ON CERTAIN DUALITY OF NÉRON-SEVERI LATTICES OF SUPERSINGULAR $K^3$ SURFACES AND ITS APPLICATION TO GENERIC SUPERSINGULAR $K^3$ SURFACES

SHIGEYUKI KONDÔ AND ICHIRO SHIMADA

ABSTRACT. Let $X$ and $Y$ be supersingular $K^3$ surfaces defined over an algebraically closed field. Suppose that the sum of their Artin invariants is 11. Then there exists a certain duality between their Néron-Severi lattices. We investigate geometric consequences of this duality. As an application, we classify genus one fibrations on supersingular $K^3$ surfaces with Artin invariant 10 in characteristic 2 and 3, and give a set of generators of the automorphism group of the nef cone of these supersingular $K^3$ surfaces. The difference between the automorphism group of a supersingular $K^3$ surface $X$ and the automorphism group of its nef cone is determined by the period of $X$. We define the notion of genericity for supersingular $K^3$ surfaces in terms of the period, and prove the existence of generic supersingular $K^3$ surfaces in odd characteristics for each Artin invariant larger than 1.

1. INTRODUCTION

A $K^3$ surface defined over an algebraically closed field $k$ is said to be supersingular (in the sense of Shioda) if its Picard number is 22. Supersingular $K^3$ surfaces exist only when $k$ is of positive characteristic. Let $X$ be a supersingular $K^3$ surface in characteristic $p > 0$, and let $S_X$ denote its Néron-Severi lattice. Artin [1] showed that the discriminant group of $S_X$ is a $p$-elementary abelian group of rank $2\sigma$, where $\sigma$ is an integer such that $1 \leq \sigma \leq 10$. This integer $\sigma$ is called the Artin invariant of $X$. The isomorphism class of the lattice $S_X$ depends only on $p$ and $\sigma$ (Rudakov and Shafarevich [27]). Moreover supersingular $K^3$ surfaces with Artin invariant $\sigma$ form a $(\sigma - 1)$-dimensional family, and a supersingular $K^3$ surface with Artin invariant 1 in characteristic $p$ is unique up to isomorphisms (Ogus [24], [25], Rudakov and Shafarevich [27]).

Recently many studies of supersingular $K^3$ surfaces in small characteristics with Artin invariant 1 have appeared. For example, for $p = 2$, Dolgachev and Kondo [8], Katsura and Kondo [12], Elkies and Schütt [11]; for $p = 3$, Katsura and Kondo [13], Kondo and Shimada [18], Sengupta [28]; and for $p = 5$, Shimada [33]. On the other hand, geometric properties of supersingular $K^3$ surfaces with big Artin invariant are not so much known (e.g. Rudakov and Shafarevich [26], [27], Shioda [35], Shimada [31], [32]).

The first author was partially supported by JSPS Grant-in-Aid for Scientific Research (S) No.22224001. The second author was partially supported by JSPS Grants-in-Aid for Scientific Research (B) No.20340002.
In this paper, we present some methods to investigate supersingular $K3$ surfaces with big Artin invariant by means of the following simple observation. Let $X_{p,\sigma}$ be a supersingular $K3$ surface in characteristic $p$ with Artin invariant $\sigma$, and let $S_{p,\sigma}$ denote its Néron-Severi lattice.

**Lemma 1.1.** Suppose that $\sigma + \sigma' = 11$. Then $S_{p,\sigma'}$ is isomorphic to $S_{p,\sigma}^\vee(p)$, where $S_{p,\sigma}^\vee(p)$ is the lattice obtained from the dual lattice $S_{p,\sigma}^\vee$ of $S_{p,\sigma}$ by multiplying the symmetric bilinear form with $p$.

Lemma 1.1 is proved in Section 3. We use this duality between $S_{p,\sigma}$ and $S_{p,\sigma'}$ in the study of genus one fibrations and the automorphism groups of supersingular $K3$ surfaces.

First, we apply Lemma 1.1 to the classification of genus one fibrations. Note that the Néron-Severi lattice $S_Y$ of a $K3$ surface $Y$ is a hyperbolic lattice. The orthogonal group $O(S_Y)$ contains the stabilizer subgroup $O^+(S_Y)$ of a positive cone of $S_Y \otimes \mathbb{R}$ as a subgroup of index 2.

**Definition 1.2.** Let $Y$ be a $K3$ surface, and let $\phi : Y \to \mathbb{P}^1$ be a genus one fibration. We denote by $f_\phi \in S_Y$ the class of a fiber of $\phi$. Let $\psi : Y \to \mathbb{P}^1$ be another genus one fibration on $Y$. We say that $\phi$ and $\psi$ are Aut-equivalent if there exist $g \in \text{Aut}(Y)$ and $\tilde{g} \in \text{Aut}(\mathbb{P}^1)$ such that $\phi \circ g = \tilde{g} \circ \psi$ holds, while we say that $\phi$ and $\psi$ are lattice equivalent if there exists $g \in O^+(S_Y)$ such that $f_{\phi,\psi} = f_\psi$. We denote by $E(Y)$ the set of lattice equivalence classes of genus one fibrations on $Y$, and by $[\phi] \in E(Y)$ the lattice equivalence class containing $\phi$.

Many combinatorial properties of a genus one fibration $\phi : Y \to \mathbb{P}^1$ depend only on the lattice equivalence class $[\phi]$. See Proposition 4.1. Moreover, when $\sigma = 10$, the classification of genus one fibrations by Aut-equivalence seems to be too fine, as is suggested by Proposition 9.2. Therefore, we concentrate upon the study of lattice equivalence classes.

Using Lemma 1.1, we prove the following:

**Theorem 1.3.** Suppose that $\sigma + \sigma' = 11$. Then there exists a canonical one-to-one correspondence $[\phi] \mapsto [\phi']$ between $E(X_{p,\sigma})$ and $E(X_{p,\sigma'})$.

We say that a genus one fibration is Jacobian if it admits a section.

**Theorem 1.4.** Suppose that a genus one fibration $\phi : X_{p,\sigma} \to \mathbb{P}^1$ is a Jacobian fibration, and let $\phi' : X_{p,\sigma'} \to \mathbb{P}^1$ be a genus one fibration on $X_{p,\sigma'}$ with $\sigma' = 11 - \sigma$ such that $[\phi'] \in E(X_{p,\sigma'})$ corresponds to $[\phi] \in E(X_{p,\sigma})$ by Theorem 1.3. Then $\phi'$ does not admit a section.

Elkies and Schütt [11] proved the following:

**Theorem 1.5 ([11]).** Any genus one fibration on $X_{p,1}$ admits a section.

Therefore we obtain the following:
Corollary 1.6. There exist no Jacobian fibrations on $X_{p,10}$.

By an $ADE$-type, we mean a finite formal sum of the symbols $A_i$ ($i \geq 1$), $D_i$ ($j \geq 4$) and $E_k$ ($k = 6, 7, 8$) with non-negative integer coefficients. For a genus one fibration $\phi : Y \to \mathbb{P}^1$ on a $K3$ surface $Y$, we have the $ADE$-type of reducible fibers of $\phi$. This $ADE$-type depends only on the lattice equivalence class $[\phi] \in \mathcal{E}(Y)$ (see Proposition 4.1). Therefore we can use $R_{[\phi]}$ to denote the $ADE$-type of the reducible fibers of $\phi$.

From the classification of lattice equivalence classes of genus one fibrations of $X_{2,1}$ by Elkies and Schütt [11], and that of $X_{3,1}$ by Sengupta [28], we obtain the classification of lattice equivalence classes of genus one fibrations on $X_{2,10}$ and $X_{3,10}$. In particular, we obtain the list of $ADE$-types $R_{[\phi]}$ of the reducible fibers of genus one fibrations $\phi'$ on $X_{2,10}$ and $X_{3,10}$. See Theorems 4.8 and 4.9.

In Elkies and Schütt [11] and Sengupta [28] mentioned above, they also obtained explicit defining equations of the Jacobian fibrations. Besides [11] and [28], there have been many works on the classification of $Aut$-equivalence classes and lattice equivalence classes of Jacobian fibrations on a $K3$ surface (e.g. Oguiso [23], Nishiyama [22], Shimada and Zhang [34], Shimada [29], Kloosterman [16]). In particular, the lattice equivalence classes of all extremal (quasi-) elliptic fibrations (i.e., Jacobian fibrations with Mordell-Weil rank zero) on supersingular $K3$ surfaces are classified in Shimada [30].

As the second application of Lemma 1.1, we investigate the automorphism group of the nef cone of a supersingular $K3$ surface. For a $K3$ surface $Y$, let $\text{Nef}(Y) \subset S_Y \otimes \mathbb{R}$ denote the nef cone. We denote by $\text{Aut}(\text{Nef}(Y)) \subset O^+(S_Y)$ the group of isometries of $S_Y$ that preserve $\text{Nef}(Y)$. Since $\text{Aut}(X_{p,\sigma})$ acts on $S_{p,\sigma}$ faithfully (Rudakov and Shafarevich [27, Section 8, Proposition 3]), we have

$$\text{Aut}(X_{p,\sigma}) \subset \text{Aut}(\text{Nef}(X_{p,\sigma})) \subset O^+(S_{p,\sigma}).$$

Using the description of $\text{Aut}(X_{2,1})$ by Dolgachev and Kondo [8], and that of $\text{Aut}(X_{3,1})$ by Kondo and Shimada [18], we give a set of generators of $\text{Aut}(\text{Nef}(X_{2,10}))$ and $\text{Aut}(\text{Nef}(X_{3,10}))$ in Theorems 6.4 and 6.9, respectively.

Suppose that $p$ is odd. We fix a lattice $N$ isomorphic to $S_{p,\sigma}$. Then a quadratic space $(N_0, q_0)$ of dimension $2\sigma$ over $\mathbb{F}_p$ is defined by

$$N_0 := pN^\vee/pN \quad \text{and} \quad q_0(px \mod pN) := px^2 \mod p \quad (x \in N^\vee).$$

We fix a marking $\eta : N \simeq S_{p,\sigma}$ for a supersingular $K3$ surface $X := X_{p,\sigma}$ defined over $k$. Then $\text{Aut}(\text{Nef}(X))$ acts on $(N_0, q_0)$, and the period $K_{(X, \eta)} \subset N_0 \otimes k$ of the marked supersingular $K3$ surface $(X, \eta)$ is defined as the Frobenius pull-back of the kernel of the natural homomorphism

$$N \otimes k \to S_X \otimes k \to H^2_{\text{DR}}(X/k).$$
In virtue of Torelli theorem for supersingular $K3$ surfaces by Ogus [24], [25], the subgroup $\text{Aut}(X)$ of $\text{Aut}(\text{Nef}(X))$ is equal to the stabilizer subgroup of the period $K_{(X,\eta)}$. In particular, the index of $\text{Aut}(X_{p,\sigma})$ in $\text{Aut}(\text{Nef}(X_{p,\sigma}))$ is finite. On the other hand, the classification of 2-reflective lattices due to Nikulin [21] implies that $\text{Aut}(\text{Nef}(X_{p,\sigma}))$ is infinite. Hence, at least when $p$ is odd, $\text{Aut}(X_{p,\sigma})$ is an infinite group. See Sections 5 and 7 for details. Moreover, Lieblich and Maulik [19] proved that, if $p > 2$, then $\text{Aut}(X_{p,\sigma})$ is finitely generated and its action on $\text{Nef}(X_{p,\sigma})$ has a rational polyhedral fundamental domain.

We say that a supersingular $K3$ surface $X$ is generic if there exists a marking $\eta : N \cong S_X$ such that the isometries of $(N_0, q_0)$ that preserve the period $K_{(X,\eta)} \subset N_0 \otimes k$ are only scalar multiplications (see Definition 7.5). Using the surjectivity of the period mapping proved by Ogus [25], we prove the following:

**Theorem 1.7.** Suppose that $p$ is odd and $\sigma > 1$. Then there exist an algebraically closed field $k$ and a supersingular $K3$ surface $X$ with Artin invariant $\sigma$ defined over $k$ that is generic.

Suppose that $X_{3,10}$ is generic. From the generators of $\text{Aut}(\text{Nef}(X_{3,10}))$ given in Theorem 6.9, we can obtain a finite set of generators of $\text{Aut}(X_{3,10})$. However, the computation would be very heavy. See Remarks 7.7 and 7.8.

As the third application, we show by an example that a lattice equivalence class of genus one fibrations on $X_{3,10}$ can contain a very large number of $\text{Aut}$-equivalence classes, provided that $X_{3,10}$ is generic. An analogous result for a generic complex Enriques surface was obtained by Barth and Peters [2].

This paper is organized as follows. In Section 2, we fix notation and terminologies about lattices and $K3$ surfaces. In Section 3, Lemma 1.1 is proved by means of the fundamental results of Rudakov and Shafarevich [27] on the Néron-Severi lattices of supersingular $K3$ surfaces. In Section 4, we study genus one fibrations on supersingular $K3$ surfaces, and prove Theorems 1.3 and 1.4. Moreover, the bijections $\mathbb{E}(X_{p,1}) \cong \mathbb{E}(X_{p,10})$ for $p = 2$ and 3 are given explicitly in Tables 4.1 and 4.2. In Section 5, we review the classical method to investigate the orthogonal group of a hyperbolic lattice by means of a chamber decomposition of the associated hyperbolic space, and fix some notation and terminologies. We then apply this method to the nef cone of a supersingular $K3$ surface. In Section 6, we give a set of generators of $\text{Aut}(\text{Nef}(X_{2,10}))$ and $\text{Aut}(\text{Nef}(X_{3,10}))$. In Section 7, we review the theory of the period mapping and Torelli theorem for supersingular $K3$ surfaces in odd characteristics due to Ogus [24], [25], and describe the relation between $\text{Aut}(X_{p,\sigma})$ and $\text{Aut}(\text{Nef}(X_{p,\sigma}))$. In Section 8, we prove Theorem 1.7. In the last section, we illustrate that the number of $\text{Aut}$-equivalence classes of genus one fibrations on $X_{3,10}$ is intractably large if $X_{3,10}$ is generic.

