SELF-DUAL KOORNWINDER-MACDONALD POLYNOMIALS

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Abstract. We prove certain duality properties and present recurrence relations for a four-parameter family of self-dual Koornwinder-Macdonald polynomials. The recurrence relations are used to verify Macdonald’s normalization conjectures for these polynomials.

1. Introduction

In a to date unpublished but well-known manuscript, Macdonald introduced certain families of multivariable orthogonal polynomials associated with (admissible pairs of integral) root systems and conjectured the values of the normalization constants turning these polynomials into an orthonormal system [M1]. Recently, Cherednik succeeded in verifying Macdonald’s normalization conjectures in the case of reduced root systems (and admissible pairs of the form $(R, R^\vee)$) using a technique involving so-called shift operators [C1]. Previously, this same technique had enabled Opdam to prove the normalization conjectures for a degenerate case ($q \to 1$) of the Macdonald polynomials known as the Heckman-Opdam-Jacobi polynomials [O, H].

Meanwhile, a generalization of Macdonald’s construction for the nonreduced root system $BC_n$—resulting in a multivariable version of the famous Askey-Wilson polynomials [AW]—was presented by Koornwinder [K2]. It turns out that all Macdonald polynomials associated with classical (i.e., non-exceptional) root systems may be seen as special cases of these multivariable Askey-Wilson polynomials [D, Sec. 5] (type $A$ by picking the highest-degree homogeneous parts of the polynomials and types $B$, $C$, $D$, and $BC$, by specialization of the parameters).

In the present paper, we will prove certain duality properties and recurrence relations for (a four-parameter subfamily of) the Koornwinder-Macdonald multivariable Askey-Wilson polynomials, which enable one to verify the corresponding Macdonald conjectures for the (ortho)normalization constants also in this (more general) situation. Our approach does not involve shift operators but rather exploits the fact that...
the polynomials are joint eigenfunctions of a family of commuting difference operators that was introduced by the author in Ref. [D1] (see also Ref. [D3]). By duality, these difference operators give rise to a system of recurrence relations from which, in turn, the normalization constants follow.

The same method employed here was used already several years ago by Koornwinder when verifying similar duality properties and normalization constants for the Macdonald polynomials related to the root system $A_n$ [K1, M3]. (In this special case, though, the validity of the normalization conjectures had also been checked by Macdonald himself.) The $A_n$-type Macdonald polynomials constitute a multivariable generalization of the $q$-ultraspherical polynomials [AW] (to which they reduce for $n = 1$). The present paper may thus be regarded as an extension of Koornwinder’s methods in Ref. [K1] (see also Ref. [M3, Ch. 6]) to the multivariable Askey-Wilson level, or, if one prefers, as an extension from type $A$ root systems to type $BC$ root systems.

2. Koornwinder-Macdonald Polynomials

The Koornwinder-Macdonald multivariable Askey-Wilson polynomials are characterized by a weight function of the form

$$
\Delta(x) = \prod_{1 \leq j < j' \leq n} d_v(\varepsilon x_j + \varepsilon' x_{j'}) \prod_{1 \leq j \leq n} d_w(\varepsilon x_j),
$$

(2.1)

where

$$
d_v(z) = \frac{(e^{-\alpha z}; q)_\infty}{(q; q)_\infty}, \quad q = e^{-\alpha \beta},
$$

$$
d_w(z) = \frac{(e^{-\alpha z}, -e^{-\alpha z}, q^{1/2} e^{-\alpha z}, -q^{1/2} e^{-\alpha z}; q)_\infty}{(q^{g_0} e^{-\alpha z}, -q^{g_1} e^{-\alpha z}, q^{(g_2 + 1/2)} e^{-\alpha z}, -q^{(g_3 + 1/2)} e^{-\alpha z}; q)_\infty},
$$

and the $q$-shifted factorials are defined, as usual, by $(a; q)_\infty = \prod_{l=0}^{\infty} (1 - aq^l)$ and $(a_1, \ldots, a_k; q)_\infty = (a_1; q)_\infty \cdots (a_k; q)_\infty$. To ensure the convergence of the infinite products contained in $\Delta$ (2.1) it will be assumed that $\alpha, \beta > 0$ (so $1 < q < 1$); in addition, we will also assume $g, g_r \geq 0$, $r = 0, 1, 2, 3$.

Let $\{m_\lambda(x)\}_{\lambda \in \Lambda}$ denote the basis consisting of even and permutation symmetric exponential monomials (or monomial symmetric functions)

$$
m_\lambda(x) = \sum_{\lambda' \in W \lambda} c^\lambda \sum_{j=1}^n \lambda'_j x_j, \quad \lambda \in \Lambda = \{\lambda \in \mathbb{Z}^n \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0\},
$$

(2.2)

with $W$ being the group generated by permutations and sign flips of $x_j$, $j = 1, \ldots, n$ ($W \cong S_n \ltimes (\mathbb{Z}_2)^n$). The monomial basis can be partially ordered by defining for
\( \lambda, \lambda' \in \Lambda \) \hfill (2.2)

\[
\lambda' \leq \lambda \quad \text{iff} \quad \sum_{1 \leq j \leq m} \lambda'_j \leq \sum_{1 \leq j \leq m} \lambda_j \quad \text{for} \quad m = 1, \ldots, n \tag{2.3}
\]

(and \( \lambda' < \lambda \) iff \( \lambda' \leq \lambda \) and \( \lambda' \neq \lambda \)). The Koornwinder-Macdonald polynomials \( p_\lambda(x) \), \( \lambda \in \Lambda \) can now be introduced as the (unique) trigonometric polynomials satisfying

i. \( p_\lambda(x) = m_\lambda(x) + \sum_{\lambda' \in \Lambda, \lambda' < \lambda} c_{\lambda, \lambda'} m_{\lambda'}(x), \quad c_{\lambda, \lambda'} \in \mathbb{C}; \)

ii. \( \langle p_\lambda, m_{\lambda'} \rangle_{\Delta} = 0 \) if \( \lambda' < \lambda \),

where \( \langle \cdot, \cdot \rangle_{\Delta} \) denotes the inner product determined by

\[
\langle m_\lambda, m_{\lambda'} \rangle_{\Delta} = \left( \frac{\alpha}{2\pi} \right)^n \int_{-\pi/\alpha}^{\pi/\alpha} \cdots \int_{-\pi/\alpha}^{\pi/\alpha} m_\lambda(ix) \overline{m_{\lambda'}(ix)} \Delta(ix) \, dx_1 \cdots dx_n \tag{2.4}
\]

(and extended by bilinearity). In other words, the polynomial \( p_\lambda(ix) \) consists of the monomial \( m_\lambda(ix) \) minus its orthogonal projection in \( L^2([-\pi/\alpha, \pi/\alpha]^n, \Delta(ix)dx_1 \cdots dx_n) \) onto the finite-dimensional subspace span\( \{m_{\lambda'}(ix)\}_{\lambda' \in \Lambda, \lambda' < \lambda} \).

It is of course possible to extend this construction of multivariable polynomials determined by Conditions i. and ii. to a more general class of weight functions than the one considered here. In general, however, the resulting polynomials will not be orthogonal (except for \( n = 1 \)) because the ordering in Eq. (2.3) is not a total ordering (unless \( n = 1 \)). (A priori the construction only guarantees that \( p_\lambda(x) \) and \( p_{\lambda'}(x) \) be orthogonal if \( \lambda \) and \( \lambda' \) are comparable with respect to the ordering in Eq. (2.3).) Still, it turns out \[K2\] that for the weight function \( \Delta \) (2.1) the corresponding polynomials indeed do constitute an orthogonal system for arbitrary \( n \):

**Orthogonality**

\[
\langle p_\lambda, p_{\lambda'} \rangle_{\Delta} = 0 \quad \text{if} \quad \lambda \neq \lambda'. \tag{2.5}
\]

This feature should be looked upon as a very restrictive property of the weight function \( \Delta \) (2.1).

**Remark:** The relation between our parameters and the parameters employed by Koornwinder reads (cf. \[K2\], Eqs. (5.1), (5.2))

\[
t = q^g, \quad a = q^{g_0}, \quad b = -q^{g_1}, \quad c = q^{(g_2+1/2)}, \quad d = -q^{(g_3+1/2)}. \tag{2.6}
\]

Furthermore, Koornwinder fixes the period the trigonometric functions to be \( 2\pi(i) \), i.e., he puts \( \alpha = 1 \).

