Gravitational constraints of $dS$ branes in $AdS$ Einstein-Brans-Dicke bulk

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We derive the full projected Einstein-Brans-Dicke gravitational equations associated with a $n$-dimensional brane embedded in a $(n + 1)$-dimensional bulk. By making use of general conditions, as the positivity of the Brans-Dicke parameter and the effective Newton gravitational constant as well, we are able to constrain the brane cosmological constant in terms of the brane tension, the Brans-Dicke scalar field, and the trace of the stress tensor on the brane, in order to achieve a $dS$ brane. Applying these constraints to a specific five-dimensional model, a lower bound for the scalar field on the brane is elicited without solving the full equations. It is shown under which conditions the brane effective cosmological constant can be ignored in the brane projected gravitational field equations, suggesting a different fine tuning between the brane tension and the bulk cosmological.

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I. INTRODUCTION

Soon after the seminal papers of Randall and Sundrum [1,2], the amount of works dealing with the theoretical possibility of large extra dimensions in theories describing the Universe increased significantly. The possibility of solving the hierarchy problem in the Randall-Sundrum (RS) scenario, for instance, is one of the major reasons for such an increase. There are even other outstanding and prominent features arising in the scope of the gravitational aspects related to RS-like models, accruing from a non-factorizable geometry and $Z_2$ symmetry. The basic setup concerning the RSI model formalism [1] is composed by two mirror domain walls — 3-branes — as boundaries of a five-dimensional $AdS$ bulk. The extra dimension is represented by the orbifold $S^1/Z_2$. It is shown in [2] that the model is still consistent, if one of the branes is taken to infinity.

Nevertheless, despite of all the RS models success, it may be not the final word in braneworld models. In the light of the string theory advances, there are at least two distinguished features that the RS models cannot provide: models dealing with more than five dimensions and also other fields mediating the gravitational phenomena. In fact, unification models such as supergravity, superstrings, and $M$-theory [3] effectively predict the existence of a scalar gravitational field acting as a gravitational interaction mediator, together with the usual rank-2 tensor field. In this vein, it is worthwhile to consider the gravitational field equations in a scalar-tensor theory context. In this work we shall deal with the simplest consistent scalar-tensor theory found in the literature: the Brans-Dicke theory [4].

There are several models concerning the accomplishment of braneworld scenarios, in many different dimensions. For instance, going forward in the program consisting of the use of topological defects in order to generate all the bulk/brane structure, the global cosmic string was used in ordinary General Relativity [5], where a six-dimensional spacetime is obtained. Moreover, the use of a local cosmic string in Brans-Dicke gravity was scrutinized in [6], while a global cosmic string, again in Brans-Dicke theory, was analyzed in [7]. Besides, in General Relativity, this type of six-dimensional scenario was investigated with an explicit local vortex source [8] and a blown-up brane was analyzed in [9, 10], where a $Z_2$ symmetry connects the solutions inside and outside the brane.

A consistent mode to investigate the gravitational aspects and effects about the presence of extra dimensions is to look at the Gauss-Codazzi equations [11], resulting from a foliation of the bulk spacetime and the projection of the gravitational field equations from the bulk into the brane. It was, in fact, applied to RS-like braneworlds in General Relativity in [12] with $Z_2$ symmetry. Such a procedure allows the investigation of a wide variety of brane gravitational phenomena [13]. Without the $Z_2$ symmetry, a similar but more involved procedure was implemented in [14].
In the context of Brans-Dicke gravity, the Gauss-Codazzi formalism was implemented in a six-dimensional braneworld with $\mathbb{Z}_2$ symmetry in reference [12], and without such a symmetry in [16]. This dimensionality ($D = 6$) is maintained in such papers in order to be consistent with the works aforementioned in the previous paragraph. In the present paper we should leave this constraint apart, working, instead, in arbitrary dimensions and scrutinizing a five-dimensional example explicitly. A key characteristic for the implementation of the Gauss-Codazzi formalism to braneworlds is the codimension — the number of extra dimensions out the brane — one. In the case of a codimension bigger than one it is still possible to choose a direction to foliate the bulk, nevertheless the projection of the bulk geometric quantities cannot be implemented, since one needs the concept of different “sides” of the hypersurface that the equations are projected on.

The aim of this paper is twofold: firstly, we are concerned to find out the full projected equations from a $(n+1)$-dimensional bulk into a $n$-dimensional brane with $\mathbb{Z}_2$ symmetry, in the context of Brans-Dicke theory. It has been never accomplished before in the Brans-Dicke theory and since it is the first step to extract some physical information about such system, we shall generalize the Gauss-Codazzi formalism to this case, for arbitrary dimensions. Secondly and most important, by exploring the general form of the resulting equations, we are able to provide some conductive formal constraints which must be satisfied by a braneworld model in Brans-Dicke gravity in arbitrary dimensions, in order to generate a specific scenario as, for instance, a $dS$ brane embedded into an $AdS$ bulk. After establishing the general picture, we scrutinize a concrete example of a 3-brane in a five-dimensional bulk. It is shown that (for the vacuum on the brane) we are able to find out a lower bound of the Brans-Dicke scalar field — usually called the “dilaton” hereon — on the brane, even without solving the full Einstein-Brans-Dicke equation. We also discuss about the possibility of discarding the brane effective cosmological constant term in the brane projected equations.

