Total Ordering Defined on the set of all Intuitionistic Fuzzy Numbers

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Abstract. L.A.Zadeh introduced the concept of fuzzy set theory as the generalisation of classical set theory in 1965 and further it has been generalised to intuitionistic fuzzy sets (IFSs) by Atanassov in 1983 to model information by the membership, non membership and hesitancy degree more accurately than the theory of fuzzy logic. The notions of intuitionistic fuzzy numbers in different contexts were studied in literature and applied in real life applications. Problems in different fields involving qualitative, quantitative and uncertain information can be modelled better using intuitionistic fuzzy numbers introduced in \cite{17} which generalises intuitionistic fuzzy values \cite{1,17}, interval valued intuitionistic fuzzy number (IVIFN) \cite{10} than with usual IFNs \cite{5,11,21}. Ranking of fuzzy numbers have started in early seventies in the last century and a complete ranking on the class of fuzzy numbers have achieved by W.Wang and Z.Wang only on 2014. A complete ranking on the class of IVIFNs, using axiomatic set of membership, non membership, vague and precise score functions has been introduced and studied by Geetha et al. \cite{10}. In this paper, a total ordering on the class of IFNs \cite{17} using double upper dense sequence in the interval \([0,1]\) which generalises the total ordering on fuzzy numbers (FNs) is proposed and illustrated with examples. Examples are given to show the proposed method on this type of IFN is better than existing methods and this paper will give the better understanding over this new type of IFNs.

Keywords: Double upper dense sequence, total order relation, intuitionistic fuzzy number, interval Valued intuitionistic fuzzy number, trapezoidal intuitionistic fuzzy number.

1. Introduction

Information system (IS) is a decision model used to select the best alternative from all the alternatives in hand under various attributes. The data collected from the experts may be incomplete or imprecise numerical quantities. To deal with such data, the theory of IFS provided by Atanassov \cite{1} aids better. In information system, dominance relation rely on ranking of data, ranking of intuitionistic fuzzy numbers is inevitable.

Many researchers have been working in the area of ranking of IFNs since last century. Different ranking methods for intuitionistic fuzzy values, interval valued intuitionistic fuzzy numbers have been studied in \cite{4,9,12,13,15,18,19,20,26,29,31,33,34}. But till date, there exists no single method or combination of methods available to rank any two arbitrary IFNs. The difficulty of defining total ordering on the class of intuitionistic fuzzy numbers is that there is no effective tool to identify an arbitrarily given intuitionistic fuzzy number by finitely many real-valued parameters. In this work, by establishing a new decomposition theorem for IFSs, any IFN can be identified by infinitely many but countable number of parameters. A new decomposition theorem for intuitionistic fuzzy sets is established by the use of an double upper dense sequence defined in \([0,1]\). Actually there are many methods for intuitionistic fuzzy values, interval valued intuitionistic fuzzy numbers have been studied in \cite{4,9,12,13,15,18,19,20,26,29,31,33,34}. But till date, there exists no single method or combination of methods available to rank any two arbitrary IFNs. The difficulty of defining total ordering on the class of intuitionistic fuzzy numbers is that there is no effective tool to identify an arbitrarily given intuitionistic fuzzy number by finitely many real-valued parameters. In this work, by establishing a new decomposition theorem for IFSs, any IFN can be identified by infinitely many but countable number of parameters. A new decomposition theorem for intuitionistic fuzzy sets is established by the use of an double upper dense sequence defined in \([0,1]\). Actually there are many
double upper dense sequences available in the interval
\([0, 1]\). Since the choice of a double upper dense sequence is
considered as the necessary reference systems for defining a complete ranking, infinitely
total orderings on the set of all IFNs can be
defined based on each choice of double upper
dense sequence. After introduction, some necessary
fundamental knowledge on ordering and intuitionistic
fuzzy numbers, interval valued intuitionistic fuzzy
numbers is introduced in Section 2. In Section 3, a new
decomposition theorem for intuitionistic fuzzy sets is
establised using double upper dense sequence defined
in the interval \([0, 1]\). Section 4 is used to define total
ordering on the set of all intuitionistic fuzzy numbers
by using double upper dense sequence in the interval
\([0, 1]\). Several examples are given in Section 5 to show
how the total ordering on IFNs can be used for ranking
better than some other existing methods. Application
of our proposed method in solving intuitionistic fuzzy
information system problem is shown in section 5 by
developing a new algorithm. Finally conclusions are
given in section 6.

1.1. Motivation

The capacity to handle dubious and uncertain data
is more effectively done by stretching out intuitionistic
fuzzy values to TrIFNs because the membership and
non-membership degrees are better expressed as
trapezoidal values rather than exact values. TrIFNs
are generalisation of intuitionistic fuzzy values and
IVIFNs. As a generalisation, the set of TrIFNs should
contain the set of all intuitionistic fuzzy values and
IVIFNs. But the existing definition for TrIFNs [11,21]
does not contain the set of intuitionistic fuzzy values
which means the existing definition for TrIFN
is not the real generalisation of intuitionistic fuzzy
values. Hence the study about new structure for
intuitionistic fuzzy number [17] is essential. In the
application point of view our proposed method is
more applicable and more natural when it is compared
with the existing methodology. More precisely, the
existing definition for Trapezoidal intuitionistic fuzzy
numbers (TrIFN) present in the literature [11,21]
is defined as \(A = \langle (a, b, c, d)(e, f, g, h) \rangle\) with \(e \leq a \leq f \leq b \leq c \leq g \leq d \leq h\) does not
generalise even the intuitionistic fuzzy value of the
kind \(A = \langle a, c \rangle\) with \(a + c \leq 1\) and \(a < c\). That
is, if we write \(A = \langle a, c \rangle\) in trapezoidal intuitionistic
form, we get \(A = \langle (a, a, a, a)(c, c, c, c) \rangle\) with \(a < c\)
which contradicts the above definition for TrIFNs. So
till today the real generalization of intuitionistic fuzzy
values and interval valued intuitionistic fuzzy numbers
have not been studied in detail. This problem motivate
us to study this type of intuitionistic fuzzy numbers
[17] and its ordering principles for ranking.

2. Preliminaries

Here we give a brief review of some preliminaries.

Definition 2.0.1. (Atanassov [1]). Let \(X\) be a nonempty
set. An intuitionistic fuzzy set (IFS) \(A\) in \(X\) is defined
by \(A = (\mu_A, \nu_A)\), where \(\mu_A : X \to [0, 1]\) and
\(\nu_A : X \to [0, 1]\) with the conditions \(0 \leq \mu_A(x) + \nu_A(x) \leq 1, \forall x \in X\). The numbers \(\mu_A(x), \nu_A(x) \in [0, 1]\)
denote the degree of membership and non-membership
of \(x\) to lie in \(A\) respectively. For each intuitionistic fuzzy
subset \(A\) in \(X\), \(\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)\) is called
hesitancy degree of \(x\) to lie in \(A\).

Definition 2.0.2. (Atanassov & Gargov, [2]). Let
\(D[0, 1]\) be the set of all closed subintervals of the
interval \([0, 1]\). An interval valued intuitionistic fuzzy
set on a set \(X \neq \phi\) is an expression given by \(A = \{ (x, \mu_A(x), \nu_A(x)) : x \in X \}\) where
\(\mu_A : X \to D[0, 1], \nu_A : X \to D[0, 1]\) with the
condition \(0 < \sup_x \mu_A(x) + \sup_x \nu_A(x) \leq 1\).

