DECAY OF THE BOLTZMANN EQUATION WITH THE SPECULAR BOUNDARY CONDITION IN NON-CONVEX CYLINDRICAL DOMAINS

CHANWOO KIM AND DONGHYUN LEE

Abstract. A basic question about the existence and stability of the Boltzmann equation in general non-convex domain with the specular reflection boundary condition has been widely open. In this paper, we consider cylindrical domains whose cross sections are general non-convex analytic planar domain. We establish the global-wellposedness and asymptotic stability of the Boltzmann equation with the specular reflection boundary condition in such domains. Our method consists of sharp classification of billiard trajectories which bounce infinitely many times or hit the boundary tangentially at some moment, and a delicate construction of an $\varepsilon$-tubular neighborhood of such trajectories. Analyticity of the boundary is crucially used. Away from such $\varepsilon$-tubular neighborhood, we control the number of bounces of trajectories and its’ distance from singular sets in a uniform fashion. The worst case, sticky grazing set, can be excluded by cutting off small portion of the temporal integration. Finally we apply a method of [14] by the authors and achieve a pointwise estimate of the Boltzmann solutions.

Contents

1. Introduction
  1.1. Uniform number of bounce on analytic domain
  1.2. Sticky grazing set on analytic domain
  1.3. $L^p-L^\infty$ bootstrap and double iteration
2. Domain decomposition and notations
  2.1. Analytic non-convex domain and notations for trajectory
  2.2. Decomposition of the grazing set and the boundary $\partial\Omega$
3. $L^\infty$ estimate
  3.1. Inflection grazing set
  3.2. Dichotomy of sticky grazing
  3.3. Grazing set
  3.4. Transversality and double Duhamel trajectory
4. $L^2$-Coercivity via contradiction method
5. Linear and Nonlinear decay
  5.1. Linear $L^2$ decay
  5.2. Nonlinear $L^\infty$ decay
6. Appendix: Example of sticky grazing point
References

1. Introduction

The Boltzmann equation is a mathematical model for dilute gas which describes a probability density function of particles. In addition to free transport of a particle, a collision effect is also considered. If there is no external force or self-generating force, probability density function $F(t,x,v)$ is governed by

$$
\partial_t F + v \cdot \nabla_x F = Q(F,F), \quad F(0,x,v) = F_0(x,v),
$$

where the position $x \in U \subset \mathbb{R}^3$ and velocity $v \in \mathbb{R}^3$ at time $t \geq 0$. The collision operator $Q(F_1,F_2)$ takes the form of

$$
Q(F_1,F_2) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v-u,\omega) \left[ F_1(u')F_2(v') - F_1(u)F_2(v) \right] d\omega du,
$$

where $u' = u + (v-u)\cdot\omega$, $v' = v - (v-u)\cdot\omega$. For collision kernel, we choose so-called the hard sphere model $B(v-u,\omega) = |(v-u)\cdot\omega|$. We study (1.1) when $F$ is near the Maxwellian $\mu = e^{-\frac{|v|^2}{2}}$. 

Date: November 8, 2018.
When the gas contacts with the boundary, we need to impose boundary condition for $F$ on $\partial U$, the boundary of the domain $U$. In this paper, we impose the specular reflection boundary condition, which is one of the most basic conditions

$$F(t, x, v) = F(t, x, R_xv), \quad x \in \partial U,$$

where $R_x := I - 2n(x) \otimes n(x)$ and $n(x)$ is the outward unit normal vector at $x \in \partial U$. Note that the Maxwellian is an equilibrium state (or a steady solution) of (1.1) with (1.2).

Despite extensive developments in the study of the Boltzmann theory, many basic boundary problems, especially regarding the specular reflection BC with general domains, have remained open. In 1977, in [15], Shizuta and Asano announced the global existence of the Boltzmann equation with the specular boundary condition in smooth convex domain without a complete proof. The first mathematical proof of such problem was given by Guo in [9], but with a strong extra assumption that the boundary should be a level set of a real analytic function. Very recently the authors proved the unique existence and asymptotic stability of the specular boundary problem for general smooth convex domains (with or without external potential) in [14], using triple iteration method and geometric decomposition of particle trajectories. This marks a complete resolution of a 40-years open question after [15].

Meanwhile, there were even fewer results for general non-convex domains with the specular boundary condition. An asymptotic stability of the global Maxwellian is established in [3], provided certain a-priori strong Sobolev estimates can be verified. However, such strong estimates seem to fail especially when the domain is non-convex (13, 8, 11). Actually we believe that the solution cannot be in $C^1$ (but in $C^{0,\alpha}$) when the domain is non-convex. To the best of our knowledge, our work is the first result on the global well-posedness and decay toward Maxwellian results for any kind of non-convex domains with the specular boundary condition! One of the intrinsic difficulties of the non-convex domain problem is the (billiard) trajectory is very complicated to control (e.g. infinite bouncing, grazing). The problems of general smooth non-convex domains or three-dimensional non-convex domains are still open.

In the case of the specular reflection boundary condition, we have the total mass and energy conservations as

$$\iint_{U \times \mathbb{R}^3} F(t) \, dt = \iint_{U \times \mathbb{R}^3} F_0, \quad \iint_{U \times \mathbb{R}^3} \frac{|v|^2}{2} F(t) = \iint_{U \times \mathbb{R}^3} \frac{|v|^2}{2} F_0. \quad (1.3)$$

By normalization, we assume that

$$\iint_{U \times \mathbb{R}^3} F_0(x, v) = \iint_{U \times \mathbb{R}^3} \mu, \quad \iint_{U \times \mathbb{R}^3} \frac{|v|^2}{2} F_0(x, v) = \iint_{U \times \mathbb{R}^3} \frac{|v|^2}{2} \mu. \quad (1.4)$$

In general, the total momentum is not conserved. However, in the case of axis-symmetric domains, we have an angular momentum conservation, i.e. if there exist a vector $x_0$ and an angular velocity $\varpi$ such that

$$\{(x - x_0) \times \varpi\} \cdot n(x) = 0 \quad \text{for all } x \in \partial U,$$

then we have a conservation of the angular momentum as

$$\iint_{U \times \mathbb{R}^3} \{(x - x_0) \times \varpi\} \cdot vF(t) = \iint_{U \times \mathbb{R}^3} \{(x - x_0) \times \varpi\} \cdot vF_0. \quad (1.6)$$

In this case, we assume

$$\iint_{U \times \mathbb{R}^3} \{(x - x_0) \times \varpi\} \cdot vF_0(x, v) = 0. \quad (1.7)$$

In this paper, we deal with periodic cylindrical domain with non-convex analytic cross section. A domain $U$ is given by

$$U = \Omega \times [0, H], \quad (x_1, x_3) \in \Omega \quad \text{and} \quad x_2 \in [0, H] \quad \text{for} \ (x_1, x_2, x_3) \in U \quad (1.8)$$

where $\Omega \subset \mathbb{R}^2$ is the cross section. See Figure 1. We assume that $F$ is periodic in $x_2$, i.e. $F(t, (x_1, 0, x_3), v) = F(t, (x_1, H, x_3), v)$. For the boundary of $U$, we denote $\partial U := \partial \Omega \times [0, H]$. We are interested in non-convex analytic cross section $\Omega \subset \mathbb{R}^2$.

**Definition 1.** Let $\Omega \subset \mathbb{R}^2$ be an open connected bounded domain and there exist simply connected subsets $\Omega_i \subset \mathbb{R}^2$, for $i = 0, 1, 2, \ldots, M < \infty$ such that

$$\Omega = \Omega_0 \backslash \bigcup_{i=0}^M \Omega_i,$$

where

1. $\Omega_i \supset \Omega_j$ for all $i = 1, 2, \ldots, M$, and $\partial \Omega_i \cap \partial \Omega_j = \emptyset$ for all $i \neq j$ and $i, j = 0, 1, 2, \ldots, M$,
2. for each $\Omega_i$, there is a closed regular analytic curve $\alpha_i : [a_i, b_i] \to \mathbb{R}^2$ such that $\partial \Omega_i$ is an image of $\alpha_i$.
3. $\partial \Omega = \bigcup_{i=0}^M \partial \Omega_i$,

where $\bigcup$ means disjoint union.
Theorem 1. Let $w(v) = (1 + |v|)^\beta$ with $\beta > \frac{5}{2}$. We assume periodic cylindrical domain $U$ defined in (1.8), where analytic non-convex cross section with punctures $\Omega$ is defined in Definition 1. We assume (1.4) and also assume (1.7) if the cross section $\Omega$ is axis-symmetric (1.5). Then, there exist $0 < \delta \ll 1$ such that if $F_0 = \mu + \sqrt{\mu} f_0 \geq 0$ and $\|w f_0\|_\infty < \delta$, then the Boltzmann equation (1.1) with the specular BC (1.2) has a unique global solution $F(t) = \mu + \sqrt{\mu} f(t) \geq 0$. Moreover, there exist $\lambda > 0$ such that $\sup_{t \geq 0} e^{\lambda t} \|w f(t)\|_\infty \lesssim \|w f_0\|_\infty$, with conservations (1.3). In the case of axis-symmetric domain (1.5), we have additional angular momentum conservation (1.6).

From (1.1), the perturbation $f$ satisfies
\begin{equation}
\partial_t f + v \cdot \nabla_x f + L f = \Gamma(f, f),
\end{equation}
and $f(t, x, v) = f(t, x, R_x v)$, for $x \in \partial U$ where
\begin{equation}
L f = -\frac{1}{\sqrt{\mu}} [Q(\mu, f \sqrt{\mu}) + Q(f \sqrt{\mu}, \mu)], \quad \Gamma(f, f) = \frac{1}{\sqrt{\mu}} Q(f \sqrt{\mu}, f \sqrt{\mu}).
\end{equation}
The linear operator $L f$ can be decomposed into $L f = \nu(v) f - K f$, where the collisional frequency $\nu(v)$ is defined
\begin{equation}
\nu(v) := \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} |(v - u) \cdot \omega| \sqrt{\mu}(u) d\omega du,
\end{equation}
with estimate $C_0(v) \leq \nu(v) \leq C_1(v)$, where $\langle v \rangle := \sqrt{1 + |v|^2}$ for some $C_0, C_1 > 0$. The linear operator $K f$ is a compact operator on $L^2(\mathbb{R}^3)$ with kernel $k(v, \cdot)$,
\begin{equation}
K f(v) := \int_{\mathbb{R}^3} k(v, u) f(u) du.
\end{equation}

We explain main scheme of the proof of Theorem 1. To apply $L^p - L^\infty$ bootstrap argument, we claim the uniform number of bounce for a finite travel length. Also, we classify some singular sets especially where trajectories belong to grazing sets on the boundary.

1.1. Uniform number of bounce on analytic domain. Let us denote backward trajectory of a particle as $X(s; t, x, v)$ and $V(s; t, x, v)$, where $X$ and $V$ are position and velocity of the particle at time $s$, which was at position $x$ with velocity $v$ at time $t \geq s$. Also we use $(t^k, x^k, v^k) = (t^k(x, x), x^k(x, x), v^k(x, x))$ to denote $k$-th bouncing time, position, and velocity backward in time. From the specular BC, dynamics in $x_2$ direction (axial direction) is very simple, because we have $X_2(s; t, x, v) = x_2 - (t - s)v_2$ and $V_2(s; t, x, v) = v_2$. So we suffice to analyze trajectory projected onto two-dimensional cross section $\Omega$, with $(x_1, x_3) \in \Omega$. We also consider finite time interval $[0, T_0]$ and...
velocity \( v \in \mathbb{R}^2 \) with \( \frac{1}{N} \leq |v| \leq N \) so that maximal travel length is uniform bounded by \( NT_0 \). Unlike to strictly convex domain, trajectory \((X(s), V(s))\) can graze at some bouncing time \( t_k \). We split grazing set \( \{(t^k, x^k, v^k) : v^k \cdot n(x^k) = 0\} \) into three types: convex grazing, concave grazing, and inflection grazing, depending on whether \( x^k \in \partial \Omega \) belongs to convex region, concave region, and inflection points. See Definition \ref{def:grazing} for explicit definitions. The following simplified lemma is the crucial tool to control the number of bounce.

**Simple version of Lemma 2** If a trajectory does not belong to inflection grazing set, infinite number of bouncing cannot happen for a finite travel length.

We prove this lemma via contradiction argument. If infinite number of bounce happens for a finite travel length, we have converging sequence of boundary points \( x^k \to x^\infty \). By analyticity, all inflection points on the boundary \( \partial \Omega \) are finite and distinct.

(i) If \( x^\infty \) is a point in convex or concave part of \( \partial \Omega \), we can choose small boundary neighborhood \( B(x^\infty, \varepsilon) \cap \partial \Omega \) so that boundary is uniformly convex or concave in the neighborhood. If it is concave, trajectory does not stay in this small neighborhood. If it is uniformly convex, it is well-known that normal component of \( v^k \) is always uniformly comparable and there exist at most finite number of bounce for a finite time interval (or equivalently finite travel length). We refer \cite{14} and \cite{9}.

(ii) Therefore, the only possible case is when \( x^\infty \) is an inflection point. By analyticity, every inflection points are isolated and we consider a sequence \( x^k \) which converges to \( x^\infty \) through convex region, because the trajectory leave the small neighborhood if it is in concave region. Using analyticity and properties of inflection points, profile of \( x^\infty \) near inflection points is nearly linear, i.e. \( \alpha''(\tau^\infty) = 0 \). Using this linear property, we obtain \( |x^{k-1} - x^k| \leq |x^k - x^{k+1}| \) which is contradiction to our assumption \( x^k \to x^\infty \). See the first picture in Figure 2.

Above Simple version of Lemma 2 is in sharp contrast to non-analytic general smooth domain, where infinite bounce in finite travel length is possible. We refer section 3 in \cite{10} for an example of infinite number of bounce for finite travel length.

Meanwhile, a trajectory with the specular boundary condition is always deterministic and we can collect all possible trajectories including inflection grazing set. For each points on these trajectories, we uniformly cut corresponding velocities off. From compactness argument, we can define infinite bouncing set

\[
\mathcal{IB} = \bigcup_{i=1}^{ib} \{B(x_1^i, r_1^i) \times O_i^{IB}\},
\]

where \( \bigcup_{i=1}^{ib} B(x_1^i, r_1^i) \) is an open cover for \( \Omega \), and corresponding open sets \( O_i^{IB} \) are sufficiently small in velocity phase. Moreover, the trajectory from \((x, v) \in \{c(l) \times \{\frac{1}{N} \leq |v| \leq N\}\} \setminus \mathcal{IB} \) is uniformly away from grazing bounce, where \( c(l) \) means closure of \( l \) in standard \( \mathbb{R}^2 \) topology.

Since we excluded inflection grazing and convex grazing, the only possibility is concave grazing. When a trajectory has concave grazing, \((t^k, x^k, v^k)\) is not continuous function of \((x, v)\). However, away from grazing points, the trajectory is always continuous in \((x, v)\). Therefore, for small perturbation \(|(y, u) - (x, v)| < 1\), bouncing phase \((x^k(y, u), v^k(y, u))\) must be very close to some \((x^\ell(x, v), v^\ell(x, v))\), where \( \ell \geq k \) can be different to \( k \) by multiple concave grazings (e.g. Figure 3), and this implies finite number of bouncing. At last, from compactness of \( \{c(l) \times \{\frac{1}{N} \leq |v| \leq N\}\} \setminus \mathcal{IB} \), we derive the uniform number of bounce for given finite travel length.

### 1.2. Sticky grazing set on analytic domain.

For concave grazing set, we should consider another type of singular points. Let \( \{\alpha(\tau) : \tau_1 < \tau < \tau_2\} \) be a local parametrization for concave boundary. Then \((\alpha(\tau), \alpha'(\tau))\) belongs to concave grazing set. Let us consider a set of trajectory

\[
X(s; t, \alpha(\tau), \alpha'(\tau)), V(s; t, \alpha(\tau), \alpha'(\tau)), \quad \tau_1 < \tau < \tau_2.
\]

If \( (x^1(\alpha(\tau), \alpha'(\tau)), v^1(\alpha(\tau), \alpha'(\tau))) \) is not grazing phase, we can use rigidity of analytic function to show that **there could be a fixed point** \( x_1^1 \in c(l) \) (we call \( x_1^1 \) a sticky grazing point) such that

\[
x_1^1 \in x_1^1(\alpha(\tau), \alpha'(\tau))x_2(\alpha(\tau), \alpha'(\tau)), \quad \text{for all } \tau \in (\tau_1, \tau_2).
\]

See second picture in Figure 2. Also see Appendix for a concrete example of sticky grazing point. This implies that backward trajectory from

\[
(x_1^1, -v^1(\alpha(\tau), \alpha'(\tau))),
\]

has grazing phase in the second bouncing for all \( \tau_1 < \tau < \tau_2 \).

Similarly, for each \( k \), if \( (x^i(\alpha(\tau), \alpha'(\tau)), v^i(\alpha(\tau), \alpha'(\tau))) \) is not grazing for all \( 1 \leq i \leq k \), there could be a sticky grazing point \( x_k^k \in c(l) \) such that

\[
x_k^k \in x_k^k(\alpha(\tau), \alpha'(\tau)x^{k+1}(\alpha(\tau), \alpha'(\tau)), \quad \text{for all } \tau \in (\tau_1, \tau_2).
\]
Now, we pick a point $x$ in
$$\{ x \in cl(\Omega) : (x,v) \in cl(\Omega) \times \{ \frac{1}{N} \leq |v| \leq N \} \} \setminus \mathcal{B},$$
for some $v \in \{ \frac{1}{N} \leq |v| \leq N \}$.

For above $x$ and $v \in \{ \frac{1}{N} \leq |v| \leq N \}$, we have at most finite $K$ bounces for fixed travel length, so we can uniformly exclude a set of velocity so that trajectory $(X(s,t,x), V(s,t,x,v))$ avoids all concave grazings. Then the trajectory avoid all three types of grazing. Moreover, from the uniform number of bounce, set of all possible sticky grazing points, $\mathcal{S}G$, contains finite points at most. Excluding all small neighborhoods of the points in $\mathcal{S}G$ from (1.13), we can state the following lemma:

**Simple version of Lemma**\[Excluding uniform $\varepsilon$ neighborhood of $\mathcal{S}G$, let us consider\]

$$\bigcup_{x \in \mathcal{S}G} B(x,\varepsilon).$$

Then we have a finite open cover $\bigcup_{i=1}^{\mathcal{I}C} B(x^C_i, r^C_i)$ and corresponding small velocity sets $\{ \mathcal{O}_i^{IB} \}_{i=1}^{\mathcal{I}C}$ such that if

$$(x,v) \in B(x^C_i, r^C_i) \times \{ v : \frac{1}{N} \leq |v| \leq N \} \setminus \mathcal{O}_i^{IB},$$

then $(x^k(x,v), v^k(x,v))$ is uniformly non-grazing, i.e.

$$|v^k(x,v) \cdot n(x^k(x,v))| > \delta > 0, \quad \text{for all} \quad 1 \leq k \leq K,$$

where $K$ is uniformly finite number of bounce.

### 1.3. $L^p - L^\infty$ bootstrap and double iteration.

We consider linear Boltzmann equation,

$$\partial_t f + v \cdot \nabla_x f + v(v)f = Kf,$$  \hspace{1cm} (1.14)

with the specular boundary condition. To apply $L^p - L^\infty$ bootstrap argument our aim is to claim

$$\| f \|_{L^\infty} \lesssim \| f_0 \|_{L^\infty} + \int_0^T \| f \|_{L^2}.$$  \hspace{1cm} (1.15)

This is obtained by trajectory analysis and change of variable. Let us explain using simplified version of (1.14),

$$\partial_t f + v \cdot \nabla_x f + f = \int_{|u| \leq N} f du, \quad x \in \Omega.$$  \hspace{1cm} (1.16)

In the aspect of transport, $\partial_t f + v \cdot \nabla_x f$ means simple transport of $f$ along trajectory and $f$ on the LHS means exponential decay effect along the trajectory. Therefore, Duhamel’s principle gives

$$f(t,x,v) = e^{-t} f_0(x,v) + \int_0^t e^{-(t-s)} \int_{|u| \leq N} f(s, X(s;t,x,v), u) du ds.$$
Applying this formula again (double iteration) to \(f(s, X(s; t, x, v), u)\), we get
\[
f(t, x, v) = \text{initial datum’s contributions} + O\left(\frac{1}{N}\right) + \int_0^t e^{-(t-s)} \int_0^{\alpha - \varepsilon} e^{-(s-s')} \int_{|u| \leq N, |v|' \leq N} f(s', X(s'; s, X(s; t, x, v), u), u') du'duds'ds.
\]
The key step is to prove that the change of variable from \(u\) to \(X(s'; s, X(s; t, x, v), u)\) is valid. We apply geometric decomposition of trajectories as introduced in [14] to study a Jacobian marix of
\[
\frac{X(s')}{du}, \quad \text{where } X(s') = X(s'; s, X(s), u) \quad \text{and} \quad X(s) = X(s; t, x, v),
\]
We note that direct computations in Lemma 12 in this paper is quite similar as Lemma 2.3 in [14], because Lemma 12 assumes non-grazing bounce of a trajectory. Then Lemma 12 can be used to compute Jacobian of \(\frac{X(s')}{du}\). Note that, unlike to triple iteration scheme in [14], we suffice to perform double iteration. Since we are assuming cylindrical domains, dynamics of axial component is very simple and \(\frac{X(s')}{du}\) gives rank one clearly. Then the problem is changed into claiming rank two in two-dimensional cross section \(\Omega\). By decomposing \((u_1, u_3)\) into speed and another independent directional variable, we can claim rank two.

Another main difference between strictly convex case (e.g. [9] and [14]) and non-convex case is the existence of sticky grazing set \(SG\). When trajectory \(X(s')\) hits boundary \(\partial \Omega\), we cannot perform change of variable so we should exclude such points, which should be chosen sufficiently small. However, for sticky grazing point \(x \in SG\), non-small portion of velocity phase should be excluded. Therefore, we cannot cut this bad set off in velocity phase. Instead, we exclude \(SG\) using small intervals in temporal integration. Because \(SG\) contains finite set, if a particle speed is uniformly nonzero, say \(|v| \geq \frac{1}{r}\), then we can choose sufficiently small neighborhoods near points in \(SG\) so that a trajectory stays in these small neighborhoods of \(SG\) only for very short time at most.

2. Domain decomposition and notations

2.1. Analytic non-convex domain and notations for trajectory. Throughout this paper, cross section \(\Omega\) is a connected and bounded open subset in \(\mathbb{R}^2\). In this section, we denote the spatial variable \(x = (x_1, x_3) \in \text{cl}(\Omega) \subset \mathbb{R}^2\), where \(\text{cl}(\Omega)\) denotes the closure of \(\Omega\) in the standard topology of \(\mathbb{R}^2\), and the velocity variable \(v = (v_1, v_3) \in \mathbb{R}^2\). We also define standard inner product using dot product notation: \(a \cdot b := (a_1, a_3) \cdot (b_1, b_3) = a_1b_1 + a_3b_3\).

The cross section boundary \(\partial \Omega\) is a local image of some smooth regular curve. More precisely, for each \(x \in \partial \Omega\), there exists \(r > 0\) and \(\delta_1 < 0 < \delta_2\) and a curve \(\alpha := (\alpha_1, \alpha_3) : \{\tau \in \mathbb{R} : \delta_1 < \tau < \delta_2\} \to \mathbb{R}^2\) such that
\[
\partial \Omega \cap B(x, r) = \{\alpha(\tau) \in \mathbb{R}^2 : \tau \in (\delta_1, \delta_2)\}, \quad (2.1)
\]
where \(B(x, r) := \{y \in \mathbb{R}^2 : |y - x| < r\}\) and \(|\dot{\alpha}(\tau)| = |(\dot{\alpha}_1(\tau))^2 + (\dot{\alpha}_3(\tau))^2|^{1/2} = \left[\left(\frac{d\alpha_1(\tau)}{d\tau}\right)^2 + \left(\frac{d\alpha_3(\tau)}{d\tau}\right)^2\right]^{1/2} \neq 0\), for all \(\tau \in (\delta_1, \delta_2)\). Without loss of generality, we can assume that \(\alpha(\tau)\) is regularly parametrized curve, i.e. 
\(|\dot{\alpha}(\tau)| = 1\). For a smooth regularized curve \(\alpha(\tau) = (\alpha_1(\tau), \alpha_3(\tau)) \in \mathbb{R}^2\), we define the signed curvature of \(\alpha\) at \(\tau\) by
\[
k(\tau) := \ddot{\alpha}(\tau) \cdot \mathbf{n}(\alpha(\tau)) = \ddot{\alpha}_1(\tau)\alpha_3(\tau) - \ddot{\alpha}_3(\tau)\alpha_1(\tau), \quad (2.2)
\]
where \(\mathbf{n}(\alpha(\tau)) = (\alpha_3(\tau), -\alpha_1(\tau))\) is outward unit normal vector on \(\alpha(\tau) \in \partial \Omega\).

Meanwhile, we assume that the curvature of \(\partial \Omega\) is uniformly bounded from above, so (2.1) should be understood as simply connected curve, i.e. we can choose sufficiently small \(r > 0\) so that \(\partial \Omega \cap B(x, r)\) is simply connected curve for all \(x \in \partial \Omega\). Throughout this paper, we assume that a local parametrization of boundary satisfies (2.1) as a simply connected curve.

We define convexity and concavity of \(\alpha\) by the sign of \(k\):

**Definition 2.** Let \(\Omega \subset \mathbb{R}^2\) be an open connected bounded subset of \(\mathbb{R}^2\) and let the boundary \(\partial \Omega\) be an image of smooth regular curve \(\alpha \in C^3\) in (2.1). For \(\partial \Omega \cap B(x, r) = \{\alpha(\tau) | \delta_1 < \tau < \delta_2\}\), if
\[
k(\tau) < 0, \quad \delta_1 < \tau < \delta_2,
\]
then we say \(\partial \Omega \cap B(x, r)\) is locally convex. Otherwise, if \(k(\tau) > 0\), we say it is locally concave.
We denote the phase boundary of the phase space \( \Omega \times \mathbb{R}^3 \) as \( \gamma := \partial \Omega \times \mathbb{R}^3 \), and split into the outgoing boundary \( \gamma_{+} \), the incoming boundary \( \gamma_{-} \), and the grazing boundary \( \gamma_{0} \):

\[
\gamma_{0} := \{ (x, v) \in \partial \Omega \times \mathbb{R}^3 : \mathbf{n}(x) \cdot v = 0 \}, \\
\gamma_{+} := \{ (x, v) \in \partial \Omega \times \mathbb{R}^3 : \mathbf{n}(x) \cdot v > 0 \}, \\
\gamma_{-} := \{ (x, v) \in \partial \Omega \times \mathbb{R}^3 : \mathbf{n}(x) \cdot v < 0 \}.
\]  

(2.3)

Let us define trajectory. Given \((t, x, v) \in [0, \infty) \times \text{cl}(\Omega) \times \mathbb{R}^3\), we use \([X(s), V(s)] = [X(s; t, x, v), V(s; t, x, v)]\) to denote position and velocity of the particle at time \(s\) which was placed at \(x\) at time \(t\). Along this trajectory, we have

\[
\frac{d}{ds} X(s; t, x, v) = V(s; t, x, v), \quad \frac{d}{ds} V(s; t, x, v) = 0.
\]

with the initial condition: \((X(t; t, x, v), V(t; t, x, v)) = (x, v)\).

**Definition 3.** We recall the standard notations from [7]. We define

\[
t_b(t, x, v) := \sup \{ s \geq 0 : X(\tau; t, x, v) \in \Omega \quad \text{for all } \tau \in (t - s, t) \},
\]

\[
x_b(t, x, v) := X(t - t_b(t, x, v); t, x, v),
\]

\[
v_b(t, x, v) := \lim_{s \to t_b(t, x, v)} V(t - s; t, x, v),
\]

and similarly,

\[
t_f(t, x, v) := \sup \{ s \geq 0 : X(\tau; t, x, v) \in \Omega \quad \text{for all } \tau \in (t, t + s) \},
\]

\[
x_f(t, x, v) := X(t + t_f(t, x, v); t, x, v),
\]

\[
v_f(t, x, v) := \lim_{s \to t_f(t, x, v)} V(t + s; t, x, v).
\]

Here, \(t_b\) and \(t_f\) are called the backward exit time and the forward exit time, respectively. We also define the specular cycle as in [7]. We set \((t^0, x^0, v^0) = (t, x, v)\). When \(t_b > 0\), we define inductively

\[
t^k = t^{k-1} - t_b(t^{k-1}, x^{k-1}, v^{k-1}),
\]

\[
x^k = X(t^k; t^{k-1}, x^{k-1}, v^{k-1}),
\]

\[
v^k = R_x V(t^k; t^{k-1}, x^{k-1}, v^{k-1}),
\]

(2.4)

where

\[
R_x V(t^k; t^{k-1}, x^{k-1}, v^{k-1}) = V(t^k; t^{k-1}, x^{k-1}, v^{k-1}) - 2(\mathbf{n}(x^k) \cdot V(t^k; t^{k-1}, x^{k-1}, v^{k-1}))\mathbf{n}(x^k).
\]

Since \(t^k, x^k, \text{ and } v^k\) depend on initial phase \((x, v) = (x^0, v^0)\), we use \(t^k(x, v), x^k(x, v), \text{ and } v^k(x, v)\) when we should denote initial phase.

We define the specular characteristics as

\[
X_{cl}(s; t, x, v) = \sum_k \mathbf{1}_{s \in (t^k+1, t^k]} X(s; t^k, x^k, v^k),
\]

\[
V_{cl}(s; t, x, v) = \sum_k \mathbf{1}_{s \in (t^k+1, t^k]} V(s; t^k, x^k, v^k).
\]

(2.5)

For the sake of simplicity, we abuse the notation of [2.5] by dropping the subscript \(\text{cl}\) in this section.

### 2.2. Decomposition of the grazing set and the boundary \(\partial \Omega\)

In order to study the effect of geometry on particle trajectory, we further decompose the grazing boundary \(\gamma_0\) (which was defined in (2.3)) more carefully:

**Definition 4.** Using disjoint union symbol \(\sqcup\), we decompose grazing set \(\gamma_0\):

\[
\gamma_0^C = \gamma_0^C \sqcup \gamma_0^V \sqcup \gamma_0^I, \quad \gamma_0^C = \gamma_0^+ \sqcup \gamma_0^-.
\]

\(\gamma_0^C\) is concave(singular) grazing set:

\[
\gamma_0^C := \{ (x, v) \in \gamma_0 : t_b(x, v) \neq 0 \text{ and } t_b(x, -v) \neq 0 \}.
\]

\(\gamma_0^V\) is convex grazing set:

\[
\gamma_0^V := \{ (x, v) \in \gamma_0 : t_b(x, v) = 0 \text{ and } t_b(x, -v) = 0 \}.
\]

\(\gamma_0^I\) is grazing set:

\[
\gamma_0^I := \{ (x, v) \in \gamma_0 : t_b(x, v) \neq 0 \text{ and } t_b(x, -v) = 0 \}.
\]
\( \gamma_0^L \) is outward inflection grazing set:

\[ \gamma_0^L = \{(x,v) \in \gamma_0 : t_b(x,v) \neq 0 \text{ and } t_b(x,-v) = 0 \text{ and } \exists \delta > 0 \text{ such that } x + \tau v \in \mathbb{R}^2 \setminus cl(\Omega) \text{ for } \tau \in (0, \delta)\}. \]

\( \gamma_0^L \) is inward inflection grazing set:

\[ \gamma_0^L = \{(x,v) \in \gamma_0 : t_b(x,v) = 0 \text{ and } t_b(x,-v) \neq 0 \text{ and } \exists \delta > 0 \text{ such that } x - \tau v \in \mathbb{R}^2 \setminus cl(\Omega) \text{ for } \tau \in (0, \delta)\}. \]

Recall that \( \Omega := \Omega_0 \setminus \{\Omega_1 \cup \cdots \cup \Omega_M\} \), where each \( \Omega_i \) is an image of a unit-speed analytic curve \( \alpha_i : [a_i, b_i] \to \mathbb{R}^2 \).