This paper is organized as follows: in the next Section the basic notations and conventions used in this work are introduced. In Section III we indite the full projected Einstein-Brans-Dicke equation on the brane for an arbitrary dimension. Those two Sections are devoted to a brief revision and also delve into a generalization of the Gauss-Codazzi formalism to an arbitrary bulk dimension, in the scope of Brans-Dicke gravity. Thereafter, in Section IV the general constraints necessary to implement a specific model are derived, namely a $dS$ brane embedded into a $AdS$ bulk. In addition, we analyze a concrete case by investigating a five-dimensional RS-like braneworld. It constitutes a prominent example of how the constraint obtained is useful in order to evince physical gravitational information about the braneworld model dealt with. In the final Section we conclude, summarizing our results, and point out some future perspectives in this research line.

II. PRELIMINARIES

This Section is devoted to recall the implementation of the Gauss-Codazzi formalism geometric part, starting from a $(n+1)$-dimensional bulk. It was accomplished elsewhere [19] in the General Relativity framework, but we shall keep this Section in order to guarantee some sequential readability of the paper. Moreover, the implementation of such formalism to the Brans-Dicke case was accomplished only for specific scenarios, as the six-dimensional case [15, 16], but not regarding an arbitrary dimension. As we shall see, this generalization provides a rigorous method to explore some physical properties of the braneworld we are dealing with.

We start by regarding the brane as a $n$-dimensional time-like hypersurface embedded in a $(n+1)$-dimensional bulk, namely, a codimension one braneworld. Denoting $n^\mu$ the components of an unitary vector orthogonal to the brane and $g_{\mu\nu}$ the components of the bulk metric tensor of signature $(−, +, +, \ldots, +)$, the components of the induced metric on the brane is given by $q_{\mu\nu} = g_{\mu\nu} − n_\mu n_\nu$. Denoting the bulk covariant derivative by $\nabla_\mu$, the Gauss equation forthwith reads

$$
(n)R^\alpha_{\beta\gamma\delta} = (n+1)R^\mu_{\nu\rho\sigma}q^\alpha_\mu q^\nu_\beta q^\sigma_\gamma q^\delta_\delta + K^\alpha_{\gamma} K_{\beta\delta} − K^\alpha_{\delta} K_{\beta\gamma},
$$

(1)

where $K_{\mu\nu} = q^\rho_\mu q^\sigma_\nu \nabla_\alpha n_\beta$ denotes the extrinsic curvature, indication the way how the brane is embedded in the bulk. Equation (1) basically asserts that the brane curvature tensor is given by the projection of the bulk curvature tensor and corrections coming from extrinsic curvature terms. The Codazzi equation is given by

$$
D_\nu K^\nu_{\mu} − D_\mu K = (n+1)R_{\rho\sigma} n^\sigma q^\rho_\mu,
$$

(2)

\[1\] There are important results concerning codimension two braneworld models. In [13], the addition of a Gauss-Bonnet term in the Lagrangian allows the extraction of physical information about the system. Recently, a rigorous approach to codimension two was derived in [18].
where $D_\mu$ denotes the covariant derivative on the brane. From Equation (1) it is immediate to see that the $n$-dimensional Ricci tensor is given by

\[
(n)R_{\beta\delta} = (n+1)R_{\nu\sigma}(\beta^\nu q^\sigma_\delta - (n+1)R^\mu_{\nu\rho}n_\mu n^\rho q^\sigma_\delta) + KK_{\beta\delta} - K_\delta^7 K_{\beta\gamma},
\]

(3)

while the scalar curvature is given by

\[
(n)R = q^{\beta\delta}(n)R_{\beta\delta} = q^{\beta\delta}(n+1)R_{\beta\delta} - (n+1)R^\mu_{\nu\rho}n_\mu n^\rho q^\nu_\sigma + K^2 - K^{\alpha\beta}K_{\alpha\beta}.
\]

(4)

