INVARIANT MEASURES OF MODIFIED LR AND L+R SYSTEMS

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Abstract. We introduce a class of dynamical systems having an invariant measure, the modifications of well known systems on Lie groups: LR and L+R systems. As an example, we study modified Veselova nonholonomic rigid body problem, considered as a dynamical system on the product of the Lie algebra so(n) with the Stiefel variety V_{n,r}, as well as the associated νL+R system on so(n) × V_{n,r}. In the 3-dimensional case, these systems model the nonholonomic problems of a motion of a ball and a rubber ball over a fixed sphere.

1. Introduction

This paper describes a class of dynamical systems allowing an invariant measure, a modification of LR systems introduced by Veselov and Veselova [21] and L+R systems introduced by Fedorov (see [14, 13]). In particular, they model the nonholonomic problems of rolling the Chaplygin ball and the rubber Chaplygin ball over a spherical surface.

Recall that the motion of the Chaplygin (balanced, dynamically asymmetric) ball over a fixed spherical surface is described by the equations

\begin{align}
\frac{d}{dt} \vec{k} &= \vec{k} \times \vec{\omega}, \\
\frac{d}{dt} \vec{\gamma} &= \epsilon \vec{\gamma} \times \vec{\omega},
\end{align}

where \( \vec{\omega} \) is the angular velocity of the ball, \( \vec{\gamma} \) is the unit vector directed from the center of the fixed sphere to the point of contact, \( \vec{k} = I \vec{\omega} + D\vec{\omega} - D(\vec{\omega}, \vec{\gamma}) \vec{\gamma} \) is the momentum of the ball with respect the contact point, and \( I = \text{diag}(I_1, I_2, I_3) \) is the inertia operator of the ball (e.g., see [6, 7]).

The parameters \( D \) and \( \epsilon \) are equal to \( m^2 \rho \) and \( \sigma/(\sigma \pm \rho) \), where \( m \) is the mass of the rolling ball, \( \rho \) its radius and \( \sigma \) is the radius of the fixed sphere. The sign “+” denotes the rolling over outer surface of the fixed sphere, while the sign “−” denotes either the rolling over inner surface of the fixed sphere (\( \sigma > \rho \)), or the case where the fixed sphere is within the rolling ball (\( \sigma < \rho \), in this case the rolling ball...
is actually a spherical shell. As $\sigma$ tends to infinity, $\epsilon$ tends to 1, and we obtain the equation of the rolling of the Chaplygin ball over a horizontal plane.

In the space $(\vec{\omega}, \vec{\gamma})$, the density $\mu$ of an invariant measure is equal to
\[
\mu = \sqrt{\det(I + D\mathcal{E})(I - D(\vec{\gamma}, (I + D\mathcal{E})^{-1}\vec{\gamma}))},
\]
the expression given by Chaplygin for $\epsilon = 1$ [9], and by Yaroshchuk for $\epsilon \neq 1$ [22].

Through the paper the operator $\mathcal{E}$ denotes the identity operator on the appropriate spaces.

Similarly, the motion of the rubber Chaplygin’s ball over a fixed spherical surface is described by the equations (Ehlers and Koiller [11], see also Borisov and Mamaev [5, 6])
\[
\frac{d}{dt}\vec{m} = \vec{m} \times \vec{\omega} + \lambda \vec{\gamma}, \quad \frac{d}{dt}\vec{\gamma} = \epsilon \vec{\gamma} \times \vec{\omega}, \quad (\vec{\omega}, \vec{\gamma}) = 0,
\]
where $\vec{m} = (I + D\mathcal{E})\vec{\omega} = \mathcal{L}\vec{\omega}$, $\lambda = (\vec{m}, I^{-1}\vec{\gamma})/(\vec{\gamma}, I^{-1}\vec{\gamma})$. Here the constraint $(\vec{\omega}, \vec{\gamma}) = 0$ model the “rubber” property of the ball: the rotations of the ball around the normal to the spherical surface at the contact point are forbidden.

Note that the above system is well defined on the whole phase space $(\vec{m}, \vec{\gamma})$, and $(\vec{\omega}, \vec{\gamma}) = const$ denotes its first integral. In the space $(\vec{m}, \vec{\gamma})$, or $(\vec{\omega}, \vec{\gamma})$, the system has an invariant measure with the density $\mu_{\epsilon}$ given by
\[
\mu_{\epsilon} = (I^{-1}\vec{\gamma}, \vec{\gamma})^{\frac{1}{2}},
\]
see [21, 10] for $\epsilon = 1$, and [11] for $\epsilon \neq 1$.

The existence of an invariant measure for some nonholonomic systems is an important property related to the geometry of the problem, its Hamiltonization, as well as to the possibility to solve the problem by quadratures (e.g., see [1, 2, 3, 10, 15, 16, 20, 23]).

We shall prove that the densities [2] and [11] are particular cases of the densities of invariant measures of $\epsilon$–modified L+R and LR systems, respectively (see Theorems 1, 3 and 5, and Examples 1 and 2). Further, as a specific example, we give the expression of an invariant measure for the modified Veselova nonholonomic rigid body problem (Theorem 4), considered as a dynamical system on the product of the Lie algebra $so(n)$ with the Stiefel variety $V_{n,r}$, as well as the expression of an invariant measure for the associated $\epsilon$L+R system on $so(n) \times V_{n,r}$ (Theorem 6). For $r = 1$, these systems represent natural multidimensional generalizations of the equations [1] and [3].

