Hyper-Kähler geometries and nonlinear supermultiplets

Č. Burdík\textsuperscript{1}, S. Krivonos\textsuperscript{2}, A. Shcherbakov\textsuperscript{2}

\textsuperscript{1} Department of Mathematics, Czech Technical University, Trojanova 13, 120 00 Prague 2, Czech Republic
\textsuperscript{2} Bogoliubov Laboratory of Theoretical Physics, JINR, 141980 Dubna, Russia

Abstract

It is presented a method of construction of sigma-models with target space geometries different from conformally flat ones. The method is based on a treating of a constancy of a coupling constant as a dynamical constraint following as an equation of motion. In this way we build $N = 4$ and $N = 8$ supersymmetric four-dimensional sigma-models in $d = 1$ with hyper-Kähler target space possessing one isometry, which commutes with supersymmetry.

Introduction

One-dimensional theories (i.e. mechanics) with four and eight supercharges stand out among all one-dimensional theories with extended supersymmetry. This is due to existence of linear $N = 4$ and $N = 8$ irreducible representations having no auxiliary fields \cite{11}. Construction of sigma-model actions corresponding to these models turns out to be quite easy task to do. A detailed analysis of sigma-model geometries of arising bosonic manifolds was performed in papers \cite{2, 3} and revealed an interesting fact: under quite general assumptions concerning the structure of sigma-model actions arising bosonic manifolds are to be conformally flat. This is a direct evidence that those considerations seem to overlook some points relating to other possible geometries, because a dimensional reduction of four-dimensional $N = 2$ sigma-models \cite{12} down to $d = 1$ is known to lead to hyper-Kähler bosonic manifolds. It is quite easy to understand what was missed: in one dimension there exists a wide class of nonlinear off-shell supermultiplets. These are the very supermultiplets that play a crucial role in constructing supersymmetric sigma-models.
with different type of target space geometry \cite{5,6,8,9}. Unlike to linear supermultiplets, construction and classification of nonlinear off-shell ones are much more difficult problem which is complicated by their absence in higher dimensions.

In this paper we describe a dualization of a coupling constant procedure to construct a nonlinear realization of supersymmetry that allows us to build the most general $N = 4,8$ supersymmetric four-dimensional sigma-models with one triholomorphic isometry.

The idea of the coupling constant dualization can be most easily demonstrated by an example of a conformal mechanics which is governed by the action

$$S = \int dt \left[ \dot{x}^2 - \frac{g^2}{x^2} \right],$$

with a bosonic field $x(t)$ depending on time $t$ only and $g$ being a coupling constant. The constraint $g = \text{const}$ can obviously be interpreted as a solution to a differential equation

$$\frac{dg}{dt} = 0.$$  \hspace{1cm} (2)

Thus in such an approach we have got a system (1) with a constraint (2). Alternative to solving the constraint (2) is including it into the action (1) with a Lagrange multiplier $\phi(t)$

$$S = \int dt \left[ \dot{x}^2 - \frac{g^2}{x^2} - 2\dot{\phi} \right],$$

with a quantity $g(t)$ being no more a constant but some function of $t$. Varying (3) over $\phi$ we will get just (2), while the “equation of motion” for $g$ reads

$$g = x^2 \dot{\phi}.$$  \hspace{1cm} (4)

Eliminating the “coupling constant” $g$ we get the following action

$$S = \int dt \left[ \dot{x}^2 + x^2 \dot{\phi}^2 \right],$$

which is easily recognized as the action of a $D = 2$ free particle written in the polar coordinates.

1 Constructing $N = 4$ hyper-Kähler $\sigma$-manifold

In this section we construct an $N = 4$ supersymmetric $\sigma$-model with a hyper-Kähler geometry of its scalar manifold using dualization of a coupling constant in a way described above. Since the coupling constant dualization increases the number of the physical scalars...
by one, therefore we should start from a model with at least a three-dimensional target space to have a four-dimensional one as a result. Appropriate three-dimensional model is based on a linear $(3, 4, 1)$ supermultiplet with four scalars (three physical and one auxiliary) and four fermions [10] [11] [12].

