CRPO: A New Approach for Safe Reinforcement Learning with Convergence Guarantee

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Abstract

In safe reinforcement learning (SRL) problems, an agent explores the environment to maximize an expected total reward and meanwhile avoids violation of certain constraints on a number of expected total costs. In general, such SRL problems have nonconvex objective functions subject to multiple nonconvex constraints, and hence are very challenging to solve, particularly to provide a globally optimal policy. Many popular SRL algorithms adopt a primal-dual structure which utilizes the updating of dual variables for satisfying the constraints. In contrast, we propose a primal approach, called constraint-rectified policy optimization (CRPO), which updates the policy alternatingly between objective improvement and constraint satisfaction. CRPO provides a primal-type algorithmic framework to solve SRL problems, where each policy update can take any variant of policy optimization step. To demonstrate the theoretical performance of CRPO, we adopt natural policy gradient (NPG) for each policy update step and show that CRPO achieves an $O(1/\sqrt{T})$ convergence rate to the global optimal policy in the constrained policy set and an $O(1/\sqrt{T})$ error bound on constraint satisfaction. This is the first finite-time analysis of primal SRL algorithms with global optimality guarantee. Our empirical results demonstrate that CRPO can outperform the existing primal-dual baseline algorithms significantly.

1. Introduction

Reinforcement learning (RL) has achieved great success in solving complex sequential decision-making and control problems such as Go (Silver et al., 2017), StarCraft (DeepMind, 2019) and recommendation system (Zheng et al., 2018), etc. In these settings, the agent is allowed to explore the entire state and action space to maximize the expected total reward. However, in safe RL (SRL), in addition to maximizing the reward, an agent needs to satisfy certain constraints. Examples include self-driving cars (Fisac et al., 2018), cellular network (Julian et al., 2002), and robot control (Levine et al., 2016). The global optimal policy in SRL is the one that maximizes the reward and at the same time satisfies the cost constraints.

The current safe RL algorithms can be generally categorized into the primal and primal-dual approaches. The primal-dual approaches (Tessler et al., 2018; Ding et al., 2020a; Stooke et al., 2020; Yu et al., 2019; Achiam et al., 2017; Yang et al., 2019a; Altman, 1999; Borkar, 2005; Bhatnagar & Lakshmanan, 2012; Liang et al., 2018; Paternain et al., 2019a) are most commonly used, which convert the constrained problem into an unconstrained one by augmenting the objective with a sum of constraints weighted by their corresponding Lagrange multipliers (i.e., dual variables). Generally, primal-dual algorithms apply a certain policy optimization update such as policy gradient alternatively with a gradient descent type update for the dual variables. Theoretically, (Tessler et al., 2018) has provided an asymptotic convergence analysis for primal-dual method and established a local convergence guarantee. (Paternain et al., 2019b) showed that the primal-dual method achieves zero duality gap. Recently, (Ding et al., 2020a) proposed a primal-dual type proximal policy optimization (PPO) and established the regret bound for linear constrained MDP. The convergence rate of primal-dual method based on a natural policy gradient algorithm was characterized in (Ding et al., 2020b).

The primal type of approaches (Liu et al., 2019b; Chow et al., 2018; 2019; Dalal et al., 2018a) enforce constraints via various designs of the objective function or the update process without an introduction of dual variables. The primal algorithms are much less studied than the primal-dual approach. Notably, (Liu et al., 2019b) developed an interior point method, which applies logarithmic barrier functions for SRL. (Chow et al., 2018; 2019) leveraged Lyapunov functions to handle constraints. (Dalal et al., 2018a) introduced a safety layer to the policy network to enforce

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A New Algorithm: We propose a novel primal approach called Constraint-Rectified Policy Optimization (CRPO) for SRL, where all updates are taken in the primal domain. CRPO applies unconstrained policy maximization update w.r.t. the reward on the one hand, and if any constraint is violated, momentarily rectifies the policy back to the constraint set along the descent direction of the violated constraint also by applying unconstrained policy minimization update w.r.t. the constraint function. From the implementation perspective, CRPO can be implemented as easy as unconstrained policy optimization algorithms. Without introduction of dual variables, it does not suffer from hyperparameter tuning of the learning rates to which the dual variables are sensitive, nor does it require initialization to be feasible. Further, CRPO involves only policy gradient descent for both objective and constraints, whereas the primal-dual approach typically requires projected gradient descent, where the projection causes higher complexity to implementation as well as hyperparameter tuning due to the projection thresholds.

To further explain the advantage of CRPO over the primal-dual approach, CRPO features immediate switches between optimizing the objective and reducing the constraints whenever constraints are violated. However, the primal-dual approach can respond much slower because the control is based on dual variables. If a dual variable is nonzero, then the policy update will descend along the corresponding constraint function. As a result, even if a constraint is already satisfied, there can often be a significant delay for the dual variable to iteratively reduce to zero to release the constraint, which slows down the algorithm. Our experiments in Section 5 validates such a performance advantage of CRPO over the primal-dual approach.

Theoretical Guarantee: To provide the theoretical guarantee for CRPO, we adopt NPG as a representative policy optimizer and investigate the convergence of CRPO in two settings: tabular and function approximation, where in the function approximation setting the state space can be infinite. For both settings, we show that CRPO converges to a global optimum at a convergence rate of $O(1/\sqrt{T})$. Furthermore, the constraint violation also converges to zero at a rate of $O(1/\sqrt{T})$. To the best of our knowledge, we establish the first provably global optimality guarantee for a primal SRL algorithm of CRPO.

In this paper, we will provide the affirmative answers to the above questions, thus establishing appealing advantages of the primal approach for SRL.

1.1. Main Contributions

A New Algorithm: We propose a novel primal approach called Constraint-Rectified Policy Optimization (CRPO) for SRL, where all updates are taken in the primal domain. CRPO applies unconstrained policy maximization update w.r.t. the reward on the one hand, and if any constraint is violated, momentarily rectifies the policy back to the constraint set along the descent direction of the violated constraint also by applying unconstrained policy minimization update w.r.t. the constraint function. From the implementation perspective, CRPO can be implemented as easy as unconstrained policy optimization algorithms. Without introduction of dual variables, it does not suffer from hyperparameter tuning of the learning rates to which the dual variables are sensitive, nor does it require initialization to be feasible. Further, CRPO involves only policy gradient descent for both objective and constraints, whereas the primal-dual approach typically requires projected gradient descent, where the projection causes higher complexity to implementation as well as hyperparameter tuning due to the projection thresholds.

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1.2. Related Work

Safe RL: Algorithms based on primal-dual methods have been widely adopted for solving constrained RL problems, such as PDO (Chow et al., 2017), RCPO (Tessler et al., 2018), OPDOP (Ding et al., 2020a) and CPPO (Stooke et al., 2020). Constrained policy optimization (CPO) (Achiam et al., 2017) extends TRPO to handle constraints, and is later modified with a two-step projection method (Yang et al., 2019a). The effectiveness of primal-dual methods is justi-
A discounted Markov decision process (MDP) is a tuple $(S, A, c_0, P, \xi, \gamma)$, where $S$ and $A$ are state and action spaces; $c_0 : S \times A \times S \rightarrow \mathbb{R}$ is the reward function; $P : S \times A \times S \rightarrow [0,1]$ is the transition kernel, with $P(s'|s,a)$ denoting the probability of transitioning to state $s'$ from previous state $s$ given action $a$; $\xi : S \rightarrow [0,1]$ is the initial state distribution; and $\gamma \in (0,1)$ is the discount factor. A policy $\pi : S \rightarrow \mathcal{P}(A)$ is a mapping from the state space to the space of probability distributions over the actions, with $\pi(\cdot|s)$ denoting the probability of selecting action $a$ in state $s$. When the associated Markov chain $P(s'|s) = \sum_A P(s'|s,a)\pi(a|s)$ is ergodic, we denote $\mu_\pi$ as the stationary distribution of this MDP, i.e. $\int P(s'|s)\mu_\pi(ds) = \mu_\pi(s')$. Moreover, we define the visitation measure induced by the policy $\pi$ as $\nu_\pi(s,a) = (1-\gamma)\sum_{t=0}^{\infty} \gamma^t P(s_t = s, a_t = a)$.

For a given policy $\pi$, we define the state value function as $V^\pi_\nu(s) = \mathbb{E}[\sum_{t=0}^{\infty} \gamma^t c_0(s_t, a_t, s_{t+1})|s_0 = s, \pi]$, the state-action value function as $Q^\pi_\nu(s,a) = \mathbb{E}[\sum_{t=0}^{\infty} \gamma^t c_0(s_t, a_t, s_{t+1})|s_0 = s, a_0 = a, \pi]$, and the advantage function as $A^\pi_\nu(s,a) = Q^\pi_\nu(s,a) - V^\pi_\nu(s)$. In reinforcement learning, we aim to find an optimal policy that maximizes the expected total reward function defined as $J_0(\pi) = \mathbb{E}[\sum_{t=0}^{\infty} \gamma^t c_0(s_t, a_t, s_{t+1})] = \mathbb{E}_\pi[V^\pi_\nu(s)] = \mathbb{E}_\pi[Q^\pi_\nu(s,a)]$.

2.2. Safe Reinforcement Learning (SRL) Problem

The SRL problem is formulated as an MDP with additional constraints that restrict the set of allowable policies. Specifically, when taking action at some state, the agent can incur a number of costs denoted by $c_1, \cdots, c_p$, where each cost function $c_i : S \times A \times S \rightarrow \mathbb{R}$ maps a tuple $(s,a,s')$ to a cost value. Let $J_i(\pi)$ denotes the expected total cost function with respect to $c_i$ as $J_i(\pi) = \mathbb{E}[\sum_{t=0}^{\infty} \gamma^t c_i(s_t, a_t, s_{t+1})]$. The goal of the agent in SRL is to solve the following constrained problem

$$\max_{\pi} J_0(\pi), \text{ s.t. } J_i(\pi) \leq d_i, \forall i = 1, \cdots, p, \quad (1)$$

where $d_i$ is a fixed limit for the $i$-th constraint. We denote the set of feasible policies as $\Omega_C \equiv \{\pi : \forall i, J_i(\pi) \leq d_i\}$, and define the optimal policy for SRL as $\pi^* = \text{arg min}_{\pi \in \Omega_C} J_0(\pi)$. For each cost $c_i$, we define its corresponding state value function $V^i_\nu$, state-action value function $Q^i_\nu$, and advantage function $A^i_\nu$ analogously to $V^\pi_\nu$, $Q^\pi_\nu$, and $A^\pi_\nu$, with $c_i$ replacing $c_0$, respectively.

2.3. Policy Parameterization and Policy Gradient

In practice, a convenient way to solve the problem eq. (1) is to parameterize the policy and then optimize the policy over the parameter space. Let $\{\pi_w : S \rightarrow \mathcal{P}(A)|w \in \mathcal{W}\}$ be a parameterized policy class, where $\mathcal{W}$ is the parameter space. Then, the problem in eq. (1) becomes

$$\max_{w \in \mathcal{W}} J_0(\pi_w), \text{ s.t. } J_i(\pi_w) \leq d_i, \forall i = 1, \cdots, p. \quad (2)$$

The policy gradient of the function $J_i(\pi_w)$ has been derived by (Sutton et al., 2000) as $\nabla J_i(\pi_w) = \mathbb{E}(Q^i_w(s,a)\phi_w(s,a))$, where $\phi_w(s,a) := \nabla_w \log \pi_w(a|s)$ is the score function. Furthermore, the natural policy gradient was defined by (Kakade, 2002) as $\Delta_i(w) = F(w)^T \nabla J_i(\pi_w)$, where $F(w)$ is the Fisher information matrix defined as $F(w) = \mathbb{E}_{\nu_{\pi(w)}}[\phi_w(s,a)\phi_w(s,a)^T]$. 
Algorithm 1 Constraint-Rectified Policy Optimization (CRPO)

1: Initialize: initial parameter \( w_0 \), empty set \( N_0 \)
2: for \( t = 0, \ldots, T - 1 \) do
3:   Policy evaluation under \( \pi_{w_t} \): 
4:   Sample \( (s_j, a_j) \in B_t \sim \xi \cdot \pi_{w_t} \), compute constraint estimation \( \hat{J}_{i, B_t} = \sum_{j \in B_t} \rho_{j, t} Q^i_j(s_j, a_j) \) for \( i = 0, \ldots, p \), \( \rho_{j, t} \) is the weight
5:   if \( J_{i, B_t} \leq d_i + \eta \) for all \( i = 1, \ldots, p \), then
6:     Add \( w_t \) into set \( N_0 \)
7:     Take one-step policy update towards maximize \( J_0(w_{t+1}) \): 
8:       \( w_t \rightarrow w_{t+1} \)
9:   end if
10: end for
11: Output: \( w_{\text{out}} \) uniformly chosen from \( N_0 \)

3. Constraint-Rectified Policy Optimization (CRPO) Algorithm

In this section, we propose the CRPO approach (see Algorithm 1) for solving the SRL problem in eq. (2). The idea of CRPO lies in updating the policy to maximize the unconstrained objective function \( J_0(\pi_{w_t}) \) of the reward, alternatingly with rectifying the policy to reduce a constraint function \( J_i(\pi_{w_t}) \) (\( i \geq 1 \)) (along the descent direction of this constraint) if it is violated. Each iteration of CRPO consists of the following three steps.

Policy Evaluation: At the beginning of each iteration, we estimate the state-action value function \( Q^i(\pi_{w_t}) \approx Q^i_{\pi_{w_t}}(s, a) \) for both reward and costs under current policy \( \pi_{w_t} \).

Constraint Estimation: After obtaining \( Q^i_{\pi_{w_t}} \), the constraint function \( J_i(w_t) = \mathbb{E}_{\xi \cdot \pi_{w_t}}[Q^i_{w_t}(s, a)] \) can then be approximated via a weighted sum of approximated state-action value function: 

\[
J_{i, B_t} = \sum_{j \in B_t} \rho_{j, t} Q^i_j(s_j, a_j).
\]

Note this step does not take additional sampling cost, as the generation of samples \( (s_j, a_j) \in B_t \) from distribution \( \xi \cdot \pi_{w_t} \) does not require the agent to interact with the environment.

Policy Optimization: We then check whether there exists an \( i_t \in \{1, \ldots, p\} \) such that the approximated constraint \( J_{i_t, B_t} \) violates the condition \( J_{i_t, B_t} \leq d_i + \eta \), where \( \eta \) is the tolerance. If so, we take one-step update of the policy towards minimizing the corresponding constraint function \( J_{i_t}(\pi_{w_t}) \) to enforce the constraint. If multiple constraints are violated, we can choose to minimize any one of them. If all constraints are satisfied, we take one-step update of the policy towards maximizing the objective function \( J_0(\pi_{w_t}) \).

To apply CRPO in practice, we can use any policy optimization update such as natural policy gradient (NPG) (Kakade, 2002), trust region policy optimization (TRPO) (Schulman et al., 2015), proximal policy optimization (PPO) (Schulman et al., 2017), ACKTR (Wu et al., 2017), DDPG (Lillicrap et al., 2015) and SAC (Haarnoja et al., 2018), etc, in the policy optimization step (line 7 and line 10).

The advantage of CRPO over the primal-dual approach can be readily seen from its design. CRPO features immediate switches between optimizing the objective and reducing the constraints whenever they are violated. However, the primal-dual approach can respond much slower because the control is based on dual variables. If a dual variable is nonzero, then the policy update will descend along the corresponding constraint function. As a result, even if a constraint is already satisfied, there can still be a delay (sometimes a significant delay) for the dual variable to iteratively reduce to zero to release the constraint, which yields unnecessary sampling cost and slows down the algorithm. Our experiments in Section 5 validates such a performance advantage of CRPO over the primal-dual approach.

From the implementation perspective, CRPO can be implemented as easy as unconstrained policy optimization such as unconstrained policy gradient algorithms, whereas the primal-dual approach typically requires the projected gradient descent to update the dual variables, which is more complex to implement. Further, without introduction of the dual variables, CRPO does not suffer from hyperparameter tuning of the learning rates and projection threshold of the dual variables, whereas the primal-dual approach can be very sensitive to these hyperparameters. Nor does CRPO require initialization to be feasible, whereas the primal-dual approach can suffer significantly from bad initialization. We also empirically verify that the performance of CRPO is robust to the value of \( \eta \) over a wide range, which does not cause additional tuning effort compared to unconstrained algorithms. More discussions can be referred to Section 5.

CRPO algorithm is inspired by, yet very different from the cooperative stochastic approximation (CSA) method (Lan & Zhou, 2016) in optimization literature. First, CSA is designed for convex optimization subject to convex constraint, and is not readily capable of handling the more challenging SRL problems eq. (2), which are nonconvex optimization subject to nonconvex constraints. Second, CSA is designed to handle only a single constraint, whereas CRPO can handle multiple constraints with guaranteed constraint satisfaction and global optimality. Thus, the finite-time analysis for CSA and CRPO feature different approaches due to the aforementioned differences in their designs.

4. Convergence Analysis of CRPO

In this section, we take NPG as a representative optimizer in CRPO, and establish the global convergence rate of CRPO.
in both the tabular and function approximation settings. Note that TRPO and ACKTR update can be viewed as the NPG approach with adaptive stepsize. Thus, the convergence we establish for NPG implies similar results for CRPO that takes TRPO or ACKTR as the optimizer.

4.1. Tabular Setting

In the tabular setting, we consider the softmax parameterization. For any $w \in \mathbb{R}^{|S| \times |A|}$, the corresponding softmax policy $\pi_w$ is defined as

$$\pi_w(a|s) := \frac{\exp(w(s,a))}{\sum_{a' \in A} \exp(w(s,a'))}, \quad \forall (s,a) \in S \times A. \quad (3)$$

Clearly, the policy class defined in eq. (3) is complete, as any stochastic policy in the tabular setting can be represented in this class.

**Policy Evaluation:** To perform the policy evaluation in Algorithm 1 (line 3), we adopt the temporal difference (TD) learning, in which a vector $\theta^i \in \mathbb{R}^{|S| \times |A|}$ is used to estimate the state-action value function $Q^i_{\pi_w}$ for all $i = 0, \ldots, p$. Specifically, each iteration of TD learning takes the form of

$$\theta^i_{k+1}(s,a) = \theta^i_k(s,a) + \beta_k[c(s,a,s') + \gamma \theta^i_k(s',a') - \theta^i_k(s,a)], \quad (4)$$

where $s \sim \mu_{\pi_w}, a \sim \pi_w(\cdot|s), s' \sim P(\cdot|s,a), a' \sim \pi_w(\cdot|s')$, and $\beta_k$ is the learning rate. In line 3 of Algorithm 1, we perform the TD update in eq. (4) for $K_{\text{it}}$ iterations. It has been shown in (Sutton, 1988; Bhandari et al., 2018; Dalal et al., 2018b) that the iteration in eq. (4) of TD learning converges to a fixed point $\theta^i_k(\pi_w)$ in $\mathbb{R}^{|S| \times |A|}$, where each component of the fixed point is the corresponding state-action value: $\theta^i_k(\pi_w)(s,a) = Q^i_{\pi_w}(s,a)$. After performing $K_{\text{it}}$ iterations of TD learning as eq. (4), we let $Q^i_{\pi_w}(s,a) = \theta^i_{K_{\text{it}}}(s,a)$ for all $(s,a) \in S \times A$ and all $i = \{0, \ldots, p\}$.

