Tailoring steep density profile with unstable points

Shun Ogawa†
Laboratory for Neural Computation and Adaptation, RIKEN Center for Brain Science, 2-1 Hirosawa Wako Saitama 351-0198, Japan

Xavier Leoncini
Aix Marseille Univ, Université de Toulon, CNRS, CPT, Marseille, France†

Alexei Vasiliev
Space Research Institute, Profsoyuznaya 84/32, Moscow 117997, Russia

Xavier Garbet
CEA, IRFM, F-13108 St. Paul-lez-Durance cedex, France

The mesoscopic properties of a plasma in a cylindrical magnetic field are investigated from the view point of test particle dynamics. When the system has enough time and spatial symmetries, a Hamiltonian of a test particle is completely integrable and can be reduced to a single degree of freedom Hamiltonian for each initial state. The reduced Hamiltonian sometimes has unstable fixed points (saddle points) and associated separatrices. To choose among available dynamically compatible equilibrium states of the one particle density function of these systems we use a maximum entropy principle and discuss how the unstable fixed points affect the density profile or a local pressure gradient, and are able to create a steep profile that improves plasma confinement.

Let us quickly review the single particle motion and adiabatic chaos. We consider a model of charged particle moving in a non-uniform cylindrical magnetic field $B(r) = V \wedge A_0(r)$. The vector potential $A_0(r)$ is given by

$$
A_0(r) = \frac{B_0 r}{2} \mathbf{e}_0 - B_0 F(r) \mathbf{e}_z, \quad F(r) = \int_0^r \frac{r \, dq}{R_{\text{per}}(q(r))},
$$

where the cylinder is parametrized with the coordinate $(r, \theta, z)$, $B_0$ is strength of the magnetic field, $z$ has $2\pi R_{\text{per}}$-periodicity, $\mathbf{e}_0$ and $\mathbf{e}_z$ are basic units for each direction, and $q(r)$ is a winding number called a safety factor of magnetic field lines. The Hamiltonian of the particle $H = ||v||^2/2$, where $v$ denotes particle’s velocity, has three constants of motion, the energy, the angular momentum, and the momentum, associated with time, rotational, and translational symmetry of the system respectively, so that the Hamiltonian $H$ on the six dimensional phase space is reduced into the single-degree-of-freedom Hamiltonian on the two dimensional phase space, $(r, p_r)$-plane [11][12].

$$
H_{\text{eff}}(r, p_r) = p_r^2/2 + V_{\text{eff}}(r),
$$

$$
V_{\text{eff}}(r) = \frac{v_0^2}{2} + \frac{v_z^2}{2} = \left( \frac{p_{\theta} r^{-1} - B_0 r/2}{2} \right)^2 + \left( p_z + B_0 F(r) \right)^2,
$$

where $p_i$ stands for the conjugate momentum for $i = r, z$, and $\theta$ respectively, and where $v_0 = r \dot{\theta}$ and $v_z = \dot{z}$. The upper dot denotes $d/dt$. Invariants $p_r$ and $p_\theta$ are fixed by the initial condition of the particle in the 6-dimensional phase space. Appropriately setting the safety factor $q$ and choosing initial condition, we can find an unstable fixed point in $(r, p_r)$ phase plane, which can induce the adiabatic chaos [13][17] when the weak magnetic perturbation or the curvature effect added to the flat torus (cylinder) exist[11][12].