We consider a supersymmetry algebra with four odd generators. Appropriate superspace $\mathbb{R}^{1|2}$ may be equipped with covariant spinor derivatives which satisfy the following relations
\[
\{ D, \bar{D} \} = 2i \frac{d}{dt}, \quad \{ D, D \} = \{ \bar{D}, \bar{D} \} = 0.
\]
In terms of $N = 2$ superfields the supermultiplet $(3, 4, 1)$ is given by a real $N = 2$ superfield $V(t, \theta, \bar{\theta})$, a chiral $N = 2$ one $\Phi(t, \theta, \bar{\theta})$ and its conjugated $\bar{\Phi}(t, \theta, \bar{\theta})$
\[
\bar{V} = V, \quad D\Phi = \bar{D}\Phi = 0. \quad (6)
\]
To maintain $N = 4$ supersymmetry this $N = 2$ formulation has to be augmented by an extra $N = 2$ supersymmetry transformation which mixes the superfields $V$ and $\Phi$
\[
\delta V = -\epsilon \bar{D}\Phi - \bar{\epsilon} D\Phi, \quad \delta \Phi = \epsilon \bar{D}V, \quad \delta \bar{\Phi} = \bar{\epsilon} DV. \quad (7)
\]
The standard $N = 4$ supersymmetric $\sigma$-model action for the supermultiplet $(3, 4, 1)$ looks like [10]
\[
S_1 = \int dt d^2 \theta G \left( DV \bar{D}V + D\Phi \bar{D}\Phi \right), \quad (8)
\]
with a metric $G$ being an arbitrary function of the superfields $V, \Phi, \bar{\Phi}$. The action (8) is invariant with respect to the additional $N = 4$ supersymmetry transformations (7). To apply the above mentioned dualization one should have a constant, but we still miss it. To overcome this problem we just add a potential term
\[
S_2 = g \int dt d^2 \theta H(V, \Phi, \bar{\Phi}), \quad g = \text{const} \quad (9)
\]
to the $\sigma$-model action $S_1$. The potential term contains a dimensional constant $g$ and is invariant under manifest $N = 2$ supersymmetry. To enlarge it to the $N = 4$ one, we should require the invariance of $S_2$ under the transformations (7). This results in restriction on the function $H$ to be a harmonic one
\[
H_{VV} + H_{\Phi\bar{\Phi}} = 0. \quad (10)
\]
Therefore, $N = 4$ $d = 1$ supersymmetric action we will deal with acquires the following form
\[
S = S_1 + S_2 = \int dt d^2 \theta \left[ G \left( DV \bar{D}V + D\Phi \bar{D}\Phi \right) + gH \right] \quad (11)
\]
with the function $H$ satisfying Laplace equation (10). After integration over the Grassmann coordinates we get the component form of the action

$$S = \int dt \left[ - (G_{\dot{v}v} + G_{v\dot{v}}) \psi \bar{\psi} \xi \bar{\xi} + i \bar{\psi} (G_{\dot{v}\phi} \xi \bar{\psi} + G_{v\dot{\phi}} \bar{\xi} \psi) + g (H_{\dot{v}v} \xi \bar{\psi} + H_{v\dot{v}} \psi \bar{\xi}) - A (G_{\dot{v}v} (\psi \bar{\psi} - \xi \bar{\xi}) + G_{v\dot{v}} \xi \bar{\psi} - G_{v\dot{v}} \bar{\xi} \psi + g H_{v\dot{v}} (\psi \bar{\psi} - \xi \bar{\xi})) - i \dot{\phi} (G_{\phi} (\psi \bar{\psi} - \xi \bar{\xi}) - 2 G_{\dot{v}v} \psi \bar{\xi}) + i \dot{\bar{\phi}} (G_{\bar{\phi}} (\psi \bar{\psi} - \xi \bar{\xi}) - 2 G_{v\dot{v}} \bar{\xi} \psi) + G (\dot{\psi}^2 + 4 \dot{\phi} \bar{\phi} + A^2 + i \dot{\psi} \bar{\psi} - i \psi \bar{\psi} + i \dot{\phi} \bar{\phi} - i \xi \bar{\xi} + i \xi \bar{\xi} - 4 i \xi \bar{\xi} - i g (H_{\phi} \dot{\bar{\phi}} - H_{\bar{\phi}} \dot{\phi}) \right].$$

