What Good Are
Quantum Field Theory Infinities?

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Abstract

A lesson for the new millennium from quantum field theory: Not all field-theoretic infinities are bad. Some give rise to finite, symmetry-breaking effects, whose consequences are observed in Nature.

Quantum field theory is the most successful theoretical structure in physics, with applications that range from the short distances of subatomic particles to the microscopic dimensions characterizing atomic, chemical, and condensed matter physics, and onto the astronomical distances where quantum field theory fuels “inflation” – a speculative but completely physical analysis of early universe cosmology. Remarkably, no experimental observation has contradicted the predictions that are made by appropriate field theoretical models for the relevant phenomena. When accurate calculation is feasible and precision experiments are available, numerical agreement between theory and experiment extends to many significant places, as for example in the ground-state energies of simple atoms like hydrogen and helium, or in the magnetic moments of electrons and muons.

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Nevertheless, this gloriously successful invention of the human mind is logically defective in that some well-posed questions cannot be answered – a computation that should resolve the question can yield ambiguous or meaningless answers. This happens because the available methods of calculation encounter infinities that either persist, leading to meaningless results, or cancel among themselves, leaving ambiguous, undetermined “finite” parts. There is no reason to suppose that this defect should be attributed to the method of (approximate) computation – it appears to be intrinsic to interacting quantum field theory when excitations are point particles and interactions are local. (More specifically, I am referring to ultraviolet infinities, which arise because various integrals over intermediate energies diverge at their high-energy end, which corresponds to short distances in position space. There are also other infinities, like diverging perturbative series or infrared/large-distance singularities associated with long-range forces. But these infinities are less troublesome, because they are attributed to the approximation method and are not viewed as intrinsic defects of quantum field theory.)

For physically relevant models, but not including gravity theory, it has been possible to isolate the infinities by the “renormalization” procedure, which hides them and also permits unambiguous calculation of quantities not contaminated by the infinities. Within this framework definite numerical results have been obtained, which in principle explain all observed fundamental processes. (Failure to tame infinities in quantum gravity has thus far been irrelevant for practical purposes, because all presently observed manifestations of gravitational forces are described by the classical Newton-Einstein theory.)

In spite of the great success of quantum field theory, its infinities notwithstanding, there are many who remain unconvinced by the pragmatism of renormalization. Dirac and Schwinger, who count among the creators of quantum field theory and renormalization theory, respectively, ultimately rejected their constructs because of the infinities. But even those who accept renormalization disagree about its ultimate efficacy at well-defining a theory. Some argue that sense can be made only of “asymptotically free” renormalizable field theories – in these theories the interaction strength decreases with increasing energy. On the contrary, it is claimed that asymptotically nonfree models, like electrodynamics and $\phi^4$-theory, do not define quantum theories, even though they are renormalizable – it is said “they do not exist.” Yet electrodynamics is the most precisely verified quantum field theory, while the
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\( \phi^4 \)-model is a necessary component of the “standard model” for elementary particle interactions, which thus far has met no experimental contradiction.

The ultraviolet infinities appear as a consequence of space-time localization of excitations and of their interactions. (Sometimes it is claimed that field-theoretic infinities arise from the unhappy union of quantum theory with special relativity. But this does not describe all cases – later I shall discuss a nonrelativistic, ultraviolet-divergent, and renormalizable field theory.) Therefore choosing models with extended excitations and interactions provides a way for avoiding ultraviolet infinities. These days “string theory” is a model with precisely such extended features, and all quantum effects — including gravitational ones — are ultraviolet finite. This very desirable state of affairs has persuaded many that fundamental physical theory in the next millennium should be based on the string paradigm (generalized to encompass even more extended structures, like membranes and so on). This will replace quantum field theory, which although marred by its ultraviolet infinities has served us well in the twentieth century.

