Abstract. In this article we describe properties of the 2-functor from the 2-category of comonads to the 2-category of functors that sends a comonad to its forgetful functor. This allows us to describe contexts where algebras over a monad are enriched tensored and cotensored over coalgebras over a comonad.

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Introduction

This is the first of a series of articles about categories enriched over a category describing some notion of a coalgebra. This article is devoted to coalgebras over a comonad. The next article will deal with coalgebras over an operad and the last article will focus on coalgebras in the context of chain complexes.

Let us start with Sweedler’s theory ([Swe69], [AJ13]). For any two differential graded coassociative coalgebras $(V, w_V, \tau_V)$ and $(W, w_W, \tau_W)$, the tensor product $V \otimes W$ inherits the structure of a coassociative coalgebra as follows

$$V \otimes W \xrightarrow{w_V \otimes w_W} V \otimes V \otimes W \otimes W \cong V \otimes W \otimes V \otimes W.$$ 

This gives a symmetric monoidal structure on the category of differential graded coalgebras. Moreover, from the existence of cofree coalgebras [Ane14], one can show that this monoidal structure is closed. Besides, for any differential graded associative algebra $(A, m)$ and any differential graded coalgebra $(V, w)$, the mapping chain complex $[V, A]$ has the canonical structure of an algebra

$$[V, A] \cong [V \otimes V, A \otimes A] \xrightarrow{[w, m]} [V, A].$$ 

It is usually called the convolution algebra of $V$ and $A$. From the existence of free algebras and cofree coalgebras, one can build a left adjoint $V \boxtimes -$ to the functor $[V, -]$ and a left adjoint $\{ -, A \}$ to the functor $[-, A]$. These three bifunctors

$$[-, -] : \text{Coalgebras}^{\text{op}} \times \text{Algebras} \to \text{Algebras}$$

$$- \boxtimes - : \text{Coalgebras} \times \text{Algebras} \to \text{Algebras}$$

$$\{ -, - \} : \text{Algebras}^{\text{op}} \times \text{Algebras} \to \text{Coalgebras}$$

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make the category of algebras tensored, cotensored and enriched over the category of coalgebras.

This may be reinterpreted in the language of monads and comonads since associative dg algebras and coassociative dg coalgebras are respectively algebras over a monad $M$ and coalgebras over a comonad $Q$ on chain complexes (see [Anel14] for a description of $Q$). Indeed, the structure of a monoidal category on coalgebras is related to the structure of a Hopf comonad on $Q$, that is the data of a natural map

$$Q(X) \otimes Q(Y) \to Q(X \otimes Y)$$

satisfying dual conditions as those of a Hopf monad (see [Moe02]). Moreover, the lifting of the mapping chain complex bifunctor $X, Y \mapsto [X, Y]$ to a bifunctor from $\text{Coalgebras}^{op} \times \text{Algebras}$ to $\text{Algebras}$ is related to the structure of a Hopf module monad on $M$ with respect to the Hopf comonad $Q$, that is the data of a natural map

$$M([X, Y]) \to [Q(X), M(Y)]$$

satisfying again some conditions. All these relations between monoidal structures on categories of algebras and coalgebras and structures on monads and comonads are encoded in a single 2-functor from the 2-category $\text{Comonads}$ of categories with comonads and oplax morphisms to the 2-category $\text{Functors}$ made up of functors between two categories.

**Theorem.** The construction that sends a category $C$ with a comonad $Q$ to the forgetful functor $U_Q$ from $Q$-coalgebras to $C$ induces a 2-functor from the two category $\text{Comonads}$ to the 2-category $\text{Functors}$ that is strictly fully faithful and preserves strict finite products.

This theorem shows in particular that these relations between monoidal structures on categories of algebras and coalgebras and structures on monads and comonads are bijections. For instance, a monoidal structure on the category of $M$-algebras that lifts that of chain complexes is equivalent to the structure of a Hopf monad on $M$ (see again [Moe02]). Moreover, the existence of the two bifunctors $- \otimes -$ and $\{-,-\}$ described above is then just a consequence of Johnstone’s adjoint lifting theorem ([Joh75]).

**Layout.** In the first section, we describe a strict version of a monoidal symmetric monoidal category that we call a monoidal context. In the second section, we describe the monoidal context of comonads and relate it to the monoidal context of functors. In the third section, we apply this relation and the adjoint lifting theorem to describe mapping coalgebras.

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**Universes.** Let us consider three universes $\mathcal{U} \in \mathcal{V} \in \mathcal{W}$. A set is called $\mathcal{U}$-small if it is an element of $\mathcal{U}$ and it is called $\mathcal{U}$-large if it is a subset of $\mathcal{U}$. The notion of smallness and largeness are defined similarly for the other universes. We thus have a hierarchy of sizes of sets

$$\mathcal{U} \text{ small sets} \subset \mathcal{U} \text{ large sets} \subset \mathcal{V} \text{ small sets} \subset \mathcal{V} \text{ large sets} \subset \mathcal{W} \text{ small sets} \subset \mathcal{W} \text{ large sets}.$$ 

Besides, a $\mathcal{U}$-category is a category whose set of objects is $\mathcal{U}$-large and whose hom sets are all $\mathcal{U}$-small. Such a $\mathcal{U}$-category is called $\mathcal{U}$-small if its set of objects is $\mathcal{U}$-small. We have similar notions of $\mathcal{V}$ and $\mathcal{W}$.

Finally, we will use the following aliases:

- a set, also called small set will be a $\mathcal{W}$-small set;
- a large set will be a $\mathcal{W}$-large set;
- a category will be a $\mathcal{W}$-category;
- a small category will be a $\mathcal{W}$-small category.
Some notations.

- For any natural integer $n$, the set of permutations of the set
  $$\mathcal{S}_n = \{1, \ldots, n\}$$
  will be denoted $S_n$.
- For any natural integer $n$, any permutation $\sigma \in S_n$ and any category $C$, we denote $\sigma^*$ the following functor
  $$C^n = \text{Fun}(\mathcal{S}_n, C) \xrightarrow{\sigma} \text{Fun}(\mathcal{S}_n, C) = C^n.$$
- For a comonad $Q$ on a category $E$, the induced comonadic adjunction relating $Q$-coalgebras to $E$ will be denoted $U_Q \dashv L_Q$.
- For a monad $M$, we will sometimes denote $T_M \dashv U^M$ the induced monadic adjunction relating $M$-algebras to $E$. Otherwise, if the context is clear, we will use the same notation for the monad $M$ and the left adjoint functor from the ground category to the category of algebras.

1. Monoidal context

In this section, we describe a semi-strict notion of a symmetric monoidal 2-category that we call a monoidal context. Then, we describe some usual algebraic notions in a monoidal context.

We refer to [DS97] and [SP09] for broader notions of symmetric monoidal 2-categories. In particular, the PhD thesis of Chris Schommer-Pries describes in details a "full" version of a symmetric monoidal bicategory.

Let us fix a universe (except in the last subsection); for instance $\mathcal{U}$. Then $\mathcal{U}$-small sets will be called small sets or just sets and $\mathcal{U}$-large sets will be called large sets.

1.1. Strict 2-categories.

1.1.1. The definition of a 2-category.

**Definition 1.** A strict 2-category $\mathcal{C}$ is a category enriched in categories, that is the data of

- a large set of objects $\text{Ob}(\mathcal{C})$;
- for any two objects $X, Y$, a small category $\mathcal{C}(X, Y)$;
- a unital associative composition $\mathcal{C}(Y, Z) \times \mathcal{C}(X, Z) \to \mathcal{C}(Y, Z)$ whose units are elements $1_x \in \mathcal{C}(X, X)$.

A strict 2-functor $F$ between two strict 2-categories $\mathcal{C}$ and $\mathcal{D}$ is a morphism of categories enriched in categories, that is the data of

- a function $F(-) : \text{Ob}(\mathcal{C}) \to \text{Ob}(\mathcal{D})$;
- functors $F(X, Y) : \mathcal{C}(X, Y) \to \mathcal{D}(F(X), F(Y))$ that commute with the composition and send units to units.

**Definition 2.** A strict 2-category is called small if its set of objects is small. The data of small strict 2-categories and strict 2-functors form the category $\mathcal{Cats}$.

**Remark 1.** From now on, we will call strict 2-functors just 2-functors.

**Definition 3.** Given a strict 2-category $\mathcal{C}$ and two objects $X, Y \in \text{Ob}(\mathcal{C})$,

- an object of $\mathcal{C}(X, Y)$ will be called a morphism of $\mathcal{C}$;
- a morphism in the category $\mathcal{C}(X, Y)$ will be called a 2-morphism of $\mathcal{C}$.

**Definition 4.** Let $\mathcal{C}$, be a strict 2-category and $X, Y, Z$ be three objects.

- Given morphisms $f_1, f_2 : X \to Y$ and $g_1, g_2 : Y \to Z$ and 2-morphisms $a : f_1 \to f_2$ and $b : g_1 \to g_2$, one can compose $a$ and $b$ using the functor
  $$\mathcal{C}(Y, Z) \times \mathcal{C}(X, Y) \to \mathcal{C}(X, Z)$$
  to obtain a 2-morphism from $g_1 \circ f_1$ to $g_2 \circ f_2$ that is denoted $b \circ_a a$ and is called the horizontal composition of $a$ and $b$.
- Given morphisms $f_1, f_2, f_3 : X \to Y$ and 2-morphisms $a_1 : f_1 \to f_2$ and $a_2 : f_2 \to f_3$, one can compose $a_1$ and $a_2$ as morphisms in the category $\mathcal{C}(X, Y)$ to obtain a 2-morphism from $f_1$ to $f_3$, denoted $a_2 \circ_v a_1$ and called the vertical composition of $a_1$ and $a_2$. 

...
Notation. Let us consider a 2-morphism $a$ of $C$, that is a morphism in $\text{C}(X,Y)$ for two objects $X,Y$. For any morphism $f : Y \to Z$, we will usually denote the horizontal composition $\text{Id}_f \circ a$ as $f \circ a$.

Similarly, for any morphism $g : Z \to X$, we will denote $a \circ \text{Id}_g$ as $a \circ g$.

**Definition 5.** An isomorphism in a strict 2-category $C$ is an isomorphism of the underlying category (that is the skeleton of $C$ denoted $\text{sk}(C)$). A 2-isomorphism is an invertible 2-morphism. Finally, an equivalence in $C$ is a morphism $f : X \to Y$ so that there exists a morphism $g : Y \to X$ and 2-isomorphisms

$$a : \text{Id}_X \simeq g \circ f;$$

$$b : \text{Id}_Y \simeq f \circ g.$$ 

Then, $g$ is a pseudo-inverse of $f$.

**Definition 6.** An adjunction in $C$ is the data of two objects $X,Y$ together with morphisms $l : X \to Y$ and $r : Y \to X$ and 2-morphisms $\eta : \text{Id}_X \to rl$ and $\epsilon : lr \to \text{Id}_X$ so that

$$(r \circ h \epsilon) \circ \eta \circ h r = \text{Id}_R;$$

$$(\epsilon \circ h l) \circ \eta \circ h f = \text{Id}_R.$$ 

**Definition 7.** An adjoint equivalence is an adjunction whose unit and counit are isomorphisms.

**Proposition 1.** If $f : X \to Y$ is an equivalence of a strict 2-category with pseudo-inverse $g$, then, $f$ and $g$ are part of an adjoint equivalence.

**Proof.** Let us consider two 2-isomorphisms

$$\eta : \text{Id}_Y \simeq fg;$$

$$\zeta : gf \simeq \text{Id}_X.$$ 

We can notice that the two maps $gf \circ h \zeta$ and $\zeta \circ h gf$ from $gf$ to $gf$ are equal. If we define the 2-isomorphism $\epsilon : gf \simeq \text{Id}_X$ as the composition

$$gf \xrightarrow{gf \circ h \zeta} gf \xrightarrow{gf \circ h \eta \circ h f} \text{Id}_X;$$

then, the tuple $(f,g,\eta,\epsilon)$ is an adjunction. $\square$

1.1.2. The strict 2-category of strict 2-categories. Categories are organised into a 2-category. Similarly, 2-categories form a 3-category. But in the same way one can consider the category of categories, one can also consider a 2-category of strict 2-categories. This implies that we will not consider mapping categories up to equivalences but up to isomorphisms.

**Definition 8.** Categories form a coreflexive full subcategory of strict 2-categories. We denote $\text{sk}$ the corresponding idempotent comonads that sends a strict 2-category to its underlying category (that is with the same objects and morphisms) called its skeleton.

**Definition 9.** Let $\text{Mor}$ be the category with two objects $0$ and $1$ and a non trivial morphism from $0$ to $1$. Let $\text{Nat}$ be the strict 2-category with two objects $0$ and $1$ and so that

$$\begin{cases}
\text{Nat}(0,1) = \text{Mor}; \\
\text{Nat}(0,0) = \text{Nat}(1,1) = *; \\
\text{Nat}(1,0) = \emptyset.
\end{cases}$$

Given two strict 2-categories $C, D$, their product $C \times D$ is the strict 2-categories whose

- set of objects is $\text{Ob}(C \times D) = \text{Ob}(C) \times \text{Ob}(D)$;
- categories of morphisms are

$$C \times D((X, X'), (Y, Y')) = C(X, Y) \times D(X', Y').$$
Proposition 2. The category $\text{2-Cats}^-$ of small strict 2-categories is a cartesian closed monoidal category.

Proof. It suffices to show that for any small 2-category $C$, the endofunctor $- \times C$ of $\text{2-Cats}^-$ has a right adjoint. It is given by the functor $\text{2-Fun}(C, -)$ that sends a strict 2-category $D$ to the 2-category whose

- objects are 2-functor from $C \to D$;
- morphisms are 2-functors from $\text{Mor} \times C$ to $D$;
- 2-morphisms are 2-functors from $\text{Nat} \times C$ to $D$; □

Remark 2. This product of strict 2-categories or even of bicategories does not have good homotopical properties. For instance, the product $\text{Mor} \times \text{Mor}$ is the strictly commutative square instead of the square commutative up to a natural isomorphism.

Definition 10. Given two 2-functors $F, G$ from $C$ to $D$, a strict natural transformation from $F$ to $G$ is just a morphism from $F$ to $G$ in the category $\text{2-Fun}(C, D)$.

Proposition 3. A strict natural transformation $A$ from $F$ to $G$ (that share the same source $C$ and the same target $D$) is equivalent to the data of morphisms in $D$

$$A(X) : F(X) \to G(X)$$

for any object $X \in C$ so that

- for any morphism $f : X \to Y$ in $C$, the following square diagram commutes in $D$

$$\begin{array}{ccc}
F(X) & \xrightarrow{A(X)} & G(X) \\
\downarrow{F(f)} & & \downarrow{G(f)} \\
F(Y) & \xrightarrow{A(Y)} & G(Y);
\end{array}$$

- for any 2-morphism $a : f \to g$ in $C$ where $f, g : X \to Y$, we have

$$A(Y) \circ_a F(a) = G(a) \circ_a A(X).$$

In other words, this is a natural transformation from the functor $\text{sk}(F)$ to the functor $\text{sk}(G)$ that satisfies the last condition.

Proof. Straightforward. □

Corollary 1. Two strict natural transformations $A, A' : F \to G$ are equal if and only if the underlying natural transformations from $\text{sk}(F)$ to $\text{sk}(G)$ are equal.

Proof. This just follows from the fact that a strict natural transformations from $F$ to $G$ is the data of a natural transformation from $\text{sk}(F)$ to $\text{sk}(G)$ that satisfies an additional condition. □

Definition 11. A 2-functor $F : C \to D$ is strictly fully faithful if for any objects $X, Y \in \text{Ob}(C)$, the functor

$$F(X, Y) : C(X, Y) \to D(F(X), F(Y))$$

is an isomorphism of categories.

Definition 12. A 2-functor $F : C \to D$ is strictly essentially surjective if the underlying functor $\text{sk}(F)$ is essentially surjective.
**Definition 14.** A 2-functor $F : C \to D$ is an iso-equivalence if there exists a 2-functor $G : D \to C$ and strict natural isomorphisms

$$\text{Id}_C \simeq G \circ F;$$
$$\text{Id}_D \simeq F \circ G.$$  

**Remark 4.** In other words $F$ is an iso-equivalence if it is an equivalence in the 2-category made up of 2-categories, 2-functors and strict natural transformations.

**Proposition 4.** A 2-functor $F$ is an iso-equivalence if and only if it is strictly fully faithful and strictly essentially surjective.

**Proof.** This follows from the same arguments as for categories. \qed

### 1.2. Monoidal context

We give here the definition of a monoidal context, which could also be called an almost strict symmetric monoidal strict 2-category. Indeed, this is the data of a strict 2-category together with a symmetric monoidal structure which is as strict as symmetric monoidal structures on categories.

Any 2-category is equivalent to a strict 2-category. But this is not true for 3-categories. Likewise we do not expect our notion of monoidal context to encompass all symmetric monoidal 2-categories.

**Definition 15.** A monoidal context is the data a strict 2-category $C$ together with

- a 2-functor $\otimes : C \times C \to C$;
- a 2-functor $1 : * \to C$;
- a strict natural isomorphism called the associator $(X \otimes Y) \otimes Z \simeq X \otimes (Y \otimes Z)$
- a strict natural isomorphism called the commutator $X \otimes Y \simeq Y \otimes X$;
- two strict natural isomorphisms called the unitors $1 \otimes X \simeq X \simeq X \otimes 1$

that makes the skeleton $\text{sk}(C)$ a symmetric monoidal category (for instance they satisfy the pentagon identity and the triangle identity).

**Definition 16.** A monoidal context is small if its underlying strict 2-category is small.

We now define morphisms between monoidal contexts that we call (lax) context functors. This is an almost strict version of (lax) symmetric monoidal 2-functor.

**Definition 17.** A lax context functor between two monoidal contexts $C, D$ is the data of

- a 2-functor $F : C \to D$;
- a strict natural transformation $F(X) \otimes F(Y) \to F(X \otimes Y)$;
- a strict natural transformation $1 \to F(1)$;

that makes the functor between skeletons $\text{sk}(F)$ a lax symmetric monoidal functor. Let $\text{Context}_{\text{lax}}$ be the category of small monoidal contexts and lex context functors.

**Definition 18.** A lax context functor between two monoidal contexts $F : C \to D$ whose structural strict natural transformation $F(X) \otimes F(Y) \to F(X \otimes Y), 1 \to F(1)$ are isomorphisms (resp. identities) is called a strong context functor (resp. a strict context functor).

**Remark 5.** A strict context functor is equivalently a 2-functor that commute with the monoidal context structures.

**Definition 19.** Given two lax context functors $F, G : C \to D$, a monoidal natural transformation between them is a strict natural transformation $A : F \to G$ so that the induced natural transformation from $\text{sk}(F)$ to $\text{sk}(G)$ is monoidal. In other words, $A$ commutes with the structural natural transformations of the two lax context functors.
1.3. **Mac Lane's coherence theorem.** Let us consider a monoidal context \( C \). In particular the skeleton \( \text{sk}(C) \) is a symmetric monoidal category.

**Definition 20.** For any planar tree \( t \) (see Appendix D for a precise definition) with \( n \) leaves whose nodes have arity 2 or 0 one obtains a 2-functor
\[ \otimes_t : C^n \to C \]
as follows:

- if \( t \) has no node, then \( \otimes_t = \text{Id} \);
- if \( t \) is the arity 0 corolla, then \( \otimes_t \) is the 2-functor \( * \to C \) corresponding to the object 1;
- if \( t \) is the arity 2 corolla, then \( \otimes_t = - \otimes - \);
- for any larger tree made up of an arity 2-corolla that supports a left tree \( t_l \) and a right tree \( t_r \), then
\[ \otimes_t = ( - \otimes - ) \circ ( \otimes_{t_l} \times \otimes_{t_r} ) ; \]

**Definition 21.** From any planar tree \( t \) with \( n \) leaves whose nodes have arity 2 or 0 and for any permutation \( \sigma \in S_n \), one obtains a 2-functor
\[ \otimes_{t,\sigma} : C^n \to C \]
as follows the composition
\[ C^n \xrightarrow{\sigma} C^n \otimes \xrightarrow{\sigma} C. \]

**Definition 22.** Let \( n \) be natural integer. The \( n \)-structural groupoid of the monoidal context \( C \) is the subcategory of \( 2\text{-Fun}(C^n, C) \) whose objects are the functors \( \otimes_{t,\sigma} \) for \( t \) a tree with \( n \) leaves and whose nodes have arity 2 0 and for \( \sigma \in S_n \); the morphisms are the strict natural transformations generated by the following maps

1. for any tree \( t \) and any subtree of the form \( \Uparrow \), let \( t' \) be the tree obtained from \( t \) by replacing the subtree \( \Uparrow \) by \( \Uparrow' \); then the associator of \( C \) gives us a strict natural isomorphism
\[ \otimes_{t,\sigma} \simeq \otimes_{t',\sigma}; \]
for any permutation \( \sigma \);

2. similarly as in the previous recipe one can replace subtrees of the form \( \Uparrow \) (resp. \( \Uparrow' \)) by a simple edge and use the left unitor (resp. right unitor) to obtain strict natural isomorphisms;

3. for any tree \( t \) and any binary node \( n \), let \( t_l, t_r \) be the subtrees made up edges above respectively the left edge of \( n \) and the right edge of \( n \) and let \( t' \) be the tree obtained from \( t \) by swapping \( t_1 \) and \( t_2 \). Let \( \rho \) be the permutation so that
\[ \begin{align*}
\rho(i) &= i \text{ if } i \leq a ; \\
\rho(a + j) &= a + b + j \text{ if } 1 \leq j \leq l_1 ; \\
\rho(a + l_1 + j) &= a + j \text{ if } 1 \leq j \leq l_2 ; \\
\rho(i) &= i \text{ if } i > a + l_1 + l_2 .
\end{align*} \]
The commutator of \( C \) gives us a strict natural isomorphism
\[ \otimes_{t,\sigma} \simeq \otimes_{t',\rho \circ \sigma}; \]
for any permutation \( \sigma \).

