INTEGRABLE SPINOR/QUATERNION GENERALIZATIONS
OF THE NONLINEAR SCHRÖDINGER EQUATION

STEPHEN C. ANCO\textsuperscript{1} and AHMED M.G. AHMED\textsuperscript{2} and ESMAEEL ASADI\textsuperscript{3}

\textsuperscript{1} DEPARTMENT OF MATHEMATICS AND STATISTICS
BROCK UNIVERSITY
ST. CATHARINES, ON L2S3A1, CANADA

\textsuperscript{2} DEPARTMENT OF MATHEMATICS
UNIVERSITY OF SOUTH FLORIDA
TAMPA, FL 33620, USA

\textsuperscript{3} DEPARTMENT OF MATHEMATICS
INSTITUTE FOR ADVANCE STUDIES IN BASIC SCIENCE (IASBS)
45137–66731, ZANJAN, IRAN

Abstract. An integrable generalization of the NLS equation is presented, in which the dynamical complex variable $u(t, x)$ is replaced by a pair of dynamical complex variables $(u_1(t, x), u_2(t, x))$, and $i$ is replaced by a Pauli matrix $\sigma$. Integrability is retained by the addition of a nonlocal term in the resulting 2-component system. A further integrable generalization is obtained which involves a dynamical scalar variable and an additional nonlocal term. For each system, a Lax pair and a bi-Hamiltonian formulation are derived from a zero-curvature framework that is based on symmetric Lie algebras and that uses Hasimoto variables. The systems are each shown to be equivalent to a bi-normal flow and a Schrodinger map equation, generalizing the well-known equivalence of the NLS equation to the bi-normal flow in $\mathbb{R}^3$ and the Schrodinger map equation in $S^2$. Furthermore, both of the integrable systems describe spinor/quaternion NLS-type equations with the pair $(u_1(t, x), u_2(t, x))$ being viewed as a spinor variable or equivalently a quaternion variable.

1. Introduction

The nonlinear Schrödinger (NLS) equation $u_t = i(u_{xx} + \frac{1}{2}|u|^2 u)$ is an important integrable system that arises in numerous areas of mathematical physics. As a consequence, there has been considerable mathematical and physical interest in finding and studying multi-component generalizations that retain its integrability features.

One key feature is that the NLS equation possesses a hierarchy of symmetries which are generated by a recursion operator applied to a root symmetry given by an infinitesimal phase rotation $u \rightarrow iu$. These symmetries correspond to commuting bi-Hamiltonian flows which come from a Lax pair formulated as a zero-curvature equation in the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$.

Here, a novel generalization of the NLS equation and its symmetry structure will be introduced, based on replacing the $U(1)$ phase rotation group by the larger group $SU(2)$, and replacing the complex scalar variable $u$ by a pair of complex scalar variables $(u_1, u_2)$,
with the root symmetry consisting of \((u_1, u_2) \rightarrow (u_1, u_2) \sigma\) where \(\sigma\) is a Pauli matrix. The generalization is constructed by use of a zero-curvature equation in larger Lie algebras. This framework yields a Lax pair and a pair of compatible Hamiltonian operators as well as a recursion operator.

One novelty of this generalization is that it does not retain any \(U(1)\) symmetry. In contrast, the known multi-component generalizations of the NLS equation, such as the Fordy-Kullish NLS systems \([1]\) associated to Hermitian symmetric Lie algebras and the unitarily-invariant NLS systems \([2]\) associated to Riemannian symmetric Lie algebras with a unitary subalgebra, all involve an invariance group whose center is a \(U(1)\) subgroup. Their root symmetry comes from multiplication by \(i\) on their variables. Generalizing \(U(1)\) to \(SU(2)\), with \(i\) replaced by \(\sigma\), leads to the interesting structure that the variables in the resulting \(SU(2)\) integrable system can be identified with the components of a spinor representation of \(Spin(3)\), so thus the system will describe an integrable NLS spinor equation for \(u = (u_1, u_2)\).

Another interesting feature of this generalization is that the spinor variable \(u\) can be identified with a quaternion variable \(u = u_1 + u_2j\), with \(i, j, k = ij\) being the imaginary basis quaternions. Similarly, \(\sigma\) can be identified with an imaginary unit quaternion \(q\). In terms of the quaternion variable \(u\), the NLS spinor equation can be thereby expressed as a quaternion NLS equation, which provides a novel integrable generalization of the NLS equation in which \(i\) is replaced by \(q\).

There are several zero-curvature equations from which the NLS equation can be derived. The one that will be most useful for the present work is based on the Lie algebra of Euclidean space, \(e(3) = \mathbb{R}^3 \rtimes so(3)\). This Lie algebra can be viewed in a natural way as a contraction of the symmetric Lie algebra \(so(4)\). It also is the one that underlies the Hasimoto transformation \([3]\) showing the equivalence of the NLS equation and the vortex filament equation in fluid mechanics.

For generalizing the NLS equation to an \(SU(2)\) system, the Euclidean Lie algebra will be replaced by a Lie algebra obtained from a similar contraction of the symmetric Lie algebra \(su(4)/sp(2)\). Another integrable \(SU(2)\) generalization will be derived by considering a contraction of the symmetric Lie algebra \(so(6)/u(3)\), which will lead to an NLS-type system involving a real scalar variable in addition to a pair of complex scalar variables. An important ingredient in these derivations will be the use of Hasimoto variables associated naturally to the symmetric structure of the Lie algebras, which utilizes the general results in Ref. \([4]\) yielding explicit bi-Hamiltonian operators associated to Riemannian symmetric Lie algebras.

Through the Hasimoto transformation, the NLS equation is well known to correspond to a bi-normal equation for the motion of an inelastic (non-stretching) curve in Euclidean space. In turn, this motion is well known to be equivalent to a Schrödinger map into the 2-sphere, \(S^2 \subset \mathbb{R}^3\). (For an overview, see Ref. \([5, 6]\).) Similar geometric formulations hold for the two \(SU(2)\) NLS-type systems. As a result, novel geometric generalizations of the bi-normal equation and the Schrödinger map are obtained.

In section\([2]\) the zero-curvature derivation of the NLS equation from the Lie algebra \(so(4)\) is summarized to illustrate the method and the notation. Its relationship to the \(\mathfrak{sl}(2, \mathbb{C})\) zero-curvature framework will be discussed briefly. In section\([3]\) the derivation of the first \(SU(2)\) NLS-type integrable system is presented, and it is formulated as a quaternion equation. Its
Lax pair and bi-Hamiltonian structure also are stated. In section 4, the second $SU(2)$ NLS-type integrable system and its corresponding quaternion formulation are presented, along with its Lax pair and bi-Hamiltonian structure. In section 5, the geometric formulations of the systems are summarized.

Some concluding remarks are made in section 6.

2. Zero-curvature derivation of the NLS equation

To begin, some brief general preliminaries will be summarized. (For more details, see Ref. [7, 4].) A symmetric Lie algebra, $(\mathfrak{g}, \mathfrak{h})$, has the structure $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ in which $\mathfrak{h}$ is a subalgebra and $\mathfrak{m}$ is a subspace with the bracket relations

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}. \quad (2.1)$$

There is Cartan-Killing form $K$ on $\mathfrak{g}$, where $K(\mathfrak{h}, \mathfrak{m}) = 0$. When $\mathfrak{g}$ is compact, the Cartan-Killing form provides a negative-definite inner product on $\mathfrak{m}$.

Contraction of $\mathfrak{g}$ consists of scaling $\mathfrak{m} \rightarrow \epsilon \mathfrak{m}$ and taking the singular limit $\epsilon \rightarrow 0$, which changes the third bracket relation: $[\mathfrak{m}, \mathfrak{m}] \rightarrow 0$. This yields a Lie algebra given by the semi-direct product structure $\mathfrak{m} \bowtie \mathfrak{h}$, retaining the first and second bracket relations (2.1), with $\mathfrak{m}$ becoming an abelian subalgebra.

2.1. NLS equation from $\mathfrak{so}(4)/\mathfrak{so}(3)$. Consider the symmetric Lie algebra $(\mathfrak{so}(4), \mathfrak{so}(3))$, which has $\mathfrak{m} = \mathbb{R}^3 \simeq \mathfrak{so}(4)/\mathfrak{so}(3)$ and $\mathfrak{h} = \mathfrak{so}(3)$.

