Shear viscosity $\eta$ to electric conductivity $\sigma_{el}$ ratio for the Quark-Gluon Plasma

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(Dated: July 10, 2014)

The transport coefficients of strongly interacting matter are currently subject of intense theoretical and phenomenological studies due to their relevance for the characterization of the quark-gluon plasma produced in ultra-relativistic heavy-ion collisions (uRHIC). We discuss the connection between the shear viscosity to entropy density ratio, $\eta/s$, and the electric conductivity, $\sigma_{el}$. We note that once the relaxation time is tuned to determine the shear viscosity $\eta$ to have a minimum value $\eta/s = 1/4\pi$ near the critical temperature $T_c$, one simultaneously predicts an electric conductivity $\sigma_{el}/T$ very close to recent lQCD data. More generally, we discuss why the ratio of $\eta/s$ over $\sigma_{el}/T$ supplies a measure of the quark to gluon scattering rates whose knowledge would allow to significantly advance in the understanding of the QGP phase. We also predict that ($\eta/s$)/($\sigma_{el}/T$), independently on the running coupling $\alpha_s(T)$, should increase up to about $\sim 50$ for $T \to T_c$, while it goes down to a nearly flat behavior around $\sim 3$ for $T \geq 4T_c$.

Relativistic Heavy Ion Collider (RHIC) at BNL and Large Hadron Collider (LHC) at CERN have produced a very hot and dense system of strongly interacting particles as in the early universe with energy densities and temperature largely above the transition temperature $T_c \approx 160$ MeV [1, 2] expected for the transition from nuclear matter to the Quark-Gluon Plasma (QGP) [3, 4]. The phenomenological studies by viscous hydrodynamics [5, 6] and parton transport [7, 8] of the collective behavior of such a matter has shown that the QGP has a very small shear viscosity to entropy density ratio $\eta/s$, quite close to the conjectured lower-bound limit for a strongly interacting system in the limit of infinite coupling $\eta/s = 1/4\pi$ [15]. This suggest that hot QCD matter could be a nearly perfect fluid with the smaller viscous dynamics ever observed, even less dissipative than the ultra cold matter created by magnetic traps [16, 17]. Furthermore as for atomic and molecular systems, a minimum in $\eta/s$ is expected slightly above $T_c$ [18, 19].

Being the Hot QCD matter a plasma, another key transport coefficient, yet much less studied, is the electric conductivity $\sigma_{el}$. This transport coefficient represents the linear response of the system to an applied external electric field. Several processes occurring in uRHIC as well as in the Early Universe are regulated by the electric conductivity. HICs are expected to generate very high electric and magnetic field ($\varepsilon E \simeq eB \simeq m_{\pi}^2$, with $m_{\pi}$ the pion mass) in the very early stage of the collisions [20, 21]. A large value of $\sigma_{el}$ would lead to a relaxation time of the electromagnetic field of the order of $\sim 1 - 2 fm/c$ [22, 23], which would be of fundamental importance for the strength of the Chiral-Magnetic Effect [24], a signature of the CP violation of the strong interaction. Also in mass asymmetric collisions, like Cu+Au, the electric field directed from Au to Cu induces a current resulting in charge asymmetric collective flow directly related to $\sigma_{el}$ [21]. Furthermore $\sigma_{el}$ can be directly related to the emission rate of soft photons that should be proportional to $\sigma_{el}$ [25], accounting for the exponential rise observed in their spectra [26, 27]. Despite its relevance there is yet only a poor theoretical and phenomenological knowledge of $\sigma_{el}$ and its temperature dependence. First preliminary studies in lQCD has extracted only few estimates with large uncertainties [24] and only recently more safe extrapolation from the current correlator has been developed [25, 51, 52].

In this Letter we emphasize the main elements determining the conductivity for a QGP plasma and in particular the connection with the shear viscosity $\eta$. In fact, while one may expect that the QGP is quite a good conductor due to the deconfinement of color charges, on the other hand, the very small $\eta/s$ indicates large scattering rates which can largely damp the conductivity, especially if the plasma is dominated by gluons that do not carry any electric charge.

