Estimating the Gains (and Losses) of Revenue Management

Xavier D’Haultfoeuille†  Ao Wang‡  Philippe Février§
Lionel Wilner¶

Abstract

Despite the wide adoption of revenue management in many industries such as airline, railway, and hospitality, there is still scarce empirical evidence on the gains or losses of such strategies compared to uniform pricing or fully flexible strategies. We quantify such gains and losses and identify their underlying sources in the context of French railway transportation. The identification of demand is complicated by censoring and the absence of exogenous price variations. We develop an original identification strategy combining temporal variations in relative prices, consumers’ rationality and weak optimality conditions on the firm’s pricing strategy. Our results suggest similar or better performance of the actual revenue management compared to optimal uniform pricing, but also substantial losses of up to 16.7% compared to the optimal pricing strategy. We also highlight the key role of revenue management in acquiring information when demand is uncertain.

Keywords: Revenue management, dynamic pricing, demand estimation, demand learning, moment inequalities.

JEL Codes: C61, L11, L92, R41.

*We thank people at iDTGV for providing us the data. We also thank Philippe Choné, Pierre Dubois, Carlos Daniel Santos, Laurent Lamy, Lars Nesheim, Matt Shum and seminar participants at the California Institute of Technology, University of Mannheim, Rice University, Sciences-Po, Texas A&M, Toulouse School of Economics, Yale University and EARIE in Athens, 3rd CREST-ECODEC Conference, the 5th French Econometrics Conference, the 2022 European Winter Meeting of the Econometric Society for their comments.

†CREST-ENSAE, xavier.dhaultfoeuille@ensae.fr.
‡University of Warwick and CAGE, ao.wang@warwick.ac.uk.
§CREST, fevrier@ensae.fr.
¶CREST, lionel.wilner@ensae.fr.
1 Introduction

Revenue management is a collection of operations research techniques that adjust supply to the random demand for perishable goods. It has been adopted in a number of industries, notably airline and railway sectors, as liberalization reforms supposed to increase competitive pressures.\(^1\) Adjusting prices in a flexible way is likely to increase firms’ revenues, compared to conventional uniform pricing. But it also comes at some cost, as it requires specialized teams and good algorithms. Then, heuristic rules are usually set to simplify the pricing strategy, likely resulting in lower revenues than the optimal dynamic pricing that is complex to implement. Quantifying such gains and losses, as well as identifying their causes, is crucial for understanding and improving firms’ actual revenue management. However, empirical evidence on that matter is still scarce.

We investigate these issues in the context of French high-speed railway (HSR) sector and study revenue management at iDTGV, a subsidiary of the French railway monopoly (SNCF). During its existence between 2004 and 2017, iDTGV provided low-cost trains from Paris to several cities in France, and the corresponding return trains. Our focus is motivated by three facts. First, HSR has been widely seen as a green mode of transportation and is thriving in many parts of the world (e.g., EU, UK, and Asia).\(^2\) Second, iDTGV was a marketing laboratory for SNCF and its operation environment (as will be detailed in Section 2.1) was relatively simple, allowing us to abstract from some complex issues (e.g., refunding, group booking, loyalty program)

\(^1\)Historically, revenue management was invented by major airlines in the wake of the deregulation in the US in the 1970s. It is considered among “the most successful application areas of operations research” (Talluri and van Ryzin, 2005). More recently, the liberalization of the EU railway sector and low-cost airlines’ entry in domestic routes have motivated European railway operators to adopt such techniques. See Delaplace and Dobruszkes (2015); Dressen (2018) and Abrate et al. (2016); Beria et al. (2019) for examples in France and Italy, respectively.

\(^2\)According to the International Union of Railways, passenger kilometers have more than quadrupled in the last decade, from 245.1 to 1071.8 billion. Worldwide, there are 58,839km of HSR in operation and 19,710km under construction. Even though HSR service is much limited in the US compared to Europe and East Asia, the only existing one being Acela, operated by Amtrak, several lines are under development or construction, e.g., the line between Los Angeles and San Francisco, Brightline West and Texas Central.
and focus on the key mechanism also relevant in other settings. Finally, and importantly, the revenue management at iDTGV is quantity-based, the prime approach used in airline and railway sectors (van Ryzin and Talluri, 2005; Von Martens and Hilbert, 2011). Specifically, seats were dynamically allocated to different fare classes, while the price in each fare class remains unchanged during the booking period. For the economy class on which we focus here, 12 classes of prices were defined, sorted in ascending order. At any moment before the departure of a given train, revenue managers could decide to close the current fare class and open a higher one, thus increasing the prices of the seats.

To investigate the relative benefits of this pricing strategy compared to uniform pricing or other dynamic strategies, we propose a demand and supply framework that captures typical features of transport demand and caters to the context of iDTGV. First, we propose a stochastic demand model of transportation mode for traveling on a route (e.g., Paris-Marseille on the 1st of March 2008). It allows the number of consumers searching for a route to vary with the time before departure. Moreover, we allow consumers to choose from different transportation modes, creating instantaneous competition between iDTGV and, for instance, low-cost airlines. We prove that recovering the price elasticity coefficient and parameters governing the average demand for each destination within the route is sufficient to identify a rich set of counterfactual revenues. In other words, expected revenues do not depend on how demand varies across time during reservation. This property is crucial to the paper’s empirical tasks because such information is unobserved in our case.

To identify the price elasticity, we rely on a difference-in-difference in spirit argument tailored to our application but that may apply to other contexts as well. It circumvents the typical identification challenges in the setting of quantity-based revenue management such as demand censoring (Swan, 1990; Lee, 1990; Stefanescu, 2012) and the absence of exogenous variation in fares. Concretely, we exploit the fact that

---

3See iDTGV, le laboratoire marketing de la SNCF for experimental measures at iDTGV.

4Another form of revenue management is price-based. See Talluri and van Ryzin (2005) for descriptions of different approaches in revenue management.

5In contrast, price path of a counterfactual pricing strategy depends on the time-varying demand parameters. One would need more detailed data than ours, such as instantaneous purchases at iDTGV and competitors’ prices, to identify these parameters.
revenue management is done at a route level (e.g., Paris-Toulouse), while the train serves several cities (e.g. Bordeaux and Toulouse). This means that fare classes close at the same time for all destinations within the same route. Relative prices between, e.g., Bordeaux and Toulouse, then vary simultaneously whenever a fare class closes and can be used to identify the price elasticity. This works provided that the temporal pattern of consumers arrival, but not their total amount, is the same for both destination. As in difference-in-differences, we can test this condition, and we find suggestive evidence in favor of it.

