Functoriality of HKR Isomorphisms

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Abstract: For a closed embedding of smooth schemes $X \hookrightarrow S$ with a fixed first order splitting, there are two HKR isomorphisms between the derived scheme $X \times_X \Delta X$ and the total space of the shifted normal bundle $N_{X/S}[-1]$, due to Arinkin-Căldăraru, Arinkin-Căldăraru-Hablicsek, and Grivaux. In this paper, we study functoriality properties of these two HKR isomorphisms for a sequence of closed embeddings $X \hookrightarrow Y \hookrightarrow S$. The second type of HKR isomorphism is shown to be functorial. The first type of HKR isomorphism is functorial when a certain cohomology class, which we call the Bass-Quillen class, vanishes. We obtain Lie theoretic interpretations for the two HKR isomorphisms and for the Bass-Quillen class as well.

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1. Introduction

1.1. Let $X$ be a smooth algebraic variety over a field of characteristic zero. There is a Hochschild-Kostant-Rosenberg isomorphism [S96] in the derived category of $X$

$$\Delta^* \Delta_* \mathcal{O}_X \cong \text{Sym}_{\mathcal{O}_X}(\Omega_X[1]),$$

where $\Delta$ is the diagonal embedding of $X$ in $X \times X$. 

It was observed by Kapranov and Kontsevich that there is a Lie theoretic interpretation of the HKR isomorphism. The derived loop space \( LX = X \times_R X \times X \) has the structure of a derived group scheme over \( X \) and the shifted tangent bundle \( T_X[-1] \) is its Lie algebra \([K99]\). The HKR isomorphism can be thought of as a version of the exponential map \( \mathbb{T}_X[-1] \to LX \) \([CR11]\), where \( \mathbb{T}_X[-1] \) is the total space of the shifted tangent bundle.

### 1.2.
In the case of classical Lie groups we have a commutative diagram

\[
\begin{array}{ccc}
G & \xrightarrow{f} & H \\
\exp & \downarrow & \exp \\
g & \xrightarrow{df} & h.
\end{array}
\]

A map of schemes \( X \to Y \) induces a map of derived group schemes \( LX \to LY|_X \) over \( X \). The analogous statement for derived schemes to the above Lie theoretic statement is that the diagram

\[
\begin{array}{ccc}
LX & \xrightarrow{\text{HKR}} & LY|_X \\
\mathbb{T}_X[-1] & \xrightarrow{\text{HKR}} & \mathbb{T}_Y|_X[-1]
\end{array}
\]

commutes. It will follow from the results in this paper.

### 1.3.
We would like to prove the commutativity of this diagram in a more general setting. One can still get HKR isomorphisms if one replaces the diagonal embedding by an arbitrary closed embedding of schemes \( i : X \hookrightarrow S \) with a fixed first order splitting. Arinkin, Căldăraru, Hablicsek \([AC12, ACH19]\), and Grivaux \([Gri14]\) provided two different ways to construct HKR isomorphisms between the total space of the shifted normal bundle \( N_{X/S}[-1] \) and the derived self-intersection \( X \times_R S \) from a fixed first order splitting of \( i \). In this paper we study the functoriality of these two types of HKR isomorphisms.

### 1.4.
We will be in the following setting from now on. Let \( X \hookrightarrow Y \hookrightarrow S \) be a sequence of closed embeddings of smooth schemes. Assume that \( X \) is split to first order in \( Y \), and similarly for \( X \) in \( Y \) and \( Y \) in \( S \). We want to understand if the two diagrams

\[
\begin{array}{ccc}
X \times_R Y & \xrightarrow{\cong} & X \times_R S X \\
\cong & \downarrow & \cong \\
N_{X/Y}[-1] & \xrightarrow{\cong} & N_{X/S}[-1]
\end{array}
\]

\[
\begin{array}{ccc}
Y \times_R S X & \xrightarrow{\cong} & Y \times_R S Y|_X \\
\cong & \downarrow & \cong \\
N_{Y/S}|_X[-1] & \xrightarrow{\cong} & N_{Y/S}[-1]
\end{array}
\]

are commutative.
1.5. Since there are two notions of HKR isomorphisms and we do not know if they agree, these are separate questions for each type of HKR. The construction of HKR isomorphisms in [AC12] involves a differential graded $\mathcal{O}_{X^\prime Y}$-algebra resolution of $\mathcal{O}_X$. We will call it HKR$_1$. The construction of HKR isomorphisms in [ACH19] and [Gri14] reduces all HKR isomorphisms to HKR$_1$ isomorphisms of diagonal embeddings. We will call this second type HKR$_2$. For more details see section 2.

1.6. The following question is important in this paper. The restriction of the conormal bundle $N_{Y/S}$ to the first order neighborhood $X^\prime Y$ is a vector bundle on $X^\prime Y$. One can ask whether this bundle is isomorphic to $s^*(N_{Y/S}|_X)$, where $s$ is the chosen first order splitting of $X$ in $Y$. If the answer to this question is positive, i.e., there is an isomorphism of vector bundles on $X^\prime Y$

$$N_{Y/S}|_{X^\prime Y} \cong s^*(N_{Y/S}|_X), \ (*)$$

we will say condition $(\ast)$ is satisfied.

1.7. The condition $(\ast)$ is equivalent to the vanishing of a certain cohomology class in $\text{Ext}^1(N_{X/Y} \otimes N_{Y/S}|_X, N_{Y/S}|_X)$ associated to $N_{Y/S}|_{X^\prime Y}$. One can construct a similar cohomology class $\alpha_{s, G} \in \text{Ext}^1(N_{X/Y} \otimes G^\vee|_X, G^\vee|_X)$ for any vector bundle $G$ on $X^\prime Y$ and the fixed splitting $s$. This class vanishes if and only if $G \cong s^*(G|_X)$. We will call this cohomology class the Bass-Quillen class since it is related to the Bass-Quillen conjecture as we explain below.

Suppose $X$ is a smooth algebraic variety, and $Y$ is the total space of a vector bundle $\mathcal{E}$ on $X$. Let $\mathcal{F}$ be a vector bundle on $Y$. Then one can ask if $\mathcal{F}$ is isomorphic to the pull back of some vector bundle on $X$. When $X$ is affine this is known as the Bass-Quillen problem and was answered affirmatively in [L81]. However, in the global case the answer to this question is negative. In particular, there can be no vector bundle on $X$ whose pull back is $\mathcal{F}$ if the Bass-Quillen class in $\text{Ext}^1(N_{X/Y} \otimes \mathcal{F}^\vee|_X, \mathcal{F}^\vee|_X)$ associated to $\mathcal{F}|_{X^\prime Y}$ is not zero.

Here are the main results in this paper.

1.8. Theorem A. Let $X \hookrightarrow Y \hookrightarrow S$ be a sequence of closed embeddings of smooth schemes. Further assume that there are compatible splittings on the tangent bundles

$$T_X \xleftarrow{p} T_Y|_X \xrightarrow{q|_X} T_S|_X.$$
Compatibility means that \( p \circ q|_X = \rho \).

(a) For HKR\(_2\) both squares below are commutative

\[
\begin{array}{ccc}
X \times_S^R X & \rightarrow & Y \times_S^R Y \\
\cong & & \cong \\
\mathbb{N}_{X/Y}[{-1}] & \rightarrow & \mathbb{N}_{Y/S}[{-1}]
\end{array}
\]

The horizontal maps between normal bundles are the linear ones, i.e., they are vector bundle maps.

(b) For HKR\(_1\) the left square is commutative. If condition \((*)\) is satisfied, i.e., the Bass-Quillen class of \( N_{Y/S|X|Y}^\vee \) is zero, then the right square is also commutative.

1.9. Application to orbifolds. In [CH] we will use this result to construct an associative product on the polyvector fields on a global quotient orbifold \([S/G]\) with \(G\) finite abelian. We will apply Theorem A to the setting where \(S\) admits an action of a finite group \(G\) and \(X\) is the fixed locus of \(G\) and \(Y\) is the fixed locus of a subgroup \(H \leq G\). It is easy to see that the splittings obtained from the averaging maps are compatible in this case.

1.10. There is a Lie theoretic interpretation for the second part of Theorem A. Under certain extra assumptions, all derived self-intersections in Theorem A are groups in the derived category of dg schemes. The shifted normal bundles are their Lie algebras. One can check that the natural maps

\[
X \times_Y^R X \rightarrow X \times_S^R X \rightarrow X \times_S^R Y = Y \times_S^R Y|_X
\]

are maps of groups. The map \(N_{X/Y}[-1] \rightarrow N_{Y/S|X|Y}[-1]\) respects the Lie structures in general. However, \(N_{X/S}[-1] \rightarrow N_{Y/S|X|Y}[-1]\) may not preserve the Lie brackets. The Bass-Quillen class \(\alpha_{s,N_{Y/S|X|Y}^\vee}[1]\) is precisely the obstruction to this map preserving the Lie brackets.

With no assumptions other than the choice of a first order splitting, the bundle \(N_{X/S}[-1]\) carries an anti-symmetric bracket which may not satisfy the Jacobi identity. We call this structure a pre-Lie bracket. We will provide more details in section 2.

Theorem B. In the same setting as Theorem A, the vector bundle map \(N_{X/S}[-1] \rightarrow N_{Y/S|X|Y}[-1]\) preserves the pre-Lie brackets if and only if the Bass-Quillen class \(\alpha_{s,N_{Y/S|X|Y}^\vee}[1]: N_{X/Y} \otimes N_{Y/S|X|Y} \rightarrow N_{Y/S|X|Y}[1] \) is zero.

Thus Theorem A and Theorem B provide a generalization of the original result for Lie groups to the setting of groups obtained as self-intersections.
1.11. Plan of the paper. Section 2 is reviewing background material. We recall what is known about the structure of arbitrary closed embeddings with further assumptions on (possibly higher order) splittings, and how the HKR\textsubscript{1} and HKR\textsubscript{2} isomorphisms are constructed for a closed embedding with a fixed first order splitting. Then we define the Bass-Quillen class. We also provide details about the Lie theoretic interpretation of the Bass-Quillen class and our Theorem B.

Section 3 is devoted to the proof of theorem A. To prove commutativity for HKR\textsubscript{1}, it suffices to check whether the explicit resolutions of $\mathcal{O}_X$ and $\mathcal{O}_Y$ in [AC12] are compatible. The construction of the HKR\textsubscript{2} isomorphisms in [ACH19] and [Gri14] reduces all HKR\textsubscript{2} isomorphisms to HKR\textsubscript{1} isomorphisms for diagonal embeddings, so it is easier to prove functoriality in this case.

Section 4 is about Theorem B for Lie algebras in the category of vector spaces. We consider an inclusion of Lie algebras $\mathfrak{h} \hookrightarrow \mathfrak{g}$ and the associated short exact sequence of $\mathfrak{h}$-modules

$$
0 \longrightarrow \mathfrak{h} \xrightarrow{\alpha} \mathfrak{g} \xrightarrow{\beta} n = \mathfrak{g}/\mathfrak{h} \longrightarrow 0.
$$

There is a way to construct a pre-Lie bracket on $n$ which may not be compatible with the Lie bracket on $\mathfrak{g}$. Then we prove a result similar to Theorem B using methods that make sense in the derived category. This approach will be generalized to derived category of schemes in section 6.