My goal in this essay is to argue that at least some of the divergences of quantum field theory must not be viewed as unmitigated defects. On the contrary, they convey crucially important information about the physical situation, without which most of our theories would not be physically acceptable. The stage where my unconventional considerations play a role is that of symmetry, symmetry breaking, and conserved quantum numbers, so next I have to review these ideas.

Physicists are mostly agreed that ultimate laws of Nature enjoy a high degree of symmetry. Presence of symmetry implies absence of complicated and irrelevant structure, and our conviction that this is fundamentally true reflects an ancient aesthetic prejudice — physicists are happy in the belief that Nature in its fundamental workings is essentially simple. Moreover, there are practical consequences of the simplicity entailed by symmetry — it is easier to understand the predictions of physical laws. For example, working out the details of very-many-body motion is beyond the reach of actual calculations, even with the help of computers. But taking into account the symmetries that are present allows understanding at least some aspects of the motion, and charting regularities within it.

Symmetries bring with them conservation laws — an association that is precisely formulated by Noether’s theorem. Thus time-translation symmetry, which states that physical laws do not change as time passes, ensures energy conservation; space-translation symmetry, the statement that physical laws
take the same form at different spatial locations, ensures momentum conservation. For another example, we note that the quantal description makes use of complex numbers. But physical quantities are real, so complex phases can be changed at will, without affecting physical content. This invariance against phase redefinition, called *gauge symmetry*, leads to charge conservation. The above examples show that symmetries are linked to constants of motion. Identifying such constants on the one hand satisfies our urge to find regularity and permanence in natural phenomena, and on the other hand we are provided with useful markers for ordering physical data.

Moreover, a large degree of symmetry in the mathematical formulation of physically successful quantum field theory models is desirable not only aesthetically but also practically. Symmetry facilitates unraveling the consequences of the complicated dynamical model; more importantly, the presence of symmetry is required for a successful renormalization of the infinities, so that unambiguous answers can be extracted from the formalism.

However, in spite of our preference that descriptions of Nature be enhanced by a large amount of symmetry and characterized by many conservation laws, we must recognize that actual physical phenomena rarely exhibit overwhelming regularity. Therefore, at the very same time that we construct a physical theory with intrinsic symmetry, we must find a way to break the symmetry in physical consequences of the model. Progress in physics can be frequently seen as the resolution of this tension.

In classical physics, the principal mechanism for symmetry breaking, realized already within Newtonian mechanics, is through boundary and initial conditions on dynamical equations of motion. For example, radially symmetric dynamics for planetary motion allows radially nonsymmetric, noncircular orbits with appropriate initial conditions. But this mode of symmetry breaking still permits symmetric configurations – circular orbits, which are rotationally symmetric, are allowed. In quantum mechanics, which anyway does not need initial conditions to make physical predictions, we must find mechanisms that prohibit symmetric configurations altogether.

In the simplest, most direct approach to symmetry breaking, we suppose that in fact dynamical laws are not symmetric, but that the asymmetric effects are “small” and can be ignored “in first approximation.” Familiar examples are the breaking of rotational symmetry in atoms by an external electromagnetic field or of isospin symmetry by the small electromagnetic interaction. However, this explicit breaking of symmetry is without fundamental interest for the exact and complete theory; we need more intrinsic
mechanisms that work for theories that actually are symmetric.

A more subtle idea is *spontaneous symmetry breaking*, where the dynamical laws are symmetric, but only asymmetric configurations are actually realized (because the symmetric ones are energetically unstable). This mechanism, urged on particle physicists by Heisenberg, Anderson, Nambu, and Goldstone, is readily illustrated by the potential energy profile possessing left-right symmetry and depicted in the Figure. The left-right symmetric value at the origin is a point of unstable equilibrium; stable equilibrium is attained at one of the two reflection-unsymmetric points \( \pm a \). Moreover, in quantum field theory, the energy barrier separating the two asymmetric configurations is infinite and no tunneling occurs between them. Once the system settles in one or the other location, left-right parity is absent. One says that the symmetry of the equations of motion is “spontaneously” broken by the stable solution.