**Theorem 1** (Mac Lane’s coherence theorem, \cite{Lan63}). For any natural integer \( n \), the \( n \)-structural groupoid of \( C \) is contractible.

**Proof.** Actually Mac Lane’s result deals with monoidal categories instead of monoidal contexts. Its extension to monoidal contexts is just a consequence of the fact that the \( n \)-structural groupoid of \( C \) is the same as that of \( \text{sk}(C) \).

\[ \Box \]

**Remark 6.** \cite{Kel74} In particular, in a monoidal context, the left unitor \( 1 \otimes 1 \to 1 \) applied to the unit \( 1 \) equals its right unitor. This gives \( 1 \) the structure of a strict unital commutative algebra whose unit is just the identity \( 1 = 1 \).
1.4. Multiple tensors. Let $C$ be a monoidal context. One can define multiple tensors as follows.

**Definition 23.** Let $n > 2$ be a natural integer. The 2-functor
\[
\otimes_n : (X_1, \ldots, X_n) \in C^n \to X_1 \otimes \cdots \otimes X_n \in C
\]
is the colimit of the diagram of 2-functors whose objects are the 2-functors $\otimes_t$ where $t$ is a binary tree with $n$ leaves and whose morphisms are those of the $n$-structural groupoid.

Then, for any planar tree $t$ with $m$ leaves and whose nodes have arity 0, 2 or more than 2, and any permutation $\sigma \in S_m$, one can define the 2-functors
\[
\otimes_t, \otimes_{t, \sigma} : C^m \to C
\]
in the same way as in Definition 20 and Definition 21.

**Definition 24.** Let $n$ be natural integer. The extended $n$-structural groupoid of the monoidal context $C$ is the subcategory of $\text{2-Fun}(C^n, C)$ whose objects are the functors $\otimes_{t, \sigma}$ for $t$ a tree with $n$ leaves and whose nodes have arity 0, 2 or more than 2 and for $\sigma \in S_n$, the morphisms are the strict natural transformations generated by the morphisms of the (non extended) $n$-structural groupoid and, for any permutation $\sigma$, for any tree $t$ and any binary subtree $s$ with $m$-leaves the strict natural isomorphism
\[
\otimes_{t, \sigma} \simeq \otimes_{t/s, \sigma}
\]
induced by the canonical map $\otimes_s \to \otimes_m$, where $t/s$ is the tree obtained from $t$ by contracting $s$ into a $m$-corolla.

**Corollary 2.** Let $n$ be a natural integer. The extended $n$-structural groupoid of $C$ is contractible.

**Proof.** Straightforward. \qed

Finally for any lax context functor $F : C \to D$, the lax structure induces canonical strict natural transformations
\[
\otimes_{t, \sigma} \circ F^n \to F \circ \otimes_{t, \sigma}
\]
for any tree with $n$ leaves and whose nodes have arity different from 1 and for any permutation $\sigma \in S_n$. Moreover, for any canonical isomorphism $\otimes_{t, \sigma} \simeq \otimes_{t', \sigma'}$, the following square diagram commutes
\[
\begin{array}{ccc}
\otimes_{t, \sigma} \circ F^n & \Rightarrow & F \circ \otimes_{t, \sigma} \\
\downarrow & & \downarrow \\
\otimes_{t', \sigma'} \circ F^n & \Rightarrow & F \circ \otimes_{t', \sigma'}
\end{array}
\]

1.5. Cartesian monoidal structures.

**Definition 25.** Let $C$ be a strict 2-category and $X, Y$ be two objects. The strict product of $X$ and $Y$ if it exists is an object $X \times Y$ (defined up to a unique isomorphism) equipped with two morphisms
\[
X \times Y \to X; \\
X \times Y \to Y;
\]
so that for any object $Z$ the functor
\[
C(Z, X \times Y) \to C(Z, X) \times C(Z, Y)
\]
is an isomorphism of categories.

**Definition 26.** A strict final element in a strict 2-category $C$ is an object $*$ defined up to a unique isomorphism so that for any object $Z$ the functor
\[
C(Z, \ast) \to \ast
\]
is an isomorphism of categories.

**Definition 27.** A strict 2-category is said to have strict finite products if it has strict products of any pair of objects and if it as a strict final element.

**Proposition 5.** Suppose that the strict 2-category $C$ has strict finite products. Then $C$ gets from the product and the final element the structure of a monoidal context.
Proof. The associator is the canonical natural transformation
\[(X \times Y) \times Z \simeq X \times (Y \times Z)\]
and the unitors are the canonical natural transformations
\[X \times * \simeq X \simeq * \times X.\]
\[\square\]

Definition 28. A cartesian monoidal context is a strict 2-category that has strict finite products and that is equipped with the induced structure of a monoidal context.

Definition 29. Let \(C, D\) be two strict 2-categories with strict finite products. A 2-functor \(F : C \to D\) preserve strict products if the canonical morphisms
\[F(X \times Y) \to F(X) \times F(Y);\]
\[F(*) \to *;\]
are isomorphisms.

Proposition 6. Let \(C, D\) be two cartesian monoidal contexts. A 2-functor \(F : C \to D\) that preserves strict finite products has the canonical structure of a context functor.

Proof. This structure is given by the strict natural transformation \(F(X \times Y) \simeq F(X) \times F(Y)\).
\[\square\]

1.6. Opposite structures. Let \((C, \otimes, 1)\) be a monoidal context. Since in some sense this monoidal context is a 3-categorical structure, there are three ways to inverse structures.

- one can take opposite morphisms;
- or take opposite 2-morphisms;
- or take the opposite monoidal structure.

Definition 30. Let \(C^{\text{op}}\) be the monoidal context with the same object as \(C\) and so that \(C^{\text{op}}(X, Y) = C(Y, X)\) for any two objects \(X, Y\).

Definition 31. Let \(C^{\text{co}}\) be the monoidal context with the same object as \(C\) and so that \(C^{\text{co}}(X, Y) = C(X, Y)^{\text{op}}\) for any two objects \(X, Y\).

Definition 32. Let \(C^{\text{tr}} = (C, \otimes^{\text{tr}}, 1)\) (where \(\otimes^{\text{tr}}\) stands for "transposition") be the monoidal context with the same underlying 2-category \(C\) but whose monoidal structure is defined by
\[X \otimes^{\text{tr}} Y = Y \otimes X\]
for any two objects \(X, Y\). The associator, unitors and commutator are given by that of the monoidal context \((C, \otimes, 1)\).

It is possible to apply more than one of these transformation to obtain \(C^{\text{coop}}, C^{\text{optr}}, C^{\text{cotr}}\) and \(C^{\text{cooptr}}\).

1.7. Some monoidal contexts. The main example of a monoidal context is that of categories.

Definition 33. Let \(\text{Cats}\) be the strict 2-category of small categories, functors and natural transformations. It has strict finite products and is thus a cartesian monoidal context.

Here are two other examples of monoidal contexts.

- The 2-category of small strict 2-categories, 2-functors and strict natural transformations form a cartesian monoidal context.
- The 2-category of small monoidal contexts, lax 2-functors and monoidal natural transformations form a cartesian monoidal context.

We will not deal directly with these two cartesian monoidal contexts but with their underlying cartesian symmetric monoidal categories \(2 - \text{Cats}\) and \(\text{Contexts}_{\text{lax}}\).
Definition 34. Let Functors be the strict 2-category 2-Fun(Mor, Cats), that is
\[ \text{its objects are functors } F : C \to D \text{ between small categories; } \]
\[ \text{its morphisms from } F_1 \text{ to } F_2 \text{ are pairs of functors } (S, T) \text{ so that the following square diagram } \]
of categories is strictly commutative
\[
\begin{array}{ccc}
C_1 & \rightarrow & C_2 \\
\downarrow^{F_1} & & \downarrow^{F_2} \\
D_1 & \rightarrow & D_2
\end{array}
\]
\[ \text{its 2-morphisms from } (S_1, T_1) \text{ to } (S_2, T_2) \text{ are natural transformations } A_S : S_1 \to S_2 \text{ and } \]
\[ A_T : T_1 \to T_2 \text{ so that } A_{T \circ h} F_1 = F_2 \circ h A_S. \]

Proposition 7. The 2-category Functors form a cartesian monoidal context. Moreover, the two
2-functors to categories (target and source) preserve strict finite products.

Proof. Straightforward. \( \square \)

One can also work with subuniverses \((\mathcal{U} \text{ and } \mathcal{V})\).

Definition 35. A \(\mathcal{U}\)-strict 2-category \(C\) is a strict 2-category whose set of objects is \(\mathcal{U}\)-large and so that for any two objects \(x, y\), the category \(C(x, y)\) is \(\mathcal{U}\)-small.

A \(\mathcal{U}\)-strict 2-category is \(\mathcal{U}\)-small if its set of objects is \(\mathcal{U}\)-small.

One defines similarly \(\mathcal{V}\)-strict 2-categories and \(\mathcal{V}\)-small strict 2-categories.

Definition 36. Let \(\text{Cats}_\mathcal{U}\) be the \(\mathcal{V}\)-strict 2-category (in particular it is a \(\mathcal{W}\)-small) whose
\[ \text{objects are } \mathcal{U}\text{-categories (that are in particular } \mathcal{V}\text{-small categories); } \]
\[ \text{morphisms are functors; } \]
\[ \text{2-morphisms are natural transformations.} \]
This is actually the full sub-2-category of the strict 2-category of small categories spanned by \(\mathcal{U}\)-categories.

Proposition 8. The small 2-category \(\text{Cats}_\mathcal{U}\) has strict finite products and is hence a small cartesian monoidal context.

Proof. It suffices to notice that the product of two \(\mathcal{U}\)-categories is again a \(\mathcal{U}\)-category. This follows from the fact that for any two elements \(x, y\) of \(\mathcal{U}\), the pair \((x, y)\) still belongs to \(\mathcal{U}\). \( \square \)

2. Categorical operads acting on a monoidal context

To describe algebraic structures in a monoidal context \(C\), one general way is to consider actions of another monoidal context on \(C\), that is context functors from some monoidal context \(A\) to \(C\). We choose here to deal with the restricted framework of action of operads enriched in categories that we call categorical operads.

2.1. Operads.

Definition 37. A categorical collection \(X\) is the data of a set of colours \(O = \text{Ob}(X)\) (also called objects) and a functor
\[ X : \coprod_{n \geq 0} O^n \times O \to \text{Cats}, \]
that is equivalently the data of small categories
\[ X(c_1, \ldots, c_n, c) \]
for any natural integer \(n\) and any elements \((c_1, \ldots, c_n, c) \in O^{n+1}\). A morphism \(f\) of categorical collections from \(X\) to \(Y\) is the data of a function
\[ f(-) : \text{Ob}(X) \to \text{Ob}(Y) \]
together with functors
\[ f(c_1, \ldots, c_n; c) : X(c_1, \ldots, c_n; c) \to Y(f(c_1), \ldots, f(c_n); f(c)). \]
This defines the category Collections of categorical collections.

**Definition 38.** Let \( O \) be a set of objects. A \( O \)-coloured tree is the data of a tree \( t \) and a function \( \text{col}_t : \text{edges}(t) \to O \).

**Definition 39.** Given a categorical collection \( X \), a \( \text{Ob}(X) \) coloured tree \( t \) and a node \( v \) of \( t \), we denote
\[ X(v) = X(c_1, \ldots, c_n; c) \]
where \( (c_i)_{i=1}^n \) are the colours of the ordered inputs of \( v \) and \( c \) is the colour of its output. Then, we denote
\[ X(t) = \prod_{v \in \text{nodes}(t)} X(v). \]

**Definition 40.** For any categorical collection \( X \), let \( T_{pl}(X) \) be the categorical collection with the same set of objects as \( X \) and so that
\[ T_{pl}(X)(c_1, \ldots, c_n; c) = \prod_t X(t) \]
where the coproduct is taken over the small set of equivalence classes of \( \text{Ob}(X) \)-coloured trees \( t \) whose root is coloured by \( c \) and whose leaves are coloured by \( (c_1, \ldots, c_n) \) (we respect the order on leaves). This defines an endofunctor of the category of categorical collections that has the canonical structure of a monad whose
\[ \triangleright \text{ unit is given by the inclusion of corollas into all trees;} \]
\[ \triangleright \text{ product is given by forgetting partitions in partitioned trees.} \]

**Definition 41.** Categorical planar operads \( P \) are algebras over the planar tree monad \( T_{pl} \) in categorical collections. Equivalently, a categorical planar operad \( P \) is a categorical collection equipped with compositions
\[ P(c; c) \times P(d; c) \xrightarrow{\rho} P(c \otimes d; c) \]
\[ 1_c \in P(c; c) \]
that satisfy associativity and unitality conditions and where
\[ c = (c_1, \ldots, c_n); \]
\[ d = (d_1, \ldots, d_m); \]
\[ c \otimes d = (c_1, \ldots, c_{i-1}, d_1, \ldots, d_m, c_{i+1}, \ldots, c_n). \]
A morphism of categorical planar operads from \( P \) to \( Q \) is a morphism of categorical collections \( f \) that commutes strictly with the operadic compositions and units. We denote \( \text{Operad}^\text{cat,pl} \) the category of categorical planar operads.

**Definition 42.** Let \( P \) be a categorical planar operad. Then we denote \( P_S \) the categorical planar operad with the same objects as \( P \) and so that
\[ P_S(c; c) = \prod_{\sigma \in S_n} P(\sigma^*(c); c) \times \{\sigma\} \]
where
\[ c = (c_1, \ldots, c_n); \]
\[ \sigma^*(c) = (c_{\sigma^{-1}(1)}, \ldots, c_{\sigma^{-1}(n)}). \]
Moreover, the composition
\[ P_S(c; c) \times P_S(d; c) \xrightarrow{\rho} P_S(c \otimes d; c) \]
in \( P_S \) is given by the maps
\[ P(\sigma^*(c); c) \times \{\sigma\} \times P(\mu^*(d); c) \times \{\mu\} \xrightarrow{\rho(\sigma, \mu)} P((\sigma^*(c) \otimes \mu^*(d); c) \times \{\rho\} \]
where $\rho \in S_{n+m-1}$ is the permutation given by

\[
\{1, \ldots, n + m - 1\} \\
\{1, \ldots, i - 1\} \cup \{i + m, \ldots, n + m - 1\} \cup \{i, \ldots, i + m - 1\} \\
\{1, \ldots, i - 1\} \cup \{i + 1, \ldots, n\} \cup \{1, \ldots, m\} \\
\{(1, \ldots, n) - \{i\}\} \cup \{1, \ldots, m\} \\
\{1, \ldots, \sigma(i) - 1\} \cup \{\sigma(i) + 1, \ldots, n\} \cup \{1, \ldots, m\} \\
\{(1, \ldots, n) - \{\sigma(i)\}\} \cup \{1, \ldots, m\} \\
\{1, \ldots, \sigma(i) - 1\} \cup \{\sigma(i) + m, \ldots, n + m - 1\} \cup \{\sigma(i), \ldots, \sigma(i) + m - 1\} \\
\{1, \ldots, n + m - 1\}
\]

This defines an endofunctor of the category of categorical planar operads that has the canonical structure of a monad whose

- unit is given by the inclusion of identities into all permutations; 
- product is given by the composition of permutations.

**Definition 43.** A categorical operad $P$ is an algebra over the monad $_S -$ in the category of categorical planar operads. In other words, this is the data of categorical planar operad together with isomorphisms

$\sigma^* : P(\xi; c) \to P(\xi^\sigma; c)$

for any permutation $\sigma \in \Sigma_n$, where

$\xi = (c_1, \ldots, c_n)$;

$\xi^\sigma = (c_{\sigma(1)}, \ldots, c_{\sigma(n)}) = (\sigma^{-1})^*(\xi)$;

so that $(\mu \circ \sigma)^* = \sigma^* \circ \mu^*$ and that satisfy coherence conditions with respect to the operadic composition. We denote $\text{Operad}_{\text{cat}}$ the category of categorical operads.

**Remark 7.** As for strict 2-categories, categorical operads are actually organised into a 3-category. It is even enriched in 2-operads (represented for instance by categorical operads). We will see later a lax version of this enrichment.

One has a sequence of adjunctions

Collections $\xrightarrow{T_{pl}}$ $\text{Operad}_{\text{cat},pl} \xleftarrow{\Sigma} \text{Operad}_{\text{cat}}.$

**Proposition 9.** The composite adjunction is monadic. We denote $T = T_{pl}(-)_S$ the resulting monad.

**Proof.** For instance, one can use the monadicity theorem and prove that the forgetful functor creates coequalisers of pairs of morphisms of categorical operads $f, g : P \to P'$ that are split in the category of categorical collections. □

**Definition 44.** Let $C$ be a small monoidal context. Then let $\text{End}(C)$ be the categorical operad whose set of objects is that of $C$ and so that

$\text{End}(C)(x_1, \ldots, x_n; x) = C(x_1 \otimes \cdots \otimes x_n, x).$
This defines a functor

$$\text{End}(-) : \text{Context}_{\text{atl}} \to \text{Operad}_{\text{cat}}.$$ 

**Proposition 10.** The functor $\text{End}(-)$ is fully faithful and preserves products.

**Proof.** Straightforward. 

Actually, the essential image of the functor $\text{End}(-)$ is the full subcategory spanned by operads $P$ so that the morphism $P \to u\text{Com}$ is some kind of strict Grothendieck opfibration.

**Definition 45.** Let $P$ be a categorical operad. Then, let $P^{\text{co}}$ be the categorical operad with the same objects and so that

$$P^{\text{co}}(c_1, \ldots, c_n; c) = P(c_1, \ldots, c_n; c)^{\text{op}}$$

for any colours $c_1, \ldots, c_n, c$. The compositions, units and actions of symmetric groups are defined as in $P$. This construction induces an involutive endofunctor

$$-^{\text{co}} : \text{Operad}_{\text{cat}} \to \text{Operad}_{\text{cat}}.$$ 

2.2. **Algebras.** Let $P$ be a categorical operad and let $C$ be a small monoidal context.

**Definition 46.** A $P$-algebra $A$ in $C$ is the data of a morphism of categorical operads

$$A : P \to \text{End}(C).$$

More concretely, such an algebra is the data of an object $A_c \in C$ for any colour $c \in \text{Ob}(P)$ together with functors

$$P(c_1, \ldots, c_n; c) \to C(A_{c_1} \otimes \cdots \otimes A_{c_n}, A_c)$$

$$p \mapsto A(p)$$

that commute with the units, compositions and actions of the symmetric groups.

