Fluid-electromagnetic helicities and knotted solutions of the fluid-electromagnetic equations

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Abstract: In this paper we consider an Euler fluid coupled to external electromagnetism. We prove that the Hopfion fluid-electromagnetic knot, carrying fluid and electromagnetic (EM) helicities, solves the fluid dynamical equations as well as the Abanov Wiegmann (AW) equations for helicities, which are inspired by the axial-current anomaly of a Dirac fermion. We also find a nontrivial knot solution with truly interacting fluid and electromagnetic fields. The key ingredients of these phenomena are the EM and fluid helicities. An EM dual system, with a magnetically charged fluid, is proposed and the analogs of the AW equations are written down. We consider a fluid coupled to a nonlinear generalizations for electromagnetism. The Hopfions are shown to be solutions of the generalized equations. We write down the formalism of fluids in 2+1 dimensions, and we dimensionally reduce the 3+1 dimensional solutions. We determine the EM knotted solutions, from which we derive the fluid knots, by applying special conformal transformations with imaginary parameters on un-knotted null constant EM fields.

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1 Introduction

The theory of fluids, and the theory of electromagnetism, have both a long history. An interesting set of solutions to both are knotted solutions, often solutions with a nonzero Hopf index, or “Hopfions” (they appear in other areas of physics as well\(^1\)), though writing explicit forms of the solutions is often challenging. The solutions to Maxwell’s electromagnetism without sources were written by Rañada in [1, 2], after the early work by Trautman in [3].

There are solutions which are null in terms of the Riemann-Silberstein vector \(\vec{F} = \vec{E} + i\vec{B}\), i.e., \(\vec{F}^2 = 0\), easy to describe in terms of the Bateman construction [4], or even partially null solutions. These solutions are not solitonic, since electromagnetism is linear.

On the other hand, fluid dynamics is nonlinear, so it is even more difficult to describe, so fluid knots remained an abstract, yet fertile ground for a long time [5] (see the reviews [6–8] and the book [9]). Only relatively recently we had experimental observation of fluid knots [10] and numerical constructions in [11]. In [12, 13], based on a map between electromagnetism and fluid dynamics, null pressureless fluid knots were obtained.

Both electromagnetic and fluid knots also are characterized by the existence of conserved “helicities”, which are spatial integrals of Chern-Simons like terms, in the case of electromagnetism things like \(\int d^3x e^{ijk} A_i \partial_j B_k\), obtaining helicities \(H_{ab}\), \(a, b = e, m\) (electric, magnetic) so that \(H_{ee} = H_{mm} = 1\), and in the case of fluid dynamics, \(H_f = \int d^3x e^{ijk} v_i \partial_j v_k = \int d^3x \vec{v} \cdot \vec{\omega}\) (thus analogous to \(H_{mm}\) in electromagnetism), first defined by Moffat [14]. Some special solutions with fluid helicities were studied in [15, 16], and other solutions were found in [17–19].

In Bateman’s ansatz for electromagnetism, in terms of two complex functions \(\alpha\) and \(\beta\), it was shown in [20, 21] that one can find other solutions by replacing the pair \((\alpha, \beta)\) with holomorphic transformations for them, \(f(\alpha, \beta), g(\alpha, \beta)\). In [22], it was found that one can obtain “\((p, q)\)-knotted solutions” by applying the transformations \(\alpha \rightarrow (\alpha)^p, \beta \rightarrow (\beta)^q\) on solutions with \(H_{ee} = H_{mm} = 1\), and that such topologically nontrivial solution can be obtained from topologically trivial ones by acting with special conformal transformations with complex rather than real parameters.

At the next level in complexity, one can consider fluids coupled to electromagnetism in “magnetohydrodynamics”, and knots were considered as well, just that from the point of view of the what happens to the electromagnetic knots when they are coupled to fluid, see for instance [23–27].

However, an interesting case that was considered very recently by Abanov and Wiegmann [28, 29]\(^2\) is when an Euler fluid is coupled to external electromagnetic fields, so a case when there is no feedback from the fluid to electromagnetism via Ohm’s law, yet the conductivity is assumed to be finite, so that the electromagnetic fields do not solve the vacuum Maxwell’s equations. It was found that in this case, there is a total helicity that is conserved instead of individual fluid or electromagnetic helicities, \(H_{tot} = H_f + H_{mm} + 2H_{fm}\), and there is a chiral density and current that is conserved, and is sourced by the anomaly \(\vec{E} \cdot \vec{B}\), and

\(^1\)See for instance the website http://www.hopfion.com/.
\(^2\)For earlier work considering chiral liquids, i.e., liquids with chirality with respect to electromagnetism, and the conservation of helicities in this context, see [30–32].
that the fluid and cross helicities obey some equations we have dubbed Abanov-Wiegmann equations.

In this paper, we consider the consequences of this construction for knot type solutions, and construct new knot solutions to this Euler fluid coupled to external electromagnetic fields. We also explore generalizations of this set-up, for instance with nonlinear electromagnetic fields, and new helicities, as well as extending the formalism to 2+1 dimensions. We also explore the possibility of constructing solutions via conformal invariance.

The paper is organized as follows. In the next section we review the AW formalism. In particular we discuss the helicities and the axial anomaly. We also present a covariant formulation. In section 3 we show that the decoupled fluid-electromagnetic Hopfion configuration is a solution of the fluid and AW equations, and also find a general configuration in which fluid and electromagnetic helicities are truly coupled and interacting. We start by describing knot solutions of the free Maxwell equations. We then provide a map from EM knots to a fluid knots. We discuss the question of whether there are fluid solutions with \((p,q)\) helicities. Next we prove that the fluid Hopfion configuration is a solution of the AW equations. Finally we find the truly interacting knotted fluid-electromagnetic solution with helicities changing between the fluid and electromagnetic fields. Next in section 4 we propose a magnetically charged fluid. We start with dualizing Maxwell’s equations. We then write down the four helicities associated with the ordinary EM equations and their duals. Next we derive the magnetically analogs of the AW equations and discuss possible physical systems with magnetically charged fluids. Section 5 is devoted to coupling fluid to non-linear generalization of electromagnetism. In section 6 we discuss fluid EM systems in 2+1 dimensions and dimensionally reduce the 3+1 dimensional ones. In section 7 we show how to derive the EM knot solutions, which we later map to fluid knots, from constant null electric and magnetic fields by applying special conformal transformations with imaginary parameters. We conclude and suggest several open questions in section 8.

2 Review of 4 dimensional Euler-electromagnetic formalism

Consider an Euler fluid (inviscid, barotropic) composed of electrically charged particles of charge \(e\) (so electrons, or ions), interacting with external Maxwell fields via the Lorentz force, as in [28]. We consider that there is no feedback on the electromagnetic field from the fluid, so no interaction via Ohm’s law \(j = \sigma E\), though the conductivity \(\sigma\) of the fluid is assumed to be finite (otherwise, we would have \(E + v/c \times B = 0\), so \(E \cdot B = 0\)), i.e.,

\[
\frac{d}{dt} \rho + \nabla \cdot \rho \vec{v} = 0
\]

\[
(\partial_t + \vec{v} \cdot \nabla)m\vec{v} + \nabla \mu = e\vec{E} + (e/c)\vec{v} \times \vec{B},
\]

where the first is the continuity equation, and the second is the Euler equation with a Lorentz force source, for an inviscid fluid, with shear viscosity \(\eta = 0\), and barotropic, so \(p = p(\rho)\), or rather, since \(dp = \rho d\mu\), with \(\mu = \mu(\rho)\) the chemical potential, we have \(\nabla \mu = \frac{1}{\rho} \nabla p\). Note that we will consider later also non-barotropic fluids, and then \(\mu\) must be understood formally as \(\int dp/\rho\).
2.1 Helicities and anomaly

The fluid helicity is normally defined as

$$\mathcal{H}_f = \frac{1}{\Gamma^2} \int d^3 \mathbf{x} \mathbf{v} \cdot \mathbf{\nabla} \times \mathbf{v} = \frac{m^2}{\hbar^2} \int d^3 \mathbf{x} \mathbf{\bar{v}} \cdot \mathbf{\bar{\omega}}, \quad (2.2)$$

where $\Gamma$ is a normalization constant, here taken to be $= \hbar/m$ in order for $\mathcal{H}$ to be integer valued (which is needed for the case of the superfluid), and $\omega = \mathbf{\nabla} \times \mathbf{v}$ is the vorticity.\(^3\)

This helicity is conserved, $\frac{d}{dt} \mathcal{H}_f = 0$, for a fluid without external sources.

But in the case of a fluid with electromagnetic sources, as above, [28] argue that we should replace the momentum $\mathbf{p} = m \mathbf{v}$ inside $\mathcal{H}_f$ with the canonical momentum

$$\mathbf{\pi} = m \mathbf{v} + \mathbf{A}, \quad (2.3)$$

to obtain the total (fluid plus electromagnetic, specifically magnetic-magnetic) helicity,

$$\mathcal{H}_\text{tot} = \frac{1}{\hbar^2} \int d^3 x \mathbf{\bar{\pi}} \cdot \mathbf{\nabla} \times \mathbf{\bar{\pi}} = \frac{1}{\hbar^2} \int d^3 x \left[ m^2 \mathbf{\bar{\nu}} \cdot \mathbf{\bar{\omega}} + \mathbf{\bar{A}} \cdot \mathbf{\bar{B}} + 2 m \mathbf{\bar{v}} \cdot \mathbf{\bar{B}} \right] = \mathcal{H}_f + \mathcal{H}_\text{mm} + 2 \mathcal{H}_\text{fm}, \quad (2.4)$$

where $\int d^3 x \mathbf{\bar{\pi}} \cdot \mathbf{\bar{\omega}} = \int d^3 x \mathbf{\bar{A}} \cdot \mathbf{\bar{\omega}}$ (by partial integration), so the 2 cross-terms (“cross-helicities”) are equal, giving what we called $\mathcal{H}_\text{fm}$. Note that

$$\mathcal{H}_\text{mm} = \int d^3 x \mathbf{\bar{A}} \cdot \mathbf{\bar{B}} = \int d^3 x \epsilon^{ijk} A_i \partial_k A_k \quad (2.5)$$

is the electromagnetic helicity of magnetic-magnetic type. This total helicity is found to be conserved,

$$\frac{d}{dt} \mathcal{H}_\text{tot} = 0. \quad (2.6)$$

From now on, we put $\hbar = c = 1$.

