Conformal measures and the Dobrushin-Lanford-Ruelle equations

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Abstract. We demonstrate the equivalence of two definitions of a Gibbs measure on a subshift over a countable group, namely a conformal measure and a Gibbs measure in the sense of the Dobrushin-Lanford-Ruelle (DLR) equations. We formulate a more general version of the classical DLR equations with respect to a measurable cocycle, which reduce to the classical equations when the cocycle is induced by an interaction or a potential, and show that a measure satisfying these equations must be conformal. To ensure the consistency of these results with earlier work, we review methods of constructing an interaction from a potential and vice versa, such that the interaction and the potential constructed from it, or vice versa, induce the same cocycle.

1. Introduction

This paper is concerned with two notions of a Gibbs measure on a subshift over a countable group. The first of these is defined by the Dobrushin-Lanford-Ruelle (DLR) equations, or equivalently a Gibbsian specification. This notion of a Gibbs measure appears for instance in the classical theorems of Dobrushin [5] and Lanford-Ruelle [9]. The second is the notion of a conformal measure, introduced in [16] and [4] and used for instance by Meyerovitch in [12] as the setting for a stronger Lanford-Ruelle theorem. There are other definitions in the literature, such as a Gibbs measure in the sense of Bowen, but we do not consider these here.

The purpose of the present article is to show that the two notions of Gibbs measure recalled above coincide in some generality. Our results build on those of Kimura [11], who proves two results relevant here. The first is that every conformal measure, with respect to an appropriately regular potential, satisfies the DLR equations for that potential. The second is a partial converse, namely that every measure satisfying the DLR equations for such a potential is topologically Gibbs. This is a weaker property than being conformal, although equivalent on certain subshifts, such as shifts of finite type [12]. Sarig ([18], [19]) obtains the full converse in the case of a topologically mixing one-sided shift of finite type, using martingale and Ruelle transfer operator methods.

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Our main result, Theorem 5, strengthens these partial results to a full converse in a more general setting. Specifically, we show that any measure satisfying certain equations with respect to a measurable cocycle on the Gibbs relation must also be conformal with respect to that cocycle. When the cocycle is induced either by an interaction or by a potential in the standard way, these equations reduce to the classical DLR equations. We prove this result for arbitrary subshifts with finite alphabet on an arbitrary countable group. The results of Kimura and Sarig in the forward direction (conformal implies DLR) can also be generalized to our setting; in §4 we mention the idea for the proof but refer readers to [11] for the details in Kimura’s setting, as the proof strategy changes very little.

The plan is as follows. In §2 we review the definitions and basic facts required to prove our main result in [3]. In §4 and §5 we recall well-known material on interactions and potentials, respectively, in order to show that the equations involved in our main theorem do in fact reduce to the classical DLR equations. In §6 we recall results of Muir and Kimura, elaborating on Ruelle, by which a potential can be constructed from a sufficiently regular interaction, and vice versa, with “physical” results of Muir and Kimura, elaborating on Ruelle, by which a potential can be set that yields a certain spherical growth condition, defined in §4. This condition is satisfied, for any generating set, by any group of polynomial growth, such as \( F_n \), the case of greatest physical interest. It is also satisfied by any group \( G \) isomorphic to the free group \( F_n \), with generating set of cardinality \( n \).

## 2. Cocycles and the Gibbs relation: definitions and properties

Throughout, let \( G \) be a countable group with identity \( e \). Let \( \mathcal{A} \) be a finite alphabet equipped with the discrete topology, and \( X \subseteq \mathcal{A}^G \) a subshift, i.e. a closed set in the product topology, invariant under the natural right action of \( G \) via \((x \cdot g)_h = x_{gh}\). The topology on \( X \) is generated by cylinders, i.e. sets of the form \([\omega] = \{ x | x_\Lambda = \omega \}\) for finite sets \( \Lambda \subseteq G \). This topology can be induced by a metric such that the resulting metric space is complete and separable; that is, \( \mathcal{A}^G \) is a Polish space. We equip \( X \) with the Borel \( \sigma \)-algebra \( \mathcal{S} \).

The **Gibbs relation**, also called the asymptotic relation, is the equivalence relation \( \mathfrak{T}_X \subseteq X \times X \) such that \((x,y) \in \mathfrak{T}_X\) if and only if \( x_\Lambda = y_\Lambda \) for some finite set \( \Lambda \subseteq G \). Let \((\Lambda_N)_{N=1}^\infty\) be a sequence of finite sets exhausting \( G \), i.e. \((\Lambda_N)_{N=1}^\infty\) is an increasing sequence and \( G = \bigcup_{N=1}^\infty \Lambda_N \). Define the subrelation \( \mathfrak{T}_{X,N} = \{ (x,y) : x_{\Lambda_N^c} = y_{\Lambda_N^c} \} \subseteq \mathfrak{T}_X \). Observe that, for each subrelation \( \mathfrak{T}_{X,N} \), each equivalence class is a finite set, and that \( \mathfrak{T}_X = \bigcup_{N=0}^\infty \mathfrak{T}_{X,N} \). (In the language of Borel equivalence relations, this means that \( \mathfrak{T}_X \) is **hyperfinite** [10], which we mention for context, although we do not use any theorems about hyperfiniteness in this paper.) This shows in particular that every equivalence class in \( \mathfrak{T}_X \) is at most countable. Note that we can write each subrelation as \( \mathfrak{T}_{X,N} = \cap_{N=1}^\infty \bigcup_{\omega \in \mathcal{A}^{\Lambda_N} \setminus \Lambda_N} [\omega] \times [\omega] \), which shows that \( \mathfrak{T}_{X,N} \) is a measurable subset of \( X \times X \) in the product \( \sigma \)-algebra \( \mathcal{S} \otimes \mathcal{S} \), as is \( \mathfrak{T}_X \).

For Borel sets \( A,B \subseteq X \), a **holonomy** of \( \mathfrak{T}_X \) (\( \mathfrak{T}_{X,N} \)) is a Borel isomorphism \( \psi : A \to B \) such that \((x,\psi(x)) \in \mathfrak{T}_X \) (\( \mathfrak{T}_{X,N} \)) for all \( x \in A \). We say that a holonomy \( \psi \) is **global** if \( A = B = X \). The definition for \( \mathfrak{T}_{X,N} \) is analogous, with a holonomy of \( \mathfrak{T}_{X,N} \) also a holonomy of \( \mathfrak{T}_X \), for every \( N \).
For a Borel set \(A \subseteq X\), we denote \(\mathfrak{S}_X(A) = \cup_{x \in A} \{y \in X | (x, y) \in \mathfrak{S}_X\}\), and the same for the subrelations. The saturations \(\mathfrak{S}_X(A)\) and \(\mathfrak{S}_{X,N}(A)\) are easily shown to be Borel using the fact that the diagonal in \(X \times X\) is measurable in the product \(\sigma\)-algebra, which follows as an easy exercise from the fact that \(X\) is Polish.

