GRÖBNER-SHIRSHOV BASES AND LINEAR BASES FOR FREE DIFFERENTIAL TYPE ALGEBRAS OVER ALGEBRAS

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ABSTRACT. We study a question which can be roughly stated as follows: Given a (unital or nonunital) algebra $A$ together with a Gröbner-Shirshov basis $G$, consider the free operated algebra $B$ over $A$, such that the operator satisfies some polynomial identities $\Phi$ which are Gröbner-Shirshov in the sense of Guo et al., when does the union $\Phi \cup G$ will be an operated Gröbner-Shirshov basis for $B$? We answer this question in the affirmative under a mild condition in our previous work with Wang. When this condition is satisfied, $\Phi \cup G$ is an operated Gröbner-Shirshov basis for $B$ and as a consequence, we also get a linear basis of $B$. However, the condition could not be applied directly to differential type algebras introduced by Guo, Sit and Zhang, including usual differential algebras.

This paper solves completely this problem for differential type algebras. Some new monomial orders are introduced which, together with some known ones, permit the application of the previous result to most of differential type algebras, thus providing new operated GS bases and linear bases for these differential type algebras. Versions are presented both for unital and nonunital algebras. However, a class of examples are also presented, for which the natural expectation in the question is wrong and these examples are dealt with by direct inspection.

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INTRODUCTION

In our previous work with Wang [25], we studied operated Gröbner-Shirshov (aka GS) bases for free operated algebras over associative algebras whose operator satisfies some polynomial identities. This paper continues the work of [25] and concentrates into differential type algebras.

0.1. From usual GS bases to operated GS bases.

Gröbner-Shirshov basis theory was invented by Shirshov [30] and Buchberger [6] in the sixties of last century. It becomes one of the main tools of computational algebra since then; see for instance [13, 2, 4]. In order to deal with algebras endowed with operators, Guo et al. introduced a GS basis theory in operated contexts in a series of papers [14, 38, 16, 11] (see also [3]) with the goal to attack Rota’s program [28] to classify “interesting” operators on algebras. For a global view about the state of art, we refer the reader to the survey paper [12] and for recent development, see [33, 35, 7, 34, 15, 25, 36, 37].

Guo et al. considered operators satisfying some polynomial identities, hence called operated polynomial identities (aka. OPIs). Via GS basis theory and the somewhat equivalent theory: rewriting systems, they could define when OPIs are GS. They are mainly interested into two classes of OPIs: differential type OPIs and Rota-Baxter type OPIs, which are carefully studied in [16, 38, 11].

In this paper we are interested into differential type OPIs and differential type algebras.

0.2. Free operated algebras over algebras.

Recently, there is a need to develop free operated algebras satisfying some OPIs over a fixed algebras and construct GS bases and linear bases for these free algebras as long as a GS basis is known for the given algebra. Ebrahimi-Fard and Guo [10] used rooted trees and forests to give explicit constructions of free noncommutative Rota-Baxter algebras on modules and sets; Lei and Guo [18] constructed the linear basis of free Nijenhuis algebras over associative algebras; Guo and Li [15] gave a linear basis of the free differential algebra over associative algebras by introducing the notion of differential GS bases.

As in [25], we want to consider a question which can be roughly stated as follows:

Question 0.1. Given a (unital or nonunital) algebra $A$ with a GS basis $G$ and a set $\Phi$ of OPIs, assume that these OPIs $\Phi$ are GS in the sense of [3, 16, 38, 11]. Let $B$ be the free operated algebra satisfying $\Phi$ over $A$. Under what conditions, $\Phi \cup G$ will be a GS basis for $B$?

We answer this question in the affirmative under a mild condition in [25, Theorem 5.9]. When this condition is satisfied, $\Phi \cup G$ is a GS basis for $B$ and as a consequence, we also get a linear basis of $B$. This result has been applied to all Rota-Baxter type OPIs, a class of differential type OPIs, averaging OPIs and Reynolds OPI in [25].

However, this result can NOT be applied directly to all differential type OPIs. The main reason is that under the chosen monomial order for differential type OPIs, the condition of [25, Theorem 5.9] is not satisfied.
The goal of this paper is to introduce some new monomial orders which permit to deal with nearly all free differential type algebras over algebras.

Moreover, we provide a class of counter-examples for which the expectation in Question 0.1 is wrong; see Theorems 4.12 and 4.13.

0.3. Differential type algebras over algebras.

A commutative algebra equipped with an operator satisfying the Leibniz rule, is called a differential algebra. The notion is originated by Ritt [27, 26] in the algebraic study of differential equations in the 1930s. This mathematical subject has been developed into a broad realm such as differential Galois theory [20, 32], differential algebraic geometry and differential algebraic groups [17]. In recent years, researchers began to investigate noncommutative differential algebras in order to broaden the scope of the theory to include path algebras, for instance, and to have a more meaningful differential Lie algebra theory [23, 24] and also from an operadic point of view [19, 9]. There are also some recent work dealing with other algebraic structures endowed with derivations [29, 8, 31].

As mentioned above, [25, Theorem 5.9] can NOT be applied directly to all differential type OPIs in order to obtain GS bases and linear bases for free differential type algebras over algebras. In this paper, we introduce some new monomial orders and solve completely this question.

Our method is as follows: We change a differential type OPI to an equivalent one in the sense of Definition 2.5; by choosing convenient monomial orders, we can apply [25, Theorem 5.9] to obtain a GS basis; as a consequence, we get a linear basis. We also deal with the problem in the nonunital case; see Theorem 2.15. Quite interestingly, there are some exceptional cases for which our method could not apply and we treat these cases directly.

We would like to mention that Guo and Li introduced the notion of differential GS bases to deal with the above problem in [15] and their method is completely different from ours; see Remark 5.12.

0.4. Outline of the paper.

This paper is organized as follows. In Section 1 some basic definitions and constructions of free objects in operated contexts are recalled and some new monomial orders are introduced. Section 2 contains an account of the theory of operated GS bases, which serve as a main tool in this paper; the main result of [25] is recalled and a nonunital version is presented: a remainder about the basic theory of rewriting systems. In Section 3, we consider differential type OPIs and they are replaced by some equivalent ones. Section 4 is devoted to operated GS bases and linear bases for free nonunital differential type algebras over an algebra, while the last section considers unital ones.

Notation: Throughout this paper, \( k \) denotes a base field. All the vector spaces and algebras are over \( k \).

1. New monomial orders on free operated semigroups and monoids

In this section, we recall the construction of free operated semigroups and monoids due to Guo [14]. We will define a new monomial order \( \leq_{dl} \) on free operated semigroups by combining several well orders and extend it to a monomial order \( \leq_{udl} \) on free operated monoids via Remark 1.2. The main results of this paper will highly depend on these two new monomial orders.
1.1. Free operated semigroups and monoids.

**Definition 1.1** ([14, 38, 16, 11]). An operated semigroup is a semigroup $S$ with a map $P : S \to S$ (which is not necessarily a homomorphism of semigroups). Operated monoids can be defined similarly. For operated $k$-algebra (resp. unital operated $k$-algebra), we ask additionally that the operator should be a $k$-linear map.

We use $\mathcal{OpAlg}_k$ (resp. $\mathcal{uOpAlg}_k$) to denote the category of operated $k$-algebras (resp. unital operated $k$-algebras) with obvious morphisms.

Throughout this section, $X$ denotes a set. Denote by $kX$ (resp. $S(X)$, $M(X)$) the free $k$-space (resp. free semigroup, free monoid) generated by $X$.

We recall the construction of the free operated semigroup over a given set $X$ due to Guo [14]. For a set $X$, denote by $[X]$ the set of the formal elements $[x], x \in X$. The inclusion $X \hookrightarrow X \sqcup [X]$, which identifies $X$ with the first component, induces an injective semigroup homomorphism $i_{0,1} : \Xi_0(X) := S(X) \hookrightarrow \Xi_1(X) := S(X \sqcup [X])$.

For $n \geq 2$, assume that we have constructed $\Xi_{n-2}(X)$ and $\Xi_{n-1}(X) = S(X \sqcup [\Xi_{n-2}(X)])$ endowed with an injective homomorphism of semigroups $i_{n-2,n-1} : \Xi_{n-2}(X) \hookrightarrow \Xi_{n-1}(X)$. We define the semigroup

$$\Xi_n(X) := S(X \sqcup [\Xi_{n-1}(X)])$$

and the natural injection $\text{Id}_X \sqcup [i_{n-2,n-1}] : X \sqcup [\Xi_{n-2}(X)] \hookrightarrow X \sqcup [\Xi_{n-1}(X)]$ induces an injective semigroup homomorphism $i_{n-1,n} : \Xi_{n-1}(X) = S(X \sqcup [\Xi_{n-2}(X)]) \hookrightarrow \Xi_n(X) = S(X \sqcup [\Xi_{n-1}(X)])$.

Define $\Xi(X) = \varprojlim \Xi_n(X)$ and the map sending $u \in \Xi_n(X)$ to $[u] \in \Xi_{n+1}(X)$ induces an operator $P_{\Xi(X)}$ on $\Xi(X)$.

The construction of the free operated monoid $\mathcal{M}(X)$ over a set $X$ is similar, by just replacing $S$ by $M$ everywhere in the construction.

**Remark 1.2.** We will use another construction of $\mathcal{M}(X)$. In fact, add a symbol $[1]$ to $X$ and form $\Xi(X \sqcup [1])$, then $\mathcal{M}(X)$ can be obtained from $\Xi(X \sqcup [1])$ by just adding the empty word $1$.

It is easy to see that $k\Xi(X)$ (resp. $k\mathcal{M}(X)$) is the free nonunital (resp. unital) operated algebra generated by $X$. More constructions of free objects in operated contexts can be found in [25].

1.2. Monomial orders.

We need some preliminaries about orders.

**Definition 1.3.** Let $X$ be a nonempty set.

(a) A preorder $\leq$ is a binary relation on $X$ that is reflexive and transitive, that is, for all $x, y, z \in X$, we have
   (i) $x \leq x$; and
   (ii) if $x \leq y, y \leq z$, then $x \leq z$. We denote $x =_X y$ if $x \leq y$ and $x \geq y$. If $x \leq y$ but $x \neq y$, we write $x < y$ or $y > x$.

(b) A pre-linear order $\leq$ on $X$ is a preorder $\leq$ such that either $x \leq y$ or $x \geq y$ for all $x, y \in X$.

(c) A linear order or a total order $\leq$ on $X$ is a pre-linear order $\leq$ such that $\leq$ is symmetric, that is, $x \leq y$ and $y \leq x$ imply $x =_X y$. 
(d) A preorder $\leq$ on $X$ satisfies the descending chain condition, if $x_1 \geq x_2 \geq x_3 \geq \cdots$, then there exists $N \geq 1$ such that $x_N = x_{N+1} = x_{N+2} \cdots$. A linear order satisfying the descending chain condition is called a well order.

Before giving the definition of monomial orders, we need to introduce the following notions.

**Definition 1.4** ([3, 16, 38, 11]). Let $Z$ be a set and $\star$ a symbol not in $Z$.

(a) Define $\mathfrak{M}^*(Z)$ to be the subset of $\mathfrak{M}(Z \cup \star)$ consisting of elements with $\star$ occurring only once.

(b) For $q \in \mathfrak{M}^*(Z)$ and $u \in \mathfrak{M}(Z)$, we define $q|_u \in \mathfrak{M}(Z)$ obtained by replacing the symbol $\star$ in $q$ by $u$. In this case, we say $u$ is a subword of $q|_u$.

(c) For $q \in \mathfrak{M}^*(Z)$ and $s = \sum_i c_i u_i \in k\mathfrak{M}(Z)$ with $c_i \in k$ and $u_i \in \mathfrak{M}(Z)$, we define

$$q|_s := \sum_i c_i q|_{u_i}.$$  

(d) Define $\mathfrak{E}^*(Z)$ to be the subset of $\mathfrak{E}(Z \cup \star)$ consisting of elements with $\star$ occurring only once. It is easy to see $\mathfrak{E}^*(Z)$ is a subset of $\mathfrak{M}^*(Z)$, so we also have the above definitions for $\mathfrak{E}^*(Z)$ by restriction.

**Definition 1.5** ([3, 16, 38, 11]). Let $Z$ be a set. We deonte $u < v$ if $u \leq v$ but $u \neq v$ for an order $\leq$.

(a) A monomial order on $S(Z)$ is a well-order $\leq$ on $S(Z)$ such that

$$u < v \Rightarrow uz < vz \text{ and } wu < wv \text{ for any } u, v, w, z \in S(Z);$$

a monomial order on $M(Z)$ is a well-order $\leq$ on $M(Z)$ such that

$$u < v \Rightarrow wuz < wvz \text{ for any } u, v, w, z \in M(Z).$$

(b) A monomial order on $\mathfrak{E}(Z)$ is a well-order $\leq$ on $\mathfrak{E}(Z)$ such that

$$u < v \Rightarrow q|_u < q|_v \text{ for all } u, v \in \mathfrak{E}(Z) \text{ and } q \in \mathfrak{E}^*(Z);$$

a monomial order on $\mathfrak{M}(Z)$ is a well-order $\leq$ on $\mathfrak{M}(Z)$ such that

$$u < v \Rightarrow q|_u < q|_v \text{ for all } u, v \in \mathfrak{M}(Z) \text{ and } q \in \mathfrak{M}^*(Z).$$

**Definition 1.6.**

(a) Given some preorders $\leq_{a_1}, \ldots, \leq_{a_k}$ on a set $X$ with $k \geq 2$, introduce another preorder $\leq_{a_1, \ldots, a_k}$ by imposing recursively

$$u \leq_{a_1, \ldots, a_k} v \Leftrightarrow \left\{ \begin{array}{ll} u <_{a_1} v, & \text{or} \\ u =_{a_1} v \text{ and } u \leq_{a_{i+1}, \ldots, a_k} v. & \end{array} \right.$$  

(b) Let $k \geq 2$ and let $\leq_{a_i}$ be a pre-linear order on $X_i$, $1 \leq i \leq k$. Define the lexicographical product order $\leq_{\text{lex}}$ on the cartesian product $X_1 \times X_2 \times \cdots \times X_k$ by recursively defining

$$(x_1, \ldots, x_k) \leq_{\text{lex}} (y_1, \ldots, y_k) \Leftrightarrow \left\{ \begin{array}{ll} x_1 <_{a_1} y_1, & \text{or} \\ x_1 =_{X_1} y_1 \text{ and } (x_2, \ldots, x_k) \leq_{\text{lex}} (y_2, \ldots, y_k), & \end{array} \right.$$  

where $(x_2, \ldots, x_k) \leq_{\text{lex}} (y_2, \ldots, y_k)$ is defined by induction, with the convention that $\leq_{\text{lex}}$ is the trivial relation when $k = 1$.

