Thermodynamics of interacting holographic dark energy

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August 22, 2014

Abstract

The thermodynamics of a scheme of dark matter-dark energy interaction is studied considering a holographic model for the dark energy in a flat Friedmann-Lemaître-Robertson-Walker background. We obtain a total entropy rate for a general horizon and study the Generalized Second Law of Thermodynamics. In addition, we study two horizons related to the Ricci and Ricci-like model and its effect on an interacting system.

1 Introduction

The observational data from the cosmic microwave background [1] and observations of type Ia Supernova [2, 3], among other data, strongly suggest that the universe is in a phase of late accelerated expansion. The most accepted interpretation in the context of Einstein’s General Relativity is that this phase is driven by an unknown component called dark energy [4, 5], which would account for around 73% of the total energy density in the universe today. This component is usually described as a fluid with negative pressure. Various theoretical models have been proposed to account for it. One such model, the holographic dark energy model [6], is based on the application of the holographic principle to cosmology. According to [7], the energy contained in a region of size $L$ must not exceed the mass of a black hole of the same size, which means, in terms of energy density, $\rho \leq L^{-2}$. If in a cosmological context it is considered $H = L^{-1}$, where $H$ is the Hubble parameter, the dark energy density behaves as $\rho_{DE} \sim H^2$, where $\rho_{DE}$ is the dark energy density, giving a model of holographic nature for this density. Based on this idea, M. Li [8, 9] proposed the model $\rho_{DE} = 3c^2H^2$, known as Holographic Dark Energy (HDE). Later on, the authors in [10] proposed that $\rho_{DE} \sim R$, where $R = 6\left(2H^2 + \dot{H} + k/a^2\right)$ is the Ricci scalar, model referred to as Ricci Holographic Dark Energy (RHDE). In reference [11], inspired by RHDE for a spatially flat
section, the model with \( \rho_{\text{DE}} = 3 \left( \alpha H^2 + \beta \dot{H} \right) \) was then proposed, called Ricci-like, where \( \alpha \) and \( \beta \) are constants. Models with interaction between dark energy and dark matter have been studied to explain the evolution of the Universe and the cosmic coincidence problem. The interaction is modeled by a phenomenological function \( Q \) and these models describe energy transfer between two components which are not conserved separately. Interacting models based on this generalization have been studied by [12] for Modified Holographic Ricci Dark Energy (MHRDE). Recently, a work on HDE was published focusing on sign interaction for a Ricci-like model [13].

The possibility that a change of sign in energy transfer during cosmic evolution has recently been proposed by [14], where without choosing an interaction, used an approach in which such interaction is partitioned and constant in each partition. Fitting with observational data they found that the functional crosses the non-interacting line. Following this line of research, the authors [15] proposed a variable phenomenological interaction. The sign change appears at early times when it is fitted with observational data.

There are more examples of this type of sign-change interaction in the literature, see for instance [16, 17, 18]. There are studies in which the interaction is proportional to the deceleration parameter, with the objective of obtaining a change of sign in the interaction when the universe transits from a non-accelerated phase to an accelerated one [19]. There are no thermodynamics studies to the author’s knowledge however, there are studies of HDE model with thermodynamics for a positive interaction using the Generalized Second Law (GSL). Moreover, several horizon have been tested to account for some basic model universe requirement, such as late acceleration, cosmic coincidence problem and GSL, and introduce interaction to modify those results.

In [20] the authors propose a thermodynamical description of the interaction between HDE and dark matter, and obtain an expression for the cosmological interaction from thermodynamical considerations, for the future event horizon, by obtaining a relation with the logarithmic entropy correction.

On this basis, the authors of [21] generalize a thermodynamical interpretation of interacting HDE for a non-flat Universe, and obtain a relation for the interaction in terms of the Hubble horizon; they review it later in [22]. How the interacting function behaves considering thermodynamical factors is of interest given the unknown nature of cosmological interactions.

In [23] the author reviews GSL thermodynamics of interacting HDE with the apparent horizon and express the total entropy change as a function of a ratio between energy densities for three types of Anzatzes for the interaction. The author also asks whether thermodynamics in an accelerating universe can reveal some properties of dark energy itself. In [24] the authors study GSL thermodynamics for interacting dark energy in a non-flat FRW universe enclosed by the dynamical apparent horizon, and expressing the total change in entropy as a function of variables that can be estimated today they conclude that the GSL is respected for the present time. Later on, the author in [25] generalizes this to three fluids but considering an event horizon. By expressing the total change in entropy as a function of variables that can also be measured today the author concludes that the GSL is not respected for the present time for an event horizon. In [26] the authors test thermodynamical laws considering a total energy density to obtain the change in entropy and then using three different parametrization for the variable EoS associated with HDE. A graphical analysis is performed, given that analytic expressions are not conclusive, and for one specific range one of the models respects GSL.

In [27] the authors study GSL thermodynamics at the apparent and event horizons in FRW; in the latter they establish an upper limit for the dark energy EoS parameter. Obtaining limits for this parameter from thermodynamics criteria will be investigated further in our work.
In [28] the authors study thermodynamics in Generalized Holographic and Ricci Dark Energy models at the apparent horizon, particle horizon and event horizon. They use graphical investigation due to the complexity of the expressions and find that the GSL cannot be satisfied at these horizon for this model. In [29] they study HDE focusing on the State Finder Parameters with several horizons, the interaction is given and the IR cut off for HDE is chosen as Ricci’s length scale or radius of the future event horizon. In [30] the authors consider an interacting model of RHDE and GSL for the event horizon.

