Note on the semi-continuity of the algebraic dimension.

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Abstract. In this short Note we show that the direct image sheaf $R^1\pi_*(\mathcal{O}_X)$ associated to an analytic family of compact complex manifolds $\pi : \mathcal{X} \to S$ parametrized by a reduced complex space $S$ is a locally free (coherent) sheaf of $\mathcal{O}_S$–modules. This result allows to improve a semi-continuity type result for the algebraic dimension of compact complex manifolds in an analytic family given in [B.15].

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1 Introduction

It is well known that for a compact complex manifold $X$ of the Fujiki-Varouchas class $C$ (recall that Varouches [V.89] shows that this is simply the class of compact complex manifolds which admit a Kähler modification) the number

$$h^{0,1}(X) := \dim \mathbb{C} H^1(X, \mathcal{O}_X)$$

is a topological invariant (half of the first Betti number), so it is constant in an analytic deformation inside the class $C$. In this short Note we prove that this number is invariant in any analytic deformation for any compact complex manifold. As an application, this allows us to improve a semi-continuity type result for the algebraic dimension of compact complex manifolds in an analytic family given in [B.15]. See the theorem 3.0.7 and its corollaries in section 3.

2 The result

**Theorem 2.0.1** Let $\pi : X \to S$ be a holomorphic family of compact complex connected manifolds of dimension $n$ parametrized by an irreducible complex space $S$. Then the coherent sheaf $R^1\pi_*(\mathcal{O}_X)$ on $S$ is a locally free sheaf.

The proof will use several lemmata.

**Lemma 2.0.2** Let $X$ be a compact normal connected complex space of dimension $n$ and let $L$ be a holomorphic line bundle on $X$. Then if $L$ and $L^*$ have a non trivial holomorphic section, the line bundle $L$ is holomorphically trivial.

**Proof.** Let $\sigma$ and $\tau$ the non trivial holomorphic sections of $L$ and $L^*$. Then the function $x \mapsto <\sigma(x), \tau(x)>$ is holomorphic and not identically zero. So it is a non zero constant function and we see that $\sigma$ and $\tau$ cannot vanish. Now the map $L \to X \times \mathbb{C}$ given by $\xi \mapsto (\pi(\xi), \xi/\sigma(\pi(\xi)))$ is holomorphic and linear on fibres with inverse the holomorphic map given by $(x, \lambda) \mapsto \lambda.\sigma(x)$. So $L$ is trivial. $\blacksquare$

We shall use this lemma in order to get the fact that if a holomorphic line bundle on a compact complex connected manifold is not holomorphically trivial, then $L$ or $L^*$ has no non trivial holomorphic section.

**Proposition 2.0.3** Let $M$ be a reduced complex space and $(X_s)_{s \in S}$ an analytic family of compact $n$-cycles in $M$ parametrized by a closed irreducible complex subset $S$ of $\mathcal{C}_n(M)$ the space of compact $n$-cycles in $M$. Assume that each cycle in this family is reduced, normal and connected. Let $\mathcal{L}$ be a holomorphic line bundle on $M$. Then the subset $\Sigma$ of points in $S$ such that the restriction $\mathcal{L}|_{X_s}$ is not holomorphically trivial is an open subset in $S$. 2
Let \( f : \mathcal{L} \to M \) be the projection. Then the direct image of compact \( n \)-cycles \( f_* : \mathcal{C}_n(\mathcal{L}) \to \mathcal{C}_n(M) \) is a holomorphic map (see [B-M 1] chapter IV). We can restrict this map to the subset \( Z \subset \mathcal{C}_n(\mathcal{L}) \) defined by the condition that the cycles are connected and that the direct image by \( f \) of a cycle \( C \) in \( Z \) is a cycle \( X_s \) for some \( s \in S \). These two conditions are analytic and closed thanks to the theorem IV 7.2.1 of [B-M 1] and to the assumption that \( S \) is a closed analytic subset in \( \mathcal{C}_n(M) \).

Remark that, as we assume that \( X_s \) is normal and connected, a compact connected \( n \)-cycle \( C \) in \( \mathcal{L} \) with direct image \( X_s \) by the projection \( f \) is a section of the line bundle \( \mathcal{L}|_{X_s} \).

