POLYGONS WITH PRESCRIBED EDGE SLOPES: CONFIGURATION SPACE AND EXTREMAL POINTS OF PERIMETER

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ABSTRACT. We describe the configuration space $S$ of polygons with prescribed edge slopes, and study the perimeter $P$ as a Morse function on $S$. We characterize critical points of $P$ (these are tangential polygons) and compute their Morse indices. This setup is motivated by a number of results about critical points and Morse indices of the oriented area function defined on the configuration space of polygons with prescribed edge lengths (flexible polygons). As a by-product, we present an independent computation of the Morse index of the area function (obtained earlier by G. Panina and A. Zhukova).

1. Introduction

Consider the space $L$ of planar polygons with prescribed edge lengths and the oriented area $A$ as a Morse function defined on it. It is known that generically:

- $L$ is a smooth closed manifold whose diffeomorphic type depends on the edge lengths;\footnote{The space $L$ appears in the literature as “configuration space of a flexible polygon”, or “configuration space of a polygonal linkage”, or just as “space of polygons”.}
- The oriented area $A$ is a Morse function whose critical points are cyclic configurations (that is, polygons with all the vertices lying on a circle), whose Morse indices are known, see Theorem;\footnote{2000 Mathematics Subject Classification. 52R70, 52B99. Key words and phrases. Morse index, critical point, cyclic polygon, flexible polygon.} The Morse index depends not only on the combinatorics of a cyclic polygon, but also on some metric data. Direct computations of the Morse index proved to be quite involved, so the existing proof comes from bifurcation analysis combined with a number of combinatorial tricks.
- Bifurcations of $A$ are captured by cyclic polygons $P$ whose dual polygons $P^*$ have zero perimeter; see also Lemma. That is, in a generic one-parametric family of edge lengths a critical point $P$ bifurcates whenever the perimeter of the dual tangential polygon $P^*$ vanishes.

The polygon $P^*$ is tangential (see Definition), so tangential polygons with zero perimeter play a special role in the framework of flexible polygons and
oriented area. The initial motivation of the present paper was to clarify this role.

In the paper we consider the following problem: instead of prescribing edge lengths, we prescribe the slopes of the edges. Instead of taking the oriented area as a Morse function, we take the oriented perimeter. We prove:

- The space $S$ of polygons with prescribed edge slopes is a smooth non-compact manifold (see Theorem 4 for its diffeomorphism type).
- The (oriented) perimeter $P$ is a Morse function with either zero or two critical points (Theorem 5). Critical points of $P$ are tangential polygons.
- The absence of critical points is captured by existence of tangential polygons with zero perimeter (Corollary 1). That is, in a generic one-parametric family of slopes critical points disappear whenever the perimeter of the (uniquely defined) tangential polygon vanishes.
- Although there are at most two critical points, these are not necessarily maximum and minimum of the perimeter function. The Morse index of a tangential polygon is expressed in Theorem 6. The proof is based on direct computation of the leading principal minors of the Hessian matrix.
- The Morse index of a tangential polygon depends on the combinatorics of the polygon and the sign of its perimeter only.
- Local projective duality provides an alternative proof of Theorem 2, that is, the formula for the Morse index of a cyclic polygon (related the area function).

Yet another motivation of this research is projective duality. Oversimplifying, assume that the ambient space of the polygons is the sphere $S^2$. Then projective duality takes polygons with prescribed edge lengths to polygons with prescribed angles. It also takes area to a linear function of perimeter, so the critical polygons in the two settings are mutually projectively dual and have related Morse indices. (A necessary warning: there exists only a local version of projective duality. However it is sufficient for purposes.)

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2. Definitions and setups

A polygon is an oriented closed broken line in the plane. We assume that its vertices are numbered in the cyclic order and thus induce an orientation of the polygon.

**Definition 1.** The oriented area of a polygon \( P \) with the vertices \( v_i = (x_i, y_i) \) is defined by

\[
2A(P) = (x_1y_2 - x_2y_1) + \ldots + (x_ny_1 - x_1y_n).
\]

Equivalently, one defines

\[
A(P) := \int_{\mathbb{R}^2} w(P, x)dx,
\]

where \( w(P, x) \) is the winding number of \( P \) around the point \( x \).

