Analysis of Motion Planning by Sampling in Subspaces of Progressively Increasing Dimension

Marios P. Xanthidis, Joel M. Esposito, Ioannis Rekleitis, and Jason M. O’Kane

Abstract—Despite the performance advantages of modern sampling-based motion planners, solving high dimensional planning problems in near real-time remains a challenge. Applications include hyper-redundant manipulators, snake-like and humanoid robots. Based on the intuition that many of these problem instances do not require the robots to exercise every degree of freedom independently, we introduce an enhancement to popular sampling-based planning algorithms aimed at circumventing the exponential dependence on dimensionality. We propose beginning the search in a lower dimensional subspace of the configuration space in the hopes that a simple solution will be found quickly. After a certain number of samples are generated, if no solution is found, we increase the dimension of the search subspace by one and continue sampling in the higher dimensional subspace. In the worst case, the search subspace expands to include the full configuration space — making the completeness properties identical to the underlying sampling-based planner. Our experiments comparing the enhanced and traditional version of RRT, RRT-Connect, and BidirectionalT-RRT on both a planar hyper-redundant manipulator and the Baxter humanoid robot indicate that a solution is typically found much faster using this approach and the run time appears to be less sensitive to the dimension of the full configuration space. We explore important implementation issues in the sampling process and discuss its limitations.

Index Terms—Motion and Path Planning, Redundant Robots

I. INTRODUCTION

It is well known that the general motion planning problem is PSPACE-complete [1] and that the runtime of even the best known exact algorithm is exponential in the dimension of the configuration space [2]. While sampling-based motion planning algorithms such as rapidly exploring random trees (RRTs) [3] and probabilistic roadmaps (PRMs) [4] are able to avoid explicit reconstruction of the free configuration space Canny’s algorithm [2] relies upon, they cannot avoid the curse of dimensionality. In particular, Esposito’s Conditional Density Growth Model (CDGM) for RRTs [5], predicts that the expected number of samples required for an RRT to explore a certain volume fraction of a space remains exponential in the dimension. For this reason, even though modern sampling-based planners have exhibited dramatic improvements over the original RRT and PRM techniques, computing real time solutions for systems with 10 or more degrees of freedom, such as hyper-redundant manipulators, snake-like robots or humanoids, remains a significant challenge.

1Marios P. Xanthidis, Ioannis Rekleitis, and Jason M. O’Kane are with the Computer Science and Engineering Department, University of South Carolina, United States, mariosx@email.sc.edu, [yiannnisr, jokane]@cse.sc.edu
2Joel M. Esposito is with the Weapons and Systems Engineering Department, United States Naval Academy, esposito@usna.edu

Figure 1: The simulated Baxter humanoid robot used in the experiments.

This study focuses on this path planning problem in high-dimensional configuration spaces. The intuition arises from three observations:

- For many systems, it is rare for all of the kinematic abilities to be needed for a specific task [6].
- Several studies that attempt to reduce the dimensionality in motion planning in various ways have already been done [7], [8], [9], [10], [11], [12], [13], [14].
- As shown by Šucan and Kavraki [15], even random projections of the configuration space can provide good estimates for its coverage.

Based on this intuition, we explore an algorithmic enhancement aimed at reducing the expected runtime of several common sampling-based planners. We propose beginning the search in a lower dimensional subspace of the configuration space in the hopes that a simple solution will be found quickly. After a certain number of samples are generated, if no solution is found, we increase the dimension of the search subspace by one and continue sampling in the larger subspace. We repeat this process until a solution is found. In the worst case, the search subspace expands to include the full dimensional configuration space — making completeness properties identical to the underlying sampling-based planner. In many cases our experiments indicate that a solution is found much faster using this approach and the run time appears to be less sensitive to the full dimension of the configuration space.

An important property of these subspaces is that a solution entirely lying inside these subspaces should be feasible, in the
absence of obstacles, so the initial and goal configurations must lie inside every subspace. The proposed method, by construction, generates subspaces that satisfy this constraint.

