WAVELET TIGHT FRAME AND PRIOR IMAGE-BASED IMAGE 
RECONSTRUCTION FROM LIMITED-ANGLE 
PROJECTION DATA

CHENXIANG WANG
School of Mathematical Sciences, University of Electronic Science and Technology of China
Chengdu, 611731, China
College of Mathematics and Statistics, Chongqing University
Chongqing 401331, China

LI ZENG
1
College of Mathematics and Statistics, Chongqing University
Chongqing 401331, China
Engineering Research Center of Industrial Computed Tomography
Nondestructive Testing of the Education Ministry of China, Chongqing University
Chongqing 400044, China

YUMENG GUO AND LINGLI ZHANG
College of Mathematics and Statistics, Chongqing University
Chongqing 401331, China

(Communicated by Hui Ji)

Abstract. The limited-angle projection data of an object, in some practical applications of 
computed tomography (CT), are obtained due to the restriction of scanning condition. In these 
situations, since the projection data are incomplete, some limited-angle artifacts will be 
presented near the edges of reconstructed image using some classical reconstruction algorithms, 
such as filtered backprojection (FBP). The reconstructed image can be fine approximated 
by sparse coefficients under a proper wavelet tight frame, and the quality of reconstructed 
image can be improved by an available prior image. To deal with limited-angle CT reconstruction 
problem, we propose a minimization model that is based on wavelet tight frame and a prior image, 
and perform this minimization problem efficiently by iteratively minimizing separately. Moreover, 
we show that each bounded sequence, which is generated by our method, converges to a critical or a 
stationary point. The experimental results indicate that our algorithm can efficiently suppress artifacts and 
noise and preserve the edges of reconstructed image, what’s more, the introduced prior image will not 
miss the important information that is not included in the prior image.

1. Introduction. Limited-angle computed tomography (CT) reconstruction is an important image 
reconstruction task that reconstructs a high quality image from incomplete or inconsistent 
projection data. In some practical applications of CT imaging, due to the restriction of scanning 
condition and the harm of X-ray ionizing radiation [14, 29, 33, 34, 23], the projection data of the object 
are obtained from a limited scanning range, which is less than $180^\circ$ plus a fan-angle. For example, in

2010 Mathematics Subject Classification. Primary: 90C90, 15A29, 44A12; Secondary: 94A08.
Key words and phrases. Inverse problems, image reconstruction, wavelet transform, $\ell_0$ regularized, prior information.

1The corresponding author: drlizeng@cqu.edu.cn

©2017 American Institute of Mathematical Sciences
industrial fields, CT is utilized to detect a pipeline in service or a large object size [42], and the straight-line trajectory CT [17, 12] is utilized to detect the defects of a long object. In the medical domain, there are several examples, such as C-arm CT [1], imaging of the breasts [39], and short exposure time [22]. In these situations, the filtered backprojection (FBP) algorithm (Chap.7 in [5]) does not work well because the projection data are incomplete and the reconstruction problem is severely ill-posed [27] (see the NCAT phantom [36] for limited-angle reconstruction shown in Figure 2). Figure 1 shows the scanning geometry of limited-angle CT.

![Figure 1. Scanning geometry of limited-angle CT. S denotes the X-ray source, o denotes the rotation center of object, D denotes the detector, and $\theta$ denotes the rotation angle which is less than $180^0$ plus a fan-angle.](image1)

![Figure 2. Reconstructed result of NCAT phantom using FBP algorithm for the scanning angle $[0, 120^0]$]. The limited-angle artifacts are labelled by the red rectangles.](image2)

In limited-angle CT reconstruction problem, the goal is to reconstruct the attenuation coefficient $f$ (a CT image) from the incomplete projection data with noisy

$$g = Af + \eta,$$

where $\eta \in \mathbb{R}^M$ denotes the noise, $g \in \mathbb{R}^M$, $f \in \mathbb{R}^N$, and $A \in \mathbb{R}^{M \times N} (M \ll N)$ denotes the system matrix of X-ray transform in a limited scanning range.

Due to the ill-posed for the limited-angle problem, the regularization has to be considered to stabilize the procedure of limited-angle reconstruction. Thus, in the reconstruction procedure, a priori knowledge about the solution has to be incorporated.
The classical approach is the so-called Tikhonov type regularization, where a regularized solution \( f_\lambda \) is obtained from the following optimization problem

\[
\begin{align*}
  f_\lambda & \in \arg \min_{f \in \Omega} \left\{ \frac{1}{2} \| Af - g \|^2_2 + \lambda R(f) \right\},
\end{align*}
\]

where \( \Omega \) denotes a convex set, \( \lambda > 0 \) is a regularization parameter, and \( R : \Omega \to [0, +\infty] \) is a regularization function (may be non-convex and non-continuous). The first term of (2) is the data fidelity term that controls the error of data. The second term is the penalty term or prior term that includes the object’s prior information.

According to the goal of CT image reconstruction, such as to obtain the high resolution of reconstruction image or to preserve the particular edge, there are various choices for the regularization function. For example, in CT image reconstruction, the total variation (TV) norm which is generally used for preserving edges of reconstructed image [13, 19].

Let \( f(i, j) \) be the pixel value of an image at \((i, j)\), then, the TV norm of an image can be written as

\[
\| f \|_{TV} = \sum_{i,j} \sqrt{(f_{i,j} - f_{i-1,j})^2 + (f_{i,j} - f_{i,j-1})^2}.
\]

In [35], the authors considered the sparsity of image under the gradient transform, and an adaptive steepest descent-projection onto convex sets (ASD-POCS) algorithm was proposed. The model of ASD-POCS algorithm is

\[
\begin{align*}
  \min_{f} \| f \|_{TV} \text{ s.t. } \| Af - g \|_2 \leq \varepsilon, f \geq 0,
\end{align*}
\]

where \( \varepsilon \) denotes the noise.

A high quality reconstructed image can be obtained using ASD-POCS algorithm for the sparse-view sampling over 360°. However, the limited-angle artifacts will be presented near edges of reconstructed image when the scanning angular range is seriously limited [45]. To overcome these problems, some authors have improved the CT reconstruction algorithm with TV. In [24] the authors proposed a new alternating optimization program for limited-angle reconstruction problem. In [25], the authors developed a novel iterative reconstruction algorithm using weighted TV as the objective function. In [10], the authors proposed an anisotropic TV reconstruction algorithm. Although these methods reduce the limited-angle artifacts near edges, the edges of object are still distorted [44].

In [20], the authors pointed out that TV regularization favors piecewise constant functions that may destroy relevant information. Recently, wavelet frames have been used in sparse-view CT image reconstruction [40, 48, 46, 21, 15, 47, 16], the basic idea is that the reconstructed image can be fine approximated by sparse coefficients under a proper wavelet tight frame. It is well-known that \( \ell_1 \) norm or \( \ell_0 \) quasi-norm prefers sparse solutions [48, 46], then, the sparse regularization solution \( f_\lambda \) is obtained from the following optimization problem

\[
\begin{align*}
  f_\lambda & \in \arg \min_{f \in \Omega} \left\{ \frac{1}{2} \| Af - g \|^2_2 + \lambda \| Wf \|_p \right\},
\end{align*}
\]

where \( \Omega \) is a convex set, \( \lambda > 0 \) is a regularization parameter, \( W \) is a wavelet frame, \( p = 0 \) or 1 denotes the \( \ell_1 \) norm or \( \ell_0 \) quasi-norm. A high quality reconstructed image can be obtained by this model for sparse-view CT reconstruction, however, the limited-angle artifacts will be presented near edges of reconstructed image for limited-angle problem that the scanning angular range is seriously limited.
To reconstruct a high quality image under the incomplete data or inconsistent data, a prior image has been incorporated into the CT image reconstruction by some scholars. In [8], a prior image constrained compressed sensing (PICCS) algorithm was proposed. The prior image of this algorithm, which was obtained from the union of interleaved dynamical data sets, was utilized to sparse-view image reconstruction for the individual time frames. In [38, 9], the PICCS algorithm was utilized to cardiac CT to improve the temporal resolution. In [26], the PICCS algorithm was utilized to prospectively study CT dose reduction. The PICCS algorithm can effectively preserve the edges of the reconstructed image for the sparse-view CT image reconstruction when the parameter is chosen properly. The model of PICCS algorithm [8, 38] is

\[
\lambda \in \arg \min \{ \lambda \| W_1 (f - f_p) \|_1 + (1 - \lambda) \| W_2 f \|_1 \}, \quad s.t \quad Af = g,
\]

where \( W_1 \) and \( W_2 \) denote the discrete gradient transforms, \( f_p \) denotes a prior image, and \( \lambda > 0 \) is a regularization parameter. The PICCS algorithm includes the algebraic reconstruction technique (ART) and the standard steepest descent method.

A prior image, which is used to the follow-up detection or diagnosis, can be obtained from full-scan data using FBP algorithm [8, 26, 28] or ASD-POCS algorithm before the equipment is installed or in the first diagnosis. For example, imaging of a pipeline in service which is attached to wall or installed on the ground, the pipeline is only scanned in a limited angular range due to the restriction of scanning environment. To make the follow-up detection conveniently, we obtain a prior image from full-scan data before the pipeline is installed. The CT image of pipeline in service is close to the prior image except for few small features, such as crack. In medical fields, the prior image should be obtained from full-scan data in the first diagnosis, then, the prior image can be used for same patient in the follow-up diagnosis who is scanned in a limited angular range for reducing the dose of X-Ray [26] or for the restriction of C-arm CT [28]. The CT image of same patient is close to the prior image except for few small features, such as tumor.

In this paper, to better suppress the limited-angle artifacts near edges that occur in limited-angle CT image reconstruction and preserve the edges, a new model based on a wavelet tight frame and a prior image is proposed. The model is a constrained optimization model whose objective function includes a data fidelity term and two regularization terms. One of the regularization terms is based on the sparsity of the image under a wavelet tight frame, the other is the difference between the high-frequency of prior image and that of reconstructed image.

Unlike the common procedure used in [48, 46, 8, 38, 9, 26], where the \( \ell_1 \) norm or \( \ell_0 \) quasi-norm was used to measure the sparsity of all wavelet coefficients, we utilize the \( \ell_0 \) quasi-norm to promote the sparsity of wavelet coefficients of low-frequency, which can suppress the limited-angle artifacts as the \( \ell_0 \) quasi-norm can cut off the small wavelet coefficients of low-frequency. And also different from [8, 38, 9, 26], whose all information of prior image was used to the reconstruction procedure, we only utilize the high-frequency of prior image that only includes the image detail parts and make the difference between the high-frequency of prior image and that of reconstructed image to be minimized, which can preserve the edges of the reconstructed image and make the reconstructed image be closer to the desired solution. More detail discussions of the reconstruction model will be found in Subsection 3.1.

The objective function incorporating the \( \ell_0 \) quasi-norm will make this problem
into a non-convex and non-smooth minimization problem, and it is difficult to obtain a global minimization of the object function. To deal with this non-convex and non-smooth minimization problem, we use the Moreau envelope to approximate the regularization terms of original objective function \[31\] and use hard thresholding (HT) to deal with $\ell_0$ quasi-norm \[46, 43\]. An alternative minimizing iterative is therefore implemented to solve our model. Moreover, this paper provides the convergence analysis of the alternating iterative algorithm for solving our model. We show that each bounded sequence, which is generated by iteratively minimizing iterative, converges to a critical or a stationary point. Finally, the experimental results indicate that our algorithm can efficiently suppress the limited-angle artifacts and noise and preserve the edges of reconstructed image for limited-angle CT reconstruction.

The remain of the paper is organized as follows. A brief description of mathematical principles and some definitions used in this work are provided in Section 2. In Section 3, we introduce the proposed minimization model and the corresponding numerical algorithm. In Section 4, we give the convergence analysis of the alternating iterative algorithm for solving our model under certain conditions. Finally, our numerical experimental results of limited-angle CT reconstruction are presented in Section 5 and conclusions are given in Section 6.

2. Preliminaries. In this section, a brief description of mathematical principles and some definitions used in this work are provided.

2.1. Wavelet and tight frame. A brief introduction of the wavelet tight frame and its construction is given in this subsection. More details are available in \[48, 21, 2, 6, 18, 30, 11, 7\]. Let $\mathcal{H}$ be a Hilbert space, if
\[
\|f\|^2 = \sum_n |\langle f, e_n \rangle|^2, \quad \text{for any } f \in \mathcal{H},
\]
then the sequence \(\{e_n\} \subset \mathcal{H}\) is a tight frame for $\mathcal{H}$.

