FINITE-TYPE KNOT INVARIANTS BASED ON THE BAND-PASS AND DOUBLED-DELTA MOVES

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ABSTRACT. We study generalizations of finite-type knot invariants obtained by replacing the crossing change in the Vassiliev skein relation by some other local move.

First, we represent the local moves by normal subgroups of the pure braid group $P_{\infty}$. Subgroups that are stable under the “strand-tripling” endomorphisms are shown to produce finite-type invariants with familiar properties; in particular, generalized Goussarov’s $n$-equivalence classes of knots form groups under the connected sum. (Similar results, but with a different approach, have been obtained before by Taniyama and Yasuhara.)

Treating local moves as surgeries on claspers, we study two particular cases in detail: the band-pass and the doubled delta move. While the band-pass move gives only one “new” invariant (namely, the Arf invariant), the invariants corresponding to the doubled-delta move contain information which is not available to any finite collection of Vassiliev invariants.

The complete degree 0 doubled-delta invariant is the $S$-equivalence class of the knot. In this context, we generalize a result of Murakami and Ohtsuki to show that the only primitive Vassiliev invariants of $S$-equivalence taking values in an abelian group with no 2-torsion arise from the Alexander-Conway polynomial. To this end, we introduce a discrete logarithm which transforms the coefficients of the Conway polynomial into primitive integer-valued invariants. As for the higher degree doubled-delta invariants, we start analyzing them by considering which Vassiliev invariants are degree 1 as doubled-delta invariants. We find that there is exactly one Vassiliev invariant in each odd degree which is of doubled-delta degree one, whereas in each even degree there is at most a $\mathbb{Z}_2$-valued invariant, which we show exists in degree 4. For higher doubled delta degrees, we observe that the Euler degree $n + 1$ part of Garoufalidis and Kricker’s rational lift of the Kontsevich integral is a doubled-delta degree $2n$ invariant. Finally, the doubled-delta move is a special case of Garoufalidis and Rozansky’s null-move of pairs $(M; k)$ where $M$ is a homology 3-sphere and $k$ a knot in $M$. A consequence of our work is that $n$-equivalence classes of pairs $(S^3; k)$ with respect to the null-move, do, in fact, form a group.

1. INTRODUCTION

It has been observed that the theory of finite-type knot invariants can be generalized by replacing the crossing change appearing in the Vassiliev skein relation by some other local move.

Vassiliev’s knot-space definition of finite-type invariants provides little motivation for such generalization. Indeed, a crossing change on a knot diagram corresponds to a codimension one singularity of a map from a circle to a 3-sphere, whereas other local moves correspond to singularities of higher codimension.

Nevertheless, the theories of finite-type invariants based on local moves other than crossing changes fit perfectly with the approach developed in the works of Goussarov [9, 10, 11] and later, Habiro [12] and Stanford [13]. In particular, Goussarov’s notion of $n$-equivalence can be extended to a wide class of local moves; this generalized notion of $n$-equivalence is consistent with suitably defined finite-type invariants, and the $n$-equivalence classes of knots form groups under the connected sum operation. See Stanford [14] for a particular example of such situation and Taniyama and Yasuhara [21] for a more general treatment of theories based on local moves.

The local moves considered in [19] and [21] essentially consist of replacing one fixed subtangle of a link by another fixed subtangle. A different approach to local moves can be taken by defining local moves as surgeries on claspers. These local moves can change the ambient manifold. Finite-type invariants based on
a move of this type (the “null-move”) were considered by Garoufalidis and Rozansky in [8]. It was shown in [8, 6] that the rational lift of the Kontsevich integral is a universal rational finite-type (with respect to the null-move) invariant of knots in integral homology spheres with trivial Alexander polynomial.

Apart from the null-move, the only local moves for which the corresponding finite-type invariants have been identified to some extent are the $C_k$-moves. Taniyama and Yasuhara proved in [20] that the space of primitive $C_k$-finite-type invariants of order $n$ and smaller coincides with the space of usual primitive finite-type invariants of order $kn$ and smaller.

The present paper consists of two parts. In the first part we show how local moves on knots can be interpreted via pure braid closures. The notion of a local move is replaced here by the notion of a modification by an element of a normal subgroup of the pure braid group on an infinite number of strands $P_\infty$ (This is the direct limit of finite pure braid groups, each included in the next by adding a trivial strand.) Each normal subgroup $G \subset P_\infty$ gives rise to $G$-finite-type knot invariants and to the relation of $\gamma_nG$-equivalence on the isotopy classes of knots. (Here $\gamma_n$ refers to the $n$th term of the lower central series.) Using a Markov-type theorem for the short-circuit closure of [15], we show the following

**Theorem 1.** If the subgroup $G$ is stable under “strand-tripling”, then

(i) two $G$-trivial knots are $\gamma_nG$-equivalent if and only if they cannot be distinguished by $G$-finite-type invariants of orders $n$ and smaller;

(ii) the set of $G$-trivial knots modulo $\gamma_nG$-equivalence is an abelian group.

Here a knot is said to be $G$-trivial if it is $G$-equivalent to the unknot. The precise definition of the strand-tripling endomorphisms of $P_\infty$ is given in Section 2.

The above result generalizes the well-known theorems of Goussarov [9]. It is very similar, though not immediately equivalent, to a theorem by Taniyama and Yasuhara [21]. It is probable that our result can be obtained using methods of [21]; nevertheless, we think that our approach is of independent interest.

We also re-state Theorem 1 in terms of claspers; this enables us to apply it to finite-type invariants based on the null move.

In the second part we study the finite-type invariants based on two particular local moves: the band-pass move and the doubled-delta move.

We prove that the finite-type invariants with respect to the band-pass move essentially coincide with the Vassiliev invariants. More precisely, primitive finite-type invariants with respect to the band-pass move of order $n$ coincide with primitive Vassiliev invariants of order $n$ for $n \geq 1$. The Arf invariant is the unique band-pass-finite-type invariant of order 0.

The case of the doubled-delta move turns out to be more interesting.

In what follows, we shall abbreviate “doubled-delta-finite-type invariants of order $n$” to “$\Delta\Delta_n$-invariants”. These are quite interesting already for $n = 0$: it has been proved by Naik and Stanford in [16] that two knots are $S$-equivalent if and only if one can be transformed to the other by a sequence of doubled delta moves. Therefore, $\Delta\Delta_0$-invariants coincide with the invariants of $S$-equivalence. Note that the Seifert form of a knot is a complete invariant of $S$-equivalence; in particular, $S$-trivial knots are exactly the knots with trivial Alexander polynomial.

One may ask which Vassiliev invariants are also $\Delta\Delta_0$-invariants. It was shown by Murakami and Ohtsuki in [14] that all $\mathbb{Q}$-valued Vassiliev invariants that are invariants of $S$-equivalence are polynomials in the coefficients of the Conway polynomial. Here we extend their result to primitive Vassiliev invariants with values in any abelian group $A$ with no 2-torsion.

To this end, we define a logarithm-like function

$$\log_Z : 1 + z \cdot \mathbb{Z}[[z]] \rightarrow z \cdot \mathbb{Z}[[z]]$$

which takes multiplication to addition. This is, as far as we know, a novel construction which should be quite useful in studying primitive Vassiliev invariants over the integers. Applying $\log_Z$ to the Conway polynomial $C(z)$, the coefficients are non-zero only for even degrees of $z$. Denote by $p_{2n}$ the coefficient at $z^{2n}$.
Theorem 2. For each \( n > 0 \), \( pc_{2n} \) is both a \( \Delta \Delta_0 \)-invariant and a primitive Vassiliev invariant. Every primitive Vassiliev knot invariant taking values in an Abelian group with no 2-torsion, which is also a \( \Delta \Delta_0 \)-invariant, is a linear combination of the \( pc_{2n} \). As a Vassiliev invariant, \( pc_{2n} \) has order \( 2n \), and \( pc_{2n} \) mod 2 has order \( 2n - 1 \).

Here we should clarify what it means for an \( A \)-valued invariant \( f \) to be a linear combination of the \( pc_{2n} \): this is said to be the case if there exists a homomorphism \( \phi : \mathbb{Z}^k \to A \) for some \( k \), such that \( f \) is the composition of the direct sum of the \( pc_{2n} \), for \( 0 < n \leq k \), with \( \phi \).

Proceeding to \( \Delta \Delta_1 \) invariants, we show

Theorem 3.

- The associated graded \( \mathbb{Q} \)-module of primitive rational-valued Vassiliev invariants which are also \( \Delta \Delta_1 \)-invariants, has exactly one generator in degree \( 2n + 1 \) for each \( n \geq 1 \), and no generators in even degrees.
- Any integer-valued Vassiliev invariant of order 4 is, modulo 2, a \( \Delta \Delta_1 \)-invariant.

It is possible that there are more 2-torsion invariants in larger even degrees; we shall see, however that at most one copy of \( \mathbb{Z}/2\mathbb{Z} \) can exist in each even degree.

One could assemble the \( \mathbb{Z} \)-valued invariants into a power series which is a \( \Delta \Delta_1 \)-invariant but not a Vassiliev invariant, but we leave open the question of whether there exist \( \Delta \Delta_1 \)-invariants which do not come directly from Vassiliev invariants in this way.

We shall not discuss \( \Delta \Delta_n \)-invariants with \( n > 1 \) in detail, though not for the lack of examples. Indeed, the \( \Delta \Delta \)-move is a special case of a null-move that preserves the ambient 3-manifold (see [6, 8]). Above we have mentioned that the rational lift of the Kontsevich integral, \( Z^{rat} \), is a universal rational null-finite-type invariant of \( S \)-trivial knots. This implies the following theorem.

Theorem 4. \( Z^{rat}_{1+\epsilon} \) is a \( \Delta \Delta_2 \)-invariant. Therefore, the vector space of rational \( \Delta \Delta_{2n} \)-invariants modulo \( \Delta \Delta_{2n+1} \)-invariants is infinite-dimensional.

Some notation. The band-pass and the doubled-delta moves will be denoted by BP and \( \Delta \Delta \) respectively; \( V \) will sometimes be used for the crossing change. (Here “\( V \)” stands for “Vassiliev invariants”.) The notation \( \mathbb{Z}[X] \) will be used for the free \( \mathbb{Z} \)-algebra on a monoid \( X \). The lower central series of a group \( H \) will be denoted by \( \gamma_k H \), indexed so that \( \gamma_1 H = H \). \( \mathcal{P}_N \) will denote the pure braid group on \( N \) strands; the trivial braid on \( N \) strands will be written as \( 1_N \).

2. Local Moves and Finite-Type Invariants

There are at least three different ways to define local moves on knots. The definitions we give below are not equivalent; however, all “interesting” local moves, such as the band-pass, doubled delta, \( C_k \)-moves and many other moves can be obtained from all three constructions.