2. LR AND L+R SYSTEMS

2.1. LR systems. Let $G$ be a $n$–dimensional compact connected Lie group $G$, $\mathfrak{g} = \text{Lie}(G)$ its Lie algebra, $\langle \cdot, \cdot \rangle$ an $\text{Ad}_{G}$-invariant scalar product on $\mathfrak{g}$, and let $\langle \cdot, \cdot \rangle_1$ be a left-invariant metric on $G$ given by the positive definite operator (called inertia operator) $I : \mathfrak{g} \to \mathfrak{g}^* \cong \mathfrak{g}^* : \langle \eta_1, \eta_2 \rangle_1 = \langle I(g^{-1}\eta_1), g^{-1}\eta_2 \rangle \eta_1, \eta_2 \in T_g G$. Here we identify $\mathfrak{g}$ and $\mathfrak{g}^*$ by $\langle \cdot, \cdot \rangle$. 


The LR system on \( G \) is a nonholonomic Lagrangian system \((G, L, D)\), where \( L = \frac{1}{2}(\dot{g}, \dot{g})_{\Omega} \) is a left-invariant Lagrangian and \( D \) is a right-invariant nonintegrable distribution on the tangent bundle \( TG \), determined by its restriction \( \mathfrak{d} \) to the Lie algebra \( \mathfrak{g} \). Let \( \mathfrak{h} \) be the orthogonal complement of \( \mathfrak{d} \) with respect to \( \langle \cdot, \cdot \rangle \). Then the right-invariant constraints, in the left-trivialization of \( TG \), can be written as \( \omega \in \text{Ad}_{g^{-1}} \mathfrak{d} \), or \( \langle \omega, \text{Ad}_{g^{-1}} \mathfrak{h} \rangle = 0 \), where \( \omega = g^{-1} \cdot \dot{g} \) is the angular velocity.

The LR system \((G, L, D)\) is described by the Euler–Poincaré–Chetaev equations on \( T^*G(m, g) \) (or \( TG(\omega, g) \))

\[
m = [m, \omega] + \sum_{s=1}^{k} \lambda^s e_s, \quad \dot{g} = g\omega,
\]

where \( m = [\omega] \in \mathfrak{g}^* \cong \mathfrak{g} \) is the angular momentum, \( e_s = g^{-1} \cdot E_s \cdot g, E_1, \ldots, E_k \) is the orthonormal base of \( \mathfrak{h} \), and \( \lambda^s \) are Lagrange multipliers which can be found by differentiating the constraints \( \phi_s = \langle \omega, e_s \rangle = 0 \), \( s = 1, \ldots, k \). These equations define a dynamical system on the whole cotangent bundle \( T^*G \), and the right-invariant constraint functions \( \phi_s \) are its first integrals. Also, to the system \((5)\) we can associate a closed system on \( \mathfrak{g}^{m+1}(m, e_1, \ldots, e_k) \):

\[
m = [m, \omega] + \sum_{s=1}^{k} \lambda^s e_s, \quad \dot{e}_s = [e_s, \omega].
\]

The LR system \((6)\) possesses an invariant measure \( \mu\, dm \wedge de_1 \wedge \cdots \wedge de_k \) with density (see \(21\))

\[
\mu = \sqrt{\det(e_s, \Gamma^{-1} e_l)}, \quad s, l = 1, \ldots, k,
\]

implying that the original system \((5)\) has an invariant measure \( \sqrt{\det(\Gamma^{-1}g^{-1}h_{\mathfrak{g}})}\, \Omega^n \). Here \( \Gamma^{-1}g^{-1}h_{\mathfrak{g}} \) is the restriction of the inverse inertia tensor to the linear space \( g^{-1}h_{\mathfrak{g}} \subset \mathfrak{g} \), by \( dm, de_i \) we denoted the standard measures on \( \mathfrak{g} \) with respect to the metric \( \langle \cdot, \cdot \rangle \), and \( \Omega \) is the standard symplectic form on \( T^*G \).

2.2. L+R systems. In addition to the inertia operator \( \mathbb{I} \) defining the left-invariant metric \((\cdot, \cdot)_{\mathbb{I}}\), introduce a constant linear operator \( \Pi^0 : \mathfrak{g} \to \mathfrak{g} \) defining a right-invariant scalar product \((\cdot, \cdot)_{\Pi} \) on \( G \):

\[
(\eta_1, \eta_2)_{\Pi} = (\Pi^0 \eta_1 g^{-1}, \eta_2 g^{-1}) = (\Pi^0 g \eta_1, \eta_2 g^{-1}), \quad \Pi_g = \text{Ad}_{g^{-1}} \circ \Pi^0 \circ \text{Ad}_g,
\]

\( \eta_1, \eta_2 \in T_gG \). We suppose that \( \kappa_g = \mathbb{I} + \Pi_g \) is nondegenerate and positive definite on the whole group \( G \). The L+R system on \( G \) is defined as a dynamical system

\[
\frac{d}{dt} (\mathbb{I} \omega + \Pi_g \omega) = [\mathbb{I} \omega + \Pi_g \omega, \omega], \quad \dot{g} = g \cdot \omega.
\]

This is the modification of the geodesic motion on the group \( G \) with respect to the metric \((\cdot, \cdot)_{\mathbb{I}} + (\cdot, \cdot)_{\Pi} \), by rejecting the term \( [\omega, \Pi_g \omega] \) at the right hand side of the first equation in \((5)\).

In the view of the definition of \( \Pi_g \), its evolution is given by the matrix equation \( \dot{\Pi}_g = \Pi_g \circ \text{ad}_\omega + \text{ad}_\omega^T \circ \Pi_g \). Note that for compact group we have \( \text{ad}_\omega^T = - \text{ad}_\omega \), and
therefore, we get a closed system

\[
\frac{d}{dt} ([\omega + \Pi \omega]) = [[\omega + \Pi \omega, \omega],
\]

\[
\Pi = [\Pi, \text{ad}_\omega]
\]
on \mathfrak{g} \times \text{Sym}(\mathfrak{g})\), where \(\text{Sym}(\mathfrak{g})\) is the space of symmetric operators on \(\mathfrak{g}\), which we also refer as a L+R system. It possesses the kinetic energy integral \(\frac{1}{2} \langle [\omega + \Pi \omega, \omega] \rangle\) and an invariant measure \(\mu \, d\omega \wedge d\Pi\) with density

\[
\mu = \sqrt{\det(I + \Pi)},
\]

where \(d\Pi\) is the standard measure on \(\text{Sym}(\mathfrak{g})\) [14]. It appears that every L+R system can be seen as a reduced system of a certain LR system on a direct product \(G \times \mathfrak{g} \cdot n\), where \(\mathfrak{g}\) is considered as a commutative group (see Theorem 3.3 in [18]).

