SHARP LOCAL WELL-POSEDNESS FOR QUASILINEAR WAVE EQUATIONS WITH SPHERICAL SYMMETRY

CHENGBO WANG

Abstract. In this paper, we prove a sharp local well-posedness result for spherically symmetric solutions to quasilinear wave equations with rough initial data, when the spatial dimension is three or higher. Our approach is based on Morawetz type local energy estimates with fractional regularity for linear wave equations with variable $C^1$ coefficients, which rely on multiplier method, weighted Littlewood-Paley theory, duality and interpolation. Together with weighted linear and nonlinear estimates (including weighted trace estimates, Hardy’s inequality, fractional chain rule and fractional Leibniz rule) which are adapted for the problem, the well-posed result is proved by iteration. In addition, our argument yields almost global existence for $n = 3$ and global existence for $n \geq 4$, when the initial data are small, spherically symmetric with almost critical Sobolev regularity.

1. Introduction

Let $n \geq 3$, we are interested in the local well-posedness of the spherically symmetric solutions for the Cauchy problem of the quasilinear wave equations with low regularity

\begin{align}
\Box u + g(u)\Delta u &= a(u)u_t^2 + b(u)|\nabla u|^2, \quad (t, x) \in (0, T) \times \mathbb{R}^n, \\
(1.2) \quad u(0, x) &= u_0(x) \in H^s_{rad}(\mathbb{R}^n), \quad \partial_t u(0, x) = u_1(x) \in H^{s-1}_{rad}(\mathbb{R}^n),
\end{align}

where $\Box = -\partial^2_t + \Delta$, $g, a, b$ are smooth functions, $g(0) = 0$ and such that $\Box + g(u)\Delta$ satisfy the uniform hyperbolic condition. Here, $H^s_{rad}$ stands for the space of spherically symmetric functions lying in the usual Sobolev space $H^s$.

The equation (1.1) is scale-invariant in the sense that $u_\lambda(t, x) = u(t/\lambda, x/\lambda)$ solves (1.1) for every $\lambda > 0$, provided that $u(t, x)$ is a solution. This gives us the critical homogeneous Sobolev space $\dot{H}^{s_c}$ with

$$s_c = \frac{n}{2},$$

which is known to be a lower bound of the regularity for the problem to be well-posed in $H^s$. On the other hand, another characteristic feature of the wave equations is that the propagation of singularities along the light cone, which heuristically yields ill posedness for the problem at the regularity level $s \leq s_i = (n + 5)/4$. 

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For semilinear wave equations, that is, $g \equiv 0$, it could be shown to be locally well-posed in $H^s$ for $s > n/2 + 1 - 1/q$, with $q = \max(2, 4/(n-1))$, by $L_t^q L_x^\infty$ Strichartz estimates. Moreover, it is known that the problem is locally well-posed if

$$s > \max(s_c, s_l),$$

and ill-posed in general, if $s < s_c$ or $s \leq s_l$, see Ponce and Sideris [40] ($n = 3$), Tataru [46] ($n \geq 5$), Zhou [53] ($n = 2, 4$) for positive results, and Lindblad [28, 29] ($n = 3$), Fang-Wang [8] ($n \geq 6$), Liu-Wang [32] ($n \leq 5$) for negative results. The critical well-posedness remains open for higher dimensional case ($n \geq 6$).

If the nonlinearity is of the first null-form, that is, $a(u)u_t^2 + b(u)|\nabla u|^2 = c(u)(u_t^2 - |\nabla u|^2)$ for some $c(u)$, improved local well-posed results are also available, which states that $s > s_c$ is sufficient for local well-posedness, see, e.g., Klainerman-Machedon [21] ($n = 3$), Klainerman-Selberg [26] ($n \geq 2$), see also Liu-Wang [32].

Furthermore, it is well-known that we could extend the admissible pairs for Strichartz estimates, when the initial data are spherically symmetric or have certain amount of angular regularity, see Klainerman-Machedon [23], Sterbenz [44], Machihara-Nakamura-Nakanishi-Ozawa [33], Fang-Wang [9]. With help of this observation, we could improve the radial results to $s > 3/2$ for $n = 2$ and $s \geq 2$ for $n = 3$. We see that there are still 1/2 gap of regularity, between the positive results and the scaling regularity. When $n = 3$ in the case of radial small data, by exploiting the local energy estimates and weighted fractional chain rule, the regularity assumption is improved to the almost critical assumption $s > 3/2$, in Hidano-Jiang-Lee-Wang [12], with previous results of Hidano-Yokoyama [16] for $s = 2$. In view of [12], it seems that the critical radial regularity for $n = 2$ is $s = 3/2$ instead of the scaling critical regularity $s = 1$. Concerning radial solutions to general semilinear hyperbolic systems in 3D under null condition, the global existence for small scaling invariant $\dot{W}^{2,1}(\mathbb{R}^3)$ data is known from Yin-Zhou [52].

Turning to the quasilinear problem (1.1), it is much more delicate. Based on the classical energy argument, it is locally well-posed, as long as $s > n/2 + 1$ (17)). Similar to the semilinear problem, the approach of using $L_t^q L_x^\infty$ Strichartz estimates has been intensively investigated. To make the argument work, we need to obtain Strichartz estimates for wave operators with variable coefficients. It is known that we have the full Strichartz estimates, provided that the perturbation is of $C^{1,1}$, see Smith [41], Tataru [48], as well as Kapitanski [19], Mockenhaupt-Seeger-Sogge [37] for previous results with smooth perturbation. However, in view of application to the quasilinear problem (1.1), $u \in H^s$ with $s < n/2 + 1$ will only imply $g(u) \in C^{0,s-n/2}$, by Sobolev embedding, which means that it is desirable to obtain the Strichartz estimates for wave operators with rough coefficients.

The first breakthrough was achieved through the independent works of Bahouri-Chemin [3] (Hadamard parametrix) and Tataru [47] (FBI transform), where weaker Strichartz estimates for metric with limited regularity was obtained and so is the local well-posedness for

$$s > \frac{n + 2}{2} - \begin{cases} 1/4, & n \geq 3, \\ 1/8, & n = 2. \end{cases}$$

This approach was developed further, to arrive

$$s > \frac{n + 2}{2} - \begin{cases} 1/3, & n \geq 3, \\ 1/6, & n = 2. \end{cases}$$
see Bahouri-Chemin [2], Tataru [48]. To get further improved result, it is desirable to exploit the additional geometric information on the metric $g(u)$ and solution $u$. In the work of Klainerman-Rodnianski [22], the nonlinear structure of solutions was exploited to obtain the improved result $s > 3 - \sqrt{3}/2$ when $n = 3$. Finally, the well-posedness for

$$s > \frac{n + \frac{3}{2}}{2} - \left\{ \begin{array}{ll}
1/2, & 3 \leq n \leq 5 \\
1/4, & n = 2 
\end{array} \right.,$$

was proven by Klainerman-Rodnianski [24] for the Einstein-vacuum equations ($n = 3$), and Smith-Tataru [42] for general quasilinear wave equations in general spatial dimensions, by constructing a parametrix using wave packets. Later, Q. Wang [51] gives an alternative proof for Smith-Tataru’s result when $n = 3$, by commuting vector fields approach.

When $n = 3$ and 2, we know from Lindblad [30] and Liu-Wang [32] that the well-posed result in [42] is sharp in general. However, concerning the Einstein-vacuum equations, the so-called bounded $L^2$ curvature conjecture (well-posed in $H^2$) was verified in Klainerman-Rodnianski-Szeftel [25]. In contrast, Ettinger-Lindblad [7] proved ill-posed result in $H^2$ for Einstein-vacuum equations in the harmonic gauge.

In summary, the quasilinear problem is locally well-posed in $H^s$ for

$$s > \left\{ \begin{array}{ll}
(n + 5)/4 & n \leq 3, \\
(n + 1)/2 & n = 4, 5, \\
n/2 + 2/3 & n \geq 6 
\end{array} \right.,$$

in general.

Comparing with the semilinear problem, we expect naturally that there should be improved well-posed theory, when the problem and the initial data are spherically symmetric. Actually, in [14], together with Hidano and Yokoyama, we proved that the 3-dimensional problem

$$\Box u + g(u)\Delta u = au_t^2 + b|\nabla u|^2$$

is well-posed for small radial data in $H^2_{rad} \times H^1_{rad}$, with almost global existence of solutions, up to $\exp(c/||(\nabla u_0, u_1)||_{H^1})$ (see also Zhou-Lei [54] for previous work on global existence with $a = b = 0$, for compactly supported $H^2_{rad} \times H^1_{rad}$ data). On the other hand, when $n = 2$, the improved local well-posed result for $H^s_{rad}$ with $s > 3/2$ was suggested in Fang-Wang [10]. In addition, as we have mentioned, when $n = 2, 3$, the long time well-posedness with small radial data in $H^s_{rad}$ with $s > 3/2$ is known from Hidano-Jiang-Lee-Wang [12].

1.1. Main results. Let us turn to the first main result of this paper, concerning the physically important case, $n = 3$. As we know, it is well-posed in $H^2_{rad}$ (at least for small data) and generally ill-posed in $H^s_{rad}$ with $s < s_c = 3/2$. Heuristically, comparing with the semilinear results, we may expect well-posedness in $H^s_{rad}$ for certain $s < 2$. However, we notice that $H^2_{rad}$ is the lowest possible regularity we could obtain by the approach of $(L^2_t L^\infty_x)$ Strichartz estimates, even in the radial case. To break the Sobolev regularity barrier $s = 2$, we need to circumvent the Strichartz estimates approach.

In the following result, we could prove well-posedness in $H^s_{rad}$ for any subcritical regularity, $s > 3/2$, which shows that there are no any other obstacles to well-posedness in the radial case, except scaling. As far as we know, this might be the first well-posed result for three-dimensional quasilinear wave equations, which
breaks the Sobolev regularity barrier, \(s = 2\). More precisely, we prove the following, with certain low frequency condition on \(u_1\).

**Theorem 1.1.** Let \(n = 3\), \(s \in (3/2, 2]\) and \(s_0 \in [2 - s, s - 1]\). Considering (1.1)-(1.2) with \(u_0 \in H^s_{\text{rad}}\) and \(u_1 \in H^{s-1}_{\text{rad}} \cap H^{s_0-1}_{\text{rad}}\). There exists \(T_0 > 0\) such that the problem is (unconditionally) locally well-posed in the function space

\[
(1.3) \quad \ u \in L^\infty_t H^s_{\text{rad}} \cap C^0_t H^{s-1}_t \cap C^1_t \dot{H}^{s_0}([0, T_0] \times \mathbb{R}^3), \quad \partial_t u \in C^1_t \dot{H}^{s_0-1}_t.
\]

More precisely,

1. (Existence) There exists a universal constant \(C > 0\), so that there exists a (weak) solution \(u\) satisfying (1.3) and

\[
\|\partial u\|_{L^\infty H^s} + T^{-\delta} \|\sqrt{-\Delta} D^\theta \partial u\|_{L^2(0,T) \times \mathbb{R}^3} \leq C \|\| \nabla u_0, u_1 \| H^s,
\]

for all \(\theta \in \{s_0 - 1\} \cup [0, s-1]\) and \(T \in (0, T_0]\). Here \(s = s - 3/2\), \(D = \sqrt{-\Delta}\).

2. (Uniqueness) the solution is unconditionally unique in (1.3).

3. (Persistence of regularity) Let \(T_*\) be the maximal time of existence (lifespan) for the solution in (1.3). If \((u_0, u_1) \in H^{s_1} \times H^{s_1-1}\) for some \(s_1 \geq 3\), then the solution \(u \in L^\infty H^{s_1} \times C^0_t H^{s_1-1}\) in \([0, T] \times \mathbb{R}^3\) for any \(T < T_*\).

4. (Continuous dependence) We also have continuously dependence on the data when \(s_0 < s - 1\), in the following sense: for any \(T \in (0, T_*)\), \(s_1 \in (s, s)\) and \(\epsilon > 0\), there exists \(\delta > 0\), such that whenever \(\|\nabla (u_0 - v_0), u_1 - v_1\|_{H^{s-1} \cap H^{s_0-1}} \leq \delta\), the corresponding solution \(v \in L^\infty H^{s_1} \times C^0_t H^{s_1-1}\) is well-defined in \([0, T] \times \mathbb{R}^3\)

\[
\|\partial (u - v)\|_{L^\infty(H^{s-1} \cap H^{s_0-1})} \leq \epsilon.
\]

**Remark 1.2.** The regularity assumption of the lifespan obtained in Theorem 1.1 is sharp in general. More precisely, we could not have well-posedness for data in some critical space, \(B\), and possibly non-subcritical space \(\dot{H}^s\) with \(s \leq s_c = 3/2\). Actually, let \(g = 0, a = 1, b = 0, \phi, \psi\) be given nonnegative, nontrivial, spherically symmetric \(C^\infty_{\text{rad}}\) functions, then it is well-known, see, e.g., John [18], that for classical solutions, for any \(\epsilon > 0\), the lifespan \(T_* < \infty\) for data \(u_0 = \epsilon \phi, u_1 = \epsilon \psi\). By persistence of regularity, we know that \(T_*\) is the same as the lifespan for weak solutions. If the problem is still well-posed, then by continuous dependence for the trivial solution, there exists \(\delta > 0\) such that \(T_* \geq \delta\), for any data with critical norm, \(\|(u_0, u_1)\|_B \leq \delta\) and \(\epsilon_s = \|\nabla (u_0, u_1)\|_{H^{s-1}} \leq \delta\). Let \(\epsilon \ll 1\) such that the norm \(\leq \delta\) and we obtain a solution \(u\) with \(T_* \ll \infty\). For each fixed \(\epsilon > 0\), by rescaling, we know that, for any \(0 < \lambda \leq 1\), \(u_\lambda(t,x) = u(t/\lambda, x/\lambda)\) solves the equation with rescaled data, for which we have

\[
\|(u_\lambda(0), \partial_t u_\lambda(0))\|_B = \|(u_0, u_1)\|_B \leq \delta, \quad \epsilon_s \lambda = \lambda^{s_c - s} \epsilon_s \leq \epsilon_s \leq \epsilon_0, \quad T_* \lambda = \lambda T_*.
\]

This gives us \(0 < \delta \leq T_* \lambda = \lambda T_* < \infty\) for any \(0 < \lambda \leq 1\), which is clearly a contradiction. On the other hand, we have an auxiliary low frequency regularity assumption on the initial velocity \(u_0 \in \dot{H}^{s_0-1}\), due to the second order feature of the equation and the limited regularity level \(s < 2\). This assumption plays a key role in our analysis, to close the iteration, and it will be interesting to determine if it is essential for the well-posed result or not. Notice, however, that we do not need to assume \(u_1 \in \dot{H}^{s_0-1}\) when it is compactly supported.

Next, we present our high dimensional well-posed result.
Theorem 1.3. Let \( n \geq 4 \), \( s = n/2 + \mu \) with
\[
(1.4) \quad \mu \in \begin{cases} (0, 1/2], & n \text{ odd} \\ (0, 1), & n \text{ even} \end{cases}.
\]
The problem (1.1)-(1.2) is (unconditionally) locally well-posed in the function space
\[
(1.5) \quad u \in L^\infty_t H^s_{rad} \cap C^{0,1}_t H^{s-1}_{rad} \cap C^1 \cap H^1 \cap L^2.
\]
More precisely,
1. (Existence) There exists a constant \( C > 0 \) so that for any data \((u_0, u_1) \in H^s_{rad} \times H^{s-1}_{rad}\), there exist \( T > 0 \), and a (weak) solution \( u \) in (1.5) in \([0, T] \times \mathbb{R}^n\), satisfying
\[
\|\partial u\|_{L^\infty_t H^s} + T^{-\frac{\mu}{2}} \|r^{-\frac{1+\mu}{2}} D^\theta \partial u\|_{L^2_t([0, T] \times \mathbb{R}^n)} \leq C \|\nabla u_0, u_1\|_{H^s},
\]
for all \( \theta \in [0, s - 1] \).
2. (Uniqueness) the solution is unconditionally unique in (1.5).
3. (Persistence of regularity) Let \( T_* \) be the lifespan. If \((u_0, u_1) \in H^{s_1} \times H^{s_1-1}\) for some \( s_1 \geq [(n + 4)/2] \), then the solution \( u \in L^\infty_t H^{s_1} \times C^{0,1}_t H^{s_1-1} \) in \([0, T] \times \mathbb{R}^n\) for any \( T < T_* \).
4. (Continuous dependence) We also have continuously dependence on the data, in the \( H^{s_1} \) topology, for \( s_1 \in (s_c, s) \).

When considering the small data problem, it turns out that we could give the following long time existence results.

Theorem 1.4 (Long time existence for small data). Let \( n \geq 3 \) and \( s > s_c = n/2 \). Considering (1.1)-(1.2) with \((u_0, u_1) \in H^s_{rad} \times H^{s-1}_{rad}\). When \( n = 3 \), assuming further that \( u_1 \in H^{s-2} \), there exist \( c > 0 \) and \( \delta > 0 \) such that for any data with \( \varepsilon_1 + \varepsilon_s < \delta \), the problem admits an almost global \( L^\infty_t ([0, T], \mathcal{H}^{\min(s, 2)}(\mathbb{R}^3)) \) solution, where
\[
(1.6) \quad T = \exp(c/(\varepsilon_1 + \varepsilon_s)),
\]
\[
(1.7) \quad \varepsilon_s := \|\nabla u_0, u_1\|_{H^{s-1}}, \quad \varepsilon_c = \|\nabla u_0, u_1\|_{H^{s_c-1}}.
\]
When \( n \geq 4 \), for any \( s > s_c \), there exists \( \varepsilon > 0 \) such that the problem admits global solutions whenever \( \varepsilon_s + \varepsilon_1 \leq \varepsilon \).

Remark 1.5. The lower bounds of the lifespan, (1.6), obtained in Theorem 1.4 for \( n = 3 \) is sharp in general, in terms of the order. Actually, as in Remark 1.2, for the sample case \( a = 1, g = b = 0 \), it is well known that there exists data \((\varepsilon \phi, \varepsilon \psi)\), so that, for any \( \varepsilon \in (0, 1] \), the lifespan of the classical solutions has upper bound \( T_* \leq \exp(C/\varepsilon) \) for some \( C > 0 \). By the way, it is clear from the proof of Theorems 1.1-1.3, that, when \(|g| \ll 1\), we could obtain the following lower bound of the lifespan
\[
T_* \geq c(g, a, b, \varepsilon_c) \varepsilon_s^{-1/(s-s_c)}.
\]
Moreover, when \( \varepsilon_c \ll 1 \),
\[
T_* \geq c(g, a, b) \varepsilon_s^{-1/(s-s_c)} \exp(c(g, a, b)/\varepsilon_c).
\]
On the other hand, when it satisfies certain nonlinear structures, such as the null conditions, or many case of the weak null conditions, the problem admits global solutions with small data. See, e.g., [1], [31], [54] for global results with \( a = b = 0 \). In such situations, we naturally expect that global radial results with \( s > s_c \) (or
even certain critical space like $\dot{B}^{s+1}_{2,1}$) remain hold, which is an interesting further problem.

**Remark 1.6.** Although we state only the results for scalar quasilinear wave equations, as it is clear from the proof, our results apply also for general multi-speed system of quasilinear wave equations, which permit spherically symmetric solutions. In particular, the system with multi-speeds ($c_j > 0$)

$$\partial_t^2 u^j - c_j^2(1 + g_j(u)) \Delta u^j = Q_{kl}^{\alpha \beta}(u) \partial_\alpha u^k \partial_\beta u^l, 1 \leq j \leq N,$$

is local well-posed in $H^s_{\text{rad}}(\mathbb{R}^n) \times (H^{s-1}_{\text{rad}}(\mathbb{R}^n) \cap H^{s-2}(\mathbb{R}^n))$ for $n \geq 3$ and $s > n/2$, as well as the system admits spherically symmetric solutions. Similar statement holds for

$$\partial_t^2 u^j - c_j^2(1 + g_j(u, \partial u)) \Delta u^j = Q_{kl}^{\alpha \beta}(u) \partial_\alpha u^k \partial_\beta u^l, 1 \leq j \leq N,$$

in $H^s_{\text{rad}}(\mathbb{R}^n) \times (H^{s-1}_{\text{rad}}(\mathbb{R}^n) \cap H^{s-3}(\mathbb{R}^n))$ when $n \geq 3$ and $s > (n + 2)/2$.

In addition, the quasilinear part could be replaced by the D'Alembertian $\Box g(u)$ with respect to the metric $ds^2 = -dt^2 + g(u)dx^2$, as well as to $\Box g + g(u)\Delta \Box g(t, x, u)$, when $g$ is a small, long range perturbation of the Minkowski metric:

$$g = -K_0(t, r)^2dt^2 + 2K_{01}(t, r)dt dr + K_1^2(t, r)dr^2 + r^2d\omega^2,$$

$$\|(K_0 - 1, K_{01}, K_1 - 1)\| \ll 1, \sum_{j \geq 0} \||\gamma|-|\nu|\|^{\nu}(K_0 - 1, K_{01}, K_1 - 1)\|_{L^\infty_{t, x}(1 + |x|^{-2})} \ll 1,$$

for $1 \leq |\gamma| \leq [n/2] + 1$ (and $K_{01} = 0$ when $n = 3$).

### 1.2. Idea of proof

Let us discuss the idea of proof. Basically, we rely on the approach of using Morawetz type local energy estimates, instead of Strichartz estimates. As have appeared in many works on dispersive and wave equations, Morawetz type local energy estimates have been proven to be more fundamental and robust than Strichartz estimates, in many nonlinear problems.

To make the approach work for quasilinear wave equations, similar to the approach of using Strichartz estimates, we prove a version of Morawetz type local energy estimates, Theorem 3.1, for linear wave equations with variable $C^1$ coefficients. It is this version of local energy estimates which makes it possible to relax the regularity requirement for quasilinear wave equations. The proof is based on the classical multiplier approach with well-chosen multipliers, which yields such estimates for small perturbation of the Minkowski metric. Furthermore, the property of finite speed of propagation is exploited to handle the general case of large perturbation.

With the help of the well-adapted Morawetz type local energy estimates (weighted space-time $L^2$ estimates), we are naturally led to develop the corresponding linear and nonlinear estimates involving weighted functions. Among others, we prove weighted Sobolev type estimates (including weighted trace estimates, Proposition 2.2, and weighted Hardy's inequality, Lemma 2.7), weighted fractional chain rule (Theorem 2.3 and Proposition 2.8), as well as the weighted Leibniz rule (Theorem 2.4).

The Morawetz type local energy estimates, Theorem 3.1, is at the regularity level of $H^1$. To make the approach work, we need to develop the corresponding version of local energy estimates, at the regularity level $\dot{H}^s$ with $s > n/2$. With the help of interpolation, Littlewood-Paley theory involving weighted functions, together with the weighted Sobolev type estimates from Proposition 2.2, Lemma 2.7 and Lemma
and 3.6 and 3.9.

Equipped with all these linear and nonlinear estimates, we could then use the standard iteration argument to establish local existence and uniqueness, as well as the long time existence. In particular, for the case of \( n = 3 \), to prove convergence of approximate solutions, we develop local energy estimates with negative regularity, and need to assume certain low frequency requirement on the initial velocity, due to the second order feature of the equation and the limited regularity level \( s < 2 \).

Recall that, in the approach of using Strichartz estimates, the proof of local existence immediately implies the persistence of regularity and continuous dependence on the data, through Gronwall’s inequality. Unlike the approach of using Strichartz estimates, in our approach, the proofs for persistence of regularity and continuous dependence on the data are not so direct. For example, concerning Strichartz estimates, in our approach, the proofs for persistence of regularity and continuous dependence on the data are not so direct. For example, concerning persistence of regularity, we prove first the result from regularity index \( s \in (s_c, ([n + 2]/2)) \), and then prove the persistence to \( s = ([n + 4]/2) \), which is sufficient to conclude persistence of higher regularity.

This paper is organized as follows. In the next section, we recall and prove various basic linear and nonlinear estimates, including weighted Sobolev type estimates, weighted trace estimates, weighted Hardy’s inequality, as well as the weighted fractional chain rule and the weighted Leibniz rule. In Section 3, we present a version of Morawetz type local energy estimates, for linear wave equations with variable coefficients, as well as the estimates with fractional regularity. In Sections 4 and 5, by iteration argument, we prove local existence and uniqueness, for \( n = 3 \) and \( n \geq 4 \). In Section 6, we show that persistence of regularity for the weak solutions, when the initial data have higher regularity, as well as the continuous dependence on the data. Next, in Sections 7 and 8, we present the proof of almost global existence and global existence for \( n = 3 \) and \( n \geq 4 \), when the initial data are small. Finally, in the appendix, we present the fundamental Morawetz type estimates, by elementary multiplier approach, with carefully chosen multipliers.

1.3. Notations. We close this section by listing the notations.

- \( F(f) \) and \( f \) denote the Fourier transform of \( f \). \( D = \sqrt{-\Delta} := F^{-1}[\cdot]F \) and \( P_j = \phi(2^{-j}D) \) is the (homogeneous) Littlewood-Paley projection on the space-variable, \( j \in \mathbb{Z} \).
- \( r = |x| \), \( \langle r \rangle = \sqrt{2 + r^2} \), \( \partial = (\partial_1, \nabla_x) = (\partial_1, \nabla) \), \( \partial u = (\partial u, u/r) \), \( |\nabla^k u| = \sum_{|\gamma| = k} |\nabla^\gamma u| \) for multi-index \( \gamma \).
- \( L^p(\mathbb{R}^n) \) denotes the usual Lebesgue space, and \( L^p(\mathbb{R}^+) = L^p(\mathbb{R}^+: r^{n-1}dr) \).
- \( L^p_x L^q_t \) is Banach space defined by the following norm
\[
\|f\|_{L^p_x L^q_t} = \|\|f(r\omega)\|_{L^q_t}\|_{L^p_x}.
\]
- \( H^s, \dot{H}^s (B^{s}_{p,q}, \dot{B}^{s}_{p,q}) \) are the usual inhomogeneous and homogeneous Sobolev (Besov) spaces on \( \mathbb{R}^n \).
- With parameters \( \mu, \mu_1 \in (0, 1) \) and \( T \in (0, \infty) \), we define
\[
\|u\|_{L^\infty_T} = \|\partial u\|_{L^\infty_T L^2} + \|r^{-\frac{n-\mu}{2}} \langle r \rangle^{-\frac{n+\mu}{2}} \partial u\|_{L^\infty_T L^2} + (\langle |\nabla(T)| \rangle)^{-\frac{1}{2}} \|r^{-\frac{n-\mu}{2}} \langle r \rangle^{-\frac{\mu}{2}} \partial u\|_{L^\infty_T L^2}.
\]
for functions on \([0, T] \times \mathbb{R}^n\). In the limit case \(T = \infty\), we set

\[
\|u\|_{L^E} = \|\partial u\|_{L^E_t L^2_x} + \|r^{-\frac{1}{2} - \frac{s}{p}} r^{-\frac{1}{2} + \frac{\alpha}{p}} \partial u\|_{L^2_t x} + \sup_{T > 0} \|u\|_{L^E_T}.
\]

In addition, for fixed \(\mu \in (0, 1),\)

\[
(1.9) \quad \|u\|_{X_T} := \|u\|_{L^\infty_T L^2_x} + T^{-\frac{1}{p}} \|r^{-\frac{1}{2} - \frac{s}{p}} u\|_{L^2_t x},
\]

\[
(1.10) \quad \|F\|_{X_T^q} := \inf_{F = F_1 + F_2} (\|F_1\|_{L^1 T L^q_x} + T^{\frac{q}{2}} \|r^{-\frac{1}{2} - \frac{s}{p}} F_2\|_{L^2_t x}).
\]

For \(q \in [1, \infty),\) we introduce the Besov version as follows

\[
\|u\|_{X_T, q} := \|u\|_{L^\infty_T B^0_{q, 2}} + T^{-\frac{1}{p}} \|r^{-\frac{1}{2} - \frac{s}{p}} P_j u\|_{L^2_t x}.
\]

2. Sobolev type and nonlinear estimates

In this section, we recall and prove various basic estimates to be used.