**Definition 47.** Let $A$ and $B$ be two $P$-algebras in $C$. A lax $P$-morphism from $A$ to $B$ is the data of morphisms in $C$

$$f_c \in C(A_c, B_c), \quad c \in \text{Ob}(P)$$

and natural transformations

$$P(c_1, \ldots, c_n; c) \xrightarrow{A} C(B_{c_1} \otimes \cdots \otimes B_{c_n}, B(c))$$

$$C(A_{c_1} \otimes \cdots \otimes A_{c_n}, A(c)) \xleftarrow{f_c} C(A_{c_1} \otimes \cdots \otimes A_{c_n}, B(c))$$

for any colours $c_1, \ldots, c_n, c$ of $P$, where $f^{\otimes n}$ and $f$ actually stands for $f_{c_1} \otimes \cdots \otimes f_{c_n}$ (or $\text{Id}_1$ if $n = 0$) and $f_c$. We also require that these natural transformations behave coherently with respect to the units, compositions and actions of the symmetric groups of the operads $P$ and $\text{End}(C)$ in the sense that the following diagrams commute

$$B(1_c) \circ f \xrightarrow{B(\text{Id})} f \circ A(1_c)$$

$$B(p^2) \circ f^{\otimes n} \xrightarrow{B(\text{Id}) \otimes \text{Id}} f \circ A(p^2)$$

$$B(p) \circ (\text{Id}^{\otimes n-1} \otimes B(p') \otimes \text{Id}^{\otimes n-i}) \circ f^{n+m-1} \xrightarrow{B(p) \circ (\text{Id}^{\otimes m} \otimes A(p') \otimes \text{Id}^{\otimes n-i})} B(p) \circ f^{\otimes n} \circ (\text{Id}^{\otimes i-1} \otimes A(p') \otimes \text{Id}^{\otimes n-i})$$

$$f \circ A(p) \circ \text{Id}^{\otimes n-1} \otimes A(p') \otimes \text{Id}^{\otimes n-i}$$

$$B(p \circ_i p') \circ f^{\otimes n+m-1} \xrightarrow{B(p \circ_i p') \circ f^{\otimes n+m-1}} f \circ A(p \circ_i p')$$

for any operations $p \in P(c_1, \ldots, c_n; c)$ and $p' \in P(c'_1, \ldots, c'_m; c_i)$ and any permutation $\sigma \in S_n$. 
Remark 8. The lax morphisms are actually algebras over a categorical operad $P \otimes_{BV,\text{lax}} \mathbb{1}$ which a "lax variation" of the Boardman–Vogt tensor product of $P$ with the poset $\mathbb{1}$.

Definition 48. A strong $P$-morphism (resp. strict $P$-morphism) is a lax $P$-morphism $f$ so that the structural natural transformations are invertible (resp. are identities).

Definition 49. Let $f, g : A \to B$ be two lax $P$-morphisms. A $P$-2-morphism $a$ from $f$ to $g$ is the data of 2-morphisms in $C$

$$a_c : f_c \to g_c$$

for any colour $c \in \text{Ob}(P)$ so that the following diagram commutes

$$\begin{array}{ccc}
B(p) & \xrightarrow{f\otimes^n} & f \circ A(p) \\
\downarrow & & \downarrow \\
B(p) & \xrightarrow{g\otimes^n} & g \circ A(p)
\end{array}$$

for any element $p$ of $P(c_1, \ldots, c_n, c)$.

Definition 50. Let $\text{ALG}_C(P)_{\text{lax}}$ be the strict 2-category whose objects are $P$-algebras in $C$, whose morphisms are lax $P$-morphisms, and whose 2-morphisms are $P$-2-morphisms. The composition of two lax $P$-morphisms $f : A \to A'$ and $g : A' \to A''$ is given by the composition of morphisms in $C$

$$(g \circ f)_c = g_c \circ f_c$$

and the 2-morphism

$$A''(p) \circ (g \circ f)\otimes^n = A''(p) \circ g\otimes^n \circ f\otimes^n \to g \circ A'(p) \circ f\otimes^n \to g \circ f \circ A(p).$$

The vertical composition and the horizontal composition of $P$-2-morphisms are just given by the vertical composition and the horizontal composition of 2-morphisms in $C$. Besides, one defines similarly the strict 2-categories $\text{ALG}_C(P)_{\text{strong}}$, and $\text{ALG}_C(P)_{\text{strict}}$ replacing mutatis mutandis lax $P$-morphisms by respectively strong ones and strict ones.

2.3. The monoidal context of algebras. Let $C$ be a monoidal context and let $P$ be a categorical operad.

Proposition 11. The strict 2-category $\text{ALG}_C(P)_{\text{lax}}$ has the structure of a monoidal context so that the forgetful 2-functor

$$\text{ALG}_C(P)_{\text{lax}} \to C^{\text{Ob}(P)}$$

is a strict context functor.

Proof. The monoidal unit is given by the following $P$-algebra obtained from the monoidal unit of $C$

$$P \to \text{uCom} \xrightarrow{1_P} \text{End}(C).$$

The tensor product $- \otimes -$ is the bifunctor that sends a pair of $P$-algebra $A, A'$ to the algebra $A \otimes A'$ given by

$$P \to P \times P \xrightarrow{A \times A'} \text{End}(C) \times \text{End}(C) \xrightarrow{\otimes} \text{End}(C \times C) \xrightarrow{\text{End}(- \otimes -)} \text{End}(C \times C)$$

and that sends a pair of lax morphisms $f : A \to B$ and $g : A' \to B'$ to the morphism $f \otimes g$ given by the maps

$$f_c \otimes g_c : A_c \otimes B_c \to A'_c \otimes B'_c$$
and the lax structure

\[(A \otimes B)(p) \circ (f \otimes g)^{\otimes n}\]

\[\sigma\]

and the lax structure

\[\frac{(A(p) \circ f^{\otimes n}) \otimes (B(p) \circ g^{\otimes n}) \circ \sigma}{(f \circ A'(p)) \otimes (g \circ B'(p)) \circ \sigma}\]

\[\circ \sigma\]

\[(f \otimes g) \circ (A \otimes B)(p),\]

where \(\sigma\) is the isomorphism

\[A_c \otimes B_{c_1} \otimes \cdots \otimes A_{c_n} \otimes B_{c_2} \simeq A_{c_1} \otimes \cdots \otimes A_{c_n} \otimes B_{c_1} \otimes \cdots \otimes B_{c_2}.\]

For any two 2-morphisms \(M : f \to f'\) and \(N : g \to g'\) in \(\text{ALG}\_C(P)\_{\text{lax}}\), one can check that the 2-morphisms \((M_c \otimes N_c)_{c \in \text{Ob}(P)}\) in \(C\) form a 2-morphism in \(\text{ALG}\_C(P)\_{\text{lax}}\).

Now, for any three algebras \(A, A', A''\), the associators, unitors and commutators in \(C\)

\[(A_c \otimes A'_c) \otimes A''_c \simeq A_c \otimes (A'_c \otimes A''_c) ; \quad A_c \otimes 1 \simeq 1 \otimes A_c ; \quad A_c \otimes A''_c \simeq A'_c \otimes A_c\]

form natural strict morphisms of \(P\)-algebras. These natural transformations satisfy the axioms of a monoidal context since their images into \(\text{C}^{\text{Ob}(P)}\) do and since the forgetful functor

\[\text{sk}(\text{ALG}\_C(P)\_{\text{strict}}) \to \text{sk}(\text{C}^{\text{Ob}(P)})\]

is faithful. \(\square\)

**Corollary 3.** The strict 2-categories \(\text{ALG}\_C(P)\_{\text{strong}}\) and \(\text{ALG}\_C(P)\_{\text{strict}}\) have canonical structures of monoidal contexts and the 2-functors

\[\text{ALG}\_C(P)\_{\text{strict}} \to \text{ALG}\_C(P)\_{\text{strong}} \to \text{ALG}\_C(P)\_{\text{lax}} \to \text{C}^{\text{Ob}(P)}\]

are strict context functors.

### 2.4. Naturality of algebras.

**Proposition 12.** The construction that sends a categorical operad \(P\) and a monoidal context \(C\) to the monoidal context \(\text{ALG}\_C(P)\_{\text{lax}}\) induces a functor

\[\text{ALG}\_\(-(-)\)_{\text{lax}} : \text{Operad}_{\text{cat}}^{\text{op}} \times \text{Context}_{\text{lax}} \to \text{Context}_{\text{lax}}.\]

that has the canonical structure of a lax monoidal functor. One obtains similar functors lax monoidal functors by replacing lax morphisms by strong ones or strict ones. Then, one has canonical monoidal natural transformations of lax monoidal functors

\[\text{ALG}\_\(-(-)\)_{\text{strict}} \to \text{ALG}\_\(-(-)\)_{\text{strong}} \to \text{ALG}\_\(-(-)\)_{\text{lax}} \to ((P, C) \to \text{C}^{\text{Ob}(P)}).\]

**Proof.** Given a morphism of operads \(f : Q \to P\) and a lax context functor \(F : C \to D\), one gets a lax context functor

\[\text{ALG}\_C(P)_{\text{lax}} \to \text{ALG}\_D(Q)_{\text{lax}}\]

that sends an object, that is a morphism of operads \(A : P \to \text{End}(C)\) to the morphism

\[\text{End}(F) \circ A \circ f.\]

The image of lax morphisms and 2-morphisms are defined accordingly using \(f\) and \(F\). Moreover, the structure of a lax context functor is given by the that of \(F\).

Finally, the lax monoidal structure on the functor \(\text{ALG}\_\(-(-)\)_{\text{lax}}\) is given by the natural strict context functor

\[\text{ALG}\_C(P)_{\text{lax}} \times \text{ALG}\_D(Q)_{\text{lax}} \to \text{ALG}\_{C \times D}(P \times D)_{\text{lax}}\]

that sends the pair \((A, B)\) of a \(P\)-algebra \(A\) and a \(Q\)-algebra \(B\) to the family of objects

\[(A, B)_{c, c'} = (A_c, B_{c'}) \in C \times D\]
indexed by $\text{Ob}(P) \times \text{Ob}(Q)$ that are equipped with the structure of a $P \times Q$-algebra given by

$$\begin{align*}
P(c_1, \ldots, c_n; c') & \times Q(c'_1, \ldots, c'_n; c') \\
\downarrow & \\
C(A_{c_1} \otimes \cdots \otimes A_{c_n}, A_c) \times D(B_{c'_1} \otimes \cdots \otimes B_{c'_n}, B_{c'}) \\
\downarrow & \\
C \times D((A_{c_1}, B_{c'_1}) \otimes \cdots \otimes (A_{c_n}, B_{c'_n})), (A_c, B_{c'})).
\end{align*}$$

Remark 9. Again, by only considering a functor (and not a 3-functor here), we are discarding a lot of higher information.

Corollary 4. Let $F : C \to D$ be a strong (resp. strict) context functor and let $f : P \to Q$ be a morphism of operads. Then, the resulting 2-functor

$$\text{ALG}_C(Q) \to \text{ALG}_D(P)$$

is a strong (resp. strict) context functor.

Proposition 13. Let $P$ be a categorical operad and let $F : C \to D$ be a strictly fully faithful strong context functor. Then, the square

$$\begin{array}{ccc}
\text{ALG}_C(P)_{\text{lax}} & \to & \text{ALG}_D(P)_{\text{lax}} \\
\phantom{i} & \downarrow & \phantom{i} \\
C^{\text{Ob}(P)} & \to & D^{\text{Ob}(P)}
\end{array}$$

is a pullback square in the category of strict 2-categories. In particular, the 2-functor $\text{ALG}_C(P)_{\text{lax}} \to \text{ALG}_D(P)_{\text{lax}}$ is strictly fully faithful.

Proof. It amounts to prove that the underlying square of sets given by objects is a pullback (which means that a $P$-algebra in $C$ is the same thing as objects $A = (A_c)_{c \in \text{Ob}(P)}$ in $C$ together with the structure of a $P$-algebra on $F(A)$) and that the 2-functor $\text{ALG}_C(P)_{\text{lax}} \to \text{ALG}_D(P)_{\text{lax}}$ is strictly fully faithful.

Corollary 5. Let $F : C \to D$ be a strong context functor that is also an iso-equivalence. Then, for any categorical operad $P$, the 2-functor

$$\text{ALG}_C(P)_{\text{lax}} \to \text{ALG}_D(P)_{\text{lax}}$$

is an iso-equivalence.

Proof. We already know that this 2-functor is strictly fully faithful. It suffices then to show that it is strictly essentially surjective. Let us consider a $P$-algebra $B$ in $D$. Since $F$ is strictly essentially surjective, let us consider objects $(A_c)_{c \in \text{Ob}(P)}$ in $C$ together with isomorphisms

$$F(A_c) \simeq B_c, \ c \in \text{Ob}(P).$$

Using these isomorphisms, one can build a structure of a $P$-algebra $A'$ on the objects $(F(A_c))_{c \in \text{Ob}(P)}$ so that these isomorphisms form an isomorphism of $P$-algebras $A' \simeq B$ in $D$. By Proposition 13, this new $P$-algebra $A'$ is the image of a $P$-algebra in $C$. □

2.5. A long remark about categorical operads. One has actually a functor

$$\mathcal{A}(-)_{\text{lax}} : \text{Operad}^\text{op}_{\text{cat}} \times \text{Operad}^\text{op}_{\text{cat}} \to \text{Operad}^\text{op}_{\text{cat}}$$

so that the following diagram commutes

$$\begin{array}{ccc}
\text{Context}^\text{lax} & \to & \text{Operad}^\text{op}_{\text{cat}} \\
\downarrow & \downarrow & \downarrow \\
\text{Context}^\text{lax} & \to & \text{Operad}^\text{op}_{\text{cat}}
\end{array}$$
2.5.1. The lax mapping operad.

**Definition 51.** Let $P, Q$ be two categorical operads. A $P$-algebra in $Q$ is the data of a morphism of categorical operads

$$A : P \rightarrow Q.$$ 

**Definition 52.** Let $P, Q$ be two categorical operads and let $A_1, \ldots, A_n, B$ be $P$-algebras in $Q$. A lax operation from $(A_1, \ldots, A_n)$ to $B$ is the data of elements

$$m_c \in Q(A_1(c), \ldots, A_n(c); B(c)), \quad c \in \text{Ob}(P)$$

together with natural transformations $l(c_1, \ldots, c_N; c)$

$$P(c_1, \ldots, c_N; c) \xrightarrow{B} Q(B(c_1), \ldots, B(c_N); B(c))$$

$$\xrightarrow{(A_i)_{i=1}^N} Q \left( \left( A_i(c_j) \right)_{i=1,j=1}^{n,N} ; B(c) \right)$$

$$\xrightarrow{\xi_{n,N}} \prod_{i=1}^n Q(A_i(c_1), \ldots, A_i(c_N); A_i(c))$$

for any colours $c_1, \ldots, c_N, c$ of $P$, where $\xi_{n,N} \in S_{n \times N}$ is the permutation

$$(j-1)n + i \mapsto (i-1)N + j, \quad 1 \leq i \leq n, 1 \leq j \leq N.$$ 

Such natural transformations are required to behave coherently with respect to the operadic structure of $P$ and $Q$ in the sense that the following diagrams commute

$$B \left( (p \circ (p_j)_{j=1}^N) \circ (m_{c_{ij}})_{k=1,j=1}^{N,j} \right) \xrightarrow{B(p) \circ \left( (B(p_j) \circ (m_{c_{ij}})_{k=1,j=1}^{N,j} ) \right)_{j=1}^N} \xrightarrow{B(p) \circ \left( (m_c \circ (A_i(p_j))_{j=1}^n \circ (\xi_{n,N})_{j=1}^N \right)_{j=1}^N}$$

$$\xrightarrow{(B(p) \circ (m_c)_{j=1}^N) \circ (A_i(p))_{j=1}^n \circ (\xi_{n,N})_{j=1}^N} \xrightarrow{(m_c \circ (A_i(p))_{j=1}^n) \circ (\xi_{n,N})_{j=1}^N \circ (\xi_{n,N})_{j=1}^N}$$

for any operations $p \in P(c_1, \ldots, c_N; c)$ and $p_j \in P((c_{k,j})_{k=1}^N; c_j)$ for $j = 1, \ldots, N$ and where $\mu \in S_{n \times N}$ is the permutation

$$(j-1)n + i \mapsto (\sigma(j) - 1) + i, \quad 1 \leq j \leq N, 1 \leq i \leq n.$$
A morphism of lax operations from \((m_c)_{c \in \text{Ob}(P)}\) to \((m'_c)_{c \in \text{Ob}(P)}\) is the data of a morphism \(m_c \to m'_c\) in \(Q(A_1(c), \ldots, A_n(c); B(c))\) for any colour \(c \in \text{Ob}(P)\) so that the following diagram commutes
\[
\begin{array}{ccc}
B(p) \circ (m_{c_1}, \ldots, m_{c_k}) & \longrightarrow & (m_c \circ (A_i(p))_{i=1}^n) \\
\downarrow & & \downarrow \\
B(p) \circ (m'_{c_1}, \ldots, m'_{c_k}) & \longrightarrow & (m'_c \circ (A_i(p))_{i=1}^n)
\end{array}
\]
for any operation \(p \in P(c_1, \ldots, c_k; c)\).

Such lax operations and their morphisms form a category.

**Definition 53.** For any two categorical operads \(P, Q\), let \(\mathcal{A}(P, Q)_{\text{lax}}\) be the categorical operad whose objects are \(P\)-algebras in \(Q\) and so that for any \(P\)-algebras in \(Q\) \(A_1, \ldots, A_n, B\), \(\mathcal{A}(P, Q)_{\text{lax}}(A_1, \ldots, A_n; B)\) is the category of lax operations from \((A_1, \ldots, A_n)\) to \(B\). The composition, units and actions of symmetric groups in \(\mathcal{A}(P, Q)_{\text{lax}}\) are given by those of the operad \(Q\).

**Proposition 14.** The construction \(Q, P \mapsto \mathcal{A}(P, Q)_{\text{lax}}\)

induces a functor
\[
\text{Operad}^{\text{cat}}_{\text{cat}} \times \text{Operad}^{\text{op}}_{\text{cat}} \to \text{Operad}^{\text{cat}}_{\text{cat}}
\]
that has the canonical structure of a lax monoidal structure with respect to the cartesian monoidal structures on these categories.

**Proof.** Given a pair of morphism of categorical operads \(f : P' \to P\) and \(g : Q \to Q'\), one obtains a morphism of categorical operads
\[(g^*, f^*) : \mathcal{A}(P, Q)_{\text{lax}} \to \mathcal{A}(P', Q')_{\text{lax}}\]
that sends
\[\triangleright \text{ a colour of } \mathcal{A}(P, Q)_{\text{lax}} \text{ that is a morphism } A : P \to Q \text{ to the morphism } g \circ A \circ f : P' \to Q';\]
\[\triangleright \text{ a lax operation } (m_c \in Q(A_1(c), \ldots, A_n(c); B(c)))_{c \in \text{Ob}(P)} \text{ to the lax operation } g(m_{c'})_{c' \in \text{Ob}(P)};\]
\[\triangleright \text{ a morphism of lax operations to its image through } g.\]

The lax monoidal structure of the functor \(\mathcal{A}(-, -)_{\text{lax}}\) is given by natural morphisms of operads
\[\mathcal{A}(P, Q)_{\text{lax}} \times \mathcal{A}(P', Q')_{\text{lax}} \to \mathcal{A}(P \times P', Q \times Q')_{\text{lax}}\]
that sends
\[\triangleright \text{ a pair of colours } A, A' \text{ to the morphism } A \times A' : P \times P' \to Q \times Q';\]
\[\triangleright \text{ a pair of lax operations } (m_c \in Q(A_1(c), \ldots, A_n(c); B(c)))_{c \in \text{Ob}(P)}, (m'_{c'} \in Q'(A'_1(c'), \ldots, A'_n(c'); B'(c')))_{c' \in \text{Ob}(P')} \text{ to the lax operation } (m_c, m'_{c'}) \in Q(A_1(c), \ldots, A_n(c); B(c)) \times Q'(A'_1(c'), \ldots, A'_n(c'); B'(c'))_{(c, c') \in \text{Ob}(P) \times \text{Ob}(P')}.\]

**Proposition 15.** The following diagram commutes
\[
\begin{array}{ccc}
\text{Context}_{\text{lax}} \times \text{Operad}^{\text{op}}_{\text{cat}} & \xrightarrow{\text{ALG}(-, -)_{\text{lax}}} & \text{Context}_{\text{lax}} \\
\downarrow & & \downarrow \\
\text{Operad}_{\text{cat}} \times \text{Operad}^{\text{op}}_{\text{cat}} & \xrightarrow{\mathcal{A}(-, -)_{\text{lax}}} & \text{Operad}_{\text{cat}}
\end{array}
\]
up to a canonical natural isomorphism.

**Proof.** This follows from a straightforward checking. \qed
2.5.2. **Grothendieck strict opfibrations.**

**Definition 54.** Let \( f : P \to Q \) be a morphism of operads. An element \( p \in P(c; c) \) is strictly \( f \)-cartesian if for any colour \( c' \) and any two tuples \( c', c'' \) of colours of \( P \), the functor from \( P(c', c, c''; c) \) to the pullback

\[
P(c', c', c''; c) \times_{Q(f(c', c', c''); f(c'))} Q(f(c', c, c''); f(c'))
\]

given by the square diagram

\[
\begin{array}{ccc}
P(c', c, c''; c) & \xrightarrow{-\text{op}} & P(c', c', c''; c) \\
\downarrow & & \downarrow \\
Q(f(c', c, c''; f(c')) & \xrightarrow{-\text{op}(p)} & Q(f(c', c', c''; f(c'))
\end{array}
\]

is an isomorphism of categories.

**Definition 55.** A morphism of operads \( f : P \to Q \) is a strict Grothendieck opfibration if for any colours \( c_1, \ldots, c_n \) of \( P \) and any operation \( q \in Q(f(c_1), \ldots, f(c_n); d) \) there exists a colour \( c \) of \( P \) so that \( f(c) = d \) and a strictly \( f \)-cartesian element \( p \in P(c_1, \ldots, c_n; c) \) so that \( f(p) = q \). Such an element \( p \) is called a cartesian lift of \( q \) below \( (c_1, \ldots, c_n) \).

