Integrable reductions of $Spin(7)$ and $G_2$ invariant self-dual Yang–Mills equations and gravity

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Abstract

There is remarkable relation between self-dual Yang–Mills and self-dual Einstein gravity in four Euclidean dimensions. Motivated by this we investigate the $Spin(7)$ and $G_2$ invariant self-dual Yang–Mills equations in eight and seven Euclidean dimensions and search for their possible analogs in gravitational theories. The reduction of the self-dual Yang–Mills equations to one dimension results into systems of first order differential equations. In particular, the $Spin(7)$-invariant case gives rise to a 7-dimensional system which is completely integrable. The different solutions are classified in terms of algebraic curves and are characterized by the genus of the associated Riemann surfaces. Remarkably, this system arises also in the construction of solutions in gauged supergravities that have an interpretation as continuous distributions of branes in string and M-theory. For the $G_2$ invariant case we perform two distinct reductions, both giving rise to 6-dimensional systems. The first reduction, which is a complex generalization of the 3-dimensional Euler spinning top system, preserves an $SU(2) \times SU(2) \times Z_2$ symmetry and is fully integrable in the particular case where an extra $U(1)$ symmetry exists. The second reduction we employ, generalizes the Halphen system familiar from the dynamics of monopoles. Finally, we analyze massive generalizations and present solitonic solutions interpolating between different degenerate vacua.
1 Introduction

In an important paper almost two decades ago, the self-duality equations for the gauge field strength of Yang–Mills (YM) theories were examined in a general $D$-dimensional Euclidean space \[^{[1]}\]. These authors classified all possible cases for $D = 5, 6, 7, 8$ in terms of the maximal subgroups of the rotation group $SO(D)$ that leave invariant a 4-index totally antisymmetric tensor. The eight-dimensional case with invariance subgroup the $Spin(7)$ of $SO(8)$ is quite distinct in the sense that it generalizes more closely the four-dimensional self-duality. The work of \[^{[1]}\] was based on group theory, but nevertheless, the analogy with four-dimensional self-dual YM equations was pushed further when the eight-dimensional analogue of the four-dimensional instanton solution was found \[^{[2]}\]. This solution can also be embedded in heterotic string theory \[^{[3]}\]: The seven-dimensional case with invariance subgroup the $G_2$ of $SO(7)$ is also quite interesting and can be discussed in parallel with the eight-dimensional $Spin(7)$ case \[^{[1]}\]. Also in this case a seven-dimensional instanton solution was found, as well as its embedding in heterotic string theory \[^{[4]}\] (see also \[^{[5]}\]).

In this paper we reexamine in detail the self-dual YM equations in eight and seven dimensions with the invariance groups $Spin(7)$ and $G_2$, respectively, that we mentioned. In particular, we consider the systems arising from reducing them to one dimension, when all fields depend only on one space variable. Our motivation to perform this kind of reduction stems from the fact that, in a similar setting in four Euclidean dimensions, there is a remarkable relation between self-dual YM and self-dual gravity. We recall that the four-dimensional self-dual YM equations

$$F_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F_{\rho\sigma} ,$$

with the gauge choice $A_4 = 0$ and in the ansatz that the remaining fields $A_i$, $i = 1, 2, 3$ depend only on the variable $x_4 \equiv \tau$, reduce to the Nahm system

$$\frac{dA_i}{d\tau} = \frac{1}{2} \epsilon_{ijk}[A_j, A_k] ,$$

which appeared in the theory of static non-abelian monopoles \[^{[6]}\]. Choosing $SU(2)$ as the gauge group one can parametrize the gauge fields as $A_j = -i \omega_j \sigma_j / 2$ (no sum over $j$), where the $\sigma_i$’s are Pauli matrices and derive the system of equations (see, for instance, \[^{[7]}\])

$$\frac{d\omega_1}{d\tau} = \omega_2 \omega_3 , \quad \text{(and cyclic perms.)} .$$

This is the well known Lagrange system that also coincides with the Euclidean continuation of the three-dimensional Euler top equations describing the free motion of a rigid body with one point fixed. There is a inequivalent parametrization of the gauge field that results instead of \[^{(1.3)}\] to \[^{[6, 8]}\]

$$\frac{d\omega_1}{d\tau} = -\omega_2 \omega_3 + \omega_1 (\omega_2 + \omega_3) , \quad \text{(and cyclic perms.)} ,$$

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which is known as the Halphen system. A well known remarkable fact is that, both of
the above systems also arise in self-dual four-dimensional Einstein gravity with metrics
having an $SO(3)$ isometry. As famous examples we mention, the Eguchi–Hanson metric
[9], described by the system (1.3) and the Taub–NUT and Atiyah–Hitchin metrics [10]
described by (1.4) [11].

Recently, there is quite a bit of interest on special holonomy $Spin(7)$ and $G_2$, metrics
of eight- and seven-dimensional Einstein gravity due to their relevance not only in math-
ematics, but also in physics (see, for instance, [12]-[14]). In practice, the computation
reduces mathematically to solving a system of first order ordinary differential equations
of the type in (1.3) and (1.4), but typically much more complicated since in general they
involve six or seven unknown functions. Although there are a few scattered important
solutions [15]-[19] no systematic study of these systems exists in the literature and finding
new solutions proves a very difficult task. We believe that making connection with the
eight- or seven-dimensional self-dual YM equations is a promising avenue for such a sys-
tematic approach towards their integrability. In fact, there is at least one case where this
correspondence in precise and complete. We will see that for the $Spin(7)$ case the resulting
7-dimensional system coincides with one that arose before in the construction of domain
wall solutions in gravity coupled to scalars theories [20, 21]. These theories are sectors of
gauged supergravities in four, five and seven dimensions and the domain wall solutions,
when viewed from a string or an M-theory point of view, represent the gravitational field
of a large number of continuously distributed $p$-branes. As such this system turns out to
be completely integrable with the help of algebraic curves and the various solutions are
characterized by the genus of associated Riemann surfaces.

The rest of the paper is organized as follows: In section 2 we consider the $Spin(7)$
invariant eight-dimensional self-dual YM equations dimensionally reduced to one space
dimension. We present the general solution in terms of an auxiliary function and reduce
the problem mathematically to the study of a non-linear differential equation this function
obeys. In section 3 we give a number of elementary, albeit non-trivial, examples. In
section 4 we associate the differential equation that we mentioned above with algebraic
curves and the corresponding Riemann surfaces which then are used in order to classify
all inequivalent solutions. We also present our main results in tables and make the precise
connection with solutions of gauged supergravity in various dimensions and their lift to
M- and string theory. In section 5 we consider a massive extension of the $Spin(7)$ self-dual
invariant eight-dimensional self-dual YM equations. We study the vacuum structure of this
theory and show that there are solitonic solutions interpolating between degenerate isolated
vacua. In the simplest case we obtain the usual kink solution in a theory of one scalar self-
interacting with a “mexican hat” potential, but we also exhibit other examples. In section
6 we consider the $G_2$ invariant seven-dimensional self-dual YM equations again reduced to one dimension. We perform two distinct reductions and obtain six-dimensional systems of differential equations that are complex generalizations of the Lagrange and Halphen systems (1.3) and (1.4) above. We provide constants on motion and in one particular case the full solution. We also present a massive generalization based on the analogy we develop with weak $G_2$ holonomy metrics. We end the paper with a few concluding remarks and some feature directions of this work in section 7. We have also written an appendix with some useful properties of the octonionic structure constants and related tensors.

2 8D self-dual YM with $Spin(7)$ invariance

Consider the eight-dimensional self-duality equations [1]

$$F_{a\beta} = \lambda \Psi_{a\beta\gamma\delta} F_{\gamma\delta} ,$$

(2.1)

where $\alpha = 1, 2, \ldots, 8$ and the gauge field strength is $F_{a\beta} = \partial_a A_\beta - \partial_\beta A_a - [A_a, A_\beta]$. The totally antisymmetric 4-index tensor $\Psi_{a\beta\gamma\delta}$ is invariant under the $Spin(7)$ subgroup of the rotational group $SO(8)$. Its components are constructed in terms of the structure constants of the octonionic algebra. Some useful properties of these tensors are collected in the appendix. Note also that, solutions of the self-duality equations (2.1) automatically provide solutions to the equations of motion for the gauge fields, since the latter are reduced to the Bianchi identity due to the antisymmetry of the tensor $\Psi_{a\beta\gamma\delta}$. As already noted in [1], consistency of (2.1) requires either one of the four values

$$\lambda = \frac{\epsilon}{2} , \quad \lambda = -\frac{\epsilon}{6} , \quad \epsilon = \pm 1 .$$

(2.2)

We pick a particular direction, say the eighth, and split the index $\alpha = (a, 8)$, where the $a = 1, 2, \ldots, 7$. We break the 28 independent conditions in (2.1) into

$$F_{a8} = \lambda \psi_{abc} F_{bc} ,$$

(2.3)

representing 7 conditions and

$$F_{ab} = 2\lambda \psi_{abc} F_{c8} + \lambda \psi_{abcd} F_{cd} ,$$

(2.4)

representing the remaining 21 conditions. One can show that the 21 conditions in (2.4) imply the 7 conditions in (2.3) if the parameter $\lambda$ takes either one of the four values in (2.2). However, the 7 conditions in (2.3) imply the 21 conditions in (2.4) only for the value $\lambda = \epsilon/2$.