### 3. Difference equations

Another special property of the polynomials associated with \( \Delta \) (2.1) is that they satisfy a second order difference equation \[K2\]. (This difference equation is in fact
instrumental in the orthogonality proof.) It can be written as

\begin{equation}
\sum_{1 \leq j \leq n} \left( w(x_j) \prod_{k \neq j} v(x_j + x_k) v(x_j - x_k) \right) \left[ p_\lambda(x + \beta e_j) - p_\lambda(x) \right] + w(-x_j) \prod_{k \neq j} v(-x_j + x_k) v(-x_j - x_k) \left[ p_\lambda(x - \beta e_j) - p_\lambda(x) \right] = E(\rho + \lambda) p_\lambda(x),
\end{equation}

with

\begin{align*}
v(z) &= \frac{\text{sh}^{\frac{\alpha}{2}}(\beta g + z)}{\text{sh}(\frac{\alpha}{2} z)}, \\
w(z) &= \frac{\text{sh}^{\frac{\alpha}{2}}(\beta g_0 + z) \text{ch}^{\frac{\alpha}{2}}(\beta g_1 + z) \text{sh}^{\frac{\alpha}{2}}(\beta g_2 + \frac{\beta}{2} + z) \text{ch}^{\frac{\alpha}{2}}(\beta g_3 + \frac{\beta}{2} + z)}{\text{sh}(\frac{\alpha}{2} z) \text{ch}(\frac{\alpha}{2} z) \text{sh}^{\frac{\alpha}{2}}(\frac{\beta}{2} + z) \text{ch}^{\frac{\alpha}{2}}(\frac{\beta}{2} + z)},
\end{align*}

and

\begin{equation}
E(y) = 2 \sum_{1 \leq j \leq n} \left( \text{ch}(\alpha \beta y_j) - \text{ch}(\alpha \beta \rho_j) \right), \\
\rho = \sum_{1 \leq j \leq n} \rho_j e_j, \quad \rho_j = (n - j) g + (g_0 + g_1 + g_2 + g_3)/2.
\end{equation}

(The vector $e_j$ denotes the $j$-th unit element of standard basis in $\mathbb{R}^n$.) For $n = 1$ this difference equation reduces to the well-known difference equation for the Askey-Wilson polynomials [AW].

In Ref. [D1] it was shown that for arbitrary number of variables $n$ the above difference equation can be extended to a system of $n$ independent difference equations of order $2r$, $r = 1, \ldots, n$, respectively. This system is explicitly given by:

\textit{Difference equations}

\begin{align}
\sum_{J \subset \{1, \ldots, n\}, 0 \leq |J| \leq r, \epsilon_j = \pm 1, j \in J} U_{r-|J|}(x) V_{\epsilon,J}(x) p_\lambda(x + \beta \epsilon_j e_J) &= E_r(\rho + \lambda) p_\lambda(x), \quad r = 1, \ldots, n,
\end{align}

(3.2)
with
\[ V_{\varepsilon,J,K}(x) = \prod_{j \in J} w(\varepsilon_j x_j) \prod_{j,j' \in J, j < j'} v(\varepsilon_j x_j + \varepsilon_{j'} x_{j'}) v(\varepsilon_j x_j + \varepsilon_{j'} x_{j'} + \beta) \times \prod_{j \in J} v(\varepsilon_j x_j + x_k) v(\varepsilon_j x_j - x_k), \]

\[ U_{K,p}(x) = (-1)^p \sum_{L \subseteq K, |L| = p} \prod_{l \in L} w(\varepsilon_l x_l) \prod_{l,l' \in L, l < l'} v(\varepsilon_l x_l + \varepsilon_{l'} x_{l'}) v(-\varepsilon_l x_l - \varepsilon_{l'} x_{l'} - \beta) \times \prod_{l \in L} v(\varepsilon_l x_l + x_k) v(\varepsilon_l x_l - x_k), \]

and
\[ E_r(y) = 2^r \sum_{J \subseteq \{1, \ldots, n\}} (-1)^{r-|J|} \left( \prod_{j \in J} \text{ch}(\alpha\beta y_j) \sum_{r \leq l_1 \leq \cdots \leq l_{r-|J|} \leq n} \text{ch}(\alpha\beta \rho_{l_1}) \cdots \text{ch}(\alpha\beta \rho_{l_{r-|J|}}) \right). \]

In the above formulas $|J|$ represents the number of elements of $J \subseteq \{1, \ldots, n\}$ and
\[ e_{\varepsilon J} = \sum_{j \in J} \varepsilon_j e_j \quad (\varepsilon_j = \pm 1). \] (3.3)

Furthermore, we used the conventions that empty products are equal to one, $U_{K,p} = 1$ if $p = 0$, and the second sum in $E_r(y)$ is equal to one if $|J| = r$. For $r = 1$, the difference equation in Eq. (3.2) reduces to that of Eq. (3.1).

Remarks: i. Equation (3.2) may be interpreted as a system of eigenvalue equations
\[ (D_r p_{\lambda})(x) = E_r(\rho + \lambda) p_{\lambda}(x), \] (3.4)

for a family of $n$ independent commuting difference operators of the form
\[ D_r(x) = \sum_{J \subseteq \{1, \ldots, n\}, 0 \leq |J| \leq r} U_{Jc, r-|J|}(x) V_{\varepsilon,J,Jc}(x) T_{\varepsilon,J,\beta}, \quad r = 1, \ldots, n, \] (3.5)

with $T_{\varepsilon,J,\beta} = \prod_{j \in J} T_{\varepsilon_j,\beta}$ and
\[ (T_{\pm\beta,\varepsilon}(f))(x_1, \ldots, x_n) = f(x_1, \ldots, x_{j-1}, x_j \pm \beta, x_{j+1}, \ldots, x_n). \]

The commutativity of $D_1(x), \ldots, D_n(x)$ follows [11] from the fact that the difference operators are simultaneously diagonalized by the basis $\{p_{\lambda}(x)\}_{\lambda \in \Lambda}$. Two other properties of the operators that follow immediately from their diagonalization by the Koornwinder-Macdonald polynomials are the triangularity with respect to the partially ordered monomial basis $\{m_{\lambda}(x)\}_{\lambda \in \Lambda}$, and the symmetry with respect to the
inner product $\langle \cdot, \cdot \rangle_\Delta$ (2.4) (the eigenvalues $E_\nu(\rho + \lambda)$ are real). More precisely, one has (Triangularity)

$$(D_r m_\lambda)(x) = E_r(\rho + \lambda) m_\lambda(x) + \sum_{\lambda' \in \Lambda, \lambda' < \lambda} [D_r]_{\lambda, \lambda'} m_{\lambda'}(x), \quad [D_r]_{\lambda, \lambda'} \in \mathbb{C}, \quad (3.6)$$

and (Symmetry)

$$\langle D_r m_\lambda, m_{\lambda'} \rangle_\Delta = \langle m_\lambda, D_r m_{\lambda'} \rangle_\Delta. \quad (3.7)$$

ii. For $r = 1$ the operator $D_r(x)$ (3.3) reduces to (cf. the l.h.s. of Eq. (3.1))

$$D(x) = \sum_{1 \leq j \leq n} \left( w(x_j) \prod_{k \neq j} v(x_j + x_k) v(x_j - x_k) (T_{j, \beta} - 1) + w(-x_j) \prod_{k \neq j} v(-x_j + x_k) v(-x_j - x_k) (T_{-j, \beta} - 1) \right). \quad (3.8)$$

With the aid of this operator the following useful representation for the Koornwinder-Macdonald polynomials can be given (cf. Ref. [M1])

$$p_\lambda(x) = \left( \prod_{\lambda' \in \Lambda, \lambda' < \lambda} \frac{D(x) - E(\rho + \lambda')}{E(\rho + \lambda) - E(\rho + \lambda')} \right) m_\lambda(x). \quad (3.9)$$

(Using the Triangularity (3.6) and Symmetry (3.7) of $D(x)$ (3.8), it is not difficult to verify that the r.h.s. of Eq. (3.9) satisfies the Conditions i. and ii. in Section 2. Notice also that the denominators in the r.h.s. of Eq. (3.9) do not vanish because $E(\rho + \lambda') < E(\rho + \lambda)$ if $\lambda' < \lambda$ (cf. Ref. [D1, Lemma 5.1]).) It follows from this representation for $p_\lambda(x)$ that the Koornwinder-Macdonald polynomials are rational in $\exp(\alpha \beta g)$ and $\exp(\alpha \beta g_r)$, $r = 0, 1, 2, 3$. Hence, they may be extended uniquely to nonnegative (or even complex) values of the parameters $g, g_r$. In view of the analytic dependence on the parameters it is clear that the resulting polynomials then satisfy the Difference equations (3.2) for all the values of $g, g_r, r = 0, 1, 2, 3$.

4. Duality and recurrence relations

Note: In this section, we will drop the condition that the parameters $g, g_r, r = 0, 1, 2, 3$ be nonnegative (cf. Remark ii. of the previous section).

In order to describe the duality relations it is convenient to introduce certain dual polynomials $p_\lambda^*(x)$, $\lambda \in \Lambda$. These dual polynomials are again Koornwinder-Macdonald polynomials but with a slightly different parametrization. Specifically, the parameters of $p_\lambda^*(x)$ are related to those of $p_\lambda(x)$ by

$$\alpha^* = \beta, \quad \beta^* = \alpha, \quad g^* = g,$n

$$(g_0^* \quad g_1^* \quad g_2^* \quad g_3^*) = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} g_0 \\ g_1 \\ g_2 \\ g_3 \end{pmatrix}. \quad (4.1)$$
Notice that the reparametrization in Eq. \ref{eq:reparametrization} is involutive, i.e., \( p_\lambda^\ast(x) = p_\lambda(x) \).

