GEOMETRIC APPROACH TO PARABOLIC INDUCTION

DAVID KAZHDAN AND YAKOV VARSHAVSKY

Abstract. In this note we construct a “restriction” map from the cocenter of a reductive group $G$ over a local non-archimedean field $F$ to the cocenter of a Levi subgroup. We show that the dual map corresponds to the parabolic induction and deduce that a parabolic induction preserves stability. We also give a new (purely geometric) proof that a character of the normalized parabolic induction does not depend on a parabolic subgroup. In the appendix, we use similar argument to extend a theorem of Lusztig–Spaltenstein on induced unipotent classes to all infinite fields.

To Joseph Bernstein, with great admiration

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Introduction

Let $G$ be a linear algebraic group over a local non-archimedean field $F$, and let $\hat{C}^G(G)$ be the space of invariant generalized functions on $G = G(F)$. By definition, $\hat{C}^G(G)$ is the dual space of $\mathcal{H}(G)_G$, where $\mathcal{H}(G)$ denotes the Hecke algebra of $G$,
and \((\cdot)_G\) denotes the coinvariants. To every admissible representation \(\pi\) of \(G\), one can associate its character \(\chi_\pi \in \hat{C}^G(G)\).

Now assume that \(G\) is connected reductive, \(P \subset G\) is a parabolic subgroup, \(M \subset P\) is a Levi subgroup, and \(U \subset P\) is the unipotent radical. Then to every admissible representation \(\rho\) of \(M = M(F)\) one can associate an admissible representation \(\pi = i_{P,M}(\rho)\), called the normalized parabolic induction of \(\rho\).

The main goal of this work is to construct a continuous map \(i_{P,M}^G : \hat{C}^M(M) \to \hat{C}^G(G)\), satisfying \(i_{P,M}^G(\chi_\rho) = \chi_\pi\) for every \(\rho\) as above. Namely, we construct a “restriction” map \(r_{P,M}^G : \mathcal{H}(G)_G \to \mathcal{H}(M)_M\), and define \(i_{P,M}^G\) to be the dual of \(r_{P,M}^G\).

Then we show that \(i_{P,M}^G\) does not depend on \(P \supset M\). From this we conclude that for each \(\rho\) as above, the composition factors of \(i_{P,M}^G(\rho)\) do not depend of \(P\), thus giving a geometric proof of \([\text{BDK}, \text{Lem 5.4 (iii)}]\).

Finally, we show that \(i_{P,M}^G\) preserves stability. This implies that a parabolic induction of a stable representation is stable. This result is considered to be well-known by specialists, but does not seem to appear in a written form.

Note that if the characteristic of \(F\) is zero, then the span of characters of smooth irreducible representations of \(G\) is dense in \(\hat{C}^G(G)\). Namely, this follows from a combination of a theorem of Kazhdan \([\text{Ka}, \text{Appendix, Thm 1}]\) and a group analog of a theorem of Harish–Chandra \([\text{HC}, \text{Thm 3.1}]\). Therefore, in this case, the independence assertion for \(i_{P,M}^G(\rho)\) do not depend of \(P\), thus giving a geometric proof of \([\text{BDK}, \text{Lem 5.4 (iii)}]\).

In the appendix, we study a related question, motivated by the work of Lusztig–Spaltenstein \([\text{LS}]\).

Let \(G\) and \(P, M\) and \(U\) be as above, but over an arbitrary infinite field \(F\). To every unipotent conjugacy class \(C \subset M\) we associate an \(\text{Ad} G\)-invariant subset \(C_{P,G} := \cup_{g \in G} g(C \cdot U)g^{-1} \subset G\). Then \(C_{P,G}\) is a union of unipotent conjugacy classes in \(G\), and a natural question is to what extend the set \(C_{P,G}\) depends on \(P \supset M\).

In their work, Lusztig and Spaltenstein showed, using the representation theory, that the Zariski closure of \(C_{P,G}\) does not depend on \(P\). This result can be thought as the assertion that \(C_{P,G}\) is “essentially” independent of \(P\), if \(F\) is algebraically closed.

In the appendix we extend this result to an arbitrary \(F\). Namely, for every algebraic variety \(X\) over \(F\) and a subset \(A \subset X = X(F)\) we define a “saturation” \(\text{sat}(A) \subset X\). The main result of the appendix asserts that the saturation \(\text{sat}(G_{P,G}) \subset G\) does not depend on \(P\).

Since every Zariski closed subset is saturated, our result is an extension of a theorem of Lusztig–Spaltenstein. Similarly, when \(F\) is a local field, our result implies that the closure of \(C_{P,G}\) in the analytic topology does not depend on \(P\).
The result of the appendix indicates that the independence of $P$ of a normalized parabolic induction has a purely algebraic flavour. We also believe that a notion of a saturation is interesting for its own right and deserves to be studied.

Our paper is organized as follows. In Section 1 we describe general properties of reductive groups and so-called generalized Grothendieck–Springer resolutions. In Section 2 we study non-vanishing top degree differential forms, which are basic tools for this work. Note that in these two sections the ground field $F$ is arbitrary, while starting from Section 3 the field $F$ is local non-archimedean.

In Section 3 we introduce smooth measures with compact support and carry out our construction of the restriction $r^{G}_{P,M}$ and the induction $i^{G}_{P,M}$. In Section 4 we show that the induction map sends a character of a representation to a character of the induced representation. Then, in Section 5 we construct another restriction map $R^{G}_{H}$, defined for a connected equal rank subgroup $H$ of $G$, and show that $R^{G}_{H}$ is compatible with $r^{G}_{P,M}$. Finally, in Section 6 we deduce from the results of Section 5 that the normalized parabolic induction is independent of $P$ and preserves stability.

Notice that in Sections 1-5, we work with non-normalized induction, which has a purely geometric interpretation, while we pass to normalized induction only in Section 6.

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1. Preliminaries on algebraic groups

1.1. The Chevalley map. (a) Let $G$ be a linear algebraic group over a field $F$, let $c_{G} := \text{Spec } F[G]^{G}$ be the Chevalley space of $G$, and let $\nu_{G} : G \to c_{G}$ be the morphism, dual to the inclusion $F[G]^{G} \hookrightarrow F[G]$.

(b) A homomorphism of linear algebraic groups $H \to G$ induces a morphism $\pi_{H,G} : c_{H} \to c_{G}$ of the Chevalley spaces, making the following diagram commutative

$$
\begin{array}{ccc}
H & \xrightarrow{\nu_{H}} & c_{H} \\
\downarrow \pi_{H,G} & & \downarrow \\
G & \xrightarrow{\nu_{G}} & c_{G}.
\end{array}
$$

(c) Let $G$ be connected reductive and split, $T \subset G$ a maximal split torus, and $W_{G} = W_{G,T}$ the Weyl group of $G$. Then the restriction $\nu_{G}|_{T} : T \to c_{G}$ is surjective and induces an isomorphism $W_{G}\backslash T \xrightarrow{\sim} c_{G}$ (see [St, Cor 6.4]).

1.2. Notation. Let $G$ be as in 1.1 and let $H \subset G$ be a closed subgroup.

(a) Let $\lambda_{G} : G \to G_{m}$ be the homomorphism $g \mapsto \det \text{Ad} g$ and let $\Delta_{H,G} \in F[H]^{H} = F[c_{H}]$ be the Ad $H$-invariant function $h \mapsto \det(\text{Ad} h^{-1} - 1, \text{Lie } G / \text{Lie } H)$. 


1.3. The generalized Grothendieck–Springer resolution.

(a) Let $H^{\text{reg}/G} \subset H$ (resp. $c_{H}^{\text{reg}/G} \subset c_{H}$) be the open subscheme defined by the condition $\Delta_{H,G} \neq 0$. By construction, $H^{\text{reg}/G} = \nu_{H}^{-1}(c_{H}^{\text{reg}/G})$.

(c) Let $K \subset H$ be a closed subgroup. Then $\Delta_{K,G} = \Delta_{K,H} \cdot (\Delta_{H,G}|_{K})$, thus $K^{\text{reg}/G} = K^{\text{reg}/H} \cap H^{\text{reg}/G}$.

1.4. Simple properties. (a) Every regular $\text{Ad}H$-invariant morphism $f : H \to X$ gives rise to the $\text{Ad}G$-equivariant morphism $\tilde{G}_{H} \to X : [g,h] \mapsto f(h)$. In particular, $\lambda_{H}$ gives rise to a morphism $\lambda_{\tilde{G}_{H}} : \tilde{G}_{H} \to G$, and $\Delta_{H,G}$ gives rise to a function $\Delta_{\tilde{G}_{H}} : \tilde{G}_{H} \to \mathbb{A}^{1}$.

(b) By (a), the morphism $\nu_{H} : H \to c_{H}$ gives rise to a morphism $\nu_{\tilde{G}_{H}} : \tilde{G}_{H} \to c_{H}$. Moreover, by (1.2(b)), we have a commutative diagram

$$
\begin{array}{c}
\tilde{G}_{H} \\
\downarrow a_{H,G} \\
G
\end{array}
\begin{array}{c}
\nu_{\tilde{G}_{H}} \\
\downarrow z_{H,G} \\
\nu_{G}
\end{array}
\quad \\
\quad \\
\quad \quad c_{H} \\
\quad \quad c_{G},
$$

which induces a morphism $\tilde{G}_{H} \to G \times_{c_{G}} c_{H}$.

(c) Set $\tilde{G}_{H}^{\text{reg}} := \nu_{\tilde{G}_{H}}^{-1}(c_{H}^{\text{reg}/G}) \subset \tilde{G}_{H}$. Explicitly, $\tilde{G}_{H}^{\text{reg}} = \text{Ind}_{H}^{G}(H^{\text{reg}/G})$. By definition, the morphism of (b) induces a morphism $\iota_{H,G} : \tilde{G}_{H}^{\text{reg}} \to G \times_{c_{G}} c_{H}^{\text{reg}/G}$.

(d) Let $K \subset H$ be a closed subgroup. Then the morphism $a_{K,H} : \tilde{H}_{K} \to H$ induces a morphism $\tilde{G}_{K} = \text{Ind}_{H}^{G} (\tilde{H}_{K}) \to \text{Ind}_{H}^{G}(H) = \tilde{G}_{H}$, which we again denote by $a_{K,H}$. Note that $a_{H,G} \circ a_{K,H} = a_{K,G}$. In addition, by (1.2(c)), $a_{K,H}$ induces a morphism $G_{K}^{\text{reg}} \to \tilde{G}_{H}^{\text{reg}}$.

(e) We also set $\tilde{H}_{K}^{\text{reg}/G} := \nu_{\tilde{G}_{K}}^{-1}(c_{K}^{\text{reg}/G})$. 

1.5. The equal rank case. (a) Let $G$ be connected reductive, and let $H \subset G$ be a connected equal rank subgroup, by which we mean that a maximal torus $T \subset H$ is a maximal torus of $G$. Let $U = U_H \subset H$ be the unipotent radical of $H$, and fix a maximal torus $T \subset H$, defined over $F$.

(b) The group $H$ has a Levi subgroup (see [Bo, 11.22]). Moreover, there exists a unique Levi subgroup $M \subset H$, containing $T$.

Proof. By uniqueness, we can extend scalars to a finite separable extension, thus assuming that $T$ is split. In this case, the subgroup $H \subset G$ is generated $T$ and roots subgroups $U_\alpha \subset G$ for $\alpha \in \Phi(H, T)$. Indeed, this follows from the fact that $H$ is generated by its Borel subgroups $B \supset T$, and that the corresponding assertion for solvable $H$ follows from [Bo, Prop 14.4].

