CONFIGURATION SPACES OF A KINEMATIC SYSTEM AND MONSTER TOWER OF SPECIAL MULTI-FLAGS

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Abstract. In this note we show that the configuration spaces of the kinematic system constructed in [4] and [12] gives rise to a natural tower of sphere bundles. Moreover, we prove that, each tower of projective bundles associated to special multi-flags (cf [1], [13], [2], [3]), we can associate such a tower of sphere bundles which is a two-fold covering of the previous one. In particular we give a positive answer of some conjecture proposed in [3].

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1. Introduction

A special multi-flag of step \( m \geq 1 \) and length \( k \geq 1 \) is a sequence (see [5]):

\[ F : D = D_k \subset D_{k-1} \subset \cdots \subset D_j \subset \cdots \subset D_1 \subset D_0 = TM \]

of distributions of constant rank on a manifold \( M \) of dimension \((k+1)m + 1\) which satisfies the following conditions:

(i) \( D_{j-1} = [D_j, D_j] \) is the distribution generated by all Lie bracket of sections of \( D_j \).

(ii) \( D_j \) is a distribution of constant rank \((k-j+1)m + 1\).

(iii) Each Cauchy characteristic subdistribution\(^2\) \( L(D_j) \) of \( D_j \) is a subdistribution of constant corank one in each \( D_{j+1} \), for \( j = 1, \ldots, k-1 \), and \( L(D_k) = 0 \).

(iv) there exists a completely integrable subdistribution \( F \subset D_1 \) of corank one in \( D_1 \).

(see section 2.1 for a more precise definition)

The notion of special multi-flags is described in some ways in [10] and [6]. Furthermore, for \( m \geq 2 \), it is proved in [1] and [13] that the existence of a completely integrable subdistribution \( F \) of \( D_1 \) implies property (iii), and when such a distribution \( F \) exists, it is than unique (see Theorem 2.1). When \( m = 1 \) a special multi-flag is a Goursat flag, and, in this case, the conditions (iii) and (iv) are automatically satisfied but for such a distribution \( F \) is not unique. One fundamental result on Goursat flags is the existence of locally universal Goursat distributions proved by R. Montgomery and M. Zhitomirskii in [8]: the “monster Goursat manifold” which is constructed by applying Cartan prolongations \( k \) times. On the other hand, the kinematic system of a car with \( k \) trailers can be described by an appropriate Goursat distribution \( \Delta_k \) on \( \mathbb{R}^2 \times (\mathbb{S}^1)^k \) and moreover, this configuration space is diffeomorphic to the Cartan prolongation of the distribution \( \Delta_{k-1} \) on \( \mathbb{R}^2 \times (\mathbb{S}^1)^{k-1} \) (see Appendix D of [8]).

The essential result of this note is to proved a generalization of this last result for special multi-flags of step \( m \geq 2 \)

More precisely, special multi-flags can be considered as a generalization of the notion of Goursat flags and the fundamental result of [1] and [13] is again obtained by Cartan prolongation (see also [6]). So, in this situation, we can also define a ”monster tower” by successive Cartan prolongations of \( T\mathbb{R}^{m+1} \) (see for instance[1], [13], [2] or [3] ). On the other hand, we can construct a kinematic system, called articulated arm in [12], and also called system of rigid bars in [4]. The configuration space \( C^k(m) \) of such a kinematic system is diffeomorphic to \( \mathbb{R}^{m+1} \times (\mathbb{S}^m)^k \), and in this case is characterized by a distribution \( D_k \) which generates a special multi-flags of length \( k \) (see section 3.1).

On one hand, by Cartan prolongations, we have a tower of projective bundles: (see section 2.2)

\[ \cdots \to P^k(m) \to P^{k-1}(m) \to \cdots \to P^3(m) \to P^2(m) := \mathbb{R}^{m+1} \]

On the other hand we can also define a natural notion of ”spherical prolongation” which also gives rise to a tower of sphere bundles (see section 2.3)

\[ \cdots \to \hat{P}^k(m) \to \hat{P}^{k-1}(m) \to \cdots \to \hat{P}^3(m) \to \hat{P}^2(m) := \mathbb{R}^{m+1} \]

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2see section 2.1
Note that we have a canonical 2-fold covering

\[ \hat{P}^k(m) \rightarrow P^k(m) \]

for any \( k \geq 1 \) and \( m \geq 2 \).

The essential result of this note is:

**Theorem 1.**

Let be \( \hat{\Delta}_k \) the canonical distribution obtained on \( \hat{P}^k(m) \) after \( k \)-fold "spherical prolongation". Then we have:

- For each \( k \geq 1 \) and \( m \geq 2 \), there exists a diffeomorphism \( \Phi^k \) from \( \hat{P}^k(m) \) on \( C^k(m) \) such that:
  1. If \( \hat{\pi}^k : \hat{P}^k(m) \rightarrow P^{k-1}(m) \) and \( \pi^k : C^k(m) \rightarrow C_{k-1}(m) \) are the canonical projections, we have:
  
  \[ \hat{\pi}^k \circ \Phi^k = \Phi^{k-1} \circ \hat{\pi}^k \]
  
  \[ \Phi^k(\hat{\Delta}_k) = D_k \]

  In particular, this result gives a positive answer to a conjecture proposed in section 6 of [3].

Here is a short description of this note. The section 2 recall, in the first part, the context of special multi-flags, Cartan prolongation and the construction of the tower of projective bundles (1). In the second part we give a definition of "spherical prolongation" and the construction of the tower of sphere bundle (2). In a first part of section 3, we expose the context of the kinematic system of an articulated arm and its properties. The proof of Theorem 1 is given in the last part of this section.

2. Preliminaries

2.1. Special multi-flags.

A distribution \( D \) on a manifold \( M \) is an assignment \( D : x \rightarrow D_x \subset TM \) of subspace \( D_x \) of the tangent space \( T_xM \). A local vector field \( X \) on \( M \) is tangent to \( D \) if for any \( X(x) \) belongs to \( D_x \) for all \( x \) in the open set on which \( X \) is defined. A distribution is called a smooth distribution if there exists a set \( \mathcal{X} \) of local vector fields such that \( D_x \) is generated by the set \( \{ X(x), \; x \in \mathcal{X} \} \). Then say that \( D \) is generated by \( \mathcal{X} \).

In this paper any distribution will be smooth and we denote by \( \Gamma(D) \) the set of all local vector fields which are tangent to \( D \). Such a distribution will be called a distribution of constant rank if \( D \) defines a subbundle of \( TM \). According to [1] and [13], any pair \( (M, D) \) of a distribution of constant rank on a smooth manifold \( M \) will be called a differential system. Given two differential systems \( (M, D) \) and \( (N, \Delta) \), we will say that \( (M, D, x) \) is locally equivalent to \( (N, \Delta, y) \) if there exists an open neighbourhood \( U \) of \( x \) in \( M \) and a diffeomorphism \( \phi \) from \( U \) onto an open neighbourhood \( V \) of \( y = \phi(x) \) in \( N \) so that \( \phi_* (D|_U) = \Delta|_V \).

The Lie square of a distribution \( D \) is the distribution denoted \( D^2 \) which is generated by the sets \( \Gamma(D) \) and \( \{ [X, Y], \; X, Y \in \Gamma(D) \} \). The Cauchy characteristic distribution \( L(D) \) of a distribution \( D \) is the distribution generated by the set vector fields \( \{ [X, Y], \; X, Y \in \Gamma(D) \} \) such that \( [X, Y]|_x \in D_x \). If \( L(D) \) is a distribution of constant rank, then it is an integrable distribution.

A special multi-flag of step \( m \geq 2 \) and length \( k \geq 1 \) is a sequence (see [5]):

\[ F : D = D_k \subset D_{k-1} \subset \cdots \subset D_1 \subset D_0 = TM \]

of distributions of constant rank on a manifold \( M \) of dimension \((k + 1)m + 1\) which satisfies the following conditions:

- (i) \( D_{j-1} = (D_j)^2 \).
- (ii) \( D_j \) is a distribution of constant rank \((k + j)m + 1\).
- (iii) Each Cauchy characteristic subdistribution \( L(D_j) \) of \( D_j \) is a subdistribution of constant corank one in each \( D_{j+1} \), for \( j = 1, \cdots, k - 1 \), and \( L(D_k) = 0 \).
- (iv) there exists a completely integrable subdistribution \( F \subset D_1 \) of corank one in \( D_1 \).

In the following, a flag \( F \) which satisfies conditions (i), (ii) but not conditions (iii) and (iv) will be just called a multi-flag of step \( m \) or a \( m \)-flag and we say that \( F \) is generated by \( D \).

The necessary and sufficient condition of a multi-flag to be a special multi-flag is given by the following result (see [1] Proposition 1.3 and [13] Theorem 6.2).

**Theorem 2.1.** \([1], [13]\) for \( k \geq 2 \), a \( m \)-flag

\[ F : D = D_k \subset D_{k-1} \subset \cdots \subset D_j \subset \cdots \subset D_1 \subset D_0 = TM \]
is a special multi-flag if and only if there exists a completely integrable subbundle $F$ of $D_1$ of corank 1. Moreover, if such a subbundle $F$ exists, $F$ is unique.

According to the previous definition of a special multi-flag, we obtain the following sandwich diagram:

$$D_k \subset D_{k-1} \subset \cdots \subset D_1 \subset D_0 = TM$$

All vertical inclusions in this diagram are of codimension one, while all horizontal inclusions are of codimension $k$. The squares built by these inclusions can be perceived as certain sandwiches, i.e each “subdiagram” number $j$ indexed by the upper left vertices $D_j$:

$$D_j \subset \cdots \subset D_{j-1}$$

$L(D_{j-1}) \subset L(D_{j-2})$ is called sandwich number $j$.

We can read the length $s$ of the special multi-flag by adding one to the total number of sandwiches in the sandwich diagram.

In a sandwich number $j$, at each point $x \in M$, in the $(m+1)$ dimensional vector space $D_{j-1}/L(D_{j-1})(x)$ we can look for the relative position of the $m$ dimensional subspace $L(D_{j-2})/L(D_{j-1})(x)$ and the 1 dimensional subspace $D_j/L(D_{j-1})(x)$:

- either $L(D_{j-2})/L(D_{j-1})(x) \oplus D_j/L(D_{j-1})(x) = D_{j-1}/L(D_{j-1})(x)$
- or $D_j/L(D_{j-1})(x) \subset L(D_{j-2})/L(D_{j-1})(x)$.

We say that $x \in M$ is a regular point if the first situation is true in each sandwich number $j$, for $j = 1, \cdots, k$. Otherwise $x$ is called a singular point.

2.2. Cartan prolongation and tower of projective bundles.

Let be $D$ a distribution of constant rank $m + 1$ on a manifold $M$ of dimension $n$. Classically the Grassmannian bundle $G(D, 1)$ on $M$ is the set

$$G(D, 1, M) := \bigcup_{x \in M} P(D(x), 1)$$

where $P(D(x), 1)$ is the projective space of the vector space $D(x)$. So we have a bundle $\pi : G(D, 1) \to M$ whose fiber $\pi^{-1}(x)$ is diffeomorphic to the projective space $\mathbb{RP}^m$. The rank one Cartan prolongation is the distribution $D^{(1)}$ defined in the following way: given a point $(x, \lambda) \in G(D, 1)$ then

$$D^{(1)}_{(x, \lambda)} := d\pi^{-1}(\lambda) \subset T_{(x, \lambda)}G(D, 1, M)$$

where $\lambda$ is a direction of $D(x)$. Then $D^{(1)}$ is a distribution on $G(D, 1, M)$ of constant rank $m + 1$. As in [13], for any $m \geq 2$ and $k \geq 1$ we obtain inductively a tower of bundles:

$$\cdots \to P^k(m) \to P^{k-1}(m) \to \cdots \to P^1(m) \to P^0(m) := \mathbb{R}^{m+1}$$

where, for any $j = 0, \cdots, k$, $P^j(m)$ is a manifold of dimension $(j + 1)m + 1$, and on each $P^j(m)$, $\Delta_j$ is a distribution which are defined inductively by:

$P^j(k) = G(\Delta_j-1, 1, P^{j-1}(m))$ and $\Delta_j = (\Delta_{j-1})^{(1)}$ for $j = 1, \cdots, k$ and $\Delta_0 = TR^{m+1}$. We have then the following result:

Theorem 2.2. [13]

1. On $P^k(m)$, the distribution $\Delta_k$ generates a special multi-flag of step $m$ and length $k$.
2. let be $F : D = D_k \subset D_{k-1} \subset \cdots \subset D_j \subset \cdots \subset D_1 \subset D_0 = TM$ a special multi-flag of step $m \geq 2$ and length $k \geq 1$. Then, for any $x \in M$, there exists $y \in P^k(m)$ for which the differential system $(P^k(m), \Delta_x, y)$ is locally equivalent to the differential system $(M, D, x)$.

