Real-analytic geodesics in the Mabuchi space of Kähler metrics and quantization

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January 25, 2024

Abstract

We prove the convergence of quantized Bergman geodesics to the Mabuchi geodesics for the initial value problem, in the case of real-analytic initial data and in short time. This partially solves a conjecture of Y. Rubinstein and the last author. We also argue against the existence of a solution to the boundary value problem, generically in real-analytic regularity.

To this end, we introduce non-self-adjoint Fourier Integral Operators, and prove that they are satisfactory approximations to the Bergman geodesics, that is, solutions of a semiclassical Schrödinger equation with skew-adjoint Hamiltonian.

1 Setting and main results

1.1 The Mabuchi metric

Let $(M, J, \omega_0)$ be a Kähler manifold of complex dimension $d$. We let $\mathcal{H}$ denote the space of $C^{1,1}$ changes of Kähler potentials on $(M, J, \omega_0)$, that is, the following open subset of $C^{1,1}(M, \mathbb{R})$:

$$\mathcal{H} = \{ \phi \in C^{1,1}(M, \mathbb{R}), \omega_\phi := \omega_0 + i \partial \bar{\partial} \phi > 0 \}.$$ 

We endow the infinite-dimensional space $\mathcal{H}$ with a Riemannian metric named after Mabuchi [24, 31, 16]: at $(\phi, v) \in T\mathcal{H} \simeq \mathcal{H} \times C^{1,1}(M, \mathbb{R})$, the squared norm of $v$ is

$$\int_M |v|^2 \omega_\phi^d. \quad (1)$$

The geodesic equation associated to the Mabuchi metric is

$$\ddot{\phi}(t) = |\partial \dot{\phi}(t)|_t^2, \quad (2)$$

where $|\cdot|_t$ is the norm on $TM$ induced by the Kähler metric $(M, J, \omega_{\phi(t)})$ and $\partial$ is the holomorphic gradient.
In this way, projector $\Pi_k$ is Hermitian line bundle ($\mathcal{L}, h$). Suppose that ($\varphi$) convergence for small times if $\omega_0$ of this article (Theorems A and B) are a partial proof of this claim, in the real-analytic case, by showing for (2) as long as the latter exists; this conjecture also appears, in a different form, in [2]. The main results of its relationship with actual solutions and motivates the study of Bergman metrics, and $\varphi: \mathcal{L} \rightarrow \mathcal{M}$ as geodesics in the approximating spaces $\mathcal{B}_k$ are symmetric spaces of the form $SL(d_k, \mathbb{C})/SU(d_k)$, where the dimension $d_k$ tends to infinity with $k$, and they have their own geodesics, known as Bergman rays. Two natural maps $\text{Hilb}_k : \mathcal{H} \rightarrow \mathcal{B}_k$ and $\text{FS}_k : \mathcal{B}_k \rightarrow \mathcal{H}$ allow to compare Mabuchi geodesics and Bergman rays with each other.

Since $SL(d_k, \mathbb{C})/SU(d_k)$ is finite-dimensional, Bergman rays can always be extended in infinite time. Every Bergman ray corresponds to a subsolution of (2), that is, for every $k \in \mathbb{N}$, if $A : \mathbb{R} \rightarrow \mathcal{B}_k$ is a geodesic and $\phi_k(t) = \text{FS}_k(A(t))$, then

$$
\dot{\phi}_k(t) \geq |\partial \phi_k(t)|^2.
$$

This result provides a natural way to construct a subsolution of the initial value problem as the limsup of the Bergman rays as $k \rightarrow +\infty$. This subsolution is always defined in infinite time, which raises the question of its relationship with actual solutions and motivates the study of Bergman metrics.

It was conjectured in [27] that this subsolution matches the actual solution of the initial value problem for (2) as long as the latter exists; this conjecture also appears, in a different form, in [2]. The main results of this article (Theorems A and B) are a partial proof of this claim, in the real-analytic case, by showing convergence for small times if $\omega_0$ and $\phi(0)$ are real-analytic and describing the asymptotic structure of these geodesics as $k \rightarrow +\infty$.

### 1.2 Holomorphic sections of line bundles

Suppose that $(\mathcal{M}, J, \omega_0)$ is polarised, that is, $\omega_0 \in H^2(\mathcal{M}, 2\pi \mathbb{Z})$. There exists a (not necessarily unique) Hermitian line bundle $(\mathcal{L}, h_0) \rightarrow \mathcal{M}$ with curvature $-i\omega_0$, called a prequantum line bundle.

For every $k \in \mathbb{N}$, the space of holomorphic sections $H^0(\mathcal{M}, \mathcal{L}^\otimes k)$ inherits a Hilbert space structure from the Hermitian metric $h_0$ on $\mathcal{L}$:

$$
\|u\|_0^2 := \int_{\mathcal{M}} \|u(x)\|_{h_0}^2 \omega_0^{|dx|}.
$$

In this way, $H^0(\mathcal{M}, \mathcal{L}^\otimes k)$ sits as a finite-dimensional subspace of $L^2(\mathcal{M}, \mathcal{L}^\otimes k)$. The associated orthogonal projector $\Pi_k : L^2(\mathcal{M}, \mathcal{L}^\otimes k) \rightarrow H^0(\mathcal{M}, \mathcal{L}^\otimes k)$ is called the Bergman projector.

Let $\phi \in \mathcal{H}$. The symplectic structure

$$
\omega_\phi = \omega_0 + i\partial \bar{\partial} \phi
$$

belongs to the same cohomology class as $\omega_0$; if we perform the same construction with $\omega_\phi$ instead of $\omega_0$, we find the same topological bundle $\mathcal{L}$ with a different Hermitian structure:

$$
\| \cdot \|_{h_0}^2 := \| \cdot \|_{h_0}^2 e^{-\phi}.
$$

In this way, one can naturally quantize an element of $\mathcal{H}$ into a sequence of Hilbert space structures on the spaces $H^0(\mathcal{M}, \mathcal{L}^\otimes k)$, where the squared norm of an element $u$ is now

$$
\|u\|_\phi^2 := \int_{\mathcal{M}} \|u(x)\|_{h_0^\otimes k}^2 \omega_\phi^{|dx|}.
$$

(5)
The new Kähler structure $\phi$ then allows to define a new Bergman projector $\Pi_{k}^{\phi}$.

We let $\mathcal{B}_k$ denote the set of Hilbert structures (scalar products) on $H^0(M,L^{\otimes k})$, and we call $\text{Hilb}_k$ the map $\phi \mapsto \|\cdot\|_{\phi}^2$.

Conversely, elements of $\mathcal{B}_k$ allow to define Kähler structures on $M$ via a map $\text{FS}_k$ defined as follows: let $H \in \mathcal{B}_k$ and let $(s_j)$ be an orthonormal basis of $H$, then we define

$$\text{FS}_k(H) : x \mapsto \frac{1}{k} \log \sum_j \|s_j(x)\|_{h_0}^2 - d \frac{\log(k)}{k}.$$

$\text{FS}_k$ is asymptotically a left inverse to $\text{Hilb}_k$; see Proposition 2.13 for details.[1]

Given a “reference” element $H_0 \in \mathcal{B}_k$, the set $\mathcal{B}_k$ can be identified with the space $S^{++}(H_0)$ of positive definite $H_0$-self-adjoint operators via the formula

$$\langle u,v \rangle_H = \langle u, Av \rangle_{H_0}.$$ (6)

Indeed, any $A \in S^{++}(H_0)$ defines a new Hilbert structure $H$ in this way, and from $H$ one uniquely recovers $A$ – for instance, choosing an orthonormal basis $(e_j)$ for $H_0$, the matrix coefficients of $A$ are $((e_j,e_k)_H)_{j,k}$. This identification allows to understand the tangent space $T_{H_0}\mathcal{B}_k$ as the space of $H_0$-self-adjoint operators on $H^0(M,L^{\otimes k})$. A natural Riemannian metric on $\mathcal{B}_k$ is then given by prescribing the norm of an element $(H,A) \in T\mathcal{B}_k$ as the Hilbert–Schmidt norm of $A$ as a $H$-self-adjoint operator:

$$\|A\|^2_H = \text{tr}(A^2).$$ (7)

Geodesics, for this metric, are one-parameter families $\gamma(t)$ of scalar products, such that

$$\langle \cdot , \cdot \rangle_{\gamma(t)} = \langle \cdot , e^{tA}\cdot \rangle_{\gamma(0)},$$

where $A$ is a $\gamma(0)$-self-adjoint operator. The curvature element between normalised elements $A_1,A_2$ of $T\mathcal{B}_k$ over the same base point is

$$K(A_1,A_2) = \frac{1}{4} \text{tr}([A_1,A_2]^2).$$ (8)

It is already somewhat pleasant to compare this formula with (3).

Given $(\phi, v) \in T\mathcal{H}$, one can use (5) to compute $d\text{Hilb}_k(v)$: indeed, as $\epsilon \in \mathbb{R}$ tends to zero, given $u_1,u_2 \in H^0(M,L^{\otimes k})$,

$$\langle u_1, u_2 \rangle_{\text{Hilb}_k(\phi+\epsilon v)} = \int_M \langle u_1(x), u_2(x) \rangle_{h_0^{\otimes k}} e^{-\epsilon kv} (\omega_\phi + i\epsilon d\bar{\partial} v)^{\wedge d}$$

$$= \frac{1}{k} \int_M \langle u_1(x), u_2(x) \rangle_{h_0^{\otimes k}} (\omega_\phi^{\wedge d} - \epsilon kv \omega_\phi^{\wedge d} + \epsilon i d\bar{\partial} v \wedge \omega_\phi^{d-1} + O_k(\epsilon^2))$$

$$= \langle u_1, u_2 \rangle_{\text{Hilb}_k(\phi)} + \epsilon \int_M \langle u_1(x), u_2(x) \rangle_{h_0^{\otimes k}} (-kv + \Delta_\phi v) \omega_\phi^{d} + O_k(\epsilon^2)$$

$$= \langle u_1, u_2 \rangle_{\text{Hilb}_k(\phi)} + \epsilon \langle u_1, \Pi_{k}^{\phi}((-kv + \Delta_\phi v)u_2) \rangle_{\text{Hilb}_k(\phi)} + O_k(\epsilon^2).$$

Introducing Berezin–Toeplitz operators

$$L^\infty(M,\mathbb{R}) \ni v \mapsto T_k^{\phi}(v) := \Pi_{k}^{\phi} v \Pi_{k}^{\phi}$$ (9)

[1] The usual definition of $\text{FS}_k$ does not contain the asymptotically vanishing $-d \log(k)/k$ term. We choose this definition to compensate for the universal subleading term in $\text{FS}_k(\text{Hilb}_k(\phi))$ in the usual definition.
which are $\text{Hilb}_k(\phi)$-self-adjoint operators, we obtain
\[ d\text{Hilb}_k(v) = T^\phi_k \left( -kv + \Delta_\phi v \right). \tag{10} \]

The map $\text{Hilb}_k$ between the infinite-dimensional space $\mathcal{H}$ and the finite-dimensional space $\mathcal{B}_k$ cannot be injective. However, it approximately preserves, at the infinitesimal level, the Riemannian data (up to a rescaling) thanks to the classical-quantum correspondence for Berezin–Toeplitz operators.

**Proposition 1.1** ([4,6]). Let $\phi \in \mathcal{H}$ be such that $\omega_\phi \in C^\infty$ and let $v_1, v_2 \in C^\infty(M, \mathbb{R})$. Then
\[
\text{tr}(T^\phi_k(v_1)^2) = \frac{k^d}{\pi^d} \int |v_1|^2 \omega^{\wedge d}_\phi + O(k^{d-1}).
\]

Thus, up to a scale factor $C_d k^{\frac{d}{2}+1}$, the geometry of $\mathcal{B}_k$ is presumed to reflect that of $\mathcal{H}$: indeed, after using equation (10), the norms (11) and the curvature elements (3) match up to a relative error $O(k^{-1})$. Using this fact, it was proved that the image by $\text{Hilb}_k$ of geodesics almost solve the geodesic equation.

**Proposition 1.2.** ([12], Proposition 3.5) Let $t \mapsto \phi(t)$ denote a geodesic path of smooth Kähler structures in $\mathcal{H}$. For every $t$ in the domain of this path, let $c_k(t) = \text{Hilb}_k(\phi(t))$. Then, as $k \to +\infty$, the curve $c_k$ almost satisfies the geodesic equation on $\mathcal{B}_k$:
\[ \| \nabla c_k \| = o(k^{\frac{d}{2}+1}). \]

In this article, we study the initial value problem before and after application of $\text{Hilb}_k$, and prove that the distance between the projected geodesic and the actual geodesic is small. Unfortunately, because the spaces $\mathcal{H}$ and $\mathcal{B}_k$ are negatively curved with very large or unbounded curvature, one cannot apply a Grönwall-type lemma, in the spirit of [22], Corollary 4.6.1, to prove this claim using only Proposition 1.2. Nevertheless, under hypotheses of real-analyticity, we are able to prove closeness of the two geodesics for short times.

**Theorem A.** Suppose that $\omega_0$ is real-analytic and let $\phi_0 \in C^\infty(M, \mathbb{R})$. Let $\phi(t)$ denote the geodesic in $\mathcal{H}$ with initial value $(0, \phi_0)$, which is well-defined for short time. For all $k \in \mathbb{N}$, let
\[ c_k : t \mapsto \text{Hilb}_k(\phi(t)). \]

Let also $\gamma_k(t)$ be the geodesic on $\mathcal{B}_k$ with initial value $(\text{Hilb}_k(0), d\text{Hilb}_k(\phi_0)) = (c_k(0), \dot{c}_k(0))$. Then there exists $t_0 > 0$ and $C > 0$ such that, uniformly on $t \in [0, t_0]$, as $k \to +\infty$, one has
\[
\text{dist}_{\mathcal{B}_k}(c_k(t), \gamma_k(t)) \leq C k^{\frac{d}{2}} \quad \text{dist}_{\mathcal{H}}(\phi(t), FS_k(\gamma_k(t))) \leq C k^{-1}. \quad \tag{11, 12}
\]

Recalling from Proposition 1.1 that the natural scaling between the distances on $\mathcal{B}_k$ and $\mathcal{H}$ is $k^{-\frac{d}{2}-1}$, so that (11) and (12) express the same magnitude of distances between the considered objects.

