Geometric quantization of topological gauge theories

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Abstract

We study the symplectic quantization of Abelian gauge theories in 2 + 1 space-time dimensions with the introduction of a topological Chern-Simons term.

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1. Introduction

Gauge theories developed in (2+1) spacetime dimensions have interesting and intriguing properties. This occurs due to the introduction of a topological term in the Lagrangian in order that the theory can be completely formulated. These terms are called Chern-Simons (CS) in the literature [1].

Among the above-mentioned features we refer to the generation of a (topological) mass for the gauge field [1,2] and the possibility of appearing exotic statistics when there is a coupling with matter fields [3]. More recently, the interest in CS theories has grown up by virtue of a Witten work [4] where it was shown the connection between three-dimensional topological field theories and conformal field theories in two dimensions.

On the other hand, one can say that quantization of CS theories is a fascinating subject by its own right. This occurs due to its peculiar structure of constraints. The canonical quantization in the non-covariant Weyl (temporal) and Coulomb gauges was first achieved by Deser et al. [5]. A manifest covariant canonical quantization was carried out more recently [6] and also the quantization by means of operator formalism [7]. The canonical quantization based on the Dirac Hamiltonian procedure [8] was developed by Lin and Ni [9], using the temporal gauge, and Martínez-Fernández and Wotzasek [10], using the Coulomb gauge.

The purpose of the present work is to use the symplectic formalism [11,12] to quantize this interesting constrained system [13,14]. We give the following organization to our paper: In Sec. 2 we deal with the pure CS theory. In Sec. 3 we consider Maxwell plus CS. Sec. 4 contains some concluding remarks.

2. Pure Abelian CS theory

The pure CS Lagrangian density is

\[ \mathcal{L} = \frac{k}{4\pi} \epsilon^{\mu\nu\rho} \partial_\mu A_\nu A_\rho \] (2.1)
where $\kappa$ is a dimensionless coupling constant. We adopt the following conventions $\eta_{\mu\nu} = \text{diag}(1,-1,-1)$ and $\epsilon_{012} = \epsilon^{012} = 1$.

Developing the CS Lagrangian we may write

$$\mathcal{L}^{(0)} = \frac{\kappa}{4\pi} \epsilon^{ij} A_j \dot{A}_i - V^{(0)}$$

where the superscript $^{(0)}$ means an initial Lagrangian, i.e., without the introduction of constraint terms. The quantity $\epsilon^{ij}$ is the projection of $\epsilon^{\mu\nu\rho}$ in the subspace spanned by space indices and

$$V^{(0)} = -\frac{\kappa}{2\pi} \epsilon^{ij} \partial_i A_j A_0$$

From expression (2.2) we identify the symplectic coefficients [11]

$$a^{(0)i}(\vec{x}, t) = \frac{\kappa}{4\pi} \epsilon^{ij} A_j(\vec{x}, t)$$

$$a^{(0)0}(\vec{x}, t) = 0$$

This permit us to calculate the matrix elements

$$f^{(0)ij}(\vec{x}, \vec{y}) = \frac{\delta a^{(0)j}(\vec{y})}{\delta A_i(\vec{x})} - \frac{\delta a^{(0)i}(\vec{x})}{\delta A_j(\vec{y})}$$

$$= -\frac{\kappa}{2\pi} \epsilon^{ij} \delta(\vec{x} - \vec{y})$$

$$f^{(0)0j}(\vec{x}, \vec{y}) = 0 = f^{(0)00}(\vec{x}, \vec{y})$$

Now and throughout it will be understood that all quantities are taken at the same time. The matrix $f^{(0)}$ reads

$$f^{(0)} = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{\kappa \epsilon^{ij}}{2\pi} \delta(\vec{x} - \vec{y}) \end{pmatrix}$$
which is obviously singular. This means that this system has constraints. Let us consider that the general form of the eigenvector with zero eigenvalue is [13,14]

\[ v^{(0)} = \begin{pmatrix} v_0^{(0)} \\ v_j^{(0)} \end{pmatrix} \] (2.7)

