A Convolution Bound Implies Tolerance to Time-variations and Unmodelled Dynamics

Mohamad T. Shahab, Daniel E. Miller

Dept. of Electrical and Computer Engineering,
University of Waterloo
Waterloo, ON, Canada N2L 3G1

Abstract

Recently it has been shown, in several settings, how to carry out adaptive control for an LTI plant so that a convolution bound holds on the closed-loop behavior; this, in turn, has been leveraged to prove robustness of the closed-loop system to time-varying parameters and unmodelled dynamics. The goal of this paper is to show that the same is true for a large class of finite-dimensional, nonlinear plant and controller combinations.

Keywords: Adaptive Control, Robustness, Convolution Bounds, Time-variations, Unmodelled Dynamics

1. Introduction

In control system design, a common requirement is that the closed-loop system not only be stable, but also be robust, in the sense that it tolerates, at the very least, small time-variations in the plant parameters and a small amount of unmodelled dynamics. Of course, if the plant and controller are both linear and time-invariant, then such robustness follows from closed-loop stability—see [9], [1]. On the other hand, if either the plant or controller is nonlinear, this is often not the case and/or it is not easy to prove.

Recently it has been proven, in both the pole placement and first order one-step-ahead settings, that if discrete-time adaptive control is carried out in just the right way, then a (stable) convolution bound can be obtained on the closed-loop behavior—see [4] and [6]: hence, the closed-loop system acts ‘linear-like’, and the convolution bound can be leveraged in a modular fashion\(^1\) to prove that tolerance to small time-variations and a small amount of unmodelled dynamics follows. Of course, there the controller is nonlinear and the nominal plant is single-input, single-output, and LTI. The goal of this paper is to generalize this result to a larger class of multi-input multi-output plants and controllers.

To this end, here we consider a class of finite-dimensional, nonlinear plant and controller combinations; if a convolution bound holds, then we prove that tolerance to small time-variations in the plant parameters and a small amount of unmodelled dynamics follows. An immediate application of this result is to prove robustness of our recently designed multi-estimator switching adaptive controllers presented in [6] and [8]. This result should also prove useful in extending our work on the adaptive control of LTI plants [5], [6], [8], [7] to that of nonlinear plants, allowing us to focus on the ideal plant model in our analysis, knowing that robustness will come for free. Last of all, this result has the potential for use in other non-adaptive (but nonlinear) contexts.

We denote \(\mathbb{Z}, \mathbb{Z}^+\) and \(\mathbb{N}\) as the sets of integers, non-negative integers and natural numbers, respectively. We will denote the Euclidean-norm of a vector and the induced norm of a matrix by the subscript-less default notation \(\|\cdot\|\). We let \(\mathcal{S}(\mathbb{R}^{p \times q})\) denote the set of \(\mathbb{R}^{p \times q}\)-valued bounded sequences. We also let \(\mathcal{L}_\infty(\mathbb{R}^{p \times q})\) denote the set of \(\mathbb{R}^{p \times q}\)-valued bounded sequences. If \(\Omega \subset \mathbb{R}^{p \times q}\) is a bounded set, we define \(\|\Omega\| := \sup_{x \in \Omega} \|x\|\).

Throughout this paper, we say that a function \(\Gamma : \mathbb{R}^p \to \mathbb{R}^q\) has a bounded gain if there exists a \(\nu > 0\) such that for all \(x \in \mathbb{R}^p\), we have \(\|\Gamma(x)\| \leq \nu \|x\|\); the smallest such \(\nu\) is the gain, and is denoted by \(\|\Gamma\|\).

For a closed and convex set \(\Omega \subset \mathbb{R}^p\), the function \(\text{Proj}_\Omega(\cdot) : \mathbb{R}^p \to \Omega\) denotes the projection onto \(\Omega\); it is well known that the function \(\text{Proj}_\Omega\) is well defined.

2. The Setup

Here the nominal plant is multi-input multi-output with finite memory and an additive disturbance, such that the uncertain plant parameter enters linearly. To this end,
with an output \( y(t) \in \mathbb{R}^r \), an input \( u(t) \in \mathbb{R}^m \), a disturbance \( w(t) \in \mathbb{R}^r \), a modeling parameter of \( \theta^* \in \mathcal{S} \subset \mathbb{R}^{p \times r} \), and a vector of input-output data of the form

\[
\phi(t) = \begin{bmatrix}
y(t)
y(t-1)
\vdots
y(t-n_y+1)
u(t)
u(t-1)
\vdots
u(t-n_u+1)
\end{bmatrix} \in \mathbb{R}^{n_y \times r + n_u \times m},
\]

we consider the plant

\[
y(t+1) = \theta^* \top f(\phi(t)) + w(t), \quad \phi(t_0) = \phi_0;
\]

we assume that \( f : \mathbb{R}^{n_y \times r + n_u \times m} \rightarrow \mathbb{R}^p \) has a bounded gain and that \( \mathcal{S} \) is a bounded set; both requirements are reasonable given that we will require uniform bounds in our analysis. We represent this system by the pair \((f, \mathcal{S})\).

Here we consider a large class of controllers which subsumes LTI ones as well as a large class of adaptive ones. To this end, we consider a controller with its state partitioned into two parts:

- \( z_1(t) \in \mathbb{R}^{l_1} \) and
- \( z_2(t) \in \mathbb{R}^{l_2} \),

an exogenous signal \( r(t) \in \mathbb{R}^q \) (typically a reference signal), together with equations of the form

\[
\begin{align*}
z_1(t+1) &= g_1(z_1(t), z_2(t), \phi(t), y(t+1), r(t), t, t_0), & z_1(t_0) = z_{10} \\
z_2(t+1) &= g_2(z_1(t), z_2(t), \phi(t), y(t+1), r(t), t, t_0), & z_2(t_0) = z_{20},
\end{align*}
\]

\[
u(t) = h(z_1(t), z_2(t), \phi(t), r(t)).
\]

Here we assume that

\[
g_2 : \mathbb{R}^{l_1} \times \mathcal{X} \times \mathbb{R}^{n_y \times r + n_u \times m} \times \mathbb{R}^r \times \mathbb{R}^q \times \mathbb{Z} \rightarrow \mathcal{X},
\]

i.e. if \( z_2 \) is initialized in \( \mathcal{X} \), then it remains in \( \mathcal{X} \) throughout.

**Remark 1.** This class subsumes finite-dimensional LTI controllers: simply set \( l_2 = 0 \) so that the sub-state \( z_2 \) disappears, and make the functions \( g_1 \) and \( h \) to be linear.

**Remark 2.** This class subsumes many adaptive controllers: simply set \( l_1 = 0 \) and let \( z_2 \) be the state of a parameter estimator constrained to the set \( \mathcal{X} \).