**Theorem A** is a consequence of a more technical result about the integral kernels of $c_k(t)$ and $\gamma_k(t)$ which is interesting in its own right and which we describe now. Recall from (10) that both $c_k(t)$ and $\gamma_k(t)$ can be cast as positive symmetric operators on $H^0(M, L^{\otimes k})$ endowed with the scalar product $\text{Hilb}_k(\phi(0))$. We prove that $c_k(t)$ and $\gamma_k(t)$ have integral kernels that are relatively close to each other.

**Theorem B.** Suppose that $\omega_0$ is real-analytic and let $\phi_0 \in C^\infty(M, \mathbb{R})$. Let $\phi(t)$ denote the geodesic in $\mathcal{H}$ with initial value $(0, \phi_0)$, which is well-defined for short time and let $c_k(t) = \text{Hilb}_k(\phi(t))$. Let $\gamma_k(t)$ be the geodesic on $\mathcal{B}_k$ with initial value $(\text{Hilb}_k(0), d\text{Hilb}_k(\phi_0))$. Then there exists
• a time $t_0 > 0$
• an open neighbourhood $V$ of the diagonal of $M \times M$
• for each $t \in [-t_0, t_0]$, a non-vanishing holomorphic section $\Phi(t) \in L \otimes T$ over $V$, with analytic dependence on $t$ (see Definition 2.3 of the line bundle $L \otimes T$)
• two uniformly bounded sequences of real-analytic functions $(a_k(t))_{k \in \mathbb{N}}, (b_k(t))_{k \in \mathbb{N}}$ on $V$
• constants $C > 0, c > 0$

such that for every $t \in [-t_0, t_0]$, for every $u, v \in H^0(M, L^{\otimes k})$, one has

$$\left| \langle u, v \rangle_{\mathcal{A}_k(t)} - k^d \int_V \langle \Phi(t, x, y)^{\otimes k}, u(x) \otimes v(y) \rangle_{h_0^{\otimes k} \times h_0^{\otimes k}} \omega_0^\wedge (dx) \omega_0^\wedge (dy) \right| \leq Ce^{-ck} \|u\|_{\mathcal{A}_k(t)} \|v\|_{\mathcal{A}_k(t)}$$

(13)

$$\left| \langle u, v \rangle_{\mathcal{B}_k(t)} - k^d \int_V \langle \Phi(t, x, y)^{\otimes k}, u(x) \otimes v(y) \rangle_{h_0^{\otimes k} \times h_0^{\otimes k}} \omega_0^\wedge (dx) \omega_0^\wedge (dy) \right| \leq Ce^{-ck} \|u\|_{\mathcal{B}_k(t)} \|v\|_{\mathcal{B}_k(t)}.$$  

(14)

The proof of Theorem 13 and of the link with Theorem 11 necessitates a calculus (approximate composition and inversion) of families of operators on $H^0(M, L^{\otimes k})$ with integral kernels (in the sense given by (13) or (14)) of the form

$$(x, y) \mapsto k^d \Phi(x, y)^{\otimes k} a_k(x, y),$$

which we will call analytic Fourier Integral Operators. We develop this theory in Section 2.

Equation (13) boils down to the structure of the Szegő projector $\Pi_k^\phi$, which in the analytic case is now well-known to exponential precision [28, 14, 18, 8, 15]. Equation (14) is new: it concerns the integral kernel of an operator of the form $e^{it\mathcal{L}(v)}$, where $v = -\dot{\phi}(0) + \frac{1}{k} \Delta \phi(0)$. It is known that the imaginary time kernel $e^{it\mathcal{L}(v)}$ has the structure of a complex Fourier Integral operator [35, 9], at least up to $O(k^{-\infty})$; in the limit $k \to +\infty$ one recovers the Hamilton flow of $-\dot{\phi}(0)$, but usual $(C^\infty)$ techniques are limited to the analysis of unitary operators. A semiclassical analysis of the kernel of non-unitary propagators was only previously known in the case of quadratic symbols [20, 27].

Formula (14) is expected to generalise to propagators associated with more general non-self-adjoint Berezin–Toeplitz operators, with, hopefully, interesting consequences for the description of the dynamics and the spectrum of these operators.

1.3 The boundary value problem

The boundary value problem for (2), namely the problem of finding a geodesic on $\mathcal{H}$ with fixed endpoints, is seemingly better behaved than the initial value problem. In fact, this problem is formally equivalent to an elliptic nonlinear boundary value problem, the Homogeneous Complex Monge-Ampère (HCMA) equation (16). It was progressively proved [13, 3, 13] that any two points in $\mathcal{H}$ (i.e. $C^{1,1}$ Kähler metrics in the same cohomology class) are joined by a unique shortest length geodesic of $C^{1,1}$ metrics (in the HCMA sense). Weak convergence of the Bergman rays (with projected boundary values by Hilb_k) to the geodesic was proved in [25] and strong convergence in [12]. As a byproduct, the distances and angles on $\mathcal{H}$ are asymptotically preserved by the map Hilb_k. In these results, the main ingredient from asymptotic analysis is the study of the properties of the kernel of the Bergman projector $\Pi_k^\phi$ on and the diagonal as $k \to +\infty$ (Propositions 2.13 and 2.11).

As the regularity increases, the boundary value problem becomes harder to solve. Generically, two $C^k$ metrics can be joined by a $C^{\frac{k}{4}+1}$ geodesic for $k \geq 5$ [11], but there exist smooth metrics which cannot be
joined by a $C^2$ geodesic \[23\] and there exist arbitrarily close analytic metrics which cannot be joined by a smooth geodesic \[21\]. It is conjectured in \[11\] that smooth metrics close to each other are generically joined by a smooth metric. Because of the link with the space of Hamiltonian diffeomorphisms, we make a conjecture in the opposite direction for analytic metrics.

**Conjecture 1.** Let $E$ and $E'$ be two analytic function spaces on $M$ (i.e. Banach spaces of analytic functions containing all analytic functions with sufficiently small radius of injectivity).

Then there exists an open dense subset of $(E \cap \mathcal{H})^2$, in which no pair of elements is linked by a geodesic in $E'$.

This conjecture has for immediate consequence (by choosing a countable sequence of analytic function spaces $E'$ containing all analytic functions) that there exists a countable intersection of open dense sets in $(E \cap \mathcal{H})^2$ of points not linked to each other by an analytic geodesic. In this sense, the opposite of the conjecture in \[11\] would hold for real-analytic metrics.

Because of the apparent loss of a fraction $\frac{1}{4}$ of derivatives appearing in \[11\], we conjecture that generic (in some sense) analytic metrics are linked by a $\frac{4}{3}$-Gevrey geodesic.

### 1.4 Techniques and perspectives

We emphasize that there are many formal arguments indicating that Theorems \[A\] and \[B\] should be true. The essential problem in this article is that the formal arguments are based on approximate propagators, or parametrices. It is often straightforward to prove a somewhat formal convergence for these approximations, but the degree of precision of the parametrix, even in the smooth case, is not enough. More precisely, we wish to obtain a good description of the integral kernel of the Bergman geodesic

$$U(t, x, y) = e^{ikT_k(-v+k^{-1}\Delta v)}(x, y).$$

This is the analytic continuation in $t$ of the Schrödinger propagator

$$U(i\tau, x, y) = e^{ikT_k(-v+k^{-1}\Delta v)}(x, y),$$

which is well-understood as a Fourier Integral Operator \[35, 9\] if the initial data is smooth. However, the precision of this description is $O(k^{-\infty})$, whereas exponential precision $O(e^{-ck})$, for some $c > 0$, is needed for our purposes. One reason for this is the Duhamel formula, associated with the fact that for $t \in \mathbb{R}$ one has

$$\log \|U(t)\|_{H^0 \to H^0} \sim_k \max(t \sup(v), t \inf(v)).$$

An exponential level of precision is only available in special cases, including (for short times) in the real-analytic case, using the recently developed framework of Berezin–Toeplitz quantization in real-analytic regularity \[28, 14, 18, 8, 15\].

Our proof of Theorem \[B\] uses a representation of both $c_k(t)$ and $\gamma_k(t)$ as Fourier Integral Operators with complex, real-analytic phase. The crucial point is that, in this representation, both $c_k(t)$ and $\gamma_k(t)$ have the same canonical relation (the same $\Phi$). In particular, we interpret a real-analytic change of Kähler structure $\phi$ on $M$ as a biholomorphism $\mathcal{L}$ between neighborhoods of $M$ in its complexification; a path of Kähler structures $\phi(t)$ is a path of biholomorphisms $\mathcal{L}(t)$. The image of a geodesic for the Mabuchi metric is then a piece of one-parameter subgroup of biholomorphisms. This interpretation (which, formally, stems from Proposition \[11\]) was one of the motivations for the introduction of Berezin–Toeplitz quantization in the treatment of the Mabuchi problem.

Conjecture \[H\] is also linked to the interpretation of changes of Kähler structures as complex symplectomorphisms. Geodesics correspond to autonomous Hamiltonian flows, and to support this claim, we use the fact that, among real Hamiltonian diffeomorphisms, the autonomous ones are non-generic.
The link between the geometry of Mabuchi space and that of Hamiltonian diffeomorphisms (where geodesics are autonomous flows) uses a complexification argument. Thus, it is not surprising that, as the regularity of the data increases, the behaviours of the two problems become closer.

This link is also interesting from the perspective of optimal transport. Mabuchi geodesics, that is, solutions of the complex Monge-Ampère equation, are instances of time-dependent, $\omega$-preserving maps from the complexification of $M$ to itself; they form a totally real subspace of this space $\mathcal{H}^C$, and a complementary subspace is formed by $\omega$-preserving maps from $M$ to itself. This mimics the usual optimal transport situation where optimal transport maps, solving a real Monge-Ampère equation, are “transverse” to solutions of the incompressible Euler equation inside a larger space of maps. This parallel suggests several relevant questions: is there a “polar decomposition” where a general map is the composition of a change of Kähler metric and a real Hamiltonian diffeomorphism? Since the boundary value problem for the geodesic equation among real Hamiltonian differentiations is not well-posed, can one suitably generalise the problem in order to find a minimal length geodesic? Some of these questions were already addressed in [17]; in real-analytic regularity we hope to be able to develop this theory.

In Section 2 we recall the basic ingredients from (complexified) Kähler geometry and semiclassical analysis in real-analytic regularity which we will need. We use in particular the structure of the Bergman kernel in real-analytic regularity (Proposition 2.11). Section 3 develops basic tools for the treatment of non-unitary Fourier Integral Operators. One can compose and invert such operators, and change the reference Kähler metric with respect to which they are defined, as long as all involved objects are close to the “classical” case of the Bergman kernel. Then in Section 4 we use these tools to prove our main claims; we notably prove in Proposition 4.2 that quantum propagators of skew-adjoints operators are Fourier Integral Operators.

We conclude this introduction with a remark about positivity and canonical bundles: letting $K$ be the canonical bundle over $M$, another convention for Berezin–Toeplitz operators and the Hilb$_k$ and FS$_k$ maps, is to consider holomorphic sections of $L^\otimes k \otimes K$ rather than $L^\otimes k$. In terms of semiclassical analysis, the difference between the two cases is subprincipal, and although the particular identities (notably Proposition 2.15) that we use here should be modified, our main three claims should also hold in the case of a twist by the canonical bundle (or, in fact, any fixed line bundle). A notable difference, and the reason why we focus on the untwisted case, is that the property that Bergman geodesics are subsolutions of the Mabuchi equation (equation (4)) is specific to the untwisted case. In fact, there is a reverse inequality in the canonical twist case, where the image by Hilb$_k$ of a Mabuchi geodesic is a sub-$B^\text{twist}_k$ geodesic [1]. Our end goal is to study natural candidates for Mabuchi geodesics after explosion, and unfortunately the liminf of the twisted Bergman geodesics is not, in general, a super-solution of (4).

2 Semiclassical analysis of Berezin-Toeplitz operators

2.1 Polarisation, complexification, and complex symplectic geometry

Recall that $(M,J,\omega_0)$ is a real-analytic Kähler manifold: $J$ is a complex structure, $\omega_0$ is a symplectic form which is real-analytic in the $J$-holomorphic charts, and $\omega_0(\cdot,J\cdot)$ is a Riemannian metric. Recall also that there is a line bundle $L \rightarrow M$ and a Hermitian metric $h_0$ on $L$ such that $\text{curv}(h_0) = -i\omega_0$. Associated with $(L,h_0)$ is a Hermitian connection $\nabla$, whose curvature is also $-i\omega_0$.

Concretely, in holomorphic charts for $L$ (in which holomorphic sections of $L$ are simply holomorphic C-valued functions), given $v \in L$ one has $||v||_{h_0} = e^{-\frac{\psi_0}{2} |v(x)|}$ and given a section $s$ and $(x,\dot{x}) \in TM$, one has $\nabla s(\dot{x}) = \frac{d}{d\epsilon}(e^{-\frac{\psi_0}{2}} s) e^{\frac{\psi_0}{2}}$, where $\psi_0$ is a local Kähler potential: $\partial_\bar{\partial} \psi_0 = -i\omega_0$.

Starting with this data, we will construct several manifolds and bundles. These definitions serve two goals: first, to properly define the association between integral kernels and operators as used in Theorem 3 second, to give a geometric interpretation of the phase $\Phi$ appearing in Theorem 3.
\textbf{Definition 2.1.} Define $\overline{M}$ as the Kähler manifold $(M, -J, -\omega_0)$, with flipped complex and symplectic structure. Observe that $-\omega_0(\cdot, -J \cdot) = \omega_0(\cdot, J \cdot)$ so that the Riemannian structure on $\overline{M}$ is identical to that on $M$. The natural line bundle $\overline{L}$ over $\overline{M}$ is constructed as follows: given a $J$-holomorphic atlas of $M$ and (holomorphic) transition charts for $L$, the same atlas is $-J$-holomorphic for $\overline{M}$ and we set the transition charts of $\overline{L}$ to be the complex conjugate of that for $L$. In this way, $\overline{L}$ is indeed a holomorphic bundle over $\overline{M}$.