3. Modified LR systems

As the sphere–sphere problems [1], [3] suggest, it is natural to consider modifications of the equations (6) and (9). We define the \(\epsilon\)LR system on the space \(\mathfrak{g}^{k+1}(\omega, e_1, \ldots, e_k)\), or \(\mathfrak{g}^{k+1}(m, e_1, \ldots, e_k)\), by the equations

\[
\dot{m} = [m, \omega] + \Lambda, \quad \Lambda = \sum_{i=1}^{k} \lambda_i e_i, \quad m = I\omega,
\]

\[
\dot{e}_i = \epsilon [e_i, \omega],
\]

\(i = 1, \ldots, k\). The term \(\Lambda\) can be interpreted as a reaction force and it is determined from the condition that the trajectories \((\omega(t), e_1(t), \ldots, e_k(t))\) satisfy constraints

\[
\phi_i = \langle \omega, e_i \rangle = e_i = \text{const}, \quad i = 1, \ldots, k.
\]

By differentiating (12), we obtain the linear system

\[
\epsilon \langle [e_i, \omega], \omega \rangle + \langle e_i, I^{-1}[m, \omega] \rangle + \sum_j \lambda_j A_{ij} = 0, \quad A_{ij} = \langle e_i, I^{-1}e_j \rangle,
\]

and we get the Lagrange multipliers in the form

\[
\lambda_i = -\sum_j \langle e_j, I^{-1}[m, \omega] \rangle A_{ij}^j.
\]

Here \(A_{ij}^j\) is the inverse of the matrix \(A_{ij} = \langle I^{-1}e_i, e_j \rangle\), \(i, j = 1, \ldots, k\). In particular, \(\Lambda\) does not depend on \(\epsilon\).

**Theorem 1.** The \(\epsilon\)LR system (10), (11), (14) has an invariant measure

\[
\mu_{\epsilon} \, dm \wedge de_1 \wedge \cdots \wedge de_k, \quad \text{i.e.,} \quad \mu_{\epsilon} \, d\omega \wedge de_1 \wedge \cdots \wedge de_k
\]

where the density is given by

\[
\mu_{\epsilon} = (\Delta)^{\frac{1}{2}}, \quad \Delta = \det(A_{ij}) = \det((I^{-1}e_i, e_j)), \quad i, j = 1, \ldots, k.
\]
Proof. Consider the vector field
\[ X = (\dot{m}, \dot{e}_1, \ldots, \dot{e}_k) = ([m, \omega] + \Lambda, e[e_1, \omega], \ldots, e[e_k, \omega]). \]
The volume form \((15)\) is invariant with respect to the flow of \((10)\) if and only if
\[ \mathcal{L}_X \mu_\epsilon \, dm \wedge de_1 \wedge \cdots \wedge de_k = 0, \]
i.e., if \(\mu_\epsilon\) satisfies the Liouville equation
\[ \mu_\epsilon (\text{div}(\dot{m}) + \text{div}(\dot{e}_1) + \cdots + \text{div}(\dot{e}_k)) + \dot{\mu}_\epsilon = 0, \]
where \(\text{div}(\cdot)\) is the standard divergence on \(g\). It is clear that \(\text{div}(\dot{e}_i) = 0, i = 1, \ldots, k\).
In Theorem 1 [21], it is proved that
\[ \text{div}(\dot{m}) = \text{div}(\Lambda) = -\sum_{i,j=1}^k A^{ij} \langle [I^{-1} e_i, I^{-1} m], e_j \rangle. \]
For any regular matrix \(A\) we have the identity
\[ \frac{d}{dt} \det A = \det A \text{tr}(A^{-1} \dot{A}). \]
Thus,
\[
\dot{\mu}_\epsilon = \frac{1}{2\epsilon} (\Delta)^{\frac{1}{2}} \dot{\Lambda} = \frac{1}{2\epsilon} \mu_\epsilon \sum_{i,j=1}^k A^{ij} \frac{d}{dt} \dot{A}_{ji} \\
= \frac{1}{2} \mu_\epsilon \sum_{i,j=1}^k A^{ij} \left( \langle I^{-1} e_i, [e_j, \omega] \rangle + \langle I^{-1} e_j, [e_i, \omega] \rangle \right) \\
= \mu_\epsilon \sum_{i,j=1}^k A^{ij} \langle I^{-1} e_i, [e_j, \omega] \rangle,
\]
which together with \((18)\) implies \((17)\). Since \(dm = \det I \cdot d\omega = \text{const} \cdot d\omega\), if instead of \(m\) we take the variable \(\omega\), the expression for an invariant measure remains the same. \(\square\)

