Research Article

On the Coefficients of the Singularities of the Solution of Maxwell’s Equations near Polyhedral Edges

Boniface Nkemzi

Department of Mathematics, University of Buea, Buea, Cameroon

Correspondence should be addressed to Boniface Nkemzi; nkemzi@yahoo.com

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1. Introduction

Unlike the regularity analysis for elliptic boundary value problems in domains with geometric singularities, where there exists a unified theory based on the shift theorem (see [1–3]), the regularity analysis of the solution of Maxwell's equations has several interpretations. Most papers are concerned with the $H^1$-regularity for nonconvex domains, and it is shown that the main singularity is the gradient of singular functions associated with the Laplace equation; see, for example, Birman and Solomyak [4], Bonnet-Ben Dhia et al. [5], Moussaoui [6], Hazard and Lohrengel [7], Hazard and Lenoir [8], and Lohrengel [9]. Costabel in [10] went further to address the $H^{1/2}$-regularity for Lipschitz domains.

The $H^2$-regularity of the solution of Maxwell's equations has been considered, for example, by Nkemzi [11–14]. To be more precise, the asymptotic behaviour of the solution near axisymmetric edges and their efficient numerical treatment by means of the Fourier-finite-element method on graded meshes were analyzed in [11, 14]. In [12] the problem was considered in axisymmetric domains with conical points and the asymptotic behaviour of the solution near the conical points was analyzed. Here explicit representation formulas for the coefficients of the singularities were derived. In [13] Maxwell's equations in polygonal domains were considered and formulas for the coefficients of the singularities were derived. The present paper considers Maxwell's equations in three-dimensional domains with polyhedral edges and the main focus is on the explicit description of the coefficients of the singularities. Unlike in two-dimensional case and the case of conical points where the space of the singular solutions is finite dimensional and the coefficients of the singularities are some real numbers, in three-dimensional domains with edges the space of the singular solutions is infinite dimensional and the coefficients of the singularities are functions defined along the edges. Here the space of the singular solutions along polyhedral edges is completely described.

It should be noted that a more rigorous regularity analysis based on the shift theorem for the Maxwell equations in plane domains with corners and in polyhedral domains has been
2. The Boundary Value Problem and Functional Tools

We consider as model problem the electromagnetic fields \( \{E, B\} \) (\(E\) the electric field and \(B\) the magnetic field) of time-harmonic Maxwell’s equations in a simply connected and bounded domain \( \Omega \subset \mathbb{R}^3 \) with Lipschitz boundary \( \Gamma \) containing an isotropic and homogeneous medium subject to perfect conductor boundary conditions [5, 19, 20]:

\[
-\varepsilon \mu \omega E + \text{curl } B = \mu J + \mu \sigma E \quad \text{in } \Omega, \\
\omega B + \text{curl } E = 0 \quad \text{in } \Omega, \\
E \wedge n = 0, \\
B \cdot n = 0, \\
on \Gamma,
\]

where the domain related parameters \( \varepsilon > 0, \mu > 0, \) and \( \sigma > 0 \) are, respectively, the dielectric permittivity, magnetic permeability, and electric conductivity, \( J \) is a given divergence-free current density, that is, \( \text{div} J = 0, \omega \neq 0 \) is the pulsation of the electromagnetic fields, \( i \) is the imaginary unity, and \( n \) denotes the unit outward normal on the boundary \( \Gamma \).

If we suppose temporarily that the vector function \( J \) is sufficiently smooth, then system (1) can be written as two decoupled systems in terms of the magnetic field \( B \) and the electric field \( E \) as follows:

\[
\text{curl curl } B - \alpha^2 B = h \quad \text{in } \Omega, \\
B \cdot n = 0 \quad \text{on } \Gamma, \\
\text{curl curl } E - \alpha^2 E = f \quad \text{in } \Omega, \\
E \wedge n = 0 \quad \text{on } \Gamma,
\]

where \( h = \mu \text{curl } J, \alpha^2 = \omega^2 \mu (\varepsilon - i \sigma / \omega), \) and \( f = -i \omega \mu J \).

It follows directly from (1), (2), and (3) that the solution of problem (2) can be derived from the solution of problem (3) and vice versa. Thus it suffices to solve or analyze only one of the problems. Subsequently we will dedicate our analysis to the boundary value problem (3). For the Hilbert space formulation of (3) we introduce the function spaces; see [19, 21]:

\[
\mathcal{H} (\text{curl}, \Omega) = \{v \in (L^2_\cdot(\Omega))^3 : \text{curl } v \in (L^2_\cdot(\Omega))^3\}, \\
\mathcal{H}_0 (\text{curl}, \Omega) = \{v \in \mathcal{H} (\text{curl}, \Omega) : v \wedge n = 0 \text{ on } \Gamma\},
\]

equipped with the norm

\[
\|v\|_{\mathcal{H} (\text{curl}, \Omega)} = \left\| \|v\|_{(L^2_\cdot(\Omega))^3}^2 + \|\text{curl } v\|_{(L^2_\cdot(\Omega))^3}^2 \right\|^{1/2}.
\]

The variational formulation of problem (3) is as follows.

Find \( u \in \mathcal{H}_0 (\text{curl}, \Omega) \) such that

\[
a (u, v) := (\text{curl } u, \text{curl } v) - \alpha^2 (u, v) = (f, v) = : f (v),
\]

\( \forall v \in \mathcal{H}_0 (\text{curl}, \Omega), \)
where $(\cdot, \cdot)$ denotes the usual scalar product in the Hilbert space $L^2(\Omega)$ of complex-valued functions. The well-posedness of problem (6) is addressed by the following theorem; see also [19, 22, 23].

**Theorem 1.** Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with at least Lipschitz continuous boundary $\Gamma$. Suppose $\alpha \neq 0$ and $\alpha^2$ is not an eigenvalue of the operator curl curl with electric boundary condition. Then, for any $f \in (L^2(\Omega))^3$, there exists a unique solution $u \in \mathcal{H}_0(\text{curl}, \Omega)$ of the variational problem (6). Moreover, the solution satisfies the estimate

$$
\|u\|_{(L^2(\Omega))^3}^2 + \|\text{curl} u\|_{(L^2(\Omega))^3}^2 \leq C \|f\|_{(L^2(\Omega))^3}^2.
$$

We will always assume where necessary, without explicitly stating so, that the conditions of Theorem 1 are satisfied.

We observe that the operator $E \mapsto \text{curl} \text{curl} E - \alpha^2 E$ from (2) and (3) is not elliptic. However, a widely used alternative formulation of the boundary value problem (3) is the so-called regularized formulation of the Maxwell equations; see [5, 7–9]. In fact, it is easily seen that the boundary value problem (3) is equivalent to the boundary value problem

$$
\text{curl} \text{curl} E - \text{grad} \text{div} E - \alpha^2 E = f \quad \text{in} \Omega,
$$

$$
E \wedge n = 0 \quad \text{on} \Gamma,
$$

$$
\text{div} E = 0 \quad \text{on} \Gamma,
$$

in the sense that the solution of (3) solves (8) and vice versa. We notice that the operator $E \mapsto \text{curl} \text{curl} E - \text{grad} \text{div} E - \alpha^2 E$ is elliptic. The associated Hilbert space formulation for the boundary value problem (8) is as follows.

Find $u \in \mathcal{H}_0(\text{curl}, \text{div}, \Omega)$ such that

$$
a(u, v) = f(v), \quad \forall v \in \mathcal{H}_0(\text{curl}, \text{div}, \Omega),
$$

where

$$
a(u, v) := (\text{curl} u, \text{curl} v) + (\text{div} u, \text{div} v) - \alpha^2 (u, v),
$$

$$
f(v) = (f, v),
$$

$$
\mathcal{H}_0(\text{curl}, \text{div}, \Omega)
$$

$$
= \{ v \in \mathcal{H}_0(\text{curl}, \Omega) : \text{div} v \in L^2(\Omega) \},
$$

$$
\|v\|_{\mathcal{H}_0(\text{curl}, \text{div}, \Omega)}
$$

$$
= \left( \|v\|_{(L^2(\Omega))^3}^2 + \|\text{curl} v\|_{(L^2(\Omega))^3}^2 + \|\text{div} v\|_{L^2(\Omega)}^2 \right)^{1/2}.
$$

The following theorem addresses the question of well-posedness of problem (9); see also [5, 9].

**Theorem 2.** Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with at least Lipschitz continuous boundary $\Gamma$. Suppose $\alpha \neq 0$ and $\alpha^2$ is not an eigenvalue of the Dirichlet-Laplace operator on $\Omega$.

Then, for any $f \in (L^2(\Omega))^3$, there exists a unique solution $u \in \mathcal{H}_0(\text{curl}, \text{div}, \Omega)$ of the variational problem (9). Moreover, the solution satisfies the estimate

$$
\|u\|_{\mathcal{H}_0(\text{curl}, \text{div}, \Omega)}^2 + \|\text{curl} u\|_{\mathcal{H}_0(\text{curl}, \Omega)}^2 + \|\text{div} u\|_{L^2(\Omega)}^2
$$

$$
\leq C \|f\|_{(L^2(\Omega))^3}^2.
$$

The rest of this paper is dedicated to a rigorous regularity analysis of the solution of time-harmonic Maxwell equations in simply connected and bounded domains $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, with Lipschitz boundary containing an isotropic and homogeneous medium subject to perfect conductor boundary conditions. We will systematically use the regularized formulation (8) and all derivatives should always be understood in the sense of distributions. First we state here one regularity result that is frequently quoted in the literature; see [21]. We will use the notation

$$
H_N(\Omega) = \left\{ v \in \left( H^1(\Omega) \right)^d : v \wedge n = 0 \text{ on } \Gamma \right\}.
$$

**Theorem 3.** Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with boundary $\Gamma$. If $\Gamma$ is of class $C^{1,1}$ or if $\Omega$ is a convex polygon in $\mathbb{R}^2$ or a convex polyhedron in $\mathbb{R}^3$, then the relation

$$
H_N(\Omega) = \mathcal{H}_0(\text{curl}, \text{div}, \Omega)
$$

holds and on these spaces the norms $\| \cdot \|_{(H^1(\Omega))^d}$ and $\| \cdot \|_{\mathcal{H}_0(\text{curl}, \text{div}, \Omega)}$ are equivalent.

On the other hand, if $\Omega \subset \mathbb{R}^d$ is a nonconvex polygon or polyhedron, then the space $H_N(\Omega)$ is a proper closed subset of the space $\mathcal{H}_0(\text{curl}, \text{div}, \Omega)$ and the following result holds; see [4, 6].

**Theorem 4.** The space $\mathcal{H}_0(\text{curl}, \text{div}, \Omega)$ can be split as a direct sum of the form

$$
\mathcal{H}_0(\text{curl}, \text{div}, \Omega) = H_N(\Omega) \oplus \text{grad} S,
$$

where $S$ denotes the space of functions spanned by the singular functions associated with the Dirichlet boundary value problem for the Laplace equation in $\Omega \subset \mathbb{R}^d$.