Note that we have a closed embedding \( j : S \to Z \) which associates to \( s \in S \) the reduced \( n \)-cycle in \( Z \) equal to the zero section of the line bundle \( \mathcal{L}|_{X_s} \).

Now if the line bundle \( \mathcal{L}|_{X_s} \) has a non trivial holomorphic section the cycle \( j(s) \) can move in \( Z \cap f_*^{-1}(j(s)) \) by homotheties in an analytic 1-dimensional family containing \( j(s) \). So we have

\[
\dim_{j(s)}(Z \cap f_*^{-1}(j(s))) \geq 1.
\]

But the subset \( W \) of points \( w \) in \( Z \) such the inequality \( \dim_{w}[Z \cap f_*^{-1}(f_*(w))] \geq 1 \) is a closed analytic subset in \( Z \). So the subset \( \Sigma_0 := j^{-1}(W) \) is a closed analytic subset in \( S \). Then the complement of \( \Sigma_0 \) is an open set in \( S \). So if \( L|_{X_0} \) has no non trivial holomorphic section, for \( s \) in this open set, \( L|_{X_s} \) is not holomorphically trivial. If \( L^* \) has no non trivial holomorphic section we obtain in the same way an open set around 0 such that, for any \( s \) in it, \( L|_{X_s} \) is not holomorphically trivial. The case when \( L \) and \( L^* \) have both a non trivial holomorphic section is excluded by the lemma 2.0.2.

**Lemma 2.0.4** Let \( \pi : \mathcal{X} \to \Delta \) a proper holomorphic submersion of a complex manifold \( \mathcal{X} \) onto an open disc \( \Delta \) with center 0 in \( \mathbb{C} \), with \( n \)-dimensional connected fibres. Let \( L \) be a line bundle on \( \mathcal{X} \) and assume that \( L \) is holomorphically trivial on each \( X_s, \forall s \in \Delta \). Then \( L \) is trivial on \( \mathcal{X} \).

**Proof.** Consider the following data : an open disc \( \Delta_1 \) in \( \Delta \), an open set \( \mathcal{U} \) in \( \pi^{-1}(\Delta_1) \), a holomorphic trivialization \( t : L|_{\mathcal{U}} \to \mathcal{U} \times \mathbb{C} \) and a holomorphic section \( \gamma : \Delta_1 \to \mathcal{U} \) of \( \pi \). Of course, choosing first a local trivialization of \( L \) on an open set in \( \mathcal{X} \) we can find such data with any point in \( \Delta \) as the center of the (small) disc \( \Delta_1 \). Let \( Z \subset \mathcal{C}_n(L) \) the analytic subset of connected compact \( n \)-cycles \( C \) in \( L \) such the direct image cycle \( f_* (C) \) of \( C \) by the projection \( f : L \to \mathcal{X} \) is one of the fibres of \( \pi \).

So we have a holomorphic map \( g : Z \to \Delta \) defined by \( f_*(C) = X_{g(C)} \). Denote now by \( Z_1 \) the subset in \( Z \) of cycles \( C \in g^{-1}(\Delta_1) \) such that the cycle \( L_*(C \cap f^{-1}(\mathcal{U})) \) contains the point \((\gamma(g(C)), 1) \in \mathcal{U} \times \mathbb{C} \). We want to prove the following assertions:

1) The subset \( Z_1 \) is a closed analytic subset of the open set \( g^{-1}(\Delta_1) \subset Z \).

2) The projection on \( L|_{\pi^{-1}(\Delta_1)} \) of the graph \( \Gamma_1 \subset Z_1 \times L|_{\pi^{-1}(\Delta_1)} \) of the analytic family of compact \( n \)-cycles in \( L \) parametrized by \( Z_1 \) is a closed embedding of a
complex sub-manifold in \( L_{\pi^{-1}(\Delta_1)} \) which is disjoint of the zero section and gives a holomorphic section of \( L_{|\pi^{-1}(\Delta_1)} \).

As a consequence, we shall obtain that \( L_{\pi^{-1}(\Delta_1)} \) is trivial on \( \pi^{-1}(\Delta_1) \). And, as this is true for any given point \( s_1 \) in \( \Delta \) and a small enough open disc \( \Delta_1 \) with center \( s_1 \), the conclusion will follow because \( H^1(\Delta, \mathcal{O}_\Delta) = \{1\} \).