The zero-curvature framework in this Lie algebra is set up as follows. Choose a constant element $e$ belonging to a Cartan subspace in $\mathfrak{m}$, and let $H_\parallel \subset SO(3)$ denote the linear isotropy group that preserves $e$, namely $\text{Ad}(H_\parallel)e = e$. The corresponding Lie subalgebra will be denoted $\mathfrak{h}_\parallel \subset \mathfrak{h}$, where $\text{ad}(\mathfrak{h}_\parallel)e = 0$. There is an orthogonal decomposition $\mathfrak{h} = \mathfrak{h}_\parallel \oplus \mathfrak{h}_\perp$, where $K(\mathfrak{h}_\parallel, \mathfrak{h}_\perp) = 0$. Similarly, there is an orthogonal decomposition $\mathfrak{m} = \mathfrak{m}_\parallel \oplus \mathfrak{m}_\perp$, where $\text{ad}(\mathfrak{m}_\parallel)e = 0$ and $K(\mathfrak{m}_\parallel, \mathfrak{m}_\perp) = 0$.

A matrix representation of this structure is given by

$$\mathfrak{m} = \begin{pmatrix} 0 & a_1 & a_2 & a_3 \\ -a_1 & 0 & 0 & 0 \\ -a_2 & 0 & 0 & 0 \\ -a_3 & 0 & 0 & 0 \end{pmatrix} \simeq \mathbb{R}^3, \quad \mathfrak{h} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & b_1 & b_2 \\ 0 & -b_1 & 0 & b_3 \\ 0 & -b_2 & -b_3 & 0 \end{pmatrix} \simeq \mathfrak{so}(3) \quad (2.2)$$

and

$$e = \chi \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{m}. \quad (2.3)$$

Here $\chi$ is arbitrary non-zero constant related to norm of $e$: $K(e, e) = -2\chi^2$. Note that, because $(\mathfrak{so}(4), \mathfrak{so}(3))$ has rank 1 [7], its Cartan subspaces have dimension 1 and are equivalent to $\text{span}(e)$ up to the action of $\text{Ad}(SO(3))$.  


The components of this matrix equation yield the system which are taken to satisfy
\[ h = \begin{pmatrix} 0 & a_1 & 0 & 0 \\ -a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \simeq \mathbb{R}, \quad m = \begin{pmatrix} 0 & 0 & a_2 & a_3 \\ 0 & 0 & 0 & 0 \\ -a_2 & 0 & 0 & 0 \\ -a_3 & 0 & 0 & 0 \end{pmatrix} \simeq \mathbb{R} \times \mathbb{R} \simeq \mathbb{C}, \quad (2.4) \]

\[ h_\parallel \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_3 \\ 0 & 0 & -b_3 & 0 \end{pmatrix} \simeq \mathfrak{so}(2) = \mathfrak{u}(1), \quad h_\perp = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & b_1 & b_2 & 0 \\ 0 & -b_1 & 0 & 0 \\ 0 & -b_2 & 0 & 0 \end{pmatrix} \simeq \mathbb{R} \times \mathbb{R} \simeq \mathbb{C}. \quad (2.5) \]

It will be useful to consider the scaled space
\[ h' = \frac{1}{\chi} \mathfrak{ad}(e) m_\perp = -\frac{1}{\chi} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & b_1 & b_2 & 0 \\ 0 & -b_1 & 0 & 0 \\ 0 & -b_2 & 0 & 0 \end{pmatrix} \simeq h_\perp. \quad (2.6) \]

Associated to \( h' \) will be the flow variables \( (h_1', h_2') \in \mathbb{R} \times \mathbb{R} \simeq \mathbb{C} \). The Hasimoto variables \( [4] \) are defined by belonging to \( h_\perp \); \((u_1, u_2) \in \mathbb{R} \times \mathbb{R} \simeq \mathbb{C} \). Under the action of \( \mathfrak{so}(2) \), the Hasimoto variables undergo a rotation generated by \((u_1, u_2) \to (-u_2, u_1)\). Then the zero-curvature equation in \( \mathfrak{so}(4) \) is given by the matrices
\[ U = \begin{pmatrix} 0 & \chi & 0 & 0 \\ -\chi & 0 & u_1 & u_2 \\ 0 & -u_1 & 0 & 0 \\ 0 & -u_2 & 0 & 0 \end{pmatrix} \in \mathfrak{m} \oplus \mathfrak{h}, \quad V = \begin{pmatrix} 0 & \chi & 0 & 0 \\ -\chi & 0 & w_1 & w_2 \\ \chi & w_1 & 0 & 0 \\ \chi & w_2 & 0 & 0 \end{pmatrix} \in \mathfrak{m} \oplus \mathfrak{h}. \quad (2.7) \]

which are taken to satisfy
\[ D_t U - D_x V - [U, V] = 0. \quad (2.8) \]

The components of this matrix equation yield the system
\[ D_x h_\parallel = u_1 h_1' + u_2 h_2', \quad D_x w_\parallel = u_1 w_1 - u_2 w_2, \quad (2.9a) \]
\[ w_1 = h_1 u_1 + D_x h_1', \quad w_2 = h_2 u_2 + D_x h_2', \quad (2.9b) \]
\[ u_1 = D_x w_1 - w_1 u_2 + \chi^2 h_1', \quad u_2 = D_x w_2 + w_1 u_1 + \chi^2 h_2'. \quad (2.9c) \]

The isotropy group \( \mathcal{H} = SO(2) \) acts as a rotation on \((u_1, u_2), (w_1, w_2, h_1^\perp, h_2^\perp)\). Hence, it is natural to go to complex variables
\[ u = u_1 + iu_2, \quad w = w_1 + iw_2, \quad h^\perp = h_1^\perp + ih_2^\perp. \quad (2.10) \]

on which the isotropy group acts a \( U(1) \) phase rotation. In particular, this is a symmetry group of the system \((2.9)\).

In terms of the complex variables \((2.10)\), the system has the form
\[ D_x h_\parallel = \text{Re}(\bar{u} h^\perp), \quad (2.11a) \]
\[ w = h_\parallel u + D_x h^\perp, \quad (2.11b) \]
\[ D_x w_\parallel = \text{Im}(\bar{u} w), \quad (2.11c) \]
\[ u_t = D_x w + iw_\parallel u + \chi^2 h^\perp. \quad (2.11d) \]
The action of the $U(1)$ symmetry consists of a phase rotation $e^{i\phi}$. Its generator is now used to define a flow (see Ref. [4,2]) by putting

$$h^\perp = iu.$$  

(2.12)

The system (2.11) thereby yields $h_\parallel = c_1$, $w = iu_x + c_1u$, $w_\parallel = \frac{1}{2}u\bar{u} + c_2$, and

$$u_t = i(u_{xx} + \frac{1}{2}|u|^2u) + c_1u_x + (c_2 + \chi^2)iu$$

(2.13)

where $c_1, c_2$ are arbitrary constants. Thus, the NLS equation is obtained for $c_1 = 0$ and $c_2 = -\chi^2$.

The integrability structure of the NLS equation is encoded directly in the zero-curvature system (2.11).

First, through elimination of the variables $h_\parallel$ and $w_\parallel$, the system can be expressed in the operator form

$$u_t = \mathcal{H}(w), \quad w = \mathcal{J}(h^\perp)$$

(2.14)

with $\mathcal{H} = D_x + iuD_x^{-1}\text{Im} \bar{u}$ and $\mathcal{J} = D_x + uD_x^{-1}\text{Re} \bar{u}$. The general theorem in Ref. [4] establishes that $\mathcal{H}$ is a Hamiltonian operator and $\mathcal{J}$ is a compatible symplectic operator, which is well known. Their composition $\mathcal{R} = \mathcal{HJ}$ is a symmetry recursion operator. This yields the Hamiltonian formulation $u_t = \mathcal{H}(\delta \mathcal{J}/\delta \bar{u}) = \mathcal{E}(\delta \mathcal{E}/\delta \bar{u})$, where $\mathcal{E} = \mathcal{RH}$ is Hamiltonian operator compatible with $\mathcal{H}$. The first Hamiltonian is given by $\mathcal{H} = \int \text{Im}(\bar{u}_x u) \, dx$; the second Hamiltonian is $\mathcal{E} = 0$, using $D_x^{-1}(0) = c =$ constant, with $c = 1$.

Second, the matrices (2.7) and the zero-curvature equation (2.8) define a matrix Lax pair, with $h^\perp = iu$, $h_\parallel = 0$, $w = iu_x$, $w_\parallel = \frac{1}{2}u\bar{u} - \chi^2$, as given by solving the zero-curvature system (2.11):

$$U = \begin{pmatrix} 0 & \chi & 0 & 0 \\ -\chi & 0 & \text{Re} u & \text{Im} u \\ 0 & -\text{Re} u & 0 & 0 \\ 0 & -\text{Im} u & 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 0 & \chi \text{Im} u & -\chi \text{Re} u \\ 0 & 0 & -\text{Im} u_x & -\text{Re} u_x \\ -\chi \text{Im} u_x & \text{Im} u_x & 0 & \frac{1}{2}|u|^2 - \chi^2 \\ \chi \text{Re} u_x & \text{Re} u_x & -\frac{1}{2}|u|^2 + \chi^2 & 0 \end{pmatrix}.$$  

(2.15)

In this Lax pair, $\chi$ represents the spectral parameter.