The intervals \(\mu_A(x)\) and \(\nu_A(x)\) denote, respectively,
the degree of belongingness and non-belongingness
of the element \(x\) to the set \(A\). Thus for each \(x \in X\),
\(\mu_A(x)\) and \(\nu_A(x)\) are closed intervals whose lower
and upper end points are, respectively, denoted by
\(\mu_{A_L}(x), \mu_{A_U}(x)\) and \(\nu_{A_L}(x), \nu_{A_U}(x)\). We denote
\(A = \{ (x, \mu_{A_L}(x), \mu_{A_U}(x), \nu_{A_L}(x), \nu_{A_U}(x)) : x \in X \}\) where \(0 < \mu_{A_L}(x) + \nu_{A_U}(x) \leq 1\).

For each element \(x \in X\), we can compute the
unknown degree (hesitance degree) of belongingness
\(\pi_A(x)\) to \(A\) as \(\pi_A(x) = 1 - \mu_A(x) - \nu_A(x) = 1 - \mu_{A_L}(x) - \nu_{A_U}(x) - (\mu_{A_U}(x) - \nu_{A_L}(x))\). We denote
the set of all IVIFSs in \(X\) by IVIFS(\(X\)). An IVIF value
is denoted by \(A = \langle [a, b], [c, d] \rangle\) for convenience.

Definition 2.0.3. (Atanassov & Gargov, [2]). The
complement \(A^c\) of \(A = \langle x, \mu_A(x), \nu_A(x) : x \in X \rangle\) is
given by \(A^c = \langle x, \nu_A(x), \mu_A(x) : x \in X \rangle\).

Definition 2.0.4. (Lakshmana et.al [17]). An intuitionistic
fuzzy set (IFS) \(A = (\mu_A, \nu_A)\) of \(R\) is said to
be an intuitionistic fuzzy number if \(\mu_A\) and \(\nu_A\) are
fuzzy numbers. Hence \(A = (\mu_A, \nu_A)\) denotes an
intuitionistic fuzzy number if \(\mu_A\) and \(\nu_A\) are fuzzy
numbers with \(\nu_A \leq \mu_A\), where \(\mu_A\) denotes the
complement of \(\mu_A\).
An intuitionistic fuzzy number
\[ A = \{ (a, b_1, b_2, c), (e, f_1, f_2, g) \} \] with \( (e, f_1, f_2, g) \leq (a, b_1, b_2, c) \) is shown in fig(1).

**Definition 2.0.5.** (Lakshmana et al. [17]) A trapezoidal intuitionistic fuzzy number \( A \) is defined by
\[ A = \{ (\mu_A(x), \nu_A(x)) \mid x \in \mathbb{R} \}, \]
where \( \mu_A \) and \( \nu_A \) are trapezoidal fuzzy numbers with \( \mu_A(x) \leq \nu_A(x) \).

We note that the condition \( (e, f_1, f_2, g) \leq (a, b_1, b_2, c) \) of the trapezoidal intuitionistic fuzzy number \( A = \{ (a, b_1, b_2, c), (e, f_1, f_2, g) \} \) where \( (a, b_1, b_2, c) \) and \( (e, f_1, f_2, g) \) are membership and nonmembership fuzzy numbers of \( A \) with either \( e \geq b_2 \) and \( f_1 \geq c \) or \( f_2 \leq a \) and \( g \leq b_1 \) on the legs of trapezoidal intuitionistic fuzzy number.

A trapezoidal intuitionistic fuzzy number \( A = \{ (a, b_1, b_2, c), (e, f_1, f_2, g) \} \) with \( e \geq b_2 \) and \( f_1 \geq c \) is shown in fig(2).

**Definition 2.0.6.** (Lakshmana et al. [16]) For any IVIFN \( A = \{ (a, b], [c, d] \}, \) the membership score function is defined as
\[ L(A) = \frac{a+b-c-d+ac+bd}{2}. \]

**Definition 2.0.7.** (Geetha et al. [10]) For any IVIFN \( A = \{ (a, b], [c, d] \}, \) the imprecise score function is defined as
\[ IP(A) = \frac{a-b-c+d-ac+bd}{2}. \]

**Definition 2.0.8.** (Geetha et al. [10]) For any IVIFN \( A = \{ (a, b], [c, d] \}, \) the vague score function is defined as
\[ V(A) = \frac{a-b+c+d-ac+bd}{2}. \]

**Definition 2.0.9.** (Geetha et al. [10]) For any IVIFN \( A = \{ (a, b], [c, d] \}, \) the imprecise score function is defined as
\[ IP(A) = \frac{a+b-c-d+ac+bd}{2}. \]

**Definition 2.0.10.** Let \( X \) be a non-empty set. Any subset of the cartesian product \( X \times X \) is called a relation, denoted by \( R \), on \( X \). We write \( aRb \) iff \( (a, b) \in R \). A relation \( R \) is called a partial ordering on \( X \) if it is reflexive, antisymmetric, and transitive. A partial ordering \( R \) on \( X \) is called total ordering if either \( aRb \) or \( bRa \) for any \( a, b \in X \). Two total orderings \( R_1 \) and \( R_2 \) are different iff there exist \( a, b \in X \) with \( a \neq b \) such that \( aR_1b \) but \( bR_2a \) or \( aR_2b \) but \( bR_1a \). For any given total ordered infinite set, there are infinitely many ways to redefine a new total ordering on it. A relation \( R \) is called an equivalence relation if it is reflexive, symmetric and transitive.

3. **Total Ordering defined on Intuitionistic Fuzzy Number**

Geetha et.al [10] have achieved the total ordering on the set of IVIFN using membership, nonmembership, vague and precise score functions. Let us recall the total ordering defined by Geetha et.al [10] on IVIFN.

Let \( A = [a_1, b_1], [c_1, d_1], B = [a_2, b_2], [c_2, d_2] \) be two IVIFNs. The total ordering \( \leq \) on IVIFN may be defined to one of the following criteria:

1. \( L(A) < L(B), \)
2. \( L(A) = L(B) \) but \( -L(A) < -L(B) \),
3. \( L(A) < L(B) \) and \( LG(A) = LG(B) \) but \( P(A) < P(B) \),
4. \( L(A) = L(B) \), \( LG(A) = LG(B) \) and \( P(A) = P(B) \) but \( -IP(A) \leq -IP(B) \).

The above way of defining a total ordering is often referred to as lexicographic in literature [8].

In this section a new total ordering is defined on the set of all intuitionistic fuzzy numbers using the above ordering and a new decomposition theorem on intuitionistic fuzzy sets which is given in the following subsection.

3.1. **Upper Dense sequence in (0, 1]**

The major difficulty in defining total ordering on the set of all fuzzy numbers is that there is no
effectivenot to identify an arbitrarily given fuzzy number by only finitely many real-valued parameters. To overcome this difficulty, W.Wang, Z.Wang [27] has introduced the concept of upper dense sequence in the interval (0, 1), and defined the total ordering on the set of all fuzzy numbers using this sequence. This upper dense sequence gives values for α in the α-cut of FNs. But for IFNs, two sequences are needed to give values for α & β in the (α, β)-cut where α ∈ (0, 1) & β ∈ [0, 1]. For convenience upper dense sequence of α is denoted by S₁ = {α; i = 1, 2,...} and upper dense sequence of β is denoted by S₂ = {β; i = 1, 2,...}. In this section, the upper dense sequence in (0, 1] is briefly reviewed.

