The Generating Function of the Embedding Capacity for 4-dimensional Symplectic Ellipsoids

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Abstract

Quite recently, McDuff showed that the existence of a symplectic embedding of one four-dimensional ellipsoid into another can be established by comparing their corresponding sequences of ECH capacities. In this note we show that these sequences can be encoded in a generating function, which gives several new equivalent formulations of McDuff’s theorem.

1. Embedding 4-dimensional Symplectic Ellipsoids. We consider ellipsoids

$$E(a, b) := \left\{ z \in \mathbb{C}^2 : \frac{|z_1|^2}{a} + \frac{|z_2|^2}{b} \leq 1 \right\}$$

equipped with the standard symplectic structure $\omega_0 = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$ of Euclidean space $\mathbb{R}^4$. The embedding problem in symplectic geometry asks if for given integers $a, b, c, d > 0$ there exists a symplectic embedding $E(a, b) \hookrightarrow E(c, d)$. Since each such embedding preserves the volume, an immediate obstruction for existence is $ab \leq cd$.

There are further obstructions which have their origin in embedded contact homology. Namely, define $N(a, b)$ to be the sequence of numbers from the set

$$S(a, b) := \{ ka + lb : k, l \in \mathbb{Z} \text{ and } k, l \geq 0 \}$$
aranged in nondecreasing order with repetitions. For example, we have

$$N(2, 3) = (0, 2, 3, 4, 5, 6, 6, 7, 8, 8, 9, 9, \ldots).$$

For sequences of numbers $A$ and $B$ define a partial ordering by saying $A \preceq B$ if, for all $n \geq 0$, the $n$-th entry of $A$ is not larger than the $n$-th entry of $B$. Hutchings showed in [9] that an obstruction for the embedding problem is given by $N(a, b) \preceq N(c, d)$. Indeed, as conjectured by Hofer and recently proved by McDuff in [12], this is the only obstruction.

THEOREM 1. There is a symplectic embedding $E(a, b) \hookrightarrow E(c, d)$ if and only if

$$N(a, b) \preceq N(c, d).$$

Hence the embedding problem for symplectic ellipsoids can be reduced to studying the sequences $N(a, b)$. Define a new sequence $L(a, b)$ by

$$L_n(a, b) := \max \{ j : N_j(a, b) \leq n \} = \# \{ m \in S(a, b) : m \leq n \}.$$

From the definition it is clear, that

$$L(a, b) \geq L(c, d) \iff N(a, b) \preceq N(c, d). \quad (1.1)$$

Geometrically, $L_n(a, b)$ corresponds to the number of lattice points in the triangle $T_{a,b}^n$ bounded by $x = 0$, $y = 0$ and $ax + by = n$, including points on its boundary (Figure 1).

The aim of this note is to remark that the generating function of $L(a, b)$ is given by a surprisingly simple formula.

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Proposition 1. For $0 \leq z < 1$ we have the expansion
\[ \frac{1}{(1-z)(1-z^a)(1-z^b)} = \sum_{n=0}^{\infty} \mathcal{L}_n(a, b) z^n. \] (1.2)

Proof: We have
\[
\frac{1}{(1-z)(1-z^a)(1-z^b)} = \left( \sum_{k=0}^{\infty} z^k \right) \left( \sum_{l=0}^{\infty} z^{al} \right) \left( \sum_{m=0}^{\infty} z^{bm} \right)
= \sum_{n=0}^{\infty} \left( \# \{ (k, l, m) \in \mathbb{Z}^3 : k, l, m \geq 0 \text{ and } k + al + bm = n \} \right) z^n
= \sum_{n=0}^{\infty} \left( \# \{ (l, m) \in \mathbb{Z}^2 : l, m \geq 0 \text{ and } al + bm \leq n \} \right) z^n = \sum_{n=0}^{\infty} \mathcal{L}_n(a, b) z^n.
\]