The electric conductivity can be formally derived from the Green-Kubo formula and it is related to the relaxation of the current-current correlator for a system in thermal equilibrium. It can be written as $\sigma_{el} = V/T \langle \vec{J}(t = 0) \cdot \vec{J}(t = 0) \rangle \cdot \tau$, where $\tau$ is the relaxation time of the correlator and the initial value can be related to the thermal average $\frac{\vec{J}^2}{T}$ [33]. Generalizing to the case of QGP one can write:

$$\sigma_{el} = \frac{e^2}{3T} \langle \vec{p}^2 / E^2 \rangle \sum_{j=q, \bar{q}} f_j^2 \tau_j \rho_j = \frac{e^2}{3T} \langle \vec{p}^2 / E^2 \rangle \tau \rho_q,$$

(1)

where $e^2 = e^2 \sum_{q, \bar{q}} f_j^2 = 4e^2/3$ with $f_j$ the fractional charge. Eq. (1) in the non-relativistic limit reduces to the Drude formula $\sigma_{el} \frac{e^2}{m}$. The relaxation time of a particle of species $j$ in terms of cross-sections and particle densities can be written in the relaxation time approximation (RTA) as $\tau_j = \sum_{q, \bar{q}} \langle \rho_j v_{rj}^{(i)} \sigma_{eij}^{(i)} \rangle$, where $j = q, \bar{q}$ while the sum runs over all particle species with $\rho_i$ the density of species $i$, $v_{rj}^{(i)}$ is the relative velocity and $\sigma_{eij}^{(i)}$ is the transport cross-section. In Ref. [34] it
has been shown that RTA is able to describe with quite good approximation $\sigma_{\text{eq}}$ as can be numerically evaluated by measuring the current induced by an external electric field. As usually done within the Hard-Thermal-Loop (HTL) approach, we will consider the cross section regulated by a screening Debye mass $m_D = g(T)T$, with $g(T)$ being the strong coupling:

$$\sigma_{\text{tot}}^j = \beta^j \sigma(s) = \beta^j \frac{\pi \alpha_s^2}{m_D^2} \frac{s}{s + m_D^2}$$

(2)

where $\alpha_s = g^2/4\pi$ and the coefficient $\beta^j$ depends on the pair of interacting particles: $\beta^{qq} = 16/9$, $\beta^{qg} = 8/9$, $\beta^{gg} = 2$, $\beta^{gg} = 9$. We notice that this factor is directly related to the quark and gluon Casimir factor, for example $\beta^{qg}/\beta^{gg} = (C_F/C_A)^2 = (4/9)^2$.

The shear viscosity $\eta$ is known from the Green-Kubo relation to be given by $\eta = V/\tau (\Pi^{xy}(t = 0)\Pi^{xy}(t = 0)) \cdot \tau$, where $\tau$ is the initial value of the correlator of the transverse components of the energy-momentum tensor can be written as $\frac{\beta^j}{15}(p^4/E^2)$ [45, 54]. Hence for a system with different species it can be written in the RTA approximation [38, 39] as:

$$\eta = \frac{1}{15T} \left( \frac{p^4}{E^2} \right) \left( \tau_q p_q^\text{tot} + \tau_g p_g \right)$$

(3)

where the relaxation time $\tau_q$ has a similar expression as above with $j = g$. The thermodynamical part entering this expression, as the one in Eq. (1), will be fixed employing a quasi-particle (QP) scheme tuned to reproduce the bulk thermodynamics evaluated by lQCD [40], similarly to [41, 42]. The $p^4/E^2$ in a massless approximation is simply $4\pi T/\rho$, an expression that remains valid with good approximation in the QP model employed, meaning that the first term in Eq. (3) is essentially determined by the lQCD thermodynamics. The QP model, as an effective way to describe microscopically the QGP, has become a quite solid approach at least for $T > 2 - 3 T_c$, especially in the improved NNLO HTLpt version [15].

However here it is merely exploited as a flexible tool to fix the thermal averages entering the transport coefficients, while not necessarily endorsing it as the most appropriate microscopic description of the QGP. We also notice that a self-consistent dynamical model (DQPM), that includes also the pertinent spectral function, has been developed in [12] and leads to nearly identical behavior of the strong coupling $g(T)$. The simple QP model has the advantage to handle simpler analytical expression to pin down the core physics. We will consider the DQPM explicitly, showing that the considerations elaborated in this Letter are quite general and can be only marginally affected by particle width. The quark and gluon masses are given by $m_q^2 = 3/4 g^2 T^2$ and $m_g^2 = 1/3 g^2 T^2$ in terms of a running coupling $g(T)$ that is determined by a fit to the lattice energy density, which allows to well describe also the pressure $P$ and entropy density $s$ above