Next we observe that the set of roots $\Phi(U, T)$ consists of all $\alpha \in \Phi(H, T)$ such that $-\alpha \notin \Phi(H, T)$. Therefore a Levi subgroup $M \supset T$ of $H$ has to coincide with the subgroup $M' \subset G$, generated by $T$ and roots subgroups $U_\alpha \subset G$ for all $\alpha \in \Phi' := \Phi(H, T) \setminus \Phi(U, T)$.

It remains to show that $M' \subset H$ is indeed a Levi subgroup. Since $M'U = H$, it suffices to show that $\Phi(M', T) \subset \Phi'$. Recall that $G$ has an automorphism $\iota$ such that $\iota(T) = T$ and $\iota(U_\alpha) = U_{-\alpha}$ for every $\alpha \in \Phi(G, T)$. Since the subset $\Phi' \subset \Phi(G, T)$ is stable under the map $\alpha \mapsto -\alpha$, we conclude that $M' \subset H \cap \iota(H)$, thus $\Phi(M', T)$ is contained in $\Phi(H, T) \cap \Phi(\iota(H), T)) = \Phi'$.

□

1.6. Remark. In this work we only consider the case when $H$ is either a parabolic subgroup $P \subset G$ or a Levi subgroup $M \subset P$. In these cases, the assertion [1.5](b) is well-known (see [Bo, Cor 14.19]). On the other hand, we believe that the context of equal rank subgroups is the “correct framework” to work in.

1.7. Properties. Assume that we are in the situation of [1.5] Let $p$ be the projection $H \to H/U \cong M$, and let $i$ be the inclusion $M \hookrightarrow H$.

(a) Set $H' := \pi^{-1}(M_{\text{reg}}/G) \subset H$. Then the morphism $\text{Ind}^H_M(M_{\text{reg}}/G) \to H'$, induced by $a_{M,H}$, is an isomorphism. Indeed, using the equality $H = U \rtimes M$, it remains to show that for every $m \in M_{\text{reg}}/G$ the map $f_m : U \to U : u \mapsto m^{-1}umu^{-1}$ is an isomorphism.

Consider the upper (or lower) filtration $U^{(i)}$ of $U$. It remains to show that $f_m$ induces an isomorphism on each quotient $U^{(i)}/U^{(i+1)}$. Using a natural isomorphism $U^{(i)}/U^{(i+1)} \cong \text{Lie}U^{(i)}/U^{(i+1)}$, it remains to show that $\text{Ad}(m^{-1}) - \text{Id}$ is invertible on $\text{Lie}U^{(i)}/U^{(i+1)}$. Since $m \in M_{\text{reg}}/G$, the map $\text{Ad}(m^{-1}) - \text{Id}$ is invertible on $\text{Lie}G/\text{Lie}M$. Hence it is invertible on $\text{Lie}U = \text{Lie}H/\text{Lie}M$, and the assertion follows.

(b) The morphisms $c_H \to c_M$ and $c_M \to c_H$, induced by $p$ and $i$, are isomorphisms. Since $p \circ i = \text{Id}_M$, it suffices to show that the pullback $i^* : F[H]^H \to F[M]^M$
is injective. By (a), the induced map $\iota^* : F[H]^H \to F[M^{\text{reg}/G}]^M$ is injective. Since $H' \subset H$ is dense, the restriction map $F[H] \to F[H']$ is injective, and the assertion follows.

(c) Assume that $T$ is split. Then, by definition, an element $t \in T$ belongs to $H^{\text{reg}/G}$ if and only if $\alpha(t) \neq 0$ for every root $\alpha \in \Phi(G, T) \setminus \Phi(H, T)$. Equivalently, this happens if and only if the connected stabilizer $Z_G(t)^0$ equals $Z_H(t)^0$. If, in addition, the derived group of $G$ is simply connected, then the stabilizer $Z_G(t)$ is connected. In this case, for every $t \in T \cap H^{\text{reg}/G}$, we have $Z_G(t) = Z_H(t) \subset H$.

(d) The isomorphism $c_M \sim c_H$ from (b) induces an isomorphism $c_M^{\text{reg}/G} \sim c_H^{\text{reg}/G}$. Indeed, extending scalars to a finite separable extension, we can assume that $T$ is split. Then, by (a), it remains to show that $T \cap M^{\text{reg}/G} = T \cap H^{\text{reg}/G}$. But this follows from the explicit description of both sides, given in (c).

(e) By (d), we have the equality $p^{-1}(M^{\text{reg}/G}) = H^{\text{reg}/G}$. Hence, by (a), the morphism $\text{Ind}^{H^M}_{M}(M^{\text{reg}/G}) \to H^{\text{reg}/G}$, induced by $\alpha_{M,H}$, is an isomorphism. Therefore the morphism $G_M^{\text{reg}} \to G_H^{\text{reg}}$, induced by $\alpha_{M,H}$ (see 1.1(d)), is an isomorphism as well.

(f) Set $e_G := \nu_G(1) \in c_G$ and similarly for $H$. Then the morphism $\pi_{H,G} : c_H \to c_G$ (see 1.1(b)) satisfies $\pi_{H,G}^{-1}(e_G) = e_H$. Indeed, by (b), we can replace $H$ by $M$, thus assuming that $H$ is reductive. Extending scalars, we can assume that $T$ is split. In this case $\pi_{H,G} : c_H \to c_G$ is the projection $W_H \setminus T \to W_G \setminus T$ (see 1.1(c)), and the assertion is immediate.

Lemma 1.8. In the situation of 1.7 assume that the derived group of $G$ is simply connected. Then

(a) the morphism $\pi_{H,G} : c_H^{\text{reg}/G} \to c_G$ is étale.

(b) the morphism $\iota_{H,G} : \tilde{G}_H^{\text{reg}} \to G \times_{c_G} c_H^{\text{reg}/G}$ from 1.7(c) is an isomorphism.

Proof. Extending scalars to a finite separable extension of $F$, we can assume that $T$ is split.

(a) By 1.7(d), we can replace $H$ by its Levi subgroup $M$, thus we can assume that $H$ is reductive. Then $\pi_{H,G} : c_H \to c_G$ is the projection $W_H \setminus T \to W_G \setminus T$ (see 1.1(c)). Thus, it remains to show that for every $t \in T \cap H^{\text{reg}/G}$ we have the equality of stabilizers $\text{Stab}_{W_H}(t) = \text{Stab}_{W_G}(t)$. But our assumption on $G$ implies that $Z_H(t) = Z_G(t)$ (see 1.7(c)), so the assertion follows.

(b) Note that $\tilde{G}_H^{\text{reg}}$ is étale over $G$ by Corollary 2.6 below, while $G \times_{c_G} c_H^{\text{reg}/G}$ is étale over $G$ by (a). Hence $\iota_{H,G}$ is étale. Thus, in order to show that $\iota_{H,G}$ is an isomorphism, it suffices to show that $\iota_{H,G}$ is a bijection (on $F$-points).

First we show that $\iota_{H,G}$ is surjective. Since $\nu_H|_T : T \to c_H$ is surjective, every element of $G \times_{c_G} c_H^{\text{reg}/G}$ has a form $(g, \nu_H(t))$ for some $g \in G$ and $t \in T \cap H^{\text{reg}/G}$ such that $\nu_G(g) = \nu_G(t)$. Let $g = su$ be the Jordan decomposition. Then $\nu_G(s) = \nu_H(u)$.
\( \nu_G(g) = \nu_G(t) \) (see [St] Cor 6.5). Since \( s \) and \( t \) are semisimple, they are \( G \)-conjugate (by [St] Cor 6.6]). Since \( \iota_{G,H} \) is \( G \)-equivariant, we can replace \( g \) by its conjugate, thus assuming that \( s = t \).

Since \( u \) is unipotent and \( us = su \), we get that \( u \in Z_G(s)^0 \). Hence \( g \in H \) (use 1.7(c)), thus \( g = su \in H_{\text{reg}}/G \), and \( \iota_{H,G}([1,g]) = ([g], \nu_H(t)) \).

To show the injectivity, assume that two elements \( \tilde{g} = [g,h] \) and \( \tilde{g}' = [g',h'] \) of \( \widetilde{G}_H^\text{reg} \) satisfy \( \iota_{H,G}(\tilde{g}) = \iota_{H,G}(\tilde{g}') \). Since \( \iota_{G,H} \) is \( G \)-equivariant, we can replace \( g \) and \( g' \) by their \( g'^{-1} \)-conjugates, thus assuming that \( g' = 1 \). In this case, identity \( \nu_H(h) = \nu_H(h') \) implies that \( \nu_H(h) = ghg^{-1} = h' \). It suffices to show that \( g \in H_0 \), hence \( \tilde{g} = [1, ghg^{-1}] = \tilde{g}' \).

Let \( h = su \) and \( h' = s'u' \) be the Jordan decompositions. Then \( gsg^{-1} = s' \) and \( \nu_H(s) = \nu_H(s') \). Thus, \( s \) and \( s' \) are \( H \)-conjugates. Hence, replacing \( (g,h) \) by \( (gx^{-1}, xhx^{-1}) \) for some \( x \in H_0 \), we may assume that \( s = s' \), thus \( g \in Z_G(s) \). Since \( h \in H_{\text{reg}}/G \), we get \( s \in H_{\text{reg}}/G \). By 1.7(c), we conclude that \( Z_G(s) \subset H_0 \), thus \( g \in H_0 \).

Corollary 1.9. In the situation of 1.7, the morphism \( \iota_{H,G} : \widetilde{G}_H^\text{reg} \to G \times_{c_g} c_{H,G}^\text{reg} \) from 1.7(c) is finite and surjective.

Proof. Let \( G^{sc} \) be the simply connected covering of the derived group of \( G \). Consider the natural isogeny \( \pi : G' := G^{sc} \times Z(G)^0 \to G \), and set \( H' := \pi^{-1}(H) \subset G' \). Then we have a commutative diagram

\[
\begin{array}{ccc}
\widetilde{G}_H^{\text{reg}} & \xrightarrow{\iota_{H,G}} & G' \times_{c_{G'}} c_{H,G}^{\text{reg}} \\downarrow & & \downarrow \\widetilde{G}_H^{\text{reg}} & \xrightarrow{\iota_{H,G}} & G \times_{c_g} c_{H,G}^{\text{reg}} \\
G' & & G' \\
\end{array}
\]

where vertical arrows are finite surjective morphisms, induced by \( \pi \). Now, \( \iota_{H,G} : G' \to G \times_{c_g} c_{H,G}^{\text{reg}} \) is an isomorphism by Lemma 1.8. Therefore \( \iota_{H,G} \) is finite and surjective.

1.10. Remark. Though morphism \( \iota_{H,G} \) from Corollary 1.7 is not an isomorphism in general, it is an isomorphism over a “strongly regular locus”. Moreover, the whole morphism \( \iota_{H,G} \) “can be made an isomorphism”, if one replaces (singular) Chevalley spaces \( c_G \) and \( c_H \) by their smooth Artin stack versions.

2. Top degree differential forms

2.1. Notation. (a) For every smooth algebraic (or analytic) variety \( X \) over \( F \) of dimension \( d \), we denote by \( K_X = \Omega_X^d \) the line bundle of top degree differential forms, called the canonical bundle.
(b) For every map of smooth algebraic (or analytic) varieties \( f : X \to Y \) over \( F \) of dimension \( d \) we have a natural morphism \( i_f : f^*(K_Y) \to K_X \) of line bundles, which corresponds to the pullback of differential forms.

2.2. The group case. Let \( G \) be an algebraic group over \( F \).

(a) For each \( g \in G \) the left multiplication \( L_g : G \to G : x \mapsto gx \) induces an isomorphism \( \text{Lie} \, G = T_1(G) \sim T_g(G) \) of tangent spaces. These isomorphisms for all \( g \) induce a trivialization \( G \times \text{Lie} \, G \sim T(G) \) of the tangent bundle of \( G \).