Being able to sustain a steep density profile in hot magnetized plasma is one of the major key points to achieve magnetically confined fusion devices. These steep profiles are typically associated with the emergence in the plasma of so-called internal transport barriers (ITB) [1][2]. Both the creation and study of these barriers have generated numerous investigations mostly numerical using either a fluid, or magnetic field or kinetic perspective or combining some of these. In this paper starting from the direct study of particle motion, we propose a simple mechanism to set up a steep profile which may not have been fully considered yet. Indeed, charged particle motion in a non-uniform magnetic field is one of main classical issues of physics of plasmas in space or in fusion reactors. To tackle this problem the guiding center [7] and the gyrokinetic [8] theories are developed to trace the particle’s slower motion by averaging the faster cyclotron motion. These reductions suppress computational cost and they are widely used to simulate the magnetically confined plasmas in fusion reactors [6]. These reduction theories assume existence of an invariant or an adiabatic invariant of motion associated with the magnetic moment. Meanwhile, this assumption does not always hold true. Then, recently, studies on full particle orbits without any reductions are done to look into phenomena ignored by these reductions and to interpolate the guiding center orbit. There exists a case that a guiding center trajectory and a full trajectory are completely different [10]. Further, it is found that the assumption of the invariant magnetic moment breaks [11][12] due to the chaotic motion of the test particles.
This Letter aims to exhibit one possibility that the unstable fixed point inducing chaotic motion modifies mesoscopic properties of plasmas, local density and pressure gradients which are believed to be associated with the internal transport barriers (ITBs) [1] [2] a feature missed by gyrokinetics, or a pure magneto-hydrodynamic approach. For this purpose, we shall compute an equilibrium radial density function ρ(r) from stationary kinetic distribution. When neglecting the feedback on the fields of the motion of the particles governed by the Hamiltonian [2] computing a stationary state of an ensemble of particles, i.e. a stationary one-particle density function, resumes to find a density function \( f_0 \) on the phase space, which commutes with Hamiltonian [2]. Since the motion is integrable, these solutions correspond after a local change to action-angle variables to functions depending only on the actions of the Hamiltonian with a uniform distribution of the associated angles (see for instance for a similar situation [23]). As a consequence, there exist infinitely many steady states. In order to choose one, we may assume a vanishing collisionality, and consider that the one maximizing the information entropy under suitable constraint conditions is picked out [22] [24]. In principle, we should consider a Vlasov-Maxwell system consisting of the collisionless Boltzmann equation describing a temporal evolution of single particle density functions of ions and electrons, coupled with the Maxwell equation determining a self-consistent particle density functions of ions and electrons, and is independent of \( \theta \) and \( z \). We then obtain a radial density function \( \rho (r) \) given by

\[
\rho (r) = \frac{\int \! n(q) r d\theta dz}{\int \! r d\theta dz} = \frac{1}{4\pi^2 r^2 R_{per}} \int \! n(q) r d\theta dz, \tag{9}
\]

as

\[
\rho (r) = \frac{\exp \left( -ar^2 - bF(r) \right)}{\int_0^\infty \exp \left( -ar^2 - bF(r) \right) dr}, \tag{10}
\]

\[a = \frac{\gamma_0}{2} \left( B_0 - \frac{\gamma_0}{\beta} \right), \quad b = -\gamma_z B_0.\]

We can notice from Eq. (10) that the equilibrium profile is not flat as soon as \( \gamma_0 \) is not zero and that it depends on the poloidal magnetic field configuration when \( \gamma_z \neq 0 \). Given the definitions, this means as soon as the plasma moves the profiles are not flat. Since we are considering an equilibrium configuration, we may as well end up with a non-flat temperature profile, but here we have to consider the local radial kinetic temperature of the particles rather than the thermodynamic one [6], so this would correspond to the average of the energy at constant radius. In the same spirit as for the density we can compute the spatial energy density function \( \varepsilon (q) \) is deduced from this result as

\[
\varepsilon (q) = \int \! f_0 H_{eff} d^3p = \int \! f_0 H_{eff} r^{-1} d\theta d\phi, \tag{11}
\]

Thus, we notice that

\[
\varepsilon (q) = \frac{\partial n(q)}{\partial \beta} \tag{12}
\]
so the kinetic temperature profile $T(r)$ is proportional to $\rho(r)$. In the same spirit it should be noted that the local pressure $P(r)$ is proportional to the radial density $\rho(r)$, because we assume the equation of state $P(r) = N\rho(r)T_0$ holds locally true, where $N$ is the number of particles and here we consider the equilibrium temperature $T_0$.

