FOUR DIMENSIONAL BLACK HOLES
AND DUALITY IN SUPERSTRING THEORY

C.P. BURGESS[1], R.C. MYERS

Physics Department, McGill University
3600 University St., Montréal, Québec, Canada, H3A 2T8.

and

F. QUEVEDO[2]

Institut de Physique, Université de Neuchâtel
1 Rue A.L. Breguet, CH-2000, Neuchâtel, Switzerland.

ABSTRACT

Some recent results on the applications of duality (and related) transformations to general four-dimensional, spherically symmetric, asymptotically flat and time-independent string configurations are summarized. Two classes of results have been obtained. First, these transformations are used to generate the general such solution to the lowest-order field equations in the $\alpha'$ expansion. Second, the action and implications of duality (based on time-translation) on the general configuration is determined. It is found to interchange two pairs of the six parameters which label these configurations, namely: (1) the mass with the dilaton charge, and (2) the axion charge with the Taub-NUT parameter. For the special case of the Schwarzschild black hole this implies the relation $M \rightarrow -k/\alpha'M$, where $k$ is a known, positive, dimensionless number. It is argued that, in some circumstances, dual theories need not be equivalent in the simplest sense.

[1]Invited talk presented to the Sixth Canadian Conference on General Relativity and Relativistic Astrophysics, Fredericton, NB, May 1995.

[2]Address after Sept. 1, 1995: Theory Division, CERN, CH-1211, Genève 23, Switzerland.
1 Introduction

This article is meant to describe some preliminary applications of duality to unravelling the implications of string theory for black hole physics in four dimensions \cite{Bur94a, Bur94b}. Before summarizing the points to be outlined, some motivation is in order as to why duality transformations, and their applications to black hole physics, are worth thinking about.

Duality transformations \cite{Giv94, gives an extensive review} are a particular type of change of variable which have come to play increasingly important roles in extracting the physical content of string theory, and of some ordinary quantum field theories. Field theories which are related by duality transformations are thought to be completely equivalent, even though they may appear to be very different. Since it is sometimes true that intractible strongly-coupled theories turn out to be dual to calculable weakly-coupled ones, the physical equivalence of the dual theories can be exploited to infer the behaviour of otherwise unsolvable systems.

The development of this new tool may open new approaches to studying old problems of compelling physical interest which have hitherto been too difficult to crack. In particular, a current hope is that they may be useful for extracting the predictions of string theory for the dynamics of spacetimes having very strong, or even singular, curvatures, such as are believed to arise in the final state of runaway gravitational collapse, or in the earliest moments of the universe. The predictions of string theory are of particular interest for this long-standing puzzle, since string theory is the only known theory which gives sensible results for the simpler, but related, problem of the scattering of particles in flat space at energies at and above the Planck-scale.

The good news (so far) is that there are preliminary indications that string theory has qualitatively new features which may figure importantly in our final understanding of gravitational collapse. The main new feature to emerge so far is the realization that string theories appear to be quite forgiving in their notion of what constitutes a physically unacceptable singularity. What might be a malignantly singular field configuration from the point of view of ordinary point-particle field theories, can be completely benign as a background for string propagation. This understanding has emerged as more solutions have been constructed \cite{Tse94, for a recent review} to the full string equations, including some singular ones. It is also indicated by the existence of duality transformations which relate spacetimes with singularities to duals which are equivalent, and yet are absolutely nonsingular (including even Minkowski space). Some of the dual spacetimes which have been related
in this way even include black hole spacetimes [Wit 91, Giv 91, Dij 92], although admittedly only in two spacetime dimensions.

All of this motivates further study of black-hole-type field configurations within the context of string theory. This article reports on the results of two lines of inquiry in this direction. These two lines may be summarized as follows:

1. The first investigation [Bur 94a] uses duality transformations, and some of their extensions, to generate new classical solutions to the string equations of motion, starting from simple initial solutions. In particular, they are used to systematically construct all possible spherically-symmetric, time-independent and asymptotically-flat solutions in four dimensions. There are two new features to these results. (i) First, because using duality to generate new solutions is a purely algebraic procedure, it is much simpler than a direct attempt to solve the relevant set of nonlinear, coupled, partial-differential equations. This simplicity has permitted the inclusion of more nontrivial background fields than previously had been tractable. (ii) Second, keeping in mind the broad-mindedness of string theory towards singularities, all solutions are presented, including some that are singular. Singular solutions were often omitted in previous constructions.

2. The second application [Bur 94b] is to critically analyze the situation under which dual configurations can be expected to be physically equivalent. In particular, an important class of duality transformations — including most of those applied to black holes — are argued to not relate physically-equivalent spacetimes in the usual sense. The same considerations also lead to restrictions on the boundary conditions which must be satisfied by some fields in order for the duality transformations to take their usual form.

These two developments are presented in the remainder of this article in the following way. §2 sets the stage by briefly presenting a reminder of the connection between string field configurations, and two-dimensional conformal field theories. This is followed by the two applications listed above. First, §3 through §5 present the use of duality-related transformations for constructing solutions to the low-energy field equations. §3 summarizes the transformations themselves, while §4 gives the simplest — i.e. dilaton-metric — solutions. §5 then applies the transformations of §3 to the field configurations of §4, thereby generating solutions also incorporating nonzero axion and gauge fields.

The second line of argument is the topic of §6 through §8. §6 states the form of the duality transformations in a manner which is sufficiently
general for the desired applications. Their implications for the general spherically-symmetric, time-independent and asymptotically-flat solutions, including in particular the black hole solutions, are the topic of §7. The result, as applied to black hole configurations, makes it difficult to see how the dual solutions can be physically equivalent. §8 is dedicated to the resolution of this puzzle.

The conclusions which follow from these sections are finally summarized in §9.

2 Strings and 2D Field Theories

This section is meant to review the connection between string field configurations, and two-dimensional conformal field theories, since this underlies all of what follows. This connection allows the application of results for two-dimensional field theories to draw conclusions about solutions to the string field equations.

