The probability for a wave packet to remain in a disordered cavity.

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We show that the probability that a wave packet will remain in a disordered cavity until the time $t$ decreases exponentially for times shorter than the Heisenberg time and log-normally for times much longer than the Heisenberg time. Our result is equivalent to the known result for time-dependent conductance; in particular, it is independent of the dimensionality of the cavity. We perform non-perturbative ensemble averaging over disorder by making use of field theory. We make use of a one-mode approximation which also gives an interpolation formula (arccosh-normal distribution) for the probability to remain. We have checked that the optimal fluctuation method gives the same result for the particular geometry which we have chosen. We also show that the probability to remain does not relate simply to the form-factor of the delay time. Finally, we give an interpretation of the result in terms of path integrals.

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The interest of experimentalists in open quantum dots motivates the computation of the probability to remain in a weakly disordered cavity. This problem is similar to the computation of the time-dependent conductance of weakly disordered samples. However the geometry of our problem allows a simple computation scheme, which is similar to the zero-dimensional non-linear $\sigma$-model.

Let us assume that we have a weakly disordered open cavity and the whole system is filled by non-interacting fermions at zero temperature. At time $t = 0$ the Fermi level outside of the cavity decreases and particles begin to escape from the cavity. The probability for the particles to remain in the cavity of volume $S$ till time $t$ is given by the ratio

$$p(t) = \frac{\int_S \rho(\vec{r}, t) d\vec{r}}{\int_S \rho(\vec{r}, 0) d\vec{r}},$$

where $\rho(\vec{r}, t)$ is the density of the particles on the Fermi surface, and $\rho(\vec{r}, 0) = 1$ for $\vec{r} \in S$ and 0 elsewhere. On a short time scale this is the standard problem of the diffusion emission. The time evolution of the distribution function can be found by solving the diffusion equation

$$\left[ \frac{\partial}{\partial t} - D \nabla^2 \right] \rho(\vec{r}, t) = 0$$

with the boundary conditions $n \vec{\nabla} \rho = 0$ at the walls of the cavity ($\vec{n}$ is normal to the boundary) and $\rho + 0.71 \ell n \vec{\nabla} \rho = 0$ at the open edge of the cavity. Here $\ell$ is the mean free path and the scattering is uniform. In further calculations we will use the open edge boundary condition $\rho = 0$, assuming that $\ell$ is much smaller than all the characteristic lengths of the system.

The solution of the diffusion equation can be represented as a sum over diffusion modes, each of them decaying exponentially at its own rate. The “lowest” diffusion mode is computed in Appendix for two examples of circular and spherical cavities with the contact in the middle, see Fig. 1. Therefore the probability to remain in the cavity behaves like $p(t) = e^{-\gamma t}$, where $\gamma$ is the decay rate of the “lowest” diffusion mode. This escape rate is proportional to the size of the contact and may vary from zero to the inverse diffusion time through the system $E_c/\hbar$. In the rest of the paper we will use units where $\hbar = 1$, and then we have in general

$$0 \leq \gamma \leq E_c.$$

![FIG. 1. Boundary conditions for the diffusion mode in the circular cavity with the contact in the middle.](image-url)

In quantum mechanics the evolution of the density matrix is given by the product of two exact Green functions

$$\rho(\vec{r}_1, \vec{r}_4, t) = \int d\vec{r}_2 d\vec{r}_3 G^R(\vec{r}_1, \vec{r}_2, t) G^A(\vec{r}_3, \vec{r}_4, t) \rho(\vec{r}_2, \vec{r}_3, 0).$$

(4)
In order to compute the probability to remain in the cavity we should prepare our system in the state

\[
\rho(\vec{r}_2, \vec{r}_3, 0) = \int d\vec{p} e^{i\vec{p}(\vec{r}_2 - \vec{r}_3)} \rho \left( \frac{\vec{r}_2 + \vec{r}_3}{2} \right),
\]

\[
\delta(E_{F'} - \mathcal{H}(\vec{p}, \frac{\vec{r}_2 + \vec{r}_3}{2}))
\]

where \(\mathcal{H}(\vec{p}, \vec{r})\) is the Hamiltonian of the system, and \(\rho(\vec{r}, 0)\) is one inside the cavity and zero outside. The quantum-mechanical expression for the probability to remain is therefore

\[
p(t) = \frac{\int_S \rho(\vec{r}, \vec{r}, t) d\vec{r}}{\int_S \rho(\vec{r}, \vec{r}, 0) d\vec{r}} = \frac{1}{2\pi \nu S} \int \frac{d\omega}{2\pi} e^{-i\omega t}
\]

\[
\times \int_S d\vec{r}_1 d\vec{r}_2 G^{R}_{E_{F'} + \omega} (\vec{r}_1, \vec{r}_2) G^{A}_{E_{F'} - \omega} (\vec{r}_2, \vec{r}_1),
\]

where \(\nu\) is the density of states on the Fermi surface.