We now provide a definition of the desired linear-like closed-loop property:

**Definition 1.** We say that (2) provides a convolution bound for \((f, \mathcal{S})\) with gain \( c \geq 1 \) and decay rate \( \lambda \in (0, 1) \) if, for every \( \theta^* \in \mathcal{S} \), \( t_0 \in \mathbb{Z} \), \( \phi_0 \in \mathbb{R}^{n_y \times r + n_u \times m} \), \( z_{10} \in \mathbb{R}^{l_1} \), \( z_{20} \in \mathcal{X} \subset \mathbb{R}^{l_2} \), \( w \in \mathcal{S}(\mathbb{R}^r) \) and \( r \in \mathcal{S}(\mathbb{R}^q) \), when (2) is applied to (1), the following holds:

\[
\begin{align*}
\|\phi(t)\|_2 &\leq c_{\lambda, r} \|\phi(\tau)\|_2 + \\
\sum_{\tau=\tau_0}^{t-1} c_2 \lambda^{t-\tau-1} (\|r(\tau)\|_2 + \|w(\tau)\|_2)\|r(\tau)\|_2, & t \geq \tau \geq t_0.
\end{align*}
\]

**Remark 3.** The reason why we do not focus on the exponential stability aspect of (3) is that the \( c_\lambda \lambda^{-\tau} \|\phi(\tau)\|_2 \|z_1(\tau)\|_2 \) term can be viewed, in essence, as the effect of the past inputs on the future, in much the same way as the ‘zero-input-response’ can be viewed in the analysis of LTI systems. More specifically, the \( c_\lambda \lambda^{-\tau} \|\phi(\tau)\|_2 \|z_1(\tau)\|_2 \) term can be viewed as having arisen from a convolution of the past inputs (before time \( \tau \)) with \( c_\lambda \lambda^n \), so this term can be viewed as a convolution sum in its own right.

### 3. Tolerance to Time-Variation

We now consider plants with a possibly time-varying parameter vector \( \theta^*(t) \) instead of a static \( \theta^* \):

\[
y(t+1) = \theta^*(t) \top f(\phi(t)) + w(t), \quad \phi(t_0) = \phi_0.
\]

With \( c_0 \geq 0 \) and \( \epsilon > 0 \), let \( \mathcal{S}(\mathcal{X}, c_0, \epsilon) \) denote the subset of \( \mathcal{E}_\infty(\mathbb{R}^{n_y \times r}) \) whose elements \( \theta^* \) satisfy:

- \( \theta^*(t) \in \mathcal{S} \) for every \( t \in \mathbb{Z} \),
- and

\[
\sum_{t=t_1}^{t_2-1} \|\theta^*(t+1) - \theta^*(t)\|_2 \leq c_0 + \epsilon (t_2 - t_1), \quad t_2 > t_1, \quad t_1 \in \mathbb{Z}.
\]

The above time-variation model encompasses both slow variations and/or occasional jumps; this class is well-known in the adaptive control literature, e.g. see [2]. We can extend Definition 1 in a natural way to handle time-variations.
Definition 2. We say that (2) provides a convolution bound for \((f, s(S, c_0, \epsilon))\) with gain \(c \geq 1\) and decay rate \(\lambda \in (0, 1)\) if, for every \(\theta^* \in s(S, c_0, \epsilon), t_0 \in \mathbb{Z}\), \(\phi_0 \in \mathbb{R}^{r+n+u_m}, z_{t_0} \in \mathbb{R}^l, z_{t_0} \in \mathcal{X} \subset \mathbb{R}^l, w \in S(\mathbb{R}^r)\) and \(r \in S(\mathbb{R}^q)\), when (2) is applied to (4), the following holds:

\[
\left\| \begin{bmatrix} \phi(t) \\ \z(t) \end{bmatrix} \right\| \leq c \lambda^{t-t_0} \left\| \begin{bmatrix} \phi(\tau) \\ \z(\tau) \end{bmatrix} \right\| + \sum_{j=t_0}^{t-1} c \lambda^{t-j-1} (\|r(j)\| + \|w(j)\|) + \epsilon \|r(t)\|, \\
\quad t \geq t_0.
\] (5)

We now will show that if a controller (2) provides convolution bounds for the plant (1), then the same will be true for the time-varying plant (4), as long as \(\epsilon\) is small enough. We consider two cases: one where there is a desired decay rate, and one where there is not.

Theorem 1. Suppose that the controller (2) provides a convolution bound for (1) with gain \(c \geq 1\) and decay rate \(\lambda \in (0, 1)\). Then for every \(\lambda_1 \in (\lambda, 1)\) and \(c_0 > 0\), there exist a \(c_1 \geq c\) and \(\epsilon > 0\) so that (2) provides a convolution bound for \((f, s(S, c_0, \epsilon))\) with gain \(c_1\) and decay rate \(\lambda_1\).

Remark 4. This proof is based, in part, on the proof of Theorem 2 of [6], which deals with a much simpler setup.

Proof of Theorem 1. Suppose the controller (2) provides a convolution bound for (1) with gain \(c \geq 1\) and a decay rate of \(\lambda\). Fix \(\lambda_1 \in (\lambda, 1)\) and \(c_0 > 0\); let \(t_0 \in \mathbb{Z}\), \(\phi_0 \in \mathbb{R}^{r+n+u_m}, z_{t_0} \in \mathbb{R}^l, z_{t_0} \in \mathcal{X}, w \in S(\mathbb{R}^r)\) and \(r \in S(\mathbb{R}^q)\) be arbitrary.

Now fix \(m \in \mathbb{N}\) to be any number satisfying

\[
m \geq \frac{\ln(c) + \frac{4w^2(2\|f\|/\|\phi(0)\|) + \ln(2) - \ln(\lambda + \lambda_1)}{2\ln(\lambda_1) - \ln(\lambda + \lambda_1)}}{2\lambda_1 - \lambda_1},
\]

(the rationale for this choice will be more clear shortly), and set

\[
\epsilon = \frac{c_0}{m^2}.
\]

Let \(\theta^* \in s(S, c_0, \epsilon)\) be arbitrary and apply the controller (2) to the time-varying plant (4). To proceed, we analyze the closed-loop system behavior on intervals of length \(m\), which we further analyze in groups of \(m^2\).

