The DLCQ Spectrum of $\mathcal{N} = (8, 8)$ Super Yang-Mills

F. Antonuccio, O. Lunin, S. Pinsky

Department of Physics,
The Ohio State University,
Columbus, OH 43210, USA

H.-C. Pauli, S. Tsujimaru

Max-Planck-Institut für Kernphysik,
69029 Heidelberg, Germany

Abstract

We consider the 1+1 dimensional $\mathcal{N} = (8, 8)$ supersymmetric matrix field theory obtained from a dimensional reduction of ten dimensional $\mathcal{N} = 1$ super Yang-Mills. The gauge groups we consider are U($N$) and SU($N$), where $N$ is finite but arbitrary. We adopt light-cone coordinates, and choose to work in the light-cone gauge. Quantizing this theory via Discretized Light-Cone Quantization (DLCQ) introduces an integer, $K$, which restricts the light-cone momentum-fraction of constituent quanta to be integer multiples of $1/K$. Solutions to the DLCQ bound state equations are obtained for $K = 2, 3$ and $4$ by discretizing the light-cone supercharges, which preserves supersymmetry manifestly. We discuss degeneracies in the massive spectrum that appear to be independent of the light-cone compactification, and are therefore expected to be present in the decompactified limit $K \to \infty$. Our numerical results also support the claim that the SU($N$) theory has a mass gap.
1 Introduction

The non-perturbative properties of super Yang-Mills theories have received a lot of attention lately. In a seminal paper by Witten [1], it was shown that the low energy dynamics of $N$ coincident $Dp$-branes could be described by $p + 1$ dimensional $U(N)$ super Yang-Mills. This insight was instrumental in motivating the M(atrix) theory conjecture [2], and also played a role in the AdS/CFT correspondence recently proposed by Maldacena [3].

In summary, theorists are now grappling with the rather surprising fact that Yang-Mills theories seem to know more about the dynamics of string theory than previously conceived. Moreover, physics in many space-time dimensions may be described consistently by low dimensional Yang-Mills theories. There is therefore renewed interest in studying the non-perturbative properties of low dimensional super Yang-Mills theories.

Motivated by these developments, we consider the $1 + 1$ dimensional supersymmetric matrix field theory obtained from a dimensional reduction of ten dimensional $\mathcal{N} = 1$ super Yang-Mills, which results in a two dimensional gauge theory with $\mathcal{N} = (8,8)$ supersymmetry. The possible gauge groups are $U(N)$ and $SU(N)$, where $N$ is finite but arbitrary. A similar theory with $\mathcal{N} = (1,1)$ supersymmetry was studied recently in [4].

After introducing light-cone coordinates, and adopting the light-cone gauge, it is a straightforward procedure to implement Discrete Light-Cone Quantization (DLCQ) in order to extract numerical bound state solutions [5]. As was pointed out in the earlier work [6], exact supersymmetry may be preserved in the DLCQ spectrum if we choose to discretize the light-cone supercharges rather than the light-cone Hamiltonian.

The complexity of the $\mathcal{N} = (8,8)$ model far exceeds any other two dimensional theories studied in the context of DLCQ (see [4] for an extensive review), since there are now eight boson and eight fermion fields that propagate as physical modes. In practice, this means we can only probe the theory for rather crude discretizations ($K \leq 4$, where $1/K$ is the smallest unit of light-cone momentum). Despite this shortcoming, we are able to resolve some interesting features of the decompactified ($K \to \infty$) theory. In particular, we are able to count degeneracies of certain states in the massive spectrum, and establish evidence for the existence of a mass gap in the $SU(N)$ theory.

The organization of the paper may be summarized as follows; in Section 2 we introduce the $1 + 1$ dimensional $\mathcal{N} = (8,8)$ supersymmetric gauge theory, which we formulate in light-cone coordinates. Explicit expressions for the quantized light-cone supercharges are
written down, followed by a discussion on the DLCQ formulation of the theory. In Section 3 we tabulate the results of our numerical DLCQ analysis, highlighting the degeneracies observed in the spectrum. We also argue why the numerical results are consistent with the existence of a mass gap; i.e. there are no normalizable massless states in the SU(N) theory other than the light-cone vacuum. A summary of our observations, and further discussion, appears in Section 4. The formulation of ten-dimensional super Yang-Mills theory in light-cone coordinates is presented in Appendix A.

2 Light-Cone Quantization and DLCQ at Finite N

The two dimensional \( \mathcal{N} = (8,8) \) supersymmetric gauge theory we are interested in may be formally obtained by dimensionally reducing 9 + 1 dimensional \( \mathcal{N} = 1 \) super Yang-Mills to 1 + 1 dimensions. For the sake of completeness, we review the underlying ten dimensional light-cone Yang-Mills theory in Appendix A — in perhaps more detail than is customary — although the ideas should be familiar to many readers.

Dimensional reduction of the ten dimensional Yang-Mills action (42) given in Appendix A is carried out by stipulating that all fields are independent of the (eight) transverse coordinates \( x^I, I = 1, \ldots, 8 \). We may therefore assume that the fields depend only on the light-cone variables \( \sigma^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^9) \). The resulting two dimensional theory may be described by the action

\[
S_{1+1}^{LC} = \int d\sigma^+ d\sigma^- \operatorname{tr} \left( \frac{1}{2} D_\alpha X_I D^\alpha X_I + \frac{g^2}{4} [X_I, X_J]^2 - \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} + i \theta_R^T D_+ \theta_R + i \theta_L^T D_- \theta_L - \sqrt{2} g \theta_L^T \gamma^I [X_I, \theta_R] \right),
\]

where the repeated indices \( \alpha, \beta \) are summed over light-cone indices \( \pm \), and \( I, J \) are summed over transverse indices \( 1, \ldots, 8 \). The eight scalar fields \( X_I(\sigma^+, \sigma^-) \) represent \( N \times N \) Hermitian matrix-valued fields, and are remnants of the transverse components of the ten dimensional gauge field \( A_\mu \), while \( A_\pm(\sigma^+, \sigma^-) \) are the light-cone gauge field components of the residual two dimensional U(N) or SU(N) gauge symmetry. The spinors \( \theta_R \) and \( \theta_L \) are remnants of the right-moving and left-moving projections of a sixteen component real spinor in the ten dimensional theory. The components of \( \theta_R \) and \( \theta_L \) transform in the adjoint representation of the gauge group. \( F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha + \)

\( \) \( ^1 \)The space-time points in ten dimensional Minkowski space are parametrized by coordinates \((x^0, x^1, \ldots, x^9)\).
$ig[A_\alpha, A_\beta]$ is the two dimensional gauge field curvature tensor, while $D_\alpha = \partial_\alpha + ig[A_\alpha, \cdot]$ is the covariant derivative for the (adjoint) spinor fields. The eight $16 \times 16$ real symmetric matrices $\gamma^I$ are defined in Appendix A.