To evaluate this approach, we modified three well-established planners, RRT [3], RRT-Connect [16], and BidirectionalT-RRT [17] to produce RRT+, RRT+-Connect, and BidirectionalT-RRT+, with the + symbol indicating that the planners are enhanced using the idea described above. All three planners were compared to the originals and to KPIECE [13] and STRIDE [10]. These planners were tested on a planar hyper-redundant arm and on a simulated Baxter humanoid robot, shown in Figure 1 through OMPL [36] and the MoveIt! framework [19]. The results show that even a seemingly straightforward method for defining such subspaces can still have superior planning time performance for many problems compared to state-of-the-art planners.

The remainder of this paper is structured as follows. Section II reviews other approaches on planning for high-dimensional systems. Section III provides technical details of the enhancement, considering important issues such as how the search subspaces is selected, how to generate the samples, and when to expand the search dimension. Section IV presents our experience enhancing three well established planners RRT [3], RRT-Connect [16], and BidirectionalT-RRT [17]. Finally, Section V concludes with a discussion of future work.

II. RELATED WORK

The problem of motion planning has been proven to be PSPACE-hard [11]. During the late 1990’s, sampling-based methods were introduced and shown to be capable of solving challenging motion planning problems, but without guarantees of finding the solution in finite time [20]. The two most prominent representatives of those algorithms are probabilistic roadmaps (PRMs) by Kavraki et al. [4] that are useful for multiple queries in a known, generally stable environment, and RRTs by LaValle [3], that are more suitable for single query applications. Two other variations of RRTs, RRT-Connect by Kuffner and LaValle [16] that extends the tree more aggressively in each iteration, and T-RRT by Jailliet et al. [17] that plans efficiently in costmaps, were used in this study.

Although the performance of these techniques can be affected substantially by the degrees of freedom, some studies have used them successfully for high-dimensional configuration spaces. A method that uses PRMs and finds collision-free paths for hyper-redundant arms was presented by Park et al. [21]. Other studies use RRTs for motion planning of redundant manipulators, such as the work of Bertram et al. [22], which solves the inverse kinematics in a novel way. Weghe et al. [23] apply RRTs to redundant manipulators without the need to solve the inverse kinematics of the system. A study by Qian and Rahmani [24] combines RRTs and inverse kinematics in a hybrid algorithm in a way that drives the expansion of RRTs by the Jacobian pseudo-inverse.

Additionally, some works use RRTs for mobile manipulators. Vannoy et al. [25] propose an efficient and flexible algorithm for operating in dynamic environments. The work of Berenson et al. [26] provides an application of their technique to a 10-DoF mobile manipulator.

For multi-robot systems, many sampling-based algorithms have been proposed. The study of van den Berg and Overmars [7] uses a PRM and presents a prioritized technique for motion planning of multiple robots. Other studies use RRT-based algorithms such as the study by Carpin and Pagello [27] which introduced the idea of having multiple parallel RRTs for multi-robot systems. The work of Wagner [8] plans for every robot individually and, if needed, coordinates the motion in higher dimensional spaces. Other studies propose efficient solutions by using a single RRT [28, 29].

Furthermore, motion planning for humanoid robots with sampling-based algorithms has been explored; Kuffner et al. presented algorithms for motion planning on humanoid robots with both the use of PRMs [30] and RRTs [31]. Other studies, such as the work of Liu et al. [32], use RRTs for solving the stepping problem for humanoid robots.

The work of Vernaza and Lee tried to extract structural symmetries in order to reduce the dimensionality of the problem [9] by also providing near-optimal solutions. This technique is more time efficient than the traditional RRT, mostly for very high-dimensional configuration spaces. Yershova et al. [33] proposed an approach to focus sampling in the most relevant regions.

Finally, other approaches attempt to deal with the dimensionality in different ways. Gipson et al. developed STRIDE [10] which samples non-uniformly with a bias toward unexplored areas of the configuration space. KPIECE [18] by Şucan and Kavraki uses random 2D and 3D projections to estimate the coverage of . Additionally, Gochev et al. [11] proposed a motion planner that decreases the dimensionality by recreating a configuration space with locally adaptive dimensionality. Yoshida’s work [12] tries to sample in ways that exploit the redundancy of a humanoid system. Kim et al. [13] present an RRT-based algorithm for articulated robots that reduces the dimensionality of the problem by projecting each sample into subspaces that are defined by a metric. Shkolnik and Tedrake [34] plan for highly redundant manipulators in the dimensional task space with the use of Jacobian transpose and Voronoi bias. Wells and Plaku [14] reduce the dimensionality for 2-D hyper-redundant manipulators by modeling the end-effector as a single mobile robot, and the other links as trailers that are getting pulled. Lastly, the work of Bayazit et al. [35] where a PRM was used to plan in subspaces of the configuration space, creates paths that solve an easier problem than the original and then iteratively optimize the solution until the solution becomes valid.