We say that a measure \(\mu\) on \(X\) (by which we always mean a Borel probability measure) is \(\mathfrak{S}_X\)-nonsingular if for every Borel \(A \subset X\) with \(\mu(A) = 0\), we have \(\mu(\mathfrak{S}_X(A)) = 0\). Note that if \(\mu\) is \(\mathfrak{S}_X\)-nonsingular and \(\psi : A \to B\) is a holonomy of \(\mathfrak{S}_X\), then whenever \(E \subset A\) has \(\mu(E) = 0\), we have \(\mu(\psi(E)) \leq \mu(\mathfrak{S}_X(E)) = 0\). In particular, the Radon-Nikodym derivative \(\frac{d\mu_{\psi\circ\mu}}{d\mu}\) is well-defined. The same holds with \(\mathfrak{S}_X\) replaced by \(\mathfrak{S}_{X,N}\).

A (real-valued) cocycle on \(\mathfrak{S}_X\) is a Borel measurable function \(\phi : \mathfrak{S}_X \to \mathbb{R}\) such that \(\phi(x, y) + \phi(y, z) = \phi(x, z)\) for all \(x, y, z \in X\) with \((x, y), (y, z) \in \mathfrak{S}_X\) (so that \((x, z) \in \mathfrak{S}_X\) as well). Any cocycle on \(\mathfrak{S}_X\) clearly restricts to a cocycle on \(\mathfrak{S}_{X,N}\), for any given \(N\). Given a \(\mathfrak{S}_X\)-nonsingular measure \(\mu\) on \(X\), we say that a Borel function \(D : \mathfrak{S}_X \to \mathbb{R}\) is a Radon-Nikodym cocycle on \(\mathcal{R}\) with respect to \(\mu\) if the pushforward of \(\mu\) by any holonomy \(\psi : A \to B\) of \(\mathfrak{S}_X\) satisfies \(\frac{d(\mu_{\psi\circ\mu})}{d\mu}(x) = D(x, \psi(x))\) for \(\mu\)-a.e. \(x \in A\). It is routine to show that any \(\mathfrak{S}_X\)-nonsingular measure \(\mu\) on \(X\) has a \(\mu\)-a.e. unique Radon-Nikodym cocycle. Indeed, if \(\psi_1, \psi_2 : A \to B\) are two holonomies that agree \(\mu\)-a.e., then they yield equal derivatives \(\frac{d(\mu_{\psi_1\circ\mu})}{d\mu}(x) = \frac{d(\mu_{\psi_2\circ\mu})}{d\mu}(x)\) for \(\mu\)-a.e. \(x \in A\), so in particular, given a holonomy \(\psi : X \to X\), the value \(\frac{d(\mu_{\psi\circ\mu})}{d\mu}(x)\) depends, except for at most a null set of points \(X\), on the pair \((x, \psi(x))\); we can therefore take \(D(x, y) = \frac{d(\mu_{\psi\circ\mu})}{d\mu}(x)\) for some holonomy \(\psi\) with \(\psi(x) = y\).

**Definition 1** (conformal measure). Let \(\mu\) be a \(\mathfrak{S}_X\)-nonsingular Borel probability measure on \(X\), and let \(\phi : \mathfrak{S}_X \to \mathbb{R}\) be a cocycle. We say that \(\mu\) is \((\phi, \mathfrak{S}_X)\)-conformal if for any holonomy \(\psi : A \to B\) of \(\mathfrak{S}_X\), and \(\mu\)-a.e. \(x \in A\), we have \(D_{\mu,\mathfrak{S}_X}(x, \psi(x)) = \exp(\phi(x, \psi(x)))\).

**Remark.** The name “conformal measure” was given in [4], motivated by Patterson’s study [15] of measures on the limit sets of a particular groups of conformal mappings of the unit disc in the complex plane, or more generally of hyperbolic space. In the case of identically zero cocycle, conformal measures were also studied in [16] under the name \(\mathfrak{S}_X\)-invariant measures.

**Definition 2** (DLR equations for a cocycle). Let \(X \subseteq \mathcal{A}^G\) be a subshift, \(\phi\) a cocycle on \(\mathfrak{S}_X\), and \(\mu\) a measure on \(X\). For a Borel set \(A \subseteq X\) and a finite set \(\Lambda \subset G\), the DLR equation for \(x \in X\) is as follows:

\[
(1) \quad \mu(A | \mathcal{F}_{\Lambda^c})(x) = \sum_{\eta \in \mathcal{A}^A} \left[ \sum_{\zeta \in \mathcal{A}^A} \exp(\phi(\eta x_{\Lambda^c}, \zeta x_{\Lambda^c})) \chi_x(\zeta x_{\Lambda^c}) \right]^{-1} \chi_A(\eta x_{\Lambda^c})
\]

We say that \(\mu\) is DLR with respect to \(\phi\) if, for any Borel \(A \subseteq X\) and any \(\Lambda \subset G\), (1) holds for \(\mu\)-a.e. \(x \in X\).

To prove our main result, we will need the following lemma:

**Lemma 1.** There exists a countable group \(\Gamma\) of global holonomies of \(X\) such that

\[\mathfrak{S}_X = \{(x, \gamma(x)) : x \in X, \gamma \in \Gamma\}\].

In other words, \(\Gamma\) generates \(\mathfrak{S}_X\).
Proof. The group Γ can be described explicitly as a countable increasing union of finite groups Γₜ. For each t, the group Γₜ generates Σₓₜ and is isomorphic to the symmetric group of order |A³|t|. Take Γₜ to be generated by holonomies ψ of the following form: given ω, η ∈ A³, define ψω,η : X → X by

\[
\psi_{\omega,\eta}(x) = \begin{cases} 
\eta x\Lambda_{\omega}^t & x\Lambda_{\omega}^t \in X \\
\omega x\Lambda_{\eta}^t & x\Lambda_{\eta}^t \in X \\
x & \text{otherwise}
\end{cases}
\]

That is, ψω,η exchanges ω and η, wherever possible, and otherwise does nothing. These involutions were considered in [12], for slightly different purposes. Observe that (x, y) ∈ Σₓₜ if and only if there exists ψ ∈ Γₜ with ψ(x) = y, so Σₓₜ is precisely the orbit relation of Γₜ. The result for Γ is immediate.