**Convention.** We use $\text{Aut}$ to denote automorphism groups of lattice theoretic objects, and $\text{Aut}$ to denote automorphism groups of geometric objects on $K3$ surfaces.
2. Preliminaries

2.1. Lattices. A \( \mathbb{Q} \)-lattice is a free \( \mathbb{Z} \)-module \( L \) of finite rank equipped with a non-degenerate symmetric bilinear form \( \langle \cdot, \cdot \rangle_L : L \times L \to \mathbb{Q} \). We omit the subscript \( L \) in \( \langle \cdot, \cdot \rangle \) if no confusions will occur. If \( \langle \cdot, \cdot \rangle_L \) takes values in \( \mathbb{Z} \), we say that \( L \) is a lattice. For \( x \in L \otimes \mathbb{R} \), we call \( x^2 := \langle x, x \rangle \) the norm of \( x \). A vector in \( L \otimes \mathbb{R} \) of norm \( n \) is sometimes called an \( n \)-vector. A lattice \( L \) is said to be even if \( x^2 \in 2\mathbb{Z} \) holds for any \( x \in L \).

Let \( L \) be a free \( \mathbb{Z} \)-module of finite rank. A submodule \( M \) of \( L \) is primitive if \( L/M \) is torsion free. A non-zero vector \( v \in L \) is primitive if the submodule of \( L \) generated by \( v \) is primitive.

Let \( L \) be a \( \mathbb{Q} \)-lattice of rank \( r \). For a non-zero rational number \( m \), we denote by \( L(m) \) the free \( \mathbb{Z} \)-module \( L \) with the symmetric bilinear form \( \langle x, y \rangle_{L(m)} := m \langle x, y \rangle_L \). The signature of \( L \) is the signature of the real quadratic space \( L \otimes \mathbb{R} \). We say that \( L \) is negative definite if the signature of \( L \) is \((0, r)\), and \( L \) is hyperbolic if the signature is \((1, r-1)\). A Gram matrix of \( L \) is an \( r \times r \) matrix with entries \( \langle e_i, e_j \rangle \), where \( \{e_1, \ldots, e_r\} \) is a basis of \( L \). The determinant of a Gram matrix of \( L \) is called the discriminant of \( L \).

For an even lattice \( L \), the set of \((-2)\)-vectors is denoted by \( \mathcal{R}(L) \). A negative definite even lattice \( L \) is called a root lattice if \( L \) is generated by \( \mathcal{R}(L) \). Let \( R \) be an ADE-type. The root lattice of type \( R \) is the root lattice whose Gram matrix is the Cartan matrix of type \( R \). Suppose that \( L \) is negative definite. By the \( ADE \)-type of \( \mathcal{R}(L) \), we mean the \( ADE \)-type of the root sublattice \( \langle \mathcal{R}(L) \rangle \) of \( L \) generated by \( \mathcal{R}(L) \). (See, for example, Ebeling [10] for the classification of root lattices.)

Let \( L \) be an even lattice and let \( L^\vee := \text{Hom}(L, \mathbb{Z}) \) be identified with a submodule of \( L \otimes \mathbb{Q} \) with the extended symmetric bilinear form. We call this \( \mathbb{Q} \)-lattice \( L^\vee \) the dual lattice of \( L \). The discriminant group of \( L \) is defined to be the quotient \( L^\vee/L \), and is denoted by \( A_L \). We define the discriminant quadratic form of \( L \)

\[ q_L : A_L \to \mathbb{Q}/2\mathbb{Z} \]

by \( q_L(x \mod L) := x^2 \mod 2\mathbb{Z} \). The order of \( A_L \) is equal to the discriminant of \( L \) up to sign. We say that \( L \) is unimodular if \( A_L \) is trivial, while \( L \) is \( p \)-elementary if \( A_L \) is \( p \)-elementary. An even 2-elementary lattice \( L \) is said to be of type \( I \) if \( q_L(x \mod L) \in \mathbb{Z}/2\mathbb{Z} \) holds for any \( x \in L^\vee \). Note that \( L \) is \( p \)-elementary if and only if \( pG_L^{-1} \) is an integer matrix, where \( G_L \) is a Gram matrix of \( L \).

Let \( O(L) \) denote the orthogonal group of a lattice \( L \), that is, the group of isomorphisms of \( L \) preserving \( \langle \cdot, \cdot \rangle_L \). We assume that \( O(L) \) acts on \( L \) from right, and the action of \( g \in O(L) \) on \( v \in L \otimes \mathbb{R} \) is denoted by \( v \mapsto v^g \). Similarly \( O(q_L) \) denotes the group of isomorphisms of \( A_L \) preserving \( q_L \). There is a natural homomorphism \( O(L) \to O(q_L) \).

Let \( L \) be a hyperbolic lattice. A positive cone of \( L \) is one of the two connected components of

\[ \{ x \in L \otimes \mathbb{R} \mid x^2 > 0 \} \].
Let \( P_L \) be a positive cone of \( L \). We denote by \( O^+(L) \) the group of isometries of \( L \) that preserve \( P_L \). We have \( O(L) = O^+(L) \times \{\pm 1\} \). For a vector \( v \in L \otimes \mathbb{R} \) with \( v^2 < 0 \), we put
\[
(v)^\perp := \{ x \in P_L \mid \langle x, v \rangle = 0 \},
\]
which is a real hyperplane of \( P_L \). An isometry \( g \in O^+(L) \) is called a reflection with respect to \( v \) or a reflection into \( (v)^\perp \) if \( g \) is of order 2 and fixes each point of \( (v)^\perp \). An element \( r \) of \( R(L) \) defines a reflection \( s_r : x \mapsto x + \langle x, r \rangle r \) with respect to \( r \). We denote by \( W(S_L) \) the subgroup of \( O^+(L) \) generated by the set of these reflections \( \{s_r \mid r \in R(L)\} \). It is obvious that \( W(S_L) \) is normal in \( O^+(L) \).

2.2. \( K3 \) surfaces. Let \( Y \) be a \( K3 \) surface, and let \( S_Y \) denote the Néron-Severi lattice of \( Y \). A smooth rational curve on \( Y \) is called a \((-2)\)-curve. We denote by \( \mathcal{P}(Y) \subset S_Y \otimes \mathbb{R} \) the positive cone containing an ample class of \( Y \). Recall that the nef cone \( \text{Nef}(Y) \) of \( Y \) is defined by
\[
\text{Nef}(Y) := \{ x \in S_Y \otimes \mathbb{R} \mid \langle x, [C] \rangle \geq 0 \text{ for any curve } C \text{ on } Y \},
\]
where \([C] \in S_Y \) is the class of a curve \( C \subset Y \). Then \( \text{Nef}(Y) \) is contained in the closure \( \overline{\mathcal{P}(Y)} \) of \( \mathcal{P}(Y) \) in \( S_Y \otimes \mathbb{R} \). We put
\[
\text{Nef}^\circ(Y) := \text{Nef}(Y) \cap \mathcal{P}(Y) = \{ x \in \text{Nef}(Y) \mid x^2 > 0 \}.
\]
The following is well-known. See, for example, Rudakov and Shafarevich [27, Section 3].

**Proposition 2.1.** (1) We have
\[
\text{Nef}(Y) = \{ x \in S_Y \otimes \mathbb{R} \mid \langle x, [C] \rangle \geq 0 \text{ for any } (-2)\text{-curve } C \text{ on } Y \}.
\]
(2) If \( v \in S_Y \) is contained in \( \overline{\mathcal{P}(Y)} \), then there exists \( g \in W(S_Y) \) such that \( v^g \in \text{Nef}(Y) \).

3. Néron-Severi lattices of supersingular \( K3 \) surfaces

Let \( X_{p,\sigma} \) be a supersingular \( K3 \) surface with Artin invariant \( \sigma \) in characteristic \( p > 0 \). Then the isomorphism class of the Néron-Severi lattice \( S_{p,\sigma} \) of \( X_{p,\sigma} \) depends only on \( p \) and \( \sigma \), and is characterized as follows (see Rudakov-Shafarevich [27, Sections 3,4 and 5] for the proof).

**Theorem 3.1** ([27]). (1) The lattice \( S_{p,\sigma} \) is an even hyperbolic \( p \)-elementary lattice of rank 22 with discriminant \(-p^{2\sigma}\). Moreover, \( S_{2,\sigma} \) is of type \( I \).

(2) Suppose that \( N \) is an even hyperbolic \( p \)-elementary lattice of rank 22 with discriminant \(-p^{2\sigma}\). When \( p = 2 \), we further assume that \( N \) is of type \( I \). Then \( N \) is isomorphic to \( S_{p,\sigma} \).

Using this theorem, we can prove Lemma 1.1 easily.
Proof of Lemma 1.1. It is enough to show that \( S'_{p,\sigma}(p) \) is an even \( p \)-elementary lattice of discriminant \(-p^{2\sigma'}\), and that \( S'_{2,\sigma}(2) \) is of type I. Since \( S_{p,\sigma} \) is \( p \)-elementary, we have \( pS'_{p,\sigma} \subset S_{p,\sigma} \). Therefore \( S'_{p,\sigma}(p) \) is a lattice. Let \( G_{p,\sigma} \) be a Gram matrix of \( S_{p,\sigma} \). Then the determinant of the Gram matrix \( pG_{p,\sigma}^{-1} \) of \( S'_{p,\sigma}(p) \) is equal to \( p^{22} \cdot \det(G_{p,\sigma})^{-1} = -p^{2\sigma'} \). Therefore the discriminant of \( S'_{p,\sigma}(p) \) is \(-p^{2\sigma'}\). Since \( p(pG_{p,\sigma}^{-1})^{-1} = G_{p,\sigma} \) is an integer matrix, \( S'_{p,\sigma}(p) \) is \( p \)-elementary. Suppose that \( p \) is odd. Then, for any \( \xi \in S'_{p,\sigma} \), we have \( p\xi \in S_{p,\sigma} \) and hence \( (p\xi, p\xi)_{S_{p,\sigma}} = p(\xi, \xi)_{S'_{p,\sigma}(p)} \) is even. Therefore \( S'_{p,\sigma}(p) \) is even. Suppose that \( p = 2 \). Then, for any \( \xi \in S'_{2,\sigma} \), we have \( (\xi, \xi)_{S'_{2,\sigma}} \in \mathbb{Z} \), because \( S_{2,\sigma} \) is of type I. Therefore \( S'_{2,\sigma}(2) \) is even. Moreover, for any \( \eta \in (S'_{2,\sigma}(2))^\vee \), we have \( (\eta, \eta)_{S_{2,\sigma}(1/2)} \in \mathbb{Z} \), because \( S_{2,\sigma} \) is even. Therefore \( S'_{2,\sigma}(2) \) is of type I. \( \square \)

Corollary 3.2. Suppose that \( \sigma + \sigma' = 11 \). Then there exists an embedding of \( \mathbb{Z} \)-modules

\[ j : S_{p,\sigma} \hookrightarrow S_{p,\sigma'} \]

that induces an isomorphism of lattices \( S'_{p,\sigma}(p) \cong S_{p,\sigma'} \). This embedding induces an isomorphism

\[ j_* : O(S_{p,\sigma}) \cong O(S_{p,\sigma'}) \]

Moreover such an embedding \( j \) is unique up to compositions with elements of \( O(S_{p,\sigma'}) \).

Remark 3.3. Suppose that \( v \in S_{p,\sigma} \) satisfies \( v^2 \geq 0 \). Then, by Proposition 2.1(2), we can choose \( j : S_{p,\sigma} \hookrightarrow S_{p,\sigma'} \) in Corollary 3.2 in such a way that \( j(v) \) is contained in \( \text{Nef}(X_{p,\sigma'}) \).

4. Genus One Fibrations

Let \( Y \) be a \( K3 \) surface defined over an algebraically closed field of arbitrary characteristic. Recall that \( f_\phi \in S_Y \) is the class of a fiber of a genus one fibration \( \phi : Y \rightarrow \mathbb{P}^1 \), \( E(Y) \) is the set of lattice equivalence classes of genus one fibrations on \( Y \), and \([\phi] \in E(Y)\) is the class containing \( \phi \). We summarize properties of a genus one fibration \( \phi : Y \rightarrow \mathbb{P}^1 \) that depends only on the class \([\phi]\). See Sections 3 and 4 of Rudakov and Shafarevich [27], and Shioda [36] for the proof.

1. The fibration \( \phi \) admits a section if and only if there exists a \((-2)\)-vector \( z \in S_Y \) such that \( \langle f_\phi, z \rangle = 1 \).

2. Note that \( f_\phi \in S_Y \) is primitive of norm 0, and that \( \langle f_\phi \rangle^\perp/\langle f_\phi \rangle \) is an even negative definite lattice, where \( \langle f_\phi \rangle^\perp \) is the orthogonal complement in \( S_Y \) of the lattice \( \langle f_\phi \rangle \) of rank 1 generated by \( f_\phi \). The ADE-type of the reducible fibers of \( \phi \) is equal to the ADE-type of the set \( \mathcal{R}(\langle f_\phi \rangle^\perp/\langle f_\phi \rangle) \) of \((-2)\)-vectors in \( \langle f_\phi \rangle^\perp/\langle f_\phi \rangle \).

3. Suppose that \( \phi \) admits a section \( Z \subset Y \). Then \( f_\phi \) and \([Z] \in S_Y \) generate an even unimodular hyperbolic lattice \( U_\phi \) of rank 2 in \( S_Y \). Let \( K_\phi \) denote the orthogonal complement of \( U_\phi \) in \( S_Y \). We have an orthogonal direct-sum decomposition

\[ S_Y = U_\phi \oplus K_\phi, \]
and the lattice $\langle f_\phi \rangle^\perp/\langle f_\phi \rangle$ is isomorphic to $K_\phi$. Then the Mordell-Weil group of $\phi$ is isomorphic to $K_\phi/\langle \mathcal{R}(K_\phi) \rangle$, where $\langle \mathcal{R}(K_\phi) \rangle$ is the root sublattice of $K_\phi$ generated by the $(-2)$-vectors in $K_\phi$.

(4) In characteristic 2 or 3, $\phi$ is quasi-elliptic if and only if $\langle \mathcal{R}(K_\phi) \rangle$ is $p$-elementary of rank 20.

