A system of $k$ Sylvester-type quaternion matrix equations with $3k + 1$ variables

Qing-Wen Wang$^{a,*}$, Mengyan Xie$^a$,

Abstract: In this paper, we provide some solvability conditions in terms of ranks for the existence of a general solution to a system of $k$ Sylvester-type quaternion matrix equations with $3k + 1$ variables $A_i X_i + Y_i B_i + C_i Z_i D_i + F_i Z_{i+1} G_i = E_i$, $i = 1, k$. As applications of this system, we present rank equalities as the necessary and sufficient conditions for the existence of a general solution to some systems of quaternion matrix equations $A_i X_i + (A_i X_i)_\phi + C_i Z_i (C_i)_\phi + F_i Z_{i+1} (F_i)_\phi = E_i$, $i = 1, k$, involving $\phi$-Hermicity.

Keywords: Quaternion matrix equation; $\phi$-Hermitian; General solution; Solvability

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1. Introduction

Sylvester matrix equation and its generalizations are closely related with many problems in robust control [11], neural network [13], output feedback control [10], graph theory [1], and so on. Recently, some researchers have considered Sylvester-type matrix equations over the quaternion algebra. Quaternion algebra is an associative and noncommutative division algebra over the real number field. Nowadays quaternion has applications in signal and color image processing, quantum physics, and so on.

In this paper, we aim to consider the solvable conditions for the existence of a general solution to the systems of quaternion matrix equations:

\[
\begin{align*}
A_1 X_1 + Y_1 B_1 + C_1 Z_1 D_1 + F_1 Z_2 G_1 &= E_1 \\
A_2 X_2 + Y_2 B_2 + C_2 Z_2 D_2 + F_2 Z_3 G_2 &= E_2 \\
&\vdots \\
A_k X_k + Y_k B_k + C_k Z_k D_k + F_k Z_{k+1} G_k &= E_k,
\end{align*}
\]

where $A_k, B_k, C_k, D_k, F_k, G_k$ and $E_k$ are given matrices, $X_k, Y_k, Z_k, Z_{k+1}$ are unknowns. We give a practical necessary and sufficient conditions for the existence of a solution to the system \((1.1)\) in terms of ranks. The system \((1.1)\) have been firstly investigated by [12] when $k = 1$. He [5] give the solvable conditions and general solution to system \((1.1)\) when $k = 2$. However, it is hard to solve the Theorem 3.1 by using the same approach in \((1.1)\). We will use another approach to prove Theorem 3.1.

Since Rodman [9] first presented the definition of $\phi$-Hermitian quaternion matrix, there have been several papers to discuss the quaternion matrix equations involving $\phi$-Hermicity (e.g., [2],

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1This research was supported by the National Natural Science Foundation of China (Grant no. 11971294). Email address: wqw@t.shu.edu.cn

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3, 5, 6). As applications of Theorem 3.1 we present rank equalities as the solvable conditions for the system of quaternion matrix equations involving \( \phi \)-Hermicity

\[
\begin{aligned}
A_1X_1 + (A_1X_1)\phi + C_1Z_1(C_1)\phi + F_1Z_2(F_1)\phi &= E_1, \\
A_2X_2 + (A_2X_2)\phi + C_2Z_2(C_2)\phi + F_2Z_3(F_3)\phi &= E_2, \\
&\vdots \\
A_kX_k + (A_kX_k)\phi + C_kZ_k(C_k)\phi + F_kZ_{k+1}(F_k)\phi &= E_k, \\
Z_i = (Z_i)\phi.
\end{aligned}
\tag{1.2}
\]

The remainder of this paper is organized as follows. In Section 2, we review some definitions of quaternion algebra and consider a quaternion matrix equation. In Section 3, we present some practical necessary and sufficient conditions for the existence of a solution to the system of quaternion algebra and consider a quaternion matrix equation. In Section 3, we present rank equalities as the solvable conditions to the system (1.2) involving \( \phi \)-Hermicity.

2. Preliminaries

Let \( \mathbb{R} \) and \( \mathbb{H}^{m\times n} \) stand, respectively, for the real field and the space of all \( m \times n \) matrices over the real quaternion algebra

\[
\mathbb{H} = \{ a_0 + a_1i + a_2j + a_3k \mid i^2 = j^2 = k^2 = ijk = -1, a_0, a_1, a_2, a_3 \in \mathbb{R} \}.
\]

The symbols \( r(A) \) and \( A^\dagger \) stand for the rank of a given quaternion matrix \( A \) and the conjugate transpose of \( A \) and the transposed of \( A \), respectively. \( I \) and 0 are the identity matrix and zero matrix with appropriate sizes, respectively. The Moore-Penrose inverse \( A^\dagger \) of a quaternion matrix \( A \), is defined to be the unique matrix \( A^\dagger \), such that

(i) \( AA^\dagger A = A \), (ii) \( A^\dagger AA^\dagger = A^\dagger \), (iii) \( (AA^\dagger)^* = AA^\dagger \), (iv) \( (A^\dagger A)^* = A^\dagger A \).

Furthermore, \( L_A \) and \( R_A \) stand for the projectors \( L_A = I - A^\dagger A \) and \( R_A = I - AA^\dagger \) induced by \( A \), respectively. For more definitions and properties of quaternions, we refer the reader to the book [9].

In order to solve the system (1.1), we need the solvability conditions and general solution to the quaternion matrix equation

\[
A_1X_1 + Y_1B_1 + C_1Z_1D_1 + F_1Z_2G_1 = E_1,
\]

where \( A_1, B_1, C_1, D_1, F_1, G_1, E_1 \) are given matrices, and \( X_1, Y_1, Z_1, Z_2 \) are unknowns.

