The Conditional Probability Interpretation of the Hamiltonian Constraint.

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Abstract. The Conditional Probability Interpretation (CPI), first introduced by Page and Wootters [4], is reviewed and refined. It is argued that in its refined form the CPI is capable of answering various past criticisms [9–11]. In particular, questions involving more than one clock time are described in detail, resolving the problems raised in Kuchař’s “reduction ad absurdum” [5, 10]. In the case of Parametrized Particle Dynamics, conventional quantum mechanics is recovered in the ideal clock limit. When $E = 0$ is among the continuous spectrum of the Hamiltonian, the induced inner product [25–29, 31] is used to construct the physical Hilbert space $\mathcal{H}_{ph}$ from the generalized eigenvectors in (the topological dual of) $\mathcal{H}_{aux}$. This allows the CPI to be applied to these ‘continuous-spectrum’ cases in a more rigorous fashion than that described previously [5–8]. The discrete spectrum case is also treated.

1. Introduction

Canonical quantization of General Relativity leads [1] to the well-known ‘Hamiltonian constraint’

$$\hat{H}|\psi\rangle = 0$$

(on the physical Hilbert space $\mathcal{H}_{ph}$), known as the Wheeler-DeWitt equation. This constraint results from the Dirac-quantization [2, 3] of any reparametrization-invariant theory (or from the canonical quantization of generally covariant theories), and poses immediate problems for the role of time in such theories. Combined with the Schrödinger equation, it dictates that physical states do not change with time! The Heisenberg picture offers no respite, since physical operators must satisfy

$$[\hat{H}, \hat{A}]|\psi\rangle = 0 \quad \text{on} \quad \mathcal{H}_{ph}$$

which ensures that all physical properties are constants of motion! Attempts to understand this requirement, and to reconcile it with the ever changing evolution we see around us, began as soon as the requirement was noticed [1], and quickly developed into a busy field of investigation. Reviews of this topic [9, 10] provide a useful categorisation of the various ‘problems of time’, and of the various methods proposed to understand or circumvent them. Among those proposals are the so called ‘timeless interpretations’, which accept equations (1), (2) unaltered, and accept that coordinate time has no
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foundational role to play in the quantum theory. An important task of such approaches is to explain how a phenomenological notion of time-evolution can emerge in appropriate circumstances. Why, as observers within such a system, we can observe changes in the world around us, and importantly, why the changes in isolated subsystems can be described by the time-dependent Schrödinger equation.

It is one such timeless interpretation, the conditional probability interpretation (advanced first by Page and Wootters [4] and later by Page [5–8]) that will be the subject of this article. This proposal (herein called CPI) received strong criticism, most notably by Kuchař [10] in his 1992 review article (see also [9]). While no proposal escaped criticism in that review (with Kuchař concluding that “In my opinion, none of us has so far succeeded in proposing an interpretation of quantum gravity that would either solve or circumvent the problems of time”) his criticism of the CPI amounted to “a reduction ad absurdum of the conditional probability interpretation.” In this article we review both the CPI, and the various criticisms of it. We propose a refinement of the CPI, and explain how, at least in its refined form, the CPI is capable of answering these criticisms. In particular, it yields reasonable predictions when comparing more than one ‘clock time’.

Section 2 describes briefly how the Hamiltonian constraint emerges in reparametrization-invariant particle models, both in classical and quantum treatments. We describe how the analogous classical ‘problem of time’ is resolved in these models, and lay the groundwork for tackling the quantum mechanical case. This is intended as a brief introduction to the concepts involved. We stress that later sections are not limited to reparametrization-invariant particle models, and apply more generally to systems subject to a Hamiltonian constraint. We assume the existence of a Hilbert space $H_{\text{tot}}$ for the unconstrained system, on which the Hamiltonian operator $\hat{H}$ exists and is Hermitian. In Section 3 we assume also that $E = 0$ is among the discrete spectrum of $\hat{H}$. We present the refined CPI as it applies to such systems, and explain how it differs from the original presentation [4,5]. In particular, questions involving more than one ‘clock time’ are described, and Kuchař’s ‘reduction ad absurdum’ is addressed and fixed. Section 4 describes the case when $E = 0$ is among the continuous spectrum of the Hamiltonian. The ‘induced’ or ‘spectral analysis’ inner product [25–29, 31], is used, along with the closely related ‘group averaging’ procedure [32–34] to construct the physical Hilbert space $H_{\text{ph}}$ from the generalized eigenvectors in (the topological dual of) $H_{\text{tot}}$ (only pertinent aspects of group averaging and the induced inner product are described in Section 4 - further rigor is presented elsewhere [26, 31, 33, 34]). This allows the CPI to be applied almost unchanged to these ‘continuous-spectrum’ cases. The case of Parametrized Particle Dynamics is briefly addressed, and conventional quantum mechanics is recovered in the ‘ideal clock’ limit. Conclusions are presented in Section 5.
2. Background

2.1. Timelessness in Classical Mechanics

As an example of reparametrization invariant particle dynamics, consider the Jacobi Action (presented for instance in [36] pg 142, and discussed in detail in [37, 38]).

\[ S = 2 \int \sqrt{E - V(x^i)} \sqrt{T(x^i)} d\lambda \]  

(3)

where \[ T \equiv \sum_i \frac{m_i}{2} \left( \frac{dx^i}{d\lambda} \right)^2 \]  

(4)

The resulting Lagranges equations are:

\[ \frac{d}{d\lambda} \left( \sqrt{E - V \over T} m_i \dot{x}^i \right) = -\sqrt{E - V \over T} \frac{\partial V}{\partial x^i} \]  

(5)

They uniquely determine (for given end points) physical paths in configuration space, but leave the parametrization of those paths arbitrary. The transformation \( x^i(\lambda) \to x^i(\lambda') \) has the status of a gauge transformation, and gauge-invariance leads to the requirement that physical observables be reparametrization invariant. Hence physical observables must be constant on physical motions.

In the Hamiltonian formulation (see [2,3] for instance) we start with the observation that \( \frac{\partial L}{\partial \dot{x}^i} \dot{x}^i - L \) is identically zero. This leads to the primary (first class) constraint:

\[ \phi = \sum_i \frac{p_i^2}{2m_i} + V(x^i) - E = 0 \]  

(6)

in terms of which the Hamiltonian is simply \( H = u\phi \) where \( u \) is an arbitrary phase space function. Physical observables are then functions \( A(p_i, x^i) \) satisfying:

\[ \{ A, \phi \} \approx 0 \]  

(7)

where \( \approx \) means ‘equal on the constraint surface’. This is sufficient to ensure that:

\[ \{ A, H \} \approx 0 \]  

(8)

and hence that all physical observables are constants of the motion. We note in passing that not all physical observables \( A \neq B \) are distinct, since they may satisfy \( A \approx B \), in which case they represent the same physical observable.

We see that even in the classical description of reparametrization invariant systems, physical observables have no dependence on ‘coordinate time’ \( \lambda \). To understand how change can still be described in these timeless situations, consider a physical path \( \gamma \) in configuration space, and consider questions such as:

- Does the path pass through a given region \( V \in Q \)?
- Does \( x^2 = X \) anywhere on \( \gamma \)? If so, what is \( x^1 \) when \( x^2 = X \)?
These are well-posed questions, the answers to which are independent of the parametrization of \( \gamma \). In the second question, we could consider how the value of \( x^1 \) changes as we vary \( X \). By so doing we can describe how properties of the system (\( x^1 \) here) change w.r.t. other properties (\( x^2 \) here) of the system. Sufficiently large complex systems will typically include many subsystems (clocks, planetary motions etc) which could be used to describe changes in other subsystems in essentially this way. In laboratory physics for instance, we measure how a chosen (sub)subsystem changes (‘evolves’) w.r.t. changes in clock time. Since the clock is part of the total system, then this change is just like the second question above. It depends only on the path in configuration space (which includes the clocks configuration), and not on the parametrization of that path. It is this kind of change, which does not require ‘coordinate time’ \( \lambda \), which we seek to describe in the quantized theory.