An immediate consequence of the regularity Theorems 3 and 4 is that, in nonconvex polygons or polyhedrons, the weak solution of the boundary value problem (8) does not belong to the space $H^1$ and can therefore not be approximated by means of the usual $H^1$-conforming nodal finite-element method.

### 3. Corner and Edge Singularities for Maxwell’s Equations

In this section we formulate the main results of this paper. The proofs which are very lengthy in nature will be carried out in subsequent sections.
3.1. Corner Singularities for Maxwell’s Equations. Here we consider the electric field \( \mathbf{u} \) of time-harmonic Maxwell’s equations in a simply connected and bounded domain \( \Omega \subset \mathbb{R}^2 \) with Lipschitz boundary \( \Gamma \), formally the variational solution of the boundary value problem

\[
\text{curl} \cdot \text{curl} \mathbf{u} - \text{grad} \cdot \text{div} \mathbf{u} - \alpha^2 \mathbf{u} = \mathbf{f} \quad \text{in} \ \Omega, \\
\mathbf{u} \wedge \mathbf{n} = 0 \quad \text{on} \ \Gamma, \\
\text{div} \mathbf{u} = 0 \quad \text{on} \ \Gamma,
\]

where \( \mathbf{f} \in (L_2(\Omega))^2 \) and the parameter \( \alpha \neq 0 \) are given.

Now, suppose that the boundary \( \Gamma \) of \( \Omega \) consists of finitely many disjoint analytic arcs \( \Gamma_j, j = 1, \ldots, J \), such that \( \Gamma = \bigcup_{j=1}^J \Gamma_j \), where the segments \( \Gamma_j \) are numbered according to the positive orientation, that is, in anticlockwise direction. Let the endpoints of each \( \Gamma_j \) be denoted by \( A_j \) and let the solid angle at \( A_j \) be denoted by \( \omega_j \), where \( 0 < \omega_j < 2\pi \). We denote by \( r_j \) and \( \theta_j \) (resp., \( x_j \) and \( y_j \)) local polar (resp., Cartesian) coordinates attached to the vertex \( A_j \), such that

\[
x_j = r_j \cos(\theta_j), \\
y_j = r_j \sin(\theta_j),
\]

that is, \( \Gamma_j \) is supported by the line \( \theta_j = \omega_j \) and \( \Gamma_{j+1} \) is on the line \( \theta_j = 0 \). Suppose that the domain \( \Omega \) coincides near each singular point \( A_j \) with a circular sector \( \overline{K}_j \) with radius \( R_j \) and angle \( \omega_j \); that is,

\[
\overline{K}_j = \{(x, y) \in \Omega : x_j = r_j \cos(\theta_j), \ y_j = r_j \sin(\theta_j), \ 0 < r_j < R_j, \ 0 < \theta_j < \omega_j\}.
\]

The boundary \( \partial \overline{K}_j \) will be represented subsequently as \( \partial \overline{K}_j = \Gamma_j \cup \Gamma_{j+1} \cup \Gamma_{j0} \), where \( \Gamma_j \) is given by

\[
\Gamma_j = \{(x, y) : x_j = r_j \cos(\theta_j), \ y_j = r_j \sin(\theta_j), \ \theta_j = \omega_j, \ 0 < r_j < R_j\},
\]

\[
\Gamma_{j+1} = \{(x, y) : x_j = r_j \cos(\theta_j), \ y_j = r_j \sin(\theta_j), \ 0 < r_j < R_j, \ \theta_j = 0\},
\]

\[
\Gamma_{j0} = \{(x, y) : x_j = r_j \cos(\theta_j), \ y_j = r_j \sin(\theta_j), \ r_j = R_j, \ 0 < \theta_j < \omega_j\}.
\]

We define with respect to the vertex \( A_j \) a smooth truncation function \( \eta_j \in \mathcal{D}(\overline{\Omega}) \) which depends only on the distance \( r_j \) from \( A_j \) by

\[
\eta_j(x_j, y_j) = \begin{cases} 
1, & \text{for } 0 \leq r_j \leq \frac{R_j}{3}, \\
0, & \text{for } r_j \geq \frac{2R_j}{3}, \\
0 \leq \eta_j(r_j) \leq 1,
\end{cases}
\]

where \( R_j \) is taken from (17); that is, \( \text{supp}(\eta_j) \subset \overline{K}_j \). Furthermore, we define on each sector neighbourhood \( \overline{K}_j \) of the vertex \( A_j \) the functions

\[
f_{1j}(x_j, y_j) = \eta_j(f_1 + \alpha^2 u_1) - u_1 \Delta x_{x_1} \eta_j - 2V x_{x_1} \eta_j - \cdot \nabla x_{x_1} u_1, \\
f_{2j}(x_j, y_j) = \eta_j(f_2 + \alpha^2 u_2) - u_2 \Delta x_{x_1} \eta_j - 2V x_{x_1} \eta_j - \cdot \nabla x_{x_1} u_2,
\]

where \( f = (f_1, f_2) \), \( u = (u_1, u_2) \) and the parameter \( \alpha \) are taken from (15), \( \eta_j \) is as defined in (19), and

\[
\Delta x_{x_1} = \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right); \\
V x_{x_1} = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right).
\]

Our main result on corner singularity is the following.

**Theorem 5.** For each \( f = (f_1, f_2) \in (L_2(\Omega))^2 \) and \( \alpha \neq 0 \), let \( \mathbf{u} \in \mathcal{H}_0(\text{curl, div,} \ \Omega) \) be the variational solution of the boundary value problem (15). Let \( \lambda_{jk} = k\pi/\omega_j \), \( k \in \mathbb{N}, \omega_j \neq \pi, \ j = 1, \ldots, J \). If \( \lambda_{jk} \neq 2, k \in \mathbb{N}, j = 1, \ldots, J \), then there exist coefficients \( \gamma_{jk} \) such that the solution \( \mathbf{u} \) can be split as a sum in the form

\[
\mathbf{u} = \mathbf{w} + \sum_{j=1}^J \sum_{0 < \lambda_{jk} < 2} \gamma_{jk} r_j^{\lambda_{jk}-1} \left( \sin(\lambda_{jk} - 1) \theta_j \right),
\]

where \( \mathbf{w} = (w_1, w_2) \in (H^2(\Omega))^2 \). The coefficients \( \gamma_{jk} \) of the asymptotic expansion (22) are given explicitly by the formula

\[
\gamma_{jk} = \frac{R_j^{21-\lambda_{jk}}}{2\omega_j (\lambda_{jk} - 1)} \int_{\overline{K}_j} \left( f_{1j}(x_j, y_j) \sin((\lambda_{jk} - 1) \theta_j) + f_{2j}(x_j, y_j) \cos((\lambda_{jk} - 1) \theta_j) \right) r_j^{\lambda_{jk}-1} \, dx_1 \, dx_2,
\]
where the function $f_j = (f_{1j}, f_{2j})$ is defined in (20). The constants $R_j$ and $\omega_j$ and the local Cartesian and polar coordinates $x_j$, $y_j$ and $r_j$, $\theta_j$ are as specified in (17) and (16). Moreover, there exists a constant $C > 0$ independent of $f$ and $u$ such that

$$|y_j| + \|w_H(f_j, u_j)\|_{L^2(\Omega)} \leq C \|f\|_{L^2(\Omega)}.$$  \hspace{1cm} (24)

The proof of Theorem 5 will be carried out in Section 4; see Theorem 9.

3.2. Edge Singularities for Maxwell’s Equations. Let $Q \subset \mathbb{R}^3$ be a simply connected and bounded domain with Lipschitz boundary $\partial Q$. For $f \in (L_2(Q))^3$ and $\alpha \neq 0$, we consider the variational solution $u = (u_1, u_2, u_3) \in H_0(\text{curl, div}, Q)$ of the boundary value problem

$$\text{curl curl } u - \alpha^2 u = f \quad \text{in } Q,$$

$$u \wedge n = 0 \quad \text{on } \partial Q, \hspace{1cm} (25)$$

$$\text{div } u = 0 \quad \text{on } \partial Q.$$

Since we are interested only in the asymptotic behaviour of the solution $u$ near straight edges of the domain $Q$, we can assume, without loss of generality, that the domain $Q$ is a prismatic cylinder; that is, it has the form $Q = \Omega \times (0, l)$ with a real constant $l > 0$ and a bounded domain $\Omega \subset \mathbb{R}^2$ with piecewise smooth boundary $\Gamma$ and such that for each $(x_1, x_2, x_3) \in Q$, $(x_1, x_2) \in \Omega$, and $x_3 \in (0, l)$. In this way we can use the same notations as in Section 3.1 for $\Omega$. In particular, the edges of the domain $Q$ are $E_j = A_j \times (0, l)$ and the measure of the interior angle along the edge $E_j$ is $\omega_j$, $j = 1, \ldots, J$. We associate with each edge $E_j$ a wedge $G_j = \mathbb{R}_+ \times (0, l)$, where $\mathbb{R}_+$ is as defined in (17). Further we introduce on $G_j$ the functions

$$f_{1j}^* (x, y, x_3) = \eta_j (f_1 + \alpha^2 u_1) - u_1 \Delta_{x_1, x_2} \eta_j - 2 \nabla_{x_1, x_2} \eta_j \cdot \nabla_{x_1, x_2} u_1$$

$$- \frac{nn}{l} \left( \frac{\eta_j \mu_1}{r_j} \right)$$

$$- \frac{nn}{l} \left( \cos \theta_j \frac{\partial (\eta_j u_1)}{\partial x_1} + \sin \theta_j \frac{\partial (\eta_j u_1)}{\partial x_2} \right)$$

$$- \left( \frac{nn}{l} \right)^2 (\eta_j u_1),$$

$f_{2j}^* (x, y, x_3) = \eta_j (f_2 + \alpha^2 u_2) - u_2 \Delta_{x_1, x_2} \eta_j - 2 \nabla_{x_1, x_2} \eta_j \cdot \nabla_{x_1, x_2} u_2$

$$- \frac{nn}{l} \left( \frac{\eta_j \mu_2}{r_j} \right)$$

where the functions $f = (f_1, f_2, f_3)$, $u = (u_1, u_2, u_3)$, and the parameter $\alpha$ are taken from (25), $\eta_j$ is from (19), and $\Delta_{x_1, x_2}$, $\nabla_{x_1, x_2}$ are as defined in (21). Obviously $f_j = (f_{1j}^*, f_{2j}^*, f_{3j}^*) \in (L_2(G_j))^3$.

Our main result on edge singularity is the following.