(12)

Here the components of the superfields $V$ and $\Phi$ are defined as follows

$$v(t) = V, \quad \dot{v}(t) = DV, \quad \bar{\psi}(t) = -\bar{D}V, \quad A(t) = \frac{1}{2} [D, \bar{D}] V,$$

$$\phi(t) = \Phi, \quad \dot{\phi}(t) = \bar{\Phi}, \quad \xi(t) = D\Phi, \quad \bar{\xi}(t) = -\bar{D}\Phi$$

with the right hand sides being evaluated at $\theta = \bar{\theta} = 0$. As it was previously described, to dualize the constant $g$ we just add a term, which provides the constancy of $g$, to the action (13)

$$S \rightarrow S - \int dt y(t) \dot{y}.$$  

(13)

In contrast to the previously considered example with a conformal mechanics, now the coupling constant $g$ is involved into the action $S$ only linearly. Thus, its interpretation now is as a Lagrange multiplier for the some constraint on the field content of our theory which includes now one additional bosonic field $y(t)$. It is easy to see that this additional constraint expresses the auxiliary field $A$ in terms of the $y(t)$

$$A = \frac{1}{H_{\dot{v}v}} \left[ H_{v\dot{v}} (\psi \bar{\psi} - \xi \bar{\xi}) - i (H_{\phi} \dot{\bar{\phi}} - H_{\bar{\phi}} \dot{\phi}) - \dot{y} + H_{v\phi} \xi \bar{\psi} + H_{v\bar{\phi}} \psi \bar{\xi} \right].$$

(14)

Therefore, the number of the physical scalars is increased and we arrived at a supermultiplet $(4, 4, 0)$. It consists of the for physical bosons $v, \phi, \bar{\phi}$ and $y$ and four fermions $\psi, \bar{\psi}$ and $\xi, \bar{\xi}$. With respect to the full $N = 4$ supersymmetry they transform as follows

$$\delta v = \eta \bar{\psi} - \bar{\eta} \psi + \epsilon \xi - \bar{\epsilon} \bar{\xi}, \quad \delta \phi = -\bar{\eta} \xi - \epsilon \bar{\psi}, \quad \delta \bar{\phi} = \eta \xi + \epsilon \bar{\psi},$$

$$\delta y = -i\eta (H_{v\phi} \bar{\psi} + H_{\phi \bar{\psi}} \bar{\xi} - i\bar{\eta} (H_{v\phi} \psi + H_{\phi \bar{\psi}} \xi) + i\epsilon (H_{v\phi} \xi - H_{\phi \bar{\psi}} \bar{\psi}) + i\bar{\epsilon} (H_{v\phi} \bar{\xi} - H_{\bar{\phi}} \bar{\psi}),$$

$$\delta \psi = -i \eta \bar{\psi} + \epsilon A + 2i \eta \bar{\phi}, \quad \delta \bar{\psi} = i \bar{\eta} \psi + \epsilon A - 2i \bar{\eta} \bar{\phi},$$

$$\delta \xi = -i \epsilon \bar{\psi} - \bar{\epsilon} A - 2i \eta \bar{\phi}, \quad \delta \bar{\xi} = i \bar{\epsilon} \psi - \bar{\epsilon} A + 2i \eta \bar{\phi},$$

(15)

where $\eta$ and $\epsilon$ are the supersymmetry parameters and expression for $A$ is given by formula (14).
Now we see the main distinction of this new \((4,4,0)\) supermultiplet from the known ones: transformations \((15)\) are highly nonlinear and involve an arbitrary harmonic function \(H\). As well as the known ones, the constructed nonlinear supermultiplet is defined off-shell.