Another simple example of a reparametrization invariant system is provided by ‘Parametrized Particle Dynamics’. In this case we start with a (non-relativistic) Lagrangian, such as \( \tilde{L}(x^i, \frac{dx^i}{dt}) = \sum_i m_i \left( \frac{dx^i}{dt} \right)^2 - V(x^i) \). By introducing an arbitrary parameter \( \lambda \) we can write the action as:

\[
S = \int d\lambda L(x^i, t, \dot{x}, \dot{t})
\]

where \( L(x^i, t, \dot{x}, \dot{t}) \equiv i\tilde{L}(x^i, \frac{\dot{x}^i}{\dot{t}}) \) and \( \dot{\cdot} \equiv \frac{d}{d\lambda} \) (10)

The coordinate \( t \) is now part of the configuration space. It must represent an observable property of the system - a ‘clock variable’. The action is reparametrization invariant w.r.t. the independent variable \( \lambda \), which gives the Hamiltonian constraint:

\[
H_{\text{tot}} = p_t + \tilde{H}(x^i, p_i) \approx 0
\]

where \( \tilde{H}(x^i, p_i) \) is the Hamiltonian appropriate to the original Lagrangian. Equation (8) requires that physical observables be independent of \( \lambda \), but allows dependence on \( t \). Equation (11) ensures that the dependence on \( t \) agrees with that ascribed by \( \tilde{H}(x^i, p_i) \).

2.2. Quantizing Reparametrization-Invariant Systems.

The quantization of constrained systems is described in various texts \([2, 3, 35]\). Our treatment is brief, and covers only salient features:

(i) Find a mapping \( F \rightarrow \hat{F} \) from functions on phase space to operators on some vector space \( \mathcal{H}_{\text{aux}} \), satisfying \([\hat{F}, \hat{G}] = i\hbar \{F, G\} + O(\hbar^2)\).

(ii) The constraint \( H \approx 0 \), is replaced by the operator equation \( \hat{H}\psi = 0 \), which defines a subspace \( \mathcal{H}_{\text{ph}} \) of \( \mathcal{H}_{\text{aux}} \) - the space of physical states. Equation (9) leads to equation (2), and hence to the requirement that physical operators map physical states onto physical states. We assume for convenience that there is only one constraint (the Hamiltonian constraint) left to be solved in obtaining \( \mathcal{H}_{\text{ph}} \). If instead there is a
family $\hat{C}_i$ of constraint operators then, provided they generate a unitary group [34], a minor generalization of our treatment can still apply.

(iii) An inner product is defined on $\mathcal{H}_{ph}$ such that real observables $A$ map onto Hermitian operators $\hat{A}_{ph}$ on $\mathcal{H}_{ph}$. This normally allows a unitary evolution to be defined on $\mathcal{H}_{ph}$ by the well-known Schrödinger equation $i\hbar \frac{d}{d\tau} |\psi\rangle = \hat{H} |\psi\rangle$, in terms of which the standard probabilistic interpretation follows. Combined with step (ii) however, this just tells us that physical states do not evolve with respect to $\lambda$. The relevant probabilistic interpretation of the theory is then more subtle, and is described in the next section.

Notice that step (iii) only requires $\mathcal{H}_{ph}$ (and not $\mathcal{H}_{aux}$) to be Hilbert. We will assume however (as is commonly the case) that the unconstrained space $\mathcal{H}_{aux}$ is equipped with an inner product with respect to which $\hat{H}$ is Hermitian. This is the case for instance in the ‘connection representation’ of quantum gravity [34, 35] and in Loop Quantum Gravity [39, 40], where a Hilbert space $\mathcal{H}_{aux}$ has been found on which the Gauss and diffeomorphism constraints have been solved, and only the Hamiltonian constraint remains to be rigourously solved. (While the Hamiltonian constraint in those theories depends non-trivially on an arbitrary smearing function $N(x)$, Thiemann has recently proposed a resolution of this difficulty, by replacing the constraints with one unique ‘Master Constraint’ $\hat{M}$ [17].)

For simplicity we also assume in Section 3 that the constraint equation specifies a ‘bound’ eigenspace, in the sense that $E = 0$ is among the discrete spectrum of $\hat{H}$. In Section 4 we describe how the CPI can be applied also when $E = 0$ is among the continuous spectrum of $\hat{H}$.

3. The Refined CPI for the ‘Bound’ Case

If $E = 0$ is among the discrete spectrum of $\hat{H}$, then $\mathcal{H}_{ph}$ is a subspace of $\mathcal{H}_{aux}$; the inner product on $\mathcal{H}_{ph}$ is the same as that on $\mathcal{H}_{aux}$. The projection operator $\hat{P}^{ph} : \mathcal{H}_{aux} \rightarrow \mathcal{H}_{ph}$ can be used to generate a physical state $|\psi_{ph}\rangle \equiv \hat{P}^{ph} |\psi\rangle$ from any state $|\psi\rangle \in \mathcal{H}_{aux}$ and to generate a physical operator:

$$\hat{A}^{ph} \equiv \hat{P}^{ph} \hat{A} \hat{P}^{ph}$$

(12)

from any operator $\hat{A}$ on $\mathcal{H}_{aux}$. However, distinct operators $\hat{A}, \hat{B}$ on $\mathcal{H}_{aux}$ only generate distinct physical properties if $\hat{A}^{ph} \neq \hat{B}^{ph}$. Projection operators $\hat{P}_A$ on $\mathcal{H}_{aux}$ generate POVM’s $\hat{P}_A^{ph}$ on $\mathcal{H}_{ph}$.

It is sometimes convenient to write the projection operator $\hat{P}^{ph}$ as:

$$\hat{P}^{ph} |\psi\rangle = \lim_{\tau \rightarrow \infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} da e^{-i\hat{H}a} |\psi\rangle$$

(13)

which is related to the process of ‘group averaging’ often used to provide a formal solution of the constraint equation [33, 34] (see also [40] for a similar construction in Loop Quantum Gravity), as will be described further in Section 4. Equation (13) is
easily verified by expanding $|\psi\rangle$ in terms of the spectrum of $\hat{H}$. While physical states have no dependence on coordinate time, equation (13) shows that they can be written as the time-average of Schrödinger-evolved states. In this sense the physical state $|\psi\rangle$ is associated with an entire history of the system, rather than with any point on that history. It is the quantum analog of the ‘path in configuration space’ discussed in the last section. The integration variable $\tau$ in equation (13) is nothing more than a variable of integration. (While it is sometimes convenient to introduce a ‘lapse function’ $N(\lambda)$ and put $\mathrm{d}a = N(\lambda)\mathrm{d}\lambda$ [30], this is not necessary.) Physical density operators can similarly be written as:

$$\hat{\rho}^{\text{ph}} \equiv \hat{P}^{\text{ph}} \hat{\rho}^{\text{ph}} = \lim_{\tau \to \infty} \frac{1}{(2\tau)^2} \int_{-\tau}^{\tau} \mathrm{d}a \int_{-\tau}^{\tau} \mathrm{d}a' e^{-i\hat{H}a} \rho e^{-i\hat{H}a'}$$ (14)

Given a projection operator $\hat{P}_A$ on $\mathcal{H}_{\text{aux}}$ and a physical density operator $\hat{\rho}^{\text{ph}}$ we identify

$$P(A; \hat{\rho}) \equiv \frac{\text{Tr}(\hat{P}_A^{\text{ph}} \hat{\rho}^{\text{ph}})}{\text{Tr}(\hat{\rho}^{\text{ph}})} = \frac{\text{Tr}(\hat{P}^{\text{ph}} \hat{P}_A^{\text{ph}} \hat{\rho})}{\text{Tr}(\hat{P}^{\text{ph}} \hat{\rho})}$$ (15)

with the apriori probability of $A$ in $\hat{\rho}^{\text{ph}}$. This does not refer to the probability of $A$ ‘at some given time’, and at this stage bears little resemblance to the probabilities of our intuition. In semiclassical terms, it is perhaps best thought of as representing the ‘proportion of the physical path (on configuration space) on which $A$ is true’. While we can replace $\hat{P}_A^{\text{ph}}$ with $\hat{P}_A$ in (15) without altering the result, we should not forget that it is the physical operator $\hat{P}_A^{\text{ph}}$ that we are measuring. Similarly, we could replace $\hat{\rho}^{\text{ph}}$ with $\hat{\rho}$ in equation (15), without change. (We cannot replace both $\hat{P}_A^{\text{ph}}$ and $\hat{\rho}^{\text{ph}}$ by their unphysical counterparts.)