The proof is easiest in a formulation in terms of 4-dimensional objects (though not Lorentz invariant) to be studied next. However, Abanov and Wiegmann derive the following equations for the fluid- and cross-helicity densities, what we will call Abanov-Wiegmann equations in the following:

$$\partial_t (m^2 \mathbf{\bar{v}} \cdot \mathbf{\bar{\omega}}) + \mathbf{\nabla} \cdot \left[ \mathbf{\bar{v}} (m^2 \mathbf{\bar{v}} \cdot \mathbf{\bar{\omega}}) + m \mathbf{\bar{\omega}} \left( \mu - \frac{m^2}{2} \right) + m \mathbf{\bar{v}} \times (\mathbf{\bar{E}} + \mathbf{\bar{v}} \times \mathbf{\bar{B}}) \right] = 0$$

$$\partial_t (m \mathbf{\bar{v}} \cdot \mathbf{\bar{B}}) + \mathbf{\nabla} \cdot \left[ m \mathbf{\bar{v}} (m \mathbf{\bar{v}} \cdot \mathbf{\bar{B}}) + \mathbf{\bar{B}} \left( \mu - \frac{m^2}{2} \right) - m \mathbf{\bar{v}} \times (\mathbf{\bar{E}} + \mathbf{\bar{v}} \times \mathbf{\bar{B}}) \right] + m \mathbf{\bar{\omega}} (\mathbf{\bar{E}} + \mathbf{\bar{v}} \times \mathbf{\bar{B}}) = 0, \quad (2.7)$$

from which one finds the “local anomaly equation”,

$$\dot{\rho}_A + \mathbf{\nabla} \cdot \mathbf{j}_A = 2 \mathbf{\bar{E}} \cdot \mathbf{\bar{B}}, \quad (2.8)$$

\(^3\)Note that [28] define it with an $m$, so $\omega = m \mathbf{\nabla} \times \mathbf{v}$, but we use the usual definition.
where the “fluid chirality density” is defined as
\[ \rho_A = m\vec{v} \cdot (m\vec{\omega} + 2\vec{B}) = \frac{\mathcal{H}_{\text{tot}}^{d} - \mathcal{H}_{\text{mm}}^{d}}{m}, \] (2.9)
i.e., the difference between the total helicity density and the magnetic-magnetic helicity density, while the fluid chirality current \( \vec{j}_A \) is found to be
\[ \vec{j}_A = \rho_A \vec{v} + (m\vec{\omega} + 2\vec{B}) \left( \mu - \frac{m\vec{v}^2}{2} \right) + m\vec{v}(\vec{E} + \vec{v} \times \vec{B}). \] (2.10)

### 2.2 4-dimensional formalism

The equations are much easier to derive in a 4 dimensional formulation, though without Lorentz invariance.

One first can check that the Euler equation, the second equation in (2.1), can be rewritten as
\[ \rho(\dot{\pi} - \vec{\nabla}\pi_0) - \rho \vec{v} \times (\vec{\nabla} \times \vec{\pi}) = 0, \] (2.11)
where \( \pi_0 \) is the Bernoulli function,
\[ \pi_0 = \Phi + A_0, \quad -\Phi = \mu + \frac{m\vec{v}^2}{2}. \] (2.12)

Then, defining the 4-current \( j^\mu = (\rho, \rho v^i) \) and canonical 4-momentum \( \pi^\mu = (\pi_0, \pi_i) \), we see that the above rewriting of the Euler equation, can be compactly rewritten as
\[ j^\mu \Omega_{\mu\nu} = 0, \quad \Omega_{\mu\nu} \equiv \partial_\mu \pi_\nu - \partial_\nu \pi_\mu, \] (2.13)
which is seen to be understood in form language. Indeed, the \( i \) component of this is the rewritten Euler equation, and the 0 component is \( v^i \times \vec{\nabla} \times \vec{\pi} \) times the same (taking into account that the second term vanishes, since we have \( \epsilon^{ijk} v^i v_j \)).

Then the helicity density 3-form is
\[ h = \pi \wedge d\pi = \pi \wedge \Omega. \] (2.14)
Its components are the total helicity density,
\[ \mathcal{H}_{\text{tot}}^{d} = h_0 = \vec{\pi} \cdot (\vec{\nabla} \times \vec{\pi}) = \rho_A + \vec{A} \cdot \vec{B}, \] (2.15)
and the total helicity flux,
\[ \vec{h} = \vec{\pi} \times (\vec{\pi} - \vec{\nabla}\pi_0) - \pi_0(\vec{\nabla} \times \vec{\pi}) = h_0 \vec{v} - (\vec{\nabla} \times \vec{\pi})(\vec{\pi} \cdot \vec{v} + \pi_0). \] (2.16)

Then the conservation of the total helicity density is almost trivial in form language,
\[ h = \pi \wedge d\pi = \pi \wedge d\Omega \Rightarrow dh = \Omega \wedge \Omega = 0, \] (2.17)
or in components, the continuity equation,
\[ h_0 + \vec{\nabla} \cdot \vec{h} = 0. \] (2.18)
Integrating it over space with vanishing boundary conditions at infinity and using Gauss’s law, we find the conservation law

$$\frac{d}{dt} H_{\text{tot}} = 0.$$ 

To find the chirality equation, consider the fluid chirality current form (based on the extension of the 0th components, the densities)

$$j_A = \pi \wedge d\pi - A \wedge dA = (\pi - A) \wedge (d\pi + dA).$$ \hspace{1cm} (2.19)

Then this gives the 3-vector fluid chirality current,

$$j_A^i = \epsilon^{ijk} [(\pi_0 - A_0)(d\pi)_{jk} + (d\pi)(A)_{jk} - (\pi_j - A_j)(d\pi)_{0k} + (d\pi)(A)_{0k}]$$

$$= - [\epsilon^{ijk} m\dot{v}^k (\nabla^k \Phi - 2E^k) + \Phi \epsilon^{ijk} m\dot{v}_j v_k + 2(dA)_{jk}],$$ \hspace{1cm} (2.20)

so

$$\vec{j}_A = m\vec{v} \times (m\dot{v} - \nabla \Phi - 2\vec{E}) - \Phi (m\vec{\omega} + 2\vec{B}).$$ \hspace{1cm} (2.21)

Then, by substituting $m\dot{v}$ from the Euler equation, we indeed obtain the 3-vector fluid chirality current (2.10).

3 Hopfion knotted solutions of the fluid + electromagnetism and Abanov-Wiegmann equations

We want to find solutions to the Euler+eletromagnetism equations, as well as to the Abanov-Wiegmann equations for the helicities. In particular, we would like to find knotted solutions, that have a nonzero Hopf number, which is usually associated with a nonzero electromagnetic magnetic-magnetic helicity (see for instance [22, 33]).

3.1 Electromagnetic knots

In [12, 13] a map was given between electromagnetism and a null ($\vec{v}^2 = 1$) pressureless fluid, which was used to map the electromagnetic knot into a fluid knot.

The electromagnetic knot in the Bateman formulation is written in terms of the complex Riemann-Silberstein vector

$$\vec{F} = \vec{E} + i\vec{B},$$ \hspace{1cm} (3.1)

with the Bateman ansatz

$$\vec{F} = \vec{\nabla} \alpha \times \vec{\nabla} \beta,$$ \hspace{1cm} (3.2)

with the two complex scalar fields $\alpha, \beta \in \mathbb{C}$.

In components, the ansatz is

$$E^i = \epsilon^{ijk} (\partial_j \alpha_R \partial_k \beta_R - \partial_j \alpha_I \partial_k \beta_I)$$

$$B^i = \epsilon^{ijk} (\partial_j \alpha_R \partial_k \beta_I - \partial_j \alpha_I \partial_k \beta_R),$$ \hspace{1cm} (3.3)

where the indices $I$ and $R$ refer to the imaginary and real parts, respectively.

