Nonlinearities of the Vlasov equation in the spinodal region

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Abstract

We estimate the importance of the nonlinear terms in the Vlasov equation for the development of the unstable modes. The results allow to identify the region of wavelength where the linear evolution is justified.

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Recently the Vlasov equation has been applied for the description of the dynamics of the fragment formation in the nuclear matter unstable against the spinodal decomposition [1, 2, 3]. In the linear approximation a perturbation of the translationally invariant distribution is unstable. Unstable modes will develop exponentially in time leading to the formation of density inhomogeneities and fragmentation [1, 3]. Since the growth rate depends on the wave-vector, the wave-vector with the largest growth rate \( k \) will dominate at large times. One expects the formation of fragments of the size corresponding to the most unstable modes \( 1/k \) [4]. If the linear regime can be trusted in the development of the spinodal decomposition then the fragment size distribution should show an excess of fragments of intermediate sizes. The fragment size distribution would be different if at the fragmentation time many different modes are excited due to the nonlinear evolution of the Vlasov equation. Of course, in realistic simulations initial fluctuations and a noise term must be taken into account. This determines the strength by which different modes of the linearized Vlasov equation would be excited.

The nonlinearities in the Vlasov equation are non-negligible at large times as the deviation of the unstable modes from equilibrium becomes important. The existing estimates of this effect are based on the numerical results for the 2-Dimensional Vlasov equation [2, 3]. In this letter we shall estimate the importance of the nonlinear effects in the Vlasov equation analytically. We shall study the case where at initial time \( t = 0 \) one or two unstable modes are perturbed. One unstable mode engenders nonlinear effects at double wave-vector. Two unstable modes can lead to nonlinearities also for small wave-vector. The estimate will allow to define the regime of wave-vectors where the linear approximation can be valid.

We start with the usual Vlasov equation:

\[
\frac{\partial f(t, x, p)}{\partial t} + v \nabla_x f(t, x, p) - \nabla_x U(x) \nabla_p f(t, x, p) = 0 ,
\]

\(1\)

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where $U(x)$ is the density dependent mean field. In the following we take a Skyrme type mean field potential folded with a Gaussian:

$$U(x) = \int d^3y \ g(x - y) \left( A \frac{\rho(y)}{\rho_0} + B \left( \frac{\rho(y)}{\rho_0} \right)^\sigma \right).$$

(2)

Expanding to second order in $\delta f(t, x, p)$, the deviation from the homogeneous solution, and Fourier transforming in $x$, $\delta f(t, k, p) = \int d^3x \ \delta f(t, x, p)e^{-i\mathbf{k}\cdot\mathbf{r}}$, we obtain:

$$\frac{\partial f(t, k, p)}{\partial t} + i\mathbf{k}\mathbf{v}\delta f(t, k, p) - i\frac{\delta U_k}{\delta \rho} \mathbf{v}\frac{\partial n_0}{\partial \epsilon} \delta \rho(t, k)$$

$$- \frac{i}{2} \frac{\delta^2 U_k}{\delta \rho^2} \mathbf{v}\frac{\partial n_0}{\partial \epsilon} \int \frac{d^3l}{(2\pi)^3} \delta \rho(t, k - l) \delta \rho(t, l)$$

$$- \frac{i}{2} \int \frac{d^3l}{(2\pi)^3} \frac{\delta U_k - l}{\delta \rho} \delta \rho(t, k - l) (\mathbf{k} - \mathbf{l}) \nabla_p \delta f(t, k, p) = 0 ,$$

(3)

where $n_0$ is the equilibrium momentum distribution. The first three terms of the above equation are the linear approximation to the Vlasov equation. The remaining two terms are the second order expansion in $\delta f$ and represent a mode-mode coupling for the modes of the linear equation. The actual solution of the second order equation is as difficult as for the full Vlasov equation. However, to make an estimate of the nonlinear effects it is enough to take for $\delta f(t, k, p)$ and $\delta \rho(t, k)$ in the nonlinear terms only the contribution from the unstable modes to the evolution of an initial phase space density perturbation at $t = 0$:

$$\delta f^+(t, k, p) = \frac{\delta U_k}{\delta \rho} \frac{\partial n_0}{\partial \epsilon} \frac{\mathbf{v}}{\mathbf{v} - i\Gamma_k} \delta \rho^+(k)e^{\Gamma_k t},$$