This paper introduces the idea of iteratively searching in lower-dimensional subspaces, and emphasizes the potential of using such approach for efficient motion planning on arbitrary hyper-redundant systems. The approach operates regardless of if there is an articulated robot, a humanoid, or a heterogeneous multi-robot system, in a sense that additional limitations are not introduced in the original planners. Unlike previous works,
the approach tries to find paths that are not only confined entirely to a subspace but also in a subspace in which a solution can exist, since the initial and goal configurations are part of the subspace. Contrary to the work of Bayazit et al. [35] the approach searches in subspaces that are strictly lower-dimensional. Intuitively, instead of finding results faster by simplifying the problem, our approach tries to find plans fast by simplifying the model of the system by applying virtual constraints, and then by iteratively relaxing them.

III. METHODOLOGY

A. Problem Statement

Let \( \mathcal{C} \) denote a configuration space with \( n \) degrees of freedom, partitioned into free space \( \mathcal{C}_{\text{free}} \) and obstacle space \( \mathcal{C}_{\text{obs}} \) with \( \mathcal{C} = \mathcal{C}_{\text{free}} \cup \mathcal{C}_{\text{obs}} \). The obstacle space \( \mathcal{C}_{\text{obs}} \) is not explicitly represented, but instead can be queried using collision checks on single configurations or short path segments. Given initial and goal configurations \( q_{\text{init}}, q_{\text{goal}} \in \mathcal{C}_{\text{free}} \), we would like to find a continuous path within \( \mathcal{C}_{\text{free}} \) from \( q_{\text{init}} \) to \( q_{\text{goal}} \).

For purposes of sampling, we assume that each degree of freedom in \( \mathcal{C} \) is parameterized as an interval subset of \( \mathbb{R} \), so that

\[
\mathcal{C} = \left[ c_1^{(\text{min})}, c_1^{(\text{max})} \right] \times \cdots \times \left[ c_n^{(\text{min})}, c_n^{(\text{max})} \right] \subseteq \mathbb{R}^n .
\] (1)

Note that we treat \( \mathcal{C} \) as Euclidean only in the context of sampling; other operations such as distance calculations and the generation of local path segments utilize identifications on the boundary of \( \mathcal{C} \) as appropriate for the topology.

B. Planning in Subspaces

The proposed method is based on searching for a solution in lower dimensional subspaces of \( \mathcal{C} \), in hopes that such a path might be found faster than expanding the tree in all dimensions. The underlying idea is to exploit the redundancy of each system for each problem. To achieve this, the algorithm starts searching in the unique linear 1-dimensional subspace of \( \mathcal{C} \) that contains \( q_{\text{init}} \) and \( q_{\text{goal}} \). If this search fails, the planner expands its search subspace by one dimension. This process continues iteratively until the planner finds a path, or until it searches in all of \( \mathcal{C} \). In each subsearch the tree structure created in lower dimensions is kept and expanded in subsequent stages.

Algorithm 1: RRT+

\begin{algorithm}
\caption{RRT+}
\begin{algorithmic}[1]
\State \textbf{Input} : A configuration space \( \mathcal{C} \), an initial configuration \( q_{\text{init}} \), and a goal configuration \( q_{\text{goal}} \).
\State \textbf{Output}: RRT graph \( G \).
\State \( G.\text{init}(q_{\text{init}}) \)
\State \( \mathcal{C}_{\text{sub}} \leftarrow 1\text{-d subspace of } \mathcal{C}, \text{ through } q_{\text{init}} \text{ and } q_{\text{goal}} \)
\While{True}
\State \( q_{\text{rand}} \leftarrow \text{sample drawn from } \mathcal{C}_{\text{sub}} \)
\State \( q_{\text{new}} \leftarrow \text{NewConf}(q_{\text{near}}, q_{\text{rand}}) \)
\State \( G.\text{AddVertex}(q_{\text{new}}) \)
\State \( G.\text{AddEdge}(q_{\text{near}}, q_{\text{new}}) \)
\If{done searching \( \mathcal{C}_{\text{sub}} \) \text{ then}}
\If{dim(\( \mathcal{C}_{\text{sub}} \)) < dim(\( \mathcal{C} \))}
\State Expand \( \mathcal{C}_{\text{sub}} \) by one dimension.
\Else
\State \text{return } G
\EndIf
\EndIf
\EndWhile
\end{algorithmic}
\end{algorithm}