Since we are working in the light-cone frame, it is natural to adopt the light-cone gauge $A_- = 0$. With this gauge choice, the action (1) becomes

$$\tilde{S}_{1+1}^{LC} = \int d\sigma^+ d\sigma^- tr \left( \partial_+ X_I \partial_- X_I + i \theta_R^T \partial_+ \theta_R + i \theta_L^T \partial_- \theta_L + \frac{1}{2} (\partial_- A^+)^2 + gA_+ J^+ - \sqrt{2} g \theta_L^T \gamma^I [X_I, \theta_R] + \frac{g^2}{4} [X_I, X_J]^2 \right),$$

(2)

where $J^+ = i [X_I, \partial_- X_I] + 2 \theta_R^T \theta_R$ is the longitudinal momentum current. The (Euler-Lagrange) equations of motion for the $A_+$ and $\theta_L$ fields are now

$$\partial_+^2 A_+ = g J^+,$$

(3)

$$\sqrt{2} i \partial_- \theta_L = g \gamma^I [X_I, \theta_R].$$

(4)

These are evidently constraint equations, since they are independent of the light-cone time $\sigma^+$. The “zero mode” of the constraints above provide us with the conditions

$$\int d\sigma^- J^+ = 0, \text{ and } \int d\sigma^- \gamma^I [X_I, \theta_R] = 0,$$

(5)

which will be imposed on the Fock space to select the physical states in the quantum theory. The first constraint above is well known in the literature, and projects out the colorless states in the quantized theory. The second (fermionic) constraint is perhaps lesser well known, but certainly provides non-trivial relations governing the small-$x$ behavior of light-cone wave functions.

At any rate, equations (3),(4) permit one to eliminate the non-dynamical fields $A_+$ and $\theta_L$ in the theory, which is a particular feature of light-cone gauge theories. There are no ghosts. We may therefore write down explicit expressions for the light-cone momentum $P^+$ and Hamiltonian $P^-$ in terms of the physical degrees of freedom of the theory, which are denoted by the eight scalars $X_I$, and right-moving spinor $\theta_R$:

$$P^+ = \int d\sigma^- tr \left( \partial_- X_I \partial_- X_I + i \theta_R^T \partial_- \theta_R \right),$$

(6)

$$P^- = g^2 \int d\sigma^- tr \left( - \frac{1}{2} \frac{\partial^2}{\partial^2_+} J^+ - \frac{1}{4} [X_I, X_J]^2 \right.$$

$$\left. + \frac{i}{2} (\gamma^I [X_I, \theta_R])^T \frac{1}{\partial_-} \gamma^J [X_J, \theta_R] \right).$$

(7)

2If we introduce a mass term, such relations become crucial in establishing finiteness conditions. See 8, for example.
The light-cone Hamiltonian propagates a given field configuration in light-cone time $\sigma^+$, and contains all the non-trivial dynamics of the interacting field theory.

In the representation for the $\gamma^I$ matrices specified by (32) in Appendix A, we may write

$$\theta_R = \begin{pmatrix} u_0 \\ \end{pmatrix},$$

where $u$ is an eight component real spinor.

In terms of their Fourier modes, the fields may be expanded at light-cone time $\sigma^+ = 0$ to give

$$X^I_{pq} (\sigma^-) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{dk^+}{\sqrt{2k^+}} \left( a^I_{pq} (k^+) e^{-ik^+\sigma^-} + a^I_{qp} (k^+) e^{ik^+\sigma^-} \right), \quad I = 1, \ldots, 8;$$

$$u^\alpha_{pq} (\sigma^-) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{dk^+}{\sqrt{2}} \left( b^\alpha_{pq} (k^+) e^{-ik^+\sigma^-} + b^\alpha_{qp} (k^+) e^{ik^+\sigma^-} \right), \quad \alpha = 1, \ldots, 8.$$ (9)

For the gauge group $U(N)$, the (anti)commutation relations take the form

$$[a^I_{pq} (k^+), a^J_{rs} (k'^+) \dagger] = \delta^{IJ} \delta_{pr} \delta_{qs} \delta(k^+ - k'^+),$$

$$\{b^\alpha_{pq} (k^+), b^\beta_{rs} (k'^+) \dagger\} = \delta^{\alpha\beta} \delta_{pr} \delta_{qs} \delta(k^+ - k'^+),$$

while for $SU(N)$, we have the corresponding relations

$$[a^I_{pq} (k^+), a^J_{rs} (k'^+) \dagger] = \delta^{IJ} \delta_{pr} \delta_{qs} \delta(k^+ - k'^+) - \frac{1}{N} \delta_{pq} \delta_{rs} \delta(k^+ - k'^+),$$

$$\{b^\alpha_{pq} (k^+), b^\beta_{rs} (k'^+) \dagger\} = \delta^{\alpha\beta} \delta_{pr} \delta_{qs} \delta(k^+ - k'^+) - \frac{1}{N} \delta_{pq} \delta_{rs} \delta(k^+ - k'^+).$$ (13)

An important simplification of the light-cone quantization is that the light-cone vacuum is the Fock vacuum $|0\rangle$, defined by

$$a^I_{pq} (k^+) |0\rangle = b^\alpha_{pq} (k^+) |0\rangle = 0,$$ (15)

for all positive longitudinal momenta $k^+ > 0$. We therefore have $P^+ |0\rangle = P^- |0\rangle = 0$.

The “charge-neutrality” condition (first integral constraint from (5)) requires that all the color indices must be contracted for physical states. Thus physical states are formed by color traces of the boson and fermion creation operators $a^I \dagger$, $b^\alpha \dagger$ acting on the light-cone vacuum. A single trace of these creation operators may be identified as a single closed string, where each creation operator (or ‘parton’), carrying some longitudinal

---

\^ The symbol $\dagger$ denotes quantum conjugation, and does not transpose matrix indices.
momentum $k^+$, represents a ‘bit’ of the string. A product of traced operators is then a multiple string state, and the quantity $1/N$ is analogous to a string coupling constant.

At this point, we may determine explicit expressions for the quantized light-cone operators $P^\pm$ by substituting the mode expansions (9), (10) into equations (6), (7). The mass operator $M^2 \equiv 2P^+P^-$ may then be diagonalized to solve for the bound state mass spectrum. However, as was pointed out in [6], it is more convenient to determine the quantized expressions for the supercharges, since this leads to a regularization prescription for $P^-$ that preserves supersymmetry even in the discretized theory.

In order to elaborate upon this last remark, first note that the continuum theory possesses sixteen supercharges, which may be derived from the dimensionally reduced form of the ten dimensional $\mathcal{N} = 1$ supercurrent:

$$Q^+_{\alpha} = 2^{1/4} \int_{-\infty}^{\infty} d\sigma^- \text{tr} \left( \partial_- X_I \cdot \beta_{I\alpha} \cdot \eta \right)$$

$$Q^-_{\alpha} = g \int_{-\infty}^{\infty} d\sigma^- \text{tr} \left( -2^{3/4} J^+ \frac{1}{\partial_-} u_\alpha + 2^{-1/4} \left[ X_I, X_J \right] \cdot (\beta_I^{T \beta_J})_{\alpha \beta} \cdot u_\beta \right),$$

where $\alpha = 1, \ldots, 8$, and repeated indices are summed. The eight $8 \times 8$ real matrices $\beta_I$ are discussed in Appendix A. By explicit calculation or otherwise, these charges satisfy the following relations:

$$\{Q^+_{\alpha}, Q^+_{\beta}\} = \delta_{\alpha\beta} \cdot \frac{1}{\sqrt{2}} P^+$$

$$\{Q^-_{\alpha}, Q^-_{\beta}\} = \delta_{\alpha\beta} \cdot \frac{1}{\sqrt{2}} P^-$$