There is an antilinear correspondence between $H^0(M, L)$ and $H^0(\overline{M}, \overline{L})$: associate $s$ with $\overline{s}$ in each chart. Using this correspondence we can provide $\overline{L}$ with a Hermitian metric $h_{0,\overline{L}}$ and a Hermitian connection. One recovers then $\text{curv}(\overline{L}, h_{0,\overline{L}}) = i\omega_0$, so that $\overline{L}$ is a prequantum bundle over $\overline{M}$.

\textbf{Definition 2.2.} The line bundle $L \otimes \overline{L}$ over $M \times \overline{M}$ is the holomorphic line bundle whose fibre over a point $(x, y) \in M$ is $L_x \otimes \overline{L}_y$.

The product manifold $M \times \overline{M}$ is particularly interesting for several reasons. First of all, for the line bundle $L \otimes \overline{L}$ over $M \times \overline{M}$, the holomorphic sections will be integral kernels of operators acting on $H^0(M, L)$. Moreover, the diagonal in $M \times \overline{M}$ is a copy of $M$ which is a maximally totally real submanifold of $M \times \overline{M}$. Consequently, real-analytic data on $M$ can be extended in a unique way into holomorphic objects in small neighbourhoods of the diagonal in $M \times \overline{M}$. We can therefore define new geometric structures on neighbourhoods of the diagonal in $M \times \overline{M}$.

\textbf{Definition 2.3.} Define $\tilde{M}$ as a small neighbourhood of the diagonal in $M \times \overline{M}$, endowed with the complex structure $I = (J, -J)$.

We also extend the Hermitian line bundle $(L, h_0)$ and the connection $\nabla$ into a $\mathbb{C}$-bundle $(\tilde{L}, \tilde{h}_0)$ and a Hermitian connection $\tilde{\nabla}$ over $\tilde{M}$, by first extending the Kähler potentials $\psi_0$ in charts into $\tilde{\psi}_0$ and then setting $\|v\|_{\tilde{h}_0} = |e^{-\frac{1}{4}\tilde{\psi}_0}(x)v|$ and $\tilde{\nabla}_x(x) = \frac{d}{dx}(e^{-\frac{1}{4}\tilde{\psi}_0}e^{\frac{1}{4}\tilde{\psi}_0}).$ One sets then, in charts

$$\omega_0 := i \text{curv}(\tilde{\nabla}) = \frac{i}{2} \sum_{j,k}(\overline{h_{j,k}(z, w)}dz_j \wedge dw_k + \overline{h_{j,k}(z, w)}dz_k \wedge dw_j) \quad \text{where } h_{j,k} = \frac{\partial^2 \tilde{\psi}_0}{\partial z_j \partial \overline{z}_k}.$$

There is a second natural holomorphic structure on $\tilde{M}$, denoted by $\tilde{J} = (J, J)$. Note that $\tilde{J}$ preserves the tangent space of the diagonal of $M \times \overline{M}$, where it coincides with $J$, and $\tilde{J}$ commutes with $I$. Given a $J$-holomorphic object on $M$, its $I$-holomorphic extension to $\tilde{M}$ is also $\tilde{J}$-holomorphic.

We insist that $\tilde{\omega}_0$ is very different from the already available symplectic form on $M \times \overline{M}$ (stemming from the symplectic structure on each factor), which is a real-valued, non holomorphic, symplectic form. In the same way, $\tilde{L}$ is not $L \otimes \overline{L}$.

The curvature of $\tilde{\nabla}$ is positive in the following sense.

\textbf{Proposition 2.4.} Consider the following anti-linear involution on $T^C \tilde{M}$:

$$\sigma : \sum_j (v^1_j dz_j + v^2_j \overline{dz}_j + v^3_j dw_j + v^4_j \overline{dw}_j) \mapsto \sum_j (\overline{v^1_j} dw_j + \overline{v^2_j} \overline{dw}_j + \overline{v^3_j} dz_j + \overline{v^4_j} \overline{dz}_j).$$

Let $v \in T^C \tilde{M}$ nonzero and suppose that $Iv = iv$ (so that $v$ is an $I$-holomorphic tangent vector). Then

$$\tilde{\omega}_0(v, \sigma \tilde{J}v) > 0.$$

The involution $\sigma$ maps $I$-holomorphic vectors into $I$-antiholomorphic vectors and also maps $\tilde{J}$-holomorphic vectors into $\tilde{J}$-antiholomorphic vectors. Identifying the diagonal of $M \times \overline{M}$ with $M$, $\sigma$ coincides with the natural involution on $T^C M$. It is the only anti-linear involution with these properties.
\textbf{Proof.} In a chart, write
\[ v = \sum_j v_j^1 dz_j + \sum_j v_j^2 d\bar{w}_j. \]
Then
\[ \tilde{J} v = i \sum_j v_j^1 dz_j - i \sum_j v_j^2 d\bar{w}_j, \]
so that
\[ \tilde{\omega}_0(v, \sigma \tilde{J} v) = \frac{1}{2} \sum_{j,k} [h_{j,k}(z, \overline{w}) + h_{j,k}(z, \overline{w})](v_j^1 \overline{v_k^1} + v_j^2 \overline{v_k^2}). \]
The Hermitian matrix under brackets is a perturbation of $2h_{j,k}(z, \overline{\tau})$, which is positive definite. Therefore it is positive definite as well, hence the claim. \hfill \square

\textbf{Definition 2.5.} The program of holomorphic extension of a real-analytic Kähler manifold applied in the last two paragraphs can be applied to $M \times \overline{M}$ as well. We obtain a manifold which we will denote $\widetilde{M} \times \overline{M}$.

We will consider two natural holomorphic structures on $\widetilde{M} \times \overline{M}$. The first one is $(I, I)$, denoted also by $I$, and the second is $(\tilde{J}, -\tilde{J})$, denoted also by $\tilde{J}$. The symplectic form on $\widetilde{M} \times \overline{M}$ will be $(\tilde{\omega}_0, -\tilde{\omega}_0)$. Holomorphic objects on $M \times \overline{M}$ again enjoy the property that their $I$-holomorphic extension to $\widetilde{M} \times \overline{M}$ is also $\tilde{J}$-holomorphic.

Note that the diagonal of $\widetilde{M} \times \overline{M}$ is $I$-holomorphic but $\tilde{J}$-totally real.

Let us motivate the introduction of $\widetilde{M} \times \overline{M}$: with formula (11) as our objective, we are trying to describe the geometry underlying operators of the form $\exp(tkT_k(f))$ when $f \in C^\infty$, $t \in \mathbb{R}$ is small but fixed, and $k \to +\infty$. For every $k$, these operators have analytic dependence on $t$ and we may as well consider $\exp(-itkT_k(f))$ for $t \in \mathbb{R}$. Such operators are now well-known at least modulo $O(k^{-\infty})$: they correspond to integral kernels of the form (14), for particular sections $\Phi(it)$ which “correspond to” (in a sense we will make precise) Lagrangians of $M \times \overline{M}$. The kernel of $\exp(-itkT_k(f))$ is in fact a “Lagrangian section” of $L \otimes \overline{L}$ as introduced in [2]. These Lagrangians are the graphs of the Hamilton flow of $f$ at time $t$. It is natural to expect $\Phi(t)$ to “correspond to” (in the same sense) the graph of the Hamilton flow of $f$ at imaginary time $-it$. This graph, however, does not sit inside $M \times \overline{M}$ anymore; it will be a $I$-holomorphic Lagrangian of $\widetilde{M} \times \overline{M}$.

We now explain how some holomorphic sections of $L \otimes \overline{L}$ are associated with holomorphic Lagrangians of $\widetilde{M} \times \overline{M}$. The first holomorphic section of interest is associated with the diagonal of $\widetilde{M} \times \overline{M}$ and will be the phase of the Bergman kernel.

\textbf{Proposition 2.6.} Let $(M, J, \omega_0)$ be a quantizable Kähler manifold and let $(L, h_0)$ be a prequantum line bundle over $M$. Let $\Psi$ be the unique holomorphic section of $L \otimes \overline{L}$ over a neighbourhood of the diagonal in $M \times \overline{M}$ such that $h_0(\Psi) = 1$ on the diagonal.

Let $\tilde{\nabla}$ denote the $I$-holomorphic extension of $\Psi$ to a neighbourhood of the diagonal of $M \times \overline{M}$ in $\widetilde{M} \times \overline{M}$. Let $\tilde{\nabla}$ denote the natural connection on $L \otimes \overline{L}$.

Then (up to further restricting the neighbourhood of the diagonal) $\tilde{\nabla} \Psi$ vanishes on the diagonal of $\widetilde{M} \times \overline{M}$ and nowhere else; moreover $\tilde{\nabla} \Psi$ is a defining function for its zero set: the distance to the diagonal is comparable to $\|\tilde{\nabla} \Psi\|$.

To make sense of the conditions on $\Psi$, note that for every $x \in M$, $\Psi(x, x)$ is an element of $L_x^\ast \otimes \overline{L_x^\ast}$, on which $h_0(x)$ acts as a nonzero linear form. Usually (see for instance [3]), one uses $h_0$ to identify $L_x^\ast$ with $L_x^\prime$, and the condition becomes $\Psi(x, x) = 1$. 

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Proof. For the moment let us consider matters on \( M \times \overline{M} \), equipped with its connection \( \nabla \). Observe that since \( h_0(\Psi) = 1 \) on the diagonal, \( \nabla \Psi \) vanishes on the diagonal of \( M \times \overline{M} \) (it certainly vanishes along the diagonal directions, and moreover \( \Psi \) is holomorphic so that \( \nabla_J \Psi = i \nabla \Psi \) for all \( \dot{x} \in T(M \times \overline{M}) \); to conclude, the diagonal is totally real in \( M \times \overline{M} \). Because \( \mathrm{icurv}\nabla \) is the symplectic form on \( M \times \overline{M} \), sets of the form \( \{ x \in M, \nabla \Phi(x) = 0 \} \), for \( \Phi \) a general nonvanishing section of \( L \boxtimes \overline{L} \), have to be isotropic for this symplectic form. Therefore (up to restricting our attention to a smaller open neighbourhood of the diagonal) \( \{ \nabla \Psi = 0 \} \) is the diagonal of \( M \times \overline{M} \) and moreover (using again the curvature identity and Proposition 2.4), \( \| \nabla \Psi \| \) is comparable to the distance to the diagonal.

Letting now \( \tilde{\Psi} \) be the holomorphic extension of \( \Psi \), defined in a neighbourhood of the the diagonal of \( M \times \overline{M} \) in \( \tilde{M} \times \overline{M} \), for the same reasons the equation \( \nabla \tilde{\Psi} = 0 \) defines a I-holomorphic isotropic submanifold of \( \tilde{M} \times \overline{M} \) for the natural symplectic form; since its restriction to the real set is the Lagrangian \( \text{diag}(M) \), we conclude that \( \{ \nabla \tilde{\Psi} = 0 \} = \text{diag}(M) \), and \( \nabla \tilde{\Psi} \) is still a defining function for its zero set.

**Proposition 2.7.** Let \( \Phi \) be a holomorphic section of \( L \boxtimes \overline{L} \) over a neighbourhood of the diagonal of \( M \times \overline{M} \). Suppose that, in a topology of real-analytic functions, the function \( x \mapsto h_0(\Phi(x,x)) \) is close to 1. Then the holomorphic extension \( \tilde{\Phi} \) of \( \Phi \) to a section over a neighbourhood of the diagonal of \( M \times \overline{M} \) in \( \tilde{M} \times \overline{M} \) is such that \( \{ \nabla \tilde{\Phi} = 0 \} \) is a I-holomorphic Lagrangian, for which \( \nabla \tilde{\Phi} \) is a defining function.

Proof. Recall that \( L \boxtimes \overline{L} \) is a holomorphic line bundle over \( M \times \overline{M} \), and that the diagonal of \( M \times \overline{M} \) is a totally real manifold. Since \( \Phi \) is a holomorphic section, it is determined by its restriction to the diagonal. Under our hypotheses, \( \Phi \) is close to \( \Psi \) when restricted to the diagonal, in some suitable real-analytic topology. Therefore, in a small neighbourhood of the diagonal, \( \Phi \) is close to \( \Psi \) (as defined in Proposition 2.6) in a real-analytic topology. In turn, in a neighbourhood of the diagonal of \( M \times \overline{M} \) in \( \tilde{M} \times \overline{M} \), \( \tilde{\Phi} \) is close to \( \tilde{\Psi} \) in a real-analytic topology.

In particular, the equation \( \tilde{\nabla} \tilde{\Psi} = 0 \) still defines a I-holomorphic submanifold of half dimension of \( \tilde{M} \times \overline{M} \). Since \( \mathrm{icurv}\nabla = \tilde{\omega} \), this manifold again has to be isotropic and is therefore a Lagrangian, which lies close (in real-analytic topology) to the diagonal of \( \tilde{M} \times \overline{M} \).

Let \( \mathcal{L} \) be a I-holomorphic Lagrangian of \( \tilde{M} \times \overline{M} \), close to the diagonal in real-analytic topology. Can \( \mathcal{L} \) be realised as the vanishing set of \( \tilde{\nabla} \tilde{\Phi} \) for some holomorphic section \( \Phi \)? The answer relies on the Bohr-Sommerfeld index.