Remark 2.1.

The part 2 of Theorem 2.2 can be found precisely in [13] called "Drapeau Theorem". However, according to the definition of a special multi-flag, we can easily deduce this result from Theorem 2 of [6].
2.3. Spherical prolongation, Cartan prolongation and tower of sphere bundles.

Let be \( D \) a distribution of constant rank \( m + 1 \) on a manifold \( M \) of dimension \( n \). Choose any riemannian metric \( g \) on \( M \), and we denote by \( S(D, M, g) \) the unit sphere bundle of \( D \) associated to the induced riemannian metric on \( D \). So we get a bundle \( \tilde{\pi} : S(D, M, g) \to M \). On \( S(D, M, g) \), we consider the antipodal action of \( \mathbb{Z}_2 \). Clearly, the quotient of \( S(D, M, g) \) by this action can be identified with \( G(D, 1, M) \) and the associated projection \( \tau : S(D, M, g) \to G(D, 1, M) \) is a bundle morphism over \( M \), and also a two-fold covering. In particular \( \tau \) is a local diffeomorphism. On \( S(D, M, g) \) we consider the distribution \( D^{[1]} \) defined in the following way

\[
D^{[1]}_{(x,\nu)} := \{ v \in T_{(x,\nu)}S(D, M, g) \text{ such that } d\tilde{\pi}(v) = \lambda \nu \text{ for some } \lambda \in \mathbb{R} \}
\]

where \( \nu \) is a norm one vector in \( D(x) \).

The distribution \( D^{[1]} \) is called the **rank one spherical prolongation** of \( (M, D, g) \).

**Remark 2.2.**

In fact, the unit sphere bundle of \( S(D, M) \) is defined as soon as we fix some riemannian metric on \( D \). In this case, the distribution \( D^{[1]} \) is also well defined.

**Lemma 2.1.**

(i) we have \( \tau_* D^{[1]} = D^{(1)} \)

(ii) There exists a canonical riemannian metric \( \hat{g} \) on \( S(D, M, g) \) which is uniquely defined from the riemannian metric \( g \) on \( M \).

**Proof.**

At first we show part (i) locally. Choose a chart domain \( U \) over which \( D \) is trivial. We choose an orthonormal frame \( \{ e_0, \cdots, e_m \} \) of \( D \) over \( U \). Without loss of generality we can assume that \( D_U \equiv \mathbb{R}^n \times \mathbb{R}^{m+1} \) so, the bundle \( S(D, M, g)_U \) is isomorphic to \( \mathbb{R}^n \times S^m \) and \( G(D, 1, M)_U \) is isomorphic to \( \mathbb{R}^n \times \mathbb{R}^m \). So locally, \( \tau : \mathbb{R}^n \times S^m \to \mathbb{R}^n \times \mathbb{R}^m \) is the map \((x,\nu) \to (x,|\nu|)\) where \(|\nu|\) is the line bundle generated by \( \nu \). From the definition of \( D^{[1]}_{(x,\nu)} \) and \( D^{(1)}_{(x,|\nu|)} \) we have \( \tau_* (D^{[1]}_{(x,\nu)}) = D^{(1)}_{(x,|\nu|)} \). As \( \tau \) is a local diffeomorphism we get the part (i) locally. On the other hand, the map \( \hat{\alpha} : S(D, M, g) \to S(D, M, g) \) given by \( \hat{\alpha}(x,\nu) = (x,-\nu) \) is a diffeomorphism which commutes with \( \tau \). From the definition of \( D^{[1]} \), we get

\[
\hat{\alpha}_* (D^{[1]}_{(x,\nu)}) = D^{[1]}_{(x,-\nu)}
\]

This ends the proof of part (i).

For part (ii), denote by \( \hat{g} \) the canonical riemannian metric on \( TM \) associated to \( g \). As, \( S(D, M, g) \) can be considered as a submanifold on \( TM \) we get an natural induced riemannian metric \( \hat{g} \) on \( S(D, M, g) \).

Let be \( g_0 \) and \( g_1 \) two riemannian metrics on \( M \). We denote by \( S_i(D, M) \) the sphere bundle of \( D \) associated to the metric \( g_i \), and \( D^{[1]}_i \) the spherical prolongation of \( (M, D, g_i) \) for \( i = 0,1 \).

**Lemma 2.2.**

There exists a canonical isomorphism of sphere bundle \( \psi : S_0(D, M) \to S_1(D, M) \) such that \( \psi_*(D^{[1]}_0) = D^{[1]}_1 \)

**Proof.**

Let be \( D^o = \bigcup_{x \in M} \{ [D_x] \setminus \{ 0 \} \} \). Then \( D^o \) is an open submanifold of \( D \subset TM \). On \( D^o \) we consider the map \( \Psi(x, u) : D^o \to D^o \) defined by

\[
\Psi(x, u) = (x, \frac{u}{|g_1(u, u)|^{1/2}}).
\]

If \( \Pi : D \to M \) is the projection bundle, for any \((x, u) \in D\), there exists a neighbourhood \( \hat{U} = \Pi^{-1}(U) \cap D^o \) of \((x, u)\) in \( D^o \) such that over this open, \( TD^o|_U \) can be identified with \( \hat{U} \times T_x M \times D_x \). Then, in this context, we have:

\[
d\Psi(v, w) = (v, -\frac{g_1(u, w)}{2|g_1(u, u)|^{1/2}}).
\]

It is easy to see that \( \Psi \) is a diffeomorphism from \( D^o \) into itself which commutes with \( \Pi \) and which sends \( S_0(D, M) \) to \( S_1(D, M) \). So the restriction \( \psi \) of \( \Psi \) to \( S_0(D, M) \) is a diffeomorphism onto \( S_1(D, M) \). Moreover, from (7), \( d\Psi \) map the the linear span \( \mathbb{R}u \) into itself, for any \( u \) in the fiber \( D^{o}_x \) over \( x \). So we have

\[
\psi_*(D^{[1]}_0) = D^{[1]}_1.
\]

\( \square \)
Consider a differential system \((M', D')\) and \(\phi : M \to M'\) an injective immersion such that \(\phi_\ast(D_x) \subset D'_{\phi(x)}\) for any \(x \in M\). Given any riemannian metric \(g'\) on \(M'\), we get an induced riemannian metric \(g\) on \(M\) and we can consider the associated spherical prolongation then we have:

**Lemma 2.3.**

in the previous context, let be \(\hat{\phi} : S(D, M, g) \to S(D', M', g')\) the map defined by

\[
\hat{\phi}(x, \nu) = (\phi(x), d_x \phi(\nu)).
\]

Then \(\hat{\phi}\) is a bundle morphism over \(\phi\) which is an injective immersion and such that

(i) \(\hat{\phi}(S(D, M, g)) = S(\phi_\ast(D), \phi(M), g')\)

(ii) \(\hat{\phi}_\ast(\mathcal{D}^{[i]}) = (\phi_\ast(D))^{[i]} \subset (D')^{[i]}\).

Moreover, if \(\phi\) is a diffeomorphism such that \(\phi_\ast(D) = D'\), then \(\hat{\phi}\) is also a diffeomorphism and we have \(\hat{\phi}_\ast(\mathcal{D}^{[i]}) = (D')^{[i]}\) and the riemannian metric \(\phi_\ast g'\) is nothing but the canonical metric \(\hat{g}\) naturally associated to \(g\) on \(M\).

**Proof.**

As in Lemma, consider the map \(\hat{\phi}(x, \nu) = (\phi(x), d_x \phi(\nu))\). From our assumptions, we get a smooth map from \(S(D, M, g)\) to \(S(D', M', g')\) and from its definition, clearly \(\hat{\phi}\) is a bundle morphism over \(\phi\). As \(\phi\) is an injective immersion, it follows that at first \(\hat{\phi}\) is injective.

Note that the tangent space \(T_{(x, \nu)}S_x\) of the fiber \(S_x\) over \(x\) of \(S(D, M, g)\) can be identified with \(\{v \in D_x\ such\ that\ g(\nu, v) = 0\}\).

Now any \(V \in T_{(x, \nu)}S(D, M, g)\) can be written \(V = (u, v)\) with \(u \in T_x M\) and \(v \in T_{(x, \nu)}S_x\). So we have then:

\[
(8) \quad d_{(x, \nu)} \hat{\phi}(u, v) = (d_x \phi(u), d_x \phi(\nu))
\]

So, \(\hat{\phi}\) is an immersion, from (8).

On the other hand, as \(\phi_\ast g' = g, d_x \phi\) is an isometry on its range, and then, \(d_x \phi(S_x)\) is the fiber over \(\phi(x)\) of \(S(\phi_\ast(D), \phi(M), g')\) and we get (i).

Let be \(\bar{\pi} : S(D, M, g) \to M\) and \(\bar{\pi}' : S'(D', M', g') \to M'\) the natural projections. We have then:

\[
d\bar{\pi}' \circ d\hat{\phi} = d\phi \circ d\bar{\pi}
\]

So, we get:

\[
\{\phi_\ast(\mathcal{D}^{[i]})\}_{\hat{\phi}(x, \nu)} = \{d\phi(u, v)\}, \quad \{u, v\} \in T_{(x, \nu)}S(D, M, g) \quad d\bar{\pi}(u, v) = \lambda \nu \quad \lambda \in \mathbb{R}
\]

and

\[
\{\phi_\ast(\mathcal{D}^{[i]})\}_{\hat{\phi}(x, \nu)} = \{d\phi(u, v)\}, \quad \{u, v\} \in T_{(x, \nu)}S(D, M, g) \quad d\bar{\pi}' = d\bar{\pi} = d\phi(\nu) \quad \lambda \in \mathbb{R}
\]

This ends the proof of (ii).

Assume now that \(\phi\) is a diffeomorphism such that \(\phi_\ast(D) = D'\) and let be \(\psi = \phi^{-1}\). From the definition of \(\hat{\phi}\) and \(\psi\), it follows trivially that \(\psi \circ \hat{\phi} = Id\). On the other hand, from the definition of \([\phi_\ast(\mathcal{D})]^{[i]}\), as \(d_x \phi\) is an isomorphism, we must have \(\{(\phi_\ast(\mathcal{D}))^{[i]}\}_{\hat{\phi}(x, \nu)} = \{(\mathcal{D})^{[i]}\}_{\hat{\phi}(x, \nu)}\). Finally, by assumption, \(\phi\) is an isometry from \((M, g)\) to \((M', g')\), \(d\phi\) is also an isometry for \((TM, \hat{g})\) and \((TM', \hat{g}')\) if \(\hat{g}\) and \(\hat{g}'\) are the canonical riemannian metric on the tangent bundle induced by \(g\) and \(g'\) respectively. As by construction \(\hat{g}\) and \(\hat{g}'\) are the restriction respective of \(\hat{g}\) and \(\hat{g}'\) to \(S(D, M, g) \subset TM\) and \(S(D', M', g') \subset TM'\), we get the last property and this ends the proof of the Lemma.