The vacuum Maxwell’s equations of motion in terms of $\vec{F}$ are

$$\vec{\nabla} \cdot \vec{F} = 0; \quad \partial_t \vec{F} + i\vec{\nabla} \times \vec{F} = 0,$$ \hspace{1cm} (3.4)
where the first one is trivially satisfied by the Bateman ansatz and the second takes the form

\[ i\nabla \times (\partial_t \alpha \nabla \beta - \partial_t \beta \nabla \alpha) = \nabla \times \vec{F}, \]  
\[ (3.5) \]

which is satisfied if

\[ (\partial_t \alpha \nabla \beta - \partial_t \beta \nabla \alpha) = \vec{F}. \]  
\[ (3.6) \]

Solutions to this equation have necessarily a zero norm

\[ F^2 = (\partial_t \alpha \nabla \beta - \partial_t \beta \nabla \alpha)(\vec{\nabla} \alpha \times \vec{\nabla} \beta) = 0, \]  
\[ (3.7) \]

which implies that

\[ \vec{E} \cdot \vec{B} = 0 \quad E^2 - B^2 = 0. \]  
\[ (3.8) \]

Topological non-trivial solutions of Maxwell’s equations that are characterized by non-trivial helicity \( \mathcal{H}_{mm} \) defined in (2.5) were found in [1, 2]. The basic solution which carries \( \mathcal{H}_{mm} = 1 \) is given by

\[ \alpha = \frac{A - iz}{A + it}, \quad \beta = \frac{x - iy}{A + it}, \quad A = \frac{1}{2}(x^2 + y^2 + z^2 - t^2 + 1). \]  
\[ (3.9) \]

It is easy to check that if \( \alpha(x^\mu), \beta(x^\mu) \) is a solution of Maxwell equation then also

\[ g(\alpha(x^\mu), \beta(x^\mu)), h(\alpha(x^\mu), \beta(x^\mu)) \]  
\[ (3.10) \]

for \( g, h \) holomorphic functions are solutions. In particular if a solution with \( \alpha(x^\mu), \beta(x^\mu) \) carries a (1,1) helicity charges \( (\mathcal{H}_{mm}, \mathcal{H}_{ee}) \) defined in (2.5), (4.6),

\[ g(\alpha) = \alpha(x^\mu)^m, h(\beta(x^\mu)) = (\beta(x^\mu))^n. \]  
\[ (3.11) \]

are knotted solutions with charges \( (m, n) \).

One can write the knotted solutions also in terms of the electromagnetic 2-form field strength and its dual as (see, e.g. [33])

\[ F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu = \frac{1}{4\pi i} \frac{\partial_{\mu} \phi \partial_{\nu} \bar{\phi} - \partial_{\nu} \phi \partial_{\mu} \bar{\phi}}{(1 + |\phi|^2)^2} dx^\mu \wedge dx^\nu \]

\[ F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu = \frac{1}{4\pi i} \frac{\partial_{\mu} \bar{\theta} \partial_{\nu} \theta - \partial_{\nu} \bar{\theta} \partial_{\mu} \theta}{(1 + |\theta|^2)^2} dx^\mu \wedge dx^\nu \]

\[ \phi = \frac{Ax + t(A - 1) + i(tx - Ay)}{Ax + ty + i(A(A - 1) - tz)} \]

\[ \theta = \frac{Az + t(A - 1) + i(tz - Ay + i(A(A - 1) - tz)).}{} \]  
\[ (3.12) \]
3.2 From EM knots to fluid knots

In [12, 13] a derivation of fluid knots was worked by implementing a map from the EM knot solutions. This map is in fact a map between the energy momentum tensor of the EM theory $T_{\mu\nu}^{(EM)}$ and that of a perfect fluid $T_{\mu\nu}^{(fluid)}$. The latter takes the well known form

$$T_{\mu\nu}^{(fluid)} = \rho(u_\mu u_\nu) + P(g_{\mu\nu} + u_\mu u_\nu), \quad (3.13)$$

where $\rho$ is the fluid density, $u^\mu$ is the four velocity vector and $g_{\mu\nu}$ is the space-time metric.

The conservation of the energy momentum tensor takes the form

$$\partial^\mu T_{\mu\nu} = 0, \quad (3.14)$$

which implies that

$$u_\mu u_\nu \partial^\mu (\rho + P) + (\rho + P)(\partial^\mu u_\nu)u_\mu + (\rho + P)u_\mu \partial_\nu P = 0. \quad (3.15)$$

It turns out, as will be shown below, that the velocity four vectors that we will get from the EM knot configurations are null, namely $u^\mu \to v^\mu$,

$$v^\mu v_\mu = 0; \quad v^\mu = (1, \vec{v}); \quad (\vec{v})^2 = 1. \quad (3.16)$$

For such velocity four-vector the conservation of motion of $T_{\mu\nu}$ reduce to the continuity and Euler equations:

$$\partial_t \rho + \vec{\nabla} \cdot (\rho \vec{v}) = 0, \quad (3.17)$$

$$\rho \partial_t \vec{v} + \rho (\vec{v} \cdot \vec{\nabla}) \vec{v} + \vec{\nabla} P = 0. \quad (3.18)$$

For fluid with $\vec{\nabla} P = 0$, one gets exactly the same equations provided that we take the following map:

$$T_{00} = \frac{1}{2} (\vec{E}^2 + \vec{B}^2) \leftrightarrow \rho$$

$$T_{0i} = [\vec{E} \times \vec{B}]_i \leftrightarrow \rho v_i \quad (3.19)$$

$$T_{ij} = - \frac{1}{2} (\vec{E} \cdot \vec{B}) \delta_{ij} \leftrightarrow \rho v_i v_j \quad (3.20)$$

Solving this for $\vec{v}, \rho$, we obtain

$$\rho \leftrightarrow \frac{1}{2} (\vec{E}^2 + \vec{B}^2); \quad v_i \leftrightarrow \frac{[\vec{E} \times \vec{B}]_i}{\frac{1}{2} (\vec{E}^2 + \vec{B}^2)}, \quad (3.21)$$

which is valid only under the null condition $\vec{E} \cdot \vec{B} = 0$, $\vec{E}^2 - \vec{B}^2 = 0$, satisfied by the electromagnetic knot from the previous subsection.

It is now easy to check that for null EM fields that obey $\vec{E} \cdot \vec{B} = 0$, $\vec{E}^2 - \vec{B}^2 = 0$, we have $\vec{v}^2 = 1$, as follows:

$$v_i v^i = \frac{[E \times B]_i [E \times B]^i}{\rho^2} = \frac{\epsilon_{ijk} E^j B^k \epsilon^{ilm} E_l B_m}{E^2 B^2} = \frac{E^2 B^2 - (\vec{E} \cdot \vec{B})^2}{E^2 B^2} = 1. \quad (3.22)$$

\[ \text{---} \]
We recapitulate that $\rho$ and $\vec{v}$ that follow from this map from the null EM fields obey the continuity and usual Euler equations.

In terms of the Riemman-Silberstein vector $\vec{F}$, the velocity vector is given by

$$\vec{v} = \Im((\vec{F}^*) \times \vec{F}) = \frac{1}{i} \frac{(\vec{F}^*) \times \vec{F}}{|F|^2} = \frac{|\vec{E} \times \vec{B}|}{\frac{1}{2}(\vec{E}^2 + \vec{B}^2)}.$$ \hspace{1cm} (3.23)

We can now express also the vorticity in terms of $\vec{F}, \vec{F}^*$ as follows:

$$w^i = \epsilon^{ijk} \partial_j v_k = \frac{\partial_j F^i F^{*j} - \partial_j F^{*i} F^j}{|F|^2} - \frac{(F^{*i} F^j - F^{*j} F^i)}{(|F|^2)^2}.$$ \hspace{1cm} (3.24)

The total helicity of the system (2.4) is built from three terms

$$H_{\text{total}} = H_{mm} + H_{fm} + H_f$$ \hspace{1cm} (3.25)

By construction the fluid that follows from the EM knot has a non-trivial $H_{mm}$. On the other hand, since $\vec{v} \cdot \vec{B} = 0$, we have $H_{fm} = 0$.

The fluid helicity $H_f$ was defined in (2.2) and was argued to be conserved under certain conditions (while the total helicity is always conserved). The conservation follows from Euler’s equation; for a fluid without pressure and external force,

$$\partial_t (\vec{v} \cdot \vec{w}) = \vec{\nabla} \cdot \left( \frac{1}{2} (\vec{v})^2 \vec{w} - \vec{v} (\vec{w} \cdot \vec{v}) \right).$$ \hspace{1cm} (3.26)

This implies that indeed $H_f$ is conserved, provided the surface term in the corresponding integral vanishes. To check the conservation of the fluid helicity, we can substitute the expressions for $\vec{v}$ and $\vec{w}$ in terms of $\vec{F}, \vec{F}^*$, obtaining

$$\vec{v} \cdot \vec{w} = \frac{(\vec{F}^*) \times \vec{F}}{|F|^2} \cdot \left[ \frac{(\vec{F}^*) \times \vec{F}}{|F|^2} \right] = \epsilon^{ilm} F^i m (\partial_j F^{*i} F^j - \partial_j F^i F^{*j}) \frac{|F|^2}{|F|^2}.$$ \hspace{1cm} (3.27)

Next we integrate this result, and apply $\partial_t$, and we find that it yields a surface term.

### 3.3 The fluid Hopfion

Under the map (3.19), the basic EM Hopfion solution (3.9) transforms into a fluid knot solution, taking the form

$$v_x = \frac{2(y + x(t - z))}{1 + x^2 + y^2 + (t - z)^2}, \quad v_y = \frac{-2(x - y(t - z))}{1 + x^2 + y^2 + (t - z)^2},$$

$$v_z = \pm \sqrt{1 - v_x^2 - v_y^2} = \pm \frac{1 - x^2 - y^2 + (t - z)^2}{1 + x^2 + y^2 + (t - z)^2},$$

$$\rho = \frac{16(1 + x^2 + y^2 + (t - z)^2)^2}{(t^4 - 2t^2(x^2 + y^2 + z^2 - 1) + (1 + x^2 + y^2 + z^2)^2)^3}. \hspace{1cm} (3.28)$$

The velocity profiles at $t = 0$ for $z = 0$ and $x = 0$ are drawn in figure 1, and the density at $t = 0$ for $z = 0$ in figure 2.
Figure 1. Orthogonal sections of the velocity field for the Hopfion solution, on the $(x,y)$ plane (top) and $(y,z)$ plane (bottom). Using rotational symmetry in the $(x,y)$ directions the linked torus structure is apparent [12, 13].

Figure 2. The density of the basic (1,1) fluid Hopfion as a function of $x$ and $y$ for $t = 0, z = 0$.

