Suppression of blow-up in Patlak-Keller-Segel via shear flows

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September 12, 2016

Abstract

In this paper we consider the parabolic-elliptic Patlak-Keller-Segel models in $T^d$ with $d = 2, 3$ with the additional effect of advection by a large shear flow. Without the shear flow, the model is $L^1$ critical in two dimensions with critical mass $8\pi$; solutions with mass less than $8\pi$ are global and solutions with mass larger than $8\pi$ with finite second moment, all blow up in finite time. In three dimensions, the model is $L^{3/2}$ critical and $L^1$ supercritical; there exists solutions with arbitrarily small mass which blow up in finite time arbitrarily fast. We show that the additional shear flow, if it is chosen sufficiently large, suppresses one dimension of the dynamics and hence can suppress blow-up. In two dimensions, the problem becomes effectively $L^1$ subcritical and so all solutions are global in time (if the shear flow is chosen large). In three dimensions, the problem is effectively $L^1$ critical, and solutions with mass less than $8\pi$ are global in time (and for all mass larger than $8\pi$, there exists solutions which blow up in finite time).

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1 Introduction

Consider the parabolic-elliptic Patlak-Keller-Segel model in $T^d$ with the additional effect of a large shear flow

\[
\begin{aligned}
&\partial_t n + Au \partial_x n + \nabla \cdot (n\nabla c) = \Delta n \\
&-\Delta c = n - \bar{n} \\
n(t=0, x, y) = n_0(x, y),
\end{aligned}
\]

where $\bar{n}$ denotes the average of $n$. If $d = 3$, then we denote $y = (y_1, y_2)$. Here, $u = u(y)$ if $d = 2$ and $u = u(y_1)$ if $d = 3$, is a fixed, $C^3$ function with at most finitely many non-degenerate critical points. In the case $A = 0$, this system is one of the fundamental models for the study of aggregation via chemotaxis of certain microorganisms; see e.g. [40, 34, 32, 30]. The quantity $n$ denotes the density of microorganisms, which are executing a random walk with a bias up the gradient of the chemo-attractant $c$. The second equation describes the quasi-static equilibration and production of the chemo-attractant by the microorganisms. Patlak-Keller-Segel and its variations have received considerable mathematical attention over the years, for
example, see the review [32] or some of the representative works [20] [33] [12] [29] [13] [17] [16] [14] [15] and the references therein. The case $A \neq 0$ models the microorganisms suspended in a shear flow: the elliptic equation $-\Delta c = n - \bar{n}$ arises as the formal limit as $\epsilon \to 0$ of the advection-diffusion equation

$$\partial_t c + A \partial_x c = \epsilon^{-1}(\Delta c + n),$$

under the assumption that $\epsilon A \ll 1$. In particular, (1.1) requires that the time-scale of equilibration of $c$ is faster than the transport due to the shear flow.

One of the most well-known features of (1.1) is that it is $L^1$ critical in two dimensions and is $L^{3/2}$ critical in three dimensions. For any reasonable notion of solution, the $L^1$ norm of the density is conserved, $M := \|n(t)\|_{L^1} = \|n_{in}\|_{L^1}$ (for (1.1), this is the mass). There is also the free energy, for which (1.1) (with $A = 0$) is formally a gradient flow with respect to the $L^2$ Wasserstein metric:

$$F[n(t)] = \int n \log ndx - \frac{1}{2} \int |\nabla c|^2 dx \leq F[n_{in}].$$

In $\mathbb{R}^2$, the conservation of mass and dissipation of the free energy (the latter is a logarithmically subcritical quantity) can be used to prove that if $\|n_{in}\|_{L^1} \leq 8\pi$, then solutions exist for all time (see e.g. [17] [16] [13]). All solutions with finite second moment and $\|n_{in}\|_{L^2} > 8\pi$ are known to blow up in finite time [33] [39] [17]. On $\mathbb{T}^2$, some similar results are known [13]. On $\mathbb{R}^3$, since (1.1) is super-critical with respect to the controlled quantities, significantly less is understood. Solutions which are initially small in $L^{3/2}$ are known to exist globally and it is known that there exists blow-up solutions with arbitrarily small mass [22].

In [35] it was shown that if, instead of a shear flow, one has $A u \cdot \nabla n$ where $u$ is relaxation-enhancing — a generalization of weakly mixing introduced in [21] — then for each smooth initial datum, one can choose $A$ large enough so that the solution to (1.1) does not blow-up in finite time. Such velocity fields are very good mixers, and this ensures that any non-constant density configuration undergoes a large growth of gradients, and hence a large dissipation. The effect at work is then an enhanced dissipation. This effect has been studied previously in a variety of contexts, such as [21] [48] [3] [47] [5] [4], in the physics literature [38] [41] [25] [37] [11], and in control theory [21]: a closely related effect was also studied in [20].

Mixing due to a shear flow is quite different from that due to a relaxation-enhancing or weakly mixing flow. In particular, data which is independent of $x$ does not mix at all, and so one must separate the evolution of the zero (or low if $x \in \mathbb{R}$) frequencies in $x$ from the non-zero frequencies, which is the decomposition into the nullspace of the transport operator and its orthogonal complement. Enhanced dissipation due to shear flow was shown in [21] [4] [7] [8] [10] to be important for understanding the stability of the Couette flow in the 2D and 3D Navier-Stokes equations at high Reynolds number. For example, [6] [7] [8] show that the enhanced dissipation can suppress 3D effects and simplify the dynamics to be essentially 2D. It is intuitive then to expect that a large shear flow can also in some sense suppress one dimension in (1.1) and hence make 2D $L^1$ subcritical and 3D $L^1$ critical. This is essentially what we prove for $u \in C^3$ with finitely many non-degenerate critical points (the relevance of these hypotheses are discussed after the statements).

**Theorem 1.** Let $u \in C^3(T)$ have finitely many, non-degenerate critical points and let $n_{in} \in H^1(T^2) \cap L^\infty(T^2)$ be arbitrary. There exists an $A_0 = A_0(u, \|n_{in}\|_{H^1}, \|n_{in}\|_{L^\infty})$ such that if $A > A_0$ then the solution to (1.1) is global in time.

**Remark 1.1.** Theorem 1 extends to the cylindrical domain $T \times \mathbb{R}$ provided $u$ is bounded uniformly away from zero near $y \to \pm \infty$.

It is clear that Theorem 1 cannot hold in 3D. Indeed, consider any solution to the 3D problem which is constant in the $x$ direction: $n(t, x, y_1, y_2) = n(t, y_1, y_2)$. This solution will solve (1.1) on $\mathbb{T}^2$ with $A = 0$ and hence the $8\pi$ critical mass will still apply. Our next result shows that for $A$ large the third dimension is suppressed and $8\pi$ is indeed the critical mass for (1.1) in $\mathbb{T} \times \mathbb{R}^2$ and $\mathbb{T}^3$. As this setting is effectively critical, Theorem 2 is harder to prove than Theorem 1 (which is effectively subcritical, as [35]).

**Theorem 2.** (a) Let $u \in C^3(T)$ have finitely many, non-degenerate critical points and let $n_{in} \in H^1(T^3) \cap L^\infty(T^3)$ be arbitrary such that $\|n_{in}\|_{L^1} < 8\pi$ and for some $q > 0$, there holds $n_{in}(x) \geq q > 0$ for all $x \in T^3$. Then there exists an $A_0 = A_0(u, \|n_{in}\|_{H^1}, \|n_{in}\|_{L^\infty}, \|n_{in}\|_{L^1}, q)$ such that if $A > A_0$ then the solution to (1.1) is global in time.
(b) Suppose $u \in C^3(\mathbb{R})$ have finitely many, non-degenerate critical points and $u'$ is bounded uniformly away from zero near infinity. Let $n_{in} \in H^1(\mathbb{T} \times \mathbb{R}^2) \cap L^\infty(\mathbb{T} \times \mathbb{R}^2)$ be arbitrary such that $\|n_{in}\|_{L^1} < 8\pi$ and $I[n_{in}] := \int n_{in}(x, y) \|y\|^2 \, dx \, dy < \infty$. Consider the problem
\begin{equation}
\begin{cases}
\partial_t n + Au(y_1) \partial_x n + \nabla \cdot (\nabla cn) = \Delta n, \\
-\Delta c = n, \\
n(\cdot, 0) = n_0.
\end{cases}
\tag{1.3}
\end{equation}

Then, there exists an $A_0 = A_0(a, \|n_{in}\|_{L^\infty}, \|n_{in}\|_{H^1}, \|n_{in}\|_{L^1}, I[n_{in}])$, such that if $A > A_0$ then the solution is global in time.

Remark 1.2. It is not clear whether or not one could expect Theorem 2 to hold also in the case $\|n_{in}\|_{L^1} = 8\pi$ as in $\mathbb{R}^2$ [17].

Let us now briefly discuss the proofs of Theorems 1 and 2. By re-scaling time $t \mapsto A^{-1} t$, the system (1.1) is equivalent to
\begin{equation}
\begin{cases}
\partial_t n + u \partial_x n + \frac{1}{A} (\nabla \cdot (n \nabla c) - \Delta n) = 0, \\
-\Delta c = n - \bar{n}, \\
n(t = 0, x, y) = n_{in}(x, y),
\end{cases}
\tag{1.4}
\end{equation}

For our purposes, it is convenient to use the form (1.4). In [4], enhanced dissipation was studied for the passive scalar equation
\begin{equation}
\partial_t f + u \partial_x f = \frac{1}{A} \Delta f.
\tag{1.5}
\end{equation}

Among other things, it was shown in [4] that for $u$ satisfying the hypotheses of Theorems 1 and 2 there exists some $\epsilon > 0$ such that
\begin{equation}
\left\| f(t) - \frac{1}{2\pi} \int_{\mathbb{T}} f(t, x, \cdot) \, dx \right\|_{L^2} \lesssim e^{-\frac{\epsilon \nu_{1/2}}{1 + |\log|x|^2}} t \|f(0)\|_{L^2}.
\end{equation}

The technique employed in [4] is an energy method known as hypocoercivity, see e.g. the text [46] for an overview or [23] [24] [28] [27] [26] and the references therein. In the proof of Theorem 1 we will couple such hypocoercivity energy estimates to $H^1$ energy estimates for the zero-in-x frequency as well as to $L^p$ estimates on (1.1), similar to the estimates in [33] [17] [18], which do not see the advection term. In the proof of Theorem 2 the $x$-independent system is now formally $L^1$ critical, and hence in order to get results for mass up to $8\pi$, we need to employ the free energy in a manner similar to [17]. However, the two-dimensional free energy is not a monotonically dissipated quantity for (1.1), and hence we need to also couple an estimate on the 2D free energy to the other energy estimates we make and control the errors using the enhanced dissipation. This is particularly tricky if one is interested in the result on $\mathbb{T} \times \mathbb{R}^2$. Enhanced dissipation (or something similar) was studied via hypocoercivity also in [24] [23] [4], however, to the authors’ knowledge, this is the first work that uses hypocoercivity to obtain enhanced dissipation estimates for nonlinear problems. We remark that the Fourier analysis methods used in [8] [10] also apply to (1.1) in the specific case $u(y) = y$ and $y \in \mathbb{R}$. This approach is much simpler than the hypocoercivity methods we employ, however, the hypocoercivity methods allow us to study a much wider variety of shear flows.

1.1 Notations

1.1.1 Miscellaneous

The constants $B$ below are universal constants which have no dependence on any quantities, except perhaps $u$ and $M$. On the contrary, the dependence of the constants $C_{\ast}$ on various quantities involving $n_{in}$ is more important and will be made a little more explicit. Given quantities $X, Y$, if there exists a constant $B$ such that $X \leq BY$, we often write $X \lesssim Y$. We will moreover use the notation $\langle x \rangle := (1 + |x|^2)^{1/2}$. 

3
1.1.2 Fourier Analysis

Most of the time, we consider the Fourier transform only in the $x$ variable, and denoting it and its inverse as
\[ \hat{f}(y) := \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikx} f(x, y)dx, \quad \check{g}(x, y) = \sum_{k=-\infty}^{\infty} g_k(y)e^{ikx}. \]

Define the following orthogonal projections:
\[ f_0(t, y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t, x, y)dx, \quad f_\neq(t, x, y) = f(t, x, y) - f_0(t, y), \]
for “zero frequency” and “non-zero frequency”. For any measurable function $m(\xi)$, we define the Fourier multiplier $m(\nabla)f := (m(\xi)\hat{f}(\xi))^\vee$.

1.1.3 Functional spaces

The norm for the $L^p$ space is denoted as $|| \cdot ||_p$ or $|| \cdot ||_{L^p(\cdot)}$:
\[ ||f||_p = ||f||_{L^p} = (\int |f|^pdx)^{1/p}, \]
with natural adjustment when $p$ is $\infty$. If we need to emphasize the ambient space, we use the second notation, i.e., $||n_\neq||_{L^p(\mathbb{R}_x \times \mathbb{R}^2)}$. Otherwise, we use the first notation for the sake of simplicity. The Sobolev norm $|| \cdot ||_{H^s}$ is defined as follow:
\[ ||f||_{H^s} := ||(\nabla)^s f||_{L^2}. \]

For a function of space and time $f = f(t, x)$, we use the following space-time norms:
\[ ||f||_{L^p_t L^q_x} := ||||f||_{L^q_x}||_{L^p_t}, \]
\[ ||f||_{L^p_t H^s_x} := ||||f||_{H^s_x}||_{L^p_t}. \]

2 Proof of Theorem 1

2.1 Outline of the proof

In this section, we prove Theorem 1. The enhanced dissipation does not act in the nullspace of the advection term, and hence it is reasonable to decompose the solution as follows
\[ \partial_t n_0 + \frac{1}{A} \partial_y (\partial_y c_0 n_0) + \frac{1}{A} (\nabla \cdot (\nabla c_\neq n_\neq))_0 = \frac{1}{A} \partial_y n_0, \]
\[ -\Delta c_0 = n_0 - \pi; \tag{2.1} \]
and,
\[ \partial_t n_\neq + u(y)\partial_x n_\neq - \frac{1}{A} (n_0 - \pi)n_\neq - \frac{1}{A} n_0 n_\neq + \frac{1}{A} \nabla c_0 \cdot \nabla n_\neq + \frac{1}{A} \nabla c_\neq \cdot \nabla n_0 \]
\[ = -\frac{1}{A} (\nabla \cdot (\nabla c_\neq n_\neq))_\neq + \frac{1}{A} \Delta n_\neq, \tag{2.2} \]
\[ -\Delta c_\neq = n_\neq. \]

As in 4, it is convenient to consider (2.2) after applying the Fourier transform only in $x$. Applying to both sides of (2.2) we have,
\[ \partial_t \hat{n}_k + NL_k + L_k + u(y)ik\hat{n}_k = \frac{1}{A} (\partial_{yy} - |k|^2)\hat{n}_k, \]
\[ -(\partial_{yy} - |k|^2)c_k = n_k, \tag{2.3} \]
Moreover, in order to simplify the exposition, we introduce the following constant:

\[ NL_k := -\frac{1}{A} \sum_{\ell \neq 0} \hat{n}_{k-\ell}(y) \hat{n}_{\ell}(y) + \frac{1}{A} \sum_{\ell \neq 0} \partial_y \hat{c}_{k-\ell} \partial_y \hat{n}_{\ell} - \frac{1}{A} \sum_{\ell \neq 0} (k-\ell) \hat{c}_{k-\ell} \ell \hat{n}_{\ell}, \]  

(2.4a)  

\[ L_k := -\frac{1}{A} \hat{m}_k - \frac{2}{A} (n_0 - \hat{m}) \hat{n}_k + \frac{1}{A} \partial_y c_0 \partial_y \hat{n}_k + \frac{1}{A} \partial_y \hat{c}_k \partial_y n_0. \]  

(2.4b)

Here, the \( L \) refers to “linear with respect to the nonzero frequencies” and \( NL \) refers to “nonlinear with respect to the nonzero frequencies”.

For constants \( C_{ED}, C_{L^2}, C_{H^1}, \) and \( C_\infty \) determined by the proof, define \( T_* \) to be the end-point of the largest interval \([0, T_*]\) such that the following hypotheses hold for all \( T \leq T_* \):

1. Nonzero mode \( L_t^2 H^1_{x,y} \) estimate:

\[ \frac{1}{A} \int_0^{T_*} ||\nabla n_x||^2_2 dt \leq 8 ||n_{in}||^2_2; \]  

(2.5a)

2. Nonzero mode enhanced dissipation estimate:

\[ ||n_x(t)||^2_2 \leq 4C_{ED} ||n_{in}||^2_{H^1} e^{-\frac{4t}{A^{1/2} \log A}}, \]  

(2.5b)

where \( c \) is a small constant depending only on \( u \).

3. Uniform in time estimates on the zero mode:

\[
\begin{align*}
||n_0 - \hat{m}||_{L^\infty_t(0,T_*;L^2)} &\leq 4C_{L^2}, \\
||\partial_y n_0||_{L^\infty_t(0,T_*;L^2)} &\leq 4C_{H^1};
\end{align*}
\]  

(2.5c)

4. \( L^\infty \) estimate of the whole solution:

\[ ||n||_{L^\infty_t(0,T_*;L^\infty_{x,y})} \leq 4C_{C_\infty}. \]  

(2.5d)

Moreover, in order to simplify the exposition, we introduce the following constant:

\[ C_{2,\infty} := 1 + M + C_{ED}^{1/2} ||n_{in}||_{H^1} + C_{L^2} + C_\infty. \]  

(2.6)

**Remark 2.1.** In the above, \( C_{ED} \) is first chosen depending only on \( u \). Then, \( C_{L^2} \) is chosen depending only on the initial data \( n_{in} \) (and \( C_{ED} \)). Then \( C_{C_\infty} \) is chosen depending only on \( n_{in}, C_{L^2}, \) and \( C_{ED} \). Finally, \( C_{H^1} \) depends on \( n_{in}, C_{ED}, C_{L^2}, \) and \( C_{\infty} \). Then, \( A \) is chosen large depending on all of these parameters.

We will refer to the hypotheses (2.5a), (2.5b), (2.5c), and (2.5d) together as the **bootstrap hypotheses**, denoted as (H). Notice that by local well-posedness of mild solutions, the quantities on the left-hand sides of (2.5a), (2.5b), (2.5c), and (2.5d) take values continuously in time. Moreover, the inequalities are all satisfied with the \( 4 \)'s replaced by \( 2 \)'s for \( t \) sufficiently small. By the standard continuation criteria for (1.1), the solution exists and remains smooth on an interval \([0, t_0]\), with \( t_0 > T_* \) such that \( t_0 - T_* \) can be taken to depend only on \( ||n(T_*)||_{L^2} \). By continuity, the following proposition shows that the solution is global and satisfies the a priori estimates (H) for all time.

**Proposition 1.** For all \( n_{in} \) and \( u \), there exists an \( A_0(u, ||n_{in}||_{H^1}, ||n_{in}||_{L^\infty}) \) such that if \( A > A_0 \) then the following conclusions, referred to as (C), hold on the interval \([0, T_*]\):

1. \[ \frac{1}{A} \int_0^{T_*} ||\nabla_{x,y} n_x||_2^2 dt \leq 4 ||n_{in}||^2_2; \]  

(2.7a)

2. For all \( t < T_* \),

\[ ||n_x(t)||_2^2 \leq 2C_{ED} ||n_{in}||^2_{H^1} e^{-\frac{4t}{A^{1/2} \log A}}; \]  

(2.7b)

where \( L_k, NL_k \) are defined as follows:

\[ NL_k := -\frac{1}{A} \sum_{\ell \neq 0} \hat{n}_{k-\ell}(y) \hat{n}_{\ell}(y) + \frac{1}{A} \sum_{\ell \neq 0} \partial_y \hat{c}_{k-\ell} \partial_y \hat{n}_{\ell} - \frac{1}{A} \sum_{\ell \neq 0} (k-\ell) \hat{c}_{k-\ell} \ell \hat{n}_{\ell}, \]  

(2.4a)  

\[ L_k := -\frac{1}{A} \hat{m}_k - \frac{2}{A} (n_0 - \hat{m}) \hat{n}_k + \frac{1}{A} \partial_y c_0 \partial_y \hat{n}_k + \frac{1}{A} \partial_y \hat{c}_k \partial_y n_0. \]  

(2.4b)
For all the following estimate holds,

\[
\|n_0 - \overline{n}\|_{L^\infty((0,T);L_2^\infty)} \leq 2C_{L^2},
\]
\[
\|\partial_y n_0\|_{L^\infty((0,T);L_2^\infty)} \leq 2C_{H^1},
\]

(2.7c)

\[
\|n\|_{L^\infty((0,T);L_2^\infty)} \leq 2C_{\infty}.
\]

(2.7d)

The remainder of the section is dedicated to proving Proposition 1.