As a corollary, we obtain the following:

**Proposition 4.1.** Suppose that genus one fibrations $\phi : Y \to \mathbb{P}^1$ and $\psi : Y \to \mathbb{P}^1$ on $Y$ are lattice-equivalent. Then the following hold:

1. The fibration $\phi$ admits a section if and only if so does $\psi$.
2. The ADE-type of the reducible fibers of $\phi$ is equal to that of $\psi$.
3. Suppose that $\phi$ and $\psi$ admit a section. Then the Mordell-Weil groups for $\phi$ and for $\psi$ are isomorphic.
4. In characteristic 2 or 3, the fibration $\phi$ is quasi-elliptic if and only if so is $\psi$.

**Definition 4.2.** For a hyperbolic lattice $S$, we put

$$\tilde{E}(S) := \{ v \in S \otimes \mathbb{Q} | v \neq 0, v^2 = 0 \}/\mathbb{Q}^\times$$
and $E(S) := \tilde{E}(S)/O(S)$.

**Remark 4.3.** Let a positive cone $\mathcal{P}_S$ of $S$ be fixed. Then each element of $\tilde{E}(S)$ is represented by a unique non-zero primitive vector $v \in S$ of norm 0 that is contained in the closure $\overline{\mathcal{P}_S}$ of $\mathcal{P}_S$ in $S \otimes \mathbb{R}$.

In Sections 3 and 4 of Rudakov and Shafarevich [27], the following is proved:

**Proposition 4.4.** Let $v$ be a non-zero vector of $S_Y$. Then there exists a genus one fibration $\phi : Y \to \mathbb{P}^1$ such that $v = f_\phi$ if and only if $v$ is primitive, $v^2 = 0$, and $v \in \text{Nef}(Y)$.

Combining Propositions 2.1, 4.4 and Remark 4.3, we obtain the following:

**Corollary 4.5.** The map $\phi \mapsto f_\phi$ induces a bijection from $E(Y)$ to $E(S_Y)$.

From now on, we work over an algebraically closed field of characteristic $p > 0$.

**Proof of Theorem 1.3.** Consider the embedding $j : S_{p,\sigma} \hookrightarrow S_{p,\sigma'}$ in Corollary 3.2. Then $j$ is unique up to $O(S_{p,\sigma'})$, induces a bijection from $\tilde{E}(S_{p,\sigma})$ to $\tilde{E}(S_{p,\sigma'})$, and induces an isomorphism $O(S_{p,\sigma}) \cong O(S_{p,\sigma'})$. Hence it induces a canonical bijection from $E(S_{p,\sigma})$ to $E(S_{p,\sigma'})$. \hfill $\square$

We denote this canonical one-to-one correspondence from $E(X_{p,\sigma})$ to $E(X_{p,\sigma'})$ by $[\phi] \mapsto [\phi']$.

**Remark 4.6.** Let a genus one fibration $\phi : X_{p,\sigma} \to \mathbb{P}^1$ be given, and let $\phi' : X_{p,\sigma'} \to \mathbb{P}^1$ be a representative of $[\phi']$. Then we can choose the embedding $j : S_{p,\sigma} \hookrightarrow S_{p,\sigma'}$ inducing $S_{p,\sigma}^{\vee}(p) \cong S_{p,\sigma'}$ in such a way that $j(f_\phi)$ is a scalar multiple of $f_{\phi'}$ by a positive integer.

**Theorem 4.7.** Suppose that a genus one fibration $\phi : X_{p,\sigma} \to \mathbb{P}^1$ admits a section. Then the corresponding genus one fibration $\phi' : X_{p,\sigma'} \to \mathbb{P}^1$ does not admit a section. Moreover the ADE-type of the reducible fibers of $\phi'$ is equal to the ADE-type of $\mathcal{R}(K_{\phi}(p))$. 
Proof. Let $z \in S_{p,\sigma}$ be the class of a section of $\phi$. We choose $j : S_{p,\sigma} \rightarrow S_{p,\sigma'}$ as in Remark 4.6. Since $U_{\phi}^{\vee} = U_{\phi}$, we see that $j(f_{\phi})$ is primitive in $S_{p,\sigma'}$ and hence $j(f_{\phi}) = f_{\phi'}$. We have an isomorphism $S_{p,\sigma'} \cong U_{\phi}(p) \oplus K_{\phi}^{\vee}(p)$ such that $f_{\phi'}$ and $j(z)$ form a basis of $U_{\phi}(p)$. Since there are no vectors $v \in U_{\phi}(p) \oplus K_{\phi}^{\vee}(p)$ with $\langle v, f_{\phi'} \rangle = 1$, the fibration $\phi'$ does not admit a section. Moreover the lattice $\langle f_{\phi'} \rangle^{\perp}/\langle f_{\phi'} \rangle$ is isomorphic to $K_{\phi}^{\vee}(p)$. \qed

The list of lattice equivalence classes of genus one fibrations on $X_{2,1}$ and $X_{3,1}$ were obtained by Elkies and Schütt [11] and by Sengupta [28], respectively. From their results, we obtain the following results on supersingular $K3$ surfaces with Artin invariant 10:

**Theorem 4.8.** There exist 18 lattice equivalence classes of genus one fibrations on $X_{2,10}$. The ADE-type $R_{[\phi]}$ of the reducible fibers of each $[\phi'] \in \mathcal{E}(X_{2,10})$ is given at the last column of Table 4.1.

**Theorem 4.9.** There exist 52 lattice equivalence classes of genus one fibrations on $X_{3,10}$. The ADE-type $R_{[\phi]}$ of the reducible fibers of each $[\phi'] \in \mathcal{E}(X_{3,10})$ is given at the last column of Table 4.2.

| No. | $R_N$ | $\sigma = 1$ | $\sigma = 10$ |
|-----|-------|-------------|--------------|
|     | $R_{[\phi]}$ | $MW_{tor}$ | rank($MW$) | $R_{[\phi']}$ |
| 1   | $4A_5 + D_4$ | $4A_5$ | [3, 6] | 0 | 0 |
| 2   | $6D_4$ | $5D_4$ | [2, 2, 2, 2] | 0 | 0 |
| 3   | $2A_7 + 2D_5$ | $2A_7 + D_5$ | [8] | 1 | $A_1$ |
| 4   | $2A_9 + D_6$ | $2A_1 + 2A_9$ | [10] | 0 | $2A_1$ |
| 5   | $4D_6$ | $2A_1 + 3D_6$ | [2, 2, 2] | 0 | $2A_1$ |
| 6   | $A_{11} + D_7 + E_6$ | $A_{11} + D_7$ | [4] | 2 | $A_2$ |
| 7   | $A_{11} + D_7 + E_6$ | $A_3 + A_{11} + E_6$ | [6] | 0 | $3A_1$ |
| 8   | $4E_6$ | $3E_6$ | [3] | 2 | $A_2$ |
| 9   | $3D_8$ | $D_4 + 2D_8$ | [2, 2] | 0 | $4A_1$ |
| 10  | $A_{15} + D_9$ | $A_{15} + D_3$ | [4] | 0 | $5A_1$ |
| 11  | $A_{17} + E_7$ | $3A_1 + A_{17}$ | [6] | 0 | $A_3$ |
| 12  | $D_{10} + 2E_7$ | $3A_1 + D_{10} + E_7$ | [2, 2] | 0 | $A_3$ |
| 13  | $D_{10} + 2E_7$ | $D_6 + 2E_7$ | [2] | 0 | $6A_1$ |
| 14  | $2D_{12}$ | $D_8 + D_{12}$ | [2] | 0 | $8A_1$ |
| 15  | $D_{16} + E_8$ | $D_4 + D_{16}$ | [2] | 0 | $D_4$ |
| 16  | $D_{16} + E_8$ | $D_{12} + E_8$ | [1] | 0 | $12A_1$ |
| 17  | $3E_8$ | $D_4 + 2E_8$ | [1] | 0 | $D_4$ |
| 18  | $D_{24}$ | $D_{20}$ | [1] | 0 | $20A_1$ |

**Table 4.1.** Genus one fibrations on $X_{2,1}$ and $X_{2,10}$.
| No. | $R_N$                      | $R_{[\sigma]}$ | $\sigma = 1$ | $\sigma = 10$ |
|-----|----------------------------|----------------|--------------|--------------|
|     |                            | $R_{[\sigma]}$ | $\sigma = 1$ | $\sigma = 10$ |
| 1   | $12A_2$                    | $10A_2$        | [3, 3, 3]    | 0            | 0            |
| 2   | $8A_3$                     | $6A_3$         | [4, 4]       | 2            | 0            |
| 3   | $6A_4$                     | $2A_1 + 4A_4$  | [5]          | 2            | 0            |
| 4   | $6D_4$                     | $4D_4$         | [2, 2]       | 4            | 0            |
| 5   | $4A_5 + D_4$               | $A_2 + 3A_5$   | [3]          | 3            | 0            |
| 6   | $4A_5 + D_4$               | $3A_5 + D_4$   | [2, 6]       | 1            | $A_1$        |
| 7   | $4A_5 + D_4$               | $2A_2 + 2A_5 + D_4$ | [2]       | 2            | 0            |
| 8   | $4A_6$                     | $3A_6$         | [7]          | 2            | $A_1$        |
| 9   | $4A_6$                     | $2A_3 + 2A_6$  | [1]          | 2            | 0            |
| 10  | $2A_7 + 2D_5$              | $4A_1 + 2A_7$  | [2, 4]       | 2            | 0            |
| 11  | $2A_7 + 2D_5$              | $A_1 + A_7 + 2D_5$ | [4]     | 2            | $A_1$        |
| 12  | $2A_7 + 2D_5$              | $2A_1 + A_4 + A_7 + D_5$ | [2]   | 2            | 0            |
| 13  | $2A_7 + 2D_5$              | $2A_1 + 2D_5$  | [1]          | 2            | 0            |
| 14  | $3A_8$                     | $A_2 + 2A_8$   | [3]          | 2            | $A_1$        |
| 15  | $3A_8$                     | $2A_5 + A_8$   | [1]          | 2            | 0            |
| 16  | $4D_6$                     | $3D_6$         | [2, 2]       | 2            | $2A_1$       |
| 17  | $4D_6$                     | $2A_3 + 2D_6$  | [2, 2]       | 2            | 0            |
| 18  | $2A_9$                     | $2A_9$         | [5]          | 2            | $2A_1$       |
| 19  | $2A_9 + D_6$               | $A_3 + A_9 + D_6$ | [2]   | 2            | $A_1$        |
| 20  | $2A_9 + D_6$               | $A_3 + A_6 + A_9$ | [1]   | 2            | 0            |
| 21  | $2A_9 + D_6$               | $2A_6 + D_6$   | [1]          | 2            | 0            |
| 22  | $4E_6$                     | $A_2 + 3E_6$   | [3]          | 0            | $A_2$        |
| 23  | $4E_6$                     | $4A_2 + 2E_6$  | [3, 3]       | 0            | 0            |
| 24  | $A_{11} + D_7 + E_6$       | $A_2 + A_{11} + D_7$ | [4] | 0            | $A_2$        |
| 25  | $A_{11} + D_7 + E_6$       | $A_{11} + E_6$  | [3]          | 3            | $2A_1$       |
| 26  | $A_{11} + D_7 + E_6$       | $2A_2 + A_{11} + D_4$ | [6] | 1            | 0            |
| 27  | $A_{11} + D_7 + E_6$       | $A_5 + D_7 + E_6$ | [1]  | 2            | $A_1$        |
| 28  | $A_{11} + D_7 + E_6$       | $2A_2 + A_8 + D_7$ | [1] | 1            | 0            |
| 29  | $A_{11} + D_7 + E_6$       | $A_8 + D_4 + E_6$ | [1]  | 2            | 0            |
| 30  | $2A_{12}$                  | $A_6 + A_{12}$  | [1]          | 2            | $A_1$        |
| 31  | $2A_{12}$                  | $2A_9$         | [1]          | 2            | 0            |
| 32  | $3D_8$                     | $2A_1 + 2D_5$  | [2, 2]       | 2            | $2A_1$       |
| 33  | $3D_8$                     | $2D_5 + D_8$   | [2]          | 2            | 0            |
| 34  | $A_{15} + D_9$             | $A_3 + A_{15}$  | [4]          | 2            | $2A_1$       |
| 35  | $A_{15} + D_9$             | $A_9 + D_9$    | [1]          | 2            | $A_1$        |
| 36  | $A_{15} + D_9$             | $A_{12} + D_6$ | [1]          | 2            | 0            |

(to be continued)
In Table 4.1 (resp. Table 4.2), the lists $E(X_{2,1})$ and $E(X_{2,10})$ (resp. $E(X_{3,1})$ and $E(X_{3,10})$) are presented. Two lattice equivalence classes in the same row are the pair of $[\phi] \in E(X_{p,1})$ and its corresponding partner $[\phi^\prime] \in E(X_{p,10})$. The $ADE$-type $R_\phi$ of the reducible fibers of $\phi$, and the torsion $\text{MW}_{\text{tor}}$ and the rank of the Mordell-Weil group of $\phi$ are also given. (Recall that $\phi$ is Jacobian for any $[\phi] \in E(X_{p,1})$ by Elkies and Schütz [11].) The meaning of the entry $R_N$ is explained in the proof of Theorems 4.8 and 4.9.

**Proof of Theorems 4.8 and 4.9.** By Theorem 4.7, it is enough to calculate the $ADE$-type of $R(K_{\phi}^\vee(p))$ for $p = 2, 3$ and $[\phi] \in E(X_{p,1})$. The lattices $K_\phi$ are calculated in Elkies and Schütz [11] and Sengupta [28] by Nishiyama’s method [22]. We put

$$T := \text{the root lattice of type } \begin{cases} D_4 & \text{if } p = 2, \\ 2A_2 & \text{if } p = 3. \end{cases}$$

Then, for each $[\phi] \in E(X_{p,1})$, there exist a Niemeier lattice $N_\phi$ and a primitive embedding of $T$ into $N_\phi$ such that $K_\phi$ is isomorphic to the orthogonal complement of $T$ in $N_\phi$. The entry $R_N$ in Tables 4.1 and 4.2 indicates the $ADE$-type of $R(N_\phi)$. From a Gram matrix of $K_\phi$, we can calculate the $ADE$-type of $R(K_{\phi}^\vee(p))$ by the algorithm described in [32, Section 4] or [33, Section 3].