So, as in the context of Cartan prolongation, for any \(m \geq 2\) and \(k \geq 1\) we can define, inductively, a tower of sphere bundles (for a fixed choice of the metric \(g\) on a manifold \(M\)):

\[
\cdots \to \hat{P}^{k}(M) \to \hat{P}^{k-1}(M) \to \cdots \to \hat{P}^{1}(M) \to \hat{P}^{0}(M) := M
\]

where, for any \(j = 0, \cdots, k\), \(\hat{P}^{j}(M)\) is a manifold of dimension \((j + 1)m + 1\), and, on each \(\hat{P}^{j}(M)\) we have a canonical distribution \(\Delta_j\) and a riemannian metric \(g_j\) on \(P^j(M)\), all defined inductively in the following way:

\[
\hat{P}^{j}(M) = S(\hat{\Delta}_{j-1}, \hat{P}^{j-1}(M), g_{j-1}), \quad \hat{\Delta}_{j} = (\hat{\Delta}_{j-1})^{[i]} \quad \text{for} \quad j = 1, \cdots, k \quad \text{and} \quad \hat{\Delta}_0 = TM, \quad g_j \quad \text{is the riemannian metric} \quad \hat{g}_{j-1} \quad \text{on} \quad S(\hat{\Delta}_{j-1}, \hat{P}^{j-1}(M), g_{j-1}) \quad \text{associated to} \quad g_{j-1} \quad \text{for} \quad j = 1, \cdots, k \quad \text{and} \quad g_0 = g.
\]

Note that if \(g'\) is another riemannian metric on \(M\), according to Lemma 2.2 and Lemma 2.3, by induction we get a family of diffeomorphisms \(\psi^j\) such that, if :
\[ \cdots \rightarrow \hat{P}^k(M) \rightarrow \hat{P}^{k-1}(M) \rightarrow \cdots \rightarrow \hat{P}^1(M) \rightarrow \hat{P}^0(M) := M \]

is the tower of sphere bundles associated to the choice \( g' \) on \( M \) we have, for all \( j = 0, \ldots, k \):

\[ \psi^j(\hat{P}^j(M)) = \hat{P}^j(M) \]

\( \psi^j \) is fiber preserving

\[ \psi^j(\Delta_i) = \Delta_i' \]

So the properties of the tower (9) is independant of the choice of the riemannian metric \( g \) on \( M \). For simplicity we write \( \hat{P}^j(M) := \hat{P}^j(\mathbb{R}^{m+1}) \) for any \( j \in \mathbb{N} \). From Theorem 2.2, and Lemma 2.1 we get:

**Theorem 2.3.** Let be

\[ \cdots \rightarrow \hat{P}^k(m) \rightarrow \hat{P}^{k-1}(m) \rightarrow \cdots \rightarrow \hat{P}^1(m) \rightarrow \hat{P}^0(m) := \mathbb{R}^{m+1} \]

the tower of sphere bundles associated to the canonical metric on \( \mathbb{R}^{m+1} \).

1. we have a canonical two-fold covering \( \tau_k : \hat{P}^k(m) \rightarrow P^k(m) \) such that

\[ \tau_k(\Delta_i) = \Delta_k. \]

2. On each manifold \( \hat{P}^k(m) \), the distribution \( \hat{\Delta}_k \) generates a special multi-flag of step \( m \) and length \( k \).

3. let be \( F : D = D_k \subset D_{k-1} \subset \cdots \subset D_1 \subset D_0 = TM \) a special multi-flag of step \( m \geq 2 \) and length \( k \geq 1 \). Then, for any \( x \in M \), there exists \( y \in \hat{P}^k(m) \) for which the differential system \( \left( \hat{P}^k(m), \hat{\Delta}_x, y \right) \) is locally equivalent to the differential system \( (M, D, x) \).

The tower \( (10) \) will be called the spherical tower of special multi-flags of step \( k \).

### 3. Tower of sphere bundles associated to an articulated arm

#### 3.1. A kinematic system: articulated arm.

According to [12], an articulated arm of length \( k \) is a series of \( k \) segments \([M_i; M_{i+1}],[i = 0, \ldots, k - 1]\), in \( \mathbb{R}^{m+1} \), with \( m \geq 2 \), keeping a constant length \( l_i = 1 \) between \( M_i \) and \( M_{i+1} \), and the articulation occurs at points \( M_i \), for \( i = 1, \ldots, k - 1 \).

The kinematic evolution of the extremity \( M_0 \), under the constraint that the velocity of each point \( M_i \), \( i = 0, \ldots, k - 1 \), is collinear with the segment \([M_i; M_{i+1}]\) is completely described in [12]. Note that such a system is also studied in [4] and is called a "k-bar system". The kinematic evolution of this mechanical system can be described in hyperspherical coordinates (see [12]). We can associated to this problem a special multi-flag of step \( m \geq 2 \) and length \( k \geq 1 \) as explained in the following:

We can decompose \( (\mathbb{R}^{m+1})^{k+1} \), into a product \( \mathbb{R}^{m+1} \times \mathbb{R}^{m+1} \times \cdots \times \mathbb{R}^{m+1} \). Let \( x_i = (x_1^i, \ldots, x_{m+1}^i) \) be the canonical coordinates on the space \( \mathbb{R}^{m+1} \) which is equipped with its canonical scalar product \( < , > \). \((\mathbb{R}^{m+1})^{k+1} \) is then equipped with its canonical scalar product too.

Consider an articulated arm of length \( k \) denoted by \((M_0, \cdots, M_k)\). On \((\mathbb{R}^{m+1})^{k+1} \), consider the vector fields:

\[ Z_i = \sum_{r=1}^{m+1} (x_{i+1}^r - x_i^r) \frac{\partial}{\partial x_i^r} \quad \text{for} \quad i = 0, \ldots, k - 1 \]

From our previous assumptions, the kinematic evolution of the articulated arm is described by a controlled system:

\[ \dot{x} = \sum_{i=0}^{s-1} u_i Z_i + \sum_{r=1}^{m+1} u_{n+r} \frac{\partial}{\partial x_i^r} \]

with the following constraints:

\[ ||x_i - x_{i+1}|| = 1 \quad \text{for} \quad i = 0 \cdots k - 1 \quad \text{(see [4] or [12]).} \]

Consider the map \( \Psi_i(x_0, \cdots, x_k) = ||x_i - x_{i+1}||^2 - 1 \). Then, the configuration space \( C^k(m) \) is the set

\[ \{(x_0, \cdots, x_k), \quad \text{such that} \quad \Psi_i(x_0, \cdots, x_k) = 0 \quad \text{for} \quad i = 0, \cdots, k - 1 \} \]

For \( i = 0, \cdots, k - 1 \), the vector field:

\[ \mathcal{N}_i = \sum_{r=1}^{m+1} (x_{i+1}^r - x_i^r) \left[ \frac{\partial}{\partial x_{i+1}^r} - \frac{\partial}{\partial x_i^r} \right] \]

is proportional to the gradient of \( \Psi_i \).

So the tangent space \( T_0 C^k(k) \) is the subspace of \( T_0 (\mathbb{R}^{m+1})^{k+1} \) which is orthogonal to \( \mathcal{N}_i(q) \) for \( i = 0, \cdots, k - 1 \).
Denote by $\mathcal{E}_k$ the distribution generated by the vector fields
\[ \{ Z_0, \ldots, Z_{k-1}, \frac{\partial}{\partial x_k}, \ldots, \frac{\partial}{\partial x_{k+1}} \} . \]

**Lemma 3.1.** [12]
Let $\mathcal{D}_k$ be the distribution on $\mathcal{C}^k(m)$ defined by $\Delta(q) = T_q C \cap \mathcal{E}$. Then $\mathcal{D}_k$ is a distribution of dimension $m + 1$ generated by
\[ (x_k^r - x_{k-1}^r) \prod_{i=0}^{k-1} A_i Z_i + \frac{\partial}{\partial x_k^r} \text{ for } r = 1 \cdots m + 1 \]
where $A_j(q) = < N_j(q), N_{j-1}(q) > = -< Z_j(q), N_{j-1}(q) >$ for $j = 1, \cdots, k - 1$ and $A_k = 1$.

The properties of $\mathcal{D}_k$ are summarized in the following result. (see [12] also [4])

**Theorem 3.1.**
On $\mathcal{C}^k(m)$, the distribution $\mathcal{D}_k$ satisfies the following properties:
1. $\mathcal{D}_k$ is a distribution of rank $m + 1$.
2. The distribution $\mathcal{D}_k$ is a special multi-flag on $\mathcal{C}^k(m)$ of step $m$ and length $k$.

3.2. Articulated arm and spherical prolongation.

To an articulated arm on $\mathbb{R}^{m+1}$ ($m \geq 2$) of length $k \geq 1$ we can associate the following canonical tower of sphere bundles:
\[ \mathcal{C}^k(m) \rightarrow \mathcal{C}^{k-1}(m) \rightarrow \cdots \rightarrow \mathcal{C}^1(m) \rightarrow \mathcal{C}^0(m) := \mathbb{R}^{m+1} \]
where for $j = 1, \cdots, k$, the projection $\mathcal{C}^j(m) \rightarrow \mathcal{C}^{j-1}(m)$ is the restriction of the canonical projection
\[ \mathbb{R}_0^{m+1} \times \cdots \times \mathbb{R}_i^{m+1} \times \cdots \times \mathbb{R}_j^{m+1} \rightarrow \mathbb{R}_0^{m+1} \times \cdots \times \mathbb{R}_i^{m+1} \times \cdots \mathbb{R}_j^{m+1} \]
\[ (x_0, \ldots, x_{j-1}, x_j) \rightarrow (x_0, \ldots, x_{j-1}) \]

According to Theorem 3.1 and Theorem 2.3, we know that the differential system $(\mathcal{C}^k(m), \mathcal{D}_k)$ associated to an articulated arm of length $k$ on $\mathbb{R}^{m+1}$ is locally isomorphic to the canonical differential system $(\hat{P}^k(m), \hat{\Delta}_k)$ at some appropriate points. In fact, we have more: (see Theorem 1 in the introduction)

**Theorem 3.2.**
For each $k \geq 1$ and $m \geq 2$, there exists a diffeomorphism $\Phi^k$ from $\hat{P}^k(m)$ on $\mathcal{C}^k(m)$ such that:
1. if $\tilde{p}^k : \hat{P}^k(m) \rightarrow \hat{P}^{k-1}(m)$ and $p^k : \mathcal{C}^k(m) \rightarrow \mathcal{C}^{k-1}(m)$ are the canonical projections, we have:
\[ \tilde{p}^k \circ \Phi^k = \Phi^{k-1} \circ \tilde{p}^k \]
2. $\Psi^k(\hat{\Delta}_k) = \mathcal{D}_k$

So according to Theorem 2.3 from Theorem 3.2 we have:

**Theorem 3.3.** Let be $F : D = D_k \subset D_{k-1} \subset \cdots \subset D_j \subset \cdots \subset D_1 \subset D_0 = TM$ a special multi-flag of step $m \geq 2$ and length $k \geq 1$. Then, for any $x \in M$, there exists $y \in \mathcal{C}^k(m)$ for wich the differential system $(\mathcal{C}^k(m), \mathcal{D}_k, y)$ is locally equivalent to the differential system $(M, D, x)$.

The end of this section is devoted to the proof of Theorem 3.2. At first, we need some auxiliary results.

The following Lemma is partially proved in [12], however, for completeness of this note, we give a proof of this result.

**Lemma 3.2.**
For $k \geq 1$, consider, on $\mathcal{C}^k(m)$, the natural decomposition:
\[ [T(\mathbb{R}^{m+1})^{k+1}]_{|\mathcal{C}^k(m)} = T\mathcal{C}^k(m) \oplus [T\mathcal{C}^k(m)]^\perp \]
where $[T\mathcal{C}^k(m)]^\perp$ is the orthogonal of $T\mathcal{C}^k(m)$.
Denote by $\Pi$ the orthogonal projection of $[T(\mathbb{R}^{m+1})^{k+1}]_{|\mathcal{C}^k(m)}$ on $T\mathcal{C}^k(m)$.
1. The family of vector field $\{ \Pi(\frac{\partial}{\partial x_i^r})\}(p), r = 1, \cdots, m + 1 \}$ generates the tangent space at $p \in \mathcal{C}^k(m)$ of the sphere of equation $\Psi_i = 0, i = 0, \cdots, k - 1$. 

(2) On $C^k(m)$, let $\mathcal{L}_k$ be the involutive distribution whose leaves are defined by $\Psi_{k-1} = 0$. The distribution $\mathcal{D}_k$, is generated by $\mathcal{L}_k$ and the vector field $X_k = \sum_{i=0}^{k-1} \prod_{j=i+1}^{k} A_j \Pi(Z_i)$.

Moreover $Z_i = \Pi(Z_i)$ is tangent to the sphere of equation $\Psi_i = 0$, $i = 1, \cdots, k-1$ and $Z_0 = Z_0$.