We now move on and consider how the existence of the unstable fixed points with relevant energy level affects the obtained equilibrium density profile. For this purpose we have to discuss how the safety factor is chosen, in other words which function $F$ in Eq. (1) leads to the emergence of “practical” unstable points in the effective potential $V_{\text{eff}}$. Indeed, as a first point to pin out, if the amplitude of $F$ is large, we can expect that the term $v_z^2/2 = (p_z + F)^2/2$ in the Hamiltonian (2) becomes also large, then the unstable points appear in the phase space at so high energy level that they become physically irrelevant. Therefore, the amplitude of $F$ should be small.

Moreover if the variations of $F(r)$ are smooth and “gentle” with $r$, so does again $(p_z + F)^2/2$ in $V_{\text{eff}}$, then $V_{\text{eff}}$ has only one minimum point that is essentially governed by the term $(p_0/r - B_0 r/2)^2/2$. Thus, enough concavity of $v_z^2/2$ near but not at the minimum point of $(p_0/r - B_0 r/2)^2/2$, $r = \sqrt{2p_0/B_0}$ is necessary so that $V_{\text{eff}}$ has unstable points. These considerations are illustrated in Fig. 1. In the panel (d), we assume that there exists an $r$ such that $p_z + F(r) = 0$. We stress out as well that if $|p_z|$ is sufficiently large, it is also possible to create an unstable point, but then again the energy level is so high that it is irrelevant for the mesoscopic profiles in considered plasmas.

Given the obtained density profile (10), we can notice that a sudden fast variation of the function $F$, will lead to strong variations of the profile, as long as $r$ is not too large, for instance a step like profile should translate in a steep profile. With this in mind, since this effect is present if $\gamma_z$, related to the average velocity along the cylinder axis, we may expect that in the context of magnetic fusion with machines with large aspect ratios the presence of zonal flows along the toroidal direction is important to increase confinement. Going back to our simple model, since the safety factor $q(r)$ can be directly associated with the function $F(r)$ at the origin of the unstable fixed points, and $q(r)$ is a crucial parameter for the operation of magnetized fusion machine, let us discuss more how the constraints discussed previously translate on the $q$-profile. For instance let us consider a situation with a non-monotonous profile such that $q(r)$ has a minimum $q_0$ at $r = \alpha$ and the spatial scale is characterized with $\lambda$. Then locally $q(r)$ can be expressed as

$$q(r) = q_0 \left[ 1 + \lambda^2 (r - \alpha)^2 \right], \quad r \sim \alpha. \quad (13)$$

Recalling Eq. (1), the function $F$ is scaled as $q_0^{-1}\lambda^{-1}$, and $v_z^2/2 = (p_z + F)^2/2 \sim q_0^{-2} \lambda^{-2}$, so that this provides a typical energy level of the particles located near a separatrix. It should be noted that the width of the well of $v_z^2/2$ scales as $\lambda^{-1}$ (see Fig. 1). For a fixed value of $q_0$, a large value of $\lambda$ creates unstable fixed points with relevant energy levels for the particle whose angular momentum is $p_0 \sim B_0 a^2/2$. As $\lambda$ gets to be larger, the number of the particles with unstable fixed point increase. This is because, roughly speaking, the energy levels of unstable points get to be lower, and the one-particle density (5) is proportional to $e^{-\beta H_{\text{eff}}}$. As a consequence of these considerations we illustrate on Fig. 2 how to adjust a given $q$-profile in order to create unstable fixed points whose location is $r \sim \alpha$. One can set up unstable fixed point around $r \sim \alpha$ by modifying the $q$-profile so that it has concavity around $r = \alpha$.