Lemma 2.1. (7, 12). Consider the quaternion matrix equation (2.1). Set

\[
A_{11} = R_{A_1}C_1, \quad B_{11} = D_1L_{B_1}, \quad C_{11} = R_{A_1}F_1, \quad D_{11} = G_1L_{B_1},
\]

\[
E_{11} = R_{A_1}E_1L_{B_1}, \quad M_{11} = R_{A_{11}}C_{11}, \quad N_{11} = D_{11}L_{B_{11}}, \quad S_{11} = C_{11}L_{M_{11}}.
\]

Then the equation (2.1) is consistent if and only if

\[
R_{M_{11}}R_{A_{11}}E_{11} = 0, \quad E_{11}L_{B_{11}}L_{N_{11}} = 0, \quad R_{A_{11}}E_{11}L_{D_{11}} = 0, \quad R_{C_{11}}E_{11}L_{B_{11}} = 0.
\]
In this case, the general solution to (2.1) can be expressed as

\[
X_1 = A_1^\dagger (E_1 - C_1 Z_1 D_1 - F_1 Z_2 G_1) - T_{17} B_1 + L_{A_1} T_{16},
\]

\[
Y_1 = R_{A_1} (E_1 - C_1 Z_1 D_1 - F_1 Z_2 G_1) B_1^\dagger + A_1 T_{17} + T_{18} R_{B_1},
\]

\[
Z_1 = A_{11}^\dagger E_{11} B_{11}^\dagger - A_{11}^\dagger C_{11} M_{11}^\dagger E_{11} B_{11}^\dagger - A_{11}^\dagger S_{11} C_{11}^\dagger E_{11} N_{11}^\dagger D_{11} B_{11}^\dagger - A_{11}^\dagger S_{11} T_{112} R_{N_{11}} D_{11} B_{11}^\dagger + L_{A_{11}} T_{14} + T_{15} R_{B_{11}},
\]

\[
Z_2 = M_{11}^\dagger E_{11} D_{11}^\dagger + S_{11}^\dagger S_{11} C_{11}^\dagger E_{11} N_{11}^\dagger + L_{M_{11}} L_{S_{11}} T_{11} + L_{M_{11}} T_{12} R_{N_{11}} + T_{13} R_{D_{11}},
\]

where \( T_{11}, \ldots, T_{18} \) are arbitrary matrices over \( \mathbb{H} \) with appropriate sizes.

The following lemma is due to Marsaglia and Styan [8], which can be generalized to the quaternion algebra.

**Lemma 2.2.** (8). Let \( A \in \mathbb{H}^{m \times n}, B \in \mathbb{H}^{m \times k}, \) and \( C \in \mathbb{H}^{l \times n}, \) be given. Then

(1) \( r(A) + r(R_A B) = r(B) + r(R_B A) = r \begin{pmatrix} A & B \end{pmatrix}. \)

(2) \( r(A) + r(C L_A) = r(C) + r(A L_C) = r \begin{pmatrix} A & C \end{pmatrix}. \)

3. **Solvability conditions to the system (1.1)**

The goal of this section is to consider the solvability conditions to the system (1.1) in terms of rank equalities.

**Theorem 3.1.** The system (1.1) is consistent if and only if the following rank equalities hold for all \( i = 1, k, \) \( 1 \leq m \leq n \leq k: \)

\[
r \begin{pmatrix} E_i & A_i & C_i & F_i \\ B_i & 0 & 0 & 0 \end{pmatrix} = r \begin{pmatrix} A_i & C_i & F_i \end{pmatrix} + r(B_i), \quad r \begin{pmatrix} E_i & A_i \\ B_i & 0 \end{pmatrix} = r \begin{pmatrix} B_i & 0 \end{pmatrix} + r(A_i), \quad (3.1)
\]

\[
r \begin{pmatrix} E_i & A_i & C_i \\ B_i & 0 & 0 \\ G_i & 0 & 0 \end{pmatrix} = r \begin{pmatrix} A_i & C_i \end{pmatrix} + r \begin{pmatrix} B_i & 0 \\ G_i & 0 \end{pmatrix}, \quad r \begin{pmatrix} E_i & A_i & F_i \\ B_i & 0 & 0 \\ D_i & 0 & 0 \end{pmatrix} = r \begin{pmatrix} A_i & F_i \end{pmatrix} + r \begin{pmatrix} B_i & 0 \\ D_i & 0 \end{pmatrix}, \quad (3.2)
\]
\[ r \begin{pmatrix} C_m & F_m & D_{m+1} & A_m \\ G_m & C_{m+1} & -E_{m+1} & F_{m+1} & A_{m+1} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ B_m & B_{m+1} & \cdots & \cdots & B_n \end{pmatrix} = r \begin{pmatrix} C_m & F_m & A_m & A_{m+1} \\ C_{m+1} & F_{m+1} & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots \\ C_n & F_n & \vdots & \vdots \end{pmatrix} + r \begin{pmatrix} G_m & D_{m+1} \\ G_{m+1} & D_{m+2} \\ \vdots & \ddots \\ G_{n-1} & D_n \\ B_m & B_{m+1} \\ \vdots & \ddots \end{pmatrix}, \quad (3.3) \]

\[ r \begin{pmatrix} D_m & F_m & D_{m+1} & A_m \\ E_m & C_{m+1} & -E_{m+1} & F_{m+1} & A_{m+1} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ B_m & B_{m+1} & \cdots & \cdots & B_n \end{pmatrix} = r \begin{pmatrix} D_m & F_m & A_m & A_{m+1} \\ C_{m+1} & F_{m+1} & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots \\ C_n & F_n & \vdots & \vdots \end{pmatrix} + r \begin{pmatrix} D_m & D_{m+1} \\ G_m & G_{m+1} & \vdots \\ \vdots & \ddots & \ddots \\ \vdots & \ddots & \ddots & D_n \\ B_m & B_{m+1} & \cdots & \ddots \end{pmatrix}, \quad (3.4) \]

\[ r \begin{pmatrix} C_m & E_m & F_m & D_{m+1} & A_m \\ G_m & C_{m+1} & -E_{m+1} & F_{m+1} & A_{m+1} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ B_m & B_{m+1} & \cdots & \cdots & B_n \end{pmatrix} = r \begin{pmatrix} C_m & E_m & F_m & A_m & A_{m+1} \\ G_m & C_{m+1} & -E_{m+1} & F_{m+1} & A_{m+1} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ B_m & B_{m+1} & \cdots & \cdots & B_n \end{pmatrix} + r \begin{pmatrix} D_m & D_{m+1} \\ G_m & G_{m+1} & \vdots \\ \vdots & \ddots & \ddots \\ \vdots & \ddots & \ddots & D_n \\ B_m & B_{m+1} & \cdots & \ddots \end{pmatrix}. \]
can be expressed as

\[
\begin{pmatrix}
  C_m & F_m & C_{m+1} & A_m & A_{m+1} \\
  C_{m+1} & F_{m+1} & \cdots & C_n & A_n \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  \vdots & \vdots & \ddots & C_n & A_n \\
\end{pmatrix}
= r \begin{pmatrix}
  D_m & E_m & F_m & C_{m+1} & A_m & A_{m+1} \\
  C_{m+1} & -E_{m+1} & \cdots & C_n & A_n \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  \vdots & \vdots & \ddots & C_n & A_n \\
\end{pmatrix} + r \begin{pmatrix}
  B_m & B_{m+1} \\
  B_m & B_{m+1} \\
  \vdots & \vdots \\
  B_m & B_{m+1} \\
\end{pmatrix},
\]

(3.5)

\[
\begin{pmatrix}
  D_m & E_m & F_m & C_{m+1} & A_m & A_{m+1} \\
  C_{m+1} & -E_{m+1} & \cdots & C_n & A_n \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  \vdots & \vdots & \ddots & C_n & A_n \\
\end{pmatrix}
= r \begin{pmatrix}
  D_m & E_m & F_m & C_{m+1} & A_m & A_{m+1} \\
  C_{m+1} & -E_{m+1} & \cdots & C_n & A_n \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  \vdots & \vdots & \ddots & C_n & A_n \\
\end{pmatrix} + r \begin{pmatrix}
  B_m & B_{m+1} \\
  B_m & B_{m+1} \\
  \vdots & \vdots \\
  B_m & B_{m+1} \\
\end{pmatrix},
\]

(3.6)

where the blank entries in above rank equalities are all zeros.