**Constraint Estimation:** In the tabular setting, we let the sample set $B_t$ include all state-action pairs, i.e., $B_t = S \times A$, and the weight factor be $\rho_{j,t} = \xi(s_j)\pi_{w_t}(a_j|s_j)$ for all $t = 0, \ldots, T - 1$. Then, the estimation error of the constraints can be upper bounded as $|J_i(\theta^j_t) - J_i(w_t)| = |E[Q^j_i(s,a)] - E[Q^j_{\pi_w}(s,a)]| \leq ||Q^j(\theta^j_t) - Q^j_{\pi_w}||_2^2$. Thus, our approximation of constraints is accurate when the approximated value function $Q^j_{\pi_w}(s,a)$ is accurate.

**Policy Optimization:** In the tabular setting, it can be checked that the natural policy gradient of $J_i(\pi_w)$ is $\Delta_i(w)_{s,a} = (1 - \gamma)^{-1}Q^j_{\pi_w}(s,a)$ (see Appendix B). Once we obtain an approximation $Q^j_{\pi_w}(s,a) \approx Q^j_{\pi_w}(s,a)$, we can use it to update the policy in the upcoming policy optimization step:

$$w_{t+1} = w_t + \alpha \Delta_i, \quad \text{(line 7)}$$

or

$$w_{t+1} = w_t - \alpha \Delta_i, \quad \text{(line 10)},$$

where $\alpha > 0$ is the stepsize and $\Delta_i(s,a) = (1 - \gamma)^{-1}Q^j_{\pi_w}(s,a) \text{ (line 7)}$ or $(1 - \gamma)^{-1}Q^j_{\pi_w}(s,a) \text{ (line 10)}$.

Our main technical challenge lies in the analysis of policy optimization, which runs as a stochastic approximation (SA) process with random and dynamical switches between optimization objectives of the reward and cost targets. Moreover, since critics estimate the constraints and help actor to estimate the policy update, the interaction error between actor and critics affects how the algorithm switches between objective and constraints. The typical analysis technique for NPG (Agarwal et al., 2019) is not applicable here, because NPG has a fixed objective to optimize, and its analysis technique does not capture the overall convergence performance of an SA with dynamically switching optimization objective. Furthermore, the updates with respect to the constraint functions involve the stochastic selection of a constraint if multiple constraints are violated, which further complicates the random events to analyze. To handle these issues, we develop a novel analysis approach, in which we focus on the event in which critic returns almost accurate value function estimation. Such an event greatly facilitates us to capture how CRPO switches between objective and multiple constraints and establish the convergence rate.

The following theorem characterizes the convergence rate of CRPO in terms of the objective function and constraint error bound.

**Theorem 1.** Consider Algorithm 1 in the tabular setting with softmax policy parameterization defined in eq. (3) and any initialization $w_0 \in \mathbb{R}^{|S| \times |A|}$. Suppose the policy evaluation update in eq. (4) takes $K_{\text{it}} = \Theta(T^{1/\sigma}(1 - \gamma)^{-2/\sigma} \log^{2/\sigma}(T^{1+2/\sigma}/\delta))$ iterations. Let the tolerance $\eta = \Theta(\sqrt{|S||A|}/((1 - \gamma)^{1.5}T))$ and perform the NPG update defined in eq. (5) with $\alpha = (1 - \gamma)^{1.5}/\sqrt{|S||A|}T$. Then, with probability at least $1 - \delta$, we have

$$J_0(\pi^*) - E[J_0(w_{\text{out}})] \leq \Theta \left( \frac{\sqrt{|S||A|}}{(1 - \gamma)^{1.5}T} \right),$$

$$E[J_i(w_{\text{out}})] - d_i \leq \Theta \left( \frac{\sqrt{|S||A|}}{(1 - \gamma)^{1.5}T} \right)$$

for all $i = \{1, \ldots, p\}$, where the expectation is taken with respect to selecting $w_{\text{out}}$ from $N_0$.

As shown in Theorem 1, starting from an arbitrary initialization, CRPO algorithm is guaranteed to converge to the globally optimal policy $\pi^*$ in the feasible set $\Omega_C$ at a sublinear rate $O(1/\sqrt{T})$, and the constraint violation of the output policy also converges to zero also at a sublinear rate $O(1/\sqrt{T})$. Thus, to attain a $w_{\text{out}}$ that satisfies $J_0(\pi^*) - E[J_0(w_{\text{out}})] \leq \epsilon$ and $E[J_i(w_{\text{out}})] - d_i \leq \epsilon$ for all $1 \leq i \leq p$, CRPO needs at most $T = O(\epsilon^{-2})$ iterations, with each policy evaluation step consists of approximately...
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\( K_n = O(T) \) iterations when \( \sigma \) is close to 1. Theorem 1 is the first global convergence for a primal-type algorithm even under the nonconcave objective with nonconcave constraints.

Outline of Proof Idea. We briefly explain the idea of the proof of Theorem 1, and the detailed proof can be referred to Appendix B. The key challenge here is to analyze an SA process that randomly and dynamically switches between the target objectives of the reward and the constraint. To this end, we construct novel concentration events for capturing the impact of such a dynamic process on the update of the reward and cost functions in order to establish the high probability convergence guarantee.

More specifically, we focus on the event in which all policy evaluation step returns an estimation with high accuracy. Then we show that under the parameter setting specified in Theorem 1, either the size of the approximated feasible policy set \( \mathcal{N}_0 \) is large, or the average policies in the set \( \mathcal{N}_0 \) is at least as good as \( \pi^* \). In the first case we have enough candidate policies in the set \( \mathcal{N}_0 \), which guarantees the convergence of CRPO within the set \( \mathcal{N}_0 \). In the second case we can directly conclude that \( J(w_{\text{out}}) \geq J(\pi^*) \). To establish the convergence rate of the constraint violation, note that \( w_{\text{out}} \) is selected from the set \( \mathcal{N}_0 \), and thus the violation cost is not worse than the summation of constraint estimation error and the tolerance. □

4.2. Function Approximation Setting

In the function approximation setting, we parameterize the policy by a two-layer neural network together with the softmax policy.

\[
\pi^W_W(a|s) := \frac{\exp(\tau \cdot f((s, a); W))}{\sum_{a' \in A} \exp(\tau \cdot f((s, a'); W))},
\]

for all \((s, a) \in \mathcal{S} \times \mathcal{A}\), where \( \tau \) is the temperature parameter, and it can be verified that \( \pi^W_W(a|s) = \pi_{\tau W}(a|s) \). We define the feature mapping \( \phi_W(s, a) = [\phi_W^1(s, a)^T, \ldots, \phi_W^m(s, a)^T]^T : \mathbb{R}^d \rightarrow \mathbb{R}^{md} \) as

\[
\phi_W(s, a)^T = \frac{b_r}{\sqrt{m}} \mathbb{I}(W^T \psi(s, a) > 0) \cdot \psi(s, a),
\]

for all \((s, a) \in \mathcal{S} \times \mathcal{A}\) and for all \(r \in \{1, \ldots, m\}\).

Policy Evaluation: To estimate the state-action value function in Algorithm 1 (line 3), we adopt another neural network \( f((s, a); \theta^i) \) as an approximator, where \( f((s, a); \theta^i) \) has the same structure as \( f((s, a); W) \), with \( W \) replaced by \( \theta \in \mathbb{R}^{md} \) in eq. (7). To perform the policy evaluation step, we adopt the TD learning with neural network parametrization, which has also been used for the policy evaluation step in (Cai et al., 2019; Wang et al., 2019; Zhang et al., 2020). Specifically, we choose the same initialization as the policy neural network, i.e., \( \theta_0 = W_0 \), and perform the TD iteration as

\[
\theta_{k+1/2} = \theta_k^i + \beta (c_k(s, a, s') + \gamma f((s', a'); \theta_k^i) - f((s, a); \theta_k^i)) \nabla_a f(s, a; \theta_k^i),
\]

\[
\theta_{k+1} = \arg \min_{\theta \in \mathcal{B}} \| \theta - \theta_{k+1/2} \|^2_2,
\]

where \( s \sim \mu_{\tau W}, a \sim \pi_W(\cdot|s), s' \sim P(\cdot|s, a), \alpha' \sim \pi_W(\cdot|s') \), \( \beta \) is the learning rate, and \( \mathcal{B} = \{ \theta \in \mathbb{R}^{md} : \| \theta - \theta_0 \|_2 \leq R \} \). For simplicity, we denote the state-action pair as \( x = (s, a) \) and \( x' = (s', a') \) in the sequel. We define the temporal difference error as \( \delta_k(x, x', \theta_k^i) = f(x', \theta_k^i) - f(x, \theta_k^i) - c_k(x, x') \), stochastic semi-gradient as \( g_k(\theta_k^i) = \delta_k(x, x', \theta_k^i) \nabla f(x, \theta_k^i) \), and full semi-gradient as \( \tilde{g}_k(\theta_k^i) = \mathbb{E}_{\mu_{\tau W}}[\delta_k(x, x', \theta_k^i) \nabla f(x, \theta_k^i)] \).

We then describe the following regularity conditions on the stationary distribution \( \mu_{\pi_W} \), state-action value function \( Q_{\pi_W} \), and variance, which have been adopted widely in the analysis of TD learning with function approximation and stochastic approximation (SA) (Cai et al., 2019; Wang et al., 2019; Zhang et al., 2020; Fu et al., 2020).

Assumption 1. There exists a constant \( C_0 > 0 \) such that for any \( \tau \geq 0, x \in \mathbb{R}^d \) with \( \| x \|_2 = 1 \) and \( \pi_W \), it holds that \( P(\| x^T \psi(s, a) \|_2 \leq \tau) \leq C_0 \cdot \tau \), where \((s, a) \sim \mu_{\pi_W}\).

Assumption 2. We define the following function class:

\[
\mathcal{F}_{R, \infty} = \{ f((s, a); \theta) = f((s, a); \theta_0) + \int \mathbb{I}(\theta^T \psi(s, a) > 0) \cdot \lambda(\theta)^T \psi(s, a) dp(\theta) \}
\]
where \( f((s, a); \theta_0) \) is the two-layer neural network corresponding to the initial parameter \( \theta_0 = W_0 \), \( \lambda(\theta) : \mathbb{R}^d \to \mathbb{R}^d \) is a weighted function satisfying \( ||\lambda(w)||_\infty \leq R/\sqrt{d} \), and \( p(\cdot) : \mathbb{R}^d \to \mathbb{R} \) is the density \( D_w \). We assume that \( Q_{\pi_t}^i \in F_{R, \infty} \) for all \( \pi_t \) and \( i \in \{0, \cdots, p\} \).

**Assumption 3.** For any parameterized policy \( \pi_W \), there exists a constant \( C_\xi > 0 \) such that for all \( k \geq 0 \),

\[
\mathbb{E}_{\pi_W} \left[ \exp \left( \frac{1}{2} \left( g_k(\theta_k) - g_k(\theta_{k-1}) \right)^2 / C_\xi^2 \right) \right] \leq 1.
\]

Assumption 1 implies that the distribution of \( \psi(s, a) \) has a uniformly upper bounded probability density over the unit sphere, which can be satisfied for most of the ergodic Markov chain. Assumption 2 is a mild regularity condition on \( Q_{\pi_0}^i \), as \( F_{R, \infty} \) is a function class of neural networks with infinite width, which captures a sufficiently general family of functions. Assumption 3 on the variance bound is standard, which has been widely adopted in stochastic optimization literature (Ghadimi & Lan, 2013; Nemirovski 2009; Lan, 2012; Ghadimi & Lan, 2016).

In the following lemma, we characterize the convergence rate of neural TD in high probability, which is needed for our the analysis. Such a result is stronger than the convergence in expectation provided in (Bhandari et al., 2018; Cai et al., 2019; Wang et al., 2019; Zhang et al., 2020; Srikant & Ying, 2019), which is not sufficient for our need later on.

**Lemma 1** *(Convergence rate of TD in high probability).* Consider the TD iteration with neural network approximation defined in eq. (8). Let \( \theta_K = \frac{1}{K} \sum_{k=0}^{K-1} \theta_k \) be the average of the output from \( k = 0 \) to \( K - 1 \). Let \( Q^i_t(s, a) = f((s, a); \theta^i_{K_t}) \) be an estimator of \( Q^i_{\pi_t, \pi_t}(s, a) \). Suppose Assumptions 1-3 hold, assume that the stationary distribution \( \mu_{\pi_t, \pi_t} \) is not degenerate for all \( W \in B \), and let the stepsize \( \beta = \min \left\{ 1/\sqrt{K}, (1 - \gamma)/12 \right\} \). Then, with probability at least \( 1 - \delta \), we have

\[
\left\| Q^i_t(s, a) - Q^i_{\pi_t, \pi_t}(s, a) \right\|_{\mu_\pi}^2 \leq \Theta \left( \frac{1}{(1 - \gamma) - 1/\sqrt{K} \log \left( \frac{1}{\delta} \right)} \right)
\]

\[
+ \Theta \left( \frac{1}{(1 - \gamma)^3 m^{1/8}} \log \left( \frac{K}{\delta} \right) \right).
\]

**Policy Optimization:** In the neural softmax approximation setting, at each iteration \( t \), an approximation of the natural policy gradient can be obtained by solving the following linear regression problem (Agarwal et al., 2019; Wang et al., 2019; Xu et al., 2019b):

\[
\Delta_t(W_t) = \arg \min_{\theta \in B} \mathbb{E}_{\nu_{\pi_t, \pi_t}} \left[ (Q^i_t(s, a) - \phi_{W_t}(s, a)^\top \theta)^2 \right].
\]

Given the approximated natural policy gradient \( \Delta_t \), the policy update takes the form of

\[
\tau_{t+1} = \tau_t + \alpha \tau_t 
\]

\[
\cdot w_{t+1} = \tau_t \cdot w_t + \alpha \Delta_t (\text{line 7})
\]

or

\[
\tau_{t+1} \cdot w_{t+1} = \tau_t \cdot w_t - \alpha \Delta_t (\text{line 10}).
\]

Note that in eq. (11) we also update the temperature parameter \( \tau_{t+1} = \tau_t + \alpha \) simultaneously, which ensures \( \Delta_t \in B \) for all \( t \). The following theorem characterizes the convergence rate of Algorithm 1 in terms of both the objective function and the constraint violation.

**Theorem 2.** Consider Algorithm 1 in the function approximation setting with neural softmax policy parameterization defined in eq. (7). Suppose Assumptions 1-4 hold. Suppose the same setting of policy evaluation step stated in Lemma 1 holds, and consider performing the neural TD in eq. (8) and eq. (9) with \( K_m = \Theta((1 - \gamma)^2 \sqrt{m}) \) at each iteration. Let the tolerance \( \eta = \Theta(m (1 - \gamma)^{-1/2} T + (1 - \gamma)^{-2} m^{1/8}) \) and perform the NPG update defined in eq. (11) with \( \alpha = \Theta(1/\sqrt{T}) \). Then with probability at least \( 1 - \delta \), we have

\[
J_0(\pi^*) - \mathbb{E}[J_0(\pi_{\tau_{t+1} W_{t+1}})] \leq \Theta \left( \frac{1}{(1 - \gamma) \sqrt{T}} \right)
\]

\[
+ \Theta \left( \frac{1}{(1 - \gamma)^2 m^{1/8}} \log \left( \frac{1}{\delta} \right) \right).
\]

and for all \( i = 1, \cdots, p \), we have

\[
\mathbb{E}[J_i(\pi_{\tau_{t+1} W_{t+1}})] - d_i \leq \Theta \left( \frac{1}{(1 - \gamma) \sqrt{T}} \right)
\]

\[
+ \Theta \left( \frac{1}{(1 - \gamma)^2 m^{1/8}} \log \left( \frac{1}{\delta} \right) \right),
\]

where the expectation is taken only with respect to the randomness of selecting \( W_{out} \) from \( N_0 \).
Theorem 2 guarantees that CRPO converges to the global optimal policy \( \pi^* \) in the feasible set at a sublinear rate \( \mathcal{O}(1/\sqrt{T}) \) with a approximation error \( \mathcal{O}(m^{-1/8}) \) vanishes as the network width \( m \) increases. The constraint violation bound also converges to zero at a sublinear rate \( \mathcal{O}(1/\sqrt{T}) \) with a vanishing error \( \mathcal{O}(m^{-1/8}) \) decreases as \( m \) increase. The approximation error arises from both the policy evaluation and policy optimization due to the limited expressive power of neural networks.

To compare with the primal-dual approach in the function approximation setting, Theorem 2 shows that while the value function gap of CRPO achieves the same convergence rate as the primal-dual approach, the constraint violation of CRPO decays at a convergence rate of \( \mathcal{O}(1/\sqrt{T}) \), which substantially outperforms the rate \( \mathcal{O}(1/T^{2/3}) \) of the primal-dual approach (Ding et al., 2020b). Such an advantage of CRPO is further validated by our experiments in Section 5, which show that the constraint violation of CRPO vanishes much faster than that of the primal-dual approach.

**Remark 1.** Our convergence analysis for Theorem 2 can still hold without Assumptions 3 and 4. As a result, the convergence rate of CRPO would have polynomial dependence on \( \delta \) rather than logarithmic dependence.

**Remark 2.** Both Theorems 1 and 2 can be extended to cases with Markovian sampling, where an additional bias error due to Markovian sampling can be bounded using the standard techniques (e.g. (Bhandari et al., 2018; Tagori & Scherrer, 2015)).

**Remark 3.** The result in Theorem 2 can be extended to scenarios with a continuous action space and with a generally parametrized policy (not necessarily softmax), by leveraging the analysis in (Agarwal et al., 2019) for proving the global convergence of NPG with general function approximation.

## 5. Experiments

In this section, we conduct simulation experiments on different SRL tasks to compare our CRPO with the other baseline SRL algorithms: primal-dual optimization (PDO), constrained policy optimization (CPO), and interior point optimization (IPO). We consider two tasks based on OpenAI gym (Brockman et al., 2016) with each having multiple or a single constraints given as follows:

**Cartpole:** The agent is rewarded for keeping the pole upright, but is penalized with cost if (1) entering into some specific areas, or (2) having the angle of pole being large.

**Acrobot:** The agent is rewarded for swing the end-effector at a specific height, but is penalized with cost if (1) applying torque on the joint when the first link swings in a prohibited direction, or (2) when the second link swings in a prohibited direction with respect to the first link. In the single constraint setting, we only consider the fist penalty.