Theorem 6. For $f \in (L_2(Q))^3$ and $\alpha \neq 0$, let $u \in H_0(\text{curl, div}, Q)$ be the variational solution of the boundary value problem (25). Let $\lambda_{jk} = k \pi / \omega_j$, $k \in \mathbb{N}$, $\omega_j \neq \pi$, $j = 1, \ldots, J$. If $\lambda_{jk} \neq 2$, $k \in \mathbb{N}$, $j = 1, \ldots, J$, then there exist unique functions $\Psi_{jk} \in H^{1,\pi}(0, l)$ and $\Psi_{jk} \in H^{1,\pi}(0, l)$ such that the solution $u \in H_0(\text{curl, div}, Q)$ can be split into a regular and a singular part in the form

$$u = (\omega_2, \omega_1, \omega_3) + (s_1, s_2, s_3),$$

$$w \in (L_2(Q))^3,$$

$$s_1 (x_1, x_2, x_3)$$

$$= \sum_{j=1}^{J} \sum_{0 < \lambda_j < 2 \lambda_{jk} \neq 1} \Psi_{jk} (x_3)$$

$$* T_j (r_j, x_3) r_j^{\lambda_{jk} - 1} \sin \left( (\lambda_{jk} - 1) \theta_j \right),$$

$$s_2 (x_1, x_2, x_3)$$

$$= \sum_{j=1}^{J} \sum_{0 < \lambda_j < 2 \lambda_{jk} \neq 1} \Psi_{jk} (x_3)$$

$$* T_j (r_j, x_3) r_j^{\lambda_{jk} - 1} \cos \left( (\lambda_{jk} - 1) \theta_j \right),$$
\[ s_3(x_1, x_2, x_3) = \sum_{j=1}^{l} \Psi_j(x_3) \ast T_{3j}(r_j, x_3) r_j^{\lambda_j} \sin (\lambda_j \theta_j) \]

if \( 0 < \lambda_j 1 < 1. \) (27)

The functions \( T_j \) and \( T_{3j} \) are fixed kernels defined by

\[ T_j(r_j, x_3) := \sum_{n=1}^{\infty} e^{-i \pi n \rho r_j} \sin \frac{n \pi x_3}{l}, \]

\[ T_{3j}(r_j, x_3) := \sum_{n=1}^{\infty} e^{-i \pi n \rho r_j} \cos \frac{n \pi x_3}{l}. \] (28)

The coefficients \( \Psi_{jk} \) and \( \Psi_j \) of the asymptotic expansion (27) can be expressed in Fourier series in the form

\[ \Psi_{jk}(x_3) = \sum_{n=1}^{\infty} \gamma_{jkn} \sin \frac{n \pi x_3}{l}, \]

\[ \Psi_j(x_3) = \frac{1}{2} \rho_j + \sum_{n=1}^{\infty} \gamma_{jn} \cos \frac{n \pi x_3}{l}, \] (29)

where the Fourier coefficients \( \gamma_{jkn} \) and \( \gamma_{jn} \) are given explicitly by the formulas

\[ \gamma_{jkn} = \frac{1}{\omega_j \lambda_j - 1} \int_{\Gamma_j} \sin \left( \frac{n \pi x_3}{l} \right) \cdot e^{(n \pi n)1/2} \left\{ f_{1j}^* (x_j, y_j, x_3) \sin \left( (\lambda_j - 1) \theta_j \right) \right. \]

\[ + f_{2j}^* (x_j, y_j, x_3) \cos \left( (\lambda_j - 1) \theta_j \right) \}

\[ \left. \cdot r_j^{\lambda_j - 1} \right|_x dx_3 \right), \]

\[ \gamma_{jn} = \frac{1}{\omega_j \lambda_j} \int_{\Gamma_j} \cos \left( \frac{n \pi x_3}{l} \right) e^{(n \pi n)1/2} f_{3j}^* (x_j, y_j, x_3) \]

\[ \cdot \sin \left( \lambda_j \theta_j \right) r_j^{\lambda_j - 1} dx_3 \right|_x dx_3. \] (30)

Here, the function \( f_j^* = (f_{1j}^*, f_{2j}^*, f_{3j}^*) \) is as defined in (26). The constants \( R_j \) and \( \omega_j \) and the local Cartesian and polar coordinates \( x_j, y_j, r_j, \) and \( \theta_j \) are as defined in (17) and (16). In (27) the symbol \( \ast \) denotes convolution product in the variable \( x_3; \) that is,

\[ \Psi_{jk}(x_3) \ast T_j(r_j, x_3) = \sum_{n=1}^{\infty} \gamma_{jkn} e^{-(n \pi n)1/2} \sin \frac{n \pi x_3}{l}, \]

\[ \Psi_j(x_3) \ast T_{3j}(r_j, x_3) = \frac{1}{2} \rho_j + \sum_{n=1}^{\infty} \gamma_{jn} e^{-(n \pi n)1/2} \cos \frac{n \pi x_3}{l}. \] (31)

The proof of Theorem 6 will be carried out in Section 5; see Theorem 24.

### 4. Maxwell’s Equations in Two-Dimensional Domains with Corners

In this section, we consider in greater detail the structure of the solution of the Maxwell equations (15) in two-dimensional domains with corners and show how the results of Theorem 5 are derived.

#### 4.1. Maxwell’s Equations in a Bounded Sector

For purely mathematical reasons we consider first a slightly modified boundary value problem for the Maxwell equations in a circular sector; see Figure 1. The results of this subsection are largely found in Nkemzi [13] and will be kept very brief.

Let \( \overline{K} \) denote a circular sector in \( \mathbb{R}^2 \) with radius \( R, \) interior angle \( \omega \neq \pi, \) and boundary \( \partial \overline{K} = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2; \) see Figure 1. We assume that the Cartesian coordinate system of \( x_1 \) and \( x_2 \) is positioned such that the vertex \( A \) of \( \overline{K} \) coincides with the origin and the side \( \Gamma_1 \) is supported by the \( x_1 \)-axis.

For \( \overline{\mathbf{u}} = (u_1, u_2) \in \mathcal{H}_{d}(\text{curl}, \text{div}, \overline{K}) \) of the boundary value problem

\[ \text{curl curl } \overline{\mathbf{u}} - \text{grad div } \overline{\mathbf{u}} = \overline{\mathbf{f}} \quad \text{in } \overline{K}, \]

\[ \overline{\mathbf{u}} \cdot n = 0 \quad \text{on } \Gamma_1 \cup \Gamma_2, \]

\[ \text{div } \overline{\mathbf{u}} = 0 \quad \text{on } \Gamma_1 \cup \Gamma_2, \]

\[ \overline{\mathbf{u}} = 0 \quad \text{on } \Gamma_0. \] (32)

Local polar coordinates \( r \) and \( \theta \) in \( \overline{K} \) with respect to the vertex \( A \) are related to the Cartesian coordinates \( x_1 \) and \( x_2; \) namely,

\[ x_1 = r \cos \theta, \]

\[ x_2 = r \sin \theta, \] (33)

\[ 0 < r < R, \quad 0 < \theta < \omega. \]
Accordingly, the sector domain $\mathcal{K}$ is transformed by the one-to-one mapping into the rectangle

$$\mathcal{K} = \{(r, \theta) : 0 < r < R_0, \ 0 < \theta < \omega\}$$

in the polar coordinate system. By the transformation (33), each function $\tilde{u} = \tilde{u}(x_1, x_2)$ defined on $\mathcal{K}$ is mapped uniquely to some function $u = u(r, \theta)$ defined on $\mathcal{K}$ by

$$u(r, \theta) = \tilde{u}(r \cos \theta, r \sin \theta).$$

Similarly, each vector field $\tilde{u} = (u_1(x_1, x_2), u_2(x_1, x_2))$ defined on $\mathcal{K}$ is mapped uniquely to some vector field $u = (u_r(r, \theta), u_\theta(r, \theta))$ defined on $\mathcal{K}$ by

$$u_r(r, \theta) = u_1(x_1, x_2) \cos \theta + u_2(x_1, x_2) \sin \theta,$$

$$u_\theta(r, \theta) = -u_1(x_1, x_2) \sin \theta + u_2(x_1, x_2) \cos \theta.$$ (36)

The boundary value problem (32) can be solved explicitly. In fact, we have the following result which can be verified by direct substitution.

**Lemma 7.** The weak solution $\tilde{u} = u = (u_r(r, \theta), u_\theta(r, \theta))$ of the boundary value problem (32) can be represented in Fourier series in the form

$$u_r(r, \theta) = \sum_{k=1}^{\infty} u_{rk}(r) \sin \lambda_k \theta,$$

$$u_\theta(r, \theta) = \sum_{k=1}^{\infty} u_{\theta k}(r) \cos \lambda_k \theta,$$ (37)

where the Fourier coefficients \{\(u_k = (u_{rk}, u_{\theta k}) : k \in \mathbb{N}\)} satisfy the relations

$$u_{rk}(r) = -\frac{r^{\lambda_k-1}}{4(\lambda_k - 1) R^{2\lambda_k-2}}$$

$$\cdot \int_0^R (f_{rk}(r) + f_{\theta k}(r)) r^{\lambda_k} d\tau + \frac{r^{\lambda_k+1}}{4(\lambda_k + 1) R^{2\lambda_k+2}}$$

$$\cdot \int_0^R (-f_{rk}(r) + f_{\theta k}(r)) r^{\lambda_k+1} d\tau + \frac{r^{-(\lambda_k+1)}}{4(\lambda_k + 1)}$$

$$\cdot \int_0^R (f_{rk}(r) - f_{\theta k}(r)) r^{-\lambda_k} d\tau + \frac{r^{1-\lambda_k}}{4(\lambda_k - 1)}$$

$$\cdot \int_0^R (f_{rk}(r) + f_{\theta k}(r)) r^{\lambda_k} d\tau + \frac{r^{1+\lambda_k}}{4(\lambda_k + 1)}$$

$$\cdot \int_r^R (f_{rk}(r) - f_{\theta k}(r)) r^{-\lambda_k} d\tau, \ \lambda_k \neq 1.$$ (38)

Using the explicit representation formulas (37)-(38) for the solution of the boundary value problem (32) and taking into account relation (36) one can derive various regularity properties for the solution.

The main result of this subsection is the following; see, for example, [13, 15, 16], for the proof.

**Theorem 8.** Let $\mathcal{K}$ be a circular sector with angle $\omega \in (0, 2\pi)$, $\omega \neq \pi$. Let $\lambda_k = k\pi/\omega$, $k \in \mathbb{N}$. Then for each $\tilde{f} \in (L_2(\mathcal{K}))^2$ the solution $\tilde{u} \in \mathcal{Z}_0(\text{curl}, \text{div}, \mathcal{K})$ of the boundary value problem (32) has the following additional regularity properties:

(a) There exists a constant $C > 0$ independent of $\tilde{f}$ and $\tilde{u}$ such that the

$$\|\text{div} \tilde{u}\|_{L_2(\mathcal{K})} \leq C \|\tilde{f}\|_{(L_2(\mathcal{K}))^2}.$$ (39)

That is, the condition $\text{div} \tilde{u} \in L_2(\mathcal{K})$ is always satisfied.