Algorithm 1 summarizes the general approach, as applied to RRT. The planner starts optimistically by searching in one dimension, along the line passing through \( q_{\text{init}} \) and \( q_{\text{goal}} \). If this search fails to find a path—a certainty, unless there are no obstacles between \( q_{\text{init}} \) and \( q_{\text{goal}} \)—the search expands to a planar subspace that includes \( q_{\text{init}} \) and \( q_{\text{goal}} \), then to a 3D flat \(^1\) and so on until, in the worst case, the algorithm eventually searches all of \( \mathcal{C} \); see Figure 2.

The description in Algorithm 1 leaves three important elements unspecified. First, the algorithm needs a method for selecting and representing the subspace \( \mathcal{C}_{\text{sub}} \) (Lines 2 and 11). Second, a method is required for sampling from this subspace (Line 4). Third, the conditions that must be met before moving to the next subsearch must be defined (Line 9). The choices explored in this study are described in the next sections.

C. Representing and Sampling from Subspaces

The central idea is to search for solutions in subspaces of progressively higher dimensions. The primary constraint on

\(^1\)We use the term flat to refer to a subset of \( \mathbb{R}^n \) congruent to some lower-dimensional Euclidean space.
Algorithm 2: Prioritized Sampler

**Input**: Initial configuration \( q_{\text{init}} \), goal configuration \( q_{\text{goal}} \), set of constrained DoFs \( P_{\text{con}} \), boundaries of the random number \( r, r_{\text{min}}, r_{\text{max}} \).

**Output**: Sample configuration \( q \)

1. \( q \leftarrow \) random point in \( \mathcal{C} \) along line \( L \) from \( q_{\text{init}} \) to \( q_{\text{goal}} \) with \( r_{\text{min}} \) and \( r_{\text{max}} \).
2. for \( i = 1, \ldots, n \) do
3.  if \( i \notin P_{\text{con}} \) then
4.     \( q[i] \leftarrow \) Random(0, 1)*\((c_i^{(\text{max})} - c_i^{(\text{min})}) + c_i^{(\text{min})}) \)
5. end
6. end
7. return \( q \)

these subspaces is that they must contain both \( q_{\text{init}} \) and \( q_{\text{goal}} \). Subspaces that violate this constraint cannot, of course, contain a path connecting \( q_{\text{init}} \) to \( q_{\text{goal}} \).

In general, the algorithm’s selections for \( \mathcal{C}_{\text{sub}} \) should ideally be directed by the likelihood that a solution will exist fully within \( \mathcal{C}_{\text{sub}} \).

The choice that we investigate in this paper—one that trades some generality for simplicity—is a prioritized release of the degrees of freedom. Given a set \( P_{\text{con}} \subseteq \{1, \ldots, n\} \) of DoFs to be constrained, we can form \( \mathcal{C}_{\text{sub}} \) by constraining the DoFs in \( P_{\text{con}} \) to form a line passing from \( q_{\text{init}} \) and \( q_{\text{goal}} \) and allowing the remaining DoF to vary freely.

Next, the algorithm requires a technique for drawing samples from \( \mathcal{C}_{\text{sub}} \). The sampling happens in a very efficient linear time method that initially generates a sample within \( \mathcal{C} \) along the line between \( q_{\text{init}} \) and \( q_{\text{goal}} \) by selecting a random scalar \( r \) and applying:

\[
q_{\text{rand}}[i] = (q_{\text{goal}}[i] - q_{\text{init}}[i])r + q_{\text{init}}[i] \quad (2)
\]

The algorithm then modifies \( q_{\text{rand}} \) by inserting, for each DoF not in \( P_{\text{con}} \), a different random value within the range for that dimension; see Algorithm 3.