2.1. Polygons with prescribed edge lengths. Oriented area as a Morse function. Assume that a generic \( n \)-tuple of positive numbers \( (l_1, \ldots, l_n) \) is given. The space of all planar polygons whose consecutive edge lengths are \( (l_1, \ldots, l_n) \) (modulo translations and rotations) is called the configuration space of polygons with prescribed edge lengths. We denote it by \( L = L(l_1, \ldots, l_n) \).

Generically, \( L \) is a smooth closed manifold \([1, 2]\), and the oriented area \( A \) is a Morse function on \( L \).

**Definition 2.** A polygon \( P \) is cyclic if all its vertices \( v_i \) lie on a circle.

**Theorem 1.** Generically, \( A \) is a Morse function. At smooth points of the space \( L \), a polygon \( P \) is a critical point of the oriented area \( A \) iff \( P \) is a cyclic configuration. \[ \square \]

Before we recall a formula for the Morse index of a cyclic configuration from \([3, 4, 5, 6]\), let us fix the following notation for a cyclic polygon \( P \).

- \( \omega_P = w(P, O) \) is the winding number of \( P \) with respect to the center \( O \) of the circumscribed circle.
- \( \alpha_i \) is the half of the angle between the vectors \( \overrightarrow{Ov_i} \) and \( \overrightarrow{Ov_{i+1}} \). The angle is defined to be positive, orientation is not involved.
- \( \varepsilon_i \) is the orientation of the edge \( v_iv_{i+1} \), that is,

\[
\varepsilon_i = \begin{cases} 
1, & \text{if the center } O \text{ lies to the left of } \overrightarrow{v_iv_{i+1}}; \\
-1, & \text{if } O \text{ lies to the right of } \overrightarrow{v_iv_{i+1}}.
\end{cases}
\]

- \( e(P) \) is the number of positive entries in \( \varepsilon_1, \ldots, \varepsilon_n \), that is, \( e(P) \) is the number of positively oriented edges.
- \( \mu_P = \mu_P(A) \) is the Morse index of the function \( A \) at the point \( P \). That is, \( \mu_P(A) \) is the number of negative eigenvalues of the Hessian matrix \( Hess_P(A) \).

**Theorem 2.** Generically, for a cyclic polygon \( P \),

\[
\mu_P(A) = e(P) - 1 - 2\omega_P - \begin{cases} 
0 & \text{if } \sum_{i=1}^{n} \varepsilon_i \tan \alpha_i > 0; \\
1 & \text{otherwise}.
\end{cases}
\]

In the present paper we give an alternative proof of this theorem: we prove a slightly stronger claim, see Corollary \[2\].
Remark 3. In a continuous one-parametric family of cyclic polygons with non-vanishing edge lengths, \( \mu_P \) changes iff \( \sum_{i=1}^{n} \varepsilon_i \tan \alpha_i \) vanishes. Although \( e(P) \) and \( \omega_P \) can vary, the sum \( e(P) - 1 - 2\omega_P \) is constant.

2.2. Polygons with prescribed edge slopes. Fix \( n \) pairwise non-parallel straight lines \( s_1, \ldots, s_n \) in \( \mathbb{R}^2 \) passing through the origin and call them slope lines.

Each \( n \)-tuple of lines \( e_1, \ldots, e_n \subset \mathbb{R}^2 \) with \( e_i \) parallel to \( s_i \) yields a polygon \( P \) whose consecutive vertices are \( v_1 = e_1 \cap e_2, \ldots, v_n = e_n \cap e_1 \). We will denote this polygon by \( Q = Q(e_1, \ldots, e_n) \) and say that \( Q \) is a polygon with edge slopes \( s_1, \ldots, s_n \).

The space of all polygons \( Q \) (modulo translations) with edge slopes \( s_1, \ldots, s_n \), is denoted by \( \tilde{S} = \tilde{S}(s_1, \ldots, s_n) \).