To proceed, let \(\ell \geq t_0\) be arbitrary. Define a sequence \(\{\ell_i\}\) by

\[
\ell_i = \ell + im, \quad i \in \mathbb{Z}^+.
\]

We can rewrite the time-varying plant as

\[
y(t + 1) = \theta(\ell_i)^T f(\phi(t)) + w(t) + [\theta(t) - \theta(\ell_i)]^T f(\phi(t)),
\]

\[
t \in [\ell_i, \ell_{i+1}).
\]

On the interval \([\ell_i, \ell_{i+1}),\] we can regard the plant as time-invariant, but with an extra disturbance; so by hypothesis,

\[
\left\| \begin{bmatrix} \phi(t) \\ \z(t) \end{bmatrix} \right\| \leq c \lambda^{t-\ell_i} \left\| \begin{bmatrix} \phi(\ell_i) \\ \z(\ell_i) \end{bmatrix} \right\| + \sum_{j=\ell_i}^{t-1} c \lambda^{t-j-1} (\|r(j)\| + \|w(j)\| + \|\bar{n}_i(j)\|) + \epsilon \|r(t)\|, \\
\quad t \in [\ell_i, \ell_{i+1}), i \in \mathbb{Z}^+.
\]

To analyze this difference inequality, we first construct an associated difference equation:

\[
\psi(t + 1) = \lambda \psi(t) + \|r(t)\| + \|w(t)\| + \|\bar{n}_i(t)\|, \quad t \in [\ell_i, \ell_{i+1}).
\]

With an initial condition of

\[
\psi(\ell_i) = \left\| \begin{bmatrix} \phi(\ell_i) \\ \z(\ell_i) \end{bmatrix} \right\|.
\]

Using the fact that \(c \geq 1\), it is straightforward to prove that

\[
\left\| \begin{bmatrix} \phi(t) \\ \z(t) \end{bmatrix} \right\| \leq c \psi(t) + \epsilon \|r(t)\|, \quad t \in [\ell_i, \ell_{i+1}).
\]

(7)

Now we analyze this equation for \(i = 0, 1, \ldots, m - 1\).

Case 1: \(\|\bar{n}_i(t)\| \leq \frac{\lambda \lambda_1 - \lambda}{c_1} \|\phi(t)\|\) for all \(t \in [\ell_i, \ell_{i+1})\). Using the above bound (7) and the fact that \(\lambda_1 - \lambda \in (0, 1)\), we obtain

\[
\psi(t + 1) \leq \lambda \psi(t) + \|r(t)\| + \|w(t)\| + \|\bar{n}_i(t)\| \\
\leq \lambda \psi(t) + \|r(t)\| + \|w(t)\| + \frac{\lambda \lambda_1 - \lambda}{c_1} \|\phi(t)\| \\
\leq \lambda \psi(t) + \|r(t)\| + \|w(t)\| + \frac{\lambda \lambda_1 - \lambda}{c_1} (\|r(t)\| + \|w(t)\|) \\
\leq \frac{\lambda \lambda_1 - \lambda}{c_1} \psi(t) + 2 \|r(t)\| + \|w(t)\|, \quad t \in [\ell_i, \ell_{i+1}),
\]

which means that

\[
\psi(t) \leq \left( \frac{\lambda \lambda_1 - \lambda}{c_1} \right)^{t-\ell_i} \psi(\ell_i) + \sum_{j=\ell_i}^{t-1} \left( \frac{\lambda \lambda_1 - \lambda}{c_1} \right)^{t-1-j-1} (2 \|r(j)\| + \|w(j)\|), \\
\quad t = \ell_i, \ell_i + 1, \ldots, \ell_{i+1}.
\]

(9)

This, in turn, implies that there exists \(c_2 \geq 2c\) so that

\[
\left\| \begin{bmatrix} \phi(\ell_{i+1}) \\ \z(\ell_{i+1}) \end{bmatrix} \right\| \leq c_2 \left( \frac{\lambda \lambda_1 - \lambda}{c_1} \right)^{\ell_{i+1}-1} \left\| \begin{bmatrix} \phi(\ell_i) \\ \z(\ell_i) \end{bmatrix} \right\| + \sum_{j=\ell_i}^{\ell_{i+1}-1} c_2 \left( \frac{\lambda \lambda_1 - \lambda}{c_1} \right)^{\ell_{i+1}-j-1} (\|r(j)\| + \|w(j)\|) + \epsilon \|r(\ell_{i+1})\|.
\]

(10)
Since $\theta^*(t) \in S$ for $t \geq t_0$, we see that
\[
\|\Delta t(t)\| \leq 2\|S\| \|f(\phi(t))\| \leq 2\|f\|\|S\| \times \|\phi(t)\|, \quad t \in [t_i, t_{i+1}).
\]
This means that
\[
\psi(t + 1) = \psi(t) + \|r(t)\| + \|w(t)\| + \|\Delta t(t)\| \\
\leq \lambda \psi(t) + \|r(t)\| + \|w(t)\| + 2\|f\|\|S\| \|\phi(t)\| \\
= (1 + 2\|f\|\|S\|) \psi(t) + (1 + 2\|f\|\|S\|)\|r(t)\| + \|w(t)\|, \\
t \in [t_i, t_{i+1}),
\]
which means that
\[
|\psi(t)| \leq \gamma_3^{t-t_i}|\psi(t_i)| + \sum_{j=t_i}^{t-1} \gamma_3^{t-j-1} (\gamma_3 \|r(j)\| + \|w(j)\|),
\]
t = t_i, t_i + 1, \ldots, t_{i+1}. \tag{12}

Setting $t = t_{i+1}$ and using (7) yields
\[
\left\| \left[ \begin{array}{c} \phi(t_{i+1}) \\ z_1(t_{i+1}) \end{array} \right] \right\| \leq c\gamma_3^m \left\| \left[ \begin{array}{c} \phi(t_i) \\ z_1(t_i) \end{array} \right] \right\| + \\
\sum_{j=t_i}^{t_{i+1}-1} c\gamma_3^{t_{i+1}-j-1} \gamma_3 (\|r(j)\| + \|w(j)\|) + c\|r(t_{i+1})\| \\
\leq c\gamma_3^m \left\| \left[ \begin{array}{c} \phi(t_i) \\ z_1(t_i) \end{array} \right] \right\| + c\gamma_3 (1 + \lambda)^m \sum_{j=t_i}^{t_{i+1}-1} \gamma_3^m \gamma_3^{t_{i+1}-j-1} (\|r(j)\| + \|w(j)\|) + c\|r(t_{i+1})\|. \tag{13}
\]

This completes Case 2.