(b) If $\lambda_k > 1$, $k \in \mathbb{N}$ (i.e., $\mathcal{K}$ is convex), then $\tilde{u} \in (H^1(\mathcal{K}))^2$ and there exists a constant $C > 0$ independent of $\tilde{f}$ and $\tilde{u}$ such that

$$\|\tilde{u}\|_{H^1(\mathcal{K})} \leq C \|\tilde{f}\|_{(L_2(\mathcal{K}))^2}.$$ (40)

(c) If $\lambda_k > 2$, $k \in \mathbb{N}$ (i.e., $0 < \omega < \pi/2$), then $\tilde{u} \in (H^2(\mathcal{K}))^2$ and there exists a constant $C > 0$ independent of $\tilde{f}$ and $\tilde{u}$ such that

$$\|\tilde{u}\|_{H^2(\mathcal{K})} \leq C \|\tilde{f}\|_{(L_2(\mathcal{K}))^2}.$$ (41)
If $\lambda_k \neq 2, k \in \mathbb{N}$, then $\mathbf{u}$ can be split into a regular and a singular part in the form
\[
\mathbf{u} = \mathbf{w} + \sum_{\lambda_k < 2} \gamma_k \mathbf{v}_k \quad \text{with} \quad \mathbf{w} \in (H^2(\mathbb{R}))^2.
\]

The coefficients $\gamma_k$ are given explicitly by the formula
\[
\gamma_k = -\frac{R_{2(1-\lambda_k)}}{2\omega (\lambda_k - 1)} \int_{\mathbb{R}} \left( f_1(x_1, x_2) \sin((\lambda_k - 1) \theta) + f_2(x_1, x_2) \cos((\lambda_k - 1) \theta) \right) r^{\lambda_k - 1} dx_1 dx_2.
\]

Moreover, there exists a constant $C > 0$ independent of $f$ such that
\[
\|\gamma_k\|_{(H^2(\mathbb{R}))^2} \leq C \|f\|_{(L_2(\mathbb{R}))^2}.
\]

4.2. Maxwell's Equations in Plane Domains with Corners. We can now make definite statements on the regularity properties of the solution $\mathbf{u} \in \mathcal{H}(\text{curl}, \text{div}, \Omega)$ of the Maxwell boundary value problem (15) in bounded domains $\Omega \subset \mathbb{R}^2$ with piecewise smooth boundary $\Gamma$. We will use the same notations as in Section 3.1.

Let $\mathbf{u} \in \mathcal{H}(\text{curl}, \text{div}, \Omega)$ be the solution of (15). Then the function $\mathbf{u}_j := \eta_j \mathbf{u}$, where $\eta_j$ is the smooth truncation function from (19), belongs to the space $\mathcal{H}(\text{curl}, \text{div}, \mathbb{R}^2_{\Gamma_j})$ and is the unique weak solution of the boundary value problem
\[
\text{curl curl } \mathbf{u}_j - \text{grad div } \mathbf{u}_j = f_j \quad \text{in } \mathbb{R}^2_{\Gamma_j},
\]
\[
\mathbf{u}_j \wedge \mathbf{n} = 0 \quad \text{on } \Gamma_j \cup \Gamma_{j+1},
\]
\[
\text{div } \mathbf{u}_j = 0 \quad \text{on } \Gamma_j \cup \Gamma_{j+1},
\]
\[
\mathbf{u}_j = 0 \quad \text{on } \Gamma_0,
\]
where the function $f_j = (f_{1j}(x_j, y_j), f_{2j}(x_j, y_j)) \in (L_2(\mathbb{R}^2_{\Gamma_j}))^2$ is as defined in (20).

We observe that problems (45) and (32) are similar and therefore their solutions have the same regularity properties as described in Theorem 8. On the other hand, the solution $\mathbf{u}_j \in \mathcal{H}(\text{curl}, \text{div}, \mathbb{R}^2_{\Gamma_j})$ of problem (45) coincides near the vertex $A_j$ of $\Omega$ to the solution $\mathbf{u} \in \mathcal{H}(\text{curl}, \text{div}, \Omega)$ of problem (15). Thus the two solutions have the same asymptotic behaviour near the vertex $A_j$. Taking into consideration the fact that singularity is a local property and the technique for coupling local and global regularity properties (see [2, 18]), we obtain directly from Theorem 8 the following properties for the solution $\mathbf{u} \in \mathcal{H}(\text{curl}, \text{div}, \Omega)$ of problem (15).

Theorem 9. For each $f \in (L_2(\Omega))^2$, let $\mathbf{u} \in \mathcal{H}(\text{curl}, \text{div}, \Omega)$ be the solution of the boundary value problem (15). Let $\lambda_{j_k} = k\pi/\omega_j$, $k \in \mathbb{N}$, $j = 1, \ldots, J$, and $\omega_j \neq \pi$. Then the solution $\mathbf{u}$ has the following additional regularity properties:

(a) $\text{div } \mathbf{u} \in L_2(\Omega)$ and there exists a constant $C > 0$ independent of $f$ and $\mathbf{u}$ such that
\[
\|\text{div } \mathbf{u}\|_{L_2(\Omega)} \leq C \|f\|_{(L_2(\Omega))^2}.
\]

(b) If $\lambda_{j_k} > 1, k \in \mathbb{N}, j = 1, \ldots, J$ (i.e., $\Omega$ is convex), then $\mathbf{u} \in H^2(\Omega)$ and there exists a constant $C > 0$ independent of $f$ and $\mathbf{u}$ such that
\[
\|\mathbf{u}\|_{H^2(\Omega)} \leq C \|f\|_{(L_2(\Omega))^2}.
\]

(c) If $\lambda_{j_k} > 2, k \in \mathbb{N}, j = 1, \ldots, J$ (i.e., $0 < \omega_j < \pi/2$ for all $j$), then $\mathbf{u} \in (H^2(\Omega))^2$ and there exists a constant $C > 0$ independent of $f$ such that
\[
\|\mathbf{u}\|_{H^2(\Omega)} \leq C \|f\|_{(L_2(\Omega))^2}.
\]

(d) If $\lambda_{j_k} \neq 2, k \in \mathbb{N}, j = 1, \ldots, J$, then the solution $\mathbf{u}$ can be split into a regular and a singular part in the form
\[
\mathbf{u} = \mathbf{w} + \sum_{j=1}^{J} \sum_{\lambda_{j_k} < 2} \gamma_{j_k} \mathbf{v}_{j_k} \quad \text{with } \mathbf{w} \in (H^2(\Omega))^2.
\]

The coefficients $\gamma_{j_k}$ are given explicitly by the formula
\[
\gamma_{j_k} = -\frac{R_{2(1-\lambda_{j_k})}}{2\omega_j (\lambda_{j_k} - 1)} \int_{\mathbb{R}^2_{\Gamma_j}} \left( f_{1j}(x_j, y_j) \sin((\lambda_{j_k} - 1) \theta_j) + f_{2j}(x_j, y_j) \cos((\lambda_{j_k} - 1) \theta_j) \right) r^{\lambda_{j_k} - 1} dx_1 dx_2,
\]

where $f_j = (f_{1j}(x_j, y_j), f_{2j}(x_j, y_j))$ is defined in (20) and the local Cartesian coordinates $x_j$ and $y_j$ are defined in (16). Moreover, there exists a constant $C > 0$ independent of $f$ such that
\[
\|\gamma_{j_k}\|_{(H^2(\Omega))^2} \leq C \|f\|_{(L_2(\Omega))^2}.
\]

4.3. Some Auxiliary Boundary Value Problems with a Parameter. Let $T = (f_1(x_1, x_2), f_2(x_1, x_2)) \in (L_2(\mathbb{R}))^2$ and $\mathbf{T} = (T(x_1, x_2)) \in L_2(\mathbb{R})$ be given functions. We consider in this subsection the following boundary value problems with
parameter on the sector domain \( \overline{K} \subset \mathbb{R}^2 \) (see Figure 1), formulated in polar coordinates:

\[
-\frac{\partial^2 u}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} u \right) - \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{2}{r^2} \frac{\partial}{\partial \theta} \left( u \right) + \frac{1}{r^2} u - 2 \frac{\partial^2 u}{\partial r \partial \theta} - \frac{\xi}{r} u + \xi^2 u = f_r \quad \text{in} \ K,
\]

\[
-\frac{\partial^2 u}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} u \right) - \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{2}{r^2} \frac{\partial}{\partial \theta} \left( u \right) + \frac{1}{r^2} u - 2 \frac{\partial^2 u}{\partial r \partial \theta} - \frac{\xi}{r} u + \xi^2 u = f_\theta \quad \text{in} \ K,
\]

\[\text{(52)}\]

where the vector field \( f = (f_r(r, \theta), f_\theta(r, \theta)) \) from (52) in polar coordinates is linked to the vector field \( \bar{f} = (f_1(x_1, x_2), f_2(x_1, x_2)) \) in Cartesian coordinates according to relation (36) and the scalar function \( \bar{f} = f(r, \theta) \) from (53) is linked to the scalar function \( \bar{f} = f(x_1, x_2) \) according to relation (35). The symbol \( \xi \) denotes a positive real parameter. The interest on the boundary value problems (52) and (53) is purely mathematical and there is no evidence that these problems have any practical applications. The results obtained in this subsection will be used for the analysis of the regularity properties of the solution of Maxwell’s equations near polyhedral edges.

The main results concerning the regularity properties of the solution of the boundary value problems (52) and (53) are formulated in Theorems 10 and 11.

**Theorem 10.** Let \( \bar{f} = (f_1, f_2) \in (L^2(\overline{K}))^2 \), \( \xi > 0 \), and \( \lambda_k = k\pi/\omega_k \in \mathbb{N}, \) with \( \omega_k \neq \pi \). Then the boundary value problem (52) has a unique variational solution \( \bar{u} = (\bar{u}_1, \bar{u}_2) \in H^2(\overline{K}) \) such that \( \bar{u}(x_1, x_2) = u(r, \theta) = \bar{w}(x_1, x_2) + y_1 r^{\lambda_1} e^{\bar{r} \theta} \sin \lambda_2 \theta \) with \( \bar{w} \in H^2(\overline{K}) \).

(b) If \( 0 < \lambda_1 < 1 \) and \( \lambda_2 \neq 1 \), then there exists a coefficient \( y_1 = y_1(\xi) \) such that the solution \( \bar{u} \in H^2(\overline{K}) \) can be split as a sum of a regular and a singular part in the form

\[
\bar{u}(x_1, x_2) = u(r, \theta) = \bar{w}(x_1, x_2) + y_1 r^{\lambda_1} e^{\bar{r} \theta} \sin \lambda_2 \theta
\]

\[\text{(55)}\]

The coefficient \( y_1 = y_1(\xi) \) is given explicitly by the formula

\[
y_1(\xi) = \frac{1}{\omega \lambda_1} \int_{K} f(r, \theta) e^{\bar{r} \theta} r^{1-\lambda_1} \sin \lambda_2 \theta d\theta.
\]

Further, there exists a constant \( C > 0 \) independent of \( \bar{f} \) and \( \xi \) such that

(i) \( \xi^{1-\lambda_1} |y_1(\xi)| \leq C \| \bar{f} \|_{L^2(\overline{K})} \) as \( \xi \rightarrow \infty \),

(ii) \( \| \bar{w} \|_{H^1(\overline{K})}^2 + \xi^2 \| \bar{w} \|_{L^2(\overline{K})}^2 \leq C \| f \|_{L^2(\overline{K})}^2 \) as \( \xi \rightarrow \infty \).