Using Equations (2) and (3) it is possible to construct the $n$-dimensional Einstein tensor $(n)G_{\mu\nu}$:

\[
(n)G_{\mu\nu} = (n+1)R_{\nu\sigma}q^\mu_\sigma - \tilde{E}_{\beta\delta} + KK_{\beta\delta} - K_\delta^7 K_{\beta\gamma} - \frac{1}{2}q^{\beta\delta}q^{\nu_\sigma}(n+1)R_{\nu\sigma} - \frac{1}{2}q^{\beta\delta}(K_k - K^{\alpha\gamma}K_{\alpha\gamma}),
\]

where $\tilde{E}_{\beta\delta} = (n+1)R^\mu_{\nu\rho}n_\mu n^\rho q^\nu_\sigma$. Now, taking into account the relations

\[
q^{\beta\delta}q^{\nu_\sigma}(n+1)R_{\nu\sigma} = g^{\nu_\sigma}(n+1)Rq^\nu_\sigma - q^{\beta\delta}(n+1)R_{\nu\sigma}n_\nu n_\sigma
\]

and

\[
(n+1)R^\mu_{\nu\rho}n_\mu n^\rho q_{\nu\sigma} = (n+1)R_{\mu\rho}n_\mu n_\rho,
\]

we arrive at

\[
(n)G_{\beta\delta} = (n+1)G_{\nu\sigma}q^\nu_\sigma + (n+1)R_{\nu\sigma}n_\nu n_\sigma q^{\beta\delta} + KK_{\beta\delta} - K_\delta^7 K_{\beta\gamma}
- \frac{1}{2}q^{\beta\delta}(K_k - K^{\alpha\gamma}K_{\alpha\gamma}) - \tilde{E}_{\beta\delta}.
\]

(8)

It is useful to express the $n$-dimensional Einstein tensor in terms of the bulk Weyl tensor. After all manipulations, the Weyl tensor — denoted here by $C^\nu_{\nu\rho}$ — brings some genuine contributions from the bulk. The relation between the Riemann, the Ricci, the Weyl tensors, and the scalar curvature in $(n+1)$-dimensions is provided by

\[
(n+1)R^\mu_{\nu\rho} = (n+1)C^\mu_{\nu\rho} + \frac{2}{n-1} \left( (n+1)R^\mu_{[\nu\rho]} + (n+1)R^\mu_{\nu\rho}g^{\mu}_{[\sigma]} \right) - \frac{2}{n(n-1)}(n+1)Rg^\mu_{[\nu\rho]}.n
\]

(9)

The first important relation to notice is that $\tilde{E}_{\beta\delta}$ can be expressed in terms of $E_{\beta\delta} \equiv (n+1)C^\nu_{\nu\rho}n_\nu n^\rho q^\nu_\sigma q^{\beta\delta}$ by

\[
\tilde{E}_{\beta\delta} = E_{\beta\delta} + \frac{1}{n-1} \left( (n+1)R^\mu_{\nu\rho}n_\nu n^\rho q^{\beta\delta} + (n+1)R_{\nu\sigma}q^\nu_\sigma q^{\beta\delta} \right) - \frac{1}{n(n-1)}(n+1)Rq^{\beta\delta}.
\]

(10)

Then, after some manipulations the brane Einstein tensor can be expressed as the following:

\[
(n)G_{\beta\delta} = \frac{(n-3)}{(n-1)} \left( (n+1)G_{\nu\sigma}q^\nu_\sigma - (n+1)R_{\nu\sigma}q^{\mu_\nu_\sigma} \right) + \frac{(n^2 - 4n + 2)}{n(n-1)}(n+1)Rq^{\beta\delta} + KK_{\beta\delta} - K_\delta^7 K_{\beta\gamma}
- \frac{1}{2}q^{\beta\delta}(K_k - K^{\alpha\gamma}K_{\alpha\gamma}) - E_{\beta\delta}.
\]

(11)

The equation above encloses the main result of this Section. In the next Section we express the geometric bulk quantities — $(n+1)G_{\mu\nu}$, $(n+1)R_{\mu\nu}$, and $(n+1)R$ — in terms of the dilaton field and the bulk energy-momentum tensor.

### III. THE FULL PROJECTED EINSTEIN-BRANS-DICKE EQUATION

In order to extend the application of the Gauss-Codazzi formalism to the Brans-Dicke theory for gravity in arbitrary dimensions, we start by writing the $(n+1)$-dimensional Brans-Dicke equation

\[
(n+1)G_{\nu\rho} = \frac{8\pi}{\phi} T_{\nu\rho} + \frac{\omega}{\phi^2} \left( \nabla_\nu \phi \nabla_\rho \phi - \frac{1}{2} g_{\nu\rho} \nabla_\alpha \phi \nabla^\alpha \phi \right) + \frac{1}{\phi} \left( \nabla_\nu \nabla_\rho \phi - g_{\nu\rho} \Box^2 \phi \right).
\]

(12)
On the other hand, from Equations (2) and (15) it follows that reads (for a (n+1)-dimensional bulk) as the projection of any tensor, $X$, onto the brane by \( T^{\mu\nu} \) and contracting all the resulting equation with \( g^{\mu\nu} \) we have

\[
R = \frac{-2w}{n + (n-1)w} 8\pi \phi T + \frac{w}{\phi^2} \nabla^\nu \phi \nabla_\alpha \phi.
\]  

(13)