**Definition 2.8.**

1. Let \( \mathcal{L} \) be a Lagrangian of \( \tilde{M} \times \overline{M} \). Let \( \gamma : [0,1] \to \mathcal{L} \) be a parametrised closed loop. Let \( \Gamma(0) \in \tilde{L} \boxtimes \overline{L} \) nonzero over \( \gamma(0) \) and let \( \Gamma \) be the parallel transport of \( \Gamma(0) \) along \( \gamma \) (following the already defined connection \( \nabla \)). The **Bohr-Sommerfeld index** of \( \gamma \) is \( \Gamma(1)/\Gamma(0) \). Since parallel transport is a linear differential equation and \( \tilde{L} \boxtimes \overline{L} \) is a line bundle, the Bohr-Sommerfeld index of \( \gamma \) does not depend on \( \Gamma(0) \).

2. Since \( \mathcal{L} \) is Lagrangian, it is \( \tilde{\nabla} \)-flat; therefore the Bohr-Sommerfeld index of a loop only depends on its topology. Define the **Bohr-Sommerfeld class** as the group morphism \( \pi_1(\mathcal{L}) \to \mathbb{C}^* \).

3. We say that \( \mathcal{L} \) is a Bohr-Sommerfeld Lagrangian when the group morphism above is trivial.

**Proposition 2.9.** Let \( \mathcal{L} \) be a I-holomorphic Lagrangian of \( \tilde{M} \times \overline{M} \). The equation \( \tilde{\nabla} \tilde{\Phi} = 0 \) on \( \mathcal{L} \), with J-holomorphic and I-holomorphic unknown \( \tilde{\Phi} \), defines a rank one sheaf over \( \mathcal{L} \); nonzero local solutions are such that \( \tilde{\nabla} \tilde{\Phi} \) are defining functions for \( \mathcal{L} \).

The cohomology of this sheaf is exactly the Bohr-Sommerfeld class of \( \mathcal{L} \); in particular, a non-zero global solution exists if, and only if, \( \mathcal{L} \) is a Bohr-Sommerfeld Lagrangian.
In other terms, one can always find \( \tilde{\Phi} \) such that \( \tilde{\nabla}\tilde{\Phi} = 0 \) on \( \mathcal{L} \) locally near any point of \( \mathcal{L} \), and the solution is unique up to a multiplicative constant. The different solutions can be patched into a global function \( \tilde{\Phi} \) if and only if \( \mathcal{L} \) is Bohr-Sommerfeld.

Proof. Let us first prove that, locally, there is no obstruction for the existence of \( \tilde{\Phi} \). Fixing arbitrarily the value of \( \tilde{\Phi} \) at a point \( x_0 \in \mathcal{L} \), the equation \( \tilde{\nabla}\tilde{\Phi} = 0 \) on \( \mathcal{L} \) determines \( \tilde{\Phi} \) on a neighbourhood of \( x_0 \) in \( \mathcal{L} \), by parallel transport of \( \Phi(x_0) \) along short paths in \( \mathcal{L} \). Since \( \mathcal{L} \) is Lagrangian, the value obtained by parallel transport does not depend on the short paths but only on their endpoints. Since \( \mathcal{L} \) is a Lagrangian, it is \( J \)-totally real by Proposition 2.4, and since \( \tilde{\Phi} \) is \( J \)-holomorphic, the knowledge of \( \tilde{\Phi} \) locally on \( \mathcal{L} \) determines \( \tilde{\Phi} \) on a whole neighbourhood of \( x_0 \) in \( \tilde{M} \times \tilde{M} \); using Proposition 2.4 again, if \( \Phi(x_0) \neq 0 \) then \( \tilde{\nabla}\tilde{\Phi} \) is constrained by the curvature identities to be a defining function for \( \mathcal{L} \) near \( x_0 \). In summary, without any constraint, the data of a \( \mathcal{I} \)-holomorphic Lagrangian \( \mathcal{L} \) near any of its points determines completely \( \tilde{\Phi} \) as in the claim of Proposition 2.7 near this point up to a multiplicative constant.

Let us now study whether these local solutions can be patched together. Noticing that the restriction of \( \tilde{\Phi} \) to a loop in \( \mathcal{L} \) has to be flat (that is, it solves the parallel transport equation), by Definition 2.8 a nonzero solution \( \Phi \) exists in a neighbourhood of a loop \( \gamma \subset \mathcal{L} \) if, and only if, the Bohr-Sommerfeld index of \( \gamma \) is 1. Hence, a nonzero solution exists on a neighbourhood of \( \mathcal{L} \) if and only if \( \mathcal{L} \) is Bohr-Sommerfeld.

2.2 Analytic symbols

Here we rapidly present the analytic function spaces symbol spaces which we will rely on; we refer to [32, 19] for more in-depth introductions to analytic semiclassical analysis.

Definition 2.10. Let \( U \) be a real-analytic Riemannian manifold (possibly open) and let \( m, r, R > 0 \).

The Banach space \( H^r_m(U) \) consists of all functions \( a \) from \( U \) to \( \mathbb{C} \) such that there exists \( C > 0 \) satisfying, for every \( j \in \mathbb{N} \),

\[
\|\nabla^j a\|_{L^\infty(U)} \leq C \frac{r^j j!}{(j + 1)^m};
\]

the best constant \( C \) above is the Banach norm of \( a \).

The Banach space \( S^r_m(U) \) of \textit{formal analytic symbols} consists of all sequences \( (a_k)_{k \in \mathbb{N}} \) of elements of \( H^r_m(U) \) such that there exists \( C > 0 \) satisfying, for every \( (j, k) \in \mathbb{N}^2 \),

\[
\|\nabla^j a_k\|_{L^\infty(U)} \leq C \frac{r^j R^k(j + k)!}{(j + k + 1)^m} ;
\]

the best constant \( C \) above is the Banach norm of \( a \).

The definition of \( S^r_m \) and the Stirling formula imply that for every \( c_1 > 0, c_2 > 0 \) small enough (strictly smaller than \( \frac{e}{R} \)), there exists \( c_3 > 0, C \) such that for every \( a \in S^r_m(U) \), for every \( h > 0 \),

\[
\left\| \sum_{k=c_1 h^{-1}}^{c_2 h^{-1}} h^k a_k \right\|_{L^\infty(U)} \leq C e^{-c_3 h^{-1}} \|a\|_{S^r_m(U)} .
\]

We will call \textit{classical analytic symbol} (or simply analytic symbol, since we will only encounter classical ones in this text) a function of the form

\[
U \times (0, 1) \ni (x, h) \mapsto \sum_{k=0}^{c h^{-1}} h^k a_k(x)
\]
where \((a_k)_{k \in \mathbb{N}} \in S^{r,R}_m(U)\) and \(0 < c < \frac{c_0}{R}\); such a function will be called a realisation of \((a_k)_{k \in \mathbb{N}}\). By the above inequality, different realisations of the same formal analytic symbol are exponentially close to each other, and an analytic symbol is associated with a unique formal analytic symbol.

The point of Definition 2.10 is that (classical) analytic symbols are stable by analytic stationary phase: an expression of the form
\[
(x; h) \mapsto \int e^{i\frac{\phi(x,y)}{h}} a(x, y; h) dy
\]
is, under suitable geometric conditions on \(\phi\), if \(\phi\) is real-analytic and \(a\) is a classical real-analytic symbol, equal to \(e^{i\frac{\phi(x)}{h} N(x)} b(x; h) + O(e^{-ch^{-1}})\) where \(\psi\) is real-analytic, \(N\) is an integer, \(b\) is a classical real-analytic symbol, and \(c > 0\). We refer to [32, Théorème 2.8] for details.

### 2.3 Semiclassical analysis of the Bergman projector

In this subsection we gather the available results on Berezin–Toeplitz quantization in analytic regularity that we will use. We formulate the results in terms of the geometric definitions of Section 2.1. We present in particular covariant Berezin–Toeplitz operators, an alternative to formula (9) which is more suited to our analysis.

**Proposition 2.11.**

1. Let \((M, J, \omega)\) be a compact, real-analytic, polarised Kähler manifold; let \((L, h_0)\) be a prequantum line bundle over \(M\). Let \(U\) be a small open neighbourhood of the diagonal in \(M \times \overline{M}\) and let \(\Psi\) be the section of \(L \otimes \overline{L}\) featured in Proposition 2.7 \((\Psi\) is holomorphic on \(U\) and \(h_0(\Psi(x,x)) = 1\)). There exists a formal analytic symbol \(s \in S^{r,R}_m(U)\) with principal symbol \(s_0 = (2\pi)^{-d}\), and constants \(c > 0\), \(C > 0\) such that for any \(u, v \in H^0(M, L^{\otimes k})\),

\[
\left| \langle u, v \rangle_{\text{Hilb}_k(0)} - k^d \int_U \langle \Psi(x, y) \otimes k, \overline{u(x)} \otimes v(y) \rangle_{(h_0 \otimes h_0)^{\otimes k}} s(x, y; k^{-1}) \omega_0^{\otimes d} (dx) \omega_0^{\otimes d} (dy) \right| \leq C e^{-ck} \| u \|_{\text{Hilb}_k(0)} \| v \|_{\text{Hilb}_k(0)}
\]

where \(s(x, y; k^{-1})\) denotes any realisation of \(s\).

2. Given \(r, R, m > 0\), \(V \subset U\) containing the diagonal of \(M \times \overline{M}\) and \(a \in S^{r,R}_m(V)\), let

\[
T_k^{\text{cov}}(a) : (x, y) \mapsto 1_{(x,y) \in V} k^d \Psi^{\otimes k}(y) s(x, y; k^{-1}) a(x, y; k^{-1})
\]

where \(a(x, y; k^{-1})\) denotes any realisation of \(a\). Then there exists \(m_0 > 0\), \(c > 0\), \(C > 0\), and functions \(r_0, R_0\) such that, for every \(m, r, R\) such that \(m \geq m_0, r \geq r_0(m), R \geq R_0(r, m)\), for every \(a \in S^{m}_r, R(V)\) and \(b \in S^{m}_r, R(V)\) there exists \(a \# b \in S^{m}_r, 2R(V)\) such that

\[
\| T_k^{\text{cov}}(a) T_k^{\text{cov}}(b) - T_k^{\text{cov}}(a \# b) \|_{H^0 \to H^0} \leq C \| a \|_{S^r_m, R} \| b \|_{S^r_m, 2R} e^{-ck},
\]

and moreover the bilinear map \((a, b) \to a \# b\) is continuous:

\[
\| a \# b \|_{S^r_m, 2R} \leq C \| a \|_{S^r_m, R} \| b \|_{S^r_m, 2R}.
\]

3. Given \(r, R, m > 0, c_0 > 0\), \(V \subset U\) containing the diagonal of \(M \times \overline{M}\), there exists \(r', R', m', C, c > 0\) and \(V' \subset U\) containing the diagonal of \(M \times \overline{M}\) such that for every \(a \in S^{m}_r, R(V)\) with principal symbol \(a_0\) bounded away from zero on \(V\) with \(\| a_0 \| > c_0\), there exists \(b \in S^{m'}_{r'}, R'(V')\) with continuous dependence on \(a_0\) such that

\[
\| T_k^{\text{cov}}(a) T_k^{\text{cov}}(b) - \Pi_k \|_{H^0 \to H^0} \leq C \| a \|_{S^r_m} e^{-ck}.
\]
Proof. Equation (15) corresponds to Theorem 5.5 of [28] and Theorem A of [14].

Equation (16) is a consequence of Theorem B in [14]; let us prove how to obtain (16) from there, since the definition of covariant operators here is slightly different from that in [14]. Letting * denote the Cauchy product of formal symbols, we know from [14], Theorem B, that

\[ \| (a \# b) \ast s \|_{s^{r,2R}_m} \leq C \| a \ast s \|_{s^{r,R}_m} \| b \ast s \|_{s^{r,R}_m} \]

for \( r, R, m \) as before. Here, without loss of generality, the symbol \( s \) of the Bergman projector and its Cauchy inverse \( s^{-1} \) lie in a symbol class which injects continuously in \( S^{r,R}_m \). Thus, by continuity of the Cauchy product ([14], Proposition 3.8), we finally obtain the desired result.

The third part of the proposition is a consequence of the second part of Theorem B in [14], modulo multiplication or division by \( s \) for the Cauchy product, exactly as for (10).

\[ \square \]

Remark 2.12 (Equivalent analytic norms). There is now a large body of literature concerning Berezin–Toeplitz operators in real-analytic regularity and their symbolic calculus. The analytic norms of Definition 2.10 used in [14], lead to lengthy proofs, but to this date, it is the only way to obtain, globally, analytic norms which are relatively stable under composition: the Banach space of \( a \# b \) is the same as that of \( b \), if \( a \) is more regular. This contrasts with, e.g., Proposition 4.1 in [8], where there is a loss of regularity. We will use this fact later.

A consequence of the Bergman projector asymptotics in Proposition 2.11 (for which real-analyticity is in fact a very strong requirement) is that \( \text{FS}_k \) is approximately a left inverse to \( \text{Hilb}_k \).

Proposition 2.13. [33] [34] As \( k \to \infty \), for every \( j \in \mathbb{N} \), \( \| \text{FS}_k(\text{Hilb}_k(0)) \|_{C^j} = O(k^{-1}) \).

Remark 2.14.

1. Using the notations of Proposition 2.11 one has, for every \( x \in M \) and every \( k \) large enough,

\[ \text{FS}_k(\text{Hilb}_k(0))(x) = \frac{1}{k} \log(k^d s(x,x;k^{-1})) - \frac{d \log(k)}{k} = \frac{1}{k} \log(s(x,x;k^{-1})) \]

which motivates our alternative choice for the definition of \( \text{FS}_k \).

2. The method of proof of [34] allows to generalise Proposition 2.13 into the fact that

\[ \text{FS}_k(\text{T}_k^{\text{cov}}(f)) = O_{C^{\infty}}(k^{-1}) \]

for every \( f \) such that \( T_k^{\text{cov}}(f) \) is positive and self-adjoint, i.e. \( f \) is real and positive on the diagonal.

The map \( \text{FS}_k \circ \text{Hilb}_k \) is independent of the choice of the reference Kähler metric, so that we obtain \( \text{FS}_k(\text{Hilb}_k(\phi)) = \phi + O(k^{-1}) \) for every \( \phi \in \mathcal{H} \) corresponding to a smooth metric.

