On the Solution to the “Frozen Star” Paradox, Nature of Astrophysical Black Holes, non-Existence of Gravitational Singularity in the Physical Universe and Applicability of the Birkhoff’s Theorem

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1. Introduction

Oppenheimer and Snyder found in 1939 that gravitational collapse in vacuum produces a “frozen star”, i.e., the collapsing matter only asymptotically approaches the gravitational radius (event horizon) of the mass, but never crosses it within a finite time for an external observer. Based upon our recent publication on the problem of gravitational collapse in the physical universe for an external observer, the following results are reported here: (1) Matter can indeed fall across the event horizon within a finite time and thus BHs, rather than “frozen stars”, are formed in gravitational collapse in the physical universe. (2) Matter fallen into an astrophysical black hole can never arrive at the exact center; the exact interior distribution of matter depends upon the history of the collapse process. Therefore gravitational singularity does not exist in the physical universe. (3) The metric at any radius is determined by the global distribution of matter, i.e., not only by the matter inside the given radius, even in a spherically symmetric and pressureless gravitational system. This is qualitatively different from the Newtonian gravity and the common (mis)understanding of the Birkhoff’s Theorem. This result does not contract the “Lemaître-Tolman-Bondi” solution for an external observer.

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never witness the formation of an astrophysical BH. Given the finite age of the universe and the fact that all observers are necessarily external, the last conclusion of Ref. [1] seems to indicate that astrophysical BHs cannot be formed in the physical universe through gravitational collapse.

Recently, Vachaspati, Stojkovic & Krauss [2] have stressed that “The process of BH formation is generally discussed from the viewpoint of an infalling observer. However, in all physical settings it is the viewpoint of the asymptotic observer [i.e., O] that is relevant.” They analyzed the process of the self-collapse of a domain wall (a massive shell with no thickness) and concluded that O sees the domain wall asymptotically shrinking to \( R_H \), i.e., a BH is never formed within a finite time to O. This is a further confirmation to the conclusion of Ref. [1]. Vachaspati et al. [2] then went on to study the quantum mechanical effect of the contracting shell and found that the matter accumulating just outside \( R_H \) actually produces radiation, which they called pre-Hawking radiation. They concluded that “Evaporation by pre-Hawking radiation implies that O can never lose objects down a BH.”

Combining the above two works separated by nearly 70 years, a very surprising scenario seems inevitable: Gravitational collapse will not produce BHs, but result in complete conversion of matter into radiation. This scenario, if correct, would have profound implications to our understanding of general relativity which has long been considered to robustly predict the existence of BHs, as well as a vast amount of astronomical observations which can, and perhaps only, be understood by invoking BHs [3]. However, both of the above works are over-simplified and do not catch all the essence of gravitational collapse in the physical universe, because both investigations only considered gravitational contraction in vacuum and the work of Ref. [2] did not allow a finite thickness of the contracting shell.

To overcome the drawbacks of these two works discussed above, Liu & Zhang [4] studied the gravitational collapse of a single shell and double-shells onto a pre-existing BH; these shells can have finite thicknesses and the outer shell in the double-shell case mimics the matter outside the collapsing shell in the physical universe. The gravitational contractions studied in the two previous works can be considered as special cases of that studied in Ref. [4]. The main conclusion of Liu & Zhang [4] is that matter does not accumulate outside \( R_H \), but instead falls straight across it, within a finite time of O. In the rest of this paper, we first review briefly the main results in Ref. [4] and then discuss several issues related to the “frozen star” paradox, nature of astrophysical BHs, gravitational singularity in the physical universe, and finally applicability of the Birkhoff’s theorem. All calculations and discussions in this report are within the framework of Einstein’s general relativity.

2. Exact solutions for shells collapsing onto a pre-existing BH

We briefly review the main results in Ref. [4]. In Fig. 1, we show the initial conditions of gravitational collapse onto a BH in the comoving coordinates, for a single shell and double-shell cases, respectively. For the single shell case, \( m \) and \( m_s \) are the total
gravitating masses of the BH and the shell, respectively. \( a' \) and \( a \) are the radii of the inner and outer boundaries of the shell, respectively. Letting \( a' = 0 \) and \( m = 0 \), this recovers to the case studied in Ref. 1. Letting \( a \approx a' \) and \( m = 0 \), this recovers to the case studied in Ref. 2.