Returning to the Equation (12), the Ricci tensor is given by

\[
R_{\nu\rho} = \frac{8\pi \phi}{n + (n-1)w} T_{\nu\rho} + \frac{w}{\phi^2} \nabla^\nu \phi \nabla_\rho \phi + \frac{1}{\phi} \nabla^\nu \nabla_\rho \phi - \frac{8\pi}{\phi} q_{\nu\rho} \frac{w + 1}{n + (n-1)w} T.
\]  

(15)

Now, the first three terms on the right-hand side of Eq. (11) can be computed. The resulting equation expressed in terms of the dilaton field and the stress-tensor is given by

\[
(\nu^{(n)})G_{\beta\delta} = \frac{(n-3)}{(n-1)} \left( \frac{8\pi}{\phi} T_{\nu\rho} + \frac{w}{\phi^2} \nabla^\nu \phi \nabla_\rho \phi + \frac{1}{\phi} \nabla^\nu \nabla_\rho \phi \right) (q^{\nu\rho} - q_{\nu\rho} T_{\mu\nu}) + \frac{(n-4)}{2n} \frac{w}{\phi^2} q_{\beta\delta} \nabla_\gamma \phi \nabla^\gamma \phi
\]

\[
- \frac{8\pi}{\phi} \frac{T}{n + (n-1)w} q_{\beta\delta} \left[ (w + 1)(1 - n) + \frac{(n-4)w}{n} \right] + KK_{\beta\delta} + K^\gamma K_{\beta\gamma}
\]

\[
- \frac{1}{2} q_{\beta\delta} (K^2 - K^{\alpha\beta} K_{\alpha\beta}) - E_{\beta\delta}.
\]  

(16)

On the other hand, from Equations (2) and (15) it follows that

\[
D_\nu K^\nu - D_\mu K = \left( \frac{8\pi}{\phi} T_{\nu\rho} + \frac{w}{\phi^2} \nabla^\nu \phi \nabla_\rho \phi + \frac{1}{\phi} \nabla^\nu \nabla_\rho \phi \right) q^\nu q^{\rho}.
\]  

(17)

Herein in this Section our efforts will be focused to effectively project Equation (16) on the brane. It is accomplished by the generalization of the Israel-Darmois [20] junction condition to the Brans-Dicke case. By using standard tools of the distributional calculus, it is shown in the Appendix of reference [15] that the generalized junction condition reads (for a \((n+1)\)-dimensional bulk)

\[
[K_{\mu\nu}] - [K] q_{\mu\nu} = -\frac{8\pi}{\phi} \left( T^{\text{brane}}_{\mu\nu} - q_{\mu\nu} \frac{T^{\text{brane}}}{n + (n-1)w} \right),
\]  

(18)

where the quantity \([X]\) means \([X] = X^+ - X^-\). Here \(X^\pm\) denotes the projection of any tensor, \(X\), onto the brane by the \(\pm\) side. We refer the reader to the reference [15] for all the details. Apart from that, we stress that \(T^{\text{brane}}_{\mu\nu}\) is the energy-momentum tensor on the brane and \(T^{\text{brane}}\) its respective trace. As we will see, these terms bring contribution from the brane tension and the matter on the brane as well. From the trace of Equation (18) it is straightforward to see that

\[
[K] = \frac{8\pi}{\phi} T^{\text{brane}} \frac{w}{n + (n-1)w}.
\]  

(19)

Then, returning to (18) one finds

\[
[K_{\mu\nu}] = -\frac{8\pi}{\phi} T^{\text{brane}}_{\mu\nu} + \frac{8\pi}{\phi} q_{\mu\nu} (w + 1) T^{\text{brane}} \frac{w}{n + (n-1)w}.
\]  

(20)

Now, the quantities in brackets can be forthwith determined by the imposition of the \(Z_2\) symmetry. The role of the \(Z_2\) symmetry in braneworld models is multiple [21], but in what concerns its immediate effect on the gravitational equations, it turns out to be quite straightforward: it just changes the sign of the unitary orthogonal vector field, \(n^{\alpha}\), across the brane. Imposing such a symmetry, it implies that \(n^{\alpha} \rightarrow -n^{\alpha}\). From the extrinsic curvature definition, one
concludes that the same effect holds for $K_{\mu\nu}$, namely $K_{\alpha\beta}^+ \mapsto -K_{\alpha\beta}^-$. This result enables us to write Equations (19) and (20) as the following:

$$K^+ = \frac{4\pi}{\phi} T_{\nu\rho}^\text{brane} \frac{w}{n + (n-1)w}, \quad K_{\mu\nu}^+ = \frac{4\pi}{\phi} T_{\nu\rho}^\text{brane} + \frac{4\pi (w+1)q_{\mu\nu}}{\phi} T_{\nu\rho}^\text{brane},$$

(21)

respectively. Note that in Equation (16) the terms of extrinsic curvature are quadratic. Therefore, the values of $K$ and $K_{\mu\nu}$ can be computed at any side of the brane, in such way that the label $\pm$ can be concealed without any detriment in the theory.

