Geometrical interpretation of solutions of certain PDEs

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Abstract
In §1 the authors define the notion of harmonic map between two generalized Lagrange spaces. In §2 it is proved that for certain systems of differential or partial differential equations, the solutions belong to a class of harmonic maps between two generalized Lagrange spaces. §3 describes the main properties of the generalized Lagrange spaces constructed in §2. These spaces, being convenient relativistic models, allow us to write the Maxwell’s and Einstein’s equations.

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1 Introduction
Looking for generalizing a Poincaré problem, Sasaki tried to find a Riemannian metric on a manifold $M$ such that the orbits of an arbitrary vector field $X$ should be geodesics. This attempt was a failure, but Sasaki discovered the well known almost contact metric structures on a manifold of odd dimension [8]. After the introduction of generalized Lagrange structures [9], the problem of Poincaré-Sasaki was reconsidered by the first author [10, 11]. He succeeded to discover a Lagrange structure on $M$, depending of the given vector field $X$ and a $(1,1)$-tensor field built using $X$, a metric $g$, and the covariant derivative induced by $g$, such that the $C^2$ orbits should belong to a class of geodesics. Moreover, replacing the system of ODEs of the orbits of $X$ by a system of PDEs and the notion of geodesic by the notion of harmonic map, some open general problems appear [12], namely

1) There exist Lagrange type structures such that the solutions of certain PDEs of order one should be harmonic maps?

2) What is a harmonic map between two generalized Lagrange spaces?

Using the notion of direction dependent harmonic map between a Riemannian manifold and a generalized Lagrange space, a partial answer to the Udrişte’s questions was offered by the second author [6].

Let us introduce, in a natural way, the notion of harmonic map between two Lagrange spaces $(M, g_{\alpha\beta}(a,b))$ and $(N, h_{ij}(x,y))$, where $M$ (resp. $N$) has the dimensions $m$ (resp. $n$) and $(a,b) = (a^\mu, b^\nu)$ (resp. $(x,y) = (x^k, y^k)$) are coordinates on $TM$ (resp. $TN$).

Definition. On $M \times N$, a tensor $P$ of type $(1,2)$ with all components null excepting $P^\beta_{\alpha}(a,x)$ and $P^i_{\alpha}(a,x)$, where $\alpha, \beta = 1, m$, $i, j = 1, n$, is called tensor of connection.
Assume that the manifold $M$ is connected, compact, orientable and endowed also with a Riemannian metric $\varphi_{\alpha\beta}$. This fact ensures the existence of a volume element on $M$. In these conditions, we can define the $\left( g^P \varphi \ h \right)$-energy functional,

$$E^{P}_{g\varphi h} : C^\infty(M,N) \rightarrow \mathbb{R},$$

$$E^{P}_{g\varphi h}(f) = \frac{1}{2} \int_M g^{\alpha\beta}(a,b)h_{ij}(f(a),y)f^i_a f^j_{\beta} \sqrt{\varphi} da,$$

where

$$\begin{cases}
  f^i = x^i(f), \ f^i_a = \frac{\partial f^i}{\partial a^\alpha}, \ \varphi = \text{det}(\varphi_{\alpha\beta}), \\
  b(a) = b^\gamma(a) \frac{\partial}{\partial a^\gamma} \bigg|_{f(a)} = \varphi_{\alpha\beta}(a)f^i_a(a)P^\gamma_{\beta i}(a,f(a)) \frac{\partial}{\partial a^\gamma} \bigg|_{f(a)}, \\
  y(f(a)) = y^k(a) \frac{\partial}{\partial x^k} \bigg|_{f(a)} = \varphi_{\alpha\beta}(a)f^i_a(a)P^k_{\beta i}(a,f(a)) \frac{\partial}{\partial x^k} \bigg|_{f(a)}.
\end{cases}$$

**Definition.** A map $f \in C^\infty(M,N)$ is called $\left( g^P \varphi \ h \right)$-harmonic if $f$ is a critical point for the functional $E^{P}_{g\varphi h}$.