There are two associated operators in a tight frame, one is the analysis operator $W$ that is defined by
\[
W : f \in \mathcal{H} \rightarrow \{\langle f, e_n \rangle\} \in \ell_2(\mathbb{N}),
\]
the sequence \(\{\langle f, e_n \rangle\}\) is called the canonical frame coefficient sequence, and the other is its synthesis operator $W^T$ that is defined by
\[
W^T : \{f_n\} \in \ell_2(\mathbb{N}) \rightarrow \sum_n f_n e_n \in \mathcal{H}.
\]

Then, the sequence \(\{e_n\} \subset \mathcal{H}\) is a tight frame if and only if $W^T W = I$, where $I$ is an identical operator of $\mathcal{H}$. It implies the following canonical expansion can be obtained under a given tight frame
\[
f = \sum_n \langle f, e_n \rangle e_n, \quad \text{for any } f \in \mathcal{H}.
\]

Tight frame system is widely used in image processing, which is generated by the filters \(\{h_0\}_{r=0}\) of framelets. Let \(\{V_n\}_{n \in \mathbb{N}}\) be multiresolution analysis (MRA) derived from the refinable function $\phi$ with refinement mask $h_0$. In Fourier domain, the definition of refinable function $\phi$ is $\hat{\phi}(2 \cdot) = \hat{h}_0 \hat{\phi}$. Here $\hat{\phi}$ is the Fourier transform of $\phi$, and $\hat{h}_0$ is the Fourier series of refinement mask $h_0$. Let $X \subset L_2(\mathbb{R})$ be a
countable set, \( \Psi = \{ \psi_1, \psi_2, \ldots, \psi_r \} \subset V_1. \) The construction of a tight wavelet frames is to find \( \Psi \) that is the same as to find the filters \( \{ h_l \} \) such that
\[
\psi_l(x) = 2 \sum_{k \in \mathbb{Z}} h_l[k] \phi(2x - k).
\]
The sequences \( \{ h_l \}_{l=1}^r \) are called the high pass filters, the refinement mask \( h_0 \) is called the low pass filter. The equality above can be rewritten in the Fourier domain as
\[
\hat{\psi}_l(2\cdot) = \hat{h}_l \hat{\phi}, \quad l = 1, 2, \ldots, r,
\]
for some \( 2\pi \)-periodic functions \( \{ \hat{h}_l \}_{l=1}^r \). In [30], the Unitary Extension Principle (UEP) was proposed that can be used to construct a tight frame. One of the full UEP conditions
\[
\hat{h}_0(\omega) \hat{h}_0(\omega + t\pi) + \sum_{l=1}^r \hat{h}_l(\omega) \hat{h}_l(\omega + t\pi) = \delta_{t,0}, \quad t = 0, 1,
\]
where \( \omega \in \{ \omega \in \mathbb{R} : [\hat{\phi}, \hat{\phi}](\omega) \neq 0 \} \). The piecewise linear B-spline framelets are used in this work. The refinement mask is \( \hat{h}_0(\omega) = \cos^2(\frac{\omega}{2}) \), whose corresponding filter is \( h_0 = \frac{1}{2}[1, 2, 1] \). Two framelets are \( \hat{h}_1(\omega) = -\sqrt{2}i \sin(\omega) \) and \( \hat{h}_2(\omega) = \sin^2(\frac{\omega}{2}) \), whose corresponding filters are
\[
h_1 = \frac{\sqrt{2}}{4}[1, 0, -1], \quad h_2 = \frac{1}{4}[-1, 2, -1].
\]

2.2. Some definitions. In this subsection, we will introduce some definitions used in our work.

**Definition 1.** Let \( \Omega \subseteq \mathbb{R}^N \) be a convex set, and \( G : \Omega \to [0, +\infty] \) be a proper and lower semi-continuous function. Given \( f \in \Omega \) and \( \beta > 0 \), the Moreau envelope function associated with \( \beta \) is defined as [31]
\[
\text{env}_\beta^G(f) := \inf_{g \in \Omega} \{ G(g) + \frac{\beta}{2} \| g - f \|_2^2 \}.
\]

**Remark 1.** \( \text{env}_\beta^G(f) \to G(f) \) for any \( f \in \Omega \) when \( \beta \to +\infty \).

**Definition 2.** Let \( \Omega \subseteq \mathbb{R}^N \) be a convex set, and \( G : \Omega \to [0, +\infty] \) be a proper and lower semi-continuous function. Given \( f \in \Omega \) and \( \beta > 0 \), the proximal map associated with \( \beta \) is defined as [31]
\[
\text{prox}_\beta^G(f) := \arg \min_{g \in \Omega} \{ G(g) + \frac{\beta}{2} \| g - f \|_2^2 \}.
\]

**Definition 3.** The hard thresholding (HT) operator with threshold \( \lambda \in \mathbb{R}^+ \) is defined as [46]
\[
H_\lambda(x) = \begin{cases} 
0 & \text{if } |x| < \lambda \\
\{0, x\} & \text{if } |x| = \lambda \\
x & \text{if } |x| > \lambda 
\end{cases}
\]

**Definition 4.** The indicator function \( \delta_\Omega(f) : \mathbb{R}^N \to (-\infty, +\infty] \) of a nonempty and convex set \( \Omega \) is defined as [4]
\[
\delta_\Omega(f) = \begin{cases} 
0 & \text{if } f \in \Omega, \\
+\infty & \text{otherwise.}
\end{cases}
\]
Definition 5. Let \( f : \mathbb{R}^N \to (-\infty, +\infty) \) be a proper and lower semi-continuous function. The Fréchet subdifferential \( \partial f(x) \) is defined as \( \{ u : \liminf_{y \neq x, y \to x} \frac{f(y) - f(x) - (u, y - x)}{\|y - x\|} \geq 0 \} \), for any \( x \in \text{dom} f \) and \( \partial f(x) = \emptyset \) if \( x \notin \text{dom} f \). For each \( x \in \text{dom} f \), \( x \) is called the stationary or critical point of \( f \) if it satisfies \( 0 \in \partial f(x) \).

3. CT reconstruction model and numerical algorithm. In this section, we propose a new minimization model for limited-angle CT reconstruction problem, which is based on a wavelet tight frame and a prior image. An efficient numerical solver is also provided for the proposed minimization model.

3.1. CT reconstruction model. In limited-angle reconstruction problem, the main difficult problem is that the reconstructed image will present some limited-angle artifacts near edges because the projection data are incomplete (see Figure 2). In order to solve this problem, a priori knowledge about the solution has to be incorporated into the reconstruction model. It is known that the reconstructed images can be well sparsely approximated using a proper wavelet tight frame. Thus, the wavelet transform of image under a tight frame is sparse can be regarded as a prior knowledge and used to our reconstruction model.

Figure 3. (a) and (b) are the reconstructed results using FBP algorithm for scanning range \([0, 360^0]\) and \([0, 120^0]\), respectively.

Figure 3 shows the reconstructed results for the scanning range \([0, 360^0]\) and \([0, 120^0]\) using FBP algorithm. In limited-angle reconstruction problem, the low-frequency degradation are presented in reconstructed image (see the top left of Figures 4 and 5), in this situation, the sparsity of the wavelet coefficients of low-frequency of the reconstructed image with limited-angle artifacts is weaker than that of the desired image. To correct the limited-angle artifacts near edge, the \( \ell_0 \) quasinorm of the wavelet coefficients of low-frequency of reconstructed image is utilized that penalizes smaller wavelet coefficients (i.e., the smaller wavelet coefficients of the low-frequency of reconstructed image will be set to zero by hard thresholding, which makes the wavelet coefficients of low-frequency more sparse).

In limited-angle CT reconstruction problem, the edges of reconstructed image are distorted (see Figure 3), which corresponding to the high-frequency of reconstructed image in the wavelet domain (see Figures 4 and 5). The quality of reconstructed image can be improved by an available prior image. As seen from Figure 3, the edges of (a) are well preserved, if we regard the high-frequency of the prior image
as a prior information (the prior image is a reconstructed image from full-scan data using FBP algorithm), and minimize the $\ell_2$ norm of the difference between the high-frequency of prior image and that of reconstructed image, then, this method
will make the high-frequency of reconstructed image be closer to that of prior image, and preserve the edges of reconstructed image for limited-angle reconstruction problem. In additional, because the noise is also included in the high-frequency of reconstructed image, the noise will be effectively suppressed by minimizing the ℓ₂ norm of the difference between the high-frequency of prior image (not include the noise or include the low level of noise) and that of reconstructed image.

Therefore, the high-frequency information of prior image and the sparsity of wavelet coefficients of low-frequency are utilized to suppress the limited-angle artifacts and to preserve the edges of reconstructed image. We propose to reconstruct a high quality image by solving the following minimization model

\[
\min_{f \in \Omega} \frac{1}{2} \|Af - g\|_2^2 + \frac{\gamma}{2} \|(Wf)_G - (Wf_0)_G\|_2^2 + \lambda \|(Wf)_L\|_0,
\]

where \(Wf = ((Wf)_L, (Wf)_G)\), \(f_0\) is a prior image, \(f\) denotes the reconstructed image, \(\Omega = \{f \in \mathbb{R}^N | f \geq 0\}\) denotes a convex set, \(\|\cdot\|_0\) denotes the number of nonzero terms, \((Wf)_G\) and \((Wf)_L\) denote the high-frequency of wavelet transform of image \(f\) and \(f_0\), respectively, \((Wf)_L\) denotes the low-frequency of wavelet transform of image \(f\), \(A \in \mathbb{R}^{M \times N} (M \ll N)\) denotes the system matrix of X-ray transform in a small scanning angular range, \(\gamma\) and \(\lambda\) are the positive regularization parameters, and \(g \in \mathbb{R}^M\) denotes the projection data.

### 3.2. Numerical algorithm.

The non-convex and non-continuous of ℓ₀ quasi-norm and the nonseparable property of \(\|(Wf)_L\|_0\) inevitably make the numerical algorithm difficult in solving the minimization model (8) directly. To solve the problem (8), the Moreau envelope function (see, Definition 1) is introduced to approximate the regularization terms of original objective function (see Remark 1) [31], then the (8) becomes the following minimization problem

\[
\min_{f \in \Omega} \frac{1}{2} \|Af - g\|_2^2 + \inf_{\alpha} \left\{\frac{\gamma}{2} \|\alpha_G - (Wf_0)_G\|_2^2 + \lambda \|\alpha_L\|_0 + \frac{\beta}{2} \|Wf - \alpha\|_2^2\right\},
\]

where \(\alpha = (\alpha_L, \alpha_G)\) and \(\beta > 0\). The minimization model (9) can be reformed as an unconstrainted minimization model using an indicator function (see, Definition 4) as follows

\[
\min_{f} \frac{1}{2} \|Af - g\|_2^2 + \inf_{\alpha} \left\{\frac{\gamma}{2} \|\alpha_G - (Wf_0)_G\|_2^2 + \lambda \|\alpha_L\|_0 + \frac{\beta}{2} \|Wf - \alpha\|_2^2\right\} + \delta_{\Omega}(f).
\]

It is clear that the objective function in model (10) is close to that in model (8) when the parameter \(\beta\) is sufficiently large.

The model (10) is essentially consistent with the following model with two variables by the following proposition

\[
\min_{f, \alpha} \frac{1}{2} \|Af - g\|_2^2 + \frac{\gamma}{2} \|\alpha_G - (Wf_0)_G\|_2^2 + \lambda \|\alpha_L\|_0 + \frac{\beta}{2} \|Wf - \alpha\|_2^2 + \delta_{\Omega}(f).
\]

We use the following notes for convenience

\[
Q(f, \alpha) := \frac{1}{2} \|Af - g\|_2^2 + \frac{\gamma}{2} \|\alpha_G - (Wf_0)_G\|_2^2 + \lambda \|\alpha_L\|_0 + \frac{\beta}{2} \|Wf - \alpha\|_2^2 + \delta_{\Omega}(f),
\]

\[
H(\alpha_G) := \frac{\gamma}{2} \|\alpha_G - (Wf_0)_G\|_2^2.
\]
Proposition 1. Let $\Omega$ be a convex set, $Q(f, \alpha)$ is noted in (12) and $H(\alpha_G)$ is noted in (13), $W$ is a wavelet tight frame system. For any $\lambda > 0$, $\gamma > 0$, and $\beta > 0$, if a pair $(f^*, \alpha^*)$ is a solution of (11), then

$$\alpha^*_L \in \text{prox}_{\lambda/\beta}(\alpha_L Wf)_L,$$

$$\alpha^*_G = \text{prox}_{H(\alpha_G)}(Wf^*)_G,$$

where $\alpha^* = (\alpha^*_L, \alpha^*_G)$ and $Wf^* = ((Wf^*)_L, (Wf^*)_G)$. Moreover, $f^*$ is a solution of the model (10) with $\alpha^*$ satisfying (14) and (15) if and only if a pair $(f^*, \alpha^*)$ is a solution of (11).