Because the argument above only relies on relative demand for one destination versus another, train-specific demand shifters common to both destinations are differenced out. To recover their distribution, we build moment inequalities based on consumers’ rationality, a condition implied by our demand model, and a weak optimality condition on the firm’s side. This condition imposes that the observed revenue management is on expectation higher than any uniform pricing strategy using one of the price of existing fare classes. We obtain lower and upper bounds of train-specific effects from the moment inequalities. By combining them with the previously identified demand parameters, we set identify the counterfactual revenues, to which we compare the observed revenue.

We obtain the following key findings. First, our results suggest that the observed revenue management practice might be effective but is still substantially suboptimal. Concretely, the observed revenue management generates a gain of up to 8.1% compared to the optimal uniform pricing in an incomplete information set-up. However, we also estimate, under the same informational set-up, a loss of at least 6.8% and up to 15.1% compared to the optimal pricing strategy with the same number of ascending fare classes as the observed scenario. Actually, we estimate that simple strategies, such as 12, non necessarily ascending fare classes, already secure almost 99% of the fully unconstrained optimal pricing strategy. Second, our findings emphasize the key role of demand uncertainty on revenues, and how effective revenue management can be to mitigate it. Moving from an incomplete to a complete information set-up, we find that revenue from a uniform pricing strategy increases by 17.2%. But the informational gains are much smaller (0.22%) when considering fully flexible dynamic pricing strategies. In other words, implementing the optimal dynamic pricing strat-
egy mitigates almost entirely the loss entailed by demand uncertainty. The reason behind is that information accumulates quickly: by observing and learning from the sales of half of the available seats, the firm can already secure more than 97% of the revenue under complete information.

Related Literature. Our paper relates to several theoretical and empirical papers in operational research and economics. The theoretical literature on revenue management has investigated optimal quantity-based revenue managements, where firms segment demand by choosing either once for all or dynamically the allocation of, say, seats into fare classes in which prices are predetermined. We refer in particular to Littlewood (1972) and Brumelle and McGill (1993) for static solutions, and to Gallego and Van Ryzin (1994), Feng and Gallego (1995), Feng and Xiao (2000), Aviv and Pazgal (2002) for dynamic solutions. These last papers have studied optimal pricing strategies assuming that consumers arrive under some homogeneous Poisson process.

In our paper, we assume that consumers arrive according to a flexible, non-homogenous Poisson process, as Bitran and Mondschein (1997), Zhao and Zheng (2000), and McAfee and te Velde (2008). Our demand model is closest to McAfee and te Velde (2008), but with one key difference. Whereas they assume that the firm has complete information on demand parameters, we also consider an incomplete information set-up where only the distribution of these parameters is known. The firm then updates this distribution as consumers arrive. Such an incomplete information set-up seems more plausible when aggregate demand varies much from one train to another, as is the case here. We also generalize McAfee and te Velde (2008) by studying constrained pricing strategies close to those implemented in practice.

Our results underline the important role of information and demand learning to explain the gains and losses of revenue management. Recent studies show that learning from consumers’ characteristics or realized sales improves firm’s static pricing via third-price discrimination (Dubé and Misra, 2023), or updated market conditions (Huang et al., 2022). The importance of information and demand learning for dynamic pricing has also been made in the theoretical literature but to our knowledge, we are the first to quantify these roles using real data. Lin (2006) studies similar models to ours and allows for firm’s Bayesian learning from observed purchases or
arrivals. Instead of deriving the optimal policy, his paper focuses on a specific policy, which is shown to be nearly optimal in simulations. Aviv and Pazgal (2002) derive the optimal policy assuming some unknown constant arrival rate of consumers, and simulate the loss due to incomplete information. By contrast, we allow for heterogeneous arrival rates and study other practically relevant pricing strategies as well. Finally, in contrast to all these papers and ours, den Boer and Zwart (2015) consider another form of learning by the firm, based on maximizing the likelihood of the data at its disposal. We refer to den Boer (2015) for a complete survey on demand learning in dynamic pricing.

In the empirical literature on revenue management, the closest papers to ours are Lazarev (2013) and Williams (2022), both of which study dynamic airline pricing in a monopolistic market.\(^6\) While both papers accentuate price discrimination and its welfare effects, the main goal of our paper is to quantify the potential gains and loss due to revenue management in practice. As a result, contrary to their models, ours explicitly incorporates firm’s learning behavior from the realized demand. Moreover, we do not impose strong optimality conditions on the observed prices.\(^7\) On the other hand, while Lazarev (2013) allows travelers to be forward-looking, we abstract from any strategic considerations by consumers here, following Williams (2022) and the operation research literature. The rationale behind is that in our context, and contrary to what happens in the airline industry, prices always increase. In the absence of uncertainty on the opportunity of the journey, the consumers have no incentives to wait.

The rest of the paper is organized as follows. In Section 2, we present the context and our data. Section 3 displays the demand model and defines the various counterfactual revenues we consider afterwards. Section 4 is devoted to the identification and estimation of the demand and counterfactual revenues. Section 5 presents the results. The appendix includes a microfoundation of our demand model, results based on aggregate data, formulas used to compute counterfactual revenues, robustness checks

---

\(^6\) Another related work is Cho et al. (2018), which studies revenue management under oligopoly in hospitality industry. Their analysis focuses on the pricing behavior of “hotel 0” (from which the demand data is obtained) in a competing environment.

\(^7\) See also Cho et al. (2018, 2019) for recent examples that identify demand without imposing strong optimality conditions.
and the proofs of our identification results.

2 Institutional Background and Data

2.1 Revenue Management at iDTGV in 2007-2009

iDTGV was created in 2004 as a low-cost subsidiary of the French railway monopoly, SNCF. This decision has been motivated by low-cost airlines’ entry in the French domestic routes (e.g., EasyJet launched the line Paris-Marseille at the beginning of 2004) and the EU rail market liberalization.\(^8\) Before its disappearance in December 2017, iDTGV was highly autonomous: it owned its trains and had a pricing strategy independent from SNCF.\(^9\) Prices were generally lower than the full-rate prices of SNCF, but were also associated with a slightly lower quality of services. Namely, tickets could only be bought on Internet, they were nominative and could not be cancelled. On top of that, they could be exchanged only under some conditions and at some cost.

