The Perturbative Quantization of Gravity

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Abstract
A suggestion is made for quantizing gravity perturbatively, and is illustrated for the example of a massive scalar field with gravity.

1 The Scheme

This short talk is about the renormalizability of Einstein gravity\footnote{shiekh@ictp.trieste.it}. It is NOT renormalizable so this might indeed be a very short talk!

However the work might best be viewed as a look back over failed campaigns of the past. We are forever rushing ahead with new ideas, that perhaps sometimes we lose perspective. So let us review; and if we happen to stumble upon a trail missed before, all the better.

Renormalization is all about reabsorbing infinities into the starting Lagrangian [Ramond, 1990; Collins, 1989]. This can fail in one of two ways:

I) The terms to be reabsorbed are not present in the original Lagrangian.

II) One might keep extending the starting Lagrangian to take up the counter terms and end up with an infinite number of terms and associated

\footnote{This talk was given at the Henri Poincaré seminar held at Protvino Russia, and is a condensed report of the work “Orthodox Gravity”, presently under submission.}
undetermined coupling constants. The theory is then technically renormalized, but has lost all predictive content.

This short review complete, we can look at the problems of quantizing Einstein gravity with a massive scalar field [Veltman, 1976].

Starting from the example of a free scalar field with gravity described by the classical Lagrangian in Euclidean space:

\[ L = -\sqrt{g} \left( R + \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} m^2 \phi^2 \right) \]

(using units where \( 16\pi G = 1, \ c = 1 \))

one proceeds by extracting the Feynman rules (of which there are an infinite number, although only a finite number are used to any finite loop order). At one loop one already encounters infinities that cannot be reabsorbed into the starting Lagrangian. Einstein gravity is not renormalizable.

This very malady suggests its own solution: namely to extend the original Lagrangian so that symmetry will ensure that the counter terms all fall within the original Lagrangian (the second line carries all the higher derivatives):

\[ L_0 = -\sqrt{g_0} \left( -2\Lambda_0 + R_0 + \frac{1}{2} p_0^2 + \frac{1}{2} m_0^2 \phi_0^2 + \frac{1}{2} \phi_0^4 \lambda_0(\phi_0^2) + p_0^2 \phi_0^2 \kappa(\phi_0^2) + R_0 \phi_0^2 \gamma_0(\phi_0^2) 
+ p_0^4 a_0(p_0^2, \phi_0^2) + R_0 p_0^2 b_0(p_0^2, \phi_0^2) + R_0^2 c_0(p_0^2, \phi_0^2) + R_{0\mu\nu} R^\mu_0 R^\nu_0 d_0(p_0^2, \phi_0^2) + ... \right) \]

(where \( p_0^2 \) is shorthand for \( g_0^{\mu\nu} \partial_\mu \phi_0 \partial_\nu \phi_0 \))

which then renormalizes to:

\[ L = -\sqrt{g} \left( -2\Lambda + R + \frac{1}{2} p^2 + \frac{1}{2} m^2 \phi^2 + \frac{1}{4} \phi^4 \lambda(\phi^2) + p^2 \phi^2 \kappa(\phi^2) + R \phi^2 \gamma(\phi^2) 
+ p^4 a(p^2, \phi^2) + Rp^2 b(p^2, \phi^2) + R^2 c(p^2, \phi^2) + R_{\mu\nu} R^{\mu\nu} d(p^2, \phi^2) + ... \right) \]
Now all is lost from the start, for there are an infinite number of arbitrary coupling constants. We have played a futile game, but at least we can ‘formally’ renormalize this particular gravity theory. No progress can be made with this approach unless there is more physics at hand.

It is at this point that we make recourse to ‘desperate measures’. As a first strike one might simply cut back to the desired end theory (supposedly Einstein gravity) by setting undesired renormalized coupling constants to zero. But perhaps we can do better, and argue our way forward.

Paradoxically perhaps for science, some lines of reasoning blossom and wane with the fashion of the day. Einstein himself (after several attempts) proposed $R$ gravity, abandoning higher derivative gravity on the grounds that such theories in general violate causality. As a matter of necessity we will follow the same route, despite the modern acceptance of these pathologies (as in string theory). We go on to also abandon $R$ coupling terms on the grounds that they violate the equivalence principle, and further insist that in the flat space-time limit the resultant theory is renormalizable in the ‘traditional sense’, so abandoning such terms as $\phi^6$.

These criteria guide us to set the renormalized coupling of all but a finite number of terms to zero, leading to:

$$L = -\sqrt{g} \left( -2\Lambda + R + \frac{1}{2}p^2 + \frac{1}{2}m^2\phi^2 + \frac{1}{4!}\lambda\phi^4 \right)$$

But even if we get to this stage, we have worries about the renormalization group pulling the coupling constants around. This is an open point to which I feel one of three things might happen:

1) The couplings, set to zero at a low energy scale, might reappear around the Plank scale. Whether the resulting theory then makes sense at this scale is a matter for dispute.
II) Certain extra coupling constants should be zero, in-order that the beta functions be zero, ensuring that all the couplings set to zero stay zero. This consistency condition could be the basis of a unification scheme.