The exact Green functions depend on the realizations of the disorder potential. The non-perturbative averaging over the \(\delta\)-correlated disorder potential can be done by making use of field theory. Our main result is

\[
p(t) = \exp \left\{ -\frac{\pi \gamma}{2\Delta} \text{arccosh}^2 \left( \frac{t\Delta}{\pi} + 1 \right) \right\},
\]

(7)

where \(\Delta\) is the mean level spacing of the “closed” cavity \(1/\Delta = \nu S\) and \(\gamma\) is the classical escape rate. This formula gives exponential decay on the short time scale and \(\log - \text{normal decay on the long time scale}.\) Equation (7) was derived under the conditions

\[
\gamma \ll E_c, \quad t \gg E_c^{-1}, \quad \log(t\Delta) \ll \begin{cases} L_1 \log(L_2/L_1)/\ell & \text{2 dimensions}, \\ L_1/\ell & \text{3 dimensions}, \end{cases}
\]

for all relation of \(\gamma\) and \(\Delta\). Particularly, the exact probability to remain should have some features on the time scale \(E_c^{-1}\) near \(t = 0\) and \(t = 2\pi/\Delta\), as it takes place for spectral form-factors of closed systems. Our result Eq. (8) does not have any structure on the time scale \(E_c^{-1}\) and this is the meaning of the conditions Eqs. (8a) and (8b). If the last inequality Eq. (8c) is not fulfilled one should use the ballistic action in the field theory.

It is interesting to compare the result Eq. (8) with the form-factor of the delay time

\[
K_\gamma(t) = \frac{1}{2\pi \nu S} \int \frac{d\omega}{2\pi} e^{-i\omega t} T(E_{F'} + \frac{\omega}{2}) T(E_{F'} - \frac{\omega}{2})
\]

\[
\approx \frac{1}{2\pi \nu S} \int \frac{d\omega}{2\pi} e^{-i\omega t}
\]

\[
\times \int_S d\vec{r}_1 d\vec{r}_2 G^{R}_{E_{F'} + \omega} (\vec{r}_1, \vec{r}_1) G^{A}_{E_{F'} - \omega} (\vec{r}_2, \vec{r}_2),
\]

(9)

where the subscript \(\gamma\) means that our system is open (it has the classical escape rate \(\gamma\)). The Schrödinger equation for the cavity has solutions, which are expanding waves far away from the system. The corresponding eigenvalues of the energy are complex \(E_n - i\Gamma_n/2\), see Ref. 13, §134. The delay time is the analog of the density of states \(T(E) = \text{Im} \sum_n (E - E_n + i\Gamma_n/2)\) and it can be expressed in terms of the scattering matrix. The disorder averaging leads to

\[
K^{\text{unit}}_\gamma(t) \approx K^{\text{unit}}_0(t) p(t),
\]

(10)

\[
K^{\text{unit}}_0(t) = \min \left( \frac{t\Delta}{2\pi} - 1 \right),
\]

(11)

where the superscript “unit” means that the form-factor was computed as if the system is not symmetrical under the time reversal, and \(p(t)\) is given by Eq. (6). From the definition Eq. (6) one can see that the form factor of the delay time \(K_\gamma(t)\) approaches the form factor of the density of states \(K_0(t)\) when \(\gamma\) goes to zero and this is consistent with Eqs. (6) and (11). The decay rate of the lowest diffusion mode, \(\gamma \to 0\), when the opening of the cavity becomes smaller \(L_1 \to 0\), see Eqs. (A3) and (A7). In this case all particles remains in the cavity forever and our solution gives \(p(t) \to 1\).