Substituting the expression for the auxiliary field \(A\) back into the action \((13)\) we get

\[
S = \int dt \left[ G \left( \dot{v}^2 + 4 \dot{\phi} \dot{\bar{\phi}} \right) + \frac{G}{H^2_v} \left( \dot{y} - iH_v \dot{\phi} + iH_v \dot{\bar{\phi}} \right)^2 + \text{fermions} \right].
\]

Kinetic part of this action describes a metric of a \(\sigma\)-model manifold

\[
d_{s^2} = G \left( dv^2 + 4 d\phi d\bar{\phi} \right) + \frac{G}{H^2_v} \left( dy - iH_v d\phi + iH_v d\bar{\phi} \right)^2.
\]

The Weyl tensor constructed for this metric is different from zero, so that this manifold is genuinely not conformally-flat. Moreover, imposing an additional requirement

\[
G = H_v
\]

we get a Ricci-flat bosonic manifold with a general Gibbons–Hawking metric for a hyper-Kähler manifold with one triholomorphic isometry \((15)\). Under the condition \((17)\) the action gets the form

\[
S = \int dt \left[ H_v \left( \dot{v}^2 + 4 \dot{\phi} \dot{\bar{\phi}} \right) + \frac{1}{H_v} \left( \dot{y} - iH_v \dot{\phi} + iH_v \dot{\bar{\phi}} \right)^2 \right.
\]

\[
+ \psi \bar{\xi} \left( iH_v \dot{v} - 2iH_v \dot{\phi} - i \frac{H_v \dot{\phi} - H_{\bar{v}} \dot{\bar{\phi}}}{H_v} + \dot{y} \frac{H_{\bar{v}}}{H_v} \right)
\]

\[
+ \xi \bar{\psi} \left( -iH_v \dot{\phi} - 2iH_v \dot{\bar{\phi}} - i \frac{H_v \dot{\bar{\phi}} - H_{\bar{v}} \dot{\phi}}{H_v} + \dot{y} \frac{H_{\bar{v}}}{H_v} \right)
\]

\[
+ \left( \psi \bar{\psi} - \xi \bar{\xi} \right) \left( i\dot{v} \left( H_{\bar{v}} - \frac{H_{vv} H_{\bar{v}}}{H_v} \right) - i\dot{\bar{\phi}} \left( H_v - \frac{H_{vv} H_v}{H_{\bar{v}}} \right) + \dot{y} \frac{H_{vv}}{H_v} \right)
\]

with four-fermionic terms disappearing, as it should be for Ricci-flat \(\sigma\)-model manifolds.

Thus, presented method of dualization allowed us to construct a nonlinear supermultiplet with component structure \((4,4,0)\) and find the action based on this supermultiplet. The latter corresponds to a not conformally flat manifold and becomes a hyper-Kähler one if condition \((17)\) holds.

Let us add that the expression for the auxiliary field \(A\) \((13)\) coincides (up to a total time-derivative term) with that one previously found in \((6)\), while the idea of construction of an auxiliary field through a general superspace potential term was firstly proposed by E. Ivanov \((7)\).
Constructing $N = 8$ hyper-Kähler $\sigma$-manifold

This section is based on an $N = 4$ superfield formalism therefore we first introduce notations. A superspace $\mathbb{R}^{1|4}$ we are dealing with is parameterized by one even coordinate $t$ and four odd coordinates $\theta^i$ and $\bar{\theta}^i$ with the index $i$ being $SU(2)$ one running $i = 1, 2$. All superfields to be considered are supposed to live in this superspace. To single out an irreducible representation from a general superfield we make use of covariant spinor derivatives $D^i$ and $\bar{D}^i$ defined on $\mathbb{R}^{1|4}$ and satisfying the following relations

$$\{D^i, \bar{D}^j\} = 2i \delta^i_j \frac{d}{dt}, \quad \{D^i, D^j\} = \{\bar{D}^i, \bar{D}^j\} = 0.$$  

In a full analogy with $N = 4$ supersymmetric case we will start from an irreducible $N = 8$ supermultiplet with three physical bosons, eight fermions and five auxiliary bosons, i.e. $(3, 8, 5)$ multiplet. Such a representation is described [13] in $\mathbb{R}^{1|4}$ by a real $N = 4$ superfield $V(t, \theta, \bar{\theta})$ and a chiral $N = 4$ superfield $\Phi(t, \theta, \bar{\theta})$

$$D^i \Phi = \bar{D}^i \bar{\Phi} = 0, \quad D^i D_i V = \bar{D}^i \bar{D}_i V = 0. \quad (18)$$