Proof. The proof is by mathematical induction on \( n \). For \( n = 1 \), the statement is clear. We assume that the (3.1)–(3.5) is true for \( n = k - 1 \). So next we will show that the induction is true on \( n = k \).

We divide the system (1.1) into the following \( k \) equations:

\[
A_1X_1 + Y_1B_1 + C_1Z_1D_1 + F_1Z_2G_1 = E_1,
\]

(3.7)

\[
A_2X_2 + Y_2B_2 + C_2Z_2D_2 + F_2Z_3G_2 = E_2,
\]

(3.8)

\[
A_kX_k + Y_kB_k + C_kZ_kD_k + F_kZ_{k+1}G_k = E_k.
\]

(3.9)

Applying Lemma [2.1] the equations (3.7)–(3.9) are consistent if and only if rank equalities (3.1), (3.2) hold. Put \( A_{ii} = R_{A_i}C_i, \ B_{ii} = D_iL_{B_i}, \ C_{ii} = R_{A_i}F_i, \ D_{ii} = G_iL_{B_i}, \ E_{ii} = R_{A_i}E_iL_{B_i}, \ M_{ii} = R_{A_i}C_{ii}, \ N_{ii} = D_{ii}L_{B_{ii}}, \ S_{ii} = C_{ii}L_{M_{ii}}, \ (i = 1, k) \), the general solution of equation

\[
A_iX_i + Y_iB_i + C_iZ_iD_i + F_iZ_{i+1}G_i = E_i
\]

(3.10)

can be expressed as

\[
X_i = A_i^1(E_i - C_iZ_iD_i - F_iZ_{i+1}G_i) - T_{i7}B_i + L_{A_i}T_{i6},
\]

\[
Y_i = R_{A_i}(E_i - C_iZ_iD_i - F_iZ_{i+1}G_i)B_i^1 + A_iT_{i7} + T_{i8}R_{B_i},
\]
\begin{align*}
Z_i &= A_i^i E_i B_i^+ - A_i^i C_i M_i^i E_i B_i^+ - A_i^i S_i C_i^i E_i N_i^+ D_i B_i^+ - A_i^i S_i T_i R_i N_i^+ D_i B_i^+ \\
&\quad + L_{A_i} T_{i4} + T_{i3} R_{B_i}, \\
Z_{i+1} &= M_i^i E_i D_i^+ + S_i^i S_i C_i^i E_i N_i^+ + L_{M_i} L_{S_i} T_{i1} + L_{M_i} T_{i2} R_i N_i^+ + T_{i3} R_{D_i},
\end{align*}

where $T_{i1}, \ldots, T_{i8}$ are arbitrary matrices over $\mathbb{H}$ with appropriate sizes.

Let $Z_i$ in the $(i+1)$th equation be equal to $Z_{i+1}$ in the $i$th equation, $0 \leq i \leq k$ we can obtain the following system

\begin{equation}
\begin{pmatrix}
\hat{A}_1 \\
\hat{A}_2 \\
\vdots \\
\hat{A}_{k-1}
\end{pmatrix}
\begin{pmatrix}
T_{11} \\
T_{21} \\
\vdots \\
T_{k-1,1}
\end{pmatrix} + 
\begin{pmatrix}
T_{13} & T_{25} \\
T_{23} & T_{35} \\
\vdots & \vdots \\
T_{k-1,3} & T_{k5}
\end{pmatrix}
\begin{pmatrix}
\hat{B}_1 \\
\hat{B}_2 \\
\vdots \\
\hat{B}_{k-1}
\end{pmatrix} + 
\begin{pmatrix}
\hat{C}_1 T_{12} \hat{D}_1 + \hat{F}_1 T_{22} \hat{G}_1 = \hat{E}_1, \\
\hat{C}_2 T_{22} \hat{D}_2 + \hat{F}_2 T_{32} \hat{G}_2 = \hat{E}_2, \\
\vdots \\
\hat{C}_{k-1} T_{k-1,2} \hat{D}_{k-1} + \hat{F}_{k-1} T_{k2} \hat{G}_{k-1} = \hat{E}_{k-1},
\end{pmatrix}
\end{equation}

where

\begin{align}
\hat{A}_{k-1} &= (L_{M_{k-1,k-1}} L_{S_{k-1,k-1}} - L_{A_{kk}}), \\
\hat{B}_{k-1} &= \begin{pmatrix} R_{D_{k-1,k-1}} \\ - R_{B_{kk}} \end{pmatrix}, \\
\hat{C}_{k-1} &= L_{M_{k-1,k-1}}, \\
\hat{D}_{k-1} &= R_{N_{k-1,k-1}}, \\
\hat{F}_{k-1} &= A_{kk}^+ S_{kk}, \\
\hat{G}_{k-1} &= R_{N_{kk}} D_{kk} B_{kk}^+. 
\end{align}

\begin{align}
\hat{E}_{k-1} &= A_{kk}^+ E_{kk} B_{kk}^+ - A_{kk}^+ C_{kk} M_{kk}^+ E_{kk} B_{kk}^+ - A_{kk}^+ S_{kk} C_{kk}^+ E_{kk} N_{kk}^+ D_{kk} B_{kk}^+ \\
&\quad - M_{k-1,k-1}^+ E_{k-1,k-1} D_{k-1,k-1}^+ - S_{k-1,k-1}^+ S_{k-1,k-1} C_{k-1,k-1}^+ E_{k-1,k-1} N_{k-1,k-1}^+.
\end{align}