Furthermore, instead of working with monic polynomials we go over to a different normalization by introducing

\[
\tilde{p}_\lambda(x) = \frac{p_\lambda(\beta x)}{p_\lambda(\beta \rho^\ast)}, \quad \tilde{p}_\lambda^\ast(x) = \frac{p_\lambda^\ast(\alpha x)}{p_\lambda^\ast(\alpha \rho)}
\]  \tag{4.2}

\((\rho^\ast = \sum_{j=1}^n \rho_j^* e_j\) with \(\rho_j^* = (n-j)g^* + (g_0^* + \cdots + g_3^*)/2\), where we have also rescaled the arguments of the trigonometric polynomials such that \(\tilde{p}_\lambda(x)\) and \(\tilde{p}_\lambda^\ast(x)\) have the same (imaginary) period in \(x_j, j = 1, \ldots, n\) (viz. \(2\pi i/\alpha \beta\)). Of course, the renormalization in Eq. \ref{eq:renormalization} only makes sense provided \(p_\lambda(\beta \rho^\ast)\) and \(p_\lambda^\ast(\alpha \rho)\) do not vanish. This is guaranteed, at least for generic parameters, by the following lemma.

**Lemma 4.1.** For generic parameters one has

\[
p_\lambda(\beta \rho^\ast), p_\lambda^\ast(\alpha \rho) \neq 0.
\]

**Proof.** For \(g, g_0, \ldots, g_3 = 0\), the polynomial \(p_\lambda(x)\) reduces to the monomial symmetric function \(m_\lambda(x)\) and \(\rho^\ast = 0\). Thus, it is clear that for this special choice of the parameters \(p_\lambda(\beta \rho^\ast) \neq 0\). But then the same follows for \(g, g_0, \ldots, g_3\) in an open dense subset of \(\mathbb{R}\) (or \(\mathbb{C}\)) because of the analytic dependence of \(p_\lambda(x)\) on the parameters (cf. Remark \textit{ii.} in Section 3). The analogous statement for \(p_\lambda^\ast(\alpha \rho)\) follows by duality. \(\Box\)

In Section 5 the value of \(p_\lambda(\beta \rho^\ast)\) will be computed explicitly. We will then see that \(p_\lambda(\beta \rho^\ast)\) is positive for all nonnegative values of parameters \(g, g_r, r = 0, 1, 2, 3\).

The matrix in Eq. \ref{eq:matrix} relating \((g_0^* \ldots, g_3^*)^t\) and \((g_0, \ldots, g_3)^t\) has eigenvalues +1 (with multiplicity three) and −1 (with multiplicity one). The invariant subspace corresponding to the eigenvalue +1 consists of the hyperplane \(g_0 - g_1 - g_2 - g_3 = 0\). For parameters in this hyperplane one has \(g_r^* = g_r\) \((r = 0, 1, 2, 3)\) and \(\tilde{p}_\lambda(x) = \tilde{p}_\lambda^\ast(x)\). In other words, for these parameters the polynomials \(\tilde{p}_\lambda(x)\) are self-dual. In the rest of the paper we will always assume that the \textit{Self-duality condition}

\[
g_0 - g_1 - g_2 - g_3 = 0 \tag{4.3}
\]

is satisfied (unless explicitly stated otherwise).

After these preparations we are now ready to formulate the duality theorem, which relates the value of \(\tilde{p}_\lambda(x)\) in the point \(\rho^\ast + \mu\) to value of \(\tilde{p}_\mu^\ast(x)\) in the point \(\rho + \lambda\) \((\lambda, \mu \in \Lambda)\).

**Theorem 4.2** (duality relations). \textit{Let} \(\lambda, \mu \in \Lambda \) \textit{(2.2)}. \textit{Then the renormalized Koornwinder-Macdonald polynomials} \(\tilde{p}_\lambda(x)\) \textit{and} \(\tilde{p}_\lambda^\ast(x)\) \textit{satisfy the relation}

\[
\tilde{p}_\lambda(\rho^\ast + \mu) = \tilde{p}_\mu^\ast(\rho + \lambda). \tag{4.4}
\]

Theorem 4.2 was conjectured by Macdonald in Ref. [22] (without imposing the \textit{Self-duality condition} \ref{eq:condition}). In the present self-dual set-up it is of course not necessary to distinguish between the polynomials \(\tilde{p}_\lambda(x)\) and the dual polynomials \(\tilde{p}_\lambda^\ast(x)\) as both polynomials coincide when Condition \ref{eq:condition} holds. However, we have chosen...
to keep this distinction in our notation because it is expected that with the present formulation all results remain valid also when the Self-duality condition (4.3) is not satisfied (cf. Remark 7.2 of Section 7).

Before going to the proof of Theorem 4.2, which is relegated to Section 6, let us first discuss some important consequences of these duality relations. The main point is that the Difference equations (3.2) together with the Duality relations (4.4) imply a system of recurrence relations for the Koornwinder-Macdonald polynomials. To see this one first substitutes $x = \alpha (\rho + \lambda)$ in the difference equations for the dual polynomial $p_\mu^*(x), \mu \in \Lambda$. After dividing by $p_\mu^*(\alpha \rho)$ and invoking of Definition (4.2) one arrives at an equation of the form

$$E_r^*(\rho^* + \mu) \tilde{p}_\mu^*(\rho + \lambda) = \sum_{J \subset \{1, \ldots, n\}, 0 \leq |J| \leq r, \varepsilon_j = \pm 1, j \in J} \tilde{U}_{J^c, r - |J|}(\rho + \lambda) \tilde{V}_{\varepsilon J, J^c}(\rho + \lambda) \tilde{p}_\mu^*(\rho + \lambda + e_{\varepsilon j}),$$

(4.5)

where

$$\tilde{V}_{\varepsilon J, K}(x) = \prod_{j \in J} \tilde{w}(\varepsilon_j x_j) \prod_{j, j' \in J, j < j'} \tilde{v}(\varepsilon_j x_j + \varepsilon_j x_{j'}) \tilde{v}(\varepsilon_j x_j + \varepsilon_{j'} x_{j'}) + 1)

\times \prod_{j, j' \in J, j < j'} \tilde{v}(\varepsilon_j x_j + x_k) \tilde{v}(\varepsilon_j x_j - x_k),$$

$$\tilde{U}_{K, p}(x) = (-1)^p \sum_{L \subset K, |L| = p, \varepsilon_j = \pm 1, l \in L} \prod_{l \in L} \tilde{w}(\varepsilon_l x_l) \prod_{l, l' \in L, l < l'} \tilde{v}(\varepsilon_l x_l + \varepsilon_l x_{l'}) \tilde{v}(\varepsilon_l x_l - x_l - 1)

\times \prod_{l \in L, k \in K \setminus L} \tilde{v}(\varepsilon_l x_l + x_k) \tilde{v}(\varepsilon_l x_l - x_k),$$

with

$$\tilde{v}(z) = \frac{\text{sh} \frac{\alpha_2}{2} (g^* + z)}{\text{sh} \frac{\alpha_2}{2} z},$$

$$\tilde{w}(z) = \frac{\text{sh} \frac{\alpha_2}{2} (g_0^* + z)}{\text{sh} \frac{\alpha_2}{2} z} \frac{\text{ch} \frac{\alpha_2}{2} (g_1^* + z)}{\text{ch} \frac{\alpha_2}{2} z} \frac{\text{sh} \frac{\alpha_2}{2} (g_2^* + \frac{1}{2} + z)}{\text{sh} \frac{\alpha_2}{2} \frac{1}{2} + z} \frac{\text{ch} \frac{\alpha_2}{2} (g_3^* + \frac{1}{2} + z)}{\text{ch} \frac{\alpha_2}{2} \frac{1}{2} + z}$$

(and $E_r^*(y)$ is the dual of $E_r(y)$ in Eq. (3.2), i.e., with $\rho_j$ replaced by $\rho_j^*$). One may restrict the summation in Eq. (4.3) to those index sets $J \subset \{1, \ldots, n\}$ and configurations of signs $\varepsilon_j, j \in J$ for which $\lambda + e_{\varepsilon j} \in \Lambda$ (2.2) because of the following lemma.

**Lemma 4.3.** Let $\lambda \in \Lambda = \{ \mu \in \mathbb{Z}^n \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0 \}$. For generic parameters one has

$$\tilde{V}_{\varepsilon J, J^c}(\rho + \lambda) = 0 \quad \text{iff} \quad \lambda + e_{\varepsilon j} \notin \Lambda.$$
Proof. Suppose $\tilde{V}_{\varepsilon,J,J'}(\rho + \lambda) = 0$. Then we are in one of the following situations.

1. $\tilde{w}(\varepsilon_j(\rho_j + \lambda_j)) = 0$ for some $j \in J$.
   For generic parameters this only happens when $\lambda_j = 0$, $\varepsilon_j = -1$ (and $j = n$).
   So, $\lambda + \varepsilon J \not\in \Lambda$ because $\lambda_j + \varepsilon_j < 0$.
2. $\tilde{v}(\varepsilon_j(\rho_j + \lambda_j) + \varepsilon_{j'}(\rho_{j'} + \lambda_{j'})) = 0$ for some $j, j' \in J$ with $j < j'$.
   For generic parameters this only happens when $-\varepsilon_j = \varepsilon_{j'} = 1$, $\lambda_j = \lambda_{j'}$ (and $j' = j + 1$). So, $\lambda + \varepsilon J \not\in \Lambda$ because $\lambda_j + \varepsilon_j < \lambda_{j'} + \varepsilon_{j'}$ with $j < j'$.
3. $\tilde{v}(\varepsilon_j(\rho_j + \lambda_j) + \varepsilon_{j'}(\rho_{j'} + \lambda_{j'})) = 0$ for some $j, j' \in J$ with $j < j'$.
   For generic parameters this only happens when $-\varepsilon_j = \varepsilon_{j'} = 1$, $\lambda_j = \lambda_{j'} + 1$ (and $j' = j + 1$). So, $\lambda + \varepsilon J \not\in \Lambda$ because $\lambda_j + \varepsilon_j < \lambda_{j'} + \varepsilon_{j'}$ with $j < j'$.
4. $\tilde{v}(\varepsilon_j(\rho_j + \lambda_j) + \varepsilon_k(\rho_k + \lambda_k)) = 0$ for some $j \in J$ and $k \not\in J$.
   For generic parameters this only happens when $\varepsilon_j = -\varepsilon_k = 1$, $\lambda_j = \lambda_k$ and $j = k + 1$, or $\varepsilon_j = -\varepsilon_k = -1$, $\lambda_j = \lambda_k$ and $j = k - 1$. So, $\lambda + \varepsilon J \not\in \Lambda$ because $\lambda_j + \varepsilon_j < \lambda_k$ with $j \in J$, $k \not\in J$ and $j < k$, or $\lambda_j + \varepsilon_j > \lambda_k$ with $j \in J$, $k \not\in J$ and $j > k$.

