Reduced Description of Stellar Dynamics
by Moments of Gravitation Field

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Because of absence of time derivatives from scalar potential as a generalized coordinate of gravitation field (GF) in action of nonrelativistic gravitating system, application of the Hamilton method for description of GF mechanics was impossible. In the paper a transformation of the generalized coordinate of GF, that is based on continuity equation and minimal action principle, is proposed. A potential vector is introduced that is similar to fixing of Hamilton gauge of the electromagnetic field. This transformation gives possibility of the calculation a Hamilton function (HF), removes mathematical troubles of the Jeans theory (Jeans swindle) and allows to construct kinetic theory of GF using statistical mechanics methods.

Introduction

Nowadays we have powerful computers that allow us to calculate motion of every particle of complicated systems, but for macroscopic systems like galaxies or clusters we still need to simplify our description.

It is well known that main equations (kinetic or hydrodynamic) of stellar dynamics theory use self-consistent GF [1]. That means, all correlation moments of GF are neglected and the first moment (mean field) is considered as external field.

On the other hand, to construct more accurate theory we need HF of masses with gravitation interaction. Such function has been written many years ago by Hamilton himself, but without generalized coordinates (and freedom degrees) of GF! That is effective HF for masses only which uses special solution of Poisson equation to deliver from GF.

We will try to use analogy with longitudinal electromagnetic field [2] to construct a transformation of generalized coordinates of GF, which gives HF with GF coordinates. That will allow us to use statistical mechanics methods to describe GF not only by mean field, but also, for example, by one particle distribution functions (normal and anomalous) which are equivalent to second correlation moments of GF, that gives more information about GF and whole system.

Mechanics

We shall start from action for Newtonian GF and masses $m$ (equal for simplicity) with density $\sigma$ ($\mathbb{R}^3$ §106)

$$S = \int \int \left( \frac{m\sigma v^2}{2} - m\sigma \varphi - \frac{(\nabla \varphi)^2}{8\pi G} \right) dV dt$$

(1)

It is convenient to construct perturbation theory by interaction when we have "small charge". To this purpose make next generalized coordinate transformation $\varphi = \sqrt{G} \varphi$.

$$S = \int \int \left( \frac{m\sigma v^2}{2} - \sqrt{G} m\varphi - \frac{(\nabla \varphi)^2}{8\pi} \right) dV dt$$

(2)

And we introduce an often used gravitational charge of particle $a$: $e_a = \sqrt{G} m_a$, that gives

$$S = \int \int \left( \frac{m\sigma v^2}{2} - e_a \varphi - \frac{(\nabla \varphi)^2}{8\pi} \right) dV dt$$

(3)
This action looks like action for longitudinal electric field in fixed Coulomb gauge (with minus, of course). It is well known, that (3) gives us only effective HF, because there are no time derivatives from scalar potential of GF. That means, scalar potential is not a convenient generalized coordinate and we need transform it.

Obviously, we have the continuity equation (from charge conservation fact)

\[
\text{div} \vec{j} + \partial_t \rho = 0, \tag{4}
\]

(density and current of mass are respectively \(\rho = \sum_a e_a \delta (x - x_a)\) and \(\vec{j} = \sum_a e_a \vec{v} \delta (x - x_a)\)). This is a common physical fact, but we can not obtain it with Noether theorem from the action (3). However, we can make gauge transformation using (4) in standard way. We introduce a new function of time and coordinates \(\lambda\) and, after integration by parts, obtain

\[
\int \int \left( \text{div} \vec{j} + \partial_t \rho \right) \lambda dV dt = - \int \int \left( \vec{j} \nabla \lambda + \rho \partial_t \lambda \right) dV dt \tag{5}
\]

and subtract (5) from (3) to obtain combination \(\int \int \rho \left( \partial_t \lambda - \varphi \right) dV dt\) that allows to vanish \(\varphi = \partial_t \lambda\).

Then let introduce a potential \(\vec{A} = c \nabla \lambda\), that will be a new (dynamical) coordinate of GF (\(c\) is an arbitrary constant, introduced for analogy with electric field). Now, after using (6) in last term (3), we obtain that the action is

\[
S = \int \int \left( \frac{\rho \nu^2}{2} + \vec{j} \vec{A} / c - \left( \frac{\partial_t \vec{A} / c}{8\pi} \right)^2 \right) dV dt \tag{7}
\]

and we have generalized velocity proper to the new coordinate. We now introduce a notation for GF strength

\[
- c \vec{E} = = \frac{\partial \vec{A}}{\partial t}. \tag{8}
\]