Note that the system (3.13) is the same as (1.1) when $n = k - 1$, so we can apply the induction to (3.13). The system (3.13) is consistent if and only if

\begin{align}
r \begin{pmatrix}
\hat{E}_i \\
\hat{A}_i \\
\hat{C}_i \\
\hat{F}_i
\end{pmatrix} &= r \begin{pmatrix}
\hat{A}_i \\
\hat{C}_i \\
\hat{F}_i
\end{pmatrix} + r(\hat{B}_i), \\
r \begin{pmatrix}
\hat{E}_i \\
\hat{A}_i \\
\hat{C}_i \\
\hat{F}_i
\end{pmatrix} &= r \begin{pmatrix}
\hat{B}_i \\
\hat{D}_i \\
\hat{G}_i
\end{pmatrix} + r(\hat{A}_i), \\
r \begin{pmatrix}
\hat{E}_i \\
\hat{A}_i \\
\hat{C}_i \\
\hat{F}_i
\end{pmatrix} &= r \begin{pmatrix}
\hat{B}_i \\
\hat{D}_i \\
\hat{G}_i
\end{pmatrix} + r(\hat{A}_i), \\
r \begin{pmatrix}
\hat{E}_i \\
\hat{A}_i \\
\hat{C}_i \\
\hat{F}_i
\end{pmatrix} &= r \begin{pmatrix}
\hat{B}_i \\
\hat{D}_i \\
\hat{G}_i
\end{pmatrix} + r(\hat{A}_i).
\end{align}
\[
\begin{pmatrix}
\hat{C}_m & \hat{E}_m & \hat{F}_m & 0 & \cdots & \hat{A}_m \\
\hat{G}_m & \hat{D}_m & \hat{E}_m & \hat{F}_m & \cdots & \hat{A}_{m+1} \\
\hat{C}_{m+1} & -\hat{E}_{m+1} & \hat{F}_{m+1} & 0 & \cdots & \hat{A}_{m+1} \\
\quad & \hat{C}_m & (-1)^{n-m} \hat{E}_m \hat{F}_n & \hat{A}_n \\
\hat{B}_m & \hat{B}_{m+1} & \cdots & \hat{B}_n \\
\end{pmatrix}
\]

\[
= r \begin{pmatrix}
\hat{C}_m & \hat{E}_m & \hat{F}_m & 0 & \cdots & \hat{A}_m \\
\hat{G}_m & \hat{D}_m & \hat{E}_m & \hat{F}_m & \cdots & \hat{A}_{m+1} \\
\hat{C}_{m+1} & -\hat{E}_{m+1} & \hat{F}_{m+1} & 0 & \cdots & \hat{A}_{m+1} \\
\quad & \hat{C}_m & (-1)^{n-m} \hat{E}_m \hat{F}_n & \hat{A}_n \\
\hat{B}_m & \hat{B}_{m+1} & \cdots & \hat{B}_n \\
\end{pmatrix}
\]

\[
= r \begin{pmatrix}
\hat{G}_m & \hat{D}_m & \hat{E}_m & \hat{F}_m & \cdots & \hat{A}_n \\
\hat{B}_m & \hat{B}_{m+1} & \cdots & \hat{B}_n \\
\end{pmatrix}
\]

(3.19)

\[
\begin{pmatrix}
\hat{B}_m & \hat{E}_m & \hat{F}_m & 0 & \cdots & \hat{A}_m \\
\hat{G}_m & \hat{D}_m & \hat{E}_m & \hat{F}_m & \cdots & \hat{A}_{m+1} \\
\hat{C}_{m+1} & -\hat{E}_{m+1} & \hat{F}_{m+1} & 0 & \cdots & \hat{A}_{m+1} \\
\quad & \hat{C}_m & (-1)^{n-m} \hat{E}_m \hat{F}_n & \hat{A}_n \\
\hat{B}_m & \hat{B}_{m+1} & \cdots & \hat{B}_n \\
\end{pmatrix}
\]

\[
= r \begin{pmatrix}
\hat{F}_m & \hat{C}_{m+1} & \hat{F}_{m+1} & 0 & \cdots & \hat{A}_m \\
\hat{G}_m & \hat{D}_m & \hat{E}_m & \hat{F}_m & \cdots & \hat{A}_{m+1} \\
\hat{C}_{m+2} & (-1)^{n-m} \hat{E}_m \hat{F}_n & \hat{A}_n \\
\quad & \hat{C}_m & \hat{F}_{n-1} & \hat{A}_n \\
\hat{B}_m & \hat{B}_{m+1} & \cdots & \hat{B}_n \\
\end{pmatrix}
\]

\[
= r \begin{pmatrix}
\hat{G}_m & \hat{D}_m & \hat{E}_m & \hat{F}_m & \cdots & \hat{A}_n \\
\hat{B}_m & \hat{B}_{m+1} & \cdots & \hat{B}_n \\
\end{pmatrix}
\]

(3.20)
the following facts. Now we prove that (3.17)–(3.22) is equivalent to (3.1)–(3.6). To do this it is useful to establish

\[ \begin{align*}
\begin{pmatrix}
\hat{c}_m & \hat{f}_m & \hat{d}_{m+1} & \hat{a}_1 \\
\hat{g}_m & \hat{c}_{m+1} & -\hat{f}_{m+1} & \hat{d}_{m+2} \\
\hat{c}_n & \hat{g}_{m+1} & -\hat{d}_n & \hat{a}_n \\
\hat{b}_m & \hat{b}_{m+1} & \hat{b}_n
\end{pmatrix} \\
= r \begin{pmatrix}
\hat{c}_m & \hat{f}_m & \hat{a}_1 \\
\hat{c}_{m+1} & \hat{f}_{m+1} & \hat{a}_2 \\
\hat{c}_n & \hat{f}_n & \hat{a}_n \\
\hat{b}_m & \hat{b}_{m+1} & \hat{b}_n
\end{pmatrix} + r \begin{pmatrix}
\hat{g}_m & \hat{d}_{m+1} & \hat{a}_1 \\
\hat{g}_{m+1} & \hat{d}_{m+2} & \hat{a}_2 \\
\hat{g}_n & \hat{d}_n & \hat{a}_n \\
\hat{b}_m & \hat{b}_{m+1} & \hat{b}_n
\end{pmatrix},
\end{align*} \]

(3.21)

Now we prove that (3.17)–(3.22) is equivalent to (3.1)–(3.6). To do this it is useful to establish the following facts.