Conversely, if $\lambda' = \lambda + \varepsilon J \not\in \Lambda$ then either $\lambda'_n < 0$ or there exist a $\lambda \in \{1, \ldots, n - 1\}$ such that $\lambda'_j < \lambda'_{j+1}$. The first case, i.e. $\lambda'_n < 0$, can occur only if $n \in J$ and $\lambda_n = 0$, $\varepsilon_n = -1$. The vanishing of $\tilde{V}_{\varepsilon,J,J'}(\rho + \lambda)$ then follows from the vanishing of $\tilde{w}(\varepsilon_n(\rho_n + \lambda_n))$. In the second case, i.e. $\lambda'_j < \lambda'_{j+1}$, it is convenient to distinguish the following three situations.

1. $j, j + 1 \in J$.
   One has $\lambda'_j = \lambda_j + \varepsilon_j < \lambda_{j+1} + \varepsilon_{j+1} = \lambda'_{j+1}$ only if $0 \leq \lambda_j - \lambda_{j+1} \leq 1$ and $-\varepsilon_j = \varepsilon_{j+1} = 1$. The vanishing of $\tilde{V}_{\varepsilon,J,J'}(\rho + \lambda)$ then follows because either $\tilde{v}(\varepsilon_j(\rho_j + \lambda_j) + \varepsilon_{j'}(\rho_{j'} + \lambda_{j'})) = 0$ or $\tilde{v}(\varepsilon_j(\rho_j + \lambda_j) + \varepsilon_{j'}(\rho_{j'} + \lambda_{j'})) = 0$ (depending on whether $\lambda_j = \lambda_{j+1}$ or $\lambda_j = \lambda_{j+1} + 1$).

2. $j \in J$, $j + 1 \not\in J$.
   One has $\lambda'_j = \lambda_j + \varepsilon_j < \lambda_{j+1} = \lambda'_{j+1}$ only if $\lambda_j = \lambda_{j+1}$ and $\varepsilon_j = -1$. The vanishing of $\tilde{V}_{\varepsilon,J,J'}(\rho + \lambda)$ then follows because $\tilde{v}(\varepsilon_j(\rho_j + \lambda_j) + (\rho_{j'} + \lambda_{j'})) = 0$.

3. $j \not\in J$, $j + 1 \in J$.
   One has $\lambda'_j = \lambda_j < \lambda_{j+1} + \varepsilon_{j+1} = \lambda'_{j+1}$ only if $\lambda_j = \lambda_{j+1}$ and $\varepsilon_{j+1} = 1$. The vanishing of $\tilde{V}_{\varepsilon,J,J'}(\rho + \lambda)$ then follows because $\tilde{v}(\varepsilon_j(\rho_j + \lambda_j) + (\rho_{j'} + \lambda_{j'})) = 0$.

(The a priori fourth situation $j, j + 1 \not\in J$ does not occur because in that case $\lambda'_j = \lambda_j \geq \lambda_{j+1} = \lambda'_{j+1}$.)

In order to restrict the sum in Eq. (4.5) to the index sets and signs with $\lambda + \varepsilon J \in \Lambda$ it is of course sufficient to know that $\tilde{V}_{\varepsilon,J,J'}(\rho + \lambda) = 0$ if $\lambda + \varepsilon J \not\in \Lambda$. It is clear from the proof of Lemma 4.3 that this is actually true for all values of the parameters (the genericity of the parameters was needed only when proving the converse statement).
After restricting the sum, we may apply the duality theorem (Theorem 4.2) to obtain the equation
\[ E_r^*(\rho^* + \mu) \tilde{p}_\lambda(\rho^* + \mu) = \sum_{J \subset \{1, \ldots, n\}, 0 \leq |J| \leq r} \tilde{U}_{J^c, J} (\rho + \lambda) \tilde{V}_{\varepsilon J, J^c} (\rho + \lambda) \tilde{p}_{\lambda + \varepsilon J} (\rho^* + \mu). \] (4.6)

Equation (4.6) describes an equality between trigonometric polynomials that is satisfied for all \( \rho^* + \mu, \mu \in \Lambda \). But then the equality must hold identically for all values of the argument and we arrive at the following theorem.

**Theorem 4.4** (recurrence relations). The renormalized Koornwinder-Macdonald polynomials \( \tilde{p}_\lambda(x) \) \((4.2)\), \( \lambda \in \Lambda \) \((2.2)\), satisfy a system of recurrence relations of the form
\[ E_r^*(x) \tilde{p}_\lambda(x) = \sum_{J \subset \{1, \ldots, n\}, 0 \leq |J| \leq r} \tilde{U}_{J^c, J} (\rho + \lambda) \tilde{V}_{\varepsilon J, J^c} (\rho + \lambda) \tilde{p}_{\lambda + \varepsilon J}(x), \] \( r = 1, \ldots, n \).

For \( r = 1 \) the Recurrence relation (4.7) becomes
\[ E_1^*(x) \tilde{p}_\lambda(x) = \sum_{1 \leq j \leq n, \lambda + \varepsilon_j \in \Lambda} \tilde{w}(\rho_j + \lambda_j) \prod_{k \neq j} \tilde{v}(\rho_j + \lambda_j + \rho_k + \lambda_k) \tilde{v}(\rho_j + \lambda_j - \rho_k - \lambda_k) [\tilde{p}_{\lambda + \varepsilon_j}(x) - \tilde{p}_\lambda(x)] + \sum_{1 \leq j \leq n, \lambda - \varepsilon_j \in \Lambda} \tilde{w}(-\rho_j - \lambda_j) \prod_{k \neq j} \tilde{v}(-\rho_j - \lambda_j + \rho_k + \lambda_k) \tilde{v}(-\rho_j - \lambda_j - \rho_k - \lambda_k) [\tilde{p}_{\lambda - \varepsilon_j}(x) - \tilde{p}_\lambda(x)]. \]

\((E^*_1(y) = E_1^*(y))\). In this special case the recurrence formula was conjectured by Macdonald \([M2]\); after specialization to one variable \((n = 1)\) this formula reduces to the three-term recurrence relation for the Askey-Wilson polynomials \([AW]\).

In the one-variable case both duality relations and self-duality can be easily checked directly through the explicit representation of \( \tilde{p}_\lambda(x) \) in terms of the basic hypergeometric \( 4 \phi_3 \)-series (cf. \([AW]\, Eq. \,(5.8))\) and Relation (2.3):
\[ \tilde{p}_\lambda(x) = 4 \phi_3 \left( \begin{array}{c} q^{-l}, q^{2g_0 + l}, q^{g_0 - x}, q^{g_0 + x} \\ -q^{g_0 + g_1}, q^{g_0 + g_2 + 1/2}, -q^{g_0 + g_3 + 1/2} ; q, q \end{array} \right). \] (4.8)

It is clear from Representation (4.8) that the duality relations \( \tilde{p}_\lambda(g_0 + m) = \tilde{p}_\lambda^*(g_0^* + l) \)
hold actually without restriction on the parameters (recall that for \( n = 1 \) one has \( \rho = g_0^* \) and \( \rho^* = g_0 \)) and that \( \tilde{p}_\lambda(x) \) is self-dual iff \( g_0^* = g_0 \), or equivalently, iff \( g_0 - g_1 - g_2 - g_3 = 0 \) (notice that \( g_0 + g_r = g_0^* + g_r^* \) for \( r = 1, 2, 3 \)).

