d-1} \)), and computing the square-root of the average value of \( \| \nabla_{S_{d-1}} \hat{f}(x) \|^2 \) over these 500 points. We also compute the squared test error on this test dataset.

Figure 1: (Experiments 1 – 3) Empirical verification of our proposed laws of robustness at different noise levels in the problem. Results shown are for the ReLU activation function (results for other activation functions are in the appendix). Each color corresponds to strength \( \zeta \) of the label noise. Notice the (super)linear trend between the x-axis and the y-axis of the figures, in conformity with the predictions of our theorems.

7.2 Results of the experiments

Confirmation of the robustness laws. As predicted by Theorems 3.2, 6.1, and 5.1, in Figure 1 we observe a clear linear relationship between the Sobolev-seminorm \( \mathcal{S}(\hat{f}) \) of the models \( \hat{f} \) (see Experiments 1, 2, and 3 of section 7.1 for details) and \( (\epsilon^* - \hat{\epsilon}_n(\hat{f})) \sqrt{n} \) (for infinite-width RF/NTK and finite-width RF) and \( (\epsilon^* - \hat{\epsilon}_n(\hat{f})) \sqrt{n/k} \), for finite-width NTK; a quantitative tradeoff between memorization and robustness.

Multiple-descent (MD) behavior in robustness for finite-width regimes. In Figure 2(a), we plot the Sobolev-seminorm \( \mathcal{S}(\hat{f}_{\text{NTK}}) \) of the min-norm interpolator \( \hat{f}_{\text{RF}} \) versus \( \sqrt{n/k} \) (Experiment 1). We observe a multiple-descent phenomenon (MD) whereby \( \mathcal{S}(\hat{f}_{\text{NTK}}) \sqrt{n/k} \) becomes unbounded for all the activation functions, at the point \( kd = n \) in the phase diagram. Interestingly, this singularity point (i.e. for which \( kd = n \)) in phase-space corresponds to the so-called nonlinear interpolation threshold which has been recently identified in d’Ascoli et al. (2020); Adlam and Pennington (2020). For the tanh and erf activation functions, we see a second singularity at the point \( n = k \). This corresponds to the so-called linear interpolation threshold d’Ascoli et al. (2020); Adlam and Pennington (2020).

Notice how the test error and the Sobolev-seminorm \( \mathcal{S}(\hat{f}) \) of the model follow similar multiple-descent patterns. Also notice the attenuation effect of regularization between the interpolation thresholds, we observe a linear trend between \( \mathcal{S}(\hat{f}_{\text{NTK}}) \) and \( \sqrt{n/k} \) as predicted by Theorem 6.1.

In the case of RF, in Figure 2(b) we plot the Sobolev-seminorm interpolator \( \hat{f}_{\text{RF}} \), for different activation functions and label noise levels. We observe a singularity point in the phase space along \( n = d \) for all the activation functions, and another one at \( n = k \) for the tanh and erf activation functions. Outside the interpolation thresholds, we confirm the linear law predicted by Theorem 5.1 and Theorem 5.2.

Importantly, we observe in Figure 2 that in both the finite-width RF and finite-width NTK experiments, the generalization error and nonrobustness curves have the same multiple-descent pattern. This is the first time MD is exhibited in a statistical functional (here, robustness) other generalization error. Finally, we observe that the Sobolev-seminorm \( \mathcal{S}(\hat{f}) \) is reduced with increasing ridge regularization level \( \lambda \). This also kills the multiple-descent.

The effect of the noise level. In Figure 3, we plot multiple-descent curves (again for robustness and test / generalization error) as a function of the amount of label noise in the data distribution (see 2). As would be expected, we see that MD is amplified with increasing label noise level (\( \zeta \)).
(a) (Experiment 1) Finite-width NTK with \( d = 50, k = 40, \) and \( n \in \{100, 200, 300, \ldots, 3000\} \). The vertical lines correspond to interpolation thresholds at \( n = k \) and \( n = kd \) Adlam and Pennington (2020); d’Ascoli et al. (2020).

(b) (Experiment 2) RF regime with \( d = 300 \) and \( k = 600 \) and \( n \in \{100, 200, 300, 1200\} \). The vertical lines correspond to interpolation thresholds at \( n = d \) and \( n = k \) Adlam and Pennington (2020); d’Ascoli et al. (2020).

Figure 2: Multiple-descent in robustness. The first row of each plot corresponds to training error, second row corresponds to test / generalization error, while the third row corresponds to Sobolev-seminorm of the model (our measure of nonrobustness). Columns are different choices of activation function \( \sigma \). The data is generated according to (2) with label noise level is fixed at \( \zeta = 0.2 \). The colors correspond to different values of the ridge parameter \( \lambda \). Observe how the test error and the Sobolev-seminorm of each model follow the same multiple-descent pattern.

7.3 Partial explanation of multiple-descent in robustness (in case of finite-width RF)

Refer to Figure 2. As with multiple-descent (MD) in generalization error Belkin et al. (2019); Loog et al. (2020); Adlam and Pennington (2020); d’Ascoli et al. (2020), MD in robustness we observe here is probably due to bad conditioning of the kernel gram matrix \( K_M(X, X) \) (for \( M \in \{\text{RF, NTK}\} \)) close to the interpolation thresholds. We observe that the Sobolev-seminorm \( \mathcal{E}(\hat{f}) \) is reduced with increasing ridge regularization level \( \lambda \). This also kills the multiple-descent.
(a) (Experiment 1) Finite-width NTK with $d = 50$, $k = 40$, and $n \in \{100, 200, 300, \ldots, 3000\}$. The vertical lines correspond to interpolation thresholds at $n = k$ and $n = kd$, as predicted in Adlam and Pennington (2020); d’Ascoli et al. (2020).

(b) (Experiment 2) RF regime with $d = 300$ and $k = 600$ and $n \in \{100, 200, 300, 1200\}$. The vertical lines correspond to interpolation thresholds at $n = k$ and $n = d$, predicted in Adlam and Pennington (2020); d’Ascoli et al. (2020).

Figure 3: Multiple-descent in robustness in min-norm interpolator. All plots correspond to no regularization ($\lambda = 0$). The first row of each plot corresponds to training error, second row corresponds to test / generalization error, while the third row corresponds to Sobolev-seminorm of the model (our measure of nonrobustness). Columns are different choices of activation function $\sigma$. The data is generated according to (2) with different values the level $\zeta$ of label-noise.

**Explaining MD for finite-width RF.** In the case of RF, Theorem 5.2 rigorously predicts the singularity observed in Figure 2 at $n = k$ (i.e $\gamma = 1$) for the min-norm interpolator (corresponding to $\lambda = 0$). Indeed, the prove of Theorem 5.2 reveals that if $\hat{\mathbf{v}}_{RF,\lambda} \in \mathbb{R}^k$ is the output weights vector of the RF interpolator with ridge penalty $\lambda = O(1)$, then the following holds w.p $1 - d^{-\Omega(1)}$ over the random hidden weights matrix $\mathbf{W} \in \mathbb{R}^{k \times d}$

$$\mathbb{S}(\hat{f}_{RF,\lambda}) \asymp \|\hat{\mathbf{f}}_{RF,\lambda}\| = \begin{cases} \omega_{\mathbb{P}}(\sqrt{d}) = \omega_{\mathbb{P}}(\sqrt{n}), & \text{if } (\gamma, \lambda) = (1, 0), \\ \Theta_{\mathbb{P}}(\sqrt{d}) = \Omega_{\mathbb{P}}(\sqrt{n}), & \text{else}. \end{cases}$$  

(24)
Thus, there is a singularity at $(\gamma, \lambda) = (1, 0)$, i.e. for the ridge-less interpolator at $n = k$, where in $\mathcal{S}(\hat{f}_{RF,\lambda})/\sqrt{n} = \omega_p(1) \to \infty$ in the limit $n, d, k \to \infty$ according to (18). Moreover, the above formula reveals that any multiple-descent behavior in $\|\hat{v}_{RF,\lambda}\|$, produces the exactly the same multiple-descent behavior in $\mathcal{S}(\hat{f}_{RF,\lambda})$, asymptotically in the sample size $n$.

**Still missing the full picture.** Providing rigorous explanations for the other singularities in Figures 2 and 3, namely at $n = d$ for RF, and at $n = k$ and $n = kd$ for NTK (finite-width and infinite-width) is left for future work.

### 8 Conclusion

In this work, we have derived precise laws for robustness of neural networks in both the (in)finite-width random features (RF) and (in)finite-width neural tangent kernel (NTK) regimes. Our results show a clear tradeoff between memorization and robustness, as measured by the Sobolev seminorm of the model, a new measure of (non)robustness we propose, for the min-norm interpolators, ridged interpolators, or generalizations thereof (in fact, any model in the so-called "representer subspace" of the data (7)). Empirical results confirm our theoretical findings. We also accidentally observe a new phenomenon in the finite-width regimes: multiple-descent in robustness, for which we provide a theoretical explanation in the case of finite-width RF.

### Limitations and future directions.

1. **Going beyond log-concavity.** For technical reasons, our work only considers log-concave isotropic data. A step towards removing this assumption would be to consider the student-teacher paradigm with block structure like in d’Ascoli et al. (2021).

2. **Analysis of fully-trained neural networks.** We have provided a complete picture of the fundamental tradeoffs between robustness of linearized neural networks (RF and NTK). An analysis of fully-trained neural networks would be a big next step. Exploring Theorem 4.2 could be a starting point for this.

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A More experimental results

In this section, we present additional empirical results to complement the results presented in section 7 of the manuscript.
Figure 4: (Experiment 3) Sobolev-seminorm $\mathcal{S}(\hat{f})$ of min-norm interpolator for infinite-width RF and NTK regime (11). Notice the linear relation between $\mathcal{S}(\hat{f})$ and $\sqrt{n}$ predicted by Theorems 3.2 and 3.3. For the absolute-value activation function, we see that the the curve is flat. This does not contradict Theorem 3.2 because this activation function fails to satisfy the $\phi'(0) \neq 0$ condition of the theorem. Indeed, thanks to (Louart et al., 2018, Table 1), we know that $\phi_{\text{NTK}}(t) = (2/\pi)t^2 + O(t^4)$ and $\phi_{\text{RF}}(t) = 2/\pi + (1/\pi)t^2 + O(t^4)$ for the absolute-value activation function.

Figure 5: Detailed version of Figure 1.
B Law of robustness for kernel function classes over the sphere

Notation. We will use the notation \(a_n \gtrsim b_n\) (also written \(a_n = O(b_n)\) or equivalently, \(b_n = O(n)\)) to mean that \(a_n \geq c b_n\) for some \(c > 0\) and for sufficiently large \(n\), while \(a_n \asymp b_n\) means \(a_n \gtrsim b_n \gtrsim a_n\). We will use \(\Omega(\cdot)\) to mean \(\Omega(\cdot)\) modulo log-factors. The notation \(o(1)\) will be used to denote a quantity which goes to zero with \(n\). Probabilistic versions of these notations are written with a subscript \(\mathbb{P}\), for example \(O_{\mathbb{P}}(\cdot), o_{\mathbb{P}}(\cdot)\), etc. The acronym \(a.s.\) means almost-surely, \(a.e.\) means almost-everywhere, \(w.p\) means with probability, and \(w.h.p\) means with high probability. The \(L_p\)-norm of a finite-dimensional vector \(w\) is denoted \(\|w\|_p\). We will write \(\|w\|\) to mean \(\|w\|_2\).

B.1 Proof of Theorem 2.1

Theorem 2.1 (Law of robustness for ordinary linear models). For sufficiently large \(n\) and \(d\), the following holds \(w.p\ 1 - n^{-\Omega(1)}\) over \(\mathcal{D}_n\): every linear model \(g_w\) which \(\varepsilon\)-memorizes \(\mathcal{D}_n\) verifies \(\text{Lip}(g_w) \geq \mathcal{G}(g_w) \gtrsim (\varepsilon_{\text{test}}^* - \varepsilon)\sqrt{n}\). In particular, for the high-dimensional regime \(d > n\), the min-norm / least squares interpolator \(\hat{g} = g_{\hat{w}}\), defined by setting \(\hat{w} = X^\top (XX^\top)^{-1} y \in \mathbb{R}^d\), satisfies \(\Omega(\varepsilon_{\text{test}}\sqrt{n}) \leq \mathcal{G}(\hat{g}) \leq O(\varepsilon_{\text{test}}\sqrt{n}) w.p\ 1 - n^{-\Omega(1)}\).