2.1. Local Moves and Tangles. The most intuitive approach to local moves is via tangles. The following definition can be found, for example, in [21].

Let \( T \) and \( S \) be two tangles in an oriented closed ball \( U \). Assume that neither \( T \) nor \( S \) contain closed components and denote by \( t_1, t_2, \ldots, t_k \) and \( s_1, s_2, \ldots, s_k \) the arcs of \( T \) and \( S \) respectively. Suppose that for each \( t_i \) there exists some \( s_j \) such that \( \partial t_i = \partial s_j \). Then the ordered pair \((T, S)\) is called a local move. Two local moves \((T, S)\) and \((T', S')\), defined in 3-balls \( U \) and \( U' \) respectively, are equivalent if there exists an orientation preserving homeomorphism \( h : U \to U' \) such that \( h(T) = T' \) and \( h(S) \) is ambient isotopic to \( S' \) relative to \( \partial U' \).

Let \( K_1 \) and \( K_2 \) be knots in an oriented 3-manifold \( M \). We say that \( K_2 \) is obtained from \( K_1 \) by applying a local move \((T, S)\) if there exists a ball \( U \) in \( M \) such that

(i) \( \partial U \) intersects both knots transversely,
(ii) \( K_1 \) and \( K_2 \) coincide outside \( U \),
(iii) the pair \((U \cap K_1, U \cap K_2)\) is a local move equivalent to \((T, S)\).
A theory of local moves based on this definition is developed in [20][21]. We shall not give further details here.

2.2. Local moves and braids. There are several kinds of braid closures. We shall use the “short-circuit” closure of [15]; this is a version of the plat closure which sends pure braids on an odd number of strands to (long) knots. See Figure 2. The short-circuit closure commutes with adding two unbraided strands to a braid, hence, there is a map

\[ S : \mathcal{P}_\infty \rightarrow \text{Knots}. \]
A Markov-type theorem for the short-circuit closure is proved in [15]: it says that there are two subgroups $H^T$ and $H^B$ of $P_\infty$ such that the map $\mathcal{S}$ identifies the set of isotopy classes of knots with the two-sided quotient $H^T/\mathcal{P}_\infty/H^B$.

Let $G \subset \mathcal{P}_\infty$ be a normal subgroup. Two knots $a$ and $b$ are $G$-equivalent if there exist $x, y \in \mathcal{P}_\infty$ and $h \in G$ such that $a = \mathcal{S}(x)$, $b = \mathcal{S}(y)$ and $x = hy$. The Markov Theorem for the short-circuit closure implies that $G$-equivalence is indeed an equivalence relation on the set of isotopy classes of knots. All knots are $\mathcal{P}_\infty$-equivalent.

An informal explanation of this definition in terms of local moves is that $G$ should be thought of as the subgroup of $\mathcal{P}_\infty$, consisting of the braids that can be undone by some fixed local move. For example, crossing changes can undo all braids in $\mathcal{P}_\infty$: delta moves undo the commutator subgroup $[\mathcal{P}_\infty, \mathcal{P}_\infty]$ (see [19]). Then one can think of $G$-equivalent knots as those that can be transformed into each other by a sequence of local moves that undo $G$.

Related to $G$-equivalence is the notion of a $G$-finite-type invariant. Let $I_G$ be the augmentation ideal of $G$ inside $\mathbb{Z}[G]$, and denote by $\hat{I}_G$ the ideal in $\mathbb{Z}[\mathcal{P}_\infty]$ generated by $I_G$. The ideal $\hat{I}_G$ is the kernel of the ring homomorphism $\mathbb{Z}[\mathcal{P}_\infty] \to \mathbb{Z}[\mathcal{P}_\infty]/G$ sending elements of $\mathcal{P}_\infty$ to $\mathcal{P}_\infty/G$.

A linear function $\mathbb{Z}[\text{Knots}] \to A$, where $A$ is an abelian group, is a $G$-finite-type invariant of order $n$ if it vanishes on all elements of $\mathbb{Z}[\text{Knots}]$ which are of the form $S(a)$ with $a \in \hat{I}_G^n+1$. For $G = \mathcal{P}_\infty$ this defines the usual Vassiliev invariants. Notice that all the knots within the linear combination $S(a)$ are $G$-equivalent, so it makes sense to define a $G$-finite-type invariant on a single $G$-equivalence class. In particular, $G$-finite-type invariants of order $n$ for $G$-trivial knots can be defined as those that vanish on $S(I_G^n+1)$.

Denote by $\tau_0 : \mathcal{P}_\infty \to \mathcal{P}_\infty$ the homomorphism of shifting the braid by two strands “to the right”. In other words, $\tau_0$ sends the braid $x$ to $I_2 \otimes x$. For $k > 0$ let $\tau_k$ be the homomorphism of $\mathcal{P}_\infty$ into itself that triples the $k$-th strand.

**Theorem 5.** Let $G$ be a normal subgroup of $\mathcal{P}_\infty$ such that $\tau_k(G) \subset G$ for all $k \geq 0$. Then

(i) $G$-trivial knots considered modulo $\gamma_n G$-equivalence form a group under connected sum;

(ii) two $G$-trivial knots are $\gamma_n G$-equivalent if and only if they cannot be distinguished by $G_{\leq n}$-invariants.

The proof of this theorem will be given in Section 3.

### 2.3. Local moves and claspers

We shall assume that the reader is familiar with the language of claspers. For definitions and properties of claspers we refer to [12, 14]. We shall use the terminology of [4].

Let $T$ be an abstract (i.e. not embedded) clasper of a fixed tree type. Then a basic $T$-move on a pair $(M, k)$ where $k$ is a knot embedded in a three-manifold $M$, is a surgery on an embedded $T$-clasper in $M \setminus k$, which produces a new pair $(M', k')$. A $cT$-move is a $T$-clasper surgery where the leaves bound disjoint disks (called caps) which may hit the knot. Note that such moves preserve the ambient 3-manifold. An $nT$-move is a $T$-move where the clasper’s leaves link the knot homologically trivially. (Here “$n$” stands for “null.”) Note that this makes sense when $M$ is a homology sphere. Finally an $ncT$-move is a $T$-move where the leaves bound disjoint caps that intersect the knot algebraically trivially.

Let the clasper with tree type an interval be denoted $I$, and the clasper with tree type a “wye” be denoted $Y$. Then what follows is a table of other names for the above moves. See Figure 1.

| $c$ | $n$ | $nc$ |
|-----|-----|-----|
| $I$ | Crossing change (V) | Band-pass move (BP) |
| $Y$ | Delta move ($\Delta$) | Null-move ($\Delta \text{null}$) | Doubled-delta move ($\Delta \Delta$) |

This table should be interpreted in the sense that every $cI$-move can be realized by a sequence of crossing changes and similarly for the other entries.

Let $\mu$ denote a move type of the form $cT$ or $ncT$. Then there is a descending filtration

$$\mathbb{Z}[\text{Knots}] = \mathcal{F}_0^\mu \supset \cdots \supset \mathcal{F}_n^\mu \supset \mathcal{F}_{n+1}^\mu \supset \cdots,$$
where $F^\mu_n$ is defined as follows. Given a knot $k$, and $n$ disjointly embedded $\mu$-claspers $C_1, \ldots, C_n$, with disjointly embedded caps. Let
\[ [k; C_1, \ldots, C_n] = \sum_{\sigma \subset [n]} (-1)^{|\sigma|} k_\sigma, \]
where $k_\sigma$ means “modify $k$ by the claspers $C_i$” where $i \in \sigma$. Then $F^\mu_n$ is defined to be the submodule generated by such elements $[k; C_1, \ldots, C_n]$ for all choices of $k$ and $\{C_i\}$. By Habiro’s zip construction and standard arguments, in the $cT$ case the submodule $F^\mu_n$ is actually generated by such alternating sums where each clasper is simple. (That is, each of the caps intersect the knot in a single point.) In the $ncT$ case one may assume each cap intersects the knot in a pair of algebraically canceling points.

Let $\mathbb{Z}[\text{Knots}]$ be the quotient of $\mathbb{Z}[\text{Knots}]$ by the subspace generated by the vectors
\[ k_1 + k_2 - k_1 \neq k_2 \]
for all pairs of knots. Let $\overline{F^\mu_n} \subset \mathbb{Z}[\text{Knots}]$ be the image of $F^\mu_n$. Linear functions $\mathbb{Z}[\text{Knots}] \to G$ are exactly the primitive (that is, additive) knot invariants.

**Definition 6.**

(i) An invariant $f: \text{Knots} \to A$, where $A$ is an abelian group, is said to be $\mu$-finite-type of degree (order) $n$ if $f(F^\mu_{n+1}) = 0$. We also say that $f$ is a $\mu_n$-invariant.

(ii) Two knots $k_1$ and $k_2$ are $\mu_n$-equivalent if $k_1 - k_2 \in F^\mu_{n+1}$. In other words, $\mu_n$-equivalent knots are those that cannot be distinguished by $\mu_{\leq n}$ invariants. By $G^\mu_n$ we denote the set of knots which are $\mu_0$-equivalent to the unknot, considered modulo $\mu_n$-equivalence.

Theorem 5 implies the following statement.

**Theorem 7.** Given a tree $T$ which is a $Y$ or an $I$, let $\mu$ be a move of the type $cT$ or $ncT$. Then $G^\mu_n$ is an abelian group for all $n$.

The language of claspers facilitates the treatment of the null-move $N$. One can define a null-finite-type invariant of degree $n$ to be an invariant (with values in some abelian group) of pairs $(M, k)$ which vanishes on alternating sums of $n + 1$ null moves, and analogously define $N_n$-equivalence. Let $G^N_n$ be the set of pairs $(M, k)$ which are $N_0$-equivalent to the pair $(S^3, \text{unknot})$, considered modulo $N_n$-equivalence. Note that $G^N_n$ is a monoid under the operation of connected sum, where the 3-ball that is removed hits the knot in a standard arc. Let $G^N_n(S^3)$ be the submonoid of pairs $(S^3, k)$.

**Theorem 8.** $G^N_n(S^3)$ is a group for all $n$.

**Proof.** Let $X$ denote the set of pairs $(S^3, k)$ which are $N_0$-equivalent to the pair $(S^3, \text{unknot})$. We claim that $X$ is the set of all pairs $(S^3, k)$ where $k$ has trivial Alexander polynomial. Every such pair is in $X$ by [16]. Conversely, any element in $X$ will have trivial Alexander polynomial.