We notice that this is actually true if

\[ v_j^{(0)} = 0 \] (2.8)

Possible constraints are obtained from

\[ \int d^2 \vec{x} \left( v^{(0)0}(\vec{x}) \frac{\delta}{\delta A^0(\vec{x})} + v^{(0)i}(\vec{x}) \frac{\delta}{\delta A^i(\vec{x})} \right) \int d^2 \vec{y} V^{(0)} = 0 \] (2.9)

Considering the expression for \( V^{(0)} \) given by (2.3) and the condition given by eq. (2.8), expression (2.9) leads to

\[ \epsilon^{ij} \int d^2 \vec{x} v^{(0)0}(\vec{x}) \partial_i A_j(\vec{x}) = 0 \] (2.10)

Since \( v^{(0)0} \) is an arbitrary function of \( \vec{x} \) we conclude that

\[ \epsilon^{ij} \partial_i A_j(x) = 0 \] (2.11)

The next step is to introduce this constraint into the kinetic part of the Lagrangian [13, 14]. However, looking at (2.3) we notice that it is already in the potential part. Then, what we have to do is just to transpose it from the potential to the kinetic sector. This is directly done by making

\[ A_0 = \dot{\lambda} \] (2.12)

After this replacement the Lagrangian turns to be
$$\mathcal{L}^{(1)} = \frac{\kappa}{4\pi} \epsilon^{ij} A_j \dot{A}_i + \frac{\kappa}{2\pi} \epsilon^{ij} \partial_i A_j \dot{\lambda}$$  \hfill (2.13)$$

The potential $V^{(1)}$ is obviously zero. From the Lagrangian above one identifies the coefficients

$$a^{(1)i}(x) = \frac{\kappa}{4\pi} \epsilon^{ij} A_j(x)$$
$$a^{(1)\lambda}(x) = \frac{\kappa}{2\pi} \epsilon^{ij} \partial_i A_j(x)$$  \hfill (2.14)$$

This leads to the matrix $f^{(1)}$

$$f^{(1)} = \begin{pmatrix}
    f^{(1)ij} & f^{(1)i\lambda} \\
    f^{(1)i\lambda} & f^{(1)\lambda\lambda}
\end{pmatrix}
= \frac{\kappa}{2\pi} \begin{pmatrix}
    -\epsilon^{ij} & \epsilon^{ik} \partial_k \\
    \epsilon^{ik} \partial_k & 0
\end{pmatrix} \delta(\vec{x} - \vec{y})$$  \hfill (2.15)$$

Without explicit indication, partial derivatives will always be understood to be acting on the variable $\vec{x}$.

The matrix above is still singular (*). Further, there is no possibility to find out new constraints because as we have seen $V^{(1)} = 0$. Now is the point where the gauge condition has to be chosen. Let us first consider the temporal gauge

$$A_0(x) = 0$$  \hfill (2.16)$$

which means from (2.12) that $\lambda = \text{constant}$. We then introduce this new constraint into

(* In order to confirm this fact one can show that there actually exists a nontrivial eigenvector with zero eigenvalue. Considering that a general form of this eigenvector is $v^{(1)} = (v^{(1)}_j, v^{(1)}_\lambda)$, this will be a zero mode with the condition $v^{(1)}_i - \partial_i v^{(1)}_\lambda = 0$.}

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the kinetic part of the Lagrangian by means of a Lagrange multiplier $\eta$. The result is

$$\mathcal{L}^{(2)} = \frac{\kappa}{4\pi} \epsilon^{ij} A_j \dot{A}_i + \frac{\kappa}{2\pi} \left( \epsilon^{ij} \partial_i A_j + \eta \right) \dot{\lambda}$$

(2.17)