If we substitute the mode expansions (9), (10) into equations (16), (17) for the light-cone supercharges $Q^+_{\alpha}$, we obtain the following ‘momentum representations’ for these charges:

$$Q^+_{\alpha} = 2^{-3/4} \int_{0}^{\infty} dk \sqrt{k} \cdot \beta_{I\alpha} \cdot \left( a^+_i (k) b_{ij} (k) - b^+_i (k) a_{ij} (k) \right),$$

and

$$Q^-_{\alpha} = \frac{i2^{-1/4} g}{\sqrt{\pi}} \int_{0}^{\infty} dk_1 dk_2 dk_3 \delta(k_1 + k_2 - k_3) \cdot \left\{ \frac{1}{2\sqrt{k_1 k_2}} \left( \frac{k_2 - k_1}{k_3} \right) \left[ b^+_i (k_3) a_{im} (k_1) a_{mj} (k_2) - a^+_i (k_1) a^+_{im} (k_2) b_{aij} (k_3) \right] \right. $$

$$\left. + \frac{1}{2\sqrt{k_1 k_3}} \left( \frac{k_1 + k_3}{k_2} \right) \left[ a^+_i (k_1) b^+_{amj} (k_2) a_{ij} (k_3) - a^+_i (k_3) a_{im} (k_1) b_{amj} (k_2) \right] \right\}.$$
\[ + \frac{1}{2\sqrt{k_2 k_3}} \left( \frac{k_2 + k_3}{k_1} \right) \left[ a_{ij}^\dagger(k_3) b_{aim}(k_1) a_{Imj}(k_2) - b_{i}^\dagger(k_1) a_{Imj}^\dagger(k_2)b_{ij}(k_3) \right] \]
\[ - \frac{1}{k_1} \left[ b_{\beta ij}^\dagger(k_3) b_{aim}(k_1) b_{\beta mj}(k_2) + b_{aim}^\dagger(k_1) b_{\beta mj}^\dagger(k_2) b_{ij}(k_3) \right] \]
\[ - \frac{1}{k_2} \left[ b_{\beta ij}^\dagger(k_3) b_{\beta im}(k_1) b_{aimj}(k_2) + b_{\beta im}^\dagger(k_1) b_{aimj}^\dagger(k_2) b_{ij}(k_3) \right] \]
\[ + \frac{1}{k_3} \left[ b_{\alpha ij}^\dagger(k_3) b_{\beta im}(k_1) b_{\beta mj}(k_2) + b_{\beta im}^\dagger(k_1) b_{\beta mj}^\dagger(k_2) b_{\alpha ij}(k_3) \right] \]
\[ + \left( \beta I \beta_J^T - \beta_J \beta_I^T \right)_{\alpha\beta} \times \left( \frac{1}{4\sqrt{k_1 k_2}} \left[ b_{\beta ij}^\dagger(k_3) a_{Iim}(k_1) a_{Imj}(k_2) + a_{Iim}^\dagger(k_1) a_{Imj}^\dagger(k_2) b_{ij}(k_3) \right] \right) \]
\[ + \frac{1}{4\sqrt{k_2 k_3}} \left[ a_{IIj}^\dagger(k_3) b_{\beta im}(k_1) a_{Imj}(k_2) + b_{\beta im}^\dagger(k_1) a_{Imj}^\dagger(k_2) a_{IIj}(k_3) \right] \]
\[ + \frac{1}{4\sqrt{k_3 k_1}} \left[ a_{IIIj}^\dagger(k_3) a_{Iim}(k_1) b_{\beta mj}(k_2) + a_{Iim}^\dagger(k_1) b_{\beta mj}^\dagger(k_2) a_{IIIj}(k_3) \right] \right) \right), \quad (21) \]

where repeated indices are always summed: \( \alpha, \beta = 1, \ldots, 8 \) (SO(8) spinor indices), \( I, J = 1, \ldots, 8 \) (SO(8) vector indices), and \( i, j, m = 1, \ldots, N \) (matrix indices).

In order to implement the DLCQ formulation of the bound state problem which is tantamount to imposing periodic boundary conditions \( \sigma^- \sim \sigma^- + 2\pi R \) – we simply restrict the momentum variable(s) appearing in the expressions for \( Q_\alpha^\pm \) (equations (20), (21)) to the following discretized set of momenta: \( \{ \frac{k}{K} P^+, \frac{2k}{K} P^+, \frac{3k}{K} P^+, \ldots \} \). Here, \( P^+ \) denotes the total light-cone momentum of a state, and may be thought of as a fixed constant, since it is easy to form a Fock basis that is already diagonal with respect to the quantum operator \( P^+ \) \footnote{It might be useful to consult \cite{3, 4, 11, 12} for an elaboration of DLCQ in models with adjoint fermions.}. The integer \( K \) is called the ‘harmonic resolution’, and \( 1/K \) measures the coarseness of our discretization – we recover the continuum by taking the limit \( K \to \infty \). Physically, \( 1/K \) represents the smallest positive unit of longitudinal momentum-fraction allowed for each parton in a Fock state.

Of course, as soon as we implement the DLCQ procedure, which is specified unambiguously by the harmonic resolution \( K \), the integrals appearing in the definitions for \( Q_\alpha^\pm \) are replaced by finite sums, and the eigen-equation \( 2P^+ P^- |\Psi\rangle = M^2 |\Psi\rangle \) is reduced to a finite matrix diagonalization problem. In this last step, we use the fact that \( P^- \) is proportional to the square of any one of the eight supercharges \( Q_\alpha^- \), \( \alpha = 1, \ldots, 8 \)
\footnote{We exclude the zero mode \( k^+ = 0 \) in our analysis; the massive spectrum is not expected to be affected by this omission, but there are issues concerning the light-cone vacuum that involve \( k^+ = 0 \) modes \cite{13, 14}.}.
(equation (13)), and so the problem of diagonalizing $P^-$ is equivalent to diagonalizing any one of the eight supercharges $Q^-_{\alpha}$. As was pointed out in [3], this procedure yields a supersymmetric spectrum for any resolution $K$. In the present work, we are able to perform numerical diagonalizations for $K = 2, 3$ and 4 with the help of Mathematica and a desktop PC.

The fact that we may choose any one of the eight supercharges to calculate the spectrum provides a strong test for the correctness of our computer program. As expected, we find that the spectrum we obtain by squaring the eigenvalues of any two different supercharges yields the same massive spectrum. Moreover, the spectrum turns out to be exactly supersymmetric, which is also what we require. Such tests are very convenient when studying complicated models; for example, in the expression for $Q^-_{\alpha}$ (eqn (21)), there are approximately 3500 terms.

### 3 DLCQ Bound State Solutions

We consider discretizing the light-cone supercharge $Q^-_{\alpha}$ for a particular $\alpha \in \{1, 2, \ldots, 8\}$, and for the values $K = 2, 3, 4$. For a given resolution $K$, the light-cone momenta of partons in a given Fock state must be some positive integer multiple of $P^+/K$, where $P^+$ is the total light-cone momentum of the state. For example, when $K = 2$, there are precisely 256 Fock states in the U($N$) theory that are made up from two partons:

128 Bosons:

\[
\begin{align*}
&\{ \text{tr}[a^\dagger_I(\frac{1}{2}P^+)a^\dagger_J(\frac{1}{2}P^+)]|0\rangle \quad I, J = 1, 2, \ldots, 8; \\
&\text{tr}[b^\dagger_\alpha(\frac{1}{2}P^+)b^\dagger_\beta(\frac{1}{2}P^+)]|0\rangle \quad \alpha, \beta = 1, 2, \ldots, 8; \quad (\alpha \neq \beta); \\
&\text{tr}[a^\dagger_I(\frac{1}{2}P^+)][\text{tr}[a^\dagger_J(\frac{1}{2}P^+)]|0\rangle \quad I, J = 1, 2, \ldots, 8; \\
&\text{tr}[b^\dagger_\alpha(\frac{1}{2}P^+)][\text{tr}[b^\dagger_\beta(\frac{1}{2}P^+)]|0\rangle \quad \alpha, \beta = 1, 2, \ldots, 8; \quad (\alpha \neq \beta); \\
\end{align*}
\]