Let us prove the assertion 1). As the condition for \( C \in g^{-1}(\Delta_1) \) to be in \( Z_1 \) is given by the fact that the point \((C, (\gamma(g(C)), 1))\) is in the image of the graph \( \Gamma_1 \cap (Z_1 \times L_\mu) \) by the proper embedding \( \text{id}_{Z_1} \times t \), this is clearly a closed analytic condition as \( g, \gamma \) and \( t \) are holomorphic.

To prove the assertion 2), remark first that each \( C \in Z_1 \) is the image of a holomorphic section of \( L_{|\pi^{-1}(\Delta_1)} \) which does not vanishes at the point \( \gamma(g(C)) \). As \( L_{|\pi^{-1}(\Delta_1)} \) is trivial, this section never vanishes on \( X_{g(C)} \). Remark also that \( g \) is injective in \( Z_1 \) because if \( g(C) = g(C') := s \) then \( C \) and \( C' \) in \( Z_1 \) are the images of two holomorphic sections of the trivial line bundle \( L_{|X_\ast} \) and take the same value at the point \( \gamma(s) \). So \( C = C' \) and \( g : Z_1 \rightarrow \Delta_1 \) is an isomorphism. So the analytic family of compact cycle \( (C)_{C \in Z_1} \) gives exactly one holomorphic never vanishing section of \( L_{|X_\ast} \) for each \( s \in \Delta_1 \). This is enough to prove our second assertion as the graph of this analytic family is a closed analytic subset in \( L_{\pi^{-1}(\Delta_1)} \) disjoint from the zero section and which is one to one on \( \pi^{-1}(\Delta_1) \) by the projection of \( L \) on \( \mathcal{X} \).

**Lemma 2.0.5** Let \( \pi : \mathcal{X} \rightarrow \Delta \) a proper holomorphic submersion of a complex manifold \( \mathcal{X} \) onto an open disc \( \Delta \) with center 0 in \( \mathbb{C} \), with \( n \)-dimensional connected fibres. Consider the injection of sheaves on \( \Delta \)

\[
j : R^1\pi_*\mathcal{O}_\Delta \rightarrow R^1\pi_*\mathcal{O}_\mathcal{X}.
\]

The following properties are equivalent:

1) Any section \( \sigma \) with support \( \{0\} \) of the sheaf \( R^1\pi_*\mathcal{O}_\mathcal{X} \) vanishes.

2) Any section \( \sigma \) of the sheaf \( R^1\pi_*\mathcal{O}_\mathcal{X} \) such its restriction to \( \Delta^* \) is in the image of the map \( j : H^0(\Delta^*, R^1\pi_*\mathcal{O}_\mathcal{X}) \rightarrow H^0(\Delta^*, R^1\pi_*\mathcal{O}_\mathcal{X}) \) is also in the image of the map \( j : H^0(\Delta, R^1\pi_*\mathcal{O}_\Delta) \rightarrow H^0(\Delta, R^1\pi_*\mathcal{O}_\mathcal{X}) \).

3) Any topologically trivial line bundle on \( \mathcal{X} \) which induces on \( X_0 \) a line bundle which is holomorphically trivial on each \( \mathcal{X}_s \) for any \( s \neq 0 \) near-by enough 0 induces a line bundle which is holomorphically trivial on \( X_0 \).

**Proof.** 1) \( \Rightarrow \) 2). Take any section \( \sigma \) of the sheaf \( R^1\pi_*\mathcal{O}_\mathcal{X} \) such its restriction to \( \Delta^* \) is in the image of \( j : H^0(\Delta^*, R^1\pi_*\mathcal{O}_\mathcal{X}) \rightarrow H^0(\Delta^*, R^1\pi_*\mathcal{O}_\mathcal{X}) \). As \( R^1\pi_*\mathcal{O}_\mathcal{X} \) is a constant sheaf on \( \Delta \) we have \( H^0(\Delta^*, R^1\pi_*\mathcal{O}_\mathcal{X}) = H^0(\Delta, R^1\pi_*\mathcal{O}_\Delta) \). So there exists
\( \tau \in H^0(\Delta, R^1\pi_*\mathcal{Z}) \) such that \( \sigma - j(\tau) \) vanishes on \( \Delta^* \). Then by 1) we have \( \sigma = j(\tau) \).