**Definition 3.1.1.** (W.Wang, Z.Wang [27]) A sequence S = {α; i = 1, 2,...} ⊂ (0, 1] is said to be upper dense in (0, 1] if, for every point x ∈ (0, 1] and ε > 0, there exists some α ∈ S such that α ∈ [x, x + ε) for some i. A sequence S ⊂ (0, 1] is said to be lower dense in (0, 1] if, for every point x ∈ (0, 1] and any ε > 0, there exists α ∈ S such that α ∈ (x - ε, x) for some i.

From definition 3.1.1, we note that, any upper dense sequence in (0, 1] is nothing but a dense sequence with real number 1 and it is also a lower dense sequence.

**Definition 3.1.2.** Let S₁ = {α; i = 1, 2,...} and S₂ = {β; i = 1, 2,...} be two upper dense sequences in (0, 1] then the double upper dense sequence is defined as S = (S₁, S₂) = {(α, β); i = 1, 2,...}.

Example for an upper dense sequence and double upper dense sequence in (0, 1] is given as follows.

**Example 3.1.1.** Let S₁ = {α; i = 1, 2,...} be the set of all rational numbers in (0, 1], where α₁ = 1, α₂ = \(\frac{1}{2}\), α₃ = \(\frac{3}{4}\), α₄ = \(\frac{5}{4}\), α₅ = \(\frac{1}{4}\), α₆ = \(\frac{1}{4}\), α₇ = \(\frac{1}{4}\), α₈ = \(\frac{1}{4}\), α₉ = \(\frac{1}{4}\), α₁₀ = \(\frac{1}{4}\), ....

Then sequence S₁ is an upper dense sequence in (0, 1].

If we allow a number to have multiple occurrences in the sequence, the general members in upper dense sequence S₀ = {s₀; i = 1, 2,...} can be expressed by s₀; i = 1, 2,... such that k = \(\sqrt{2i + \frac{1}{2}}\). That is, s₀₁ = 1, s₀₂ = \(\frac{1}{2}\), s₀₃ = \(\frac{1}{3}\), s₀₄ = \(\frac{1}{3}\), s₀₅ = \(\frac{1}{5}\), s₀₆ = \(\frac{1}{6}\), s₀₇ = \(\frac{1}{7}\), s₀₈ = \(\frac{1}{8}\), s₀₉ = \(\frac{1}{9}\), s₀₁₀ = \(\frac{1}{10}\), .... In sequence S₀, for instance, s₀₃ is the same real number as s₀₅.

**Example 3.1.2.** Consider S₁ as in example 3.1.1. Let S = (S₁, S₂) = {(α, β); α = β, α₀ ∈ S₁} = {(1, 1), (1/2, 1/2), (1/3, 1/3), (2/3, 2/3), (1/4, 1/4), (3/4, 3/4), (1/5, 1/5), ...}. Clearly S is a double upper dense sequence in (0, 1].

In the forthcoming sections, this double upper dense sequence will be very useful for defining total orderings on the set of IFNs.

**3.2. Decomposition theorem for intuitionistic fuzzy number using upper dense sequence**

In this section, a new decomposition theorem for IFNs is established using the double upper dense sequence defined in (0, 1]. Before establishing a new decomposition theorem for intuitionistic fuzzy sets, it is needed for us to define decomposition theorems for intuitionistic fuzzy sets using special α - β cuts.

**Definition 3.2.1.** Let X be a nonempty universal set and A = (µ_A, ν_A) be an intuitionistic fuzzy set of X with membership function µ_A and a nonmembership function ν_A. Let α, β ∈ (0, 1] such that:

1. The α - β cut, denoted by \(α^{-β}A\) is defined by \(α^{-β}A = α^{-β}μ_A × β^{-ν}_A\) where \(α^{-β}μ_A = \{x ∈ X | μ_A(x) ≥ α\} and β^{-ν}_A = \{x ∈ X | ν_A(x) ≥ β\}. Equivalently \(α^{-β}A = μ_A^{-1}((α, 1)) × ν_A^{-1}((β, 1))\) is a subset of \(ψ(X) × ψ(X)\).

2. The strong alpha beta cut, denoted by \(α^{+β}A\) is defined by \(α^{+β}A = α^{+β}μ_A × β^{+ν}_A\) where \(α^{+β}μ_A = \{x ∈ X | μ_A(x) > α\} and β^{+ν}_A = \{x ∈ X | ν_A(x) > β\}. Equivalently \(α^{+β}A = μ_A^{-1}((α, 1)) × ν_A^{-1}((β, 1))\) is a subset of \(ψ(X) × ψ(X)\).

3. The level set of A, denoted by L(A) is defined by L(A) = \(L_{μ_A} × L_{ν_A}\) where \(L_{μ_A} = \{α μ_A(x) = α, x ∈ X\}\) and \(L_{ν_A} = \{β ν_A(x) = β, x ∈ X\}\) which is a subset of \(ψ([0, 1]) × ψ([0, 1])\).

**Example 3.2.1.** Let A = ((0.17, 0.3, 0.47, 0.56), (0.05, 0.13, 0.16, 0.23)) be a trapezoidal intuitionistic fuzzy number. Then \((α^{-β})A = (0.17 + (0.3 - 0.17)α, 0.56 - (0.56 - 0.47)α), [0.05 + (0.13 - 0.05)β, 0.23 - (0.23 - 0.16)β] = (0.17 + 0.13α, 0.56 - 0.09α), [0.05 + 0.08β, 0.23 - 0.07β]\), \(∀α, β ∈ [0, 1]\).

Let \(α = 0.17 + (0.3 - 0.17)α, 0.56 - (0.56 - 0.47)α\), \(0.05 + (0.13 - 0.05)β, 0.23 - (0.23 - 0.16)β\) = (0.17 + 0.13α, 0.56 - 0.09α), (0.05 + 0.08β, 0.23 - 0.07β), \(∀α, β ∈ [0, 1]\).
One way of representing a fuzzy set is by special fuzzy sets on \( \alpha \)-cuts and another way of representing a fuzzy set is by special fuzzy sets on Strong \( \alpha \)-cuts. As a generalisation of fuzzy sets, any intuitionistic fuzzy set can also be represented by the use of special intuitionistic fuzzy sets on \( \alpha - \beta \) cuts and special intuitionistic fuzzy sets on strong \( \alpha - \beta \) cuts.

**Definition 3.2.2.** The special intuitionistic fuzzy set \( (\alpha, \beta)A = (\alpha \mu_A, \beta \nu_A) \) is defined by its membership \( \alpha \mu_A \) and non-membership function \( \beta \nu_A \) as follows,

\[
\alpha \mu_A(x) = \begin{cases} 
\alpha, & x \in \alpha A \\
0, & \text{otherwise} 
\end{cases}
\]

\[
\beta \nu_A(x) = \begin{cases} 
\beta, & x \in \beta A \\
0, & \text{otherwise} 
\end{cases}
\]

The following decomposition theorems will show the representation of an arbitrary IFS in terms of the special IFSs \((\alpha, \beta)A\).