It is straightforward to convert this Lax pair into a matrix representation for the contracted Lie algebra $\mathbb{R}^3 \times \mathfrak{so}(3)$ which is the Lie algebra of the isometry group $ISO(3)$ of $\mathbb{R}^3$. At the level of the zero-curvature system (2.11), the contraction simply removes the term $\chi^2 h^\perp$ in the flow equation for the Hasimoto variable $u$.

There is an alternative zero-curvature equation that yields the NLS equation. It uses the Lie algebra $\mathfrak{so}(3,1) \simeq \mathfrak{sl}(2,\mathbb{C})$, which is a symmetric Lie algebra where $\mathfrak{m} = \mathbb{R}^{2,1}$ is 3-dimensional Minkowski space and $\mathfrak{h} = \mathfrak{so}(2,1) \simeq \mathfrak{sl}(2,\mathbb{R})$ is the Lie algebra of rotations and boosts. The only change in the previous construction is that $e$ is chosen to be a constant element corresponding to a timelike vector in $\mathbb{R}^{2,1}$. The resulting matrices $U$, $V$ are the directly related to the well-known AKNS scheme [8] for the NLS equation.
3. First NLS-type system and its integrability

For writing down the $SU(2)$ generalization of the NLS equation, take a general element in the Lie algebra $su(2)$ and normalize it to get

$$J := \begin{pmatrix} i \cos \theta & e^{i \psi} \sin \theta \\ -e^{-i \psi} \sin \theta & -i \cos \theta \end{pmatrix} \in su(2), \quad \theta, \psi \in [0, 2\pi]$$  \hspace{1cm} (3.1)

(with $|J| = 1$ being the absolute norm given by the Cartan-Killing metric). This normalized general element has the properties

$$\bar{J}^t = -J, \quad J^2 = -id, \quad \det(\exp(\phi J)) = 1$$  \hspace{1cm} (3.2)

which describe an $SU(2)$ analog of the element $i$ in $u(1)$: $\bar{i} = -i, \quad i^2 = -1, \quad |e^{i \phi}| = 1$. (Throughout, a superscript “t” denotes the transpose; a subscript “0” denotes trace-free part of a matrix.)

The $SU(2)$ generalization of the NLS equation has the form

$$u_t = u_{xx}J + |u|^2 uJ + 2uD_x^{-1}(\bar{u}^t u_x J + J\bar{u}_x^t u)_0$$  \hspace{1cm} (3.3)

in terms of the variable

$$u := (u_1, u_2) \in \mathbb{C} \times \mathbb{C} = \mathbb{C}^2, \quad u_1, u_2 \in \mathbb{C}.$$  \hspace{1cm} (3.4)

The nonlocal $D_x^{-1}$ term can be rewritten in various ways via the identities

$$u(J\bar{u}^t u)_0 = \frac{1}{2}i \text{Im}(\bar{u} \cdot (uJ))u, \quad u(\bar{u}^t uJ)_0 = |u|^2 uJ - \frac{i}{2} \text{Im}(\bar{u} \cdot (uJ))u.$$  \hspace{1cm} (3.5)

In particular:

$$u_t - u_{xx}J = |u|^2 uJ + \frac{1}{4}i \text{Im}(\bar{u} \cdot (uJ))u + 2uD_x^{-1}[\bar{u}^t u_x, J]$$

$$= 3|u|^2 uJ - i \text{Im}(\bar{u} \cdot (uJ))u - 2uD_x^{-1}[\bar{u}_x^t u, J]$$

$$= 2|u|^2 uJ + 2uD_x^{-1}[\bar{u}^t u_x - \bar{u}_x^t u, J].$$  \hspace{1cm} (3.6)

Since $u$ naturally represents a spinor variable, the equation (3.3) describes a spinor generalization of the NLS equation. Moreover, the identifications

$$u = (u_1, u_2) \leftrightarrow u = u_1 + u_2j, \quad J \leftrightarrow q$$  \hspace{1cm} (3.7)

allows the equation (3.3) to be expressed as a quaternion NLS equation:

$$u_t = u_{xx}q + |u|^2 uq + uD_x^{-1}(\bar{u}u_x q + q\bar{u}_x u)$$  \hspace{1cm} (3.8)

where $q$ is an imaginary unit quaternion

$$q^2 = -1, \quad q\bar{q} = 1.$$  \hspace{1cm} (3.9)

3.1. Integrability structure. The $SU(2)$ NLS-type equation (3.3) is an integrable system. First, it possesses a symmetry recursion operator $R = HJ$ where

$$H = D_x + iuD_x^{-1}\text{Im} \bar{u} \cdot + 2uD_x^{-1}P_{su} \bar{u}^t + 4\bar{u}D_x^{-1}P_{so} u^t$$  \hspace{1cm} (3.10)

is a Hamiltonian operator, and where

$$J = D_x - 4uD_x^{-1}\text{Re} \bar{u}.$$  \hspace{1cm} (3.11)

is a symplectic operator. Here $P_{su}$ is a projection onto the skew-Hermitian, trace-free part of a matrix; $P_{so}$ is a projection onto the skew part of a matrix.
Second, this pair of operators (3.10)–(3.11) provides a bi-Hamiltonian formulation
\[ u_t = H(\delta\mathbf{J}/\delta\mathbf{u}^\dagger) = E(\delta\mathcal{E}/\delta\mathbf{u}^\dagger) \]  
(3.12)
where $E = \mathcal{R}H$ is Hamiltonian operator compatible with $H$. The first Hamiltonian is given by $\mathbf{J} = \int \text{Re}(\mathbf{u} \cdot (u_x J)) \, dx$; the second Hamiltonian is $E = 0$ with $D_x^{-1}(0) = 0$, $D_x^{-1}(0) = \mathbf{J}$ in $\mathcal{H}$. An interesting remark is that the $\mathfrak{su}(2)$ generator $\mathbf{J}$ is also a Hamiltonian operator, however, the $SU(2)$ equation (3.3) is not Hamiltonian with respect to this operator. This stands in contrast to the structure of the NLS equation, where $i$ is a Hamiltonian operator that is compatible with the other two.

Third, the $SU(2)$ equation (3.3) possess a Lax pair (2.8) given by the $\mathfrak{su}(4)$ matrices
\[ U = \begin{pmatrix} i\chi & u_1 & 0 & \bar{u}_2 \\ -\bar{u}_1 & -i\chi & 0 & \bar{u}_2 \\ 0 & -u_2 & i\chi & \bar{u}_1 \\ -u_2 & 0 & -u_1 & -i\chi \end{pmatrix}, \]
(3.13)
\[ V = \begin{pmatrix} iw_1 & -2i\chi(u^*_1 J_1 + 4(u^*_x J_1) & w_2 & -2i\chi(\bar{u}_1 J_2 + 4(\bar{u}_x J_2) \\ -2i\chi(\bar{u}_1 J_1 - 4(\bar{u}_x J_1) & \bar{u} \cdot (u J_1) & 2i\chi(\bar{u}_1 J_2 - 4(\bar{u}_x J_2) & \bar{w}_2 \\ -2i\chi(u^*_1 J_2 - 4(u^*_x J_2) & -\bar{w}_2 & -2i\chi(u^*_1 J_1 - 4(u^*_x J_1) & -\bar{u} \cdot (u J_1) \end{pmatrix} \]
(3.14)
with
\[ \begin{pmatrix} 0 \\ -\bar{w}_2 \\ 0 \end{pmatrix} = u^* u J - J^* u^* u, \quad \begin{pmatrix} -iw_1 \\ -w_2 \\ iw_1 \end{pmatrix} = 2D_x^{-1}(\bar{u}^* u_x J_x + J^* u^*_x u) - \chi^2 J. \]
(3.15)
The spectral parameter in this Lax pair is $\chi$.

3.2. Derivation. The $SU(2)$ equation (3.3) comes from the general zero-curvature framework outlined in section 2 using the symmetric Lie algebra $\mathfrak{su}(4), \mathfrak{sp}(2) \simeq (\mathfrak{so}(6), \mathfrak{so}(5))$. This Lie algebra has the bracket relations (2.1) with $\mathfrak{h} = \mathfrak{sp}(2)$, $\mathfrak{m} = \mathbb{R}^5 \simeq \mathfrak{su}(4)/\mathfrak{sp}(2)$, and $\mathfrak{g} = \mathfrak{su}(4)$. There is a unique choice of $\mathfrak{e}$ in a Cartan space in $\mathfrak{m}$ up to gauge freedom generated by $\text{ad}(\mathfrak{h})$, since the rank of the Lie algebra is 1 [7].