The detailed experimental setting is described in Appendix A. For all experiments, we use neural softmax policy with two hidden layers of size (128, 128). We adopt TRPO as the optimizer for CRPO, PDO and CPO, and PPO as the optimizer for IPO, which is the approach taken in the original IPO algorithm in (Liu et al., 2019b). It remains unclear how to develop an IPO approach based on TRPO. In CRPO, we let the tolerance \( \eta = 0.5 \). In PDO, we initialize the Lagrange multiplier as zero, and select the best tuned stepsize for dual variable update. In CPO, we select the best tuned size of the line search region for both the reward and cost optimization. In IPO, the regularization factor of the barrier function is set to be 20 as suggested in (Liu et al., 2019b).

### 5.1. Comparison with PDO

The learning curves for CRPO and PDO are provided in Figure 1. At each step we evaluate the performance based on two metrics: the return reward and constraint value of the output policy. We also show the learning curve of unconstrained TRPO (the green line), which, although achieves the best reward, does not satisfy the constraints.

![Figure 1. Average performance for CRPO, PDO, and unconstrained TRPO over 10 seeds. The red dot lines in (a) and (b) represent the limits of the constraints.](image-url)

In both tasks, CRPO tracks the constraint returns almost exactly to the limit, indicating that CRPO sufficiently explores the boundary of the feasible set, which results in an optimal return reward. In contrast, although PDO also outputs a constraints-satisfying policy in the end, it tends to over- or under-enforce the constraints, which results in lower return reward and unstable constraint satisfaction performance. In terms of the convergence, the constraints of CRPO drop below the thresholds (and thus satisfy the constraints) much faster than that of PDO, corroborating our theoretical comparison that the constraint violation of CRPO (given in Theorem 2) converges much faster than that of PDO.
PDO given in (Ding et al., 2020b).

We also find that the performance of CRPO is robust to the value of $\eta$ over a wide range, whereas the convergence performance of PDO is very sensitive to the stepsize of the dual variable (see additional experiments of hyperparameters comparison in Appendix A). Thus, in contrast to the difficulty of tuning PDO, CRPO is much less sensitive to hyper-parameters and is hence much easier to tune.

5.2. Comparison with CPO

Since it is very difficult for CPO to solve multi-constraint tasks as discussed in (Liu et al., 2019b), in order to compare the performance between CRPO and CPO, we focus on the ‘Acrobot’ task with a single constraint. We also add the learning curve of IPO in the plot for comparison. Figure 2 illustrates that CRPO converges faster and achieves higher reward than CPO (and IPO), although all algorithms share similar convergence behavior over the constraint values.

![Figure 2. Average performance of CRPO, CPO and IPO in 'Acrobot' with one constraint over 10 seeds.](image)

5.3. Comparison with IPO

We compare the performance between CRPO and IPO over the same setting of ‘Acrobot’ with two constraints. As discussed in (Liu et al., 2019b), IPO relies on the barrier regularization function to enforce the satisfaction of the constraints, and hence IPO is guaranteed to converge only to a suboptimal point. Such a regularization can also slow down the convergence speed of the constraint value. As shown in Figure 3, our CRPO outperforms IPO in terms of the convergence of both the reward and constraint values in a multi-constraint setting.

![Figure 3. Average performance of CRPO and IPO in 'Acrobot' with two constraints over 10 seeds.](image)

6. Conclusion

In this paper, we propose a novel CRPO approach for policy optimization for SRL, which is easy to implement and has provable global optimality guarantee. We show that CRPO achieves an $O(1/\sqrt{T})$ convergence rate to the global optimum and an $O(1/\sqrt{T})$ rate of vanishing constraint error when NPG update is adopted as the optimizer. This is the first primal SRL algorithm that has a provable convergence guarantee to a global optimum. In the future, it is interesting to incorporate various momentum schemes to CRPO to improve its convergence performance.

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Supplementary Materials

A. Experimental Setting

In our constrained Cartpole environment, the cart is restricted in the area $[-2.4, 2.4]$. Each episode length is no longer than 200 and terminated when the angle of the pole is larger than 12 degree. During the training, the agent receives a reward $+1$ if (1) entering the area $[-2.4, -2.2]$, $[-1.3, -1.1]$, $[-0.1, 0.1]$, $[1.1, 1.3]$, and $[2.2, 2.4]$; or (2) having the angle of pole larger than 6 degree.

In our constrained Acrobot environment, each episode has length 500. During the training, the agent receives a reward $+1$ when the end-effector is at a height of 0.5, but is penalized with cost $+1$ when (1) a torque with value $+1$ is applied when the first pendulum swings along an anticlockwise direction; or (2) a torque with value $+1$ is applied when the second pendulum swings along an anticlockwise direction with respect to the first pendulum.

For details about the update of PD, please refer to (Achiam et al., 2017)[Section 10.3.3]. The performance of PD is very sensitive to the stepsize of the dual variable’s update. If the stepsize is too small, then the dual variable will not update quickly to enforce the constraints. If the stepsize is too large, then the algorithm will behave conservatively and have low return reward. To appropriately select the stepsize for the dual variable, we conduct the experiments with the learning rates $\{0.0001, 0.0005, 0.001, 0.005, 0.01, 0.05\}$ for both tasks. The learning rate $0.005$ performs the best in the first task, and the learning rate $0.0005$ performs the best in the second task. Thus, our reported result of Cartpole is with the stepsize $0.005$ and our reported result of Acrobot is with the stepsize $0.0005$.

Next, we investigate the robustness of CRPO with respect to the tolerance parameter $\eta$. We conduct the experiments under the following values of $\eta \{10, 5, 2, 1, 0.5\}$ for the Acrobot environment. It can be seen from Figure 4 that the learning curves of CRPO with the tolerance parameter $\eta$ taking different values are almost the same, which indicates that the convergence performance of CRPO is robust to the value of $\eta$ over a wide range. Thus, the tolerance parameter $\eta$ does not cause much parameter tuning cost for CRPO.

![Figure 4. Comparison of CRPO in Acrobat with tolerance parameter $\eta$ taking different values.](image)

B. Proof of Theorem 1: Tabular Setting

B.1. Supporting Lemmas for Proof of Theorem 1

The following lemma characterizes the convergence rate of TD learning in the tabular setting.

**Lemma 2** ((Dalal et al., 2019)). Consider the iteration given in eq. (4) with arbitrary initialization $\theta_0$. Assume that the stationary distribution $\mu_{\pi_w}$ is not degenerate for all $w \in \mathbb{R}^{S \times A}$. Let stepsize $\beta_k = \Theta\left(\frac{1}{K}\right)$ ($0 < \sigma < 1$). Then, with probability at least $1 - \delta$, we have

$$
\left\|\theta_K - \theta^*_{\pi_w}\right\|_2 = \mathcal{O}\left(\frac{\log(|S|^2 |A|^2 K^2/\delta)}{(1 - \gamma)K^{\sigma/2}}\right).
$$

Note that $\sigma$ can be arbitrarily close to 1. Lemma 2 implies that we can obtain an approximation $\tilde{Q}_t^i$ such that $\left\|\tilde{Q}_t^i - Q_{\pi_{w_t}}^i\right\|_2 = \ldots$
where $J_\rho^i$ and $v_\rho$ denote the accumulated reward (cost) function and visitation distribution under policy $\rho$ when the initial state distribution is $\rho$.

**Lemma 4** (Lemma 5.1. (Agarwal et al., 2019)). Considering the approximated NPG update in line 7 of Algorithm 1 in the tabular setting and $i = 0$, the NPG update takes the form:

$$w_{t+1} = w_t + \frac{\alpha}{1 - \gamma} \bar{Q}_t^i,$$

and $\pi_{w_{t+1}}(a|s) = \pi_{w_t}(a|s) \frac{\exp(\alpha \bar{Q}_t^i(s,a)/(1 - \gamma))}{Z_t(s)}$, where

$$Z_t(s) = \sum_{a \in A} \pi_{w_t}(a|s) \exp\left(\frac{\alpha \bar{Q}_t^i(s,a)}{1 - \gamma}\right).$$

Note that if we follow the update in line 10 of Algorithm 1, we can obtain similar results for the case $i \in \{1, \cdots, p\}$ as stated in Lemma 4.

**Lemma 5** (Policy gradient property of softmax parameterization). Considering the softmax policy in the tabular setting (eq. (3)). For any initial state distribution $\rho$, we have

$$\nabla_w J_\rho^i(w) = \mathbb{E}_{s \sim \rho, a \sim \pi_w(\cdot|s)} \left[ \left( I_{w_{\pi}} - \sum_{a' \in A} \pi_w(a'|s) I_{a'_{\pi}} \right) Q^i_{\pi_w}(s,a) \right],$$

and

$$\|\nabla_w J_\rho^i(w)\|_2 \leq \frac{2c_{\text{max}}}{1 - \gamma},$$

where $I_{as}$ is an $|S| \times |A|$-dimension vector, with $(a,s)$-th element being one, and the rest elements being zero.

**Proof.** The first result can follows directly from Lemma C.1 in (Agarwal et al., 2019). We now proceed to prove the second result.

$$\|\nabla_w J_\rho^i(w)\|_2 \leq \mathbb{E} \left[ \left( I_{w_{\pi}} - \sum_{a' \in A} \pi_w(a'|s) I_{a'_{\pi}} \right) Q^i_{\pi_w}(s,a) \right]_2 \leq \mathbb{E} \left[ \left( I_{w_{\pi}} - \sum_{a' \in A} \pi_w(a'|s) I_{a'_{\pi}} \right) Q^i_{\pi_w}(s,a) \right]_2 \leq 2\mathbb{E} \left[ Q^i_{\pi_w}(s,a) \right] \leq \frac{2c_{\text{max}}}{1 - \gamma}.$$
We then proceed to prove Lemma 6. The performance difference lemma (Lemma 3) implies:

\[ \log Z_t(s) - \frac{\alpha}{1 - \gamma} \mathbb{E}_{s \sim \rho_t} \left( \log \mathbb{E}_{s \sim \rho_t} \left( \alpha Q_i^\pi(s,a) \right) - Q_i^\pi(s,a) \right) \]

\[ - \frac{1}{1 - \gamma} \mathbb{E}_{s \sim \rho_t} \sum_{a \in A} \pi_{w_t}(a|s) \left( \bar{Q}_i^\pi(s,a) - Q_i^\pi(s,a) \right) \]

\[ - \frac{1}{1 - \gamma} \mathbb{E}_{s \sim \rho_t} \sum_{a \in A} \pi_{w_{t+1}}(a|s) \left| Q_i^\pi(s,a) - \bar{Q}_i^\pi(s,a) \right| \]

**Proof.** We first provide the following lower bound.

\[ \log Z_t(s) - \frac{\alpha}{1 - \gamma} \mathbb{E}_{s \sim \rho_t} \left( \log \mathbb{E}_{s \sim \rho_t} \left( \alpha Q_i^\pi(s,a) \right) - Q_i^\pi(s,a) \right) - \frac{1}{1 - \gamma} \mathbb{E}_{s \sim \rho_t} \sum_{a \in A} \pi_{w_t}(a|s) \left( \bar{Q}_i^\pi(s,a) - Q_i^\pi(s,a) \right) \]

\[ - \frac{1}{1 - \gamma} \mathbb{E}_{s \sim \rho_t} \sum_{a \in A} \pi_{w_{t+1}}(a|s) \left| Q_i^\pi(s,a) - \bar{Q}_i^\pi(s,a) \right| \]

Thus, we conclude that

\[ \log Z_t(s) - \frac{\alpha}{1 - \gamma} \mathbb{E}_{s \sim \rho_t} \left( \log \mathbb{E}_{s \sim \rho_t} \left( \alpha Q_i^\pi(s,a) \right) - Q_i^\pi(s,a) \right) - \frac{1}{1 - \gamma} \mathbb{E}_{s \sim \rho_t} \sum_{a \in A} \pi_{w_t}(a|s) \left( \bar{Q}_i^\pi(s,a) - Q_i^\pi(s,a) \right) \geq 0. \]

We then proceed to prove Lemma 6. The performance difference lemma (Lemma 3) implies:

\[ J_i^\pi(w_{t+1}) - J_i^\pi(w_t) \]

\[ = \frac{1}{1 - \gamma} \mathbb{E}_{s \sim \rho_t} \sum_{a \in A} \pi_{w_{t+1}}(a|s) A_i^\pi(s,a) \]

\[ = \frac{1}{1 - \gamma} \mathbb{E}_{s \sim \rho_t} \sum_{a \in A} \pi_{w_{t+1}}(a|s) Q_i^\pi(s,a) - \mathbb{E}_{s \sim \rho_t} V_i^\pi(s) \]

\[ = \frac{1}{1 - \gamma} \mathbb{E}_{s \sim \rho_t} \sum_{a \in A} \pi_{w_{t+1}}(a|s) \left( Q_i^\pi(s,a) - \bar{Q}_i^\pi(s,a) \right) - \mathbb{E}_{s \sim \rho_t} V_i^\pi(s) \]

\[ + \frac{1}{1 - \gamma} \mathbb{E}_{s \sim \rho_t} \sum_{a \in A} \pi_{w_{t+1}}(a|s) \log \left( \frac{\pi_{w_{t+1}}(a|s) Z_t(s)}{\pi_{w_t}(a|s)} \right) \]

\[ + \frac{1}{1 - \gamma} \mathbb{E}_{s \sim \rho_t} D_{KL}(\pi_{w_{t+1}} || \pi_{w_t}) + \frac{1}{1 - \gamma} \mathbb{E}_{s \sim \rho_t} \log Z_t(s) \]

\[ + \frac{1}{1 - \gamma} \mathbb{E}_{s \sim \rho_t} \sum_{a \in A} \pi_{w_{t+1}}(a|s) \left( Q_i^\pi(s,a) - \bar{Q}_i^\pi(s,a) \right) - \mathbb{E}_{s \sim \rho_t} V_i^\pi(s) \]

\[ \geq \frac{1}{1 - \gamma} \mathbb{E}_{s \sim \rho_t} \left( \log Z_t(s) - \frac{\alpha}{1 - \gamma} V_i^\pi(s) + \frac{1}{1 - \gamma} \sum_{a \in A} \pi_{w_t}(a|s) \left| Q_i^\pi(s,a) - Q_i^\pi(s,a) \right| \right) \]
Algorithm 1 in the tabular setting when \( i \)

where (Lemma 7 stated in Lemma 6. Note that if we follow the update in line 10 of Algorithm 1, we can obtain similar results for the case \( i \in \{1, \cdots, p\} \) as stated in Lemma 6.

**Lemma 7** (Upper bound on optimality gap for approximated NPG). Consider the approximated NPG updates in line 7 of Algorithm 1 in the tabular setting when \( i = 0 \). We have

\[
J_0(\pi^*) - J_0(\pi_{w_t}) \leq \frac{1}{\alpha} E_{s \sim \nu^*} \left( D_{KL}(\pi^*||\pi_{w_t}) - D_{KL}(\pi^*||\pi_{w_{t+1}}) \right) + \frac{2\alpha c_{\max}^2 |S| |A|}{(1 - \gamma)^3} + \frac{3(1 + \alpha c_{\max})}{(1 - \gamma)^2} \|Q_{\pi_{w_t}}^0 - \bar{Q}_i^0\|_2^2.
\]

**Proof.** By the performance difference lemma (Lemma 3), we have

\[
J_i(\pi^*) - J_i(\pi_{w_t})
\]

\[
= \frac{1}{1 - \gamma} E_{s \sim \nu^*} \sum_{a \in A} \pi^*(a|s)A_{\pi_{w_t}}(s, a)
\]

\[
= \frac{1}{1 - \gamma} E_{s \sim \nu^*} \sum_{a \in A} \pi^*(a|s)Q_{\pi_{w_t}}^i(s, a) - \frac{1}{1 - \gamma} E_{s \sim \nu^*} V_{\pi_{w_t}}^i(s)
\]

\[
= \frac{1}{1 - \gamma} E_{s \sim \nu^*} \sum_{a \in A} \pi^*(a|s)\bar{Q}_i^i(s, a) + \frac{1}{1 - \gamma} E_{s \sim \nu^*} \sum_{a \in A} \pi^*(a|s)(Q_{\pi_{w_t}}^i(s, a) - \bar{Q}_i^i(s, a))
\]

\[
- \frac{1}{1 - \gamma} E_{s \sim \nu^*} V_{\pi_{w_t}}^i(s)
\]

\[
\geq \left( \frac{\alpha}{1 - \gamma} \right) E_{s \sim \nu^*} \sum_{a \in A} \pi^*(a|s) \log \frac{\pi_{w_{t+1}}(a|s)Z_i(s)}{\pi_{w_t}(a|s)} + \frac{1}{1 - \gamma} E_{s \sim \nu^*} \sum_{a \in A} \pi^*(a|s)(Q_{\pi_{w_t}}^i(s, a) - \bar{Q}_i^i(s, a))
\]

\[
- \frac{1}{1 - \gamma} E_{s \sim \nu^*} V_{\pi_{w_t}}^i(s)
\]

\[
= \frac{1}{\alpha} E_{s \sim \nu^*} (D_{KL}(\pi^*||\pi_{w_t}) - D_{KL}(\pi^*||\pi_{w_{t+1}})) + \frac{1}{\alpha} E_{s \sim \nu^*} \left( \log Z_i(s) - \frac{\alpha}{1 - \gamma} V_{\pi_{w_t}}^i(s) \right)
\]

\[
+ \frac{1}{\alpha} E_{s \sim \nu^*} \sum_{a \in A} \pi^*(a|s)(Q_{\pi_{w_t}}^i(s, a) - \bar{Q}_i^i(s, a))
\]

\[
\leq \frac{1}{\alpha} E_{s \sim \nu^*} (D_{KL}(\pi^*||\pi_{w_t}) - D_{KL}(\pi^*||\pi_{w_{t+1}}))
\]

\[
+ \frac{1}{\alpha} E_{s \sim \nu^*} \left( \log Z_i(s) - \frac{\alpha}{1 - \gamma} V_{\pi_{w_t}}^i(s) + \frac{\alpha}{1 - \gamma} \sum_{a \in A} \pi_{w_t}(a|s) \bar{Q}_i^i(s, a) - Q_{\pi_{w_t}}^i(s, a) \right)
\]
\[
+ \frac{1}{1 - \gamma} \mathbb{E}_{s \sim \nu^*} \sum_{a \in A} \pi^*(a|s) (Q^i_{\pi^*}(s, a) - \bar{Q}^i_t(s, a))
\]
\[
\leq \frac{1}{\alpha} \mathbb{E}_{s \sim \nu^*} (D_{KL}(\pi^*||\pi_{w_t}) - D_{KL}(\pi^*||\pi_{w_{t+1}}))
\]
\[
+ \frac{1}{1 - \gamma} (J^\nu_t(w_{t+1}) - J^\nu_t(w_t)) + \frac{1}{1 - \gamma} \mathbb{E}_{s \sim \nu^*} \sum_{a \in A} \pi_{w_{t+1}}(a|s) \left| Q^i_{\pi_{w_{t+1}}}(s, a) - \bar{Q}^i_t(s, a) \right|
\]
\[
+ \frac{1}{1 - \gamma} \mathbb{E}_{s \sim \nu^*} \sum_{a \in A} \pi_{w_t}(a|s) \left| Q^i_{\pi_{w_t}}(s, a) - \bar{Q}^i_t(s, a) \right|
\]
\[
\leq \frac{1}{\alpha} \mathbb{E}_{s \sim \nu^*} (D_{KL}(\pi^*||\pi_{w_t}) - D_{KL}(\pi^*||\pi_{w_{t+1}})) + \frac{2\alpha \gamma^2}{(1 - \gamma)^3} \|w_{t+1} - w_t\|_2^2 + \frac{3}{1 - \gamma} \left\| Q^i_{\pi_{w_t}} - \bar{Q}^i_t \right\|_2^2
\]
\[
\leq \frac{1}{\alpha} \mathbb{E}_{s \sim \nu^*} (D_{KL}(\pi^*||\pi_{w_t}) - D_{KL}(\pi^*||\pi_{w_{t+1}})) + \frac{2\alpha \gamma^2}{(1 - \gamma)^3} \|Q^i_{\pi_{w_t}} - \bar{Q}^i_t\|_2^2 + \frac{3(1 + \alpha \gamma^2)}{(1 - \gamma)^2} \left\| Q^i_{\pi_{w_t}} - \bar{Q}^i_t \right\|_2^2
\]
\[
\leq \frac{1}{\alpha} \mathbb{E}_{s \sim \nu^*} (D_{KL}(\pi^*||\pi_{w_t}) - D_{KL}(\pi^*||\pi_{w_{t+1}})) + \frac{2\alpha \gamma^2}{(1 - \gamma)^3} \|Q^i_{\pi_{w_t}} - \bar{Q}^i_t\|_2^2 + \frac{3(1 + \alpha \gamma^2)}{(1 - \gamma)^2} \left\| Q^i_{\pi_{w_t}} - \bar{Q}^i_t \right\|_2^2
\]

where (i) follows from Lemma 4, (ii) follows from Lemma 6 and (iii) follows from the Lipschitz property of $J^\nu_t(w)$ such that $J^\nu_t(w_{t+1}) - J^\nu_t(w_t) \leq \frac{2\gamma^2}{1 - \gamma} \|w_{t+1} - w_t\|_2$, which is proved by Proposition 1 in (Xu et al., 2020b). 