Given two commuting projection operators $\hat{P}_A$ and $\hat{P}_B$ on $\mathcal{H}_{\text{aux}}$ we can define $P(A$ when $B; \hat{\rho})$ in terms of $\mathcal{H}_{\text{ph}}$ by:

$$P(A \text{ when } B; \hat{\rho}) \equiv \frac{\text{Tr}(\hat{P}^{\text{ph}} \hat{P}_A \hat{P}_B \hat{P}^{\text{ph}} \hat{\rho})}{\text{Tr}(\hat{P}^{\text{ph}} \hat{P}_B \hat{P}^{\text{ph}} \hat{\rho})} = \frac{\text{Tr}(\hat{P}^{\text{ph}} \hat{P}_A \hat{\rho})}{\text{Tr}(\hat{P}^{\text{ph}} \hat{\rho})}$$ (16)

It can be written in terms of measurement operators (see for instance [41] pg 85 and 102) as:

$$P(A \text{ when } B; \hat{\rho}) = \frac{\text{Tr}(\hat{M}_A \hat{M}_B \hat{\rho})}{\text{Tr}(\hat{M}^{\text{ph}} \hat{M}^{\text{ph}} \hat{\rho})} = \frac{\text{Tr}(\hat{M}_A \hat{\rho})}{\sum_A \text{Tr}(\hat{M}_A \hat{\rho})}$$ (17)

where $\hat{M}_{AB} = \hat{P}_A \hat{P}_B \hat{P}^{\text{ph}}$, $\hat{M}_A = \hat{P}_A \hat{P}^{\text{ph}}$ and $\hat{M}_B = \hat{P}_B \hat{P}^{\text{ph}}$. These are measurement operators (in the sense of [41]) but are not projective measurements. We will call operators of the form $\hat{M}_X = \hat{P}_X \hat{P}^{\text{ph}}$ “physical measurement operators”, since $\hat{M}_X \hat{M}_X = \hat{P}^{\text{ph}}$ is a physical operator.

$P(A \text{ when } B; \hat{\rho})$ is best thought of as representing the proportion of the physical path on which $A$ and $B$ are simultaneously true, divided by the proportion on which $B$
is true. It is not the same as the probability of \( A \) given \( B \), \( P(A|B; \hat{\rho}) \), which is instead given by:

\[
P(A|B; \hat{\rho}) = \frac{\text{Tr}(\hat{P}_B^\text{ph} \hat{P}_B^\text{ph} \hat{\rho} \hat{P}_B^\text{ph})}{\text{Tr}(\hat{P}_B^\text{ph} \hat{P}_B^\text{ph} \hat{\rho} \hat{P}_B^\text{ph})} = \frac{\text{Tr}(\hat{M}_B^\dagger \hat{M}_B \hat{M}_B \hat{\rho} \hat{P}_B^\text{ph})}{\sum_A \text{Tr}(\hat{M}_B^\dagger \hat{M}_B \hat{M}_B \hat{\rho} \hat{P}_B^\text{ph})}
\]

(18)

\[
P(A; \hat{\rho}_B) \quad \text{where} \quad \hat{\rho}_B \equiv \frac{\hat{M}_B \hat{\rho} \hat{P}_B^\text{ph}}{\text{Tr}(\hat{M}_B^\dagger \hat{M}_B \hat{\rho} \hat{P}_B^\text{ph})}
\]

(19)

Although \( \hat{\rho}_B \) in equation (19) is unphysical, both \( P(A|B; \hat{\rho}) \) and \( P(B; \hat{\rho}) \) are physical operations, since \( \hat{M}_A^\dagger \hat{M}_A \) and \( \hat{M}_B^\dagger \hat{M}_B \) are physical. (Replacing \( \hat{\rho}_B \) with \( \hat{\rho}_B^\text{ph} \) does not alter the result.) To appreciate the logical distinction between \( P(A \text{ when } B; \hat{\rho}) \) and \( P(A|B; \hat{\rho}) \) recall that a physical state \( |\psi\rangle \) here is the quantum mechanical equivalent of a path in configuration space, not of a point on it. The question “does a path pass through volume \( V_2 \) given that it passes through volume \( V_1 \)” is clearly distinct from the question “does a path pass through volume \( V_2 \) when it passes through volume \( V_1 \)?”

In Pages original writings [4–8], equation (16) was referred to as the conditional probability \( P(A|B) \), despite acknowledging it’s distinction from equation (18). We consider that to have been misleading, and have rectified it here. Equations (16) and (19) can also be combined, to give for instance:

\[
P(A_2 \text{ when } B_2|A_1 \text{ when } B_1; \hat{\rho}) = P(A_2 \text{ when } B_2; \hat{\rho}_{A_1B_1})
\]

where \( \hat{\rho}_{A_1B_1} = \frac{\hat{M}_{A_1B_1} \hat{\rho} \hat{P}_B^\text{ph} \hat{M}_{A_1B_1}^\dagger}{\text{Tr}(\hat{M}_{A_1B_1}^\dagger \hat{M}_{A_1B_1} \hat{\rho} \hat{P}_B^\text{ph})} \)

(20)

(21)

\[
\]

\[
= \frac{\text{Tr}(\hat{\rho}_B^\text{ph} \hat{P}_B^\text{ph} \hat{P}_B^\text{ph} \hat{\rho}_B^\text{ph})}{\text{Tr}(\hat{P}_A^\text{ph} \hat{P}_B^\text{ph} \hat{P}_B^\text{ph} \hat{\rho}_B^\text{ph})} = \frac{\text{Tr}(\hat{\rho}_B^\text{ph} \hat{P}_B^\text{ph} \hat{P}_B^\text{ph} \hat{\rho}_B^\text{ph})}{\sum_A \text{Tr}(\hat{\rho}_B^\text{ph} \hat{\rho}_B^\text{ph} \hat{\rho}_B^\text{ph})}
\]

(22)

(23)

We will return to this expression later. For now, we consider equation (16) for \( P(A \text{ when } B; \hat{\rho}) \). Our interpretation of equation (16) is justified simply because, as will now be shown, it accords with our conventional notion of ‘when’ as being ‘at the same time as’, whenever a reliable ‘time’ can be defined. For this purpose, suppose now that the total Hamiltonian can be written as:

\[
\hat{H} \approx \hat{H}_s + \hat{H}_c
\]

(24)

where \( \hat{H}_c \) describes a ‘clock Hamiltonian’, \( \hat{H}_s \) a ‘system Hamiltonian’, and \([\hat{H}_s, \hat{H}_c] = 0\). Then \( \mathcal{H}_{\text{aux}} \) can be decomposed as \( \mathcal{H}_c \otimes \mathcal{H}_s \), and we can seek a 1-parameter family of clock projection operators \( \hat{P}_T \) on \( \mathcal{H}_c \) satisfying:

\[
\hat{P}_T \approx e^{-i\hat{H}_c T} \hat{P}_0 e^{i\hat{H}_c T}
\]

(25)