Next we can determine the vorticity vector at $t = 0$,

\[
\begin{align*}
    w_x &= \frac{2 \left( y \left( x^2 + z^2 + 3 \right) - 2xz + y^3 \right)}{(x^2 + y^2 + z^2 + 1)^2}, \\
    w_y &= \frac{-2 \left( x \left( y^2 + z^2 + 3 \right) - 2yz + x^3 \right)}{(x^2 + y^2 + z^2 + 1)^2}, \\
    w_z &= \frac{4 \left( z^2 + 1 \right)}{(x^2 + y^2 + z^2 + 1)^2}.
\end{align*}
\]
The vorticity has a norm of
\[(\vec{w})^2 = \frac{4 (x^2 + y^2 + 4)}{(x^2 + y^2 + z^2 + 1)^2}, \tag{3.32}\]
and the density of the fluid helicity is
\[\rho_w = \vec{v} \cdot \vec{w} = \frac{4}{x^2 + y^2 + z^2 + 1}. \tag{3.33}\]

Next we calculate explicitly the vorticity \(\omega\) and \(h_f\) at nonzero times. We find
\[
\begin{align*}
  w_x &= \frac{2 (t^2 y + 2 t (x - y z) + y (x^2 + z^2 + 3) - 2 x z + y^3)}{(t - z)^2 + x^2 + y^2 + 1} \tag{3.34}
  \\
  w_y &= -\frac{2 (x ((t - z)^2 + y^2 + 3) + 2 y (z - t) + x^3)}{(t - z)^2 + x^2 + y^2 + 1} \\
  w_z &= -\frac{4 ((t - z)^2 + 1)}{(t - z)^2 + x^2 + y^2 + 1}.
\end{align*}
\]

The helicity density at nonzero times comes out to be
\[\rho_w = \vec{v} \cdot \vec{w} = \frac{4}{(t - z)^2 + x^2 + y^2 + 1}. \tag{3.35}\]

It turns out that the fluid helicity, which is the space integral of this density,
\[\mathcal{H} = \int d^3x \vec{v} \cdot \vec{w}, \tag{3.36}\]
diverges.

If one “normalize” the result by subtracting from the velocity components their asymptotic values, namely
\[\vec{v}_n = \vec{v} - \vec{v}_{asym}, \tag{3.37}\]
one gets that the helicity density (at t=0) takes the form
\[\rho_w = \vec{v}_n \cdot \vec{w} = \frac{4 \left( x^2 (z^2 - 1) + y^2 (z^2 - 1) + (z^2 + 1)^2 \right)}{(x^2 + y^2 + z^2 + 1)^2 \sqrt{(x^2 + y^2 + z^2 - 1)^2 + 4z^2}} \tag{3.38}\]
The corresponding space integral, the helicity, is still divergent.

It is interesting to note that if we define a deformed helicity density of the form \(\vec{p} \cdot \vec{w}\), then the correspondence helicity comes out to be finite
\[\mathcal{H}_d = \int d^3x \vec{p} \cdot \vec{w} = \frac{72 \pi^2}{5}. \tag{3.39}\]

This result is based on the values of \(\vec{p}\) and \(\vec{w}\) at \(t = 0\), but as expected it is not conserved in time.
3.4 Are there fluid knot solutions with higher \((p,q)\)?

Next, in analogy to the higher \((p,q)\) knots of the EM solution we would like to explore the possibility of having also higher fluid knot solutions with higher \((p,q)\) knot numbers. For concreteness we analyze the \((2,3)\) case. We start with the Bateman configuration that corresponds to the EM \((2,3)\) knot, namely

\[
\alpha = \frac{(-t^2 + x^2 + y^2 + (z + i)^2)^2}{(-t(t - 2i) + x^2 + y^2 + z^2 + 1)^2},
\]

\[
\beta = \frac{8(x - iy)^3}{(-t(t - 2i) + x^2 + y^2 + z^2 + 1)^3}.
\]

Using (3.3) we determine the electric and magnetic fields, which are written down in appendix A. It is straightforward to check explicitly that indeed \(\vec{E} \cdot \vec{B} = 0\) and \(\vec{E}^2 = \vec{B}^2\).

Next following the steps of (3.21) we determine the density and the velocity vector. At \(t = 0\) the density takes the form

\[
\rho(\vec{x}) = E^2 = B^2 = \frac{9216 (x^2 + y^2)^2 (2z^2 (x^2 + y^2 + 1) + (x^2 + y^2 - 1)^2 + z^4)}{(x^2 + y^2 + z^2 + 1)^{10}}.
\]

The density as a function of \((x, y)\) for \(z = 0\) and of \((x, z)\) for \(y = 0\) are given in figures 3a and b, respectively.

Here we faced a surprise. Whereas the density of the \((2,3)\) solution is different from the basic solution, the velocity components are identical to those of the \((1,1)\) basic fluid knot. In fact it is easy to show that the velocity vectors of all the \((p, q)\) knots are the same. This follows from the definition of the velocity vector (3.21) under the substitution of \(\alpha^p, \beta^q\).

\[
\vec{v}_{(pq)} = \frac{1}{i} \frac{(\vec{F})^* \times \vec{F}}{|F^2|} = \frac{(\nabla \alpha^p \times \nabla \beta^q)^* \times (\nabla \alpha^p \times \nabla \beta^q)}{(\nabla \alpha^p \times \nabla \beta^q)^* \cdot (\nabla \alpha^p \times \nabla \beta^q)} = \frac{(\nabla \alpha \times \nabla \beta)^* \times (\nabla \alpha \times \nabla \beta)}{(\nabla \alpha \times \nabla \beta)^* \cdot (\nabla \alpha \times \nabla \beta)} = \vec{v}_{(11)}.\]

Thus the \((p, q)\) knot is characterized by its density but not by the helicity, which is the same for all of them.
3.5 Fluid+electromagnetic Hopfion as a solution to electromagnetic coupled fluid equations

We have seen that the fluid Hopfion (3.28) is a solution of the usual Euler equations and the continuity equation, and the electromagnetic Hopfion is a solution of the Maxwell’s equations without source. But we want a solution of the continuity and Euler equation for the electromagnetic coupled fluid (2.1), with nonzero \((\vec{v}, \vec{E}, \vec{B})\).

As a first possibility, we consider the case that the fluid \(\vec{v}\) follows via the map (3.19) from the same \(\vec{E}, \vec{B}\) that appears on the right-hand side (as a source) of the fluid equations (2.1). The map leads to a null velocity field, \(v_\mu v^\mu = 0\), or \(v_i v^i = 1\).

But under the null condition, for the fluid \(\vec{v}\) interacting with electromagnetism \(\vec{E}, \vec{B}\), we get

\[
v_i = \frac{\epsilon^{ijk} E_j B_k}{B^2} \Rightarrow (\vec{v} \times \vec{B})^i = -\epsilon^{ijk} v_j B_k = -\frac{\epsilon^{ijk} \epsilon^{ilm} E_l B_mB_k}{B^2} = E_i, \quad (3.44)
\]

so \(\vec{E} + \vec{v} \times \vec{B} = 0\), which means that if the electromagnetic field is Bateman’s knot and the velocity field is the fluid knot derived from it, the source on the right-hand side of the basic equations (2.1) is zero, leaving us with the usual Euler equation, that was already satisfied due to our map. That means that now we have a solution of the combined system, with nontrivial \(\vec{v}, \vec{E}, \vec{B}\)!

3.6 Hopfion solution to Abanov-Wiegmann equations for the helicities

As was shown above, the fluid + electromagnetic Hopfion solves the continuity and Euler equations (2.1) for the case of a divergent-less pressure \(\nabla P = 0\). Since the Abanov-Wiegmann equations are derived from them, it follows that the same fluid + electromagnetic Hopfion solves them as well.

But the condition \(\nabla P = 0\) is too restrictive, so we ask: can we relax it, if we consider only the Abanov-Wiegmann equations for the helicity densities? The answer turns out to be yes, as we now show.

By construction from the map (3.21) we obtain easily that

\[
\vec{B} \cdot \vec{v} = 0, \quad (3.45)
\]

Furthermore, since the EM field are null, we have (3.44) which means that the Lorentz force (source of the fluid equations) vanishes,

\[
\vec{E} + \vec{v} \times \vec{B} = 0. \quad (3.46)
\]

Then the second equation in the Abanov-Wiegmann equations (2.7) takes the form

\[
\nabla \left[ \vec{B} \left( \mu - \frac{1}{2} m v^2 \right) \right] = 0. \quad (3.47)
\]

Inserting the map for the velocity, we find that the second term vanishes and thus the equation holds for fluids for which (note that \(\vec{v}^2 = 1\) for the fluid solution constructed...
as above, so $\nabla v^2 = 0$, and also $\nabla \cdot \vec{B} = 0$ from Maxwell’s equations satisfied by the electromagnetic knot

$$\frac{1}{\rho} \vec{B} \cdot \nabla p = 0. \quad (3.48)$$

Note that this condition is valid even for a non-barotropic fluid.

On the other hand from the first equation in (2.7), we get

$$\partial_t (m^2 \vec{v} \cdot \vec{\omega}) + \nabla \cdot \left[ \vec{v} (m^2 \vec{v} \cdot \vec{\omega}) + m \vec{\omega} \left( \mu - \frac{m \vec{v}^2}{2} \right) \right] = 0. \quad (3.49)$$

The third term includes two terms, one of which is proportional to $\nabla \cdot \vec{w} = 0$, and a term $\nabla (v^2) = 0$, so the equation takes the form

$$\partial_t (\rho_w) + \nabla \cdot [\vec{v} (\rho_w)] + m \vec{\omega} \cdot \nabla \mu = 0, \quad (3.50)$$

where

$$\rho_w \equiv m^2 \vec{v} \cdot \vec{\omega} \equiv h_0^f \quad (3.51)$$

is the fluid helicity density.