For the proof of Theorem 10, see [24, pp. 171–174].

**Theorem 11.** Let \( \bar{f} = (f_1, f_2) \in (L^2(\overline{K}))^2 \), \( \xi > 0 \), and \( \lambda_k = k\pi/\omega_k \in \mathbb{N}, \) with \( \omega_k \neq \pi \). Then there exists a unique variational solution \( \bar{u} = (u_1, u_2) \in H^1(\overline{K}) \) of the boundary value problem (52). If \( \lambda_k \neq 2, k \in \mathbb{N}, \) then there exist coefficients \( y_{1k} = y_{1k}(\xi) \) such that the solution \( \bar{u} \) can be split as a sum of a regular and a singular part in the form

\[
\bar{u} = (\omega_1, w_2) + (s_r(r, \theta), s_\theta(r, \theta))
\]

\[\text{with} \ \bar{w} \in H^2(\overline{K})^2,
\]

\[\text{(57)}\]

\[
\bar{u} = \sum_{0 \leq \lambda_k < 2, \lambda_k \neq 1} y_{1k} r^{\lambda_k - 1} \left( \sin \lambda_k \theta, \cos \lambda_k \theta \right).
\]

The coefficients \( y_{1k} = y_{1k}(\xi) \) are given explicitly by the formula

\[
y_{1k}(\xi) = -\frac{1}{2\omega (\lambda_k - 1) R^{2\lambda_k - 2}} \int_{\overline{K}} (f_r(r, \theta) \sin \lambda_k \theta + f_\theta(r, \theta) \cos \lambda_k \theta)
\]

\[\text{e}^{\bar{r} \theta} r^{\lambda_k} d\theta.
\]

Further, there exists a constant \( C > 0 \) independent of \( \bar{f} \) and \( \xi \) such that

(i) \( \xi^{1-\lambda_k} |y_{1k}(\xi)| \leq C \| \bar{f} \|_{L^2(\overline{K})} \) as \( \xi \rightarrow \infty \),

(ii) \( \| \bar{w} \|_{H^1(\overline{K})}^2 + \xi^2 \| \bar{w} \|_{L^2(\overline{K})}^2 \leq C \| f \|_{L^2(\overline{K})}^2 \) as \( \xi \rightarrow \infty \).

For the proof of Theorem 11, see [24, pp. 171–174].
The proof of Theorem 11 is very lengthy and will be broken down into several lemmas as follows.

**Lemma 12.** The solution \( \bar{u} = u = (u_r, u_\theta) \) of problem (52) can be represented in a Fourier series in the form

\[
(u_r, (r, \theta), u_\theta (r, \theta)) = \sum_{k=1}^{\infty} (u_{rk} (r) \sin \lambda_k \theta, u_{\theta k} (r) \cos \lambda_k \theta),
\]

where the Fourier coefficients \( u_k = (u_{rk}, u_{\theta k}), k \in \mathbb{N}, \) are given explicitly by the formulas

\[
\begin{align*}
u_{rk}(r) &= - \frac{r^{\lambda_k - 1} e^{\xi r}}{4 (\lambda_k - 1) R^{\lambda_k + 2}} \\ &+ \frac{r^{\lambda_k + 1} e^{-\xi r}}{4 (\lambda_k + 1) R^{\lambda_k + 2}} \\ \cdot \int_r^R e^{\xi \tau} \left(f_{rk}(\tau) + f_{\theta k}(\tau)\right) \tau^{\lambda_k} d\tau \\
&+ \frac{r^{\lambda_k + 1} e^{-\xi r}}{4 (\lambda_k - 1) R^{\lambda_k + 2}} \\ &- \frac{r^{\lambda_k + 1} e^{-\xi r}}{4 (\lambda_k + 1) R^{\lambda_k + 2}} \\ \cdot \int_r^R e^{\xi \tau} \left(f_{rk}(\tau) + f_{\theta k}(\tau)\right) \tau^{-\lambda_k} d\tau \\
&+ \frac{r^{\lambda_k - 1} e^{\xi r}}{4 (\lambda_k - 1) R^{\lambda_k + 2}} \\ &- \frac{r^{\lambda_k + 1} e^{-\xi r}}{4 (\lambda_k + 1) R^{\lambda_k + 2}} \\ \cdot \int_r^R e^{\xi \tau} \left(f_{rk}(\tau) + f_{\theta k}(\tau)\right) \tau^{2-\lambda_k} d\tau,
\end{align*}
\]

\[
u_{\theta k}(r) = - \frac{r^{\lambda_k - 1} e^{-\xi r}}{4 (\lambda_k - 1) R^{\lambda_k + 2}} \\ - \frac{r^{\lambda_k + 1} e^{-\xi r}}{4 (\lambda_k + 1) R^{\lambda_k + 2}} \\ \cdot \int_r^R e^{\xi \tau} \left(f_{rk}(\tau) + f_{\theta k}(\tau)\right) \tau^{\lambda_k} d\tau \\
&- \frac{r^{\lambda_k + 1} e^{-\xi r}}{4 (\lambda_k - 1) R^{\lambda_k + 2}} \\ &- \frac{r^{\lambda_k + 1} e^{-\xi r}}{4 (\lambda_k + 1) R^{\lambda_k + 2}} \\ \cdot \int_r^R e^{\xi \tau} \left(f_{rk}(\tau) + f_{\theta k}(\tau)\right) \tau^{-\lambda_k} d\tau
\]

\[
|\gamma_{ik}(\xi)| \leq C (R^{\lambda_k - 1} \|f_k\|_{L^{2,1/2}(0,R)}^2). \tag{62}
\]

**Proof.** Application of Cauchy-Schwarz inequality and the substitution \( \xi \tau = s \) lead to the estimates

\[
|\gamma_{ik}|^2 \leq C \left( \int_0^R e^{\xi \tau} \left|f_{rk}(\tau) + f_{\theta k}(\tau)\right| \tau^{\lambda_k} d\tau \right)^2 \\
\leq C \int_0^R \left(|f_{rk}(\tau)|^2 + |f_{\theta k}(\tau)|^2\right) \tau d\tau \int_0^R e^{2\xi \tau} \tau^{2-\lambda_k} d\tau \tag{63}
\]

\[
\leq C \xi^{-2\lambda_k} \|f_k\|_{L^{1,1/2}(0,R)}^2 \int_0^R e^{2\xi s} s^{2\lambda_k - 1} ds \\
\leq C \xi^{-2\lambda_k} \|f_k\|_{L^{1,1/2}(0,R)}^2. \tag{64}
\]

**Lemma 13.** Let

\[
\gamma_{ik}(\xi) = - \frac{1}{4 (\lambda_k - 1) R^{2\lambda_k - 2}} \\ \cdot \int_0^R e^{\xi \tau} \left(f_{rk}(\tau) + f_{\theta k}(\tau)\right) \tau^{\lambda_k} d\tau.
\]

If \( 0 < \lambda_k < 2 \), then there exists a constant \( C > 0 \) independent of \( f_k \) and \( \xi \) such that

\[
|\gamma_{ik}(\xi)| \leq C \xi^{-\lambda_k} \|f_k\|_{L^{2,1/2}(0,R)}^2.
\]

**Lemma 14.** Let

\[
T_k(r) = r^{\lambda_k - 1} e^{-\xi r}. \tag{64}
\]

If \( \lambda_k > 2 \), then there exists a constant \( C > 0 \) independent of \( \xi \) such that

\[
(i) \quad t_1 = \int_0^R |T_k(\tau)|^2 \tau d\tau \leq C \xi^{-2\lambda_k} \text{ as } \xi \to \infty;
\]

\[
(ii) \quad t_2 = \int_0^R \left|T_k(\tau)\right|^2 + \frac{\lambda_k^2}{\tau} \left|\frac{T_k(\tau)}{\tau}\right|^2 \tau d\tau \leq C \xi^{-2\lambda_k} \text{ as } \xi \to \infty.
\]

\[\]
(iii)

\[
\begin{align*}
t_3 &= \int_0^R \left[ \left| T''_k (r) \right|^2 + 2\lambda_k^2 \left| \frac{T'_k (r)}{r} - \frac{T_k (r)}{r^2} \right|^2 \right] d\tau \\
&\quad + \left[ \left| \frac{T'_k (r)}{r} - \frac{\lambda_k^2}{r^2} T_k (r) \right|^2 \right] d\tau \leq C \xi^{4-2\lambda_k}
\end{align*}
\]

as \( \xi \to \infty \).

**Proof.** Using the substitution \( \xi r = s \) one easily verifies the estimates

\[
\begin{align*}
t_1 &= \int_0^R e^{-2\lambda_k s^{1/2}} d\tau \leq C \xi^{-2\lambda_k} \int_0^\infty e^{-2\lambda_k s^{1/2}} ds \\
&\leq C \xi^{-2\lambda_k},
\end{align*}
\]

\[
\begin{align*}
t_2 &\leq C \left( \xi^2 \int_0^R e^{-2\lambda_k s^{1/2}} d\tau + \int_0^R e^{-2\lambda_k s^{1/2}} d\tau \right) \\
&\leq C \xi^{-2\lambda_k},
\end{align*}
\]

\[
\begin{align*}
t_3 &\leq \int_0^R e^{-2\lambda_k s^{1/2}} d\tau + \xi^2 \int_0^R e^{-2\lambda_k s^{1/2}} d\tau \\
&\quad + \int_0^R e^{-2\lambda_k s^{1/2}} d\tau \leq C \xi^{4-2\lambda_k} \left( \int_0^\infty e^{-2\lambda_k s^{1/2}} ds + \int_0^\infty e^{-2\lambda_k s^{1/2}} ds \right) \\
&\quad + \int_0^\infty e^{-2\lambda_k s^{1/2}} ds \leq C \xi^{4-2\lambda_k}.
\end{align*}
\]

The following lemmas can be proved by analogy. We omit the proofs for the sake of brevity.