By the proof of [38, Lemma 5.4(a)], we have the following result:

**Lemma 1.7.**

(a) Let $k \geq 2$. Let $\leq_{a_1}, \ldots, \leq_{a_k}$ be pre-linear orders on $X$, and $\leq_{a_k}$ a linear order on $X$. Then $\leq_{a_1, \ldots, a_k}$ is a linear order on $X$.

(b) Let $\leq_{a_i}$ be a well order on $X_i$, $1 \leq i \leq k$. Then the lexicographical product order $\leq_{\text{lex}}$ is a well order on the cartesian product $X_1 \times X_2 \times \cdots \times X_k$. 
Let us first recall the well known degree lexicographical order.

**Definition 1.8.** Let $X$ be a set endowed with a well order $\leq_X$. For $u = u_1 \cdots u_r \in S(X)$ with $u_1, \ldots, u_r \in X$, define $\deg_X(u) = r$.

Introduce the degree lexicographical order $\leq_{\text{dlex}}$ on $S(X)$ by imposing, for any $u \neq v \in S(X)$, $u <_{\text{dlex}} v$ if

(a) either $\deg_X(u) < \deg_X(v)$, or

(b) $\deg_X(u) = \deg_X(v)$, and $u = mu_1n$, $v = mv_1n'$ for some $m, n, n' \in S(X)$ and $u_i, v_i \in X$ with $u_i <_X v_i$.

It is obvious that the degree lexicographic order $\leq_{\text{dlex}}$ on $S(X)$ is a well order and $\leq_{\text{dlex}}$ can be extended to $M(X)$ by setting $1$ to be the least element.

For two well ordered sets $X$ and $Y$, one obtains an extended well order on the disjoint union $X \sqcup Y$ by defining $a < b$ for all $a \in X$ and $b \in Y$.

We now define a linear order on $\mathcal{S}(X)$, by the following recursion:

(a) Let $u, v \in \mathcal{S}_{\text{dlex}_0}(X) = S(X)$. We define

\[ u \leq_{\text{dlex}_0} v \iff u \leq_{\text{dlex}} v. \]

(b) Assume that we have constructed a well order $\leq_{\text{dlex}_n}$ on $\mathcal{S}_n(X)$ for $n \geq 0$ extending all $\leq_{\text{dlex}_i}$ for any $0 \leq i \leq n - 1$. The well order $\leq_{\text{dlex}_n}$ induces a well order on $[\mathcal{S}_n(X)]$ and by imposing $a > b$ for all $a \in [\mathcal{S}_n(X)]$ and $b \in X$, we obtain an extended well order on $X \sqcup [\mathcal{S}_n(X)]$. Denote by $\leq_{\text{dlex}_{n+1}}$ the induced degree lexicographic order on $\mathcal{S}_{n+1}(X) = \mathcal{S}(X) \sqcup [\mathcal{S}_n(X)]$.

Obviously $\leq_{\text{dlex}_{n+1}}$ extends $\leq_{\text{dlex}_n}$. By a limit process, we get a preorder on $\mathcal{S}(X)$ which will be denoted by $\leq_{\text{dlex}}$.

It is easy to see that this is a linear order, but it is NOT a well order. An example is given by

\[ u[v] > [u[v]] > [[u[v]]] > \cdots \]

for $u, v \in X$.

Recall that an arbitrary element of $\mathcal{S}(X)$ has a unique expression $u = u_0u_1^*u_2^* \cdots u_r^*$, where $u_0, u_1, \ldots, u_r \in M(X)$ and $u_1^*, u_2^*, \ldots, u_r^* \in \mathcal{S}(X)$. Notice that $u_0, u_1, \ldots, u_r$ could be the empty word. Define the $P$-breath $|u|_P$ of $u = u_0u_1^*u_2^* \cdots u_r^*$ to be $r$.

**Definition 1.9** ([38, Definition 5.3]). For two elements $u, v \in \mathcal{S}(X)$,

(a) define

\[ u \leq_{\text{deg}_P} v \iff \deg_P(u) \leq \deg_P(v), \]

where the $P$-degree $\deg_P(u)$ of $u$ is the number of occurrence of $P = \lfloor \rfloor$ in $u$;

(b) define

\[ u \leq_{\text{deg}_X} v \iff \deg_X(u) \leq \deg_X(v), \]

where the $X$-degree $\deg_X(u)$ is the number of elements of $X$ occurring in $u$ (including the repetitive ones);

(c) define

\[ u \leq_{|P|} v \iff |u|_P \leq |v|_P. \]

**Definition 1.10.** For any $u \in \mathcal{S}(X)$, let $u_1, \ldots, u_n \in X$ be all the elements occurring in $u$ from left to right. If a half bracket $\lfloor \rfloor$ (resp. $\rceil$) is between $u_i$ and $u_{i+1}$, where $1 \leq i < n$, the $G$-degree of this half bracket is defined to be $i$; if there is a half bracket $\lfloor \rfloor$ (resp. $\rceil$) appearing on the left of $u_i$
(resp. on the right of \( u_n \)), we define the G-degree of this half bracket to be 0 (resp. \( n \)). We denote \( \deg_G(u) \) by the sum of the G-degree of all the half brackets in \( u \).

For \( u, v \in \mathcal{X}(X) \), define the G-degree order \( \leq_{dg} \) by

\[
u \leq_{dg} v \iff \deg_G(u) \leq \deg_G(v).
\]

It is easy to obtain the following lemma whose proof is thus omitted.

**Lemma 1.11.** The orders \( \leq_{dg} \), \( \leq_{dx} \), \( \leq_{br} \) and \( \leq_{dg} \) are pre-linear orders satisfying the descending chain condition.

By Lemma 1.7, we can define a linear order \( \leq_{dl} \) on \( \mathcal{X}(X) \),

\[
u \leq_{dl} v \iff \nu \leq_{dg}, dx, D_{lex} \nu \iff \begin{cases} u \leq_{dg} v, \text{ or } \\ u =_{dg} v \text{ and } u <_{dx} v, \text{ or } \\ u =_{dg} v, u =_{dx} v \text{ and } u <_{dg} v, \text{ or } \\ u =_{dg} v, u =_{dx} v, u =_{dg} v \text{ and } u \leq_{D_{lex}} v. \end{cases}
\]

**Lemma 1.12.** The order \( \leq_{dl} \) is a well order on \( \mathcal{X}(X) \).

**Proof.** Since \( \leq_{dl} \) is a linear order, we only need to verify that \( \leq_{dl} \) satisfies the descending chain condition. Let \( v_1 \geq_{dl} v_2 \geq_{dl} v_3 \geq_{dl} \cdots \in \mathcal{X}(X) \). Since the pre-linear order \( \leq_{dg} \), \( \leq_{dx} \) and \( \leq_{dg} \) satisfy the descending chain condition, there exist \( N \geq 1 \) and \( k \geq 0 \) such that

\[
\deg_P(v_N) = \deg_P(v_{N+1}) = \deg_P(v_{N+2}) = \cdots = k,
\]

\[
\deg_X(v_N) = \deg_X(v_{N+1}) = \deg_X(v_{N+2}) = \cdots
\]

and

\[
\deg_G(v_N) = \deg_G(v_{N+1}) = \deg_G(v_{N+2}) = \cdots.
\]

Thus all \( v_i \) with \( i \geq N \) belong to \( \mathcal{X}_k(X) \). The restriction of the order \( \leq_{D_{lex}} \) to \( \mathcal{X}_k(X) \) equals to the well order \( \leq_{D_{lex}} \), which by definition satisfies the descending chain condition, so the chain \( v_1 \geq_{dl} v_2 \geq_{dl} v_3 \geq_{dl} \cdots \) stabilizes after finite steps. \( \square \)

**Definition 1.13** ([38, Definition 5.6]). A preorder \( \leq_\alpha \) on \( \mathcal{M}(X) \) (resp. \( \mathcal{X}(X) \)) is called bracket compatible (resp. left compatible, right compatible) if

\[
u \leq_\alpha v \Rightarrow [u] \leq_\alpha [v], \text{ (resp. } wu \leq_\alpha wv, \text{ resp. } uw \leq_\alpha vw), \text{ for all } w \in \mathcal{M}(X) \text{ (resp. } \mathcal{X}(X))
\]

**Lemma 1.14** ([38, Lemma 5.7]). A well order \( \leq \) is a monomial order on \( \mathcal{M}(X) \) (resp. \( \mathcal{X}(X) \)) if and only if \( \leq \) is bracket compatible, left compatible and right compatible.

**Theorem 1.15.** The well order \( \leq_{dl} \) is a monomial order on \( \mathcal{X}(X) \).

**Proof.** Let \( u \leq_{dl} v \). It is obvious that preorders \( \leq_{dg} \) and \( \leq_{dx} \) are bracket compatible, left compatible and right compatible. This solves the case \( u \leq_{dg} v \) and that of \( u =_{dg} v \) and \( u <_{dx} v \). If \( u =_{dg} v, u =_{dx} v \text{ and } u <_{dg} v \), for \( w \in \mathcal{X}(X) \), obviously \( [u] <_{dg} [v] \), \( uw <_{dg} vw \) and \( uw <_{dg} vw \).

Now we only need to consider the case that \( u =_{dg} v, u =_{dx} v, u =_{dg} v \text{ and } u \leq_{D_{lex}} v \). Let \( \deg_P(u) = \deg_P(v) = n \). Since \( u, v \in \mathcal{X}_n(X) \), thus \( u \leq_{D_{lex}} v \). By the fact that the restriction of \( \leq_{D_{lex}} \) to \( \mathcal{X}_n(X) \) is equal to \( \leq_{D_{lex}} \), we have \( [u] \leq_{D_{lex}} [v] \) and \( [u] \leq_{dl} [v] \). Let \( w \in \mathcal{X}_m(X) \). One can obtain \( uw \leq_{D_{lex}} vw \) and \( uw \leq_{D_{lex}} vw \) for \( r = \max \{m, n\} \), so \( uw \leq_{dl} vw \) and \( uw \leq_{dl} vw \).

We are done. \( \square \)
Remark 1.16. In fact, we can define another monomial order \( \leq_{dl'} \) instead of \( \leq_{dl} \) on \( \mathfrak{Z}(Z) \), which keeps \([u,v] > [uv]\) for any \( u,v \in \mathfrak{Z}(Z) \). To this end, we only need to modify the definition of G-degree in \( \leq_{dl} \) as follows:

We modify the definition of G-degree in \( \leq_{dl} \) as follows: for any \( u \in \mathfrak{Z}(Z) \), let \( u_1, \ldots, u_n \in Z \) be all the elements occurring in \( u \) from left to right. If the half bracket \( [ \) (resp. \( ] \) ) is between \( u_i \) and \( u_{i+1} \) (1 \( \leq \) i < n), define the G-degree of this half bracket to be \( n - i \); if there is a half bracket \( [ \) (resp. \( ] \) ) appearing on the left of \( u_1 \) (resp. on the right of \( u_n \)), we define the G-degree of this half bracket to be \( n \) (resp. \( 0 \)). Denote \( \deg_G(u) \) by the sum of the G-degree of all the half brackets in \( u \).

Now we extend \( \leq_{dl} \) from \( \mathfrak{Z}(X) \) to \( \mathcal{M}(X) \).

Definition 1.17. Let \( X \) be a set with a well order. Let \( \dagger \) be a symbol which is understood to be \( [1] \) and write \( X' = X \cup \{ \dagger \} \). Consider the free operated semigroup \( \mathfrak{Z}(X') \) over the set \( X' \). The well order on \( X \) extends to a well order \( \leq \) on \( X' \) by setting \( z > \dagger \), for any \( z \in X \). Then the monomial order \( \leq_{dl} \) on \( \mathfrak{Z}(X') \) induces a well order \( \leq_{udl} \) on \( \mathcal{M}(X) = \mathfrak{Z}(X') \cup \{1\} \) (in which \( [1] \) is identified with \( \dagger \)), by setting \( u > 1 \) for any \( u \in \mathfrak{Z}(X') \).

Remark 1.18. Notice that by writing \( \dagger \) instead of \([1]\), we avoid counting the brackets of \([1]\) when computing \( P \)-degrees and G-degrees.

Theorem 1.19. The well order \( \leq_{udl} \) is a monomial order on \( \mathcal{M}(X) \) and \( \leq_{udl} \) restricts on \( \mathfrak{Z}(X) \) is the monomial order \( \leq_{dl} \).

Proof. For any \( x \in \mathcal{M}(X) \setminus \{1\} \), \( x >_{udl} 1 \). We have \([x] >_{udl} x \geq_{udl} \dagger \). Thus \( \leq_{udl} \) is bracket compatible. Clearly, \( \leq_{udl} \) is left and right compatible.

We record an important technical result.

Proposition 1.20. For any \( u,v \in \mathcal{M}(X) \setminus \{1\} \), we have

\[
\begin{align*}
(a) & \ [u,v] <_{udl} [uv] <_{udl} [u]v <_{udl} [u],[v] ; \\
(b) & \ u[1]v <_{udl} [u],v ; \\
(c) & \ [1],uv <_{udl} [u],v ; \\
(d) & \ u <_{udl} [1]u ; \\
(e) & \ u <_{udl} u[1].
\end{align*}
\]

Proof. Let \( u,v \in \mathcal{M}(X) \setminus \{1\} = \mathfrak{Z}(X') \).

(a) \( [u,v] <_{udl} [uv] \) holds because
\[
\deg_G([uv]) - \deg_G([u],v) = \deg_{X'}(v) > 0,
\]
and \( [u],v <_{udl} [u],v ] \) follows from \( [u],v <_{dgp} [u],v ] \) and \( [uv] <_{udl} u[v] \) is obtained from
\[
\deg_G([u],v] - \deg_G([uv]) = \deg_{X'}(u) > 0.
\]
(b) The inequality is valid as \( [1],v = u \dagger v u_{dgp} u[v]. \)
(c) The statement can be deduced from \( [1],uv = \dagger uv <_{dgp} u[v]. \)
(d) The assertion follows from \( \deg_{X'}(u) < \deg_{X'}(\dagger u) = \deg_{X'}([1],u). \)
(e) We infer the result from \( \deg_{X'}(u) < \deg_{X'}(\dagger u) = \deg_{X'}([1],u). \)

Next, we recall another monomial order \( \leq_{db} \) on \( \mathcal{M}(X) \) constructed by Gao, Guo, Sit and Zheng [38].