We mention for context that Lemma 1 is a special case of the main theorem of [6], which in fact asserts the same for any Borel equivalence relation on a Polish space in which every equivalence class is countable. This result was adapted to the symbolic setting in [12], with the countability of the equivalence classes established via the expansivity of the shift action. The proof is presented for subshifts over ℤₜ, but the same proof goes through for arbitrary countable groups without modification. However, since we establish Lemma 1 directly, we do not need to appeal to the theorem of [6] (nor the symbolic corollary in [12]).

3. Equivalence of the conformal and DLR properties

For us, the main value of Lemma 1 is the following lemma, which reveals in particular that to show that a given measure is conformal (such as in Theorem 5), it is sufficient to consider only global holonomies.

Lemma 2. Let μ be a Borel probability measure on X, let φ be a cocycle on Σₓ, and let Γ be a countable group generating Σₓ. Then μ is (φ, Σₓ)-conformal if and only if, for each γ ∈ Γ, the pushforward μ ◦ γ is absolutely continuous with respect to μ, with \( \frac{d(μ\circ γ)}{dμ}(x) = \exp(φ(x, γ(x))) \) for μ-a.e. x ∈ X.

Proof. The “only if” direction is immediate from the definition of conformal measure. To confirm the “if” direction, we first check nonsingularity. Let A ⊆ X be Borel with μ(A) = 0. Then Σₓ(A) = \( \bigcup_{γ ∈ Γ} γ(A) \), which is a countable union and thus has measure zero by the explicit expression for \( \frac{d(μ\circ γ)}{dμ} \).

Now let ψ : A → B be a holonomy of Σₓ and let E ⊆ A be Borel. Let Γ = (γₙ)ₙ∈ℕ be an enumeration of Γ. For each n ∈ ℕ, let Eₙ = \{x ∈ E : ψ(x) = γₙ(x)\}. To see that each Eₙ is Borel, define the map τₙ : X → X × X by τₙ(x) = (ψ(x), γₙ(x)), which is clearly measurable in the product σ-algebra. Then Eₙ = τ⁻¹ₙ(D) where D ⊆ X × X is the diagonal, which, as discussed above, is also Borel in the product σ-algebra, because X is Polish.

Now let E'ₙ₀ = E₀, and for n ≥ 1, let E'ₙ = Eₙ \ \bigcup_{k=0}^{n-1} Eₖ. The Borel sets E'ₙ partition E, so

\[
μ(ψ(E)) = \sum_{n=0}^{∞} μ(γₙ(E'ₙ)) = \int_{E} \exp(φ(x, ψ(x))) \, dμ(x)
\]

Thus \( \frac{d(μ\circ γ)}{dμ}(x) = \exp(φ(x, ψ(x))) \) for μ-a.e. x ∈ A, as required.
We will use Lemma 2 in concert with the following lemma, which reduces the question of (φ, Ξ_X)-conformality to that of conformality with respect to the finite-order subrelations.

**Lemma 3.** Let μ be a measure on X and φ a cocycle on Ξ_X. Suppose that μ is (φ, Ξ_{X,N})-conformal for each N ≥ 0. Then, μ is (φ, Ξ_X)-conformal.

**Proof.** Let ψ : X → X be a global holonomy of the Gibbs relation Ξ_X and let A ⊆ X be a Borel set. We begin by writing A as the increasing union A = ∪_{N=0}^{∞} A_N, where A_N = {x ∈ A : (x, ψ(x)) ∈ Ξ_{X,N}}. Since ψ|_{A_N} is a local holonomy of Ξ_{X,N} and μ is (φ, Ξ_{X,N})-conformal, we have

\[
\mu(\psi(A)) = \lim_{N \to \infty} \mu(\psi(A_N)) = \lim_{N \to \infty} \int_{A_N} \exp(\phi(x, \psi(x))) \, d\mu(x) = \int_A \exp(\phi(x, \psi(x))) \, d\mu(x),
\]

by dominated convergence. Thus, μ is indeed (φ, Ξ_X)-conformal. □

To echo the comment above on hyperfiniteness, we remark here that both of these results apply, with the same proofs, to any hyperfinite Borel equivalence relation on any Polish space. The following lemma, by contrast, seems to rely more specifically on the structure of X as a subshift.

**Lemma 4.** Let X ⊆ A^G be a subshift, let φ be a cocycle on X, and let μ be a DLR measure on X with respect to φ. Let N ≥ 1. Then μ is (φ, Ξ_{X,N})-conformal.

**Proof.** It is enough to show that μ(ψ([ω])) = \int_{[ω]} \exp(\phi(x, \psi(x))) \, d\mu(x) for any cylinder [ω] and (by Lemma 2) any global holonomy ψ of Ξ_{X,N}. Fix a holonomy ψ : X → X of Ξ_{X,N}. Since the equivalence classes of Ξ_{X,N} are finite, and in fact have bounded cardinality, there exists some r ≥ 0 such that ψ^r(x) = x, for all x ∈ X. Let m ≥ N and fix ω ∈ A^{λ_m}. We now partition X according to the orbits of points under ψ, in such a way that [ω] is partitioned into sets that are easy to control. Specifically, for each \( \eta = (\eta_0, \ldots, \eta_{r-1}) \) ∈ (A^{λ_m})^r, let

\[
T_\eta = \{ x ∈ X : \psi^j(x)_{λ_m} = \eta_j, 0 ≤ j ≤ r - 1 \}
\]

We then have [ω] = ∪_{\eta_0,...,\eta_{r-1}} T_\eta, and ψ(T_\eta) = T_{\sigma \eta}, where \( \sigma \eta = (\eta_1, \ldots, \eta_{r-1}, \eta_0) \) is a cyclic permutation of \( \eta \). It is enough to show that, for all \( \eta \in (A^{λ_m})^r \), we have

\[
\mu(\psi(T_\eta)) = \int_{T_\sigma} \exp(\phi(x, \psi(x))) \, d\mu(x).
\]