![Energy density](image)

Left-right symmetric energy density. The symmetric point at 0 is energetically unstable. Stable configurations are at \( \pm a \). Because field theory is defined in an infinite volume, the finite energy density separating \( \pm a \) produces an infinite energy barrier and tunneling is suppressed. The system settles into state \(+a\) or \(-a\) and left-right symmetry is spontaneously broken.

While the predictions of a theory with spontaneously broken symmetry no longer follow the patterns that one would find if the symmetry were present in the solutions, one important benefit of the symmetry remains: the
renormalization procedure is unaffected. So the mechanism of spontaneous symmetry breaking accomplishes the phenomenologically desired reduction of formal symmetries without endangering renormalization, but it does not reduce them enough. Fortunately there exists a further, even more subtle mode of symmetry breaking, with which we can further suppress symmetries, thereby bringing our theories in accord with observed phenomena. Here one crucially relies on the various ultraviolet infinities of local quantum field theory, for which the renormalization procedure (needed to make sense of the theory) cannot be carried out in a manner consistent with the symmetry. Nevertheless the symmetry breaking effects are finite, even though they arise from infinities.

This mode of symmetry breaking is called *anomalous* or *quantum mechanical*, and in order to explain it, let me begin by recalling that the quantum revolution did not erase our reliance on the earlier, classical physics. Indeed, when proposing a theory, we begin with classical concepts and construct models according to the rules of classical, prequantum physics. We know, however, such classical reasoning is not in accord with quantum reality. Therefore, the classical model is reanalyzed by the rules of quantum physics (which comprise the true laws of Nature), that is, the classical model is quantized.

Differences between the physical pictures drawn by a classical description and a quantum description are of course profound. To mention the most dramatic, we recall that dynamical quantities are described in quantum mechanics by operators, which need not commute. Nevertheless, one expects that some universal concepts transcend the classical/quantal dichotomy, and enjoy rather the same role in quantum physics as in classical physics.

For a long time it was believed that symmetries and conservation laws of a theory are not affected by the transition from classical to quantum rules. For example, if a model possesses translation and gauge invariance on the classical level, and consequently energy/momentum and charge are conserved classically, it was believed that after quantization the quantum model is still translation and gauge invariant so that the energy/momentum and charge operators are conserved within quantum mechanics, that is, they commute with the quantum Hamiltonian operator. But now we know that in general this need not be so. Upon quantization, some symmetries of classical physics may disappear when the quantum theory is properly defined in the presence of its infinities. Such tenuous symmetries are said to be *anomalously* broken; although present classically, they are absent from the quantum version of the
theory, unless the model is carefully arranged to avoid this effect.

The nomenclature is misleading. At its discovery, the phenomenon was unexpected and dubbed “anomalous.” By now the surprise has worn off, and the better name today is “quantum mechanical” symmetry breaking.

Anomalously or quantum mechanically broken symmetries play several and crucial roles in our present-day physical theories. In some instances they save a model from possessing too much symmetry, which would not be in accord with experiment. In other instances the desire to preserve a symmetry in the quantum theory places strong constraints on model building and gives experimentally verifiable predictions; more about this later.[1]

Now I shall describe two specific examples of the anomaly phenomenon. Consider first massless fermions moving in the background of an electromagnetic field. Massive, spin-$\frac{1}{2}$ fermions possess two spin states – up and down – but massless fermions can exist with only one spin state (out of two), called a helicity state, in which spin is projected along (or against) the direction of motion. So the massless fermions with which we are here concerned carry only one helicity and these are an ingredient in present-day theories of quarks and leptons. Moreover, they also arise in condensed matter physics, not because one is dealing with massless, single-helicity particles, but because a well-formulated approximation to various many-body Hamiltonians can result in a first-order matrix equation that is identical to the equation for single-helicity massless fermions, that is, a massless Dirac-Weyl equation for a spinor $\Psi$.