2.1. weighted Sobolev type estimates. We will use the following version of the weighted Sobolev estimates, which essentially are consequences of the well-known trace estimates.

Lemma 2.1 (Trace estimates). Let \(n \geq 2\) and \(s \in [0, n/2),\) then

\[
(2.1) \quad \|r^{(n-1)/2} u\|_{L^\infty T^2_x} \lesssim \|u\|_{H^{1/2}_x}, \quad \|r^{n/2 - s} u\|_{L^\infty T^{n-1/2}_x} \lesssim \|u\|_{H^1}, \quad s > 1/2,
\]

\[
(2.2) \quad \|r^{n(2-1/p) - s} f\|_{L^\infty T^2_x} \lesssim \|f\|_{H^1}, \quad 2 \leq p < \infty, \ 2 - 1/p \leq s < n/2.
\]

The estimate (2.1) is well-known, see, e.g., [11, (1.3), (1.7)] and references therein. The inequality (2.2) with \(s = 1/2 - 1/p\) is due to [27], see also [15] for alternative proof using real interpolation and (2.1).

We shall also use the following weighted variant of the trace estimates.

Proposition 2.2 (Weighted trace estimates). Let \(n \geq 2, \alpha \in (1/2, n/2)\) and \(\beta \in (\alpha - n/2, n/2)\). Then we have

\[
(2.3) \quad \|r^{n/2 - \alpha + \beta} P_j u\|_{L^\infty_T H^{\alpha - 1/2}_x} \lesssim \|r^\beta D^\alpha u\|_{L^2_x},
\]

\[
(2.4) \quad \|r^{n/2 - \alpha + \beta} u\|_{L^\infty_T H^{\alpha - 1/2}_x} \lesssim \|r^{\beta+2\alpha} P_j u\|_{L^2_x}.
\]

In addition, we have

\[
(2.5) \quad \|r^{n(2-1/p) - \alpha + \beta} u\|_{L^p T^2_x} \lesssim \|r^\beta D^\alpha u\|_{L^2_x},
\]

for any \(p \in [2, \infty), \alpha \in (1/2 - 1/p, n/2),\) and \(\beta \in (\alpha - n/2, n/2)\).

Proof. We essentially follow [50, Lemma 4.2], where (2.3) was proven for \(\alpha \in (1/2, 1)\) and \(n \geq 3\). Recall that we have the following weighted Littlewood-Paley square-function estimate

\[
(2.6) \quad \|w P_j f\|_{L^p T^2_x} \simeq \|w f\|_{L^p}, \quad w^p \in A_\rho, \ f \in L^p(w^p dx), \ p \in (1, \infty).
\]

As \(r^\beta \in A_2\) if and only if \(|\beta| < n/2,\) we get

\[
(2.7) \quad \|r^\beta D^\alpha u\|_{L^2_x} \simeq \|r^\beta D^\alpha u\|_{L^2_x}, \ \beta \in (-n/2, n/2).
\]
Based on this estimate, we observe that, by rescaling, interpolation and frequency localization, the proof of (2.3) and (2.4) can be reduced to the following estimate

\[ \|u(\omega)\|_{H^{\beta-1/2}(\beta-n)} \lesssim \|r^\beta \nabla^k u\|_{L^2}^{\alpha/k}, \alpha \in [1/2, n/2), \]

where \( \beta \in (\alpha - n/2, n/2) \), \( k \in (\alpha, \alpha + 1) \).

For the proof of (2.8), we recall the weighted Hardy-Littlewood-Sobolev estimates of Stein-Weiss

\[ \|r^\beta \alpha u\|_{2} \lesssim \|r^\beta D^\alpha u\|_{2}, \alpha \in (0, n), \beta \in (\alpha - n/2, n/2) \; . \]

Then for any \( \alpha \in (0, n) \) and \( \beta \in (\alpha - n/2, n/2) \), we have

\[ \|r^\beta \alpha u\|_{2} \lesssim \|r^\beta D^\alpha u\|_{2} \lesssim \|r^\beta \nabla^k u\|_{2}^{\alpha/k} \|r^\beta u\|_{2}^{1-\alpha/k}, \]

if \( \alpha < k \). Moreover, if \( k \in (\alpha, \alpha + 1) \cap \mathbb{N} \), we have

\[ \|r^\beta \alpha u\|_{L^2} \lesssim \|r^\beta \nabla^k u\|_{L^2}, \forall j < k, \]

for such \( \alpha, \beta \). Let \( \phi \) be a cutoff function of \( B_2 \setminus B_{1/2} \) which equals one for \( |x| = 1 \), we get from (2.1) that for \( \alpha \in [1/2, n/2) \) and \( \beta \in (\alpha - n/2, n/2) \)

\[ \|u(\omega)\|_{H^{\beta-1/2}} \lesssim \|\nabla^k (\phi u)\|_{L^2}^{\alpha/k} \|\phi u\|_{L^2}^{-\alpha/k} \lesssim (\sum_{j < k} \|r^\beta \alpha \nabla^{k-j} u\|_{L^2}^{\alpha/k} \|r^\beta u\|_{L^2}^{1-\alpha/k} + \|r^\beta \alpha u\|_{L^2}^{1-\alpha/k}, \]

where we have used (2.10) and (2.11). This gives us (2.8), and so is (2.3) and (2.4).

Finally, (2.5) follows directly from interpolation between (2.9), (2.3) and (2.4).

This completes the proof.

2.2. weighted fractional chain rule. When dealing with the nonlinear problems, it is natural to introduce the weighted fractional chain rule and Leibniz rule. We first present the following generalized version of the weighted fractional chain rule of Hidano-Jiang-Lee-Wang [12], which could be viewed as a transition from Sobolev type norm to Besov type norm, as well as a transition from space variables to space-time variables. For the weight functions, we recall the Muckenhoupt \( A_p \) class, which by definition,

\[ w \in A_1 \Leftrightarrow \mathcal{M}w(x) \leq C w(x), \text{a.e. } x \in \mathbb{R}^n \; , \]

\[ w \in A_p (1 < p < \infty) \Leftrightarrow (\int_Q w(x)dx) \left( \int_Q w^{1-p'} (x)dx \right)^{p-1} \leq C |Q|^p, \forall \text{ cubes } Q \; , \]

with \( \mathcal{M}w(x) = \sup_{r > 0} \int_{B_r(x)} w(y)dy \) denotes the Hardy-Littlewood maximal function. See, e.g., [39, §2.5.2].

**Theorem 2.3** (Weighted fractional chain rule). Let \( s \in (0, 1) \), \( \lambda \geq 1 \), \( q, q_1, q_2 \in (1, \infty) \), \( p, p_1, p_2 \in [1, \infty] \) with

\[ (2.12) \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}, \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \; . \]

Assume \( F : \mathbb{R}^k \to \mathbb{R}^l \) is a \( C^1 \) map, satisfying \( F(0) = 0 \) and

\[ (2.13) \quad |F'(\tau v + (1-\tau)w)| \leq \mu(\tau)|G(v) + G(w)|, \]

where \( \mu(\tau) \) is a non-negative function of \( \tau \) and \( G(v) = \int_0^v \|F'(z)w\|^p dz \).
with $G > 0$ and $\mu \in L^1([0,1])$. If $(w_1 w_2)^q \in A_q$, $w_1^{q_1} \in A_{q_1}$, $w_2^{q_2} \in A_{q_2}$, then we have

$$\|w_1 w_2 2^{js} P_j F(u)\|_{\ell_{j\in\mathbb{Z}}^q L_k^p L_r^s} \lesssim \|w_1 2^{js} P_j u\|_{\ell_{j\in\mathbb{Z}}^q L_k^p L_r^s} \|w_2 G(u)\|_{L_k^q L_r^s},$$

for any $[0,T) \times \mathbb{R}^n \ni (t,x) \to u(t,x) \in \mathbb{R}^k$. In addition, when $q_2 = \infty$ and $q \in (1, \infty)$, if $w_1^q, (w_1 w_2)^q \in A_q$ and $w_2^{-1} \in A_1$, we have

$$\|w_1 w_2 2^{js} P_j F(u)\|_{\ell_{j\in\mathbb{Z}}^q L_k^p L_r^s} \lesssim \|w_1 2^{js} P_j u\|_{\ell_{j\in\mathbb{Z}}^q L_k^p L_r^s} \|w_2 G(u)\|_{L_k^q L_r^s}.$$

As a comparison, we recall here that the estimates obtained from [12, Theorem 1.2] state as follows:

$$\|w_1 w_2 D^s F(u)\|_{L_k^q} \lesssim \|w_1 D^s u\|_{L_k^q} \|w_2 G(u)\|_{L_k^q},$$

(2.16)

$$\|w_1 w_2 D^s F(u)\|_{L_k^q} \lesssim \|w_1 D^s u\|_{L_k^q} \|w_2 G(u)\|_{L_k^q}.$$  

(2.17)

**Proof.** The proof proceeds similar arguments as the estimates obtained from [12, Theorem 1.2]. At first, recall that, by repeating essentially the same argument as in the proof of Taylor [49, (5.6), page 112], we can obtain

$$|P_j F(u)(x)| \lesssim \sum_{k \in \mathbb{Z}} \min(1, 2^{k-j})(\mathcal{M}(P_k u)(x) \mathcal{M}(H)(x) + \mathcal{M}(HP_k u)(x)),$$

where $H(x) \equiv G(u(x)).$

By (2.18), we know that

$$\|w_1 w_2 2^{js} P_j F(u)\|_{\ell_{j\in\mathbb{Z}}^q L_k^p L_r^s} \lesssim \|w_1 w_2 2^{js} \min(1, 2^{k-j})(\mathcal{M}(P_k u) \mathcal{M}(H) + \mathcal{M}(HP_k u))\|_{\ell_{j\in\mathbb{Z}}^q L_k^p L_r^s} \lesssim \|w_1 w_2 2^{js} \min(1, 2^{k-j})(\mathcal{M}(P_k u) \mathcal{M}(H) + \mathcal{M}(HP_k u))\|_{\ell_{j\in\mathbb{Z}}^q L_k^p L_r^s},$$

(2.19)

where we used Young’s inequality with the assumption $s \in (0,1)$ in the last inequality.

By applying Minkowski’s and Hölder’s inequalities to the last expression we have

$$\|w_1 w_2 2^{js} P_j F(u)\|_{\ell_{j\in\mathbb{Z}}^q L_k^p L_r^s} \lesssim \|w_1 w_2 2^{js} \mathcal{M}(P_k u) \mathcal{M}(H)\|_{\ell_{j\in\mathbb{Z}}^q L_k^p L_r^s} + \|w_1 w_2 2^{js} \mathcal{M}(HP_k u)\|_{\ell_{j\in\mathbb{Z}}^q L_k^p L_r^s} \lesssim \|w_1 2^{js} \mathcal{M}(P_k u) \mathcal{M}(H)\|_{L_k^p L_r^s} + \|w_1 2^{js} \mathcal{M}(HP_k u)\|_{L_k^p L_r^s},$$

for any $q \in (1, \infty)$, $p, p_1, q_2 \in [1, \infty]$, and $q_1, q_2 \in (1, \infty)$ with (2.12). The last term in the above we used weighted Hardy-Littlewood inequality, for $(w_1 w_2)^q \in A_q$ with $q \in (1, \infty)$.

If $q_2 < \infty$, recall that we have assumed $w_1^{q_1} \in A_{q_1}$, $w_2^{q_2} \in A_{q_2}$, applying Hölder’s inequality and weighted Hardy-Littlewood inequality again, we obtain

$$\|w_1 w_2 2^{js} P_j F(u)\|_{\ell_{j\in\mathbb{Z}}^q L_k^p L_r^s} \lesssim \|w_2 H\|_{L_k^q L_r^s},$$

(2.19)

which gives the desired inequality.

For the remaining case $q_2 = \infty$, a similar argument yields (2.15), if we recall the weighted $L^\infty$ estimate of [12, (2.17)]:

$$\|w^{-1} \mathcal{M}(H)\|_{L^\infty} \lesssim \|w^{-1} H\|_{L^\infty}, \forall w \in A_1,$$

This completes the proof.
2.3. **Weighted Fractional Leibniz Rule.** As closely related and useful result is the weighted fractional Leibniz rule.

**Theorem 2.4 (Weighted Fractional Leibniz Rule).** Let \( s > 0, q_0, q_1, q_2 \in (1, \infty), p_1, p_2 \in (1, \infty), s_j \in [1, \infty] \) such that

\[
\frac{1}{q_0} = \frac{1}{q_1} + \frac{1}{p_1}, \quad \frac{1}{q_2} = \frac{1}{q_1} + \frac{1}{p_2}, \quad \frac{1}{s_0} = \frac{1}{s_1} + \frac{1}{s_2} = \frac{1}{s_3} + \frac{1}{s_4}.
\]

Suppose the time-independent weight functions satisfy \( w_0 = w_1 z_1 = w_2 z_2 > 0 \), \( w_j^{(p_j)} \in A_{q_j}, z_j^{(p_j)} \in A_{p_j} \) when \( p_j < \infty \) and \( z_j^{-1} \in A_1 \) when \( p_j = \infty \), then we have

\[
\begin{aligned}
\| w_0 2^{js} P_j (uv) \|_{L_t^{s_0} L_x^{q_0}} \lesssim & \| w_1 2^{j1} P_j u \|_{L_t^{s_1} L_x^{q_1}} \| z_1 v \|_{L_t^{s_2} L_x^{q_2}} \\
& + \| w_2 2^{j2} P_j v \|_{L_t^{s_2} L_x^{q_2}} \| z_2 u \|_{L_t^{s_3} L_x^{q_3}},
\end{aligned}
\]

which yields also (for time-independent functions)

\[
\| w_0 2^{js} P_j (uv) \|_{L_t^{s_0} L_x^{q_0}} \lesssim \| w_1 2^{j1} P_j u \|_{L_t^{s_1} L_x^{q_1}} \| z_1 v \|_{L_t^{s_2} L_x^{q_2}} + \| w_2 2^{j2} P_j v \|_{L_t^{s_2} L_x^{q_2}} \| z_2 u \|_{L_t^{s_3} L_x^{q_3}}.
\]

We remark that the following weighted fractional Leibniz rule

\[
\| w_0 D^x (uv) \|_{L_t^{s_0} L_x^{q_0}} \lesssim \| w_1 D^x u \|_{L_t^{s_1} L_x^{q_1}} \| z_1 v \|_{L_t^{s_2} L_x^{q_2}} + \| w_2 D^x v \|_{L_t^{s_2} L_x^{q_2}} \| z_2 u \|_{L_t^{s_3} L_x^{q_3}}.
\]

We remark that the following weighted fractional Leibniz rule (2.20)

\[
\| w_0 D^x (uv) \|_{L_t^{s_0} L_x^{q_0}} \lesssim \| w_1 D^x u \|_{L_t^{s_1} L_x^{q_1}} \| z_1 v \|_{L_t^{s_2} L_x^{q_2}} + \| w_2 D^x v \|_{L_t^{s_2} L_x^{q_2}} \| z_2 u \|_{L_t^{s_3} L_x^{q_3}},
\]

with \( q_j, p_j \in (1, \infty) \) has been obtained, see Cruz–Uribe and Naibo [5], D'Ancona [6]. However, as is clear, in view of application, the results with \( p_j = \infty \) seem to be more desirable.

**Proof.** The proof follows from a standard para-product argument and we present only the proof of (2.20). In view of \( u = \sum P_j u, v = \sum P_j v \), we introduce the para-product and decompose \( uv \) as follows

\[
T_u v = \sum_{j-k \geq N} P_k u P_j v, \quad uv = T_u v + T_v u + R(u, v)
\]

where \( N \) is chosen such that \( P_k u P_j v \) has spectral localization in the annulus of radius \( \sim 2^j \). The estimates for \( T_u v \) and \( T_v u \) are easy:

\[
\begin{aligned}
\| w_0 2^{js} P_j (T_u v) \|_{L_t^{s_0} L_x^{q_0}} & \lesssim \sum_{|l-j| \leq N} \| w_1 2^{ls} P_l u \|_{L_t^{s_1} L_x^{q_1}} \| z_1 v \|_{L_t^{s_2} L_x^{q_2}} P_{l-N} v \|_{L_t^{s_3} L_x^{q_3}} \\
& \lesssim \sum_{|l-j| \leq N} \| w_1 2^{ls} P_l u \|_{L_t^{s_1} L_x^{q_1}} \| z_1 v \|_{L_t^{s_2} L_x^{q_2}},
\end{aligned}
\]

and so is (2.20) for \( T_u v + T_v u \), where we have applied (2.19) when \( p_1 = \infty \), as well as the facts that

\[
\| P_{l-N} v \|_{L_t^p (wdx)} \leq C \| v \|_{L_t^p (wdx)}, \quad w \in A_p, p \in (1, \infty),
\]

and \( |P_{l-N} v| \lesssim M(v) \).

It remains to control \( R(u, v) = \sum_{|l-j| \leq N} P_l u P_k v \), for which we have

\[
R(u, v) = P_j \left( \sum_{|l-j| \leq N, j-k \leq N} P_k u P_l v \right).
\]
Then it follows that
\[
\begin{align*}
\|w_\nu 2^j s P_j R(u, v)\|_{L^0_L L^0_L} &\lesssim \|w_\nu 2^j s P_k u\|_{L^0_L L^0_L} \|z_1 P_k v\|_{L^0_L L^0_L} \|P_j \geq 2 - C N (|w(-k)| \leq N) \\
&\lesssim \|w_\nu 2^j s P_k u\|_{L^0_L L^0_L} \|z_1 v\|_{L^0_L L^0_L} \\
&\lesssim \|2^{(j-k)\varepsilon} w_\nu 2^k P_k u\|_{L^0_L L^0_L} \|z_1 v\|_{L^0_L L^0_L} \\
&\lesssim \|w_\nu 2^{k\varepsilon} P_k u\|_{L^0_L L^0_L} \|z_1 v\|_{L^0_L L^0_L},
\end{align*}
\]

where we used Young's inequality in the last inequality, as well as the assumption \( s > 0 \). This completes the proof. 

We shall encounter the following weight functions, which are known to be \( A_p \) weight functions, \([12, \text{Lemma 2.5}]\).

**Lemma 2.5.** Let \( w(x) = r^{-1+2\delta_1} (r-2\delta_1-2\delta_2) \), with \( 0 \leq 1 - 2\delta_1 \leq 1 + 2\delta_2 < n \). Then \( w \in A_p(\mathbb{R}^n) \), for any \( p \in [1, \infty) \).

As a corollary of the weighted fractional Leibniz rule, Theorem 2.4, together with a weighted variant of the trace estimates, Proposition 2.2, we obtain the following inequality which will be frequently used.

**Proposition 2.6.** Let \( n \geq 3, \mu \in (0, 1) \) and \( |\theta| \leq \frac{n-2}{2} + \mu \). Then
\[
\| r^{\frac{1-\mu}{2}} \theta^\varepsilon (fg) \|_{L^2} \lesssim \| r^{\frac{1-\mu}{2}} \theta^\varepsilon f \|_{L^2} \| g \|_{H^{\frac{n-2}{2}+\mu}},
\]
whenever \( f, g \) are either spherically symmetric or first-order derivative of spherically symmetric functions. Moreover, for any \( q \in [1, \infty] \) and non-endpoint \( \theta \), i.e., \( |\theta| < \frac{n-2}{2} + \mu \), we could obtain the following estimates by interpolation
\[
\| r^{\frac{1-\mu}{2}} \theta^\varepsilon P_j (fg) \|_{L^2} \lesssim \| r^{\frac{1-\mu}{2}} \theta^\varepsilon P_j f \|_{L^2} \| g \|_{L^\infty} \| g \|_{H^{\frac{n-2}{2}+\mu}}.
\]

**Proof.** At first, we notice that it suffices to prove the result with \( \theta = \frac{n-2}{2} + \mu \), by duality and complex interpolation, if we recall the well-known fact that \( r^\alpha \in A_2 \) if and only if \( |\alpha| < n \), and so \( \| r^{\alpha/2} D^\theta f \|_{L^2} \lesssim \| r^{n/2} D^\theta f \|_{L^2} \).

By Theorem 2.4 (2.21), with \( \lambda = q_0 = 2 \), we have
\[
\| r^{\frac{1-\mu}{2}} \theta^\varepsilon (fg) \|_{L^2} \lesssim \| r^{\frac{1-\mu}{2}} \theta^\varepsilon f \|_{L^2} \| g \|_{L^\infty} \| r^{\frac{1-\mu}{2}} \theta^\varepsilon g \|_{L^\infty},
\]
provided that
\[
r^{-(1-\mu)/2}, r^{(1-\mu)} \in A_2,
\]
which is true as \( \mu \in (1-n, 1) \). By the symmetric assumption, we have
\[
\| r^{1-\mu} g \|_{L^\infty} \lesssim \| r^{1-\mu} \theta g \|_{L^\infty} \| r^{1-\mu} \|_{L^2} \lesssim \| D^\theta g \|_{L^2},
\]
\[
\| r^{\frac{1-\mu}{2}} f \|_{L^\infty} \lesssim \| r^{\frac{1-\mu}{2}} \theta f \|_{L^\infty} \| r^{\frac{1-\mu}{2}} \|_{L^2} \lesssim \| r^{\frac{1-\mu}{2}} f \|_{L^2},
\]
where we have used Lemma 2.1 and Proposition 2.2 (5). This gives us (2.24) with \( \theta = \frac{n-2}{2} + \mu \). This completes the proof.
2.4. **Inhomogeneous weight.** We will need the following weighted Hardy type estimate with inhomogeneous weight.

**Lemma 2.7 (Weighted Hardy's inequality).** Let $0 \leq \alpha \leq \beta < n/2 - s$ and $s \geq 0$. Then
\[
\|r^{-\alpha - s}(r)^{-\beta + \alpha}u\|_{L^q} \lesssim \|r^{-\alpha}(r)^{-\beta + \alpha}D^s u\|_{L^q}.
\]

**Proof.** The proof is inspired by [39, §9.3]. By Lemma 2.5, the conditions are sufficient to ensure
\[
(r^{-\alpha - s}(r)^{-\beta + \alpha})^2 \in A_2, \ \forall s \in [0, n/2).
\]
By Littlewood-Paley theory, we could reduce the proof to the case of $u = P_k u$, with $k \in \mathbb{Z}$, that is, we want to show uniform boundedness of the following operators on $L^2$
\[
T_k = 2^{-ks}r^{-\alpha - s}(r)^{-\beta + \alpha}P_k r^\alpha (r)^{\beta - \alpha}.
\]

It is equivalent to the uniform boundedness of $T_k^* T_k$, with kernel
\[
K_k(x, y) = \int 2^{-2ks}w(x)\phi_k(x-z)|z|^{-2s}w^{-1}(z)\phi_k(z-y)w(y)dz
\]
where we set $w(x) = |x|^\alpha (x)^{\beta - \alpha}$, $\phi_k(x) = 2^{kn}\phi(2^k x)$ with $\phi \in \mathcal{S}(\mathbb{R}^n)$. As $K_k(x, y) = K_k(y, x)$, by Schur’s test, we need only to prove the uniform boundedness of
\[(2.26) \quad K_k(x, y) \in L^\infty_x L^1_y.
\]
We will divide the proof into three cases: i) $|y| \lesssim |z|$, ii) $|y| \gg |z|$ and $|z| \gg 2^{-k}$, iii) $|y| \gg |z|$ and $|z| \leq 2^{-k}$.

**Case i) $|y| \lesssim |z|$.** In this case, we have $w(y) \lesssim w(z)$ and so
\[(2.27) \quad \int |K_k(x, y)|dy \lesssim \int 2^{-2ks}w(x)|\phi_k(x-z)||z|^{-2s}w^{-1}(z)dz.
\]
We consider first the sub-case: $|z| \geq 2^{-k}$, for which we have $|2^k z|^{-2s} \leq 1$, as $s \geq 0$. If $|x| \lesssim |z|$, for which we have $w(x) \lesssim w(z)$ and so,
\[
\int |K_k(x, y)|dy \lesssim \int |\phi_k(x-z)|dz \lesssim 1.
\]
Else, since $|x| \gg |z|$, we know that
\[
|\phi_k(x-z)| \lesssim 2^{kn}(2^k x)^{-N},
\]
and thus
\[
\int |K_k(x, y)|dy \lesssim \int w(x)|\phi_k(x-z)||w^{-1}(z)dz \lesssim (2^k x)^{-N}w(x)2^{kn}\int w^{-1}(z)dz.
\]
If $|x| \lesssim 1$, we have $w(x) \simeq |x|^\alpha$ and
\[
\int_{|x| \gg |z| \geq 2^{-k}} w^{-1}(z)dz \lesssim |x|^{n-\alpha} \lesssim |x|^nw^{-1}(x),
\]
else, for $|x| \gg 1$, $w(x) \simeq |x|^\beta$ and so
\[
\int_{|x| \gg |z| \geq 2^{-k}} w^{-1}(z)dz \lesssim |x|^{n-\beta} \lesssim |x|^nw^{-1}(x).
\]
In conclusion, we get (2.26) for $|x| \gg |z| \geq 2^{-k}$:
\[
\int |K_k(x, y)|dy \lesssim (2^k x)^{-N}(2^k |x|)^n \lesssim 1.
\]
We consider then another sub-case: \(|z| \ll 2^{-k}\), for which we have
\[ |\phi_k(x - z)| \ll 2^{kn} (2^k x)^{-N}, \]
and we need to control \( I = \int |z|^{-2s} w^{-1}(z) dz \). If \( k \geq 0 \), we have \(|z|^{-2s} w^{-1}(z) \approx |z|^{-2s-\alpha} \) and so \( I \lesssim 2^{-k(n-2s-\alpha)} \). Then
\[ \int |K_k(x, y)| dy \lesssim \int 2^{-2k} w(x) |\phi_k(x - z)||z|^{-2s} w^{-1}(z) dz \lesssim 2^{k\alpha} (2^k x)^{-N} w(x) \lesssim 1. \]
For the other case \( k < 0 \), \( I \lesssim 1 + 2^{-k(n-2s-\beta)} \lesssim 2^{-k(n-2s-\beta)} \) and we have (2.26) similarly.

**Case ii)** \(|y| \gg |z| \) and \(|z| \gg 2^{-k}\). At first, if \(|z| \gg \max(1, 2^{-k})\), we have \( w(y) \approx |y|^\beta \), and
\[ \int |\phi_k(z - y)| w(y) dy \lesssim \int 2^k (2^k y)^{-N} |y|^\beta dy \lesssim 2^{-k\beta} |z|^\beta \approx w(z). \]
Thus we get (2.27), which has been proven to be bounded.

Else, if \( 2^{-k} \ll |z| \ll \max(1, 2^{-k}) \), we have \( k > 0 \), \( 2^{-k} \ll |z| \ll 1 \) and \( w(z) \approx |z|^\alpha \). Then
\[ \int |\phi_k(z - y)| w(y) dy \lesssim \int_{|y| \geq 1} + \int_{|z| \leq |y| \leq 1} |\phi_k(z - y)| w(y) dy \lesssim 2^{k(n-N)} + 2^{-k\alpha} |z|^\alpha, \]
which also gives us (2.27).