Remark: Recurrence relations of the type in Theorem 4.4 are sometimes called generalized Pieri formulas (after similar formulas for the Schur functions \([M3]\)). Similarly, the duality relations of Theorem 4.2 are also referred to as symmetry relations.
5. Normalization

In this section we exploit the recurrence relations of Theorem 4.4 to evaluate the normalization constants \( p_\lambda(\rho^*) \) and \( \langle p_\lambda, p_\lambda \rangle_\Delta \) (giving rise to polynomials satisfying the Duality relations (4.4) and orthonormal polynomials, respectively). First some notation is needed. Let

\[
\hat{\Delta}_+(x) = \prod_{1 \leq j < j' \leq n} \tilde{d}_v(x_j + x_{j'}) \tilde{d}_v(x_j - x_{j'}) \prod_{1 \leq j \leq n} \tilde{d}_w(x_j), \quad (5.1)
\]

with

\[
\tilde{d}_v(z) = q^{-g^*z/2} \frac{(q^z; q)_\infty}{(q^{g^*+z}; q)_\infty} \quad (q = e^{-\alpha \beta}),
\]

\[
\tilde{d}_w(z) = q^{-(g_0^* + \cdots + g_r^*)z/2} \frac{(q^{z^2}, -q^z, q^{(1/2+z)}, -q^{(1/2+z)} ; q)_\infty}{(q^{(g_0^*+z)}, -q^{g_0^*+z}, q^{g_0^*+1/2+z}, -q^{g_0^*+1/2+z}; q)_\infty},
\]

and let

\[
\hat{\Delta}_+(x) = \prod_{1 \leq j < j' \leq n} \hat{d}_v(x_j + x_{j'}) \hat{d}_v(x_j - x_{j'}) \prod_{1 \leq j \leq n} \hat{d}_w(x_j), \quad (5.2)
\]

with

\[
\hat{d}_v(z) = q^{g^*(z+1)/2} \frac{(q^{(z+1)} ; q)_\infty}{(q^{-g^*+z+1}; q)_\infty},
\]

\[
\hat{d}_w(z) = q^{(g_0^* + \cdots + g_r^*)(z+1)/2} \frac{(q^{z+1}, -q^{z+1}, q^{(3/2+z)}, -q^{(3/2+z)} ; q)_\infty}{(q^{(-g_0^*+z+1)}, -q^{(-g_0^*+z+1)}, q^{(-g_0^*+3/2+z)}, -q^{(-g_0^*+3/2+z)}; q)_\infty}.
\]

(The function \( \hat{\Delta}_+(x) \) (5.2) is obtained from \( \hat{\Delta}_+(x) \) (5.1) by substituting \( g^* \to -g^* \), \( g_r^* \to -g_r^*(r = 0, 1, 2, 3) \), and \( z \to z + 1 \).)

From the relations \( \tilde{d}_v(z + 1) = \tilde{v}(z)\tilde{d}_v(z) \), \( \tilde{d}_w(z + 1) = \tilde{w}(z)\tilde{d}_w(z) \) and \( \hat{d}_v(z + 1) = \hat{v}(z)\hat{d}_v(z) \), \( \hat{d}_w(z + 1) = \hat{w}(z)\hat{d}_w(z) \) it follows that

\[
\frac{\hat{\Delta}_+(x + e_{\{1,\ldots,r\}})}{\Delta_+(x)} = \tilde{V}_{\{1,\ldots,r\},\{r+1,\ldots,n\}}(x) \quad (5.3)
\]

and

\[
\frac{\hat{\Delta}_+(x + e_{\{1,\ldots,r\}})}{\Delta_+(x)} = \tilde{V}_{\{1,\ldots,r\},\{r+1,\ldots,n\}}(-x - e_{\{1,\ldots,r\}}), \quad (5.4)
\]

where \( \tilde{V}_{J,K}(x) \) is the same in Eq. (4.3) (with all signs \( \varepsilon_j, j \in J \) being positive).

**Theorem 5.1** (normalization 1: duality). One has

\[
p_\lambda(\beta \rho^*) = \frac{\Delta_+(\rho + \lambda)}{\Delta_+(\rho)}. \quad (5.5)
\]
Proof. Expanding the l.h.s. and the r.h.s. of Recurrence relation (4.7) in monomial symmetric functions and comparing the coefficients of the leading monomial $m_{\lambda+e_{\{1,\ldots,r\}}}(x)$ at both sides of the equation leads to a relation between $p_{\lambda+e_{\{1,\ldots,r\}}}^{\beta}(\rho^*)$ and $p_{\lambda}(\rho^*)$:

$$1/p_{\lambda}(\rho^*) = \tilde{V}_{\{1,\ldots,r\}}(\rho + \lambda) / p_{\lambda+e_{\{1,\ldots,r\}}}^{\beta}(\rho^*).$$

(5.6)

One can rewrite this relation with the aid of Eq. (5.3) in the form

$$\frac{p_{\lambda}(\beta^*)}{\Delta_+(\rho + \lambda)} = \frac{p_{\lambda+e_{\{1,\ldots,r\}}}^{\beta}(\rho^*)}{\Delta_+(\rho + \lambda + e_{\{1,\ldots,r\}})}.$$  

(5.7)

Because any vector in the cone $\Lambda$ (2.2) can be written as a nonnegative integral combination of the vectors $e_{\{1,\ldots,r\}}$, $r = 1, \ldots, n$, it follows from (iterated application of) Eq. (5.7) that the quotient $p_{\lambda}(\beta^*)/\Delta_+(\rho + \lambda)$ does not depend on the choice of $\lambda \in \Lambda$. Evaluation of this quotient in $\lambda = 0$ (so $p_{\lambda} = 1$) then entails Eq. (5.5).

Theorem 5.1 manifestly demonstrates that $p_{\lambda}(\beta^*)$ is indeed nonzero for generic parameters (Lemma 4.1). One observes in particular that $p_{\lambda}(\beta^*)$ is positive for nonnegative values of the parameters $g, g_r$ ($r=0,1,2,3$). (For positive parameters this is immediate from the expression in Eq. (5.5), whereas for $g$ or $g_r$ equal to zero one needs to consider the limit $g \to 0_+$ or $g_r \to 0_+$, respectively).

Let us next turn to the computation of the (ortho)normalization constants $\langle p_{\lambda}, p_{\lambda} \rangle_\Delta$ (for nonnegative values of the parameters $g, g_r$, $r=0,1,2,3$). In the case $\lambda = 0$ (so $p_{\lambda} = 1$), the corresponding (generalized Selberg-type) integral was evaluated by Gustafson in Ref. [G]. In our notation, Gustafson’s result may be rephrased as (cf. Remark ii., below)

$$\langle 1, 1 \rangle_\Delta = 2^n n! \Delta_+(\rho) \Delta_+(\rho).$$

(5.8)

The following theorem generalizes this (constant term) formula to the case of Koornwinder-Macdonald polynomials of arbitrary degree.

**Theorem 5.2** (normalization 2: orthonormality). *Let the parameters $g, g_r$, $r = 0, \ldots, 3$ be nonnegative. Then one has*

$$\langle p_{\lambda}, p_{\lambda} \rangle_\Delta = 2^n n! \Delta_+(\rho + \lambda) \Delta_+(\rho + \lambda).$$

(5.9)

**Proof.** Set $\tilde{\Delta}(x) = \Delta_+(x) \Delta_+(x)$ with $g^*, g_r^*$ replaced by $g, g_r$ (so $\tilde{\Delta}(x)$ equals $\Delta_+(\beta x)$, cf. Eq. (2.1)) and let $\langle \cdot, \cdot \rangle_\Delta$ be the inner product determined by (cf. Eq. (2.4))

$$\langle \tilde{m}_{\lambda}, \tilde{m}_{\lambda'} \rangle_\Delta = \left( \frac{\alpha \beta}{2\pi} \right)^n \int_{-\pi/\alpha}^{\pi/\alpha} \cdots \int_{-\pi/\alpha}^{\pi/\alpha} \tilde{m}_{\lambda}(ix) \tilde{m}_{\lambda'}(ix) \Delta(ix) \, dx_1 \cdots dx_n$$

(5.10)

(where $\tilde{m}_{\lambda}(x) = m_{\lambda}(\beta x)/m_{\lambda}(0)$, cf. Definition (4.2)). Clearly, the polynomials $\tilde{p}_{\lambda}$ (4.2) are orthogonal with respect to $\langle \cdot, \cdot \rangle_\Delta$ (cf. Eq. (2.5)). Furthermore, one has
(using Definition (4.2) and Theorem 5.1)

\[ N_\lambda \equiv \langle \tilde{p}_\lambda, \tilde{p}_\lambda \rangle_\Delta = \frac{1}{|p_\lambda(\beta \rho^*)|^2} \langle p_\lambda, p_\lambda \rangle_\Delta = \left( \frac{\tilde{\Delta}_+(\rho)}{\Delta_+(\rho + \lambda)} \right)^2 \langle p_\lambda, p_\lambda \rangle_\Delta. \] (5.11)

By evaluating both sides of the identity

\[ \langle E_r^* \tilde{p}_\lambda, \tilde{p}_{\lambda + e_{1\ldots r}} \rangle_\Delta = \langle \tilde{p}_\lambda, E_r^* \tilde{p}_{\lambda + e_{1\ldots r}} \rangle_\Delta \] (5.12)

using Recurrence relation (4.7) and the orthogonality of the polynomials \( \tilde{p}_\lambda, \lambda \in \Lambda, \) we arrive at the following relation between \( N_\lambda (= \langle \tilde{p}_\lambda, \tilde{p}_\lambda \rangle_\Delta) \) and \( N_{\lambda + e_{1\ldots r}} : \)

\[ V_{\{1\ldots r\}, \{r+1\ldots n\}}(\rho + \lambda) N_{\lambda + e_{1\ldots r}} = V_{\{1\ldots r\}, \{r+1\ldots n\}}(-\rho - \lambda - e_{1\ldots r}) N_\lambda. \] (5.13)

With the aid of Eqs. (5.3) and (5.4) one rewrites this relation in the form

\[ N_\lambda \frac{\tilde{\Delta}_+(\rho + \lambda)}{\Delta_+(\rho + \lambda)} = N_{\lambda + e_{1\ldots r}} \frac{\tilde{\Delta}_+(\rho + \lambda + e_{1\ldots r})}{\Delta_+(\rho + \lambda + e_{1\ldots r})}, \] (5.14)

which after invoking of Eq. (5.11) goes over in

\[ \frac{\langle p_\lambda, p_\lambda \rangle_\Delta}{\Delta_+(\rho + \lambda) \Delta_+(\rho + \lambda)} = \frac{\langle p_{\lambda + e_{1\ldots r}}, p_{\lambda + e_{1\ldots r}} \rangle_\Delta}{\Delta_+(\rho + \lambda + e_{1\ldots r}) \Delta_+(\rho + \lambda + e_{1\ldots r})}. \] (5.15)

It follows from Eq. (5.15) that the quotient \( \langle p_\lambda, p_\lambda \rangle_\Delta / (\tilde{\Delta}_+(\rho + \lambda) \tilde{\Delta}_+(\rho + \lambda)) \) does not depend on \( \lambda \in \Lambda \) (2.2) (cf. the proof of Theorem 5.1). Combining this with Gustafson’s result (5.8) for \( \lambda = 0 \) yields Eq. (5.9).