(a) Let \( u,v \in \mathcal{M}_0(X) \), define
\[
\leq_{db_0} \iff \leq_{dlex} v.
\]
By a limit process, one can get a monomial order \( \phi \) said to satisfy the OPI \( \phi \geq t \) if it extends \( \leq_{\text{db}} \) on \( \mathcal{M}(X) \) such that \( \leq_{\text{db}} \) extends \( \leq_{\text{db},1} \). For \( r \geq 0 \), denote \( \mathcal{M}_{n+1,r}(X) = \{ u \in \mathcal{M}_{n+1}(X) \mid |u|_P = r \} \). Let \( u, v \in \mathcal{M}_{n+1,r}(X) \), so they can be written as

\[
\begin{align*}
    u &= u_0[u_1^1][u_2^1] \cdots [u_r^1]u, \\
    v &= v_0[v_1^1][v_2^1] \cdots [v_r^1]v,
\end{align*}
\]

where \( u_0, u_1, \ldots, u_r, v_0, v_1, \ldots, v_r \in \mathcal{M}(X) \) and \( u_1^1, u_2^1, \ldots, u_r^1, v_1^1, v_2^1, \ldots, v_r^1 \in \mathcal{M}_n(X) \). Then impose

\[
    u \leq_{\text{lex}_n} v \iff (u_1^1, u_2^1, \ldots, u_r^1, u_0, u_1, \ldots, u_r) \leq_{\text{clex}} (v_1^1, v_2^1, \ldots, v_r^1, v_0, v_1, \ldots, v_r),
\]

where \( \leq_{\text{clex}} \) is the lexicographical order on \( \mathcal{M}_n(X)^r \times \mathcal{M}(X)^{r+1} \) induced by \( \leq_{\text{db}} \). Now the well order \( \leq_{\text{db},1} \) on \( \mathcal{M}_{n+1}(X) \) is given by

\[
    u \leq_{\text{db},1} v \iff u \leq_{\text{dgp,brp,lex}_n} v \iff \begin{cases} 
    u \leq_{\text{dgp}} v, \text{ or } \\
    u \geq_{\text{dgp}} v \text{ and } u <_{\text{brp}} v, \text{ or } \\
    u \geq_{\text{dgp}} v, u =_{\text{brp}} v \text{ and } u \leq_{\text{lex}_n+1} v.
\end{cases}
\]

By a limit process, one can get a monomial order \( \leq_{\text{db}} \) on \( \mathcal{M}(X) \); for details, see [38, Theorem 5.8].

2. Operator polynomial identities and operated GS bases

In this section, we recall basic facts about free operated algebras whose operator satisfies some polynomial identities and the GS basis theory in this setup. We will also recall differential type algebras which are the main objects in this paper.

2.1. Free operated algebras and operator polynomial identities.

In this subsection, we recall some basic notions and facts related to free operated algebras and operator polynomial identities.

Let \( X \) be a set.

**Definition 2.1** ([16, 38, 11]). We call an element \( \phi(x_1, \ldots, x_n) \in k \mathcal{E}(X) \) (resp. \( k \mathcal{M}(X) \)) with \( n \geq 1, x_1, \ldots, x_n \in X \) an operator polynomial identity (aka OPI).

From now on, we always assume that OPIs are multilinear, that is, they are linear in each \( x_i \).

**Definition 2.2** ([16, 38, 11]). Let \( \phi(x_1, \ldots, x_n) \) be an OPI. A (unital) operated algebra \( (A, P) \) is said to satisfy the OPI \( \phi(x_1, \ldots, x_n) \) if \( \phi(r_1, \ldots, r_n) = 0 \), for all \( r_1, \ldots, r_n \in A \). In this case, \( (A, P) \) is called a (unital) \( \phi \)-algebra and \( P \) is called a \( \phi \)-operator.

Generally, for a family of OPIs \( \Phi \), we call a (unital) operated algebra \( (A, P) \) a (unital) \( \Phi \)-algebra if it is a (unital) \( \phi \)-algebra for any \( \phi \in \Phi \). Denote the category of \( \Phi \)-algebras (resp. unital \( \Phi \)-algebras) by \( \Phi\mathcal{M}_q \) (resp. \( \Phi\mathcal{M}_q \)-algebras).

**Definition 2.3** ([16, 38, 11]). An operated ideal of an operated algebra is an ideal of the associative algebra closed under the action of the operator. The operated ideal generated by a subset \( S \) is denoted by \( \langle S \rangle_{\mathcal{C} \mathcal{P} \mathcal{I} \mathcal{G}} \) (resp. \( \langle S \rangle_{\mathcal{U} \mathcal{C} \mathcal{P} \mathcal{I} \mathcal{G}} \)).

Obviously the quotient of an operated algebra (resp. unital operated algebra) by an operated ideal is naturally an operated algebra (resp. unital operated algebra).

From now on, \( \Phi \) denotes a family of OPIs in \( k \mathcal{E}(X) \) or \( k \mathcal{M}(X) \). For a set \( Z \) and a subset \( Y \) of \( \mathcal{M}(Z) \), introduce the subset \( S_\Phi(Y) \subseteq k \mathcal{M}(Z) \) to be

\[
    S_\Phi(Y) := \{ \phi(u_1, \ldots, u_k) \mid u_1, \ldots, u_k \in Y, \ \phi(x_1, \ldots, x_k) \in \Phi \}.
\]
We will consider free (unital) $\Phi$-algebra over an associative algebra which is defined via the usual universal property; see for example [25, Proposition 4.7]. The following result gives a construction of free (unital) $\Phi$-algebra over an associative algebra.

**Proposition 2.4** ([25, Proposition 4.8]).

(a) Let $\Phi \subseteq \mathbb{k}\Xi(X)$ and $A = \mathbb{k}S(Z)/I_A$ an algebra. Then 
\[ F_{\Phi\mathbb{k}I}(A) := \mathbb{k}\Xi(Z)/\langle S_{\Phi}(\Xi(Z)) \cup I_A \rangle_{\mathbb{C}p\mathbb{p}I} \]
is the free $\Phi$-algebra generated by $A$.

(b) Let $\Phi \subseteq \mathbb{k}\mathfrak{M}(X)$ and $A = \mathbb{k}\mathfrak{M}(Z)/I_A$ a unital algebra. Then 
\[ F_{\Phi\mathbb{k}I}(A) := \mathbb{k}\mathfrak{M}(Z)/\langle S_{\Phi}(\mathfrak{M}(Z)) \cup I_A \rangle_{\mathbb{C}p\mathbb{p}I} \]
is the free unital $\Phi$-algebra over $A$.

We introduce a notion of equivalences between families of OPIs.

**Definition 2.5.** We say two families of OPIs $\Phi_1, \Phi_2 \subseteq \mathbb{k}\Xi(X)$ are equivalent in $\mathbb{k}\Xi(X)$ if
\[ \langle S_{\Phi_1}(\Xi(Z)) \rangle_{\mathbb{C}p\mathbb{p}I} = \langle S_{\Phi_2}(\Xi(Z)) \rangle_{\mathbb{C}p\mathbb{p}I} \]
for any set $Z$; we say two families of OPIs $\Phi_1, \Phi_2 \subseteq \mathbb{k}\mathfrak{M}(X)$ are equivalent in $\mathbb{k}\mathfrak{M}(X)$ if
\[ \langle S_{\Phi_1}(\mathfrak{M}(Z)) \rangle_{\mathbb{C}p\mathbb{p}I} = \langle S_{\Phi_2}(\mathfrak{M}(Z)) \rangle_{\mathbb{C}p\mathbb{p}I} \]
for any set $Z$.

2.2. Operated GS bases for free $\Phi$-algebras.

**Definition 2.6** ([3, 16, 38, 11]). Let $Z$ be a set, $\leq$ a linear order on $\mathfrak{M}(Z)$ and $f \in \mathbb{k}\mathfrak{M}(Z)$.

(a) Let $f \notin \mathbb{k}$. The leading monomial of $f$, denoted by $\bar{f}$, is the largest monomial appearing in $f$. The leading coefficient of $f$, denoted by $c_f$, is the coefficient of $\bar{f}$ in $f$. We call $f$ monic with respect to $\leq$ if $c_f = 1$.

(b) Let $f \in \mathbb{k}$ (including the case $f = 0$). We define the leading monomial of $f$ to be 1 and the leading coefficient of $f$ to be $c_f = f$.

(c) A subset $S \subseteq \mathbb{k}\mathfrak{M}(Z)$ is called monicized with respect to $\leq$, if each nonzero element of $S$ has leading coefficient 1. Obviously, each subset $S \subseteq \mathfrak{M}(Z)$ can be made monicized if we divide each nonzero element by its leading coefficient.

We need another notation. Let $Z$ be a set. For $u \in \mathfrak{M}(Z)$ with $u \neq 1$, as $u$ can be uniquely written as a product $u_1 \cdots u_n$ with $u_i \in Z \cup \{\mathfrak{M}(Z)\}$ for $1 \leq i \leq n$, call $n$ the breadth of $u$, denoted by $|u|$; for $u = 1$, we define $|u| = 0$.

**Definition 2.7** ([3, 16, 38, 11]). Let $\leq$ be a monomial order on $\Xi(Z)$ (resp. $\mathfrak{M}(Z)$) and $f, g \in \mathbb{k}\Xi(Z)$ (resp. $\mathbb{k}\mathfrak{M}(Z)$) be monic.

(a) If there are $w, u, v \in \Xi(Z)$ (resp. $\mathfrak{M}(Z)$) such that $w = \bar{f}u = v\bar{g}$ with $\max\{|\bar{f}|, |\bar{g}|\} < |w| < |\bar{f}| + |\bar{g}|$, we call 
\[ (f, g)_{w}^{uv} := fu - vg \]
the intersection composition of $f$ and $g$ with respect to $w$.

(b) If there are $w \in \Xi(Z)$ (resp. $\mathfrak{M}(Z)$) and $q \in \Xi^*(Z)$ (resp. $\mathfrak{M}^*(Z)$) such that $w = \bar{f} = q|_{\bar{g}}$, we call
\[ (f, g)_{w}^{q} := f - q|_{\bar{g}} \]
the inclusion composition of $f$ and $g$ with respect to $w$. 


Definition 2.8 ([3, 16, 38, 11]). Let \( Z \) be a set and \( \leq \) a monomial order on \( \mathcal{E}(Z) \) (resp. \( \mathcal{M}(Z) \)). Let \( G \subseteq k\mathcal{E}(Z) \) (resp. \( k\mathcal{M}(Z) \)).

(a) An element \( f \in k\mathcal{E}(Z) \) (resp. \( k\mathcal{M}(Z) \)) is called trivial modulo \( (G, w) \) for \( w \in \mathcal{E}(Z) \) (resp. \( \mathcal{M}(Z) \)) if

\[
f = \sum c_i q_i | s_i < w, \text{ where } c_i \in k, \ q_i \in \mathcal{E}^*(Z) \text{ (resp. } \mathcal{M}^*(Z) \text{) and } s_i \in G.
\]

(b) The subset \( G \) is called a GS basis in \( k\mathcal{E}(Z) \) (resp. \( k\mathcal{M}(Z) \)) with respect to \( \leq \), for all pairs \( f, g \in G \) monicized with respect to \( \leq \), every intersection composition of the form \( (f, g)_w^\mu \) is trivial modulo \( (G, w) \), and every inclusion composition of the form \( (f, g)_w^\tau \) is trivial modulo \( (G, w) \).

To distinguish from usual GS bases for associative algebras, from now on, we shall rename GS bases in operated contexts by operated GS bases.

Proposition 2.9. Let \( Z \) be a set and \( \leq \) a monomial order on \( \mathcal{M}(X) \). Then its restriction to \( \mathcal{E}(X) \) is also a monomial order. Moreover, \( G \subseteq k\mathcal{E}(Z) \) is an operated GS basis in \( k\mathcal{E}(Z) \) with respect to the restriction of \( \leq \) is equivalent to that \( G \) is an operated GS basis in \( k\mathcal{M}(Z) \) with respect to \( \leq \).

Theorem 2.10 ([3, 16, 38, 11]). (Composition-Diamond Lemma) Let \( Z \) be a set, \( \leq \) a monomial order on \( \mathcal{M}(Z) \) and \( G \subseteq k\mathcal{M}(Z) \). Then the following conditions are equivalent:

(a) \( G \) is an operated GS basis in \( k\mathcal{M}(Z) \).

(b) Denote

\[
\text{Irr}(G) := \mathcal{M}(Z) \setminus \{ q | s \in G, \ q \in \mathcal{M}^*(Z) \}.
\]

As a \( k \)-space, \( k\mathcal{M}(Z) = k\text{Irr}(G) \oplus \langle G \rangle_{u \in \text{Irr}(G)} \) and \( \text{Irr}(G) \) is a \( k \)-basis of \( k\mathcal{M}(Z) / \langle G \rangle_{u \in \text{Irr}(G)} \).

Theorem 2.11 ([3, 16, 38, 11]). (Composition-Diamond Lemma) Let \( Z \) be a set, \( \leq \) a monomial order on \( \mathcal{E}(Z) \) and \( G \subseteq k\mathcal{E}(Z) \). Then the following conditions are equivalent:

(a) \( G \) is an operated GS basis in \( k\mathcal{E}(Z) \).

(b) Denote

\[
\text{Irr}(G) := \mathcal{E}(Z) \setminus \{ q | s \in G, \ q \in \mathcal{E}^*(Z) \}.
\]

As a \( k \)-space, \( k\mathcal{E}(Z) = k\text{Irr}(G) \oplus \langle G \rangle_{u \in \text{Irr}(G)} \) and \( \text{Irr}(G) \) is a \( k \)-basis of \( k\mathcal{E}(Z) / \langle G \rangle_{u \in \text{Irr}(G)} \).

The following result refines slightly [25, Proposition 6.7].

Proposition 2.12. Let \( Z \) be a set and \( \leq \) a monomial order on \( \mathcal{E}(Z) \) (resp. \( \mathcal{M}(Z) \)). Clearly when restricted to \( \mathcal{S}(Z) \) (resp. \( \mathcal{M}(Z) \)), it is still a monomial order. Then a GS basis \( G \subseteq k\mathcal{S}(Z) \) (resp. \( k\mathcal{M}(Z) \)) with respect to the restriction of \( \leq \) is also an operated GS basis in \( k\mathcal{E}(Z) \) (resp. \( k\mathcal{M}(Z) \)) with respect to \( \leq \).