**Theorem 3.1. First Decomposition theorem of an IFS:** Let \( X \) be a non-empty set. For an intuitionistic fuzzy subset \( A = (\mu_A, \nu_A) \) in \( X \),

\[
A = \left( \bigcup_{\alpha \in [0,1]} \alpha \mu_A, \bigcup_{\beta \in [0,1]} \beta \nu_A \right), \text{ where } \bigcup \text{ is standard union.}
\]

**Proof.** Let \( x \) be an arbitrary element in \( X \) and let \( \mu_A(x) = a \& \nu_A(x) = b \). Then

\[
\left( \left( \bigcup_{\alpha \in [0,1]} \alpha \mu_A \right)(x), \left( \bigcup_{\beta \in [0,1]} \beta \nu_A \right)(x) \right) = \left( \text{Sup}_{\alpha \in [0,1]} \alpha \mu_A(x), \text{Sup}_{\beta \in [0,1]} \beta \nu_A(x) \right) = \left( \max \left[ \text{Sup}_{\alpha \in [0,a]} \alpha \mu_A(x), \text{Sup}_{\beta \in [0,b]} \beta \nu_A(x) \right] \right).
\]

For each \( \alpha \in [0,1] \), we have \( \mu_A(x) = a \geq \alpha \), therefore \( \alpha \mu_A(x) = \alpha \). On the other hand, for each \( \alpha \in (a,1] \), we have \( \mu_A(x) = a < \alpha \) and \( \alpha \mu_A(x) = 0 \). Similarly, for each \( \beta \in (0,b] \), we have \( \nu_A(x) = b \geq \beta \), therefore \( \beta \nu_A(x) = \beta \). On the other hand, for each \( \beta \in (b,1] \), we have \( \nu_A(x) = b < \beta \) and \( \beta \nu_A(x) = 0 \).

Therefore,

\[
\left( \left( \bigcup_{\alpha \in [0,1]} \alpha \mu_A \right)(x), \left( \bigcup_{\beta \in [0,1]} \beta \nu_A \right)(x) \right) = \left( \text{Sup}_{\alpha \in [0,a]} \alpha \mu_A(x), \text{Sup}_{\beta \in [0,b]} \beta \nu_A(x) \right) = \left( \mu_A(x), \nu_A(x) \right).
\]

Hence the theorem.

To illustrate the above theorem, let us consider a trapezoidal intuitionistic fuzzy number \( A \) as in figure 3.

For each \( \alpha, \beta \in [0,1] \), the \( \alpha - \beta \) cut of \( A = \{(a_1, b_1, c_1), (e_1, f_1, g_1)\} \) is given by \( (\alpha - \beta)A = \{(a + (b_1 - a)\alpha, c - (c - b_2)\alpha), (e + (f_1 - e)\beta, g - (g - f_2)\beta)\} \) and the special intuitionistic fuzzy set \( (\alpha, \beta)A \) employed in definition 3.2.2 is defined by its membership \( \alpha \mu_A \) and non-membership function \( \beta \nu_A \) as follows:

\[
\alpha \mu_A(x) = \begin{cases} 
\alpha, & x \in [a + (b_1 - a)\alpha, c - (c - b_2)\alpha] \\
0, & \text{otherwise} 
\end{cases}
\]

\[
\beta \nu_A(x) = \begin{cases} 
\beta, & x \in [e + (f_1 - e)\beta, g - (g - f_2)\beta] \\
0, & \text{otherwise} 
\end{cases}
\]

Examples of sets \( \alpha \mu_A, \beta \nu_A, \alpha \mu_A \) and \( \beta \nu_A \) for three values of \( \alpha \) (namely \( \alpha_1, \alpha_2, \alpha_3 \)) and \( \beta \) (namely \( \beta_1, \beta_2, \beta_3 \)) are shown in figure 3.

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{figure3.png}
\caption{Illustration of First Decomposition Theorem}
\end{figure}

According to theorem 3.1, \( A \) is obtained by taking the standard fuzzy union of sets \( (\alpha \mu_A, \beta \nu_A) \) for all \( \alpha, \beta \in [0,1] \).

**Theorem 3.2. Second Decomposition theorem of an IFS:** Let \( X \) be a non-empty set. For an intuitionistic fuzzy subset \( A \) in \( X \),

\[
A = \left( \bigcup_{\alpha \in [0,1]} \alpha + \mu_A, \bigcup_{\beta \in [0,1]} \beta + \nu_A \right), \text{ where } \bigcup \text{ is standard union.}
\]

**Proof.** Let \( x \) be an arbitrary element in \( X \) and let \( \mu_A(x) = a \& \nu_A(x) = b \). Then

\[
\left( \left( \bigcup_{\alpha \in [0,1]} \alpha + \mu_A \right)(x), \left( \bigcup_{\beta \in [0,1]} \beta + \nu_A \right)(x) \right) = \left( \text{Sup}_{\alpha \in [0,1]} \alpha + \mu_A(x), \text{Sup}_{\beta \in [0,1]} \beta + \nu_A(x) \right) = \left( \max \left[ \text{Sup}_{\alpha \in [0,a]} \alpha + \mu_A(x), \text{Sup}_{\beta \in [0,b]} \beta + \nu_A(x) \right] \right).
\]

\[
\left( \left( \bigcup_{\alpha \in [0,1]} \alpha + \mu_A \right)(x), \left( \bigcup_{\beta \in [0,1]} \beta + \nu_A \right)(x) \right) = \left( \text{Sup}_{\alpha \in [0,1]} \alpha + \mu_A(x), \text{Sup}_{\beta \in [0,1]} \beta + \nu_A(x) \right) = \left( \alpha + \mu_A(x), \beta + \nu_A(x) \right).
\]

max \( \left[ \text{Sup}_{\beta \in [0,b]} \beta + \nu_A(x), \text{Sup}_{\alpha \in [0,a]} \alpha + \mu_A(x) \right] \).

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{figure4.png}
\caption{Illustration of Second Decomposition Theorem}
\end{figure}
For each $\alpha \in [0, a)$, we have $\mu_A(x) = a > \alpha$, therefore $\alpha + \mu_A(x) = \alpha$. On the other hand, for each $\alpha \in [a, 1]$, we have $\mu_A(x) = \alpha \leq \alpha$ and $\alpha + \mu_A(x) = 0$.

Similarly, for each $\beta \in [0, b)$, we have $\nu_A(x) = b > \beta$, therefore $\beta + \nu_A(x) = \beta$. On the other hand, for each $\beta \in [b, 1]$, we have $\nu_A(x) = b \leq \beta$ and $\beta + \nu_A(x) = 0$.

Therefore \( \left( \bigcup_{\alpha \in [0,1]} (\alpha + \mu_A(x)) \right), \left( \bigcup_{\beta \in [0,1]} (\beta + \nu_A(x)) \right) = (\sup_{\alpha \in [0,a]} (\alpha + \mu_A(x)), \sup_{\beta \in [0,b]} (\beta + \nu_A(x))) = (a, b) = (\mu_A(x), \nu_A(x)) \). Hence the theorem.

**Theorem 3.3. Third Decomposition theorem of an IFS:** Let $X$ be a non-empty set. For an intuitionistic fuzzy subset $A$ in $X$,

\[
A = \left( \bigcup_{\alpha \in L_A} \alpha \mu_A, \bigcup_{\beta \in L_\nu} \beta \nu_A \right)
\]

where $\bigcup$ is standard union.

**Proof.** The proof of this theorem is similar to the **Theorem 5.2**.

Regarding IFN as special intuitionistic fuzzy subset (IFS) of $\mathbb{R}$, these decomposition theorems are also available for IFNs. Since they identify an intuitionistic fuzzy number by uncountably infinite real valued parameters generally and therefore lexicography cannot be used anymore, unfortunately none of them can be used to define a total ordering on the set of IFNs. Thus, establishing a new decomposition theorem, which identifies any IFN by only countably many real valued parameters is essential.