Proof. The idea of this proof is similar to [31].

The Proposition 1 means that the solution of model (10) can be obtained by solving model (11). Thus, we will focus on solving the model (11).

To solve model (11), we can perform this minimization problem efficiently by iteratively minimizing with respect to $f$ and $\alpha$ separately. This means that we will convert the model (11) into two sub-problems that are solved iteratively in an alternating fashion.

(16) **Sub-problem 1**: $\arg\min_f \{ \frac{1}{2} \|Af - g\|_2^2 + \frac{\beta}{2} \|Wf - \alpha\|_2^2 + \delta_\Omega(f) \}$.

(17) **Sub-problem 2**: $\arg\min_\alpha \{ \frac{\gamma}{2} \|\alpha_G - (Wf_0)_G\|_2^2 + \lambda \|\alpha_L\|_0 + \frac{\beta}{2} \|Wf - \alpha\|_2^2 \}$.

In [4], the authors indicated that minimizing the sum of a smooth function with a non-smooth function using the proximal forward-backward scheme can be regarded as a proximal regularization of the smooth function linearized at a given point. Motivated by this idea, we adopt this scheme to solve the sub-problem 1 and write it as an iterative scheme. Linearizing the $\theta(f) := \frac{1}{2} \|Af - g\|_2^2$ at a given point $f^n$, we have that

$$f^{n+1} \in \arg\min_f \{ \theta(f^n) + (f - f^n, \theta'(f^n)) + \frac{\tau}{2} \|f - f^n\|_2^2 + \frac{\beta}{2} \|Wf - \alpha^n\|_2^2 + \delta_\Omega(f) \}.$$  

An equivalent form of (18) is that

$$f^{n+1} \in \arg\min_f \{ \frac{\tau}{2} \|f - f^n\|_2^2 + \frac{1}{\tau} \|\theta'(f^n)\|_2^2 + \frac{\beta}{2} \|Wf - \alpha^n\|_2^2 + \delta_\Omega(f) \}.$$  

We can use the simultaneous algebraic reconstruction technique (SART) (the system matrix $A$ need not be stored) [5, 27] in this step if we do it like this. The (19) has a closed form solution and the solution is

$$f^{n+1} = \max\{ (\tau(f^n - \frac{1}{\tau} \theta'(f^n)) + \beta W^T(\alpha^n))/((\tau + \beta), 0) \}.$$  

To solve the sub-problem 2, because $\frac{\beta}{2} \|Wf - \alpha_L\|_2^2 = \frac{\beta}{2} \|Wf - \alpha\|_2^2$, we solve the low-frequency part $\alpha_L$ and the high-frequency part $\alpha_G$ of wavelet coefficients $\alpha$ separately.

(21) **High-frequency**: $\arg\min_\alpha \{ \frac{\gamma}{2} \|\alpha_G - (Wf_0)_G\|_2^2 + \frac{\beta}{2} \|Wf_G - \alpha_G\|_2^2 \}$.

(22) **Low-frequency**: $\arg\min_\alpha \{ \lambda \|\alpha_L\|_0 + \frac{\beta}{2} \|Wf_L - \alpha_L\|_2^2 \}$.
We write (21) and (22) as an iterative scheme

\[ \alpha_{n+1}^G = \arg \min_{\alpha_G} \left\{ \gamma\|\alpha_G - (Wf_0)G\|_2^2 + \frac{\beta}{2}\|Wf_{n+1})G - \alpha_G\|_2^2 \right\}. \]

(24) Low-frequency: \( \alpha_{n+1}^L \in \arg \min_{\alpha_L} \left\{ \lambda\|\alpha_L\|_0 + \beta\|((Wf_{n+1})L - \alpha_L\|_2^2 \right\}.

The (23) has a closed form solution, which can be easily obtained by the first order optimality condition. The solution is

\[ \alpha_{n+1}^G = \frac{\gamma(Wf_0)G + \beta(Wf_{n+1})G}{\gamma + \beta}. \]

Linearizing the second term \( \frac{1}{2}\|((Wf_{n+1})L - \alpha_L\|_2^2 \) of the objective function of (24) at a given point \( \alpha_L^n \) to solve (24), a new iterative scheme can expressed as

\[ \alpha_{n+1}^L \in \arg \min_{\alpha_L} \left\{ \lambda\|\alpha_L\|_0 + \beta\|((Wf_{n+1})L - \alpha_L\|_2^2 - \beta(\langle(Wf_{n+1})L - \alpha_L, \alpha_L - \alpha_L^n\rangle) \right\}.

The (26) also has a closed form solution [46], which can be obtained using hard thresholding (see, Definition 3). The closed form solution is

\[ \alpha_{n+1}^L = H\sqrt{\frac{2\lambda}{\beta}} \left( \frac{1}{\rho}((Wf_{n+1})L - \alpha_L^n) + \alpha_L^n \right), \]

therefore, \( \alpha_{n+1} = (\alpha_{n+1}^L, \alpha_{n+1}^G) \).

Next, the process of the alternating minimization with respect to \( f \) and \( \alpha \) separately are summarized in the form of a pseudo-code. \( N_{\max} \) denotes the maximum number of iteration, \( L_0 \) and \( L_{\varphi} \) are constants. The implementation steps of the alternating minimization for CT reconstruction are presented as follows

Our algorithm:

\[ \text{Initialization:} \]

Given \( \lambda, \beta, \gamma > 0, \tau > \frac{\beta}{2} > 0, \rho > L_{\varphi} > 0, f^1 = 0, \alpha^1 = Wf^1, n = 1. \]

while \( n < N_{\max} \)

\[ \text{Step 1. updating } f: \]

Obtaining \( f^{n+1} \) using (20).

\[ \text{Step 2. updating } \alpha: \]

Solving \( \alpha^{n+1} \) by (25) and (27).

end while

4. Convergence analysis. In this section, the convergence analysis of our algorithm is provided and we need a sequence of the lemmas to establish the convergence theorem.

Lemma 1. (Descent lemma of [4]) Let \( h : \Omega \to [0, +\infty] \) be a continuously differentiable function with \( h' \) being \( L_h \)-Lipschitz continuous. Then,

\[ h(u) \leq h(v) + \langle u - v, h'(v) \rangle + \frac{L_h}{2}\|u - v\|_2^2, \forall u, v \in \Omega. \]
Let \( \eta \in (0, +\infty) \), and \( \Phi_{\eta} \) be the family of all continuous and concave functions \( F : [0, \eta) \to \mathbb{R}^+ \) which satisfy three conditions: (i) \( F(0) = 0 \); (ii) \( F \) is \( C^1 \) on \( (0, \eta) \) and continuous at \( 0 \); (iii) for all \( s \in (0, \eta) \): \( F'(s) > 0 \).

**Lemma 2.** (Uniformized KL property \cite{4}) Let \( Q : \Omega_1 \subset \mathbb{R}^d \to (-\infty, +\infty] \) be a proper and lower semi-continuous function and \( \Omega_1 \) be a compact set. Suppose that \( Q \) is constant on \( \Omega_1 \) and satisfies the KL property at each point of \( \Omega_1 \). Then, there exist \( \varepsilon_1 > 0 \), \( \eta > 0 \) and \( F \in \Phi_{\eta} \) such that for all \( \bar{z} \in \Omega_1 \) and all \( z \) in the following intersection

\[
\{ z \in \mathbb{R}^d : \text{dist}(z, \Omega_1) < \varepsilon_1 \} \cap \{ Q(\bar{z}) < Q(z) < Q(\bar{z}) + \eta \},
\]

where \( \text{dist}(z, \Omega_1) := \inf \{ \| y - z \|_2 : y \in \Omega_1 \} \). One has,

\[
F'(Q(z) - Q(\bar{z})) \text{dist}(0, \partial Q(z)) \geq 1.
\]

We use the following notations for convenience:

\[
\theta(f) := \frac{1}{2} \| Af - g \|_2^2, \quad \psi(f, \alpha) := \frac{\beta}{2} \| Wf - \alpha \|_2^2,
\]

\[
\phi(f, \alpha_G) := \frac{\beta}{2} \| (Wf)_G - \alpha_G \|_2^2,
\]

\[
\varphi(f, \alpha_L) := \frac{\beta}{2} \| (Wf)_L - \alpha_L \|_2^2.
\]

And two assumptions for the convergence analysis in the following are needed:

1. \( \theta(f) \) is a continuously differentiable function with \( \theta' \) being Lipschitz continuous with a constant \( L_\theta > 0 \);
2. \( \varphi(f, \alpha_L) \) is a continuously differentiable function about the variable \( \alpha_L \) with \( \partial_{\alpha_L} \varphi \) being Lipschitz continuous with a constant \( L_\varphi > 0 \).

**Lemma 3.** Let \( \theta(f) \) and \( \psi(f, \alpha) \) be defined as in (30), and the assumption 1 holds. Fix any \( \tau > \frac{L_\theta}{2} \). Then, for any \( f^n \in \text{dom} \psi \), \( f^n \in \Omega \) (a convex set), \( f^{n+1} \in \Omega \) and \( \alpha^n \) are the minimizations of (19), (23) and (26), respectively. We have

\[
\theta(f^{n+1}) + \psi(f^{n+1}, \alpha^n) \leq \theta(f^n) + \psi(f^n, \alpha^n) - (\tau - \frac{L_\varphi}{2}) \| f^{n+1} - f^n \|_2^2.
\]

**Proof.** Because \( f^n \in \text{dom} \psi_\beta \) and \( f^{n+1} \in \Omega \) is a minimization of (19), we have that

\[
f^{n+1} \in \arg \min \left\{ \frac{\tau}{2} \| f - f^n - \theta'(f^n) \|_2^2 + \frac{\beta}{2} \| Wf - \alpha^n \|_2^2 + \delta_\Omega(f) \right\}.
\]

Because \( f^n, f^{n+1} \in \Omega \) and \( \psi(f, \alpha^n) + \delta_\Omega(f) \) is a convex function about variable \( f \), we have that

\[
\psi(f^n, \alpha^n) \geq \psi(f^{n+1}, \alpha^n) + \langle \nu^{n+1}, f^n - f^{n+1} \rangle,
\]

for all \( \nu^{n+1} \in \partial f \psi(f^{n+1}, \alpha^n) + \partial \delta_\Omega(f^{n+1}) \). According to the optimization condition of (33) and \( f^{n+1} \in \Omega \), we have that

\[
\theta'(f^n) + \tau (f^{n+1} - f^n) + \nu^{n+1} = 0,
\]

where \( \nu^{n+1} \in \partial f \psi(f^{n+1}, \alpha^n) + \partial \delta_\Omega(f^{n+1}) \). According to (34) and (35), we have that

\[
\psi(f^n, \alpha^n) \geq \psi(f^{n+1}, \alpha^n) + \langle \theta'(f^n), f^{n+1} - f^n \rangle + \tau \| f^{n+1} - f^n \|_2^2.
\]
By Lemma 1 and (36), we have that
\[
\theta(f^{n+1}) + \psi(f^{n+1}, \alpha) \leq \theta(f^n) + \langle f^{n+1} - f^n, \theta'(f^n) \rangle + \frac{L_\theta}{2} \|f^{n+1} - f^n\|_2^2
\]
\[
+ \psi(f^{n+1}, \alpha^n)
\]
\[
\leq \theta(f^n) + \frac{L_\theta}{2} \|f^{n+1} - f^n\|_2^2 + \psi(f^n, \alpha^n) - \tau \|f^{n+1} - f^n\|_2^2
\]
\[
= \theta(f^n) + \psi(f^n, \alpha^n) - (\tau - \frac{L_\theta}{2}) \|f^{n+1} - f^n\|_2^2.
\]