We first point out that there is a uniform upper bound on \(\|n(t)\|_{L^2}\) over the initial time layer \(t \leq \delta A\) for a sufficiently small \(\delta\) depending only on \(\|n_{in}\|_{L^2}\) (as such we can always choose \(A > \delta^{-1}\)). This is an immediate consequence of the standard local existence theory of (1.1) via the time-rescaling used in (1.4), however, we include a brief sketch of the a priori estimate for completeness. Proposition 2 and standard higher regularity theory for (1.1) (see e.g. [33]) imply that (2.7d) holds over the time interval \(0 \leq t \leq \delta A\).

**Proposition 2.** For all \(n_{in} \in L^2(\mathbb{T}^2)\), there exists \(\delta = \delta(\|n_{in}\|_{L^2})\) sufficiently small such that for \(t \leq \delta A\), the following estimate holds,

\[
\|n_\#(t)\|_{L^2}^2 + \|n_0 - \overline{n}\|_{L^2}^2 = \|n(t) - \overline{n}\|_{L^2}^2 \leq 2\|n_{in} - \overline{n}\|_{L^2}^2 \leq 2\|n_{in}\|_{L^2}^2.
\]

(2.8)

**Proof.** The time derivative of the \(L^2\) norm of \(n - \overline{n}\) is estimated as follows, using a Gagliardo-Nirenberg-Sobolev inequality,

\[
\frac{1}{2} \frac{d}{dt} \|n - \overline{n}\|_{L^2}^2 = -\frac{1}{A}\|\nabla n\|_{L^2}^2 - \int \nabla \cdot (\nabla cn)(n - \overline{n})dx
\]

\[
= -\frac{1}{A}\|\nabla n\|_{L^2}^2 + \frac{1}{2A}\|n - \overline{n}\|_{L^2}^2 + \frac{1}{A}\|\overline{n}\|_{L^2}^2 - \frac{1}{A}\|\nabla n\|_{L^2}^2 + \frac{B}{A}\|\nabla n\|_{L^2}^2\|n - \overline{n}\|_{L^2}^2 + \frac{1}{A}M\|n - \overline{n}\|_{L^2}^2
\]

\[
\leq \frac{1}{A}\|\nabla n\|_{L^2}^2 + \frac{B}{A}\|\nabla n\|_{L^2}^2\|n - \overline{n}\|_{L^2}^2 + \frac{1}{A}M^2.
\]

The desired estimate follows (note that \(M \lesssim \|n_{in}\|_{L^2}\)). \(\square\)

### 2.2 Enhanced dissipation estimate, (2.7b)

Proposition 2 implies that (2.7b) holds trivially on a time-scale like \(t \lesssim A^{1/2} \log A\). In order to deduce the enhanced dissipation effect for longer times, we use the hypo coercivity technique of [1], which builds on the earlier work of [26, 3]. As outlined in [46], hypo coercivity techniques are based on finding an energy which extracts the fact that the quadratic quantity \(A^{-1}\|\nabla f\|_{L^2}^2 + \|u'\partial_x f\|_{L^2}^2\) is much ‘more coercive’ than \(A^{-1}\|\nabla f\|_{L^2}^2\). In [4] and here this is done via the following energies, defined \(k\)-by-\(k\),

\[
\Phi_k[n(t)] = \|\hat{n}_k(t)\|_{L^2}^2 + \|\sqrt{\alpha} \partial_y \hat{n}_k(t)\|_{L^2}^2 \quad (2.9)
\]

\[
\Phi[n(t)] = \sum_{k \neq 0} \Phi_k[n(t)] = \|n_\#(t)\|_{L^2}^2 + \|\sqrt{\alpha} \partial_y n_\#(t)\|_{L^2}^2 + 2\|\sqrt{\gamma} \partial_x n_\#(t)\|_{L^2}^2 + \|\sqrt{\gamma} |\partial_x| n_\#(t)\|_{L^2}^2. \quad (2.10)
\]

Here \(\alpha, \beta, \gamma\) are \(k\)-dependent constants (and hence should be interpreted as Fourier multipliers) satisfying

\[
\alpha(A, k) = \epsilon_a A^{-1/2} |k|^{-1/2}
\]

(2.11a)

\[
\beta(A, k) = \epsilon_\beta |k|^{-1}
\]

(2.11b)

\[
\gamma(A, k) = \epsilon_\gamma A^{1/2} |k|^{-3/2}
\]

(2.11c)
where $\epsilon_\alpha$, $\epsilon_\beta$, and $\epsilon_\gamma$ are small constants depending only on $u$ chosen in [4]. Among other things, these are chosen such that $8\beta^2 \leq \alpha \gamma$. Notice that in [4] for treating general situations one must also take $\alpha$, $\beta$, and $\gamma$ to be $y$-dependent, however, as suggested by [3], this is not necessary to treat shear flows with non-degenerate critical points with $y \in \mathbb{T}$ or $y \in \mathbb{R}$. The parameters $\epsilon_\alpha$, $\epsilon_\beta$, and $\epsilon_\gamma$ are tuned such that,

$$\Phi_k[u] \approx ||\hat{n}_k||^2_2 + ||\sqrt{\alpha} \partial_y \hat{n}_k||^2_2 + |k|^2 ||\sqrt{\gamma} u' \hat{n}_k||^2_2,$$

(2.12)

and hence

$$||\hat{n}_k||^2_2 + A^{-1/2} |k|^{-1/2} ||\partial_y \hat{n}_k||^2_2 \lesssim \Phi_k[u] \lesssim ||\hat{n}_k||^2_2 + |k|^{1/2} A^{1/2} ||\hat{n}_k||^2_2 + A^{-1/2} |k|^{-1/2} ||\partial_y \hat{n}_k||^2_2.$$  

(2.13)

As a result, $\Phi_k(t)$ is equivalent to the $H^1$ norm of $n_k$ but with constants that depend on $A$ and $k$. The primary step in the results of [4] is that for $u(y)$ satisfying the hypotheses in (1), then for the passive scalar equation on $\mathbb{T}^2$,

$$\partial_t f + u(y) \partial_x f = \frac{1}{A} \Delta f,$$

the norm $\Phi_k[f(t)]$ satisfies the following differential inequality for some small constant $\tilde{\epsilon}$ independent of $k$, $A$ (but depending on $u$):

$$\frac{d}{dt} \Phi_k[f(t)] \leq -\tilde{\epsilon} |k|^{1/2} A^{1/2} \Phi_k[f(t)].$$

The primary step in the proof of (2.7b) is the analogous statement (though summed over all $k$ due to the nonlinearity).

**Proposition 3.** There exists a small constant $c > 0$ depending only on $u$ such that, under the bootstrap hypotheses and for $A$ sufficiently large, there holds

$$\frac{d}{dt} \Phi[n(t)] \leq -\frac{c}{A^{1/2}} \Phi[n(t)].$$

(2.14)

By (2.13), it follows that

$$||n_{\neq}(t)||^2_{L^2} \leq \Phi(0) e^{-cA^{-1/2} t} \lesssim A^{1/2} ||n_{in}||^2_{H^1} e^{-cA^{-1/2} t}.$$  

(2.15)

**Remark 1.** Propositions 3 and 4 together imply (2.7b). Indeed, for $A$ sufficiently large:

$$||n_{\neq}(t)||^2 \lesssim ||n_{in}||^2_{H^1} \mathbf{1}_{t \leq \frac{1}{c} A^{1/2} \log A} + \mathbf{1}_{t \geq \frac{1}{c} A^{1/2} \log A} A^{1/2} ||n_{in}||^2_{H^1} e^{-\frac{cA^{-1/2} t}{2}} \lesssim ||n_{in}||^2_{H^1} e^{-\frac{cA^{-1/2} t}{2} \log A}. $$

We first compute the time derivative of $\Phi_k[n(t)]$. 

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Proposition 4. For $\varepsilon$ sufficiently small depending only on $u$, there holds,

$$
\frac{d}{dt}\Phi_k[n(t)] \leq \left\{ -\frac{\varepsilon}{2} |k|^{1/2} ||\tilde{n}_k||^2 - \frac{\varepsilon}{2} |k|^{5/2} ||\nabla \tilde{n}_k||^2 - \frac{\varepsilon}{2} |k|^{1/2} ||\nabla \partial_y \tilde{n}_k||^2 - \frac{\varepsilon}{2} |k|^{1/2} ||\nabla \partial_y \partial_z \tilde{n}_k||^2 - \frac{\varepsilon}{2} A^{1/2} ||\nabla \partial_y \tilde{n}_k||^2 - \frac{\varepsilon}{2} A^{1/2} ||\nabla \partial_y \partial_z \tilde{n}_k||^2
\right. \\
- \frac{1}{2} |k|^2 ||\nabla \partial_y \tilde{n}_k||^2 - \frac{1}{2} |k|^2 ||\tilde{n}_k||^2 - \frac{1}{4A} ||\nabla \partial_y \tilde{n}_k||^2 \\
- \frac{1}{4A} |k|^2 ||\nabla \partial_y \partial_z \tilde{n}_k||^2 \right\} \\
+ \left\{ 2Re(-L_k, \tilde{n}_k) - 2Re(\alpha \partial_y \tilde{n}_k, \tilde{n}_k, -L_k) - 2kRe([i\beta u'L_k, \partial_y \tilde{n}_k] + [i\beta u' \tilde{n}_k, \partial_y L_k]) \\
+ 2|k|^2 Re(\gamma u')^2 \tilde{n}_k, -L_k) \right\} \\
+ \left\{ -2Re(NL_k, \tilde{n}_k) + 2Re(\alpha \partial_y \tilde{n}_k, NL_k) - 2kRe([i\beta u' NL_k, \partial_y \tilde{n}_k] + [i\beta u' \tilde{n}_k, \partial_y NL_k]) \\
- 2|k|^2 Re(\gamma u')^2 \tilde{n}_k, NL_k) \right\}
$$

(2.16)

where $N_k$ refers to the negative terms. Recall that $L_k, NL_k$ are defined in (2.4b,2.4a).

Proof. The estimates from the linear terms (that is, the terms arising from the passive scalar equation (1.5)) are made in [4] and are omitted for the sake of brevity. The extra terms from the Keller-Segel nonlinearity in (1.4) are immediate. 

The remainder of the section is devoted to controlling $L$ and $NL$ by the negative terms in (2.16).

2.2.1 Estimate on the $L$ terms in (2.16)

These terms are linear in the $k$-th mode, and it accordingly makes sense to estimate these terms $k$-by-$k$. In this section we prove that for $A$ sufficiently large,

$$
L_k^1 + L_k^0 + L_k^\beta + L_k^\gamma \leq -\frac{1}{4} N_k. 
$$

(2.17)

We begin by estimating the $L_k^1$ term in (2.16). Integrating by parts and using Lemma A.1 Lemma A.2 and the bootstrap hypotheses, we have, for any fixed constant $B \geq 1$,

$$
|L_k^1| \leq \frac{2}{A} (2||n_0 - \overline{n})||\nabla \tilde{n}_k||^2 + \frac{1}{AB} ||\partial_y \tilde{n}_k||^2 + B ||\partial_y c_0||^2 ||\tilde{n}_k||^2 \\
+ \text{Re} \frac{2}{A} (\partial_y \tilde{c}_k \tilde{n}_k + \partial_y \tilde{c}_k \partial_y \tilde{n}_k, n_0 - \overline{n}) \\
\lesssim \frac{C_2^2}{A} ||\tilde{n}_k||^2 + \frac{1}{AB} ||\partial_y \tilde{n}_k||^2.
$$

Therefore, we can choose $B$ sufficiently large, and then $A$ sufficiently large, such that the following holds:

$$
|L_k^1| \lesssim \frac{C_2^2}{A} ||\tilde{n}_k||^2 + \frac{1}{AB} ||\partial_y \tilde{n}_k||^2 \leq -\frac{1}{16} N_k.
$$

and hence by the definition of $N_k$, this is consistent with (2.17).

Turn next to $L_k^0$ in (2.16), which we divide into the following four contributions:

$$
L_k^0 = -2Re(\alpha \partial_y \tilde{n}_k, \frac{1}{A} \overline{n} \tilde{n}_k + \frac{2}{A} (n_0 - \overline{n}) \tilde{n}_k - \frac{1}{A} \partial_y c_0 \partial_y \tilde{n}_k - \frac{1}{A} \partial_y \tilde{c}_k \partial_y n_0) \\
=: L_{k,0}^0 + L_{k,1}^0 + L_{k,2}^0 + L_{k,3}^0.
$$

(2.18)
For the $L_{k,1}^α$ term, we have the following by the bootstrap hypotheses, for any fixed $B \geq 1$:

$$
\left| L_{k,1}^α \right| \lesssim \frac{1}{AB} \left\| \sqrt{\alpha} \partial_{yy} \hat{n}_k \right\|_2^2 + \frac{B}{A} \left\| \sqrt{\alpha} (n_0 - \vec{n}) \hat{n}_k \right\|_2^2 \\
\lesssim \frac{1}{AB} \left\| \sqrt{\alpha} \partial_{yy} \hat{n}_k \right\|_2^2 + \frac{B}{A^{3/2}} \left\| n_0 - \vec{n}_k \right\|_2^2 \\
\lesssim \frac{1}{AB} \left\| \sqrt{\alpha} \partial_{yy} \hat{n}_k \right\|_2^2 + \frac{BC^2}{A^{3/2}} \left\| \hat{n}_k \right\|_2^2.
$$

Recalling the definition $\mathcal{N}_k$ from (2.16), it follows that by choosing $B$, then $A$, sufficiently large, we can control this term consistent with (2.17). The $L_{k,0}^α$ term is treated in the same manner; we omit the details for brevity.

Next, we estimate the second term $L_{k,2}^α$ in (2.18). Using Lemma A.2 we have the following for any $B \geq 1$:

$$
\left| L_{k,2}^α \right| \lesssim \frac{1}{BA} \left\| \sqrt{\alpha} \partial_{yy} \hat{n}_k \right\|_2^2 + \frac{B}{A^{3/2}} \left\| \partial_y \partial_0 \hat{n}_k \right\|_2^2 \\
\lesssim \frac{1}{BA} \left\| \sqrt{\alpha} \partial_{yy} \hat{n}_k \right\|_2^2 + \frac{BC^2}{A^{3/2}} \left\| \hat{n}_k \right\|_2^2.
$$

Hence, by the bootstrap hypotheses and the definition of $\mathcal{N}_k$, it follows we can choose $B$ large and then $A$ large to control this term consistent with (2.17).

Similarly, for $L_{k,3}^α$ in (2.18), by Lemma A.1

$$
\left| L_{k,3}^α \right| \lesssim \frac{1}{BA} \left\| \sqrt{\alpha} \partial_{yy} \hat{n}_k \right\|_2^2 + \frac{B}{A^{3/2}} \left\| \partial_y \hat{n}_k \right\|_2^2 \\
\lesssim \frac{1}{BA} \left\| \sqrt{\alpha} \partial_{yy} \hat{n}_k \right\|_2^2 + \frac{BC^2}{A^{3/2}} \left\| \hat{n}_k \right\|_2^2.
$$

As above, it follows we can choose $B$ large and then $A$ large to control this term consistent with (2.17).

Next, turn to the $L_{k,4}^α$ term in (2.18), which we divide into two contributions:

$$
L_{k,4}^α = 2kRe(i\beta u \hat{n}_k, \partial_y \left( \frac{1}{A} \hat{n}_k + \frac{2}{A} (n_0 - \vec{n}) \hat{n}_k - \frac{1}{A} \partial_0 c_0 \partial_y \hat{n}_k - \frac{1}{A} \partial_y \hat{c}_k \partial_y n_0 \right)) \\
+ 2kRe(i\beta u' \hat{n}_k, \partial_y \left( \frac{1}{A} \hat{n}_k + \frac{2}{A} (n_0 - \vec{n}) \hat{n}_k - \frac{1}{A} \partial_0 c_0 \partial_y \hat{n}_k - \frac{1}{A} \partial_y \hat{c}_k \partial_y n_0 \right), \partial_y \hat{n}_k) \\
=: L_{k,1}^α + L_{k,2}^α.
$$

By analogy with the $α$ terms, the first term in (2.19) is further decomposed via

$$
L_{k,1}^α = 2kRe(i\beta u'' \hat{n}_k, \partial_y \left( \frac{1}{A} \hat{n}_k + \frac{2}{A} (n_0 - \vec{n}) \hat{n}_k - \frac{1}{A} \partial_0 c_0 \partial_y \hat{n}_k - \frac{1}{A} \partial_y \hat{c}_k \partial_y n_0 \right)) \\
=: L_{k,10}^α + L_{k,11}^α + L_{k,12}^α + L_{k,13}^α.
$$

For the $L_{k,11}^α$ term in (2.20), we have the following, (for any fixed $B \geq 1$):

$$
\left| L_{k,11}^α \right| \lesssim 2kRe(i\beta u'' \hat{n}_k, \partial_y \left( \frac{2}{A} (n_0 - \vec{n}) \hat{n}_k \right)) \\
\lesssim \frac{1}{AB} \left\| \partial_y \hat{n}_k \right\|_2^2 + \frac{B}{A} |k|^2 \left\| n_0 - \vec{n} \right\|_\infty \left\| \sqrt{\beta} u' \hat{n}_k \right\|_2^2.
$$

By the bootstrap hypotheses and by choosing $B$, then $A$, large enough, this term is controlled consistent with (2.17). The $L_{k,10}^α$ term is treated in the same manner; we omit the details for the sake of brevity.
For the $L^\beta_{k,12}$ term in (2.20), using Lemma A.2 we have that for some fixed $B \geq 1$, the following holds,

$$
|L^\beta_{k,12}| \leq 2kRe\langle i\beta u'\hat{n}_k, \frac{1}{A}(n_0 - \pi)\partial_y \hat{n}_k \rangle + 2kRe\langle i\beta u'\hat{n}_k, \frac{1}{A}\partial_y c_0 \partial_y \hat{n}_k \rangle
$$

$$
\leq \frac{1}{AB}||\partial_y \hat{n}_k||^2 + \frac{B|k|^2\beta}{A}||n_0 - \pi||^2||\sqrt{\beta u'} \hat{n}_k||^2
$$

$$
\quad + \frac{1}{AB}||\sqrt{\alpha} \partial_y \hat{n}_k||^2 + \frac{B|k|^2\beta}{A\alpha}||\sqrt{\beta u'} \hat{n}_k||^2||\partial_y c_0||^2
$$

$$
\quad \leq \frac{1}{AB}||\partial_y \hat{n}_k||^2 + \frac{B|k|^2}{A}||n_0 - \pi||^2||\sqrt{\beta u'} \hat{n}_k||^2
$$

$$
\quad + \frac{1}{AB}||\sqrt{\alpha} \partial_y \hat{n}_k||^2 + \frac{B|k|^2M^2}{A1/2}||\sqrt{\beta u'} \hat{n}_k||^2
$$

As above, by the bootstrap hypotheses, for $B$ and $A$ sufficiently large, this term is controlled consistent with (2.17).

Consider next $L^\beta_{k,13}$ in (2.20), which we integrate by parts and further sub-divide as:

$$
L^\beta_{k,13} = 2kRe\langle i\beta u''\hat{n}_k + i\beta u' \partial_y \hat{n}_k, \frac{1}{A}\partial_y \hat{n}_k \partial_y n_0 \rangle =: L^\beta_{k,131} + L^\beta_{k,132}.
$$

For $L^\beta_{k,131}$, by Lemma A.1 and the definition of $\beta$, we have the following for a large constant $B \geq 1$,

$$
\left| L^\beta_{k,131} \right| \leq \frac{|k|^2B}{A}||\beta u'' \hat{n}_k||^2||\partial_y n_0||^2 + \frac{1}{AB}||\partial_y \hat{n}_k||^2
$$

$$
\quad \leq \frac{B||\partial_y n_0||^2}{A}||\n_0||^2 + \frac{B||\n_0||^2||\partial_y n_0||^2}{A}.
$$

Therefore, by the bootstrap hypotheses, for $B$, then $A$, large, this term is controlled consistent with (2.17). Using Lemma A.1 and the definition of $\beta$, the $L^\beta_{k,132}$ term in (2.21) is handled as follows for a large constant $B \geq 1$:

$$
\left| L^\beta_{k,132} \right| \leq \frac{1}{AB}||\partial_y \hat{n}_k||^2 + \frac{B}{A}||\n_0||^2||\partial_y n_0||^2
$$

Therefore, by the bootstrap hypotheses (in particular, (2.5c)), for $B$ and $A$ sufficiently large, this is consistent with (2.17).