3.1. Momentum equation. As above, let \(E_1, \ldots, E_k\) be an orthonormal base of \(\mathfrak{h}\). Further, let \(E_{k+1}, \ldots, E_n\) be an orthonormal base of \(\mathfrak{d}\) and \(O_{E_i}\) be the adjoint orbit of \(E_i, i = 1, \ldots, n\). It is clear that we can also consider the \(\epsilon LR\) system \((10), (11), (14)\) on the space
\[ M = \{ (\omega, e_1, \ldots, e_n) \in \mathfrak{g} \times O_{E_1} \times \cdots \times O_{E_n} | \langle e_i, e_j \rangle = \delta_{ij} \}, \]
simply by taking \(i = 1, \ldots, n\) in the equation \((11)\). Then it has an invariant measure
\[ \mu_\epsilon \, d\omega \wedge de_1 \wedge \cdots \wedge de_n|_M. \]
Moreover, on \(M\) we have well defined orthogonal projections \(pr_H, pr_D\) (\(pr_H + pr_D = E\)) onto the complementary subspaces of \(g\):
\[ H = \text{span}\{e_1, \ldots, e_k\}, \quad D = \text{span}\{e_{k+1}, \ldots, e_n\}, \]
which, according to (11), satisfy the equations

\[
\frac{d}{dt} \text{pr}_D = \epsilon [\text{pr}_D, \text{ad}_\omega], \quad \frac{d}{dt} \text{pr}_H = \epsilon [\text{pr}_H, \text{ad}_\omega].
\]

Note that with the above notation, we can express \(\Lambda\) in (10) as

\[
\Lambda = -A^{-1} \text{pr}_H \Pi^{-1} [I_\omega, \omega],
\]

where \(A^{-1}: H \to H\) is the inverse of the mapping \(A = \Pi^{-1}|_H = \text{pr}_H \circ \Pi^{-1}: H \to H\).

Following [15], let us introduce the momentum

\[
m = \text{pr}_D \Pi \omega + \text{pr}_H \omega = J\omega, \quad J = \text{pr}_D \Pi + \text{pr}_H = E + \text{pr}_D (\Pi - E).
\]

Consider the space

\[
\mathcal{N} = \{ (m, e_{k+1}, \ldots, e_n) \in g \times \mathcal{O}_{E_{k+1}} \times \cdots \times \mathcal{O}_{E_n} \mid \langle e_i, e_j \rangle = \delta_{ij} \}.
\]

**Proposition 2.** The \(\epsilon\)LR system on \(\mathcal{N}\) has the following form

\[
\begin{align*}
\dot{m} &= \epsilon [m, \omega] + (1 - \epsilon) \text{pr}_D [\Pi \omega, \omega], \\
\dot{e}_i &= \epsilon [e_i, \omega], \quad i = k + 1, \ldots, n.
\end{align*}
\]

**Proof.** From the equations (10) and (21), the evolution of the momentum \(m\) reads

\[
\begin{align*}
\dot{m} &= \frac{d}{dt} (\text{pr}_D \Pi \omega + \text{pr}_H \omega) \\
&= \epsilon (\text{pr}_D [\omega, \Pi \omega] - [\omega, \text{pr}_D \Pi \omega]) + \text{pr}_D ([\Pi \omega, \omega] + \Lambda) \\
&\quad + \epsilon (\text{pr}_H [\omega, \omega] - [\omega, \text{pr}_H \omega]) + \text{pr}_H \dot{\omega} \\
&= \epsilon [m, \omega] + (1 - \epsilon) \text{pr}_D [\Pi \omega, \omega] + \text{pr}_H \dot{\omega}.
\end{align*}
\]

On the other hand, from (12) it follows: \(\epsilon ([e_i, \omega], \omega) + \langle e_i, \dot{\omega} \rangle = \langle e_i, \dot{\omega} \rangle = 0\), \(i = 1, \ldots, k\). Whence

\[
\text{pr}_H \dot{\omega} = \sum_{i=1}^{k} \langle \omega, e_i \rangle e_i = 0.
\]

Finally note, since the linear operator \(J\) is invertible, \(\omega = J^{-1}m\) and we have a closed system (22) on \(\mathcal{N}\). \(\square\)

In the case \(\epsilon = 1\), the first equation in (22) reduces to the momentum equation (2.7) of [15]. Also, as in Theorem 3.2 [15], we have

\[
\det \Pi \cdot \det A = \det \Pi \cdot \det (\Pi^{-1}|_H) = \det J = \det (I|_D),
\]

where \(\Pi|_D = \text{pr}_D \circ \Pi: D \to D\). Therefore:

\[
\mu_\epsilon \, d\omega = \mu_\epsilon \, \frac{\partial \omega}{\partial m} \, dm = \left( \det (\Pi^{-1}|_H) \right)^{\frac{1}{2}} \left( \det J \right)^{-1} \, dm
\]

Combining the expression of the invariant measure (20) on the space \(\mathcal{M}\) and the above identity, we get the following statement:
Theorem 3. The $\epsilon LR$ system \((22)\) has an invariant measure
\[ \tilde{\mu} \, dm \wedge de_{k+1} \wedge \cdots \wedge de_n | \mathcal{X}, \]
where the density is given by
\[ \tilde{\mu} = (\det I|_D)^{\frac{1}{2}} = (\det (\langle I e_i, e_j \rangle))^{\frac{1}{2}}, \quad i, j = k + 1, \ldots, n. \]

Example 1. Let us consider the case when $\mathcal{H}$ is the isotropy algebra of $\gamma = e_1$: $\mathcal{H} = \{ x \in \mathfrak{g} \mid [x, \gamma] = 0 \}$. Then $\mathcal{D}$ can be identified with the tangent plane $T_\gamma O$ of the adjoint orbit $O$ through $\gamma$. Since $\mathcal{H}$ and $\mathcal{D}$ are uniquely determined by $\gamma$, we can write the closed system in variables $(m, \gamma)$ or $(\omega, \gamma)$:
\begin{align*}
\dot{m} &= \epsilon [m, \omega] + (1 - \epsilon) \text{pr}_D [\omega, \omega], \\
\dot{\gamma} &= \epsilon [\gamma, \omega],
\end{align*}
\begin{equation}
(25) \quad \dot{m} = \text{pr}_D I \omega + \text{pr}_H \omega.
\end{equation}