We then have a form of density profile (10) and a condition for $q$-profile exhibiting unstable fixed points. We next consider where the unstable points appear, and we shall exhibit that the emergence of unstable fixed points induces the presence of a local steep profile in their vicinity, i.e. their radial positions are inducing the existence of locally strong density gradients. For this purpose we simply consider the $q$-profile given by Eq. (13) with parameters $q_0 = 0.12$, $\lambda = 55$, and $\alpha = \sqrt{0.18} \approx 0.4243$, in Figs. 3. When including perturbations, we point out that the adiabatic chaos due to separatrix crossing in this magnetic field has been discussed in Ref. [11]. The results are displayed in Fig. 3 where two density profiles (10) obtained for two $q$-profiles with and without unstable fixed points are shown. The parameter $\alpha$ is changed so that they have same density in the center of cylinder. We note $\alpha$ can be changed keeping the thermodynamical temperature $\beta^{-1}$ and changing the average of angle velocity $v_\theta$. We find the steep region which corresponds to the local steep density gradient around $r = \alpha$ on which $q(r)$ satisfies $q'(r) = 0$ for the $q$-profile with unstable points. Going further on, reasoning directly with a $q$-profile

FIG. 1. Schematic picture showing how the saddle point appears around the bottom of safety factors. A Panel (a) represents a part of $v_z^2/2 = (p_0/r - B_0 r/2)^2/2$, (b) $V_{\text{eff}}$, (c) $F(r)$, and (d) $v_z^2/2 = (p_z + F(r))^2/2$. In the panel (d), we consider that there exists an $r$ such that $p_z + F(r) = 0$. The parameters $q_0$, $\lambda$, and $\alpha$ are associated with Eq. (13).
or

2. (a) and (b) correspond respectively to the density profiles with unstable fixed points. The parameters $q = 55$ induces the unstable fixed points, so does not one with $q = 2$. (a) and (b) correspond respectively to the $q$-profiles without and with unstable fixed points.

![Graph of $q(r)$](image)

**FIG. 2.** Schematic picture exhibiting how to create unstable fixed point by modifying $q$-profile. The solid curve represents the given $q$-profile. Dotted and bold curves are the graph of $q_0(1 + \lambda^2 (r - a)^2)$ for $q_0 = 0.12$, $a = \sqrt{0.18}$ and $\lambda = 2, 55$ respectively. The $q(r)$ with $\lambda = 55$ induces the unstable fixed points, so does not one with $\lambda = 2$. (a) and (b) correspond respectively to the $q$-profiles without and with unstable fixed points.

may lead to some physical problems. For a given magnetic configuration inspired from a tokamak, the toroidal component of the magnetic field is generated by the current within the plasma. In the large aspect ratio (cylindrical) limit, this readily gives a $r$ dependence of the current

$$j(r) = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial F}{\partial r} \right). \quad (14)$$

**FIG. 3.** (Color online) The solid and dotted curves express the density profiles with $q$-profiles with unstable points (shaded region) and without unstable points respectively. The parameters in a local $q$-profile [Eq. (13)] are determined as $q_0 = 0.12$, and $a = \sqrt{0.18} = 0.4243$, and $\lambda = 55$ for the solid curve and $\lambda = 2$ for the dotted one. In both profiles, $\gamma_z$ and $\beta$ are same, but $\gamma_\theta$ is different.

When looking at the current profile given by the $q$-profile giving rise to Fig. 3, we end up with two different region one, with a negative current for $r$ sufficiently large, and one with a positive one for small $r$. This is likely to be physically not realistic. Even though it may have appeared as possible, the actual stability of these configurations is doubtful, see for instance [29][31]. In order to generate a profile that could be more physically relevant, we may have again a look at Eq. (10). We can notice that a “steepest” variation of the function $F(r)$ will likely trigger a steepness in the density profile. Taking into account Eq. (14), we can look for profiles that can achieve this variation while keeping a positive current. Given the form of Eq. (14), it appears as simpler to look for just one Bessel function of the first kind $F = F_0 J_\nu (\lambda r)$ and to create only one separatrix, we consider the case when no plasma current is present, then the minimum of the effective potential $\lambda = 55, \alpha = 4, b = 1$, which leads to ex-

![Graph of $V_{eff}(r)$](image)

**FIG. 4.** (Color online) Effective potential for $F(r) = F_0 J_2(\lambda r)$. An Unstable point appears as expected around $r_0 = 0.6$. Here we choose the parameters as $F_0 = 1$, $p_0 = 0.5$, $p_2 = 1$, $B_0 = 4$. 