Section 5 is the proof of our Lie theoretic interpretation of the Bass-Quillen class $\alpha_{s,N_{Y/S}|X,Y}$. This class can be viewed as a Lie module structure map, where $N_{X/Y}[-1]$ is the Lie algebra and $N_{Y/S}|X[-1]$ is the module.

Section 6 generalizes the proofs in section 4 to the derived setting. We end this paper with an example where the Bass-Quillen class is not zero.

1.12. Conventions. All the schemes we considered in this paper are smooth over a field of characteristic zero.

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2. Background and Lie theoretic interpretations

We first discuss what is known about the HKR isomorphism for the diagonal embedding. Then we recall the definitions of the two types of HKR isomorphisms for a general closed embedding $X \hookrightarrow S$ with a fixed first order splitting. We briefly recall the Lie theoretic interpretations of general embeddings with possibly higher order splittings. We define the Bass-Quillen class and explain the Lie theoretic interpretations for the Bass-Quillen class $\alpha_{S, N_{S/S}^{(1)}|_S}$ and Theorem B at last.

2.1. The diagonal embedding. Let $X$ be a smooth algebraic variety. There is an HKR isomorphism

$$HH^*(X) \cong \bigoplus_{p+q=\ast} H^p(X, \wedge^q T_X)$$

that identifies Hochschild cohomology of $X$ with polyvector fields as vector spaces. More precisely, we have an HKR isomorphism at the level of sheaves in the derived category of $X$

$$\Delta^* \Delta_* \mathcal{O}_X \cong \text{Sym}_{\mathcal{O}_X}(\Omega_X[1]),$$

where $\Delta : X \hookrightarrow X \times X$ is the diagonal embedding. We get the desired isomorphism on cohomology by applying $R\text{Hom}(-, \mathcal{O}_X)$ to the isomorphism of sheaves.

2.2. In the world of derived schemes we consider the free loop space $LX$ of $X$, defined as the derived self-intersection $X \times^R_{X \times X} X$. Its structure complex is $\Delta^* \Delta_* \mathcal{O}_X$, and the structure complex of the total space of the shifted tangent bundle $T_X[-1] = \text{Spec}_{\mathcal{O}_X}(\text{Sym}(\Omega_X[1]))$ is $\text{Sym}(\Omega_X[1])$. We can restate the HKR isomorphism as an isomorphism of derived schemes over $X$

$$T_X[-1] \xrightarrow{\cong} LX = X \times^R_{X \times X} X.$$

It can be viewed as the exponential map from the Lie algebra $T_X[-1]$ to the group $LX$ as explained in the introduction.

2.3. General embeddings. There exist generalized HKR isomorphisms if we replace the diagonal embedding by an arbitrary closed embedding $i : X \hookrightarrow S$ of smooth schemes.

The embedding $i$ factors as

$$X \xrightarrow{\mu} X_S^{(1)} \xrightarrow{\nu} S,$$
where \( X_S^{(1)} \) is the first order neighborhood of \( X \) in \( S \). We say \( i \) splits to first order if and only if the map \( \mu \) is split, i.e., there exists a map of schemes \( \varphi : X_S^{(1)} \to X \) such that \( \varphi \circ \mu = id \). There is a bijection between first order splittings of \( i \) and splittings of the short exact sequence below \([G64, 20.5.12 (iv)]\)

\[
0 \longrightarrow T_X \longrightarrow T_{S|X} \longrightarrow N_{X/S} \longrightarrow 0.
\]

2.4. Arinkin and Căldăruț [AC12] provided a necessary and sufficient condition for \( i^*i_*O_X \) to be isomorphic to \( \text{Sym}(N_{X/S}^\vee[1]) \). In [ACH19] Arinkin, Căldăruț, and Hablicsek proved that the derived intersection \( X \times_X X \) is isomorphic to \( N_{X/S}[-1] \) over \( X \times X \) if and only if the embedding \( i \) splits to first order. Grivaux independently proved a similar result for complex manifolds in [Gri14].

2.5. Let us briefly recall how the HKR_1 isomorphism \( i^*i_*O_X \cong \text{Sym}(N_{X/S}^\vee[1]) \) was constructed in [AC12]. It is defined as the composite map

\[
\mu^*\nu^*\nu_*\mu_*O_X \longrightarrow \mu^*\mu_*O_X \longrightarrow T^c(N_{X/S}^\vee[1]) \longrightarrow \text{Sym}(N_{X/S}^\vee[1]).
\]

The left most map is given by the counit of the adjunction \( \nu^* \dashv \nu_* \). The map \( \text{exp} \) is multiplying by \( 1/k! \) on the degree \( k \) piece, and the last one is the natural projection map. The \( T^c(N_{X/S}^\vee[1]) \) is the free coalgebra on \( N_{X/S}^\vee \) with the shuffle product structure, and \( T(N_{X/S}^\vee[1]) \) is the tensor algebra on \( N_{X/S}^\vee \). The isomorphism \( \mu^*\mu_*O_X \cong T^c(N_{X/S}^\vee[1]) \) in the middle is non-trivial and needs more explanation. With the splitting \( \varphi \) one can build an explicit resolution of \( \mu_*O_X \) as an \( O_X^{(1)} \)-algebra

\[
(T^c(\varphi^*N_{X/S}^\vee[1]), d) \longrightarrow \mu_*O_X,
\]

where \( (T^c(\varphi^*N_{X/S}^\vee[1]), d) \) is the free coalgebra on \( \varphi^*N_{X/S}^\vee \) with the shuffle product structure and a differential \( d \). The differential is defined as follows. There is a short exact sequence on \( X_S^{(1)} \)

\[
0 \longrightarrow \mu_*N_{X/S} \longrightarrow O_X^{(1)} \rightarrow \mu_*O_X \rightarrow 0.
\]

Consider the composite map

\[
\varphi^*N_{X/S} \rightarrow \mu_*\mu^*\varphi^*N_{X/S} = \mu_*N_{X/S} \rightarrow O_X^{(1)},
\]
whose cokernel is $\mu_+ O_X$. Tensor the morphism above with $(\varphi^* N^\vee_{X/S})^{\otimes (k-1)}$. We get the degree $k$-th piece of the differential $d_k: (\varphi^* N^\vee_{X/S})^{\otimes k} \to (\varphi^* N^\vee_{X/S})^{\otimes (k-1)}$. The differential vanishes once we pull this resolution back on $X$ via $\mu$, so we get the desired isomorphism.

2.6. For any vector bundle $E$ on $X$, we tensor the resolution above by $\varphi^* E$. Using the projection formula and $\varphi \circ \mu = id$, one can show that we get a resolution of $\mu_+ E$

$$(T^e (\varphi^* N^\vee_{X/S}[1]) \otimes \varphi^* E, d) \to \mu_+ E.$$  

The same argument shows that $i^* i_\ast (E) \cong E \otimes \text{Sym}(N^\vee_{X/S}[1])$, i.e., that $i^* i_\ast (-) \cong (-) \otimes \text{Sym}(N^\vee_{X/S}[1])$ as dg functors. This shows that $X \times_S^R X \cong N_{X/S}[-1]$ over $X \times X$.

2.7. Let us recall how $\text{HKR}_2: N_{X/S}[-1] \cong X \times_S^R X$ was constructed in [ACH19] and [Gri14]. It is defined as the composite map

$$
\begin{align*}
N_{X/S}[-1] &\longrightarrow \mathbb{T}_S X[-1] \cong S \times_S^R S|_X \\
S \times_S^R X &\longrightarrow S \times_S^R (X \times X) \cong X \times_S^R X.
\end{align*}
$$

The dotted arrow is the splitting we fixed. The isomorphism in the middle $\mathbb{T}_S X[-1] \cong S \times_S^R S$ is the $\text{HKR}_1$ isomorphism discussed previously for the diagonal embedding $S \hookrightarrow S \times S$. There are two splittings to define $\text{HKR}_1$ for the diagonal embeddings. We always choose $p_1$, i.e., the projection onto the left factor

$$\Delta_S: S \xrightarrow{p_1} S \times S.$$

2.8. Lie theoretic interpretations for general self-intersections. Consider a closed embedding $i: X \hookrightarrow S$ of smooth schemes. The derived self-intersection $X \times_S^R X$ has an $\infty$-groupoid structure in the $(\infty, 1)$-category of dg schemes over $X$. The associated $L_\infty$ algebroid is $N_{X/S}[-1]$. Passing to the derived category, we get a groupoid in the derived category of dg schemes having $X$ as the space of objects. The target and source maps are the two projections $\pi_1$ and $\pi_2: X \times_S^R X \to X$. See [CCT14] for more details.

When $S = X \times X$ and $i$ is the diagonal embedding $\Delta: X \to X \times X$, there are two projections $p_1$ and $p_2: X \times X \to X$ such that $p_i \circ \Delta = id$. This implies that the source map $\pi_1$ and the target map $\pi_2$ are equal in the derived category in this case, so $X \times_S^R X$ becomes a group over $X$ [ACH19]. A similar argument works if the inclusion from $X$ to its formal neighborhood in $S$ splits.
Generally speaking, $N_{X/S}[-1]$ has an $L_\infty$ algebroid structure in the $(\infty, 1)$-category of dg quasi-coherent sheaves on $X$. However, the Lie bracket may not be $O_X$-linear, and it may not satisfy the Jacobi identity when we pass to the derived category of $X$. Calaque, Căldăraru, and Tu proved that the induced bracket in the derived category is $O_X$-linear if $i$ splits to first order, and it satisfies the Jacobi identity if $i$ splits to second order [CCT14]. As a consequence $N_{X/S}[-1]$ admits a natural Lie algebra structure in the derived category if $i$ splits to second order. Later, Calaque and Grivaux showed that $N_{X/S}[-1]$ has a natural Lie algebra structure if $X \hookrightarrow S$ is a tame quantized cycle [CG17], a weaker condition than splitting to second order. More precisely, for an embedding $i : X \hookrightarrow S$ with a chosen first order splitting, they described the $O_X$-linear bracket $N_{X/S} \otimes N_{X/S} \to \text{Sym}^2 N_{X/S} \to N_{X/S}[1]$ explicitly as the extension class of the short exact sequence of vector bundles on $X$

$$0 \to \text{Sym}^2 N_{X/S} \cong \frac{I_X}{I_X^2} \to \varphi_* \frac{I_X}{I_X^3} \to \frac{I_X}{I_X^5} \cong N_{X/S} \to 0,$$

where $I_X$ is the ideal sheaf of $X$ in $S$. This bracket satisfies the Jacobi identity under the tameness assumption. In the rest of the paper we will only use embeddings which are split to first order without requiring the Jacobi identity to hold for this specific bracket. We will call this type of bracket to be a pre-Lie bracket.