In the rest of the paper we restrict to the self-dual case with $\lambda = \frac{1}{2}$ (the anti-self-dual case with $\lambda = -\frac{1}{2}$ is recovered trivially) which means to the 21 of $Spin(7)$. Then, solving
the system of 7 equations in (2.3) we automatically provide a solution to (2.1) (for \( \lambda = \frac{1}{2} \)). We next make the gauge choice \( A_8 = 0 \) and we look for solutions that depend only on the eighth coordinate \( x^8 \equiv \tau \). Then (2.3) (for \( \lambda = \frac{1}{2} \)) becomes
\[
d\!A_a = \frac{1}{2} \psi_{abc} [A_b, A_c] , \quad a, b, c = 1, 2, \ldots, 7 .
\] (2.5)
which is the seven-dimensional generalization of the Nahm system (1.2). Writing \( A_a = \psi_i^a \omega_i \) (no sum over \( a \)), where \( \psi_i \) is the adjoint-like representation, defined in (A.11), we obtain the system of 7 coupled non-linear equations
\[
d\omega_a = \frac{1}{2} \psi_{abc}^2 \omega_b \omega_c , \quad a, b, c = 1, 2, \ldots, 7 ,
\] (2.6)
where we have used the properties (A.12) and (A.13). This is the generalization of the three-dimensional Lagrange system (1.3) to seven dimensions and it was first obtained in [22, 23].1 These first order equations imply the second order ones
\[
d^2 \omega_a = \frac{1}{2} \psi_{abc}^2 \psi_{bd}^e \omega_c \omega_d \omega_e .
\] (2.7)
The system (2.6) represents a flow in a seven-dimensional manifold spanned by the \( \omega_a \)’s. Moreover, it is a gradient flow in the sense that there exists a prepotential \( W \) such that the first order system is obtained as
\[
d\omega_a = g_{ab} \frac{\partial W}{\partial \omega_b} , \quad a, b = 1, 2, \ldots, 7 ,
\] (2.8)
for some metric \( g_{ab} \) in the space of the \( \omega_a \)’s. The corresponding second order equations can be derived from the Lagrangian
\[
\mathcal{L} = -\frac{1}{2} g_{ab}^{-1} \dot{\omega}_a \dot{\omega}_b - V ,
\] (2.9)
with the potential given in terms of the prepotential \( W \) as
\[
V = \frac{1}{2} g_{ab} \frac{\partial W}{\partial \omega_a} \frac{\partial W}{\partial \omega_b} .
\] (2.10)
Every solution to the first order system solves the second order Lagrange equations for (2.9). However, the reverse is of course not true. Namely, not every solution to the second order Lagrange equations satisfies the first order system (2.8). In general, a prepotential has a number of critical points for the \( \omega_a \)’s which are found by solving the algebraic system of equations \( \partial W / \partial \omega_a = 0 \). For a positive definite metric \( g_{ab} \) every critical point of the prepotential corresponds to a minimum \( V = 0 \) for the potential. Other extrema the potential itself might have, necessarily correspond to its maxima. In our case the prepotential and the metric are given by
\[
W = \frac{1}{6} \psi_{abc}^2 \omega_a \omega_b \omega_c , \quad g_{ab} = \delta_{ab} .
\] (2.11)

1It is interesting that solutions of (2.6) can be used to construct solutions of the self-dual membrane embedded in 8-dimensions [22].

4
2.1 Solving the first order equations

In order to solve (2.6) it is convenient to make the standard choice for the set of structure constants \( \psi_{abc} \) given by (A.2). Let’s next define a new set of variables as

\[
\Omega_a = M_{ab} \omega_b ,
\]

where the matrix \( M \) and its inverse are given by

\[
M = \begin{pmatrix}
0 & 1 & 1 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 1 \\
\end{pmatrix},
\]

\[
M^{-1} = \frac{1}{4} \begin{pmatrix}
-1 & 1 & -1 & 1 & 1 & -1 & 1 \\
1 & -1 & -1 & 1 & 1 & -1 & 1 \\
1 & 1 & -1 & 1 & 1 & -1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & -1 \\
1 & 1 & 1 & -1 & 1 & 1 & 1 \\
1 & 1 & -1 & 1 & -1 & 1 & 1 \\
1 & -1 & -1 & 1 & -1 & -1 & 1 \\
\end{pmatrix} .
\]

The matrix \( M \) has in each row four non-zero unit elements precisely at the column position corresponding to the indices for which the elements of \( \psi_{abcd} \), given in (A.4), are non-zero.

Then our system of differential equations (2.6) becomes

\[
\frac{d\Omega_a}{d\tau} = \frac{1}{4} \Omega \Omega_a - \Omega_a^2 , \quad a = 1, 2, \ldots, 7 , \quad \Omega \equiv \sum_{b=1}^{7} \Omega_b .
\]

This is a particular case of a more general system which is completely integrable as we will show. Namely, let us consider the \( N \)-dimensional system of first order non-linear differential equations

\[
\frac{d\Omega_a}{d\tau} = \frac{1}{\Delta} \Omega \Omega_a - \Omega_a^2 , \quad a = 1, 2, \ldots, N , \quad \Omega \equiv \sum_{b=1}^{N} \Omega_b ,
\]

where \( \Delta \) is a real constant. This is also a gradient flow in the sense of (2.8)-(2.10) with the \( \Omega_a \)'s replacing \( \omega_a \)'s and the summation extended as \( a, b = 1, 2, \ldots, N \). The prepotential, the metric and its inverse are

\[
W = -\frac{1}{6} \sum_{a=1}^{N} \Omega_a^3 + \frac{1}{4\Delta} \Omega \sum_{a=1}^{N} \Omega_a^2 - \frac{1}{12\Delta^2} \Omega^3 ,
\]

\[
g_{ab} = 2\delta_{ab} + \frac{2}{2\Delta - N} , \quad g_{a}^{-1} = \frac{1}{2}\delta_{ab} - \frac{1}{4\Delta} ,
\]

\[
M_{ac} M_{bc} = 2(\delta_{ab} + 1) ,\]

\[
\frac{1}{6} \psi_{abc} M_{ad} M_{be} M_{cf} = -\frac{1}{6} \delta_{de} \delta_{df} + \frac{1}{48}(\delta_{de} + \delta_{df} + \delta_{ef}) - \frac{1}{192} ,
\]

where, in the second identity we sum over the repeated indices \( a, b, c \) on the left hand side. There is no sum on the right hand side. In addition, we note that the system (2.15) was also derived in [24].

\[\text{[Note added: passing the useful identities]}\]
where we exclude of course the case $N = 2\Delta$ since then the metric $g_{ab}$ has no meaning.

It is remarkable that a system identical to (2.16) arose before in the completely different context of constructing domain wall solutions in gravity coupled to scalars theories, corresponding to sectors of gauged supergravities in four, five and seven dimensions [20, 21]. When lifted to string or M-theory, these solutions have the interpretation as the gravitational field of continuous distributions of $p$-branes. In this analogy, the rôles of the functions $\Omega_a$ is played by exponentials of scalars leaving in the coset $\text{SL}(N, \mathbb{R})/\text{SO}(N)$ which are used to deform the $\text{SO}(N)$ spherical symmetry of the space transverse to the branes in the ten- or eleven-dimensional supergravity solutions. Guided by these previous works we find that, for all values of $N$ and $\Delta$, the most general solution to the system (2.16) is given by

$$
\Omega_a = \frac{f^{1/\Delta}}{F - b_a}, \quad a = 1, 2, \ldots, N, \quad f \equiv \prod_{c=1}^N (F - b_c),
$$

(2.18)

where the function $F(\tau)$ satisfies the differential equation

$$
\left( \frac{dF}{d\tau} \right)^\Delta = f.
$$

(2.19)

The $b_a$’s are the $N$ constants of integration, which, without loss of generality, can be ordered as

$$
b_1 \geq b_2 \geq \cdots \geq b_N.
$$

(2.20)

Hence, the entire problem boils down to solving the differential equation (2.19). In general, this is a difficult task, to which we will shortly turn. We also mention that, a solution to the system (2.13) was also presented in [24] in terms of an auxiliary function satisfying a non-linear differential equation. Presumably it is equivalent to ours for $N = 7$ and $\Delta = 4$ by appropriate transformations and renaming of variables.

Note that, when all the $b_a$’s are equal, then the $\text{SO}(N)$ rotational symmetry in the $N$-dimensional Euclidean space spanned by the $\Omega_a$’s is preserved. This symmetry breaks into smaller subgroups when some of the $b_a$’s differ from each other and it is completely broken in the generic case when all of them are unequal. We point out that the corresponding symmetry in the space of the $\omega_i$’s is usually smaller since the transformation (2.12) doesn’t preserve rotations.

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3For work related to these type of solutions in relation to the Coulomb branch of $\mathcal{N} = 4$ SYM see also [23, 26, 27].

4Specifically, the system (2.16) coincides with the system (3.2) of [21] if the various parameters and functions of [21] are identified as: $z = -\tau$, $g = 1$, $e^{2\beta_i} \rightarrow \Omega_a f^{1/N-1/\Delta}$. Then, the solution in eq. (3.6) of [21] also coincides with (2.18).
It is consistent to set to zero in the system (2.13) some of the \( \Omega_a \)'s and then proceed to solve for the rest, as before. In such cases, the solution is still given by (2.18) and (2.19) with the same value for \( \Delta \), but with less moduli parameters \( b_a \) corresponding to a smaller value for \( N \). It is possible to recover these particular cases from our general solution (2.18) by a limiting procedure. To be concrete, let us consider the rescaling (for the generic case with \( N \neq \Delta + 1 \))

\[
F \to F(-b_N)^{-\frac{1}{N-\Delta-1}}, \quad b_a = b_a(-b_N)^{-\frac{1}{N-\Delta-1}}, \quad a = 1, 2, \ldots, N-1,
\]

and then take the limit \( b_N \to -\infty \). Then, from the solution (2.18) we obtain that \( \Omega_N = 0 \), whereas for \( a = 1, 2, \ldots, N-1 \) the expressions are the same, but with \( N \) replaced by \( N-1 \). Also, after the limit, the differential equation (2.19) does not contain in its right hand side the factor \( (F - b_N) \). This of course is the solution we would have found if we had started from the very beginning with \( \Omega_N = 0 \).

3 Examples

Let us now focus to the case of interest, the 7-dimensional Euler problem, and set the parameters to their values \( N = 7 \) and \( \Delta = 4 \). We’ve seen that the problem boils down mathematically to solving the differential equation (2.19). A more detailed study of (2.19) necessarily involves techniques of algebraic geometry. This task will be undertaken in the following section which can be read independently. In this section we will mention some general features and give a few elementary examples.

The evolution of the function \( F(\tau) \) should be such that the \( \Omega_a \)'s remain real which in turn implies that \( f \geq 0 \). The function \( F(\tau) \) can be bounded by two consecutive constants among the \( b_a \)'s, for instance \( b_2 \leq F \leq b_1 \), or it can be unbounded. In the latter case we have necessarily that \( F \geq b_1 \). For reasons that will become apparent in the discussion of solitonic solution in the massive case later in the paper, it is useful to expose the behaviour of \( F(\tau) \) near the end points. We find that

\[
F = \left( \frac{4/3}{\tau_0 - \tau} \right)^{4/3}, \quad \text{as} \quad \tau \to \tau_0^-,
\]

where \( \tau_0 \) is a constant of integration. This is a universal behaviour since it does not depend on the constants of integration \( b_i \). For \( F \to b_1^+ \) the behaviour depends crucially on the degree of degeneracy of \( b_1 \), which according to the arrangements of parameters in (2.20), is the maximum among the \( b_a \)'s. In general, let \( b_1 = b_2 = \cdots = b_n \), so that the degree of degeneracy of \( b_1 \) is \( n \). Then, for \( n \neq 4 \) there is a power law behaviour

\[
F - b_1 = \left( (1 - n/4)f_0^{1/4}(\tau - \tau_1) \right)^{1/n}, \quad \text{as} \quad \tau \to \begin{cases} -\infty & \text{if} \quad n = 5, 6, 7, \\ \tau_1 & \text{if} \quad n = 1, 2, 3. \end{cases}
\]
where \( f_0 = \prod_{a=n+1}^7 (b_1 - b_a) \) and \( \tau_1 \) is some other constant related to \( \tau_0 \) above. The precise relation requires of course the knowledge of the behaviour of \( F(\tau) \) in the entire \( \tau \)-interval. For \( n = 4 \) the behaviour is exponential

\[
F - b_1 = (\text{const.}) e^{\tau_0/4} , \quad \text{as} \quad \tau \to -\infty .
\] (3.3)

A similar analysis can be performed for the case of bounded motion, where \( b_2 \leq F \leq b_1 \), but will not present it here.