**Fact 1:** The expressions of \( \hat{E}_i \) in (3.16). Note that

\[ Z_{i+1}^1 := M_{ii}^1 E_{ii} D_{ii}^1 + S_{ii}^1 C_{ii}^1 E_{ii} N_{ii}^1 \]
and

\[ Z_{i+1}^2 := A_{i+1, i+1}^\dagger E_{i+1, i+1} + A_{i+1, i+1}^\dagger C_{i+1, i+1} M_{i+1, i+1} E_{i+1, i+1} + A_{i+1, i+1}^\dagger S_{i+1, i+1} C_{i+1, i+1} E_{i+1, i+1} + A_{i+1, i+1}^\dagger N_{i+1, i+1} D_{i+1, i+1} B_{i+1, i+1} \]

are special solutions to the equations \((3.8)\) \((k = i \text{ and } k = i + 1)\), respectively. Hence,

\[ \hat{E}_i = Z_{i+1}^2 - Z_i^1. \] (3.23)

**Fact 2:** Formulas about \( S_{ii} \): From

\[ S_{ii} - A_{ii} A_{ii}^\dagger S_{ii} = R_{A_{ii}} S_{ii} = R_{A_{ii}} C_{ii} L_{M_{ii}} = M_{ii} L_{M_{ii}} = 0, \] (3.24)

we infer that

\[ A_{ii} A_{ii}^\dagger S_{ii} = S_{ii}. \] (3.25)

**Fact 3:** The ranks of \( r\left( \begin{pmatrix} R_{D_{ii}} \\ R_{N_{ii}} \end{pmatrix} \right) \) \( r(R_{N_{ii}}) \): Applying Lemma 2.2 to \( r\left( \begin{pmatrix} R_{D_{ii}} \\ R_{N_{ii}} \end{pmatrix} \right) \) \( r(R_{N_{ii}}) \)
gives

\[ r\left( \begin{pmatrix} R_{D_{ii}} \\ R_{N_{ii}} \end{pmatrix} \right) - r(R_{N_{ii}}) = r\left( \begin{pmatrix} I & D_{ii} \\ I & 0 \end{pmatrix} \right) - r\left( \begin{pmatrix} I & N_{ii} \end{pmatrix} \right) - r(D_{ii}) = r\left( D_{ii} L_{N_{ii}} \right) - r(D_{ii}) = 0. \]

Hence, we have

\[ r\left( \begin{pmatrix} R_{D_{ii}} \\ R_{N_{ii}} \end{pmatrix} \right) = r(R_{N_{ii}}), \] (3.26)

i.e.,

\[ R_{D_{ii}} = U_i R_{N_{ii}}, \] (3.27)

where \( U_i \) is a matrix.

**Fact 4:** Formulas about \( R_{N_{i+1,i+1}} D_{i+1,i+1} B_{i+1,i+1}^\dagger B_{i+1,i+1} \): Note that

\[ R_{N_{i+1,i+1}} D_{i+1,i+1} B_{i+1,i+1}^\dagger B_{i+1,i+1} = R_{N_{i+1,i+1}} D_{i+1,i+1} L_{B_{i+1,i+1}} = R_{N_{i+1,i+1}} N_{i+1,i+1} = 0, \]

Hence, we have

\[ R_{N_{i+1,i+1}} D_{i+1,i+1} B_{i+1,i+1}^\dagger B_{i+1,i+1} = R_{N_{i+1,i+1}} D_{i+1,i+1}. \] (3.28)

Firstly, we prove that \((3.3) \iff (3.6) \iff (3.19) \iff (3.22)\) for the case \( n - m = 1 \), respectively. It follows from Lemma 2.2 that

\[ r\left( \begin{pmatrix} \hat{E}_i & \hat{C}_i & \hat{F}_i & \hat{A}_i \\ \hat{B}_i & 0 & 0 & 0 \end{pmatrix} \right) = r\left( \begin{pmatrix} \hat{A}_i & \hat{C}_i & \hat{F}_i \end{pmatrix} \right) + r(\hat{B}_i) \] (3.29)
Similarly, we have that

\[ \Leftrightarrow r \left( \begin{array}{cccc}
E_i & L_{M_i} & L_{S_{ii}} & L_{A_{i+1,i+1}} \\
R_{D_{ii}} & 0 & 0 & 0 \\
R_{B_{i+1,i+1}} & 0 & 0 & 0
\end{array} \right) \]

\[ = r \left( \begin{array}{cccc}
L_{M_i} & L_{S_{ii}} & L_{A_{i+1,i+1}} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array} \right) + r \left( \begin{array}{cccc}
R_{D_{ii}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array} \right) \]

By using (3.25)

\[ \Leftrightarrow r \left( \begin{array}{cccc}
E_i & I & I & A_{i+1,i+1}^{i+1} S_{i+1,i+1} \\
R_{D_{ii}} & 0 & 0 & 0 \\
R_{B_{i+1,i+1}} & 0 & 0 & 0
\end{array} \right) = r \left( \begin{array}{cccc}
I & I & A_{i+1,i+1}^{i+1} S_{i+1,i+1} \\
0 & M_{ii} & 0 & 0 \\
0 & 0 & B_{i+1,i+1} & 0
\end{array} \right) + r \left( \begin{array}{cccc}
I D_{ii} & 0 & 0 & 0 \\
0 & I & 0 & 0 \\
0 & 0 & B_{i+1,i+1} & 0
\end{array} \right) \]

By using (3.23)

\[ \Leftrightarrow r \left( \begin{array}{cccc}
E_i & I & I & 0 \\
R_{D_{ii}} & 0 & 0 & 0 \\
R_{B_{i+1,i+1}} & 0 & 0 & 0
\end{array} \right) = r \left( \begin{array}{cccc}
I & I & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & M_{ii} & 0
\end{array} \right) + r \left( \begin{array}{cccc}
I D_{ii} & 0 & 0 & 0 \\
0 & I & 0 & 0 \\
0 & 0 & B_{i+1,i+1} & 0
\end{array} \right) \]

By using (3.23)

\[ \Leftrightarrow r \left( \begin{array}{cccc}
E_i & I & I & 0 \\
R_{D_{ii}} & 0 & 0 & 0 \\
R_{B_{i+1,i+1}} & 0 & 0 & 0
\end{array} \right) = r \left( \begin{array}{cccc}
I & I & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & C_{i+1,i+1} & 0
\end{array} \right) + r \left( \begin{array}{cccc}
I D_{ii} & 0 & 0 & 0 \\
0 & I & 0 & 0 \\
0 & 0 & B_{i+1,i+1} & 0
\end{array} \right) \]

By using (3.23)

\[ \Leftrightarrow r \left( \begin{array}{cccc}
Z_{i+1}^{2} & Z_{i+1}^{1} & I & I \\
0 & 0 & G_{ii} & 0 \\
0 & 0 & 0 & D_{i+1} \\
0 & 0 & F_{i+1} & 0 \\
0 & 0 & 0 & 0
\end{array} \right) = r \left( \begin{array}{cccc}
I & I & 0 & 0 \\
0 & F_{i} & 0 & 0 \\
0 & C_{i+1} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array} \right) + r \left( \begin{array}{cccc}
I G_{i} & 0 & 0 & 0 \\
0 & I & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array} \right) \]

By using (3.23), (3.27), (3.28) and elementary operations

(3.29)

By using (3.23), (3.27) and elementary operations

(3.30)

By using (3.23), (3.25), (3.27) and elementary operations

(3.31)
Secondly, we prove that (3.3)–(3.6) ⇐⇒ (3.19)–(3.22) for the case \( n - m > 1 \), respectively.