Before concluding this letter we would like to make some remarks on the observed profiles which indicate the presence of what we may call an ITB although the underlying physical mechanisms are quite different. We would like to stress out the similarities our observations and the magnetic ITBs discussed for instance in [19]. One of the main difference is that if the plateau of $q(r)$ appears the transport barrier emerges even if the value of $q$ is far from rational number $m/n$ with small integers $m$ and $n$, in that sense the creation of the barrier with the $q$-profile [15] is not correlated to the existence of a resonant surface, on the other hand for
the considered example, we can notice that the location of the magnetic ITB coincides with the place on which the local density gradient exists.

Another study between the particle’s motion and the existence of an ITB using a pure field line approach and the existence of a stable magnetic tori has been performed from the viewpoint of the difference between the magnetic winding number \( q(r) \) and the effective one \( q_{\text{eff}}(r) \) for the guiding center orbit of the energetic particles [25]. In a recent study [26], it is shown that the resonance shift due to the grad-\( B \) drift and its disappearance due to the curvature drift effect can create an invariant tori in the particle dynamics while there are none for magnetic field lines, and this is confirmed both analytically and numerically with the full particle orbits around the resonance points. It should be remarked that the guiding center theory is useless to clarify it unlike Ref. [26].

The above consideration provides de facto another difference between the magnetic ITB and the effective ITB induced by the separatrices. Moreover, we can stress out that the magnetic ITB is present in both situations described in Fig. 3 while the steep profile occurs only when the hyperbolic points are present. When considering the degenerate \( q \)-profile we also note that the two unstable fixed points appear around the magnetic ITB and when the parameter \( \lambda \) in Eq. (15) gets to be large, the steep region of \( \rho(r) \) gets to be strong as the influence of the separatrix grows because the gap of \( F(r) \) between \( r < a \) and \( r > a \) becomes smaller (see Fig. 1).

To conclude, we have shown in this letter that steep equilibrium density profiles can emerge due to the presence of a separatrix in the passive particle orbits, this phenomenon is not related to the existence of a local resonant surface and the observed phenomenon is reminiscent of the presence of an ITB although the physical mechanisms inducing it are a priori quite different in interpretation. We also discussed how the \( q \)-profile can be tuned in order to generate such barriers.

We finally remark on what happens if we consider a toroidal configuration. Let us imagine our cylindrical system is an infinite toroidal radius limit of the toroidal system as Ref. [12]. The finite toroidal radius effect breaks the integrability and it thus can induces adiabatic chaos. It has been known for a while that the presence of chaos affects sometimes the density profile locally. For instance, the averaging effect in plasmas from the global chaos induced by the resonance overlapping [27] has been found and it modifies the density profile [28]. Even though we are directly tackling the passive particle motions of ions and thus a different type of localized chaos in this letter, similar things can be expected. In the present case the unstable fixed points are located around the place in which the steepness of density [10] and the local pressure gradient exist as shown in Fig. 3. Therefore the flattening effect makes them steeper locally. As a result we can expect that this steepening effect observed in the cylindrical configuration can be robust at least as long as strongly chaotic motion remains localized near each hyperbolic point.

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[1] R. C. Wolf, Plasma Phys. Control. Fusion 45 (2003) R1.

[2] J. W. Connor, T. Fukuda, X. Garbet, C. Gormezano, V. Mukhovatov, M. Wakatani, ITB Database Group, and ITPA Topical
Group on Transport and Internal Barrier Physics, Nucl. Fusion 44 (2004) R1.