**Remarks:**

\( i. \) Both the evaluations for \( p_\lambda(\beta \rho^*) \) (Theorem 5.1) and \( \langle p_\lambda, p_\lambda \rangle_\Delta \) (Theorem 5.2) were conjectured by Macdonald in Ref. [M2] (without imposing the Self-duality condition (4.3)). The formula in Theorem 5.1 is sometimes referred to as the evaluation or specialization formula/conjecture, and the formula in Theorem 5.2 is also known as the inner product identity/conjecture. In the special case that \( \lambda = 0, \) the latter formula is also called the constant term formula (as it amounts to explicit computation of the constant term in the Fourier decomposition of \( \Delta(ix) \)).

\( ii. \) Equation (5.8) boils down to an explicit evaluation of the generalized Selberg-type integral

\[ \left( \frac{\alpha}{2\pi} \right)^n \prod_{j=1}^{\pi/\alpha} \Delta(ix) \, dx_1 \cdots dx_n = 2^n n! \tilde{\Delta}_+(\rho) \tilde{\Delta}_+(\rho). \] (5.16)

By cancelling common factors in the numerator and denominator, it is not difficult to rewrite the r.h.s. of Eq. (5.16) as

\[ 2^n n! \prod_{1 \leq j \leq n} \frac{(t; q)_\infty (a_0 a_1 a_2 a_3 t^{n+j-2}; q)_\infty}{(q; q)_\infty (t^j; q)_\infty \prod_{0 \leq r < r' \leq 3} (a_r a_{r'} t^{j-1}; q)_\infty}. \] (5.17)
where \( t = q^g, a_0 = q^{0_0}, a_1 = -q^{0_1}, a_2 = q^{0_2+1/2}, \) and \( a_3 = -q^{0_3+1/2} \) (cf. Eq. (2.6)). This is the expression for the evaluation constant found by Gustafson [G].

6. PROOF OF THE DUALITY THEOREM

In this section we will prove the Duality relations (4.4) (Theorem 4.2) by performing induction on \( \mu \). It is immediate from the definition of the (renormalized) Koornwinder-Macdonald polynomials that the duality relations hold for all \( \lambda \in \Lambda \) if \( \mu = 0 \), as in that case both sides of Eq. (4.4) are identical to one. Now, let \( \omega \in \Lambda \) be nonzero and let us assume as induction hypothesis that Eq (4.4) is valid for all \( \lambda \in \Lambda \) and all \( \mu \in \Lambda \) with \( \mu < \omega \). We shall prove that this implies that Eq (4.4) also holds for \( \mu = \omega \) (and all \( \lambda \in \Lambda \)).

Since \( \omega \in \Lambda \) is nonzero, there must exist an \( s \in \{1, \ldots, n-1\} \) such that \( \omega_s > \omega_{s+1} \geq 0 \). Hence, \( \nu \equiv \omega - e_{\{1, \ldots, s\}} \in \Lambda \) (recall \( e_{\{1, \ldots, s\}} \equiv e_1 + \cdots + e_s \)). The most important step in the duality proof consists of demonstrating (of course without relying on Theorem 4.2 or Theorem 4.4) that \( \tilde{p}_\nu(x) \) satisfies the Recurrence relation (4.7) for \( r = s \). The proof of this statement mimics the derivation presented in Section 4 of the recurrence relations (Theorem 4.4) starting from the duality relations (Theorem 4.2). At some points, however, it is necessary to adapt the arguments given there since the starting point now is the above induction hypothesis rather than the duality theorem.

As before (cf. Section 4), substituting \( x = \alpha(\rho + \nu) \) in the \( s \)-th difference equation for \( p^*_\lambda(x) \) entails (after renormalizing and restricting the sum with the aid of Lemma 4.3 to those index sets and sign configurations such that \( \nu + e_{\varepsilon J} \in \Lambda \))

\[
E^*_s(\rho^* + \lambda) \tilde{p}^*_\lambda(\rho + \nu) = \sum_{0 \leq |J| \leq s, \varepsilon_{j} = 1, j \in J, \nu + e_{\varepsilon J} \in \Lambda} U_{J^c, s-|J|}(\rho + \nu) V_{\varepsilon J, J}(\rho + \nu) \tilde{p}^*_\lambda(\rho + \nu + e_{\varepsilon J}).
\]

(6.1)

Let us assume for the moment that \( \lambda \leq \omega \). Then we may use the induction hypothesis to rewrite Eq. (6.1) as

\[
E^*_s(\rho^* + \lambda) \tilde{p}^*_\mu(\rho^* + \lambda) = \sum_{0 \leq |J| \leq s, \varepsilon_{j} = 1, j \in J, \nu + e_{\varepsilon J} \in \Lambda} U_{J^c, s-|J|}(\rho + \nu) V_{\varepsilon J, J}(\rho + \nu) \tilde{p}^*_\mu(\rho + \nu + e_{\varepsilon J}).
\]

(6.2)

To obtain Eq. (6.2) from Eq. (6.1), we have used for the l.h.s. that \( \tilde{p}^*_\lambda(\rho + \nu) = \tilde{p}^*_\nu(\rho^* + \lambda) \) because \( \nu(\equiv \omega - e_{\{1, \ldots, s\}}) < \omega, \) and for the r.h.s. that \( \tilde{p}^*_\lambda(\rho + \nu + e_{\varepsilon J}) = \tilde{p}^*_\nu(\rho^* + \lambda) \) because either 1. \( \lambda < \omega \), or 2. \( \nu + e_{\varepsilon J} < \omega \) (this happens when \( e_{\varepsilon J} \neq e_{\{1, \ldots, s\}} \)), or 3. \( \lambda = \nu + e_{\varepsilon J} = \omega \) (this happens when \( \lambda = \omega \) and \( e_{\varepsilon J} = e_{\{1, \ldots, s\}} \)). In the third case one has that \( \tilde{p}^*_\lambda(\rho + \omega) = \tilde{p}^*_\omega(\rho^* + \omega) \) trivially, because of the Self-duality assumption (4.3) (which implies that \( \tilde{p}^*_\omega(x) = \tilde{p}^*_\omega(x) \) and \( \rho^* = \rho \)). It is precisely at this point (and only at this point) that we actually use the Self-duality condition (cf. Remark 7.2 of Section 7).
So far we have shown that Eq. (1.2) holds for all \( \rho^* + \lambda \) with \( \lambda \leq \omega \). To see that equality actually holds for all values of the argument (and thus proving the \( s \)-th recurrence relation for \( \tilde{p}_\nu(x) \)), we need two lemmas.

**Lemma 6.1.** One has

\[
E_s^*(x) \tilde{p}_\nu(x) = \sum_{\mu \in \Lambda, \mu \leq \omega} c_\mu \tilde{p}_\mu(x) \quad \text{with} \quad c_\mu \in \mathbb{C}. \tag{6.3}
\]

**Proof.** It is clear from the definitions that \( E_s^*(x) \tilde{p}_\nu(x) \) is an even and permutation invariant trigonometric polynomial. Hence, it can be expanded as a finite linear combination of monomials \( \tilde{m}_\lambda(x) \), \( \lambda \in \Lambda \) (where \( \tilde{m}_\lambda(x) \equiv m_\lambda(\beta x)/m_\lambda(0) \), cf. Definition (4.2)). It follows from the asymptotics at infinity that in this expansion only monomials \( \tilde{m}_\lambda(x) \) with \( \lambda \leq \omega \) occur with a nonzero coefficient. To see this we set \( x = Ry \) with \( y_1 > y_2 > \cdots > y_n > 0 \) and notice that

\[
\tilde{m}_\lambda(Ry) \sim e^{\alpha R(\lambda, y)} \quad \text{for} \quad R \to \infty \tag{6.4}
\]

(where \( \langle \lambda, y \rangle \equiv \sum_{1 \leq j \leq n} \lambda_j y_j \) and \( \sim \) denotes proportionality). One furthermore has

\[
E_s^*(Ry) \sim e^{\alpha R(e_{\{1, \ldots, s\}}, y)}, \quad \tilde{p}_\nu(Ry) \sim e^{\alpha R(\nu, y)} \quad \text{for} \quad R \to \infty. \tag{6.5}
\]

In obtaining the asymptotics for \( \tilde{p}_\nu(Ry) \) we have used the fact that

\[
\lambda' \leq \lambda \quad \text{iff} \quad \langle \lambda', y \rangle \leq \langle \lambda, y \rangle \quad \forall y \in \mathbb{R}^n \quad \text{with} \quad y_1 > \cdots > y_n > 0. \tag{6.6}
\]