At this point we combine Case 1 and 2. We would like to analyze $m$ intervals of length $m$. On the interval $[t, t + m^2]$, there are $m$ subintervals of length $m$; furthermore, because of the choice of $\epsilon$ we have that
\[
\sum_{j=t}^{t+m^2-1} \|\theta(j + 1) - \theta(j)\| \leq c_0 + \epsilon m^2 \leq 2c_0.
\]
It is easy to see that there are at most $N_1 \geq \frac{4c_0\|f\|}{\lambda_1 - \lambda}$ subintervals which fall into the category of Case 2, with the remainder falling into the category of Case 1; it is clear from the formula for $m$ that $m > N_1$. If we use (10) and (13) to analyze the behavior of the closed-loop system on the interval $[t, t + m^2]$, we end up with a crude bound of
\[
\left\| \left[ \begin{array}{c} \phi(t + m^2) \\ z_1(t + m^2) \end{array} \right] \right\| \leq c\gamma_3 N_1 \gamma_3^m \gamma_3^{m(m-N_1)} \left\| \left[ \begin{array}{c} \phi(t) \\ z_1(t) \end{array} \right] \right\| + \\
2m (\frac{2\gamma_3}{\lambda_1 - \lambda})^m \gamma_3^{m+1} \gamma_3^{m} \gamma_3^{m+1} \left( \frac{2}{\lambda_1 - \lambda} \right)^{m+1} \left\| \left[ \begin{array}{c} \phi(t) \\ z_1(t) \end{array} \right] \right\| + \\
\sum_{j=t}^{t+m^2-1} \gamma_3^{t+m^2-j-1} (\|r(j)\| + \|w(j)\|) + \\
c_2 \|r(t + m^2)\|. \tag{14}
\]
From the choice of $m$ above, it is easy to show that
\[
m^2 \ln \left( \frac{2\gamma_3}{\lambda_1 - \lambda} \right) \geq \ln(c) + N_1 \ln(\gamma_3) + N_1 \ln \left( \frac{2}{\lambda_1 + \lambda} \right);
\]
this immediately implies that
\[
c^m N_1 \gamma_3 \left( \frac{2}{\lambda_1 + \lambda} \right) \gamma_3^m \gamma_3^{m(m-N_1)} \leq \left( \frac{2\gamma_3}{\lambda_1 + \lambda} \right)^m \gamma_3^{m+1} \gamma_3^{m} \gamma_3^{m+1} \left( \frac{2}{\lambda_1 - \lambda} \right)^{m+1} \left\| \left[ \begin{array}{c} \phi(t) \\ z_1(t) \end{array} \right] \right\| + \\
\gamma_4 \sum_{j=t}^{t+m^2-1} \lambda_1^{t+m^2-j-1} (\|r(j)\| + \|w(j)\|) + \gamma_4 \|r(t + m^2)\|. \tag{15}
\]
Now let $\tau \geq t_0$ be arbitrary. By setting $\tilde{t} = \tau + m^2, \tau + 2m^2, \ldots$, in succession, it follows from (15) that
\[
\left\| \left[ \begin{array}{c} \phi(t + qm^2) \\ z_1(t + qm^2) \end{array} \right] \right\| \leq \lambda_1^{qm^2} \left\| \left[ \begin{array}{c} \phi(t) \\ z_1(t) \end{array} \right] \right\| + \\
\gamma_4 \sum_{j=\tau}^{\tau+qm^2-1} \lambda_1^{t+qm^2-j-1} (\|r(j)\| + \|w(j)\|) + \gamma_4 \|r(t + qm^2)\|, \quad q \in \mathbb{Z}^+. \tag{16}
\]
So $\left[ \begin{array}{c} \phi(t) \\ z_1(t) \end{array} \right]$ is well-behaved at $t = \tau, \tau + m^2, \tau + 2m^2, \ldots$; we can use (9) of Case 1, (12) of Case 2 and (7) to prove that nothing untoward happens between these times. We conclude that there exists a constant $c_5$ so that
\[
\left\| \left[ \begin{array}{c} \phi(t) \\ z_1(t) \end{array} \right] \right\| \leq c_5 \lambda_1^t \left\| \left[ \begin{array}{c} \phi(t) \\ z_1(t) \end{array} \right] \right\| + \\
c_5 \sum_{j=\tau}^{t-1} \lambda_1^{t-j-1} (\|r(j)\| + \|w(j)\|) + \gamma_5 \|r(t)\|, \quad t \geq \tau. \tag{17}
\]
Since $\tau \geq t_0$ is arbitrary, the desired bound is proven. \hfill \blacksquare

A careful examination of the above proof reveals that $\epsilon \to 0$ as $c_0 \to 0$ and as $c_0 \to \infty$. If we do not care about the decay rate, then we can remove this drawback.
Theorem 2. Suppose that the controller (2) provides a convolution bound for (1) with gain \( c \geq 1 \) and decay rate \( \lambda \in (0, 1) \). Then there exists an \( \epsilon > 0 \) such that for every \( c_0 \geq 0 \), there exist \( \lambda_s \in (0, 1) \) and \( \gamma > 0 \) so that (2) provides a convolution bound for \( (f, s(S, c_0, \epsilon)) \) with gain \( \gamma \) and decay rate \( \lambda_s \).

Proof of Theorem 2. Suppose the controller (2) provides a convolution bound for (1) with gain \( c \geq 1 \) and a decay rate of \( \lambda \). Fix \( \lambda_s \in (1, \lambda) \); let \( t_0 \in \mathbb{Z} \), \( \phi_0 \in \mathbb{R}^{n_x + r + n_u - m} \), \( z_{10} \in \mathbb{R}^{n_x} \), \( z_{20} \in X \), \( w \in S(\mathbb{R}^r) \) and \( r \in S(\mathbb{R}^y) \) be arbitrary. The goal is to prove that for a small-enough \( \epsilon \), the controller (2) provides a convolution bound for \( (f, s(S, c_0, \epsilon)) \) for every \( c_0 \geq 0 \). So at this point we will analyze the closed-loop system for an arbitrary \( \epsilon > 0 \), \( c_0 \geq 0 \), and \( \theta^* \in s(S, c_0, \epsilon) \).

To proceed, let \( t \geq t_0 \) be arbitrary. For \( m \in \mathbb{N} \), we will first analyze closed-loop behavior on intervals of length \( m \); define a sequence \( \{t_i\} \) by

\[
 t_i = t + im, \quad i \in \mathbb{Z}^+.
\]

We can rewrite the time-varying plant as

\[
y(t+1) = \theta(t_i) ^T f(\phi(t)) + w(t) + [\theta(t) - \theta(t_i)] ^T f(\phi(t)),
\]

where \( t \in [t_i, t_{i+1}]. \)

On the interval \( [t_i, t_{i+1}] \), we regard the plant as time-invariant, but with an extra disturbance: so we obtain

\[
|\phi(t)| \leq \epsilon \lambda^{t-t_i} |\phi(t_i)| + \sum_{j=t_i}^{t_{i+1}-1} \epsilon \lambda^{t-j-1} (|r(j)| + |w(j)| + |\bar{n}_i(j)|) + c_2 |r(t)|,
\]

\[
t \in [t_i, t_{i+1}], \quad i \in \mathbb{Z}^+.
\] (18)

Using the same idea as in the proof of Theorem 1, we define the difference equation

\[
|\phi(t)| \leq \epsilon |\phi(t)| + c_2 |r(t)|, \quad t \in [t_i, t_{i+1}] \]

it follows that

\[
|\phi(t)| \leq c_2 |\phi(t)| + c_3 |r(t)|, \quad t \in [t_i, t_{i+1}]. \]

Case 1: \(|\bar{n}_i(t)| \leq \frac{2c}{\lambda} \epsilon |\phi(t)| \) for all \( t \in [t_i, t_{i+1}] \).