2.1 String Field Configurations

A classical solution to the string theory equations of motion is equivalent to a conformally-invariant, two-dimensional field theory. The connection is simplest to see in the case that the two-dimensional field theory is a sigma model, whose two-dimensional scalar fields, \( x^\mu(\sigma) \), can be interpreted as describing the worldsheet of a string propagating within a ‘target’ spacetime. For instance, suppose the sigma model action has the form

\[
S[x^\mu] = -\frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{\gamma} \left[ G_{\mu\nu}(x) \gamma^{\alpha\beta} + B_{\mu\nu}(x) \epsilon^{\alpha\beta} \right] \partial_\alpha x^\mu \partial_\beta x^\nu - \frac{1}{8\pi} \int d^2\sigma \sqrt{\gamma} \phi(x) \mathcal{R},
\]

where \( \gamma_{\alpha\beta} \) and \( \epsilon^{\alpha\beta} \) are the two-dimensional metric and antisymmetric Levi-Civita symbol and \( \mathcal{R} \) is the curvature scalar for the metric \( \gamma_{\alpha\beta} \).

The coupling functions, \( G_{\mu\nu}(x) \), \( B_{\mu\nu}(x) \) and \( \phi(x) \), are interpreted in string theory as being background values for three fields — the metric, the ‘axion’, and the ‘dilaton’ — which represent three of the modes of the string. These particular modes would have been massless if the string were propagating through Minkowski space. There are typically other such nearly ‘massless’ modes as well in a string theory, and we shall consider later a spin-one gauge fields, \( A_\mu(x) \), in addition to the above three.
Now comes the main point. There are two ways in which one might imagine defining the equations of motion for these fields, given our presently limited understanding of string theory itself. The direct way is to infer the interactions of the various string modes by computing their tree-level scattering on simple spacetimes, such as for Minkowski space. One finds an action which reproduces these scattering amplitudes, and then computes the equation of motion using this action.

The alternative way to determine the equations of motion for $G_{\mu\nu}, B_{\mu\nu}, \phi$ and $A_\mu$ is to compute the conditions which these quantities must satisfy in order for the corresponding two-dimensional theory to be conformally invariant at the quantum level. It is an amazing fact that these two methods produce results which agree with one another — up to the ubiquitous freedom to perform field redefinitions — in all cases for which they have been compared.

The resulting field equations can be explicitly written as an expansion in powers of derivatives of the background fields times the dimensionful constant $\alpha'$. In four dimensions the action which reproduces the leading terms in these equations for the fields of interest is $[\text{Gre} 87]$:

$$\mathcal{L} = \frac{e^\phi}{8\pi\alpha'} \sqrt{-G} \left[ R(G) + (\nabla \phi)^2 - \frac{1}{12} H^{\mu\nu\lambda} H_{\mu\nu\lambda} - \frac{1}{8} F^{\mu\nu} F_{\mu\nu} \right] + \cdots,$$

(2)

where $H_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda} - \frac{1}{4} A_\mu F_{\nu\lambda} + (\text{cyclic permutations})$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ are, respectively, the field strengths for the axion and electromagnetic fields. $R(G)$ is the Ricci scalar for the metric, $G_{\mu\nu}$, and the square root involves the quantity: $G = \det G_{\mu\nu}$. This metric is often called the ‘sigma-model’ metric to distinguish it from the one for which a field redefinition has been performed to put the Einstein term into standard form. The ellipses in eq. (2) represent terms which involve other massless fields and others involving more derivatives that arise at higher orders in $\alpha'$.

Some of the higher-order corrections to these equations in the derivative expansion have also been computed. For later purposes we quote the quantity which is responsible for the higher-order corrections to configurations involving only the metric and the dilaton, which turns out to be $[\text{Gre} 87]$:

$$\delta \mathcal{L} = \frac{\lambda e^\phi}{2} \sqrt{-G} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}.$$

(3)

3By including the gauge-field-dependent Chern-Simons term, $\frac{1}{4} A F$, in the definition of $H$ we restrict our discussion of gauge fields to the heterotic string. By contrast, the discussion for $A_\mu = 0$ applies equally well to bosonic and superstrings.
Here $\lambda$ is $\frac{1}{2}$ for bosonic strings, $\frac{1}{4}$ for heterotic strings, and 0 for supersymmetric strings.

This establishes a direct connection between background string configurations and two-dimensional conformal field theories. The spectrum of string states corresponds to a class of operators in these field theories which represent the conformal group in a particular way. The scattering of these string states can be computed by evaluating appropriate correlation functions involving these operators in the conformal field theory.

Given this connection it is immediate how to interpret the equivalence between any two conformal field theories in terms of string states. If two conformal theories can be shown to have precisely the same content, then they must describe identical string scattering about identical background string configurations. Duality transformations are interesting because they imply precisely this kind of equivalence between two-dimensional field theories.

### 3 Classical Transformations

There is a broad class of transformations which are guaranteed to map conformal field theories into other conformal field theories, even though the theories which are related in this way need not be physically equivalent. That is to say, this broader class of transformations are guaranteed to take classical string vacua into other classical vacua, but the full Hilbert space of string modes constructed about these vacua can be different. Although these transformations cannot therefore be considered to be *bona-fide* string symmetries, they are nevertheless very useful for generating new classical solutions from known ones.

There are two transformations of this type which are used in what follows. We outline both of these in the following two subsections.

#### 3.1 $SL(2, \mathbb{R})$ Transformations

The first class of such transformations is a group of $SL(2, \mathbb{R})$ transformations [Sha 91, DeR 83, Sen 93] which includes the classical string $S$-duality transformations [Wit 85, Bur 86, Li 87] of the low-energy effective theory. To formulate these transformations it is useful to use the scalar variable, $a(x)$, which is dual to the three-index axion field, $H_{\mu\nu\lambda}$:

$$H_{\mu\nu\rho} = -e^{-2\phi} \epsilon_{\mu\nu\rho\kappa} \nabla^\kappa a.$$  \hspace{1cm} (4)
Here all indices are raised and lowered with the Einstein metric, $g_{\mu\nu}$, which (in four spacetime dimensions) is related to the sigma-model metric, $G_{\mu\nu}$ by: $g_{\mu\nu} = e^\phi G_{\mu\nu}$, and the Levi-Civita tensor is also constructed using $g_{\mu\nu}$. Many properties of the theory take a simple form if the field, $a$, is combined with the dilaton, $\phi$, into the complex combination, $S = a + i e^\phi$. In terms of this variable the $SL(2, \mathbb{R})$ transformation becomes

$$
S \rightarrow \frac{aS + b}{cS + d},
$$

where $S^*$ is the complex conjugate of $S$, and $(F_{\pm})_{\mu\nu} \equiv F_{\mu\nu} \pm \frac{i}{2} \epsilon_{\mu\nu\rho\kappa} F^{\rho\kappa}$. Once again it is the Einstein metric which is involved in these definitions. The quantities $a$, $b$, $c$ and $d$ are real numbers which must satisfy $ad - bc = 1$. If $S$, $g_{\mu\nu}$, and $F_{\mu\nu}$ all satisfy the string equations of motion, then so must the transformed variables as defined by eq. (5).