We could require exact equality in equation (25) and use it to define \( \hat{P}_T \) in terms of \( \hat{P}_0 \), as was done for instance in [4]. However the more general condition (25) is still
sufficient, and we note that it need only be defined for a suitable finite range of the parameter $T$ (the ‘clocks lifetime’ for instance). Given an operator $\hat{P}_A$ on $\mathcal{H}_s$ we can now calculate

$$P(A \text{ when } T; \hat{\rho}) = \frac{\text{Tr}_{\text{ph}}(\hat{P}_{\text{ph}} \hat{P}_A \hat{P}_T \hat{P}_{\text{ph}} \hat{\rho}_{\text{ph}})}{\text{Tr}_{\text{ph}}(\hat{P}_{\text{ph}} \hat{P}_T \hat{P}_{\text{ph}} \hat{\rho}_{\text{ph}})} = \frac{\text{Tr}_{\text{aux}}(\hat{P}_A \hat{P}_T \hat{\rho}_{\text{ph}})}{\text{Tr}_{\text{aux}}(\hat{P}_T \hat{\rho}_{\text{ph}})}$$  \hspace{1cm} (26)

$$\approx \frac{\text{Tr}_{\text{aux}}(\hat{P}_A \hat{P}_0 e^{i\hat{H}_c T} \hat{\rho}_{\text{ph}} e^{-i\hat{H}_c T})}{\text{Tr}_{\text{aux}}(\hat{P}_0 e^{i\hat{H}_c T} \hat{\rho}_{\text{ph}} e^{-i\hat{H}_c T})}$$  \hspace{1cm} (27)

$$\approx \frac{\text{Tr}_{\text{aux}}(\hat{P}_A \hat{P}_0 e^{-i\hat{H}_c T} \hat{\rho}_{\text{ph}} e^{i\hat{H}_c T})}{\text{Tr}_{\text{aux}}(\hat{P}_0 e^{-i\hat{H}_c T} \hat{\rho}_{\text{ph}} e^{i\hat{H}_c T})}$$  \hspace{1cm} (28)

$$= \text{Tr}_s(\hat{P}_A e^{-i\hat{H}_c T} \hat{\rho}^s e^{i\hat{H}_c T}) \text{ where } \hat{\rho}^s \equiv \frac{\text{Tr}_{\text{c}}(\hat{P}_0 \hat{\rho}_{\text{ph}})}{\text{Tr}_{\text{aux}}(\hat{P}_0 \hat{\rho}_{\text{ph}})}$$  \hspace{1cm} (29)

That is, the probability of $A$ at clock time $T$ behaves as if we Schrödinger-evolved the ‘system state’ $\hat{\rho}^s$ in $\mathcal{H}_s$. This is the essence of the Conditional Probability Interpretation.$\dagger$

It was emphasized by Kuchar [10], that neither the operator $\hat{P}_A$ nor the density operator $\hat{\rho}_s$ is physical, in the sense of equations (11) and (2). This is certainly true, but unimportant, since $\hat{P}_{\text{ph}}$ and $\rho_{\text{ph}}$ are physical, so $P(A \text{ when } T)$ is undeniably physical. It is necessary for consistency to demonstrate that this physical probability behaves as if we Schrödinger-evolved $\hat{\rho}^s$ in $\mathcal{H}_s$, but we needn’t accept the separate reality of $\hat{\rho}^s$ or of $P_A$ in order to achieve this.

In equations (26) - (29) no requirement was placed on the commutator of clock operators denoting different ‘clock times’. $[\hat{P}_{T_1}, \hat{P}_{T_2}] \neq 0$ in general, so different clock states will not distinguish ‘different times’ with absolute certainty. A desirable property for a ‘good clock’ is that $\text{Tr}_{s}(\hat{P}_{T_1} \hat{P}_{T_2})$ be sufficiently small whenever $|T_1 - T_2|$ is sufficiently large. A choice of ‘good clock’ will depend in general on what subsystem $\mathcal{H}_s$ we are probing, and on what accuracy we desire. Most choices of $\hat{H}_c$ will not admit ‘perfect clocks’ (satisfying $\hat{P}_{T_1} \hat{P}_{T_2} = \delta(T_1 - T_2)$), although one exception is Parametrized Particle dynamics, which we consider briefly in Section 4. It is a virtue of the CPI that it does not require a ‘perfect clock’. Nor does it require an ‘internal time variable’ $T(x, p)$ on phase space, or a ‘time operator’ $\hat{T}$ on $\mathcal{H}_{\text{aux}}$ (contrary to beliefs expressed elsewhere [9, 11]).

The state $\hat{\rho}^s(T) = e^{-i\hat{H}_c T} \hat{\rho}^s e^{i\hat{H}_c T}$ in equation (29) is to be interpreted as representing the state of the system when the clock reads $T$. To investigate this more fully, suppose that equations (21) and (25) are strict equalities, and that

$$\hat{P}_T = |\psi_c(T)\rangle \langle \psi_c(T)|$$  \hspace{1cm} (30)

$\dagger$ The original papers [4, 8] dealt with closed systems generally, and so used the density operator $\hat{\rho} = \lim_{T \to \infty} \frac{1}{T} \int_{-T}^{T} e^{-i\hat{H}_t} \hat{\rho} e^{i\hat{H}_t} dt$ rather than equation (14). It was acknowledged already in [8] that this should be suitably modified when considering the Hamiltonian constraint.
where $|\psi_c(T)\rangle \equiv e^{-i\hat{H}_c T}|\psi_c\rangle$ in $\mathcal{H}_c$. For convenience, suppose further that $\hat{\rho}^{ph}$ is given by:

$$\hat{\rho}^{ph} = \hat{P}^{ph}|\psi_s\rangle\langle\psi_c|\hat{P}^{ph}$$

where $|\psi_s\rangle \in \mathcal{H}_s$ is a chosen ‘initial system state’. Equation (13) can be used to write the ‘effective system density operator’ $\hat{\rho}^{s}(T)$ on $\mathcal{H}_s$ as

$$\hat{\rho}^{s}(T) = \frac{|\psi_{s}^{eff}(T)\rangle\langle\psi_{s}^{eff}(T)|}{\langle\psi_{s}^{eff}(T)|\psi_{s}^{eff}(T)\rangle}$$

where the ‘effective system state’ $|\psi_{s}^{eff}(T)\rangle$ is given by

$$|\psi_{s}^{eff}(T)\rangle = \lim_{\tau \to \infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} da f_c(a)e^{-i\hat{H}_s (T+a)}|\psi_s\rangle = e^{-i\hat{H}_s T}|\psi_{s}^{eff}\rangle$$

where $f_c(a) \equiv \langle\psi_c|\psi_c(a)\rangle = \langle\psi_c|e^{-i\hat{H}_c a}|\psi_c\rangle$.

and $|\psi_{s}^{eff}\rangle \equiv \lim_{\tau \to \infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} da f_c(a)e^{-i\hat{H}_s a}|\psi_s\rangle$

Notice that $|f_c(a)|^2 = \text{Tr}_c(\hat{P}_0 \hat{P}_a) = \text{Tr}_c(\hat{P}_T \hat{P}_{T+a})$ provides a measure of the clocks effectiveness at distinguishing different ‘clock times’. If $|\psi_c\rangle$ was a ‘good clock’ then $f_c(a)$ would be sharply peaked about $a = 0$ and we would have (up to scale) $|\psi_{s}^{eff}(T)\rangle \approx |\psi_s(T)\rangle$. The clock would have picked out the system state ‘at time T’, as desired. However, recall that throughout this Section we have assumed that the system is bounded, in the sense that $E = 0$ was among the discrete spectrum of the operator $\hat{H}$. In this case even ‘good clocks’ are not possible; $|f_c(a)|$ will generally return close to one for an infinite total range of $a$. In Section 5 we will discuss the ‘unbound’ case, where $E = 0$ is among the continuous spectrum of $\hat{H}$. In that context non-repeating clock states will be plentiful, and such ambiguities are easily avoided.

Further understanding of $|\psi_{s}^{eff}(T)\rangle$ can be obtained by examining the spectra of $\hat{H}_s$ and $\hat{H}_c$. Suppose that equation (24) is an exact equality, and that there are $N$ eigenvalues $E_i^s$ in the spectrum of $\hat{H}_s$ for which $E_i^c = -E_i^s$ is in the spectrum of $\hat{H}_c$. ($\{|E_i^s\rangle\}$ will be discrete, and will be finite whenever $\hat{H}_c$ and $\hat{H}_s$ are bounded below). Then $\hat{P}^{ph}$ can be written in terms of the spectral projectors in $\mathcal{H}_s$ and $\mathcal{H}_c$ as

$$\hat{P}^{ph} = \sum_{i=1}^{N} \hat{P}_{E_i^c} \hat{P}_{E_i^s}$$

Using this, $|\psi_{s}^{eff}(T)\rangle$ is given by:

$$|\psi_{s}^{eff}(T)\rangle = \sum_{i=1}^{N} A_i \hat{P}_{E_i^c}|\psi_s\rangle e^{-iE_i^c T}$$

where $A_i = \langle\psi_c|\hat{P}_{E_i^c}|\psi_c\rangle$

First consider the case when the spectrum of $\hat{H}_s$ and $-\hat{H}_c$ have only one eigenvalue in common. In this case $\hat{P}^{ph} = \hat{P}_{E^c} \hat{P}_{E^s}$ does not entangle states in $\mathcal{H}_s$ with states in
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$H_c$, so $|\psi_{\text{eff}}^s(T)\rangle$ is stationary, and $P(A \text{ when } T; \hat{\rho})$ is independent of $T$ for all $\hat{\rho}$ and all clocks $\hat{P}_T$. The CPI would not be viable for this choice of $\hat{H}_s$ and $\hat{H}_c$ (although this conclusion needn’t apply if equation (24) is not exact). If the eigenspaces for $E^s_i$ and $E^c_i$ are both degenerate, we could seek a better choice of $\hat{H}_c$ and $\hat{H}_s$ in equation (24) which lifts this degeneracy, and provides the necessary correlations between system and clock. If, however, the Hamiltonian constraint has a unique solution (a possibility first suggested for quantum gravity by DeWitt [1]), then this is not possible. For such a universe no Conditional Probability Interpretation would be possible in which equation (24) was exact.

