Is Hall Conductance in Hall Bar Geometry a Topological Invariant?

K. Ishikawa, N. Maeda, and K. Tadaki

Department of Physics, Hokkaido University, Sapporo 060, Japan

ABSTRACT

A deep connection between the Hall conductance in realistic situation and a topological invariant is pointed out based on von-Neumann lattice representation in which Landau level electrons have minimum spatial extensions. We show that the Hall conductance has no finite size correction in quantum Hall regime, but a coefficient of induced Chern-Simons term in QED$_3$ has a small finite size correction, although both of them are similar topological invariant.
Since the discovery of the quantum Hall effect,\(^1\) topological origin of quantization of the Hall conductance has been discussed. It was initiated by Laughlin\(^2\) who gave a simple argument for showing a quantization in a torus system. He used the fact that the torus system is invariant under a finite gauge transformation, which can be considered as adding a unit of flux to the inside of the torus. The response of the electronic system gives an average Hall conductance over a unit of flux because the flux is not infinitesimal. Thus the quantization of the average Hall conductance was shown. The idea was extended by Halperin\(^3\) and the role of edge was identified.

An idea of using topological invariant was introduced later by TKNdN.\(^4\) They have identified the Hall conductance in tight binding model as first Chern class of the fibre bundle. An argument was extended further later,\(^5\) in which average over boundary condition is taken.

For the topological invariant to be defined, the corresponding space must be compact. In the above cases, invariance of the system under gauge transformation, or the model itself ensure the compactness. They are not applied in real cases, however, because the electron system is described neither on the torus nor by the tight binding model, and average over flux or boundary condition is not taken. Instead, the system is described by continuum space model in a finite area with boundary. The conductance is defined by a derivative also.

A new idea of topological invariant in momentum space was introduced by one of the present authors\(^6,7\) and others later, and low energy theorem was given. Recently, the quantization is also shown by using Landauer formula\(^8\) for one dimensional edge current under the assumption that only the edge states are current carrying states.

The purpose of the present work is to unify all pictures and to find out if the Hall conductance is a topological invariant or it has a finite size correction in realistic systems,
where there are zero-energy one-dimensional edge states. Our conclusion is that the Hall conductance in the realistic situation is equivalent to the topological winding number in the momentum space and the quantization is thus exact in the quantum Hall regime. The role of edge states is also clarified. A similar topological invariant of three-dimensional quantum electrodynamics (QED3), a coefficient of induced Chern-Simons term, has a small finite size corrections, of order $e^{-mL}$, where $L$ is the system size and $m$ is the typical mass scale.

The roles of the magnetic field are essential in making not only the space to be compact but also the finite size effect to disappear.

We describe an infinite system first, a finite system without boundary second, and finite system with boundary at the end.

(1) Infinite plane without boundary — Before discussing the quantum Hall system, let us discuss QED3 which resembles the quantum Hall system in many respects. 9–11 The slope of the current correlation function of momentum, $\pi_{\mu\nu}(q)$, at the origin is given by

\[ \frac{1}{3!} \epsilon^{\mu\nu\rho} \frac{\partial}{\partial q_\rho} \pi_{\mu\nu}(q) \bigg|_{q=0} = \frac{1}{3!} \int \frac{d^3p}{(2\pi)^3} \epsilon^{\mu\nu\rho} \text{Tr} \left[ \frac{\partial S^{-1}(p)}{\partial p_\rho} S(p) \frac{\partial S^{-1}(p)}{\partial p_\mu} S(p) \frac{\partial S^{-1}(p)}{\partial p_\nu} S(p) \right], \]

\[ S^{-1}(p) = p_\mu \gamma^\mu - m, \quad \gamma^0 = \sigma^3, \quad \gamma^1 = i\sigma^1, \quad \gamma^2 = i\sigma^2. \]

Obviously the slope has a peculiar meaning as a winding number of the mapping defined by the propagator $S(p)$, 12 if the momentum is defined on the compact space. Arguments concerning its topological nature 13,14 and the non-renormalization theorem 15,16 have been given in the literature. It should be noted that the non-trivial topology can be defined even in infinite planar system and a torus configuration is not necessary in QED3. For non-renormalization theorem to be satisfied, current conservation and Ward-Takahashi identity are important.
It is convenient to use a representation in which base functions are localized spatially in each Landau level in order to derive Ward-Takahashi identity and related exact low energy theorem in quantum Hall system. Coherent state von-Neumann lattice representation\textsuperscript{7} in centre variables is such representation that has minimum spatial extensions allowed from commutation relations and its usefulness has been shown in previous works in showing the low energy theorem and others. We use this representation in the present work, too, and give low energy theorems of finite systems.