Note that if we follow the update in line 10 of Algorithm 1, we can obtain the following result for the case $i \in \{1, \ldots, p\}$ as stated in Lemma 7:

\[
J_i(\pi_{w_t}) - J_i(\pi^*)
\]
\[
\leq \frac{1}{\alpha} \mathbb{E}_{s \sim \nu^*} (D_{KL}(\pi^*||\pi_{w_t}) - D_{KL}(\pi^*||\pi_{w_{t+1}})) + \frac{2\alpha \gamma^2}{(1 - \gamma)^3} \|Q^i_{\pi_{w_t}} - \bar{Q}^i_t\|_2^2 + \frac{3(1 + \alpha \gamma^2)}{(1 - \gamma)^2} \left\| Q^i_{\pi_{w_t}} - \bar{Q}^i_t \right\|_2^2.
\]

**Lemma 8.** Considering CRPO in Algorithm 1 in the tabular setting. Let $K_{\alpha} = \Theta(T^{1/\sigma} \log^2/\sigma (|S|^2 |A|^2 T^{1/2}/\sigma))$. Define $\mathcal{N}_i$ as the set of steps that CRPO algorithm chooses to minimize the $i$-th constraint. With probability at least $1 - \delta$, we have

\[
\sum_{i \in \mathcal{N}_0} (J_0(\pi^*) - J_0(\pi_{w_t})) + \alpha \gamma \sum_{i = 1}^p |\mathcal{N}_i|
\]
\[
\leq \mathbb{E}_{s \sim \nu^*} D_{KL}(\pi^*||\pi_{w_0}) + \frac{2\alpha \gamma^2}{(1 - \gamma)^3} |S| |A| T + \frac{3\alpha}{1 - \gamma^2} \left\| Q^0_{\pi_{w_0}} - \bar{Q}^0_t \right\|_2^2.
\]

**Proof.** If $t \in \mathcal{N}_0$, by Lemma 7 we have

\[
\alpha (J_0(\pi^*) - J_0(\pi_{w_t}))
\]
\[
\leq \mathbb{E}_{s \sim \nu^*} (D_{KL}(\pi^*||\pi_{w_t}) - D_{KL}(\pi^*||\pi_{w_{t+1}})) + \frac{2\alpha \gamma^2}{(1 - \gamma)^3} |S| |A| T + \frac{3\alpha(1 + \alpha \gamma^2)}{(1 - \gamma)^2} \left\| Q^0_{\pi_{w_t}} - \bar{Q}^0_t \right\|_2^2.
\]

If $t \in \mathcal{N}_i$, similarly we can obtain

\[
\alpha (J_i(\pi_{w_t}) - J_i(\pi^*))
\]
\[
\leq \mathbb{E}_{s \sim \nu^*} (D_{KL}(\pi^*||\pi_{w_t}) - D_{KL}(\pi^*||\pi_{w_{t+1}})) + \frac{2\alpha \gamma^2}{(1 - \gamma)^3} |S| |A| T + \frac{3\alpha(1 + \alpha \gamma^2)}{(1 - \gamma)^2} \left\| Q^i_{\pi_{w_t}} - \bar{Q}^i_t \right\|_2^2.
\]
Taking the summation of eq. (12) and eq. (13) from $t = 0$ to $T - 1$ yields

$$
\alpha \sum_{t \in \mathcal{N}_0} (J_0(\pi^*) - J_0(\pi_{w_t})) + \alpha \sum_{i=1}^p \sum_{t \in \mathcal{N}_i} (J_i(\pi_{w_t}) - J_i(\pi^*)) \\
\leq \mathbb{E}_{s \sim \nu^*} \cdot D_{\text{KL}}(\pi^* || \pi_{w_0}) + \frac{2\alpha^2 c_{\text{max}}^2 |S| |A| T}{(1 - \gamma)^3} + \frac{3\alpha(1 + \alpha c_{\text{max}})}{(1 - \gamma)^2} \sum_{i=0}^p \sum_{t \in \mathcal{N}_i} \left\| Q^i_{\pi_{w_t}} - \bar{Q}^i_t \right\|_2^2.
$$
(14)

Note that when $t \in \mathcal{N}_i$ ($i \neq 0$), we have $J_i(\theta_i^*) > d_i + \eta$ (line 9 in Algorithm 1), which implies that

$$
J_i(\pi_{w_t}) - J_i(\pi^*) \geq J_i(\theta_i^*) - J_i(\pi^*) - |J_i(\theta_i^*) - J_i(\pi_{w_t})| \\
\geq d_i + \eta - J_i(\pi^*) - |J_i(\theta_i^*) - J_i(\pi_{w_t})| \\
\geq \eta - \|Q^i_{\pi_{w_t}} - \bar{Q}^i_t\|_2.
$$
(15)

Substituting eq. (15) into eq. (14) yields

$$
\alpha \sum_{t \in \mathcal{N}_0} (J_0(\pi^*) - J_0(\pi_{w_t})) + \alpha \eta \sum_{i=1}^p |\mathcal{N}_i| - \alpha \sum_{i=1}^p \sum_{t \in \mathcal{N}_i} \left\| Q^i_{\pi_{w_t}} - \bar{Q}^i_t \right\|_2^2 \\
\leq \mathbb{E}_{s \sim \nu^*} \cdot D_{\text{KL}}(\pi^* || \pi_{w_0}) + \frac{2\alpha^2 c_{\text{max}}^2 |S| |A| T}{(1 - \gamma)^3} + \frac{3\alpha(1 + \alpha c_{\text{max}})}{(1 - \gamma)^2} \sum_{i=0}^p \sum_{t \in \mathcal{N}_i} \left\| Q^i_{\pi_{w_t}} - \bar{Q}^i_t \right\|_2^2,
$$

which implies

$$
\alpha \sum_{t \in \mathcal{N}_0} (J_0(\pi^*) - J_0(\pi_{w_t})) + \alpha \eta \sum_{i=1}^p |\mathcal{N}_i| \\
\leq \mathbb{E}_{s \sim \nu^*} \cdot D_{\text{KL}}(\pi^* || \pi_{w_0}) + \frac{2\alpha^2 c_{\text{max}}^2 |S| |A| T}{(1 - \gamma)^3} + \frac{\alpha(2 + (1 - \gamma)^2) + 3\alpha c_{\text{max}}}{(1 - \gamma)^2} \sum_{i=0}^p \sum_{t \in \mathcal{N}_i} \left\| Q^i_{\pi_{w_t}} - \bar{Q}^i_t \right\|_2^2.
$$
(16)

By Lemma 2, we have with probability at least $1 - \delta$, the following holds

$$
\left\| Q^i_{\pi_{w_t}} - \bar{Q}^i_t \right\|_2 \leq \mathcal{O} \left( \frac{\log(|S|^2 |A|^2 K_{\text{in}}^{2/\delta})}{(1 - \gamma)K_{\text{in}}^{\sigma/2}} \right).
$$

Thus, if we let

$$
K_{\text{in}} = \Theta \left( \left( \frac{T}{(1 - \gamma)^2 |S| |A|} \right)^{\frac{\delta}{2}} \log \left( \frac{T^{\frac{\delta}{2} + 1}}{(1 - \gamma)^{\frac{\delta}{2}}} |S|^\frac{\delta}{2} - 2 |A|^\frac{\delta}{2} - 2 \right) \right),
$$

then with probability at least $1 - \delta/T$, we have

$$
\left\| Q^i_{\pi_{w_t}} - \bar{Q}^i_t \right\|_2 \leq \frac{\sqrt{(1 - \gamma) |S| |A|}}{\sqrt{T}}.
$$
(17)

Applying the union bound to eq. (17) from $t = 0$ to $T - 1$, we have with probability at least $1 - \delta$ the following holds

$$
\sum_{i=0}^p \sum_{t \in \mathcal{N}_i} \left\| Q^i_{\pi_{w_t}} - \bar{Q}^i_t \right\|_2 \leq \sqrt{(1 - \gamma) |S| |A| T},
$$
(18)

which further implies that, with probability at least $1 - \delta$, we have

$$
\alpha \sum_{t \in \mathcal{N}_0} (J_0(\pi^*) - J_0(\pi_{w_t})) + \alpha \eta \sum_{i=1}^p |\mathcal{N}_i| \\
\leq \mathbb{E}_{s \sim \nu^*} \cdot D_{\text{KL}}(\pi^* || \pi_{w_0}) + \frac{2\alpha^2 c_{\text{max}}^2 |S| |A| T}{(1 - \gamma)^3} + \alpha \sqrt{|S| |A| T (2 + (1 - \gamma)^2 + 3\alpha c_{\text{max}})} \left( \frac{1}{1 - \gamma} \right)^{1.5},
$$

which completes the proof.
Lemma 9. If
\[
\frac{1}{2} \alpha \eta T \geq \mathbb{E}_{s \sim \nu^t} D_{KL}(\pi^* || \pi_{w_0}) + \frac{2 \alpha^2 c_{max}^2 |s| |a| T}{(1 - \gamma)^3} + \frac{\alpha \sqrt{|s| |a| T} (2 + (1 - \gamma)^2 + 3 \alpha c_{max})}{(1 - \gamma)^{1.5}} \tag{19}
\]
then with probability at least 1 − δ, we have the following holds

1. \(\mathcal{N}_0 \neq \emptyset\), i.e., \(w_{out}\) is well-defined,

2. One of the following two statements must hold,
   
   (a) \(|\mathcal{N}_0| \geq T/2\),
   
   (b) \(\sum_{t \in G} (J_0(\pi^*) - J_0(w_t)) \leq 0\).

Proof. We prove Lemma 9 in the event that eq. (18) holds, which happens with probability at least 1 − δ. Under such an event, the following inequality holds, which is also the result of Lemma 8.

\[
\alpha \sum_{t \in \mathcal{N}_0} (J_0(\pi^*) - J_0(\pi_{w_t})) + \alpha \eta \sum_{i=1}^{p} |\mathcal{N}_i| \\
\leq \mathbb{E}_{s \sim \nu^t} D_{KL}(\pi^* || \pi_{w_0}) + \frac{2 \alpha^2 c_{max}^2 |s| |a| T}{(1 - \gamma)^3} + \frac{\alpha \sqrt{|s| |a| T} (2 + (1 - \gamma)^2 + 2 \alpha c_{max})}{(1 - \gamma)^{1.5}}. \tag{20}
\]

We first verify item 1. If \(\mathcal{N}_0 = \emptyset\), then \(\sum_{i=1}^{p} |\mathcal{N}_i| = T\), and eq. (20) implies that

\[
\alpha \eta T \leq \mathbb{E}_{s \sim \nu^t} D_{KL}(\pi^* || \pi_{w_0}) + \frac{2 \alpha^2 c_{max}^2 |s| |a| T}{(1 - \gamma)^3} + \frac{\alpha \sqrt{|s| |a| T} (2 + (1 - \gamma)^2 + 2 \alpha c_{max})}{(1 - \gamma)^{1.5}},
\]

which contradicts eq. (19). Thus, we must have \(\mathcal{N}_0 \neq \emptyset\).

We then proceed to verify item 2. If \(\sum_{t \in \mathcal{N}_0} (J_0(\pi^*) - J_0(w_t)) \leq 0\), then (b) in item 2 holds. If \(\sum_{t \in \mathcal{N}_0} (J_0(\pi^*) - J_0(w_t)) \leq 0\), then eq. (20) implies that

\[
\alpha \eta \sum_{i=1}^{p} |\mathcal{N}_i| \leq \mathbb{E}_{s \sim \nu^t} D_{KL}(\pi^* || \pi_{w_0}) + \frac{2 \alpha^2 c_{max}^2 |s| |a| T}{(1 - \gamma)^3} + \frac{\alpha \sqrt{|s| |a| T} (2 + (1 - \gamma)^2 + 3 \alpha c_{max})}{(1 - \gamma)^{1.5}}.
\]

Suppose that \(|\mathcal{N}_0| < T/2\), i.e., \(\sum_{i=1}^{p} |\mathcal{N}_i| \geq T/2\). Then,

\[
\frac{1}{2} \alpha \eta T \leq \alpha \eta \sum_{i=1}^{p} |\mathcal{N}_i| \\
\leq \mathbb{E}_{s \sim \nu^t} D_{KL}(\pi^* || \pi_{w_0}) + \frac{2 \alpha^2 c_{max}^2 |s| |a| T}{(1 - \gamma)^3} + \frac{\alpha \sqrt{|s| |a| T} (2 + (1 - \gamma)^2 + 3 \alpha c_{max})}{(1 - \gamma)^{1.5}},
\]

which contradicts eq. (19). Hence, (a) in item 2 holds.

\[\square\]

B.2. Proof of Theorem 1

We restate Theorem 1 as follows to include the specifics of the parameters.

**Theorem 3 (Restatement of Theorem 1).** Consider Algorithm 1 in the tabular setting. Let \(\alpha = (1 - \gamma)^{1.5} / \sqrt{|s| |a| T}\), \(\eta = 2 \sqrt{|s| |a| T} (3 + \mathbb{E}_{s \sim \nu^t} D_{KL}(\pi^* || \pi_{w_0}) + 3 c_{max}^2 + c_{max}^2)\), and

\[
K_m = \Theta \left( \left( \frac{T}{(1 - \gamma) |s| |a|} \right)^{\frac{1}{2}} \log^{\frac{1}{2}} \left( \frac{T^{\frac{3}{2} + 1}}{(1 - \gamma)^{\frac{3}{2}} |s|^{-2} |a|^{\frac{2}{2} - 2}} \right) \right).
\]
Suppose the same setting for policy evaluation in Lemma 2 hold. Then, with probability at least $1 - \delta$, we have
\[
J_0(\pi^*) - \mathbb{E}[J_0(w_{out})] = \frac{2\sqrt{|S||A|}}{(1 - \gamma)^{1.5}} \sqrt{T} \left( \mathbb{E}_{\bar{\pi} \sim \nu^*} D_{\text{KL}}(\pi^* || \pi_{w_0}) + 3 + 2c_{\text{max}}^2 + 3c_{\text{max}} \right),
\]
and for all $i \in \{1, \cdots, p\}$, we have
\[
\mathbb{E}[J_i(\pi_{w_0})] - d_i \leq \frac{2\sqrt{|S||A|}}{(1 - \gamma)^{1.5}} \sqrt{T} \left( 3 + \mathbb{E}_{\bar{\pi} \sim \nu^*} D_{\text{KL}}(\pi^* || \pi_{w_0}) + 3c_{\text{max}} + c_{\text{max}}^2 \right) + \frac{2\sqrt{(1 - \gamma)|S||A|}}{\sqrt{T}}.
\]
To prove Theorem 1 (or Theorem 3), we still consider the following event given in eq. (18) that happens with probability at least $1 - \delta$:
\[
\sum_{i=0}^p \sum_{t \in \mathcal{N}_i} \left| Q^i_{\pi_{w_0}} - Q^i_t \right|_2 \leq \sqrt{(1 - \gamma)|S||A|T},
\]
which implies
\[
\alpha \sum_{t \in \mathcal{N}_0} (J_0(\pi^*) - J_0(\pi_{w_0})) + \alpha \eta \sum_{i=1}^p |\mathcal{N}_i| \\
\leq \mathbb{E}_{\bar{\pi} \sim \nu^*} D_{\text{KL}}(\pi^* || \pi_{w_0}) + \frac{2\alpha^2 c_{\text{max}}^2 |S||A|T}{(1 - \gamma)^3} + \frac{\alpha \sqrt{|S||A|T(2 + (1 - \gamma)^2 + 3\alpha c_{\text{max}})}}{(1 - \gamma)^{1.5}}.
\]
We first consider the convergence rate of the objective function. Under the above event, the following holds
\[
\alpha \sum_{t \in \mathcal{N}_0} (J_0(\pi^*) - J_0(\pi_{w_0})) \\
\leq \mathbb{E}_{\bar{\pi} \sim \nu^*} D_{\text{KL}}(\pi^* || \pi_{w_0}) + \frac{2\alpha^2 c_{\text{max}}^2 |S||A|T}{(1 - \gamma)^3} + \frac{\alpha \sqrt{|S||A|T(2 + (1 - \gamma)^2 + 3\alpha c_{\text{max}})}}{(1 - \gamma)^{1.5}}.
\]
If $\sum_{t \in \mathcal{N}_0} (J_0(\pi^*) - J_0(\pi_{w_0})) \leq 0$, then we have $J_0(\pi^*) - J_0(\pi_{w_0}) \leq 0$. If $\sum_{t \in \mathcal{N}_0} (J_0(\pi^*) - J_0(\pi_{w_0})) \geq 0$, we have $|\mathcal{N}_0| \geq T/2$, which implies the following convergence rate
\[
J_0(\pi^*) - \mathbb{E}[J_0(\pi_{w_0})] \\
= \frac{1}{|\mathcal{N}_0|} \sum_{t \in \mathcal{N}_0} (J_0(\pi^*) - J_0(\pi_{w_0})) \\
\leq \frac{2\alpha}{T} \mathbb{E}_{\bar{\pi} \sim \nu^*} D_{\text{KL}}(\pi^* || \pi_{w_0}) + \frac{4\alpha^2 c_{\text{max}}^2 |S||A|}{(1 - \gamma)^3} + \frac{2\sqrt{|S||A|T(2 + (1 - \gamma)^2 + 3\alpha c_{\text{max}})}}{(1 - \gamma)^{1.5}} \sqrt{T} \\
\leq \frac{\sqrt{|S||A|}}{(1 - \gamma)^{1.5}} \sqrt{T} \left( 2\mathbb{E}_{\bar{\pi} \sim \nu^*} D_{\text{KL}}(\pi^* || \pi_{w_0}) + 6 + 4c_{\text{max}}^2 + 6c_{\text{max}} \right).
\]
We then proceed to bound the constrains violation. For any $i \in \{1, \cdots, p\}$, we have
\[
\mathbb{E}[J_i(\pi_{w_0})] - d_i = \frac{1}{|\mathcal{N}_0|} \sum_{t \in \mathcal{N}_0} J_i(\pi_{w_0}) - d_i \\
\leq \frac{1}{|\mathcal{N}_0|} \sum_{t \in \mathcal{N}_0} (\bar{J}_i(\theta^*_i) - d_i) + \frac{1}{|\mathcal{N}_0|} \sum_{t \in \mathcal{N}_0} |J_i(\pi_{w_0}) - \bar{J}_i(\theta^*_i)| \\
\leq \eta + \frac{1}{|\mathcal{N}_0|} \sum_{t \in \mathcal{N}_0} \sum_{t=0}^{T-1} |J_i(\pi_{w_0}) - \bar{J}_i(\theta^*_i)| \\
\leq \eta + \frac{1}{|\mathcal{N}_0|} \sum_{i=0}^p \sum_{t \in \mathcal{N}_i} \left| Q^i_{\pi_{w_0}} - Q^i_t \right|_2.
We provide the proof of supporting lemmas for Lemma 1. Suppose Assumption 1 holds. For any policy included here for completeness. For the following Lemma 11 and Lemma 12, we provide slightly different proofs from those in (Cai et al., 2019), which are

deferred to the appendix.