We follow the work of [31] which describes the thermodynamics of three interacting fluids and the Generalized Second thermodynamic Law (GSL) for the apparent horizon. Based on this work, we focus in particular on sign change interactions in order to study the conditions that are compatible with GSL. The present study was developed in the framework of a flat Friedmann-Lemaître-Robertson-Walker (FLRW) cosmology with two interacting fluids, dark energy and pressureless dark matter. The main goal of this work is to examine the validity of GSL for an interacting holographic context, considering the holography Ansatz related to the horizon itself and study its effect on the sign-change cosmological interaction.

The novelty of our work resides in the fact that in previous studies the interaction is generally given and positive, an assumption that we will not make. Furthermore, to the authors knowledge there are no previous publications considering the HDE Ansatz as an horizon to discern the nature of the cosmological interaction.

This article is organized as follows: Section 2 describes the scheme of interacting fluids and their thermodynamics. Section 3 studies the dark energy density related to a length scale $L$ and later develops some models in detail using specific horizon for HDE models. Finally, Section 4 is devoted to discussion of our results and final remarks.

## 2 Interaction and the Generalized Second Law

We begin our analysis considering a flat model with two fluids\(^1\), $\rho_{DE}$, the dark energy component, $p_{DE}$ the dark energy pressure, $\rho_{DM}$, the dark matter component with $p_{DM}$ its pressure. The field equations, considering natural units throughout this work, are

\[
3H^2 = \rho_{DE} + \rho_{DM},
\]

\[
\dot{H} = -\frac{1}{2} (\rho_{DM} + \rho_{DE} + p_{DE} + p_{DM}),
\]

where $H = \dot{a}/a$ is the Hubble parameter, $a$ is the scale factor and $\dot{a}$ denotes derivative of $a$ with respect to cosmic time $t$. This notation will be used throughout the paper. The total energy density $\rho = \rho_{DM} + \rho_{DE}$ is conserved overall and its equation of conservation is given as

\[
\dot{\rho}_{DE} + \dot{\rho}_{DM} + 3H (\rho_{DE} + \rho_{DM} + p_{DM} + p_{DE}) = 0.
\]

Assuming that in our model, dark energy interacts with dark matter through a phenomenological coupling function denoted by $Q$, the conservation equation for each constituent fluid is given as

\[
\dot{\rho}_{DE} + 3H (\rho_{DE} + p_{DE}) = -Q,
\]

\[
\dot{\rho}_{DM} + 3H (\rho_{DM} + p_{DM}) = Q.
\]

\(^1\)We neglect other components given that we are interested in late Universe dynamics
Several models have been considered in the literature where $Q$ is a function of the energy densities and the Hubble parameter $H$, usually considering a specific function that allows equations (4) and (5) to be integrated straightforwardly. In the present study, we do not assume this coupling term a priori, but rather study the system through the field equations and remain as general as possible. We consider the coincidence parameter as an auxiliary variable, defined as the ratio between the two energy densities

$$r \equiv \frac{\rho_{DM}}{\rho_{DE}}$$

This parameter is often used to address the cosmic coincidence problem: Why are energy densities of the same order of magnitude today? The interaction function in terms of this parameter can be obtained replacing $\rho_{DE} = \frac{3H^2}{1+r^r}$ in its equation of conservation (1), thus obtaining

$$Q \equiv \frac{3H^2}{1+r^r} \left[ r' - r(r+1)\frac{p_{DE}}{H^2} \right],$$

which depends only on $(H, r, p_{DE}, r')$, for the pressureless dark matter scenario; the prime in $r'$ denotes a derivative of log $a$. Given that the dark energy pressure is negative and that the term $r'$ is negative at late times for a positive decreasing $r$ function, which would alleviate cosmic coincidence problem, this interaction would be negative, as noted by [13] for a barotropic EoS in a holographic context. This configuration will be studied considering thermodynamics and the possibility that the cosmological interaction could be a slightly negative function at some point during the evolution of the Universe.

To examine the thermodynamic behavior of a cosmological scenario, the universe must be considered as a thermodynamical system. The Friedmann equations themselves arise from the first law of thermodynamics, as shown by [34]. We consider our interacting dark sector universe as a thermodynamical system with the horizon as its boundary. This follows the work of [31], where the laws of thermodynamics are valid and the universe is interpreted as a state in thermodynamical equilibrium. We will examine whether the sum of the entropy enclosed by the horizon and the entropy of the horizon itself, denoted as $S_{tot}$, is not a decreasing function of time. This principle is known as the Generalized Second Law (GSL) of Thermodynamics and is expressed through this inequality $\dot{S}_{tot} \geq 0$. After equilibrium establishes, all the fluids in the universe acquire the same temperature $T_h$ (from here on the subscript $h$ denotes quantities associated with the horizon).