$T_c = 160$ MeV. In Ref. [40] we have obtained:

$$g_{QP}^2(T) = \frac{48\pi^2}{(11 N_c - 2 N_f)} \ln \left[ \lambda \left( \frac{T - T_c}{T_q} \right) \right]$$

(4)

with $\lambda = 2.6$, $T_q/T_c = 0.57$. For its general interest and asymptotic validity for $T \rightarrow \infty$, we also consider the behavior of the QCD running coupling constant for the evaluation of transport relaxation time: $\eta_{QCD}(T) = \frac{\pi \eta s}{9} \ln^{-1} \left( \frac{2\pi T}{\Lambda_{QCD}} \right)$. On one hand, close to $T_c$, such a case misses the dynamics of the phase transition, on the other hand it allows to see explicitly what is the impact of a different running coupling. We first estimate the shear viscosity $\eta/s$ that is shown in Fig. 1. Red dashed line is the result for the QP model, blue dotted line is obtained using $g_{QCD}$ green open circles the DQPM [40] and by symbols several lQCD results (see captions for more details). The main difference between our QP model and DQPM comes from the fact that the latter assumes isotropic scatterings which decrease the relaxation time by about $30 - 40\%$. We have checked that for isotropic scatterings the results become almost identical. Anyway, the $\eta/s$ predicted is toward higher value with respect to the conjectured minimum value of $\eta/s \sim 0.08$, supported also by several phenomenological estimates [38, 40]. This means that if the $\eta/s$ is really very close to $1/4\pi$ then the relaxation time estimated is still too large. However, it has to be mentioned that, within the QP model, it has been discussed in the literature also another approach. This assumes the relaxation times $\tau_{q,g} = C_{q,g} g^T \ln(a/g^T)$ [17] and fixes $C_{q,g}$ and $a$ to reproduce asymptotically the pQCD estimate [38] and a minimum for $\eta/s(T) = 1/4\pi$. We do not report also

![FIG. 1. Shear viscosity to entropy density ratio $\eta/s$: dashed line represents QP model results, dot-dashed line is pQCD, open circles is DQPM [40]. Red thick solid line and blue thin solid line are obtained rescaling $g(T)$. Blue dotted line is Ads/CFT result from [15]. Symbols are lattice date: full squares [54], diamonds and triangles [55], open and full circles [40].](image)
this case, see \cite{40,43}, but, in the T region of interest, the result is quite similar to upscaling the coupling $g(T)$ by a k-factor in such a way to have the minimum of $\eta/s$. The corresponding curves are shown in Fig. 1 by red thick solid line for the $g_{QCD}(T)$ coupling (rescaled by $k = 1.59$) and by blue thin solid line for the $g_{QCD}(T)$ (rescaled by $k = 2.08$). One obtains $\tau_q \simeq \tau_q/2 \sim 0.2 \text{fm}/c$ and also $\eta/s(T)$ roughly linearly rising with $T$ in agreement with quenched IQCD estimates, full circles \cite{49}.

A main point we want to stress is that, once the relaxation time is set to an $\eta/s(T) = 0.08$, the $\sigma_{el}/T$ predicted, with the same $\tau_q$ as for $\eta/s$, is in quite good agreement with most of the IQCD data, shown by symbols in Fig. 2 (see caption for details). Therefore a low $\sigma_{el}/T$ is obtained at variance with the early IQCD estimate, Ref. \cite{28}, as a consequence of the small $\tau_q/g$ that are associated with $\eta/s \simeq 0.08$. In Fig. 2 we show also the predictions of DQPM (green open circles) \cite{34,40}.

In Fig. 2 we plot also the $N = 4$ Super Yang Mills (green dotted line) electric conductivity \cite{54} that predicts a constant behavior for $\sigma_{el}/T = e^2 N_c^2/(16\pi) \simeq 0.0164$. We note that in our framework one instead expects that, even if the $\eta/s$ is independent on the temperature, the $\sigma_{el}$ should still have a strong T-dependence. This can be seen noticing that one can write, with good approximation, $\eta/s \simeq T^{-2}\tau_\rho$, being $(p^4/E^2) \simeq \epsilon T/\rho$, and $\sigma_{el}/T \simeq g^{-1}(T)T^{-2}\tau_\rho$, being $(p^4/E^2) \simeq T/m(T)$, which leads to a simple relation between shear viscosity and electric conductivity $\sigma_{el}/T \simeq g^{-1}(T)\eta/s$. This leads to a steep decrease of $\sigma_{el}/T$ close to $T_c$ due to the increasing of the non-perturbative coupling for $T \rightarrow T_c$ or, from a classical point of view, to the growing of the charge carrier’s inertia.