**Corollary 4.10.** There exist no quasi-elliptic fibrations on $X_{3,10}$.
Remark 4.11. Rudakov and Shafarevich [27, Section 5] showed that there exists a quasi-elliptic fibration on $X_{2,\sigma}$ for any $\sigma$. The quasi-elliptic fibration on $X_{2,10}$ (No. 18 of Table 4.1) was discovered by Rudakov and Shafarevich [26, Section 4].

5. CHAMBER DECOMPOSITION OF A POSITIVE CONE

Let $S$ be an even hyperbolic lattice, and let $\mathcal{P}_S \subset S \otimes \mathbb{R}$ be a positive cone. In this section, we review a general method to find a set of generators of a subgroup of $O^+(S)$ by means of a chamber decomposition of $\mathcal{P}_S$, which was developed by Vinberg [37], [38], Conway [7] and Borcherds [3], [4].

Any real hyperplane in $\mathcal{P}_S$ is written in the form $(v)\perp$ by some vector $v \in S \otimes \mathbb{R}$ with negative norm. We denote by $H_S$ the set of real hyperplanes in $\mathcal{P}_S$, which is canonically identified with $\{v \in S \otimes \mathbb{R} | v^2 < 0\}/\mathbb{R}^\times$.

For a subset $V$ of $\{v \in S \otimes \mathbb{R} | v^2 < 0\}$, we denote by $V^* \subset H_S$ the image of $V$ by $v \mapsto (v)\perp$. A closed subset $D$ of $\mathcal{P}_S$ is called a chamber if the interior $D^\circ$ of $D$ is non-empty and there exists a set $\Delta_D$ of vectors $v \in S \otimes \mathbb{R}$ with $v^2 < 0$ such that $D = \{x \in \mathcal{P}_S | \langle x, v \rangle \geq 0 \text{ for all } v \in \Delta_D\}$.

A hyperplane $(v)\perp$ of $\mathcal{P}_S$ is called a wall of $D$ if $D^\circ \cap (v)\perp = \emptyset$ and $D \cap (v)\perp$ contains an open subset of $(v)\perp$. When $D$ is a chamber, we always assume that the set $\Delta_D$ is minimal in the sense that, for any $v \in \Delta_D$, there exists a point $x \in \mathcal{P}_S$ such that $\langle x, v \rangle < 0$ and $\langle x, v' \rangle \geq 0$ for any $v' \in \Delta_D \setminus \{v\}$, that is, the projection $\Delta_D \to \Delta_D^*$ is bijective and every hyperplane $(v)\perp \in \Delta_D^*$ is a wall of $D$.

For a chamber $D$, we put $Aut(D) := \{g \in O^+(S) | D^g = D\}$.

A chamber $D$ is said to be fundamental if the following hold:

(i) $\mathcal{P}_S$ is the union of all $D^g$, where $g$ runs through $O^+(S)$, and
(ii) if $D^g \cap D^h \neq \emptyset$, then $g \in Aut(D)$.

Let $\mathcal{F}$ be a family of hyperplanes in $\mathcal{P}_S$ with the following properties:

(a) $\mathcal{F}$ is locally finite in $\mathcal{P}_S$, and
(b) $\mathcal{F}$ is invariant under the action of $O^+(S)$ on $H_S$.

Then the closure of each connected component of $\mathcal{P}_S \setminus \bigcup_{\mathcal{F}} (v)\perp$ is a chamber, which we call an $\mathcal{F}$-chamber.
Proposition 5.1. \( F \) satisfies the property (i) if and only if every \( (b) \) of \( F \) satisfies the property (ii) in the definition of fundamental chambers. Moreover, \( F \) satisfies the property (i) if and only if every \( F \)-chamber is equal to \( D^g \) for some \( g \in O^+(S) \).

For each wall \((v)^\perp \in \Delta_D^\times\) of an \( F \)-chamber \( D \), there exists a unique \( F \)-chamber \( D' \) distinct from \( D \) such that \( D \cap D' \cap (v)^\perp \) contains an open subset of \((v)^\perp\). We say that \( D' \) is adjacent to \( D \) along \((v)^\perp\), and that \((v)^\perp\) is the wall between the adjacent chambers \( D \) and \( D' \).

**Proposition 5.1.** An \( F \)-chamber \( D \) is fundamental if and only if, for each \( v \in \Delta_D \), there exists \( g_v \in O^+(S) \) such that \( D^{g_v} \) is adjacent to \( D \) along \((v)^\perp\).

**Proof.** The ‘only if’ part is obvious. We prove the ‘if’ part. It is enough to show that, for an arbitrary \( F \)-chamber \( D' \), there exists \( g \in O^+(S) \) such that \( D' = D^g \). Since the family \( F \) of hyperplanes is locally finite in \( P_S \), there exists a finite chain of \( F \)-chambers \( D_0 = D, D_1, \ldots, D_N = D' \) such that \( D_i \) and \( D_{i+1} \) are adjacent. We show, by induction on \( N \), that there exists a sequence of vectors \( v_1, \ldots, v_N \in \Delta_D \) such that \( D_{i} = D^{g_{v_i}} \) holds for \( i = 1, \ldots, N \). The case \( N = 0 \) is trivial. Suppose that \( N > 0 \). Let \((w)^\perp\) be the wall between \( D_{N-1} \) and \( D_N \), and let \( v_N \in \Delta_D \) be the vector such that the wall \((v_N)^\perp\) of \( D \) is mapped to the wall \((w)^\perp\) of \( D_{N-1} \) by \( g_{v_{N-1}} \cdots g_{v_1} \). Then we have \( D_N = D^{g_{v_N}} \).

**Remark 5.2.** If an \( F \)-chamber is fundamental, then any \( F \)-chamber is fundamental.

Let \( G \) be a subset of \( F \) that is invariant under the action of \( O^+(S) \). Then \( G \) is locally finite, and any \( G \)-chamber is a union of \( F \)-chambers. If an \( F \)-chamber is fundamental, then any \( G \)-chamber is also fundamental.

**Proposition 5.3.** Let \( D \) be an \( F \)-chamber and let \( C \) be a \( G \)-chamber such that \( D \subset C \). Suppose that \( D \) is fundamental. For \( v \in \Delta_D \), let \( g_v \in O^+(S) \) be an isometry such that \( D^{g_v} \) is adjacent to \( D \) along \((v)^\perp\). We put

\[
\Gamma := \{ g_v \mid v \in \Delta_D, (v)^\perp \notin G \}.
\]

Then \( \text{Aut}(C) \) is generated by \( \text{Aut}(D) \) and \( \Gamma \).

**Proof.** If \( g_v \in \Gamma \), then \( D^{g_v} \) is contained in \( C \) because the wall \((v)^\perp\) between \( D \) and \( D^{g_v} \) does not belong to \( G \), and hence \( g_v \in \text{Aut}(C) \). Therefore the subgroup \( \langle \text{Aut}(D), \Gamma \rangle \) of \( O^+(S) \) generated by \( \text{Aut}(D) \) and \( \Gamma \) is contained in \( \text{Aut}(C) \). To prove \( \text{Aut}(C) \subset \langle \text{Aut}(D), \Gamma \rangle \), it is enough to show that, for any \( g \in \text{Aut}(C) \), there exists a sequence \( \gamma_1, \ldots, \gamma_N \) of elements of \( \Gamma \) such that \( D^g = D^{\gamma_N \cdots \gamma_1} \). There exists a sequence of \( F \)-chambers \( D_0 = D, D_1, \ldots, D_N = D^g \) such that each \( D_i \) is contained in \( C \) and that \( D_{i+1} \) is adjacent to \( D_i \) for \( i = 0, \ldots, N - 1 \). Suppose that we have constructed \( \gamma_1, \ldots, \gamma_i \in \Gamma \) such that \( D_i = D^{\gamma_i \cdots \gamma_1} \) holds. The wall \((w)^\perp\) between \( D_i \) and \( D_{i+1} \) does not belong to \( G \). Let \( v_{i+1} \) be an element of \( \Delta_D \) such that \((v_{i+1})^\perp\) of \( D \) is mapped to the wall \((w)^\perp\) of \( D_i \) by \( \gamma_i \cdots \gamma_1 \). Since \( G \) is invariant under the action of \( O^+(S) \), we have \((v_{i+1})^\perp \notin G \) and hence \( \gamma_{i+1} := g_{v_{i+1}} \) is an element of \( \Gamma \). Then \( D_{i+1} = D^{\gamma_{i+1} \cdots \gamma_1} \) holds.
Remark 5.4. Let $D$ and $C$ be as in Proposition 5.3. Let $v$ and $v'$ be elements of $\Delta_D$. Suppose that the wall $(v)\perp$ of $D$ is mapped to the wall $(v')\perp$ of $D$ by $h \in Aut(D)$. Then $D^{bg_r} = D^{h^{-1}}$ is adjacent to $D$ along $(v)\perp$. Let $\Delta_D'$ be a subset of $\Delta_D$ such that the subset $\Delta_D''$ of $\Delta_D'$ is a complete set of representatives of the orbit decomposition of $\Delta_D'$ by the action of $Aut(D)$. Then $Aut(C)$ is generated by $Aut(D)$ and $\{g_v \mid v \in \Delta_D \setminus (v)\perp \notin \mathcal{G}\}$.

Considering the case $\mathcal{G} = \emptyset$, we obtain the following:

**Corollary 5.5.** Let $D$ be an $\mathcal{F}$-chamber. If $D$ is fundamental, then $O^+(S)$ is generated by $Aut(D)$ and the isometries $g_v$ that map $D$ to its adjacent chambers.

**Example 5.6.** Recall that $W(S) \subset O^+(S)$ is the subgroup generated by $\{s_r \mid r \in R(S)\}$. Any $R(S)^*$-chamber is fundamental, because every $r \in R(S)$ defines a reflection $s_r$. It follows that $O^+(S)$ is equal to the semi-direct product of $W(S)$ and the automorphism group $Aut(D)$ of an $R(S)^*$-chamber $D$. In particular, we have

$$Aut(D) \cong O^+(S)/W(S).$$

Let $L$ be an even unimodular hyperbolic lattice, and let $\iota : S \rightarrow L$ be a primitive embedding. Let $\mathcal{P}_L$ be the positive cone of $L$ that contains $\iota(\mathcal{P}_S)$. We denote by $T_\iota$ the orthogonal complement of $S$ in $L$, and by

$$v \mapsto v_S$$

the orthogonal projection $L \otimes \mathbb{R} \rightarrow S \otimes \mathbb{R}$. Since $L$ is a submodule of $S^\vee \oplus T_\iota^\vee$, the image of $L$ by $v \mapsto v_S$ is contained in $S^\vee$. We assume the following:

$$\text{(5.1)} \quad \text{the natural homomorphism } O(T_\iota) \rightarrow O(qT_\iota) \text{ is surjective.}$$

Then we have the following:

**Proposition 5.7.** For any $g \in O^+(S)$, there exists $\tilde{g} \in O^+(L)$ such that $\iota(v^g) = \iota(v)^{\tilde{g}}$ holds for any $v \in S \otimes \mathbb{R}$.

**Proof.** See Nikulin [20, Proposition 1.6.1].

A hyperplane $(r)^\perp$ of $\mathcal{P}_L$ defined by a $(-2)$-vector $r \in R(L)$ intersects $\iota(\mathcal{P}_S)$ if and only if $r_S^2 < 0$. We put

$$\mathcal{R}(L, \iota) := \{ r_S \mid r \in R(L) \text{ and } r_S^2 < 0 \} \subset S^\vee.$$

Since $T_\iota$ is negative definite, we have $-2 \leq r_S^2$ for any $r \in R(L)$. Since $S^\vee$ is discrete in $S \otimes \mathbb{R}$, the family of hyperplanes $\mathcal{R}(L, \iota)^*$ is locally finite in $\mathcal{P}_S$. By Proposition 5.7, if $r \in R(L)$ satisfies $r_S \in \mathcal{R}(L, \iota)$, then, for any $g \in O^+(S)$, we have $r_S^g = (r_S)^g \in \mathcal{R}(L, \iota)$. Therefore $\mathcal{R}(L, \iota)$ is invariant under the action of $O^+(S)$. Note that $\mathcal{R}(L, \iota)$ contains $\mathcal{R}(S)$, and that $\mathcal{R}(S)$ is obviously invariant under the action of $O^+(S)$. Therefore, by Proposition 5.3, we can obtain a set of generators of the automorphism group $Aut(C)$ of an $\mathcal{R}(S)^*$-chamber $C$ if we find an $\mathcal{R}(L, \iota)^*$-chamber $D$ contained
in \( C \), show that \( D \) is fundamental, calculate the group \( Aut(D) \), and find isometries of \( S \) that map \( D \) to its adjacent chambers.

Let \( L_{26} \) denote an even hyperbolic unimodular lattice of rank 26, which is unique up to isomorphisms by Eichler’s theorem (see, for example, Cassels [6]). The walls of an \( \mathcal{R}(L_{26}) \)-chamber \( D \subset L_{26} \otimes \mathbb{R} \) and the group \( Aut(D) \subset O^+(L_{26}) \) were determined by Conway [7]. Then Borechers [3], [4] determined the structure of \( O^+(S) \) for some even hyperbolic lattices \( S \) of rank \(< 26 \) by embedding \( S \) into \( L_{26} \) in such a way that \( T_i \) is a root lattice.

Kondo [17] applied the Conway-Borcherds method to the study of the automorphism group of a generic Jacobian Kummer surface. Then Keum and Kondo [14] applied it to Kummer surfaces associated with the product of two elliptic curves, Dolgachev and Keum [9] applied it to quartic Hessian surfaces, Dolgachev and Kondo [8] applied it to \( X_{2,1} \), and Kondo and Shimada [18] applied it to \( X_{3,1} \).

We say that an even hyperbolic lattice \( S \) is 2-reflective if the index of \( W(S) \) in \( O^+(S) \) is finite, or equivalently, if the automorphism group of an \( \mathcal{R}(S) \)-chamber is finite (see Example 5.6). Nikulin [21] classified all 2-reflective lattices of rank \( \geq 5 \). It turns out that there are no 2-reflective lattices of rank \( > 19 \).