For concreteness we present a few explicit examples.

**Example 1:** In the simplest example all constants of integration \( b_a \) are equal and the \( SO(7) \) symmetry in the space of the \( \Omega_a \)'s remains unbroken. With no loss of generality we choose them as \( b_1 = b_2 = \cdots = b_7 = 0 \). Then

\[
F = \left( \frac{4/3}{\tau_0 - \tau} \right)^{4/3}
\] (3.4)

and

\[
\Omega_a = 4\omega_a = \frac{4/3}{\tau_0 - \tau} , \quad a = 1, 2, \ldots, 7 .
\] (3.5)

The result is in agreement with the universal behaviour (3.1) which in this case is exact for all values of \( \tau \).

**Example 2:** Consider now the case with \( b_1 = b_2 = b_3 = b_4 = 1 \) and \( b_5 = b_6 = b_7 = 0 \). Then the \( SO(7) \) breaks down to the \( SO(4) \times SO(3) \) subgroup and the number of integration constants \( b_a \) which equal the maximum one \( b_1 \), is \( n = 4 \). The function \( F(\tau) \) is determined by solving (2.19). We have two distinct cases depending on the range of \( F \), namely, either \( F \geq 1 \) or \( 0 \leq F \leq 1 \).

For \( F \geq 1 \) the solution is

\[
\ln \left( \frac{F^{1/4} + 1}{F^{1/4} - 1} \right) - 2 \cot^{-1} \left( F^{1/4} \right) = \tau_0 - \tau ,
\] (3.6)

where \( \tau_0 \) is an integration constant. The evolution takes place in the interval \( \tau \in (-\infty, \tau_0) \) and \( F(\tau) \) increases monotonically from 1 to \( +\infty \). We cannot invert (3.6) and obtain \( F(\tau) \) explicitly, except near the end points

\[
F(\tau) = \left\{ \begin{array}{ll}
\left( \frac{4/3}{\tau_0 - \tau} \right)^{4/3} & \text{as} \quad \tau \to \tau_0^- , \\
1 + 8e^{\tau} & \text{as} \quad \tau \to -\infty ,
\end{array} \right.
\] (3.7)

which is in agreement with the general expressions (3.1) and (3.3). Also the \( \Omega_a \)'s in terms of the function \( F(\tau) \) are

\[
\Omega_1 = \Omega_2 = \Omega_3 = \Omega_4 = F^{3/4} , \quad \Omega_5 = \Omega_6 = \Omega_7 = (F - 1)F^{-1/4} .
\] (3.8)
For $0 \leq F \leq 1$ the solution is
\[
\ln \left( \frac{1 + F^{1/4}}{1 - F^{1/4}} \right) + 2 \tan^{-1} \left( F^{1/4} \right) = \tau - \tau_0 ,
\] (3.9)
where, as before, $\tau_0$ is a constant of integration. The evolution now takes place in the interval $\tau \in (\tau_0, \infty)$ and $F(\tau)$ increases monotonically from 0 to 1. The behaviour near the end points is
\[
F(\tau) = \begin{cases} 
\left( \frac{\tau - \tau_0}{4} \right)^4 & \text{as } \tau \to \tau_0^+ , \\
1 - 8e^{-\tau} & \text{as } \tau \to +\infty .
\end{cases}
\] (3.10)

The $\Omega_a$’s in terms of the function $F(\tau)$ are
\[
\Omega_1 = \Omega_2 = \Omega_3 = \Omega_4 = -F^{3/4} , \quad \Omega_5 = \Omega_6 = \Omega_7 = (1 - F)F^{-1/4} .
\] (3.11)

**Example 3:** Consider the case with $b_1 = b_2 = b_3 = 0$ and $b_4 = b_5 = b_6 = b_7 = -1$, where the reality condition forces that $F \geq 0$. Hence the symmetry subgroup of $SO(7)$ that remains unbroken is still $SO(4) \times SO(3)$, but now $n = 3$. We find that
\[
\Omega_1 = \Omega_2 = \Omega_3 = F^{-1/4}(F + 1) , \quad \Omega_4 = \Omega_5 = \Omega_6 = \Omega_7 = F^{-1/4}
\] (3.12)
and
\[
\ln \left( \frac{1 + \sqrt{2}F^{1/4} + F^{1/2}}{1 - \sqrt{2}F^{1/4} + F^{1/2}} \right) + 2 \cot^{-1} \left( \frac{\sqrt{2}F^{1/4}}{F^{1/2} - 1} \right) = \sqrt{2}(\tau - \tau_0) .
\] (3.13)
The evolution occurs at a finite interval for $\tau$ as $F(\tau)$ grows from 0 to $\infty$.

**Example 4:** Consider next the case with $b_1 = b_2 = b_3 = b_4 = b_5 = 1$ and $b_6 = b_7 = 0$, where necessarily $F \geq 1$. Now $SO(7)$ is broken to the $SO(5) \times SO(2)$ subgroup and $n = 5$. The solution for $F(\tau)$ is written in terms of a hypergeometric function $2F_1$ as
\[
F^{-1/2}(F - 1)^{-1/4} - \frac{2}{3}F^{-3/4}2F_1(3/4, 1/4, 7/4, 1/F) = \frac{1}{4}(\tau_0 - \tau) .
\] (3.14)
Its behaviour near the end points is
\[
F(\tau) = \begin{cases} 
1 + (-4/\tau)^4 & \text{as } \tau \to -\infty , \\
\left( \frac{4/3}{\tau_0 - \tau} \right)^{4/3} & \text{as } \tau \to \tau_0^- .
\end{cases}
\] (3.15)
which again is in agreement with the general expressions (3.2) and (3.3). In between it grows monotonically from 1 to $+\infty$. In terms of $F(\tau)$ the expressions for the $\Omega_a$’s are
\[
\Omega_1 = \Omega_2 = \Omega_3 = \Omega_4 = \Omega_5 = (F - 1)^{1/4}F^{1/2} , \quad \Omega_6 = \Omega_7 = (F - 1)^{5/4}F^{-1/2} .
\] (3.16)

**Example 5:** Finally, consider two cases where we start from the beginning with $\Omega_7 = 0$. As we have already explained the general solution is given by (2.18) and (2.19) with $N = 6$.
and has six moduli parameters. First we choose them such that $b_1 = b_2 = b_3 = b_4 = 1$ and $b_5 = b_6 = 0$, so that the subgroup of the maximum symmetry group $SO(6)$ which is preserved is $SO(4) \times SO(2)$ and $n = 4$. We have two cases depending on whether $F \geq 1$ or $0 \leq F \leq 1$. In the former case we find the explicit solution

$$F(\tau) = \coth^2 \frac{\tau}{2}, \quad -\infty < \tau \leq 0,$$

(3.17)

where we have absorbed the integration constant into a redefinition of $\tau$. From that we compute

$$\Omega_1 = \Omega_2 = \Omega_3 = \Omega_4 = F^{1/2} = -\coth \frac{\tau}{2}, \quad \Omega_5 = \Omega_6 = (F-1)F^{-1/2} = -\frac{2}{\sinh \tau}.$$  

(3.18)

In the case with $0 \leq F \leq 1$ we find instead that

$$F(\tau) = \tanh^2 \frac{\tau}{2}, \quad 0 \leq \tau < \infty$$

(3.19)

and

$$\Omega_1 = \Omega_2 = \Omega_3 = \Omega_4 = -F^{1/2} = -\tanh \frac{\tau}{2}, \quad \Omega_5 = \Omega_6 = (1-F)F^{-1/2} = \frac{2}{\sinh \tau}.$$  

(3.20)

Note that this solution can also be obtained by analytically continue $\tau \to \tau + i\pi$ in (3.18).

**Example 6**: Finally, we consider a generalization of the previous example such that the preserved symmetry group is enhanced to $SO(2) \times SO(2) \times SO(2)$. Namely, let $b_1 = b_2$, $b_3 = b_4$ and $b_5 = b_6$. This is the case of the 3-dimensional Euler top since using (2.12) we find that $\omega_1 = \omega_2 = \omega_6 = \omega_7 = 0$ and that the remaining $\omega_4$, $\omega_5$ and $\omega_6$ obey the standard 3-dimensional Euler top equations in (1.3) (with $\omega_i$ replaced by $\omega_{i+3}$). In that case we know of course that the solution is given in terms of elliptic Jacobi functions with the help of the two independent constants of motion

$$I_1 = \omega_3^2 - \omega_4^2, \quad I_2 = \omega_4^2 - \omega_5^2.$$  

(3.21)

This will be verified in our framework. Indeed, in the case at hand, (2.19) reduces to the Weierstrass differential equation with solution

$$F(\tau) = \wp(\tau/2) + \frac{1}{3}(b_1 + b_3 + b_5),$$  

(3.22)

where $\wp(\tau/2)$ is the Weierstrass function. This is a doubly periodic function in the argument $\tau/2$ and the two half-periods are given by$^5$

$$\text{half-periods} : \quad \omega_1 = \frac{K(k)}{\sqrt{\epsilon_1 - \epsilon_3}}, \quad \omega_2 = \frac{iK(k')}{\sqrt{\epsilon_1 - \epsilon_3}},$$  

(3.23)

$^5$The half-periods $\omega_1$ and $\omega_2$ below should not be confused with the $\omega_a(\tau)$'s that parametrize the gauge fields.
where $K$ is the complete elliptic integral of the first kind with modulus $k$ and complementary modulus $k'$ given by

$$k^2 = \frac{e_2 - e_3}{e_1 - e_3}, \quad k'^2 = 1 - k^2 = \frac{e_1 - e_2}{e_1 - e_3}. \tag{3.24}$$

Here $e_1$, $e_2$ and $e_3$ are the values of the Weierstrass function at the half-periods, i.e. $\wp(\omega_1) = e_1$, $\wp(\omega_2) = e_3$ and $\wp(\omega_1 + \omega_2) = e_2$, which are expressed in our case in terms of the parameters $b_1$, $b_3$ and $b_5$ as

$$e_1 = b_1 - \frac{1}{3}(b_1 + b_3 + b_5). \tag{3.25}$$

The expressions for $\epsilon_2$ and $e_3$ are given by the above formulae after interchanging the roles of $b_1$ with $b_3$ and $b_5$, respectively.