The first law of thermodynamics for the matter content is written $T dS = P dV + dE$, and therefore the entropy variation of the fluid inside the horizon reads

$$T d(S_{DE} + S_{DM}) = (p_{DE} + p_{DM}) dV + d(E_{DE} + E_{DM})$$

where $(S_{DE}, S_{DM})$ are the entropies of the dark components and $(E_{DE}, E_{DM})$ their associated energies. The volume of the system $V = 4\pi L^3 / 3$ is bounded by the radius $L$ and thus its differential form is $dV = 4\pi L^2 dL$. In thermal equilibration, all the constituent fluids of the universe have the same temperature $T$, while their energy and pressure can, in general, be different. Dividing by $dt$ we obtain

$$T(\dot{S}_{DE} + \dot{S}_{DM}) = (p_{DE} + p_{DM}) 4\pi L^2 \dot{L} + \dot{E}_{DE} + \dot{E}_{DM}.$$
The energy associated with each fluid is defined in relation to its energy density and the length scale as

\[ E_{DE} = \frac{4\pi}{3} L^3 \rho_{DE}, \quad E_{DM} = \frac{4\pi}{3} L^3 \rho_{DM}. \]  

(10)

If we consider the derivative of these energies into Eq. (9), then we obtain

\[ \dot{S}_{DE} + \dot{S}_{DM} = \frac{4\pi}{T} L^2 \dot{L} (p_{DE} + p_{DM} + \rho_{DM} + \rho_{DE}) + \frac{4\pi}{3T} L^3 (\dot{\rho}_{DM} + \dot{\rho}_{DE}). \]  

(11)

From the total conservation equation (3) we can obtain the sum of the derivatives of the energy densities

\[ \dot{S}_{DE} + \dot{S}_{DM} = -\frac{8\pi L^2}{T} \left( \dot{L} - H\dot{L} \right) \dot{H}, \]  

(12)

introducing \( \dot{H} \) from Eq. (2). The discussion of negative entropy or negative temperature has been addressed elsewhere [31] and has been proposed as a reason to consider the GSL.

At this stage, we have to connect the temperature of the fluids \( T \), which is equal to that of the horizon \( T_h \). The temperature of the horizon is given as

\[ T_h = \frac{1}{2\pi L}. \]  

(13)

according to [31] and references therein. In this work we observe that if the temperature decreases, as we would expect in an expanding model, the length scale \( L \) will increase. The entropy associated with the horizon \( S_h \) is

\[ S_h = 8\pi^2 L^2. \]  

(14)

In order to obtain the total variation in the total entropy we add the entropies change rate (11) with the derivative with respect to cosmic time of the entropy (14)

\[ \dot{S}_{tot} = \dot{S}_{DE} + \dot{S}_{DM} + \dot{S}_h = 16\pi^2 L \dot{L} - 16\pi^2 L^3 \left( \dot{L} - H\dot{L} \right) \dot{H}. \]  

(15)

considering the definition (13) for the temperature of the sources inside the horizon, which is in equilibrium with the temperature associated within the horizon for late times. Given that the length scale \( L \) is positive we can simplify Eq. (15) and thus obtain the total entropy change of the interacting system as

\[ \frac{\dot{S}_{tot}}{16\pi^2 L} = \left( 1 - H\dot{L}^2 \right) \left( \dot{L} - H\dot{L} \right) + H\dot{L}. \]  

(16)

The sign of this expression depends only on \((H, L, \dot{H}, \ddot{L})\) and is valid regardless of the horizon chosen. The interacting term is not present nor has played any role in the calculations preceding this result. Eq. (16) is equivalent to other results where the horizon (and Ansatz) were selected before testing the GSL: in [27] for apparent and event horizons; in [26] for the event horizon; and then in [28] for a generalized RHDE model, also for both horizons.

On the other hand, using Eq. (2) for \( \dot{H} \) into Eq. (16), we obtain

\[ \frac{\dot{S}_{tot}}{8\pi^2 L^3 H^2} = \left( 3 + \frac{p}{H^2} \right) \left( \dot{L} - H\dot{L} \right) + \frac{2\dot{L}}{H^2 L^2}, \]  

(17)
which depends on \((H, L, p, \dot{L})\), where \(\dot{L}\) and \(p\) could determine the sign of the entropy change. If we consider that the horizon is not a decreasing function, as expected from Eq. (13), its derivative is positive and the range where GSL is respective will depend on the total pressure and how negative it can be. According to the length scale we can obtain a limit to dark energy pressure by requiring the right side of Eq. (17) to be positive. We will consider some specific scenarios and their generalization in the following section.

### 3 Horizons and Dark Energy

In a flat FRW model, if the horizon is related to the length scale \(L\), the dark energy as proposed by [8, 9] is written as

\[
\rho_{DE} = \frac{3c^2}{L^2},
\]

where \(c\) is a positive constant that fulfill \(c^2 < 1\) [35]. In this context, we can express the coincidence parameter \((6)\) as a function of the length scale and the Hubble parameter

\[
r = \frac{3H^2 - \rho_{DE}}{\rho_{DE}} = \frac{H^2L^2}{c^2} - 1.
\]

In order to test the validity of GSL in general, we attempt to avoid unknown function such as the interaction in the analysis; thus we obtain the length scale from (19) in terms of \(r\) and hence its derivative. Then we focus on one expression from (16)

\[
\left(\dot{L} - HL\right) = \frac{c^2}{LH} \left[\frac{r'}{2} + q(1 + r)\right]
\]

where we substitute \(\dot{H}\) in terms of the deceleration parameter \(q \equiv -\left(1 + \frac{\dot{H}}{H^2}\right)\), whose sign we would like to be negative at late times. If \(q < 0\) and, for a decreasing coincidence parameter, \(r' < 0\), we conclude that the parenthesis \((\dot{L} - HL)\) in (20) is negative at late times. This will describe a universe with late acceleration and a coincidence parameter that decreases during its evolution. Comparing this with Eq. (12) we notice that the sign of the change of entropy of the matter content will depend only on \(\dot{H}\). Substituting this in Eq. (16) we obtain

\[
\frac{H\dot{S}_{\text{tot}}}{16\pi^2c^2} = \left[1 + c^2(1 + r)(q + 1)\right] \left[\frac{r'}{2} + (1 + r)q\right] + (1 + r).
\]