If we view the spinor field $\Psi$ as an ordinary mathematical function, we recognize that it possesses a complex phase, which can be altered without changing the physical content of the equation that $\Psi$ obeys. We expect therefore that this instance of gauge invariance implies charge conservation. However, in a quantum field theory $\Psi$ is a quantized field operator, and one finds that in fact the charge operator $Q$ is not conserved; rather

$$\frac{dQ}{dt} = \frac{i}{\hbar}[H, Q] \propto \int_{\text{volume}} \mathbf{E} \cdot \mathbf{B}$$

where $\mathbf{E}$ and $\mathbf{B}$ are the background electric and magnetic fields in which our massless fermion is moving – gauge invariance is lost!

One way to understand this breaking of symmetry is to observe that our model deals with massless fermions and conservation of charge for single-helicity fermions makes sense only if there are no fermion masses. But quantum field theory is beset by its ultraviolet infinities, which must be controlled
in order to do a computation. This is accomplished by regularization and renormalization, which introduces mass scales for the fermions, and we see that the symmetry is anomalously broken by the ultraviolet infinities of the theory.

The phase-invariance of single-helicity fermions is called chiral (gauge) symmetry, and chiral symmetry has many important roles in the standard model, which involves many kinds of fermion fields, corresponding to the various quarks and leptons. In those channels where a gauge vector meson couples to the fermions, chiral symmetry must be maintained to ensure gauge invariance. Consequently, fermion content must be carefully adjusted so that the anomaly disappears. This is achieved because the proportionality constant in the above failed conservation law involves a sum over all the fermion charges, \( \sum q_n \), so if that quantity vanishes the anomaly is absent. In the standard model the sum indeed vanishes, separately for each of the three fermion families. For a single family this works out as follows:

| Fermion Type          | \( q_n \) | \( \sum q_n \) |
|-----------------------|----------|-----------------|
| three quarks          | \( \frac{2}{3} \) | 2               |
| three quarks          | \( -\frac{1}{3} \) | \(-1\)          |
| one charged lepton    | \( -1 \)  | \(-1\)          |
| one neutrino lepton   | \( 0 \)   | \( 0 \)         |

In channels to which no gauge vector meson couples, there is no requirement that the anomaly vanish, and this is fortunate. A theoretical analysis shows that chiral gauge invariance in the up-down quark channel prohibits the two-photon decay of the neutral pion (which is composed of up and down quarks). But the decay does occur with the invariant decay amplitude of \( 0.025 \pm 0.001 \text{GeV}^{-1} \). Before anomalous symmetry breaking was understood, this decay could not be fitted into the standard model, which seemed to possess the decay-forbidding chiral symmetry. Once it was realized that the relevant chiral symmetry is anomalously broken, this obstacle to phenomenological viability of the standard model was removed. Indeed since the anomaly is completely known, the decay amplitude can be completely calculated (in the approximation that the pion is massless) and one finds \( 0.025 \text{GeV}^{-1} \), in excellent agreement with experiment.
We must conclude that Nature knows about and makes use of the anomaly mechanism. On the one hand fermions are arranged into gauge-anomaly–free representations, and the requirement that anomalies disappear “explains” the charges of elementary fermions. On the other hand the pion decays into two photons because of an anomaly in an ungauged channel. It is therefore paradoxical but true that in local quantum field theory these phenomenologically desirable results are facilitated by ultraviolet divergences, which give rise to finite symmetry anomalies, derived from infinities.