2.9. We finished background materials. The rest of this section is about the definition of the Bass-Quillen class and the Lie theoretic interpretations of our results. From now on we fix smooth subvarieties $X$ and $Y$ of $S$ with closed embeddings $i$, $j$, and $f$

$$X \xhookleftarrow{f} Y \xhookrightarrow{j} S.$$

We assume that $f$, $j$, and $i$ are split to first order, and that we have fixed first order splittings of $f$, $j$, and $i$ which will be denoted as $s$, $\pi$, and $\varphi$.

2.10. It is crucial to note that all the constructions of HKR isomorphisms depend on the choice of splitting. Therefore, we need to assume some compatibility on these splittings, namely, that $p \circ q|_X = \rho$

$$T_X \xleftarrow{p} T_Y|_X \xrightarrow{q|_X} T_S|_X,$$

where $p$, $q$, and $\rho$ are splittings on the tangent bundles corresponding to the first order splittings $s$, $\pi$, and $\varphi$. 
2.11. Definition of the Bass-Quillen class. There is a class \( \alpha_{s,N_Y/S} \in \text{Ext}^1(N_{X/Y} \otimes N_{Y/S}|x, N_{Y/S}|x) \) which plays an important role in what follows. It has interpretations both in terms of Lie theory and as the obstruction for a general Bass-Quillen theorem to hold.

Consider the short exact sequence of \( O_X^{(1)} \)-modules

\[
0 \to t_s N_{X/Y}^\vee \to O_X^{(1)} \to t_s O_X \to 0,
\]

where \( t \) is the inclusion \( X \hookrightarrow X_Y^{(1)}. \)

For any vector bundle \( M \) on \( X_{Y}^{(1)}, \) tensor it with this short exact sequence. Then push-forward the sequence onto \( X \) via \( s. \) Using the fact that \( s \circ t = \text{id} \) and the projection formula, we get a short exact sequence of vector bundles on \( X \)

\[
0 \to N_{X/Y}^\vee \otimes M|_X \to s_* M \to M|_X \to 0.
\]

Dualizing, we get an extension class \( \alpha_{s,M} : N_{X/Y} \otimes M^\vee|_X \to M^\vee|_X[1]. \) We call \( \alpha_{s,M} \) the Bass-Quillen class for the pair \( (s, M) \) for the reason explained in (1.7). The class \( \alpha_{s,M} \) vanishes if and only if \( M \) is isomorphic to \( s^* M|_X [CG17]. \)

Now, choose \( M \) to be \( N_{Y/S}\), which gives the class \( \alpha_{s,N_{Y/S}} \). It admits an interpretation as a Lie module structure map. We explain this interpretation in the rest of this section.

2.12. Lie theoretic interpretations of the Bass-Quillen class and Theorem B. To make our Lie theoretic interpretation clearer, let us assume that all the three derived self-intersections \( X \times_Y^R X, X \times_S^R X, \) and \( Y \times_S^R Y \) are groups in this section. However, we will state and explain theorems and propositions later in sections 3 to 6 which only assume existence of first order splittings. One can check that the natural maps

\[
X \times_Y^R X \to X \times_S^R X \to X \times_S^R Y = Y \times_S^R Y|_X
\]

are maps of groups. All the shifted normal bundles are Lie algebras under this assumption. In this section we denote \( N_{X/Y}[-1], N_{X/S}[-1], \) and \( N_{Y/S}|x[-1] \) by \( \mathfrak{h}, \mathfrak{g}, \) and \( \mathfrak{n} \) respectively. The functoriality of HKR isomorphisms can be viewed as the functoriality of the exponential maps from Lie algebras to Lie groups.

2.13. The map \( \mathfrak{h} = N_{X/Y}[-1] \hookrightarrow \mathfrak{g} = N_{X/S}[-1] \) preserves the Lie brackets, so we are able to prove the commutativity of the left square in Theorem A for HKR1 with no difficulty. Moreover, the compatibility of the Lie brackets implies that \( \mathfrak{g} \) is an
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$h$-module, and $h \hookrightarrow g$ is a map of $h$-modules. Therefore, $n = g/h = N_{Y/S}[x][-1]$ has a natural $h$-module structure. We get a short exact sequence of $h$-modules

$$0 \rightarrow h = N_{X/Y}[-1] \rightarrow g = N_{X/S}[-1] \rightarrow n = g/h = N_{Y/S}[x][-1] \rightarrow 0.$$  

The $h$-module structure on $g/h: N_{X/Y} \otimes N_{Y/S}|X \rightarrow N_{Y/S}|X[1]$ is exactly the Bass-Quillen class $\alpha_{sN_{Y/S}|X[1]}$. We will prove this statement in section 5.

2.14. On the other hand, the map $g = N_{X/S}[-1] \rightarrow n = N_{Y/S}|X[-1]$ may not in general preserve the Lie brackets even if we assume that all the derived self-intersections are groups. This explains the difficulty for proving the functoriality of the exponential maps in the right square of Theorem A. In section 6 we will show that $h = N_{X/Y}[-1]$ acts on its module $g/h = N_{Y/S}|x[-1]$ trivially if and only if the Lie brackets are preserved, i.e., $g \rightarrow g/h = n$ is a map of Lie algebras. This is Theorem B. As a consequence the right square in Theorem A commutes when the Lie brackets are preserved.

2.15. Let us restate (2.13) and Theorem B for Lie algebras in the category of vector spaces. By abuse of notations, we use the same notations $h$ and $g$ for Lie algebras in the category of vector spaces. Consider an injective morphism of Lie algebras $h \hookrightarrow g$. The quotient $n = g/h$ is naturally an $h$-module, so we get a short exact sequence of $h$-modules

$$0 \rightarrow h \xrightarrow{\alpha} g \xrightarrow{\beta} n = g/h \rightarrow 0.$$  

There is a way to construct a pre-Lie bracket on $n$ once a splitting $n \rightarrow g$ is chosen. It becomes a Lie bracket under the tameness assumption [CG17], but we do not need this pre-Lie bracket to be a Lie bracket in our paper. The morphism $\beta$ preserves the pre-Lie brackets if and only if $h$ acts trivially on $n$. Triviality of the $h$-module structure on $n$ implies that $n$ is actually a Lie algebra, the map $\beta$ is a Lie algebra morphism, and the diagram

$$\begin{array}{ccc} G & \xrightarrow{\Phi} & N \\ \downarrow{\exp} & & \downarrow{\exp} \\ g & \xrightarrow{\beta=d\Phi} & n \end{array}$$  

is commutative.

3. The proof of Theorem A

In this section we prove Theorem A.
3.1. Suppose $i : X \hookrightarrow S$ is a closed embedding of smooth schemes with a first order splitting

$$X \xrightarrow{\varphi} X^{(1)}_S \xrightarrow{\nu} S.$$ 

In section 2 we recalled the construction of the HKR$_1$ isomorphism $i^* i_* \mathcal{O}_X \cong \text{Sym}(N_{X/S}^\vee [1])$ from [AC12]. It is defined as the composite map

$$\mu^* \nu^* \nu_* \mu_* \mathcal{O}_X \xrightarrow{= T^s(N_{X/S}^\vee [1]) \xrightarrow{\text{exp}} T(N_{X/S}^\vee [1]) \xrightarrow{= \text{Sym}(N_{X/S}^\vee [1])}}.$$

It is easy to see that all the constructions are canonical except for the isomorphism $\mu^* \mu_* \mathcal{O}_X \cong T^s(N_{X/S}^\vee [1])$ which depends on the choice of the splitting $\varphi$.

3.2. We have a commutative diagram

under the assumptions in Theorem A. The solid arrows are the obvious ones. The dotted arrows $\pi$, $s$, and $\varphi$ are the first order splittings of the closed embeddings $j : Y \to S$, $f : X \to Y$, and $i : X \to S$ respectively. Notice that $X^{(1)}_Y$ is the fiber product of $X^{(1)}_S$ and $Y$ over $Y^{(1)}_S$, so we can pull $\pi$ back along the morphism $f^{(1)}$ to define $\sigma$. The compatibility condition on the splittings means that $s \circ \sigma = \varphi$.

We remind the reader of the two claims of Theorem A which will be proved below.

(a) For HKR$_2$, both squares below are commutative

$X \times^R_Y X \xrightarrow{=} X \times^R_S X \xrightarrow{=} Y \times^R_S X \xrightarrow{=} Y \times^R_S Y|_X$

$\cong \frac{N_{X/Y}[-1]}{N_{X/S}[-1]} \xrightarrow{=} \frac{N_{X/S}[-1]}{N_{Y/S}|X[-1]}$, where the horizontal maps between the normal bundles are linear, i.e., they are vector bundle maps.
(b) For HKR$_1$, the left square is commutative. If the Bass-Quillen class $\alpha_{s,N^/_Y/S}[1]$ is zero, then the right square is also commutative.

**Proof of the second part of theorem A.** To check the commutativity of the left square in Theorem A for HKR$_1$, it suffices to show that the diagram

$$
\begin{array}{ccc}
\mu^*\mu_*\mathcal{O}_X & \rightarrow & T^c(N^/_X/S[1]) \\
\downarrow & & \downarrow \\
t^*t_*\mathcal{O}_X & \rightarrow & T^c(N^/_X/Y[1])
\end{array}
$$

is commutative since all the other constructions are canonical. The right vertical map is obtained from the natural vector bundle map $N^/_X/S \rightarrow N^/_X/Y$. The horizontal isomorphisms are constructed using the splittings, from explicit resolutions of $\mu^*\mathcal{O}_X$ and $t^*\mathcal{O}_X$ on $X^{(1)}_S$ and $X^{(1)}_Y$ respectively. These resolutions are of the form $(T^c(\varphi^*N^/_X/S[1]), d)$ and $(T^c(s^*N^/_X/Y[1]), d)$ as explained in (2.5).

We have $g^*\varphi^*N^/_X/S = s^*N^/_X/S$ using the fact that $\varphi = s \circ \sigma$ and $\sigma \circ g = id$. There is a natural map of vector bundles $g^*\varphi^*N^/_X/S = s^*N^/_X/S \rightarrow s^*N^/_X/Y$ which induces a map of complexes $g^*(T^c(\varphi^*N^/_X/S[1]), d) \rightarrow (T^c(s^*N^/_X/Y[1]), d)$. One can check carefully the induced map is indeed a map of complexes, i.e., the differentials are preserved. This proves that the diagram

$$
\begin{array}{ccc}
g^*(T^c(\varphi^*N^/_X/S[1]), d) & \rightarrow & g^*\mu^*\mathcal{O}_X \\
\downarrow & & \downarrow \\
(T^c(s^*N^/_X/Y[1]), d) & \rightarrow & t^*\mathcal{O}_X
\end{array}
$$

which relates the two explicit resolutions of $\mathcal{O}_X$ as an $\mathcal{O}_X^{(1)}$-algebra and as an $\mathcal{O}_X^{(1)}$-algebra is commutative. If we pull the natural map $g^*\varphi^*N^/_X/S = s^*N^/_X/S \rightarrow s^*N^/_X/Y$ back to $X$, we get the natural vector bundle map $N^/_X/S \rightarrow N^/_X/Y$. This proves that we get our desired commutative diagram at the beginning of the proof of the second part of Theorem A once we pull the commutative diagram above back to $X$.