By comparing the asymptotics of \( E_s^*(Ry) \tilde{p}_\nu(Ry) \sim \exp(\alpha \beta R(\omega, y)) \) (recall \( \omega = \nu + e_{\{1, \ldots, s\}} \)) with that of \( \tilde{m}_\lambda(Ry) \) (6.4), one infers—using Eq. (6.6)—that

\[
E_s^*(x) \tilde{p}_\nu(x) = \sum_{\mu \in \Lambda, \mu \leq \omega} \tilde{c}_\mu \tilde{m}_\mu(x) \quad \text{with} \quad \tilde{c}_\mu \in \mathbb{C}. \tag{6.7}
\]

The lemma is now immediate from Eq. (6.7) and the fact that the transition matrix relating the bases \( \{\tilde{p}_\lambda(x)\}_{\lambda \in \Lambda} \) and \( \{\tilde{m}_\lambda(x)\}_{\lambda \in \Lambda} \) is upper triangular (\( \tilde{p}_\lambda(x) = \sum_{\lambda', \lambda \leq \omega} \tilde{c}_{\lambda', \lambda} \tilde{m}_{\lambda'}(x) \) and \( \tilde{m}_\lambda(x) = \sum_{\lambda', \lambda \leq \omega} \tilde{c}_{\lambda', \lambda} \tilde{p}_{\lambda'}(x) \)). \( \square \)

**Lemma 6.2.** For generic parameters \( g, g_0, \ldots, g_3 \) and scale factors \( \alpha, \beta \), the matrix \( [\tilde{p}_\mu(\rho^* + \lambda)]_{\mu, \lambda} \), with \( \mu, \lambda \in \Lambda \) and \( \mu, \lambda \leq \omega \), is invertible.

**Proof.** From the asymptotics \( \tilde{m}_\mu(\rho^* + \lambda) \sim \exp(\alpha \beta \langle \mu, \rho^* + \lambda \rangle) = q^{-\langle \mu, \rho^* + \lambda \rangle} \) for \( g^*(= g) \to \infty \) and the fact that \( \det[q^{-\langle \mu, \lambda \rangle}]_{\mu, \lambda \leq \omega} \neq 0 \) for generic \( q \) (see Remark at the end of this section), it follows that the determinant \( \det[\tilde{m}_\mu(\rho^* + \lambda)]_{\mu, \lambda \leq \omega} \) is not identically zero. The same is then true for \( \det[\tilde{p}_\mu(\rho^* + \lambda)]_{\mu, \lambda \leq \omega} \), as the transition matrix relating the bases \( \{\tilde{p}_\lambda(x)\}_{\lambda \in \Lambda} \) and \( \{\tilde{m}_\lambda(x)\}_{\lambda \in \Lambda} \) (and therefore also the matrix relating \( \det[\tilde{p}_\mu(\rho^* + \lambda)]_{\mu, \lambda \leq \omega} \) and \( \det[\tilde{m}_\mu(\rho^* + \lambda)]_{\mu, \lambda \leq \omega} \)) is (upper) triangular (cf. the proof of Lemma (6.1)). Thus, in view of the analyticity in the parameters and in the
scale factors $\alpha, \beta$, it is clear that $\det[\tilde{p}_\mu(\rho + \lambda)]_{\mu, \lambda \leq \omega} \neq 0$ generically, which proves the Lemma.

With these two lemmas we are finally in the position to show that Eq. (6.2) not only holds for $\rho^* + \lambda, \lambda \leq \omega$ but in fact for all values of the argument. Substituting $x = \rho^* + \lambda$ in Expansion (6.3) and comparing with Eq. (6.2), tells us—using Lemma 6.2—that the expansion coefficient $c_\mu$ equals $\tilde{U}_{j_\nu, s-|J|}(\rho + \nu) \tilde{V}_{\varepsilon J}(\rho + \nu)$ if $\mu = \nu + e_{\varepsilon J}$ (for some index set $J$ with $0 \leq |J| \leq s$ and configuration of signs $\varepsilon_j, j \in J$), and that $c_\mu = 0$ otherwise. It is clear that with such expansion coefficients $c_\mu$, Eq. (6.3) goes over in the $s$-th recurrence relation for $\tilde{p}_\nu(x)$:

$$E_s^*(x) \tilde{p}_\nu(x) = \sum_{0 \leq |J| \leq s, \varepsilon_j = \pm 1, j \in J} \tilde{U}_{j_\nu, s-|J|}(\rho + \nu) \tilde{V}_{\varepsilon J}(\rho + \nu) \tilde{p}_{\nu + e_{\varepsilon J}}(x).$$

(6.8)

It follows in particular from the Recurrence relation (6.8) that Eq. (6.2) is valid for all $\lambda \in \Lambda$ (and not just for $\lambda \leq \omega$). Furthermore, if we subtract Eq. (6.3) from Eq. (6.2), then all terms in the difference—except the leading term in the r.h.s. corresponding to the index set $J = \{1, \ldots, s\}$ with all signs $\varepsilon_j (j \in J)$ positive—manifestly cancel each other in view of the induction hypothesis (recall $\tilde{p}_s^*(\rho + \nu) = \tilde{p}_\nu(\rho^* + \lambda)$ because $\nu < \nu + e_{\{1, \ldots, s\}} = \omega$, and $\tilde{p}_s^*(\rho + \nu + e_{\varepsilon J}) = \tilde{p}_{\nu + e_{\varepsilon J}}(\rho^* + \lambda)$ for $e_{\varepsilon J} \neq e_{\{1, \ldots, s\}}$ because $\nu + e_{\varepsilon J} < \nu + e_{\{1, \ldots, s\}} = \omega$ if $e_{\varepsilon J} \neq e_{\{1, \ldots, s\}}$). Hence, we end up with the equation (recall $U_{K, 0} = 1$)

$$0 = V_{\{1, \ldots, s\}, \{s+1, \ldots, n\}}(\rho + \nu) (\tilde{p}_s^*(\rho + \omega) - \tilde{p}_\omega(\rho^* + \lambda)).$$

(6.9)

For generic parameters $V_{\{1, \ldots, s\}, \{s+1, \ldots, n\}}(\rho + \nu)$ is nonzero (Lemma 1.3), so we find that $\tilde{p}_s^*(\rho + \omega) = \tilde{p}_\omega(\rho^* + \lambda)$ for all $\lambda \in \Lambda$ (first for generic parameters and then for all parameters in view of the analyticity in the parameters).

This completes the induction step and thus the proof of Theorem 1.2.

Remark: In the proof of Lemma 6.2 we have used that $\det[q^{-\langle \mu, \nu \rangle}]_{\mu, \nu \leq \omega} \neq 0$ for generic $q$. This follows from a more general result due to Koornwinder [K1] (see also Ref. [M3, Ch. 6]) stating that for distinct vectors $v_1, \ldots, v_N \in \mathbb{R}^n$ (and generic $q$)

$$\det[q^{\langle v_j, v_k \rangle}]_{1 \leq j, k \leq N} = \sum_{\sigma \in S_N} (-1)^\sigma q^{\langle v_{\sigma(j)}, v_j \rangle} \neq 0.$$

(6.10)

To see that the terms $q^{\langle v_j, v_\sigma(j) \rangle}$ in Eq. (6.10) cannot all cancel each other one uses the estimate

$$\sum_{1 \leq j \leq N} \langle v_j, v_\sigma(j) \rangle \leq \sum_{1 \leq j \leq N} \langle v_j, v_j \rangle$$

(6.11)

(immediate from the Cauchy-Schwarz inequality (twice)), with equality holding only when $\sigma = \text{id}$. It follows from this estimate, first for $q \to \infty$ and hence for generic (say positive) $q$ by analyticity, that the determinant in Eq. (6.10) is indeed nonzero.
7. Concluding remarks

7.1. Structure of the difference equations. It has already been pointed out in Remark 3.1 of Section 3.2 that the structure of the Difference equations (3.2) is that of a system of eigenvalue equations for a family of commuting difference operators $D_r$ (3.3) ($r = 1, \ldots, n$). The highest-order part of $D_r$ is of the form (recall $U_{K,0} = 1$)

$$\sum_{J \subset \{1, \ldots, n\}, |J| = r} V_{\varepsilon J, J^c}(x) T_{\varepsilon J, \beta}. \quad (7.1)$$

The coefficients $V_{\varepsilon J, J^c}(x)$ (3.2) are related to the weight function $\Delta(x)$ (2.1) via (cf. Eq. (5.3))

$$V_{\{1, \ldots, r\}, \{r+1, \ldots, n\}}(x) = \Delta_+(x + \beta e_{\{1, \ldots, r\}})/\Delta_+(x), \quad (7.2)$$

where $\Delta_+(x) = \tilde{\Delta}_+(\beta x)$ with $g^*, g_r^*$ replaced by $g, g_r$ (so $\Delta(x) = \Delta_+(x)\Delta_+(-x)$).

The lower-order terms of $D_r$ (3.3) consist of a similar sum over the index sets $J \subset \{1, \ldots, n\}$ with cardinality smaller than $r$ and with each term $V_{\varepsilon J, J^c}(x) T_{\varepsilon J, \beta}$ being multiplied by the function $U_{\varepsilon J, r-|J|}(x)$.