Let \( Y \) be a \( K3 \) surface with the Néron-Severi lattice \( S_Y \) and the positive cone \( P(Y) \) containing an ample class. Then the closed subset \( \text{Nef}(Y) = \text{Nef}(Y) \cap P(Y) \) of \( P(Y) \) is an \( \mathcal{R}(S_Y) \)-chamber by Proposition 2.1(1), and hence we have

\[
\text{Aut}(\text{Nef}(Y)) = \text{Aut}(\text{Nef}^0(Y)) \cong O^+(S_Y)/W(S_Y).
\]

Combining this fact with Nikulin’s classification of 2-reflective lattices, we obtain the following:

**Corollary 5.8.** For any supersingular \( K3 \) surface \( X_{p,\sigma} \), the group \( \text{Aut}(\text{Nef}(X_{p,\sigma})) \) is infinite.

### 6. THE GROUPS \( \text{Aut}(\text{Nef}(X_{2,10})) \) AND \( \text{Aut}(\text{Nef}(X_{3,10})) \)

#### 6.1. The group \( \text{Aut}(\text{Nef}(X_{2,10})) \)

By Lemma 1.1, the result of Dolgachev and Kondo [8], and the method of the previous section, we obtain a set of generators of \( \text{Aut}(\text{Nef}(X_{2,10})) \).

First we recall the results of [8]. As a projective model of \( X_{2,1} \), we consider the minimal resolution \( X \) of the inseparable double cover \( Y \to \mathbb{P}^2 \) of \( \mathbb{P}^2 \) defined by

\[
w^2 = x_0x_1x_2(x_0^3 + x_1^3 + x_2^3).
\]

Note that the projective plane \( \mathbb{P}^2(\mathbb{F}_4) \) defined over \( \mathbb{F}_4 \) contains 21 points \( p_1, \ldots, p_{21} \) and 21 lines \( \ell_1, \ldots, \ell_{21} \). The inseparable double cover \( Y \) has 21 ordinary nodes over the 21 points in \( \mathbb{P}^2(\mathbb{F}_4) \) and hence \( X \) has 21 disjoint (-2)-curves. We denote by \( e_1, \ldots, e_{21} \in S_{2,1} \) the classes of these (-2)-curves, by \( h \in S_{2,1} \) the class of the pullback of a line on \( \mathbb{P}^2 \), and by \( f_1, \ldots, f_{21} \in S_{2,1} \) the
We define a chamber $w_M := \frac{1}{3} \sum_{i=1}^{21} (e_i + f_i)$
has the property $w_M \in S_X, \ w_M^2 = 14, \ \langle w_M, e_i \rangle = \langle w_M, f_i \rangle = 1.$

The complete linear system associated with the line bundle corresponding to $w_M$ defines an embedding of $X$ into $\mathbb{P}^2 \times \mathbb{P}^2$, and its image $X_M \subset \mathbb{P}^2 \times \mathbb{P}^2$ is defined by
\[
\begin{cases}
    x_0 y_0^2 + x_1 y_1^2 + x_2 y_2^2 = 0, \\
    x_0^2 y_0 + x_1^2 y_1 + x_2^2 y_2 = 0.
\end{cases}
\]

Six points on $\mathbb{P}^2(\mathbb{F}_4)$ are said to be general if no three points of them are collinear. There exist 168 sets of general six points in $\mathbb{P}^2(\mathbb{F}_4)$. If $I = \{p_{i_1}, \ldots, p_{i_6}\}$ is a set of general six points, then the $(−1)$-vector
\[c_I := h - \frac{1}{2}(e_{i_1} + \cdots + e_{i_6})\]
is contained in $S_{2,1}^\vee$. Note that each $c_I$ defines a reflection
\[x \mapsto x + 2\langle x, c_I \rangle c_I\]
in $O^+(S_{2,1})$ because $c_I \in S_{2,1}^\vee$. Let $P(X_{2,1})$ be the positive cone of $S_{2,1}$ containing an ample class, and let $\Delta(X_{2,1})$ be the set consisting of $e_1, \ldots, e_21, f_1, \ldots, f_21$ and the $(−1)$-vectors $c_I$ defined above. We define a chamber $D(X_{2,1})$ in $P(X_{2,1})$ by
\[D(X_{2,1}) := \{ x \in P(X_{2,1}) \mid \langle x, v \rangle \geq 0 \ \text{for all} \ v \in \Delta(X_{2,1}) \}.
\]

Then, for each $v \in \Delta(X_{2,1})$, the hyperplane $(v)^\perp$ is a wall of $D(X_{2,1})$. Moreover the ample class $w_M$ is contained in the interior of $D(X_{2,1})$. Recall that $L_{26}$ is the even unimodular hyperbolic lattice of rank 26. There exists a primitive embedding $\iota : S_{2,1} \hookrightarrow L_{26}$, which is unique up to $O(L_{26})$. The orthogonal complement $T_i$ of $S_{2,1}$ in $L_{26}$ is isomorphic to the root lattice of type $D_4$, and hence satisfies the hypothesis (5.1).

**Proposition 6.1.** The chamber $D(X_{2,1})$ is an $\mathcal{R}(L_{26}, \iota)^\ast$-chamber contained in the $\mathcal{R}(S_{2,1})^\ast$-chamber $\text{Nef}^\ast(X_{2,1})$. An isometry $g \in O^+(S_{2,1})$ belongs to $\text{Aut}(D(X_{2,1}))$ if and only if $w_M^g = w_M$.

Thus we can apply Proposition 5.3 to the pair of chambers $D(X_{2,1})$ and $\text{Nef}^\ast(X_{2,1})$ for the study of $\text{Aut} (\text{Nef}(X_{2,1}))$ and $\text{Aut}(X_{2,1})$.

We have the following elements in $\text{Aut}(X_{2,1})$ and $O^+(S_{2,1})$. Since $\text{Aut}(X_{2,1})$ is naturally embedded in $O^+(S_{2,1})$, we use the same letter to denote an element of $\text{Aut}(X_{2,1})$ and its image in $O^+(S_{2,1})$. 
• The action of $\text{PGL}(3, \mathbb{F}_4)$ on $\mathbb{P}^2$ induces automorphisms of the inseparable double cover $Y$ of $\mathbb{P}^2$, and hence automorphisms of $X_{2,1}$. Their action on $S_{2,1}$ preserves $D(X_{2,1})$.

• The interchange of the two factors of $\mathbb{P}^2 \times \mathbb{P}^2$ preserves $X_M \subset \mathbb{P}^2 \times \mathbb{P}^2$, and hence it induces an involution $sw \in \text{Aut}(X_{2,1})$, which we call the switch. Its action on $S_{2,1}$ preserves $D(X_{2,1})$.

• For each set $I$ of general six points in $\mathbb{P}^2(\mathbb{F}_4)$, the linear system of plane curves of degree 5 that pass through the points of $I$ and are singular at each point of $I$ defines a birational involution of $\mathbb{P}^2$, and this involution lifts to an involution of $Y$. Hence we obtain an involution $Cr_I \in \text{Aut}(X_{2,1})$, which we call a Cremona automorphism of $X_{2,1}$. The action of $Cr_I$ on $S_{2,1}$ is the reflection with respect to $c_I \in S_{2,1}^\vee$.

• The Frobenius action of $\text{Gal}(\mathbb{F}_4/\mathbb{F}_2)$ on $X_M$ induces an isometry $Fr$ of $S_{2,1}$, which preserves $D(X_{2,1})$.

• We have the reflections $s_{e_i}$ and $s_{f_i}$ with respect to the $(-2)$-vectors $e_i$ and $f_i$.

By the reflections $Cr_I$, $s_{e_i}$, and $s_{f_i}$, we see that the chamber $D(X_{2,1})$ is fundamental.

**Theorem 6.2** ([8]). (1) The projective automorphism group $\text{Aut}(X_{2,1}, w_M)$ of $X_M \subset \mathbb{P}^2 \times \mathbb{P}^2$ is generated by $\text{PGL}(3, \mathbb{F}_4)$ and the switch $sw$.

(2) The group $\text{Aut}(D(X_{2,1}))$ is generated by $\text{Aut}(X_{2,1}, w_M)$ and Fr.

(3) The automorphism group $\text{Aut}(X_{2,1})$ is generated by $\text{Aut}(X_{2,1}, w_M)$ and the 168 Cremona automorphisms $Cr_I$.

(4) The group $\text{Aut}(\text{Nef}(X_{2,1}))$ is generated by $\text{Aut}(X_{2,1})$ and Fr.

(5) The group $\text{O}^+(S_{2,1})$ is generated by $\text{Aut}(\text{Nef}(X_{2,1}))$ and the $21 + 21$ reflections $s_{e_i}$ and $s_{f_i}$.

We then study $\text{Aut}(\text{Nef}(X_{2,10}))$. By Corollary 3.2, we have an embedding

$$j : S_{2,1} \hookrightarrow S_{2,10}$$

that induces $S_{2,1}^\vee(2) \cong S_{2,10}$. Composing $j$ with some element of $W(S_{2,10}) \times \{ \pm 1 \}$, we can assume that $j(w_M)$ is contained in $\text{Nef}(X_{2,10})$ (Proposition 2.1(2)). The isomorphism $j_* : \text{O}^+(S_{2,1}) \cong \text{O}^+(S_{2,10})$ induced by $j$ is denoted by

$$g \mapsto g'. $$

The $j(\mathcal{R}(L_{26, \iota}))^\ast$-chamber $j(D(X_{2,1}))$ is fundamental, and we have

$$\text{Aut}(j(D(X_{2,1}))) = \text{Aut}(D(X_{2,1})).$$

**Lemma 6.3.** The set $j(\mathcal{R}(L_{26, \iota}))$ contains $\mathcal{R}(S_{2,10})$. Hence the $j(\mathcal{R}(L_{26, \iota}))^\ast$-chamber $j(D(X_{2,1}))$ is contained in the $\mathcal{R}(S_{2,10})^\ast$-chamber $\text{Nef}^\ast(X_{2,10})$.

**Proof.** It is enough to show that, if $v \in S_{2,1}^\vee$ satisfies $v^2 = -1$, then $v \in \mathcal{R}(L_{26, \iota})$, that is, there exists $u \in T_{x}^\vee$ such that $u^2 = -1$ and that $u + v$ is contained in the submodule $L_{26}$ of $S_{2,1}^\vee \oplus T_{x}^\vee$. By
Nikulin [20, Proposition 1.4.1], the submodule $L_{26}/(S_{2,1} \oplus T_i)$ of $(S_{2,1}' \oplus T_i')/(S_{2,1} \oplus T_i) = A_{S_{2,1}} \oplus A_{T_i}$ is the graph of an isomorphism

$$q_{S_{2,1}} \cong -q_{T_i}.$$ 

Hence it is enough to show that, for any $\bar{u} \in A_{T_i}$ with $q_{T_i}(\bar{u}) = 1$, there exists $u \in T_i'$ such that $u^2 = -1$ and $u \mod T_i = \bar{u}$. Since $T_i$ is a root lattice of type $D_4$, we can confirm this fact by direct computation. The set of $(-1)$-vectors in $T_i'$ consists of 24 vectors, and its image by the natural projection $T_i' \to A_{T_i}$ is the set of all non-zero elements of $A_{T_i} \cong \mathbb{P}_2^2$.

The set of walls of $j(D(X_{2,1}))$ is equal to

$$\{(j(e_i))^\perp | i = 1, \ldots, 21\} \cup \{(j(f_i))^\perp | i = 1, \ldots, 21\} \cup \{(j(c_i))^\perp | I \text{ is a set of general six points}\}.$$ 

Note that the $21 + 21$ vectors $j(e_i)$ and $j(f_i)$ are of norm $-4$ and the 168 vectors $j(c_i)$ are of norm $-2$. Note also that neither $(j(e_i))^\perp$ nor $(j(f_i))^\perp$ are contained in $\mathcal{R}(S_{2,10})^*$, because there are no rational numbers $\lambda$ such that $(-4)\lambda^2 = -2$. By Proposition 5.3, Theorem 6.2 and Lemma 6.3, we obtain the following:

**Theorem 6.4.** The group $\text{Aut}(\text{Nef}(X_{2,10}))$ is generated by $\text{PGL}(3, \mathbb{F}_4)\,'$, $sw'$, $Fr'$, $s_{e_i}'$ and $s_{f_i}'$.

6.2. **The group $\text{Aut}(\text{Nef}(X_{3,10}))$.** By the same argument as above, we obtain a set of generators of $\text{Aut}(\text{Nef}(X_{3,10}))$ from the result of Kondo and Shimada [18].

We consider the Fermat quartic surface

$$X_{\mathbb{F}_Q} : x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0$$

in characteristic $3$. Then $X_{\mathbb{F}_Q}$ is isomorphic to $X_{3,1}$. The surface $X_{\mathbb{F}_Q}$ contains 112 lines, and their classes $l_1, \ldots, l_{112}$ span $S_{3,1}$. We denote by $h_{\mathbb{F}_Q} \in S_{3,1}$ the class of a hyperplane section of $X_{\mathbb{F}_Q}$.

There exists a primitive embedding $i : S_{3,1} \hookrightarrow L_{26}$, which is unique up to $O(L_{26})$. The orthogonal complement $T_i$ is isomorphic to the root lattice of type $2A_2$, and hence satisfies the hypothesis (5.1). We calculated an $\mathcal{R}(L_{26}, i)^*$-chamber $D(X_{3,1})$ that contains $h_{\mathbb{F}_Q}$ in its interior, and found

648 vectors $u_j \in S_{3,1}'$ of norm $-4/3$, and 5184 vectors $w_k \in S_{3,1}'$ of norm $-2/3$

such that the walls of $D(X_{3,1})$ consist of the 112 hyperplanes $(l_i)^\perp$, the 648 hyperplanes $(u_j)^\perp$ and the 5184 hyperplanes $(w_k)^\perp$. Note that the $\mathcal{R}(L_{26}, i)^*$-chamber $D(X_{3,1})$ is contained in the $\mathcal{R}(S_{3,1})^*$-chamber $\text{Nef}^0(X_{3,1})$, because $h_{\mathbb{F}_Q} \in D(X_{3,1})^0$. Moreover, since $28 h_{\mathbb{F}_Q} = \sum i_i$, the following holds:

**Proposition 6.5.** An isometry $g \in O^+(S_{3,1})$ belongs to $\text{Aut}(D(X_{3,1}))$ if and only if $h_{\mathbb{F}_Q}^g = h_{\mathbb{F}_Q}$.