The subspace of \( \tilde{S}(s_1, \ldots, s_n) \) consisting of polygons with \( |\mathcal{A}(Q)| = 1 \) is called the configuration space of polygons with prescribed edge slopes. We denote it by \( S = S(s_1, \ldots, s_n) \). It splits into a disjoint union \( S_- \sqcup S_+ \) where the index indicates the sign of \( \mathcal{A} \).

![Figure 1. Slope lines (left) and a polygon with these edge slopes (right).](image)

Note that: (1) the condition \( |\mathcal{A}(Q)| = 1 \) means that we factor out dilations; (2) fixing slopes is the same as fixing angles and factoring out rotations.

Now fix a direction on each of the slope lines \( s_i \). Take \( Q = Q(e_1, \ldots, e_n) \in S(s_1, \ldots, s_n) \), and orient the lines \( e_1, \ldots, e_n \) consistently. Denote the oriented lines by \( \vec{e}_1, \ldots, \vec{e}_n \).

**Definition 3.** The perimeter of a polygon \( Q \in \tilde{S}(s_1, \ldots, s_n) \) is defined as follows:

\[
\mathcal{P}(Q) = \sum_{i=1}^{n} \text{sign}_Q(i) |v_i v_{i+1}|,
\]

where \( \text{sign}_Q(i) = \begin{cases} 1, & \text{if } \vec{e}_i \text{ is codirected with } \vec{v}_i \vec{v}_{i+1}; \\ -1, & \text{otherwise.} \end{cases} \)
Thus defined, perimeter may be negative or vanish.

3. Topology of the configuration space of polygons with prescribed slopes

Let the angle $\angle(r, s)$ between two lines $r$ and $s$ be the minimal positive angle such that the counterclockwise rotation by $\angle(r, s)$ takes $r$ to $s$.

Assuming that an $n$-tuple of slope lines $s_1, \ldots, s_n$ in $\mathbb{R}^2$ is fixed, set $t(s_1, \ldots, s_n) := \sum_{i=1}^{n-1} \angle(s_i, s_{i+1}) + \angle(s_n, s_1)$.

The example for $n = 3$ will be useful in the sequel:

**Example 1.** $t(s_1, s_2, s_3)$ is either $\pi$ or $2\pi$. In the first (respectively, second) case the area of each nondegenerate triangle in $\widehat{S}$ is negative (respectively, positive).

**Lemma 1.**

1. $t(s_1, \ldots, s_n)$ takes values in $\{\pi, 2\pi, \ldots, (n-1)\pi\}$.
2. $t(s_1, \ldots, s_n) = t(s_1, \ldots, s_{n-1}) + t(s_1, s_{n-1}, s_n) - \pi$. $\square$

The informal meaning of the following lemma is: the area and perimeter behave additively with respect to homological sum of polygons.

**Lemma 2.** For a polygon $Q = Q(e_1, \ldots, e_n) \in \widehat{S}(s_1, \ldots, s_n)$, we have

1. $A(Q) = A(Q_{1, \ldots, n}) = A(Q_{1, \ldots, n-1}) + A(Q_{1,n-1,n})$;
2. $P(Q) = P(Q_{1, \ldots, n}) = P(Q_{1, \ldots, n-1}) + P(Q_{1,n-1,n})$,

where $Q_{i_1, \ldots, i_k} = Q(e_{i_1}, \ldots, e_{i_k}) \in \widehat{S}(s_{i_1}, \ldots, s_{i_k})$. $\square$

![Figure 2. A summand of $A(Q_{1, \ldots, n})$.](image)

**Theorem 4.** The configuration space $S(s_1, \ldots, s_n)$ is homeomorphic to disjoint union of products of a sphere and a disc:

$$
S^{n-k-2} \times \mathbb{D}^{k-1} \sqcup S^{k-2} \times \mathbb{D}^{n-k-1},
$$

where $t(s_1, \ldots, s_n) = k\pi$.

The left-hand part corresponds to $S_-$ and the right-hand part corresponds to $S_+$. 
Proof. We shall prove that $S_+$ is homeomorphic to $S^{k-2} \times \mathbb{D}^{n-k-1}$. The proof for $S_-$ is analogous.