(22)

128 Fermions:

\[
\begin{align*}
&\{ \text{tr}[b^\dagger_\alpha(\frac{1}{2}P^+)b^\dagger_\beta(\frac{1}{2}P^+)]|0\rangle \quad I, \alpha = 1, 2, \ldots, 8; \\
&\text{tr}[a^\dagger_I(\frac{1}{2}P^+)][\text{tr}[b^\dagger_\alpha(\frac{1}{2}P^+)]|0\rangle \quad I, \alpha = 1, 2, \ldots, 8; \\
\end{align*}
\]

(23)

Of course, there are an additional 16 single particle states: eight bosons of the form $\text{tr}[a^\dagger_I(P^+)]|0\rangle$ and eight fermions of the form $\text{tr}[b^\dagger_\alpha(P^+)]|0\rangle$. This gives a total of $128 + 8$ bosons and $128 + 8$ fermions in the DLCQ Hilbert space for the U($N$) theory. If we calculate the matrix representation of $Q^-_{\alpha}$ (for any $\alpha$) with respect to this finite basis, we find that the masses $M^2 \sim (Q^-_{\alpha})^2$ of all these states are zero. In fact, this is what we expect. First of all, it can be shown that the the light-cone supercharge $Q^-_{\alpha}$ for the U($N$) gauge group is identical to the expression for the SU($N$) supercharge. This is tantamount to saying that the U(1) part of the U($N$) theory decouples completely as a
free field theory, and is identically zero for the light-cone Hamiltonian. The U(1) states in the U(N) DLCQ Fock space are readily identified; they are precisely those states that are made from a product of one-particle Fock states. The remaining states – consisting of 64 bosons and 64 fermions – belong to the SU(3) Fock space, and must therefore be single-trace states of two partons. Since the supercharge changes the number of partons in a Fock state by one, it must annihilate any SU(N) Fock state, which can only have two partons when \( K = 2 \).

The decoupling of the U(1) degrees of freedom in the U(N) theory provides trivial examples of massless states, and implies that all the non-trivial dynamics is contained in the SU(N) gauge theory. In particular, investigating the existence (or not) of massless states in the SU(N) theory is a highly non-trivial problem for \( K \geq 3 \). We will therefore restrict our attention to the SU(N) gauge theory.

To begin, we list all two parton states in the SU(N) gauge theory for \( K = 3 \):

128 Bosons:
\[
\left\{ \begin{array}{l}
\text{tr}[a_I^\dagger(\frac{1}{3}P^+)+a_J^\dagger(\frac{2}{3}P^+)]|0\rangle I, J = 1, 2, \ldots, 8; \\
\text{tr}[b_\alpha^\dagger(\frac{1}{3}P^+)+b_\beta^\dagger(\frac{2}{3}P^+)]|0\rangle \alpha, \beta = 1, 2, \ldots, 8;
\end{array} \right.
\]

128 Fermions:
\[
\left\{ \begin{array}{l}
\text{tr}[a_I^\dagger(\frac{1}{3}P^+)+b_\alpha^\dagger(\frac{2}{3}P^+)]|0\rangle I, \alpha = 1, 2, \ldots, 8; \\
\text{tr}[a_I^\dagger(\frac{2}{3}P^+)+b_\alpha^\dagger(\frac{1}{3}P^+)]|0\rangle I, \alpha = 1, 2, \ldots, 8.
\end{array} \right.
\]

Thus, there are 128 bosons and 128 fermions that consist of two partons. For three parton states, where the momentum is shared equally among each parton, the states take the following form:

688 Bosons:
\[
\left\{ \begin{array}{l}
\text{tr}[a_I^\dagger(\frac{1}{3}P^+)+a_J^\dagger(\frac{2}{3}P^+)+a_K^\dagger(\frac{1}{3}P^+)]|0\rangle I, J, K = 1, 2, \ldots, 8; \\
\text{tr}[a_I^\dagger(\frac{1}{3}P^+)+b_\alpha^\dagger(\frac{2}{3}P^+)+b_\beta^\dagger(\frac{1}{3}P^+)]|0\rangle I, \alpha, \beta = 1, 2, \ldots, 8;
\end{array} \right.
\]

688 Fermions:
\[
\left\{ \begin{array}{l}
\text{tr}[a_I^\dagger(\frac{1}{3}P^+)+b_\alpha^\dagger(\frac{2}{3}P^+)+b_\beta^\dagger(\frac{1}{3}P^+)]|0\rangle \alpha, \beta, \gamma = 1, 2, \ldots, 8; \\
\text{tr}[a_I^\dagger(\frac{1}{3}P^+)+a_J^\dagger(\frac{2}{3}P^+)+b_\alpha^\dagger(\frac{1}{3}P^+)]|0\rangle I, J, \alpha = 1, 2, \ldots, 8.
\end{array} \right.
\]

More specifically, there are 176 boson states of the form \( \text{tr}[a_I^\dagger(\frac{1}{3}P^+)+a_J^\dagger(\frac{2}{3}P^+)+a_K^\dagger(\frac{1}{3}P^+)]|0\rangle \), and \( 8 \times 8 \times 8 = 512 \) states of the form \( \text{tr}[a_I^\dagger(\frac{1}{3}P^+)+b_\alpha^\dagger(\frac{2}{3}P^+)+b_\beta^\dagger(\frac{1}{3}P^+)]|0\rangle \). Therefore, the SU(N) \( K = 3 \) DLCQ Hilbert space consists of 816 bosons and 816 fermions. It is indeed satisfying to find that our computer algorithm generates precisely this number of states.

The results of our DLCQ numerical diagonalization of \((Q^\alpha)^2\) is summarized in Table

---

\[^6\] We use Polya Theory to count these states; we think of a necklace with three beads, where each bead may be colored in eight distinct ways. The permutation symmetry involving only rotations is \(Z_3\), and the ‘cyclic index polynomial’ is therefore \(\frac{1}{3}[x_1^3 + 2x_3]\). Hence there are \(\frac{1}{3}[8^3 + 2 \cdot 8] = 176\) distinct configurations modulo cyclic rotations.
To test our numerical algorithms, we diagonalize different supercharges, and find the same spectrum – which is consistent with supersymmetry.

| Bound State Masses $M^2$ for $K = 3$ | Mass Degeneracy |
|------------------------------------|-----------------|
| $M^2$                              |                 |
| 0                                  | $560 + 560$     |
| 18                                 | $128 + 128$     |
| 72                                 | $112 + 112$     |
| 126                                | $16 + 16$       |

Table 1: SU($N$) bound state masses $M^2$ in units $g^2 N/\pi$ for resolution $K = 3$. When expressed in these units, the masses are independent of $N$ (i.e. there are no 1/$N$ corrections at this resolution), and so these results are applicable for any $N > 1$. The notation ‘128 + 128’ above implies a 256-fold mass degeneracy in the spectrum with 128 bosons and 128 fermions.