(b) Consider the one-dimensional vector space \( V \) equipped with \( \text{Lie} \, G \). Let \( \omega \) be an \( \mathcal{O}_G \otimes_F V_G \)-module such that \( \omega \) is nonvanishing, if \( v \neq 0 \).

(c) Similarly, for every \( v \in V_G \), there exists a unique right invariant differential form \( \omega^*_G(v) \) on \( G \) such that \( \omega^*_G(v)|_{g=1} = v \). Moreover, \( \omega^*_G(v) \) is nonvanishing, if \( v \neq 0 \).

2.3. The canonical bundle on \( \tilde{G}_H \).

(a) For every \([g] \in G/H\), we have natural identifications \( T_{[g]}(G/H) \sim \text{Lie} \, G / \text{Lie} \, H_g \), and \( \text{pr}^{-1}_{G/H}([g]) \sim H_g : [g, h] \mapsto ghg^{-1} \) (see 1.3(c)). Thus, for every \([g, h] \in \tilde{G}_H\), we have an exact sequence

\[
0 \to T_{ghg^{-1}}(H_g) \to T_{[g,h]}(\tilde{G}_H) \to \text{Lie} \, G / \text{Lie} \, H_g \to 0.
\]

(b) Using the identification \( \text{Lie} \, H_g \sim T_{ghg^{-1}}(H_g) \), induced by \( L_{ghg^{-1}} \) as in 2.2(a), we have a natural isomorphism

\[
\det T_{[g,h]}(\tilde{G}_H) \cong \det \text{Lie} \, H_g \otimes \det(\text{Lie} \, G / \text{Lie} \, H_g) \cong \det \text{Lie} \, G.
\]

(c) By (b), we have a natural isomorphism \( i : \mathcal{O}_{\tilde{G}_H} \otimes_F V_G \sim \mathcal{K}_{\tilde{G}_H} \). In particular, every \( 0 \neq v \in V_G \) defines a nonvanishing differential form \( \omega^{\tilde{G}_H}_G(v) := i(v) \) on \( \tilde{G}_H \).

(d) Using (c) and the isomorphism \( \mathcal{K}_G \sim \mathcal{O}_G \otimes_F V_G \) from 2.2(b), we get a natural isomorphism

\[
\tilde{i} : a^*_{H,G}(\mathcal{K}_G) \sim a^*_{H,G}(\mathcal{O}_G \otimes_F V_G) = \mathcal{O}_{\tilde{G}_H} \otimes V_G \sim \mathcal{K}_{\tilde{G}_H}.
\]

2.4. Remarks. (a) Recall that the isomorphism \( \text{Lie} \, H_g \sim T_{ghg^{-1}}(H_g) \) was constructed in 2.3(b) using left multiplication. Therefore the differential form \( \omega^{\tilde{G}_H}_G(v) \) from 2.3(c) is left-invariant, that is, the restriction of \( \omega^{\tilde{G}_H}_G(v) \) to each \( \text{pr}^{-1}_{G/H}([g]) \) is left invariant.
(b) Instead, we could construct an isomorphism $\text{Lie} H_g \cong T_{ghg^{-1}}(H_g)$, using the right multiplication. As a result, for every $v \in V_G$ we would get a right invariant differential form $\omega_{G_H}(v)$ on $G_H$ such that $\omega_{G_H}(v) = \lambda_{G_H} \cdot \omega_{G_H}(v)$ (compare 2.2(c)).

(c) Each differential form $\omega_{G_H}(v)$ is $G$-invariant. Indeed, it suffices to show that for every $g \in G$ and $h \in H$ the following diagram is commutative

$$
\begin{array}{c}
\text{Lie} H \\
\downarrow \text{Ad}_g
\end{array}
\begin{array}{c}
T_h(H)
\end{array}
\begin{array}{c}
\downarrow \text{Ad}_g
\end{array}
\begin{array}{c}
\text{Lie} H_g
\end{array}
\begin{array}{c}
\downarrow L_{ghg^{-1}}
\end{array}
\begin{array}{c}
T_{ghg^{-1}}(H_g)
\end{array}
$$

But this follows from the identity $g(hx)g^{-1} = (ghg^{-1})(gxg^{-1})$.

**Lemma 2.5.** The differential form $a_{H,G}(\omega_G(v))$ equals $\Delta_{G_H} \cdot \omega_{G_H}(v)$.

**Proof.** We have to show that after we identify all fibers of $\omega_G(v)$ and $\omega_{G_H}(v)$ with $v$ as in 2.3(b) and 2.2(b), then the Jacobian of the map $a_{H,G} : G_H \to G$ at $[g, h]$ is $\Delta_{H,G}(h)$. By the $G$-equivariance, we can assume that $g = 1$.

Using 2.3(a), we have two exact sequences of tangent spaces

$$
0 \to \text{Lie} H \to T_{[1,1]}(G_H) \to \text{Lie} G / \text{Lie} H \to 0
$$

$$
0 \to \text{Lie} H \to T_h(G) = \text{Lie} G \to \text{Lie} G / \text{Lie} H \to 0.
$$

It suffice to show that the differential $da_{H,G}|_{[1,1]} : T_{[1,1]}(G_H) \to T_h(G)$ induces the identity on Lie $H$ and the map $\text{Ad} h^{-1} - \text{Id}$ on Lie $G / \text{Lie} H$.

For the first assertion, notice that the restriction of $a_{H,G}$ to $\text{pr}_{G,H}^{-1}([1])$ is the inclusion $H \hookrightarrow G$. For the second one, notice that the endomorphism of Lie $G / \text{Lie} H$ induced by $da_{H,G}|_{[1,1]}$ is induced by the differential of the map $G \to G : g \mapsto ghg^{-1}$ at $g = 1$. Since $\omega_{G}(v)$ is left invariant, the last differential coincides with the differential of the map $G \to G : g \mapsto (h^{-1}gh)g^{-1}$, which equals $\text{Ad} h^{-1} - \text{Id}$. □

**Corollary 2.6.** The open subvariety $G_H^{\text{reg}} \subset G_H$ is the largest open subvariety, where the map $a_{H,G} : G_H \to G$ is étale.

2.7. **The parabolic case.** Let $G$ be connected reductive, $P \subset G$ a parabolic subgroup, $M \subset P$ a Levi subgroup, and $U \subset P$ the unipotent radical.

(a) For every $m \in M$, we have the identity

$$
\Delta_{P,G}(m)^2 = (-1)^{\dim U} \Delta_{M,G}(m) \cdot \lambda_P(m).
$$

Indeed, both sides are regular functions on $M$, thus we may assume that $m$ lies in a maximal torus $T \subset M$. Now we can decompose $\text{Lie} G / \text{Lie} M$ as a sum of root subspaces, and the equality follows from the identity $(t - 1)^2 = (-1)(t - 1)(t - 1 - 1)t$. 


(b) Let \( P^- \) be the opposite parabolic of \( P \), and let \( U^- \subset P^- \) be the unipotent radical. Then the multiplication map \( m : U^- \times P \to G \) is an open embedding. Hence the maps \( U^- \to G/P: u \mapsto [u] \) and \( j : U^- \times P \to \tilde{G}_P: (u, x) \mapsto [u, x] \) are open embeddings. In addition, the differential \( dm|_{(1,1)} : \text{Lie} U^- \oplus \text{Lie} P \to \text{Lie} G \) is an isomorphism. Hence it induces an isomorphism \( V_{U^- \times P} \to V_G \).

**Lemma 2.8.** In the notation of (2.7(b), for each \( v \in V_G \) we have equalities \( j^*(\omega^t_{G_P}(v)) = \omega^t_{U^- \times P}(v) \) and \( m^*(\omega_G(v)) = \omega^t_{U^- \times P}(v) \).

**Proof.** Since the fibers of both \( j^*(\omega^t_{G_P}(v)) \) and \( \omega^t_{U^- \times P}(v) \) at 1 is \( v \), for the first equality it remains to show that \( j^*(\omega^t_{G_P}(v)) \) is left \((U^- \times P)\)-invariant. For the \( U^- \)-invariance, notice that \( j \) is \( U^- \)-equivariant, and \( \omega^t_{G_P}(v) \) is \( G \)-invariant (see 2.4(c)).

For the \( P \)-invariance notice that \( \omega^t_{G_P}(v) \) is left \( P \)-invariant (see 2.4(a)).

For the second equality, one has to show that \( m^*(\omega_G(v)) \) is right \((U^- \times P)\)-invariant. Since \( \omega_G(v) \) is \( G \times G \)-invariant, the pullback \( m^*(\omega_G(v)) \) is right \( P \)-equivariant and left \( U^- \)-equivariant. Then \( m^*(\omega_G(v)) \) is also right \( U^- \)-invariant, since \( U^- \) is unipotent, thus \( \lambda_{U^-} = 1 \).

---

3. Smooth measures, restriction, and induction

From now on, let \( F \) be a local non-archimedean field, and let \( | \cdot | : F^\times \to \mathbb{R}^\times \) be the norm map. For every compact analytic subgroup \( K \) over \( F \), we denote by \( \delta_K \) the Haar measure on \( K \) with total measure 1.

**3.1. Smooth measures.** Let \( X \) be a smooth analytic variety over \( F \).

(a) Let \( C^\infty_c(X) \) be the space of smooth (complex-valued) functions (resp. with compact support), and let \( \mathcal{M}(X) \) be the dual space of \( C^\infty_c(X) \). Every non-vanishing \((F,\mathbb{R}^\times)\)-valued analytic function \( f \) on \( X \) induces a smooth function \( |f| \in C^\infty(X) \), defined by \( |f|(x) := |f(x)| \) for every \( x \in X \).

(b) We say that a measure \( \chi \in \mathcal{M}(X) \) is smooth, and write \( \chi \in \mathcal{M}^\infty(X) \), if for every \( x \in X \) there exists an open neighborhood \( U \subset X \) of \( x \) and an analytic isomorphism \( \phi : \mathcal{O}_F^n \to U \) such that \( \phi^*(\chi|_U) \in \mathcal{M}(\mathcal{O}_F^n) \) is a multiple of a Haar measure on \( \mathcal{O}_F^n \).

(c) By a construction of Weil [We], every non-vanishing top degree differential form \( \omega \) on \( X \) defines a smooth measure \( |\omega| \in \mathcal{M}^\infty(X) \).

Namely, for every open analytic embedding \( \phi : \mathcal{O}_F^n \to U \subset X \), the differential form \( \phi^*(\omega) \) equals \( f dx_1 \wedge \ldots \wedge dx_n \) for some non-vanishing analytic function \( f \) on \( \mathcal{O}_F^n \). Then \( |f| \in C^\infty(\mathcal{O}_F^n) \), and \( |\omega| \) is characterized by the condition that the pullback \( \phi^*(|\omega|) \) equals \( |f| \delta_{\mathcal{O}_F} \in \mathcal{M}^\infty(\mathcal{O}_F^n) \).

Now we give an equivalent (more geometric) definition of smooth measures.
3.2. An alternative definition. Let $K^{-1}_X$ be the dual line bundle of $K_X$, and let $\Sigma_X \to X$ be the $F^\times$-torsor of non-vanishing sections of $K^{-1}_X$. We claim that the space $\mathcal{M}^\infty(X)$ is canonically identified with the space $C^\infty(\Sigma_X, | \cdot |)$ of smooth functions $f : \Sigma_X \to \mathbb{C}$ satisfying $f(bx) = |b|f(x)$ for every $b \in F^\times$.

Since both $\mathcal{M}^\infty(X)$ and $C^\infty(\Sigma_X, | \cdot |)$ are defined as global sections of certain sheaves on $X$, we can construct an isomorphism $\mathcal{M}^\infty(X) \to C^\infty(\Sigma_X, | \cdot |)$ locally on $X$. Thus we can assume there exists a non-vanishing differential form $\omega \in \Gamma(X, K_X)$.

