Euclidean thermal spinor Green’s function in the spacetime of a straight cosmic string

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Abstract

Within the framework of the quantum field theory at finite temperature on a conical space, we determine the Euclidean thermal spinor Green’s function for a massless spinor field. We then calculate the thermal average of the energy-momentum tensor of a thermal bath of massless fermions. In the high-temperature limit, we find that the straight cosmic string does not perturb the thermal bath.

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1 Introduction

A straight cosmic string immersed in a thermal bath of massless bosons must modify the thermal average of the energy-momentum tensor \[ T_{\mu\nu} \]. Within the quantum field theory at finite temperature, we explicitly determined the Euclidean thermal scalar Green’s function on a conical space for a massless scalar field \[ \mathbb{R} \]. We then enabled us to evaluate the thermal average of the energy-momentum tensor in the case of a conformally invariant scalar field, in particular in the high-temperature limit. Recently, Frolov et al \[ 4 \] extended this work to an arbitrary massless scalar field and Guimarães \[ 5 \] has furthermore assumed the existence of a magnetic flux running through the cosmic string.

The aim of this paper is to study the gravitational influence of a straight cosmic string on a thermal bath of massless fermions, for instance neutrinos. To this purpose, we will determine the Euclidean thermal spinor Green’s function for a massless spin-\( \frac{1}{2} \) field within the quantum field theory at finite temperature. From this, we will be able to calculate the thermal average of the energy-momentum tensor of this thermal bath.

In general relativity, the spacetime describing a straight cosmic string possesses a conical-type line singularity \[ \mathbb{R} \]; the metric can be written as

\[
ds^2 = d\rho^2 + B^2 \rho^2 d\varphi^2 + dz^2 - dt^2
\]

in a coordinate system \((\rho, \varphi, z, t)\) with \(\rho \geq 0\) and \(0 \leq \varphi < 2\pi\) where the constant \(B\) is related to the linear mass density \(\mu\) of the cosmic string by \(B = 1 - 4G\mu (0 < B \leq 1)\). By performing a Wick rotation

\[
\tau = -it
\]

metric \(\mathbb{I}\) takes a Riemannian form

\[
ds^2 = d\rho^2 + B^2 \rho^2 d\varphi^2 + dz^2 + d\tau^2
\]

in which we can consider the Euclidean Green’s functions.

The Euclidean spinor Green’s function \(S_{E\beta}\) at finite temperature \(T\) in metric \(\mathbb{II}\) is characterised by the property to be antiperiodic in coordinate \(\tau\) with period \(\beta\) where \(\beta = 1/T\). We use units in which \(c = \hbar = k_B = 1\). The spinor Green’s functions can be derived from the scalar Green’s functions following a procedure that we have already used \[ 4 \] in metric \(\mathbb{I}\). In
the present case, the determination of the thermal spinor Green’s function requires the knowledge of the scalar Green’s function $G_A^{(γ)}$ which is antiperiodic in coordinate $τ$ with period $β$, i.e.

$$G_A^{(γ)}(τ + β) = -G_A^{(γ)}(τ)$$ (4)

and which is subject to the following boundary condition

$$G_A^{(γ)}(ϕ + 2π) = \exp(2πiγ)G_A^{(γ)}(ϕ)$$ (5)

for a constant $γ$ ($0 ≤ γ < 1$) that we will subsequently give the value in function of the constant $B$.

The plan of the work is as follows. In section 2, we determine the scalar Green’s function $G_A^{(γ)}$ in a handy form for a massless field, hereafter denoted $D_A^{(γ)}$. We show in section 3 that we can derive from this an expression of the spinor Green’s function $S_{Eβ}$ which is locally the sum of the usual thermal spinor Green’s function in Euclidean space and a regular part specifically induced by the global geometry of the spacetime of a straight cosmic string. In section 4, by using the Wick rotation, we calculate the thermal average of the energy-momentum of a thermal bath of massless fermions. We add some concluding remarks in section 5.

## 2 Antiperiodic scalar Green’s function

For a massive scalar field of mass $m$, the scalar Green’s function $G_A^{(γ)}$ obeys the equation

$$(□ - m^2)G_A^{(γ)}(x, x_0; m) = -δ^{(4)}(x, x_0)$$ (6)

where $□$ is the Laplacian operator and $δ^{(4)}$ the Dirac distribution in metric (3) and furthermore it satisfies conditions (4) and (5).