In our coherent state von-Neumann lattice representation\textsuperscript{7}, the kinetic term and the conserved electromagnetic current are written as,

\[
\int d^2x \psi^\dagger(x) \left( \frac{\tilde{\mathbf{p}} + e \mathbf{A}}{2m} \right)^2 \psi(x) = \sum_{l,m,n} E_l b_l(m,n) a_l(m,n),
\]

\[
\partial_x A_y - \partial_y A_x = B, \quad E_l = \frac{eB}{m}(l + \frac{1}{2}),
\]

\[
\hat{j}_\mu(x) = \frac{eB}{m} \sum_{l,m,n} b_{l1}(m_1,n_1) a_{l2}(m_2,n_2) \int \frac{d^2k}{(2\pi)^2} \langle \tilde{R}_{m_1,n_1} | e^{i\mathbf{k} \cdot \mathbf{X}} | R_{m_2,n_2} \rangle (f_{l1} \xi e^{i(k_x \xi + k_y \eta)} f_{l2}) e^{-i\mathbf{k} \cdot \mathbf{x}},
\]

(3)

where $\xi_\mu = (1,-\eta,\xi)$, $eB\xi = p_y + eA_y$, $eB\eta = -(p_x + eA_x)$, $X = x - \xi$, $Y = y - \eta$, and electron field is expressed by quantized operators $a_l(m,n)$ and $b_l(m,n)$, and c-number functions of relative coordinates and centre coordinates:

\[
\psi = \sum a_l(m,n) f_l(\xi,\eta) |R_{m,n}\rangle,
\]

\[
\psi^\dagger = \sum b_l(m,n) f_l(\xi,\eta) \langle \tilde{R}_{m,n} |,
\]

(4)
\[
\frac{e^2 B^2}{2m} (\xi^2 + \eta^2) f_\ell(\xi, \eta) = E_\ell f_\ell(\xi, \eta),
\]

\[
(X + iY) |R_{m,n}\rangle = a(m + in) |R_{m,n}\rangle,
\]

\[
\langle \tilde{R}_{m_1,n_1} | R_{m_2,n_2}\rangle = \delta_{m_1,m_2} \delta_{n_1,n_2} - \frac{1}{\sum_{m,n}},
\]

where \( R_{m,n} = (ma, na) \) and \( m, n \) are integers. The coherent states in centre coordinates become minimum set of complete set \(^{17}\) if lattice spacing \( a \) is equal to \( \sqrt{2\pi/eB} \), which we use hereafter. The dual basis \( \langle \tilde{R}_{m,n}\rangle \) is obtained by a linear combination of \( \langle R_{m,n}\rangle \) with suitable coefficients, as was given in Refs.\(^7\) and \(^{18}\). One-particle states, such as localized states around short range impurity, edge states, and extended states under periodic potentials, were studied in this representation before.\(^{18}\)

The electron field operators, \( a_\ell(m,n) \) and \( b_\ell(m,n) \), have lattice coordinates and Landau level indices. Hence the electron propagator and the vertex part become matrices which have Landau level indices and momentum defined on the torus. The momentum space is thus compact in the spatial directions.

The matrix relation between the propagator and the vertex part is derived from the current conservation and commutation relations and becomes to

\[
d^\mu \Gamma_\mu(p_1, p_2) = S^{-1}(p_1) R(p_1, p_2) - L(p_1, p_2) S^{-1}(p_2),
\]

\[
R_{l_1,l_2}(p_1, p_2) = \delta_{l_1,l_2} + iq_j [d_j(p_2) \delta_{l_1,l_2} + \tilde{d}_j(l_1,l_2)],
\]

\[
L_{l_1,l_2}(p_1, p_2) = \delta_{l_1,l_2} + iq_j [d'_j(p_2) \delta_{l_1,l_2} + \tilde{d}'_j(l_1,l_2)],
\]

where Landau level dependent terms in the above matrices satisfy,

\[
[\tilde{d}_x, \tilde{d}_y] = [\tilde{d}'_x, \tilde{d}'_y] = -i/eB.
\]
The vertex part of the same momenta thus satisfies a modified Ward-Takahashi identity,
\[ \Gamma_\mu(p, p) = \frac{\partial}{\partial p_\mu} S^{-1}(p) + S^{-1}(p) D_\mu - D'_\mu S^{-1}(p), \]
\[ D_0 = D'_0 = 0, \quad D_i = i(d_i + \bar{d}_i), \quad D'_i = i(d'_i + \bar{d}'_i). \]  