Note that $\omega$ induces an isomorphism $i_\omega : \Sigma_X \xrightarrow{\sim} F^\times \times X : s \mapsto s \otimes \omega$, and every $f \in C^\infty(X)$ defines a function $\tilde{f} \in C^\infty(F^\times \times X, | \cdot |) : (a, f) \mapsto |a|f(x)$. The map $f|\omega| \mapsto i_\omega^*(\tilde{f})$ defines an isomorphism $\mathcal{M}^\infty(X) \to C^\infty(\Sigma_X, | \cdot |)$, independent of $\omega$.

3.3. Smooth measures with compact support. Let $\mathcal{M}_c(X)$ (resp. $\mathcal{H}(X) = \mathcal{M}^\infty_c(X)$) be the space of measures (resp. smooth measures) on $X$ with compact support. If $G$ is a group acting on $X$, then $G$ acts on the space $\mathcal{H}(X)$, and we denote by $\mathcal{H}(X)^G$ and $\mathcal{H}(X)_G$ the spaces of $G$-invariants and $G$-coinvariants, respectively.

3.4. Pullback of smooth measures. (a) Assume that we are given a morphism $f : X \to Y$ of smooth analytic varieties and an isomorphism $i : f^*(K_Y) \xrightarrow{\sim} K_X$ of line bundles. Then $i$ induces an isomorphism of line bundles $K^{-1}_X \xrightarrow{\sim} f^*(K_Y^{-1})$, hence a morphism of $F^\times$-torsors $(f, i) : \Sigma_X \xrightarrow{\sim} X \times_Y \Sigma_Y \to \Sigma_Y$.

By (3.2) $(f, i)$ induces a pullback map $(f, i)^* : \mathcal{M}^\infty(Y) \to \mathcal{M}^\infty(X)$. If, in addition, $f$ is proper, then $(f, i)^*$ induces a pullback $(f, i)^* : \mathcal{H}(Y) \to \mathcal{H}(X)$.

(b) If $f : X \to Y$ is a local isomorphism, then the morphism $i_f : f^*(K_Y) \to K_X$ from (3.1)(b) is an isomorphism. So it induces a pullback map $f^* = (f, i_f)^* : \mathcal{M}^\infty(X) \to \mathcal{M}^\infty(Y)$. In particular, if $X \to Y$ is an open (resp. open and closed) embedding, then we have a natural restriction map $\text{res} : \mathcal{M}^\infty(Y) \to \mathcal{M}^\infty(X)$ (resp. $\text{res} : \mathcal{H}(Y) \to \mathcal{H}(X)$).

The following simple lemma is basic for what follows.

**Lemma 3.5.** (a) Let $f : X \to Y$ be a smooth map between smooth analytic varieties. Then the pushforward map $f_! : \mathcal{M}_c(X) \to \mathcal{M}_c(Y)$ satisfies $f_!(\mathcal{H}(X)) \subset \mathcal{H}(Y)$, that is, $f_!(h) \in \mathcal{H}(Y)$ for every $h \in \mathcal{H}(X)$.

(b) Let $G$ be an analytic group, and let $f : X \to Y$ be a principal $G$-bundle. Then the map $f_!$ induces an isomorphism $\mathcal{H}(X)_G \xrightarrow{\sim} \mathcal{H}(Y)$.

**Proof.** (a) The question is local in $X$ and $Y$, so we may assume that $X = O_F^{n+m}$, $Y = O_F^n$, $f$ is the projection, and $h = \delta_X$. In this case, $f_!(h) = \delta_Y$.

(b) The assertion is local in $Y$, and $f$ is locally trivial, so we may assume that $X = G \times Y$. Choose a compact open subgroup $K \subset G$. Then the map $\mathcal{H}(Y) \to \mathcal{H}(X)_G : h \mapsto [h \boxtimes \delta_K]$ is the inverse of $f_! : \mathcal{H}(X)_G \xrightarrow{\sim} \mathcal{H}(Y)$. \qed
3.6. The induced space. (a) Let $G$ an analytic group, $H \subset G$ a closed analytic subgroup, and let $H$ acts on a smooth analytic variety $X$. Then the product $G \times X$ is equipped with an action of $G \times H$ defined by $(g, h)(g', x) := (gg' h^{-1}, h(x))$, and the quotient $\text{Ind}^G_H(X) := G \times X$ is smooth and equipped with an action of $G$.

(b) Consider diagram $\text{Ind}^G_H(X) \xrightarrow{p_1} G \times X \xrightarrow{p_2} X$, where $p_1$ and $p_2$ are natural projections. Since $p_1$ is a $G$-equivariant $H$-bundle, while $p_2$ is a $H$-equivariant $G$-bundle, Lemma 3.5(b) implies that we have a natural isomorphism $\varphi^G_H := (p_2)_! \circ (p_1)_!^{-1} : \mathcal{H}(\text{Ind}^G_H(X)) \leftarrow \mathcal{H}(G \times X) \xrightarrow{\sim} \mathcal{H}(X)_H$.

(c) By construction, isomorphism $\varphi^G_H$ from (b) can be characterized as a unique map $\varphi^G_H : \mathcal{H}(\text{Ind}^G_H(X)) \to \mathcal{H}(X)_H$ such that $\varphi^G_H \circ (p_1)_! = (p_2)_!$.

(d) Assume that $H$ acts on $X$ trivially. Then $\text{Ind}^G_H(X) = (G/H) \times X$, and the natural projection $\text{pr} : (G/H) \times X \to X$ satisfies $\text{pr} \circ p_1 = p_2$. Thus, by (c), the map $\varphi^G_H$ is induced by $\text{pr}_! : \mathcal{H}((G/H) \times X) \to \mathcal{H}(X)$.

(e) Assume that $H$ is a retract of $G$, and let $p : G \to H$ be a homomorphism such that $p|_H = \text{Id}_H$. Then $p$ induces a map $\tilde{p} : \text{Ind}^G_H(X) \to \text{Ind}^H_H(X) = X$ such that $p \circ p_1 = p_2$. Thus, by (c), the map $\varphi^G_H$ is induced by $p_! : \mathcal{H}(\text{Ind}^G_H(X)) \to \mathcal{H}(X)$.

3.7. Notation. For every (smooth) algebraic variety $X$ over $F$ we denote by $X$ the corresponding (smooth) analytic variety $X(F)$. In particular, we have $G = G(F)$, $\tilde{G} = \tilde{G}_H(F)$, etc. We also denote by $a_{H,G} : \tilde{G}_H \to G$, $\Delta_{H,G} : H \to F$, $\lambda_G : G \to F^{\times}$, etc. the maps induced by $a_{H,G}$, $\Delta_{H,G}$ and $\lambda_G$, respectively.

3.8. The restriction map. In the situation of 2.7 set $\tilde{G} := \tilde{G}_P$ and $a := a_{P,G}$.

(a) In 2.3 we constructed an $\text{Ad}$-$G$-equivariant isomorphism $\tilde{\iota} : a^*(\mathcal{K}_G) \xrightarrow{\sim} \mathcal{K}_{\tilde{G}}$, which by 3.1 induces a pullback map $\tilde{a}^* = (a, \tilde{\iota})^* : \mathcal{M}^\infty(G) \to \mathcal{M}^\infty(\tilde{G})$.

Explicitly, for every $v \in V_G$ and $f \in C^\infty(G)$, the map $\tilde{a}^*$ is given by the formula $\tilde{a}^*(f|\omega_G(v)) := a^*(f)|\omega_{\tilde{G}}(v)$. Moreover, since $a : \tilde{G} \to G$ is proper, the map $\tilde{a}^*$ induces a $G$-equivariant map $\tilde{a}^* : \mathcal{H}(G) \to \mathcal{H}(\tilde{G})$.

(b) Since $(G/P)(F)$ equals $G(F)/P(F) = G/P$ (see [Bo] Prop 20.5]), the set $\tilde{G} = \tilde{G}(F)$ equals $\text{Ind}^G_P(P)$. Hence, by 3.6(b), we have a natural isomorphism $\varphi^G_P : \mathcal{H}(\tilde{G})(G) = \mathcal{H}(\text{Ind}^G_P(P)) \xrightarrow{\sim} \mathcal{H}(P)_P$.

We denote by $\text{Res}^G_P : \mathcal{H}(G)_G \to \mathcal{H}(P)_P$ the composition map $\varphi^G_P \circ \tilde{a}^*$.

(c) Let $p : P \to M$ be the projection (see 1.7). Since $p$ is smooth, the induced map $p : P \to M$ is smooth. Thus, $p$ induces a map $p_! : \mathcal{H}(P)_P \to \mathcal{H}(M)_M$, and we denote by $\text{Res}^G_{P,M} : \mathcal{H}(G)_G \to \mathcal{H}(M)_M$ the composition $p_! \circ \text{Res}^G_P$. 
3.9. Generalized functions and induction. (a) Let $X$ be a smooth analytic variety over $F$, and let $\hat{C}(X)$ be the dual space $\mathcal{H}(X)^*$ of $\mathcal{H}(X)$, equipped with a weak topology. Elements of $\hat{C}(X)$ are called generalized functions.

(b) For a group $G$ acting on $X$, we denote by $\hat{C}^G(X) \subset \hat{C}(X)$ the subspace of invariant generalized functions with the induced topology. Equivalently, $\hat{C}^G(X)$ is the dual space of $\hat{\mathcal{H}}(X)_G$.

(c) By Lemma 3.5(a), every smooth map $f : X \to Y$ induces a continuous map $f^* : \hat{C}(Y) \to \hat{C}(X)$ dual to $f$. If, in addition, $f$ is $G$-equivariant, then $f^*$ induces a continuous map $\hat{C}^G(Y) \to \hat{C}^G(X)$.

(d) In the situation of 3.6 we have a linear homeomorphism $\hat{\mathcal{C}}^H(X) \cong \hat{\mathcal{C}}^H(\text{Ind}_H^G(X))$, dual to $\varphi_H^*$.

(e) In the situation of 3.8 we denote by $\text{Ind}_{P,M}^G : \hat{\mathcal{C}}^M(M) \to \hat{\mathcal{C}}^G(G)$ (resp. $\text{Res}_{P,M}^G : \hat{\mathcal{C}}^P(P) \to \hat{\mathcal{C}}^G(G)$) the map dual to $\text{Res}_{P,M}^G$ (resp. $\text{Res}_{P,M}^G$).

(f) Let $H$ be an algebraic group over $F$. Then to every admissible representation $\pi$ of $H = \mathcal{H}(F)$ we can associate its character $\chi_\pi \in \hat{\mathcal{C}}^H(H)$.

4. Relation to characters of induced representations

Assume that we are in the situation of 3.8.

Proposition 4.1. Let $\tau$ be an admissible representation of $P$, and let $\pi = \text{Ind}_P^G(\tau)$ be the induced representation. Then we have the equality $\chi_\tau = \text{Ind}_P^G(\chi_{\pi})$.

To prove the result, we are going to calculate $\chi_\tau$ and $\text{Ind}_P^G(\chi_{\pi})$ explicitly.

4.2. Notation. (a) Let $K \subset G$ be a compact open subgroup, set $K_P := K \cap P$, and let $\mu^K$ (resp. $\mu^K_P$) be the left invariant Haar measure on $G$ (resp. $P$) normalized by the condition that $\mu^K(K) = 1$ (resp. $\mu^K(P) = 1$).