But then $X$ is precisely the set of knots in $S^3$ which are $\Delta\Delta_0$ equivalent to the unknot. $G^N_n$ is an abelian group, by Theorem 7 which projects onto the monoid $G^N_n(S^3)$. The latter is then forced to be a group. □

**Problem 1.** Is $G^N_n$ a group?

On the level of manifolds, this is true. The monoid of homology 3-spheres is a group after modding out by alternating sums of $Y$ moves as above [10] [22].

2.4. **Translating between claspers and braids.** Given a tree $T$, a simple $cT$ clasper (with respect to a given knot) is defined to be a $cT$ clasper where each leaf bounds a disk which hits the knot in one point.

**Definition 9.** A clasper $C$ embedded in the complement of the trivial braid $1_n$ is said to be braid-like, if

(i) Surgery along the clasper $(1_n)_C$ is a braid.

(ii) Regarding $1_n$ as a subset of a cube in the usual way, the complement of $1_n$ in the cube is a regular neighborhood of the clasper.

**Lemma 10.** If $T$ is an interval or a $Y$, then there is a braid-like $T$ clasper.
Proof. The case of an interval is obvious. A braid-like Y-clasper is illustrated below.

\begin{center}
\includegraphics[width=0.5\textwidth]{braid_clasper.png}
\end{center}

Definition 11.

(i) Let $G^{cT}$ be the subgroup of $\mathcal{P}_\infty$ normally generated by elements $(1_N)_{C_1 \cup \cdots \cup C_i}$, where $C_1 \cup \cdots \cup C_i$ is a union of $cT$ claspers which have the property that $(1_N)_{C_1 \cup \cdots \cup C_i}$ is a braid.

(ii) Denoting by $\eta: P_\infty \to P_\infty$ the homomorphism which doubles every strand, define $G^{ncT}$ as the group normally generated by $\eta(G^{cT})$ inside $B_\infty$, the direct limit of braid groups on a finite number of strands. Note that $G^{ncT}$ is indeed a subgroup of $P_\infty$.

Proposition 12. The subgroups $G^{ncT}$ and $G^{cT}$ are invariant under the strand tripling homomorphism $\tau_k$.

Proof. Consider a generating element $(1_N)_{C_1 \cup \cdots \cup C_i}$ of $G^{cT}$. If we triple the $k$th strand, this will either miss the caps of the claspers or it will convert a single intersection point to three intersection points. The claspers will remain capped. Also, the homomorphism $\tau_k$ commutes with the clasper surgery, indicating that $\tau_k((1_N)_{C_1 \cup \cdots \cup C_i})$ is again a braid. We note that the inverse of a generator is also a generator, since it is formed by mirror reflection, and the mirror reflection of a clasper surgery is also a clasper surgery of the same tree type. (Note though that the clasper itself is not the mirror reflection, since orientation data is important.)

Now consider a generator $\eta((1_N)_{C_1 \cup \cdots \cup C_i})$ of $G^{ncT}$. Applying $\tau_k$, we are tripling one strand of the double of a strand in $\eta((1_N)_{C_1 \cup \cdots \cup C_i})$. That is we are doubling all the strands of the braid except one which we are quadrupling. So $\tau_k\eta((1_N)_{C_1 \cup \cdots \cup C_i}) = \eta((1_N+1)_{C'_1 \cup \cdots \cup C'_i})$ where the claspers $C'_i$ are formed from $C_i$ by doubling the appropriate strand of $1_N$.

Theorem 13. Suppose $T$ is a tree for which a braid-like clasper exists. For $\mu = cT$ or $\mu = ncT$, $\mu_n$-invariants coincide with $G^{nT}_n$-invariants.

Proof. Let us first consider the case $\mu = cT$. Suppose $f: \text{Knots} \to G$ is a $cT_n$-invariant. Note that $\hat{I}^{n+1}_{G^{cT}}$ is generated by elements of the form

$$g_1(h_1 - 1)g_2(h_2 - 1) \cdots g_{n+1}(h_{n+1} - 1)g_{n+2},$$

where $h_i \in G^{cT}$ and $g_i \in P_\infty$. One may assume that each $h_i$ is a normal generator of $G^{cT}$: $h_i = 1_{\overline{C}_i}$, where $\overline{C}_i$ is a union of $cT$ claspers.

Letting

$$x = S \left( g_1(1_{\overline{C}_1} - 1)g_2(1_{\overline{C}_2} - 1) \cdots g_{n+1}(1_{\overline{C}_{n+1}} - 1)g_{n+2} \right)$$

we must show that $f(x) = 0$. But this follows since

$$x = (-1)^{n+1}[S(g_1 \cdots g_{n+2})],$$

and in the usual way one can write an alternating sum of surgeries on unions of claspers as a linear combination of alternating sums of surgeries on individual claspers.

Conversely, let $C_1, \ldots, C_{n+1}$ be a union of simple $cT$ claspers on a knot $k$. We want to deform the Morse function on $\mathbb{R}^3$ to get a knot $S(b)$ where the claspers $C_i$ satisfy the following properties:

- Each $C_i$ sits in an interval $[s_i, t_i]$ with respect to the Morse function, and all these intervals are pairwise disjoint.
The restriction of the knot to each interval is the trivial braid.
• Each $C_i$ is braid-like.

To do this, using condition (2) in the definition of “braid-like,” position the claspers $C_i$ so that they match their braid-like embedding with respect to the morse function for $1_n$. Now poke the maxima and minima of the knot up and down, to get a short-circuit closure. Since a surgery on a braid-like clasper is equivalent to a modification by an element of $G^{cT}$, the case $\mu = cT$ is settled.

Consider now the case $\mu = ncT$. Showing that an $ncT_n$ invariant vanishes on $S(I^+_{G^{ncT}})$ is proved exactly analogously. We have that $\hat{I}^+_{G^{ncT}}$ is generated by elements

$$g_1(h_1-1)g_2(h_2-1)\cdots g_{n+1}(h_{n+1}-1)g_{n+2}$$

where $h_i \in \eta(G^{cT})$, and $g_i \in B_\infty$. So each $h_i$ is equal to $1_{C_i}$, where $C_i$ is a union of $ncT$-claspers, and the argument indeed proceeds as above.

Now for the converse. Let $C_1, \ldots, C_{n+1}$ be a union of $ncT$ claspers on $k$ where each leaf bounds a disk hitting $k$ in precisely two algebraically canceling points. As above, we can deform the Morse function so that each $C_i$ sits in between two heights $s_i, t_i$, all the intervals $[s_i, t_i]$ are disjoint, in each interval $C_i$ sits in the complement of a trivial braid such that $C_i$ is obtained from a braid-like embedding by doubling some strands. So each $C_i$ converts the identity element $1_N$ to a pure braid $b$. Assume that the strands of $1_N$ are numbered from left to right. The leaves of $C_i$ each grab two antiparallel strands. If this pair of strands is of the form $(2j-1, 2j)$ we have $1_{C_i} = \eta(1_{C_i'})$ with $C_i'$ a braid-like clasper. If this is not the case, let $p$ be a braid such that in $p1C_i p^{-1}$, the leaves of $C_i$ do grab pairs of consecutive strands of the form $(2j-1, 2j)$. So then $C_i$ turns $1_N = p^{-1}p1_Np^{-1}p$ into $b_i' = p^{-1}(p\eta_{C_i'})p$. Now $p\eta_{C_i'} = \eta(1_{C_i'})$ for a braid-like $cT$-clasper $C_i'$, and so $b_i' \in G^{ncT}$. As before, the argument is now straightforward. Assume, for notational ease, that the claspers $C_i$ occur in order on the braid which closes up to give $k$. Write

$$k = S(g_1 \cdot 1 \cdot g_2 \cdot 1 \cdot \cdots \cdot g_{n+2})$$

where $C_i$ turns the 1 following $g_i$ into the braid $b_i' \in G^{ncT}$. Then

$$[k; C_1, \ldots, C_{n+1}] = S(g_1(1-b_i') \cdot \cdots \cdot g_{n+1}(1-b'_{n+1})g_{n+2}) \in \hat{I}^+_{G^{ncT}}.$$

\[
3. \text{ Groups of knots via the short-circuit closure}
\]

3.1. Proof of Theorem 5. We shall think of $P_\infty$ as consisting of braids on finite odd number of strands. Given $x \in P_\infty$, the long knot $S(x)$ is obtained by joining together lower and upper ends of the strands in turns, and extending to infinity the upper end of the first strand and the lower end of the last strand. There are two subgroups $H^T, H^B \subset P_\infty$ such that $x, y \in P_\infty$ satisfy $S(x) = S(y)$ if and only if there exist $t \in H^T$ and $b \in H^B$ with $x = tgb$, see [15].

Recall the definition of the endomorphisms $\tau_k$ of $P_\infty$: $\tau_0$ shifts the braids by two strands “to the right” and, for $k > 0$, $\tau_k$ triples the $k$-th strand. It is clear that $S(\tau_0(x)) = S(x)$ for all $x \in P_\infty$, thus $\tau_0(x) = txb$, where $t \in H^T$ and $b \in H^B$.

Lemma 14. For $x \in G$ there exist $t \in H^T \cap G$ and $b \in H^B \cap G$ such that $\tau_0(x) = txb$.

Proof. Let $t_{2k-1} = \tau_{2k-1}(x)(\tau_{2k}(x))^{-1}$, and let $b_{2k} = (\tau_{2k+1}(x))^{-1}\tau_{2k}(x)$. Notice that $t_{2k-1} \in H^T \cap G$ and $b_{2k} \in H^B \cap G$. We have

$$\tau_{2k-1}(x) = t_{2k-1}\tau_{2k}(x),$$

$$\tau_{2k}(x) = \tau_{2k+1}(x)b_{2k}.$$  

There exists $N$ such that $\tau_{2N+1}(x) = x$. Thus the following equality holds:

$$\tau_0(x) = t_1 \cdots t_{2N-1}xb_{2N} \cdots b_0,$$

and this completes the proof. \qed

Denote by $G_n$ the set of $\gamma_0 G$-equivalence classes of $G$-trivial knots. The operation of connected sum descends to an addition on $G_n$ as $\tau_0(\gamma_0 G) \subset \gamma_0 G$.\qed
Lemma 15. $G_n$ is an abelian group.

Proof. The connected sum of knots is commutative and associative and the neutral element is provided by the equivalence class of the unknot. Thus it is only necessary to check that every element of $G_n$ has an inverse.