If we require GSL in (21), the range for the variables \((r, q, r')\) is

\[
- c^2(q + 1) > \frac{1}{1 + r} + \frac{1}{r'/2 + (1 + r)q}.
\]

The upper or lower limit of the holographic parameter \(c^2\) will depend on the deceleration parameter and whether \(q > -1\) or \(q < -1\).

Turning to (16), we could also derivative the expression (18), then use (4) for \(\dot{\rho}_{DE}\). Deriving (18)
again, but for $\rho_{DE}$ results in an expression that can be used to obtain $\dot{L}$ as a function of the interaction

$$
\dot{L} = \frac{HL}{2} \left[ 3 + \frac{L^2}{c^2} \left( p_{DE} + \frac{Q}{3H} \right) \right],
$$

(23)

where the total energy density was obtained from Eq. (1). Replacing this in Eq. (17) for a pressureless dark matter component we obtain

$$
\frac{\dot{S}_{tot}}{4\pi^2 H L^2} = 4 + \left( 2 + 3H^2L^2 + p_{DE}L^2 \right) \left( 1 + \frac{L^2}{3c^2} \left( 3p_{DE} + \frac{Q}{H} \right) \right),
$$

(24)

which depends only on $(H, L, p_{DE}, Q)$, with no explicit time derivative. In addition, we consider the coincidence parameter (19) into Eq. (24) and we rewrite the total entropy change in terms of $r$ as

$$
\frac{\dot{S}_{tot}}{4\pi^2 H L^2} = 4 + \left[ (3 + \tilde{p}_{DE}) c^2 (1 + r) + 2 \right] \left[ 1 + (1 + r) \left( \tilde{p}_{DE} + \tilde{Q} \right) \right],
$$

(25)

a function of $(r, \tilde{p}_{DE}, \tilde{Q})$, using $\tilde{p}_{DE} = \frac{p_{DE}}{H^2}$ and $\tilde{Q} = \frac{Q}{3H^3}$. If GSL is required, the right side of equation (25) must be positive, which leads to inequalities in terms of the interaction according to the sign of each individual term. Considering the sign of expression (20) into this equation we already know that the second square parenthesis in (25) is negative, then $\tilde{p}_{DE} + \tilde{Q} < -1/(1 + r)$. Introducing two auxiliary parameters

$$
Q_* = -\tilde{p}_{DE} - \frac{1}{(1 + r)}, \quad Q_* = -\frac{4}{(1 + r) \left( (3 + \tilde{p}_{DE}) c^2 (1 + r) + 2 \right)},
$$

(26)

whose sign will be dependent on the pressure. For instance, for barotropic pressures the sign of parameter $Q_*$ will depend only on the EoS of $p_{DE}$. This can be seen considering $p_{DE} = \omega_{DE} \rho_{DE}$ in Eq. (26), where we obtain $Q_* = -(1 + 3\omega_{DE})/(1 + r)$, whose sign will depend on whether $\omega_{DE}$ is greater or less than $-1/3$. Then, using the dynamical parameters from (26), we can rewrite Eq. (25) as

$$
\frac{\dot{S}_{tot}}{8\pi^2 H L^2} = -\frac{4}{Q_*} \left( \tilde{Q} - Q_* - Q_* \right),
$$

(27)

where $\dot{L} - HL \sim \tilde{Q} - Q_*$ is a negative quantity. It should be noted that if quantity $Q_*$ is negative, then the interaction between the fluids in the dark sector $\tilde{Q}$ has to be negative also in order for the model to respect GSL and alleviate cosmic coincidence.

If GSL is required, the right side of equation (27) must be positive, which leads to inequalities in terms of the interaction $\tilde{Q}$ where the horizon does not play any role. Considering the sign of expression (20) into this equation we already know that $(\tilde{Q} - Q_*)$ is negative. Thus, all possible remaining scenarios are:

- If $Q_* > 0$, which is to say $(3 + \tilde{p}_{DE}) c^2 (1 + r) + 2 < 0$, then we obtain
  $$
  \tilde{Q} < Q_* < Q_* + Q_*,
  $$

(28)

an upper limit on the interaction $\tilde{Q}$. On the other hand if $Q_* < 0$, which is to say $(3 + \tilde{p}_{DE}) c^2 (1 + r) + 2 > 0$, then we obtain

$$
Q_* > \tilde{Q} > Q_* + Q_*,
$$

(29)
an upper and lower limit to the interacting function. These inequalities do not depend on the length scale, given that $Q_*$ and $Q_\ast$ defined in (26) do not.

As an application, we consider a specific horizon related to holographic dark energy (18) and study the implications described above for when the GSL is satisfied.