A matrix representation is given by
\[ \mathfrak{h} = \begin{pmatrix} C & D \\ -D & C \end{pmatrix} \simeq \mathfrak{sp}(2), \quad C \in \mathfrak{u}(2), D \in \mathfrak{s}(2, \mathbb{C}) \]  
(3.16)
\[ \mathfrak{m} = \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \simeq \mathbb{R}^5, \quad A \in \mathfrak{su}(2), B \in \mathfrak{so}(2, \mathbb{C}) \]  
(3.17)
and
\[ \mathfrak{e} = \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix} \in \mathfrak{m}, \quad E = \chi \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in \mathfrak{su}(2) \]  
(3.18)
where $\chi$ is an arbitrary non-zero real constant related to norm of $\mathfrak{e}$ (namely, $K(e,e) = -4\chi^2$). The isotropy group preserving $\mathfrak{e}$ is $H_1 = SU(2) \times Sp(1) \subset Sp(2)$, whose Lie algebra is $\mathfrak{h}_1 \simeq \mathfrak{su}(2) \oplus \mathfrak{sp}(1) \subset \mathfrak{sp}(2)$. This Lie algebra contains the normalized generator (3.1):
\[ J \simeq \begin{pmatrix} C_j & D_j \\ -D_j & C_j \end{pmatrix}, \quad C_j = \begin{pmatrix} \cos \theta & 0 \\ 0 & -\cos \theta \end{pmatrix}, \quad D_j = \begin{pmatrix} e^{-i\omega} \sin \theta & 0 \\ 0 & 0 \end{pmatrix}. \]  
(3.19)
The orthogonal decomposition \( \mathbf{m} = \mathbf{m}_\parallel \oplus \mathbf{m}_\perp \) corresponds to \( \mathbf{A} = \mathbf{A}_\parallel + \mathbf{A}_\perp \) and \( \mathbf{B} = \mathbf{B}_\parallel + \mathbf{B}_\perp \):

\[
\mathbf{A}_\parallel = \begin{pmatrix} -ia_\parallel & 0 \\ 0 & ia_\parallel \end{pmatrix} \in \mathfrak{su}(2), \quad \mathbf{B}_\parallel = \mathbf{0}, \quad a_\parallel \in \mathbb{R} \quad (3.20)
\]

\[
\mathbf{A}_\perp = \begin{pmatrix} 0 & a_\perp \\ -a_\perp & 0 \end{pmatrix} \in \mathfrak{su}(2), \quad \mathbf{B}_\perp = \begin{pmatrix} 0 & b_\perp \\ -b_\perp & 0 \end{pmatrix} \in \mathfrak{so}(2, \mathbb{C}), \quad a_\perp, b_\perp \in \mathbb{C}. \quad (3.21)
\]

Likewise, the orthogonal decomposition \( \mathfrak{h} = \mathfrak{h}_\parallel \oplus \mathfrak{h}_\perp \) corresponds to \( \mathbf{C} = \mathbf{C}_\parallel + \mathbf{C}_\perp \) and \( \mathbf{D} = \mathbf{D}_\parallel + \mathbf{D}_\perp \):

\[
\mathbf{C}_\parallel = \begin{pmatrix} ic_\parallel & 0 \\ 0 & iC_\parallel \end{pmatrix} \in \mathfrak{u}(2), \quad \mathbf{D}_\parallel = \begin{pmatrix} d_\parallel & 0 \\ 0 & D_\parallel \end{pmatrix} \in \mathfrak{s}(2, \mathbb{C}), \quad c_\parallel, C_\parallel \in \mathbb{R}, d_\parallel, D_\parallel \in \mathbb{C} \quad (3.22)
\]

\[
\mathbf{C}_\perp = \begin{pmatrix} 0 & c_\perp \\ -c_\perp & 0 \end{pmatrix} \in \mathfrak{u}(2), \quad \mathbf{D}_\perp = \begin{pmatrix} 0 & d_\perp \\ d_\perp & 0 \end{pmatrix} \in \mathfrak{s}(2, \mathbb{C}), \quad c_\perp, d_\perp \in \mathbb{C}. \quad (3.23)
\]

The scaled space \( \mathfrak{h}^\perp := \frac{1}{\chi^2} \text{ad}(e)\mathbf{m}_\perp \simeq \mathfrak{h}_\perp \) is given by the correspondence

\[
\mathbf{C}_\perp^\perp := \frac{2}{\chi} \begin{pmatrix} 0 & i a_\perp \\ i a_\perp & 0 \end{pmatrix}, \quad \mathbf{D}_\perp^\perp := \frac{2}{\chi} \begin{pmatrix} 0 & i b_\perp \\ i b_\perp & 0 \end{pmatrix}. \quad (3.24)
\]

The matrices in the zero-curvature equation (2.28) in \( \mathfrak{su}(4) \) are given by

\[
U = \begin{pmatrix} \mathbf{E} + \mathbf{U}_1 & \mathbf{U}_2 \\ -\mathbf{U}_2 & \mathbf{E} + \mathbf{U}_1 \end{pmatrix}, \quad V = \begin{pmatrix} \mathbf{H}_1 + \mathbf{W}_1 & \mathbf{H}_2 + \mathbf{W}_2 \\ \mathbf{H}_2 - \mathbf{W}_2 & -\mathbf{H}_1 + \mathbf{W}_1 \end{pmatrix}, \quad (3.25)
\]

where

\[
\mathbf{U}_1 = \begin{pmatrix} 0 & u_1 \\ -\bar{u}_1 & 0 \end{pmatrix}, \quad \mathbf{U}_2 = \begin{pmatrix} 0 & \bar{u}_2 \\ \bar{u}_2 & 0 \end{pmatrix}, \quad u_1, u_2 \in \mathbb{C} \quad (3.26)
\]

\[
\mathbf{W}_1 = \begin{pmatrix} iw_1 & w_1 \perp \\ -\bar{w}_1 & i w_1 \perp \end{pmatrix}, \quad \mathbf{W}_2 = \begin{pmatrix} w_2 & \bar{w}_2 \perp \\ \bar{w}_2 & \bar{w}_2 \perp \end{pmatrix}, \quad w_{1\parallel}, W_{1\parallel} \in \mathbb{R}, w_{1\perp}, w_{2\perp}, W_{2\parallel} \in \mathbb{C} \quad (3.27)
\]

\[
\mathbf{H}_1 = \begin{pmatrix} -i \chi \mathbf{h}_\perp & -i \frac{1}{2} \chi \mathbf{h}_1 \perp \\ -i \frac{1}{2} \chi \mathbf{h}_1 \perp & i \chi \mathbf{h}_\parallel \end{pmatrix}, \quad \mathbf{H}_2 = \begin{pmatrix} 0 & i \frac{1}{2} \chi \mathbf{h}_2 \perp \\ -i \frac{1}{2} \chi \mathbf{h}_2 \perp & 0 \end{pmatrix}, \quad h_\parallel \in \mathbb{R}, h_1 \perp, h_2 \perp \in \mathbb{C}. \quad (3.28)
\]

Here \((u_1, u_2)\) are the Hasimoto variables, and \((h_1 \perp, h_2 \perp)\) are the flow variables associated to the space \( \mathfrak{h}^\perp \) (see Ref. [4]).