Our method is based on the fact that the scalar Green’s functions can be deduced from the heat kernels within the Schwinger-De-Witt formalism. In the present case, we have

$$G_A^{(γ)}(x, x_0; m) = \int_0^∞ K_A^{(γ)}(x, x_0; s)ds$$ (7)

where the heat kernel $K_A^{(γ)}$ satisfies the condition corresponding to (8)

$$K_A^{(γ)}(τ + β) = -K_A^{(γ)}(τ)$$ (8)

3
and the condition corresponding to (5)

\[ K_A^{(\gamma)}(\varphi + 2\pi) = \exp(2\pi i \gamma)K_A^{(\gamma)}(\varphi) \]  

(9)

We recall that the Euclidean heat kernel \( K_A^{(\gamma)} \) obeys the equation

\[ \left( \frac{\partial}{\partial s} - \Box + m^2 \right) K_A^{(\gamma)} = 0 \quad (s > 0) \quad \text{with} \quad \lim_{s \to 0} K_A^{(\gamma)}(x, x_0; s) = \delta^{(4)}(x, x_0) \]

For an ultrastatic metric, i.e. static and \( g_{\tau \tau} = 1 \), Braden [8] has proved that \( K_A^{(\gamma)} \) may be factorized with the aid of the following theta function

\[ \theta_4(z | \tau) = 1 + \sum_{n=-\infty}^{\infty} \exp(i\pi n^2 \tau + 2nz) \quad (\Im \tau > 0) \]

under the form

\[ K_A^{(\gamma)}(x, x_0; m) = \theta_4(i\frac{\beta(\tau - \tau_0)}{4s} | i\frac{\beta^2}{4\pi s})K_E^{(\gamma)}(x, x_0; m) \]  

(10)

where \( K_E^{(\gamma)} \) is the zero-temperature heat kernel which vanishes when the points \( x \) and \( x_0 \) are infinitely separated. In metric (3), formula (7) yields the scalar Green’s function \( G_E^{(\gamma)} \) for a charged scalar field interacting with a magnetic flux \( \gamma \); it has been explicitly determined by Guimarães [5].

We now recall the expression of \( K_E^{(\gamma)} \) because it will be needed in the next calculations. We confine ourselves to the case \( B > 1/2 \). In the subset of the considered space defined by

\[ \frac{\pi}{B} - 2\pi < \varphi - \varphi_0 < 2\pi - \frac{\pi}{B} \]

(11)

Guimarães [5] has found an expression of \( K_E^{(\gamma)} \) which is the sum of the usual heat kernel

\[ K_{E\text{ usual}}^{(\gamma)}(x, x_0; s) = \frac{1}{16\pi^2 s^2} \exp\left(-\frac{r_4^2}{4s} - m^2 s\right) \]

(12)

with \( r_4 = \sqrt{(\tau - \tau_0)^2 + (z - z_0)^2 + \rho^2 + \rho_0^2 - 2\rho \rho_0 \cos[B(\varphi - \varphi_0)]} \) and a regular part which has the integral expression

\[ K_{E}^{(\gamma)^*}(x, x_0; s) = \frac{\exp(-m^2 s)}{32\pi^3 B s^2} \int_{0}^{\infty} \exp\left[-\frac{R_4^2(u)}{4s}\right] F_B^{(\gamma)}(u, \varphi - \varphi_0) du \]  

(13)
with $R_4(u) = \sqrt{(\tau - \tau_0)^2 + (z - z_0)^2 + \rho^2 + \rho_0^2 + 2\rho\rho_0 \cosh u}$ and where the function $F_B^{(\gamma)}(u, \psi)$ has been determined \[3, 4\] under the form

$$F_B^{(\gamma)}(u, \psi) = \frac{e^{i(\psi + \pi/B)\gamma} \cosh[u(1 - \gamma)/B] - e^{-i(\psi + \pi/B)(1-\gamma)} \cosh[u\gamma/B]}{\cosh(u/B) - \cos(\psi + \pi/B)}$$

$$-i\frac{e^{i(\psi - \pi/B)\gamma} \cosh[u(1 - \gamma)/B] - e^{-i(\psi - \pi/B)(1-\gamma)} \cosh[u\gamma/B]}{\cosh(u/B) - \cos(\psi - \pi/B)}$$

(14)