**Case iii)** \(|y| \gg |z| \) and \(|z| \leq 2^{-k}\). In this case, we have
\[ |\phi_k(x - z)| \ll 2^{kn} (2^k x)^{-N}, \quad |\phi_k(z - y)| \ll 2^{kn} (2^k y)^{-N}. \]
Consider first the case \( k \geq 0 \), we have \( w(z) \approx |z|^\alpha \) and
\[ \int |K_k(x, y)| dy \approx \int_{|z| \leq |y|} 2^{-2k(s-n)} w(x) (2^k x)^{-N} (2^k y)^{-N} |y|^{-2s} w^{-2}(z) w(y) dz dy \]
\[ \approx 2^{-2k(s-n)} w(x) (2^k x)^{-N} \int (2^k y)^{-N} |y|^{n-2(s+\alpha)} w(y) dy \]
\[ \approx 2^{-2k(s-n)} w(x) (2^k x)^{-N} 2^{k(2s+\alpha-2n)} \lesssim w(2^k x) (2^k x)^{-N} \lesssim 1. \]
For the remaining case of \( k < 0 \), we consider three sub-cases: \(|z| \geq 1, |y| \leq 1, \) and \(|z| < 1 < |y| \).

When \(|z| \geq 1\), we get
\[ \int |K_k(x, y)| dy \approx \int_{1 \leq |z| \leq |y|} 2^{-2k(s-n)} w(x) (2^k x)^{-N} (2^k y)^{-N} |z|^{-2(s+\beta)} |y|^\beta dz dy \]
\[ \approx 2^{-2k(s-n)} w(x) (2^k x)^{-N} \int (2^k y)^{-N} |y|^{n-2s-\beta} dy \]
\[ \approx 2^{-2k(s-n)} w(x) (2^k x)^{-N} 2^{k(2s+\beta-2n)} \lesssim w(2^k x) (2^k x)^{-N} \lesssim 1, \]
where we have used the assumption that \( 2(s + \beta) < n \).

On the other hand, if \(|y| \leq 1\), we have \( (2^k y) \approx 1 \) and so
\[ \int |K_k(x, y)| dy \approx \int_{|z| \leq |y| \leq 1} 2^{-2k(s-n)} w(x) (2^k x)^{-N} |y|^{-2(s+\alpha)} |y|^\alpha dz dy \]
\[ \approx 2^{-2k(s-n)} w(x) (2^k x)^{-N} \int_{|y| \leq 1} |y|^{n-2s-\alpha} dy \]
\[ \approx 2^{-2k(s-n)} w(x) (2^k x)^{-N} \lesssim 1, \]
where we have used the fact that \(-2(s - n) \geq \beta \geq \alpha\).

Finally, when \(|z| < 1 < |y|\), we see that

\[
\int |K_k(x, y)| \, dy \lesssim \int_{|z| < 1 < |y|} 2^{-2k(s-n)} w(x) \langle 2k \rangle^{-N} \langle 2k \rangle^{-N} |z|^{-2(s+\alpha)} |y|^\beta \, dz \, dy
\]

\[
\lesssim 2^{-2k(s-n)} w(x) \langle 2k \rangle^{-N} \int_{|y| \geq 1} |y|^\beta \, dy
\]

\[
\lesssim 2^{k(n-2s-\beta)} w(x) \langle 2k \rangle^{-N} \leq 1,
\]

where we have used the assumption that \(n - 2s - \beta \geq \beta \geq \alpha\) in the last inequality. This completes the proof. \(\blacksquare\)

Based on Theorems 2.3 and 2.4, we could obtain weighted fractional chain rule with higher regularity. For simplicity and future reference, we present the result with the inhomogeneous weight \(r^{-\alpha}(r)^{-(\beta - \alpha)}\) as in Lemma 2.5.

**Proposition 2.8** (Weighted fractional chain rule, higher regularity). Let \(\theta \in \mathbb{R}_+, k = [\theta] \in [0, n/2), 0 \leq \alpha \leq \beta < n/2 - k\), then we have

\[
\|r^{-\alpha}(r)^{-(\beta - \alpha)} D^\theta f(u)\|_{L^2} \lesssim f(\max_{j \leq k} \|r^j \nabla^j u\|_{L^\infty}) \|r^{-\alpha}(r)^{-(\beta - \alpha)} D^\theta u\|_{L^2},
\]

for any \(f \in C^\infty\).

**Proof.** Let \(w := r^{-\alpha}(r)^{-(\beta - \alpha)}\), by Lemma 2.5, the assumptions on \(\alpha, \beta\) ensure

\(w^2, r^{-2k}w^2 \in A_2, r^{-\frac{j}{2}} \in A_1, \forall j \in [0, k]\).

The case with \(k = 0\) follows directly from Theorem 2.3. In the following, we assume \(k \geq 1\).

Let \(\theta = k + \tau\) with \(\tau \in [0, 1)\) and \(k \geq 1\), we have

\[
\|w D^\theta f(u)\|_{L^2} \lesssim \|w \nabla^k D^\tau f(u)\|_{L^2}
\]

\[
\lesssim \sum_{|\beta| = k, |\beta| \geq 1} \|w D^\tau (f^{(j)}(u) \Pi_{l=1}^j \nabla^\beta u)\|_{L^2}.
\]

For each term, we know from Theorem 2.3 and Theorem 2.4 that

\[
\|w D^\tau (f^{(j)}(u) \Pi_{l=1}^j \nabla^\beta u)\|_{L^2} \lesssim \|r^{-k} w D^\tau (f^{(j)}(u) - f^{(j)}(0))\|_{L^2} \|r^{k} \Pi_{l=1}^j \nabla^\beta u\|_{L^\infty}
\]

\[
+ \sum_{l=0} C(f, \|u\|_{L^\infty}) \|r^{-k} w D^\tau u\|_{L^2} \Pi_{l \neq l_0} \|r^{k} \nabla^\beta u\|_{L^\infty}
\]

\[
\lesssim C(f, \max_{j \leq k} \|r^j \nabla^j u\|_{L^\infty}) \|w D^\theta u\|_{L^2}.
\]

where we have also used the weighted Hardy’s inequality, Lemma 2.7, in the last two inequalities. \(\blacksquare\)

**Lemma 2.9** (Weighted trace estimate). Let \(n \geq 2, 0 \leq \alpha \leq \beta \leq (n - 1)/2\). Then, for any \(p \in [2, \infty)\),

\[
\|r^{-\alpha+(n-1)(\frac{1}{p} - \frac{1}{2})} (r)^{\alpha - \beta} \phi\|_{L^p L^2} \lesssim \|r^{-\alpha}(r)^{\alpha - \beta} D^\frac{1}{2 - \frac{1}{p}} \phi\|_{L^2}.
\]
Proof. Let \( w = r^{-\alpha}(r)^{\alpha-\beta} \) with \( w^2 \in A_2 \). As before, by interpolation, we need only to prove the endpoint case:

\[
\| r^{n-\frac{1}{2}} w \phi \|_{L^\infty_t L^2_x} \lesssim \| w P_j 2^{j/2} \phi \|_{L^1_t L^2_x},
\]

which follows from

\[
\| r^{n-\frac{1}{2}} w \phi \|_{L^\infty_t L^2_x}^2 \lesssim \| w \phi \|_{L^2_x} \| w \nabla \phi \|_{L^2_x}.
\]

The proof is elementary, by observing that \( r^{n-1}w^2 \) is essentially increasing:

\[
\int_{S^{n-1}} w(R)^2 R^{n-1} \phi(R) \omega^2 d\omega = \int_{S^{n-1}} \int_{\mathbb{R}} w(R)^2 R^{n-1} \partial_r \phi^2(r \omega) dr d\omega \\
\lesssim \int_{\mathbb{R}^n} w^2 |\partial_r \phi'| dx \\
\lesssim \| w \phi \|_{L^2_x} \| w \nabla \phi \|_{L^2_x},
\]

which completes the proof.

3. Morawetz type local energy estimates

In this section, we present a version of Morawetz type local energy estimates, involving fractional derivatives, for linear wave equations with small, variable \( C^1 \) coefficients. It is this version of local energy estimates which makes it possible to decrease the regularity requirement for quasilinear wave equations.

The similar estimates for linear wave equations with small \( C^2 \) coefficients have been well-known, see Metcalfe-Tataru [35]. There the authors employ the paradifferential calculus and positive commutator method to obtain a microlocal version of the local energy estimates. However, it is well-known that for applications to quasilinear problems, \( C^2 \) requirement is simply too strong to apply for the problem with low regularity. In particular, in the current setting, we are working with the regularity level \( s < 2 \) (in the most physical related case of \( n = 3 \)) and the most we could require is a local energy estimate with \( C^{1,\alpha} \) \((\alpha \leq 1/2)\) metric, even in the spherically symmetric case.

Here, we present an approach to yield certain weaker but still strong enough variant of the Morawetz type local energy estimates, which apply for linear wave equations with small \( C^1 \) coefficients. The approach is remarkably simple, which is relied basically on the multiplier method with well-chosen multiplier, and interpolation, without consulting paradifferential calculus. The multiplier method has been well-developed in Metcalfe-Sogge [34] and Hidano-Wang-Yokoyama [14, Section 2] (with more general weights which we will mainly follow), for small perturbations of \( \Box \). As we shall see, the Morawetz type local energy estimates we shall use are also closely related with the KSS estimates, which appear first in Keel-Smith-Sogge [20].

Let \( T \in (0, \infty), S_T = [0, T) \times \mathbb{R}^n, h_{\alpha\beta} \in C^1(S_T) \) with \( h_{\alpha\beta} = h_{\beta\alpha}, 0 \leq \alpha, \beta \leq n \) satisfying the following uniform hyperbolic condition

\[
(3.1) \quad \delta_0 (\sigma^k) < (h^{jk}(t, x)) < \delta_0^{-1} (\sigma^k), \quad h^{00} = -1, \quad |h^{0j}| \leq \delta_0^{-1},
\]

for some \( \delta_0 \in (0, 1) \). Set \( \tilde{h}^{\alpha\beta} = h^{\alpha\beta} - m^{\alpha\beta} \) where \( m^{\alpha\beta} \) is the flat Minkowski metric component, \((m^{\alpha\beta}) = Diag(-1, 1, 1, \cdots, 1)\). Consider the linear wave equation with
variable coefficients (with the summation convention for repeated upper and lower indices)

\[ \Box_h u := (h^{\alpha \beta}(t,x)\partial_\alpha \partial_\beta)u = F(t,x) \quad \text{in} \quad (0,T) \times \mathbb{R}^n, \]

with the initial data

\[ u(0,\cdot) = u_0, \quad \partial_t u(0,\cdot) = u_1. \]

**Theorem 3.1** (Morawetz type estimates). Let \( n \geq 3, \mu \in (0,1) \) and consider the initial value problem (3.2)-(3.3) with \( h^{0 \gamma} = 0, \ h^{jk} \in C^\infty(S_T) \) satisfying (3.1) and

\[ \|r^{1-\mu}\partial_h\|_{L^\infty_{t,x}(0,T_0) \times \mathbb{R}^n} \leq \delta_0^{-1}. \]

Then, there exist \( \delta_1 \in (0,\min(\delta_0, T_0)) \) and \( C_0 \geq 1 \), such that for any \( T \in (0, \delta_1] \), we have

\[ \|\tilde{\partial}_t u\|_{X_T} \leq C_0(\|\nabla u_0, u_1\|_{L^2(\mathbb{R}^n)} + \|F\|_{X_T^r}), \]

where \( X_T \) and \( X_T^r \) are defined in (1.9) and (1.10).

### 3.1. Morawetz type estimates for small perturbations of fixed background.

We begin with a standard energy estimates.

**Lemma 3.2** (Energy estimates). For any solutions \( u \in C^\infty([0,T),C^\infty(\mathbb{R}^n)) \) to the uniformly hyperbolic equation (3.2) in \( S_T \). Let \( e^0 = (h^{jk}u_1u_k - h^{00}(u_1)^2)/2 \approx |u_1|^2 + |\nabla u|^2 \), then there exist a uniform constant \( C \) depending only on \( \delta \) and \( n \) such that we have, for \( E(t) = \int_{\mathbb{R}^n} e^0 \, dx \),

\[ \frac{d}{dt} E(t) \leq C \int_{\mathbb{R}^n} (|F||u_1| + |\partial h||e^0|) \, dx. \]

**Proof.** The result is classical, which follows from a multiplier argument:

\[
\begin{align*}
\partial_t(h^{\alpha \beta}(t,x)\partial_\alpha \partial_\beta)u &= \partial_\alpha(h^{\alpha \beta}u_\beta u_t) - \partial_\alpha(h^{\alpha \beta})u_\beta u_t - (\partial_\alpha \partial_t u)h^{\alpha \beta}u_\beta \\
&= \partial_\alpha(h^{\alpha \beta}u_\beta u_t) - \partial_\alpha(h^{\alpha \beta})u_\beta u_t - \partial_t(h^{\beta \alpha}u_\alpha u_\beta)/2 + 1/2(\partial_\alpha h^{\beta \alpha})u_\beta u_\alpha \\
&= \partial_\alpha E^0 + R,
\end{align*}
\]

where

\[ E^0 = -\frac{h^{\alpha \beta}u_\alpha u_\beta}{2} + h^{0 \beta}u_\beta u_t = \frac{h^{00}(u_1)^2 - h^{jk}u_1u_k}{2}, \quad R = -\partial_\alpha(h^{\alpha \beta})u_\beta u_t + (\partial_t h^{\alpha \beta})u_\alpha u_\beta/2. \]

We observe that we have \( e^0 = -E^0 \approx u_1^2 + |\nabla u|^2 \), as long as we have (3.1), which gives us (3.6) in view of the divergence theorem.

**Lemma 3.3** (Morawetz type estimates, for small perturbation of Minkowski). Let \( n \geq 3, \mu \in (0,1) \) and consider the initial value problem (3.2)-(3.3) for \( h^{\alpha \beta} \in C^\infty(S_T) \) satisfying (3.1). Then, there exist \( \delta \in (0,\delta_0) \) and \( C \geq 1 \), such that for any \( T > 0 \) with

\[ \|r^{1-\mu}\partial h^{\alpha \beta}\|_{L^\infty_{t,x}(0,T) \times \mathbb{R}^n} \leq \delta T^{-\mu}, \quad \|\partial h^{\alpha \beta}\|_{L^\infty(S_T)} \leq \delta, \]

we have

\[ \|\tilde{\partial}_t u\|_{X_T} \leq C(\|\nabla u_0, u_1\|_{L^2(\mathbb{R}^n)} + \|F\|_{X_T^r}). \]
To prove this result, we need the following fundamental Morawetz type estimates, which follows from the elementary multiplier approach, with carefully chosen multipliers. We leave the tedious proof to the appendix.

**Theorem 3.4 (Morawetz type estimates, multiplier approach).** Let \( n \geq 3, \mu \in (0, 1) \) and consider the initial value problem (3.2)-(3.3) for \( h^{\alpha \beta} \in C^\infty(S_T) \) satisfying the condition (3.1). Then there exists \( C \geq 1 \), which is independent of \( T \in (0, \infty) \), such that we have

\[
\|\hat{u}\|_{L^2_T} \leq C\|\nabla u_0, u_1\|_{L^2_\infty}^2 + C \int_0^T \int_{\mathbb{R}^n} |\hat{u}| \left( |F| + |\partial u| \left( |\partial h| + \frac{|\hat{h}|}{r^\mu (r + T)^\mu} \right) \right) dx dt ,
\]

for any solutions \( u \in C^\infty([0, T], C^\infty_0(\mathbb{R}^n)) \) to (3.2)-(3.3), with \( F \in C^\infty([0, T], C^\infty_0) \). In addition, for \( T \in (0, \infty) \), we have

\[
\|u\|^2_{L^2_T} \leq C\|\nabla u_0, u_1\|^2_{L^2_\infty} + C \int_0^T \int_{\mathbb{R}^n} |\hat{u}| \left( |F| + |\partial u| \left( |\partial h| + \frac{|\hat{h}|}{r^\mu (r + T)^\mu} \right) \right) dx dt .
\]

With the help of (3.9) and the Cauchy-Schwarz inequality, Lemma 3.3 follows directly from the assumption

\[
|\partial h| + \frac{|\hat{h}|}{r^{1-\mu} (r + T)^\mu} \ll r^{\mu-1} T^{-\mu} .
\]

**3.2. Proof of Theorem 3.1.** With the help of Lemma 3.2 and Lemma 3.3, we are ready to present the proof of Theorem 3.1.

Let \( \delta_i > 0 \) to be determined. At first, without loss of generality, we may assume that the speed of propagation does not exceed \( \delta_0^{-1} \), and then for any \( x_0 \in \mathbb{R}^n \setminus B_{\delta_0} \), the solution \( u \) in

\[
\Lambda_{\delta_i}(x_0) = \{(t, x) : t \in [0, \delta_1], |x - x_0| < 2\delta_1 + \delta_0^{-1}(\delta_1 - t)\},
\]

depends only on \( h, F \) in \( \Lambda_{\delta_i}(x_0) \), and the data in \( B_{(2\delta_i^{-1})\delta_i}(x_0) \).

To apply Lemma 3.3, we need the estimate of perturbation, in \( \Lambda_{\delta_i}(x_0) \). Let \( x(s) = x_0 + s(x - x_0) \) with \( s \in [0, 1] \), we have either \( |x(s)| \geq |x_0|/2 \) or \( |x(s)| \leq |x_0|/2 \). In the second case, there exists \( s_0 \in [0, 1] \) such that \( |x(s_0)| = \inf |x(s)| \leq |x_0|/2 \). Then we have \( x - x_0 \perp x(s_0), |x - x_0| \geq |x - x(s_0)| \geq |x_0|/2 \) and

\[
|x(s)|^2 = |x(s_0)|^2 + (s - s_0)^2 |x - x_0|^2 \geq (s - s_0)^2 |x - x_0|^2 \geq \frac{(s - s_0)^2 |x_0|^2}{4} .
\]

Notice that for the first case, we also have \( |x(s)| \geq |x_0|/2 \geq s |x_0|/2 \), and we see that in either cases, we have

\[
|x(s)| \geq \frac{|s - s_0||x_0|}{2}
\]

for some \( s_0 \in [0, 1] \).
In view of (3.4) and (3.11), the perturbation of $h$ in $\Lambda_{\delta_1}(x_0)$ could be controlled as follows

$$
|h(t, x) - h(0, x_0)| \leq \left| \int_0^1 \nabla h(t, x(s)) \cdot (x - x_0) ds \right| + t \| \partial_t h(s, x_0) \|_{L^\infty(s \in [0, t])}
$$

$$
\leq (|x - x_0| \int_0^1 |x(s)|^{\mu - 1} + t|x_0|^{\mu - 1}) \| r^{1-\mu} \partial h \|_{L^\infty_{t,x}([0, \delta_1] \times \mathbb{R}^n)}
$$

$$
\lesssim (|t| + |x - x_0|) |x_0|^{\mu - 1} \| r^{1-\mu} \partial h \|_{L^\infty_{t,x}([0, \delta_1] \times \mathbb{R}^n)}
$$

$$
\lesssim \delta_1^{-1} \| r^{1-\mu} \partial h \|_{L^\infty_{t,x}([0, \delta_1] \times \mathbb{R}^n)}.
$$

Thus, $h^{\alpha\beta}$ could be viewed as a small perturbation of $h^{\alpha\beta}(0, x_0)$, in $\Lambda_{\delta_1}(x_0)$, when $\delta_1 \ll 1$. If $h^{\alpha\beta}(0, x_0) = m^{\alpha\beta}$, we could apply Lemma 3.3 in $\Lambda_{\delta_1}(x_0)$.

In general, as $h^{\mu\nu}$ are uniform elliptic, there exists a linear transform $M : \mathbb{R}^n \rightarrow \mathbb{R}^n$ so that in the new coordinates, $h^{\mu\nu}(0, x_0)$ reduces to the Euclidean metric. Suppose that in the new coordinates, $y = Mx$, we have

$$
H^{\alpha\beta}(t, y) \partial_\alpha \partial_\beta u = F.
$$

$H^{00} = -1, H^{0j} = 0, H^{jk}(t, Mx_0) = \delta^{jk}$. Notice that there exists an uniform $C > 0$ such that

$$
\| r^{1-\mu} \partial H \|_{L^\infty_{t,x}([0, T] \times \mathbb{R}^n)} \leq C \| r^{1-\mu} \partial h \|_{L^\infty_{t,x}([0, T] \times \mathbb{R}^n)}.
$$

Thus, when $\delta_1 \ll 1$, we have the following variant of (3.7) with $T \leq \delta_1$,

$$
(3.12) \quad \| r^{1-\mu} \partial H^{\alpha\beta} \|_{L^\infty_{t,y}([0, \delta_1] \times \mathbb{R}^n)} \leq \delta_1^{\mu}, \quad \| H^{\alpha\beta} - m^{\alpha\beta} \|_{L^\infty(M\Lambda_{\delta_1}(x_0))} \leq \delta,
$$

from which we conclude (3.8) in $M\Lambda_{\delta_1}(x_0)$, with $T \leq \delta_1$, from Lemma 3.3. Transforming back to the original variable, we obtain for some uniform $C > 0$,

$$
(3.13) \quad \| \chi_{\Lambda_{\delta_1}(x_0)} \partial u \|_{X_T} \leq C \| (\nabla u_0, u_1) \|_{L^2(B_{\delta_1^{-1}}(x_0))} + \| \chi_{\Lambda_{\delta_1}(x_0)} F \|_{X_T},
$$

for any $|x_0| \geq \delta_1 \geq T$.

Finally, we choose $x_0 \in \{z_j\}_{j=1}^\infty$ so that $\cup_j \Lambda_{\delta_1}(z_j) = S_{\delta_1}$ while $\Lambda_{\delta_1}(z_j)$ satisfy finite overlapping property. Thus we conclude (3.5) from (3.13).

3.3. Local energy estimates with fractional regularity, $\theta \in [0, 1]$. Based on Theorem 3.1, we obtain the following local energy estimates with fractional regularity.

**Proposition 3.5** (Local energy estimates with positive regularity). Let $n \geq 3$, $\mu \in (0, 1)$, $h$ \in $C^1$ with $h^{0j} = 0$, (3.1) and (3.4). Then there exist $\delta_2 \in (0, \delta_1]$ and a constant $C_1 > 4C_0$, such that any $T \in (0, \delta_1]$ such that

$$
(3.14) \quad T^{\mu} \| r^{1-\mu} \partial h \|_{L^\infty_{t,x}([0, T] \times \mathbb{R}^n)} \leq \delta_2,
$$

and solutions to (3.2) with data $(u_0, u_1)$, we have

$$
(3.15) \quad \| \partial D^\theta u \|_{X_T} \leq C_1 (\| (\nabla u_0, u_1) \|_{H^\theta} + \| D^\theta F \|_{X_T}), \quad \forall \theta \in [0, 1],
$$

$$
(3.16) \quad \| \partial D^{1/2} u \|_{X_{T,1}} \leq C_1 (\| (\nabla u_0, u_1) \|_{\dot{B}^{1/2}_{2,1}} + T^{\mu} \| r^{1-\mu} \|_{L^\infty_{t,x}([0, T] \times \mathbb{R}^n)}).
$$

**Proof.** At first, by approximation, we could assume $h \in C^\infty, u, F \in C^\infty_{t,x}$ so that we could apply Theorem 3.1.
Let us begin with proving a higher order estimate of (3.5). Applying spatial derivative $\partial_j$ to the equation (3.2), we get
\begin{equation}
(-\partial_t^2 + \Delta + \tilde{h}^{mk}\partial_m \partial_k)\partial_j u = \partial_j F(t, x) - (\partial_j h^{mk})\partial_m \partial_k u.
\end{equation}
By (3.5), we see that
\begin{align*}
\|\tilde{\partial} \nabla u\|_{X_T} &\lesssim \|u_0\|_{\tilde{H}^2} + \|u_1\|_{\tilde{H}^1} + \|\nabla F\|_{X_T} + T^{\frac{2}{p'}} \parallel (\nabla h) \nabla^2 u\parallel_{L^2_t} \\
&\lesssim \|u_0\|_{\tilde{H}^2} + \|u_1\|_{\tilde{H}^1} + \|\nabla F\|_{X_T} + T^{\frac{2}{p'}} \parallel r^{-\frac{1}{p'}} \nabla h\parallel_{L^\infty} \|\partial \nabla u\|_{X_T}.
\end{align*}
In view of (3.14) for some $\delta_2 \ll 1$, we could absorb the last term and have
\begin{equation}
\|\tilde{\partial} \nabla u\|_{X_T} \lesssim \|u_0\|_{\tilde{H}^2} + \|u_1\|_{\tilde{H}^1} + \|\nabla F\|_{X_T}.
\end{equation}

Notice that all of the weights occurred in $\|\tilde{\partial} u\|_{X_T}$ and $X_T$ are among the functions $w = r^{-\frac{1}{p'}}$, $r^{-\frac{1}{2p'}}$ and their reciprocals, which share the property that $w^2 \in A_2$. Based on this fact, we know (2.6) holds for $p = 2$, and so is the complex interpolation satisfied by the weighted Sobolev space of fractional order (see e.g. [4, Theorem 6.4.3], [36, Lemma 4.6] for similar results)
\begin{equation}
[H^0_\tilde{w},\tilde{H}^1_\tilde{w}]_\theta = \tilde{H}^\theta_\tilde{w}, \theta \in [0, 1], \|f\|_{\tilde{H}^\theta_\tilde{w}} := \|w^{D\theta} f\|_{L^2}.
\end{equation}

With the help of (2.6), we see that (3.18) gives us (3.15) with $\theta = 1$. As (3.5) is just (3.15) with $\theta = 0$, the general estimate (3.15) with $\theta \in [0, 1]$ follows, in view of (3.19). Finally, with the help of the (3.15), (2.6), and real interpolation with $\theta = 1/2$, we obtain (3.16).

From basically the same argument, based on Theorem 3.4, we could also get the following local energy estimates with fractional regularity, for small perturbation of Minkowski.

**Proposition 3.6.** Let $n \geq 3$, $\mu \in (0, 1)$ and $h \in C^1$ with (3.1). There exists a constant $C > 1$ such that if
\begin{equation}
\|(r^{1-\mu}\partial h, \tilde{h})\|_{L^\infty_t([0, T] \times B_1)} + (\ln (T))\|(r^{\mu} \partial h, \tilde{h})\|_{L^\infty_{t,x}([0, T] \times B_1^c)} \leq \frac{1}{C},
\end{equation}
then for any weak solutions to (3.2) with data $(u_0, u_1)$, we have
\begin{equation}
\|D^\theta u\|_{L^\infty_T} \leq C(\|(\nabla u_0, u_1)\|_{\tilde{H}^\theta} + (\ln (T))^{\frac{1}{2}} \parallel r^{\frac{1-\mu}{2}} (r)^{\frac{\mu}{2}} D^\theta F\parallel_{L^2_{t,x}}),
\end{equation}
for any $\theta \in [0, 1]$. Similarly, if instead of (3.20), we assume
\begin{equation}
\|(r^{\mu_1} (r^{1-\mu} (r^{\mu} \partial h, \tilde{h}))\|_{L^\infty_{t,x}([0, \infty) \times \mathbb{R}^n)} \leq \frac{1}{C},
\end{equation}
then we have
\begin{equation}
\|D^\theta \phi\|_{L^\infty} \leq C(\|(\nabla u_0, u_1)\|_{\tilde{H}^\theta} + (\ln (T)) \parallel r^{\frac{1-\mu}{2}} (r)^{\frac{\mu+\mu_1}{2}} D^\theta F\parallel_{L^2_{t,x}}), \forall \theta \in [0, 1].
\end{equation}

### 3.4 Local energy estimates with negative regularity

It is well-known that the quasilinear problems endure the issue of loss of regularity, which naturally occurs when we try to prove the convergence of the Picard iteration series. More precisely, we will need to control some term like $(g(u) - g(v)) \Delta v$, for which one standard way to bypass is to prove the convergence in certain weaker topology. One typical choice will be the standard energy norm, for which we are led to the requirement $s \geq 2$ for the regularity. In this sense, to break the regularity barrier 2 (for dimension three), it is very natural to consider energy type estimates with
negative regularity. To obtain such estimates, as we have limited regularity for \( h \), it is natural to work for equations in divergence form.