(4)

and

$$\delta \rho^+(k) = \delta \rho^+(k)\Theta(k_{max} - |k|)e^{\Gamma_k t} ,$$

(5)

with $k_{max}$ being the maximal wave-vector with imaginary frequency $[1]$ and

$$\delta \rho^+(k) = \int \frac{d^3p}{(2\pi)^3} \frac{\delta f(0, k, p)}{\mathbf{v} - i\Gamma_k} .$$

(6)

Inserting the forms (5) and (6) in the mode-mode coupling term in eq. (3) and taking the one-sided Fourier transform:

$$\delta f(\omega, k, p) = \int_0^\infty dt \ \delta f(t, k, p)e^{i\omega t} ,$$

(7)

with $Im \ \omega > 2\Gamma_{max}$, one obtains:

$$i(\mathbf{v} - \omega)\delta f(\omega, k, p) - \frac{\delta U_k}{\delta \rho} \frac{\partial n_0}{\partial \epsilon} \delta \rho(\omega, k)$$

$$= \delta f(0, k, p) - \frac{i}{2} \frac{\delta^2 U_k}{\delta \rho^2} \frac{\mathbf{v}}{\mathbf{v} - i\Gamma_k} \int \frac{d^3l}{(2\pi)^3} \delta \rho^+(k - l)\delta \rho^+(l)$$

$$\frac{1}{i\omega + \Gamma_{k-l} + \Gamma_l}$$

$$- \frac{i}{2} \int \frac{d^3l}{(2\pi)^3} \frac{\delta U_{k-l}}{\delta \rho} \frac{\delta U_l}{\delta \rho} \delta \rho^+(k - l)\delta \rho^+(l) \frac{1}{i\omega + \Gamma_{k-l} + \Gamma_l}$$

$$(\mathbf{k} - \mathbf{l}) \nabla_p \left( \frac{1}{\mathbf{v} - i\Gamma_l} \frac{\partial n_0}{\partial \epsilon} \right) .$$

(8)

Dividing by $\mathbf{v} - \omega$ and integrating over $p$ an equation for the density perturbation is obtained:

$$\delta \rho(\omega, k) = \frac{-iG(\omega, k)}{\epsilon(\omega, k)}$$
\[-\frac{1}{2\epsilon(\omega, k)} \frac{\delta^2 U_k}{\delta \rho^2} \int \frac{d^3p}{(2\pi)^3} \frac{k}{k - \omega} \frac{\partial n_0}{\partial \epsilon} \int \frac{d^3l}{(2\pi)^3} \frac{\delta \rho^+(k - l)\delta \rho^+(l)}{i\omega + \Gamma_{k-l} + \Gamma_l} = -\frac{1}{2\epsilon(\omega, k)} \int \frac{d^3l}{(2\pi)^3} \frac{\delta U_{k-l}}{\delta \rho} \frac{\delta \rho^+(k - l)\delta \rho^+(l)}{i\omega + \Gamma_{k-l} + \Gamma_l} \int \frac{d^3p}{(2\pi)^3} \frac{1}{k - \omega} \left( k - l \right) \nabla_p \left( \frac{1}{k - \omega - i\Gamma_l} \frac{\partial n_0}{\partial \epsilon} \right), \tag{9}\]

with

\[G(\omega, k) = \int \frac{d^3p}{(2\pi)^3} \frac{\delta f(0, k, p)}{k - \omega} \tag{10}\]

and

\[\epsilon(\omega, k) = 1 - \frac{\partial U_k}{\partial \rho} \int \frac{d^3p}{(2\pi)^3} \frac{k}{k - \omega} \frac{\partial n_0}{\partial \epsilon}. \tag{11}\]

The time dependence of the density perturbation \(\delta \rho\) can be found using the inverse transform:

\[\delta \rho(t, k) = \int_{-\infty}^{\infty} d\omega \frac{\epsilon(\omega, k)}{2\pi} e^{-i\omega t} \left( \frac{1}{k - \omega - i\Gamma_l} \frac{\partial n_0}{\partial \epsilon} \right), \tag{12}\]

de the integration path in the inverse Fourier transform laying above any singularity of the integrand.