The observation that infinities of quantum field theory lead to anomalous symmetry breaking allows comprehending a second example of quantum-mechanical breaking of yet another symmetry – scale invariance. Like the space-time translations mentioned earlier, which lead to energy-momentum conservation, scale transformations also act on space-time coordinates, but in a different manner. They dilate the coordinates, thereby changing the units of space and time measurements. Such transformations will be symmetry operations in models that possess no fundamental parameters with time or space dimensionality, and therefore do not contain an absolute scale for units of space and time. Our quantum chromodynamical (QCD) model for quarks is free of such dimensional parameters, and it would appear that this theory is scale invariant – but Nature certainly is not! The observed variety of different objects with different sizes and masses exhibits many different and inequivalent scales. Thus if scale symmetry of the classical field theory, which underlies the quantum field theory of QCD, were to survive quantization, experiment would have grossly contradicted the model, which therefore would have to be rejected. Fortunately, scale symmetry is quantum mechanically broken, owing to the scales that are introduced in the regularization and renormalization of ultraviolet singularities. Once again a quantum field-theoretic pathology has a physical effect, a beneficial one – an unwanted symmetry is anomalously broken, and removed from the theory.

A different perspective on the anomaly phenomenon comes from the path integral formulation of quantum theory, where one integrates over classical paths the phase exponential of the classical action:

$$\text{Quantum Mechanics} \iff \int_{\text{(measure on paths)}} e^{i\text{(classical action)}/\hbar}.$$  

When the classical action possess a symmetry, the quantum theory will respect that symmetry if the measure on paths is unchanged by the relevant transformation. In the known examples (chiral symmetry, scale symmetry)
anomalies arise precisely because the measure fails to be invariant and this failure is once again related to infinities. The measure is an infinite product of measure elements for each point in the space-time where the quantum (field) theory is defined; regulating this infinite product destroys its apparent invariance.

Yet another approach to chiral anomalies, which arise in (massless) fermion theories, makes reference to the first instance of regularization/renormalization, used by Dirac to remove the negative-energy solutions to his equation. Recall that to define a quantum field theory of fermions, it is necessary to fill the negative-energy sea and to renormalize the infinite mass and charge of the filled states to zero. In modern formulations this is achieved by “normal ordering”, but for our purposes it is better to remain with the more explicit procedure of subtracting the infinities, that is, renormalizing them.

It can then be shown that in the presence of an external gauge field, the distinction between “empty” positive-energy states and “filled” negative-energy states cannot be drawn in a gauge-invariant manner, for massless, single-helicity fermions. Within this framework, the chiral anomaly comes from the gauge noninvariance of the infinite negative-energy sea. Since anomalies have physical consequences, we must assign physical reality to this infinite negative-energy sea.

Actually, in condensed matter physics, where a Dirac-type equation governs electrons, owing to a linearization of dynamical equations near the Fermi surface, the negative-energy states do have physical reality. They correspond to filled, bound states, while the positive energy states describe electrons in the conduction band. Consequently, chiral anomalies also have a role in condensed matter physics, when the system is idealized so that the negative-energy sea is taken to be infinite.

In this condensed matter context another curious, physically realized, and infinity-driven phenomenon has been identified. When the charge of the filled negative states is renormalized to zero, one is subtracting an infinite quantity, and rules have to be agreed upon so that no ambiguities arise when infinite quantities are manipulated. With this agreed-upon subtraction procedure, the charge of the vacuum is zero, and filled states of positive energy carry integer units of charge. Into the system one can insert a soliton – a localized structure that distinguishes between different domains of the condensed matter. In the presence of such a soliton, one needs to recalculate charges using the agreed-upon rules for handling infinities and one finds, surprisingly, a noninteger result, typically half-integer: the negative-energy
sea is distorted by the soliton to yield a half-unit of charge. The existence of fractionally charged states in the presence of solitons has been experimentally identified in polyacetylene. We thus have another example of a physical effect emerging from infinities of quantum field theory.

Let me conclude my qualitative discussion of anomalies with an explicit example from quantum mechanics, whose wave functions provide a link between particle and field-theoretic dynamics. My example also dispels any suspicion that ultraviolet divergences and the consequent anomalies are tied to the complexities of relativistic quantum field theory. The nonrelativistic example shows that locality is what matters.