Turn next to $L^\beta_{k,2}$, which we sub-divide as follows:

$$
L^\beta_{k,2} = 2kRe\left\langle i\beta u' \frac{1}{\sqrt{\alpha}} \n_0 \partial_y \hat{n}_k, \partial_y \hat{n}_k \right\rangle + 2kRe\left\langle i\beta u' \frac{1}{\sqrt{\alpha}} \n_0 - \pi \partial_y \hat{n}_k, \partial_y \hat{n}_k \right\rangle
$$

$$
\quad - 2kRe\left\langle i\beta u' \frac{1}{\sqrt{\alpha}} \partial_y c_0 \partial_y \hat{n}_k, \partial_y \hat{n}_k \right\rangle - 2kRe\left\langle i\beta u' \frac{1}{\sqrt{\alpha}} \partial_y c_0 \partial_y n_0, \partial_y \hat{n}_k \right\rangle
$$

$$
\quad =: L^\beta_{k,20} + L^\beta_{k,21} + L^\beta_{k,22} + L^\beta_{k,23}.
$$

By anti-symmetry, $L^\beta_{k,22} = 0$ (note this is simply the observation that $\langle u' \partial_y c_0 \sqrt{\alpha} \partial_y n_0, \sqrt{\alpha} \partial_y n \rangle = 0$). For the $L^\beta_{k,21}$ term, we use the following straightforward estimate for a constant $B \geq 1$:

$$
\left| L^\beta_{k,21} \right| \leq \frac{1}{AB}||\partial_y \hat{n}_k||^2 + \frac{B||n_0 - \pi||^2}{A}||k||^2||\sqrt{\beta u'} \hat{n}_k||^2
$$

This is consistent with (2.17) by the bootstrap hypotheses and $B,A$ large. The $L^\beta_{k,20}$ is treated similarly, we skip the detail for brevity. The $L^\beta_{k,23}$ term can be estimated in the same manner as $L^\beta_{k,132}$ above in (2.22) and hence is omitted for brevity. This completes the treatment of the $L^\beta_k$ term in (2.16).
Finally, we estimate $L_k^γ$ in (2.10). We first sub-divide into four parts:

$$L_k^γ = 2|k|^2 \text{Re}(\gamma(x')^2 \hat{n}_k + \frac{1}{A} \hat{m}_k + \frac{2}{A} (\bar{m}_0 - \pi) \hat{n}_k - \frac{1}{A} \partial_x c_0 \partial_y \hat{n}_k - \frac{1}{A} \partial_y \hat{c}_k \partial_y n_0)$$

$$= L_{k,0}^γ + L_{k,1}^γ + L_{k,2}^γ + L_{k,3}^γ. \tag{2.24}$$

The second term in (2.24) is estimated as follows for a fixed constant $B \geq 1$,

$$\left| L_{k,1}^γ \right| \lesssim \frac{B^γ}{A^β} \| \sqrt{\beta} u' \hat{n}_k \|_{2} \| n_0 - \pi \|_∞^2 + \frac{|k|^4}{AB} \| \sqrt{\gamma} u' \hat{n}_k \|_2^2 \lesssim \frac{B}{A^{1/2}} \| \sqrt{\beta} u' \hat{n}_k \|_{2} \| n_0 - \pi \|_∞^2 + \frac{|k|^4}{AB} \| \sqrt{\gamma} u' \hat{n}_k \|_2^2.$$ 

As above, this is consistent with (2.17) by the bootstrap hypotheses and $B,A$ large. The term $L_{k,0}^γ$ is treated similarly, hence, we omit the details for the sake of brevity. The term $L_{k,2}^γ$ in (2.24) is similar. Indeed, by Lemma A.3 we have for $B \geq 1$ large,

$$\left| L_{k,2}^γ \right| \lesssim \frac{B^γ}{A^β} \| k \| \| \sqrt{\beta} u' \hat{n}_k \|_{2} \| \partial_y c_0 \|_∞^2 + \frac{|k|^2}{AB} \| \sqrt{\gamma} u' \partial_y \hat{n}_k \|_2^2 \lesssim \frac{B}{A^{1/2}} \| k \| \| \sqrt{\beta} u' \hat{n}_k \|_{2} \| n_0 - \pi \|_∞^2 + \frac{|k|^2}{AB} \| \sqrt{\gamma} u' \partial_y \hat{n}_k \|_2^2.$$ 

As usual, this is consistent with (2.17) by the bootstrap hypotheses and $B,A$ large. The $L_{k,3}^γ$ term in (2.24), is estimated slightly differently; using Lemma A.1 we have for $B \geq 1$ large,

$$\left| L_{k,3}^γ \right| \lesssim \frac{1}{A^{1/2}} \| k \| \| \sqrt{\beta} u' \hat{n}_k \|_{2}^2 + \frac{B}{A^{3/2}} \| k \| \| \sqrt{\gamma} \partial_y \hat{c}_k \|_∞^2 \| \partial_y n_0 \|_2^2 \lesssim \frac{1}{A^{1/2}} \| k \| \| \sqrt{\beta} u' \hat{n}_k \|_{2}^2 + \frac{B}{A} \| \hat{n}_k \|_2^2 \| \partial_y n_0 \|_2^2.$$ 

This is consistent with (2.17) by the bootstrap hypotheses and $B,A$ large. This completes the proof of (2.17), and hence, under the bootstrap hypotheses, the contributions of the $L$ terms in (2.16) is absorbed by the $N_k$ terms for $A$ chosen sufficiently large.

### 2.2.2 Estimate on $NL$ terms

As these terms are nonlinear in non-zero frequencies, it is more natural to consider all of the frequencies at once. For the $NL_k^γ$ term in (2.10), writing,

$$- \sum_{k \neq 0} 2 \text{Re}(NL_k, \hat{n}_k) = - \left( \frac{1}{A} \nabla \cdot (n_φ \nabla c_φ), n_φ \right) = \frac{1}{A} \left( n_φ \nabla c_φ, \nabla n_φ \right) \leq \frac{1}{A} \| \nabla c_φ \|_∞ \| \nabla n_φ \|_2 \| n_φ \|_2.$$ 

By (A.3), for some constant $B > 0$,

$$- \sum_{k \neq 0} 2 \text{Re}(NL_k, \hat{n}_k) \lesssim \frac{1}{AB} \| \nabla n_φ \|_2^2 + \frac{B}{A} C_2^∞ \| n_φ \|_2^2.$$ 

By first choosing $B$ large relative to the implicit constant, and then choosing $A$ large (relative to constants and $B$), these terms are absorbed by the negative terms in (2.10).

For the $NL_k^γ$ term in (2.16), we use (A.3) and the bootstrap hypotheses to deduce (using the definition of $α$; recall that $α$ is a Fourier multiplier in $x$),

$$2 \text{Re} \sum_{k \neq 0} \left( \alpha(\partial_x) \partial_{yy} \hat{n}_k, NL_k \right) = \frac{2}{A} \left( \alpha(\partial_x) \partial_{yy} n_φ, \nabla \cdot (n_φ \nabla c_φ) \right) \lesssim \frac{1}{A^{3/4}} \| \sqrt{\alpha} \partial_{yy} n_φ \|_2 \| \nabla n_φ \|_2 \| \nabla c_φ \|_∞ + \| n_φ \|_2 \| n_φ \|_∞ \| n_φ \|_2 \lesssim \frac{1}{A^{3/4}} \| \sqrt{\alpha} \partial_{yy} n_φ \|_2 \left( C_2^∞ \| n_φ \|_2 + C_∞ \| n_φ \|_2 \right) \lesssim \frac{1}{A^{3/4}} \| \sqrt{\alpha} \partial_{yy} n_φ \|_2^2 + \frac{C_2^∞}{A^{3/4}} \| n_φ \|_2^2 + \frac{C_∞}{A^{3/4}} \| n_φ \|_2^2.$$
and choosing $A$ large, these terms are absorbed by the negative terms in (2.10).

There are two terms in $NL_k^\beta$ in (2.10); we estimate the first as follows (using that $\beta(k) \lesssim |k|^{-1}$ and defines a self-adjoint operator, Lemma A.3, and that $u$ does not depend on $x$):

$$-2k \sum_{k \neq 0} \text{Re}(\langle \beta(\partial_x) u' N L_k, \partial_y \vec{n}_k \rangle) = -\frac{2}{A} \langle \beta(\partial_x) u' \partial_x n_\beta, \partial_y n_\beta \rangle$$

$$\lesssim \frac{1}{A} \frac{1}{2} \|n_\beta\|_2 \|n_\beta\|_\infty \|\partial_y n_\beta\|_2 + \frac{1}{A} \|\beta u' \partial_x \partial_y n_\beta\|_2 \|\nabla c_\beta\|_\infty \|\nabla n_\beta\|_2$$

$$\lesssim \frac{C_0}{A} \|n_\beta\|_2 \|\partial_y n_\beta\|_2 + \frac{C_2}{A^{5/4}} \|\sqrt{\gamma} u' \partial_x \partial_y n_\beta\|_2 \|\nabla n_\beta\|_2$$

$$\lesssim \frac{C_0}{A^{5/4}} \|n_\beta\|_2^2 + \frac{1}{A} \|\nabla n_\beta\|_2^2 + \frac{C_2}{A^{5/4}} \|\sqrt{\gamma} u' \partial_x \partial_y n_\beta\|_2^2.$$  (2.25)

Choosing $A$ large, these terms are absorbed by the negative terms in (2.10). For the second term in $NL_k^\beta$ we use

$$2 \text{Re} \sum_{k \neq 0} \langle \beta(\partial_x) u' k \vec{n}_k, \partial_y NL_k \rangle = 2 \frac{A}{A} \langle \beta(\partial_x) u' \partial_x n_\beta, \partial_y \nabla \cdot (n_\beta \nabla c_\beta) \rangle$$

$$- \frac{2}{A} \langle \beta(\partial_x) u' \partial_x n_\beta, \partial_y \nabla \cdot (n_\beta \nabla c_\beta) \rangle - \frac{2}{A} \langle \beta(\partial_x) u' \partial_x n_\beta, \nabla \cdot (n_\beta \nabla c_\beta) \rangle$$

$$= NL_k^\beta_{k,1} + NL_k^\beta_{k,2}.$$  

Using Lemma A.3 $\beta(\partial_x) = \epsilon |\partial_x|^{-1}$, and that $u$ does not depend on $x$, we have,

$$|NL_k^\beta_{k,1}| \lesssim \frac{1}{A} \frac{1}{2} \|\beta u' \partial_x n_\beta\|_2 (\|\nabla n_\beta\|_2 \|\nabla c_\beta\|_\infty + \|n_\beta\|_2 \|\nabla n_\beta\|_\infty)$$

$$\lesssim \frac{C_2}{A} \|n_\beta\|_2 (\|n_\beta\|_2 + \|n_\beta\|_2)$$

$$\lesssim \frac{1}{BA} \|\nabla n_\beta\|_2^2 + \frac{BC_2}{A} \|n_\beta\|_2^2,$$  

yielding terms which are absorbed by the negative terms in (2.10) for $A$ sufficiently large. The treatment of $NL_k^\beta_{k,2}$ is similar to (2.25), hence it is omitted for the sake of brevity.

Turn finally to term $NL_k^\gamma$ in (2.10) associated with $\gamma$:

$$-2 \text{Re} \sum_{k \neq 0} \langle |k|^2 \gamma(k) u' n_k, u' NL_k \rangle = -\frac{2}{A} \langle \gamma(\partial_x) u' \partial_x n_\beta, u' \partial_x \nabla \cdot (n_\beta \nabla c_\beta) \rangle$$

$$= \frac{2}{A} \langle \gamma(\partial_x) u' \partial_x n_\beta, u' \partial_x \nabla c_\beta \rangle + \frac{4}{A} \langle \gamma(\partial_x) u' u' \partial_x n_\beta, \partial_x (n_\beta \partial_y c_\beta) \rangle$$

$$=: NL_k^\gamma_{k,1} + NL_k^\gamma_{k,2}.$$  (2.26)

Then we use $\gamma(\partial_x) = \epsilon A^{1/2} |\partial_x|^{-3/2}$, interpolation, and Lemma A.3 to deduce the following bound for $NL_k^\gamma_{k,1}$:

$$NL_k^\gamma_{k,1} \lesssim \frac{1}{A} \|\sqrt{\gamma} u' \partial_x \nabla n_\beta\|_2 \|\sqrt{\gamma} \partial_x (u' n_\beta \nabla c_\beta)\|_2$$

$$\lesssim \frac{1}{A^{3/4}} \|\sqrt{\gamma} u' \partial_x \nabla n_\beta\|_2 \|\partial_x |^{1/4} (u' n_\beta \nabla c_\beta)\|_2$$

$$\lesssim \frac{1}{A^{3/4}} \|\sqrt{\gamma} u' \partial_x \nabla n_\beta\|_2 \|u' n_\beta \nabla c_\beta\|^{3/4} \|\partial_x (u' n_\beta \nabla c_\beta)\|^{1/4}$$

$$\lesssim \frac{1}{A^{3/4}} \|\sqrt{\gamma} u' \partial_x \nabla n_\beta\|_2 \|u' n_\beta\|^{3/4} \|\nabla c_\beta\|^{3/4} \left(\|u' \partial_x n_\beta\|_2^{1/4} \|\nabla c_\beta\|_\infty^{1/4} + \|n_\beta\|_2^{1/4} \|\partial_x \nabla c_\beta\|_2^{1/4}\right)$$

$$\lesssim \frac{C_2}{A^{3/4}} \|\sqrt{\gamma} u' \partial_x \nabla n_\beta\|_2 \|u' n_\beta\|^{3/4} \left(\|u' \partial_x n_\beta\|_2^{1/4} + \|n_\beta\|_2^{1/4}\right)$$

$$\lesssim \frac{1}{BA} \|\sqrt{\gamma} u' \partial_x \nabla n_\beta\|_2^2 + \frac{BC_2}{A^{1/2}} \|u' n_\beta\|_2^2 \left(\|u' \partial_x n_\beta\|_2^{1/2} + \|n_\beta\|_2^{1/2}\right).$$
Hence, for $B$ chosen large, then $A$ chosen large, we may absorb these contributions in the negative terms in (2.1b).

Next we estimate the $NL_{k,2}^c$ term in (2.20),

$$NL_{k,2}^c \lesssim \frac{1}{A^{3/4}} \|\sqrt{\gamma} u' \partial_x \|^{5/4} \|n_x\|_2 \|n_x \nabla c_x\|_2 \lesssim \frac{1}{A^{3/4}} \|\sqrt{\gamma} u' \partial_x \|^{5/4} \|n_x\|_2^2 + C^2 \frac{\infty}{A^{3/4}} \|n_x\|_2^2.$$  

Hence, for $A$ chosen large, we may absorb these contributions in the negative terms in (2.1b). This finishes the estimate of the $NL$ terms.

### 2.3 Nonzero mode $L_t^2 \hat{H}_{x,y}^1$ estimate (2.7a)

The nonzero mode $L_t^2 \hat{H}_{x,y}^1$ estimate (2.7a) comes from an estimate on the $\frac{d}{dt} \|\hat{n}_x\|_2^2$ and the knowledge that $\|\hat{n}_x\|_2^2$ is bounded by $4C_{ED} \|n_{in}\|_{\hat{H}}^2$ from Hypothesis (2.5a). Indeed, from (1.1) and Lemma A.2 there holds for some universal constant $B$,

$$\frac{1}{2} \frac{d}{dt} \|n_x\|_2^2 = \frac{1}{A} \Delta n_x + \frac{1}{A} m_n + \frac{1}{A} \Delta n_0 \cdot \gamma n_x - \frac{1}{A} \nabla c_0 \cdot \nabla n_x - \frac{1}{A} \nabla c_x \cdot \nabla n_0 - \frac{1}{A} (\nabla \cdot (\nabla c_\epsilon(n_x)))_\epsilon$$

$$\leq - \frac{1}{2A} \|\nabla n_x\|_4^2 + \frac{1}{A} \|n_x\|_2^2 \left( \|n_0 - m\|_\infty + \|n_0\|_\infty + \|\nabla c_0\|_4 \right) + \frac{1}{A} \|\partial_x n_x\|_\infty \|\partial_x n_0\|_2 \|n_x\|_2 + \frac{2}{A} \|\nabla n_x\|_2 \|\nabla c_x\|_4 \|n_x\|_4$$

$$\leq - \frac{1}{2A} \|\nabla n_x\|_2^2 + \frac{B(C^2_{\infty} + C^2_{H^1})}{A} \|n_x\|_2^2 + \frac{2}{A} \|\nabla n_x\|_2 \|\nabla c_x\|_4 \|n_x\|_4. \quad (2.27)$$

By the Gagliardo-Nirenberg-Sobolev inequality, we obtain

$$\|\nabla n_x\|_2 \|\nabla c_x\|_4 \|n_x\|_4 \lesssim \|c_x\|_1 \|\nabla c_x\|_1 \|\nabla n_x\|_2 \lesssim \|n_x\|_2 \|\nabla n_x\|_2^3 \lesssim \|n_x\|_2 \|\nabla n_x\|_2^3,$$

which implies the following (possibly adjusting $B$),

$$\frac{1}{2} \frac{d}{dt} \|n_x\|_2^2 \leq - \frac{1}{4A} \|\nabla n_x\|_2^2 + \frac{B(C^2_{\infty} + C^2_{H^1})}{A} \|n_x\|_2^2 + \frac{B}{A} \|n_x\|_2^6. \quad (2.28)$$

By (2.25a), the time integral of $\frac{1}{A} \|n_x\|_2^2$ is estimated as

$$\int_0^{T_N} \frac{1}{A} \|n_x(t)\|_2^2 dt \lesssim \log A + \frac{A}{A^{1/2}}. \quad (2.29)$$

Hence, by applying (2.5a), integrating (2.28), and choosing $A$ large, there holds

$$\frac{1}{A} \int_0^{T_N} \|\nabla n_x\|_2^2 dt \leq \frac{1}{A^{1/4}} + 2 \|n_{in}\|_2^2 \leq 4 \|n_{in}\|_2^2.$$

As a result, we have proved (2.7a).