The constraints (18) leave among the components of the superfields $V$ and $\Phi$ the following independent ones:

$$v(t) = V, \quad \psi_i(t) = -i \bar{D}_i V, \quad \bar{\psi}^i(t) = -i D^i V, \quad A_{ij}(t) = i [\bar{D}_i, D_j] V, \quad \varphi(t) = \Phi, \quad \xi_i(t) = -i \bar{D}_i \Phi, \quad \bar{\xi}^i(t) = -i D^i \bar{\Phi}, \quad B(t) = D^i D_i \bar{\Phi}. \quad (19)$$

The right hand sides of the above expressions are supposed to be taken with vanishing $\theta_i$ and $\bar{\theta}^i$.

One should note that the constraints (18) impose the following restrictions on the superfield $V$ [14]

$$\frac{\partial}{\partial t}[D^i, \bar{D}_i] V = 0 \quad \Rightarrow \quad [D^i, \bar{D}_i] V = 2g, \quad g = \text{const.} \quad (19)$$

If $g \neq 0$ it appears in the $\theta$’s decomposition of the superfield $V$

$$V(t, \theta, \bar{\theta}) = v(t) + i \theta_i \bar{\psi}^i(t) + i \bar{\theta}^i \psi_i(t) + \frac{1}{2} \theta^i \bar{\theta}^j (i A_{ij}(t) - \varepsilon_{ij} g) + \ldots \quad (20)$$

Having these $N = 4$ superfields one can easily build a supersymmetric action as an integral of a real superfunction over the whole superspace

$$S = \int dt L = - \int dt d^2 \theta d^2 \bar{\theta} F(V, \Phi, \bar{\Phi}). \quad (21)$$
Being constructed in terms of manifest \( N = 4 \) superfields the action \( (21) \) is just \( N = 4 \) supersymmetric, not \( N = 8 \). To promote it to \( N = 8 \) one should require the action to be invariant with respect to an additional \( N = 4 \) supersymmetry

\[
\delta V = \eta_i D_i \Phi + \bar{\eta} \bar{D}_i \Phi, \quad \delta \Phi = -\eta_i D^i V, \quad \delta \bar{\Phi} = -\bar{\eta}^i \bar{D}_i V. \tag{22}
\]

This additional \( N = 4 \) supersymmetry commutes with the manifest one and extends it to \( N = 8 \). The invariance of the action \( (21) \) with respect \( (22) \) puts the restricting the prepotential \( F \) to be a harmonic function

\[
\frac{\partial^2 F}{\partial V \partial \bar{V}} + \frac{\partial^2 F}{\partial \Phi \partial \bar{\Phi}} = 0. \tag{23}
\]

Finally, after performing Grassmann integration in eq. \( (21) \) one gets the following expression for the Lagrangian

\[
L = F,_{\psi \bar{\psi}} \left( \bar{\psi}^2 + 4 \bar{\psi} \bar{\psi} \dot{\bar{\psi}} - i \bar{\psi} \bar{\psi} \dot{\bar{\psi}} + i \bar{\psi} \bar{\psi} \dot{\bar{\psi}} + i \bar{\psi} \bar{\psi} \dot{\bar{\psi}} + i \bar{\psi} \bar{\psi} \dot{\bar{\psi}} \right) + F,_{\psi \bar{\psi}} \left( \psi^2 \bar{\psi}^2 + 4 \psi \bar{\psi} \psi \bar{\psi} - i \psi \bar{\psi} \psi \bar{\psi} - i \psi \bar{\psi} \psi \bar{\psi} \right) + F,_{\psi \bar{\psi}} \left( \psi^2 \bar{\psi}^2 - \psi \bar{\psi} \psi \bar{\psi} + i \psi \bar{\psi} \psi \bar{\psi} \right) + F,_{\psi \bar{\psi}} \left( \psi^2 \bar{\psi}^2 - \psi \bar{\psi} \psi \bar{\psi} + i \psi \bar{\psi} \psi \bar{\psi} \right) + F,_{\psi \bar{\psi}} \left( \psi^2 \bar{\psi}^2 - \psi \bar{\psi} \psi \bar{\psi} + i \psi \bar{\psi} \psi \bar{\psi} \right) + F,_{\psi \bar{\psi}} \left( \psi^2 \bar{\psi}^2 - \psi \bar{\psi} \psi \bar{\psi} + i \psi \bar{\psi} \psi \bar{\psi} \right)
\]