Consider, on the other hand, that $N$ is large. If the clock state $|\psi_c\rangle$ is chosen such that all $A_i$ are equal, then

$$|\psi_{\text{eff}}^s(T)\rangle \propto |\psi_{\text{relevant}}^s(T)\rangle = \sum_{i=1}^N \hat{P}_{E^c_i} |\psi^s_s(T)\rangle$$

(39)

which is as close to $|\psi_s(T)\rangle$ as is possible for this choice of $\hat{H}_s, \hat{H}_c$. For other choices of clock state, the effective system state $|\psi_{\text{eff}}^s(T)\rangle$ could bear little resemblance to $\psi^s_s(T)\rangle$. Even then however, no violation of the Schrödinger equation would be observed in $H_s$, since we still have $|\psi_{\text{eff}}^s(T)\rangle = e^{-i\hat{H}_s T} |\psi_{\text{eff}}^s(T)\rangle$ (equation (33)). The only limitation is that the effective initial state $|\psi_{\text{eff}}^s\rangle$ is not the ‘chosen system state’ $|\psi_s\rangle$. An imperfect clock limits our ability to prepare a perfect ‘initial state’, but does not affect the apparent evolution of this state.

We now turn to the main objection to the Conditional Probability Interpretation, made by Kuchar in 1992 [10] and recorded also in [5]. To quote Kuchar [5] “You always apply the conditional probability formula to calculate the conditional probability of a projector at a single instant of an internal clock time. You never apply it to answering the fundamental DYNAMICAL question of the internal Schrödinger interpretation, namely “If one finds the particle at $Q'$ at the time $T'$, what is the probability of finding it at $Q'$ at the time $T''$?”. By your formula, that conditional probability differs from zero only if $T' = T''$ and $Q' = Q''$. In short, your interpretation prohibits the time to flow and the system to move!”

This is an important objection, which Kuchar refers to as a “reduction ad absurdum of the condition probability proposal”. Pages response (also recorded in [5]) was “In my viewpoint, only quantities at a single instant of time are directly accessible, and so one cannot directly test the two-time probability you discuss.” We do not intend to defend this response here. It is the opinion of the author that, if Kuchar’s claim were correct, it would indeed amount to a reduction ad absurdum of the CPI. However, lets investigate how this conclusion was reached. The formula used by Kuchar was (equation (13.19) of [10], which we have adjusted to our notation, and to states that are not necessarily pure)

$$P_{\text{Kuchar}}(Q'' \text{ when } T'' | Q' \text{ when } T'; \hat{\rho}) = \frac{\text{Tr}(\hat{P}_{Q'T'} \hat{P}_{Q''T''} \hat{P}_{Q'T'} \hat{\rho}_{\text{ph}})}{\text{Tr}(\hat{P}_{Q'T'} \hat{\rho}_{\text{ph}})}$$

(40)
This is in disagreement with equations (20) - (23), and was never advocated by any proponent of the CPI. The first time equation (40) appeared was in Kuchar’s criticism of it [10]. It cannot be written in terms of physical measurement operators \(M_{Q''T''}\) and \(M_{Q'T'}\), so it cannot be written as \(P(Q''T''; \hat{\rho}_{Q'T'})\) (in the sense of equation (13)) for any state \(\hat{\rho}_{Q'T'}\). In classical terms, the numerator represents the proportion of the physical path on which \(Q'', T'', Q', T'\) all occur simultaneously. It is not surprising therefore, that it is zero unless \(Q' = Q''\) or that, for perfect clocks, (which Kuchar was referring to in the earlier quote) it is zero whenever \(T' \neq T''\!\).

Let us investigate then, the predictions of equations (20) - (23). We consider the more general probability \(P(\psi'_s; T_2 | \psi'_s; T_1; \hat{\rho})\) where the projection \(\hat{P}_{\psi'_s; T_1} = |\psi'_s\rangle\langle \psi'_s|\) on \(\mathcal{H}_s\) specifies the ‘initial system state’, and \(\hat{P}_T = |\psi_c(T)\rangle\langle \psi_c(T)|\) on \(\mathcal{H}_c\), just as in equation (30). The projection operator \(\hat{P}_{\psi'_s; T_1}\) specifies a unique state \(|\psi'_s\rangle|\psi_c(T_1)\rangle \equiv |\psi'_s, T_1\rangle\) in \(\mathcal{H}_{aux}\) so we have:

\[
\hat{P}_{\psi'_s; T_1} \hat{\rho} \hat{P}_{\psi'_s; T_1} \propto \hat{P}^{ph}_{\psi'_s; T_1}
\]

regardless of \(\hat{\rho}^{ph}\). Equation (23) gives:

\[
P(\psi'_s; T_2 | \psi'_s; T_1; \hat{\rho}) = \frac{\langle \psi'_s, T_2 | \hat{P}^{ph}_{\psi'_s; T_1} | \psi'_s, T_1 \rangle^2}{\sum_i \langle \psi'_s, T_2 | \hat{P}^{ph}_{\psi'_s; T_1} | \psi'_s, T_1 \rangle^2}
\]

where \(i\) runs over a basis of \(\mathcal{H}_s\). Equation (13) allows us to write:

\[
\langle \psi'_s, T_2 | \hat{P}^{ph}_{\psi'_s; T_1} | \psi'_s, T_1 \rangle = \lim_{\tau \to \infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} da \langle \psi_c | e^{-iH_s(a+T_1-T_2)} | \psi_c \rangle \langle \psi'_s | e^{-iH_s a} | \psi'_s \rangle
\]

\[
= \lim_{\tau \to \infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} da f_c(a) \langle \psi'_s | e^{-iH_s(a+T_2-T_1)} | \psi'_s \rangle
\]

\[
= \langle \psi'_s | e^{-iH_s(T_2-T_1)} | \psi'_s \rangle
\]

where the LHS is evaluated in \(\mathcal{H}_{aux}\) and the RHS in \(\mathcal{H}_s\). The conditional probability is then given by:

\[
P(\psi'_s; T_2 | \psi'_s; T_1; \hat{\rho}) = \frac{|\langle \psi'_s | e^{-iH_s(T_2-T_1)} | \psi'_s \rangle|^2}{\langle \psi'_s | \psi'_s \rangle}
\]

It is an exact transition probability in \(\mathcal{H}_s\) between the out-state \(|\psi'_s\rangle\) and the effective in-state \(|\psi'_s \rangle\). Again, the choice of clock state \(|\psi_c\rangle\) determines the choice of effective in-state \(|\psi'_s \rangle\), which undergoes an ‘evolution’ with respect to clock time that is indistinguishable from Schrödinger evolution in \(\mathcal{H}_s\). It is the physical projection operator \(\hat{P}^{ph}\) in equation (13) (absent in Kuchar’s proposal (10)) which imposes the correlations between clock and state that dictate this effective Schrödinger evolution in \(\mathcal{H}_s\).

This shows that the CPI, at least in its refined form, is perfectly applicable to comparing more than one clock time, and that it makes appropriate predictions in such cases. We now turn our attention to systems which are ‘unbounded’ in the sense that \(E = 0\) is among the continuous spectrum of the Hamiltonian.
4. The Refined CPI for ‘Unbounded’ systems.