2) \( \Rightarrow \) 3). As \( \Delta \) is Stein and contractible and we know that for each \( i \geq 0 \) the sheaves \( R^i\pi_*\mathcal{O}_\mathcal{X} \) are coherent and the sheaves \( R^i\pi_*\mathcal{Z} \) are constant sheaves, the Leray spectral sequence gives natural isomorphisms \( H^i(\mathcal{X}, \mathcal{O}_\mathcal{X}) \cong H^0(\Delta, R^i\pi_*\mathcal{O}_\mathcal{X}) \) and \( H^i(\mathcal{X}, \mathcal{Z}) \cong H^0(\Delta, R^i\pi_*\mathcal{Z}) \) for each \( i \geq 0 \). Then we have:

\[
\text{Coker} j := H^0(\Delta, R^1\pi_*\mathcal{O}_\mathcal{X}) / j(H^0(\Delta, R^1\pi_*\mathcal{Z})) \cong H^1(\mathcal{X}, \mathcal{O}_\mathcal{X}) / H^1(\mathcal{X}, \mathcal{Z})
\]

which classifies the holomorphic line bundles on \( \mathcal{X} \) which are topologically trivial, up to isomorphism. So the isomorphism class of a given topologically trivial line bundle \( L \) is defined by the image in \( \text{Coker} j \) of some \( \sigma \in H^0(\Delta, R^1\pi_*\mathcal{O}_\mathcal{X}) \).

Now take a line bundle \( L \) on \( \mathcal{X} \) which is topologically trivial. Assume that \( L \) is holomorphically trivial on each \( X_s \) for \( s \in \Delta^* \). So, thanks to the lemma 2.0.4, this implies that the section \( \sigma \) corresponding to the isomorphism class of \( L \) is such that \( \sigma|_{\Delta^*} \) is in \( j(H^0(\Delta^*, R^1\pi_*\mathcal{Z})) \). So by 2) we obtain that \( \sigma \) gives 0 in \( \text{Coker} j \) and then the line bundle \( L \) is holomorphically trivial. So the restriction to \( X_0 \) is holomorphically trivial and 3) is proved.

3) \( \Rightarrow \) 1). If \( \sigma \in H^0(\Delta, R^1\pi_*\mathcal{O}_\mathcal{X}) \) vanishes on \( \Delta^* \) this implies that the corresponding line bundle on \( \mathcal{X} \) is trivial on each \( X_s, \forall s \in \Delta^* \). If \( L_{X_0} \) is also trivial, then the lemma 2.0.4 implies that \( L \) is trivial on \( \mathcal{X} \). So there exists some \( \tau \in H^0(\Delta, R^1\pi_*\mathcal{Z}) \) such that \( \sigma = j(\tau) \). But as \( j \) is injective and as \( R^1\pi_*\mathcal{Z} \) is a constant sheaf, we have \( \tau = 0 \) and then \( \sigma = 0 \). So if \( \sigma \neq 0 \) the restriction \( L|_{X_0} \) cannot be holomorphically trivial and then 3) gives a contradiction.

\[\boxed{\text{Corollary 2.0.6} \quad \text{Let} \ \pi : \mathcal{X} \to \Delta \ \text{a proper holomorphic submersion of a complex manifold} \ \mathcal{X} \ \text{onto an open disc} \ \Delta \ \text{with center} \ 0 \ \text{in} \ \mathbb{C}, \ \text{with} \ n-\text{dimensional connected fibres. Then the coherent sheaf} \ R^1\pi_*\mathcal{O}_\mathcal{X} \ \text{is locally free.}}\]

**Proof.** It is enough to prove that the coherent sheaf \( R^1\pi_*\mathcal{O}_\mathcal{X} \) has no torsion so that the property 1) in the previous lemma is satisfied. But he property 3) of the previous lemma is given by the proposition 2.0.3.

**Proof of the Theorem 2.0.1.** It is enough to prove that this sheaf is \( S-\text{flat} \). But the classical “curve test” for flatness is clearly satisfied thanks to the corollary 2.0.6.