### 2.4 Zero mode estimate (2.7c)

First, by non-negativity, note that $\|n_x\|_{L^1_t} = \|n\|_{L^1_t} = M$ is constant in time. We begin by estimating $\|n_0 - \bar{m}\|_2^2$, then go on to estimate $\|\partial_x n_0\|_2^2$. From (1.1) we have, by Minkowski’s inequality,

$$\frac{d}{dt} \|n_0 - \bar{m}\|_2^2 = \|n_0 - \bar{m}\|_2^2 + \frac{1}{A} \|\partial_x n_0\|_2^2 \|n_0\|_2^2 + \frac{1}{A} \|\partial_x n_0\|_2^2 \|\partial_x n_0\|_2^2$$

$$\leq \frac{1}{2A} \|\partial_x n_0\|_2^2 + \frac{BM^2}{A} \|n_0\|_2^2 + \frac{1}{A} \|\partial_x c_x n_x\|_2^2 \|n_0\|_2^2.$$  

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Recall the following Nash inequality on $T$, under the assumption that $\int_T \rho dx = 0$:

$$||\rho||_{L^2(T)} \lesssim ||\rho||_{L^1(T)}^{2/3}||\partial_y \rho||_{L^2(T)}^{1/3}. \quad (2.30)$$

Hence,

$$-||\partial_y n_0||^2_2 \lesssim -\frac{||n_0 - \pi||^2_2}{||n - \pi||_1} \lesssim -\frac{||n_0 - \pi||^2_2}{M^4}.$$ 

Therefore,

$$\frac{d}{dt} ||n_0 - \pi||^2_2 \lesssim -\frac{1}{A} ||n_0 - \pi||^2_2 + \frac{1}{|A|} ||n_0 - \pi||^2_2 (||n_0 - \pi||^2_2 + M^2) + \frac{1}{|A|} ||\nabla n_\#||_2 ||n_\#||^2_2$$

$$\lesssim -\frac{1}{A} ||n_0 - \pi||^2_2 (||n_0 - \pi||^2_2 - M^4 ||n_0 - \pi||^2_2 - M^6) + \left\{ \frac{1}{|A|} ||\nabla n_\#||^2_2 + \frac{B}{|A|} ||n_\#||^2_2 \right\}.$$ 

Define the following quantity $G$ to be the time integral of the terms in the $\{\}$:

$$G(t) := \int_0^t \frac{1}{AB} ||\nabla n_\#||^2_2 + \frac{B}{A} ||n_\#||^6_2 dt, \quad t \geq 0. \quad (2.31)$$

By the bootstrap hypotheses, there holds $0 \leq G \lesssim ||n_{in}||^2_2 + ||n_{in}||^6_{H^1} A^{-1/2} \log A$. Applying this definition,

$$\frac{d}{dt} (||n_0 - \pi||^2_2 - G(t)) \lesssim -\frac{1}{A} ||n_0 - \pi||^2_2 (||n_0 - \pi||^2_2 - M^8 - M^6)$$

$$\lesssim -\frac{1}{A} ||n_0 - \pi||^2_2 (||n_0 - \pi||^2_2 - G(t) - \sqrt{M^8 + M^6} (||n_0 - \pi||^2_2 + \sqrt{M^8 + M^6})).$$ 

Choosing $A$ large relative to $||n_{in}||^6_{H^1}$ and universal constants, we have

$$||n_0 - \pi||^2_2 \lesssim \pi^2 + G(t) + \sqrt{M^8 + M^6} + ||n_{in}||^2_2$$

$$\lesssim M^2 + ||n_{in}||^2_2 + \sqrt{M^8 + M^6}$$

$$=: C^2_{\pi} (||n_{in}||^2_2, M). \quad (2.32)$$

This completes the estimate on $||n_0||^2_2$, which implies the first estimate in conclusion (2.23).

Next, we use (2.32) to estimate $||\partial_y n_0||^2_2$. From (1.1) and Minkowski’s integral inequality, we have for some $B > 0$,

$$\frac{1}{2} \frac{d}{dt} ||\partial_y n_0||^2_2$$

$$= \langle \partial_y n_0, \partial_y \left( \frac{1}{A} \partial_y n_0 - \frac{1}{A} \partial_y \partial_y c_0 n_0 - \frac{1}{A} (\nabla \cdot (\nabla c_\# n_\#)) \right) \rangle$$

$$\leq -\frac{1}{2|A|} ||\partial_y n_0||^2_2 + \frac{B}{|A|} ||\partial_y c_0 n_0||^2_2 + \frac{B}{|A|} ||\partial_y c_\# n_\#||_2 ||n_\#||^2_{L^2(T)} + \frac{B}{|A|} ||\partial_y c_\# n_\#||_2 ||n_\#||^2_{L^2(T)}$$

$$\leq -\frac{1}{2|A|} ||\partial_y n_0||^2_2 + \frac{B}{|A|} ||\partial_y c_0 n_0||^2_2 + \frac{B}{|A|} ||\partial_y c_\# n_\#||_2 ||n_\#||^2_{L^2(T)} + \frac{B}{|A|} ||\partial_y c_\# n_\#||_2 ||n_\#||^2_{L^2(T)}$$

$$+ \frac{B}{|A|} ||\partial_y c_\#||^2_{L^\infty(T^2)} ||\partial_y n_\#||^2_{L^2(T^2)}. \quad (2.33)$$

Using (A.3) in the above estimate (2.33), we have for some $B$ (possibly adjusted from above),

$$\frac{1}{2} \frac{d}{dt} ||\partial_y n_0||^2_2 \leq -\frac{1}{2|A|} ||\partial_y n_0||^2_2 + \frac{B}{|A|} ||\partial_y c_0 n_0||^2_2 + \frac{B}{|A|} ||\partial_y c_\# n_\#||^2_2 + \frac{B}{|A|} ||\partial_y n_\#||^2_{L^2(T^2)} + \frac{B}{A} ||\partial_y n_\#||^2_{L^\infty(T^2)}$$

$$+ \frac{BC^2_{\pi}}{A} ||\partial_y n_\#||^2_{L^2(T^2)}. \quad (2.34)$$
Analogously to (2.31), we define
\[ G(t) := \int_0^t \frac{B}{A} ||n_\neq||_{L^2((T_2^\infty T^2))}^2 + \frac{BC^2}{A} ||\partial_y n_\neq||_{L^2((T_2^\infty T^2))}^2 dt, \quad \forall t \in [0, T_*]. \]  

By the bootstrap hypothesis (2.5a), (2.5b) and (2.5d) and choosing \( A \) large, there holds:
\[ G(t) \lesssim \int_0^{T_*} \frac{C_4^2}{A} e^{-\frac{1}{A^2}t} \left( \frac{1}{A} \right)^{\frac{1}{A}} C_2^2 ||\partial_y n_\neq||_{L^2(T_2^\infty T^2)}^2 dt \lesssim C_2^4 e^C. \]

Therefore, from (2.34), we have for some \( B > 0 \) (using also \( ||\partial_y n_0||_2 \lesssim ||n_0||_2^{1/2} ||\partial_y n_0||_2^{1/2} \)),
\[ \frac{d}{dt} (||\partial_y n_0||_2^2 - 2G(t)) \leq - \frac{||\partial_y n_0||^4_2}{ABC^2_{L^2}} + \frac{B}{A} ||n_0||^2_2 ||\partial_y n_0||_2^2 + \frac{B}{A} ||n_0 - \pi||^2_2 ||\partial_y n_0||_2^2 \]
\[ \leq - \frac{1}{ABC^2_{L^2}} ||\partial_y n_0||^2_2(||\partial_y n_0||_2^2 - 2G(t) - C_4^4 B^2). \]

Integrating and applying (2.35) implies the following:
\[ ||\partial_y n_0||_2^2 \leq 2G(t) + C_4^4 B + ||\partial_y n_{in}||_2^2 \lesssim C_2^4 + ||\partial_y n_{in}||_2^2. \]

Hence, by choosing \( C^2_{H_1} \gg C_2^4 + ||\partial_y n_{in}||_2^2 \), we complete the proof of (2.7c).

### 2.5 \( L^\infty \) uniform control (2.7d)

By the bootstrap hypothesis (2.5b) and (2.5c), it follows that \( ||n||_2^2 \lesssim ||n_{in}||_{H^1}^2 + C^2_4 (||n_{in}||_2, M) < \infty \). As the \( L^2 \) norm is subcritical for 2D Patlak-Keller-Segel, it is standard (see e.g. [33 36 18] and the references therein) that this implies a uniform-in-time \( L^\infty \) bound which depends only on \( ||n||_{L^\infty(0, T_*; L^2)} \). Therefore, by choosing \( C_\infty \) appropriately, we have (2.7d):
\[ ||n||_{L^\infty(0, T_*; L^\infty)} \leq 2C_\infty = 2C_\infty (||n_{in}||_{H^1}). \]

This completes the proof of Proposition 11 and hence Theorem 1.

### 3 Proof of Theorem 2 in the case \( \mathbb{T}^3 \)

Next we turn to the 3D case. Heuristically, we expect the problem to be effectively \( L^1 \) critical with critical mass \( 8\pi \). As in e.g. [17], we will need to use the free energy to obtain such a precise control.

#### 3.1 Basic setting and bootstrap

Consider the Patlak-Keller-Segel equation with advection on \( \mathbb{T}^3 \):
\[ \begin{cases} 
\partial_t n + u(y_1) \partial_x n + \frac{1}{2} \nabla \cdot (\nabla cn) = \frac{1}{4} \Delta n, \\
-\Delta c = n - \pi, \\
n(\cdot, 0) = n_{in},
\end{cases} \]

(3.1)

where \( (x, y_1, y_2) \in \mathbb{T}^3 \). We use the notation
\[ (x, y_1, y_2) \in \mathbb{T} \times \mathbb{R}^2, \]
\[ dy = dy_1 dy_2, \]
\[ \nabla_y = (\partial_{y_1}, \partial_{y_2}), \]
\[ \Delta_y = \partial_{y_1 y_1} + \partial_{y_2 y_2}. \]
As above, the bootstrap argument is applied to prove Theorem 2. For constants $C_{ED}, C_{L^2}, C_{H^1}, C_{\infty}$ determined by the proof, define $T_*$ to be the end-point of the largest interval $[0, T_*]$ such that the following hypotheses hold for all $T \leq T_*:

(1) Nonzero mode $L_t^2 H_{x,y}^1$ estimates:

$$\frac{1}{A} \int_0^{T_*} \| \nabla_{x,y} n_{\neq} \|_{L_t^2(T^3)} dt \leq 8 \| n_{in} \|_{H^1}^2; \quad (3.2a)$$

(2) Nonzero mode enhanced dissipation estimate:

$$\| n_{\neq} \|_{L_t^2(T^3)} \leq 4C_{ED} \| n_{in} \|_{H^1}^2 e^{-\frac{t^2}{A^{1/2} \log A}}, \quad (3.2b)$$

where $c$ is a small number independent of $A$;

(3) Zero mode time independent estimate:

$$\| n_0 \|_{L_t^\infty (0, T_*; L_x^2)} \leq 4C_{L^2}, \quad (3.3c)$$

$$\| \partial_y n_0 \|_{L_t^\infty (0, T_*; L_x^2)} \leq 4C_{H^1}; \quad (3.4c)$$

(4) $L_t^4 L_{x,y}^\infty$ estimate of the whole solution:

$$\| n \|_{L_t^\infty (0, T_*; L_{x,y}^\infty)} \leq 4C_{\infty}. \quad (3.4d)$$

As in the two-dimensional case, we introduce the following constant:

$$C_{2,\infty} := 1 + M + C_{ED}^{1/2} \| n_{in} \|_{H^1} + C_{L^2} + C_{\infty}. \quad (3.5)$$

Here $C_{ED}$ just depends on the properties of the shear flow $u$, $C_{L^2}$ just depends on the initial data $n_{in}$, $C_{\infty}$ depends on $n_{in}$ and $C_{L^2}$, and $C_{H^1}$ depends on $n_{in}$, $C_{L^2}$, and $C_{\infty}$. Recall that we assume that the data is initially bounded strictly away from zero from below:

$$\min_{(x, y_1, y_2) \in T^3} n_{in}(x, y_1, y_2) \geq q > 0. \quad (3.4)$$

As in [2] by local well-posedness of mild solutions, the quantities on the left-hand sides of (3.2a), (3.2b), (3.2c), and (3.2d) take values continuously in time. Moreover, the inequalities are all satisfied with the 4’s replaced by 2’s for $t$ sufficiently small. By the standard continuation criteria for (1.1), the solution exists and remains smooth on an interval $(0, t_0)$, with $t_0 > T_*$ such that $t_0 - T_*$ can be taken to depend only on $\| n(T_*) \|_{L_t^2}$. By continuity, the following proposition shows that the solution is global and satisfies the a priori estimates (H) for all time.

**Proposition 5.** For all $n_{in}$ and $u$, if the condition (3.5) and the above bootstrap hypothesis (H) are satisfied, there exists an $A_0(\| n_{in} \|_{L^\infty}, \| n_{in} \|_{H^1}, M, q)$ such that if $A > A_0$ then the following conclusions, referred to as (C), hold on the interval $[0, T_*]$:

(1)

$$\frac{1}{A} \int_0^{T_*} \| \nabla_{x,y} n_{\neq} \|_{L_t^2(T^3)} dt \leq 4 \| n_{in} \|_{H^1}^2; \quad (3.5a)$$

(2)

$$\| n_{\neq} \|_{L_t^2(T^3)} \leq 2C_{ED} \| n_{in} \|_{H^1} e^{-\frac{t^2}{A^{1/2} \log A}}; \quad (3.5b)$$

(3)

$$\| n_0 \|_{L_t^\infty (0, T_*; L_x^2)} \leq 2C_{L^2}, \quad (3.5c)$$

$$\| \partial_y n_0 \|_{L_t^\infty (0, T_*; L_x^2)} \leq 2C_{H^1};$$

(4)

$$\| n \|_{L_t^\infty (0, T_*; L_{x,y}^\infty)} \leq 2C_{\infty}. \quad (3.5d)$$
The main new difficulty in the 3D case arises in the proof of (4.4c): even if non-zero modes could be
neglected entirely, the evolution of \( n_0 \) would be given by the \( L^1 \) critical parabolic-elliptic Patlak-Keller-
Segel. In [17], the free energy, together with the logarithmic Hardy-Littlewood-Sobolev inequality (see e.g.
[19]), was applied to prove global existence up to the critical mass. Similarly, here we will estimate the 2D
free energy of \( n_0 \) (no longer a conserved quantity) and apply the 2D logarithmic Hardy-Littlewood-Sobolev
inequality on \( n_0 \). We are met with a small difficulty in estimating the effect of non-zero f requencies on the
free energy in regions of low density; to help deal with this, we utilize a pointwise lower bound on the solution
(See Lemma 3.1 below).

3.2 Estimate on the zero mode (3.5c)
The idea of the proof is to exploit the fact that the shear flow strongly damps the nonzero frequencies.
Hence, even though the equation (3.1) is posed on \( \mathbb{T}^3 \), we can approximate the evolution as the classical
Keller-Segel equation in \( \mathbb{T}^2 \) with a rapidly decaying perturbation \((\nabla \cdot (\nabla c \neq n \neq 0))\) coming from the nonzero
modes.

First we derive an exponentially decreasing lower bound for \( n_0 \).

Lemma 3.1. Under the bootstrap hypotheses (H) and (3.4), there holds the following pointwise lower bound
on the solution for all \( t \in [0, T^*] \)

\[
\left\| \frac{1}{n_0(t)} \right\|_\infty \leq \left\| \frac{1}{n(t)} \right\|_\infty \leq q^{-1} e^{-\frac{t}{T^*}}. 
\] (3.6)

Proof. The equation (3.1) implies that at the point \((x_{\min}(t), y_{\min}(t))\) where the minimum in space of the
solution is achieved, the following inequality is satisfied:

\[
(\partial_t n)(x_{\min}, y_{\min}) = \frac{1}{A} (\Delta n)(x_{\min}, y_{\min}) + \frac{1}{A} (n(x_{\min}, y_{\min}) - \bar{n}) n(x_{\min}, y_{\min})
\geq - \frac{1}{A} n(x_{\min}, y_{\min}),
\]

which implies that

\[
\frac{d}{dt} n_{\min}(t) \geq - \frac{1}{A} n_{\min}(t).
\]

Combining this differential inequality with (3.4), this yields

\[
n_{\min}(t) \geq q e^{-\frac{t}{T^*}},
\]

which completes the lemma.

Next, we study the classical 2D free energy of \( n_0 \) on \( \mathbb{T}^2 \):

\[
\mathcal{F}[n_0] = \int_{\mathbb{T}^2} n_0 \log n_0 - \frac{1}{2} (n_0 - \bar{n}) c \, dy.
\]

Lemma 3.2. Under the bootstrap hypotheses (H) and (3.4), for \( A \) sufficiently large, there holds the following
uniform bound on \( t \in [0, T^*] \),

\[
\mathcal{F}[n(t)] \leq 2 \mathcal{F}[n_{\min}].
\] (3.8)

Proof. By applying the hypothesis (3.2b,3.2d), Minkowski’s integral inequality, and (3.6), the time derivative
of \( F[n_0] \) can be estimated as follows

\[
\frac{d}{dt} F[n_0] = - \frac{1}{A} \int n_0 |\nabla_y \log n_0 - \nabla_y c_0|^2 dy - \frac{1}{A} \int (\nabla_y c_{\neq n_0})_0 \cdot (\nabla_y \log n_0 - \nabla_y c_0) dy
\]

\[
\leq - \frac{1}{2A} \int |n_0 |\nabla_y \log n_0 - \nabla_y c_0|^2 dy + \frac{1}{2A} \int \frac{1}{n_0} \left| \int_0^1 \frac{|\nabla_y c_{\neq n_0}(t, y)|^2}{n_0} dt \right| dy
\]

\[
\leq - \frac{1}{2A} \int |n_0 |\nabla_y \log n_0 - \nabla_y c_0|^2 dy + \frac{1}{2A} \left( \log n_0 \right)_0 \| \nabla_y c_{\neq n_0} \|^2_{L^2(\mathbb{T}^3)} \| n_0 \|^2_{L^\infty(\mathbb{T}^3)}
\]

\[
\lesssim - \frac{1}{2A} \int |n_0 |\nabla_y \log n_0 - \nabla_y c_0|^2 dy + \frac{C^2A}{2A} e^{C^2 \log A} (\pi^{-1/2 \log A})^t.
\]

Note that for \( A \) sufficiently large yields:

\[
\int_0^\infty \frac{C^2A}{2A} e^{-t/4} dt \leq \int_0^\infty \frac{C^2A}{2A} e^{-2t/4 \log A} dt \lesssim \frac{C^2A}{2A} A^{1/2} \log A.
\]

Combining (3.9) and (3.10) yields the uniform time (3.8).

Next, we use (3.8) to get a bound on the entropy:

**Lemma 3.3.** If (3.8) holds and \( A \) is chosen large enough, there exists a constant \( C_L \log L(n_{in}) \) such that

\[
\int n_0 \log^+ n_0 dy \leq C_L \log L(n_{in}).
\]

**Proof.** The following logarithmic Hardy-Littlewood-Sobolev inequality on a compact manifold is needed:

**Theorem 1.** Let \( M \) be a two-dimensional, Riemannian, compact manifold. For all \( M > 0 \), there exists a constant \( C(M) \) such that for all non-negative functions \( f \in L^1(M) \) such that \( f \log f \in L^1 \), if \( \int_M f dx = M \), then

\[
\int_M f \log f dx + \frac{2}{M} \int_M \int_{M \times M} f(x) f(y) \log d(x, y) dx dy \geq -C(M),
\]

where \( d(x, y) \) is the distance on the Riemannian manifold.