The naturalness of the preceding definitions comes from the following particular cases:

i) If $g_{\alpha\beta}(a,b) = \varphi_{\alpha\beta}(a)$ and $h_{ij}(x,y) = h_{ij}(x)$ are Riemannian metrics, it recovers the classical definition of a harmonic map between two Riemannian manifolds.

ii) If $M = [a,b] \subset \mathbb{R}$, $\varphi_{11} = g_{11} = 1$ and $P = (P^1_{1}, \delta^1_1)$ we shall find $C^\infty(M,N) = \{c : [a,b] \rightarrow N | c - C^\infty \text{differentiable} \}$ not $\Omega_{a,b}(N)$ and the energy functional will be

$$E^{P}_{11\delta_{1}}(c) = \frac{1}{2} \int_a^b h_{ij}(c(t),c(t)) \frac{dc^i}{dt} \frac{dc^j}{dt} dt, \ \forall c \in \Omega_{a,b}(N).$$

In conclusion, the $\left( \begin{array}{cc} P & \delta \\ 1 & h \end{array} \right)$-harmonic curves are exactly the geodesics of the generalized Lagrange space $(N,h_{ij}(x,y))$.

iii) If we take $N = \mathbb{R}$, $h_{11} = 1$ and $P = (\delta^i_1, P^1_{1})$ we shall obtain $C^\infty(M,N) = \mathcal{F}(M)$ and the energy functional becomes

$$E^{P}_{\varphi 1}(f) = \frac{1}{2} \int_M g^{\alpha\beta}(a, \text{grad}_x f)f^i_a f^j_{\beta} \sqrt{\varphi} da, \ \forall f \in \mathcal{F}(M).$$

Obviously, the Euler-Lagrange equations of the energy functional $E$ will be the equations of harmonic maps, that is,

$$(H) \quad \sqrt{\varphi} \frac{\partial L}{\partial f^i} + \frac{\partial}{\partial a^\alpha} \left( \sqrt{\varphi} \frac{\partial L}{\partial f^i_a} \right) = 0, \ \forall i = 1, \ldots, n,$$

where $L(a^\alpha, f^i, f^i_a) = \frac{1}{2} g^{\mu\nu}(a^\nu, b^\gamma)h_{kl}(x^p, y^p)f^k_{\alpha} f^l_{\mu}$. 

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In the particular cases when the metric tensors are $g_{\alpha\beta}(a, b) = e^{-2\sigma(a, b)}\varphi_{\alpha\beta}(a)$ and $h_{ij}(x, y) = e^{2\tau(x, y)}\psi_{ij}(x)$, where $\sigma : TN \to R$, $\tau : TN \to R$ are smooth functions and $\psi_{ij}$ is a pseudo-Riemannian metric on $N$, we shall obtain

$$\begin{align*}
\frac{\partial L}{\partial x^i} &= e^{2\sigma + 2\tau} \varphi^{\gamma\mu} \varphi^{\delta\nu} \psi_{kl} \left[ \frac{\partial P^{\nu}_{\tau \epsilon}}{\partial x^\epsilon} \frac{\partial \sigma}{\partial y^\nu} + \frac{1}{2} g^{\gamma\mu} \frac{\partial h_{kl}}{\partial x^i} x^k x^l \right] \\
\frac{\partial L}{\partial x_{\alpha}^i} &= e^{2\sigma + 2\tau} \left\{ \varphi^{\gamma\mu} \varphi^{\delta\nu} \psi_{kl} \left[ \frac{\partial P^{\nu}_{\tau \epsilon}}{\partial x^\epsilon} \frac{\partial \sigma}{\partial y^\nu} + \frac{1}{2} g^{\gamma\mu} \frac{\partial h_{kl}}{\partial x^i} x^k x^l \right] + \frac{1}{2} g^{\gamma\mu} \frac{\partial h_{kl}}{\partial x^i} x^k x^l \right\}.
\end{align*}$$

These expressions will be reduced if we consider the following more particular cases:

1) $\sigma = \sigma(a)$ and $P = (P_{\alpha\beta}^a, A_{\beta}(a)\delta_x^i)$, where $\{A_{\beta}\}$ are the components of a covector $A$ on $M$. In this situation, we shall obtain

$$\begin{align*}
\frac{\partial L}{\partial x^i} &= \frac{1}{2} g^{\gamma\mu} \frac{\partial h_{kl}}{\partial x^i} x^k x^l \\
\frac{\partial L}{\partial x_{\alpha}^i} &= e^{2\sigma + 2\tau} \left\{ \varphi^{\gamma\mu} \varphi^{\delta\nu} \psi_{kl} A_{\beta} \frac{\partial \tau}{\partial y^\gamma} x^k x^l + \varphi^{\gamma\mu} x_{\alpha}^i x^k \right\}.
\end{align*}$$