By the equality ψ(T_\eta) = T_{\sigma \eta}, we have

\[
\mu(\psi(T_\eta)) = \int_X \mu(T_{\sigma \eta} \mid F_{λ_m}) \, d\mu(x)
\]

For any x ∈ X, we know that

\[
1_{T_{\sigma \eta}}(x_{λ_m}) = 1_{T_{\eta}}(x_{λ_m})
\]
By this identity, as well as the DLR hypothesis and the defining property of a
cocycle, we have the following manipulations:

$$
\mu(T_{\sigma \eta} | F_{\Lambda c m})(x) = \sum_{\zeta \in A^{\Lambda m}} \exp(\phi(\eta_1 x_{\Lambda c m}, \zeta x_{\Lambda c m})) 1_X(\zeta x_{\Lambda c m})^{-1} 1_{T_{\sigma \eta}}(\eta_1 x_{\Lambda c m})
$$

$$
= \sum_{\zeta \in A^{\Lambda m}} \exp(\phi(\eta_0 x_{\Lambda c m}, \zeta x_{\Lambda c m})) 1_X(\zeta x_{\Lambda c m})^{-1} \times 1_{T_{\sigma \eta}}(\eta_0 x_{\Lambda c m}) \exp(\phi(\eta_0 x_{\Lambda c m}, \eta_1 x_{\Lambda c m}))
$$

Integrating this equation yields the result. □

We have therefore done all the work required to prove the following:

**Theorem 5.** Let $X \subseteq A^G$ be a subshift, $\phi$ a cocycle on $X$, and $\mu$ a DLR
measure on $X$ with respect to $\phi$. Then $\mu$ is $(\phi, T_X)$-conformal.

**Proof.** By Lemma 4, $\mu$ is $(\phi, T_{X, N})$-conformal for each $N$. The result is then immediate from Lemma 3. □

Theorem 5 is the converse of the following result proven by Kimura ([11],
Theorem 5.30), in the special case where $G = \mathbb{Z}^d$ and the cocycle $\phi$ is induced by
a potential, in the manner that we discuss in Proposition 8 below.

**Theorem 6.** Let $X \subseteq A^G$ be a subshift, $\phi$ a cocycle on $X$, and $\mu$ a $(\phi, T_X)$-
conformal measure on $X$. Then $\mu$ is DLR with respect to $\phi$.

The proof is a straightforward adaptation of Kimura’s methods. The rough idea
is to show that two cylinder sets have conditional measures with the appropriate
ratio by considering the holonomy that exchanges them, as in the proof of Lemma
1 above, then applying the conformal hypothesis. The main difference required to
adapt the proof is that the version stated here concerns the DLR equations for an
arbitrary measurable cocycle, not necessarily one induced by a potential.

4. Interactions

In this section, we show that, when a cocycle is induced by an interaction, the
DLR equations for the cocycle reduce to those for the interaction.

**Definition 3 (interaction).** An interaction is a family $\Phi = (\Phi_\Lambda)_{\Lambda \in G}$ of func-
tions $\Phi_\Lambda : X \to \mathbb{R}$ such that for each $\Lambda \in G$, $\Phi_\Lambda$ is $F_\Lambda$-measurable, and for all
$\Lambda \in G$, $x \in X$, the Hamiltonian series

$$
H^\Phi_\Lambda(x) = \sum_{\Delta \in G \setminus \Lambda \neq \emptyset} \Phi_\Delta(x)
$$

converges in the sense that there exists a real number $H^\Phi_\Lambda(x)$ and, for every $\epsilon > 0$,
there exists some $F \subseteq G$ such that, for all $F' \supseteq F$,

$$
\left| H^\Phi_\Lambda(x) - \sum_{\Delta \subseteq F' \setminus \Lambda \neq \emptyset} \Phi_\Delta(x) \right| < \epsilon
$$
Proposition 7. Let $\Phi$ be an interaction. For each $(x, y) \in \mathfrak{T}_X$, the series

$$
\sum_{\Lambda \in G} [\Phi_\Lambda(x) - \Phi_\Lambda(y)]
$$

converges in the same sense as the Hamiltonian series. Moreover, the function $\phi_\Phi : \mathfrak{T}_X \to \mathbb{R}$ defined by

$$
\phi_\Phi(x, y) = \sum_{\Lambda \in G} [\Phi_\Lambda(x) - \Phi_\Lambda(y)]
$$

is a cocycle on $\mathfrak{T}_X$.

Proof. Let $(x, y) \in \mathfrak{T}_X$ be such that $x_{\Delta^c} = y_{\Delta^c}$. We claim that

$$
\sum_{\Lambda \in G} [\Phi_\Lambda(x) - \Phi_\Lambda(y)] = H^\Phi_\Delta(x) - H^\Phi_\Delta(y)
$$

with the equality understood in the sense of convergence discussed in the statement of the proposition. Indeed, choose $\varepsilon > 0$. By the definition of an interaction, there exists some $F' \in G$ sufficiently large that whenever $F \subseteq F' \subseteq G$, we have (noting that $\Phi_E(x) - \Phi_E(y) = 0$ when $E \cap \Delta = \emptyset$),

$$
\left| H^\Phi_\Delta(x) - \sum_{E \subseteq F'} [\Phi_E(x) - \Phi_E(y)] \right| \leq H^\Phi_\Delta(x) - \sum_{E \subseteq F' \setminus \Delta \neq \emptyset} \Phi_E(x) + \left| H^\Phi_\Delta(y) - \sum_{E \subseteq F' \setminus \Delta \neq \emptyset} \Phi_E(y) \right| < \varepsilon
$$

This establishes that the series converges, in the sense claimed, to a real number $\phi_\Phi(x, y) = H^\Phi_\Delta(x) - H^\Phi_\Delta(y)$. Moreover, this energy difference expression makes it obvious that $\phi_\Phi$ is a cocycle, concluding the proof. 