**Proposition 3.7** (Local energy estimates with negative regularity). Let \( n \geq 3 \), \( \mu \in (0, 1/2] \), \( h^{\alpha \beta} \in C^1 \) with \( h^{00} = 0 \), (3.1) and (3.4). Then there exist \( \delta_3 \in (0, \delta_2] \) and a constant \( C > 0 \), such that for any \( T \in [0, \delta_1] \) with

\[
T^\mu \| r^{1-\mu} \partial h \|_{L^\infty_t(\mathbb{R}^n)} \leq \delta_3 ,
\]

we have

\[
\| \partial D^{-\theta} u \|_{X_T} \leq C\| D^{-\theta} F \|_{X_T^*}, \quad \forall \theta \in [0, 1] ,
\]

for any weak solutions to

\[
(3.26) \quad (\partial_\alpha h^{\alpha \beta} \partial_\beta) u = F ,
\]

with vanishing data. Here \( X_T \) and \( X_T^* \) are defined in (1.9)-(1.10). In addition, if \( \theta \in [(4 - n)/2 - \mu, 1] \cap [0, 1] \), and \( h^{jk} \) are spherically symmetric, then we have

\[
\| \partial D^{-\theta} u \|_{X_T^*} \lesssim \| \partial u(0) \|_{H^{-\theta}} + T^\mu \| h \|_{H^{\infty}_t H^{\infty}_x} \| \partial u(0) \|_{H^{1-\theta}} + \| D^{-\theta} F \|_{X_T^*}
\]

for any spherically symmetric weak solutions to (3.26).

**Remark 3.8.** Here, as is clear from the local energy estimate (3.5), when \( \theta = 0 \), the second term on the right of (3.27) is not necessary. We do not know, however, if it is necessary to have such a term for general \( \theta \).

**Proof.** At first, we observe that the local energy estimate (3.5) applies also for the wave operator in the divergence form

\[
\partial_\alpha h^{\alpha \beta}(t, x) \partial_\beta ,
\]

as the difference of these two operators are just a term like \( (\partial_\alpha h^{\alpha \beta}) \partial_\beta \), which could be absorbed to the left hand by (3.24) with small \( \delta_3 \), and gives us (3.25) with \( \theta = 0 \), which particularly give us

\[
\| D u \|_{X_T^*} \lesssim \| F \|_{X_T^*}.
\]

By duality, we obtain

\[
(3.28) \quad \| \nabla D^{-1} u \|_{X_T^*} \lesssim \| u \|_{X_T} \lesssim \| D^{-1} F \|_{X_T^*} .
\]

By interpolation, for the proof of (3.25), it remains to give the estimate for \( \partial_t u \) with \( \theta = 1 \), for which we shall also argue by duality. Observe that the difference between \( (\Box + \partial_j \tilde{h}^{jk} \partial_k) u \) and \( (\Box + \partial_j \tilde{h}^{jk} \partial_k) u \) is given by \( \partial_j((\partial_k h^{jk}) u) = \partial_j((\partial_k h^{jk}) u) \), which is an admissible error term thanks to (3.4) and (3.28), as we have

\[
\| D^{-1} \partial_j((\partial_k h^{jk}) u) \|_{X_T^*} \lesssim \| (\partial_k h^{jk}) u \|_{X_T^*} \lesssim T^\mu \| r^{1-\mu} \partial h \|_{L^\infty_t} \| u \|_{X_T} \lesssim \| D^{-1} F \|_{X_T^*} .
\]

It is then reduced to the proof of the estimate for \( (\Box + \partial_j \tilde{h}^{jk} \partial_k) u = F \). Recall that, for any \( G \in C^\infty_0 \) with \( \| DG \|_{X_T^*} \leq 1 \), we have

\[
\| \partial_j \tilde{h}^{jk} \partial_k w \|_{X_T} + \| D \partial_j w \|_{X_T} \lesssim \| D \partial w \|_{X_T} \lesssim \| DG \|_{X_T^*} \leq 1
\]

for any solutions to \( (\Box + \tilde{h}^{jk} \partial_j \partial_k) w = G \) with vanishing data on time \( t = T \), which follows directly from the estimate (3.18). Now, for the purpose of duality, we observe the fact that

\[
\frac{d}{dt} \int_{\mathbb{R}^n} \left( w_t u_t + \nabla w \cdot \nabla u - u \tilde{h}^{jk} \partial_j \partial_k w \right) dx + \int_{\mathbb{R}^n} w_t F + u_t G - u(\partial_t h^{jk}) \partial_j \partial_k w dx = 0 ,
\]
and so
\[
\int_{S_T} G \partial_t u dt dx = \int_{S_T} ((\partial_t h^j k) u \partial_j \partial_k w - F \partial_k w) dt dx.
\]
Then, thanks to (3.24) and (3.25) with \( \nabla D^{-\theta} u \), we obtain that
\[
\int_{S_T} G \partial_t u dt dx \leq \|D \partial_t w\|_{L^1} \|D^{-1} F\|_{X_T^\infty} + T^{\mu} \|r^{1-\mu} \partial h\|_{L^1_{T,x}} \|u\|_{X_T^\infty} \|\partial_j \partial_k w\|_{X_T^\infty},
\]
which, by duality, gives us the desired estimate:
\[
\|D^{-1} \partial_t u\|_{X_T^\infty} \lesssim \|D^{-1} F\|_{X_T^\infty},
\]
and completes the proof of (3.25).

Finally, we give the proof of the homogeneous estimates, for (\( \partial_n h^{\alpha \beta} \partial_{\beta} u \)) = 0. For this purpose, we introduce the homogeneous solution for the standard d’Alembertian \( \Box w = 0 \), \( w(0) = u(0) \), \( w_t(0) = u_t(0) \). Then it follows from (3.5) and \([\Box, D^\theta] = 0\) that
\[
\|D^\theta \partial w\|_{X_T^\infty} \leq C_0 \|\partial u(0)\|_{\dot{H}^s},
\]
for any \( \theta \in \mathbb{R} \). Next, we want to estimate the difference \( v = u - w \), for which we observe that it satisfies \( v(0) = \partial_t v(0) = 0 \) and
\[
(\partial_n h^{\alpha \beta} \partial_{\beta} v) = -\partial_j (\tilde{h} j k \partial_k w).
\]
Applying (3.25) for \( v \), we obtain that
\[
\|D^{-\theta} \partial v\|_{X_T^\infty} \lesssim \|T^{\mu} \|r^{1-\mu} D^{-\theta} (\tilde{h} j k \partial_k w)\|_{L^2_{t,x}} + T^{\mu} \|r^{1-\mu} \tilde{h} \nabla w\|_{L^2_{t,x}} + T^{\mu} \|\tilde{h}\|_{L^\infty_{T,x} H^{n-2}_x} \|\partial u(0)\|_{H^{1-s}},
\]
provided that \( \theta \in [(n-3)/2 - \mu, 1] \cap [0, 1] \) so that \( |1 - \theta| \leq (n-2)/2 + \mu \) and \( \theta \in [0, 1] \), where we have used Lemma 2.6 in the third inequality, since \( h \) and \( u \) are assumed to be spherically symmetric. Gluing all these estimates together, we obtain (3.27) and this completes the proof. \( \blacksquare \)

3.5. **Local energy estimates with high regularity, radial case.** Considering spherically symmetric equations and solutions, we have the following version of local energy estimates with high regularity.

**Proposition 3.9** (Local energy estimates with high regularity). Let \( n \geq 3 \), \( \mu \in (0, 1) \), \( h(t, x) = h(t, |x|) \in (\delta_0 - 1, \delta_0^{-1} - 1) \) and consider radial solutions for
\[
(-\partial_t^2 + \Delta + h \Delta) \phi = F, \phi(0) = u_0, \phi(0) = u_1.
\]
Then there exists \( \delta > 0 \), such that we have
\[
(3.29) \quad \|\Box D^\theta \phi\|_{X_T^\infty} \lesssim \|D^\theta (\nabla u_0, u_1)\|_{L^2} + \|D^\theta F\|_{X_T^\infty}, \quad \theta \in [0, [n/2]],
\]
\[
(3.30) \quad \|\partial D^{n/2-1} \phi\|_{X_{T,1}} \lesssim \|\nabla u_0, u_1\|_{\dot{B}^{n/2-1}_{2,1}} + T^{\frac{n}{2}} \|r^{\frac{n}{2}(n-2)} P_j F\|_{L_{j,T}^2},
\]
\[
(3.31) \quad \|\partial D^{n/2} \phi\|_{X_{T,1}} \lesssim \|\nabla u_0, u_1\|_{\dot{B}^{n/2}_{2,1}} + T^{\frac{n}{2}} \|r^{\frac{n}{2}(n-2)} P_j F\|_{L_{j,T}^2}.
\]
for any classical solutions to (3.29), provided that
\begin{equation}
T^\mu \| \partial h \|_{L_t^\infty H_n^{n/2-1+\mu}} \leq \delta.
\end{equation}
In addition, when (3.32) is satisfied, for \( k = 1 + [n/2] \), we have
\begin{equation}
\| \partial T \|_{L_t^\infty H_n^{n/2-1+\mu}} \leq 2,
\end{equation}
if \( n \) is odd or \( \mu > 1/2 \), and
\begin{equation}
\| \partial T \|_{L_t^\infty H_n^{n/2-1+\mu}} \leq 2,
\end{equation}
if \( \mu > 1/2 \). Similarly, when \( n \geq 4, \mu \in (0, 1/2), \mu_1 \in (0, \mu), \), there exists \( \delta' > 0 \), such that we have
\begin{equation}
\| D^\phi \|_{L_{t,x}^\infty} \leq \| D^\phi(\partial^u, u_1) \|_{L^2} + \| r^{1/2} \langle r \rangle^{1/2} D^\phi \|_{L_{t,x}^2} \theta \in [0, [(n-1)/2]],
\end{equation}
for any classical solutions to (3.29), provided that
\begin{equation}
\| r^{1-\mu} \partial \phi \|_{L_t^\infty H_n^{n/2-1+\mu}} \leq \| \partial h \|_{L_t^\infty H_n^{n/2-1+\mu}} \leq \delta'.
\end{equation}

**Proof.** As in Proposition 3.5, the estimates (3.30) and (3.31) could be reduced to the proof of (3.30) with \( D^\phi \) replaced by \( \nabla^k \), with \( k \in \mathbb{N} \). When \( \theta = 0, 1 \), it has been proven from (3.15) of Proposition 3.5, by recalling the trace estimates:
\begin{equation}
\| r^{1-\mu} \partial \phi \|_{L_t^\infty H_n^{n/2-1+\mu}} \leq \| \partial h \|_{L_t^\infty H_n^{n/2-1+\mu}}.
\end{equation}
The general case follows from the similar strategy. By applying \( \nabla^\alpha \) with \( |\alpha| = k \geq 2 \), we have
\begin{equation}
(-\partial_t^2 + \Delta + 1/h\Delta) \nabla^\alpha \phi = \nabla^\alpha F + [h, \nabla^\alpha]\Delta \phi = \nabla^\alpha F + \sum_{j=1}^{k} O(|\nabla^j h| \nabla^{k-j} \Delta \phi) .
\end{equation}
At first, if \( 1 \leq j \leq k-1 \), we have \( n/2 + \mu - j \in (1/2, n/2) \), and
\begin{align*}
\| (\nabla^j h) \nabla^{k-j} \Delta \phi \|_{L_t^\infty H_n^{n/2-1+\mu}} & \lesssim T^{\mu/2} \| r^{-\mu} \nabla^j h \|_{L^\infty} \| r^{\mu/2} \nabla^{k-j} \Delta \phi \|_{L^2} \\
& \lesssim T^{\mu/2} \| h \|_{L_t^\infty H_n^{n/2+\mu}} \| r^{-(1-\mu)/2} \nabla^{k-1} \Delta \phi \|_{L^2},
\end{align*}
where we have used (2.9) and trace estimate. For the term with \( j = k \), we could proceed similarly, if we have \( n/2 + \mu - k \in (1/2, n/2) \), that is, \( k < (n-1)/2 + \mu \). Thus all these commutator terms could be absorbed to the left, in view of (3.32), which proves (3.30) with \( 0 \leq k < (n-1)/2 + \mu \).

For the remaining case, \( (n-1)/2 + \mu \leq k \leq [n/2] \), it only happens when \( n \) is even and \( k = n/2 \) with \( \mu \in (0, 1/2] \). Then we have \( k - 1 \geq 1 \),
\begin{align*}
\| (\nabla^k h) \Delta \phi \|_{L_t^\infty H_n^{n/2-1+\mu}} & \lesssim T^{n/2} \| r^{-\mu} \nabla^k h \|_{L_t^\infty L^2} \| r^{(1+\mu)/2} \Delta \phi \|_{L_t^2 L^\infty} \\
& \lesssim T^{n/2} \| h \|_{L_t^\infty H_n^{k+\mu}} \| r^{-(1-\mu)/2} \nabla^{k-1} \Delta \phi \|_{L^2},
\end{align*}
where we have used (2.4) in Proposition 2.2 and Hardy’s inequality. This gives us (3.30) with \( k = n/2 \).

Turning to the proof of (3.33) and (3.34), in which case we have \( (n-1)/2 + \mu \leq k < (n+1)/2 + \mu \). Notice that we still have \( n/2 + \mu - j \in (1/2, n/2) \) for \( 1 \leq j \leq k-1 \).
and, as before, these commutator terms are good terms. For the case $j = k$, we have $(n + 2)/2 + \mu - k > 1/2$ and so

$$\|(\nabla^k h)\Delta \phi\|_{X_T^+} \lesssim T^{n/2}\|r^{1 - \mu + k - 2}\nabla^k h\|_{L^\infty} \|r^{- (1 - \mu)/2 + 2 - k} \Delta \phi\|_{L^2},$$

$$\lesssim T^{n/2}\|h\|_{L^\infty H^{(n+2)/2+\mu}} \|r^{- (1 - \mu)/2} \nabla^k \phi\|_{L^2}$$

$$\lesssim T^{n/2}\|h\|_{L^\infty H^{(n+2)/2+\mu}} \|\nabla^k \phi\|_{X_T},$$

which gives us (3.33). Similarly, for (3.34), the term with $j = k$ could be controlled as follows

$$\|(\nabla^k h)\Delta \phi\|_{X_T^+} \lesssim T^{n/2}\|r^{n/2-1}\nabla^k h\|_{L^\infty} \|r^{-(1-\mu)/2 + 2 - k} \Delta \phi\|_{L^2}$$

$$\lesssim T^{n/2}\|h\|_{L^\infty H^{(n+2)/2+\mu}} \|r^{-(1-\mu)/2} D^{n/2+\mu} \phi\|_{L^2}$$

$$\lesssim T^{n/2}\|h\|_{L^\infty H^{(n+2)/2+\mu}} \|D^{n/2+\mu} \phi\|_{X_T},$$

where we used $n/2 + \mu \geq 2$ and (2.9), which completes the proof of (3.34).

Finally, we treat (3.35), for which we follow the similar strategy, by reducing it to $\nabla^k$ with $k = [(n - 1)/2]$. At first, for $1 \leq j \leq \lfloor n/2 \rfloor - 1$, we notice that $n/2 - \mu_1 - j, n/2 + \mu - j \in (1/2, n/2)$, and so

$$\|r^{(1 - \mu)/2}r^{n/2+\mu_1} (\nabla^j h)\nabla^{k-j} \Delta \phi\|_{L^2}$$

$$\lesssim \|r^{j-\mu} r^{\mu + \mu_1} \nabla^j h\|_{L^\infty} \|r^{-(1-\mu)/2}r^{-(j-1)(1/2 - \mu_1)/2} \nabla^{k-j} \Delta \phi\|_{L^2}$$

$$\lesssim \|h\|_{L^\infty H^{n/2+\mu_1}} \|r^{-(1-\mu)/2}r^{-(j+\mu_1)/2} \nabla^{k-j} \Delta \phi\|_{L^2},$$

where we have used Lemma 2.7. For the remaining terms with $j > \lfloor n/2 \rfloor - 1$, we see that $n$ is odd, $j = k = (n-1)/2$ and so $n/2 + \mu - k = 1/2 + \mu, n/2 - \mu_1 - k = 1/2 - \mu_1$. Notice that (2.2) gives us that

$$\|r^{(n-1)(1/2 - \mu_1)} h\|_{X_T^+} \lesssim \|h\|_{X_T^+}, \|r^{(n-1)(1/2 - \mu_1)} - \mu_1 h\|_{X_T^+} \lesssim \|h\|_{X_T^+},$$

that is,

$$(3.37) \quad \|r^{(n-1)(1/2 - \mu_1)} - \mu_1 h\|_{L^\infty} \lesssim \|h\|_{L^\infty} + \|h\|_{L^\infty}$$

With the help of (3.37), we obtain, for $1/q = 1/2 - \mu_1$,

$$\|r^{1/2} r^{-(1/2 - \mu_1)} \nabla^k h\|_{L^2}$$

$$\lesssim \|r^{(n-1)(1/2 - \mu_1)} - \mu_1 h\|_{L^\infty} \|r^{- \frac{1}{2} - \frac{\mu}{2}} \nabla^{k-1} D^{\mu_1} \Delta \phi\|_{L^q}$$

$$\lesssim \|h\|_{L^\infty H^{n/2+\mu_1} H^{n/2+\mu_1}} \|r^{- \frac{1}{2} - \frac{\mu}{2}} \nabla^{k-1} D^{\mu_1} \Delta \phi\|_{L^2}$$

$$\lesssim \|h\|_{L^\infty H^{n/2+\mu_1} H^{n/2+\mu_1}} \|r^{- \frac{1}{2} - \frac{\mu}{2}} \nabla^{k-1} D^{\mu_1} \Delta \phi\|_{L^2},$$

where we have used Lemma 2.9, Lemma 2.7 and the assumption $n \geq 4$ so that we have $(n - 3)/2 \geq \mu_1$. This gives us (3.35).

4. LOCAL EXISTENCE AND UNIQUENESS FOR DIMENSION THREE

With the help of Propositions 3.5 and 3.7, we are able to prove the local existence and uniqueness part of Theorem 1.1.
4.1. Approximate solutions. Firstly, we fix a spherically symmetric function \( \rho \in C^\infty_c(\mathbb{R}^3) \) which equals 1 near the origin and \( \int_{\mathbb{R}^3} \rho(x) dx = 1 \). and set \( \rho_0(x) = 2^{3k} \rho(2^k x) \). Based on \( \rho \), we define standard sequence of \( C^\infty \), spherically symmetric, approximate functions to \((u_0, u_1)\),

\[
(4.1) \quad u^{(k)}_0(x) = \rho_k * u_0(x), \quad u^{(k)}_1(x) = \rho_k * u_1(x), \quad k \geq 3.
\]

As is clear, we know that

\[
\| (\nabla u^{(k)}_0, u^{(k)}_1) \|_{H^{s}_\text{rad}} \leq \| (\nabla u_0, u_1) \|_{H^{s}_\text{rad}}, \quad \forall \theta \in \mathbb{R},
\]

\[
\| (\nabla u^{(k)}_0, u^{(k)}_1) \|_{B^s_{2,1}} \leq \| (\nabla u_0, u_1) \|_{B^s_{2,1}}, \quad \forall \theta \in \mathbb{R}.
\]

Since \((u_0, u_1) \in H^s_\text{rad} \times (H^{s-1}_\text{rad} \cap H^{s_0-1}_\text{rad})\) with \( s \in (3/2, 2] \) and \( s_0 \in [2-s, s-1] \), we have

\[
\lim_{k \to \infty} (\| u^{(k)}_0 - u_0 \|_{H^s_\text{rad}} + \| u^{(k)}_1 - u_1 \|_{H^{s-1}_\text{rad} \cap H^{s_0-1}_\text{rad}}) = 0.
\]

In addition, for any \( \theta \in [s_0 - 3, s - 1] \), we know that

\[
\sum_{k=3}^{\infty} \left( \| \nabla u^{(k)}_0 - \nabla u^{(k+1)}_0 \|_{\dot{H}^s(\mathbb{R}^3)} + \| u^{(k)}_1 - u^{(k+1)}_1 \|_{\dot{H}^s(\mathbb{R}^3)} \right)^2 < \infty.
\]

Indeed, we can easily check this property by using the fact that \( \| \rho_k * \varphi - \varphi \|_{L^2} \leq C2^{-\theta k} \| \varphi \|_{H^s} \) for any \( \theta \in [0, 2] \). Moreover, there exists subsequence \( \{ j_k \} \), so that

\[
u^{(j_k)}_0 - u^{(j_{k+1})}_0 \|_{\dot{H}^s(\mathbb{R}^3)} + \| u^{(j_k)}_1 - u^{(j_{k+1})}_1 \|_{\dot{H}^{s-1}(\mathbb{R}^3)} \leq 2^{-k},
\]

and we also have (4.2) for \((u^{(j_k)}_0, u^{(j_k)}_1)\) with \( \theta = s - 1 \). Furthermore, we could cut-off the data so that they are compactly supported smooth functions, while all of these properties remain valid (with possible augment of the constants). We still denote the sequence (after cut-off) as \((u^{(j_k)}_0, u^{(j_k)}_1)\).

With \((u^{(j_k)}_0, u^{(j_k)}_1)\) as data, we use a standard iteration to define the sequence of approximate solutions. Let \( F(u) = a(u)u_2^2 + b(u) |\nabla u|^2, \quad u_2 \equiv 0 \) and define \( u_k \) (\( k \geq 3 \)) recursively by solving

\[
\left\{ \begin{array}{l}
\Box u_k + g(u^{(k-1)}_0) \Delta u_k = F(u^{(k-1)}_0), \quad (t, x) \in (0, T) \times \mathbb{R}^3, \\
u^{(j_k)}_0(0, \cdot) = u^{(j_k)}_0(\cdot), \quad \partial_t u_k(0, \cdot) = u^{(j_k)}_1(\cdot).
\end{array} \right.
\]

By Proposition 3.5, together with a standard existence, uniqueness and regularity theorem, we will see that, there exists some uniform \( T(u_0, u_1) \in (0, \infty) \) to be determined, so that, for all \( k \geq 2 \), \( u_k(0, \cdot) \) is well defined, spherically symmetric, and satisfies \( u_k \in C^\infty(S_T) \)

\[
\| \partial u_k \|_{L^{\infty} H^s} \leq \| \partial D^\theta u_k \|_{X_T} \leq 2C_1 \| (\nabla u^{(j_k)}_0, u^{(j_k)}_1) \|_{H^s}, \quad \forall \theta \in [0, 1],
\]

\[
\| \partial u_k \|_{L^\infty \dot{B}^{1/2}_2} \leq \| \partial D^{1/2} u_k \|_{X_{T,1}} \leq 2C_1 \| (\nabla u_0, u_1) \|_{\dot{B}^{1/2}_2} = 2C_1 \epsilon_c.
\]

4.2. Uniform boundedness of \( u_k \). In this subsection, we prove the uniform boundedness of the sequence, (4.5) and (4.6).

**Lemma 4.1.** Let \( s \in (3/2, 2] \), \( \epsilon_s := \| (\nabla x u_0, u_1) \|_{\dot{H}^{s-1}} \) and set \( s = 3/2 + \mu \). Then there exists \( c = c(g, a, b, \epsilon_c) \) such that the spherically symmetric functions \( u_k \in C^\infty \cap CH^{\theta}_{2,1} \) are well-defined on \( S_T \) for any \( k \geq 2, \theta \geq 3 \) and enjoy the uniform bounds (4.5) and (4.6), for any \( T \in (0, T_0] \) with \( T_0 = \min(\delta_1, c(g, a, b, \epsilon_c) \epsilon_s^{-1/\mu}) \).
Proof. The proof proceeds by induction. At first, the result is trivial for $k = 2$. Then we make the inductive assumption that for some $m \geq 3$, we have for any $2 \leq k \leq m - 1$, $u_k \in C^\infty \cap CH^\theta \cap C^1 H^{\theta - 1}$ for any $\theta \geq 3$ with the bounds (4.5)- (4.6) satisfied.

Recall the Sobolev inequality
\[ \| \phi \|_{L^\infty} \leq C \| \phi \|_{B^3_{1,1}} \]
and in view of (4.6) for $u_{m-1}$, we see that
\begin{equation}
(4.7) \quad \| u_{m-1} \|_{L^\infty_{t,x}} \leq C \| u_{m-1} \|_{L^\infty_{t,x} B^3_{1,1}} \leq 2CC_1 \| (\nabla u_0, u_1) \|_{B^3_{1,1}} = 2CC_1 \varepsilon_c.
\end{equation}

As $u_{m-1} \in C^\infty \cap CH^\theta \cap C^1 H^{\theta - 1}$ for any $\theta \geq 3$ with the bounds (4.5) and (4.6), we see that $F(u_{m-1}) \in L^1([0, T]; H^{\theta - 1})$ and $g(u_{m-1}) \in C^\infty$. Based on this information, we see from the classical local existence theorem that the equation (4.4) is solvable with solution $u_m$ well-defined, smooth in $[0, T] \times \mathbb{R}^n$ and $u_m \in CH^\theta \cap C^1 H^{\theta - 1}$ for any $\theta \geq 3$.