**3.3.** Similarly, to prove the commutativity of the right square of Theorem A for HKR$_1$, it suffices to show that the diagram

$$
\begin{array}{ccc}
\mu^*\mu_*\mathcal{O}_X & \rightarrow & T^c(N^/_X/S[1]) \\
\downarrow & & \downarrow \\
f^*b^*b_*\mathcal{O}_Y & \rightarrow & f^*T^c(N^/_Y/S[1])
\end{array}
$$
is commutative. The right vertical map is induced by the natural map of vector bundles $N_{Y/S}\mid X \to N_{X/S}$. If the Bass-Quillen class $\alpha_{s,N_Y^\vee|X}$ vanishes, then we have an isomorphism between $a^* N_Y^\vee$ and $s^*(N_Y^\vee|X)$. The latter maps to $s^* N_X^\vee$ naturally. Therefore, we get a map $\sigma^* a^* N_Y^\vee \cong \sigma^* s^* N_Y^\vee|X \to \sigma^* s^* N_X^\vee = \varphi^* N_X^\vee/S$. Notice that $a \circ \sigma = \pi \circ f^{(1)}$ by the definition of $\sigma$, so we get a map $f^{(1)*} \pi^* N_Y^\vee = \sigma^* a^* N_Y^\vee \to \sigma^* s^* N_X^\vee = \varphi^* N_X^\vee/S$. This map induces a map of complexes $f^{(1)*}(T^* (\pi^* N_Y^\vee[1]), d) \to (T^* (\varphi^* N_X^\vee[1]), d)$.

As a consequence the diagram of resolutions of $\mu_* \mathcal{O}_X$ and $b_* \mathcal{O}_Y$ is commutative. We recover the map of vector bundles $N_Y^\vee|X \to N_X^\vee$ if we pull the natural map $\sigma^* a^* N_Y^\vee = f^{(1)*} \pi^* N_Y^\vee \to \varphi^* N_X^\vee/S = \sigma^* s^* N_X^\vee$ back to $X$. This proves that we get our desired commutative diagram at the beginning of (3.3) once we pull the commutative diagram above back to $X$.

\[\Box\]

### 3.4. Proof of the first part of theorem A.

The construction of HKR$_2 : \mathbb{N}_{X/S} [-1] \cong X \times_S X$ in [ACH19] was explained in section 2. It is defined as the composite map

\[\mathbb{N}_{X/S} [-1] \xrightarrow{\rho_{[-1]}} T_S|X [-1] \xrightarrow{\cong} S \times_S S|X \xrightarrow{\cong} S \times_S S X \xrightarrow{\cong} S \times_S S (X \times X) \xrightarrow{\cong} X \times_S X.\]

### 3.5. The left square.

Focus on the commutativity of the left square for HKR$_2$ in Theorem A

\[X \times_Y X \xrightarrow{\cong} X \times_S X \]

Expand the two HKR$_2$ isomorphisms in detail

\[\mathbb{N}_{X/Y} [-1] \xrightarrow{\cong} Y \times_{Y \times Y} X \xrightarrow{\cong} Y \times_{Y \times Y} (X \times X) \cong X \times_S X\]

\[\mathbb{N}_{X/S} [-1] \xrightarrow{\cong} S \times_S S X \xrightarrow{\cong} S \times_S S (X \times X) \cong X \times_S X.\]
It is clear that the splittings
\[ 0 \rightarrow T_X \rightarrow T_{Y|X} \rightarrow N_{X/Y} \rightarrow 0 \]
and
\[ 0 \rightarrow T_X \rightarrow T_{S|X} \rightarrow N_{X/S} \rightarrow 0 \]
are compatible. Therefore we reduce the question to the functoriality of the diagonal HKR\(_1\) isomorphisms
\[ T_Y[-1] \xrightarrow{\sim} Y \times_{Y \times Y} Y \]
\[ T_S[-1] \xrightarrow{\sim} S \times_{S \times S} S|_Y. \]
The proof of the functoriality of the diagonal HKR\(_1\) isomorphisms is similar to the ones in (3.2) and (3.3). With the commutative diagram
\[ \begin{array}{ccc}
Y & \xrightarrow{\sim} & Y \times Y \\
\downarrow & & \downarrow \\
S & \xrightarrow{\sim} & S \times S,
\end{array} \]
one can verify that the two resolutions used to define HKR\(_1\) for \( Y \hookrightarrow Y \times Y \) and \( S \hookrightarrow S \times S \) are compatible. As a consequence the diagonal HKR\(_1\) is functorial.

3.6. The right square. We focus on the right square for HKR\(_2\) in Theorem A. We have a map \( \Psi : N_{Y/S}|_X \rightarrow T_{S|X} \rightarrow N_{X/S} \) because \( Y \) splits to first order in \( S \). As a consequence the following short exact sequence splits
\[ 0 \rightarrow N_{X/Y} \rightarrow N_{X/S} \rightarrow N_{Y/S}|_X \rightarrow 0. \]
The HKR\(_2\) isomorphism \( \Phi_Y : N_{Y/S}|[-1] \cong Y \times_S Y \) is by definition the composite map
\[ N_{Y/S}|[-1] \xrightarrow{\Phi_Y} S \times_{S \times S} (Y \times Y) \cong Y \times_S Y. \]
We restrict the isomorphism \( \Phi_Y \) above to \( X \), so we get the HKR\(_2\) isomorphism \( \Phi_Y|_X : N_{Y/S}|_X[-1] \cong Y \times_S Y|_X = Y \times_S X. \)
3.7. Proposition. The HKR$_2$ map above $\Phi_Y|_X : N_{Y/S}|_X[-1] \cong Y \times^R_S X$ is the same as the composite map $\Theta \circ \Psi[-1]$, where $\Theta$ is the map

$$\Theta : N_{X/S}|[-1] \xrightarrow{\Phi_X} X \times^R_S X \xrightarrow{id_X \times f} X \times^R_S Y,$$

and $\Psi$ is the splitting in (3.6).

Proof. Expand the HKR$_2$ isomorphism $\Phi_X : N_{X/S} \cong X \times^R_S X$ in the definition of $\Theta$

$$\Theta \circ \Psi[-1] : N_{Y/S}|[-1] \xrightarrow{\Psi[-1]} N_{X/S}|[-1] \xrightarrow{\rho[-1]} T_S|X[-1] \cong S \times^R_S S|X \xrightarrow{id_X \times f} X \times^R_S Y.$$ 

These two constructions give the same HKR$_2$ isomorphism due to the compatibility on the splittings, i.e., $q|_X = \rho \circ \Psi$. 

3.8. We get a diagram

$$\begin{array}{ccc}
N_{X/Y}|[-1] & \xrightarrow{\Psi[-1]} & N_{X/S}|[-1] \\
\downarrow & & \downarrow \\
X \times^R_S Y. & & X \times^R_S Y. \\
\Theta & \xrightarrow{\Phi_Y|_X = \Theta \circ \Psi[-1]} & \Theta \circ \Psi[-1]
\end{array}$$

The map $\Phi_Y|_X$ is defined as the composition $\Theta \circ \Psi[-1]$. Since $\Psi[-1] \circ \Lambda$ need not be the identity, there is no a prior reason for the triangle of solid arrows above to commute. However, the question of the commutativity of the right square in Theorem A for HKR$_2$ is precisely the question of whether $\Phi_Y|_X \circ \Lambda = \Theta$. This will turn out to be equivalent to a statement similar to $\Theta \circ \Psi = 0$, by an argument that generalizes a similar result in abelian categories. Note that one can think of the top row above as a short exact sequence.

3.9. We need the following explanation before we prove the commutativity of the triangle above. Suppose $\tilde{X}$ and $\tilde{Y}$ are derived schemes that are affine over a classical scheme $T$. The structure sheaf $O^*_X$ of $\tilde{X}$ is a differential graded commutative $O_T$-algebra. The smart truncation $\tau^{\leq -1}(O^*_X)$ is the maximal homogenous ideal of this algebra. The 0-th cohomology $\mathcal{H}^0(O^*_X)$ of $O^*_X$ is an $O_T$-algebra concentrated in degree zero. We call $X^0 = \text{Spec}_{O_T} \mathcal{H}^0(O^*_X)$ the underlying classical scheme of $\tilde{X}$. A map of derived schemes $\tilde{X} \to \tilde{Y}$ over $T$ induces a map $O^*_Y \to O^*_X$ of dg $O_T$-algebras.
We will say that this map of schemes factors through the underlying classical scheme if there exists a map \( \tilde{X} \to Y^0 \) such that the diagram

\[
\begin{array}{ccc}
\tilde{X} & \longrightarrow & \tilde{Y} \\
\downarrow & & \downarrow \\
Y^0 & \longrightarrow & Y^0 \\
\end{array}
\]

commutes. It is equivalent to saying that the induced map of dg algebras is zero on the maximal homogeneous ideal of \( \mathcal{O}_{\tilde{Y}}^* \), i.e., the map \( \tau ^{\leq -1} (\mathcal{O}_{\tilde{Y}}^*) \subset \mathcal{O}_{\tilde{Y}}^* \to \mathcal{O}_{\tilde{X}}^* \) is zero.

For example, if \( \omega : E \to F \) is a map of vector bundles over \( T \), the induced map \( E[~-1] \to F[~-1] \) factors through the underlying classical scheme if and only if \( \omega \) is zero.

To prove the commutativity of the triangle of solid arrows in (3.8), it suffices to prove that the induced map on the kernel \( \Theta \circ \Upsilon : \mathbb{N}_{X/Y}[-1] \to X \times^R_S Y \) factors through \( X \). Equivalently, it suffices to prove that \( X \times^R_Y X \to Y \times^R_S Y |_X \) factors through \( X \) since we have proved that the diagram

\[
\begin{array}{ccc}
X \times^R_Y X & \longrightarrow & X \times^R_S X \\
\downarrow & & \downarrow \\
\mathbb{N}_{X/Y}[~-1] & \longrightarrow & \mathbb{N}_{X/S}[~-1] \\
\end{array}
\]

is commutative. This will follow from the lemma below.

**3.10. Lemma.** The map \( X \times^R_Y X \to Y \times^R_S Y \) factors through \( Y \).

**Proof.** By the universal property of the derived fibre product

\[
\begin{array}{ccc}
X \times^R_Y X & \longrightarrow & Y \\
\downarrow & & \downarrow \\
X \times^R_Y Y & \longrightarrow & Y \times^R_S Y \\
\end{array}
\]
we get a commutative diagram

\[
\begin{array}{ccc}
X \times^R_X X & \longrightarrow & X \times^R_X X \\
\downarrow & & \downarrow \\
Y & \longrightarrow & Y \times^R_X Y.
\end{array}
\]

This is a derived version of the following commutative diagram if we think of it in terms of structure sheaves

\[
\begin{array}{ccc}
A/I \otimes_A A/I & \longrightarrow & A/I \otimes_B A/I \\
\downarrow & & \downarrow \\
A & \longrightarrow & A \otimes_B A
\end{array}
\]

where \(A\) is \(O_Y\), \(B\) is \(O_S\), \(A/I\) is \(O_X\), and \(m\) is the multiplication map.