Recall that \( \kappa \) stands the signed curvature in Definition \[2.2\]. Since the curvature \( \kappa \) is continuous, the set \( \{\tau \in [a_i, b_i] : \kappa(\tau) > 0\} \) is an open subset of the interval \([a_i, b_i]\) and therefore it is a countable union of disjoint open intervals, i.e.

\[ \{\tau \in [a_i, b_i] : \kappa(\tau) > 0\} = \bigcup_{j=1}^{\infty} \{\tau \in (a_{ij}, b_{ij}) : a_{ij} < \tau < b_{ij}\}. \]

It is clear that \( \kappa(a_{ij}) = 0 = \kappa(b_{ij}) \) for all \( i, j \). Suppose not, then there exists \( \epsilon > 0 \) such that \( (a_{ij} - \epsilon, b_{ij} + \epsilon) \in \{\tau \in [a_i, b_i] : \kappa(\tau) > 0\} \) which is a contradiction.

On the other hand, the signed curvature \( \kappa \) is analytic since the curve \( \alpha_i \) is analytic. If \( \kappa \) is identically zero then \( \alpha_i \) is a straight line so that \( \partial \Omega_i \) cannot be a boundary of a bounded set \( \Omega \). Since the analytic function \( \kappa \) have at most finite zeroes on a compact set \([a_i, b_i]\), there is a finite number \( N_i < \infty \) such that

\[ \{\tau \in [a_i, b_i] : \kappa(\tau) > 0\} = \bigcup_{j=1}^{N_i} \{\tau \in (a_{ij}, b_{ij}) : a_{ij} < \tau < b_{ij}\}, \]

which is a finite union of disjoint open intervals.

Now we consider the closure of \( \{\tau \in [a_i, b_i] : \kappa(\tau) < 0\} \) which is a union of closed intervals and there may exist two closed intervals which have same end point. For example [1,1,1,1] and [1,2,1,2] could have the same end point as \( b_{1,1} = a_{1,2} \). In this case we can rewrite \([a_{1,1}, b_{1,1}] \cup [a_{1,2}, b_{1,2}] \equiv [\tilde{a}_{1,1}, \tilde{b}_{1,1}] \) with \( \tilde{a}_{1,1} = a_{1,1} \) and \( \tilde{b}_{1,1} = b_{1,2} \). Therefore we can decompose the closure of \( \{\tau \in [a_i, b_i] : \kappa(\tau) < 0\} \) as the disjoint union of \( M_i(\leq N_i) \)'s closed intervals:

\[ cl\{\{\tau \in [a_i, b_i] : \kappa(\tau) > 0\}\} = \bigcup_{j=1}^{N_i} [a_{ij}, b_{ij}] = \bigcup_{j=1}^{M_i} [\tilde{a}_{ij}, \tilde{b}_{ij}]. \]

For simplicity, we abuse the notation \( a_{ij} = \tilde{a}_{ij} \) and \( b_{ij} = \tilde{b}_{ij} \).

**Definition 5.** Let \( \Omega \subset \mathbb{R}^2 \) be an analytic non-convex domain in Definition \[1\]. We decompose the boundary \( \partial \Omega \) into three parts:

\[ \partial \Omega_C := \bigcup_{i=0}^{M} \bigcup_{j=1}^{N_i} \{\alpha_i(\tau) : \tau \in (a_{ij}, b_{ij})\}, \quad \partial \Omega_I := \bigcup_{i=0}^{M} \bigcup_{j=1}^{N_i} \{\alpha_i(a_{ij}), \alpha_i(b_{ij})\}, \quad \partial \Omega_V := \partial \Omega \setminus (\partial \Omega_C \cup \partial \Omega_I). \]

The number \( M_C = \sum_{i=0}^{M} N_i \) and the \( l \)-th concave part \( \partial \Omega_C^l \) for \( l = 1, 2, \ldots, M_C \) is a renumbered sequence of \( \{\alpha_i(\tau) : \tau \in [a_{ij}, b_{ij}]\} \) for \( i = 0, 1, \ldots, M \) and \( j = 1, 2, \ldots, N_i \). Therefore, we can define \( M_C \) number of parametrization \( \tilde{\alpha}_l \) with \( l = 1, 2, \ldots, M_C \) such that

\[ \partial \Omega_C^l = \{\tilde{\alpha}_l(\tau) : \tau \in (\tilde{a}_l, \tilde{b}_l)\}. \quad \text{(Concave boundary)} \]

We further split \( \partial \Omega_I = \partial \Omega_{I^+} \cup \partial \Omega_{I^-} \) where \( \partial \Omega_{I^+} := \bigcup_{i=0}^{M} \partial \Omega_{I^+}^i \) and \( \partial \Omega_{I^-} := \bigcup_{i=0}^{M} \partial \Omega_{I^-}^i \) with

\[ \partial \Omega_{I^+}^i = \{\alpha_i(\tau) \in \partial \Omega_C^i : \exists \varepsilon > 0 \text{ such that } \kappa_i(\tau) < 0 \text{ for } \tau \in (\tau - \varepsilon, \tau) \text{ and } \kappa_i(\tau) > 0 \text{ for } \tau' \in (\tau, \tau + \varepsilon)\}, \]

\[ \partial \Omega_{I^-}^i = \{\alpha_i(\tau) \in \partial \Omega_C^i : \exists \varepsilon > 0 \text{ such that } \kappa_i(\tau') > 0 \text{ for } \tau' \in (\tau - \varepsilon, \tau) \text{ and } \kappa_i(\tau) < 0 \text{ for } \tau' \in (\tau, \tau + \varepsilon)\}. \]

Note that the following decomposition is compatible with those of Definition \[4\].

\[ \gamma_0^C = \{(x,v) \in \gamma_0 : x \in \partial \Omega_C^0\}, \quad \text{(Concave grazing set)} \]

\[ \gamma_0^I = \{(x,v) \in \gamma_0 : x \in \partial \Omega_I^0\}, \quad \text{(Inflection grazing set)} \]

\[ \gamma_0^V = \{(x,v) \in \gamma_0 : x \in \partial \Omega_V^0\}. \quad \text{(Convex grazing set)} \]
Remark that from the definition, it is clear that
\[
\text{cl}(\partial \Omega^C) = \bigcup_{i=0}^{M} \bigcup_{j=1}^{N} \text{cl}(\{ \alpha_i(\tau) : \tau \in (a_{i,j}, b_{i,j}) \}) = \bigcup_{i=0}^{M} \bigcup_{j=1}^{N} \{ \alpha_i(\tau) : \tau \in [a_{i,j}, b_{i,j}] \}.
\]

3. \( L^\infty \) estimate

3.1. Inflection grazing set. Trajectory of a particle is very simple for axial direction,
\[
V_2(s; t, x, v) = v_2, \quad X_2(s; t, x, v) = x_2 - (t - s)v_2.
\]
Therefore, the characteristics of trajectories come from dynamics in two-dimensional cross section \( \Omega \). In this subsection, we analyze trajectories in \( \Omega \subset \mathbb{R}^2 \). First, for fixed \( N \gg 1 \), we define the admissible set of velocity:
\[
\forall^N := \{ v \in \mathbb{R}^2 : 2 \frac{N}{2} \leq |v| \leq \frac{N}{2} \}.
\]
And \( m_2 : P(\Omega) \rightarrow \mathbb{R} \) is standard Lebesgue measure in \( \mathbb{R}^2 \).

We control collection of bad phase sets those are nearly grazing set for each open covers containing boundary \( \partial \Omega \).

**Lemma 1.** Let \( \Omega \subset \mathbb{R}^2 \) be an analytic non-convex domain, defined in Definition [1]. For \( \varepsilon \ll 1, N \gg 1 \), there exist finite points
\[
\{ x_1^{n_1}, \cdots, x_{l_n}^{n_1} \} \subset \text{cl}(\Omega),
\]
and their open neighborhoods
\[
B(x_1^{n_1}, r_1^{n_1}), \cdots, B(x_{l_n}^{n_1}, r_{l_n}^{n_1}) \subset \mathbb{R}^2,
\]
as well as corresponding open sets
\[
\Omega_1^{n_1}, \cdots, \Omega_{l_n}^{n_1} \subset \forall^N,
\]
with \( m_2(\Omega_i^{n_B}) \leq \varepsilon \) for all \( i = 1, \cdots, l_n \) such that for every \( x \in \text{cl}(\Omega) \) there exists \( i \in \{ 1, \cdots, l_n \} \) with \( x \in B(x_i^{n_B}, r_i^{n_B}) \) and satisfies either
\[
B(x_i^{n_B}, r_i^{n_B}) \cap \partial \Omega = \emptyset \quad \text{or} \quad |v' \cdot n(x')| > \varepsilon/N^4,
\]
for all \( x' \in B(x_i^{n_B}, r_i^{n_B}) \cap \partial \Omega \) and \( v' \in \forall^N \setminus \Omega_i^{n_B} \).

**Proof.** By Definition [1], \( \partial \Omega \subset \mathbb{R}^2 \) is a compact set in \( \mathbb{R}^2 \) and a union of the images of finite curves. For \( x \in \Omega \), we define \( r_x > 0 \) such that \( B(x, r_x) \setminus \partial \Omega = \emptyset \). For each \( x \in \partial \Omega \), we can define the outward unit normal direction \( n(x) \) and the outward normal angle \( \theta_n(x) \in [0, 2\pi) \) specified uniquely by \( n(x) = (\cos \theta_n(x), \sin \theta_n(x)) \). Using the smoothness and uniform boundedness of curvature of the boundary \( \partial \Omega \), there exist uniform \( r_{\varepsilon,N} > 0 \) such that for \( r_x \leq r_{\varepsilon,N} \),
\[
|\theta_n(x') - \theta_n(x)| < \varepsilon/2N^2 \quad \text{for all} \quad x' \in B(x, r_x) \cap \partial \Omega,
\]
and \( B(x, r_x) \cap \partial \Omega \) is a simply connected curve.

By compactness, we have finite integer \( l_n > 0 \), points \( \{ x_i^{n_B} \}_{i=1}^{l_n} \), and positive numbers \( \{ r_i^{n_B} \}_{i=1}^{l_n} \) such that
\[
\text{cl}(\Omega) \subset \bigcup_{i=1}^{l_n} B(x_i^{n_B}, r_i^{n_B}), \quad r_i^{n_B} \leq r_{\varepsilon,N}.
\]
By above construction, for each \( 1 \leq i \leq l_n \), we have either
\[
B(x_i^{n_B}, r_i^{n_B}) \cap \partial \Omega = \emptyset,
\]
or
\[
x_i^{n_B} \in \partial \Omega \quad \text{and} \quad r_i^{n_B} < r_{\varepsilon,N} \quad \text{so that} \quad (3.1) \quad \text{holds.}
\]
For \( i \) with case (3.2), we set \( \Omega_i^{n_B} = \emptyset \). For \( i \) with case (3.3), we define
\[
\Omega_i^{n_B} := \{ v \in \forall^N : v = (|v|\cos \theta, |v|\sin \theta) \quad \text{where} \quad \theta \in \left( \frac{\theta_i - \pi}{2} - \frac{\varepsilon}{N^3}, \frac{\theta_i + \pi}{2} + \frac{\varepsilon}{N^3} \right) \},
\]
where we abbreviated \( \theta_n(x_i^{n_B}) = \theta_i \). Obviously, \( m_2(\Omega_i^{n_B}) \leq \pi \frac{N^2 \varepsilon/2N^3}{\pi} \leq \varepsilon \) and
\[
|v' \cdot n(x')| \geq |v'| \times \left| (\cos \theta', \sin \theta') \cdot (\cos \theta_n(x'), \sin \theta_n(x')) \right|
\geq \frac{2}{N} \times \left| \cos \left( \frac{\pi}{2} + \frac{\varepsilon}{N^3} \right) \right| = \frac{2}{N} \left| \sin \left( \frac{\varepsilon}{N^3} \right) \right|, \quad \varepsilon/N^3 \ll 1,
\]
for \( x' \in B(x_i^{n_B}, r_i^{n_B}) \) and \( v' = |v'|(\cos \theta', \sin \theta') \in \forall^N \setminus \Omega_i^{n_B} \).

\[\square\]
We state critical property of analytical boundary for non-convergence of consecutive specular bouncing points. We use notation of the specular cycles \((x^i, v^i)\) defined in (2.4).

**Lemma 2.** Assume \(Ω ⊂ \mathbb{R}^2\) is the analytic non-convex domain of Definition 1. Choose \(x ∈ cl(Ω)\) and nonzero \(v ∈ V^N\). If \([x^i(x,v), v^{i−1}(x,v)] \notin \gamma_0^\ast\) for all \(i = 0, 1, 2, \cdots\), then
\[
\sum_{i=0}^{∞} |x^i(x,v) − x^{i+1}(x,v)| = ∞.
\]

**Proof.** We prove this lemma by contradiction argument: suppose \([x^i(x,v), v^i(x,v)] \notin \gamma_0^\ast\) for all \(i = 0, 1, 2, \cdots\)

Then \(x^i(x,v) \to x^∞\) and \(x^∞ = \lim_{i→∞} α(τ_i) = α(τ_∞) ∈ ∂Ω\) using that \(∂Ω\) is closed set. For \(i ≥ 1\), we assume \(x^i(x,v) ∈ \{α_j(τ) : τ ∈ [a_j, b_j]\}\) for some fixed \(j ∈ N\) in Definition 1. Otherwise \(x^i(x,v)\) can not converge because \(dist(∂Ω_j_1, ∂Ω_j_2) > δ > 0\) for \(j_1 ≠ j_2\). Therefore we drop index \(j\) and denote \(α(τ_i) = α_j(τ_i) = x^i(x,v)\) in this proof.

**Step 1.** Let us drop notation of fixed \((x,v)\) and assume that
\[
x^i = α(τ_i), \quad x^{i+1} = α(τ_{i+1}), \quad x^{i+2} = α(τ_{i+2}).
\]

We claim that if \(τ_i < τ_{i+1}\), then \(τ_{i+1} < τ_{i+2}\) for sufficiently large \(i ≥ 1\). As explained in (2.1), we can find \(r^* < 1\) such that if \(r ≤ r^*\), then \(B(x,r) ∩ \partialΩ\) is simply connected curve for \(x ∈ ∂Ω\). Also for \(x ∈ ∂Ω\), we can find \(r^*> 1\) such that if \(r ≤ r^*\), then \(B(x,r) ∩ \partialΩ\) \(∩ N(x) = \{x\}\) where \(N(x) = \{cn(x) : c ∈ \mathbb{R}\}\), the normal line crossing \(x ∈ ∂Ω\) for \(r = min(r^*, r^*)\). We can decompose
\[
B(x^{i+1},r) ∩ ∂Ω = \left\{α(τ) : τ < τ_{i+1}\right\} ∩ ∂Ω \bigcup \left\{α(τ) : τ > τ_{i+1}\right\} ∩ ∂Ω. \tag{3.5}
\]

From (3.4), for any \(ε < \frac{1}{2} min(r^*, r^*)\), we can choose \(R ≥ 1\) such that
\[
|x^i − x^{i+1}| < ε, \quad ∀i > R. \tag{3.6}
\]

If we consider \(B(x^{i+1}, min(r^*, r^*))\), both \(x^i\) and \(x^{i+2}\) are in \(B(x^{i+1}, min(r^*, r^*)) \cap ∂Ω\) by (3.6). If \(τ_i < τ_{i+1}\), then \(τ_i ∈ B_−.\) Combining this fact with disjoint decomposition (3.5), we know that \(v^{i+1} · α(τ_{i+1}) > 0\). Therefore, \(x^{i+2} ∉ B_−\) and we already know that \(x^{i+2} ≠ x^{i+1}\). Finally we get
\[
x^{i+2} ∈ B(x^{i+1}, min(r^*, r^*)) \cap ∂Ω \setminus \{B_− ∪ \{x^{i+1}\}\} := B_+.
\]

By definition of \(B_+, \tau_i < \tau_{i+2}\).

**Step 2.** We split \(τ_∞\) into three cases and study possible cases for (3.4). Without loss of generality, we assume that \(ε\) and \(i > R\) in the rest of this proof satisfy (3.6).

(i) If \(κ(τ_∞) < 0, \exists ε > 0\) such that \(κ(τ) < 0\) for \(τ ∈ (τ_∞ − ε, τ_∞ + ε)\). While boundary is convex, we can apply velocity lemma, Lemma 1 in [9] or Lemma 2.6 in [14]. From the velocity lemma, normal velocity at bouncing points are equivalent, especially,
\[
e^{-C_0(|v|+2)|τ|^2}(v^i · n(x^i)) ≤ e^{-C_0(|v|+2)|τ|^2}(v^{i+1} · n(x^{i+1}))
\]
\[
e^{-C_0(|v|+2)|τ|^2}(v^i · n(x^i)) \geq e^{-C_0(|v|+2)|τ|^2}(v^{i+1} · n(x^{i+1})). \tag{3.7}
\]

Since nonzero speed \(|v|\) is constant, (3.4) implies finite time stop of the trajectory. From (3.7), \(v^i · n(x^i)\) cannot be zero at finite time. So this is contradiction.

(ii) If \(κ(τ_∞) > 0, \exists ε > 0\) such that \(κ(τ) > 0\) for \(τ ∈ (τ_∞ − ε, τ_∞ + ε)\). Without loss of generality, we choose \(ε ≤ min(r^*, r^*)\) which as chosen in Step 1. By concavity,
\[
(α(τ) − x^{i+1}) · n(x^{i+1}) > 0 \quad \text{for} \quad τ ∈ (τ_∞, τ_∞ + ε) \quad \text{where} \quad R_{x^i+1}v^{i+1} · n(x^{i+1}) < 0.
\]

This implies, \(τ_i ∈ (τ_{i+1} − ε, τ_{i+1})\) then \(τ_{i+2} ∉ [τ_{i+1}, τ_{i+1} + ε)\). This is contradiction.

(iii) If \(κ(τ_∞) = 0, κ(τ) > 0\) for \(τ ∈ (τ_∞ − ε, τ_∞)\), this case is exactly same as case (ii).

(iv) If \(κ(τ_∞) = 0, κ(τ) = 0\) for \(τ ∈ (τ_∞ − ε, τ_∞)\), then \(κ(τ) = 0\) for \(τ ∈ [a_j, b_j]\) by analyticity. So, \(Ω\) must be half plane and we get contradiction.
(v) Assume $\kappa(\tau_\infty) = 0$ and $\kappa(\tau) < 0$ for $\tau \in (\tau_\infty - \varepsilon, \tau_\infty]$.

**Step 3.** We derive contradiction for the last case (v) by claiming

$$l_{i+1} = |x_{i+1} - x_i| \leq |x_{i+2} - x_{i+1}| = l_{i+2} < \varepsilon, \quad i \geq R,$$

for $\varepsilon$ and $R$ is what we have chosen in (3.6). As explained in (2.1), we can assume that $B(x_\infty, \varepsilon) \cap \partial\Omega$ is a graph of analytic function $\varphi(s)$. From the argument of Step 1, we assume $s_{\infty} - \varepsilon < s_1 < s_{i+1} < s_{i+2} < s_{\infty}$. Moreover, up to translation and rotation, we can assume that $\varphi(s_{i+1}) = \varphi'(s_{i+1}) = 0$ and $\varphi''(s) > 0$ on $s \in (s_{\infty} - \varepsilon, s_{\infty})$. There exist $n_0 \in \mathbb{N}$ such that

$$\frac{d^{n_0}}{ds^{n_0}} \varphi(s_{\infty}) \neq 0 \quad \text{and} \quad \frac{d}{ds} \varphi(s_{\infty}) = 0 \quad \text{for all } 0 \leq i < n_0.$$

If $n_0 = \infty$, $\partial\Omega$ is straight line so contradiction as explained in (iv) of Step 2. Also by definition of inflection point, $n_0 \geq 3$. For finite $n_0 \in \mathbb{N}$, for $|s| < \varepsilon \ll 1$,

$$\varphi''(s) = c_{n_0-2}(s - s_\infty)^{n_0-2}(1 + O(|s - s_\infty|)) \to 0 \quad \text{as} \quad s \to s_\infty.$$

To claim $|x_{i+1} - x_i| \leq |x_{i+2} - x_{i+1}|$, we suffice to claim $s_{i+1} - s_i \leq s_{i+2} - s_{i+1}$, because absolute values of slopes of $x_{i+1}x_{i+2}$ and $x_{i+1}x_{i+2}$ are same by the specular boundary condition. Since we assume $\varphi'(s_{i+1}) = 0$, from the specular boundary condition,

$$\frac{\varphi(s_{i+2}) - \varphi(s_{i+1})}{s_{i+2} - s_{i+1}} = \frac{\varphi(s_i) - \varphi(s_{i+1})}{s_{i+1} - s_i},$$

$$\frac{1}{s_{i+2} - s_{i+1}} \int_{s_{i+1}}^{s_{i+2}} \varphi''(r) dr dt = \frac{1}{s_{i+1} - s_i} \int_t^{s_{i+1}} \varphi''(r) dr dt$$

(3.10)

It is important that near inflection point, from (3.9), $\varphi'' > 0$ is monotone decreasing to zero on $s \in (s_{\infty} - \varepsilon, s_{\infty})$ for $\varepsilon \ll 1$. Therefore,

$$\frac{1}{s_{i+2} - s_{i+1}} \int_{s_{i+1}}^{s_{i+2}} \varphi''(r) dr dt \leq \frac{1}{2}(s_{i+2} - s_{i+1})\varphi''(s_{i+1}),$$

(3.11)

From (3.10) and (3.11), we get $s_{i+1} - s_i \leq s_{i+2} - s_{i+1}$ and justify (3.8). We proved contradictions for all possible cases listed in Step 2 and finish the proof.

Remark that this fact is non-trivial because we can observe the infinitely many bounds of the specular cycles in a finite time interval even in some convex domains [10]. Moreover in the case of non-convex domains we need to treat carefully the trajectories hit the inflection part (Definition 5) tangentially. The analyticity assumption is essential in the proof.

Using Lemma 2, we define and control bad phase sets where their cycles may hit inflection grazing sets $\gamma_0^1$, defined in Definition 4 or 5.

**Lemma 3.** Let $\Omega \subset \mathbb{R}^2$ be an analytic non-convex domain in Definition 1. For $T_0 > 0, \varepsilon \ll 1, N \gg 1$, there exist finite points

$$\{x_{1,n}^1, \cdots, x_{n,n}^1\} \subset \text{cl}(\Omega),$$

and open balls

$$B(x_{1,n}^1, r_{1,n}^1), \cdots, B(x_{n,n}^1, r_{n,n}^1) \subset \mathbb{R}^2,$$

as well as corresponding open sets

$$O_{1,n}^1, \cdots, O_{n,n}^1 \subset \mathbb{V}^N,$$

with $m_2(O_{1,n}^1) \leq \varepsilon$ for all $i = 1, \cdots, n$ such that for every $x \in \text{cl}(\Omega)$ there exists $i \in \{1, \cdots, n\}$ with $x \in B(x_{i,n}^1, r_{i,n}^1)$ and, for $v \in \mathbb{V}^N \setminus O_{1,n}^1$, the following holds.

$$[X(s;T_0, x, v), V(s;T_0, x, v)] \notin \gamma_0^1 \quad \text{for all } s \in [0, T_0].$$

**Proof.** With the specular boundary condition, an particle trajectory is always reversible in time. Therefore, we track backward in time trajectory which depart from inflection grazing phase. Recall from Definition 5 that the inflection boundary $\partial\Omega^I$ is a set of finite points and denote $\partial\Omega^I = \{x_1^I, x_2^I, \cdots, x_{M^I}^I\}$. Define

$$\bigcup_{j=1}^{M^I} \left\{(X(s; T_0, x_j^I, v), V(s; T_0, x_j^I, v)) \in \text{cl}(\Omega) \times \mathbb{R}^2 : s \in [0, T_0], \ (x_j^I, v) \in \gamma_0^I, \ v \in \mathbb{V}^N \right\}. $$

11
Now we fix one point of the inflection boundary \( x_j^f \in \partial \Omega^f \) and a velocity \( v_j^f \in \mathbb{R}^2 \) with \( |v_j^f| = 1 \) such that \((x_j^f, v_j^f) \in \gamma_0^f\). More precisely, for \( x_j^f = \alpha_i(\tau) \in \partial \Omega_i^f \) with some \( i = 1, \ldots, M \) in Definition 5, we choose \( v_j^f = -\hat{\alpha}_i(\tau) \), and for \( x_j^f = \alpha_i(\tau) \in \partial \Omega_i^{-f} \) we choose \( v_j^f = \hat{\alpha}_i(\tau) \) so that \((x_j^f, v_j^f) \in \gamma_0^f\) and backward in time trajectory is well-defined for short time \((T_0 - \varepsilon, T_0)\), \( \varepsilon \ll 1 \) at least.

Since \( |V(s;T_0,x_j^f,v_j^f)| = |v_j^f| \leq \frac{N}{2} \) for \( v_j^f \in \mathbb{V}^N \), possible total length of the specular cycles is bounded by \( \frac{NT_0}{2} \).

By Lemma 2, number of bounce cannot be infinite for finite travel length without hitting inflection grazing phase. Moreover, if trajectory hit inward inflection grazing phase, \( \gamma_0^{-f} \), particle cannot propagate anymore. Therefore, number of bounce for finite travel length is always bounded. This implies

\[
m(x_j^f) := \inf \left\{ m \in \mathbb{N} : \sum_{i=1}^m |x^i(x_j^f, v_j^f) - x^{i+1}(x_j^f, v_j^f)| > \frac{NT_0}{2} \right\} < +\infty,
\]

which actually depends on \( N \) for fixed \( \Omega \) and \( T_0 \). Therefore the set \((3.1)\) is a subset of

\[
A := \bigcup_{j=1}^{M^f} \bigcup_{i=0}^{m(x_j^f)} \left\{ (y, u) \in \text{cl}(\Omega) \times \mathbb{V}^N : y \in x^i(x_j^f, v_j^f)x^{i+1}(x_j^f, v_j^f) \text{ and } \frac{u}{|u|} = \pm v^i(x_j^f, v_j^f) \right\},
\]

which is a set of all particle paths from all inflection grazing phase. Now, we define projection of \( A \) on spatial dimension,

\[
\mathcal{P}_x(A) := \bigcup_{j=1}^{M^f} \bigcup_{i=0}^{m(x_j^f)} \left\{ y \in \text{cl}(\Omega) : y \in x^i(x_j^f, v_j^f)x^{i+1}(x_j^f, v_j^f) \right\}.
\]

Now we construct open coverings: For \( x \in \text{cl}(\Omega) \setminus \mathcal{P}_x(\text{cl}(A)) \), we pick \( r_x > 0 \) so that \( B(x, r_x) \cap \mathcal{P}_x(\text{cl}(A)) = \emptyset \).

For \( x \in \mathcal{P}_x(\text{cl}(A)) \), we pick \( r_x > 0 \) to generate covering for \( \mathcal{P}_x(\text{cl}(A)) \). By compactness, we have finite open covering \( B(x_{i1}^f, r_{i1}^f), \ldots, B(x_{iM^f}^f, r_{iM^f}^f) \). From above construction, for each \( 1 \leq i \leq l_{i1} \), we have either

\[
B(x_{i1}^f, r_{i1}^f) \cap \mathcal{P}_x(\text{cl}(A)) = \emptyset,
\]

or

\[
x_{i1}^f \in \mathcal{P}_x(\text{cl}(A)).
\]

For \( x_i \) with \((3.12)\) case, we set \( O_{i1}^f = \emptyset \). For \( i \) with \((3.13)\) case, there are finite number of straight segments (may intersect each other) of \( \mathcal{P}_x(A) \). This number of segments are bounded by \( M^f \times \max_m m(x_j^f) \) for \( i = 1, \ldots, M^f \). By \( O_{i1}^f \) with \( i \) satisfies \((3.13)\), we mean

\[
O_{i1}^f = \left\{ u \in \mathbb{V}^N : |u/|u| \pm v^i(x_j^f, v_j^f)| < C_N \varepsilon, \quad \forall (i,j) \text{ s.t } x^i(x_j^f, v_j^f)x^{i+1}(x_j^f, v_j^f) \cap B(x_{i1}^f, r_{i1}^f) \neq \emptyset \right\}.
\]

Obviously \( m_2(O_i^f) \leq \pi N^2 \frac{C_N}{2\varepsilon} M^f \times \max_m m(x_j^f) \leq \varepsilon \) by choosing \( C_N \leq \frac{1}{M^f} \) for sufficiently large \( N \gg 1 \).

Now we prove \((3.14)\). Since trajectory is reversible in time, \( [X(s;T_0,x,v), V(s;T_0,x,v)] \notin \gamma_0^f \) if \( (x,v) \notin A \). By definition of \((3.14)\), if \( x \in B(x_{i1}^f, r_{i1}^f), v \in \mathbb{V}^N \setminus O_i^f \), and \( s \in [0,T_0] \), then \( (x,v) \notin A \). This finishes proof.

\[\]

The following lemma comes from Lemma 1 and Lemma 3.

**Lemma 4.** Consider \( \Omega \) as defined in Definition 2. For \( \varepsilon \ll 1, N \gg 1 \), there exist finite points

\[
\{x_1^{IB}, \ldots, x_{l_{1B}}^{IB}\} \subset \text{cl}(\Omega),
\]

and open balls

\[
B(x_1^{IB}, r_1^{IB}), \ldots, B(x_{l_{1B}}^{IB}, r_{l_{1B}}^{IB}) \subset \mathbb{R}^2,
\]

as well as corresponding open sets

\[
O_1^{IB}, \ldots, O_{l_{1B}}^{IB} \subset \mathbb{V}^N
\]

with \( m_2(O_i^{IB}) < C \varepsilon \) (for uniform constant \( C > 0 \)) for all \( i = 1, \ldots, l_{1B} \) such that for every \( x \in \text{cl}(\Omega) \), there exists \( i \in \{1, \ldots, l_{1B}\} \) with \( x \in B(x_i^{IB}, r_i^{IB}) \) and, for \( v \in \mathbb{V}^N \setminus O_i^{IB} \),

\[
|v \cdot n(x)| > \frac{\varepsilon}{N^4},
\]

for all \( x \in \partial \Omega \cap B(x_i^{IB}, r_i^{IB}) \) and

\[
(X(t_k;T_0,x,v), V(t_k;T_0,x,v)) \notin \gamma_0^f \quad \text{for all } t_k \in [0,T_0].
\]
Using above lemma, we define the infinite-bounces set $\mathfrak{B}$ as

$$
\mathfrak{B} := \left\{ (x, v) \in \text{cl}(\Omega) \times \mathbb{N}^N : v \in O_i^B \text{ for some } i \in \{1, 2, \ldots, l_B\} \text{ satisfying } x \in B(x_i^{1B}, r_i^{1B}) \right\}.
$$

(3.15)

The most important property of the infinite-bounces set (3.15) is that the bouncing number of the specular backward trajectories on $\{ \text{cl}(\Omega) \times \mathbb{N}^N \}\setminus\mathfrak{B}$ is uniformly bounded.