To apply Proposition 3.5 for $u_m$, we need to check (3.14) for $h^{ij} = g(u_{m-1})$ and $h^{i\alpha j} = 0$ with $\alpha \neq \beta$. As $\mu \in (0, 1/2]$ and $u_{m-1}$ is spherically symmetric, by the inequality (2.1), we have
\begin{equation}
(4.8) \quad \| r^{1-\mu} \partial g(u_{m-1}) \|_{L^\infty} \leq \| r^{1-\mu} g'(u_{m-1}) \partial u_{m-1} \|_{L^\infty} \leq \| \partial u_{m-1} \|_{L^\infty} \leq \| u_{m-1} \|_{H^{1+2\mu}_{1,1}} \leq \| u_{m-1} \|_{L^\infty_{t,x}} \leq \| \partial u_{m-1} \|_{L^\infty_{t,x}} \leq \| u_{m-1} \|_{L^\infty_{t,x}}.
\end{equation}
where we have used (4.5) and (4.7) for $u_{m-1}$. Here we notice that the implicit constant may depend on $g$ and $\varepsilon_c$ through $\| g'(u_{m-1}) \|_{L^\infty}$. Thus, with
\begin{equation}
(4.9) \quad T_0 = c(g, \varepsilon_c) \varepsilon_s^{-1/\mu},
\end{equation}
for some small constant $c$, which may depend on $\varepsilon_c$ and $g$, we have (3.14) for $g(u_{m-1})$ and could apply Proposition 3.5 for $u_m$. In conclusion, we get for $\theta \in [0, 1]$ and $T \in (0, \delta_1]$,
\begin{equation}
(4.10) \quad \| \partial \Delta^\theta u_m \|_{X_T} \leq C_1 \| (\nabla u_0, u_1) \|_{B^3_{1,1}} + T_0^n \| r^{1-\mu} D^\theta F(u_{m-1}) \|_{L^2_{t,x}}.
\end{equation}
and
\begin{equation}
(4.11) \quad \| \partial D^\theta_{1/2} u_m \|_{X_T} \leq C_1 \| (\nabla u_0, u_1) \|_{B^3_{1,1}} + T_0^{n/2} \| r^{1-\mu} \partial D^\theta F(u_{m-1}) \|_{L^2_{t,x}}.
\end{equation}

To control the right hand side, we will exploit Lemma 2.6, the weighted fractional chain rule, Theorem 2.3, as well as the weighted fractional Leibniz rule, Theorem 2.4. Without loss of generality, we assume $b = 0$ and write
\begin{equation}
(4.12) \quad F(u) = a(u) u^2 = \tilde{a}(u) u^2 + a(0) u^2 := F_1(u) + F_2(u).
\end{equation}
We first give the estimate for $F_2(u)$ for any $\theta \in [0, 1]$: \begin{equation}
\| r^{1-\mu} 2^{j\theta} P_j F_2(u) \|_{L^{\infty}_{t,x}} \leq \| r^{1-\mu} 2^{j\theta} P_j u_t \|_{L^{\infty}_{t,x}} \| r^{1-\mu} u_t \|_{L^{\infty}_{t,x}} \leq \| r^{1-\mu} 2^{j\theta} P_j u_t \|_{L^{\infty}_{t,x}} \| u_t \|_{L^{\infty}_{t,x} H^{1+2\mu}}.
\end{equation}

\begin{equation}
by Lemma 2.5, (2.20) and (2.1). For the first term $F_1(u)$, we use similar argument, followed by (2.20), (2.15), and (2.1) to obtain for $\theta \in [0, 1]$,\begin{equation}
\| r^{1-\mu} 2^{j\theta} P_j F_1(u) \|_{L^{\infty}_{t,x}} \leq \| \tilde{a}(u) u_t \|_{L^{\infty}_{t,x}} + \| r^{1-\mu} 2^{j\theta} P_j \tilde{a}(u) \|_{L^{\infty}_{t,x}} \| r^{2-\mu} u^2 \|_{L^{\infty}_{t,x}} \leq C(\| \partial u \|_{L^{\infty}_{t,x}}) \| u_t \|_{L^{\infty}_{t,x} H^{1+2\mu}} \| r^{1-\mu} 2^{j\theta} P_j u_t \|_{L^{\infty}_{t,x}}."
For the last term occurred in the last inequality, we recall that we have
\[ \|r^{-\frac{3m}{4}}2^{\delta \theta}P_j u\|_{\ell_t^2 L_x^r} \lesssim \|r^{-\frac{m}{4}}D^\theta u\|_{\ell_t^2 L_x^r} \lesssim \|r^{-\frac{1}{4}}D^{1+\theta} u\|_{\ell_t^2 L_x^r} \lesssim \|r^{-\frac{1}{4m}}D^\theta \nabla u\|_{\ell_t^2 L_x^r}, \]
in the case of $q = 2$ and $\theta \in [0, 1]$, which follows directly from the weighted Hardy-Littlewood-Sobolev inequality. The general result for $q$ with non-endpoint $\theta \in (0, 1)$ follows then from the real interpolation, that is, we have
\[ \|r^{-\frac{m}{4}}2^{\delta \theta}P_j u\|_{\ell_t^2 L_x^r} \lesssim \|r^{-\frac{1}{4m}}2^{\delta \theta}P_j \nabla u\|_{\ell_t^2 L_x^r}. \]
In conclusion, we have proved that, for $q = 2, \theta \in [0, 1]$ or $q = 1, \theta = 1/2$
(4.13) \[ \|r^{-\frac{1}{4m}}2^{\delta \theta}P_j F(u)\|_{\ell_t^2 L_x^r} \leq C(a, ||\partial u||_{L_t^\infty B_x^{s_1}})T^\delta \|\partial D^\theta u\|_{X_{T,q}} \||\partial u||_{L_t^\infty H_x^{s+\mu}}, \]
and from which we get
\[ \|\partial D^\theta u_m\|_{X_{T,q}} \leq 2C_1 \|(\nabla u_0^{(m)}, u_1^{(m)})\|_{\dot{B}^{s_1}_{\infty,q}}, \]
provided that we set $T \leq \min(\delta_1, T_0)$ where $T_0$ is given in (4.9) with possibly smaller $c = c(g, a, b, \varepsilon_c) > 0$ such that $4C_1^2C(a, 2C_1\varepsilon_c)T^\mu \leq 1$. This completes the proof by induction. \(\blacksquare\)

To prove convergence of the approximate solutions, when $s < 2$, we will also require bounds for the solutions in Sobolev space of negative order.

**Lemma 4.2.** Under the same assumption as in Lemma 4.1. Let $(u_0, u_1) \in H^s_{rad}(\mathbb{R}^3) \times (H^{s-1}_{rad} \cap H^{-1}_{rad})(\mathbb{R}^3)$ with $s_0 \in [2-s, s-1]$. Then there exist some $c = c(g, a, b, \varepsilon_c) \in (0, 1)$ and $C > 0$ such that for any $\theta \in [s_0, s-1]$, we have
(4.14) \[ \|D^{\theta-1} \partial u_k\|_{X_{T}} \leq C \varepsilon_\theta + C T^\mu \varepsilon_{\theta+1} \varepsilon_{s-1}, \forall k \geq 2, \]
provided that
(4.15) \[ T \leq \min(\delta_1, \varepsilon_\theta^{1/\mu}). \]
In particular, with $\theta = s-1$, we have
(4.16) \[ \|\partial u_k\|_{L_t^\infty H_x^{s-2}} \leq \|D^{s-2} \partial u_k\|_{X_{T}} \leq 2C \|(\nabla u_0, u_1)\|_{\dot{H}^{s-2}}. \]
**Proof.** As in the proof of Lemma 4.1, we proceed by induction. At first, the result is trivial for $k = 2$. Then we make the inductive assumption that for some $m \geq 3$, we have (4.14) satisfied by $u_k$, for any $2 \leq k \leq m - 1$.

To apply Proposition 3.7, we write the equation (4.4) of $u_m$ in the equivalent divergence form for $(t, x) \in (0, T) \times \mathbb{R}^3$:
(4.17) \[ \left\{ \begin{array}{l}
\Box u_k + \nabla \cdot (g(u_k) \nabla u_k) = \nabla (g(u_k)) \cdot \nabla u_k + F(u_k), \\
u_k(0, \cdot) = u_0^{(j_k)}, \quad \partial_t u_k(0, \cdot) = u_1^{(j_k)}.
\end{array} \right. \]
As we see from the proof of Lemma 4.1, (4.7) and (4.8), we know that (3.24) is satisfied and we could apply Proposition 3.7 to obtain for $\theta \in [1/2 - \mu, 1/2 + \mu]$:
\[ \|D^{-\theta} \partial u_k\|_{X_T} \lesssim \|\partial u_k(0)\|_{H^{-\theta}} + T^\mu \|g(u_k-1)\|_{L_t^\infty H_x^{s+\mu}} \|\partial u_k(0)\|_{H_1^0} + T^\mu \|r^{\frac{1}{4m}} D^\theta \nabla (g(u_k-1)) \cdot \nabla u_k\|_{L_t^2 L_x^r} + T^\mu \|r^{\frac{1}{4m}} D^\theta F(u_k-1)\|_{L_t^2 L_x^r} \lesssim \|\partial u_k(0)\|_{H^{-\theta}} + C(g, \|u_k-1\|_{L_t^\infty}) T^\mu \|u_k-1\|_{L_t^\infty H_x^{s+\mu}} \|\partial u_k(0)\|_{H_1^0} + T^\mu \|D^{-\theta} \partial(u_k, u_k-1)\|_{X_T} \||\nabla (g(u_k-1)), (a(u_k-1), b(u_k-1))\|_{H_1^0} \|\partial u_k(0)\|_{L_t^\infty H_x^{s+\mu}}, \]
where, in the last inequality, we have used Proposition 2.6 and fractional chain rule based on the fact that \(g(0) = 0\). To control the last term, as \(\nabla (g^k u_{k-1}) = g'(u_{k-1}) \nabla u_{k-1}\), we see that all terms are of the form of \(f(u) \partial u\), for which we could use the classical fractional Leibniz and chain rule to conclude,

\[
\|f(u) \partial u\|_{H^{s+\mu}} \lesssim \|f(0)\| \|\partial u\|_{H^{s+\mu}} + \|f(u)\|_{L^\infty} \|\partial u\|_{H^{s+\mu}} + \|\tilde{f}(u)\|_{W^{s+\mu,\theta}} \|\partial u\|_{L^3}
\]

(4.18)

where \(\tilde{f}(u) = f(u) - f(0)\).

In view of the boundedness (4.5) and (4.6), for \(\theta \in [1/2 - \mu, 1/2 + \mu]\), we see that

\[
\|D^\theta \partial u_k\|_{X_T} \leq C_2 \varepsilon_{1-\theta} + \varepsilon_{s} \varepsilon_{1-\theta} + \|D^\theta \partial (u_k, u_{k-1})\|_{X_T}.
\]

Thus, for \(T\) satisfying (4.15) with sufficiently small \(c\), we get by the inductive assumption that

\[
\|D^\theta \partial u_k\|_{X_T} \leq C\varepsilon_{1-\theta} + C(g, a, b, \varepsilon_c) T \varepsilon_{s-1} \varepsilon_{2-\theta},
\]

which completes the proof.

4.3. Convergence in \(\hat{C}H^{s_0}\). In this subsection, we show that the approximate solutions are convergent, in the weaker topology \(\hat{C}H^{s_0}\), so that the desired solution of the quasilinear problem is given by the limit.

**Lemma 4.3.** Under the same assumption as in Lemma 4.1. Let \((u_0, u_1) \in H^s_\text{rad} \times (H^{s-1}_\text{rad} \cap \hat{H}^{s_0-1}_\text{rad})\) with \(s_0 \in [2 - s, s - 1]\), and \(\{u_k\}_{k \geq 2}\) be the approximate solutions defined in (4.4), or equivalently, (4.17), which satisfy the bounds (4.5), (4.6) and (4.14), for any \(k \geq 2\). Then there exists \(c = c(g, a, b, \varepsilon_c) \in (0, 1)\), such that for any \(T\) with

\[
T \leq \min(\delta_1, c(\varepsilon_s + \varepsilon_{s-1})^{-1/\mu})
\]

(4.19)

we have \(u_k\) is Cauchy in the space \(C([0, T]; \hat{C}H^{s_0}) \cap C^{0,1}([0, T]; \hat{H}^{s_0-1})\), with

\[
\sum_{k \geq 3} \|D^{s_0-1} \partial (u_{k+1} - u_k)\|_{X_T} \lesssim \varepsilon_{s_0} + \varepsilon_{s_0+1} + \sum_{k \geq 3} \|\partial (u_{k+1} - u_k)(0)\|_{\hat{H}^{s_0-1} \cap \hat{H}^{s_0}}.
\]

(4.20)

Here the right hand side is bounded because of (4.2) - (4.3).

**Proof.** If we set \(w_k = u_{k+1} - u_k\), it satisfies

\[
\square w_k + \nabla \cdot (g(u_k) \nabla w_k) = \nabla \cdot ((g(u_{k-1}) - g(u_k)) \nabla u_k) + \nabla (g(u_k)) \cdot \nabla u_{k+1} - \nabla (g(u_{k-1})) \cdot \nabla u_k + F(u_k) - F(u_{k-1}),
\]

for which we denote the right hand side by \(G\).

As we see from the proof of Lemma 4.1, we know that (3.4) is satisfied for \(h = g(u_k)\) and we could apply Proposition 3.7 with \(\theta \geq 1/2 - \mu = 2 - s\) to obtain

\[
\|D^\theta \partial w_k\|_{X_T} \lesssim \|\partial w_k(0)\|_{H^{-s}} + T^\mu \|g(u_k)\|_{L^\infty H^{s-1}} \|\partial w_k(0)\|_{H^{1-s}} + \|D^\theta G\|_{X_T}.
\]

(4.21)

For the term involving \(g(u_k)\), we know from Theorem 2.3, \(g(0) = 0\), (4.6) and (4.16) that

\[
\|g(u_k)\|_{L^\infty H^{s-1}} \leq C(g, \varepsilon_c) \|u_k\|_{H^{s-1}} \lesssim \varepsilon_{s-1}.
\]

(4.22)
The main part of the proof is to deal with $G$. We will write it into a combination of favorable terms and deal with each term separately. For this purpose, we set $G_1 = \nabla \cdot (g(u_{k-1}) - g(u_k))\nabla u_k)$, $G_2 = F(u_k) - F(u_{k-1})$ and then

$$G - G_1 - G_2 = g'(u_k)\nabla u_k \cdot \nabla u_{k+1} - g'(u_{k-1})\nabla u_{k-1} \cdot \nabla u_k$$

$$= g'(u_k)\nabla u_k \cdot \nabla w_k + g'(u_k)\nabla w_{k-1} \cdot \nabla u_k + (g'(u_k) - g'(u_{k-1}))\nabla u_{k-1} \cdot \nabla u_k$$

$$= G_3 + G_4 + G_5.$$ 

For $G_j$ with $j \geq 2$, we observe that they fall into the following two categories:

$$\tilde{G}_2 = (f(u_k) - f(u_{k-1}))\partial (u_{k-1}, u_k)\partial (u_{k-1}, u_k),$$

$$\tilde{G}_3 = f(u_k)\partial (u_{k-1}, u_k)\partial (w_{k-1}, w_k).$$

For all these terms, we claim that we have for any $\theta \in [2-s, s-1]$,

$$||D^{-\theta}G||_{X_T} \leq C(g, a, b, \varepsilon_c)T^\mu \varepsilon_s ||D^{-\theta}\partial (w_{k-1}, w_k)||_{X_T}.$$ 

Before presenting the proof of (4.23), we apply it to prove (4.20). Actually, by (4.21) and (4.22), we have

$$||D^{-\theta}\partial w_k||_{X_T} \leq C(\|\partial w_k(0)\|_{H^{s-1}} + \|\partial w_k(0)\|_{H^s}) + 1/4 ||D^{s_0-1}\partial (w_{k-1}, w_k)||_{X_T},$$

and so

$$||D^{s_0-1}\partial w_k||_{X_T} \leq 2C(\|\partial w_k(0)\|_{H^{s_0-1}} + \|\partial w_k(0)\|_{H^{s_0}}) + 1/2 ||D^{s_0-1}\partial w_{k-1}||_{X_T},$$

for any $k \geq 3$. However, recall that $w_2 = u_3 - u_2 = u_3$, we know from (4.14) and (4.19) that

$$||D^{s_0-1}\partial w_2||_{X_T} \leq C(\varepsilon_{s_0} + T^\mu \varepsilon_{s_0+1}\varepsilon_{s-1}) \leq C(\varepsilon_{s_0} + \varepsilon_{s_0+1}).$$

Thus an iteration argument gives us that $\sum ||D^{s_0-1}\partial w_k||_{X_T}$ is convergent and we have (4.20). Actually, for any $j \in [3, \infty)$, for finite summation from 3 to $j$, we have

$$\sum_{k=3}^{j} ||D^{s_0-1}\partial w_k||_{X_T} \leq 2C \sum_{k=3}^{j} \|\partial w_k(0)\|_{H^{s_0-1} \cap H^{s_0}} + \sum_{k=3}^{j-1} 1/2 ||D^{s_0-1}\partial w_k||_{X_T},$$

and so

$$\sum_{k=3}^{j} ||D^{s_0-1}\partial w_k||_{X_T} \leq 4C \sum_{k=3}^{j} \|\partial w_k(0)\|_{H^{s_0-1} \cap H^{s_0}} + ||D^{s_0-1}\partial w_2||_{X_T}$$

$$\leq 4C \sum_{k=3}^{j} \|\partial w_k(0)\|_{H^{s_0-1} \cap H^{s_0}} + C(\varepsilon_{s_0} + \varepsilon_{s_0+1}).$$

Letting $j$ goes to $\infty$, we obtain (4.20).

It remains to prove (4.23), for which we divide it into three terms, $G_1, \tilde{G}_2$ and $\tilde{G}_3$. 


i) **first term**: $G_1 = \nabla \cdot ((g(u_{k-1}) - g(u_k))\nabla u_k)$. For the first term $G_1$, we see that
\[
\|D^{-\theta}G_1\|_{L^\infty_T} \leq T_{\theta} \|r^{-\frac{1}{2\mu}}D^{-\theta}G_1\|_{L^2_{t,x}}
\]
\[
\lesssim T_{\theta} \|r^{-\frac{1}{2\mu}}D^{1-\theta}(g(u_{k-1}) - g(u_k))\|_{L^2_{t,x}}
\]
\[
\lesssim T_{\theta} \|r^{-\frac{1}{2\mu}}D^{1-\theta}(g(u_{k-1}) - g(u_k))\|_{L^2_{t,x}} \|\nabla u_k\|_{L^\infty_t H^{s-1}}
\]
where, as $\theta \in [2 - s, s]$, in the last inequality, we have used Proposition 2.6 with $|1 - \theta| \leq s - 1$. To control the term involving $g(u_{k-1}) - g(u_k)$, we observe that
\[
g(u) - g(v) = \int_0^1 g'(v + \lambda(u - v))(u - v)d\lambda
\]
and so
\[
\|r^{-\frac{1}{2\mu}}D^{1-\theta}(g'(v + \lambda(u - v))(u - v))\|_{L^2_{t,x}}
\]
\[
\lesssim \|r^{-\frac{1}{2\mu}}D^{1-\theta}(u - v)\|_{L^2_{t,x}} \|g'(v + \lambda(u - v))\|_{L^\infty_{t,x}}
\]
\[
+ \|r^{-\frac{1}{2\mu}}(u - v)\|_{L^2_{t,x}} \|D^{1-\theta}g'(v + \lambda(u - v))\|_{L^\infty_{t,x}}
\]
\[
\lesssim \|g'(v + \lambda(u - v))\|_{L^\infty_{t,x}} \|r^{-\frac{1}{2\mu}}D^{1-\theta}(u - v)\|_{L^2_{t,x}}
\]
\[
\lesssim C(g, \|(u, v)\|_{L^\infty_t B^{3/2}_{2,1}})T_{\theta} \|D^{1-\theta}(u - v)\|_{X_T}
\]
where we have used Theorem 2.4 and (2.5) in the first and second inequalities, for $\theta \in [0, 1]$. In summary, we have proved that
\[
(4.24) \quad \|r^{-\frac{1}{2\mu}}D^{1-\theta}(g(u) - g(v))\|_{L^2_{t,x}} \leq C(g, \|(u, v)\|_{L^\infty_t B^{3/2}_{2,1}})T_{\theta} \|D^{1-\theta}(u - v)\|_{X_T},
\]
which gives us
\[
(4.25) \quad \|D^{-\theta}G_1\|_{X^*_T} \leq C(g, \varepsilon)T^{\mu} \|D^{1-\theta}u_{k-1}\|_{X_T} \|\nabla u_k\|_{L^\infty_t H^{s-1}}.
\]

ii) **second category of terms**: $\tilde{G}_2$. Recall that
\[
\tilde{G}_2 = (f(u_k) - f(u_{k-1}))\partial(u_{k-1}, u_k)\partial(u_{k-1}, u_k).
\]
Let us present the proof for the typical term $\tilde{G}_2 = (f(u_k) - f(u_{k-1}))\partial y \partial z$, for which we know that, as $\theta \in [0, s - 1]$, $\|D^{-\theta}\tilde{G}_2\|_{X^*_T}$ is bounded by
\[
T_{\theta} \|r^{-\frac{1}{2\mu}}D^{-\theta}(f(u_k) - f(u_{k-1}))\partial y\|_{L^2_{t,x}} \|\partial z\|_{L^\infty_t H^{s-1}}
\]
\[
\lesssim T_{\theta} \|r^{-\frac{1}{2\mu} + \theta}(f(u_k) - f(u_{k-1}))\partial y\|_{L^2_{t,x}} \|\partial z\|_{L^\infty_t H^{s-1}}
\]
\[
\lesssim T_{\theta} \|r^{-\frac{1}{2\mu} - 1 + \theta}(f(u_k) - f(u_{k-1}))\|_{L^2_{t,x}} \|\partial y\|_{L^\infty_t B^{3/2}_{2,1}} \|\partial z\|_{L^\infty_t H^{s-1}}
\]
\[
\lesssim C(f, \|(u_{k-1}, u_k)\|_{L^\infty_{t,x}})T_{\theta} \|r^{-\frac{1}{2\mu} - 1 + \theta}u_{k-1}\|_{L^2_{t,x}} \|\partial y\|_{L^\infty_t B^{3/2}_{2,1}} \|\partial z\|_{L^\infty_t H^{s-1}}
\]
\[
\lesssim C(f, \|(u_{k-1}, u_k)\|_{L^\infty_{t,x}})T_{\theta} \|r^{-\frac{1}{2\mu}}D^{1-\theta}u_{k-1}\|_{L^2_{t,x}} \|\partial y\|_{L^\infty_t B^{3/2}_{2,1}} \|\partial z\|_{L^\infty_t H^{s-1}}.
\]
That is, we have
\[
(4.26) \quad \|D^{-\theta}\tilde{G}_2\|_{X^*_T} \leq C(f, \varepsilon)T^{\mu} \|D^{1-\theta}u_{k-1}\|_{X_T} \|\partial(u_{k-1}, u_{k-1})\|_{L^\infty_t H^{s-1}}.
\]
iii) third category of terms: $\hat{G}_3 = f(u_k) \partial(u_{k-1}, u_k) \partial(w_{k-1}, w_k)$. In this case, with the help of Proposition 2.6, we see that $\|D^{-\theta} \hat{G}_3\|_{X_T^{r}}$ is bounded by

$$ T^\theta \|D^{-\theta} \hat{G}_3\|_{X_T^{r}} \leq C \|D^{-\theta} \partial(w_{k-1}, w_k)\|_{L_T^r} \|f(u_k) \partial(u_{k-1}, u_k)\|_{L_T^r}. $$

Similar to the proof of (4.18) in Lemma 4.2, we know that

$$ \|f(u_k) \partial(u_{k-1}, u_k)\|_{L_T^r} \leq C \|\partial(u_{k-1}, u_k)\|_{L_T^{r,B_{2,1}}} \|\partial(u_{k-1}, u_k)\|_{L_T^{r,H^{s-1}}} , $$

and so

$$ \|D^{-\theta} \hat{G}_3\|_{X_T^{r}} \leq C \|D^{-\theta} \partial(w_{k-1}, w_k)\|_{X_T} \|\partial(u_{k-1}, u_k)\|_{L_T^{r,H^{s-1}}} . $$

In summary, in view of (4.25), (4.26) and (4.27), as well as the uniform bounds (4.5) and (4.6), we complete the proof of (4.23) and Lemma 4.3.  

4.4. Local wellposedness in $H^s$. Equipped with Lemma 4.1, Lemma 4.2, and Lemma 4.3, we are ready to prove the (unconditionally) local wellposedness.

**Lemma 4.4.** Let $n = 3$, $s = 3/2 + \mu \in (3/2, 2]$ and $s_0 \in [2 - s, s - 1]$. Considering the initial value problem (1.1)-(1.2), with $(u_0, u_1) \in H^s \times (H^{s-1} \cap H^{s_0-1})$. Then, for any $T$ satisfying (4.19), there exists a unique weak solution

$$ u = u_T \in L_t^\infty H^s \cap \cap C_t^0 H^{s-1} \cap C_t^1 H^{s_0} \cap C_t^1 H^{s_0-1} $$

in $[0, T] \times \mathbb{R}^3$ for the initial value problem (1.1)-(1.2). Moreover, there exists $C_2 > 0$ such that the solution satisfies $\partial u \in C([0, T]; H^{s_0-1})$ for any $\theta \in [s_0, s)$.

**Proof.** By Lemma 4.1 and Lemma 4.2, the approximate solutions $u_k$ are well defined and satisfy the bounds (4.5), (4.6) and (4.14). Moreover, Lemma 4.3 tells us that $u_k$ is Cauchy in the space $C([0, T]; H^{s_0}) \cap C^0([0, T]; H^{s_0-1})$, for which we denote the limit by $u \in C([0, T]; H^{s_0}) \cap C^0([0, T]; H^{s_0-1})$. By Helly’s selection theorem, we see that there is a subsequence of $u_k$ which is weak star convergent to $u$, in $L_t^\infty ([0, T]; H^{s_0}) \cap C_t^0([0, T]; H^{s_0-1})$, and so we have (4.28). Then, it is clear that $\partial u_k$ is convergent to $\partial u$ in $C([0, T]; H^{s_0-1})$ for any $\theta \in [s_0, s)$, which follows directly, from the boundedness for $\theta = s$ and continuity for $\theta = s_0$. Consequently, in view of the definition of $u_k$, (4.4), $u$ is the desired weak solution for the initial value problem (1.1)-(1.2), as well as the bounds (4.29)-(4.31).

It remains to prove the unconditional uniqueness. Suppose there is a solution

$$ v \in L_t^\infty H^s \cap C_t^0 H^{s-1} \cap C_t^1 H^{s_0} \cap C_t^1 H^{s_0-1}, $$

in $[0, T] \times \mathbb{R}^3$ for the initial value problem (1.1)-(1.2), for some $T_1 \in (0, T]$. The key observation here is that by (4.32), we have

$$ D^{s_0-1} \partial v \in L^\infty H^1 $$

and so by Hardy’s inequality,

$$ \|e^{-\frac{i}{\delta} \int_0^t D^{s_0-1} \partial v dt}\|_{L^2([0, T_1] \times \mathbb{R}^3)} \lesssim \|D^{s_0-1} \partial v\|_{L^\infty([0, T_1]; H^{s_0})} \lesssim \|D^{s_0-1} \partial v\|_{L^\infty([0, T_2]; H^1)}. $$

The proof is complete.
for any \( T_2 \in (0, T_1] \). In other words, we see that \( D^{s_0 - 1} \partial w \in X_{T_2} \) for any \( T_2 \in (0, T_1] \).

Similar to the proof of Lemma 4.3, we set \( w = u - v \) with \( D^{s_0 - 1} \partial w \in X_{T_2} \), and write the equation for \( w \) as follows

\[
- \partial_t^2 w + \Delta w + \nabla \cdot (g(u) \nabla w) = \nabla \cdot ((g(v) - g(u)) \nabla v) + \nabla (g(u)) \cdot \nabla u - \nabla (g(v)) \cdot \nabla v + F(u) - F(v) := G(u, w),
\]

together with \( w(0, x) = 0, \partial_t w(0, x) = 0 \).

As \( u \) is constructed as limit of \( u_k \), we could apply Proposition 3.7 with \( \theta = 1 - s_0 \) for the wave operator \( - \partial_t^2 + \Delta + \nabla \cdot g(u) \nabla \). That is, we have

\[
(4.33) \quad \| D^{s_0 - 1} \partial w \|_{X_T} \leq \| D^{s_0 - 1} G \|_{X_T}.
\]

With the help of (4.33), applied to \( w = u - v \), together with the similar proof as (4.25), (4.26) and (4.27), we get that

\[
\| D^{s_0 - 1} \partial w \|_{X_{T_2}} \leq \| D^{s_0 - 1} G(u, w) \|_{X_{T_2}}
\]

\[
\leq C(g, a, b, \| \partial (u, v) \|_{L^\infty_t L^{2/3}_{x, q}}) T^u \| D^{s_0 - 1} \partial w \|_{X_{T_1}} \| \partial (u, v) \|_{L^\infty_t \dot{H}^{s_1}}
\]

\[
\leq T^u \| D^{s_0 - 1} \partial w \|_{X_T}.
\]

Thus, with \( T_2 \in (0, T_1] \) sufficiently small, we see that \( \| D^{s_0 - 1} \partial w \|_{X_{T_2}} = 0 \) and so \( w \equiv 0 \) in \([0, T_2] \times \mathbb{R}^3 \), in view of \( w(0, x) = 0 \). After a simple iteration argument, this proves that \( u \equiv v \) in \([0, T_3] \times \mathbb{R}^3 \), which completes the proof of unconditional uniqueness.

\[ \qed \]

5. High dimensional well-posedness

Let \( n \geq 4, s = n/2 + \mu \) with \( \mu \) as in (1.4), and \( \varepsilon_s, \varepsilon_c \) be as in (1.7). In this section, we prove the existence and uniqueness part of Theorem 1.3, following the similar approach as in Section 4.