**Theorem 3.4. Fourth Decomposition theorem of an IFS:** Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subset of $X$, and $S = (S_1, S_2)$ be a given double upper dense sequence in $[0, 1]$. Then $A = \left( \bigcup_{\alpha \in S_1} \alpha \mu_A, \bigcup_{\beta \in S_2} \beta \nu_A \right)$.

**Proof.** Let $x$ be an arbitrary element in $X$ and let $\mu_A(x) = a \& \nu_A(x) = b$. Since $S_1 \subseteq [0,1]$ & $S_2 \subseteq [0,1]$, thus we have $\bigcup_{\alpha \in S_1} \alpha \mu_A \subseteq \bigcup_{\alpha \in [0,1]} \alpha \mu_A = \mu_A$ and $\bigcup_{\beta \in S_2} \beta \nu_A \subseteq \bigcup_{\beta \in [0,1]} \beta \nu_A = \nu_A$ .

So $\left( \bigcup_{\alpha \in S_1} \alpha \mu_A, \bigcup_{\beta \in S_2} \beta \nu_A \right) \subseteq A$ \hspace{1cm} (1)

Now we have to show that $\mu_A(x) \leq \bigcup_{\alpha \in S_1} \alpha \mu_A(x)$ and $\nu_A(x) \leq \bigcup_{\beta \in S_2} \beta \nu_A(x)$ to prove the theorem. From second decomposition theorem we know that, for each $x \in X$, $(\mu_A(x), \nu_A(x)) = (\sup_{\alpha \in [0,1]} (\alpha + \mu_A(x)), \sup_{\beta \in [0,1]} (\beta + \nu_A(x))) = (\sup_{\alpha \in [0,a]} (\alpha + \mu_A(x)), \sup_{\beta \in [0,b]} (\beta + \nu_A(x)))$.

For each $\alpha \in [0, a)$, since $S_1$ is upper dense in $[0, 1]$. We may find a real number $K_1 \in S_1$ such that $K_1 \geq \alpha$ , which implies that $K_1 \in [\alpha, a]$ and $\alpha + \mu_A(x) < K_1 \mu_A(x)$.

Similarly for each $\beta \in [0, b)$, since $S_2$ is upper dense in $[0, 1]$. We may find a real number $K_2 \in S_2$ such that $K_2 \geq \beta$ , which implies that $K_2 \in [\beta, b]$ and $\beta + \nu_A(x) < K_2 \nu_A(x)$.

Thus taking the supremum with respect to $\alpha \in (0, \mu_A(x))$, we obtain $\mu_A(x) = \sup_{\alpha \in (0,a)} \alpha + \mu_A(x) \leq \sup_{\alpha \in S_1} \alpha + \mu_A(x) = \sup_{\alpha \in S_1} \alpha \mu_A(x)$, i.e., $\mu_A(x) \leq \bigcup_{\alpha \in S_1} \alpha \mu_A(x)$.

Similarly for each $\beta \in (0, b)$, since $S_2$ is upper dense in $[0, 1]$. We may find a real number $K_2 \in S_2$ such that $K_2 \geq \beta$ , which implies that $K_2 \in (\beta, b)$ and $\beta + \nu_A(x) < \sup_{\beta \in (0,b)} \beta \nu_A(x) \leq \sup_{\beta \in S_2} \beta \nu_A(x)$.

Hence $x(x) = (\mu_A(x), \nu_A(x)) \subseteq \left( \bigcup_{\alpha \in S_1} \alpha \mu_A(x), \bigcup_{\beta \in S_2} \beta \nu_A(x) \right).$ \hspace{1cm} (2)

(1) and (2) concludes the proof.

**4. Total Ordering on the set of all Intuitionistic Fuzzy Numbers**

The new decomposition theorem established in section 3 identifies an arbitrary intuitionistic fuzzy number by a countably many real-valued parameters. It provides us with a powerful tool for defining total order in the class of IFN, by extending lexicographic ranking relation defined in Geetha et al [10].

**Note 4.0.1.** Let $A$ be an intuitionistic fuzzy number. Let $S = \{(a_i, b_i) | i = 1, 2, \ldots \} \subseteq [0, 1]$ be an double upper dense sequence. The $(\alpha_i - \beta_i)$-cut of an IFN $A$ at each $\alpha_i, \beta_i, i = 1, 2, \ldots,$ is a combination of two closed intervals. Denote this intervals by \([a_i, b_i], [c_i, d_i]\), where $[a_i, b_i]$ is the $\alpha_i$-cut of the membership function of $A$ and $[c_i, d_i]$ is the $\beta_i$-cut of the non-membership function of $A$ and $C_{4i-3} = a_i+b_i-c_i-d_i+\frac{3}{2}a_i+c_i$, $C_{4i-2} = a_i+b_i-c_i-d_i-a_i+c_i-b_i$, $C_{4i-1} = a_i-b_i-c_i+d_i+a_i+c_i+b_i$, $C_{4i} = a_i-b_i-c_i-d_i+a_i+c_i-b_i$, $i = 1, 2, \ldots$. 

By fourth decomposition theorem these countably many parameters \(\{C_i\}_{j = 1, 2, \ldots} \) identify the intuitionistic fuzzy number. Using these parameters, we define a relation on the set of all intuitionistic fuzzy number as follows.

**Definition 4.0.3. (Ranking Principle)** Let $A$ and $B$ be any two IFNs. Consider any double upper dense sequence $S = \{(a_i, b_i) | i = 1, 2, \ldots \} \subseteq [0, 1]$. using
The following example shows the importance of double upper dense sequence. In the previous examples, different IFNs are ranked by means of $C_1$ and $C_2$. But in general $C_1$ and $C_2$ alone need not be enough to rank the entire class of intuitionistic fuzzy number which is shown in example 4.0.4.

**Example 4.0.4.** Let $A = \{(0.3,0.35,0.4,0.5), (0.1,0.2,0.25,0.3)\}$ $B = \{(0.35,0.35,0.4,0.55), (0.0,0.2,0.25,0.35)\}$ be two TrIFNs. The ordering $<$ defined by using double upper dense sequence $S$ given in example 3.1.2 and the way shown in definition $\textbf{4.0.3}$ are now adapted. We have $(\alpha^-)A = \{(0.35 + 0.5\alpha , \alpha , 0.35 + 0.1\beta , \alpha )\}$, $(\alpha^-)B = \{(0.35 + 0.5\beta , \beta , 0.35 + 0.1\alpha , \beta )\}$, $(\alpha^-)C = \{(0.35 + 0.2\alpha , 0.35, 0.35 + 0.1\beta , 0.35)\}$. For $i = 1$, $(\alpha_1, \alpha_1) = (1, 1)$, $C_1(A) = -0.085$, $C_1(B) = 0.1084$, $C_1(C) = -0.05$. i.e., $C_1(A) < C_1(B) < C_1(C)$. Hence $A < B < C$.

**Theorem 4.1.** Relation $<$ is a total ordering on the set of all intuitionistic fuzzy number.

**Proof.** Claim: $<$ is total ordering on the set of IFN.

To prove $<$ is total ordering we need to show the following (1). $<$ is Partial ordering on the set of IFN (2). Any two elements of the set of IFNs are comparable. (1). $<$ is partial ordering on the set of IFN:

(i) $<$ is reflexive: It is very clear that the relation $<$ is reflexive for any $A$.