Let \( y \in \mathbb{T}^2 \) be fixed. Define the cut-off function \( \varphi_y(z) \in C^\infty \) such that

\[
\text{supp}(\varphi_y) = B(y, 1/4), \quad \varphi_y(z) \equiv 1, \forall z \in B(y, 1/8), \quad \text{supp}(\nabla \varphi_y(z)) \subset \overline{B}(y, 1/4) \backslash B(y, 1/8).
\]

By extending \( n_0(z) \) and \( c_0(z) \) periodically to \( \mathbb{R}^2 \), we can rewrite the equation \( -\Delta c_0 = n_0 - \pi \) on \( \mathbb{T}^2 \) such that it is posed on \( \mathbb{R}^2 \):

\[
-\Delta_z (\varphi_y c_0(z)) = (n_0(z) - \pi) \varphi_y(z) - 2 \nabla_z \varphi_y(z) \cdot \nabla_z c_0(z) - \Delta_z \varphi_y(z) c_0(z).
\]

Using the fundamental solution of the Laplacian on \( \mathbb{R}^2 \):

\[
c_0(y) = c_0(y) \varphi_y(y)
\]

\[
= \frac{1}{2\pi} \int_{\mathbb{R}^2} \log |y - z| \left( (n_0(z) - \pi)c_0(z) - 2 \nabla_z c_0(z) \cdot \nabla_z c_0(z) - \Delta_z \varphi_y(z) c_0(z) \right) dz
\]

\[
= \frac{1}{2\pi} \int_{|y - z| \leq \frac{1}{4}} \log |y - z| (n_0(z) - \pi)c_0(z) dz - \frac{1}{\pi} \int_{|y - z| \leq \frac{1}{4}} \nabla_z \cdot (\log |y - z| \nabla_z \varphi_y(z)) c_0(z) dz
\]

\[
+ \frac{1}{2\pi} \int_{|y - z| \leq \frac{1}{4}} \log |y - z| \Delta_z \varphi_y(z) c_0(z) dz.
\]
Due to the support of $\varphi_y$, we can identify the above with an analogous integral on $\mathbb{T}^2$ with $|y - z|$ replaced by $d(y, z)$. Therefore, we have the following estimate on the interaction energy,

$$\frac{-1}{2} \int_{\mathbb{T}^2} (n_0(y) - \bar{\pi}) c(y) dy$$

$$= \frac{1}{4\pi} \iint_{\mathbb{T}^2 \times \mathbb{T}^2} \log d(y, z)(n_0(y) - \bar{\pi})(n_0(z) - \bar{\pi}) \varphi_y(z) dz dy + \frac{1}{2\pi} \iiint_{\mathbb{T}^2 \times \mathbb{T}^2} (n_0(y) - \bar{\pi}) \nabla_z \cdot (\log d(y, z) \nabla_z \varphi_y(z)) c_0(z) dz dy$$

$$- \frac{1}{4\pi} \iint_{\frac{1}{4} \leq d(y, z) \leq \frac{1}{4}} (n_0(y) - \bar{\pi}) \log d(y, z) \Delta_z \varphi_y(z) c_0(z) dz dy$$

$$= \frac{1}{4\pi} \iint_{\frac{1}{4} \leq d(y, z) \leq \frac{1}{4}} \log d(y, z)(n_0(y) - \bar{\pi})(n_0(z) - \bar{\pi}) dz dy + \frac{1}{4\pi} \iiint_{\frac{1}{4} \leq d(y, z) \leq \frac{1}{4}} \log d(y, z)(n_0(y) - \bar{\pi})(n_0(z) - \bar{\pi}) \varphi_y(z) dz dy$$

$$+ \frac{1}{2\pi} \iint_{\frac{1}{4} \leq d(y, z) \leq \frac{1}{4}} (n_0(y) - \bar{\pi}) \nabla_z \cdot (\log d(y, z) \nabla_z \varphi_y(z)) c_0(z) dz dy - \frac{1}{4\pi} \iiint_{\frac{1}{4} \leq d(y, z) \leq \frac{1}{4}} (n_0(y) - \bar{\pi}) \log d(y, z) \Delta_z \varphi_y(z) c_0(z) dz dy.$$

The 2nd, 3rd, 4th, 5th terms in the last line are bounded below by $-BM^2$ for some constant $B > 0$. The 6th and 7th terms are bounded below by $-BM||c_0||_{L^1} ||n_0 - \bar{\pi}||_{L^1} \leq M$. Denoting $K$ to be the fundamental solution of the Laplacian on $\mathbb{T}^2$, by Young’s inequality, we have

$$||c_0||_{L^1(\mathbb{T}^2)} = ||K \ast (n_0 - \bar{\pi})||_{L^1(\mathbb{T}^2)} \leq ||K||_{L^1(\mathbb{T}^2)} ||n_0 - \bar{\pi}||_{L^1(\mathbb{T}^2)} \leq M.$$

The calculation above hence implies the following for some constant $B > 0$,

$$-\frac{1}{2} \int (n_0 - \bar{\pi}) (-\Delta)^{-1} (n_0 - \bar{\pi}) dy \geq \frac{1}{4\pi} \iint_{\mathbb{T}^2 \times \mathbb{T}^2} \log d(z, y)n_0(y)n_0(y) dz dy - BM^2.$$

Combining this estimate with (3.8) yields

$$2\mathcal{F}[n_{in}] \geq \left(1 - \frac{M}{8\pi}\right) \int_{\mathbb{T}^2} n_0 \log n_0 dy + \frac{M}{8\pi} \left(\int_{\mathbb{T}^2} n_0 \log n_0 dy + \frac{2}{M} \int_{\mathbb{T}^2} n_0 \log d(z, y)n_0(y) dz dy\right) - BM^2.$$

Applying (3.12) in the above estimate, we obtain

$$2\mathcal{F}[n_{in}] \geq \left(1 - \frac{M}{8\pi}\right) \int_{\mathbb{T}^2} n_0 \log n_0 dy - C(M) - BM^2,$$

which results in

$$\int_{\mathbb{T}^2} n_0 \log n_0 dy \leq \frac{2\mathcal{F}[n_{in}]}{1 - \frac{M}{8\pi}} + C(M) + BM^2.$$
As \( x \log x \) is bounded below, this implies the following for a suitable constant \( C_L \log L \) depending only on the initial data due to \( y \in \mathbb{T}^2 \):
\[
\int_{\mathbb{T}^2} n_0 \log^+ n_0 \, dy \leq C_L \log L(n_{in}) < \infty.
\]
This completes the proof of the lemma.

\[\square\]

### 3.3 Enhanced dissipation estimate, (3.5b)

There are only a few differences with (2.2) which we focus on below. Analogous to (2.2) we define the energy \( \Phi[n] \) on the torus \( \mathbb{T}^3 \) as follows:
\[
\Phi_k[n(t)] = ||\hat{n}_k(t)||_2^2 + ||\sqrt{\alpha} \partial_y \hat{n}_k(t)||_2^2 + 2kRe(\langle i\beta u' \hat{n}_k(t), \partial_y \hat{n}_k(t) \rangle) + ||k||_2^2 ||\sqrt{\gamma} u' \hat{n}_k(t)||_2^2;
\]  \( (3.13) \)
\[
\Phi[n(t)] = \sum_{k \neq 0} \Phi_k[n(t)] = ||n_\neq(t)||_2^2 + ||\sqrt{\alpha} \partial_y n_\neq(t)||_2^2 + 2\langle \beta u' \partial_x n_\neq(t), \partial_y n_\neq(t) \rangle + ||\sqrt{\gamma} u' \partial_x n_\neq(t)||_2^2.
\]  \( (3.14) \)

Here \( \alpha, \beta, \) and \( \gamma \) are chosen as in (2.11). Analogously, we have
\[
\Phi_k[n] \approx ||\hat{n}_k||_2^2 + ||\sqrt{\alpha} \partial_y \hat{n}_k||_2^2 + ||k||_2^2 ||\sqrt{\gamma} u' \hat{n}_k||_2^2,
\]  \( (3.15) \)
and hence
\[
||\hat{n}_k||_2^2 + A^{-1/2} |k|^{-1/2} ||\partial_y \hat{n}_k||_2^2 \leq \Phi_k[n] \leq ||\hat{n}_k||_2^2 + |k|^{1/2} A^{1/2} ||\hat{n}_k||_2^2 + A^{-1/2} |k|^{-1/2} ||\partial_y \hat{n}_k||_2^2.
\]  \( (3.16) \)

Our goal in this section is to prove the following proposition:

**Proposition 6.** There exists a small constant \( c > 0 \) depending only on \( u \) such that, under the bootstrap hypotheses and for \( A \) sufficiently large depending only on \( u, \|n_{in}\|_H^1 \) and \( \|n_{in}\|_\infty \), there holds
\[
\frac{d}{dt} \Phi[n(t)] \leq -\frac{c}{A^{1/2}} \Phi[n(t)].
\]  \( (3.17) \)

By (2.13), it follows that
\[
\|n_\neq\|_{L^2}^2 \leq \Phi(0)e^{-cA^{-1/2}t} \leq A^{1/2} \|n_{in}\|_{H^1}e^{-cA^{-1/2}t}.
\]  \( (3.18) \)

**Remark 2.** Same as in the proof of Theorem 4, Proposition 6 implies (3.5b).

On \( \mathbb{T}^3 \), the analogue of estimate (2.14) holds.

**Proposition 7.** For \( \bar{c} \) sufficiently small depending only on \( u \), there holds,
\[
\frac{d}{dt} \Phi_k[n](t) \leq \left\{ \begin{array}{c}
-\frac{\bar{c}}{2} \frac{|k|^2}{A^{1/2}} ||\hat{n}_k||_2^2 - \frac{\bar{c}}{2} |k|^{5/2} \frac{A^{1/2}}{A^{1/2}} ||\sqrt{\gamma} u' \hat{n}_k||_2^2 - \frac{1}{4A} ||\nabla_y \hat{n}_k||_2^2 \\
- \frac{1}{2} |k|^2 ||\beta u' \hat{n}_k||_2^2 - \frac{1}{4A} ||k||_2^2 ||\nabla_y \hat{n}_k||_2^2 - \frac{1}{4A} ||\sqrt{\gamma} u' \nabla_y \hat{n}_k||_2^2 \\
+ \left\{ 2Re(\langle -L_k, \hat{n}_k \rangle) - 2Re(\langle \alpha \partial_y^2 \hat{n}_k, -L_k \rangle) - 2kRe(\langle i\beta u' L_k, \partial_y \hat{n}_k \rangle) + \langle i\beta u' \hat{n}_k, \partial_y L_k \rangle \\
+ 2|k|^2 Re(\langle \gamma(u')^2 \hat{n}_k, -L_k \rangle) \\
+ \left\{ -2Re(\langle NL_k, \hat{n}_k \rangle) + 2Re(\langle \alpha \partial_y^2 \hat{n}_k, NL_k \rangle) - 2kRe(\langle i\beta u' NL_k, \partial_y \hat{n}_k \rangle) + \langle i\beta u' \hat{n}_k, \partial_y NL_k \rangle \\
- 2|k|^2 Re(\langle \gamma(u')^2 \hat{n}_k, NL_k \rangle) \right\} \right\}
\end{array} \right.
\]  \( (3.19) \)
where \( \mathcal{N}_k \) refers to the negative terms. Recall that \( L_k, NL_k \) are defined in (2.46, 2.46).
Proof. The first term in $\partial_t \Phi_k[n](t)$ is:

$$\frac{d}{dt}||\tilde{n}_k(t)||^2 = -\frac{2}{A} |k|^2 ||\tilde{n}_k||^2 + \frac{2}{A} ||\nabla \tilde{n}_k||^2 + 2Re\langle -NL_k, \tilde{n}_k \rangle + 2Re\langle -L_k, \tilde{n}_k \rangle. \quad (3.20)$$

The second term, $\frac{d}{dt}||\sqrt{\alpha}\partial_y \tilde{n}_k(t)||^2$, gives:

$$\frac{d}{dt}||\sqrt{\alpha}\partial_y \tilde{n}_k(t)||^2 = 2Re\langle \alpha \partial_y \tilde{n}_k, \partial_y \tilde{n}_k \rangle = 2Re\langle \alpha \partial_y \tilde{n}_k, \partial_y \tilde{n}_k \rangle \left( \frac{1}{A} (\Delta_y - |k|^2) \tilde{n}_k - i u(y) k \tilde{n}_k - NL_k \right)$$

$$= -\frac{2}{A} |k|^2 ||\sqrt{\alpha}\partial_y \tilde{n}_k||^2 + \frac{2}{A} ||\nabla \tilde{n}_k||^2 - 2kRe\langle \alpha i u \tilde{n}_k, \partial_y \tilde{n}_k \rangle$$

$$+ 2Re\langle \alpha \partial_y \tilde{n}_k, \partial_y (-L_k - NL_k) \rangle$$

$$= -\frac{2}{A} |k|^2 ||\sqrt{\alpha}\partial_y \tilde{n}_k||^2 + \frac{2}{A} ||\nabla \tilde{n}_k||^2 - 2kRe\langle \alpha i u \tilde{n}_k, \partial_y \tilde{n}_k \rangle$$

$$- 2Re\langle \alpha \partial^2 \tilde{n}_k, -L_k - NL_k \rangle. \quad (3.21)$$

The third term, the term involving $\beta$, can be treated as follows:

$$\frac{d}{dt}(2kRe\langle i \beta u \tilde{n}_k(t), \partial_y \tilde{n}_k(t) \rangle) = 2kRe\langle i \beta u \partial_y \tilde{n}_k(t), \partial_y \tilde{n}_k(t) \rangle + 2kRe\langle i \beta u \tilde{n}_k(t), \partial_y \tilde{n}_k(t) \rangle$$

$$= \frac{4k^3}{A} Re\langle i \beta u \tilde{n}_k, \partial_y \tilde{n}_k \rangle + \frac{4k}{A} Re\langle i \beta u \partial_y \tilde{n}_k, \partial_y \tilde{n}_k \rangle$$

$$+ \frac{2k}{A} Re\langle i \beta u \tilde{n}_k, \partial_y \tilde{n}_k \rangle - 2kRe\langle i \beta u \tilde{n}_k, u \tilde{n}_k \rangle$$

$$+ 2kRe\langle i \beta u (-NL_k - L_k), \partial_y \tilde{n}_k \rangle + 2kRe\langle i \beta u \tilde{n}_k, \partial_y (-NL_k - L_k) \rangle$$

$$+ \frac{4k}{A} Re\langle i \beta u \partial_y \partial_y \tilde{n}_k(t), \partial_y \tilde{n}_k(t) \rangle$$

$$\leq -\frac{4k^3}{A} Re\langle i \beta u \tilde{n}_k, \partial_y \tilde{n}_k \rangle + \frac{4k}{A} Re\langle i \beta u \partial_y \tilde{n}_k, \partial_y \tilde{n}_k \rangle$$

$$+ \frac{2k}{A} Re\langle i \beta u \tilde{n}_k, \partial_y \tilde{n}_k \rangle - 2|k|^2 \||\nabla \tilde{n}_k||^2$$

$$+ 2kRe\langle i \beta u (-NL_k - L_k), \partial_y \tilde{n}_k \rangle + 2kRe\langle i \beta u \tilde{n}_k, \partial_y (-NL_k - L_k) \rangle$$

$$+ \frac{1}{A} ||\nabla \partial_y \partial_y \tilde{n}_k||^2 + \frac{8|k|^2 \beta^2}{2A \alpha \gamma} ||\sqrt{\gamma} u \partial_y \tilde{n}_k(t)||^2.$$  

Using that $\frac{\beta^2}{\alpha \gamma} \leq \frac{1}{A}$ (recall, this is ensured in [3]), the corresponding terms in (3.21) and (3.22) absorb the last two terms. Other terms are treated as in (3.21) and (3.22). Finally, for the term $\frac{d}{dt}||\sqrt{\gamma} u \tilde{n}_k(t)||^2$, we have

$$\frac{d}{dt}||\sqrt{\gamma} u \tilde{n}_k(t)||^2 = -\frac{2}{A} |k|^2 ||\sqrt{\gamma} u \tilde{n}_k||^2 + \frac{2}{A} |k|^2 Re\langle \gamma u \tilde{n}_k, \partial_y \tilde{n}_k \rangle$$

$$- \frac{2}{A} |k|^2 ||\nabla \gamma \tilde{n}_k||^2$$

$$+ 2|k|^2 Re\langle \gamma (u)^2 \tilde{n}_k, -L_k - NL_k \rangle. \quad (3.22)$$

Combining the above terms yields the result. □

As in (3.22) the remainder of the section is devoted to controlling $L$ and $NL$ by the negative terms in (3.19).

### 3.3.1 Estimate on the $L$ terms in (3.19)

In this section we prove that for $A$ sufficiently large,

$$L_k^1 + L_k^\alpha + L_k^\beta + L_k^2 \leq -\frac{1}{4} N_k. \quad (3.23)$$
We begin by estimating the $L_k^1$ term in (3.19). Using (A.4) and the bootstrap hypotheses (H), we have, for any fixed constant $B \geq 1$,

$$L_k^1 = \frac{2}{A} \Re \langle \alpha \partial_y^3 \hat{n}_k \rangle + \Re \langle \nabla_y c_0 \cdot \nabla_y \hat{n}_k \rangle - \frac{2}{A} \Re \langle \nabla_y \hat{c}_k \cdot \nabla_y n_k, \hat{n}_k \rangle - \frac{2}{A} \Re \langle \nabla_y \hat{c}_k \cdot \nabla_y n_0, \hat{n}_k \rangle$$

$$\leq \frac{2}{A} \left( ||n_0 - \hat{n}_||_\infty + ||n_0||_\infty - \hat{n}_k \right)^2 + \frac{1}{AB} ||\nabla_y \hat{n}_k||_2^2 + \frac{B}{A} ||\nabla_y c_0||_\infty ||\hat{n}_k||_2^2 + \frac{2}{A} ||n_0||_\infty ||\Delta_y \hat{c}_k||_2 ||\hat{n}_k||_2$$

$$+ \frac{2}{A} ||n_0||_\infty ||\nabla_y \hat{c}_k||_2 ||\nabla_y \hat{n}_k||_2$$

$$\leq \frac{1}{AB} ||\nabla_y \hat{n}_k||_2^2 + \frac{BC_{\infty}^2}{A} ||\hat{n}_k||_2^2.$$

Therefore, by the bootstrap hypotheses, we can choose $B$ sufficiently large, and then $A$ sufficiently large, such that the following holds:

$$|L_k^1| \leq -\frac{1}{16} N_k,$$

which is consistent with (3.23).

We turn next to $L_k^\alpha$ in (3.19), which we divide into the following:

$$L_k^\alpha = -2 \Re \langle \alpha \partial_y^3 \hat{n}_k \rangle + \frac{1}{A} \hat{n}_k + \frac{2}{A} (n_0 - \hat{n}_) \hat{n}_ - \frac{1}{A} \nabla_y c_0 \cdot \nabla_y \hat{n}_k - \frac{1}{A} \nabla_y \hat{c}_k \cdot \nabla_y n_0$$

$$= L_{k,0}^\alpha + L_{k,1}^\alpha + L_{k,2}^\alpha + L_{k,3}^\alpha. \quad (3.24)$$

The treatment of the $L_{k,0}^\alpha$ and $L_{k,1}^\alpha$ terms are analogous to the treatment in (3.22) and hence we omit it for the sake of brevity. Next we estimate $L_{k,2}^\alpha$ in (3.24). Using (A.5) and the hypotheses, we have the following for any $B \geq 1$:

$$|L_{k,2}^{\alpha}| \lesssim \frac{1}{BA} ||\nabla y \partial_y^3 \hat{n}_k||_2^2 + \frac{B}{A^{3/2}} ||\nabla_y c_0||_\infty ||\nabla_y \hat{n}_k||_2^2 \lesssim \frac{1}{BA} ||\nabla y \partial_y^3 \hat{n}_k||_2^2 + \frac{BC_{\infty}^2}{A^{3/2}} ||\nabla_y \hat{n}_k||_2^2.$$

Hence, by the bootstrap hypotheses and the definition of $N_k$, it follows we can choose $B$ large and then $A$ large to control this term consistent with (3.26). Similarly, for $L_{k,3}^\alpha$ in (3.24), by (A.4) and the hypothesis (3.2c), we have that

$$|L_{k,3}^{\alpha}| \lesssim \frac{1}{BA} ||\nabla y \partial_y^3 \hat{n}_k||_2^2 + \frac{B}{A^{3/2}} ||\nabla_y \hat{c}_k||_\infty ||\nabla_y \hat{n}_0||_2^2$$

$$\lesssim \frac{1}{BA} ||\nabla y \partial_y^3 \hat{n}_k||_2^2 + \frac{B}{A^{3/2}} ||\hat{n}_k||_2 ||\nabla_y \hat{n}_0||_2 ||\nabla_y \hat{n}_0||_2^2$$

$$\lesssim \frac{1}{BA} ||\nabla y \partial_y^3 \hat{n}_k||_2^2 + \frac{B}{A^{3/2}} ||\nabla_y \hat{n}_k||_2^2 + \frac{BC_{\infty}^2}{A^{3/2}} ||\hat{n}_k||_2^2.$$

As above, it follows we can choose $B$ large and then $A$ large to control this term consistent with (3.26).