**Definition 56.** Let \( \text{Operad}_{\text{cat, fib}} \) be the full subcategory of \( \text{Operad}_{\text{cat}} \) spanned by categorical operads \( P \) so that the morphism \( P \to u\text{Com} \) is a strict Grothendieck opfibration.

**Definition 57.** Let \( P \) be a categorical operad so that the morphism \( P \to u\text{Com} \) is a strict Grothendieck opfibration. The straightening of \( P \) is the small monoidal context \( s(P) \) defined as follows:

- the underlying 2-category of \( s(P) \) is the underlying 2-category of \( P \) in the sense that its objects are the colours of \( P \) \( (\text{Ob}(s(P)) = \text{Ob}(P)) \) and for any two objects \( x, y, s(P)(x, y) = P(x; y) \); the composition and the units of \( s(P) \) are defined by those of \( P \);
- for any objects \( x_1, \ldots, x_n \), we choose a cartesian lift of the unique element of \( u\text{Com} \) below \( (x_1, \ldots, x_n) \) that we denote \( m(x_1, \ldots, x_n) \) and whose target colour is denoted \( x_1 \otimes \cdots \otimes x_n \):

\[
m(x_1, \ldots, x_n) \in P(x_1, \ldots, x_n, x_1 \otimes \cdots \otimes x_n);
\]

in particular, for \( n = 0 \), this gives us a colour of \( P \) that is the monoidal unit of \( s(P) \); for \( n = 2 \), we obtain the tensor product \( x_1 \otimes x_2 \);
- the naturality of the tensor product is given by the composite functor

\[
s(P)(x, x') \times s(P)(y, y') \xrightarrow{\approx} P(x; x') \times P(y; y')
\]

one can check that this makes \(- \otimes - \) a strict 2-functor;
the associator is the image of the identity of \((x_1 \otimes x_2) \otimes x_3\) through the map
\[
P((x_1 \otimes x_2) \otimes x_3; (x_1 \otimes x_2) \otimes x_3)
\]
\[
\cong
\]
\[
P(x_1 \otimes x_2, x_3; (x_1 \otimes x_2) \otimes x_3)
\]
\[
\cong
\]
\[
P(x_1, x_2, x_3; (x_1 \otimes x_2) \otimes x_3)
\]
\[
\cong
\]
\[
P(x_1, x_2 \otimes x_3; (x_1 \otimes x_2) \otimes x_3);
\]
the commutator is the image of the identity of \(x_1 \otimes x_2\) through the map
\[
P(x_1 \otimes x_2; x_1 \otimes x_2) \simeq P(x_1, x_2, x_1 \otimes x_2) \xrightarrow{\ (1,2) \ } P(x_2, x_1, x_1 \otimes x_2) \simeq P(x_2 \otimes x_1; x_1 \otimes x_2);
\]
the unitors are the images of the identity of \(x\) through the maps
\[
P(x; x) \simeq P(1, x; x) \simeq P(1 \otimes x; x),
\]
\[
P(x; x) \simeq P(x, 1; x) \simeq P(x \otimes 1; x).
\]

**Proposition 16.** The image through the functor \(\text{End}\) of a small monoidal context belongs to the full subcategory \(\text{Operad}_{\text{cat, fib}}\). Moreover, the construction
\[
P \in \text{Operad}_{\text{cat, fib}} \mapsto s(P)
\]
induces a functor
\[
s : \text{Operad}_{\text{cat, fib}} \to \text{Context}_{\text{lax}}
\]
that is a pseudo-inverse of the functor \(\text{End}\).

**Proof.** Straightforward. \(\square\)

**Corollary 6.** A lax context functor \(F : C \to D\) is strong if and only if the morphism of operads above \(\text{uCom}\)
\[
\text{End}(F) : \text{End}(C) \to \text{End}(D)
\]
sends cartesian lifts to cartesian lifts.

**Proof.** Straightforward. \(\square\)

2.6. Opposite structure. Let \(C\) be a small monoidal context and let \(P\) be a categorical operad.

**Definition 58.** A \(P\)-coalgebra in \(C\) is the data of a \(P\)-algebra in \(C^{\text{op}}\). Then, we define
\[
\text{COG}_C(P) = \text{ALG}_{C^{\text{op}}}(P)^{\text{op}}.
\]

**Definition 59.** Let \(A, B\) be two \(P\)-algebras. They are actually morphisms of categorical operads
\[
A, B : P \to \text{End}(C).
\]
They, equivalently may be described as morphism from \(P^{\text{co}}\) to \(\text{End}(C^{\text{co}}) = \text{End}(C)^{\text{co}}\). Then, an oplax morphism from \(A\) to \(B\) is a morphism in \(\text{ALG}_{C^{\text{co}}}(P^{\text{co}})^{\text{lax}}\). We define
\[
\text{ALG}_C(P)_{\text{oplax}} = \text{ALG}_{C^{\text{co}}}(P^{\text{co}})^{\text{lax}}^{\text{co}}
\]
Then, one can notice that
\[
\triangleright\text{ the opposite "op" construction swaps algebras and coalgebras;}
\]
\[
\triangleright\text{ the "co" construction swaps lax morphisms/modules into oplax morphisms/modules.}
\]
One cannot compose a lax \(P\)-morphism with an oplax \(P\)-morphism, but one can rewrite a sequence of a lax \(P\)-morphism followed by an oplax \(P\)-morphism into a sequence of an oplax \(P\)-morphism followed by a lax \(P\)-morphism.
Definition 60. Let us consider the following square diagram in $C^{\text{Ob}(P)}$

\[
\begin{array}{ccc}
A & \xrightarrow{f} & A' \\
\downarrow{f'} & & \downarrow{g'} \\
B & \xrightarrow{g} & B'
\end{array}
\]

where the objects are equipped with structures of $P$-algebras, where $f$ and $g'$ are equipped with structures of lax $P$-morphisms and where $f'$ and $g$ are equipped with structures of oplax $P$-morphisms. Then, the 2-morphisms

\[ a_c : g_c \circ f_c \to g'_c \circ f'_c, \quad c \in \text{Ob}(P) \]

form a rewriting if the following diagram commutes

\[
\begin{array}{ccc}
B'(p) & \xleftarrow{g' \circ f'} & g \circ A'(p) \circ f' \circ A(p) \\
\downarrow & & \downarrow \\
B'(p) & \xrightarrow{g \circ f} & g' \circ B(p) \circ f \circ A(p)
\end{array}
\]

for any operation $p$ of $P$. One can define in a similar way a rewriting from $g' \circ f'$ to $g \circ f$.

2.7. Categorical operads with a fixed set of colours. Let us restrict our attention to categorical operads with a fixed set of colours.

Definition 61. For any set of colours $O$, let

\[
\text{Operad}_{\text{cat},O}, \text{Operad}_{\text{cat,pl},O}, \text{Collections}_O
\]

be the fibers over $O$ of the $\text{Ob}(-)$ functors

Lemma 1. Let $O$ be a set. The three monadic adjunction

\[
\text{Collections} \xleftarrow{T_{\text{pl}}} \text{Operad}_{\text{cat,pl}} \xrightarrow{s} \text{Operad}_{\text{cat}}
\]

restrict to three monadic adjunction on the $O$-fibers $\text{Collections}_O, \text{Operad}_{\text{cat,pl},O},$ and $\text{Operad}_{\text{cat},O}$.

Proof. Straightforward.

Proposition 17. Let $O$ be a set. The category $\text{Collections}_O$ is cocomplete and the forgetful functors

\[
\text{Operad}_{\text{cat},O} \to \text{Operad}_{\text{cat,pl},O} \to \text{Collections}_O
\]

create and preserve sifted colimits. Hence the categories $\text{Operad}_{\text{cat},O}, \text{Operad}_{\text{cat,pl},O}$ are cocomplete.

Proof. Straightforward with Appendix C.

Lemma 2. The functor

\[
\text{Ob}(-) : \text{Operad}_{\text{cat}} \to \text{Set}
\]

is a Grothendieck fibration. The result still holds for categorical planar operads.

Proof. Given a categorical operad $P$ and a function $\phi : O \to \text{Ob}(P)$, the cartesian lifting of $P$ along $\phi$ is the categorical operad $\phi^*(P)$ whose set of colours is $O$ and so that

\[ \phi^*(P)(c_1, \ldots, c_n; c) = P(\phi(c_1), \ldots, \phi(c_n); \phi(c)) \]

for any colours $c_1, \ldots, c_n, c \in O$.

Proposition 18. Let $O$ be a set of objects. The functor

\[
\text{Operad}_{\text{cat},O} \to \text{Operad}_{\text{cat}}
\]

preserves colimits of connected diagrams. The result still holds for categorical planar operads.

Proof. Let us consider a connected diagram

\[
D : I \to \text{Operad}_{\text{cat},O}
\]

whose colimit is denoted $Q$, and a categorical operad $P$. Then, the following data are equivalent:
a cocone of $D$ in the category $\text{Operad}_{\text{cat}}$ whose target is $P$;
▷ a function $\phi : O \rightarrow \text{Ob}(P)$ and a cocone of $D$ in $\text{Operad}_{\text{cat},O}$ whose target is $\phi^*(P)$ (the cartesian lifting of $P$ along $\phi$);
▷ a function $\phi : O \rightarrow \text{Ob}(P)$ and a morphism of categorical operads above $O$ from $Q$ to $\phi^*(P)$;
▷ a morphism of categorical operads from $Q$ to $P$.

Hence, $Q$ is the colimit of $D$ in the category $\text{Operad}_{\text{cat}}$. □

2.8. Categorical operads by generators and relations.

**Proposition 19.** The category of categorical (planar) operads is complete and cocomplete.

**Proof.** Since the category of categorical collections is complete then that of categorical operads is also complete (see Appendix C).

It remains to show that $\text{Operad}_{\text{cat}}$ is cocomplete. We will actually show how to compute colimits of categorical operads.

On the one hand, coproducts are simple to compute; indeed, for a family of categorical operads $(P_i)_{i \in I}$, $\coprod_i P_i$ is the categorical operad so that

$$\text{Ob}\left(\coprod_i P_i\right) = \coprod_i \text{Ob}(P_i)$$

and for any colours $c_0, c_1, \ldots, c_n$

$$\left(\coprod_i P_i\right)(c_1, \ldots, c_n; c_0) = \begin{cases} P_j(c_1, \ldots, c_n; c_0) & \text{if } \exists j, c_0, c_1, \ldots, c_n \in \text{Ob}(P_j); \\ \emptyset & \text{otherwise}. \end{cases}$$

On the other hand, let us consider a pair of morphism of categorical operads $f, g : P \rightarrow P'$ and let $X$ be its coequaliser in the category of categorical collections. Let $\text{coeq}$ be the category with three objects $0, 1, 2$ and so that

$$\text{hom}_{\text{coeq}}(i, j) = \begin{cases} \emptyset & \text{if } i > j, \\ \{a, b\} & \text{if } i = 0, j = 1, \\ * & \text{otherwise}. \end{cases}$$

Let us contemplate the following diagram that represents a functor $\text{coeq} \times \text{coeq} \rightarrow \text{Operad}_{\text{cat}}$

\[
\begin{array}{ccc}
TTP & \xrightarrow{TTf} & TTP' \\
\downarrow & & \downarrow \\
TP & \xrightarrow{Tf} & TP' \\
\downarrow & & \downarrow \\
P & \xrightarrow{f} & P \\
\downarrow & & \downarrow \\
Q & & \text{TTX} & \xrightarrow{TTg} & \text{TX} & \xrightarrow{g} & Q
\end{array}
\]

where $Q$ is the coequaliser in $\text{Operad}_{\text{cat},\text{Ob}(X)}$ (and hence in $\text{Operad}_{\text{cat}}$) of the pair of maps $\text{TTX} \xrightarrow{g} \text{TX}$. Since the three columns of this diagram are colimiting and since the two first rows are also colimiting, then the last row is colimiting. Hence $Q$ is the coequaliser of $f, g$. □

**Remark 10.** One can notice that the category of categorical collections is presentable and that the monad $T$ on categorical collections preserve filtered colimits. Hence, using [AR94, Corollary 2.47 and Paragraph 2.78], the category of categorical operads is presentable and the forgetful functors towards categorical collections preserve filtered colimits.

**Corollary 7.** Let $D$ be a category. Then a functor

$$F : \text{Operad}_{\text{cat}} \rightarrow D$$

preserves colimits if an only if the following conditions are satisfied

▷ $F$ preserves coproducts;
the composite functor
\[ \text{Collections} \to \text{Operad}_{\text{cat}} F \to D \]
preserves coequalisers;
> for any set \( O \), the composite functor
\[ \text{Operad}_{\text{cat},O} \to \text{Operad}_{\text{cat}} F \to D \]
preserves coequalisers.

Proof. On the one hand, a functor that preserves colimits clearly satisfies these conditions.
On the other hand, if \( F \) satisfies these conditions, then it preserves coproducts and coequalisers
(this is a consequence of the way coequalisers are computed in \( \text{Operad}_{\text{cat}} \) as described in the proof of
Proposition 19). Hence it preserves colimits.

\[ \square \]

Corollary 8. The functor \( \text{Ob}(-) \) from categorical operads to sets preserves colimits.

2.9. Generators of operads and algebras. Let \( C \) be a small monoidal context.

Proposition 20. Let \( X \) be a categorical collection and let \( P = T(X) \).
> The \( P \)-algebras in \( C \) are the morphisms of categorical collections
\[ X \to \text{End}(C), \]
that is the data of objects
\[ A_c \in C, \ c \in \text{Ob}(X) \]
and functors
\[ X(c_1, \ldots, c_n; c) \to C(A_{c_1} \otimes \cdots \otimes A_{c_n}, A_c). \]
> The lax morphisms \( f : A \to B \) are the are the data of morphisms in \( C \)
\[ f_c : A_c \to B_c, \ c \in \text{Ob}(X) \]

with natural transformations
\[
\begin{align*}
& X(c_1, \ldots, c_n; c) \quad C(B_{c_1} \otimes \cdots \otimes B_{c_n}, B_c) \\
& \downarrow \quad \downarrow
\end{align*}
\]
\[
\begin{align*}
& C(A_{c_1} \otimes \cdots \otimes A_{c_n}, A_c) \quad C(A_{c_1} \otimes \cdots \otimes A_{c_n}, B_c).
\end{align*}
\]
for any operation \( x \) of \( X \).

Proof. Straightforward.

\[ \square \]

Lemma 3. Let \( f : P \to Q \) be a morphism of categorical operad that is surjective in the sense that
the underlying function on colours is surjective and for any colours \( c_1, \ldots, c_n, c \) of \( P \), the functor
\[ P(c_1, \ldots, c_n; c) \to Q(f(c_1), \ldots, f(c_n); f(c)) \]
is surjective on objects and morphisms. Then the 2-functor
\[ \text{ALG}_C(Q)_{\text{lax}} \to \text{ALG}_C(P)_{\text{lax}} \]
is injective on objects, morphisms and 2-morphisms. The result still holds if we replace lax morphisms
by oplax morphisms, strong morphisms or strict morphisms.

Proof. Straightforward with the definitions.

\[ \square \]
Proposition 21. The functor

\[ \text{Operad}_{\text{cat}}^{\text{op}} \xrightarrow{\text{ALG}_C(-)_{\text{lax}}} \text{Context}_{\text{strict}} \]

preserves small limits. The result still holds if we replace lax morphisms by oplax morphisms, strong morphisms or strict morphisms.

Proof. Following Corollary 7, we need to show that

\( \bullet \) the functor \( \text{ALG}_C(-)_{\text{lax}} \) preserves products;
\( \bullet \) the composite functor \( \text{Collections}^{\text{op}} \rightarrow \text{Operad}_{\text{cat}}^{\text{op}} \xrightarrow{\text{ALG}_C(-)_{\text{lax}}} \text{Context}_{\text{strict}} \)

preserves equalisers;
\( \bullet \) for any set of colours \( O \), the composite functor \( \text{Operad}_{\text{cat},O}^{\text{op}} \rightarrow \text{Operad}_{\text{cat}}^{\text{op}} \xrightarrow{\text{ALG}_C(-)_{\text{lax}}} \text{Context}_{\text{strict}} \)

preserves equalisers.

The two first points follow from straightforward checkings (using Proposition 20 for the second point).

Now, let \( O \) be a set and let us prove the third point. Let us consider a coequaliser of categorical operads over the set \( O \)

\[ P' \rightrightarrows P \twoheadrightarrow Q. \]

Since the coequaliser is computed in categorical collections over \( O \), then the map \( \pi : P \rightarrow Q \) is surjective. We also get a diagram of monoidal contexts and strict context functors

\[ \text{ALG}_C(Q)_{\text{lax}} \rightarrow \text{ALG}_C(P)_{\text{lax}} \rightrightarrows \text{ALG}_C(P)_{\text{lax}}. \]

Let us denote \( L \) the equaliser of the two right strict context functors (this is actually the equaliser in the category of small strict 2-categories) and let us contemplate the 2-functors

\[ \text{ALG}_C(Q)_{\text{lax}} \rightarrow L \rightarrow \text{ALG}_C(P)_{\text{lax}}. \]

They are both injective on objects, morphisms and 2-morphisms (Lemma 3). Then, for a \( P \)-algebra \( A \), the following assertions are equivalent

\( \bullet \) \( A \) is in \( \text{ALG}_C(Q)_{\text{lax}} \);
\( \bullet \) the map \( A : P \rightarrow \text{End}(C) \) factorises (uniquely) through the quotient \( Q \);
\( \bullet \) both maps from \( P' \) to \( \text{End}(C) \)

\[ P' \rightrightarrows P \twoheadrightarrow \text{End}(C) \]

are equal;
\( \bullet \) \( A \) is in \( L \).

Then, for any lax \( P \)-morphism \( f : A \rightarrow B \) between algebras \( A, B \) that belong to \( L \), the following assertions are equivalent:

\( \bullet \) \( f \) is a lax \( Q \)-morphism;
\( \bullet \) the maps

\[ B(p) \circ f^{\otimes n} \rightarrow f \circ A(p) \]

are equal for any two operations \( p, p' \) of \( P \) that have the same image in \( Q \);
\( \bullet \) the maps

\[ B(p) \circ f^{\otimes n} \rightarrow f \circ A(p) \]

are equal for any two operations \( p, p' \) of \( P \) that are the two images of the same operation of \( P' \);
\( \bullet \) \( f \) is a morphism of \( L \).
Finally, a $P$ 2-morphism $a : f \to g$ between morphisms that are in $L$ belongs to $L$ and is actually a $Q$ 2-morphism. Thus, the 2-functor $\text{ALG}_C(Q)_{\text{lax}} \to L$ is an isomorphism. To conclude, a strict context functor whose underlying 2-functor is an isomorphism of strict 2-categories is an isomorphism in $\text{Context}_{\text{strict}}$. □

Remark 11. Another way to understand the proposition just above is to remember that lax $P$-morphisms are morphisms over a categorical operad $P \otimes_{\text{BV,lax}} [1]$ (see Remark 8) and to notice that the construction

$$P \mapsto P \otimes_{\text{BV,lax}} [1]$$

preserves colimits.

2.10. Example of monoidal structures in a monoidal context. Let us consider a small monoidal context $C$. Our goal in this subsection is to describe some types of algebras encoded by categorical operads.

2.10.1. Pairing.

**Definition 62.** A $n$-pairing in the monoidal context $C$ is the data of $n + 1$ objects $(X_1, \cdots, X_n, Y)$ and a morphism

$$p : X_1 \otimes \cdots \otimes X_n \to Y.$$  

A $n$-pairing will be denoted in this subsection using a corolla with $n$-input $\cdots$.

2.10.2. Pseudo-monoids.

**Definition 63.** A pseudo monoid in $C$ is an object $A$ together with pairings

$$m : A \otimes A \to A$$

$$u : 1 \to A$$

which we can represent as corollas $m = \cdots$, $u = \cdots$ and together with 2-isomorphisms

$$\cdots \simeq \cdots ;$$

$$\cdots \simeq \text{Id}_A \simeq \cdots ;$$

called respectively the associator, the left unitor and the right unitor, so that

- the associator satisfies the pentagon identity, that is the following diagram commutes

$$\begin{array}{ccc}
\cdots & \to & \cdots \\
\downarrow & & \downarrow \\
\cdots & \to & \cdots \\
\downarrow & & \downarrow \\
\cdots & \to & \cdots
\end{array}$$

- the unitors satisfy the triangle identity, that is the following diagram commutes

$$\begin{array}{ccc}
\cdots & \to & \cdots \\
\downarrow & & \downarrow \\
\cdots & \to & \cdots
\end{array}$$

We denote $A_\infty$ the categorical planar operad that encodes pseudo monoids.
2.10.3. Pseudo commutative monoids. Let us denote
\[ \kappa(X, Y) : X \otimes Y \simeq Y \otimes X \]
the commutator of the monoidal context \( \mathcal{C} \).