We first solve the field equations in the comoving coordinates, following Ref. 1; the metric or extrinsic curvature is required to be continuous across the boundaries of the three regions. We then transform the solutions in the three regions to the Schwarzschild coordinates (for \( \mathcal{O} \)). In each of the three regions shown in the left panel of Fig. 1, the metric can be expressed in the Schwarzschild-like form,

\[
ds^2 = h_i(1 - \frac{2M(r)}{r})dt^2 - \left(1 - \frac{2M(r)}{r}\right)^{-1}dr^2 - r^2 d\Omega^2,
\]

where \( M(r) \) is the total gravitational mass within \( r \), and \( i = 1, 2, \) or 3, specifies region I, II, or III, respectively. For region III, obviously \( h_3 = 1 \), i.e., the metric is exactly Schwarzschild. In region II, \( h_2 = h_2(t, r) < 1 \), though its analytic form cannot be obtained generally. In region I, the continuity condition ensures that \( h_1 = h_1(t) < 1 \). Therefore the metric in either region I or II is not Schwarzschild, because both \( h_1 \) and \( h_2 \) change as the shell falls in.

The motion of the shell is shown in Fig. 2, in both the comoving and external coordinates. Clearly the shell falls into the BH within a finite comoving time. For \( \mathcal{O} \), the body of the shell also crosses \( \mathcal{R}_H \) within a finite time, except for its outer boundary which asymptotically approaches to \( \mathcal{R}_H \). This result is consistent with that of Ref. 1 but qualitatively different from that of Ref. 2. The difference is due to the finite thickness of the shell in Fig. 1, in contrast to the domain wall assumption in Ref. 2. In the case of a finite thickness of the shell, the increase of \( \mathcal{R}_H \) swallows

![Figure 1](image-url)

**Fig. 1.** Initial conditions of gravitational collapse onto a BH in the comoving coordinates. **Left panel:** the case for one shell. \( m \) and \( m_s \) are the total gravitating masses of the BH and the shell, respectively. \( a' \) and \( a \) are the radii of the inner and outer boundaries of the shell, respectively. **Right panel:** the case for two shells. \( m, m_1 \) and \( m_2 \) are the total gravitating masses of the BH, shells 1 and 2, respectively. \( a'_1 \) and \( a_1 \) are the radii of the inner and outer boundaries of shell 1, respectively. \( a'_2 \) and \( a_2 \) are the radii of the inner and outer boundaries of shell 2, respectively.
the shell; obviously this cannot happen if the shell has no thickness. We therefore dismiss the conclusion of Ref. [2] because there is no matter available outside the event horizon to produce any radiation, for a physical shell with non-zero thickness.

However the calculations shown in Fig. 2 still neglects one important fact for gravitational collapse in the physical universe. There is always some additional matter between the observer and the infalling shell being observed (we call it shell 1 Fig. 1 (right)), and the additional matter (we call it shell 2 in Fig. 1 (right)) is also attracted to fall inwards by the inner shell and the BH. We thus calculate the motion of the double-shell system. For $O$, the metric in each of the five regions still takes the form of Eq. (1), with $h = 1$ in region V, but $h < 1$ and is also time-dependent in all other four regions. The motions of both shells are shown in Fig. 3. In this case, shell 1 can cross $R_H$ completely even for $O$.

Fig. 2. The solution for the one shell case. (a) and (b) are evolution curves for $a = 5r_0$, $a' = 2.5r_0$, and $r'_0 = 1/8r_0$ with comoving time and coordinate time, respectively. The evolution of the event and apparent horizons are also shown. Here $G = c = 1$, $r'_0 = 2m$ and $r_0 = 2(m + m_s)$.

Fig. 3. The solution for the double-shell case. (a) and (b) are evolution curves for $a_2 = 10r_0$, $a'_2 = 8r_0$, $a_1 = 5r_0$, $a'_1 = 2.5r_0$, $r'_0 = 1/3r_0$, and $r''_0 = 2/3r_0$ with comoving time and coordinate time, respectively. Here $G = c = 1$, $r'_0 = 2m$, $r''_0 = 2(m + m_1)$, and $r_0 = 2(m + m_1 + m_2)$. 
3. “Frozen Star” or Black Hole?

The asymptotic behavior of the gravitational collapse is related to a well-known novel phenomenon predicted by general relativity, i.e., O sees a test particle falling towards a BH moving slower and slower, becoming darker and darker, and is eventually frozen near the event horizon of the BH. This process was also vividly described and presented in many popular science writings and textbooks. Because of this, the object of a complete gravitational collapse has been called a “frozen star”. A fundamental question can then be asked: “Does a gravitational collapse form a frozen star or a BH?” Alternatively one can also ask: “Can any matter ever fall into a BH, even if it does exist?” In both questions, the clock of O is referred to.

The answers to the above questions have been debated for decades. One answer is that since the comoving observer indeed has observed the test particle falling through the event horizon and reaching the singularity point, then in reality matter indeed has fallen into the BH and reached the singularity point. However, since O has no way to communicate with the comoving observer once matter crosses $R_H$, O has no way to ‘know’ if the test particle has fallen into the BH. The other answer is to invoke quantum effects. It has been argued that quantum effects may eventually bring matter into a BH, as seen by O. However, as pointed out recently, even in that case the BH will still take an infinite time to form and the pre-Hawking radiation will be generated by the accumulated matter just outside the event horizon. Thus both answers fail in the real world.