By Lemma 2

$$A(Q_1,\ldots, n) = \sum_{i=2}^{n-1} A(Q_{1,i,i+1}) = \sum_{i=2}^{n-1} \text{sign} \left( t(s_1, s_i, s_{i+1}) - \frac{3\pi}{2} \right) \cdot c_i \cdot \text{dist}(v_i, e_1)^2,$$

where $c_i = c_i(s_1, s_i, s_{i+1})$ is some positive constant depending only on the edge slopes (see Example 1 and Fig. 2). Thus we can parameterize $\tilde{S}(s_1, \ldots, s_n)$ by

$$\{x \in \mathbb{R}^{n-2} | \sum_{j \in A} x_j^2 - \sum_{j \in B} x_j^2 = 1\},$$

where $|A| = k - 1$, $A \sqcup B = \{1, \ldots, n-2\}$. This is homeomorphic to

$$\{x \in \mathbb{R}^{n-2} | \sum_{j \in A} x_j^2 = 1, \sum_{j \in B} x_j^2 < 1\}.$$

Indeed, for $k > 1$ the homeomorphism $h: x \mapsto x / \sqrt{\sum_{j \in A} x_j^2}$ is well defined and appropriate, whereas for $k = 1$ both sets are empty. The claim follows. □

4. Critical points of the perimeter

Assume that an $n$-tuple of directed slope lines $\vec{s}_1, \ldots, \vec{s}_n$ in $\mathbb{R}^2$ is fixed.

Definition 4. (1) A polygon $Q(e_1, \ldots, e_n) \in \tilde{S}(s_1, \ldots, s_n)$ is tangential if there exists a circle $\sigma$ such that

(a) each of $e_i$ is tangent to $\sigma$, and

(b) either $\sigma$ lies on the left with respect to all of $\vec{e}_i$, or $\sigma$ lies on the right with respect to all of $\vec{e}_i$, see Fig. 3.

In this case we say that the circle $\sigma$ is inscribed in $Q$ and write $\sigma = \sigma(Q)$.

(2) By the radius $r = r(\sigma(Q))$ of the inscribed circle $\sigma$ we mean the usual radius taken with the sign “+” if $\sigma$ lies on the left of $\vec{e}_i$, and with the sign “−” otherwise.

Lemma 3. The area and perimeter of a tangential polytope $Q$ satisfy:

$$A(Q) = \frac{1}{2} P(Q) \cdot r(\sigma(Q)).$$

Theorem 5. $Q \in S$ is a critical point of $P$ iff $Q$ is tangential.

Proof. Let $r_i$ be the radius of the (uniquely defined) circle inscribed in the triangle $Q_{1,i+1,i+2}$, $i = 1, \ldots, n-2$. Denote by $p_i$ the perimeter of the triangle homothetic to $Q_{1,i+1,i+2}$ (that is, with the same edge slopes) whose inscribed...
radius equals 1. If \( r_i \neq 0 \), \( p_i \) is the perimeter of \( Q_{1,i+1,i+2} \). Then by Lemma 2 and Lemma 3 we have

\[
\mathcal{P}(Q) = \sum_{i=1}^{n-2} p_i \cdot r_i, \quad \text{and}
\]

\[
\pm 1 = \mathcal{A}(Q) = \frac{1}{2} \sum_{i=1}^{n-2} p_i \cdot r_i^2.
\]

The collection of radii gives a coordinate system on \( \tilde{S} \). W.l.o.g. we assume \( r_1 > 0 \). Locally on \( S \), the second row implicitly defines a function \( r_1(r_2, \ldots, r_{n-2}) \). Let us take \( j \)-th partial derivatives of the both rows, \( j = 2, \ldots, n-2 \).

Since \( 0 = \frac{\partial \mathcal{A}}{\partial r_j}(Q) = p_1 r_1 \frac{\partial r_1}{\partial r_j} + p_j r_j, \)

we have \( \frac{\partial r_1}{\partial r_j} = -\frac{p_j r_j}{p_1 r_1}. \)

Therefore, \( \frac{\partial \mathcal{P}}{\partial r_j}(Q) = p_1 \frac{\partial r_1}{\partial r_j} + p_j = p_j (1 - \frac{r_j}{r_1}). \)

Thus the gradient of the perimeter is zero iff \( r_1 = r_2 = \ldots = r_{n-2} = r \).