Lemma 4. Let \( \varphi(f, \alpha_L) \) be defined as in (32) and the assumption 2 holds. Fix any \( \rho > L_\varphi \). Then, for any \( \alpha_L^n \in \text{dom}(\lambda \|\alpha_L\|_0) \), \( f^{n+1} \) and \( \alpha^{n+1} \) are the minimizations of (19), (23) and (26), respectively. We have
\[
\lambda \|\alpha^{n+1}_L\|_0 + \varphi(f^{n+1}, \alpha^{n+1}_L) \leq \lambda \|\alpha^n_L\|_0 + \varphi(f^{n+1}, \alpha^n_L) - \frac{\rho - L_\varphi}{2} \|\alpha^{n+1}_L\|_2^2.
\]

Proof. Because \( \partial_{\alpha_L} \varphi \) is Lipschitz continuous with a constant \( L_\varphi > 0 \), by Lemma 1, we have that
\[
\varphi(f^{n+1}, \alpha^{n+1}_L) \leq \varphi(f^{n+1}, \alpha^n_L) + \langle \partial_{\alpha_L} \varphi(f^{n+1}, \alpha^n_L), \alpha^{n+1}_L - \alpha^n_L \rangle + \frac{L_\varphi}{2} \|\alpha^{n+1}_L - \alpha^n_L\|_2^2.
\]
According to the (26), we obtain that
\[
\lambda \|\alpha^{n+1}_L\|_0 + \varphi(f^{n+1}, \alpha^n_L) + \langle \partial_{\alpha_L} \varphi(f^{n+1}, \alpha^n_L), \alpha^{n+1}_L - \alpha^n_L \rangle + \frac{\rho}{2} \|\alpha^{n+1}_L - \alpha^n_L\|_2^2
\]
\[
\leq \lambda \|\alpha^n_L\|_0 + \varphi(f^{n+1}, \alpha^n_L).
\]
According to the inequality (38) and (37), we can obtain that
\[
\lambda \|\alpha^{n+1}_L\|_0 + \varphi(f^{n+1}, \alpha^{n+1}_L) \leq \lambda \|\alpha^{n+1}_L\|_0 + \varphi(f^{n+1}, \alpha^n_L)
\]
\[
+ \langle \partial_{\alpha_L} \varphi(f^{n+1}, \alpha^n_L), \alpha^{n+1}_L - \alpha^n_L \rangle + \frac{L_\varphi}{2} \|\alpha^{n+1}_L - \alpha^n_L\|_2^2
\]
\[
+ \frac{\rho}{2} \|\alpha^{n+1}_L - \alpha^n_L\|_2^2 + \frac{\rho}{2} \|\alpha^{n+1}_L - \alpha^n_L\|_2^2
\]
\[
= \lambda \|\alpha^{n+1}_L\|_0 + \varphi(f^{n+1}, \alpha^n_L)
\]
\[
+ \langle \partial_{\alpha_L} \varphi(f^{n+1}, \alpha^n_L), \alpha^{n+1}_L - \alpha^n_L \rangle + \frac{\rho}{2} \|\alpha^{n+1}_L - \alpha^n_L\|_2^2
\]
\[
- \frac{\rho}{2} \|\alpha^{n+1}_L - \alpha^n_L\|_2^2 + \frac{L_\varphi}{2} \|\alpha^{n+1}_L - \alpha^n_L\|_2^2
\]
\[
\leq \lambda \|\alpha^n_L\|_0 + \varphi(f^{n+1}, \alpha^n_L) - \frac{\rho - L_\varphi}{2} \|\alpha^{n+1}_L - \alpha^n_L\|_2^2.
\]

\[\square\]

Let \( Q(f, \alpha), \theta(f), \psi(f, \alpha), \varphi(f, \alpha_L) \), and \( \phi(f, \alpha_G) \) be defined as above, then for all \( f \in \Omega \) and \( \alpha \), we have
\[
\partial Q(f, \alpha) = (\theta'(f) + \partial_f \psi(f, \alpha) + \partial \delta_{\Omega}(f), \partial_{\alpha_L} Q(f, \alpha), \partial_{\alpha_G} Q(f, \alpha)).
\]

Lemma 5. Suppose that assumptions 1 and 2 hold. Let \( \{f^n := (f^n, \alpha^n)\} \) be a sequence generated by our algorithm which is assumed to be bounded. For each positive integer \( n \), define
\[ A_f^n := \theta' (f^{n+1}) - \theta' (f^n) - \tau (f^{n+1} - f^n) + \partial_f \psi (f^{n+1}, \alpha^{n+1}) - \partial_f \psi (f^{n+1}, \alpha^n), \]
\[ A_{\alpha_L}^n := \rho \beta (\alpha_L^n - \alpha_L^{n+1}) - \partial_{\alpha_L} \varphi (f^{n+1}, \alpha_L^n) + \partial_{\alpha_L} \varphi (f^{n+1}, \alpha_L^{n+1}). \]

Then \((A_f^n, A_{\alpha_L}^n, 0) \in \partial Q(f^{n+1}, \alpha^{n+1})\) and there exists \(M > 0\) and \(M' > 0\) such that
\[ \|(A_f^n, A_{\alpha_L}^n, 0)\|_2 \leq (M + \tau + M') \|z^{n+1} - z^n\|_2, \]
where \(M := 2 * \max \{\beta + \rho \beta, M'\}\).

**Proof.** From (19), we have the following result by the optimality condition
\[ (40) \]
where \(\hat{\nu}^{n+1} \in \partial \delta_Q(f^{n+1}). \)

From (26), we have the following result by the optimality condition
\[ (41) \]
where \(\nu^{n+1} \in \partial (\lambda \|\alpha_L^{n+1}\|_0)\).

From (23), we have the following result by the optimality condition
\[ (42) \]
It is clear that
\[ \theta' (f^{n+1}) + \partial_f \psi (f^{n+1}, \alpha^{n+1}) + \hat{\nu}^{n+1} \in \partial_f Q(f^{n+1}, \alpha^{n+1}), \]
and
\[ (\partial_{\alpha_L} Q(f^{n+1}, \alpha_L^{n+1}), \partial_{\alpha_G} Q(f^{n+1}, \alpha_G^{n+1})) = \partial_{\alpha} Q(f^{n+1}, \alpha^{n+1}), \]
where
\[ \partial_{\alpha_L} Q(f^{n+1}, \alpha_L^{n+1}) = \partial_{\alpha_L} \varphi (f^{n+1}, \alpha_L^{n+1}) + \partial (\lambda \|\alpha_L^{n+1}\|_0), \]
\[ \partial_{\alpha_G} Q(f^{n+1}, \alpha_G^{n+1}) = \partial_{\alpha_G} \phi (f^{n+1}, \alpha_G^{n+1}) + \gamma (\alpha_G^{n+1} - (W f_0)_G) = 0. \]

According to (40), (41), and (42), we obtain that
\[ (A_f^n, A_{\alpha_L}^n, 0) \in \partial Q(f^{n+1}, \alpha^{n+1}). \]

Because \(\{(f^n, \alpha^n)\}\) is bounded, and using the Mean Value Theorem and (30), there exists \(M, M' > 0\) such that
\[ \|A_f^n\|_2 \leq \tau \|f^{n+1} - f^n\|_2 + \|\partial_f \psi (f^{n+1}, \alpha^{n+1}) - \partial_f \psi (f^{n+1}, \alpha^n)\|_2 \]
\[ + \|\theta' (f^{n+1}) - \theta' (f^n)\|_2, \]
\[ \leq \tau \|f^{n+1} - f^n\|_2 + M' \|\alpha^{n+1} - \alpha^n\|_2 + M \|f^{n+1} - f^n\|_2 \]
\[ \leq (M + \tau) \|f^{n+1} - f^n\|_2 + M' \|\alpha^{n+1} - \alpha^n\|_2. \]

On the other hand, using the (13), we have that
\[ \|A_{\alpha_L}^n\|_2 \leq \rho \beta \|\alpha_L^{n+1} - \alpha_L^n\|_2 + \|\partial_{\alpha_L} \varphi (f^{n+1}, \alpha_L^{n+1}) - \partial_{\alpha_L} \varphi (f^{n+1}, \alpha_L^n)\|_2 \]
\[ = \rho \beta \|\alpha_L^{n+1} - \alpha_L^n\|_2 + \beta \|\alpha_G^{n+1} - \alpha_G^n\|_2 \]
\[ = (\beta + \rho \beta) \|\alpha_G^{n+1} - \alpha_G^n\|_2. \]

So we have that
According to (23), we have that
\begin{equation}
\| (A_f^n, A_{oL}^n, 0) \|_2 \\
\leq (M + \tau) \| f^{n+1} - f^n \|_2 + (\beta + \rho \beta) \| \alpha_{L}^{n+1} - \alpha_{L}^n \|_2 + M \| \alpha^{n+1} - \alpha^n \|_2 \\
\leq (M + \tau) \| f^{n+1} - f^n \|_2 + (\beta + \rho \beta) \| \alpha^{n+1} - \alpha^n \|_2 + M' \| \alpha^{n+1} - \alpha^n \|_2 \\
\leq (M + \tau) \| f^{n+1} - f^n \|_2 + \hat{M} \| \alpha^{n+1} - \alpha^n \|_2 \\
\leq (M + \tau + \hat{M}) \| z^{n+1} - z^n \|_2,
\end{equation}
where \( \hat{M} := 2 \max \{ \beta + \rho \beta, M' \} \). This completes the proof. \( \Box \)

Next, we will establish a convergence theorem for our reconstruction algorithm.

**Theorem 1.** Suppose \( \rho > L_{\varphi} \) and \( \tau > \frac{L_{\theta}}{2} \). Let \( \phi(f, \alpha_G) \) and \( Q(f, \alpha) \) be defined in (31) and (12), respectively. \( W \) is a wavelet tight frame system. For the sequence \( \{ z^n := (f^n, \alpha^n) \} \) generated by our algorithm is assumed to be bounded, then, the following assertions hold:

1. \( \hat{\rho} := \min \{ \frac{2\tau - L_{\theta}}{2}, \frac{\rho - L_{\varphi}}{2} \} \);
2. \( \lim_{n \to +\infty} \| z^{n+1} - z^n \|_2 = \lim_{n \to +\infty} \| f^{n+1} - f^n \|_2 = \lim_{n \to +\infty} \| \alpha^{n+1} - \alpha^n \|_2 = 0; \)
3. There exists a constant \( C \), such that
   \[ \lim_{n \to +\infty} Q(f^n, \alpha^n) = Q(f^*, \alpha^*) = C, \]
   where \( (f^*, \alpha^*) = \lim_{q \to +\infty} (f^{n_q}, \alpha^{n_q}) \) and \( \{ (f^{n_q}, \alpha^{n_q}) \} \) is a sub-sequence of \( \{ (f^n, \alpha^n) \} \);
4. Moreover, \( (f^n, \alpha^n) \) converges to a critical point or a stationary point \( (f^*, \alpha^*) \) of \( Q(f, \alpha) \).