When $E = 0$ is among the continuous spectrum of $\hat{H}$ the ‘eigenstates’ $|E, k\rangle$ are generalized states. They are only $\delta$-function normalisable:

$$\langle E, k|E', k'\rangle = 2\pi \delta(E - E')\delta(k - k')$$

where the index $k$ (possibly dependent on $E$, and possibly discrete) resolves the degeneracy of the relevant eigenspace (the factor $2\pi$ is for later convenience). Hence, none of the physical states $|0, k\rangle$ are actually in $\mathcal{H}_{aux}$ - they all have infinite norm!

Accordingly, equation (13) gives zero in the limit for any $|\psi\rangle \in \mathcal{H}_{aux}$. We could choose to continue regardless (as was advocated in the original treatments of the CPI - see pg 151 of [8] for a discussion of this point), accepting that equations such as (26) and (42) will always be of the form $0 = 0$. Physically, this is not entirely unreasonable - it can be traced back to the fact that, if the physical path is infinite in length, then the proportion associated with any finite portion (such as our lifetime) will be zero, even though questions about that portion may be physically consistent. That said, the absence of a finite inner product on physical states, and the ill-defined nature of equations such as (13), is at best disconcerting, and a more rigorous treatment is clearly desirable. This task has been considered by various authors [25–34] under various names - the ‘induced’ or ‘Rieffel induced’ inner product, ‘Refined Algebraic Quantization’, ‘Group Averaging’ or the ‘Spectral Analysis’ inner product. While research is ongoing on these topics (see [33] for a progress report) a substantial level of rigor has already been achieved, and the equivalence and uniqueness of these procedures has been established for typical cases [31, 32]. We will present only a brief outline of the induced Hilbert space here - the reader is referred elsewhere [31–34] for more detail and rigor.

Loosely, the induced Hilbert space $\mathcal{H}_{ph}$ is constructed from the generalized states $|0, k\rangle$ by replacing the inner product (47) with the definition:

$$\langle 0, k|0, k'\rangle_{ph} \equiv \delta(k - k')$$

which is equation (17) ‘divided by $2\pi \delta(0)$’. More generally, write two generalized eigenstates $|\phi\rangle_{ph}, |\psi\rangle_{ph}$ in terms of normalisable states $|\phi_0\rangle, |\psi_0\rangle$ in $\mathcal{H}_{aux}$ through the action of the ‘operator’ $\hat{P}^{ph}$ (no longer strictly a projection on $\mathcal{H}_{aux}$):

$$|\phi\rangle_{ph} = \hat{P}^{ph}|\phi_0\rangle = \int dE dk |E, k\rangle \delta(E)\langle E, k|\phi_0\rangle$$

Whereas $\langle \hat{P}^{ph}\psi_0|\hat{P}^{ph}\phi_0\rangle$ contains the infinite factor $2\pi \delta(0)$, we can write the finite ‘induced’ inner product on $\mathcal{H}_{ph}$ as:

$$\langle \psi|\phi\rangle_{ph} \equiv \langle \psi|\hat{P}^{ph}|\phi\rangle_{aux} = \int dE \delta(E) \int dk \langle \psi_0|E, k\rangle \langle E, k|\phi_0\rangle$$

which is independent of the particular states $|\psi_0\rangle, |\phi_0\rangle$ chosen to represent $|\psi\rangle_{ph}$ and $|\phi\rangle_{ph}$. In many situations (see [32–34] for instance) $\hat{P}^{ph}$ can also be written in terms of the ‘group averaging’ procedure, as:

$$\hat{P}^{ph}|\phi\rangle = \int da e^{-iH_a}|\phi\rangle$$

and equation (51) becomes:

$$\langle \phi | \psi \rangle_{\text{ph}} = \int \text{d}a \langle \phi | e^{-i\hat{H}a} | \psi \rangle_{\text{aux}}$$

Equation (51) can be compared with equation (13). It projects out the zero-frequency component of the Schrödinger-evolved state. To gain more familiarity with equation (51) consider $\mathcal{H}_{\text{aux}} = \mathcal{H}_c \otimes \mathcal{H}_s$ and consider system and clock states contained entirely within the continuous spectrum of $\mathcal{H}_s$, $\mathcal{H}_c$ respectively. Then we can decompose the Schrödinger states $|\psi_s(a)\rangle = e^{-i\hat{H}_a}|\psi_s\rangle$ and $|\psi_c(a)\rangle = e^{-i\hat{H}_a}|\psi_c\rangle$ in terms of their frequencies as:

$$|\psi_s(a)\rangle = \int \frac{\text{d}E}{2\pi} \tilde{\psi}_s(E)e^{-iEa} \quad |\psi_c(a)\rangle = \int \frac{\text{d}E}{2\pi} \tilde{\psi}_c(E)e^{-iEa}$$

In terms of these we find:

$$\hat{P}^\text{ph} |\psi_c\rangle |\psi_s\rangle = \int \text{d}a \frac{\text{d}E}{2\pi} \frac{\text{d}E'}{2\pi} \tilde{\psi}_c(E)\tilde{\psi}_s(E')e^{-i(E+E')a} \quad \text{(54)}$$

which is the continuum equivalent of equation (36).

Given a suitable projection operator $\hat{P}_A$ on $\mathcal{H}_{\text{aux}}$, we can define $\hat{P}^\text{ph}_A$ on $\mathcal{H}_{\text{ph}}$ by:

$$\langle \psi | \hat{P}^\text{ph}_A | \phi \rangle_{\text{ph}} \equiv \langle \psi_0 | \hat{P}^\text{ph}_A \hat{P}_A \hat{P}^\text{ph} | \phi_0 \rangle_{\text{aux}} \quad \text{(56)}$$

$$= \int \text{d}EdE' \delta(E)\delta(E') \int \text{d}kd\kappa' \langle \psi_0 | E\kappa | \hat{P}_A | E'\kappa' \rangle_{\text{aux}} \langle E'\kappa' | \phi_0 \rangle_{\text{aux}} \quad \text{(57)}$$

$$= \int \text{d}a \langle \psi_0 | e^{-i\hat{H}_a} \hat{P}_A e^{-i\hat{H}_a} | \phi_0 \rangle_{\text{aux}} \quad \text{(58)}$$

where $\hat{P}_A$ is ‘suitable’ if equations (57) and (58) converge for all $|\phi_0\rangle$, $|\psi_0\rangle$ in $\mathcal{H}_{\text{aux}}$ (or in some suitably large subspace $\Phi$ of $\mathcal{H}_{\text{aux}}$ [31]). This requirement on $\hat{P}_A$ excludes the possibility that $\hat{P}_A$ already commutes with the constraint. (It corresponds physically to the requirement that $\hat{P}_T$ project onto a ‘finite portion of the physical path’.) If $[\hat{P}_A, \hat{H}] = 0$, then $\langle E, \kappa | \hat{P}_A | E', \kappa' \rangle_{\text{aux}}$ contains a factor of $\delta(E - E')$ and (57) becomes ill-defined. An alternative way to induce operators on $\mathcal{H}_{\text{ph}}$ is to consider only operators which commute with $\hat{H}$ and to define $\hat{P}^\text{ph}_A$ by omitting one of the factors of $\hat{P}^\text{ph}$ in (56). That is effectively the strategy in [31] for instance. That strategy is consistent with the strategy above, in the sense that, given an operator $\hat{P}_A$ satisfying the requirement above, we can define an operator $\hat{P}^\text{new}_A$ on $\mathcal{H}_{\text{aux}}$ by

$$\langle \psi_0 | \hat{P}^\text{new}_A | \phi_0 \rangle_{\text{aux}} = \int \text{d}EdE' \delta(E - E') \text{d}kd\kappa' \langle \psi_0 | E\kappa | \hat{P}_A | E'\kappa' \rangle_{\text{aux}} \langle E'\kappa' | \phi_0 \rangle_{\text{aux}} \quad \text{(59)}$$

This satisfies $[\hat{H}, \hat{P}^\text{new}_A] = 0$, making it suitable for the procedure described in [31]. That procedure applied to $\hat{P}^\text{new}_A$ then leads to the operator $\hat{P}^\text{ph}_A$ on $\mathcal{H}_{\text{ph}}$ as defined above.