[3] H. Alfvén, Ark. Mat. Astron. Fys. 27A (1940) 1; Cosmological electrodynamics, (Oxford university press, London, 1950).

[4] T. G. Northrop, Ann. Phys. 15 (1961) 79.

[5] R. G. Littlejohn, Phys. Fluids 24 (1981) 1730.

[6] A. H. Boozer, Rev. Mod. Phys. 76 (2004) 1071.

[7] J. R. Cary and A. J. Brizard, Rev. Mod. Phys. 81 (2009) 693.

[8] A. J. Brizard and T. S. Hahm, Rev. Mod. Phys., 79 (2007) 421.

[9] J. D. Jackson, Classical Electrodynamics, 3rd ed. (Wiley, USA, 1998).

[10] D. Pfefferlé, J. P. Graves, and W. A. Cooper, Plasma Phys. Controlled Fusion 57 (2015) 054017.

[11] S. Ogawa, B. Cambon, X. Leoncini, M. Vittot, D. del-Castillo-Negrete, G. Dif-Pradalier, and X. Garbet, Phys. Plasmas 23 (2016) 072506.

[12] B. Cambon, X. Leoncini, M. Vittot, R. Dumont, and X. Garbet, Chaos 24 (2014) 033101.

[13] A. I. Neishtadt, Sov. J. Plasma Phys. 12, 568 (1986).

[14] A. I. Neishtadt, Prikl. Matem. Mekhan. USSR 51 (1987) 586.

[15] J. L. Tennyson, J. R. Cary, and D. F. Escande, Phys. Rev. Lett. 56 (1986) 2117.

[16] J. R. Cary, D. F. Escande, and J. L. Tennyson, Phys. Rev. A 34 (1986) 4256.

[17] X. Leoncini, A. Neishtadt, and A. Vasiliev, Phys. Rev. E 79 (2009) 026213.

[18] R. Balescu, Phys. Rev. E 58 (1998) 3781.

[19] D. Constantinescu and M.-C. Firpo, Nucl. Fusion 52 (2012) 054006.

[20] A. A. Vlasov, Zh. Eksp. Ther. Fiz. 8 (1938) 291; Sov. Phys. Uspekhi 93 (1968).

[21] L. P. Pitaevskii and E.M. Lifshitz, Physical Kinetics (Butterworth-Heinemann, Oxford, 1981).

[22] T. M. Rocha Filho, A. Figueiredo, and M A. Amato, Phys. Rev. Lett. 95 (2005) 190601.

[23] X. Leoncini, T. L. Van den Berg and D. Fanelli, EPL 86 (2009) 20002.

[24] D. Zubarev, V. Morozov, and G. Röpke, Statistical Mechanics of Nonequilibrium Processes, Volume 1: Basic concepts, kinetic theory (Academic Verlag, Berlin, 1996).

[25] G. Fiksel, B. Hudson, D. J. Den Hartog, R. M. Magee, R. O’Connell, and S. C. Prager, Phys. Rev. Lett. 95 (2005) 125001.

[26] S. Ogawa, X. Leoncini, G. Dif-Pradalier, and X. Garbet, Phys. Plasmas 23 (2016) 122510.

[27] B. V. Chirikov, Phys. Rep. 52 (1979) 263.

[28] R.B. White, Commun. Nonlinear Sci. Numer. Simulat. 17 (2012) 2200; Plasma Phys. Control. Fusion 53 (2011) 085018.

[29] G. T. A. Huysmans, T. C. Hender, N. C. Hawkes, and X. Litaudon, Phys. Rev. Lett. 87 (2001) 245002.

[30] G. W. Hammett, S. C. Jardin, and B. C. Stratton, Phys. Plasmas 10 (2003) 4048.

[31] Yu. I. Pozdnyakov, Plasmas 12 (2005) 084503.