**Definition 6.** When $L > 0$, $x \in \text{cl}(\Omega) \subset \mathbb{R}^2$, and nonzero $v \in \mathbb{R}^2$ are given, we consider a set

$$
\{ k \in \mathbb{N} : \sum_{j=1}^{k} |x^{j-1}(x, v) - x^j(x, v)| > L \} \subset \mathbb{N}.
$$

If this set is not empty, then we define $\mathfrak{M}(x, v, L) \in \mathbb{N}$ as following,

$$
\mathfrak{M}(x, v, L) := \inf \left\{ k \in \mathbb{N} : \sum_{j=1}^{k} |x^{j-1}(x, v) - x^j(x, v)| > L \right\}.
$$

Otherwise, if the set is empty, it means backward trajectory is trapped in $\gamma_{10}^L$, so we define

$$
\mathfrak{M}(x, v, L) := \inf \{ i \in \mathbb{N} : (x^i(x, v), v^{i-1}(x, v)) \in \gamma_{10}^L \}.
$$

From Lemma 4 we have $\mathfrak{M}(x, v, N\tau_0^V) < \infty$ for $(x, v) \in \{ \text{cl}(\Omega) \times \mathbb{N}^N \}\setminus\mathfrak{B}$. To improve this finite result into uniform bound, we use compactness and continuity arguments.

**Lemma 5.** Let $(x, v) \in \text{cl}(\Omega) \times \mathbb{N}^N$. Then $(x^k(x, v), v^k(x, v))$ is a locally continuous function of $(x, v)$ if

$$(x^i(x, v), v^i(x, v)) \notin \gamma_0, \quad \forall i \in \{1, 2, \ldots, k\},$$

i.e. for any $\varepsilon > 0$, there exist $\delta_{x,v,\varepsilon} > 0$ such that if $|(x, v) - (y, u)| < \delta_{x,v,\varepsilon}$, then

$$
|(x^i(x, v), v^i(x, v)) - (x^i(y, u), v^i(y, u))| < \varepsilon, \quad \forall i \in \{1, 2, \ldots, k\}.
$$

Moreover $(x^i(x, v), v^i(y, u)) \notin \gamma_0$ for $i \in \{1, \ldots, k\}$.

**Proof.** First we claim continuity of $(x^i(x, v), v^i(x, v))$. Using trajectory notation and lower bound of speed in $\mathbb{N}^N$, we know

$$
x^1(x, v) = X(t_b(t, x, v); t, x, v), \quad t_b \leq CN,
$$

for uniform $C$ which depend on the size of $\Omega$. Let us assume that $|(x, v) - (y, u)| \leq \delta$. Then

$$
|x^1(x, v) - x^1(y, u)| \leq |x^1(x, v) - x^1(x, u)| + |x^1(x, u) - x^1(y, u)|
$$

(3.16)

Let $x^1(x, v) = \alpha_j(\tau^*).$ Since $(x^1(x, v), v) \notin \gamma_0$, $\frac{|\alpha_j(\tau^*)|}{|\alpha_j(\tau^*)|} < 1.$ Then we can choose sufficiently small $r_{x,v} \ll 1$ such that $\partial \Omega \cap B(x^1(x, v), r_{x,v})$ is simply connected and intersects with line $\{x + su : s \in \mathbb{R}\}$ in only one point non-tangentially, because $\alpha_j(x^1(x, v))$ is not parallel to $v$. Since $x + su$ is continuous on $x$, $x + su$ must intersect to $\partial \Omega \cap B(x^1(x, v), r_{x,v})$ at some $\alpha_j(\tau) \in \partial \Omega \cap B(x^1(x, v), r)$ whenever $|u - v| \ll \delta_{x,v}.\,$ This shows $|x^1(x, v) - x^1(x, u)| < O(\delta_{x,v}).$ And

$$
\frac{|u}{|u|} \cdot \hat{\alpha}(\tau) - \frac{v}{|v|} \cdot \hat{\alpha}(\tau^*) = \frac{|u}{|u|} \cdot \hat{\alpha}(\tau) - \frac{v}{|v|} \cdot \hat{\alpha}(\tau^*) + \frac{|u}{|u|} \cdot \hat{\alpha}(\tau^*) - \frac{v}{|v|} \cdot \hat{\alpha}(\tau^*)
$$

$$
\leq C|\tau - \tau^*| + N\delta
$$

(3.17)

$$
\leq C_N \delta \leq \frac{1}{2} \left( 1 - \frac{v}{|v|} \cdot \hat{\alpha}(\tau^*) \right)
$$

for sufficiently small $\delta \ll 1$. This implies $\frac{|u}{|u|} \cdot \hat{\alpha}(\tau) < 1$, i.e. $(x^1(x, v), u) \notin \gamma_0$.

Now, there exist small $r_{x,u} \ll 1$ such that $\partial \Omega \cap B(x^1(x, u), r_{x,u})$ is simply connected and intersects with line $\{x + su : s \in \mathbb{R}\}$ in only one point non-tangentially by (3.17). So there exist $\delta_{x,u,\varepsilon} \ll 1$ such that line $y + su$ hits $\partial \Omega \cap B(x^1(x, u), r_{x,u})$ if $|x - y| < \delta_{x,u,\varepsilon}$. It is obvious that $|x^1(x, u) - x^1(y, u)| < r_{x,u}$. By far we showed continuity of $x^1(x, \cdot)$ and $x^1(\cdot, u)$. So continuity of $x^1$ follows from (3.16).

To claim continuity of $v^1(x, v)$, we use continuity of $x^1(x, v)$. When $|(x, v) - (y, u)| < \delta_{x,v,\varepsilon} \ll 1$, we have $|x^1(x, v) - x^1(y, u)| = O(\delta_{x,v,\varepsilon})$ and therefore $\tau_1 - \tau_2 = O(\delta_{x,v,\varepsilon})$, where $\alpha(\tau_1) = x^1(x, v)$ and $\alpha(\tau_2) = x^2(x, v)$. By smoothness of $\alpha : \tau \to \mathbb{R}^2$, $\mathbf{n}(\alpha(\tau_1)) - \mathbf{n}(\alpha(\tau_2))$ is size of $O(\delta_{x,v,\varepsilon})$ as well as $\dot{\alpha}(\tau_1) - \dot{\alpha}(\tau_2)$. By the specular boundary condition, we have

$$
|v^1(x, v) - v^1(y, u)| \leq |R_{\alpha(\tau_1)} v - R_{\alpha(\tau_2)} u|
$$

$$
\leq |(I - 2\mathbf{n}(\alpha(\tau_1)) \otimes \mathbf{n}(\alpha(\tau_1))) v - (I - 2\mathbf{n}(\alpha(\tau_2)) \otimes \mathbf{n}(\alpha(\tau_2))) u| 
$$

$$
\leq O(\delta_{x,v,\varepsilon}).
$$
Moreover, $v^i \cdot n_{\tau_i}$ is also continuous function of $(x,v)$, so $(x^1(y,u),v^1(y,u)) \notin \gamma_0$ when $(y,u)$ are sufficiently close to $(x,v)$. Case of $i = 2, \cdots, k$ are easily gained by chain rule, applying above argument several times.

**Lemma 6.** Let $\Omega \subset \mathbb{R}^2$ satisfies Definition 7 Then

$$\mathcal{N}_{\varepsilon,N,T_0} := \sup_{(x,v) \in (cl(\Omega) \times \mathbb{V}) \setminus \mathcal{B}} \mathcal{N}(x,v,N,T_0) \leq C_{\varepsilon,N,T_0},$$

where $\mathcal{N}(x,v,N,T_0)$ is defined in Definition 5 and $\varepsilon$-dependence comes from $\mathcal{O}^{\mathcal{B}}_{i} \setminus \mathcal{B}_{i}$, which was defined in Lemma 4.

**Proof.** From Lemma 2 and 4, trajectory does not belongs to inflection grazing set during time $[0,T_0]$. $\mathcal{N}(x,v,\cdot)$ is nondecreasing function for fixed $(x,v)$ and we can assume $|v| = 1$, because $NT_0$ is fixed maximal travel length during time interval $[0,T_0]$ with $v \in \mathbb{V}$. 

**Step 1.** We study cases depending on concave grazing.

(Case 1) If $\mathbf{n}(x^1(x,v),v^1(x,v)) \neq 0$ for $i = 1, \cdots, \mathcal{N}(x,v,N,T_0)$, trajectory $(X(s,T_0,x,v),V(s,T_0,x,v))$ is continuous in $(x,v)$ by Lemma 5. Therefore, we can choose $\delta_{x,v,\varepsilon,NT_0} \leq 1$, such that if $|(x,v) - (y,u)| < \delta_{x,v,\varepsilon,NT_0}$, then $|x_b(x,v) - x_b(y,u)| < O(\delta_{x,v,\varepsilon,NT_0})$, where $O(\delta_{x,v,\varepsilon,NT_0}) \to 0$ as $\delta_{x,v,\varepsilon,NT_0} \to 0$. Therefore,

$$\mathcal{N}(y,u,NT_0) \leq 1 + \mathcal{N}(x,v,N,T_0),$$

for $|(x,v) - (y,u)| < \delta_{x,v,\varepsilon,NT_0} \ll 1$. Moreover, we have

$$|x^i(x,v) - x^i(y,u)| < O(\delta_{x,v,\varepsilon,NT_0}),$$

for $i = 1, \cdots, \mathcal{N}(x,v,N,T_0)$.

(Case 2) Assume that $(x^1(x,v),v^1(x,v))$ belongs to grazing set $\gamma_0$ for some $i \in \{1, \cdots, \mathcal{N}(x,v,N,T_0)\}$. Especially, $(x^1(x,v),v^1(x,v)) \in \gamma_0^C$, because $\gamma_0^C$ is not gained from $\{cl(\Omega) \times \mathbb{V}\} \setminus \mathcal{B}$, as proved in Lemma 4. $\gamma_0^C$ is the stopping index satisfying $(x^1(x,v),v^1(x,v)) \in \gamma_0^C$. Even though there are consecutive convex grazings, it must stop at some $(x^k(x,v),v^k(x,v))$, because $\Omega$ is analytic and bounded domain, i.e. there exist $i,k \in \mathbb{N}$ such that

$$(x^j(x,v),v^j(x,v)) \notin \gamma_0^C, \quad \forall j < i,$$

$$(x^j(x,v),v^j(x,v)) \in \gamma_0^C, \quad i \leq j \leq k - 1,$$

$$(x^j(x,v),v^j(x,v)) \notin \gamma_0^C, \quad j = k.$$

When $j < i$, the bouncing number can be counted similar as Step 1, 

$$\mathcal{N}(y,u,N(T_0 - t_{i-1}(x,v))) \leq 1 + \mathcal{N}(x,v,N(T_0 - t_{i-1}(x,v))),$$

for $|(x,v) - (y,u)| < \delta_{x,v,\varepsilon,NT_0}$ for some $\delta_{x,v,\varepsilon,NT_0} \ll 1$. Now we consider consecutive multiple grazing.

When $i \leq j \leq k - 1$, (consecutive convex grazing), we split into two cases, Case 2-1 and Case 2-2.

**Case 2-1** We assume $\mathbf{n}(x^j(x,v)) = \mathbf{n}(x^{j+1}(x,v)) = \cdots = \mathbf{n}(x^{k-1}(x,v))$. When $|(x,v) - (y,u)| < \delta_{x,v,\varepsilon,NT_0} \ll 1$, we have

$$|(x^{i-1}(x,v),v^{i-1}(x,v)) - (x^{i-1}(y,u),v^{i-1}(y,u))| < O(\delta_{x,v,\varepsilon,NT_0}) \ll 1$$

from Lemma 5. When trajectory $(X(s;y,u,T_0),V(s;y,u,T_0))$ passes near $x^i(x,v)$, we split into several cases.

We claim that

$$\mathcal{N}(y,u,N(T_0 - t_k(x,v))) \leq 1 + \mathcal{N}(x,v,N(T_0 - t_k(x,v))),$$

holds for all following $(i)$ and $(iv)$ cases.

(i) If $x^{i-1}(x,v)x^k(x,v)$ does not bounce near $x^i(x,v)$ for all $j \in \{i, \cdots, k - 1\}$, then obviously we get (3.19).

If case (i) does not hold, we can assume that the backward trajectory $(X(s;y,u,T_0),V(s;y,u,T_0))$ hits near $x^i(x,v)$ without hitting near $x^j(x,v)$ for $i \leq j \leq \ell - 1$. Without loss of generality, we parametrize $B(x^i(x,v),\varepsilon) \cap \partial \Omega, \varepsilon \ll 1$ by regularized curve

$$\{\beta^{\ell}(\tau): \tau^\ell - \delta_1 < \tau < \tau^\ell + \delta_2, \quad \beta^{\ell}(\tau) = x^\ell(x,v)\}, \quad 0 < \delta_1, \delta_2 < 1.$$
(ii) Let \( x^i(y, u) = \beta^i(\tau) \) with \( \tau^f - \delta_1 < \tau < \tau^f \). Without loss of generality, we assume multigrazing dashed line as \( x \)-axis. By the specular BC, the trajectory \((X(s; y, u, T_0), V(s; y, u, T_0))\) must be above tangential line \( \{x^i(y, u) + s\beta^i(\tau) \mid s \in \mathbb{R}\} \) near \( x^i(y, u) \). Moreover, from the specular BC,

\[
\frac{|v^i(y, u)|}{|v^{i-1}(y, u)|} \cdot \beta^i(\tau) = \frac{|v^{i-1}(y, u)|}{|v^{i-1}(y, u)|} \cdot \beta^i(\tau) \\
\leq \left| \left( \frac{v^{i-1}(y, u)}{|v^{i-1}(y, u)|} - \frac{v^i(x, v)}{|v^i(x, v)|} \right) \cdot \beta^i(\tau) \right| + \left| \frac{v^i(x, v)}{|v^i(x, v)|} \cdot (\beta^i(\tau) - \beta^i(\tau^f)) \right| + \left| \frac{v^i(x, v)}{|v^i(x, v)|} \cdot \beta^i(\tau^f) \right| \\
\leq 1 + O(\delta_{x,v,e,NT_0}).
\]

This implies that the angle between \( v^{i-1}(y, u) \) and tangential line \( \{x^i(y, u) + s\beta^i(\tau) \mid s \in \mathbb{R}\} \) are very small, so we can apply the argument of (i) again and we obtain (3.19).

(iii) When \( x^i(y, u) = \beta^i(\tau^f) \), we must have

\[
\left| \frac{v^i(y, u)}{|v^i(y, u)|} \cdot \beta^i(\tau^f) \right| = \left| \frac{v^{i-1}(y, u)}{|v^{i-1}(y, u)|} \cdot \beta^i(\tau^f) \right| = \left| \frac{v^{i-1}(y, u)}{|v^{i-1}(y, u)|} \cdot \frac{v^i(x, v)}{|v^i(x, v)|} \right| = 1 + O(\delta_{x,v,e,NT_0}).
\]

So the angle between \( v^{i-1}(y, u) \) and \( v^i(x, v) \) are very small. Moreover, trajectory \((X(s; y, u, T_0), V(s; y, u, T_0))\) must be above dash tangential line, we can apply (i) to derive (3.19).

(iv) When \( x^i(y, u) = \beta^i(\tau) \) with \( \tau^f < \tau < \tau^f + \delta_2 \), angle between \( \beta^i(\tau) \) and \( \beta^i(\tau^f) \) is very small, since \( \delta_2 \ll 1 \). Moreover, angle between \( v^{i-1}(y, u) \) and \( \beta^i(\tau) \) is also small from (3.20). Therefore the angle between \( v^{i-1}(y, u) \) and \( \beta^i(\tau^f) \) is also small, i.e. \( v^{i-1}(y, u) \) is nearly parallel with dashed line in Fig 1. Therefore only (i) and (ii) cases are possible for \( x^{i+1}(y, u) \). For both cases, we gain (3.19).
(Case 2-2) Assume that there exist \( \{p_1, p_2, \ldots, p_q \} \in \{i + 1, \ldots, k - 1 \} \) with \( p_1 < p_2 < \cdots < p_q \) such that

\[
\begin{align*}
N &:= n(x(x, v)) = n(x^{i+1}(x, v)) = \cdots = n(x^{p_1-1}(x, v)) \\
-N &= n(x^{p_1}(x, v)) = n(x^{p_1+1}(x, v)) = \cdots = n(x^{p_2-1}(x, v)) \\
N &= n(x^{p_2}(x, v)) = n(x^{p_2+1}(x, v)) = \cdots = n(x^{p_3-1}(x, v)) \\
-N &= n(x^{p_3}(x, v)) = n(x^{p_3+1}(x, v)) = \cdots = n(x^{p_4-1}(x, v)) \\
&\cdots
\end{align*}
\]

\[
\mathfrak{N}(y, u, N(T_0 - t_k(x, v))) \leq 1 + \mathfrak{N}(x, v, N(T_0 - t_k(x, v))),
\]

holds for all cases.

First we define \( T_{p_1} := (t^{p_1-1}(x, v) - t^{p_1}(x, v))/2, 1 \leq \ell \leq q, \) and choose \( \delta_{x,v,e,NT_0} \) so that

\[
\delta_{x,v,e,NT_0} < \frac{T_{p_1}}{N}, \quad \text{for all } \ell \in \{1, \ldots, q\},
\]

which implies that traveling time (or distance) between \( x^{p_1}(x, v) \) and \( x^{p_1-1}(x, v) \) is sufficiently larger than the size of \( \delta_{x,v,e,NT_0} \). We split into two cases (v) and (vi) as following.

(v) If \( x'(y, u) \) does not hit near any of \( x'(x, v), \ldots, x^{p_1-1}(x, v) \), we have

\[
\left| (X(T_{p_1}; y, u, T_0), V(T_{p_1}; y, u, T_0)) - (X(T_{p_1}; x, v, T_0), V(T_{p_1}; x, v, T_0)) \right| \leq O(\delta_{x,v,e,NT_0}),
\]

by Lemma 5.

(vi) If \( x'(y, u) \) hits near one of \( x'(x, v), \ldots, x^{p_1-1}(x, v) \), then we can apply (ii), (iii), or (iv) of Case 1 to claim that there are at most 2 bouncings before trajectory \( (X(s; y, u, T_0), V(s; y, u, T_0)) \) approaches \( x^{p_1}(x, v) \). Moreover, in any case of (ii), (iii), and (iv), (assuming 2 bouncings WLOG),

\[
|x^{i+2}(y, u) - x^i(x, v)| = |v^{i+2}(y, u) - v^i(x, v)| = O(\delta_{x,v,e,NT_0}).
\]

And, since trajectory \( X(s; y, u, T_0) \) is very close to \( X(s; x, v, T_0) \),

\[
|X(s; y, u, T_0) - X(s; x, v, T_0)| \leq O(\delta_{x,v,e,NT_0}), \quad t^{i-1}(x, v) \leq s \leq T_{p_1}.
\]

Using above two estimates for both velocity and position, \([3.18]\) also holds for case (vi).

Now let us derive uniform number of bounce of the second case in \([3.18]\). For (Case 2-1), we proved that \([3.19]\) holds. For (Case 2-2) case, we change index \( p_1 - 1 \leftrightarrow k - 1 \), and then apply the same argument of (Case 2-1) to derive

\[
\mathfrak{N}(y, u, N(T_0 - T_{p_1}(x, v))) \leq \mathfrak{N}(x, v, N(T_0 - T_{p_1}(x, v))),
\]

Figure 4. Case 2-2

We split into cases and claim that

\[
\mathfrak{N}(y, u, N(T_0 - t_k(x, v))) \leq 1 + \mathfrak{N}(x, v, N(T_0 - t_k(x, v))),
\]

holds for all cases.
During \( t^{p_2}(x,v), t^{p_1}(x,v) \), we can also apply same argument of (Case 2-1) with help of (3.22) and (3.23) to obtain
\[
\mathcal{R}(y,u,N(T_0 - T_{p_1}(x,v))) \leq \mathcal{R}(x,v,N(T_0 - T_{p_1}(x,v))).
\]

We iterate this process until \( T_{p_q} \) to obtain
\[
\mathcal{R}(y,u,N(T_0 - T_{p_q}(x,v))) \leq \mathcal{R}(x,v,N(T_0 - T_{p_q}(x,v))).
\]

And since \((x,v)\) is non-grazing, we have
\[
\mathcal{R}(y,u,N(T_0 - t^k(x,v))) \leq 1 + \mathcal{R}(x,v,N(T_0 - t^k(x,v))),
\]
by applying (Case 2-1) for traveling from near \( x^{p_0}(x,v) \) to \( x^{k}(x,v) \).

**Step 2.** When we encounter second consecutive convex grazings after \( t^k(x,v) \), we can follow **Step 1** to derive similar estimate as (3.24). Finally there exist \( \delta_{x,v,\varepsilon,N T_0} \ll 1 \) such that
\[
\mathcal{R}(y,u,N T_0) \leq 1 + \max_{1 \leq i \leq \ell} \mathcal{R}(x_i,v_i,N T_0) \leq C_{\varepsilon,N T_0}.
\]

**Lemma 7.** Let \( \Omega \subset \mathbb{R}^2 \) satisfies Definition \[7\]. For any \((x,v) \in \{ \text{cl}(\Omega) \times \mathbb{V}^N \} \setminus \mathcal{A} \), trajectory \((X(s;T_0,x,v),V(s;T_0,x,v))\) for \( s \in [0,T_0) \) is uniformly away from inflection grazing set \( \gamma^I_0 \), i.e. there exists \( \rho_{x,v,N T_0} > 0 \) such that
\[
D_T(x,v) := \text{dist}(\partial \Omega^2, X(s;T_0,x,v)) + |n(X(s;T_0,x,v)) \cdot V(s;T_0,x,v)| \geq \rho_{x,v,N T_0} > 0,
\]
for all \( s \in [0,T_0) \) such that \( X(s;T_0,x,v) \in \partial \Omega \).

**Proof.** By definition of \( \mathcal{A} \) and Lemma \[3\]
\[
(X(s;T_0,x,v),V(s;T_0,x,v)) \notin \gamma^I_0.
\]
Therefore,
\[
\min_{t_j \in [0,T_0]} D_T(t^j(x,v),x,v) > 0,
\]
where \( D_T(t^j(x,v),x,v) \) is defined in (3.26). To derive uniform positivity, we use compactness argument again. From Lemma \[3\] for \((x,v) \in \{ \text{cl}(\Omega) \times \mathbb{V}^N \} \setminus \mathcal{A} \), we know that
\[
\mathcal{R}(x,v,N T_0) \leq C_{\varepsilon,N T_0}.
\]
Therefore,
\[
\min_{t_j \in [0,T_0]} D_T(t^j(x,v),x,v) = \min_{1 \leq j \leq C_{x,v,\varepsilon,N T_0}} D_T(t^j(x,v),x,v) := \rho_{x,v,\varepsilon,N T_0} > 0.
\]

for some uniform positive constant \( \rho_{x,v,\varepsilon,N T_0} > 0 \). Now we split into two cases.

**Case 1.** If \((X(s;T_0,x,v),V(s;T_0,x,v)) \notin \gamma_0\), we have local continuity from Lemma \[5\] so there exist \( r_{x,v,\varepsilon,N T_0} \ll 1 \) such that if \(|(x,v) - (y,u)| < r_{x,v,\varepsilon,N T_0}\),
\[
\min_{1 \leq j \leq C_{x,v,\varepsilon,N T_0}} |D_T(t^j(x,v),x,v) - D_T(t^j(y,u),y,u)| < \frac{\rho_{x,v,\varepsilon,N T_0}}{2}.
\]

From (3.27) and (3.28),
\[
\min_{1 \leq j \leq C_{x,v,\varepsilon,N T_0}} D_T(t^j(y,u),y,u) > \frac{\rho_{x,v,\varepsilon,N T_0}}{2},
\]
which implies uniform nonzero on a ball \( \text{cl}(\mathcal{B}(B((x,v),r_{x,v,\varepsilon,N T_0}))) \). By compactness, we have a finite open cover for \( \{ \text{cl}(\Omega) \times \mathbb{V}^N \} \setminus \mathcal{A} \), which is written by \( \bigcup_{k=1}^q B((x_i,v_i),r_{x,v,\varepsilon,N T_0}) \) for some finite \( q \). Finally, we pick uniform positive number
\[
\rho_{\varepsilon,N T_0} := \min_{1 \leq \ell} \frac{\rho_{x,v,\varepsilon,N T_0}}{2} > 0,
\]
to finish the proof.
Case 2. If \((X(s; T_0, x, v), V(s; T_0, x, v)) \in \gamma_0\) for some \(s \in [0, T_0]\), it must be concave grazing by definition of \(\mathfrak{W}\) and consider consecutive concave grazing cases of \((\text{Case 2-1})\) in the proof of Lemma [6] again with Figure 1. Let us assume \((3.18)\).

When \(j < i\), using Lemma [5] we have \(r_{x,v,\varepsilon,NT_0} \ll 1\) such that if \(|(x, v) - (y, u)| < r_{x,v,\varepsilon,NT_0}\),

\[
\min_{1 \leq j < i-1} |D_I(t^j(x,v), x,v) - D_I(t^j(y,u), y,u)| < \frac{\rho_{x,v,\varepsilon,NT_0}}{2}.
\]

When \(i \leq j \leq k - 1\), it is not reasonable to compare with same bouncing index, because we have discontinuity by convex grazing. However, since \(D_I\) is uniformly bounded from below by \((3.27)\), we suffice to compare \(D_I(t^j(y,u), y,u)\) with the nearest \(D_I(t^j(x,v), x,v)\) for some \(j \leq \ell\).

(i) If \(x^{j-1}(x,v)x^k(x,v)\) does not bounce near \(x^j(x,v)\) for all \(j \in \{i, \cdots, k-1\}\), then from Lemma [5] again, we can redefine \(r_{x,v,\varepsilon,NT_0} \ll 1\) so that if \(|(x, v) - (y, u)| < r_{x,v,\varepsilon,NT_0}\),

\[
|D_I(t^j(x,v), x,v) - D_I(t^j(y,u), y,u)| < \frac{\rho_{x,v,\varepsilon,NT_0}}{2},
\]

holds. This implies

\[
D_I(t^j(y,u), y,u) \geq D_I(t^j(x,v), x,v) - \frac{\rho_{x,v,\varepsilon,NT_0}}{2} > \frac{\rho_{x,v,\varepsilon,NT_0}}{2},
\]

from \((3.27)\).

(ii) From Lemma [5], there exist \(r_{x,v,\varepsilon,NT_0} \ll 1\) so that if \(|(x, v) - (y, u)| < r_{x,v,\varepsilon,N}\), \(|x^j(y,u) - x^i(x,v)| = O(r_{x,v,\varepsilon,N})\). Moreover, from \((3.20)\), \(|v^j(y,u) - v^i(x,v)| = O(r_{x,v,\varepsilon,N})\) also holds.

\[
|D_I(t^j(x,v), x,v) - D_I(t^j(y,u), y,u)| < \frac{\rho_{x,v,\varepsilon,NT_0}}{2},
\]

holds and therefore, \((3.29)\) also holds by \((3.27)\).

(iii) Obviously, \(|x^j(y,u) - x^i(x,v)| = 0\) and \(|v^j(y,u) - v^i(x,v)| = O(r_{x,v,\varepsilon,N})\) also holds by \((3.21)\), so yields \((3.29)\), similarly.

(iv) Near \(x^i(y,u)\) (near \(x^j(x,v)\)) and \(x^{j+1}(y,u)\) (near \(x^{j+1}(x,v)\)), we use argument of (i) for both bouncings to claim that

\[
D_I(t^j(y,u), y,u), D_I(t^{j+1}(y,u), y,u) \geq \frac{\rho_{x,v,\varepsilon,NT_0}}{2},
\]

if \(|(x, v) - (y, u)| < r_{x,v,\varepsilon,NT_0}\), for some small \(r_{x,v,\varepsilon,NT_0} \ll 1\).

From Step 2 in proof of Lemma [6] number of interval of consecutive grazing is uniformly bounded because we assume Definition 1. And whenever we encounter consecutive grazing, we can split into cases \((i) \sim (iv)\) to gain uniform positivity of \(D_I(t^j(y,u), y,u)\) for \(0 \leq t^j(y,u) \leq T_0\). And then we apply compactness argument of Case 1 in the proof of this Lemma to finish the proof.

\[\square\]

3.2. Dichotomy of sticky grazing.

**Lemma 8.** Assume \(\Omega \subset \mathbb{R}^2\) as defined in Definition 4. Assume that \((\alpha_j(\tau), \alpha_j'(\tau)) \in \gamma_0^\ast\) for some \(j \in \{1, \cdots, M\}\) and \(\tau \in (\tau^\ast - \delta, \tau^\ast + \delta) \subset [a_j, b_j]\). Also we assume that

\[(X(s; T_0, \alpha_j(\tau), \alpha_j'(\tau)), V(s; T_0, \alpha_j(\tau), \alpha_j'(\tau))) \notin \gamma_0\]

for \(s \in [0, T_0]\). Let us simplify notation:

\[x^i(\tau) := x^i(\alpha_j(\tau), \alpha_j'(\tau)), \quad v^i(\tau) := v^i(\alpha_j(\tau), \alpha_j'(\tau)), \quad t^i(\tau) := t^i(\alpha_j(\tau), \alpha_j'(\tau)),\]

for \(\tau \in (\tau^\ast - \delta, \tau^\ast + \delta) \subset [a_j, b_j]\). Then we have the following dichotomy. For each \(k\),

(a) There exist unique \(x^* \in c(\Omega)\) such that \(x^* = \frac{x^k(\tau)x^{k+1}(\tau)}{v^k(\tau)}\) for all \(\tau \in (\tau^* - \delta, \tau^* + \delta) \subset [a_j, b_j]\).

(b) For each \(x \in c(\Omega)\), the following set is finite

\[\left\{ \frac{v^k(\tau)}{v^k(\tau)} : x \in \frac{x^k(\tau)x^{k+1}(\tau)}{v^k(\tau)}, \tau \in (\tau^* - \delta, \tau^* + \delta) \right\}.\]
Proof. Assume that we have some $x^*$ satisfying (a). If there exist another $y^* \neq x^*$,
\[
x^k(\tau) - x^* = |x^k(\tau) - x^*| \frac{v^k(\tau)}{|v^k(\tau)|}, \quad x^k(\tau) - y^* = |x^k(\tau) - y^*| \frac{v^k(\tau)}{|v^k(\tau)|}, \quad \tau \in (\tau^* - \delta, \tau^* + \delta).
\]
This gives
\[
x^* - y^* = \left( |x^k(\tau) - y^*| - |x^k(\tau) - x^*| \right) \frac{v^k(\tau)}{|v^k(\tau)|}.
\]
Therefore, $\frac{v^k(\tau)}{|v^k(\tau)|}$ is constant unit vector for $\tau \in (\tau^* - \delta, \tau^* + \delta)$. And since $(x^k(\tau), v^k(\tau))$ is not grazing, $x^k(\tau)$ is also constant for all $\tau \in (\tau^* - \delta, \tau^* + \delta)$. Since trajectory is deterministic forward/backward in time, $\alpha_j(\tau)$ should be constant for $\tau \in (\tau^* - \delta, \tau^* + \delta)$ which implies $\alpha_j(\tau)$ is a part of straight line locally. This is contradiction, because $\Omega$ is analytic bounded domain.