Next, turn to the $L_k^\beta$ term in (3.19), which we divide into two contributions:

$$L_k^\beta = 2k \Re \langle i \beta u \nabla y \hat{n}_k, \partial_y \left( \frac{1}{A} \hat{n}_k + \frac{2}{A} (n_0 - \hat{n}_) \hat{n}_ - \frac{1}{A} \nabla_y c_0 \cdot \nabla_y \hat{n}_k - \frac{1}{A} \nabla_y \hat{c}_k \cdot \nabla_y n_0 \rangle$$

$$+ 2k \Re \langle i \beta u \left( \frac{1}{A} \hat{n}_k + \frac{2}{A} (n_0 - \hat{n}_) \hat{n}_ - \frac{1}{A} \nabla_y c_0 \cdot \nabla_y \hat{n}_k - \frac{1}{A} \nabla_y \hat{c}_k \cdot \nabla_y n_0 \rangle, \partial_y \hat{n}_k \rangle$$

$$= : L_{k,1}^\beta + L_{k,2}^\beta. \quad (3.25)$$

The first term in (3.25) is further decomposed via

$$L_{k,1}^\beta = 2k \Re \langle i \beta u \nabla y \hat{n}_k, \partial_y \left( \frac{1}{A} \hat{n}_k + \frac{2}{A} (n_0 - \hat{n}_) \hat{n}_ - \frac{1}{A} \nabla_y c_0 \cdot \nabla_y \hat{n}_k - \frac{1}{A} \nabla_y \hat{c}_k \cdot \nabla_y n_0 \rangle$$

$$= : L_{k,10}^\beta + L_{k,11}^\beta + L_{k,12}^\beta + L_{k,13}^\beta. \quad (3.26)$$
The treatment of the $L_{10}^\beta$ and $L_{11}^\beta$ terms are analogous to the treatment in (3.2) and are hence we omitted for the sake of brevity. For the $L_{12}^\beta$ term in (3.28), we first estimate,

$$
|L_{12}^\beta| \leq 2k \text{Re}(i\beta u'' \hat{\nu}_k, \frac{1}{A} \nabla_y c_0 \cdot \nabla_y \hat{\nu}_k) + 2k \text{Re}(i\beta u' \partial_y \hat{\nu}_k, \frac{1}{A} \nabla_y c_0 \cdot \nabla_y \hat{\nu}_k)
$$

$$
= L_{1211}^\beta + L_{1212}^\beta.
$$

For $L_{1211}^\beta$, we use (A.3), the definition of $\beta$, and the bootstrap hypotheses to deduce,

$$
\left| L_{1211}^\beta \right| \leq \frac{1}{AB} \left\| \nabla_y \hat{\nu}_k \right\|^2 + \frac{B|k|^2 \|\beta u'' \hat{\nu}_k\|^2}{\|\nabla_y c_0\|^2} \frac{\|\nabla_y \hat{\nu}_k\|^2}{A}.
$$

Hence, we may choose $B$ large and then $A$ large to make these terms consistent with (3.23). Next we turn to $L_{1212}^\beta$ in (3.27). Applying integration by parts, (A.3), the definition of $\beta$ and the bootstrap hypotheses, we have

$$
\left| L_{1212}^\beta \right| \leq \frac{2k}{A} \text{Re}(i\beta u' \partial_y \nabla_y \hat{\nu}_k \cdot \nabla_y c_0, \hat{\nu}_k) + \frac{2k}{A} \text{Re}(i\beta u' \partial_y (\Delta c_0), \hat{\nu}_k)
$$

$$
\leq \frac{1}{AB} \left\| \nabla_y \partial_y \hat{\nu}_k \right\|^2 + \frac{1}{AB} \left\| \nabla_y \hat{\nu}_k \right\|^2 + \frac{|k|^2 B}{A} \left\| \beta u'' \hat{\nu}_k \right\|^2 + \frac{B|k|^2}{A} \left\| \nabla \hat{\nu}_k \right\|^2 + \frac{|\beta u'' \hat{\nu}_k|^2}{\|\nabla_y c_0\|^2} \left\| \nabla_y \hat{\nu}_k \right\|^2 + \frac{|k|^2 B \|\nabla \hat{\nu}_k\|^2}{A} \left\| \beta u'' \hat{\nu}_k \right\|^2 + \frac{B|k|^2}{A} \left\| \nabla \hat{\nu}_k \right\|^2 + \frac{BC^2}{A} \left\| \nabla \hat{\nu}_k \right\|^2.
$$

Hence, we may choose $B$ large and then $A$ large to make these terms consistent with (3.23). Consider next $L_{13}^\beta$ in (3.28), which we integrate by parts and further sub-divide as:

$$
L_{13}^\beta = 2k \text{Re}(i\beta u'' \hat{\nu}_k + i\beta u' \partial_y \hat{\nu}_k, \frac{1}{A} \nabla_y \hat{c}_k \cdot \nabla_y n_0) =: L_{1311}^\beta + L_{1312}^\beta.
$$

For $L_{1311}^\beta$, by (A.3), the definition of $\beta$ and the bootstrap hypotheses, we have the following for a large constant $B \geq 1$

$$
\left| L_{1311}^\beta \right| \leq \frac{|k|^2 B}{A} \left\| \beta u'' \hat{\nu}_k \right\|^2 \left\| \nabla_y n_0 \right\| + \frac{1}{AB} \left\| \nabla_y \hat{c}_k \right\|^2 \left\| \nabla_y n_0 \right\| + \frac{1}{AB} \left\| \nabla_y \hat{\nu}_k \right\|^2 \left\| \nabla_y n_0 \right\|.
$$

Therefore, for $B$, then $A$, large, this term is controlled consistent with (3.23). Using (A.3), the $L_{1312}^\beta$ term in (3.28) is handled as follows for a large constant $B \geq 1$

$$
\left| L_{1312}^\beta \right| \leq \frac{|k|^2 B}{A} \left\| \nabla_y \partial_y \hat{\nu}_k \right\|^2 + \frac{B \beta^2}{A |k|^2} \left\| \nabla_y \hat{c}_k \right\|^2 \left\| \nabla_y n_0 \right\| + \frac{1}{AB} \left\| \nabla_y \hat{\nu}_k \right\|^2 \left\| \nabla_y n_0 \right\| + \frac{B^2 C^2}{A} \left\| \nabla_y \hat{\nu}_k \right\|^2.
$$
Therefore, by the bootstrap hypotheses, for $B$ and $A$ sufficiently large, this is consistent with (3.23).

Turn next to $L_{k,2}^\beta$ in (3.25), which we sub-divide as follows:

$$
L_{k,2}^\beta = 2kR\text{e}^{i\beta u'} \frac{1}{A} \nabla_y \tilde{n}_k + 2kR\text{e}^{i\beta u}_{\frac{2}{A}} (n_0 - \pi) \tilde{n}_k, \quad \partial_y \tilde{n}_k
- 2kR\text{e}^{i\beta u'} \frac{1}{A} \nabla_y \gamma \cdot \nabla_y \tilde{n}_k, \quad \partial_y \tilde{n}_k
- 2kR\text{e}^{i\beta u'} \frac{1}{A} \nabla_y c_k \cdot \nabla_y n_0, \quad \partial_y \tilde{n}_k
= : L_{k,20}^\beta + L_{k,21}^\beta + L_{k,22}^\beta + L_{k,23}^\beta.
$$

The $L_{k,22}^\beta$ term can be handled in the same manner as $L_{k,122}^\beta$. For the $L_{k,21}^\beta$ term, we use the following straightforward estimate for a constant $B \geq 1$:

$$
L_{k,21}^\beta \lesssim \frac{1}{AB} ||\nabla_y \tilde{n}_k||_2^2 + \frac{BC^2}{A_{\infty}} \frac{1}{k^2} ||\sqrt{\beta u'} \tilde{n}_k||_2^2.
$$

As above, this is consistent with (3.23) by the bootstrap hypotheses and $B, A$ large. The $L_{k,20}^\beta$ term is treated in the same way, so we skip the details for the sake of brevity. The $L_{k,23}^\beta$ term can be estimated in the same manner as $L_{k,132}^\beta$ above (3.29) and hence is omitted for brevity. This completes the treatment of the $L_k^\beta$ term in (3.19).

Finally, we estimate $L_k^\gamma$ in (3.19). We first sub-divide:

$$
L_k^\gamma = 2|k|^2 R\text{e}^{i(u')2} \nabla_y \tilde{n}_k, \quad \frac{1}{A} \nabla_y \tilde{n}_k + \frac{2}{A} (n_0 - \pi) \tilde{n}_k - \frac{1}{A} \nabla_y c_0 \cdot \nabla_y \tilde{n}_k - \frac{1}{A} \nabla_y c_k \cdot \nabla_y n_0
= : L_{k,0}^\gamma + L_{k,1}^\gamma + L_{k,2}^\gamma + L_{k,3}^\gamma.
$$

The first and second term in (3.31) are estimated as in (3.22) we omit the details for brevity. For $L_{k,2}^\gamma$ in (3.31), by (3.5) and the hypotheses, we have for $B \geq 1$ large,

$$
L_{k,2}^\gamma \lesssim \frac{B_\gamma}{A^\beta} |k|^2 ||\sqrt{\beta u'} \tilde{n}_k||_2^2 ||\nabla_y c_0||_2^2 + \frac{|k|^2}{AB} ||\sqrt{\gamma} \nabla_y \tilde{n}_k||_2^2
\lesssim \frac{BC^2}{A_{1/2}} |k|^2 ||\sqrt{\beta u'} \tilde{n}_k||_2^2 + \frac{|k|^2}{AB} ||\sqrt{\gamma} \nabla_y \tilde{n}_k||_2^2.
$$

As usual, this is consistent with (3.23) by the bootstrap hypotheses and $B, A$ large. The $L_{k,3}^\gamma$ term in (3.31), is estimated slightly differently; using (3.4) and the hypotheses, we have for $B \geq 1$ large,

$$
L_{k,3}^\gamma \lesssim \frac{1}{A^{1/2}B} |k|^{5/2} ||\sqrt{\gamma} u' \tilde{n}_k||_2^2 + \frac{B}{A^{1/2}} |k|^{3/2} ||\nabla_y c_k||_\infty ||\nabla_y n_0||_2^2
\lesssim \frac{1}{A^{1/2}B} |k|^{5/2} ||\sqrt{\gamma} u' \tilde{n}_k||_2^2 + \frac{B}{A} ||\tilde{n}_k||_2 ||\nabla_y \tilde{n}_k||_2 ||\nabla_y n_0||_2^2
\lesssim \frac{1}{A^{1/2}B} |k|^{5/2} ||\sqrt{\gamma} u' \tilde{n}_k||_2^2 + \frac{1}{AB} ||\nabla_y \tilde{n}_k||_2^2 + \frac{B^3}{A} ||\tilde{n}_k||_2^2 C_{H_1}^4,
$$

this is consistent with (3.23) by the bootstrap hypotheses and $B, A$ large. This completes the proof of (3.24), and hence, under the bootstrap hypotheses, the contributions of the $L$ terms in (3.19) is absorbed by the $N_k$ terms for $A$ chosen sufficiently large.

### 3.3.2 Estimate on NL terms

The treatment of these terms is essentially the same as (3.24). For example, for the $NL_k^1$ term in (3.19), we estimate via,

$$
- \sum_{k \neq 0} 2Re<NL_k, \tilde{n}_k> = -\langle \frac{2}{A} \nabla \cdot (n_\neq \nabla c_\neq), n_\neq \rangle = \frac{2}{A} (n_\neq \nabla c_\neq, \nabla n_\neq) \leq \frac{2}{A} ||\nabla c_\neq||_{\infty} ||\nabla n_\neq||_2 ||n_\neq||_2.
$$
Applying (A.3) (together with the bootstrap hypotheses), gives the following for any constant \(B > 1\),

\[
- \sum_{k \neq 0} 2Re(\langle NL_k, \tilde{n}_k \rangle) \lesssim \frac{1}{AB} ||\nabla n_\neq||^2_2 + \frac{BC_{2, c}^s}{A} ||n_\neq||^2_2.
\]

By first choosing \(B\) big, and then choosing \(A\) large (relative to constants and \(B\)), these terms are absorbed by the negative terms in (3.19). As the other terms are similarly analogous, we omit the details for the sake of brevity.

### 3.4 Nonzero mode \(L^2_t \dot{H}^1_{x,y}\) estimate (3.5a)

Computing \(\frac{d}{dt}||n_\neq||^2_2\) and applying (A.3),

\[
\frac{1}{2} \frac{d}{dt} ||n_\neq||^2_2 = \langle n_\neq, \frac{1}{A} \Delta n_\neq + \frac{1}{A} (n_0 - \bar{n}) n_\neq + \frac{1}{A} n_\neq n_0 - \frac{1}{A} \nabla c_0 \cdot \nabla n_\neq - \frac{1}{A} \nabla c_\neq \cdot \nabla n_0 - u(y) \partial_x n_\neq - \frac{1}{A} (\nabla \cdot (\nabla c_\neq n_\neq)) \rangle
\]

\[
\lesssim - \frac{1}{2A} ||\nabla n_\neq||^2_2 + \frac{B}{A} ||n_\neq||^2_2 ||\nabla c_0||^2_\infty + \frac{B}{A} ||\nabla c_\neq||_\infty ||n_\neq||_2 ||\nabla n_0||_2
\]

\[
+ \frac{1}{A} ||n_\neq||^2_2 ||\nabla n_\neq||_2 - \frac{1}{A} ||n_\neq||^2_2 ||\nabla c_\neq||_4 ||n_\neq||_4
\]

\[
\lesssim - \frac{1}{2A} ||\nabla n_\neq||^2_2 + \frac{C_{2, c}^s}{A} ||n_\neq||^2_2 + \frac{C_{2, c}^s C_{H^1}}{A} ||n_\neq||_2 + \frac{1}{A} ||\nabla n_\neq||_2 ||\nabla c_\neq||_4 ||n_\neq||_4.
\]

Note that, due to the bootstrap hypothesis (3.2b), there holds

\[
\int_0^{T^*} \frac{C_{2, c}^s}{A} ||n_\neq||^2_2 + \frac{C_{2, c}^s C_{H^1}}{A} ||n_\neq||_2 \leq \frac{\log A}{A^{1/2}} C_{E, D} (1 + ||n_{in}||^2_{H^1}) (C_{2, c}^s + C_{2, c}^s C_{H^1}) ,
\]

which can be made arbitrarily small by choosing \(A\) large. The latter term is treated via the Gagliardo-Nirenberg-Sobolev inequality, s

\[
\frac{1}{A} ||\nabla n_\neq||_2 ||\nabla c_\neq||_4 ||n_\neq||_4 \leq \frac{1}{A} ||\nabla n_\neq||_2 ||\nabla c_\neq||_2^{1/4} ||\nabla^2 c_\neq||_2^{3/4} ||n_\neq||_2^{1/4} ||\nabla n_\neq||_2^{3/4}
\]

\[
\lesssim \frac{1}{AB} ||\nabla n_\neq||^2_2 + \frac{B}{A} ||n_\neq||^2_2
\]

\[
\lesssim \frac{1}{AB} ||\nabla n_\neq||^2_2 + \frac{BC_{2, c}^s}{A} ||n_\neq||^2_2.
\]

Hence, by choosing \(B\) then \(A\) sufficiently large, we have following \(L^2_t \dot{H}^2_{x,y}\) estimate:

\[
\frac{1}{A} \int_0^{T^*} ||\nabla n_\neq||^2_2 dt \leq \frac{1}{A^{1/4}} + 2 ||n_{in}||^2_2 \leq 4 ||n_{in}||^2_2.
\]

As a result, we have proven (3.5a).

### 3.5 Remainder of the proof of Theorem 2 in the case \(T^3\)

The remaining steps in the proof of Proposition 4 are the proofs of (3.5d) and (3.5d). Since \(L^2\) is subcritical in 3D, the proof of (3.5d) follows as in (2.4) by standard methods. The proof of (3.5d) is a slightly easier variation of the arguments carried out in (1.1). These arguments are carried out below and hence are not repeated here. This completes the proof of Proposition 4 and hence also Theorem 2 in the \(T^3\) case.

### 4 Proof of Theorem 2 in the case \(\mathbb{T} \times \mathbb{R}^2\)

The main difference between (4) and (3) is that we can no longer propagate a lower bound on the solution, which makes an estimate on the free energy more delicate. Here, we instead use an approximate free energy for which it is easier to make estimates on the effect of low densities.
4.1 Basic setting and bootstrap argument

In this section, analyse the equation:

\[
\begin{cases}
\partial_t n + u(y_1)\partial_x n + \frac{1}{\epsilon^2} \nabla \cdot (\nabla cn) = \frac{1}{\epsilon^2} \Delta n, \\
-\Delta \epsilon = n, \\
n(\cdot, 0) = n_0,
\end{cases}
\]  
(4.1)

in the space $T \times \mathbb{R}^2$. Note that the equation for $c$ is slightly different in $\mathbb{R}^2$. The basic idea behind the proof of the main theorem is the same as [33] however, we cannot use the true 2D free energy, and instead make a more complicated estimate on an approximate free energy. For constants $C$'s determined by the proof, define $T_\ast$ to be the end-point of the largest interval $[0, T_\ast]$ such that the following hypotheses hold for all $T \leq T_\ast$:

1. Nonzero mode $L_t^2(0, T_\ast; \dot{H}_{x,y}^1)$ estimate:

\[
\frac{1}{A} \int_0^{T_\ast} ||\nabla n_\neq||_2^2 (\mathbb{T} \times \mathbb{R}^2) dt \leq 8 ||n_{in}||_2^2; 
\]  
(4.2a)

2. Nonzero mode enhanced dissipation estimate:

\[
||n_\neq||_2^2 (\mathbb{T} \times \mathbb{R}^2) \leq 4C_{ED} ||n_{in}||_2^2 e^{-\frac{c}{\epsilon^2} \log\epsilon}, 
\]  
(4.2b)

where $c$ is a small number depending only on $u$ (in particular, independent of $A$);

3. Zero mode uniform in time estimate:

\[
||n_0||_{L_t^\infty(0, T_\ast; L_x^2)} \leq 4C_{L^2}, \\
||\partial_y n_0||_{L_t^\infty(0, T_\ast; L_x^2)} \leq 4C_{H^1}; 
\]  
(4.2c)

4. $L_t^\infty(0, T_\ast; L_{x,y}^\infty)$ estimate of the whole solution:

\[
||n||_{L_t^\infty(0, T_\ast; L_{x,y}^\infty)} \leq 4C_{\infty}. 
\]  
(4.2d)

As in the two-dimensional case, we introduce the following constant:

\[
C_{2,\infty} := 1 + M + C_{1/2}^1 ||n_{in}||_{H^1} + C_{L^2} + C_{\infty}. 
\]  
(4.3)

The constant $C_{ED}$ depends only on $u$. The constant $C_{L^2}$ depends only on the initial data $n_{in}$. The constant $C_{\infty}$ depends on $n_{in}$ and $C_{L^2}$. Finally, the constant $C_{H^1}$ depends on $n_{in}$, $C_{L^2}$ and $C_{\infty}$.

As in [2] and [3] the proof of Theorem 2 (b) is completed by the following proposition.

**Proposition 8.** For all $n_{in}$ and $u$, there exists an $A_0(||n_{in}||_{H^1}, ||n_{in}||_{\infty})$ such that if $A > A_0$ then the following conclusions, referred to as (C), hold on the interval $[0, T_\ast]$:

1. Nonzero mode $L_t^2(0, T_\ast; \dot{H}_{x,y}^1)$ estimate:

\[
\frac{1}{A} \int_0^{T_\ast} ||\nabla_{x,y} n_\neq||_2^2 dt \leq 4 ||n_{in}||_2^2; 
\]  
(4.4a)

2. Nonzero mode enhanced dissipation estimate:

\[
||n_\neq(t)||_2^2 \leq 2C_{ED} ||n_{in}||_{H^1}^2 e^{-\frac{c}{\epsilon^2} \log \epsilon}; 
\]  
(4.4b)

3. Zero mode uniform in time estimate:

\[
||n_0||_{L_t^\infty(0, T_\ast; L_x^2)} \leq 2C_{L^2}, \\
||\partial_y n_0||_{L_t^\infty(0, T_\ast; L_x^2)} \leq 2C_{H^1}; 
\]  
(4.4c)

4. Whole solution $L_t^\infty(0, T_\ast; L_{x,y}^\infty)$ estimate:

\[
||n||_{L_t^\infty(0, T_\ast; L_{x,y}^\infty)} \leq 2C_{\infty}. 
\]  
(4.4d)

The remaining part of this section is organized as follows: in section 3.2, we prove the estimate on the zeroth mode [4.3c]; in section 3.3, we give some remark about the proof of (4.4a), (4.4b) and (4.4c).
4.2 Estimate on the zero mode (4.4c)

For the case $y \in \mathbb{R}^2$, it is not clear how to estimate the contribution to $\frac{d}{dt} F$ from the non-zero frequencies at small values of $n_0$. The idea is to find a new (approximate) free energy which is better adapted. We use the following as the new approximate free energy:

$$F_T[n_0] = \int n_0 \Gamma(n_0) - \frac{n_0 c_0}{2} dy,$$  \hspace{1cm} (4.5)

where $\Gamma$ is defined as

$$\Gamma(n_0) = \begin{cases} \log n_0, & n_0 \geq 1, \\ (n_0 - 1) - \frac{(n_0 - 1)^2}{2}, & n_0 < 1. \end{cases}$$ \hspace{1cm} (4.6)

The $\Gamma$ function is chosen such that it matches $\log$ when $n_0$ is large but is bounded from below when $n_0$ is small. Here, we have replaced the function $\log(1 + (n_0 - 1))$ by its degree two Taylor expansion centered at 1 when $n_0 < 1$ and use the original log function when $n_0 \geq 1$.