Let $k = S(y)$ be the short-circuit closure of $y \in \gamma_n(G \cap P_{2N-1})$. If $m \geq n$, the knot $k$ is $\gamma_n$-equivalent to the trivial knot and defines the identity in $G_n$. Assume that $m < n$. Consider the knot $k \# S(y^{-1}) = S(y \tau_0^N(y^{-1}))$. By Lemma 14 the braid $\tau_0^N(y^{-1})$ can be written as $ty^{-1}b$ for $t \in H^T \cap G$ and $b \in H^b \cap G$, so

$$y \tau_0^N(y^{-1}) = yty^{-1}b = t(t^{-1}yty^{-1})b$$

and

$$S(y \tau_0^N(y^{-1})) = S(t(t^{-1}yty^{-1})b) = S(t^{-1}yty^{-1}) = S(y_1)$$

where $y_1 \in \gamma_{m+1}G \cap P_{4N-1}$.

Repeating the construction one sees that $k \# S(y_1) \# S(y_1) \# \ldots \# S(y_{q-1})$ is $\gamma_{(m+q)}G$-equivalent to the unknot. In particular, the knot $S(y^{-1}) \# \ldots \# S(y_1^N_{1-m})$ is an inverse for $k$ in $G_n$. \hfill $\square$

In fact, the same proof works to show that $G$-trivial knots modulo the derived series of $G$ form a group.

Problem 2. Are these groups modulo the derived series related to known knot invariants? Do they contain any concordance information?

The value of any $G_{<n}$-invariant on a knot depends only on the $\gamma_nG$-equivalence class of the knot. Indeed, $\gamma_nG - 1 \subset I_0^G$.

In order to prove Theorem $5$ it remains to show that different $\gamma_nG$-equivalence classes can be distinguished by finite-type invariants. This is established by the following lemma.

Lemma 16. The quotient map $S(G) \to G_k$ is a $G_k$-invariant.

Proof. We have to show that $S(I_0^G)$ is mapped to zero in $G_{n-1}$.

Define a relator of degree $n$ and length $m$ in $G$ as an element of $Z[S(G)]$ of the form

$$(*) \quad S((x_1 - 1)(x_2 - 1) \ldots (x_m - 1)y)$$

with $y \in G$, $x_i \in \gamma_nG$ and $\sum x_i = n$. The greatest $n$ such that a relator is of degree $n$ will be called the exact degree of a relator. A composite relator is an element of $Z[S(G)]$ of the form $k_1 \# k_2 - k_1 - k_2 + 1$ where $k_1, k_2$ are knots. Notice that a connected sum of two relators of non-zero degree is a linear combination of composite relators.

The kernel of the map $Z[S(G)] \to G_n$ contains all the relators of length 1 and degree $n$ and all the composite relators. On the other hand, an element of $S(I_0^G)$ is a linear combination of relators of length $n$ and, hence, of degree $n$. Thus if we show that any relator of degree $n$ is a linear combination of relators of degree $n$ and length 1 and composite relators, the lemma is proved.

Suppose that there exist relators of degree $n$ which cannot be represented as linear combinations of the above form. Among such relators choose the relator $R$ of minimal length and, given the length, of maximal exact degree. (If one of the $n_i$ in a relator exceeds $n$, then clearly the relator is in the kernel of the quotient map, so there is indeed an upper bound on the exact degree of such a relator.)

Assume that $R$ is of the form $(*)$ as above, with $y, x_i \in P_{2N-1} \cap G$. Choose $t \in H^T$ and $b \in H^B$ such that the braid $tx_1b$ coincides with the braid obtained from $x_1$ by shifting it by $2N$ strands to the right, that is, with $\tau_0^N(x_1)$. By Lemma 14 the braids $t$ and $b$ can be taken to belong to the same term of the lower central series of $G$ as the braid $x_1$. The relator

$$R' = S((tx_1b - 1)(x_2 - 1) \ldots (x_m - 1)y)$$

is a connected sum of two relators and, hence, is a combination of composite relators. On the other hand,

$$R' - R = S((tx_1b - x_1)(x_2 - 1) \ldots (x_m - 1)y)$$

$$= S(x_1(b - 1)(x_2 - 1) \ldots (x_m - 1)y)$$
Notice now that \((b - 1)\) can be exchanged with \((x_i - 1)\) and \(y\) modulo relators of shorter length or higher degree. Indeed,
\[
(b - 1)y = y(b - 1) + ([b, y] - 1)yb
\]
and
\[
(b - 1)(x_i - 1) = (x_i - 1)(b - 1) + ([b, x_i] - 1)(x_ib - 1) + ([b, x_i] - 1).
\]
Thus, modulo relators of shorter length or higher degree
\[
S(x_1(b - 1)(x_2 - 1)\ldots(x_m - 1)y) = S(x_1(x_2 - 1)\ldots(x_m - 1)y(b - 1)) = 0.
\]
and this means that \(R\) is a linear combination of composite relators and relators of length 1 and degree \(n\).

\[\square\]

4. Band-pass invariants

In this section we analyze band-pass invariants, completely characterizing the primitive ones.

**Theorem 17.** For \(n \geq 1\), primitive \(BP_n\)-invariants coincide with primitive Vassiliev invariants of order \(n\). The unique primitive \(BP_0\)-invariant is the Arf invariant. In other words, \(G_n^{BP}\) is the index 2 subgroup of \(G_n^V\) consisting of those knots with trivial Arf invariant.

We first prove some lemmas.

**Lemma 18.**

(i) For all \(n\), \(\mathcal{F}_n^{BP} \subset \mathcal{F}_n^V\).

(ii) For \(n > 2\), \(\overline{\mathcal{F}}_n^V \subset \overline{\mathcal{F}}_{n-2}^{BP}\).

**Proof.** Part (i) follows by writing a band-pass move as a union of crossing changes.

For part (ii), we use [12, Prop. 6.10]. The quoted result implies that \(\mathcal{F}_n^V\) is generated by elements of the form \(k - k_C\) where \(C\) is a simple (which implies capped) tree clasper of degree \(n\) on a knot \(k\), and by elements which lie in the second power of the augmentation ideal. However, these latter elements vanish upon passing to \(\mathbb{Z}[\text{Knots}]\). By [14, Thm. 28], we can assume the underlying tree of \(C\) is a tree where every trivalent vertex shares an edge with at least one univalent vertex.

Consider such a tree clasper \(C\) on a knot \(k\). Decompose \(C\) in the following way: insert two Hopf pairs of leaves along all edges which are not incident to a univalent vertex. This gives us a union of \(I\) and \(Y\) claspers. Perform a surgery of \(k\) along the \(Y\) claspers to get a knot \(k_Y\). Note that \(k_Y\) is isotopic to \(k\). There are exactly \(n - 2\) of the \(I\) claspers, denote them by \(I_1, \ldots, I_{n-2}\). Then
\[
(-1)^{n-2}[k_Y; I_1, \ldots, I_{n-2}] = k_C - k.
\]
Since the \(I_j\) are of the form \(ncI\), \(k_C - k \in \mathcal{F}_{n-2}^{BP}\), completing the proof. \[\square\]

**Lemma 19.**

(i) For \(n > 2\), \(\mathcal{F}_n^{BP}\) surjects onto \(\mathcal{F}_n^V / \mathcal{F}_n^V_{n+1}\).

(ii) \(\overline{\mathcal{F}}_2^{BP}\) surjects onto \(\mathbb{Z} \subset \mathbb{Z} \cong \overline{\mathcal{F}}_2^V / \mathcal{F}_3^V\).

**Proof.** Assume first \(n > 2\). Then \(\mathcal{F}_n^V / \mathcal{F}_n^V_{n+1}\) is generated by chord diagrams with \(n\) chords (see, for example, [2]). By [17], this is generated by wheels attached to the Wilson loop by some permutation, and by separated diagrams. Passing to \(\overline{\mathcal{F}}_n^V / \mathcal{F}_n^V_{n+1}\), the separated diagrams vanish. Thus it suffices to realize a wheel with \(n\) legs attached to the Wilson loop by some alternating sum of band-pass moves. An easy way to do this is to realize the wheel by an embedded clasper, producing band-pass moves as in the proof of Lemma 18 (ii).

If \(n = 2\) the same argument shows that the subspace generated by the wheel with two legs is realized. This is indeed equal to twice the generator. Finally, the generator itself cannot be realized since it has nontrivial Arf invariant, and Arf is a band-pass degree zero invariant [12], and so vanishes on \(\overline{\mathcal{F}}_2^{BP}\).

\[\square\]

**Proposition 20.** For \(n > 2\) we have \(\overline{\mathcal{F}}_n^{BP} = \overline{\mathcal{F}}_n^V\).
Proof. By Lemma 18 (i), it suffices to prove the inclusion $\mathcal{F}_n^V \subset \mathcal{F}_n^{BP}$. Let $v_n \in \mathcal{F}_n^V$. By Lemma 19 (ii), there is some $v_{n+1} \in \mathcal{F}_{n+1}^V$ such that $v_n + v_{n+1} \in \mathcal{F}_n^{BP}$. Again by Lemma 19 there is some $v_{n+2} \in \mathcal{F}_{n+2}^V$ such that $v_{n+1} + v_{n+2} \in \mathcal{F}_{n+1}^{BP} \subset \mathcal{F}_n^{BP}$. But by Lemma 19 (iii) $v_{n+2} \in \mathcal{F}_n^{BP}$. Therefore $v_n \in \mathcal{F}_n^{BP}$ as desired. □

Proof of Theorem 17. The first statement is a direct corollary of Proposition 20. The fact that the unique degree zero invariant is the Arf invariant was proven by Kauffman [13]. Now suppose $f$ is a primitive BP-finite-type invariant of degree 1. Then it is of Vassiliev degree 3. Now $\mathcal{F}_2^{BP} \supset \mathcal{F}_3^{BP} \rightarrow \mathcal{F}_3^V / \mathcal{F}_1^V$, implying that $f$ is actually of Vassiliev degree 2. But $\mathcal{F}_2^{BP} \rightarrow 2\mathbb{Z} \subset \mathbb{Z} \cong \mathcal{F}_2^V / \mathcal{F}_1^V$. Thus $f$ is either zero or the mod 2 reduction of the degree 2 Vassiliev invariant. That is, it must be zero or the Arf invariant. □

By the theory of Hopf algebras, the set of rational-valued finite-type invariants is a polynomial algebra over the primitive invariants. The same may not be true for $\mathbb{Z}$-valued invariants, leaving open the possibility that there are non-primitive BP-finite-type invariants which are not finite-type in the standard sense. Thus we pose the following problem.

Problem 3. Are BP-finite-type invariants always products of primitive invariants?