The routes of iDTGV were all between Paris and other towns in France. For each of those towns and every day, one train was leaving Paris and another coming to Paris. Table 1 presents the routes we observe in our data from May 2007 till March 2009. These routes have several stops, but to simplify the analysis, we gather them so as to form a single intermediate stop and a single final stop. We aggregate the cities according to the price schedule. For instance, we group Aix-en-Provence and Avignon together in the Paris-Marseille route since the corresponding prices are always the same. This gathering is consistent with Assumption 1 below, as our demand model remains valid after aggregation of cities.\(^10\)

Different routes may share the same intermediate destination. For instance, Bordeaux is the intermediate destination of Paris-Côte basque and Paris-Toulouse. Importantly,

\(^8\) See the quotation from iDTGV’s CEO, Paul Sessego, in iDTGV, le laboratoire marketing de la SNCF.

\(^9\) Due to internal strategic considerations at SNCF, iDTGV was replaced by Ouigo, the new low-cost service at SNCF, in 2007.

\(^10\) We also estimate demand without the grouping. The results are quantitatively similar. See Columns II and IV of Table 3.
no tickets were sold between the intermediate and the final destination, e.g. no Bordeaux-Toulouse tickets are sold on the Paris-Toulouse route. The reason for this surprising practice was that tickets were only controlled on the departure platform by a service firm, rather than by iDTGV employees in the trains (as is usually done by SNCF). This experimental measure was one of many others at iDTGV aiming at reducing cost.

Table 1: Routes with intermediate and final destinations

| Route name          | Final stop(s)                  | Intermediate stop(s)          | Nb. of trains |
|---------------------|--------------------------------|--------------------------------|---------------|
| Côte d’Azur         | Cannes, Saint-Raphaël, Nice    | Avignon                       | 452           |
| Marseille           | Marseille                      | Aix-en-Provence/Avignon       | 453           |
| Perpignan           | Perpignan                      | Nîmes, Montpellier            | 689           |
| Côte basque         | St Jean de Luz, Bayonne,       | Bordeaux                       | 405           |
|                     | Biarritz, Hendaye              |                                |               |
| Toulouse            | Toulouse                       | Bordeaux                       | 411           |
| Mulhouse            | Mulhouse                       | Strasbourg                     | 499           |
| **Total**           |                                |                                | **2,909**     |

Notes: we have different number of observations for the different routes because the period we cover varies slightly from one route to another.

The trains are split into economy class and business class cars of fixed sizes. Revenue management was implemented almost independently between the two classes, i.e. under the sole constraint that prices in economy class are always lower than in business class. This constraint was very seldom binding in practice, so we ignore it hereafter. We focus on the economy class, which represents most of the sales (roughly 70% in terms of seats and 67% in terms of revenue). In this category, there are 12 fare classes corresponding to 12 prices sorted in ascending order. The price of a given fare class, at a peak time or off peak and for some origin-destination trip (e.g. Paris-Bordeaux) remained constant for several months (e.g. from 03/01/2007 to 10/31/2007) before being adjusted marginally, mostly to account for inflation. Contrary to SNCF, iDTGV did not make any third-degree price discrimination, so there was no discount for young people, old people or families.
The quantity-based revenue management at iDTGV consists in deciding in real time to maintain the current fare class or to close it and move to a higher one, resulting in a price increase. Coming back to a previous fare is impossible; thus, there are no last minute drops in ticket prices for trains that have still several empty seats. Also, revenue managers could decide to never open the first fare classes and begin to sell directly tickets in a higher fare class. Symmetrically, the last fare class may never be reached. In practice, revenue management was operated through a Computerized Reservation System (CRS). Before the beginning of sales, it fixes a seat allocation planning for all fare classes, using the history of purchases on past trains. During sales, the CRS uses the number of tickets sold up to now to make recommendations on the size of subsequent fare classes. Revenue management managers can nevertheless always intervene, both on the initial and on subsequent seat allocations, according to their experience on past trains.

Finally, and crucially for our identification strategy, the revenue management did not use separate fare classes for a given train with several destinations. For instance, in a Paris-Toulouse train, the closure of the first fare class occurred exactly at the same moment for both Bordeaux and Toulouse. Hence, price changes of Paris-Bordeaux and Paris-Toulouse tickets happened exactly at the same time, for all trains. According to discussions with people in the revenue management department, this was to limit the number of decisions to be taken at each moment.

2.2 Data and descriptive statistics

Our data cover iDTGV trains between May 2007 and March 2009 in economy class and for journeys from Paris to the rest of France. We first observe basic characteristics of the trains: all the stops, departure and arrival time, day of departure (e.g. May 2, 2008) and whether it corresponds to a peak time or not. We also observe the price grid used for that train for each fare class. For each route and type of period (peak

\footnote{According to Mariton (2008), a congressional report on the pricing policy of the SNCF, reopening lower fare classes when approaching the departure may cause strong dissatisfaction among travelers who have purchased seats with higher prices before, and could thus harm firm’s reputation.}

\footnote{Manager intervention in automatized revenue management also exists in other industries, e.g. hospitality industry (Cho et al., 2018).}
time or off peak), there are a limited number of such grids, as they change these grids only a few times during the period we observe (e.g. 3 times for the Paris-Toulouse). We also observe the sales of each fare classes of all trains. On the other hand, we do not observe the purchasing dates, nor the opening moments of each fare class. For a given route, capacity is defined as the maximal number \( n \) such that for at least three trains, \( n \) seats were sold.\(^{13}\)

Table 2 presents some descriptive statistics on our data. We observe a substantial amount of price dispersion within trains. For instance on the Côte d’Azur line, the minimal price paid by consumers on average over the different trains (19.3€) was three times and a half lower than the average maximal price (68.4€). We also observe substantial variations on the average load across routes. While trains in Paris-Marseille were always nearly full, with an average load above 95%, this was far from being the case on the Côte basque line, with an average load of only 65.4%.

| Route          | Capacity | Avg % final Load | % final dest. | Avg | Avg min. | Avg max. | Prices max/min |
|----------------|----------|------------------|---------------|-----|----------|----------|----------------|
| Côte d’Azur    | 324      | 85.4%            | 81.5%         | 50.3| 19.3     | 68.4     | 3.54           |
| Marseille      | 324      | 95.5%            | 60.0%         | 49.5| 19.0     | 70.5     | 3.71           |
| Perpignan      | 324      | 88.6%            | 27.4%         | 50.0| 20.2     | 72.6     | 3.59           |
| Côte basque    | 350      | 65.4%            | 64.1%         | 37.3| 19.7     | 53.3     | 2.71           |
| Toulouse       | 350      | 87.3%            | 55.3%         | 43.6| 19.4     | 67.2     | 3.46           |
| Mulhouse       | 238      | 79.4%            | 24.1%         | 35.0| 19.4     | 50.0     | 2.58           |

Notes: Avg min. and max. are the average of the minimal and maximal prices charged for each train, for the final destination. max/min is the ratio between the two previous columns.