The Hubble horizon in the FRW flat universe is given by

$$L_H = H^{-1}. \quad (30)$$

which is the same as the apparent horizon for a flat configuration. Considering the Hubble Horizon as the length scale and the dark energy in relation to the Hubble parameter for this case, $\rho_{DE} = 3c^2H^2$, in Eq. (16), we obtain

$$H \frac{\dot{S}_{tot}}{16\pi^2} = \left( H^{-2} \dot{H} \right)^2 = \left( \frac{\dot{\rho}_{DE}}{6c^2H^3} \right)^2 = \left[ \frac{-3H(\rho_{DE} + p_{DE}) - Q}{6c^2H^3} \right]^2, \quad (31)$$

where the interaction plays no role in the sign of the total entropy change and $\dot{S}_{tot}$ is always defined positive regardless of the number of components. Thus the GSL of thermodynamics is fulfilled in a region enclosed by the Hubble horizon throughout the evolution of the Universe. This result is a particular case of the work of [21] for a flat model.

For the horizon (30) the cosmic coincidence parameter (6) is given as $r = \frac{1}{c^2} - 1$, a positive constant and a scaling regime, considering that $c^2 < 1$ [35]. This HDE therefore does not alleviate the cosmic coincidence problem. It has been suggested that a variable $c$ might solve this and other issues, see for instance [36]. For completeness, we calculate the deceleration parameter for this horizon which is given as

$$q = \frac{1}{2c^2} \left( c^2 + \tilde{p}_{DE} + \tilde{Q} \right). \quad (32)$$

This parameter will be negative, thus representing a cross from a non-accelerated to an accelerated Universe, if the dark energy pressure is sufficiently negative. The parameters defined in (26) for this horizon are

$$Q_* = -\tilde{p}_{DE} - c^2, \quad Q_\ast = -\frac{4c^2}{(5 + \tilde{p}_{DE})}. \quad (33)$$

The cosmological event horizon $L_E$ is defined as

$$L_E = a \int_{t}^{\infty} \frac{dt}{a} = a \int_{t}^{\infty} \frac{da}{Ha^2}, \quad (34)$$

therefore the derivative of the event horizon with respect to time simplifies as

$$\dot{L}_E = HL_E - 1. \quad (35)$$

Considering horizon (34) in Eq. (16), we obtain

$$\frac{\dot{S}_{tot}}{16\pi^2L_E} = \dot{H}L_E^2 + HL_E - 1. \quad (36)$$
This expression depends on \((H, L_E, \dot{H})\) and its sign will be determined by the term \(\dot{H}\). Expressing the total entropy change from Eq. (36) in terms of the coincidence parameter defined in (6), which is a positive evolving function, we obtain

\[
\frac{\dot{S}_{\text{tot}}}{16\pi^2 L_E} = 2c^2(1 + r)^2 r' - c^2(1 + r) - 1.
\] (37)

This expression depends on \((r, r')\); if \(r\) is a decreasing function over late times, then \(r'\) is a negative function and the GSL is not respected. No interaction or variable EoS could have change that result. This result coincides with the work of authors in [25] for their chosen Ansatz and it was briefly mentioned at the end of the reference [31].

Now, we consider a linear combination of the Hubble horizon \(L_H\) and the Event Horizon \(L_E\) assumed by [37] while investigating thermodynamics and the phantom barrier. This is given as

\[
L = \alpha L_H + \beta L_E,
\] (38)

where \(L_H\) and \(L_E\) are defined in (30) and (34) respectively, and \(\alpha\) and \(\beta\) are both positive constants \(^2\). The derivative of (38) with respect to time simplifies as

\[
\dot{L} = HL + \alpha - \beta.
\] (39)

This expression can be positive if \(\alpha > \beta\), thus describing an increasing horizon in an expanding universe, while it could be negative for some time if \(\alpha < \beta\). Comparing it with (20) we conclude that it must be \(\alpha < \beta\).

Considering this horizon as the length scale and expressing the total entropy change from Eq. (36) in terms of the coincidence parameter defined in (6) and its derivative, we can obtain \(\dot{H}\) in terms of \((r, r')\) and replacing this term into (16), we get the entropy change

\[
\frac{\dot{S}_{\text{tot}}}{16\pi^2 L} = \left[1 - c^2 r' + c^2(r + 1) + c(\alpha - \beta)\sqrt{r + 1}\right](\alpha - \beta) + c\sqrt{r + 1}
\] (40)

an expression that depends only on \((r, r')\); if \(r\) is a decreasing function over late times, then \(r'\) is a negative function and the GSL can be respected depending on the relation between \(\alpha\) and \(\beta\). Considering the analysis below (39), \(\alpha < \beta\), in which case the GSL is conditioned by the evolution of the parameters.

Henceforth, we consider in Eq. (18) the constant \(c\) is included in the respective constants for each horizon. Now, the Ricci dark energy density is

\[
\rho_{\text{DE}} = 3\alpha(2H^2 + \dot{H}),
\] (41)

where \(\alpha\) is a positive constant as proposed by [10]. This formulation considering an Ansatz on the dark energy density is equivalent to an Ansatz for the dark energy pressure given as

\[
p_{\text{DE}} = H^2 - \frac{2}{3\alpha}\rho_{\text{DE}},
\] (42)

\(^2\)This parameters should not be confused with the parameters in the Ricci-like Ansatz, since they are given in a different context.
+ for pressureless dark matter component $p_{DM} = 0$. In \cite{38} there is an expression for the pressure as a function of $p = p \left( H^2, \dot{H} \right)$. This association differs from the results from the previous section and is a consequence of the $\dot{H}$ term within the Ansatz and therefore present in the length scale.