5. Doubted-Delta Invariants

In this section, we explore $\Delta\Delta_n$-invariants. $\Delta\Delta_0$-equivalence is precisely the same as $S$-equivalence, the condition that two knots have Seifert surfaces with identical Seifert forms, see [16]. However, the question still arises as to which Vassiliev invariants are also $\Delta\Delta_n$-invariants. Murakami and Ohtsuki [14] have shown that in the case of $\mathbb{Q}$-valued invariants, they must be polynomials in $c_{2n}$, the coefficients of the Conway polynomial. We prove the analogous result for a more general target group, namely abelian groups with no 2-torsion, but restricting to primitive invariants. (Corollary 29). This still leaves open the following question:

Problem 4. Do all (not necessarily primitive) Vassiliev invariants which are also $S$-equivalence invariants come from the Conway polynomial? (This is true for $\mathbb{Q}$-valued invariants by [14], and for primitive invariants taking values in an abelian group with no 2-torsion by Corollary 29.)

Moving on to the analysis of primitive $\Delta\Delta_1$-invariants, we show that for each $n \geq 1$ there exists exactly one integer-valued Vassiliev invariant of order $2n + 1$ which is also a $\Delta\Delta_1$-invariant, and that there are no integer-valued $\Delta\Delta_1$-invariants that are Vassiliev of even order. On the other hand, we shall see that any integer-valued Vassiliev invariant of order 4 is, modulo 2, a $\Delta\Delta_1$-invariant.

There may be more 2-torsion invariants in larger even degrees, but there are two stumbling blocks to proving this. First, we are not sure whether the 2-torsion exists on the level of weight systems, although we can show that no more than one copy of $\mathbb{Z}/2\mathbb{Z}$ exists in each even degree. The second stumbling block is more serious. Once we have a weight system, we do not have a mod 2 Kontsevich integral to produce a knot invariant, which is how we prove the first part of Theorem 3.

Finally, as observed in the introduction, the Euler degree $n + 1$ part of the Kontsevich integral, $Z_{n+1}^{rat}$, is a surjective $\Delta\Delta_{2n}$-invariant. Thus the kernels

$$\ker \left( G_{2n}^{\Delta\Delta} \rightarrow G_{2n-1}^{\Delta\Delta} \right)$$

are infinitely generated for all $n \geq 1$.

5.1. $\Delta\Delta_n$-equivalence and diagrams. Let $\mathcal{A}_n$ be the $\mathbb{Z}$-module of diagrams (called “Chinese character diagrams” in [2]). These are connected trivalent graphs with a distinguished oriented cycle (Wilson loop, or outer circle) and a cyclic ordering of edges at each vertex not on the Wilson loop. The edges that are not part of the Wilson loop form the internal graph of the diagram.

Let $\mathcal{A}_n^I$ be $\mathcal{A}_n$ modulo separated diagrams [5]. Any primitive $V_n$-invariant induces a weight system on $\mathcal{A}_n^I$.

An insulated vertex in a closed diagram of $\mathcal{A}_n$ is a vertex which neither lies on the Wilson loop, nor is connected to it by an edge. Let $\mathcal{A}_n^{I''}$ be the submodule of $\mathcal{A}_n^I$ generated by those diagrams which have an insulated vertex.
Proposition 21. Suppose \( f \) is a Vassiliev invariant which is also a \( \Delta \Delta_0 \)-invariant. Then the corresponding weight system \( \hat{f} \) vanishes on diagrams with insulated vertices.

Proof. One way to prove this is through claspers. (Compare [14].) Given a diagram, \( D \), with an insulated vertex, let \( C_1, \ldots, C_k \) be simple claspers on the unknot, which are embedded versions of the connected components of the inner graph of the diagram. Then \( \hat{f}(D) = \sum_{\sigma \subset \{C_i\}} (-1)^{|\sigma|} f(\text{unknot}_\sigma) \). (See [5, Lemma 2.1].) Let \( C_{i_0} \) be the clasper corresponding to the component with an insulated vertex. Let \( k \) be unknot modified by a fixed collection of \( C_i \) except \( C_{i_0} \). Then it suffices to show that surgery on the component \( C_{i_0} \) does not change the \( S \)-equivalence class of \( k \), because then all of the terms in the alternating sum cancel in pairs.

A vertex, \( v \), of a diagram \( D \) is called good if \( D \setminus v \) is a connected graph (where we include the Wilson loop.) Otherwise the vertex is called bad. Our strategy will be to show that a diagram that has an insulated vertex is either zero or has a good insulated vertex.

Suppose \( D \) has a bad vertex. That means that there is a separating edge in the diagram. Consider a separating edge which separates the diagram into two pieces \( G \) and \( H \), where \( H \) is the piece containing the Wilson loop, and where \( G \) does not itself contain a separating edge. If \( G \) has a trivalent vertex, then it is a good insulated vertex. Otherwise \( G \) is a loop, and so the diagram is zero, since it has a loop edge.\footnote{A diagram with a loop edge is zero even over \( \mathbb{Z} \).}

Now, let \( v_0 \) be a good insulated vertex. Break the clasper \( C_{i_0} \) into a union of claspers by inserting Hopf pairs of leaves on every edge incident to \( v_0 \). Let the \( Y \) clasper which contains \( v_0 \) be called \( C_Y \), and let the union of the other claspers be called \( C_O \). Note that each of the claspers in \( C_O \) has a leaf on the knot. Thus \( k_{C_{i_0}} = (k_{C_O})_{C_Y} \). Now \( C_Y \) is a capped \( Y \)-clasper on the knot \( k_{C_O} \) whose leaves link the knot trivially. Therefore \( C_Y \) can be realized as a union of doubled delta moves. (To see this, use Habiro’s zip construction to write the \( Y \) as a union of capped \( Y \) claspers each of which hits the knot in two algebraically canceling points.)

The argument is completed by noticing that \( k_{C_Y} = k \) since the claspers in \( C_O \) each have a leaf bounding a disk disjoint from the knot. \( \square \)

Next, we shall construct a weight system which vanishes on diagrams with insulated vertices. One way to construct weight systems is through the intersection graph of a chord diagram. This graph is defined by drawing a vertex for every chord, and putting an edge between vertices if the chords cross. (Two chords are said to cross if the four endpoints of the two chords alternate between the two chords as you travel around the Wilson loop.)

A Hamiltonian cycle in the intersection graph is a cycle which goes through every vertex exactly once without repeating any edges.

Definition 22. Let \( \mathcal{H}: A_n \to \mathbb{Z}/2 \) be the function defined as the number of unoriented Hamiltonian cycles in the intersection graph, modulo 2.

Lemma 23. The function \( \mathcal{H} \) is a primitive weight system.

Proof. We need to show that \( \mathcal{H} \) vanishes on 4T relators and separated diagrams, for \( n \geq 4 \).

It is clear that \( \mathcal{H} \) vanishes on separated diagrams since the intersection graph of a separated diagram is not connected.

As for the 4T relation, let \( \alpha \) and \( \beta \) be the two chords affected by our 4T relation. There are three complementary regions of the Wilson loop. Let chords joining each of the three pairs be called \( A, B, C \). There are also chords that don’t join two regions. Call those \( D \). This is illustrated below:
In the four terms of our relation we have the following intersections between \( \alpha, \beta \) and the other chords:

1: \((\alpha, A), (\alpha, C), (\beta, B), (\beta, C)\)
2: \((\alpha, A), (\alpha, C), (\beta, B), (\beta, C), (\alpha, \beta)\)
3: \((\alpha, A), (\alpha, B), (\beta, B), (\beta, C)\)
4: \((\alpha, A), (\alpha, B), (\beta, B), (\beta, C), (\alpha, \beta)\)

We show that there are an even number of Hamiltonian cycles in these four terms. First, there is evidently a \(1 - 1\) correspondence between cycles in the first intersection graph and cycles that don’t go through \((\alpha, \beta)\) in the second. (Similarly for third and fourth.) Thus the total number of Hamiltonian cycles that don’t go through the bridge \((\alpha, \beta)\) is even. Next, we show there is a \(1 - 1\) correspondence between cycles that go through the bridge in two and those that go through the bridge in four, and that change letters as they do so. The first type of cycle must have a subchain of the form \((A \text{ or } C, \alpha, \beta, B \text{ or } C)\) and the second of the form \((A \text{ or } B, \alpha, \beta, B \text{ or } C)\). One can easily check that the input-output unordered pairs of different letters are in correspondence in both cases. Finally, we show that there is an even number of cycles that cross the bridge back into the same letter, say \(C\). \(\alpha\) and \(\beta\) must be connected to distinct vertices \(c_1, c_2\) of \(C\), else we would only have a 3-cycle. That means that the endpoints of the bridge each connect to \(c_1, c_2\). If \(c_1, \alpha, \beta, c_2\) is part of a Hamiltonian cycle, so is \(c_2, \alpha, \beta, c_1\), and we have shown that the number of Hamiltonian cycles is even.

**Lemma 24.** \(\mathcal{H} = 1\) on every wheel attached to the Wilson loop by some permutation.

**Proof.** Let \(F_1\) and \(F_2\) be two graphs with the same set of vertices, and assume that \(F_2\) is obtained from \(F_1\) by adding one edge. Then we can depict the formal sum \(F_2 + F_1\) by a graph that has the same edges and vertices as \(F_2\), but where the edge which is missing in \(F_1\) is dashed. Similarly, a graph with \(n\) dashed edges is defined as a sum of \(2^n\) graphs.

Now, apply the STU relation to every leg of the wheel. The result is a sum of \(2^n\) chord diagrams (since we are working over \(\mathbb{Z}/2\mathbb{Z}\) the signs are irrelevant); in all these diagrams each chord comes from an edge of the circle part of the wheel. The corresponding \(2^n\) intersection graphs all share a common part, and there are \(n\) edges, corresponding to adjacent edges in the wheel, which may or may not be contained in the intersection graph. Hence, the sum of the \(2^n\) intersection graphs can be written as a graph with \(n\) dashed edges. Note that the dashed edges in this graph form a Hamiltonian cycle, and all other Hamiltonian cycles in this graph come in pairs.

Indeed, if a Hamiltonian cycle does not go through some dashed edge, then there are an even number of summands with this cycle, since the edge being dashed means that we can either insert it or not. Thus we are reduced to counting the Hamiltonian cycles that go through every dashed edge, of which there is exactly one!

**Lemma 25.** \(\mathcal{H}\) vanishes on diagrams with at least one insulated vertex.

**Proof.** Any diagram with an insulated vertex can be rewritten as a linear combination of diagrams with an insulated vertex and connected internal graph, and of separated diagrams. Since \(\mathcal{H}\) vanishes on the latter, we now consider non-separated diagrams.

If a diagram has one loop and one insulated vertex, then it can be reduced to an even number of wheels, by applying IHX to eliminate trees growing off the loop.

If a diagram has two loops, then breaking one of the loops apart via IHX and STU we can write it as a sum of an even number of diagrams with one loop. These are either all wheels, or they are diagrams that have one loop and also an insulated vertex.

In the former case, the fact that \(\mathcal{H} = 1\) on wheels finishes the argument, whereas the latter case has already been covered.