**Definition 64.** A pseudo commutative monoid is a pseudo-monoid \( A \) equipped with a 2-isomorphism \( m_A \simeq m_A \circ \kappa(A, A) \) also written
\[ \simeq \]
and called the commutator so that the following diagrams commute

We denote \( \mathcal{E}_{\infty} \) the categorical operad that encodes pseudo commutative monoids.

2.10.4. Modules.

**Definition 65.** Given a pseudo monoid \( A \in \mathcal{C} \), a lax left \( A \)-module is an object \( M \in \mathcal{C} \) equipped with
\[ \triangleright \]
and 2-morphisms \( \triangleright \)
so that the following diagrams commute

Pairs of a pseudo-monoid \( A \) and a lax left module \( M \) are algebras over a categorical planar operad \( \mathcal{L} \mathcal{M}_{\text{ lax}} \) with two colours.

**Definition 66.** One can define similarly
\[ \triangleright \]
where the 2-morphisms are required to satisfy mutatis mutandis the same conditions as those in the definition of a lax left module.

Remark 12. One can notice that the transposition "tr" construction swaps left modules and right modules.

**Definition 67.** A lax left (resp. right) module \( M \) over a pseudo monoid \( A \) is called strong if the structural 2-morphisms are invertible. Pairs of a pseudo-monoid \( A \) and a strong left (resp. right) module \( M \) are algebras over a categorical planar operad \( \mathcal{L} \mathcal{M}_{\text{strong}} \) (resp. \( \mathcal{R} \mathcal{M}_{\text{strong}} \)) with two colours.
2.10.5. *Mac Lane’s coherence for operads.*

**Proposition 22.** [ML63]

- The categorical planar operad $A_{\infty}$ has one colour and $A_{\infty}(n)$ is the groupoid equivalent to the point category whose objects are (isomorphism classes of) planar trees with only arity 2 and arity 0 nodes; the composition is computed by grafting trees.
- The categorical operad $E_{\infty}$ has one colour and $E_{\infty}(n)$ is the groupoid equivalent to the point category whose objects are pairs $(t, \sigma)$ of a (isomorphism class of) planar trees with only arity 2 and arity 0 nodes and a permutation $\sigma \in S_n$; the composition is computed by grafting trees.

**Proof.** This follows from the same arguments as those used to prove MacLane’s coherence theorem [ML63].

2.11. **Doctrinal adjunction.** Let $\mathcal{C}$ be a small monoidal context and let $\mathcal{P}$ be a categorical operad. Such an operad induces a doctrine (that is a 2-monad) on $\mathcal{C}$ (see [Kel74]) whose definition of (op)lax morphisms matches with that of Definition 47. Along these lines of thought, Kelly’s results on doctrinal adjunctions apply in the framework of operads.

Remember that, using the same notation as in Definition 47, an oplax $\mathcal{P}$-morphism from $A$ to $B$ is the data of morphisms in $\mathcal{C}$

$$f_c : A_c \rightarrow B_c$$

for any colour $c \in \text{Ob}(\mathcal{P})$, and 2-morphisms

$$f \circ A(p) \rightarrow B(p) \circ f^{\otimes n}$$

for any element $p \in \mathcal{P}(c_1, \ldots, c_n; c)$ that satisfy some conditions.

**Theorem 2.** [Kel74] Let us consider two $\mathcal{P}$-algebras $A, B$ in the monoidal context $\mathcal{C}$ and, for any colour $c \in \text{Ob}(\mathcal{P})$, an adjunction

$$\begin{array}{c}
A_c \\
\downarrow \epsilon_c \\
B_c
\end{array}$$

in $\mathcal{C}$ whose unit and counit are denoted respectively $\eta_c$ and $\epsilon_c$. Then,

1. the set of structures of an oplax $\mathcal{P}$-morphism on $l$ and the set of structures of a lax $\mathcal{P}$-morphism on $r$ are canonically related by a bijection that we call Kelly’s bijection;
2. given a structure of an oplax $\mathcal{P}$-morphism on $l$ and a structure of a lax $\mathcal{P}$-morphism on $r$, they are related to each other through Kelly’s bijection if and only if the 2-morphisms $\text{Id} \circ \text{Id} \rightarrow r_c \circ l_c$ and $l_c \circ r_c \rightarrow \text{Id} \circ \text{Id}$ are rewritings;
3. if $l$ and $r$ are equipped with structures of lax $\mathcal{P}$-morphisms then the 2-morphisms $\eta_c$ and $\epsilon_c$ are $\mathcal{P}$ 2-morphisms if and only if $l$ is a strong $\mathcal{P}$-morphism (hence oplax $\mathcal{P}$-morphism) and its oplax structure is related to the lax structure on $r$ through Kelly’s bijection;
4. if $l$ and $r$ are equipped with structures of oplax $\mathcal{P}$-morphisms then the 2-morphisms $\eta_c$ and $\epsilon_c$ are $\mathcal{P}$ 2-morphisms if and only if $r$ is a strong $\mathcal{P}$-morphism (hence a lax $\mathcal{P}$-morphism) and its lax structure is related to the oplax structure on $l$ through Kelly’s bijection.

**Proof.** Let us first prove (1). Given the structure of an oplax $\mathcal{P}$-morphism on $l$, the related structure of a lax $\mathcal{P}$-morphism on $r$ is given by morphisms of the form

$$A(p) \circ r^{\otimes n} \rightarrow r \circ l \circ A(p) \circ r^{\otimes n} \rightarrow r \circ B(p) \circ f^{\otimes n} \circ r^{\otimes n} \rightarrow r \circ B(p).$$

Conversely, given the structure of a lax $\mathcal{P}$-morphism on $r$, the related structure of an oplax $\mathcal{P}$-morphism on $l$ is given by morphisms of the form

$$l \circ A(p) \rightarrow l \circ A(p) \circ r^{\otimes n} \circ f^{\otimes n} \rightarrow l \circ r \circ B(p) \circ f^{\otimes n} \rightarrow B(p) \circ f^{\otimes n}.$$

A straightforward check shows that these formulas do define respectively a lax and an oplax structure and that they are inverse to each other. This defines Kelly’s bijection.
Now, let us prove (2). The fact that the 2-morphisms \( \text{Id} \circ \text{Id} \to r_c \circ l_c \) and \( l_c \circ r_c \to \text{Id} \circ \text{Id} \) are rewrites means that the following squares are commutative

\[
\begin{array}{ccc}
  l \circ A(p) \circ r^\otimes & \to & l \circ r \circ B(p) \\
  \downarrow & & \downarrow \\
  B(p) \circ (l \circ r)^\otimes & \to & B(p) \\
\end{array}
\quad
\begin{array}{ccc}
  A(p) & \to & r \circ l \circ A(p) \\
  \downarrow & & \downarrow \\
  A(p) \circ (r \circ l)^\otimes & \to & r \circ B(p) \circ l^\otimes \\
\end{array}
\]

for any operation \( p \) of the operad \( P \) (actually, if one square is commutative, then the other one is also commutative). A straightforward diagram chasing shows that the commutation of these diagrams is equivalent to the fact that the oplax structure on \( l \) and the lax structure on \( r \) are related through Kelly’s bijection.

Then, let us prove (3). In that context, \( l \) is equipped with the structure of a lax \( P \)-morphism and with the structure of an oplax \( P \)-morphism. Then, using the right commutative square just above, it is straightforward to check that the fact that the natural transformations \( \text{Id} \to r_c l_c \) form a \( P \) 2-morphism is equivalent to the fact that the composite map

\[
l \circ A(p) \to A(p) \circ l^\otimes \to l \circ A(p)
\]

is the identity of \( l \circ A(p) \) for any \( p \). Similarly, using the left commutative square just above, it is straightforward to check that the fact that the natural transformations \( l_c r_c \to \text{Id} \) form a \( P \) 2-morphism is equivalent to the fact that the composite map

\[
A(p) \circ l^\otimes \to l \circ A(p) \to A(p) \circ l^\otimes
\]

is the identity of \( A(p) \circ l^\otimes \) for any \( p \).

Finally, (4) may be proven using the same arguments as (3). \( \square \)

**Proposition 23.** A lax \( P \)-morphism is an isomorphism (resp. an equivalence) in the 2-category of \( P \)-algebras, lax \( P \)-morphisms and \( P \) 2-morphisms if and only if it is a strong \( P \)-morphism and the underlying morphism in \( \text{Ob}(P) \) is an isomorphism (resp. an equivalence). We have the same result when considering oplax morphisms instead of lax morphisms.

**Proof.** Let \( f : A \to B \) be a lax \( P \)-morphism.

Let us suppose that \( f \) is an equivalence in the 2-category of \( P \)-algebras and lax morphisms, with right adjoint \( g \). It is clear that the underlying morphism of \( f \) in \( \text{Ob}(P) \) is an equivalence (with pseudo-inverse the underlying morphism of \( g \)). Moreover, by doctrinal adjunction, \( f \) is a strong \( P \)-morphism.

Conversely, let \( f : A \to B \) be a strong \( P \)-morphism whose underlying morphism in \( \text{Ob}(P) \) is an equivalence, with right adjoint \( g : B \to A \). Then, again by doctrinal adjunction, \( g \) inherits the structure of a lax \( P \)-morphism so that the adjunction in \( \text{Ob}(P) \) relating \( f \) and \( g \) lifts to an adjunction in the 2-category of algebras. Since the unit and the counit of this adjunction are 2-isomorphisms, this is an adjoint equivalence. \( \square \)

**2.12. Examples.** In the small cartesian monoidal context of \( \mathcal{V} \)-small strict 2-categories, pseudo commutative monoids are precisely monoidal contexts and lax \( E_\infty \)-morphisms are lax context functors.

In the small cartesian monoidal context \( \text{Cats}_{\mathcal{V}} \)-small of \( \mathcal{V} \)-small categories, one can check that:

- pseudo (commutative) monoids are (symmetric) monoidal categories;
- (op)lax morphisms of pseudo monoids are (op)lax monoidal functors;
- (op)lax morphisms of pseudo commutative monoids are (op)lax symmetric monoidal functors;
- \( A_\infty \) 2-morphisms are monoidal natural transformations;
- \( E_\infty \) 2-morphisms are symmetric monoidal natural transformations;
- lax left modules are categories tensored over a monoidal category;
- lax morphisms of lax left modules over a lax monoidal functor are functors equipped with a strength;
- a cotensorisation of a category \( D \) by a monoidal category \( C \) is the structure of a oplax right \( C^{\text{op}} \)-module on \( D \), or equivalently, the structure of a lax right \( C \)-module on \( D^{\text{op}} \).

In the small cartesian monoidal context \( \text{Functors}_{\mathcal{V} \text{-small}} \) (that is the full sub 2-category of \( \text{Functors} \) spanned by functors between \( \mathcal{V} \)-small categories, that is stable through finite products):
the pseudo (commutative) monoids are given by pairs of (symmetric) monoidal categories and strict monoidal functors.

- the (op) lax monoidal functors from $F_1 : C_1 \rightarrow D_1$ to $F_2 : C_2 \rightarrow D_2$ are pairs of (op) lax monoidal functors $S : C_1 \rightarrow C_2$ and $T : D_1 \rightarrow D_2$ so that $F_2 \circ S = T \circ F_1$ as (op) lax monoidal functors;

- the lax left modules over a pseudo monoid $A_1 \rightarrow A_2$ are the data of a category $M_1$ tensored over $A_1$, a category $M_2$ tensored over $A_2$ and a functor $M_1 \rightarrow M_2$ that commutes with all the tensoring structures.

3. Comonads and monoidal structures

In this section, we describe the monoidal context of comonads and show how it is related to the monoidal context of functors.

From now on, a set or a small set is a $U$-small set and a large set is a $U$-large set. Moreover, otherwise stated, a category is a $U$-category. Then, we will denote $\Delta$ will no more denote the cartesian monoidal context of $\mathcal{W}$-small categories but that of $U$-categories;

$\Delta$ Functors will denote the cartesian monoidal context $2$-$\text{Fun}(\text{Mor}, \mathcal{Cats}_U)$ instead of $2$-$\text{Fun}(\text{Mor}, \mathcal{Cats}_{\mathcal{W}-\text{small}})$.

3.1. The monoidal context of comonads.

**Definition 68.** Let $\Delta_{\text{act}}$ be the sub category of $\Delta$ made up of active morphisms. More precisely, its objects are the posets $[n] := (0 < 1 < \cdots < n)$ for $n \in \mathbb{N}$ and its morphisms from $[n]$ to $[m]$ are the morphisms of posets (that is functors) $f$ so that $f(0) = 0$ and $f(n) = m$. This is a strict monoidal category with tensor product $[n] \otimes [m] = [n + m]$.

Moreover, let $B(\Delta_{\text{act}})$ be the delooping of the monoidal category $\Delta_{\text{act}}$, that is the strict 2-category with one object $\ast$ and so that $B(\Delta_{\text{act}})(\ast, \ast) = \Delta_{\text{act}}$.

**Definition 69.** A category with comonad is a pair $(C, Q)$ of a category equipped with a comonad. Equivalently, this is a 2-functor from $B(\Delta_{\text{act}})$ to $\mathcal{Cats}$.

In particular, categories with comonads are algebras over a categorical operad (this is just the 2-category $B(\Delta_{\text{act}})$ seen as an operad) in the monoidal context $\mathcal{Cats}$. Thus, one can then define (op) lax morphisms between categories with comonads.

**Definition 70.** A (op) lax comonad functor between two categories with comonads $(C, Q)$ and $(D, R)$ is a (op) lax morphism of $B(\Delta_{\text{act}})$-algebras. For instance an oplax comonad functor is the data of a functor $F : C \rightarrow D$ and a natural transformation $F \circ Q \xrightarrow{A} R \circ F$ so that the the following diagrams commute

$$
\begin{array}{ccc}
FQ & \xrightarrow{\text{RF}} & RF \\
\downarrow & & \downarrow \\
FQQ & \xrightarrow{\text{RR}} & FRR
\end{array}
$$

so that $F \circ Q \xrightarrow{\text{RR}} R \circ F$.

**Definition 71.** Let $\text{Comonads}$ be the strict 2-category made up of $B(\Delta_{\text{act}})$-algebras, oplax $B(\Delta_{\text{act}})$-morphisms and $B(\Delta_{\text{act}})$ 2-morphisms. More precisely, its

- objects are categories with comonads $(C, Q)$;
- morphisms are oplax comonad functors;
2-morphisms between morphisms \((F, A)\) and \((F', A')\) from \((C, Q)\) to \((D, R)\) are natural transformations \(F \to F'\) so that the following diagram commutes

\[
\begin{array}{ccc}
FQ & \rightarrow & RF \\
\downarrow & & \downarrow \\
F'Q & \rightarrow & RF'.
\end{array}
\]

**Proposition 24.** The 2-category **Comonads** has strict finite products and hence is a cartesian monoidal context.

**Proof.** Straightforward. □

**Definition 72.** Given a category \(C\) and a comonad \(Q\) on it, let \(\text{Cog}_C(Q)\) be the category of \(Q\)-coalgebras. Equivalently, this is the mapping category

\[
\text{Cog}_C(M) = \text{Comonads} \left( \text{*Monads}, (C, M) \right).
\]

This defines a 2-functor from **Comonads** to **Cats**.

**Proposition 25.** The 2-functor \(\text{Cog}\) preserves strict finite products and hence is a context functor.

**Proof.** This follows from the fact that \(\text{*Monads}\) is a cocommutative coalgebra. □

**Proposition 26.** The forgetful 2-functor **Comonads** \(\to\) **Cats** preserves strict finite products and is hence a context functor.

**Proof.** Straightforward. □

### 3.2. Monoidal structures in the monoidal context of comonads

Our goal in this subsection is to describe pairings, pseudo-monoids and their modules in the monoidal context of comonads.

#### 3.2.1. Pairings

A 2-pairing in the monoidal context of comonads from the pair \((C, Q), (D, O)\) to \((E, R)\) consists in a bifunctor

\[
C \times D \to E
\]

\[(X, Y) \mapsto X \otimes Y;\]

together with a natural transformation

\[
Q(X) \otimes O(Y) \to R(X \otimes Y)
\]

so that the following diagrams commute

\[
\begin{array}{ccc}
Q(X) \otimes O(Y) & \rightarrow & R(X \otimes Y) \\
\downarrow & & \downarrow \\
QQ(X) \otimes OO(Y) & \rightarrow & RR(X \otimes Y)
\end{array}
\]

\[
\begin{array}{ccc}
Q(X) \otimes O(Y) & \rightarrow & R(X \otimes Y) \\
\downarrow & & \downarrow \\
X \otimes Y & \rightarrow & X \otimes Y.
\end{array}
\]

#### 3.2.2. Hopf comonads

A pseudo-monoid in the monoidal context of comonads consists in a monoidal category \(C\) together with a Hopf comonad; this notion is dual to that of a Hopf monad (see [Moe02]).

**Definition 73.** A Hopf comonad on a monoidal category \((C, \otimes, 1)\) is the data of a comonad \((Q, w, n)\) on \(C\) together with a structure of a lax monoidal functor on \(Q\)

\[
Q(X) \otimes Q(Y) \to Q(X \otimes Y); \quad \mathbb{1} \to Q(1);
\]

so that the natural transformations \(w : Q \to QQ\) and \(n : Q \to \text{Id}\) are monoidal natural transformations.

Then, a pseudo commutative monoid in the monoidal context of comonads is given by a symmetric monoidal category equipped with a commutative Hopf comonad.

**Definition 74.** Let \(Q\) be a Hopf comonad in a symmetric monoidal category \(C\). It is said to be commutative if the structure of a lax monoidal functor on \(Q\) is symmetric.

Then, one can notice that
3.2.3. Comonads comodules. A pair of a pseudo monoid together with a left lax module in the monoidal context of comonads is the data of a Hopf comonad \((Q, w, \tau)\) on a monoidal category \(E\) together with a category \(C\) tensored over \(E\) and a comonad \(R\) on \(C\) equipped with the structure of a Hopf \(Q\)-comodule as defined in the following definition.

**Definition 75.** A structure of Hopf comodule comonad on the comonad \((R, w', \tau')\) is the data of a strength on the functor \(R\) with respect to the lax monoidal functor \(Q\)

\[
Q(X) \otimes B(Y) \to B(X \otimes Y)
\]

so that the natural transformation \(w'\) is strong with respect to the monoidal natural transformation \(w\) and so that \(\tau'\) is strong with respect to \(\tau\).

3.3. Monads. We have a canonical isomorphism of monoidal contexts

\[
\text{Cats} \simeq \text{Cats}^{\text{co}}
\]

that sends a category \(C\) to its opposite category \(C^{\text{op}}\). This isomorphism lifts to an isomorphism between the monoidal context Comonads and the monoidal context of monads.

3.3.1. The monoidal context of monads.

**Definition 76.** A category with monad \((C, M)\) is the data of a category \(C\) and a monad \(M\) on \(C\). Equivalently, this is a 2-functor from \(B(\Delta^{\text{op}}_{\text{act}})\) to \(\text{Cats}\). Moreover, a lax monad functor between two categories with monads \((C, M)\) and \((D, N)\) is a lax morphism of algebras over \(B(\Delta^{\text{op}}_{\text{act}})\), that is the data of a functor \(F : C \to D\) and a natural transformation

\[
N \circ F \xrightarrow{A} F \circ M
\]

so that the following diagrams commute

\[
\begin{array}{ccc}
NNF & \to & NFM \\
\downarrow & & \downarrow \\
NF & \to & FM
\end{array}
\quad
\begin{array}{ccc}
NFM & \to & FMM \\
\downarrow & & \downarrow \\
FM & \to & FM
\end{array}
\quad
\begin{array}{ccc}
IdF & \to & FId \\
\downarrow & & \downarrow \\
FId & \to & FId
\end{array}
\]

**Definition 77.** Let Monads be the strict 2-category of \(B(\Delta^{\text{op}}_{\text{act}})\)-algebras, lax \(B(\Delta^{\text{op}}_{\text{act}})\)-morphisms and \(B(\Delta^{\text{op}}_{\text{act}})\) 2-morphisms.

**Proposition 27.** The strict 2-category Monads form a cartesian monoidal context and the construction

\[
C \in \text{Cats} \mapsto C^{\text{op}}
\]

induces a canonical isomorphism of monoidal contexts

\[
\text{Monads} \simeq \text{Comonads}^{\text{co}}.
\]

**Proof.** Straightforward. \(\square\)
3.3.2. Hopf monads and module monad over a Hopf comonad. A pseudo-monoid in the monoidal context of monads is a monoidal category equipped with a Hopf monad.