III) Some work [Culumovic et al., 1990; Leblanc et al., 1991] suggests that for traditionally unrenormalizable theories a consistency condition arises that fixes the renormalization group parameter, supposedly at the Plank scale for gravity. This idea is very tentative and not conclusive.

In some sense we have come to the end of the trail and the scheme for quantizing gravity is now presented. However, I would like to continue on a diverse, but related track.

2 The Method

There is an ‘unsung hero’ in the guise of operator regularization [McKeon and Sherry, 1987; McKeon et al., 1987; McKeon et al., 1988; Mann, 1988; Mann et al., 1989; Culumovic et al., 1990; Shiekh, 1990]. Operator regularization is an n-loop generalization of the one loop Zeta function analytic continuation technique [Salam and Strathdee, 1975; Dowker and Critchley, 1976; Hawking, 1975]. As such, quantum field theory is finite from start to finish under this technique. Since one does not change the number of space-time dimensions, it even maintains more symmetries than dimensional regularization and calculations are almost identical (c.f. the added complexity of the Pauli-Villars regulator).

I am a keen supporter of the operator regularization technique, although I will level a criticism at its present formulation, which consists of the replacement of a divergent part by the analytic continuation given by:
\[ \Omega^{-m} = \lim_{\varepsilon \to 0} \frac{d^n}{d\varepsilon^n} \left( \frac{\varepsilon^n}{n!} \Omega^{-\varepsilon-m} \right) \]

where \( n \) is chosen sufficiently large to eliminate the infinities (the loop order is sufficient). Its use will be explicitly illustrated later. Actually, operator regularization is a bit of a misnomer, since it need not be applied to an operator and does not just regulate, but also renormalizes all in one.

However, under this form of the method all theories are finite and predictive (gravity included). A little playing shows that the above is simply an automated system for minimal subtraction, and this realized, the general form is easily located, and is given by:

\[ \Omega^{-m} = \lim_{\varepsilon \to 0} \frac{d^n}{d\varepsilon^n} \left( \left(1 + \alpha_1 \varepsilon + \ldots + \alpha_n \varepsilon^n\right) \frac{\varepsilon^n}{n!} \Omega^{-\varepsilon-m} \right) \]

(\textit{the alphas being ambiguous})

This form is not too powerful, and gravity must again be dealt with the desperate measures of before.

With all this machinery in hand one might want to walk through a simple example of a divergent one loop diagram. So begin with an investigation of a massive scalar theory in its own induced gravitational field, described by the Lagrangian in Euclidean space given by:

\[ L = -\sqrt{g} \left( -2\Lambda + R + \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} m^2 \phi^2 + \frac{1}{4!} \lambda \phi^4 \right) \]

The Euclidean Feynman rules (of which there are an infinite number) we explicitly list; the gauged graviton propagator being derived from the gravitational, \( R \), Lagrangian [Veltman, 1976], in the harmonic gauge\(^2\).

\(^2\)Postscript figures of the Feynman rules and diagrams are available from the author at shiekh@ictp.trieste.it
\[ FIG.1 = \frac{\delta_{\mu\alpha} \delta_{\nu\beta} + \delta_{\mu\beta} \delta_{\nu\alpha} - \delta_{\mu\nu} \delta_{\alpha\beta}}{p^2} \]

The scalar propagator is given by:

\[ FIG.2 = \frac{1}{p^2 + m^2} \]

and the first interaction vertex by:

\[ FIG.3 = \frac{1}{2} \delta_{\mu\nu} \left( p \cdot q - m^2 \right) - p_\mu q_\nu \]

etc., using units where \( \hbar = 1 \).

Although there are an infinite number of Feynman diagrams, only a finite number are used to any finite loop order.

Divergent one loop diagram example:

Set about a one loop investigation with matter particles on the external legs:

\[ FIG.4 = \int_{-\infty}^{\infty} \frac{d^4 l}{(2\pi)^4} \left( \frac{1}{l^2 + m^2} \right)^{\frac{1}{2}} \delta_{\mu\nu} \left( p \cdot q - m^2 \right) - p_\mu q_\nu \]

Expand out the indices to yield:

\[ = \int_{-\infty}^{\infty} \frac{d^4 l}{(2\pi)^4} \frac{1}{l^2 + m^2} \left( \frac{1}{l^2 + p^2} + 2m^2 p \cdot l - 2m^4 \right) \]

and then introduce the standard Feynman parameter ‘trick’:

\[
\frac{1}{D_1^{a_1} D_2^{a_2} ... D_k^{a_k}} = \frac{1}{\Gamma(a_1 + a_2 + ... a_k)} \int_0^1 \ldots \int_0^1 dx_1 \ldots dx_k \frac{\delta(1-x_1-...-x_k)x_1^{a_1-1}...x_k^{a_k-1}}{(D_1 x_1 + ... D_k x_k)^{a_1+...+a_k}}
\]

to yield:
\[
\int_{-\infty}^{\infty} \frac{d^4l}{(2\pi)^4} \int_{0}^{1} dx \frac{p^2l^2 + 2m^2p \cdot l - 2m^4}{[l^2 + m^2x + p^2(1-x) + 2l \cdot p(1-x)]^2}
\]