Replace the notations in (3.14)–(3.17) into (3.20), we obtain:

\[
\begin{align*}
\begin{pmatrix}
\hat{C}_m & \hat{E}_m & \hat{F}_m & \hat{G}_m & \hat{D}_{m+1} \\
\hat{C}_{m+1} & -\hat{E}_{m+1} & \hat{F}_{m+1} & \hat{G}_{m+1} & \hat{D}_{m+2} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\hat{C}_n & (-1)^{n-m} \hat{E}_n & \hat{F}_n & \hat{G}_n & \hat{D}_n \\
\hat{B}_m & \hat{B}_{m+1} & \cdots & \hat{B}_n \\
\hat{A}_m & \hat{A}_{m+1} & \cdots & \hat{A}_n \\
\end{pmatrix} 
\end{align*}
\]

\[
= r \begin{pmatrix}
\hat{C}_m & \hat{E}_m & \hat{F}_m & \hat{A}_m \\
\hat{C}_{m+1} & \hat{F}_{m+1} & \hat{A}_{m+1} \\
\vdots & \vdots & \vdots \\
\hat{C}_n & \hat{F}_n & \hat{A}_n \\
\hat{B}_m & \hat{B}_{m+1} & \cdots & \hat{B}_n \\
\end{pmatrix} + r 
\begin{pmatrix}
\hat{B}_m & \hat{B}_{m+1} & \cdots & \hat{B}_n \\
\hat{A}_m & \hat{A}_{m+1} & \cdots & \hat{A}_n \\
\end{pmatrix}
\]

Replace the notations in (3.14)–(3.17)
\[
\begin{pmatrix}
L_{m,m} & \tilde{E}_m & A_{m+1,m+1}^I S_{m+1,m+1} & -L_{m+1,m+1} \\
R_{m+1,m+1} D_{m+1,m+1} B_{m+1,m+1}^I & \tilde{C}_{m+1} & R_{m+1,m+1} & \tilde{F}_{m+1} \\
\cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}
\]

\[
\Leftrightarrow r
\begin{pmatrix}
R_{D_{mm}} & \tilde{B}_{m+1} \\
-R_{B_{m+1,m+1}} \\
\cdots & \cdots \\
\end{pmatrix}
\]

\[
= r
\begin{pmatrix}
L_{m,m} & A_{m+1,m+1}^I S_{m+1,m+1} & -L_{m+1,m+1} \\
\tilde{C}_{m+1} & \tilde{F}_{m+1} & \tilde{A}_{m+1} \\
\cdots & \cdots & \cdots \\
\end{pmatrix}
\]

\[
+ r
\begin{pmatrix}
R_{D_{mm}} & \tilde{B}_{m+1} \\
-R_{B_{m+1,m+1}} \\
\cdots & \cdots \\
\end{pmatrix}
\]

\[
\Leftrightarrow r
\begin{pmatrix}
L_{m+1,m+1} & L_{m,m} & R_{D_{mm}} & R_{B_{m+1,m+1}} \\
R_{m+1,m+1} D_{m+1,m+1} B_{m+1,m+1}^I & \tilde{E}_m & A_{m+1,m+1}^I S_{m+1,m+1} & R_{m+1,m+1} \\
L_{m+1,m+1} & R_{D_{m+1,m+1}} & R_{B_{m+1,m+1}} \\
\cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}
\]

\[
= r
\begin{pmatrix}
L_{m+1,m+1} & L_{m,m} & A_{m+1,m+1}^I S_{m+1,m+1} \\
L_{m+1,m+1} & A_{m+2,m+2}^I S_{m+2,m+2} & L_{m+2,m+2} \\
\cdots & \cdots & \cdots \\
\end{pmatrix}
\]

\[
+ r
\begin{pmatrix}
R_{D_{mm}} & R_{B_{m+1,m+1}} \\
R_{m+1,m+1} D_{m+1,m+1} B_{m+1,m+1}^I & R_{m+1,m+1} \\
R_{D_{m+1,m+1}} & R_{B_{m+1,m+1}} \\
\cdots & \cdots \\
\end{pmatrix}
\]
By (3.25)
\[
\begin{align*}
\begin{pmatrix}
\begin{array}{cccc}
L_{mm} & I & I & D_{m+1,m+1} \\
E_m & -E_{m+1} & I & I \\
C_{m+1,m+1} & R_{b_{m+2,m+2}} & A_{m+1,m+1} & I \\
& & \ddots & \ddots & \ddots \\
& & & A_{m+2,m+2} & I \\
& & & & B_{m+1,m+1}
\end{array}
\end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
\begin{pmatrix}
\begin{array}{cccc}
L_{mm} & I & I & D_{m+1,m+1} \\
R_{d_{mm}} & I & I & D_{m+1,m+1} \\
R_{b_{m+2,m+2}} & I & I & B_{m+1,m+1} \\
I & & \ddots & \ddots \\
& & & A_{m+2,m+2} \\
& & & \ddots & \ddots \\
& & & & B_{m+1,m+1}
\end{array}
\end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
\begin{pmatrix}
\begin{array}{cccc}
I & I & I & D_{mm} \\
E_m & -E_{m+1} & I & I \\
C_{m+1,m+1} & C_{m+2,m+2} & A_{m+1,m+1} & I \\
& & \ddots & \ddots & \ddots \\
& & & A_{m+2,m+2} & I \\
C_{mm} & & \ddots & \ddots & \ddots \\
& & & \ddots & \ddots \\
I & I & B_{m+1,m+1} & \vdots & \vdots \\
I & I & \vdots & \vdots & \vdots \\
C_{mm} & A_{mm} & \vdots & \vdots & \vdots \\
& & \ddots & \ddots & \ddots \\
& & & B_{m+2,m+2} & \vdots \\
& & & \vdots & \vdots \\
& & & & B_{m+2,m+2}
\end{array}
\end{pmatrix}
\end{align*}
\]
\[
\begin{pmatrix}
I & I \\
I & I & 2^2_{m+1} - 2^1_{m+1} \\
I & I & 2^2_{m+2} - 2^1_{m+2} \\
C_{m+1} & F_{m+1} & F_{m+2} C_{m+2} & A_{m+1} \\
F_m & F_{m+1} & F_{m+2} C_{m+2} & A_{m+1}
\end{pmatrix}
\]