Finally, if the diagram is a tree with an insulated vertex, then modulo diagrams with one loop and one insulated vertex we can slide the endpoints of the tree into any position. In particular we can make two edges emanating from a trivalent vertex terminate in adjacent positions on the Wilson loop. It is known (see, for example, \([17]\)) that the resulting triangle formed by these two edges and the arc of the Wilson loop joining them, can be collapsed to a point and turned into a bubble inserted at any 3-valent vertex. (In other
words, the trivalent vertex is replaced by a circle to which three edges attach.) This puts us back in the case of a diagram with a loop and an insulated vertex.

Proposition 26. If \( n \) is even \( A_n^I/A_n^{iv} = \mathbb{Z} \), and if \( n \) is odd then \( A_n^I/A_n^{iv} = \mathbb{Z}/2 \).

Proof. First, we look at \( n \geq 3 \). We know that \( A_n^I \) is generated by wheels attached to the Wilson loop. Modulo diagrams with two loops (which must have an insulated vertex) the order in which the legs hit the Wilson loop is irrelevant. If \( n \) is odd, there is an orientation reversing symmetry, so that \( A_n^I/A_n^{iv} \) is at most \( \mathbb{Z}/2 \). However, the weight system \( \mathcal{H} \) detects the wheel, so that we get \( \mathbb{Z}/2 \) on the nose. When \( n \) is even, the \( \mathbb{Z} \) is detected by an integer lift of the weight system \( \mathcal{H} \).

If \( n = 2 \) the only diagrams with insulated vertices are already zero modulo IHX, and the resulting space is just \( \mathbb{Z} \).

5.2. \( \Delta \Delta_0 \)-invariants. Propositions [21] and [26] imply that for each degree there is at most one primitive Vassiliev invariant which is also a \( \Delta \Delta_0 \)-invariant, since such an invariant must give rise to a specific weight system.

In this section, we will show that there are primitive invariants coming from the Conway polynomial that do indeed give rise to these weight systems.

5.2.1. A discrete version of the logarithm. Define a function

\[
\exp_{\mathbb{Z}} : x \cdot \mathbb{Z}[x] \to 1 + x \cdot \mathbb{Z}[x]
\]

by the rule \( \exp_{\mathbb{Z}}(\sum_i a_i x^i) = \prod_i (1 + (-x)^i a_i) \). Indeed

\[
\exp_{\mathbb{Z}}(\sum_i a_i x^i) = 1 - a_1 x + \left( a_2 + \left( a_1^2 \right) \right) x^2 - \left( a_3 + a_1 a_2 + \left( a_1^3 \right) \right) x^3 + \ldots
\]

In general, the coefficient of \( x^i \) is given by \( (-1)^{\ell} \sum_{a_1, a_2, \ldots, a_{\ell} = 1} a_j \prod_{i_j = 1}^{k} a_j \). In particular the coefficient of \( x^i \) is equal to \( \pm a_i \) plus a polynomial in the lower coefficients. This implies that \( \exp_{\mathbb{Z}} \) is a bijection. Let \( \log_{\mathbb{Z}} \) be the inverse function.

Lemma 27. Writing \( \log_{\mathbb{Z}}(1 + \sum_{i=1}^{\infty} b_i x^i) = \sum_{i=1}^{\infty} a_i x^i \), we have \( a_i = (-1)^i b_i + q_i(b_1, \ldots, b_{i-1}) \) where \( q_i \) is a polynomial of graded degree \( i \).

Proof. We have

\[
1 + \sum_{i=1}^{\infty} b_i x^i = \exp_{\mathbb{Z}}(\sum_{i=1}^{\infty} a_i x^i) = 1 + \sum_{i=1}^{\infty} (-1)^i (a_i + Q_i(a_1, \ldots, a_{i-1})) x^i
\]

where \( Q_i \) is a polynomial of graded degree \( i \). The equation \( b_i = (-1)^i (a_i + Q_i(a_1, \ldots, a_{i-1})) \) can be recursively solved. Assume that all \( a_j \) for \( j < i \) are of the form \( a_j = (-1)^j b_j + q_j(b_1, \ldots, b_{j-1}) = (-1)^j b_j + q_j \). Then \( a_i = (-1)^i b_i - (-1)^i Q_i(b_1, \ldots, (-1)^i b_{i-1} + q_{i-1}) = (-1)^i b_i + q_i \).

Now define the invariants \( pc_{2i} \) by the equation

\[
\log_{\mathbb{Z}} C(z) = \sum_i pc_{2i} z^{2i}
\]

where \( z^2 = x \) in the definition of \( \log_{\mathbb{Z}} \). Recall that \( C(z) \) is the Alexander-Conway polynomial.

Then

\[
pc_{2i} = (-1)^i c_{2i} + q(c_2, c_4, \ldots, c_{2i-2}),
\]

where \( q \) is a polynomial of graded degree \( 2i \). Thus these invariants \( pc_{2i} \) are primitive Vassiliev invariants of order \( 2i \). In fact, we have

Proposition 28.
(i) $pc_{2n}$ is a Vassiliev invariant of order $2n$ for all $n$.
(ii) $pc_{2n} \mod 2$ is a Vassiliev invariant of order $2n - 1$ for all $n \geq 2$.

Moreover $\frac{1}{2}pc_{2n}$ induces the weight system $\hat{H}: A_{2n}^I \to \mathbb{Z}$ and $pc_{2n} \mod 2$ induces the weight system $H: A_{2n-1}^I \to \mathbb{Z}_2$. Here $\hat{H}$ is the integer lift of $H$ defined in [1].

**Corollary 29.** Suppose $f: Knots \to A$ is a primitive Vassiliev invariant of order $k$ which is also a $\Delta\Delta_0$-invariant. Suppose further that $A$ is an abelian group with no 2-torsion. Then $k$ is even and there is a homomorphism $\phi: \mathbb{Z}^{\oplus k/2} \to A$ such that

$$f = \frac{1}{2^{k/2}} \phi \left( \bigoplus_{i=1}^{k/2} pc_{2i} \right).$$

**Proof of Corollary 29.** We prove this by induction on the Vassiliev degree of $f$. Suppose $f: Knots \to A$ is a Vassiliev invariant of order $k$ which is primitive and invariant under the doubled-delta move. Let $\tilde{f}: A_k^I \to A$ be the induced weight system. By Proposition 26, $k$ must be even and there is a homomorphism $\phi: \mathbb{Z} \to A$ such that $2\tilde{f} = \phi \hat{H}$. So $2f - \phi(pc_k)$ is a Vassiliev invariant of order $k - 1$ (and therefore of order $k - 2$) which is also a $\Delta\Delta_0$ invariant. In this way, we can write

$$2^{k/2}f = \phi_2(pc_2) + \phi_4(pc_4) + \cdots + \phi_k(pc_k),$$

where $\phi_i$ is a homomorphism from $\mathbb{Z}$ to $A$. Tensoring with $\mathbb{Z}[\frac{1}{2}]$, which is an injection on $A$, we get that

$$f = \frac{1}{2^{k/2}}(\phi_2(pc_2) + \phi_4(pc_4) + \cdots + \phi_k(pc_k)) \in A.$$

This theorem leaves open the possibility that there are invariants taking values in an abelian 2 group which do not come from the Conway polynomial.

**Problem 5.** Are there any 2-torsion Vassiliev invariants which are also $\Delta\Delta_0$ invariants, that do not arise from the Conway polynomial?

Before we prove Proposition 28 we need a lemma.

**Lemma 30.** The degree $2n$ weight system for $pc_{2n}$ is equal to $\pm 2$ on wheels attached to the Wilson loop. The weight system for $pc_{2n}$ in degree $2n - 1$ is odd on wheels.

**Proof.** First we note that the permutation by which the wheel is attached to the Wilson loop is irrelevant. The difference of two wheels that differ by a permutation lies in the subspace with an insulated vertex. Thus the Alexander polynomial will not see the difference. In fact the only thing that matters is the orientation of the wheel. Therefore, we consider a clasper surgery on the unknot by a wheel with $n$ legs attached by the simplest permutation. Pictured below is the case of $n = 8$.

Let $k_n$ be the knot produced by this clasper surgery. The knot $k_8$ is pictured in Figure 8

We prove that
1. $pc_{2n}(k_{2n}) = 2$.
2. $pc_{2n}(k_{2n-1}) \mod 2$ is nonzero in $\mathbb{Z}/2$.

Because $k_{2n-1}$ and $k_{2n}$ both share finite-type invariants of order $\leq (2n - 2)$ with the unknot, it follows that $pc_{2n} = c_{2n}$ on these knots. The knot $k_n$ has an evident genus $n$ Seifert surface with the following Seifert matrix:

$$\Theta = \begin{pmatrix} 0 & M \\ MT^T + I & 0 \end{pmatrix},$$

where $M$ is the $n \times n$ matrix

$$M = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$  

Now the Alexander polynomial can be calculated via the following formula

$$A(k_n) = \det(t^{\frac{1}{2}}\Theta - t^{-\frac{1}{2}}\Theta^T)$$

which yields the result

$$A(k_n) = (1 + (1 - t)^n)(1 + (1 - t^{-1})^n).$$

See [7]. The Conway polynomial $C(z)$ is obtained via the substitution $z = t^{\frac{1}{2}} - t^{-\frac{1}{2}}$. Thus we have

$$A(k_n) = 1 + (t^{-n/2} + (-1)^n t^{n/2})z^n + (-1)^nz^{2n}.$$  

Now, when $n = 2k$, we get $1 + (t^{-k} + t^k)z^{2k} + z^{4k}$. Hence it suffices to show that $t^{-k} + t^k$ is a function of $z$ with 2 as the leading term. First, this is function of $z$ because it is symmetric with respect to $t$ and $t^{-1}$. To get the leading term, plug in $z = 0$ which is the same as $t = 1$.

For $n = 2k + 1$, we get $1 + (t^{-k} - t^{k+1/2})z^{2k+1} - z^{4k+2}$. Thus it suffices to show that $\phi(z) = t^{-k-1/2} - t^{k+1/2} = -(2k+1)z + \text{higher order terms}$. Again, $t^{-k-1/2} - t^{k+1/2}$ is a function of $z$. But

$$t^{-k-1/2} - t^{k+1/2} = (t^{-1/2} - t^{1/2})(t^{-k} + t^{-k+1} + \ldots + t^{-k-1} + t^k)$$

$$= -z(2k + 1 + h.o.t.)$$

This completes the proof. $\square$
Proof of Proposition 28. The fact that $pc_{2n}$ is a Vassiliev invariant of order $2n$ has already been observed. We now show that $pc_{2n} \mod 2$ is Vassiliev of order $2n - 1$.