We also recall the following subprincipal identities concerning Berezin–Toeplitz quantization in the covariant and contravariant case.

Proposition 2.15 ([6], page 4 and [14], Proposition 4.11). Let \( f \) and \( g \) be analytic symbols defined near the diagonal of \( M \). Let \( b \) be the holomorphic extension of \( (\partial f, \overline{\partial} g)_{Q^{(0,1)}(M), Q^{(1,0)}(M)} \) to a neighbourhood of the diagonal in \( M \times \overline{M} \). Then

\[ T_k^{\text{cov}}(f)T_k^{\text{cov}}(g) = T_k^{\text{cov}}(fg) + k^{-1}T_k^{\text{cov}}(b) + O(k^{-2}) \]

(17)

\[ T_k^{\text{cov}}(f) = T_k(f) + k^{-1}T_k(\Delta \phi f) + O(k^{-2}) \]

(18)

More generally, for every analytic symbol \( f^{\text{cov}} \) such that

\[ T_k^{\text{cov}}(f^{\text{cov}}) = T_k(f) \].

13
3 Analytic Fourier Integral Operators close to identity

By formula (15), the Bergman projector $\Pi_k$ has, asymptotically and near the diagonal, an expression of Wentzel-Kramers-Brillouin type, with a phase (fixed quantity to the power $k$) multiplied by an analytic symbol. Let us generalise this expression into a definition.

**Definition 3.1.** Recall the analytic symbol $s$ of the Bergman kernel from Proposition 2.11.

An analytic Fourier Integral Operator (FIO) close to identity is a sequence of sections of $L \boxtimes L$ of the form

$$I^V_k \Phi(a) : (x,y) \mapsto k^d \mathbf{1}_{(x,y) \in V} \Phi^\otimes k (x,y) s(x,y; k^{-1}) a(x,y; k^{-1}),$$

where

- $V$ is a neighbourhood of the diagonal in $M \times \overline{M}$.
- $\Phi$ is a holomorphic section of $L \boxtimes L$ over $V$.
- For all $(x,y) \in \partial V$, one has $\| \Phi(x,y) \|_{h_0 \otimes h_0} < 1$ (and therefore by compactness $\| \Phi \|_{h_0 \otimes h_0}$ is bounded away from 1 on the boundary).
- $s$ is the symbol of the Bergman kernel of Proposition 2.11.
- $a(x,y; k^{-1})$ is a (classical) analytic symbol on $V$, and is holomorphic.

By convention, $I^V_k \Phi(a)$ also denotes the associated operator on $H^0(M, L^\otimes k)$ defined as follows: given two holomorphic sections $u, v$ of $L^\otimes k$,

$$\langle u, I^V_k \Phi(a) v \rangle_{\text{Hilb}_k(0)} := \int_V \langle I^V_k \Phi(a), u(x) \otimes v(y) \rangle_{(h_0 \otimes h_0)^{\otimes k} \omega^d \omega \wedge d \omega} (dx) \omega \wedge d \omega. \quad (19)$$

Covariant Toeplitz operators are examples of Analytic FIOs with $\Phi = \Psi$.

This section is devoted to the general properties of Analytic Fourier Integral Operators. As in the usual (self-adjoint) case, they are associated with Lagrangians, which allow to understand geometrically and manipulate these operators: composition, inversion, change of reference Kähler metric, and associated metric under $\text{FS}_k$. All these manipulations will be useful in the proof of Theorem [3].

Let us first prove a rather weak bound on the operator norm of these operators.

**Proposition 3.2.** Let $I^V_k \Phi(a)$ be an analytic FIO close to identity. Then, for the operator norm given by $\text{Hilb}_k(0)$, the operator norm of $I^V_k \Phi(a)$ satisfies

$$\forall \epsilon, \exists C, \forall k, \| I^V_k \Phi(a) \|_{H^0 \to H^0} \leq C \left( \sup_{x,y} | \Phi(x,y) |_{h_0 \otimes h_0} \right)^{k(1+\epsilon)}.$$

**Proof.** In Formula (19), if both $u$ and $v$ are normalised in $\text{Hilb}_k(0)$ we want to bound the right-hand side from above. First of all, since $u$ and $v$ are holomorphic we obtain

$$\sup_{x,y} \| u(x) \otimes v(y) \|_{h_0 \otimes h_0} \leq C_0 k^{2d}. $$

The integral on the right-hand-side of (19) is therefore bounded (in charts) by

$$C_0 k^{3d} \sup |a| \left( \sup_{x,y} \| \Phi \|_{h_0 \otimes h_0} \right)^k.$$ 

\[ \square \]
Definition 3.1 is relative to the particular Kähler structure \((M,J,\omega_0)\). One can in fact change the Kähler structure to \(\omega_\phi\) for \(\phi\) close to 0 and obtain another analytic Fourier Integral Operator. The relationship between the two operators is not explicit at this stage, but one can compute the first two derivatives of the map between the phases.

**Proposition 3.3.** Let \(I_k^{V,\Phi}\) be an analytic Fourier Integral operator close to identity. There exists \(c > 0\) such that the following is true. Let \(\phi \in H\) be real-analytic and sufficiently close to 0 in real-analytic topology.

Then there exists a smaller open neighbourhood \(V'\) of the diagonal in \(M \times \overline{M}\), a holomorphic section \(\Phi'\) of \(L \otimes \overline{L}\) over \(V'\), a holomorphic classical analytic symbol \(b\) on \(V'\), such that for every two holomorphic sections \(u, v\) of \(L^{\otimes k}\),

\[
\langle u, I_k^{V,\Phi}(a)v \rangle_{\text{Hilb}_k(0)} = \langle u, I_k^{V',\Phi'}(b)v \rangle_{\text{Hilb}_k(0)} + O(e^{-ck\|u\|_{\text{Hilb}_k(0)}\|v\|_{\text{Hilb}_k(0)}}).
\]

If \(\phi(t)\) is a differentiable one-parameter family of real-analytic Kähler potentials and \(\Phi(t)\) is a differentiable one-parameter family of real-analytic sections of \(L \otimes \overline{L}\) with \(\phi(0) = 0\) and \(\Phi(0) = \Psi\) (the phase of the Bergman kernel), then \(\Phi'(0) = \Psi\) and on the diagonal

\[
\frac{d\log \Phi'}{dt} \bigg|_{t=0} = \frac{d\log \Phi}{dt} \bigg|_{t=0} + 2 \frac{d\phi}{dt} \bigg|_{t=0}.
\]

If \(\Phi\) and \(\phi\) are twice-differentiable, then so is \(\Phi'\), and if in addition \(\frac{d\log \Phi'}{dt} \big|_{t=0}\) is real, then on the diagonal

\[
\frac{d^2 \log \Phi'}{dt^2} \bigg|_{t=0} = \frac{d^2 \log \Phi}{dt^2} \bigg|_{t=0} + 2 \frac{d^2 \phi}{dt^2} \bigg|_{t=0} + 4Re(\partial \frac{d \log \Phi}{dt} \big|_{t=0}, \overline{\partial} \frac{d \phi}{dt} \big|_{t=0}) + 2(\partial \frac{d \phi}{dt} \big|_{t=0}, \overline{\partial} \frac{d \phi}{dt} \big|_{t=0});
\]

all scalar products are taken with respect to the Kähler structure \((M,J,\omega_0)\).

**Proof.** One can of course write, for \(u, v \in H^0\),

\[
\langle u, I_k^{V,\Phi}(a)v \rangle_{\text{Hilb}_k(0)} = \int \langle (\Phi(x,y)e^{\phi(x)+\phi(y)}\otimes k, u(x) \otimes \overline{v(y)} \rangle_{(h_\phi \otimes h_\phi)^\otimes k} a(x,y;k^{-1})V(x)V(y)\omega^d_\phi(dx)\omega^d_\phi(dy),
\]

where \(V(x)\) is the ratio between the volume forms \(s_0\omega^d_\phi\) and \(s_\phi\omega^d_\phi\). This expression, however, involves a non-holomorphic section of \(L \otimes \overline{L}\), since neither \(\phi\) nor \(V\) are holomorphic. Nonetheless, by definition of the Bergman kernel \(\Pi_k^\phi\), and introducing

\[
K : (x,y) \mapsto \int_{(z,w) \in V} \langle (\Pi_k^\phi(x,z) \otimes \Pi_k^\phi(y,w), \Phi(z,w)\otimes k e^{k\phi(z)+k\phi(w)} \rangle_{(h_\phi \otimes h_\phi)^\otimes k} a(z,w)V(z)V(w)dzdw,
\]

\(K\) is a holomorphic section of \(L \otimes \overline{L}\), and by definition of \(\Pi_k^\phi\),

\[
\langle u, I_k^{V,\Phi}(a)v \rangle_{\text{Hilb}_k(0)} = \int \langle (K(x,y), u(x) \otimes \overline{v(y)} \rangle_{(h_\phi \otimes h_\phi)^\otimes k} \omega^d_\phi(dx)\omega^d_\phi(dy).
\]

It remains to show that \(K\) is exponentially small away from the diagonal and of FIO form near the diagonal. The spirit of the proof is that in charts, one can compute \(K\) from the formula (22) by stationary phase. Let us do so explicitly enough to recover formulas (20) and (21). We will write down (22) in a Hermitian chart for \(h_\phi\). First of all, we write

\[
\Phi(z,w) = \Psi(z,w) \exp(f(z,w)),
\]

where \(f\) is a holomorphic function on \(V\) close to 0. We know that, in a Hermitian chart for \(h_0\), given by a Kähler potential \(\psi_0\), the phase \(\Psi\) of the Bergman kernel \(\Pi_k\) reads

\[
\Psi(z,w) = \exp(-\frac{\psi_0(z)}{2} + \overline{\psi_0(z,w)} - \frac{\psi_0(w)}{2}).
\]
Here as usual \( \tilde{\psi} \) denotes the holomorphic extension on \( V \) of \( \psi_0 \).

The kernel of \( \Pi^0 \) has a phase of the same form, in the Hermitian charts for \( h_0 \), where \( \psi_0 \) is now replaced by \( \tilde{\psi}_0 = \psi_0 + \phi \).

The transition function from the Hermitian chart for \( h_0 \) to the Hermitian chart for \( h_\phi \) is given by multiplication by \( e^{-\frac{1}{2} \phi} \). Consequently, in the Hermitian chart for \( h_\phi \), the section \( (z, w) \mapsto \Phi(z, w)e^{\phi(z) + \phi(w)} \) is written as
\[
\exp(-\frac{\psi_0(z)}{2} + \tilde{\psi}_0(z, w) - \frac{\psi_0(w)}{2}) \exp(f(z, w) - \tilde{\phi}(z, w) + \phi(z) + \phi(w)).
\]

We are now ready to compute the phase \( \Phi' \) in the case where both \( f \) and \( \phi \) are infinitesimally close to 0. If \( x \) and \( y \) are close enough, we obtain, computing every section in the Hermitian chart for \( h_\phi \),
\[
K(x, y) = \int_{W(x, y)} e^{kF(x, z, w, y)} A(x, z, w, y) dz dw
\]
where \( W(x, y) \) is a neighbourhood of \( \{x\} \) and where the holomorphic extension of \( F \) reads
\[
\tilde{F}(x, z, w, \bar{w}, y) = -\frac{\psi_0(x, \bar{z})}{2} + \tilde{\psi}_0(x, z) - \tilde{\psi}_0(z, \bar{w}) + \tilde{\psi}_0(w, y) - \frac{\psi_0(y, \bar{w})}{2} + \tilde{f}(z, \bar{w}) - \tilde{\phi}(z, \bar{w}) + \tilde{\phi}(z, \bar{z}) + \tilde{\phi}(w, \bar{w}).
\]

Suppose \((x, \bar{x}) = (y, \bar{y})\) lies on the real locus of \( \tilde{M} \). Then, the second line is close to 0 in real-analytic topology, and the first line is a positive phase function of \((z, w)\) with a unique critical point. This critical point is \((z, \bar{z}, w, \bar{w}) = (x, \bar{x}, x, \bar{x})\), where the first line is equal to 0.

For \( f \) and \( \phi \) small enough, and for \((x, \bar{x}, y, \bar{y})\) close to the diagonal of \( M \), the function \((23)\) is still a positive phase function of \((z, \bar{z}, w, \bar{w})\) with a unique critical point. Therefore we can apply analytic stationary phase \((\text{[32], Théorème 2.8})\) and obtain a formula of the form
\[
K(x, y) = \Phi'(x, y)^{\otimes k} b(x, y; k^{-1}) + O(e^{-ck})
\]
for \((x, y)\) close to the diagonal, where \( \Phi' \) is a section of \( L \otimes \mathcal{T} \), \( b \) is an analytic symbol, and \( c > 0 \). The section \( \Phi' \) will be close in (real-analytic topology) to \( \Psi \), and the neighbourhood of the diagonal of \( M \times \tilde{M} \) on which one can perform analytic stationary phase can be considered fixed for \( \phi, f \) small, so that if \( \phi, f \) are even smaller, \( |\Phi'|_{h_\phi \otimes h_0} < 1 \) on the boundary of this domain. If \((x, y)\) lies away from the diagonal of \( M \times \tilde{M} \), then in \((23)\) either \((x, z)\), \((z, w)\), or \((w, y)\) are away from the diagonal, and since \( \Phi \) close to \( \Psi \), the integrand is then \( O(e^{-ck}) \).

To prove formula \((20)\) we look at \((23)\) again: if \( \phi \) and \( f \) are infinitesimally close to 0 then the critical point \((z_0(x), \bar{z}_0(x), w_0(x), \bar{w}_0(x))\) is close to \((x, \bar{x}, x, \bar{x})\) so that the value of \( F \) at the critical point is close to \( f(x) + \phi(x) \). This identity holds in the Hermitian chart for \( h_\phi \), and therefore in the (fixed) Hermitian chart for \( h_0 \) the value of \( F \) at the critical point is close to \( f(x) + 2\phi(x) \). Therefore \((20)\) holds.