(ii) $<$ is antisymmetric: claim: If $A < B$ and $B < A$ then $A = B$.

Suppose $A \neq B$, then from the hypothesis $A < B$ and $B < A$. From the definition $\textbf{4.0.3}$ we can find $j_1$ such that $C_{j_1}(A) < C_{j_1}(B)$ and $C_{j_1}(A) = C_{j_1}(B)$ for all positive integers $j < j_1$. Similarly we are able to find $j_2$ such that $C_{j_2}(A) < C_{j_2}(B)$ and $C_{j_2}(A) = C_{j_2}(B)$ for all positive integers $j < j_2$. Then $j_1$ & $j_2$ must be the same, let it to be $j_0$. But $C_{j_0}(A) < C_{j_0}(B)$, and $C_{j_0}(B) < C_{j_0}(A)$ this contradicts our hypothesis. Therefore our assumption $A \neq B$ is wrong. Hence $A = B$.

(iii) $<$ is transitive: To prove (iii), we have to show that if $A < B$ and $B < C$ then $A < C$.

Let $A, B, C$ be three IFNs. Let us assume $A < B$ and $B < C$. Then $A < C$.

Therefore from $A < B$, we can find a positive integer $k_1$ such that $C_{k_1}(A) < C_{k_1}(B)$ and $C_{k_1}(A) = C_{k_1}(B)$ for all positive integer $k < k_1$. Similarly from $B < C$, we can find a positive integer $k_2$ such that $C_{k_2}(B) < C_{k_2}(C)$ and $C_{k_2}(B) = C_{k_2}(C)$ for all positive integer $k < k_2$. Now taking $j_0 = \min(k_1, k_2)$, we have $C_{j_0}(A) < C_{j_0}(C)$ and $C_{j_0}(A) = C_{j_0}(C)$ for all positive integer $k < j_0$, i.e., $A < C$. Hence $<$ is Transitive.
Therefore from (i), (ii), and (iii), we proved the relation \( \prec \) is Partial Ordering on the set of all IFNs. 

(2). Any two elements of the set of IFNs are comparable. For any two IFNs \( A \) and \( B \), they are either \( A = B \), or \( A \not\prec B \). In the latter case, there are some integers \( j \) such that \( C_j(A) \neq C_j(B) \). Let \( J = \{ j | C_j(A) \neq C_j(B) \} \). Then \( J \) is lower bounded 0 and therefore, according to the well ordering property, \( J \) has unique smallest element, denoted by \( j_0 \). Thus we have \( C_j(A) = C_j(B) \) for all positive integers \( j < j_0 \) and either \( C_{j_0}(A) < C_{j_0}(B) \) or \( C_{j_0}(A) > C_{j_0}(B) \), that is, either \( A < B \) or \( B < A \) in this case. So, for these two IFNs, either \( A < B \) or \( B < A \). This means that partial ordering \( \prec \) is a total ordering on the set of all intuitionistic fuzzy numbers. Hence the proof. \( \square \)

Similar to the case of total orderings on the real line \( (-\infty, +\infty) \) on the total ordering on the sets of special types of IFNs shown in section 3, infinitely many different total orderings on the set of IFNs can be defined. Even using a given upper dense sequence in \([0, 1]\), there are still infinitely many different ways to determine a total ordering on the set of IFNs. A notable fact is that each of them is consistent with the natural ordering on the set of all real numbers. This can be regarded as a fundamental requirement for any practice ordering method on the set of all IFNs.

5. Significance of the proposed method

Many researchers have proposed different ranking methods on IFNs, but none of them has covered the entire class of IFNs, and also almost all the methods have disadvantage that at some point of time they ranked two different numbers as the same. In this paper a special type of IFNs which is shown in figure 1 which generalizes IFN more natural in real scenario. Problems in different fields involving qualitative, quantitative and uncertain information can be modelled better using this type of IFNs when compared with usual IFNs. Our proposed ranking method on this type of IFN will give the better results over other existing methods, and this paper will give the better understanding over this new type of IFNs. This type of IFNs are very much important in real life problems and this paper will give the significant change in the literature. Modeling problems using this type of IFN will give better result. In this subsection our proposed method is compared with the total score function defined in Lakshmana et al.\cite{Lakshmana}.

5.1. Comparison between our proposed method with the score function defined in Lakshmana et al.\cite{Lakshmana}:

In this subsection, our proposed method is compared with the total score function defined in Lakshmana et al.\cite{Lakshmana}, which is explained here with an illustrative example.

Definition 5.1.1. (Note 1.2\cite{Definition})

The membership score of the triangular intuitionistic fuzzy number (TIFN) \( M = \{ (a, b, c) | (e, f, g) \} \) is defined by \( T(M) = \frac{1+R(M)-L(M)}{2} \), where \( L(M) = \frac{1-a}{1+b-c} \) and \( R(M) = \frac{1-b}{1+c-e} \).

Definition 5.1.2. (Lakshmana et al.\cite{Lakshmana})

The new membership score of the triangular intuitionistic fuzzy number \( M = \{ (a, b, c) | (e, f, g) \} \) is defined by \( NT(M) = \frac{e+NL(M)-NR(M)}{g} \), where \( NL(M) = \frac{1-e}{1+f-c} \) and \( NR(M) = \frac{1-f}{1+c-e} \).

Definition 5.1.3. (Lakshmana et al.\cite{Lakshmana})

The nonmembership score of the triangular intuitionistic fuzzy number \( M = \{ (a, b, c) | (e, f, g) \} \) is defined by \( NT_c(M) = 1 - NT(M) \).

Definition 5.1.4. (Lakshmana et al.\cite{Lakshmana})

Let \( M = \{ (a, b, c) | (e, f, g) \} \) be an triangular intuitionistic fuzzy number. If \( e \geq b \) and \( f \geq c \), then the score of the intuitionistic fuzzy number \( M \) is defined by \( (T, NT_c) \), where \( T \) is the membership score of \( M \) which is obtained from \( (a, b, c) \) and \( NT_c \) is the nonmembership score of \( M \) obtained from \( (e, f, g) \).

Definition 5.1.5. (Lakshmana et al.\cite{Lakshmana})

Ranking of Intuitionistic Fuzzy Numbers:

Let \( M_1 = \{ (a_1, b_1, c_1), (e_1, f_1, g_1) \} \) and \( M_2 = \{ (a_2, b_2, c_2), (e_2, f_2, g_2) \} \) be two triangular intuitionistic fuzzy numbers with \( e_i \geq b_i \) and \( f_i \geq c_i \). Then \( M_1 \preceq M_2 \) if the membership score of \( M_1 \) is less than the membership score of \( M_2 \) and the nonmembership score of \( M_1 \) is greater than the nonmembership score of \( M_2 \).

In this example definitions 5.1.1 to 5.1.5 are demonstrated and also the illogicality of Lakshmana etal’s \cite{Lakshmana} method is shown.

Example 5.1.1. Let \( M = \{ (0, 0.2, 0.4), (0.4, 0.45, 0.5) \} \) with \( (0.0, 0.2, 0.4)^C \leq (0.0, 0.45, 0.5) \) and \( 0.2 \geq 0.45 \geq 0.4 \) and \( N = \{ (0.25, 0.25, 0.25), (0.4, 0.45, 0.5) \} \) with \( (0.25, 0.25, 0.25)^C \geq (0.4, 0.45, 0.5) \) and \( 0.4 \geq 0.25 \). Let \( L(M) = 0.8333 \) and \( R(M) = 0.3333 \) which
5.3. Trapezoidal Intuitionistic Fuzzy Information System (TrIFIS)

Information system (IS) is a decision model used to select the best alternative from the all the alternatives in hand under various attributes. The data collected from the experts may be incomplete or imprecise numerical quantities. To deal with such data the theory of IFS provided by Atanassov [1] aids better. In information systems, dominance relation rely on ranking of data, ranking of intuitionistic fuzzy numbers is inevitable.