There exists exactly one (up to a dilation) pair of mutually symmetric tangential polygons \( Q \) and \( -Q \). For them we have: \( \mathcal{A}(-Q) = \mathcal{A}(Q), \mathcal{P}(-Q) = -\mathcal{P}(Q), \) and \( r(-Q) = -r(Q) \), see Fig. 3 for example. If the area is non-zero, scaling gives \( |\mathcal{A}| = 1 \).

**Corollary 1.** If the area of a tangential polygon is zero, there are no critical points of \( \mathcal{P} \) on the configuration space \( S \). Otherwise there are exactly two critical points on the configuration space \( S \). Either they both lie in \( S_+ \), or they both lie in \( S_- \).
A configuration space $S(s_1, \ldots, s_n)$ with no critical points is called exceptional.

5. Morse index of a tangential polygon

Now compute the Hessian matrix of $P$ at a critical point. Assume that $Q$ is a tangential polygon. In notation of the previous section, we have:

$$\frac{\partial^2 P}{\partial r_j^2}(Q) = p_1 \frac{\partial^2 r_1}{\partial r_j^2} = p_1 \frac{\partial}{\partial r_j} \left( -\frac{p_j r_j}{p_1 r_1} \right) = -p_j \left( \frac{1}{r_1} - \frac{r_j}{r_1^2} \frac{\partial r_1}{\partial r_j} \right) = -\frac{p_j}{r_1} \left( 1 + \frac{p_j r_j}{r_1} \right).$$

Since at the critical point all $r_i$ are equal,

$$H_{jj}(Q) = \frac{\partial^2 P}{\partial r_j^2}(Q) = -\frac{p_j}{r_1} (p_1 + p_j).$$

In the same way, for $j \neq k$

$$\frac{\partial^2 P}{\partial r_j \partial r_k} = p_1 \frac{\partial^2 r_1}{\partial r_j \partial r_k} = p_1 \frac{\partial}{\partial r_k} \left( -\frac{p_j r_j}{p_1 r_1} \right) = -p_j r_j \left( 1 - \frac{1}{r_1^2} \frac{\partial r_1}{\partial r_k} \right) = -\frac{p_j r_j}{r_1^2} \frac{p_k r_k}{p_1 r_1}.$$  

Thus:

$$H_{jk}(Q) = \frac{\partial^2 P}{\partial r_j \partial r_k} = -\frac{p_j p_k}{r_1^2}.$$

To compute its determinant we do the following:

1. add all the columns to the first one;
2. subtract the first row from the $i$-th row ($i = 2, 3, \ldots, n - 2$) taken with the coefficient $\frac{p_{i+1}}{p_2}$;
3. subtract all rows from the first row with coefficient $\frac{p_2}{p_1}$.

We get:

$$r^{n-3} \det(H) = \begin{vmatrix} -\frac{p_2}{p_1} & 0 & \ldots & 0 \\ 0 & -p_3 & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & -p_{n-2} \end{vmatrix},$$

where $\Pi = p_1 + \ldots + p_{n-2} = \frac{P(Q)}{r}$.

**Definition 5.** For an ordered pair of slopes $\vec{s}_i$ and $\vec{s}_j$ we say that we have the right turn (left turn, respectively), if $\vec{s}_j$ is obtained from $\vec{s}_i$ by a clockwise (counterclockwise, respectively) turn by an angle smaller than $\pi$, see Fig. 4.