We now observe that the DLR equations for the cocycle $\phi_\Phi$, in the sense of Definition 1, are equivalent to the classical DLR equations for the interaction $\Phi$. Indeed, if $\mu$ is a DLR measure with respect to $\phi_\Phi$, then for any $\Lambda \in G$, any Borel $A \subseteq X$, and $\mu$-a.e. $x \in X$, we have

$$
\mu(A \mid \mathcal{F}_{\Delta^c}) (x) = \sum_{\zeta \in A^\Lambda} \left[ \sum_{\eta \in A^\Lambda} \exp (\phi_\Phi(\zeta x_{\Lambda^c}, \eta x_{\Lambda^c})) 1_X(\zeta x_{\Lambda^c}) \right]^{-1} 1_A(\zeta x_{\Lambda^c})
$$

$$
= \sum_{\zeta \in A^\Lambda} \left[ \sum_{\eta \in A^\Lambda} \exp \left( H^\Phi_\Lambda(\zeta x_{\Lambda^c}) - H^\Phi_\Lambda(\eta x_{\Lambda^c}) \right) 1_X(\eta x_{\Lambda^c}) \right]^{-1} 1_A(\zeta x_{\Lambda^c})
$$

$$
= \frac{1}{Z^\Phi_\Lambda(x)} \sum_{\zeta \in A^\Lambda} \exp \left( - H^\Phi_\Lambda(\zeta x_{\Lambda^c}) \right) 1_A(\zeta x_{\Lambda^c})
$$

where

$$
Z^\Phi_\Lambda(x) = \sum_{\eta \in A^\Lambda} \exp \left( - H^\Phi_\Lambda(\eta x_{\Lambda^c}) \right) 1_X(\eta x_{\Lambda^c})
$$
By Theorem 5 if $\mu$ satisfies these (classical) DLR equations for $\Phi$, then $\mu$ is $(\phi_\Phi, \Sigma_X)$-conformal.

5. Potentials

In this section and the next, we restrict to finitely generated groups $G$ satisfying a certain uniform spherical growth condition, which we will need in order to construct a cocycle from a potential in a way that is compatible with interactions, in a sense to be made precise in §6. The condition is as follows. For a finite generating set $S \subset G$, consider the balls $B_k$ of radius $k$ centered at the identity in the Cayley graph of $G$ with respect to $S$. We are concerned with the spherical growth function $|B_k \setminus B_{k-1}|$, which is a basic quantity studied in geometric group theory, discussed for instance in [3], §VI.A). Specifically, we require that, for each $n \geq 1$,

$$\sup_{m \geq 1} \frac{|B_{m+n} \setminus B_{m+n-1}|}{|B_m \setminus B_{m-1}|} < +\infty.$$  

We refer to the finiteness of this supremum as the spherical growth property. Note that if the supremum is finite for $n = 1$ then in fact it is finite for all $n$.

Remark. Two different natural growth conditions on $G$ imply this spherical growth property, by easy calculations. The first is polynomial growth, which, by theorems of Gromov [7] and Wolf [20], Bass [1], and Guivarch [8], implies that there exists $d \in \mathbb{N}$ such that, for any word metric on $G$, there exist $0 < c < C$ with $c \leq |B_n|/n^d \leq C$ for all $n \geq 1$. In fact, by a stronger result of Pansu [14], we can take $C/c$ arbitrarily close to 1 by taking the supremum only over $n$ sufficiently large. The spherical growth property then holds by an easy calculation. In particular, in $\mathbb{Z}^d$, the standard sequence of balls $B_n = \mathbb{Z}^d \cap [-n,n]^d$ is fine, as is the sequence of balls for any other word metric on $\mathbb{Z}^d$.

For groups of exponential growth, the spherical growth property holds if there exist $a > 1$ and $0 < c < C$ with $c \leq |B_n|/a^n \leq C$ for all $n \geq 1$, by a calculation very similar to the polynomial case. This property, which we might call *exact exponential growth*, is satisfied, for example, by a countable free group with the metric induced by the usual generating set. Unlike the polynomial case, however, exact exponential growth can fail for some groups of exponential growth, at least for some generating sets, and indeed we believe it can hold for one generating set and fail for another.

When we work over a group $G$ in this section and the next, we are therefore restricting to a group $G$ that satisfies the spherical growth property with respect to some generating set, and considering the geometry on $G$ with respect to that fixed generating set.

We now turn our attention to potentials.

For a function $f : X \to \mathbb{R}$ and $k \geq 1$, define the variation of $f$ on $B_{k-1}$ as

$$v_{k-1}(f) := \sup \{|f(y) - f(x)| \mid x, y \in X, x_{B_{k-1}} = y_{B_{k-1}}\}.$$  

It is convenient to define $B_{-1} = \emptyset$ so that $v_{-1}(f) = \|f\|_\infty$. We define the shell norm $\|\cdot\|_{\text{ShVar}}$ by

$$\|f\|_{\text{ShVar}} := \sum_{k=0}^{\infty} |B_k \setminus B_{k-1}| v_{k-1}(f).$$
and the space $\text{ShReg}(X)$ as the space of \textit{shell-regular potentials}, i.e. functions $f : X \to \mathbb{R}$ with $\| f \|_{\text{ShVar}} < \infty$. It is elementary to show that shell-regularity implies continuity, and that $\text{ShReg}(X)$ is a Banach space.

\textbf{Remark.} In earlier work on subshifts over $\mathbb{Z}^d$ \cite{12}, the space of potentials under consideration is known as $\text{SV}_d(X)$, the space of potentials with $d$-summable variation, defined by the norm $\| f \|_{\text{SV}_d} = \sum_{k=1}^{\infty} k^{d-1} v_{k-1}(f)$. This space is also known as $\text{Reg}_{d-1}(X)$ \cite{13}. With $B_n = \mathbb{Z}^d \cap [-n, n]^d$, we have $|B_k \setminus B_{k-1}| = 2^d (1 + o(1)) k^{d-1}$. Thus, on $\mathbb{Z}^d$, we have $\text{ShReg}(X) = \text{SV}_d(X)$, with the identity a continuous linear map.

\textbf{Proposition 8.} For $f \in \text{ShReg}(X)$ and any $(x, y) \in \Sigma_X$, the series

$$
\sum_{g \in G} [f(y \cdot g) - f(x \cdot g)]
$$

converges absolutely and defines a cocycle $\phi_f$ on $\Sigma_X$.