Finally, we examine how (full) trains were yielded. Intuitively, a likely reason for selling out all seats is high demand. For full trains, one would thus expect that high fare classes are used and contribute more to total sales, were the observed revenue

\(^{13}\)We use this definition (rather than the maximal number of seats sold across all trains of a given route) to take into account rare cases of overbooked trains. With this definition, we observe 5 cases of overbooking, over the 2,909 trains of our dataset. Note that capacity can be assumed to be fixed for a given route because the number of coaches in economy class is fixed.
management optimal. To investigate this, we compare the 677 full trains to the 2,232 non-full trains in two different ways.

First, Figure 1 presents the distribution of the highest observed fare class, namely the highest fare class with at least one sale for either the intermediate or the final destination. If nearly 30% of full trains end up with the 12\textsuperscript{th} fare class, for almost one third (32.5%) of these trains, no ticket is sold beyond the 7\textsuperscript{th} fare class. For this fare class, the price is between 28\% and 48\% lower than that of the 12\textsuperscript{th} fare class across all routes.

Figure 1: Distribution of Observed Highest Fare Class, Full and Non-Full Trains

Second, Figure 2 displays the share of sales corresponding to each fare class, again for full and non-full trains. Surprisingly, both histograms look similar. In particular, around 66\% (resp. 78\%) of the sales for full (resp. non-full) trains are obtained by the first six classes, while the three highest fare classes only contribute up to 13\% (resp. 6\%). Figures 1 and 2 together suggest that the actual pricing may not be optimal: when demand is high, chosen prices may be too low, and the openings of higher fare classes may happen too late in the booking period.
This pattern is even more striking for routes that opened in 2008 (Côte d’Azur, Marseille), as opposed to 2007 (Perpignan, Côte basque, Toulouse, and Mulhouse). Restricting to non-full trains, the 12th fare class opened in 14.5% of and generates similar sales (around 1.41%) for both types of routes. On the other hand, this fare class opened in 35.8% of full trains, generating 8.66% of sales, for routes opened in 2007, as opposed to 21.2% of full trains, generating 4.51% of sales, for routes opened in 2008. This suggests that managers acquired more information from past purchases for routes with a longer history, and are thus more able to discriminate between trains under high and low demand. In turn, this is another clue that the actual pricing may be suboptimal.

3 Theoretical model and parameters of interest

We consider a demand model close to that of McAfee and te Velde (2008). A train $T$ is defined by its route $r(T)$ (e.g. Paris-Toulouse) and its day of departure (e.g. May 2, 2008). For each route $r$, we denote by $a_r$ the intermediate destination and by $b_r$ the final destination. To simplify notation and in the absence of ambiguity, we just denote the destinations of a train $T$ by $a$ and $b$ instead of $a_{r(T)}$ and $b_{r(T)}$. For any train $T$ departing at $t = 1$, tickets are sold between the normalized dates $t = 0$ and $t = 1$. We denote the fare classes by $k \in \{1, \ldots, K\}$. Within fare class $k$, tickets for
train \( T \) and destination \( d \in \{a,b\} \) are sold at price \( p_{dkT} \). Recall that \( p_{dkT} \) belongs to a grid of \( K \) prices that remains fixed for several months and depends only on the destination \( d \) and whether the train leaves at a peak time or not. Finally, we denote by \( V_{dT}(A,B) \) the number of consumers who search during time interval \( A \subset [0,1] \) for traveling to destination \( d \) at \( t = 1 \) with willingness to pay for train \( T \) belonging to the subset \( B \) of \([0, \infty)\). Similarly, let \( D_{dT}(t,t';p_d) \) denote the demand between dates \( t \) and \( t' \) (with \( (t,t') \in [0,1]^2 \)) for traveling to destination \( d \) by train \( T \) when the price between \( t \) and \( t' \) is constant and equal to \( p_d \). Then \( D_{dT}(t,t';p_d) = V_{dT}([t,t'),[p_d,\infty)) \).

**Assumption 1 (Consumers’ demand)** For all \( T \) and \( d \in \{a,b\} \), there exists \( \varepsilon > 1 \) and \( b_T(\cdot) \) on \([0,1]\), continuous and satisfying \( \min_{u \in [0,1]} b_T(u) > 0 \) almost surely, such that conditional on \( \xi_{dT} \) and \( b_T(\cdot) \):

1. \( V_{dT} \) is a Poisson process with intensity \( I_{dT}(t,p) = \xi_{dT}b_T(t)\varepsilon p^{-\varepsilon} \) for \((t,p) \in [0,1] \times [0,\infty)\). Without loss of generality, we let \( \int_0^1 b_T(u)du = 1 \).

2. \( V_{dT} \) and \( V_{dT} \) are independent.

In Appendix A, we provide microfoundations for the demand model in Assumption 1, which work as follows. First, consumers arrive at time \( t \in [0,1] \) according to a non-homogeneous Poisson process and search for traveling to \( d \) at time 1 among available modes of transportation (e.g., iDTGV, low-cost airline). Second, observing the prices of different modes of transportation for their travel, each consumer chooses the one with the highest indirect utility.