We now calculate explicitly the vorticity $\omega$ and $h_0^f$, and check explicitly the above equation. We find, as for the plain fluid Hopfion,

$$w_x = \frac{2 \left( t^2 y + 2 t (x - y z) + y (x^2 + z^2 + 3) - 2 x z + y^3 \right)}{(t - z)^2 + x^2 + y^2 + 1} \quad (3.52)$$

and, for the velocity solution with minus sign in $v_z$ in (3.28), we find

$$\rho_w = \frac{4}{(t - z)^2 + x^2 + y^2 + 1}. \quad (3.53)$$

Inserting this expression into (3.50), we find that

$$\partial_t (\rho_w) + \nabla \cdot [\vec{v} (\rho_w)] = 0. \quad (3.54)$$

Thus the first equation in (2.7) is fulfilled, provided the pressure obeys

$$\vec{\omega} \cdot \frac{\nabla p}{\rho} = 0. \quad (3.55)$$

Again, we understand this to be true even in the non-barotropic case, since the terms with $\mu = \int dp/\rho$ have already cancelled.

Thus we conclude that indeed the null fluid solution combined with the null electromagnetic configuration form a solution of the Abanov-Wiegman helicity equations, provided
that the gradient of the pressure is perpendicular to both $\vec{B}$ and $\vec{w}$, which means a potentially more general solution (with pressure) than the one for the (2.1) equations.

We can ask: can the Hopfion itself be extended in the above way? The answer is that it cannot be extended with a nonconstant barotropic pressure, $p = p(\rho)$. For a null fluid obtained from electromagnetism (so $\vec{E} \cdot \vec{B} = \vec{E}^2 - \vec{B}^2$ and $\vec{\nabla} \cdot \vec{E} = \vec{\nabla} \cdot \vec{B} = 0$), we obtain

$$\vec{\omega} \cdot \vec{\nabla} p = \frac{dp}{\rho d\rho} \vec{\omega} \cdot \vec{\nabla} \rho = \frac{dp}{\rho d\rho} \frac{1}{E^2} \left( (\vec{E} \cdot \vec{\nabla})\vec{B} - (\vec{B} \cdot \vec{\nabla})\vec{E} \right) \cdot \vec{\nabla} E^2,$$

(3.56)

which is nonzero in general. Moreover, even for the Hopfion solution, we obtain, at $t = 0$,

$$\vec{\omega} \cdot \vec{\nabla} \rho = \frac{-16z}{(x^2 + y^2 + z^2 + 1)^2} \left[ -3 + \frac{4}{(x^2 + y^2 + z^2 + 1)^2} \right] \neq 0. \quad (3.57)$$

So we can try for specific fluid+electromagnetic solutions other than the Hopfion, or otherwise consider a non-barotropic extension, i.e., consider $\vec{\nabla} p$ independent of $\vec{\nabla} \rho$.

### 3.7 General knotted solution with nontrivial fluid-electromagnetic interaction

In the solutions found until now, the Lorentz force, giving the interaction between the fluid and electromagnetism, was zero, so the solutions were not so satisfactory. We can ask: are there truly interacting solutions, such that the Lorentz force is nonzero, and there is a chance for the helicity to change between the fluid and electromagnetism?

The answer turns out to be yes, with a caveat. In the equations (2.1), the Maxwell fields are considered as external fields, so strictly speaking they don’t need to satisfy Maxwell’s equations. More precisely, they need to satisfy the Bianchi identity, $\partial_{[\mu} F_{\nu\rho]} = 0$, which is just a statement that the electromagnetic fields are generated by a gauge field $A_\mu$, but they do not need to satisfy the equations of motion, which can be thought to be in the presence of some unknown (not considered in the fluid equations) source,

$$\partial^\mu F_{\mu\nu} = \tilde{j}_\nu \neq 0. \quad (3.58)$$

Under this caveat, we can construct a solution as follows.

First, calculate a velocity field from the electromagnetic knot solution in (3.12), using our electromagnetism to fluid map,

$$v^i = \frac{\epsilon^{ijk} E^j B^k}{\frac{1}{2} (E^2 + B^2)} = \frac{2 F^{0j} F^{ij}}{(F^{0k})^2 + \left( \frac{1}{2} \epsilon^{klm} F_{lm} \right)^2}$$

$$= \frac{2 (\partial^0 \phi \partial^j \phi - \partial^j \phi \partial^0 \phi) (\partial^0 \bar{\phi} \partial^j \bar{\phi} - \partial^j \bar{\phi} \partial^0 \bar{\phi})}{(\partial^0 \phi \partial^k \phi - \partial^k \phi \partial^0 \phi)^2 + (\epsilon^{klm} \partial^k \phi \partial^l \phi)^2}. \quad (3.59)$$

This satisfies

$$u^\mu F_{\mu\nu} = 0 :$$

$$v^j F_{0i} = 0 \Rightarrow \vec{v} \cdot \vec{E} = 0$$

$$F_{0i} + v^j F_{ji} = 0 \Rightarrow - (\vec{E} + \vec{v} \times \vec{B})^i = 0,$$

(3.60)

where $u^\mu = (1, v^i)$ and $F_{\mu\nu}$ is the solution in (3.12).
But, if in (3.12) we replace on the left-hand side \( F_\mu \) by

\[
\Omega_{\mu\nu} = \partial_\mu \pi_\nu - \partial_\nu \pi_\mu = F_{\mu\nu} + \mathcal{V}_{\mu\nu},
\]

where

\[
\mathcal{V}_{0i} = m(\partial_0 v_i - v_j \partial_i v^j) \quad \mathcal{V}_{ij} = m(\partial_i v_j - \partial_j v_i),
\]

meaning that we now have

\[
\Omega = \frac{1}{2}\Omega_{\mu\nu} dx^\mu \wedge dx^\nu = \frac{1}{4\pi i} \frac{\partial_\mu \phi \partial_\nu \phi - \partial_\nu \phi \partial_\mu \phi}{(1 + |\phi|^2)^2} dx^\mu \wedge dx^\nu
\]

\[
\tilde{\Omega} = \frac{1}{2}\tilde{\Omega}_{\mu\nu} dx^\mu \wedge dx^\nu = \frac{1}{4\pi i} \frac{\partial_\mu \theta \partial_\nu \theta - \partial_\nu \theta \partial_\mu \theta}{(1 + |\theta|^2)^2} dx^\mu \wedge dx^\nu,
\]

it follows that we now satisfy (remember that \( j^\mu = (\rho, \rho v^i) = \rho u^\mu \))

\[
u^\mu \Omega_{\mu\nu} = 0 \Rightarrow j^\mu \Omega_{\mu\nu} = 0,
\]

i.e., the fluid Euler equations. Since the velocity field is defined by (3.60) and \( \Omega_{\mu\nu} \) by (3.63), we can subtract the velocity field \( \mathcal{V}_{\mu\nu} \) from \( \Omega_{\mu\nu} \), which will define the electromagnetic field \( F_{\mu\nu} \).

This solution will not satisfy the vacuum equation of motion \( \partial^\mu F_{\mu\nu} = 0 \). But that is fine, since \( E \) and \( B \) were considered as external fields, not as Maxwell fields, so we can define the right-hand side as the current that generates them, so \( \partial^\mu F_{\mu\nu} \equiv j^\nu \).

All we need to satisfy are the Bianchi identities for \( F_{\mu\nu} \), and those are satisfied, since \( \Omega_{\mu\nu} \) satisfies them by construction, \( \partial_\mu \Omega_{\nu\rho} = 0 \), and so does \( \mathcal{V}_{\mu\nu} \), since

\[
\mathcal{V}_{ij} = m(\partial_i v_j - \partial_j v_i)
\]

\[
\mathcal{V}_{0i} = m \partial_0 v_i - \partial_i \Phi = m \left( \partial_0 v_i + \partial_i \frac{\vec{v}^2}{2} \right) + \partial_i \mu,
\]

so obviously satisfies \( \partial_\mu \mathcal{V}_{\nu\rho} = 0 \), and therefore their difference, \( F_{\mu\nu} \) also does:

\[
\partial_\mu F_{\nu\rho} = 0.
\]

4 Euler-dual-Maxwell system of magnetically charged fluid

In this section, we consider the (Poincaré) dualization of the previous construction, and then speculate about possible physical application. Before that, however, we review the definition and properties of the dual electromagnetic theory.