Definition 2.13 ([11]). (a) Let \( \Phi \subseteq k\mathcal{E}(X) \) be a family of OPIs. Let \( Z \) be a set and \( \leq \) a monomial order on \( \mathcal{E}(Z) \). We call \( \Phi \) GS on \( k\mathcal{E}(Z) \) with respect to \( \leq \) if \( S_\Phi(\mathcal{E}(Z)) \) is an operated GS basis in \( k\mathcal{E}(Z) \) with respect to \( \leq \).

(b) Let \( \Phi \subseteq k\mathcal{M}(X) \) be a family of OPIs. Let \( Z \) be a set and \( \leq \) a monomial order on \( \mathcal{M}(Z) \). We call \( \Phi \) GS on \( k\mathcal{M}(Z) \) with respect to \( \leq \) if \( S_\Phi(\mathcal{M}(Z)) \) is an operated GS basis in \( k\mathcal{M}(Z) \) with respect to \( \leq \).

For a given unital algebra \( A \) and a GS family of OPIs \( \Phi \) under some monomial order, one can give an operated GS basis of the free unital \( \Phi \)-algebra over \( A \) by [25, Theorem 5.9].
**Theorem 2.14** ([25, Theorem 5.9]). Let $X$ be a set and $\Phi \subseteq k\mathfrak{M}(X)$ a system of OPIs. Let $A = k\mathcal{M}(Z)/I_A$ be a unital algebra with generating set $Z$. Assume that $\Phi$ is operated GS on $Z$ with respect to a monomial order $\leq$ in $\mathfrak{M}(Z)$ and that $G$ is a GS basis of $I_A$ in $k\mathcal{M}(Z)$ with respect to the restriction of $\leq$ to $\mathcal{M}(Z)$.

Suppose that the leading monomial of any OPI $\phi(x_1, \ldots, x_n) \in \Phi$ has no subword in $\mathcal{M}(X)\setminus X$, and that for all $u_1, \ldots, u_n \in \mathfrak{M}(Z)$, $\phi(u_1, \ldots, u_n)$ vanishes or its leading monomial is still $\overline{\phi}(u_1, \ldots, u_n)$. Then $S_{\phi}(\mathfrak{M}(Z)) \cup G$ is an operated GS basis of $\langle S_{\phi}(\mathfrak{M}(Z)) \cup I_A \rangle_{A \in \text{G rings}}$ in $k\mathfrak{M}(Z)$ with respect to $\leq$.

We also have a nonunital version of this result, whose proof is similar to that of Theorem 2.14, so we omit it.

**Theorem 2.15.** Let $X$ be a set and $\Phi \subseteq k\mathfrak{Z}(X)$ a system of OPIs. Let $A = k\mathcal{S}(Z)/I_A$ be an algebra with generating set $Z$. Assume that $\Phi$ is operated GS on $Z$ with respect to a monomial order $\leq$ in $\mathfrak{Z}(Z)$ and that $G$ is a GS basis of $I_A$ in $k\mathcal{S}(Z)$ with respect to the restriction of $\leq$ to $\mathcal{S}(Z)$.

Suppose that the leading monomial of any OPI $\phi(x_1, \ldots, x_n) \in \Phi$ has no subword in $\mathcal{S}(X)\setminus X$, and that for all $u_1, \ldots, u_n \in \mathfrak{Z}(Z)$, $\phi(u_1, \ldots, u_n)$ vanishes or its leading monomial is still $\overline{\phi}(u_1, \ldots, u_n)$. Then $S_{\phi}(\mathfrak{Z}(Z)) \cup G$ is an operated GS basis of $\langle S_{\phi}(\mathfrak{Z}(Z)) \cup I_A \rangle_{A \in \text{G rings}}$ in $k\mathfrak{Z}(Z)$ with respect to $\leq$.

### 2.3. Remainder on rewriting systems.

We need some basic notions about rewriting systems in order to introduce OPIs of differential type. The basic references about rewriting systems are, for instance, [1] and the recent lecture notes [21].

**Definition 2.16.** Let $V$ be a $k$-space with a $k$-basis $Z$.

- (a) For $f = \sum_{w \in Z} c_w w \in V$ with $c_w \in k$, the support $\text{Supp}(f)$ of $f$ is the set $\{w \in Z \mid c_w \neq 0\}$. By convention, we take $\text{Supp}(0) = \emptyset$.
- (b) Let $f, g \in V$. We use $f + g$ to indicate the property that $\text{Supp}(f) \cap \text{Supp}(g) = \emptyset$. If this is the case, we say $f + g$ is a direct sum of $f$ and $g$, and use $f + g$ also for the sum $f + g$.
- (c) For $f \in V$ and $w \in \text{Supp}(f)$ with the coefficient $c_w$, write $R_w(f) := c_w w - f \in V$. So $f = c_u w - R_u(f)$.

**Definition 2.17.** Let $V$ be a $k$-space with a $k$-basis $Z$.

- (a) A term-rewriting system $\Pi$ on $V$ with respect to $Z$ is a binary relation $\Pi \subseteq Z \times V$. An element $(t, v) \in \Pi$ is called a (term-)rewriting rule of $\Pi$, denoted by $t \rightarrow v$.
- (b) The term-rewriting system $\Pi$ is called simple with respect to $Z$ if $t + v$ for all $t \rightarrow v \in \Pi$.
- (c) If $f = c_t - R_v(f) \in V$, using the rewriting rule $t \rightarrow v$, we get a new element $g := c_t v - R_v(f) \in V$, called a one-step rewriting of $f$ and denoted by $f \rightarrow_{\Pi} g$.
- (d) The reflexive-transitive closure of $\rightarrow_{\Pi}$ (as a binary relation on $V$) is denoted by $\rightarrow_{\Pi}^*$ and, if $f \rightarrow_{\Pi}^* g$, we say $f$ rewrites to $g$ with respect to $\Pi$.
- (e) Two elements $f, g \in V$ are joinable if there exists $h \in V$ such that $f \rightarrow_{\Pi}^* h$ and $g \rightarrow_{\Pi}^* h$; we denote this by $f \downarrow_{\Pi} g$.
- (f) A term-rewriting system $\Pi$ on $V$ is called terminating if there is no infinite chain of one-step rewriting $f_0 \rightarrow_{\Pi} f_1 \rightarrow_{\Pi} f_2 \cdots$.
- (g) A term-rewriting system $\Pi$ is called compatible with a linear order $\geq$ on $Z$, if $t > \bar{v}$ for each $t \rightarrow \bar{v} \in \Pi$. 
The following theorem supplies a method to show that a single OPI $\phi$ is GS by rewriting system. For convenience, we will use the notation $u = (u_1, \ldots, u_k) \in \mathbb{M}(Z)^k$ (similar for $u \in \Xi(Z)^k$), and $\phi(u)$ for $\phi(u_1, \ldots, u_k)$ in the following. We say that an element $f \in k\mathbb{M}(Z)$ is in $\phi$-normal form if no monomial of $f$ contains any subword of the form $\phi(u)$ with $u \in \mathbb{M}(Z)^k$.

**Theorem 2.18** ([11, Theorem 4.1]). Let $\phi(x_1, \ldots, x_k) \in k\mathbb{M}(X)$ be a multilinear OPI such that $\phi = \bar{\phi} - R(\phi)$, where $R(\phi)$ is in $\phi$-normal form. Suppose that, for any set $Z$, there is a monomial order $\preceq$ on $\mathbb{M}(Z)$, such that the following two conditions hold:

1. if $\phi(u), \phi(v) \in S_{\phi}(\mathbb{M}(Z))$ are such that $\bar{\phi}(u) = \alpha \beta$ and $\bar{\phi}(v) = \beta \gamma$ for some $\alpha, \beta, \gamma \in \mathbb{M}(Z)$ and $u, v \in \mathbb{M}(Z)^k$, then $R(\phi(u)) \gamma \downarrow_{\Pi_k(Z)} \alpha R(\phi(v))$, where the term-rewriting system

$$\Pi(\phi)(Z) := \left\{ q|_{\bar{\phi}(u)} \rightarrow q|_{R(\phi(u))} \mid u \in \mathbb{M}(Z)^k, q \in \mathbb{M}^*(Z) \right\}$$

is compatible with $\preceq$;

2. if $\bar{\phi}(u) = q|_{\phi(u)}$ for some $\star \neq q \in \mathbb{M}^*(Z)$ and $u, v \in \mathbb{M}(Z)^k$, then $\phi(v)$ is a subword of some $u_i$, $1 \leq i \leq k$.

Then the OPI $\phi$ is GS with respect to the order $\preceq$.

**Lemma 2.19.** Assume that $X = \{x_1, \ldots, x_n\}$. Let $\Phi \subset k\Xi(X)$ be a system of OPIs, $Z$ a set and $\preceq$ a monomial order on $\Xi(Z)$. Consider the rewriting system defined as follows

$$\Pi(\phi)(Z) := \left\{ q|_{\bar{\phi}(u)} \rightarrow q|_{\phi(u)} \mid \phi \in \Phi, u \in \Xi(Z)^k, q \in \Xi^*(Z) \right\}.$$

If $f, g \in k\Xi(Z)$ are joinable, then for any $w \in \Xi(Z)$ such that $w > \bar{f}$ and $w > \bar{g}$, we have $f - g$ is trivial modulo ($S_{\phi}(\Xi(Z)), w)$.

**Proof.** By the assumption, there exists $h \in \Xi(Z)$ such that $f \rightarrow_{\Pi_k(Z)} h$ and $g \rightarrow_{\Pi_k(Z)} h$. By the definition of rewriting, there exist $l_1, \ldots, l_s, q_1, \ldots, q_s, q'_1, \ldots, q'_t \in \Xi^*(Z)$ such that $f - h = \sum_{i=1}^s q_i|_{l_i}$ and $g - h = \sum_{j=1}^t q'_j|_{l'_j}$, where $q_i|_{l_i} \preceq f$, $1 \leq i \leq s$ and $q'_j|_{l'_j} \preceq g$, $1 \leq j \leq t$. Thus,

$$f - g = f - h + h - g = \sum_{i=1}^s q_i|_{l_i} - \sum_{j=1}^t q'_j|_{l'_j}$$

is trivial modulo ($S_{\phi}(\Xi(Z)), w)$.

3. Differential type OPIs

In this section, we recall differential OPIs due to Guo et al. [16, 11] and we will show that differential type OPIs appearing in their classification are equivalent to some new OPIs to which one can apply Theorem 2.15 and Theorem 2.14.

From now on $X = \{x, y\}$ is a set with two elements.

3.1. Unital differential type OPIs.

**Definition 3.1** ([16, 11]). An OPI $\phi \in k\mathbb{M}(X)$ is said to be of unital differential type if $\phi$ is of the form $[xy] - N(x, y)$, where $N(x, y)$ satisfies the following conditions:

1. $N(x, y)$ is linear in $x$ and $y$;
2. no monomial of $N(x, y)$ contains any subword of the form $[uv]$ for any $u, v \in \mathbb{M}(\{x, y\})\{1\}$;
(c) for any set Z and u, v, w ∈ ℳ(Z)\{1},

\[ N(uv, w) - N(u, vw) \rightarrow_{\Pi_{\phi}(Z)} 0, \]

where \( \Pi_{\phi}(Z) := \{ q|_{uv} \rightarrow q|_{N(u, v)} | u, v \in ℳ(Z)\{1}, \ q \in ℳ^*(Z) \}. \)

It is showed in [16, Theorem 5.7] that for an OPI \( \phi \) of differential type, \( S_{\phi}(ℳ(Z)) \) is an operated GS basis of \( \langle S_{\phi}(ℳ(Z)) \rangle_{U \in ℳ^p} \) with respect to some monomial order, under which the leading monomial of \( \phi \) is \([xy]\).

Guo, Sit and Zheng gave a list of OPIs of differential type and they conjectured that these are all possible OPIs of differential type [16].

**Conjecture 3.2** ([16, Conjecture 4.7]). The OPIs appearing in the list below are all the OPIs of differential type:

- **(UI1)** \([xy] - a(x[y] + [x]y) - b[x][y] - cxy \) with \( a^2 = a + bc \),
- **(UI2)** \([xy] - ab^2xy - bxy - a[y][x] + ab(y[x] + [y]x) \),
- **(UI3)** \([xy] - x[y] - a(x[1]y - [1]xy) \),
- **(UI4)** \([xy] - [x]y - a(x[1]y - xy[1]) \),
- **(UI5)** \([xy] - x[y] - [x]y - ax[1]y - bxy \),
- **(UI6)** \([xy] - \sum_{i,j \geq 0} \lambda_{ij}[1]^i[x][1]^j \) with the convention that \([1]^0 = 1 \),

where \( a, b, c, \lambda_{ij} \in k, \ i, j \geq 0 \).

Guo, Sit and Zheng proved this conjecture under the condition \( N(x, y) \in ℳ_c(X) \) in [16].

In the following result, we will replace each OPI (except possibly Type (UI6)) defined in Conjecture 3.2 by an equivalent one in the sense of Definition 2.5. Notice that we only consider (UI6) type in the case \( \lambda_{ij} = 0 \) unless \( i + j \leq 1 \).

**Theorem 3.3.** In the following statements, \( \lambda, \mu, \nu \in k \) are scalars:

(a) The OPI (UI1) is equivalent to one of

\[ (UI1'_a) \quad [x][y] - \lambda(x[y] + [x]y) + \mu[x]y + \nu xy, \quad \text{where} \quad \lambda^2 = \lambda \mu + \nu, \]

\[ (UI1'_b) \quad x[y] - [xy] + [x]y - \lambda xy, \]

\[ (UI1'_c) \quad |x] - \lambda x; \]

(b) the OPI (UI2) is equivalent to one of

\[ (UI2'_a) \quad [x][y] - \lambda(x[y] + [x]y) + \mu[yx] + \lambda^2 xy - \lambda \mu yx, \]

\[ (UI2'_b) \quad [x] - \lambda x; \]

(c) the OPIs (UI3) and (UI4) are equivalent to

\[ (UI3') = (UI4') \quad [x] + \lambda x[1] - (\lambda + 1)[1]x; \]

(d) the OPI (UI5) is equivalent to one of

\[ (UI5'_a) \quad x[y] - [xy] + [x]y - \lambda xy, \]

\[ (UI5'_b) \quad x[y] - [xy] + [x]y - x[1]y, \]

\[ (UI5'_c) \quad v; \]

(e) the OPI (UI6) is equivalent to one of

\[ (UI6'_a) \quad [x] + \lambda x[1] - (\lambda + 1)[1]x, \]

\[ (UI6'_b) \quad [x] - \lambda x, \]

\[ (UI6'_c) \quad x. \]
Proof. (a) For OPI \((U1)\), if \(b = 0\), we have \(a^2 = a\). If \(a = 0\), we get \((U1')\) with \(\lambda = c\); if \(a = 1\), we get \((U1'_a)\) still with \(\lambda = c\).