Substituting the equations above into the Equation (16), suppressing the labels $\pm$, it follows that

$$G_{\beta\delta}^{(n)} = \frac{(n-3)}{(n-1)} \left( \frac{8\pi T_{\nu\rho}}{\phi^2} \right. \frac{w}{\phi^2} \nabla_\nu \phi \nabla_\rho \phi + \left. \frac{1}{\phi^2} \nabla_\nu \nabla_\rho \phi \right) (q_{\beta\delta}^\nu - q_{\beta\delta} q^{\nu\rho}) + \frac{(n-4)w}{2n} q_{\beta\delta} \nabla_\alpha \phi \nabla_\alpha \phi \phi \right)$$

$$- \frac{8\pi}{\phi} \frac{q_{\beta\delta} T}{n + (n-1)w} \left[ (w+1)(1-n) + \frac{(n-4)w}{n} \right] + \left( \frac{16\pi^2}{\phi^2} \right) \left\{ \frac{1}{2} q_{\beta\delta} T_{\nu\rho}^\text{brane} \alpha^\tau \gamma \right\}$$

$$- \frac{(w+2)}{n + (n-1)w} T_{\nu\rho}^\text{brane} \frac{(w+1)(3n-2) - w^2}{2[n + (n-1)w]^2} - E_{\beta\delta}. \quad (22)$$

We remark that the quantities involved in Equation (22) are interpreted in the limit of the extra transverse dimension approaching the brane. Equation (22) is written in a far from suitable way, in order to distinguish between the two stress-tensors, i.e. the two distinct quantities playing the role of sources, $T_{\mu\nu}$ and $T_{\nu\rho}^\text{brane}$. As a matter of fact, $T_{\mu\nu} \supset T_{\nu\rho}^\text{brane}$ and to complete the analysis we have to specify the form of the stress-tensors. Ignoring any type of bulk source, except a cosmological constant and the brane itself, one can write the bulk stress tensor as

$$T_{\mu\nu} = -\Lambda q_{\mu\nu} + \delta(y - y_b) T_{\mu\nu}^\text{brane}, \quad (23)$$

where $\delta(y)$ is necessary in order to position the brane (generically assumed in $y = y_b$) in the bulk, and

$$T_{\nu\rho}^\text{brane} = -\lambda q_{\mu\nu} + \tau_{\mu\nu}, \quad (24)$$

where $\lambda$ denotes the brane tension and $\tau_{\mu\nu}$ the contribution of any matter field to the brane energy-momentum tensor. Substituting Equations (23) and (24) into (22), and taking into account that all the quantities are computed in the limit approaching the brane, the following result accrues:

$$G_{\beta\delta}^{(n)} = \frac{(n-3)}{(n-1)} \left( \frac{w}{\phi^2} \nabla_\nu \phi \nabla_\rho \phi + \frac{1}{\phi^2} \nabla_\nu \nabla_\rho \phi \right) (q_{\beta\delta}^\nu - q_{\beta\delta} q^{\nu\rho}) - \Lambda_n q_{\beta\delta} + 8\Omega \tau_{\beta\delta} + 16\pi^2 \Sigma_{\beta\delta} - E_{\beta\delta}, \quad (25)$$

where $\Omega, \Lambda_n, \Sigma_{\beta\delta}$ are given, respectively, by

$$\Omega = \frac{2\pi \lambda}{\phi^2} \frac{[4n + (3n-2)w]}{n + (n-1)w}, \quad (26)$$

$$\Lambda_n = \frac{8\pi (-\Lambda)(1 - 3n)(w+1) - 4w]}{\phi n[n + (n-1)w]} - \frac{(n-4)w}{2n} \phi^2 \nabla^\alpha \phi \nabla_\alpha \phi - \frac{16\pi^2}{\phi^2} \lambda \tau - \frac{n(n-1)w^2}{2\phi^2} \lambda \tau$$

$$\times \left\{ (n-1)(3n^2 + n + 2) - n(8n^2 - 5n + 2) - n^2(5n - 2) \right\} - \frac{16\pi^2}{\phi^2} \lambda \tau$$

$$\times \left\{ nw^2 + (w+1)w[4n(1-n) + w(-4n^2 + 5n - 2)] \right\} \quad (27)$$

and

$$\Sigma_{\beta\delta} = \frac{1}{\phi^2} \left( \frac{[w+1)^2(3n-2) - w^2]}{2[n + (n-1)w]^2} q_{\beta\delta} \tau^2 + \frac{1}{2} q_{\beta\delta} \tau \alpha \gamma - \left( \frac{w+2}{n + (n-1)w} \right) \tau_{\beta\delta} - \tau^2 \tau_{\beta\gamma} \right) \quad (28)$$