Next, we apply a sequence of lemmas to prove that under the bootstrap hypothesis (4.2a), (4.2b) and (4.2c), the conclusion on the zero-mode (4.4c) is true given $A$ sufficiently large. The first is the following.

**Lemma 4.1.** The time derivative of the approximate free energy $F_T[n_0]$, defined in (4.5), satisfies the following estimate:

$$\frac{d}{dt} F_T[n_0(t)] \lesssim \frac{1}{A} ||(\nabla_y e^T n_\neq) ||^2_{L^2(\mathbb{R}^2)} + \frac{1}{A} ||(\nabla_y e^T n_\neq) ||_{L^{4/3}(\mathbb{R}^2)} ||n_0||_{L^{4/3}(\mathbb{R}^2)}.$$ \hspace{1cm} (4.7)

Furthermore, the following quantity is bounded:

$$- \int_{n_0 < 1} n_0 \Gamma(n_0) dy \leq \frac{3}{2} M.$$ \hspace{1cm} (4.8)

**Proof.** Taking the time derivative of $F_T[n_0(t)]$ yields

$$\frac{d}{dt} \left( \int n_0 \Gamma(n_0) - \frac{n_0 c_0}{2} dy \right) = \int (n_0 \Gamma(n_0) - c_0) dy + \int n_0 (\Gamma(n_0) - c_0) dy$$

$$= \frac{1}{A} \left\{ - \int (n_0 \nabla_y \log n_0 - \nabla_y c_0 n_0) \cdot (n_0 \nabla y n_0 - \nabla y c_0) dy \right\}$$

$$- \int \nabla_y (n_0 \Gamma'(n_0)) \cdot (n_0 \nabla_y \log n_0 - \nabla_y c_0 n_0) dy$$

$$+ \frac{1}{A} \left\{ \int \nabla_y (n_0 \Gamma'(n_0)) \cdot (n_0 \nabla_y n_0 - \nabla y c_0) dy + \int \nabla_y (n_0 \Gamma'(n_0)) \cdot (n_0 \nabla_y \log n_0 - \nabla y c_0 n_0) dy \right\}$$

$$\leq \frac{1}{A} T_0 + \frac{1}{A} T_\neq.$$ \hspace{1cm} (4.9)

The proof of the lemma is completed once we show that the term $T_0$ is non-positive and the term $T_\neq$ is controlled in an appropriate way. Using the definition of $\Gamma$, we have,

$$T_0 = - \int_{n_0 < 1} (n_0 \nabla_y \log n_0 - \nabla_y c_0 n_0) \cdot \frac{1}{n_0} \nabla_y n_0 - \nabla y c_0) dy$$

$$- \int_{n_0 < 1} (n_0 \nabla_y \log n_0 - \nabla_y c_0 n_0) \cdot ((2 - n_0) \nabla y n_0 - \nabla y c_0) dy$$

$$- \int_{n_0 < 1} (2 - n_0) \nabla y n_0 \cdot (n_0 \nabla_y \log n_0 - \nabla y c_0 n_0) dy + \int_{n_0 < 1} n_0 \nabla y n_0 \cdot (n_0 \nabla_y \log n_0 - \nabla y c_0 n_0) dy.$$

Notice the following inequality:

$$\sup_{n_0 < 1} \sqrt{(-3n_0 + 4)n_0} \leq \frac{2}{\sqrt{3}} < 2,$$
which implies,

\[ T_0 = -\int_{n_0 \geq 1} n_0|\nabla y \log n_0 - \nabla y c_0|^2 dy - \int_{n_0 < 1} (4 - 3n_0)|\nabla y n_0|^2 dy \]

\[ + \int_{n_0 < 1} \sqrt{(-3n_0 + 4)n_0 \nabla_y c_0 \cdot \nabla y n_0} dy - \int_{n_0 < 1} n_0|\nabla y c_0|^2 dy + \int_{n_0 < 1} \nabla y n_0 \cdot \nabla y c_0 dy \]

\[ \leq -\int_{n_0 \geq 1} n_0|\nabla y \log n_0 - \nabla y c_0|^2 dy - \int_{n_0 < 1} (4 - 3n_0)|\nabla y n_0|^2 dy \]

\[ + \frac{2}{\sqrt{3}} \int_{n_0 < 1} \sqrt{(-3n_0 + 4)n_0 |\nabla y c_0| |\nabla y n_0|} dy - \int_{n_0 < 1} n_0|\nabla y c_0|^2 dy + \int_{n_0 < 1} \nabla y n_0 \cdot \nabla y c_0 dy. \]

Completing a square using the 2nd, 3rd, 4th terms in the last line yields

\[ T_0 \leq -\int_{n_0 \geq 1} n_0|\nabla y \log n_0 - \nabla y c_0|^2 dy - \frac{2}{3} \int_{n_0 < 1} (4 - 3n_0)|\nabla y n_0|^2 dy \]

\[ - \int_{n_0 < 1} \left( \sqrt{4 - 3n_0} \frac{1}{\sqrt{3}} |\nabla y n_0| - \frac{1}{\sqrt{3n_0}} |\nabla y c_0| \right)^2 dy + \int_{n_0 < 1} \nabla y n_0 \cdot \nabla y c_0 dy. \]

(4.10)

Now the key is to prove that the term \( \int_{n_0 < 1} \nabla y n_0 \cdot \nabla y c_0 dy \) in (4.10) is negative. We want to integrate by parts to use the relation \(-\Delta c_0 = n_0\) and the divergence theorem, however, the level set of a smooth function is not necessarily a smooth sub-manifold. We recall the Sard theorem and the inverse image theorem of differential topology:

**Theorem 2.** (Sternberg [43], Theorem II.3.1; Sard [42]) Let \( f : \mathbb{R}^n \to \mathbb{R}^m \) be \( C^k \), (that is, \( k \) times continuously differentiable), where \( k \geq \max\{n - m + 1, 1\} \). Let \( X \) denote the critical set of \( f \), which is the set of points \( x \in \mathbb{R}^n \) at which the Jacobian matrix of \( f \) has rank \( < m \). Then the image \( f(X) \) has Lebesgue measure \( 0 \) in \( \mathbb{R}^m \). (In other words, almost all points in the image is a regular value.)

**Theorem 3.** ([21], page 14) Let \( W \) in \( \mathbb{R}^n \) be an open set and \( f : W \to \mathbb{R}^q \) a \( C^r \) map, \( 1 \leq r \leq \infty \). Suppose \( y \in f(W) \) is a regular value of \( f \); this means that \( f \) has rank \( q \) at every point of \( f^{-1}(y) \). (Therefore \( q \leq n \) ) Then the subset \( f^{-1}(y) \) is a \( C^r \) submanifold of \( \mathbb{R}^n \) of codimension \( q \).

Since the solution \( n(t) \) is \( C^\infty \) for \( t \in (0, T_c) \), if one is not a regular value for \( n_0 \), by Sard’s theorem and the inverse image theorem, we may find a sequence of \( K_j \) such that \( K_j \nearrow 1 \) and that the level set \( \{n_0 = K_j\} \) smooth. Therefore, we may integrate by parts:

\[ \int_{n_0 \leq K_j < 1} \nabla y n_0 \cdot \nabla y c_0 dy = -\int_{n_0 \leq K_j} n_0 \nabla y c_0 dy + \int_{n_0 = K_j} n_0 \nabla y c_0 \cdot \nu ds \]

\[ = \int_{n_0 \leq K_j} n_0^2 dy + K_j \int_{n_0 = K_j} \nabla y c_0 \cdot \nu ds \]

\[ = \int_{n_0 \leq K_j} n_0^2 dy + K_j \int_{n_0 \leq K_j} \Delta c_0 dy \]

\[ \leq \int_{n_0 \leq K_j} n_0^2 dy - K_j \int_{n_0 \leq K_j} n_0 dy \]

\[ \leq 0. \]

As we have \( |\nabla y n_0 \cdot \nabla y c_0| \in L^1 \) by the Lebesgue dominated convergence theorem we deduce that \( \int_{n_0 < 1} \nabla y n_0 \cdot \nabla y c_0 dy \leq 0 \). Therefore, we deduce from (4.10) that,

\[ T_0 \leq -\int_{n_0 \geq 1} n_0|\nabla y \log n_0 - \nabla y c_0|^2 dy - \frac{2}{3} \int_{n_0 < 1} (4 - 3n_0)|\nabla y n_0|^2 dy. \]

(4.11)

This finishes the treatment of \( T_0 \) in (4.9).
Now we come to the treatment of the $T_\neq$ in (4.9). The idea is to use the negative terms in (4.11) and the fast decay from the bootstrap hypotheses to control part of the influence from $T_\neq$. By Young’s inequality, we estimate $T_\neq$ as follows:

$$T_\neq = \int_{n_0 \geq 1} (\nabla_y c_\neq n_\neq) \cdot (\nabla_y \log n_0 - \nabla_y c_0) dy + \int_{n_0 < 1} (\nabla_y c_\neq n_\neq) \cdot ((2 - n_0) \nabla_y n_0 - \nabla_y c_0) dy$$

$$+ \int_{n_0 < 1} (2 - n_0) \nabla_y n_0 \cdot (\nabla_y c_\neq n_\neq) dy$$

$$\leq \int_{n_0 \geq 1} \frac{|(\nabla_y c_\neq n_\neq)|}{\sqrt{n_0}} \sqrt{n_0} \nabla \log n_0 - \nabla c_0 |dy + \int_{n_0 < 1} (\nabla_y c_\neq n_\neq) \cdot ((4 - 3n_0) \nabla_y n_0) dy$$

$$+ \int_{n_0 < 1} |(\nabla_y c_\neq n_\neq)| \nabla_y c_0 |dy$$

$$\leq \int_{n_0 \geq 1} B |(\nabla_y c_\neq n_\neq)| \nabla_y n_0 |^2 dy + \frac{1}{B} \int_{n_0 \geq 1} n_0 |\nabla y \log n_0 - \nabla y c_0 |^2 dy$$

$$+ \frac{1}{B} \int_{n_0 < 1} (4 - 3n_0) |\nabla_y n_0 |^2 dy + \| |(\nabla_y c_\neq n_\neq)| |_{L^{4/3}(R^2)} \| |\nabla_y c_0 | |_{L^{4}(R^2)}.$$

Finally, the $|\nabla_y c_0 | |_{L^4(R^2)}$ in the last line is estimated using the Hardy-Littlewood-Sobolev inequality:

$$|\nabla_y c_0 | |_{L^4(R^2)} \lesssim \| n_0 | |_{L^{4/3}(R^2)}.$$

Combining (4.9), (4.11), (4.12) and (4.13) yields (4.17). Estimate (4.8) follows from the fact that the function $\Gamma$ is bounded from below. This finishes the proof of Lemma 4.1.

**Lemma 4.2.** For $A$ sufficiently large, the approximate free energy is bounded via the following

$$\mathcal{F}_\Gamma[n_0(t)] \leq 2\mathcal{F}_\Gamma[n_0(0)].$$

Moreover, there is a constant $C_{L \log L}$ which depends only on $\mathcal{F}_\Gamma[n_0(0)]$ and $M$ such that the following holds independent of time:

$$\int n_0 log^+ n_0 dy \leq C_{L \log L} (\mathcal{F}_\Gamma[n_0(0)], M).$$

**Proof.** By an argument similar to the 2D case, we have that $\| n_0 | |_{L^{1}(T \times \mathbb{R}^2)} = M$ is preserved. Next, we estimate the right-hand side of (4.17).

By Minkowski’s inequality, and applying the elliptic estimate (A.3), we have

$$\frac{1}{A} \| (\nabla_y c_\neq n_\neq) |_{L^2(R^2)} \leq \frac{1}{A} \| |\nabla_y c_\neq n_\neq | |_{L^2(T \times \mathbb{R}^2)}$$

$$\leq \frac{1}{A} \| |\nabla_y c_\neq | |_{L^\infty} | |n_\neq | |_2 \leq \frac{C_2}{A} | |n_\neq | |_2.$$

By (2.5b), the time integral of this contribution can be made arbitrarily small by choosing $A$ sufficiently large. Next, we estimate the second term on the right hand side of (4.17). Combining Minkowski’s integral inequality, Hölder’s inequality, and the Gagliardo-Nirenberg-Sobolev inequality, we obtain

$$\frac{1}{A} \| n_0 | |_{L^{4/3}(R^2)} \| |(\nabla_y c_\neq n_\neq) | |_{L^{4/3}(T \times \mathbb{R}^2)} \leq \frac{1}{A} \| |n_0 | |_{L^{4/3}(T \times \mathbb{R}^2)} \| |\nabla_y c_\neq n_\neq | |_{L^{4/3}(T \times \mathbb{R}^2)}$$

$$\leq \frac{1}{A} \| |n_0 | |_{L^1} | |n_0 | |_{L^2} | |\nabla_y c_\neq | |_{L^2} | |n_\neq | |_4$$

$$\leq \frac{1}{A} \| |n_0 | |_{L^1} | |n_0 | |_{L^2} | |n_\neq | |_{L^2} | |n_\neq | |_{L^{1/4}} | |\nabla y n_\neq | |_{L^{3/4}}$$

$$\leq \frac{C_2}{A^{3/4}} | |n_\neq | |_2^2 + \frac{1}{A^{5/4}} | |\nabla y n_\neq | |_2.$$

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Plugging the estimates (4.16) and (4.17) in (4.17) yields

$$\frac{d}{dt} \int n_0 \Gamma(n_0) - \frac{n_0 C_0}{2} dy \lesssim \frac{C_2}{A^{\eta/4}} \|n\|_2^2 + \frac{1}{A^{\alpha/4}} \|\nabla n\|_2^2. \quad (4.18)$$

It follows from the hypotheses (4.2a), (4.2b) and (4.2c), that the time integral of the right hand side of (4.18) can be made arbitrarily small by choosing $A$ large relative to the quantities $\|n_0\|_2, \|n_0\|_{H^1}, C_{2,\infty}$, and hence (4.14) follows.

A uniform in time bound on the free energy (4.14) can be translated to a uniform in time bound on the entropy (4.15) by using the logarithmic Hardy-Littlewood-Sobolev inequality, which we recall here (see e.g. [19]):

**Theorem 4** (Logarithmic Hardy-Littlewood-Sobolev Inequality). For all $M > 0$, there exists a constant $C(M)$ such that for all be a nonnegative functions in $f \in L^1(\mathbb{R}^2)$ such that $f \log f$ and $f \log(1 + |x|^2)$ belong to $L^1(\mathbb{R}^2)$. If $\int_{\mathbb{R}^2} f dx = M$, then

$$\int f \log f dx + \frac{2}{M} \int_{\mathbb{R}^2 \times \mathbb{R}^2} f(x)f(y) \log |x-y| dy \geq -C(M). \quad (4.19)$$

We rewrite the approximate free energy so that the inequality (4.19) can be applied, (4.14):

$$2F_\Gamma[n_0] \geq \int n_0 \Gamma(n_0) - \int \frac{n_0 C_0}{2} dy$$

$$\quad = \int n_0 \log^+ n_0 dy + \int_{n_0 \leq 1} n_0 \Gamma(n_0) dy + \frac{1}{4 \pi} \int \int \log |w - y| n_0(w)n_0(y) dwdy$$

$$\quad = \left(1 - \frac{M}{8 \pi}\right) \int n_0 \log^+ n_0 dy + \int_{n_0 \leq 1} n_0 \Gamma(n_0) dy$$

$$\quad + \frac{M}{8 \pi} \left(\int n_0 \log^+ n_0 dy + \frac{2}{M} \int \int \log |w - y| n_0(w)n_0(y) dwdy\right).$$

Applying the log-HLS (4.19) and (4.18) yield:

$$2F_\Gamma[n_0] \geq \left(1 - \frac{M}{8 \pi}\right) \int n_0 \log^+ n_0 dy + \int_{n_0 \leq 1} n_0 \Gamma(n_0) dy - C(M) \frac{M}{8 \pi}$$

$$\geq \left(1 - \frac{M}{8 \pi}\right) \int n_0 \log^+ n_0 dy - \frac{3}{2} M - C(M) \frac{M}{8 \pi},$$

which leads to a bound on the entropy

$$\int n_0 \log^+ n_0 dy \leq \frac{8 \pi}{8 \pi - M} \left(2F[n_0] + \frac{3}{2} M + C(M) \frac{M}{8 \pi}\right).$$

This concludes the proof of Lemma 4.2.

**Lemma 4.3.** The bound on the entropy (4.15) yields a uniform in time $L^2$ bound of $n_0$, that is,

$$\|n_0\|_{L^2} \leq C \eta \left(n_0\right). \quad (4.20)$$

**Proof.** The proof is a small variation of classical Patlak-Keller-Segel techniques (see e.g. [33] [17]); we sketch the proof here for completeness.

Let $K > 1$ be a constant, to be chosen later. Observe that (4.15),

$$\int (n_0 - K)^+ dy \leq \int_{n_0 > K} n_0 dy \leq \frac{1}{\log(K)} \int_{n_0 > K} n_0 \log^+(n_0) dy \leq \frac{C \log L}{\log(K)}. \quad (4.21)$$

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Next, via (1.1),
\[
\frac{1}{2} \frac{d}{dt} \int (n_0 - K)^2 \, dy
\]
\[
= \frac{1}{A} \int (n_0 - K)_+ [\Delta_y n_0 - \nabla_y : (n_0 \nabla c_0) - \nabla_y : (\nabla_y c_\neq 0)] dy
\]
\[
= - \frac{1}{A} \int |\nabla ((n_0 - K)_+)|^2 \, dy + \frac{1}{2A} \int (n_0 - K)^2_+ \, dy + \frac{3K}{2A} \int (n_0 - K)^2_+ \, dy + \frac{K^2}{A} \int (n - K)_+ \, dy
\]
\[
+ \frac{1}{A} \int \nabla (n_0 - K)_+ : (\nabla_y c_\neq 0) \, dy
\]
\[
\leq - \frac{7}{8A} \int \nabla (n_0 - K)_+ |^2 \, dy + \frac{1}{2A} \int (n_0 - K)^2_+ \, dy + \frac{3K}{2A} \int (n_0 - K)^2_+ \, dy + \frac{K^2M}{A}
\]
\[
+ \frac{B}{A} \| (\nabla_y c_\neq 0) \|_{L^2(\mathbb{R}^2)}^2.
\] (4.22)

Starting with the second term in (4.22), applying the Gagliardo-Nirenberg-Sobolev inequality yields (see e.g. [17] and the references therein)
\[
\int |(n_0 - K)_+|^2 \, dy \lesssim \int |\nabla (n_0 - K)_+|^2 \, dy \int (n_0 - K)_+ \, dy.
\]

From (4.21) that we can choose \( K \) depending only on \( C_{L, \log L} \) such that:
\[
- \frac{7}{8A} \int \nabla (n_0 - K)_+ |^2 \, dy + \frac{1}{2A} \int (n_0 - K)^2_+ \, dy \leq - \frac{1}{2A} \int |\nabla (n_0 - K)_+|^2 \, dy.
\] (4.23)

Next, we apply Minkowski’s inequality, the elliptic estimate (A.5), and the hypothesis (4.2d) to control the non-zero mode contribution \( \| (\nabla_y c_\neq 0) \|_{L^2(\mathbb{R}^2)}^2 \) in (4.22):
\[
\frac{1}{A} \| (\nabla_y c_\neq 0) \|_{L^2(\mathbb{R}^2)}^2 \lesssim \frac{1}{A} \| \nabla_y c_\neq \|_{\infty}^2 \| n_\neq \|_2^2 \lesssim \frac{1}{A} C_{2, \infty}^2 \| n_\neq \|_2^2.
\] (4.24)

Plugging (4.23) and (4.24) into (4.22) yields
\[
\frac{1}{2} \frac{d}{dt} \int (n_0 - K)^2_+ \, dy \lesssim - \frac{1}{2A} \int |\nabla ((n_0 - K)_+)|^2 \, dy + \frac{3K}{2A} \int (n_0 - K)^2_+ \, dy
\]
\[
+ \frac{K^2M}{A} + \frac{C_{2, \infty}^2}{A} \| n_\neq \|_2^2.
\] (4.25)

Applying the Nash inequality
\[
\| v \|_{L^2(\mathbb{R}^2)}^2 \lesssim \| \nabla v \|_{L^2(\mathbb{R}^2)} \| v \|_{L^1(\mathbb{R}^2)}
\]
in the estimate (4.26), we obtain
\[
\frac{1}{2} \frac{d}{dt} \| (n_0 - K)_+ \|_2^2 \lesssim - \frac{1}{2A} \| (n_0 - K)_+ \|_2^2 + \frac{3K}{2A} \| (n_0 - K)_+ \|_2^2 + \frac{K^2M}{A} + \frac{1}{A} C_{2, \infty} \| n_\neq (t) \|_2^2.
\] (4.26)

Applying an argument similar to the one used in Section 2.3 to deduce (2.32), by choosing \( A \) sufficiently large implies \( \int (n - K)^2_+ \, dy \leq C(n_{in}) \). Recall the following classical inequality (see e.g. [33] [18])
\[
\| n_0 \|_{L^2} \lesssim \| (n_0 - K)_+ \|_{L^2} + K^{1/2} M^{1/2},
\]
where the implicit constant is independent of \( K \) and \( M \). The inequality (4.20) hence follows.