In this set-up, time-varying \( b_T(t) \) captures both “market size” and competition effects. It allows for potentially heterogeneous search intensity of consumers across time: the more consumers arrive and search at time \( t \), the higher \( b_T(t) \) is, increasing the demand for iDTGV. Also, in the presence of instantaneous competition between different modes of transportation, the more expensive other modes of transportation are, the greater \( b_T(t) \) is, increasing the demand for iDTGV to the same destination relative to other modes of transportation. The term \( \xi_{dT} \) captures train-destination specific overall demand shocks. For instance, demand to Cannes may increase a lot during the Cannes Film Festival. The elasticity \( \varepsilon \) is consumer’s sensitivity to price change of an iDTGV seat, reflecting both their disutility of price and the extent to which
The non-price cost of competing modes of transportation is low.\textsuperscript{14} The functional form of price effects, namely $p^{1-\varepsilon}$, is a result of a Pareto distribution assumption on non-price costs of mode of transportation in the proposed microfoundation. This specific functional form is not essential to our identification strategy, but it facilitates the practical estimation given the limited variation we have on observed prices.

The form of the intensity $I_{dT}$ in Assumption 1 has two implications. First, the non-price components of $I_{aT}$ and $I_{bT}$ have the same shape, since they are just shifted by some multiplicative destination-train specific constant $\xi_{dT}$. This restriction can be tested, an important point on which we come back in Section 4.1. Second, Assumption 1 implies that the demand for destination $d$ on the time interval $[t_1, t_2]$ satisfies

$$D_{dT}(t_1, t_2; p)|b_T(\cdot) \sim \mathcal{P}\left(\xi_{dT}p^{-\varepsilon}\int_{t_1}^{t_2} b_T(u) du\right),$$

where $\mathcal{P}$ refers to Poisson process. Like McAfee and te Velde (2008), we thus assume that the price elasticity does not vary over time.\textsuperscript{15} We nonetheless test that assumption and consider an augmented model that allows for time-varying elasticities in Section 5.3.

**Parameters of interest.** Let $n_{dkT}$ denote the number of sales for train $T$, fare class $k \in \{1, \ldots, K\}$ and destination $d \in \{a, b\}$. Besides, denote by $W_T$ a vector of observed characteristics of train $T$ (e.g., whether the train operates on a rush hour or not) and $X_{dT}$ a vector of observed characteristics of destination $d$ served by train $T$, e.g., the travel time from Paris to $d$ by train $T$.

Our main goal is to compare current revenues with several counterfactual ones corresponding to different pricing strategies, from the most basic to the most sophisticated ones. The first one, uniform pricing, consists in fixing the price of each route in a given train once and for all; we let $R_u$ denote optimal counterfactual revenues, averaged over

\textsuperscript{14}Intuitively, if consumers’ cost of traveling to an airport is on average lower (e.g., better public transport connection to the airport), then increasing the iDTGV price drives away more consumers to airline competitors, leading to a higher price elasticity (in absolute value).

\textsuperscript{15}Instead, we allow later the price elasticity to depend on train characteristics (e.g., rush hour) and therefore be heterogeneous across trains. See Appendix A for details, as well as Table 3 (resp. Table 9) for the corresponding demand estimates (resp. counterfactual revenues).
all trains, under this pricing regime. At the other extreme, in “full” dynamic pricing, prices can be changed any time: $R_f$ then corresponds to optimal counterfactual revenues in this set-up. We also study pricing strategies, called stopping-time strategies hereafter, where prices can be changed only after a ticket is sold: corresponding optimal revenues are denoted by $R_s$. Finally, we consider constrained stopping-time strategies close to what was implemented in practice, by assuming that only $M$ fares (resp. $M$ increasing fares), are allowed. The corresponding optimal revenues are denoted by $R_{sM}$ (resp. $R_{sM+}$).

To compute these counterfactual revenues, we maintain Assumption 1, which implies that pricing strategies of competing modes of transportation are unchanged. As a result, our counterfactual revenues shall be interpreted as partial equilibrium outcomes. Despite the lack of potential strategic reactions of competing modes of transportation, we can quantify the compensation in competitors’ prices that offset the gains (or losses) of the counterfactual revenue relative to observed ones, thanks to the demand model we develop in Appendix A. We refer to this appendix and Footnote 22 for more details.\footnote{Another implication of maintaining Assumption 1 in our counterfactuals is that for pricing strategies allowing for price decreases, we still rule out any strategic anticipations of such decreases by consumers.}

Finally, regarding information available to the revenue managers, we consider two scenarios:

1. (Complete information) Revenue managers fully know the expected demand for each train. Thus, they observe $\varepsilon$, $b_T(\cdot)$, $\xi_{aT}$ and $\xi_{bT}$ for each train $T$;

2. (Incomplete information) Revenue managers observe $\varepsilon$, $b_T(\cdot)$, $(X_{aT}, X_{bT}, W_T)$ but only $f_{\xi_{aT}, \xi_{bT}|X_{aT}, X_{bT}, W_T}$. As time goes by, revenue managers update their information on $(\xi_{aT}, \xi_{bT})$ according to Bayes’ rule.

The complete information case should be seen as a benchmark. It is useful in particular to quantify the value of information and contrast the gains of revenue management in both complete and incomplete information set-ups. The case of incomplete information is probably more realistic. In this scenario, revenue managers know the pattern of $b_T(\cdot)$ over time for each train, but do not know exactly the aggregate demand for
each destination \((\xi_{aT} \text{ and } \xi_{bT})\). Assuming \(b_T(\cdot)\) to be known makes especially sense if \(b_T(\cdot)\) does not depend on \(T\), in which case revenue managers may have learned how consumers arrive over time from previous observations.

If the scenario of incomplete information holds in practice, the differences between counterfactual and observed revenues ones shall be interpreted as potential gains (or losses) of the optimal revenue management under different constraints compared to the actual ones. Hereafter, we use exponents \(c\) and \(i\) to specify the two information set-ups. For instance, \(R_c^u\) refers to the counterfactual optimal revenue under uniform pricing and complete information.

4 Identification and inference

We observe \((n_{dkT}, p_{dkT}, X_{dT}, W_T)\) for \(d \in \{a,b\}, k \in \{1,\ldots,K\}\), and all \(T\)’s, and aim at identifying and estimating demand parameters that are sufficient for simulating the counterfactual revenues of interest, \(R_I^r\) for \(I \in \{c,i\}\) and \(r \in \{u,f,s,sM,sM+\}\). The asymptotic framework we consider below is obtained by letting the number of trains tend to infinity (recall that in our application, we observe 2,909 trains).