In desperation, we may take the attitude of “who cares?” When the test particle is sufficiently close to $R_H$, the redshift is so large that practically almost no signals from the test particle can be seen by O and apparently the test particle has no way of turning back, therefore the “frozen star” does appear “black” and is an infinitely deep “hole”. For practical purposes we may still call it a “BH”, whose total mass is also increased by the infalling matter. Apparently this is the view taken by most people in the astrophysical community: this is demonstrated by those similar arguments in many well-known textbooks and popular science writings by many well-known scientists. This is the reason that the “frozen star” terminology has almost disappeared completely from professional literature, although the issue had not been fully understood until very recently.

For example, recently Vachaspati et al. pointed out that matter accumulating just outside $R_H$ would produce pre-Hawking radiation. More than that, when two such “frozen stars” merge together, electromagnetic radiations will be released, in sharp contrast to the merging of two genuine BHs [i.e., all their masses are within $R_H$]; the latter can only produce gravitational wave radiation. Therefore the physical properties of “frozen stars” are fundamentally different from BHs. We therefore must answer these questions definitively.

Finally these questions are answered definitely and the above “frozen star” paradox is solved completely by Liu & Zhang. As shown in Figs. 2 and 3 taken from
Ref.4 matter cannot accumulate outside $R_H$, due to the increase of $R_H$ which swallows the matter falling in. The fundamental reason for the asymptotic behavior of a test particle is due to the negligence of the influence of the test particle to the global properties of the whole gravitating system, therefore $R_H$ would not change during the infalling process of the test particle. Therefore a BH can indeed be formed from gravitational collapse, and “frozen stars” cannot exist in the physical universe.

4. Black Hole or Singularity?
A BH has always been considered as a spacetime singularity. However Zhang classified BHs into three classes: mathematical BHs, physical BHs or astrophysical BHs. A mathematical BH is the vacuum solution of Einstein’s field equations of a point-like object, whose mass is completely concentrated at the center of the object, i.e., the singularity point. A physical BH is an object whose mass and charge are all within $R_H$, regardless of the distribution of matter within; consequently a physical BH is not necessarily a mathematical BH. Finally an astrophysical BH is a physical BH, which can be formed through astrophysical processes in the physical universe and within a time much shorter than or at most equal to the age of the universe.

From Figs. 2 and 3, it is clear that matter can never arrive at the singularity point, according to the clock of $O$. This means that astrophysical BHs in the physical universe are not mathematical BHs. Given that we do not yet know for sure if there are other channels (other than through gravitational collapse of matter) of forming BHs in the physical universe, we therefore suggest that spacetime singularity does not exist. This conclusion may sound surprising and against the common understanding of general relativity and BH physics. However we do not seem to have other alternatives, because we can only observe and study the formation process of an astrophysical BH from outside $R_H$, and thus for us, as external observers, matter can never arrive at the singularity point even after crossing $R_H$ and loss communications from us.

5. Applicability of the Birkhoff’s Theorem
The Birkhoff’s theorem states that the metric in the vacuum for a spherically distributed gravitational system is static and Schwarzschild. This means that only the exterior metric of a spherically distributed gravitational system is Schwarzschild, e.g., the metric in region I of Fig. 1 (left) and regions I & III of Fig. 1 (right) is not necessarily Schwarzschild, although there is no matter in these regions. Actually the proof of the Birkhoff’s Theorem requires that there is no matter between the given location to infinity; otherwise the time coordinate cannot be taken in the same way as that in the Schwarzschild metric. This is generally not well appreciated by researchers. For example, in doing gravitational lensing calculations, the metric everywhere is always taken as Schwarzschild, i.e., $h = 1$ in Eq. 1, if the system is spherically symmetric. This is obviously incorrect, as $h < 1$ except in the exterior region, i.e., the real vacuum. If, on the other hand, one forces $h = 1$ even in region...
I of Fig. 1 (left), the metric across the boundary between regions I and II would be discontinuous, i.e., non-physical.

Because Shapiro delay normally refers to the case \( h = 1 \) and \( h < 1 \) means extra delay, we call the time delay in the case of \( h < 1 \) Generalized Shapiro delay, which should be considered in calculating the light propagation time through, e.g., the dark matter halos of galaxies or clusters of galaxies. This means that negligence of this extra delay would over-estimate the mass of the system under investigation.