There is also a geometric interpretation behind this formula, which will be explained in the next section. Note that $\mathcal{L}_n(a, b)$ corresponds to the number of partitions of $n$ into parts of size $1$, $a$ or $b$ which is known as a denumerant problem. In this case one always obtains a rational generating function with poles that are roots of unity. Multiplying both sides of (1.2) by the denominator and comparing coefficients leads to the linear recurrence relation
\[ \mathcal{L}_n(a, b) = \mathcal{L}_{n-1}(a, b) + \mathcal{L}_{n-a}(a, b) + \mathcal{L}_{n-b}(a, b) + \mathcal{L}_{n-a-b-1}(a, b) - \mathcal{L}_{n-a-1}(a, b) - \mathcal{L}_{n-b-1}(a, b) - \mathcal{L}_{n-a-b}(a, b) \]
for $n > 0$. To initiate we take $\mathcal{L}_0(a, b) = 1$ and set $\mathcal{L}_n(a, b) := 0$ for $n < 0$. The following relation can be proved in an elementary way (see [6], section 5.6).

Proposition 2. For $n > 0$ we have
\[ \mathcal{L}_n(a, b) = \mathcal{L}_{n-1}(a, b) + \left\lfloor \frac{n}{ab} \right\rfloor + \varepsilon(n) \] (1.3)

where $\varepsilon(n)$ is either 0 or 1 and its value just depends on the remainder
\[ [n] \in \mathbb{Z}_{ab\mathbb{Z}}. \]

In some sense the whole information of $\mathcal{L}(a, b)$ is therefore stored in its first $ab$ terms. Moreover, one obtains the asymptotic behaviour
\[ \mathcal{L}_n(a, b) \sim \frac{n^2}{2ab}. \]
In the following, we denote the generating function by

\[ g_{a,b}(z) = \frac{1}{(1 - z)(1 - z^a)(1 - z^b)}. \]

Denote further by \( f^{(k)} \) the \( k \)-th derivative of a function \( f \). Via Cauchy’s integral formula we compute

\[ \mathcal{L}_n(a,b) = \frac{g_{a,b}^{(n)}(0)}{n!} = \frac{1}{2\pi i} \int_\gamma \frac{g_{a,b}(\xi)}{\xi^{n+1}} = \frac{1}{2\pi i} \int_\gamma \frac{d\xi}{(1 - \xi)(1 - \xi^a)(1 - \xi^b)\xi^{n+1}}, \]

which might be useful for numerical purposes.

On the space \( C^\infty((-1,1), \mathbb{R}) \) consider the partial ordering by saying \( f \succeq g \) iff \( f^{(k)}(x) \leq g^{(k)}(x) \) for all \( k \geq 0 \) and \( x \in [0,1) \). Putting things together we obtain the following

**Corollary 1.** There is a symplectic embedding \( \text{int} E(a,b) \hookrightarrow E(c,d) \) if and only if one of the following equivalent conditions is fulfilled:

1. \( \mathcal{N}(a,b) \preceq \mathcal{N}(c,d) \)
2. \( \mathcal{L}(a,b) \succeq \mathcal{L}(c,d) \)
3. \( g_{a,b} \succeq g_{c,d} \)

**Proof:** The equivalence of (a) and (b) was already noticed in [1]. Now (b) implies for any integer \( k \geq 0 \) and \( z \in [0,1) \)

\[ g_{a,b}^{(k)}(z) = \sum_{n=k}^{\infty} k! \binom{n}{k} \mathcal{L}_n(a,b) z^{n-k} \geq \sum_{n=k}^{\infty} k! \binom{n}{k} \mathcal{L}_n(c,d) z^{n-k} = g_{c,d}^{(k)}(z). \]

On the other hand (c) leads to

\[ \mathcal{L}_k(a,b) = \frac{g_{a,b}^{(k)}(0)}{k!} \geq \frac{g_{c,d}^{(k)}(0)}{k!} = \mathcal{L}_k(c,d). \]