Proof of Theorem 2.1. By direction computation, we have
\[
\mathcal{G}(g_w)^2 := \mathbb{E}_{x \sim \mathcal{D}_d}[\|\nabla g_w(x)\|^2 - (x^\top \nabla g_w(x))^2] = \mathbb{E}_{x \sim \mathcal{D}_d}[\|w\|^2 - (x^\top w)^2] = \|w\|^2(1 - 1/d),
\]
and so \(\mathcal{G}(g_w) \approx \|w\| = \text{Lip}(g_w)\) for large \(d\) (high dimensions). Thus, the analysis of the robustness of the linear model \(g_w\) is reduced to the analysis of how the norm of \(w\) varies with overfitting.

- First part (lower-bound). Fix any \(r \geq 0\), and let \(C_d(r) := \{w \in \mathbb{R}^d \mid \mathcal{G}(g_w) \leq r\}\) and \(B_d(r') := \{w \in \mathbb{R}^d \mid \|w\| \leq r'\}\) the closed ball of radius \(r' := r\sqrt{1 - 1/d}\) in \(\mathbb{R}^d\). Thanks to the computation 25, it is clear that \(C_d(r) \subseteq B_d(r')\). Let \(\mathcal{B}_n(C_d(r))\) be the Rademacher complexity of \(C_d(r)\) w.r.t the sample \(x_1, \ldots, x_n\). We deduce that \(\mathcal{R}_n(C_d(r)) \leq \mathcal{B}_n(B_d(r')) = r'/\sqrt{n} \lesssim r/\sqrt{n}\). Invoking standard results on \(L_p\)-loss generalization bounds for bounded function classes (see Boucheron, Stéphane et al. (2005), for example\(^3\)), we obtain: \(w.p\ 1 - \delta\), it holds for all \(w \in C_d(r)\) that
\[
\varepsilon_{\text{test}} \leq \varepsilon(g_w) \leq \hat{\varepsilon}_n(g_w) + \frac{r}{\sqrt{n}} + r\sqrt{\frac{\log(2/\delta)}{n}}.
\]
The first part of the result then follows by taking \(\delta = n^{-c}\), for any constant \(c > 0\), and the rearranging (while ignoring factors which are logarithmic in \(n\)).

- Second part (tightness). Note that if \(n < d\), then \(XX^\top\) is invertible \(w.p\ 1\). Also, by construction, the min-norm interpolator has zero training error, i.e. \(\hat{\varepsilon}_n(\hat{g}) = 0\). It follows from the first part that \(\mathcal{G}(\hat{g}) \geq \Omega(\varepsilon_{\text{test}}\sqrt{n}) w.p 1 - n^{-\Omega(1)}\). We now show that \(\|\hat{w}\| \leq O(\varepsilon_{\text{test}}\sqrt{n}) w.p 1 - n^{-\Omega(1)}\). Indeed, by standard random matrix theory (RMT) Vershynin (2012), \(w.p 1 - e^{-\Omega(n)}\) over \(X\), all the eigenvalues of the gram matrix \(XX^\top\) are contained in in interval \([c_1, c_2]\), for absolute constant \(c_1, c_2 > 0\). Let \(z = (z_1, \ldots, z_n)\) be the iid \(\zeta^2\)-subGaussian noise vector of the dataset, so that \(y_i = w_0^\top x_i + z_i\) for all \(i \in [n]\). We deduce that \(w.p\ 1 - e^{-\Omega(n)}\)
\[
\|\hat{w}\|^2 = y^\top (XX^\top)^{-1} y = z^\top (XX^\top)^{-1} z + w_0^\top X (XX^\top)^{-1} X w_0 - 2w_0^\top X (XX^\top)^{-1} z,
\]
By standard concentration standard concentration for the sub-Gaussian random vector \(z\) combined with previous remark on the eigenvalues of \(XX^\top\), the first and last terms in the above display are \(w.p\ 1 - e^{-\Omega(n)}\) at most \(O(n\zeta^2)\), from which the second part of result follows. \(\square\)

B.2 RKHS norm of a memorizer

We now extend Theorem 2.1 to general kernel function classes. For \(r \geq 0\), let \(B_K = B_K(r) := \{f \in \mathcal{H}_K \mid \|f\|_{\mathcal{H}_K} \leq r\}\) be the ball of radius \(r\) in \(\mathcal{H}_K\). Also, let \(T_K : L^2(\tau_d) \to L^2(\tau_d)\) be the
\[^3\text{Because the noise is sub-Gaussian, we use a standard truncation argument to argue as if the squared loss as bounded.}\]
induced integral operator defined for every \( f \in L^2(\sigma_d) \) by
\[
T_K f : \mathcal{S}_{d-1} \to \mathbb{R}, (T_K f)(x) = \int_{\mathcal{S}_{d-1}} K(x, x') f(x') d\sigma_d(x').
\] (26)
This is a compact positive operator and thus has countably many eigenvalues \( \lambda_1 \geq \lambda_2 \geq \ldots \), all of which are nonnegative.

We start with the following auxiliary lemma which establishes that w.h.p. and function \( f \in \mathcal{H}_K \) which memorizes even a fraction of the generic dataset \( \mathcal{D}_n \) must have RKHS norm at least \( \sqrt{n} \).

**Lemma B.1.** It holds w.p. \( 1 - n^{-\Omega(1)} \) (independent of \( \varepsilon \)) over the generic dataset \( \mathcal{D}_n \) that: every \( f \in \mathcal{H}_K \) which memorizes \( \mathcal{D}_n \) satisfies
\[
\|f\|_{\mathcal{H}_K} \geq \Omega\left(\frac{\varepsilon^{\text{test}}_{\text{test}} - \varepsilon}{\sqrt{\text{Tr}(T_K)} \sqrt{n}}\right).
\] (27)

**Remark B.1.** We make the following important notes above the above theorem.

- In the above lower-bound, thanks to concentration arguments the trace term \( \text{Tr}(T_K) \) can be replaced by a sample version \( (1/n) \sum_{i=1}^n K(x_i, x_i) \).

- \cite{Belkin2018}, Theorem 1 establishes a lower-bound of the form \( \|f\|_{\mathcal{H}_K} \geq A e^{B n^{1/d}} \), for absolute constants \( A, B > 0 \). For fixed \( d = O(1) \), this bound is better than the \( \Omega(\sqrt{n}) \) bound above, but becomes unspecial when \( d \) goes to infinity, say at the same rate as \( n \). Indeed, for such \( d \), the bound in \cite{Belkin2018} predicts \( \|f\|_{\mathcal{H}_K} \geq A e^{B} \), a lower-bound which is \( O(1) \), while our bound in Lemma B.1 ensures \( \|f\|_{\mathcal{H}_K} \gtrsim \sqrt{n} \to \infty \).

**Proof of Lemma B.1.** First note that one has \( \sup_{x \in \mathcal{S}_{d-1}} K(x, x) < \infty \) since \( \mathcal{S}_{d-1} \) is compact and \( K : \mathcal{S}_{d-1} \times \mathcal{S}_{d-1} \to \mathbb{R} \) is continuous by hypothesis. Now, for any \( r \geq 0 \), the Rademacher complexity of the RKHS ball \( \mathcal{B}_K(r) \) is upper-bounded by \( r \sqrt{E_{\omega \sim \tau_d} K(x, x)} / \sqrt{n} = r \sqrt{\text{Tr}(T_K)} / \sqrt{n} \). By classical theory of generalization theory for \( L_p \)-losses \( \text{see Boucheron, Stéphane et al. (2005)} \), it holds w.p. \( 1 - \delta \) over the dataset \( \mathcal{D}_n \) that
\[
\varepsilon^{\text{test}}_{\text{test}} \leq \varepsilon(f) \leq \varepsilon(n)(f) + C \frac{r}{\sqrt{n}} \left( \sqrt{\text{Tr}(T_K)} + \sqrt{2 \log(1/\delta)} \right), \forall f \in \mathcal{B}_K(r).
\]
The claim then follows with \( \delta = n^{-c} \) for an absolute constant \( c > 0 \), and then simplifying to get \( r \geq \Omega((\varepsilon^{\text{test}}_{\text{test}} - \varepsilon) / \sqrt{n} / \log n) \geq \Omega(\varepsilon^{\text{test}}_{\text{test}} \sqrt{n}) \) w.p. \( 1 - n^{-c} \). \( \square \)

**Not game over yet.** Our concern is robustness in the ambient space \( \mathcal{S}_{d-1} \) in which the data lives. A priori, there is no direct implication between large norm does not imply large Sobolev-seminorm (though the converse is true\(^4\)). That is, a priori, we cannot directly salvage a lower-bounds for the nonrobustness of a memorizer \( f \) by exploiting the lower-bound on its RKHS norm given by Lemma B.1. For this we need to exploit the geometric structure of the specific kernel \( K \). This will allow us convert the lower-bound on \( \|f\|_{\mathcal{H}_K} \) into lower-bounds on nonrobustness \( \mathcal{S}(f) \). The rest of the manuscript is more or less dedicated to this.

### B.3 Quantitative tradeoff between memorization and robustness

**Definition B.1.** Let \( C_K(X) \in \mathbb{R}^{n \times n} \) be the covariance matrix of the random vector \( (\sqrt{n} K(x, x_1), \ldots, \sqrt{n} K(x, x_n)) \in \mathbb{R}^n \) for \( x \sim \tau_d \) independent of the \( x_i \)'s. Also define the following "condition number" of the design matrix \( X \) relative to the kernel \( K \) by
\[
\alpha_K(X) := \frac{\lambda_{\max}(C_K(X))}{\lambda_{\min}(C_K(X))} \frac{1}{n} \sum_{i=1}^n K(x_i, x_i).
\] (28)

\(^4\)Indeed, \( \mathcal{S}(f) \leq \text{Lip}(f) \leq \|f\|_{\mathcal{H}_K} \), where the last inequality is classical (see Bietti and Mairal (2019b), for example).
Note that $\alpha_K(X)$ is a random variable, since it depends on the design matrix $X$. Recall the definition of memorization in Definition 1.2. The following generic result will be the main stepping stone for most of the results in the remainder of this section and the next. As before, let $\varepsilon_{\text{test}}^*$ be the Bayes-optimal error for the problem and let $\varepsilon$ be any error threshold in the interval $[0, \varepsilon_{\text{test}}^*)$.

**Theorem B.1** (Law of robustness for the "representer" subspace). The following holds w.p $1 - n^{-\Omega(1)}$ over the generic dataset $\mathcal{D}_n$: every $f \in \text{span}_K(X)$ which $\varepsilon$-memorizes $\mathcal{D}_n$ satisfies

$$\mathcal{G}(f) \geq \tilde{\Omega}((\varepsilon_{\text{test}}^* - \varepsilon)\sqrt{\frac{n}{\alpha_K(X)}}). \quad (29)$$

In particular, if $K(X, X)$ is invertible (which necessarily implies we are in a high-dimensional regime $d \geq n$), then for min-norm interpolator $\hat{f}_n \in \text{span}_K(X)$, it holds that $\varepsilon_n(\hat{f}_n) = 0$ almost-surely and $\mathcal{G}(\hat{f}_n) \geq \tilde{\Omega}(\varepsilon_{\text{test}}^* \sqrt{\frac{n}{\alpha_K(X)}})$ w.p $1 - n^{-\Omega(1)}$.

**Proof.** For any $c \in \mathbb{R}^n$, let $f_c : S_{d-1} \rightarrow \mathbb{R}$ be the function defined by $f_c(x) := \sum_{i=1}^n c_i K(x_i, x) = c^\top K(X, x)$. The Poincaré inequality for the probability space $(S_{d-1}, \tau_d)$ gives

$$\mathcal{G}(f_c)^2 \geq d \cdot \text{Var}_{X \sim \tau_d}(f_c(x)) = d \cdot \text{Var}_{X \sim \tau_d}(c^\top K(X, x))$$

$$= c^\top \text{Cov}(\sqrt{d}K(X, x))c := c^\top C_K(X)c.$$

From the well-known identity $\|f_c\|^2_{H_K} = c^\top K(X, X)c$, one deduces

$$\mathcal{G}(f_c)^2 \geq c^\top C_K(X)c \geq \|c\|^2 \frac{\lambda_{\min}(C_K(X))}{\lambda_{\max}(X, X)} \geq \|f_c\|^2_{H_K} \frac{\lambda_{\min}(C_K(X))}{\lambda_{\max}(X, X)}$$

$$= \|f_c\|^2_{H_K} \frac{\text{Tr}(T_K)}{\alpha_K(X)} \quad (30)$$

Invoking Lemma B.1 with $\delta = n^{-\Omega(1)}$ ensures $\|f_c\|^2_{H_K} \geq \tilde{\Omega}((\varepsilon_{\text{test}}^* - \varepsilon_n(f_c))\sqrt{n})$. The first part of the result follows upon combining with (30).