4.1 Dualizing Maxwell equations

In the absence of sources, the electromagnetic theory can be dualized. In a covariant formulation, the dual of

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu
\]

(4.1)
is
\[ G_{\mu\nu} \equiv \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} = \partial_\mu C_\nu - \partial_\nu C_\mu. \] (4.2)

The duality of the electric and magnetic fields is manifested by defining a field \( \vec{C} \) in a similar way to \( \vec{A} \),
\[ \vec{E} = \vec{\nabla} \times \vec{C}; \quad \vec{B} = \vec{\nabla} \times \vec{A}. \] (4.3)

In the Bateman formulation [4], as used before, one defines the complex four-vector
\[ H_\mu \equiv \frac{1}{2} (\alpha \partial_\mu \beta - \beta \partial_\mu \alpha), \quad C_\mu = \text{Re}(H_\mu), \quad A_\mu = \text{Im}(H_\mu). \] (4.4)

4.2 Conserved “helicities”

In electromagnetism, we can introduce helicities, that can be conserved under conditions to be defined shortly. We already introduced the magnetic helicity, that was part of the conserved total, fluid-electromagnetic, helicity \( H_{\text{tot}} \),
\[ H_{mm} = \int d^3 x \vec{A} \cdot \vec{B} = \int d^3 x \varepsilon_{ijk} A_i \partial_j A_k. \] (4.5)

As we see, the helicities are spatial integrals of Chern-Simons forms, \( H_{mm} \) being the integral of the Chern-Simons form of \( \vec{A} \). The electric-magnetic dual of the above is the electric helicity, defined as an integral of a Chern-Simons form of \( \vec{C} \),
\[ H_{ee} = \int d^3 x \vec{C} \cdot \vec{E} = \int d^3 x \varepsilon_{ijk} C_i \partial_j C_k. \] (4.6)

We can also define the integrals of BF forms, the cross helicities, the electromagnetic one,
\[ H_{em} = \int d^3 x \vec{C} \cdot \vec{B} = \int d^3 x \varepsilon_{ijk} C_i \partial_j A_k, \] (4.7)
and its electromagnetic dual, the magnetolectric one,
\[ H_{me} = \int d^3 x \vec{A} \cdot \vec{E} = \int d^3 x \varepsilon_{ijk} A_i \partial_j C_k, \] (4.8)
though of course for fields that vanish at infinity they are equal by partial integration.

The helicities are interesting, since although they are defined in terms of \( \vec{A} \) and \( \vec{C} \), they are gauge invariant under the 3-dimensional gauge transformations generated by \( \alpha(\vec{x}) \), since they are Chern-Simons and BF form. Of course, that is true only if the transformation of the Abelian fields \( \vec{A}(\vec{x}, t) \) is not a large gauge transformation, so that there are no global issues. Under large gauge transformations, as is the case for any CS or BF integrals, they can change by an integer times \( 2\pi \).

In the cases relevant for us, with time dependence, so \( \vec{A}(\vec{x}, t) \), like we are considering in this paper, conservation of these helicities in time is not guaranteed, and neither is an integer value for them (though, as we saw, for the superfluid case, the total helicity \( H_{\text{tot}} \), containing \( H_{mm} \), is quantized)
Using the Maxwell’s equations and partial integrations, the time evolution of the electromagnetic helicities is found to be

\[
\partial_t H_{mm} = \int d^3x (\partial_t \vec{A} \cdot \vec{B} + \vec{A} \cdot \partial_t \vec{B}) = -\int d^3x (\vec{E} \cdot \vec{B} + \vec{A} \cdot (\vec{\nabla} \times \vec{E}))
\]

\[
= -2 \int d^3x \vec{E} \cdot \vec{B}
\]

\[
\partial_t H_{ee} = \int d^3x (\partial_t \vec{C} \cdot \vec{E} + \vec{C} \cdot \partial_t \vec{E}) = -\int d^3x (\vec{B} \cdot \vec{E} + \vec{C} \cdot (\vec{\nabla} \times \vec{B}))
\]

\[
= -2 \int d^3x \vec{E} \cdot \vec{B}
\]

\[
\partial_t H_{me} = \int d^3x (\partial_t \vec{A} \cdot \vec{E} + \vec{A} \cdot \partial_t \vec{E}) = \int d^3x (-\vec{E} \cdot \vec{E} + \vec{A} \cdot (\vec{\nabla} \times \vec{B}))
\]

\[
= \int d^3x (\vec{E}^2 - \vec{B}^2)
\]

\[
\partial_t H_{me} = \int d^3x (\partial_t \vec{C} \cdot \vec{B} + \vec{C} \cdot \partial_t \vec{B}) = \int d^3x (-\vec{B} \cdot \vec{E} + \vec{C} \cdot (\vec{\nabla} \times \vec{E}))
\]

\[
= \int d^3x (\vec{E}^2 - \vec{B}^2)
\] (4.9)

Thus the helicities $H_{mm}$ and $H_{ee}$ are conserved for configurations for which $\vec{E} \cdot \vec{B} = 0$, and $H_{em}$ and $H_{me}$ are conserved provided that $\vec{E}^2 - \vec{B}^2 = 0$. For the null configurations defined in (3.7) these two conditions are obeyed, so for them all the four helicities are conserved.

### 4.3 Magnetically charged fluid

In the case of the Euler fluid coupled to electromagnetism via electric charges, i.e., a fluid made up of electrically charged particles, we saw that only the total helicity, fluid plus $H_{mm}$, is conserved.

But we can also consider a fluid of magnetically charged particles, where the source in the Euler equation is the magnetically charged Lorentz force, so it obeys the equations

\[
\frac{d}{dt} \rho + \nabla \cdot \rho \vec{v} = 0
\]

\[
(\partial_t + \vec{v} \cdot \nabla)v = g \vec{B} - g\vec{v} \times \vec{E}.
\] (4.10)

Formally, the derivation of [28], reviewed in section 2, can be repeated for a “magnetically charged” fluid, by using the electric-magnetic dual conjugate momentum,

\[
\vec{\pi}_c = m\vec{v} + \vec{C}.
\] (4.11)

Now the total helicity of the fluid plus the electric helicity $H_{ee}$ is defined as

\[
H_{tot}^m = \frac{1}{h^2} \int d^3x \vec{\pi}_c \cdot \nabla \times \vec{\pi}_c
\] (4.12)

so that

\[
H_{tot}^m = \frac{1}{h^2} \int d^3x \left[ m\vec{v} \cdot \vec{\omega} + \vec{C} \cdot \vec{E} + 2m\vec{v} \cdot \vec{E} \right] = H_f + H_{ee} + 2H_{fe}.
\] (4.13)
Following the same steps of the derivation of the equations for electrically charged fluid, we find for the magnetically charged one that the new total helicity $H_{\text{tot}}^m$ is conserved, and moreover we find the analog of the Abanov-Wiegmann equations,

$$
\partial_t (m^2 \vec{v} \cdot \vec{\omega}) + \vec{\nabla} \cdot \left[ \vec{v}(m^2 \vec{v} \cdot \vec{\omega}) + m \vec{\omega} \left( \mu - \frac{m^2 v^2}{2} \right) + m \vec{v} \times (\vec{B} - \vec{v} \times \vec{E}) \right] = 0
$$

$$
\partial_t (m \vec{v} \cdot \vec{E}) + \vec{\nabla} \cdot \left[ \vec{v}(m \vec{v} \cdot \vec{E}) + \vec{E} \left( \mu - \frac{m^2 v^2}{2} \right) - m \vec{v} \times (\vec{B} - \vec{v} \times \vec{E}) \right] + m \vec{\omega} (\vec{B} - \vec{v} \times \vec{E}) = 0.
$$

### 4.4 Possible physical applications

It may seem that it is a bit abstract to talk about a fluid of magnetically charged particles, as in the previous subsection, when magnetic monopoles haven’t even been observed yet. One possibility would be that we will find somewhere in the Universe some place where magnetic monopoles abound and form a fluid, and that is certainly one application of the previous formalism.

However, really, what we want are effective particles. So, for instance, if we consider a type II superconductor close to the phase transition, it will be composed of many parallel magnetic flux tubes, almost overlapping.

So from the point of view of the 2+1 dimensional reduced theory, the flux tubes look like particles, with some radius, interacting, which is kind of a description of a fluid anyway. So we can consider a 2+1 dimensional fluid of such tubes in the superconductor, interacting with external electric fields, to be governed by the above description.

Hence the above electric/magnetic case also be dimensionally reduced to 2+1 dimensions in exactly the same way as we do for the $H_{\text{em}}$ helicity, and we can use it for the “effective fluid” of flux tubes in a type II superconductor.

However, there is a trick when reducing this dual, magnetically charged fluid to 2+1 dimensions. In 2+1 dimensions, $B$ is a scalar, understood to be transverse to the 2-dimensional plane, so the magnetic Lorentz force becomes a scalar as well,

$$
g(B - \epsilon^{ab} v_a E_b),
$$

and, since this is understood as the force in the perpendicular direction, where for a consistent reduction the left-hand side in (4.10) must be zero, which means that the right-hand side must be zero as well, so the above Lorentz force must vanish.

Then, for this consistent reduction, we have that the (quantized) magnetic flux is understood as a 2+1 dimensional helicity involving the electric field,

$$
\Phi_m = \frac{e}{h} \int d^2 x B = \frac{1}{g} \int d^2 x \epsilon^{ab} v_a E_b \equiv H_e,
$$

where we also used Dirac quantization, $eg = h$.

Thus we reinterpret the 2+1 dimensional case of type II superconductor as a dual magnetically charged fluid, with this electric helicity $H_e$. 
5 Non-linear generalization of the Abanov Wiegmann system

In [12, 13] it was proven that not only are the knotted solutions like the Hopfion solutions of Maxwell’s theory, they are also solutions of any nonlinear theories that reduce to electromagnetism at small fields, theories like the Born-Infeld theory, or theories obtained by integrating out any light fields interacting with electromagnetism, like the Euler-Heisenberg Lagrangian. This was also discussed in [34].

Based on this insight and the fact that, as we have seen, the electromagnetic Hopfion and the corresponding velocity in (3.28) form a solution of the Abanov-Weigmann equations, we would like now to generalize these equations of motion to those that one gets upon replacing Maxwell’s theory with one of its non-linear generalizations.