5.1. Approximate solutions. As in subsection 4.1, we could construct a sequence of spherically symmetric, compactly supported, smooth functions \((u_{0}^{(k)}, u_{1}^{(k)}) \rightarrow (u_{0}, u_{1}) \) in \( H^s_{rad} \times H^{s-1}_{rad} \), such that

\[
(5.1) \quad \| (\nabla u_{0}^{(k)}, u_{1}^{(k)}) \|_{B^s_{2,q}} \leq C_{\theta, q} \| (\nabla u_{0}, u_{1}) \|_{B^s_{2,q}}, \quad \forall \theta \in \mathbb{R}, \ q \in [1, \infty),
\]

\[
(5.2) \quad \| \nabla u_{0}^{(k)} - \nabla u_{0}^{(k+1)} \|_{H^s_{rad}(\mathbb{R}^n)} + \| u_{1}^{(k)} - u_{1}^{(k+1)} \|_{H^{s-1}_{rad}(\mathbb{R}^n)} \leq 2^{-k}.
\]

Let \( F(u) = a(u) u_t^2 + b(u) |\nabla u|^2 \), \( u_2 \equiv 0 \) and define \( u_k \ (k \geq 3) \) recursively by solving

\[
(5.3) \quad \left\{ \begin{array}{l}
\Box u_k + g(u_{k-1}) \Delta u_k = F(u_{k-1}), \quad (t, x) \in (0, T) \times \mathbb{R}^n, \\
u_k(0, \cdot) = u_{0}^{(k)}, \quad \partial_t u_k(0, \cdot) = u_{1}^{(k)}.
\end{array} \right.
\]

5.2. Uniform boundedness of \( u_k \). Let \( C \) be the implicit constant in the estimates of Proposition 3.9. We claim that we have the uniform bounds

\[
(5.4) \quad \| \partial D^\theta u_k \|_{X_T} \leq 2C \| D^\theta (\nabla u_0, u_1) \|_{L^2}, \quad \theta \in [0, s - 1],
\]

\[
(5.5) \quad \| \partial D^\frac{s}{2} u_k \|_{X_{T, 1}} \leq 2C \| (\nabla u_0, u_1) \|_{B^\frac{s}{4}_{2,1}}
\]

for any \( T > 0 \) satisfying

\[
(5.6) \quad T^u f(C \varepsilon_c) \varepsilon_s \leq c,
\]

for some increasing function \( f \), and constants \( c \ll 1 \ll C \).
We prove the bounds by induction. It is trivially true when $k = 2$. Assuming for some $m \geq 2$, it is true for any $k \leq m$, then, for $h = g(u_m)$, we have
\begin{equation}
T^\mu \|\partial h\|_{L^\infty H^{-\frac{m}{2}+\mu}} \leq C(\varepsilon) T^\mu \varepsilon \leq \delta,
\end{equation}
and so is the requirement (3.32) of Proposition 3.9 satisfied.

Based on Proposition 3.9, we know that
\[
\|\hat{D}^\theta u_m \|_{X_T} \leq C \|D^\theta (\nabla u_0, u_1)\|_{L^2} + C \|D^\theta F(u_m)\|_{X_T}, \quad \theta \in [0, \frac{m}{2} [],
\]
\[
\|\hat{D}^\theta-1 u_m \|_{X_{T,1}} \leq C \|\nabla u_0, u_1\|_{B_{\tilde{Z}_1}^{\frac{m}{2}-1}} + C T^{\frac{\mu}{2}} \|r^{\frac{1}{2} \mu} 2^{(\frac{m}{2}-1)} P_j F(u_m)\|_{L^2_{t,x}}.
\]
To control the nonlinear term, we apply Proposition 2.6 to obtain, for the sample term $F(u) = a(u) u^2$,
\[
\|D^\theta F(u)\|_{X_T} \lesssim T^{\frac{\mu}{2}} \|r^{\frac{1}{2} \mu} D^\theta \partial u\|_{L^2_{t,x}} \|a(u) \partial u\|_{L^\infty H^{-\frac{m}{2}+\mu}} \lesssim T^\mu \|\hat{D}^\theta u\|_{X_T} \tilde{a}(\|u\|_{L^\infty B_{\tilde{Z}_1}^{\frac{m}{2}}} \|\partial u\|_{L^\infty H^{-\frac{m}{2}+\mu}})
\]
wherever $\theta \in [0, s - 1]$, where, since $s - 1 < n/2$, we have the following well-known consequence of the fractional Leibniz rule and chain rule
\[
\|(a(u) - a(0)) v\|_{H^{s-1}} \lesssim \|a(u) - a(0)\|_{B_{\tilde{Z}_1}^{\frac{m}{2}}} \|v\|_{H^{s-1}} \lesssim C(\|u\|_{L^\infty}) \|u\|_{B_{\tilde{Z}_1}^{\frac{m}{2}}} \|v\|_{H^{s-1}}.
\]
Similarly, by (2.25), we have
\[
\|r^{\frac{1}{2} \mu} 2^{(\frac{m}{2}-1)} P_j F(u)\|_{L^2_{t,x}} \lesssim \|r^{\frac{1}{2} \mu} 2^{(\frac{m}{2}-1)} P_j \partial u\|_{L^2_{t,x}} \|a(u) \partial u\|_{L^\infty H^{-\frac{m}{2}+\mu}} \lesssim T^{\frac{\mu}{2}} \|\hat{D}^\theta-1 u\|_{X_T} \tilde{a}(\|u\|_{L^\infty B_{\tilde{Z}_1}^{\frac{m}{2}}} \|\partial u\|_{L^\infty H^{-\frac{m}{2}+\mu}})
\]
Based on the induction assumption and (5.6), we get (5.4) and (5.5) for $k = m + 1$, if we set $c > 0$ to be sufficiently small. This completes the proof by induction.

**5.3. Convergence in $CH^1 \cap C^{0,1} L^2$.** Let $w_k = u_{k+1} - u_k$, it satisfies
\[
\Box w_k + g(u_k) \Delta w_k = (g(u_{k+1}) - g(u_k)) \Delta u_k + F(u_k) - F(u_{k-1}).
\]
Thus, by Theorem 3.1, we have
\[
\|\partial w_k\|_{X_T} \lesssim \|\partial w_k(0)\|_{L^2} + \|(g(u_{k-1}) - g(u_k)) \Delta u_k\|_{X_T} + \|F(u_k) - F(u_{k-1})\|_{X_T}.
\]
Notice that $\|\partial u_k\|_{X_T} = \|g(u_k) - g(u_{k-1})\|_{X_T}$ is controlled by
\[
T^{\frac{\mu}{2}} C(\varepsilon) \|r^{\frac{\mu}{2} - 1} u_{k-1}\|_{L^\infty} \|r^{-\frac{\mu}{2}} (\frac{\mu}{2} - 2) \Delta u_k\|_{L^2} \lesssim T^{\frac{\mu}{2}} C(\varepsilon) \|\partial w_{k-1}\|_{X_T} \|r^{-\frac{\mu}{2}} D^s u_k\|_{L^2} \lesssim T^{\mu} \varepsilon C(\varepsilon) \|\partial w_{k-1}\|_{X_T}.
\]
Similarly, for the sample term $F(u) = a(u) u^2$, we have
\[
\|F(u_k) - F(u_{k-1})\|_{X_T} \lesssim T^{\frac{\mu}{2}} \|r^{\frac{\mu}{2} - 1} (a(u_k) - a(u_{k-1}))\|_{L^\infty} \|r \partial u_k\|_{L^\infty} \|r^{-\frac{\mu}{2}} \Delta u_k\|_{L^2}
\]
\[
\lesssim T^{\frac{\mu}{2}} \|r^{\frac{\mu}{2} - 1} (a(u_k) - a(u_{k-1}))\|_{L^\infty} \|r \partial u_k\|_{L^\infty} \|r^{-\frac{\mu}{2}} \partial u_k\|_{L^2}
\]
\[
\lesssim T^{\mu} \varepsilon C(\varepsilon) \|\partial w_{k-1}\|_{X_T}.
\]
By letting $c$ in (5.6) be even smaller, we could conclude
\[
\|\partial w_k\|_{X_T} \leq C \|\partial w_k(0)\|_{L^2} + \frac{1}{2} \|\partial w_{k-1}\|_{X_T},
\]
which yields convergence in \( C\dot{H}^1 \cap C^{0,1} L^2 \), thanks to (5.2).

5.4. **Local well-posedness.** Let \( u \in C\dot{H}^1 \cap C^{0,1} L^2 \) be the limit of \( u_k \), by weak star compactness, we have \( \partial u \in L^\infty \dot{H}^{s-1} \) and \( \partial u_k \) is convergent to \( \partial u \) in \( C([0,T]; \dot{H}^{s-1}) \) for any \( \theta \in [1,s) \). Then, in view of the definition of \( u_k \), (5.3), it is clear that \( u \) is a weak solution for the initial value problem (1.1)-(1.2).

The unconditional uniqueness follows from the similar argument as that in Lemma 4.4, and we omit the proof.

6. **Persistence of regularity**

In this section, we show that persistence of regularity for the weak solutions, when the initial data have higher regularity, as well as the continuous dependence on the data.

In Sections 4 and 5, for data in \( H^s \times H^{s-1} \), with additional requirement in \( \dot{H}^{s_0-1} \) for initial velocity when \( n = 3 \), we have constructed solutions in \( H^s \), when

\[
s \in \left\{ \frac{n}{2}, \frac{n+1}{2}, \frac{n}{2}, \frac{n+2}{2} \right\}, \quad n \text{ odd},
\]

\[
(\frac{n}{2}, \frac{n+2}{2}), \quad n \text{ even}.
\]

Recall the classical energy argument shows local well-posedness in \( H^{s_1} \) for any \( s_1 > n/2 + 1 \), together with the persistence of higher regularity. Keeping this fact in mind, we need only to prove the persistence of regularity in \( H^{s_1} \), with \( s_1 = \left\lfloor \frac{n+4}{2} \right\rfloor \).

6.1. **Persistence of regularity: a weaker version.** At first, we prove a weaker version of persistence of regularity, that is, when the data has slightly better regularity \( s_2 = n/2 + \mu_2 \), \((u_0, u_1) \in H^{s_2} \times H^{s_2-1} \), with

\[
\mu_2 \in \left\{ \mu, \frac{1}{2}, \mu, \frac{\mu+1}{2} \right\}, \quad n \text{ odd},
\]

\[
\mu, \frac{\mu+1}{2}, \quad n \text{ even},
\]

where \( \mu = s - n/2 \).

Fix \( T < T_\delta \), we have uniform bound on \( \| \partial u \|_{L^\infty_t \dot{H}^{s-1} \cap L^\infty_t B_2^{n/2-1}([0,T] \times \mathbb{R}^n)} \). With \( \delta_3 > 0 \) to be determined, by dividing \([0,T]\) into finitely many small, disjoint, adjacent intervals \( I_j \), we have \( \|I_j\|\|\partial u\|_{L^\infty_t \dot{H}^{s-1}(I_j \times \mathbb{R}^n)} \leq \delta_3 \), so that

\[
(6.1) \quad \|I_j\|^\|\partial g(u)\|_{L^\infty_t \dot{H}^{s-1}(I_j \times \mathbb{R}^n)} \leq \delta
\]

for each \( I_j = [T_j, T_{j+1}] \), with \( \Delta T_j = |I_j| \), where \( \delta \) is that occurred in (3.32) of Proposition 3.9. In addition, we could possibly shrink \( I_j \), so that we could apply the iteration argument in \( I_j \) to obtain uniform bound in \( H^s \), for data \((u(T_j), \partial_t u(T_j))\) at \( t = T_j \).

By recasting the iteration argument for local well-posedness on \( I_j \), we obtain for the iterative \( C^\infty \) sequence \( u_k \) on \( I_j \), with

\[
\|\partial u_k\|_{L^\infty_t \dot{H}^{s-1}(I_j \times \mathbb{R}^n)} \leq C_j\|\partial u(T_j)\|_{\dot{H}^{s-1}}, \quad \|\partial u_k\|_{L^\infty_t B_2^{n/2-1}} \leq C_j\|\partial u(T_j)\|_{B_2^{n/2-1}},
\]

\[
\lim_{k \to \infty} \|\partial (u_k - u)\|_{L^\infty_t L^2(I_j \times \mathbb{R}^n)} = 0.
\]

Assuming, by induction in \( j \), that

\[
(6.2) \quad \|\partial u_k(T_j)\|_{\dot{H}^{s_2-1}} \leq C\|\partial u(T_j)\|_{\dot{H}^{s_2-1}} \leq \tilde{C}_j\|\partial u(0)\|_{\dot{H}^{s_2-1}}.
\]

Applying Proposition 3.9 with \( \theta = s_2 - 1 \), we have

\[
(6.3) \quad \|D^\theta u_{k+1}\|_{L^E_{I_j,\mu}} := \|\partial D^\theta u_{k+1}\|_{X_{\Delta T_j}(I_j)} \lesssim \|\partial u(T_j)\|_{\dot{H}^\theta} + \|D^\theta F(u_k)\|_{X_{\Delta T_j}(I_j)}.
\]
As for the nonlinear term, we have

**Lemma 6.1.** Let \( n \) be odd or \( \mu_2 < (\mu + 1)/2 \), \( F(u) = a(u)u_t^2 + b(u)\|\nabla u\|^2 \), then for radial functions \( u \),

\[
(6.4) \quad \|D^{s_2-1}F(u)\|_{\mathcal{B}_{s_2-1}^2} \lesssim C(\|\partial u\|_{L^2_tH_{s_2-1}^n}) I_j^\mu \|D^{s_2-1}u\|_{\mathcal{E}_{j-1}^{\mu}} \|\partial u\|_{L^2_tH_{s_2-1}^n}.
\]

**Proof.** As in (4.12), without loss of generality, we deal with \( F_1(u) \) and \( F_2(u) = u_t^2 \).

For \( F_2(u) = u_t^2 \), we have

\[
\|r^{\frac{1-\mu}{2}}D^\theta F_2(u)\|_{L_{t,x}^2} \lesssim \|r^{\frac{1-\mu}{2}}D^\theta u_t^2\|_{L_{t,x}^2} \|r^{2-\mu}F_2(u)\|_{L_{t,x}^\infty}
\]

by Theorem 2.4 and (2.1). Concerning the other term \( F_1(u) = \tilde{a}(u)u_t^2 = \tilde{a}(u)F_2(u) \),

with \( \tilde{a}(0) = 0 \), we get from Theorem 2.4 that

\[
\|r^{\frac{1-\mu}{2}}D^\theta F_1(u)\|_{L_{t,x}^2} \lesssim \|\tilde{a}(u)\|_{L_{t,x}^2} \|r^{\frac{1-\mu}{2}}D^\theta F_2(u)\|_{L_{t,x}^2} \|r^{2-\mu}F_2(u)\|_{L_{t,x}^\infty}.
\]

where we have used Theorem 2.3 with \( \theta = s_2 - 1 \in (0, 1) \) when \( n = 3 \), in the last inequality.

For odd \( n \geq 5 \), the inequality still holds for \( \mu_2 < 1/2 \). Actually, as \( \theta = s_2 - 1 \) with \( k = \lfloor \theta \rfloor = (n-3)/2 \geq 1 \). With \( \alpha = \frac{n}{2} - \frac{3}{2} \mu \), we see that

\[
\alpha < \frac{n}{2} k - \alpha < \frac{n}{2},
\]

and so we could apply Proposition 2.8, together with Lemma 2.1, to obtain

\[
(6.5) \quad \|r^{\frac{1-\mu}{2}}D^\theta \tilde{a}(u)\|_{L_{t,x}^2} \lesssim C(\max_{l \leq k} \|r^l \nabla^l u\|_{L_{t,x}^\infty}) \|r^{\frac{1-\mu}{2}}D^\theta u\|_{L_{t,x}^2}.
\]

Alternatively, when \( \mu_2 = 1/2 \) and so \( \theta = (n-1)/2 \geq 2 \), it could be estimated directly, as follows

\[
\|r^{\frac{1-\mu}{2}}D^\theta \tilde{a}(u)\|_{L_{t,x}^2} \lesssim \sum_{|\Sigma \beta_i = \theta, \beta_1 \geq |\beta_1| \geq 1} \|r^{\frac{1-\mu}{2}}I_{l=1}^\beta u\|_{L_{t,x}^2}
\]

\[
\lesssim \sum_{1 \leq |\beta_1| \leq \theta} \|r^{\frac{1-\mu}{2}}I_{l=2}^{1+|\beta_1|} \nabla^{|\beta_1|} u\|_{L_{t,x}^2} \|r^{\frac{1-\mu}{2}}|\nabla^{|\beta_1|} u\|_{L_{t,x}^\infty}
\]

\[
\lesssim C(\|\partial u\|_{L_{t,x}^\infty}) \|r^{\frac{1-\mu}{2}}D^\theta u\|_{L_{t,x}^2}.
\]

For the case of even \( n \), we have \( k = \lfloor \theta \rfloor = (n-2)/2 \), \( \tau = \theta - k \), \( n/2 < k - \alpha < n/2 + 1 \). Let \( q, p \in (2, \infty) \) to be determined, such that \( 1/q + 1/p = 1/2 \), the similar argument in Proposition 2.8 gives us the desired bound, except the following term

\[
\sum_{|\Sigma \beta_i = k, |\beta_i| \geq 1} \|r^{\tau - \frac{n}{2}}D^\tau \tilde{a}^{(2)}(u) - \tilde{a}^{(2)}(0)\|_{L_{t,x}^q} \|r^{\alpha - \frac{n}{2}}I_{l=1}^\beta u\|_{L_{t,x}^p}.
\]
As \(-n < \tau q - n < n(q - 1)\), we have \(r^{\tau q - n} \in A_q\) and so
\[
\|r^{-\frac{2}{3}} D^r(\tilde{a}^{(j)}(u) - \tilde{a}^{(j)}(0))\|_{L_x^q} \lesssim C(\|\partial u\|_{\beta_{2,1}^{2-q}})\|r^{-\frac{2}{3}} D^r u\|_{L_x^q}
\lesssim C(\|\partial u\|_{\beta_{2,1}^{2-q}})\|u\|_{\beta_{2,1}^{2-q}},
\]
where we have used Theorem 2.3 and Lemma 2.1. For another term, we let \(p\) be sufficiently close to 2 such that \(\theta - \beta_1 \in (1/2 - 1/p, n/2)\). Because of the assumption that \(\alpha - \mu_2 + 2 = \mu_1 + 1/2 - \mu_2 > 0\), we also have
\[
\alpha < \frac{n}{2}, \quad \alpha - (\theta - |\beta_1|) > -\frac{n}{2},
\]
and thus we could apply (2.5) to obtain
\[
\|r^{\alpha - \frac{2}{3}} \Pi_{i=1}^j \nabla^{\beta_i} u\|_{L_x^2} \lesssim \|r^{\alpha - \frac{2}{3} + |\beta_1|} \nabla^{\beta_1} u\|_{L_x^q} \|r^{k-|\beta_1|} \Pi_{i=2}^j \nabla^{\beta_i} u\|_{L_x^p}
\lesssim \|r^{\alpha - \frac{2}{3}} D^\theta u\|_{L_x^2} \|u\|_{\beta_{2,1}^{2-q}}.
\]
Thus, we still have (6.5), which completes the proof.

In view of (6.3) and Lemma 6.1, we have
\[
\|D^{s_2-1} u_{k+1}\|_{L^\infty_{E_{t,j},r}} \lesssim \|\partial u(T_j)\|_{H^{s_2-1}} + \|I_{j}^\mu\|_{L^\infty_{E_{t,j},r}} \|D^{s_2-1} u_k\|_{L^\infty_{E_{t,j},r}} \|\partial u_k\|_{H^{s_2-1}};
\]
for any \(k \geq 2\). Then, with \(\delta_3 > 0\) sufficiently small, we could conclude with the uniform bound
\[
\|D^{s_2-1} u_{k+1}\|_{L^\infty_{E_{t,j},r}} \lesssim \|\partial u(T_j)\|_{H^{s_2-1}},
\]
for any \(k \geq 2\), which, combined with the induction assumption (6.2), gives us the desired bound
\[
\|D^{s_2-1} u\|_{L^\infty_{E_{t,j},r}} \lesssim \|\partial u(T_j)\|_{H^{s_2-1}} \lesssim \|\partial u(0)\|_{H^{s_2-1}}.
\]
As (6.2) is trivial when \(T_j = 0\), by induction, we see that (6.2) holds for any \(j\) and thus
\[
\|D^{s_2-1} u\|_{L^\infty_{E_{t,j},r}} \lesssim \|\partial u(0)\|_{H^{s_2-1}}.
\]
This completes the proof of \(\partial u \in L^\infty_{E_{t,j},r} H^{s_2-1}([0, T] \times \mathbb{R}^n)\). As it is true for any \(T < T_\ast\), we see that \(\partial u \in L^\infty_{E_{t,j},r} H^{s_2-1}([0, T_\ast] \times \mathbb{R}^n)\).

Notice also that in the case of even \(n\), the result could be iterated to show that for any \(s_2 \in (s, n/2 + 1)\), we have persistence of regularity.

6.2. Persistence of regularity for odd \(n\). Now we could prove persistence of regularity to \(H^{s_1}\) with \(s_1 = \lfloor (n + 4)/2 \rfloor\). Let us begin with the case of odd \(n\), when \(s_1 = (n + 3)/2\).

As we see from Subsection 6.1, we could assume we have \(H^k\) solution, where \(k = (n + 1)/2 = \lfloor (n + 2)/2 \rfloor\) and \(\mu = 1/2\). Also, it suffices for us to prove
\[
\|\partial u\|_{L^\infty_{E_{t,j},r} H^{k}([0, T] \times \mathbb{R}^n)} \lesssim 1 + \|\partial u(0)\|_{H^{k}},
\]
for any \(T\) such that
\[
T^{1/2} \|\nabla^k u\|_{X_T} \lesssim 1,
\]
\[
\|u\|_{L^\infty_{E_{t,j},r} H^{k}([0, T] \times \mathbb{R}^n)} + \|\partial_t u\|_{L^\infty_{E_{t,j},r} H^{k-1}([0, T] \times \mathbb{R}^n)} \lesssim 1.
\]
For simplicity, we will just illustrate the proof for solutions, instead for the approximate solutions.
By (3.33), we have
\[ (6.7) \quad \| \hat{u} \nabla^k u \|_{X_T} \leq \| \partial u(0) \|_{H_k} + \| \nabla^k F \|_{X_T} + T^{1/2} \| \nabla^k u \|_{X_T} \| g(u) \|_{L_T^\infty H^{k+1}}. \]
Recall the classical Schauder estimates yield
\[ \| g(u) \|_{L_T^\infty H^{k+1}} \leq C(\| u \|_{L_T^\infty}) \| u \|_{L_T^\infty H^{k+1}}, \]
which shows that the last term on the right of (6.7) is admissible.

Then, to finish the proof of (6.6), we need only to prove a nonlinear estimate, for the nonlinear term \(\| \nabla^k F \|_{X_T} \), which is provided by the following

**Lemma 6.2.** Let \( n \) be odd and \( k = (n + 1)/2 \), \( F(u) = a(u)u_t^2 + b(u)|\nabla u|^2 \), then for radial functions \( u \),
\[ (6.8) \quad \| r^{1/4} \nabla^k F(u) \|_{L_T^2} \lesssim C(\| \partial u \|_{L_T^\infty H^{k-1}}) \| r^{1/4} \nabla^k \partial u \|_{L_T^2} \| \partial u \|_{L_T^\infty H^{k-1}}. \]

**Proof.** At first, when there are no derivatives acting on \( a(u) \) or \( b(u) \), we need only to control
\[ \| r^{1/4} \nabla^k (\partial u)^2 \|_{L_T^2} \lesssim \| r^{-1/4} \nabla^k \partial u \|_{L_T^2} \| r^{1/2} \partial u \|_{L_T^\infty} \lesssim \| r^{-1/4} \nabla^k \partial u \|_{L_T^2} \| \partial u \|_{L_T^\infty H^{k-1}}, \]
by Theorem 2.4 and (2.1).

For the remaining case, thanks to the uniform boundedness of \( u \), we are reduced to control
\[ \| r^{1/4} \Pi_j^{(j)} \nabla^\alpha_j \partial u \|_{L_T^2} \]
where \( l \geq 3 \), \( \sum |\alpha_j| = k + 2 - l. \) Without loss of generality, we assume \( |\alpha_j| \) is non-increasing. Notice then that
\[ k + 2 - l = \sum |\alpha_j| \geq 2|\alpha_2| \Rightarrow |\alpha_2| \leq \frac{n - 1}{4} \Rightarrow |\alpha_2| \leq \frac{n - 3}{2}, \]
where we used the fact that \( |\alpha_2| \) must be integer. Then we see from (2.1) that
\[ \| r^{1/4} \nabla^\alpha_j \partial u \|_{L_T^\infty} + \| r^{\alpha_j + 1} \nabla^\alpha_j \partial u \|_{L_T^\infty} \lesssim \| \partial u \|_{L_T^\infty H^{k-1}}, \]
for any \( j \geq 2. \)

When \( |\alpha_1| \geq 1 \), we have \(-1/4 - k + |\alpha_1| > -n/2 \), and
\[ \| r^{1/4} \Pi_{j=1}^{(j)} \nabla^\alpha_j \partial u \|_{L_T^2} \lesssim \| r^{1/4 + 1/2 - \sum j \geq 2 (|\alpha_j| + 1)} \nabla^\alpha_1 \partial u \|_{L_T^2} \]
\[ \times \| r^{1/2 + |\alpha_2|} \nabla^\alpha_2 \partial u \|_{L_T^\infty} \Pi_{j=3}^{(j)} \| r^{1 + |\alpha_j|} \nabla^\alpha_j \partial u \|_{L_T^\infty} \]
\[ \lesssim \| \partial u \|_{L_T^\infty H^{k-1}} \| r^{-1/4 - k + |\alpha_1|} \nabla^\alpha_1 \partial u \|_{L_T^2} \]
\[ \lesssim \| \partial u \|_{L_T^\infty H^{k-1}} \| r^{-1/4} \nabla^k \partial u \|_{L_T^2}, \]
by (2.9). While in the case \( |\alpha_1| = 0 \), we have \( l = k + 2, \ |\alpha_j| = 0 \), and
\[ \| r^{1/4} \Pi_{j=1}^{(j)} \nabla^\alpha_j \partial u \|_{L_T^2} \lesssim \| r^{1/4} \Pi_{j=1}^{(k+2)} \partial u \|_{L_T^2} \]
\[ \lesssim \| \partial u \|_{L_T^\infty H^{k-1}} \| r^{1/4 - (k+1)/2} \langle r \rangle^{-(k+1)/2} \partial u \|_{L_T^2} \]
\[ \lesssim \| \partial u \|_{L_T^\infty H^{k-1}} \| r^{-1/4 - k} \partial u \|_{L_T^2} \]
\[ \lesssim \| \partial u \|_{L_T^\infty H^{k-1}} \| r^{-1/4} \partial u \|_{L_T^2}. \]
This completes the proof. \( \blacksquare \)
6.3. Persistence of regularity for even $n$. When $n$ is even, we use similar argument. Here $s_1 = [(n+4)/2] = n/2 + 2$, $k = n/2 + 1$ and $\mu \in (1/2, 1)$. We need only to prove
\begin{equation}
\|\partial u\|_{L^\infty H^k([0,T] \times \mathbb{R}^n)} \lesssim 1 + \|\partial u(0)\|_{H^k},
\end{equation}
for any $T \ll 1$ such that
\begin{equation*}
T^\mu \|D^{n/2-1+\mu}\partial u\|_{X_T} \ll 1, \|(u, \partial u)\|_{L^\infty H^{n/2-1+\mu}} + \|\partial u\|_{X_T} \lesssim 1.
\end{equation*}

By (3.34), we have
\begin{equation}
\|\partial \nabla^k u\|_{X_T} \lesssim \|\partial u(0)\|_{H^k} + \|\nabla F\|_{X_T} + T^\mu \|g(u)\|_{L^\infty H^{k+1}} \|D^{n/2+\mu} u\|_{X_T},
\end{equation}
where, as before, the last term is admissible, thanks to Schauder estimates.