(3) The distribution $\mathcal{D}_k$ is also generated by the vector fields

$$(x_k^i - x_{k-1}^i)X_k + \Pi(\frac{\partial}{\partial x_k^i}) \text{ for } r = 1 \cdots m + 1$$

Proof.

At first, for any $p \in C^k(m)$, note that

$$\Delta_p = \{v \in E_p \text{ such that } <v, \nu> = 0, \forall \nu \in [T_p C^k(m)]^\perp\}$$

so $\mathcal{D}_p$ is nothing but $\Pi(E_p)$.

Note that, at each point $p = (x_0, \cdots, x_k) \in C^k(m)$, the vector $\frac{\partial}{\partial x_{i+1}}(p)$ satisfies

$$dx_i^j(\frac{\partial}{\partial x_{i+1}}) = 0, \text{ for any } r, s = 1 \cdots, m + 1 \text{ and } j = 0, \cdots, i + 2, \cdots, k$$

So, the family of vector field $\{\Pi(\frac{\partial}{\partial x_{i+1}})(p), r = 1, \cdots, m + 1\}$ generates the tangent space at $p$ of the sphere of equation $\Psi_i = 0$ which proves (1).

The integrable distribution $\mathcal{F}_k$, on $(\mathbb{R}^{m+1})^{k+1}$, generated by $\{\frac{\partial}{\partial x_k^i}, r = 1, \cdots, m + 1\}$, is contained in $\mathcal{E}$, and the distribution $\mathcal{L}_k = \Pi(\mathcal{F}_k)$ induced on $C^k(m)$ by $\mathcal{F}_k$ is also integrable and, of course is contained in $\mathcal{D}_k$. In particular, $\mathcal{L}_k$ is generated by $\{\Pi(\frac{\partial}{\partial x_k^i}), r = 1, \cdots, m + 1\}$. From Lemma 3.1, the vector field

$$Y = \prod_{i=0}^{k-1} \prod_{j=i+1}^{k} A_j [\sum_{r=1}^{m+1} \frac{\partial}{\partial x_k^r} (x_k^r - x_{k-1}^r)]$$

is an isometry between $\mathcal{D}_k$ and $\mathcal{L}_k$. Moreover, $\mathcal{L}_k$ is a distribution of constant rank $m + 1$ and $\mathcal{L}_k$ is an (integrable) subdistribution of rank $m$. It follows that $\mathcal{D}_k$ is generated by $\mathcal{L}_k$ and $Y$.

On the other hand, as $Y$ is tangent to $C^k(m)$ we have:

$$Y = \Pi(Y) = \prod_{i=0}^{k-1} \prod_{j=i+1}^{k} A_j \Pi(Z_i) + \Pi(\sum_{r=1}^{m+1} (x_k^r - x_{k-1}^r) \frac{\partial}{\partial x_k^r})$$

As $\prod_{r=1}^{k+1} \frac{\partial}{\partial x_k^r} (x_k^r - x_{k-1}^r)$ is tangent to $\mathcal{L}_k$, it follows that $\mathcal{D}_k$ is generated by $\mathcal{L}_k$ and

$$X_k = \sum_{i=0}^{k-1} \prod_{j=i+1}^{k} A_j \Pi(Z_i).$$

On the other hand we have

$$Z_i = \Pi(Z_i) = \sum_{r=1}^{m+1} (x_{r+1}^i - x_i^r) \Pi(\frac{\partial}{\partial x_i^r})$$

So, from (1), $Z_i$ is tangent to the sphere of equation $\Psi_i = 0$.

On the other hand, according to Lemma 3.1, $\mathcal{D}_k$ and using same arguments as previously, we obtain that $\mathcal{D}_k$ is also generated by

$$(x_k^i - x_{k-1}^i)X_k + \Pi(\frac{\partial}{\partial x_k^i}) \text{ for } r = 1 \cdots m + 1.$$

$\square$

**Proposition 3.1.**

(1) There exists a bundle isomorphism $\hat{\Psi} : \mathcal{D}_k \to C^k(m) \times \mathbb{R}^{m+1}$

(2) Let be $\gamma_k$ a riemannian metric on the bundle $\mathcal{D}_k$ so that the morphism $\hat{\Psi}$ is an isometry between $\mathcal{D}_k$ and $C^k(m) \times \mathbb{R}^{m+1}$ (for the canonical euclidian product on the fiber $\mathbb{R}^{m+1}$). Then $\hat{\Psi}$ induces a diffeomorphism

$$\Psi : S(\mathcal{D}_k, C^k(m), \gamma_k) \to C^{k+1}(m)$$

such that

(i) $\Psi$ commutes with the canonical projections $S(\mathcal{D}_k, C^k(m), \gamma_k) \to C^k(m)$ and $C^{k+1}(m) \to C^k(m)$.

(ii) $\Psi_*([\mathcal{D}_k]^1] = \mathcal{D}_{k+1}$. 


we get the result for projection, for any \( v \) to \( \ast \mathbb{R} \) so

Note that with the canonical the projections:

\[ \Gamma(x_0, x_1, \ldots, x_k, z) = (x_0, x_1, \ldots, x_k, x_k + z) \]

is a diffeomorphism. So the restriction \( \Gamma \) to \( C^k(m) \times \mathbb{S}^m \) is a diffeomorphism \( \Gamma : C^k(m) \times \mathbb{S}^m \to C^{k+1}(m) \).

Finally, \( \hat{\Psi} = \Gamma \circ \hat{\Phi} \) induces, by restriction, a diffeomorphism \( \Psi : S(D_k, C^k(m), \gamma_k) \to C^{k+1}(m) \) which commutes with the canonical the projections:

\[ S(D_k, C^k(m), \gamma_k) \to C^k(m) \quad \text{and} \quad C^{k+1}(m) \to C^k(m). \]

On \( D_k \), we have a riemannian metric so that the global basis given in (17) is orthonormal. It follows that the map \( \Psi : S(D_k, C^k(m), \gamma_k) \to C^{k+1}(m) \) is given by

\[ \Psi((x_0, x_1, \ldots, x_k, \nu)) = ((x_0, x_1, \ldots, x_k, x_k + \nu)). \]

So, in the global chart of \( S(D_k, C^k(m), \gamma_k) \) defined by \( \Psi \), according to (17), the spherical prolongation \( (D_k)^{[2]} \) of \( D_k \) is generated by the tangent space to the sphere centered at \( x_k \) and the vector field

\[ \sum_{r=1}^{m+1} (x_{k+1} - x_k) (x_k - x_{k-1}) X_k + \sum_{r=1}^{m+1} (x_{k+1} - x_k) \Pi(\frac{\partial}{\partial x_k}) \]

Note that \( \sum_{r=1}^{m+1} (x_{k+1} - x_k) \frac{\partial}{\partial x_k} = Z_{k+1} \) According to Lemma 3.2 part (2) applied at level \( k+1 \), we have then:

\[ \sum_{r=1}^{m+1} (x_{k+1} - x_k) (x_k - x_{k-1}) X_k + \Pi(Z_{k+1}) = A_{k+1} X_k + \Pi(Z_{k+1}) = X_{k+1} \]

Again from Lemma 3.2 part (2), we get:

\[ \Psi_*[(D_k)^{[2]}] = D_{k+1} \]

Proof of Theorem 3.2

Note that on the tangent bundle \( T\mathbb{R}^{m+1} \), we can put the global chart defined by the map \( (x_1, x_2) \to (x_1, x_2 - x_1) \). On the other hand, the riemannian metric \( g_1 \) on \( T\mathbb{R}^{m+1} \) induces by the canonical metric \( g \) is again the canonical metric on the product \( \mathbb{R}^{m+1} \times \mathbb{R}^{m+1} \). It follows that \( S(T\mathbb{R}^{m+1}, \mathbb{R}^{m+1}) \subset T\mathbb{R}^{m+1} \) can be identified with \( \mathbb{R}^{m+1} \times S^k \) as submanifold of \( \mathbb{R}^{m+1} \times S^k \) So we have

\[ T_{(x,u)}S(T\mathbb{R}^{m+1}, \mathbb{R}^{m+1}) = \{(u, v) \in T_{(x,u)}(\mathbb{R}^{m+1} \times \mathbb{R}^{m+1}) \mid \text{ such that } g(v, u) = 0 \}. \]

Recall that \( Z_0 = \sum_{r=1}^{m+1} (x^r_1 - x^r_0) \frac{\partial}{\partial x^r_0} \) on \( \mathbb{R}^{m+1} \times \mathbb{R}^{m+1} \). In the previous coordinates, any tangent vector of \( \mathbb{R}^{m+1} \) at a point \( x_1 \) can be written \( (x_1, x_2 - x_1) \). So, \( Z_0 \) defines a global section of the unit bundle associated to \( T\mathbb{R}^{m+1} \)

According to Theorem 3.2, the distribution \( D_1 \) is generated by \( \mathbb{R}Z_0 \) and \( T\mathbb{S}^m \) in \( T\mathbb{R}^{m+1} \times T\mathbb{S}^m \to TS(T\mathbb{R}^{m+1}, \mathbb{R}^{m+1}) \). If \( \Pi_1 : \mathbb{R}^{m+1} \times \mathbb{S}^m \to \mathbb{R}^{m+1} \) denote the natural projection, for any \( v \in \mathbb{R}Z_0 \) with \( w \in T\mathbb{S}^k \), we have \( d\Pi_1(v) = \lambda \nu \) so, \( \Delta_1 = (T\mathbb{R}^{m+1})^{[2]} \) and we get the result for \( k = 1 \).

Assume that we have a diffeomorphism \( \Phi^k : \hat{P}^k(m) \to C^k(m) \) which satisfies the properties (i), and (ii) of Theorem 2.3.

From Proposition 3.1, we have diffeomorphism \( \Psi : S(D_k, C^k(m), \gamma_k) \to C^{k+1}(m) \) so that \( \Psi_*[(D_k)^{[2]}] = D_{k+1} \) and which commutes with the natural projections

\[ S(D_k, C^k(m), \gamma_k) \to C^k(m) \quad \text{and} \quad C^{k+1}(m) \to C^k(m) \]

According to previous induction, we can put on \( \hat{P}^k(m) \), the riemannian metric \( \hat{\gamma}_k = (\Psi)^*(\gamma_k) \). From Lemma 2.3, we can extend \( \Phi^k : \hat{P}^k(m) \to C^k(m) \) into a diffeomorphism \( \Phi^k : \hat{P}^k(m) \to C^k(m) \) which satisfies the properties (i), and (ii) of Theorem 2.3.
Finally, according to Lemma 2.2, when we put on \( \hat{\Phi} : \hat{\text{tower bundle}} \) (9), we also have a diffeomorphism \( \Phi : \hat{\Phi} \) such that \( \Phi_*(\hat{\Delta}_k) = \hat{\Delta}_k^{[1]} \).

\[
S(\hat{\Delta}_k, \hat{\Phi}(m), \hat{\gamma}_k) \to S(D_k, C_k, \gamma_k) \quad \text{such that} \quad \Phi_*(\hat{\Delta}_k) = (D_k)^{[1]} \quad \text{and which commutes with the natural projections}
\]

\[
S(\hat{\Delta}_k, \hat{\Phi}(m), \hat{\gamma}_k) \to \hat{\Phi}(m) \quad \text{and} \quad C^{k+1}(m) \to C^k(m)
\]

Finally, according to Lemma 2.2, when we put on \( \hat{\Phi}(m) \) the riemannian metric induces by induction on the tower bundle (9), we also have a diffeomorphism \( \Phi : \hat{\Phi}(m) \to S(\hat{\Delta}_k, \hat{\Phi}(m), \hat{\gamma}_k) \) which commutes with the canonical projections

\[
\hat{\Phi}(m) \to \hat{\Phi}(m) \quad \text{and} \quad S(\hat{\Delta}_k, \hat{\Phi}(m), \hat{\gamma}_k) \to \hat{\Phi}(m)
\]

and such that \( \Phi_*(\hat{\Delta}_{k+1}) = \hat{\Delta}_k^{[1]} \).

\[\Box\]

4. Conclusion

According to Theorem 2.3 and Theorem 3.2, from towers (10) and (15) we get the following diagram each vertical map is a 2-fold covering for \( k \geq 1 \):

\[
\cdots \to C^k(m) \to C^{k-1}(m) \to \cdots \to C^1(m) \to C^0(m) := \mathbb{R}^{m+1}
\]

\[\Box\]

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CONFIGURATION SPACES OF A KINEMATIC SYSTEM AND MONSTER TOWER OF SPECIAL MULTI-FLAGS

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ABSTRACT. In this note we show that the configuration spaces of the kinematic system constructed in [4] and [12] gives rise to a natural tower of sphere bundles. Moreover, we prove that, each tower of projective bundles associated to special multi-flags (cf [1], [13], [2], [3]), we can associate such a tower of sphere bundles which is a two-fold covering of the previous one. In particular we give a positive answer of some conjecture proposed in [3]

AMS classification: 53, 58, 70, 93.