**Proof.** We begin with proving the Item 1. By Lemma 3 and Lemma 4, we have that
\begin{equation}
\theta(f^{n+1}) + \psi(f^{n+1}, \alpha^n) \leq \theta(f^n) + \psi(f^n, \alpha^n) - (\tau - \frac{L_{\theta}}{2}) \| f^{n+1} - f^n \|_2,
\end{equation}
\begin{equation}
\lambda \| \alpha_{L}^{n+1} \|_0 + \varphi(f^{n+1}, \alpha_{L}^{n+1}) \leq \lambda \| \alpha_{L}^n \|_0 + \varphi(f^{n+1}, \alpha_{L}^n) - \frac{\rho - L_{\varphi}}{2} \| \alpha_{L}^{n+1} - \alpha_{L}^n \|_2^2.
\end{equation}
According to (23), we have that
\begin{equation}
\frac{\gamma}{2} \| \alpha_{G}^{n+1} - (W f_0) G \|_2^2 + \phi(f^{n+1}, \alpha_{G}^{n+1}) \leq \frac{\gamma}{2} \| \alpha_{G}^n - (W f_0) G \|_2^2 + \phi(f^{n+1}, \alpha_{G}^n).
\end{equation}
Because the sequence \( f^n \) and \( f^{n+1} \) are generated by our algorithm, therefore, \( f^{n+1} \in \Omega \) and \( f^n \in \Omega \), adding (43), (44) and (45) together, and using the fact that 
\[ \phi(f, \alpha_G) + \varphi(f, \alpha_L) = \psi(f, \alpha), \]
we have that
\begin{equation}
Q(f^{n+1}, \alpha^{n+1}) \leq Q(f^n, \alpha^n) - \left( \frac{2\tau - L_{\theta}}{2} \| f^{n+1} - f^n \|_2^2 + \frac{\rho - L_{\varphi}}{2} \| \alpha_{L}^{n+1} - \alpha_{L}^n \|_2^2 \right).
\end{equation}
According to (46), we have that
\begin{equation}
\frac{2\tau - L_{\theta}}{2} \| f^{n+1} - f^n \|_2^2 + \frac{\rho - L_{\varphi}}{2} \| \alpha_{L}^{n+1} - \alpha_{L}^n \|_2^2 \leq Q(f^n, \alpha^n) - Q(f^{n+1}, \alpha^{n+1}).
\end{equation}
Because $\rho \geq L_\varphi$, we have that
\begin{equation}
\frac{2\tau - L_\theta}{2} \|f^{n+1} - f^n\|^2 \leq Q(f^n, \alpha^n) - Q(f^{n+1}, \alpha^{n+1}).
\end{equation}

According to (25), we have that
\begin{align*}
\|\alpha_G^{n+1} - \alpha_G^n\|^2 &= \frac{\beta^2}{(\beta + \gamma)^2} \| (W f^{n+1})_G - (W f^n)_G \|^2 \\
&\leq \frac{\beta^2}{(\beta + \gamma)^2} \| W f^{n+1} - W f^n \|^2 \\
&= \frac{\beta^2}{(\beta + \gamma)^2} \| f^{n+1} - f^n \|^2,
\end{align*}
where the last equality follows that $W$ is a wavelet tight frame system [32]. According to (47), we have that
\begin{equation}
\frac{2\tau - L_\theta}{2} \times \frac{(\beta + \gamma)^2}{\beta^2} \| \alpha_G^{n+1} - \alpha_G^n \|^2 + \frac{\rho - L_\varphi}{2} \| \alpha_L^{n+1} - \alpha_L^n \|^2 \leq Q(f^n, \alpha^n) - Q(f^{n+1}, \alpha^{n+1}).
\end{equation}

Let $\hat{\rho} := \min\{(2\tau - L_\theta) \times \frac{(\beta + \gamma)^2}{\beta^2}, \rho - L_\varphi\}$, we have that
\begin{equation}
\frac{\hat{\rho}}{2} \| \alpha_G^{n+1} - \alpha_G^n \|^2 + \frac{\hat{\rho}}{2} \| \alpha_L^{n+1} - \alpha_L^n \|^2 \leq Q(f^n, \alpha^n) - Q(f^{n+1}, \alpha^{n+1}).
\end{equation}

The (51) can be written as follows:
\begin{equation}
\frac{\hat{\rho}}{2} \| \alpha^{n+1} - \alpha^n \|^2 \leq Q(f^n, \alpha^n) - Q(f^{n+1}, \alpha^{n+1}).
\end{equation}

According to (48) and (52), we have that
\begin{equation}
\frac{2\tau - L_\theta}{4} \| f^{n+1} - f^n \|^2 + \frac{\hat{\rho}}{4} \| \alpha^{n+1} - \alpha^n \|^2 \leq Q(f^n, \alpha^n) - Q(f^{n+1}, \alpha^{n+1}).
\end{equation}

Let $\tilde{\rho} := \min\{\frac{2\tau - L_\theta}{2}, \frac{\hat{\rho}}{2}\} = \min\{\frac{2\tau - L_\theta}{2}, \frac{\rho - L_\varphi}{2}\}$, we have that
\begin{equation}
\frac{\tilde{\rho}}{2} \| z^{n+1} - z^n \|^2 = \frac{\tilde{\rho}}{2} \| f^{n+1} - f^n \|^2 + \frac{\tilde{\rho}}{2} \| \alpha^{n+1} - \alpha^n \|^2 \leq Q(f^n, \alpha^n) - Q(f^{n+1}, \alpha^{n+1}).
\end{equation}

Thus, we finish the proof of Item 1.

Next, we will prove the Item 2. According to the definition of $Q(f, \alpha)$, we can obtain that $Q(f, \alpha) \geq 0$. Because $\tau \geq \frac{L_\theta}{2}$ and $\rho \geq L_\varphi$ in our algorithm, we can obtain that $Q(f^{n+1}, \alpha^{n+1}) \leq Q(f^n, \alpha^n)$, therefore, we have that the sequence $\{Q(f^n, \alpha^n)\}$ converges, we have that
\begin{equation}
\lim_{n \to +\infty} \| z^{n+1} - z^n \|^2 = \lim_{n \to +\infty} \| f^{n+1} - f^n \|^2 = \lim_{n \to +\infty} \| \alpha^{n+1} - \alpha^n \|^2 = 0.
\end{equation}

This proves Item 2.

Next, we will prove the Item 3. Because $\{(f^n, \alpha^n)\}$ is bounded, if $\{(f^*, \alpha^*)\}$ is a limited point of $\{(f^n, \alpha^n)\}$, it implies that there exists a sub-sequence $\{(f^{n_r}, \alpha^{n_r})\}$ such that $\lim_{q \to +\infty} (f^{n_q}, \alpha^{n_q}) = (f^*, \alpha^*)$. We claim that the $\lambda\|\alpha_L\|$ term of $Q(f, \alpha)$ is lower semi-continuous. If $\lim_{n \to +\infty} \alpha_L^n = \alpha_L^*$, we have that $\text{supp}(\alpha_L^n) \subseteq \text{supp}(\alpha_L^*)$
where \( \text{supp}(x) = \{i|x_i \neq 0\} \), it implies that \( \|\alpha^n_L\|_0 \geq \|\alpha^*_{+}\|_0 \). Therefore, we have that
\[
\liminf_{n \to +\infty} \lambda\|\alpha^n_L\|_0 \geq \liminf_{n \to +\infty} \lambda\|\alpha^*_{+}\|_0 = \lambda\|\alpha^*_{+}\|_0.
\]
It implies that the \( \lambda\|\alpha_L\|_0 \) term of \( Q(f, \alpha) \) is lower semi-continuous by the definition of lower semi-continuous. Thus, we obtain that
\[
\text{(56)} \quad \lim_{q \to +\infty} \lambda\|\alpha^n_{+}\|_0 \geq \lambda\|\alpha^*_{+}\|_0.
\]
From (26), we obtain that
\[
\lambda\|\alpha^n_{L+1}\|_0 = \beta[(Wf^{n+1})_L - \alpha^n_L, \alpha^n_{L+1} - \alpha^n_L] - \frac{\rho}{2}\|\alpha^n_{L+1} - \alpha^n_L\|^2 \leq \lambda\|\alpha^n_L\|_0 - \beta[(Wf^{n+1})_L - \alpha^n_L, \alpha^n_L - \alpha^n_L] - \frac{\rho}{2}\|\alpha^n_L - \alpha^n_L\|^2.
\]
Let \( n = n_q - 1 \) in the above inequality and let \( q \to \infty \), and using (55), we have that
\[
\text{(57)} \quad \limsup_{q \to +\infty} \lambda\|\alpha^n_{+}\|_0 \leq \lambda\|\alpha^*_{+}\|_0.
\]
From (56) and (57), we have that
\[
\lim_{q \to +\infty} \lambda\|\alpha^n_{+}\|_0 = \lambda\|\alpha^*_{+}\|_0.
\]
Because the rest terms of objective function \( Q(f, \alpha) \) are quadratic function, we have that
\[
\lim_{q \to +\infty} Q(f^{n_q}, \alpha^{n_q}) = \lim_{q \to +\infty} \{\theta(f^{n_q}) + \psi(f^{n_q}, \alpha^{n_q}) + \frac{\gamma}{2}\|\alpha^{n_q}_G - (Wf_0)_G\|^2 + \lambda\|\alpha^{n_q}_G\|_0\} = Q(f^*, \alpha^*).
\]
According to Item 1, we have that the sequence \( \{Q(f^n, \alpha^n)\} \) converges, then, there exists a constant \( C \), such that
\[
\text{(58)} \quad \lim_{n \to +\infty} Q(f^n, \alpha^n) = Q(f^*, \alpha^*) = C.
\]
This proves the Item 3.

Next, we will prove the last Item. By Lemma 5, we have that
\[
(A^*_f, A^n_{+L}, 0) \in \partial Q(f^{n+1}, \alpha^{n+1}),
\]
and
\[
\|(A^*_f, A^n_{-L}, 0)\|_2 \leq (M + \tau + \bar{M})\|z^{n+1} - z^n\|_2.
\]
Let \( n \to +\infty \), using the Item 2 and the Remark 1 of [4], we have that
\[
(0, 0, 0) \in \partial Q(f^*, \alpha^*).
\]
This implies that \( (f^*, \alpha^*) \) is a critical or a stationary point of \( Q(f, \alpha) \).

Next, we will prove that \( (f^n, \alpha^n) \) converges to a critical point or a stationary point \( (f^*, \alpha^*) \). According to the proof of Item 3, we know that there exists a subsequence \( \{(f^{n_q}, \alpha^{n_q})\} \) satisfying \( \lim_{q \to +\infty} (f^{n_q}, \alpha^{n_q}) = (f^*, \alpha^*) \). So we need to prove that \( \{z^n := (f^n, \alpha^n)\} \) is a convergent sequence.

According to the Item 1, we have that \( \{Q(f^n, \alpha^n)\} \) is a non-increasing sequence, it implies that \( Q(f^*, \alpha^*) < Q(f^n, \alpha^n) \) for all \( n > 0 \) by (58). Again from (58), for any \( \eta > 0 \), there exists a \( n_0 \in \mathbb{N}^+ \) such that \( Q(f^n, \alpha^n) < Q(f^*, \alpha^*) + \eta \) for all \( n > n_0 \). Let \( M = \{z^n \in \mathbb{R}^N \times \mathbb{R}^P : \exists \text{ an increasing sequence of integers } \{n_l\}, \text{ such that } z^{n_l} \to z^* \text{ as } l \to +\infty\} \), and \( \omega \) can be regarded as an intersection of compact sets \( \omega = \bigcap_{q \in \mathbb{N}} \bigcup_{n > q} z^n \), thus, \( \omega \) is compact set. It is clear that \( \lim_{n \to \infty} \text{dist}(z^n, \omega) = 0 \), it implies
that for any \( \varepsilon_1 \) there exists a positive integer \( n_1 \) such that \( \text{dist}(z^n, \omega) < \varepsilon_1 \) for all \( n > n_1 \). Summing up all these facts, for all \( n > l := \max\{n_0, n_1\} \), we have that

\[
z^n \in \{ z \in \mathbb{R}^N \times \mathbb{R}^P : \text{dist}(z, \Omega_1) < \varepsilon_1 \} \cap [Q(z) < Q(z) < Q(\bar{z}) + \eta].
\]

Since \( \omega \) is compact and nonempty, and since \( Q \) is finite and constant on \( \omega \), and \( Q \) is a proper and lower semi-continuous function (the proof of Item 3), \( Q \) satisfies the KL property at each point of \( \omega \) (the examples 2 and 3 of [4]). According to the Lemma 2, for all \( n > l \), we can obtain

\[
F'(Q(z^n) - Q(z^*)) \text{dist}(0, \partial Q(z^n)) \geq 1.
\]

According to Lemma 5, we have that

\[
F'(Q(z^n) - Q(z^*)) \geq \frac{1}{M + \tau + M} \|z^n - z^{n-1}\|_2^{-1}.
\]

From the concavity of \( F \) we have that

\[
F(Q(z^n) - Q(z^*)) - F(Q(z^{n+1}) - Q(z^*)) \geq F'(Q(z^n) - Q(z^*)) (Q(z^n) - Q(z^{n+1})).
\]

From Item 1, (60), and (61), we have that

\[
\Delta_{n,n+1} \geq \frac{\|z^{n+1} - z^n\|^2}{C \|z^n - z^{n-1}\|_2},
\]

where \( \Delta_{n,n+1} = F(Q(z^n) - Q(z^*)) - F(Q(z^{n+1}) - Q(z^*)) \) and \( C = \frac{2(M + \tau + M)}{\rho} \).