\footnote{When necessary, equation (52) is understood as $\lim_{\epsilon \to 0^+} \int \text{d}a g_\epsilon(a) \langle \phi | e^{-i\hat{H}a} | \psi \rangle_{\text{aux}}$, where $g_\epsilon(a)$ is a positive integrable function for which $\lim_{\epsilon \to 0^+} g_\epsilon(a) = 1$ for all $a$.}
The definition in equations (57), (58) above is more suitable for our purposes, since we wish to construct physical operators $\hat{P}_T^{ph}$ from clock projection operators $\hat{P}_T$ satisfying $[\hat{P}_T, \hat{H}] \neq 0$.

Operators such as $\hat{P}_{BC}^{ph}$ are obtained by replacing $\hat{P}_A$ in equations (56) - (58) with $\hat{P}_B \hat{P}_C$ while products $\hat{P}_B^{ph} \hat{P}_C^{ph}$ of physical projection operators can be obtained by replacing $\hat{P}_A$ in equations (56) - (58) with $\hat{P}_B \hat{P}_C^{ph}$, to obtain:

$$\langle \psi | \hat{P}_B^{ph} \hat{P}_C^{ph} | \phi \rangle_{ph} \equiv \langle \psi_0 | \hat{P}_B^{ph} \hat{P}_C^{ph} \hat{P}_C^{ph} | \phi_0 \rangle_{aux}$$

$$= \int d\alpha' d\alpha'' \langle \psi_0 | e^{-i\hat{H}_a} \hat{P}_B e^{-i\hat{H}_a} \hat{P}_C e^{-i\hat{H}_a'} | \phi_0 \rangle_{aux}$$

These equations extend to density operators in the obvious way:

$$\text{Tr}_{ph}(\hat{\rho}) = \text{Tr}_{aux}(\hat{\rho}_0) = \int \text{d}a \text{Tr}_{aux}(e^{-i\hat{H}_a}\hat{\rho})$$

$$\text{Tr}_{ph}(\hat{P}_A^{ph} \hat{\rho}) = \text{Tr}_{aux}(\hat{P}_A^{ph} \hat{P}_B^{ph} \hat{\rho}_0) = \int \text{d}a' \text{Tr}_{aux}(e^{-i\hat{H}_a} \hat{P}_A e^{-i\hat{H}_a'} \hat{\rho})$$

The formulae of Section 3 (with $\hat{P}_B^{ph} \hat{P}_C^{ph}$ identified with $\hat{P}_B^{ph}$ throughout) now carry over almost unchanged to the unbounded case. For instance:

$$P(A \text{ when } B; \hat{\rho}) = \frac{\text{Tr}_{aux}(\hat{P}_B^{ph} \hat{P}_A^{ph} \hat{P}_B^{ph} \hat{\rho}_0)}{\text{Tr}_{aux}(\hat{P}_B^{ph} \hat{P}_B^{ph} \hat{\rho}_0)}$$

The derivation of equation (29) remains unchanged. If we assume equations (30) and (31), then we are again lead to equation (32), with the effective system state $|\psi_s^{eff}(T)\rangle$ in $\mathcal{H}_s$ given by:

$$|\psi_s^{eff}(T)\rangle = \int \text{d}a f_c(a) |\psi_s(T + a)\rangle = e^{-i\hat{H}_T} |\psi_s^{eff}\rangle$$

which is just equation (33) without the factor $\frac{1}{2}$. It is generally straightforward to find a clock state $|\psi_c\rangle$ such that $|f_c(a)| = |\langle \psi_c | e^{-i\hat{H}_a} |\psi_c\rangle|$ is integrable. Such a clock can be said to ‘read zero for a finite Schrödinger time’. This is consistent with the ‘good clock requirement’, which requires also that $|f_c(a)|$ be sharply peaked about $a = 0$. It ensures that $\hat{P}_T$ is ‘suitable’ in the sense of equations (57) and (58), and hence ensures that $|\psi_s^{eff}\rangle$ has finite norm in $\mathcal{H}_s$.

Equation (22) is also unchanged, and leads to equation (40) just as in the ‘bounded’ case. The LHS of equation (43), which is evaluated in $\mathcal{H}_{aux}$, is by definition the induced inner product $\langle \psi_{out}, T_2 | \psi_{in}, T_1 \rangle_{ph}$ between $|\psi_{in}, T_1 \rangle_{ph}$ and $|\psi_{in}, T_1 \rangle_{ph}$. Equation (45) shows that this equates to the S-Matrix element $\langle \psi_{s}^{out} | e^{-i\hat{H}_s(T_2 - T_1)} |\psi_{s}^{in}\rangle$ in $\mathcal{H}_s$ between the system states $|\psi_{s}^{in}\rangle$ and $|\psi_{s}^{out}\rangle$.

The simplest application of this construction is to Parametrized Particle Dynamics, with $\hat{H}_s = \hat{H}(\hat{p}_t, \hat{x}_t)$, and $\hat{H}_c = \hat{p}_t = -i\frac{\partial}{\partial t}$. The system Hilbert space $\mathcal{H}_s$ is then the
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standard Hilbert space of non-relativistic quantum mechanics, while the clock Hilbert space \( \mathcal{H}_c \) contains square integrable functions of \( t \). The clock function \( f_c(a) \) is given by:

\[
f_c(a) = \int dt' \psi_c^*(t') \psi_c(t' - a) \tag{66}
\]

where \( \psi_c(t) = \langle t | \psi_c \rangle \) is the initial clock state in ‘position representation’ (in \( \mathcal{H}_c \)). In the ideal clock limit \( \psi_c(t) \to \delta(t) \), so that \( f_c(a) \to \delta(a) \) and equation (66) gives \( |\psi_{\text{eff}}\rangle = |\psi_s\rangle \). Conventional quantum mechanics is exactly retrieved in this limit.

We have now shown how the CPI can be applied successfully whenever \( E = 0 \) is among the continuous or the discrete spectrum of the Hamiltonian \( \hat{H} \). These cases are of course not exhaustive - The \( E = 0 \) eigenstates of the Hydrogen atom being an example that is neither of these cases. Constructing an ‘induced inner product’ for these more difficult cases is considered in [31]. We now consider briefly how the CPI connects with other ‘timeless interpretations’ of the Hamiltonian constraint. At the time of the original debate about the CPI [9, 10] there were two other prominent attempts at understanding the Hamiltonian constraint - Hartle’s ‘consistent histories’ approach [18,19], and Rovelli’s ‘Evolving Constants of Motion’ [20,21,23] (see also [22,24] for early criticisms). The consistent histories approach has been extensively developed since then [12–16], while the work of Marolf [28–30] refined and developed the ‘evolving constants’ approach. Marolf’s construction is in many ways similar to the refined CPI presented in this Section. The induced inner product is used to construct operators on the induced Hilbert space; the steps from \( \omega \to \Omega \to \Omega_{\text{phys}} \) in [30] for instance directly parallel the construction \( \hat{P}_A \to \hat{P}_A^{\text{new}} \to \hat{P}_A^{\text{phys}} \) described after equation (59). However, the operators used to ‘keep time’ in [28,30] are different to those used here. They choose a configuration space variable \( Q \) and seek answers to the question “what is the value of \( A \) when \( Q = \tau \)?”. This involves constructing a time projection operator from the spectral projections of the Hamiltonian-evolved operator \( \hat{Q}(t) \), while the integral over \( Q \) plays a role analogous to that of \( \mathcal{H}_c \). This agrees with our construction for the case of Parametrized Particle Dynamics, but is more restrictive in general.

We have considered the role of clock projection operators \( \hat{P}_T \) throughout this paper, in order to describe how equation (16) can be used to answer questions of the form “What is the probability of \( A \) when the clock reads \( T \)” and to describe how, in conjunction with standard logical operations such as and and given it can be used to tackle more general questions, such as the two-time probability described in equation (20). However, the formalism is of course more general than this, and needn’t be restricted to questions involving specified ‘clock times’. A projection \( \hat{P}_V \) onto a region \( V \) of configuration space could be used in equation (15) for instance, to associate a probability with a given region of phase space, answering the quantum equivalent of the question “what proportion of the observers worldline is in \( V \)”.