Now we assume \(b \neq 0\). Then \((U1)\) is equivalent to

\[
[x][y] + \frac{a}{b}([x][y] + [x][y]) - \frac{1}{b}[xy] + \frac{c}{b}xy,
\]

with \(a^2 = a + bc\). Replacing \(\frac{a}{b}, \frac{1}{b}\) and \(\frac{c}{b}\) by \(-\lambda, -\mu\) and \(\nu\) respectively, we get \((U1'_a)\).

(b) For OPI \((U2)\), if \(a = 0\), we get \((U2'_a)\) by replacing \(b\) with \(\lambda\), which is the same as \((U1'_a)\). If \(a \neq 0\), then \((U2)\) is equivalent to

\[
[x][y] - b(x[y] + [x][y]) - \frac{1}{a}[xy] + b^2xy + \frac{b}{a}yx.
\]

Replace \(b\) and \(-\frac{1}{a}\) by \(\lambda\) and \(\mu\) respectively, then we get \((U2'_a)\).

(c) For OPI \((U4)\), take \(x = 1\) and replace \(y\) by \(x\), we get the OPI \((U4')\):

\[
[x] + ax[1] - (a + 1)[1]x.
\]

Conversely, given \((U4')\), by replacing \(x\) by \(xy\), we get

\[
[x][y] + axy[1] - (a + 1)[1]xy,
\]

and by multiplying \((U4')\) by \(y\) on the right, we get

\[
[x]y + ax[1]y - (a + 1)[1]xy;
\]

the difference of the above two formulae is just the OPI \((U4)\):

\[
[x][y] - [x]y - a(x[1]y - xy[1]).
\]

So \((U4)\) and \((U4')\) are equivalent.

Similarly, the OPI \((U3)\) is equivalent to the OPI \((U4')\).

(d) For the OPI \((U5)\), take \(y = 1\), then we get

\[(R1)\quad (a + 1)x[1] + bx.\]

If \(a = -1\) and \(b \neq 0\), the OPI \((R1)\) is reduced to \(x\) which is the OPI \((U5'_a)\). It is obvious that the OPI \((U5'_a)\) induces \((U5)\).

If \(a = -1\) and \(b = 0\), the original OPI becomes

\[
[xy] - x[y] - [x]y + x[1]y,
\]

which is exactly \((U5'_a)\).

Now we assume \(a \neq -1\), then

\[(U5) + \frac{a}{a + 1}(R1)\cdot y = [xy] - x[y] - [x]y + \lambda xy,
\]

where \(\lambda = -\frac{b}{a + 1}\), which gives \((U5'_a)\). The other direction follows from the fact that by imposing \(y = 1\), \((U5'_a)\) becomes \(x[1] - \lambda x\), which is exactly \((R1)\). We have shown that for \(a \neq -1\), \((U5)\) is equivalent to \((U5'_a)\).

(e) For the OPI \((U6)\), we only consider the case \(\lambda_{ij} = 0\) unless \(i + j \leq 1\).

Now the OPI \((U6)\)

\[
[xy] - \lambda_{10}[1]xy - \lambda_{01}xy[1] - \lambda_{00}xy
\]

is equivalent to

\[
[x] - \lambda_{10}[1]x - \lambda_{01}x[1] - \lambda_{00}x.
\]
Take $x = 1$, then we get

$$\text{(R2)} \quad (1 - \lambda_{10} - \lambda_{01})[1] - \lambda_{00}. $$

If $\lambda_{10} + \lambda_{01} \neq 1$, by taking $\lambda = \frac{\lambda_{00}}{1 - \lambda_{10} - \lambda_{01}}$, (R2) is equivalent to $[1] - \lambda$. Now it is easy to see (U6) is equivalent to $[x] - \lambda x$, which is the OPI (U6').

If $\lambda_{10} + \lambda_{01} = 1$ and $\lambda_{00} = 0$, by taking $\lambda_{10} = -\lambda$, we have $\lambda_{01} = \lambda + 1$ and

$$[x] = \lambda_{10}[1]x - \lambda_{01} x[1] - \lambda_{00} x = [x] + \lambda[1]x - (\lambda + 1)x[1],$$

which is the OPI (U6').

If $\lambda_{10} + \lambda_{01} = 1$ and $\lambda_{00} \neq 0$, the OPI (U6) is equivalent to $x$ which is (U6').

3.2. Nonunital differential type OPIs.

We will also consider the nonunital version of differential OPIs and the corresponding classification problem.

**Definition 3.4.** An OPI $\phi \in k \Xi(X)$ is said to be of nonunital differential type if $\phi$ is of the form $[xy] - N(x, y)$, where $N(x, y)$ satisfies the following conditions:

(a) $N(x, y)$ is linear in $x$ and $y$;

(b) no monomial of $N(x, y)$ contains any subword of the form $[uv]$ for any $u, v \in \Xi(X)$;

(c) for any set $Z$ and $u, v, w \in \Xi(Z)$,

$$N(uv, w) - N(u, vw) \to_{\Pi_{\phi}(Z)} 0,$$

where $\Pi_{\phi}(Z) := \{ q \mid q \to_{\Pi_{\phi}(Z)} 0 \}$.

Since any nonunital differential type OPI can be seen as a unital differential type OPI, we can also propose the following conjecture:

**Conjecture 3.5.** Each nonunital differential type OPI $\phi$ is one of the OPIs in the following list:

- (N1) $[xy] - a[x]y + [x]y - b[x][y] - cxy$ where $a^2 = a + bc$,
- (N2) $[xy] - ab^2xy - bxy - a[y][x] + ab(y[x] + [y]x)$,
- (N3) $[xy] - x[y],$
- (N4) $[xy] - [x]y$,

where $a, b, c, \lambda_{ij} \in k$, $i, j \geq 0$.

**Remark 3.6.** Notice that the cases (N1) (resp. (N2)) are exactly (U1) (resp. (U2)). The cases (N3) (resp. (N4)) are obtained from (U3) (resp. (U4)) by deleting monomials containing $[1]$. However, when all $[1]$ disappear, the cases (N5) and (N6) become special cases of (N1).

We also have a nonunital version of Theorem 3.3, whose proof is similar and omitted. Notice that there is some difference in each case, since we can not take $x$ or $y$ to be 1.

**Theorem 3.7.** We have the following statements:

(a) The OPI (N1) is equivalent to one of

- (N1'$_1$) $[x][y] - \lambda [x][y] + [x][y] + \mu [xy] + \nu xy$, where $\lambda^2 = \lambda \mu + \nu$,
- (N1'$_2$) $x[y] - [xy] + [x][y] - \lambda xy$,
- (N1'$_3$) $[xy] - \lambda xy$;

(b) the OPI (N2) is equivalent to one of

- (N2'$_1$) $[x][y] - \lambda [x][y] + [x][y] + \mu [xy] + \lambda^2 xy - \lambda \mu xy$,
- (N2'$_2$) $[xy] - \lambda xy$;
(c) the OPI $(N3)$ is equivalent to

$$(N3') \quad x[y] - [xy];$$

(d) the OPI $(N4)$ is equivalent to

$$(N4') \quad [x]y - [xy].$$

4. Operated GS bases of free nonunital differential type algebras

In this section, we apply Theorem 2.15 to OPIs displayed in Theorem 3.7, under the monomial order $\leq_{dl}, \leq_{dr}$ or $\leq_{db}$, we obtain an operated GS basis as well as a linear basis for the free nonunital differential type algebra over a nonunital k-algebra. However, this method can not be applied to OPIs of type $(N1)_{f} = (N2)_{f}$ in Theorem 3.7, so we deal with this case directly; see Theorem 4.12 and Corollary 4.13.

4.1. Nonunital GS OPIs of differential type.

We first deal with the types $(N1)_{f}$ and $(N3')$. Our result reads as follows:

**Proposition 4.1.** Let $\phi \in k \Xi(X)$ be a nonunital differential OPI of type $(N1)_{f}$ or $(N3')$ in Theorem 3.7. Let $Z$ be a set. Then $S_{\phi}(\Xi(Z))$ is an operated GS basis in $k \Xi(Z)$ with respect to $\leq_{dl}$.

We deduce this result from Proposition 4.2 and Lemma 4.3.

For an OPI $\phi$ of type $(N1)_{f}$ or $(N3')$ appearing in Theorem 3.7, we may write

$$\phi(x, y) = x[y] - R(x, y).$$

By Theorem 1.19 and Proposition 1.20, for each set $Z$ and for all $u, v \in \Xi(Z)$, $\phi(u, v)$ vanishes or its leading monomial is $u[v]$ with respect to the monomial order $\leq_{dl}$ on $\Xi(Z)$.

**Proposition 4.2.** Let $\phi = x[y] - R(x, y) \in k \Xi(X)$ be an OPI. Let $Z$ be a set such that $\Xi(Z)$ is endowed with a monomial order. Assume that for all $u, v \in \Xi(Z)$, $\phi(u, v)$ vanishes or its leading monomial is $u[v]$. Denote the rewriting system

$$\Pi_{\phi}(Z) := \{q|u[v] \rightarrow q|R(u, v) \mid u, v \in \Xi(Z), \ q \in \Xi^*(Z)\}.$$ 

If $R(uw, w) \downarrow_{\Pi_{\phi}(Z)} uR(v, w)$, for any $u, v, w \in \Xi(Z)$, then $S_{\phi}(\Xi(Z))$ is an operated GS basis in $k \Xi(Z)$ with respect to the monomial order.

**Proof.** We prove the statement by checking all inclusion compositions and intersection compositions are trivial modulo $(S_{\phi}(\Xi(Z)), w)$.

We firstly consider inclusion compositions. Let $f = \phi(u, v), g \in S_{\phi}(\Xi(Z))$ for $u, v \in \Xi(Z)$, such that $w = f = g|_{\Xi}$ for some $w \in \Xi(Z)$ and $q \in \Xi^*(Z)$. We have the following three cases:

(a) If $q$ has a right factor $[v']$ with $v' \in \Xi(Z)$. Since $\bar{f} = u[v], \bar{v} = v'$ and thus there exists $q' \in \Xi^*(Z)$ such that $g = q'[v]$ and $u = q'|_{\Xi}$. So we have

$$\begin{align*}
(f, g)^q_w &= f - q|_w \\
&= \phi(u, v) - \phi(q'|_{\Xi}, v) + \phi(q'|_{\Xi}, v) - q|_w \\
&= \phi(q'|_{\Xi}, v) - \phi(q'|_{\Xi}, v) + q'|_{\Xi}[v] - R(q'|_{\Xi}, v) - q|_w \\
&= \phi(q'|_{\Xi}, v) - R(q'|_{\Xi}, v).
\end{align*}$$

By the monomial order, we have $\overline{\phi(q'|_{\Xi}, v)} < \overline{\phi(q'|_{\Xi}, v)} = w$ and $\overline{R(q'|_{\Xi}, v)} < \overline{\phi(q'|_{\Xi}, v)} = w$. Then since $\phi(q'|_{\Xi}, v)$ is trivial modulo $(S_{\phi}(\Xi(Z)), w)$ and $g \in S_{\phi}(\Xi(Z))$, the inclusion composition $(f, g)^q_w$ is trivial modulo $(S_{\phi}(\Xi(Z)), w)$.
(b) If \( q \) has a right factor \([v']\) with \( v' \in \mathcal{Z}(Z) \). Since \( f = u[v] \), we have \( v = v'|_g \) and \( q = u[v'] \), thus

\[
(f, g)_w^{a, b} = f - q|_g \\
= \phi(u, v'|_g) - u|v'|_g \\
= \phi(u, v'|_g) - \left( u|v'|_g - R(u, v'|_g) \right) - R(u, v'|_g) \\
= \phi(u, v'|_g) - \phi(u, v'|_g) - R(u, v'|_g) \\
= \phi(u, v'|_g - R(u, v'|_g),
\]

which is trivial modulo \((S_{\phi}(\mathcal{Z}(Z)), w)\).

(c) If \( q \) has no right factor of the form \([v']\), then \( \star \) is a right factor of \( q \) and \( \overline{g} \) has the right factor \([v]\), i.e., \( q = t\star \) and \( \overline{g} = s[v] \) for some \( t, s \in \mathcal{Z}(Z) \). So we have \( t\overline{g} = t\overline{f} \) and \( g = \phi(s, v) \). By assumption, we have \( R(ts, v) \downarrow_{\Pi_{\phi}(Z)} tR(s, v) \) and by Lemma 2.19 the inclusion composition \((f, g)_w^a\) is trivial modulo \((S_{\phi}(\mathcal{Z}(Z)), w)\).

Next we consider the intersection compositions. Given \( f = \phi(u_1, u_2) \) and \( g = \phi(v_1, v_2) \) with \( u_1, u_2, v_1, v_2 \in \mathcal{Z}(Z) \), assume there exist \( a, b, w, w' \in \mathcal{Z}(Z) \) such that \( w = t\overline{a} = b\overline{g} \), i.e., \( w = u_1|u_2|a = bv_1|v_2 \) and \( \max\{t|f|, |g|\} < |w| < |\overline{f}| + |\overline{g}| \). We will write the intersection composition \((f, g)_w^{a, b}\) into two inclusion compositions. Consider the polynomial \( h = \phi(bv_1, v_2) \). We have \( w = \overline{h} = q_1|_f = q_2|_g \), where \( q_1 = \star a \) and \( q_2 = b\star \). As we have shown above, the inclusion compositions \((h, f)_w^{a, f}\) and \((h, g)_w^{b, \star}\) are trivial modulo \((S_{\phi}(\mathcal{Z}(Z)), w)\). And since

\[
(f, g)_w^{a, b} = f - g = -(h - f) + (h - g) = -(h, f)_w^{a, \star} + (h, g)_w^{b, \star},
\]

the intersection composition \((f, g)_w^{a, b}\) is trivial modulo \((S_{\phi}(\mathcal{Z}(Z)), w)\).

We need a technical lemma which enables us to apply Proposition 4.2.