Thus, the new coefficients are

$$a^{(2)i} = \frac{\kappa}{4\pi} \epsilon^{ij} A_j$$
$$a^{(2)} \lambda = \frac{\kappa}{2\pi} \left( \epsilon^{ij} \partial_i A_j + \eta \right)$$
$$a^{(2)} \eta = 0$$

(2.18)

which leads to the matrix

$$f^{(2)} = \frac{\kappa}{2\pi} \begin{pmatrix} -\epsilon^{ij} & \epsilon^{ik} \partial_k & 0 \\ \epsilon^{jk} \partial_k & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \delta(\vec{x} - \vec{y})$$

(2.19)

where rows and columns follow the order $A^i$, $\lambda$ and $\eta$. The above matrix is not singular and, consequently, it can be identified as the symplectic tensor. Its inverse reads

$$f^{(2)^{-1}} = 2\pi \kappa \begin{pmatrix} \epsilon^{ij} & 0 & \partial_i \\ 0 & 0 & 1 \\ \partial_j & -1 & 0 \end{pmatrix} \delta(\vec{x} - \vec{y})$$

(2.20)

The Dirac brackets of the theory correspond to the elements of this inverse matrix [11, 13, 14]. We thus have

$$\{ A_i(\vec{x}), A_j(\vec{y}) \} = \frac{2\pi}{\kappa} \epsilon_{ij} \delta(\vec{x} - \vec{y})$$
$$\{ A_i(\vec{x}), \eta(\vec{y}) \} = \frac{2\pi}{\kappa} \partial_i \delta(\vec{x} - \vec{y})$$
$$\{ \lambda(\vec{x}), \eta(\vec{y}) \} = \frac{2\pi}{\kappa} \delta(\vec{x} - \vec{y})$$

(2.21)

The first bracket above was the same found Lin and Ni [9].
Let us next choose the Coulomb gauge

$$\partial_i A_i = 0 \quad (2.22)$$

Taking the time derivative of this constraint and introducing the result into the Lagrangian (2.13) by means of a Lagrange multiplier, we get

$$\mathcal{L}^{(2)} = \frac{\kappa}{2\pi} \left( \frac{1}{2} \epsilon^{ij} A_j + \partial^i \zeta \right) \dot{A}_i + \frac{\kappa}{2\pi} \epsilon^{ij} \partial_i A_j \dot{\lambda} \quad (2.23)$$

Following the same previous procedure we find the matrix

$$f^{(2)} = \frac{\kappa}{2\pi} \begin{pmatrix} -\epsilon^{ij} & \epsilon^{ik} \partial_k & -\partial^i \\ \epsilon^{jk} \partial_k & 0 & 0 \\ -\partial^j & 0 & 0 \end{pmatrix} \delta(\vec{x} - \vec{y}) \quad (2.24)$$

whose inverse is

$$f^{(2)}^{-1} = \frac{2\pi}{\kappa} \begin{pmatrix} 0 & -\epsilon_{jk} \partial^k & \partial_j \\ -\epsilon^{ik} \partial_k & 0 & \frac{1}{\nabla^2} \\ \partial_k & \frac{1}{\nabla^2} & 0 \end{pmatrix} \delta(\vec{x} - \vec{y}) \quad (2.25)$$

where rows and columns follow the order $A^i, \lambda$ and $\zeta$. From (2.25) we identify the nonvanishing brackets

$$\{A_i(\vec{x}), \lambda(\vec{y})\} = -\frac{2\pi \epsilon_{ik} \partial^k}{\kappa \nabla^2} \delta(\vec{x} - \vec{y})$$

$$\{A_i(\vec{x}), \zeta(\vec{y})\} = \frac{2\pi \partial_i}{\kappa \nabla^2} \delta(\vec{x} - \vec{y})$$

$$\{\lambda(\vec{x}), \zeta(\vec{y})\} = \frac{2\pi}{\kappa \nabla^2} \delta(\vec{x} - \vec{y}) \quad (2.26)$$