C. Proof of Lemma 1 and Theorem 2: Function Approximation Setting

For notation simplicity, we denote the state action pairs \((s, a)\) and \((s', a')\) as \(x\) and \(x'\), respectively. We define the weighted norm \(\|f\|_D = \sqrt{\int f(x)^2 dD(x)}\) for any distribution \(D\) over \(|S| \times |A|\). We will write \(\theta_k^*\) as \(\theta_k\) whenever there is no confusion in this subsection. We define

\[
f_0(x, \theta) = \frac{1}{\sqrt{m}} \sum_{r=1}^{m} b_r \mathbb{I}(\theta_0^r \psi(x) > 0) \theta_r^T \psi(x)
\]

as the local linearization of \(f(x, \theta)\) at the initial point \(\theta_0\). We denote the temporal differences as \(\delta_0(x, x', \theta_k) = f_0((s', a'); \theta_k) - f_0((s, a); \theta_k) - r(s, a, s')\) and \(\delta_k(x, x', \theta_k) = f((s', a'); \theta_k) - f((s, a); \theta_k) - r(s, a, s')\). We define the stochastic semi-gradient \(g_k(\theta_k) = \delta_k(x_k, x_k', \theta_k) \nabla \theta f(x_k, \theta_k)\), and the full semi-gradients \(\bar{g}_0(\theta_k) = \mathbb{E}_{\mu_{\pi}} [\delta_0(x, x', \theta_k) \nabla \theta f_0(x, \theta_k)]\), and \(\bar{g}_k(\theta_k) = \mathbb{E}_{\mu_{\pi}} [\delta_k(x, x', \theta_k) \nabla \theta f(x, \theta_k)]\). The approximated stationary point \(\theta^*\) satisfies \(\bar{g}_0(\theta^*) (\theta - \theta^*) \geq 0\) for any \(\theta \in B\). We define the following function spaces

\[
\mathcal{F}_{0, m} = \left\{ \frac{1}{\sqrt{m}} \sum_{r=1}^{m} b_r \mathbb{I}(\theta_0^r \psi(x) > 0) \theta_r^T \psi(x) : \|\theta - \theta_0\|_2 \leq R \right\},
\]

and

\[
\mathcal{F}_{0, m} = \left\{ \frac{1}{\sqrt{m}} \sum_{r=1}^{m} b_r \mathbb{I}(\theta_0^r \psi(s) > 0) \theta_r^T \psi(x) : \|\theta_r - \theta_0\|_\infty \leq R/\sqrt{md} \right\},
\]

and define \(f_0(x, \theta^*_n)\) as the projection of \(Q_{\pi}(x)\) onto the function space \(\mathcal{F}_{0, m}\) in terms of \(\|\cdot\|_{\mu_{\pi}}\) norm. Without loss of generality, we assume \(0 < \delta < \frac{1}{e}\) in the sequel.

C.1. Supporting Lemmas for Proof of Lemma 1

We provide the proof of supporting lemmas for Lemma 1.

Lemma 10 (Rahimi & Recht, 2009). Let \(f \in \mathcal{F}_{0, \infty}\), where \(\mathcal{F}_{0, \infty}\) is defined in Assumption 2. For any \(\delta > 0\), it holds with probability at least \(1 - \delta\) that

\[
\left\| \Pi_{\mathcal{F}_{0, m}} f - f \right\|_D^2 \leq \frac{4R^2 \log(\frac{1}{\delta})}{m},
\]

where \(D\) is any distribution over \(S \times A\).

For the following Lemma 11 and Lemma 12, we provide slightly different proofs from those in (Cai et al., 2019), which are included here for completeness.

Lemma 11. Suppose Assumption 1 holds. For any policy \(\pi\) and all \(k \geq 0\), it holds that

\[
\mathbb{E}_{\mu_{\pi}} \left[ \frac{1}{m} \sum_{r=1}^{m} \mathbb{I}(\theta_k^r \psi(x) > 0) - \mathbb{I}(\theta_0^r \psi(x) > 0) \right] \leq \frac{C_0 R}{d_1 \sqrt{m}}.
\]
\[ |\theta_{0,r}^T \psi(x) - \theta_{0,0}^T \psi(x)| \leq \|\theta_{k,r} - \theta_{0,r}\|_2, \]

which further implies

\[ |1(\theta_{k,r}^T \psi(x) > 0) - 1(\theta_{0,r}^T \psi(x) > 0)| \leq 1(|\theta_{0,r}^T \psi(x)| \leq \|\theta_{k,r} - \theta_{0,r}\|_2). \]  \hfill (21)

Then, we can derive the following upper bound

\[
\begin{align*}
\mathbb{E}_{\mu_*} \left[ \frac{1}{m} \sum_{r=1}^{m} |1(\theta_{k,r}^T \psi(x) - 1(\theta_{0,r}^T \psi(x) > 0)| \right] \\
\leq \mathbb{E}_{\mu_*} \left[ \frac{1}{m} \sum_{r=1}^{m} 1(|\theta_{0,r}^T \psi(x)| \leq \|\theta_{k,r} - \theta_{0,r}\|_2) \right] \\
= \frac{1}{m} \sum_{r=1}^{m} \mathbb{P}_{\mu_*} (|\theta_{0,r}^T \psi(x)| \leq \|\theta_{k,r} - \theta_{0,r}\|_2) \\
\leq \frac{(i) \ C_0}{m} \sum_{r=1}^{m} \|\theta_{k,r} - \theta_{0,r}\|_2 \|\theta_{0,r}\|_2 \\
\leq \frac{C_0}{m} \left( \sum_{r=1}^{m} \|\theta_{k,r} - \theta_{0,r}\|_2^2 \right)^{1/2} \left( \sum_{r=1}^{m} \frac{1}{\|\theta_{0,r}\|_2^2} \right)^{1/2} \\
\leq \frac{(ii) \ C_0 R}{d_1 \sqrt{m}}. \quad (22)
\end{align*}
\]

where (i) follows from Assumption 1 and (ii) follows from the fact that \( \|\theta_{0,r}\|_2 \geq d_1 \). \( \square \)

**Lemma 12.** Suppose Assumption 1 holds. For any policy \( \pi \) and all \( k \geq 0 \), it holds that

\[
\mathbb{E}_{\mu_*} \left[ |f((s,a); \theta_k) - f_0((s,a); \theta_k)|^2 \right] \leq \frac{4C_0R^3}{d_1 \sqrt{m}}.
\]

**Proof.** By definition, we have

\[
\begin{align*}
|f((s,a); \theta_k) - f_0((s,a); \theta_k)| \\
= \frac{1}{\sqrt{m}} \left| \sum_{r=1}^{m} (1(\theta_{k,r}^T \psi(x) > 0) - 1(\theta_{0,r}^T \psi(x) > 0)) b_r \theta_{k,r}^T \psi(x) \right| \\
\leq \frac{1}{\sqrt{m}} \left| \sum_{r=1}^{m} (1(\theta_{k,r}^T \psi(x) > 0) - 1(\theta_{0,r}^T \psi(x) > 0)) \|b_r\| \|\theta_{k,r}^T \psi(x)\|_2 \right| \\
\leq \frac{(i) \ 1}{\sqrt{m}} \sum_{r=1}^{m} 1(|\theta_{0,r}^T \psi(x)| \leq \|\theta_{k,r} - \theta_{0,r}\|_2 \|\theta_{0,r}^T \psi(x)\|_2) \\
\leq \frac{1}{\sqrt{m}} \sum_{r=1}^{m} 1(|\theta_{0,r}^T \psi(x)| \leq \|\theta_{k,r} - \theta_{0,r}\|_2 \|\theta_{0,r} - \theta_{k,r}\|_2 + \|\theta_{0,r}^T \psi(x)\|_2) \\
\leq \frac{2}{\sqrt{m}} \sum_{r=1}^{m} 1(|\theta_{0,r}^T \psi(x)| \leq \|\theta_{k,r} - \theta_{0,r}\|_2 \|\theta_{0,r} - \theta_{k,r}\|_2 \|\theta_{0,r}^T \psi(x)\|_2) \quad (24)
\end{align*}
\]

where (i) follows from eq. (21). We can then obtain the following upper bound.

\[
\mathbb{E}_{\mu_*} \left[ |f((s,a); \theta_k) - f_0((s,a); \theta_k)|^2 \right]
\]
We first upper bound the term

\[
\frac{4}{m} \mathbb{E}_{\mu^x} \left[ \left( \sum_{r=1}^m 1 \left( |\theta_{0,r}^T \psi(x)| \leq \|\theta_{k,r} - \theta_{0,r}\|_2 \right) \right)^2 \right]
\]

which implies

\[
\mathbb{E}_{\mu^x} \left[ \sum_{r=1}^m 1 \left( |\theta_{0,r}^T \psi(x)| \leq \|\theta_{k,r} - \theta_{0,r}\|_2 \right) \right] \leq \frac{4}{m} \sum_{r=1}^m \|\theta_{0,r} - \theta_{k,r}\|_2^2
\]

where (i) follows from Holder’s inequality, and (ii) follows from the derivation in Lemma 11 after eq. (22).

**Lemma 13.** Suppose Assumption 1 holds. For any policy \( \pi \) and all \( k \geq 0 \), with probability at least \( 1 - \delta \), we have

\[
\|g_k(\theta_k) - \bar{g}_0(\theta_k)\|_2 \leq \Theta \left( \frac{\sqrt{\log(1/\delta)}}{(1 - \gamma)m^{1/4}} \right).
\]

**Proof.** By definition, we have

\[
\|g_k(\theta_k) - \bar{g}_0(\theta_k)\|_2 = \mathbb{E}_{\mu^x} \left[ \left| \delta_k(x, x', \theta_k) \nabla_\theta f(x, \theta_k) - \delta_0(x, x', \theta_k) \nabla_\theta f_0(x, \theta_k) \right| \right]
\]

which follows from the fact that \( \|\nabla_\theta f(x, \theta_k)\|_2 \leq 1 \). Then, eq. (26) implies that

\[
\|g_k(\theta_k) - \bar{g}_0(\theta_k)\|_2 \leq 2 \mathbb{E}_{\mu^x} \left[ \left| \delta_k(x, x', \theta_k) - \delta_0(x, x', \theta_k) \right| \right] + 2 \mathbb{E}_{\mu^x} \left[ \left| \nabla_\theta f(x, \theta_k) - \nabla_\theta f_0(x, \theta_k) \right| \right]
\]

where (i) follows from Holder’s inequality, and (ii) follows from the derivation in Lemma 11 after eq. (22).

\[
\|g_k(\theta_k) - \bar{g}_0(\theta_k)\|_2 \leq \Theta \left( \frac{\sqrt{\log(1/\delta)}}{(1 - \gamma)m^{1/4}} \right).
\]

We first upper bound the term \( \mathbb{E}_{\mu^x} \left[ \left| \delta_k(x, x', \theta_k) - \delta_0(x, x', \theta_k) \right| \right] \). By definition, we have

\[
|\delta_k(x, x', \theta_k) - \delta_0(x, x', \theta_k)| = |f(x, \theta_k) - f_0(x, \theta_k) - \gamma(f(x', \theta_k) - f_0(x', \theta_k))| \leq |f(x, \theta_k) - f_0(x, \theta_k)| + |f(x', \theta_k) - f_0(x', \theta_k)|.
\]

which implies

\[
\mathbb{E}_{\mu^x} \left[ \left| \delta_k(x, x', \theta_k) - \delta_0(x, x', \theta_k) \right| \right] \leq 2 \mathbb{E}_{\mu^x} \left[ \left| f(x, \theta_k) - f_0(x, \theta_k) \right| \right] + 2 \mathbb{E}_{\mu^x} \left[ \left| f(x', \theta_k) - f_0(x', \theta_k) \right| \right] = 4 \mathbb{E}_{\mu^x} \left[ \left| f(x, \theta_k) - f_0(x, \theta_k) \right| \right] \leq \frac{16C_0R^2}{\sqrt{d_1}}.
\]

where (i) follows from Lemma 12. We then proceed to bound the term \( \mathbb{E}_{\mu^x} \left[ \left| \nabla_\theta f(x, \theta_k) - \nabla_\theta f_0(x, \theta_k) \right| \right] \). By definition, we have

\[
\|\nabla_\theta f(x, \theta_k) - \nabla_\theta f_0(x, \theta_k)\|_2
\]
\begin{align*}
&= \frac{1}{\sqrt{m}} \left\| \sum_{i=1}^{m} \left[ \mathbb{1} \left( \theta_{k,r}^\top x > 0 \right) - \mathbb{1} \left( \theta_{k,0}^\top x > 0 \right) \right] b_i \theta_{k,r}^\top x \right\|_2 \\
&\leq \frac{1}{\sqrt{m}} \sum_{i=1}^{m} \mathbb{1} \left( \theta_{k,r}^\top x > 0 \right) - \mathbb{1} \left( \theta_{k,0}^\top x > 0 \right) \| \theta_{0,r} \|_2 \\
&\leq \frac{1}{\sqrt{m}} \sum_{i=1}^{m} \mathbb{1} \left( \theta_{0,r}^\top x \right) - \| \theta_{k,r} - \theta_{0,r} \|_2 \| \theta_{0,r} \|_2 ,
\end{align*}

where \((i)\) follows because \(|b_i| \leq 1\) and \(\|\psi(s)\|_2 \leq 1\), and \((ii)\) follows from eq. (21). Further, eq. (29) implies that
\begin{align*}
E_{\mu^s} [\|\nabla \theta f(x, \theta_k) - \nabla \theta f_0(x, \theta_k)\|_2^2] &\leq 1 \frac{1}{m} E_{\mu^s} \left[ \left( \sum_{i=1}^{m} \mathbb{1} \left( \theta_{0,r}^\top x \right) \right) \left( \sum_{i=1}^{m} \| \theta_{0,r} \|_2 \right) \right] \\
&\leq \frac{R^2}{m} \sum_{i=1}^{m} \mathbb{1} \left( \theta_{0,r}^\top x \right) \leq \| \theta_{k,r} - \theta_{0,r} \|_2 \\
&\leq C_0 R^3 \frac{d_1}{d_1} m ,
\end{align*}

where \((i)\) follows from the derivation in Lemma 11 after eq. (22).

Finally, we upper-bound \(E_{\mu^s} [\| \delta_0(x, x', \theta_k) \|_2^2] \). We proceed as follows.
\begin{align*}
&= \frac{1}{\sqrt{m}} \left\| \sum_{i=1}^{m} \left[ \mathbb{1} \left( \theta_{k,r}^\top x > 0 \right) - \mathbb{1} \left( \theta_{k,0}^\top x > 0 \right) \right] b_i \theta_{k,r}^\top x \right\|_2 \\
&\leq \frac{1}{\sqrt{m}} \sum_{i=1}^{m} \mathbb{1} \left( \theta_{k,r}^\top x > 0 \right) - \mathbb{1} \left( \theta_{k,0}^\top x > 0 \right) \| \theta_{0,r} \|_2 \\
&\leq \frac{1}{\sqrt{m}} \sum_{i=1}^{m} \mathbb{1} \left( \theta_{0,r}^\top x \right) - \| \theta_{k,r} - \theta_{0,r} \|_2 \| \theta_{0,r} \|_2 ,
\end{align*}

where \((i)\) follows from the fact that \(Q(x) \leq \frac{\max_{\theta \in \mathcal{F}} \| \theta \|_2}{R} \text{ and } \| \theta^*_\pi \|_2 \leq R\).