As in the general setting of quantity-based revenue management, we face two main issues for recovering the demand parameters. First, the observed demand may only be an lower bound of the total demand, due to censoring. This point was already made in similar contexts by Swan (1990), Lee (1990), and Stefanescu (2012). The second issue is the absence of usual instruments for prices. As mentioned above, prices only vary within a predetermined grid, which only (slightly) changes once every several months. Also, fare classes are unlikely to be closed for exogenous reasons unrelated to demand. In other words, there are no obvious exogenous variation of prices and usual strategies to identify the demand function are unlikely to apply here. Instead, we propose a novel identification strategy based on specific features of the observed revenue management, which may apply to other contexts.
4.1 Identification of $\varepsilon$ and $\xi_{dT}$

We first formalize the feature already discussed in Section 2.1 that the revenue management at iDTGV is operated at a route level (e.g. Paris-Toulouse) rather than for each destination of this route (e.g. Paris-Bordeaux and Paris-Toulouse for the route Paris-Toulouse).

**Assumption 2** *(Revenue management at route level)* The opening time of fare class $k \in \{1, \ldots, K\}$, $\tau_k$, is a stopping time with respect to the process $t \mapsto N_{aT}(t) + N_{bT}(t)$, where $N_{dT}(t)$ is the number of purchases for $d$ made before $t$.

Assumption 2 states that the decision of opening a new fare class depends only on past total purchases, rather than on the repartition between purchases for $a$ and for $b$. Such an assumption is fully in line with the fact that a single fare class is used for the two destinations of each route. It was also confirmed by discussions we had with the revenue management department.

Assumptions 1 and 2 turn out to be sufficient to identify $\varepsilon$ by exploiting variations in the relative prices $p_{bT}/p_{aT}$ between the two destinations and from one fare class to another. To understand why this is the case, let us first assume, for the sake of exposition, that $\tau_k$ and $\tau_{k+1}$ are deterministic. Then, by Assumption 1, $D_{aT}(\tau_k, \tau_{k+1}; p_{aT})$ and $D_{bT}(\tau_k, \tau_{k+1}; p_{bT})$ are independent conditional on $\xi_{aT}, \xi_{bT}$ and $\int_{\tau_k}^{\tau_{k+1}} b_T(u)du$. Moreover, they both follow Poisson distributions. Note that $n_{dT} = D_{dT}(\tau_k, \tau_{k+1}; p_{dT})$. As a result,

$$n_{dT} | n_{aT} + n_{bT} = n, \xi_{aT}, \xi_{bT} \sim \text{Binomial}(n, \Lambda(\ln(\xi_{bT}/\xi_{aT}) - \varepsilon \ln(p_{bT}/p_{aT})))$$

where $\Lambda(x) = 1/(1 + \exp(-x))$. The term $\ln(\xi_{bT}/\xi_{aT})$ may be seen as a train fixed effect. Hence, this model boils down to a fixed effect logit model, and $\varepsilon$ is identified as long as there are variations across fare classes $k = 1, \ldots, K$ in the relative prices $p_{bT}/p_{aT}$. In the data, we do observe such variations. In Paris-Toulouse for instance, $p_{bT}/p_{aT}$ vary from 1 for $k = 1$ to 1.18 for $k = 12$.

In fact, we prove that Equation (1) still holds with stochastic stopping times $(\tau_k)_{k=1}^{K}$, provided that they satisfy Assumption 2. This, in turn, implies that $\varepsilon$ is identified under variations in relative prices. However, because the number of available seats for
any given train $T$ is fixed, one cannot point identify the train fixed effect $\ln(\xi_{bT}/\xi_{aT})$ from (1) without further conditions.\footnote{Note that (1) is solely determined by $\ln(\xi_{bT}/\xi_{aT})$ when $\varepsilon$ is identified. If $n \to \infty$, we can then identify the mean of the binomial distribution and $\ln(\xi_{bT}/\xi_{aT})$. This is, however, not the case in our setting because $n$ is at most equal to the number of available seats in train $T$.} This motivates the following assumption.

**Assumption 3** For $d = \{a, b\}$, $\xi_{dT}$ satisfies:

(i). $\xi_{dT} = \exp\{X_{dT}'\beta_0\}g_0(W_T)\eta_{dT}$ where $\eta_{aT}$, $\eta_{bT}$ and $(X_{aT}, X_{bT}, W_T)$ are independent.

(ii). $\eta_{dT} \sim \Gamma(\lambda_{d0}, 1)$, where $\Gamma$ refers to Gamma distribution.

Assumption 3(i) specifies $\xi_{dT}$ as the product of a function of $X_{dT}$, $g_0(W_T)$, and a remainder term $\eta_{dT}$. It restricts $\xi_{aT}$ and $\xi_{bT}$ to be dependent through the observed variables $(X_{aT}, X_{bT}, W_T)$, rather than $(\eta_{aT}, \eta_{bT})$. This is plausible as long as one includes sufficient controls in $(X_{aT}, X_{bT}, W_T)$. Importantly, we leave the function $g_0(.)$, which determines how train-specific characteristics affect the demand, unrestricted.

Assumption 3(ii) imposes that conditional on $(X_{dT}, g_0(W_T))$, $\xi_{dT}$ follows a gamma distribution. Since we include a $d$-specific constant term in $X_{dT}$, we can normalize the scale parameter of the gamma distribution to 1. As detailed below, the assumption of a gamma distribution does not matter for identification. It is rather made for computational reasons: that the gamma and Poisson distributions are conjugate significantly simplifies the computation of counterfactual revenues of interest. But we also consider log-normality as a robustness check for some counterfactual revenues.

The following result formalizes the discussion above.