**Lemma 4.3.** Let \( \phi(x, y) = x[y] - R(x, y) \) be a nonunital differential OPI of type \((N1)'\) or \((N3')\) given in Theorem 3.7. Let \( Z \) be a set. Consider the rewriting system

\[
\Pi_{\phi}(Z) := \{ q|_{u[v]} \rightarrow q|_{R(u, v)} \mid u, v \in \mathcal{Z}(Z), q \in \mathcal{Z}^*(Z) \}.
\]

Then \( R(uv, w) \downarrow_{\Pi_{\phi}(Z)} uR(v, w) \), for any \( u, v, w \in \mathcal{Z}(Z) \).

**Proof.** For \( \phi \) of type \((N1)'\), the OPI \( \phi \) can be written as \( x[y] - R(x, y) \), where \( R(x, y) = [xy] - [x]y + cxy \). We have

\[
uR(v, w) = u[vw] - u[v]w + cuv \]

\[
\rightarrow [uvw] - [u]vw + cuv - ([uv] - [u]v + cuv)w + cuv \]

\[
= [uvw] - [uv]w + cuv \]

\[
R(uv, w).
\]

For the type \((N3')\) in Theorem 3.7, \( \phi(x, y) = x[y] - [xy] \) with \( R(x, y) = [xy] \). We have

\[
uR(v, w) = u[vw] \rightarrow [uvw] = R(uv, w).
\]

So in either case, \( R(uv, w) \downarrow_{\Pi_{\phi}(Z)} uR(v, w) \), for any \( u, v, w \in \mathcal{Z}(Z) \).

Proposition 4.2 and Lemma 4.3 implies immediately Proposition 4.1. This finishes our discussion for the types \((N1)'\) and \((N3')\).

With a proof similar to that of Type \((N3')\) under the order \( \leq_{dl} \) given in Remark 1.16, one can treat the case \((N4')\).
Lemma 4.4. For the OPI $\phi(x, y) = [x]y - [xy]$ of type (N4') in Theorem 3.7, $S_\phi(\mathbb{Z}(Z))$ is an operated GS basis in $k\mathbb{Z}(Z)$ with respect to the monomial order $\leq_{d_{\text{lex}}}$ for any set $Z$.

Proposition 4.5. Consider an OPI $\phi$ of type $(N1'_0)$ (resp. $(N2'_0)$). For each set $Z$, $S_\phi(\mathbb{Z}(Z))$ is an operated GS basis in $k\mathbb{Z}(Z)$ with respect to $\leq_{d_{\text{db}}}$.

Proof. The case $(N1'_0)$ (resp. $(N2'_0)$) follows from Lemma 5.1 (resp. Lemma 5.2) and the case $(N1'_1) = (N2'_1)$ can be obtained by direct inspection.

We have proved the following result:

Theorem 4.6. Each OPI in Theorem 3.7 is nonunital GS with respect to either $\leq_{d_{\text{dl}}}$, $\leq_{d_{\text{lf}}}$ or $\leq_{d_{\text{db}}}$.

4.2. Operated GS bases of free nonunital differential type algebras over algebras.

By Theorem 4.6, Theorem 2.15 can be applied directly to all differential type OPIs in Theorem 3.7 (except Type $(N1'_1) = (N2'_1)$) to obtain operated GS bases for free nonunital differential algebras over algebras.

Theorem 4.7. Let $Z$ be a set, $A = kS(Z)/I_A$ an algebra with a GS basis $B$ with respect to $\leq_{d_{\text{lex}}}$. Then by Theorem 4.6, Proposition 4.5.

(a) Let $\phi_1$ be an OPI of type $(N1'_0)$ or $(N2'_0)$. Then $S_{\phi_1}(\mathbb{Z}(Z)) \cup G$ is an operated GS basis of $\langle S_{\phi_1}(\mathbb{Z}(Z)) \cup I_A \rangle_{c_{\text{full}}} \subset k\mathbb{Z}(Z)$ with respect to $\leq_{d_{\text{db}}}$.

(b) Let $\phi_2$ be an OPI of type $(N1'_1)$ or $(N3')$. Then $S_{\phi_2}(\mathbb{Z}(Z)) \cup G$ is an operated GS basis of $\langle S_{\phi_2}(\mathbb{Z}(Z)) \cup I_A \rangle_{c_{\text{full}}} \subset k\mathbb{Z}(Z)$ with respect to $\leq_{d_{\text{rl}}}$.

(c) Let $\phi_3$ be an OPI of type (N4'). Then $S_{\phi_3}(\mathbb{Z}(Z)) \cup G$ is an operated GS basis of $\langle S_{\phi_3}(\mathbb{Z}(Z)) \cup I_A \rangle_{c_{\text{full}}} \subset k\mathbb{Z}(Z)$ with respect to $\leq_{d_{\text{db}}}$.

Then by Theorem 2.11, we obtain linear bases for these algebras. In the statement of the following result, write $[u]^1 = [u]$ and $[u]^{k+1} = [[u]^{k}]$ for $k \geq 1$. For a subset $G \subset S(Z)$, we also introduce two notations

$$\text{Irr}_G := \{ g \in G \mid g \not\in \text{M}(Z) \}$$

and

$$\text{Irr}_M(G) := \{ g \in G \mid g \not\in \text{M}(Z) \}.$$  

Note that $\text{Irr}_M(G) = \text{Irr}_G(G) \cup \{1\}$. In the case that $G$ is a GS basis of $S(Z)$ (resp. $M(Z)$), $\text{Irr}_G(G)$ (resp. $\text{Irr}_M(G)$) is a linear basis of $A = kS(Z)/(G)$ (resp. $A = kM(Z)/(G)$).

Theorem 4.8. With the same setup as in Theorem 4.7, we have the following statements:

(a) Let $B_0 = \text{Irr}(G)$ and for $n \geq 1$, define the set $B_n$ inductively as

$$B_0 \cup \bigcup_{r \geq 1} \{ u_0, v_1, \ldots, v_r | u_0, u_r \in \text{Irr}_M(G), u_i \in \text{Irr}_S(G), 1 \leq i \leq r, v_j \in B_{n-1}, 1 \leq j \leq r \}.$$  

Then the set $B = \bigcup_{n \geq 0} B_n$ is a linear basis of the free $\phi_1$-algebra $\mathcal{F}_{\phi_1, \text{full}}(A)$ over $A$.

(b) The union of $\text{Irr}(G)$ with

$$\bigcup_{r \geq 1} \{ [u]^{k} | u \in \text{Irr}_G(G), u_i \in \text{Irr}_S(G), 1 \leq i \leq r \}$$

is a linear basis of the free $\phi_2$-algebra $\mathcal{F}_{\phi_2, \text{full}}(A)$ over $A$.

(c) The union of $\text{Irr}(G)$ with

$$\bigcup_{r \geq 1} \{ [u_0, u_1, \ldots, [u_{r-1}, u_r]^{k} | u_0 \in \text{Irr}_G(G), u_i \in \text{Irr}_S(G), k_i \geq 1, 1 \leq i \leq r \}$$

is a linear basis of the free $\phi_3$-algebra $\mathcal{F}_{\phi_3, \text{full}}(A)$ over $A$.  


The proof is straightforward and is left to the reader.

**Remark 4.9.** Notice that the linear basis of a free nonunital differential type algebra over a nonunital algebra given in Theorem 4.8 (a) has the same form as that of free nonunital Rota-Baxter type algebras over algebras, because the corresponding GS bases share the same set of leading monomials, consisting of all \([u]v\) and \(g\) with \(u, v \in \mathfrak{Z}(Z), g \in G\). The phenomenon also occurs for the unital case; see Theorem 5.10 (a) and [25, Theorem 6.7].

As a special case of Rota-Baxter type, the same construction of linear bases for the free nonunital Nijenhuis algebras over algebras appeared in [18].

Notice that one cannot apply Theorem 2.15 to \([xy] - \lambda xy\) which is of type \((\mathcal{N}1) = (\mathcal{N}2)\) in Theorem 3.7, as each term has a subword \(xy\). Nevertheless, we could deal with this case directly.

We need to introduce the notions of linearly self-reduced sets and bases.

**Definition 4.10 ([5, Definition 1.2.1.2]).** Let \(V\) be a vector space with a chosen well ordered basis. Each nonzero element of \(V\), expressed as a linear combination of basis elements, has a leading basis element which is maximal among all basis elements having nonzero coefficients in this linear combination.

Let \(S\) be a subset of \(V\). A basis element is said to be linearly reduced with respect to \(S\) if it is not a leading basis element of any element of \(S\). More generally, an element \(f \in V\) is said to be linearly reduced with respect to \(S\), if all basis monomials having nonzero coefficients in \(f\) are linearly reduced with respect to \(S\).

A subset \(S \subset V\) is said to be linearly self-reduced if each element \(s \in S\) has its leading basis element with coefficient 1 and is linearly reduced with respect to \(S \setminus \{s\}\).

**Proposition 4.11 ([5, Proposition 1.2.1.6]).** Let \(V\) be a vector space with a chosen well ordered basis. Every finite dimensional subspace \(S \subset V\) has a unique linearly self-reduced basis.

Now we can consider OPIs of the type \((\mathcal{N}1) = (\mathcal{N}2)\).

**Theorem 4.12.** Let \(Z\) be a set and \(A = k\mathfrak{S}(Z)/I_{A}\) an algebra with a finite GS basis \(G\) with respect to \(\leq_{\text{dlex}}\). Let \(\phi_{4} = [xy] - \lambda xy\) be an OPI of type \((\mathcal{N}1) = (\mathcal{N}2)\) in Theorem 3.7. Denote by \(\pi : k\mathfrak{S}(Z) \to k\mathfrak{S}(Z) \to k\mathfrak{S}(Z)\) the canonical projection. Then \(\pi(G)\) is just the set of linear pairs of elements of \(G\). Let \(k\pi(G)\) be the subspace of \(k\mathfrak{S}(Z)\) spanned by \(\pi(G)\).

Let \(G^{(1)}\) be the unique linearly self-reduced basis of \(k\pi(G)\) and define

\[G' = \{|g| - \lambda g \mid g \in G^{(1)}\}.\]

Then \(S_{\phi_{4}}(\mathfrak{Z}(Z)) \cup G \cup G'\) is an operated GS basis of \(\langle S_{\phi_{4}}(\mathfrak{Z}(Z)) \cup I_{A} \rangle_{\mathfrak{c}} \setminus \mathfrak{l}_{\mathfrak{g}}\) in \(k\mathfrak{Z}(Z)\) with respect to \(\leq_{\text{db}}\).

**Proof.** We first prove that \(S_{\phi_{4}}(\mathfrak{Z}(Z)) \cup G \cup G'\) is a subset of \(\langle S_{\phi_{4}}(\mathfrak{Z}(Z)) \cup I_{A} \rangle_{\mathfrak{c}} \setminus \mathfrak{l}_{\mathfrak{g}}\). Since \(G^{(1)} \subset k\pi(G)\), any element in \(G' = \{|g| - \lambda g \mid g \in G^{(1)}\}\) can be written as a linear combination of the elements in the set \([\pi(g)] - \lambda \pi(g) \mid g \in G\). And for any \(g \in G\),

\[|\pi(g)| - \lambda \pi(g) = ([\pi(g) - g] - \lambda (\pi(g) - g)) + |g| - \lambda g \in \langle S_{\phi_{4}}(\mathfrak{Z}(Z)) \cup I_{A} \rangle_{\mathfrak{c}} \setminus \mathfrak{l}_{\mathfrak{g}},\]

since \([\pi(g) - g] - \lambda (\pi(g) - g) \in S_{\phi_{4}}(\mathfrak{Z}(Z))\) and \(|g| - \lambda g \in \langle I_{A} \rangle_{\mathfrak{c}} \setminus \mathfrak{l}_{\mathfrak{g}}\). It follows that \(G' \subset \langle S_{\phi_{4}}(\mathfrak{Z}(Z)) \cup I_{A} \rangle_{\mathfrak{c}} \setminus \mathfrak{l}_{\mathfrak{g}}\).

Now we show \(S_{\phi_{4}}(\mathfrak{Z}(Z)) \cup G \cup G'\) is an operated GS basis. Notice that for any \(u, v \in \mathfrak{Z}(Z)\), the leading monomial of \(\phi_{4}(u, v) = [uv] - \lambda uv\) is \([uv]\) with respect to \(\leq_{\text{db}}\). For any elements \(a_{1} \in S_{\phi_{4}}(\mathfrak{Z}(Z)), a_{2} \in G\) and \(a_{3} \in G'\), the leading monomials are of the forms \(\overline{a}_{1} = [uv], \overline{a}_{2} = w\)
and \( \overline{a}_3 = [z] \), where \( u, v \in \mathcal{E}(Z) \), \( w \in G \subset S(Z) \) and \( z \in G' \subset Z \). Hence, these three leading monomials have no overlap with each other. And since each of the sets \( S \phi_i(\mathcal{E}(Z)) \), \( G \) and \( G' \) is an operated GS basis, we only need to check the triviality of any inclusion composition \((f, g)^\eta_w \) modulo \((S \phi_i(\mathcal{E}(Z)) \cup G \cup G', w)\), for the cases \((f) \in S \phi_i(\mathcal{E}(Z)), g \in G\), \((f) \in S \phi_i(\mathcal{E}(Z)), g \in G'\) and \((f) \in G', g \in G\).

(a) For the case \( w = \overline{g} = q_\|\text{g} \) with \( f \in S \phi_i(\mathcal{E}(Z)), g \in G\). There exists \( q' \in \mathcal{E}^*(Z) \) such that \( q' = [q']_\|\) and \( f = [q'_\|\] - \( q'\|\_\)\).

If \( g \in kZ \), there exists a factor of \( q' \) in \( S(Z) \). Then we have

\[
(f, g)^\eta_w = f - q\|\_\| = [q'_\|\] - \( q'\|\_\) - \( q\|\_\) = ([q'_\|\] - \( q\|\_\)) - \lambda q'\|\_\).
\]

It is trivial modulo \((S \phi_i(\mathcal{E}(Z)) \cup G \cup G', w)\) since \( q'\|\_\) - \( q\|\_\) is trivial modulo \((S \phi_i(\mathcal{E}(Z)), w)\) and \( g \in G \).