The second part is a direct consequence of the first part and the fact that $\hat{f}_n(x_i) = y_i$ for all $i \in [n]$ and so $\varepsilon_n(\hat{f}_n) = 0$.

### B.4 Example: Ordinary linear models

As an example, consider the RKHS on the unit-sphere $S_{d-1}$, induced by the trivial kernel $K_{\text{id}}(x, x') = x^\top x'$. One immediately computes $K_{\text{id}}(X, X) = C_{K_{\text{id}}}(X) = XX^\top$ and the representer subspace is $\text{span}_{K_{\text{id}}}(X) = \text{span}(X) := \{x \mapsto \sum_{i=1}^n c_i x_i | c \in \mathbb{R}^n\}$. Also, one computes $\text{Tr}(T_K) = \mathbb{E}_{x \sim \tau_d}[K_{\text{id}}(x, x)] = \mathbb{E}_x ||x||^2 = 1$ and

$$\alpha_{K_{\text{id}}}(X) := \frac{\lambda_{\max}(K_{\text{id}}(X, X))}{\lambda_{\min}(K_{\text{id}}(X))} \frac{\text{Tr}(T_K)}{\lambda_{\max}(X^\top)} = \frac{\lambda_{\max}(X X^\top)}{\lambda_{\min}(X X^\top)} =: \text{cond}(X)^2,$$

where $\text{cond}(X)$ is the condition number (the one from classical linear algebra) of the design matrix.

Theorem B.1 then predicts that w.p $1 - o(1)$ over the generic dataset $\mathcal{D}_n$, every $f \in \text{span}_{K_{\text{id}}}(X)$ which $\varepsilon$-memorizes $\mathcal{D}_n$ must verify

$$\mathcal{G}(f) \geq \frac{(\varepsilon_{\text{test}}^* - \varepsilon)}{\text{cond}(X)} \tilde{\Omega}(\sqrt{n}). \quad (31)$$

From standard random matrix theory (RMT) Vershynin (2012), we know that if $n \asymp d$ are sufficiently large with $n/d \leq \gamma_1 < 1$, then $\text{cond}(X) = \Theta(1)$ w.p $1 - e^{-\Omega(d)}$. Putting things together we obtain the following corollary to Theorem B.1.

**Corollary B.1.** If large $n \asymp d$ such that $n/d \leq \gamma_1 < 1$, then it holds w.p $1 - n^{-\Omega(1)}$ that over the dataset $\mathcal{D}_n$: every linear model $f \in \text{span}_{K_{\text{id}}}(X)$ which $\varepsilon$-memorizes $\mathcal{D}_n$ satisfies $\mathcal{G}(f) \geq \tilde{\Omega}((\varepsilon_{\text{test}}^* - \varepsilon)\sqrt{n})$. In particular the min-norm interpolator $\hat{f}_n(x) := x^\top X(X X^\top)^{-1}y$ verifies $\mathcal{G}(\hat{f}_n) \geq \tilde{\Omega}(\varepsilon_{\text{test}}^* \sqrt{n})$ w.p $1 - n^{-\Omega(1)}$. 

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Gaussian kernel. Consider the case of a Laplace type kernel on $\mathcal{S}_{d-1}$ defined by $K_{\beta}(x, x') := \exp(-s^{-1}\|x - x'\|^2)$, with smoothness parameter $\beta > 0$ and bandwidth parameter $s > 0$. This is of course a dot-product kernel with $\phi(t) = e^{-(2(1-t))\beta/2}/s$.

In the next subsection, we will extend this corollary to RKHS function classes corresponding to infinite-width neural networks for certain activation functions.

C Laws of robustness for dot-product kernels

Suppose the kernel $K$ is a dot-product function in the following sense

**Definition C.1 (Kernels of dot-product type).** A kernel $K : \mathcal{S}_{d-1} \times \mathcal{S}_{d-1} \to \mathbb{R}$ is said to be a radial or dot-product kernel if there exists a continuous function $\phi : [-1, 1] \to \mathbb{R}$ such that

$$K(x, x') = \phi(x^\top x'), \forall x, x' \in \mathcal{S}_{d-1}. \quad (32)$$

Examples of dot-product kernels are abundant in machine learning. To name a few, let us mention:

- The gaussian kernel $K_{\text{Gauss}}(x, x') := \phi_{\text{Gauss}}(x^\top x')$, where $\phi_{\text{Gauss}}(t) = e^{-(2-2t)/s}$ for some bandwidth parameter $s > 0$. This kernel is known to the kernel corresponding to the infinite-width random Fourier features networks [Rahimi and Recht (2008)].
- The Laplace kernel $K_{\text{Lap}}(x, x') := \phi_{\text{Lap}}(x^\top x')$, with $\phi_{\text{Lap}}(t) = e^{-\sqrt{2t}}$.
- General exponential-type kernels given by $K_s(x, x') := \phi_{\text{Exp}(s)}(t)$, for which $\phi_{\text{Exp}(s)}(t) = e^{-(2-2t)/s}$. Note that the gaussian and Laplace kernels correspond respectively to $s = 2$ and $s = 1$.
- Polynomial kernels $K_{c,p}(x, x') := (c + x^\top x')^p$, where $p > 0$ is the degree (allowed to be fractional!) and $c \geq 0$ is an offset parameter. These are indeed dot-product kernels with $\phi(t) = (c + t)^p$. The kernel considered in section B.4 is a linear kernel with $c = 0$ and $p = 1$.

In the case of dot-product kernels, many things simplify. For example, for the associated kernel integral operator $T_K$, one has

$$\|T_K\|_{op} \leq \text{Tr}(T_K) = \mathbb{E}_{x \sim \tau_d}[K(x, x)] = \phi(1). \quad (33)$$

**Theorem C.1.** Suppose Condition 3.1 holds and $n, d \to \infty$ such that $n/d \to \gamma_1 \in [0, 1)$. Then

$$\lambda_{\text{min}}(C_K(X)) \geq \Omega_P(1), \quad \lambda_{\text{max}}(K(X, X)) \leq O_P(1), \quad \alpha_K(X) \leq O_P(1). \quad (34)$$

Theorem C.1 provides us with a lower-bound on the condition number $\alpha_K(X)$ using very macroscopic information about the dot-product function $\phi$. To proof this theorem we will need the following auxiliary result (proved in section XXX) which is important in its own right, and is therefore stated as a theorem.

**Theorem C.2.** Let $V$ be an $m \times d$ random matrix with iid rows sampled from $\tau_d$. Consider the random mapping mapping $\Phi : \mathcal{S}_{d-1} \to \mathbb{R}^m$ defined by $\Phi(x) = \varphi(Vx)$, where $\varphi : [-1, 1] \to \mathbb{R}$ is a continuous function which is thrice continuously-differentiable at 0. For $x \sim \tau_d$, let $C_\Phi \in \mathbb{R}^{m \times m}$ be the covariance matrix of the random vector $\sqrt{d}\Phi(x) \in \mathbb{R}^m$. In the limit $m, d \to \infty$ such that $m/d \leq \gamma_1 < 1$, it holds that $\lambda_{\text{min}}(C_\Phi) \to \varphi'(0)^2(1 - \sqrt{\gamma_1})^2$ almost-surely.

**Proof of Theorem C.1.** From Theorem C.2, with $m = n$, $V = X$, and $\varphi = \phi$ (the dot-product function of the kernel $K$), we have $\lambda_{\text{min}}(C_K(X)) = \phi'(0)^2(1 - \sqrt{\gamma_1})^2 - o_P(1) \geq \Omega_P(1)$. It remains to upper-bound $\lambda_{\text{max}}(K(X, X))$. For this, it suffices to apply [El Karoui, 2010, Theorem 2.1] to obtain that $\|K(X, X) - K(X, X)\text{lin}\|_{op} = o_P(1)$, where the matrix $K(X, X)\text{lin} \in \mathbb{R}^{n \times n}$ has entries

$$K(X, X)\text{lin}_{i,\ell} := \phi(0) + \frac{\phi''(0)}{2n} + \phi'(0)x_i^\top x_\ell + \nu\delta_{i,\ell},$$

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with \( \nu := \phi(1) - \phi(0) - \phi'(0) \) and \( \delta_{i,\ell} = 1 \) if \( i = \ell \) and \( \delta_{i,\ell} = 0 \) otherwise. Noting that the finite-rank (here rank 1) perturbations do not affect the limiting spectral distribution of a random matrix, we deduce that

\[
\lambda_{\max}(K(X, X)) = \|K(X, X)\|_{\text{op}} + o_p(1) = \|\phi'(0)XX^T + \nu I_n\|_{\text{op}} + o_p(1)
\]

\[
\leq |\phi'(0)|(1 - \sqrt{n})^2 + |\nu| + o_p(1), \quad \text{by Bin-Yai (to do: add ref)}
\]

Finally, we deduce that

\[
\alpha_K(X) := \frac{\lambda_{\max}(K(X, X))\phi(1)}{\lambda_{\min}(C_K(X))} \leq \frac{O_p(1)}{\Omega_p(1)} = O_p(1).
\]

The following result which extends Corollary B.1 is a Corollary to B.1. We state it as a theorem because it is important in its own right, and will in turn give laws of robustness for kernels induced by certain infinite-width neural networks, for example (section E).

**Theorem 3.1** (Law of robustness for dot-product kernels). **Under Condition 3.1, in the limit \( n, d \to \infty \) such that \( n/d \leq \gamma_1 < 1 \), it holds w.p. tending to 1 that: every \( f \in \text{span}_K(X) \) which \( \varepsilon \)-memorizes \( D_n \) satisfies \( \mathcal{S}(f) \geq \Omega((\varepsilon_{\text{test}}^* - \varepsilon)\sqrt{n}) \). In particular, if the kernel gram matrix \( K(X, X) \) is nonsingular, then the min-norm interpolator \( \tilde{f}_n(x) := K(X,x)^TK(X,X)^{-1}y \) satisfies \( \mathcal{S}(\tilde{f}_n) \geq \Omega(\varepsilon_{\text{test}}^*\sqrt{n}) \).

**Proof.** Follows Theorems B.1 and C.1.

**Example: Exponential-type kernels.** As an example, consider an exponential-type kernel \( K_{\beta}(x, x') := e^{-\|x-x'\|^2/s} \), where \( \beta > 0 \) is a "smoothness" parameter and \( s \) is a bandwidth parameter. As discussed in the paragraph just after (32), such is a dot-product kernel with dot-product function \( \phi_{\beta}(t) := e^{-(2-2t)\beta/2/s} \) which is infinitely continuously differentiable with \( \phi'(0) = -\beta^{2/3}e^{-2\beta/3}/s \neq 0 \). Thus, such kernels satisfy Condition 3.1, and we deduce the following Corollary to Theorem 3.1.

**Corollary C.1** (Law of robustness for exponential-type kernels). **In the limit \( n, d \to \infty \) such that \( n/d \leq \gamma_1 < 1 \) the following holds in probability: every \( f \in \text{span}_{K_{\beta}}(X) \) which \( \varepsilon \)-memorizes \( D_n \) satisfies \( \mathcal{S}(f) \gtrsim (\varepsilon_{\text{test}}^* - \varepsilon)\sqrt{n} \). In particular, if the gram matrix \( K_{\beta}(X, X) \) is nonsingular, then for the min-norm interpolator \( \tilde{f}_n \in \text{span}_{K_{\beta}}(X) \), it holds in probability that \( \mathcal{S}(\tilde{f}_n) \gtrsim \varepsilon_{\text{test}}^*\sqrt{n} \).

In particular, the above theorem applies to

- Two-layer infinite-width neural networks with random Fourier features, the corresponding RKHS is precisely that induced by the Gaussian kernel Rahimi and Recht (2008, 2009).
- Certain infinite-width neural networks in RF / NTK regime.
- etc.

An analogous result holds for polynomial kernels with positive degree.