**Lemma 15.** Let

\[
\delta_k = \frac{1}{4 (\lambda_k + 1) R^{2\lambda_k + 2}}
\]

\[
\cdot \int_0^R e^{\xi r} \left( f_{\delta_k} (r) + f_{\delta_0} (r) \right) r^{\lambda_k + 2} d\tau.
\]

Then there exists a constant \( C > 0 \) independent of \( f_k \) and \( \xi \) such that

\[
\left| \delta_k (\xi) \right| \leq C \xi^{-(\lambda_k + 2)} \left\| f_k \right\|_{L_{2,1/2}(0,R)}^2.
\]

**Lemma 16.** Let

\[
F_k (r) = r^{\lambda_k + 1} e^{-\xi r}.
\]

Then there exists a constant \( C > 0 \) independent of \( \xi \) such that

\[
\begin{align*}
s_1 &= \int_0^R \left| f_k (r) \right|^2 \tau d\tau \leq C \xi^{-(\lambda_k + 2)} \quad \text{as } \xi \to \infty,
\end{align*}
\]

\[
\begin{align*}
s_2 &= \int_0^R \left[ \left| F_k' (r) \right|^2 + \lambda_k^2 \left| \frac{F_k (r)}{r} - \frac{G_k (r)}{r^2} \right|^2 \right] \tau d\tau \leq C \xi^{-(\lambda_k + 1)}
\end{align*}
\]

as \( \xi \to \infty \).

**Lemma 17.** Let

\[
G_k (r; \xi)
\]

\[
= \frac{r^{-(\lambda_k + 1)} e^{-\xi r}}{4 (\lambda_k + 1)} \int_0^\infty e^{\xi r} \left( f_{\delta_k} (r) - f_{\delta_0} (r) \right) r^{\lambda_k + 2} d\tau
\]

\[
- \frac{r^{-(\lambda_k + 1)} e^{-\xi r}}{4 (\lambda_k + 1)} \int_r^\infty e^{\xi r} \left( f_{\delta_k} (r) + f_{\delta_0} (r) \right) r^{\lambda_k + 2} d\tau.
\]

Then there exists a constant \( C > 0 \) independent of \( f_k \) and \( \xi \) such that

\[
\begin{align*}
g_1 &= \int_0^R \left| G_k (r) \right|^2 \tau d\tau \leq C \xi^{-2} \left\| f_k \right\|_{L_{2,1/2}(0,R)}^2 \\
&\quad \text{as } \xi \to \infty,
\end{align*}
\]

\[
\begin{align*}
g_2 &= \int_0^R \left[ \left| G'_k (r) \right|^2 + \lambda_k^2 \left| \frac{G_k (r)}{r} - \frac{G_k (r)}{r^2} \right|^2 \right] \tau d\tau \\
&\leq C \xi^{-2} \left\| f_k \right\|_{L_{2,1/2}(0,R)}^2 \quad \text{as } \xi \to \infty,
\end{align*}
\]

\[
\begin{align*}
g_3 &= \int_0^R \left[ \left| G''_k (r) \right|^2 + 2\lambda_k^2 \left| \frac{G'_k (r)}{r} - \frac{G_k (r)}{r^2} \right|^2 \right] \tau d\tau \\
&\leq C \xi^{-2} \left\| f_k \right\|_{L_{2,1/2}(0,R)}^2 \\
&\quad \text{as } \xi \to \infty.
\end{align*}
\]
Lemma 18. Let
\[ I_k(r) = \frac{r^{1-\lambda_k} e^{-r}}{4(\lambda_k - 1)} \int_0^r e^{\xi \tau} (f_{rk}(\tau) + f_{\theta k}(\tau)) \tau^{\lambda_k - 1} d\tau + \frac{r^{\lambda_k - 1} e^{-r}}{4(\lambda_k - 1)} \int_r^R e^{\xi \tau} (f_{rk}(\tau) + f_{\theta k}(\tau)) \tau^{2-\lambda_k} d\tau. \]

Then there exists a constant \( C > 0 \) independent of \( \xi \) and \( \xi \) such that
(i) \[ i_1 = \int_0^R |I_k(\tau)|^2 \tau d\tau \leq C\xi^{-4} \| f_k \|_{L_{2,1/2}^4(0,R)}^2 \quad \text{as} \quad \xi \rightarrow \infty, \]
(ii) \[ i_2 = \int_0^R \left\{ I_k'(\tau) \right\}^2 + \lambda_k^2 \left\{ \frac{I_k(\tau)}{\tau} \right\}^2 \tau d\tau \leq C\xi^{-2} \| f_k \|_{L_{2,1/2}^4(0,R)}^2 \quad \text{as} \quad \xi \rightarrow \infty, \]
(iii) \[ i_3 = \int_0^R \left\{ I_k''(\tau) \right\}^2 + 2\lambda_k^2 \left\{ \frac{I_k'(\tau) - \lambda_k I_k(\tau)}{\tau} \right\}^2 \tau d\tau \leq C\xi^{-2} \| f_k \|_{L_{2,1/2}^4(0,R)}^2 \quad \text{as} \quad \xi \rightarrow \infty. \]

Proof (Theorem 11). The results of Theorem 11 follow by combining Lemmas 12–18, taking note of the definition of the representation of norms of functions as series of norms of their Fourier coefficients by means of generalized Parseval identities; see, for example, [13, 18].

5. Maxwell’s Equations in Domains with Polyhedral Edges

In this section, we consider and analyze the Maxwell equations (25) in three-dimensional domains with polyhedral edges and prove Theorem 6.

5.1. Maxwell’s Equations in a Three-Dimensional Wedge. We consider first a three-dimensional domain of the form \( \Omega = \mathbb{R} \times (0, l) \), where \( l > 0 \) is a real constant and \( \mathbb{R} \subset \mathbb{R}^3 \); see Figure 2. We will use the same notations as in Section 4.1 for \( \mathbb{K} \). Thus the boundary \( \Gamma \) of \( \Omega \) can be represented in the form \( \Gamma = T_0 \cup T_1 \cup T_2 \cup T_3 \cup T_4 \), where \( T_0 = \Omega \times \{0\} \), \( T_4 = \Omega \times \{l\} \), and \( T_j = T_j \times (0, l) \), \( j = 0, 1, 2 \); see Figure 2.

For a given vector field \( f \in (L_2(\Omega))^3 \), let \( u \in H_0^1(\Omega) \), \( \text{curl} \), \( \text{div}, \Omega \) be the variational solution of the boundary value problem
\[ \text{curl} \text{ curl} u - \text{grad} \text{ div} u = f \quad \text{in} \quad \Omega, \]
\[ u \wedge n = 0 \quad \text{on} \quad T_1 \cup T_2 \cup T_3 \cup T_4, \]
\[ \text{div} u = 0 \quad \text{on} \quad T_1 \cup T_2 \cup T_3 \cup T_4, \]
\[ u = 0 \quad \text{on} \quad T_0. \]

We observe that the systems of trigonometric functions \{\sin(n\pi x_l)/l : n \in \mathbb{N}\} and \{\cos(n\pi x_l)/l : n \in \mathbb{N}_0\} (\mathbb{N}_0 = \{0, 1, 2, \ldots\}) are orthogonal and complete in \( L_2(\mathbb{K}) \); see [25, 26]. Thus functions \( v \in (L_2(\Omega))^3 \) can be characterized by their Fourier coefficients as follows.

Lemma 19. (1) Let \( v = (v_1, v_2, v_3) \in (L_2(\Omega))^3 \). Then there exist in \( (L_2(\mathbb{K}))^3 \) Fourier coefficients \( \{v_n = (v_{1n}, v_{2n}, v_{3n}) : n \in \mathbb{N}_0\} \) of \( v \) defined by
\[ v_{1n}(x_1, x_2) = \frac{2}{l} \int_0^l v_1(x_1, x_2, x_3) \sin \frac{n\pi x_3}{l}, \quad i = 1, 2, \]
\[ v_{3n}(x_1, x_2) = \frac{2}{l} \int_0^l v_3(x_1, x_2, x_3) \cos \frac{n\pi x_3}{l}. \]
and satisfying the relations
\[ v_1(x_1, x_2, x_3) = \sum_{n=1}^{\infty} v_{1n}(x_1, x_2) \sin \frac{n\pi x_3}{l}, \]
\[ v_2(x_1, x_2, x_3) = \sum_{n=1}^{\infty} v_{2n}(x_1, x_2) \sin \frac{n\pi x_3}{l}, \]
\[ v_3(x_1, x_2, x_3) = \frac{1}{2} v_{30}(x_1, x_2) + \sum_{n=1}^{\infty} v_{3n}(x_1, x_2) \cos \frac{n\pi x_3}{l}. \]

Moreover, Parseval’s identity holds in the form
\[ \|v_i\|_{L^2(\Omega)}^2 = \frac{1}{4} \|v_{0i}\|_{L^2(\Omega)}^2 + \frac{1}{2} \sum_{n=1}^{\infty} \|v_{ni}\|_{L^2(\Omega)}^2 < \infty, \]

(85)

(2) For \( \mathbf{v} \in \mathcal{H}_0(\text{curl, div, } \Omega) \), relations (84), (85), and (86) hold and additionally
\[ \|\mathbf{v}\|_{\mathcal{H}_0(\text{curl, div, } \Omega)}^2 = \frac{1}{4} \|\mathbf{v}_0\|_{H^1(\Omega)}^2 + \frac{1}{2} \sum_{n=1}^{\infty} \|\mathbf{v}_n\|_{H^1(\Omega)}^2, \]

(87)

For the study of the Fourier coefficients \( \{\mathbf{u}_n = (u_{1n}, u_{2n}, u_{3n}) : n \in \mathbb{N}_0\} \) of the solution \( \mathbf{u} \in \mathcal{H}_0(\text{curl, div, } \Omega) \) of the boundary value problem (83), we introduce on \( \mathbb{R}^3 \) the spaces
\[ \mathcal{Y}(\mathbb{R}^3) = \left\{ \mathbf{w} = (w_1, w_2, w_3) \in (L^2(\mathbb{R}^3))^3 : \frac{\partial w_2}{\partial x_1}, \frac{\partial w_3}{\partial x_1}, \frac{\partial w_2}{\partial x_2}, \frac{\partial w_3}{\partial x_2} \in L^2(\mathbb{R}^3), \right\}, \]
\[ \mathcal{Y}_0(\mathbb{R}^3) = \left\{ \mathbf{w} = (w_1, w_2, w_3) \in \mathcal{Y}(\mathbb{R}^3) : w_1n_2 \right\} \]
\[ \mathcal{Y}_0(\mathbb{R}^3) = \left\{ \mathbf{w} = (w_1, w_2, w_3) \in \mathcal{Y}(\mathbb{R}^3) : w_1n_2, w_2 = w_3 = 0 \right\} \]
\[ = 0 \text{ on } \Gamma_1 \cup \Gamma_2, \left\{ \begin{array}{l} w_1 = w_2 = w_3 = 0 \text{ on } \Gamma_0 \end{array} \right\}, \]
\[ \text{where } n = (n_1, n_2) \text{ denotes the unit outward normal on the boundary } \Gamma_1 \cup \Gamma_2. \]

These spaces are equipped with the norm
\[ \|\mathbf{w}\|_{\mathcal{Y}(\mathbb{R}^3)} = \left( \|\mathbf{w}\|_{L^2(\mathbb{R}^3)}^2 + \left\| \frac{\partial w_2}{\partial x_1} \right\|_{L^2(\mathbb{R}^3)}^2 + \left\| \frac{\partial w_3}{\partial x_1} \right\|_{L^2(\mathbb{R}^3)}^2 \right)^{1/2}. \]