Our results are inconsistent with the random matrix theory, which predicts a power law decay of both \(K_\gamma(t)\) and \(p(t)\) if the number of open channels is much smaller than the dimensionality of the Hamiltonian. However, on the short time scale, \(t\Delta \ll 1\), one has from Eq. (8)\n
\[
- \log(p(t)) = \gamma t \left( 1 - \frac{t\Delta}{6\pi} \right)
\]

(12)

and this is similar to the numeric results of Ref. 16 and the random matrix theory calculations of Ref. 17. The relation between \(K_\gamma(t)\) and \(p(t)\) similar to Eq. (11) appears in the random matrix theory too.

Before going on to explain the averaging procedure let us give the semiclassical path – integral interpretation of the above result. Expansion of the Green function in a sum over classical paths \(j\) from the point \(\vec{r}_1\) to \(\vec{r}_2\) is

\[
G^{R,A}_{E'} (\vec{r}_1, \vec{r}_2) = \sum_j A_j \exp \left\{ \pm iS_j (\vec{r}_1, \vec{r}_2, E) \right\}
\]

(13)

where \(S_j (\vec{r}_1, \vec{r}_2, E)\) is the action along the path and \(A_j\) are some coefficients. Therefore, the probability to remain is the double sum over trajectories, which can be separated into the diagonal and off-diagonal parts. Stationary phase integration over \(\omega\) in Eq. (11) gives

\[
p_{\text{diag}} (t) = \int_S d\vec{r}_1 d\vec{r}_2 \sum_j \frac{A_j^2}{\nu S} \delta[t - T_j (\vec{r}_1, \vec{r}_2, E_F)],
\]

(14)

where \(T_j\) is the time which takes the particle with energy \(E_F\) to go from the point \(\vec{r}_1\) to the point \(\vec{r}_2\) along the path \(j\). Equation (14) is equivalent to Eq. (11) and therefore \(p_{\text{diag}} (t) = e^{-\gamma t}\) is the classical probability to remain.
Therefore, the log-normal tail of the quantum probability to remain represented in Eq. (5) is determined by the off-diagonal part of the sum over trajectories in Eqs. (1) and (3).

It is known that the integrals over coordinates in Eq. (4) can be computed by using the skeleton of the periodic orbit. Eq. (14) can be computed by using the skeleton of the periodic orbit. The stationary phase integration over \( Q \) is determined by the position of the integral. The probability to return should be

\[
\frac{v}{v^{-1}} = \begin{pmatrix}
1 + 2\kappa\kappa^* & 2i\kappa \\
-2i\kappa & 1 - 2\kappa\kappa^*
\end{pmatrix},
\]

where \( \theta, \theta_1, \phi, \chi \) are commuting variables and \( \kappa, \kappa^*, \eta, \eta^* \) are anticommuting variables which parameterize the matrix \( Q \). Let us write explicitly the elements of the matrix \( Q \) which appear in the integrand in Eq. (12).

\[
Q^{12} = i\left(1 - 2\eta\eta^*\right)\left(1 + 2\kappa\kappa^*\right)e^{-i\phi}\sin \theta - 4\kappa\eta^*e^{i\chi}\sinh \theta_1 - 2\kappa\eta^*\kappa e^{i\phi}\sin \theta - 2\kappa\eta^*\kappa e^{i\phi}\sin \theta_1
\]

\[
Q^{21} = -i\left(1 + 2\kappa\kappa^*\right)\left(1 - 2\eta\eta^*\right)e^{i\phi}\sin \theta - 4\kappa\eta^*e^{-i\chi}\sin \theta_1 - 2\kappa\eta^*\kappa e^{i\phi}\sin \theta + 2\kappa\eta^*\kappa e^{-i\chi}\sin \theta_1 - 2\kappa\eta^*\kappa e^{-i\phi}\sin \theta - 2\kappa\eta^*\kappa e^{-i\chi}\sin \theta_1
\]

Our purpose is to reduce the functional integral over the super-matrix \( Q \) to the conventional integral. The minimum of the action is reached when \( u \) and \( v \) are independent of coordinates \( \bar{r} \) and the action becomes

\[
F = \frac{\pi\nu}{2} \int_S \left(D(\nabla \theta)^2 + D(\nabla \theta_1)^2 + 2i(\omega - 0)(\cos \theta - \cosh \Theta)\right)d\bar{r},
\]

where \( \theta = 0 \) at the contact and \( \bar{n}\nabla \theta = 0 \) at the walls. The same boundary conditions were applied to Eq. (2) and therefore we can use the diffusion modes for computing the functional integral over \( \theta(r) \). Due to the condition Eq. (26) only the lowest diffusion mode contributes and the functional integral becomes a conventional integral over the amplitude of this mode \( \Theta \). The lowest diffusion mode is almost uniform, see Appendix A, and therefore