As a first case we take only one unstable mode:

\[\rho^+(k) = (2\pi)^3 \delta^3(k - k_0) \delta A_{k_0} \rho. \tag{13}\]

The nonlinear term is nonzero only for the mode with the doubled wave-vector \(2k_0\):

\[\delta \rho(\omega, k) = \frac{(2\pi)^3 \delta^3(k - 2k_0) \delta A_{2k_0}^2 \rho^2}{8\pi^2} \frac{1}{\epsilon(\omega, k)} \frac{\delta^2 U_{2k_0}}{\delta \rho^2} \int p^2 dp \left( 2 - \frac{\omega}{2k_0 v} \ln \left( \frac{\omega + 2k_0 v}{\omega - 2k_0 v} \right) \right) \frac{\partial n_0}{\partial \epsilon} + \frac{k_0^2}{4\pi^2} \left( \frac{\delta U_{k_0}}{\delta \rho} \right)^2 \int p^2 dp \left( \frac{-2\omega}{(4k_0 v)^2 - \omega^2} \right) \frac{\partial n_0}{\partial \epsilon} + \frac{2\Gamma_{k_0}}{k_0 v^2} \arctan \left( \frac{k_0 v}{\Gamma_{k_0}} \right) \frac{\partial n_0}{\partial \epsilon} + \frac{i\Gamma_{k_0}}{k_0 v^2} \ln \left( \frac{\omega + 2k_0 v}{\omega - 2k_0 v} \right) \frac{\partial n_0}{\partial \epsilon}. \tag{14}\]

The integral (12) can be calculated closing the integration path in the lower half-plane. Its value is determined by the singularities of the integrand. The inverse susceptibility \(1/\epsilon(\omega, k)\) has two poles corresponding the solutions of the dispersion relation:

\[\epsilon(\omega, k)|_k = 0. \tag{15}\]

The integrand also has a pole at \(\omega = 2i\Gamma_{k_0}\) and a cut on the real axis in \(\omega [5]\). The contribution from the pole at \(2i\Gamma_{k_0}\) is dominant at large times. Even if the the solution of (15) is imaginary we always have \(2\Gamma_{k_0} > \Gamma_{2k_0}\). The cut contribution is always bounded and can be neglected at large times in comparison to the growing components. The result is:

\[\delta \rho(t, k) = \frac{(2\pi)^3 \delta^3(k - 2k_0) \delta A_{2k_0}^2 \rho^2}{8\pi^2} \frac{1}{\epsilon(2i\Gamma_{k_0}, 2k_0)} e^{2i\Gamma_{k_0} t} \left[ \frac{\delta^2 U_{2k_0}}{\delta \rho^2} \frac{\delta U_{k_0}}{\delta \rho} \frac{\partial n_0}{\partial \epsilon} \right] + \frac{k_0^2}{8\pi^2} \left( \frac{\delta U_{k_0}}{\delta \rho} \right)^2 \int p^2 dp \left( \frac{-\Gamma_{k_0}}{(k_0 v)^2 + \Gamma_{k_0}^2} \frac{\partial n_0}{\partial \epsilon} \right) + \frac{i\Gamma_{k_0}}{k_0 v^2} \ln \left( \frac{\omega + 2k_0 v}{\omega - 2k_0 v} \right) \frac{\partial n_0}{\partial \epsilon}. \tag{16}\]
In Fig.1 is shown the ratio of the amplitude $F_{2k_0}(T)$ of the mode at $2k_0$ to $\rho$ at the instability time $T$ which is defined as $|\delta A_{k_0} \exp(\Gamma k_0 T)| \approx 1$. The result shows that the long wavelength modes are always nonlinear. Before the time the instability drives the mode $k_0$ to large values the mode with double wave-vector becomes larger. However, the amplitude of the linear modes with large wave-vectors ($k > 4 fm^{-1}$) is large $|\delta A_k \exp(\Gamma k T)| \approx \rho$ before the nonlinearly driven mode at $2k_0$ becomes important. In particular in our case the most unstable mode is in the region of wave-vectors, where the linear regime is valid up to the instability time $T$. Similar observation were made in the 2-Dimensional numerical solutions of the Vlasov equation, the small wave-vector modes become nonlinear at smaller amplitude than the large wave-vector modes. In Fig.2 is shown the growth time for the amplitude of the mode to reach $\rho$. We see that for the long-wavelength the growth time for the nonlinear mode at $2k_0$ is smaller than the growth time for the linear mode at $k_0$. The unphysically long values of the growth times depends of course on the initial perturbation chosen (here $\delta A = 1/50$).