2) $\tau = \tau(x)$ and $P = (\delta_x^\alpha \xi(x), P_{\beta}^a)$, where $\{\xi\}$ are the components of an 1-form $\xi$ on $N$. Now, we shall find

$$\begin{align*}
\frac{\partial L}{\partial x^i} &= e^{2\sigma + 2\tau} \varphi^{\gamma\mu} \varphi^{\delta\nu} \psi_{kl} \frac{\partial \xi_{\alpha}^i}{\partial x^\alpha} \frac{\partial \sigma}{\partial y^\nu} x^k x^l + \frac{1}{2} g^{\gamma\mu} \frac{\partial h_{kl}}{\partial x^i} x^k x^l \\
\frac{\partial L}{\partial x_{\alpha}^i} &= e^{2\sigma + 2\tau} \left\{ \varphi^{\gamma\mu} \varphi^{\delta\nu} \psi_{kl} \frac{\partial \xi_{\alpha}^i}{\partial x^\alpha} x^k x^l + \varphi^{\gamma\mu} x_{\alpha}^i x^k \right\}.
\end{align*}$$

2 Geometrical interpretation

By the above notions, we shall offer some beautiful geometrical interpretations of $C^2$ solutions of certain PDEs of order one.

We start with a smooth map $f \in C^\infty(M, N)$. This map induces the following tensor $\delta f = \frac{\partial f}{\partial y^j} \bigg|_{f(x)} \in \Gamma(T^*M \otimes f^{-1}(TN))$. On $M \times N$, let $T$ be a tensor of type $(1, 1)$ with all components null excepting $(T_{a}^{i})$, $\alpha \leq \Gamma_{1, m}$. These objects determine the system of PDEs,

$$\delta f = T \text{ expressed locally by } \frac{\partial f_{a}}{\partial a^{\alpha}} = T_{a}^{i}(a, f).$$

If $(M, \varphi_{a\beta})$ and $(N, \psi_{ij})$ are Riemannian manifolds we can build a scalar product on $\Gamma(T^*M \otimes f^{-1}(TN))$, namely $\langle T, S \rangle = \varphi^{\alpha\beta} \psi_{ij} T_{a}^{i} S_{b}^{j}$, where $T = T_{a}^{i} da^{\alpha} \otimes \frac{\partial}{\partial y^i}$ and $S = S_{b}^{j} db^{\beta} \otimes \frac{\partial}{\partial y^j}$. Obviously, the Cauchy-Schwartz inequality

$$\langle T, S \rangle \leq \| T \|^2 \| S \|^2, \forall S, T \in \Gamma(T^*M \times f^{-1}(TN)).$$
is an equality iff there exists $K \in \mathcal{F}(M)$ such that $T = KS$.

In these conditions, we prove the following

**Theorem.** If $(M, \varphi), (N, \psi)$ are Riemannian manifolds and the smooth map $f \in C^\infty(M, N)$ is solution of the system $(E)$, then $f$ is an extremal of functional

$$
\mathcal{L}_T : C^\infty(M, N) \\{ \exists a \in M \text{ such that } < \delta f, T >= (a) = 0 \} \to R_+,
$$

$$
\mathcal{L}_T(f) = \frac{1}{2} \int_M < \parallel \delta f \parallel^2, T >^2 \sqrt{\varphi} da = \frac{1}{2} \int_M \frac{\parallel T \parallel^2}{< \delta f, T >^2} \varphi^{\alpha\beta} \psi_{ij} f^\alpha f^\beta \sqrt{\varphi} da.
$$

**Proof.** Let $f$ be an arbitrary map from the definition domain of $\mathcal{L}_T$. Applying the above Cauchy-Schwarz inequality, we get

$$
\mathcal{L}_T(f) = \frac{1}{2} \int_M < \parallel \delta f \parallel^2, T >^2 \sqrt{\varphi} da \geq \frac{1}{2} \int_M \sqrt{\varphi} da = \frac{1}{2} \text{Vol}_\varphi(M).
$$

Obviously, if $f$ is solution of the system $(E)$ it follows $\mathcal{L}_T(f) = \frac{1}{2} \text{Vol}_\varphi(M)$, that is, $f$ is a global minimum point of the functional $\mathcal{L}_T$. In conclusion, the map $f$ verifies the Euler-Lagrange equations of $\mathcal{L}_T$.