**D Proof of Theorem 4.1**

**Theorem 4.1.** The following holds w.p. \( 1 - n^{-\Omega(1)} \) over the generic dataset \( D_n \): every \( f \in \text{span}_{K_\Phi}(X) \) which \( \varepsilon \)-memorizes the generic dataset \( D_n \) satisfies \( \mathcal{S}(f) \geq \Omega((\varepsilon_{\text{test}}^* - \varepsilon)\sqrt{n/\alpha_\Phi}) \).

In particular, if the gram matrix \( K_\Phi(X, X) := \Phi(X)^T\Phi(X) \) is invertible (which necessarily implies \( n \leq d \)), then w.p. \( 1 - n^{-\Omega(1)} \) over \( D_n \) it holds that the least-squares model \( \tilde{f}_\Phi(x) := K_\Phi(X,x)^TK_\Phi(X,X)^{-1}y \) satisfies \( \mathcal{S}(\tilde{f}_\Phi) \geq \Omega((\varepsilon_{\text{test}}^* - \varepsilon)\sqrt{n/\alpha_\Phi}) \).
Proof. One computes the variance of any \( f_c \in \text{span}_{K_{\Phi}}(X) \) for random \( x \sim \tau_d \), as follows

\[
\mathbb{E}(f_c)^2 \geq d \cdot \text{var}_{x \sim \tau_d}(f_c(x)) = d \cdot \text{var}_{x \sim \tau_d}(c^\top \Phi(x)\Phi(x)) \\
= c^\top \Phi(x)\text{Cov}_{x \sim \tau_d}(\sqrt{d}\Phi(x))\Phi(x)^\top c \\
\geq \lambda_{\min}(\text{Cov}_{x \sim \tau_d}(\sqrt{d}\Phi(x)))\|\Phi(x)^\top c\|^2 \\
= \lambda_{\min}(C_{\Phi})\|f_c\|^2_{\mathcal{H}_{K_{\Phi}}},
\]

where the \( m \times m \) psd matrix \( C_{\Phi} \) is the covariance matrix of the random vector \( \sqrt{d}\Phi(x) \in \mathbb{R}^m \).

We apply Lemma B.1. For this, we need to compute the trace of the kernel integral operator \( T_{K_{\Phi}} \), which equals

\[
\text{Tr}(T_{K_{\Phi}}) = \mathbb{E}_{x \sim \tau_d}[K_{\Phi}(x,x)] = \mathbb{E}_{x \sim \tau_d}\|\Phi(x)\|^2 = \|\Phi\|^2_{L^2(\tau_d)}. 
\]

The result then follows upon invoking Lemma B.1 to lower-bound \( \|f_c\|_{\mathcal{H}_{\Phi}} \) and then invoking (35) to lower-bound \( \mathbb{E}(f_c) \).

\[ \square \]

E. Neural networks in infinite-width RF and NTK regimes

We now place ourselves in the exact kernel regimes (where \( k = \infty \)), for two-layer neural networks in RF and NTK regimes.

E.1 Proof of Theorem 3.3 (tightness of lower-bound in Theorem 3.2)

We recall the following lemma, needed for the proof.

Lemma 3.1 (Invertibility of RF kernel gram matrix). For the ReLU activation function and for sufficiently large \( n \) and \( d \) with \( n/d \leq \gamma_1 < 1 \), it holds w.p. \( 1 - d^{-1+o(1)} \) that the eigenvalues \( \lambda_1 \geq \ldots \geq \lambda_n \) of the RF kernel gram matrix \( K_{\text{RF}}^\infty(X,X) \) satisfy \( c \leq \lambda_n \leq \ldots \leq \lambda_1 \leq C \), for constants \( c,C > 0 \) which only depend on \( \gamma_1 \). In particular, \( K_{\text{RF}}^\infty(X,X) \) is invertible w.p. \( 1 - d^{-1+o(1)} \).

Proof. Follows from Theorems C.1 and C.2. \[ \square \]

We now prove Theorem 3.3, namely the tightness of the lower-bound in Theorem 3.2.

Theorem 3.3 (Tightness of lower-bound in Theorem 3.2). For sufficiently large \( n \) and \( d \) such that \( n/d \leq \gamma_1 < 1 \), it holds w.p. \( 1 - n^{-\Omega(1)} \) over \( \mathcal{D}_n \), that the RF min-norm interpolator \( \tilde{f}_{\text{RF}}^\infty \) defined in (11) with ReLU activation function verifies \( \mathbb{E}(\tilde{f}_{\text{RF}}^\infty) \leq \text{Lip}(\tilde{f}_{\text{RF}}^\infty) \leq O(\varepsilon_{\text{test}}^*/\sqrt{n}) \).

Proof. Let \( \mathcal{H} \) be the RKHS induced by the infinite-width ReLU random features kernel. Note that the coefficients of \( \tilde{f}_{\text{RF}}^\infty \) in the representer subspace \( \text{span}_{K_{\text{RF}}^\infty}(X) \subseteq \mathcal{H} \) are given by \( \hat{c} := K_{\text{RF}}^\infty(X,X)^{-1}y \in \mathbb{R}^n \), so that \( \tilde{f}_{\text{RF}}^\infty(x) = \hat{c}^\top K_{\text{RF}}^\infty(X,x) = \hat{c}^\top \phi_1(x) \), where \( \phi_1(x) := (\phi_1(x^1_1), \ldots, \phi_1(x^d_n)) \in \mathbb{R}^n \) and \( \phi_1 \) is the order-1 arc-cosine dot-product function defined in (10). Thanks to (Bietti and Mairal, 2019a, Lemma 1), we know that the Lipschitz constant of \( \tilde{f}_{\text{RF}}^\infty \) is upper-bounded by its RKHS norm in \( \mathcal{H} \). Thus, one computes

\[
\text{Lip}(\tilde{f}_{\text{RF}}^\infty)^2 \leq \|\tilde{f}_{\text{RF}}^\infty\|_{\mathcal{H}} = \hat{c}^\top K_{\text{RF}}^\infty(X,X)\hat{c} = y^\top K_{\text{RF}}^\infty(X,X)^{-1}y \\
\leq \lambda_{\max}(K_{\text{RF}}^\infty(X,X)^{-1})\|y\|^2 \leq \frac{\|y\|^2}{\lambda_{\min}(K_{\text{RF}}^\infty(X,X))}.
\]

By Lemma 3.1, we know that \( \lambda_{\min}(K_{\text{RF}}^\infty(X,X)) \geq \Omega(1) \) w.p. \( 1 - o(1) \). Also, each label \( y_i \) in the dataset \( \mathcal{D}_n \) is \( \zeta^2 \)-sub-Gaussian around \( x_i^\top w_0 \) with \( \|w_0\| \leq 1 \) and \( \|x_i\| = 1 \), we know that \( \|y\|^2 = O(\zeta^2n) \) w.p. \( 1 - o(1) \). Putting things together gives the result. \[ \square \]
F  Finite-width random features regime

F.1  Simplifying the matrix $C_\sigma(W)$, the covariance matrix of $\sqrt{d}\sigma(Wx)$ for $x \sim \tau_d$

Let us restrict our attention to the following class of activation functions $\sigma$. For concreteness, the reader may think of the ReLU of the absolute value activation functions.

**Condition F.1.** The activation function $\sigma$ is 1-Lipschitz and positive-homogeneous of order 1.

The following remarkable property of positive-homogeneous functions will be very helpful in the sequel.

**Proposition F.1** (Kernel function induced by homogeneous activations Buchweitz (2016)). If $h : \mathbb{R} \to \mathbb{R}$ is positive-homogeneous of order $p \geq 0$, then for every $a, b \in S_{d-1}$ we have the identity

$$\mathbb{E}_{x \sim \tau_d}[h(x^\top u)h(x^\top v)] = \frac{C_{2,2p}}{C_{d,2p}} \phi_h(u^\top v), \text{ with } \phi_h(t) := \frac{1}{2\pi} \int_0^{2\pi} h(\cos u)h(\cos(u - \arccos t))du,$$

where $C_{d,p} := 2^{p/2-1} \cdot \frac{d \cdot \Gamma((d + p)/2)}{\Gamma((d + 2)/2)}$.

By the above proposition, the order-1 positive-homogeneity of the activation function $\sigma$ implies the existence $\phi_\sigma : [-1, 1] \to \mathbb{R}$ such that if the rows if $u, v \in S_{d-1}$ (i.e $u$ and $v$ are unit-vectors), then

$$\mathbb{E}_{x \sim \tau_d}[\sigma(x^\top u)\sigma(x^\top v)] = \frac{1}{d} \phi_\sigma(u^\top v).$$

(38)

For example, if $\sigma$ is the ReLU activation function, then

$$\phi_{\text{ReLU}}(t) = \frac{1}{2\pi}(t \arccos(-t) + \sqrt{1 - t^2}), \forall t \in [-1, 1].$$

(39)

Importantly, the function $\phi_\sigma$ depends on the activation function $\sigma$ alone (and not on problem parameters like $n, d, k$, etc.). The following lemma is a first step towards a spectrally consistent linearization of the troublesome matrix $C_\sigma(W)$.

**Lemma F.1.** Suppose the function $\phi_\sigma$ appearing in (38) is thrice continuously-differentiable at 0 and the rows of $W$ are unit-vectors, then

$$C_\sigma(W) = \phi_\sigma(WW^\top) - \phi_\sigma(0)1_k 1_k^\top + E,$$

(40)

for some $k \times k$ matrix with $\|E\|_{op} = \mathcal{O}(1/d)$.

**Proof.** The $(j, \ell)$ entry of $C_\sigma(W)$ is given by

$$c_{j,\ell} = \mathbb{E}_{x \sim \tau_d}[\sigma(x^\top w_j)\sigma(x^\top w_\ell)] - \mathbb{E}_{x \sim \tau_d}[\sigma(x^\top w_j)]\mathbb{E}_{x \sim \tau_d}[\sigma(x^\top w_\ell)],$$

$$= \phi_\sigma(w_j^\top w_\ell) - d \cdot \mathbb{E}_{x \sim \tau_d}[\sigma(x^\top w_j)]\mathbb{E}_{x \sim \tau_d}[\sigma(x^\top w_\ell)].$$

On the other hand, because $w_j$ and $w_\ell$ are unit-vectors and the distribution of $x$ is isotropic, we may write

$$\mathbb{E}_{x \sim \tau_d}[\sigma(x^\top w_j)]\mathbb{E}_{x}[\sigma(x^\top w_\ell)] = \mathbb{E}_{x \sim \tau_d}[\sigma(x^\top w_j)]^2 = \mathbb{E}_{w \sim \tau_d}\mathbb{E}_{x \sim \tau_d}[\sigma(x^\top w_j)]^2$$

$$= \mathbb{E}(x^\top z)_{\sim \tau_d \otimes \tau_d} \mathbb{E}_w[\sigma(x^\top w)\sigma(z^\top w)]$$

$$= \mathbb{E}(x^\top z)_{\sim \tau_d \otimes \tau_d} [\phi_\sigma(x^\top z)] = \phi_\sigma(0) + \mathcal{O}(\frac{1}{d^2}),$$

where the last step is thanks to a Taylor expansion of $\phi_\sigma$ around 0 and the fact that $\mathbb{E}[x^\top z] = 0$ due to isotropy and independence of $x$ and $z$. Putting things together then gives

$$c_{j,\ell} = \phi_\sigma(w_j^\top w_\ell) - \phi_\sigma(0) + \mathcal{O}(\frac{1}{d^2}),$$

from whence the result follows.  \[\square\]
### F.2 Spectrally consistent linearizations of $\tilde{C}_\sigma(W)$ for random $W$

Let $W \in \mathbb{R}^{k \times d}$ be a random matrix with independent rows uniformly on the unit-sphere $S_{d-1}$ (i.e. according to the uniform distribution $\tau_d$ thereupon), and let $\tilde{C}_\sigma(W)$ be the $k \times k$ psd matrix defined in (17). If the input dimension $d$ is sufficiently large, then for distinct $j, \ell \in [k]$, it is clear that $w_j^T w_\ell = O(1/d)$ w.h.p. Thus, if we suppose the function $\phi_\sigma$ defined in (38) is sufficiently smooth in a neighborhood of $t = 0$, one can hope to Taylor-expand $\tilde{C}_\sigma(W)$ entry-wise. In El Karoui (2010); Liang and Rakhlin (2020), such arguments are made more precise and quantitative estimates for the extreme eigenvalues of $\tilde{C}_\sigma(W)$ are obtained via a linearization trick.