Definition 5.3.1. An Information System

\[ S = (U, AT, V, f) \] with \( V = \cup_{a \in AT} V_a \) where \( V_a \) is a domain of attribute \( a \) is called trapezoidal intuitionistic fuzzy information system if \( V \) is a set of TrIFN. We denote \( f(x,a) \in V_a \) by 
\[ f(x,a) = ((a_1, a_2, a_3, a_4), (a'_1, a'_2, a'_3, a'_4)) \] where \( a_i, a'_i \in [0, 1] \).

The numerical illustration is given in example 5.5.1.

Definition 5.3.2. An TrIFIS, \( S = (U, AT, V, f) \) together with weights \( W = \{w_a | a \in AT\} \) is called Weighted Trapezoidal Intuitionistic Fuzzy Information System (WTrIFIS) and is denoted by \( S = (U, AT, V, f, W) \).

Definition 5.3.3. Let \( a \in AT \) be a criterion. Let \( x, y \in U \). If \( f(x,a) > f(y,a) \) (as per definition 4.0.3) then \( x >_a y \) which indicates that \( x \) is better than (outranks) \( y \) with respect to the criterion \( a \). Also \( x =_a y \) means that \( x \) is equally good as \( y \) with respect to the criterion \( a \), if \( f(x,a) = f(y,a) \).

Definition 5.3.4. Let \( S = (U, AT, V, f, W) \) be an WTrIFIS and \( A \subseteq AT \).

Let \( B_A(x,y) = \{a \in A | x >_a y\} \) and let \( C_A(x,y) = \{a \in A | x =_a y\} \). The weighted fuzzy dominance relation \( WR_A(x,y) : U \times U \rightarrow [0, 1] \) is defined by 
\[ WR_A(x,y) = \sum_{a \in B_A(x,y)} w_a + \frac{\sum_{a \in C_A(x,y)} w_a}{2}. \]

Definition 5.3.5. Let \( S = (U, AT, V, f, W) \) be an WTrIFIS and \( A \subseteq AT \). The entire dominance degree of each object is defined as
\[ WR_A(x_i) = \frac{1}{|U|} \sum_{j=1}^{|U|} WR_A(x_i, y_j) \]

5.4. Algorithm for Ranking of objects in WTrIFIS

Let \( S = (U, AT, V, f, W) \) be an WTrIFIS. The objects in \( U \) are ranked using following algorithm.

Algorithm 5.3

1. Using Definition 4.0.3 find \( C_i^r \) accordingly, to decide whether \( x_i >_a x_j \) or \( x_i >_a x_j \) or \( x_i =_a x_j \) for all \( a \in A (A \subseteq AT) \) and for all \( x_i, x_j \in U \).
2. Enumerate \( B_A(x_i, x_j) \) using
3. Calculate the weighted fuzzy dominance relation using $WRA(x, y) = \sum_{i=1}^{|U|} w_a \times CA_i(x, y)$. The object $x$ dominates $y$ if $WRA(x, y) = \sum_{i=1}^{|U|} w_a \times CA_i(x, y)$.

4. Calculate the entire dominance degree of each object using $WRA(x) = \frac{1}{|U|} \sum_{i=1}^{|U|} WRA(x, j)$.

5. The objects are ranked using the entire dominance degree. The larger the value of $WRA(x)$, the better is the object.

### 5.5. Numerical Illustration

In this subsection, Algorithm 5.4 is illustrated by an example 5.5.1.

### Example 5.5.1

In this example we consider a selection problem of the best supplier for an automobile company from the available alternatives $x_i | i = 1 \rightarrow 10$ of pre evaluated 10 suppliers, based on $WTRHIS$ with attributes $\{a_j | j = 1 \rightarrow 5\}$ as product quality, reputation closeness, delivery performance, social responsibility and legal issue. An $TRHIS$ with $U = \{x_1, x_2, ..., x_{10}\}$, $AT = \{a_1, a_2, a_3, a_4, a_5\}$ is given in Table 2 and weights for the each attribute $w_a$ is given by $W = \{w_a | a \in AT\} = \{0.3, 0.2, 0.15, 0.17, 0.18\}$.
from the criteria 'delivery performance' \( \alpha, \beta \). \( \langle 0, 0.2, 0.4 \rangle \) in Table 2. \( \Sigma_x \) is evaluated under the criteria 'product
acceptance' \( \alpha, \beta \). \( \langle 0, 0.2, 0.4 \rangle \) is found wherever required.

The bold letters are used in Table 3 to represent the
with acceptance of \((a_1, a_2, a_3, a_4)\) and nonacceptance of \((c_1, c_2, c_3, c_4)\). For example, \(f(x_1, a_1)\) denotes supplier \(x_1\) is evaluated under the criteria 'product quality' \((a_1)\) with "20% of acceptance and 40% of non acceptance". \(f(x_2, a_3)\) denotes the supplier \(x_2\) is evaluated under the criteria 'delivery performance' \((a_3)\) with "around 30% to 35% of acceptance and around 45% to 50% of non acceptance" and \(f(x_9, a_1)\) denotes the supplier \(x_9\) is evaluated under the criteria 'product quality' \((a_1)\) with "20% to 40% of acceptance and 40% to 60% of non acceptance".

Step 1 For \(i = 1\), \((\alpha, \beta) = (1, 1)\): By step 1, \(C_1(f(x_1, a_1))\) using definition 4.0.3 and note 4.0.1 for all \(a_1 \in AT\) and for all \(x_1 \in U\) is found and tabulated in Table 3. If \(C_1(f(x_i, a_j)) = C_1(f(x_j, a_j))\) for any alternatives \(x_i, x_j\) then \(C_2\) and other necessary score functions \(C_3\) and \(C_4\) are found wherever required. The bold letters are used in Table 3 to represent the equality of scores. From Table 3 we observe that in many places \(C_3\)'s are not distinguishable for different IFNs.

Hence from Table 3 we do not get the best alternative. Therefore the same procedure which is explained in step 1 is repeated for \(i = 2\), \((\alpha, \beta) = (1/2, 1/2)\) and it is shown in Table 4.

The weighted fuzzy dominance relation using
\[
WRA(x, y) = \sum_{a \in BA(x, y)} w_a + \frac{\sum_{a \in C(x, y)} w_a}{2}
\]
is calculated and is tabulated in Table 5. For example, \(BA(x_1, x_2) = \{a_1, a_2, a_3\}\) and \(CA(x_1, x_2) = \{\}\) and
methods, and may be adopted in decision making with alone or as a supplementary means with other ranking method can order intuitionistic fuzzy numbers, either by Wei Wang, Zhenyuan Wang [27]. Therefore this on IFNs generalises the total ordering on FNs defined ordering of real numbers. Actually this total ordering IFNs is achieved. The total orderings introduced and hence \( WR_A(x_1, x_2) = 0.3 + 0.2 + 0.15 = 0.65. \) Now the entire dominance degree of each object using \( WR_A(x_i) = \frac{1}{|U|} \sum_{j=1}^{|U|} WR_A(x_i, y_j) \) is found by definition[5,3,4]. For example, \( WR_A(x_1) = \frac{1}{10} \sum_{j=1}^{10} WR_A(x_1, x_j) = \) 0.409.