Let us now consider resolution $K = 4$. For the sake of definiteness, we enumerate carefully the SU($N$) DLCQ Fock space. Firstly, bosonic Fock states with only two partons take the following form:

192 bosons (2 partons):

$$
\begin{align*}
\text{tr}[a_I^\dagger \frac{1}{4} P^+] a_J^\dagger \frac{2}{4} P^+] |0\rangle & \quad I, J = 1, 2, \ldots, 8; \\
\text{tr}[b_\alpha^\dagger \frac{1}{4} P^+] b_\beta^\dagger \frac{2}{4} P^+] |0\rangle & \quad \alpha, \beta = 1, 2, \ldots, 8; \\
\text{tr}[a_I^\dagger \frac{1}{4} P^+] a_J^\dagger \frac{2}{4} P^+] |0\rangle & \quad I, J = 1, 2, \ldots, 8; \\
\text{tr}[b_\alpha^\dagger \frac{1}{4} P^+] b_\beta^\dagger \frac{2}{4} P^+] |0\rangle & \quad \alpha, \beta = 1, 2, \ldots, 8; \quad (\alpha \neq \beta);
\end{align*}
$$

It is straightforward to verify that there are $64 + 64 + 36 + 28 = 192$ such states. Similarly, bosonic Fock states with three partons take the form

2048 bosons (3 partons):

$$
\begin{align*}
\text{tr}[a_I^\dagger \frac{1}{4} P^+] a_J^\dagger \frac{1}{4} P^+] a_K^\dagger \frac{2}{4} P^+] |0\rangle & \quad I, J, K = 1, 2, \ldots, 8; \\
\text{tr}[a_I^\dagger \frac{1}{4} P^+] b_\alpha^\dagger \frac{1}{4} P^+] b_\beta^\dagger \frac{2}{4} P^+] |0\rangle & \quad I, \alpha, \beta = 1, 2, \ldots, 8; \\
\text{tr}[a_I^\dagger \frac{1}{4} P^+] b_\alpha^\dagger \frac{1}{4} P^+] b_\beta^\dagger \frac{1}{4} P^+] |0\rangle & \quad I, \alpha, \beta = 1, 2, \ldots, 8; \\
\text{tr}[a_I^\dagger \frac{1}{4} P^+] b_\alpha^\dagger \frac{1}{4} P^+] b_\beta^\dagger \frac{1}{4} P^+] |0\rangle & \quad I, \alpha, \beta = 1, 2, \ldots, 8;
\end{align*}
$$

and it is easily shown that there are $4 \times 8^3 = 2048$ such states. Enumerating all four parton bosonic Fock states requires additional effort. Firstly, we consider all single-trace bosonic Fock states with four partons; these are listed below:

8192 bosons (4 partons):

$$
\begin{align*}
\text{tr}[a_I^\dagger \frac{1}{4} P^+] a_J^\dagger \frac{1}{4} P^+] a_K^\dagger \frac{1}{4} P^+] a_L^\dagger \frac{2}{4} P^+] |0\rangle & \quad I, J, K, L = 1, \ldots, 8; \\
\text{tr}[a_I^\dagger \frac{1}{4} P^+] a_J^\dagger \frac{1}{4} P^+] b_\alpha^\dagger \frac{1}{4} P^+] b_\beta^\dagger \frac{2}{4} P^+] |0\rangle & \quad I, J, \alpha, \beta = 1, \ldots, 8; \\
\text{tr}[a_I^\dagger \frac{1}{4} P^+] b_\alpha^\dagger \frac{1}{4} P^+] a_J^\dagger \frac{1}{4} P^+] b_\beta^\dagger \frac{2}{4} P^+] |0\rangle & \quad I, J, \alpha, \beta = 1, \ldots, 8; \\
\text{tr}[b_\alpha^\dagger \frac{1}{4} P^+] b_\beta^\dagger \frac{1}{4} P^+] b_\gamma^\dagger \frac{1}{4} P^+] b_\delta^\dagger \frac{2}{4} P^+] |0\rangle & \quad \alpha, \beta, \gamma, \delta = 1, \ldots, 8.
\end{align*}
$$

10
To begin, all two-parton fermionic states have the form double-trace bosonic Fock states, yielding 14528 bosons in total. The remaining four-parton bosonic Fock states are formed from a product of two two-parton Fock states:

\[
\begin{align*}
&\text{4096 bosons (4 parts)}: \\
&\{ \text{tr}[a_I^\dagger (\frac{1}{4} P^+) a_J^\dagger (\frac{2}{4} P^+) ] \text{tr}[a_K^\dagger (\frac{1}{4} P^+) a_L^\dagger (\frac{1}{4} P^+) ] |0\} \quad I, J, K, L = 1, \ldots, 8; \\
&\text{tr}[a_I^\dagger (\frac{1}{4} P^+) a_J^\dagger (\frac{1}{4} P^+) ] \text{tr}[b_I^\dagger (\frac{1}{4} P^+) b_J^\dagger (\frac{1}{4} P^+) ] |0\} \quad I, J = 1, \ldots, 8; \\
&\text{tr}[a_I^\dagger (\frac{1}{4} P^+) b_I^\dagger (\frac{1}{4} P^+) ] \text{tr}[a_J^\dagger (\frac{2}{4} P^+) b_J^\dagger (\frac{2}{4} P^+) ] |0\} \quad I, J = 1, \ldots, 8; \\
&\text{tr}[b_I^\dagger (\frac{1}{4} P^+) b_J^\dagger (\frac{2}{4} P^+) ] \text{tr}[b_I^\dagger (\frac{1}{4} P^+) b_J^\dagger (\frac{2}{4} P^+) ] |0\} \quad \alpha, \beta, \gamma, \delta = 1, \ldots, 8.
\end{align*}
\]

Straightforward counting techniques yield 666 states of the first type listed above, 1008 states of the second type, 2016 states of the third type, and 406 states of the fourth type, giving a total of 4096 bosons.

We therefore conclude that there are 10432 single-trace bosonic Fock states, and 4096 double-trace bosonic Fock states, yielding 14528 bosons in total.

We now enumerate all the fermions, which turns out to be a much simpler calculation. To begin, all two-parton fermionic states have the form

\[
\begin{align*}
&\text{192 fermions (2 parts)}: \quad \begin{cases} \\
\text{tr}[a_I^\dagger (\frac{1}{4} P^+) b_I^\dagger (\frac{3}{4} P^+) ] |0\} \quad I, \alpha = 1, 2, \ldots, 8; \\
\text{tr}[a_I^\dagger (\frac{3}{4} P^+) b_I^\dagger (\frac{1}{4} P^+) ] |0\} \quad I, \alpha = 1, 2, \ldots, 8; \\
\text{tr}[a_I^\dagger (\frac{2}{4} P^+) b_I^\dagger (\frac{2}{4} P^+) ] |0\} \quad I, \alpha = 1, 2, \ldots, 8,
\end{cases}
\end{align*}
\]

and it is straightforward to check that there are 64 + 64 + 64 = 192 such states. Note that this equals the number of two-parton bosonic states. The enumeration of all three-parton fermionic states is listed below:

\[
\begin{align*}
&\text{2048 fermions (3 parts)}: \quad \begin{cases} \\
\text{tr}[a_I^\dagger (\frac{1}{4} P^+) a_J^\dagger (\frac{1}{4} P^+) b_I^\dagger (\frac{3}{4} P^+) ] |0\} \quad I, J, \alpha = 1, \ldots, 8; \\
\text{tr}[b_I^\dagger (\frac{1}{4} P^+) b_J^\dagger (\frac{1}{4} P^+) b_I^\dagger (\frac{2}{4} P^+) ] |0\} \quad \alpha, \beta, \gamma = 1, \ldots, 8; \\
\text{tr}[a_I^\dagger (\frac{2}{4} P^+) a_J^\dagger (\frac{2}{4} P^+) b_I^\dagger (\frac{3}{4} P^+) ] |0\} \quad I, J, \alpha = 1, \ldots, 8; \\
\text{tr}[a_I^\dagger (\frac{1}{4} P^+) a_J^\dagger (\frac{2}{4} P^+) b_I^\dagger (\frac{1}{4} P^+) ] |0\} \quad I, J, \alpha = 1, \ldots, 8,
\end{cases}
\end{align*}
\]

\[\text{Note:} \quad (I, \alpha) = (J, \beta). \]

We use Polya theory as before: The cyclic permutation symmetry of a necklace with four beads, each of which can be colored in eight distinct ways, is $\mathbb{Z}_4$, and gives rise to the cyclic index polynomial $\frac{1}{8} [x_1^4 + x_2^4 + 2 x_4]$. Thus, there are $\frac{1}{8} [8^4 + 8^2 + 2 \cdot 8] = 1044$ distinct configurations.