Now, the bracket $\{A_i(\vec{x}), A_j(\vec{y})\} = 0$ is zero. This result is in agreement with the one found in [10].
To conclude this section we mention that the use of the axial gauge, namely,

$$A_2 \approx 0 \quad (2.27)$$

gives

$$\{ A_i(\vec{x}), A_j(\vec{y}) \} = (\epsilon_{ij} - \epsilon_{ik}\delta^k_2 + \epsilon_{jk}\delta^k_2) \delta(\vec{x} - \vec{y})$$

$$= 0 \quad (2.28)$$

and the following ones involving Lagrange multipliers

$$\{ A_i(\vec{x}), \lambda(\vec{y}) \} = \frac{2\pi}{\kappa} \epsilon_{i2}\partial_2^{-1} \delta(\vec{x} - \vec{y})$$

$$\{ A_i(\vec{x}), \zeta(\vec{y}) \} = \frac{4\pi}{\kappa} \partial_i\partial_2^{-1} \delta(\vec{x} - \vec{y})$$

$$\{ \lambda(\vec{x}), \zeta(\vec{y}) \} = \frac{4\pi}{\kappa} \partial_2^{-1} \partial_2 \delta(\vec{x} - \vec{y}) \quad (2.29)$$

3. Maxwell plus CS

The initial Langrangian in this case is

$$\mathcal{L}^{(0)} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{\kappa}{4\pi} \epsilon^{\mu\nu\rho} \partial_\mu A_\nu A_\rho \quad (3.1)$$

Using the momentum $\pi^\mu$ conjugate to $A_\mu$ as an auxiliary field to linearize the Lagrangian, we get

$$\mathcal{L}^{(0)} = \pi^i \dot{A}_i - V^{(0)} \quad (3.2)$$

where
\[ V^{(0)} = -\frac{1}{2} \pi^i \pi_i - \frac{\kappa^2}{32\pi^2} A^i A_i + \left( \partial_i A_0 + \frac{\kappa}{4\pi} \epsilon_{ij} A^j \right) \pi^i + \frac{\kappa}{4\pi} \epsilon_{ij} \partial^i A^j + \frac{1}{2} \left( \epsilon^{ij} \partial_i A_j \right)^2 \]  

This permits us to obtain the matrix

\[ f^{(0)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \delta^{ij} \\ 0 & -\delta^{ij} & 0 \end{pmatrix} \delta(\vec{x} - \vec{y}) \]  

Here, rows and columns follow the order \( A_0, A_i \) and \( \pi_i \). This matrix is singular and the corresponding zero mode gives the constraint

\[ \partial^i \left( \pi_i + \frac{\kappa}{4\pi} \epsilon_{ij} A^j \right) \approx 0 \]  

Introducing it into the kinetic part of \( \mathcal{L}^{(0)} \) by means of a Lagrange multiplier we have

\[ \mathcal{L}^{(1)} = \left( \pi_i + \frac{\kappa}{4\pi} \epsilon_{ij} \partial^j \lambda \right) \dot{A}^i + \partial_i \lambda \dot{\pi}^i - V^{(1)} \]  

where

\[ V^{(1)} = -\frac{1}{2} \pi^i \pi_i - \frac{\kappa^2}{32\pi^2} A^i A_i - \frac{\kappa}{4\pi} \epsilon_{ij} A^j \pi^i - \frac{1}{2} \left( \epsilon^{ij} \partial_i A_j \right)^2 \]  