Also, in the special case when $\mathcal{H}$ is one dimensional, spanned by $\gamma = e_1$, we have the equations
\begin{align*}
\dot{m} &= [m, \omega] + \lambda \gamma, \\
\dot{\gamma} &= \epsilon [\gamma, \omega],
\end{align*}
\begin{equation}
(26) \quad \lambda = -\langle \gamma, \mathbb{I}^{-1}[m, \omega] \rangle / \langle \mathbb{I}^{-1} \gamma, \gamma \rangle, \quad m = \mathbb{I} \omega.
\end{equation}

The above examples coincide in the case of the Lie algebra $\text{so}(3)$. Under the usual isomorphism between the Euclidian space $\mathbb{R}^3$ and $\text{so}(3)$
\begin{equation}
(27) \quad \vec{X} = (X_1, X_2, X_3) \mapsto X = \begin{pmatrix} 0 & -X_3 & X_2 \\ X_3 & 0 & -X_1 \\ -X_2 & X_1 & 0 \end{pmatrix},
\end{equation}
replacing the inertia operator $I$ by $I = \mathbb{I} + D \mathbb{E}$, the equations \((26)\) recover the equations \((3)\) of the rubber Chaplygin ball, while the expression for the density of the invariant measure \((16)\) recovers the density \((4)\). Also, according to \((25)\), we can write the rubber Chaplygin ball equations in the equivalent form
\begin{align*}
\frac{d}{dt} \vec{m} &= \epsilon \vec{m} \times \vec{\omega} + (1 - \epsilon) \left( I \vec{\omega} \times \vec{\omega} - (I \vec{\omega} \times \vec{\omega}, \gamma) \vec{\gamma} \right), \\
\frac{d}{dt} \vec{\omega} &= \epsilon \vec{\gamma} \times \vec{\omega},
\end{align*}
\begin{equation}
(28) \quad \vec{m} = I \vec{\omega} + (\vec{\gamma}, \omega - I \vec{\omega}) \vec{\gamma}.
\end{equation}

3.2. Modified Veselova problem. Following \cite{15}, we define $\epsilon$–modified Veselova nonholonomic rigid body problem as follows. Let $e_1, \ldots, e_n$ be a (moving) orthonormal frame of the Euclidean space $(\mathbb{R}^n, (\cdot, \cdot))$. Consider the orthogonal decomposition
\begin{equation}
(29) \quad \text{so}(n) = \mathcal{H}_r \oplus \mathcal{D}_r,
\end{equation}
\begin{equation*}
\mathcal{H}_r = \text{span} \{ e_p \wedge e_q \mid r < p < q \leq n \}, \quad \mathcal{D}_r = \text{span} \{ e_i \wedge e_j \mid 1 \leq i \leq r, 1 \leq j \leq n \}.
\end{equation*}
Then
\begin{equation*}
\text{pr}_{\mathcal{D}_r} \eta = \Gamma \eta + \eta \Gamma - \Gamma \eta \Gamma, \quad \eta \in \text{so}(n),
\end{equation*}
where
\begin{equation*}
\Gamma = e_1 \otimes e_1 + \cdots + e_r \otimes e_r.
\end{equation*}
Theorem 4. The $\epsilon$-modified Veselova system \cite{30}, \cite{31} has an invariant measure

$$
\left(\sum_I a_{i_1}\cdots a_{i_r}(\mathbf{e}_1\wedge\cdots\wedge\mathbf{e}_r)^2\right)^{\frac{1}{2(r-1)}} \dim\wedge d\mathbf{e}_1\wedge\cdots\wedge d\mathbf{e}_r|_{\text{so}(n)\times V_{n,r}},
$$

where the summation is over all $r$-tuples $I = \{1 \leq i_1 < \cdots < i_r \leq n\}$, and $(\mathbf{e}_1\wedge\cdots\wedge\mathbf{e}_r)_I$ are the Plücker coordinates of the $r$-form $\mathbf{e}_1\wedge\cdots\wedge\mathbf{e}_r$. In the case $r = 1$, the density is proportional to $(\mathbf{e}_1, A\mathbf{e}_1)^{\frac{1}{2(n-1)}}(n-2)$.

Here $d\mathbf{e}_1\wedge\cdots\wedge d\mathbf{e}_r$ is the standard volume form on $\mathbb{R}^{nr}$. Note that, for the inertia operator $\mathbb{I}$ replaced by $\mathbb{I} = \mathbb{I} + D\mathbb{E}$, $r = 1$ and $n = 3$, the equations \cite{30} give another form of the rubber Chaplygin ball equations \cite{28}. 

Along the trajectory $(\mathbf{m}(t), \mathbf{e}_1(t), \ldots, \mathbf{e}_r(t))$ of the modified Veselova problem \cite{30}, the angular velocity matrix $\omega(t)$ has the form

$$
\omega(t) = \begin{pmatrix}
0 & \cdots & \omega_1(t) & \cdots & \omega_n(t) \\
\vdots & \ddots & \vdots & & \vdots \\
-\omega_1(t) & \cdots & 0 & \cdots & \omega_{rn}(t) \\
\vdots & & \vdots & & C_r \\
-\omega_1(t) & \cdots & -\omega_{rn}(t)
\end{pmatrix},
$$

where $\omega_{ij}(t) = \langle \mathbf{e}_i(t) \wedge \mathbf{e}_j(t), \omega(t) \rangle$ and $C_r$ is a constant $(n-r) \times (n-r)$ matrix.

Next, define the left-invariant metric via the relation

$$
\mathbb{I}(\mathbf{E}_i \wedge \mathbf{E}_j) = a_ia_j\mathbf{E}_i \wedge \mathbf{E}_j,
$$

where $\mathbf{E}_1, \ldots, \mathbf{E}_n$ is the standard orthonormal base of $\mathbb{R}^n$ and $A = \text{diag}(a_1, \ldots, a_n)$ is positive definite.