The symmetry here is the subgroup $\mathbb{Z}_2$ of $\mathbb{Z}_4$, and the resulting cyclic index polynomial is $\frac{1}{8} [x_1^4 + x_2^4]$. Thus, there are $\frac{1}{8} [8^4 + 8^2] = 2080$ distinct states modulo cyclic permutations. However, 64 of these states have zero norm, and may be identified as $\mathbb{Z}_4$ cyclic symmetry for which $(I, \alpha) = (J, \beta)$. Subtracting these states, we are left with $2080 - 64 = 2016$ distinct states of the third type.

The counting here is the same as in the first type because of the $\mathbb{Z}_4$ cyclic symmetry, but we must also subtract zero-norm states, which are precisely those states with $\alpha = \beta = \gamma = \delta$. There can only be 8 such states, and so we have $1044 - 8 = 1036$ distinct states overall.
and it is easy to verify that there are $4 \times 8^3 = 2048$ such states. Once again, this precisely matches the number of three-parton bosonic states. Four-parton fermionic states may consist of a single trace or a product of two traces. The single-trace Fock states take the form

$$8192 \text{ fermions (4 partons): } \left\{ \begin{array}{l} \text{tr} [a_J^\dagger (\frac{1}{4} P^+) a_I^\dagger (\frac{1}{4} P^+) a_K^\dagger (\frac{1}{4} P^+) b_{\alpha}^\dagger (\frac{1}{4} P^+)] |0\rangle \quad I, J, K, \alpha = 1, \ldots, 8; \\
\text{tr} [a_J^\dagger (\frac{1}{4} P^+) b_{\alpha}^\dagger (\frac{1}{4} P^+) b_{\beta}^\dagger (\frac{1}{4} P^+) b_{\gamma}^\dagger (\frac{1}{4} P^+)] |0\rangle \quad I, \alpha, \beta, \gamma = 1, \ldots, 8,
\end{array} \right.$$ 

and there are $2 \times 8^4 = 8192$ such states. This number agrees exactly with the number of single-trace bosonic states with four partons, although we recall that the counting of bosonic states was significantly more complicated.

Finally, four-parton fermionic states with two traces take the form

$$4096 \text{ fermions (4 partons): } \left\{ \begin{array}{l} \text{tr} [a_J^\dagger (\frac{1}{4} P^+) a_I^\dagger (\frac{1}{4} P^+) a_K^\dagger (\frac{1}{4} P^+) b_{\alpha}^\dagger (\frac{1}{4} P^+)] |0\rangle \quad I, J, K, \alpha = 1, \ldots, 8; \\
\text{tr} [a_J^\dagger (\frac{1}{4} P^+) b_{\alpha}^\dagger (\frac{1}{4} P^+) b_{\beta}^\dagger (\frac{1}{4} P^+) b_{\gamma}^\dagger (\frac{1}{4} P^+)] |0\rangle \quad I, \alpha, \beta, \gamma = 1, \ldots, 8.
\end{array} \right.$$ 

One may now verify that there are $36 \times 64 = 2304$ states of the first type, and $64 \times 28 = 1792$ states of the second type, yielding 4096 states overall. This of course agrees with the number of double-trace bosonic states calculated earlier.

We have thus verified that there are precisely an equal number of bosons and fermions in the $K = 4$ DLCQ Hilbert space of the SU($N$) theory. The total number of states is precisely $14528 + 14528 = 29056$. This reflects an important feature of DLCQ; namely, DLCQ preserves supersymmetry.

We remark here that the computer algorithm we use for constructing the DLCQ Fock states involves choosing an arbitrary set of input Fock states, and then repeatedly acting on this set by a preassigned number of supercharges until no new states are formed. These supercharges may then be diagonalized on this sub-space of Fock states. It is reassuring to find that this algorithm generates precisely the number of states that we counted above.

In order to determine the bound state spectrum, we need to diagonalize a particular supercharge $Q_{\alpha}$ on the DLCQ Hilbert space. Fortunately, because of the sixteen supersymmetries, we can reduce the problem of diagonalizing a $29056 \times 29056$ matrix to the problem of diagonalizing sixteen $1816 \times 1816$ block matrices. These block matrices may be reduced further; the double-trace states are already diagonal with respect to the mass-squared operator $M^2$, and are massless, so they decouple from the dynamics of single-trace Fock states. Therefore, the block matrix involving only single trace Fock states has dimensions $1304 \times 1304$, and is easily handled by a desk top PC.

The results of our numerical diagonalizations are presented in Table 2. Note that
there are 4096 + 4096 massive states; for \( K = 3 \), there were 256 + 256 massive bound states.

| Bound State Masses \( M^2 \) for \( K = 4 \) | Mass Degeneracy  |
|--------------------------------------------|------------------|
| 0                                          | 10432 + 10432    |
| 24                                         | 560 + 560        |
| 29.668                                     | 128 + 128        |
| 32                                         | 432 + 432        |
| 53.0605*                                   | 128 + 128        |
| 56                                         | 16 + 16          |
| 72                                         | 768 + 768        |
| 73.7982                                    | 16 + 16          |
| 80                                         | 768 + 768        |
| 88                                         | 336 + 336        |
| 90.3875*                                   | 112 + 112        |
| 96                                         | 336 + 336        |
| 114.332                                    | 128 + 128        |
| 120                                        | 112 + 112        |
| 141.612                                    | 112 + 112        |
| 151.091*                                   | 16 + 16          |
| 157.606                                    | 128 + 128        |

Table 2: SU(\(N\)) bound state masses \( M^2 \) in units \( g^2 N/\pi \) for resolution \( K = 4 \). When expressed in these units, the masses are independent of \( N \) (i.e. there are no \( 1/N \) corrections at this resolution), and so these results are applicable for any \( N > 1 \). Masses labeled with * correspond to the states observed at the lower resolution \( K = 3 \) (Table 1). To make this identification, it is necessary to study the Fock state expansion of these bound states.

4 Discussion

It is evident from the DLCQ bound state masses summarized in Tables 1 and 2 that there are a large number of massless states. At first, this seems to be at odds with the claim that the SU(\(N\)) gauge theory is expected to have a mass gap \[1\]. However, to determine whether there is a mass gap or not, we need to investigate whether there are normalizable states with zero mass in the continuum limit \( K \to \infty \). In our present study, we only considered the values \( K = 2, 3 \) and 4, and so it would seem hopeless at first to
make any statements about the continuum theory. It turns out, however, that there is already suggestive evidence of a mass gap which can be obtained at these low resolutions.