While the electric conductivity appears to be consistent with a minimal $\eta/s$, the T dependence of both $\sigma_{el}$ and $\eta/s$ are similarly regulated by the specific behavior of $\tau_{q,g}$. We note that the ratio $(\eta/s)/\sigma_{el}/T$ can be written, from Eq. 1 and Eq. 10, as:

$$\frac{\eta/s}{\sigma_{el}/T} = \frac{6}{5} \frac{T(p^2/E^2)}{\epsilon^2} \left(1 + \frac{\tau_q}{\tau_q}\right).$$

A main feature of such a ratio is its independence on the $k$-factor introduced above, and more importantly even on the T dependence of the $g(T)$ coupling. As we can see writing explicitly the transport relaxation time for quarks and gluons:

$$\tau_{q^{-1}} = \langle (s)v_{rel}\rangle (\rho_q \sum_{u,d,s} \beta_{q}^{q'1} + \rho_g \beta_{q}^{g99})$$

$$\tau_{g^{-1}} = \langle (s)v_{rel}\rangle (\rho_q^{tot} \beta_{q}^{q99} + \rho_g \beta_{q}^{g99})$$

(6)

where the $\beta^{ij}$ were defined above. Hence the ratio of transport relaxation times appearing in Eq. 5 can be written as:

$$\tau_{q/g} = \frac{C_q^9 + \rho_u}{C_q^9 + \rho_g}$$

(7)

where the coefficient $C_q^9 = (\beta_{q}^{q99} + \beta_{q}^{q99} + 2\beta_{q}^{q99} + 2\beta_{q}^{q99})/\beta_{q}^{q99}$ and $C_g^9 = \beta_{g}^{g99}/\beta_{q}^{q99}$ is the relative magnitude between quark-(anti-)quark and gluon-gluon with respect to gluon-quark scatterings. Using the standard pQCD factors for $\beta_{q}^{ij}$, $C_q^{pQCD} = \frac{C_q^{9}}{2} \simeq 3.1$ and $C_g^{pQCD} = \frac{C_g^{9}}{2}$.  

In Fig. 3 we show $(\eta/s)/\sigma_{el}/T)$ as a function of $T/T_c$; the red solid line is the prediction for the ratio using $g_{QCD}(T)$, but it is clear from the Eq. 6 that the
ratio is completely independent on the running coupling itself; the result for $g_{\rho QCD}(T)$ is shown by blue dashed line. The ratio is instead sensitive just to the relative strength of the quark (anti-quark) scatterings with respect to the gluonic ones, hence we suggest that a measurement in QCD can shed light on the relative scattering rates of quarks and gluons, providing an insight into their relative role. It is not known if such ratios, linked to the Casimir factors of SU(3)$_c$, are kept also in the non-perturbative regime, which may be not so unlikely. On the other hand at least at large temperatures ($T > 5 - 10 T_c$) this should be the case and deviation from the obtained value, $(\eta/s)/(\sigma_{el}/T) \approx 3$, would be quite surprising. As $T \to T_c$, a steep increase is predicted that is essentially regulated by $(g^2/E^2)$. It is interesting to notice that in the massless limit the factor before the parenthesis becomes a temperature independent constant and hence also the ratio. This is in quite close agreement with the AdS/CFT prediction shown by dotted line in Fig. 2.