Thus the embedding question \( \text{int} E(a,b) \hookrightarrow E(c,d) \) relates to the problem if all coefficients of

\[ G_{a,b,c,d}(z) := \frac{(1 - z^a)(1 - z^b) - (1 - z^a)(1 - z^b)}{(1 - z)(1 - z^a)(1 - z^b)(1 - z^d)} = g_{a,b}(z) - g_{c,d}(z) = \sum_{n=0}^{\infty} (\mathcal{L}_n(a,b) - \mathcal{L}_n(c,d)) z^n \]

are nonnegative. Since \( G_{a,b,c,d} \) is again a rational function, its coefficients satisfy a linear recurrence. In [3], Conjecture 2 it is conjectured that each rational function, whose dominating poles (i.e. the ones of maximal modulus) do not lie on \( \mathbb{R}_+ \), has infinitely many positive and infinitely many negative coefficients in its power series expansion. Of course, we cannot apply this to \( G_{a,b,c,d} \), since all of its poles have modulus 1 and 1 \( \in \mathbb{R}_+ \) occurs among them. One of the most celebrated results in the theory of linear recurrence sequences is the Skolem-Mahler-Lech theorem. It asserts that if a sequence \( (a_n) \) satisfies a linear recurrence relation, then the zero set

\[ \{ n \in \mathbb{N} : a_n = 0 \} \]

is the union of a finite set and finitely many arithmetic progressions.

Let us use the approach via generating functions to check algebraically that for each positive integer \( n \in \mathbb{N} \) there is a symplectic embedding

\[ \text{int} E(1, n^2) \hookrightarrow B(n). \]

Here the latter denotes the ball \( B(n) := E(n,n) \) of radius \( n \). Geometrically, this corresponds to a filling of \( B(n) \) by \( n^2 \) equal symplectic balls (Proposition 2.2 in [10]). The possibility of such a filling can be quite easily observed via toric models. For details we refer the reader to the survey paper [10].
With the lattice count interpretation we have
\[ L_k(n, n) = d\left(\left\lfloor \frac{k}{n} \right\rfloor \right), \]
where \( d(k) := \frac{1}{2}(k+1)(k+2) \) denotes the \( k \)-th triangle number. Consequently, by Proposition 1
\[ g_{n,n}(z) = \frac{1}{(1-z)(1-z^n)^2} = \sum_{k=0}^{\infty} d\left(\left\lfloor \frac{k}{n} \right\rfloor \right) z^k. \]
For integers \( k \geq 0 \) set
\[ c(k) = \begin{cases} 1 & \text{if } k \equiv 0 \pmod{n}, \\ -1 & \text{if } k \equiv 1 \pmod{n}, \\ 0 & \text{otherwise}. \end{cases} \]
Then
\[ \frac{(1-z^n)^2}{(1-z)(1-z^{n^2})} = 1 - z^n \cdot (1-z^n) \sum_{k=0}^{\infty} z^{kn^2} = (1 + z + \ldots + z^{n-1}) \sum_{k=0}^{\infty} (z^{kn^2} - z^{(kn+1)n}) \]
\[ = \sum_{k=0}^{\infty} c\left(\left\lfloor \frac{k}{n} \right\rfloor \right) z^k, \]
such that
\[ g_{1,n}(z) = \frac{g_{1,n^2}(z)}{g_{n,n}(z)} : g_{n,n}(z) = \frac{(1-z^n)^2}{(1-z)(1-z^{n^2})}, g_{n,n}(z) = \left( \sum_{k=0}^{\infty} c\left(\left\lfloor \frac{k}{n} \right\rfloor \right) z^k \right) \left( \sum_{l=0}^{\infty} d\left(\left\lfloor \frac{l}{n} \right\rfloor \right) z^l \right). \]
In view of (1.2) it suffices to show for each nonnegative integer \( N \)
\[ \sum_{k=0}^{N} c\left(\left\lfloor \frac{k}{n} \right\rfloor \right) d\left(\left\lfloor \frac{N-k}{n} \right\rfloor \right) \geq d\left(\left\lfloor \frac{N}{n} \right\rfloor \right). \] (1.4)
For given \( N \geq 0 \) we pick integers \( 0 \leq p, q, r \) with \( q,r < n \) such that \( N = pn^2 + qn + r \). Setting \( d(-1) = d(-2) := 0 \), we obtain from the periodicity of \( c(k) \)
\[ \sum_{k=0}^{p} c\left(\left\lfloor \frac{k}{n} \right\rfloor \right) d\left(\left\lfloor \frac{N-k}{n} \right\rfloor \right) = \sum_{j=0}^{p} ((r+1)d(jn+q) + (n-r-1)d(jn+q-1)) \]
\[ - \sum_{j=0}^{p} ((r+1)d(jn+q-1) + (n-r-1)d(jn+q-2)) \]
\[ = \sum_{j=0}^{p} ((r+1)(jn+q+1) + (n-r-1)(jn+q)) \]
\[ = \frac{p(p+1)}{2}n^2 + (p+1)qn + (p+1)(r+1) = (p+1)(N+1) - \frac{p(p+1)}{2}n^2. \]
For \( q < n, n \geq 2 \) we have
\[ \frac{3q}{2} + \frac{q^2}{2} = \frac{q(q+1)}{2} + q \leq \frac{nn}{2} + \frac{nn}{2}, \]
such that \( \frac{3p}{2} + \frac{p^2}{2} \leq qn \) holds for all nonnegative integers \( q < n \). One also easily checks that \( \frac{3pn}{2} \leq \frac{pm^2}{2} + p \) holds for all nonnegative integers \( p, n \). Thus
\[ (p+1)(N+1) \geq (p+1)(pn^2 + qn + 1) = p^2n^2 + pn^2 + pqn + qn + p + 1 \]
\[ \geq p^2n^2 + \frac{pm^2}{2} + \frac{3pn}{2} + pqn + \frac{q^2}{2} + \frac{3q}{2} + 1 = \frac{(pn + q + 1)(pn + q + 2)}{2} + \frac{p(p+1)}{2}n^2 \]
\[ = d\left(\left\lfloor \frac{N}{n} \right\rfloor \right) + \frac{p(p+1)}{2}n^2. \]
shows that (1.4) is valid.