We have the following elements in $\text{Aut}(X_{3,1})$ and $O^+(S_{3,1})$. Note that, for a polarization $h \in S_{3,1}$ of degree 2, we have the deck transformation $\tau(h) \in \text{Aut}(X_{3,1})$ of the generically finite morphism $X_{3,1} \to \mathbb{P}_2$ of degree 2 induced by the the complete linear system associated with $h$. 

• The subgroup PGU(4, ℂ) of PGL(4, ℂ) = Aut(ℙ³) acts on \( X_{FQ} \). Its action on \( S_{3,1} \) preserves \( D(X_{3,1}) \). Moreover, the action of PGU(4, ℂ) on \( S_{3,1}^v \) is transitive on each of the set of 112 vectors \( l_i \), the set of 648 vectors \( u_j \) and the set of 5184 vectors \( w_k \).

• There exists a polarization \( h_{648} \in S_{3,1} \) of degree 2 such that the deck transformation \( \tau(h_{648}) \in Aut(X_{3,1}) \) maps \( D(X_{3,1}) \) to an \( R(L_{26}, \iota)^* \)-chamber adjacent to \( D(X_{3,1}) \) along one of the 648 walls \( (u_j) \).

• There exists a polarization \( h_{5184} \in S_{3,1} \) of degree 2 such that the deck transformation \( \tau(h_{5184}) \in Aut(X_{3,1}) \) maps \( D(X_{3,1}) \) to an \( R(L_{26}, \iota)^* \)-chamber adjacent to \( D(X_{3,1}) \) along one of the 5184 walls \( (w_k)^* \).

• The Frobenius action of Gal(ℂ/F_3) on \( X_{FQ} \) gives rise to an element \( Fr \in Aut(D(X_{3,1})) \) of order 2.

• We have the reflections \( s_{l_i} \) with respect to the classes \( l_i \) of the 112 lines on \( X_{FQ} \).

Remark 6.6. The actions of the involutions \( \tau(h_{648}) \) and \( \tau(h_{5184}) \) on \( S_{3,1} \) are not reflections.

Thus \( D(X_{3,1}) \) is fundamental, and hence we have the following:

**Theorem 6.7** (18). (1) The projective automorphism group \( Aut(X, h_{FQ}) \) of the Fermat quartic surface \( X_{FQ} \subset \mathbb{P}^3 \) is equal to PGU(4, ℂ).

(2) The group \( Aut(D(X_{3,1})) \) is generated by \( Aut(X, h_{FQ}) \) and \( Fr \).

(3) The automorphism group \( Aut(X_{3,1}) \) is generated by \( Aut(X, h_{FQ}) \) and the two involutions \( \tau(h_{648}) \) and \( \tau(h_{5184}) \).

(4) The group \( Aut(Nef(X_{3,1})) \) is generated by \( Aut(X_{3,1}) \) and \( Fr \).

(5) The group \( O^+(S_{3,1}) \) is generated by \( Aut(Nef(X_{3,1})) \) and the 112 reflections \( s_{l_i} \).

By Corollary 3.2, we have an embedding
\[
j : S_{3,1} \hookrightarrow S_{3,10}
\]
that induces \( S_{3,1}^v(3) \cong S_{3,10} \). By Proposition 2.1(2), we can assume that \( j(h_{FQ}) \) is contained in \( Nef(X_{3,10}) \). The isomorphism \( j_* : O^+(S_{3,1}) \cong O^+(S_{3,10}) \) induced by \( j \) is denoted by \( g \mapsto g' \). The \( j(R(L_{26}, \iota))^* \)-chamber \( j(D(X_{3,1})) \) is fundamental, and \( Aut(j(D(X_{3,1}))) \) is equal to \( Aut(D(X_{3,1}))^* \).

**Lemma 6.8.** The set \( j(R(L_{26}, \iota)) \) contains \( R(S_{3,10}) \). Hence the \( j(R(L_{26}, \iota))^* \)-chamber \( j(D(X_{3,1})) \) is contained in the \( R(S_{3,10})^* \)-chamber Nef^0(X_{3,10}).

**Proof.** It is enough to show that, if \( v \in S_{3,1}^v \) satisfies \( v^2 = -2/3 \), then there exists \( u \in T_{16}^v \) such that \( u^2 = -4/3 \) and that \( u + v \) is contained in \( L_{26} \subset S_{3,1}^v \oplus T_{16}^v \). For this, it suffices to prove that, for any \( \bar{u} \in A_T \) with \( q_T(\bar{u}) = -4/3 \), there exists \( u \in T_{16}^v \) such that \( u^2 = -4/3 \) and \( u \mod T_{16} = \bar{u} \). Since \( T_{16} \) is a root lattice of type \( 2A_2 \), we can confirm this fact by direct computation. \( \square \)
The set of walls of $j(D(X_{3,1}))$ is equal to
\[
\{(j(l_i))^{\perp} \mid i = 1, \ldots, 112\} \cup \{(j(u_j))^{\perp} \mid j = 1, \ldots, 648\} \cup \\
\{(j(w_k))^{\perp} \mid k = 1, \ldots, 5184\}.
\]

Note that the vectors $j(l_i)$ are of norm $-6$, the vectors $j(u_j)$ are of norm $-4$, and the vectors $j(w_k)$ are of norm $-2$. Note also that neither $(j(l_i))^{\perp}$ nor $(j(u_j))^{\perp}$ are contained in $\mathcal{R}(S_{3,10})^*$. By Proposition 5.3, Theorem 6.7 and Lemma 6.8, we obtain the following:

**Theorem 6.9.** The group $\text{Aut}(\text{Nef}(X_{3,10}))$ is generated by $\text{PGU}(4, \mathbb{F}_9)'$, $F_3'$, $s_1'$ and $\tau(h_{648})'$.

7. **Torelli theorem for supersingular $K3$ surfaces**

We review the theory of period mapping and Torelli theorem for supersingular $K3$ surfaces in odd characteristics by Ogus [24], [25]. Throughout this section, we assume that $p$ is odd.

We summarize results on quadratic spaces over finite fields. See, for example, Kitaoka [15, Section 1.3]. Let $\mathbb{F}_q$ be a finite extension of $\mathbb{F}_p$. There exist exactly two isomorphism classes of non-degenerate quadratic forms in $2\sigma$ variables $x_1, \ldots, x_{2\sigma}$ over $\mathbb{F}_q$. They are represented by
\[
(7.1) \quad f_+ := x_1x_2 + \cdots + x_{2\sigma-1}x_{2\sigma}, \quad \text{and}
\]
\[
(7.2) \quad f_- := x_1^2 + cx_1x_2 + x_2^2 + x_3x_4 + \cdots + x_{2\sigma-1}x_{2\sigma},
\]
where $c$ is an element of $\mathbb{F}_q$ such that $t^2 + ct + 1 \in \mathbb{F}_q[t]$ is irreducible. The quadratic form $f_+$ (resp. $f_-$) is called neutral (resp. non-neutral). The group $O(\mathbb{F}_q^{2\sigma}, f_\epsilon)$ of the self-isometries of the quadratic space $(\mathbb{F}_q^{2\sigma}, f_\epsilon)$, where $\epsilon = \pm 1$, is of order
\[
2q^{\sigma(\sigma-1)}(q^\sigma - \epsilon) \prod_{i=1}^{\sigma-1} (q^{2i} - 1).
\]

Let $N$ be an even hyperbolic $p$-elementary lattice of rank $22$ with discriminant $-p^{2\sigma}$. We define a quadratic space $(N_0, q_0)$ over $\mathbb{F}_p$ by (1.2). It is known that $q_0$ is non-degenerate and non-neutral. We denote by $O(N_0, q_0)$ the group of the self-isometries of $(N_0, q_0)$. Note that the scalar multiplications in $O(N_0, q_0)$ are only $\pm 1$. Let $k$ be a field of characteristic $p$. We put
\[
\varphi := \text{id}_{N_0} \otimes F_k : N_0 \otimes k \to N_0 \otimes k,
\]
where $F_k$ is the Frobenius map of $k$.

**Definition 7.1.** A subspace $K$ of $N_0 \otimes k$ with $\dim K = \sigma$ is said to be a characteristic subspace of $(N_0, q_0)$ if $K$ is totally isotropic with respect to the quadratic form $q_0 \otimes k$ and $\dim(K \cap \varphi(K)) = \sigma - 1$ holds.
Suppose that $k$ is algebraically closed. Let $X$ be a supersingular $K3$ surface with Artin invariant $\sigma$ defined over $k$. An isomorphism

$$\eta : N \cong S_X$$

of lattices is called a marking of $X$. We fix a marking $\eta$ of $X$. The composite of the marking $\eta$ and the Chern class map $S_X \rightarrow H^2_{\text{DR}}(X/k)$ defines a linear homomorphism

$$\bar{\eta} : N \otimes k \rightarrow H^2_{\text{DR}}(X/k).$$

It is known that $\text{Ker} \bar{\eta}$ is contained in $N_0 \otimes k$, and is totally isotropic with respect to $q_0 \otimes k$. We put

$$K(X, \eta) := \varphi^{-1}(\text{Ker} \bar{\eta}),$$

and call $K(X, \eta)$ the period of the marked supersingular $K3$ surface $(X, \eta)$. Then it is proved by Ogus [24, 25] that $K(X, \eta)$ is a characteristic subspace of $(N_0, q_0)$. We denote by $\eta^* : O(S_X) \cong O(N)$ the isomorphism induced by the marking $\eta$, and let

$$\bar{\eta}^* : O(S_X) \rightarrow O(N_0, q_0)$$

be the composite of $\eta^*$ with the natural homomorphism $O(N) \rightarrow O(N_0, q_0)$. As a corollary of Torelli theorem by Ogus [25, Corollary of Theorem II"], we have the following:

**Corollary 7.2.** Let $\eta$ be a marking of $X$. Then, as a subgroup of $O^+(S_X)$, the automorphism group $\text{Aut}(X)$ of $X$ is equal to

$$\{ g \in \text{Aut}(\text{Nef}(X)) \mid K^\eta(g) = K(X, \eta) \}.$$ 

In particular, the index of $\text{Aut}(X)$ in $\text{Aut}(\text{Nef}(X))$ is at most $|O(N_0, q_0)/\{\pm 1\}|$.

Combining Corollaries 5.8 and 7.2, we obtain the following:

**Corollary 7.3.** The automorphism group $\text{Aut}(X)$ is infinite.

**Remark 7.4.** When $p = 3$ and $\sigma = 1$, the group $O(N_0, q_0)$ is of order 8, while the index of $\text{Aut}(X_{3,1})$ in $\text{Aut}(\text{Nef}(X_{3,1}))$ is 2 by Theorem 6.7.

**Definition 7.5.** We say that a supersingular $K3$ surface $X$ with Artin invariant $\sigma$ is generic if there exists a marking $\eta$ for $X$ such that the subgroup

$$\{ \gamma \in O(N_0, q_0) \mid K^{\gamma}(X, \eta) = K(X, \eta) \}$$

of $O(N_0, q_0)$ consists of only scalar multiplications $\pm 1$.

If $X$ is generic, then the subgroup (7.3) consists of only scalar multiplications for any marking $\eta$. The existence of generic supersingular $K3$ surfaces with Artin invariant $> 1$ (Theorem 1.7) is proved in the next section.
Recall that $A_{S_X}$ is the discriminant group of $S_X$, and $q_{S_X} : A_{S_X} \to \mathbb{Q}/2\mathbb{Z}$ is the discriminant quadratic form. We will regard $A_{S_X}$ as a $2\sigma$-dimensional vector space over $\mathbb{F}_p$. Note that the image of $q_{S_X}$ is contained in $(2/p)\mathbb{Z}/2\mathbb{Z}$. We define $\tilde{q}_{S_X} : A_{S_X} \to \mathbb{F}_p$ by

$$\tilde{q}_{S_X}(x \mod S_X) := p \cdot q_{S_X}(x) \mod p.$$ 

Then we obtain a quadratic space $(A_{S_X}, \tilde{q}_{S_X})$ over $\mathbb{F}_p$. Note that we can recover $q_{S_X}$ from $\tilde{q}_{S_X}$. We have natural homomorphisms

$$(7.4) \quad O(S_X) \to O(q_{S_X}) \cong O(A_{S_X}, \tilde{q}_{S_X}) \to \text{PO}(A_{S_X}, \tilde{q}_{S_X}) := O(A_{S_X}, \tilde{q}_{S_X})/\{\pm 1\}.$$ 

Let $\eta : N^\vee \cong S^\vee_X$ be the isomorphism induced by a marking $\eta$. Then the map

$$px \mod pN \in N_0 \mapsto \eta(x) \mod S_X \in A_{S_X} \quad (x \in N^\vee)$$

induces an isomorphism of quadratic spaces from $(N_0, q_0)$ to $(A_{S_X}, \tilde{q}_{S_X})$. By Corollary 7.2, we obtain the following:

**Corollary 7.6.** Suppose that $X$ is generic. Then $\text{Aut}(X)$ is equal to the kernel of the homomorphism $\Phi : \text{Aut}(\text{Nef}(X)) \to \text{PO}(A_{S_X}, \tilde{q}_{S_X})$ obtained by restricting (7.4) to $\text{Aut}(\text{Nef}(X)) \subset O(S_X)$.

**Remark 7.7.** Suppose that $X$ is generic, and that we are given a subset $\{g_1, \ldots, g_n\}$ of $\text{Aut}(\text{Nef}(X))$ that generate $\text{Aut}(\text{Nef}(X))$. Then a finite set of generators of $\text{Aut}(X)$ is obtained by the following procedure. We construct a finite directed graph $(V, E)$ as follows. The set $V$ of vertices is the image of $\Phi$, that is, the subgroup of $\text{PO}(A_{S_X}, \tilde{q}_{S_X})$ generated by $\Phi(g_1), \ldots, \Phi(g_n)$. The set $E$ of directed edges is the set of triples

$$\alpha = (s_\alpha, g_\alpha, t_\alpha),$$

where $s_\alpha, t_\alpha \in V$ and $s_\alpha \Phi(g_\alpha) = t_\alpha$. The edge $\alpha$ is directed from $s_\alpha$ to $t_\alpha$ and labelled with a generator $g_\alpha$. We put $\alpha^{-1} := (t_\alpha, g_\alpha^{-1}, s_\alpha)$. We use the identity element $e \in V$ as a base point of the 1-dimensional CW-complex $\Gamma$ associated with $(V, E)$. Then the fundamental group $\pi_1(\Gamma, e)$ is a free group of finite rank, and its generators are calculated from the graph $(V, E)$. Consider a loop

$$\gamma = \alpha_0^{\varepsilon_0} \cdots \alpha_m^{\varepsilon_m}$$

of $\Gamma$ from $e$ to $e$, where $\varepsilon_i = \pm 1$ and $\alpha_i^{\varepsilon_i} = (v_j, g_j^{\varepsilon_j}, v_{j+1})$. Then we have $v_0 = v_{m+1} = e$, and

$$\tilde{\gamma} := g_{i_0}^{\varepsilon_0} \cdots g_{i_m}^{\varepsilon_m} \in \text{Aut}(\text{Nef}(X))$$

is mapped to $e$ by $\Phi$. If $\pi_1(\Gamma, e)$ is generated by loops $\gamma_1, \ldots, \gamma_l$, then $\text{Aut}(X) = \text{Ker} \Phi$ is generated by $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_l$.  