**Definition 78.** [Moe02] A Hopf monad on a monoidal category \((C, \otimes, 1)\) is the data of a monad \(M\) together with the structure of a Hopf comonad on the related comonad on \(C^{op}\). Equivalently, this is the data of a monad \(M\) on \(C\) together with a structure of an oplax monoidal functor on \(M\)
\[
M(X \otimes Y) \to M(X) \otimes M(Y) \\
M(1) \to 1
\]

so that the natural transformations \(m : MM \to M\) and \(u : \text{Id} \to M\) are monoidal natural transformations.

Let us consider a Hopf comonad \(Q\) on a monoidal category \(C\), a category \(D\) cotensored over \(C\) through a bifunctor \(D \times C^{op} \to D\)
\[
\langle X, Y \rangle \mapsto \langle X, Y \rangle;
\]
and a monad \(M\) on \(D\). A structure of a lax right \((C, Q)\)-module on \((D^{op}, M)\) in the monoidal context of comonads that enhances the cotensorisation of \(D\) by \(C\) corresponds to the structure of a Hopf \(Q\)-module monad on \(M\).

**Definition 79.** A Hopf \(Q\)-module monad is the data of a monad \((M, m, u)\) on \(D\) together with a strength on the functor \(M\) with respect to the lax monoidal functor \(Q\)
\[
M(\langle X, Y \rangle) \to \langle M(X), Q(Y) \rangle
\]

so that the following diagrams commute
\[
\begin{array}{ccc}
MM(X, Y) & \longrightarrow & M(M(X), Q(Y)) \\
\downarrow & & \downarrow \\
M(X, Y) & \longrightarrow & \langle MM(X), QQ(Y) \rangle;
\end{array}
\]
\[
\begin{array}{ccc}
\langle X, Y \rangle & \longrightarrow & \langle MM(X), QQ(Y) \rangle; \\
\downarrow & & \downarrow \\
M(X, Y) & \longrightarrow & \langle M(X), Q(Y) \rangle.
\end{array}
\]

3.4. From comonads to coalgebras and back to comonads. Let us consider two pairs \((C, Q)\) and \((D, R)\) of a categories with comonads (that is objects in \(\text{Comonads}\)).

**Proposition 28.** Let \(F : C \to D\) be a functor. Then there is a canonical bijection between

1. the set of functors \(F_{cog} : \text{Cog}_C(Q) \to \text{Cog}_D(R)\) that lifts \(F : C \to D\) (that is \(FU_Q = U_RF_{cog}\));
2. the set of oplax comonad structures on \(F\)
\[
F \circ Q \xrightarrow{\beta} R \circ F,
\]

with respect to \(Q\) and \(R\).

**Proof.** Given a functor \(F_{cog} : \text{Cog}_C(Q) \to \text{Cog}_D(R)\) that lifts \(F\), the equality \(FU_Q = U_RF_{cog}\) gives us by adjunction a morphism
\[
F_{cog}L^Q \to L^RF
\]
and then a morphism
\[
FQ = FU_QL^Q = U_RF_{cog}L^Q \to U_RL^RF = RF.
\]
One can check that the resulting map \(FQ \to RF\) is an oplax comonad structure on \(F\).

Conversely, given an oplax comonad structure \(FQ \to RF\) on \(F\), then for any \(Q\)-algebra \(V\), the object \(F(V)\) has the structure of a \(R\)-coalgebra given by the map
\[
F(V) \to FQ(V) \to RF(V).
\]
This construction is natural and defines the expected lifting functor. A straightforward check shows that the two constructions are inverse to each other.

Corollary 9. Given a monad $M$ on $C$ and a monad $N$ on $D$, there is a canonical bijection between
1. the set of functors $F_{\text{alg}} : \text{Alg}_C() \to \text{Alg}_D()$ that lifts $F : C \to D$ (that is $FU^M = U^N F_{\text{alg}}$);
2. the set of lax monad structures on $F$
\[ N \circ F \Rightarrow F \circ M, \]
with respect to $M$ and $N$.

Proof. This follows from the same arguments as those used to prove Proposition 28. In particular, for any $M$-algebra $A$, the object $F(A)$ has the structure of a $N$-algebra given by the map $NF(A) \to FM(A) \to F(A)$.

Proposition 29. Let us consider two functors $F, G : C \to D$ together with liftings $F_{\text{cog}}, G_{\text{cog}} : \text{Cog}_C(Q) \to \text{Cog}_D(R)$ to the categories of coalgebras, that correspond to oplax comonad structures on $F$ respectively denoted $\alpha$ and $\beta$. Moreover, let $A : F \to G$ be a natural transformation. Then, the following assertions are equivalent:
1. the natural transformation $A$ lifts to a 2-morphism in $\text{Comonads}$ from $(F, \alpha)$ to $(G, \beta)$;
2. the natural transformation $A$ lifts to a 2-morphism in $\text{Functors}$ from $(F, F_{\text{cog}})$ to $(G, G_{\text{cog}})$;
3. for any $Q$-coalgebra $V$, the map $A(V) : F(V) \to G(V)$ is a morphism of $R$-coalgebras.

Proof. The assertion (2) is clearly equivalent to (3). Let us prove that (3) is equivalent to (1).
On the one hand, let us assume (1). Then for any $Q$-coalgebra $V$, the following diagram commutes
\[
\begin{array}{ccc}
F(V) & \rightarrow & FQ(V) \\
\downarrow \quad A(V) & & \downarrow A(Q(V)) \\
G(V) & \rightarrow & GQ(V)
\end{array}
\]
\[
\begin{array}{ccc}
& & RF(V) \\
\downarrow R(A(V)) & & \\
& & RG(V)
\end{array}
\]
Thus, the map $A(V) : F(V) \to G(V)$ is a morphism of $R$-coalgebra.
Conversely, let us assume (3). Then, the square diagram
\[
\begin{array}{ccc}
FQ(X) & \rightarrow & RF(X) \\
\downarrow \quad A(Q(X)) & & \downarrow \quad R(A(X)) \\
GQ(X) & \rightarrow & RG(X)
\end{array}
\]
decomposes as
\[
\begin{array}{ccc}
FQ(X) & \rightarrow & FQQ(X) & \rightarrow & RFQ(X) & \rightarrow & RF(X) \\
\downarrow \quad A(Q(X)) & & \downarrow \quad A(QQ(X)) & & \downarrow \quad R(A(Q(X))) & & \downarrow \quad R(A(X)) \\
GQ(X) & \rightarrow & GQQ(X) & \rightarrow & RGQ(X) & \rightarrow & RG(X)
\end{array}
\]
The left square and the right square are commutative by naturality. The middle square is commutative since $Q(X)$ is a $Q$-coalgebra. Hence, the whole square is commutative, which shows (1).

Theorem 3. The construction that sends a category with a comonad $(C, Q)$ to the functor $\text{Cog}_C(Q) \to C$ canonically induces a 2-functor from $\text{Comonads}$ to $\text{Functors}$ that is strictly fully faithful and that preserves strict finite products.
Proof. Such a 2-functor sends a morphism (that is an oplax comonad functor) \((F, A)\) to the pair of functors \((F, F_{\text{cop}})\) defined in Proposition \(28\) and a 2-morphism (that is a natural transformation) \(A' : (F, A) \to (G, B)\) to the pair of natural transformation \((A'', A')\) whose first component is defined in Proposition \(29\).

It is strictly fully faithful by Proposition \(28\) and Proposition \(29\) and it preserves strict finite products because both 2-functors \((C, Q) \mapsto \text{Cog}_C(Q)\) and \((C, Q) \mapsto C\) do. \(\square\)

Hence, any algebraic structure inside the 2-category \(\text{Comonads}\) may equivalently be described using forgetful functors from categories of coalgebras to the ground category.

**Corollary 10.** The construction that sends a category with a monad \((C, M)\) to the functor \(\text{Alg}_C(M) \to C\) canonically induces a 2-functor from \(\text{Monads} to \text{Functors}\) that is strictly fully faithful and that preserves strict finite products.

### 3.5. Consequences

One can draw several consequences from Theorem \(3\).

#### 3.5.1. Monoidal categories

Given a monoidal category \((C, \otimes, 1)\) and a comonad \((Q, w, n)\) (resp. a monad \((M, m, u)\)), there is a canonical bijection between

1. the set of structures of a monoidal category on \(Q\)-algebras (resp. \(M\)-algebras) that lift that of \(C\) (that is the forgetful functor \(\text{Cog}_C(Q) \to C\) is strict monoidal);
2. the set of structures of a Hopf comonad on \(Q\) (resp. structures of a Hopf monad on \(M\)).

Indeed, given a structure of a Hopf comonad on \(Q\), the tensor product of two \(Q\)-coalgebras \(V, W\) and the unit \(1\) inherit structures of \(Q\)-coalgebras through the formulas

\[
V \otimes W \to Q(V) \otimes Q(W) \to Q(V \otimes W); \, 1 \to Q(1).
\]

Conversely, from a structure of a monoidal category on \(Q\)-coalgebras that lifts that of \(C\), one obtain the structure of a Hopf comonad on \(Q\) by lifting the natural map

\[
Q(X) \otimes B(Y) \xrightarrow{\tau(X), \tau(Y)} X \otimes Y
\]
to \(Q(X \otimes Y)\).

#### 3.5.2. Lax monoidal functors

Now, let us consider Hopf comonads \(Q\) and \(O\) on monoidal categories respectively \(C\) and \(D\) and a lax monoidal functor \(F : C \to D\). The two following assertions are equivalent

1. the natural transformation \(FQ \to OF\) is monoidal;
2. the natural map in \(F\)

\[
F(V) \otimes F(W) \to F(V \otimes W)
\]

induced by the structure of a lax monoidal functor on \(F\) is a morphism of \(O\)-coalgebras for any two \(Q\)-coalgebras \(V, W\).

If these assertions are true, then the structure of a lax monoidal functor on \(F : C \to D\) induces a structure of a lax monoidal functor on \(F_{\text{cop}} : \text{Cog}_C(Q) \to \text{Cog}_O(O)\).

#### 3.5.3. Modules

Let \((C, Q)\) be monoidal category and a Hopf comonad and let \(R\) be a comonad on a category \(D\) tensored by \(C\). Then, there is a canonical bijection between

1. the set of tensorisations of the category of \(R\)-coalgebras by the monoidal category of \(Q\)-coalgebras that lifts the tensorisation of \(D\) by \(C\);
2. the set of structures of a Hopf \(Q\)-module comonad on \(R\).

Similarly, if \(M\) is a monad on \(E\) which is cotensored by \(C\), then there is a canonical bijection between

1. the set of cotensorisations of the category of \(M\)-algebras by the monoidal category of \(Q\)-coalgebras that lifts the cotensorisation of \(E\) by \(C\);
2. the set of structures of a Hopf \(Q\)-module monad on \(M\).
3.6. The adjoint lifting theorem. In this subsection, we recall the adjoint lifting theorem and its link with (op)lax (co)monad functors.

Let us consider an oplax comonad functor \((L, A) : (C, Q) \to (D, O)\). Let us assume that the functor \(L\) has a right adjoint \(R\). Then, the structure of an oplax comonad functor \(LQ \to OL\) on \(L\) induces by doctrinal adjunction the structure of a lax comonad functor on the right adjoint \(R\)

\[QR \to RLQR \to ROLR \to RO.\]

Thus, for any \(O\)-coalgebra \(W\), let us consider the two following morphisms of \(Q\)-coalgebras from \(LQ(R(W))\) to \(LQRO(W)\):

- on the one hand, the morphism induced by the map \(W \to O(W)\);
- on the other hand, the composite morphism

\[LQ(W) \to LQQR(W) \to LQRO(W).\]

This gives us a coreflexive pair of maps

\[(1) \quad LQ(R(W)) \Rightarrow LQRO(W)\]

with common left inverse induced by the map \(O(W) \to W\).

**Theorem 4** (Adjoint lifting theorem, [Joh75]). The functor \(L_{\text{cog}}\) has a right adjoint if and only if the pair of maps just above in diagram (1) has an equaliser for any \(O\)-coalgebra \(W\). Then, such a limit defines the value of this right adjoint functor on \(W\).

**Proof.** Straightforward. □

Besides, one can factorise the oplax comonad functor \((L, A)\) from \((C, Q)\) to \((D, O)\) as follows.

**Proposition 30.** The endofunctor \(LQR\) of \(D\) has the canonical structure of a comonad. Moreover, the oplax comonad functor \((L, A)\) factorises as

\[(C, Q) \xrightarrow{(L, A')} (D, LQR) \xrightarrow{(\text{Id}, A'')} (D, O)\]

where \(A'\) and \(A''\) are respectively the natural maps

\[LQ \xrightarrow{LQA} LQR;\]

\[LQR \xrightarrow{A} OLR \xrightarrow{O\varepsilon} O.\]

**Proof.** The structure of a comonad on \(LQR\) is given by the maps

\[LQR \to LQQR \to LQRLQL;\]

\[LQR \to LR \to \text{Id}.\]

Proving that these maps do define a comonad and the rest of the proposition follow from a straightforward checking. □

3.7. More on comonad functors. This subsection deals with the subset of oplax comonad morphisms spanned by strong morphisms.

**Definition 80.** A strong comonad functor between categories with comonads is an oplax comonad functor \((F, A) : (C, Q) \to (D, R)\) so that \(A\) is a natural isomorphism.

**Remark 13.** By the result of Kelly on doctrinal adjunctions, for any adjunction in the 2-category \text{Comonads}, the right adjoint is a strong comonad functor.

**Proposition 31.** An oplax comonad functor \((F, A) : (C, Q) \to (D, R)\) is a comonad functor if and only if the induced natural transformation

\[F_{\text{cog}}LQ \to LR F\]

is an isomorphism.

**Proof.** This is a direct consequence of the way the morphisms \(F_{\text{cog}}LQ \to LR F\) and \(FQ \to RF\) are related in the proof of Proposition 28. □
Let us consider an oplax comonad functor

\[(F, A) : (C, Q) \to (D, R).\]

We suppose that the categories C and D are complete and that the comonads Q, R preserve coreflective equalisers. Hence, the categories of coalgebras over these comonads are complete (see Appendix C).

**Proposition 32.** Suppose that F preserves limits and that A is a natural isomorphism (hence, \((F, A)\) is a strong comonad functor). Then, \(F_{\text{cog}}\) preserves limits.

**Proof.** Since \(U_R \circ F_{\text{cog}} = F \circ U_Q\), \(U_Q, F\) preserve coreflective equalisers and \(U_R\) create coreflective equalisers, then \(F_{\text{cog}}\) also preserve coreflective equalisers.

Let us consider a family of \(Q\)-coalgebras \((V_i)_{i \in I}\) and the following diagram

\[
\begin{array}{ccc}
F_{\text{cog}}(\prod_i V_i) & \overset{\text{F}_{\text{cog}}L_Q(\prod_i U_Q(V_i))}{{\longrightarrow}} & L^R(\prod_i U_RF_{\text{cog}}(V_i)) \\
\downarrow & & \downarrow \\
F_{\text{cog}}L_Q(\prod_i U_Q(V_i)) & \overset{\text{F}_{\text{cog}}L_Q(\prod_i QU_Q(V_i))}{{\longrightarrow}} & L^R(\prod_i RU_RF_{\text{cog}}(V_i))
\end{array}
\]

which represents a natural transformation between the two vertical subdiagrams. The middle horizontal arrow and the bottom horizontal arrow decompose respectively as

\[
F_{\text{cog}}L_Q(\prod_i U_Q(V_i)) \to L^R\text{F}(\prod_i U_Q(V_i)) \to L^R(\prod_i U_RF_{\text{cog}}(V_i)) = L^R(\prod_i U_RF_{\text{cog}}(V_i));
\]

\[
F_{\text{cog}}L_Q(\prod_i QU_Q(V_i)) \to L^R\text{F}(\prod_i QU_Q(V_i)) \to L^R(\prod_i FQU_Q(V_i)) \to L^R(\prod_i RU_RF_{\text{cog}}(V_i));
\]

all these maps are isomorphisms since \(F\) preserves products and the morphism \(FQ \to RF\) is an isomorphism. Hence, the middle horizontal arrow and the bottom horizontal arrow are also isomorphisms. Since the two vertical subdiagrams are limiting, then the top horizontal arrow is also an isomorphism.

To conclude, \(F_{\text{cog}}\) preserves coreflective equalisers and products and hence preserves all limits.

**Corollary 11.** Let us consider a lax monad functor between categories with monads \((G, B) : (C, M) \to (D, N)\) where C and D are cocomplete and M and N preserve reflexive coequalisers. If \(F\) preserves colimits and \(B\) is a natural isomorphism, then the induced functor between algebras \(F_{\text{alg}}\) preserves colimits.

4. Mapping coalgebras

In this section, we use the adjoint lifting theorem to describe contexts where some categories of (co)algebras over a (co)monad are enriched tensored and cotensored over the category of coalgebras over another comonad.

4.1. Pairing adjoints.

4.1.1. The situation. Let us consider a 2-pairing in the monoidal context of comonads

\[\langle - \circ - : (C, Q) \times (D, R) \to (E, O)\].

It is given by a bifunctor \(- \circ - : C \times D \to E\) together with a natural map \(Q(X) \circ R(Y) \to O(X \circ Y)\) that satisfy some commutation conditions with respect to the counits and decompositions of comonads. We know that such a natural map satisfying such conditions is equivalent to the data of a bifunctor

\[\langle - \circ_{\text{cog}} - : \text{Cog}_C(Q) \times \text{Cog}_D(R) \to \text{Cog}_E(O)\]

that lifts \(- \circ -\).

Let us suppose that for any \(X \in C\), the functor \(X \circ - : D \to E\) has a right adjoint that is denoted \(\langle -, X \rangle : E \to D\).
Then, by naturality, we obtain a bifunctor

\[ (-, -) : E \times C^{\text{op}} \to D; \]

together with a natural isomorphism

\[ \text{hom}_E (X \boxtimes Y, Z) \cong \text{hom}_D (Y, \langle Z, X \rangle) \]

for any \((X, Y, Z) \in C \times D \times E\).

4.1.2. The adjoint. Let \(V\) be a \(Q\)-coalgebra. The functor \(V \boxtimes_{\text{cog}} \cdot \) from \(R\)-coalgebras to \(O\)-coalgebras lifts the functor \(U_Q(V) \boxtimes - \) from \(D\) to \(E\). This corresponds to the structure of an oplax comonad functor on \(U_Q(V) \boxtimes - \) with respect to \(R\) and \(O\) given by the map

\[ U_Q(V) \boxtimes R(-) \to QU_Q(V) \boxtimes R(-) \to O(U_Q(V) \boxtimes -). \]

By doctrinal adjunction, the adjoint functor \((-U_Q(V))\) inherits the structure of a lax comonad functor given by a natural map

\[ R((-U_Q(V))) \to (O(-), U_Q(V)). \]

(See Subsection 3.6) For any \(O\)-coalgebra \(Z\), let us consider the following two \(R\)-coalgebra morphisms from \(L^R(\langle Z, V \rangle)\) to \(L^R(O(\langle Z, V \rangle))\):

1. on the one hand, the morphism induced by the structural morphism \(Z \to O(Z)\);
2. on the other hand the composite morphism

\[ L^R(\langle Z, V \rangle) \to L^R(\langle Z, V \rangle) \to L^R(\langle O(Z), V \rangle). \]

They also share a left inverse induced by the counit map \(O(Z) \to Z\). We thus obtain a coreflexive pair of morphisms

\[ L^R(\langle Z, V \rangle) \Rightarrow L^R(\langle O(Z), V \rangle). \]

**Proposition 33.** The functor

\[ V \boxtimes_{\text{cog}} - : \text{Cog}_D (R) \to Cog_E (O) \]

admits a right adjoint, that we denote \((-V)_R\), if and only if the category of \(R\)-coalgebras admits limits of the diagram \([2]\) for any \(O\)-coalgebra \(Z\).

**Proof.** This is an application of the adjoint lifting theorem (Theorem 4). \(\Box\)

**Corollary 12.** If such an adjoint \((-V)_R\) exists for any \(Q\)-coalgebra \(V\), then it yields a bifunctor

\[ (\cdot, -)_R : \text{Cog}_E (O) \times Cog_C (Q)^{\text{op}} \to \text{Cog}_D (R). \]

and a natural isomorphism

\[ \text{hom}_{\text{Cog}_E (O)} (V \boxtimes_{\text{cog}} W, Z) \cong \text{hom}_{\text{Cog}_D (R)} (W, \langle Z, V \rangle_R). \]

**Corollary 13.** If the category \(D\) admits coreflexive equalisers and if they are preserved by \(R\), then the functor \(V \boxtimes_{\text{cog}} - \) has a right adjoint for any \(Q\)-coalgebra \(V\).