and it implies that

\[
\|z^{n+1} - z^n\|^2 \leq C \Delta_{n,n+1} \|z^n - z^{n-1}\|_2,
\]

From the fact that \( 2\sqrt{ab} = a + b \) for all \( a, b \geq 0 \), we can obtain that

\[
2\|z^{n+1} - z^n\|_2 \leq C \Delta_{n,n+1} + \|z^n - z^{n-1}\|_2.
\]

Summing up (64) for \( i = l + 1, ..., n \), we have that

\[
2 \sum_{i=l+1}^{n} \|z^{i+1} - z^i\|_2 \leq \sum_{i=l+1}^{n} \|z^i - z^{i-1}\|_2 + C \sum_{i=l+1}^{n} \Delta_{i,i+1}
\]

\[
= \sum_{i=l+1}^{n} \|z^{i+1} - z^i\|_2 + \|z^{l+1} - z^l\|_2 + C \sum_{i=l+1}^{n} \Delta_{i,i+1}
\]

\[
= \sum_{i=l+1}^{n} \|z^{i+1} - z^i\|_2 + \|z^{l+1} - z^l\|_2 + C \Delta_{l+1,n+1}.
\]

Since \( F \geq 0 \), for all \( n > l \), we have that

\[
\sum_{i=l+1}^{n} \|z^{i+1} - z^i\|_2 \leq \|z^{l+1} - z^l\|_2 + C \cdot F(Q(z^{l+1}) - Q(z^*)).
\]

This easily shows that

\[
\sum_{n=1}^{\infty} \|z^{n+1} - z^n\|_2 < \infty.
\]
The \((67)\) implies that \(\sum_{n=\ell+1}^{\infty} \|z^{n+1} - z^n\|_2 \to 0\) as \(\ell \to \infty\). For \(q > p > \ell\), we have that
\[
\|z^q - z^p\|_2 = \|\sum_{n=p}^{q-1} (z^{n+1} - z^n)\|_2 \leq \sum_{n=p}^{q-1} \|z^{n+1} - z^n\|_2 \to 0.
\]
It implies that \(\{z^n\}\) is a Cauchy sequence, which is a convergent sequence, then, we have that \((f^n, \alpha^n) \to (f^*, \alpha^*)\). So we finish the proof of Item 4.

5. Numerical experimental. In this section, we conduct some numerical experiments, which include simulated and practical data, to test the performance of our algorithm for limited-angle CT reconstruction, and we compare our algorithm from limited-angle data with FBP algorithm from full-scan data due to the FBP algorithm has been widely used in commercially CT and can exactly reconstruct the object from full-scan data. In additional, our algorithm is used to compare with PICCS algorithm. All experiments are implemented on a 3.40 GHz intel(R) Core(TM) i3-4130 CPU processor with 8G memory.

5.1. Parameters and iteration number selections. Influenced by some factors, such as noise level, reconstructed object and projection data, etc., optimizing the parameters and the iteration number are a difficult task in CT reconstruction, which empirically selected by visual inspection. In the experiments, the parameters and the iteration number of our algorithm are selected by trial and error.

5.2. Quantitative characterization of the reconstructed image. In this paper, we quantitatively characterize the reconstruction quality by considering the root mean square error (RMSE) \([46]\), peak signal-to-noise ratio (PSNR) \([37]\), and mean structural similarity index (MSSIM) \([41]\), which can be used for measuring the degree of similarity between the reconstructed image from limited-angle projection data and the reference image that is reconstructed from full-scan data using ASD-POCS or desired image. It should be noted that the different quantitatively characterization results will be obtained if we use the different reference images.

5.3. Reconstruction from simulated data with Gaussian noise. In limited-angle CT image reconstruction, a digital NURBS based cardiac-torso (NCAT) phantom \([36]\), which was used in nuclear medicine research, is used to test our algorithm. The scanning parameters of the simulated limited-angle CT are given in Table 1. The scanning ranges \([0, 100^0]\), \([0, 80^0]\) and \([0, 60^0]\) are investigated, and the number of projection views are 100, 80, 60, respectively. The standard deviation and the average value of Gaussian noise added to the projection data are \(0.5\%\|g\|_\infty (1\%\|g\|_\infty)\) and zero, respectively.

| The distance between source and object center | 981 mm |
|--------------------------------------------|--------|
| The angle interval of two adjacent projection views | 1° |
| The angle interval of two adjacent rays | 0.00329° |
| The diameter of field of view | 143.6222 mm |
| Detector numbers | 256 |
| Pixel size | \(0.5632 \times 0.5632\) mm² |
| Image size | \(256 \times 256\) |
For the noise levels $0.5\% \|g\|_\infty$, the reconstruction parameters of our algorithm are $\rho = 1$, $\tau = 128$, $\beta = 0.8 \times 128$, $\lambda = 8 \times 128$, and $\gamma = 8 \times 128$. For the noise levels $1\% \|g\|_\infty$, the reconstruction parameters of our algorithm are $\rho = 1$, $\tau = 128$, $\beta = 0.8 \times 128$, $\lambda = 8 \times 128$, and $\gamma = 10 \times 128$. The maximum iteration number is $N_{\text{ite}} = 1500$ for all situations.

Figure 6 shows the NCAT phantom and prior image that are used for our algorithm and PICCS algorithm.

![Figure 6. NCAT phantom and prior image. The first column is NCAT phantom, the second column is the prior image and the third column is the absolute value of the difference between phantom and prior image. The prior image is the same with NCAT phantom.](image1)

![Figure 7. The reconstructed results for different scanning ranges using FBP algorithm, PICCS algorithm and our algorithm. The display window is [0, 255]. The noise levels are $0.5\% \|g\|_\infty$.](image2)
The reconstructed images for different scanning ranges using FBP algorithm, PICCS algorithm and our algorithm are shown in Figure 7. The noise levels are $0.5\%\|g\|\infty$. The first row is the reconstructed results using FBP algorithm. The subsequent rows are the reconstructed results using PICCS algorithm and our algorithm. From left to right, Figure 7 shows the reconstructed results for different scanning ranges $[0, 60^0]$, $[0, 80^0]$, $[0, 100^0]$ and $[0, 360^0]$ in each columns. As seen from Figure 7, the limited-angle artifacts occur in the reconstructed images using FBP algorithm and the reconstructed images are distorted near edges of the object, however, the limited-angle artifacts are better suppressed and the edges of reconstructed image are better preserved using PICCS algorithm and our algorithm for limited-angle reconstruction problem.

We also consider the high noise levels $1\%\|g\|\infty$. Figure 8 shows the reconstructed images for different scanning ranges using FBP algorithm, PICCS algorithm and our algorithm. The first row is the reconstructed results using FBP algorithm. The subsequent rows are the reconstructed results using PICCS algorithm and our algorithm. From left to right, Figure 8 shows the reconstructed results for different scanning ranges $[0, 60^0]$, $[0, 80^0]$, $[0, 100^0]$ and $[0, 360^0]$ in each columns. As seen from Figure 8, both PICCS algorithm and our algorithm can better suppress the limited-angle artifacts and preserve the edges of reconstructed image for limited-angle reconstruction problem.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure7.png}
\caption{The reconstructed results for different scanning ranges using FBP algorithm, PICCS algorithm and our algorithm. The noise levels are $0.5\%\|g\|\infty$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure8.png}
\caption{The reconstructed results for different scanning ranges using FBP algorithm, PICCS algorithm and our algorithm. The display window is $[0, 255]$. The noise levels are $1\%\|g\|\infty$.}
\end{figure}
In Table 2, we summarize the characterized quantitatively results of reconstruction quality. From Table 2, we can obtain that our algorithm, for scanning ranges \([0, 100^\circ]\), \([0, 80^\circ]\), and \([0, 60^\circ]\), achieves the minimum RMSE and maximum PSNR (MMSIM) compared to the FBP algorithm for full-scan range and the PICCS algorithm. It implies that a higher image quality can be reconstructed using our algorithm because our model makes the \(\ell_2\) norm of the difference between the high-frequency of reconstructed image and that of prior image to be minimized, and uses the \(\ell_0\) quasi-norm to promote the sparsity of wavelet coefficients of low-frequency.

### Table 2. Quantitatively characterize the reconstruction quality.

| Scanning ranges \(0 \sim 360^\circ\) | Variances \(\|g\|_\infty\) | Algorithm | RMSE  | PSNR  | MSSIM |
|-------------------------------|-----------------------------|-----------|-------|-------|-------|
| \(0 \sim 360^\circ\)          | 0.5\%\(\|g\|_\infty\)     | FBP       | 8.403 | 29.64 | 0.9921|
| \(0 \sim 100^\circ\)          | 1\%\(\|g\|_\infty\)       | FBP       | 12.88 | 25.93 | 0.9800|
| \(0 \sim 100^\circ\)          | 0.5\%\(\|g\|_\infty\)     | our algorithm | 2.116 | 41.62 | 0.9995|
| \(0 \sim 80^\circ\)           | 1\%\(\|g\|_\infty\)       | PICCS    | 3.743 | 36.67 | 0.9985|
| \(0 \sim 80^\circ\)           | 0.5\%\(\|g\|_\infty\)     | our algorithm | 4.747 | 34.60 | 0.9973|
| \(0 \sim 60^\circ\)           | 1\%\(\|g\|_\infty\)       | PICCS    | 5.953 | 32.64 | 0.9953|
| \(0 \sim 60^\circ\)           | 0.5\%\(\|g\|_\infty\)     | our algorithm | 2.240 | 41.12 | 0.9994|
| \(0 \sim 60^\circ\)           | 1\%\(\|g\|_\infty\)       | PICCS    | 4.087 | 35.90 | 0.9984|
| \(0 \sim 60^\circ\)           | 0.5\%\(\|g\|_\infty\)     | our algorithm | 4.258 | 35.55 | 0.9978|
| \(0 \sim 60^\circ\)           | 1\%\(\|g\|_\infty\)       | PICCS    | 4.915 | 32.69 | 0.9953|

Next, a modified chest phantom, which includes three circles but the prior image [49] does not includes them (see Figure 9), is used to test our algorithm. The scanning parameters of the simulated CT system are given in Table 3. The scanning ranges \([0, 80^\circ]\), \([0, 100^\circ]\) and \([0, 120^\circ]\) are investigated, and the number of projection views are 114, 143 and 171, respectively. The standard deviation and average value of Gaussian noise are 0.1\%\(\|g\|_\infty\) and zero, respectively.

**Figure 9.** Phantom, prior image, and the absolute value of the difference between phantom and prior image. Three circles are labelled by red rectangles.
Table 3. Geometrical scanning parameters of simulated CT system

| Parameter                                      | Value          |
|------------------------------------------------|----------------|
| The distance between source and object center | 981 mm         |
| The angle interval of two adjacent projection views | 0.703°         |
| The angle interval of two adjacent rays        | 0.0005°        |
| The diameter of field of view                  | 279.5 mm       |
| Detector numbers                               | 560            |
| Pixel size                                     | 0.5 × 0.5 mm²  |
| Image size                                     | 512 × 512      |

For the scanning angular ranges [0, 120°], [0, 100°] and [0, 80°], the reconstruction parameters of our algorithm are \( \rho = 1 \), \( \tau = 280 \), \( \beta = 0.8 \times 280 \), \( \lambda = 0.002 \times 280 \), and \( \gamma = 20 \times 280 \). The maximum iteration number is \( N_{ite} = 1000 \).

The reconstructed results for the different scanning ranges using FBP algorithm, PICCS algorithm and our algorithm are shown in Figure 10. The first column is the reconstructed results for the scanning ranges [0, 80°]. The subsequent columns are the reconstructed results for the scanning ranges [0, 100°], [0, 120°] and [0, 360°]. From top to bottom, Figure 10 shows the reconstructed results using FBP algorithm, PICCS algorithm and our algorithm in each rows.

![Figure 10](image-url)

**Figure 10.** The reconstructed results for the different scanning ranges using FBP algorithm, PICCS algorithm and our algorithm. Three circles are labelled by red rectangles. The display window is [0, 255].
From Figure 10, with the decrease of the scanning range, the quality of the reconstructed images becomes deteriorate. The limited-angle artifacts are better suppressed and the edges of reconstructed image are better preserved using our algorithm and PICCS algorithm for limited-angle reconstruction problem. And the three circles, which are not included in prior image, can also be better reconstructed for the scanning ranges $[0, 120^\circ]$.