4. Theorem B in classical Lie theory

As a warm-up to proving Theorem B, we present here an analogous result in Lie theory. We give a proof of this result using techniques that can be adapted to the derived setting of Theorem B.

4.1. Consider an injective map of Lie algebras in vector spaces \(\alpha : \mathfrak{h} \hookrightarrow \mathfrak{g}\). There is a short exact sequence of \(\mathfrak{h}\)-modules

\[
0 \longrightarrow \mathfrak{h} \overset{\alpha}{\longrightarrow} \mathfrak{g} \overset{\beta}{\longrightarrow} n = \mathfrak{g}/\mathfrak{h} \longrightarrow 0.
\]

Given a vector space map \(\gamma : n \longrightarrow \mathfrak{g}\) splitting \(\beta\) we define a pre-Lie bracket on \(n\) by the formula \([x, y]_n = \beta((\gamma(x), \gamma(y)))\) for any \(x, y \in n\). In general, the map \(\beta\) may not respect the pre-Lie brackets.

We define a map \(\mathfrak{g} \otimes n \rightarrow n\)

\[
\sum_i x_i \otimes y_i \rightarrow \sum_i \beta([x_i, \gamma(y_i)]),
\]

for \(x_i \in \mathfrak{g}\) and \(y_i \in n\). This map may not define a \(\mathfrak{g}\)-module structure on \(n\) if \(\beta\) is not a morphism of Lie algebras. We state a proposition which is important in sections 4 and 6. Its proof is left to the reader.
4.2. Proposition. In general, we do not expect \( g \otimes g \to \id \otimes \beta \) to be commutative. However, the diagram
\[
\begin{array}{ccc}
\wedge^2 g &=& g \\
\downarrow \beta & & \downarrow \beta \\
\wedge^2 h \oplus \wedge^2 n \oplus (h \otimes n) & \to & g \\
\downarrow \beta & & \\
(h \otimes n) \oplus \wedge^2 n & \to & n
\end{array}
\]
is commutative if we identify \( g \) with \( h \oplus n \) as a direct sum of vector spaces via \( \gamma \).

Therefore the right hand side square of the diagram
\[
\begin{array}{ccc}
g \otimes g & \to & \wedge^2 g \\
\downarrow \beta & & \downarrow \beta \\
g \otimes n & \to & (h \otimes n) \oplus \wedge^2 n \\
\downarrow \beta & & \\
g \otimes n & \to & n
\end{array}
\]
is commutative, but the square on the left is not.

Here is the analogous theorem to Theorem B.

4.3. Theorem. The map \( \beta \) preserves the pre-Lie brackets of \( g \) and \( n \) if and only if \( h \) acts trivially on \( n \). This is also equivalent to saying that \( \beta \) is a morphism of Lie algebras.

Proof. It is easy to see that the map \( g \otimes n \to n \) has the following properties.

(1) It is compatible with the \( h \)-module structure on \( n \). Equivalently, the diagram
\[
\begin{array}{ccc}
h \otimes n & \to & n \\
\downarrow \alpha \otimes \id & & \downarrow \id \\
g \otimes n & \to & n
\end{array}
\]
is commutative. This follows from the observation that the \( h \)-module structure on \( n \) can be defined as
\[
x \otimes y \to \beta([\alpha(x), \gamma(y)]),
\]
for any \( x \in h \) and \( y \in n \).
(II) It defines a $g$-module structure on $n$ if $\beta$ is a morphism of Lie algebras. Equivalently, the diagram

$$
\begin{array}{ccc}
g \otimes n & \longrightarrow & n \\
\downarrow \beta \otimes id & & \downarrow id \\
n \otimes n & \longrightarrow & n
\end{array}
$$

is commutative if $\beta$ is a morphism of Lie algebras.

(III) The diagram

$$
\begin{array}{ccc}
n \otimes n & \longrightarrow & \wedge^2 n & \longrightarrow & n \\
\downarrow \gamma \otimes id & & & & \downarrow \\
g \otimes n & \longrightarrow & (h \otimes n) \oplus \wedge^2 n & \longrightarrow & n
\end{array}
$$

is commutative.

We first prove that $h$ acts on $n$ trivially if $\beta$ preserves the pre-Lie brackets. Compose the two commutative diagrams in (I) and (II) together

$$
\begin{array}{ccc}
h \otimes n & \longrightarrow & n \\
\downarrow \alpha \otimes id & & \downarrow id \\
g \otimes n & \longrightarrow & n \\
\downarrow \beta \otimes id & & \downarrow id \\
n \otimes n & \longrightarrow & n.
\end{array}
$$

The map $h \otimes n \rightarrow n$ is zero because $\beta \circ \alpha = 0$.

Let us turn to prove that $\beta$ preserves the pre-Lie brackets if the Lie module structure map $h \otimes n \rightarrow n$ is zero. Identify $g$ with $h \oplus n$ via $\gamma$. With property (I) and (III) one can conclude that the map $g \otimes n = (h \otimes n) \oplus (n \otimes n) \rightarrow n$ that we defined at the beginning of this section is the Lie module structure map $h \otimes n \rightarrow n$ plus the pre-Lie bracket on $n$: $n \otimes n \rightarrow n$.

If $h \otimes n \rightarrow n$ is zero, then the diagram

$$
\begin{array}{ccc}
g \otimes n & \longrightarrow & (h \otimes n) \oplus \wedge^2 n & \longrightarrow & n \\
\downarrow & & \downarrow & & \downarrow \\
n \otimes n & \longrightarrow & \wedge^2 n & \longrightarrow & n
\end{array}
$$
is commutative. Put the commutative diagram in Proposition 4.2 and the diagram above together

\[
\begin{array}{ccc}
\wedge^2 g & \rightarrow & g \\
\downarrow & & \downarrow \\
(h \otimes n) \oplus \wedge^2 n & \rightarrow & n \\
\downarrow & & \downarrow \\
\wedge^2 n & \rightarrow & n.
\end{array}
\]

We conclude that \( \beta \) is a map of Lie algebras. \( \square \)

5. The Bass-Quillen class as a Lie module structure map

In this section we prove the Lie theoretic interpretation of the Bass-Quillen class that we explained in section 2. We begin by stating the result under the assumption that closed embeddings split to first order only. Then we provide explanations in Lie theoretic terms. We turn to the proof at last.

Most of this section will be devoted to constructing a map \( \kappa : N_{X/Y} \otimes N_{X/S} \rightarrow N_{X/S}[1] \). This map will be given by the extension class of an explicit short exact sequence. Using this map we will prove the following proposition.

5.1. Proposition. In the same setting as Theorem A, the vector bundle map \( N_{X/Y}[-1] \rightarrow N_{X/S}[-1] \) preserves the pre-Lie brackets constructed by Calaque and Grivaux [CG17]. There exists a map \( \kappa : N_{X/Y} \otimes N_{X/S} \rightarrow N_{X/S}[1] \) defined explicitly by the extension class of a short exact sequence. The diagram

\[
\begin{array}{ccc}
N_{X/Y} \otimes N_{Y/S}|x & \rightarrow & N_{Y/S}|x[1] \\
\downarrow & & \downarrow \\
N_{X/Y} \otimes N_{X/S} & \xrightarrow{\kappa} & N_{X/S}[1] \\
\downarrow & & \downarrow \\
N_{X/S} \otimes N_{X/S} & \rightarrow & N_{X/S}[1]
\end{array}
\]

is commutative. Here all the vertical maps are the obvious maps of vector bundles. The top horizontal map is the Bass-Quillen class \( \alpha_{x,N_{Y/S}|x[1]} \), and the bottom horizontal map is the pre-Lie bracket.
5.2. To explain in Lie theoretic terms the meaning of Proposition 5.1, assume for simplicity that the embedding $X \hookrightarrow S$ satisfies the additional conditions that make $\mathfrak{g} = N_{X/S}[-1]$ a Lie algebra. Then denote by $\mathfrak{h} = N_{X/S}[-1]$. It is a subalgebra of $\mathfrak{g}$ because the map $\mathfrak{h} \hookrightarrow \mathfrak{g}$ preserves the brackets by Proposition 5.1. The bundle $N_{Y/S}|_X[-1]$ can be identified with $\mathfrak{g}/\mathfrak{h}$. Then the diagram in Proposition 5.1 becomes

\[
\begin{array}{ccc}
\mathfrak{h} \otimes \mathfrak{g}/\mathfrak{h} & \xrightarrow{\kappa} & \mathfrak{g}/\mathfrak{h} \\
\downarrow & & \downarrow \\
\mathfrak{h} \otimes \mathfrak{g} & \xrightarrow{\kappa} & \mathfrak{g} \\
\downarrow & & \downarrow \\
\mathfrak{g} \otimes \mathfrak{g} & \xrightarrow{\kappa} & \mathfrak{g} \\
\end{array}
\]

The commutativity of

\[
\begin{array}{ccc}
N_{X/Y} \otimes N_{X/S} & \xrightarrow{\kappa} & N_{X/S}[1] \\
\downarrow & & \downarrow \\
N_{X/S} \otimes N_{X/S} & \xrightarrow{\kappa} & N_{X/S}[1] \\
\end{array}
\]

says that the morphism $\kappa$ is the structure map of the natural $\mathfrak{h}$-module structure on $\mathfrak{g} = N_{X/S}[-1]$, where $\mathfrak{h} = N_{X/Y}[-1]$ is the Lie algebra.

The commutativity of the diagram

\[
\begin{array}{ccc}
N_{X/Y} \otimes N_{Y/S}|_X & \xrightarrow{\kappa} & N_{Y/S}|_X[1] \\
\downarrow & & \downarrow \\
N_{X/Y} \otimes N_{X/S} & \xrightarrow{\kappa} & N_{X/S}[1] \\
\end{array}
\]

says that the Bass-Quillen class $\alpha_{s,N_{Y/S}|_X[-1]}$ at the top of the diagram is the structure map of the $\mathfrak{h}$-module structure on $\mathfrak{g}/\mathfrak{h} = N_{Y/S}|_X[-1]$.