Keywords: special multi-flags distributions, Cartan prolongation, spherical prolongation, articulated arm.

1. INTRODUCTION

A special multi-flag of step $m \geq 1$ and length $k \geq 1$ is a sequence (see [5]):

$$F : D = D_k \subset D_{k-1} \subset \cdots \subset D_2 \subset D_1 \subset D_0 = TM$$

of distributions of constant rank on a manifold $M$ of dimension $(k + 1)m + 1$ which satisfies the following conditions:

(i) $D_{j-1} = [D_j, D_j]$ is the distribution generated by all Lie bracket of sections of $D_j$.

(ii) $D_j$ is a distribution of constant rank $(k - j + 1)m + 1$.

(iii) Each Cauchy characteristic subdistribution $L(D_j)$ of $D_j$ is a subdistribution of constant corank one in each $D_{j+1}$, for $j = 1, \ldots, k - 1$, and $L(D_k) = 0$.

(iv) there exists a completely integrable subdistribution $F \subset D_1$ of corank one in $D_1$.

(see section 2.1 for a more precise definition)

The notion of special multi-flags is described in some ways in [10] and [6]. Furthermore, for $m \geq 2$, it is proved in [1] and [13] that the existence of a completely integrable subdistribution $F$ of $D_j$ implies property (iii), and when such a distribution $F$ exists, it is than unique (see Theorem 2.1). When $m = 1$ a special multi-flag is a Goursat flag, and, in this case, the conditions (iii) and (iv) are automatically satisfied but for such a distribution $F$ is not unique. One fundamental result on Goursat flags is the existence of locally universal Goursat distributions proved by R. Montgomery and M. Zhitomirskii in [8]; the "monster Goursat manifold" which is constructed by applying Cartan prolongations $k$ times. On the other hand, the kinematic system of a car with $k - 1$ trailers can be described by an appropriate Goursat distribution $\Delta_k$ on $\mathbb{R}^2 \times (\mathbb{S}^1)^k$ and moreover, this configuration space is diffeomorphic to the Cartan prolongation of the distribution $\Delta_{k-1}$ on $\mathbb{R}^2 \times (\mathbb{S}^1)^{k-1}$ (see Appendix D of [8]).

The essential result of this note is to proved a generalization of this last result for special multi-flags of step $m \geq 2$

More precisely, special multi-flags can be considered as a generalization of the notion of Goursat flags and the fundamental result of [1] and [13] is again obtained by Cartan prolongation (see also [6]). So, in this situation, we can also defined a "monster tower" by succesive Cartan prolongations of $T \mathbb{R}^{m+1}$ (see for instance[1], [13], [2] or [3]). On the other hand, we can construct a kinematic system, called articulated arm in [12], and also called system of rigid bars in [4]. The configuration space $C^k(m)$ of such a kinematic system is diffeomorphic to $\mathbb{R}^{m+1} \times (\mathbb{S}^1)^k$, and and this system is characterized by a distribution $D_k$ which generates a special multi-flags of length $k$ (see section 3.1).

On one hand, by Cartan prolongations, we have a tower of projective bundles: (see section 2.2)

$$\cdots \to P^k(m) \to P^{k-1}(m) \to \cdots \to P^1(m) \to P^0(m) := \mathbb{R}^{m+1}$$

On the other hand we can also defined a natural notion of "spherical prolongation" which also gives rise to a tower of sphere bundles (see section 2.3)

$$\cdots \to \hat{P}^k(m) \to \hat{P}^{k-1}(m) \to \cdots \to \hat{P}^1(m) \to \hat{P}^0(m) := \mathbb{R}^{m+1}$$

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2See section 2.1
Note that we have a canonical 2-fold covering
\[ \hat{P}^k(m) \to P^k(m) \]
for any \( k \geq 1 \) and \( m \geq 2 \).

The essential result of this note is:

**Theorem 1.**

Let be \( \hat{\Delta}_k \) the canonical distribution obtained on \( \hat{P}^k(m) \) after \( k \)-fold "spherical prolongation". Then we have:

For each \( k \geq 1 \) and \( m \geq 2 \), there exists a diffeomorphism \( \Phi^k \) from \( \hat{P}^k(m) \) on \( C^k(m) \) such that:

(i) if \( \bar{\pi}^k : \hat{P}^k(m) \to \hat{P}^{k-1}(m) \) and \( \bar{p}^k : C^k(m) \to C^{k-1}(m) \) are the canonical projections, we have:

\[ \bar{p}^k \circ \Phi^k = \Phi^{k-1} \circ \bar{\pi}^k \]

(ii) \( \Psi^k(\hat{\Delta}_k) = D_k \)

In particular, this result gives a positive answer to a conjecture proposed in section 6 of [3].

Here is a short description of this note. The section 2 recall, in the first part, the context of special multi-flags, Cartan prolongation and the construction of the tower of projective bundles (1). In the second part we give a definition of "spherical prolongation" and the construction of the tower of sphere bundle (2). In a first part of section 3, we expose the context of the kinematic system of an articulated arm and its properties. The proof of Theorem 1 is given in the last part of this section.

2. **Preliminaries**

2.1. **Special multi-flags.**

A distribution \( D \) on a manifold \( M \) is an assignement \( D : x \to D_x \subset TM \) of subspace \( D_x \) of of the tangent space \( T_x M \). A local vector field \( X \) on \( M \) is tangent to \( D \) if for any \( X(x) \) belongs to \( D_x \) for all \( x \) in the open set on which \( X \) is defined. A distribution is called a smooth distribution if there exists a set \( \{ X(x), X \in \mathcal{X} \} \), we then say that \( D \) is generated by the set \( \{ X(x), X \in \mathcal{X} \} \). In this paper any distribution will be smooth and we denote by \( \Gamma(D) \) the set of all local vector fields which are tangent to \( D \). Such a distribution will be called a distribution of constant rank if \( D \) defines is a subbunddle of \( TM \). According to [1] and [13], any pair \( (M, D) \) of a distribution of constant rank on a smooth manifold \( M \) will be called a **differential system**. Given two differential systems \( (M, D) \) and \( (N, \Delta) \), we will say that \( (M, D, x) \) is **locally equivalent** to \( (N, \Delta, y) \) if there exists an open neighbourhood \( U \) of \( x \) in \( M \) and a diffeomorphism \( \phi \) from \( U \) onto an open neighbourhood \( V \) of \( y = \phi(x) \) in \( N \) so that \( \phi_*(D|_U) = \Delta|_V \).

The **Lie square** of a distribution \( D \) is the distribution denoted \( D^2 \) which is generated by the sets \( \Gamma(D) \) and \( \{ [X,Y], X, Y \in \Gamma(D) \} \). The **Cauchy characteristic distribution** \( L(D) \) of a distribution \( D \) is the distribution generated by the set vector fields \( \{ [X,Y], X, Y \in \Gamma(D) \} \) such that \( [X,Y]|(x) \in D_x \). If \( L(D) \) is a distribution of constant rank, then it is an integrable distribution.

A **special multi-flag** of step \( m \geq 2 \) and length \( k \geq 1 \) is a sequence (see [5]):

\[ \mathcal{F} : D = D_k \subset D_{k-1} \subset \cdots \subset D_j \subset \cdots \subset D_1 \subset D_0 = TM \]

of distributions of constant rank on a manifold \( M \) of dimension \( (k+1)m+1 \) which satisfies the following conditions:

(i) \( D_{j-1} = (D_j)^2 \).

(ii) \( D_j \) is a distributon of constant rank \( (k-j+1)m+1 \).

(iii) Each Cauchy characteristic subdistribution \( L(D_j) \) of \( D_j \) is a subdistribution of constant corank one in each \( D_{j+1} \), for \( j = 1, \cdots, k-1 \), and \( L(D_k) = 0 \).

(iv) there exists a completely integrable subdistribution \( F \subset D_1 \) of corank one in \( D_1 \).

In the following, a flag \( \mathcal{F} \) which satisfies conditions (i), (ii) but not conditions (iii) and (iv) will be just called a **multi-flag** of step \( m \) or a **m-flag** and we say that \( \mathcal{F} \) is generated by \( D \).

The necessary and sufficient condition of a multi-flag to be a special multi-flag is given by the following result (see [1] Proposition 1.3 and [13] Theorem 6.2)

**Theorem 2.1.**[1],[13] for \( k \geq 2 \), a **m-flag**

\[ \mathcal{F} : D = D_k \subset D_{k-1} \subset \cdots \subset D_j \subset \cdots \subset D_1 \subset D_0 = TM \]
is a special multi-flag if and only if there exists a completely integrable subbundle $F$ of $D_1$ of corank 1. Moreover, if such a subbundle $F$ exists, $F$ is unique.

According to the previous definition of a special multi-flag, we obtain the following sandwich diagram:

\[
\begin{align*}
D_k & \subset D_{k-1} \subset \ldots \subset D_j \subset \ldots \subset D_1 \subset D_0 = TM \\
\cup & \quad \cup & \quad \cup & \quad \cup & \quad \cup \quad \cup \quad \cup \quad \cup \quad \cup \quad \cup \quad \cup \quad \cup \quad \cup \quad \cup \quad \cup \quad \cup \quad \cup \quad \cup \quad \cup
L(D_{k-1}) & \subset L(D_{k-2}) \subset \ldots \subset L(D_{j-1}) \subset \ldots \subset F
\end{align*}
\]

All vertical inclusions in this diagram are of codimension one, while all horizontal inclusions are of codimension $k$. The squares built by these inclusions can be perceived as certain sandwiches, i.e., each “subdiagram” number $j$ indexed by the upper left vertices $D_j$:

\[
\begin{align*}
D_j & \subset D_{j-1} \\
\cup & \quad \cup \\
L(D_{j-1}) & \subset L(D_{j-2})
\end{align*}
\]

is called sandwich number $j$.

We can read the length $s$ of the special multi-flag by adding one to the total number of sandwiches in the sandwich diagram.

In a sandwich number $j$, at each point $x \in M$, in the $(m+1)$ dimensional vector space $D_{j-1}/L(D_{j-1})(x)$ we can look for the relative position of the $m$ dimensional subspace $L(D_{j-2})/L(D_{j-1})(x)$ and the 1 dimensional subspace $D_j/L(D_{j-1})(x)$:

either $L(D_{j-2})/L(D_{j-1})(x) \oplus D_j/L(D_{j-1})(x) = D_{j-1}/L(D_{j-1})(x)$

or $D_j/L(D_{j-1})(x) \subset L(D_{j-2})/L(D_{j-1})(x)$.

We say that $x \in M$ is a regular point if the first situation is true in each sandwich number $j$, for $j = 1, \ldots, k$. Otherwise $x$ is called a singular point.