Since \(\mathcal{F}_{0,m} \subset \mathcal{F}_{0,m}\). Lemma 10 implies that with probability at least \(1 - \delta\), we have
\begin{align*}
E_{\mu^s} [f_0(x, \theta^*_\pi) - Q_\pi(x)]^2 &\leq \frac{4R^2 \log \left( \frac{1}{\delta} \right)}{m} \leq 4R^2 \log \left( \frac{1}{\delta} \right) .
\end{align*}

Thus, with probability at least \(1 - \delta\), we have
\begin{align*}
E_{\mu^s} [\| \delta_0(x, x', \theta_k) \|_2^2] &\leq 18R^2 + \frac{21C_{\max}}{(1-\gamma)^2} + 72R^2 \log \left( \frac{1}{\delta} \right) .
\end{align*}

Combining eq. (28), eq. (30) and eq. (33), we can obtain that, with probability at least \(1 - \delta\), we have
\begin{align*}
\| \bar{g}_k(\theta_k) - \tilde{g}_0(\theta_k) \|_2^2 &\leq \Theta \left( \frac{\log \left( \frac{1}{\delta} \right)}{(1-\gamma)^2 \sqrt{m}} \right) ,
\end{align*}

which implies that with probability at least \(1 - \delta\), we have
\begin{align*}
\| \bar{g}_k(\theta_k) - \tilde{g}_0(\theta_k) \|_2 &\leq \Theta \left( \frac{\sqrt{\log \left( \frac{1}{\delta} \right)}}{(1-\gamma)m^{1/4}} \right) ,
\end{align*}

which completes the proof.
C.2. Proof of Lemma 1

We consider the convergence of $\theta_k^i$ for a given $i$ under a fixed policy $\pi$. For the iteration of $\theta_k$, we proceed as follows.

\[
\|\theta_{k+1} - \theta^*\|_2^2 \\
= \|\Pi_B(\theta_k - \beta g_k(\theta_k)) - \Pi_B(\theta^* - \beta \tilde{g}_0(\theta^*))\|_2^2 \\
\leq \|\theta_k - \theta^*\|_2^2 - 2\beta \|g_k(\theta_k) - \tilde{g}_0(\theta^*)\| + \beta^2 \|g_k(\theta_k) - \tilde{g}_0(\theta^*)\|_2^2 \\
= \|\theta_k - \theta^*\|_2^2 - 2\beta \|g_k(\theta_k) - \tilde{g}_0(\theta^*)\|_2^2 + 2\beta \|\tilde{g}_k(\theta_k) - g_k(\theta_k)\|_2^2 + 2\beta \|\tilde{g}_0(\theta_k) - \tilde{g}_0(\theta^*)\|_2^2 \\
\leq \|\theta_k - \theta^*\|_2^2 - 2\beta \|g_k(\theta_k) - \tilde{g}_0(\theta^*)\|_2^2 + 2\beta \|g_k(\theta_k) - \tilde{g}_0(\theta^*)\|_2^2 + 2\beta \|\tilde{g}_k(\theta_k) - g_k(\theta_k)\|_2^2 + 2\beta \|\tilde{g}_0(\theta_k) - \tilde{g}_0(\theta^*)\|_2^2 \\
+ 2\beta \|\tilde{g}_0(\theta_k) - \tilde{g}_0(\theta_k)\|^2 + 3\beta^2 \|\tilde{g}_k(\theta_k) - g_k(\theta_k)\|_2^2 + 3\beta^2 \|\tilde{g}_0(\theta_k) - \tilde{g}_0(\theta_k)\|^2 \\
(i) \leq \|\theta_k - \theta^*\|_2^2 - 2\beta \|g_k(\theta_k) - \tilde{g}_0(\theta^*)\|_2^2 + 2\beta \|\tilde{g}_k(\theta_k) - g_k(\theta_k)\|_2^2 + 2\beta \|\tilde{g}_0(\theta_k) - \tilde{g}_0(\theta^*)\|_2^2 + 3\beta^2 \|\tilde{g}_k(\theta_k) - g_k(\theta_k)\|_2^2 + 3\beta^2 \|\tilde{g}_0(\theta_k) - \tilde{g}_0(\theta_k)\|^2 \\
\leq \|\theta_k - \theta^*\|_2^2 - 2\beta \|g_k(\theta_k) - \tilde{g}_0(\theta^*)\|_2^2 + 2\beta \|\tilde{g}_k(\theta_k) - g_k(\theta_k)\|_2^2 + 2\beta \|\tilde{g}_0(\theta_k) - \tilde{g}_0(\theta^*)\|_2^2 + 3\beta^2 \|\tilde{g}_k(\theta_k) - g_k(\theta_k)\|_2^2 + 3\beta^2 \|\tilde{g}_0(\theta_k) - \tilde{g}_0(\theta_k)\|^2, \\
\text{where (i) follows from the fact that} \\
(\tilde{g}_0(\theta_k) - \tilde{g}_0(\theta^*))\top (\theta_k - \theta^*) \\
\geq (1 - \gamma) \mathbb{E}_{\mu^*} [(f_0((s, a); \theta_k) - f_0((s, a); \theta^*))^2] - R \|\tilde{g}_k(\theta_k) - \tilde{g}_0(\theta_k)\|_2^2, \\
\text{and (ii) follows from the fact that} \\
\|\tilde{g}_0(\theta_k) - \tilde{g}_0(\theta^*)\|^2 \leq 4 \mathbb{E}_{\mu^*} [(f_0((s, a); \theta_k) - f_0((s, a); \theta^*))^2].
\]

Rearranging eq. (34) yields

\[
[2\beta(1 - \gamma) - 12\beta^2] \mathbb{E}_{\mu^*} [(f_0((s, a); \theta_k) - f_0((s, a); \theta^*))^2] \\
\leq \|\theta_k - \theta^*\|_2^2 - \|\theta_{k+1} - \theta^*\|_2^2 + 2\beta \|g_k(\theta_k) - \tilde{g}_0(\theta_k)\|_2^2 + 2\beta \|\tilde{g}_k(\theta_k) - g_k(\theta_k)\|_2^2 + 2\beta \|\tilde{g}_0(\theta_k) - \tilde{g}_0(\theta_k)\|^2 \\
+ 3\beta^2 \|\tilde{g}_k(\theta_k) - g_k(\theta_k)\|_2^2 + 3\beta^2 \|\tilde{g}_0(\theta_k) - \tilde{g}_0(\theta_k)\|^2. \\
\text{(35)}
\]

Taking summation of eq. (35) over $t = 0$ to $K - 1$ yields

\[
[2\beta(1 - \gamma) - 12\beta^2] \sum_{t=0}^{K-1} \mathbb{E}_{\mu^*} [(f_0((s, a); \theta_k) - f_0((s, a); \theta^*))^2] \\
\leq \|\theta_0 - \theta^*\|_2^2 - \|\theta_K - \theta^*\|_2^2 + 2\beta \sum_{t=0}^{K-1} (\tilde{g}_k(\theta_k) - g_k(\theta_k))\top (\theta_k - \theta^*) + 2\beta \sum_{t=0}^{K-1} \|\tilde{g}_k(\theta_k) - g_k(\theta_k)\|_2^2 \\
+ 3\beta^2 \sum_{t=0}^{K-1} \|g_k(\theta_k) - \tilde{g}_k(\theta_k)\|^2 + 3\beta^2 \sum_{t=0}^{K-1} \|\tilde{g}_0(\theta_k) - \tilde{g}_0(\theta_k)\|^2 \\
(i) \leq R^2 + 2\beta \sum_{t=0}^{K-1} \xi_k(\theta_k)\top (\theta_k - \theta^*) + 3\beta^2 \sum_{t=0}^{K-1} \|\xi_k(\theta_k)\|_2^2 + 2\beta \sum_{t=0}^{K-1} \|\xi_k(\theta_k)\|_2^2 + 3\beta^2 \sum_{t=0}^{K-1} \|\xi_k(\theta_k)\|^2,
\]
where in (i) we define \( \zeta_k(\theta_k) = \bar{g}_k(\theta_k) - g_k(\theta_k) \) and \( \xi_k(\theta_k) = \bar{g}_k(\theta_k) - \bar{g}_0(\theta_k) \).

We first consider the term \( \sum_{t=0}^{K-1} \|\zeta_k(\theta_k)\|_2^2 \). We proceed as follows.

\[
P_{\mu^*} \left( \sum_{t=0}^{K-1} \|\zeta_k(\theta_k)\|_2^2 \geq (1 + A) C_\zeta^2 K \right)
= P_{\mu^*} \left( \sum_{t=0}^{K-1} \|\zeta_k(\theta_k)\|_2^2 \geq 1 + A \right)
= P_{\mu^*} \left( \exp \left( \sum_{t=0}^{K-1} \|\zeta_k(\theta_k)\|_2^2 \right) \geq \exp(1 + A) \right)
\leq \frac{1}{K} \sum_{t=0}^{K-1} \mathbb{E}_{\mu^*} \left[ \exp \left( \frac{\|\zeta_k(\theta_k)\|_2^2}{C_\zeta^2} \right) \right] / \exp(1 + A)
\leq \exp(-A),
\tag{36}
\]

where (i) follows from Markov’s inequality, (ii) follows from Assumption 3. Then, eq. (36) implies that with probability at least \( 1 - \delta_1 \), we have

\[
\sum_{t=0}^{K-1} \|\zeta_k(\theta_k)\|_2^2 \leq \left( 1 + \log \left( \frac{1}{\delta_1} \right) \right) C_\zeta^2 K \leq 2 \log \left( \frac{1}{\delta_1} \right) C_\zeta^2 K.
\tag{37}
\]

We then consider the term \( \sum_{t=0}^{K-1} \zeta_k(\theta_k)^\top (\theta_k - \theta^*) \). Note that for any \( 0 \leq k \leq K - 1 \), we have

\[
|\zeta_k(\theta_k)^\top (\theta_k - \theta^*)|^2 \leq \|\zeta_k(\theta_k)\|_2^2 \|\theta_k - \theta^*\|_2^2 \leq R^2 \|\zeta_k(\theta_k)\|_2^2,
\]

which implies

\[
\mathbb{E}_{\mu^*} \left[ \exp \left( \frac{|\zeta_k(\theta_k)^\top (\theta_k - \theta^*)|^2}{B^2 C_\zeta^2} \right) \right] \leq \mathbb{E}_{\mu^*} \left[ \exp \left( \frac{\|\zeta_k(\theta_k)\|_2^2}{C_\zeta^2} \right) \right] \leq \exp(1).
\]

Applying Bernstein’s inequality for martingale (Ghadimi & Lan, 2013)[Lemma 2.3], we can obtain

\[
P_{\mu^*} \left( \left| \sum_{t=0}^{K-1} \zeta_k(\theta_k)^\top (\theta_k - \theta^*) \right| \geq \sqrt{2(1 + A) C_\zeta \sqrt{K}} \right) \leq \exp(-A^2/3),
\]

which implies that with probability at least \( 1 - \delta_2 \), we have

\[
\left| \sum_{t=0}^{K-1} \zeta_k(\theta_k)^\top (\theta_k - \theta^*) \right| \leq \sqrt{2 \left( 1 + \sqrt{3 \log \left( \frac{1}{\delta_2} \right) } \right) C_\zeta \sqrt{K}} \leq 5C_\zeta \sqrt{\log \left( \frac{1}{\delta_2} \right) \sqrt{K}}.
\tag{38}
\]

We then consider the terms \( \sum_{t=0}^{K-1} \|\xi_k(\theta_k)\|_2 \) and \( \sum_{t=0}^{K-1} \|\xi_k(\theta_k)\|_2^2 \). Lemma 13 implies that with probability at least \( 1 - \delta_3/K \), we have

\[
\|\xi_k(\theta_k)\|_2 \leq \Theta \left( \frac{\sqrt{\log(K)}}{(1 - \gamma)m^{1/4}} \right).
\]
Applying then union bound we can obtain that with probability at least $1 - \delta_3$, we have
\begin{equation}
\sum_{t=0}^{K-1} \|\xi_k(\theta_k)\|_2 \leq \Theta \left( \frac{K \sqrt{\log \left( \frac{K}{\delta_3} \right)}}{(1 - \gamma)m^{1/4}} \right).
\end{equation}

Similarly, we can obtain that with probability at least $1 - \delta_3$, we have
\begin{equation}
\sum_{t=0}^{K-1} \|\xi_k(\theta_k)\|^2_2 \leq \Theta \left( \frac{K \log \left( \frac{K}{\delta_3} \right)}{(1 - \gamma)^2 m^{1/2}} \right).
\end{equation}

Combining eq. (37), eq. (38), eq. (39) and eq. (40) and applying the union bound, we can obtain that with probability at least $1 - (\delta_1 + \delta_2 + \delta_3 + \delta_4)$, we have
\begin{equation}
[2\beta(1 - \gamma) - 12\beta^2] \sum_{t=0}^{K-1} \mathbb{E}_{\mu_\gamma} [(f_0((s, a); \theta_k) - f_0((s, a); \theta^*))^2] \leq R^2 + 10\beta C_\zeta \sqrt{\log \left( \frac{1}{\delta_2} \right)} \sqrt{K} + 6\beta^2 \log \left( \frac{1}{\delta_1} \right) C_\zeta K + \beta K \Theta \left( \frac{\log \left( \frac{K}{\delta_3} \right)}{(1 - \gamma)m^{1/4}} \right) + \beta^2 K \Theta \left( \frac{\log \left( \frac{K}{\delta_3} \right)}{(1 - \gamma)^2 m^{1/2}} \right).
\end{equation}

Divide both sides of eq. (41) by $[2\beta(1 - \gamma) - 12\beta^2]K$. Recalling that the stepsize $\beta = \min \{1/\sqrt{K}, (1 - \gamma)/12 \}$, which implies that $\frac{1}{\sqrt{K}[2\beta(1 - \gamma) - 12\beta^2]} \leq \frac{12}{(1 - \gamma)^{2}}$. Then, with probability at least $1 - (\delta_1 + \delta_2 + \delta_3 + \delta_4)$, we have
\begin{equation}
\|f_0((s, a); \bar{\theta}_K) - f_0((s, a); \theta^*)\|_{\mu_\gamma}^2 \leq \frac{1}{K} \sum_{t=0}^{K-1} \mathbb{E}_{\mu_\gamma} [(f_0((s, a); \theta_k) - f_0((s, a); \theta^*))^2] \leq \Theta \left( \frac{\sqrt{\log \left( \frac{K}{\delta_3} \right)}}{(1 - \gamma)^2 \sqrt{K}} \right) + \Theta \left( \frac{\log \left( \frac{K}{\delta_3} \right)}{(1 - \gamma)^3 m^{1/4}} \right) + \Theta \left( \frac{1}{(1 - \gamma)^2 \sqrt{K}} \log \left( \frac{1}{\delta_1} \right) \right) + \Theta \left( \frac{1}{(1 - \gamma)^3 m^{1/4}} \log \left( \frac{1}{\delta_2} \right) \right) + \Theta \left( \frac{1}{(1 - \gamma)^2 \sqrt{K}} \sqrt{\log \left( \frac{1}{\delta_1} \right)} \right) + \Theta \left( \frac{\sqrt{\log \left( \frac{K}{\delta_3} \right)}}{(1 - \gamma)^3 m^{1/4}} \sqrt{\log \left( \frac{1}{\delta_2} \right)} \right) + \Theta \left( \frac{\log \left( \frac{K}{\delta_3} \right)}{(1 - \gamma)^3 m^{1/4}} \log \left( \frac{K}{\delta_4} \right) \right) + \Theta \left( \frac{1}{(1 - \gamma)^3 m^{1/4}} \left( \sqrt{\log \left( \frac{K}{\delta_3} \right)} + \log \left( \frac{K}{\delta_4} \right) \right) \right).
\end{equation}
Finally, we upper bound \( \| f((s,a); \hat{\theta}_K) - Q_\pi(s,a) \|^2_{\mu_\pi} \). We proceed as follows

\[
\begin{align*}
\| f((s,a); \hat{\theta}_K) - Q_\pi(s,a) \|^2_{\mu_\pi} & \leq 3 \| f((s,a); \hat{\theta}_K) - f_0((s,a); \hat{\theta}_K) \|^2_{\mu_\pi} + 3 \| f_0((s,a); \hat{\theta}_K) - f_0((s,a); \theta^*) \|^2_{\mu_\pi} \\
& + 3 \| f_0((s,a); \theta^*) - Q_\pi(s,a) \|^2_{\mu_\pi} \\
& \leq \Theta \left( \frac{1}{\sqrt{m}} \right) + 3 \| f_0((s,a); \hat{\theta}_K) - f_0((s,a); \theta^*) \|^2_{\mu_\pi} + \frac{3}{1 - \gamma} \| f_0((s,a); \theta^*) - Q_\pi(s,a) \|^2_{\mu_\pi},
\end{align*}
\]  

where (i) follows from Lemma 12 and the fact that

\[
\| f_0((s,a); \theta^*) - Q_\pi(s,a) \|^2_{\mu_\pi} \leq \frac{1}{1 - \gamma} \| f_0((s,a); \theta^*) - Q_\pi(s,a) \|^2_{\mu_\pi},
\]

which is given in (Cai et al., 2019). Then, eq. (32) implies that, with probability at least \( \delta_5 \), we have

\[
\| f_0((s,a); \theta^*) - Q_\pi(s,a) \|^2_{\mu_\pi} \leq \frac{4R^2 \log \left( \frac{1}{\delta_5} \right)}{m}.
\]

Substituting eq. (42) and eq. (44) into eq. (43), we have with probability at least \( 1 - (\delta_1 + \delta_2 + \delta_3 + \delta_4 + \delta_5) \), the following holds:

\[
\begin{align*}
\| f((s,a); \hat{\theta}_K) - Q_\pi(s,a) \|^2_{\mu_\pi} & \leq \Theta \left( \frac{1}{(1 - \gamma)^2 \sqrt{K}} \left( \log \left( \frac{1}{\delta_1} \right) + \sqrt{\log \left( \frac{1}{\delta_1} \right)} \right) \right) \\
& \quad + \Theta \left( \frac{1}{(1 - \gamma)^3 m^{1/4}} \left( \log \left( \frac{K}{\delta_3} \right) + \sqrt{\log \left( \frac{K}{\delta_3} \right)} \right) \right) \\
& \quad + \Theta \left( \frac{1}{(1 - \gamma) m \log \left( \frac{1}{\delta_5} \right)} \right).
\end{align*}
\]

Letting \( \delta_1 = \delta_2 = \delta_3 = \delta_4 = \delta_5 = \frac{\delta}{5} \), we have with probability at least \( 1 - \delta \), the following holds:

\[
\begin{align*}
\| f((s,a); \hat{\theta}_K) - Q_\pi(s,a) \|^2_{\mu_\pi} & \leq \Theta \left( \frac{1}{(1 - \gamma)^2 \sqrt{K}} \left( \log \left( \frac{1}{\delta} \right) \right) \right) + \Theta \left( \frac{1}{(1 - \gamma)^3 m^{1/4}} \left( \log \left( \frac{K}{\delta} \right) \right) \right),
\end{align*}
\]

which completes the proof.