The nonlinearities become extremely important for long wave-length (Fig.1). However in this limit one should take into account both the pole at $2\Gamma k_0$ and at $i\Gamma k_0$, since the two poles merge in that limit ($\Gamma_k \sim k$). The time dependence is now different:

$$\delta \rho(t, k) = \frac{(2\pi)^3 \delta^3(k - 2k_0)\delta A_{k_0}^2 \rho^2}{i\epsilon (\omega, 2k_0)|_{\omega = 2\Gamma k_0}} t e^{2\Gamma k_0 t} \left[ \frac{\delta^2 U_{2k_0}}{\rho^2} \frac{2\delta U_{k_0}}{\rho} \right] + \frac{k_0^2}{8\pi^2} \frac{\delta U_{k_0}}{\rho} \int \rho^2 dp \frac{-\Gamma_{k_0}}{((k_0 v)^2 + \Gamma_{k_0}^2)^2} \partial_{\eta_0} \rho = (2\pi)^3 \delta^3(k - 2k_0) L_{k_0} \rho \delta A_{k_0}^2 t e^{2\Gamma t}. \quad (17)$$

Now, the amplitude $F_{2k_0}(T)$ at the instability time $T$ depends on the value of the initial perturbation $\delta A_{k_0}$:

$$F_{2k_0}(T) = \frac{L_{k_0}}{\Gamma_{k_0}} \ln(|\delta A_{k_0}|) \rho. \quad (18)$$

This shows that it is not the same to use larger initial fluctuations evolved on shorter time to using small initial fluctuations evolved for longer times. Although, the linear evolution would give the same result $|\delta A \exp(\Gamma T)| = 1$, the nonlinear modes at double wave-vector would have a different strength. In long wavelength limit the procedure of putting instead of the noise term in the Vlasov equation a stronger initial perturbation gives different results due to nonlinearities of the Vlasov equation. However, the most unstable modes are behaving according to eq. (17). Moreover, the numerical coefficient is such that the nonlinearities have no time to build up until the amplitude of the most unstable modes becomes of the order $\rho$. The results from the correct limiting expression for the limit $\Gamma (k) \sim k$ (eq. [17]) is also shown in Fig.2. The correct dependence of the growth time $T_{growth} \sim 1/k$ in the long wavelength limit is recovered.

Another case is when two unstable modes develop. We shall study the case when the wave-vectors of the unstable modes are in opposite direction:

$$\rho^+(k) = (2\pi)^3 \left( \delta^3(k - l)\delta A_l + \delta^3(k + l - k_0)\delta A_{k_0 - l} \right) \rho, \quad (19)$$

with $k_0 < l$ and $l$ corresponding to a strongly growing mode. Besides the contributions at doubled wave-vector $2l$ and $2(k_0 - l)$ we find a nonlinear mode located at $k_0$. The mechanism responsible for the growth of the mode at $k_0$ is different from the simple wave-vector doubling. As previously the dominant mode is given by the pole of $\delta A$ at $i(\Gamma_l + \Gamma_{k_0 - l})$, leading to the behavior:

$$F_{k_0}(t) \sim \delta A_{k_0 - l} e^{(\Gamma_l + \Gamma_{k_0 - l})t}. \quad (20)$$