(b) Set $\tilde{K} := K \times P = \text{Ind}_{K_P}^K(P)$. Then $\tilde{K} \subset \tilde{G}$ is an open and closed subset, and we set

\[ \text{Res}_{K_P}^K : \mathcal{H}(G)_K \xrightarrow{\tilde{a}^*} \mathcal{H}(\tilde{G})_K \xrightarrow{\text{res}} \mathcal{H}(\tilde{K})_K \xrightarrow{\varphi_{K_P}^K} \mathcal{H}(P)_{K_P}. \]

(c) For every $h \in \mathcal{H}(G)_K$, we define $f := h/\mu^K \in C_c^{\infty}(G)$, $f_P := f|_P \in C_c^{\infty}(P)$ and $h_P := f_P \mu^K_P \in \mathcal{H}(P)$.

(d) For every $g \in G$, we set $K_g := gKg^{-1}$, $h_g := (\text{Ad} g^{-1})^*(h) \in \mathcal{H}(G)_{K_g}$, $K_{g,P} := K_g \cap P$ and $h_{g,P} := (h_g)|_P$ (see (c)).

(e) Fix a set of representatives $A \subset G$ of double classes $P\backslash G/K$, which is finite, because $P\backslash G$ is compact.
Lemma 4.3. In the notation of 4.2, we have equalities:
(a) $\text{Res}_P^G([h]) = \sum_{g \in A} \text{Res}_{K_g,P}^G([h_g])$ in $\mathcal{H}(P)_P$;
(b) $\text{Res}_{K_P}^G([h]) = [h_P]$ in $\mathcal{H}(P)_{K_P}$;
(c) $\text{Res}_P^G([h]) = \sum_{g \in A} [h_{g,P}]$ in $\mathcal{H}(P)_P$.

Proof. (a) By definition, $\text{Res}_P^G([h]) = \varphi_P^G(\tilde{a}^*(h))$. Notice that the decomposition $G = \sqcup_{g \in A} Kg^{-1}P$ into open and closed subsets induces a decomposition $\tilde{G} = \sqcup_{g} \tilde{G}_g$, where $\tilde{G}_g := Kg^{-1}P \times P$. Therefore we get a decomposition $\tilde{a}^*(h) = \sum_{g} h_g$, where $\{h\}_g := \tilde{a}^*(h)|_{\tilde{G}_g} \in \mathcal{H}(\tilde{G}_g) \subset \mathcal{H}(\tilde{G})$. Thus it remains to show that for every $g \in G$, we have the equality $\varphi_P^G([h_g]) = \text{Res}_{K_g,P}^G([h_g])$.

Note that $g\tilde{G}_g = Kg^{-1}P \times P = K_gK_{g,P} \times P = \tilde{K}_g$ and $(g^{-1})^*\{h_g\} = \tilde{a}^*(h_g)|_{\tilde{K}_g}$. Using the identity, $\{h_g\} = [(g^{-1})^*\{h_g\}] \in \mathcal{H}(\tilde{G}),$ the equality $\varphi_P^G([\{h_g\}]) = \text{Res}_{K_g,P}^G([h_g])$ can be rewritten as $\varphi_P^G(\tilde{a}^*(h_g)|_{\tilde{K}_g}) = \text{Res}_{K_g,P}^G([h_g])$.

The latter equality follows from the fact that the diagram

$$
\begin{array}{ccc}
\mathcal{H}(\tilde{G}) & \xrightarrow{\varphi_P^G} & \mathcal{H}(P)_P \\
(1) \uparrow & & (2) \uparrow \\
\mathcal{H}(\tilde{K}_g) & \xrightarrow{\varphi_{K_g,P}^G} & \mathcal{H}(P)_{K_g,P},
\end{array}
$$

where (1) is a natural inclusion, while (2) is a natural projection, is commutative.

(b) Recall that $\varphi_{K_P}^G$ is a composition $\mathcal{H}(\tilde{K})_K \xrightarrow{\tilde{\varphi}} \mathcal{H}(K \times P)_{K \times K_P} \xrightarrow{\phi} \mathcal{H}(P)_{K_P}$, corresponding to the diagram $\tilde{K} \xrightarrow{\phi_1} K \times P \xrightarrow{\phi_2} P$. Since element $\delta_K \otimes h_P$ of $\mathcal{H}(K \times P)$ satisfies $(\phi_2)_!(\delta_K \otimes h_P) = h_P$, it suffices to show the equality $(\phi_1)_!(\delta_K \otimes h_P) = \tilde{a}^*(h)|_{\tilde{K}}$.

Note that $\tilde{a}^*(h) = a^*(f)\tilde{a}^*(\mu^K)$, while $\delta_K \otimes h_P = (1_K \otimes f_P)(\delta_K \otimes \mu^{K_P})$. Moreover, since $f$ is Ad $K$-invariant, we have $p_1((a^*(f)) = 1_K \otimes f_P$. Thus it remains to show the equality $(\phi_1)_!(\delta_K \otimes \mu^{K_P}) = \tilde{a}^*(\mu^K)|_{\tilde{K}}$.

Fix $0 \neq v \in V_G$ and $0 \neq v' \in V_P$. Using identities $|\omega_G(v)| = |\omega_G(v)|(K) \cdot \mu^K$ and $|\omega_P(v')| = |\omega_P(v')|(K_P) \cdot \mu^{K_P}$, it remains to show the equality

$$
(4.1) \quad (\phi_1)_!(|\omega_K(v)| \otimes |\omega_P(v')|) = |\omega_P(v')|(K_P) \cdot |\omega_P(v)|,
$$

where we set $\omega_P(v) = \omega_G(v)|_{\tilde{K}}$. Using the notation of 2.7(b), we set $K^- := K \cap U^-$, and consider open embeddings $j : K^- \times P \hookrightarrow \tilde{K}$, $m : K^- \times K_P \hookrightarrow K$ and $K^- \times P \hookrightarrow G$. Since $p_1$ is $K$-equivariant, it remains to show the restriction of the equality (4.1) to $K^- \times P$ under $j$. 

Note that $p_1^{-1}(K^- \times P) = K^- \times K_P \times P$, while $j^*(\omega_K(v)) = \omega_{K^- \times P}(v)$ and $m^*(\omega_K(v)) = \omega_{K^- \times K_P}(v)$ (by Lemma 4.3). Thus, the assertion follows from the fact that $|\omega_{K^- \times K_P}(v)| = |\omega_{K^- \times K_P}(v)|$, because $K^- \times K_P$ is compact.

(c) follows immediately from (a) and (b). □

4.4. Proof of Proposition 4.1. We have to show that for every $h \in \mathcal{H}(G)$ we have $\chi_\pi(h) = \chi_\tau(\text{Res}_P^G(h))$. Choose an open compact subgroup $K \subset G$ such that $h$ is $K \times K$-invariant. Then, $h \in \mathcal{H}(G)^K$, so by Lemma 4.3 we have to show that

$$\text{Tr}(h, \pi) = \sum_{g \in A} \text{Tr}(h_{g,P}, \tau).$$

Though the result is well-known and is an immediate generalization of the corresponding result for finite groups, we sketch the argument for completeness.

Let $W$ (resp. $V$) be the space of $\tau$ (resp. $\pi$). Then $V^K$ is the space of functions $f : G \to W$ satisfying $f(xyk) = \tau(x)(f(y))$ for all $x \in P, y \in G$ and $k \in K$. For every $g \in G$ we denote by $V^K_g \subset V^K$ the subspace of functions $f : G \to W$ from $V^K$, supported on $PgK$. Then $V^K = \bigoplus_{g \in A} V^K_g$.

For every $g \in G$, consider the endomorphism $h_{\{g\}} : V^K_{g'} \hookrightarrow V^K \xrightarrow{h} V^K \to V^K_g$, induced by $h$. Then $\text{Tr}(h, \pi) = \sum_{g \in A} \text{Tr}(h_{\{g\}})$, so it remains to show that $\text{Tr}(h_{\{g\}}) = \text{Tr}(h_{g,P}, W)$ for every $g \in G$.

Since $gK = Kg_g$, the map $f \mapsto f(g)$ induces a linear isomorphism $V^K_g \cong W^{K_{g,P}}$. It remains to show that this isomorphism identifies $h_{\{g\}} \in \text{End} V^K_{g'}$ with $\tau(h_{g,P}) \in \text{End} W^{K_{g,P}}$, that it, for every $f \in V^K_g$ we have the equality $h(f)(g) = \tau(h_{g,P})(f)(g)$.

We claim that the latter equality holds for every right $K$-invariant $h \in \mathcal{H}(G)$. Indeed, we may assume that $h = \delta_{xK}$ for some $x \in G$. Set $x_g := gxg^{-1}$. Then $h_g = \delta_{xg,K_g}$, and $h(f)(g) = f(xg) = f(x_gg)$. Assume first that $x_g \in PK_g$. Then $x_gK_g = x'K_g$ for some $x' \in P$. In this case, we have $h_{g,P} = \delta_{x'K_g,P}$, and

$$h(f)(g) = f(x_gg) = f(x'g) = \tau(x')(f(g)) = \tau(h_{g,P})(f(g)).$$

Finally, if $x_g \notin PK_g$, then $h(f)(g) = f(x_gg) = 0$, and $h_{g,P} = 0$. □

4.5. Parabolic induction. Let $\rho$ be an admissible representation of $M$. Recall that a non-normalized parabolic induction $\pi = \text{Ind}_P^M(\rho)$ is the induced representation $\text{Ind}_P^G(\tau)$, where $\tau \in \text{Rep}(P)$ is the inflation of $\rho$.

Corollary 4.6. We have the equality of characters $\chi_\pi = \text{Ind}_P^G(\chi_\rho)$.

Proof. Since the character of the inflation $\chi_\tau \in \hat{C}^P(P)$ equals $p^*(\chi_\rho)$, the assertion follows from Proposition 4.1. □
5. Restriction to an equal rank subgroup

5.1. Smooth measures with relatively compact support.

(a) Let $f : X \to Z$ be a morphism of analytic varieties over $F$, where $X$ is smooth. We denote by $\mathcal{H}(X/Z) \subset \mathcal{M}^\infty(X)$ the subspace consisting of measures, whose support is proper over $Z$. In particular, $\mathcal{H}(X/X) = \mathcal{M}^\infty(X)$. Notice that $\mathcal{H}(X) \subset \mathcal{H}(X/Z)$, if $Z$ is Hausdorff, and $\mathcal{H}(X) = \mathcal{H}(X/Z)$, if $Z$ is compact.

(b) Every smooth morphism $f : X \to Y$ over $Z$ induces canonical a push-forward map $f_* : \mathcal{H}(X/Z) \to \mathcal{H}(Y/Z)$. Indeed, we can construct the map $f_*$ locally on $Z$, thus may assume that $Z$ is compact. Now the assertion follows from Lemma 3.5.

(c) Assume that we are in the situation of 3.6, where $H$ acts on $X$ over $Z$, that is, $f(hx) = f(x)$ for every $h \in H$ and $x \in X$. Then the arguments of (b) and 3.6 imply that we have a natural isomorphism $\varphi_H^\mathcal{G} : \mathcal{H}(\text{Ind}_H^G(X)/Z)_G \simeq \mathcal{H}(X/Z)_H$.

(d) Assume that we are given a commutative diagram of analytic spaces over $F$

\[
\begin{array}{ccc}
X & \longrightarrow & Y \\
\downarrow b & & \downarrow \\
X' & \longrightarrow & Y',
\end{array}
\]

such that $X$ and $X'$ are smooth, $b$ is a local isomorphism, and the induced map $X \to X' \times_Y Y$ is proper. Then the pullback $b^* : \mathcal{M}^\infty(X') \to \mathcal{M}^\infty(X)$ (see 3.4(b)) satisfies $b^*(\mathcal{H}(X'/Y')) \subset \mathcal{H}(X/Y)$.

(e) An important particular case of (d) is when the diagram is Cartesian and $Y \to Y'$ is an open embedding. In this case, $b : X \to X'$ is an open embedding as well, and we denote $b^*$ by res and call it the restriction map.