Then we find that

$$\Omega_1 = \Omega_2 = \sqrt{\frac{(F - b_3)(F - b_5)}{F - b_1}} = (b_1 - b_5)^{1/2} \frac{\text{dnu}}{\text{cnu snu}},$$

$$\Omega_3 = \Omega_4 = \sqrt{\frac{(F - b_1)(F - b_5)}{F - b_3}} = (b_1 - b_5)^{1/2} \frac{\text{cnu}}{\text{dnu snu}},$$

$$\Omega_5 = \Omega_6 = \sqrt{\frac{(F - b_1)(F - b_3)}{F - b_5}} = (b_1 - b_5)^{1/2} \frac{\text{cnu dnu}}{\text{snu}},$$

$$u \equiv -\frac{1}{2}(b_1 - b_5)^{1/2} \tau + u_0, \quad \tau \leq 0, \tag{3.26}$$

with $u_0$ being a constant of integration and where we have used well known properties of the Weierstrass function to express the final result in terms of the Jacobi functions snu, cnu and dnu. Also, using (2.12) we find that

$$\omega_3 = \frac{1}{2}(\Omega_1 + \Omega_5 - \Omega_3), \quad \omega_4 = \frac{1}{2}(\Omega_1 + \Omega_3 - \Omega_5), \quad \omega_5 = \frac{1}{2}(\Omega_3 + \Omega_5 - \Omega_1), \tag{3.27}$$

and zero for the rest. It is easily verified, using properties of the Jacobi functions, that the constants of motion (3.24) are $I_1 = b_5 - b_3$ and $I_2 = b_1 - b_5$.

In the particular case with $b_1 = b_2 = b_3 = b_4 = 1$ and $b_5 = b_6 = 0$ the complementary modulus $k' \to 0$ and we may use the approximate expressions for the Jacobi functions snu $\simeq \tanh u$ and dnu $\simeq \text{cnu} \simeq 1/\cosh u$. Choosing the constant $u_0$ to be real and then absorbing it into a redefinition of $\tau$ we recover (3.18) as we should. In the case with $b_1 = b_2 = 1$ and $b_3 = b_4 = b_5 = b_6 = 0$ the modulus $k \to 0$ and we may use instead the approximate expressions snu $\simeq \sin u$, cnu $\simeq \cos u$ and dnu $\simeq 1$. Then we find that

$$\Omega_1 = \Omega_2 = -\frac{2}{\sin \tau}, \quad \Omega_3 = \Omega_4 = \Omega_5 = \Omega_6 = -\cot \frac{\tau}{2}, \quad -\frac{\pi}{2} \leq \tau \leq 0, \tag{3.28}$$
which is the trigonometric counterpart of (3.18). The solution (3.20) can also be obtained in this limit if we choose the constant of integration purely imaginary, namely \( u_0 = i\pi/2 \). This essentially amounts to the analytic continuation in \( \tau \), we noted before, that relates (3.18) and (3.20).

4 Algebraic curve classification

In this section we systematically study the differential equation (2.19) using classical techniques from algebraic geometry. We will see that there are cases where this approach is not only mathematically elegant, but also serves as a practical tool in explicit computations. We basically follow the discussion in [20], where such an equation arose in the construction of domain wall solutions in gauged supergravity. An approach, though not quite as systematic, based on algebraic curves was also followed in [24].

The non-linear differential equation (2.19) can be viewed, when the parameter \( \tau \) and the unknown function \( F(\tau) \) are extended to the complex domain, as a Christoffel– Schwarz transformation from a closed polygon in the \( \tau \)-plane onto the upper-half \( F \)-plane. In this case the perimeter of the polygon is mapped to the real \( F \)-axis, whereas its vertices are mapped to points parametrized by the moduli \( b_i \). Letting

\[
x = F(\tau) , \quad y = \frac{dF(\tau)}{d\tau} ,
\]

we arrive at the algebraic curve in \( \mathbb{C}^2 \)

\[
y^4 = \prod_{a=1}^{7} (x - b_a) .
\]

Given this algebraic curve we are faced with the problem of multi-valuedness since the corresponding Riemann surface is pictured geometrically by gluing four sheets together along their branch cuts. This is resolved in a standard way by uniformizing the algebraic curve in terms of the so called uniformizing complex parameter, say \( u \). As a first step one constructs birational invertible transformations \( x(v, w), y(v, w) \) such that the algebraic curve (4.2) assumes in terms of the new set of variables \( (v, w) \) the canonical standard form according to the genus \( g \) of the associated to the algebraic curve Riemman surface. For instance, according to algebraic geometry every \( g = 0 \) Riemman surface can be brought into the form \( u = v \). Then we may use as uniformizing parameter \( u = v = w \). Similarly, every \( g = 1 \) Riemman surface can be brought into the Weierstrass forms \( w^2 = 4v^3 - g_2v - g_3 \) or \( u^2 = 4(v-e_1)(v-e_2)(v-e_3) \). Then the uniformization problem is solved as \( v = \wp(u) \) and \( w = \wp'(u) \), where \( \wp(u) \) is the Weierstrass function and \( u \) denotes again the uniformizing parameter. In general, every Riemann surface with \( g \leq 2 \) can brought into the hyperelliptic
form which is still tractable. However, as the genus of the Riemann surface increases the
uniformization problem becomes increasingly more difficult to solve. In order to relate
the uniformizing parameter \( u \) to \( \tau \), we form, after solving the uniformization problem, 
the functions \( x = x(u) \) and \( y = y(u) \) and restrict the domain of values for \( u \), so that
\( x = F(\tau) = x(u) \) is a real function. Then we may obtain \( u(\tau) \) by inverting the solution of 
The differential equation

\[
\frac{d\tau}{du} = \frac{1}{y(u)} \frac{dx(u)}{du},
\]

which is derived using (4.1).

The genus of the curve can be easily determined via the Riemann–Hurwitz relation. Recall that for any curve of the form

\[
y^m = (x - \lambda_1)^{\alpha_1}(x - \lambda_2)^{\alpha_2} \ldots (x - \lambda_n)^{\alpha_n},
\]

with integers \( m \) and \( \alpha_i \) having no common factors, and all \( \lambda_i \)'s being unequal, the genus \( g \) can be found by first writing the ratios

\[
\frac{\alpha_1}{m} = \frac{d_1}{c_1}, \ldots, \frac{\alpha_n}{m} = \frac{d_n}{c_n}; \quad \frac{\alpha_1 + \cdots + \alpha_n}{m} = \frac{d_0}{c_0}
\]

in terms of relatively prime numbers and then using the relation

\[
g = 1 - m + \frac{m}{2} \sum_{i=0}^{n} \left( 1 - \frac{1}{c_i} \right).
\]

According to this, the genus of our surface turns out to be \( g = 9 \) when all \( b_i \) are unequal, and so it is difficult to determine explicitly the solution in the general case. However, when some the parameters among the \( b_i \)'s are equal the genus becomes smaller since that corresponds to degenerating the surface along certain cycles, thus reducing its genus. Similarly, the symmetry group of the solution (2.18) gets enhanced into larger subgroups of \( SO(7) \). In such cases the problem becomes more tractable. The complete list of the various algebraic curves and the genus of the associated Riemann surfaces is given by the table below. It is identical to the table 4 of [28] corresponding to distributions of D2-branes.
Table 1: Curves, their genus and symmetry groups for $N = 7$ moduli.

| Genus | Irreducible Curve                                                                 | Isometry Group          |
|-------|----------------------------------------------------------------------------------|-------------------------|
| 9     | $y^4 = (x - b_1)(x - b_2)\cdots(x - b_6)(x - b_7)$                              | None                    |
| 7     | $y^4 = (x - b_1)(x - b_2)\cdots(x - b_5)(x - b_6)^2$                            | $SO(2)$                |
| 6     | $y^4 = (x - b_1)(x - b_2)\cdots(x - b_5)(x - b_6)^3$                            | $SO(3)$                |
| 5     | $y^4 = (x - b_1)\cdots(x - b_3)(x - b_4)^2(x - b_5)^2$                          | $SO(2) \times SO(2)$   |
| 4     | $y^4 = (x - b_1)(x - b_2)(x - b_3)^2(x - b_4)^5$                                | $SO(2) \times SO(3)$   |
| 3     | $y^4 = (x - b_1)(x - b_3)(x - b_4)^4$                                           | $SO(4)$                |
|       | $y^4 = (x - b_1)(x - b_2)^2(x - b_3)^5$                                         | $SO(5)$                |
|       | $y^4 = (x - b_1)(x - b_2)^3(x - b_3)^3$                                         | $SO(3) \times SO(3)$   |
|       | $y^4 = (x - b_1)(x - b_2)^2(x - b_3)^2(x - b_4)^2$                              | $SO(2)^3$              |
| 2     | $y^4 = (x - b_1)(x - b_2)^2(x - b_3)^3$                                         | $SO(2) \times SO(2) \times SO(3)$ |
| 1     | $y^4 = (x - b_1)(x - b_2)^6$                                                    | $SO(6)$                |
|       | $y^4 = (x - b_1)(x - b_2)^2(x - b_3)^4$                                         | $SO(2) \times SO(4)$   |
|       | $y^4 = (x - b_1)(x - b_2)^2(x - b_3)^5$                                         | $SO(2) \times SO(5)$   |
| 0     | $y^4 = (x - b_1)^3(x - b_2)^4$                                                  | $SO(3) \times SO(4)$   |
|       | $y^4 = (x - b_1)^7$                                                             | $SO(7)$ (maximal)       |

According to this table, the first three examples we considered explicitly in section 3 correspond to $g = 0$ Riemann surfaces, whereas the fourth one to a surface with $g = 1$. This is also reflected in the fact that in the $g = 0$ cases the relation between $F$ and $\tau$ is via elementary functions, whereas in the $g = 1$ case the hypergeometric special function is involved.

Similarly, we may consider the case of $N = 6$ moduli parameters. Then the generic case when all the moduli parameters are unequal corresponds to a $g = 7$ Riemann surface. The resulting table 2 below arises also in studies of domain wall solutions in five-dimensional gauged supergravity that correspond to continuous distributions of D3-branes in type-IIB string theory \cite{20}.
Table 2: Curves, their genus and symmetry groups for $N = 6$ moduli.