By a transformation of the basis with suitable momentum dependent unitary matrices,
\[ \tilde{S}(p) = V(\vec{p}) S(p) U(\vec{p}), \]
\[ \tilde{\Gamma}_\mu(p_1, p_2) = U^{-1}(\vec{p}_1) \Gamma_\mu(p_1, p_2) V^{-1}(\vec{p}_2), \]
\[ U(\vec{p}) \frac{\partial}{\partial p_i} U^{-1}(\vec{p}) = -D'_i, \quad (\frac{\partial}{\partial p_i} V^{-1}(\vec{p})) V(\vec{p}) = D_i, \]
the second term and third term in right hand side of Eq.(8) are absorbed and the relation becomes to the standard form,
\[ \tilde{\Gamma}_\mu(p, p) = \frac{\partial}{\partial p_\mu} \tilde{S}^{-1}(p). \]  

The Hall conductance is the slope of the current correlation function, \( \pi_{\mu\nu}(q) \), at the origin and is expressed, by using Eq.(11), as a topologically invariant form,
\[ \sigma_{xy} = \frac{e^2}{3!} \epsilon^{\mu\nu\rho} \frac{\partial}{\partial q_\rho} \pi_{\mu\nu}(q) \bigg|_{q=0} = \frac{e^2}{2\pi} N_w, \]
\[ N_w = \frac{1}{24\pi^2} \int d^3 p \epsilon^{\mu\nu\rho} \text{Tr} \left[ \frac{\partial \tilde{S}^{-1}(p)}{\partial p_\rho} \tilde{S}(p) \frac{\partial \tilde{S}^{-1}(p)}{\partial p_\mu} \tilde{S}(p) \frac{\partial \tilde{S}^{-1}(p)}{\partial p_\nu} \tilde{S}(p) \right]. \]  

We have the momentum independent propagator from Eq.(2),
\[ S^{(0)}(p)_{l_1, l_2} = (p_0 - E_{l_1})^{-1} \delta_{l_1, l_2} \]
in the free system without disorder, boundary, and interactions. The \( \sigma_{xy} \) thus calculated depends on \( E_F \), as is shown in Fig.1. The integrand is independent from the spatial component of the momenta from Eqs.(7) and (9). This property becomes important when we discuss finite system.
In the presence of short range disorders and interactions, the above formula is valid, as well, because the current conservation and the commutation relations are kept intact. Moreover the value of $\sigma_{xy}$ is not modified by these effects in the localized state region and in the energy gap region. More details are given in Ref.7.

The Hall conductance in the infinite planar system is expressed with a topologically invariant formula and is quantized in the integer filling region and neighboring localized state region and is not modified by interactions. These low energy theorems concerning the Hall conductance have been given in the present representation. The matrix form of the Ward-Takahashi identity between the vertex part and the propagator plays the essential roles.

(2) Finite plane without boundary — In the formula of the Hall conductance, Eq.(12), integration regions of the momentum is from $-\pi/a$ to $\pi/a$. Now suppose the configuration space to be a torus of length $L$. The momentum then becomes discrete and the integration is replaced with a summation over discrete momentum. We show in the following that the topological invariant, Eq.(12), is unchanged and has the exactly same value but another topological invariant, Eq.(1), changes the value. We assume $L/2a$ being integer and study the $L$ dependence. The momentum then becomes $2\pi n/L$ with an integer $n$ in the range from $-L/2a$ to $L/2a$.

The Hall conductance in a torus geometry in $y$ direction is given by

$$\sigma_{xy} = \frac{e^2}{2\pi} N'_w,$$

where $N'_w$ is obtained by replacing $p_y$ integration of $N_w$, Eq.(12), with a summation over discrete $p_y$. Since the integrand is independent from $p_y$, the above $N'_w$ is the same as the
previous $N_w$, Eq.(12), and is an exact integer under integer Hall effect condition. The absence of finite size correction in the Hall conductance is partly due to the special form of the modified propagator, Eq.(9), and commutation relation, Eq.(7), which have an origin in the magnetic field.