### 3.2 The $O(1, 1)$ Transformations

There is a second transformation which can be used to generate new classical string solutions from old ones [Cec 88, Mei 91, Sen 92, Gas 91, Has 92]. In the simplest case of a field configuration that is independent of a single coordinate, $s$, there is an $O(1, 1)$ group of such transformations. The action of these transformations is most easily written when the background fields are written as the following $9 \times 9$ matrix

$$
M = \begin{pmatrix}
K^T G^{-1} K_- & K^T G^{-1} K_+ & -K^T G^{-1} A \\
K^T G^{-1} K_- & K^T G^{-1} K_+ & -K^T G^{-1} A \\
-A^T G^{-1} K_- & -A^T G^{-1} K_+ & A^T G^{-1} A
\end{pmatrix}
$$

where

$$
(K_{\pm})_{\mu\nu} = -B_{\mu\nu} - G_{\mu\nu} \pm \eta_{\mu\nu},
$$

and $\eta_{\mu\nu}$ is the flat Minkowski metric in four dimensions. In these expressions the quantity $G_{\mu\nu}$ is defined by: $G_{\mu\nu} = G_{\mu\nu} + \frac{1}{2} A_\mu A_\nu$. For a detailed statement of the conventions used, see ref. [Bur 94a].

With these variables the $O(1, 1)$ transformations can be expressed in matrix form, $M \rightarrow \Omega M \Omega^T$, where the transformation matrix is given by

$$
\Omega = \begin{pmatrix}
I_7 & 0 & 0 \\
0 & x & \sqrt{x^2 - 1} \\
0 & \sqrt{x^2 - 1} & x
\end{pmatrix}.
$$

7
Here $I_7$ is the $7 \times 7$ unit matrix, and $x$ is a parameter satisfying $x^2 \geq 1$. The action of the $O(1,1)$ transformations on the dilaton is given by

$$e^\phi \to \left( \frac{\det G}{\det G'} \right)^{\frac{1}{2}} e^\phi,$$

where $G'_{\mu\nu}$ denotes the transformed sigma-model metric.

4 Dilaton-Metric Configurations

The goal now is to use these transformations to generate the most general spherically-symmetric, asymptotically-flat and static solutions to the string equations. This will be done by applying the transformations to a particularly simple class of solutions. The first step — identifying this initial simple class of solutions — is the topic of the present section.

4.1 The Lowest-Order Solution

The starting point is the most general four-dimensional, asymptotically flat, static and spherically-symmetric configuration which solves the string equations to leading order in $\alpha'$, and which involves only the metric and dilaton fields. We therefore directly solve the equations:

$$R_{\mu\nu}(g) = \frac{1}{2} \nabla_\mu \phi \nabla_\nu \phi$$
$$\nabla^2 \phi = 0,$$

using the ansatz

$$g_{\mu\nu} dx^\mu dx^\nu = -f(r) dt^2 + g(r) dr^2 + h^2(r) (d\theta^2 + \sin^2 \theta d\phi^2),$$
$$\phi = \phi(r),$$

The result is a family of field configurations whose study actually predates the earliest advent of string theory itself [Buc 59, Jan 69]:

$$f = \frac{1}{g} = \Lambda^0(r)$$
$$h^2 = r^2 \Lambda^{1-\delta}(r)$$
$$e^\phi = e^{\phi_0} \Lambda^7(r),$$

where $\Lambda(r)$ is a shorthand for the function $1 - (\ell/r)$. $\ell$ is the only dimensionful parameter in the solutions, and so it simply sets their
overall scale. We assume both $\ell$ and $r$ to be positive in what follows. The two dimensionless quantities, $\delta$ and $\gamma$, are the arbitrary parameters which label the solutions, subject to the condition $\delta^2 + \gamma^2 = 1$.

$\phi_0$ is the asymptotic value which is obtained by the dilaton field as $r \to \infty$. It is convenient in what follows to shift $\phi$ by a constant to ensure that it tends to zero for large $r$, and so to completely remove $\phi_0$ from the solutions. This is always possible in string theory, with the general result — for classical string solutions — that $\phi_0$ only appears as an overall factor in the low-energy lagrangian of eqs. (2) and (3). It can therefore be absorbed into the definition of Newton’s constant, which we take (in four dimensions) to be $G_N = \frac{1}{2} e^{-\phi_0} \alpha'$.

The choice $(\delta, \gamma) = (1, 0)$ yields the Schwarzschild metric with constant dilaton field, and $\ell$ is in this case related to Newton’s constant, $G_N$, and the black hole mass, $M$, by $\ell = 2G_N M$. So long as $\gamma \ell \neq 0$, however, the metric of eq. (12) has a real curvature singularity at $r = \ell$.

### 4.2 The Conserved Charges

The two independent parameters which label this family of solutions have a useful interpretation in terms of the asymptotic behaviour, as $r \to \infty$, of the fields involved. In general, for large $r$, the solutions take the form:

\[
\begin{align*}
    f(r) & = 1 - \frac{A}{r} + \cdots; \\
    g(r) & = 1 + \frac{B}{r} + \cdots; \\
    h(r) & = r^2 \left[1 + O\left(\frac{1}{r}\right)\right], \\
    e^{\phi(r)} & = 1 - \frac{Q_D}{r} + \cdots,
\end{align*}
\]

with constants, $A = B = \delta \ell$ and $Q_D = \gamma \ell$.

These constants have a physical interpretation as specifying the corresponding conserved charges which are carried by the solutions. That is, the conserved (ADM [Arn 62]) mass, $M$, of the solution is related to the constant $A$ by $2G_N M = A$. The constant, $Q_D$, similarly defines a dilaton charge for the solution.