Clearly, the spaces \( \mathcal{Y}(\mathbb{R}^3) \) and \( \mathcal{Y}_0(\mathbb{R}^3) \) endowed with the norm (89) are Hilbert spaces. Indeed, we observe the identities
\[ \mathcal{Y}(\mathbb{R}^3) = \mathcal{H}(\text{curl, div, } \mathbb{R}^3) \times H^1(\mathbb{R}^3), \]
\[ \mathcal{Y}_0(\mathbb{R}^3) = \mathcal{H}_0(\text{curl, div, } \mathbb{R}^3) \times H^1_0(\mathbb{R}^3). \]

Lemma 20. Let \( \{v_n = (v_{1n}, v_{2n}, v_{3n}) : n \in \mathbb{N}_0\} \) denote the Fourier coefficients of a function \( \mathbf{v} \in \mathcal{H}_0(\text{curl, div, } \Omega) \) defined according to relation (84). Then \( \mathbf{v}_n \in \mathcal{Y}_0(\mathbb{R}^3) \) for any \( n \in \mathbb{N}_0. \)

Proof. It follows from (84) that \( v_0 = (0,0,v_{30}). \) The completeness relationship (87) infers then that \( v_{30} \in H^1(\mathbb{R}^3) \) and consequently \( \mathbf{v}_0 \in \mathcal{Y}(\mathbb{R}^3). \) Further, with the help of the triangle inequality and relation (87) we get the estimates
\[ \|\partial v_{3n}\|_{L^2(\mathbb{R}^3)} \leq \left( \left\| \frac{\partial v_{3n}}{\partial x_1} \right\|_{L^2(\mathbb{R}^3)} + \frac{n\pi}{T} \|v_{3n}\|_{L^2(\mathbb{R}^3)} \right) < \infty, \]
\[ \|\partial v_{3n}\|_{L^2(\mathbb{R}^3)} \leq \left( \left\| \frac{\partial v_{3n}}{\partial x_2} \right\|_{L^2(\mathbb{R}^3)} + \frac{n\pi}{T} \|v_{3n}\|_{L^2(\mathbb{R}^3)} \right) < \infty, \]
\[ \|\partial v_{3n}\|_{L^2(\mathbb{R}^3)} \leq \left( \left\| \frac{\partial v_{3n}}{\partial x_2} \right\|_{L^2(\mathbb{R}^3)} + \frac{n\pi}{T} \|v_{3n}\|_{L^2(\mathbb{R}^3)} \right) < \infty. \]

Hence, taking into account the definition of \( \mathcal{Y}(\mathbb{R}^3) \) and the completeness relationship (87) we get \( \mathbf{v}_n \in \mathcal{Y}(\mathbb{R}^3), n \in \mathbb{N}_0. \) The boundary conditions follow from the boundary conditions in \( \mathcal{H}_0(\text{curl, div, } \Omega). \)

Lemma 21. For \( f \in (L^2(\Omega))^3, \) let \( \mathbf{u} \in \mathcal{H}_0(\text{curl, div, } \Omega) \) be the solution of the boundary value problem (83). Let \( \{u_n = (u_{1n}, u_{2n}, u_{3n}) : n \in \mathbb{N}_0\} \) and \( \{f_n = (f_{1n}, f_{2n}, f_{3n}) : n \in \mathbb{N}_0\} \) denote the Fourier coefficients.
of $u$ and $f$, respectively, defined according to (84). Then the functions $u_n = (\mathbf{u}_n, u_{3n})$, $n \in \mathbb{N}_0$, satisfy the relations $\mathbf{u}_n \in \mathcal{H}_0(\text{curl, div}, \overline{K})$ and $u_{3n} \in H_0^2(\overline{K})$ and are the solutions of the boundary value problems

\[ \nabla \times \nabla \times \mathbf{u}_n - \text{grad div} \mathbf{u}_n + \left( \frac{n \pi}{l} \right)^2 \mathbf{u}_n = \mathbf{f}_n \text{ in } \overline{K}, \]

\[ u_{3n}|_{r=2} - u_{2n}|_{r=1} = 0, \quad \text{div} \mathbf{u}_n = 0 \quad \text{on } \Gamma_1 \cup \Gamma_2, \tag{92} \]

\[ \mathbf{u}_n = 0 \quad \text{on } \Gamma_0, \]

\[ -\Delta u_{3n} + \left( \frac{n \pi}{l} \right)^2 u_{3n} = f_{3n} \text{ in } \overline{K}, \]

\[ u_{3n} = 0 \quad \text{on } \partial \overline{K}. \]

**Proof.** Problems (92) are obtained directly from (83) by substituting the functions $u$ and $f$ by their respective Fourier series defined according to (85), differentiating term by term and comparing coefficients. The assertions $\mathbf{u}_n \in \mathcal{H}_0(\text{curl, div}, \overline{K})$ and $u_{3n} \in H_0^2(\overline{K})$ follow from Lemma 20.

We observe that in local polar coordinates $r$ and $\theta$ problems (92) take the form

\[ -\frac{\partial^2 u_m}{\partial r^2} - \frac{1}{r} \frac{\partial u_m}{\partial r} - \frac{1}{r^2} \frac{\partial^2 u_m}{\partial \theta^2} + \frac{2}{r^2} \frac{\partial u_{3n}}{\partial \theta} + \frac{1}{r^2} u_n = f_m \text{ in } \overline{K}, \]

\[ -\frac{\partial^2 u_{0n}}{\partial r^2} - \frac{1}{r} \frac{\partial u_{0n}}{\partial r} - \frac{1}{r^2} \frac{\partial^2 u_{0n}}{\partial \theta^2} + \frac{2}{r^2} \frac{\partial u_{3n}}{\partial \theta} + \frac{1}{r^2} u_{3n} = f_{3n} \text{ in } \overline{K}, \]

\[ \frac{\partial u_m}{\partial r} + \frac{1}{r} u_m + \frac{1}{r} \frac{\partial u_{0n}}{\partial \theta} = 0, \quad u_m = 0 \text{ on } \Gamma_1 \cup \Gamma_2, \]

\[ |u_m(r, \theta)|_{r=0} < \infty, \quad |u_{0n}(r, \theta)|_{r=0} < \infty, \]

\[ u_m(r, \theta) = u_{3n}(R, \theta) = 0, \]

\[ -\frac{\partial^2 u_{3n}}{\partial r^2} - \frac{1}{r} \frac{\partial u_{3n}}{\partial r} - \frac{1}{r^2} \frac{\partial^2 u_{3n}}{\partial \theta^2} + \frac{n^2 \pi^2}{l^2} u_{3n} = f_{3n} \text{ in } \overline{K}, \]

\[ |u_{3n}(r, \theta)|_{r=0} < \infty, \quad u_{3n}(r, \theta)|_{r=R} = 0, \]

\[ u_{3n}(r, \theta)|_{r=0} = 0, \]

\[ u_{3n}(r, \theta)|_{r=\infty} = 0. \tag{93} \]

We will need the following notations:

\[ f_m^* = f_m - 2 \frac{n \pi}{l} \frac{\partial u_m}{\partial r} - \frac{n \pi}{l} \frac{u_m}{r} - 2 \frac{n^2 \pi^2}{l^2} u_m, \]

\[ f_{3n}^* = f_{3n} - \frac{n \pi}{l} \frac{\partial u_{3n}}{\partial r} - \frac{n \pi}{l} \frac{u_{3n}}{r} - 2 \frac{n^2 \pi^2}{l^2} u_{3n}, \tag{94} \]

where $f_n = (f_m, f_{3n})$ and $u_n = (u_m, u_{3n})$, $n \in \mathbb{N}_0$, are taken from (93). Obviously $f_n^* = (f_m^*, f_{3n}^*) \in (L_2(\overline{K}))^3$.

**Theorem 22.** Let $\overline{K}$ be a circular sector with angle $\omega \in (0, 2\pi)$, $\omega \neq \pi$, and let $\lambda_k = k \pi / \omega$, $k \in \mathbb{N}$. For each $f_n = (f_m, f_{3n}) \in (L_2(\overline{K}))^3$, let $u_n = (u_m, u_{3n}) \in Y_0(\overline{K})$, $n \in \mathbb{N}_0$, be the solutions of the boundary value problems (92). If $\lambda_k \neq 2$, $k \in \mathbb{N}$, then there exist coefficients $\gamma_n$ and $\gamma_n^*$ such that the solutions $u_n, n \in \mathbb{N}_0$, can be represented in the form

\[ u_n = (u_m, u_{2n}, u_{3n}) + (s_m, s_{0n}, s_{3n}) \]

with $w_n \in (L_2(\overline{K}))^3$.

\[ s_m(r, \theta) = \sum_{0 \leq k < 2 \lambda_k} \gamma_n e^{i(\omega/k)r} \lambda_k^{-1} \sin \lambda_k \theta, \]

\[ s_{0n}(r, \theta) = \sum_{0 \leq k < 2 \lambda_k} \gamma_n e^{i(\omega/k)r} \lambda_k^{-1} \cos \lambda_k \theta, \]

\[ s_{3n}(r, \theta) = \gamma_n e^{i(\omega/k)r} \lambda_1 \sin \lambda_1 \theta \quad \text{if } 0 < \lambda_1 < 1. \]

The coefficients $\gamma_n$ and $\gamma_n^*$ of the expansion (95) are given explicitly by the formulas

\[ \gamma_n = -\frac{R^2 - 2 \lambda_k}{2 \omega (\lambda_k - 1)} \int_K e^{i(\omega/k)r} \left( f_m^* (r, \theta) \sin \lambda_k \theta + f_{3n}^* (r, \theta) \cos \lambda_k \theta \right) r^{\lambda_k} dr d\theta, \]

\[ \gamma_n^* = \frac{1}{\omega \lambda_1} \int_K e^{i(\omega/k)r} f_{3n}^* (r, \theta) \sin (\lambda_1 \theta) r^{1-\lambda_1} dr d\theta, \]
where the functions $\mathbf{f}_n^* = (f_{1n}^*, f_{2n}^*, f_{3n}^*)$, $n \in \mathbb{N}_0$, are as defined in (94). Moreover, there exists a constant $C > 0$ independent of $\mathbf{f}_n$ such that

$$
\left( \frac{m \pi}{T} \right)^{\lambda_3} |\gamma_n| \leq C \| \mathbf{f}_n^* \|_{L_2(\hat{\Omega})}^2,
$$

(97a)

$$
\left( \frac{m \pi}{T} \right)^{1-\lambda_3} |\gamma_n| \leq C \| f_{3n}^* \|_{L_2(\hat{\Omega})},
$$

(97b)

$$
\| \mathbf{w}_n \|_{H^1(\hat{\Omega})}^2 + \frac{m \pi}{T} \| \mathbf{w}_n \|_{H^1(\hat{\Omega})}^2
+ \left( \frac{m \pi}{T} \right)^2 \| \mathbf{w}_n \|_{L_2(\hat{\Omega})} \leq C \| \mathbf{f}_n^* \|_{L_2(\hat{\Omega})}^2
$$

(97c)

$$
\| w_{3n} \|_{H^1(\hat{\Omega})} + \frac{m \pi}{T} \| w_{3n} \|_{H^1(\hat{\Omega})}^2
+ \left( \frac{m \pi}{T} \right)^2 \| w_{3n} \|_{L_2(\hat{\Omega})} \leq C \| f_{3n}^* \|_{L_2(\hat{\Omega})}.
$$

(97d)

Proof. Relations (95), (96), (97a), (97b), (97c), and (97d) are obtained by a straightforward application of Theorems 10 and 11 to the boundary value problems (93) with $\xi = m \pi / l$ and taking note of the modified right hand side defined in (94).