The number of right turns $RT = RT(\vec{s}_1, \ldots, \vec{s}_n)$ for a slope collection $(\vec{s}_1, \ldots, \vec{s}_n)$ is the number of right turns of the pairs $(\vec{s}_1, \vec{s}_2)$, $(\vec{s}_2, \vec{s}_3)$, ..., $(\vec{s}_n, \vec{s}_1)$. The number of left turns $LT$ is defined analogously.
Theorem 6. Assume that for a tangential polygon $Q \in S$, the radius $r$ of the inscribed circle $\sigma(Q)$ is positive. Then $Q$ is a Morse point, and its Morse index of $P$ is equal to

$$
\mu_Q(P) = RT - 1 + 2\omega - \left\{ \begin{array}{ll} 1 & \text{if } \mathcal{P}(Q) > 0; \\ 0 & \text{otherwise.} \end{array} \right.
$$

In the case of negative radius it is equal to

$$
\mu_Q(P) = n - 3 - \mu_{-Q}(P) = LT - 1 - 2\omega - \left\{ \begin{array}{ll} 1 & \text{if } \mathcal{P}(Q) > 0; \\ 0 & \text{otherwise.} \end{array} \right.
$$

Example 2. For the polygon depicted in Fig. 3 (a), we have $\omega = 0$, $RT = 2$, and $m = 0$.

For the polygon depicted in Fig. 5 we have $\omega = 1$, $RT = 1$, and $m = 1$.

Proof. For now denote the right-hand side of (2) as $m(Q)$ and assume $r > 0$. 

Figure 4. Right and left turns

Figure 5. This tangential polygon is a saddle critical point of the perimeter.
Let us prove (2) by induction on \( n \). For \( n = 3 \) the value of \( m \) is always zero, so is the Morse index.

Prove the claim for \( n + 1 \) assuming it is true for all the numbers smaller or equal than \( n \geq 3 \).

Recall that the number of negative eigenvalues of the matrix equals the number of sign changes in the sequence of its leading principal minors. The \( k \)-th leading principal minor of \( H(Q) \) is the determinant of \( H(Q_1,\ldots,k+2) \).

So we have:

- \( m(Q) = m(Q_1,\ldots,n) + 1 \) whenever the sign of the determinant (1) is different for \( n \) and \( n + 1 \);
- \( m(Q) = m(Q_1,\ldots,n) \) whenever the sign of the determinant (1) is the same for \( n \) and \( n + 1 \).

The change of sign of the determinant (1) depends only on the sign of \( p_{n-1} \) and the sign change of \( \Pi \).

Note that 
\[
RT(s_1,\ldots,s_{n+1}) = RT(s_1,\ldots,s_n) + RT(s_1,s_n,s_{n+1}) - 1
\]
and
\[
\omega_{Q_1,\ldots,n+1} = \omega_{Q_1,\ldots,n} + \omega_{Q_1,n,n+1}.
\]

Therefore,
\[
m(Q_1,\ldots,n+1) - m(Q_1,\ldots,n) = m(Q_1,n,n+1) + 1_P(Q_1,n,n+1) > 0 - 1_P(Q)>0 + 1_P(Q_1,\ldots,n) > 0.
\]

The first summand is zero by the base of induction. So we get:
\[
m(Q_1,\ldots,n+1) - m(Q_1,\ldots,n) = 1_{p_{n-1}>0} - 1_{p_1+\ldots+p_{n-1}>0} + 1_{p_1+\ldots+p_{n-2}>0},
\]
which is exactly what we require.

The case of \( r < 0 \) follows from \( P(-Q) = -P(Q) \).

6. Cyclic polygons and tangential polygons meet

Theorem 2 motivates the following definition:

**Definition 6.** A cyclic polygon is a **bifurcating polygon** if \( \sum_{i=1}^{n} \varepsilon_i \tan \alpha_i = 0 \).

Bifurcating polygons correspond to bifurcations of the area function.

**Definition 7.** Given a cyclic polygon \( P = \{v_1,\ldots,v_n\} \), define its **dual polygon** \( P^* \) (Fig. 6) as a closed broken line with orientations on the edges constructed as follows:

1. Take the lines \( e_1,\ldots,e_n \) tangential to the circle at the points \( \{v_1,\ldots,v_n\} \).
2. Take the intersection points of \( e_i \) and \( e_{i+1} \).
3. Orient each of the lines so that the circle lies to the left of the line.