Arguing in an identical manner to the proof of Theorem 1, we obtain the following two bounds:

\[
|\psi(t)| \leq \frac{\epsilon |\phi(t)|}{2c} \psi(t_i) + \sum_{j=t_i}^{t_{i+1}-1} \frac{\epsilon |\phi(t)|}{2c} \psi(t_{i+1}) + (2|\phi(t)| + |w(j)|),
\]

\[
t = t_i, t_{i+1}, t_{i+1} + 1, \ldots, t_{i+1}, \quad \lambda = \lambda_s - \lambda.
\]

(20)

This, in turn, implies that there exists \( c_2 > c \) so that

\[
|\phi(t)| \leq \epsilon \lambda^{t-t_i} |\phi(t_i)| + \sum_{j=t_i}^{t_{i+1}-1} \epsilon \lambda^{t-j-1} (|r(j)| + |w(j)|) + c_2 |r(t_{i+1})|.
\] (21)

Case 2: \(|\bar{n}_i(t)| \leq \frac{2c}{\lambda} \epsilon \lambda^{t-t_i} |\phi(t)| \) for some \( t \in [t_i, t_{i+1}]. \)

Arguing in an identical manner to the proof of Theorem 1, we obtain the following two bounds: there exists \( \gamma_3 > 0 \) so that

\[
|\psi(t)| \leq \gamma_3 \epsilon \lambda^{t-t_i} |\phi(t)| + \sum_{j=t_i}^{t_{i+1}-1} \gamma_3 \epsilon \lambda^{t-j-1} (|r(j)| + |w(j)|),
\]

\[
t = t_i, t_{i+1}, t_{i+1} + 1, \ldots, t_{i+1}, \quad \lambda = \lambda_s - \lambda.
\]

(22)

This completes Case 2.

At this point we combine Case 1 and 2. We would like to analyze \( N \in \mathbb{N} \) intervals of length \( m \); for now we let \( N \) be free. We see that

\[
\sum_{j=t}^{t+mN-1} |\theta(j + 1) - \theta(j)| \leq c_0 + \epsilon mN.
\]

Let \( N_1 \) denote the number of intervals of the form \( [t_i, t_{i+1}] \) which lie in \( [t, t + mN] \) which fall into Case 2; it is easy to see that \( N_1 \) satisfies

\[
N_1 \leq \left( \frac{\epsilon mN}{\lambda_1 - \lambda} \right) \times c_0 \times \left( \frac{\epsilon mN}{\lambda_1 - \lambda} \right) \times \epsilon \times mN.
\]

(24)
observe that \( N_1 \) depends on both \( c_0 \) and \( \epsilon \). Using (21) and (23) we obtain
\[
\begin{align*}
\left\| \phi(t + m N) \right\| & \leq c \gamma_3 m N \left\| \phi(t) \right\| + 2 N \left( \frac{2 \gamma_3}{1 + \lambda} \right)^N \left( m N_1 \right) \left\| \phi(t) \right\| \\
& + 2 N \sum_{j=t}^{t+2N-1} (m N_j - 1) \left( \left\| r(j) \right\| + \left\| w(j) \right\| \right) + c_2 \left\| r(t_{q+1}) \right\|.
\end{align*}
\]

At this point, we will choose quantities \( m, \epsilon \) and \( \bar{N} \), in that order, so that the key gain \( c \gamma_3 m N_1 \left\| \phi(t) \right\| \leq 1 \). First of all, we apply the bound on \( N_1 \) given in (24) to this key gain:
\[
c \left( \frac{2 \gamma_3}{1 + \lambda} \right)^N \left( m N_1 \right) \left\| \phi(t) \right\| \leq \left( \left( \frac{2 \gamma_3}{1 + \lambda} \right)^N \left( \frac{m N}{\lambda} \right) \right) \left\| \phi(t) \right\|.
\]

Now choose \( m \) so that \( \left( \frac{2 \gamma_3}{1 + \lambda} \right)^N \leq \lambda_2 < 1 \), i.e. any \( m > \frac{\ln(\epsilon)}{m(2) - \ln(\lambda_1 + \lambda)} \). So rewriting (26), we now obtain
\[
c \gamma_3 m N_1 \left\| \phi(t) \right\| \leq \left( \left( \frac{2 \gamma_3}{1 + \lambda} \right)^N \left( m N_1 \right) \right) \left\| \phi(t) \right\|.
\]

Now observe that
\[
\lim_{\epsilon \to 0} \left( \left( \frac{2 \gamma_3}{1 + \lambda} \right)^N \left( \frac{m N}{\lambda} \right) \right) \leq 1,
\]
so now choose \( \epsilon > 0 \) so that
\[
\left( \left( \frac{2 \gamma_3}{1 + \lambda} \right)^N \left( \frac{m N}{\lambda} \right) \right) \leq \lambda_3 < 1;
\]
notice that \( \epsilon \) is independent of \( c_0 \). With this choice we now have
\[
c \gamma_3 m N_1 \left\| \phi(t) \right\| \leq \lambda_3 \left\| \phi(t) \right\|.
\]

Last of all, now choose \( \bar{N} \) so that
\[
\left( \left( \frac{2 \gamma_3}{1 + \lambda} \right)^N \left( \frac{m N}{\lambda} \right) \right) \leq \lambda_3 \left\| \phi(t) \right\|.
\]
any \( \bar{N} > \frac{2 c \gamma_3 m N_1 \left\| \phi(t) \right\|}{\lambda (\lambda_1 + \lambda)} \) will do. Observe that \( \bar{N} \) depends on \( c_0 \).

Now let \( \tau \geq t_0 \) be arbitrary. By setting \( t = \tau, \tau + m \bar{N}, \tau + 2m \bar{N}, \ldots \), in succession, with \( \lambda_5 := \max \left\{ \frac{\lambda_4 m N}{1 + \lambda}, \frac{\lambda_4 m N}{1 + \lambda} \right\} \) (which clearly depends on \( c_0 \) via \( \bar{N} \)) it follows from (27) that
\[
\begin{align*}
\left\| \phi(t + q \bar{N} m) \right\| & \leq \lambda_5^q m \left\| \phi(t) \right\| + \\
\left\| \phi(t) \right\| & \leq \lambda_5^q \left\| \phi(t) \right\| + \\
\gamma_4 m \left\| r(t + q \bar{N} m) \right\|, \quad q \in \mathbb{Z}^+.
\end{align*}
\]

4. Tolerance to Unmodelled Dynamics

We now consider the time-varying plant (4) with the term \( d(t) \) added to represent unmodelled dynamics:
\[
y(t + 1) = \theta^*(t) f(\phi(t)) + w(t) + d(t), \quad \phi(t_0) = \phi_0.
\]

Here we consider (a generalized version of) a class of unmodelled dynamics which is common in the adaptive control literature—see [3] and [6]. With \( g : \mathbb{R}^n + r + n_s \to \mathbb{R} \) a map with a bounded gain, \( \beta \in (0, 1) \) and \( \mu > 0 \), we consider
\[
m(t + 1) = \beta m(t) + \beta g(\phi(t)), \quad m(t_0) = m_0 \quad \text{and} \quad \mu(m(t)) = \mu(g(\phi(t))), \quad t \geq t_0.
\]
It turns out that this model subsumes a large class of classical additive uncertainty, multiplicative uncertainty, and uncertainty in a coprime factorization, with a strict causality constraint; see [6] for a more detailed explanation. We will now show that if the controller (2) provides a convolution bound for \((f, s(S, \epsilon, \epsilon))\), then a degree of tolerance to unmodelled dynamics can be proven.