According to this table, the fifth example we gave in section 3 corresponds to a $g = 0$ surface, whereas the sixth one to a $g = 1$ surface. In these cases the uniformization procedure that we have advocated and described in this section is not necessary as the solution can be readily found by elementary methods. However, finding solutions to the other cases requires that the uniformization program, which turns out to be non-trivial, is carried out completely. As an example let’s consider the other symmetric case with $g = 1$ with algebraic curve $y^4 = (x - 1)^3 x^3$, where for convenience we have chosen $b_1 = 1$ and $b_2 = 0$ (this can always be done with appropriate rescalings and shifts). We find that the birational transformation

$$x = 1 - \frac{1}{v}(v + 1/4)^2, \quad y = \frac{w^3}{8v^3}, \quad (4.7)$$

brings the curve into the form $w^2 = 4v^3 - v/4$ which is of the standard Weierstrass form with

$$g_3 = 0, \quad g_2 = \frac{1}{4}, \quad e_1 = \frac{1}{4}, \quad e_2 = 0, \quad e_3 = -\frac{1}{4},$$

$$k = k' = \frac{1}{\sqrt{2}}, \quad \omega_1 = -i\omega_2 = \frac{\Gamma(1/4)^2}{2\sqrt{2}\pi}, \quad (4.8)$$

where we have included the values for the modulus, its equal complementary modulus and the two half periods. We have then that $x = \wp(u)$ and $y = \wp'(u)$, where $u$ is determined as a function of $\tau$ after solving (4.3). In this case it turns out that we get the simple relation $u = -\tau/2$. Hence, we found that

$$F(\tau) = 1 - \frac{1}{\wp(\tau/2)(\wp(\tau/2) + 1/4)^2} = 1 - \frac{2}{\text{sn}^2 \left( \frac{\tau}{2\sqrt{2}} \right) \text{dn}^2 \left( \frac{\tau}{2\sqrt{2}} \right)}, \quad (4.9)$$

where we have used the evenness of the Weierstrass function and, in the last step, its relation to the Jacobi functions. For completeness we also mention the inverse of the transformation (4.7)

$$v = \frac{1}{4} \frac{y^2 - (x - 1)x^2}{y^2 + (x - 1)x^2}, \quad w = \frac{(x - 1)x}{2y} \frac{y^2 - (x - 1)x^2}{y^2 + (x - 1)x^2}. \quad (4.10)$$

The reader is referred to section 4 of [20] for the other low genus cases $g = 0, 1, 2$ of table 2, where the uniformization procedure has been done explicitly.

Similarly, for the cases with $N = 5$ moduli parameters we have formed table 3. In the same spirit as before, this is identical to table 3 of [21], that arose in the construction of domain-wall solutions of seven-dimensional gauged supergravity corresponding to continuous distributions of M5-branes in M-theory.
As an example, consider the $g = 1$ curve $y^4 = (x - 1)^2 x^3$. The transformation

$$x = 1 - \frac{w^2}{4v^3}, \quad y = \frac{w}{v} \left(1 - \frac{w^2}{4v^3}\right),$$

with inverse

$$v = -\frac{x(x - 1)}{4y^2}, \quad w = -\frac{x - 1}{4y},$$

brings the curve into the same standard form $w^2 = 4v^3 - v/4$ as in the previous example in this section. Hence $v = \wp(u)$ and $y = \wp'(u)$, whereas the various parameters are still given by (4.8). It also turns out that $u = -\tau/2$ as before. Finally, the result for $F(\tau)$ is

$$F(\tau) = \frac{1}{16\wp'(\tau/2)^2} = \frac{\text{sn}^4 \left(\frac{\tau}{2\sqrt{2}}\right)}{4\text{dn}^4 \left(\frac{\tau}{2\sqrt{2}}\right)}.$$  

(4.13)

For other examples with $g = 0, 1$, where the uniformization program has been carried out explicitly, we refer the reader to section 7 of [21].

Similarly, the case with $N = 4$ moduli parameters leads to table 4 below and the generic Riemann surface has $g = 3$. It coincides with table 5 of [28] that arose in the construction of supergravity solutions corresponding to continuous distributions of NS5- and D5-branes in string theory.

| Genus | Irreducible Curve | Isometry Group |
|-------|-------------------|----------------|
| 3     | $y^4 = (x - b_1)(x - b_2)(x - b_3)(x - b_4)$ | None |
| 1     | $y^4 = (x - b_1)(x - b_2)(x - b_3)^2$ | $SO(2)$ |
| 0     | $y^4 = (x - b_1)(x - b_2)^3$ | $SO(3)$ |
|       | $y^2 = (x - b_1)(x - b_2)$ | $SO(2) \times SO(2)$ |
|       | $y = (x - b_1)$ | $SO(4)$ (maximal) |

Table 4: Curves, their genus and symmetry groups for $N = 4$ moduli.
Similarly, the case of \( N = 3 \) moduli parameters leads to table 5 below and the generic Riemann surface has again \( g = 3 \). It coincides with table 6 of \cite{28} that arose in the construction of solutions representing D6-branes in string theory.

| Genus | Irreducible Curve | Isometry Group |
|-------|------------------|----------------|
| 3     | \( y^4 = (x - b_1)(x - b_2)(x - b_3) \) | None           |
| 1     | \( y^4 = (x - b_1)(x - b_2)^2 \) | \( SO(2) \)   |
| 0     | \( y^4 = (x - b_1)^3 \) | \( SO(3) \) (maximal) |

Table 5: Curves, their genus and symmetry groups for \( N = 3 \) moduli.

Finally, we include in table 6 below the case of \( N = 2 \) moduli parameters that generically corresponds to a \( g = 1 \) Riemann surface.

| Genus | Irreducible Curve | Isometry Group |
|-------|------------------|----------------|
| 1     | \( y^4 = (x - b_1)(x - b_2) \) | None           |
| 0     | \( y^2 = (x - b_1) \) | \( SO(2) \) (maximal) |

Table 6: Curves, their genus and symmetry groups for \( N = 2 \) moduli.

In this case the transformation

\[
x = -\frac{1}{v}(v - 1/4)^2, \quad y = \frac{w}{2v},
\]

with inverse

\[
v = \frac{1}{4}(2y^2 - 2x + 1), \quad w = \frac{1}{2}y(2y^2 - 2x + 1),
\]

brings the \( g = 1 \) curve \( y^4 = (x - 1)x \) into the Weierstrass form \( w^2 = 4v^2 - v/4 \) and hence \( v = \phi(u) \), \( w = \phi'(u) \) solve the uniformization problem with parameters given by (4.8). The differential equation (4.3) takes in this case the form \( d\tau/du = -\frac{1}{2}\phi'(u)^2/\phi(u)^2 \).

Integrating leads to an expression for \( \tau(u) \) in terms of elliptic integrals. However, inverting this expression and obtaining the function \( u(\tau) \) explicitly is not possible.

5 The massive case and solitons

Let us modify the first order equations (2.7) by a mass term as

\[
\frac{dA_m}{d\tau} = \frac{1}{2}\psi_{abc}[A_b, A_c] + m_{ab}A_b,
\]

17
where $m_{ab} = m_{ba}$ is a constant matrix (this mass term is similar to the symmetry breaking term in [29] for membranes embedded in 8-dimensions). Writing as before that $A_a = \psi_a \omega_a$ (no num) we obtain the system

$$\frac{d\omega_a}{d\tau} = \frac{1}{2} \psi_{abc} \omega_b \omega_c + m_{ab} \omega_b ,$$

(5.2)

which can be derived from the prepotential

$$W = \frac{1}{6} \psi_{abc} \omega_a \omega_b \omega_c + \frac{1}{2} m_{ab} \omega_a \omega_b .$$

(5.3)

The corresponding potential can be computed using (2.10).

The solution of (5.2) in the case of a matrix proportional to the identity, i.e. $m_{ab} = m \delta_{ab}$, can be obtained from that of the corresponding massless case (2.6). Indeed, after performing the change of variables

$$\omega_a = e^{m \tau} \bar{\omega}_a , \quad \eta = \frac{e^{m \tau}}{m} ,$$

(5.4)

we observe that the $\bar{\omega}_a$’s obey

$$\frac{d\bar{\omega}_a}{d\eta} = \frac{1}{2} \psi_{abc} \bar{\omega}_b \bar{\omega}_c , \quad a, b, c = 1, 2, \ldots, 7 ,$$

(5.5)

which is just the system (2.6) that we have solved. For more general mass matrices we know of no such similar transformation. We remark also that (5.4) is similar to the transformation employed in [30] in order to convert the BPS condition for dimensionally reduced $\mathcal{N} = 1$ YM to the usual Nahm system (1.2).

Adding a mass term has a physical motivation. A diagonal matrix $m_{ab} = m \delta_{ab}$ arises naturally when the self-duality conditions are defined on an 8-dimensional Euclidean $AdS_8$ space with metric $ds^2 = d\tau^2 + e^{-2m \tau} \sum_{a=1}^{7} dx_a^2$, corresponding to a cosmological constant $\Lambda \sim -m^2$. We remark that, unlike the 4-dimensional case [8], the energy momentum tensor is not zero when the gauge field strength satisfies the self-duality conditions in 8-dimensions in some gravitational background of Euclidean signature. Hence the Einstein’s equations in the presence of a negative cosmological constant are not satisfied with an $AdS_8$ metric. Therefore, in the statement above, $AdS_8$ is considered as a fixed background.