The reconstruction quality is characterized quantitatively in Table 4. From Table 4, we can obtain that our algorithm achieves the minimum RMSE and maximum PSNR (MMSIM) compared to FBP algorithm for full-scan range and PICCS algorithm, however, the quality of the three circles reconstructed using FBP algorithm for full-scan range is better than that using our algorithm and PICCS algorithm for limited-angle range by visual inspection.

**Table 4. Quantitatively characterize the reconstruction quality.**

| Scanning ranges | Algorithm | RMSE  | PSNR  | MSSIM |
|-----------------|-----------|-------|-------|-------|
| $0 \sim 360^\circ$ | FBP       | 4.978 | 34.19 | 0.9961 |
| $0 \sim 120^\circ$ | our algorithm | 3.607 | 36.99 | 0.9979 |
|                  | PICCS     | 4.208 | 35.65 | 0.9972 |
| $0 \sim 100^\circ$ | our algorithm | 3.901 | 36.31 | 0.9976 |
|                  | PICCS     | 4.190 | 35.69 | 0.9972 |
| $0 \sim 80^\circ$ | our algorithm | 3.693 | 36.78 | 0.9979 |
|                  | PICCS     | 4.177 | 35.71 | 0.9972 |

This experiment indicates that our algorithm and PICCS algorithm will not miss some important information that is not included in the prior image, and the reconstructed information is not included in the prior image (the circles) which becomes vestigial for the scanning angular range $[0, 100^\circ]$ and $[0, 80^\circ]$ using PICCS algorithm and our algorithm since the projection data are incomplete.

In additional, a simulated phantom, which includes three cracks with different directions but the prior image dose not include them (see Figure 11), is used to test our algorithm. The scanning parameters of the simulated limited-angle CT are given in Table 1. The parameters of simulated phantom are listed in Table 5, $I$ denotes the ellipse intensity value, $h$ denotes the horizontal semi-axis of the ellipse’s length, $v$ denotes the vertical semi-axis of the ellipse’s length, $x_0$ denotes the x-axis center of the ellipse, $y_0$ denotes the y-axis center of the ellipse, and $r$ denotes the angle between the horizontal semi-axis and the x-axis. The standard deviation and the average value of Gaussian noise added to the projection data are $0.1\% \| g \|_{\infty}$ and zero, respectively. In the experiments, we consider the scanning ranges $[0, 100^\circ]$ and $[0, 120^\circ]$, and the number of projection views are 100 and 120, respectively.

For the scanning angular ranges $[0, 100^\circ]$ and $[0, 120^\circ]$, the reconstruction parameters of our algorithm are $\rho = 1$, $\tau = 128$, $\beta = 1 \ast 128$, $\lambda = 0.04 \ast 128$, and $\gamma = 20 \ast 128$. The maximum iteration number is $N_{ite} = 1000$.

The reconstructed results for the different scanning ranges using PICCS algorithm and our algorithm are shown in Figure 12. The first row is the reconstructed results for the scanning ranges $[0, 120^\circ]$. The subsequent row is the reconstructed results for the scanning ranges $[0, 100^\circ]$. From left to right, Figure 12 shows the reconstructed results using PICCS algorithm and our algorithm in each column.

Figure 12 indicates that our algorithm and PICCS algorithm not only take advantage of the prior image but also not miss some important information that is
Figure 11. Phantom and prior image. The first column is the phantom, the subsequent columns are the prior image and the absolute value of the difference between phantom and prior image.

Table 5. The parameters of simulated phantom.

| I  | h  | v  | x₀  | y₀  | r  |
|----|----|----|-----|-----|----|
| 1  | 0.74 | 0.74 | 0   | 0   | 0  |
| -1 | 0.5  | 0.5  | 0   | 0   | 0  |
| -1 | 0.1  | 0.1  | 0.43| 0.43| 0  |
| -1 | 0.1  | 0.1  | -0.43| -0.43| 0  |
| -1 | 0.1  | 0.1  | -0.43| 0.43| 0  |
| -1 | 0.1  | 0.1  | 0.43 | -0.43| 0  |
| -1 | 0.12 | 0.006| 0.25| 0.55| -18 |
| -1 | 0.08 | 0.006| 0.25| -0.55| -240 |
| -1 | 0.08 | 0.006| -0.55| 0.3 | 20 |

Figure 12. The reconstructed results using PICCS algorithm and our algorithm. The display window for the first row is [0, 1] and for the second row is [0.8, 0.9].

not included in the prior image. And the cracks becomes vestigial for the scanning
angular range $[0, 100^\circ]$ using PICCS algorithm and our algorithm, which are not included in the prior image, since the projection data are incomplete.

The reconstruction quality is characterized quantitatively in Table 6. From Table 6, we can obtain that our algorithm achieves the minimum RMSE and maximum MMSIM compared to PICCS algorithm, however, the PSNR using our algorithm is smaller than that using PICCS algorithm.

Table 6. Quantitatively characterize the reconstruction quality.

| Scanning ranges | Algorithm  | RMSE  | PSNR  | MMSIM |
|-----------------|------------|-------|-------|-------|
| $0 \sim 120^\circ$ | our algorithm | 0.040 | 27.97 | 0.9949 |
|                 | PICCS      | 0.057 | 28.26 | 0.9897 |
| $0 \sim 100^\circ$ | our algorithm | 0.0378 | 28.45 | 0.9954 |
|                 | PICCS      | 0.0546 | 28.65 | 0.9906 |

Lastly, compared with the $\frac{\gamma}{2} \| (Wf)_G - (Wf_0)_G \|_2^2$ term, we also consider the $\frac{\gamma}{2} \| (Wf)_G - (Wf_0)_G \|_1$ term in the model (8), then, the soft thresholding method is utilized to solve the $\ell_1$ minimization sub-problem, which is named as “$\ell_1 - \ell_0$ algorithm” according to the regularization terms. Likewise, “our algorithm” is named as “$\ell_2 - \ell_0$ algorithm”. In the following experiments, we will test the performance of $\ell_1 - \ell_0$ algorithm for limited-angle CT reconstruction, and the reconstructed results are compared with $\ell_2 - \ell_0$ algorithm.

A modified chest phantom (see Figure 9) is used to test the performance of $\ell_1 - \ell_0$ algorithm, the scanning parameters of the simulated limited-angle CT are given in Table 3. The scanning ranges $[0, 100^\circ]$ and $[0, 120^\circ]$ are investigated, and the number of projection views are 143 and 171, respectively. The standard deviation and the average value of Gaussian noise added to the projection data are $0.1\% \| g \|_{\infty}$ and zero, respectively.

Figure 13. The reconstructed results for the different scanning ranges using $\ell_1 - \ell_0$ algorithm and $\ell_2 - \ell_0$ algorithm. Three circles are labelled by red rectangles. The display window is [0, 255].
The reconstructed results for the different scanning ranges using $\ell_1 - \ell_0$ algorithm and $\ell_2 - \ell_0$ algorithm are shown in Figure 13. The first column is the reconstructed results for the scanning ranges $[0, 100^\circ]$. The subsequent column is the reconstructed results for the scanning ranges $[0, 120^\circ]$. From top to bottom, Figure 13 shows the reconstructed results using $\ell_1 - \ell_0$ algorithm and $\ell_2 - \ell_0$ algorithm in each rows. From Figure 13, the limited-angle artifacts are better suppressed and the edge of reconstructed image is better preserved using $\ell_1 - \ell_0$ algorithm and $\ell_2 - \ell_0$ algorithm for limited-angle reconstruction problem. And the reconstruction quality using $\ell_2 - \ell_0$ algorithm is as well as that using $\ell_1 - \ell_0$ algorithm.

The reconstruction quality is characterized quantitatively in Table 7. From Table 7, the RMSE, PSNR, and MSSIM value obtained using $\ell_2 - \ell_0$ algorithm are close to that using $\ell_1 - \ell_0$ algorithm. The reason of this result is because the information of reference image, which is close to the desire solution, is introduced into the reconstruction model.

**Table 7.** Quantitatively characterize the reconstruction quality.

| Scanning ranges | Algorithm | RMSE  | PSNR  | MSSIM |
|-----------------|-----------|-------|-------|-------|
| $0 \sim 120^\circ$ | $\ell_1 - \ell_0$ | 3.725 | 36.71 | 0.9978 |
| $0 \sim 120^\circ$ | $\ell_2 - \ell_0$ | 3.607 | 36.99 | 0.9979 |
| $0 \sim 100^\circ$ | $\ell_1 - \ell_0$ | 3.856 | 36.41 | 0.9977 |
| $0 \sim 100^\circ$ | $\ell_2 - \ell_0$ | 3.901 | 36.31 | 0.9976 |

A simulated phantom (see Figure 11) is used to test the performance of $\ell_1 - \ell_0$ algorithm, the scanning parameters are given in Table 1. The scanning ranges $[0, 100^\circ]$ and $[0, 120^\circ]$ are investigated, and the number of projection views are 100 and 120, respectively. The standard deviation and the average value of Gaussian noise added to the projection data are $0.1\|g\|_{\infty}$ and zero, respectively.

The reconstructed results for the different scanning ranges using $\ell_1 - \ell_0$ algorithm and the $\ell_2 - \ell_0$ algorithm are shown in Figure 14. The first row is the reconstructed results for the scanning ranges $[0, 120^\circ]$. The subsequent row is the reconstructed results for the scanning ranges $[0, 100^\circ]$. From left to right, Figure 14 shows the reconstructed results using $\ell_1 - \ell_0$ algorithm and $\ell_2 - \ell_0$ algorithm in each columns.

The reconstruction quality is characterized quantitatively in Table 8. From Table 8, the RMSE, PSNR, and MSSIM value obtained using $\ell_2 - \ell_0$ algorithm are close to that using $\ell_1 - \ell_0$ algorithm. The reason of this result is because the information of reference image, which is close to the desire solution, is introduced into the reconstruction model.

**Table 8.** Quantitatively characterize the reconstruction quality.

| Scanning ranges | Algorithm | RMSE  | PSNR  | MSSIM |
|-----------------|-----------|-------|-------|-------|
| $0 \sim 120^\circ$ | $\ell_1 - \ell_0$ | 0.0404 | 27.86 | 0.9948 |
| $0 \sim 120^\circ$ | $\ell_2 - \ell_0$ | 0.0400 | 27.97 | 0.9949 |
| $0 \sim 100^\circ$ | $\ell_1 - \ell_0$ | 0.0383 | 28.34 | 0.9954 |
| $0 \sim 100^\circ$ | $\ell_2 - \ell_0$ | 0.0378 | 28.45 | 0.9954 |
Figure 14. The reconstructed results of simulation phantom for the different scanning ranges using $\ell_1 - \ell_0$ algorithm and $\ell_2 - \ell_0$ algorithm. The display window for the first row is $[0, 1]$ and for the second row is $[0.8, 0.9]$.

5.4. Reconstruction from practical Data. To verify the effectiveness and stability of our algorithm for practical applications, we also test our algorithm using practical projection data of a gear, which are extracted from a normal full-scan, in the angle range $[0, 80^0]$, the number of projection views is 114. The size of the reconstructed image is $512 \times 512$. The reconstruction parameters of our algorithm are $\rho = 1$, $\tau = 1950$, $\beta = 0.8 \times 1950$, $\lambda = 0.0000000001 \times 1950$, and $\gamma = 1 \times 1950$. The maximum iteration number is $N_{itc} = 1400$.

Figure 15 shows the reconstructed images from practical data using different algorithms, region of interest (ROI) is labelled by red rectangle. The image on the left column is the prior image. The subsequent columns are the reconstructed results using FBP algorithm for scanning ranges $[0, 80^0]$ and $[0, 360^0]$, and the reconstructed result using our algorithm. As seen from Figure 15, the reconstructed images using our algorithm, the slope artifact can be better suppressed and the edges of the object can be better preserved for limited-angle problem. We can obtain a better quality of reconstruction image using our algorithm for limited-angle problem as that using FBP algorithm for full-scan range. As seen from Figure 16, the reconstructed result using our algorithm for limited-angle scanning range is better than that using FBP algorithm for the full-scan range (i.e. $[0, 360^0]$) in terms of suppressing the noise.