The construction can be easily reversed: given a tangential polygon \( Q \), there exists a cyclic polygon \( P \) such that \( P^* = Q \).
Lemma 4.  

(1) For a cyclic polygon $P$, the perimeter of $P^*$ (in the sense of Definition 3) equals $\sum_{i=1}^{n} \varepsilon_i \tan \alpha_i$.

(2) In particular, the perimeter of $P^*$ vanishes iff $P$ is a bifurcating polygon.

(3) The oriented area of $P^*$ vanishes iff $P$ is a bifurcating polygon. \hfill \Box

To summarize, if a polygon $P$ is a bifurcation of the area, then its dual $Q = P^*$ yields an exceptional configuration space $S(s_1, ..., s_n)$. 

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7. Spherical polygons, local projective duality, and an alternative proof of Theorem 2

7.1. Spherical polygons. A spherical polygon is an oriented closed broken line lying on the sphere of radius $R$. We always assume that its edges are the unique shortest geodesics, that is, $l_i < \pi R$, so a spherical polygon is uniquely defined by the (circular) sequence of its vertices. 

One fails to correctly define the area function on the space of spherical polygons with prescribed edge lengths. One also fails to define the space of spherical polygons with prescribed angles together with the perimeter function. However we shall make use of their local versions. 

Here is how it goes: 

Definition 8. Let $P_0, Q_0$ be spherical polygons.

(1) Consider the space of all spherical polygons with the same edge lengths lying close\footnote{With respect to any reasonable metric} to $P_0$ subject to rotation of the sphere. This space is called
the local configuration space of spherical polygons with prescribed edge lengths \( L_{\text{loc}}(P_0) \).

(2) Elimination of a point from the sphere allows to define the winding numbers for curves in the sphere. So fix a point \( \infty \notin P_0 \) (that is, not lying on the broken line) and define the oriented area of \( P \in L_{\text{loc}}(P_0) \) as the integral

\[
A(P) := \int_{S^2} w(P, x) dx.
\]

(3) Analogously, we define the local configuration space of spherical polygons with prescribed angles \( S_{\text{loc}}(Q_0) \). Once we set some fixed orientations on the edges, we have a well defined perimeter function \( P \) on the space \( S_{\text{loc}}(Q_0) \). Note that the area \( A \) is constant on \( S_{\text{loc}}(Q_0) \).

**Proposition 1.** Assume that a spherical polygon \( P \) fits in a hemisphere not containing \( \infty \). \( P \) is a critical point of the area function \( A \) iff it is a cyclic polygon, that is, its vertices lie on a circle.

Proof. (1) Prove first the statement for polygons with four edges. If a polygon bounds a spherically convex region, then the statement is classical: a cyclic convex 4-gon exhibits either the maximum or the minimum point of the area, depending on orientation of \( P \).

("If") Assume that \( P \) intersects itself and is a cyclic polygon with vertices 1, 2, 3, 4. Add a new point 5 on the circle together with two new bars as is shown in Fig. 7. Then \( A(1234) = A(1254) - A(2543) \). A local flex of the polygon \( P \) induces flexes of the polygons (1254) and (2543). Since the latter are critical, the claim follows.

("Only if") Assume that \( P \) intersects itself and is a critical point of \( A \), but not a cyclic polygon. Take a circle superscribing 412 and add a new point 5 on the circle together with two new bars as we did above. Now (1254) is a critical polygon, and (2543) is not. \( A(1234) = A(1254) - A(2543) \) completes the proof.

(2) The general case (any number of edges) is obtained by verbatim repeating the reasonings from \( \Box \).

![Figure 7. Notation for Proposition](image-url)
Local duality of $L^{loc}(P)$ and $S^{loc}(P^o)$. Assume that $P$ is a spherical cyclic polygon. It fits in a hemisphere, and so do all the polygons from $L^{loc}(P)$. Assume also that a point $\infty$ lies beyond the hemisphere, so $A$ is well-defined on $L^{loc}(P)$, and $P$ is a critical point.