Recall first the basic dynamical equation of quantum mechanics: the time independent Schrödinger equation for a particle of mass $m$ moving in a potential $V(r)$ with energy $E$:

$$
\left(-\nabla^2 + \frac{2m}{\hbar^2} V(r)\right)\psi(r) = \frac{2m}{\hbar^2} E \psi(r).
$$

In its most important physical applications, this equation is taken in three spatial dimensions and $V(r)$ is proportional to $1/r$ for the Coulomb force relevant in atoms. Here we want to take a different model with potential that is proportional to the inverse square, so that the Schrödinger equation is presented as

$$
\left(-\nabla^2 + \frac{\lambda}{r^2}\right)\psi(r) = k^2 \psi(r), \quad k^2 \equiv \frac{2m}{\hbar^2} E.
$$

In this model, transforming the length scale is a symmetry: because the Laplacian scales as $r^{-2}$, $\lambda$ is dimensionless and in the above there is no intrinsic unit of length. A consequence of scale invariance is that the scattering phase shifts and the $S$ matrix, which in general depend on energy, that is, on $k$, are energy independent in scale-invariant models. And indeed when the above Schrödinger equation is solved, one verifies this prediction of the symmetry by finding an energy-independent $S$ matrix. Thus scale invariance is maintained in this example – there are no surprises.

Let us now look to a similar model, but in two dimensions with a $\delta$-function potential, which localizes the interaction at a point:

$$
\left(-\nabla^2 + \lambda \delta^2(r)\right)\psi(r) = k^2 \psi(r).
$$

Since in two dimensions the two-dimensional $\delta$-function scales as $1/r^2$, the above model also appears scale invariant; $\lambda$ is dimensionless. But in spite
of the simplicity of the local contact interaction, the Schrödinger equation suffers a short-distance, ultraviolet singularity at $r=0$, which must be renormalized. Here is not the place for a detailed analysis, but the result is that only the $s$-wave possesses a nonvanishing phase shift $\delta_0$, which shows a logarithmic dependence on energy:

$$\cot \delta_0 = \frac{2}{\pi} \ln kR + \frac{1}{\lambda}$$

$R$ is a scale that arises in the renormalization, and scale symmetry is decisively and quantum mechanically broken. The scattering is nontrivial solely as a consequence of anomalously broken scale invariance. (It is easily verified that the two-dimensional $\delta$-function in classical theory, where there are no anomalies and it is scale invariant, produces no scattering.) To make sense of the above phase shift in the limit $R \to \infty$, one must “renormalize” the bare coupling constant $\lambda$, allowing it to depend on $R$ in just such a way that $\cot \delta_0$ is $R$-independent (for large $R$). Alternatively, one recognizes that the $S$ matrix $e^{2i\delta_0}$ possesses a pole, corresponding to a bound state with energy

$$E_B = -\frac{\hbar^2}{2mR^2} e^{-\pi/\lambda}.$$ 

Therefore one may reexpress $\delta_0$ in terms of $E_B$, rather than $R$. With this substitution, dependence on $\lambda$ disappears (as it must since $\lambda$ is $R$-dependent) and the dimensionless (infinite) coupling constant $\lambda$, has been traded for a dimensional and physical parameter $E_B$:

$$\cot \delta_0 = \frac{1}{\pi} \ln \frac{E}{|E_B|}$$

Similar anomalous breaking of scale invariance occurs in relativistic field theory, and perhaps explains the appearance of a dimensional mass parameter in QCD as a replacement for the dimensionless, but renormalization dependent, coupling constant.  

I believe that as the millennium draws to a close, and we look forward eagerly to the new physics ideas that will flourish in the new era, one very important lesson we should take from quantum field theory is not to banish all its infinities. Apparently the mathematical language with which we are describing Nature cannot account for all natural phenomena in a clear fashion. Recourse must be made to contradictory formulations involving
infinites, which nevertheless lead to accurate descriptions of experimental facts in finite terms. It will be most interesting to see how string theory and its evolutions, which purportedly are completely finite and consistent, will handle this issue, which has been successfully, if paradoxically, resolved in quantum field theory.

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