Remove divergences using operator regularization:

\[
\Omega^{-m} = \lim_{\varepsilon \to 0} \frac{d^n}{d\varepsilon^n} \left( (1 + \alpha_1 \varepsilon + \ldots + \alpha_n \varepsilon^n) \frac{\varepsilon^n}{n!} \Omega^{-\varepsilon - m} \right)
\]

\(n\) being chosen sufficiently large to cancel the infinities. For the case in hand \(n = 1\) is adequate.

\[
\Omega^{-2} = \lim_{\varepsilon \to 0} \frac{d}{d\varepsilon} \left( (1 + \alpha \varepsilon) \varepsilon \Omega^{-\varepsilon - 2} \right)
\]

This leads to:

\[
= \int_{0}^{1} dx \lim_{\varepsilon \to 0} \frac{d}{d\varepsilon} \int_{-\infty}^{\infty} \frac{d^4l}{(2\pi)^4} \left( (1 + \alpha \varepsilon) \frac{p^2l^2 + 2m^2p \cdot l - 2m^4}{[l^2 + m^2x + p^2(1-x) + 2l \cdot p(1-x)]^{\varepsilon + 2}} \right)
\]

Then performing the momentum integrations using [Ramond, 1990]:

\[
\int_{-\infty}^{\infty} \frac{d^2\omega}{(2\pi)^2} \frac{l_\mu}{(l^2 + M^2 + 2l \cdot p)^4} = \frac{1}{(4\pi)^3} \frac{\Gamma(A - \omega)}{M^2 - p^2} A^{-\omega}
\]

\[
\int_{-\infty}^{\infty} \frac{d^2\omega}{(2\pi)^2} \frac{l_\mu l_\nu}{(l^2 + M^2 + 2l \cdot p)^4} = -\frac{1}{(4\pi)^3} \frac{\Gamma(A - \omega)}{M^2 - p^2} A^{-\omega}
\]

\[
\int_{-\infty}^{\infty} \frac{d^2\omega}{(2\pi)^2} \frac{l_\mu l_\nu l_\sigma}{(l^2 + M^2 + 2l \cdot p)^4} = \frac{1}{(4\pi)^3} \frac{\Gamma(A - \omega)}{M^2 - p^2} A^{-\omega} \left[ P_\mu P_\nu \frac{\Gamma(A - \omega)}{M^2 - p^2} A^{-\omega} + \frac{\delta_{\mu\nu}}{2} \frac{\Gamma(A - \omega - 1)}{M^2 - p^2} A^{-\omega - 1} \right]
\]

yields the finite expression:

\[
= \frac{1}{(4\pi)^2} \int_{0}^{1} dx \lim_{\varepsilon \to 0} \frac{d}{d\varepsilon} \left( \frac{\varepsilon (1 + \alpha \varepsilon)}{\Gamma(\varepsilon + 2)} \left( \frac{p^4(1-x)^2 \Gamma(x)}{|m^2x + p^2x(1-x)|} + 2 \frac{p^2(1-x) \Gamma(x)}{|m^2x + p^2x(1-x)|} \right) - \frac{2m^2p^2(1-x) \Gamma(x)}{|m^2x + p^2x(1-x)|} \right)
\]

Doing the \(\varepsilon\) differential using:
\[
\lim_{\varepsilon \to 0} \frac{d}{d\varepsilon} \left( \frac{\varepsilon (1 + \alpha \varepsilon)}{\Gamma(\varepsilon + 2)} \left( a \frac{\Gamma(\varepsilon)}{\chi^\varepsilon} + b \frac{\Gamma(\varepsilon - 1)}{\chi^{\varepsilon-1}} \right) \right) = -a + (a - b\chi) (\alpha - \ln(\chi))
\]
yields:

\[
= \frac{1}{(4\pi)^2} \int_0^1 dx \left( \left( (2m^4 + 2m^2p^2 - p^4) + p^4x (4 - 3x) \right) \left( \ln( m^2 x + p^2 x (1 - x)) - \alpha \right) \right)
\]
and finally performing the \( x \) integration gives rise to the final result in Euclidean space, namely:

\[
FIG.4 = \frac{m^4}{(4\pi)^2} \left( \frac{3 + \frac{p^2}{m^2} + \frac{m^2}{p^2}}{\frac{1}{6} \frac{p^2}{m^2} + 2 \left( 1 + \frac{p^2}{m^2} \right) \left( \ln(\frac{m^2}{\mu^2}) - \alpha \right)} \right)
\]
where there is no actual divergence at \( p = 0 \), and it should be commented that the use of a computer mathematics package can in general greatly reduced ‘calculator’ fatigue. The renormalization group factor \( \mu \) appears on dimensional grounds.

3 Comments

The mere existence of a finite perturbative formulation of quantum gravity might give us reason to return once again to Einstein gravity in a quantum context. It would at least be a start.

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