The symplectic capacity function \( c : [1, \infty) \to \mathbb{R} \) defined by
\[
c(a) := \inf \left\{ \mu : \text{int } E(1, a) \overset{\delta}{\to} B(\mu) \right\}
\]
is studied in detail in [11]. We just computed \( c(a^2) = a \) for positive integers \( a \). Indeed, \( c(a) = \sqrt{a} \) holds for \( a \in \mathbb{N} \) if \( a \) is 1, 4 or \( \geq 9 \). The other values for integral \( a \) are given by
\[
c(2) = c(3) = 2, \quad c(5) = c(6) = \frac{5}{2}, \quad c(7) = \frac{8}{3}, \quad c(8) = \frac{17}{6}.
\]
We finish this section by remarking that Theorem 1 does not hold in higher dimensions. Counterexamples are due to Guth [5] and Hind-Kerman [7]. Even worse, embedded contact homology only exists in dimension 4 and there is so far no good guess of what a criterion for embedding ellipsoids could be.

2. Counting Lattice Points in Polyhedra. Let \( P \subset \mathbb{R}^d \) be a polyhedron. In order to count the lattice points in \( P \) one associates the generating function
\[
\sum_{m \in P \cap \mathbb{Z}^d} x^m \quad \text{with} \quad x^m = x_1^{\mu_1} \cdots x_d^{\mu_d}
\]
for \( m = (\mu_1, \ldots, \mu_d) \). The total number of lattice points in \( P \) is then given by the value of the generating function at \( x = (1, \ldots, 1) \). The advantage of this approach is that these generating functions can still be computed for cones \( K \subset \mathbb{R}^d \), which actually contain an infinite number of lattice points. A cone is characterized by the property that \( 0 \in K \) and for every \( x \in K \) and \( \lambda \geq 0 \) one has \( \lambda x \in K \). For example, the generating function of the non-negative orthant is given by
\[
\sum_{m \in (\mathbb{R}^d_+ \cap \mathbb{Z}^d) \times \mathbb{Z}^d} x^m = d \prod_{i=1}^d \frac{1}{1-x_i}.
\]
The generating function of a polyhedron \( P \) is calculated as the sum of generating functions of tangent cones at the vertices of \( P \), for details see [2].