Then, from Theorem 3 and Theorem 5.1 of \cite{15}, we get:

Theorem 4. The $\epsilon$-modified Veselova system \cite{30}, \cite{31} has an invariant measure
4. Modified L+R systems

The $\epsilon L+R$ system on the space

$$g \times \text{Sym}(g)(\omega, \Pi) \quad \text{or} \quad g \times \text{Sym}(g)(k, \Pi)$$

is defined by

$$\dot{k} = [k, \omega], \quad \dot{\Pi} = \epsilon [\Pi, \text{ad}_\omega], \quad k = \Pi\omega + \Pi\omega.$$

**Theorem 5.** The $\epsilon L+R$ system possesses an invariant measure

$$\mu \, d\omega \wedge d\Pi$$

i.e.,

$$\mu = \epsilon^{-1} d k \wedge d\Pi,$$

with density same as in the case of the usual L+R systems:

$$\mu = \sqrt{\det(I + \Pi)},$$

where $d\Pi$ is the standard measure on $\text{Sym}(g)$.

**Proof.** The proof is a modification of the corresponding statement for L+R systems (see [14, 13]). We have

$$\dot{k} = (I + \Pi)\dot{\omega} + \Pi\dot{\omega} = (I + \Pi)\dot{\omega} + \epsilon\Pi[\omega, \omega] - \epsilon[\omega, \Pi\omega].$$

Thus, we can represent the equations (32) in the form

$$\dot{\omega} = (I + \Pi)^{-1} ([\Pi\omega, \omega] + (1 - \epsilon)[\Pi\omega, \omega]), \quad \dot{\Pi} = \epsilon[I, \text{ad}_\omega].$$

The volume form (33) is invariant with respect to the flow of (34) if and only if

$$\mu\left(\text{div}(\dot{\omega}) + \text{div}(\dot{\Pi})\right) + \dot{\mu} = 0,$$

where we take the standard divergence on $g$ and $\text{Sym}(g)$:

$$\text{div}(\dot{\omega}) = \sum_i \frac{\partial \dot{\omega}_i}{\partial \omega_i}, \quad \text{div}(\dot{\Pi}) = \sum_{i \leq j} \frac{\partial \dot{\Pi}_{ij}}{\partial \Pi_{ij}}.$$ 

Here we take coordinates of $\omega$ and $\Pi$ with respect to the orthonormal base $E_1, \ldots, E_n$ of $g$ and the associated base $E_i \otimes E_j$ of linear operators on $g$.

It is clear that $\text{div}(\dot{\Pi}) = 0$. Define $n \times n$ matrix $\Omega$:

$$\Omega_{ij} = \frac{\partial((I + \Pi)\dot{\omega})_i}{\partial \omega_j} = \frac{\partial([\Pi\omega, \omega] + (1 - \epsilon)[\Pi\omega, \omega])_i}{\partial \omega_j}, \quad i, j = 1, \ldots, n.$$ 

Then

$$\Omega = - \text{ad}_\omega \circ \Pi + \text{ad}_\omega + (1 - \epsilon)(- \text{ad}_\omega \circ \Pi + \text{ad}_\Pi \omega)$$

and we can write

$$\text{div}(\dot{\omega}) = \text{tr}((I + \Pi)^{-1}\Omega).$$

In view of symmetry of $I + \Pi$, the skew symmetric part of $\Omega$ does not contribute to the expression for the divergence.

Taking into account (32), the symmetric part of $\Lambda$ has the form

$$\Omega_+ = \frac{1}{2}(\Omega^T + \Omega) = \frac{1}{2}(I \circ \text{ad}_\omega - \text{ad}_\omega \circ I + (1 - \epsilon)(I \circ \text{ad}_\omega - \text{ad}_\omega \circ \Pi))$$

$$= \frac{1}{2} \left((I + \Pi) \circ \text{ad}_\omega - \text{ad}_\omega \circ (I + \Pi) - \Pi\right).$$
As a result, we obtain
\[
\text{div}(\dot{\omega}) = \text{tr}((I + \Pi)^{-1} \Omega) = \frac{1}{2} \text{tr} \left( (I + \Pi)^{-1} \left((I + \Pi) \circ \text{ad}_\omega - \text{ad}_\omega \circ (I + \Pi) - \dot{\Pi}\right) \right) = -\frac{1}{2} \text{tr} \left( (I + \Pi)^{-1} \frac{d}{dt} (I + \Pi) \right) = -\frac{1}{2} \text{det}(I + \Pi) \frac{d}{dt} \text{det}(I + \Pi),
\]
where we used the unimodularity condition for compact groups $\text{tr} \text{ad}_\omega = 0$ and the well-known identity (19). We conclude that $\mu = \sqrt{\text{det}(I + \Pi)}$ satisfies the Liouville equation (35) which establishes the theorem. □