5.2. The algebraic case. (a) Let $H \subset G$ be a closed subgroup. Since the projection map $G \to G/H$ is smooth, every $G$-orbit in $(G/H)(F)$ is open. Therefore $G/H = G(F)/H(F)$ is an open and closed subset of $(G/H)(F)$.

(b) Let $H$ acts on a smooth algebraic variety $X$. Then, by (a), the induced space $\text{Ind}_H^G(X)$ is an open and closed subset of $\text{Ind}_H^G(X)(F)$. Hence we have a natural restriction map $\text{res} : \mathcal{H}(\text{Ind}_H^G(X)(F))_G \to \mathcal{H}(\text{Ind}_H^G(X))_G$, and we denote the composition $\varphi_H^\mathcal{G} \circ \text{res} : \mathcal{H}(\text{Ind}_H^G(X)(F))_G \to \mathcal{H}(X)_H$ simply by $\varphi_H^\mathcal{G}$.

(c) Assume that $H$ acts on $X$ over $Z$ (compare 5.1(c)). Then, generalizing (b), we have a map $\varphi_H^\mathcal{G} : \mathcal{H}(\text{Ind}_H^G(X)(F)/Z)_G \to \mathcal{H}(X/Z)_H$.

5.3. Notation. Let $K \subset H$ be two closed subgroups of $G$.

(a) We set $\mathcal{H}(H)^\text{reg}/G := \mathcal{H}(H^{\text{reg}}/G/e_H^{\text{reg}}/G)$, where $H^{\text{reg}}/G = H^{\text{reg}}/G(F)$ and $e_H^{\text{reg}}/G = e_H^{\text{reg}}/G(F)$ (see 1.2(b)) and similarly, $\mathcal{H}(G_H)^{\text{reg}} := \mathcal{H}(G_H^{\text{reg}}/G^{\text{reg}}/G)$ and $\mathcal{H}(\tilde{H}_K)^{\text{reg}}/G := \mathcal{H}(\tilde{H}_K^{\text{reg}}/G^{\text{reg}}/G)$ (see 1.4(e)).
(b) By 5.1(a) and 5.1(e), we have a restriction map $\text{res} : \mathcal{H}(H) \hookrightarrow \mathcal{H}(H/cH) \to \mathcal{H}(H)_{\text{reg}}/G$

(c) Recall that the map $a_{H,G} : \widetilde{G}\text{reg}H \to G$ is $G$-equivariant and étale (see Corollary 2.6), thus the corresponding map $a_{H,G} : \widetilde{G}\text{reg}H \to G$ is a local isomorphism. Thus we have a pullback map $a^*_{H,G} : \mathcal{M}_\infty(G) \to \mathcal{M}_\infty(\widetilde{G}\text{reg}H)$ (see 5.4(b)).

5.4. The restriction map. Assume that we are in the situation of 1.5.

(a) Since the morphism $\iota_{H,G} : \widetilde{G}\text{reg}H \to G \times cG c\text{reg}/G$ is proper (by Corollary 1.9), the corresponding morphism $\iota_{H,G} : \widetilde{G}\text{reg}H \to G \times cG c\text{reg}/G$ is proper. Thus, by 5.3(c) and 5.1(d) we have a pullback map $a^*_{H,G} : \mathcal{H}(G) \to \mathcal{H}(G/cGc) \to \mathcal{H}(\widetilde{G}H)_{\text{reg}}^G$.

(b) We denote by $R^G_H : \mathcal{H}(G) \to \mathcal{H}(\widetilde{G}\text{reg}H) \to \mathcal{H}(G/cGc)$ the composition of the map $a^*_{H,G}$ from (a) and the map $\phi_{G,H}$ from 5.2(c).

5.5. Set up. Assume that we are in the situation of 3.8 and that $H$ is a connected equal rank subgroup of $M$, hence also of $G$.

(a) Then $c_{H\text{reg}/G} \subset c_{M\text{reg}/G}$ is an open subscheme (see 1.2(c)), thus we have defined a restriction map $\text{res} : \mathcal{H}(H)_{\text{reg}/M} \to \mathcal{H}(H)_{\text{reg}/G}$ (see 5.1(e)).

(b) Using 1.2(c) again, we conclude that $H\text{reg}/G \subset P\text{reg}/G$. Thus the restriction of $\Delta_{P,G}$ to $H\text{reg}/G$ is non-vanishing. Hence $\Delta_{P,G}$ gives rise to a smooth function $|\Delta_{P,G}| \in \mathcal{M}_\infty(H\text{reg}/G)$.

Lemma 5.6. In the situation of 5.5 the following diagram is commutative

\[
\begin{array}{ccc}
\mathcal{H}(G)_G & \xrightarrow{R^G_H} & \mathcal{H}(H)_{H\text{reg}/G} \\
\text{Res}_{P,M}^G \downarrow & & \uparrow |\Delta_{P,G}| \cdot \text{res} \\
\mathcal{H}(M)_M & \xrightarrow{R^M_H} & \mathcal{H}(H)_{H\text{reg}/M}
\end{array}
\]

Proof. The assertion follows from a rather straightforward diagram chase. Namely, using the inclusion $H\text{reg}/G \subset M\text{reg}/G$, we observe that diagram (5.1) decomposes as

\[
\begin{array}{ccc}
\mathcal{H}(G)_G & \xrightarrow{R^G_M} & \mathcal{H}(M)_{M\text{reg}/G} \\
\text{Res}_{P,M}^G \downarrow & & \uparrow |\Delta_{P,G}| \cdot \text{res} \\
\mathcal{H}(M)_M & \rightarrow & \mathcal{H}(H)_{H\text{reg}/M}
\end{array}
\]

Since the right inner square of (5.2) is commutative by functoriality, it remains to show the commutativity of the left inner square.
Observe that the diagram

\[
\begin{array}{ccc}
\mathcal{H}(P)_P & \xrightarrow{\text{res}} & \mathcal{H}(P)_{P/G} \\
p^! & \downarrow & \downarrow p^! \\
\mathcal{H}(M)_M & \xrightarrow{\text{res}} & \mathcal{H}(M)_{M/G}
\end{array}
\]

(5.3)

is commutative. Indeed, the left inner square of (5.3) is commutative by (1.7(e)), and the right inner square is commutative, because \(\varphi^p_M = p_!\) (see 3.6(e)) and \(a^*_{P,M}\) is an isomorphism (by (1.7(e))).

Therefore the left inner square of (5.2) decomposes as

\[
\begin{array}{ccc}
\mathcal{H}(G)_G & \xrightarrow{\hat{\alpha}^*_{P,G}} & \mathcal{H}(G)_G \\
\hat{\alpha}^*_{P,G} & \downarrow & \downarrow a^*_{P,G} \\
\mathcal{H}(\hat{G})_G & \xrightarrow{\Delta_{P,G} \cdot a^*_{P,M}} & \mathcal{H}(\hat{G})_G \\
\varphi^p_P & \downarrow & \varphi^p_P \\
\mathcal{H}(P)_P & \xrightarrow{\Delta_{P,G} \cdot a^*_{P,M}} & \mathcal{H}(P)_{P/G}
\end{array}
\]

(5.4)

We claim that all inner squares of (5.4) are commutative. Indeed, the top right square is commutative by the definition of \(R^G_M\), the bottom left square is commutative by the functoriality of \(\varphi^G_P\), and the bottom right square is commutative by the equality \(\varphi^p_M \circ \varphi^G_P = \varphi^G_P\).

Finally, the commutativity of the top left square of (5.4) follows from the equality \(a^*_{M,G} = a^*_{M,P} \circ a^*_{P,G}\) (see 1.4(d)) and Lemma 2.5.

6. Normalized induction, independence of \(P\), and stability

6.1. Normalized restriction and induction.

(a) Recall that in the construction of the restriction map \(\text{Res}^G_{P,M}\) in 3.8(c) we used the isomorphism \(O_G \otimes_F V_G \xrightarrow{\sim} K_G : v \mapsto \omega_G(v)\) (see 2.3(c)). Instead we could use the isomorphism \(O_G \otimes_F V_G \xrightarrow{\sim} K_G : v \mapsto \omega_G(v)\) (see 2.4(b)). Since \(\omega_G(v) = \lambda_G \cdot \omega_G(v)\), the resulting restriction map would be \(|\lambda_P| \cdot \text{Res}^G_{P,M}\).

(b) We denote by \(r^G_{P,M} : \mathcal{H}(G)_G \to \mathcal{H}(M)_M\) the map \(|\lambda_P|^{1/2} \cdot \text{Res}^G_{P,M}\), and call it the normalized restriction map. Let \(i^G_{P,M} : \hat{C}^M(M) \to \hat{C}^G(G)\) be the dual map of \(r^G_{P,M}\), called the normalized induction map. Explicitly, \(i^G_{P,M}(\chi) = \text{Ind}_{P,M}^G(|\lambda_P|^{1/2} \cdot \chi)\) for every \(\chi \in \hat{C}^M(M)\).
(c) For an admissible representation $\rho$ of $M$, we denote by $\mathcal{I}_{P,M}^G(\rho)$ the representation $\text{Ind}_P^G(\rho \otimes |\lambda_P|^{1/2})$ of $G$ and call it the \textit{normalized parabolic induction}.

6.2. Remark. If $G$ is semisimple and simply connected, and $P$ is a Borel subgroup of $G$, then the normalized restriction map $r_{P,M}^G$ has a geometric interpretation.

Indeed, in this case, the homomorphism $\lambda_P : P \to \mathbb{G}_m$ has a unique square root $\lambda_P^{1/2} : P \to \mathbb{G}_m$. Furthermore, $\lambda_P^{1/2}$ gives rise to a morphism $\lambda_P^{1/2} : \tilde{G} \to \mathbb{G}_m$ (by 1.4(a)), hence to an isomorphism $\mathcal{O}_G \otimes_F V_G \cong K_G : v \mapsto \lambda_P^{1/2} \cdot \omega_G(v)$, and $r_{P,M}^G$ is obtained from this isomorphism by construction 3.8.

The following result follows immediately from Corollary 4.6.

\textbf{Corollary 6.3.} We have the equality of characters $\chi_{\mathcal{I}_{P,M}^G(\rho)} = i_{P,M}^G(\chi_{\rho})$.

Next, we write a version of Lemma 5.6 for the normalized restriction.

\textbf{Corollary 6.4.} In the situation of 5.6, the following diagram is commutative

\begin{equation}
\begin{array}{ccc}
\mathcal{H}(G)_G & \xrightarrow{R_H^G} & \mathcal{H}(H)^{\text{reg}/G}_H \\
\downarrow r_{P,M}^G & & \downarrow |\Delta_{M,G}|^{1/2} \cdot \text{res} \\
\mathcal{H}(M)_M & \xrightarrow{R_H^M} & \mathcal{H}(H)^{\text{reg}/M}_H.
\end{array}
\end{equation}

\textbf{Proof.} By 2.7(a), we have the identity $|\Delta_{M,G}|^{1/2} = |\Delta_{P,G}| \cdot |\lambda_P|^{-1/2}$, so the assertion follows from Lemma 5.6. \hfill \Box

\textbf{Lemma 6.5.} The restriction map $\text{res} : \mathcal{H}(M)_M \to \mathcal{H}(M)^{\text{reg}/G}_M$ from 3.3(b) is injective.

\textbf{Proof.} By definition, the connected center $Z(M)^0 \subset Z(M)$ acts on $\text{Lie } G / \text{Lie } M$ by a direct sum of non-trivial characters. Therefore for every $m \in M$, the locus of $z \in Z(M)$ such that $zm \in M^{\text{reg}/G}$, is open and Zariski dense. Similarly, the action of $Z(M)$ on $M$ induces an action of $Z(M)$ on $c_M$, and for every $m \in c_M$, the locus of $z \in Z(M)$ such that $zm \in c_M^{\text{reg}/G}$, is open and Zariski dense. Thus, for every open subgroup $U \subset Z(M)$ we have $U(c_M^{\text{reg}/G}) = c_M$.