The finite size correction exists in the topological invariant of QED$_3$, Eq.(1). It changes the value under the replacement of momentum integration to summation over discrete momentum and becomes to \( \coth(mL/2) \). It has a finite size correction of order \( e^{-mL} \).

(3) **Finite plane with boundary (edge states)** — We discuss the quantum Hall effect in a realistic situation, in which electrons are confined in a finite area by a potential barrier. We concentrate to find out the effect of the potential barrier here. The single-particle Hamiltonian is composed of three diagonal terms,

\[
H_0 = \int d^2x \bar{\psi}(x) \left[ \frac{(\vec{p} + e\vec{A})^2}{2m} + V(x) \right] \psi(x)
\]

\[
= \sum_{\text{inside}} E_l b_l^\dagger(X) a_l(X) + \sum_{\text{outside}} (E_l + V_0) b_l(X) a_l(X) + \sum_{\text{boundary}} (E_l + \Delta E^{(\alpha)}) b^{(\alpha)} a^{(\alpha)},
\]

\[(15)\]

\[
0 \leq \Delta E^{(\alpha)} \leq V_0, \quad V(x) = \begin{cases} 0, & \text{inside,} \\ V_0, & \text{outside.} \end{cases}
\]

Eigenstates from the first and second term have the energy of Landau level, or the combined energy of Landau level and the potential energy, respectively. The third term gives the continuum energy band in an energy range between $E_l$ and $E_l + V_0$ and eigenstates are extended along the boundary. They correspond to the edge states, which may modify the previous results.

It is convenient to write the above Hamiltonian, in order to find the effect of edge states,
\[ H_0 = \sum_{x<x_0+\Delta} E_l b_l(X) a_l(X) + \sum_{x \geq x_0+\Delta} (E_l + V_0) b_l(X) a_l(X) + \sum \Delta E_{l',l}(X, X') b_l(X) a_{l'}(X'), \]

(16)

where the spatial region in the first includes inside and edge region, and \( \Delta E \) of the last term vanishes when the coordinates \( \vec{X} \) or \( \vec{X}' \) is located away from the boundary. Hence we are able to treat the last term perturbatively. The second term gives no effect except the virtual effect, if the potential height \( V_0 \) is much larger than the chemical potential. We treat the first term in the same manner as before and include the last term to it perturbatively.

Due to the last term in Eq.(16), the propagator, the vertex, and the current correlation function are not invariant under the translation. Even so, the translational invariant expression is valid. Green’s functions have invariant terms and non-invariant terms. The invariant terms contribute to the Hall conductance and non-invariant terms do not contribute to the conductance. The invariant term, \( \pi^{(0)}_{\mu\nu}(q) \), and non-invariant terms, \( \pi^{(1)}_{\mu\nu}(q_1, q_2; l_1) \), \( \pi^{(2)}_{\mu\nu}(q_1, q_2; l_1, l_2) \cdots \), of current correlation functions are given in Fig.2, where dashed lines show the perturbative edge terms which carry momentum \( l_i \). \( \pi^{(1)}_{\mu\nu}(q_1, q_2; l_1) \) satisfies

\[ q_1^\mu \pi^{(1)}_{\mu\nu}(q_1, q_2; l_1) = \pi^{(1)}_{\mu\nu}(q_1, q_2; l_1) q_2^\nu = 0. \]

If \( q_1 \) and \( q_2 \) are independent arbitrary momentum, then \( \pi^{(1)}_{\mu\nu}(q_1, q_1; 0) \) has no linear term in \( q_1 \) if this is obtained smoothly by a limit \( q_2 \to q_1 \), by Coleman-Hill argument.\(^{15}\) Other general term \( \pi^{(n)}_{\mu\nu}(q_1, q_2; l_1, \cdots l_n) \) satisfies the same relation and the same properties for general \( l_i \). The amplitude of present specific case, in which momentum along one direction is conserved, is obtained as a limit of general cases and should satisfies the same properties, as far as there is no singularity and the limit is taken smoothly. We find that each term is non-singular and the combined edge term is also non-singular. The most serious part is from edge state and is computed from one-dimensional chiral mode. We
find that it is non-singular. Consequently only the one loop diagram, \( \pi^{(0)}_{\mu\nu}(q) \), has linear term in \( q \) and contributes to \( \sigma_{xy} \). This diagram was studied in the previous part and was shown to give exactly quantized value in the quantum Hall regime. In other words, both of the edge states and the bulk states contribute to the quantized value of \( \sigma_{xy} \). The value is stable even though there are zero-energy one-dimensional edge states. In Landauer formula, the zero-energy states are important.\(^8\) Since the zero-energy states are one-dimensional in the quantum Hall regime, the \( \sigma_{xy} \) is quantized from this view point, too. Short range impurities and an electron interactions do not modify our conclusions at all, because all ingredients we used are still valid.