Using the pressure (42) and the energy density (18), we can obtain $\dot{H}$ from (2). Therefore in this model the deceleration parameter is

$$q = 1 - \frac{1}{\alpha L^2 H^2} = 1 - \frac{1}{\alpha (1 + r)},$$

(43)

which is negative for $0 < \alpha (1 + r) < 1$ and is implicitly related with the coincidence parameter that can be written as

$$\alpha (r + 1) (1 - q) = 1.$$  

(44)

Given that $\alpha$ and $r$ are positive, $\frac{1}{\alpha (1 + r)} > 0$, then $q < 1$ for this scenario, in accord with late acceleration. Furthermore, if we replace the dark energy pressure (42) in (23) we obtain

$$\dot{L} = \frac{H L}{2} \left( 3 - \frac{2}{3 \alpha} + H^2 L^2 + \frac{L^2}{3 H} Q \right).$$  

(45)

This differs from the previous section because $\dot{L}$ has a direct dependence on the interaction, which was not the case in (30), (35) and (39). Replacing $\dot{L}$ into Eq. (16) results in

$$\frac{\dot{S}_{\text{tot}}}{8 \pi^2 H L^2} = 2 + \left( 1 - \frac{1}{\alpha} + 2 H^2 L^2 \right) \left( 1 - \frac{2}{3 \alpha} + H^2 L^2 \left( 1 + \frac{Q}{3 H^3} \right) \right)$$

(46)

where the sign of this expression depends only on $(H, L, Q)$. This result resembles \cite{28} for its non-interacting model explicit in the $z$ parameter.

We can also study the change of entropy (46) as a function of the coincidence parameter $r$. Replacing the coincidence parameter (6) in Eq. (46) we obtain

$$\frac{\dot{S}_{\text{tot}}}{8 \pi^2 H L^2} = \left( 3 - \frac{1}{\alpha} + 2 r \right) \left[ 1 - \frac{2}{3 \alpha} + (1 + r) \left( 1 + \tilde{Q} \right) \right] + 2.$$  

(47)

This equation depends on $(r, \tilde{Q})$. The second square parenthesis is negative, therefore the interaction has the following upper limit $\tilde{Q} < -1 - \frac{1}{1+r} + \frac{2}{3 \alpha (1+r)}$. Eq. (47) implies that the interaction must fulfill one of the ranges presented in (28) or (29) and $\alpha$ and $r$ are bounded, which is summarized as $\alpha (1 + r) \gtrless \frac{1}{2} (1 - \alpha)$. Considering the value of the coincidence parameter today as $r_0 = 0.27/0.72 \approx 0.375$, then $\alpha \gtrsim 0.268$. In addition, we are studying an epoch where there is a dominance of dark energy, therefore if $0 < r < 1$, a positive parenthesis would give $\alpha > \frac{1}{3}$ and a negative parenthesis would give $\alpha < \frac{1}{3}$.

Now we consider the system without the interaction in terms of $r$ and $r'$. For instance $\dot{H}$ and the dark energy density can always be written as a function of $(r, H)$. With this we can use the derivation $L^2 = \frac{1 + r}{H^2}$ and obtain $\dot{L}$ written as a function of $(r', r)$. Therefore, Eq. (16) is given as

$$\frac{H \dot{S}_{\text{tot}}}{16 \pi^2} = \left[ 1 - \frac{1}{\alpha} + 2 (1 + r) \right] \left[ \frac{r'}{2} - \frac{1}{\alpha} + (1 + r) \right] + (1 + r),$$

(48)
an expression depending only on \((r, r')\) and equivalent to Eq. (47). In \([10]\) they estimate the value of \(\alpha\) as of the order of 0.46 (for a non-flat model), therefore the right side of equation (48) is positive for \(-0.233 < r' < 0\).

For the pressure (42) the parameters (26) are given as
\[
Q_* = 3 + \frac{1 - \alpha}{\alpha(1 + r)}, \quad Q_\star = \frac{-4}{(1 + r)(\frac{1}{\alpha} + 2)},
\]
where we see that the auxiliary parameters \(Q_*\) and \(Q_\star\) are positive and negative respectively. The latter implies that the inequality (28) is discarded and we are left with only (29), an upper and lower positive limit to the interacting function
\[
\frac{1 + \alpha(2 + 3r)}{\alpha(1 + r)} > \tilde{Q} > \frac{1 + \alpha(4 + 3r) + 6r\alpha^2}{\alpha(1 + r)(1 + 2\alpha)},
\]
therefore this interaction is always positive at late times for Ricci Dark Energy.