**Theorem 3.** Suppose that the controller (2) provides a convolution bound for \((f, s(S, \epsilon, \epsilon))\) with a gain \(c_1\) and decay rate \(\lambda_1 \in (0, 1)\). Then for every \(\beta \in (0, 1)\) and \(\lambda_2 \in (\max(\lambda_1, \beta), 1)\), there exist \(\mu > 0\) and \(c_2 > 0\) so that for every \(\theta^* \in s(S, \epsilon, \epsilon), \mu \in (0, \mu), t_0 \in \mathbb{Z}\), \(\phi_0 \in \mathbb{R}^{n_y + n_u + m}, z_{10} \in \mathbb{R}^l, z_{20} \in X \subset \mathbb{R}^l, r \in \mathbb{R}(\mathbb{R}^n)\) and \(w \in \mathbb{R}(\mathbb{R}^n)\), when the controller (2) is applied to the plant (30) with \(d_\Delta\) satisfying (31), the following holds:

\[
\left\| \begin{bmatrix} \phi(t) \\ z_1(t) \\ m(t) \end{bmatrix} \right\| \leq c_2 \lambda_2^{t-t_0} \left\| \begin{bmatrix} \phi_0 \\ z_{10} \\ m_{10} \end{bmatrix} \right\| + \sum_{j=t_0}^{t-1} c_2 \lambda_2^{j-t_0} \left( \|r(j)\| + \|w(j)\| \right) + c_2 \|r(t)\|, \quad t \geq t_0.
\]

(32)

**Remark 5.** This proof is based, in part, on the proof of Theorem 3 of [6], which deals with a much simpler setup.

**Proof of Theorem 3.** Fix \(\beta \in (0, 1)\) and \(\lambda_2 \in (\max(\lambda_1, \beta), 1)\) and let \(\theta^* \in s(S, \epsilon, \epsilon), t_0 \in \mathbb{Z}\), \(\phi_0 \in \mathbb{R}^{n_y + n_u + m}, z_{10} \in \mathbb{R}^l, z_{20} \in X, w \in \mathbb{R}(\mathbb{R}^n)\) and \(r \in \mathbb{R}(\mathbb{R}^n)\) be arbitrary. So by hypothesis:

\[
\left\| \begin{bmatrix} \phi(t) \\ z_1(t) \\ m(t) \end{bmatrix} \right\| \leq c_1 \lambda_1^{t-t_0} \left\| \begin{bmatrix} \phi(\tau) \\ z_1(\tau) \end{bmatrix} \right\| + \sum_{j=t}^{t-1} c_1 \lambda_1^{t-j-1} \left( \|r(j)\| + \|w(j)\| + \|d_\Delta(j)\| \right) + c_1 \|r(t)\|, \quad t \geq t_0.
\]

(33)

To convert this inequality to an equality, we consider the associated difference equations

\[
\ddot{\phi}(t+1) = \lambda_1 \ddot{\phi}(t) + c_1 \|r(t)\| + c_1 \|w(t)\| + c_1 \mu \ddot{m}(t) + c_1 \mu \|g\| \ddot{\phi}(t),
\]

\[
\ddot{\phi}(t_0) = c_1 \left\| \begin{bmatrix} \phi_0 \\ z_{10} \end{bmatrix} \right\|,
\]

and together with the difference equation based on (31a):

\[
\ddot{m}(t+1) = \beta \ddot{m}(t) + \beta \|g\| \ddot{\phi}(t), \quad \ddot{m}(t_0) = |m_0|.
\]

Using induction together with (33), (31a), and (31b), we can prove that

\[
\left\| \begin{bmatrix} \phi(t) \\ z_1(t) \end{bmatrix} \right\| \leq \ddot{\phi}(t) + c_1 \|r(t)\|.
\]

(34a)

If we combine the difference equations for \(\ddot{\phi}(t)\) and \(\ddot{m}(t)\), we obtain

\[
\begin{bmatrix} \ddot{\phi}(t+1) \\ \ddot{m}(t+1) \end{bmatrix} = \begin{bmatrix} \lambda_1 + c_1 \|g\| \mu c_1 \mu & c_1 \mu \|g\| \\ \beta \|\gamma\| & \beta \end{bmatrix} \begin{bmatrix} \ddot{\phi}(t) \\ \ddot{m}(t) \end{bmatrix} + \begin{bmatrix} \gamma \\ddot{\phi}(t) \\ \gamma \ddot{m}(t) \end{bmatrix} = A_{\gamma}(\mu) \begin{bmatrix} \ddot{\phi}(t) \\ \ddot{m}(t) \end{bmatrix} + \begin{bmatrix} \gamma \ddot{\phi}(t) \\ \gamma \ddot{m}(t) \end{bmatrix},
\]

(35)

Now we see that

\[
A_{\gamma}(\mu) = \begin{bmatrix} \lambda_1 & 0 \\ \beta \|\gamma\| & \beta \end{bmatrix}
\]

as \(\mu \to 0\), and this matrix has eigenvalues of \(\{\lambda_1, \beta\}\) which are both less that \(\lambda_2 < 1\). Using a standard Lyapunov argument, it is easy to prove that there exist \(\mu > 0\) and \(\lambda_1 > 0\) such that for all \(\mu \in (0, \mu),\) we have

\[
\|A_{\gamma}(\mu)^k\| \leq \gamma_1 \lambda_2^k, \quad k \geq 0;
\]

if we use this in (35) and then apply the bound in (34), it follows that

\[
\left\| \begin{bmatrix} \phi(t) \\ z_1(t) \\ m(t) \end{bmatrix} \right\| \leq c_1 \lambda_1^{t-t_0} \left\| \begin{bmatrix} \phi_0 \\ z_{10} \\ m_{10} \end{bmatrix} \right\| + \sum_{j=t_0}^{t-1} c_1 \lambda_1^{t-j-1} \left( \|r(j)\| + \|w(j)\| + \|d_\Delta(j)\| \right) + c_1 \|r(t)\|, \quad t \geq t_0
\]

(36)

as desired.

\[\blacksquare\]

5. Applications

In this section, we will apply Theorems 1–3 to various adaptive control problems. In these examples, it turns out that we do not need \(z_1\) as part of the controller.

5.1. First-Order One-Step-Ahead Adaptive Control

Here we consider the 1st-order linear time-invariant plant

\[
y(t+1) = ay(t) + bu(t) + w(t),
\]

\[
\begin{bmatrix} a \\ b \end{bmatrix} = \theta^*^T \begin{bmatrix} y(t) \\ w(t) \end{bmatrix} + w(t), \quad y(t_0) = y_0.
\]

(37)

We have \(y(t) \in \mathbb{R}\) as the output, \(u(t) \in \mathbb{R}\) as the input, and \(w(t) \in \mathbb{R}\) as the noise or disturbance. Here, \(\theta^*\) is unknown but lies in closed and bounded set \(S \subset \mathbb{R}^2\); to ensure controllability we require that \(\begin{bmatrix} a \\ 0 \end{bmatrix} \notin S\) for any \(a \in \mathbb{R}\). The control objective is to track a reference signal
$y^*(t)$ asymptotically; we assume that we know it one step ahead.