So by step 5, \( x_8 \) is selected as the best object from the weighted trapezoidal intuitionistic fuzzy information system is seen from table[6]

6. Conclusion

In this paper, the total ordering on the set of all IFNs is achieved. The total orderings introduced and discussed in this paper are consistent with the natural ordering of real numbers. Actually this total ordering on IFNs generalises the total ordering on FNs defined by Wei Wang, Zhenyuan Wang [27]. Therefore this is an appropriate generalization of the total ordering on the set of all real numbers to the set of IFNs. This method can order intuitionistic fuzzy numbers, either alone or as a supplementary means with other ranking methods, and may be adopted in decision making with fuzzy information.

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References

[1] Atanassov, K. T., Intuitionistic fuzzy sets. Fuzzy Sets and Systems, 20 (1986), 87-96.
[2] Atanassov, K. T., Gargov, G., Interval-valued intuitionistic fuzzy sets. Fuzzy Sets and Systems, 31(3) (1989), 343 - 349.
[3] Boender, C. G. E., J. G. de Graan, Lootsma, F. A., Multi-criteria decision analysis with fuzzy pairwise comparisons, Fuzzy Sets and Systems, 29 (1989), 133-143 .
[4] Chen, S.M., Tan, J.M., Handling multi-criteria fuzzy decision-making problems based on vague set theory. Fuzzy Sets and Systems, 67 (1994), 163-172.
[5] Deng-Feng Li, A ratio ranking method of triangular intuitionistic fuzzy numbers and its application to MADM problems. Computers and Mathematics with Applications, 60 (2010), 1557-1570.
[6] Dubois, D., Prade, H., Fuzzy Sets and Systems: Theory and Applications, Academic press, Newyork, 1980.
[7] Du, W., Hu, B., Zhao, Y., Attribute reductions in intuitionistic fuzzy information systems based on dominance relations. Operations Research and Fuzziology, 1 (2011), 1 - 5.
[8] Farhadinia, B., Ranking of fuzzy numbers on lexicographical ordering. International Journal of Applied Mathematics and Computer Science, 5(4) (2009), 248-251.

[9] Geetha Sivaraman, Lakshmana Gomathi Nayagam, V., Ponalagusamy, R., Intuitionistic fuzzy interval information system. International Journal of Computer Theory and Engineering, 4(3) (2012), 459 - 461.

[10] Geetha Sivaraman, Lakshmana Gomathi Nayagam, V., Ponalagusamy, R., A Complete ranking of incomplete interval information. Expert Systems with Applications, (2014), 1947 - 1954.

[11] Hassan Mishmasi Nehi, Hamid Reza Maleki, Intuitionistic Fuzzy Numbers and It’s Applications in Fuzzy Optimization Problem, 9th WSEAS CSCC Multiconference Vouliagmeni, Athens, Greece, July 11-16, 2005.

[12] Hong, D.H., Choi, C.H., Multi-criteria fuzzy decision-making problems based on vague set theory. Fuzzy Sets and Systems, 114 (2000), 103-113.

[13] Jun Ye, Multi-criteria fuzzy decision making method based on a novel accuracy score function under interval valued intuitionistic fuzzy environment. Expert Systems with Applications, 36 (2009), 6899-6902.

[14] Klir, G.J., Yuan, B. Fuzzy Sets and Fuzzy Logic: Theory and Applications, Prentice Hall.1995.

[15] Lakshmana Gomathi Nayagam, V., Geetha, Sivaraman, Ranking of intervalvalued intuitionistic fuzzy numbers. Applied Soft Computing, 11(4) (2011), 3368 - 3372.

[16] Lakshmana Gomathi Nayagam, V., Muralikrishnan, S., Geetha, Sivaraman, Multi Criteria decision making method based on interval valued intuitionistic fuzzy sets. Expert Systems with Applications, 38(3) (2011), 1464 - 1467.

[17] Lakshmana Gomathi Nayagam, V., Venkateshwari, G., Geetha, Sivaraman., Ranking of intuitionistic fuzzy numbers. In Proceedings of the IEEE international conference on fuzzy systems (IEEE FUZZ 2008),2008, 1971 - 1974.

[18] Lin, L.G., Xu, L.Z., Wang, J.Y., Multi-criteria fusion decision-making method based on vague set. Computer Engineering, 31 (2005), 11-13 (in Chinese).

[19] Lin, L., Yuan, X.H., Xia, Z.Q., Multicriteria fuzzy decision-making methods based on intuitionistic fuzzy sets. Journal of Computer and System Sciences, 73 (2007), 84-88.

[20] Liu, H.W., Vague set methods of multi-criteria fuzzy decision making. Systems Engineering - Theory & Practice, 24 (2004), 103-109 (in Chinese).

[21] Nehi, H. M., "A new ranking method for intuitionistic fuzzy numbers," International Journal of Fuzzy Systems, 12 (2010), 80-86.

[22] Pawlak, Z., Rough set theory and its applications to data analysis. Cybernetics and Systems, 29(7) (1998), 667-688.

[23] Pawlak, Z., Rough sets and intelligent data analysis. Information Sciences, 147(1-4)(2002), 1 - 12.

[24] Qian, Y. H., Liang, J. Y., Dang, C. Y., Interval ordered information systems. Computers and Mathematics with Applications, 56(8)(2008), 1994 - 2009.

[25] Song, Peng, Liang, Jiye, Qian, Yuhua, A two-grade approach to ranking interval data. Knowledge-Based Systems, 27(2012), 234 - 244.

[26] Wang, J., Zhang, J., Liu, S.Y., A new score function for fuzzy MCDM based on vague set theory. International Journal of Computational Cognition, 4 (2006), 44-48.

[27] Wang, W., Z.Wang, Total ordering defined on the set of all Fuzzy numbers. Fuzzy Sets and Systems, 243 (2014), 131-141.

[28] WeiBo, Lee., Novel method for ranking interval-valued intuitionistic fuzzy numbers and its application to decision making. In International conference on intelligent human-machine systems and cybernetics, IHMSC 2009, 282 - 285.

[29] Xu, Z. S., Intuitionistic fuzzy aggregation operators. IEEE Transactions on Fuzzy Systems, 15(6)(2007), 1179 - 1187.

[30] Xu, Z. S., Yager, R. R., Some geometric aggregation operators based on intuitionistic fuzzy sets. International Journal of General System, 35 (2006), 417 - 433.

[31] Ye, J., Using an improved measure function of vague sets for multicriteria fuzzy decision-making. Expert Systems with Applications, 37 (2010), 4706-4709.

[32] Zadeh, L. A., Fuzzy sets. Information and Control, 8(3) (1965), 338 - 356.

[33] Zhou, Z., Wu, Q.Z., (2005), Multicriteria fuzzy decision making method based on vague set. Mini-Micro Systems, 26, 1350-1353 (in Chinese).

[34] Zhang, H., Yu, L., MADM method based on cross-entropy and extended TOPSIS with interval-valued intuitionistic fuzzy sets. Knowledge-Based Systems, 30 (2012), 115-120.

[35] Zhang, W. X., Mi, J. S., Incomplete information system and its optimal selections. Computers & Mathematics with Applications, 48(5-6)(2004), 691 - 698.