The utility of labelling the solutions by their values for $M$ and $Q_D$ is that these quantities are equally well defined for the complete solution to the string equations. Given these asymptotic expressions, the complete equations may be solved order by order in $\alpha'$, giving a unique solution for any choice of $M$ and $Q_D$. The same is not true
for the parameters $\delta$ and $\gamma$, which are defined in terms of the specific form of the solution to these equations only at lowest order in $\alpha'$. For instance, if we focus on the black hole solution — which we take to be the solution for which the potential singularity at $r = \ell$ is only a coordinate artifact — and work to next order in $\alpha'$ by including eq. (3) into the low-energy string action, then the result is still characterized by the quantities $M$ and $Q_D$, but with [Cal 89]:

$$2G_N M = A = B = \ell + \frac{11 \lambda \alpha'}{6 \ell} + O(\alpha'^2),$$
$$Q_D = -\frac{2 \lambda \alpha'}{\ell} + O(\alpha'^2).$$

(14)

Recall that the constant $\lambda$ appearing here is $\frac{1}{2}$ in the bosonic string, $\frac{1}{4}$ in the heterotic string, and 0 in the superstring.

5 More General Classical Solutions

It is now possible to compute more general classical solutions, simply by repeatedly applying the $SL(2, \mathbb{R})$ and $O(1, 1)$ transformations of the previous sections to the above solutions. Before doing so explicitly, it is first worth characterizing the possible parameters which can describe these solutions by first analyzing the asymptotic behaviour which is permitted for the fields at large $r$.

5.1 The Conserved Charges

Once the metric and dilaton fields are supplemented by nonzero axion and gauge fields, more complicated solutions become possible. We take the following spherically-symmetric and time-independent ansatz for the axion and gauge fields:

$$a = a(r), \quad F_{tr} = E(r), \quad F_{\theta \varphi} = B(r) \sin \theta.$$  

(15)

and we generalize, for later convenience, our metric ansatz to include stationary, but not static, metrics:

$$g_{\mu\nu} dx^\mu dx^\nu = -f(r)(dt+2N \cos \theta \ d\phi)^2 + g(r) dr^2 + h^2(r)(d\theta^2 + \sin^2 \theta \ d\varphi^2).$$

(16)

The parameter $N$ here is called the NUT parameter, due to the similarity of eq. (16) with the Taub-NUT metric [Tau 53, New 63, Mis 67]. The corresponding new conserved charges can be inferred by examining the large-$r$ behaviour that is implied by the field equations for
these fields. This is given, for the dilaton and metric, by eqs. (13), and for the axion and gauge fields by:

\[ a(r) = -\frac{Q_A}{r} + \cdots; \]
\[ A_t = \frac{Q_E}{r} + \cdots, \]
\[ A_\varphi = -Q_M \cos \theta + \cdots. \] (17)

The constants \( Q_A, Q_E \) and \( Q_M \) are the solution’s axion, electric and magnetic charges, respectively.

5.2 Dilaton-Axion-Metric Solutions

The simplest generalization is the inclusion of a nonzero axion field in addition to the original dilaton and metric configurations. These may be generated by applying the \( SL(2, \mathbb{R}) \) transformations, eq. (5), to the solution of eq. (12), being careful to preserve the boundary conditions for \( \phi \) and to ensure that \( a \to 0 \) at large \( r \). (There is no loss of generality in choosing this boundary condition for the axion field, since the definition, eq. (4), only defines \( a(r) \) up to an additive constant.)

Performing the \( SL(2, \mathbb{R}) \) transformation, we obtain the same metric configuration as before, but the new dilaton and axion fields, \( \hat{\phi} \) and \( \hat{a} \):

\[ e^{\hat{\phi}} = (1 + \omega^2) \frac{\Lambda^\gamma(r)}{\omega^2 \Lambda^{2\gamma}(r) + 1}, \quad \hat{a} = \frac{\omega [\Lambda^{2\gamma}(r) - 1]}{\omega^2 \Lambda^{2\gamma}(r) + 1}, \] (18)

\( \omega \) is the new real parameter of the solution, while \( \delta \) and \( \gamma \) are the labels of the original dilaton-metric configuration. The starting dilaton-metric solution is re-obtained as the special case \( \omega = 0 \). These parameters are related to the three conserved charges of the dilaton-axion-metric system by:

\[ 2G_N M = \delta \ell, \quad Q_D = \frac{1 - \omega^2}{1 + \omega^2} \gamma \ell \quad \text{and} \quad Q_A = \frac{2\omega \gamma \ell}{1 + \omega^2}. \] (19)

All of these solutions have real singularities at \( r = \ell \), except for the special case \( \gamma \ell = 0 \). An interesting special limit of these solutions is the case \( \omega = \pm 1 \), for which the dilaton charge, \( Q_D \), vanishes. Notice that even for this choice, however, the dilaton field is nonvanishing due to the nontrivial axion configuration.
5.3 Including a Gauge Potential

It is conceptually no more difficult, although algebraically more tedious, to generate the general dilaton–axion–metric–gauge-potential configuration. This is obtained by repeatedly performing $SL(2, \mathbb{R})$ and $O(1,1)$ transformations to the previous solutions. Three new parameters, $x$, $\epsilon$ and $\rho$, enter the solution in this way before these transformations start just regenerating previously-obtained configurations. These three parameters are related to the electric and magnetic charges, $Q_E$ and $Q_M$, of the background electromagnetic fields, and to the NUT parameter, $N$, of the metric (which is defined by eq. (16)).

The result of this process is a fairly complicated field configuration, whose explicit form is not particularly illuminating and so is not repeated here. Detailed expressions are given in [Bur 94a]. Instead, we turn to the action of duality on these solutions, and on their extensions to higher order in the $\alpha'$ expansion.

6 Duality Transformations

The remainder of this article is devoted to exploring the implications of duality transformations for the black-hole, and singular, solutions just constructed. Some of the results we obtain also apply to the exact solutions to the string equations.

6.1 The Transformation Rules

The first step is to define what is meant by duality transformations. A reasonably general algorithm has emerged with which it is always possible to systematically generate dual field theories from a given one [Roc 92, Alv 94a, Alv 94b], generalizing an earlier construction which had been developed for the earliest known string duality [Bus 87]. The algorithm applies to any field theory which admits a continuous symmetry.