Accordingly to standard procedure a generalized momentum is \(P_n (x,t) = \frac{1}{4\pi c} E_n (x,t)\) and we construct the nonrelativistic HF of masses and GF \(\hat{H} = \hat{H}_m + \hat{H}_t + \hat{H}_1 + \hat{H}_2\), where terms with GF are

\[
\hat{H}_t = - \int d^3 x \frac{\vec{E} (x)^2}{8\pi}, \quad \hat{H}_1 = - \frac{1}{c} \int d^3 x \hat{A}_n (x) \hat{j}_m (x), \quad \hat{H}_2 = \frac{1}{8\pi c \rho} \int d^3 x \hat{A}_n^2 (x) \Omega^2 (x), \tag{9}
\]

and \(\Omega = \sqrt{4\pi \rho}\) is Jeans frequency, \(\hat{H}_m\) is HF of free masses (without nonrelativistic GF).

**Dynamical equations for GF moments**

A system with macroscopic number of particles (stars) can be described by distribution function which satisfy Liouville equation \(\partial_t \rho(t) = \{ \hat{H}, \rho(t) \}\). Then dynamical (time) equation for arbitrary physical value can be written as \(\dot{\hat{A}}_n (x,t) = - S \rho \rho (t) \{ \hat{H}, \hat{A}_n (x) \}\). For potential and strength of GF we have

\[
\dot{\hat{A}} = - c \hat{E}, \quad \dot{\hat{E}} = 4\pi \vec{J} \tag{10}
\]

where average current of mass \(\vec{J} (x,t) = S \rho \rho (t) \vec{J} (x)\) has such microscopic structure \(\dot{\vec{J}} (x) \equiv \dot{j}_m (x) - \frac{1}{4\pi c \rho} \hat{A}_n (x) \Omega (x)\). And we can write in such a manner equations for any moments of GF. Let us consider
GF kinetic theory approximation, that means we must construct equations for the second space correlation moments of GF [2, 3]:

\[
\frac{\partial \left( \hat{A}_n \hat{A}'_m \right)}{\partial t} = -c \left( \hat{E}_n \hat{A}'_m \right) - c \left( \hat{A}_n \hat{E}'_m \right), \quad \frac{\partial \left( \hat{E}_n \hat{E}'_m \right)}{\partial t} = -c \left( \hat{E}_n \hat{A}'_m \right) + 4\pi \left( \hat{J}_n \hat{A}'_m \right)
\]

Sources are correlations between subsystems of GF and particles Spp \((t) \hat{J}_n(x) \hat{A}_m(x') = (J_n A'_m)\).

Gravitational interaction is weak and then we can use Bogolyubov boundary condition of complete correlation weakening [5] to find solution of Liouville equation in perturbation theory. We can divide HF into main part and weak interaction. GF main part of HF is

\[
\hat{H}_f = -\int d^3 x \frac{\hat{E}(x)^2}{8\pi} + \frac{1}{8\pi c} \int d^3 x \int d^3 x' \hat{A}(x) \hat{A}(x') \omega^2(x - x'),
\]

where the function \(\omega^2(x - x')\) will be obtained as a solution of dispersion equation of linear part of time equations.

Let suppose for simplicity that subsystem of free particles is thermostat. Then we obtain with the help of reduced description method a distribution function in the first order of nonrelativistic interaction (see [2]):

\[
\rho(t) = \rho_f(t) w - \frac{1}{c} \int_{-\infty}^{0} dt \int d^3 x \{ \hat{A}_n(x, \tau) \hat{J}_n(x, \tau), \rho_f(t) w \}.
\]

Fourier-component of GF potential in interaction picture is \(\hat{A}_{nk}^{\dagger}(\tau) = ch(\omega_k \tau) \hat{A}_{nk}^{\dagger} + \frac{\omega_k}{\omega} \hat{E}_n \hat{A}_{nk}^{\dagger}\), where longitudinal vector is \(\hat{A}_{nk}^{\dagger} = \hat{A}_{mk} \hat{k}_m \hat{k}_n, \hat{k}_n = \frac{\vec{k}}{c}\).