By \( r \)

\[
= r
\begin{pmatrix}
I & I \\
I & I & 2^2_{m+1} - 2^1_{m+1} \\
I & I & 2^2_{m+2} - 2^1_{m+2} \\
C_{m+1} & F_{m+1} & F_{m+2} C_{m+2} & A_{m+1} \\
F_m & F_{m+1} & F_{m+2} C_{m+2} & A_{m+1}
\end{pmatrix}
\]

\[
\Leftrightarrow r
\begin{pmatrix}
I & I & D_{m+1} & G_m \\
I & I & D_{m+2} & G_m \\
I & I & D_{m+2} & A_{m+1} \\
C_{m+1} & F_{m+1} & F_{m+2} C_{m+2} & A_{m+1} \\
F_m & F_{m+1} & F_{m+2} C_{m+2} & A_{m+1}
\end{pmatrix}
\]

\[
= r
\begin{pmatrix}
I & I & D_{m+1} & G_m \\
I & I & D_{m+1} & G_m \\
I & I & D_{m+1} & G_m \\
C_{m+1} & F_{m+1} & F_{m+2} C_{m+2} & A_{m+1} \\
F_m & F_{m+1} & F_{m+2} C_{m+2} & A_{m+1}
\end{pmatrix}
\]

\[
\Leftrightarrow r
\begin{pmatrix}
C_m & E_m & F_m & D_{m+1} & A_m \\
G_m & C_{m+1} & -E_{m+1} F_{m+1} & A_{m+1} \\
B_m & B_{m+1} & \ldots & \ldots
\end{pmatrix}
\]

\[
= r
\begin{pmatrix}
C_m & F_m & A_m & A_{m+1} \\
C_{m+1} & F_{m+1} & A_{m+1} \\
B_m & B_{m+1} & \ldots & \ldots
\end{pmatrix}
\]

\[
+ r
\begin{pmatrix}
G_m & D_{m+1} & D_{m+2} & G_{m-2} D_m \\
G_{m+1} & D_{m+1} & D_{m+2} & G_{m-2} D_m \\
B_m & B_{m+1} & \ldots & \ldots
\end{pmatrix}
\]
Following this way, we obtain that
\[
\begin{pmatrix}
C_m & E_m & F_m & A_m \\
G_m & C_{m+1} & D_{m+1} & F_{m+1} \\
C_m & G_{m+1} & -E_{m+1} & F_{m+1} \\
B_m & C_n & (-1)^{n-m}E_n & F_n \\
B_m & B_{m+1} & C_n & A_n
\end{pmatrix}
\]
\[= r \begin{pmatrix}
C_m & F_m & A_m & \cdots & \cdots \\
C_{m+1} & F_{m+1} & A_{m+1} & \cdots & \cdots \\
\vdots & \vdots & \vdots & \ddots & \cdots \\
C_n & F_n & A_n & \cdots & \cdots \\
B_m & B_{m+1} & \cdots & \cdots & B_n
\end{pmatrix}
+ r \begin{pmatrix}
G_m & D_{m+1} & D_{m+2} & \cdots & \cdots \\
G_{m+1} & D_{m+2} & D_{m+3} & \cdots & \cdots \\
\vdots & \vdots & \vdots & \ddots & \cdots \\
G_{n-2} & D_{n-1} & D_{n} & \cdots & \cdots \\
G_n & \cdots & \cdots & \cdots & B_n
\end{pmatrix}.
\]

Similarly, it can be found that:

\[ (3.3) \iff (3.20), \ (3.4) \iff (3.21), \ (3.5) \iff (3.22), \]

for case \(n - m > 1\).

\[\square\]

4. Some systems of quaternion matrix equations involving \(\phi\)-Hermicity

In this section, we will consider the system of quaternion matrix equations involving \(\phi\)-Hermicity

\[
\begin{align*}
A_1X_1 & + (A_1X_1)_{\phi} + C_1Z_1(C_1)_{\phi} + F_1Z_2(F_1)_{\phi} = E_1, \\
A_2X_2 & + (A_2X_2)_{\phi} + C_2Z_2(C_2)_{\phi} + F_2Z_3(F_3)_{\phi} = E_2, \\
& \quad \vdots \\
A_kX_k & + (A_kX_k)_{\phi} + C_kZ_k(C_k)_{\phi} + F_kZ_{k+1}(F_k)_{\phi} = E_k, \\
Z_1 & = (Z_1)_{\phi}, \ Z_2 = (Z_2)_{\phi}, \cdots, \ Z_k = (Z_k)_{\phi}, \ Z_{k+1} = (Z_{k+1})_{\phi},
\end{align*}
\]

where \(A_i, C_i, F_i, E_i\) are given, and \(E_i\) are \(\phi\)-Hermitian matrices (\(i = 1, k\)). We derive some solvability conditions to the system (4.1) in terms of ranks involved. We first give the definition of nonstandard involution and \(\phi\)-Hermitian.

A map \(\phi: \mathbb{H} \rightarrow \mathbb{H}\) is called an involution if \(\phi(xy) = \phi(y)\phi(x)\), \(\phi(x + y) = \phi(x) + \phi(y)\) and \(\phi(\phi(x)) = x\) for all \(x, y \in \mathbb{H}\). Moreover, the matrix representing of \(\phi\), with respect to the basis \(\{1, i, j, k\}\) is

\[
\begin{pmatrix}
1 & 0 \\
0 & T
\end{pmatrix},
\]

where either \(T = -I_3\) or \(T\) is a \(3 \times 3\) real orthogonal symmetric matrix with eigenvalues 1, 1, \(-1\) (see Theorem 2.4.4 in [9]). If \(T = -I_3\), then \(\phi\) is the standard involution, and for the latter case, \(\phi\) is called a nonstandard involution (see Definition 2.4.5 in [9]). We in this paper consider only the nonstandard involution.
For \( A \in \mathbb{H}^{m \times n} \), we denote by \( A_{\phi} \) the \( n \times m \) matrix obtained by applying \( \phi \) entrywise to \( A^T \), where \( \phi \) is a nonstandard involution ([9], page 44). Now we recall the definition of the \( \phi \)-Hermitian matrix.