Theorem 23. Let $\Omega = \mathbb{R} \times (0, l)$ be a three-dimensional wedge and $\lambda_k = k \pi / \omega$, $k \in \mathbb{N}$, $\omega \neq \pi$. For each $\mathbf{f} = (f_1, f_2, f_3) \in (L_2(\Omega))^3$, let $\mathbf{u} = (u_1, u_2, u_3) \in \mathcal{H}_0(\text{curl}, \text{div}, \Omega)$ be the solution of the boundary value problem (83). If $\lambda_k \neq 2$, $k \in \mathbb{N}$, then there exist unique functions $\Psi_k \in H^{1,1}(0, l)$ and $\Psi_3 \in H^{1,1}(0, l)$ such that the solution $\mathbf{u} \in \mathcal{H}_0(\text{curl}, \text{div}, \Omega)$ can be split in the form

$$
\mathbf{u} = (w_1, w_2, w_3) + (s_1, s_2, s_3)
$$

with $w = (w_1, w_2, w_3) \in (H^2(\Omega))^3$,

$$
w_1(x_1, x_2, x_3) = \sum_{n=1}^{\infty} w_{1n}(x_1, x_2) \sin \frac{m n \pi x_3}{l},
$$

$$
w_2(x_1, x_2, x_3) = \sum_{n=1}^{\infty} w_{2n}(x_1, x_2) \sin \frac{m n \pi x_3}{l},
$$

$$
w_3(x_1, x_2, x_3) = \frac{1}{2} w_{30}(x_1, x_2) + \sum_{n=1}^{\infty} u_{3n}(x_1, x_2)
\cdot \cos \frac{m n \pi x_3}{l},
$$

$$
s_1(x_1, x_2, x_3) = \sum_{0 < \lambda_k < 2 \lambda_1 \neq 1} (\Psi_k(x_3) \ast T_1(r, x_3)) r^{\lambda_3-1}
\cdot \sin ((\lambda_k - 1) \theta),
$$

$$
s_2(x_1, x_2, x_3) = \sum_{0 < \lambda_k < 2 \lambda_1 \neq 1} (\Psi_k(x_3) \ast T_1(r, x_3)) r^{\lambda_3-1}
\cdot \cos ((\lambda_k - 1) \theta),
$$

$$
s_3(x_1, x_2, x_3) = (\Psi_3(x_3) \ast T_3(r, x_3)) r^1 \sin \lambda_3 \theta
$$

if $0 < \lambda_3 < 1$,

(98)

where

$$
\Psi_k(x_3) = \sum_{n=1}^{\infty} \gamma_n \sin \frac{m n \pi x_3}{l},
$$

$$
\Psi_3(x_3) = \frac{1}{2} \gamma_0 + \sum_{n=1}^{\infty} \gamma_n \cos \frac{m n \pi x_3}{l},
$$

$$
T_1(r, x_3) = \sum_{n=1}^{\infty} e^{-m n \pi r} \sin \frac{m n \pi x_3}{l},
$$

$$
T_3(r, x_3) = \frac{1}{2} + \sum_{n=1}^{\infty} e^{-m n \pi r} \cos \frac{m n \pi x_3}{l}.
$$

In (98), the symbol "\ast" denotes the convolution product in the variable $x_3$; that is,

$$
\Psi_k(x_3) \ast T_1(r, x_3) = \sum_{n=1}^{\infty} \gamma_n e^{-m n \pi r} \sin \frac{m n \pi x_3}{l},
$$

$$
\Psi_3(x_3) \ast T_3(r, x_3) = \frac{1}{2} \gamma_0 + \sum_{n=1}^{\infty} \gamma_n e^{-m n \pi r} \cos \frac{m n \pi x_3}{l}.
$$

The coefficients $\gamma_n$ and $\gamma_n'$ are as defined in Theorem 22.

Proof. The expressions (98), (99), and (100) are direct consequences of Lemma 21 and Theorem 22, taking into consideration relations (35) and (36). The inequalities (97c) and (97d) and Parseval’s identity (86) imply that the Fourier coefficients $w_n = (w_{1n}, w_{2n}, w_{3n})$, $n \in \mathbb{N}_0$, satisfy the estimate

$$
\sum_{n=1}^{\infty} \left\{ \| w_n \|_{H^1(\hat{\Omega})}^2 + \left( \frac{m \pi}{T} \right)^2 \| w_n \|_{H^1(\hat{\Omega})}^2 
+ \left( \frac{m \pi}{T} \right)^4 \| w_n \|_{L_2(\hat{\Omega})}^2 \right\} \leq C \| \mathbf{f}_n^* \|_{L_2(\hat{\Omega})}^2 < \infty
$$

(101)

Hence, $w \in (H^2(\Omega))^3$; see [24, Theorem 3.2]. The inequalities (97a) and (97b) lead to the estimates

$$
\sum_{n=1}^{\infty} \left( \frac{m \pi}{T} \right)^{\lambda_3} |\gamma_n|^2 \leq C \sum_{n=0}^{\infty} \| f_{3n}^* \|_{L_2(\hat{\Omega})}^2 < \infty,
$$

$$
\sum_{n=1}^{\infty} \left( \frac{m \pi}{T} \right)^{1-\lambda_3} |\gamma_n|^2 \leq C \sum_{n=0}^{\infty} \| f_{3n}^* \|_{L_2(\hat{\Omega})}^2 < \infty.
$$
which by the generalized Riesz-Fischer theorem imply the existence of functions \( \Psi_k \in H^{3}(0, l) \) and \( \Psi_3 \in H^{3}(0, l) \) whose Fourier coefficients are \( \{ y_{nk} : n \in \mathbb{N} \} \) and \( \{ y_n : n \in \mathbb{N}, j \} \), respectively.

5.2. Singularities near Polyhedral Edges. In this subsection, we consider and analyze the regularity properties of the solution of the Maxwell equations (25) in general three-dimensional domains with straight edges bounded by plain faces, that is, polyhedral edges. In fact it would be sufficient for us to consider three-dimensional domains of the form \( Q = \Omega \times (0, l) \), where \( l > 0 \) is a real constant and \( \Omega \subset \mathbb{R}^2 \) is a general bounded domain with piecewise smooth boundary \( \Gamma \). We will use the same notations as in Section 3 for \( \Omega \) and \( Q \).

Thus the edges of \( Q \) are \( E_j = A_j \times (0, l) \), \( j = 1, \ldots, J \), and the boundary \( \partial Q \) is the union of the disjoint faces \( T_j = A_j \times (0, l) \), \( j = 1, \ldots, J \), and the two bases \( T_{j+1} = \omega \times \{ 0 \} \) and \( T_{j+2} = \Omega \times \{ l \} \). We assume that, for \((x_1, x_2, x_3) \in Q \), \((x_1, x_2) \in \Omega \) and \( x_3 \in (0, l) \). We associate with each edge \( E_j \) a three-dimensional wedge \( G_j = \overline{K}_j \times (0, l) \), where \( K_j \subset \Omega \) is a circular sector; see (17) and (18). Thus the boundary \( \partial G_j \) of \( G_j \) is the union of the disjoint faces \( T_{j0} = G_j \times (0, l) \), \( T_{j1} = \Gamma_j \times (0, l) \), \( T_{j2} = \Gamma_j \times (0, l) \), \( T_{j3} = \overline{K}_j \times \{ 0 \} \), and \( T_{j4} = \overline{K}_j \times \{ l \} \). Furthermore, we define on each \( G_j \) a smooth truncation function \( \eta_j = \eta_j(x_1, x_2) = \eta_j(r_j) \); see (19).

For \( f \in (L_2(Q))^3 \) and \( \alpha \neq 0 \), let \( u \in \mathcal{H}_0(\text{curl, div}, Q) \) be the variational solution of the Maxwell equations (25). Then the function \( u_j = \eta_j u \) which is defined on the wedge \( G_j \) belongs to the space \( \mathcal{H}_0(\text{curl}, G_j) \) and is the unique weak solution of the boundary value problem

\[ \text{curl} \text{ curl} u_j - \text{grad} \text{ div} u_j = f_j \quad \text{in} \ G_j, \]

\[ u_j \wedge n = 0 \quad \text{on} \ T_{j1} \cup T_{j2} \cup T_{j3} \cup T_{j4}, \]

\[ \text{div} u_j = 0 \quad \text{on} \ T_{j1} \cup T_{j2} \cup T_{j3} \cup T_{j4}, \]

\[ u_j = 0 \quad \text{on} \ T_{j0}, \]

where the function \( f_j = (f_{1j}, f_{2j}, f_{3j}) \in (L_2(G_j))^3 \) is given by

\[ f_{1j}(x_1, x_2, x_3) = \eta_j \left( f_1 + \alpha^2 u_2 \right) - u_1 \Delta (x_1, x_2) \eta_j - 2 \nabla_{x_1 x_2} \eta_j \cdot \nabla_{x_1 x_2} u_4, \]

\[ f_{2j}(x_1, x_2, x_3) = \eta_j \left( f_2 + \alpha^2 u_1 \right) - u_2 \Delta (x_1, x_2) \eta_j - 2 \nabla_{x_1 x_3} \eta_j \cdot \nabla_{x_1 x_3} u_4, \]

\[ f_{3j}(x_1, x_2, x_3) = \eta_j \left( f_3 + \alpha^2 u_3 \right) - u_3 \Delta (x_1, x_2) \eta_j - 2 \nabla_{x_1 x_3} \eta_j \cdot \nabla_{x_1 x_3} u_4. \]
The coefficients $\Psi_{jk}$ and $\Psi_j$ are defined explicitly by
\[
\Psi_{jk}(x_3) = \sum_{n=1}^{\infty} \gamma_j n_k \sin \frac{n \pi x_3}{l},
\]
(107)
\[
\Psi_j(x_3) = \frac{1}{2} \gamma_j + \sum_{n=1}^{\infty} \gamma_{jn} \cos \frac{n \pi x_3}{l},
\]
where the coefficients $\gamma_{jnk}$ and $\gamma_{jn}$ are as defined in (30).

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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