To do this, it suffices to show that $pc_{2n}$ changes by an even number after performing a degree $2n$ simple clasper surgery (Goussarov [9], Habiro [12]). The difference between $pc_{2n}$ before and after the surgery is given by the value of $pc_{2n}$ on the diagram $A_{2n}$ coming from the clasper. Thus it suffices to show that the weight system for $pc_{2n}$ is even on $A_{2n}$. But this follows from Lemma 30 since wheels generate $A_{2n}$.

Lemma 30 also implies that the weight system for $pc_{2n} \mod 2$ is given by $\mathcal{H}$ and that $pc_{2n}$ induces the weight system $2\hat{\mathcal{H}}$. □

5.3. Degree one. Now we consider the question of which primitive Vassiliev invariants are also $\Delta\Delta_1$-invariants.

Let $\mathcal{A}_{2iv}$ be the submodule generated by diagrams with two nonadjacent insulated vertices.

Proposition 31. Let $f: \text{Knots} \rightarrow A$ be a Vassiliev invariant of order $n$ which is also a $\Delta\Delta_1$-invariant. Then its weight system $\hat{f}$ vanishes on $\mathcal{A}_{2iv}$.

Proof. Proceed as in the proof of Proposition 21. If the two insulated vertices are on different clasper components $C_\circ, C_{i_1}$, then the alternating sum is a sum of terms which are themselves alternating sums of doing the $C_\circ, C_{i_1}$ clasper surgeries. Since $C_\circ$, and $C_{i_1}$ are each sums of $\Delta\Delta$ moves, then $f$ must vanish on these alternating sums.

The argument from Proposition 21 also works when the two insulated vertices $v_0, v_1$ are on the same component, provided that this pair of vertices is good. Again good means that the complement of the two vertices, including the Wilson loop, is connected.

We will show that every diagram with 2 insulated vertices either has a good pair, or is equivalent, modulo IHX, to one which has a good pair.

The first step is that we can assume that our diagram has no separating edges. In the argument from Proposition 21 we pushed the vertex into a trivalent graph $G$ which was a part of the graph that has no separating edges, and is connected to the rest of the diagram by a single separating edge. We argued $G$ had to have at least one vertex, else the graph would have a loop edge, which is zero modulo IHX. Indeed any symmetrical graph $G$ will have this property, implying that $G$ has to have at least 6 vertices. Hence $G$ will have at least two vertices $v_0$ and $v_1$ not joined by an edge, which we can take to be our good pair.

So we may assume that the diagram minus the two vertices has exactly two components. Let $H$ be the one not containing the Wilson loop. Assume, without loss of generality, that the two vertices cut off an $H$ of minimal size. Then every (trivalent) vertex of $H$ is adjacent to both of $v_0$ or $v_1$. This leaves two possibilities: $H$ is a “Y” or $H$ is an “H”.

These two possibilities are pictured below:

This first possibility means that we have a diagram $D$, with an insulated vertex and a bubble inserted at an edge incident to that vertex. The second possibility means that we have a diagram with an edge not adjacent to the Wilson loop into which a theta graph has been inserted.

Let us take care of the first possibility. It is known that modulo IHX a bubble inserted in any edge is the same as inserting it in any other edge (See for example [17]). Hence insert a bubble on an edge incident to the Wilson loop. One of the vertices of this bubble, $v^*$, is insulated. If this does not form a good pair with any other insulated vertex, find another insulated vertex which, together with $v^*$, separates off a minimal graph $H^*$. If $H^*$ is an “H”, we are in the case of possibility 2. Otherwise we have a graph with two bubbles. Find an edge incident to the Wilson loop and insert the bubbles on each of the two edges joining it at a
trivalent vertex. The two endpoints of the bubbles near this trivalent vertex form a good pair, as pictured below:

Now we consider the second possibility. The theta we inserted on an edge is formed by first inserting a bubble in an edge and then inserting a bubble in one of the two created vertices. By [17], inserting a bubble at a vertex is independent of the vertex.

Find a place where the diagram hits the knot. Let $v$ be the adjacent trivalent vertex. Insert a bubble at a vertex adjacent to $v$ and insert a bubble in the opposite edge connecting into $v$. The two vertices of the bubbles closest to $v$ are a good pair:

□

We also have something of a converse.

**Proposition 32.** The inclusion $\mathcal{F}_k^{\Delta} \subset \mathcal{F}_k^Y$ holds. Indeed, any alternating sum of $k$ doubled-delta moves corresponds to a diagram in $\mathcal{A}_{2k}$ with $k$ insulated vertices, no two of which are joined by an edge.

**Proof.** A fact which we have been using repeatedly is that any simple clasper surgery of degree $n$ will produce a diagram in $\mathcal{A}_n$ which is essentially gotten by interpreting the clasper as the internal graph and the knot as the Wilson loop. (See [3, 12].) Using Habiro’s zip construction, any capped clasper will produce a signed sum of diagrams where each leaf is turned into a univalent vertex which can attach to any of the arcs intersecting the leaf’s cap. The sign is determined by the orientation of the cap and of the arc.

Hence each doubled-delta move produces a sum of “$Y$’s” attached to the Wilson loop in 8 possible ways, by distributing the endpoints of the $Y$ to two antiparallel arcs in the doubled-delta move. One can visualize this as a $Y$ with two dashed edges, called *tines*, emanating from each univalent vertex and connecting to the appropriate places on the Wilson loop. The two tines have opposite sign, corresponding to the antiparallelity. Each pair of tines is called a *fork*. So every doubled-delta move produces a $Y$ with a fork at each univalent vertex.

We simplify a diagram containing a $Y$ with forks by sliding one tine of a fork along the knot until it is adjacent to the other tine. The resulting diagram is zero because it is a difference of two identical diagrams (with forks). Now such a slide can be thought of as a sum of diagrams, one for every edge attaching to the Wilson loop between the fork’s tines, where the fork is placed so as to straddle each of these intervening edges. However, by the STU relation, such a straddling fork is the same as attaching the corresponding $Y$ endpoint to the intervening edge. Doing this for the other two forks in the $Y$ we get a sum of diagrams where the endpoints of the $Y$ are connected to other edges. So the $Y$’s central vertex is insulated. Do this for all the other $Y$’s sequentially. In the end we get $k$ insulated vertices. Furthermore, no two will be joined by an edge.

□

For the next argument, let $Z: \text{Knots} \to \mathcal{A} \otimes \mathbb{Q}$ be the Kontsevich integral, and let $\mathcal{A}_{2\ell}^{\text{liv}}$ be the submodule of $\mathcal{A} \otimes \mathbb{Q}$ generated by diagrams with $\ell$ insulated vertices, no two of which share an edge. (When $\ell = 2$ this is a (minor) abuse of notation.)
Proposition 33. Let $C_1, \ldots, C_\ell$ be disjoint $nY$-claspers on a knot $k$. Then
\[
Z[k; C_1, \ldots, C_\ell] \in \mathcal{A}^{2iv}.
\]

Proof. Let $T$ be a tangle comprised of three straight vertical strands, and let $C$ be a braid-like $Y$-clasper whose leaves are meridians to the three strands. Now let $T^d$ be the same tangle, but with each strand doubled into a pair of antiparallel strands. Assume (in the same spirit as the Aarhus papers) that these antiparallel strands are infinitesimally close.

Fact: Let $\Xi = Z(T^d_2 - T^d)$. Then $\Xi = d(Z(T_C - T))$, where $d$ is the map that takes the endpoint of a diagram on $T$ and sums over distributing it to the two new components. Since one of the strands is oriented oppositely, it gets a minus sign.

The proof of this fact follows directly from Kontsevich’s original definition of the Kontsevich integral.

Note that the components of $\Xi$ each contain a trivalent vertex because $C$ does not alter linking numbers.

Now $Z[k; C_1, \ldots, C_\ell]$ can be calculated as follows. Fix a height function and isotop the claspers $C_i$ so that they are surrounded by balls which look like the model tangle $(T^d, C)$ above. Then the Kontsevich integral can be calculated locally and then the resulting diagrams glued together to give the Kontsevich integral of the entire knot. In this way, we see that the Kontsevich integral will contain $\ell$ copies of $\Xi$, one for each of the balls surrounding each $C_i$. Now each copy of $\Xi$ is a “diagram with forks” as in the proof of Proposition 32, where the diagram contains at least one vertex. This gives rise to $k$ insulated vertices as in that proof.

In order to identify $\mathcal{A}^{iv}_{n}/\mathcal{A}^{2iv}_{n} \otimes \mathbb{Q} \cong \mathcal{B}^{iv}_{n}/\mathcal{B}^{2iv}_{n}$, we first attack the question of what happens rationally. Let $\mathcal{B}^{iv}_{n}$ be the rational vector space spanned by connected, vertex-oriented, unitrivalent graphs with $2n$ vertices, modulo the antisymmetry and IHX relations. Let $\mathcal{B}^{2iv}_{n}$ be the subspace spanned by graphs which have two vertices, neither of which is joined by an edge to a univalent vertex, and which don’t share an edge with each other.

Proposition 34. $\mathcal{A}^{iv}_{n}/\mathcal{A}^{2iv}_{n} \otimes \mathbb{Q} \cong \mathcal{B}^{iv}_{n}/\mathcal{B}^{2iv}_{n}$

Proof. According to [3], there is an algebra isomorphism $\chi \circ \partial_3: \mathcal{B} \to \mathcal{A} \otimes \mathbb{Q}$, where $\mathcal{B}$ is the space of not-necessarily-connected unitrivalent graphs, modulo $IHX$ and $AS$. Let $\mathcal{B}^{2iv}$ be the subspace generated by graphs which have two trivalent vertices not connected by an edge, neither of which is connected by an edge to a univalent vertex.

We argue that $\chi \circ \partial_3(\mathcal{B}^{2iv}) = \mathcal{A}^{2iv} \otimes \mathbb{Q}$. Clearly $\chi \circ \partial_3(\mathcal{B}^{2iv}) \subset \mathcal{A}^{2iv} \otimes \mathbb{Q}$, by the definitions of $\chi$ and $\partial_3$. For the reverse direction, note that $\chi^{-1} = \sigma$ (See [2]), is defined by an iterative process in which diagrams keep getting more things tacked onto them. In particular, insulated vertices will stay insulated. Thus $\chi^{-1}(\mathcal{A}^{2iv} \otimes \mathbb{Q}) \subset \mathcal{B}^{2iv}$. Now we wish to show that $\partial_3^{-1}(\mathcal{B}^{2iv}) \subset \mathcal{B}^{2iv}$. So let $A$ be such that $\partial_3(A) \in \mathcal{B}^{2iv}$. Suppose, toward a contradiction, that $A \notin \mathcal{B}^{2iv}$. Consider the smallest degree where $A$ has terms not in $\mathcal{B}^{2iv}$. Let $s$ be the sum of terms in $A$ which are of this degree. Then $\partial_3(s) = s + h.o.t$. Hence the terms in $s$ which are not in $\mathcal{B}^{2iv}$ cannot cancel with anything else, which is the desired contradiction.