Now, consider the $k \times k$ matrix $\tilde{C}_\sigma(W)^{\text{lin}}$ with entries given by

$$
\tilde{C}_\sigma(W)^{\text{lin}}_{j,\ell} := \phi_\sigma(0) + \phi_\sigma'(0) w_j^T w_\ell + (\phi_\sigma(1) - \phi_\sigma(0) - \phi_\sigma'(0)) \delta_{j,\ell}.
$$

We now show that that the curvature coefficients $\beta_0(\sigma) \geq 0$, $\beta_1(\sigma) \geq 0$, and $\beta_\ast(\sigma) \in \mathbb{R}$ defined in (16) are precisely the low-order coefficients in the above polynomial.

**Lemma F.2.** We have the following identities

$$
\begin{align*}
\beta_0(\sigma) &= \phi_\sigma(0), \\
\beta_1(\sigma) &= \phi_\sigma'(0), \\
\beta_\ast(\sigma) &= \phi_\sigma(1) - \phi_\sigma(0) - \phi_\sigma'(0).
\end{align*}
$$

**Proof.** By definition, $\phi_\sigma(t) := d \cdot E_{x \sim \tau_d} [\sigma(x^T w_j) \sigma(x^T w_\ell)]$, where $t := w_j^T w_\ell$. If $t = 0$, then $w_j$ and $w_\ell$ are orthogonal, and $x^T w_j$ and $x^T w_\ell$ are (statistically) independent, with the same distribution, which is approximately $N(0, 1/d)$ (the approximation error in Kolmogorov distance is of order $O(1/\sqrt{d})$). We deduce that $\phi_\sigma(0) = E_{z \sim N(0, 1)} [\sigma(z)]^2 + O(1/d) = \beta_0(\sigma) + O(1/d)$, by definition of $\beta_1(\sigma)$. One can use analogous arguments to obtain $\phi_\sigma'(0) = \beta_1(\sigma) + O(1/d)$.

If $t = 1$, then $w_j^T w_\ell = 1$, and so $w_j = w_\ell$. Thus, one computes

$$
\phi_\sigma(1) = d \cdot E_{x \sim \tau_d} [\sigma(x^T w_j) \sigma(x^T w_j)] = d \cdot E_{x \sim \tau_d} [\sigma(\sqrt{d}, x^T w_j)^2] = E_{z}[\sigma(z)^2] + O(1/d) =: \beta_\ast(\sigma) + O(1/d),
$$

which completes the proof. \hfill \square

The following lemma which is a direct consequence of a result of Liang and Rakhlin (2020) (see also previous work in El Karoui (2010)), establishes that $\tilde{C}_\sigma(W)^{\text{lin}}$ is a linearization of $\tilde{C}_\sigma(W)$, which keeps the main spectral information of the former.

**Lemma F.3 (Linear approximation of $\tilde{C}_\sigma(W)$).** For sufficiently large $d$, it holds w.p $1 - d^{-\Omega(1)}$ over the choice of $W$ that $\|C_\sigma(W) - \tilde{C}_\sigma(W)^{\text{lin}}\|_{\text{op}} = o(1)$.

**Proof.** The proof is based on (Liang and Rakhlin, 2020, Proposition A.2) which is itself a non-asymptotic / quantitative version of (El Karoui, 2010, Theorem 2.1). One may write $w_j = \Sigma_d^{-1/2} z_j$, where $\Sigma_d = I_d$ and $z_j$ is uniformly distributed on the sphere of radius $\sqrt{d}$ in $\mathbb{R}^d$, as thus is 1-subGaussian. thanks to Lemma F.1. In (Liang and Rakhlin, 2020, Proposition A.2), noting that $\text{Tr}(\Sigma_d) = \text{Tr}(\Sigma_d^2)$, and taking $m = \infty$ (i.e $\theta = 1/2$) (since the $z_j$’s are 1-subGaussian isotropic random vectors), we deduce that for $\delta$ sufficiently small and $d$ sufficiently large, it holds w.p $1 - \delta - d^{-2}$ that $\|\tilde{C}_\sigma(W) - \tilde{C}_\sigma(W)^{\text{lin}}\|_{\text{op}} \leq d^{-1/2}(\delta^{-1/2} + \log^{0.51} d)$. It then suffices to take $\delta = d^{-c}$ for any $0 < c < 1$ to complete the proof. \hfill \square

### F.3 Proof of Theorem 5.1 (Law of robustness in RF regime with finite width)

For the proof of the theorem, we shall need a specialized corollary to Lemma F.3 to give probabilistic estimates for the extreme eigenvalues of $C_\sigma(W) \in \mathbb{R}^{k \times k}$, the covariance matrix of $\sqrt{d} \sigma(Wx)$ for $x \sim \tau_d$. Recall the definition of the curvature coefficients $\beta_0(\sigma)$, $\beta_1(\sigma)$, and $\beta_\ast(\sigma)$ from (16).
Corollary F.1 (Extreme eigenvalues of $C_\sigma(W)$). If Condition 5.1 holds, then for sufficiently large $d$ and $k$ with $k \gg d$, it holds $1 - d^{-\Omega(1)} \leq \lambda^\sigma(C_\sigma(W)) \leq C^\sigma(C_\sigma(W)) \leq C^\sigma(C_\sigma(W)) \\ where $c, C > 0$ are constants which only depend on the ratio $k/d$ and the activation function $\sigma$.

Proof. Using Lemma F.3 and the fact that $C_\sigma(W) = \phi_\sigma(\sigma)(W) - (\beta(\sigma)W + \beta(\sigma)I_d)_{\text{op}} = o(1)$ w.p $1 - d^{-\Omega(1)}$. On the other hand, standard RMT Vershynin (2012) guarantees the existence of universal constants $c', C' > 0$ such that $c' \leq \lambda^\sigma_{\min}(W) \leq \lambda^\sigma_{\max}(W) \leq C' w.p 1 - e^{-\Omega(d)}$. The result then follows upon taking into account Condition 5.1.

We are now ready to establish a law of robustness for finite-width neural two-layer neural networks in the random features regime. We restate the theorem for convenience. As before, let $\varepsilon^\text{test}$ be the Bayes-optimal error for the problem and let $\varepsilon$ be any error threshold in the interval $[0, \varepsilon^\text{test}]$.

Theorem 5.1 (Law of robustness for finite-width RF). Assume $k \gg d$ and Condition 5.1. Then it holds w.p $1 - (n \land d)^{-\Omega(1)}$ over $W$ and the generic dataset $D_n$ that every $f \in \text{span}_{K_{RF}}(X)$ which $\varepsilon$-memorizes $D_n$ verifies $\mathbb{E}(f) \geq \Omega(\sqrt{\varepsilon^\text{test} - \varepsilon} \sqrt{n})$.

Proof. From Corollary F.1, we know that $\lambda^\sigma_{\min}(C_\sigma(W)) = \Omega(1)$ w.p $1 - d^{-\Omega(1)}$. The result then follows directly upon combining with Theorem 4.2 and the fact that $\|W\|_F = \sqrt{k}$ because the rows of $W$ are on the unit-sphere $S_{d-1}$.

F.4 Proof of Theorem 5.2 (tightness of lower-bound in Theorem 5.1)

We shall now prove that the $\sqrt{n}$ lower-bound in Theorem 5.2 is tight: it is achieved by the min-norm interpolator.

Theorem 5.2 (Upper-bound for nonrobustness in finite-width RF regime). For a large class of activation functions including the ReLU, tanh, gaussian error-function (erf), and the absolute-value, we have the following. In the limit when $n, d, k \to \infty$ in the sense of (18) and fixed ridge parameter $\lambda \geq 0$, it holds w.p tending to 1 that $\mathbb{E}(\hat{f}_{RF,\lambda}) \approx \varepsilon^\text{test}_k \sqrt{n}$ if $(\gamma, \lambda) \neq (1, 0)$ and $\mathbb{E}(\hat{f}_{RF,\lambda})/(\varepsilon^\text{test}_k \sqrt{n}) \to \infty$ otherwise.

We will make use of the following result from Mei and Montanari (2019).

Proposition F.2 (Theorem 6 of Mei and Montanari (2019), specialized to the case of positive-homogeneous activation functions). Assume Condition 5.5. In the limit when $n, d, k \to \infty$ in the sense of (18), the following hold.

- **Memorization.** There is a constant $L(\gamma_1, \gamma_2, \theta^2, \lambda) \geq 0$ which is increasing in $\lambda$ with $L(\gamma_1, \gamma_2, \theta^2, 0) = 0$, such that $\mathbb{E}_{X, W}|\text{MSE}(\hat{f}_{RF,\lambda}) - L(\gamma_1, \gamma_2, \theta^2, \lambda)| = o(1)$.

- **Norm of min-norm interpolator.** There is a constant $A(\gamma_1, \gamma_2, \theta^2, \lambda) \in [0, \infty)$ satisfying
  - $A(\gamma_1, \gamma_2, \theta^2, \lambda)$ is decreasing in $\lambda$,
  - $A(\gamma_1, \gamma_2, \theta^2, 0)$ is finite and increasing in $\gamma := \gamma_2/\gamma_1$, for $\gamma \in (0, 1)$,
  - $\lim_{\gamma \to 1} A(\gamma_1, \gamma_2, \theta^2, 0) = \infty$,
  - $A(\gamma_1, \gamma_2, \theta^2, 0)$ is finite and increasing in $\gamma$, for $\gamma \in (1, \infty)$,
  such that $\mathbb{E}_{X, W}(|\beta^2|/d)_{\text{RF,}\lambda}^2 - A(\gamma_1, \gamma_2, \theta^2, \lambda)| = o(1)$.

Remark F.1. The following remarks are in place.

- We have restated the result of Mei and Montanari (2019) for our purposes. In particular, the authors proved a stronger statement in which the labels are not entirely independent of the data (i.e positive SNR). The version stated above corresponds to noise-only regime where the SNR is zero.
• The $1/d$ factor in $\|\hat{\nu}_{RF,\lambda}\|^2$ in the above proposition accommodates for the fact that we work on the unit-sphere $S_{d-1}$ while the results of Mei and Montanari (2019) were stated for $\sqrt{d}S_{d-1}$. The above version of their result is then obtained via a simple change of activation function $\tilde{\sigma}(t) := \sigma(t)/\sqrt{d}$ by 1-homogeneity of $\sigma$, from where we obtain the relations $\beta_*^2(\tilde{\sigma}) = (\beta_*^2(\sigma)/d)$ and $\theta^2(\tilde{\sigma}) = \theta^2(\sigma)$.

• It was also observed (empirically) in Mei and Montanari (2019) that when $\gamma \to \infty$, $A(\gamma_1, \gamma_2, \theta^2, \lambda)$ converges to a positive finite constant which does not depend on any of $\gamma_1$, $\gamma_2$, $\beta_*^2$, or $\theta^2$.

**Proof of Theorem 5.2.** The memorization part of the theorem is a direct consequence of Proposition F.2. Still by Proposition F.2, we know that $(\beta_*^2/d)\|\hat{\nu}_{RF,\lambda}\|^2 = A(\gamma_1, \gamma_2, \theta^2, \lambda) + o_p(1)$ for a constant $A(\gamma_1, \gamma_2, \theta^2, \lambda)$ satisfying all the properties in the proposition. Since, $n$ is proportional to $d$ and $\beta_*^2 > 0$ by hypothesis, we conclude upon invoking Theorem H.6, that

$$\mathcal{G}(\hat{f}_{RF,\lambda}) \simeq \|\hat{\nu}_{RF,\lambda}\| = \begin{cases} \omega_p(\sqrt{d}) = \omega_p(\sqrt{n}), & \text{if } (\gamma, \lambda) = (1, 0), \\ \Theta_p(\sqrt{d}) = \Omega_p(\sqrt{n}), & \text{else}, \end{cases}$$

(44)

which concludes the proof. \qed

## G Finite-width NTK regime

### G.1 Proof of Theorem 6.1

**Theorem 6.1** (Law of robustness for finite-width NTK). Assume Condition 6.1 holds. For sufficiently large $n, d,$ and $k$ such that $d \asymp k \lesssim O(n)$, the following holds w.p. $1 - d^{-\Omega(1)}$ over $W$ and the generic dataset $D_n$: every $f \in \text{span}_{\Phi_{\text{NTK}}}(X)$ which $\varepsilon$-memorizes $D_n$ satisfies $\mathcal{G}(f) \geq \Omega((\varepsilon_{\text{test}} - \varepsilon)\sqrt{\frac{n}{k}})$.