To ensure that the original sample along the line \( L \) between \( q_{\text{init}} \) and \( q_{\text{goal}} \) is within \( \mathcal{C} \), we compute \( r_{\text{min}} \) and \( r_{\text{max}} \) using Algorithm 3 and select \( r \) randomly from the interval \([r_{\text{min}}, r_{\text{max}}]\). The ComputeBoundaryValues function finds the intersections of this line all the different \( c_i^{(\text{min})} \) and \( c_i^{(\text{max})} \) flats, and defines the limits of \( r \) from the values of \( r \) producing the two intersected points that lie inside the \( \mathcal{C} \). The ComputeBoundaryValues function is called only once before the planning loop.

The prioritized method provides an efficient and easy way to sample from projections of arbitrary dimensionality, while, as stated before, it provides valuable understanding that serves the purpose of our study.

D. Terminating the subsearches

The only remaining detail to be discussed is how long the search in each subspace should continue. A set of timeouts \( \{t_1, t_2, \ldots, t_n\} \) should be generated where \( t_i \) corresponds to the amount of time spent in the \( i^{th} \) iteration. Ideally, the planner should stop searching in subspaces that seem unlikely to provide a solution. For simplicity in this paper we precompute the timeouts by assuming that the \( t_i \) follow a geometric progression. The idea is to exponentially increase the number of samples in each successive subsearch, acknowledging the need for more samples in higher dimensions.

The proposed approach uses two different parameters specified by the user. The first parameter \( T \) is the timeout for the entire algorithm. The second parameter \( \alpha > 1 \) is a factor describing a constant ratio of the runtime between successive subsearches. The total time \( T \) available to the algorithm can be expressed in terms of \( \alpha \) and a base time \( t_0 \):

\[
T = \sum_{i=1}^{n} t_0 \alpha^i \quad (3)
\]

Solving for \( t_0 \) we obtain:

\[
t_0 = \frac{\alpha - 1}{\alpha^\alpha - 1} T \quad (4)
\]

Using this \( t_0 \) every \( t_i \) can be computed as:

\[
t_i = \alpha t_{i-1} \quad (5)
\]

In this study values for \( \alpha \) ranging between 1 and 2 and \( T \) scaling linearly with the dimensionality of the configuration space seem to lead to good performance.

IV. EXPERIMENTS

For the experiments three new planners were developed using the OMPL framework [36]. These planners work by applying the technique to three RRT variants: (1) RRT [13] with default goal bias of 0.05, (2) RRT-Connect [16], and (3) the bidirectional T-RRT [17] by assuming a uniform costmap. The T-RRT variant is intended to test the effect of our method to a powerful costmap planner.
Figure 3: The two different environments, called Cluttered Random and Horn, used in [10], (a-b) with red indicating $q_{init}$ and green $q_{goal}$, and two different solutions for those environments (c-d) with RRT$^+$-Connect, respectively.

A. Experiments with a 2D hyper-redundant manipulator

In order to test the ability of the new planners to adapt to different problems, the prioritization was chosen randomly for each run, although an optimization might be expected to give results faster. We demonstrate that given enough redundancy even a random prioritization will provide results much faster. A computer with 6th Generation Intel Core i7-6500U Processor (4MB Cache, up to 3.10 GHz) and 16GB of DDR3L (1600MHz) RAM was used.

The OMPL’s standard example of a 2D hyper-redundant manipulator, created for STRIDE [10], was used in order to test and compare the enhanced planners in a standard way. Two different environments were tested 100 times each for a kinematic chain with varying 12 to 20 degrees of freedom: A Cluttered Random environment and a Horn environment; see Figure 3. The initial and goal configurations were the same as shown in Figure 3, so a qualitative comparison in problems with different redundancies could be done.

In all the cases, the enhanced versions of each planner were faster and in most cases significantly faster than the original ones. The average and the median times for the Random Cluttered environment are presented in Figure 4 and in Figure 5 for the Horn environment. Although the plots of the path length are not provided, our planners produced solutions with length — after the standard simplification method provided by OMPL. — very similar, if not slightly better, than the other planners.

The BiT-RRT$^+$ not only outperformed the BiT-RRT by a wide margin but also outperformed all the other planners using uniform sampling. Additionally, it provided competitive results to robust planners with biased sampling such as KPIECE [18] and STRIDE [10], which as expected perform much better in less redundant environments.