The crucial observation is that all the massless states in the DLCQ spectrum are made up of partons carrying the smallest positive unit of light-cone momentum allowed at the given resolution. For example, at $K = 2$, we saw that the SU($N$) Hilbert space consisted of two-parton Fock states – 64 bosons and 64 fermions (all massless) – where each parton carried the smallest integer unit of light-cone momentum. For $K = 3$, we find that all the massless states are a superposition of only three-parton Fock states, so each parton carries one unit of light-cone momentum. The states made from a superposition of two-parton Fock states, which were massless at $K = 2$, acquire a mass at the higher resolution $K = 3$. Similarly, after studying carefully the DLCQ bound states at resolution $K = 4$, we find that the massless states are superpositions of only four-parton Fock states. Each parton in these Fock states carries precisely one unit of light-cone momentum. There are no massless states involving Fock states with two or three partons at $K = 4$, so the massless states we observe at $K = 2$ and $K = 3$ have evidently acquired a mass at the higher resolution.

This pattern is very suggestive; namely, we expect that at a given resolution $K$, the massless states in the DLCQ spectrum will be a superposition of only $K$-parton Fock states, so that each parton carries a single unit of light-cone momentum. It is clear, then, that as we take the continuum limit $K \to \infty$, these massless states do not converge to any well-defined massless state in the continuum, which contrasts what is observed in a two dimensional supersymmetric model with (1, 1) supersymmetry [4]. Of course, this assumption is not enough to establish the existence of a mass gap, since it is possible that lighter massive states may appear at higher resolutions, and possibly converge to zero in the limit $K \to \infty$ [4]. However, we note that the lightest massive states at $K = 4$ are heavier than the ones observed at $K = 3$, and so increasing the resolution does not appear to introduce lighter massive states. Evidently, it would be desirable to probe larger values of $K$ to help clarify this issue, and we leave this for future work. Nevertheless, our results clearly support the existence of a mass gap in the continuum SU($N$) supersymmetric gauge theory.

There is also additional information about the continuum theory that emerges from our DLCQ results. First of all, the massive states observed at $K = 3$ (see Table [4]) are also observed at $K = 4$ (Table [2]) with the same mass degeneracy. We therefore expect these degeneracies to be preserved for all values of $K$, including the continuum limit.
$K \to \infty$. Our numerical results therefore indicate mass degeneracies that are expected to be present in the spectrum of the continuum theory.

We finally comment on possible connections between the DLCQ $\mathcal{N} = (8,8)$ model studied here and various string-related models. It has already been claimed that at resolution $K$ one finds massless states made up of $K$-parton Fock states, so that each parton carries precisely one unit of light-cone momentum. If one thinks of $K$ as being large but finite, then these states become string-like states made up of many ‘bits’. One also finds that the lightest massive states at $K = 4$ are composed of mainly three and four-parton Fock states, and so, in general, one expects the low energy spectrum to be dominated by string-like states – a property that is in fact observed for two dimensional $(1,1)$ super Yang-Mills \cite{13}. This suggests that the DLCQ model studied here might be closely related to the ‘string-bit’ models originally proposed by Thorn \cite{15}. Perhaps more intriguing is the possible connection with matrix string theory \cite{16}. In the DLCQ model we compactify a light-like direction, while for matrix string theory, one works with the same Lagrangian, but chooses instead to compactify a space-like coordinate, which originates from the geometry of closed strings in Type IIA string theory. It would be very interesting to compare these two schemes, and possibly relate them. Perhaps understanding the origin of quantized electric flux in the context of light-cone quantized gauge theories \cite{13,14} will pave the way to a better understanding of the significance of the DLCQ model studied here and the dynamics of non-perturbative string theory.

Acknowledgments

F.A. is grateful to Jungil Lee for assistance with computer work. S.T. is grateful for hospitality during his visit at Ohio State.

A Appendix: Super Yang-Mills in Ten Dimensions

Let’s start with $\mathcal{N} = 1$ super Yang-Mills theory in 9+1 dimensions with gauge group $\text{U}(N)$:

$$S_{9+1} = \int d^{10}x \text{tr}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{i}{2}\bar{\Psi}\Gamma^\mu D_\mu \Psi\right),$$  \hspace{1cm} (28)

where

\begin{align*}
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu], \hspace{1cm} (29) \\
D_\mu \Psi &= \partial_\mu \Psi + ig[A_\mu, \Psi], \hspace{1cm} (30)
\end{align*}
and \( \mu, \nu = 0, \ldots, 9 \). The Majorana spinor \( \Psi \) transforms in the \textit{adjoint} representation of \( U(N) \). The \textit{(flat)} space-time metric \( g_{\mu\nu} \) has signature \((-,-,\ldots,-)\), and we adopt the normalization \( \text{tr}(T^a T^b) = \delta^{ab} \) for the generators of the \( U(N) \) gauge group.

In order to realize the ten dimensional Dirac algebra \( \{ \Gamma_{\mu}, \Gamma_{\nu} \} = 2 g_{\mu\nu} \) in terms of Majorana matrices, we use as building blocks the reducible \( 8_s + 8_c \) representation of the \( \text{spin}(8) \) Clifford Algebra. In block form, we have

\[
\gamma^I = \begin{pmatrix} 0 & \beta_I \\ \beta_I^T & 0 \end{pmatrix}, \quad I = 1, \ldots, 8,
\]

where the \( 8 \times 8 \) real matrices, \( \beta_I \), satisfy \( \{ \beta_I, \beta_J^T \} = 2 \delta_{IJ} \). This automatically ensures the \( \text{spin}(8) \) algebra \( \{ \gamma^I, \gamma^J \} = 2 \delta^{IJ} \) for the \( 16 \times 16 \) real-symmetric matrices \( \gamma^I \). An explicit representation for the \( \beta_I \) algebra may be given in terms of a tensor product of Pauli matrices \[10\]. In the present context, we may choose a representation such that a ninth matrix, \( \gamma^9 = \gamma^1 \gamma^2 \cdots \gamma^8 \), which anti-commutes with the other eight \( \gamma^I \)'s, takes the explicit form

\[
\gamma^9 = \begin{pmatrix} 1_8 & 0 \\ 0 & -1_8 \end{pmatrix}.
\]

We may now construct \( 32 \times 32 \) pure imaginary (or Majorana) matrices \( \Gamma^\mu \) which realize the Dirac algebra for the Lorentz group \( \text{SO}(9,1) \):

\[
\Gamma^0 = \sigma_2 \otimes 1_{16}, \quad (33)
\]

\[
\Gamma^I = i \sigma_1 \otimes \gamma^I, \quad I = 1, \ldots, 8; \quad (34)
\]

\[
\Gamma^9 = i \sigma_1 \otimes \gamma^9. \quad (35)
\]

The Majorana spinor therefore has 32 real components, and since it transforms in the \textit{adjoint} representation of \( U(N) \), each of these components may be viewed as an \( N \times N \) Hermitian matrix.

An additional matrix \( \Gamma_{11} = \Gamma^0 \cdots \Gamma^9 \), which is equal to \( \sigma_3 \otimes 1_{16} \) in the representation specified by \((32)\), is easily seen to anti-commute with all other gamma matrices, and satisfies \( (\Gamma_{11})^2 = 1 \). It is also real, and so the Majorana spinor field \( \Psi \) admits a chiral decomposition via the projection operators \( \Lambda_{\pm} \equiv \frac{1}{2}(1 \pm \Gamma_{11}) \):

\[
\Psi = \Psi_+ + \Psi_-, \quad \Psi_{\pm} = \Lambda_{\pm} \Psi. \quad (36)
\]

We will therefore consider only spinors with positive chirality \( \Gamma_{11} \Psi = +\Psi \) (Majorana-Weyl):

\[
\Psi = 2^{1/4} \begin{pmatrix} \psi \\ 0 \end{pmatrix}, \quad (37)
\]
where $\psi$ is a sixteen component real spinor, and the numerical factor $2^{1/4}$ is introduced for later convenience.