\[
L = \int dt \ y(t) \dot{y}(t) \quad \tag{25}
\]

with the help of a Lagrange multiplier \( y(t) \). Now we are ready to eliminate the all set of auxiliary fields \( A_{ij}(t) \), \( B(t) \) and \( g(t) \) from the modified action \( (25) \) using their equations of motion. This results in the action written in terms of physical fields only

\[
S = \int dt \ [K - U] \quad \tag{26}
\]

\[1\] The bilinear products \( \psi^2 \) and \( \bar{\psi}^2 \) stands for \( \psi^i \psi_i \) and \( \bar{\psi}^i \bar{\psi}_i \) respectively.
with the kinetic term equal to

\[ K = F_{vv} \left( \dot{\psi}^2 + 4 \dot{\phi} \dot{\bar{\phi}} - i \psi^i \dot{\bar{\psi}}_i + i \psi^i \dot{\psi}_i - i \dot{\xi}^i \dot{\bar{\xi}}_i + i \xi^i \dot{\bar{\xi}}_i \right) + \frac{(y - i F_{v \psi} \dot{\phi} + i F_{v \bar{\psi}} \dot{\bar{\phi}})^2}{F_{vv}} \]

and potential one

\[
U = \frac{1}{4} \left( \psi^2 \bar{\psi}^2 + \xi^2 \bar{\xi}^2 - 4 \xi^i \bar{\psi}_i \bar{\bar{\psi}}^i \right) \left( F_{vvvv} + \frac{F_{vvvF_{v \psi \bar{\bar{\psi}}}} - 2 F_{v \psi \bar{\bar{\psi}}}}{F_{vv}} \right) \\
+ \frac{1}{2} \xi^2 \bar{\xi}^2 \left( F_{v \psi \bar{\bar{\psi}}} - \frac{3 F_{v \bar{\bar{\psi}}}}{F_{v \psi}} \right) + \frac{1}{2} \bar{\xi}^2 \bar{\bar{\psi}}^2 \left( F_{v v \bar{\bar{\psi}}} - \frac{3 F_{v \psi \bar{\bar{\psi}}}}{F_{vv}} \right) \\
+ \frac{1}{2} \left( \psi^2 \bar{\xi}^i \bar{\psi}_i - \xi^2 \bar{\bar{\psi}}^i \right) \left( F_{v v \psi \bar{\bar{\psi}}} - \frac{3 F_{v \psi \bar{\bar{\psi}}}}{F_{vv}} \right) - 2 i F_{v \psi} \left( \dot{\phi} \bar{\xi} \psi^i - \dot{\bar{\phi}} \xi \bar{\psi}^i \right) \\
+ i \left( F_{v \psi \bar{\bar{\psi}}} \dot{\bar{\psi}}_i - F_{v \psi \bar{\bar{\psi}}} \dot{\psi}^i \right) - i \left( F_{v \psi \bar{\bar{\psi}}} - F_{v \bar{\psi} \psi} \right) \left( \psi^i \bar{\psi}_i - \xi^i \bar{\xi}_i \right) \\
- \frac{y - i F_{v \psi \bar{\bar{\psi}}} + i F_{v \bar{\psi} \psi}}{F_{vv}} \left( \psi^i \bar{\psi}_i - \xi^i \bar{\xi}_i \right) + F_{v \psi \bar{\bar{\psi}}} \xi^i \bar{\psi}_i + F_{vv \psi \bar{\bar{\psi}}} \xi^i \bar{\xi}_i. \]

The kinetic term defines the following metric of the bosonic manifold

\[ ds^2 = F_{vv} \left( d\psi^2 + 4 d\phi d\bar{\phi} \right) + \frac{1}{F_{vv}} \left( dy - i F_{v \psi} d\phi + i F_{v \bar{\psi}} d\bar{\phi} \right)^2. \tag{27} \]

The metric (27) is of Gibbons–Hawking form \[15\] corresponding to the most general four-dimensional hyper-Kähler manifold with one triholomorphic isometry, which is realized as a shift along the coordinate \( y \).