Fig. 4 (left) shows that \( h < 1 \) and is a function of time, i.e., the metric in region I is neither schwarzschild nor stationary, although there is no matter there. Fig. 4 (right) further shows that the motion of shell 1 (the inner shell) is influenced by the existence and motion of shell 2 (the outer shell). This is clearly against the common misconception that metric is only determined by the interior mass; this is a fundamental difference between Einstein’s general relativity and Newtonian gravity. However, this conclusion seems to contradict the well-known “Lemaître-Tolman-Bondi” metric inside a spherically symmetric distribution of matter,

\[
ds^2 = dT^2 - \frac{(\sqrt{E} - \frac{2M}{r}dT + dr)^2}{1 + E} - r^2 d\Omega^2, \tag{2}\]

which is expressed in the Painlévé-Gullstrand coordinates, and where \( E = E(T, r) \) is the energy function of the shell at \( r \) and \( M = M(T, r) \) is the gravitational mass inside \( r \). Clearly the metric at \( r \) is fully determined by \( E(T, r) \) and \( M(T, r) \).

To understand this apparent conflict, it is necessary to transform \( T \) in the Painlévé-Gullstrand coordinates to \( t \) in the Schwarzschild coordinates by,

\[
\left(\frac{\partial T}{\partial t}\right)^2 = 1 + E \quad \text{and} \quad 1 - \frac{2M}{r} \frac{\partial T}{\partial r} = \sqrt{\frac{2M}{r} + E}. \tag{3}\]

![Fig. 4](image)

Fig. 4. Left panel: the evolution of \( h(t) \) with the position of the outer boundary of the shell for the one shell case with \( a = 5r_0, a' = 2.5r_0, \) and \( r_0' = 1/8r_0 \). Right panel: the comparison of the evolution of the outer boundary of shell 1 between the case with (solid) or without (dashed) shell 2. The parameters of the shells are \( a_2 = 10r_0, a'_2 = 6r_0, a'_1 = 2.5r_0, a_1 = 5r_0, r_0' = 1/5r_0, \) and \( r_0'' = 2/5r_0 \), where \( r_0 \) is the Schwarzschild radius corresponding to the total gravitating mass of the system. The inset is the ratio of the solid line to the dashed line.
Solving these partial differential equations requires integrals from \( r \) to infinity, or to the outer boundary of the system to match the Schwarzschild metric. Therefore eventually the metric at \( r \) includes both \( E \) and \( M \) outside \( r \), if one uses the clock of \( O \). For example, \( h(r) \) in Eq. 1 has to be calculated inside the system when tracing light through it. Therefore, taking \( h(r) = 1 \), as commonly done, is only an approximation.

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References

1. J. R. Oppenheimer and H. Snyder, *Phys. Rev.* 56 (1939) 455.
2. T. Vachaspati, D. Stojkovic and L. M. Krauss, *Phys. Rev. D* 76 (2007) 024005.
3. S. N. Zhang, Astrophysical Black Holes in the Physical Universe, to appear in THE HEAVENS UNVEILED: Engaging Big Questions in Astronomy and Cosmology, eds. D. G. York, O. Gingerich, S.-N. Zhang, C. L. Harper, Jr. (Taylor & Francis Group LLC/CRC Press, 2010) (arXiv:1003.0291)
4. Y. Liu and S. N. Zhang, *Physics Letters B* 679 (2009) 88.
5. R. Ruffini and J. A. Wheeler, *Physics Today.* (January 1971) 30.
6. J. P. Luminet, *Black Holes* (Cambridge University Press, 1992)
7. K. S. Thorne, *Black Holes & Time Warps - Einstein’s Outrageous Legacy.* (W.W. Norton & Company, 1994)
8. M. C. Begelman, M. J. Rees, *Gravity’s fatal attraction - black holes in the universe.* (Scientific American Library, New York, 1998)
9. C. W. Misner, K. S. Thorne, J. A. Wheeler, *Gravitation.* (W.H. Freeman, New York, 1973)
10. S. Weinberg, *Gravitation And Cosmology: Principles and Applications of the General Theory of Relativity.* (Basic Books, New York:, 1977)
11. S. L. Shapiro and S. A. Teukolsky, *Black Holes, White Dwarfs and Neutron Stars.* (John Wiley & Sons, New York, 1983).
12. B. F. Schutz, *A first course in general relativity.* (Cambridge University Press, 1990)
13. P. Townsend, Lecture notes for a ‘Part III’ course ‘Black Holes’ given in DAMTP, Cambridge [gr-qc/9707012] (1997)
14. D. Raine and E. Thomas, *Black Holes - An Introduction.* (Imperial College Press, 2005)
15. V. P. Frolov and I.D. Novikov, *Black Hole Physics.* (Kluwer Academic Publishers, Dordrecht, 1998)
16. S. W. Hawking and G. F. R. Ellis, *The large scale structure of space-time.* (Cambridge University Press, 1973)
17. T. Vachaspati, [arXiv:0706.1203v1] (2007)
18. P. D. Lasky, A. W. C. Lun and R. B. Burston, *ANZIAM J.* 49 (1) (2007) 53.