The components of the zero-curvature equation yield the system

\[
D_x h_\parallel = - \text{Re}(\bar{u}_1 h_1 \perp + \bar{u}_2 h_2 \perp), \quad (3.29a)
\]

\[
D_x w_1 = - \text{Im}(\bar{u}_1 w_1 \perp - \bar{u}_2 w_2 \perp), \quad D_x W_1 = \text{Im}(\bar{u}_1 w_1 \perp + \bar{u}_2 w_2 \perp), \quad (3.29b)
\]

\[
D_x w_2 = 2(\bar{u}_2 w_{1\perp} - u_1 \bar{w}_{2\perp}), \quad D_x W_2 = 2(\bar{u}_1 \bar{w}_{2\perp} - \bar{u}_2 \bar{w}_{1\perp}), \quad (3.29c)
\]

\[
w_{1\perp} = \frac{1}{4} D_x h_1 \perp - u_1 h_\parallel, \quad w_{2\perp} = \frac{1}{4} D_x h_2 \perp - u_2 h_\parallel, \quad (3.29d)
\]

\[
u_{1t} = D_x w_{1\perp} + i u_1 (W_{1\parallel} - w_{1\parallel}) + u_2 w_2 \perp - \bar{u}_2 \bar{W}_2 \parallel + \chi^2 h_1 \perp, \quad (3.29e)
\]

\[
u_{2t} = D_x w_2 \perp + i u_2 (W_{1\parallel} + w_{1\parallel}) - u_1 w_2 \perp + \bar{u}_1 \bar{W}_2 \parallel + \chi^2 h_2 \perp. \quad (3.29f)
\]
This system is invariant under an SU(2) symmetry coming from the action of the isotropy group \( H_\parallel = SU(2) \) on \((u_1, u_2), (w_{1\perp}, w_{2\perp}), (h^\parallel_1, h^\parallel_2)\). The symmetry action looks simplest by going to the 2-component spinor variable (3.4) along with the similar variables

\[
w := (w_{1\perp}, w_{2\perp}) \in \mathbb{C} \times \mathbb{C}, \quad h^\parallel := (h^\parallel_1, h^\parallel_2) \in \mathbb{C} \times \mathbb{C},
\]

and the matrix variables

\[
\begin{align*}
w_\parallel &:= \begin{pmatrix} -i w_{1\parallel} & -w_{2\parallel} \\ w_{2\parallel} & i w_{1\parallel} \end{pmatrix} \in \mathfrak{su}(2), \quad W_\parallel := \begin{pmatrix} 0 & \bar{W}_\parallel^\perp \\ -\bar{W}_\parallel & 0 \end{pmatrix} \in \mathfrak{so}(2, \mathbb{C}).
\end{align*}
\]

Also, for ease of notation, put \( W_\parallel := W_1 \parallel \).

Thus, the system (3.32) takes the invariant form

\[
\begin{align*}
D_x h_\parallel &= -\text{Re}(\bar{u} \cdot h^\perp), \\
w &= \frac{1}{4} D_x h^\perp + h_\parallel u, \\
D_x W_\parallel &= 2(u^4 w - w^4 u), \quad D_x W_\parallel = \text{Im}(\bar{u} \cdot w), \\
u_t &= D_x w + i W_\parallel u + u w_\parallel + \bar{u} W_\parallel + \chi^2 h^\perp
\end{align*}
\]

with the normalized SU(2) symmetry generator (3.19) being \( u \rightarrow uJ \) on the Hasimoto spinor variable \( u \).

Now, the SU(2) symmetry generator is used to define a flow

\[
h^\perp = uJ
\]

The system (3.32) thereby yields

\[
\begin{align*}
h_\parallel &= c_1, \\
w &= \frac{1}{4} u_x J + c_1 u, \\
W_\parallel &= \frac{1}{4} (u^4 uJ - (uJ)^4 u) + C_1, \\
W_\parallel &= \frac{1}{4} u_x \bar{u}_x J + J \bar{u}_x u_0 + C_2
\end{align*}
\]

and

\[
u_t = \frac{1}{4} u_x u J + \frac{1}{4} |u|^2 u J + \frac{1}{2} u D_x^{-1} (\bar{u}^4 u_x J + J \bar{u}_x u_0)
+ c_1 u_x + C_1 i u + u (C_2 + \chi^2 J) + \bar{u} C_1,
\]

where \( c_1, C_1 \) are arbitrary constants, \( C_1 \) is an arbitrary constant matrix in \( \mathfrak{so}(2, \mathbb{C}) \), and \( C_2 \) is an arbitrary constant matrix in \( \mathfrak{su}(2, \mathbb{C}) \). This yields the SU(2) NLS-type equation (3.3) after a scaling of \( t \), with \( c_1 = C_1 = 0, C_1 = 0, \) and \( C_2 = -\chi^2 J \).

The integrability structure is obtained directly from the system (3.32) by the same method shown for the NLS equation in section 2.

4. Second NLS-type system and its integrability

The second SU(2) generalization of the NLS equation has the form

\[
\begin{align*}
v_t &= 2 \text{Im}(\bar{u}_x \cdot (uJ)) \\
u_t &= u_x u J - i v_x u J + v^2 u J + \frac{1}{2} i \text{Im}(\bar{u} \cdot (uJ)) u \\
&\quad + u D_x^{-1} (\bar{u}^4 u_x J + J \bar{u}_x u_0) + i u D_x^{-1} (v [J, \bar{u}^4 u])
\end{align*}
\]
where \( u \) is the spinor variable \([3.4]\), and \( v \) is a real scalar variable. Through the identities \([3.5]\), the nonlocal \( D_x^{-1} \) terms can be rewritten in several ways:

\[
\begin{align*}
    u_t - u_{xx}J + iv_xuJ - v^2uJ &= i \Im(\bar{u} \cdot (uJ))u + uD_x^{-1}[\bar{u}^\dagger u_x, J] + iuD_x^{-1}(v[J, \bar{u}^\dagger u]) \\
    &= |u|^2uJ - uD_x^{-1}[\bar{u}^\dagger u_x, J] + iuD_x^{-1}(v[J, \bar{u}^\dagger u]) \\
    &= \frac{1}{2}|u|^2uJ + \frac{i}{2} \Im(\bar{u} \cdot (uJ))u \\
    &\quad + \frac{1}{2}uD_x^{-1}[\bar{u}^\dagger u_x - \bar{u}^\dagger u, J] + iuD_x^{-1}(v[J, \bar{u}^\dagger u]).
\end{align*}
\] (4.2)

The system (4.1) describes a scalar-spinor generalization of the NLS equation. Through the identifications \([3.7]\), it can be expressed as a quaternion NLS system:

\[
\begin{align*}
    v_t &= 2 \Im(\bar{u}_xuq), \\
    u_t &= u_{xx}q - iv_xuq + v^2uq + \frac{1}{2}i[q, \bar{u}u]u \\
    &\quad + uD_x^{-1}(\bar{u}u_xq + q\bar{u}_xu) + \frac{1}{2}uD_x^{-1}(v[q, \bar{u}u]).
\end{align*}
\] (4.3)

where \( q \) is an imaginary unit quaternion \([3.9]\).

4.1. **Integrability structure.** This \( SU(2) \) NLS-type system \([4.1]\) has an integrability structure similar to that of equation \([3.3]\).

It possesses a symmetry recursion operator \( R = HJ \) given in terms of a Hamiltonian operator

\[
H = \begin{pmatrix}
D_x & \Im \bar{u} \\
-iu & D_x + iv + 2iuD_x^{-1} \Im \bar{u} \cdot + uD_x^{-1} P_{su} \bar{u}^\dagger
\end{pmatrix}
\] (4.4)

and a symplectic operator

\[
J = \begin{pmatrix}
\frac{1}{4}D_x + vD_x^{-1}v & \frac{1}{2} \Im \bar{u} \cdot + vD_x^{-1} \Re \bar{u} \\
-\frac{1}{2}iu + uD_x^{-1}v & D_x - iv + uD_x^{-1} \Re \bar{u}
\end{pmatrix}
\] (4.5)

The pair of operators \([4.4]-[4.5]\) provides a bi-Hamiltonian formulation

\[
\begin{pmatrix}
v_t \\
u_t
\end{pmatrix} = H \begin{pmatrix}
\delta \tilde{H} / \delta v \\
\delta \tilde{H} / \delta \bar{u}
\end{pmatrix} = E \begin{pmatrix}
\delta \mathcal{E} / \delta v \\
\delta \mathcal{E} / \delta \bar{u}
\end{pmatrix}
\] (4.6)

where \( \mathcal{E} = RH \) is Hamiltonian operator compatible with \( H \). The first Hamiltonian is given by \( \tilde{H} = \int \left( \Re(\bar{u} \cdot (u_x)) + v \Im(\bar{u} \cdot (uJ)) \right) dx \), while the second Hamiltonian is similar to the one for the \( SU(2) \) equation \([3.3]\). Likewise, system \([4.1]\) is not Hamiltonian with respect to \( J \).