If there does not exist $x^*$ which satisfies (a) for $\tau \in (\tau^* - \delta, \tau^* + \delta)$,
\[
\{ \tau \in (\tau^* - \delta, \tau^* + \delta) \mid x^k(\tau) - x^* = |x^k(\tau) - x^*| \frac{v^k(\tau)}{|v^k(\tau)|} \}
\]
is a finite set for any $x^* \in cl(\Omega)$ by rigidity of analytic function. This yields (b).

\[\square\]

3.3. Grazing set. In this section, we characterize the points of $\{ cl(\Omega) \times \mathbb{V}^N \} \setminus \mathcal{IB}$ whose specular backward cycle grazes the boundary (hits the boundaries tangentially) at some moment. By definition of $\mathcal{IB}$, this grazing cannot be inflection grazing $\gamma^\prime_0$. Moreover, Lemma 4 guarantees that convex grazing does not happen neither. Therefore, the only possible grazing is concave grazing $\gamma_0^\prime$. We will classify this concave grazing sets depending on the first(backward in time) concave grazing time.

**Definition 7.** For $T_0 > 0$ and $(x, v) \in cl(\Omega) \times \mathbb{R}^2$, we define grazing set:
\[
\mathcal{G} := \left\{ (x, v) \in cl(\Omega) \times \mathbb{V}^N \right\} \setminus \mathcal{IB} : \exists s \in [0, T_0) \ s.t \ (X(s; T_0, x, v), V(s; T_0, x, v)) \in \gamma_0^\prime \right\},
\]
which is a set of phase $(x, v)$ whose trajectory grazes at least once for time interval $[0, T_0]$. We also define $\mathcal{G}^C$, $\mathcal{G}^V$, and $\mathcal{G}^I$ by grazing type, i.e.
\[
\mathcal{G}^C := \left\{ (x, v) \in cl(\Omega) \times \mathbb{V}^N \right\} \setminus \mathcal{IB} : \exists s \in [0, T_0) \ s.t \ (X(s; T_0, x, v), V(s; T_0, x, v)) \in \gamma^C_0 \right\},
\]
\[
\mathcal{G}^V := \left\{ (x, v) \in cl(\Omega) \times \mathbb{V}^N \right\} \setminus \mathcal{IB} : \exists s \in [0, T_0) \ s.t \ (X(s; T_0, x, v), V(s; T_0, x, v)) \in \gamma^V_0 \right\},
\]
\[
\mathcal{G}^I := \left\{ (x, v) \in cl(\Omega) \times \mathbb{V}^N \right\} \setminus \mathcal{IB} : \exists s \in [0, T_0) \ s.t \ (X(s; T_0, x, v), V(s; T_0, x, v)) \in \gamma^I_0 \right\}.
\]

By definition of $\mathcal{IB}$, we know that $\mathcal{G}^V = \mathcal{G}^I = \emptyset$. Therefore, we rewrite and decompose $\mathcal{G}$ as
\[
\mathcal{G} = \mathcal{G}^C := \bigcup_{l=1}^{M^C} \mathcal{G}^{C,l} := \bigcup_{l=1}^{M^C} \mathcal{G}_l^C = \bigcup_{l=1}^{M^C} \mathcal{G}^{C,l},
\]
where
\[
\mathcal{G}^{C,l} := \left\{ (x, v) \in \mathcal{G}^C : (x^l(x, v), v^l(x, v)) \in \gamma^C_0 \right\},
\]
\[
\mathcal{G}_l^C := \left\{ (x, v) \in \mathcal{G}^C : \exists k \ s.t \ (X(x, v), V(x, v)) \in \gamma_0^C \ and \ x^k(x, v) \in \partial \Omega_0^C \right\},
\]
\[
\mathcal{G}^{C,l} := \left\{ (x, v) \in \mathcal{G}^{C,l} : x^l(x, v) \in \partial \Omega_0^C \right\},
\]
where $l \in \{1, \ldots, M^C\}$ which is defined in \[2.6\].

**Remark 1.** Let us use renumbered notation \[2.6\] and the sets defined in Definition 7. If $(x, v) \in \mathcal{G}_l^C$, then there exists $\tau \in (\tilde{a}_l, \tilde{b}_l)$ and $k$ such that $(x^k(x, v), v^k(x, v)) \in \gamma_0^C$ and $x^k(x, v) = \tilde{a}_l(\tau)$. Due to Lemma 4, such $\tau$ cannot be arbitrarily close to the end points $\tilde{a}_l, \tilde{b}_l$ which are inflection points $\kappa = 0$. Lemma 4 implies that there exists $\tilde{a}_l^* > \tilde{a}_l$ and $\tilde{b}_l^* > \tilde{b}_l$ for each $l \in \{1, \ldots, M^C\}$ such that
\[
\left\{ \tau \in (\tilde{a}_l, \tilde{b}_l) : (X(s; T_0, x, v), V(s; T_0, x, v)) \in \gamma_0^C, \ X(s; T_0, x, v) = \tilde{a}_l(\tau) \ for \ (x, v) \in \mathcal{G}_l^C \right\} \subset [\tilde{a}_l^*, \tilde{b}_l^*]. \quad (3.30)
\]
Throughout this subsection, we use some temporary symbols. Inspired by (2.4), we can also define $k$-th backward/forward exit time:

\[
\begin{align*}
t_b(x,v) & := t_1^b(t,x,v), \\
t_b^k(x,v) & := t - t^b_k(t,x,v), \\
x_b(x,v) & := x^k(t,x,v), \\
t_f(x,v) & := t_F^k(t,x,v), \\
t_f^k(x,v) & := -t^k(0,x,-v), \\
x_f^k(x,v) & := x^k(t,x,-v).
\end{align*}
\]

3.3.1. **Grazing Set, $\mathcal{G}_{i,1}^C$.** Let us use renumbered notation for concave part (2.6). From the definition of $\mathcal{G}_{i,1}^C$ and (3.30),

\[
\mathcal{G}_{i,1}^{C_1} \subset \bigcup_{p=\pm 1} \left\{ (\bar{\alpha}_i(t) + sp|v|\tilde{\alpha}_i(t), p|v|\tilde{\alpha}_i(t)) \in \left\{ (\text{cl}(\Omega) \times \nu^N) \cap \mathcal{B} \right\} : \right. \\
& \left. \tau \in [\bar{\alpha}_i^*, \bar{\beta}_i^*], \ v \in \nu^N, \ s \in [0, t_f(\bar{\alpha}_i(t), p|v|\tilde{\alpha}_i(t))] \right\}.
\]

Since the signed curvature $\kappa$ is positive and bounded, but finite points, $\mathbb{S}^1 \cap \{v \in \mathbb{R}^2 : (x,v) \in \mathcal{G}_{i,1}^{C_1} \}$ has at most two points for fixed $x$. Since $M^C$ is uniformly bounded, $\mathbb{S}^1 \cap \{v \in \mathbb{R}^2 : (x,v) \in \mathcal{G}_{i,1}^{C_1} \}$ contains at most $2 \times M^C$ points and therefore,

\[
m_2\{v \in \mathbb{R}^2 : (x,v) \in \mathcal{G}_{i,1}^{C_1} \} = 0. \tag{3.31}
\]

**Lemma 9.** For any $\varepsilon > 0$, there exist an open cover $\bigcup_{i=1}^{l_1} B(x_i^{C_1}, r_i^{C_1})$ for $\mathcal{P}_x(\{\text{cl}(\Omega) \times \nu^N \} \cap \mathcal{B})$, where $\mathcal{P}_x$ is projection on spatial space, and corresponding velocity set $\mathcal{O}_{x_i}^{C_1} \subset \nu^N$ with $m_2(\mathcal{O}_{x_i}^{C_1}) < \varepsilon$ such that

1. For any $(x,v) \in \{\text{cl}(\Omega) \times \nu^N \} \cap \mathcal{B}$, there exists $x_i^{C_1}$, $r_i^{C_1}$, and $\delta_i^{C_1} > 0$ such that $x \in B(x_i^{C_1}, r_i^{C_1})$ and
2. $\phi^1(x,v) = |v \cdot n(x_b(x,v))| > \delta_i^{C_1} > 0$ holds for $v \in \nu^N \setminus \mathcal{O}_{x_i}^{C_1}$, for some uniformly positive $\delta_i^{C_1} > 0$.

From above, we define $\varepsilon$- neighborhood of $\mathcal{G}_{i,1}^{C_1}$:

\[
(\mathcal{G}_{i,1}^{C_1})_\varepsilon := \bigcup_{i=1}^{l_1} \{ B(x_i^{C_1}, r_i^{C_1}) \times \mathcal{O}_{x_i}^{C_1} \}.
\]

**Proof.** Let $x \in \mathcal{P}_x(\{\text{cl}(\Omega) \times \nu^N \} \cap \mathcal{B})$. Then, there exist at most $2M^C$ distinct unit velocity $\frac{v_i}{|v_i|}$, $i \in \{1, \cdots, 2M^C\}$ such that $(x,v_i) \in \mathcal{G}_{i,1}^{C_1}$. We define

\[
\mathcal{O}_{x_i}^{C_1} := \left\{ v \in \nu^N : \left| \frac{v}{|v|} - \left| \frac{v}{|v|} \right| < C_1(\varepsilon) \right. \right\}.
\]

When $v \in \nu^N \setminus \mathcal{O}_{x_i}^{C_1}$, we can apply Lemma 5 to show that

\[
\phi^1(x,v) := |v \cdot n(x_b(x,v))| \quad \text{is well-defined and locally smooth, since } (x,v) \in \left\{ (\text{cl}(\Omega) \times \nu^N \} \setminus \mathcal{B} \right\} \setminus \mathcal{G}_{i,1}^{C_1}.
\]

Using local continuity of Lemma 5 again, we can find $r_{x_i}^{C_1} \leq 1$ such that

\[
\phi^1(x,v) > \delta_{x_i}^{C_1} > 0, \quad \text{for } (x,v) \in \text{cl}(B(x,r_{x_i}^{C_1})) \times \nu^N \setminus \mathcal{O}_{x_i}^{C_1}.
\]

By compactness, we can find finite open cover $\bigcup_{i=1}^{l_i} B(x_i^{C_1}, r_i^{C_1})$ for $\mathcal{P}_x(\{\text{cl}(\Omega) \times \nu^N \} \cap \mathcal{B})$ and corresponding $\mathcal{O}_{x_i}^{C_1}$ with small measure $m_2(\mathcal{O}_{x_i}^{C_1}) < \varepsilon$ by choosing (3.32) with some proper small $C_1(\varepsilon)$. Finally we choose

\[
\delta_i^{C_1} := \min_{1 \leq i \leq l_i} \delta_{x_i}^{C_1} > 0,
\]

to finish the proof. \[\square\]
3.3.2. 2nd Grazing Set $\mathcal{G}^{C,2}$. From the definition of $\mathcal{G}^{C,2}$ and (3.30), the set $\mathcal{G}^{C,2} \setminus (\mathcal{G}^{C,1})_\epsilon$ is a subset of

$$\bigcup_{l=1}^{M^C} \bigcup_{p=1}^{L^l} \left\{ \left( x^l_I(\bar{\alpha}_l(\tau), p|v|\dot{\alpha}_l(\tau)), s^l_I(\bar{\alpha}_l(\tau), p|v|\dot{\alpha}_l(\tau)), v^l_I(\bar{\alpha}_l(\tau), p|v|\dot{\alpha}_l(\tau)) \right) \in \{cl(\Omega) \times \mathbb{V}^N \} \setminus \mathcal{B} : \tau \in [\bar{a}_l^*, \bar{b}_l^*], v \in \mathbb{V}^N, s \in [0, (t^l_I - t^l_I')(\bar{\alpha}_l(\tau), p|v|\dot{\alpha}_l(\tau))] \right\} \right\} \setminus (\mathcal{G}^{C,1})_\epsilon. \tag{3.33}$$

Without loss of generality, we suffice to consider only $p = 1$ case of (3.33), since $p = -1$ does not change any argument.

**Step 1** Fix $p = 1$ and $l \in \{1, \ldots, M^C\}$. First, we remove 1st-grazing set by complementing $(\mathcal{G}^{C,1})_\epsilon$.

Let us consider $(\bar{x}, \bar{v}) \in \mathcal{G}^{C,1} \cap \mathcal{G}^{C,2}$ and we write $\bar{\alpha}_l(\tau) = x^2(\bar{x}, \bar{v})$. Then, from Lemma 9 and Lemma 5, there exist $i \in \{1, \ldots, l\}$ such that $(\bar{x}, \bar{v}) \in B(x^C_{i, 1}, r^C_{i, 1}) \times O^C_{i, 1}$ and

$$\left\{ x^2(x, v) \in \partial O^C_{i, 1} : \forall (x, v) \in cl(B(x^C_{i, 1}, r^C_{i, 1}) \times O^C_{i, 1}) \right\}$$

$$\subset \left[ \bar{\alpha}_l(\bar{r} - \delta_n), \bar{\alpha}_l(\bar{r} + \delta_n) \right], \quad \text{for } \delta_n = O(r_{l, 1, \varepsilon}) \ll 1.$$

Excluding (3.34) from $[\bar{a}_l^*, \bar{b}_l^*]$ for all $(\bar{x}, \bar{v}) \in \mathcal{G}^{C,1} \cap \mathcal{G}^{C,2}$ yields a union of countable open connected intervals $I$, i.e.

$$I := [\bar{a}_l^*, d_{i,1}] \cup [c_{l,2}, d_{l,2}] \cup \cdots \cup [\bar{a}_l^*, b_l^*], \quad \bar{a}_l^* < d_1 < c_2 < d_2 < \cdots .$$

Now we claim that $I$ contains only finite subintervals. If this union is not finite, there exist infinitely many distinct $\{\tau_n\}_{n=1}^\infty$, $\tau_1 < \tau_2 < \cdots$ such that

$$\dot{\alpha}_l(\tau_n) \cdot n(x^I(\bar{\alpha}_l(\tau_n), \bar{\alpha}_l(\tau_n))) = 0, \quad i \in \mathbb{N}.$$}

We pick monotone increasing sequence $\tau_1, \tau_2, \cdots, \tau_n, \cdots$ by choosing a point $\tau_n$ for each disjoint closed interval. Since $\tau_n \leq d_1^+$ for all $n \in \mathbb{N}$, there exist a $\tau_n$ such that $\tau_n \to \tau_\infty$ up to subsequence. Let us assume that

$$\left( x^I(\bar{\alpha}_l(\tau_n), \bar{\alpha}_l(\tau_n)), \bar{\alpha}_l(\tau_n) \right) \in \gamma_0^C : x^I(\bar{\alpha}_l(\tau_n), \bar{\alpha}_l(\tau_n)) \in \partial O_{p}^C.$$}

Since we have chosen $\tau_n$’s from each distinct intervals, there exist $\tau', \tau_n < \tau' < \tau_{n+1}$ such that

$$\left( x^I(\bar{\alpha}_l(\tau'), \bar{\alpha}_l(\tau')) \right) \notin \gamma_0^C.$$}

By monotonicity of $\{\tau_1, \cdots, \tau_\infty\}$ the fact that $\tau_\infty$ is accumulation implies that we have accumulating concave grazing phase $\{ (x^I(\bar{\alpha}_l(\tau_n), \bar{\alpha}_l(\tau_n)), \bar{\alpha}_l(\tau_n)) \}_{n=1}^\infty$ near $\{ (x^I(\tau(\tau_\infty), \bar{\alpha}_l(\tau(\tau_\infty)), \bar{\alpha}_l(\tau(\tau_\infty))) \}$. This is contradiction because $\partial \Omega$ is analytic domain. Finally we can write $I$ as disjoint union of finite $m^C_{l, 2}$ intervals, i.e.

$$I := [\bar{a}_l^*, d_{i,1}] \cup [c_{l,2}, d_{l,2}] \cup \cdots \cup [\bar{a}_l^*, b_l^*]. \tag{3.35}$$}

**Step 2** Since we have chosen $\delta_\pm$ as nonzero in (3.34), we can include boundary points of each subinterval of (3.35). Therefore, $\mathcal{G}^{C,2} \setminus (\mathcal{G}^{C,1})_\epsilon$ is a subset of

$$\bigcup_{l=1}^{M^C} \bigcup_{p=1}^{L^l} \left\{ \left( x^l_I(\bar{\alpha}_l(\tau), |v|\dot{\alpha}_l(\tau)), s^l_I(\bar{\alpha}_l(\tau), |v|\dot{\alpha}_l(\tau)), v^l_I(\bar{\alpha}_l(\tau), |v|\dot{\alpha}_l(\tau)) \right) \in \{cl(\Omega) \times \mathbb{V}^N \} \setminus \mathcal{B} : \tau \in [\bar{a}_l^*, d_{i,1}] \cup [c_{l,2}, d_{l,2}] \cup \cdots \cup [c_{l,m_{l,2}^l-1}, d_{l,m_{l,2}^l-1}] \cup [c_{l,m_{l,2}^l}, \bar{b}_l^*], \right.$$}

$$v \in \mathbb{V}^N, s \in [0, (t^l_I - t^l_I')(\bar{\alpha}_l(\tau), |v|\dot{\alpha}_l(\tau))] \right\} \setminus (\mathcal{G}^{C,1})_\epsilon \right\} \right\} \setminus (\mathcal{G}^{C,1})_\epsilon. \tag{3.36}$$

where $\delta_{C,1}$ was found in Lemma 9. Moreover, we can choose these subintervals so that measure of each punctures $\{(d_{i,1}, c_{i,1+1})\}_{i=1}^{m_{l,2}^l-1}$ are arbitrary small, because we can choose $\delta_\pm > 0$ arbitrary small in (3.34).

**Step 3** We construct 2nd-Sticky Grazing Set $\mathcal{G}_{C,2}$ where all grazing rays from non-measure zero subset of $[\bar{a}_l^*, d_{i,1}] \cup [c_{l,2}, d_{l,2}] \cup \cdots \cup [c_{l,m_{l,2}^l-1}, d_{l,m_{l,2}^l-1}] \cup [c_{l,m_{l,2}^l}, \bar{b}_l^*]$ intersect at a fixed point in $\mathcal{P}_\mathcal{B}(\{cl(\Omega) \times \mathbb{V}^N \} \setminus \mathcal{B})$ where $\mathcal{P}_\mathcal{B}$ is projection on spital domain. Choose any $i \in \{1, \ldots, m_{l,2}^l\}$ and corresponding sub interval $[c_{l,i}, d_{l,i}]$. We define

$$\mathcal{G}_{C,2} : \left\{ \left( x^l_I(\bar{\alpha}_l(\tau), |v|\dot{\alpha}_l(\tau)), s^l_I(\bar{\alpha}_l(\tau), |v|\dot{\alpha}_l(\tau)), v^l_I(\bar{\alpha}_l(\tau), |v|\dot{\alpha}_l(\tau)) \right) \in \{cl(\Omega) \times \mathbb{V}^N \} \setminus \mathcal{B} : \tau \in [c_{l,i}, d_{l,i}], v \in \mathbb{V}^N, s \in [0, (t^l_I - t^l_I')(\bar{\alpha}_l(\tau), |v|\dot{\alpha}_l(\tau))] \right\} \right\} \setminus (\mathcal{G}^{C,1})_\epsilon.$$}

Fix $x^* \in \overline{\Omega}$. If there does not exist $\tau \in [c_{l,i}, d_{l,i}]$ and $s \in [t^l_I(\alpha(I), \alpha(I')), t^l_I(\alpha(I'), \alpha(I'))]$ satisfying $x^* = x^l_I(\alpha(I), \alpha(I')) + s^l_I(\alpha(I), \alpha(I'))$ then $\forall v \in \mathbb{R}^2 : (x^*, v) \in \mathcal{G}_{C,2} = \emptyset$ with zero measure. Now suppose that
there exist such $\tau$ and $s$.

Due to Lemma 8 there are only two cases: (i) **sticky grazing**: for all $\tau \in [c_{l,i}, dt_2]$, there exists $s = s(\tau) \in [t^*_f(\alpha^i(\tau), \bar{\alpha}^i(\tau)), t^*_f(\alpha^i(\tau), \bar{\alpha}^i(\tau))]$ and fixed $x^* \in cl(\Omega)$ such that

$$x^* = x^*_f(\alpha^i(\tau), \bar{\alpha}^i(\tau)) + s v^*_f(\alpha^i(\tau)),$$

(3.37) 

or (ii) **isolated grazing**: there exists $\delta_-, \delta_+ > 0$ so that for $\tau' \in (\tau - \delta_-, \tau + \delta_+) \setminus \{\tau\}$, there is no $s$ satisfying (3.37).

We define the $2^{nd}$ sticky grazing set $SG^{C,2}$ as collection of all such $x^* \in cl(\Omega)$ points, i.e.

**Definition 8.** Consider (3.36) and disjoint union of intervals $[a^*_1, c_{l,i}] \cup [c_{l,i}, dt_2] \cup \cdots \cup [c_{l,m^*_1-1}, dt_{i,m^*_1-1}] \cup [c_{l,m^*_1}, \bar{b}^*_1]$. There are finite $i \in I_{sg,l}^2 \subset \{1, 2, \ldots, M'_2\}$ such that case (i) **sticky grazing** holds:

$$\bigcup_{\tau \in [c_{l,i},dt_2]} x^*_f(\alpha^i(\tau), \bar{\alpha}^i(\tau)) x^*_f(\alpha^i(\tau), \bar{\alpha}^i(\tau)) = x^*_{sg,i} \text{ which is a point in } cl(\Omega),$$

by writing $\bar{a}^*_1 = c_{l,1}$, $\bar{b}^*_1 = d_{i,m^*_1}$. The $2^{nd}$ sticky grazing set is the collection of such points:

$$SG^{C,2} := \bigcup_{l=1}^{M^C} \bigcup_{i=1}^{M^C} \{x^*_{sg,i} \in cl(\Omega) : i \in I_{sg,l}^2\}. \tag{3.38}$$

Note that $SG^{C,2}$ is a set of finite points, from finiteness of $M^C$ and Lemma 8.

**Step 4** We claim

$$m_2\{v \in \mathbb{R}^2 : (x, v) \in \mathcal{G}^{C,2} \setminus \mathcal{O}^{C,2}\} = 0,$$

(3.39)

for all $x \in \mathcal{P}_x(\{cl(\Omega) \times \mathcal{V}^N\} \setminus \mathcal{O}^N(\mathbb{B}))$, $\mathcal{S}G^{C,2}$. Consider again the set (3.36) and fix $l \in \{1, \ldots, M^C\}$. For any $i \in \{1, \ldots, M'_2\} \setminus I_{sg,l}^2$, we apply case (b) of Lemma 8 to say that

$$\{v \in \mathbb{R}^2 : (x, v) \in \mathcal{G}^{C,2} \setminus \mathcal{O}^{C,2}\} \cap S^1 = \text{finite points},$$

which gives (3.39).

**Lemma 10.** For any $\varepsilon > 0$, there exist an open cover

$$\bigcup_{i=1}^{I_2} B(x_i^{C,2}, r_i^{C,2}) \cup \{y \in SG^{C,2} : B(y, \varepsilon)\}$$

for $\mathcal{P}_x(\{cl(\Omega) \times \mathcal{V}^N\} \setminus \mathcal{O}^N(\mathbb{B}))$ and corresponding velocity sets $O^{C,2}_i \subset \mathcal{V}^N$ with $m_2(O^{C,2}_i) < \varepsilon$ such that

1. For any $(x, v) \in \{cl(\Omega) \times \mathcal{V}^N\} \setminus \mathcal{O}^N(\mathbb{B})$, $x \in B(x_i^{C,2}, r_i^{C,2})$ or $x \in B(y, \varepsilon)$,

for some $x_i^{C,2}$, $r_i^{C,2}$, and $y \in SG^{C,2}$.

2. Moreover, if $x \notin \bigcup_{i \in SG^{C,2}} B(y, \varepsilon)$, $x \in B(x_i^{C,2}, r_i^{C,2})$, and $v \in \mathcal{V}^N \setminus O^{C,2}_i$, then

$$\phi^2(x, v) = |v^1(x, v) \cdot n(x^2(x, v))| > \delta^{C,2} > 0 \quad \text{and} \quad \phi^1(x, v) = |v \cdot n(x_1(x, v))| > \delta^{C,1} > 0,$$

for some uniformly positive $\delta^{C,1}, \delta^{C,2} > 0$.

From above, we define $\varepsilon$-neighborhood of $\mathcal{O}^{C,2}$:

$$\mathcal{O}^{C,2}_x := \left\{ v \in \mathcal{V}^N : \left| \frac{v^1}{|v|} - \frac{v}{|v|} \right| < C_2(N)\varepsilon, \forall v_i \text{ s.t } (x, v_i) \in \mathcal{G}^{C,1} \cup \mathcal{G}^{C,2} \right\}.$$
When \( v \in \mathbb{V}^N \setminus \mathcal{O}_{C,2}^i \), trajectory does not graze within second bounces, so both
\[
\phi^1(x,v) = |v \cdot n(x_0(x,v))|, \quad \phi^2(x,v) := |v^1(x,v) \cdot n(x^2(x,v))|
\]
are well-defined and locally smooth, because \((x,v) \in \{ cl(\Omega) \times \mathbb{V}^N \} \setminus \mathcal{M}_{2}^i \) \( \Rightarrow \mathcal{O}_{C,1}^i \cup \mathcal{O}_{C,2}^i \) implies that trajectory does not graze in first two bounces. Using local continuity of Lemma 5 again, we can find \( r_{C,2}^k \ll 1 \) such that
\[
\phi^1(x,v) > \delta_1^2 > 0, \quad \phi^2(x,v) > \delta_2^2 > 0, \quad \text{for} \quad (x,v) \in cl(B(x_r^{C,2} r_1^k)) \setminus \mathbb{V}^N \setminus \mathcal{O}_{C,2}^i.
\]
By compactness, we can find finite open cover \( \bigcup_{i=1}^{d_1^k} B(x_r^{C,2} r_1^k) \) for \( \mathcal{P}_x \{ cl(\Omega) \times \mathbb{V}^N \} \setminus \mathcal{M}_{2}^i \) \( \cup \) \( y \in \mathcal{S}^G C, k \) \( B(y, \varepsilon) \) and corresponding \( \mathcal{O}_{C,2}^i \) with small measure \( m_2(\mathcal{O}_{C,2}^i) < \varepsilon \) by choosing (3.32) with sufficiently small \( C_2(N) \). Finally we choose
\[
\delta_1 := \min_{1 \leq i \leq l_1} \delta_{1,C,2}^i > 0, \quad \delta_2 := \min_{1 \leq i \leq l_1} \delta_{2,C,2}^i > 0,
\]
to finish the proof.

3.3.3. kth—Grazing Set, \( \mathcal{S}^G C, k \). Now we are going to construct, for \( k > 2 \), the \( k^{th} \) Grazing Set and it’s \( \varepsilon \)-neighborhood. We construct such sets via the mathematical induction. We assume Lemma 10 holds for \( \mathcal{S}^G C, k-1 \), i.e.

**Assumption 1.** For any \( \varepsilon > 0 \), there exist \( \mathcal{S}_{C,k-1}^G \) which contains finite points in \( cl(\Omega) \), and an open cover
\[
\left\{ \bigcup_{i=1}^{l_{k-1}} B(x_i^{C,k-1}, r_i^{C,k-1}) \right\} \bigcup \left\{ \bigcup_{y \in \mathcal{S}_{C,k-1}^G} B(y, \varepsilon) \right\}
\]
for \( \mathcal{P}_x \{ cl(\Omega) \times \mathbb{V}^N \} \setminus \mathcal{M}_{2}^i \) and corresponding velocity sets \( \mathcal{O}_{C,k-1}^i \subset \mathbb{V}^N \) with \( m_2(\mathcal{O}_{C,k-1}^i) < \varepsilon \) such that

1. For any \((x,v) \in cl(\Omega) \times \mathbb{V}^N \) \( \setminus \mathcal{M}_{2}^i \),
\[
\begin{align*}
 x & \in B(x_i^{C,k-1}, r_i^{C,k-1}) \quad \text{or} \quad x \in B(y, \varepsilon), \\
 for \ some \ x_i^{C,k-1}, r_i^{C,k-1}, \text{and} \ y \in \mathcal{S}_{C,k-1}^G.
\end{align*}
\]
2. Moreover, if \( x \notin \bigcup_{y \in \mathcal{S}_{C,k-1}^G} B(y, \varepsilon) \), \( x \in B(x_i^{C,k-1}, r_i^{C,k-1}) \), and \( v \in \mathbb{V}^N \setminus \mathcal{O}_{C,k-1}^i \), then
\[
\phi^s(x,v) = |v^{s-1}(x,v) \cdot n(x^s(x,v))| > \delta_{C,s} > 0,
\]
for all \( s \in \{ 1, \ldots, k-1 \} \) some uniformly positive \( \delta_{C,1}, \delta_{C,2}, \ldots, \delta_{C,k-1} > 0 \).

We define \( \varepsilon \)-neighborhood of \( \mathcal{S}_{C,k-1}^G \):
\[
(\mathcal{S}_{C,k-1}^G)_\varepsilon = \left\{ \bigcup_{i=1}^{l_{k-1}} B(x_i^{C,k-1}, r_i^{C,k-1}) \times \mathcal{O}_{C,k-1}^i \right\} \bigcup \left\{ \bigcup_{y \in \mathcal{S}_{C,k-1}^G} B(y, \varepsilon) \times \mathbb{V}^N \right\}.
\]
Now, under above assumption, we follow the steps in \( \mathcal{S}_{C}^G \). From the definition of \( \mathcal{S}_{C,k}^G \) and (3.30), the set \( \mathcal{S}_{C,k}^G \setminus (\mathcal{S}_{C,k-1}^G)_\varepsilon \) is a subset of
\[
\bigcup_{\tau = [\tilde{a}_i^+, \tilde{b}_i^+]}^{M_C} \bigcup_{\nu = [\tilde{a}_i^-, \tilde{b}_i^-]}^{t_{k-1}} \left\{ (x^{k-1}(\tilde{a}_i(\tau), p|v|\tilde{a}_i(\tau)) + s v^{k-1}_f(\tilde{a}_i(\tau), p|v|\tilde{a}_i(\tau)), v^{k-1}_f(\tilde{a}_i(\tau), p|v|\tilde{a}_i(\tau))) \right\} \in cl(\Omega) \times \mathbb{V}^N \setminus \mathcal{M}_{2}^i : \quad (3.40)
\]
Without loss of generality, we suffice to consider only \( p = 1 \) case of (3.40).