We start with an auxiliary lemma that will be crucial for the proof of the Theorem.

**Lemma G.1.** For $x \sim \tau_d$, the covariance matrix of $\sqrt{d}\Phi_{\text{NTK}}(x)$ is given by

$$C_{\Phi_{\text{NTK}}} = \frac{1}{k} (\tilde{C}_{\sigma'}(W) \otimes I_d) \in \mathbb{R}^{kd \times kd},$$

where $\tilde{C}_{\sigma'}(W)$ is the $k \times k$ PSD matrix with entries given by $C_{\sigma'}(W)_{j,k} = \mathbb{E}_{x \sim \tau_d}[\sigma'(x^\top w_j)\sigma'(x^\top w_k)]$.

**Proof.** Let $x \sim \tau_d$ and $z(x) := (1/\sqrt{k})\sigma'(W x) := ((1/\sqrt{k})\sigma'(x^\top w_1), \ldots, (1/\sqrt{k})\sigma'(x^\top w_k)) \in \mathbb{R}^k$, and observe $\Phi_{\text{NTK}}(x) = z(x) \otimes x \in \mathbb{R}^{kd}$, the Kronecker product of $z(x)$ and $x$. On the other hand, it is clear that $z(x)$ and $x$ are independent\(^5\). Thanks to Lemma G.1, we then obtain

$$C_{\Phi_{\text{NTK}}} = d \cdot \text{Cov}(z(x) \otimes x) = \mathbb{E}[z(x)z(x)^\top] \otimes \text{Cov}(\sqrt{d}x) = \frac{1}{k} \tilde{C}_{\sigma'}(W) \otimes I_d,$$

as claimed. \qed

Note that under Condition F.1, $\sigma'$ is positive-homogeneous of order 0, and thus by Proposition F.1, there exists a continuous function $\phi_{\sigma'} : [-1, 1] \to \mathbb{R}$ such that

$$\mathbb{E}_{x \sim \tau_d}[\sigma'(x^\top u)(\sigma'(x^\top v))] = \phi_{\sigma'}(u^\top v).$$

(45)

The following Lemma can be easily proved by differentiating through formula 37.

**Lemma G.2.** We have the functional identity: $\phi_{\sigma'} = (\phi_{\sigma})'$.\(^5\)

For example, if $\sigma$ is the RELU activation function, then

$$\phi_{\text{ReLU'}}(t) = (\phi_{\text{ReLU}})'(t) = \frac{\arccos(-t)}{2\pi},$$

by differentiating equation (39).

\(^5\)Because $W x$ and $x$ are independent, since $W$ and $x$ are.
Proof of Theorem 6.1. One may upper-bound the energy of $\Phi_{\text{NTK}}$ like so
\[
\|\Phi_{\text{NTK}}\|_2^2(\tau_d) = \frac{1}{k} \mathbb{E}_{x \sim \tau_d}[\|\sigma'(Wx) \otimes x\|^2] \leq \frac{1}{k} \mathbb{E}[\|\sigma'(Wx)\|^2\|x\|^2] = \frac{1}{k} \mathbb{E}_x[\|\sigma'(Wx)\|^2] \leq 1,
\]
where the last step is because $\sigma$ is 1-Lipschitz. Combining with Corollary F.1 and Lemma G.1, gives
\[
\alpha_{\Phi_{\text{NTK}}} := \frac{\|\Phi_{\text{NTK}}\|_2^2(\tau_d)}{\lambda_{\min}(C_{\Phi_{\text{NTK}}})} \leq \frac{O(1)}{\Omega(1/k)} \leq O(k),
\]
w.p $1 - d^{-\Omega(1)}$ over the random matrix $W$. The result then follows from Theorem 4.1.

### H Misc: Arbitrary / nonhomogeneous activation functions

We now drop the homogeneity assumption on the activation function $\sigma$. In this scenario, we cannot carry out computations as in section H.2. Given a neural network $f = f_{W,v} \in \mathcal{F}_{d,k}(\sigma)$, analysing the Lipschitz constant $\text{Lip}_{\text{S}_{d-1}}(f)$ of a function, or even the lower-bound $\mathcal{G}(f)$ thereof, is difficult as the parameters $W$ and $v$ enter the definition of $f$ in a rather complex manner (due to the nonlinearity $\sigma$). Fortunately, the Poincaré inequality is there for the rescue: we can bound the later quantity via the variance of $f$, which leads to quadratic-form in $v$ by means of a kernel matrix generated by $W$ and $\sigma$. As we shall see, this will lead to the emergence of another kernel matrix $C'_{\Phi}(W) := \text{Cov}_{x \sim \tau_d}(\sqrt{d}Wx)$ which will take over the role of $C_{\Phi}(W)$ introduced in (61).

#### H.1 Poincaré inequality on the sphere and the emergence of another kernel matrix

Recall that, for uniform-distribution $\tau_d$ on the unit-sphere $S_{d-1}$ (assumed in the definition of generic data), the Poincaré inequality tells us that, for any continuously-differentiable function $f : \mathbb{R}^d \to \mathbb{R},$
\[
\mathcal{G}(f)^2 \geq c \cdot (d - 1) \text{var}_\tau_d(f) \geq d \cdot \text{var}_\tau_d(f),
\]
where $c > 0$ is an absolute constant (with a concrete value like 1 or 2, independent of the dimension $d$ and the test function $f$), and $\text{var}_\tau_d(f) := \|f - \mathbb{E}_\tau_d[f]\|_2^2(\tau_d)$ is the variance of $f$, with $\mathbb{E}_\tau_d[f] = \mathbb{E}_{x \sim \tau_d}[f(x)] := \int_{S_{d-1}} f(x) d\tau_d(x)$ being the average value of $f$ w.r.t the measure $\tau_d$. The factor $d - 1$ in (48) is optimal; it is the (optimal) Poincaré constant for the uniform distribution $\tau_d$ on the unit-sphere $S_{d-1}$. We refer the reader to standard monographs on the subject, like Ledoux (1999); Boucheron et al. (2013); Gozlan et al. (2015).

Let $\Phi : S_{d-1} \to \mathbb{R}^m$ be a measurable function $a \in \mathbb{R}^m$, and consider a general linear model $f : S_{d-1} \to \mathbb{R}$ given by
\[
f(x) = a^\top \Phi(x).
\]
The vector $\Phi(x) \in \mathbb{R}^m$ are the features of the example $x$. Note that we allow for cases where the feature mapping $\Phi : S_{d-1} \to \mathbb{R}$ is learnable. This subsamples feed-forward linear neural networks, and in particular, the class $\mathcal{F}_{d,k}(\sigma)$ of two layer neural networks $f : x \mapsto v^\top \sigma(Wx)$ with activation function $\sigma$, by taking $a = v$ and $\Phi(x) = \sigma(Wx)$. Least squares estimators in general RKHSs are also an instance of (49). Let $\mu_{\Phi}$ and $C_{\Phi}$ be the mean and the covariance (resp.) w.r.t $x \sim \tau_d$ of the feature vector $\sqrt{d}\Phi(x)$, i.e
\[
\mu_{\Phi} := \mathbb{E}[\Phi(x)] \in \mathbb{R}^m, \text{ and } C_{\Phi} := \text{cov}_x(\sqrt{d}\Phi(x)) := \mathbb{E}_x[\sqrt{d}\Phi(x)\sqrt{d}\Phi(x)^\top] - \mu_{\Phi}\mu_{\Phi}^\top \in \mathbb{R}^{m \times m}.
\]

**Theorem H.1.** For any function $f : S_{d-1} \to \mathbb{R}$ of the form (49), we have the lower-bound
\[
\mathcal{G}(f)^2 \geq a^\top C_{\Phi} a \geq \|a\|^2 \lambda_{\min}(C_{\Phi}).
\]
This result will be heavily used in subsequent sections to analyze robustness analysis of random features and NTK regimes induced by general / non-homogeneous activation functions.
Proof of Theorem H.1. Using standard formulae for expectations of quadratic forms, one computes
\[
d \cdot \mathbb{E}_x[f(x)^2] = \mathbb{E}_x[(a^\top \sqrt{d}\Phi(x))^2] = \mathbb{E}_x[\sqrt{d}\Phi(x)^\top a a^\top \sqrt{d}\Phi(x)] = a^\top C_\Phi a + (a^\top \mu_\Phi)^2
\]
Thus, the variance of \( f \) w.r.t. to \( x \sim \tau_d \) is given by the following quadratic form in \( V \)
\[
d \cdot \text{var}_x(f(x)) := d \cdot \mathbb{E}_x[f(x)^2] - d \cdot (\mathbb{E}_x[f(x)])^2 = a^\top C_\Phi a.
\]
(52)
Combining with the Poincaré inequality (48), this proves the following template result linking the Lipschitz constant of \( f \) with the \( L_2 \)-norm w.r.t. the covariance matrix feature \( C_{\Phi} \), of the parameter vector \( a \in \mathbb{R}^m \).

H.2 Spectral analysis of \( C_{\Phi} \), for embeddings of the form \( \Phi(x) = \varphi(Vx) \)

Suppose the embedding function \( \Phi : S_{d-1} \rightarrow \mathbb{R}^m \) is of the form
\[
\Phi(x) = \varphi(Vx) := (\varphi(x^\top v_1), \ldots, (\varphi(x^\top v_m)),
\]
for some continuous scalar function \( \varphi : [-1, 1] \rightarrow \mathbb{R} \) and \( m \times d \) matrix \( V \) with rows \( v_1, \ldots, v_m \in \mathbb{R}^d \). This is the case of exact two-layer neural networks where \( \varphi = \sigma \) (the activation function), \( m = k \) (the number of hidden neurons), and \( V = W \in \mathbb{R}^{k \times d} \) (the hidden weights matrix).

In view of applying Theorem H.1 to get lower-bounds on the nonrobustness of the model \( f : S_{d-1} \rightarrow \mathbb{R} \), \( x \mapsto a^\top \Phi(x) = a^\top \varphi(Vx) \), one must lower-bound the smallest eigenvalue of \( C_{\Phi} \), the covariance matrix of the random vector \( \sqrt{d}\Phi(x) \in \mathbb{R}^m \), for \( x \sim \tau_d \). This is the purpose of the next theorem.

Theorem H.2 (Lower-bound on \( \lambda_{\text{min}}(C_{\Phi}) \)). Suppose \( \varphi \) is thrice continuously-differentiable at zero, with Maclaurin expansion \( \varphi(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + O(t^4) \). Then, we have
\[
(C_{\Phi})_{j,\ell} = (\overline{C}_{\Phi})_{j,\ell} - \frac{a_2^2}{d} + O(\frac{1}{d^3}),
\]
where \( \overline{C}_{\Phi} := c_d V V^\top + \tilde{c}_d V V^\top \circ V V^\top \in \mathbb{R}^{m \times m} \), and \( c_d \) and \( \tilde{c}_d \) are defined by
\[
c_d := a_1^2 + \frac{6a_1 a_3}{d}, \quad \tilde{c}_d := \frac{2a_2^2}{d}.
\]
(55)

For the proof of Theorem H.2, we will need the following lemma.

Lemma H.1 (Correlation functions of coordinates of uniform random vector on sphere). Suppose \( \varphi \) is thrice continuously-differentiable at zero. If \( \varphi(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + O(t^4) \) is its Maclaurin expansion, then for every \( u, v \in S_{d-1} \), and \( x \sim \tau_d \), we have the approximation
\[
\mathbb{E}_x[\varphi(x^\top u)\varphi(x^\top v)] - \mathbb{E}_x[\varphi(x^\top u)]\mathbb{E}_x[\varphi(x^\top v)] = -\frac{a_2^2}{d^2} + \left(\frac{a_1^2}{d} + \frac{6a_1 a_3}{d^2}\right)u^\top v + \frac{2a_2^2}{d^2}(u^\top v)^2 + O(\frac{1}{d^3}).
\]
(56)
In particular, if \( u \) and \( v \) are perpendicular, then
\[
\mathbb{E}_x[\varphi(x^\top u)\varphi(x^\top v)] - \mathbb{E}_x[\varphi(x^\top u)]\mathbb{E}_x[\varphi(x^\top v)] = -\frac{a_2^2}{d^2} + O(\frac{1}{d^3}).
\]
(57)

The proof of the lemma is given in Appendix I.