Generally, the global minimum points of the functional $\mathcal{L}_T$ are solutions of the system $\delta f = KT$, where $K \in \mathcal{F}(M)$.

Now, we remark that, in certain particular cases, the functional $\mathcal{L}_T$ becomes exactly a functional of type $(P^1_1 h)$-energy and, consequently, the Euler-Lagrange equations reduce to equations of harmonic maps. This fact allows the following geometrical interpretations:

1. **Orbits**

Taking $M = ([a, b], 1)$ and $T = \xi \in \Gamma(c^{-1}(TN))$, the PDEs system $(E)$ reduces to the system of orbits

$$
\frac{dc^i}{dt} = \xi^i(c(t)), \ c : [a, b] \to N,
$$

and the functional $\mathcal{L}_\xi$ comes to

$$
\mathcal{L}_\xi(c) = \frac{1}{2} \int_a^b \frac{\parallel \xi \parallel^2_{\psi}}{[\xi^b(c)]^2} \psi_{ij} \frac{dc^i}{dt} \frac{dc^j}{dt} dt,
$$

where $\xi^b = \xi_i dx_i = \psi_{ij} \xi^j dx^i$. Hence the functional $\mathcal{L}_\xi$ is a $(P \begin{pmatrix} g & P \\ \varphi & h \end{pmatrix})$-energy, where the Lagrange metric tensor $h_{ij} : TN \\{ y | \xi^b(y) = 0 \} \to R$ is defined by

$$
h_{ij}(x, y) = \frac{\parallel \xi \parallel^2_{\psi}}{[\xi^b(y)]^2} \psi_{ij}(x) = \psi_{ij}(x) \exp \left[ 2 \ln \frac{\parallel \xi \parallel_{\varphi}}{\parallel \xi \parallel_{\varphi^b(y)}} \right].
$$

This case was studied, in other way, by Udriște [34-36].

Replacing $\sigma = 0$, $\tau(x, y) = \ln(\parallel \xi \parallel_{\psi}/|\xi^b(y)|)$, $\varphi_{11} = 1$ and $A_1 = 1$ in the equations $(\ast)$, the following equations will be the equations of these harmonic curves,

$$
\frac{\partial L}{\partial c^i} + \frac{d}{dt} \frac{\partial L}{\partial \dot{c}^i} = 0, \ \forall i = 1, n,
$$

where $L = \delta f^T$, $\dot{c}^i = \dot{c}^i(c(t))$, and $\dot{\xi}^i = \xi^i(c(t))$.
where \( \frac{\partial L}{\partial c^i} = \frac{1}{2} h_{kl} c^k c^l \) and \( \frac{\partial L}{\partial c^i} = e^{2\tau} \left\{ \psi_{kl} \frac{\partial \tau}{\partial c^i} c^k c^l + \psi_{ik} c^i \right\} \).

2. Pfaff systems

If we put \( N = (R, 1) \) and \( T = A \in \Lambda^1(T^*M) \), the PDEs system \((E)\) becomes the Pfaffian system
\[
df = A, \ f \in F(M)
\]
and the functional \( \mathcal{L}_T \) is
\[
\mathcal{L}_T(f) = \frac{1}{2} \int_M \frac{\|A\|^2}{|A(\text{grad}_f f)|^2} \varphi \alpha \beta f_{\alpha} f_{\beta} \sqrt{\varphi} da.
\]
Consequently, the functional \( \mathcal{L}_T \) is a \( \left( g^P \varphi_h \right) \)-energy, where \( g_{\alpha \beta} : TM \backslash \{b \in 0 \} \to R \) is defined by
\[
g_{\alpha \beta}(a, b) = \frac{|A(b)|^2}{\|A\|_{\varphi}^2} \varphi_{\alpha \beta}(a) = \varphi_{\alpha \beta}(a) \exp \left[ 2 \ln \frac{|A(b)|}{\|A\|_{\varphi}} \right].
\]