**Step 1** Fix \( p = 1 \) and \( l \in \{ 1, \ldots, M_C \} \). First, we remove \( k-1 \) th-grazing set by complementing \( (\mathcal{S}_{C,k-1}^G)_\varepsilon \).
Let us consider \((\bar{x}, \bar{v}) \in \mathcal{S}_{C,k-1}^G \cap \mathcal{S}_{C,k}^G \) and write \( \tilde{a}_i(\bar{x}, \bar{v}) = x^k(\bar{x}, \bar{v}) \). Then, from Assumption 1 there exist \( i \in \{ 1, \ldots, l_{k-1} \} \) such that \((\bar{x}, \bar{v}) \in B(x_i^{C,k-1}, r_i^{C,k-1}) \times \mathcal{O}_{C,k-1}^i \) and
\[
\begin{align*}
\left\{ x^k(\bar{x}, \bar{v}) \in \partial \mathcal{O}_{C,k}^i : \forall (x,v) \in cl(B(x_i^{C,k-1}, r_i^{C,k-1}) \times \mathcal{O}_{C,k-1}^i), \text{ where} \ (\bar{x}, \bar{v}) \in B(x_i^{C,k-1}, r_i^{C,k-1}) \times \mathcal{O}_{C,k-1}^i \right\} \\
\subset \left[ \tilde{a}_i(\bar{x} - \delta_-), \tilde{a}_i(\bar{x} + \delta_+) \right], \quad \text{for} \quad 0 < \delta_{k-1} = O(r_{k-1,i}^k, \varepsilon) \ll 1.
\end{align*}
\]
Excluding (3.41) from \( \tilde{a}_i^+, \tilde{b}_i^- \) for all \((\bar{x}, \bar{v}) \in \mathcal{S}_{C,k-1}^G \cap \mathcal{S}_{C,k}^G \) yields a union of countable open connected intervals \( \mathcal{T}^k \), i.e.
\[
\mathcal{T} := \tilde{a}_i^+ \cup d_{i,2}^k \cup \cdots \subset \tilde{a}_i^+, \tilde{b}_i^- \], \quad \tilde{a}_i^+ < d_1 < c_2 < d_2 < \cdots .
\]
Using exactly same argument of Step 1 in 2nd–Grazing Set $\Theta^{C,2}$, we know that this should be finite union of subintervals and write
\[ I^k := [\tilde{a}_1^k, d_{l,1}^k] \cup (c_{l,2}^k, d_{l,2}^k) \cup \cdots \cup (c_{l,m_k^k-1}^k, d_{l,m_k^k-1}^k) \cup (c_{l,m_k^k}^k, \tilde{b}_1^l). \tag{3.42} \]

**Step 2** Since we have chosen $\delta_k$ as nonzero in (3.4.1), we can include boundary points of each subinterval of (3.42). Therefore, $\Theta^{C,k} \setminus (\Theta^{C,k-1})_l$ is a subset of
\[ \bigcup_{l=1}^{M^C} \left\{ \left( x_l^{-1}(\tilde{a}_l(\tau), |v|\tilde{a}_l(\tau)) + st_{l}^{-1}(\tilde{a}_l(\tau), |v|\tilde{a}_l(\tau)), v_l^{-1}(\tilde{a}_l(\tau), |v|\tilde{a}_l(\tau)) \right) \in (cl(\Omega) \times \mathbb{V}^N) \setminus \mathcal{B} : \right. \]
\[ \tau \in [\tilde{a}_l^k, d_{l,1}^k] \cup [c_{l,2}^k, d_{l,2}^k] \cup \cdots \cup [c_{l,m_l^k-1}^k, d_{l,m_l^k-1}^k] \cup [c_{l,m_l^k}^k, \tilde{b}_1^l], \]
\[ v \in \mathbb{V}^N, s \in [0, (t_{l_k}^{-1} - t_{l_k}^{-1})(\tilde{a}_l(\tau), |v|\tilde{a}_l(\tau))], \]
and for all $\tau \in [\tilde{a}_l^k, d_{l,1}^k] \cup [c_{l,2}^k, d_{l,2}^k] \cup \cdots \cup [c_{l,m_l^k-1}^k, d_{l,m_l^k-1}^k] \cup [c_{l,m_l^k}^k, \tilde{b}_1^l],$
\[ |\tilde{a}_l(\tau) \cdot \mathbf{n}(x_l(\tilde{a}_l(\tau), \tilde{a}_l(\tau)))| > \min_{i=1, \ldots, k-1} \delta_{l,i}^k > 0, \]
where $\delta_{l,i}^k > 0$ were found in Assumption [1]. Moreover, we can choose $\delta_k > 0$ arbitrary small in (3.41).

**Step 3** We construct $k^{th}$–Sticky Grazing Set $SG^{C,k}$ where all grazing rays from non-measure zero subset of $[\tilde{a}_l^k, d_{l,1}^k] \cup [c_{l,2}^k, d_{l,2}^k] \cup \cdots \cup [c_{l,m_l^k-1}^k, d_{l,m_l^k-1}^k] \cup [c_{l,m_l^k}^k, \tilde{b}_1^l]$ intersect at a fixed point in $\mathcal{P}_x \left( (cl(\Omega) \times \mathbb{V}^N) \setminus \mathcal{B} \right)$, where $\mathcal{P}_x$ is projection on spatial domain. Choose any $i \in \{1, \ldots, m_k^i\}$ and corresponding sub interval $[c_{l,i}^k, d_{l,i}^k]$. We define
\[ \Theta_{l,i}^{C,k} := \left\{ \left( x_l^{-1}(\tilde{a}_l(\tau), |v|\tilde{a}_l(\tau)) + st_{l}^{-1}(\tilde{a}_l(\tau), |v|\tilde{a}_l(\tau)), v_l^{-1}(\tilde{a}_l(\tau), |v|\tilde{a}_l(\tau)) \right) \in (cl(\Omega) \times \mathbb{V}^N) \setminus \mathcal{B} : \right. \]
\[ \tau \in [c_{l,i}^k, d_{l,1}^k] \cup [c_{l,2}^k, d_{l,2}^k] \cup \cdots \cup [c_{l,m_l^k-1}^k, d_{l,m_l^k-1}^k] \cup [c_{l,m_l^k}^k, \tilde{b}_1^l], \]
\[ v \in \mathbb{V}^N, s \in [0, (t_{l_k}^{-1} - t_{l_k}^{-1})(\tilde{a}_l(\tau), |v|\tilde{a}_l(\tau))], \]
Fix $x^* \in \tilde{O}$. If there does not exist $\tau \in [c_{l,i}^k, d_{l,1}^k]$ and $s \in [t_{l_k}^{-1}(\tilde{a}_l(\tau), \tilde{a}_l(\tau)), t_{l_k}^{-1}(\tilde{a}_l(\tau), \tilde{a}_l(\tau))]$ satisfying $x^* = x_l^{-1}(\tilde{a}_l(\tau), \tilde{a}_l(\tau)) + s t_{l_k}^{-1}(\tilde{a}_l(\tau), \tilde{a}_l(\tau))$, then $\{v \in \mathbb{R}^2 : (x^*, v) \in \Theta_{l,i}^{C,k} \} = \emptyset$ with zero measure. Now suppose there exist such $\tau$ and $s$.

Due to Lemma [8] there are only two cases: (i) **sticky grazing:** for all $\tau \in [c_{l,i}^k, d_{l,1}^k]$, there exists $s = s(\tau) \in [t_{l_k}^{-1}(\tilde{a}_l(\tau), \tilde{a}_l(\tau)), t_{l_k}^{-1}(\tilde{a}_l(\tau), \tilde{a}_l(\tau))]$ and points $\{x^{*,r}\}_{r=1}^{k-1} \in cl(\Omega)$ such that
\[ x^{*,r} = x_l^{-1}(\tilde{a}_l(\tau), \tilde{a}_l(\tau)) + s t_{l_k}^{-1}(\tilde{a}_l(\tau), \tilde{a}_l(\tau)), \quad \text{for all } \tau \in [c_{l,i}^k, d_{l,1}^k] \tag{3.44} \]
or (ii) **isolated grazing:** there exists $\delta_-, \delta_+ > 0$ so that for $\tau' \in (\tau - \delta_-, \tau + \delta_+), \{\tau\}$ there is no $s$ satisfying (3.44). We define $k^{th}$–sticky grazing set $SG^{C,k}$ as collection of all such $x^{*,r} \in cl(\Omega)$ points.

**Definition 9.** Consider $\{3.43\}$ and disjoint union of intervals $[\tilde{a}_l^*, d_{l,1}^*] \cup [c_{l,2}^*, d_{l,2}^*] \cup \cdots \cup [c_{l,m_l^*}^*, d_{l,m_l^*}^*] \cup [c_{l,m_l^*}^*, \tilde{b}_1^l]$. There are finite $i \in I_{sg,i}^{k-1} \subset \{1, 2, \ldots, (k-1)m_k^l\}$ such that case (i) **sticky grazing** holds:
\[ \bigcap_{\tau \in [c_{l,i}^k, d_{l,1}^k]} x_l^{-1}(\tilde{a}_l(\tau), \tilde{a}_l(\tau)) = x_{sg,i}^{k-1}(\tilde{a}_l(\tau), \tilde{a}_l(\tau)) = x_{sg,i}^{k-1} \in cl(\Omega), \quad \text{for some } r = 1, \ldots, k-1, \]
by writing $\tilde{a}_l^* = c_{l,1}^k, \tilde{b}_1^l = d_{l,m_k^l}$. When above intersection is nonempty we collect all those points to obtain $k^{th}$–sticky grazing set:
\[ SG^{C,k} := \bigcup_{l=1}^{M^C} SG_i^{C,k} := \bigcup_{l=1}^{M^C} \bigcup_{i=1}^{k-1} \bigcup_{r=1}^{k-1} \{x_{sg,i}^{k-1} \in cl(\Omega)\}. \tag{3.45} \]
Note that $SG^{C,k}$ has at most $(k-1)M^C m_k^l$ points, from index $i, l$, and $r$.

**Step 4** We claim
\[ m_k^i \{v \in \mathbb{R}^2 : (x, v) \in \Theta^{C,k} \setminus (\Theta^{C,k-1})_l \} = 0, \tag{3.46} \]
for all $x \in \mathcal{P}_x \left( (cl(\Omega) \times \mathbb{V}^N) \setminus \mathcal{B} \right) \setminus SG^{C,k}$. Consider again the set $\{3.43\}$ and fix $l \in \{1, \ldots, M^C\}$. For any point $x \in cl(\Omega)$ such that $i \in \{1, 2, \ldots, (k-1)m_k^l\} \setminus I_{sg,i}$, we apply case (b) of Lemma [8] to say that
\[ \{v \in \mathbb{R}^2 : (x, v) \in \Theta^{C,k} \setminus (\Theta^{C,k-1})_l \} \cap S^1 = \text{finite points}, \]
which gives \((3.46)\).

**Lemma 11.** We assume Assumption \([7]\) Then, for any \(\varepsilon > 0\), there exist an open cover
\[
\left\{ \bigcup_{i=1}^{l_k} B(x_i^{C,k}, r_i^{C,k}) \right\} \cup \left\{ \bigcup_{y \in SG^{C,k}} B(y, \varepsilon) \right\}
\]
for \(\mathcal{P}_x((\{cl(\Omega) \times \mathbb{V}^N\} \setminus \mathcal{I} \mathcal{B})\) and corresponding velocity sets \(\mathcal{O}_i^{C,k} \subset \mathbb{V}^N\) with \(m_2(O_i^{C,k}) < \varepsilon\) such that

1. For any \((x, v) \in (\{cl(\Omega) \times \mathbb{V}^N\} \setminus \mathcal{I} \mathcal{B})\),
\[x \in B(x_i^{C,k}, r_i^{C,k}) \text{ or } x \in B(y, \varepsilon),\]
for some \(x_i^{C,k}, r_i^{C,k}\), and \(y \in SG^{C,k}\).

2. Moreover, if \(x \notin \bigcup_{y \in SG^{C,k}} B(y, \varepsilon), x \in B(x_i^{C,k}, r_i^{C,k}), \) and \(v \in \mathbb{V}^N \setminus \mathcal{O}_i^{C,k}\), then
\[
\phi^r(x, v) = |v^{r-1}(x, v) \cdot n(x^r(x, v))| > \delta^{C,r} > 0, \quad r = 1, \ldots, k,
\]
for some uniformly positive \(\delta^{C,r} > 0\), \(r = 1, \ldots, k\).

From above, we define \(\varepsilon\)- neighborhood of \(\mathcal{G}^{C,k}\):
\[
(\mathcal{G}^{C,k}, \varepsilon) = \left\{ \bigcup_{i=1}^{l_k} B(x_i^{C,k}, r_i^{C,k}) \times \mathcal{O}_i^{C,k} \right\} \cup \left\{ \bigcup_{y \in SG^{C,k}} B(y, \varepsilon) \times \mathbb{V}^N \right\}.
\]

**Proof.** We suffice to follow the scheme of proof of Lemma \([10]\) From \((3.45)\), \(SG^{C,k}\) has finite points so we make a cover with finite balls, \(\bigcup_{y \in SG^{C,k}} B(y, \varepsilon)\) for \(SG^{C,k}\).

For \(x \in \mathcal{P}_x((\{cl(\Omega) \times \mathbb{V}^N\} \setminus \mathcal{I} \mathcal{B}) \setminus \bigcup_{y \in SG^{C,k}} B(y, \varepsilon)\), there at most finite (at most \(2M^{C} + 2 \sum_{r=1}^{k} \sum_{i=1}^{M^{r}} m_i\)) unit vectors \(\hat{v}_n / |\hat{v}_n|\) such that
\[
(x, v_i) \in \mathcal{G}_C^k \cup \mathcal{G}_C^{k-1} \cup \cdots \cup \mathcal{G}_C^1,
\]
from \((3.46)\). So we define
\[
\mathcal{O}_x^{C,k} := \left\{ v \in \mathbb{V}^N : \left| \frac{v_i}{|v_i|} - \frac{v}{|v|} \right| < C_k(N) \varepsilon, \forall v_i \text{ s.t } (x, v_i) \in \bigcup_{r=1}^{k} \mathcal{G}_C^r \right\}.
\]

When \(v \in \mathbb{V}^N \setminus \mathcal{O}_x^{C,k}\), trajectory does not graze within second bounces, so
\[
\phi^r(x, v) = |v \cdot n(x^r(x, v))|, \quad 1 \leq r \leq k
\]
are well-defined and locally smooth, because \((x, v) \in \{\{cl(\Omega) \times \mathbb{V}^N\} \setminus \mathcal{I} \mathcal{B}\} \setminus (\bigcup_{r=1}^{k} \mathcal{G}_C^r)\) implies that trajectory does not graze in first \(k\) bounces. Using local continuity of Lemma \([5]\) again, we can find \(r_x^{C,k} \ll 1\) such that
\[
\phi^r(x, v) > \delta^r_x > 0, \quad \text{ for } 1 \leq r \leq k \text{ and } (x, v) \in cl(B(x, r_x^{C,k})) \times \mathbb{V}^N \setminus \mathcal{O}_x^{C,k}.
\]
By compactness, we can find an open cover \(\bigcup_{i=1}^{l_k} B(x_i^{C,k}, r_i^{C,k})\) for \(\mathcal{P}_x((\{cl(\Omega) \times \mathbb{V}^N\} \setminus \mathcal{I} \mathcal{B}) \setminus \bigcup_{y \in SG^{C,k}} B(y, \varepsilon)\) and corresponding \(\mathcal{O}_i^{C,k}\) with small measure \(m_2(O_i^{C,k}) < \varepsilon\) by choosing \((3.47)\) with sufficiently small \(C_k(N)\). Finally we choose
\[
\delta^r := \min_{1 \leq r \leq l_k} \delta^r_{C,k} > 0, \quad 1 \leq r \leq k,
\]
to finish the proof.

**Proposition 1.** For any \(\varepsilon > 0\), we have the \(\varepsilon\)- neighborhood of \(\mathcal{G}\):
\[
(\mathcal{G}, \varepsilon) = \left\{ \bigcup_{i=1}^{l_g} B(x_i^{C}, r_i^{C}) \times \mathcal{O}_i^{C} \right\} \cup \left\{ \bigcup_{j=1}^{l_y} B(y_j^{C}, \varepsilon) \times \mathbb{V}^N \right\},
\]
with \(\mathcal{O}_i^{C} \subset \mathbb{V}^N\), \(m_2(O_i^{C}) < \varepsilon\) for all \(i = 1, 2, \cdots, l_g < \infty\), and \(j = 1, 2, \cdots, l_y < \infty\). For any \((x, v) \in \{cl(\Omega) \times \mathbb{V}^N\} \setminus \mathcal{I} \mathcal{B}\),
\[x \in B(x_i^{C}, r_i^{C}) \text{ or } x \in B(y_j^{C}, \varepsilon),\]
for some \(x_i^{C}\) or \(y_j^{C}\). Moreover, if \(x \notin \bigcup_{j=1}^{l_y} B(y_j^{C}, \varepsilon), x \in B(x_i^{C}, r_i^{C}), \) and \(v \in \mathbb{V}^N \setminus \mathcal{O}_i^{C}\), then
\[|v^{k-1}(T_0, x, v) \cdot n(x^k(T_0, x, v))| > \delta > 0, \quad \forall t^k(T_0, x, v) \in [0, T_0].\]
Proof. We use mathematical induction. We already proved \( k = 1 \) case in Lemma 9 when there is no sticky grazing set. From \( k = 2 \), sticky grazing set appears and we proved Lemma 10. From Assumption 1 and Lemma 11, we know that Lemma 11 holds for any finite \( k \in \mathbb{N} \). Moreover, number of bounce is uniformly bounded from Lemma 6. So we stop mathematical induction in the maximal possible number of bouncing on \([0, T_0] \).

\[ \square \]

3.4. Transversality and double Duhamel trajectory. We introduce local parametrization for \( U = \Omega \times [0, H] \). Since we should treat three-dimensional trajectory from this subsection, we introduce the following notation to denote two-dimensional points in cross section,

\[
x = (x_1, x_3), \quad v = (v_1, v_3).
\]

where missing \( x_2 \) and \( v_2 \) are components for axis direction. So we can write

\[
x = (x, x_2) \in \Omega, \quad v = (v, v_2) \in \mathbb{R}^3.
\]

Especially for the points near boundary, we define local parametrization, i.e. for \( p \in \partial \Omega \),

\[
\eta_p : \{x_p \in \mathbb{R}^3 : x_{p,3} < 0 \} \cap B(0, \delta_1) \rightarrow \Omega \cap B(p, \delta_2),
\]

\[
x_p = (x_p, x_{p,2}) \rightarrow x := \eta_p(x_p),
\]

\[
\eta_p(0, 0, 0) = p, \quad x = \eta_p(x_p) = (\eta_p(x_{p,1}), x_{p,2}),
\]

\[
x = \eta_p(x_p) = \eta_p(x_{p,1}, 0) + x_{p,3}n(\eta_p(x_{p,1}, 0)),
\]

and \( \eta_p(x_p) \in \partial \Omega \) if and only if \( x_{p,3} = 0 \). \( n(\eta_p(x_{p,1}, 0)) \) is outward unit normal vector at \( (\eta_p(x_{p,1}, 0), x_{p,2}) \in \partial \Omega \). Since \( \Omega \) is cylindrical, unit normal vector \( \mathbf{n} \) is independent to \( x_{p,2} \). We use the following derivative symbols,

\[
\frac{\partial}{\partial x_1}, \quad \frac{\partial}{\partial x_2}, \quad \frac{\partial}{\partial x_3}, \quad \frac{\partial}{\partial x_i}.
\]

\[
\nabla = (\nabla_1, \nabla_2), \quad \nabla_x = (\nabla_{x_1}, \nabla_{x_2}),
\]

where \( x \in cl(\Omega) \) and \( x_p \in \eta_p^{-1}(\Omega) \). Note that it is easy to check \( \eta_p \) is locally triple orthogonal system, i.e.

\[
(\partial_i \eta_p, \partial_j \eta_p) = 0, \quad \text{for all} \quad i \neq j, \quad x \in \{x_p \in \mathbb{R}^3 : x_{p,3} < 0 \} \cap B(0, \delta_1).
\]

We also use standard notations \( g_{p,ij} := (\partial_i \eta_p, \partial_j \eta_p) \) and transformed velocity \( \mathbf{v}_p \) is defined by

\[
\mathbf{v}_{p,i}(x_p) := \frac{\partial \eta_p(x_p)}{\sqrt{g_{p,ii}(x_p)}} \cdot v,
\]

or equivalently,

\[
\mathbf{v}_p = \begin{bmatrix} v_{p,1} \\ v_{p,2} \\ v_{p,3} \end{bmatrix} = Q^T \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = Q^T v, \quad \text{where} \quad Q := \begin{bmatrix} \frac{\partial \eta_{p,1}}{\sqrt{g_{p,11}}} & \frac{\partial \eta_{p,1}}{\sqrt{g_{p,21}}} & \frac{\partial \eta_{p,1}}{\sqrt{g_{p,31}}} \\ \frac{\partial \eta_{p,2}}{\sqrt{g_{p,12}}} & \frac{\partial \eta_{p,2}}{\sqrt{g_{p,22}}} & \frac{\partial \eta_{p,2}}{\sqrt{g_{p,32}}} \\ \frac{\partial \eta_{p,3}}{\sqrt{g_{p,13}}} & \frac{\partial \eta_{p,3}}{\sqrt{g_{p,33}}} & \frac{\partial \eta_{p,3}}{\sqrt{g_{p,33}}} \end{bmatrix}.
\]

We compute transversality between two consecutive bouncings using local parametrization (3.48) and transformed velocity (3.50). To denote bouncing index, we define

\[
x_{p,k} := (x_{p,k+1}, x_{p,k+2}, 0) \text{ such that } x^k = \eta_{p,k}(x_{p,k}),
\]

\[
v_{p,k,i} := \frac{\partial_i \eta_{p,k}(x_{p,k})}{\sqrt{g_{p,k,ii}(x_{p,k})}} \cdot v^k,
\]

where \( p^k \) is a point on \( \partial \Omega \) near bouncing point \( x^k \).

Since dynamics in \( x_2 \) direction is independent to the dynamics in cross section, we focus on the dynamics of two-dimensional cross section \( \Omega \), for fixed \( x_2 \).

Lemma 12. Assume that \( \Omega \) are \( C^2 \) (not necessarily convex) and \( |\mathbf{v}_{p,k,3}|, |\mathbf{v}_{p,k,3}^{k+1}| > 0 \). Consider \( (t^{k+1}, x_{p,k+1}^{k+1}, x_{p,k+1}^{k+1}) \) as a function of \( (t^{k+1}, x_{p,k+1}^{k+1}, x_{p,k+1}^{k+1}) \)

\[
\frac{\partial(t^k - t^{k+1})}{\partial x_{p,k,1}^{k+1}} = \frac{-1}{\mathbf{v}_{p,k+1,3}^{k+1}} \frac{\partial \eta_{p,k+1}(x_{p,k+1}^{k+1})}{\sqrt{g_{p,k+1,33}(x_{p,k+1}^{k+1})}} \cdot \left[ \frac{\partial_i \eta_{p,k}(x_{p,k+1}^{k+1}, 0) - (t^k - t^{k+1}) \frac{\partial v^k}{\partial x_{p,k,1}^{k+1}}} \right],
\]

(3.52)
\[
\frac{\partial x_{p+1}^{k+1}}{\partial x_{p,1}} = \frac{1}{\sqrt{g_{p,11}(x^k)}} \left[ \frac{\partial_{x_{p+1}} \eta_{p+1}^{k+1}(x^k)}{\sqrt{g_{p,11}(x^k)}} + \frac{v_{p+1,1}^{k+1}}{\sqrt{g_{p,13}(x^k)}} \frac{\partial_{x_{p+1}} \eta_{p+1}^{k+1}(x^k)}{\sqrt{g_{p,13}(x^k)}} \right].
\]

(3.53)

\[
\frac{\partial v_{p+1}^{k+1}}{\partial x_{p,1}} = \frac{v_{p,1}}{\sqrt{g_{p,11}(x^k)}} \frac{\partial_{x_{p+1}} \eta_{p+1}^{k+1}(x^k)}{\sqrt{g_{p,11}(x^k)}} + \frac{\partial_{x_{p,1}} \eta_{p+1}^{k+1}(x^k)}{\sqrt{g_{p,11}(x^k)}},
\]

(3.54)

\[
\frac{\partial v_{p+1}^{k+1}}{\partial x_{p,1}} = \frac{v_{p,1}}{\sqrt{g_{p,13}(x^k)}} \frac{\partial_{x_{p+1}} \eta_{p+1}^{k+1}(x^k)}{\sqrt{g_{p,13}(x^k)}},
\]

(3.55)

where

\[
\frac{\partial v_{p,1}}{\partial x_{p,1}} = \sum_{\ell=1}^{3} v_{p,\ell} \left[ \sqrt{g_{p,\ell r}(x^k)} \frac{\partial_{x_{p+1}} \eta_{p+1}^{k+1}(x^k)}{\sqrt{g_{p,\ell r}(x^k)}} \right],
\]

(3.56)

For \( i = 1 \) and \( j = 1, 3, \)

\[
\frac{\partial (t^k - t^{k+1})}{\partial v_{p,1}^{k+1}} = \frac{\partial_{x_{p,1}} \eta_{p,1}^{k+1}(x^k)}{\sqrt{g_{p,11}(x^k)}} + \frac{\partial_{x_{p,1}} \eta_{p,1}^{k+1}(x^k)}{\sqrt{g_{p,13}(x^k)}},
\]

(3.57)

(3.58)

(3.59)

(3.60)

Proof. Proof of (3.52). By the definitions (3.48), (2.4), and our setting (3.51) and (2.1),

\[
x_{p,1}^{k+1}(x_{p+1,1,1}^{k+1}, 0) = \eta_{p,1}^{k}(x_{p,1}^{k}, 0) + \int_{t^k}^{t^{k+1}} v^k.
\]

(3.61)

We take \( \frac{\partial}{\partial x_{p,1}} \) to above equality to get

\[
\left. \frac{\partial x_{p,1}^{k+1}}{\partial x_{p,1}^{k+1}} \frac{\partial x_{p,1}^{k+1}}{\partial x_{p,1}^{k+1}} \right|_{x^{k+1}} = -(t^k - t^{k+1}) \frac{\partial v^k}{\partial x_{p,1}^{k+1}} + \frac{\partial (t^k - t^{k+1})}{\partial v_{p,1}^{k+1}} v^k + \partial_{x_{p,1}} \eta_{p,1}^{k}(x_{p,1}^{k}, 0).
\]

(3.62)
and then an inner product with \( \frac{\partial y_{p,k+1}^{k+1}}{\sqrt{g_{p,k+1,33}}} \bigg|_{x^{k+1}} \) to have

\[
\frac{\partial y_{p,k+1}^{k+1}}{\partial x_{p,k+1}^{k+1}} \bigg|_{x^{k+1}} \cdot \frac{\partial y_{p,k+1}^{k+1}}{\sqrt{g_{p,k+1,33}}} \bigg|_{x^{k+1}} = -(t^k - t^{k+1}) \frac{\partial y_{p,k+1}^{k+1}}{\partial x_{p,k}^{k+1}} \bigg|_{x^{k+1}} - \frac{\partial (t^k - t^{k+1})}{\partial x_{p,k}^{k+1}} v^k \cdot \frac{\partial y_{p,k+1}^{k+1}}{\sqrt{g_{p,k+1,33}}} \bigg|_{x^{k+1}} + \partial y_{p,k+1}^{k+1} (x_{p,k+1,1}^k,0) \cdot \frac{\partial y_{p,k+1}^{k+1}}{g_{p,k+1,11}} \bigg|_{x^{k+1}},
\]

where we abbreviated \( X(k) = X(s; t^k, z^k, \eta^k) \) and \( V(s) = V(s; t^k, z^k, \eta^k) \). Due to (3.49) the LHS equals zero. Now we consider the RHS. From (3.50), we prove (3.56). We also note that

\[
\lim_{s \downarrow t^{k+1}} V(s; t^k, z^k, \eta^k) = v^k.
\]

Therefore, from (2.4) and (3.51),

\[
v^k \cdot \frac{\partial y_{p,k+1}^{k+1}}{g_{p,k+1,11}} \bigg|_{x^{k+1}} = -v_{p,k+1,3}^{k+1}.
\]

Dividing both sides by \( v^k \cdot \frac{\partial y_{p,k+1}^{k+1}}{g_{p,k+1,11}} \bigg|_{x^{k+1}} = \frac{v_{p,k+1,3}^{k+1}}{g_{p,k+1,11}} \), we get (3.52).

**Proof of (3.53).** We take inner product with \( \frac{\partial y_{p,k+1}^{k+1}}{g_{p,k+1,11}} \bigg|_{x^{k+1}} \) to (3.62) to have

\[
\frac{\partial x_{p,k+1}^{k+1}}{\partial x_{p,k}^{k+1}} \bigg|_{x^{k+1}} \cdot \frac{\partial y_{p,k+1}^{k+1}}{g_{p,k+1,11}} \bigg|_{x^{k+1}} = \frac{\partial x_{p,k+1}^{k+1}}{g_{p,k+1,11}} \bigg|_{x^{k+1}} - \frac{\partial (t^k - t^{k+1})}{\partial x_{p,k}^{k+1}} \bigg|_{x^{k+1}} + \frac{\partial y_{p,k+1}^{k+1}}{g_{p,k+1,11}} \bigg|_{x^{k+1}}.
\]

Since

\[
v^k \cdot \frac{\partial y_{p,k+1}^{k+1}}{g_{p,k+1,11}} \bigg|_{x^{k+1}} = -v_{p,k+1,3}^{k+1},
\]

from (3.49) and (3.52),

\[
\frac{\partial x_{p,k+1}^{k+1}}{\partial x_{p,k}^{k+1}} \bigg|_{x^{k+1}} = \frac{1}{v_{p,k+1,3}^{k+1}} \cdot \frac{\partial y_{p,k+1}^{k+1}(x^{k+1})}{\sqrt{g_{p,k+1,33}(x^{k+1})}} \cdot \frac{\partial y_{p,k+1}^{k+1}}{g_{p,k+1,11}} \bigg|_{x^{k+1}} + \frac{\partial y_{p,k+1}^{k+1}}{g_{p,k+1,11}} \bigg|_{x^{k+1}} \cdot \frac{\partial y_{p,k+1}^{k+1}}{g_{p,k+1,11}} \bigg|_{x^{k+1}} + \frac{\partial y_{p,k+1}^{k+1}}{g_{p,k+1,11}} \bigg|_{x^{k+1}}.
\]

This ends the proof of (3.53).

**Proof of (3.54) and (3.55).** From (2.4) and (3.51),

\[
v_{p,k+1,1}^{k+1} = \frac{\partial y_{p,k+1}^{k+1}}{g_{p,k+1,11}} \bigg|_{x^{k+1}} \cdot \lim_{s \downarrow t^{k+1}} V(s; t^k, z^k, \eta^k),
\]

\[
v_{p,k+1,3}^{k+1} = -\frac{\partial y_{p,k+1}^{k+1}}{g_{p,k+1,33}} \bigg|_{x^{k+1}} \cdot \lim_{s \downarrow t^{k+1}} V(s; t^k, z^k, \eta^k).
\]

From (3.64),

\[
\frac{\partial x_{p,k+1}^{k+1}}{\partial x_{p,k}^{k+1}} \bigg|_{x^{k+1}} = \frac{\partial y_{p,k+1}^{k+1}(x^{k+1})}{\sqrt{g_{p,k+1,33}(x^{k+1})}} \cdot \frac{\partial y_{p,k+1}^{k+1}}{g_{p,k+1,11}} \bigg|_{x^{k+1}} + \frac{\partial y_{p,k+1}^{k+1}}{g_{p,k+1,11}} \bigg|_{x^{k+1}} \cdot \frac{\partial y_{p,k+1}^{k+1}}{g_{p,k+1,11}} \bigg|_{x^{k+1}} + \frac{\partial y_{p,k+1}^{k+1}}{g_{p,k+1,11}} \bigg|_{x^{k+1}} \cdot \lim_{s \downarrow t^{k+1}} V(s; t^k, z^k, \eta^k),
\]

(3.65)
\[
\frac{\partial v_{p,k+1}^{k+1}}{\partial x_{p,k+1}} = -\frac{\partial g_{p,k+1}}{\sqrt{g_{p,k+1}}} \frac{\partial x_{p,k+1}}{\partial x_{p,k+1}} \frac{\partial v_{k}^{k}}{\partial x_{p,k+1}} - \frac{\partial x_{p,k+1}}{\partial x_{p,k+1}} \left( \frac{\partial g_{p,k+1}}{\sqrt{g_{p,k+1}}} \right) \bigg|_{x^{k+1}} \cdot \lim_{s \rightarrow t^{k+1}} V(s; t^{k}, x^{k}, t^{k}) \frac{\partial v_{k}^{k}}{\partial x_{p,k+1}}.
\]

From (3.63), we prove (3.54) and (3.55).