2.2. Cartan prolongation and tower of projective bundles.

Let be $D$ a distribution of constant rank $m + 1$ on a manifold $M$ of dimension $n$. Classically the Grassmannian bundle $G(D, 1)$ on $M$ is the set

\[
G(D, 1, M) := \bigcup_{x \in M} P(D(x), 1)
\]

where $P(D(x), 1)$ is the projective space of the vector space $D(x)$. So we have a bundle $\pi : G(D, 1) \to M$ whose fiber $\pi^{-1}(x)$ is diffeomorphic to the projective space $\mathbb{R}P^m$. The rank one Cartan prolongation is the distribution $D^{(1)}$ defined in the following way: given a point $(x, \lambda) \in G(D, 1)$ then

\[
D^{(1)}_{(x, \lambda)} := d\pi^{-1}(\lambda) \subset T_{(x, \lambda)}G(D, 1, M)
\]

where $\lambda$ is a direction of $D(x)$. Then $D^{(1)}$ is a distribution on $G(D, 1, M)$ of constant rank $m + 1$. As in [13], for any $m \geq 2$ and $k \geq 1$ we obtain inductively a tower of bundles:

\[
\cdots \to P^k(m) \to P^{k-1}(m) \to \cdots \to P^1(m) \to P^0(m) := \mathbb{R}^{m+1}
\]

where, for any $j = 0, \ldots, k$, $P^j(m)$ is a manifold of dimension $(j + 1)m + 1$, and on each $P^j(m)$, $\Delta_j$ is a distribution which are defined inductively by:

$P^j(k) = G(\Delta_{j-1}, 1, P^{j-1}(m))$ and $\Delta_j = (\Delta_{j-1})^{(1)}$ for $j = 1, \ldots, k$ and $\Delta_0 = TR^{m+1}$. We have then the following result:

**Theorem 2.2.** [13]

1. On $P^k(m)$, the distribution $\Delta_k$ generates a special multi-flag of step $m$ and length $k$.
2. let be $F : D = D_k \subset D_{k-1} \subset \ldots \subset D_1 \subset \ldots \subset D_1 \subset D_0 = TM$ a special multi-flag of step $m \geq 2$ and length $k \geq 1$. Then, for any $x \in M$, there exists $y \in P^k(m)$ for which the differential system $(P^k(m), \Delta_x, y)$ is locally equivalent to the differential system $(M, D, x)$.

**Remark 2.1.**

The part 2 of Theorem 2.2 can be found precisely in [13] called "Drapeau Theorem". However, according to the definition of a special multi-flag, we can easily deduce this result from Theorem 2 of [6].
2.3. Spherical prolongation, Cartan prolongation and tower of sphere bundles.

Let be $D$ a distribution of constant rank $m+1$ on a manifold $M$ of dimension $n$. Choose any riemannian metric $g$ on $M$, and we denote by $S(D, M, g)$ the unit sphere bundle of $D$ associated to the induced riemannian metric on $D$. So we get a bundle $\hat{\pi} : S(D, M, g) \to M$. On $S(D, M, g)$, we consider the antipodal action of $\mathbb{Z}_2$. Clearly, the quotient of $S(D, M, g)$ by this action can be identified with $G(D, 1, M)$ and the associated projection $\tau : S(D, M, g) \to G(D, 1, M)$ is a bundle morphism over $M$, and also a two-fold covering. In particular $\tau$ is a local diffeomorphism. On $S(D, M, g)$ we consider the distribution $D^{[1]}$ defined in the following way

\begin{equation}
D^{[1]}_{(x, \nu)} := \{ v \in T_{(x, \nu)} S(D, M, g) \text{ such that } d\hat{\pi}(v) = \lambda \nu \text{ for some } \lambda \in \mathbb{R} \}
\end{equation}

where $\nu$ is a norm one vector in $D(x)$.

The distribution $D^{[1]}$ is called the rank one spherical prolongation of $(M, D, g)$.

Remark 2.2.

In fact, the unit sphere bundle of $S(D, M)$ is defined as soon as we fix some riemannian metric on $D$. In this case, the distribution $D^{[1]}$ is also well defined.

Lemma 2.1.

(i) we have $\tau_* D^{[1]} = D^{[1]}$

(ii) There exists a canonical riemannian metric $\hat{g}$ on $S(D, M, g)$ which is uniquely defined from the riemannian metric $g$ on $M$.

Proof. At first we show part (i) locally. Choose a chart domain $U$ over which $D$ is trivial. We choose an orthonormal frame $\{e_0, \cdots, e_m\}$ of $D$ over $U$. Without loss of generality we can assume that $D_U \equiv \mathbb{R}^m \times \mathbb{R}^{m+1}$ so, the bundle $S(D, M, g)|_U$ is isomorphic to $\mathbb{R}^n \times S^m$ and $G(D, 1, M)|_U$ is isomorphic to $\mathbb{R}^n \times \mathbb{R}^{m+1}$. So locally, $\tau : \mathbb{R}^n \times S^m \to \mathbb{R}^n \times \mathbb{R}^{m+1}$ is the map $(x, \nu) \mapsto (x, [\nu])$ where $[\nu]$ is the line bundle generated by $\nu$. From the definition of $D^{[1]}_{(x, \nu)}$ and $\tau_*(D^{[1]}_{(x, \nu)})$ we have $\tau_*(D^{[1]}_{(x, \nu)}) = D^{[1]}_{(x, [\nu])}$.

As $\tau$ is a local diffeomorphism we get the part (i) locally. On the other hand, the map $\hat{\alpha} : S(D, M, g) \to S(D, M, g)$ given by $\hat{\alpha}(x, \nu) = (x, -\nu)$ is a diffeomorphism which commutes with $\tau$. From the definition of $D^{[1]}$, we get

$$\hat{\alpha}_*(D^{[1]}_{(x, \nu)}) = D^{[1]}_{(x, -\nu)}$$

This ends the proof of part (i).

For part (ii), denote by $\hat{g}$ the canonical riemannian metric on $TM$ associated to $g$. As, $S(D, M, g)$ can be considered as a submanifold on $TM$ we get an natural induced riemannian metric $\hat{g}$ on $S(D, M, g)$.

Let be $g_0$ and $g_1$ two riemannian metrics on $M$. We denote by $S_i(D, M)$ the sphere bundle of $D$ associated to the metric $g_i$, and $D^{[1]}_i$ the spherical prolongation of $(M, D, g_i)$ for $i = 0, 1$.

Lemma 2.2.

There exists a canonical isomorphism of sphere bundle $\psi : S_0(D, M) \to S_1(D, M)$ such that $\psi_*(D^{[1]}_0) = D^{[1]}_1$.

Proof. Let $D^0 = \bigcup_{x \in M} [D_x \setminus \{0]\}$. Then $D^0$ is an open submanifold of $D \subset TM$. On $D^0$ we consider the map $\Psi(x, u) : D^0 \to D^0$ defined by $\Psi(x, u) = (x, \frac{u}{[g_1(u, u)]^{1/2}})$.

If $\Pi : D \to M$ is the projection bundle, for any $(x, u) \in D$, there exists a neighbourhood $\hat{U} = \Pi^{-1}(U) \cap D^0$ of $(x, u)$ in $D^0$ such that over this open, $TD^0|_U$ can be identified with $\hat{U} \times T_x M \times D_x$. Then, In this context, we have:

\begin{equation}
d\Psi(v, w) = (v, -\frac{g_1(u, w)}{2[g_1(u, u)]^{1/2}}).
\end{equation}

It is easy to see that $\Psi$ is a diffeomorphism from $D^0$ into itself which commutes with $\Pi$ and which sends $S_0(D, M)$ to $S_1(D, M)$. So the restriction $\psi$ of $\Psi$ to $S_0(D, M)$ is a diffeomorphism onto $S_1(D, M)$. Moreover, from (7), $d\Psi$ map the the linear span $\mathbb{R}u$ into itself, for any $u$ in the fiber $D^0_x$ over $x$. So we have

$$\psi_*(D^{[1]}_0) = D^{[1]}_1.$$
Consider a differential system \((M', D')\) and \(\phi : M \to M'\) an injective immersion such that \(\phi_\ast(D_x) \subset D_{\phi(x)}\) for any \(x \in M\). Given any riemannian metric \(g'\) on \(M'\), we get an induced riemannian metric \(g\) on \(M\) and we can consider the associated spherical prolongation then we have:

**Lemma 2.3.**

in the previous context, let be \(\hat{\phi} : S(D, M, g) \to S(D', M', g')\) the map defined by
\[
\hat{\phi}(x, \nu) = (\phi(x), d_x \phi(\nu)).
\]

Then \(\hat{\phi}\) is a bundle morphism over \(\phi\) which is an injective immersion and such that

(i) \(\hat{\phi}(S(D, M, g)) = S(\phi_\ast(D), \phi(M), g')\)

(ii) \(\hat{\phi}_\ast(D^{[1]})) = (\phi_\ast(D))^{[1]} \subset (D')^{[1]}\).

Moreover, if \(\phi\) is a diffeomorphism such that \(\phi_\ast(D) = D'\), then \(\hat{\phi}\) is also a diffeomorphism and we have \(\phi_\ast(D^{[1]})) = (D')^{[1]}\) and the riemannian metric \(\phi_\ast g'\) is nothing but the canonical metric \(\hat{g}\) naturally associated to \(g\) on \(M\).

**Proof.**

As in Lemma, consider the map \(\hat{\phi}(x, \nu) = (\phi(x), d_x \phi(\nu))\). From our assumptions, we get a smooth map from \(S(D, M, g)\) to \(S(D', M', g')\) and from its definition, clearly \(\hat{\phi}\) is a bundle morphism over \(\phi\). As \(\phi\) is an injective immersion, it follows that at first \(\hat{\phi}\) is injective.

Note that the tangent space \(T_{(x,\nu)}S_x\) of the fiber \(S_x\) over \(x\) of \(S(D, M, g)\) can be identified with \(\{v \in D_x \ such \ that \ g(\nu, v) = 0\}\).

Now any \(V \in T_{(x,\nu)}S(D, M, g)\) can be written \(V = (u, v)\) with \(u \in T_x M\) and \(v \in T_{(x,\nu)}S_x\). So we have then:
\[
(8) \quad d_{(x,\nu)} \hat{\phi}(u, v) = (d_x \phi(u), d_x \phi(\nu))
\]

So, \(\hat{\phi}\) is an immersion, from (8).

On the other hand, as \(\phi_\ast g' = g, d_x \phi\) is an isometry on its range, and then, \(d_x \phi(S_x)\) is the fiber over \(\phi(x)\) of \(S(\phi_\ast(D), \phi(M), g')\) and we get (i).

Let be \(\pi : S(D, M, g) \to M\) and \(\pi' : S(D', M', g') \to M'\) the natural projections. We have then:
\[
d\pi' \circ d\hat{\phi} = d\phi \circ d\pi
\]

So, we get:
\[
\{\hat{\phi}_\ast(D^{[1]})) = \{(d\phi_\ast)_\ast(D^{[1]}))\}_{\hat{\phi}(x, \nu)} = \{(D')^{[1]}\}_{\phi(x, \nu)}.
\]

Finally, by assumption, \(\phi\) is an isometry from \((M, g)\) to \((M', g')\), \(d\phi\) is also an isometry for \((TM, \hat{g})\) and \((TM', \hat{g}')\) if \(\hat{g}\) and \(\hat{g}'\) are the canonical riemannian metric on the tangent bundle induced by \(g\) and \(g'\) respectively. As by construction \(\hat{g}\) and \(\hat{g}'\) are the restriction respective of \(\hat{g}\) and \(\hat{g}'\) to \(S(D, M, g) \subset TM\) and \(S(D', M', g') \subset TM'\) we get the last property and this ends the proof of (ii).

Assume now that \(\phi\) is a diffeomorphism such that \(\phi_\ast(D) = D'\) and let be \(\psi = \phi^{-1}\). From the definition of \(\hat{\phi}\) and \(\hat{\psi}\), it follows trivially that \(\psi \circ \hat{\psi} = Id\). On the other hand, from the definition of \([\phi_\ast(D)]^{[1]}\), as \(d_x \phi\) is an isomorphism, we must have \(\{\phi_\ast(D)]^{[1]}\}_{\phi(x, \nu)} = \{(D')^{[1]}\}_{\phi(x, \nu)}\). Finally, by assumption, \(\phi\) is an isometry from \((M, g)\) to \((M', g')\), \(d\phi\) is also an isometry for \((TM, \hat{g})\) and \((TM', \hat{g}')\) if \(\hat{g}\) and \(\hat{g}'\) are the canonical riemannian metric on the tangent bundle induced by \(g\) and \(g'\) respectively. As by construction \(\hat{g}\) and \(\hat{g}'\) are the restriction respective of \(\hat{g}\) and \(\hat{g}'\) to \(S(D, M, g) \subset TM\) and \(S(D', M', g') \subset TM'\) we get the last property and this ends the proof of the Lemma.