The form of harmonic maps equations are obtained, in this case, replacing \( \tau = 0 \), \( \sigma(a, b) = \ln(\|A\|_{\varphi}/|A(b)|) \), \( \psi_{11} = h_{11} = 1 \) and \( \xi_{1} = 1 \) in \((**)\). These will be the equations of harmonic maps \((H)\) with \( n = 1 \), where
\[
\frac{\partial L}{\partial f_{\alpha}} = e^{2\tau} \left\{ \varphi^{\gamma \mu} \varphi^{\alpha \beta} \frac{\partial \sigma}{\partial b^x} f_{\gamma} f_{\mu} + \varphi^{\gamma \alpha} f_{\gamma} \right\}, \ \frac{\partial L}{\partial f} = 0.
\]

3. Pseudolinear functions

We suppose that \( T^k(x, a) = \xi^k(x) A_{\beta}(a) \), where \( \xi^k \) is vector field on \( N \) and \( A_{\beta} \) is 1-form on \( M \). In this case the functional \( \mathcal{L}_T \) is expressed by
\[
\mathcal{L}_T(f) = \frac{1}{2} \int_M \frac{\|\xi\|^2}{\|A\|^2} \|A\|^2_{\varphi} \varphi^{\alpha \beta} \psi_{ij} f_{\alpha} f_{\beta} \sqrt{\varphi} da =
\]
\[
= \frac{1}{2} \int_M g^{\alpha \beta}(a, b) h_{ij}(f(a)) f_{\alpha} f_{\beta} \sqrt{\varphi} da,
\]
where \( P_{ij\alpha}(x) = \delta^\beta_\alpha \xi_i(x), \ b^\gamma = \varphi^{\alpha \beta} f^i f^j P_{ij\alpha}, \ h_{ij}(x) = \|\xi\|^2 \psi_{ij}(x) \) and the Lagrange metric tensor \( g_{\alpha \beta} : TM \backslash \{b \in 0 \} \to R \) is defined by
\[
g_{\alpha \beta}(a, b) = \frac{|A(b)|^2}{\|A\|^2_{\varphi}} \varphi_{\alpha \beta}(a) = \varphi_{\alpha \beta}(a) \exp \left[ 2 \ln \frac{|A(b)|}{\|A\|_{\varphi}} \right].
\]

It follows that the functional \( \mathcal{L}_T \) becomes a \( \left( g^P \varphi_h \right) \)-energy.

The equations of harmonic maps can be computed, putting
\[
\tau = \ln \|\xi\|_{\psi}, \ \sigma(a, b) = \ln(\|A\|_{\varphi}/|A(b)|), \ P_{\beta i}^\gamma = \delta^\gamma_\beta \xi_i^\gamma(x),
\]
in \((**)\), where \( \xi_i^\gamma = \psi_{ij} \xi^j \).
In the particular case when we have $M = (R^n, \varphi = \delta)$ and $N = (R, \psi = 1)$, supposing that $\nu f(a) \neq 0$, $\forall a \in M$, the solutions of the above system are the well known pseudolinear functions. These functions have the property that all hypersurfaces of constant level $M \times N$ are totally geodesic. Consequently, the pseudolinear functions are examples of harmonic maps between the generalized Lagrange spaces of type $(M, \gamma)$. Moreover, such spaces are harmonic maps between generalized Lagrange spaces of type $(M, \gamma, \phi)$. \[ \langle \nabla f, \partial \rangle = 1 \]

4. Continuous groups of transformations

The fundamental PDEs system of the group having the infinitesimal generators $\xi_r$ are

$$\frac{\partial f^i}{\partial \theta^a} = \sum_{r=1}^t \xi^i_r(f) A^a_r(a),$$

where $\{\xi_r\}_{r=1}^t \subset \mathfrak{X}(N)$ are vector fields on $N$ and $\{A^a_r\}_{r=1}^t \subset \Lambda^1(M)$ is a family of covector fields on $M$. The geometrical interpretation of solutions via harmonic maps theory is still an open problem, though the Lagrangian

$$L(a^\alpha, f^i, f^i_a) = \frac{1}{2} g^{ij}(a^\alpha, b^\alpha) h_{kl}(x^\mu, y^\nu) f^k_i f^l_j$$

is what we need.