**Definition 4.1 (\( \phi \)-Hermitian Matrix).** [9] \( A \in \mathbb{H}^{n \times n} \) is said to be \( \phi \)-Hermitian if \( A = A_{\phi} \), where \( \phi \) is a nonstandard involution.

Some properties of \( A_{\phi} \), rank and Moore-Penrose inverse of \( A_{\phi} \) are given as follows.

**Property 4.1.** [9] Let \( \phi \) be a nonstandard involution. Then,

1. \( (\alpha A + \beta B)_{\phi} = A_{\phi}\phi(\alpha) + B_{\phi}\phi(\beta) \), \( \alpha, \beta \in \mathbb{H} \), \( A, B \in \mathbb{H}^{m \times n} \).
2. \( (A\alpha + B\beta)_{\phi} = \phi(\alpha)A_{\phi} + \phi(\beta)B_{\phi} \), \( \alpha, \beta \in \mathbb{H} \), \( A, B \in \mathbb{H}^{m \times n} \).
3. \( (AB)_{\phi} = B_{\phi}A_{\phi} \), \( A \in \mathbb{H}^{m \times n}, B \in \mathbb{H}^{n \times p} \).
4. \( (A_{\phi})_{\phi} = A \), \( A \in \mathbb{H}^{n \times n} \).
5. If \( A \in \mathbb{H}^{n \times n} \) is invertible, then \( (A_{\phi})^{-1} = (A^{-1})_{\phi} \).
6. \( r(A) = r(A_{\phi}) \), \( A \in \mathbb{H}^{m \times n} \).
7. \( I_{\phi} = I, \ 0_{\phi} = 0 \).
8. \( (A_{\phi})^{\dagger} = (A^{\dagger})_{\phi} \), [9].
9. \( (L_{A})_{\phi} = R_{A_{\phi}}, \ (R_{A})_{\phi} = L_{A_{\phi}} \), [9].

For more properties of \( \phi \)-Hermitian quaternion matrix, we refer the reader to the recent book [9] and the papers [3] and [6]. The following theorem gives some solvability conditions to the system (4.1) in terms of ranks.

**Theorem 4.1.** The system (4.1) is consistent if and only if

\[
r \begin{pmatrix} E_i & A_i & C_i & F_i \\ (A_i)_{\phi} & 0 & 0 & 0 \end{pmatrix} = r \begin{pmatrix} A_i & C_i & F_i \end{pmatrix} + r(A_i), \tag{4.3}
\]

\[
r \begin{pmatrix} E_i & A_i & C_i \\ (A_i)_{\phi} & 0 & 0 \end{pmatrix} = r \begin{pmatrix} A_i & C_i \end{pmatrix} + r \begin{pmatrix} A_i \\ F_i \end{pmatrix}, \tag{4.4}
\]

\[
r \begin{pmatrix} C_m & E_m \\ (F_m)_{\phi} & F_m \end{pmatrix} \begin{pmatrix} (C_{m+1})_{\phi} \\ -E_{m+1} (F_m)_{\phi} \end{pmatrix} A_{m+1} \begin{pmatrix} A_m \\ \cdots \\ A_n \end{pmatrix} \\
\begin{pmatrix} (C_{m+1})_{\phi} \\ -E_{m+1} (F_m)_{\phi} \end{pmatrix} A_{m+1} \begin{pmatrix} A_m \\ \cdots \\ A_n \end{pmatrix} \\
\begin{pmatrix} \cdots \\ \cdots \\ \cdots \end{pmatrix} \\
\begin{pmatrix} \cdots \\ \cdots \\ \cdots \end{pmatrix} \begin{pmatrix} C_n & F_n \\ C_n & F_n \end{pmatrix} \begin{pmatrix} A_m \\ \cdots \\ A_n \end{pmatrix} + r \begin{pmatrix} F_m & C_{m+1} \\ F_{m+1} & C_{m+2} \\ \cdots \\ F_{n-1} & C_n \end{pmatrix}, \tag{4.5}
\]
results can be viewed as special cases of the ones obtained in this paper. To solve the system of quaternion matrix equations involving system (1.1), we have presented some necessary and sufficient conditions for the existence of a solution to the system of quaternion matrix equations (1.1) in terms of ranks. Based on the results of the system (1.2), we can give the solvability conditions to the system (1.2) by Theorem 3.1.

\[
\begin{pmatrix}
(C_m)_\phi & F_m \\
\vdots & \ddots \\
(C_{m+1})_\phi & F_{m+1} \\
A_m & \ldots \\
\end{pmatrix} 
= r 
\begin{pmatrix}
F_m \\
\vdots \\
F_{m+1} \\
A_m \\
\end{pmatrix} 
+ r 
\begin{pmatrix}
C_m \\
\vdots \\
C_{m+1} \\
A_m \\
\end{pmatrix},
\]

(4.6)

**Proof.** We prove that the system (4.1) has a solution if and only if the system of quaternion matrix equations

\[
\begin{align*}
A_1 \hat{X}_1 + \hat{Y}_1(A_1)_\phi + C_1 \hat{Z}_1(C_1)_\phi + F_1 \hat{Z}_2(F_1)_\phi &= E_1, \\
A_2 \hat{X}_2 + \hat{Y}_2(A_2)_\phi + C_2 \hat{Z}_2(C_2)_\phi + F_2 \hat{Z}_3(F_2)_\phi &= E_2, \\
&\vdots \\
A_k \hat{X}_k + \hat{Y}_k(A_k)_\phi + C_k \hat{Z}_k(C_k)_\phi + F_k \hat{Z}_{k+1}(F_k)_\phi &= E_k,
\end{align*}
\]

(4.7)

has a solution. If the system (4.1) has a solution, it can be expressed as

\[
X_i = \frac{\hat{X}_i + (\hat{Y}_i)_\phi}{2}, Z_i = \frac{\hat{Z}_i + (\hat{Z}_i)_\phi}{2}, Z_{i+1} = \frac{\hat{Z}_{i+1} + (\hat{Z}_{i+1})_\phi}{2}, (1 \leq i \leq k).
\]

We can give the solvability conditions to the system (1.2) by Theorem 3.1.

**5. Conclusions**

We have provided some necessary and sufficient conditions for the existence of a solution to the system of quaternion matrix equations (1.1) in terms of ranks. Based on the results of the system (1.1), we have presented some necessary and sufficient conditions for the existence of a solution to the system of quaternion matrix equations involving \(\phi\)-Hermicity (1.2). Some known results can be viewed as special cases of the ones obtained in this paper.

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