Remark 7.8. Suppose that $X_{3,10}$ is generic. Applying the procedure in Remark 7.7 to the generators of $\text{Aut}(\text{Nef}(X_{3,10}))$ given in Theorem 6.9, we can obtain a finite set of generators of $\text{Aut}(X_{3,10})$. However, a naive application of the procedure would be inexecutable, because, when $p = 3$ and $\sigma = 10$, the order of $O(N_0,q_0)$ is

$$2^{36} \cdot 3^{90} \cdot 5^6 \cdot 73 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 37 \cdot 41^2 \cdot 61 \cdot 73 \cdot 193 \cdot 547 \cdot 757 \cdot 1093 \cdot 1181,$$

which is about $7.886 \times 10^{90}$.

For a non-zero vector $v \in S_X \otimes \mathbb{Q}$, we denote by $(v)_{\mathbb{Q}}$ the linear subspace of $S_X \otimes \mathbb{Q}$ spanned by $v$, and put

$$\bar{v} := ((v)_{\mathbb{Q}} \cap S_X^\vee )/( (v)_{\mathbb{Q}} \cap S_X),$$

which is a linear subspace of $A_{S_X} \cong \mathbb{P}_{\mathbb{F}_p}^{2\sigma}$. When $\bar{v} \neq 0$, we denote by

$$[\bar{v}] \in \mathbb{P}(A_{S_X})$$

the corresponding point of the projective space $\mathbb{P}(A_{S_X})$ over $\mathbb{F}_p$. We consider the action of $O(S_X)$ on $\mathbb{P}(A_{S_X})$.

Remark 7.9. By definition, the reflections $s_r$ with respect to $r \in \mathcal{R}(S_X)$ act on $A_{S_X}$ trivially. Hence the restriction $\Phi$ of the homomorphism (7.4) to the subgroup $\text{Aut}(\text{Nef}(X))$ of $O(S_X)$ is also obtained by passing to the quotient $O(S_X)/(W(S_X) \times \{\pm 1\}) \cong \text{Aut}(\text{Nef}(X))$. Thus the orbit of $[\bar{v}]$ under the action of $\text{Aut}(\text{Nef}(X))$ is equal to the orbit of $[\bar{v}]$ under the action of $O(S_X)$.

Corollary 7.10. Suppose that $X$ is generic. Let $v \in S_X$ be a vector such that $\bar{v} \subset A_{S_X}$ is not zero. Let $m$ be the cardinality of the orbit of $[\bar{v}] \in \mathbb{P}(A_{S_X})$ under the action of $O(S_X)$. Then the number of $\text{Aut}(X)$-orbits contained in the $O(S_X)$-orbit of $v$ in $S_X$ is at least $m$.

8. Existence of generic supersingular K3 surfaces

We prove Theorem 1.7. For the proof, we recall the construction by Ogus [24] of the scheme $\mathcal{M}$ parameterizing characteristic subspaces of the $2\sigma$-dimensional quadratic space $(N_0,q_0)$ over $\mathbb{F}_p$. This scheme $\mathcal{M}$ plays the role of the period domain for supersingular K3 surfaces. We continue to assume that $p$ is odd.

Let $\text{Grass}(\nu,N_0)$ denote the Grassmannian variety of $\nu$-dimensional subspaces of $N_0$, and let $\text{Isot}(\nu,q_0)$ be the subscheme of $\text{Grass}(\nu,N_0)$ parameterizing $\nu$-dimensional totally isotropic subspaces of $(N_0,q_0)$. We put

$$\text{Gen} := \text{Isot}(\sigma,q_0),$$

where Gen is for “generatrix”. Note that $\text{Isot}(\nu,q_0)$ is defined over $\mathbb{F}_p$ for any $\nu$. Let $k$ be a field of characteristic $p$. For a subspace $L$ of $N_0 \otimes k$ with dimension $\nu$, we denote by $[L]$ the $k$-valued point of $\text{Grass}(\nu,N_0)$ corresponding to $L$. We then have the following:

1. If $\nu < \sigma$, then $\text{Isot}(\nu,q_0)$ is geometrically connected.
The scheme $\text{Gen} \otimes \mathbb{F}_{p^2}$ has two connected components $\text{Gen}_+$ and $\text{Gen}_-$, each of which is geometrically connected. Since $q_0$ is non-neutral, the action of $\text{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p)$ interchanges the two connected components.

Let $K$ and $K'$ be two $\sigma$-dimensional totally isotropic subspaces of $(N_0, q_0) \otimes k$. Suppose that $\dim(K \cap K') = \sigma - 1$. Then the $k$-valued points $[K]$ and $[K']$ belong to distinct connected components of $\text{Gen}$.

Suppose that $k$ is algebraically closed. Then, for each $k$-valued point $[L]$ of the scheme $\text{Isot}(\sigma - 1, q_0)$, there exist exactly two $\sigma$-dimensional totally isotropic subspaces of $(N_0, q_0) \otimes k$ that contain $L$.

Let $P$ be the subscheme of $\text{Gen} \times \text{Gen}$ parameterizing pairs $(K, K')$ such that $\dim(K \cap K') = \sigma - 1$. Then the scheme $P \otimes \mathbb{F}_{p^2}$ has two connected components, each of which is isomorphic to $\text{Isot}(\sigma - 1, q_0)$ over $\mathbb{F}_{p^2}$. The action of $\text{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p)$ interchanges the two connected components.

Consider the graph

$$\text{id} \times \varphi : \text{Gen} \to \text{Gen} \times \text{Gen}$$

of the Frobenius morphism $\text{Gen} \to \text{Gen}$ given by $K \mapsto \varphi(K)$. The subscheme $\mathcal{M}$ of $\text{Gen}$ that parametrizes the characteristic subspaces of $(N_0, q_0)$ is defined by the fiber product

$$\mathcal{M} \hookrightarrow \text{Gen} \quad \downarrow \quad \downarrow \text{id} \times \varphi$$

$$P \quad \hookrightarrow \text{Gen} \times \text{Gen}.$$  

Ogus [24] proved the following:

**Theorem 8.1.** The scheme $\mathcal{M}$ defined over $\mathbb{F}_p$ is smooth and projective of dimension $\sigma - 1$. The scheme $\mathcal{M} \otimes \mathbb{F}_{p^2}$ has two connected components $\mathcal{M}_+ = \mathcal{M} \cap \text{Gen}_+$ and $\mathcal{M}_- = \mathcal{M} \cap \text{Gen}_-$, each of which is geometrically connected. The action of $\text{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p)$ interchanges $\mathcal{M}_+$ and $\mathcal{M}_-$.

**Proof of Theorem 1.7.** Let $\kappa$ be an algebraic closure of the function field of the scheme $\mathcal{M}_+$ over $\mathbb{F}_{p^2}$, and let $[K_\kappa]$ be the geometric generic point of $\mathcal{M}_+$. By the surjectivity of the period mapping for supersingular $K3$ surfaces (Ogus [25, Theorem III′]), there exist a supersingular $K3$ surface $X$ of Artin invariant $\sigma$ defined over $\kappa$ and a marking $\eta : N \cong S_X$ such that $K_{(X, \eta)} = K_\kappa$. We prove that this $X$ is generic, that is,

$$G_\kappa := \{ \gamma \in O(N_0, q_0) \mid K^\gamma_\kappa = K_\kappa \}$$

is equal to $\{ \pm 1 \}$. Note that the closure of the point $[K_\kappa]$ coincides with $\mathcal{M}_+$. Therefore we have the following: If a field $k$ contains $\mathbb{F}_{p^2}$, then the action of $G_\kappa$ leaves $K$ invariant for any $k$-valued point $[K]$ of $\mathcal{M}_+$. 


Suppose that $\sigma \geq 3$. Let $u$ be an arbitrary non-zero isotropic vector of $N_0$. We will prove that $u$ is an eigenvector of $G_{\kappa}$. Let

$$b_0 : N_0 \times N_0 \to \mathbb{F}_p$$

denote the symmetric bilinear form obtained from $q_0$. There exists a vector $v \in N_0$ such that $q_0(v) = 0$ and $b_0(u, v) = 1$, and hence $(N_0, q_0)$ has an orthogonal direct-sum decomposition

$$N_0 = U^\perp \oplus U,$$

where $U$ is the subspace spanned by $u$ and $v$. Repeating this procedure and noting that $q_0$ is non-neutral, we obtain a basis $a_1, \ldots, a_{2\sigma}$ of $N_0$ with $u = a_{2\sigma}$ such that $q_0(x_1 a_1 + \cdots + x_{2\sigma} a_{2\sigma})$ is equal to the quadratic polynomial $f_-$ in (7.2). Let $\alpha$ and $\bar{\alpha} = \alpha^p$ be the roots in $\mathbb{F}_{p^2}$ of the irreducible polynomial $t^2 + ct + 1 \in \mathbb{F}_p[t]$. We consider the basis

$$b_i^{(-1)} := \alpha a_1 + a_2, \quad b_i^{(1)} := \bar{\alpha} a_1 + a_2, \quad \text{and} \quad b_i^{(1)} := a_{2i-1}, \quad b_i^{(-1)} := a_{2i} \quad (i = 2, \ldots, \sigma)$$

of $N_0 \otimes \mathbb{F}_{p^2}$. Note that each $b_i^{(\pm 1)}$ is isotropic, and that

$$b_0(b_i^{(\alpha)}, b_j^{(\beta)}) = 0 \quad \text{if} \quad i \neq j, \quad b_0(b_i^{(1)}, b_i^{(-1)}) = \begin{cases} \begin{array}{ll} (1 - c^2)/2 & \text{if } i = 1, \\ 1/2 & \text{if } i \geq 2. \end{array} \end{cases}$$

We put

$$E := \{1, -1\}^\sigma.$$ 

For $e = (\varepsilon_1, \ldots, \varepsilon_\sigma) \in E$, we denote by $K_e$ the linear subspace of $N_0 \otimes \mathbb{F}_{p^2}$ spanned by

$$b_1^{(\varepsilon_1)}, \ldots, b_\sigma^{(\varepsilon_\sigma)}.$$ 

It is obvious that $K_e$ is isotropic. Moreover, since

$$\varphi(b_i^{(\varepsilon)}) = b_i^{(-\varepsilon)} \quad \text{and} \quad \varphi(b_i^{(\varepsilon)}) = b_i^{(\varepsilon)} \quad \text{if} \quad i \geq 2,$$

we have $\dim(K_e \cap \varphi(K_e)) = \sigma - 1$. Therefore $K_e$ is a characteristic subspace of $(N_0, q_0)$. Suppose that $e$ and $e' \in E$ differ only at one component. Then we have $\dim(K_e \cap K_{e'}) = \sigma - 1$, and hence the $\mathbb{F}_{p^2}$-valued points $[K_e]$ and $[K_{e'}]$ of $\mathcal{M}$ belong to distinct connected components. We put

$$E_+ := \{e \in E \mid \text{the number of } -1 \text{ in } e \text{ is even} \}, \quad 1 := (1, \ldots, 1) \in E_+.$$ 

Interchanging $\alpha$ and $\bar{\alpha}$ if necessary, we can assume that $[K_1]$ is an $\mathbb{F}_{p^2}$-valued point of $\mathcal{M}_+$, and hence $[K_e]$ is an $\mathbb{F}_{p^2}$-valued point of $\mathcal{M}_+$ for any $e \in E_+$. It follows that $K_e$ is invariant under the action of $G_{\kappa}$ for any $e \in E_+$. Let $b_i^{(\alpha)}$ be an arbitrary element among the basis (8.1). Recall that we have assumed $\sigma \geq 3$. Therefore, for each element $b_j^{(\beta)}$ among the basis (8.1) that is distinct from $b_i^{(\alpha)}$, there exists $e(j, \beta) = (\varepsilon_1, \ldots, \varepsilon_\sigma) \in E_+$ such that $\varepsilon_i = \alpha$ and $\varepsilon_j \neq \beta$. Since

$$\bigcap_{\langle j, \beta \rangle \neq \langle i, \alpha \rangle} K_{e(j, \beta)} = \langle b_i^{(\alpha)} \rangle.$$
is invariant under the action of $G_\kappa$, we see that $b_i^{(a)}$ is an eigenvector of $G_\kappa$. In particular, the isotropic vector $u = a_{2\sigma} = b_\sigma^{(1)}$ given at the beginning is an eigenvector of $G_\kappa$.

Let

$$\lambda_i^{(a)} : G_\kappa \to \mathbb{F}_p^\times$$

be the character defined by $b_i^{(a)}$. Suppose that $i, j \geq 2$ and $i \neq j$. Then $b_i^{(a)} + b_j^{(b)}$ is an isotropic vector of $N_0$ for any choice of $\alpha, \beta \in \{ \pm 1 \}$, and hence is an eigenvector of $G_\kappa$. Therefore we have

\begin{equation}
(8.2) \quad \lambda_i^{(a)} = \lambda_j^{(b)} \quad \text{if } i, j \geq 2 \text{ and } i \neq j.
\end{equation}

Since the cardinality of $\{ x^2 | x \in \mathbb{F}_p \}$ is $(p + 1)/2$, there exist $\xi, \eta \in \mathbb{F}_p$ such that

$$\begin{align*}
(4 - c^2) + \xi^2 + \eta^2 &= 0.
\end{align*}$$

Then

$$b_1^{(1)} + b_1^{(-1)} + \xi(b_2^{(1)} + b_2^{(-1)}) + \eta(b_3^{(1)} + b_3^{(-1)})$$

is also an isotropic vector of $N_0$, and hence is an eigenvector of $G_\kappa$. Therefore we have

\begin{equation}
(8.3) \quad \lambda_1^{(1)} = \lambda_1^{(-1)} = \lambda_2^{(-1)} \quad \text{or} \quad \lambda_1^{(1)} = \lambda_1^{(-1)} = \lambda_3^{(1)} = \lambda_3^{(-1)}.
\end{equation}

Combining (8.2) and (8.3), we see that all the characters $\lambda_i^{(a)}$ are equal to each other. Thus $G_\kappa$ consists of only scalar multiplications.