Remark 1. Note that the kinetic energy $H = \frac{1}{2} \langle k, \omega \rangle$ is conserved for $\epsilon$-L+R systems
\[
\frac{d}{dt} \langle k, \omega \rangle = 2\langle (I + \Pi) \omega, \omega \rangle + \langle \frac{d}{dt} (I + \Pi) \omega, \omega \rangle = 2\langle [\omega, \omega] + (1 - \epsilon)\Pi \omega, \omega \rangle + \epsilon \langle \Pi \omega, \omega \rangle - \langle [\omega, \Pi \omega], \omega \rangle = 0,
\]
while, for the $\epsilon$LR systems, the kinetic energy $H = \frac{1}{2} \langle \omega, \omega \rangle$ is conserved only on the invariant submanifold $\phi_s = \langle \omega, e_s \rangle = 0$, i.e., when $\text{pr}_\mathcal{H} \omega = 0$. However in the case $\epsilon = 1$, we have also the preservation of the following modification of the kinetic energy:
\[
F = \frac{1}{2} \langle \omega, \omega \rangle - \langle \text{pr}_\mathcal{H} \omega, \omega \rangle.
\]
Indeed, $\frac{d}{dt} H = \langle \omega, \Lambda \rangle$, and from (21), (23) we have
\[
\frac{d}{dt} \langle \text{pr}_\mathcal{H} \omega, \omega \rangle = \langle \epsilon \text{pr}_\mathcal{H} \omega, \omega \rangle - \epsilon \langle [\omega, \text{pr}_\mathcal{H} \omega], \omega \rangle + \langle \text{pr}_\mathcal{H} \omega, [\omega, \omega] + \Lambda \rangle = (1 - \epsilon) \langle \text{pr}_\mathcal{H} \omega, [\omega, \omega] \rangle + \langle \omega, \Lambda \rangle = \langle \omega, \Lambda \rangle \quad \text{for} \quad \epsilon = 1.
\]

Remark 2. The ($\epsilon$-modified) LR and L+R systems can be considered on non-compact groups, when we have an invariant measure for unimodular groups as well. Recall that the group $G$ is unimodular if $\text{tr} \text{ad}_\omega = 0$.

Example 2. As in Example 1, consider the orthogonal decomposition $g = \mathcal{H} \oplus D$, where $\mathcal{H}$ is the isotropy algebra of $\gamma = e_1$. Let us take $\Pi = D \text{pr}_D$. Then the modified L+R system (32) takes the form
\[
\dot{k} = [k, \omega], \quad \dot{\gamma} = \epsilon [\gamma, \omega], \quad k = [\omega, D \text{pr}_D \omega].
\]
After the identification $\text{so}(3) \cong \mathbb{R}^3$ given by (27), Theorem 3 recovers the invariant measure (2). Another natural choice for the operator $\Pi$ is
\[
\Pi = D[[\gamma, \omega], \gamma]
\]
(see [17, 19]), which also leads, for $g = \text{so}(3)$, to the sphere-sphere problem (1).
4.1. \(\epsilon\)L+R system on \(so(n) \times V_{n,r}\). Here we use the notation of Subsection 3.2. Consider the decomposition of \(so(n)\) given by (29), and take \(\Pi = D\text{pr}_{D_{r}}\). As a result, we obtain the modified L+R system on the space \(so(n) \times V_{n,r}\)

\[
\dot{k} = [k, \omega],
\]

\[
\dot{e}_{i} = -\epsilon \omega e_{i}, \quad i = 1, \ldots, r,
\]

\[
\dot{k} = I\omega + D(\Gamma \omega + \omega \Gamma - \Gamma \omega), \quad \Gamma = e_{1} \otimes e_{1} + \cdots + e_{r} \otimes e_{r}.
\]

For \(r = 1\) and \(\epsilon = 1\), the equations (36) model the problem of rolling without slipping of a Chaplygin ball over the hyperplane in \(\mathbb{R}^{n}\), orthogonal to \(e_{1}\) (see [13, 19]). As in [19] (see eq. (49) of [19]), consider the metric defined by in inertia operator (37)

\[
I(E_{i} \wedge E_{j}) = \frac{D a_{i} a_{j}}{D - a_{i} a_{j}} E_{i} \wedge E_{j},
\]

where \(0 < a_{i} a_{j} < D, i, j = 1, \ldots, n\).

**Theorem 6.** The \(\epsilon\)L+R system (36), with the inertia operator given by (37) has an invariant measure

\[
\left(\sum_{i} \frac{(e_{1} \wedge \cdots \wedge e_{r})^{2}}{a_{i_{1}} \cdots a_{i_{r}}}\right)^{-\frac{1}{2}(n-r-1)} \ dk \wedge de_{1} \wedge \cdots \wedge de_{r}|_{so(n) \times V_{n,r}},
\]

For \(r = 1\), the density is proportional to \((e_{1}, A^{-1} e_{1})^{-\frac{1}{2}(n-2)}\).

**Proof.** Motivated by the relationship between 3–dimensional Veselova problem and the Chaplygin ball problem established by Fedorov [12], define the operator \(I\) and matrixes \(w, m\) by

\[
I = E + D\Pi^{-1}, \quad w = I\omega, \quad m = \text{pr}_{D_{r}} Iw + \text{pr}_{H_{r}} w.
\]

Then

\[
m = \text{pr}_{D_{r}} I\omega + \text{pr}_{D_{r}} D\Pi^{-1} I\omega + \text{pr}_{H_{r}} I\omega = \omega + D \text{pr}_{D_{r}} \omega = k.
\]

Therefore, by using (24), we get

\[
det(I + \Pi) = det \frac{\partial k}{\partial \omega} = det \frac{\partial m}{\partial w} = det \frac{\partial w}{\partial \omega} = det(I|_{\Pi}) \cdot det\Pi.
\]

Now, the definition of \(\Pi\) can be seen as follows: it implies the identity \(I(E_{i} \wedge E_{j}) = Da_{i}^{-1} a_{j}^{-1} E_{i} \wedge E_{j}\). By combining the above expressions with Theorem 5 and Theorem 5.1 of [15] we obtain the statement. \(\square\)

**Remark 3.** The system (11) is integrable by the Euler–Jacobi theorem [1] for \(\epsilon = 1\) (Chaplygin [9], see also [1, 6]) and for \(\epsilon = -1\) (Borisov and Fedorov [4], see also [6, 7]). Similarly, in the case of the rubber rolling, we have integrability for \(\epsilon = 1\) (see 21 [10]) and \(\epsilon = -1\) (see [5, 6]). The problem with the addition of potential forces is studied in details in [8]. On the other hand, the integrable models of the rolling of the Chaplygin ball (on the zero level set of the \(SO(n - 1)\)–momentum map) and the rubber Chaplygin ball over a hyperplane in \(\mathbb{R}^{n}\) are given in [19] and [15, 18], respectively. It would be interesting to study appropriate problems where we have a spherical surface instead of a hyperplane. The natural \(n\)–dimensional
variants of the equation (1) and (3) are, respectively, the equations (36) and (30) (with \( I \) replaces by \( I = I + DE \)), where we set \( r = 1 \).