So, as in the context of Cartan prolongation, for any \(m \geq 2\) and \(k \geq 1\) we can define, inductively, a tower of sphere bundles (for a fixed choice of the metric \(g\) on a manifold \(M\)):
\[
\cdots \to \tilde{P}^k(M) \to \tilde{P}^{k-1}(M) \to \cdots \to \tilde{P}^1(M) \to \tilde{P}^0(M) := M
\]

where, for any \(j = 0, \cdots, k\), \(\tilde{P}^j(M)\) is a manifold of dimension \((j + 1)m + 1\), and, on each \(\tilde{P}^j(M)\) we have a canonical distribution \(\Delta_j\) and a riemannian metric \(g_j\) on \(P^j(M)\), all defined inductively in the following way:
\[
\tilde{P}^j(M) = S(\Delta_{j-1}, \tilde{P}^{j-1}(M), g_{j-1}), \Delta_j = (\Delta_{j-1})^{[1]} \text{ for } j = 1, \cdots, k \text{ and } \Delta_0 = TM,
g_j \text{ is the riemannian metric } \tilde{g}_{j-1} \text{ on } S(\Delta_{j-1}, \tilde{P}^{j-1}(M), g_{j-1}) \text{ associated to } g_{j-1} \text{ for } j = 1, \cdots, k \text{ and } g_0 = g.
\]

Note that if \(g'\) is another riemannian metric on \(M\), according to Lemma 2.2 and Lemma 2.3, by induction we get a family of diffeomorphisms \(\psi^j\) such that, if :
The tower of sphere bundles associated to the canonical metric on $\mathbb{R}^{m+1}$.

1. We have a canonical two-fold covering $\tau_k : \hat{P}^k(m) \to P^k(m)$ such that
   $\tau_k(\hat{\Delta}) = \Delta_k$.

2. On each manifold $\hat{P}^k(m)$, the distribution $\hat{\Delta}_k$ generates a special multi-flag of step $m$ and length $k$.

3. Let be $\mathcal{F} : D = D_k \subset D_{k-1} \subset \cdots \subset D_1 \subset D_0 = TM$ a special multi-flag of step $m \geq 2$ and length $k \geq 1$. Then, for any $x \in M$, there exists $y \in \hat{P}^k(m)$ for which the differential system $(\hat{P}^k(m), \hat{\Delta}, y)$ is locally equivalent to the differential system $(M, D, x)$.

The tower (10) will be called the spherical tower of special multi-flags of step $k$.

3. Tower of sphere bundles associated to a kinematic system

3.1. A kinematic system for special multi-flags.

We locate us in the context of [4] and [12]. Consider a serie of $k$ segments $[M_i ; M_{i+1}]$, $i = 0, \cdots, k - 1$, in $\mathbb{R}^{m+1}$, with $m \geq 2$, keeping a constant length $l_i = 1$ between $M_i$ and $M_{i+1}$, and the articulation occurs at points $M_i$ for $i = 1, \cdots, k - 1$.

Such a system is called a "k-bar system" in [4] and an "articulated arm of length $k$" in [12]. The kinematic evolution of the extremity $M_0$, under the constraint that the velocity of each point $M_i$, $i = 0, \cdots, k - 1$, is collinear with the segment $[M_i, M_{i+1}]$ is completely described in terms of hyperspherical coordinate in [12] and result of flatness and controllability for such a system are proved in in [4]. We can associated to this problem a special multi-flag of step $m \geq 2$ and length $k \geq 1$ as explained in the following:

We can decompose $(\mathbb{R}^{m+1})^{k+1}$ into a product $\mathbb{R}^{m+1}_0 \times \cdots \times \mathbb{R}^{m+1}_k$. Let $x_i = (x_i^1, \cdots, x_i^{m+1})$ be the canonical coordinates on the space $\mathbb{R}^{m+1}_k$ which is equipped with its canonical scalar product $< , >$. $(\mathbb{R}^{m+1})^{k+1}$ is then equipped with its canonical scalar product too.

Consider an articulated arm of length $k$ denoted by $(M_0, \cdots M_k)$. On $(\mathbb{R}^{m+1})^{k+1}$, consider the vector fields:

$$Z_i = \sum_{r=1}^{m+1} (x_{i+1}^r - x_i^r) \frac{\partial}{\partial x_i^r} \text{ for } i = 0, \cdots, k - 1$$

From our previous assumptions, the kinematic evolution of the articulated arm is described by a controlled system:

$$\dot{q} = \sum_{i=0}^{k-1} u_i Z_i + \sum_{r=1}^{m+1} u_{n+r} \frac{\partial}{\partial x_i^r}$$

with the following constraints:

$$||x_i - x_{i+1}|| = 1 \text{ for } i = 0 \cdots k - 1 \text{ (see [4] or [12]).}$$

Consider the map $\Psi_i(x_0, \cdots, x_k) = ||x_i - x_{i+1}||^2 - 1$. Then, the configuration space $C^k(m)$ is the set

$$\{ (x_0, \cdots, x_k), \text{ such that } \Psi_i(x_0, \cdots, x_k) = 0 \text{ for } i = 0, \cdots, k - 1 \}$$

For $i = 0, \cdots, k - 1$, the vector field:

$$N_i = \sum_{r=1}^{m+1} (x_{i+1}^r - x_i^r) \left[ \frac{\partial}{\partial x_i^r} - \frac{\partial}{\partial x_{i+1}^r} \right]$$

is proportional to the gradient of $\Psi_i$.

So the tangent space $T_q C^k (m)$ is the subspace of $T_q (\mathbb{R}^{m+1})^{k+1}$ which is orthogonal to $N_i(q)$ for $i = 0, \cdots, k - 1$. 

Denote by $\mathcal{C}_k$ the distribution generated by the vector fields
\[
\{Z_0, \ldots, Z_{k-1}, \frac{\partial}{\partial x_k^1}, \ldots, \frac{\partial}{\partial x_k^{m+1}}\}.
\]

**Lemma 3.1.** [12]

Let $D_k$ be the distribution on $\mathcal{C}^k(m)$ defined by $\Delta(q) = T_qC \cap E$. Then $D_k$ is a distribution of dimension $m+1$ generated by
\[
(x_k^r - x_{k-1}^r)\sum_{i=0}^{k-1} \prod_{j=i+1}^k A_j Z_i + \frac{\partial}{\partial x_k^r} \text{ for } r = 1 \cdots m + 1
\]
where $A_j(q) = N_j(q), N_{j-1}(q) > Z_j(q), N_{j-1}(q)$ for $j = 1, \ldots, k-1$ and $A_k = 1$.

**Remark 3.1.** : according to notations of Lemma, we set
\[
Y_k = \sum_{i=0}^{k-1} \prod_{j=i+1}^k A_j Z_i = \sum_{i=0}^{k-1} \prod_{j=i+1}^k A_j Z_i + Z_{k-1} \text{ and } Y_{k-1} = \sum_{i=0}^{k-2} \prod_{j=i+1}^{k-2} A_j Z_i + Z_{k-2}
\]

So we have $Y_k = \sum_{r=1}^{m+1} (x_k^r - x_{k-1}^r)(x_{k-1}^r - x_{k-2}^r)]Y_{k-1} + Z_{k-1}$. Moreover, $D_k$ is generated by the family
\[
\{(x_k^r - x_{k-1}^r)Y_k + \frac{\partial}{\partial x_k^r}, r = 1 \cdots m + 1\}
\]

The properties of $D_k$ are summarized in the following result. (see [12] also [4])

**Theorem 3.1.**

On $\mathcal{C}^k(m)$, the distribution $D_k$ satisfies the following properties:

1. $D_k$ is a distribution of rank $m + 1$.
2. The distribution $D_k$ is a special multi-flag on $\mathcal{C}^k(m)$ of step $m$ and length $k$.

### 3.2. Kinematic system and spherical prolongation.

To an articulated arm on $\mathbb{R}^{m+1}$ ($m \geq 2$) of length $k \geq 1$ we can associate the following canonical tower of sphere bundles:

\[
\mathcal{C}^k(m) \rightarrow \mathcal{C}^{k-1}(m) \rightarrow \cdots \rightarrow \mathcal{C}^1(m) \rightarrow \mathcal{C}^0(m) := \mathbb{R}^{m+1}
\]

where for $j = 1, \ldots, k$, the projection $\mathcal{C}^j(m) \rightarrow \mathcal{C}^{j-1}(m)$ is the restriction of the canonical projection

\[
\mathbb{R}^{m+1} \times \cdots \times \mathbb{R}^{m+1} \times \cdots \times \mathbb{R}^{m+1} \rightarrow \mathbb{R}^{m+1} \times \cdots \times \mathbb{R}^{m+1} \times \cdots \times \mathbb{R}^{m+1}
\]

\[
(x_0, \ldots, x_{j-1}, x_j) \rightarrow (x_0, \ldots, x_{j-1})
\]

According to Theorem 3.1, and Theorem 2.3, we know that the differential system $(\mathcal{C}^k(m), D_k)$ associated to an articulated arm of length $k$ on $\mathbb{R}^{m+1}$ is locally isomorphic to the canonical differential system $(\hat{P}^k(m), \hat{\Delta}_k)$ at some appropriate points. In fact, we have more: (see Theorem 1 in the introduction)

**Theorem 2.2.**

For each $k \geq 1$ and $m \geq 2$, there exists a diffeomorphism $\Phi^k$ from $\hat{P}^k(m)$ on $\mathcal{C}^k(m)$ such that:

(i) if $\hat{\pi}^k : \hat{P}^k(m) \rightarrow \hat{\Delta}^{k-1}(m)$ and $p^k : \mathcal{C}^k(m) \rightarrow \mathcal{C}^{k-1}(m)$ are the canonical projections, we have:

\[
p^k \circ \Phi^k = \Phi^{k-1} \circ \hat{\pi}^k
\]

(ii) $\Psi^k(\hat{\Delta}_k) = D_k$

So according to Theorem 2.3 from Theorem 3.2 we have:

**Theorem 3.3.**

Let be $F : D = D_k \subset D_{k-1} \subset \cdots \subset D_1 \subset \cdots \subset D_1 \subset D_0 = TM$ a special multi-flag of step $m \geq 2$ and length $k \geq 1$. Then, for any $x \in M$, there exists $y \in \mathcal{C}^k(m)$ for wich the differential system $(\mathcal{C}^k(m), D_k, y)$ is locally equivalent to the differential system $(M, D, x)$.

The end of this section is devoted to the proof of Theorem 3.2. At first, we need some auxiliary results.
Lemma 3.2. 
For $k \geq 1$, consider, on $C^k(m)$, the natural decomposition:
\[ [T(R^{m+1})^k(m)] \oplus [T^r(m)^k(m)] \]
where $[T^r(m)^k(m)]$ is the orthogonal of $T^r(m)^k(m)$ and denote by $\Pi_k$ the orthogonal projection of $[T(R^{m+1})^k(m)]$ onto $T^r(m)^k(m)$. On the other hand, let $L$ be the involutive distribution whose leaves are the fibers of the natural fibration of $C^k(m)$ onto $C^{k-1}(m)$.

1. The family vector fields $\{\Pi_k(\frac{\partial}{\partial x^r_k}), r = 1, \ldots, m + 1\}$ generates the distribution $L$.

2. The distribution $D_k$ is generated by $L$ and the vector field $X_k = \sum_{i=0}^{k-1} \prod_{j=1}^{k} A_j Z_i + V_k$

where $V_k = \sum_{i=1}^{m+1} (x_k^r - x_i^r) \frac{\partial}{\partial x_k^r}$.

3. The distribution $D_k$ is also generated by the family of vector fields

\[(x_k^r - x_i^r)X_k + \Pi_k(\frac{\partial}{\partial x_k^r})\]

Proof.
Let $H_k$ the subdistribution of $C_k$ generated by the family vector fields $\{\frac{\partial}{\partial x_k^r}, r = 1, \ldots, m + 1\}$. So $H_k \cap T^r(m)^k(m)$ is a distribution on $C^k(m)$ which is contained in $D_k$. In fact, we have

$\mathcal{L}_k = \ker d\Psi_{k-1} \cap H_k = \Pi_k(H_k)$.