Lemma H.2 (Eigenvalues of perturbed matrix). If \( A \) and \( E \) are \( N \times N \) matrices with \( |e_{i,j}| \leq \varepsilon \) for all \( i, j \in [N] \), then
\[
\sup_{1 \leq i \leq N} |\tau_i(A + E) - \tau_i(A)| \leq N\varepsilon,
\]
where \( \tau_1(A) \geq \tau_2(A) \geq \ldots \geq \tau_N(A) \) are the singular-values of \( A \) (and similarly for \( A + E \)).

Proof. Its is well-known that \( \sup_{1 \leq i \leq N} |\tau_i(A + E) - \tau_i(A)| \leq \|E\| \). It then suffices to observe that \( \|E\|_{op} \leq \|E\|_F \leq \sqrt{N^2\varepsilon^2} = N\varepsilon \).
Proof of Theorem H.2. From Lemma H.1, we know that
\[
(C\Phi)_{j,\ell} = d \cdot (\mathbb{E}_x[\varphi(x^T v_j)\varphi(x^T v_\ell)] - \mathbb{E}_x[\varphi(x^T v_j)\mathbb{E}_x[\varphi(x^T v_\ell)])
\]
\[
= d \cdot (-\frac{a^2_\Phi}{d^2} + \frac{a_\Phi}{d} v_j^T v_\ell + \frac{3}{d} (v_j^T v_\ell)^2 + \mathcal{O}(\frac{1}{d^3})) = -\frac{a^2_\Phi}{d} + (C\Phi)_{j,\ell} + E_{j,\ell},
\]
where \(E := C\Phi - C\Phi - (a^2_\Phi/d)1_m 1_m^T\) with \(\|E\|_{op} = d \cdot \mathcal{O}(1/d^2) = \mathcal{O}(1/d)\), thanks to the above display and Lemma H.2. One then derives that
\[
\lambda_{min}(C\Phi) = \lambda_{min}(C\Phi + E - \frac{a^2_\Phi}{d}1_m 1_m^T) \geq \lambda_{min}(C\Phi - a^2_\Phi 1_m 1_m^T) = \|E\|_{op},
\]
where the first inequality is thanks to the **Cauchy-Weyl interlacing inequality** to compare the eigenvalues of psd matrices \(C\Phi + (a^2_\Phi/d)1_m 1_m^T\) and \(C\Phi + E\), the third is by definition of \(\|E\|_{op}\), and the last inequality uses the fact that \(VV^T \circ VV^T\) is psd (thanks to the **Shur product theorem**). This proves part (A) of the theorem. Part (B) is a direct consequence of Theorem H.1.

The following Corollary to Theorem H.2 will be crucial for our analysis of finite-width RF models and infinite-width RF / NTK models.

**Corollary H.1** (Theorem C.2 restated). Suppose \(\varphi\) is thrice continuously-differentiable at zero. Suppose the rows of \(V\) are drawn iid from an isotropic 1-subGaussian distribution in \(\mathbb{R}^d\). If \(m, d \to \infty\) such that \(m/d \to \gamma \in [0, 1)\), then \(\lambda_{min}(C\Phi) \to \varphi'(0)^2 (1 - \sqrt{\gamma})^2\) almost-surely.

**Proof.** Thanks to Bai-Yin, we know that \(\lambda_{min}(VV^T) \to (1 - \sqrt{\gamma})^2\) a.s. Invoking Theorem H.2 and the fact that finite-rank perturbations do not affect the limiting spectral distribution of random matrices, we deduce that \(\lambda_{min}(C\Phi) \to \varphi'(0)^2 (1 - \sqrt{\gamma})^2\) a.s as claimed.

**H.3 An analytic formula for \(G(f_{W,v})\)**

Suppose the activation function \(\sigma\) is positively-homogeneous of order 1. As an example, the reader may think of the ReLU or the absolute-value activation function. For any two-layer neural network \(f = f_{W,v} \in \mathcal{F}_{d,k}(\sigma)\), may compute the squared Sobolev-seminorm of \(f\) as follows
\[
G(f)^2 := \mathbb{E}_x[\|\nabla f(x)\|^2] = \mathbb{E}_x[\|\nabla f(x)\|^2 - \mathbb{E}_x(x^T \nabla f(x))^2]
\]
\[
= \mathbb{E}_x[\|\nabla f(x)\|^2 - \mathbb{E}_x[\|f(x)\|^2],
\]
where we have used the order-1 positive-homogeneity of the activation function \(\sigma\) in the last step (**Euler’s Theorem**). We now compute each term of the rightmost side separately.

**The first term.** Now, \(f(x) := v^T \sigma(Wx) = \sum_{j=1}^k v_j \sigma(x^T w_j)\) and so
\[
\mathbb{E}_x[\|\nabla f(x)\|^2 = \mathbb{E}_x[\sum_{j=1}^k v_j \sigma'(x^T w_j) w_j] = \mathbb{E}_x \sum_{j=1}^k v_j \sigma'(x^T w_j) i w_j w_j^T w_j = \sum_{j=1}^k \sum_{\ell=1}^k v_j v_\ell \mathbb{E}_x[\sigma'(x^T w_j) \sigma'(x^T w_\ell)] w_j w_\ell = \|v\|_{C_{\phi'}(W; W)}^2\]
\]
where \(C_{\phi'}(W)\) is the \(k \times k\) matrix with entries given by
\[
(C_{\phi'}(W))_{j,\ell} := \mathbb{E}_{x \sim \tau_{\gamma}}[\sigma'(x^T w_j) \sigma'(x^T w_\ell)] = \phi_{\sigma'}(w_j^T w_\ell) = \phi_{\sigma'}(w_j^T w_\ell),
\]
where \(\phi_{\sigma'} : [-1, 1] \to \mathbb{R}\) is the continuous function whose existence is guaranteed by Proposition F.1.

**The second term.** One computes \(\|f(x)\|^2 = \sum_{j=1}^k \sum_{\ell=1}^k v_j v_\ell \sigma(x^T w_j) \sigma(x^T w_\ell)\), and so
\[
\mathbb{E}_x[f(x)^2] = \sum_{j=1}^k \sum_{\ell=1}^k v_j v_\ell \mathbb{E}_x[\sigma(x^T w_j) \sigma(x^T w_\ell)] = \|v\|_{C_{\phi}(W)}^2.
\]

where $\tilde{C}_\sigma(W)$ is the $k \times k$ psd matrix with entries given by
\[
(\tilde{C}_\sigma(W))_{j,\ell} := \mathbb{E}_{x \sim \tau_d}[\sigma(x^\top w_j)\sigma(x^\top w_\ell)].
\] (60)

Let $G_\sigma(W)$ be the $k \times k$ psd matrix with entries given for all $j, \ell \in [k]$ by
\[
(G_\sigma(W))_{j,\ell} := (WW^\top \circ \tilde{C}_\sigma(W))_{j,\ell} - (\tilde{C}_\sigma(W))_{j,\ell} = \tilde{\phi}_\sigma(w_j^\top w_\ell),
\] (61)

where $\tilde{\phi}_\sigma : [-1, 1] \to \mathbb{R}$ is defined by $\tilde{\phi}_\sigma(t) := t\phi_\sigma(t) - \phi_\sigma(t)$. Putting things together, we obtain the following result which gives an analytic formula for the $\mathcal{S}(f_{W,v})$ as a quadratic form in $v$, with coefficient matrix $G_\sigma(W)$. Thanks to (Louart et al., 2018, Table 1), we obtain Table 2 below which summaries the Maclaurin expansion of $\tilde{\phi}_\sigma$ for a certain number of common activation functions.

| $\sigma$ | $\tilde{\phi}_\sigma(t) := t\phi_\sigma(t) - \phi_\sigma(t)$ | Maclaurin expansion of $\tilde{\phi}_\sigma$ |
| --- | --- | --- |
| ReLU | $\frac{t \arccos(-t)}{2\pi} - \frac{t \arccos(-t) + \sqrt{1-t^2}}{2\pi d}$ | $\frac{1}{4}t + \frac{1}{2}t^2 + O(t^3, \frac{1}{d})$ |
| abs | $\frac{2t \arcsin(t)}{\pi} - \frac{2t \arcsin(t) + 2\sqrt{1-t^2}}{\pi d}$ | $\frac{2}{\pi}t^2 + O(t^3, \frac{1}{d})$ |
| erf | $\frac{4t}{\pi \sqrt{9-4t^2}} - \frac{2}{\pi d} \arcsin(2t/3)$ | $\frac{4}{3\pi}t + O(t^3, \frac{1}{d})$ |

Table 2: Table of the function $\tilde{\phi}_\sigma$ defined in (61), for the ReLU and absolute-value activation functions.

**Theorem H.3** (Analytic formula for Sobolev norm of two-layer neural network). For any $f = f_{W,v} \in \mathcal{F}_{d,k}(\sigma)$, we have the identity $\mathcal{S}(f) = v^\top G_\sigma(W)v$.

Thanks to the definition of extreme singular-values of matrices, we know that
\[
\lambda_{\min}(G_\sigma(W))||v||^2 \leq v^\top G_\sigma(W)v \leq \lambda_{\max}(G_\sigma(W))||v||^2.
\] (62)

Thus, in virtue of Theorem H.3, to get lower- and upper-bounds for $\mathcal{S}(f_{W,v})$, it suffices to

- control the extreme singular-values of $G_\sigma(W)$, and
- control the $L_2$-norm of the output weights vector $||v||$.

**Theorem H.4.** Let the activation function $\sigma$ be the ReLU and let the rows of the hidden weights matrix $W \in \mathbb{R}^{k \times d}$ be drawn iid from $\tau_d$. For sufficiently large $k \gg d$, it holds w.p $1 - d^{-\Omega(1)}$ that
\[
||G_\sigma(W) - \frac{1}{4}(WW^\top + I_k)||_{op} = o(1).
\] (63)

In particular, there exist constants $C \geq c \geq 1/4$ (only depending on the ratio $k/d$) such that $c \leq \lambda_{\min}(G_\sigma(W)) \leq \lambda_{\max}(G_\sigma(W)) \leq C$ w.p $1 - d^{-\Omega(1)}$.

**Proof.** The first part is completely analogous to the proof of Lemma F.3, with $\phi_\sigma$ replaced with $\tilde{\phi}_\sigma$. We also make use of Table 2 for the computations for extracting the Maclaurin coefficients of $\tilde{\phi}_\sigma$. The second part follows from standard RMT Vershynin (2012).

**Theorem H.5.** Let the activation function $\sigma$ be the ReLU, absolute-value, gaussian rf, or tanh, and let the rows of the hidden weights matrix $W \in \mathbb{R}^{k \times d}$ be drawn iid from $\tau_d$. Then, there exist constants $c, C > 0$ (only depending on the ratio $k/d$) such that $c \leq \lambda_{\min}(G_\sigma(W)) \leq \lambda_{\max}(G_\sigma(W)) \leq C$ w.p $1 - d^{-\Omega(1)}$.

**Theorem H.6.** Let $\sigma$ and $W$ be as in Theorem H.5. Then, for sufficiently large $k \asymp d$ it holds w.p $1 - d^{-\Omega(1)}$ over $W$ that $\mathcal{S}(f_{W,v}) \asymp ||v||$, for every $f_{W,v} \in \mathcal{F}_W(\sigma)$.

Thus, any lower / upper-bound on the output weights of a neural network in $\mathcal{F}_W$ immediate translate to a comparable lower / upper-bound on $\mathcal{S}(f)$.

**Proof.** Proof follows from combining Theorems H.3 and H.5.
I Technical proofs

I.1 Proof of Lemma H.1

We will need the following auxiliary lemma proved further below.