If \( g \in G \setminus kZ \), it can be written as \( g = g_1 + \pi(g) \in k(S(Z) \setminus Z) \) and we have \( \overline{g} = \overline{g}_1 \). The inclusion composition \((f, g)^\eta_w \) can be written as

\[
(f, g)^\eta_w = f - q\|\_\| = [q'_\|\] - \( q'\|\_\) - \( q\|\_\) = ([q'_\|\] - \( q\|\_\)) - \lambda q'\|\_\).
\]

Since \( q'\|\_\) - \( q\|\_\) is trivial modulo \((S \phi_i(\mathcal{E}(Z)), w)\) and \( g \in G \), the first two terms are trivial modulo \((S \phi_i(\mathcal{E}(Z)) \cup G \cup G', w)\). Now we only need to consider the third term. Since \( G^1 \) is the linearly self-reduced basis of \( k\pi(G) \), we have \( \pi(g) = \sum_{i=1}^n k_i g_i \) where \( g_i \in G^1 \). Notice that we have \( \overline{g}_i \leq \overline{\pi(g)} \) for \( 1 \leq i \leq n \). Otherwise, there exist \( g_i, g_j \in G^1 \) such that \( \overline{g}_i = \overline{g}_j \), which contradicts to that \( G^1 \) is linearly self-reduced. If \( q' \) has a factor in \( S(Z) \), we have \( q'\|\_\) - \( q\|\_\) is trivial modulo \((S \phi_i(\mathcal{E}(Z)), w)\); otherwise, \( q' = \# \) and thus \( q'\|\_\) - \( q\|\_\) is trivial modulo \((G', w)\). So we have shown that the inclusion composition \((f, g)^\eta_w \) is trivial modulo \((S \phi_i(\mathcal{E}(Z)) \cup G \cup G', w)\).

(b) For the case \( w = \overline{g} = q_\|\text{g} \) with \( f \in S \phi_i(\mathcal{E}(Z)), g \in G' \). There exists \( q' \in \mathcal{E}^*(Z) \) such that \( q' = [q'_\|\] - \( q\|\_\)\). Since \( \overline{g} \in [Z] \), there exists a factor of \( q' \) in \( S(Z) \).

Thus we have

\[
(f, g)^\eta_w = f - q\|\_\| = [q'_\|\] - \( q'\|\_\) - \( q\|\_\) = ([q'_\|\] - \( q\|\_\)) - \lambda q'\|\_\).
\]

which is trivial modulo \((S \phi_i(\mathcal{E}(Z)) \cup G \cup G', w)\) since \( q'\|\_\) - \( q\|\_\) is trivial modulo \((S \phi_i(\mathcal{E}(Z)), w)\) and \( g \in G' \).

(c) For the case \( w = \overline{g} = q_\|\text{g} \) with \( f \in G' \), \( g \in G \). Write \( f = [f_1] - \lambda f_1 \), where \( f_1 \in G^1 \).

Since \( \overline{g} = q_\|\text{g} \in [Z] \), we have \( q' = \# \), \( f_1 = g \) and \( g = \pi(g) \in \pi(G) \). Since \( G^1 \) is the linearly self-reduced basis of \( k\pi(G) \), we have \( f_1 - g = \sum_{i=1}^n k_i g_i \), where \( g_i \in G^1 \) and \( \overline{g}_i < \overline{f}_i \) for \( 1 \leq i \leq n \). So the inclusion composition

\[
(f, g)^\eta_w = [f_1] - \lambda f_1 - [g] = ([f_1 - g] - \lambda (f_1 - g)) - \lambda g
\]

is trivial modulo \((S \phi_i(\mathcal{E}(Z)) \cup G \cup G', w)\), since \( [f_1 - g] - \lambda (f_1 - g) = \sum_{i=1}^n k_i ([g_i] - \lambda g_i) \) is trivial modulo \((G', w)\) and \( g \in G \).

This finishes the proof. \( \Box \)

By Theorem 2.11, we obtain a linear basis.
Theorem 4.13. Within the same setup as in Theorem 4.12, the union of $\text{Irr}(G)$ with
$$
\bigcup_{r \geq 1} \left\{ u_0[v_1]^{(k_1)} u_1 \cdots [v_r]^{(k_r)} u_r \mid u_i \in \text{Irr}(G) \cup \{1\}, 0 \leq i \leq r, v_j \in Z, [\tilde{g} \mid g \in G^{(1)}], k_j \geq 1, 1 \leq j \leq r \right\}
$$
is a linear basis of the free $\phi_A$-algebra $\mathcal{F}_{\mathfrak{a}_0}^{\phi_A}(A)$ over $A$.

Proof. By Theorem 4.12, the set
$$
\text{Irr}(S_{\phi_A}(\mathfrak{S}(Z) \cup G \cup G')) = \mathfrak{S}(Z) \setminus \left\{ q_{[s], q_{[uv]}} \mid s \in G \cup G', q \in \mathfrak{S}^{*}(Z), u, v \in \mathfrak{S}(Z) \right\}
$$
is a linear basis of $\mathcal{F}_{\mathfrak{a}_0}^{\phi_A}(A)$. Besides elements of $\text{Irr}(G)$, for a monomial $u_0[v'_1] u_1[v'_2] \cdots [v'_r] u_r$, with $r \geq 1, u_i \in \text{Irr}(G) \cup \{1\}, 0 \leq i \leq r, v'_j \in S(Z), 1 \leq j \leq r$, we should avoid all subwords of the form $[x y], x, y \in S(Z)$ and $[z]$ for $z \in Z$ being the leading basis element of an element of $G^{(1)} \subset kZ$. The result can be deduced from this observation. \hfill \square

5. Operated GS bases of free unital differential type algebras

In this section, we will show that under some monomial order, all unital differential type OPIs in Theorem 3.3 are GS and satisfy the conditions in Theorem 2.14. As a consequence, we can obtain an operated GS basis and a linear basis for each free unital differential type algebra over a unital algebra with a given GS basis.

5.1. Unital GS OPIs of differential type.

Lemma 5.1. The OPIs of type $\langle U1' \rangle$ in Theorem 3.3 are unital GS with respect to the monomial order $\unlhd_{db}$.

Proof. We will prove the lemma by checking such OPIs satisfy the conditions in Theorem 2.18. According to the monomial order $\unlhd_{db}$, the leading monomial of such an OPI is $[x][y]$. Consider the rewriting system defined by

$$
\Pi_{\phi}(Z) := \left\{ [u][v] \rightarrow \lambda [u][v] + \lambda u[v] - \mu [uv] - uvv \mid u, v \in \mathfrak{M}(Z) \right\}.
$$

For the first condition in Theorem 2.18, assume there are $u, v, v', w, \alpha, \beta, \gamma \in \mathfrak{M}(Z)$, such that $[u][v] = \alpha \beta$ and $[v'][w] = \beta \gamma$. Then we only need to consider the case $\alpha = [u], \beta = [v] = [v']$ and $\gamma = [w]$.

On one hand, we have

$$
R(\phi(u), v)\gamma = \lambda [u][v][w] + \lambda u[v][w] - \mu [uv][w] - uvv [w]
$$
$$
\rightarrow \lambda [u][v][w] + \lambda u(\lambda [v][w] + \lambda v[w] - \mu [vw] - vuv)
$$
$$
- b(\lambda [uv][w] - \mu [uvw] - uvw) - uvv [w]
$$
$$
= \lambda [u][v][w] + \lambda^2 u[v][w] + \lambda^2 uv[w] - \lambda mu [vw] - \lambda uvw
$$
$$
- \lambda [u][w] - \lambda mu[w] + \mu^2 [uvw] + \mu uvw - vyw [w]
$$
$$
= \lambda [u][v][w] + \lambda^2 u[v][w] - \lambda mu [vw]
$$
$$
- \lambda [u][w] + \mu^2 [uvw] - (\lambda - \mu)uvw,
$$

where the last equation holds since the coefficient of $uv [w]$ is $\lambda^2 - \lambda \mu - \nu$, which vanishes by definition.
On the other hand,
\[\alpha R(\phi(v, x))\]
\[= \lambda [u] [v] w + \lambda [u] v [w] - \mu [u] [vw] - \nu [u] vw\]
\[\rightarrow \lambda (\lambda [u] v + \lambda u [v] - \mu [uv] - \nu uvw)w + \lambda [u] v [w]\]
\[-\mu (\lambda [u] vw + \lambda u [vw] - \mu [uvw] - \nu uvw) - \nu [u] vw\]
\[= \lambda^2 [u] vw + \lambda^2 u [v] w - \lambda \mu [uv] w - \lambda uvw + \lambda [u] v [w]\]
\[-\lambda \mu [u] vw - \lambda \mu u [vw] + \mu^2 [uvw] + \mu uvw - \nu [u] vw\]
\[= \lambda [u] v [w] + \lambda^2 u [v] w - \lambda \mu u [vw]\]
\[-\lambda \mu [uv] w + \mu^2 [uvw] - (\lambda - \mu) uvw \quad (\text{by } \lambda^2 = \lambda \mu + \nu).\]

We have shown that \(R(\phi(u, v)) \downarrow_{\text{OPI}(Z)} \alpha R(\phi(v, x))\). For the second condition in Theorem 2.18, assume that there are \(q \neq \star \in \mathfrak{M}(Z)\) and \(u_1, u_2, v_1, v_2 \in \mathfrak{M}(Z)\), such that \([u_1][u_2] = q|_{[v_1][v_2]}\), then \([v_1][v_2]\) is a subword of \(u_1\) or \(u_2\).

We are done. \(\square\)

With a similar proof, we have the following result:

**Lemma 5.2.** The OPIs of type \((U1^*_\lambda)\) in Theorem 3.3 are unital operated GS with respect to the monomial order \(\leq_{\text{db}}\).

Before dealing with OPI of type \((U1^*_\lambda) = (U5^*_\lambda)\), we introduce the following lemma.

**Lemma 5.3.** Let \(\phi \in k \Xi(X)\) be a differential OPI of type \((U1^*_\lambda)\) in Theorem 3.3. Then \(S_\phi(\mathfrak{M}(Z) \setminus \{1\})\) is an operated GS in \(k \Xi(Z)\) with respect to \(\leq_{\text{dir}}\).

**Proof.** Replacing \(Z\) by \(Z \sqcup \dagger\) in Theorem 4.1, \(S_\phi(\Xi(Z \sqcup \dagger))\) is an operated GS basis in \(k \Xi(Z \sqcup \dagger)\) with respect to \(\leq_{\text{dir}}\). Then by the definition of the order \(\leq_{\text{dir}}\) on \(\mathfrak{M}(Z)\), which is induced by \(\leq_{\text{dir}}\) on \(\mathfrak{M}(Z) \setminus \{1\} = \Xi(Z \sqcup \dagger)\), the assertion follows. \(\square\)

We will not distinguish \(\dagger\) from \([1]\) in the following lemma.

**Lemma 5.4.** The OPIs of type \((U1^*_\lambda) = (U5^*_\lambda)\) in Theorem 3.3 are unital operated GS with respect to the monomial order \(\leq_{\text{dir}}\).

**Proof.** Let \(\phi(x, y) = x[y] - R(x, y)\) be the OPI of type \((U1^*_\lambda) = (U5^*_\lambda)\) and denote \(Z' = Z \cup [1]\).

Firstly, we will show that \(S_\phi(\Xi(Z')) \cup \{[1] - \lambda\}\) is an operated GS basis in \(k \Xi(Z)\) with respect to \(\leq_{\text{dir}}\). By Lemma 5.3, \(S_\phi(\Xi(Z'))\) is an operated GS basis in \(k \mathfrak{M}(Z)\). So we only need to consider the inclusion composition of \(f \in S_\phi(\Xi(Z'))\) and \(g = [1] - \lambda\). Let \(w = f = u[v] = q|_{[1]}\) where \(u, v \in \Xi(Z')\), then \(q\) has a left factor \(u\) or a right factor \([v]\), i.e., \(q = u[q']\) or \(q = q'[v]\) with \(q' \in \mathfrak{M}(Z)\).

For the first case, we have \(v = q'|_{[1]}\), and the inclusion composition
\[
(f, g)^\phi_{uw} = f - q|_{g}
\]
\[= \phi(u, q'|_{[1]}) - u[q'|_{[1]}] + u[q'|_{1}]
\[= -R(u, q'|_{[1]} - \lambda (u[q'|_{1}] - R(u, q'|_{1}))
\[= -R(u, q'|_{[1]} - \lambda \phi(u, q'|_{1}).
\]
If \( q' \) has a factor in \( S(Z) \), then \( \phi(u, q'_1) \) is trivial modulo \( \langle S(\mathcal{E}(Z')), w \rangle \): otherwise we have \( q' = \star \), thus \( \phi(u, q'_1) = u[1] - \lambda u \) is trivial modulo \( \langle [1] - \lambda \rangle, w \). And since \( R(u, q'_1) \) is trivial modulo \( \langle [1] - \lambda \rangle, w \), it follows that \( (f, g)_w^q \) is trivial modulo \( \langle S(\mathcal{E}(Z')) \cup \{ [1] - \lambda \} \rangle, w \).

For the second case, we have \( u = q'_1 \), and similar to the first case, the inclusion composition
\[
(f, g)_w^q = -R(u, q'_1) + \lambda \phi(q'_1, v)
\]
is trivial modulo \( \langle S(\mathcal{E}(Z')) \cup \{ [1] - \lambda \}, w \rangle \). So we have shown that \( S(\mathcal{E}(Z')) \cup \{ [1] - \lambda \} \) is an operated GS basis in \( k\mathcal{M}(Z) \).

Now we show that \( S(\mathcal{M}(Z)) \) is also an operated GS basis in \( k\mathcal{M}(Z) \). Since \( \mathcal{M}(Z) = \mathcal{E}(Z') \cup \{ 1 \} \), one can write
\[
S(\mathcal{M}(Z)) = \{ \phi(u, v) \mid u, v \in \mathcal{M}(Z) \}
\]
\[
= \{ \phi(u, v) \mid u, v \in \mathcal{E}(Z') \} \cup \{ \phi(u, 1), \phi(1, u), \phi(1, 1) \mid u \in \mathcal{E}(Z') \}
\]
\[
= S(\mathcal{E}(Z')) \cup \{ u[1] - \lambda u, [1]u - \lambda u, [1] - \lambda \mid u \in \mathcal{E}(Z') \}.
\]
So it is easy to see \( \langle S(\mathcal{E}(Z')) \cup \{ [1] - \lambda \} \rangle_{u \in \mathcal{E}(Z')} = \langle S(\mathcal{M}(Z)) \rangle_{u \in \mathcal{E}(Z')} \) and \( \text{Irr}(S(\mathcal{E}(Z')) \cup \{ [1] - \lambda \}) = \text{Irr}(S(\mathcal{M}(Z))) \). By Theorem 2.10, the set \( \text{Irr}(S(\mathcal{E}(Z')) \cup \{ [1] - \lambda \}) \) is a linear basis of the operated algebra \( k\mathcal{M}(Z)/(\mathcal{E}(Z')) \cup \{ [1] - \lambda \})_{u \in \mathcal{E}(Z')} \), i.e., \( \text{Irr}(S(\mathcal{M}(Z))) \) is a linear basis of \( k\mathcal{M}(Z)/(\mathcal{M}(Z))_{u \in \mathcal{E}(Z')} \). By Theorem 2.10 again, the set \( S(\mathcal{M}(Z)) \) is also an operated GS basis in \( k\mathcal{M}(Z) \) with respect to \( \leq \).