It is simple to state the result for the case of the sigma model, eq. (1), when the symmetry corresponds to the independence of one of the coordinate directions, of the fields $G_{\mu \nu}$, $B_{\mu \nu}$, $\phi$ and $A_\mu$. Denoting this direction by $s$, then the sigma model which is dual to the original one involves the fields $\tilde{G}_{\mu \nu}$, $\tilde{B}_{\mu \nu}$, $\phi$ and $A_\mu$, where [Bus 87, Giv 89, Sha 89, Alv 95]:

\[ \tilde{G}_{ss} = 1/G_{ss}, \quad \tilde{G}_{s\mu} = -B_{s\mu}/G_{ss}, \]
\[ \tilde{G}_{\mu\nu} = G_{\mu\nu} - \left[ \frac{G_{sp}G_{sv} + B_{sp}B_{sv}}{G_{ss}} \right] \]

\[ \tilde{B}_{sp} = -\frac{G_{sp}}{G_{ss}}, \quad \tilde{B}_{\mu\nu} = B_{\mu\nu} + \frac{G_{sp}B_{sv} - G_{sv}B_{sp}}{G_{ss}} \]

\[ \tilde{A}_s = -A_s \frac{G_{ss}}{G_{ss}}, \quad \tilde{A}_\mu = A_\mu - A_s \frac{G_{sp} - B_{sp}}{G_{ss}} \]

\[ e^{\phi} = e^\phi \left( \frac{{\det} G}{{\det} \tilde{G}} \right)^{1/2} . \]

As before, the quantity \( G_{\mu\nu} \) is defined by: \( G_{\mu\nu} = G_{\mu\nu} + \frac{1}{4} A_\mu A_\nu \), and the index \( \mu \) runs over all values except for \( \mu = s \). (Ref. [Alv 94b] presents the duality transformations in a more manifestly covariant way.) These transformations in general can acquire higher-derivative corrections in powers of \( \alpha' \) as well.

### 6.2 An Example: The Torus

Perhaps the simplest example brings out many of the main features of these duality transformations\(^4\) The simplest example consists of a toroidal spacetime, for which the metric is flat and all other background fields are taken to be trivial. If we base the duality on the symmetry of translations along one of the compact coordinate directions of the torus — call it \( s \), say — then the general expression of eq. (20) simply reduces to the replacement:

\[ G_{ss} \rightarrow \frac{1}{G_{ss}}, \quad \phi_0 \rightarrow \phi_0 + \log \left| G_{ss} \right| . \]  

Here both \( G_{ss} \) and \( \phi_0 \) are constants. The dual configuration is once more a torus, but eq. (21) states that if the circumference in the \( s \) direction is initially \( R \), then for the dual theory it becomes \( \tilde{R} \propto \alpha'^2 / R \).

The beauty of the toroidal example is that for this background the complete spectrum of string fluctuations is known, and the correspondence of states in the dual theories can be explicitly followed. States turn out to be labelled by two integers, \( m \) and \( n \), in addition to other quantum numbers, where \( m \) labels the quantized momentum in the \( s \) direction and \( n \) gives the number of times which the string winds around this direction. It turns out that duality acts to interchange \( m \)

---

\(^4\)See ref. [Giv 94] for references to the extensive literature on toroidal duality.
and \( n \), while leaving all of the other quantum numbers fixed. The important role which is played by the winding modes, for which \( n \neq 0 \), emphasizes the intrinsically ‘stringy’ nature of the duality symmetry.

When the direction associated with the symmetry coordinate is compact, such as for the translations of a torus just considered, then the sigma models having the dual field configurations can be shown to be completely equivalent, and so describe exactly the same string physics [Roč 92]. Eqs. (20) can therefore be considered as full quantum symmetries of string theory.

The relation of the dual theories is less clear when the relevant symmetry coordinate is not compact. We argue in what follows that the equivalence need not be true in the noncompact case.

7 Applications of Duality

We now explore the implications of duality for general spherically-symmetric, time-independent and asymptotically-flat field configurations.

7.1 The Dilaton-Metric Configuration

To get an idea for what is implied by a duality transformation, we first apply one to the dilaton-metric configuration considered in §4.1, above. We base the duality transformation on the symmetry of time translation of the original solution. This solution, given by eq. (12), is characterized by the two constants, \( \delta \) and \( \gamma \), together with the dimensional quantity \( \ell \). Applying the duality transformation, eq. (20), to this configuration leads to another solution of the form as eq. (12), but with the constants \( \delta \) and \( \gamma \) interchanged:

\[ \delta \leftrightarrow \gamma. \]  

(22)

Given the relation between these parameters and the conserved charges we see that eq. (22) implies:

\[ 2G_N M \leftrightarrow Q_D. \]  

(23)

As will now be discussed, this last way of writing the duality transformation law applies equally well to the solutions at higher-orders in \( \alpha' \).

7.2 Application to General Configurations

Since part of the promise of studying duality transformations lies in their potential for leading to exact information concerning the theory
involved, it is of enormous interest to understand how duality acts on the exact string solutions, rather than simply having its action on the approximate solutions to low orders in the derivative expansion. The obvious difficulty lies in determining this action when explicit expressions for the solutions, and the transformation laws, are themselves not known beyond the leading order in $\alpha'$. It is nonetheless possible to draw general conclusions for the case of time-independent, spherically-symmetric, asymptotically flat field configurations, since these are in principle completely characterized by their values for the conserved charges $M, N, Q_D, Q_A, Q_E,$ and $Q_M$. Furthermore, these charges are themselves completely determined by the behaviour of the fields in the asymptotic, large-$r$, regime for which all fields are very slowly-varying. As a result, it is possible to determine the action of duality on the conserved charges of a solution using just the leading expressions in the derivative expansion.

By applying the transformation law, eq. (20), to the asymptotic forms for the background fields, eqs. (13) and (17), the action of duality on the space of four-dimensional, asymptotically-flat, time-independent and spherically-symmetric string solutions may be determined in general. We find that the solution which is labelled by the charges $M, N, Q_D, Q_A, Q_E,$ and $Q_M$ becomes mapped to the solution whose charges are $\tilde{M}, \tilde{N}, \tilde{Q}_D, \tilde{Q}_A, \tilde{Q}_E,$ and $\tilde{Q}_M$, where:

$$2G_N\tilde{M} = Q_D, \quad \tilde{Q}_D = 2G_N M, \quad \tilde{Q}_A = 2N, \quad 2\tilde{N} = Q_A,$$  (24)

and

$$\tilde{Q}_E = Q_E, \quad \tilde{Q}_M = Q_M.$$  (25)

That is, duality simply interchanges the mass with the dilaton charge, as well as interchanging the axion charge with the NUT parameter, while leaving the electric and magnetic charges untouched.