**Theorem 4.1** Suppose that Assumptions 1 and 2 hold. Moreover, with positive probability, $k \mapsto p_{bKT}/p_{aKT}$ is not constant. Then,

1. Equation (1) holds and $\varepsilon$ is point identified;

2. If Assumptions 3(i) and 5 in the appendix further hold, $\beta_0$ and the distribution of $\eta_{bT}/\eta_{aT}$ are identified. Then, if Assumption 3(ii) further holds, $(\lambda_{a0}, \lambda_{b0})$ is also identified.
Four remarks on this theorem are in order. Firstly, the identification of $\varepsilon$ follows from a kind of difference-in-differences strategy, where demand for destinations may vary but we assume that the temporal pattern of consumers’ arrival is the same. This parallel also suggests that the underlying condition may be tested, something we do in Section 5.3 below. Secondly, Equation (1) does not hold for any possible random stopping times. We can easily build counterexamples by making $(\tau_k)_{k=1}^K$ depend solely on $N_{aT}(.)$, for instance. Such situations are however ruled out by Assumption 2. Under this condition, intuitively, the stopping times will be independent of the proportion of consumers buying tickets for $a$ (versus $b$). Thirdly, the distribution of $\eta_{bT}/\eta_{aT}$ in the second statement of Theorem 4.1 is identified, which implies the identification of $(\lambda_{a0}, \lambda_{b0})$ under Assumption 3(ii). Fourthly, we made in Assumption 3(i) the simplifying assumption that trains only had two destinations, an intermediate $a$ and a final one $b$. But recall from Table 1 that most of them serve more than just two cities, so $a$ or $b$ actually correspond to more than one city. If so, we modify (i) by assuming that

$$\xi_{dT} = \left[ \sum_{c \in d} \exp(X'_{cT} \beta_0) \right] g_0(W_T) \eta_{dT},$$

where $c$ is an index for cities belonging to either $a$ or $b$. For instance, in a train to Côte d’Azur, $c$ corresponds to Avignon for destination $a$ whereas $b$ includes Cannes, Saint-Raphaël and Nice, see again Table 1. Note, on the other hand, that all cities $c \in d$ are priced equally, so we do not need to take into account price variations between cities.

Beyond the identification of $(\varepsilon, \beta_0)$ and the distribution of $\eta_{bT}/\eta_{aT}$, Equation (1) can be the basis of testing some implications of our demand model. First, the separability between $b_T(\cdot)$ and $\xi_{dT}$ in Assumption 1 implies that if $p_{bkT} = p_{akT}$ for several fare classes $k$, we should observe similar proportions $n_{bT}/(n_{aT} + n_{bT})$ for the corresponding $k$. Second, we can also test for the fact that price elasticities do not evolve over time, by considering more general specifications than (1), e.g., different elasticities for early and late purchases proxied by low and high fare classes. Third, we have imposed thus far that the price elasticity was constant for all routes. We can however allow for variation in the elasticity according to the day and hour of departure and according to groups of routes sharing the same sections. We consider all these extensions and
robustness checks in Sections 5.1 and 5.3 below.

Under Assumption 3, $g_0(W_T)$ is common to $\xi_{aT}$ and $\xi_{bT}$. As a result, it drops out of $\xi_{bT}/\xi_{aT}$ in Equation (1) and cannot be identified using the strategy behind Theorem 4.1. We now build moment inequalities based on consumers’ rationality (Assumption 1.1) and a weak optimality condition on the actual revenue management to construct bounds on $g_0(W_T)$.

**Consumers’ rationality and a lower bound on demand.** By Assumption 1.1, all consumers who bought a ticket for $d$ at price $p_{djT}$ for $j \geq k$ would have also bought it at price $p_{dkT}$. Therefore, for all $k = 1, \ldots, K$ and $d \in \{a, b\}$,

$$D_{dT}(p_{dkT}; g_0(W_T), X_{dT}) \geq \sum_{j=k}^{K} n_{djT},$$

where we now index total demand $D_{dT}(p_{dk})$ by $g_0(W_T)$ and $X_{dT}$. Let $C_T$ denote the capacity of train $T$. Then we also have $C_T \geq \sum_{j=k}^{K} n_{djT}$. Combining these inequalities and integrating over trains with given $W_T$ yields, for all $k = 1, \ldots, K$ and $d \in \{a, b\}$,

$$\mathbb{E} \left[ \sum_{j=k}^{K} n_{djT} - C_T \wedge D_{dT}(p_{dkT}; g_0(W_T), X_{dT}) \bigg| W_T \right] \leq 0. \quad (3)$$

We assume hereafter that $X_{dT}$ is a deterministic function of $W_T$. This holds in our context where $W_T$ includes route dummies and $X_{dT}$ includes time-invariant destination variables and interactions between such variables and $W_T$. Under that condition, the function $g \mapsto \mathbb{E}[C_T \wedge D_{dT}(p_{dk}; g, X_{dT})|W_T]$ is strictly increasing. Denoting by $Q_k^{-1}(\cdot; W_T, \theta_0)$ its inverse, we get

$$g_0(W_T) \geq Q_k^{-1} \left( \mathbb{E} \left[ \sum_{j=k}^{K} n_{djT} \bigg| W_T \right] ; W_T, \theta_0 \right).$$

Then, we obtain a lower bound for $g_0(W_T)$:

$$g_0(W_T) \geq g^L_0(W_T) := \max_{d=a, b} \left\{ Q_k^{-1} \left( \mathbb{E} \left[ \sum_{j=k}^{K} n_{djT} \bigg| W_T \right] ; W_T, \theta_0 \right) \right\}. \quad (4)$$

While $Q_k^{-1}$ does not have a closed form, we can compute it easily through simulations.
**Weak optimality condition and an upper bound on demand.** To start with, let $R_T(p_a, p_b)$ denote the maximal revenue for train $T$ under a uniform pricing of $(p_a, p_b)$ for destinations $a$ and $b$ respectively. This maximal revenue is obtained by considering the optimal quotas $C_{aT}$ and $C_{bT}$ of tickets sold for destinations $a$ and $b$ respectively, with $C_{aT} + C_{bT} = C_T$, the total capacity of train $T$. Then, under Assumptions 1-3, we have (see Appendix E.3.2, section “uniform pricing”, for details)

$$
\mathbb{E}[R_T(p_a, p_b)|W_T] = \max_{(C_{aT}, C_{bT})} \left\{ \sum_{d \in \{a,b\}} p_d \int_0^\infty \mathbb{E} \left[ D(\exp\{X'_dT\beta_0\}p_d^{-\varepsilon}g_0(W_T)z) \right] \wedge C_{dT}|W_T \right\} \times g_{\lambda_0,1}(z)dz, 
$$

where $D(u) \sim \mathcal{P}(u)$ and $g_{\lambda_0,1}$ is the density of a $\Gamma(\lambda_0, 1)$. The weak optimality condition we consider is the following:

**Assumption 4 (Weak optimality of actual revenue management)** We have

$$
\max_{k=1,\ldots,K} \mathbb{E}[R_T(p_{akT}, p_{bkT})|W_T] \leq \mathbb{E}\left[R_{T}^{obs}|W_T\right]. 
$$

By conditioning on $W_T$, which only includes coarse proxies of the true demand, we allow for the possibility that revenue managers use limited information for their pricing strategy. In reality, it seems credible that they have access to additional signals on the true demand for a specific train. For instance, they could use the past number of purchases in each fare class on previous years for the exact same train. If so, we would expect that (6) also holds conditional on this information, so that Assumption (4) is still satisfied.