3 Maxwell and Einstein equations

Finally, we remark that, in all above cases, the solutions of the system $\delta f = T$ are harmonic maps between generalized Lagrange spaces of type $(M^n, e^{2\sigma(x,y)} \gamma_{ij}(x))$, where $\sigma : TM \setminus \{\text{Hyperplane}\} \to R$ is a smooth function. These spaces, endowed with the non-linear connection $N^j_i(x, y) = \Gamma^j_{ik}(x)y^k$, where $\Gamma^j_{ik}(x)$ are the Christoffel symbols for the Riemannian metric $\gamma_{ij}(x)$, verify a constructive axiomatic formulation of General Relativity due to Ehlers, Pirani and Schild. Moreover, such spaces represent convenient relativistic models because they have the same conformal and projective properties as the Riemannian space $(M, \gamma_{ij})$.

Denoting by $r^j_{kl}$ the curvature tensor field of the metric $\gamma_{ij}$, by $\gamma^{ij}$ the inverse matrix of $\gamma_{ij}$, $r_{ij} = r_{ijk}^k$, $r = \gamma^{ij} r_{ij}$, $\delta/\delta x^i = \partial/\partial x^i - N^i_j(\partial/\partial y^j)$, $\sigma_i = \delta \sigma/\delta x^i$ and $\bar{\sigma}_i = \partial \sigma/\partial y^i$, we shall use the following notations

$$\sigma^H = \gamma^{kl} \sigma_k \sigma_l, \quad \sigma_{ij} = \sigma_{ij} + \sigma_i \sigma_j - \gamma_{ij} \sigma^H/2, \quad \bar{\sigma} = \gamma^{ij} \sigma_{ij},$$

$$\sigma^V = \gamma^{ab} \bar{\sigma}_a \bar{\sigma}_b, \quad \bar{\sigma}_{ab} = \bar{\sigma}_{ab} = \bar{\sigma}_a \bar{\sigma}_b - \gamma_{ab} \sigma^V/2, \quad \bar{\sigma} = \gamma^{ab} \bar{\sigma}_{ab},$$

where $|i|$ (resp. $|a|$) represents the $h$- (resp. $v$-) covariant derivative induced by the non-linear connection $N^i_j$.

Developing the formalism presented in \[ \[ \text{4}, \text{3} \] \], the following Maxwell’s equations hold

$$\begin{cases}
F_{ij|k} + F_{jk|i} + F_{ki|j} = \sum_{ijkl} g_{ip} q_{jk} \sigma_{h} y^p y^q, \\
F_{ij} + F_{jk|i} + F_{ki|j} = -(F_{ij|k} + f_{jk|i} + f_{ki|j}), \\
f_{ij} = f_{jk|i} + f_{ki|j} = 0,
\end{cases}$$

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where the electromagnetic tensors $F_{ij}$ and $f_{ij}$ are
\[ F_{ij} = (g_{ip} \sigma^p_j - g_{jp} \sigma^p_i) y^p, \quad f_{ij} = (g_{ip} \dot{\sigma}^p_j - g_{jp} \dot{\sigma}^p_i) y^p. \]

Also, the Einstein’s equations will take the form
\[
\begin{cases} 
   r_{ij} - \frac{1}{2} r^r_{rj} + t_{ij} = K T^H_{ij}, \\
   \gamma_{ij} = K T^V_{ij}, 
\end{cases}
\]
where $T^H_{ij}$ and $T^V_{ij}$ are the $h$- and $v$- components of the energy momentum tensor field, $K$ is the gravific constant and
\[
t_{ij} = (n - 2) (\gamma_{ij} - \sigma_{ij}) + \gamma_{ij} r_{st} y^a \gamma_{ip} \dot{\sigma}^p_r + \dot{\sigma}^a_{ij} r_{ta} y^t - \gamma_{ij} \gamma^{ap} \sigma_{r p} \dot{\sigma}_{t a} y^t.
\]

\textbf{Remark.} For the form of generalized Einstein-Yang-Mills equations in a space $(M, c^{2\sigma(x,y)} \gamma_{ij}(x))$, see [1, 5].

Consequently, in certain particular case, it is posible to build a generalized Lagrange geometry naturally attached to a system of PDEs.

\textbf{Open problem.} Is it possible to build a unique generalized Lagrange geometry naturally asociated to a given PDEs system, in the large?

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