Note that $Z(M)$ acts smoothly on $\mathcal{H}(M)$, and induces a smooth action on $\mathcal{H}(M)_M$. Fix $h \neq 0$, and let $U \subset Z(M)$ be the stabilizer of $h$. Since $U(c_M^{\text{reg}/G}) = c_M$, we conclude that $h|_{c_M^{\text{reg}/G}} \neq 0$, thus $\text{res}(h) \neq 0$. \hfill \Box

Now we show that the normalized induction map does not depend on $P$.

\textbf{Corollary 6.6.} (a) The normalized restriction map $r_{P,M}^G : \mathcal{H}(G)_G \to \mathcal{H}(M)_M$ and induction map $i_{P,M}^G : \hat{C}^M(M) \to \hat{C}^G(G)$ do not depend on $P$. 

(b) For every admissible representation ρ of M, the family of composition factors of \( i_{P,M}^G(ρ) \) does not depend on P.

Proof. (a) It suffices to show the assertion for \( r_{P,M}^G \). By Lemma 6.5, it suffices to show that the composition \( \text{res} \circ r_{P,M}^G : \mathcal{H}(G)_G \to \mathcal{H}(M)^{\text{reg}/G}_M \) does not depend on P. On the other hand, by Corollary 6.4 for \( H = M \), this composition equals \( |Δ_{M,G}|^{-1/2} \cdot R_M^G \).

(b) It is enough to show that the character of \( i_{P,M}^G(ρ) \) do not depend on P. But this follows from (a) and Corollary 6.3.

6.7. Notation. By Corollary 6.6(a), we can now denote \( r_{P,M}^G \) and \( i_{P,M}^G \), simply by \( r_M^G \) and \( i_M^G \), respectively.

6.8. (Stable) orbital integrals. Let \( G^{\text{rss}} \subset G \) be the set of regular semisimple elements.

(a) In the situation of 5.4, let \( H = S \subset G \) be a maximal torus defined over \( F \). In this case, \( c_S^{\text{reg}/G} \) equals \( S^{\text{rss}}_G := S \cap G^{\text{rss}}_S \) and \( R_S^{G} \) is the map \( \mathcal{H}(G) \to \mathcal{M}^∞(S^{\text{rss}}_G) \). We denote by \( O_{S,G} : \mathcal{M}^∞(S^{\text{rss}}_G)^* \to \hat{C}^G(G) \) the dual of \( R_S^{G} \) and call it the orbital integral map.

(b) Explicitly, let \( \text{pr} : (G/S) \times S^{\text{rss}}_G \to S^{\text{rss}}_G \) be the projection. Since \( S \) acts on \( S^{\text{rss}}_G \) trivially, it follows from 3.6(d) that \( R_S^{G} \) equals the composition

\[
\mathcal{H}(G)_G \overset{a_S,G}{\to} \mathcal{H}([G/S] \times S^{\text{rss}}_G/S^{\text{rss}}_G)_G \overset{\text{pr}}{\to} \mathcal{M}^∞(S^{\text{rss}}_G).
\]

(c) Denote by \( (R_S^{G})^* : \mathcal{H}(G)_G \to \mathcal{M}^∞(S^{\text{rss}}_G) \) the composition

\[
\mathcal{H}(G)_G \overset{a_S,G}{\to} \mathcal{H}([G/S](F) \times S^{\text{rss}}_G/S^{\text{rss}}_G)_G \overset{\text{pr}}{\to} \mathcal{M}^∞(S^{\text{rss}}_G),
\]

and let \( O_{S,G}^* : \mathcal{M}^∞(S^{\text{rss}}_G)^* \to \hat{C}^G(G) \) be the dual map, called the stable orbital integral.

6.9. Comparison. (a) Let \( [g_1], \ldots, [g_n] \in (G/S)(F) \) be a set of representatives of the set of \( G \)-orbits. For every \( j \), let \( S_j \subset G \) be the stabilizer of \( [g_j] \). Then \( S_j \) is a maximal torus of \( G \), and there is a canonical isomorphism \( i_j : S \to S_j \). Explicitly, if \( g \in G(F) \) is any representative of \( [g_j] \), then \( i_j \) is the map \( s \mapsto g_j s g_j^{-1} \). By construction, we have the equality \( O_{S,G}^* = \sum_{j=1}^n O_{S_j,G} \circ (i_j)_* \).

(b) Fix a Haar measure \( |ω|_S \) on \( S \). Then for every \( γ \in S^{\text{rss}}_G \), we can consider a “δ-function” \( δ_γ \in \mathcal{M}^∞(S^{\text{rss}}_G)^* \), defined by the formula \( δ_γ(f|ω|_S) = f(γ) \) for every \( f \in C^∞(S^{\text{rss}}_G) \). Then construction 6.8 gives us “classical” (stable) orbital integrals \( O_{γ,G} := O_{S,G}(δ_γ) \in \hat{C}^G(G) \) (resp. \( O_{S,G}^* := O_{S,G}^*(δ_γ) \in \hat{C}^G(G) \)). For example, observation (a) implies that a stable orbital integral is a sum of orbital integrals.
Corollary 6.10. In the situation of 6.8, the following diagram is commutative

\[
\begin{array}{ccc}
\mathcal{M}^\infty(S_G^{rss})^* & \xrightarrow{O^\text{st}_{S,G}} & \hat{\mathcal{C}}^G(G) \\
\downarrow (|\Delta_{M,G}|^{1/2} \cdot \text{res})^* & & \downarrow i^G_M \\
\mathcal{M}^\infty(S_M^{rss})^* & \xrightarrow{O^\text{st}_{S,M}} & \hat{\mathcal{C}}^M(M),
\end{array}
\]

and similarly for \(O_{S,G}\) and \(O_{S,M}\).

Proof. The assertion for orbital integrals is simply the dual of Corollary 6.4 for \(\mathbf{H} = \mathbf{S}\). The assertion for stable orbital integrals follows from that for orbital integrals, by observation 6.9(a) and the equality \((G/M)(F) = G/M\).

\[ \square \]

6.11. Stable generalized functions and representations.

(a) We denote by \(\hat{\mathcal{C}}^{\text{st}}(G) \subset \hat{\mathcal{C}}^G(G)\) the closure of the image of \(O^\text{st}_G := \oplus_{S \subset G} O^\text{st}_{S,G} : \mathcal{M}^\infty(S_G^{rss})^* \to \hat{\mathcal{C}}^G(G)\).

Elements of \(\hat{\mathcal{C}}^{\text{st}}(G)\) are called stable generalized functions.

(b) An admissible representation \(\pi\) of \(G\) is called stable, if its character \(\chi_\pi\) is stable.

Corollary 6.12. (a) The induction map \(i^G_M : \hat{\mathcal{C}}^M(M) \to \hat{\mathcal{C}}^G(G)\) sends stable generalized functions into stable ones.

(b) The induction functor \(\mathcal{L}_P^G\) sends stable representations into stable ones.

Proof. (a) Since \(S_G^{rss} \subset S_M^{rss}\) is dense, the restriction map \(\text{res} : \mathcal{M}^\infty(S_M^{rss}) \to \mathcal{M}^\infty(S_G^{rss})\) is injective. Thus the dual map \(\text{res}^*\) is surjective. Hence, by Corollary 6.10, we get \(i^G_M(\text{Im} O^\text{st}_{S,M}) \subset \hat{\mathcal{C}}^{\text{st}}(G)\) for every \(S \subset G\). Since \(i^G_M\) is continuous, the assertion follows.

(b) Follows from (a) and Corollary 6.3 \[ \square \]

Appendix A. A Generalization of a Theorem of Lusztig–Spaltenstein

A.1. Notation. Let \(F\) be an infinite field. All algebraic varieties and all morphisms of algebraic varieties are over \(F\).

(a) Let \(G\) be a connected reductive group, \(P \subset G\) a parabolic subgroup, \(U \subset P\) the unipotent radical, and \(M \subset P\) a Levi subgroup.

(b) For an algebraic variety \(X\), we denote the set \(X(F)\) by \(X\). In particular, we have \(G = G(F), \tilde{G}_P = \tilde{G}_P(F)\), etc (compare 3.7).

(c) For an \(\text{Ad} P\)-invariant subset \(D \subset P\), we set \(\text{Ind}_{P}^{G}(D) := G^{P,\text{ad}} \times D \subset \tilde{G}_P\).
(d) For an Ad $M$-invariant subset $C \subset M$, we set $\text{Ind}_{M}^{G}(C) := G^{M,\text{ad}} \times C \subset \tilde{G}_{M}$, $C_{P} := U \cdot C \subset P$, $\text{Ind}_{P}^{G}(C_{P}) \subset \tilde{G}_{P}$ and $C_{P,G} := a_{P,G}(\text{Ind}_{P}^{G}(C_{P})) \subset G$, where $a_{P,G}$ is defined in 1.3.

From now on, we assume that $C \subset M$ is a unipotent $M$-conjugacy class.

A.2. Question. Does the set $C_{P,G}$ depend on a choice of $P \supset M$?

A.3. Remarks. (a) $C_{P,G}$ is a union of unipotent conjugacy classes in $G$.

(b) Let $F$ be algebraically closed. By a theorem of Chevalley, $C_{P,G} \subset G$ is a constructible set, whose Zariski closure $\overline{C_{P,G}}$ is irreducible. This case was considered by Lusztig and Spaltenstein in [LS], and they showed that $\overline{C_{P,G}}$ does not depend on $P$, using representation theory.

The goal of the appendix is to generalize this result to other fields.

A.4. Saturation. Let $X$ be an algebraic variety over $F$, and let $A \subset X$ be a subset.

(a) We denote by $\text{sat}'(A) = \text{sat}'_{X}(A) \subset X$ the union $\bigcup_{(V,x,f)} f(x)$, taken over triples $(V,x,f)$, where $V \subset A^{1}$ is an open subvariety, $x \in V$, $V' := V \setminus x$, and $f : V \to X$ is a morphism such that $f(V') \subset A$.

(b) We say that a subset $A \subset X$ is saturated, if $\text{sat}'(A) = A$.

(c) Let $\text{sat}(A) \subset X$ be the smallest saturated subset, containing $A$.

Theorem A.5. The saturation $\text{sat}(C_{P,G})$ does not depend on $P$.

A.6. Remarks. (a) The notion of saturation is only reasonable, if the variety $X$ is rationally connected.

(b) For every closed subvariety $Y \subset X$, the subset $Y(F) \subset X$ is saturated. Also, if $F$ is a local field, then every closed subset of $X$ is saturated.

(c) If $X = A^{1}$, then a subset $A \subset X$ is saturated if and only if either $A = X$ or $X \setminus A$ is infinite.

(d) By (c), saturated subsets of $X$ are not closed under finite unions. Therefore the set $X$ does not have a topology, whose closed subsets are precisely the saturated subsets. On the other hand, the proof of Theorem A.5 indicate that in some respects saturated sets behave like closed subsets in some topology.

Lemma A.7. Let $X$ and $Y$ be algebraic varieties.

(a) For a morphism $f : X \to Y$ and a subset $A \subset X$, we have an inclusion $f(\text{sat}'(A)) \subset \text{sat}(f'(A))$.

(b) For every $A \subset X$ and $B \subset Y$, we have $\text{sat}'(A \times B) = \text{sat}'(A) \times \text{sat}'(B)$.

(c) Let $H$ be an algebraic group, and let $f : X \to Y$ be a principal $H$-bundle, locally trivial in the Zariski topology. Then for every subset $A \subset Y$ we have the equality $\text{sat}'(f^{-1}(A)) = f^{-1}(\text{sat}'(A))$. 
(d) For an Ad $P$-invariant subset $A \subset P$, the corresponding subset $\text{Ind}_P^G(A) \subset \text{Ind}_P^G(X)$ satisfies $\text{sat}'(\text{Ind}_P^G(A)) = \text{Ind}_P^G(\text{sat}'(A))$.