A Lax pair \([2.8]\) for system \([4.1]\) is given by the \( so(6) \) matrices

\[
U = \begin{pmatrix}
E + \Re U & \Im U \\
\Im U & E + \Re U
\end{pmatrix}, \quad V = \begin{pmatrix}
\Re(H + W) & \Im(H + W) \\
\Im(H - W) & -\Re(H - W)
\end{pmatrix}
\] (4.7)
where

\[
E = \begin{pmatrix}
0 & \chi & 0 \\
-\chi & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\quad
U = \begin{pmatrix}
iv & 0 & u_1 \\
0 & iv & u_2 \\
-u_1 & -u_2 & 0
\end{pmatrix},
\quad\quad(4.8)
\]

\[
H = \begin{pmatrix}
0 & 0 & -\chi \text{Re}(uJ)_2 \\
0 & 0 & \chi \text{Re}(uJ)_1 \\
\chi \text{Re}(uJ)_2 & -\chi \text{Re}(uJ)_1 & 0
\end{pmatrix},
\quad(4.9)
\]

\[
W = \begin{pmatrix}
\frac{1}{2} \text{Im}(\bar{u} \cdot (uJ)) + iw_1 & w_2 \\
-\bar{w}_2 & \frac{1}{2} \text{Im}(\bar{u} \cdot (uJ)) - iw_1 \\
-(\bar{u}_x \bar{J} + iv\bar{u} \bar{J})_1 & -\bar{w}_2 \bar{J}_2 + i \text{Im}(\bar{u} \cdot (uJ))
\end{pmatrix},
\quad(4.10)
\]

with

\[
\begin{pmatrix}
-w_1 \\
-w_2 \\
iw_1 \\
iw_2
\end{pmatrix} = D_x^{-1}(\bar{u}^t \cdot u_x J + J\bar{u}^t \cdot u + iv[J, \bar{u}^t \cdot u])_0 - \chi^2 J.
\quad(4.11)
\]

The spectral parameter in the Lax pair is \(\chi\).

4.2. Derivation. The \(SU(2)\) system \((4.1)\) comes from applying the general zero-curvature framework outlined in section 2 to the symmetric Lie algebra \((\mathfrak{so}(6), u(3))\). This Lie algebra has the bracket relations \((2.1)\) with \(h = u(3), m = \mathbb{R}^6 \simeq \mathfrak{so}(6)/u(3), \text{and } g = \mathfrak{so}(6)\). Its rank is 1 \([7]\), and so there is a unique choice of \(e\) in a Cartan space in \(m\) up to gauge freedom generated by \(\text{ad}(h)\).

A matrix representation is given by

\[
g = \begin{pmatrix}
\text{Re}(A + B) & \text{Im}(A + B) \\
\text{Im}(A - B) & -\text{Re}(A - B)
\end{pmatrix} \simeq \mathfrak{so}(6), \quad \text{Re } B, \text{Re } A, \text{Im } A \in \mathfrak{so}(3), \text{Im } B \in \mathfrak{s}(3)
\quad(4.12)
\]

\[
h = \begin{pmatrix}
\text{Re } B & \text{Im } B \\
-\text{Im } B & \text{Re } B
\end{pmatrix} \simeq \mathfrak{u}(3), \quad B \in \mathfrak{u}(3)
\quad(4.13)
\]

\[
m = \begin{pmatrix}
\text{Re } A & \text{Im } A \\
\text{Im } A & -\text{Re } A
\end{pmatrix} \simeq \mathbb{C}^3 \simeq \mathbb{R}^6, \quad A \in \mathfrak{so}(3, \mathbb{C})
\quad(4.14)
\]

and

\[
e = \begin{pmatrix}
E & 0 \\
0 & -E
\end{pmatrix}
\quad(4.15)
\]

where \(\chi\) is arbitrary non-zero real constant related to norm of \(e\) (namely, \(K(e, e) = -4\chi^2\)). Note \(\mathfrak{s}(n)\) denotes the space of symmetric \(n \times n\) matrices.

The isotropy group preserving \(e\) is \(H_\parallel = SU(2) \times U(1) \subset U(3)\), whose Lie algebra is \(\mathfrak{h}_\parallel \simeq \mathfrak{su}(2) \oplus \mathfrak{u}(1) \in \mathfrak{u}(3)\). This Lie algebra contains the normalized generator \((3.1)\):

\[
J \simeq \begin{pmatrix}
\text{Re } B_J & \text{Im } B_J \\
-\text{Im } B_J & \text{Re } B_J
\end{pmatrix}, \quad B_J = \begin{pmatrix}
ic \theta & -e^{-i\psi} \sin \theta & 0 \\
e^{i\psi} \sin \theta & -\cos \theta & 0 \\
0 & 0 & 0
\end{pmatrix}.
\quad(4.16)
\]
The orthogonal decompositions $m = m_\parallel \oplus m_\perp$ and $h = h_\parallel \oplus h_\perp$ correspond to $A = A_\parallel + A_\perp$ and $B = B_\parallel + B_\perp$:

$$A_\parallel = \begin{pmatrix} 0 & a_\parallel & 0 \\ -a_\parallel & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{so}(3, \mathbb{C}), \quad a_\parallel \in \mathbb{R} \quad (4.17)$$

$$A_\perp = \begin{pmatrix} 0 & ia_\perp & a_\perp \\ -ia_\perp & 0 & a_\perp \\ -a_\perp & -a_\perp & 0 \end{pmatrix} \in \mathfrak{so}(3, \mathbb{C}), \quad a_\perp \in \mathbb{R}, a_\perp, a_\perp \in \mathbb{C} \quad (4.18)$$

$$B_\parallel = \begin{pmatrix} ib_\parallel & b_2 \parallel & 0 \\ -b_2 \parallel & -ib_\parallel & 0 \\ 0 & 0 & ib_\parallel \end{pmatrix} \in \mathfrak{u}(3), \quad b_\parallel, b_\parallel \in \mathbb{R}, b_2 \parallel, b_2 \parallel \in \mathbb{C}, \quad (4.19)$$

$$B_\perp = \begin{pmatrix} ib_\perp & 0 & b_\perp \\ 0 & ib_\perp & 0 \\ -b_\perp & -b_\perp & 0 \end{pmatrix} \in \mathfrak{u}(n), \quad b_\perp, b_\perp \in \mathbb{R}, b_\perp, b_\perp \in \mathbb{C} \quad (4.20)$$

The scaled space $h^\perp := \frac{1}{\chi} \text{ad}(e) m_\perp \simeq h_\perp$ is given by the correspondence

$$B^\perp := \frac{1}{\chi} \begin{pmatrix} -2ia_\perp & 0 & a_\perp \\ 0 & -2ia_\perp & -a_\perp \end{pmatrix} \quad (4.21)$$

The matrices in the zero-curvature equation (2.18) in $\mathfrak{so}(6)$ have the form (4.17) with

$$U = \begin{pmatrix} iv & 0 & u_1 \\ 0 & iv & u_2 \\ -u_1 & -u_2 & 0 \end{pmatrix}, \quad v \in \mathbb{R}, u_1, u_2 \in \mathbb{C} \quad (4.22)$$

$$H = \begin{pmatrix} 0 & \chi h_\parallel + i \chi H^\perp & -\chi h_2 \perp \\ -\chi h_\parallel & 0 & \chi h_1 \perp \\ \chi h_2 \perp & -\chi h_1 \perp & 0 \end{pmatrix}, \quad h_\parallel, H^\perp \in \mathbb{R}, h_1 ^\perp, h_2 ^\perp \in \mathbb{C} \quad (4.23)$$

$$W = \begin{pmatrix} iw_\perp + iw_1 \perp & w_2 \parallel & w_\perp \perp \\ -\bar{w}_2 \parallel & iw_\perp + iw_1 \perp & w_2 \perp \perp \\ -\bar{w}_1 \perp & -\bar{w}_2 \perp & iw_\parallel \perp \end{pmatrix}, \quad w_1 \parallel, W_\parallel, W_\perp \in \mathbb{R}, w_2 \parallel, w_2 \perp \in \mathbb{C} \quad (4.24)$$

Here $(v, u_1, u_2)$ are the Hasimoto variables, and $(H^\perp, h_1 ^\perp, h_2 ^\perp)$ are the flow variables associated to the space $h^\perp$ (see Ref. [4]). The components of the zero-curvature equation yield the system

$$D_x h_\parallel = v H^\perp + \text{Re}(\bar{u}_1 h_1 ^\perp + \bar{u}_2 h_2 ^\perp), \quad (4.25a)$$

$$D_x w_1 \parallel = -\text{Im}(\bar{u}_1 w_1 \perp - \bar{u}_2 w_2 \perp), \quad D_x w_2 \parallel = u_1 \bar{w}_2 \perp - \bar{u}_2 w_1 \perp, \quad (4.25b)$$

$$D_x W_\parallel = 2 \text{Im}(u_1 w_1 \perp + u_2 w_2 \perp), \quad (4.25c)$$

$$W_\perp = \frac{1}{4} D_x H^\perp + v h_\parallel + \frac{1}{2} \text{Re}(\bar{u}_1 h_1 ^\perp + \bar{u}_2 h_2 ^\perp), \quad (4.25d)$$

$$w_1 \perp = D_x h_1 ^\perp + u_1 (h_\parallel - \frac{i}{2} H^\perp) - iv h_1 \perp, \quad w_2 \perp = D_x h_2 ^\perp + u_2 (h_\parallel - \frac{i}{2} H^\perp) - iv h_2 \perp, \quad (4.25e)$$

$$v_\ell = D_x W_\perp + \text{Im}(\bar{u}_1 w_1 \perp + \bar{u}_2 w_2 \perp) + \chi^2 H^\perp, \quad (4.25f)$$

$$u_1 \ell = D_x w_1 \perp + iv w_1 \perp - iv_1 (W_\perp - W_\parallel + w_1 \parallel) - u_2 w_2 \perp + \chi^2 h_1 ^\perp, \quad (4.25g)$$
\[ u_{2t} = D_x w_{2\perp} + i v w_{2\perp} - i u_2 (W_{\perp} - W_{\parallel} - w_{1\parallel}) + u_1 \bar{w}_{2\parallel} + \chi^2 h_{\perp}^2. \] (4.25h)