\textbf{Proof.} Fix $(x, y) \in \Sigma_X$, and let $n \geq 1$ be such that $x B^n = y B^n$. If $g \in G$ and $m \geq 1$ are such that $B_{m-1} \subseteq g^{-1} B^n$, then $(x \cdot g)|_{B_{m-1}} = (x \cdot g)|_{B_{m-1}}$ so $|f(y \cdot g) - f(x \cdot g)| \leq v_{B_{m-1}}(f)$. For $m \geq 1$ and $g \in B_k \setminus B_{k-1}$, the triangle inequality guarantees that $g^{-1} B_{m-1} \subseteq B^n_k$ if $k - n \geq m$. Since the shells $B_k \setminus B_{k-1}$ partition $G$, we then have that

$$
\sum_{g \in G} |f(y \cdot g) - f(x \cdot g)| \leq 2 |B_n| \| f \|_\infty + \sum_{k=n+1}^{\infty} |B_k \setminus B_{k-1}| v_{k-1}(f)
$$

so indeed the cocycle is well-defined by an absolutely convergent series. \qed

Just as in the case of an interaction, this expression for the cocycle $\phi_f$ allows us to rewrite the DLR equations in a more classical form. Let $f \in \text{ShReg}(X)$. It follows from a simple manipulation that for any $x, y \in \Sigma_X$, we have

$$
\exp(\phi_f(x, y)) = \lim_{m \to +\infty} \exp \left( \sum_{g \in B_m} [f(y \cdot g) - f(x \cdot g)] \right) = \lim_{m \to +\infty} \frac{\exp f_m(y)}{\exp f_m(x)},
$$

where $f_m(z) = \sum_{g \in B_m} f(z \cdot g)$. Now, let $\mu$ be a measure on $X$, and let $A \subseteq X$. If $\mu$ is a DLR measure with respect to $\phi_f$, then for $\mu$-a.e. $x \in X$, we have

$$
\mu(A \big| \mathcal{F}_{\Lambda^c}) = \sum_{\eta \in \Lambda^c} \left[ \sum_{\zeta \in \Lambda^c} \exp(\phi_f(\eta x_{\Lambda^c}, \zeta x_{\Lambda^c})) 1_X(\zeta x_{\Lambda^c}) \right]^{-1} 1_A(\eta x_{\Lambda^c})
$$

$$
\mu(A \big| \mathcal{F}_{\Lambda^c}) = \sum_{\eta \in \Lambda^c} \left[ \sum_{\zeta \in \Lambda^c} \lim_{m \to +\infty} \frac{\exp f_m(\zeta x_{\Lambda^c})}{\exp f_m(\eta x_{\Lambda^c})} 1_X(\zeta x_{\Lambda^c}) \right]^{-1} 1_A(\eta x_{\Lambda^c})
$$

$$
\mu(A \big| \mathcal{F}_{\Lambda^c}) = \lim_{m \to +\infty} \frac{\sum_{\eta \in \Lambda^c} e^{f_m(\eta x_{\Lambda^c})} 1_A(\eta x_{\Lambda^c})}{\sum_{\zeta \in \Lambda^c} e^{f_m(\zeta x_{\Lambda^c})} 1_X(\zeta x_{\Lambda^c})}
$$

These are the DLR equations as found in Kimura \cite{11}. Applying Theorem\cite{10} therefore shows that any DLR measure with respect to a potential $f \in \text{ShReg}(X)$ is
necessarily \((\phi_f, \mathcal{F}_X)\)-conformal, providing the full converse for Kimura’s result described in the introduction.

6. Potentials induced by interactions, and vice versa

We have seen that the DLR property implies the conformal property for an arbitrary cocycle on the Gibbs relation, with Gibbs measures for interactions and for potentials as two special cases. These cases are not independent. In this section, we adapt the methods and results of Muir \[13\] and Ruelle \[17\] to construct a potential from an interaction in various physically equivalent ways, and, for sufficiently regular potentials, to construct an interaction. The novelty in this section is in the greater generality of the group \(G\), and in clarifying a condition on the support of an interaction necessary for the calculations to go through.

In this section, all interactions are translation-invariant, i.e. for any \(\Lambda \in G\) and any \(x \in X\), we require that \(\Phi_{\gamma^{-1}\Lambda}(x \cdot g) = \Phi_\Lambda(x)\). We recall a classical space of particularly well-behaved interactions:

**Definition 4.** For an interaction \(\Phi\), let

\[
\|\Phi\|_B = \sum_{\Lambda \in G} \|\Phi_\Lambda\|_\infty
\]

We define \(B\) as the normed space of absolutely summable interactions \(\Phi\), i.e. those for which \(\|\Phi\|_B < \infty\).

It is routine to check that \((B, \| \cdot \|_B)\) is a Banach space. Moreover, for an absolutely summable interaction \(\Phi \in B\), we in fact have absolute convergence of the series defining the cocycle, since for any \((x,y) \in \mathcal{F}_X\) with \(x_\Delta = y_\Delta\) for some \(\Delta \in G\), we have

\[
\sum_{\Lambda \in G} |\Phi_\Lambda(x) - \Phi_\Lambda(y)| \leq 2 \sum_{\substack{\Lambda \in G \\ \Lambda' \Delta \neq \emptyset}} \|\Phi_\Lambda\|_\infty
\]

\[
\leq 2|\Delta| \sum_{\Lambda \in G} \|\Phi_\Lambda\|_\infty
\]

\[
= 2|\Delta| \|\Phi\|_B < \infty
\]

We introduce a family of linear maps that convert interactions into potentials.

**Definition 5 (translate-weighting maps).** Let \((a_\Lambda)_{\Lambda \in G, e \in \Lambda}\) be a collection of nonnegative real coefficients such that, for each \(\Lambda \in G\) with \(e \in \Lambda\), we have \(\sum_{g \in \Lambda} a_{g^{-1}\Lambda} = 1\). Then, for an interaction \(\Phi\), define the potential \(A_\Phi\) via

\[
A_\Phi(x) = -\sum_{\substack{\Lambda \in G \\ e \in \Lambda}} a_\Lambda \Phi_\Lambda(x)
\]

The map \(\Phi \mapsto A_\Phi\) is clearly linear. We refer to this map as the translate-weighting map determined by the weights \((a_\Lambda)_{\Lambda \in G, e \in \Lambda}\).

**Remark.** Two important examples are the following.

- The uniform map, where \(a_\Lambda \in \{0, 1/|\Lambda|\}\) for every nonempty \(\Lambda \in G\). Muir uses the letter \(A\) to denote this specific operator, i.e. \(A(\Phi) = A_\Phi\).
• The class of dictator maps, where $a_\Lambda \in \{0, 1\}$ for every $\Lambda \in G$. For instance, on $\mathbb{Z}^d$, Ruelle studies the operator for which $a_\Lambda = 1$ if and only if 0 is the middle element, or more precisely the $[(|\Lambda|+1)/2]$-th element, of $\Lambda$ in lexicographic order. In [13], Muir refers to this operator as $A^\Lambda$.