Interestingly, for each problem, the single fastest solution across all trials was generated by BiT-RRT$^+$. This suggests that a good choice of prioritization may give results faster in a consistent way.

Additional experiments with the fastest planners were done from 12 to 30 dimensions for the Cluttered environment. The performance of the BiT-RRT$^+$ in Figure 6, it deals much better with the dimensionality. Also the BiT-RRT$^+$ provides the fastest results among all the planners for the most part. Even with random prioritization only after the 29 dimensions due to insufficient prioritizations the BiT-RRT$^+$ performs slower than KPIECE and STRIDE.

B. Experiments with Baxter

Experiments are presented for the Baxter humanoid robot (Figure 1) of 14 degrees of freedom using the OMPL [36] and MoveIt! framework [19] with an Intel i7-7700 8-core processor (3.6GHz), and 32 GiB RAM.

Instead of choosing the prioritizations randomly, a generic task independent policy was used to show that even naive policies can eliminate the outliers observed in the other experiments and lead to superior performance. The policy was giving priority to the joints closer to the base.

We tested the enhanced planners in a cluttered workspace, consisted by a table and four parallel pillars. As shown in Figure 7, Baxter starts with the manipulators in the relaxed position below the table and the goal is to reach the configuration that both arms fit between the pillars. Moreover, in order to
Figure 5: The average (a) and the median (b) time for the Horn environment.

Figure 6: Comparison of the median time for the Cluttered Random environment from 12 to 30 dimensions of BiT-RRT and BiT-RRT+, KPIECE and STRIDE. Interestingly, BiT-RRT+ is more than 200 times faster than BiT-RRT for 30 DoF, making the problem more challenging for the enhanced planners, even when they were failing in the case of RRT+, produced solutions very quickly showing the advantage of sampling in lower-dimensional subspaces. Similarly to the experiments presented in the previous section, the fastest planner among all was the BiTRRT+. Dominantly the solutions from RRT+- were found in a 8-dimensional subspace, and from RRT+-Connect and BiTRRT+ in a 6-dimensional.

V. CONCLUSION

We studied a novel method for potentially accelerating motion planning in high dimensional configuration spaces by sampling in subspaces of progressively increasing dimension. The method provides, on average, solutions much faster than the original RRT-based methods. The approach is general enough to be applied to a broad variety of motion planning problems. For example, our experiments show potential for planning with costmaps via adaptation of the bidirectional T-RRT. The enhanced planners can also be adjusted to each problem and provide results even faster. The study clearly shows that such methods have great potential in solving very fast, seemingly difficult problems and further investigation is needed.

Future work will target a number of important questions. First, it would be very interesting to find an efficient way to choose subspaces that are more likely to contain solutions. Second, it is important to provide a general way to identify when a new iteration should begin, using a metric of the expansion of the tree, and eliminating the parameters. Prior work [5] provides an estimation of the coverage in an efficient analytical way. Moreover, by using the previous metric, it is possible to identify when the tree overcame a difficult area and then reduce the dimensionality of the search, in order to accelerate the results further. For both of the above problems discrete methods such as the one of Şucan and Kavraki [15] should be also considered.

Currently, there are two planners in OMPL that in some cases outperform the RRT+ planners in our experiments: KPIECE which uses random 2D or 3D projections to estimate the coverage efficiently and STRIDE which samples non-uniformly with a bias to narrow spaces. Our study can extend the first planner to sample strictly in subspaces of $C$ and...
Figure 8: The histograms comparing the enhanced planners, with the original ones. The red line at 60 seconds indicates the Timeout, and the results after that line should be considered failures.

use 2D, 3D or 4D projections to estimate the coverage, and the second planner, can be used with the enhancement to accelerate the solutions in the lower dimensional subspaces where narrow passages are expected to be common. The integration of the ideas underlying the proposed method with those planners to accelerate the results is left for future work.

Lastly, we plan to explore the ability of the method to efficiently produce paths that satisfy some natural constraints of each system. Since the subspaces are defined only by simple constraints between the DoFs, subspaces that satisfy some dynamic constraints can be explored by defining dynamic constraints between the different DoFs. Similarly, the ability of the method to find alternative solutions by avoiding exploring non-desired areas shows some potential.

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