We also briefly want to mention that one possible scenario could be that when the QGP approaches the phase transition, the confinement dynamics becomes dominant and the di-quark $qq$ scattering, precursors of mesonic states, and di-quark $qq$ states, precursors of baryonic states, are strongly enhanced by a resonant scattering respect to other channels, as found in a T-matrix approach in the heavy quark sector. For this reason, we explore the sensitivity of the ratio $(\eta/s)/(\sigma_{el}/T)$ on the magnitude of $C^q$ and $C^g$. The orange solid line shows the behavior for an enhancement of the quark scatterings, $C^q = 10 C^q_{\rho QCD}$. We can see in Fig. 4 that this would lead to an enhancement of the ratio by about a 40%. We also see that instead the ratio is not very sensitive to a possible enhancement of only the $gg$ scattering with respect to the $qq, qg, gg$; in fact even for $C^q = 10 C^q_{\rho QCD}$ one obtains the thin black solid line. This is due to the fact that already in the pQCD case $\tau_g/\tau_q \sim 0.3 - 0.4$. Furthermore the gluon density at equilibrium is quite smaller than the total quark one, in the massless limit $\rho_g/\rho_q^{tot} \approx d_g/d_{q+\bar{q}} = 4/9$ even not dwelling on the details of the QP model where the larger gluon mass further decreases this ratio. Therefore the second term in parenthesis in Eq. 5 is of the order of $10^{-1}$ in the pQCD case and a further decrease of its value would not be visible because the ratio is anyway dominated by the first term equal to one. Also for this reason the $(\eta/s)/(\sigma_{el}/T)$ should be a quite stable predictions and large deviation from the predicted value would be a signature of unknown and unexpected properties of the QGP. We reported in Fig. 4 also the ratio from the DQPM model, as deduced from [44] and we can see that, even if it is not evaluated through Eq. 5, it is in very good agreement with our general prediction. In Fig. 2 we also display by symbols the ratio evaluated from the available IQCD data, considering for $\eta/s$ those smaller than four times the minimum value, while for $\sigma_{el}/T$ we choose red diamonds as a lower limit (filled symbols) and the others in Fig. 2 as an upper limit (open symbols), excluding only the grey squares. In order to be able to compute the ratio $(\eta/s)/(\sigma_{el}/T)$ at the same temperature of $\eta/s$ lattice data, we do an interpolation between the data point of $\sigma_{el}$. We warn to consider these estimates only as a first rough indications, in fact the lattice data collected of $\eta/s$ and $\sigma_{el}$ are obtained with different actions among them and also with respect to the most realistic one for the EoS, that we employed to tune the QP model.

**Conclusions** - In this Letter we point out the direct relation between the shear viscosity $\eta$ and the electric conductivity $\sigma_{el}$. In particular, we have discussed why most recent IQCD data predicting an electric conductivity $\sigma_{el} \approx 10^{-2} T$ for $T < 2 T_c$, appears to be consistent with a fluid at the minimal conjectured viscosity $4 \pi \eta/s \approx 1$, while the data of Ref. [28] appear to be hardly reconcilable with it. Also a steep rise of $\sigma_{el}/T$, in agreement with IQCD data, appears quite naturally in the quasi-particle approach as inverse of the self-energy determining the effective masses needed to correctly reproduce the IQCD thermodynamics. This result is at variance with the AdS/CFT, but our analysis suggests that it is regulated by constraints coming from thermodynamics. It would be quite interesting to see if an AdS/QCD approach, able to correctly describe the interaction measure of IQCD, will modify the AdS/CFT prediction.

We identify the dimensionless ratio $(\eta/s)/(\sigma_{el}/T)$ as independent on the uncertainties in the running coupling $g(T)$. Moreover due to the fact that gluons do not carry an electric charge, the ratio is regulated by the relative strength and chemical composition of the QGP through the term $1 + \tau_g/\tau_q (\tau_q^{tot})$. Our analysis provides the baseline of such a ratio that in this decade will most likely be more safely evaluated thanks to the developments of IQCD techniques. This will provide a first and pivotal insight into the understanding of the relative role of quarks and gluons in the QGP also if deviations from our predictions for $(\eta/s)/(\sigma_{el}/T)$ will be seen at high temperature $T \sim 5 - 10 T_c$.

V.G. acknowledge the support of the ERC-StG Grant under the QGPdyn project. We thanks M. Ruggieri for carefully reading the manuscript.

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[1] B. V. Jacak and B. Muller, *Science* **337**, 310 (2012).
[2] E. Shuryak, *Prog.Part.Nucl.Phys.* **53**, 273 (2004).
[3] S. Borsanyi et al., *JHEP* 1009 073 (2010).
[4] A. Bazavov, T. Bhattacharya, M. Cheng, C. DeTar, H. T. Ding, S. Gottlieb, R. Gupta and P. Hegde et al.,