**Definition 9.**

1. In the above setting, define the dual polygon $P^0$:
   - (a) Fix the hemisphere containing $P$ centered at the center of the superscribed circle.
   - (b) Assume that $P = \{p_1, \ldots, p_n\}$. Take the big circles $C_1, \ldots, C_n$ projectively dual to the points $\{p_1, \ldots, p_n\}$.
   - (c) Connect the intersection points $q_i$ of $l_i$ and $l_{i+1}$ lying in the hemisphere by short geodesics.
   - (d) Orient each of the lines such that the circle lies on the left from each of the lines.

2. Continuously extend the duality to $L^{loc}(P)$. The extension is uniquely defined by the condition that edges of the dual polygon lie on dual lines to the vertices of the initial polygon.

**Lemma 5.** For a cyclic $P$,

1. The polygon $P^o$ is tangential.
2. $\mu_P(A) = n - 3 - \mu_{P^o}(P)$.

Proof. (1) is straightforward.

Projective duality (on the unit sphere) takes edge lengths of a polygon to the exterior angles of the dual, and vice versa. Since $A(P) = Const - R \cdot P(P^o)$,

\[ A(P) = Const - R \cdot P(P^o), \]

the claims (2) follows. \[ \square \]

**Morse indices: planar vs spherical.** Let $P_0$ be a planar cyclic polygon. It is uniquely defined by the circumscribed circle $\sigma$ and the ordered sequence of its points. Put the circle $\sigma$ with the $n$ points on the sphere of radius $R$, provided that $R > r(\sigma)$. It defines a spherical cyclic polygon $P^R_0$ fitting in the hemisphere centered at the center of $\sigma$. In turn, the spherical polygon $P^R_0$ gives rise to the local configuration space $L^{loc}(P^R_0)$ of polygons with prescribed edge lengths. Clearly, the bigger $R$ is, the smaller is the distortion of edge lengths and angles.

**Lemma 6.** The Morse index of the spherical polygon $P^R_0$ with respect to the area function $A_R$ is the same as the Morse index of $P_0$.

Proof. We have the one parametric family of spherical polygons and their local configuration spaces $P^R_0, L^{loc}(Q^R_0)$. The area function $A_R$ is well defined on $L^{loc}(P^R_0)$, and $P^R_0$ is its critical point. As the radius $R$ tends to infinity, the polygon $P^R_0$ deforms and tends to $P_0$. Since $P^R_0$ is the unique critical point in the neighborhood, $P^R_0$ does not bifurcate, so its Morse index does not change.
Besides, by standard arguments, the Morse index of $\mathcal{P}_R$ converges to the Morse index of the planar polygon $\mathcal{P}_0$.

Now we are ready to prove Theorem 2. Take a planar cyclic polygon $\mathcal{P}$. Take $\mathcal{P}_R$ for some big $R$. By the above lemma, $\mu_{\mathcal{P}_R}(\mathcal{A}_R) = \mu_{\mathcal{P}}(\mathcal{A})$. Gradually make $R$ smaller, such that the circumscribed circle tends to an equator of the sphere. The Morse index stays the same. Now take the projectively dual polygon $\mathcal{Q}_R := (\mathcal{P}_R)^\circ$. It is a small (and therefore, almost planar) tangential polygon.

By Lemma 5
\[ \mu_{\mathcal{P}_R}(\mathcal{A}_R) = n - 3 - \mu_{\mathcal{Q}_R}(\mathcal{P}_R). \]
Replace $\mathcal{Q}_R$ by a planar polygon $\mathcal{Q}$. On the one hand, the Morse index $\mu_{\mathcal{Q}}(\mathcal{P})$ stays the same. On the other hand, we know the Morse index by Theorem 2. It remains to observe that local projective duality maintains the winding number, and takes left turns to positively oriented edges.

This approach also gives the following fact which exceeds Theorem 2:

Corollary 2. Assume we have a cyclic polygon $\mathcal{P}$ such that (1) no two consecutive vertices are antipodal (with respect to the superscribed circle), and (2) the polygon does fit in a straight line. Then $\mathcal{P}$ is a Morse point of the oriented area function iff it is not a bifurcating polygon.

Proof. The dual polygon is a non-degenerate Morse point, see Theorem 2.

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