### 7.3 The Black Hole Special Case

This transformation law has some odd consequences, as may be seen by focussing on the black-hole solutions. At lowest order in $\alpha'$ these are characterized by the parameters $\delta = 1$ and $\gamma = 0$, and so their image under duality must have $\tilde{\delta} = 0$ and $\tilde{\gamma} = 1$. If we use the expressions for $M$ and $Q_D$, as given to $O(\alpha')$ in eqs. (14), then we find for the dual configuration:

$$2G_N\tilde{M} = -\frac{2\lambda\alpha'}{\ell} + O(\alpha'^2),$$

$$\tilde{Q}_D = \ell + \frac{11\lambda\alpha'}{6\ell} + O(\alpha'^2).$$  (26)
This transformation rule contains the seeds of a puzzle. Notice, in this regard, the mapping from positive to negative mass:
\[
\tilde{M} = -\frac{k}{\alpha' M} + O\left(\frac{1}{\alpha'^2 M^3}\right),
\]  
(27) where \(k = \frac{\lambda \alpha'^2}{2G_N^2} = 2\lambda e^{2\phi_0}\) is a dimensionless constant. (In the special case of the superstring, for which \(\lambda = 0\), it is necessary to work to still higher order in \(\alpha'\) to infer the sign of \(\tilde{M}\), again giving the result [Bur 94b] that \(M\) and \(\tilde{M}\) have opposite signs.) The puzzle is to understand how the physical equivalence of dual solutions can be reconciled with a change of sign of the solution’s mass. The next section is devoted to the resolution of this puzzle.

8 Equivalence Under Duality

Intuitively, it seems absurd that a background field configuration having negative mass could be physically equivalent to one for which the mass is positive. The goal of this section is to pin down this intuition, in order to better understand the circumstances under which dual spacetimes can be equivalent.

A more precise reason for doubting the equivalence of spacetimes having opposite masses would be the expectation that such spacetimes should gravitationally scatter incident test particles differently. After all, based on experience with the Newtonian force law for universal gravitation, a positive mass should exert an attractive gravitational force on a distant, slowly-moving particle, while the force due to a negative mass should be repulsive in this limit. One might worry, however, whether the presence of other massless background fields — such as the dilaton or axion etc. — may invalidate any intuition which is based on the Newtonian limit of pure gravity.

8.1 Test-Particle Scattering

It is straightforward to test these ideas, by computing the scattering of test particles which remain at large \(r\), and so which experience only slowly varying fields. This calculation may be performed explicitly in string theory by choosing massless string states as test particles. The result is particularly simple in the limit for which the wavelength of the incident state is both longer than the Planck length, and shorter than the typical distances over which the background fields vary. This is the regime of geometric optics, for which simple arguments show that
massless string states simply follow the null geodesics of the metric, regardless of the presence of axion and dilaton fields [Bur 94b].

For example, in this limit the scattering angle, $\Delta \varphi$, of a photon which remains at large $r$ throughout its interaction with the background field is given (for vanishing NUT parameter) by [Bur 94b]

$$\Delta \varphi \approx -\frac{4G_N M}{r_0}, \quad (28)$$

where $r_0$ is the smallest radial coordinate that is attained by the photon. As advertised, this result depends only on the mass $M$ of the source, and not on its other charges. The result for the dual spacetime is given by the same expression, with $M \to \tilde{M}$.5

8.2 A Better Argument

Convincing as the previous calculation may seem, it leaves room for doubt concerning the inequivalence of the dual field configurations we have considered. This is because it leaves open a loophole, whose existence becomes clear once the well-understood example of toroidal duality is reconsidered. As was stated in §6.2, duality acts to interchange the labels $m$ and $n$ which respectively characterize the momentum and winding number along the symmetry direction. It is the scattering of ‘winding modes’ ($n \neq 0$) in the dual theory which is equal to the scattering of ‘momentum modes’ ($m \neq 0$) in the original theory. The problem with the scattering calculation of §8.1 arises because it is not clear whether the photon in the original theory is mapped to the same photon in the dual theory. If not there is no reason for a result like eq. (28) to be the same for both the original spacetime and its dual.

In fact, since the symmetry on which duality is based for the configurations considered here is time translation, it is not completely clear precisely how the winding and momentum modes should be defined. As a result, an alternative argument in favour of the inequivalence of the dual solutions is now presented which does not rely on being able to trace the detailed action of duality on the various string states. This is done by identifying a one-parameter family of equivalent background field configurations, and showing that these dualize to a one-parameter family of inequivalent configurations. We are led to the conclusion that eqs. (20) need not be a string symmetry, at least when the coordinate ‘s’ labels a noncompact direction.

5This assumes only that the dual NUT parameter also vanishes, which implies the vanishing of the axion charge, $Q_A$, of the original solution. This class of configurations is amply big for the present purposes.
The key point on which this argument relies is the observation that the transformations of eqs. (20) depend in detail upon the boundary condition which is satisfied at large \( r \) by the gauge potential in the time direction, \( A_t \). In particular, the duality transformations obtained in §7.2 for the conserved charges of the spherically-symmetric solutions are only correct if it is assumed that \( A_t \) falls off like \( 1/r \). This is required if the quantity \( G_{tt} = G_{tt} + \frac{1}{4} A_t A_t \) is to have the same asymptotic form as does \( G_{tt} \) — a result which is required in deriving eqs. (24) and (25).

Suppose, then, we instead assume the following asymptotic behaviour for \( A_t \):

\[
A_t = 2v + \frac{Q_E}{r} + \cdots ,
\]

where \( v \) is a constant. This form does not affect the asymptotic behavior of the electric and magnetic fields, and so does not change the values of the electric and magnetic charges. In order to not change the asymptotic form for the field strength, \( H = dB - \frac{1}{4} A dA \), it is also necessary to alter the asymptotic behavior of the Kalb-Ramond field, \( B \), which we assume to have done in what follows.