### 5.1 Solitons

In the massless case there are no isolated degenerate vacua corresponding to minima of the potential. However, such vacua develop when a mass term is turned on and in these cases we expect that there are solitonic solutions to the equations of motion which interpolate between them. The size of these solitons will of course be dictated by the interplay between
the various mass parameters. To be concrete, we will consider the most symmetric case of a diagonal mass matrix $m_{ab} = m\delta_{ab}$, but a similar discussion can be made for other choices as well. Then, the prepotential has seven critical points labeled by the integer $n = 0, 1, 2, 3, 5, 6, 7$. They are found by solving the system of seven algebraic equations obtained by setting the right hand side of (2.15) to zero. Without loss of generality these critical points can be arranged to be

$$
\Omega^{(n)}_a = \frac{4}{4-n} m, \quad a = 1, 2, \ldots, n, \\
\Omega^{(n)}_{n+1} = \Omega^{(n)}_{n+2} = \cdots = \Omega^{(n)}_7 = 0,
$$

(5.6)

up to a renaming of the index $a$. Hence in the $n$th vacuum the maximal symmetry $SO(7)$ is broken to $SO(n) \times SO(7-n)$. In terms of the $\omega_a$’s, using (2.12), we find that these critical points are

\begin{align*}
  n = 0 : & & \vec{\omega}^{(0)} &= (0, 0, 0, 0, 0, 0), & \text{SO}(7) , \\
  n = 1 : & & \vec{\omega}^{(1)} &= \frac{m}{3} (-1, 1, 1, 1, -1, -1, 1), & \text{SO}(4) \times \text{SO}(3) , \\
  n = 2 : & & \vec{\omega}^{(2)} &= m (0, 0, 1, 1, -1, 0, 0), & \text{SO}(4) \times \text{SO}(2) , \\
  n = 3 : & & \vec{\omega}^{(3)} &= m (-1, -1, 1, 3, -1, 1, 1), & \text{SO}(3) \times \text{SO}(3) , \\
  n = 5 : & & \vec{\omega}^{(5)} &= m (-1, 1, -1, -3, -1, 1, -1), & \text{SO}(4) \times \text{SO}(2) , \\
  n = 6 : & & \vec{\omega}^{(6)} &= m (0, 0, -1, -1, -1, 0, 0), & \text{SO}(4) \times \text{SO}(3) , \\
  n = 7 : & & \vec{\omega}^{(7)} &= -\frac{m}{3} (1, 1, 1, 1, 1, 1), & \text{SO}(7) .
\end{align*}

(5.7)

In the last column above we indicated the subgroup of $SO(7)$ which is preserved by the corresponding vacuum. Notice that, the symmetry group is the same for the $n$th and the $(7-n)$th vacua. Also, the symmetry of a vacuum in the space of the $\omega_a$’s is different than that in the space of $\Omega_a$’s we stated above. The reason, as we have mentioned, is that the transformation (2.12) breaks part of this symmetry. One can expand perturbatively around each vacuum separately and derive the corresponding mass matrix. The matrix elements for the $n$th vacuum are given by

$$
M_{ab}^{(n)} = m\delta_{ab} + \psi_{abc}^{(n)} \omega_c^{(n)}, \quad a, b, c = 1, 2, \ldots, 7 .
$$

(5.8)

These mass matrices can be diagonalized in each case separately. Their eigenvalues and their degeneracies are given by

\begin{align*}
  n = 0 : & & m \ (7\text{-fold}) , \\
  n = 1, 7 : & & -m, \frac{4}{3} m \ (6\text{-fold}) ,
\end{align*}

19
\[ n = 2, 6 : \quad -m, -2m, 2m \ (5\text{-fold}) , \]
\[ n = 3, 5 : \quad -m, -4m \ (2\text{-fold}), 4m \ (4\text{-fold}) . \]

Note that, the mass spectra around vacua having the same symmetry in (5.7) (for instance, for \( n = 1 \) and \( n = 6 \)), are not identical. There exists instead an identity between the mass spectrum of the \( n \)th and the \((8 - n)\)th vacuum for \( n = 1, 2, 3, 5, 6, 7 \).

It can be easily seen that the on-shell action for a solitonic solution interpolating between the \( n \)th and the \( k \)th vacua above, as \( \tau \) goes from \(-\infty \) to \( \infty \), is finite and given in terms of the values of the prepotential \( W \) at \( \tau = \pm \infty \) as
\[ S_{n \to k} = -W_k + W_n , \quad W_n = W(\tau = -\infty) = -\frac{n(8 - n)}{3(n - 4)^2} m^3 . \] (5.10)
The expression for \( W_k = W(\tau = \infty) \) is similar to the one above for \( W_n \) with \( n \) replaced by \( k \). Note also the symmetry \( W_n = W_{8 - n} \) for \( n = 1, 2, 3, 5, 6, 7 \), which is similar to the symmetry we noted for the perturbative mass spectra in (5.9).

### 5.2 Examples

The simplest solitonic solution is that relating the vacuum with \( \omega_a = 0 \), \( a = 1, 2, \ldots, 7 \) which is labeled by the integer \( n = 0 \) in (5.7), with any other vacuum labeled with \( n = 1, 2, 3, 5, 6, 7 \). Let us consider for the system (5.2) any of the following 6 truncations
\[ \vec{\omega}(\tau) = (x(\tau) + \frac{1}{2})\vec{\omega}^{(n)} , \quad n = 1, 2, 3, 5, 6, 7 , \] (5.11)
where \( x(\tau) \) is the single function whose evolution we would like to determine. It is easy to verify that all 7 equations in (5.2) give rise to the same equation for \( x(\tau) \) and therefore the above truncations are consistent. We find that
\[ \frac{dx}{d\tau} = m(1/4 - x^2) \quad \implies \quad x(\tau) = \frac{1}{2} \tanh \left(\frac{m(\tau - \tau_0)}{2}\right) . \] (5.12)
This is the usual kink solution centered at \( \tau = \tau_0 \) and interpolating between the two vacua at \( x(-\infty) = -\frac{1}{2} \) (equivalently \( \vec{\omega}^{(0)} = \vec{0} \)) and \( x(+\infty) = +\frac{1}{2} \) (equivalently \( \vec{\omega}^{(n)} = \vec{0} \)) with \( n \) assuming one of the values \( n = 1, 2, 3, 5, 6, 7 \). The equation in (5.12) follows from the prepotential \( W = m(x/4 - x^3/3) \) and therefore the Lagrangian for the theory is (we discard a numerical factor that depends on \( n \) times \( m^2 \))
\[ \mathcal{L} = -\frac{1}{2} \dot{x}^2 - \frac{m^2}{2} (x^2 - 1/4)^2 , \] (5.13)
where the familiar “mexican hat” potential is computed using (2.10). The equation of motion for this Lagrangian is well known to admit the soliton solution in (5.12). The
novelty of our approach is that we found them by solving a first order equation instead of
the second order one that follows from the Lagrangian (5.13). We also note that, the kink
solution in (5.12) also arises by starting from the solution (3.5), i.e. \( \bar{\omega}_a = -\frac{1}{3n} \) and then
using (5.4).

Another example follows from the consistent truncation to two fields \( x(\tau) \) and \( y(\tau) \) as
\[
\Omega_1 = \Omega_2 = \cdots = \Omega_5 = \frac{4}{\sqrt{15}} x + \frac{2}{\sqrt{3}} y, \quad \Omega_6 = \Omega_7 = \sqrt{3} y, \quad (5.14)
\]
where the choice of the arithmetic factors is such that the two fields are canonically nor-
malized. Indeed, the Lagrangian for the theory takes the form
\[
\mathcal{L} = -\frac{1}{2}(\dot{x}^2 + \dot{y}^2) - V, \\
V = \frac{1}{6} y^2(\sqrt{3}m + \sqrt{5}x + y)^2 + \frac{1}{120}(2\sqrt{15}mx + 2x^2 + 5y^2)^2, \quad (5.15)
\]
where the potential has been computed using (2.10) with prepotential
\[
W = \frac{\sqrt{3}}{9} y^3 + \frac{\sqrt{15}}{90} (2x^3 + 15xy^2) + \frac{m}{2}(x^2 + y^2). \quad (5.16)
\]
Our anzatz (5.14), implies that this prepotential should have as critical points those cor-
responding to the cases with \( n = 0, 2, 5, 7 \) in (5.6). Indeed, we find
\[
n = 0 : \quad x = 0, \quad y = 0, \\
n = 2 : \quad x = -\sqrt{\frac{5}{3}} m, \quad y = \frac{2}{\sqrt{3}} m, \\
n = 5 : \quad x = -\sqrt{15} m, \quad y = 0, \\
n = 7 : \quad x = -\frac{1}{3} \sqrt{\frac{5}{3}} m, \quad y = -\frac{4}{3\sqrt{3}} m. \quad (5.17)
\]
Finding solitonic solutions is easy using the replacement (5.4). For instance, the solitonic
solution connecting the vacua labeled by \( n = 0 \) and \( n = 5 \) is
\[
\Omega_1 = \cdots = \Omega_5 = e^{m\tau} (F(\eta) - 1)^{1/4} F(\eta)^{1/2}, \quad \Omega_6 = \Omega_7 = e^{m\tau} \frac{(F(\eta) - 1)^{5/4}}{F(\eta)^{1/2}}, \\
\eta = e^{m\tau}/m, \quad m < 0, \quad (5.18)
\]
where the function \( F(\eta) \) given by (3.14) after we replace \( \tau \) by \( \eta \). Indeed, with the help
of (3.15), we verify that, for \( \tau \to -\infty \) we approach the vacuum labeled by \( n = 5 \) in (5.6)
and (5.17). In addition, it is easily seen that for \( \tau \to +\infty \) we approach the \( n = 0 \) vacuum.
The shape of \( \Omega_1 = \cdots = \Omega_5 \) is similar to an anti-kink, whereas that of \( \Omega_6 = \Omega_7 \) is similar
to a lump.
6 7D self-dual YM with $G_2$ invariance

In this section we consider the self-dual YM equations in seven dimensions in the case that their group of invariance is the $G_2$ subgroup of the rotation group $SO(7)$. As noted in [1] this can be obtained from the $Spin(7)$ invariant system (2.1) (with $\lambda = \frac{1}{2}$) if we set to zero all the components of $F_{\alpha\beta}$ that have one subscript equal to 8. Indeed, then (2.3) gives a total of 7 conditions leaving the 21 equations in (2.4), appropriate for the 7-dimensional system which reads

$$F_{ab} = \frac{1}{2} \psi_{abcd} F_{cd}, \quad a = 1, 2, \ldots, 7.$$  \hspace{1cm} (6.1)

It is useful to split the index $a = (7, i, \hat{i})$ with $i = 1, 2, 3$ and $\hat{i} = i + 3$ and represent the octonionic structure constants and the 4-index totally antisymmetric tensor as

$$\psi_{ijk} = \epsilon_{ijk}, \quad \psi_{i\hat{j}k} = -\epsilon_{ijk}, \quad \psi_{i\hat{i}j} = \delta_{ij},$$

$$\psi_{\hat{i}j\hat{k}} = \epsilon_{ijk}, \quad \psi_{i\hat{i}\hat{j}k} = -\epsilon_{ijk}, \quad \psi_{ij\hat{m}\hat{n}} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}.$$  \hspace{1cm} (6.2)

Similarly to the $Spin(7)$ invariant case, we can show that a solution to a system of 7 equations suffices to construct a solution of the entire set of self-dual equations (6.1). These are the 6 equations arising after choosing for the index $b = 7$ in (6.1)

$$F_{i7} = \frac{1}{2} \psi_{i\hat{a}7} F_{\hat{a}b} = \frac{1}{2} \epsilon_{ijk}(F_{\hat{j}k} - F_{jk}),$$

$$F_{i7} = \frac{1}{2} \psi_{i\hat{a}7} F_{\hat{a}b} = -\epsilon_{ijk} F_{jk}$$  \hspace{1cm} (6.3)

and in addition, the single equation

$$F_{i\hat{i}} = 0,$$  \hspace{1cm} (6.4)

which is the only independent one from (6.3) when the indices in the left hand side in (6.1) restrict to the values $a, b = 1, 2, \ldots, 6$. In order to reduce into one-dimensional systems we make the gauge choice $A_7 = 0$ and we seek solutions where the remaining fields $A_i$ and $\hat{A}_i \equiv A_{i+3}$ depend only on $x^7 \equiv \tau$. Then we obtain the system

$$\frac{d\hat{A}_i}{d\tau} = \frac{1}{2} \epsilon_{ijk}([\hat{A}_j, \hat{A}_k] - [A_j, A_k]),$$

$$\frac{dA_i}{d\tau} = -\epsilon_{ijk}[A_j, \hat{A}_k]\quad (6.5)$$

with solutions subject to the constraint

$$[A_i, A_j] = 0,$$  \hspace{1cm} (6.6)

\footnote{In this way we restrict to the 14 of $G_2$. In complete analogy with the case of weak holonomy metrics investigated recently in [31], this can be relaxed as we show in subsection 6.1 below, leading to massive theories.}
which follows from (6.4). An equivalent useful complex form is

$$\frac{d\bar{S}_i}{d\tau} = \frac{1}{2} \epsilon_{ijk} [S_j, S_k] , \quad S_j = \hat{A}_j + iA_j , \quad (6.7)$$

and $\bar{S}_i$ denotes its complex conjugation. The constraint becomes

$$[S_i, \bar{S}_i] = 0 \quad (6.8)$$

Note that this system is a complex extension of the Nahm’s system in (1.2). It reduces to that in two cases: if $A_i = 0$, $i = 1, 2, 3$ or if $A_i = \sqrt{3} \hat{A}_i$, $i = 1, 2, 3$.