This system is invariant under an SU(2) symmetry that comes from the action of the isotropy group \( H = SU(2) \) on \((v, u_1, u_2), (W_{\perp}, w_{1\perp}, w_{2\perp}), (H^\perp, h_{\perp}^1, h_{\perp}^2)\). By going to the 2-component spinor variables \((\mathbf{3.4}}, \mathbf{3.31})\), along with the similar matrix variable

\[ \mathbf{w} = \begin{pmatrix} -i w_1 \parallel & \bar{w}_2 \parallel \\ -i w_2 \parallel & i w_1 \parallel \end{pmatrix}, \] (4.26)

the system \((\mathbf{4.25})\) takes the simpler form

\[ D_x h_{\parallel} = v H^\perp + \text{Re}(\bar{u} \cdot h^\perp), \] (4.27a)

\[ W = \frac{1}{2} D_x H^\perp + v h_{\parallel} + \frac{1}{2} \text{Im}(\bar{u} \cdot h^\perp), \] (4.27b)

\[ w = D_x h_{\perp} - i v h_{\perp} + u(h_{\parallel} - \frac{1}{2} i H^\perp), \] (4.27c)

\[ D_x W_{\parallel} = 2 \text{Im}(\bar{u} \cdot w), \quad D_x \mathbf{w} = (\bar{u}^\dagger \cdot w - w^\dagger \cdot u)_0, \] (4.27d)

\[ v_1 = D_x W + \text{Im}(\bar{u} \cdot w) + \chi^2 H^\perp, \] (4.27e)

\[ u_t = D_x w + i v w + i u(W_{\perp} - W) + u \mathbf{w} + \chi^2 h_{\perp}, \] (4.27f)

where, for ease of notation, \( W := W_{\perp} \). Specifically, the normalized SU(2) symmetry generator \((\mathbf{4.16})\) becomes \((v, u) \rightarrow (0, u \sigma)\), which acts trivially on \(v\).

Now, the SU(2) symmetry generator is used to define a flow

\[ H^\perp = 0, \quad h^\perp = u J. \] (4.28)

The system \((\mathbf{4.27})\) then yields

\[ h_{\parallel} = c_1, \] (4.29)

\[ W = \frac{1}{2} \text{Im}(\bar{u} \cdot (u J)) + c_1 v, \quad w = u_x J - i v u J + c_1 u, \] (4.30)

\[ W_{\parallel} = \text{Im}(\bar{u} \cdot (u J)) + C_1, \quad \mathbf{w} = D_x^{-1}(\bar{u}^\dagger u_x J + J \bar{u}^\dagger u)_0 + i D_x^{-1}(v[J, \bar{u}^\dagger u]) + C_2, \] (4.31)

and

\[ v_1 = 2 \text{Im}(\bar{u}_x \cdot (u J)) + c_1 v_x, \]

\[ u_t = u_{xx} J - i v_x u J + v^2 u J + \frac{1}{2} \text{Im}(\bar{u} \cdot (u J)) u \\
+ u D_x^{-1}(\bar{u}^\dagger u_x J + J \bar{u}^\dagger u)_0 + i u D_x^{-1}(v[J, \bar{u}^\dagger u]) + c_1 u_x + u (i C_1 + C_2 + \chi^2 J), \] (4.32)

where \(c_1, C_1\) are arbitrary constants, and \(C_2\) is an arbitrary constant matrix in \(su(2)\). Hence, the SU(2) NLS-type system \((\mathbf{4.1})\) is obtained for \(c_1 = C_1 = 0\) and \(C_2 = -\chi^2 J\).

The integrability structure is obtained directly from the system \((\mathbf{4.27})\) similarly to that for the previous SU(2) equation.

5. Geometric flows

A primary geometric formulation of the NLS equation is given by the bi-normal flow of an inelastic curve \(\mathbf{r}(x)\) in \(\mathbb{R}^3\), where \(x\) is arclength. The bi-normal flow equation consists of \(\mathbf{r}_t = \mathbf{r}_x \times \mathbf{r}_{xx}\). Its well-known equivalence to the NLS equation comes from the Hasimoto transformation \((\mathbf{3})\) \(u = \kappa e^{i \int \tau dx}\) in terms of the curvature \(\kappa\) and torsion \(\tau\) of the curve (see Ref. \([\mathbf{6}, \mathbf{5}]\) for details).

For the later generalizations, a useful observation is that the bi-normal equation in \(\mathbb{R}^3\) can be expressed in the form \(\mathbf{r}_t = I_{\mathbf{r}} (D_x \mathbf{r})\) with \(I_{\mathbf{r}} = \mathbf{r}_x \times\) being a geometric representation of
where \( \nabla \) can be shown to have the form of a generalized \( SU_u \) element is the generator of a

5.2. Geometric

\( u \) is determined (up to a sign) by the properties \( J \) being the curvature.

These identifications can be derived from the general formulation of inelastic curve flows

\( \gamma \) is the complex structure tensor on \( SU \) from which an equivalent

\( \gamma \) connection) on \( SU \) is the tangent vector; \( N = \frac{1}{2} D_x T \) is the principal normal vector; \( B = T \times N \) is the bi-normal vector. This formulation leads directly to a Schrodinger map equation by

The Hasimoto transformation is carried out by means of the following identifications:

\[ \begin{align*}
\vec{r}_x &\leftrightarrow U_m, \quad \vec{t}_x \leftrightarrow V_{m_{\perp}}, \quad D_x \vec{r}_x \leftrightarrow [U_m, U_b]
\end{align*} \]

(5.1)

where \( U \) and \( V \) are the matrices in the Lax pair, with the subscripts denoting projections. These identifications can be derived from the general formulation of inelastic curve flows in

\( \gamma \) is a Hermitian symmetric

\( \gamma \) is the covariant derivative (Riemannian connection) on \( SU \), and where \( J \) is the complex structure tensor on \( S^2 \), satisfying \( \nabla J = 0 \). Here \( \nabla_x \) := \( \gamma_x \nabla \) is the covariant derivative along \( \gamma \).

There is a similar geometric formulation of the \( SU(2) \) integrable systems (3.3) and (4.1).

The Hasimoto transformation is carried out by means of the following identifications:

\[ \begin{align*}
\vec{r}_x &\leftrightarrow U_m, \quad \vec{t}_x \leftrightarrow V_{m_{\perp}}, \quad D_x \vec{r}_x \leftrightarrow [U_m, U_b]
\end{align*} \]

(5.1)

where \( U \) and \( V \) are the matrices in the Lax pair, with the subscripts denoting projections. These identifications can be derived from the general formulation of inelastic curve flows in symmetric spaces presented in Ref. [4, 10]. They will produce a \( SU(2) \) bi-normal equation, from which an equivalent \( SU(2) \) Schrodinger map equation is readily obtained. A key difference compared to the NLS case is that the \( u(1) \) generator given by the complex structure tensor on \( S^2 \) will be replaced by an \( su(2) \) generator that is attached to normal space of the curve \( \gamma \).

5.1. Geometric \( SU(2) \) flow in \( su(4)/sp(2) \). Consider an inelastic curve \( \vec{r}(x) \) in \( \mathbb{R}^5 \cong su(4)/sp(2) \), where \( x \) is the arclength. A geometric representation of the \( SU(2) \) symmetry generator \( J \) is given by a vector operator \( J_{\vec{r}} \) that has the properties \( J_{\vec{r}}(\vec{r}_x) = 0 \) and \( J_{\vec{r}}^2 = -P_{\vec{r}}^\perp \), where \( P_{\vec{r}}^{\perp} \) is the projection operator onto the 4-dimensional normal plane with respect to \( \vec{r}_x \).

These two properties determine \( J_{\vec{r}} \) up to a sign.

The \( SU(2) \) bi-normal flow is given by

\[ \vec{r}_t = J_{\vec{r}}(D_x \vec{r}_x). \]

(5.2)

An equivalent formulation consists of \( \vec{r}_t = \kappa B \) where \( B = J_{\vec{r}}(N) \) is a bi-normal vector given in terms of the principal normal vector \( N = \frac{1}{\kappa} D_x T \) and the tangent vector \( T = \vec{r}_x \), with \( \kappa \) being the curvature.