Thus, in a such simple manner we construct \( N = 8 \) supersymmetric hyper-Kähler \( \sigma \)-model.

**Conclusion**

In the paper we presented a simple idea for constructing \( N = 4 \) and \( N = 8 \) supersymmetric hyper-Kähler \( \sigma \)-models by dualizing a coupling constant, which may either be present in the superfield decomposition or serve as an coupling constant for the potential term.

The idea of such a dualization is based on an ambivalent interpretation of coupling constants in one dimensions: on the one hand it is just a constant, on the other hand – it may be interpreted as some constant values of angular momenta. Dualized system contains one additional scalar field and describes a mechanics with an arbitrary value of such momenta. In particular, dualization turns the tensor supermultiplet into the nonlinear
hypermultiplet for the case of $N = 4$. The most essential point is that transformation properties of constructed hypermultiplet is nonlinear. Moreover, in the case of $N = 4$ dualization includes one additional harmonic function and gives a nonlinear hypermultiplet defined off-shell whose transformations properties under supersymmetry crucially depend on this function. The case of $N = 8$ is a little bit different and the question whether the hypermultiplet is off-shell requires more detailed study.

There are some questions yet to be solved: it unclear dualization of what constants is essential, when constructed nonlinear supermultiplets are defined off-shell or on-shell, how to describe such supermultiplets in terms of superfield approach, etc.

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References

[1] S. J. Gates, Jr., L. Rana: Phys.Lett. B 342 (1995), 132–137, A. Pashnev, F. Toppan: J. Math. Phys. 42 (2001), 5257–5271.

[2] G. W. Gibbons, G. Papadopoulos, K. S. Stelle: Nucl. Phys. B 508 (1997), 623–658.

[3] C. M. Hull: “The Geometry of Supersymmetric Quantum Mechanics,” hep-th/9910028.
  G. Papadopoulos: Class. Quant. Grav. 17 (2000), 3715–37141,
  J. Michelson, A. Strominger: Commun. Math. Phys. 213 (2000), 1–17,
  J. Michelson, A. Strominger: JHEP 9909 (1999), 005,
  R. A. Coles: G. Papadopoulos Class. Quantum Grav. (7), 1990427–438.

[4] L. Alvarez-Gaumé, D. Freedman: Phys. Lett. B 94 (1980), 171–173.

[5] Č. Burdík, S. Krivonos, A. Shcherbakov: Czechoslovak Journal of Physics 55 (2005), 1357–1364.

[6] S. Krivonos, A. Shcherbakov: Phys. Lett. B 637 (2006), 119–122.

[7] E. Ivanov, private communication.

[8] S. Bellucci, S. Krivonos, A. Shcherbakov: Phys. Rev. D 73 (2006), 085014.

[9] F. Delduc, E. Ivanov: “Gauging $N = 4$ Supersymmetric Mechanics,” hep-th/0605211.
[10] Ivanov E., Smilga A.: Phys.Lett. B. 257 (1991), 79-82.

[11] Berezovoi V., Pashnev A.: Class. Quant. Grav. 8 (1991), 19.

[12] Ivanov E., Krivonos S., Lechtenfeld O.: Class. Quant. Grav. 21 (2004), 1031-1050.

[13] S. Bellucci, E. Ivanov, S. Krivonos, O. Lechtenfeld: Nucl. Phys. B 699 (2004), 226–252.

[14] E. Ivanov, S. Krivonos, V. Leviant: J. Phys. A 22 (1989), 4201.

[15] Gibbons, S. W. Hawking: Phys. Lett. B 78 (1978), 430–432.