\[
\frac{1}{S} \int_S \theta^2 d\bar{r} = \Theta^2,
\]

\[
\frac{1}{S} \int_S \theta_1^2 d\bar{r} = \Theta_1^2,
\]

\[
\frac{1}{S} \int_S D(\nabla \theta)^2 d\bar{r} = \gamma \Theta^2,
\]

\[
\frac{1}{S} \int_S \cosh \theta d\bar{r} = \cosh \Theta,
\]

\[
\frac{1}{S} \int_S \sinh \theta d\bar{r} = \sinh \Theta,
\]

and so on for other functions of \( \theta \) and \( \theta_1 \). Particularly the pre-exponential factor in the expression for the return probability becomes
\[ Q_{21}^{12}Q_{12}^{21} = -8\eta^*\kappa^*(\sin^2\Theta + \sinh^2\Theta_1) \]  \hspace{1cm} (21)

We computed this factor explicitly from the expressions for \( Q^{12} \) and \( Q^{21} \). This expression was computed in Appendix 3 of Ref. but the numeric coefficients are different.

The model reduces to the conventional integral

\[ p(t) = -2(\pi\nu)^2 \int \frac{d\omega}{2\pi} e^{-i\omega t} \int \frac{8\eta^*\kappa^*(\sin^2\Theta + \sinh^2\Theta_1)}{Q_{21}^{12}Q_{12}^{21}} \]

\[ \times e^{-F} \frac{2^{-8}(\cosh\Theta_1 - \cos\Theta)^{-2} dydy'dx'dx}{J} \]

\[ \times \left( \frac{2/\pi}{\sin\Theta}\sin\Theta_1 \right) dy\,d\Theta d\Theta_1 = (\pi\nu)^2 \int \frac{d\omega}{2\pi} e^{-i\omega t} \]

\[ \times \left( \frac{(2/\pi)^2}{\sin\Theta}\sin\Theta_1 \right) dy\,d\Theta d\Theta_1 = (\pi\nu)^2 \int \frac{d\omega}{2\pi} e^{-i\omega t} \]

\[ \times \left( \frac{(2/\pi)^2}{\sin\Theta}\sin\Theta_1 \right) dy\,d\Theta d\Theta_1 = (\pi\nu)^2 \int \frac{d\omega}{2\pi} e^{-i\omega t} \]

\[ \times \left( \frac{F}{\Delta} \right) = \left\{ \frac{\Theta_1 + \Theta_1}{2} + (i\omega - 0)(\cos\Theta - \cosh\Theta) \right\} . \]  \hspace{1cm} (23)

where we have kept Efetov’s notations for differentials and Jacobians, see Ref. page 107.

The standard change of variables \( \lambda = \cos\Theta \) and \( \lambda_1 = \cosh\Theta_1 \) leads to the relatively simple expression for the probability to remain

\[ p(t) = \frac{1}{2} \int_{\max(-1,1-t\Delta/\pi)}^{1} d\lambda \left( \frac{t\Delta/\pi + 2\lambda}{t\Delta/\pi} \right) \]

\[ \times e^{-\frac{\pi}{2\Delta}(\arccosh^2(\lambda) + \arccosh^2(t\Delta/\pi + \lambda))} . \]  \hspace{1cm} (24)

For both short times \( t \ll \pi/\Delta \) and long times \( t \gg \pi/\Delta \) we can put \( \lambda = 1 \) in the exponent on the right hand side of Eq. (24). Then the integral can be computed exactly and we arrive at our interpolation formula Eq. (4). The computation of the form-factor is similar, and Eq. (4) matches both short and long time asymptotes of Eq. (24). Both results Eqs. (24) and (25) where derived under the condition Eq. (14) which becomes Eq. (8a) in the one-mode approximation.

The functional integral over \( \hat{\theta}(r) \) in the theory with action Eq. (29) can be computed by the optimal fluctuation method. The calculations are straightforward and give the same results, see Appendix 3 where we have checked the three dimensional case. In the same way one can also check the two dimensional result.

In the present work we have assumed that the opening or contact is small. If it is not the case for some system the result for the probability to remain may be different. In such a system one should consider the optimal fluctuations of the random potential. In our geometry this method can be used for very long times, which broke the condition Eq. (8a).