5.3. Before we prove Proposition 5.1, we have to define the morphism $\kappa : N_{X/Y} \otimes N_{X/S} \rightarrow N_{X/S}[1]$ that appears in the middle of the diagram in Proposition 5.1. We hope to describe the morphism $\kappa$ explicitly as the extension class of a short exact sequence. There is a technical detail we need to deal with. As we can see, the pre-Lie bracket is defined as $\text{Sym}^2 N_{X/S} \rightarrow N_{X/S}[1]$ instead of $N_{X/S} \otimes N_{X/S} \rightarrow N_{X/S}[1]$. It is easy to describe the short exact sequence corresponding to $\text{Sym}^2 N_{X/S} \rightarrow N_{X/S}[1]$. However, it is hard to describe explicitly what short exact sequence the morphism $N_{X/S} \otimes$
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$N_X/S \rightarrow N_X/S[1]$ corresponds to. The same phenomenon appears when we try to define $N_X/Y \otimes N_X/S \rightarrow N_X/S[1]$. There is an anti-symmetric part $\wedge^2 N_X/Y \hookrightarrow N_X/Y \otimes N_X/S$. We can only define our desired Lie module structure map $\kappa$ via the extension class of a short exact sequence after we kill this anti-symmetric part. Lemma 5.4 below describes how to kill the anti-symmetric part of $N_X/Y \otimes N_X/S \cong (N_X/Y \otimes N_X/Y) \oplus (N_X/Y \otimes N_Y/S|X)$ canonically.

5.4. Lemma. The vector bundle $\frac{I_X^2}{I_X^3 + I_Y^2}$ on $X$ is isomorphic to $\text{Sym}^2 N_X^\vee \oplus (N_X^\vee \otimes N_Y^\vee|X)$, where $I_X$ and $I_Y$ are the ideal sheaves of $X$ and $Y$ in $S$.

Proof. The cokernel of $\text{Sym}^2 N_Y^\vee|X \hookrightarrow \text{Sym}^2 N_X^\vee$ is isomorphic to $\text{Sym}^2 N_X^\vee \oplus (N_X^\vee \otimes N_Y^\vee|X)$ using the splitting in (3.6).

There is a commutative diagram on $X$

\[
\begin{array}{cccccc}
0 & \rightarrow & I_X^2 & \rightarrow & I_X^3 & \rightarrow & I_X^2/I_X^3 + I_Y^2 & \rightarrow & 0 \\
\cong & & \downarrow & & \cong & & \\
0 & \rightarrow & \text{Sym}^2 N_Y^\vee|X & \rightarrow & \text{Sym}^2 N_X^\vee & \rightarrow & \text{Sym}^2 N_X^\vee \oplus (N_X^\vee \otimes N_Y^\vee|X) & \rightarrow & 0.
\end{array}
\]

The two vertical maps above are isomorphisms, so we can complete the diagram above as an isomorphism of short exact sequences. This implies our desired isomorphism. □

5.5. Definition. Define the morphism $\kappa : N_X/Y \otimes N_X/S \rightarrow N_X/S[1]$ as follows

$N_X/Y \otimes N_X/S \rightarrow \text{Sym}^2 N_X/Y \oplus (N_X/Y \otimes N_Y/S|X) \cong \left(\frac{I_X^2}{I_X^3 + I_Y^2}\right)^\vee \rightarrow N_X/S[1],$

where the map on the left is the obvious map under the identification $N_X/S \cong N_X/Y \oplus N_Y/S|X$ in (3.6), and the map on the right is given by the extension class of the short exact sequence

\[
0 \rightarrow \frac{I_X^2}{I_X^3 + I_Y^2} \rightarrow \varphi_* \left(\frac{I_X}{I_X^3 + I_Y^2}\right) \rightarrow \frac{I_X}{I_X^2} \rightarrow 0.
\]

5.6. We will focus on the proof of Proposition 5.1. The result will follow from Lemma 5.8 and Propositions 5.10 and 5.11 below.
5.7. Lemma. Sym\(^2\)\(N_{X/Y}^\vee\) is isomorphic to \(\frac{I_X^2}{I_X^2 + I_XI_Y}\).

Proof. The ideal sheaf of \(X\) in \(Y\) is \(\frac{I_X}{I_Y} \subseteq \mathcal{O}_Y = \mathcal{O}_S/I_Y\). Note that \(\frac{I_X}{I_Y} \neq \left(\frac{I_X}{I_Y}\right)^n \subseteq \mathcal{O}_S/I_Y\). It is easy to show that \(\left(\frac{I_X}{I_Y}\right)^n \simeq \frac{I_X^n + I_Y}{I_Y} \subseteq \mathcal{O}_Y = \mathcal{O}_S/I_Y\). Therefore,

\[\text{Sym}^2N_{X/Y}^\vee \equiv \left(\frac{I_X}{I_Y}\right)^2 \equiv \frac{I_X^2 + I_Y}{I_Y} \equiv \frac{I_X^2}{I_X^2 + I_Y} \cap (I_X + I_Y)\]

We have \(I_X^2 \cap (I_X^2 + I_Y) = (I_X^2 \cap I_Y^3) + (I_X^2 \cap I_Y)\) because \(I_X^3 \subseteq I_X^2\). The equality \(I_X^2 \cap I_Y = I_XI_Y\) is due to the injective map below

\[N_{Y/S}|_X = \frac{I_Y}{I_YI_X} \hookrightarrow N_{X/S}^\vee = \frac{I_X}{I_X^2}\]

\(5.8.\) Lemma. The map of vector bundles \(N_{X/Y}[-1] \to N_{X/S}[-1]\) preserves the pre-Lie brackets.

Proof. One can check that the two short exact sequences

\[
\begin{array}{ccccccccc}
0 & \rightarrow & I_X^2 & \rightarrow & I_X^2 & \rightarrow & I_X & \rightarrow & I_X & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \frac{I_X^3}{I_X^3} & \rightarrow & \frac{I_X}{I_X^2} & \rightarrow & \frac{I_X}{I_X^2} & \rightarrow & \frac{I_X}{I_X^2} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \frac{I_X^3}{I_X^3} & \rightarrow & \frac{I_X}{I_X^2} & \rightarrow & \frac{I_X}{I_X^2} & \rightarrow & \frac{I_X}{I_X^2} & \rightarrow & 0 \\
\end{array}
\]

are compatible. \(\square\)

5.9. On the other hand, the map \(N_{X/S}[-1] \to N_{Y/S}|_X[-1]\) may not preserve the pre-Lie brackets because there is no map \((\pi_*\frac{I_Y}{I_Y})|_X \to \varphi_*\frac{I_X}{I_X^2}\) generally.

We prove the commutativity of the two diagrams in Proposition 5.1. It is divided into two propositions below.
**5.10. Proposition.** The map in Definition 5.5 is compatible with the pre-Lie bracket of $N_{X/S}[−1]$. It is equivalent to saying that the diagram

$$
\begin{array}{c}
N_{X/Y} \otimes N_{X/S} \\
\downarrow \\
N_{X/S} \otimes N_{X/S}
\end{array} \longrightarrow
\begin{array}{c}
N_{X/S}[1] \\
\downarrow \\
N_{X/S}[1]
\end{array}
$$

is commutative.

**Proof.** We need to show that the three small diagrams

$$
\begin{array}{c}
N_{X/Y} \otimes N_{X/S} \\
\downarrow \\
N_{X/S} \otimes N_{X/S}
\end{array} \longrightarrow
\begin{array}{c}
\text{Sym}^2 N_{X/Y} \oplus (N_{X/Y} \otimes N_{Y/S}|_{X}) \\
\downarrow \\
\text{Sym}^2 N_{X/S}
\end{array} \cong
\begin{array}{c}
\left( \frac{I_X^2}{I_X^3 + I_Y^3} \right)^* \\
\downarrow \\
\left( \frac{I_X^2}{I_X} \right)^*
\end{array} \longrightarrow
\begin{array}{c}
N_{X/S}[1] \\
\downarrow \\
N_{X/S}[1]
\end{array}
$$

are commutative. Clearly the one on the left is commutative. The commutativity of the isomorphism in the middle follows from the compatibility of the two short exact sequences in Lemma 5.4. The diagram on the right commutes because the two short exact sequences

$$
\begin{array}{c}
0 \\
\downarrow \\
0
\end{array} \longrightarrow
\begin{array}{c}
\frac{I_X^2}{I_X^3 + I_Y^2} \\
\downarrow \\
\frac{I_X^2}{I_X^3}
\end{array} \longrightarrow
\begin{array}{c}
\frac{I_X}{I_X^3 + I_Y} \\
\downarrow \\
\frac{I_X}{I_X^3}
\end{array} \longrightarrow
\begin{array}{c}
\frac{I_X}{I_X^2} \\
\downarrow \\
0
\end{array}
$$

are compatible. \qed

**5.11. Proposition.** There is a commutative diagram

$$
\begin{array}{c}
N_{X/Y} \otimes N_{X/S} \\
\downarrow \\
N_{X/S}[1]
\end{array} \longrightarrow
\begin{array}{c}
N_{X/Y} \otimes N_{Y/S}|_{X} \\
\downarrow \\
N_{Y/S}[1]
\end{array}
$$

where the left vertical map is in Definition 5.5, and the right vertical map is the Bass-Quillen class $\alpha_{s,N_{Y/S}^1}|_{I_X^1}$.
Proof. We need to prove that the three small diagrams in the diagram (1)

\[
\begin{align*}
N_{X/Y} \otimes N_{X/S} &\to N_{X/Y} \otimes N_{Y/S}|_X \\
\text{Sym}^2 N_{X/Y} \oplus (N_{X/Y} \otimes N_{Y/S}|_X) &\to N_{X/Y} \otimes N_{Y/S}|_X \\
\left(\frac{I_X}{I_X^2} + I_Y\right)^\vee &\to \left(\frac{I_X}{I_X^2} + I_Y\right)^\vee \otimes \left(\frac{I_Y}{I_YI_X}\right)^\vee \\
N_{X/S}[1] &\to N_{Y/S}[1]
\end{align*}
\]

are commutative. Obviously the one on the top is commutative.