We can now obtain the desired expression of $G_A^{(\gamma)}$ valid in subset (11). In the case $m = 0$, it has a handy form. By substituting (14) into (7), we obtain $D_A^{(\gamma)}$ as the sum of the antiperiodic scalar Green’s function in an Euclidean space

$$D_A^{\text{insel}}(x, x_0) = \int_0^{\infty} \theta_4(i\frac{\beta(\tau - \tau_0)}{4s} | i\frac{\beta^2}{4\pi s}) \frac{1}{4s^2} \exp(-\frac{R_4^2}{4s}) ds$$

(15)

and a regular part which has the integral expression

$$D_A^{(\gamma)*}(x, x_0) = \frac{1}{32\pi^3 B} \int_0^{\infty} \int_0^{\infty} \theta_4(i\frac{\beta(\tau - \tau_0)}{4s} | i\frac{\beta^2}{4\pi s})$$

$$\times \frac{1}{s^2} \exp[-\frac{R_4^2(u)}{4s}] F_B^{(\gamma)}(u, \varphi - \varphi_0) duds$$

(16)

By changing the variable of integration $x = 1/s$, we rewrite (16) as

$$D_A^{(\gamma)*}(x, x_0) = \frac{1}{32\pi^3 B} \int_0^{\infty} \int_0^{\infty} \theta_4(i\frac{\beta(\tau - \tau_0)}{4x} | i\frac{\beta^2}{4\pi x})$$

$$\times \exp[-\frac{R_4^2(u)}{4x}] F_B^{(\gamma)}(u, \varphi - \varphi_0) dxds$$

(17)

The $x$-integration in expression (17) can be performed by using the formula

$$\int_0^{\infty} \theta_4(i\pi k x | i\pi x) \exp[-(k^2 + l^2)x] dx = \frac{\sinh l}{l(\cosh l - \cos k)} - \frac{\sinh 2l}{l(\cosh 2l - \cos 2k)}$$

This result is proved with the help of the identity between the theta functions

$$\theta_4(z | \tau) = 2\theta_3(2z | 4\tau) - \theta_3(z | \tau)$$
already noticed by Dowker and Schofield [10], and then by using the following integral
\[
\int_0^\infty \theta_3(i\pi kx | i\pi x) \exp[-(k^2 + l^2)x] dx = \frac{\sinh 2l}{l(\cosh 2l - \cos 2k)}
\]
that we have already considered [3]. Thus, the expression of the antiperiodic scalar Green’s function \(D_A^{(\gamma)}\), valid in subset \(\Pi\), is the sum of the antiperiodic scalar Green’s function in an Euclidean space
\[
D_A^{\text{usual}}(x, x_0) = \frac{1}{4\pi^2 r_3} \left\{ \frac{\sinh(\pi r_3/\beta)}{\cosh(\pi r_3/\beta) - \cos[\pi(\tau - \tau_0)/\beta]} - \frac{\sinh(2\pi r_3/\beta)}{\cosh(2\pi r_3/\beta) - \cos[2\pi(\tau - \tau_0)/\beta]} \right\}
\]
with \(r_3 = \sqrt{(z - z_0)^2 + \rho^2 + \rho_0^2 - 2\rho\rho_0 \cos[B(\varphi - \varphi_0)]}\) and a regular part which has the integral expression
\[
D_A^{(\gamma)\ast}(x, x_0) = \frac{1}{8\pi^2 \beta B} \int_0^\infty \frac{1}{R_3(u)} \left\{ \frac{\sinh[\pi R_3(u)/\beta]}{\cosh[\pi R_3(u)/\beta] - \cos[\pi(\tau - \tau_0)/\beta]} - \frac{\sinh[2\pi R_3(u)/\beta]}{\cosh[2\pi R_3(u)/\beta] - \cos[2\pi(\tau - \tau_0)/\beta]} F_B^{(\gamma)}(u, \varphi - \varphi_0) du \right\}
\]
with \(R_3(u) = \sqrt{(z - z_0)^2 + \rho^2 + \rho_0^2 + 2\rho\rho \cosh u}\) where \(F_B^{(\gamma)}(u, \psi)\) is given by \([14]\).