We follow here the steps taken in [12, 13] and we start with the generalization of the Maxwell electromagnetic theory to the formalism of Born and Infeld [35] (see also [36] for a generalization of this analysis). Defining the quantities

\[ F = \frac{F_{\mu\nu}F^{\mu\nu}}{2b^2} = \frac{1}{b^2}(\vec{B}^2 - \vec{E}^2); \]
\[ G = \frac{1}{8b^2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} = \frac{1}{b^2} \vec{E} \cdot \vec{B}, \]

where \( b \) is a dimensionful constant of mass dimension 2, the BI Lagrangian is

\[ \mathcal{L} = -b^2 [\sqrt{1 + F - G^2} - 1]. \]

We define the conjugate quantities analogous to the ones of electromagnetism in a medium,

\[ \vec{H} = -\frac{\partial \mathcal{L}}{\partial \vec{B}}, \quad \vec{D} = +\frac{\partial \mathcal{L}}{\partial \vec{E}}. \]

The Maxwell equations in terms of them take the same form as for electromagnetism in a medium,

\[ \vec{\nabla} \times \vec{E} + \partial_0 \vec{B} = 0, \quad \vec{\nabla} \cdot \vec{B} = 0 \]
\[ \vec{\nabla} \times \vec{H} - \partial_0 \vec{D} = 0, \quad \vec{\nabla} \cdot \vec{D} = 0. \]

In the BI theory case, we have

\[ \vec{H} = \frac{\vec{B} - G\vec{E}}{\sqrt{1 - F + G^2}}; \quad \vec{D} = \frac{\vec{E} + G\vec{B}}{\sqrt{1 - F + G^2}}. \]

We see explicitly that for \( F = G = 0 \), like we have for the Hopfion and for the null knotted solutions, we obtain \( \vec{H} = \vec{B} \) and \( \vec{D} = \vec{E} \), and therefore the Maxwell equations reduce to the ones in vacuum.

From the relations \( \vec{H}(\vec{E}, \vec{B}), \vec{D}(\vec{E}, \vec{B}) \), we find that the same is true for any Lagrangian that is only a function of \( F \) and \( G \), and which contains electromagnetism as a small fields limit, i.e., a Lagrangian that can be written as an expansion containing the Maxwell term plus higher order terms,

\[ \mathcal{L} = b^2 \left[ -\frac{F}{2} + \sum_{n\geq2} \sum_{m\geq0} c_{n,m} F^n G^m \right]. \]
Indeed, in this case we obtain
\[ \vec{H} = \vec{B} + \mathcal{O}(F,G); \quad \vec{D} = \vec{E} + \mathcal{O}(F,G), \] (5.7)
therefore if \( F = G = 0 \) we obtain the usual Maxwell’s equations in terms of \( \vec{E} \) and \( \vec{B} \).

However, introducing the replacement
\[ \vec{E} \rightarrow \vec{D}, \quad \vec{B} \rightarrow \vec{H}, \] (5.8)
in the Maxwell’s equations, given that the particle coupling to electromagnetism is still \( q \int dx^\mu A_\mu \), leading to the same \( qF_{\mu\nu}u^\nu \) Lorentz force, so in terms of the same \( \vec{E}, \vec{B} \), the DBI- Euler equations of motion are actually the same as in (2.1), and from them we can find the same Abanov-Wiegmann equations (2.7).

6 Two-dimensional formalism and solutions

Next, we consider the 2+1 dimensional version of the fluid equations, and of the 4 dimensional fluid formalism, and seek equations to them.

6.1 2+1 dimensional generalization of the 4 dimensional formalism and equations

The 2+1 dimensional version of the continuity and Euler equations is
\[ \partial_t \rho + \partial_a (\rho v_a) = 0 \]
\[ (\partial_t + v^c \partial_c) mv^a + \partial^a \mu = eE^a + e \epsilon^{ab} v_b B. \] (6.1)

Note that in 2 spatial dimensions we have
\[ E^a = F^{0a} = -(\partial_0 A_a - \partial_a A_0), \quad B = \epsilon^{ab} \partial_a A_b. \] (6.2)

That means, since \( \omega = \epsilon^{ab} \partial_a v_b, B = \epsilon^{ab} \partial_a A_b \), that the only possible definition of a (reduced) 2+1 dimensional helicity is
\[ H_{\text{tot,red}} = \frac{1}{\hbar} \int d^2 x \epsilon^{ab} \partial_a \pi_b = \int d^2 x (m \omega + eB). \] (6.3)

Note that the normalization was chosen such that the purely magnetic helicity, equal to the magnetic flux through the plane, has integer values. Indeed, we know that the fluxon is \( \Phi_0 = \hbar/e \), so \( \frac{\hbar}{e} \int d^2 x B = k \in \mathbb{Z} \).

Then, construct
\[ \pi_a = mv_a + eA_a \]
\[ \pi_0 = \Phi + eA_0 \]
\[ -\Phi = \mu + \frac{mv^2}{2}. \] (6.4)

The Euler equation is rewritten as
\[ \rho (\partial_t \pi^a - \partial^a \pi_0) - \rho e^{ab} v_b (m \omega + eB) = 0, \] (6.5)
as we can easily check (use $\epsilon^{ab}\epsilon^{cd} = \delta^a_c\delta^b_d - \delta^a_d\delta^b_c$), where
\[
\vec{\nabla} \times \vec{\pi} \rightarrow \epsilon^{ab}\partial_a\pi_b = \mathcal{H}^d_{\text{tot}} = m\omega + eB.
\]

This rewriting of the Euler equation can be further written in a compact form, first defining
\[
\Omega_{\mu\nu} = \partial_\mu\pi_\nu - \partial_\nu\pi_\mu,
\]
and then $j^\mu = (\rho, \rho v^a)$, as before,
\[
j^\mu\Omega_{\mu\nu} = 0.
\]

The continuity equation is $\partial_\mu j^\mu = 0$, also as before.

We define the helicity density 2-form and its 1-form dual,
\[
h = \ast d\pi = \ast \Omega.
\]

Then the total helicity density is its 0 component,
\[
\mathcal{H}^d_{\text{tot}} = h_0 = \frac{1}{2} \epsilon^{ab}\Omega_{ab} = \epsilon^{ab}\partial_a\pi_b,
\]
and the helicity flux are the $a$ components,
\[
\vec{h} : h^a = \epsilon^{ab}\Omega_{0a} = \epsilon^{ab}(\partial_0\pi_a - \partial_a\pi_0).
\]

The conservation of helicity is once again trivial,
\[
d \ast h = dd\pi = 0,
\]
in components
\[
h_0 + \partial^a h_a = 0 \Rightarrow \partial_t \int d^2x h_0 = 0,
\]
with appropriate boundary conditions at infinity.

Now define the time dependence of the electromagnetic helicities. As we said, in 2+1 dimensions, we can define the magnetic helicity which is just magnetic flux,
\[
H_m = \frac{e}{\hbar} \int d^2x B,
\]

but, unlike in 3+1 dimensions, there are no other possibilities, since the electric field $\vec{E}$, unlike $B$, is a vector, so cannot be used to contract with $B$. So we cannot even define a dual helicity $H_e$; since in 2+1 dimensions, the dual to the vector $A_\mu$ is a scalar, call it $C$ (similar to the $\vec{C}$ in 4 dimensions).

Then, the time derivative of $H_m$ (which in 4 dimensions is proportional to $\int d^3x \vec{E} \cdot \vec{B}$) is now, via the 2+1 dimensional Maxwell’s equations,
\[
\frac{d}{dt} H_m \propto \frac{d}{dt} \int_S d^2x B dS = -\int_S d^2x \vec{\nabla} \times \vec{E} dS = -\oint_{C=\partial S} \vec{E} \cdot d\vec{l},
\]

and in 2+1 dimensions
\[
\vec{\nabla} \times \vec{E} = \epsilon^{ab}\partial_a E_b.
\]
6.2 2+1 dimensional reduction of solutions

In [12, 13], it was also shown how to dimensionally reduce a fluid solution (of the Euler equation) to 2+1 dimensions.

First, one notices that the velocity for the fluid knot (3.28), obtained from the map from the electromagnetic knot, can be rewritten in general (for \(t-z \neq 0\)) as the vortex-like solution

\[
v^a = \epsilon^{ab} \partial_b \psi + (t-z) \partial^a \psi, \quad \psi = \log(1 + x^2 + y^2 + (t-z)^2).
\]

(6.17)

For it, the (reduced) 2+1 dimensional fluid helicity, the integral of the vorticity, is constant,

\[
H_{f,\text{red}} = \int d^2 x \omega = 4\pi,
\]

(6.18)

though it should be written in units of \(\frac{h}{4\pi m}\), for consistency with the magnetic flux.

This fluid knot (“Hopfion”) solution satisfies the continuity equation

\[
\partial_t \rho + \partial_a (\rho v^a) = 0,
\]

(6.19)

\[
\partial^- v^a + \beta^b \partial_b v^a = 0, \quad \beta^a = \frac{v^a}{1 - v_z},
\]

Here \(\partial^-\) now stands for \(\partial_t\), and taking \(\epsilon^{ab} \partial_b\) on the above equation, we obtain \(\partial^- \omega + \partial_a (\epsilon^{ab} \beta^c \partial_c v_b) = 0\), which when integrated gives \(\partial^- \int d^2 x \omega + \oint S_{\psi} (\ldots) = 0\), or \(\partial_t H_{f,\text{red}} = 0\). That is, indeed the 2+1 dimensional fluid helicity is conserved in (reduced) time, \(x^-\).