\(^1\)A geometric way to get this is to consider the linear space associated to this coherent sheaf: then on any curve it has constant rank by corollary 2.0.6 so it is a vector bundle and the sheaf is locally free.
3 Application

As an immediate consequence of the theorem 2.0.1 we can suppress the hypothesis on the continuity of the $h^{0,1}(s)$ in the theorem 1.0.3 of [B.15] and obtain the following semi-continuity result for the algebraic dimension.

**Theorem 3.0.7** Let $\pi : X \to S$ be a holomorphic family of compact complex connected manifolds of dimension $n$ parametrized by an irreducible complex space $S$. Assume that there exists a dense Zariski open set $S'$ in $S$ such that for each $s$ in $S'$ the manifold $X_s$ satisfies the $\partial \bar{\partial}$–lemma\(^2\) and such that there exists a (smooth) relative sG-form for the family $\pi|_{S'} : X|_{S'} \to S'$.

Then if $a := \inf_{s \in S'} [a(X_s)]$ we have $a(X_s) \geq a$ for each $s \in S$. ■

**Remark.** A simpler statement (see remark 3 following the theorem 1.0.3 in [B.15]) which is a special case of the previous one, is obtained by assuming that the restriction of $\pi$ to $\pi^{-1}(S')$ is a weakly kähler morphism in the sense of F. Campana (see for instance [C.81]); this implies the $\partial \bar{\partial}$–lemma assumption and the existence of a smooth relative sG-form for the restriction of $\pi$ over $S'$. □

As it is not so easy to show that a proper map is weakly Kähler (and we need less : each fibre in $S'$ has a sG-form and satisfies the $\partial \bar{\partial}$–lemma is enough) let me recall the following results from [B.15]

**Lemma 3.0.8** Let $\pi : X \to S$ be a proper holomorphic family of compact connected complex manifolds of dimension $n$ parametrized by an irreducible complex space $S$. Assume that for a point $s_0 \in S$, the manifold $X_{s_0} := \pi^{-1}(s_0)$ has a sG-form $\omega_0$. Then we can find a small open neighbourhood $S'$ of $s_0$ in $S$ and a relative sG-form $\omega$ on $\pi^{-1}(S')$ inducing $\omega_0$ on $X_{s_0}$.

**Theorem 3.0.9** Let $\pi : X \to S$ be a holomorphic family of compact complex connected manifolds of dimension $n$ parametrized by an irreducible complex space $S$. Let $s_0$ in $S$ such that the manifold $X_{s_0}$ admits a (smooth) sG-form. Then there exists an open neighbourhood $S_0$ of $s_0$, a countable union $\Sigma$ of closed irreducible analytic subsets in $S_0$ with no interior point and a non negative integer $a$ such that

(i) For any $s \in S_0$ we have $a(X_s) \geq a$.

(ii) For any $s \in S_0 \setminus \Sigma$ we have $a(X_s) = a$.

Then the following corollaries are immediate from the theorem 3.0.7 and 3.0.9

\(^2\)See for instance [Va.86].
Corollary 3.0.10 Let $\pi : \mathcal{X} \to S$ be a holomorphic family of compact complex connected manifolds of dimension $n$ parametrized by an irreducible complex space $S$. Assume that there exists a dense Zariski open set $S'$ in $S$ such that for each $s$ in $S'$ the manifold $X_s$ is Kähler. Then if $a := \inf_{s \in S'} [a(X_s)]$ we have $a(X_s) \geq a$ for each $s \in S$.

Corollary 3.0.11 Let $\pi : \mathcal{X} \to S$ be a holomorphic family of compact complex connected manifolds of dimension $n$ parametrized by an irreducible complex space $S$. Assume that there exists a dense Zariski open set $S'$ in $S$ such that for each $s$ in $S'$ the manifold $X_s$ is projective. Then for each $s \in S$ the manifold $X_s$ is Moishezon.

We conclude by noticing that there exists an analytic family of smooth complex compact surfaces of the class VII (not Kähler) parametrized by a disc $\Delta$ such that the central fibre has algebraic dimension 0 and all other fibres have algebraic dimension 1. See [F-P.09].

This shows that in our theorem 3.0.7 some Kähler type assumption on the general fibre $X_s$ cannot be avoided in order that the “general” algebraic dimension gives a lower bound for the algebraic dimensions of all fibres.

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