To prove \((21)\) we need to study the term of order 2 in \( \phi \) and \( f \) of the value at the critical point of \( F \). For this we first write down the equations for the critical point as
\[
0 = -\partial \tilde{\psi}_0(z, \bar{z}) + \partial \tilde{\psi}_0(z, \bar{w}) + \partial f(z, \bar{w})
\]
\[
0 = \bar{\partial} \tilde{\psi}_0(x, \bar{z}) + \bar{\partial} \tilde{\psi}_0(x, \bar{w}) - \bar{\partial} f(x, \bar{w})
\]
\[
0 = \bar{\partial} \tilde{\psi}_0(w, \bar{z}) + \bar{\partial} \tilde{\psi}_0(w, \bar{w}) - \bar{\partial} f(w, \bar{w})
\]
\[
0 = -\bar{\partial} \tilde{\psi}_0(w, \bar{w}) + \bar{\partial} \tilde{\psi}_0(z, \bar{w}) + \bar{\partial} f(z, \bar{w}),
\]
these equations being obtained by differentiating \((23)\) with respect to \( z, \bar{z}, w, \bar{w} \) respectively.
To first order in $f$ and $\phi$, the critical point is then given by
\[
\begin{align*}
 z^* &= x + A_0(x)^{-1} \overline{\phi}(x) + O(|f|^2 + |\phi|^2) \\
 z^* &= \overline{x} + A_0(x)^{-1} \overline{\phi}(x) + O(|f|^2 + |\phi|^2) \\
 w^* &= x + A_0(x)^{-1} \overline{\phi}(x) + O(|f|^2 + |\phi|^2),
\end{align*}
\]
where $A_0(x)$ is the positive definite matrix $\partial \overline{\partial} \psi_0(x, x)$.

If $f$ is real, the computations for the second order contribution become quite symmetrical, and we find
\[
\tilde{F}(x, \overline{x}, x^*, \overline{x}^*, w^*, \overline{w}^*, x, \overline{x}) = f(x) + \phi(x) + \text{Re}(\partial f \cdot A_0(x)^{-1} \overline{\phi}) + 2\partial \phi \cdot A_0(x)^{-1} \overline{\phi} + O(|f|^3 + |\phi|^3).
\]
The products involved are exactly the scalar products between (anti)-holomorphic gradients for the Kähler structure $\omega_0$. Remembering to add $\phi(x)$ to $F$ to obtain an expression in the Hermitian chart for $h_0$, the proof is complete. \square

If $\Phi$ is close to $\Psi$ in a real-analytic topology, then by the results of the previous section, to $\Phi$ one can associate a holomorphic Lagrangian of $\widetilde{M} \times \overline{\widetilde{M}}$. This Lagrangian encodes several properties of $I_k^{V, \Phi}(a)$, and allows for instance to understand the composition law on analytic Fourier Integral operators.

**Proposition 3.4.** Let $V$ be a neighborhood of the diagonal in $\widetilde{M}$. Let $\Psi$ denote the phase of the Bergman kernel in Proposition 2.15. For every $r, R, m, c > 0, C$ and a neighborhood $V'$ of the diagonal in $\widetilde{M}$ such that the following is true. For every holomorphic sections $\Phi_1$ and $\Phi_2$ of $L \otimes \overline{L}$ over $V$ such that such that
\[
\|\Phi_1/\Psi - 1\|_{H^m_\alpha(V)} < \delta \quad \|\Phi_2/\Psi - 1\|_{H^m_\alpha(V)} < \delta,
\]
there exists a holomorphic section $\Phi$ over $V'$ such that
\[
\|\Phi_1/\Psi - 1\|_{H^m_\alpha(V')} < \epsilon
\]
(and the map $(\Phi_1, \Phi_2) \mapsto \Phi$ is continuous), and for every $a_1, a_2 \in S^r_{m,R}(V)$ there exists $a \in S^r_{m', R'}(V')$ (depending continuously on $a_1, a_2, \Phi_1, \Phi_2$), satisfying
\[
\|a\|_{S^r_{m', R'}(V')} \leq C \|a_1\|_{S^r_{m,R}(V)} \|a_2\|_{S^r_{m,R}(V)}
\]
and such that
\[
\left\|I_{k}^{V, \Phi_1}(a_1) \circ I_{k}^{V, \Phi_2}(a_2) - I_{k}^{V', \Phi}(a)\right\|_{L^2 \to L^2} \leq Ce^{-ck}.
\]
Moreover, denoting
\[
\mathcal{L}_1 = \{\nabla \Phi_1 = 0\} \quad \mathcal{L}_2 = \{\nabla \Phi_2 = 0\} \quad \mathcal{L} = \{\nabla \Phi = 0\},
\]
one has
\[
\mathcal{L} = \mathcal{L}_1 \circ \mathcal{L}_2 := \{(x, \overline{z}) \in \widetilde{M} \times \overline{\widetilde{M}}, \exists y \in \widetilde{M}, (x, \overline{y}) \in \mathcal{L}_1, (y, \overline{z}) \in \mathcal{L}_2\}.
\]
**Proof.** We will use the analytic stationary phase theorem ([22], Théorème 2.8) to study, given $(x, z) \in \widetilde{M}$, the integral
\[
I_{k}^{V, \Phi_1}(a_1) \circ I_{k}^{V, \Phi_2}(a_2) : (x, z) \mapsto k^{2d} \int_{W(x, z)} h_0(\Phi_1(x, \overline{y}) \otimes \Phi_2(y, \overline{z})) k s(x, y, k^{-1}) s(y, z, k^{-1}) a_1(x, \overline{y}, k^{-1}) a_2(y, \overline{z}, k^{-1}) dv(y).
\]
Here and in the rest of the proof

$$W(x,\bar{v}) = \{ y \in M, (x, \bar{y}) \in V, (y, \bar{v}) \in V \}$$

and $v$ is a volume form.

Let us first study the case where $\Phi_1 = \Phi_2 = \Psi$ and $x = z$. The real-analytic function

$$y \mapsto h_0(\Psi(x, \bar{y}) \otimes \Psi(y, \bar{v}))$$

is non-zero; it is equal to 1 at $y = x$ where its derivative vanishes. Its modulus is strictly less than 1 elsewhere, and as $y$ tends to $x$,

$$1 - |h_0(\Psi(x, y) \otimes \Psi(y, x))| \sim \text{dist}(x, y)^2.$$ 

Therefore the conditions to apply analytic stationary phase are met: without any change of contour, we have an integral of the form

$$\int_U e^{k\phi(y)}u(y;k^{-1})dy$$

where $U$ is a relatively compact open set, $u$ is a classical analytic symbol on $U$, $\phi$ is real-analytic on $U$, $\text{Re}(\phi) \geq 0$ everywhere with equality only at $y = y_0$ which is a critical point of $\phi$ where $\phi(y_0) = 0$ and a non-degenerate maximum point of $\text{Re}(\phi)$; all of this data has real-analytic dependence on a parameter $x$.

Modulo contour deformations and computations of the value of the phase at the critical point, all the properties above are stable under small analytic deformation of the phase $\phi$, and in particular we may deform $(\Psi, \Psi)$ into closeby analytic sections $(\Phi_1, \Phi_2)$ and move $(x, z)$ to a small neighbourhood of the diagonal. Hence, given general $\Phi_1, \Phi_2, x, z$, there exists a contour in $\tilde{M}$ homotopic to $W(x, z)$ (with boundary fixed), on which the function

$$y \mapsto h_0(\Phi_1(x, \bar{y}) \otimes \Phi_2(y, \bar{v}))$$

is such that its modulus reaches a unique maximum at a point $y_0$ in a non-degenerate way; moreover $y_0$ is a critical point for the function.

Applying this contour deformation and [32], Théorème 2.8, we obtain, if $(x, z)$ is close to the diagonal,

$$I^V_kV\Phi_1(a_1) \circ I^V_kV\Phi_2(a_2)(x, \bar{v}) = k^d \Phi(x, \bar{v}) \otimes b(x, \bar{v}, k^{-1}).$$

Here, $b$ is the realisation of a classical analytic symbol, which continuously depends on $a, \Phi_1, \Phi_2$ for some analytic symbol topologies.

Since the starting integral is $(J, -J)$-holomorphic with respect to $(x, \bar{v})$, so are $\Phi$ and $b$; moreover $\Phi(x, \bar{v})$ is close to $\Psi(x, \bar{v}) = 1$ therefore $\Phi$ is close to $\Psi$ everywhere.

If $x$ and $z$ are too far away from the diagonal, then $I^V_kV\Phi_1(a_1) \circ I^V_kV\Phi_2(a_2)(x, \bar{v})$ is small anyway; indeed we’ve already seen that, in a fixed size neighbourhood of the diagonal,

$$|h_0(\Psi(x, \bar{y}) \otimes \Psi(y, \bar{v})))| = |\Psi(x, \bar{y})(\Psi(y, \bar{v})|_{h_0} \leq 1 - \text{dist}(x, z)^2/2$$

and therefore, under the hypotheses of the claim, if $\text{dist}(x, z)^2 > 4\delta$, then

$$|\Psi(x, \bar{y})|_{h_0} \leq 1 - \text{dist}(x, z)^2/4.$$ 

Therefore (up to reducing the allowed value of $\delta$) the manifold $M \times \overline{M}$ decomposes into a neighbourhood $V'$ of the diagonal, where one can apply the stationary phase argument above, and its complement set, where $I^V_kV\Phi_1(a_1) \circ I^V_kV\Phi_2(a_2)(x, \bar{v})$ is exponentially small.

It remains to prove the stated identity between the Lagrangians associated with $\Phi_1, \Phi_2, \Phi$. 

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Let us first prove that
\[ \mathcal{L}_1 \circ \mathcal{L}_2 \subseteq \mathcal{L}. \]
To this end, let \((\tilde{x}, \tilde{y}, \tilde{z}) \in \tilde{M}\), close to each other, and such that
\[ \nabla \Phi_1(\tilde{x}, \tilde{y}) = 0 \quad \text{and} \quad \nabla \Phi_2(\tilde{y}, \tilde{z}) = 0. \]
The holomorphic extension of
\[ M^3 \ni (x, y, z) \mapsto h_0(\Phi_1(x, y) \otimes \Phi_2(y, z)) \]
is
\[ \tilde{M}^3 \ni (x, \tilde{y}, y, z, \tilde{z}) \mapsto \tilde{h}_0(\tilde{\Phi}_1(x, y) \otimes \tilde{\Phi}_2(y, z)). \]
Therefore (because \(\nabla\tilde{V}\) is Hermitian for \(\tilde{h}_0\)) at the considered point one has
\[ \nabla \tilde{z} \tilde{h}_0(\tilde{\Phi}_1(x, y) \otimes \tilde{\Phi}_2(y, z)] = 0 \]
and
\[ d_\tilde{y} \tilde{h}_0(\tilde{\Phi}_1(x, y) \otimes \tilde{\Phi}_2(y, z)] = 0 \]
\[ \nabla \tilde{z} \tilde{h}_0(\tilde{\Phi}_1(x, y) \otimes \tilde{\Phi}_2(y, z)] = 0. \]
In particular, \(\Phi(x, \tilde{z}) = \tilde{h}_0(\tilde{\Phi}_1(x, y) \otimes \tilde{\Phi}_2(y, z))\) (because of the second equation), and then, at this point,
\[ \nabla \Phi(x, \tilde{z}) = 0. \]
We now argue that \(\mathcal{L}_1 \circ \mathcal{L}_2 = \mathcal{L}\). Because \(\Phi_1, \Phi_2, \Phi\) are close to \(\Psi\), the manifolds \(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}\) are close to the diagonal. In particular, in \((\tilde{M} \times \tilde{M})^2\), the manifolds \(\mathcal{L}_1 \times \mathcal{L}_2\) and \(\{(x, y, y, z), (x, y, z) \in \tilde{M}^3\}\) are transverse. Therefore \(\mathcal{L}_1 \circ \mathcal{L}_2\) and \(\mathcal{L}\) have the same dimension, and both are close (in analytic regularity) to the diagonal. Therefore the inclusion \(\mathcal{L}_1 \circ \mathcal{L}_2 \subseteq \mathcal{L}\) is an equality. \(\square\)

One can not only multiply analytic Fourier Integral Operators but also invert them, as long as their principal symbols are bounded away from zero.

**Proposition 3.5.** Let \(V\) be a neighborhood of the diagonal in \(\tilde{M}\). Let \(\Psi\) denote the phase of the Bergman kernel in Proposition 2.13. For every \(r, R, m, c, 0, c > 0\), there exists \(\delta > 0, c > 0, C, r', R', m'\) and a neighborhood \(V'\) of the diagonal in \(\tilde{M}\) such that the following is true.

For every holomorphic section \(\Phi_1\) of \(L \boxtimes L\) over \(V\) such that
\[ \|\Phi_1/\Psi - 1\|_{H^0_r(V)} < \delta \]
there exists a holomorphic section \(\Phi_2\) of \(L \boxtimes L\) over \(V'\) such that
\[ \|\Phi_2/\Psi - 1\|_{H^0_{r'}(V')} < \epsilon \]
(and in this topology the dependence on \(\Phi_1\) is continuous) and for every \(a_1 \in S^{r,R}_m(V)\) holomorphic whose principal symbol \(a_{1,0}\) satisfies \(|a_{1,0}| > c_0\), there exists \(a_2 \in S^{r',R'}_{m'}\) depending continuously on \(a_1\) and \(\Phi_1\), such that
\[ \left\| I_k^{V, \Phi_1(a_1)} I_k^{V', \Phi_2(a_2)} - \Pi_k \right\|_{L^2 \rightarrow L^2} \leq C e^{-ck}. \]
Proof. Let \( \mathcal{L} \) be the canonical relation of \( \Phi_1 \). Let

\[
\mathcal{L}^{-1} = \{ (x, \overline{y}) \in \widetilde{M} \times \widetilde{M}, (y, \overline{x}) \in \mathcal{L} \}.
\]

Then \( \mathcal{L} \circ \mathcal{L}^{-1} \) is the diagonal of \( \widetilde{M} \times \widetilde{M} \) (indeed the diagonal is clearly included in \( \mathcal{L} \circ \mathcal{L}^{-1} \), and since \( \mathcal{L} \) and \( \mathcal{L}^{-1} \) are close to the diagonal, they are both graphs, and then so is \( \mathcal{L} \circ \mathcal{L}^{-1} \).