As for the nonlinear term, we have the following estimates, which is sufficient to conclude the proof of (6.9).

**Lemma 6.3.** Let $n \geq 4$ be even, $k = n/2 + 1$, $\mu = 2/3$, and $F(u) = a(u)u^2 + b(u)\nabla u^2$, then for radial functions $u$,
\begin{equation}
\|\nabla F\|_{X_T} \lesssim C(\|\partial u\|_{L^\infty H^{n/2-1+\mu}}) T^\mu (\|\partial u\|_{X_T} + \|D^\mu \partial u\|_{X_T}) \|\partial u\|_{L^\infty H^{n/2-1+\mu}}.
\end{equation}

**Proof.** The proof proceeds similar as that of Lemma 6.2. At first, when there are no derivatives acting on $a(u)$ or $b(u)$, we need only to control
\begin{equation*}
\|\nabla^k (\partial u)^2\|_{X_T} \lesssim T^{n/2} \|r^{-(1-\mu)/2} \nabla \partial u\|_{L^\infty_{t,x}} \|r^{-1-\mu} \partial u\|_{L^\infty_{t,x}}
\end{equation*}
by Theorem 2.4 and (2.1).

For the remaining case, thanks to the uniform boundedness of $u$, we are reduced to control
\begin{equation*}
\|r^{(1-\mu)/2} \Pi_{j=1}^l \nabla^{\alpha_j} \partial u\|_{L^2_T}
\end{equation*}
where $l \geq 3$, $\sum |\alpha_j| = k + 2 - l$ and $|\alpha_j|$ is non-increasing.

When $|\alpha_2| = 0$, we see from (2.1) that
\begin{equation*}
\|r^{1-\mu} \partial u\|_{L^\infty_{t,x}} \lesssim \|\partial u\|_{L^\infty_{t,x} H^{n/2-1+\mu}},
\end{equation*}
for any $j \geq 2$. As $3 \leq l \leq k+2$, $|\alpha_1| = k+2-l$, $k-\mu(l-2) \in [(1-\mu)k,k]$, and
\begin{equation*}
(1-\mu)/2 - (l-1)(1-\mu) > -n/2,
\end{equation*}
we have
\begin{equation*}
\|r^{(1-\mu)/2} \Pi_{j=1}^l \nabla^{\alpha_j} \partial u\|_{L^2_T} \lesssim \|r^{(1-\mu)/2 - (l-1)(1-\mu)} \nabla^{\alpha_1} \partial u\|_{L^\infty_{t,x}} \|r^{1-\mu} \partial u\|_{L^\infty_{t,x}},
\end{equation*}
\begin{equation*}
\lesssim \|r^{(1-\mu)/2} D^{l-2}(1-\mu) \nabla^{\alpha_1} \partial u\|_{L^2_T} \|\partial u\|_{L^\infty_{t,x}},
\end{equation*}
\begin{equation*}
\lesssim \|r^{-(1-\mu)/2} D^{(l-2)(1-\mu)} \partial u\|_{L^\infty_{t,x}} \|\partial u\|_{L^\infty_{t,x}}.
\end{equation*}

On the other hand, if $|\alpha_1| = |\alpha_2| = 1$, as $\mu > 1/2$, we have $n/2 + \mu - 1 - |\alpha_2| > 1/2$, and then
\begin{equation*}
\|r^{|\alpha_j| + 1-\mu} \nabla^{\alpha_j} \partial u\|_{L^\infty_{t,x}} \lesssim \|\partial u\|_{H^{n/2-1+\mu}},
\end{equation*}
for any $j \geq 2$. Thus
\begin{equation*}
\|r^{(1-\mu)/2} \Pi_{j=1}^l \nabla^{\alpha_j} \partial u\|_{L^2_T} \lesssim \|r^{(1-\mu)/2} D^{k-2-l |\alpha_1|} \nabla^{\alpha_1} \partial u\|_{L^\infty_{t,x}} \|r^{1-\mu} \partial u\|_{L^\infty_{t,x}}
\end{equation*}
\begin{equation*}
\lesssim \|r^{-(1-\mu)/2} D^{k-2-l |\alpha_1|} \partial u\|_{L^\infty_{t,x}} \|\partial u\|_{L^\infty_{t,x}}.
\end{equation*}
where in the last inequality, we have used the fact $\mu(l-2) \geq \mu > (1 - \mu)/2$, thanks to $\mu \in (1/2, 1)$, so that we could apply (2.9).

It remains to consider the case $|\alpha_1| \geq 2$, then

$$-(1 - \mu)/2 - (k - |\alpha_1|) > -n/2,$$

and so

$$\|r^{-(1-\mu)/2} - (k - |\alpha_1|)\|_{L^2} \lesssim r^{-(1-\mu)/2} \nabla^k \partial u\|_{L^2}.$$ 

Also, notice that

$$|\alpha_2| \leq k + 2 - l - |\alpha_1| \leq k + 2 - 3 - 2 = \frac{n}{2} - 2,$$

we see from (2.1) that

$$\|r^{|\alpha_2| + 1 - \mu} \nabla^{\alpha_2} \partial u\|_{L^2} \lesssim \|\partial u\|_{\dot{H}_{n/2 - 1}^{s_1}}, \quad \|r^{|\alpha_2| + 1 - \mu} \nabla^{\alpha_2} \partial u\|_{L^2} \lesssim \|\partial u\|_{\dot{H}_{n/2 - 1}^{s_1}},$$

for any $j \geq 3$. Thus

$$\|r^{(1-\mu)/2} L_{j=1}^{\alpha_j} \partial u\|_{L^2} \lesssim \|r^{(1-\mu)/2} L_{j=1}^{\alpha_j} \partial u\|_{L^2} \lesssim \|r^{(1-\mu)/2} L_{j=1}^{\alpha_j} \partial u\|_{L^2},$$

This completes the proof.  

### 6.4. Continuous dependence.

The continuous dependence property is essentially included in the proofs of convergence of the approximate solutions, Lemma 4.3, and the unconditional uniqueness.

Let $T_s > 0$ be the lifespan of the solution $u$, with data $(u_0, u_1)$. Fix $T < T_s$ and $s_1 \in (s_c, s)$, we have uniform bound on $\|\partial u\|_{L^\infty_0 T \times \mathbb{R}^3}$, when $n = 3$, as $s_0 < s - 1$, without loss of generality, we could assume $s_0 = s - 1$ and also have the uniform bound $\|\partial u\|_{L^\infty_0 \dot{H}^{s_1 - 2}}$. As the proof for $n \geq 4$ is relatively easier, we present only the proof for $n = 3$.

#### 6.4.1. Short time continuity.

Before proving the full continuous dependence property, we present a result of short time continuous dependence, for data with regularity $\tau_0$ to solution with regularity $\tau_1$, with $s \geq \tau_0 > \tau_1 \geq s_1$. Suppose $\|\partial u(0)\|_{\dot{H}^{s_1 - 1} \cap \dot{H}^{s_1 - 2}} \leq M < \infty$, $\tau_0 - \tau_1 \geq \varepsilon > 0$, we would like to find $T > 0$, with the following property: for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$, such that whenever $\|((\nabla(u_0 - v_0), u_1 - v_1))\|_{\dot{H}^{s_1 - 1} \cap \dot{H}^{s_1 - 2}} \leq \delta$, the corresponding solution $v \in L^\infty_0 H^{\tau_0} \times C^{0,1}_t H^{\tau_0 - 1}$ is well-defined in $[0, T] \times \mathbb{R}^3$ and

$$\|\partial(u - v)\|_{L^\infty_0 \dot{H}^{s_1 - 1} \cap \dot{H}^{s_1 - 2}} \leq \varepsilon.$$ 

Here, $T$ could be chosen to be independent of the specific choices of $\tau_0, \tau_1$.

At first, by assuming $\varepsilon \leq 1$, we could always assume $\|\partial u(0)\|_{\dot{H}^{s_1 - 1} \cap \dot{H}^{s_1 - 2}} \leq M + 1 < \infty$. Based on this, we know that

$$\varepsilon(\tau_0 + \varepsilon^{\tau_0 - 1} - (\tau_0 - s_c)) \geq (2(M + 1))^{-1/(s_1 - s_c)} > 0,$$

and thus, in view of Lemma 4.4 and (4.19), the corresponding solution $v \in L^\infty_0 H^{\tau_0} \times C^{0,1}_t H^{\tau_0 - 1} \cap C^1_t H^{s_1 - 1} \cap C^{0,1}_t H^{s_1 - 2}$ is well-defined in $[0, T_4] \times \mathbb{R}^3$, together with a uniform bound in $L^\infty_0 H^{\tau_0} \times C^{0,1}_t H^{\tau_0 - 1}$, where

$$T_4 := \min(\delta_c, (2(M + 1))^{-1/(s_1 - s_c)}, T_0).$$
We need to give the estimate of
\[ \| \partial (u - v) \|_{L^\infty(\dot{H}^{r_1-1} \cap \dot{H}^{r_1-2})}, \]
in terms of the norm of \( \partial (u - v)(0) \). For this purpose, we give firstly the estimate of \( \| \partial (u - v) \|_{L^\infty \dot{H}^{r_1-2}} \). Let \( w = u - v \) and \( \mu_0 = \tau_0 - s_c \), it satisfies
\[
\begin{align*}
\Box w + \nabla \cdot (g(u) \nabla w) &= \nabla \cdot ((g(v) - g(u)) \nabla v) \\
&+ \nabla (g(u)) \cdot \nabla u - \nabla (g(v)) \cdot \nabla v + F(u) - F(v) := G(u, w),
\end{align*}
\]
together with \( w(0, x) = u_0 - v_0, \partial_t w(0, x) = u_1 - v_1 \). Notice that \( T^{\mu_0} \| g(u) \|_{L^\infty \dot{H}^{r_0}} \lesssim 1 \), then by Proposition 3.7, and arguing as in Lemma 4.3, we obtain that for any \( R \in (0, T_4] \),
\[
\begin{align*}
\| D^{r_1-2} \partial w \|_{X_R} &\lesssim \| \partial w(0) \|_{H^{r_1-2}} + R^{\mu_0} \| g(u) \|_{L^\infty \dot{H}^{r_0-1}} \| \partial w_j(0) \|_{H^{r_1-1}} + \| D^{r_1-2} G(u, w) \|_{X_R} \\
&\lesssim \| \partial w(0) \|_{H^{r_1-2} \cap H^{r_1-1}} \\
&+ C(g, a, b, \| \partial (u, v) \|_{L_{t \in [0, T]} \dot{H}^{r_1-1}}) R^{\mu_0} \| D^{r_1-2} \partial w \|_{X_R} \| \partial (u, v) \|_{L^\infty \dot{H}^{r_0-1}} \\
&\lesssim \| \partial w(0) \|_{H^{r_1-2} \cap H^{r_1-1}} + R^{\mu_0} \| D^{r_1-2} \partial w \|_{X_R},
\end{align*}
\]
where the implicit constant is independent of \( R \in (0, T_4] \). Thus by choosing \( R \) small enough, we obtain
\[
\| \partial w \|_{L^\infty([0, R]; \dot{H}^{r_1-2})} \lesssim \| D^{r_1-2} \partial w \|_{X_R} \lesssim \| \partial w(0) \|_{H^{r_1-2} \cap H^{r_1-1}}.
\]
Iterating this argument finite many times (\( \sim T_4/R \)), we obtain
\[
(6.13) \quad \| \partial w \|_{L^\infty([0, T_4]; \dot{H}^{r_1-2})} \lesssim \| \partial w(0) \|_{H^{r_1-2} \cap H^{r_1-1}}.
\]
Combined with the uniform bounds, as that in Lemma 4.4,
\[
\| \partial w \|_{L^\infty([0, T_4]; \dot{H}^{r_0-1})} \leq \| \partial u \|_{L^\infty([0, T_4]; \dot{H}^{r_0-1})} + \| \partial v \|_{L^\infty([0, T_4]; \dot{H}^{r_0-1})} \leq 2C_2(M + 1),
\]
we obtain, for any \( t \in [0, T_4] \),
\[
(6.14) \quad \| \partial w(t) \|_{H^{r_1-1}} \leq \| \partial w(t) \|_{\dot{H}^{r_0-1}}^{1-\theta} \| \partial w(t) \|_{H^{r_1-2}}^{\theta} \lesssim \| \partial w(0) \|_{H^{r_1-2} \cap H^{r_1-1}}^{\theta},
\]
where \( \tau_0(1-\theta) + (s_1 - 1)\theta = \tau_1 \).

6.4.2. Long time continuity. With short time continuity available, it is easy to conclude long time continuity. Actually, as \( T < T_* \), there exists \( M < \infty \) such that
\[
\| \partial u \|_{L^\infty([0, T]; \dot{H}^{r_1-1} \cap \dot{H}^{r_1-2})} \lesssim M < \infty.
\]
For fixed \( s_1 \in (s_c, s) \), we have uniform \( T_4 \) so that we have short time continuity, in any interval with length less than \( T_4 \), around the solution \( u \). Thus, we could divide \([0, T]\) into finite, say, \( N \), adjacent intervals \( \{I_j\}_{j=1}^N \), with \( |I_j| < T_4 \), \( I_j = [t_{j-1}, t_j] \), \( t_0 = 0, t_N = T \).

Let \( \tau_j = s - j(s - s_1)/N \), we could apply short time continuity, from \( t_{j-1} \) to the interval \( I_j \). Gluing together, we obtain the long time continuity.
7. Three dimensional almost global existence with small data

In this section, when \( n = 3 \), we show that the lower bound of the lifespan, available from local results, could be improved to almost global existence, Theorem 1.4. Without loss of generality, we assume \( s = 3/2 + \mu \) with \( \mu \in (0, 1/2] \) and the solution lies in \( CH^s \cap C^1 [H^{s-1}, 1.4] \).

Let \( I \subset J := [0, T_s(u_0, u_1)) \) be the subset such that for any \( T \in I \), we have

\[
\|D^{s-1}u\|_{LE_T} \leq 10C_3\varepsilon_s, \quad \|u\|_{LE_T} \leq 10C_3\varepsilon_1 ,
\]

where \( C_3 \) denotes the constant in (3.21). It is clear that \( I \) is non-empty and closed set in \( J \). By bootstrap argument, to show existence up to \( \exp(c/(\varepsilon_1 + \varepsilon_s)) \), we need only to show that (7.1) holds for \( 5C_3 \) instead of \( 10C_3 \), for any \( T \in I \cap [0, \exp(c/(\varepsilon_1 + \varepsilon_s))] \), provided that \( \varepsilon_1 + \varepsilon_s < \delta \) for some sufficiently small \( \delta > 0 \).

By Sobolev embedding, we see that

\[
\|u\|_{L^\infty(S_T)} \leq C\|\nabla u\|_{L^{\infty}H^{s-1}(S_T)} \leq 10CC_3(\varepsilon_1 + \varepsilon_s) \leq 1 ,
\]

\( g'(u) = O(1) \), and so

\[
\|g(u)\|_{L^{\infty}(S_T)} \leq \|u\| \int_0^1 g'(\lambda u)d\lambda \leq \varepsilon_1 + \varepsilon_s .
\]

Moreover, we have

\[
\|r^{1-\mu}\partial g(u)\|_{L^{\infty}} \leq \|r^{1-\mu}\partial u\|_{L^{\infty}} \leq \varepsilon_1 + \varepsilon_s ,
\]

\[
\|r\partial g(u)\|_{L^{\infty}} \leq \|r\partial u\|_{L^{\infty}} \leq \varepsilon_1 + \varepsilon_s .
\]

From these estimates, we see that (3.20) is satisfied when \( T \leq \exp(c/\varepsilon_c) \) with \( c + \varepsilon_1 + \varepsilon_s \ll 1 \).

Recall that \( u \) is constructed through approximation of \( C^{\infty}_1C^\infty \) solutions of approximate equations, Proposition 3.6 applies for \( u \) as well, which gives us

\[
\|D^{s-1}u\|_{LE_T} \leq C_3\varepsilon_s + C_3(\ln(T))^{\frac{1}{2}}\|r^{1-\mu}(r)^{\frac{\mu}{2}}D^{s-1}F(u)\|_{L^{\infty}_{t,x}} ,
\]

\[
\|u\|_{LE_T} \leq C_3\varepsilon_1 + C_3(\ln(T))^{\frac{1}{2}}\|r^{1-\mu}(r)^{\frac{\mu}{2}}F(u)\|_{L^{\infty}_{t,x}} .
\]

When \( F(u) = a(u)u_t^2 \), in view of (7.2) and (7.3), we have

\[
\|r^{1-\mu}(r)^{\frac{\mu}{2}}F(u)\|_{L^{\infty}_{t,x}} \leq \|r^{1-\mu}(r)^{-\frac{\mu}{2}}u_t\|_{L^{\infty}_{t,x}}\|r^{1-\mu}(r)^{\mu}a(u)u_t\|_{L^{\infty}_{t,x}} \leq (\ln(T))^{1/2}(\varepsilon_1 + \varepsilon_s) .
\]
For the term with \( D^{s-1} = D^{1/2+\mu} \), by Theorem 2.4 and Theorem 2.3, together with Lemma 2.5, we have

\[
\| r^{1-\mu} (r)^{-\frac{3}{2}} D^{s-1} F(u) \|_{L^2_{t,x}} \lesssim \| r^{1-\mu} (r)^{-\frac{3}{2}} D^{s-1} u \|_{L^2_{t,x}} \| r^{1-\mu} (r)^{\mu} (|a(u)| + |a(0)|) w \|_{L^\infty_{t,x}} \\
+ \| r^{-\frac{3}{2}(1-\mu)} (r)^{-\frac{3}{2}} D^{s-1} (a(u) - a(0)) \|_{L^2_{t,x}} \| r^{2(1-\mu)} (r)^{\mu} u^2 \|_{L^\infty_{t,x}} \\
\lesssim \| r^{1-\mu} (r)^{-\frac{3}{2}} D^{s-1} u \|_{L^2_{t,x}} \| \partial u \|_{L^\infty_{t,x} H^{s-1}} \\
+ \| r^{-\frac{3}{2}(1-\mu)} (r)^{-\frac{3}{2}} D^{s-1} u \|_{L^2_{t,x}} \| \partial u \|_{L^\infty_{t,x} H^{s-1}} \| \partial u \|_{L^\infty_{t,x} H^{s-1}} \\
\lesssim (\ln(T))^{1/2} \varepsilon_s (\varepsilon_1 + \varepsilon_s) + \| r^{-\frac{1}{2}} \langle r \rangle^{-\frac{3}{2}} D^{s-1} \|_{L^2_{t,x}} \| \partial u \|_{L^\infty_{t,x} H^{s-1}} \\
\lesssim (\ln(T))^{1/2} \varepsilon_s (\varepsilon_1 + \varepsilon_s) ,
\]

where in the second to the last inequality, we have used Lemma 2.7.

Then, combined with (7.4) and (7.5), we arrive at

\[
\| D^\lambda u \|_{L^\infty_T} \leq C_3 \varepsilon_\lambda + C(\varepsilon_1 + \varepsilon_s) \varepsilon_\lambda \ln(T), \quad \lambda = 1, s ,
\]

and so

\[
\| D^{s-1} u \|_{L^\infty_T} \leq 5C_3 \varepsilon_s, \quad \| u \|_{L^\infty_T} \leq 5C_3 \varepsilon_1 ,
\]

for any \( T \in I \cap [0, \exp(c_1/(\varepsilon_1 + \varepsilon_s))] \), where \( c_1 = \min(c, 1/(4C)) \).

8. HIGH DIMENSIONAL GLOBAL WELL-POSEDNESS

In this section, we show that when \( \varepsilon_s + \varepsilon_1 \) is small enough, the lower bound of the lifespan could be improved to global existence, when \( n \geq 4 \).

For any \( s > s_c \), there exists \( \mu \in (0, 1/3) \) such that \( s > s_c + \mu \). Without loss of generality, we assume \( s = s_c + \mu \) and the solution lies in \( CH^s \cap C^1 H^{s-1} \). Let \( I \subset J := [0, T_s(u_0, u_1)) \) be the subset such that for any \( T \in I \), we have

\[
\| D^{s-1} u \|_{L^\infty_T} \leq 10C \varepsilon_s, \quad \| u \|_{L^\infty_T} \leq 10C \varepsilon_1 ,
\]

where \( C \) is the constant occurred in Proposition 3.9 (3.35). It is clear that \( I \) is non-empty and closed set in \( J \). By bootstrap argument, to show global existence, we need only to show that (8.1) holds for \( 5C \) instead of \( 10C \), for any \( T \in I \), provided that \( \varepsilon_1 + \varepsilon_s < \delta \) for some sufficiently small \( \delta > 0 \).

By Sobolev embedding, we see that

\[
\| u \|_{L^\infty_{t,x}(S_T)} \lesssim \| \partial u \|_{L^\infty_{t,x}(S_T)} \lesssim \varepsilon_1 + \varepsilon_s ,
\]

\[
g'(u) = O(1) ,
\]

and so

\[
\| (r)^{\mu} g(u) \|_{L^\infty(S_T)} \lesssim \| (r)^{\mu u} \|_{L^\infty(S_T)} \| u \|_{L^\infty(H^{-\frac{3}{2} - \varepsilon_1} \cap H^{\frac{3}{2} + \varepsilon_1}(S_T))} \lesssim \varepsilon_1 + \varepsilon_s ,
\]

provided that \( \mu \leq \mu_1 \). Moreover, we have

\[
\| r^{1-\mu} \partial g(u) \|_{L^\infty} \lesssim \| r^{1-\mu} \partial u \|_{L^\infty} \lesssim \| \partial u \|_{L^\infty_{t,x} H^{s-1}} \lesssim \varepsilon_s ,
\]

\[
\| r^{1+\mu_1} \partial g(u) \|_{L^\infty} \lesssim \| r^{1+\mu_1} \partial u \|_{L^\infty} \lesssim \| \partial u \|_{L^\infty_{t,x} H^{s-2-\mu_1}} \lesssim \varepsilon_1 + \varepsilon_s ,
\]

and for any \( 0 \leq j \leq \lfloor (n-1)/2 \rfloor \),

\[
\| r^j \nabla^j u \|_{L^\infty_{t,x}} \sim \| u \|_{H^{s,j} \lesssim \varepsilon_1 + \varepsilon_s} .
\]

From these estimates, we see that (3.36) is satisfied when \( \varepsilon_1 + \varepsilon_s \ll 1 \).
Recall that \( u \) is constructed through approximation of \( C_l^\infty C_c^\infty \) solutions of approximate equations, Proposition 3.9 (3.35) applies for \( u \) as well, which gives us
\[
\|u\|_{L^\infty_T} \leq C\varepsilon_1 + C\|wF(u)\|_{L^2_{t,x}} ,
\]
and in the case of odd \( n \),
\[
\|D^{s-1}u\|_{L^\infty_T} \leq C\varepsilon_s + C\|wD^{s-1}F(u)\|_{L^2_{t,x}} ,
\]
where we set \( w = r^{\frac{1}{2+\mu}}(r) \). We claim that the following variant of (8.5)
\[
\|D^{s-1}u\|_{L^\infty_T} \leq C\varepsilon_s + C\|wD^{s-1}F(u)\|_{L^2_{t,x}} + \tilde{C}\varepsilon_s(\varepsilon_1 + \varepsilon_s) ,
\]
for some \( \tilde{C} \), applies for even \( n \) as well. Before presenting the proof of (8.6), let us use it to conclude the global existence.

At first, we have
\[
\|wF(u)\|_{L^2_{t,x}} \lesssim \|w^{-1}\partial u\|_{L^2_{t,x}}\|w^2(|a(u)| + |b(u)|)\partial u\|_{L^\infty_{t,x}} \\
\lesssim \|w^{-1}\partial u\|_{L^2_{t,x}} C(\|u\|_{L^\infty_{t,x}})\|\partial u\|_{L^\infty_{t,x} H^{s-1}} \\
\lesssim \varepsilon_1 + \varepsilon_s \varepsilon_1 .
\]
Concerning the part with \( D^{s-1} \), when \( F(u) = a(u)u^2 = (a(0) + \tilde{a}(u))u^2 \) and \( n \) is odd, by Theorem 2.4, Proposition 2.8 with \([s-1] = k = (n-3)/2\) and \( k + (1 - \mu)/(1 + \mu_1)/2 < n/2 \), together with Lemma 2.5, we see that \( \|wD^{s-1}F(u)\|_{L^2_{t,x}} \) is controlled by
\[
\|w^{-1}D^{s-1}u_t\|_{L^2_{t,x}}\|w^2u_t\|_{L^\infty_{t,x}} + \|r^{-(1-\mu)}w^{-1}D^{s-1}\tilde{a}(u)\|_{L^2_{t,x}}\|r^\mu wu_t^2\|_{L^\infty_{t,x}} \\
\lesssim \|D^{s-1}u\|_{L^\infty_T}\|\partial u\|_{L^\infty_{t,x} H^{s-1}} + \|r^{-(1-\mu)}w^{-1}D^{s-1}u\|_{L^2_{t,x}}\|\partial u\|_{L^\infty_{t,x} H^{s-1}} \\
\lesssim \|D^{s-1}u\|_{L^\infty_T}\|\partial u\|_{L^\infty_{t,x} H^{s-1}} + \|w^{-1}D^{s-1}\mu u\|_{L^2_{t,x}}\|\partial u\|_{L^\infty_{t,x} H^{s-1}} \\
\lesssim \varepsilon_1 + \varepsilon_s \varepsilon_s ,
\]
where in the second inequality, we have used Lemma 2.7.

When \( n \) is even, we have \([s-1] = n/2 - 1\), and we could apply Proposition 2.8 only if \( \mu > (1 + \mu_1)/2 \). For the remaining case \( 0 < \mu \leq (1 + \mu_1)/2 \), we notice that \( 1 - \mu + \frac{\mu_1}{2} < n/2 \), and we could apply Lemma 2.7 to obtain
\[
\|w^{-1}r^{\mu-\mu}\varepsilon^{s-1}\tilde{a}(u)\|_{L^2_{t,x}} \lesssim \|w^{-1}D^{s-1}\tilde{a}(u)\|_{L^2_{t,x}} \lesssim \|w^{-1}\nabla^\frac{2}{3}\tilde{a}(u)\|_{L^2_{t,x}} .
\]
Notice that
\[
|\nabla^\frac{2}{3}\tilde{a}(u)| \lesssim \sum_{\sum |\beta_i| = \frac{2}{3}, |\beta_i| \geq 1 \geq 1} \Pi_{\beta_i} |\nabla^\beta_i u| \lesssim \sum_{|\beta_i| < n/2, l \geq 2} r^{l|\beta_i| - \frac{2}{3}} |\nabla^\beta_i u| \Pi_{\beta_i} r^{l|\beta_i| - \frac{2}{3}} |\nabla^\beta_i u| ,
\]
we get
\[
\|w^{-1}\nabla^\frac{2}{3}\tilde{a}(u)\|_{L^2_{t,x}} \lesssim \sum_{1 \leq j \leq \frac{2}{3}} \|w^{-1}r^{j-\frac{2}{3}} \nabla^j u\|_{L^2_{t,x}} \lesssim \|w^{-1}D^{\frac{2}{3}} u\|_{L^2_{t,x}} \lesssim \varepsilon_1 + \varepsilon_s ,
\]
and thus we have the same estimate as for the odd spatial dimension.