In Fact 7.8 in [13], it is claimed that $A_\Phi \in \text{ShReg}(X)$ for every translate-weighting map and every $\Phi \in \mathcal{B}$. This claim is incorrect, as we demonstrate with an example below. However, the argument presented for this claim is correct in the case of what Muir calls “cubic-type” interactions. Here we reproduce a version of this proof for a broader class of interactions and give them a different name, suggested by the geometric reason for their necessity.

**Definition 6.** An interaction $\Phi$ is full-dimensional if there exists some $C > 0$ such that, for all $\Lambda \in G$ with $e \in \Lambda$ and $\Phi_\Lambda \neq 0$, we have the bound

$$\sup\{|B_n| : n \in \mathbb{N}, \Lambda \cap B_{n-1}^c \neq \emptyset\} \leq C|\Lambda|$$

**Proposition 9.** If $\Phi \in \mathcal{B}$ is full-dimensional, then $A_\Phi \in \text{ShReg}(X)$, where $A_\Phi$ is the image of $\Phi$ under an arbitrary translate-weighting map.

**Proof.** We first estimate $v_{k-1}(A_\Phi)$:

$$v_{k-1}(A_\Phi) = \sup \left\{ \sum_{\Lambda \in \mathcal{G}} a_\Lambda |\Phi_\Lambda(x) - \Phi_\Lambda(y)| : x, y \in X, x_{B_{k-1}} = y_{B_{k-1}} \right\}$$

$$\leq 2 \sum_{\Lambda \in \mathcal{G}} a_\Lambda \|\Phi_\Lambda\|_\infty$$

We can now estimate the shell norm by an exchange of summations:

$$\|A_\Phi\|_{\text{ShVar}} \leq 2 \sum_{k=0}^{\infty} |B_k \setminus B_{k-1}| \sum_{\Lambda \in \mathcal{G}} a_\Lambda \|\Phi_\Lambda\|_\infty$$

$$= 2 \sum_{\Lambda \in \mathcal{G}} a_\Lambda \|\Phi_\Lambda\|_\infty \sum_{k \geq 0} |B_k \setminus B_{k-1}|$$

Observe that

$$\sum_{k \geq 0} |B_k \setminus B_{k-1}| = \sup\{|B_n| : n \in \mathbb{N}, \Lambda \cap B_{n-1}^c \neq \emptyset\} \leq C|\Lambda|$$

so in fact

$$\|A_\Phi\|_{\text{ShVar}} \leq 2C \sum_{\Lambda \in \mathcal{G}} a_\Lambda |\Lambda| \|\Phi_\Lambda\|_\infty$$

We need to rearrange this sum. For a given $\Lambda \in G$, consider the set of translates of $\Lambda$ containing the identity, denoted $T(\Lambda) = \{g^{-1}\Lambda, g \in \Lambda\}$. For instance, in $\mathbb{Z}$, if $\Lambda = \{0, 1\}$, then $T(\Lambda) = \{\{-1, 0\}, \{0, 1\}\}$. Let $\mathcal{T}$ denote the set of such sets of translates, i.e. $\mathcal{T} = \{T(\Lambda) : \Lambda \in G, e \in \Lambda\}$. Note that $\mathcal{T}$ is a partition of the set $\{\Lambda \in G, e \in \Lambda\}$. Observe furthermore that $|T| = |\Lambda|$ for any $\Lambda \in T$. 
For any given $T \in T$, the value $|\Lambda| \|\Phi_\Lambda\|_\infty$ is the same for any $\Lambda \in T$, i.e. any $\Lambda$ such that $T = T(\Lambda)$. so we denote it by $c_T$. We can then express the bound on $\|A_\Phi\|_{\text{ShVar}}$ by summing over $T \in T$, as follows:

$$\sum_{\Lambda \in G \cap e \in \Lambda} a_\Lambda |\Lambda| \|\Phi_\Lambda\|_\infty = \sum_{T \in T} \sum_{\Lambda \in T} a_\Lambda c_T$$

$$= \sum_{T \in T} c_T \sum_{\Lambda \in T} a_\Lambda$$

$$= \sum_{T \in T} |\Lambda| \|\Phi_\Lambda\|_\infty$$

$$= \sum_{T} \sum_{\Lambda \in T} \|\Phi_\Lambda\|_\infty$$

$$= \sum_{T} c_T \sum_{\Lambda \in T} a_\Lambda$$

$$= \sum_{T} \sum_{\Lambda \in T} \|\Phi_\Lambda\|_\infty$$

Thus $\|A_\Phi\|_{\text{ShVar}} \leq 2C\|\Phi\|_B < \infty$. □

**Remark.** The following example, due to Nishant Chandgotia (personal communication), shows that if $\Phi \in B$ is not full-dimensional, then $A_\Phi$ can fail to be shell-regular.

**Example.** Let $X = \{0, 1\}^Z$, with the standard metric on $Z$, so $B_k = [-k, k]$. Define $\Phi = (\Phi_\Lambda)_{\Lambda \in Z}$ as follows: for any $i, j \in Z$, $\Phi_{\{i,j\}}(x) = \frac{1}{2}$ if $x_i = x_j = 1$ and 0 otherwise; and $\Phi_\Lambda \equiv 0$ for all other $\Lambda \in G$. Clearly $\Phi$ is translation-invariant.

We claim that $\Phi \in B$ but $A_\Phi \notin \text{ShReg}(X)$, where $A_\Phi$ is the image of $\Phi$ under the dictator map that ignores $\Lambda \in Z$ unless $0 = \inf \Lambda$. Indeed, $\|\Phi\|_B = 2 \sum_{j=1}^{\infty} \frac{1}{j} < \infty$, but

$$v_k(A_\Phi) = \sum_{l=k+1}^{\infty} \frac{1}{l^2} \geq \frac{1}{k+1}$$

which implies that

$$\|A_\Phi\|_{\text{ShVar}} \geq 2 \sum_{k=1}^{+\infty} \frac{1}{k+1} = +\infty$$

The next two propositions establish that for any full-dimensional interaction $\Phi \in B$, the images $A_\Phi$ and $A_\Phi'$ of $\Phi$ under any two translate-weighting maps are equivalent in a similar sense to that described in (13, p.118). That is, $A_\Phi$ and $A_\Phi'$ induce the same cocycle, so they have all of the same Gibbs measures (in either sense); and they have the same integral under any translate-invariant measure, so they have all of the same equilibrium measures, for a given notion of measure-theoretic entropy. (On non-amenable groups, such as the free group, various entropies that are equivalent in the amenable setting can fail to coincide [2].)

