A Euclidean Ramsey result in the plane

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Abstract

An old question in Euclidean Ramsey theory asks, if the points in the plane are red-blue coloured, does there always exist a red pair of points at unit distance or five blue points in line separated by unit distances? An elementary proof answers this question in the affirmative.

1 Introduction

Many problems in Euclidean Ramsey theory ask, for some \( d \in \mathbb{Z}^+ \), if \( E^d \) is coloured with \( r \geq 2 \) colours, does there exist a colour class containing some desired geometric structure? Research in Euclidean Ramsey theory was surveyed in [2–4] by Erdős, Graham, Montgomery, Rothschild, Spencer, and Straus; for a more recent survey, see Graham [5].

Say that two geometric configurations are congruent iff there exists an isometry (distance preserving bijection) between them. For \( d \in \mathbb{Z}^+ \) and geometric configurations \( F_1, F_2 \), let the notation \( E^d \to (F_1, F_2) \) mean that for any red-blue coloring of \( E^d \), either the red points contain a congruent copy of \( F_1 \), or the blue points contain a congruent copy of \( F_2 \). For a positive integer \( i \), denote by \( \ell_i \) the configuration of \( i \) collinear points with distance 1 between consecutive points. One of the results in [3] states that

\[
E^2 \to (\ell_2, \ell_4).
\] (1)

In the same paper, it was asked if \( E^2 \to (\ell_2, \ell_5) \), or perhaps a weaker result holds: \( E^3 \to (\ell_2, \ell_5) \).

The result (1) was generalised by Juhász [7], who proved that if \( T_4 \) is any configuration of 4 points, then \( E^2 \to (\ell_2, T_4) \). Juhász (personal communication, 10 February 2017) informed the author that Iván’s thesis [6] contains a proof that for any configuration \( T_5 \) of 5 points, \( E^3 \to (\ell_2, T_5) \) (which implies that \( E^3 \to (\ell_2, \ell_5) \)). Arman and Tsaturian [1] proved that \( E^3 \to (\ell_2, \ell_6) \).

In this paper, it is proved that \( E^2 \to (\ell_2, \ell_5) \):

**Theorem 1.1.** Let the Euclidean space \( \mathbb{E}^2 \) be coloured in red and blue so that there are no two red points distance 1 apart. Then there exist five blue points that form an \( \ell_5 \).

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2 Proof of Theorem 1.1

The proof is by contradiction; it is assumed that there are no five blue points forming an $\ell_5$. The following lemmas are needed.

**Lemma 2.1.** Let $\mathbb{E}^2$ be coloured in red and blue so that there is no red $\ell_2$. If there is no blue $\ell_5$, then there are no three blue points forming an equilateral triangle with side length 3 and with a red centre.

**Proof.** Suppose that $\mathbb{E}^2$ is coloured in red and blue so that there is no red $\ell_2$ and no blue $\ell_5$. Suppose that blue points $A$, $B$ and $C$ form an equilateral triangle with side length 3 and with red centre $O$. Consider the part of the unit triangular lattice shown in Figure 1(a). The points $D$, $E$, $F$, $G$ are blue, since they are distance 1 apart from $O$. The point $X$ is red; otherwise $XADEB$ is a red $\ell_5$. Similarly, $Y$ is red (to prevent red $YAFGC$). Then $X$ and $Y$ are two red points distance 1 apart, which contradicts the assumption. □

![Figure 1](image1)

Figure 1: Red points are denoted by diamonds, blue points are denoted by discs.

**Lemma 2.2.** Let $\mathbb{E}^2$ be coloured in red and blue so that there is no red $\ell_2$. If there is no blue $\ell_5$, then there are no three red points forming an equilateral triangle with side length $\sqrt{3}$ and with a red centre.

**Proof.** Suppose that $\mathbb{E}^2$ is coloured in red and blue so that there is no red $\ell_2$ and no blue $\ell_5$. Suppose that blue points $A$, $B$ and $C$ form an equilateral triangle with side length 3 and with red centre $O$. Let $A'$, $B'$, $C'$ be the images of $A$, $B$ and $C$, respectively, under a rotation about $O$ so that $AA' = BB' = CC' = 1$ (see Figure 1(b)). Then $A'$, $B'$, $C'$ are blue and form an equilateral triangle with side length $\sqrt{3}$ and red centre $O$, which contradicts the result of Lemma 2.1. □
Define $\mathcal{T}_3$, $\mathcal{T}_4$, $\mathcal{T}_5$, $\mathcal{T}_6$, $\mathcal{T}_7$ to be the configurations of three, four, five, six and seven points (respectively), depicted in Figure 2 (all the smallest distances between the points are equal to $\sqrt{3}$).

**Lemma 2.3.** Let $\mathbb{E}^2$ be coloured in red and blue so that there is no red $\ell_2$. If there is no blue $\ell_5$, then there are no seven red points forming a $\mathcal{T}_7$.