For a consistent reduction of the equations satisfied by the fluid knot to 2+1 dimensional Euler fluid equations, we have two options:

1. The first one is to multiply the equation in (6.19) by \((1 - v_z)\), and note that we can define \(\partial_a p \equiv (1 - v_z) \partial^- v_a\) and consider constant density \(\rho = 1\), in which case we get the usual static (in \(x^-\) time) Euler equation with pressure in 2+1d,

\[
v^b \partial_b v^a + \partial^a p = 0,
\]

(6.20)

leading to the solution

\[
v_x = \frac{2y}{1 + x^2 + y^2}, \quad v_y = \frac{-2x}{1 + x^2 + y^2}, \quad p = p_{\infty} - \frac{2}{1 + x^2 + y^2}.
\]

(6.21)

2. The other option is to define new velocities,

\[
\beta^a = \frac{v^a}{1 - v_z},
\]

(6.22)

in terms of which we have the continuity and Euler equations with constant pressure in 2+1 dimensions, in terms of \(x^-\) time,

\[
\partial^- \rho + \partial_a (\rho \beta^a) = 0, \quad \partial^- \beta^a + \beta^b \partial_b \beta^a = 0,
\]

(6.23)

integrated to (\(\rho\) the same as in (3.28) and)

\[
\beta^a = \epsilon^{ab} \partial_b \tilde{\psi} + (t-z) \partial^a \tilde{\psi}, \quad \tilde{\psi} = \log(x^2 + y^2 - 1 - (t-z)^2).
\]

(6.24)

Note the signs different inside the log in \(\tilde{\psi}\) with respect to \(\psi\) in (6.17).
To find knot (or rather, vortex) solutions of the 2+1 dimensional equations (6.1) solutions, as in 3+1 dimensions, it would suffice if we would have zero Lorentz force,

$$E^a + \epsilon^{ab} v_b B = 0.$$  \hfill (6.25)

This would be obtained from the 2+1 dimensional version of the fluid map, namely if we would have $E^a E^a = B^2$ (note that since $B$ basically means $B_z$, by definition we have $\vec{E} \cdot \vec{B} = 0$) and

$$v^a = \frac{\epsilon^{ab} E_b}{\sqrt{E^c E_c}}.$$  \hfill (6.26)

Of course, while at case 1 we would have velocities $v^a$, at case 2 we would have velocities $\beta^a$, and in both cases time is $x^- = t - z$.

7 Solutions via conformal transformations

In [22] it was shown how to construct knotted solutions form un-knotted solutions like constant and plane wave electric and magnetic fields via complex special conformal transformations. The proof of this statement and several applications to construct knotted solutions was done using the Bateman variables [22]. Whereas the special conformal transformation of the electric and magnetic field are complicated, in terms of $\alpha$ and $\beta$ one has to perform only the transformation of the coordinates.

The prototype example is the derivation of the basic Hopfion solution (3.9) from the configuration of constant perpendicular electric and magnetic fields given by

$$\alpha = 2i(t + z), \quad \beta = 2(x - iy),$$  \hfill (7.1)

which corresponds to electromagnetic fields

$$\vec{E} = (-4, 0, 0), \quad \vec{B} = (0, 4, 0).$$  \hfill (7.2)

Under the special conformal transformation

$$x^\mu \rightarrow \frac{x^\mu + b^\mu x_\nu x^\nu}{1 + 2b^\mu x_\mu + b^\mu b^\nu x_\mu x_\nu},$$  \hfill (7.3)

with

$$b^\mu = i(1, 0, 0, 0),$$  \hfill (7.4)

we obtain the Hopfion solution written in (3.9), namely with

$$\alpha = \frac{A - iz}{A + it}, \quad \beta = \frac{x - iy}{A + it}, \quad A = \frac{1}{2}(x^2 + y^2 + z^2 - t^2 + 1).$$  \hfill (7.5)

Since, as was explained in section 3, we construct the velocity $\vec{v}$ from the electric and magnetic fields, then we can also generate novel knotted fluid configurations by applying complex special conformal transformations on $\alpha$ and $\beta$ that yields novel EM knot configuration from which we get the fluid configurations.
We now demonstrate this method by deriving the \((p, q)\) knotted EM and the corresponding fluid configurations. For this case we start with

\[
\alpha = [2i(t + z)]^p \quad \beta = [2(x - iy)]^q. \tag{7.6}
\]

Next we apply the same special conformal transformation as the one given above in (7.3), to find

\[
\alpha = \frac{(-t^2 + x^2 + y^2 + (z + i)^2)^p}{(-t(t - 2i) + x^2 + y^2 + z^2 + 1)^p}, \tag{7.7}
\]

\[
\beta = \frac{8(x - iy)^q}{(-t(t - 2i) + x^2 + y^2 + z^2 + 1)^q}. \tag{7.8}
\]

The corresponding fluid density at \(t = 0\) has a denominator that is of the form \((x^2 + y^2 + z^2 + 1)^{2(p+q)}\).

A similar path can be applied on the variables \(\alpha\) and \(\beta\) that corresponds to an EM plane-wave,

\[
\alpha = e^{i(z-t)}; \quad \beta = x + iy, \tag{7.9}
\]

which after the above complex special conformal transformation reads

\[
\alpha = \exp \left(-1 + \frac{i(t + z - i)}{(2A + it)} \right); \quad \beta = \frac{x + iy}{(2A + it)}. \tag{7.10}
\]

In a similar manner to applying complex SCT on the basic configurations of the constant and plane wave electric and magnetic fields, one can also apply conformal transformations on the Hopfion itself, in particular time translations, space-translations, rotations, boosts and scale transformations with imaginary parameter.

### 8 Conclusions and discussion

In this paper we have considered knotted solutions to the Euler fluid plus external electromagnetism equations, and to the equations for the helicities of the fluid and electromagnetism (AW equations). We have found that a map from electromagnetism to a null fluid can be used to find knotted solutions, with helicities, to the coupled system, and to the Abanov-Wiegmann helicity equations. An electromagnetic dual case, for a magnetically charged fluid, was found to be similar. The case of nonlinear electromagnetism was treated similarly, with similar results. The 3+1 dimensional formalism for the fluid, used in the derivation of the conservation of the total helicity, was extended to 2+1 dimensions, and the solutions in 3+1 dimensions were dimensionally reduced, to find 2+1 dimensional solutions. Using conformal transformations, we were able to obtain the knotted solutions from unknotted ones.

There are many open questions related to this topic. Here we list several of them:

- The systems discussed here did not incorporate the back-reaction of the fluid on the EM fields. The latter were taken to be external fields. It will be very interesting to find fluid and EM configurations that solve the coupled equations. This is of particular interest since so far all the EM knots have been solutions only to the free Maxwell’s equations with not currents and charges.
• The fluids involved in the knotted solutions did not permit gradients of the pressure. It will be interesting to explore the possibility to find fluid knots in the presence of non-constant pressure.

• A very interesting question is whether one can also construct gauge and fluid knots associated with non-abelian gauge symmetry. This implies searching for solutions of the YM equation generalizing the AW equations to incorporate non-abelian charges.

• EM knots can easily be constructed using special conformal transformations with imaginary parameters. It is very natural in the context of Bateman formulation of the EM theory. It will be interesting to study these transformation in a analogous formulation of fluid dynamics.

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A The (2,3) EM fluid knot solution

Performing the complex special conformal transformations discussed in section 7 we derive the components of the electric and magnetic fields. At \( t = 0 \) they are given by

\[
E_x = \frac{96 (-x^6 + 5x^4y^2 - 4x^3yz + x^2 (5y^4 + 6y^2 (z^2 - 1) + z^4 - 6z^2 + 1))}{(x^2 + y^2 + z^2 + 1)^6}
+ \frac{96 (4xyz (3y^2 + 2z^2 - 2) - y^2 (2 (y^2 - 3) z^2 + (y^2 - 1)^2 + z^2))}{(x^2 + y^2 + z^2 + 1)^6}
\]

\[
E_y = \frac{192 (2x^3y - 2x^2z + 3x^3y (z^2 - 1) - 2x^2z (-3y^2 + z^2 - 1))}{(x^2 + y^2 + z^2 + 1)^6}
- \frac{192 (xy (-2y^4 - (y^2 + 6) z^2 + y^2 + z^4 + 1) + 2y^2z (z^2 - 1))}{(x^2 + y^2 + z^2 + 1)^6}
\]

\[
E_z = \frac{192 (xz^3 (x^2 - 3y^2) - 3yz^2 (y^2 - 3x^2))}{(x^2 + y^2 + z^2 + 1)^6}
- \frac{192 (xz (x^2 - 3y^2) (x^2 + y^2 - 3) - y (y^2 - 3x^2) (x^2 + y^2 - 1))}{(x^2 + y^2 + z^2 + 1)^6}
\]

\[
B_x = \frac{192 (2x^3y + x^3y (z^2 - 1) + 2x^2z (3y^2 + z^2 - 1))}{(x^2 + y^2 + z^2 + 1)^6}
+ \frac{192 (-xy (2y^4 + 3y^2 (z^2 - 1) + z^4 - 6z^2 + 1) - 2y^2z (y^2 + z^2 - 1))}{(x^2 + y^2 + z^2 + 1)^6}
\]
\begin{equation}
\begin{aligned}
B_y &= -\frac{96 \left(x^6 + x^4(-5y^2 + 2z^2 - 2) + 12x^3yz + x^2(-5y^4 - 6(y^2 + 1)(z^2 + 6y^2 + z^4 + 1))\right)}{(x^2 + y^2 + z^2 + 1)^6}
- \frac{96 \left(-4xyz(y^2 - 2z^2 + 2) + y^2(y^4 - z^4 + 6z^2 - 1)\right)}{(x^2 + y^2 + z^2 + 1)^6} \\
B_z &= -\frac{192 \left(x^5 - 3x^4yz + x^3(-2y^2 + 3z^2 - 1) + x^2yz(-2y^2 - 3z^2 + 9)\right)}{(x^2 + y^2 + z^2 + 1)^6}
- \frac{192 \left(-3xy^2(y^2 + 3z^2 - 1) + y^3z(y^2 + z^2 - 3)\right)}{(x^2 + y^2 + z^2 + 1)^6}.
\end{aligned}
\tag{A.1}
\end{equation}

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