In [4] the case of $\mathcal{S}$ being convex is considered. An adaptive controller is designed based on the ideal projection algorithm, and it is proven that a convolution bound is provided. In that paper this is leveraged to prove a degree of tolerance to time-variation and unmodelled dynamics, though the results there are not quite as strong as those provided by Theorems 1–3.

Now we turn to the more general case of $\mathcal{S}$ not convex. This was considered in [8] and a convolution bound was proven\(^2\), but nothing was proven about robustness to time-variation and unmodelled dynamics. Here we will show that the controller proposed there fits into the framework of this paper, so that Theorems 1–3 can be applied. In this case, it is proven in [8] that $\mathcal{S}$ can be covered by two convex and compact sets $\mathcal{S}_1$ and $\mathcal{S}_2$ so that, for every $a \in \mathcal{S}_1 \cup \mathcal{S}_2$ we have that $b \neq 0$. To proceed, we use two parameter estimators—one for $\mathcal{S}_1$ and one for $\mathcal{S}_2$—and then use a switching adaptive controller to switch between the estimates as necessary. For each $i \in \{1, 2\}$ and given an estimate $\hat{\theta}_i(t)$ at time $t > t_0$, we have a prediction error of

$$e_i(t + 1) := y(t + 1) - \hat{\theta}_i(t)^T \phi(t);$$

estimator updates are computed by

$$\hat{\theta}_i(t + 1) = \begin{cases} \hat{\theta}_i(t) & \text{if } \phi(t) = 0 \\ \hat{\theta}_i(t) + \frac{\phi(t)}{\|\phi(t)\|^2} e_i(t + 1) & \text{otherwise} \end{cases} \quad (38)$$

$$\hat{\theta}_i(t + 1) = \text{Proj}_{\mathcal{S}_i} \{ \hat{\theta}_i(t + 1) \}. \quad (39)$$

We partition $\hat{\theta}_i(t)$ in a natural way by $\hat{\theta}_i(t) = \begin{bmatrix} \hat{a}_i(t) \\ \hat{b}_i(t) \end{bmatrix}$. We define a switching signal $\sigma : \mathbb{Z} \to \{1, 2\}$ to choose which parameter estimates to use in the control law at any point in time. Namely, with $\sigma(t_0) \in \{1, 2\}$ to choose is

$$\sigma(t + 1) = \arg \min_{i \in \{1, 2\}} |e_i(t + 1)|, \quad t \geq t_0, \quad (40)$$

i.e. it is the index corresponding to the smallest prediction error. Next we apply the Certainty Equivalence Principle to yield

$$u(t) = -\hat{a}_{\sigma(t)}(t) \hat{b}_{\sigma(t)}(t) y(t) + \frac{1}{\hat{b}_{\sigma(t)}(t)} y^*(t + 1). \quad (41)$$

We observe here that the controller (38)–(41) fits into the paradigm of Section 2: we set

$$\mathcal{X} = \mathcal{S}_1 \times \mathcal{S}_2 \times \{1, 2\},$$

$$z_1(t) = \varnothing,$$

$$z_2(t) = \begin{bmatrix} \hat{a}_1(t) \\ \hat{a}_2(t) \\ \sigma(t) \end{bmatrix},$$

$$r(t) = y^*(t + 1).$$

In [8] it is proven that (38)–(41) provides a convolution bound for (37); by Theorems 1–3 we immediately see that the same is true in the presence of time-variation and/or unmodelled dynamics.

5.2. Pole-Placement Adaptive Control

In this section, we consider the Pole-Placement Adaptive Control problem. We consider the $n$th-order linear time-invariant plant

$$y(t + 1) = \sum_{j=0}^{n-1} a_{j+1} y(t - j) + \sum_{j=0}^{n-1} b_{j+1} u(t - j) + w(t)$$

$$= \begin{bmatrix} y(t) \\ y(t - n + 1) \\ \vdots \\ u(t) \\ u(t - n + 1) \end{bmatrix}^T \begin{bmatrix} a_1 \\ \vdots \\ a_n \\ b_1 \\ \vdots \\ b_n \end{bmatrix} + w(t), \quad t \geq t_0 \quad (42)$$

with $\phi(t_0) = \phi_0$. We have $y(t) \in \mathbb{R}$ as the output, $u(t) \in \mathbb{R}$ as the input, and $w(t) \in \mathbb{R}$ as the noise or disturbance. Here, $\theta^*$ is unknown but lies in a known set $\mathcal{S} \subset \mathbb{R}^n$. Associated with this plant model are the polynomials

$$\mathbf{A}(z^{-1}) = 1 - a_1 z^{-1} - a_2 z^{-2} - \ldots - a_n z^{-n},$$

$$\mathbf{B}(z^{-1}) = b_1 z^{-1} + b_2 z^{-2} + \ldots + b_n z^{-n};$$

We impose the following assumption:

**Assumption 1.** $\mathcal{S}$ is compact, and for each $\theta^* \in \mathcal{S}$, the corresponding polynomials $\mathbf{A}(z^{-1})$ and $\mathbf{B}(z^{-1})$ are coprime.

The objective here is to obtain some form of stability with a secondary objective that of asymptotic tracking of a reference signal $y^*(t)$; the plant may be non-minimum phase, which limits the tracking goal.

In [6] the case of $\mathcal{S}$ convex is considered. An adaptive controller is designed based on a modified version of the ideal projection algorithm, and it is proven that a convolution bound is provided; this is leveraged there to prove a degree of tolerance to time-variation and unmodelled dynamics, much like that provided by Theorems 1 and 3.

Now we turn to the more general case of $\mathcal{S}$ not convex. This was also considered in [6] subject to

**Assumption 2.** $\mathcal{S} \subset \mathcal{S}_1 \cup \mathcal{S}_2$ with $\mathcal{S}_1$ and $\mathcal{S}_2$ compact and convex, and for each $\theta^* \in \mathcal{S}_1 \cup \mathcal{S}_2$, the corresponding polynomials $\mathbf{A}(z^{-1})$ and $\mathbf{B}(z^{-1})$ are coprime.
A convolution bound was proven, but nothing was proven about robustness to time-variation and to unmodelled dynamics. Here we will show that the controller proposed there fits into the framework of this paper, so that Theorems 1–3 can be applied. To proceed, we use two parameter estimators—one for $S_1$ and one for $S_2$, and then use a switching adaptive controller to switch between these estimates as necessary; to prove that the approach works, all closed-loop poles are placed at the origin.