Suppose that $\sigma = 2$. In this case, the scheme $\mathcal{M}$ coincides with $\text{Isot}(2, q_0)$, which is the scheme parametrizing lines on the smooth quadratic surface $Q_0 = \{ q_0 = 0 \}$ in the projective space $\mathbb{P}_x N_0 = \text{Grass}(1, N_0)$. Hence $\mathcal{M}_+$ and $\mathcal{M}_-$ correspond to the two rulings of $Q_0$. Let $g$ be an element of $G_\kappa$. Then $g$ leaves every line in the ruling of $Q_0$ corresponding to $\mathcal{M}_+$ invariant. Since $g$ is defined over $\mathbb{F}_p$ and $\text{Gal}(\mathbb{F}_p / \mathbb{F}_p)$ interchanges $\mathcal{M}_+$ and $\mathcal{M}_-$, we see that $g$ also leaves every line in the other ruling of $Q_0$ invariant. Therefore $g$ fixes every point of $Q_0$, and hence every point of $\mathbb{P}_x N_0$. \hfill \square

9. LATTICE EQUIVALENCE CLASSESVERSUS AUT-EQUIVALENCE CLASSES ON $X_{3,10}$

Suppose that $p > 2$ and $\sigma + \sigma' = 11$. We denote by $A_{p,\sigma'}$ the discriminant group $S_{p,\sigma'}/S_{p,\sigma'}$ of $S_{p,\sigma'}$, and use the notation in Section 7.

Let $\phi : X_{p,\sigma} \to \mathbb{P}^1$ be a genus one fibration, and let $\phi' : X_{p,\sigma'} \to \mathbb{P}^1$ be a genus one fibration whose lattice equivalence class $[\phi'] \in \mathbb{E}(X_{p,\sigma'})$ corresponds to $[\phi] \in \mathbb{E}(X_{p,\sigma})$ by Theorem 1.3. By Remark 4.6, we have an embedding $j : S_{p,\sigma} \hookrightarrow S_{p,\sigma'}$ inducing $S_{p,\sigma'}(p) \cong S_{p,\sigma'}$, such that $j(f_\phi)$ is a positive scalar multiple of $f_{\phi'}$. Suppose that

$$\overline{f_{\phi'}} = j(f_\phi) = ((f_{\phi'})_Q \cap S_{p,\sigma'}^\vee) / ((f_{\phi'})_Q \cap S_{p,\sigma'}) \subset A_{p,\sigma'}$$

is not zero. Let $m$ be the cardinality of the orbit of $\overline{f_{\phi'}} \in \mathbb{P}(A_{p,\sigma'})$ by the action of $O(S_{p,\sigma'})$ (or equivalently, in virtue of Remark 7.9, by the action of $\text{Aut}(\text{Nef}(X_{p,\sigma'}))$). By Corollary 7.10, the
number of $\text{Aut}$-equivalence classes of genus one fibrations contained in the lattice equivalence class $[\phi']$ is at least $m$, provided that $X_{p,\sigma}$ is generic.

**Remark 9.1.** We can regard $S_{p,\sigma'}$ as a submodule of $S_{p,\sigma} \otimes \mathbb{Q}$ by $j$. Then $S_{p,\sigma'}^{\vee}$ is equal to $(1/p)S_{p,\sigma}$. Hence $(1/p)j(f_{\phi})$ is contained in $S_{p,\sigma'}^{\vee}$.

As a consequence of the fact that $\text{Aut}(\text{Nef}(X_{3,10}))$ contains the subgroup $\text{PGU}(4, \mathbb{F}_9)'$ of order $13063680$, we obtain the following:

**Proposition 9.2.** Suppose that $X_{3,10}$ is generic. Then there exists a genus one fibration on $X_{3,10}$ whose lattice equivalence class contains at least $6531840$ $\text{Aut}$-equivalence classes.

**Proof.** Let $(w, x, y)$ be the affine coordinates of the Fermat quartic surface

$$X_{\mathbb{F}_9} = \{ w^4 + x^4 + y^4 + 1 = 0 \}$$

in characteristic $3$, and let $i$ denote $\sqrt{-1} \in \mathbb{F}_9$. Consider the following ten lines on $X_{\mathbb{F}_9} \cong X_{3,1}$:

- $\ell_1 := \{ w + (1 + i) = x + (1 + i)y = 0 \}$
- $\ell_2 := \{ w + (1 + i) = x + (1 - i)y = 0 \}$
- $\ell_3 := \{ w + iy - i = x + iy + i = 0 \}$
- $\ell_4 := \{ w + iy + 1 = x + iy - 1 = 0 \}$
- $\ell_5 := \{ w - y + 1 = x - y - 1 = 0 \}$
- $\ell_6 := \{ w - iy - 1 = x + iy - i = 0 \}$
- $\ell_7 := \{ w + (1 - i) = x + (1 + i)y = 0 \}$
- $\ell_8 := \{ w - (1 - i) = x + (1 + i) = 0 \}$
- $\ell_9 := \{ w + (1 + i) = x + (1 - i) = 0 \}$
- $\ell_{10} := \{ w + iy - 1 = x - iy - 1 = 0 \}$

They form a configuration of $(-2)$-curves whose dual graph is the affine Dynkin diagram of type $\tilde{A}_9$. Then the class $f_{\phi} := \sum_{k=1}^{10} \ell_k$ defines a genus one fibration $\phi : X_{3,1} \to \mathbb{P}^1$ in the lattice equivalence class No. 20 of Table 4.2. The line defined by $\{ w + y + 1 = x + iy - i = 0 \}$ provides us with a section of $\phi$ that intersects $\ell_{10}$.

Let $\phi' : X_{3,10} \to \mathbb{P}^1$ be a genus one fibration corresponding to $\phi$ by Theorem 1.3. Since the Néron-Severi lattice of $X_{\mathbb{F}_9}$ is generated by the classes of lines, we can calculate the action of $\text{PGU}(4, \mathbb{F}_9)$ on $S_{3,1}$ from the permutaions of lines induced by $\text{PGU}(4, \mathbb{F}_9)$, and thus we can calculate the action of $\text{PGU}(4, \mathbb{F}_9)'$ on $S_{3,10}$. By computer, we calculate the action of $\text{PGU}(3, \mathbb{F}_4)'$ on the vector space $A_{3,10} \cong \mathbb{F}_3^{20}$. It turns out that the stabilizer subgroup of the non-zero vector

$$(1/3)j(f_{\phi}) \mod S_{3,10} \in A_{3,10}$$

is trivial. Hence the orbit of $[f_{\phi}] \in \mathbb{P}(A_{3,10}) \cong \mathbb{P}^{19}(\mathbb{F}_3)$ by the action of $\text{PGU}(4, \mathbb{F}_9)'$ contains at least $|\text{PGU}(4, \mathbb{F}_9)|/|\mathbb{F}_3^*|$ points. □

**References**

[1] M. Artin. Supersingular $K3$ surfaces. *Ann. Sci. École Norm. Sup. (4)*, 7:543–567 (1975), 1974.

[2] W. Barth and C. Peters. Automorphisms of Enriques surfaces. *Invent. Math.*, 73(3):383–411, 1983.

[3] R. E. Borcherds. Automorphism groups of Lorentzian lattices. *J. Algebra*, 111(1):133–153, 1987.
[4] R. E. Borcherds. Coxeter groups, Lorentzian lattices, and $K^3$ surfaces. *Internat. Math. Res. Notices*, (19):1011–1031, 1998.

[5] N. Bourbaki. *Éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitre IV: Groupes de Coxeter et systèmes de Tits. Chapitre V: Groupes engendrés par des réflexions. Chapitre VI: systèmes de racines*. Actualités Scientifiques et Industrielles, No. 1337. Hermann, Paris, 1968.

[6] J. W. S. Cassels. *Rational quadratic forms*, volume 13 of *London Mathematical Society Monographs*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], London, 1978.

[7] J. H. Conway. The automorphism group of the 26-dimensional even unimodular Lorentzian lattice. *J. Algebra*, 80(1):159–163, 1983.

[8] I. Dolgachev and S. Kondo. A supersingular $K^3$ surface in characteristic 2 and the Leech lattice. *Int. Math. Res. Not.*, (1):1–23, 2003.

[9] I. Dolgachev and J. Keum. Birational automorphisms of quartic Hessian surfaces. *Trans. Amer. Math. Soc.*, 354(8):3031–3057 (electronic), 2002.

[10] W. Ebeling. *Lattices and codes*. Advanced Lectures in Mathematics. Friedr. Vieweg & Sohn, Braunschweig, revised edition, 2002. A course partially based on lectures by F. Hirzebruch.

[11] N. Elkies and M. Schütt. Genus 1 fibrations on the supersingular $K^3$ surface in characteristic 2 with Artin invariant 1, 2012. arXiv:1207.1239v1.

[12] T. Katsura and S. Kondo. A note on a supersingular $K^3$ surface in characteristic 2. In *Geometry and Arithmetic, Series of Congress Reports*, pages 243–255. European Math. Soc., 2012.

[13] T. Katsura and S. Kondo. Rational curves on the supersingular $K^3$ surface with Artin invariant 1 in characteristic 3. *J. Algebra*, 352:299–321, 2012.

[14] J. Keum and S. Kondo. The automorphism groups of Kummer surfaces associated with the product of two elliptic curves. *Trans. Amer. Math. Soc.*, 353(4):1469–1487 (electronic), 2001.

[15] Y. Kitaoka. *Arithmetic of quadratic forms*, volume 106 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1993.

[16] R. Kloosterman. Classification of all Jacobian elliptic fibrations on certain $K^3$ surfaces and their Mordell-Weil groups. *Japan. J. Math. (N.S.)*, 22(2):293–347, 1996.

[17] K. Nishiyama. The Jacobian fibrations on some $K^3$ surfaces and their Mordell-Weil groups. *Japan. J. Math. (N.S.)*, 22(2):293–347, 1996.

[18] K. Oguiso. On Jacobian fibrations on the Kummer surfaces of the product of nonisogenous elliptic curves. *J. Math. Soc. Japan*, 41(4):651–680, 1989.
SUPERSINGULAR K3 SURFACES

[24] A. Ogus. Supersingular K3 crystals. In *Journées de Géométrie Algébrique de Rennes (Rennes, 1978)*, Vol. II, volume 64 of *Astérisque*, pages 3–86. Soc. Math. France, Paris, 1979.

[25] A. Ogus. A crystalline Torelli theorem for supersingular K3 surfaces. In *Arithmetic and geometry, Vol. II*, volume 36 of *Progress in Math.*, pages 361–394. Birkhäuser Boston, Boston, MA, 1983.

[26] A. N. Rudakov and I. R. Shafarevich. Supersingular K3 surfaces over fields of characteristic 2. *Izv. Akad. Nauk SSSR Ser. Mat.*, 42(4):848–869, 1978. Reprinted in I. R. Shafarevich, Collected Mathematical Papers, Springer-Verlag, Berlin, 1989, pp. 614–632.

[27] A. N. Rudakov and I. R. Shafarevich. Surfaces of type K3 over fields of finite characteristic. In *Current problems in mathematics, Vol. 18*, pages 115–207. Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Informatsii, Moscow, 1981. Reprinted in I. R. Shafarevich, Collected Mathematical Papers, Springer-Verlag, Berlin, 1989, pp. 657–714.

[28] T. Sengupta. Elliptic fibrations on supersingular K3 surface with Artin invariant 1 in characteristic 3. preprint, 2012, arXiv:1105.1715.

[29] I. Shimada. On elliptic K3 surfaces. *Michigan Math. J.*, 47(3):423–446, 2000.

[30] I. Shimada. Rational double points on supersingular K3 surfaces. *Math. Comp.*, 73(248):1989–2017 (electronic), 2004.

[31] I. Shimada. Supersingular K3 surfaces in characteristic 2 as double covers of a projective plane. *Asian J. Math.*, 8(3):531–586, 2004.

[32] I. Shimada. Supersingular K3 surfaces in odd characteristic and sextic double planes. *Math. Ann.*, 328(3):451–468, 2004.

[33] I. Shimada. Projective models of the supersingular K3 surface with Artin invariant 1 in characteristic 5. Preprint, 2012, arXiv:1201.4533

[34] I. Shimada and De-Qi Zhang. Classification of extremal elliptic K3 surfaces and fundamental groups of open K3 surfaces. *Nagoya Math. J.*, 161:23–54, 2001.

[35] T. Shioda. Supersingular K3 surfaces with big Artin invariant. *J. Reine Angew. Math.*, 381:205–210, 1987.

[36] T. Shioda. On the Mordell-Weil lattices. *Comment. Math. Univ. St. Paul.*, 39(2):211–240, 1990.

[37] È. B. Vinberg. Some arithmetical discrete groups in Lobačevskiĭ spaces. In *Discrete subgroups of Lie groups and applications to moduli (Internat. Colloq., Bombay, 1973)*, pages 323–348. Oxford Univ. Press, Bombay, 1975.

[38] È. B. Vinberg. Hyperbolic groups of reflections. *Uspekhi Mat. Nauk*, 40(1(241)):29–66, 255, 1985.

GRADUATE SCHOOL OF MATHEMATICS, NAGOYA UNIVERSITY, NAGOYA, 464-8602, JAPAN

E-mail address: kondo@math.nagoya-u.ac.jp

DEPARTMENT OF MATHEMATICS, GRADUATE SCHOOL OF SCIENCE, HIROSHIMA UNIVERSITY, HIGASHI-HIROSHIMA, 739-8526, JAPAN

E-mail address: shimada@math.sci.hiroshima-u.ac.jp