**Lemma I.1.** Let \( u \) and \( v \) be fixed and \( x \) be uniformly random on the unit-sphere \( S_{d-1} \). Let \( p \) and \( q \) be nonnegative integers and define \( c_{p,q}(u,v) := \mathbb{E}_x[(x^\top u)^p(x^\top v)^q] \). If \( p \) and \( q \) have different parities, then \( c_{p,q}(u,v) = 0 \). Otherwise, we have the formula

\[
c_{p,q}(u,v) = \frac{p!q!\Gamma\left(\frac{d}{2}\right)}{2^{p+q}\Gamma\left(d+\frac{p+q}{2}\right)} \sum_t \frac{2^t}{t!(p-1)!(q-1)!} (u^\top v)^t,
\]

where the sum is over all \( t \) between 0 and \( p \wedge q \) inclusive, that have the same parity as \( p \) and \( q \). The formula is simplified in the table below for special values of \( p \) and \( q \).

| \( p \) | \( q \) | \( c_{p,q}(u,v) \) | Comment |
|-------|-------|-----------------|---------|
| 2m    | 0     | \( \frac{C_m}{d_m} \) | \( C_m > 0 \) only depends on \( m \) |
| odd   | 1     | \( \frac{u^\top v}{d} \) | Opposite parity |
| 2     | 2     | \( \frac{(u^\top v)^2}{d(d+2)} \) | |
| 1     | 3     | \( \frac{3u^\top v}{d(d+2)} \) | |

Table 3: Table of formulae for \( c_{p,q}(u,v) := \mathbb{E}_x[(x^\top u)^p(x^\top v)^q] \), where \( u, v \in S_{d-1} \) (fixed) and \( x \) is uniformly random over the unit-sphere \( S_{d-1} \). This table is a direct application of Lemma I.1. Note that \( c_{p,q}(u,v) \) is symmetric in \( (p,q) \) and in \( (u,v) \).

**Proof of Lemma H.1.** WLOG, assume \( a_0 = 0 \). Thanks to Lemma I.1, one may compute

\[
(\mathbb{E}_x[h(x^\top u)])^2 = \left( \frac{a_2}{d} + \mathcal{O}\left(\frac{1}{d^2}\right) \right)^2 = \frac{a_2^2}{d^2} + \mathcal{O}\left(\frac{1}{d^3}\right),
\]

and similarly

\[
\mathbb{E}_x[h(x^\top u)h(x^\top v)] = \sum_{p=0}^3 \sum_{q=0}^3 a_p a_q \mathbb{E}_x[(x^\top u)^p(x^\top v)^q] + \mathcal{O}\left(\frac{1}{d^3}\right)
\]

\[
= \frac{a_2^2}{d} u^\top v + \frac{2a_2^2}{d(d+2)} (u^\top v)^2 + \frac{6a_1 a_3}{d(d+2)} u^\top v + \mathcal{O}\left(\frac{1}{d^3}\right)
\]

\[
= \left( \frac{a_2^2}{d} + \frac{6a_1 a_3}{d^2} \right) u^\top v + \frac{2a_2^2}{d^2} (u^\top v)^2 + \mathcal{O}\left(\frac{1}{d^3}\right),
\]

and the claim follows after subtracting the previous display. □

**Proof of Lemma I.1.** Let \( z \sim \mathcal{N}(0,I_d) \) and \( x \sim \tau_d \). Then by using (hyper)spherical coordinates

\[
\mathbb{E}_z[(z^\top u)^p(z^\top v)^q] = (2\pi)^{-\frac{d}{2}} \int_0^\infty dr \text{vol}(S^{d-1}) r^{d-1+p+q} e^{-r^2} \mathbb{E}_z[(x^\top u)^p(x^\top v)^q]
\]

\[
= \frac{2^\frac{d+p+q}{2} \Gamma\left(\frac{p+q+d}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} \mathbb{E}_x[(x^\top u)^p(x^\top v)^q]
\]

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We need $p + q$ even for a non-zero result, so we assume that. The Gaussian expectation $\mathbb{E}_x[(z^\top u)^p(z^\top v)^q]$ can be computed with the Isserlis-Wick Theorem. It amounts to a sum over complete matchings of a set with $p + q$ elements subdivided into two compartments, one of size $p$ and one of size $q$. Let me organize the count according to $k$, the number of matched pairs joining the two compartments. We then get

$$
\mathbb{E}_x[(z^\top u)^p(z^\top v)^q] = \sum_k \frac{p!}{2^p \Gamma \left( \frac{p}{2} \right)} \frac{q!}{2^q \Gamma \left( \frac{q}{2} \right)} \frac{(p-t)!}{2^{p-t} \Gamma \left( \frac{p-t}{2} \right)} \frac{(q-t)!}{2^{q-t} \Gamma \left( \frac{q-t}{2} \right)} (u^\top u)^t (v^\top v)^{q-t} 
$$

$$
= \sum_k \frac{p!}{2^p \Gamma \left( \frac{p}{2} \right)} \frac{q!}{2^q \Gamma \left( \frac{q}{2} \right)} \frac{(p-t)!}{2^{p-t} \Gamma \left( \frac{p-t}{2} \right)} \frac{(q-t)!}{2^{q-t} \Gamma \left( \frac{q-t}{2} \right)} (u^\top v)^t 
$$

where the sum is over $0 \leq t \leq \min(p, q)$ of same parity as $p$ and $q$.

Finally, after some cleanup,

$$
\mathbb{E}_x[(x^\top u)^p(x^\top v)^q] = \frac{p! q!}{2^{p+q} \Gamma \left( \frac{p+q}{2} \right)} \sum_t \frac{2^t}{t!} \left( \frac{2^t}{t!} \right) (u^\top v)^t 
$$

with the same range of summation for $t$.

\[\square\]

### 1.2 Covariance matrix of outer product of independent random vectors

**Lemma 1.2** (Covariance of outer product of independent random vectors). If $z$ and $x$ are independent random vectors, at least one of which has zero mean, then $\text{Cov}(\text{vec}(z \otimes x)) = E[z(z^\top) \otimes E[xx^\top]]$.

**Proof.** Let $r$ and $s$ be the dimensionality of $z$ and $x$ be the dimensionality of $x$. Every index $I \in [rs]$ can be identified with a pair $(i, j) \in [r] \times [s]$ of indices in an obvious way so that $(z x^\top)_I = z_i x_j$. For $I, I' \in [rs]$, compute the $(I, I')$th entry of the covariance matrix of $z \otimes x \in \mathbb{R}^{rs}$ as

$$
\mathbb{E}[(zz^\top)_I (zz^\top)_I'] = \mathbb{E}[(zz^\top)_I] = \mathbb{E}[z_i z_j x_i x_j] = \mathbb{E}[z_i] \mathbb{E}[z_j] \mathbb{E}[x_i] \mathbb{E}[x_j] = \mathbb{E}[z_i z_j] \mathbb{E}[x_i x_j] = (\mathbb{E}[z z^\top] \otimes \mathbb{E}[xx^\top])_I I',
$$

where in the last but one step, we have used the fact that on of $\mathbb{E}[z_i]$ and $\mathbb{E}[x_j]$ equals zero. We conclude that $\text{Cov}(z \otimes x) = \mathbb{E}[z z^\top] \otimes \mathbb{E}[xx^\top]$ as claimed.

\[\square\]

### J Alternative proof of Theorem B.1 (removing the hidden log-factors)

Let $K : S^{d-1} \times S^{d-1} \rightarrow \mathbb{R}$ be a Mercer kernel and $\mathcal{H}_K$ be the induced RKHS. We are interested in lower-bounding the RKHS norm of functions in $\mathcal{H}_K$, which memorize the generic dataset $D_n$. To this end, define the random variable $\eta_K(n, r) \geq 0$ by

$$
\eta_K(n, r) := \inf_{f \in B_{K(r)}} \frac{1}{n} \| f(X) - y \|^2.
$$

By the generalized representer theorem (see Proposition 2.1), every minimizer in the above problem is an element of the representor subspace $\mathcal{K}(X) := \{ f_c : x \mapsto \sum_{i=1}^n c_i K(x, \cdot) \mid c \in \mathbb{R}^n \} \subseteq \mathcal{H}_K$.

Recall that the RKHS norm of every $f_c \in \text{span}_K(X)$ writes $\| f_c \|_{\mathcal{H}_K} = \sqrt{c^\top G c} \leq \| G \|_{op} \| c \|^2$, where $G := K(X, X) \in \mathbb{R}^{n \times n}$ is the kernel gram matrix. Let $\zeta := \sqrt{\| w_0 \|^2 / n + \zeta^2} \geq \zeta$, where $w_0$ and $\zeta$ are as in the noisy linear data generating process (2). One computes

$$
\inf_{\| c \| \leq \zeta} \frac{1}{n} \| f_c(X) - y \|^2 = \inf_{\| c \| \leq \zeta} \frac{1}{n} \| G c - y \|^2 = \inf_{\| c \| \leq \zeta} \frac{1}{n} \| G c - y \|^2 \geq (i) \left( \frac{\| y \|}{\sqrt{n}} - \frac{\| G \|_{op}}{\sqrt{n}} \right)^2 + (ii) \left( \zeta - \frac{\| G \|_{op}}{\sqrt{n}} \right)^2,
$$

where $(i)$ is an application of Lemma J.1 and $(ii)$ is thanks to the Law of Large Numbers (and the convergence is in probability). Thus, if $f_c \in \text{span}_K(X)$ memorizes $D_n$ then

$$
\| c \| \geq \Omega \left( \frac{\sqrt{n}}{\| G \|_{op}} \right).
$$

(66)
Thanks to (30), if we can control \( \|G\|_{op} \) in (66), then we’d immediately get lower-bound on the Sobolev-seminorm \( \mathcal{S}(f_c) \) of any memorizer \( f_c \in \text{span}_K(X) \). The name of the game is then to upper-bound \( \|G\|_{op} \), the operator norm of the kernel gram matrix \( G \). Below, we sketch a number of examples where this can be done without difficulty.

Thanks to (66), the name of the game is then to upper-bound the operator norm of the kernel gram matrix \( G \). We sketch a number of examples where this can be done without difficulty.

**Infinite-width RF and NTK.** As an example, in the case of infinite-width RF or NTK with \( d \approx n \to \infty \), we know that \( \|G\|_{op} = \mathcal{O}(1) \) w.p 1 - \( d^{-\Omega(1)} \). Thus, w.p 1 - \( d^{-\Omega(1)} \), every \( f = f_c \in \text{span}_K(X) \) which memorizes \( D_n \) must verify \( \|f\|_{\mathcal{H}_K}, \|c\| \geq \Omega(\sqrt{n}) \). Accordingly, this would remove all log-factors from the lower-bound in Theorem 3.2.

**Ordinary linear models.** Here, the gram matrix is \( G = XX^\top \) and for \( n \approx d \to \infty \), one has \( \|G\|_{op} = \mathcal{O}(1) \). Accordingly, this would remove all the log-factors from the lower-bound in Theorem 2.1.

**Finite-width RF with proportionate scaling** (18). It is a classical result (e.g. see Pennington and Worah (2017)) that \( \|G\|_{op} = \mathcal{O}(1) \) in this scenario. Accordingly, this would remove all the log-factors from the lower-bound in Theorem 5.1.

### J.1 A useful lemma

**Lemma J.1.** Let \( A : \mathcal{H}_1 \to \mathcal{H}_2 \) be a compact operator between Hilbert spaces and let \( b \in \mathcal{H}_2 \). We have the following inequalities

\[
\left( \|b\| - \|A\|_{op} \right)_+ \leq \inf_{u \in \mathcal{H}_1, \|u\| \leq 1} \|Au - b\| \leq \left( \|b\| - \|A^\top\|_{\text{min}} \right)_+ \leq \|b\|,
\]

(67)

where \( \|A^\top\|_{\text{min}} \) is the infimum of the singular-values of the adjoint operator \( A^\top \).

**Proof.** Let \( B_j \) be the unit-ball of \( \mathcal{H}_j \). By duality of norms, one has

\[
\inf_{u \in B_1} \|Au - b\| = \inf_{u \in B_1} \sup_{v \in B_2} \langle v, b - Au \rangle = \sup_{v \in B_2} \langle v, b \rangle - \inf_{v \in B_2} \langle u, A^\top v \rangle
\]

\[
= \sup_{v \in B_2} \langle v, b \rangle - \|A^\top v\|
\]

The result follows by noting that

- \( \|A^\top\|_{\text{min}}\|u\| \leq \|A^\top v\| \leq \|A\|_{op}\|u\| \), and
- \( \sup_{v \in B_2} \langle v, b \rangle - r\|v\| = (\|b\| - r)_+ \) for any \( r \in \mathbb{R} \). To see this, note that the optimal \( v \in B_2 \) must point in the same direction as \( b \). Now, set \( v = (R/\|b\|)b \) and optimize over \( R \in [0, 1] \).

\( \square \)