Since \( \mathcal{L} \) is Bohr-Sommerfeld and the involution \( (x, \overline{y}) \mapsto (y, \overline{x}) \) preserves \( \widetilde{M}_0 \), then \( \mathcal{L}^{-1} \) is also Bohr-Sommerfeld, so that it corresponds to a holomorphic section \( \Phi_2 \) over a neighbourhood \( V_1 \) of the diagonal of \( M \times M \), by Proposition 2.11. In some analytic norms, \( \Phi_2 \) depends continuously on \( \mathcal{L}^{-1} \), which depends continuously on \( \mathcal{L} \), which depends continuously on \( \Phi_1 \).

By Proposition 3.3 there exists an analytic symbol \( b \) and a neighbourhood \( V_2 \) of the diagonal such that

\[
I^V_{k, \Phi_1}(a_1) \circ I^V_{k, \Phi_2}(1) = I^V_{k, \Psi}(b).
\]

We now recall from Proposition 2.11 that \( I^V_{k, \Psi}(b) \) is of the form \( T^\text{cov}(r) \) where \( r \) is an analytic symbol. Moreover, an examination of the proof of Proposition 3.3 shows that \( r \) is obtained from \( a_1 \) by stationary phase, and therefore its principal symbol \( r_0 \) is bounded away from 0 if the principal symbol of \( a_1 \) is bounded away from 0.

From there, one can apply Proposition 2.11 again to obtain that there exists an analytic symbol \( q \) such that \( T^\text{cov}(r) \circ T^\text{cov}(q) = \Pi_k \) up to an exponentially small error.

Now, we apply Proposition 3.3 one last time to

\[
I^V_{k, \Phi_2}(1) \circ T^\text{cov}(q)
\]

which is an analytic Fourier Integral Operator with phase \( \Phi_2 \), since \( \mathcal{L}^{-1} \circ \text{diag} = \mathcal{L}^{-1} \).

The notion of Lagrangian associated with a phase also allows us to find the Fubini-Study metric associated with the phase \( \Phi \) of a Fourier integral operator. The Fubini-Study metric associated with \( \Phi \) corresponds to the Kähler potential \( \phi \) such that, in the setting of Proposition 3.3 \( \Phi' \) is the phase of the Bergman projector associated with \( \phi \). Solving this equation directly is probably possible using Nash-Moser type arguments (especially since we understand the differential by Proposition 3.3), but it turns out that \( \phi \) has a rather direct geometric interpretation from the Lagrangian \( \Lambda \) of \( \Phi \).

**Proposition 3.6.** Let \( V \) be a neighbourhood of the diagonal in \( M \times \overline{M} \). Let \( \Phi \) be a section of \( L \otimes L \) over \( V \) which is close to the section \( \Psi \) of the Bergman kernel and which is self-adjoint, in the sense that for every \( (x, y) \in M^2 \),

\[
\overline{\Phi(x, y)} = \Phi(y, \overline{x}).
\]

Then the Lagrangian \( \Lambda = \{ \nabla \Phi = 0 \} \) is close to \( \{(x, y, x, y), (x, y) \in \widetilde{M} \} \) and invariant under the symmetry \( (x, z, w, y) \mapsto (y, w, z, x) \). In particular, the intersection of \( \Lambda \) with the “1=4 diagonal” \( \{ y = x \} \) is of the form \( \{(x, \kappa(x), \kappa(x), x), x \in M \} \), for some \( J \)-holomorphic diffeomorphism \( \kappa \).

Let \( \phi \) be a Kähler potential corresponding to the initial Kähler structure pulled back by \( \kappa \) (unique up to an additive constant). Then, with the notations of Proposition 3.3 \( \Phi' \) is the phase of the Bergman projector associated with \( \phi + C_0 \) for some \( C_0 \in \mathbb{R} \). In particular, for every analytic symbol \( a \) holomorphic on \( V \) such that \( a(x, x; k^{-1}) \in \mathbb{R} \) and \( \lambda_0(x, x) > 0 \) for every \( x \in M \), the operator \( I^V_k(\Phi)(a) \) belongs to \( S^{+\infty} \) and

\[
\text{FS}_k(I^V_k(\Phi)(a)) = \phi + C_0 + O_C(\kappa^{-1}).
\]

**Proof.** Recall the notations of Proposition 3.3 but switch the roles played by \( \omega_0 \) and \( \omega_\phi \), so as to obtain, for every holomorphic section \( \Phi_1 \) close to \( \Psi_0 \), a holomorphic section \( \Phi_2 \) close to \( \Psi_0 \) such that

\[
\langle u, I^V_k(\Phi_2)(a) v \rangle_{\text{Hilb}_k(0)} \approx \langle u, I^V_k(\Phi_1)(b) v \rangle_{\text{Hilb}_k(\phi)}.
\]
Suppose now that \( \Phi_1 \) is the phase of the Bergman kernel for \( \phi \). Then the section \( \Phi_2 \) is self-adjoint; moreover, in a Hermitian chart associated with a local Kähler potential \( \psi_0 \), its holomorphic extension to \( \tilde{M} \) reads as
\[
\Phi_2(x, \overline{x}, y, \overline{y}) = \exp \left[ -\frac{\psi_0(x, \overline{x})}{2} + \psi_0(x, \overline{x}) - \psi_0(z, \overline{z}) - \phi(z, \overline{z}) + \psi_0(z, \overline{y}) - \frac{\psi_0(y, \overline{y})}{2} \right],
\]
where the point \((z, \overline{z}) \in \tilde{M} \) is determined by \((x, \overline{y})\) as the unique critical point of the expression above with respect to \((z, \overline{z})\). Consequently, in this chart,
\[
\begin{align*}
\nabla_x \log \Phi_2 &= \partial \psi_0(x, \overline{x}) - \partial \psi_0(x, \overline{x}) = \partial \psi_0(z, \overline{z}) - \partial \psi_0(x, \overline{x}) \\
\nabla_y \log \Phi_2 &= \overline{\partial} \psi_0(y, \overline{y}) - \overline{\partial} \psi_0(y, \overline{y}) = \overline{\partial} \psi_0(x, \overline{z}) - \overline{\partial} \psi_0(y, \overline{y}).
\end{align*}
\]
The first identity on each line yields \( \Lambda_{\Phi_2} = \{(x, z, \overline{z}, \overline{y})\} \), and the second yields the claimed link between \( \Lambda_{\Phi_2} \) and the pulled-back metric.

To conclude, given \( \Phi \) close to \( \Psi \) and self-adjoint, there exists a Kähler potential \( \psi \) such that, in the notations of Proposition 3.3, the Lagrangian of \( \Phi' \) is the identity. This means that \( \Phi' \) coincides with the phase \( \Psi_\phi \) of the appropriate Bergman kernel up to a multiplicative factor, so that
\[
\langle u, e^{tV_{\Phi'}(b)}v \rangle_{\text{Hilb}_k(\phi)} = e^{kC_0(u, T_{k}^{\text{cov}, \phi}(b)v)_{\text{Hilb}_k(\phi)}}. \tag{24}
\]
The sesquilinear form associated with \( T_k^{\Phi}(\phi) \) is symmetric if and only if \( a \) is real-valued on the diagonal (that is, if and only if \( a(x, y) = a(y, x) \)). In turn, this holds if and only if \( b \) is real-valued on the diagonal. By the above considerations and analytic stationary phase, the principal symbols of \( a \) and \( b \) are related by a formula of the form \( b_0(z, \overline{z}) = v(z) a_0(x, x) \), where \( z \) is the critical point above and \( v : \tilde{M} \rightarrow \mathbb{C}^* \). Since \( a \) is real if and only if \( b \) is real, \( v \) is real-valued, and then since \( v = 1 \) when \( \phi = 0 \) and \( \mathcal{H} \) is connected, we obtain that \( v > 0 \). This allows to conclude: \( a \) is real with \( a_0 > 0 \), if and only if the corresponding sesquilinear form is definite positive, and then, by [21] and Remark 2.14, the proof is complete. \( \square \)

## 4 Approximate geodesics

It turns out that all the objects on \( B_k \) considered in the introduction are analytic FIOs close to identity on \((M, J, \omega_0)\), provided the associated geometric data is close to \( \omega_0 \). Using the techniques developed in Section 3, we can then prove the main claims.

### 4.1 Geodesics as Fourier Integral Operators

**Proposition 4.1.** Let \( \phi \in \mathcal{H} \) be analytic and close to 0. Then \( \text{Hilb}_k(\phi) \) is an analytic Fourier Integral Operator close to identity, up to \( O(e^{-ck}) \).

**Proof.** The proof consists simply in reverting the roles played by \( \omega_0 \) and \( \omega_\phi \) in Proposition 3.3 and applying this result to the Bergman kernel \( \Pi_k^\phi \), known to be a Fourier Integral Operator for \( \omega_\phi \) by Proposition 2.11. \( \square \)

We now turn to what is in fact the technical core of this article, namely that Bergman rays (i.e. imaginary time Schrödinger propagators) are, in short time, analytic FIOs close to identity.

**Proposition 4.2.** Let \( f \in C^\infty(M, \mathbb{R}) \). Then, for \( t \in \mathbb{C} \) small, \( e^{ikt_k^{\text{cov}}(f)} \) is an analytic Fourier Integral Operator close to identity.

The proof of this Proposition occupies the rest of Subsection 4.1.

We begin by identifying the phase of the FIO. To begin with, let \( \kappa_t \) be the time flow of the Hamiltonian \( \mathcal{H} \) on \( \tilde{M} \). In short times, the graph \( \mathcal{L}(t) \) of \( \kappa_t \) is a complex Lagrangian, close to identity.
Proposition 4.3. For $t$ small, the Lagrangian $\mathcal{L}(t)$ is Bohr-Sommerfeld.

Proof. Since $\mathcal{L}(t)$ is a graph, loops in $\mathcal{L}(t)$ are of the form $\{((\gamma(s), s) \in [0, 1])$ for some loop $\gamma : [0, 1] \to \tilde{L}$. The Bohr-Sommerfeld index of this curve is then the Bohr-Sommerfeld index of $\gamma$ minus that of $\kappa_t(\gamma)$. By Stokes’ theorem, this difference is the integral of $\tilde{\omega}$ over a surface with boundaries $\gamma$ and $\kappa_t(\gamma).

To prove that this integral is zero, we remark that it is equal to 0 when $t = 0$, and moreover the derivative of this integral with respect to $t$ is $\int_{\kappa_t(\gamma)} tX_j \omega = i \int_{\kappa_t(\gamma)} df = 0$.

Moreover, (in short time, and restricted to the vicinity of $M$) it satisfies the group morphism property

$$\mathcal{L}(t) \circ \mathcal{L}(s) = \mathcal{L}(t + s).$$

By Proposition 2.9, the family of Lagrangians corresponds to a family of holomorphic sections $\Phi(t)$, with $\Phi(0) = \Psi$. Then, by Proposition 4.4, composition of an analytic FIO with phase $\Phi(t)$ and one with phase $\Phi(t + s)$, up to an exponentially small error, and provided that $t$ and $s$ are small.

Let us introduce a neighborhood $V$ of the diagonal in $\tilde{\mathcal{M}}$ and a first candidate for the propagator:

$$U_0(t) = I_k^{V, b(t)}(1).$$

The fact that the canonical relation of $U_0(t)$ is the flow of $i\partial_t$ translates into the following identity.

Lemma 4.4. There exists $c > 0$ and, for $t$ small, there exists an analytic symbol $g(t)$ (with holomorphic dependence on $t$) such that

$$U_0(-t) \left( \frac{\partial U_0(t)}{\partial t} - T_k^{\text{cov}}(f)U_0(t) \right) = T_k^{\text{cov}}(g(t)) + O(e^{-ck}).$$

Proof. By Proposition 3.4, both $k^{-1}U_0(t)\frac{\partial U_0(t)}{\partial t}$ and $T_k^{\text{cov}}(f)U_0(t)$ are analytic covariant Toeplitz operators, whose symbol has holomorphic dependence on $t$. One only has to show that their principal symbols are equal; however, this is true for imaginary time $t$, where $\mathcal{L}(t)$ is the complexification of a real Lagrangian. Indeed, in this setting, one has a parametrix for the propagator at any order whose phase is exactly $\phi(t)$. Since the principal symbols agree for imaginary time, and have holomorphic dependence on $t$, they agree everywhere. This concludes the proof.

Since $\mathcal{L}(t) \circ \mathcal{L}(-t)$ is the diagonal, Proposition 3.4 gives

$$U_0(-t)U_0(t) = T_k^{\text{cov}}(b(t)) + O(e^{-ck}),$$

for some analytic symbol $b(t)$ with nonvanishing principal symbol with analytic dependence on $t$, the same constant $c > 0$ as in Lemma 4.4 and all $t$ small enough.

For $t$ small enough, one has also in operator norm, by Proposition 3.2,

$$\|U_0(t)\| + \|(U_0(t))^{-1}\| \leq C e^{\tilde{t}k}$$

and

$$\|e^{tkT_k(f)}\| \leq C e^{\tilde{t}k}.$$
At $t = 0$, one has of course $e^{-tkT_k(f)}U_0(t) = T_k^{\text{cov}}(1) + O(e^{-ck})$. Uniformly for $t$ close to 0, $F(t)$ belongs to some analytic class $S_m^{2r,2R}(V')$ for some neighborhood $V'$ of the diagonal.