The Ricci-like dark energy density is given by
\[
\rho_{DE} = 3(\alpha H^2 + \beta \dot{H}),
\]
where \(\alpha\) and \(\beta\) are both positive constants as proposed by \([11]\) and for non-interacting cosmic fluids both constants are positive and less than one according to the comparison performed by \([39]\). Comparing with (18) we obtain a relation between the horizon and the Hubble parameter and its derivative
\[
L^2 = \left(\alpha H^2 + \beta \dot{H}\right)^{-1}.
\]
This Ansatz (51) has an implicit equivalence with other models for the dark energy pressure described in \([40]\), which can be clearly seen obtaining \(\dot{H}\) from the Ansatz (51) and replacing it into Eq. (2) for a pressureless dark matter configuration, as
\[
p_{DE} = -\frac{2}{3\beta} \rho_{DE} + (2\alpha - 3\beta) \frac{H^2}{\beta} = -\frac{1}{L^2} \left[ \frac{2}{\beta} - (2\alpha - 3\beta) \frac{H^2L^2}{\beta} \right],
\]
This pressure will be negative if \(2 - (2\alpha - 3\beta)H^2L^2 > 0\) and its the generalization of the pressure for RDE (42). The expression for the pressure (53) is not polytropic but resembles inhomogeneous fluid cosmology, where the pressure is dependent not only on the dark energy density but also on the Hubble parameter. This is caused by the \(\dot{H}\) term in (52) as was the case in the Ricci Ansatz. By considering the definition of the deceleration parameter and \(\dot{H}\) from (52) as a function of \((H, L)\), we obtain
\[
q = \frac{1}{\beta} \left( \alpha - \beta - H^2L^2 \right) = \frac{1}{\beta} \left( \alpha - \beta - \frac{1}{1 + r} \right),
\]
whereas if we have an Ansatz for the coincidence parameter \(r(a)\), the deceleration parameter \(q(a)\) can be obtained, and thus we can study whether the model is accelerated in late times. It should be noted that for a constant \(r\), the deceleration parameter \(q\) would also be constant, thus there would be no evolution from a non-accelerated Universe to an accelerated one. By requiring the
deceleration parameter in (54) to be negative, we obtain an inequality \((\alpha - \beta)(r + 1) < 1\).

With the pressure (53) and (18) we can rewrite equation (23) as

\[
\dot{L} = \frac{HL}{2} \left[ 3 - \frac{2}{\beta} + H^2 L^2 \left( 2\alpha + 3\beta + \frac{Q}{3H^2} \right) \right].
\]

(55)

Transferring this into Eq. (16), we obtain an equation that depends only on \((H, L, Q)\). Considering this equation in terms of the coincidence parameter (6) we obtain

\[
\frac{\dot{S}_{\text{tot}}}{8\pi^2 HL^2} = \left[ 1 - \frac{1}{\beta} (1 - \alpha(1 + r)) \right] \left[ 1 - \frac{2}{\beta} + (1 + r) \left( 2\alpha + 3\beta + \tilde{Q} \right) \right] + 2,
\]

(56)

an expression that depends on \((r, \tilde{Q})\). The sign on the first parenthesis will depend on \(\alpha \gtrless \frac{1-\beta}{1+r}\) and the second parenthesis is negative at late times, independent of the horizon. Considering the first parenthesis negative or positive, and GSL, we obtain the ranges for \(r\) and \(\tilde{Q}\) defined in (28) or (29). This gives an upper limit on the value of the interaction, and in some cases whether is positive or negative.

Given that we do not yet know the sign of the interaction we attempt a different approach in order to obtain an expression independent of \(Q\). Rewriting \(\dot{H}\) in terms of \(r\) and \(r'\) instead of \(Q\) or \(p_{DE}\), the dark energy density (and thus \(L\)) can always be written as a function of \((r, H)\) as

\[
L^2 = 1 + \frac{r}{H^2}
\]

and its derivative is given as

\[
\dot{L} = \frac{1}{2LH^2} \dot{r} - \frac{1}{LH\beta} (1 - \alpha(1 + r)).
\]

(57)

Therefore we obtain

\[
\beta^2 H \frac{\dot{S}_{\text{tot}}}{16\pi^2} = \left[ \beta - (1 - \alpha(r + 1)) \right] \left[ \frac{\beta}{2} r' - 1 + (\alpha - \beta)(1 + r) \right] + \beta^2(r + 1),
\]

(58)

an expression depending only on \((r, r')\). We can observe that studying an Ansatz on \(r\) would immediately allow us to analyze whether this Ansatz respects the GSL, as in the previous cases.

The possibility of sign change in \(Q\) for different stages of evolution is allowed for several ranges. We recall that the interaction can be written in terms of \(p_{DE}\) and \(r\) according to (7); however in this context by choosing a horizon we are inherently choosing a pressure. Equation (7) can therefore be simplified using the pressure (53) where \(p_{DE}\) is a function of \(r\)

\[
\tilde{p}_{DE} = -\frac{1}{\beta} \left[ \frac{2}{(1+r)} - (2\alpha - 3\beta) \right],
\]

(59)

then the interaction (7) can be written as

\[
\tilde{Q} = \frac{\beta \, r' + r \, \left( 2 - (2\alpha - 3\beta)(1 + r) \right)}{\beta(1+r)^2},
\]

(60)

a function of \((r, r')\). The interaction \(\tilde{Q}\) has been formulated as a function of derivates of \(r\) before in [42] and references therein, but that result differ from ours because the EoS is given as a constant bound to cross the phantom barrier that determines the sign of \(Q\). In our case the sign of \(Q\) remains undetermined so far. Let us note that the interaction is null for a particular Ansatz of
the coincidence parameter \( r = \frac{(-2+\epsilon)}{-\epsilon + \alpha(2-\epsilon)} \), where \( \epsilon = 2\alpha - 3\beta \). In the present paper, the result obtained for the pressure (59) indicates that the evolution of \( r \) could be determined by the evolution of \( p_{DE} \), relation obtained from the holographic context in (51).