Next, we prove the higher regularity estimate (4.2c) using (4.20).
Lemma 4.4. For $A$ sufficiently large, provided \((4.20)\) holds, the following improvement to \((4.2c)\) holds on $[0,T_1)$ for a suitable choice of $C_{H_1}$:

$$\|\nabla_y n_0\|_{L^2(R^2)} \leq 2C_{H_1}. \quad (4.27)$$

Proof. We employ the following standard multi-index notation:

$$\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2,$$

$$\partial_\alpha^y = \partial_1^{\alpha_1} \partial_2^{\alpha_2},$$

$$||\partial_\alpha^y n_0||_2^2 = \sum_{|\alpha|=s} ||\partial_\alpha^y n_0||_2^2.$$  

Let $\alpha$ be such that $|\alpha| = 1$. Computing the time derivative of $||\partial_\alpha^y n_0||_2^2$ and applying $\epsilon$-Young’s inequality:

$$\frac{1}{2} \frac{d}{dt} ||\partial_\alpha^y n_0||_2^2 = -\frac{1}{A} \int |\partial_\alpha^y \nabla c_0|^{2} dy + \frac{1}{A} \int |\partial_\alpha^y \nabla c_0 \cdot (\nabla c_0 n_0) + \frac{1}{A} \int |\partial_\alpha^y \nabla c_0 \cdot (\nabla c_0 n_0) dy$$

$$= -\frac{1}{A} \int |\partial_\alpha^y \nabla c_0|^{2} dy + \frac{1}{A} \int |\partial_\alpha^y \nabla c_0 \cdot (\nabla c_0 n_0) dy$$

$$+ \frac{1}{A} \int |\partial_\alpha^y \nabla c_0 \cdot (\nabla c_0 n_0) dy + \frac{1}{A} \int |\partial_\alpha^y \nabla c_0 \cdot (\nabla c_0 n_0) dy$$

$$\leq -\frac{7}{8A} \int |\partial_\alpha^y \nabla c_0|^{2} dy + \frac{B}{A} ||\partial_\alpha^y \nabla c_0 n_0||_2^2 + \frac{B}{A} ||\nabla c_0 \partial_\alpha^y n_0||_2^2$$

$$=: -\frac{7}{8A} \int |\partial_\alpha^y \nabla c_0|^{2} dy + T_1 + T_2 + NZ.$$  

We first estimate the term $T_1$ in \((4.28)\). Combining the bound \((4.20)\), the Gagliardo-Nirenberg-Sobolev inequality, and the $L^4$ boundedness of the Riesz transform yields:

$$T_1 \lesssim \frac{1}{A} ||\nabla c_0||_4^2 \lesssim \frac{1}{A} ||n_0||_4 \lesssim \frac{1}{A} ||n_0||_2 \|\nabla c_0 n_0\|_2 \lesssim \frac{1}{A} C_{H_1}^2 ||\nabla c_0 n_0\|_2^2. \quad (4.29)$$

Next for the second term $T_2$ in \((4.28)\), combining the elliptic estimate \((4.3)\) and the hypothesis \((4.2a)\) yields:

$$T_2 \leq \frac{B}{A} ||\nabla c_0\|_\infty^2 ||\partial_\alpha^y n_0||_2^2 \lesssim \frac{1}{A} C_{H_1}^2 ||\nabla y n_0\|_2^2. \quad (4.30)$$

Similar to the two dimensional case, the $NZ$ term in \((4.28)\) is estimated using Minkowski’s inequality, the elliptic estimate \((4.3)\), \((4.2a)\), and \((4.2b)\) as follows:

$$NZ \leq \frac{B}{A} ||n_\alpha\|_2^2 ||n_\alpha\|_\infty^2 + \frac{B}{A} ||\nabla c_\alpha\|_\infty^2 ||\nabla y n_\alpha\|_2^2$$

$$\lesssim \frac{BC_{H_1}^2}{A} ||n_\alpha\|_2^2 + \frac{BC_{H_1}^2}{A} ||\nabla y n_\alpha\|_2^2. \quad (4.31)$$

Combining the the above estimates \((4.28), (4.20), (4.30), (4.31)\) and summing over $\alpha = 1, 2$ yield:

$$\frac{1}{2} \frac{d}{dt} ||\nabla y n_0||_2^2 \leq -\frac{1}{2A} ||\partial_\alpha^y n_0||_2^2 + \frac{B}{A} C_{H_1}^2 ||\nabla y n_0||_2^2 + G(t), \quad (4.32)$$

where $G(t)$ is defined as:

$$G(t) := \int_0^t \frac{BC_{H_1}^2}{A} ||n_\alpha\|_2^2 + \frac{BC_{H_1}^2}{A} ||\nabla y n_\alpha\|_2^2 d\tau, \quad \forall t \in [0,T_*].$$

Applying an argument similar to the one used in Section 2.3 to prove \((2.32)\), choosing $A$ sufficiently large implies that:

$$||\nabla y n_0||_2^2 \lesssim C_{H_1}^4 \quad (4.33)$$

which is independent of $A$ and $C_{H_1}$. Note that we still have the freedom to pick our $C_{H_1}$, and we choose it such that $C_{H_1}^4$ is much bigger than the right hand side of \((4.33)\). This finishes the proof of Lemma 4.4 and the conclusion \((4.4c)\) follows. 

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A Appendix

A.1 $\nabla c$ estimates

We have applied various estimates on $\nabla c_0, \nabla c_\neq$; while all are standard, we sketch the proofs here for the readers' convenience.

Lemma A.1. In the two-dimensional case, the following estimate holds for uniformly for all $k \in \mathbb{Z} \setminus \{0\}$ and $(k^2 - \partial_{yy})\hat{c}_k = \hat{n}_k$:

$$|k|^{1/2}||\partial_y \hat{c}_k||_{L^\infty(T)} \lesssim ||\hat{n}_k||_{L^2(T)}.$$  \hspace{1cm} (A.1)

Proof. By the Gagliardo-Nirenberg-Sobolev inequality,

$$|k|^{1/2}||\partial_y \hat{c}_k||_{L^\infty} \lesssim |k|^{1/2}||\hat{c}_k||_{L^2}^{1/4}||\partial_{yy} \hat{c}_k||_{L^2}^{3/4} \lesssim ||k^2 \hat{c}_k||_{L^2}^{1/2}||\partial_{yy} \hat{c}_k||_{L^2}^{3/4} \lesssim ||\hat{n}_k||_2.$$  

This completes the proof of the lemma. \hfill \Box

Lemma A.2. In the two-dimensional case, the following estimate on $\nabla c_0$ holds:

$$||\partial_y c_0||_{L^\infty(T)} \lesssim ||n_0 - \overline{n}||_{L^1(T)}.$$ \hspace{1cm} (A.2)

Proof. By the fundamental theorem of calculus, $||\partial_y c||_{L^\infty} \leq ||\partial_{yy} c_0||_{L^1(T)}$, and hence the lemma follows. \hfill \Box

Lemma A.3. In the two-dimensional case, the following elliptic estimate holds:

$$||\nabla (c_\neq)||_{L^\infty(T^2)} \lesssim ||n_\neq||_{L^1(T^2)}.$$ \hspace{1cm} (A.3)

Proof. By Morrey's inequality, there holds for any $p > 2$,

$$||\nabla c_\neq||_{L^\infty(T^2)} \lesssim_p ||\nabla c||_{L^p} + ||\nabla^2 c||_{L^p}.$$  

The lemma follows from the Calderon-Zygmund inequality and the lack of low frequencies. \hfill \Box

In the 3-dimensional case, we need the following lemmas.

Lemma A.4. In the 3-dimensional case, the following mode by mode estimates are true:

$$||\nabla_y \hat{c}_k||_{L^\infty(\mathbb{R}^2)} \lesssim ||\hat{n}_k||_{L^2(\mathbb{R}^2)}^{1/2}||\nabla_y \hat{n}_k||_{L^2(\mathbb{R}^2)}^{1/2}$$ \hspace{1cm} (A.4)

$$||\nabla_y \hat{c}_k||_{L^\infty(T^2)} \lesssim ||\hat{n}_k||_{L^2(T^2)}^{1/2}||\nabla_y \hat{n}_k||_{L^2(T^2)}^{1/2}.$$  

Proof. From the Gagliardo-Nirenberg-Sobolev inequality,

$$||\nabla c_k||_{L^\infty} \lesssim ||\nabla c_k||_{L^2}^{1/2}||\nabla^3 c_k||_{L^2}^{1/2},$$  

from which the result follows. \hfill \Box

Other than the lemma above, we need the following 3D elliptic estimates.

Lemma A.5. In the three-dimensional case, the following elliptic estimates are true:

$$||\nabla c_\neq||_{L^\infty(T \times \mathbb{R}^2)} \lesssim ||n_\neq||_{L^4(T \times \mathbb{R}^2)},$$  

$$||\nabla c_\neq||_{L^\infty(T^3)} \lesssim ||n_\neq||_{L^4(T \times \mathbb{R}^2)},$$  

$$||\nabla_y c_0||_{L^\infty(\mathbb{R}^2)} \lesssim ||n_0||_{L^1(\mathbb{R}^2)}^{1/4}||n_0||_{L^3(\mathbb{R}^2)}^{1/4},$$  

$$||\nabla_y c_0||_{L^\infty(T^2)} \lesssim ||n_0 - \overline{n}||_{L^3(T^2)}.$$ \hspace{1cm} (A.5)

Proof. The first two inequalities follow from Morrey’s inequality and the Calderon-Zygmund inequality, as above. Similarly, as does the last inequality. The third inequality follows from a standard argument: optimizing over the choice of $R$ we have,

$$\nabla_y c_0(y) \lesssim \int_{|y-y'| \geq R} \frac{y-y'}{|y-y'|^2} n_0(y') dy' + \int_{|y-y'| < R} \frac{y-y'}{|y-y'|^2} n_0(y') dy'$$

\[
\lesssim \frac{1}{R} \|n_0\|_{L^1} + \|n_0\|_{L^3} \left( \int_{|y'| < R} \frac{1}{|y'|^{3/2}} dy' \right)^{2/3} \\
\lesssim \frac{1}{R} \|n_0\|_{L^1} + \|n_0\|_{L^3} R^{1/3} \\
\lesssim \|n\|_{L^1}^{1/4} \|n_0\|_{3/4}^{3/4}.
\]

Acknowledgments

The authors would like to thank Michele Coti Zelati for helpful discussions and Eitan Tadmor for helpful discussions and for suggesting this problem.

References

[1] K. Beauchard. Null controllability of kolmogorov-type equations. *Mathematics of Control, Signals, and Systems*, 26(1):145–176, 2014.

[2] K. Beauchard and E. Zuazua. Some controllability results for the 2D Kolmogorov equation. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 26(5):1793–1815, 2009.

[3] M. Beck and C. Wayne. Metastability and rapid convergence to quasi-stationary bar states for the two-dimensional Navier–Stokes equations. *Proc. Royal Soc. of Edinburgh: Sec. A Mathematics*, 143(05):905–927, 2013.

[4] J. Bedrossian and M. Coti Zelati. Enhanced dissipation, hypoellipticity, and anomalous small noise inviscid limits in shear flows. *arXiv:1510.08098*, 2015.

[5] J. Bedrossian, M. Coti Zelati, and N. Glatt-Holtz. Invariant measures for passive scalars in the small noise inviscid limit. *To appear in Comm. Math. Phys. (arXiv:1505.07356)*, 2015.

[6] J. Bedrossian, P. Germain, and N. Masmoudi. Dynamics near the subcritical transition of the 3D Couette flow I: Below threshold. *arXiv:1506.03720*, 2015.

[7] J. Bedrossian, P. Germain, and N. Masmoudi. Dynamics near the subcritical transition of the 3D Couette flow II: Above threshold. *arXiv:1506.03721*, 2015.

[8] J. Bedrossian, P. Germain, and N. Masmoudi. On the stability threshold for the 3D Couette flow in Sobolev regularity. *arXiv:1511.01373*, 2015.

[9] J. Bedrossian, N. Masmoudi, and V. Vicol. Enhanced dissipation and inviscid damping in the inviscid limit of the Navier-Stokes equations near the 2D Couette flow. *Arch. Rat. Mech. Anal.*, 216(3):1087–1159, 2016.

[10] J. Bedrossian, V. Vicol, and F. Wang. The Sobolev stability threshold for 2D shear flows near Couette. *To appear in J. Non. Sci. (arXiv:1604.01831)*, 2016.

[11] A. Bernoff and J. Lingevitch. Rapid relaxation of an axisymmetric vortex. *Phys. Fluids*, 6(3717), 1994.

[12] P. Biler. The Cauchy problem and self-similar solutions for a nonlinear parabolic equation. *Studia Math.*, 114(2):181–192, 1995.
[13] P. Biler, G. Karch, P. Laurençot, and T. Nadzieja. The $8\pi$-problem for radially symmetric solutions of a chemotaxis model in the plane. Math. Meth. Appl. Sci, 29:1563–1583, 2006.

[14] A. Blanchet, V. Calvez, and J. Carrillo. Convergence of the mass-transport steepest descent scheme for subcritical Patlak-Keller-Segel model. SIAM J. Num. Anal., 46:691–721, 2008.

[15] A. Blanchet, E. Carlen, and J. Carrillo. Journal of Functional Analysis, 262(5):2142–2230, 2012.

[16] A. Blanchet, J. Carrillo, and N. Masmoudi. Infinite time aggregation for the critical Patlak-Keller-Segel model in $\mathbb{R}^2$. Comm. Pure Appl. Math., 61:1449–1481, 2008.

[17] A. Blanchet, J. Dolbeault, and B. Perthame. Two-dimensional Keller-Segel model: Optimal critical mass and qualitative properties of the solutions. E. J. Diff. Eqns, 2006(44):1–32, 2006.

[18] V. Calvez and J. Carrillo. Volume effects in the Keller-Segel model: energy estimates preventing blow-up. J. Math. Pures Appl., 86:155–175, 2006.

[19] E. Carlen and M. Loss. Competing symmetries, the logarithmic HLS inequality and Onofri’s inequality on $S^n$. Geom. Func. Anal., 2(1):90–104, 1992.

[20] S. Childress and J. Percus. Nonlinear aspects of chemotaxis. Math. Biosci., 56:217–237, 1981.

[21] P. Constantin, A. Kiselev, L. Ryzhik, and A. Zlatoš. Diffusion and mixing in fluid flow. Ann. of Math. (2), 168:643–674, 2008.

[22] L. Corrias, B. Perthame, and H. Zaag. Global solutions of some chemotaxis and angiogenesis systems in high space dimensions. Milan J. Math., 72:1–28, 2004.

[23] L. Desvillettes, C. Villani, et al. On the trend to global equilibrium in spatially inhomogeneous entropy-dissipating systems. Part I: the linear Fokker-Planck equation. Comm. Pure Appl. Math, 54(1):1–42, 2001.

[24] J. Dolbeault, C. Mouhot, and C. Schmeiser. Hypocoercivity for linear kinetic equations conserving mass. Trans. Amer. Math. Soc., 367(6):3807–3828, 2015.

[25] B. Dubrulle and S. Nazarenko. On scaling laws for the transition to turbulence in uniform-shear flows. Euro. Phys. Lett., 27(2):129, 1994.

[26] I. Gallagher, T. Gallay, and F. Nier. Spectral asymptotics for large skew-symmetric perturbations of the harmonic oscillator. International Mathematics Research Notices, page rnp013, 2009.

[27] F. Hérau. Short and long time behavior of the Fokker-Planck equation in a confining potential and applications. J. Funct. Anal., 244(1):95–118, 2007.

[28] F. Hérau and F. Nier. Isotropic hypoellipticity and trend to equilibrium for the fokker-planck equation with a high-degree potential. Archive for Rational Mechanics and Analysis, 171(2):151–218, 2004.

[29] M. Herrero and J. Velázquez. Singularity patterns in a chemotaxis model. Math. Ann., 306:583–623, 1996.

[30] T. Hillen and K. J. Painter. A user’s guide to PDE models for chemotaxis. J. Math. Biol., 58(1-2):183–217, 2009.

[31] M. W. Hirsch. Differential Topology, volume 33 of Graduate Texts in Mathematics. Springer-Verlag New York, first edition, 1976.

[32] D. Horstmann. From 1970 until present: the Keller-Segel model in chemotaxis and its consequences. I, Jahresber. Deutsch. Math.-Verein, 105(3):103–165, 2003.

[33] W. Jäger and S. Luckhaus. On explosions of solutions to a system of partial differential equations modelling chemotaxis. Trans. Amer. Math. Soc., 329(2):819–824, 1992.
[34] E. F. Keller and L. Segel. Model for chemotaxis. *J. Theor. Biol.*, 30:225–234, 1971.

[35] A. Kiselev and X. Xu. Suppression of chemotactic explosion by mixing. *Archive for Rational Mechanics and Analysis, arXiv:1508.05333v3 [math.AP]*, pages 1–36, 2015.

[36] R. Kowalczyk. Preventing blow-up in a chemotaxis model. *J. Math. Anal. Appl.*, 305:566–588, 2005.

[37] M. Latini and A. Bernoff. Transient anomalous diffusion in Poiseuille flow. *J. of Fluid Mech.*, 441:399–411, 2001.

[38] T. Lundgren. Strained spiral vortex model for turbulent fine structure. *Phys. Fl.*, 25:2193, 1982.

[39] T. Nagai. Blow-up of radially symmetric solutions to a chemotaxis system. *Adv. Math. Sci. Appl.*, 5(2):581–601, 1995.

[40] C. S. Patlak. Random walk with persistence and external bias. *Bull. Math. Biophys.*, 15:311–338, 1953.

[41] P. Rhines and W. Young. How rapidly is a passive scalar mixed within closed streamlines? *J. of Fluid Mech.*, 133:133–145, 1983.

[42] A. Sard. The measure of the critical values of differentiable maps. *Bulletin of the American Mathematical Society*, 48(12):883890, 1942.

[43] T. Senba and T. Suzuki. Weak solutions to a parabolic-elliptic system of chemotaxis. *J. Func. Anal.*, 191:17–51, 2002.

[44] I. Shafrir and G. Wolansky. The logarithmic hls inequality for systems on compact manifolds. *J. Func. Anal.*, 227:200226, 2006.

[45] S. Sternberg. *Lectures on differential geometry*. Englewood Cliffs, NJ: Prentice-Hall, 1964.

[46] C. Villani. *Hypocoercivity*. American Mathematical Soc., 2009.

[47] J. Vukadinovic, E. Deditis, A. C. Poje, and T. Schäfer. Averaging and spectral properties for the 2D advection–diffusion equation in the semi-classical limit for vanishing diffusivity. *Phys. D*, 310:1–18, 2015.

[48] A. Zlatoš. Diffusion in fluid flow: dissipation enhancement by flows in 2D. *Comm. Partial Differential Equations*, 35(3):496–534, 2010.