**Reduction 1:** A quite general ansatz for metrics with $G_2$ holonomy that preserve an $SU(2) \times SU(2) \times Z_2$ symmetry was made in [18] and [32]. The $G_2$ holonomy constraints for the closure and co-closure of the associative three-form result into a six-dimensional first order system of equations to which a special solution having an extra $U(1)$ symmetry was found in [18]. Since there is no systematic study of this system, we hope that making contact with solutions of self-dual YM will give some new insight in this direction as well. Hence, we seek solutions of (6.7) that preserve at least an $SU(2) \times SU(2) \times Z_2$ symmetry.

Let us introduce two commuting sets of Pauli matrices $\{\sigma_i\}$ and $\{\Sigma_i\}$ with $i = 1, 2, 3$ obeying

$$[\sigma_i, \sigma_j] = 2i \epsilon_{ijk} \sigma_k , \quad \text{Tr}(\sigma_i \sigma_j) = 2 \delta_{ij} \quad (6.9)$$

and similarly for $\Sigma_i$. Our $SU(2) \times SU(2) \times Z_2$ invariant ansatz for the gauge fields is

$$A_j(\tau) = -\frac{i}{2} \omega_j(\tau)(\sigma_j - \Sigma_j) , \quad \hat{A}_j(\tau) = -\frac{i}{2} \hat{\omega}_j(\tau)(\sigma_j + \Sigma_j) \quad (6.10)$$

where $\omega_j, \hat{\omega}_j$ are 6 functions to be determined. The equivalently complex form is

$$S_j(\tau) = -\frac{i}{2} s_j(\tau) \sigma_j - \frac{i}{2} \bar{s}_j(\tau) \Sigma_j , \quad s_j = \omega_j + i\hat{\omega}_j . \quad (6.11)$$

Our reduction is similar to the reduction of the Nahm system (1.2) to the Lagrange system (1.3) which was considered in [7], [8]. We find that (6.7) reduce to

$$\frac{d\bar{s}_1}{d\tau} = s_2 \bar{s}_3 \quad \text{(and cyclic perms.)} \quad (6.12)$$

whereas the constraint (6.8) is trivially satisfied. In terms of the real components we have

$$\frac{d\hat{\omega}_1}{d\tau} = \hat{\omega}_2 \hat{\omega}_3 - \omega_2 \omega_3 , \quad \frac{d\omega_1}{d\tau} = -\hat{\omega}_2 \omega_3 - \omega_2 \hat{\omega}_3 \quad \text{(and cyclic perms.)} \quad (6.13)$$

We emphasize that this system cannot be obtained as a particular case of (2.3) since the underline group structure is quite different. Nevertheless, similarly to (2.4), our system is
also a gradient flow with the non-vanishing metric components, prepotential and potential given by

\[ g_{s_i s_j} = \delta_{ij}, \quad W = s_1 s_2 s_3 + \bar{s}_1 \bar{s}_2 \bar{s}_3, \]
\[ V = |s_1 s_2|^2 + |s_1 s_3|^2 + |s_2 s_3|^2. \]  

(6.14)

The system (6.12) is the complex extension of the 3-dimensional Lagrange or Euler system (1.3) and it has an obvious symmetry under cyclic permutation in 1, 2, 3. However, there is a less obvious discrete symmetry present only in this case which acts as

\[ \tau \rightarrow -\tau, \quad (\omega_1, \omega_2, \omega_3) \rightarrow (\hat{\omega}_1, \hat{\omega}_2, -\omega_3), \quad (\hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_3) \rightarrow (\omega_1, \omega_2, \hat{\omega}_3). \]  

(6.15)

Equivalent representations are obtained by cyclic permutation in the indices 1, 2 and 3. This discrete symmetry originates from the automorphism of the octonionic algebra (A.1) under the discrete transformation obtained from (6.15) if we replace the \( \omega_i \)'s by the \( e_i \)'s, the \( \hat{\omega}_i \)'s by the \( e_{i+3} \)'s and \( \tau \) by \( e_7 \), as it can be readily verified.\(^7\) In order to, at least partially, integrate the system (6.12) it is useful to recognize the constants of motion. It can be easily seen that

\[ I_1 = |s_1|^2 - |s_2|^2 = \omega_1^2 + \omega_2^2 - \omega_3^2, \]
\[ I_2 = |s_2|^2 - |s_3|^2 = \omega_2^2 + \omega_3^2 - \omega_1^2, \]
\[ I_3 = i \frac{1}{2}(s_1 s_2 s_3 - \bar{s}_1 \bar{s}_2 \bar{s}_3) = \omega_1 \omega_2 \omega_3 - \omega_1 \hat{\omega}_2 \hat{\omega}_3 - \omega_2 \hat{\omega}_3 \hat{\omega}_1 - \omega_3 \hat{\omega}_1 \hat{\omega}_2, \]  

(6.16)

and

\[ I_3 = i \frac{1}{2}(s_1 s_2 s_3 - \bar{s}_1 \bar{s}_2 \bar{s}_3) = \omega_1 \omega_2 \omega_3 - \omega_1 \hat{\omega}_2 \hat{\omega}_3 - \omega_2 \hat{\omega}_3 \hat{\omega}_1 - \omega_3 \hat{\omega}_1 \hat{\omega}_2, \]  

(6.17)

are indeed constants of motion. The first two are simply complex extensions of the two independent integrals of motion of the 3-dimensional Euler top case. We were unable to find additional independent constants of motion towards further integrating the system (6.12). In that respect it is important to investigate if it admits a Lax pair representation as it is the case for the usual Lagrange system (1.3) (see, for instance, [8]). Nevertheless, we remark that in the case with \( \omega_1 = \omega_2 \) and \( \hat{\omega}_1 = \hat{\omega}_2 \), where our ansatz (6.10) develops an extra \( U(1) \) invariance, we have 2 independent constants of motion \( I_2 \) and \( I_3 \) which can be used to fully integrate the system for the remaining 4 functions.\(^8\) Indeed, if we change

\[^{7}A similar discrete symmetry leaves invariant the full first order system for 6 functions of [18, 32] obtained in the construction of \( G_2 \) holonomy metrics having an \( SU(2) \times SU(2) \times Z_2 \) invariance (joint work with I. Bakas). The reason for this similarity is that, the system of [18, 32] can also be obtained from imposing the self-duality condition on the spin connection \( \omega_{ab} = \frac{i}{4} \epsilon_{abcd} \hat{\omega}_{cd} \) [31].
\[^{8}There is an analogy for this in the 6 function system of [18, 32] describing the \( G_2 \) holonomy metrics preserving an \( SU(2) \times SU(2) \times Z_2 \) symmetry. In the case that an extra \( U(1) \) symmetry factor develops the system reduces to one for 4 functions and a special solution to it was found [18].\)
variables as $s_j = e^{i \phi_j} r_j$ we can rewrite the system (6.12) as

$$\frac{dr_1}{d\tau} = r_2 r_3 \cos(\phi_1 + \phi_2 + \phi_3) , \quad \text{(and cyclic perms.)},$$

$$\frac{d\phi_1}{d\tau} = -\frac{r_2 r_3}{r_1} \sin(\phi_1 + \phi_2 + \phi_3) , \quad \text{(and cyclic perms.)}. \quad (6.18)$$

In the case that $\phi_1 = \phi_2$ and $r_1 = r_2$ we can easily show that all functions are determined in terms of a single function $x(\tau)$ as

$$r_1^2 = r_2^2 = x + \frac{I_2}{3} , \quad r_3^2 = x - \frac{2I_2}{3} ,$$

$$\frac{d\phi_1}{d\tau} = \frac{d\phi_2}{d\tau} = \frac{I_3}{x + I_2/3} , \quad \frac{d\phi_3}{d\tau} = \frac{I_3}{x - 2I_2/3} . \quad (6.19)$$

The function $x(\tau)$ obeys the differential equation

$$\left(\frac{dx}{d\tau}\right)^2 = 4x^3 - \frac{4}{9} I_2^2 x - 4I_3^2 - \frac{8}{27} I_2^3 , \quad (6.20)$$

which is of the Weierstrass form and therefore its solution is given in terms of the corresponding elliptic functions.