We shall conclude this Section providing some useful interpretations about the results encoded in Equations (25) – (28). We write the full projected equation in an Einstein tensor-like form. The first term of (25) brings the
specific contribution of the scalar field dynamics, and it must be carefully analyzed in any cosmological application of Equation (29). In fact, we expect that such a term play an important role in cosmological scenarios. The second term is given by the effective brane cosmological constant, \( \Lambda_n \), which depends on the bulk cosmological constant, the brane tension, the dilaton field, and on the stress-tensor trace of matter on the brane. We shall analyze carefully this term in the next Section. The term proportional to \( \tau_{\mu \nu} \) is analogue to the usual source term of Einstein equation, where \( \Omega \) is the effective Newtonian constant. The penultimate term is quadratic on the brane stress-tensor and could play a very important role in early stages of the cosmological evolution. The last term proportional to the Weyl tensor brings genuine contribution coming from the bulk, and has no analogue in the usual four-dimensional case.

Note that, from Equation (29), the sign of the Newtonian effective gravitational constant strongly depends on the sign of the brane tension \( \lambda \). It will be our starting point to derive the constraints in the next Section. As we will see, the functional form of the scalar field plays an important role in the dependence of \( \Omega \) on the brane tension. Before beginning such analysis, however, we want to comment two more remarks about \( \Omega \). First, its dependence on \( \lambda \) tell us that it could not be possible to define gravity in some cosmological era before the formation of structures. Apart from that, the dependence on \( 1/\phi^2 \) turns explicit the importance of the dilaton field stabilization, in order to guarantee the agreement with usual gravity on the brane. In some artificial way, this situation can be used to fix the brane position along the transverse extra dimension, leading then to the right value to the scalar field. In a more rigorous way, the stabilization of the dilaton field can be accomplished, for instance, by the introduction of a well behaved potential in the Brans-Dicke part of the action (22).

IV. GENERAL CONSTRAINTS: A FIVE-DIMENSIONAL EXAMPLE

In this Section we shall derive some general constraints relating the bulk cosmological constant, the brane tension, and the scalar field, in order to lead to a specific scenario, namely, a \( dS \) brane (\( \Lambda_n \geq 0 \)), accordingly to the low expansion observations, with a positive Newton’s effective gravitational constant (24) embedded into an \( AdS \) bulk.

Before proceeding, let us state a relevant remark about Equation (29). Note that the brane tension must be positive in order to engender the right sign to the effective gravitational constant. It is in accordance with the results previously obtained in the literature, and is also consistent with a physical gravitational object, since a negative tension brane is intrinsically unstable. It is purposeful to remark that for certain compactification models, the Brans-Dicke parameter may be negative. In this case, it is still possible to have a positive brane tension and \( \Omega > 0 \), provided that \( |\omega| < 3/(2 - 1/n) \). Henceforward we restrict the analysis for positive Brans-Dicke parameter.

Now, we focus on a scenario consisting of an \( AdS \) (\( \Lambda < 0 \)) bulk with an embedded \( dS \) (\( \Lambda_n \geq 0 \)) brane. As we shall see, the functional form for the dilaton field on the brane can be extracted, providing an important boundary condition of such scalar degree of freedom. By looking at the explicit form of the effective cosmological constant (29) and imposing the above conditions, it implies — for a non-vanishing scalar field — that

\[
\frac{\Lambda}{\phi} \frac{[(1 - 3n)(w + 1) - 4w]}{n[n + (n - 1)w]} \geq \frac{(n - 4)}{2n} \frac{w}{8\pi\phi^2} \frac{\lambda^2}{\phi^2} \frac{\lambda^2}{n + (n - 1)w^2} \\
\times \{(n - 1)(3n^2 + n + 2)w^2 + n(8n^2 - 5n + 2)w + n^2(5n - 2)\} + \frac{2\pi}{\phi^2} \frac{\lambda^2}{n + (n - 1)w^2} \\
\times \{nw^2 + (w + 1)[4n(1 - n) + w(-4n^2 + 5n - 2)]\}.
\]

(29)

The equation above is written in a form that completely obscures the associated physical content. Nevertheless, it may be useful by the specification of a particular scenario. Let us investigate the example of a RS-like model, composed by a 3-brane (\( n = 4 \)) embedded in the bulk. Note that in such dimensionality there is a great simplification, since the term \( \nabla_\alpha \phi \nabla^\alpha \phi \) appearing in (29) vanishes. Just by investigating this general constraint, it is obtained a lower bound for the dilaton field, in terms of the bulk cosmological constant, the brane tension and the Brans-Dicke parameter, without solving the full projected equation. Going further, we show what type of relationship between the bulk cosmological constant and the brane tension is necessary for \( \Lambda_4 \) to be discarded of the projected effective brane gravitational equation. Comparing with the case in General Relativity, this sufficient condition is a genuine output of the Brans-Dicke bulk gravity. As will be shown, this type of interplay between the bulk cosmological constant and the brane tension is intrinsically different from the usual RS fine tuning.