When performing the duality transformation, eq. (20), using the asymptotic form of eq. (29) it is necessary to rescale the time coordinate to preserve the limit \( G_{tt} \rightarrow -1 \) as \( r \rightarrow \infty \), and to shift the dilaton by a constant in order to recover its previous limiting value at infinity. With these choices the action of duality on the asymptotic charges is no longer given by eqs. (24) and (25), but rather by [Bur 94b]:

\[
\begin{align*}
2\tilde{G}_N\tilde{M} &= \frac{1}{1 - v^2} \left[ Q_D - 2v^2 G_N M - v Q_E \right] \\
\tilde{Q}_D &= \frac{1}{1 - v^2} \left[ 2G_N M - v^2 Q_D + v Q_E \right] \\
2\tilde{N} &= \frac{1}{1 - v^2} \left[ Q_A - 2v^2 N \right] \\
\tilde{Q}_A &= \frac{1}{1 - v^2} \left[ (1 - 2v^2)2N + v^2 Q_A \right] \\
\tilde{Q}_E &= \frac{1}{1 - v^2} \left[ (1 + v^2)Q_E + 2v(2G_N M - Q_D) \right] \\
\tilde{Q}_M &= Q_M + \frac{v}{1 - v^2} \left[ 2N - Q_A \right].
\end{align*}
\]

It is assumed in these equations that \( v^2 < 1 \). Notice that the previous results are obtained as \( v \rightarrow 0 \). The ‘tilde’ is written over Newton’s constant in the first line to emphasize that this constant is modified because of the shift which was required of the dilation field in order to preserve its boundary conditions at large \( r \).
We are now in a position to argue against the necessity of physical equivalence between dual solutions when the duality which relates the solutions is based on a noncompact symmetry. The first step is the observation that the asymptotic value, \( v \), which is taken by \( A_t \) drops out of all physical quantities because it can be changed by performing a gauge transformation. The one-parameter family of field configurations which differ only in their asymptotic value for \( A_t \) are therefore all physically equivalent to one another. In particular, all would agree on their prediction for the scattering of an incident massless string mode.

Now, if duality based on eqs. (20) were always to generate physically equivalent string configurations, it would follow that the one-parameter family of dual configurations which are obtained from the original gauge-equivalent family must also all be physically equivalent. But eqs. (30) show that the dual configurations have conserved charges which depend explicitly on the value of the parameter, \( v \). They therefore differ in their physical predictions, such as for the scattering of massless string modes, and so cannot be physically equivalent. We conclude that for the class of configurations considered here, eqs. (20) are not symmetries of string theory.

Notice that the compactness of the symmetry plays a role in this argument. If the symmetry direction were compact, the above argument would fail since it would no longer be possible to shift \( A_t \) by an arbitrary constant value simply by performing a gauge transformation. Different constant values for \( A_t \) would be physically distinct since they could be characterized by different values for a gauge-invariant quantity: the Wilson lines around the compact direction.

### 8.3 Flat Space Revisited

The above considerations can be further focussed by reconsidering the case of flat space and toroidal compactifications. We can avoid the complication of understanding the meaning of winding and momentum modes in the time direction by considering instead duality based on translations in one of the noncompact spatial directions. It is instructive, in this case, to consider this direction as the limit of a compact, circular, direction as its circumference tends to infinity.

For any finite circumference, duality indeed represents an exact string symmetry, with all momentum modes dualizing to winding modes in the symmetry direction, and vice versa. For infinite circumference the momentum states degenerate into a continuum of permitted eigenvalues, while the masses of all of the winding modes tend to infinity. As required by duality, precisely the opposite happens as the circumference tends to zero: winding states form a continuum and momentum states
become infinitely massive. So long as both winding- and momentum-modes are both kept in the infinite (and zero-) circumference limit, duality continues to relate equivalent theories.

But string theory, as it is normally defined on (noncompact) flat Minkowski space, is the path integral over all possible string embeddings in this space. This corresponds to what is obtained from a compact space in the infinite-circumference limit provided that the winding modes are dropped in this limit. Once the winding modes are omitted, the equivalence of dual configurations is no longer guaranteed.

This picture is similar to that obtained by Ref. [Alv 94b], who analyze more systematically the equivalence of solutions related by duality based on a noncompact symmetry. They find equivalence, but only if one of the dual pair of solutions is quantized as a vortex gas. In this unconventional string theory, there are no local vertex operators carrying momentum in the symmetry direction, but instead one constructs nonlocal ‘vortex’ operators carrying ‘winding-number’ for this direction.

This line of argument indicates that a string theory is not uniquely specified by listing its configuration of background fields, since additional choices — such as whether or not to include nonlocal vortex operators — must also be made. Although duality can always produce an equivalent string theory from any given one, the two string theories generically will not both be interpretable as the sum over string embeddings in the given background field configurations. In comparing the dual theories, one must keep in mind this distinction. For example, for two-dimensional black holes, the propagation and interactions of tachyons near the singularity are equivalent to the propagation and interactions near the horizon of some dual vortex states, rather than of tachyons.

9 Conclusions

The purpose here has been to present some results of a preliminary investigation [Bur 94a, Bur 94b] into the consequences of duality, and related, transformations on our understanding of string propagation through black-hole-like spacetimes. This has been accomplished by outlining two qualitatively different types of results.

The first class of result simply uses these transformations to generate new solutions to the string equations. This technique has been used to generate the most general static, spherically-symmetric and asymptotically-flat string configuration which solves the string equations to lowest order in $\alpha'$. Solutions of this form are now known which
involve all of the metric, dilaton, axion and gauge fields — a broader class than had hitherto been constructed. They are labelled by a collection of five conserved quantities: the mass $M$, and the dilaton, axion, electric and magnetic charges: $Q_D$, $Q_A$, $Q_E$ and $Q_M$. If the static condition is relaxed to include some stationary metrics, then a sixth charge, the NUT parameter, $N$, is also required.

Using duality, and related, transformations to generate these solutions has the practical advantage of requiring the use of only algebraic techniques, instead of having to solve several coupled, nonlinear, partial-differential equations. This makes its application relatively straightforward to apply to more complicated, less symmetric, field configurations.

Most of the solutions which were generated in this way have real curvature singularities at fixed, nonzero $r$, and these singularities are in some cases naked. We regard it to be premature to discard these solutions until it is better understood precisely what configurations string theory considers to be singular.

The second type of investigation reported here concerns the action and implications of duality on these low-energy solutions in particular, and on static, spherically-symmetric and asymptotically-flat string configurations in general.

The result can be presented in a very general form. By investigating the action of duality on the asymptotic form of a general field configuration for large $r$, it becomes clear that duality simply interchanges the configuration’s mass and dilaton charge, as well as interchanging the axion charge with the NUT parameter. The electric and magnetic charges do not change. These results are very robust — applying equally well once higher orders in $\alpha'$ are included — depending as they do only on the large-$r$ behaviour of the solution.