By Proposition 2.11 we are able to apply the Picard–Lindelöf theorem to the following (linear) Cauchy problem:

$$a'(t) = a(t) \mathcal{F}(t) \quad a(0) = 1 \quad a(t) \in S_m^{2r,2R}(V'),$$

where $\mathcal{F}$ denotes the symbol product of covariant Berezin–Toeplitz operators. There exists a unique solution $a(t)$ to this Cauchy problem, and one has, for some $c' > 0$,

$$\frac{\partial}{\partial t} T_k^{\text{cov}}(a(t)) = T_k^{\text{cov}}(a(t))T_k(F(t)) + O(e^{-c'k}).$$

The true solution of the equation

$$A'(t) = A(t)T_k^{\text{cov}}(F(t)) \quad A(0) = \Pi_k$$

is uniformly bounded (in operator norm), along with its inverse, as $k \to +\infty$. Thus, by the Duhamel formula, one has both

$$T_k^{\text{cov}}(a(t)) = A(t) + O(e^{-c'k})$$

$$e^{-tkT_k(f)}U_0(t) = A(t) + O(e^{-c'k})$$

and consequently, for small times,

$$e^{tkT_k(f)} = (T_k^{\text{cov}}(a(t)))^{-1}U_0(t) + O(e^{-\frac{c'}{2}k})$$

$$= I_k^{V,\psi(t)}(b(t)) + O(e^{-\frac{c'}{2}k})$$

where we applied again Propositions 3.3 and 3.5. This concludes the proof of Proposition 4.2.

**Remark 4.5.** The ability to find an analytic symbol in the propagator for Proposition 4.2 relies on an application of the Picard-Lindelöf theorem on a space of analytic symbols; it is essential that multiplication by a more regular symbol leaves invariant an analytic class.

### 4.2 Geodesic equations

To conclude the proof of Theorem 3.1, it remains to show that the phases of the Fourier Integral Operators appearing in Propositions 4.1 and 4.2 are identical whenever $\phi(t)$ is a real-analytic Mabuchi geodesic with $\phi(0) = 0$ and $f = \dot{\phi}(0)$.

**Proposition 4.6.** Let $V$ be a neighbourhood of the diagonal in $M \times \overline{M}$. Let $\phi(0) \in C^\omega(M)$. Let $\phi(t)$ be the Mabuchi geodesic with initial value $(0, \phi(0))$ and let $\Psi(t), a(t)$ be respectively a holomorphic section of $L \otimes \overline{L}$ over $V$ and a holomorphic analytic symbol on $V$ such that, as in Proposition 4.1,

$$\text{Hilb}_k(\phi(t)) = I_k^{V,\Psi(t)}(a(t)) + O(e^{-ck})$$

for some $c > 0$, uniformly for $t$ in a neighbourhood of 0.

Let also $\Psi(t), b(t)$ be respectively a holomorphic section of $L \otimes \overline{L}$ over $V$ and a holomorphic analytic symbol on $V$ such that, as in Proposition 4.2,

$$\exp(tkT_k^{\text{cov}}(-\dot{\phi}(0))) = I_k^{V,\Phi(t)}(b(t)) + O(e^{-ck})$$

for some $c > 0$, uniformly for $t$ in a neighbourhood of 0.

Then $\Phi = \Psi$. 

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Proof. Letting \( A(t) = \exp(tkT_k^{\text{cov}}(-\dot{\phi}(0))) \), then \( t \mapsto A(t) \) is the representation as Hilb\(_k\)(0)-self-adjoint operators of a geodesic \( \gamma_k(t) \) in \( B_k \). Representants \( A^{\psi_{\text{ref}}}(t) \) of this geodesic at any base point Hilb\(_k\)(ψ_{\text{ref}}) satisfy

\[
A^{\psi_{\text{ref}}}(t) = A^{\psi_{\text{ref}}}(t)(A^{\psi_{\text{ref}}}(t))^{-1} A^{\psi_{\text{ref}}}(t).
\]  

(25)

By Proposition 3.13 and 4.4 if \( \psi_{\text{ref}} \) is real-analytic and close to 0 then \( A^{\psi_{\text{ref}}} \) and its time derivatives are analytic Fourier Integral Operators close to identity.

Moreover, for any \( t \in \mathbb{R} \) close to 0, since \( \exp(tkT_k^{\text{cov}}(-\dot{\phi}(0))) \) is self-adjoint, then \( \Phi(t) \) is self-adjoint, meaning that

\[
\Phi(t,x,y) = \Phi(t,y,x).
\]

Following Proposition 3.6 there exists a path of elements of \( \mathcal{H} \), denoted \( t \mapsto \phi_1(t) \), such that, for every \( t \),

\[
A^{\phi_1(t)}(t) = T_k^{\text{cov},\phi_1(t)}(g(t))
\]

for some real-analytic symbol \( g(t) \). More generally, one can write

\[
A^{\phi(T)}(t) = I_k^{\phi(T)}(g(t,T))
\]

where \( \Phi^T \) is the phase of the Bergman kernel associated with \( \phi_1(T) \) and \( g \) is a real-analytic symbol.

By successive differentiation, we obtain

\[
\frac{d}{dt}A^{\phi(T)}(t)|_{t=T} = kT_k^{\text{cov},\phi_1(T)}(\frac{d\log\Phi^T}{dt}|_{t=T} + k^{-1}\frac{dg}{dt}|_{t=T})
\]

\[
\frac{d^2}{dt^2}A^{\phi(T)}(t)|_{t=T} = k^2T_k^{\text{cov},\phi_1(T)}([\frac{d\log\Phi^T}{dt}|_{t=T}]^2 + k^{-1}\frac{d^2\log\Phi^T}{dt^2}|_{t=T} + 2k^{-1}\frac{d\log\Phi^T}{dt}|_{t=T} \frac{dg}{dt}|_{t=T} + k^{-2}\frac{d^2g}{dt^2}|_{t=T}).
\]

Applying now the geodesics equation (25) and the subprincipal calculus identity (17), we obtain that

\[
\frac{d^2}{dt^2}\phi_1(t)|_{t=T} = -\frac{d^2\phi_1(T)}{dt^2} - 2\frac{d\phi_1(T)}{dt} \frac{dg}{dt}|_{t=T} + k^2T_k^{\text{cov},\phi_1(T)}([\frac{d\log\Phi^T}{dt}|_{t=T}]^2 + k^{-1}\frac{d^2\log\Phi^T}{dt^2}|_{t=T} + 2k^{-1}\frac{d\log\Phi^T}{dt}|_{t=T} \frac{dg}{dt}|_{t=T} + k^{-2}\frac{d^2g}{dt^2}|_{t=T}).
\]

Applying (20), and Proposition 2.11 there holds \( \frac{d\log\Phi^T}{dt}|_{t=T} = -\dot{\phi}_1(t) \), and then by (21),

\[
\frac{d^2\phi_1}{dt^2} = -\frac{d^2\phi_1(T)}{dt^2} + 2\frac{d\phi_1(T)}{dt} \frac{dg}{dt}|_{t=T}.
\]

Thus \( \phi_1 \) satisfies the equation

\[
\ddot{\phi}_1(t) = \|\dot{\phi}_1\|_{\phi_1}
\]

which is exactly the Mabuchi geodesic equation (2). At \( t = 0 \) one has \( \frac{d\phi_1}{dt} = kT_k^{\text{cov}}(-\dot{\phi}(0)) \) so that \( \phi_1(0) = \dot{\phi}(0) \). By the Cauchy-Kovalevskaya theorem, one can conclude that \( \phi_1 = \phi \), and therefore by Proposition 4.3 that \( \Psi(t) = \Phi(t) \). \( \square \)

We are now also in position to complete the proof of Theorem 1.

Proposition 4.7. Given a real-analytic Kähler manifold \((M,J,\omega_0)\) and given \( r > 0, C_0 > 0, m \in \mathbb{R} \), there exists \( \epsilon > 0 \) and \( C_1 > 0 \) such that, for all \((\phi_1, \phi_1) \in TH \) such that

\[
\|\phi_1\|_{H(r,m)} + \|\dot{\phi}_1\|_{H(r,m)} \leq C_0
\]

the Mabuchi geodesic equation with initial conditions \((\phi_1, \dot{\phi}_1)\) has a solution \( \phi(t) \) as a real-analytic curve on times \((-\epsilon, \epsilon)\). Moreover, for every \( k \in \mathbb{N} \), the geodesic \( \gamma_k \) on \( B_k \) whose initial condition is \((\text{Hilb}_k(\phi_1), \text{dHilb}_k(\phi_1))\) satisfies, for all \( t \in (-\epsilon, \epsilon) \),

\[
\text{dist}_{B_k}(\gamma_k(t), \text{Hilb}_k(\phi(t))) \leq C_1k^{\frac{d}{2}}
\]

and

\[
\text{dist}_H(\text{FS}_k(\gamma_k(t)), \phi(t)) \leq C_1k^{-1}.
\]

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Proof. Let \( \phi(t) \) be a geodesic in Mabuchi space, and let \( H(t) \) be the geodesic associated with the projected initial value problem. Then there exists \( \Phi \) and two analytic symbols \( a \) and \( b \) such that

\[
\text{Hilb}_k(\phi(t)) : (x, y) \mapsto k^d \Phi^k(x, y)a(t, x, y; k^{-1}) + O(e^{-ck})
\]

\[
H(t) : (x, y) \mapsto k^d \Phi^k(x, y)b(t, x, y; k^{-1}) + O(e^{-ck}).
\]

In particular, by Proposition 3.4

\[
\text{Hilb}_k(\phi(t))(H(t))^{-1} = T_k^{\text{cov}}(r) + O(e^{-ck}),
\]

where \( r \) is an analytic symbol. Hence

\[
dist_{B_k}(\text{Hilb}_k(\phi(t)), H(t)) = dist_{B_k}(I, T_k^{\text{cov}}(r) + O(e^{-ck})) = O(k^\frac{1}{2}).
\]

This concludes the proof of (11).

We now turn to the proof of (12). By Proposition 2.13

\[
\text{FS}_k(\text{Hilb}_k(\phi(t))) = \phi(t) + O_{C^\infty}(k^{-1}).
\]

Moreover, since \( H(t) \) is a Fourier Integral Operator with the same phase, by Proposition 3.6 its FS\(_k\) is the same up to \( O_{C^\infty}(k^{-1}) \). Noticing that \( \|\phi_1 - \phi_2\|_{C^2} \geq \text{dist}_H(\phi_1, \phi_2) \), the proof is complete. \( \Box \)

5 No analytic geodesic between analytic endpoints

To conclude this article, we argue for the non-existence of a solution to the boundary value problem in real-analytic regularity, that is, Conjecture 1

Let \( E, E' \) be two analytic function spaces. Let \( (M, J, \phi_0) \) be a Kähler metric in \( E \). Let \( \phi_1 \in \mathcal{H} \cap E \) and suppose that \( \phi_2 \) is joined to \( \phi_0 \) by an analytic geodesic \( \phi_t \) in \( E' \) such that \( \dot{\phi}_t \in E' \).

There exists \( \epsilon > 0 \) such that, for all \( t_0 \in [0, 1] \) and all \( f \in B_{E'}(0, \epsilon) \), the conclusion of Theorem A applies up to time 1 for the Cauchy problem with initial data \((\phi_t, f)\). Thus, there are well-defined complex Lagrangians \( L_{t_0}(t - t_0) \) corresponding to the change from \( \phi_{t_0} \) to \( \phi_t \), for all \( |t - t_0| < \frac{\epsilon}{2} \), where \( R = \sup_{t \in [0, 1]} \|\phi_t\|_{E'} \).

The imaginary time Lagrangians \( L_{t_0}(i\tau) \) can be extended to \( t \in \mathbb{R} \); they are the graphs of the flow of the time-independent Hamiltonian \( \phi_{t_0} \). In this respect, they form a closed set with empty interior amongst the graph of flows of time-dependent Hamiltonians in \( E' \) (this classical result goes as follows: graphs of flows of time-dependent Hamiltonians generically intersect each other cleanly, so that their periodic points form a set of dimension 0; however periodic points of \( L_{t_0}(i\tau) \) correspond to closed orbits of \( \phi_{t_0} \), which form a set of dimension 1).

Close to \( \phi_{t_0} \) in the norm of \( E' \), the map which to a Kähler potential associates its Lagrangian should be a well-defined diffeomorphism. Hence, there exists \( \delta > 0 \) such that, for all \( t_0 \), once \( \phi_{t_0} \) is fixed, the functions \( \phi_{t_0 + \tau} \), for \( |\tau| < 2\delta \), belong to a closed set with empty interior, sitting in the totally real submanifold of Kähler potentials whose Lagrangians (with respect to \( \phi_{t_0} \)) are real.

One should be able to conclude that the functions \( \phi_{t_0 + \tau} \), for \( |\tau| < 2\delta \), belong to a closed set with empty interior among real-valued Kähler potentials. To prove this, the missing link is a continuous map, near \( \phi_{t_0} \), between analytic real-valued Kähler potentials and Kähler potentials whose Lagrangians are real, and which sends solutions of the Mabuchi geodesic equation to autonomous flows. A candidate for such a map, using Berezin–Toeplitz quantization, is \( \lim_{k \to +\infty} \text{FS}_k \circ \mathcal{J} \circ \text{Hilb}_k \), where \( \mathcal{J} : SL(d_k, \mathbb{C})/SU(d_k, \mathbb{C}) \to SU(d_k, \mathbb{C}) \) is the exponentiation of the multiplication by \( i \) between tangent spaces at \( \text{Hilb}_k(\phi(t_0)) \). By Theorem A this map is indeed well-defined at the end-point of an analytic geodesic, but the behaviour of the limit for general closely analytic potentials is not clear.

Non-genericity for short times would be enough to prove the conjecture: if we can prove progressively that the family \( (\phi_{\delta}, \phi_{2\delta}, \ldots, \phi_{|\delta^{-1}j\delta| \delta}, \phi_1) \), belongs to a closed set with empty interior, then \( \phi_1 \) itself satisfies this property.
6 Acknowledgements

This work was supported by the National Science Foundation under Grant No. DMS-1440140 while A. Deleporte was in residence at the Mathematical Sciences Research Institute in Berkeley, California, during the Fall 2019 semester, by the Centre National de la Recherche Scientifique under a PEPS JCJC grant during 2021, as well as by the NSF grant no. DMS-1810747 for the whole duration of this project.

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