Now $\chi \circ \partial_3$ will send disconnected graphs to separated graphs in $\mathcal{A} \otimes \mathbb{Q}$, thus inducing an isomorphism $\mathcal{B}^{iv}_{n}/\mathcal{B}^{2iv}_{n} \to \mathcal{A}^{iv}_{n}/\mathcal{A}^{2iv}_{n} \otimes \mathbb{Q}$. □

Proposition 35. There is an isomorphism $\mathcal{B}^{iv}_{n}/\mathcal{B}^{2iv}_{n} \cong \mathbb{Q}$.

Proof. Elements of $\mathcal{B}^{iv}_{n}$ consist of a trivalent core graph with edges that have a univalent vertex attached to some of the edges. These edges with univalent vertices are called hairs.

If the core graph is not a circle, a theta, or two loops joined by an edge (barbell graph), then we have an element of $\mathcal{B}^{2iv}$. However, the IHX relation can be used to show that a barbell graph with hairs is always zero. (If there is no hair on the non-loop edge, apply an IHX relation to that edge. If there is at least one hair on the non-loop edge, apply an IHX relation to distribute a hair close to a loop in two canceling ways.)

Thus $\mathcal{B}^{iv}_{n}/\mathcal{B}^{2iv}_{n}$ is generated by a wheel together with graphs that have theta as a core graph. When $n$ is even, this latter type of graph has an orientation reversing symmetry, so we are left with just the wheel. So we get $\mathbb{Q}$ in that case. On the other hand, if $n$ is odd, the wheel has an orientation reversing symmetry, and disappears. Thus we need to analyze the space of theta graphs with hairs attached to the edges, modulo $\mathcal{B}^{2iv}_{n}$. These can be characterized as unordered triplets $(a, b, c)$, where $a, b, c$ represent the number of hairs
attached to each edge. Hence \(a + b + c = n - 1\). There are two types of IHX relation. The first kind can occur when \(a, b, \) or \(c\) is zero, and alters the structure of the underlying core graph. However this type of IHX relation merely asserts that a certain barbell graph is zero; the two terms with a theta as core graph cancel.

The second kind of IHX relation turns into the following relation \((a + 1, b, c) + (a, b + 1, c) + (a, b, c + 1) = 0\) for any \(a + b + c = n - 2\). Also \((a, b, c) \in B_n^{2iv}\) if all of \(a, b, c\) are nonzero. Thus the only relations that don’t lie entirely in \(B_n^{2iv}\) are for triples \((a, b, c)\), where \(a = 0\). If \(b = 0\), as well, the ensuing relation is just \((0, 0, n - 1) + (0, 1, n - 2) = 0\). Otherwise, modulo \(B_n^{2iv}\), the relation is just \((0, a, b) = (0, a - 1, b + 1)\), where \(b > a \geq 2\). Thus we see that \(B_n^{2iv}/B_n^{2iv} \cong \mathbb{Q}\), generated by \((0, 1, n - 2)\).

This allows us to do a partial calculation over \(\mathbb{Z}\).

**Proposition 36.** We have

\[
\mathbb{Z} \oplus \mathbb{Z}_2 \to A_{2n}/A_{2n}^{2iv} \\
\mathbb{Z} \cong A_{2n-1}/A_{2n-1}^{2iv}
\]

**Proof.** As always, we use that \(A_{2n}^I\) is generated by wheels attached to the Wilson loop by some permutation. Let \(Y \in A_{2n}^I\) be a theta graph with a single hair on one edge, no hairs on the second edge and \(n - 2\) hairs on the third edge, attached to the Wilson loop by some permutation. Modulo \(A_{2n}^{2iv}\), the order in which the legs is attached to the Wilson loop is irrelevant. So, abusing notation, let these all be called \(Y\). Then any two ways of attaching a wheel to the Wilson loop will differ, modulo \(A_{2n}^{2iv}\) by multiples of \(Y\). Let \(X\) be the wheel attached by the identity permutation. Then we have just shown that \(X\) and \(Y\) generate.

Now in the case when \(n\) is even, \(Y\) has an odd number of trivalent internal vertices. Hence, flipping \(Y\) over (and reordering the legs accordingly) implies that \(Y = -Y\), or that \(2Y = 0\). On the other hand, we know \(X\) has infinite order, since it corresponds to \(pc_n/2\). This establishes the first statement above.

Suppose, then, that \(n\) is odd. Flip \(X\) over, and rearrange the legs to be the identity permutation again. Counting the number of \(Y\)’s we get as error terms, we get the equation \(X = -X + (2n - 3)Y\), implying \(2X = (2n - 3)Y\). Thus \(A_{2n-1}/A_{2n-1}^{2iv}\) is a quotient of \(\mathbb{Z} \oplus \mathbb{Z}/(2, 2n - 3) \cong \mathbb{Z}\). On the other hand, we know that tensoring with \(\mathbb{Q}\) gives us \(\mathbb{Q}\), so that in fact no further relations are possible. \(\square\)

This proves the first part of Theorem 3 of the introduction.

**Corollary 37.** There exist Vassiliev invariants of order \(2n - 1\) \(v_{2n-1}\): Knots \(\to \mathbb{Q}\) which are \(\Delta\Delta_1\)-invariants.

**Proof.** Let \(W_{2n-1}^I: A_{2n-1}^I \to \mathbb{Z}\) be a weight system that vanishes on \(A_{2n-1}^{2iv}\) which detects \(A_{2n-1}^I/A_{2n-1}^{2iv} \cong \mathbb{Z}\). Let \(v_{2n-1}^I = W_{2n-1}^I \circ Z_{2n-1}\). Clearly \(v_{2n-1}^I\) is a Vassiliev invariant of order \(2n - 1\). Let \(C_1, C_2\) be two \(\Delta\Delta\)-moves on a knot \(k\). Then \(Z_{2n-1}(k; C_1, C_2) \in A_{2n-1}^{2iv}\) by Proposition 33. \(\square\)

**Proposition 38.** There is an isomorphism \(A_{2n}^I/A_{2n}^{2iv} \cong \mathbb{Z} \oplus \mathbb{Z}/2\).

**Proof.** One uses the generators \(X\) and \(Y\), where \(X\) is a wheel, say, attached by the identity permutation, and \(Y\) is a tree of degree 4 with no crossings. By the usual sliding argument, any diagram can be realized, modulo separated diagrams, as a sum of diagrams with at least one loop. Moreover, this will preserve any insulated vertices, and possibly create new ones. So we list all nonzero graphs that have a core equal to a trivalent graph, \(G\). The possible \(G’s\) are:

Add forests to these cores in all ways, and enumerate those that have two insulated vertices. All of them are an even multiple of \(Y\), and indeed the graph which is a tetrahedron with one hair on two adjacent edges, is equal to \(\pm 2Y\). \(\square\)
Corollary 39. Any integer-valued finite-type invariant of degree 4 is, modulo two, a doubled-delta invariant of degree 1.

Proof. According to Proposition 32 it suffices to notice that any degree 4 finite-type invariant vanishes on $A^{2\text{niv}}$ modulo 2.

This establishes the second part of Theorem 3 of the introduction.

REFERENCES

[1] D. Bar-Natan and S. Garoufalidis, On the Melvin-Morton-Rozansky conjecture, Inventiones Mathematicae 125 (1996), no. 1, 103–133.
[2] D. Bar-Natan, On the Vassiliev knot invariants, Topology 34 (1995), no. 2, 423–472.
[3] D. Bar-Natan, T. T. Q. Le and D. Thurston, Two applications of elementary knot theory to Lie algebras and Vassiliev invariants, Geometry and Topology 7 (2003), 1–31.
[4] J. Conant and P. Teichner, Grope cobordism of classical knots, Topology 43 (2004), No. 1, 119–156.
[5] J. Conant and P. Teichner, Grope cobordism and Feynman diagrams, Mathematische Annalen 328 (2004), 135–171.
[6] S. Garoufalidis and A. Kricker, A rational noncommutative invariant of boundary links, Geometry and Topology 8 (2004), 115–204.
[7] S. Garoufalidis and J. Levine, Concordance and 1-loop clovers, Algebraic and Geometric Topology, Vol. 1, (2001), 687-697
[8] S. Garoufalidis and L. Rozansky, The loop expansion of the Kontsevich integral, the null-move and $S$-equivalence, Topology 43 (2004), no. 5, 1183–1210.
[9] M. Gusarov, On n-equivalence of knots and invariants of finite degree, Topology of manifolds and varieties, 173–192, Adv. Soviet Math., 18, Amer. Math. Soc., Providence, RI, 1994.
[10] M. Gusarov, Finite type invariants and n-equivalence of 3-manifolds, C. R. Acad. Sci. Paris Sér.I Math. 329 (1999), no. 6, 517–522.
[11] M. N. Gusarov, Variations of knotted graphs. The geometric technique of $n$-equivalence, Algebra i Analiz 12 (2000), no. 4, 79–125; translation in St. Petersburg Math. J. 12 (2001), no. 4, 569–604 ;
[12] K. Habiro, Claspers and finite type invariants of links, Geometry and Topology 4 (2000), 1–83.
[13] L. Kauffman, Formal knot theory, Princeton University Press (1983), Princeton NJ.
[14] H. Murakami and T. Ohtsuki, Finite type invariants of knots via their Seifert matrices, Asian Journal of Mathematics 5 (2001), No. 2, 379–386.
[15] J. Mostovoy and T. Stanford, On a map from pure braids to knots, Journal of Knot Theory and its Ramifications 12 (2003), no. 3, 417–425.
[16] S. Naik and T. Stanford, A move on diagrams that generates S-equivalence of knots, Journal of Knot Theory and its Ramifications 12 (2003), no. 5, 717–724.
[17] K. Y. Ng, Groups of ribbon knots, Topology 37 (1998), No. 2, 441–458.
[18] T. Stanford Vassiliev invariants and knots modulo pure braid subgroups, math.GT/9805092
[19] T. Stanford, Braid commutators and delta finite-type invariants, in Knots in Hellas ’98 (Delphi), 471–476, World Sci. Publishing, River Edge, NJ, 2000.
[20] K. Taniyama and A. Yasuhara, Band description of knots and Vassiliev invariants, Mathematical Proceedings of Cambridge Philosophical Society 133(2002), 325–343.
[21] K. Taniyama and A. Yasuhara, Local moves on spatial graphs and finite type invariants, Pacific Journal of Mathematics 211, (2003), 183–200.

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