To prove the isomorphism in the middle is compatible, we construct a commutative diagram

\[
\begin{align*}
N_{X/Y}^\vee \otimes N_{Y/S}|_X &\to N_{X/S}^\vee \otimes N_{Y/S}|_X \\
\text{Sym}^2 N_{X/S} &\to \text{Sym}^2 N_{X/Y} \oplus (N_{X/Y}^\vee \otimes N_{Y/S}|_X) \\
\frac{I_X}{I_X^2} \otimes \frac{I_Y}{I_YI_X} &\to \frac{I_X}{I_X^2} \otimes \frac{I_X}{I_X^2} \otimes \frac{I_Y}{I_YI_X} \\
N_{X/S}[1] &\to N_{Y/S}[1]
\end{align*}
\]

where the dotted arrows are defined by the splitting in (3.6), and the right square commutes as mentioned in the proof of Proposition 5.10. Clearly, the left and middle squares are commutative. We hope to prove that this big commutative diagram is exactly dual to the one in the middle of (1). It suffices to show that the map

\[
\frac{I_X}{I_X^2 + I_Y} \otimes \frac{I_Y}{I_YI_X} \to \frac{I_X}{I_X^2} \otimes \frac{I_X}{I_X^2} \otimes \frac{I_Y}{I_YI_X}
\]

defined using the splitting is equal to the natural map

\[
\frac{I_X}{I_X^2 + I_Y} \otimes \frac{I_Y}{I_YI_X} \to \frac{I_X^2}{I_X^2 + I_Y^2},
\]

\[ (a \otimes b) \to ab, \text{ for } a \in \frac{I_X}{I_X^2 + I_Y}, \text{ and } b \in \frac{I_Y}{I_YI_X}. \]

One can check this easily.
Let us focus on the commutativity of the bottom square in (1)

\[
\begin{array}{c}
\frac{I_X^2}{I_X^3 + I_Y^2} \\
\Downarrow \\
\frac{I_X^2}{I_X^3 + I_Y^2} \otimes \frac{I_Y^2}{I_Y I_X} \\
\Downarrow \\
N_{X/S}[1]
\end{array}
\]

The bottom horizontal map is defined by a short exact sequence

\[
0 \rightarrow N_{X/Y}^\vee \otimes N_{Y/S}^\vee |X \rightarrow s_* a^* N_{Y/S}^\vee \rightarrow N_{Y/S}^\vee \rightarrow 0.
\]

Moreover, we have a morphism of short exact sequences

\[
\begin{array}{c}
0 \\
\Downarrow \\
I_X I_Y \\
\Downarrow \\
I_Y(I_X^2 + I_Y) \\
\Downarrow \\
s_* \\
I_Y(I_X^2 + I_Y) \\
\Downarrow \\
I_Y I_X \\
\Downarrow \\
0.
\end{array}
\]

This implies that \(N_{X/Y}^\vee \otimes N_{Y/S}^\vee \cong \frac{I_Y I_Y}{I_Y(I_X^2 + I_Y)}\). It suffices to show that the short exact sequence \(0 \rightarrow \frac{I_X I_Y}{I_Y(I_X^2 + I_Y)} \rightarrow s_* \frac{I_Y}{I_Y I_X} \rightarrow \frac{I_Y}{I_Y I_X} \rightarrow 0\) is compatible with the short exact sequence \(0 \rightarrow \frac{I_X^2}{I_X^3 + I_Y^2} \otimes \frac{I_Y^2}{I_Y I_X} \rightarrow \varphi_* \frac{I_X}{I_X^2 + I_Y^2} \rightarrow \frac{I_X}{I_X^2 + I_Y^2} \rightarrow 0\).

There is a natural map of sheaves \(g_* \frac{I_Y}{I_Y(I_X^2 + I_Y)} \rightarrow \frac{I_X}{I_X^3 + I_Y^2}\). We get \(s_* \frac{I_Y}{I_Y I_X} \rightarrow \varphi_* \frac{I_X}{I_X^2 + I_Y^2}\) by applying \(\varphi_*\) on both sides, so the two short exact sequences

\[
\begin{array}{c}
0 \\
\Downarrow \\
I_X I_Y \\
\Downarrow \\
I_Y(I_X^2 + I_Y) \\
\Downarrow \\
s_* \\
I_Y(I_X^2 + I_Y) \\
\Downarrow \\
I_Y I_X \\
\Downarrow \\
0.
\end{array}
\]

\[
\begin{array}{c}
0 \\
\Downarrow \\
\frac{I_X^2}{I_X^3 + I_Y^2} \\
\Downarrow \\
\frac{I_X^2}{I_X^3 + I_Y^2} \otimes \frac{I_Y^2}{I_Y I_X} \\
\Downarrow \\
\frac{I_X}{I_X^2 + I_Y^2} \\
\Downarrow \\
\frac{I_X}{I_X^2 + I_Y^2} \\
\Downarrow \\
0
\end{array}
\]

are compatible.
6. The proof of Theorem B

We generalize the proof in section 4 to prove Theorem B. We first define a morphism $N_{X/S} \otimes N_{Y/S}|_X \rightarrow N_{Y/S}|_X[1]$ which is analogous to the map $g \otimes n \rightarrow n$ in section 4. Then we prove similar statements to properties (I) and (II), and Proposition 4.2 in section 4. We prove Theorem B at last.

6.1. The first thing that we need is the map $N_{X/S} \otimes N_{Y/S}|_X \rightarrow N_{Y/S}|_X[1]$ which is the analogue of the map $g \otimes n \rightarrow n$ from section 4. We need to deal with the same technical issue that appears in section 5. Using the splitting in (3.6) we see that there is an anti-symmetric part $\wedge^2 N_{Y/S}|_X$ in $N_{X/S} \otimes N_{Y/S}|_X = (N_{X/Y} \oplus N_{Y/S}|_X) \otimes N_{Y/S}|_X$. We need to kill this anti-symmetric part canonically. Lemma 6.2 and 6.3 describe how to do this.

6.2. Lemma. $I^2_Y \cap I^2_X Y = I^2_Y X$.

Proof. We have $I^2_Y \cap I^2_X = I^2_Y X$ because the map

$$
\text{Sym}^2 N^Y_{Y/S}|_X = \frac{I^2_Y}{I^2_Y X} \rightarrow \text{Sym}^2 N^X_{X/S} = \frac{I^2_X}{I^2_X}
$$

is injective. Then we have

$$
I^2_X X \subset I^2_Y \cap I^2_X Y \subset I^2_Y \cap I^2_X = I^2_Y X.
$$

6.3. Lemma. There is an isomorphism of vector bundles $(\frac{I_X Y}{I^2_X Y})^Y \cong \text{Sym}^2 N_{Y/S}|_X \oplus (N_{X/Y} \otimes N_{Y/S}|_X)$ on $X$.

Proof. There is a morphism of short exact sequences

$$
0 \rightarrow \frac{I_Y}{I_Y X} \otimes I_Y I_X \rightarrow \frac{I_X}{I_X} \otimes I_Y I_X \rightarrow \frac{I_Y}{I_Y X} \otimes \frac{I_X}{I_X} \longrightarrow 0
$$

$$
0 \rightarrow \frac{I^2_Y}{I^2_Y X} \rightarrow I_X I_Y \rightarrow \frac{I_X I_Y}{I^2_X I_Y} \longrightarrow 0.
$$

Everything above is clear except for the injectivity of $\frac{I^2_Y}{I^2_Y X} \rightarrow \frac{I_X I_Y}{I^2_X I_Y}$. This is due to Lemma 6.2.
The short exact sequence on the top is the dual of the sequence of the normal bundles tensored with $N_{Y/S}|_X$, so it splits naturally. The map $u$ is an isomorphism as mentioned in the proof of Proposition 5.11. One can construct a splitting $v \circ \tau \circ u^{-1}$ for the short exact sequence on the bottom. Therefore $I_XI_Y \cong \frac{I_Y^2}{I_Y^2I_X} \oplus \frac{I_XI_Y}{I_Y(I_X^2 + I_Y)} \cong \text{Sym}^2N_{Y/S}|_X \oplus (N_{X/Y} \otimes N_{Y/S}|_X)$. The diagram of two short exact sequences above says that there is a commutative diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & N_{Y/S}|_X \otimes N_{Y/S}|_X & \rightarrow & N_{X/S} \otimes N_{Y/S}|_X & \rightarrow & N_{X/Y} \otimes N_{Y/S}|_X & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \text{Sym}^2N_{Y/S}|_X & \rightarrow & \text{Sym}^2N_{Y/S}|_X \oplus (N_{Y/S}|_X \otimes N_{X/Y}) & \rightarrow & N_{X/Y} \otimes N_{Y/S}|_X & \rightarrow & 0 \\
& & \cong & & \cong & & \cong & & \\
0 & \rightarrow & \frac{I_Y^2}{I_Y^2I_X} & \rightarrow & \frac{I_XI_Y}{I_YI_X} & \rightarrow & \frac{I_XI_Y}{(I_X^2 + I_Y)} & \rightarrow & 0.
\end{array}
\]

6.4. Definition. Define the morphism $N_{X/S} \otimes N_{Y/S}|_X \rightarrow N_{Y/S}[1]$ which is analogous to $g \otimes n \rightarrow n$ in section 4 as follows

$N_{X/S} \otimes N_{Y/S}|_X \rightarrow \text{Sym}^2(N_{Y/S}|_X) \oplus (N_{X/Y} \otimes N_{Y/S}|_X) \cong (\frac{I_XI_Y}{I_YI_X})^\vee \rightarrow N_{Y/S}|_X[1],$

where the map on the left hand side is the obvious one under the identification $N_{X/S} \cong N_{X/Y} \oplus N_{Y/S}|_X$, and the map on the right hand side is given by the extension class of the short exact sequence

$0 \rightarrow \frac{I_XI_Y}{I_YI_X} \rightarrow \frac{I_Y}{I_YI_X} \rightarrow \frac{I_Y}{I_YI_X} \rightarrow 0.$

The following proposition is analogous to property (I) in section 4.

6.5. Proposition. There is a commutative diagram

\[
\begin{array}{ccc}
N_{X/S} \otimes N_{Y/S}|_X & \rightarrow & N_{Y/S}[1] \\
\uparrow & & \uparrow \\
N_{X/Y} \otimes N_{Y/S}|_X & \rightarrow & N_{Y/S}|_X[1],
\end{array}
\]

where the horizontal map at top is in Definition 6.4, and the horizontal map at bottom is the Bass-Quillen class $\alpha_{s,N_{Y/S}|_X}$.
Proof. The isomorphism $N_{X/Y}^\vee \otimes N_{Y/S}^\vee|_X \cong \frac{I_X I_Y}{(I_X^2 + I_Y)I_Y}$ is mentioned in the proof of Proposition 5.11. It suffices to prove that the three small diagrams

\[
\begin{array}{ccc}
N_{X/S} \otimes N_{Y/S}|_X & \longrightarrow & \text{Sym}^2 N_{Y/S}|_X \oplus (N_{X/Y} \otimes N_{Y/S}|_X) \\
\uparrow & \downarrow & \uparrow \\
N_{X/Y} \otimes N_{Y/S}|_X & \longrightarrow & N_{X/Y} \otimes N_{Y/S}|_X \\
\uparrow & \downarrow & \uparrow \\
N_{X/Y} \otimes N_{Y/S}|_X & \longrightarrow & N_{X/Y} \otimes N_{Y/S}|_X \\
\end{array}
\]

are commutative. Obviously, the left one is commutative. Commutativity of the one in the middle is due the the compatibility of short exact sequence in Lemma 6.3. The rest of our proof is devoted to the commutativity of the diagram on the right.

There is a natural map $I_Y \frac{I_Y}{I_Y I_X^2} \rightarrow g_* \frac{I_Y}{(I_Y^2 + I_Y)I_Y}$. We get $\varphi_* \frac{I_Y}{I_Y I_X^2} \rightarrow s_* \frac{I_Y}{(I_Y^2 + I_Y)I_Y}$ by applying $\varphi_*$ on both sides. This gives the two compatible short exact sequences

\[
\begin{array}{cccccccc}
0 & \longrightarrow & I_X I_Y & \longrightarrow & I_Y & \longrightarrow & 0 \\
\downarrow & & \varphi_* & \downarrow & \varphi_* & \downarrow & \varphi_* \\
0 & \longrightarrow & I_X I_Y & \longrightarrow & I_Y & \longrightarrow & 0 \\
\end{array}
\]

which proves that the diagram on the right is commutative. \hfill $\square$

The following proposition is similar to Proposition 4.2.