Usually a cone \( K \) is given as a span of vectors \( u_1, \ldots, u_k \in \mathbb{R}^d \),
\[
K = \text{co}(u_1, \ldots, u_k),
\]
meaning that every vector \( v \in K \) can be written as a sum \( v = \sum \lambda_i v_i \) with \( \lambda_i \geq 0 \). A cone \( K \) is called unimodular, if it is spanned by \( u_1, \ldots, u_d \in \mathbb{Z}^d \) and these vectors form a basis of the lattice. Generating functions for unimodular cones are particularly easy to calculate. Unfortunately, all tangent cones of the triangle \( T^n_{a,b} \) are unimodular only if \( a = b \). Hence we cannot expect an easy formula for \( a \neq b \), also we have already seen that the number of lattice points in \( T^n_{a,a} \) is given by
\[
d \left( \left\lfloor \frac{n}{a} \right\rfloor \right).
\]
Instead consider \( T^n_{a,b} = \{ x, y \in \mathbb{R}^2_{\geq 0} : ax + by \leq n \} \) as lying in the hyperplane \( z \equiv n \) in \( \mathbb{R}^3 \). Then
\[
\bigcup_{n \geq 0} T^n_{a,b} \cap \mathbb{Z}^3 = \text{co} \left( \begin{pmatrix} 1 \\ 0 \\ a \end{pmatrix} , \begin{pmatrix} 0 \\ 1 \\ b \end{pmatrix} , \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) \cap \mathbb{Z}^3.
\]
The latter cone is unimodular and has generating function
\[
f(x, y, z) = \frac{1}{(1-xz^n)(1-yz^n)(1-z)}.
\]
In particular, the number of lattice points in \( T^n_{a,b} \) corresponds to the coefficient of \( z^n \) of the expansion of \( f \) restricted to \( x = y = 1 \). This explains formula (1.2).
3. Scale Invariance. The condition in Theorem 1 is scale invariant, meaning that for each real \( \lambda > 0 \) one has

\[
N(a, b) \leq N(c, d) \iff N(\lambda a, \lambda b) \leq N(\lambda c, \lambda d).
\]

Unfortunately, this scale invariance does not descend to the generating functions. Thus \( g_{a,b} \geq g_{c,d} \) does not imply \( g_{\lambda a, \lambda b} \geq g_{\lambda c, \lambda d} \) and it does not make sense to extend our notion of generating functions to real parameters \( a, b \). For rational \( a, b, c, d \in \mathbb{Q} \), the best one could do is to choose \( N \in \mathbb{N} \) such that \( Na, Nb, Nc, Nd \) are integers and then compare the generating functions \( g_{Na,Nb} \) and \( g_{Nc,Nd} \).

The embedding condition \( g_{a,b} \geq g_{c,d} \) requires

\[
g_{a,b}(z) \geq g_{c,d}(z)
\]

for all \( z \in [0,1) \). But (3.1) is scale invariant, since it is equivalent to

\[
\frac{(1 - z^c)(1 - z^d)}{(1 - z^a)(1 - z^b)} \geq 1
\]

and one may substitute \( z = w^λ \) with \( w \in [0,1) \) on the left hand side. Therefore it corresponds to an embedding obstruction which extends to real parameters \( a, b \). The following lemma shows that at least in the case of embeddings into a ball this obstruction is the volume constraint.

**Lemma 1.** Let \( a, b, c, d \in \mathbb{R} \) be positive, such that \( b \leq \min(c,d) \). Then the inequality

\[
g_{a,b}(z) \geq g_{c,d}(z)
\]

holds for all \( z \in [0,1) \) if and only if \( a \) is chosen such that \( ab \leq cd \).