The unit vector \( T \) in \( \mathbb{R}^5 \) can be identified with a map \( \gamma(t, x) \) into \( S^4 \). The resulting flow can be shown to have the form of a generalized \( SU(2) \) Schrodinger map

\[ \gamma_t = J_{\gamma}(\nabla_x \gamma_x) \]

(5.3)

where \( \nabla \) is the covariant derivative (Riemannian connection) on \( S^4 \), and where \( J_{\gamma} \) is the \( su(2) \) generator in the normal subspace of the tangent space along \( \gamma \) in \( S^4 \).

5.2. Geometric \( SU(2) \) flow in \( so(6)/u(3) \). Similarly, consider an inelastic curve \( \vec{r}(x) \) in \( \mathbb{R}^6 \cong so(6)/u(3) \), where \( x \) is arclength. Recall that \( (so(6), u(3)) \) is a Hermitian symmetric Lie algebra, and let \( J \in u(3) \) denote the element representing the hermitian structure. This element is the generator of a \( u(1) \) subalgebra in \( u(3) \), which acts on \( so(6)/u(3) \cong \mathbb{R}^6 \) via \( \text{ad}(J) \).

A geometric representation of \( SU(2) \) symmetry generator \( J \) is given by the operator \( J_{\vec{r}} \) that is determined (up to a sign) by the properties \( J_{\vec{r}}(\vec{r}_x) = 0 \), \( J_{\vec{r}}(\text{ad}(J)\vec{r}_x) = 0 \), and \( J_{\vec{r}}^2 = -P_{\vec{r}}^\perp \), where \( P_{\vec{r}}^\perp \) is the projection operator onto the 4-dimensional plane that is orthogonal to the
span of $\vec{r}_x$ and $\text{ad}(J)\vec{r}_x$. Namely, this plane is the complexified normal plane defined relative to the complexified tangent space of the curve.

A remark here is that $J$ does not belong to the isotropy Lie subalgebra $\mathfrak{h}_\parallel \cong \mathfrak{su}(2) \oplus \mathfrak{u}(1)$; accordingly, note that the $\mathfrak{u}(1)$ generator in $\mathfrak{h}_\parallel$ does not coincide with the hermitian structure $J$.

Using these structures, the resulting $SU(2)$ bi-normal flow has the form (5.2). This flow is equivalent to the $SU(2)$ Schrodinger map equation (5.3) on $S^5$.

6. Concluding remarks

The zero-curvature framework used to obtain the two novel spinor/quaternion NLS-type systems (3.3) and (4.1) utilizes the structure of the isotropy Lie subalgebra $\mathfrak{h}_\parallel \supset \mathfrak{su}(2)$ in a symmetric Lie algebra $\mathfrak{g}/\mathfrak{h}$ with a choice of a constant element $e$ in the Cartan space $\mathfrak{m} \cong \mathfrak{g}/\mathfrak{h}$.

For the two symmetric Lie algebras $\mathfrak{su}(4)/\mathfrak{sp}(2) \cong \mathfrak{so}(6)/\mathfrak{so}(5)$ and $\mathfrak{so}(6)/\mathfrak{u}(3)$, there is a unique choice of $e$ up to the gauge action of $\text{ad}(\mathfrak{h}_\parallel)$. The isotropy subalgebra in the second symmetric Lie algebra contains a $\mathfrak{u}(1)$ factor, and its generator can be used to derive an integrable $U(1)$ NLS-type system. This system turns out to be a 2-component version of the integrable Yajima-Oikawa (YO) system [9]. In contrast, the isotropy subalgebra in the first symmetric Lie algebra has no $\mathfrak{u}(1)$ factor, and so no $U(1)$ NLS-type system can be derived.

A picture of the relationship holding between these symmetric Lie algebras and NLS-type systems is summarized in Table 1.

| symmetric Lie algebra $\mathfrak{g}/\mathfrak{h}$ | isotropy subalgebra $\mathfrak{h}_\parallel \subset \mathfrak{h}$ | symmetry generator | Hasimoto variables | integrable system |
|-----------------------------------------------|-----------------------------------------------|-------------------|-------------------|------------------|
| $\mathfrak{so}(4)/\mathfrak{so}(3)$         | $\mathfrak{so}(2) \cong \mathfrak{u}(1)$     | $i \in \mathfrak{u}(1)$ | $u \in \mathbb{C}$ | NLS equation |
| $\mathfrak{so}(6)/\mathfrak{u}(3)$         | $\mathfrak{u}(1) \oplus \mathfrak{su}(2)$    | $i \in \mathfrak{u}(1)$ | $v \in \mathbb{R}$, $u \in \mathbb{C} \times \mathbb{C}$ | 2-component YO  |
| $\mathfrak{so}(6)/\mathfrak{u}(3)$         | $\mathfrak{u}(1) \oplus \mathfrak{su}(2)$    | $J \in \mathfrak{su}(2)$ | $v \in \mathbb{R}$, $u \in \mathbb{C} \times \mathbb{C}$ | (4.1)            |
| $\mathfrak{su}(4)/\mathfrak{sp}(2)$       | $\mathfrak{su}(2) \oplus \mathfrak{sp}(1)$    | $J \in \mathfrak{su}(2)$ | $u \in \mathbb{C} \times \mathbb{C}$ | (3.3)            |
| $\cong \mathfrak{so}(6)/\mathfrak{so}(5)$ | $\cong \mathfrak{su}(2) \oplus \mathfrak{so}(3)$ |                   |                   |                  |

Multi-component versions of the spinor/quaternion NLS-type systems can be obtained straightforwardly by extending the zero-curvature framework to the symmetric Lie algebras $\mathfrak{su}(2n-2)/\mathfrak{sp}(n-1)$ and $\mathfrak{so}(2n)/\mathfrak{u}(n)$. In the resulting systems, $u_1$, $u_2$ will become complex vectors with $n-2$ components, whereby $(u_1, u_2)$ can be viewed as a vectorial spinor or equivalently a vectorial quaternion.

Other integrable systems can be derived in a similar way with $SU(2)$ replaced by a larger symmetry group. An especially interesting case would be the group $\text{SL}(2, \mathbb{C})$ which is associated to spinors/quaternions in 4-dimensional Minkowski space.

An interesting question for future work will be to study the soliton solutions of the two new integrable spinor/quaternion systems (3.3) and (4.1). Their Lax pairs are the starting point for the construction of an inverse scattering transform that can be used analytically to solve the initial value problem.
Acknowledgements

S.C.A. is supported by an NSERC research grant. E.A. thanks the Mathematics & Statistics Department of Brock University for support during the period in which part this work was initiated.

References

[1] A.P. Fordy and P.P. Kullish, Nonlinear Schrödinger equations and simple Lie algebras, Commun. Math. Phys. 89 (1983), 427–443.
[2] Unitarily-invariant integrable systems and geometric curve flows in SU(n+1)/U(n) and SO(2n)/U(n), A. Ahmed, S.C. Anco, E. Asadi, J. Phys. A: Math. Theor. 51 (2018) 065205 (35 pages).
[3] H. Hasimoto, Soliton on a vortex filament, J. Fluid Mech. 51 (1972), 477–485.
[4] S.C. Anco, Group-invariant soliton equations and bi-Hamiltonian geometric curve flows in Riemannian symmetric spaces, J. Geom. Phys. 58 (2008), 1–37.
[5] S.C. Anco and R. Myrzakulov, Integrable generalizations of Schrödinger maps and Heisenberg spin models from Hamiltonian flows of curves and surfaces, J. Geom. Phys. 60 (2010), 1576–1603.
[6] G. Mari Beffa, J. Sanders, J.-P. Wang, On integrable systems in 3-dimensional Riemannian geometry, J. Nonlinear Sci. 12 (2002), 143–167.
[7] S. Helgason, Differential Geometry, Lie Groups, and Symmetric Spaces (Amer. Math. Soc., Providence, 2001).
[8] M.J. Ablowitz, D.J. Kaup, A.C. Newell, H. Segur, The inverse scattering transform — Fourier analysis for nonlinear problems, Stud. Appl. Math. 53 (1974), 249–315.
[9] N. Yajima, M. Oikawa, Formation and interaction of sonic-Langmuir solitons, Prog. Theor. Phys. 56(6) (1976), 1719–1739.
[10] S.C. Anco, E. Asadi, Hasimoto variables, generalized vortex filament equations, Heisenberg models and Schrödinger maps arising from group-invariant NLS systems, J. Geom. Phys. 144 (2019), 324–357.