Now we consider (3.57)-(3.60) for \(v\)-derivatives.

**Proof of (3.57).** We take \(\frac{\partial}{\partial v_{p,k}}\) to (3.61) for \(j = 1, 3\) to get

\[
\frac{\partial x_{p,k+1}}{\partial v_{p,k,j}} \frac{\partial x_{p,k+1}^{k+1}}{\partial v_{p,k,j}} \bigg|_{x^{k+1}} = -(t^{k} - t^{k+1}) \frac{\partial v_{k}^{k}}{\partial v_{p,k,j}} - \frac{\partial (t^{k} - t^{k+1})}{\partial v_{p,k,j}} \frac{\partial v_{k}^{k}}{\partial v_{p,k,j}} \bigg|_{x^{k+1}},
\]

and then take an inner product with \(\frac{\partial n_{p,k+1}}{\sqrt{g_{p,k+1}}} \bigg|_{x^{k+1}}\) to have

\[
\frac{\partial x_{p,k+1}}{\partial v_{p,k,j}} \frac{\partial x_{p,k+1}^{k+1}}{\partial v_{p,k,j}} \bigg|_{x^{k+1}} \frac{\partial \eta_{p,k+1}^{k+1}}{\sqrt{g_{p,k+1}}} \bigg|_{x^{k+1}} = \left\{ -(t^{k} - t^{k+1}) \frac{\partial v_{k}^{k}}{\partial v_{p,k,j}} - \frac{\partial (t^{k} - t^{k+1})}{\partial v_{p,k,j}} \lim_{s \rightarrow t^{k+1}} V(s; t^{k}, x^{k}, t^{k}) \right\} \frac{\partial \eta_{p,k+1}^{k+1}}{\sqrt{g_{p,k+1}}} \bigg|_{x^{k+1}}.
\]

Due to (3.49), the LHS equals zero. Now we consider the RHS. From (3.50),

\[
\frac{\partial v_{k}^{k}}{\partial v_{p,k,j}} = \frac{\partial g_{p,k+1}(x_{p,k+1}, 0)}{\sqrt{g_{p,k+1}(x_{p,k+1}, 0)}}.
\]

Using (3.64), (3.66), and (3.67), we prove (3.57).

**Proof of (3.58).** For \(j = 1, 3\), we take inner product with \(\frac{\partial n_{p,k+1}}{\sqrt{g_{p,k+1}}} \bigg|_{x^{k+1}}\) to (3.65) to have

\[
\frac{\partial x_{p,k+1}}{\partial v_{p,k,j}} = \left\{ -(t^{k} - t^{k+1}) \lim_{s \rightarrow t^{k+1}} V(s; t^{k}, x^{k}, t^{k}) - (t^{k} - t^{k+1}) \frac{\partial v_{k}^{k}}{\partial v_{p,k,j}} \right\} \frac{\partial \eta_{p,k+1}^{k+1}}{\sqrt{g_{p,k+1}}} \bigg|_{x^{k+1}}.
\]

From (3.67) and (3.57), we prove (3.58).

**Proof of (3.59) and (3.60).** For \(j = 1, 3\), from (3.64),

\[
\frac{\partial x_{p,k+1}}{\partial v_{p,k,j}} = \frac{\partial x_{p,k+1}}{\partial v_{p,k,j}} \frac{\partial g_{p,k+1}}{\sqrt{g_{p,k+1}}} \bigg|_{x^{k+1}} \lim_{s \rightarrow t^{k+1}} V(s; t^{k}, x^{k}, t^{k}) + \frac{\partial t_{p,k+1}}{\sqrt{g_{p,k+1}}} \bigg|_{x^{k+1}} \frac{\partial \eta_{p,k+1}^{k+1}}{\sqrt{g_{p,k+1}}} \bigg|_{x^{k+1}}.
\]

From (3.57) and (3.58), we prove (3.59). The proof of (3.60) is also very similar as above from (3.64). 

\[\Box\]
\[
\frac{\partial x_{p,1}^j}{\partial v_j} = -t_b e_j \cdot \frac{1}{\sqrt{g_{p,11}(x^1)}} \left[ \frac{\partial \eta_{p,1}^j(x^1)}{\sqrt{g_{p,11}(x^1)}} + \frac{\partial \eta_{p,1}^j(x^1)}{\sqrt{g_{p,13}(x^1)}} \right] \quad j = 1, 3, \tag{3.71}
\]
\[
\frac{\partial v_{p,i}}{\partial x_j} = \frac{\partial x_{p,1}^i}{\partial x_j} \frac{\partial (\frac{\partial \eta_{p,1}^i}{\sqrt{g_{p,11}}} | x^1 \cdot V(t - t_b)} = \frac{\partial x_{p,1}^i}{\partial x_j} \frac{\partial (\frac{\partial \eta_{p,1}^i}{\sqrt{g_{p,11}}} | x^1 \cdot v^i \cdot v^i, i = 1, 3, j = 1, 3, \tag{3.72}
\]
\[
\frac{\partial v_{p,i}}{\partial v_j} = \frac{\partial x_{p,1}^i}{\partial v_j} \frac{\partial (\frac{\partial \eta_{p,1}^i}{\sqrt{g_{p,11}}} | x^1 \cdot e_j + \frac{\partial x_{p,1}^i}{\partial v_j} \frac{\partial (\frac{\partial \eta_{p,1}^i}{\sqrt{g_{p,11}}} | x^1 \cdot v^i, i = 1, 3, j = 1, 3. \tag{3.73}
\]

Here, \( e_j \) is the \( j \)th directional standard unit vector in \( \mathbb{R}^3 \).

Moreover,
\[
\frac{\partial |x^1_p|}{\partial x_j} = 0, \tag{3.74}
\]
\[
\frac{\partial |x^1_p|}{\partial v_j} = \lim_{s \downarrow t^1} \frac{V(s; t, x, v)}{|V(s; t, x, v)|}. \tag{3.75}
\]

Proof. We have
\[
\lim_{s \downarrow t^1} V(s; t, x, v) = v, \quad X(t^1; t, x, v) = x + v(t^1 - t). \tag{3.76}
\]

Especially, when \( \tau = t^1 \), we get
\[
X(t^1; t, x, v) = x + v(t^1 - t). \tag{3.77}
\]

From (3.76), we have
\[
\lim_{s \downarrow t^1} \frac{\partial V(s; t, x, v)}{\partial x_j} = 0.
\]

To prove (3.68) - (3.73), these estimates are very similar with those of Lemma [12]. We are suffice to choose global euclidean coordinate instead of \( \eta_{p,k} \). Therefore we should replace
\[
\eta_{p,k+1} \rightarrow \eta_{p,k}, \quad \eta_{p,k} \rightarrow x, \quad t^k \rightarrow t, \quad t^{k+1} \rightarrow t - t_b = t^1, \quad \partial_x \xi = e_j.
\]

Proof of (3.68). For \( j = 1, 3 \), we apply \( \partial x_j \) to (3.77) and take \( \frac{\partial v_{p,1}}{\sqrt{g_{p,33}} | x^1} \). In this case, we have \( \frac{\partial v_{p,3}}{\partial x_j} = 0 \). Then we get
\[
\frac{\partial t_b}{\partial x_j} = -\frac{1}{v_{p,3} \sqrt{g_{p,33}(x^1)}} \cdot e_j.
\]

Proof of (3.69). For \( j = 1, 2 \), we apply \( \partial v_j \) to (3.77) and take \( \frac{\partial \eta_{p,1}}{\sqrt{g_{p,33}} | x^1} \). Then we get
\[
0 = \frac{\partial x_{p,1}^1}{\partial v_j} \frac{\partial \eta_{p,1}}{\partial x_{p,1}^1} \frac{\partial \eta_{p,1}}{\partial \eta_{p,1}^{x^1}} \frac{\partial \eta_{p,1}^{x^1}}{\sqrt{g_{p,11}} \xi^1} = \left\{ - \frac{\partial (t - t^1)}{\partial v_j} \lim_{s \downarrow t^1} V(s; t, x, v) - \frac{\partial v_t}{\partial v_j} \right\} \frac{\partial \eta_{p,1}}{\sqrt{g_{p,11}} | x^1}.
\]

We use (3.76) to get (3.69).

Proof of (3.70). For \( j = 1, 3 \), we apply \( \partial x_j \) to (3.77) and take \( \frac{\partial \eta_{p,1}}{\sqrt{g_{p,11}}} \xi^1 \). And then,
\[
\frac{\partial x_{p,1}^1}{\partial x_j} = \frac{1}{v_{p,3}} \frac{\partial \eta_{p,1}^{x^1}}{\sqrt{g_{p,11}} \xi^1} \frac{\partial \eta_{p,1}^{x^1}}{\sqrt{g_{p,11}} \xi^1} e_j + \frac{1}{v_{p,1} \sqrt{g_{p,11}}} e_j
\]

This yields (3.70).

Proof of (3.71). For \( j = 1, 3 \), we apply \( \partial v_j \) to (3.77) and take \( \frac{\partial \eta_{p,1}}{\sqrt{g_{p,11}}} \xi^1 \). And then,
\[
\frac{\partial x_{p,1}^1}{\partial v_j} = \left\{ - \frac{\partial (t - t^1)}{\partial v_j} \lim_{s \downarrow t^1} V(s; t, x, v) - \frac{\partial v_t}{\partial v_j} \right\} \frac{\partial \eta_{p,1}}{\sqrt{g_{p,11}} | x^1}.
\]

And then we get (3.71).
Proof of (3.72). For $j = 1, 3$, we apply $\partial x_j$ to
\[
\begin{align*}
v_{p^1,1} & = \frac{\partial_1 \eta_{p^1}}{\sqrt{g_{p^1,11}} z^1} \cdot \lim_{s \uparrow t} V(s; t, x, v), \\
v_{p^1,3} & = -\frac{\partial_3 \eta_{p^1}}{\sqrt{g_{p^1,33}} z^1} \cdot \lim_{s \uparrow t} V(s; t, x, v).
\end{align*}
\] (3.78)

From (3.64),
\[
\begin{align*}
\frac{\partial v_{p^1,1}}{\partial x_j} & = \frac{\partial x_{p^1,1}}{\partial x_j} \frac{\partial}{\partial x_{p^1,1}} \left( \frac{\partial_1 \eta_{p^1}}{\sqrt{g_{p^1,11}}} \right) \bigg|_{z^1} \cdot \lim_{s \uparrow t} V(s; t, x, v), \\
\frac{\partial v_{p^1,3}}{\partial x_j} & = -\frac{\partial x_{p^1,1}}{\partial x_j} \frac{\partial}{\partial x_{p^1,1}} \left( \frac{\partial_3 \eta_{p^1}}{\sqrt{g_{p^1,33}}} \right) \bigg|_{z^1} \cdot \lim_{s \uparrow t} V(s; t, x, v).
\end{align*}
\]

From (3.76), (3.70), and (3.68), we prove (3.57).

Proof of (3.73). Similar as above, we apply $\partial v_j$ to (3.78) and then use (3.76), (3.71), and (3.69). We skip detail.

Proof of (3.74). Since there is no external force speed is constant, so result is obvious.

Proof of (3.75). Note that $|v_{p^1,1}| = \lim_{s \uparrow t} |V(s; t, x, v)|$ and
\[
2|v_{p^1,1}| \frac{\partial v_{p^1,1}}{\partial v_j} = 2 \lim_{s \uparrow t} V(s; t, x, v) \cdot \lim_{s \uparrow t} \partial v_j V(s; t, x, v),
\]
so we have
\[
\frac{\partial |v_{p^1,1}|}{\partial v_j} = \lim_{s \uparrow t} \frac{V(s; t, x, v)}{V(s; t, x, v)} \cdot \lim_{s \uparrow t} \partial v_j V(s; t, x, v).
\] (3.79)

Since
\[
\lim_{s \uparrow t} \frac{\partial V(s; t, x, v)}{\partial v_j} = e_j,
\] (3.80)
we combine (3.79), (3.80), and (3.69) to derive (3.75).

Lemma 14. Assume $\Omega$ satisfies Definition 7 and $\frac{1}{N} \leq |v| \leq N$, for $1 \ll N$. Also we assume $|t^k - t^{k+1}| \leq 1$ and $|v_{p^{k+1},3}|, |v_{p^{k+1},3}| > 0$. Then
\[
| \det \left[ \begin{array}{ccc} \partial x_{p^{k+1},1} & \partial x_{p^{k+1},1} & \partial x_{p^{k+1},1} \\ \partial x_{p^{k+1},1} & \partial x_{p^{k+1},1} & \partial x_{p^{k+1},1} \end{array} \right] | = \frac{\sqrt{g_{p^{k+1},11}(\pm k^2)}}{\sqrt{g_{p^{k+1},11}(\pm k^{k+1})}} |v_{p^{k+1},3}|.
\]

for the mapping $(x_{p^{k+1},1}, v_{p^{k+1}}) \mapsto (x_{p^{k+1},1}, v_{p^{k+1}})$.

Proof. We note that Lemma [12] holds for nonconvex domain and result is exactly same as Lemma 26 in [14], without external potential. Then simplified two-dimensional version directly yields above result.

Lemma 15. We define,
\[
\dot{v}_{p^{k+1},1} := \frac{v_{p^{k+1},1}}{\sqrt{|v_{p^{k+1}}|}}, \quad |v_{p^{k+1}}| = \sqrt{(v_{p^{k+1},1})^2 + (v_{p^{k+1},3})^2},
\]
where $v_{p^{k+1}} = v_{p^{k+1}}(t, x, v)$ are defined in (3.57). Assume $\frac{1}{N} \leq |v| \leq N$ and $|v_{p^{k+1},3}|, |v_{p^{k+1},3}| > \delta_2 > 0$ for $1 \ll N$ and $k \leq \Omega, N, \delta_2$. If $|t - t^k| \leq 1$, then
\[
| \det \left[ \begin{array}{ccc} \partial x_{p^{k+1},1} & \partial x_{p^{k+1},1} & \partial x_{p^{k+1},1} \\ \partial x_{p^{k+1},1} & \partial x_{p^{k+1},1} & \partial x_{p^{k+1},1} \end{array} \right] | > \epsilon_{\Omega, N, \delta_2}, \quad (3.81)
\]

\]
where \( t^i = t^i(t,x,v) \), \( x^{1}_{p^i,j} = x^{1}_{p^i,j}(t,x,v) \), \( \dot{v}^{1}_{p^i,j} = \dot{v}^{1}_{p^i,j}(t,x,v) \), and
\[
\begin{align*}
    x^{k}_{p^i,j} &= x^{k}_{p^i,j}(t^1, x^{1}_{p^i,j}, \dot{v}^{1}_{p^i,j}, |\mathbf{v}^1_p|), \\
    \dot{v}^{k}_{p^i,j} &= \dot{v}^{k}_{p^i,j}(t^1, x^{1}_{p^i,j}, \dot{v}^{1}_{p^i,j}, |\mathbf{v}^1_p|).
\end{align*}
\]

Here, the constant \( \epsilon_{\Omega,N,\delta_2} > 0 \) does not depend on \( t \) and \( x \).

**Proof.** Step 1. We compute
\[
J^{i+1} = \begin{pmatrix} \frac{\partial(x^i_{p^i,j}, \mathbf{v}^i_{p^i,j})}{\partial(x^{i+1}_{p^i,j}, \mathbf{v}^{i+1}_{p^i,j}, |\mathbf{v}^{i+1}_p|)} \\
\frac{\partial(x^i_{p^i,j}, \mathbf{v}^i_{p^i,j})}{\partial(x^{i+1}_{p^i,j}, \mathbf{v}^{i+1}_{p^i,j}, |\mathbf{v}^{i+1}_p|)} \\
\frac{\partial(x^i_{p^i,j}, \mathbf{v}^i_{p^i,j})}{\partial(x^{i+1}_{p^i,j}, \mathbf{v}^{i+1}_{p^i,j}, |\mathbf{v}^{i+1}_p|)} \end{pmatrix}.
\tag{3.82}
\]

For \( Q_i \),
\[
Q_i = \begin{bmatrix} 1 & 0 & 0 \\
0 & \frac{\partial \mathbf{v}^1_{p^i,j}}{\partial \mathbf{v}^1_{p^i,j}} & \frac{\partial \mathbf{v}^1_{p^i,j}}{\partial \mathbf{v}^1_{p^i,j}} \\
0 & \frac{\partial \mathbf{v}^1_{p^i,j}}{\partial \mathbf{v}^1_{p^i,j}} & \frac{\partial \mathbf{v}^1_{p^i,j}}{\partial \mathbf{v}^1_{p^i,j}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\
0 & \frac{\partial \mathbf{v}^1_{p^i,j}}{\partial \mathbf{v}^1_{p^i,j}} & \frac{\partial \mathbf{v}^1_{p^i,j}}{\partial \mathbf{v}^1_{p^i,j}} \\
0 & \frac{\partial \mathbf{v}^1_{p^i,j}}{\partial \mathbf{v}^1_{p^i,j}} & \frac{\partial \mathbf{v}^1_{p^i,j}}{\partial \mathbf{v}^1_{p^i,j}} \end{bmatrix}.
\tag{3.83}
\]

For \( Q_{i+1} \),
\[
Q_{i+1} = \begin{bmatrix} 1 & 0 & 0 \\
0 & \frac{\partial \mathbf{v}^{i+1}_{p^i,j}}{\partial \mathbf{v}^{i+1}_{p^i,j}} & \frac{\partial \mathbf{v}^{i+1}_{p^i,j}}{\partial \mathbf{v}^{i+1}_{p^i,j}} \\
0 & \frac{\partial \mathbf{v}^{i+1}_{p^i,j}}{\partial \mathbf{v}^{i+1}_{p^i,j}} & \frac{\partial \mathbf{v}^{i+1}_{p^i,j}}{\partial \mathbf{v}^{i+1}_{p^i,j}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\
0 & \frac{\partial \mathbf{v}^{i+1}_{p^i,j}}{\partial \mathbf{v}^{i+1}_{p^i,j}} & \frac{\partial \mathbf{v}^{i+1}_{p^i,j}}{\partial \mathbf{v}^{i+1}_{p^i,j}} \\
0 & \frac{\partial \mathbf{v}^{i+1}_{p^i,j}}{\partial \mathbf{v}^{i+1}_{p^i,j}} & \frac{\partial \mathbf{v}^{i+1}_{p^i,j}}{\partial \mathbf{v}^{i+1}_{p^i,j}} \end{bmatrix}.
\tag{3.84}
\]

Note that
\[
\frac{\partial \mathbf{v}^{i+1}_{p^i,j}}{\partial \mathbf{v}^{i+1}_{p^i,j}} = \mathbf{v}^{i+1}_{p^i,j} \frac{\partial}{\partial \mathbf{v}^{i+1}_{p^i,j}} (1) = \mathbf{v}^{i+1}_{p^i,j} \frac{\mathbf{v}^{i+1}_{p^i,j} + \mathbf{v}^{i+1}_{p^i,j}}{|\mathbf{v}^{i+1}_p|^3} = \mathbf{v}^{i+1}_{p^i,j} \frac{\mathbf{v}^{i+1}_{p^i,j} + \mathbf{v}^{i+1}_{p^i,j}}{|\mathbf{v}^{i+1}_p|^3},
\]
and for \( k = 1, 3 \),
\[
\frac{\partial \mathbf{v}^{i+1}_{p^i,j}}{\partial \mathbf{v}^{i+1}_{p^i,j}} = -\frac{\mathbf{v}^{i+1}_{p^i,j}}{|\mathbf{v}^{i+1}_p|^3}.
\]

From (3.83), (3.84), and (3.85),
\[
\det Q_{i+1} = \frac{1}{|\mathbf{v}^{i+1}_p|} \left( -\frac{\mathbf{v}^{i+1}_{p^i,j} + \mathbf{v}^{i+1}_{p^i,j}}{|\mathbf{v}^{i+1}_p|^3} + \mathbf{v}^{i+1}_{p^i,j} \mathbf{v}^{i+1}_{p^i,j} \mathbf{v}^{i+1}_{p^i,j} \mathbf{v}^{i+1}_{p^i,j} |\mathbf{v}^{i+1}_p|^3 - \mathbf{v}^{i+1}_{p^i,j} \mathbf{v}^{i+1}_{p^i,j} |\mathbf{v}^{i+1}_p|^3 \right).
\tag{3.86}
\]

By taking inverse, we get
\[
\det Q_i = -\frac{|\mathbf{v}^{i+1}_p|^4}{(\mathbf{v}^{i+1}_p, \mathbf{v}^{i+1}_p, \mathbf{v}^{i+1}_p, \mathbf{v}^{i+1}_p)^3}.
\tag{3.87}
\]

From (3.82), (3.87), (3.86), and Lemma 14 we get
\[
\det \begin{bmatrix} \frac{\partial x^{i+1}_{p^i,j}}{\partial x^{i+1}_{p^i,j}} & \frac{\partial x^{i+1}_{p^i,j}}{\partial x^{i+1}_{p^i,j}} & \frac{\partial x^{i+1}_{p^i,j}}{\partial x^{i+1}_{p^i,j}} \\
\frac{\partial x^{i+1}_{p^i,j}}{\partial x^{i+1}_{p^i,j}} & \frac{\partial x^{i+1}_{p^i,j}}{\partial x^{i+1}_{p^i,j}} & \frac{\partial x^{i+1}_{p^i,j}}{\partial x^{i+1}_{p^i,j}} \\
\end{bmatrix} = \frac{\sqrt{g_{p^i,j,11}(x^{i+1}_p)}}{\sqrt{g_{p^i,j,11}(x^{i+1}_p)}} |\mathbf{v}^{i+1}_p, \mathbf{v}^{i+1}_p, \mathbf{v}^{i+1}_p, \mathbf{v}^{i+1}_p|.
\]

Therefore,
\[
\left| \det J^{i+1} \right| = \left| \det Q_i \det P_i \det Q_{i+1} \right| = \left| \frac{\mathbf{v}^{i+1}_p}{(\mathbf{v}^{i+1}_p, \mathbf{v}^{i+1}_p, \mathbf{v}^{i+1}_p, \mathbf{v}^{i+1}_p)^3} \frac{\sqrt{g_{p^i,j,11}(x^{i}_p)}}{\sqrt{g_{p^i,j,11}(x^{i+1}_p)}} \left( \mathbf{v}^{i+1}_p, \mathbf{v}^{i+1}_p, \mathbf{v}^{i+1}_p, \mathbf{v}^{i+1}_p \right)^3 \right| = \left| \frac{\sqrt{g_{p^i,j,11}(x^{i}_p)}}{\sqrt{g_{p^i,j,11}(x^{i+1}_p)}} \left( \mathbf{v}^{i+1}_p, \mathbf{v}^{i+1}_p, \mathbf{v}^{i+1}_p, \mathbf{v}^{i+1}_p \right)^3 \right| = \left| \frac{\sqrt{g_{p^i,j,11}(x^{i}_p)}}{\sqrt{g_{p^i,j,11}(x^{i+1}_p)}} \left( \mathbf{v}^{i+1}_p, \mathbf{v}^{i+1}_p, \mathbf{v}^{i+1}_p, \mathbf{v}^{i+1}_p \right)^3 \right|.
\]
and we get

\[ |\det J_1^k| \leq \frac{\sqrt{g_{p_i,11}}^3}{g_{p_i,11}} |\v^k_{p_i,3}|^2 \]  

(3.88)

**Step 2.** From (3.63),

\[
2\left[ N_{p_i+1} \right] \frac{\partial |\v^{i+1}_{p_i}|}{\partial v^i_{p_i,n}} = \frac{\partial V(t^{i+1}; t^i, \v^i, \v^i)}{\partial v^i_{p_i,n}} = 2 \frac{\partial V(t^{i+1}; t^i, \v^i, \v^i)}{\partial v^i_{p_i,n}} \cdot V(t^{i+1}; t^i, \v^i, \v^i)
\]

Therefore, we get

\[
\frac{\partial |\v^{i+1}_{p_i}|}{\partial v^i_{p_i,n}} = \frac{\v^{i+1}_{p_i,n}}{|\v^{i+1}_{p_i}|} \quad \text{for } n = 1, 3.
\]  

(3.89)

Since speed is conserved, for \( n = 1, 3 \),

\[
\frac{\partial |\v^{i+1}_{p_i}|}{\partial v^i_{p_i,n}} = 0, \quad \text{for } n = 1, 3.
\]  

(3.90)

Also, by conservation,

\[
\frac{\partial |\v^{i+1}_{p_i}|}{\partial v^i_{p_i}} = 1.
\]  

(3.91)

**Step 3.** From (3.88), (3.89), (3.90), and (3.91),

\[
|\det J_1^k| = \frac{\sqrt{g_{p_i,11}}^3}{g_{p_i,11}} |\v^k_{p_i,3}|^2
\]

\[
= \left| \det \begin{bmatrix}
\frac{\partial \v^k_{p_i,1}}{\partial v^i_{p_i,1}} & \frac{\partial \v^k_{p_i,1}}{\partial v^i_{p_i,1}} & 0 \\
\frac{\partial \v^k_{p_i,1}}{\partial v^i_{p_i,1}} & \frac{\partial \v^k_{p_i,1}}{\partial v^i_{p_i,1}} & 0 \\
0 & 0 & 1
\end{bmatrix} \right|.
\]

Therefore, we conclude (3.81) by (3.88).

Now we study lower bounded of \( \frac{dX}{dt} \). Instead of Euclidean variable \( \v = (v_1, v_3) \), we introduce new variables via geometric decomposition. In two-dimensional cross section, we split velocity \( \v \) into speed and direction,

\[
|\v| \quad \text{and} \quad \hat{v}_1 := \frac{v_1}{|\v|}.
\]

Note that \( \{\partial_1, \partial_3\} \) are independent if \( v_3 \geq \frac{1}{N} > 0 \). So under assumption of \( v_3 \geq \frac{1}{N} > 0 \), we perform \( \partial_1, \partial_3 \), instead of \( \partial_x, \partial_y \). We assume \( \frac{1}{N} \leq |\v| \leq N \), \( t^{k+1}(t, x, v) < s < t^k(t, x, v) \), and \( |v^i_{p_i,3}| > \delta_2 > 0 \) (nongrazing) for \( 1 \leq i \leq k \).

When we differentiate \( X \) by speed \( |\v| \),

\[
\partial_{|\v|} X(s; t, x, v) = \partial_{|\v|} \left( \eta_{p_k}(x^k_{p_i,1}, 0) - (t^k - s)|\v| \hat{v}^k \right), \quad \hat{v}^k := \frac{v^k}{|\v|}
\]

\[
= \partial_{|\v|} x^k_{p_i,1} \partial_1 \eta_{p_k}(x^k_{p_i,1}, 0) + \partial_{|\v|} \left[ (t - t^k)|\v| \right] \hat{v}^k - (t - s) \partial_{|\v|} |\v| \hat{v}^k
\]

(3.92)

where we used \( \partial_{|\v|} x^k_{p_i,1} = 0 \), \( \partial_{|\v|} \left[ (t - t^k)|\v| \right] = 0 \), \( \partial_{|\v|} \hat{v}^k = 0 \), and \( \partial_{|\v|} |\v| = 1 \). Note that this is because, bouncing position \( x^k \), travel length until \( x^k \), direction of \( \v^k \) are independent to \( |\v| \).

On the other hand, differentiating \( X \) by \( \hat{v}_1 \),

\[
\partial_{\hat{v}_1} X(s; t, x, v) = \partial_{\hat{v}_1} x^k_{p_i,1} \partial_1 \eta_{p_k}(x^k_{p_i,1}, 0) - \partial_{\hat{v}_1} t^k |\v| \hat{v}^k - (t - s) |\v| \partial_{\hat{v}_1} \hat{v}^k.
\]  

(3.93)
To compute the last term \( \partial_{k} \hat{v}^{k} \), we use \( \psi_{p^{k},3}^{k} = \sqrt{1 - |\psi_{p^{k},1}^{k}|^2} \) and \( |\psi_{p^{k},1}^{k}| > 0 \) to get

\[
\partial_{k} \hat{v}^{k} = \partial_{k} \left[ \frac{\partial \eta_{p^{k}}^{k}}{\sqrt{g_{p^{k},11}}} (x_{p^{k},1}^{k}, 0) \psi_{p^{k},1}^{k} + \frac{\partial \eta_{p^{k}}^{k}}{\sqrt{g_{p^{k},33}}} (x_{p^{k},1}^{k}, 0) \sqrt{1 - |\psi_{p^{k},1}^{k}|^2} \right]
\]

\[
= \partial_{k} x_{p^{k},1}^{k} \sum_{\ell = 1,3} \partial_{k} \left[ \frac{\partial \eta_{p^{k}}^{k}}{\sqrt{g_{p^{k},\ell \ell}}} (x_{p^{k},1}^{k}, 0) \psi_{p^{k},\ell}^{k} + \frac{\partial \eta_{p^{k}}^{k}}{\sqrt{g_{p^{k},11}}} (x_{p^{k},1}^{k}, 0) \partial_{k} \hat{v}_{p^{k},1}^{k} \right]
\]

\[
- \frac{\partial \eta_{p^{k}}^{k}}{\sqrt{g_{p^{k},33}}} (x_{p^{k},1}^{k}, 0) \frac{1}{\sqrt{1 - |\psi_{p^{k},1}^{k}|^2}} \left[ \hat{v}_{p^{k},1}^{k} \partial_{k} \hat{v}_{p^{k},1}^{k} \right]
\]

\[
= \left( \sum_{\ell = 1,3} \partial_{k} \left[ \frac{\partial \eta_{p^{k}}^{k}}{\sqrt{g_{p^{k},\ell \ell}}} (x_{p^{k},1}^{k}, 0) \psi_{p^{k},\ell}^{k} \right] \partial_{k} x_{p^{k},1}^{k} \right)
\]

\[
+ \left[ \frac{\partial \eta_{p^{k}}^{k}}{\sqrt{g_{p^{k},11}}} (x_{p^{k},1}^{k}, 0) - \frac{\partial \eta_{p^{k}}^{k}}{\sqrt{g_{p^{k},33}}} (x_{p^{k},1}^{k}, 0) \frac{1}{\sqrt{1 - |\psi_{p^{k},1}^{k}|^2}} \right] \partial_{k} \hat{v}_{p^{k},1}^{k}.
\]

Combining (3.93) and (3.94), we get

\[
\partial_{k} \left[ X(s; t, x, v) \right] = -\partial_{k} (t^{k} v^{k}) + \partial_{k} x_{p^{k},1}^{k} \partial_{k} \eta_{p^{k}}^{k} (x_{p^{k},1}^{k}, 0)
\]

\[
- (t^{k} - s) \psi_{p^{k},1}^{k} \left[ \sum_{\ell = 1,3} \partial_{k} \left[ \frac{\partial \eta_{p^{k}}^{k}}{\sqrt{g_{p^{k},\ell \ell}}} (x_{p^{k},1}^{k}, 0) \psi_{p^{k},\ell}^{k} \right] \right] \partial_{k} x_{p^{k},1}^{k}
\]

\[
- (t^{k} - s) \psi_{p^{k},1}^{k} \left[ \frac{\partial \eta_{p^{k}}^{k}}{\sqrt{g_{p^{k},11}}} (x_{p^{k},1}^{k}, 0) - \frac{\partial \eta_{p^{k}}^{k}}{\sqrt{g_{p^{k},33}}} (x_{p^{k},1}^{k}, 0) \frac{1}{\sqrt{1 - |\psi_{p^{k},1}^{k}|^2}} \right] \partial_{k} \hat{v}_{p^{k},1}^{k}.
\]
and
\[
\left| \frac{\partial \eta_{x^k}}{\sqrt{|g_{p^k,11}|}} \cdot e_1 \right| > \frac{1}{N} > 0,
\] (3.101)
for some uniform \(\delta_2 > 0\). Then there exist at least one \(i \in \{1, 2\}\) such that
\[
|R_{i-1}^{k,p_k}(t, x, v)| > \varrho_{i,N,\delta_2},
\] (3.102)
for some constant \(\varrho_{i,N,\delta_2} > 0\).