The most general probability that can be constructed from the rules described in Section 3 is of the form:

\[
P_{\alpha_0,A} = \frac{\Tr(\hat{C}^\dagger_{\alpha_0} \hat{C}_{\alpha_0} \hat{\rho})}{\sum_{\alpha \in A} \Tr(\hat{C}_{\alpha_0}^\dagger \hat{C}_{\alpha_0} \hat{\rho})} \tag{67}
\]
where \( \hat{C}_\alpha = \hat{M}_{\alpha_1} \hat{M}_{\alpha_2} \ldots \) (68)

and the operators \( \hat{M}_{\alpha_i} \) are physical measurement operators. Although the individual operators \( \hat{C}_\alpha \) do not commute with the constraint (and are not hermitian), \( \hat{C}_\alpha^\dagger \hat{C}_\beta \) commutes with the constraint for any \( \alpha, \beta \). The operators \( \hat{C}_\alpha \) can be identified with the class operators of the consistent histories approach \([12, 18, 19]\), allowing connection to be made between the CPI and the consistent histories approach. (We haven’t considered the decoherence functional \( \text{Tr}(\hat{C}_\alpha^\dagger \hat{C}_\beta \hat{\rho}) \) here, although we acknowledge that in specific cases this should be considered before treating the ‘histories’ as a family of decoherent alternatives.) Decoherent histories approaches have been developed based on the induced inner product \([13, 14]\) and on the Klein Gordon inner product \([12, 16]\). The detailed connection between the CPI and these specific approaches is yet to be investigated.

Another point worth addressing is the fact that in practice the ‘system’ \( \mathcal{H}_s \) and the ‘clock’ \( \mathcal{H}_c \) do not together constitute the whole universe. Consider for instance, replacing equation (24) with:

\[
\hat{H}_\text{tot} \approx \hat{H}_s + \hat{H}_c + \hat{H}_\text{rest}
\]

where \([\hat{H}_\text{rest}, \hat{H}_s] = 0 = [\hat{H}_\text{rest}, \hat{H}_c]\). This results in the minor change:

\[
P(A \text{ when } T) = \text{Tr}_s(\hat{P}_s e^{-i\hat{H}_c T} \hat{\rho}^s e^{i\hat{H}_c T}) \quad \text{where} \quad \hat{\rho}^s = \frac{\text{Tr}_\text{rest}(\hat{P}_0 \hat{\rho}^{ph})}{\text{Tr}_\text{aux}(\hat{P}_0 \hat{\rho}^{ph})}
\]

(70)

The system still behaves as if \( \hat{\rho}_s \) is Schrödinger evolved in \( \mathcal{H}_s \) with the only difference being the trace over \( \mathcal{H}_\text{rest} \) in (70). We might ask whether this trace can destroy the correlations between clock and system that are necessary for the clock to ‘keep time’. Fortunately this is not so. Consider for instance the simple case described in equation (31), with \( \hat{\rho}^{ph} = \hat{P}^{ph} |\psi_s\rangle |\psi_c\rangle \langle \psi_c| \hat{P}^{ph} \). The ‘effective initial density operator’ \( \hat{\rho}^s \) in equation (70) is then given by:

\[
\hat{\rho}^s \propto \int da \, f(a) f^*(a') \text{Tr}_{\text{rest}}(e^{-i\hat{H}_c(a-a')} \langle \psi_s| \hat{P}^{ph} |\psi_s\rangle \langle \psi_s| \hat{P}^{ph} |\psi_s\rangle )
\]

(71)

If there is no trace over \( \mathcal{H}_\text{rest} \) then equation (71) factorizes, to give \( \hat{\rho}^s \propto |\psi^\text{eff}_{s}\rangle \langle \psi^\text{eff}_{s}| \) as in equation (32). If on the other hand \( \mathcal{H}_\text{rest} \) is very large as is typical in everyday systems, then we would expect in general that \( \text{Tr}_{\text{rest}}(e^{-i\hat{H}_c(a-a')} \) would be sharply peaked about \( a = a' \). Then \( \hat{\rho}^s \) is given, up to an overall factor, by:

\[
\hat{\rho}^s \approx \int da |f(a)|^2 |\psi_s(a)\rangle \langle \psi_s(a)|
\]

(72)

This state is no longer pure, but is still a good approximation of the initial system state whenever \( f(a) \) is sharply peaked around \( a = 0 \). The trace over \( \mathcal{H}_\text{rest} \) has not destroyed the correlations necessary for the clock to ‘keep time’, because these correlations are established in \( \hat{P}_0 \hat{\rho}^{ph} \). Since the trace is taken over \( \hat{P}_0 \hat{\rho}^{ph} \) and not over \( \hat{\rho}^{ph} \) itself, then it does not affect time-keeping. For ‘repeating’ clocks, equation (72) is in some ways more natural than equation (32). Suppose for instance that \( |f(a)| \)
was sharply peaked about two different values of $a$; such a clock would ‘read zero on two occasions’. Then equation (72) registers these occasions as classical alternatives, whereas equation (32) would include interference terms between the two occasions.

5. Conclusion

The Conditional Probability Interpretation (CPI) has been reviewed, and minor refinements proposed. We have explained how, at least in it’s refined form, the CPI is capable of answering various past criticisms [9–11]. In particular, questions involving more than one clock time were described in detail, resolving the problems raised in Kuchař’s “reduction ad absurdum” [5,10]. In the case of Parametrized Particle Dynamics, conventional quantum mechanics was exactly recovered in the ideal clock limit. Situations were addressed where $E = 0$ was among the continuous spectrum of the Hamiltonian. The induced inner product [25–29,31] was used to construct the physical Hilbert space $\mathcal{H}_{\text{ph}}$ from the generalized eigenvectors in (the topological dual of) $\mathcal{H}_{\text{aux}}$. This allowed the CPI to be applied to these ‘continuous-spectrum’ cases in a more rigorous fashion than that described previously [5–8]. This induced construction was described in outline only here - more rigor in that construction is possible, desirable, and current [31–33]. A useful feature of this induced construction, particularly in conjunction with group averaging techniques [32–34] is that, once the existence of $\mathcal{H}_{\text{ph}}$ has been established, we can proceed to work entirely in $\mathcal{H}_{\text{aux}}$. We needn’t solve the eigenvalue equation directly, since this is achieved implicitly through the integral representation of $\hat{P}_{\text{ph}}$. Nor must we find operators $\hat{P}_A$ which have clear physical meaning and which directly commute the Hamiltonian. The definition (52) allows us to define such physical operators implicitly, and helps to suggest their physical interpretation. The induced Hilbert space and group averaging techniques are also prominent in current approaches to quantum gravity [17,34,35,39,40], while the auxiliary Hilbert space plays a similar role to that played here, suggesting that the CPI could be ideally suited for interpreting the role of time in those theories.

The Conditional Probability Interpretation does not require a ‘global time coordinate’ on the configuration space, or a ‘time operator’ $\hat{T}$ on the total Hilbert space. The choice of clock projection operators $\hat{P}_T$ can be tailored to the subsystem $\mathcal{H}_s$ we are interested in, just as in ordinary physics, where different time-keeping devices (atomic clocks, wrist-watches, planetary motions, or the universes scale factor $a$) are appropriate for measuring change in different situations. Neither the physical Hilbert space $\mathcal{H}_{\text{ph}}$ nor the auxiliary Hilbert space $\mathcal{H}_{\text{aux}}$ depend in any way on our choice of time-keeping device. This flexibility in the choice of time-keeping device allows the CPI to be fully compatible with situations where no unique ‘time coordinate’ exists, thus avoiding the “multiple choice” and “global time” problems described in [10]. Indeed, equation (10) allows us to give meaning to the statement ‘$A$ when $B$’ even when neither $A$ nor $B$ are ‘clocks’ (in the sense of equation (25)). The apparent unitary evolution of $\rho_s(T)$ in $\mathcal{H}_s$ stems directly from equation (25). The Hamiltonian constraint then forces the
correlations between clock and system that ensure the effective Schrödinger-evolution in $\mathcal{H}_s$. This is why, as observers within a constrained system, the correlations between ourselves and the world around us allow us to observe changes in that world, and why isolated subsystems change with respect to ‘clock time’ in a way that is compatible with the time-dependent Schrödinger equation.

6. References

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