The parameter estimation is projection-algorithm-based and similar to that of the previous sub-section. For $i \in \{1,2\}$ and given an estimate $\hat{\theta}_i(t)$ at time $t > t_0$, we have a prediction error of

$$e_i(t+1) := y(t+1) - \hat{\theta}_i(t)\top \phi(t);$$

evaluator updates are computed by

$$\hat{\theta}_i(t+1) = \begin{cases} \hat{\theta}_i(t) + \frac{\phi(t)}{\|\phi(t)\|^2} e_i(t+1) & \text{if } \|\phi(t)\| \neq 0 \\ \hat{\theta}_i(t) & \text{otherwise} \end{cases}$$

We partition $\hat{\theta}_i(t)$ as

$$\hat{\theta}_i(t) := [\hat{a}_{i,1}(t) \cdots \hat{a}_{i,n}(t) \hat{b}_{i,1}(t) \cdots \hat{b}_{i,n}(t)]\top;$$

associated with $\hat{\theta}_i(t)$ are the polynomials

$$\hat{A}_i(t, z^{-1}) = 1 - \hat{a}_{i,1}(t)z^{-1} - \hat{a}_{i,2}(t)z^{-2} \cdots - \hat{a}_{i,n}(t)z^{-n}, \quad \text{and}$$

$$\hat{B}_i(t, z^{-1}) = \hat{b}_{i,1}(t)z^{-1} + \hat{b}_{i,2}(t)z^{-2} \cdots + \hat{b}_{i,n}(t)z^{-n}. $$

We design a strictly proper controller by choosing its denominator and numerator polynomials, respectively, by

$$\hat{L}_i(t, z^{-1}) = 1 + \hat{\ell}_i(t)z^{-1} + \hat{\ell}_i,2(t)z^{-2} \cdots + \hat{\ell}_i,n(t)z^{-n}, \quad \text{and}$$

$$\hat{P}_i(t, z^{-1}) = \hat{p}_i,1(t)z^{-1} + \hat{p}_i,2(t)z^{-2} \cdots + \hat{p}_i,n(t)z^{-n}$$

satisfying

$$\hat{A}_i(t, z^{-1})\hat{L}_i(t, z^{-1}) + \hat{B}_i(t, z^{-1})\hat{P}_i(t, z^{-1}) = 1, \quad (45)$$

i.e. we place the closed-loop poles at zero.

A switching signal $\sigma : \mathbb{Z} \to \{1,2\}$ is used to choose which parameter estimator to use in the control law at any point in time. We update $\sigma(t)$ only every $N \geq 2n$ steps; to this end, we define a sequence of switching times as follows: we initialize $\hat{t}_0 := t_0$ and then define

$$\hat{t}_\ell := t_0 + \ell N, \quad \ell \in \mathbb{N}.$$ 

The switching signal is given by

$$\sigma(t) = \sigma(\hat{t}_\ell), \quad t \in [\hat{t}_\ell, \hat{t}_{\ell+1}), \quad \ell \in \mathbb{Z}^+.$$ 

Now define the control gains $\hat{K}_i(t) \in \mathbb{R}^{2n}$ that are also only updated every $N \geq 2n$ steps:

$$\hat{K}_i(t) := [-\hat{p}_{i,1}(t) \cdots - \hat{p}_{i,n}(t) - \hat{l}_{i,1}(t) \cdots - \hat{l}_{i,n}(t)], \quad t \in [\hat{t}_\ell, \hat{t}_{\ell+1}), \quad \ell \in \mathbb{Z}^+; \quad (47)$$

define the filtered reference signal

$$r_2(t) := \sum_{j=1}^n \hat{p}_{\sigma(\hat{t}_\ell),j}(t)g^r(t - j + 1), \quad t \in [\hat{t}_\ell, \hat{t}_{\ell+1}), \quad \ell \in \mathbb{Z}^+. $$

For each $i$, define a performance signal

$$J_i(\hat{t}_\ell) := \begin{cases} 0 & \text{if } \hat{\sigma}(j) > 0 \text{ for all } j \in [\hat{t}_\ell, \hat{t}_{\ell+1}) \\ \max_{j \in [\hat{t}_\ell, \hat{t}_{\ell+1})} \{J_i(\hat{t}_\ell)\} & \text{otherwise.} \end{cases} \quad (48)$$

With $\sigma(\hat{t}_0) \in \{1,2\}$, we set

$$J_i(\hat{t}_{\ell+1}) = \arg \min_{i \in \{1,2\}} J_i(\hat{t}_\ell), \quad \ell \in \mathbb{Z}^+,$$

and define the control law by

$$u(t) = \hat{K}_{\sigma(\hat{t}_\ell)}(t - 1)\phi(t - 1) + r_2(t - 1). \quad (50)$$

We observe here that the controller (43),(44), (45), (48), (49) and (50) fits into the paradigm of Section 2; we can rewrite the controller in the form of (2) as follows. First we set

$$\mathcal{X} = \mathbb{R}^N \times \mathbb{R}^N \times S_1 \times S_2 \times \{1,2\}.$$ 

For $t \geq t_0$, we then set

$$z_1(t+1) = \hat{K}_{\sigma(t)}(t)\phi(t) + r(t), \quad z_2(t) = \begin{bmatrix} z_2(t) \\ \hat{\theta}_1(t) \\ \hat{\theta}_2(t) \\ \sigma(t) \end{bmatrix}, \quad u(t) = z_1(t),$$

with $r(t) = r_2(t)$; for $t \geq t_0$ and $i = 1, 2$, we then set

$$z_2(t+1) = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix} z_2(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \cdot \phi(t) \neq 0 \text{ otherwise,}$$

and for $t > t_0$, we set

$$\sigma(t) = \begin{cases} \sigma(t-1) & \hat{t}_0 \notin \mathbb{N} \\ \arg \min_{i \in \{1,2\}} \|z_2(t)\|_{\infty} = J_i(t-1) \notin \mathbb{N} \end{cases} \quad (51)$$

Here we use $\|z_2(t)\|_{\infty}$ to denote the $\infty$-norm of the vector $z_2(t)$.
In [6] it is proven\(^4\) that this adaptive controller provides a convolution bound for (42); by Theorems 1–3 we see that the same is true in the presence of time-variation and/or unmodelled dynamics.

6. Summary and Conclusion

In this paper we have shown that for a class of nonlinear plant and controller combinations, if a convolution bound on the closed-loop behavior can be proven, then tolerance to small time-variations in the plant parameters and a small amount of unmodelled dynamics follows immediately. We applied the result to prove robustness of our recently designed multi-estimator switching adaptive controllers presented in [6] and [8]. We expect this to be applicable to other adaptive control paradigms, such as the adaptive control of nonlinear plants; this will allow one to focus on the ideal plant in the analysis knowing that robustness will come for free. This result also has the potential to be applied in more general nonlinear contexts.

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\(^4\)Technically speaking, the bound (3) is only proven for \(t \geq \tau = t_0\). However, since the controller is periodic of period \(N \geq 2n\), it follows immediately that the same bound (3) holds for \(t \geq \tau \geq t_0\) for all \(\tau \in \{t_0+N, t_0+2N, t_0+3N, \ldots\}\). Since the controller has a bounded gain, nothing untoward can happen for other \(\tau\)’s; it is easy to prove that (3) will still hold for a suitably larger choice of \(c\) (but with the same \(\lambda\).