To deal with OPI of type \( (\mathcal{U}5)' \), we need a unital version of Proposition 4.2, whose proof is similar, so we omit it.

**Proposition 5.5.** Let \( \phi = x[y] - R(x, y) \in k\mathcal{M}(X) \) be an OPI. Let \( Z \) be a set such that \( \mathcal{M}(Z) \) is endowed with a monomial order. Assume that for all \( u, v \in \mathcal{M}(Z) \), \( \phi(u,v) \) vanishes or its leading monomial is still \( u[v] \). Let
\[
\Pi(\phi) := \{ q_{u[v]} \rightarrow q_{R(u,v)} \mid u, v \in \mathcal{M}(Z), q \in \mathcal{M}^*(Z) \}.
\]
If \( R(u,v) \downarrow \Pi(\phi) \) \( uR(v,w) \), for any \( u,v,w \in \mathcal{M}(Z) \), then \( S(\mathcal{M}(Z)) \) is an operated GS basis in \( k\mathcal{M}(Z) \) with respect to \( \leq \).

**Lemma 5.6.** The OPI of type \( (\mathcal{U}5)' \) in Theorem 3.3 is unital operated GS with respect to the monomial order \( \leq_{\text{udlt}} \).

**Proof.** For OPI \( \phi = x[y] - (\langle xy \rangle - [x]_y + x[1]_y) \) of type \( (\mathcal{U}5)' \), let \( R(x,y) = \langle xy \rangle - [x]_y + x[1]_y \).
Consider the rewriting system
\[
\Pi(\phi) := \{ q_{u[v]} \rightarrow q_{R(u,v)} \mid u, v \in \mathcal{M}(Z), q \in \mathcal{M}^*(Z) \}.
\]
Firstly we show \( R(u,v) \downarrow \Pi(\phi) \) \( uR(v,w) \). We have
\[
\begin{align*}
uR(v,w) &= u[vw] - u[v]w + uv[1]w \\
&\rightarrow [uvw] - [u]vw + u[1]vw - ([uv]w - [u]vw + u[1]vw) + uv[1]w \\
&= [uvw] - [uv]w + uv[1]w \\
&= R(u,v,w).
\end{align*}
\]
If \( u = 1 \) or \( v = 1 \), then \( \phi(u,v) = 0 \), and for \( u, v \in \mathcal{M}(Z) \), \( u[v] \) is always the leading monomial of \( \phi(u,v) \) by Proposition 1.20. It follows from Proposition 5.5 that \( S(\mathcal{M}(Z)) \) is an operated GS basis in \( k\mathcal{M}(Z) \).
Notice that we can not apply Proposition 5.5 to the proof of Lemma 5.4, since for the OPI φ of type (U1′) = (Us′) in Theorem 3.3, φ(u, v) ≠ u|v| if we take u = 1 and v ≠ 1. In fact, φ(1, v) = [1]v ≠ [v].

**Lemma 5.7.** Each OPI of the types (U1′) = (Us′), (U3′) = (U4′) and (Us′) = (Us′) in Theorem 3.3 is unital operated GS with respect to the monomial order ≤db.

**Proof.** It is easy to see each of these OPIs satisfies the conditions in Theorem 2.18, so the assertion follows.

We have considered all OPI in Theorem 3.3.

**Theorem 5.8.** Each OPI in Theorem 3.3 is unital operated GS with respect to either the monomial order ≤udl or ≤db.

### 5.2. Operated GS bases of free unital differential type algebras over algebras.

By Theorem 5.8, Theorem 2.14 implies immediately the following result.

**Theorem 5.9.** Let Z be a set and A = kM(Z)/IA a unital algebra with a GS basis G with respect to ≤dlex.

(a) Let φ1 be an OPI of type (U1′) or (Us′). Then Sφ1(M(Z)) ∪ G is an operated GS basis of (Sφ1(M(Z)) ∪ IA)uCset α in kM(Z) with respect to ≤db.

(b) Let φ2 be an OPI of type (U1′) = (Us′) and (Us′). Then Sφ2(M(Z)) ∪ G is an operated GS basis of (Sφ2(M(Z)) ∪ IA)uCset α in kM(Z) with respect to ≤udl.

(c) Let φ3 be an OPI of type (U3′) = (U4′) = (Us′). Then Sφ3(M(Z)) ∪ G is an operated GS basis of (Sφ3(M(Z)) ∪ IA)uCset α in kM(Z) with respect to ≤db.

(d) Let φ4 be an OPI of type (U1′) = (Us′) = (Us′). Then the free unital φ4-algebra Fφ4(M(Z))(A) is isomorphic to A.

(e) Let φ5 be an OPI of type (Us′) = (Us′). Then the free unital φ5-algebra Fφ5(M(Z))(A) vanishes.

By Theorem 2.10, we obtain linear bases for these algebras with a simple proof.

**Theorem 5.10.** With the same setup as in Theorem 4.7 then we have the following statements:

(a) Let Bn = IrrM(G) and for any n ≥ 1, define the set Bn inductively as follows

\[ \bigcup_{r≥0} \left\{ u_0[v_1]u_1 \cdots [v_r]u_r \mid u_0, u_r \in \text{Irr}_M(G), u_i \in \text{Irr}_S(G), 1 ≤ i ≤ r - 1, v_j \in B_{n-1}, 1 ≤ j ≤ r \right\}. \]

Then the set \( B = \bigcup_{n≥0} B_n \) is a linear basis of the free unital φ1-algebra \( F_{φ_1}(M(Z))(A) \) over the unital algebra A.

(b) The set

\[ \bigcup_{r≥0} \left\{ [\cdots [[u_r]^{(k_r)}u_{r-1}]^{(k_{r-1})} \cdots u_j]^{(k_j)}u_0 \mid u_0, u_i \in \text{Irr}_M(G), k_i ≥ 1, 1 ≤ i ≤ r \right\} \]

is a linear basis of the free unital φ2-algebra \( F_{φ_2}(M(Z))(A) \) over A.

(c) The set

\[ \bigcup_{r≥0} \left\{ u_0[1]^{k_1}u_1[1]^{k_2} \cdots [1]^{k_r}u_r \mid u_0, u_i \in \text{Irr}_M(G), u_i \in \text{Irr}_S(G), 1 ≤ i ≤ r - 1, k_j ≥ 1, 1 ≤ j ≤ r \right\} \]

is a linear basis of the free unital φ3-algebra \( F_{φ_3}(M(Z))(A) \) over A.
Example 5.11. By taking $a = 1$ and $c = 0$ in the type ($\Omega 1$) of Conjecture 3.2, we get the OPI:

$$\phi_{\text{Diff}}(x, y) = [xy] - x [y] - [x] y - b [x] [y].$$

A $\phi_{\text{Diff}}$-algebra is called a differential algebra of weight $b$. Given a unital algebra $A$ with a GS basis $G$, then we get a linear basis for the free unital differential algebra of nonzero weight over $A$ by Theorem 5.10 (a) and by Theorem 5.10 (b) in weight zero case.

Remark 5.12. In [15], Guo and Li introduced the notion of differential GS bases in order to study free differential algebras over algebras. They showed that the free differential algebra $F_{\phi_{\text{Diff}}-\text{uAlg}}(A)$ over a unital $k$-algebra $A = kM(X)/I_A$ is the same as the free differential algebra over the set $X$ modulo the differential ideal generated by the ideal $I_A$ [15, Proposition 2.7], which can be deduced from Proposition 2.4 as well. They also gave a linear basis of $F_{\phi_{\text{Diff}}-\text{uAlg}}(A)$ [15, Proposition 3.2].

Our method is completely different from theirs.

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References

[1] F. Baader, T. Nipkow, Term Rewriting and All That, Cambridge U. P., Cambridge, 1998.
[2] L. A. Bokut and Y. Chen, Gröbner-Shirshov bases and their calculations, Bull. Math. Sci. 4 (2014), 325-395.
[3] L. A. Bokut, Y. Chen and J. Qiu, Gröbner-Shirshov bases for associative algebras with multiple operators and free Rota-Baxter algebras, J. Pure Appl. Algebra 214 (2010) 89-110.
[4] L. A. Bokut, Y. Chen, K. Kalorkoti, P. Kolesnikov and V. Lopatkin, Gröbner-Shirshov Bases: Normal Forms, Combinatorial and Decision Problems in Algebra, World Scientific 2020.
[5] M.R. Bremner and V. Dotsenko, Algebraic operads: an algorithmic companion, Chapman and Hall/CRC, 2016.
[6] B. Buchberger, An algorithm for finding a basis for the residue class ring of a zero-dimensional polynomial ideal, Ph.D. thesis, University of Innsbruck (1965) (in German).
[7] D. Chen, Y.-F. Luo, Y. Zhang, Y.-Y. Zhang, Free $\Omega$-Rota-Baxter algebras and Gröbner-Shirshov bases, Internat. J. Algebra Comput. 30 (2020), no. 7, 1359-1373.
[8] A. Das, Leibniz algebras with derivations, J. Homotopy Relat. Struct. 16 (2021), no. 2, 245-274.
[9] M. Doubek and T. Lada, Homotopy derivations, J. Homotopy Relat. Struct. 11 (2016), no. 3, 599-630.
[10] K. Ebrahimi-Fard and L. Guo, Free Rota-Baxter algebras and rooted trees, J. Algebra Appl. 7 (2008), no. 2, 167-194.
[11] X. Gao and L. Guo, Rota’s Classification Problem, rewriting systems and Gröbner-Shirshov bases, J. Algebra 470 (2017), 219-253.
[12] X. Gao, L. Guo and H. Zhang, Rota’s program on algebraic operators, rewriting systems and Gröbner-Shirshov bases, arXiv:2108.11823.
[13] E. Green, An introduction to noncommutative Groebner bases, Lect. Notes Pure Appl. Math., 151 (1993), 167-190.
[14] L. Guo, Operated semigroups, Motzkin paths and rooted trees, J. Algebra Comb. 29 (2009), 35-62.
[15] L. Guo and Y. Li, Construction of free differential algebras by extending Gröbner-Shirshov bases, J. Symbolic Comput. 107 (2021), 167-189.
[16] L. Guo, W. Y. Sit and R. Zhang, Differential type operators and Gröbner-Shirshov bases, J. Symbolic Comput. 52 (2013), 97-123.
[17] E. R. Kolchin, Differential Algebras and Algebraic Groups, Academic Press, New York, 1973.
[18] P. Lei and L. Guo, Nijenhuis algebras, NS algebras, and N-dendriform algebras, Front. Math. China 7 (2012), no. 5, 827-846.
[19] J.-L. Loday, On the operad of associative algebras with derivation, Georgian Math. J. 17 (2010), 347-372.
[20] A. R. Magid, Lectures on differential Galois theory, University Lecture Series 7, American Mathematical Society, 1994.

[21] P. Malbos, Lectures on Algebraic Rewriting, available at http://math.univ-lyon1.fr/~malbos/ens.html.

[22] A. I. Shirshov, Some algorithmic problem for $\varepsilon$-algebras, Sibirsk. Mat. Z. 3 (1962), 132-137. (in Russian)

[23] L. Ponsot, Differential (Lie) algebras from a functorial point of view, Adv. Appl. Math. 72 (2016), 38-76.

[24] L. Ponsot, Differential (monoid) algebra and more, In: Post-Proceedings of Algebraic and Algorithmic Differential and Integral Operators Session, AADIOS 2012, Lecture Notes in Comput. Sci. 8372, Springer, 2014, 164-189.

[25] Z. Qi, Y. Qin, K. Wang and G. Zhou, Free objects and Gröbner-Shirshov bases in operated contexts, J. Algebra 584 (2021), 89-124.

[26] J. F. Ritt, Differential Algebra, Amer. Math. Soc. Colloq. Pub. 33 (1950), Amer. Math. Soc., New York.

[27] J. F. Ritt, Differential equations from the algebraic standpoint, Amer. Math. Soc. Colloq. Pub. 14 (1934), Amer. Math. Soc., New York.

[28] G. C. Rota, Baxter algebras and combinatorial identities I, II, Bull. Amer. Math. Soc. 75 (1969) 325-329, pp. 330-334.

[29] R. Tang, Y. Frégier and Y. Sheng, Cohomologies of a Lie algebra with a derivation and applications, J. Algebra 534 (2019), 65-99.

[30] A. I. Shirshov, Some algorithmic problems for $\varepsilon$-algebras, Sibirsk. Mat. Z. 3 (1962), 132-137.

[31] S. Xu, Cohomology, derivations and abelian extensions of 3-Lie algebras, J. Algebra Appl. 18 (2019), no. 7, 1950130, 26 pp.

[32] M. van der Put and M. Singer, Galois Theory of Linear Differential Equations, Grundlehren der Mathematischen Wissenschaften 328, Springer, Berlin, 2003.

[33] J. Zhang and X. Gao, Free operated monoids and rewriting systems, Semigroup Forum 97 (2018), no. 3, 435-456.

[34] T. Zhang, X. Gao and L. Guo, Reynolds algebras and their free objects from bracketed words and rooted trees, J. Pure Appl. Algebra 225 (2021), no. 12, Paper No. 106766, 28 pp.

[35] Y. Zhang and X. Gao, Free Rota-Baxter family algebras and (tri)dendriform family algebras, Pacific J. Math. 301 (2019), no. 2, 741-766.

[36] Z. Zhu. H. Zhang and X. Gao, Free weighted (modified) differential algebras, free (modified) Rota-Baxter algebras and Gröbner-Shirshov bases, arXiv:2108.03563.

[37] H. Zhang, X. Gao and L. Guo, Rota’s program on algebraic operators for Lie algebras, in preparation.

[38] S. Zheng, X. Gao, L. Guo and W. Sit, Rota-Baxter type operators, rewriting systems and Gröbner-Shirshov bases, J. Symb. Comput. (2021), in press, arXiv:1412.8055v1.