C.3. Supporting Lemmas for Proof of Theorem 2

For the two-layer neural network defined in eq. (6), we have the following property: \( \tau \cdot f(x, W) = f(x, \tau W) \). Thus, in the sequel, we write \( \pi_W(a|s) = \pi_{\tau W}(a|s) \). In the technical proof, we consider the following policy class:

\[
\pi_W(a|s) := \sum_{a'} \frac{\exp \left( f((s,a); W) \right)}{\sum_{a'} \exp \left( f((s,a'); W) \right)}, \quad \forall (s,a) \in S \times A,
\]

and \( J_t(W) \) as the accumulated cost with policy \( \pi_W \). We denote \( \phi_W(s,a) = \nabla_W f_t((s,a), W) \). We define the diameter of \( B_W \) as \( R_W \). When performing each NPG update, we will need to solve the linear regression problem specified in eq. (10). As shown in (Wang et al., 2019), when the neural network for the policy parametrization and value function approximation share the same initialization, \( \hat{\theta}_t \) is an approximated solution of the problem eq. (10). Thus, instead of solving the problem eq. (10) directly, here we simply use \( \hat{\theta}_t \) as the approximated NPG update at each iteration:

\[
\tau_{t+1} \cdot W_{t+1} = \tau_t \cdot W_t + \frac{\alpha}{1 - \gamma} \hat{\theta}_t.
\]
Without loss of generality, we assume that for the visitation distribution of the global optimal policy $\nu^*$, there exists a constants $C_{RN}$ such that for all $\pi_w$, the following holds

$$\int_x \left( \frac{d\nu^*(x)}{d\mu_{\pi_w}(x)} \right)^2 d\mu_{\pi_w}(x) \leq C_{RN}^2. \tag{46}$$

**Lemma 14.** For any $\theta, \theta' \in B$ and $\pi$, we have

$$\|\phi_\theta(s, a)^\top \theta' - \phi_{\theta_0}(s, a)^\top \theta'\|_{\mu_\pi}^2 \leq \frac{4C_0R^3}{d_1\sqrt{m}}. \tag{47}$$

**Proof.** By definition, we have

$$\phi_\theta(s, a)^\top \theta' - \phi_{\theta_0}(s, a)^\top \theta'$$

$$= \frac{1}{\sqrt{m}} \left| \sum_{r=1}^m \left( \mathbb{I}(\theta_r^\top \psi(x) > 0) - \mathbb{I}(\theta_{0,r}^\top \psi(x) > 0) \right) b_r \theta_r^\top \psi(x) \right|$$

$$\leq \frac{1}{\sqrt{m}} \sum_{r=1}^m \left| \left( \mathbb{I}(\theta_r^\top \psi(x) > 0) - \mathbb{I}(\theta_{0,r}^\top \psi(x) > 0) \right) b_r \right| \|\theta_r^\top \psi(x)\|_2$$

$$\leq \frac{1}{\sqrt{m}} \sum_{r=1}^m \left( \mathbb{I}(\theta_{0,r}^\top \psi(x) \leq \|\theta_r - \theta_{0,r}\|_2) \right) \|\theta_r^\top \psi(x)\|_2$$

$$\leq \frac{1}{\sqrt{m}} \sum_{r=1}^m \left( \mathbb{I}(\theta_{0,r}^\top \psi(x) \leq \|\theta_r - \theta_{0,r}\|_2) \right) \left( \|\theta_r^\top \psi(x)\|_2 + \|\theta_{0,r}^\top \psi(x)\|_2 \right)$$

$$\leq \frac{1}{\sqrt{m}} \sum_{r=1}^m \left( \mathbb{I}(\theta_{0,r}^\top \psi(x) \leq \|\theta_r - \theta_{0,r}\|_2) \right) \left( \|\theta_r^\top \psi(x)\|_2 + \|\theta_{0,r}^\top \psi(x)\|_2 \right), \tag{47}$$

where (i) follows from eq. (21). Following from Holder’s inequality, we obtain from eq. (47) that

$$\left|\phi_\theta(s, a)^\top \theta' - \phi_{\theta_0}(s, a)^\top \theta'\right|^2$$

$$\leq \frac{1}{m} \left( \sum_{r=1}^m \mathbb{I}(\theta_{0,r}^\top \psi(x) \leq \|\theta_r - \theta_{0,r}\|_2) \right) \left( \sum_{r=1}^m \left( \|\theta_r^\top \psi(x)\|_2 + \|\theta_r^\top \psi(x)\|_2 \right)^2 \right)$$

$$\leq \frac{2}{m} \left( \sum_{r=1}^m \mathbb{I}(\theta_{0,r}^\top \psi(x) \leq \|\theta_r - \theta_{0,r}\|_2) \right) \left( \sum_{r=1}^m \|\theta_r - \theta_{0,r}\|^2 + \sum_{r=1}^m \|\theta_r - \theta_{0,r}\|^2 \right)$$

$$\leq \frac{4R^2}{m} \left( \sum_{r=1}^m \mathbb{I}(\theta_{0,r}^\top \psi(x) \leq \|\theta_r - \theta_{0,r}\|_2) \right),$$

which implies

$$\|\phi_\theta(s, a)^\top \theta' - \phi_{\theta_0}(s, a)^\top \theta'\|_{\mu_\pi}^2 = \mathbb{E}_{\mu_\pi}[\left|\phi_\theta(s, a)^\top \theta' - \phi_{\theta_0}(s, a)^\top \theta'\right|^2] \leq \frac{4C_0R^3}{d_1\sqrt{m}}, \tag{48}$$

where (i) follows from the derivation in Lemma 11 after eq. (22).

**Lemma 15** (Upper bound on optimality gap for neural NPG). Consider the approximated NPG updates in the neural network approximation setting. We have

$$\alpha(1 - \gamma)(J_0(\pi^*) - J_0(\pi_{\tau, W_1}))$$
Applying the Lipschitz property of \(E = \alpha \leq \alpha^* \alpha + \alpha^* \nu \sqrt{\nu} \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \n
where (i) follows from Holder’s inequality, and (ii) follows from eq. (46). Similarly, we can obtain

$$\int_x h(x) d(\nu^* \pi_W)(x) \le C_{\text{RN}}^2 \left\| h(x) \right\|_{\mu^* W}.$$  \hspace{1cm} (51)

Substituting eq. (50) and eq. (51) into eq. (49) and using the fact that \(\left\| \bar{\theta}_t \right\|_2 \le R + \sqrt{md_2}\) yield

$$\mathbb{E}_{\nu^*} \left[ D_{\text{KL}}(\pi^* || \pi_{\tau_t W_t}) \right] - \mathbb{E}_{\nu^*} \left[ D_{\text{KL}}(\pi^* || \pi_{\tau_{t+1} W_{t+1}}) \right]$$

$$\ge \alpha(1 - \gamma)(J_0(\pi^*) - J_0(\pi_{\tau_t W_t})) - \alpha C_{\text{RN}} \sqrt{\mathbb{E}_{\nu^*} \left[ (\phi_W(s, a)^\top \bar{\theta}_t - f((s, a), \bar{\theta}_t))^2 \right]}$$

$$- \alpha C_{\text{RN}} \left\| (Q_{\pi_{\tau_t W_t}}(s, a') - f((s, a'), \bar{\theta}_t))^2 \right\|$$

$$- \alpha C_{\text{RN}} \left\| (f((s, a'), \bar{\theta}_t) - \phi_W(s, a')^\top \bar{\theta}_t)^2 \right\| - \alpha^2 L_f (R^2 + md_2^2)$$

$$= \alpha(1 - \gamma)(J_0(\pi^*) - J_0(\pi_{\tau_t W_t})) - 2\alpha C_{\text{RN}} \left\| (\phi_W(s, a)^\top \bar{\theta}_t - f((s, a), \bar{\theta}_t))^2 \right\|$$

$$- 2\alpha C_{\text{RN}} \left\| (f((s, a), \bar{\theta}_t) - Q_{\pi_{\tau_t W_t}}(s, a))^2 \right\|$$

$$- 2\alpha C_{\text{RN}} \left\| (f((s, a), \bar{\theta}_t) - Q_{\pi_{\tau_t W_t}}(s, a))^2 \right\|$$

$$\le \alpha(1 - \gamma)(J_0(\pi^*) - J_0(\pi_{\tau_t W_t}))$$

$$- 2\alpha C_{\text{RN}} \left\| (\phi_W(s, a)^\top \bar{\theta}_t - f((s, a), \bar{\theta}_t))^2 \right\|$$

$$+ 2\alpha C_{\text{RN}} \left\| (\phi_W(s, a)^\top \bar{\theta}_t - f((s, a), \bar{\theta}_t))^2 \right\|$$

$$\le \frac{16 C_{\text{c}} R^3}{d_1 \sqrt{m}}.$$  \hspace{1cm} (53)

where (i) follows from Lemma 12 and Lemma 14. Substituting eq. (53) into eq. (52) yields

$$\mathbb{E}_{\nu^*} \left[ D_{\text{KL}}(\pi^* || \pi_{\tau_t W_t}) \right] - \mathbb{E}_{\nu^*} \left[ D_{\text{KL}}(\pi^* || \pi_{\tau_{t+1} W_{t+1}}) \right]$$

$$\le \alpha(1 - \gamma)(J_0(\pi^*) - J_0(\pi_{\tau_t W_t}))$$

$$- \frac{8\alpha C_{\text{RN}} \sqrt{C_0 R^{1.5}}}{\sqrt{d_1^2 m^{1/4}}} - \alpha^2 L_f (R^2 + md_2^2)$$

$$- 2\alpha C_{\text{RN}} \left\| (f((s, a), \bar{\theta}_t) - Q_{\pi_{\tau_t W_t}}(s, a))^2 \right\|_{\mu^* W_t}.$$  \hspace{1cm} (54)

Rearranging the above inequality yields the desired result. 

Note that when we follow the update in line 10 of Algorithm 1, we can obtain similar results for the case \(i \in \{1, \cdots, p\}\) as stated in Lemma 15:

$$\alpha(1 - \gamma)(J_i(\pi_{\tau_t W_t}) - J_i(\pi^*))$$

$$\le \mathbb{E}_{\nu^*} \left[ D_{\text{KL}}(\pi^* || \pi_{\tau_t W_t}) \right] - \mathbb{E}_{\nu^*} \left[ D_{\text{KL}}(\pi^* || \pi_{\tau_{t+1} W_{t+1}}) \right]$$

$$+ \frac{8\alpha C_{\text{RN}} \sqrt{C_0 R^{1.5}}}{\sqrt{d_1^2 m^{1/4}}} + \alpha^2 L_f (R^2 + md_2^2)$$

$$+ 2\alpha C_{\text{RN}} \left\| (f((s, a), \bar{\theta}_t) - Q_{\pi_{\tau_t W_t}}(s, a))^2 \right\|_{\mu^* W_t}.$$
Lemma 16. Considering the CRPO update in Algorithm 1 in the neural network approximation setting. Let $K_m = C_1((1 - \gamma)^2 \sqrt{m})$ and $N = T \log(2T/\delta)$. With probability at least $1 - \delta$, we have

$$
\alpha(1 - \gamma) \sum_{i \in \mathcal{N}_0} (J_0(\pi^*) - J_0(\pi_{w_i})) + \alpha(1 - \gamma)\eta \sum_{i=1}^p |\mathcal{N}_i|
\leq \mathbb{E}_{s \sim \nu^*} D_{KL}(\pi^* || \pi_{w_0}) + C_3 \left( \frac{\alpha T}{m^{1/4}} \right) + C_4 (\alpha^2 mT)
+ C_5 \left( \frac{\alpha T}{(1 - \gamma)^{1.5} m^{1/8}} \log^4 \left( \frac{T^3}{\delta} \right) \right) + C_6 \left( \alpha(1 - \gamma) \sqrt{T} \right).
$$

where $C_3 = \frac{8\mathbb{E}_{R^1 \mathbb{R}^2} R^{1.5}}{\sqrt{d_1}}$, $C_4 = L_f (R^2 + d_2^2)$, $C_5 = 3\alpha C_2 C_{RN}$, $C_6 = 2C_f$ and $C_2$ is a positive constant depend on $C_1$.

Proof. We define $\mathcal{N}_i$ as the set of steps that CRPO algorithm chooses to minimize the $i$-th constraint. If $t \in \mathcal{N}_0$, by Lemma 15 we have

$$
\alpha(1 - \gamma) (J_0(\pi^*) - J_0(\pi_{\tau, W_i}))
\leq \mathbb{E}_{\nu^*} [D_{KL}(\pi^* || \pi_{\tau, W_i})] - \mathbb{E}_{\nu^*} [D_{KL}(\pi^* || \pi_{\tau, W_{i+1}})] + \frac{8\alpha C_{RN} \sqrt{C_0 R^{1.5}}}{\sqrt{d_1} m^{1/4}} + \alpha^2 L_f (R^2 + m d_2^2)
+ 2\alpha C_{RN} \left\| f_i((s, a), \theta_i) - Q^i_{\pi_{\tau, W_i}}(s, a) \right\|_{\pi_{\tau, W_i}}. \tag{54}
$$

If $t \in \mathcal{N}_i$, similarly we can obtain

$$
\alpha(1 - \gamma) (J_i(\pi_{\tau, W_i}) - J_i(\pi^*))
\leq \mathbb{E}_{\nu^*} [D_{KL}(\pi^* || \pi_{\tau, W_i})] - \mathbb{E}_{\nu^*} [D_{KL}(\pi^* || \pi_{\tau, W_{i+1}})] + \frac{8\alpha C_{RN} \sqrt{C_0 R^{1.5}}}{\sqrt{d_1} m^{1/4}} + \alpha^2 L_f (R^2 + m d_2^2)
+ 2\alpha C_{RN} \left\| f_i((s, a), \theta_i) - Q^i_{\pi_{\tau, W_i}}(s, a) \right\|_{\pi_{\tau, W_i}}. \tag{55}
$$

Taking summation of eq. (12) and eq. (13) from $t = 0$ to $T - 1$ yields

$$
\alpha(1 - \gamma) \sum_{t \in \mathcal{N}_0} (J_0(\pi^*) - J_0(\pi_{w_t})) + \alpha(1 - \gamma) \sum_{i=1}^p \sum_{t \in \mathcal{N}_i} (J_i(\pi_{w_t}) - J_i(\pi^*))
\leq \mathbb{E}_{s \sim \nu^*} D_{KL}(\pi^* || \pi_{w_0}) + \frac{8\alpha C_{RN} \sqrt{C_0 R^{1.5}}}{\sqrt{d_1} m^{1/4}} + \alpha^2 L_f (R^2 + m d_2^2) T
+ 2\alpha C_{RN} \sum_{i=0}^p \sum_{t \in \mathcal{N}_i} \left\| f_i((s, a), \theta_i) - Q^i_{\pi_{\tau, W_i}}(s, a) \right\|_{\pi_{\tau, W_i}}. \tag{56}
$$

Note that when $t \in \mathcal{N}_i (i \neq 0)$, we have $\bar{J}_i(\theta^i_t) > d_i + \eta$ (line 9 in Algorithm 1), which implies that

$$
J_i(\pi_{\tau, W_i}) - J_i(\pi^*) \geq \bar{J}_i(\theta^i_t) - J_i(\pi^*) - |\bar{J}_i(\theta^i_t) - J_i(\pi_{\tau, W_i})|
\geq d_i + \eta - J_i(\pi^*) - |\bar{J}_i(\theta^i_t) - J_i(\pi_{\tau, W_i})|
\geq \eta - |\bar{J}_i(\theta^i_t) - J_i(\pi_{\tau, W_i})|. \tag{57}
$$

To bound the term $|\bar{J}_i(\theta^i_t) - J_i(\pi_{\tau, W_i})|$, we proceed as follows

$$
|\bar{J}_i(\theta^i_t) - J_i(\pi_{\tau, W_i})|
= |\bar{J}_i(\theta^i_t) - \mathbb{E}_{\nu^{\pi_{\tau, W_i}}} [f_i((s, a), \theta_i)] + \mathbb{E}_{\nu^{\pi_{\tau, W_i}}} [f_i((s, a), \theta_i)] - J_i(\pi_{\tau, W_i})|
\leq |\bar{J}_i(\theta^i_t) - \mathbb{E}_{\nu^{\pi_{\tau, W_i}}} [f_i((s, a), \theta_i)]| + \left\| f_i((s, a), \theta_i) - Q^i_{\pi_{\tau, W_i}}(s, a) \right\|_{\pi_{\tau, W_i}}.
$$
We then upper bound the term

\[ (i) \left| J_i(\theta_t^i) - \mathbb{E}_{\nu_{\pi_{\tau Wt}}} f_i((s, a), \tilde{\theta}_t) \right| + C_{RN} \left\| f_i((s, a), \tilde{\theta}_t) - Q^i_{\pi_{\tau Wt}}(s, a) \right\|_{\mu_{\pi_{\tau Wt}}}, \]

(58)

where \((i)\) can be obtained by following steps similar to those in eq. (50). Substituting eq. (58) into eq. (57) yields

\[
J_i(\pi_{\tau Wt}) - J_i(\pi^*) \\
\geq \eta \left( J_i(\theta_t^i) - \mathbb{E}_{\nu_{\pi_{\tau Wt}}} f_i((s, a), \tilde{\theta}_t) \right) + C_{RN} \left\| f_i((s, a), \tilde{\theta}_t) - Q^i_{\pi_{\tau Wt}}(s, a) \right\|_{\mu_{\pi_{\tau Wt}}}. \tag{59}
\]

Then, substituting eq. (59) into eq. (56) yields

\[
\alpha(1 - \gamma) \sum_{t \in \mathbb{N}_0} (J_0(\pi^*) - J_0(\pi_{\tau Wt})) + \alpha(1 - \gamma) \eta \sum_{i=1}^p |N_t| \\
\leq \mathbb{E}_{s \sim \nu^*} D_{KL}(\pi^*||\pi_{\tau Wt}) + \frac{8\alpha C_{RN} \sqrt{C_0 R^1.5 T}}{\sqrt{d_1} m^{1/4}} + \alpha^2 L_f (R^2 + m d_2) T \\
+ 3\alpha C_{RN} \sum_{i=0}^p \sum_{t \in \mathbb{N}_t} \left\| f_i((s, a), \tilde{\theta}_t) - Q^i_{\pi_{\tau Wt}}(s, a) \right\|_{\mu_{\pi_{\tau Wt}}} \\
+ \alpha(1 - \gamma) \sum_{i=1}^p \sum_{t \in \mathbb{N}_t} \left| J_i(\theta_t^i) - \mathbb{E}_{\nu_{\pi_{\tau Wt}}} f_i((s, a), \tilde{\theta}_t) \right| \\
\leq \mathbb{E}_{s \sim \nu^*} D_{KL}(\pi^*||\pi_{\tau Wt}) + \frac{8\alpha C_{RN} \sqrt{C_0 R^1.5 T}}{\sqrt{d_1} m^{1/4}} + \alpha^2 L_f (R^2 + m d_2) T \\
+ 3\alpha C_{RN} \sum_{t=0}^{T-1} \left\| f_i((s, a), \tilde{\theta}_t) - Q^i_{\pi_{\tau Wt}}(s, a) \right\|_{\mu_{\pi_{\tau Wt}}} \\
+ \alpha(1 - \gamma) \sum_{t=0}^{T-1} \left| J_i(\theta_t^i) - \mathbb{E}_{\nu_{\pi_{\tau Wt}}} f_i((s, a), \tilde{\theta}_t) \right|. \tag{60}
\]

We then upper bound the term \( \sum_{t=0}^{T-1} \left\| f_i((s, a), \tilde{\theta}_t) - Q^i_{\pi_{\tau Wt}}(s, a) \right\|_{\mu_{\pi_{\tau Wt}}} \). Lemma 1 implies that if we let \( K_{in} = C_1 (1 - \gamma)^2 \sqrt{m} \), then with probability at least \( 1 - \delta_1 / T \), we have

\[
\left\| f((s, a); \tilde{\theta}_K) - Q_{\pi}(s, a) \right\|_{\mu_{\pi}} \leq C_2 \left( \frac{1}{(1 - \gamma)^{1.5} m^{1/8}} \log^2 \left( \frac{(1 - \gamma)^{2T} \sqrt{m}}{\delta_1} \right) \right),
\]

where \( C_1 \) and \( C_2 \) are positive constant. Applying the union bound, we have with probability at least \( 1 - \delta_1 \),

\[
\sum_{t=0}^{T-1} \left\| f_i((s, a), \tilde{\theta}_t) - Q^i_{\pi_{\tau Wt}}(s, a) \right\|_{\mu_{\pi_{\tau Wt}}} \leq C_2 \left( \frac{T}{(1 - \gamma)^{1.5} m^{1/8}} \log^2 \left( \frac{(1 - \gamma)^{2T} \sqrt{m}}{\delta_1} \right) \right). \tag{61}
\]