Ultra-violet divergence is generated from virtual effect and renormalization is made. The results still hold, due to the same reasons.

As a summary, we find that the small finite size correction exists in QED\(_3\), but the finite size correction does not exist in the normal quantum Hall region. It is the magnetic field that gives these phenomena to occur.

The present work is partially supported by the special Grant-in-Aid for promotion of Education and Science in Hokkaido University Provided by the Ministry of Education, Science and Culture, a Grant-in-Aid for general Scientific Research(03640256), and the Scientific Research on Priority Area(04231101), the Ministry of Education, Science and Culture, Japan.
REFERENCES

1. K. v. Klitzing, G. Dorda, and M. Pepper, Phys. Rev. Lett. 45, 494 (1980).

2. R. B. Laughlin, Phys. Rev. B 23, 5632 (1981).

3. B. I. Halperin, Phys. Rev. Lett. 25, 2185 (1982).

4. D. J. Thouless, M. Kohmoto, M. P. Nightingale and M. den Nijs, Phys. Rev. Lett. 49, 405 (1982); For a recent work, see Y. Hatsugai, Phys. Rev. Lett. 71, 3697 (1993).

5. A. Niu, D. J. Thouless, and Y. S. Wu, Phys. Rev. B 31, 3372 (1985).

6. K. Ishikawa and T. Matsuyama, Z. Phys. C 33, 41 (1986); Nucl. Phys. B 280, 523 (1987).

7. N. Imai, K. Ishikawa, T. Matsuyama, and I. Tanaka, Phys. Rev. B42, 10610 (1990).

8. R. Landauer, IBM J. Res. Dev. 1, 223 (1957); M. Büttiker, Phys. Rev. B 38, 9375 (1988).

9. K. Ishikawa, Phys. Rev. Lett. 53, 1615 (1984); Phys. Rev. D 31, 1432 (1985).

10. R. Jackiw, Phys. Rev. D 29, 2375 (1984).

11. A. Niemi and G. Semenoff, Phys. Rev. Lett. 51, 2088 (1983); A. Redlich, Phys. Rev. Lett. 52, 18 (1984).

12. K. Ishikawa, in Proceedings of the 3rd International Symposium of Fundation of Quantum Mechanics, the Physical Society of Japan, 1990, p.70.

13. T. Matsuyama, Prog. Theor. Phys. 77, 711 (1987).

14. H. So, Prog. Theor. Phys. 74, 585 (1985).
15. S. Coleman and B. Hill, Phys. Lett. **159B**, 184 (1985); Y. C. Kao and M. Suzuki, Phys. Rev. **D 29**, 2137 (1985).

16. O. Abe and K. Ishikawa, in *Rationale of Being: Festschrift in honor of Gyo Takeda*, edited by K. Ishikawa *et al.* (World Scientific, Singapore, 1986), p.137.

17. A. M. Perelomov, Theor. Mat. Fiz. **6**, 213 (1971); V. Bargmann *et al.*, Rep. Math. Phys. **2**, 221 (1971).

18. K. Ishikawa, N. Maeda, and K. Tadaki, Hokkaido University preprint, EPHOU-94-002 (unpublished).
FIGURE CAPTIONS

1) Hall conductance, from Eq.(12), has a step like dependence on the Fermi energy. The value is stable and has no correction from impurities and interactions in the gap region and localized region (shaded region).

2) Feynman diagram of current-current correlation function is shown. The perturbative edge terms carry momentum and are shown by dashed lines. The wavy lines stand for the currents. Since the edge terms carry momentum, two momenta, $q_1$ and $q_2$, are different generally.