**Proposition 10.** Let $\Phi \in B$ be full-dimensional. Then $\Phi$ and $A_\Phi$ induce the same cocycle, i.e. $\phi_{A_\Phi} = \phi_\Phi$, where $A_\Phi$ is the image of $\Phi$ under an arbitrary translate-weighting map, with weights $a_\Lambda$. [13]
**Proof.** Suppose that \((x, y) \in \mathcal{X}_X\) with \(x_{\Delta^c} = y_{\Delta^c}\). Observe that

\[
\phi_{\Phi}(x, y) = \sum_{\Lambda \in G, \Lambda \cap \Delta \neq \emptyset} [\Phi_{\Lambda}(x) - \Phi_{\Lambda}(y)]
\]

To compute \(\phi_{A_{\Phi}}\), we first observe that, since \(\Phi_{\Lambda}(x \cdot g) = \Phi_{g\Lambda}(x)\), we have the following convenient expression for \(A_{\Phi}(x \cdot g)\):

\[
A_{\Phi}(x \cdot g) = - \sum_{\Lambda \in G, e \in \Lambda, \Lambda \cap g^{-1}\Delta \neq \emptyset} a_{g^{-1}\Lambda} \Phi_{\Lambda}(x)
\]

We then compute:

\[
\phi_{A_{\Phi}}(x, y) = \sum_{g \in G} \sum_{\Lambda \in G, \Lambda \cap \Delta \neq \emptyset} a_{g^{-1}\Lambda} [\Phi_{\Lambda}(x) - \Phi_{\Lambda}(y)]
\]

\[
= \sum_{\Lambda \in G, \Lambda \cap \Delta \neq \emptyset} \left( \sum_{g \in \Lambda} a_{g^{-1}\Lambda} \right) \Phi_{\Lambda}(x) - \Phi_{\Lambda}(y)]
\]

\[
= \phi_{\Phi}(x, y)
\]

Crucially, the interchange of infinite summations in the second equality from last was justified by the absolute convergence of the series defining the cocycles \(\phi_{A_{\Phi}}\) and \(\phi_{\Phi}\), implied by the regularity of \(\Phi\) and \(A_{\Phi}\). \(\square\)

Proposition 11 is similar to Theorem 5.42 in [11], which is stated for Ruelle’s operator \(A\), using specifications rather than cocycles.

**Proposition 11.** Let \(\mu\) be a \(G\)-invariant measure on \(X\), let \(\Phi \in \mathcal{B}\) be full-dimensional. Let \(A_{\Phi}\) be the image of \(\Phi\) under an arbitrary translate-weighting map determined by weights \((a_{\Lambda})_{\Lambda \in G, e \in \Lambda}\). Then the integral \(\int_X A_{\Phi} \, d\mu\) depends only on \(\Phi\) and \(\mu\), and not on the weights \(a_{\Lambda}\).

**Proof.** As in the proof of Proposition 10 for each finite \(\Lambda \in G\) with \(e \in \Lambda\), let \(T(\Lambda) = \{g^{-1}\Lambda \mid g \in \Lambda\}\). For any given \(T\), the quantity \(\int_X \Phi_{\Lambda} \, d\mu\) is constant as \(\Lambda\) ranges over \(T\), so we denote it by \(b_T\). We now compute:

\[
\int_X A_{\Phi} \, d\mu = \int_X \sum_{T \in T} \sum_{\Lambda \in T} a_{\Lambda} \Phi_{\Lambda} \, d\mu
\]

\[
= \sum_{T \in T} b_T \sum_{\Lambda \in T} a_{\Lambda}
\]

\[
= \sum_{\Lambda \in G, e \in \Lambda} \frac{1}{|\Lambda|} \int_X \Phi_{\Lambda} \, d\mu
\]

which does not depend on the weights \(a_{\Lambda}\), and in addition clearly expresses the integral \(\int_X A_{\Phi} \, d\mu\) as the average energy at the identity due to the interaction \(\Phi\).
To justify exchanging the integral and the infinite sum over sets of translates \( T \) in the second equality, observe that the sum converges absolutely to a continuous function, which is therefore bounded since \( X \) is compact and thus integrable since \( \mu \) is a probability measure. Indeed, let \( |\Phi| \) be the interaction given by \( |\Phi|_A = |\Phi_A| \). Then \( |\Phi| \) is clearly still full-dimensional, with \( \|\Phi\|_B = \|\Phi\|_B \), so

\[
\sum_{T \in T} \sum_{\Lambda \in T} a_{\Lambda} |\Phi_{\Lambda}| = A|\Phi| \in \text{ShReg}(X)
\]

by Proposition \( \square \).

Finally, we introduce a smaller Banach space \( \text{VolReg}(X) \) of \textit{volume-regular functions}, defined analogously to \( \text{ShReg}(X) \) by a volume norm rather than a shell norm. That is, \( \text{VolReg}(X) = \{ f : X \to \mathbb{R} : \|f\|_{\text{VolVar}} < \infty \} \) where we define

\[
\|f\|_{\text{VolVar}} := \sum_{k=0}^{\infty} \|B_k\|v_{k-1}(f)
\]

Volume-regularity clearly implies shell-regularity. The following result of Muir ([13], proof of Fact 7.6) is stated for \( \mathbb{Z}^d \), with the name \( \text{Reg}_d(X) \) for \( \text{VolReg}(X) \), but is valid, with the same proof, on any finitely generated group (or indeed any countable group, with volume-regularity defined with respect to some exhausting sequence of finite sets, rather than balls as in the finitely generated case).

**Theorem 12.** Let \( f \in \text{VolReg}(X) \) be a volume-regular potential. Then there exists an absolutely summable \( \Phi \in B \) with \( A_{\Phi} = f \) where \( A_{\Phi} \) is the image of \( \Phi \) under some dictator map.

In particular, any Gibbs measure (in either sense) for \( f \in \text{VolReg}(X) \) is also a Gibbs measure for any potential \( \Phi \in B \) with \( A_{\Phi} = f \), and vice versa. We remark that this applies in particular to any local potential, i.e. any potential \( f \) such that for some \( \Lambda \subset G \), \( f(x) \) is determined by \( x_{\Lambda} \). The interaction \( \Phi \) guaranteed in Theorem \( \text{12} \) then has bounded range.

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