Equation (7) can be also simplified, using \( r \) as a function of \( p_{DE} \) from (59)

\[
r + 1 = \frac{2}{(2\alpha - 3\beta) - \beta \tilde{p}_{DE}},
\]

and its derivative in the interaction (7) we obtain

\[
\tilde{Q} = \frac{1}{2} (2\alpha - \beta (3 + \tilde{p}_{DE})) \left( \frac{\beta \tilde{p}_{DE}'}{2} + \tilde{p}_{DE} \right) - \tilde{p}_{DE},
\]

which depends on \((\tilde{p}_{DE}, \tilde{p}_{DE}')\). In the expressions (60) and (62), it can be observed that the sign of the interaction can change according to the choice of Ansatz \((p_{DE} \text{ or } r)\) and the sign of its respective derivatives. It is equivalent to study the expression for interaction described by (60) as it is the interaction as a function of the pressure \( p_{DE} \).

For the pressure (59) the parameters (26) are given as

\[
Q^* = \frac{2 - \beta}{\beta (1+r)} - \frac{2\alpha - 3\beta}{\beta}, \quad Q_* = \frac{-2\beta/(1+r)}{\beta - 1 + \alpha(1+r)},
\]

where its sign will depend on the holography parameter \( \alpha \) and \( \beta \). For the parameter \( Q_* \) to be negative, the coincidence parameter should fulfill \( r > \frac{1-\alpha + \beta}{\alpha - 3/2\beta} \), and the holography constants \( \beta < \frac{2}{3}\alpha \). The sign of this interaction can be negative or positive, depending on

\[
Q_* + Q^* = \frac{2(1 - (1 + r)\alpha)^2 + 3\beta r(1 - \beta - \alpha(1 + r))}{\beta(1 + r)(1 - \beta - \alpha(1 + r))}.
\]

This function is studied in the parameter space of \((\alpha, \beta)\) for \( r_0 = 0.375 \) in Figure 1, where it can be negative in a certain region (black) of this space. This negative interaction region that can respect GSL is introduced due to considering the HDE as the horizon.

4 Discussion

In this paper we studied the effect of GSL in a flat FRW universe with holographic dark energy interacting with non-relativistic dark matter. The interaction \( Q \) is not a phenomenological function given \textit{a priori}, but an unknown variable to be studied according to the thermodynamics of the system in equilibrium conditions. We examined the validity of the GSL of thermodynamics considering the universe as a thermodynamical system bounded by the length scale \( L \). One of the results obtained in this work, Eq. (7), is the possibility of a change in the role of \( Q \) interaction; this equation is independent of the choice of horizon and depends on the coincidence parameter, its derivative and the EoS. Considering this possible negativity of \( Q \), is of interest to analyze whether this interacting model respects the GSL of thermodynamics. When choosing the holographic dark energy density (18), the associated dark energy pressure determines ranges for the interaction function where the GSL is valid. This is equivalent to when the sign change in entropy is determined by \( \dot{L} \) and \( p \). When we applied the GSL of thermodynamics to a specific Horizon and
Figure 1: The graph represents the positive (white) and negative (black) values of the sum of the parameters $Q_\star$ and $Q_\ast$ for $r_0 = 0.375$ in the phase space of $(\alpha, \beta)$.

Dark Energy Models in Eq. (26), we obtain two ranges where the interacting function respects this law; for instance Eq. (28) imposes an upper limit for the interacting function, which allows this function to take positive or negative values, or a transit between the two options and still respect the GSL. An example of the latter behavior can be found in [13] and references therein. Regarding Eqs. (28) and (29), we can conclude that the interaction can take negative values for a negative $Q_\ast$ within a certain range determined by the pressure and coincidence parameter but respect the GSL of thermodynamics independent of the horizon chosen. This also has implications for the EoS of dark energy and the acceleration of the universe; from Eq. (22) it can be seen that the holographic parameter has an upper or lower limit depending on whether the deceleration parameter is greater or less than $-1$ respectively. To examine the validity of the GSL, we considered different horizons. Unlike other works, we used the horizon for the holographic dark energy density as a cosmological horizon. It appears that the variation with respect to the cosmic time of the total entropy can be positive or negative depending on the horizon chosen when this horizon includes a term of $\dot{H}$ that redefines the pressure. These results are verified in Ricci and Ricci-like horizons, where there is an equivalence between the coincidence parameter $r$ and the dark energy pressure $p_{DE}$. For these cases the cosmic coincidence problem is alleviated, given that when the pressure is a variable function, $r$ is also a variable function. For the RDE model, the analysis indicates that the interaction must be positive in order to respect the GSL, however the Ricci-like HDE is allowed to be negative and respect GSL but only for a certain range. The aspects introduced by each model at different phases in the evolution of the Universe will be studied in future research.
Acknowledgments

This work is dedicated to the memory of our recently deceased colleague Sergio del Campo, for his contributions to Cosmology in Chile. This work has been supported by Comisión Nacional de Ciencias y Tecnología through Fondecyt Grants 3130736 (FA). (FP) acknowledges DI14-0007 of Dirección de Investigación y Desarrollo, Universidad de La Frontera. (PC) acknowledges partial financial support from the Master of Physics Program of Universidad de La Frontera. (FA) would like to thank A. Cid and P. Mella for helpful references.

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