So $\mathcal{L}_k$ is generated by the family vector fields $\{\Pi_k(\frac{\partial}{\partial x_k^r}), r = 1, \ldots, m + 1\}$. On the other hand, as $H_k$ is the vertical bundle of the canonical projection

\[ \mathbb{R}_{m+1} \times \mathbb{R}_{m+1} \rightarrow \mathbb{R}_{m+1} \times \cdots \times \mathbb{R}_{m+1} \]

\[(x_0, \ldots, x_{k-1}, x_k) \rightarrow (x_0, \ldots, x_{k-1}) \]

it follows that $\mathcal{L}_k$ is also the vertical bundle of the induced projection of $C^k(m)$ onto $C^{k-1}(m)$. On the other hand, the fiber over $q \in C^{k-1}(m)$ of the previous fibration is the unit sphere $S_q = \{(q, x_k) : \Psi_{k-1}(q, x_k) = 0\}$ which proves (1).

The vector field $V_k = \sum_{i=1}^{m+1} (x_k^r - x_i^r) \frac{\partial}{\partial x_k^r}$ is vertical for the projection (16) and is orthogonal to each $S_q$. As $\|V_k\| = 1$ we thus have:

\[ \Pi_k(\frac{\partial}{\partial x_k^r}) = \frac{\partial}{\partial x_k^r} - (x_k^r - x_i^r)V_k \]

From Lemma 3.1, $D_k$ is generated by the family $\{(x_k^r - x_i^r) \sum_{i=0}^{k-1} \prod_{j=1}^{k} A_j Z_i + \frac{\partial}{\partial x_k^r} \}$ for $r = 1, \ldots, m + 1$.

So the vector field $X_k = \sum_{i=0}^{k-1} \prod_{j=1}^{k} A_j Z_i + \sum_{r=1}^{m+1} (x_k^r - x_i^r) \frac{\partial}{\partial x_k^r}$ is tangent to $D_k$ but clearly this vector fields is not tangent to $L$. As $D_k$ is a distribution of constant rank $m + 1$ and $L$ is an (integrable) subdistribution of rank $m$, it follows that $D_k$ is generated by $L$ and $X_k$ which proves (2).

On the other hand, according to (17), each vector field $(x_k^r - x_i^r) \sum_{i=0}^{k-1} \prod_{j=1}^{k} A_j Z_i + \frac{\partial}{\partial x_k^r}$ can be written

\[(x_k^r - x_i^r)X_k + \Pi_k(\frac{\partial}{\partial x_k^r})\]

According to Lemma 3.1 we get (3). \hfill \square

Proposition 3.1.
(1) There exists a bundle isomorphism $\hat{\Psi} : D_k \rightarrow C^k(m) \times \mathbb{R}^{m+1}$

(2) Let be $\gamma_k$ a riemannian metric on the bundle $D_k$ so that the morphism $\hat{\Psi}$ is an isometry between $D_k$ and $C^k(m) \times \mathbb{R}^{m+1}$ (for the canonical euclidian product on the fiber $\mathbb{R}^{m+1}$). Then $\hat{\Psi}$ induces a diffeomorphism $\hat{\Psi} : S(D_k, C^k(m), \gamma_k) \rightarrow C^{k+1}(m)$ such that

(i) $\hat{\Psi}$ commutes with the canonical projections $S(D_k, C^k(m), \gamma_k) \rightarrow C^k(m)$ and $C^{k+1}(m) \rightarrow C^k(m)$. 
(ii) \( \Psi_*[(D_k)^{[1]}] = D_{k+1} \).

Proof.
From Lemma 3.2 part (3), the bundle \( D_k \) has \( m + 1 \) non-zero global sections

\[
(x_k^r - x_{k-1}^r)X_k + \Pi_k \left( \frac{\partial}{\partial x_k^r} \right) \text{ for } r = 1 \cdot m + 1
\]

so \( D_k \) is a trivial bundle. It follows that there exists a bundle isomorphism \( \tilde{\Psi} : D_k \to \mathcal{C}^k(m) \times \mathbb{R}^{m+1} \) which ends part (1).

Put on \( D_k \) the riemannian metric \( \gamma_k = \tilde{\Psi}^*g \) where \( g \) is the canonical euclidian product on \( \mathbb{R}^{m+1} \). Now the map \( \Gamma : (\mathbb{R}^{m+1})^{k+1} \times (\mathbb{R}^{m+1})^{k+2} \) defined by

\[
\Gamma(x_0, x_1, \ldots, x_k, z) = (x_0, x_1, \ldots, x_k, x_k + z)
\]
is a diffeomorphism. So the restriction \( \hat{\Gamma} \) of \( \Gamma \) to \( \mathcal{C}^k(m) \times \mathbb{R}^m \) is a diffeomorphism \( \Gamma : \mathcal{C}^k(m) \times \mathbb{R}^m \to \mathcal{C}^{k+1}(m) \).

Finally, \( \Psi = \Gamma \circ \tilde{\Psi} \) induces, by restriction, a diffeomorphism \( \Psi : S(D_k, \mathcal{C}^k(m), \gamma_k) \to \mathcal{C}^{k+1}(m) \) which commutes with the canonical the projections:

\[
S(D_k, \mathcal{C}^k(m), \gamma_k) \to \mathcal{C}^k(m) \quad \text{and} \quad \mathcal{C}^{k+1}(m) \to \mathcal{C}^k(m).
\]

On \( D_k \), we have a riemannian metric so that the global basis given in (18) is orthonormal. It follows that the map \( \Psi : S(D_k, \mathcal{C}^k(m), \gamma_k) \to \mathcal{C}^{k+1}(m) \) is given by

\[
\Psi((x_0, x_1, \ldots, x_k, \nu) = ((x_0, x_1, \ldots, x_k, x_k + \nu).
\]

So, in the global chart of \( S(D_k, \mathcal{C}^k(m), \gamma_k) \) defined by \( \Psi \), according to (18), the spherical prolongation \( (D_k)^{[1]} \) of \( D_k \) is generated by the tangent space to the sphere centered at \((x_0, \ldots, x_k)\) and the vector field:

\[
Y_{k+1} = \sum_{r=1}^{m+1} \left( (x_{k+1}^r - x_k^r)X_k + \Pi_k \left( \frac{\partial}{\partial x_k^r} \right) \right) + \sum_{r=1}^{m+1} \left( (x_{k+1}^r - x_k^r)X_k + \Pi_k \left( \frac{\partial}{\partial x_k^r} \right) \right)
\]

According to (17) and the vector field can be written

\[
\sum_{r=1}^{m+1} \left( (x_{k+1}^r - x_k^r)X_k + \Pi_k \left( \frac{\partial}{\partial x_k^r} \right) \right) = \frac{\partial}{\partial x_k^r}
\]

Note that

\[
\sum_{r=1}^{m+1} \left( (x_{k+1}^r - x_k^r) \right) \frac{\partial}{\partial x_k^r} = Z_k.
\]

So according to Remark 3.1, applied at level \( k+1 \), we have

\[
Y_{k+1} = \left( \sum_{r=1}^{m+1} (x_{k+1}^r - x_k^r)X_k + \Pi_k \left( \frac{\partial}{\partial x_k^r} \right) \right) Y_k + Z_k.
\]

and the distribution \( D_{k+1} \) is generated by

\[
\{(x_{k+1}^r - x_k^r)Y_{k+1} + \frac{\partial}{\partial x_k^r} \mid r = 1 \cdot m + 1 \}
\]

Again from Lemma 3.2 part (2), at level \( k+1 \) we get:

\[
\Psi_*[(D_k)^{[1]}] = D_{k+1}
\]

Proof of Theorem 3.2
Note that on the tangent bundle \( T \mathbb{R}^{m+1} \), we can put the global chart defined by the map \((x_1, x_2) \to (x_1, x_2 - x_1)\). On the other hand, the riemannian metric \( g_1 \) on \( T \mathbb{R}^{m+1} \) induces by the canonical metic \( g \) is again the canonical metic on the product \( \mathbb{R}^{m+1} \times \mathbb{R}^{m+1} \). It follows that \( S(T \mathbb{R}^{m+1}, \mathbb{R}^{m+1}) \subset T \mathbb{R}^{m+1} \) can be identified with \( \mathbb{R}^{m+1} \times S^k \) as submanifold of \( \mathbb{R}^{m+1} \times \mathbb{R}^{m+1} \). So we have

\[
T(x, v) S(T \mathbb{R}^{m+1}, \mathbb{R}^{m+1}) = \{(u, v) \in T(x, v)(\mathbb{R}^{m+1} \times \mathbb{R}^{m+1}) \mid \text{such that } g(v, u) = 0 \}.
\]

Recall that \( Z_0 = \sum_{r=1}^{m+1} (x_1^r - x_0^r) \frac{\partial}{\partial x_0^r} \) on \( \mathbb{R}^{m+1} \times \mathbb{R}^{m+1} \). In the previous coordinates, any tangent vector of \( \mathbb{R}^{m+1} \) at a point \( x_1 \) can be written \((x_1, x_2 - x_1)\). So, \( Z_0 \) defines a global section of the unit bundle associated to \( T \mathbb{R}^{m+1} \)

According to Lemma 3.2, the distribution \( D_1 \) is generated by
\[ \mathbb{R} \mathbb{Z}_0 \text{ and } T \mathbb{S}^m \text{ in } TR^{m+1} \times T \mathbb{S}^m \equiv TS(TR^{m+1}, \mathbb{R}^{m+1}). \] If \( \Pi_1 : \mathbb{R}^{m+1} \times \mathbb{S}^m \to \mathbb{R}^{m+1} \) denote the natural projection, for any \( v = \lambda \mathbb{Z}_0 + w \in \{ D_1 \}_{x, \nu} \) with \( w \in T \mathbb{S}^k \), we have \( d\Pi_1(v) = \lambda \nu \) so, \( \Delta_1 = (TR^{m+1})[1] \) and we get the result for \( k = 1 \).

Assume that we have a diffeomorphism \( \Phi^k : \hat{P}^k(m) \to C^k(m) \) which satisfies the properties (i), and (ii) of Theorem 2.3.

From Proposition 3.1, we have diffeomorphism \( \Psi : S(D_k, C^k(m), \gamma_k) \to C^{k+1}(m) \) so that \( \Psi_{\ast}((D_k)[1]) = D_{k+1} \) and which commutes with the natural projections
\[ S(D_k, C^k(m), \gamma_k) \to C^k(m) \quad \text{and} \quad C^{k+1}(m) \to C^k(m) \]

According to previous induction, we can put on \( \hat{P}^k(m) \), the riemannian metric \( \hat{\gamma}_k = (\Psi\Psi^k)^{\ast}(\gamma_k) \). From Lemma 2.3, we can extend \( \Phi^k : \hat{P}^k(m) \to C^k(m) \) into a diffeomorphism \( \Phi^k : S(\Delta_k, \hat{P}^k(m), \hat{\gamma}_k) \to S(D_k, C^k(m), \gamma_k) \) such that \( \Phi^k(\Delta_k)[1] = (D_k)[1] \) and which commutes with the natural projections
\[ S(\Delta_k, \hat{P}^k(m), \hat{\gamma}_k) \to \hat{P}^k(m) \quad \text{and} \quad C^{k+1}(m) \to C^k(m) \]

Finally, according to Lemma 2.2, when we put on \( \hat{P}^k(m) \) the riemannian metric induces by induction on the tower bundle (9), we also have a diffeomorphism \( \Phi : \hat{P}^{k+1}(m) \to S(\Delta_k, \hat{P}^k(m), \hat{\gamma}_k) \) which commutes with the canonical projections
\[ \hat{P}^{k+1}(m) \to \hat{P}^k(m) \quad \text{and} \quad S(\Delta_k, \hat{P}^k(m), \hat{\gamma}_k) \to \hat{P}^k(m) \]

and such that \( \Phi_{\ast}(\Delta_k) = \Delta_k[2] \).

\[ \square \]

4. Conclusion

According to Theorem 2.3 and Theorem 3.2, from towers (10) and (15) we get the following diagram each vertical map is a 2-fold covering for \( k \geq 1 \):
\[ \cdots \to C^k(m) \to C^{k-1}(m) \to \cdots \to C^i(m) \to C^0(m) := \mathbb{R}^{m+1} \]
\[ \cdots \to P^k(m) \to P^{k-1}(m) \to \cdots \to P^i(m) \to P^0(m) := \mathbb{R}^{m+1} \]

Taking in account the results of [2] and [3], in a forthcoming paper ([7]) we will give the direct links between the "RVT" code for curves and the definitions of singularity classes given in [5] and [6] and also their interpretation in the previous kinematic system (articulated arms) which are defined in [11].

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