Let us make a brief comment about the approach outlined in the previous paragraph. The basic idea is to derive gravitational constraints, which provides physical information about the system encoded in the Equations (29) – (28). Here, two points should be stressed. Firstly, the generality of the Gauss-Codazzi procedure rests upon the fact that we do not need to make any consideration about the specific functional form of the metric. In this vein, the derivation of simple, and informative, gravitational constraints without solving the projected equation appears to be an interesting approach. Besides, the solution of the \( (\phi, q_{\mu \nu}) \)-system may be far from trivial even in the simplest case. Let us analyze...
an short example in order to make more clear this second point. Suppose the dilaton depending only on the extra transverse dimension. In view of the Equation (13), it is clear that \( C^2 \phi = \alpha \), being \( \alpha \) a constant involving the brane tension and the bulk cosmological constant. This scenario may be achieved by the consideration of the brane vacuum, for instance. So, the resulting differential equation reads

\[ \frac{d^2 \phi}{dy^2} + f(x^\mu) \frac{d\phi}{dy} = \alpha, \]

where \( f(x^\mu) \) is given by \( f(x^\mu) = \frac{1}{2} g^{\mu\nu} \partial_y g_{\mu\nu} \) and the bulk metric is diagonal. The solution for the Equation (30) may be settled in the form

\[ \phi = \int \left\{ \alpha \int \exp \left( \int f(x^\mu)dy \right) + C_1 \right\} \exp \left( -\int f(x^\mu)dy \right) dy + C_2, \]

being \( C_{1,2} \) constants. Even being the \( g_{yy} \) component trivial, the brane metric may still have a complicated dependence on the extra dimension. It is, of course, the entire point of non-factorizable (warped) geometries of which a RS-like model is a simple example. Hence, the integration of \( f(x^\mu) \) may not be trivial. Apart from that, the robustness of this approach is, as remarked, in the generality of the unknown metric. Thus, we shall not to restrict the argumentation by specifying the brane metric.

Now, going further in our example, it is immediate to note that for \( n = 4 \), Equation (29) reads

\[ -|\Lambda| \leq \frac{2 \pi}{\phi^2 (4 + 3w)(11 + 15w)} \left\{ 4(21w^2 + 55w + 36)\lambda - (21w^2 + 47w + 24)\tau \right\}. \]

We shall derive a result comparable with the standard procedure developed in the General Relativity realm. In this vein we turn our attention to the brane vacuum. Note, however, that it appears inconsistent for a positive dilaton field on the brane, since the resulting constraint is \(-|\Lambda| \geq Q\), being \( Q \) a strictly positive quantity. So, one are led to consider a negative dilaton field. In this vein, for a vacuum on the brane it is straightforward to derive, from (32), a lower bound for the dilaton field projected on the brane

\[ |\phi| \gtrsim \frac{64\pi \lambda^2}{|\Lambda|} \frac{21w^2 + 55w + 36}{45w^2 + 93w + 44}. \]

Equation (33) asserts relevant information that shall be remarked. The general form \(|\phi| \sim \frac{\pi \lambda^2}{|\Lambda|}\), is quite simple, nevertheless it is far from trivial. Again, the scalar field obviously is completely determined by the full solution of the bulk field equations, which may be a very difficult task. Instead, by simple inputs (necessary for the fixation of a particular scenario, but keeping considerable generality\(^2\)) it is possible to derive a constraint that gives the boundary value for the scalar field. We remark that, from Equation (33), it is possible to achieve an upper bound for the effective Newtonian constant on the brane, as well as its correct dependence on the brane tension in such scalar-tensor gravity scenario. In fact, taking into account Equation (20) for \( n = 4 \) it follows that

\[ \Omega \lesssim \frac{|\Lambda|^2}{\lambda^3} \Delta(w), \]

where \( \Delta(w) = \frac{(15w^2 + 56w + 48)(11 + 15w)^2}{(32(21w^2 + 55w + 36))^2} \). The Equation (34) stress once again the impossibility about the definition of gravity before the structure formation. Note also, that it keeps the dependence of \( \Omega \) on the \( \lambda \) signal.

Now, it is possible to further explore the condition making the approximation \( \Lambda_4 \sim 0 \) valid. This approximation is widely used in the application of braneworld models to cosmological problems as, for instance, the fitting of the galactic rotation curves without dark matter and corrections in the black hole area in General Relativity \([13, 27, 28]\).