Now, for $\lambda = 0$ the r.h.s. of the Difference equation (3.2) vanishes because $E_r(\rho) = 0$ [D1]. Equivalently, one could say that the difference operators $D_1, \ldots, D_n$ (3.3) annihilate constant functions. (For $D = D_1$ this is clear from Eq. (3.8).) This property of the difference operators/equations gives rise to the following set of relations between functions $V_{\varepsilon J, I}$ and $U_{K, p}$:

$$\sum_{J \subset \{1, \ldots, n\}, 0 \leq |J| \leq r} U_{\varepsilon J, r-|J|} V_{\varepsilon J, J^c} = 0, \quad r = 1, \ldots, n. \quad (7.3)$$

These functional relations (together with the condition that $U_{K, p} = 1$ for $p = 0$) actually determine $U_{K, p}$ completely in terms of $V_{\varepsilon J, I}$. For instance, for $r = 1$ Eq. (7.3) entails

$$U_{K,1} = -\sum_{k \in K} \left( V_{\{k\}, K \setminus \{k\}} + V_{-\{k\}, K \setminus \{k\}} \right)$$

$$= -\sum_{k \in K} \left( w(x_k) \prod_{k' \in K, k' \neq k} v(x_k + x_{k'}) v(x_k - x_{k'}) + w(-x_k) \prod_{k' \in K, k' \neq k} v(-x_k + x_{k'}) v(-x_k - x_{k'}) \right).$$

(Eq. (7.3) immediately yields $U_{K,1}$ for $K = \{1, \ldots, n\}$; the case of general $K$ then follows by renumbering.)
More generally, one obtains $U_{K,p}$ for general $p$ from Eq. (7.3) with $r = p$ by performing induction on $p$. This way one finds

$$U_{K,p} = \sum_{\emptyset \subseteq L_1 \subseteq \cdots \subseteq L_m \subseteq K, \ 1 \leq m \leq p} (-1)^m V_{\varepsilon_{L_1}}K \setminus L_1 V_{\varepsilon(L_2 \setminus L_1)}K \setminus L_2 \cdots V_{\varepsilon(L_m \setminus L_{m-1})}K \setminus L_m.$$  

It turns out that this expression for $U_{K,p}$ can be rewritten in the more compact form that was used in Eq. (3.2) \[D2, D3\]. The equality of both expressions for $U_{K,p}$ hinges on a system of functional equations for $v(z)$ satisfied by $v(z) = \text{sh} \frac{\alpha^2}{2}(\beta g + z)/\text{sh} \left( \frac{\alpha^2}{2}z \right)$.

### 7.2. Dropping the self-duality condition

In Section 3 we provided a proof of the duality relations for the Koornwinder-Macdonald polynomials (Theorem 4.2) with parameters subject to the Self-duality condition (4.3). This condition effectively reduces the number of independent parameters from five to four (not counting the scale factors $\alpha$ and $\beta$). It is expected (and conjectured by Macdonald [M2]), however, that the Duality relations (4.4) are true for the full five-parameter Koornwinder-Macdonald family (thus generalizing the state of affairs for $n = 1$). Should one succeed in proving the Duality theorem 4.2 without restrictions on the parameters, then automatically all other results of Sections 4 and 5 carry over to this slightly more general situation (with the proofs given applying verbatim).

A careful examination of the proof of Theorem 4.2 given in Section 3 reveals that the only step requiring invoking of the Self-duality condition (4.3) has been the derivation of Eq. (6.2) from Eq. (6.1) (with the aid of the induction hypothesis). At that point we needed that $\tilde{p}_\omega^*(\rho + \omega) = \tilde{p}_\omega(\rho^* + \omega)$, which is trivial for self-dual polynomials (because then $\tilde{p}_\omega^*(x) = \tilde{p}_\omega(x)$ and $\rho^* = \rho$), but which requires a proof when the self-duality condition is dropped. If one would be able to prove the relation $\tilde{p}_\omega^*(\rho + \omega) = \tilde{p}_\omega(\rho^* + \omega)$ ($\omega \in \Lambda$) for arbitrary parameters, then Theorem 4.2 (and thus all other theorems in Sections 4 and 5) would follow immediately for the complete five-parameter Koornwinder-Macdonald family.

There might be an alternative approach. If one chooses to define the renormalized Koornwinder-Macdonald polynomials $\tilde{p}_\lambda(x)$ as $p_\lambda(\beta x)$ divided by the constant in the r.h.s. of Eq. (5.5), then the derivation of Eq. (6.8) can be established starting from Eq. (6.1) by assuming $\lambda < \omega$ and applying the induction hypothesis to arrive at Eq. (6.2) (because $\lambda < \omega$ it is now not necessary to invoke the self-duality condition). Next we bring all terms in the r.h.s. of Eq. (6.2) with $|J| = s$ to the l.h.s. and employ a version of Lemma 6.1 and Lemma 7.2 with strict inequalities ($\mu, \lambda < \omega$) to arrive at Eq. (8.8). (By bringing the terms with $|J| = s$ to the other side one ensures that the resulting function in the l.h.s. can be expanded in polynomials $\tilde{p}_\mu(x)$ with $\mu < \omega$.) This proves the induction step. However, to check now that the duality relations hold for $\mu = 0$ amounts to proving Theorem 5.1. Hence, the upshot is that all results of the paper can be extended to arbitrary parameters once Eq. (5.5) (Theorem 5.1) has
been verified. Or, in other words, the Duality relations (4.4) and the Evaluation (or specialization) formula (5.5) follow from each other.

7.3. **Affine Hecke algebras.** In Ref. [C1], Cherednik proved Macdonald’s orthonormalization conjectures for the Macdonald polynomials related to reduced root systems by means of certain Hecke-algebraic techniques (leading to shift operators). More precisely, Cherednik considered the case of admissible pairs of the form \((R, R^\vee)\) with \(R\) reduced. Recently, these algebraic methods also resulted in a proof of the corresponding duality relations and renormalization formulas (i.e., the evaluation/specialization formulas, cf. Remark 1. of Section 5) [C2]. For the type \(A\) root system this approach reproduces (albeit in a completely different manner) the results of Koornwinder and Macdonald [K1, M3]. For the remaining classical reduced root systems (i.e., the types \(B, C,\) and \(D\)), the \((R, R^\vee)\)-type Macdonald polynomials amount to (self-dual) Koornwinder-Macdonald polynomials with special parameters \(g\) (type \(B\): \(g_0 = g_2, g_1, g_3 = 0\); type \(C\): \(g_0 = g_1, g_2, g_3 = 0\); type \(D\): \(g_0, g_1, g_2, g_3 = 0\)). (For the root systems \(B_n\) and \(D_n\) one obtains only half the polynomials via the above specialization of the parameters; the other half of the polynomials is obtained by specializing to parameters that are not self-dual, cf. Ref. [D1, Section 5].) It hence follows that for the above parameters Refs. [C1, C2] provide an alternative approach towards the proof of the Theorems 4.2, 5.1, and 5.2.

Meanwhile, both Noumi [N2] and Macdonald [M4] independently announced that Cherednik’s Hecke-algebraic techniques may be extended to the full five-parameter Koornwinder-Macdonald family. Hopefully this will eventually lead to an alternative proof of the Theorems 4.2, 5.1, and 5.2 valid for all parameters.

One aspect of the present paper seems hard to achieve via Hecke algebras, though. It does not seem very likely that such algebraic methods will independently reproduce our explicit formulas for the difference equations and the recurrence relations for the Koornwinder-Macdonald polynomials. At present Cherednik’s techniques permit concluding the existence of such difference equations and recurrence relations (of course for special parameters) without providing much clue as regards to their explicit form except in the simplest cases, e.g. when \(r = 1\) (cf. in this connection also the statements in second paragraph of the proof of Theorem 5.1 of Cherednik’s paper Ref. [C1]).

7.4. **Quantum groups.** Another algebraic framework in which the Koornwinder-Macdonald polynomials appear is the representation theory of (compact) quantum groups. Specifically, it has been demonstrated recently by Noumi and Sugitani that for certain special values of the parameters the Koornwinder-Macdonald polynomials may be interpreted as zonal spherical functions on compact quantum symmetric spaces of classical type [NS] (see also Ref. [N1] for a detailed treatment of the type \(A\) case).
In addition, it turned out that for the root system $A_{n-1}$ Macdonald's polynomials may also be realized as vector valued characters of $U_q(sl_n)$ [EK1]. This observation has led to yet another (representation-theoretic) proof of Koornwinder's duality and recurrence relations for the $A$-type Macdonald polynomials [EK2].

7.5. Integrable systems. It is possible to view the commuting difference operators $D_1, \ldots, D_n$ (3.5) as a complete set of quantum integrals for an integrable quantum mechanical $n$-particle model [D1, D2]. Similar integrable systems associated with and diagonalized by the Macdonald polynomials related to classical root systems are obtained via limit transitions (type $A$) or specialization of the parameters (type $B$, $C$, $D$, and $BC$) [D3]. For the type $A$ root systems the commuting difference operators (quantum integrals) of the model were already found independently by Ruijsenaars [R1] and Macdonald [M1]. In this special case, Ruijsenaars also studied in great detail the properties of the corresponding classical mechanical systems [R2]. It is interesting to note that also at the classical level duality relations, which were actually known even before their quantum counterparts were discovered, play a crucial role in solving the system.

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