Somewhat surprisingly we find that the dual field configuration depends in an important way on the assumed asymptotic form for the electrostatic potential, $A_t$. The conclusions of the previous paragraph assumed $A_t$ to be chosen to vanish asymptotically.

For the special case of the four-dimensional ‘Schwarzschild-like’ solutions — i.e. those that are nonsingular at the Schwarzschild radius — a black hole of mass $\tilde{M}$ is mapped by duality onto a singular configuration whose mass, $\tilde{\tilde{M}}$, is given by $\tilde{\tilde{M}} = -k/(\alpha' M)$, where $k = \lambda \alpha'^2/2G_N^2 = 2\lambda e^{2\phi_0}$. $\lambda$ here is $\frac{1}{2}$ in the bosonic, and $\frac{1}{4}$ in the heterotic string. For the superstring, it happens that $\lambda = k = 0$, and so $O(\alpha'^3)$ corrections are required in order to determine the dual mass.

The result in this case turns out to be: $\tilde{\tilde{M}} = -k'/(\alpha'^3 M^5)$, where $k' = 3\zeta(3) e^{6\phi_0}$. 

We have also addressed the potential inequivalence of the two dual solutions in the case where the symmetry on which duality is based is not compact. An argument was presented which led to one of the following two options: (i) dual solutions, defined by eqs. (20), can be physically inequivalent; or (ii) constant shifts of the electrostatic potential, $A_t$, can change the physical content of the theory. Arguments in favour of the first option were presented, using the well-understood example of flat space.

The potential implications of duality are just beginning to be explored, both for black holes, and for other physical systems. May the rewards be both rich and varied!

10 Acknowledgements

C.B. would like to thank the organizers for their kind invitation to speak at the conference. Our funds have been provided by N.S.E.R.C. of Canada, les Fonds F.C.A.R. du Québec and the Swiss National Foundation.

References

[Alv 94a] E. Alvarez, L. Alvarez-Gaumé and Y. Lozano, preprint CERN-TH-7486, hep-th/9410237.

[Alv 94b] E. Alvarez, L. Alvarez-Gaumé, J. Barbón and Y. Lozano, *Nucl. Phys.* B415 (1994) 71.

[Alv 95] E. Alvarez, L. Alvarez-Gaumé and I. Bakas, preprint CERN-TH-95-201, hep-th/9507112.

[Arn 62] R. Arnowitt, S. Deser and C.W. Misner, in *Gravitation: An Introduction to Current Research*, ed. by L. Witten (New York, Wiley, 1962).

[Buc 59] H. Buchdahl, *Phys. Rev.* 115 (1959) 1325.

[Bur 86] C.P. Burgess, A. Font and F. Quevedo, *Nucl. Phys.* B272 (1986) 661.

[Bur 94a] C.P. Burgess, R.C. Myers and F. Quevedo, *Nucl. Phys.* B442 (1995) 75.
[Bur 94b] C.P. Burgess, R.C. Myers and F. Quevedo, *Nucl. Phys.* **B442** (1995) 97.

[Bus 87] T. Buscher, *Phys. Lett.* **B194** (1987) 59; *Phys. Lett.* **B201** (1988) 466.

[Cal 89] C.G. Callan, R.C. Myers and M.J. Perry, *Nucl. Phys.* **B311** (1989) 673.

[Cec 88] S. Cecotti, S. Ferrara and L. Girardello, *Nucl. Phys.* **B308** (1988) 436.

[DeR 85] M. De Roo, *Nucl. Phys.* **B255** (1985) 515.

[Dij 92] R. Dijkgraaf, E. Verlinde and H. Verlinde, *Nucl. Phys.* **B371** (1992) 269.

[Gas 91] M. Gasperini, J. Maharana and G. Veneziano, *Phys. Lett.* **B272** (1991) 277.

[Giv 89] A. Giveon, E. Rabinovici and G. Veneziano, *Nucl. Phys.* **B322** (1989) 167.

[Giv 91] A. Giveon, *Mod. Phys. Lett.* **A6**, (1991) 2843.

[Giv 94] For a recent review see, e.g. A. Giveon, M. Porrati and E. Rabinovici, *Phys. Rept.* **244** (1994) 77.

[Gre 87] See, e.g. M. Green, J. Schwarz and E. Witten *Superstring Theory I* (1987), (Cambridge University Press).

[Has 92] S. Hassan and A. Sen, *Nucl. Phys.* **B375** (1992) 103.

[Jan 69] A.I. Janis, D.C. Robinson and J. Winicour, *Phys. Rev.* **186** (1969) 1729.

[Li 87] S.P. Li, R. Peschanski and C. Savoy, *Phys. Lett.* **B194** (1987) 226; *Phys. Lett.* **B178** (1986) 193.

[Mei 91] K. Meissner and G. Veneziano, *Phys. Lett.* **B267** (1991) 33.

[Mis 67] C.W. Misner, ‘Taub-NUT space as a counterexample to almost anything’, in *Relativity Theory and Astrophysics I, Relativity and Cosmology*, ed. by J. Ehlers, Lectures in Applied Mathematics, Vol. 8 (American Mathematical Society, 1967) 160.
[New 63] E. Newman, L. Tamburino and T. Unti, *J. Math. Phys.* **4** (1963) 915.

[Roč 92] M. Roček and E. Verlinde, *Nucl. Phys.* **B373** (1992) 630.

[Sen 92] A. Sen, *Phys. Lett.* **B274** (1992) 34.

[Sen 93] A. Sen, *Nucl. Phys.* **B404** (1993) 109.

[Sha 89] A. Shapere and F. Wilczek, *Nucl. Phys.* **B320** (1989) 669.

[Sha 91] A. Shapere, S. Trivedi and F. Wilczek, *Mod. Phys. Lett.* **A6**, (1991) 2677.

[Tau 53] A.H. Taub, *Ann. Math.* **53** (1951) 472.

[Tse 94] A.A. Tseytlin, to appear in: Proceedings of the 2nd Journée Cosmologie, Observatoire de Paris, June 2-4, 1994 (World Scientific).

[Wit 85] E. Witten, *Phys. Lett.* **B155** (1985) 151.

[Wit 91] E. Witten, *Phys. Rev.* **D44** (1991) 314.