Proof. First we claim that
\[
|\det(S_{k,p^k})| > \varrho_{i,N,\delta_2}.
\]
We suffice to compute diagonal entries. From (3.97),
\[
|\mathcal{S}_{k,p^k}^1| = |\sqrt{|g_{p^k,11}|} \cdot \frac{\partial \eta_{x^k}}{\sqrt{|g_{p^k,11}|}}| |v^k^3| = \sqrt{|g_{p^k,11}|} |v^k_{p^k,3}| |v^k|,
\]
and
\[
|\mathcal{S}_{k,p^k}^2| = \left[ \frac{\partial \eta_{x^k}}{\sqrt{|g_{p^k,11}|}} - \frac{\partial \eta_{x^k}}{\sqrt{|g_{p^k,33}|}} \right] |v^k_{p^k,3}| |v^k_{p^k,1}| |v^k|,
\]
which implies uniform invertibility of \(2 \times 2\) matrix \(S_{k,p^k}^{k,p^k}\). To consider \(2 \times 1\) vector on the RHS of (3.99), we compute
\[
\begin{pmatrix}
\frac{\partial \eta_{x^k}}{\sqrt{|g_{p^k,11}|}} \\
\frac{\partial \eta_{x^k}}{\sqrt{|g_{p^k,33}|}}
\end{pmatrix}
\begin{pmatrix}
\frac{\partial \eta_{x^k}}{\sqrt{|g_{p^k,11}|}} \\
\frac{\partial \eta_{x^k}}{\sqrt{|g_{p^k,33}|}}
\end{pmatrix} = \left[ \begin{array}{c}
\frac{\partial \eta_{x^k}}{\sqrt{|g_{p^k,11}|}} \\
\frac{\partial \eta_{x^k}}{\sqrt{|g_{p^k,33}|}}
\end{array} \right] = \left[ \begin{array}{c}
\frac{\partial \eta_{x^k}}{\sqrt{|g_{p^k,11}|}} \\
\frac{\partial \eta_{x^k}}{\sqrt{|g_{p^k,33}|}}
\end{array} \right] = \left[ \begin{array}{c}
\frac{\partial \eta_{x^k}}{\sqrt{|g_{p^k,11}|}} \\
\frac{\partial \eta_{x^k}}{\sqrt{|g_{p^k,33}|}}
\end{array} \right]
\]
where we used (3.72) and (3.73). Determinant of \(A\) is uniformly nonzero from (3.81) in Lemma 15. From elementary row operation for \(B\),
\[
\text{det } B = \text{det } \begin{pmatrix}
\frac{\partial \eta_{x^k}}{\sqrt{|g_{p^k,11}|}} \\
0
\end{pmatrix} = \left| \frac{\partial \eta_{x^k}}{\sqrt{|g_{p^k,11}|}} \cdot e_1 \right|.
\]
From (3.70), (1, 1) entry of matrix \(B\) is computed by
\[
\left| \frac{\partial \eta_{x^k}}{\sqrt{|g_{p^k,11}|}} \cdot \left[ \frac{\partial \eta_{x^k}}{\sqrt{|g_{p^k,11}|}} + \frac{\partial \eta_{x^k}}{\sqrt{|g_{p^k,33}|}} \right] \right| = \left| \frac{1}{\sqrt{|g_{p^k,11}|}} \left( \frac{1}{|v_{p^k,3}^3|} \cdot e_1 \right) \right| = \left| \frac{1}{|g_{p^k,11}(x^k)|} \right| |v_{p^k,3}|.
\]
Therefore, from (3.100) and (3.101), determinant of \(B\) is uniformly nonzero and thus LHS of (3.103) has also uniformly nonzero determinant. This yields uniform nonzeroness of second column, i.e., \(\left[ \begin{array}{c}
\frac{\partial \eta_{x^k}}{\sqrt{|g_{p^k,11}|}} \\
\frac{\partial \eta_{x^k}}{\sqrt{|g_{p^k,33}|}}
\end{array} \right] \). From uniform invertibility of matrix \(S_{k,p^k}^{k,p^k}\) and (3.99), we finish the proof. \(\square\)
Lemma 17. Assume that \( b(z), c(z) \) are continuous-functions of \( z \in \mathbb{R}^n \) locally. We consider \( G(z, s) := b(z)s + c(z) \).

(i) Assume \( \min |b| > 0 \). Define

\[
\varphi_1(z) := -\frac{c(z)}{b(z)}. \tag{3.104}
\]

Then \( \varphi_1(z) \in C^1_{t,x,v} \) with \( \|\varphi_1\|_{C^1_{t,x,v}} \leq C(\min |b|, \|b\|_{C^1_{t,x,v}}, \|c\|_{c^1_{t,x,v}}) \). Moreover, if \( |s| \leq 1 \) and \( |s - \varphi_1(z)| > \delta \), then \( |G(z, s)| \geq \min |b| \times \delta \).

(ii) Assume \( \min |c| > 0 \). Define

\[
\varphi_2(z) := \frac{1}{|b(z)|>\min|c|} -\frac{c(z)}{b(z)}. \tag{3.105}
\]

Then \( \varphi_2(z) \in C^1_{t,x,v} \) with \( \|\varphi_2\|_{C^1_{t,x,v}} \leq C(\min |b|, \|b\|_{C^1_{t,x,v}}, \|c\|_{c^1_{t,x,v}}) \). Moreover, if \( |s| \leq 1 \) and \( |s - \varphi_5(z)| > \delta \), then \( |G(z, s)| \geq \min \left\{ \frac{\min|c|}{2}, \frac{|\min|c|}{2} \times \delta \right\} \).

Proof. Now we consider (i). Clearly \( \varphi_1 \) is \( C \) for this case. And

\[
|G(z, s)| \geq \min \{|b(z)(\frac{c(z)}{b(z)} \delta + c(z)), |b(z)(\frac{-c(z)}{b(z)} \delta + c(z))|\} \geq \min |b| \times \delta.
\]

Now we consider (ii). First, if \( |b| < \frac{\min|c|}{2} \) then \( |\varphi_2(z)| \geq \frac{|c(z)|}{\min|c|/2} \geq 2 \). Therefore,

\[
|G(z, s)| \geq \min \{|G(z, 1)|, |G(z, -1)|\} \geq |c(z)| - |b(z)| \geq \frac{\min|c|}{2}.
\]

Consider the case of \( |b| > \frac{\min|c|}{2} \). If \( |s - \varphi_2(s)| > \delta \) then

\[
|G(z, s)| \geq \min \{|b(z)(\frac{-c(z)}{b(z)} \delta + c(z)), |b(z)(\frac{-c(z)}{b(z)} \delta + c(z))|\}
= \min |b| \times \delta \geq \frac{\min|c|}{2} \times \delta.
\]

\( \square \)

Lemma 18. Fix \( k \in \mathbb{N} \) with \( t^k \geq t \geq 1 \). Assume \( \Omega \) is \( C^2 \) and \( \|t^k - t\| \leq \epsilon \). Let \( t^0 \geq 0 \), \( x^0, x^1 \in \Omega \), \( b^0 \in \mathbb{R}^2 \), and assume

\[
0 \leq |x^0| \leq N, \quad \min \{\|t^1\|, \|t^0\| \} > \delta_2 > 0, \quad \forall 1 \leq i \leq k,
\tag{3.106}
\]

and \( \|x^1 - x^0\| \leq \delta_2 > 0 \) in Lemma 16, where \( (x^0, x^1) := (x^0(t^0, x^0, v^0), x^1(t^0, x^0, v^0)) \). Then there exists \( \epsilon > 0 \) and \( C^1_{t,x,v} \)-functions \( \psi_1^k, \psi_2^k : B_{\epsilon}(t, x, v) \to \mathbb{R} \) with \( \max |\psi_1^k|_{C^1_{t,x,v}} \lesssim_{\delta_2,\Omega,N} 1 \) and there exists a constant \( \epsilon_{\delta_2,\Omega,N} > 0 \), such that

\[
\frac{\min i=1,2}{\psi_1^k(t, x, v)} > \delta_s
\]

and \( s; t, x, v \) \in \([\max\{t - 1, t^k + 1\}, \min\{t - 1/N, t^k\}] \times B_{\epsilon}(t^0, x^0, v^0) \), \( t^k \). Then \( |\partial_2 X(s; t, x, v) \times \partial_1 X(s; t, x, v)| > \epsilon_{\delta_2,\Omega,N,\delta_s} \).

It is important that this lower bound \( \epsilon_{\delta_2,\Omega,N} \) does not depend on time \( t \).

Proof. Step 1. Fix \( k \) with \( |t^{k}(t, x, v) - t| \leq 1 \). Then we fix the orthonormal basis \( \{e_0^k, e_{\perp,1}^k\} \) of \( 3.97 \) with \( z^k = z^k(t, x, v) \), \( z^k = z^k(t, x, v) \). Note that this orthonormal basis \( \{e_0^k, e_{\perp,1}^k\} \) depends on \( (t, x, v) \).

For \( s < t^k \), recall the forms of \( \frac{\partial X(s)}{\partial t} \) and \( \frac{\partial X(s)}{\partial v} \) in \( 3.92 \) and \( 3.95 \), where \( \overline{X(s)} = \overline{X(s; t^k, x^k, v^k)} \). Using the specular basis \( 3.97 \), we rewrite \( 3.92 \) and \( 3.95 \) as

\[
\begin{bmatrix}
\frac{\partial X(s)}{\partial t} \cdot e_0^k \\
\frac{\partial X(s)}{\partial v} \cdot e_0^k \\
\frac{\partial X(s)}{\partial t} \cdot e_{\perp,1}^k \\
\frac{\partial X(s)}{\partial v} \cdot e_{\perp,1}^k
\end{bmatrix} = \begin{bmatrix}
(t - s) \frac{\partial X(s)}{\partial t} \cdot e_0^k \\
0 \frac{\partial X(s)}{\partial v} \cdot e_0^k \\
(t - s) \frac{\partial X(s)}{\partial t} \cdot e_{\perp,1}^k \\
0 \frac{\partial X(s)}{\partial v} \cdot e_{\perp,1}^k
\end{bmatrix}.
\]

Note that \( (2,2) \) component is written by

\[
\frac{\partial X(s)}{\partial v} \cdot e_{\perp,1}^k = \mathcal{R}_{1,p}^k - (t^k - s)\mathcal{R}_{2,p}^k,
\]

by \( 3.95 \) and \( 3.98 \), where \( \mathcal{R}_{1,p}^k \) are defined in \( 3.99 \). By the direct computation, determinant becomes

\[
\partial_2 \overline{X}(s) \times \partial_1 \overline{X}(s) = -(t - s) \{ \mathcal{R}_{1,p}^k - (t^k - s)\mathcal{R}_{2,p}^k \}.
\tag{3.107}
\]
Here $R_i^{k,p^k}, t^k, v_{p^k}^k,$ and $e_{i,j}^k$ depend on $(t, x, y)$, but not $s$.

Step 2. Recall Lemma 16. From (3.106), we can choose non-zero constants $\delta_2$ for a large $N \gg 1$. Applying Lemma 16 and (3.109), we conclude that, for some $i \in \{1, 2\}$,

$$|R_i^{k,p^k}(t, x, y)| > \frac{\Omega N \delta_2}{2}.$$  \hspace{1cm} (3.108)

Also, we can claim that $R_i^{k,p^k}(t, x, y) \in C^1_{t,x,y}$. From (3.106), all bouncings are non-grazing. We use Lemma 5, (3.71) and (3.73) in Lemma 13, and (3.99) with regularity of $\Omega$ to derive $R_i^{k,p^k}(t, x, y) \in C^1_{t,x,y}$. Finally we choose a small constant $\varepsilon > 0$ such that, for some $i \in \{1, 2\}$ satisfying (3.108),

$$|R_i^{k,p^k}(t, x, y)| > \frac{\Omega N \delta_2}{2} \quad \text{for } |(t, x, y) - (t^0, x^0, y^0)| < \varepsilon.$$  \hspace{1cm} (3.109)

Step 3. With $N \gg 1$, from (3.109), we divide the cases into the follows

$$|R_i^{k,p^k}| > \frac{\Omega N \delta_2}{2} \quad \text{and} \quad |R_2^{k,p^k}| \geq \frac{\Omega N \delta_2}{2}.$$  \hspace{1cm} (3.110)

We split the first case (3.110) further into two cases as

$$|R_i^{k,p^k}| > \frac{\Omega N \delta_2}{2} \quad \text{and} \quad |R_2^{k,p^k}| \geq \frac{\Omega N \delta_2}{4N},$$

and

$$|R_i^{k,p^k}| > \frac{\Omega N \delta_2}{2} \quad \text{and} \quad |R_2^{k,p^k}| \geq \frac{\Omega N \delta_2}{4N}.$$  \hspace{1cm} (3.111)

Set the other case

$$|R_2^{k,p^k}| \geq \frac{\Omega N \delta_2}{2}.$$  \hspace{1cm} (3.112)

Then clearly (3.111) and (3.112) cover all the cases.

Step 4. We consider the case of (3.111). Then, from (3.107),

$$|\partial_x X(s) \times \partial_{\bar{v}} X(s)| \geq |u^k| |R_2^{k,p^k}(t - s) - R_1^{k,p^k}|(t - s)$$

$$= |u^k| |R_2^{k,p^k}(t - s) + [ - R_1^{k,p^k} + (t^k - t)|u^k| |R_2^{k,p^k}|]|(t - s).$$  \hspace{1cm} (3.113)

We define

$$\bar{s} = t - s,$$  \hspace{1cm} (3.114)

and set

$$b := |u^k| |R_2^{k,p^k} \quad \text{and} \quad c := - R_1^{k,p^k} + (t^k - t)|u^k| |R_2^{k,p^k}|.$$  \hspace{1cm} (3.115)

Note that $R_i^{k,p^k}, R_2^{k,p^k}, |u^k|,$ and $t^k$ only depend on $(t, x, y)$.

Hence we regard the underbraced term of (3.113) as an affine function of $\bar{s}$

$$b(t, x, y)\bar{s} + c(t, x, y).$$

Note that from (3.111)

$$|c(t, x, y)| \geq \frac{\Omega N \delta_2}{2} - N \frac{\Omega N \delta_2}{4N} \geq \frac{\Omega N \delta_2}{4}.$$  \hspace{1cm} (3.116)

Now we apply (ii) of Lemma 17. With $\varphi_2(t, x, y)$ in (3.105), if $|\bar{s} - \varphi_2(t, x, y)| > \delta_*$, then $|b(t, x, y)\bar{s} + c(t, x, y)| \geq \frac{\Omega N \delta_2}{4} \times \delta_*$. We set

$$\psi_2(t, x, y) = t - \varphi_2(t, x, y).$$

From (3.114),

$$|s - \psi_2(t, x, y)| > \delta_* \quad \text{then} \quad |b(t, x, y)(s - \psi_2(t, x, y)) + c(t, x, y)| \geq \frac{\Omega N \delta_2}{4} \times \delta_*.$$  \hspace{1cm} (3.117)

Now we consider the case of (3.112). From (3.107),

$$|\partial_x X(s) \times \partial_{\bar{v}} X(s)| \geq |u^k| |R_2^{k,p^k}(t - s) - R_1^{k,p^k}|(t - s)$$

$$= |u^k| |R_2^{k,p^k}(t - s) + [ - R_1^{k,p^k} + (t^k - t)|u^k| |R_2^{k,p^k}|]|(t - s).$$  \hspace{1cm} (3.118)

We set $\bar{s}$ as (3.114) and

$$b := |u^k| |R_2^{k,p^k} \quad \text{and} \quad c := - R_1^{k,p^k} + (t^k - t)|u^k| |R_2^{k,p^k}|.$$  \hspace{1cm} (3.119)

From (3.112) and (3.117)

$$|b(t, x, y)| \geq \frac{\Omega N \delta_2}{8N \delta_*}.$$  \hspace{1cm} (3.120)

We apply (i) of Lemma 17 to this case: With $\varphi_1(t, x, y)$ in (3.104), we set

$$\psi_4(t, x, y) = t - \varphi_1(t, x, y).$$
and

\[ \text{if } |s - \psi_1(t, x, v)| > \delta_*, \text{ then } |b(t, x, v)(t - s) + c(t, x, v)| \geq \frac{\rho_{1,N,\delta}}{8N^2} \times \delta_. \] (3.118)

Finally, from (3.115), (3.113), (3.118), and (3.116), we conclude the proof of Lemma 18.

Now we return to three-dimensional cylindrical domain \( U := \Omega \times (0, H) \subset \mathbb{R}^3 \). We state a theorem about uniform positivity of determinant of \( \frac{\partial X}{\partial v} \).

**Proposition 2.** Let \( t \in [T, T + 1] \),

\[ (x, v) = (x, \xi, v_2) \in U \times \mathbb{V}^N \times \{v_2 \in \mathbb{R} : \frac{1}{N} \leq v_2 \leq N\}. \]

Recall \( \varepsilon, \delta \) in Lemma 4. For each \( i = 1, 2, \cdots, l_G \), there exists \( \delta_2 > 0 \) and \( C_{t, \xi, v}^i \)-function \( \psi_{\ell_0, \ell, i}^i \) for uniform bound \( k \leq C_{\varepsilon, N} \), where \( \psi_{\ell_0, \ell, i}^i \) is defined locally around \( (T + \delta_2 \ell_0, X(T + \delta_2 \ell_0; t, x, v), (\delta_2 \ell_0, u_2)) \) with \( (\ell_0, \ell, \ell) \in \{0, 1, \cdots, \lfloor \frac{1}{N} \rfloor + 1\} \times \{-\frac{1}{N}, -1, \cdots, 0, \cdots, \lfloor \frac{N}{2} \rfloor + 1\}^2 \) and \( \|\psi_{\ell_0, \ell, i}^i\|_{c_{t, \xi, v}^i} \leq C_{N, \Omega, \delta, \delta_2} < \infty \).

For \((X(s; t, x, v), u) \in \{c(\Omega) \times \mathbb{V}^N\}\backslash \mathfrak{I}_\mathcal{B}, \) if

\[ u_2 \geq \frac{1}{N}, \] (3.119)

\[ \left| \frac{\partial_1 \eta_{p_1}}{\sqrt{\eta_{p_1^3}}^{11}} \right|_{x'(X(s; t, x, v), v) \cdot e_1} > \frac{1}{N} > 0, \] (3.120)

\[ X(s; t, x, v) \notin \bigcup_{j=1}^{l_G} B(y_i^c, \varepsilon), \quad \text{sticky grazing set defined in Lemma 4} \] (3.121)

\[ (X(s; t, x, v), u) \in B(x_i^c, r_i^c) \times \mathbb{V}^N \backslash \mathcal{O}_i^C \text{ for some } i = 1, 2, \cdots, l_G, \] (3.122)

\[ (s, u) \in [T + (\ell_0 - 1)\delta_2, T + (\ell_0 + 1)\delta_2] \times B(\delta_2 \ell_0, 2\delta_2), \] (3.123)

\[ |s - s'| \geq \delta_2, \] (3.124)

\[ s' \in \left[ 0, 1 \right] \left[ T + \delta_2 \ell_0; X(T + \delta_2 \ell_0; t, x, v), \delta_2 \ell_0 \right] + \frac{1}{N}, \] (3.125)

and

\[ \left| s' - \psi_{\ell_0, \ell, i}^i(T + \delta_2 \ell_0; X(T + \delta_2 \ell_0; t, x, v), \delta_2 \ell_0) \right| > N^2(1 + \|\psi_{\ell_0, \ell, i}^i\|_{c_{t, \xi, v}^i})\delta_2, \] (3.126)

then

\[ \text{det} \left( \frac{\partial X(s'; s, X(s; t, x, v), u)}{\partial u} \right) > \epsilon'_{t, \Omega, \varepsilon, \delta, \delta_2} > 0, \] (3.127)

where \( B(x_i^c, r_i^c) \times \mathbb{V}^N \backslash \mathcal{O}_i^C \) was constructed in Lemma 4. Also note that \( \epsilon'_{t, \Omega, \varepsilon, \delta, \delta_2} \) does not depend on \( T, t, x, v \).

**Proof.** Step 1. First we extend two-dimensional analysis into three dimension case. For \( v_2 \) direction, dynamics is very simple, i.e.

\[ X_2(s; t, x, v) = x_2 - (t - s)v_2, \]

so we have

\[ \frac{dX_2}{dv_2} = -(t - s). \]

Note that it is obvious that \( v_2 \) directional dynamics is independent to two-dimensional trajectory which is projected on cross section \( \Omega \), because of cylindrical domain with the specular boundary condition.

Step 2. Fix \( t \in [T, T + 1] \), \( (x, v) \in \Omega \times \mathbb{V}^N \) and assume \((X(s; t, x, v), u) \in \{c(\Omega) \times \mathbb{V}^N\}\backslash \mathfrak{I}_\mathcal{B} \). Assume that \( s \in [T, t] \),

\[ X(s; t, x, v) \notin \bigcup_{j=1}^{l_G} B(y_i^c, \varepsilon) \quad \text{and} \quad (X(s; t, x, v), u) \in B(x_i^c, r_i^c) \times \mathbb{V}^N \backslash \mathcal{O}_i^C, \]

38
for some $i = 1, \ldots, l_G$. Due to Lemma 11, $(X(s'; s, X(s; t, x, y), u), V(s'; s, X(s; t, x, y), u))$ is well-defined for all $s' \in [T, s]$ and

$$|n(x^k(s, X(s; t, x, y), u)) \cdot \psi^k(s, X(s; t, x, y), u)| > \delta,$$

for all $k$ with $|t - t^k(s, X(s; t, x, y), u)| \leq 1$.

From $X(s; t, x, y) = X(s; t, x, y) + \int_t^s V(\tau; t, x, y) d\tau$,

$$|\psi^k(s, X(s; t, x, y), u) - \psi^k(s, X(s; t, x, y), u)| \leq \|\psi^k\|_{C_{i \pm 2}} \{ s - \bar{s} \} + |X(s; t, x, y) - X(s; t, x, y)| + |u - \bar{u}|$$

(3.128)

For $0 < \delta_2 \ll 1$ we split

$$[T, T+1] = \bigcup_{\ell_0=0}^{\lceil \delta_2^{-1} \rceil + 1} [T + (\ell_0 - 1)\delta_2, T + (\ell_0 + 1)\delta_2],$$

$$\mathcal{V}_i \setminus \mathcal{O}_i^\mathcal{C} = \bigcup_{|\ell| = 0}^{N/\delta_2^2 + 1} B(\ell_1 \delta_2, \ell_3 \delta_2, 2\delta_2) \cap \mathcal{V}_i \setminus \mathcal{O}_i^\mathcal{C}.$$

From (3.128), if $(s, u) \in [T + (\ell_0 - 1)\delta_2, T + (\ell_0 + 1)\delta_2] \times \{B(\ell_1 \delta_2, \ell_3 \delta_2, 2\delta_2) \cap \mathcal{V}_i \setminus \mathcal{O}_i^\mathcal{C}\}$, then

$$|\psi^k(T + \ell_0 \delta, X(T + \ell_0 \delta; t, x, y), (\ell_1 \delta, \ell_3 \delta)) - \psi^k(s, X(s; t, x, y), u)| \leq \|\psi^k\|_{C_{i \pm 2}} (2 + N)\delta_2.$$

Therefore, if (3.126) holds,

$$|s' - \psi^k(s, X(s; t, x, y), u)| \geq (N^2 - N)\|\psi^k\|_{C_{i \pm 2}} \delta_2 \geq N\|\psi^k\|_{C_{i \pm 2}} \delta_2.$$

Step 3. Consider the three-dimensional mapping $u \mapsto X(s'; s, X(s; t, x, y), u)$. Note that from Lemma 11 we verify the condition of Lemma 18. From Lemma 18 and 6, we construct $C_{i \pm 2}$-function $\psi^k : B_c(s, X(s; t, x, y), u) \to \mathbb{R}$ for uniform bound $k \leq C_{e,N}$ such that if $|s' - \psi^k(s, X(s; t, x, y), u)| \geq N\Omega, \delta \delta_2$, then

$$\left| \det \left( \frac{\partial X(s'; s, X(s; t, x, y), u)}{\partial u} \right) \right| = \left| \frac{dX_2}{dv_2} \left| \partial_{u_1} X(s'; s, X(s; t, x, y), u) \times \partial_{u_2} X(s'; s, X(s; t, x, y), u) \right| \right| > |s - s'| \epsilon_{\Omega, N, \delta, \delta_2} > \epsilon'_{\Omega, N, \delta, \delta_2} > 0.$$

Now we study $L^\infty$ estimate via trajectory and Duhamel’s principle.

**Lemma 19.** Let $f$ solves linear boltzmann equation (1.14). For $h := w f$ with $w = (1 + |v|)^\beta$, $\beta > 5/2$, we have the following estimate.

$$\|h(t)\|_\infty \lesssim e^{-\nu t} \|h(0)\|_\infty + \int_0^t \|f(s)\|_2 ds.$$

**Proof.** Since $L = \nu(v) - K$,

$$\partial_t f + v \cdot \nabla f + \nu f = Kf.$$

For $h := w f$

$$\partial_t h + v \cdot \nabla h + \nu h = K_w h, \quad K_w h := w K(\frac{h}{w}).$$
We define,
\[ E(v, t, x) := \exp \left\{ - \int_t^s \nu(V(\tau)) \right\}. \]

Along the trajectory,
\[
\frac{d}{ds} \left( E(v, t, s)h(s, X(s; t, x), V(s; t, x, v)) \right) = E(v, t, s) \left[ K_w h \right](s, X(s; t, x, v), V(s; t, x, v)).
\]

By integrating from 0 to \( t \), we obtain
\[
h(t, x, v) = E(v, t, 0)h(0, X(0), V(0)) + \int_0^t E(v, t, s) \int_{\mathbb{R}^3} k_w(u, V(s))h(s, X(s; t, x, v), u)du ds.
\]

(3.129)

Recall the standard estimates (see Lemma 4 and Lemma 5 in [7])
\[
\int_{\mathbb{R}^3} |k_w(v, u)|du \leq C_K(v)^{-1}.
\]

We apply Duhamel’s formula (3.129) two times, for sufficiently small \( 0 < \delta \ll 1 \), and cut a part of domain where change of variable does not work. Especially, we use Lemma 4 and split sticky grazing set.

\[
h(t, x, v) = E(v, t, 0)h(0, X(0), V(0)) + \int_0^t E(v, t, s) \int_{\mathbb{R}^3} k_w(u, V(s))h(s, X(s; t, x, v), u)du ds
\]

\[
\leq E(v, t, 0)h(0) + \int_0^t E(v, t, s) \int_{\mathbb{R}^3} k_w(u, v)E(u, s, 0)h(0) + ||(E_1) + ||(E_2)|| + ||(E_3)|| + ||(E_4)|| + ||(E_5)||,
\]

(3.130)

where
\[
(E_k) := \int_0^t E(v, t, s) \int_{\mathbb{R}^3} k_w(u, v) \int_{\mathbb{R}^3} k_w(u', v)h(s', X(s'), u') 1_{E_k}(X(s), u), \quad k = 1, 2, 3, 4, 5.
\]

(3.131)

Note that we abbreviated notations
\[
X(s) := X(s; t, x, v), \quad X(s') := X'(s'; s, X(s; t, x, v), u),
\]

and \( E_k \) in characteristic functions in (3.131) are defined as

\[
E_1 := \{ (X(s), u) \in \mathbb{R}^3 \times \mathbb{R}^3 : u \in \mathbb{R}^2 \setminus \nu^N \text{ or } |u_2| \in \mathbb{R}\left| \frac{1}{N}, N \right| \},
\]

\[
E_2 := \{ (X(s), u) \in \mathbb{R}^3 \times \mathbb{R}^3 : (u, u_2) \in \nu^N \times [\frac{1}{N}, N], (X(s), u) \in \mathcal{B} \},
\]

\[
E_3 := \{ (X(s), u) \in \mathbb{R}^3 \times \mathbb{R}^3 : (u, u_2) \in \nu^N \times [\frac{1}{N}, N], (X(s), u) \in \{d(\Omega) \times \nu^N\} \setminus \mathcal{B},
\]

\[
X(s) \in \bigcup_{j=1}^{l_{sg}} B(y_j^C, \varepsilon)
\]

\[
E_4 := \{ (X(s), u) \in \mathbb{R}^3 \times \mathbb{R}^3 : (u, u_2) \in \nu^N \times [\frac{1}{N}, N], (X(s), u) \in \{d(\Omega) \times \nu^N\} \setminus \mathcal{B},
\]

\[
(X(s), u) \in \left\{ \bigcup_{i=1}^{l_{sg}} B(x_i^C, r_i^C) \times \mathcal{C}_i^C \right\} \setminus \left\{ \bigcup_{j=1}^{l_{sg}} B(y_j^C, \varepsilon) \times \nu^N \right\},
\]

and

\[
E_5 := \{ (X(s), u) \in \mathbb{R}^3 \times \mathbb{R}^3 : (u, u_2) \in \nu^N \times [\frac{1}{N}, N], (X(s), u) \in \{d(\Omega) \times \nu^N\} \setminus \mathcal{B},
\]

\[
(X(s), u) \in \left\{ B(x_i^C, r_i^C) \times \nu^N \setminus \mathcal{C}_i^C \right\} \setminus \left\{ \bigcup_{j=1}^{l_{sg}} B(y_j^C, \varepsilon) \times \nu^N \right\} \text{ for some } i = 1, \ldots, l_{sg}
\]