Since $\gamma^9$ anti-commutes with the other eight $\gamma^I$’s, and satisfies $(\gamma^9)^2 = 1$, we may construct further projection operators $P_R \equiv \frac{1}{2}(1 + \gamma^9)$ and $P_L \equiv \frac{1}{2}(1 - \gamma^9)$ which project out, respectively, the right-moving and left-moving components of the sixteen component spinor $\psi$ defined in (37):

$$
\psi = \psi_R + \psi_L, \quad \psi_R = P_R \psi, \quad \psi_L = P_L \psi.
$$

(38)

This decomposition is particularly useful when working with light-cone coordinates, since in the light-cone gauge one can express the left-moving component $\psi_L$ in terms of the right-moving component $\psi_R$ by virtue of the fermion constraint equation. We will derive this result shortly. In terms of the usual ten dimensional Minkowski space-time coordinates, the light-cone coordinates are given by

$$
\begin{align*}
x^+ &= \frac{1}{\sqrt{2}}(x^0 + x^9), \quad \text{“time coordinate”} \\
x^- &= \frac{1}{\sqrt{2}}(x^0 - x^9), \quad \text{“longitudinal space coordinate”} \\
x^\perp &= (x^1, \ldots, x^8), \quad \text{“transverse coordinates”}
\end{align*}
$$

(39) (40) (41)

Note that the ‘raising’ and ‘lowering’ of the $\pm$ indices is given by the rule $x^\pm = x_{\mp}$, while $x^I = -x_I$ for $I = 1, \ldots, 8$, as usual. It is now a routine task to demonstrate that the Yang-Mills action (28) for the positive chirality spinor (37) is equivalent to

$$
S_{9+1}^{\text{LC}} = \int dx^+ dx^- dx^\perp \text{tr} \left( \frac{1}{2} F^2_{++} + F_{+I} F_{-I} - \frac{1}{4} F^2_{IJ} \\
+ i\bar{\psi}_R^T D_+ \psi_R + i\bar{\psi}_L^T D_- \psi_L + i\sqrt{2} \bar{\psi}_L^T \gamma^I D_I \psi_R \right),
$$

(42)

where the repeated indices $I, J$ are summed over $(1, \ldots, 8)$. Some surprising simplifications follow if we now choose to work in the light-cone gauge $A^+ = A_- = 0$. In this gauge $D_- \equiv \partial_-$, and so the (Euler-Lagrange) equation of motion for the left-moving field $\psi_L$ is simply

$$
\partial_- \psi_L = -\frac{1}{\sqrt{2}} \gamma^I D_I \psi_R,
$$

(43)

which is evidently a non-dynamical constraint equation, since it is independent of the light-cone time. We may therefore eliminate any dependence on $\psi_L$ (representing unphysical degrees of freedom) in favor of $\psi_R$, which carries the eight physical fermionic
degrees of freedom in the theory. In addition, the equation of motion for the $A_+$ field yields Gauss’ law:

$$\partial^2 A_+ = \partial_- \partial_I A_I + g J^+$$

(44)

where $J^+ = i[A_I, \partial_- A_I] + 2\psi_R^T \psi_R$, and so the $A_+$ field may also be eliminated to leave the eight bosonic degrees of freedom $A_I$, $I = 1, \ldots, 8$. Note that the eight fermionic degrees of freedom exactly match the eight bosonic degrees of freedom associated with the transverse polarization of a ten dimensional gauge field, which is of course consistent with the supersymmetry. We should emphasize that unlike the usual covariant formulation of Yang-Mills, the light-cone formulation here permits one to remove explicitly any unphysical degrees of freedom in the Lagrangian (or Hamiltonian); there are no ghosts.

References

[1] E.Witten, *Bound States Of Strings And p-Branes*, Nucl.Phys. B460, (1996), 335-350, [hep-th/9510135](hep-th/9510135).

[2] T.Banks, W.Fischler, S.Shenker and L.Susskind, *M Theory As A Matrix Model: A Conjecture*, Phys.Rev. D55, (1997), 5112-5128, [hep-th/9610043](hep-th/9610043).

[3] Juan M. Maldacena, *The Large N Limit of Superconformal Field Theories and Supergravity*, [hep-th/9711200](hep-th/9711200).

[4] F.Antonuccio, O.Lunin, S.Pinsky, *Non-Perturbative Spectrum of Two Dimensional (1,1) Super Yang-Mills at Finite and Large N*, [hep-th/9803170](hep-th/9803170) (to appear in Phys.Rev. D); F.Antonuccio, O.Lunin, S.Pinsky, *Bound States of Dimensionally Reduced SYM_{2+1} at Finite N*, [hep-th/9803027](hep-th/9803027) (To appear in Phys.Lett. B).

[5] H.-C. Pauli and S.J.Brodsky, *Phys.Rev. D32* (1985) 1993, 2001.

[6] Yoichiro Matsumura, Norisuke Sakai, Tadakatsu Sakai, *Mass Spectra of Supersymmetric Yang-Mills Theories in 1 + 1 Dimensions*, [hep-th/9504150](hep-th/9504150), Phys. Rev. D52 (1995) 2446.

[7] S.J. Brodsky, H.C. Pauli, and S.S. Pinsky, *Quantum Chromodynamics and Other Field Theories on the Light Cone* (To appear in Phys.Rept.), [hep-ph/9705477](hep-ph/9705477).
[8] F. Antonuccio, S. Brodsky and S. Dalley, *Light-cone Wavefunctions at Small x*, Phys. Lett. B412 (1997) 104-110, hep-ph/9705413.

[9] S. Dalley and I. Klebanov, *String Spectrum of 1+1-Dimensional Large N QCD with Adjoint Matter*, hep-th/9209049, Phys. Rev. D47 (1993) 2517-2527; G. Bhanot, K. Demeterfi and I. Klebanov, *1+1-Dimensional Large N QCD coupled to Adjoint Fermions*, hep-th/9307111, Phys. Rev. D48 (1993) 4980-4990.

[10] M.B. Green, J.H. Schwarz, and E. Witten, *Superstring Theory*, Vol.1, CUP (1987).

[11] F. Antonuccio, S.S. Pinsky, Phys. Lett B397:42-50, 1997, hep-th/9612021.

[12] S. Pinsky, “The Analog of the t’Hooft Pion with Adjoint Fermions” Invited talk at New Nonperturbative Methods and Quantization of the Light Cone, Les Houches, France, 24 Feb - 7 Mar 1997, hep-th/9705242.

[13] S. Pinsky, Phys. Rev D56:5040-5049, 1997, hep-th/9612073.

[14] S. Pinsky and D. Robertson, Phys. Lett B 379 (1996) 169-178; G. McCartor, D. G. Robertson and S. Pinsky Phys. Rev D56:1035-1049, 1997 hep-th/9612083.

[15] C.B. Thorn, Phys. Rev D19 (1979) 639; C.B Thorn, Reformulating String Theory with the 1/N Expansion, hep-th/9405069.

[16] L. Motl, Proposals on Non-Perturbative Superstring Interactions, hep-th/9701027; T. Banks and N. Seiberg, Strings from Matrices, Nucl. Phys. B497 (1997) 41, hep-th/9702187; R. Dijkgraaf, E. Verlinde, H. Verlinde, Matrix String Theory, Nucl. Phys. B500 (1997) 43, hep-th/9703030.