3 Thermal spinor Green’s function

In order to write down the Dirac operator in metric \(\Pi\), we choose the following vierbein
\[
e^\mu_\alpha = (0, \frac{1}{B\rho}, 0, 0) \quad e^\mu_\alpha = \delta^\mu_\alpha \quad \alpha \ne 2
\]
which is different of one often used \([11]\). In the study of the vacuum polarization, we determined the ordinary spinor Green’s function \(S_E\) with this choice \([7]\). The thermal spinor Green’s function \(S_{E\beta}\) for a massive spin-\(\frac{1}{2}\) field obeys the equation
\[
(e^\mu_\alpha \gamma^\alpha + \frac{\gamma^\perp}{2\rho} + mI)S_{E\beta}(x, x_0; m) = -I\delta^{(4)}(x, x_0)
\]
where the $\gamma^a$ are the Dirac matrices such that $\gamma^a \gamma^b = -\delta^{ab}$ and $I$ is the unit matrix. As explained in [7], due to choice (20) of vierbein, the spinor Green’s function is well defined if we impose the boundary condition

$$S_E(\varphi + 2\pi) = -S_E(\varphi)$$  \hspace{1cm} (22)$$

The thermal character is expressed by the property of antiperiodicity in the coordinate $\tau$ with period $\beta$, i.e.

$$S_E(\tau + \beta) = -S_E(\tau)$$  \hspace{1cm} (23)$$

The spinor Green’s function $S_E$ satisfying (21), (22) and (23) can be written as

$$S_E(x, x_0; m) = (\gamma^\mu \gamma^\alpha \partial_\mu + \frac{\gamma^1}{2\rho} - mI)G_A(x, x_0; m)$$  \hspace{1cm} (24)$$

where we have set

$$G_A(x, x_0; m) = (i\Re + \gamma^1\gamma^2)\exp(iB\varphi - \varphi_0)G^{(\gamma)}_A(x, x_0; m)$$  \hspace{1cm} (25)$$

in which $G^{(\gamma)}_A$ is the antiperiodic scalar Green’s function for a parameter $\gamma$ given by the equation

$$\gamma = \frac{1}{2} - \frac{B}{2}$$  \hspace{1cm} (26)$$

We have $0 \leq \mu < 1/4$ since $1/2 < B \leq 1$. Hence, we will obtain in subset (11) the thermal spinor Green’s function under the form

$$S_{E\beta}(x, x_0; m) = S_{E\beta}^{\text{usual}}(x, x_0; m) + S_{E\beta}^*(x, x_0; m)$$  \hspace{1cm} (27)$$

where $S_{E\beta}^*$ can be calculated from formulas (24) and (25) when one knows $G^{(\gamma)}_A$ or $D^{(\gamma)}_A$ in the case $m = 0$. This regular part is specifically induced by the global geometry of the space describing a conical-type line singularity. In the limit where $\beta$ tends to the infinity, i.e. at zero temperature, $S_{E\infty}$ coincides with $S_E$. 

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4 Thermal average of the energy-momentum tensor

Within the quantum field theory at finite temperature, a renormalization is performed by removing the zero-temperature Green’s function. The thermal average of the energy-momentum tensor of a thermal bath of massless fermions can be thus computed by the formula

\[ < T^{\mu\nu} (x) >_\beta = T^{(1/2)}_{\mu\nu} (S_{E\beta}(x, x_0) - S_{E\infty}(x, x_0)) |_{x=x_0} \] (28)

where \( T^{(1/2)}_{\mu\nu} \) is the following differential operator in \( x \) and \( x_0 \) suivant

\[ T^{(1/2)}_{\mu\nu} = \frac{1}{4} tr[\gamma^a (e_{\mu\nu}(\partial_\nu - \partial_\nu_0) + e_{\nu\mu}(\partial_\mu - \partial_\mu_0))] \] (29)

in which we need to know only the Green’s function for points \( x \) and \( x_0 \) sufficiently near. By considering form (27) of the spinor Green’s function, the energy-momentum tensor can be written down as the sum of the usual form in an Euclidean space

\[ < T^{\mu\nu} (x) >_{\beta} = T^{(1/2)}_{\mu\nu} (S_{\text{usuel}}^{\beta}(x, x_0) - S_{E\infty}(x, x_0)) |_{x=x_0} \] (30)

and a regular term

\[ < T^{\mu\nu} (x) >^*_{\beta} = T^{(1/2)}_{\mu\nu} S_{E\beta}^* (x, x_0) |_{x=x_0} \] (31)

which results from the existence of the conical defect.