We do not quote explicitly the lengthy formula for the coefficient, given by residue at the pole of [1]. In Fig.3 is shown the ratio $|F_{k_0}(T)/F_{k_0-2l}(T)|$ at the time $T$ when the amplitude of the most unstable mode $l \sim 0.7fm^{-1}$ reaches $\rho$. For $|k_0 - l| < 0.3fm^{-1}$ the linear mode at $k_0 - l$ is weaker than the nonlinear
mode at \( k_0 > 0.3 fm^{-1} \). Similar nonlinear effect were observed in numerical simulations [2]. In the case of a strongly excited mode at relatively small wave-vector and a weakly excited mode with strong growth rate several modes at intermediate wave-vectors appear, leading to strong nonlinearities. Of course if the most unstable mode \( l \) is strongly excited it will dominate both the linear mode at \( k_0 - l \) and the nonlinear mode at \( k_0 \). When \( k_0 \to l \) the contribution from the pole of \((20)\) at \( \Gamma_{k_0} \) cannot be neglected. This changes the behavior of the time dependence of the nonlinear mode from \((20)\) to a form analogous to \((17)\). Thus the singularity in Fig.3 when \( k_0 \to l = 0.7 fm^{-1} \) is spurious.

In summary, we have studied the importance of the nonlinearities in the Vlasov equation for the development of the spinodal instabilities. The mechanism of wave-vector doubling is important only in the regime of wave-vector \( k < 0.35 fm^{-1} \) (this value is determined mainly by the range of the interaction). For the most unstable modes the nonlinear effects at double wave-vector are small. For the case of two exponentially growing modes with wave-vectors in opposite direction, we have found that nonlinear modes at intermediate wave-vectors can be excited. Thus, a small excitation of a rapidly growing mode \( \sim 0.7 fm^{-1} \) together with a strong excitation of the mode \( -0.3 fm^{-1} \) leads to strong nonlinear effects at wave-vectors \( \sim 4 fm^{-1} \). If for a given wave-vector the nonlinearities are found to be important, then the perturbative approach leading from \((8)\) to \((5)\) becomes invalid at some time smaller than the instability time. From the one and two mode cases studied in this letter any initial one-dimensional perturbation of the homogeneous solution can be constructed. In the second order in \( \delta f(t,k,p) \) the corresponding nonlinearities from mode-mode coupling can be calculated. Those estimates explain the observations made in numerical simulations of the 2-Dimensional Vlasov equation [2, 3] and are the first estimate of nonlinear effects in 3-Dimensions.

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The ratio of the amplitude $|F_{2k}(T)|$ of the nonlinear mode at double wave-vector $2k$ to the average density $\rho$ as a function of the wave-vector at the instability time $T = \ln(1/|\delta A_k|)/\Gamma_k$. The calculation is done at zero temperature with the parameters of the interaction $A = -356\,MeV$, $B = 303\,MeV$, $\sigma = 7/6$, $\rho = \rho_0/3$ and the range of the Gaussian equal to $0.9\,fm$. 

Fig.1

The growth time $T_{\text{growth}}$ for the amplitude of a mode to reach the average density $\rho$ with $\delta A_k = 1/50$, as a function of the wave-vector. The solid line is for the nonlinear mode at $2k$, the dashed-dotted line is for the linear mode at $k$ and the dotted line is for the nonlinear mode at $2k$ in the regime $\Gamma_k \sim k$. The parameters used are the same as in Fig.1.
Fig. 3
The ratio of the amplitude $F_k(T)$ of the nonlinear mode at $k$ to the amplitude $F_{k-l}(T)$ of the linear mode at $k-l$ as a function of the wave-vector at the time $T = \ln(1/|\delta A_l|)/\Gamma_l$ ($l = 0.7 \text{fm}^{-1}$), i.e. when the amplitude of the other linear mode reaches the average density. The parameters used are the same as in Fig. 1.