(e) Let $Y \subset X = \mathbb{A}^n$ be an open dense subvariety. Then $\text{sat}'_X(Y) = X$.

Proof. (a) is clear.

(b) The inclusion $\subset$ follows from (a). Conversely, assume that $a \in \text{sat}'(A)$ and $b \in \text{sat}'(B)$ are defined using triples $(V_a, x_a, f_a)$ and $(V_b, x_b, f_b)$, respectively, where $V_a$ and $V_b$ are open subsets of $\mathbb{A}^1$. Then we can assume that $V_a = V_b \subset \mathbb{A}^1$ and $x_a = x_b$, which implies that $(a,b) \in \text{sat}'(A \times B)$.

(c) Since the saturation $\text{sat}'$ is local in the Zariski topology, we can assume that $X = Y \times H$. In this case the assertion follows from (b).

(d) follows from (c) (compare [3.6]).

(e) It suffices to show that for every $x \in \mathbb{A}^n(F)$ there exists a line $L \subset \mathbb{A}^n$, defined over $F$, such that $x \in L$ and $L \cap Y \neq \emptyset$. Consider the variety $\mathbb{P}_x$ of lines $L \subset \mathbb{A}^n$ such that $x \in L$. Since $Y \subset \mathbb{A}^n$ is Zariski dense, the set of $L \in \mathbb{P}_x$ such that $L \cap Y \neq \emptyset$, is Zariski dense. Since $\mathbb{P}_x \cong \mathbb{P}^{n-1}$, while $F$ is infinite, the subset $\mathbb{P}_x(F) \subset \mathbb{P}_x$ is Zariski dense, and the assertion follows.

\section*{A.8. Remark.} All the properties of $\text{sat}'$, formulated in Lemma A.7, have natural analogs for $\text{sat}$.

\section*{A.9. Relative saturation.} Let $h : X \to Y$ be a morphism and $A \subset X$.

(a) Denote by $\text{sat}'(h; A) \subset \text{sat}'(h(A))$ the union $\cup_{(V, x, f)}(f(x))$, taken over all triples $(V, x, f)$ in the definition of $\text{sat}'(h(A))$ such that $f|_{V'} : V' \to Y$ has a lift $\tilde{f'} : V' \to X$ with $\tilde{f'}(V') \subset A$.

(b) If $h$ is proper, then $\text{sat}'(h; A) = h(\text{sat}'(A))$. Indeed, the valuative criterion implies that every pair $(f, \tilde{f'})$ as in (a) defines a unique morphism $\tilde{f} : V \to X$ such that $h \circ \tilde{f} = f$ and $\tilde{f}|_{V'} = \tilde{f'}$.

(c) Let $X' \subset X$ be an open subvariety such that $A \subset X'$, and let $h' : h|_{X'} : X' \to Y$ be the restriction. By definition, $\text{sat}'(h'; A) = \text{sat}'(h; A)$.

\section*{A.10. Notation.} (a) $a_{M,G} : \tilde{G}^\text{reg}_M \to G$ be the map defined in [1.3] and [1.4] let $\nu_G : G \to c_G$ be the Chevalley map (see [I.Ia]), and set $e := \nu_G(1)$.

(b) Let $G^\text{der} \subset G$ be the derived group, $Z(M)$ the center of $M$, and set $Z_M := (Z(M) \cap G^\text{der})^0$. Then $Z_M$ is a split torus over $F$. Set $Z_M^\text{reg} := Z_M \cap M^\text{reg}/G$. Notice that since $Z_M$ acts on $\text{Lie} G / \text{Lie} M$ by a direct sum of nontrivial characters, the subset $Z_M^\text{reg} \subset Z_M$ is open and dense.

(c) Set $C_Z^\text{reg} := C \cdot Z_M^\text{reg} \subset M$. Since $C \subset M$ consists of unipotent elements, and $Z_M \subset Z(M)$, we have $C_Z^\text{reg} \subset M^\text{reg}/G$. Also $C_Z^\text{reg}$ is Ad $M$-invariant, so we can form a subset $\text{Ind}_M^G(C_Z^\text{reg}) \subset G^\text{reg}_M$.

(d) Set $C_{P,Z}^\text{reg} := p^{-1}(C_Z^\text{reg}) = C_P \cdot Z_M^\text{reg} \subset P^\text{reg}/G$ (see [1.7](c)).
A.11. Proof of Theorem A.5. Consider the subset \( D_P := a_{P,G}(\text{sat}'(\text{Ind}_P^G(C_P))) \) of \( G \). Since \( C_{P,G} = a_{P,G}(\text{Ind}_P^G(C_P)) \), we have inclusions \( C_{P,G} \subset D_P \subset \text{sat}(C_{P,G}) \) (see Lemma A.7(a)), thus \( \text{sat}(D_P) = \text{sat}(C_{P,G}) \). It suffices to show that \( D_P \) does not depend on \( P \). But this follows from the following description of \( D_P \).

Claim A.12. We have the equality \( D_P = \text{sat}'(a_{M,G}; \text{Ind}_M^G(C_Z^\text{reg})) \cap \nu_G^{-1}(e) \).

Proof. Recall (see 1.4(d)) that morphism \( a_{M,G} \) factors as \( \tilde{G}_M^\text{reg} \xrightarrow{a_{M,P}} \tilde{G}_P \xrightarrow{a_{P,G}} G \). Notice that \( a_{M,P} : \tilde{G}_M^\text{reg} \to \tilde{G}_P \) is an open embedding (use 1.7(e)) and satisfies \( a_{M,P}(\text{Ind}_M^G(C_Z^\text{reg})) = \text{Ind}_P^G(C_{P,Z}^\text{reg}) \). Therefore, by A.9(c), we have the equality

\[
\text{sat}'(a_{M,G}; \text{Ind}_M^G(C_Z^\text{reg})) = \text{sat}'(a_{P,G}; \text{Ind}_P^G(C_{P,Z}^\text{reg})).
\]

Next, since \( a_{P,G} \) is proper, we conclude from A.9(b) that

\[
\text{sat}'(a_{P,G}; \text{Ind}_P^G(C_{P,Z}^\text{reg})) = a_{P,G}(\text{sat}'(\text{Ind}_P^G(C_{P,Z}))).
\]

Thus, it suffices to show the equality

\[
a_{P,G}(\text{sat}'(\text{Ind}_P^G(C_P))) = a_{P,G}(\text{sat}'(\text{Ind}_P^G(C_{P,Z}^\text{reg}))) \cap \nu_G^{-1}(e).
\]

Using the commutative diagram from 1.4(b) for \( H = P \) and equality \( \pi_{P,G}^{-1}(e) = e \) (see 1.7(f)), we conclude that \( a_{P,G}(A) \cap \nu_G^{-1}(e) = a_{P,G}(A \cap \nu_G^{-1}(e)) \) for every subset \( A \subset \tilde{G}_P \). Thus, it suffices to show the equality

\[
\text{sat}'(\text{Ind}_P^G(C_{P,Z}^\text{reg})) \cap \nu_{G_P}^{-1}(e) = \text{sat}'(\text{Ind}_P^G(C_P)) \subset \tilde{G}_P.
\]

Using Lemma A.7(d), it suffices to show the equality

\[
(A.1) \quad \text{sat}'(C_{P,Z}^\text{reg}) \cap \nu_P^{-1}(e) = \text{sat}'(C_P) \subset P.
\]

Set \( P_{\text{un}} := \nu_P^{-1}(e) \subset P \), and \( P_{Z_M} := \nu_P^{-1}(\nu_P(Z_M)) \subset P \). Since the map \( \nu_P|_{Z_M} : Z_M \to \mathcal{C}_P \cong \mathcal{C}_M \) is a closed embedding, the multiplication map induces an isomorphism \( P_{\text{un}} \times Z_M \cong P_{Z_M} \). Moreover, it induces a bijection \( C_P \times Z_M^\text{reg} \cong C_{P,Z}^\text{reg} \). Thus, formula (A.1) follows from the equality

\[
\text{sat}'_{P_{\text{un}} \times Z_M}(C_P \times Z_M^\text{reg}) = \text{sat}'_{P_{\text{un}}}(C_P) \times \text{sat}'_{Z_M}(Z_M^\text{reg}) = \text{sat}'_{P_{\text{un}}}(C_P) \times Z_M,
\]

which follows from Lemma A.7(b),(e). □

Corollary A.13. (a) If \( F \) is algebraically closed, then the closure \( \overline{\text{cl}}(C_{P,G}) \subset G \) of \( C_{P,G} \) in the Zariski topology does not depend on \( P \).

(b) If \( F \) is a local field, then the closure \( \overline{\text{cl}}(C_{P,G}) \subset G \) of \( C_{P,G} \) in the analytic topology does not depend on \( P \).
Proof. In both cases, every closed subset in $G$ is saturated. Therefore we have inclusions $C_{P,G} \subset \text{sat}(C_{P,G}) \subset \text{cl}(C_{P,G})$, which implies that $\text{cl}(C_{P,G}) = \text{cl}(\text{sat}(C_{P,G}))$. Thus the assertion follows from Theorem [A.5].

A.14. Notation. For an Ad$G$-invariant subset $D \subset G$, we denote by $D^\circ \subset D$ the union of $G$-conjugacy classes, which are Zariski dense in (the Zariski closure of) $D$.

A.15. Question. Is it true that $C_{P,G}^\circ$ is independent of $P$?

A.16. Remarks. (a) Let $F$ be algebraically closed. Since the number of unipotent conjugacy classes in $G$ is finite, we conclude that $C_{P,G}^\circ$ is a single conjugacy class.

(b) Let $F$ be general. Then, by (a), $C_{P,G}^\circ$ is a union of unipotent conjugacy classes, belonging to a single conjugacy class over $\overline{F}$.

Lemma A.17. Let $F$ be either algebraically closed or local. Then for every Ad$G$-invariant subset $D \subset G$, we have $D^\circ = \text{sat}(D)\circ$.

Proof. Let $\text{cl}(D) \subset G$ be the closure of $D$ in the Zariski topology, if $F$ is algebraically closed, and analytic topology, if $F$ is local. Then, as in the proof of Corollary [A.13] we have $D \subset \text{sat}(D) \subset \text{cl}(D)$. Thus, it suffices to show that $D^\circ = \text{cl}(D)^\circ$.

Let $O \subset \text{cl}(D)$ be a Zariski dense $G$-conjugacy class, let $\overline{D} \subset G$ be the Zariski closure of $D$, and choose $x \in O$. Then the morphism $G \to \overline{D} : g \mapsto gxg^{-1}$ is dominant. Therefore, in both cases, the corresponding map $G \to \text{cl}(D)$ is open. Thus $O \subset \text{cl}(D)$ is open, hence $O \subset D$.

Corollary A.18. Let $F$ be either algebraically closed or local. Then the subset $C_{P,G}^\circ \subset G$ does not depend on $P$.

Proof. This follows immediately from Theorem [A.5] and Lemma [A.17].

A.19. Remark. We do not expect that the conclusion Lemma [A.17] holds for an arbitrary field $F$. We wonder whether the equality $\text{sat}(C_{P,G})^\circ = C_{P,G}^\circ$ always holds.

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Institute of Mathematics, The Hebrew University of Jerusalem, Givat-Ram, Jerusalem, 91904, Israel

E-mail address: kazhdan@math.huji.ac.il

Institute of Mathematics, The Hebrew University of Jerusalem, Givat-Ram, Jerusalem, 91904, Israel

E-mail address: vyakov@math.huji.ac.il