6.6. Proposition. There is a commutative diagram

\[
\begin{array}{ccc}
\text{Sym}^2 N_{X/S} & \longrightarrow & N_{X/S}[1] \\
\downarrow & & \downarrow \\
\text{Sym}^2 N_{Y/S}|_X \oplus (N_{X/Y} \otimes N_{Y/S}|_X) & \cong & (I_X I_Y)^\vee \frac{I_X I_Y}{I_Y I_X} \longrightarrow N_{Y/S}[1],
\end{array}
\]

where the bottom horizontal map is in Definition 6.4.
However, we do not expect the following big diagram is commutative generally. One can check that the left square is not commutative as mentioned in Proposition 4.2.

Proof. We need to prove that the two small diagrams in diagram (2) are commutative. The diagram on the right of (2) commutes because the two short exact sequence

are compatible. Let us prove that the left diagram in (2) commutes. We construct a
diagram below

\[
\begin{array}{c}
\text{Sym}^2 N^\vee_{Y/S} \oplus (N^\vee_{X/Y} \otimes N^\vee_{Y/S} |_X) \xrightarrow{\epsilon \times \zeta} \text{Sym}^2 N^\vee_{Y/S} \\
\text{Sym}^2 N^\vee_{Y/S} \xrightarrow{\epsilon'} \text{Sym}^2 N^\vee_{X/S} \\
\end{array}
\]

where all the vertical maps are natural isomorphisms. The dotted arrows are constructed by splittings in the proof of Lemma 6.3, and the solid arrows are the obvious ones which also appear in the proof of Lemma 6.3. The two short exact sequences and their splittings in the proof of Lemma 6.3 are compatible, so the diagram above is commutative. Taking the direct sum and direct product of the maps above, we get a commutative diagram

\[
\begin{array}{c}
\text{Sym}^2 N^\vee_{Y/S} \oplus (N^\vee_{X/Y} \otimes N^\vee_{Y/S} |_X) \xrightarrow{\epsilon \times \zeta} \text{Sym}^2 N^\vee_{Y/S} \oplus (N^\vee_{X/Y} \otimes N^\vee_{Y/S} |_X) \xrightarrow{\epsilon' \oplus (\theta' \circ \xi') \oplus \delta'} \text{Sym}^2 N^\vee_{X/S} \\
\text{Sym}^2 N^\vee_{Y/S} \xrightarrow{\epsilon'} \text{Sym}^2 N^\vee_{X/S} \\
\end{array}
\]

We hope to prove that the diagram above is dual to the left square in (2). This says that we need to prove the following statement (3):

The map we constructed above \(((\theta' \circ \xi') \oplus \delta') \circ (\xi \times \delta)\) is equal to the natural map \(I_X I_Y \to \frac{P^2_X}{P^2_X + I_Y}\).
Consider the following diagram

\[
\begin{array}{c}
\begin{array}{ccc}
I_XI_Y & \xrightarrow{\delta} & I^2_Y \\
I^2_Y & \xrightarrow{\delta'} & I^2_X \\
\end{array}
\end{array}
\]

where the dotted arrows are the splittings in the proof of Lemma 6.3. The diagram above is commutative because the two short exact sequences and their splittings in the proof of Lemma 6.3 are compatible.

Notice that \( N_{X/S} \otimes N_{Y/S}|X \rightarrow \frac{I_XI_Y}{I^2_X} \) is surjective. To prove the statement (3), it suffices to show that the map \((\theta' \oplus (\theta' \circ \lambda')) \circ (\theta \times \lambda) : N_{X/S} \otimes N_{Y/S}|X = \frac{I_X}{I^2_X} \otimes \frac{I_Y}{I_YI_X} \rightarrow \) constructed via the splittings is equal to the natural map

\[
\frac{I_X}{I^2_X} \otimes \frac{I_Y}{I_YI_X} \xrightarrow{\text{Sym}^2N_{X/S}} \frac{I^2_Y}{I_YI_X}
\]

\[
(a \otimes b) \rightarrow ab, \text{ for } a \in \frac{I_X}{I^2_X}, \text{ and } b \in \frac{I_Y}{I_YI_X}.
\]

One can verify it easily. \( \square \)

The following lemma is analogous to property (II) in section 4.

6.7. Lemma. If \( N_{X/S}[-1] \rightarrow N_{Y/S}|X[-1] \) preserves the pre-Lie brackets, then there is a commutative diagram

\[
\begin{array}{ccc}
\text{Sym}^2N_{Y/S}|X & \longrightarrow & N_{Y/S}|X[1] \\
\downarrow & & \downarrow \text{id} \\
\text{Sym}^2N_{Y/S} \oplus (N_{X/Y} \otimes N_{Y/S}|X) & \longrightarrow & N_{Y/S}|X[1],
\end{array}
\]
where the bottom horizontal map is in Definition 6.4.

Proof. Put what we want to prove into a larger diagram

\[
\begin{array}{ccc}
\text{Sym}^2 N_{Y/S}|X & \xrightarrow{\chi} & N_{Y/S}|X[1] \\
\downarrow \psi & & \downarrow \phi \\
\text{Sym}^2 N_{Y/S}|X \oplus (N_{X/Y} \otimes N_{Y/S}|X) & \xrightarrow{\iota} & N_{Y/S}|X[1].
\end{array}
\]

The outer square commutes because we assume that the brackets are preserved. We want to show that \(\chi \circ \psi = \phi\). The commutativity of the outer square and Proposition 6.6 show that \(\chi \circ \psi \circ \iota = \varphi \circ \iota\). The map \(\iota\) splits naturally, so we have our desired result.

There should be a statement analogous to property (III). It will appear in the proof of Theorem B below.

**Proof of Theorem B.** We first prove that the Bass-Quillen Lie module map \(\alpha_{s,N_{Y/S}|X}^{(1)}\) is zero if the pre-Lie brackets are preserved. There is a commutative diagram

\[
\begin{array}{ccc}
N_{Y/S}|X \otimes N_{Y/S}|X & \xrightarrow{\text{id}} & \text{Sym}^2 N_{Y/S}|X \\
\downarrow \text{id} & & \downarrow \text{id} \\
\text{Sym}^2 N_{Y/S}|X \oplus (N_{X/Y} \otimes N_{Y/S}|X) & \xrightarrow{\text{id}} & N_{Y/S}|X[1] \\
\downarrow \text{id} & & \downarrow \text{id} \\
N_{X/Y} \otimes N_{Y/S}|X & \xrightarrow{\text{id}} & N_{Y/S}|X[1]
\end{array}
\]

due to Lemma 6.7 and Proposition 6.5. The Bass-Quillen class \(\alpha_{s,N_{Y/S}|X}^{(1)}\) that appears at the bottom of the diagram above vanishes because the composition \(N_{X/Y} \rightarrow N_{X/S} \rightarrow N_{Y/S}|X\) is zero.

Let us turn to prove that the vector bundle map \(N_{X/S}[-1] \rightarrow N_{Y/S}|X[-1]\) preserves the pre-Lie brackets if the Bass-Quillen class \(\alpha_{s,N_{Y/S}|X}^{(1)}\) is zero. The proof is similar to the
one in section 4. In [CG17] Calaque and Grivaux showed that the pre-Lie brackets on $N_{X/S}[-1]$ and $N_{Y/S}[X_{-1}]$ defined by the extension classes of short exact sequences can be also defined as follows:

$$
\text{Sym}^2 N_{X/S} \rightarrow \text{Sym}^2 T_S \rightarrow T_S[1] \rightarrow N_{X/S}[1].
$$

$$
\text{Sym}^2 N_{Y/S} \rightarrow \text{Sym}^2 T_S \rightarrow T_S[1] \rightarrow N_{Y/S}[1].
$$

The dotted arrow is due the fact that $f : X \hookrightarrow S$ and $j : Y \hookrightarrow S$ split to first order. The map in the middle is the Atiyah class.

Using the compatibility condition on splittings of tangent bundles and the fact above, we conclude that the diagram

$$
\begin{array}{ccc}
\text{Sym}^2 N_{Y/S}|X & \rightarrow & \text{Sym}^2 T_S|X \\
\downarrow & & \downarrow \\
\text{Sym}^2 N_{X/S} & \rightarrow & \text{Sym}^2 T_S|X
\end{array}
$$

is commutative. The diagram above and Proposition 6.6 say that we have a commutative diagram

$$
\begin{array}{ccc}
\text{Sym}^2 N_{Y/S}|X & \rightarrow & N_{Y/S}|X[1] \\
\downarrow & & \downarrow \\
\text{Sym}^2 N_{Y/S}|X \oplus (N_{Y/S}|X \otimes N_{X/Y}) & \rightarrow & N_{Y/S}|X[1]
\end{array}
$$

which is analogous to property (III) in section 4. Based on the diagram above and Proposition 6.5 it is clear that the diagram

$$
\begin{array}{ccc}
\text{Sym}^2 N_{Y/S}|X \oplus (N_{Y/S}|X \otimes N_{X/Y}) & \rightarrow & N_{Y/S}|X[1] \\
\downarrow & & \downarrow \\
\text{Sym}^2 N_{Y/S}|X & \rightarrow & N_{Y/S}|X[1]
\end{array}
$$

is commutative if the Bass-Quillen class $\alpha_{s,N_{Y/S}|X_{Y}}$ is zero. Compose the diagram
above with the one in Proposition 6.6. We get a commutative diagram

\[
\begin{array}{ccc}
\text{Sym}^2 N_{X/S} & \rightarrow & N_{X/S}[1] \\
\downarrow & & \downarrow \\
\text{Sym}^2 N_{Y/S}|_X \oplus (N_{Y/S}|_X \otimes N_{X/Y}) & \rightarrow & N_{Y/S}|_X[1] \\
\downarrow & & \downarrow \text{id} \\
\text{Sym}^2 N_{Y/S}|_X & \rightarrow & N_{Y/S}|_X[1],
\end{array}
\]

so we conclude that the pre-Lie brackets are preserved if the Bass-Quillen class \( \alpha_{s,N_{Y/S}|_X^{(1)}} \) is zero.

\[ \Box \]

6.8. We end this section by providing an example where the Bass-Quillen class is not zero. Consider the embeddings

\[
X \xrightarrow{\Delta_X} X \times X = Y \xrightarrow{\Delta_X \times X} S = X \times X \times X \times X
\]

for a smooth scheme \( X \), where all the inclusions are the diagonal embeddings, and all the splittings are the projections to the first factor. The normal bundle \( N_{Y/S} \) is \( p_1^* T_X \oplus p_2^* T_X \) in this case, where \( p_1 \) and \( p_2 \) are the two projections: \( X \times X \rightarrow X \). It is clear that the Bass-Quillen class \( \alpha_{p_1,N_{Y/S}|_X^{(1)}} : T_X \otimes (T_X \oplus T_X) \rightarrow T_X[1] \) is zero plus the Atiyah class: \( T_X[-1] \otimes T_X[-1] \rightarrow T_X[-1] \). It is not zero in general.

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