\( A_0 \) has been absorbed in \( \dot{\lambda} \). Now, the corresponding matrix is

\[ f^{(1)} = \begin{pmatrix} 0 & \delta^{ij} & \frac{\kappa}{4\pi} \epsilon^{ik} \partial_k \\ -\delta^{ij} & 0 & \partial^j \\ \frac{\kappa}{4\pi} \epsilon^{ik} \partial_k & \partial^j & 0 \end{pmatrix} \delta(\vec{x} - \vec{y}) \]  

which is also singular. However the zero modes do not generate new constraints.
As we have done in the previous section, let us fix the gauge. Following the same procedure as before we have that for the temporal gauge the result is

\[
\{ A_i(x), A_j(y) \} = 0 \\
\{ A_i(x), \pi_j(y) \} = -\delta_{ij} \delta(x - y)
\] (3.9)

Plus other brackets involving the Lagrange multipliers. The result above is also in agreement with the work of ref. [9]. Considering now the Coulomb gauge, the brackets just involving \( A_i \) and \( \pi_j \) are

\[
\{ A_i(x), A_j(y) \} = 0 \\
\{ A_i(x), \pi_j(y) \} = -\left( \delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2} \right) \delta(x - y)
\] (3.10)

that are also in agreement with the results found in [10]. Finally, for the axial gauge, the result is

\[
\{ A_i(x), A_j(y) \} = 0 \\
\{ A_i(x), \pi_j(y) \} = \left( -\delta_{ij} + \delta_{2j} \partial_i \partial_2^{-1} \right) \delta(x - y)
\] (3.11)

### 4. Quantization and propagators

Since there are no problem with ordering operators, all the previous Dirac brackets, obtained by means of the symplectic formalism, can be transformed to commutators by means of the usual rule \( \{ \text{Dirac brackets} \} \rightarrow -i\hbar [\text{commutators}] \). With these quantum relations one can calculate the propagators. We list below the results (we just consider propagators among gauge fields)
(i) CS and axial gauge

\[ G_{ij}(k) = 0 \] (4.1)

(ii) CS and temporal gauge

\[ G_{ij}(k) = i \frac{2\pi}{\kappa} \frac{\epsilon_{ij}}{k_0} \] (4.2)

(iii) CS and Coulomb gauge

\[ G_{ij}(k) = 0 \] (4.3)

(iv) Maxwell plus CS and axial gauge

\[ G_{11}(k) = \frac{1}{k^2 - \frac{\kappa^2}{4\pi^2}} \left( 1 + \frac{k_1 k_2}{k_0^2} \right) \] (4.4)

(v) Maxwell plus CS and temporal gauge

\[ G_{ij}(k) = \frac{1}{k^2 - \frac{\kappa^2}{4\pi^2}} \left( \delta_{ij} - \frac{k_i k_j}{k_0^2} - \frac{i\kappa k_0}{2\pi k^2} \left( \epsilon_{ij} - \frac{\epsilon_{il} k_l k_i}{k_0^2} + \frac{\epsilon_{jl} k_j k_i}{k_0^2} \right) \right) \] (4.5)

(vi) Maxwell plus CS and Coulomb gauge

\[ G_{ij} = \frac{1}{k^2 - \frac{\kappa^2}{4\pi^2}} \left( \delta_{ij} - \frac{k_i k_j}{k^2} - \frac{i\kappa k_0}{2\pi \left(k^2 - \frac{\kappa^2}{4\pi^2}\right)} \left( \epsilon_{ij} - \frac{\epsilon_{il} k_l k_i}{k} + \frac{\epsilon_{jl} k_j k_i}{k} \right) \right) \] (4.6)
4. Conclusion

We have studied the quantization of Abelian gauge theories in 2+1 space-time dimensions by using the symplectic formalism. These theories, developed in odd dimensions, have the inclusion of a topological term in order to be consistently defined. The presence of this term gives us an interesting constrained system where the symplectic method could be verified. The results we have obtained are in agreement with those one previously obtained by means of the standard Dirac procedure.

In this paper we have just used Abelian gauge fields. Extensions to nonabelian ones can be done in a straightforward way without any great difficulty and the results do not present any special feature that justify to be presented here.

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