**Proof.** Suppose that $\mathbb{E}^2$ is coloured in red and blue so that there is no red $\ell_2$ and no blue $\ell_5$. Suppose that $A, B, C, D, E, F$ and $G$ are red points forming a $\mathcal{T}_7$ (as in Figure 3). Let $X$ be the reflection of $F$ in $BC$. Let $X'$, $A'$, $F'$ be the images of $X$, $A$, $F$, respectively, under the clockwise rotation about $B$ such that $XX' = AA' = FF' = 1$. Since $A$ and $F$ are red, $A'$ and $F'$ are blue. If $X'$ is blue, then $X'A'F'$ is a blue equilateral triangle with side length 3 and red center $B$, which contradicts the result of Lemma 2.1. Therefore, $X'$ is red. Let $X''$, $D''$, $F''$ be the images of $X$, $D$, $F$, respectively, under the clockwise rotation about $C$ such that $XX'' = DD'' = FF'' = 1$. Since $D$ and $F$ are red, $D''$ and $F''$ are blue. If $X''$ is blue, then $X''D''F''$ is a blue equilateral triangle with side length 3 and red center $C$, which contradicts the result of Lemma 2.1. Therefore, $X''$ is red. Since $X'$ can be obtained from $X''$ by the clockwise rotation through 60° about $X$, $XX'X''$ is a unit equilateral triangle, hence $X'X''$ is a red $\ell_2$, which contradicts the assumption of the lemma. $\square$
Lemma 2.4. Let $\mathbb{E}^2$ be coloured in red and blue so that there is no red $\ell_2$. Let $A, B, C$ be three red points forming a $\Sigma_3$. If there is no blue $\ell_5$, then there exists a red $\Sigma_6$ that contains $\{A, B, C\}$ as a subset.

Proof. Suppose that $\mathbb{E}^2$ is coloured in red and blue so that there is no red $\ell_2$ and no blue $\ell_5$. Let $A, B, C$ be three red points forming a $\Sigma_3$. Consider the unit triangular lattice depicted in Figure 4.

Suppose that there is no red point $D$ such that $A, B, C, D$ form a $\Sigma_4$. Then points $X, Y, Z$ are blue. Points $E, F, G, H, I, J$ are blue, since each of them is distance 1 apart from a red point. If the point $K$ is red, then the points $L$ and $M$ are blue and $LMGYH$ is a blue $\ell_5$. Therefore, $K$ is blue. Then $N$ is red (otherwise $KJIZN$ is a blue $\ell_5$), hence $P$ and $Q$ are blue, which leads to a blue $\ell_5$ $PQFEX$. A contradiction is obtained, therefore there exists a red point $D$ such that $A, B, C, D$ form a $\Sigma_4$. 

Figure 3

![Figure 3](image3.png)

Figure 4

![Figure 4](image4.png)
Let $A, B, C, D$ form a red $T_4$. Consider the part of the unit triangular lattice depicted in Figure 5. Suppose that there is no red point $E$ such that $A, B, C, D, E$ form a $T_5$. Then the points $X, F$ and $G$ are blue. Points $H, I, K, L, M, N$ are blue, since each of them is distance 1 apart from a red point. Point $P$ is red (otherwise $FHIGP$ is a blue $ℓ_5$), therefore $Q$ and $R$ are red. Then $X, N, M, Q, R$ form a blue $ℓ_5$, which gives a contradiction. Hence, there exists a red point $E$ such that $A, B, C, D, E$ form a $T_5$.

Let $A, B, C, D, E$ form a $T_5$ (Figure 6). Suppose that $F$ is blue. By Lemma 2.2, points $X$ and $Y$ are blue (otherwise $X, E, C$ ($Y, A, D$) form a red triangle with side length 3 and red center $B$). Points $G, H, I, J, K, L, M, N$ are blue, since each one of them is at distance 1 from a red point. If point $P$ is blue, then $Q$ is red (otherwise $QPKLF$ is a blue $ℓ_5$), $U$ and $T$ are blue and form a blue $ℓ_5$ with points $G, H$ and $X$. Therefore, $P$ is red. Similarly, $R$ is red (otherwise $S$ is red and $VWJIY$ is a blue $ℓ_5$). Then $A, B, C, D, E, P$ and $R$ form a red $T_7$, which is not possible by Lemma 2.3. Therefore, $F$ is red and $A, B, C, D, E, F$ form a red $T_6$.

Figure 5

Figure 6
Lemma 2.5. Let $\mathbb{E}^2$ be coloured in red and blue so that there is no red $\ell_2$ and no blue $\ell_5$. Let $\mathcal{L}$ be a unit triangular lattice that contains three red points forming a $\mathfrak{T}_3$. If there is no blue $\ell_5$, then the colouring of $\mathcal{L}$ is unique (up to translation or rotation by a multiple of $60^\circ$), and is depicted in Figure 7.