**Proof:** By scale invariance it suffices to show that under the assumption \( b \leq \min(1,c) \) the inequality

\[
(1 - z)(1 - z^c) \geq (1 - z^a)(1 - z^b)
\]

(3.2) holds for all \( z \in (0,1) \) if and only if \( a \leq \frac{b}{c} \).

We first consider the case \( a = \min(c,d) \), such that \( b \leq 1 \leq a \). Then we have

\[
ab \leq \min(a, ab + 1) \leq \max(a, ab + 1) \leq a + b.
\]

The function \( f(x) = x^z \) is convex and monotone decreasing for fixed \( z \in (0,1) \) and \( x \in (0,\infty) \). Hence the segment from \((ab, z^{ab})\) to \((a + b, z^{a+b})\) lies above the segment from \((a, z^a)\) to \((ab + 1, z^{ab+1})\). Comparing the heights of intersection of these segments with the horizontal line \( x = \frac{b(ab+1)+a}{b+1} \) yields the estimate

\[
\frac{b}{b+1}z^{ab+1} + \frac{1}{b+1}z^a \leq \frac{b}{b+1}z^{ab} + \frac{1}{b+1}z^{a+b}.
\]

Considering the function \( F : (1, \infty) \to \mathbb{R} \),

\[
F(a) = z^{ab+1} + z^a + z^b - z^{ab} - z^{a+b} - z
\]

for fixed \( z \in (0,1) \) and \( b \leq 1 \), the previous inequality implies that \( f \) is monotone increasing in \( a \). Consequently, \( F(a) \geq F(1) = 0 \). This tells us that (3.2) holds for all \( z \in (0,1) \) if \( c = ab \). Since increasing \( c \) only increases the left hand side of (3.2), we have shown that this inequality is satisfied for all \( z \in (0,1) \) if \( c \geq ab \).

Now we fix any \( 0 < \lambda < 1 \) and consider the case \( c = \lambda ab \). Let

\[
C := \frac{\lambda^2 ab^2 + a + b}{\lambda b + 1} > \frac{1 + b}{\lambda b + 1} > 1.
\]

Choose \( \delta > 0 \) small enough, such that

\[
z^{(1-\delta)\lambda b} \geq \frac{(a + b)^2}{4(1-\lambda)b} \log z
\]
This shows that (3.2) is violated for \( c > z \) is monotone decreasing for \( 1 \leq \tau \leq a \).

We now apply the inequality for \( 1 \leq \mu \leq \tau \):

\[
\frac{\lambda b}{\lambda b + 1} z^{\lambda b + 1} + \frac{1}{\lambda b + 1} z^\tau \geq f \left( \frac{\lambda^2 \tau b^2 + \tau + \lambda b}{\lambda b + 1} \right) \geq f \left( \frac{\lambda^2 \tau b^2 + \tau + b}{\lambda b + 1} \right) - \frac{(1 - \lambda)b}{\lambda b + 1} f' \left( \frac{\lambda^2 \tau b^2 + \tau + b}{\lambda b + 1} \right).
\]

Consequently, the function \( f \) defined by

\[
\frac{\lambda b}{\lambda b + 1} z^{\lambda b + 1} + \frac{1}{\lambda b + 1} z^\tau \geq f (\mu(\lambda b) + (1 - \mu)(\tau + b)) + \frac{|\lambda b - (\tau + b)|^2}{8} \max_{\xi \in [\lambda b, \tau + b]} f''(\xi)
\]

for \( 1 \leq \tau \leq a \) and \( z \in (1 - \delta, 1) \). Consequently, the function \( F_\lambda : [1, a] \rightarrow \mathbb{R} \) defined by

\[
F_\lambda(\tau) = z^{\lambda \tau b + 1} + z^\tau + z^b - z^{\lambda b} - z^{\tau + b} - z
\]

is monotone decreasing for \( z \in (1 - \delta, 1) \). Hence for these values of \( z \) we have

\[
F_\lambda(\tau) \leq F_\lambda(1) = (1 - z)(z^b - z^{ab}) < 0.
\]

This shows that (3.3) is violated for \( c = \lambda ab \) with \( 0 < \lambda < 1 \).

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