We regard the prior image as reference image, the reconstruction quality is characterized quantitatively in Table 9. From Table 9, we can obtain that our algorithm achieves the minimum RMSE and maximum PSNR (MMSIM) compared to the FBP algorithm for full-scan range.

Table 9. Quantitatively characterize the reconstruction quality.

| Scanning ranges | Algorithm  | RMSE  | PSNR  | MSSIM  |
|-----------------|------------|-------|-------|--------|
| $0 \sim 360^0$  | FBP        | 6.559 | 31.79 | 0.9965 |
| $0 \sim 80^0$   | our algorithm | 4.543 | 34.98 | 0.9983 |
Figure 15. The reconstructed results of gear using FBP algorithm and our algorithm. ROI is labelled by red rectangle. The display window is $[0, 0.0096] \text{mm}^{-1}$.

Figure 16. The zoom-in view of ROI of Figure 15.

6. **Conclusions and discussion.** To suppress the limited-angle artifacts of limited-angle CT reconstruction, we propose a minimization model that is based on wavelet tight frame and prior image. We perform this minimization problem efficiently by iteratively minimizing. Moreover, we show that each bounded sequence, which is generated by iteratively minimizing separately, converges to a critical or stationary point. The experimental results indicate that our method can suppress the artifacts in reconstructed images, can recover the edge structure information more effectively and can obtain a better quality of reconstructed image for limited-angle problem as that using FBP algorithm for full-scan range. But above all, the introduced prior image will not miss the important information that is not included in the prior image, and the information, using PICCS algorithm and our algorithm, which becomes vestigial for the scanning angular range is seriously limited because the projection data are incomplete. What’s more, the chosen prior image is very important for our algorithm, the error solution will be obtained using our algorithm if the chosen prior image is inappropriate. For example, more boundary of a prior image does not match with the reconstructed object. Thus, an image, which is chosen as a prior image, should be close to the underline image except for few small features. A prior image, which is used to the follow-up detection or diagnosis, can be obtained from full-scan data using FBP algorithm or ASD-POCS algorithm before the equipment is installed or the first diagnosis. This research has investigated only in fan-beam limited-angle CT and the parameters and iteration number of our algorithm are selected by trial and error. In the future, we will investigate this model for other applications and consider how to optimize the parameters and iterations.
Acknowledgments. The authors wish to thank the reviewer and the editor for helpful suggestions and comments for improving this paper. This work is supported by the National Natural Science Foundation of China (61271313 and 61771003) and National Instrumentation Program of China (2013YQ030629).

REFERENCES

[1] G. Bachar, J. H. Siewerdsen, M. J. Daly, D. A. Jaffray and J. C. Irish, Image quality and localization accuracy in C-arm tomosynthesis-guided head and neck surgery, Med. Phys., 34 (2007), 4664–4677.

[2] C. Bao, H. Ji and Z. Shen, Convergence analysis for iterative data-driven tight frame construction scheme, Appl. Comput. Harmon. Anal., 38 (2015), 510–523.

[3] J. G. Bian, J. H. Siewerdsen, X. Han, E. Y. Sidky, J. L. Prince, C. A. Pelizzari and X. C. Pan, Evaluation of sparse-view reconstruction from flat-panel-detector cone-beam CT, Physics in Medicine and Biology, 55 (2010), 6575–6599.

[4] J. Bolte, S. Sabach and M. Teboulle, Proximal alternating linearized minimization for non-convex and nonsmooth problems, Mathematical Programming, 146 (2014), 459–494.

[5] T. M. Buzug, Computed Tomography: From Photon Statistics to Modern Cone-beam CT, 1st edition, Springer-Verlag, Berlin Heidelberg, 2008.

[6] J. F. Cai, H. Ji, Z. W. Shen and G. B. Ye, Data-driven tight frame construction and image denoising, Appl. Comput. Harmon. Anal., 37 (2014), 89–105.

[7] J. F. Cai, S. Osher and Z. Shen, Split Bregman methods and frame based image restoration, Multiscale Modeling and Simulation, 8 (2009), 337–369.

[8] G. H. Chen, J. Tang and S. Leng, Prior image constrained compressed sensing (PICCS): A method to accurately reconstruct dynamic CT images from highly undersampled projection data sets, Medical Physics, 35 (2008), 660–663.

[9] G. H. Chen, et al., Time-resolved interventional cardiac C-arm cone-beam CT: An application of the PICCS algorithm, IEEE Transactions on Medical Imaging, 31 (2012), 907–923.

[10] Z. Q. Chen, X. Jin, L. Li and G. Wang, A limited-angle CT reconstruction method based on anisotropic TV minimization, Phys. Med. Biol., 58 (2013), 2119–2141.

[11] B. Dong and Z. Shen, MRA-based wavelet frames and applications: Image segmentation and surface reconstruction, In SPIE Defense, Security, and Sensing, International Society for Optics and Photonics, (2012), 840102.

[12] M. M. Eger and P. E. Danielsson, Scanning of logs with linear cone-beam tomography, Computers and Electronics in Agriculture, 41 (2003), 45–62.

[13] J. M. Fadili and G. Peyré, Total variation projection with first order schemes, IEEE Transactions on Image Processing, 20 (2011), 657–669.

[14] J. Frikel, Sparse regularization in limited angle tomography, Appl. Comput. Harmon. Anal., 34 (2013), 117–141.

[15] H. Gao, J. F. Cai, Z. W. Shen and H. Zhao, Robust principal component analysis-based four-dimensional computed tomography, Phys. Med. Biol., 56 (2011), 3781–3798.

[16] H. Gao, R. Li, Y. Lin and L. Xing, 4D cone beam CT via spatiotemporal tensor framelet, Medical Physics, 39 (2012), 6943–6946.

[17] H. Gao, L. Zhang, Z. Chen, Y. Xing, J. Cheng and Z. Qi, Direct filtered-backprojection-type reconstruction from a straight-line trajectory, Optical Engineering, 46 (2007), 057003.

[18] B. Han, G. Kutyniok and Z. Shen, Adaptive multiresolution analysis structures and shearlet systems, SIAM. J. Numer. Anal., 49 (2011), 1921–1946.

[19] P. C. Hansen, E. Y. Sidky and X. C. Pan, Accelerated gradient methods for total-variation-based CT image reconstruction, arXiv:1105.4002.

[20] G. T. Herman and R. Davidi, Image reconstruction from a small number of projections, Inverse Problems, 24 (2008), 045011.

[21] X. Jia, B. Dong, Y. Lou and S. B. jiang, GPU-based iterative cone-beam CT reconstruction using tight frame regularization, Phys. Med. Biol., 56 (2010), 3787–3806.
[22] V. Kolehmainen, S. Siltanen, S. Jarvenpaa, J. P. Kaipio, P. Koistinen, M. Lassas, J. Pirttila and E. Somersalo, Statistical inversion for medical x-ray tomography with few radiographs: II. Application to dental radiology, Phys. Med. Biol., 48 (2003), 1465–1490.

[23] S. J. LaRoque, E. Y. Sidky and X. C. Pan, Accurate image reconstruction from few-view and limited-angle data in diffraction tomography, JOSA A, 25 (2008), 1772–1782.

[24] X. Lu, Y. Sun and Y. Yuan, Image reconstruction by an alternating minimisation, Neurocomputing, 74 (2011), 661–670.

[25] X. Lu, Y. Sun and Y. Yuan, Optimization for limited angle tomography in medical image processing, Phys. Med. Biol., 44 (2011), 2427–2435.

[26] M. G. Lubner, et al., Prospective evaluation of prior image constrained compressed sensing (PICCS) algorithm in abdominal CT: A comparison of reduced dose with standard dose imaging, Abdominal Imaging, 40 (2015), 207–221.

[27] F. Natterer, The Mathematics of Computed Tomography, 1nd edition, B.G. Teubner, Stuttgart, 1986.

[28] B. Nett, J. Tang, S. Leng and G. H. Chen, Tomosynthesis via total variation minimization reconstruction and prior image constrained compressed sensing (PICCS) on a C-arm system, Medical Imaging. International Society for Optics and Photonics, 6913 (2008), 1–10.

[29] E. T. Quinto, Exterior and limited-angle tomography in non-destructive evaluation, Inverse Problems, 14 (1998), 1339–1353.

[30] A. Ron and Z. Shen, Affine systems in $L^2(R^d)$: The analysis of the analysis operator, Journal of Functional Analysis, 148 (1997), 408–447.

[31] L. Shen, Y. Xu and X. Zeng, Wavelet inpainting with the $\ell_0$ sparse regularization, Appl. Comput. Harmon. Anal., 41 (2016), 26–53.

[32] Z. Shen, Wavelet frames and image restorations, In Proceedings of the International congress of Mathematicians, 4 (2010), 2834–2863.

[33] E. Y. Sidky, C. M. Kao and X. C. Pan, Accurate image reconstruction from few-views and limited-angle data in divergent-beam CT, arXiv:0904.4495.

[34] E. Y. Sidky and X. C. Pan, Accurate image reconstruction in circular cone-beam computed tomography by constrained, total-variation minimization: A preliminary investigation, IEEE Nuclear Science Symposium Conference Record, 5 (2006), 2904–2907.

[35] E. Y. Sidky and X. C. Pan, Image reconstruction in circular cone-beam computed tomography by total variation minimization: A preliminary investigation, Physics in Medicine and Biology, 53 (2008), 3572277–3572284.

[36] W. P. Segars, D. S. Lalush and B. M. Tsui, A realistic spline-based dynamic heart phantom, IEEE Trans. Nucl. Sci., 46 (1999), 503–506.

[37] M. Storath, A. Weinmann, J. Frikel and M. Unser, Joint image reconstruction and segmentation using the Potts model, Inverse Problems, 31 (2015), 025003.

[38] J. Tang, J. Hsieh and G. H. Chen, Temporal resolution improvement in cardiac CT using PICCS (TRI-PICCS): Performance studies, Medical Physics, 37 (2010), 4377–4388.

[39] A. Tingberg, X-ray tomosynthesis: A review of its use for breast and chest imaging, Radiation Protection Dosimetry, 139 (2010), 100–107.

[40] H. K. Tuy, An inversion formula for cone-beam reconstruction, SIAM J. Appl. Math., 43 (1983), 546–552.

[41] Z. Wang, A. Bovik, H. Sheikh and E. Simoncelli, Image quality assessment: From error visibility to structural similarity, IEEE Trans Image Process, 13 (2004), 600–612.

[42] Z. Wang, Z. Huang, Z. Chen, L. Zhang, X. Jiang, K. Kang, H. Yin, Z. Wang and M. Stampanoni, Low-dose multiple-information retrieval algorithm for x-ray grating-based imaging, Nuclear Instruments and Methods in Physics Research Section A: Accelerators, Spectrometers, Detectors and Associated Equipment, 635 (2011), 103–107.

[43] F. Yang, Y. Shen and Z. S. Liu, The proximal alternating iterative hard thresholding method for $\ell_0$ minimization, with complexity $O(\frac{1}{\sqrt{\epsilon}})$, Journal of Computational and Applied Mathematics, 311 (2017), 115–129.

[44] W. Yu and L. Zeng, $\ell_0$ gradient minimization based image reconstruction for limited-angle computed tomography, PLoS ONE, 10 (2015), e0130793.
[45] L. Zeng, J. Q. Guo and B. D. Liu, Limited-angle cone-beam computed tomography image reconstruction by total variation minimization and piecewise-constant modification, *Journal of Inverse and Ill-Posed Problems*, 21 (2013), 735–754.

[46] Y. Zhang, B. Dong and Z. S. Lu, ℓ₀ Minimization for wavelet frame based image restoration, *Mathematics of Computation*, 82 (2013), 995–1015.

[47] B. Zhao, H. Gao, H. Ding and S. Molloi, Tight-frame based iterative image reconstruction for spectral breast CT, *Medical Physics*, 40 (2013), 031905.

[48] W. Zhou, J. F. Cai and H. Gao, Adaptive tight frame based medical image reconstruction: a proof-of-concept study for computed tomography, *Inverse problems*, 29 (2013), 1–18.

[49] chest phantom website, [http://lgdv.cs.fau.de/External/vollib/](http://lgdv.cs.fau.de/External/vollib/).

Received December 2015; revised August 2017.

E-mail address: 20110602082@cqu.edu.cn
E-mail address: drlizeng@cqu.edu.cn
E-mail address: 20092194@cqu.edu.cn
E-mail address: evralingli@yeah.net