(3.132)
Also note that
\[
(\mathcal{G})_z := \left\{ \bigcup_{i=1}^{l_G} B(x_i^C, \rho_i^C) \times O_i^C \right\} \cup \left\{ \bigcup_{j=1}^{l_{lg}} B(y_j^C, \epsilon) \times \mathbb{V}^N \right\}
\]
was defined in Lemma 1 and we have
\[
E(v, t, s) \leq e^{-\nu(s)(t-s)}.
\]
On the RHS of (3.130), every term except \((E_1), (E_2), (E_3), (E_4),\) and \((E_5)\), are controlled by
\[
Ce^{-\nu t} \|h(0)\|_{\infty}
\]
We claim smallness of \((E_1) \sim (E_4)\). From \(\int u \: 1_{\{u \in \mathbb{R}^2 \setminus \mathbb{V}^N \text{ or } |u_2| \leq \frac{1}{N} \}}(u) \sqrt{\mu} du = O(\frac{1}{N})\),
\[
(E_1) \leq O\left(\frac{1}{N}\right) \sup_{0 \leq s \leq t} \|h(s)\|_{\infty}.
\]
From Lemma 4 \(m_2(O_i^{IB}) \leq \epsilon\) for \(1 \leq i \leq l_{IB}\). Therefore,
\[
(E_2) \leq O(\epsilon) \sup_{0 \leq s \leq t} \|h(s)\|_{\infty}.
\]
For \((E_3)\), we also have similar estimate because
\[
(E_3) \leq \int_0^t ds \sup_{s \in [0, t]} \left| h(s) \right| \left| h(s) \right|_{\infty}
\]
\[
\leq C \epsilon \sup_{0 \leq s \leq t} \left| h(s) \right|_{\infty}, \text{ since } \nu \geq \frac{2}{N},
\]
\[
\leq C \epsilon N \sup_{0 \leq s \leq t} \left| h(s) \right|_{\infty} \leq O\left(\frac{1}{N}\right) \sup_{0 \leq s \leq t} \left| h(s) \right|_{\infty}.
\]
For estimate for \((E_4)\), since \(m_2(O_i^{C}) < \epsilon\) from Lemma 1,
\[
(E_4) \leq O(\epsilon) \sup_{0 \leq s \leq t} \left| h(s) \right|_{\infty}.
\]
For \((E_5)\), we choose \(m(N)\) so that
\[
k_{w,m}(u, v) := 1_{\{|u-v| \geq \frac{1}{N}, \ |u| \leq m\}} k_w(u, v),
\]
satisfies \(\int_{\mathbb{R}^3} |k_{w,m}(u, v) - k_w(u, v)| \: du \leq \frac{1}{N}\) for sufficiently large \(N \geq 1\). Then, by splitting \(k_w\),
\[
(E_5) \leq \int_0^t \int_0^s e^{-\nu(s')(s'-s)} \int u k_{w,m}(u, v) \int u' k_{w,m}(u', u) h(s', X'(s'), u') \: duds'ds
\]
\[
+ O\left(\frac{1}{N}\right) \sup_{0 \leq s \leq t} \left| h(s) \right|_{\infty}.
\]
We define following sets for fixed \(n, \tilde{n}, i, k,\), where Proposition 2 does not work.
\[
R_1 := \{ u \mid u \notin B(\tilde{n}, 2\delta) \cap \{ \mathbb{R}^2 \setminus O_i^C \} \},
\]
\[
R_2 := \{ s' \mid |s - s'| \leq \delta \},
\]
\[
R_3 := \{ s' \mid \max_{i=1,2} |s' - \psi_{1,\tilde{n},k}(n\delta, X(n\delta; t, x, \nu), (\tilde{n}, u_2))| \leq N \delta \left| \psi_1 \right|_{\infty} \},
\]
\[
R_4 := \{ s' \mid \left| |s' - t^k(n\delta, X(n\delta; t, x, \nu), (\tilde{n}, u_2))| \leq N \delta \left| \psi_1 \right|_{\infty} \},
\]
\[
R_5 := \{ u \mid |u_3| \leq \frac{1}{N} \},
\]
\[
R_6 := \{ u \in \mathbb{R}^2 \mid \left| \frac{\partial \psi_{1}}{\sqrt{g_{t11}}} \right| \leq \frac{1}{N} \}. \quad (3.139)
\]
Using (3.139), we write (**) as

$$
(**) = \sum_{n=0}^{[t/\delta]+1} \sum_{|\bar{\nu}| \leq N} \sum_{k} \int_{(n-1)\delta}^{(n+1)\delta} \int_{k+1}^{t} e^{-\nu(v)(t-s')} \times \int_{|u| \leq N, |w| \leq N} k_{w,m}(u, v) k_{w,m}(u', u) |h(s', X(s'), u')| \\
\times 1_{R_1^c \cap R_2^c \cap R_3^c \cap R_4^c \cap R_5^c \cap R_6^c} f_E (X(s), u)_{(\text{MAIN})} + R,
$$

(3.140)

where \( R \) corresponds to where \((u, s')\) is in one of \(R_1 \sim R_6\). We replace \(1_{R_1^c \cap R_2^c \cap R_3^c \cap R_4^c \cap R_5^c \cap R_6^c}\) into \(1_{R_1 \cup R_2 \cup R_3 \cup R_4 \cup R_5 \cup R_6}\) in (MAIN). For \( R \), we have the following smallness estimate:

$$
R \leq \int_0^t \int_0^s e^{-\frac{1}{2} \nu(v)(t-s')} \int_{|u| \leq N} k_{w,m}(u, v) \int_{|w| \leq N} k_{w,m}(u', u) h(s', X(s'), u') \times 1_{R_1 \cup R_2 \cup R_3 \cup R_4 \cup R_5 \cup R_6} \\
\leq C_N (\delta + \varepsilon + O(\frac{1}{N})) \sup_{0 \leq s \leq t} \| h(s) \|_\infty,
$$

(3.141)

by choosing sufficiently small \( \delta \ll \frac{1}{N} \). Note that smallness from \( R_3 \) to \( R_5 \) are trivial. For \( R_6 \), we note that by analyticity and boundness of \( \Omega \), there are only finite points \( x \) such that \( \frac{\partial X_{n}^{e}}{\sqrt{\rho^{i} \pi \eta}} \bigg|_{x \in \partial \Omega} \cdot e_1 = 0 \), so \( R_6 \) gives smallness \( O(\frac{1}{N}) \).

Let us focus on (MAIN) in (3.140). From (3.132) and (3.139), all conditions (3.119)-(3.126) in Proposition 2 are satisfied and

$$
\exists s \in \{1, 2, \cdots, l_G \} \quad \text{such that} \quad X(s) \in B(x_s^C, r_s^C).
$$

Under the condition of \((u, s') \in R_1^c \cap R_2^c \cap R_3^c \cap R_4^c \cap R_5^c \cap R_6^c\), indices \( n, \bar{n}, i_s, k \) are determined so that

$$
\begin{align*}
\bar{X}(s; t, \bar{x}, v) & \in B(x_s^C, r_s^C), \\
v & \in B(\bar{n} \delta, 2\delta) \cap \{V^N \setminus O_i^C\},
\end{align*}
$$

and (3.127) in Proposition 2 gives local time-independent lower bound

$$
| \det \left( \frac{\partial X(s')}{\partial u} \right) | > \epsilon_0 > 0.
$$

If we choose sufficiently small \( \delta \), there exist small \( r_{\delta, n, \bar{n}, i_s, k} \) such that there exist one-to-one map \( M \),

$$
M : B(\bar{n} \delta, 2\delta) \cap \{V^N \setminus O_i^C\} \rightarrow B(\bar{X}(s'; s, \bar{X}(s; t, \bar{x}, v), v), r_{\delta, n, \bar{n}, i_s, k}).
$$

So we perform change of variable for (MAIN) in (3.140) to obtain
We collect (3.130), (3.133), (3.134)–(3.137), (3.138), (3.140), (3.141), and (3.142) with sufficiently large $\varepsilon > 0$ and $\delta > 0$ to conclude

$$
\|h(t)\|_\infty \lesssim e^{-\varepsilon_0 t}\|h(0)\|_\infty + \int_0^t \|f(s)\|_2 ds.
$$

(3.143)

\[\square\]

4. $L^2$-Coercivity via Contradiction Method

We start with a lemma which was proved in Lemma 5.1 in [14].

Lemma 20. Let $g$ be a (distributional) solution to

$$
\partial_t g + v \cdot \nabla_x g = G.
$$

Then, for a sufficiently small $\varepsilon > 0$,

$$
\int_0^1 \int_{U \cap B(\gamma, \varepsilon)} |1_{\text{dist}(x, \partial U) < \varepsilon}1_{|x| > \varepsilon} g(t)|^2 dx dt \leq \int_0^1 \int_{U \cap B(\gamma, \varepsilon)} |1_{\text{dist}(x, \partial U) > \varepsilon^2/2} g(t)|^2 dx dt + \int_0^1 \|gG\|_2^2 dt.
$$

Proposition 3. Assume that $f$ solves linear Boltzmann equation

$$
\partial_t f + v \cdot \nabla f + Lf = 0,
$$

(4.1)

and satisfies the specular reflection BC and (1.3) for $F = \mu + \sqrt{\mu} f$. Furthermore, for an axis-symmetric domain, we assume (1.6). Then there exists $C > 0$ such that, for all $N \in \mathbb{N}$,

$$
\int_0^N \|P f(t)\|_2^2 dt \leq C \int_0^N \|(I - P) f(t)\|_2^2 dt.
$$

(4.2)

Proof. We will use contradiction method which is used in [9] and also in [14] with some modification. Instead of full detail, we describe scheme of proof following [14].

Step 1. First, (4.1) is translation invariant in time, so it suffices to prove coercivity for finite time interval $t \in [0, 1]$ and so we claim (4.2) for $N = 0$. Now assume that Proposition 3 is wrong. Then, for any $m \geq 1$, there exists a solution $f^m$ to (4.1) with specular reflection BC, which solves

$$
\partial_t f^m + v \cdot \nabla f^m + Lf^m = 0, \text{ for } t \in [0, 1]
$$

(4.3)
and satisfies
\[ \int_0^1 \| \mathbf{P} f^m(t) \|_2^2 dt \geq m \int_0^1 \| (\mathbf{I} - \mathbf{P}) f^m(t) \|_2^2 dt. \]

Defining normalized form of \( f^m \) by
\[ Z^m(t, x, v) := \frac{f^m(t, x, v)}{\sqrt{\int_0^1 \| \mathbf{P} f^m(t) \|_2^2 dt}}. \tag{4.4} \]

Then \( Z^m \) also solves \([4.3]\) with specular BC and
\[ \frac{1}{m} \geq \int_0^1 \| (\mathbf{I} - \mathbf{P}) Z^m(t) \|_v^2 dt. \tag{4.5} \]

**Step 3.** Therefore, the sequence \( \{ Z^m \}_{m \geq 1} \) is uniformly bounded in \( \sup_{0 \leq t \leq 1} \| g(t) \|_v^2 \). By the weak compactness of \( L^2 \)-space, there exists weak limit \( Z \) such that
\[ Z^m \rightharpoonup Z \text{ in } L^\infty([0,1]; L^2(\mathbb{R}^3)) \cap L^2([0,1]; L^2_v(\mathbb{R}^3)). \]

Therefore, in the sense of distributions, \( Z \) solves \([4.1]\) with the specular BC. See the proof of Proposition of 1.4 in [14] to see that \( Z \) also satisfies the specular BC. Moreover, it is easy to check that weak limit \( Z \) satisfies conservation laws:
\[ \iint_{\mathbb{R}^3} Z(t) \sqrt{\mu} = 0, \quad \iint_{\mathbb{R}^3} Z(t) \frac{|v|^2}{2} \sqrt{\mu} = 0, \quad 0 \leq t \leq 1. \tag{4.7} \]

In the case of axis-symmetry \([1.5]\),
\[ \int_{\mathbb{R}^3} \{(x - x^0) \times \omega \} \cdot v Z(t) \sqrt{\mu} = 0. \tag{4.8} \]

On the other hand, since
\[ \mathbf{P} Z^m \rightarrow \mathbf{P} Z \quad \text{and} \quad (\mathbf{I} - \mathbf{P}) Z^m \rightarrow 0 \quad \text{in} \quad \int_0^1 \| \cdot \|_v^2 dt, \]
we know that weak limit \( Z \) has only hydrodynamic part, i.e.
\[ Z(t, x, v) = \{ a(t, x) + v \cdot b(x, v) + \frac{|v|^2 - 3}{2} c(t, x) \} \sqrt{\mu}, \tag{4.9} \]
and
\[ \int_0^1 \| Z \|_v^2 dt \leq \liminf_{m \to \infty} \int_0^1 \| Z^m \|_v^2 dt \leq 1 + \frac{1}{m} \to 1. \]

**Step 4. Compactness.** For interior compactness, let \( \chi_{\varepsilon} : \partial \Omega \rightarrow [0,1] \) be a smooth function such that \( \chi_{\varepsilon}(x) = 1 \) if \( \text{dist}(x, \partial U) > 2\varepsilon^4 \) and \( \chi_{\varepsilon}(x) = 0 \) if \( \text{dist}(x, \partial U) < \varepsilon^4 \). From \([4.1]\) with \( Z^m \),
\[ [\partial_t + v \cdot \nabla_x] (\chi_{\varepsilon} Z^m) = v \cdot \nabla_x \chi_{\varepsilon} Z^m - L(\chi_{\varepsilon} Z^m). \]

From the standard Average lemma, \( \chi_{\varepsilon} Z^m \) is compact i.e.
\[ \chi_{\varepsilon} Z^m \rightarrow \chi_{\varepsilon} Z \text{ strongly in } L^2([0,1]; L^2_v(\mathbb{R}^3)). \tag{4.10} \]

For near boundary compactness for non-grazing part, we claim that
\[ \int_{\varepsilon}^{1-\varepsilon} \| (Z^m(t, x, v) - Z(t, x, v)) \mathbf{1}_{\text{dist}(x, \partial U) < \varepsilon^4} \mathbf{1}_{|n_x(x)| < \varepsilon} \|_v^2 \leq \int_0^1 \| (Z^m(t, x, v) - Z(t, x, v)) \mathbf{1}_{\text{dist}(x, \partial U) > \varepsilon^4} \|_v^2 + O\left(\frac{1}{\sqrt{m}}\right). \tag{4.11} \]
We are looking up the equation of $Z^m - Z$. From \[4.9\],
$$[\partial_t + v \cdot \nabla_x](Z^m - Z) + LZ^m = 0.$$  \hfill (4.12)

We apply Lemma \[20\] to \[4.12\] by equating $g$ and $G$ with $Z^m - Z$ and the RHS of \[4.12\] respectively. Then

$$
\int_0^1 \|1_{\text{dist}(x, \partial U) < \varepsilon} 1_{|n(x)| \cdot v> \varepsilon}(Z^m - Z)(t)\|_2^2 dt \\
\leq \int_0^1 \|1_{\text{dist}(x, \partial U) < \varepsilon^2/2}(Z^m - Z)(t)\|_2^2 dt + \int_0^1 \iint_{U \times \mathbb{R}^3} |Z^m - Z| \langle 1 \rangle (I - P) Z^m| dt \\
\leq \sqrt{m} \int_0^1 \|1_{\langle 1 \rangle (I - P) Z^m\|_2^2} + \frac{1}{\sqrt{m}} \int_0^1 \|Z^m\|_2^2 + \|Z\|_2^2.
$$

By \[4.6\] and \[4.5\], we conclude \[4.11\].

On the other hand, from \[4.5\], \[4.9\], and \[4.6\],

$$\int_0^1 \|1_{|n(x)| \cdot v \leq \varepsilon}(Z^m - Z)\|_2^2 dt \leq \int_0^1 \|1_{\langle 1 \rangle (I - P) Z^m\|_2^2} + O(\varepsilon) \int_0^1 \|PZ^m\|_2^2 + \|PZ\|_2^2
$$

\hfill (4.13)

**Step 6. Strong convergence.** For given $\varepsilon > 0$, we can choose $m \gg \varepsilon$ such that

$$
\int_0^1 \iint_{U \times \mathbb{R}^3} |Z^m - Z|^2 dt \\
\leq \int_0^1 \iint_{U \times \mathbb{R}^3} + \int_0^1 \iint_{U \times \mathbb{R}^3} + \int_0^1 \iint_{U \times \mathbb{R}^3}
$$

$$
+ \int_0^1 \iint_{U \times \mathbb{R}^3} \cap \{ |n(x)| \cdot v \leq \varepsilon \} + \int_0^1 \iint_{U \times \mathbb{R}^3} \cap \{ |n(x)| \cdot v \geq \varepsilon \} \\
< C \varepsilon,
$$

where we have used \[4.6\], \[4.10\], \[4.11\], and \[4.13\]. Therefore, we conclude that $Z^m \to Z$ strongly in $L^2([0, 1] \times U \times \mathbb{R}^3)$ and hence

$$\int_0^1 \|Z\|_2^2 = 1. \hfill (4.14)$$

**Step 8. We claim $Z = 0$.** Plugging \[4.9\] into linearized Boltzmann equation, we get

$$
\partial_t c = 0, \\
\partial_b c + \partial_b b_i = 0, \\
\partial_b b_j + \partial_b b_i = 0, \quad i \neq j, \\
\partial_b b_i + \partial_b a = 0, \\
\partial_b a = 0.
$$

\hfill (4.15)

Using the first equations and direct computation of Lemma 12 in [2],

$$b(t, x) = -\partial_t c(t) x + \varpi(t) \times x + m(t).$$

From the second equation in \[4.15\] and the specular BC,

$$c(t, x) = c_0, \quad b = \varpi(t) \times x + m(t).$$

We split into two cases $\varpi = 0$ and $\varpi \neq 0$.

**Case of $\varpi = 0$.** $b(t) = m(t)$ and from specular BC, we deduce that

$$b(t) \equiv m(t) \equiv 0.$$

And, from the fourth and the last equations of \[4.15\], we can derive

$$a(t, x) = a_0.$$

Since $a(t, x)$ and $c(t, x)$ are constant, from \[4.7\], we derive $a_0 = c_0 = 0$, and hence $Z = 0$.

**Case of $\varpi \neq 0$.** From the specular BC,

$$b(t, x) \cdot n(x) = (\varpi(t) \times x + m(t)) \cdot n(x) = 0.$$

Since $m(t)$ is fixed vector for given $t$, we decompose $m(t)$ into the parallel and orthogonal components to $\varpi(t)$ as

$$m(t) = a(t) \varpi(t) - \varpi(t) \times x_0(t).$$
Then
\[
  b(t, x) \cdot n(x) = (\varpi(t) \times x + m(t)) \cdot n(x)
  = (\varpi(t) \times (x - x_0(t))) \cdot n(x) + \alpha(t) \varpi(t) \cdot n(x) = 0, \quad \forall x \in \partial U.
\]  
(4.16)
Choose \( t \) with \( \varpi(t) \neq 0 \). We can pick \( x' \in \partial U \) such that \( \varpi(t) \| n(x') \). Then the first term of the RHS in (4.16) is zero. Hence we deduce
\[
  \alpha(t) = 0 \text{ and } b(t, x) = \varpi(t) \times (x - x_0(t)).
\]
(4.17)
This yields
\[
  (\varpi(t) \times (x - x_0(t))) \cdot n(x) = 0, \quad \forall x \in \partial U.
\]
(4.18)
The equality (4.18) implies that \( U \) is axis-symmetric with the origin \( x_0(t) \) and the axis \( \varpi(t) \). From (4.8) and (4.17),
\[
  0 = \int_U |\varpi \times (x - x_0(t))| \cdot v^2 dxdv.
\]
Therefore, we conclude that \( b(t, x) \equiv 0 \). Then using conservation laws (mass and energy) again, we deduce \( Z = 0 \).

**Step 9.** Finally we deduce a contradiction from (4.14) and \( Z = 0 \) of Step8. This finishes the proof.

\[ \square \]

5. **Linear and Nonlinear decay**

5.1. **Linear \( L^2 \) decay.** We use coercivity estimate Proposition 3 to derive exponential linear \( L^2 \) decay of linear boltzmann equation (1.14) with the specular boundary condition.

**Corollary 1.** Assume \( f \) solves linear boltzmann equation with the specular BC so that \( f \) satisfies Proposition 3. Then there exists \( \lambda > 0 \) such that a solution of (1.14) satisfies
\[
  \sup_{0 \leq t} e^{\lambda t} \| f(t) \|_2^2 \leq \| f_0 \|_2^2.
\]
(5.1)

**Proof.** Assume that \( 0 \leq t \leq 1 \). From the energy estimate of (4.1) in a time interval \([0, N]\),
\[
  \| f(N) \|_2^2 + \int_0^N \iint_{U \times R^3} fL f \leq \| f(0) \|_2^2.
\]
From (4.1), for any \( \lambda > 0 \)
\[
  [\partial_t + v \cdot \nabla_x] (e^{\lambda t} f) + L(e^{\lambda t} f) = \lambda e^{\lambda t} f.
\]
By the energy estimate,
\[
  \| e^{\lambda t} f(N) \|_2^2 + \int_0^N \iint_{U \times R^3} e^{2\lambda s} fL f - \lambda \int_0^N \iint_{U \times R^3} |e^{\lambda s} f(s)|^2 \leq \| f(0) \|_2^2.
\]
(5.2)
Firstly we consider (1) in (5.2). From semi-positiveness of operator \( L \), the term (1) in (5.2) is bounded below by
\[
  (1) \geq \delta L \int_0^N \iint_{U \times R^3} \langle v \rangle |e^{\lambda s}(I - P)f|^2 \geq \delta L \int_0^N \| e^{\lambda s}(I - P)f \|_{L^2}^2.
\]
By time translation, we apply (4.2) to obtain
\[
  (1) \geq \frac{\delta L}{2} \int_0^N \| e^{\lambda s}(I - P)f \|_{L^2}^2 + \frac{\delta L}{2C} \int_0^N \| e^{\lambda s}P f \|_{L^2}^2 \geq \frac{\delta L}{2C} \int_0^N \| e^{\lambda s} f \|_{L^2}^2.
\]
Therefore, we derive
\[
  e^{2\lambda N} \| f(N) \|_2^2 + \left( \frac{\delta L}{2C} - \lambda \right) \int_0^N \| e^{\lambda s} f \|_2^2 \leq \| f(0) \|_2^2.
\]
(5.3)
On the other hand, from the energy estimate of (4.1) in a time interval \([N, t]\), using semi-positiveness of \( L \), we have
\[
  \| f(t) \|_2^2 \leq \| f(N) \|_2^2.
\]
(5.4)
Finally choosing \( \lambda \ll 1 \), from (5.3) and (5.4), we conclude that
\[
  e^{\lambda N} \| f(t) \|_2^2 = e^{\lambda(t-N)} e^{\lambda N} \| f(N) \|_2^2 \leq 2 \| f(0) \|_2^2,
\]
and obtain (5.1).
5.2. **Nonlinear $L^\infty$ decay.** We use $L^\infty - L^2$ bootstrap form \[3.143\], Duhamel’s principle, and Corollary 1 to derive nonlinear $L^\infty$ decay.

**Proof of Theorem 1** From \[3.143\],

$$
\sup_{s \in [T,t]} \|h(s)\|_{\infty} \lesssim e^{-\nu_0(t-T)}\|h(T)\|_{\infty} + \int_T^t \|f(s)\|_2 ds.
$$

We assume that $m \leq t < m + 1$ and define $\lambda^* := \min\{\nu_0, \lambda\}$, where $\lambda$ is some constant from Corollary 1. We use \[3.143\] repeatedly for each time step, $[k, k + 1]$, $k \in \mathbb{N}$ and Corollary 1 to perform $L^2 - L^\infty$ bootstrap,

$$
\|h(t)\|_{\infty} \lesssim e^{-m\nu_0}\|h(0)\|_{\infty} + \sum_{k=0}^{m-1} e^{-k\nu_0} \int_{m-1-k}^m \|f(s)\| ds
$$

$$
\lesssim e^{-m\nu_0}\|h(0)\|_{\infty} + \sum_{k=0}^{m-1} e^{-k\nu_0} \int_{m-1-k}^m e^{-\lambda(m-1-k)}\|f(0)\| ds \lesssim e^{-\lambda^*t}\|h(0)\|_{\infty}.
$$

For nonlinear problem from Duhamel principle,

$$
h := U(t)h_0 + \int_0^t U(t-s)w\Gamma\left(\frac{h}{w}, \frac{h}{w}\right)(s)ds,
$$

$$
\|h(t)\|_{\infty} \lesssim e^{-\lambda^*t}\|h(0)\|_{\infty} + \left\| \int_0^t U(t-s)w\Gamma\left(\frac{h}{w}, \frac{h}{w}\right)(s)ds \right\|_{\infty},
$$

where $U(t)$ is linear solver for linearized Boltzmann equation. Inspired by [9], we use Duhamel’s principle again:

$$
U(t-s) = G(t-s) + \int_s^t G(t-s_1)K_wU(s_1 - s)ds_1,
$$

where $G(t)$ is linear solver for the system

$$
\partial_t h + v \cdot \nabla_x h + \nu h = 0, \quad \|G(t)h_0\| \leq e^{-\nu_0t}\|h_0\|.
$$

For the last term in (5.5),

$$
\left\| \int_0^t U(t-s)w\Gamma\left(\frac{h}{w}, \frac{h}{w}\right)(s)ds \right\|_{\infty}
$$

$$
\leq \left\| \int_0^t G(t-s)w\Gamma\left(\frac{h}{w}, \frac{h}{w}\right)(s)ds \right\|_{\infty} + \left\| \int_0^t \int_s^t G(t-s_1)K_wU(s_1 - s)w\Gamma\left(\frac{h}{w}, \frac{h}{w}\right)(s)ds_1 ds \right\|_{\infty}
$$

$$
\leq C e^{-\lambda^*t}\left(\sup_{0 \leq s \leq \infty} e^{\lambda s}\|h(s)\|_{\infty}\right)^2.
$$

Therefore, for sufficiently small $\|h_0\|_{\infty} \ll 1$, we have uniform bound

$$
\sup_{0 \leq t \leq \infty} e^{\lambda^*t}\|h(t)\|_{\infty} \ll 1,
$$

hence we get global decay and uniqueness. Also note that positivity of $F$ is standard by linear solvability and solution sequence $F^\ell$:

$$
\partial_t F^{\ell+1} + v \cdot \nabla F^{\ell+1} = Q_+(F^\ell, F^\ell) - \nu(F^\ell)F^{\ell+1}, \quad F|_{t=0} = F_0,
$$

$$
F^{\ell+1}(t, x, v) = F^{\ell+1}(t, x, R_x v) \quad \text{on } \partial U.
$$

From $F_0 \geq 0$ and $F^\ell \geq 0$, we have $F^{\ell+1} \geq 0$.

\[\square\]

6. **Appendix: Example of sticky grazing point**

Let us consider backward in times trajectories which start from $(1, 1)$ with velocity $v$ between $(1, 1)$ and $(1, 1 + \delta)$, $0 < \delta \ll 1$. Then all the trajectories are part of set of rays

$$\{(x, y) : y = (1 + \delta)(x - 1) + 1, \ 0 < \delta \leq \varepsilon \ll 1\}.
$$

We consider the trajectories bounce on the curve $f(x) = \frac{1}{2}x^2$. When $\delta = 0$, trajectory bounce on $(0, 0)$ with collision angle $\frac{\pi}{4}$. When $0 < \delta \ll 1$, bouncing point on $f(x) = \frac{1}{2}x^2$ is

$$
(\delta_+ \frac{1}{2} \delta^2 x^*), \quad \text{where } \delta_* = (1 + \delta) - \sqrt{(1 + \delta)^2 - 2\delta}.
$$

47
Using the specular BC, bounced trajectory with \( v^1 \) direction is part of set of rays

\[
\{(x, y) : y = L(\delta)(x - \delta_x) + \frac{1}{2} \delta_y^2\}, \quad L(\delta) = \frac{(1 + \delta)(1 + \delta_x^2) - 2\sqrt{1 + \delta^2}}{1 + \delta_x^2 + 2\delta_y \sqrt{1 + \delta^2}}.
\]

We parametrize convex grazing boundary with parameter \( \delta \),

\[
(X(\delta), Y(\delta)), \quad X(0) = -Y(0) < 0.
\]

Considering tangential line on \( X(\delta), Y(\delta) \), it is easy to derive two conditions from concave grazing.

\[
\frac{Y'(\delta)}{X'(\delta)} = L(\delta),
\]

\[
-L(\delta) \delta_x + \frac{1}{2} \delta_y^2 = \frac{Y'(\delta)}{X'(\delta)} X(\delta) + Y(\delta).
\]

(6.1)

We differentiate second equation and combine with first equation to get

\[
\frac{d}{d\delta} \left( -L(\delta) \delta_x + \frac{1}{2} \delta_y^2 \right) = -L'(\delta) X(\delta) - L(\delta) X'(\delta) + Y'(\delta)
\]

\[
= -L'(\delta) X(\delta).
\]

(6.2)

It is easy to check \( L' > 0 \) locally \( 0 < \delta \ll 1 \). (6.2) gives \( X(\delta) \) and this is analytic from analyticity of \( L(\delta) \) and \( \delta_y^2 \), \( X(\delta) \) is analytic function of \( \delta \) for local \( 0 < \delta \ll 1 \). Using the first equation of (6.1), we obtain ODE for \( Y(\delta) \) with \( Y(0) = -X(0) \). Since \( X(\delta) \) is analytic, \( Y(\delta) \) is also analytic. Moreover, we can check concavity of \( (X(\delta), Y(\delta)) \) by

\[
\frac{d}{d\delta} \left( \frac{Y'(\delta)}{X'(\delta)} \right) = L'(\delta) > 0.
\]

Acknowledgements. The authors thank Yan Guo for stimulating discussions. Their research is supported in part by NSF-DMS 1501031, WARF, and the Herchel Smith fund. They thank KAIST Center for Mathematical Challenges and ICERM for the kind hospitality.

References

[1] Cercignani, C., Illner, R., and Pulvirenti, M. The mathematical theory of dilute gases. Applied Mathematical Sciences, 106. Springer-Verlag, New York, 1994.

[2] Chernov, N. and Markarian, R. Chaotic Billiards. Mathematical Surveys and Monographs, 127. American Mathematical Society, Providence, RI, 2006.

[3] Devillettes, L. and Villani, C. On the trend to global equilibrium for spatially inhomogeneous kinetic systems: the Boltzmann equation. Invent. Math. 159 (2005), no. 2, 245–316.

[4] Do Carmo, M. Differential Geometry of Curves and Surfaces. Prentice Hall, 1976.

[5] Esposito, R., Guo, Y., Kim, C., and Marra, R. Non-Isothermal Boundary in the Boltzmann Theory and Fourier Law. Comm. Math. Phys. 323 (2003), no. 1, 177–239.

[6] Esposito, R., Guo, Y., Kim, C., and Marra, R. Stationary solutions to the Boltzmann equation in the Hydrodynamic limit, submitted.

[7] Guo, Y., Kim, C., Tonon, D., and Trescases, A. Regularity of the Boltzmann Equation in Convex Domains. Invent. Math. 207 (2017), no. 1, 115–290.

[8] Guo, Y., Kim, C., Tonon, D., and Trescases, A. BV-Regularity of the Boltzmann Equation in Non-convex Domains. Arch. Rational Mech. Anal. 220 (2016), no. 3, 1045–1093.

[9] Guo, Y. Decay and Continuity of Boltzmann Equation in Bounded Domains. Arch. Rational Mech. Anal. 197 (2010), no. 3, 713–809.

[10] Halpern, B.: Strange billiard tables. Tran. Amer. Math. Soc. 232 (1977), 297–305.

[11] Kim, C. Formation and propagation of discontinuity for Boltzmann equation in non-convex domains. Comm. Math. Phys. 308 (2011), no. 3, 641–701.

[12] Kim, C. Boltzmann equation with a large external field. Comm. PDE. 39 (2014), no. 8, 1393–1423.

[13] Kim, C. and Yun, S. The boltzmann equation near a rotational local maxwellian. SIAM J. Math. Anal. 44 (2012), no. 4, 2560–2598.

[14] Kim, C. and Lee, D. The Boltzmann equation with specular boundary condition in convex domains. Comm. Pure Appl. Math. accepted.

[15] Shizuta, Y. and Asano, K. Global solutions of the Boltzmann equation in a bounded convex domain. Proc. Japan Acad. 53A (1977), 3–5.