The result encoded in the Equation (33) reveals that if the dilaton obeys such a constraint, then the desired behavior for the projected cosmological constant is achieved (\( \Lambda_4 \geq 0 \)). It is immediate to see that for a huge bulk cosmological constant in comparing to the brane tension, namely \( |\Lambda| \gg \lambda^2 \), the constraint (33) is easily satisfied. Note that it is not so trivial as it may sound. If we compare to the four-dimensional case in usual General Relativity, the effective brane cosmological constant (\( \Lambda_{GR4} \)) is given by

\[ \Lambda_{GR4} = \frac{\kappa_5^2}{2} \left( \Lambda + \frac{1}{6} \kappa_5^2 \lambda^2 \right). \]

\(^2\) Note that we do not assume any form for the metric.
where $\kappa_5$ is the five-dimensional gravitational coupling. So, the imposition of $|\Lambda| \gg \lambda^2$ in Equation (35) is not enough to guarantee the approximation $^{(4)}G_{\mu\nu} = -E_{\mu\nu}$, since there is another term, proportional to the brane tension itself. This is closely related to the fine tuning between the bulk cosmological constant and the brane tension of the RS model. These two quantities must be of the same order for the cancellation of the $\Lambda_4$ contribution. In the Brans-Dicke gravity context, however, the dynamics of the scalar field — the analogue to $1/\kappa_5^2$ — enables us to bound both terms of $\Lambda_4$ in an unique term and, since the behavior of the dilaton field obeys $\phi \sim 1/|\Lambda|$, it is quite enough a huge bulk cosmological constant.

It is important to stress, besides, another appreciable departure of the Einstein-Brans-Dicke case to the usual scenario in General Relativity. Going further in the vacuum on the brane case, Equation (25) forthwith reads

$$^{(4)}G_{\beta\delta} = \frac{1}{3} \left( \frac{w}{\phi^2} \partial_\nu \phi \partial_\rho \phi + \frac{1}{\phi} \nabla_\nu \partial_\rho \phi \right) \left( q^\nu_\beta q^\rho_\delta - q^\beta_\delta q^\rho_\nu \right) - \Lambda_4 q_{\beta\delta} - E_{\beta\delta}. \quad (36)$$

In this case then, even discarding the effective cosmological constant, we shall expect some subtle but substantial modifications coming genuinely from the scalar field dynamics in the analysis of specific cosmological systems, specially in the context of a weak dilaton field on the brane. In other words, the first term of the Equation (36) can provide a good laboratory for the study of General Relativity departures in the braneworld context.

As an aside remark, we call some attention to fact that the first two terms of the Equation (36) may obey a specific balance in order to provide a null brane scalar of curvature. From (36) we have

$$^{(4)}R = \left( \frac{w}{\phi^2} \partial_\nu \phi \partial_\rho \phi + \frac{1}{\phi} \nabla_\nu \partial_\rho \phi \right) q^{\nu\rho} + 4\Lambda_4. \quad (37)$$

Therefore, this two terms may cancel out one each other resulting in a large scale flat brane. This possibility, however, may preclude a departure of Brans-Dicke brane scenarios from the usual General Relativity ones, and, obviously, shall be based upon some dynamical mechanism. Perhaps, an similar analysis to the one carried out in reference [30] shall shed some light in such a mechanism. In any case, the possibility raised in this paragraph needs further study.

V. FINAL REMARKS AND OUTLOOK

Before weaving some final remarks we shall briefly summarize the main procedures and results of this paper. Motivated by string theory low energy recovered gravity, we started finding out the full projected Einstein-Brans-Dicke equation in an arbitrary dimension via the application of the Gauss-Codazzi mechanism. Up to our knowledge, it was never accomplished before, for arbitrary dimensions, to the Brans-Dicke gravity case. After that, we found the necessary condition — relating the scalar field, the bulk cosmological constant, and the brane tension — under which we obtain a $dS$ brane embedded in an $AdS$ bulk. Applying our results to a concrete five-dimensional model we are able to set the profile of the dilaton field on a brane vacuum.

Using the profile of the dilaton field on the brane we derived the general form of the effective brane cosmological constant, arriving at the sufficient condition ($|\Lambda| \gg \lambda^2$) that makes possible to discard the $\Lambda_4$ term of the projected gravitational equation. This fact is a genuine output of the Brans-Dicke gravity. It indicates a different type of fine tuning between the bulk cosmological constant and the brane tension in the Brans-Dicke gravity framework.

As a remark aside, we stress that in our analysis, we did not treated extensively the possibility of a negative Brans-Dicke parameter. It is well known that to some specific compactification models, the interplay between string theory low energy gravity and Brans-Dicke theory is given by a negative $w$. However, by keeping the usual Brans-Dicke gravity motivation we restrict our analysis to the positive $w$ case.

This work is a first step in order to investigate and to delve into some cosmological properties of Brans-Dicke braneworld models. As our results indicates, it is expected subtle but important departures of those cases analyzed in the scope of General Relativity. In this context, specific problems as the variation of quasar luminosity and corrections on the black holes areas [28], just to enumerate some physically interesting and relevant systems, which are normally investigated due the presence of extra dimensions, deserve more attention in the context of Brans-Dicke braneworld gravity.
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