We then bound the term \( \sum_{t=0}^{T-1} \left| J_i(\theta_t^i) - \mathbb{E}_{\nu_{\pi_{\tau Wt}}} f_i((s, a), \tilde{\theta}_t) \right| \). For simplicity, we denote \( J_i^*(\theta_t^i) = \mathbb{E}_{\xi_{\mu_{\pi_{\tau Wt}}}} f_i((s, a), \tilde{\theta}_t) \). Recall that \( J_i(\theta_t^i) = \frac{1}{N} \sum_{j=1}^N f_i((s_j, a_j), \tilde{\theta}_t) \). For each \( t \geq 0 \), we bound the error \( J_i(\theta_t^i) - J_i^*(\theta_t^i) \) as follows:

\[
P \left( \left( \frac{1}{N} \sum_{j=1}^N f_i((s_j, a_j), \tilde{\theta}_t) - J_i^*(\theta_t^i) \right)^2 \geq \frac{(1 + A) C_f^2}{N} \right) \leq P \left( \frac{1}{N} \sum_{j=1}^N \left[ f_i((s_j, a_j), \tilde{\theta}_t) - J_i^*(\theta_t^i) \right]^2 \geq 1 + A \right) \]
Then with probability at least $\frac{1}{2}$, we have

$$P \left( \frac{1}{N} \sum_{j=1}^{N} \left[ f_i((s_j, a_j), \tilde{\theta}_t) - J'_i(\tilde{\theta}_t) \right]^2 / C_f^2 \geq 1 + \lambda \right) \leq \frac{1}{N} \sum_{j=1}^{N} \exp \left( \frac{\left[ f_i((s_j, a_j), \tilde{\theta}_t) - J'_i(\tilde{\theta}_t) \right]^2}{C_f^2} \right) \geq 1 + \lambda \right)$$

where $(i)$ follows from Markov’s inequality. Then, eq. (62) implies that with probability at least $1 - \sigma_3/T$, we have

$$\left| \frac{1}{N} \sum_{j=1}^{N} f_i((s_j, a_j), \tilde{\theta}_t) - J'_i(\tilde{\theta}_t) \right| \leq C_f \frac{T}{\sqrt{N}} \left( 1 + \sqrt{\log \frac{T}{\sigma_3^2}} \right).$$

Applying the union bound, we have with probability at least $1 - \sigma_2$,

$$\sum_{i=1}^{T-1} \left| J_i(\theta_i^t) - \mathbb{E}_{n \sim \tau_n, \tilde{w}_i} \left[ f_i((s, a), \tilde{\theta}_i) \right] \right| \leq C_f T \frac{T}{\sqrt{N}} \left( 1 + \sqrt{\log \frac{T}{\sigma_3^2}} \right).$$

Letting $\delta_1 = \delta_2 = \frac{\delta}{T}, \ N = T \log(2T/\delta)$, and combing eq. (61) and eq. (63), we have with probability at least $1 - \delta$

$$\alpha(1 - \gamma) \sum_{i \in N_0} \left( J_0(\pi^*) - J_0(\pi_{w_0}) \right) + \alpha(1 - \gamma) \eta \sum_{i=1}^{P} |N_i|$$

$$\leq \mathbb{E}_{s \sim \nu} D_KL(\pi^* || \pi_{w_0}) + C_3 \left( \frac{\alpha T}{m^{1/4}} \right) + C_4 (\alpha^2 mT)$$

$$+ C_5 \left( \frac{\alpha T}{(1 - \gamma) \log \frac{1}{\delta}} \right) \left( \frac{(1 - \gamma)^2 T \sqrt{m}}{\delta} \right) + C_6 \left( \alpha (1 - \gamma) \sqrt{T} \right),$$

where $C_3 = \frac{8C_{RN} \sqrt{C_f} R^{1.5}}{\sqrt{d_t^*}}, C_4 = L_f (R^2 + d_2^2), C_5 = 3\alpha C_2 C_{RN}$, and $C_6 = 2C_f$ are positive constants.

**Lemma 17.** Let $K_{in} = C_1((1 - \gamma)^2 \sqrt{m}), N = T \log(2T/\delta)$, and

$$\frac{1}{2} \alpha(1 - \gamma) \eta T \geq \mathbb{E}_{s \sim \nu} D_KL(\pi^* || \pi_{w_0}) + C_3 \left( \frac{\alpha T}{m^{1/4}} \right) + C_4 (\alpha^2 mT)$$

$$+ C_5 \left( \frac{\alpha T}{(1 - \gamma) \log \frac{1}{\delta}} \right) \left( \frac{(1 - \gamma)^2 T \sqrt{m}}{\delta} \right) + C_6 \left( \alpha (1 - \gamma) \sqrt{T} \right).$$

Then with probability at least $1 - \delta$, we have the following holds

1. $N_0 \neq \emptyset$, i.e., $w_{out}$ is well-defined,

2. One of the following two statements must hold,

(a) $|N_0| \geq T/2$,

(b) $\sum_{i \in G} (J_0(\pi^*) - J_0(w_i)) \leq 0$.

**Proof.** Under the event given in Lemma 16, which happens with probability at least $1 - \delta$, we have

$$\alpha(1 - \gamma) \sum_{i \in N_0} (J_0(\pi^*) - J_0(\pi_{w_i})) + \alpha(1 - \gamma) \eta \sum_{i=1}^{P} |N_i|$$
\[
\begin{align*}
\mathbb{E}_{s \sim \nu^*} D_{KL}(\pi^* || \pi_{w_0}) + C_3 \left( \frac{\alpha T}{m^{1/4}} \right) + C_4 (\alpha^2 mT) \\
+ C_5 \left( \frac{\alpha T}{(1 - \gamma)^{1.5} m^{1/8}} \log \frac{1}{\delta} \left( \frac{(1 - \gamma)^2 T \sqrt{m}}{\delta} \right) \right) + C_6 \left( \alpha (1 - \gamma) \sqrt{T} \right).
\end{align*}
\] (65)

We first verify item 1. If \( \mathcal{N}_0 = \emptyset \), then \( \sum_{i=1}^{p} |\mathcal{N}_i| = T \), and Lemma 16 implies that

\[
\alpha (1 - \gamma) \eta T \leq \mathbb{E}_{s \sim \nu^*} D_{KL}(\pi^* || \pi_{w_0}) + C_3 \left( \frac{\alpha T}{m^{1/4}} \right) + C_4 (\alpha^2 mT) \\
+ C_5 \left( \frac{\alpha T}{(1 - \gamma)^{1.5} m^{1/8}} \log \frac{1}{\delta} \left( \frac{(1 - \gamma)^2 T \sqrt{m}}{\delta} \right) \right) + C_6 \left( \alpha (1 - \gamma) \sqrt{T} \right).
\]

which contradicts eq. (64). Thus, we must have \( \mathcal{N}_0 \neq \emptyset \).

We then proceed to verify the item 2. If \( \sum_{t \in \mathcal{G}} (J_0(\pi^*) - J_0(w_t)) \leq 0 \), then (b) in item 2 holds. If \( \sum_{t \in \mathcal{G}} (J_0(\pi^*) - J_0(w_t)) \leq 0 \), then eq. (65) implies that

\[
\frac{1}{2} \alpha (1 - \gamma) \eta T \leq \mathbb{E}_{s \sim \nu^*} D_{KL}(\pi^* || \pi_{w_0}) + C_3 \left( \frac{\alpha T}{m^{1/4}} \right) + C_4 (\alpha^2 mT) \\
+ C_5 \left( \frac{\alpha T}{(1 - \gamma)^{1.5} m^{1/8}} \log \frac{1}{\delta} \left( \frac{(1 - \gamma)^2 T \sqrt{m}}{\delta} \right) \right) + C_6 \left( \alpha (1 - \gamma) \sqrt{T} \right),
\]

which contradicts eq. (64). Hence, (a) in item 2 holds.

C.4. Proof of Theorem 2

We restate Theorem 2 as follows to include the specific of the parameters.

**Theorem 4** (Restatement of Theorem 2). Consider Algorithm 1 in the neural network approximation setting. Suppose Assumptions 1-4 hold. Let \( \alpha = \frac{1}{2C_4 \sqrt{T}} \) and

\[
\begin{align*}
\eta &= \frac{4C_4 \mathbb{E}_{s \sim \nu^*} D_{KL}(\pi^* || \pi_{w_0})}{(1 - \gamma) \sqrt{T}} + \frac{2C_3}{(1 - \gamma)^{1/4} m^{1/4}} + \frac{m}{(1 - \gamma) \sqrt{T}} \\
&+ 2C_5 \left( \frac{\alpha T}{(1 - \gamma)^{2.5} m^{1/8}} \log \frac{1}{\delta} \left( \frac{(1 - \gamma)^2 T \sqrt{m}}{\delta} \right) \right) + \frac{2C_6}{\sqrt{T}}.
\end{align*}
\]

Suppose performing neural TD with \( K_{in} = C_1 (1 - \gamma)^2 \sqrt{m} \) iterations at each iteration of CRPO. Then, with probability at least \( 1 - \delta \), we have

\[
J_0(\pi^*) - \mathbb{E}[J_0(w_{w_{t+1}})] \leq \frac{C_7 m}{(1 - \gamma) \sqrt{T}} + \frac{C_8}{(1 - \gamma)^{2.5} m^{1/8}} \log \frac{1}{\delta} \left( \frac{(1 - \gamma)^2 T \sqrt{m}}{\delta} \right),
\]

where

\[
C_7 = \frac{4C_4 D_{KL}(\pi^* || \pi_{w_0})}{m} + \frac{2(1 - \gamma)C_6}{m} + 1,
\]

and

\[
C_8 = 2C_5 + \frac{2C_6 (1 - \gamma)^{1.5}}{m^{1/8}}.
\]
For all $i \in \{1, \cdots, p\}$, we have

$$
\mathbb{E}[J_i(\pi_{\text{waq}})] - d_i \leq \frac{4C_4E_{\pi \sim \nu^*}D_{\text{KL}}(\pi^* || \pi_{\text{waq}})}{(1 - \gamma)\sqrt{T}} + \frac{2C_3}{(1 - \gamma)m^{1/4}} + \frac{m}{(1 - \gamma)\sqrt{T}}
+ 2(C_2 + C_5) \left( \frac{\alpha T}{(1 - \gamma)^{2.5}m^{1/8}} \log \left( \frac{(1 - \gamma)^2T\sqrt{m}}{\delta} \right) \right)
+ \frac{4C_6}{\sqrt{T}}.
$$

To proceed the proof of Theorem 2/Theorem 4, we consider the event given in Lemma 16, which happens with probability at least $1 - \delta$:

$$
\alpha(1 - \gamma) \sum_{i \in \mathcal{N}_0} (J_0(\pi^*) - J_0(\pi_{\text{waq}})) + \alpha(1 - \gamma)\eta \sum_{i=1}^p |\mathcal{N}_i|
\leq E_{\pi \sim \nu^*}D_{\text{KL}}(\pi^* || \pi_{\text{waq}}) + C_5 \left( \frac{\alpha T}{m^{1/4}} \right) + C_4(\alpha^2mT)
+ C_5 \left( \frac{\alpha T}{(1 - \gamma)^{1.5}m^{1/8}} \log \left( \frac{(1 - \gamma)^2T\sqrt{m}}{\delta} \right) \right)
+ C_6 \left( \alpha(1 - \gamma)\sqrt{T} \right).
\tag{66}
$$

We first consider the convergence rate of the objective function. Under the aforementioned event, we have the following holds:

$$
\alpha(1 - \gamma) \sum_{i \in \mathcal{N}_0} (J_0(\pi^*) - J_0(\pi_{\text{waq}}))
\leq E_{\pi \sim \nu^*}D_{\text{KL}}(\pi^* || \pi_{\text{waq}}) + C_5 \left( \frac{\alpha T}{m^{1/4}} \right) + C_4(\alpha^2mT)
+ C_5 \left( \frac{\alpha T}{(1 - \gamma)^{1.5}m^{1/8}} \log \left( \frac{(1 - \gamma)^2T\sqrt{m}}{\delta} \right) \right)
+ C_6 \left( \alpha(1 - \gamma)\sqrt{T} \right).
$$

If $\sum_{i \in \mathcal{N}_0} (J_0(\pi^*) - J_0(\pi_{\text{waq}})) \leq 0$, then we have $J_0(\pi^*) - J_0(\pi_{\text{waq}}) \leq 0$. If $\sum_{i \in \mathcal{N}_0} (J_0(\pi^*) - J_0(\pi_{\text{waq}})) \geq 0$, we have $|\mathcal{N}_0| \geq T/2$, which implies the following convergence rate

$$
J_0(\pi^*) - \mathbb{E}[J_0(\pi_{\text{waq}})] = \frac{1}{|\mathcal{N}_0|} \sum_{i \in \mathcal{N}_0} (J_0(\pi^*) - J_0(\pi_{\text{waq}}))
\leq \frac{2E_{\pi \sim \nu^*}D_{\text{KL}}(\pi^* || \pi_{\text{waq}})}{\alpha(1 - \gamma)T} + \frac{2C_3}{(1 - \gamma)m^{1/4}} + \frac{2C_4\alpha m}{1 - \gamma}
+ 2C_5 \left( \frac{\alpha T}{(1 - \gamma)^{2.5}m^{1/8}} \log \left( \frac{(1 - \gamma)^2T\sqrt{m}}{\delta} \right) \right)
+ \frac{2C_6}{\sqrt{T}}.
$$

Letting $\alpha = \frac{1}{2C_4\sqrt{T}}$, we can obtain the following convergence rate

$$
J_0(\pi^*) - \mathbb{E}[J_0(\pi_{\text{waq}})] \leq \frac{C_7m}{(1 - \gamma)\sqrt{T}} + \frac{C_8}{(1 - \gamma)^{2.5}m^{1/8}} \log \left( \frac{(1 - \gamma)^2T\sqrt{m}}{\delta} \right),
$$

where

$$
C_7 = \frac{4C_4D_{\text{KL}}(\pi^* || \pi_{\text{waq}})}{m} + \frac{2(1 - \gamma)C_6}{m} + 1,
$$

and

$$
C_8 = 2C_5 + \frac{2C_3(1 - \gamma)^{1.5}}{m^{1/8}}.
$$

We then proceed to bound the constraint violation. For any $i \in \{1, \cdots, p\}$, we have

$$
\mathbb{E}[J_i(\pi_{\text{waq}})] - d_i \leq \frac{1}{|\mathcal{N}_0|} \sum_{i \in \mathcal{N}_0} J_i(\pi_{\text{waq}}) - d_i
$$
\[
\frac{1}{|\mathcal{N}_0|} \sum_{t \in \mathcal{N}_0} (\bar{J}_t - d_i) + \frac{1}{|\mathcal{N}_0|} \sum_{t \in \mathcal{N}_0} |J_t(\pi_{w_0}) - \bar{J}_t| \leq \eta + \frac{1}{|\mathcal{N}_0|} \sum_{t = 0}^{T-1} |J_t(\pi_{w_0}) - \bar{J}_t| \\
\leq \eta + \frac{1}{|\mathcal{N}_0|} \sum_{t = 0}^{T-1} |\bar{J}_t| - E_{\nu_{\pi_{\tau_i}w_t}} [f_i((s, a), \tilde{\theta}_t)] \\
+ C_{RN} \sum_{t = 0}^{T-1} \left\| f_i((s, a), \tilde{\theta}_t) - Q^i_{\pi_{\tau_i}w_t}(s, a) \right\|_{\mu_{\pi_{\tau_i}w_t}}.
\]

Recalling eq. (61) and eq. (63), under the event defined in eq. (66), we have

\[
\sum_{t=0}^{T-1} |\bar{J}_t| - E_{\nu_{\pi_{\tau_i}w_t}} [f_i((s, a), \tilde{\theta}_t)] \leq C_6 \sqrt{T},
\tag{67}
\]

and

\[
\sum_{t=0}^{T-1} \left\| f_i((s, a), \tilde{\theta}_t) - Q^i_{\pi_{\tau_i}w_t}(s, a) \right\|_{\mu_{\pi_{\tau_i}w_t}} \leq C_2 \left( \frac{T}{(1-\gamma)^{1.5} m^{1/8}} \log^3 \left( \frac{(1-\gamma)^2 T \sqrt{m}}{\delta} \right) \right).
\tag{68}
\]

Let the value of the tolerance \(\eta\) be

\[
\eta = \frac{4C_4 \mathbb{E}_{\nu_{\pi_0}} D_{KL}(\pi^* || \pi_{w_0})}{(1-\gamma)\sqrt{T}} + \frac{2C_3}{(1-\gamma)m^{1/4}} + \frac{m}{(1-\gamma)\sqrt{T}} + 2C_5 \left( \frac{\alpha T}{(1-\gamma)^{1.5} m^{1/8}} \log^4 \left( \frac{(1-\gamma)^2 T \sqrt{m}}{\delta} \right) \right) + \frac{2C_6}{\sqrt{T}}.
\tag{69}
\]

We have

\[
\frac{1}{2} \alpha (1-\gamma) \eta T \geq \mathbb{E}_{\nu_{\pi_0}} D_{KL}(\pi^* || \pi_{w_0}) + C_3 \left( \frac{\alpha T}{m^{1/4}} \right) + C_4 (\alpha^2 m T) + C_5 \left( \frac{\alpha T}{(1-\gamma)^{1.5} m^{1/8}} \log^4 \left( \frac{(1-\gamma)^2 T \sqrt{m}}{\delta} \right) \right) + C_6 \left( \alpha (1-\gamma) \sqrt{T} \right),
\]

which satisfies the requirement specified in Lemma 17. Combining eq. (67), eq. (68) and eq. (69), and using Lemma 17, we have with probability at least \(1 - \delta\) at least one of the following holds:

\[
\mathbb{E}[J_t(\pi_{w_{out}})] - d_i \leq 0,
\]

or \(|\mathcal{N}_0| \geq T/2\), which further implies

\[
\mathbb{E}[J_t(\pi_{w_{out}})] - d_i \leq \frac{4C_4 \mathbb{E}_{\nu_{\pi_0}} D_{KL}(\pi^* || \pi_{w_0})}{(1-\gamma)\sqrt{T}} + \frac{2C_3}{(1-\gamma)m^{1/4}} + \frac{m}{(1-\gamma)\sqrt{T}} + 2(C_2 + C_5) \left( \frac{\alpha T}{(1-\gamma)^{2.5} m^{1/8}} \log^4 \left( \frac{(1-\gamma)^2 T \sqrt{m}}{\delta} \right) \right) + \frac{4C_6}{\sqrt{T}}.
\]