Then, combined with (8.4) and (8.5), we arrived at
\[
\|D^{s-1}u\|_{L^\infty_T} \leq C\varepsilon_s + \tilde{C}\varepsilon_s(\varepsilon_1 + \varepsilon_s) , \|u\|_{L^\infty_T} \leq C\varepsilon_1 + \tilde{C}\varepsilon_1(\varepsilon_1 + \varepsilon_s) .
\]
Consequently, with \( \varepsilon_1 + \varepsilon_s \ll 1 \), we have
\[
\|D^{s-1}u\|_{L^\infty_T} \leq 2C\varepsilon_s , \|u\|_{L^\infty_T} \leq 2C\varepsilon_1 ,
\]
for any $T \in I$, which gives us (8.1) holds for $2C$. By continuity, we see that $T_s = \infty$ and this completes the proof.

8.1. (8.6) for even spatial dimension. In the case of even $n$ with $s = n/2 + \mu \geq 2$, we could apply (3.35) with $\theta = s - 2$, for the equation of $\nabla u$,

$$ \Box + g(u)\Delta \nabla u = \nabla F(u) - (\nabla g(u))\Delta u , $$

which gives us

(8.8) \[ \| D^{s-1}u \|_{L^\infty_T L^2_x} \lesssim \varepsilon_s + \| w D^{s-1} F(u) \|_{L^2_T} + \| w D^{s-2}((\nabla g(u))\Delta u) \|_{L^2_T} . \]

When $n \geq 6$, we have $n/2 - 2 - \mu_1 > 1/2$ so that

$$ \| w^2 r \Delta u \|_{L^\infty_T} \lesssim \| \Delta u \|_{H^{n-2}_T} \lesssim \varepsilon_1 + \varepsilon_s, \quad \| r^{1-\mu} \nabla u \|_{L^\infty_T} \lesssim \varepsilon_s . $$

Moreover, by Lemma 2.7, as $2 - \mu + (1 + \mu_1)/2 < n/2$, we have the following similar estimate as that of (8.7),

$$ \| w^{-1} r^{-2} D^{s-2} g'(u) \|_{L^2_T} \lesssim \| w^{-1} D^\delta g'(u) \|_{L^2_T} \lesssim \| w^{-1} D^\delta u \|_{L^2_T} \lesssim \varepsilon_1 + \varepsilon_s , $$

which gives us

(8.9) \[ \| w^{-1} r^{-2} D^{s-2} g'(u) \|_{L^2_T} \lesssim \varepsilon_1 + \varepsilon_s . \]

Then, by Theorem 2.4, we get

$$ \| w D^{s-2}((\nabla g(u))\Delta u) \|_{L^2_T} \lesssim \| w^{-1} D^{s-2} \Delta u \|_{L^2_T} \| w^{2} g'(u) \nabla u \|_{L^\infty_T} + \| w^{-1} r^{-1} D^{s-2} \nabla u \|_{L^2_T} \| w^{2} r g'(u) \Delta u \|_{L^\infty_T} + \| w^{-1} r^{-2} D^{s-2} g'(u) \|_{L^2_T} \| w^{2} r^{2-\mu} \nabla u \Delta u \|_{L^\infty_T} \lesssim \varepsilon_s (\varepsilon_1 + \varepsilon_s) . $$

Turning to the case of $n = 4$, for which we have $s = 2 + \mu$. Let $q = 2/(1 - 2\mu)$ so that $1/q + \mu = 1/2$, and

$$ \| r^{3\mu} \Delta u \|_{L^\infty_T L^2} \lesssim \| D^\mu \Delta u \|_{L^\infty_T L^2} \lesssim \varepsilon_s . $$

Moreover, we claim that

(8.10) \[ \| w r^{-3\mu} D^\mu \nabla g(u) \|_{L^2_T L^1_T} \lesssim \varepsilon_1 + \varepsilon_s . \]

Thus we have

$$ \| w D^\mu((\nabla g(u))\Delta u) \|_{L^2_T} \lesssim \| w^{-1} D^\mu \Delta u \|_{L^2_T} \| w^{2} \nabla g(u) \|_{L^\infty_T} + \| w r^{-3\mu} D^\mu \nabla g(u) \|_{L^2_T L^1_T} \| r^{3\mu} \Delta u \|_{L^\infty_T L^2} \lesssim \varepsilon_s (\varepsilon_1 + \varepsilon_s) . $$

It remains to give the proof of the claim (8.10). Actually, notice that

$$ w r^{-3\mu} = r^{-1-4\mu} r^{\mu+1} w^{-1} \lesssim r^{3(\frac{1}{2} - \mu)} (w^{-1} r^{-\frac{1}{2} - \mu} + w^{-1} r^{\mu+\frac{1}{2}}) , $$

an application of Lemma 2.9 gives us that

$$ \| w^{-3\mu} D^\mu \nabla g(u) \|_{L^2_T L^1_T} \lesssim \| w^{-1} r^{-\frac{1}{2} - \mu} D^\frac{1}{2} \nabla g(u) \|_{L^\infty_T} + \| w^{-1} r^{\mu+\frac{1}{2}} D^\frac{1}{2} \nabla g(u) \|_{L^2_T} , $$

where we have used the assumption $\mu \leq 1/3$ to ensure $-(1 + \mu_1)/2 - 1/2 - \mu \geq -(n-1)/2$. The second term on the right could be controlled by using Proposition 2.8 and Lemma 2.7, as follows

$$ \| w^{-1} r^{-\frac{1}{2} - \mu} D^\frac{1}{2} \nabla g(u) \|_{L^\infty_T} \lesssim \| w^{-1} r^{-\frac{1}{2} + \mu} D^\frac{1}{2} u \|_{L^2_T} \lesssim \| w^{-1} D^{2-\mu} u \|_{L^2_T} \lesssim \varepsilon_1 + \varepsilon_s . $$
Instead, concerning the first term on the right, we use Lemma 2.7 to obtain
\[
\|w^{-1}r^{-\frac{3}{2}}M^2 \nabla g(u)\|_{L^2_{t,x}} \lesssim \|w^{-1}r^{-\mu} \Delta g(u)\|_{L^2_{t,x}} \\
\lesssim \|w^{-1}r^{-\mu} \Delta u\|_{L^2_{t,x}} + \|w^{-1}r^{-1} \nabla u\|_{L^2_{t,x}} \|r^{1-\mu} \nabla u\|_{L^\infty_{t,x}} \lesssim \varepsilon_1 + \varepsilon_a.
\]

9. Appendix: Proof of Morawetz type estimates, Lemma 3.4

In this section, we are interested in proving the fundamental Morawetz type estimates, Theorem 3.4. Let \( S_T = [0, T) \times \mathbb{R}^n \) with \( n \geq 3 \), we consider the linear wave equations (3.2), that is
\[
\Box u := (-\partial_t^2 + \Delta + \text{b}^{abc}(t, x)\partial_\alpha \partial_\beta) u = F,
\]
where we assume \( \text{b}^{abc} = \text{b}^{ac} = \text{b}^{ac} = \text{h}^{abc}, \Delta _{00} = \text{h}^{00} = 0 \) and \( \Box \) is uniform hyperbolic, in the sense of (3.1).

**Lemma 9.1** (Morawetz type estimates). Let \( f = f(r) \) be any fixed differential function. For any solutions \( u \in C^\infty([0, T], C^\infty(\mathbb{R}^n)) \) to the equation (9.1) in \( S_T \) with (3.1) and \( n \geq 3 \), we have
\[
\Box u := (-\partial_t^2 + \Delta + \text{b}^{abc}(t, x)\partial_\alpha \partial_\beta) u = \partial_\gamma P_h^{\gamma} - Q,
\]
where \( X = (\partial_t + \frac{a}{2r}) u, P_h^0 = \text{h}^{abc} \partial_\alpha \partial_\beta uX, P_h^1 = O([f^2 + |r|f'] + |\text{b}h|)|\partial u|^2), \)
\[
Q = Q_0 + O([\frac{|\text{b}h|}{r} + |\partial(\text{h}h)|]|\partial u| |\partial u|),
\]
and \( |\Box u|^2 = |\nabla u|^2 - |\partial_t u|^2 \).

This is essentially coming from multiplying \( f(r) \) \( \partial_t + \frac{a}{2r} \) \( u \) to the wave equation and a tedious calculation of integration by parts. See e.g. [34, P199-200], [14, P273 (2.10)-(2.11)]. Typically, \( f \) is chosen to be differential functions satisfying
\[
f \leq 1, 2f \geq r f'(r) \geq 0, -\Delta (f/r) \geq 0,
\]
which ensure that \( Q_0 \) is positive semidefinite. In literatures, some of the typical choices are \( f = 1, 1 - (3 + r)^{-\delta} (\delta > 0), 45, r/(R + r), 43, 44, (r/(R + r))^\mu \)
(\( \mu \in (0,1), [14, 13] \)).

9.1. Details: general case. Let \( \omega = \omega_j = x^j/r \). As \( \partial_j = \omega_j \partial_r + \nabla_j, \partial_r = \omega_j \partial_j, \partial_r = \omega_j \partial_j, 2X = \partial_r - \partial_r = \omega_j \partial_j \partial_j = \omega_j \partial_j + (n-1)/r \), we have \( [X, \partial_r] = 0, \)
\[
[X, \partial_k] = [\omega_j, \partial_k] \partial_j + \frac{n-1}{2} \frac{1}{r}, \partial_k = -\frac{\delta^j_k - \omega_j \omega_k}{r} \partial_j + \frac{n-1}{2r^2} \omega_k = \frac{1}{r} (-\partial_k + \omega_k X).
\]
Notice that
\[
\partial_\alpha \partial_\beta uXu = \partial_\alpha (\partial_\beta uXu) - \partial_\beta uX \partial_\alpha uXu = \partial_\alpha (\partial_\beta uXu) - \partial_\beta uX |Xu| - \partial_\beta uX \partial_\alpha u,
\]
as \( u_{\alpha \beta} = u_{\beta \alpha} \), we obtain
\[
2 \partial_\alpha \partial_\beta uXu = \partial_\alpha (\partial_\beta uXu) + \partial_\beta (\partial_\alpha uXu) - \partial_\beta uX |Xu| - \partial_\beta uX \partial_\alpha u,
\]
\[
- \partial_\beta uX \partial_\alpha u - \partial_\alpha uX \partial_\beta u.
\]
Notice also that $\partial_j((\omega^jF)G) = FXG + GXF$, we get

$$2\partial_\alpha\partial_\beta u Xu = \partial_\alpha(\partial_\beta u Xu) + \partial_\beta(\partial_\alpha u Xu) - \partial_\beta u[\partial_\alpha, X]u - \partial_\alpha u[\partial_\beta, X]u - \partial_j(\omega^j\partial_\beta u\partial_\alpha u).$$

To be specific, we have

$$\partial^2_t u Xu = \partial_t(\partial_t u Xu) - \partial_j(\omega^j\frac{u^2_j}{2}),$$

$$2\partial_\alpha\partial_\beta u Xu = \partial_t(\partial_\beta u Xu) + \partial_\beta(\partial_\alpha u Xu) - \partial_k(\omega^k\partial_t u\partial_j u) + \frac{u_t(-\partial_k + \omega_k X)u}{r},$$

$$2\partial_j\partial_k u Xu = \partial_j(\partial_k u Xu) + \partial_k(\partial_j u Xu) - \partial_m(\omega^m\partial_j u\partial_k u) + \frac{\partial_j u(-\partial_k + \omega_k X)u + \partial_k u(-\partial_j + \omega_j X)u}{r}.$$  

In summary, we have

$$(\partial_\alpha\partial_\beta u) Xu = \partial_\gamma P^\gamma_{\alpha\beta} + Q_{\alpha\beta},$$

with $P^\gamma_{\alpha\beta} = O(||\partial u||\partial u||), rQ_{\alpha\beta} = O(||\partial u||\partial u||)$.

9.2. Details: general multiplier. By (9.5), when we multiply $f Xu$ against $\Box u$, we get

$$\partial^\alpha\partial_\alpha u f Xu = \partial_\gamma (f m^{\alpha\beta} P^\gamma_{\alpha\beta}) - f' \omega_k m^{\alpha\beta} P^k_{\alpha\beta} + m^{\alpha\beta} Q_{\alpha\beta},$$

where

$$m^{\alpha\beta} P^0_{\alpha\beta} = -P^0 + \sum_j P^0_{jj} = -\partial_t u Xu,$$

$$m^{\alpha\beta} P^k_{\alpha\beta} = -P^k_0 + \sum_j P^k_{jj} = \omega^k u^2_j + \sum_j (\delta^j_k u_j X u - \frac{1}{2} \omega^k u^2_j) = \omega^k u^2_j - |\nabla u|^2 + u_k X u,$$

$$m^{\alpha\beta} Q_{\alpha\beta} = -Q_0 + \sum_j Q_{jj} = \sum_j \partial_j u\frac{1}{r}(-\partial_j + \omega_j X) u = \frac{1}{r}(-|\nabla u|^2 + u^2) + \frac{n - 1}{2r^2} u \partial_t u.$$

Notice that $\frac{u^2_j}{r} = \frac{1}{r} \partial_t u^2 = -\frac{1}{2} \nabla r^{-1} \cdot \nabla u^2$,

$$2 f \frac{\partial_t u^2}{r^2} = -\nabla \frac{f}{r} \cdot \nabla u^2 + \frac{f'}{r} \partial_t u^2 = -\nabla \left( u^2 \nabla \frac{f}{r} \right) + u^2 \Delta \left( \frac{f}{r} \right) + \frac{f'}{r} \partial_t u^2,$$

we obtain

$$-f' \omega_k m^{\alpha\beta} P^k_{\alpha\beta} = -f'(u^2_j - \frac{|\nabla u|^2}{2}) + u_r X u,$$

and

$$f m^{\alpha\beta} Q_{\alpha\beta} = -f \frac{|\nabla u|^2}{r} + \frac{n - 1}{4} \left( \Delta \left( \frac{f}{r} \right) u^2 + \frac{f'}{r} \partial_t u^2 \right) + \partial_j F^j,$$

with $F^j = O((||f|| + ||rf'||)r^{-2} u^2)$.

In summary, we have

$$(\partial^\alpha\partial_\alpha u) f Xu = \partial_\gamma (f m^{\alpha\beta} P^\gamma_{\alpha\beta}) - f' \omega_k m^{\alpha\beta} P^k_{\alpha\beta} + m^{\alpha\beta} Q_{\alpha\beta},$$

$$= \partial_\gamma \tilde{P}^\gamma - \frac{f}{r} |\nabla u|^2 + \frac{n - 1}{4} \Delta \left( \frac{f}{r} \right) u^2 - f'(\frac{u^2_j - |\nabla u|^2}{2} + u^2_r),$$

$$= \partial_\gamma \tilde{P}^\gamma - f'(\frac{u^2_j + u^2_r}{2} + \frac{n - 1}{4} \Delta \left( \frac{f}{r} \right) u^2 - \frac{(2f - rf')|\nabla u|^2}{2r},$$

$$= \partial_\gamma \tilde{P}^\gamma - Q_0,$$

where $\tilde{P}^\gamma = O((||f|| + ||rf'||)\partial u|^2), \tilde{P}^0 = -fu_t X u.$
For perturbation, we have
\begin{equation}
(9.6) \quad f\hat{h}^{\alpha \beta}(\partial_\alpha \partial_\beta u) Xu = \partial_\gamma(f\hat{h}^{\alpha \beta} P_{\alpha \beta}^{\gamma}) - P_{\alpha \beta}^{\gamma} \partial_\gamma(f\hat{h}^{\alpha \beta}) + f\hat{h}^{\alpha \beta} Q_{\alpha \beta}.
\end{equation}
In summary, we obtained (9.2).

9.3. Choice of multiplier function \( f \). To prove the Morawetz type estimates, Lemma 3.4, we will choose two kinds of the multiplier functions \( f \), with parameter \( R > 0 \),
\begin{equation}
(9.7) \quad f = \frac{r}{R + r},
\end{equation}
\begin{equation}
(9.8) \quad f = \left(\frac{r}{R + r}\right)^\mu = \left(1 - \frac{R}{R + r}\right)^\mu, \quad \mu \in (0, 1).
\end{equation}
Of course, (9.7) could be viewed as the limit case of (9.8) when \( \mu = 1 \).

Now we do the calculation for \( f \) given in (9.8) with \( \mu \in (0, 1] \). We first notice that
\begin{equation}
(9.9) \quad f'(r) = \mu \left(\frac{r}{R + r}\right)^{\mu - 1} \frac{R}{(R + r)^2} = \mu \frac{R r^{\mu - 1}}{(R + r)^{\mu + 1}} \geq 0,
\end{equation}
\begin{equation}
(9.10) \quad \frac{f(r)}{r} - f'(r) = \frac{r^{\mu - 1}}{(R + r)^\mu} \left(1 - \frac{\mu R}{R + r}\right) \geq 0,
\end{equation}
and
\begin{equation}
(9.11) \quad \frac{2f(r) - r f'(r)}{r} \geq f(r) \geq f'(r).
\end{equation}
In order to compute \(-\Delta (f(r)/r)\), we recall that
\begin{equation}
-\Delta \left(\frac{f(r)}{r}\right) = -r^{1-n} \partial_\gamma \left(r^{n-1} \partial_\gamma \left(\frac{f(r)}{r}\right)\right) = r^{1-n} \partial_\gamma \left(r^{n-2} \left(\frac{f(r)}{r} - f'(r)\right)\right).
\end{equation}
Using this identity and (9.10), we see that \(-\Delta (f(r)/r)\) equals to
\begin{align*}
r^{1-n} \partial_\gamma \left(r^{n+\mu-3} \left(1 - \frac{\mu R}{R + r}\right)\right) \\
= \left(\frac{n + \mu - 3}{R + r}\right) - \frac{\mu^{\mu-2+\mu}}{(R + r)^{\mu + 1}} \left(1 - \frac{\mu R}{R + r}\right) + \frac{\mu^{\mu-2+\mu}}{(R + r)^{\mu + 2}} \\
= \frac{r^{-\mu + 2+\mu}}{(R + r)^\mu} \left(n + \mu - 3 - \frac{\mu R}{R + r}\right) \left(1 - \frac{\mu R}{R + r}\right) + \frac{\mu^{\mu-2+\mu}}{(R + r)^{\mu + 2}} \\
= \frac{r^{-\mu + 2+\mu}}{(R + r)^\mu} \left(n - 3 + \frac{\mu R}{R + r}\right) \left(1 - \frac{\mu R}{R + r}\right) + \frac{\mu^{\mu-2+\mu}}{(R + r)^{\mu + 2}},
\end{align*}
from which we see that, as \( n \geq 3 \),
\begin{equation}
(9.12) \quad -\Delta \left(\frac{f(r)}{r}\right) \geq (1 - \mu) \frac{\mu R^2 r^{-3+\mu}}{(R + r)^{\mu + 2}} + \frac{\mu^{\mu-2+\mu}}{(R + r)^{\mu + 2}} \geq 0.
\end{equation}
In summary, we see that when \( \mu \in (0, 1) \), we get \( Q_0 \) from (9.3) is non-negative and has the following lower bound for \( r \leq R \)
\begin{equation}
(9.13) \quad Q_0 \geq f' \frac{\partial u}{2} - \frac{n-1}{4} \Delta \left(\frac{f}{r}\right) u^2 \geq \frac{\mu |\partial u|^2}{R^{\mu + 2}}.
\end{equation}
where the implicit constant depends only on \(n\) and \(\mu \in (0,1)\), and in particular, independent of \(R > 0\). On the other hand, when \(\mu = 1\), \(Q_0\) from (9.3) is still non-negative and has the following lower bound for \(R/2 \leq r \leq R\)

\[
(9.14) \quad Q_0 \geq f \left( \frac{\partial u}{r} \right)^2 - \frac{n-1}{4} \Delta \left( \frac{f}{r} \right) u^2 \geq \frac{1}{8R} |\partial u|^2 + \frac{n-1}{32} \frac{1}{R^2r} u^2 \geq \frac{|\partial u|^2}{r}.
\]

9.4. **Proof of Morawetz type estimates, Theorem 3.4.** Equipped with Lemma 3.2 and Lemma 9.1, together with the observations (9.13)-(9.14), we could give the proof of Morawetz type estimates, Lemma 3.4.

Let us begin with the proof of (3.10). At first, applying (9.13) with \(R = 1\), and (9.14) with \(R \geq 1\), that is we use \(f = \left( \frac{1}{\pi r} \right)^\mu\) and \(\pi r\) with \(R \geq 1\), we get

\[
\int_{r \leq 1} \frac{\tilde{\partial u}}{r^{1-\mu}} \, dx \, dt + \sup_{R \geq 1} \int_{R/2 \leq r \leq R} \frac{\tilde{\partial u}}{r} \, dx \, dt
\]

\[
\lesssim \sup_{f = \left( \frac{1}{\pi r} \right)^\mu, \pi r \geq 1} \int_{S_T} Q_0 \, dx \, dt
\]

\[
\lesssim \sup_{f = \left( \frac{1}{\pi r} \right)^\mu, \pi r \geq 1} \left( - \int_{S_T} F \left( \frac{\partial u}{r} + \frac{n-1}{2r} \right) u \, dx \, dt + \int_{R^n} P^0_1(t, \cdot) \, dx \right)_{t=0}^T
\]

\[
+ \int_{S_T} Q_0 - Q \, dx \, dt
\]

\[
\lesssim \int_{S_T} |F\tilde{\partial u}| \, dx \, dt + \int_{R^n} |\partial u(T)||\tilde{\partial u}(T)| \, dx + \int_{R^n} |\partial u(0)||\tilde{\partial u}(0)| \, dx
\]

\[
+ \sup_{f = \left( \frac{1}{\pi r} \right)^\mu, \pi r \geq 1} \int_{S_T} \left( \frac{|\tilde{\partial h}|}{r} + \frac{|\tilde{h}|}{r} \right) |\partial u||\tilde{\partial u} \, dx \, dt
\]

\[
\lesssim \int_{S_T} |F\tilde{\partial u}| \, dx \, dt + \|\tilde{\partial u}(t)\|_{L^2(S_T)} + \int_{S_T} \left( |\tilde{\partial h}| + \frac{|\tilde{h}|}{r^{1-\mu}(r)} \right) |\partial u||\tilde{\partial u} \, dx \, dt
\]

where we have used (9.2) in the second inequality, the facts \(|f| \leq 1, 0 \leq f' \leq f/r\),

\[
|Q - Q_0| \lesssim \left( \frac{|\tilde{\partial h}|}{r} + \frac{|\tilde{h}|}{r} \right) |\partial u||\tilde{\partial u}| \leq \left( |\tilde{\partial h}| + \frac{|\tilde{h}|}{r} \right) |\partial u||\tilde{\partial u}|,
\]

and \(|P^0| \leq |\partial u||\tilde{\partial u}|\) in the third inequality. By Lemma 3.2 and Hardy’s inequality, we see that

\[
\|u\|_{X_1}^2 := \int_{r \leq 1} \frac{\tilde{\partial u}^2}{r^{1-\mu}} \, dx \, dt + \sup_{R \geq 1} \int_{R/2 \leq r \leq R} \frac{\tilde{\partial u}^2}{r} \, dx \, dt + \|\partial u(t)\|_{L^\infty L^2(S_T)}^2
\]

\[
\lesssim \int_{S_T} |F\tilde{\partial u}| \, dx \, dt + \|\tilde{\partial u}(0)\|_{L^2}^2 + \int_{S_T} \left( |\tilde{\partial h}| + \frac{|\tilde{h}|}{r^{1-\mu}(r)} \right) |\partial u||\tilde{\partial u} \, dx \, dt.
\]

Thus to give (3.10), we need only to show that

\[
(9.15) \quad \|u\|_{L^4 T} \lesssim \|u\|_{X_1},
\]

which essentially follows from a standard argument of Keel-Smith-Sogge [20]. Here for completeness, we write down the proof. The first and second terms are trivial
to control. For the remaining two terms, with \( \alpha \in [0, \mu] \), we have

\[
\| r^{-\frac{1}{2} + \mu} \partial_t \partial_x u \|_{L^2_{t,x}(S_T)}^2 \lesssim \| r^{-\frac{1}{2} + \mu} \partial_t u \|_{L^2_{t,x}(r \leq 1)}^2 + \sum_{0 \leq j \leq \ln(T)} \| r^{-\frac{1}{2} + \mu} \partial_t u \|_{L^2_{t,x}(r \geq 2^j)}^2 + \| r^{-\frac{1}{2} + \mu} \partial_t u \|_{L^2_{t,x}(r \geq T)}^2 \lesssim \| u \|_{X_1}^2 + \sum_{0 \leq j \leq \ln(T)} 2^j(\mu - \alpha) \| u \|_{X_1}^2 + (T)^{\mu - \alpha} \| \partial_t u \|_{L^\infty_{t,x} L^2_{t,x}}^2 \lesssim C_\alpha(T) \| u \|_{X_1}^2 ,
\]

where

\[
C_\alpha(T) = \left\{ \begin{array}{ll}
\ln(T) & \alpha = \mu , \\
(T)^{\mu - \alpha} & \alpha \in [0, \mu) .
\end{array} \right.
\]

This completes the proof of (9.15), and so is (3.10).

Turning to the proof of (3.9), we will use \( f = \left( \frac{r}{T + r} \right)^\mu \leq 1 \). Applying (9.13), we get as before,

\[
\int_{r \leq T} \frac{|\partial_t u|^2}{T^{r(1 - \mu)}} dx dt \lesssim \int_{S_T} \left( |F| + |f \partial_t \| \partial_x u| \right) |\partial_t u| dx dt + \| \partial_t u(t) \|_{L^\infty_{t,x}}^2 \gtrsim \int_{S_T} \left( |F| + \left( |\partial_t | + \frac{2}{1 - \mu} \right) |\partial_x u| \right) |\partial_t u| dx dt + \| \partial_t u(t) \|_{L^\infty_{t,x}}^2 .
\]

Together with Lemma 3.2, we see that

\[
\| u \|_{X_2}^2 := \int_{r \leq T} \frac{|\partial_t u|^2}{T^{r(1 - \mu)}} dx dt + \| \partial_t u(t) \|_{L^\infty_{t,x}}^2 \lesssim \int_{S_T} \left( |F| + \left( |\partial_t | + \frac{2}{1 - \mu} \right) |\partial_x u| \right) |\partial_t u| dx dt + \| \partial_t u(0) \|_{L^2_{t,x}}^2 .
\]

Moreover, we have

\[
\| r^{-\frac{1}{2} + \mu} \partial_t u \|_{L^2_{t,x}} \lesssim \| r^{-\frac{1}{2} + \mu} \partial_t u \|_{L^2_{t,x}(|x| \leq T)} + \| r^{-\frac{1}{2} + \mu} \partial_t u \|_{L^2_{t,x}(|x| \geq T)} \lesssim \| r^{-\frac{1}{2} + \mu} \partial_t u \|_{L^2_{t,x}(|x| \leq T)} + T^{\frac{\alpha}{2}} \| \partial_t u \|_{L^\infty_{t,x} L^2_{t,x}(|x| \geq T)} \lesssim \| r^{-\frac{1}{2} + \mu} \partial_t u \|_{L^2_{t,x}(|x| \leq T)} + T^{\frac{\alpha}{2}} \| \partial_t u \|_{L^\infty_{t,x} L^2_{t,x}} \lesssim T^{\frac{\alpha}{2}} \| u \|_{X_2}
\]

which gives us (3.9).

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School of Mathematical Sciences, Zhejiang University, Hangzhou 310027, P. R. China

*Email address*: wangcbo@zju.edu.cn

*URL*: http://www.math.zju.edu.cn/wang