Proof. Suppose that $\mathbb{E}^2$ is coloured in red and blue so that there is no red $\ell_2$ and no blue $\ell_5$. Suppose there exist three red points of $\mathcal{L}$ that form a $\mathfrak{T}_3$. By Lemma 2.4, it may be assumed that there is a red $\mathfrak{T}_6$. Denote its points by $A, B, C, D, E, F$ (see Figure 8). It will be proved that the translate $A'B'C'D'E'F'$ of $ABCDEF$ by the vector of length 5 collinear to $\overrightarrow{AB}$ is red.

Consider the points shown in Figure 8. Since $A, D$ and $F$ are red, by Lemma 2.2, $I$ is blue. Since $C, F$ and $D$ are red, by Lemma 2.2, $J$ is blue. Points $K, L, M, N$ are blue, since each one is distance 1 apart from a red point. If $R$ is red, then both $P$ and $Q$ are blue and form a blue $\ell_5$ with $K, L$ and $I$. Therefore $R$ is blue. Then the point $A'$ is red (otherwise $A'JNMR$ is a red $\ell_5$).

Since $S_1, S_2, S_3, S_4$ are blue (as distance 1 apart from red points $D$ and $A'$), $B'$ is red. Similarly, $F'$ is red. Points $V$ and $W$ are blue as they are distance 1 apart from $C$. Points $U$ is blue by Lemma 2.2 (since $A, D$ and $B$ are red). If $X$ is red, then $X_1$ and $X_2$ are blue and a blue $\ell_5 UVWX_1X_2$ is formed. Therefore, $X$ is blue. Similarly, $Y$ is blue. By Lemma 2.4, $A'B'C'D'E'F'$ must be contained in a red $\mathfrak{T}_6$, and since $X$ and $Y$ are blue, the only possible such $\mathfrak{T}_6$ is $A'B'C'D'E'F'$. Hence, $A', B', C', D', E', F'$ are blue.

Similarly, the translates of $ABCDEF$ by vectors of length 5 collinear to $\overrightarrow{EB}$.
and $\overline{CF}$ are red. By repeatedly applying the same argument to the new red translates, it can be seen that all the translates of $ABCDEF$ by a multiple of 5 in $\mathcal{L}$ are red. All the other points are blue, as each one is distance 1 apart from a red point. Hence, the colouring as in Figure 7 is obtained.

\[ \square \]

**Figure 8**

**Lemma 2.6.** Let $\mathbb{E}^2$ be coloured in red and blue so that there is no red $\ell_2$. Let $\mathcal{L}$ be a unit triangular lattice that does not contain three red points forming a $T_3$. If there is no blue $\ell_5$, then the colouring of $\mathcal{L}$ is unique (up to translation or rotation by a multiple of $60^\circ$), and is depicted in Figure 9.

**Figure 9**
Proof. Suppose that $\mathbb{E}^2$ is coloured in red and blue so that there is no red $\ell_3$ and no blue $\ell_5$. If $\mathcal{L}$ does not contain a red point, then any $\ell_5$ is blue, therefore $\mathcal{L}$ contains a red point $A$. By Lemma 2.1 one of the points of $\mathcal{L}$ at distance $\sqrt{3}$ to $A$ is red (otherwise the three such points form a blue triangle with side length 3 and red centre $A$). Denote this point by $B$ (Figure 10). Since $\mathcal{L}$ does not contain a red $\ell_3$, the points $D$ and $G$ are blue. Points $E, F, I, H, K, J$ are blue, since they are distance 1 apart from $B$. Then the point $B'$ is red (otherwise blue $\ell_5$ $DEFGB'$ is formed). Point $N$ is 1 apart from $B'$, hence blue. Then $C$ and $A'$ are red (otherwise a blue $\ell_5$ is formed).

By repeating the same argument for points $B$ and $C$, $B$ and $A$ (instead of $A$ and $B$), and so on, it can be shown that any node of $\mathcal{L}$ on the line $AB$ is red. Similarly, since $A'$ and $B'$ are both red, any node of $\mathcal{L}$ on the line $A'B'$ is red. By the same argument, $A''$, $B''$ and any node on the line containing them is red; $A'''$, $B'''$ and any node on the line containing them is red, and so on. By colouring all point distance 1 apart form red points blue, the colouring in Figure 9 is obtained.

Figure 10

Proof of Theorem 1.1. Let the Euclidean space $\mathbb{E}^2$ be coloured in red and blue so that there are no two red points distance 1 apart. Suppose that there are no five blue points that form an $\ell_5$. Then there is a red point $A$. Consider two points $B$ and $C$, both distance 5 apart from $A$, such that $|BC| = 1$. At least one of the points $B$ and $C$ (say, $B$) is blue. Consider the unit triangular lattice $\mathcal{L}$ that contains $A$ and $B$. By Lemma 2.1 and Lemma 2.6 $\mathcal{L}$ is coloured either as in Figure 7 or as in Figure 9. But neither one of the colourings contains two points of different colour distance 5 apart, which gives a contradiction. Therefore, there exist five blue points that form an $\ell_5$. 

\[ \square \]
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