In summary, we have obtained the arccosh-normal distribution for the probability to remain in a disordered cavity. We made use of the one-mode approximation, which is similar to both the zero-dimensional \( \sigma \)-model and the optimal fluctuation method. Our result is not consistent with the random matrix theory prediction.

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\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2.pdf}
\caption{Radial dependence of the diffusion mode for \( t \ll L_1 \ll L_2 \).}
\end{figure}

\section*{APPENDIX A: COMPUTATION OF THE DIFFUSION MODE.}

Let us compute the diffusion mode for the circular cavity with the circular contact in the middle, as shown in Fig. 4. The lowest mode is independent of the polar angle and the radial dependence is expressed in terms of the Bessel functions

\[ \varrho(r) \propto \text{Im} \left[ H_0^{(1)}(r\sqrt{\frac{\gamma}{D}})H_1^{(2)}(L_2\sqrt{\frac{\gamma}{D}}) \right] \]  \hspace{1cm} (A1)

\[ 0 = \text{Im} \left[ H_0^{(1)}(L_1\sqrt{\frac{\gamma}{D}})H_1^{(2)}(L_2\sqrt{\frac{\gamma}{D}}) \right] \]  \hspace{1cm} (A2)

where \( L_2\sqrt{\frac{\gamma}{D}} \) has to be less that the first root of \( J_1(x) \) and this is possible for roughly \( L_1/L_2 < 1/2 \). The solution for \( L_1 \ll L_2 \) is shown in Fig. 3 and in this case

\[ \gamma = \frac{D}{L_2^2 \log \frac{L_2}{L_1}} . \]  \hspace{1cm} (A3)
We see that the mode is non-uniform for roughly $L_1 < r < L_2/2$. Therefore the presence of the contact strongly affects the diffusion mode in a quarter of the cavity area.

In the case of the three dimensional sphere it might be difficult to maintain the contact in the middle, but it is still interesting to compute the lowest diffusion mode. In three dimensions the lowest mode is independent of polar and azimuthal angles and depends only on the distance $r$ from the center. The lowest mode in three dimensions is

$$\varrho(r) \propto \frac{1}{r} \sin \left( \frac{1}{3} \sqrt{3L_1^2 - (r - L_1)^2} \right), \quad (A4)$$

$$L_2 \sqrt{\frac{\gamma}{D}} = \tan \sqrt{\frac{(L_2 - L_1)^2}{D}}, \quad (A5)$$

where $L_2$ is the radius of the cavity and $L_1$ is the radius of the contact. For the small values of the ratio $L_1/L_2$, this mode is also uniform in the most of the volume. In this case the three dimensional mode becomes

$$\varrho(r) \propto \frac{r - L_1}{r} - \frac{L_1}{2r} \left( \frac{r - L_1}{L_2} \right)^2, \quad (A6)$$

$$\gamma = 3L_1 D/L_2^2 \quad (A7)$$

and its shape is similar to that which is shown in Fig. 3.

**APPENDIX B: THE OPTIMAL FLUCTUATION METHOD IN THREE DIMENSIONS.**

According to the optimal fluctuation method the probability to remain is given by the following system of equations:

$$p(t) \propto e^{-\frac{\pi \delta^2}{L_1^2} \int_S (\nabla \varrho)^2 d\vec{r}}, \quad (B1)$$

$$t = \pi \nu \int_S \cosh(\vartheta) d\vec{r}, \quad (B2)$$

$$D \nabla^2 \vartheta = -i\omega \sinh(\vartheta), \quad (B3)$$

and the boundary conditions for the last equation were written after Eq. (20). This very problem was solved by Falko and Efetov, Chap. VI and VII. In three dimensions one has

$$\vartheta = A(1 - \frac{L_1}{r}), \quad (B4)$$

$$\frac{A/e^A}{3L_1 D} \ll 1, \quad (B5)$$

$$t = \frac{\pi}{A} e^A \quad (B6)$$

$$p(t) \propto e^{-\frac{\pi \delta^2}{L_1^2} \frac{\varrho^2}{r^2}} = e^{-\frac{\pi \delta^2}{L_1^2} \log^2 \frac{r}{\varrho}} \quad (B7)$$

where $\gamma$ is given by Eq. (A7). This result is precisely equal to our one mode result Eq. (7) in the long time limit.

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