**Kinetic coefficients in the GF equations**

Averaging with distribution function \([13]\) sources in \([10]\) and \([11]\), we obtain current up to the second order, which is linear on GF parameters

\[
J_n(x, t) = \int d^3 x' \sigma_{nl} (x - x') E_l(x', t) + \int d^3 x' \lambda_{nl} (x - x') A_l(x', t),
\]

where kinetic coefficients of second order for GF are

\[
\sigma_{nl,k} = \left( \hat{G}^l (k, -i\omega) - \hat{G}^l (k, i\omega) \right) \vec{k}_n \vec{k}_l / 2\omega, \quad \lambda_{nl,k} = \left( \Omega^2 / 2\pi + \hat{G}^l (k, i\omega) + \hat{G}^l (k, -i\omega) \right) \vec{k}_n \vec{k}_l / 2c.
\]

Here we have used a Fourier-transformed Green function

\[
G_{nl}(x, \omega) = \delta^d (k, \omega) \delta^d (k, \omega) G_{nl} (\vec{k}, \omega) e^{i\vec{k}\vec{x} - \omega t}
\]

for homogeneous and isotropic medium \(G_{nl} (\vec{k}, \omega) = G^l (k, \omega) \delta_{nl} + G^l (k, \omega) \delta_{nl} \vec{k}_l\).

For average GF-particles correlations we obtain

\[
(A_{nl}^x J_{nl}^x)^t = \int d^3 x'' \{ \sigma_{lm} (x' - x'') (A_{nl}^x E_{nm}^{x''})^t + \lambda_{lm} (x' - x'') (A_{nl}^x A_{nm}^{x''})^t \} + S_{nl} (x - x'),
\]

\[
(E_{nl}^x J_{nl}^x)^t = \int d^3 x'' \{ \sigma_{lm} (x' - x'') (E_{nl}^x E_{nm}^{x''})^t + \lambda_{lm} (x' - x'') (E_{nl}^x A_{nm}^{x''})^t \} + T_{nl} (x - x').
\]

where free terms look like Langevin force correlations of phenomenological theories

\[
S_{nl}(k) = -4\pi T c^2 \lambda_{nl}(k) / \omega_k^2, \quad T_{nl}(k) = 4\pi T \sigma_{nl}(k).
\]
Solutions of the GF equations

We shall search a solution as $A(t) = A e^{i\omega t}$. Then from (10) with (14) we obtain

$$\omega_k A_{nk}^l = -c E_{nk}^l, \quad \omega_k E_{nk}^l = -4\pi \epsilon \left( \sigma_k E_{nk}^l + \lambda_k A_{nk}^l \right)$$

(19)

Dispersion equation for (19) is $\omega_k = 4\pi \left( \sigma_k - \lambda_k \omega \right)$ or after using (15) $\omega_k^2 = \Omega^2 + 4\pi G^l (k, i\omega_k)$. It gives well known results for example with Maxwell distribution function of particles. For big wave vector and one type of particles we have $\omega_k = -1 - \frac{k^2}{\sqrt{\pi^2 k^2}}$, where $v_T = T/m$ and $r_D = v_T/\Omega$. For small wave vector we have increment $\omega_k = \pm \sqrt{\Omega^2 - 3k^2 v_T^2}$.

When we have external current with increment $\omega_k$, we must change in (12) $\omega^2(x - x') \rightarrow \omega^2$ and go on to obtain $\omega A_{nk}^l = -c E_{nk}^l, \quad \omega E_{nk}^l = 4\pi \left( \sigma_k E_{nk}^l + \lambda_k A_{nk}^l + J_{nk}^{\text{ext}} \right)$. If we return to standard scalar potential $E = -\nabla \varphi$ then for point mass and zero frequency we obtain gravitation screening $\varphi(r) = -\sqrt{GmT/2}$. It is well known that Jeans theory in equilibrium state requires $\varphi = 0$ (6 p. 273), but Poisson equation forbids it (Jeans swindle). Using (10) in hydrodynamical medium we have equation $\dot{E}_n = 4\pi \rho u_n$ that has zero equilibrium solution.

But in equilibrium state GF is not absent. It has nonzero values of second correlation moments. For example, energy of GF is proportional to $(E_n E_m(x' - x))$ correlation moment and from (11) we obtain $(EE)^k = -4\pi T$, where $T$ is thermostat temperature.

Solution for linear equations (11) with (17) has the form of combination frequencies $A(t) = A e^{(\omega_k + \omega_k') t}$. For homogeneous and isotropic but nonstationary GF, when the first moments are zero, for the second moments we obtain time dependence with $2\omega_k$.

Results and Conclusions

So, we have transformed nonrelativistic action for masses and GF using continuity equation and introduced potential vector as a new generalized coordinate of GF. That allows us to obtain real (not effective!) HF for this system with generalized coordinate and momentum of GF. Using Bogolyubov reduced description method with special GF main HF we have obtained time equations for the first and the second correlation moments of GF, that are linear in perturbation theory of weak gravitation interaction. Equations for mean GF give zero equilibrium solution, unlike the Poisson equation, that removes mathematical troubles of the Jeans theory (Jeans swindle). Expressions for kinetic coefficients in terms of the Green function of the currents are constructed. Solutions for GF equations in the case of Maxwell particles distribution are found.

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