Remark 14. If the functor \(- \boxtimes Y\) has a right adjoint \([Y, -]\) for any \(Y \in D\), one gets the same phenomenon, that is the functor \(- \boxtimes_{\text{cog}} W\) has a right adjoint if and only if equalisers of pairs of maps of the form

\[ L^Q([W, Z]) \Rightarrow L^Q([W, O(Z)]) \]

exist in \(Q\)-coalgebras. This just follows from considering the composite pairing

\[ (D, R) \times (C, Q) \Rightarrow (C, Q) \times (D, R) \xrightarrow{\text{equiv}} (E, O) \]

and applying the same results.
4.2. **Enrichment.** Let $C$ be a monoidal category and let $D$ be a category enriched, tensored and cotensored over $C$. Let us suppose that $C$ have all coreflexive equalisers and that $D$ have all coreflexive equalisers and all reflexive coequalisers.

Let us consider a Hopf comonad $Q$ on $C$ that preserves coreflexive equalisers, a comonad $R$ on $D$ that preserves coreflexive equalisers and a monad $M$ on $D$ that preserves reflexive coequalisers.

**Theorem 5.** Given a tensorisation

\[ \text{Cog}_C(Q) \times \text{Cog}_D(R) \to \text{Cog}_D(R) \]

that lifts that of $C$ on $D$, then $R$-coalgebras are enriched, tensored and cotensored over $Q$-coalgebras.

**Theorem 6.** Given a cotensorisation

\[ \text{Alg}_D(M) \times \text{Cog}_C(Q) \to \text{Cog}_D(M) \]

that lifts that of $C$ on $D$, then $M$-algebras are enriched, tensored and cotensored over $Q$-coalgebras.

4.3. **Pairing transfer.** Let us consider two 2-pairings in the monoidal context of comonads

\[ - \boxtimes - : (C, Q) \times (D, R) \to (E, O) \]

\[ - \boxtimes ': (C', Q') \times (D', R') \to (E', O') \]

together with a lax morphism of pairings, that is the data of functors

\[ F_C : C \to C'; \]
\[ F_D : D \to D'; \]
\[ F_E : E \to E'; \]

that are lifted to the level of coalgebras by functors respectively denoted $F_{C,cog}$, $F_{D,cog}$ and $F_{E,cog}$ and together with a natural morphism

\[ F_E(X \boxtimes Y) \to F_C X \boxtimes' F_D Y \]

that also lifts to the level of coalgebras.

Let us suppose that the functors

\[ X \boxtimes : D \to E; \quad X' \boxtimes' : D' \to E'; \]

\[ V \boxtimes_{cog} : \text{Cog}_D(R) \to \text{Cog}_E(O); \quad V' \boxtimes'_{cog} : \text{Cog}_D'(R') \to \text{Cog}_E'(O'); \]

all have right adjoints for any objects $X, X' \in C \times C'$, any $Q$-coalgebra $V$ and any $Q'$-coalgebra $V'$. We denote these right adjoints $\langle -, X \rangle$, $\langle -, X' \rangle'$, $\langle -, V \rangle_R$ and $\langle -, V' \rangle_R$.

**Proposition 34.** Let us suppose that

\begin{itemize}
  \item the natural transformations $F_D R \to R' F_D$ and $F_E O \to O' F_E$ are isomorphisms;
  \item the canonical morphism
    \[ F_D((Z, X)) \to (F_E(Z), F_C(X))' \]

    is an isomorphism for any objects $X, Z \in C \times E$ ;
  \item the image through the functor $F_{D,cog}$ of the limiting diagram
    \[ \langle Z, V \rangle_R \to \langle L^0 Z, V \rangle_R \Rightarrow \langle L^0 O Z, V \rangle_R \]

    is limiting for any $O$-coalgebra $Z$ and any $Q$-coalgebra $V$.
\end{itemize}

Then, for any two coalgebras $Z, V \in \text{Cog}_E(O) \times \text{Cog}_C(Q)$, the canonical morphism

\[ F_{D,cog}((Z, V)_R) \to \langle F_{E,cog}(Z), F_{C,cog}(V) \rangle_R \]

is an isomorphism.
Proof. First, let us prove the result in the case where $Z$ is cofree, that is $Z = L^0 K$.

From the lifting property of the functor $F_C, F_D, F_E$ and of the natural transformation $F_E(X \boxtimes Y) \rightarrow F_C X \boxtimes F_D Y$ we get a commutative diagram of functors

$$
\begin{array}{c}
U_O \circ (F_{cog}(V) \boxtimes -) \circ F_{D,cog} \\
\downarrow \\
F_E \circ U_O \circ (V \boxtimes -) \\
\end{array}
\begin{array}{c}
(F_C U_Q(V) \boxtimes -) \circ U_R \circ F_{D,cog} \\
(F_C U_Q(V) \boxtimes -) \circ F_R \circ U_R \\
\end{array}
\begin{array}{c}
F_E \circ U_O \circ (V \boxtimes -) \\
\end{array}
$$

By adjunction, we thus get a commutative diagram of functors

$$
\begin{array}{c}
F_{D,cog} \circ \langle - , V \rangle R \circ \langle - , U_Q(V) \rangle R \\
\downarrow \\
L^R \circ F_D \circ \langle - , U_Q(V) \rangle R \\
\end{array}
\begin{array}{c}
\langle - , F_{cog}(V) \rangle_R \circ F_{E,cog} \circ L^O \\
\downarrow \\
\langle - , F_{cog}(V) \rangle_R \circ F_E \circ \langle - , U_Q(V) \rangle R \\
\downarrow \\
\langle - , F_{cog}(V) \rangle_R \circ L^Q \circ F_E \\
\end{array}
$$

The horizontal arrows are isomorphisms since for any isomorphism of left adjoint functors the induced morphism of right adjoint functors is also an isomorphism. Then, by the hypothesis, the two left vertical arrows and the bottom right vertical arrow are all isomorphisms. Hence the map

$$
F_{D,cog} \circ \langle - , V \rangle R \circ L^O \rightarrow \langle - , F_{cog}(V) \rangle_R \circ F_{E,cog} \circ L^O
$$

is also an isomorphism. This proves the result for $Z = L^0(K)$.

In the general case, let us consider the following diagram

$$
\begin{array}{c}
F_{D,cog}(L^0 Q(Z), V)_R \rightarrow \langle F_{E,cog} L^0 Q(Z), F_{cog}(V) \rangle_R \\
\uparrow \\
F_{D,cog}(L^0(Q), V)_R \rightarrow \langle F_{E,cog} L^0(Z), F_{cog}(V) \rangle_R \\
\uparrow \\
F_{D,cog}(Z, V)_R \rightarrow \langle F_{E,cog}(Z), F_{cog}(V) \rangle_R
\end{array}
$$

The left vertical part is limiting as well as the left vertical part (by hypothesis). Moreover, the two first horizontal arrows are isomorphisms. Hence, the bottom horizontal map is also an isomorphism. This proves the result. □

Remark 15. The third condition of Proposition 34 is true if in particular, the categories $D$ and $D'$ have coreflexive equalisers that are preserved by $F_D, R$ and $R'$.

Remark 16. One has the same result for right adjoints of the functors $- \boxtimes Y$ and $- \boxtimes_{cog} W$. It suffices to apply the result to the opposite pairing

$$
Y \boxtimes_{op} X = X \boxtimes Y.
$$

4.4. The example of chain complexes. Let $Ch$ be the category of chain complexes of modules over a ring $K$ and let $\text{Mod}^{gr}_K$ be the category of $Z$-graded $K$-modules. Let us denote by $U_d$ the forgetful functor from chain complexes to graded modules. Let us notice that the categories $Ch$ and $\text{Mod}^{gr}_K$ are complete and cocomplete and that $U_d$ preserves limits and colimits.
Definition 81. We call a monad $M$ (resp. a comonad $Q$) on chain complexes "computed on graded modules" if there exists a monad $M^{gr}$ (resp. a comonad $Q^{gr}$) on graded modules so that we have equalities

$$U_d \circ M = M^{gr} \circ U_d$$
$$U_d \circ Q = Q^{gr} \circ U_d$$

that are consistent with respect to the structural morphisms of monads and comonads. This determines uniquely $M^{gr}$ and $Q^{gr}$.

The following diagrams of categories are commutative

$$
\begin{array}{ccc}
\text{Ch} \times \text{Ch} & \cong & \text{Ch} \\
\downarrow & & \downarrow \\
\text{Mod}_{K}^{gr} \times \text{Mod}_{K}^{gr} & \cong & \text{Mod}_{K}^{gr};
\end{array}
\quad
\begin{array}{ccc}
\text{Ch} \times \text{Ch} & \cong & \text{Ch} \\
\downarrow & & \downarrow \\
(\text{Mod}_{K}^{gr})^{\text{op}} \times \text{Mod}_{K}^{gr} & \cong & (\text{Mod}_{K}^{gr})^{\text{op}}; \\
\downarrow & & \downarrow \\
\text{Mod}_{K}^{gr} \times \text{Mod}_{K}^{gr} & \cong & \text{Mod}_{K}^{gr}.
\end{array}
$$

Moreover, the functors $X \otimes -$, $- \otimes X$, have the same right adjoint given by $[X, -]$ and the functor $[-, X]$ from $\text{Ch}^{\text{op}}$ to $\text{Ch}$ has a left adjoint given actually by the same formula $[X, -]$.

Let us consider three comonads on chain complexes $Q, R, O$ and two monads $M, N$. We suppose that these monads and comonads are computed on graded modules, that the comonads preserve coreflexive equalisers and that the monads preserve reflexive coequalisers.

4.4.1. Coalgebras in chain complexes. Let us consider a bifunctor

$$- \otimes - : \text{Cog}_{\text{Ch}} (Q) \times \text{Cog}_{\text{Ch}} (R) \to \text{Cog}_{\text{Ch}} (O)$$

that lifts the tensor product of chain complexes. Since these comonads are computed at the level of graded modules, the bifunctor between categories of coalgebras described above lifts another bifunctor

$$- \otimes - : \text{Cog}_{\text{Mod}_{K}^{gr}} (Q^{gr}) \times \text{Cog}_{\text{Mod}_{K}^{gr}} (R^{gr}) \to \text{Cog}_{\text{Mod}_{K}^{gr}} (O^{gr}),$$

that lifts itself the tensor product of graded modules. By the adjoint lifting theorem as used in this section, we obtain four bifunctors

$$(-, -) : \text{Cog}_{\text{Ch}} (O) \times \text{Cog}_{\text{Ch}} (Q) \to \text{Cog}_{\text{Ch}} (R);$$

$$(-, -) : \text{Cog}_{\text{Ch}} (R) \times \text{Cog}_{\text{Ch}} (O) \to \text{Cog}_{\text{Ch}} (Q);$$

$$(-, -)^{gr} : \text{Cog}_{\text{Mod}_{K}^{gr}} (O^{gr}) \times \text{Cog}_{\text{Mod}_{K}^{gr}} (Q^{gr}) \to \text{Cog}_{\text{Mod}_{K}^{gr}} (R^{gr});$$

$$(-, -)^{gr} : \text{Cog}_{\text{Mod}_{K}^{gr}} (R^{gr}) \times \text{Cog}_{\text{Mod}_{K}^{gr}} (Q^{gr}) \to \text{Cog}_{\text{Mod}_{K}^{gr}} (O^{gr});$$

together with natural isomorphisms

$$\text{hom} (W, \{Z, V\}) \simeq \text{hom} (V \otimes W, Z) \simeq \text{hom} (V, \{W, Z\});$$

$$\text{hom} (W', \{Z', V'\}^{gr}) \simeq \text{hom} (V' \otimes W', Z') \simeq \text{hom} (V', \{W', Z'\}^{gr});$$

for any $Q, R, O, Q^{gr}, R^{gr}, O^{gr}$-coalgebras respectively $V, W, Z, V', W', Z'$. From Proposition 34 we get that the natural morphisms

$$U_{d, \text{cog}}(Z, V) \to (U_{d, \text{cog}}(Z), U_{d, \text{cog}}(V))^{gr}$$
$$U_{d, \text{cog}}(\{W, Z\}) \to \{U_{d, \text{cog}}(W), U_{d, \text{cog}}(Z)\}^{gr}$$

are isomorphisms.

4.4.2. Algebras in chain complexes. Let us consider a bifunctor

$$[-, -] : \text{Alg}_{\text{Ch}} (Q) \times \text{Alg}_{\text{Ch}} (M) \to \text{Alg}_{\text{Ch}} (N)$$

that lifts the internal hom of chain complexes. Since $M, N, Q$ are computed at the level of graded modules, the bifunctor between categories of algebras and coalgebras described above lifts another bifunctor

$$[-, -] : \text{Cog}_{\text{Mod}_{K}^{gr}} (Q^{gr}) \times \text{Cog}_{\text{Mod}_{K}^{gr}} (M^{gr}) \to \text{Cog}_{\text{Mod}_{K}^{gr}} (N^{gr}).$$
that lifts itself the internal hom of graded modules. Thus, from the adjoint lifting theorem, we obtain
four bifunctors

$- \boxtimes - : \text{Cog}_\text{Ch}(Q) \times \text{Cog}_\text{Ch}(N) \to \text{Alg}_\text{Ch}(M)$;

$\{-,-\} : \text{Alg}_\text{Ch}(N) \text{op} \times \text{Alg}_\text{Ch}(M) \to \text{Cog}_\text{Ch}(Q)$;

$- \boxtimes _{gr} : \text{Cog}_\text{Mod}_{gr}^i(Q) \times \text{Alg}_\text{Mod}_{gr}^i(N) \to \text{Cog}_\text{Mod}_{gr}^i(M)$;

$\{-,-\}_{gr} : \text{Alg}_\text{Mod}_{gr}^i(N) \text{op} \times \text{Alg}_\text{Mod}_{gr}^i(M) \to \text{Cog}_\text{Mod}_{gr}^i(Q)$;

together with natural isomorphisms

$\text{hom}(V \boxtimes B, A) \simeq \text{hom}(B, [V, A]) \simeq \text{hom}(V, \{B, A\})$;

$\text{hom}(V' \boxtimes _{gr} B', A') \simeq \text{hom}(B', [V', A']) \simeq \text{hom}(V', \{B', A'\}_{gr})$;

for any $Q, Q_{gr}$-coalgebras $V, V'$ and any $M, N, M_{gr}$, $N_{gr}$-algebras $A, B, A', B'$. From Proposition 34,
we get that the natural morphisms

$U_{d,\text{cog}}(V) \boxtimes _{gr} U_{d,\text{alg}}(B) \to U_{d,\text{alg}}(V \boxtimes B)$

$U_{d,\text{cog}}(\{B, A\}) \to \{U_{d,\text{alg}}(B), U_{d,\text{alg}}(A)\}_{gr}$

are isomorphisms.

Appendix A. Symmetric monoidal categories

In this first appendix, we just recall the main definitions related to symmetric monoidal categories.

A.1. Monoidal categories.

Definition 82. A monoidal category is the data of a category $\mathcal{C}$ equipped with a bifunctor

$- \otimes - : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$

an object $1$ and natural isomorphisms

$X \otimes (Y \otimes Z) \simeq (X \otimes Y) \otimes Z, \quad 1 \otimes X \simeq X, \quad X \otimes 1 \simeq X$

called respectively the associator, the left unitor and the right unitor and so that the following diagrams commute

$((U \otimes Y) \otimes X) \otimes Z \quad \rightarrow \quad (U \otimes X) \otimes (Y \otimes Z)$

$(X \otimes 1) \otimes Y \quad \rightarrow \quad X \otimes (1 \otimes Y)$

$U \otimes ((X \otimes Y) \otimes Z) \quad \rightarrow \quad U \otimes (X \otimes (Y \otimes Z))$

$X \otimes Y.$

Definition 83. Given two monoidal categories $\mathcal{C}, \mathcal{D}$, a lax monoidal functor from $\mathcal{C}$ to $\mathcal{D}$ is the data
of a functor $f : \mathcal{C} \to \mathcal{D}$ together with a natural transformation

$f(X) \otimes f(Y) \rightarrow f(X \otimes Y)$

and a map $1 \rightarrow f(1)$ so that the following diagrams commute

$(f(X) \otimes f(Y)) \otimes f(Z) \quad \rightarrow \quad f(X) \otimes (f(Y) \otimes f(Z))$

$f(X \otimes Y) \otimes f(Z) \quad \rightarrow \quad f(X) \otimes f(Y \otimes Z)$

$f((X \otimes Y) \otimes Z) \quad \rightarrow \quad f(X \otimes (Y \otimes Z))$
Definition 84. Given lax monoidal functors between monoidal categories $f, g : C \to D$, a monoidal natural transformation between them is a natural transformation between the functor $f, g$ so that the following diagrams commute

\[ \begin{align*}
1 \otimes f(X) & \longrightarrow f(1) \otimes f(X) & f(X) \otimes 1 & \longrightarrow f(X) \otimes f(1) \\
f(X) & \longrightarrow f(1 \otimes X), & f(X) & \longrightarrow f(X \otimes 1).
\end{align*} \]

A.2. Symmetric monoidal categories.

Definition 85. A symmetric monoidal category is the data of a category $C$ equipped with a bifunctor $- \otimes - : C \times C \to C\newline$

an object $1$ and natural isomorphisms

\[ X \otimes (Y \otimes Z) \simeq (X \otimes Y) \otimes Z, \quad X \otimes Y \simeq Y \otimes X, \quad 1 \otimes X \simeq X, \quad X \otimes 1 \simeq X \]

called respectively the associator, the commutator, the left unitor and the right unitor and so that the following diagrams commute

\[ \begin{align*}
(U \otimes X) \otimes Y & \longrightarrow (U \otimes Y) \otimes Z & (X \otimes Y) \otimes Z & \longrightarrow (Y \otimes X) \otimes Z \\
(U \otimes (X \otimes Y)) \otimes Z & \longrightarrow U \otimes (X \otimes (Y \otimes Z)) & Z \otimes (X \otimes Y) & \longrightarrow Y \otimes (X \otimes Z) \\
U \otimes ((X \otimes Y) \otimes Z) & \longrightarrow U \otimes (X \otimes (Y \otimes Z)) & (Z \otimes X) \otimes Y & \longrightarrow Y \otimes (Z \otimes X) \\
X \otimes Y & \longrightarrow Y \otimes X & 1 \otimes X & \longrightarrow X \otimes 1 & (X \otimes 1) \otimes Y & \longrightarrow X \otimes (1 \otimes Y) \\
X \otimes Y & \longrightarrow Y \otimes X & 1 \otimes X & \longrightarrow X \otimes 1 & (X \otimes 1) \otimes Y & \longrightarrow X \otimes (1 \otimes Y) \\
\end{align*} \]

Definition 86. Given two symmetric monoidal categories $C, D$, a lax symmetric monoidal functor from $C$ to $D$ is the data of a functor $f : C \to D$ together with a natural transformation $f(X) \otimes f(Y) \to f(X \otimes Y)$

and a map $1 \to f(1)$ so that the following diagrams commute

\[ \begin{align*}
(f(X) \otimes f(Y)) \otimes f(Z) & \longrightarrow f(X) \otimes (f(Y) \otimes f(Z)) \\
f(X \otimes Y) \otimes f(Z) & \longrightarrow f(X) \otimes (f(Y) \otimes f(Z)) \\
f((X \otimes Y) \otimes Z) & \longrightarrow f(X \otimes (Y \otimes Z)) \\
f(X) \otimes f(Y) & \longrightarrow f(Y \otimes f(X)) \\
1 \otimes X & \longrightarrow X \otimes 1 & (X \otimes 1) \otimes Y & \longrightarrow X \otimes (1 \otimes Y) \\
\end{align*} \]
Definition 87. Given lax symmetric monoidal functors between symmetric monoidal categories $f, g : C \to D$, a monoidal natural transformation between them is a natural transformation between the functor $f, g$ so that the following diagrams commute

\[
\begin{array}{ccc}
  f(X) \otimes f(Y) & \longrightarrow & f(X \otimes Y) \\
  \downarrow & & \downarrow \\
  g(X) \otimes g(Y) & \longrightarrow & g(X \otimes Y)
\end{array}
\]

\[
\begin{array}{ccc}
  f(1) & \longrightarrow & f(1) \\
  \downarrow & & \downarrow \\
  g(1) & \longrightarrow & g(1)
\end{array}
\]

Appendix B. Category enriched tensored and cotensored over a monoidal category

B.1. Category tensored over a monoidal category.

Definition 88. A category $C$ is said to be tensored over a monoidal category $(E, \otimes, 1)$ if it is equipped with the structure of a lax $E$-module, that is there exists an oplax monoidal functor $E \to \text{Fun}(C, C)$.

Equivalently, there exists a bifunctor $- \boxtimes - : E \times C \to C$ together with natural transformations

\[
\begin{array}{ccc}
  1 \boxtimes X & \longrightarrow & X \\
  (V' \otimes V) \boxtimes X & \longrightarrow & V' \boxtimes (V \boxtimes X)
\end{array}
\]

that satisfy coherences.

Let $E$ and $F$ be two monoidal categories and let $G : E \to F$ be a lax monoidal functor.