Since the energy-momentum tensor of a thermal bath for massless spin-\( \frac{1}{2} \) field is traceless and conserved in metric (I), we get thereby

\[ < T^{\nu}_{\mu} (x) >_{\beta} = - < T^{tt}_{\mu} (x) >_{\beta} \text{ diag} (-1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \] (32)

where the energy density \( < T^{tt}_{\mu} > \) is given by \( - < T^{\tau\tau}_{\tau\tau} > \) by virtue of the Wick rotation (2). By substituting (24) into (28), we obtain the expression of the energy density

\[ < T^{tt}_{\mu} (x) >_{\beta} = - 4 \partial^{\tau}_{\tau} \Re D^{(\gamma)}_{\lambda}(x, x_0) |_{x=x_0} \] (33)
The application of formula (30) gives the usual energy-momentum tensor for a thermal bath of massless fermions in an Euclidean space. By observing that

$$D_A^{\text{usual}}(\tau - \tau_0) \big|_{\rho=\rho_0,\varphi=\varphi_0,z=z_0} \sim \frac{1}{4\pi^2(\tau - \tau_0)^2} - \frac{1}{24\beta^2} - \frac{7\pi^2(\tau - \tau_0)^2}{480\beta^4}$$

when \((\tau - \tau_0) \to 0\), we find from (33) that the energy density is

$$<T^{tt}(x)>_{\text{usual}} \beta^2 = \frac{7\pi^2}{60\beta^4}$$

(34)

The application of formula (31) gives the perturbation of the energy-momentum tensor due to the straight cosmic string. From (33), we find

$$<T^{tt}(x)>^{*}_{\beta} = \frac{1}{4\beta^3} \int_0^\infty \frac{1}{\cosh(u/2)} \times \left\{ \frac{\sinh(2\rho \cosh(u/2))}{\beta} - 4 \frac{\sinh(4\rho \cosh(u/2))}{\beta} \right\} F_B^{(\gamma)}(u,0) du$$

(35)

in which \(F_B^{(\gamma)}(u,0)\) is given by (14) with \(\psi = 0\). We recall that we have assumed that \(1/2 < B \leq 1\).

We firstly examine the limit of result (35) when \(\beta \to \infty\), i.e. at zero temperature. By setting

$$x = \frac{2\pi \rho \cosh(u/2)}{\beta}$$

it becomes

$$<T^{tt}(x)>^*_{\beta} = \frac{1}{32\pi^3 \rho^4 B} \int_0^\infty \frac{1}{\cosh(u/2)^4} \times \left[ \frac{\sinh x}{\cosh x - 1} - 4 \frac{\sinh 2x}{\cosh 2x - 1^2} \right] F_B^{(\gamma)}(u,0) du$$

When \(\beta \to \infty\), we easily see that

$$<T^{tt}(x)>^*_{\infty} = \frac{1}{16\pi^3 \rho^4} \int_0^\infty \frac{1}{\cosh(u/2)^4} F_B^{(\gamma)}(u,0) du$$

(36)
But contribution (34) to the energy density of the thermal bath disappears at zero temperature, therefore part (36) gives the vacuum energy-momentum tensor within the vacuum polarization which agrees with the one of Frolov and Serebriany [12] as one can prove it [13, 9].

We secondly examine the asymptotic behavior of result (35) when \( \beta \to 0 \), i.e. in the high-temperature limit. Taking into account the properties of the hyperbolic functions, we see that

\[
< T_{tt}(x) >_\beta^* \sim \frac{1}{4\beta^3 \rho B} \int_0^\infty \frac{1}{\cosh(u/2)}
\times \{ \exp\left[ -\frac{2\pi \rho \cosh(u/2)}{\beta} \right] - 4 \exp\left[ -\frac{4\pi \rho \cosh(u/2)}{\beta} \right] \} F_B^{(\gamma)}(u, 0) du
\]

when \( \beta \to 0 \). Thus, the energy density \( < T_{tt} >_\beta^* \) is exponentially decreasing in the variable \( 1/\beta \); only contribution (34) to energy density of the thermal bath of massless fermions survives.

## 5 Conclusion

In the spacetime of a straight cosmic string, we have in fact derived a general expression of the Euclidean thermal Green’s function for a massive spin-\( \frac{1}{2} \) field but we have really got a handy form for a massless field. We have calculated the thermal average of the energy-momentum tensor of a thermal bath in the case of a massless spinor field.

There exists a sharp contrast between the case of thermal bath of massless bosons and the present situation: the energy-momentum tensor of the thermal bath of massless fermions is not perturbed in high-temperature limit by the presence of the straight cosmic string.
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