**Reduction 2:** There is alternative reduction one can perform that resembles that of [7, 8] and their reduction of the Nahm system (1.2) to the Halphen system (1.4). In order to proceed we need to introduce the group element $g \in SU(2)$ and construct the components $L^i_\mu$ of the left invariant Maurer–Cartan 1-forms $L^i$ and the matrix $C_{ij}$ as

$$L^i_\mu = -i \text{Tr}(g^{-1} \partial_\mu g \sigma_j) , \quad C_{ij} = \frac{1}{2} \text{Tr}(\sigma_i g \sigma_j g^{-1}) , \quad (6.21)$$

where $x^\mu, \mu = 1, 2, 3$ represent the variables that we use to parametrize the group element $g \in SU(2)$. Some useful properties

$$\partial_\mu L^i_\nu - \partial_\nu L^i_\mu = 2 \epsilon_{ijk} L^j_\mu L^k_\nu ,$$

$$C_{ik} C_{jk} = \delta_{ij} , \quad C_{im} C_{jn} \epsilon_{mnl} = \epsilon_{ijk} C_{kl} . \quad (6.22)$$

Also we recall the left-invariant $SU(2)$ vector fields $X_i, i = 1, 2, 3$ satisfying

$$[X_i, X_j] = -2 \epsilon_{ijk} X_k , \quad [X_k, C_{ij}] = 2 C_{il} \epsilon_{ijk} . \quad (6.23)$$

In terms of the inverses $L^\mu_i$ of the $L^i_\mu$’s they are can be represented as the differential operators $X_i = L^i_\mu \partial_\mu$. Then the alternative parametrization of the gauge fields is

$$A_i(\tau) = \frac{1}{2} \omega_j(\tau) C_{ij} X_j , \quad \hat{A}_i(\tau) = \frac{1}{2} \hat{\omega}_j(\tau) C_{ij} X_j , \quad S_i(\tau) = \frac{1}{2} s_j(\tau) C_{ij} X_j . \quad (6.24)$$
Then \((6.7)\) gives rise to the system
\[
\frac{d\bar{s}_1}{d\tau} = -s_2 s_3 + s_1(s_2 + s_3), \quad \text{(and cyclic perms.)},
\]
which is a complex generalization of the Halphen system \((1.4)\). One can also show that the constraint \((6.8)\) is respected by the ansatz \((6.24)\). For the real components we have
\[
\frac{d\omega_1}{d\tau} = \omega_2 \omega_3 - \omega_1(\omega_2 + \omega_3) - \hat{\omega}_2 \hat{\omega}_3 + \omega_1(\hat{\omega}_2 + \hat{\omega}_3), \quad \text{(and cyclic perms.)},
\]
\[
\frac{d\omega_1}{d\tau} = \hat{\omega}_2 \omega_3 + \hat{\omega}_3 \omega_2 - \omega_1(\omega_2 + \omega_3) - \omega_1(\hat{\omega}_2 + \hat{\omega}_3), \quad \text{(and cyclic perms.)}. \quad (6.26)
\]
As previously this system is a gradient flow with non-vanishing metric components, prepotential and potential given by
\[
g_{s_i \bar{s}_j} = 1 - 2 \delta_{ij}, \quad W = s_1 s_2 s_3 + \bar{s}_1 \bar{s}_2 \bar{s}_3,
\]
\[
V = |s_1 s_2 + s_1 s_3 + s_2 s_3|^2 - 2(|s_1 s_2|^2 + |s_1 s_3|^2 + |s_2 s_3|^2). \quad (6.27)
\]
Note that the expression \((6.17)\) is a constant of motion in this case as well. However, we were not able to find additional constants of motion. Also notice that, the discrete transformation \((6.15)\) is no longer a symmetry of the system \((6.26)\).

### 6.1 The massive case

We may choose in passing from the 8-dimensional self-duality conditions \((2.3)\) and \((2.4)\) to the 7-dimensional ones to keep a non-zero piece proportional to \(\psi_{abc} F_{bc}\) transforming as a 7 of \(G_2\). We only require that it is proportional to the gauge field \(A_a\), i.e. \(F_{8a} = mA_a\).

This is in complete analogy with the weak holonomy metrics (a notion introduced in [33]) that were recently investigated in [31]. Then \((6.1)\) is modified as
\[
F_{ab} = -\frac{1}{2} \psi_{abcd} F_{cd} - m \psi_{abc} A_c. \quad (6.28)
\]
We emphasize that even in this case the seven conditions \(F_{7i} = \ldots, F_{\hat{7}i} = \ldots\) and \(F_{ii} = \ldots\) suffice to satisfy the rest. Then, by performing the usual reduction to one dimension we find that the system \((6.3)\) and the constraint \((6.6)\) are modified as
\[
\frac{d\hat{A}_i}{d\tau} = \frac{1}{2} \epsilon_{ijk} ([\hat{A}_j, \hat{A}_k] - [A_j, A_k]) + mA_i,
\]
\[
\frac{dA_i}{d\tau} = -\epsilon_{ijk} [A_j, \hat{A}_k] - m \hat{A}_i, \quad (6.29)
\]
and
\[
[A_i, A_j] = -mA_7. \quad (6.30)
\]
This shows clearly that the effect of keeping the 7 is to produce a mass term. However, we will not proceed in investigating further this system in the present paper.
7 Concluding remarks

We have shown that certain reductions to one dimension of the $\text{Spin}(7)$ invariant 8-dimensional self-dual YM equations result into systems that can be completely integrated using techniques from algebraic geometry. The different inequivalent solutions are characterized by the genus of certain Riemann surfaces. We find it rather remarkable that the same system and its solution arose in the constructions of domain wall solutions in gravitational theories with scalars, corresponding to sectors of gauged supergravities in diverse dimensions. These solutions were constructed before \[20, 21\] in studies of the Coulomb branch of SYM theories in the context of the AdS/CFT correspondence. We have also considered reductions to one dimension of the $G_2$ invariant 7-dimensional self-dual YM equations that preserve an $SU(2) \times SU(2) \times Z_2$ symmetry. This system has the same symmetries as a similar system that arose recently in the construction of $G_2$ holonomy metrics in Euclidean 7-dimensional gravity \[18, 32\]. It is very interesting to explore the possibility that there is a change of variables that maps one system to the other. We have been able to find three integrals of motion for the system (6.12) and in fact solve it in general for the case that there is an extra $U(1)$ symmetry. Hence, if such a mapping exists, it will be advantageous towards systematizing the search for new $G_2$ holonomy manifolds. We hope to report work along this direction in the feature.

Acknowledgements

This research was supported by the European Union under TMR-ERBFMRX-CT96-0045 and -0090, by the Swiss Office for Education and Science, by the Swiss National Foundation and by the contract HPRN-CT-2000-00122.
A Octonionic identities

The octonionic non-associative algebra is given by

\[
e_a \cdot e_b = -\delta_{ab}e_0 + \psi_{abc}e_c , \quad a, b, c = 1, 2, \ldots, 7 ,
\]  

(A.1)

where \(e_0\) is the unit element and \(\psi_{abc}\) are the octonionic structure constants. In the standard basis \[34\]

\[
\psi_{123} = \psi_{516} = \psi_{624} = \psi_{435} = \psi_{471} = \psi_{673} = \psi_{572} = 1 .
\]  

(A.2)

The 4-index totally antisymmetric tensor is defined as

\[
\psi_{abcd} = \frac{1}{3!} \epsilon_{abcdefg} \psi_{efg}
\]  

(A.3)

and in the standard basis is given by

\[
\psi_{1245} = \psi_{2671} = \psi_{3526} = \psi_{4273} = \psi_{5764} = \psi_{6431} = \psi_{7531} = 1 .
\]  

(A.4)

The tensors \(\psi_{abc}\) and \(\psi_{abcd}\) can be assembled in a single object \(\Psi_{\alpha\beta\gamma\delta}\) of \(SO(8)\) as:

\[
\alpha = (i, 8) , \quad \Psi_{abc8} = \psi_{abc} , \quad \Psi_{abcd} = \psi_{abcd} .
\]  

(A.5)

Then

\[
\Psi_{\alpha\beta\gamma\delta} = \frac{\epsilon}{4!} \epsilon_{\alpha\beta\gamma\delta\xi\eta\rho\kappa} \Psi_{\xi\eta\rho\kappa} , \quad e = \pm 1 .
\]  

(A.6)

For \(e = 1\) it is selfdual and for \(e = -1\) antiselfdual.

The basic identity is\[7\]

\[
\Psi_{\alpha\beta\gamma\delta} \Psi_{\xi\eta\rho\kappa} = \delta_\gamma^\xi (\delta_\alpha^\xi \delta_\beta^\eta - \delta_\alpha^\eta \delta_\beta^\xi) + \delta_\eta^\rho (\delta_\alpha^\rho \delta_\beta^\kappa - \delta_\alpha^\kappa \delta_\beta^\rho) + \delta_\xi^\kappa (\delta_\alpha^\kappa \delta_\beta^\sigma - \delta_\alpha^\sigma \delta_\beta^\kappa) - \epsilon (\Psi_{\alpha\beta} \delta_\eta^\sigma + \Psi_{\alpha\beta} \delta_\rho^\sigma + \Psi_{\alpha\beta} \delta_\kappa^\sigma)\]

\[
- \epsilon (\Psi_{\gamma\alpha} \delta_\eta^\sigma + \Psi_{\gamma\alpha} \delta_\rho^\sigma + \Psi_{\gamma\alpha} \delta_\kappa^\sigma)\]

\[
- \epsilon (\Psi_{\beta\gamma} \delta_\eta^\sigma + \Psi_{\beta\gamma} \delta_\rho^\sigma + \Psi_{\beta\gamma} \delta_\kappa^\sigma) .
\]  

(A.7)

From this we derive several other identities

\[
\Psi_{\alpha\beta\gamma\delta} \Psi_{\gamma\delta\xi\eta} = 6 (\delta_\alpha^\gamma \delta_\beta^\delta - \delta_\beta^\gamma \delta_\alpha^\delta) - 4 \epsilon \Psi_{\alpha\beta} \gamma^\delta .
\]  

(A.8)

\[9\]If \(\Psi_{\alpha\beta\gamma\delta}\) was replaced by \(\epsilon_{\alpha\beta\gamma\delta}\) in four-dimensions, only the first line below should be kept.
and

\[ \psi_{abf} \psi_{cdef} = -\epsilon \psi_{ade} \delta^c_b - \epsilon \psi_{acg} \delta^e_b + \epsilon \psi_{bed} \delta^c_a + \epsilon \psi_{bce} \delta^d_a, \]
\[ \psi_{abc} \psi_{cd} = \delta^d_c \delta^b_a - \delta^d_b \delta^c_a - 2 \epsilon \psi_{ab} \delta^c_d, \]
\[ \psi_{abcdef} \psi_{cdef} = 4(\delta^d_c \delta^b_a - \delta^d_b \delta^c_a) - 2 \epsilon \psi_{ab} \delta^c_d, \]
\[ \psi_{abcdef} \psi_{cdef} = -4 \epsilon \psi_{ab} \delta^c_d. \] (A.9)

We can define an adjoint-like representation for seven-dimensional matrices as

\[ \psi_a : \ (\psi_a)_{bc} = \psi_{abc}. \] (A.11)

Then, using the above properties we find that

\[ [\psi_a, \psi_b]_{cd} = \delta_{ad} \delta_{bc} - \delta_{ac} \delta_{bd} - 2 \epsilon \psi_{abcd}. \] (A.12)

and

\[ \psi_{abc} [\psi_a, \psi_c] = 6 \psi_a, \quad \text{Tr}(\psi_a \psi_b) = -6 \delta_{ab}, \quad \text{Tr}([\psi_a, \psi_b] \psi_c) = -6 \psi_{abc}. \] (A.13)

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