Definition 89. Let $C$ and $D$ be respectively a lax $E$-module and a lax $F$-module. A $G$-strength on $F$ is the additional structure of a lax morphism of lax modules that is a natural morphism $G(V) \boxtimes F(X) \to F(V \boxtimes X)$ for any $V \in E$ and any $X \in C$, so that the following diagrams commute

\[
\begin{array}{ccc}
  (G(V) \otimes G(V')) \boxtimes F(X) & \longrightarrow & G(V) \boxtimes (G(V') \boxtimes F(X)) & \longrightarrow & G(V) \boxtimes F(V' \boxtimes X) \\
  \downarrow & & \downarrow & & \downarrow \\
  G(V \otimes V') \boxtimes F(X) & \longrightarrow & F((V \otimes V') \boxtimes X) & \longrightarrow & F(V \boxtimes (V' \boxtimes X))
\end{array}
\]

\[
\begin{array}{ccc}
  \mathbb{I} \boxtimes F(X) & \longrightarrow & G(1) \boxtimes F(X) \\
  \downarrow & & \downarrow \\
  F(X) & \longrightarrow & F(1 \boxtimes X)
\end{array}
\]

In the case where all the structural morphisms $G(V) \boxtimes F(X) \to F(V \boxtimes X)$ are isomorphisms, one talks about $G$-tensorial strength isomorphism. In the case where $G$ is the identity functor of $E$, one talks about a $E$-tensorial strength for $F$.

Let us consider two lax monoidal functors $G, G' : E \to F$ between monoidal categories, a monoidal natural transformation $G \to G'$, two categories $C$ and $D$ tensored over respectively $E$ and $F$ and two functors $F, F' : C \to D$. Let us suppose that $F$ is equipped with a $G$-strength and that $F'$ is equipped with a $G'$-strength.

Definition 90. A strong natural transformation from $F$ to $F'$, with respect to their strength and with respect to the monoidal natural transformation $G \to G'$ is a natural transformation so that the following diagram commutes

\[
\begin{array}{ccc}
  G(V) \boxtimes F(X) & \longrightarrow & F(V \boxtimes X) \\
  \downarrow & & \downarrow \\
  G'(V) \boxtimes F'(X) & \longrightarrow & F'(V \boxtimes X)
\end{array}
\]

for any $X \in C, V \in E$. 
B.2. Tensorisation, cotensorisation and enrichment.

**Definition 91.** For any monoidal category \((E, \otimes, 1)\), we denote by \(E^{tr} = (E, \otimes, 1)\) the transposed monoidal category of \(E\), that is the same category with the opposite monoidal structure
\[
X \otimes^{tr} Y := Y \otimes X.
\]

**Definition 92 (Category cotensored over a monoidal category).** Let \((E, \otimes, 1)\) be a monoidal category and let \(C\) be a category. We say that \(C\) is cotensored over \(E\) if there exists a bifunctor\(\langle - , - \rangle : C \times E^{op} \to C\) together with natural transformations\[
X \to \langle X, 1 \rangle,
\langle \langle X, V \rangle, V' \rangle \to \langle X, V \otimes V' \rangle,
\]
that makes the category \(C\) tensored over \(E^{op,tr} = (E^{op}, \otimes^{tr}, 1)\).

**Definition 93.** A category enriched over a monoidal category \(E\), (or \(E\)-category) \(C\) is the data of a set (possibly large) called the set of objects or the set of colours, an object \(\{X,Y\} \in E\) for any two colours \(X, Y\) and morphisms\[
\gamma : \{Y, X\} \otimes \{X, Y\} \to \{X, Z\};
\eta : 1 \to \{X, X\}
\]
for any three colours \(X, Y, Z\) that define a unital associative composition. Moreover, a morphism of \(E\)-categories from \(C\) to \(D\) is the data of a function on colours \(\phi\) and morphisms\[
\{X, Y\} \to \{\phi(X), \phi(Y)\}
\]
for any two colours \(X, Y\) of \(C\) that commute with units and compositions. This defines the category \(\text{Cat}_E\) of \(E\)-categories.

**Definition 94.** A category tensored-cotensored-enriched (TCE for short) over \(E\) is the data of a category \(C\) equipped with three bifunctors:
\[
\begin{align*}
- \boxtimes - : E \times C & \to C \\
\{ - , - \} : C^{op} \times C & \to E \\
\langle - , - \rangle : C \times E^{op} & \to C,
\end{align*}
\]
with natural isomorphisms,
\[
\hom_C(V \boxtimes X, Y) \simeq \hom_E(V, \{X, Y\}) \simeq \hom_C(X, \langle Y, V \rangle),
\]
and with
\[
\begin{itemize}
\item a structure of a tensorisation on \(- \boxtimes -\),
\item or equivalently a structure of a cotensorisation on \(\langle - , - \rangle\),
\item or equivalently a structure of an enrichment on \(\{ - , - \}\).
\end{itemize}
\]

Given a tensorisation \(- \boxtimes -\), the composition and the unit on \(\{ - , - \}\) are the adjoints maps of the morphisms\[
((\{Y, Z\} \otimes \{X, Y\})) \boxtimes X \to \{Y, Z\} \boxtimes ((\{X, Y\} \boxtimes X)) \to \{Y, Z\} \boxtimes Y \to Z,
1 \boxtimes X \to X.
\]

Given a cotensorisation \(\langle - , - \rangle\), the composition and the unit on \(\{ - , - \}\) are the adjoints maps of the morphisms of the map\[
X \to \langle Y, \{X, Y\} \rangle \to \langle \langle Z, \{Y, Z\} \rangle, \{X, Y\} \rangle \to \langle Y, \{Y, Z\} \otimes \{X, Y\} \rangle,
X \to \langle X, 1 \rangle.
\]
Conversely, given an enrichment \( \{ - , - \} \), the structural tensorisation morphisms for \( - \boxtimes - \) are adjoints to the maps

\[
V \otimes V' \to \{ V' \boxtimes X, V \otimes (V' \boxtimes X) \} \otimes \{ X, V \otimes (V' \boxtimes X) \} \to \{ X, X \}
\]

and the structural cotensorisation morphisms for \( \langle - , - \rangle \) are adjoints to the maps

\[
V \otimes V' \to \{ (X, V), X \} \otimes \{ (X, V), X \} \to \{ X, X \}
\]

Remark 17. Given an adjunction between monoidal categories

\[
E \xrightarrow{L} F \xleftarrow{R} E
\]

where \( L \) is monoidal (and then \( R \) is lax monoidal), then any category \( C \) TCE over \( F \) is also TCE over \( E \).

Definition 95. A biclosed monoidal category is a monoidal category \( (E, \otimes, 1) \) TCE over itself in a way so that the tensorisation is the tensor product of \( E \). If it is symmetric, one says simply that it is closed.

Notation. When dealing with a biclosed (or a closed) monoidal category, the enrichment will usually be denoted \( \{ - , - \} \).

B.3. Strength in the context of categories enriched-tensored and cotensored. Let \( G : E \to F \) be lax monoidal functor between monoidal categories. Moreover, let \( C \) and \( D \) be two categories TCE respectively over \( E \) and \( F \) and let \( F : C \to D \) be a functor.

A \( G \)-strength on \( F \) is equivalent to the data of natural morphisms \( F(\langle X, V \rangle) \to \langle F(X), G(V) \rangle \) for any \( V \in E \) and any \( X \in C \), so that the following diagrams commute

\[
\begin{array}{c}
F(\langle X, V \rangle) \to \langle F(\langle X, V \rangle), G(W) \rangle \\
\downarrow \quad \downarrow \\
F(\langle X, V \otimes W \rangle) \to \langle F(X), G(V \otimes W) \rangle \to \langle F(X), G(V) \otimes G(W) \rangle \\
\downarrow \quad \downarrow \\
F(\langle X, 1 \rangle) \to \langle F(X), G(1) \rangle \\
\downarrow \quad \downarrow \\
F(X) \to \langle F(X), 1 \rangle.
\end{array}
\]

It is also equivalent to the data of natural morphisms \( G(\{ X, Y \}) \to \{ FX, FY \} \) for any \( X, Y \in C \), so that the following diagrams commute

\[
\begin{array}{c}
G(\{ Y, Z \}) \otimes G(\{ X, Y \}) \to \{ FY, FZ \} \otimes \{ FX, FY \} \\
\downarrow \quad \downarrow \\
G(\{ Y, Z \} \otimes \{ X, Y \}) \to \{ FX, FX \} \leftarrow G(\{ X, X \}).
\end{array}
\]

Moreover, let us consider another lax monoidal functor \( G' : E \to F \) a monoidal natural transformation \( G \to G' \), and another functor \( F' : C \to D \) equipped with a \( G' \)-strength. Then, a natural transformation \( F \to F' \) is strong if and only if the following diagram commutes

\[
\begin{array}{c}
F(\langle X, V \rangle) \to \langle F(X), G(V) \rangle \\
\downarrow \quad \downarrow \\
F'(\langle X, V \rangle) \to \langle F'(X), G'(V) \rangle \to \langle F'(X), G(V) \rangle.
\end{array}
\]
for any $X \in C, V \in E$, if and only if the following diagram commutes
\[
\begin{array}{ccc}
G(\{X, Y\}) & \longrightarrow & \{F(X), F(Y)\} \\
\downarrow & & \\
G'(\{X, Y\}) & \longrightarrow & \{F'(X), F'(Y)\}
\end{array}
\]
for any $X, Y \in C$.

**B.4. Strong adjunctions.** Let us consider an adjunction
\[
\begin{array}{ccc}
C & \xrightarrow{L} & D. \\
\downarrow & & \\
R & \xleftarrow{R} & C
\end{array}
\]
We suppose that $C$ and $D$ are TCE over a monoidal category $E$ and $F$.

**Definition 96.** The adjunction $L \dashv R$ is said to be strong if $L$ and $R$ are equipped with strengths with respect to $E$ so that the natural transformations $\text{Id} \rightarrow RL$ and $LR \rightarrow \text{Id}$ are strong.

Equivalently, by doctrinal adjunction, $L \dashv R$ is strong if $L$ is equipped with a strength so that the natural morphism
\[
V \otimes L(X) \rightarrow L(V \otimes X)
\]
is an isomorphism for any $V, X \in E \times C$. Then, the strength on $R$ is given by the formula
\[
V \otimes R(Y) \rightarrow RL(V \otimes R(Y)) \simeq R(V \otimes LR(Y)) \rightarrow R(V \otimes Y).
\]

One has also an adjunction relating the opposite categories
\[
\begin{array}{ccc}
D^\text{op} & \xrightarrow{R^\text{op}} & C^\text{op} \\
\downarrow & & \\
L^\text{op} & \xleftarrow{L^\text{op}} & D^\text{op}
\end{array}
\]
The cotensorisation of $C$ and $D$ over $E$ are actually tensorisation of $C^\text{op}$ and $D^\text{op}$ over $E^\text{tr}$.

**Proposition 35.** The structure of a strong adjunction on $L \dashv R$ with respect the tensorisation of $C$ and $D$ over $E$ is equivalent to the structure on a strong adjunction on $R^\text{op} \dashv L^\text{op}$ with respect the tensorisation of $D^\text{op}$ and $C^\text{op}$ over $E^\text{tr}$.

**Proof.** In Subsection B.3 we saw that we have a one to one correspondence between the strengths on $L$ (resp. $R$) with respect to $E$ and the strengths on $L^\text{op}$ (resp. $R^\text{op}$) with respect to $E^\text{tr}$. Then, given a pair of strength on $L$ and $R$, the natural transformations $\text{Id} \rightarrow RL$ and $LR \rightarrow \text{Id}$ are strong if and only if the natural transformations $\text{Id} \rightarrow L^\text{op}R^\text{op}$ and $R^\text{op}L^\text{op} \rightarrow \text{Id}$ are strong with respect to the induced strength on $L^\text{op}$ and $R^\text{op}$. □

**Proposition 36.** Let us suppose that the functors $L$ and $R$ are equipped with strengths. Then, the following assertions are equivalent.

1. the natural transformations $\text{Id} \rightarrow RL$ and $LR \rightarrow \text{Id}$ are strong (hence, the adjunction $L \dashv R$ is strong);
2. the natural maps
\[
\begin{array}{ccc}
\{LX, Y\} & \rightarrow & \{RLX, RY\} \\
\downarrow & & \\
\{X', Y'\} & \rightarrow & \{LX', LY'\}
\end{array}
\]
are isomorphisms inverse to each other for any $X, Y \in C \times D$.

**Proof.** We know from Subsection B.3 that the fact that the natural transformations $\text{Id} \rightarrow RL$ and $LR \rightarrow \text{Id}$ are strong is equivalent to the commutation of the following two diagrams
\[
\begin{array}{ccc}
\{X, Y\} & \longrightarrow & \{RX, RY\} \\
\downarrow & & \\
\{LRX, Y\} & \longleftarrow & \{LRX, LRY\}
\end{array}
\]
for any $X, X', Y, Y'$.
Let us suppose (1). Using the commutation of the diagram just above, it is straightforward to prove that the composite maps
\[
\{LX, Y\} \to \{X, RY\} \to \{LX, Y\},
\]
\[
\{X, RY\} \to \{LX, Y\} \to \{X, RY\},
\]
are identities.

Conversely, let us suppose (2). By naturality, the two following diagrams are commutative
\[
\begin{array}{ccc}
X, Y & \to & RLX, Y \\
\downarrow & & \downarrow \\
LRX, Y & \to & RX, RY
\end{array}
\]
\[
\begin{array}{ccc}
X, Y' & \to & RLX, Y' \\
\downarrow & & \downarrow \\
LRX, Y' & \to & RX, RY'
\end{array}
\]
for any \(X, X', Y, Y'\). Then, the commutation of the two previous squares follows from the fact that the maps
\[
\{LRX, Y\} \to \{RX, RY\}
\]
\[
\{X', RLY'\} \to \{LX', LRY'\}
\]
are respective inverses of the maps that appear in these previous diagrams. □

B.5. Homotopical enrichment.

Definition 97 (Homotopical enrichment). Let \(M\) be a model category and let \(E\) be a monoidal model category. We say that \(M\) is homotopically TCE over \(E\) if it TCE over \(E\) and if for any cofibration \(f : X \to X'\) in \(M\) and any fibration \(g : Y \to Y'\) in \(M\), the morphism in \(E\):
\[
\{X', Y\} \to \{X', Y\} \times_{\{X, Y\}} \{X, Y\}
\]
is a fibration. Moreover, we require this morphism to be a weak equivalence whenever \(f\) or \(g\) is a weak equivalence.

This is equivalent to the fact that for any cofibration \(f : X \to Y\) in \(M\) and any cofibration \(g : V \to W\) in \(E\), the morphism
\[
V \boxtimes Y \coprod_{V \boxtimes X} W \boxtimes X \to W \boxtimes Y
\]
is a cofibration and it is acyclic whenever \(f\) or \(g\) is acyclic. This is also equivalent to the fact that for any fibration \(f : X \to Y\) in \(M\) and any cofibration \(g : V \to W\) in \(E\), the morphism
\[
\langle X, W \rangle \to \langle X, V \rangle \times_{\langle Y, V \rangle} \langle Y, W \rangle
\]
is a fibration and it is acyclic whenever \(f\) or \(g\) is acyclic.

Appendix C. Monads, comonads, limits and colimits

Proposition 37. Let \(M\) be a monad on a category \(C\). Then, the functor \(UM\) preserves and creates limits that may exist in \(C\).

Proof. The functor \(UM\) preserves limits since it is right adjoint. Moreover, it reflects limits since it is conservative. Finally, for any diagram \(D : I \to \text{Alg}_C(M)\), if \(UM \circ D\) has a limit, then this limit has the structure of a \(M\)-algebra given by
\[
M(\lim UMD) \to \lim UM^D = \lim UMT_MUMD \to \lim UMD.
\]
This \(M\)-algebra is the limit of the diagram \(D\). □

Corollary 14. Let \(Q\) be a comonad on a category \(C\). Then, the functor \(UQ\) preserves and creates colimits that may exist in \(C\).

Proposition 38. Let \(M\) be a monad on a category \(C\). Let us suppose \(C\) has all reflexive coequalisers and that \(M\) preserves these reflexive coequalisers. Then the category of \(M\)-algebras has all reflexive coequalisers and these are preserved by the functor \(UM\).

Proof. Let \(I\) be the category generated by
two objects 0 and 1;
▷ three nontrivial morphisms $f, g : 0 \to 1$ and $h : 1 \to 0$

with the relation $fs = gs = \text{Id}_1$. For any diagram $D : I \to \text{Alg}_C(M)$, let $A$ be the colimit of the functor $U^M \circ D$. Then, $A$ has the structure of a $M$-algebra as follows

$$M(A) = M(\text{colim} U^M \circ D) \simeq \text{colim}(U^M U^M \circ D) \to \text{colim} U^M \circ D = A.$$  

A straightforward check shows that this defines the structure of a $M$-algebra on $A$ which gives the colimit of the diagram $D$. It is then clear that $U^M$ preserves reflexive coequalisers.

**Proposition 39.** In the context of Proposition 38, let us suppose that $C$ is cocomplete. Then the category of $M$-algebras is cocomplete.

**Proof.** Given a family of $M$-algebras $(A_i)_{i \in I}$, their coproduct is given by the reflexive coequaliser of the following diagram

$$T_M(\coprod_i M(A_i)) \rightrightarrows T_M(\prod_i A_i).$$

It is enough to have all reflexive coequalisers and all coproducts to be cocomplete. □

**Corollary 15.** Let $Q$ be a comonad on a category $C$. Let us suppose $C$ has all coreflexive equalisers and that $Q$ preserves these coreflexive equalisers. Then the category of $Q$-coalgebras has all coreflexive equalisers and these are preserved by the functor $U_Q$.

**Corollary 16.** In the context of Corollary 15, let us suppose that $C$ is complete. Then the category of $Q$-coalgebras is complete.

### Appendix D. Trees

**Definition 98.** A tree (also called planar tree) $t = (\text{edges}(t), \text{leaves}(t))$ is the data of a nonempty finite set $\text{edges}(t)$ called the set of edges of $t$ and a subset $\text{leaves}(t)$ called the set of leaves of $t$ together with two partial order on $\text{edges}(t)$

▷ the height order $\leq$, that has a minimal element called the root, so that leaves are maximal elements, and so that for any edge $e$ the poset $\text{edges}(t)_{\leq e} = \{e' \leq e\}$ is linear;
▷ the planar order $\leq_{pl}$ that is a linear order and so that $e' \leq e \implies e' \leq_{pl} e$.

**Definition 99.** Given an edge $e$ of a tree $t$, the descendants of $e$ are the edges $e'$ so that $e' > e$. The children of $e$ are its immediate descendants, that is the edges $e' > e$ so that for any other edge $e''$:

$$e' \geq e'' > e \implies e' = e''.$$

**Definition 100.** A subtree $t'$ of a tree $t$ is the data of

▷ non empty subset $\text{edges}(t') \subseteq \text{edges}(t)$ that has a minimal element (not necessarily the root of $t$) and so that for any of its element $e$, if it contains a descendant of $e$, then it contains all its children;
▷ a subset $\text{leaves}(t')$ of the set of maximal elements of edges($t'$) that contains the intersection $\text{edges}(t') \cap \text{leaves}(t')$ and that contains all the maximal elements of leaves($t'$) that are not maximal in leaves($t$).

A subtree is in particular a tree.

A node of a tree $t$ is a subtree that contains just a non leaf edge $e$ and its children which become the leaves of the node.
The following picture represents a tree whose set of edges is 
\{r, e_1, e_2, e_3, l_1, l_2, l_3, l_4\} 
whose root is \(r\) and whose set of leaves is \{l_1, l_2, l_3, l_4\}. The height order is given by the relations
\begin{align*}
l_1, l_2 &> e_1; \\
e_3, l_4 &> e_2; \\
e_1, l_3, e_2 &> r.
\end{align*}
The planar order is
\( r < e_1 < l_1 < l_2 < l_3 < e_2 < e_3 < l_4. \)
The nodes are represented using dots.

**Definition 101.** Let \( n \) be a natural integer. An \( n \) corolla is a tree whose set of edges contains exactly the root and \( n \) leaves.

**Definition 102.** Let \( t \) be a tree and let \( e \) be an edge. The height of \( e \) in \( t \) is the cardinal of the set 
\( \{e' < e\} \).
In particular the height of the root is zero. Then, the height of a node in \( t \) is the height of its root. Finally, the height of the tree \( t \) is the maximal height of its nodes plus one 
\( \text{height}(t) = \max_{\text{node}} \text{height}(n) + 1. \)
If \( t \) has no node, then its height is zero.

**Definition 103.** An isomorphism of trees from \( t \) to \( t' \) is the data of an isomorphism
\( \phi : \text{edges}(t) \simeq \text{edges}(t') \)
that sends leaves to leaves and that preserves the height order. This defines the groupoid of trees.

**Definition 104.** A planar isomorphism of trees from \( t \) to \( t' \) is an isomorphism of trees that also preserves the planar order. This defines the groupoid of planar trees.

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