Acknowledgments. I am grateful to the referee for useful suggestions. The research was supported by the Serbian Ministry of Education and Science Project 174020 Geometry and Topology of Manifolds, Classical Mechanics, and Integrable Dynamical Systems.

References

[1] Arnold V I, Kozlov V V, Neishtadt A I 1985 Mathematical aspects of classical and celestial mechanics. Itogi Nauki i Tekhniki. Sovr. Probl. Mat. Fundamental’nye Napravleniya, Vol. 3, VINITI, Moscow 1985. English transl.: Encyclopaedia of Math. Sciences, Vol.3, Springer-Verlag, Berlin 1989.

[2] P. Balseiro, L. Garcia-Naranjo 2012 Gauge Transformations, Twisted Poisson brackets and hamiltonization of nonholonomic systems, Archive for Rational Mechanics and Analysis, 205 no. 1, 267–310, arXiv:1104.0880.

[3] Bolsinov A V, Borisov A V, Mamaev I S 2012 Hamiltonization of Non-Holonomic Systems in the Neighborhood of Invariant Manifolds, Regul. Chaotic Dyn. 16, no. 5, 443-464.

[4] Borisov A V, Fedorov Yu N 1995 On two modified integrable problems in dynamics Mosc. Univ. Mech. Bull. 50, No.6, 16–18.

[5] Borisov A V, Mamaev I S 2007 Rolling of a Non-Homogeneous Ball over a Sphere Without Slipping and Twisting, Regul. Chaotic Dyn. 12, 153–159.

[6] Borisov A V, Mamaev I S 2008 Conservation Laws, Hierarchy of Dynamics and Explicit Integration of Nonholonomic systems, Regul. Chaotic Dyn. 13, 443–490.

[7] Borisov A V, Fedorov Yu N, Mamaev I S 2008 Chaplygin ball over a fixed sphere: an explicit integration Regul. Chaotic Dyn. 13, 557–571, [arXiv:0812.4718].

[8] Borisov A V, Mamaev I S, Bizyaev I A 2013 The hierarchy of dynamics of a rigid body rolling without slipping and spinning on a plane and a sphere. Regul. Chaotic Dyn. 18, no. 3, 277-328.

[9] Chaplygin S A 1903 On a rolling sphere on a horizontal plane. Mat. Sbornik 24 139-168 (Russian)

[10] Ehlers K, Koiller J, Montgomery R, Rios P 2005 Nonholonomic systems via moving frames: Cartan’s equivalence and Chaplygin Hamiltonization, The breadth of symplectic and Poisson geometry, 75–120, Progr. Math., 232, Birkhuser Boston, Boston, MA, [arXiv:math-ph/0408005].

[11] Ehlers K, Koiller J 2007 Rubber rolling over a sphere Regul. Chaotic Dyn., 12, 127-152, [arXiv:math/0612036].

[12] Fedorov Yu 1989 Two Integrable Nonholonomic Systems in Classical Dynamics, Vest. Moskov. Univ. Ser I Mat. Mekh. no 4, 38–41 (Russian).

[13] Fedorov Yu N, Kozlov V V 1995 Various aspects of n-dimensional rigid body dynamics Amer. Math. Soc. Transl. Series 2, 168 141–171.

[14] Fedorov Yu 1996 Dynamical systems with an invariant measure on the Riemannian symmetric pairs (GL(N), SO(N)). (Russian) Reg. Ch. Dyn. 1, no. 1, 38–44.

[15] Fedorov Yu N, Jovanović B 2004 Nonholonomic LR systems as Generalized Chaplygin systems with an Invariant Measure and Geodesic Flows on Homogeneous Spaces. J. Non. Sci., 14, 341-381, [arXiv:math-ph/0307016].

[16] Fedorov Yu N, Garca-Naranjo L C, Marrero, J C 2014 Unimodularity and preservation of volumes in nonholonomic mechanics, [arXiv:1304.1788].

[17] Hochgerner S 2009 Chaplygin Systems Associated to Cartan Decompositions of Semi–Simple Lie Groups, [arXiv:0907.0636] [math.DG].

[18] Jovanović B 2009 LR and L+R systems, J. Phys. A: Math. Theor. 42 No 22, 225202 (18pp), [arXiv:0902.1656] [math-ph].

[19] Jovanović B 2010 Hamiltonization and Integrability of the Chaplygin Sphere in \( \mathbb{R}^n \), J. Nonlinear. Sci. 20 569–593, [arXiv:0902.4397].

[20] Kozlov V V 1988 Invariant measures of the Euler-Poincare equations on Lie algebras , Funkt. Anal. Prilozh. 22 69–70 (Russian); English trans.: Funct. Anal. Appl. 22 (1988) 58–59.
[21] Veselov A P, Veselova L E 1988 Integrable nonholonomic systems on Lie groups Mat. zametki 44 no. 5, 604-619 (Russian); English translation: 1988 Mat. Notes 44 no. 5.

[22] Yaroshchuk V A 1992 New cases of the existence of an integral invariant in a problem on the rolling of a rigid body, Vestnik Moskov. Univ. Ser. I Mat. Mekh., no. 6, 26–30. (Russian)

[23] Zenkov D V and Bloch A M 2003 Invariant measures of nonholonomic flows with internal degrees of freedom, Nonlinearity 16 1793–1807.