Importantly, Assumption 4 does not imply that the revenue management performs better than the optimal uniform pricing, because we only impose that observed revenues exceed any uniform pricing strategy that is constrained to the grid of the 12 predetermined prices. In other words, we simply assume that observed revenues are on average higher than those one would have obtained by sticking to a single fare class over the whole booking period.

We refrain to impose optimality with respect to all dynamic strategies for several reasons. First, such an assumption would conflict with our very objective to quantify the gains or losses of the actual revenue management, compared to alternative scenarios.
By definition, assuming a strong form of optimality would result in gains against most simpler pricing strategies. Second and related to the first point, it seems restrictive in our setting to assume that the optimal dynamic strategy was adopted. As discussed in Section 2.1, the revenue management applied simplified rules (increasing fares from 12 predetermined fare classes), which can at best approach the optimal solution. The distributions of the observed highest fare class and per-class sales in Figures 1 and 2 also suggest suboptimality of the current pricing strategy. Note that seat allocation decisions were subject to the manager’s manual intervention, which could be a source of suboptimality.\footnote{See Cho et al. (2018, 2019); Phillips (2021) for evidence of suboptimality due to human management.} Also, computing the optimal dynamic strategy under the simplified rules is still a very complicated dynamic programming problem. While Feng and Xiao (2000) have proposed an algorithm for computing the solution for a homogeneous Poisson process, little has been done so far for the non-homogeneous case, to our knowledge. Finally, iDTG\V may have other objectives (e.g., social ones) or political constraints, which are not the focus of the paper but could lead to violations of revenue maximization.

Now, Theorem 4.1 and Equation (5) imply that the function $R$ defined by

$$R(g_0(W_T); X_{aT}, X_{bT}, \theta_0) := \max_{k=1, \ldots, K} \mathbb{E}[R_T(p_{akT}, p_{bkT}) | W_T]$$

is identified. Again, our notation reflects that $(X_{aT}, X_{bT})$ is a deterministic function of $W_T$. The weak optimality condition (6) is equivalent to:

$$R(g_0(W_T); X_{aT}, X_{bT}, \theta_0) \leq \mathbb{E} \left[ R^\text{obs}_T | W_T \right].$$

(7)

The function $R(\cdot; X_{aT}, X_{bT}, \theta_0)$ is strictly increasing. Denoting by $R^{-1}(\cdot; X_{aT}, X_{bT}, \theta_0)$ its inverse, we obtain the following (identified) upper bound for $g_0(W_T)$:

$$g_0(W_T) \leq g_0^U(W_T) = R^{-1}(\mathbb{E} \left[ R^\text{obs}_T | W_T \right]; X_{aT}, X_{bT}, \theta_0)$$

(8)

### 4.2 Identification of counterfactual revenues

We first show that $b_T(\cdot)$ is unnecessary for recovering counterfactual revenues under the scenarios described above.
Theorem 4.2 Suppose that Assumptions 1, 2, and 3(i) hold. Then, for any \((I, r) \in \{c, i\} \times \{u, f, s, sM, sM+\}\), \(R_{I}^r\) does not depend on \(b_T(\cdot)\).

Theorem 4.2 implies that we only need to recover the price elasticity \(\varepsilon\) and the conditional distribution of destination-train-specific effects \((\xi_{aT}, \xi_{bT})\) to identify counterfactual revenues. This property is crucial in our context where no information on purchasing dates is available. By Theorem 4.1 and the bounds on \(g_0(W_T)\) obtained above, we can then partially identify average revenues under counterfactual scenarios. Note, on the other hand, that the price pathes corresponding to counterfactual revenues do depend on \(b_T(\cdot)\) in general (see Appendix E.3.1 for details), so we cannot identify them given our data.

McAfee and te Velde (2008) obtains a similar result as Theorem 4.2 for the “full” dynamic pricing strategy under complete information and a similar demand model. We extend their results in two directions. First, we consider other types of pricing strategies, and in particular possibly constrained stopping-time strategies, which are very common in practice and correspond to the actual revenue management at iDTGV. Second, we also show a similar result in an incomplete information set-up.

We obtain Theorem 4.2 by constructing a different set-up from the true one, with a new time variable and new Poisson demand processes. This new set-up is very close to the true one, except that the Poisson processes are now homogeneous: \(b_T(\cdot)\) is replaced by \(\tilde{b}_T(\cdot) = 1\). We prove that the optimal revenues are the same in this set-up as in the true one. This shows that the optimal revenues does not depend on \(b_T(\cdot)\): it does not matter whether consumers arrive early or late, as long as on average, the same number of consumers eventually arrive. The result holds because basically, all the constraints on pricing we consider are independent of time. In this sense, Theorem 4.2 holds beyond the specific scenarios we consider here. But it would fail if time constraints were imposed on the pricing strategies, for instance if a limit on the number of price changes occurring before a given date \(t^* \leq 1\) were set.

A computational issue. In all the counterfactual scenarios, we consider a separate pricing strategy for destinations \(a\) and \(b\), more flexible than the actual practice. On the other hand, without further restrictions, the state space is large: the optimal strategy at any time \(t\) depends on the remaining seats for both destinations, a typical numerical
challenge in multiple-resource quantity-based revenue management (van Ryzin and Talluri, 2005). To reduce the state space, we approximate the original multiple-resource problem by multiple independent single-resource ones.\(^{19}\) Concretely, we fix ex ante the total number of seats available for stops \(a\) \((C_{aT},\text{say})\) and thus to \(b\) \((C_{bT} = C_T - C_{aT},\text{with } C_T \text{ the total number of seats in train } T)\). Then, depending on the scenario we consider, we either consider the optimal pre-allocation \(C_{aT}\), or fix it so that \(C_{aT}\) matches the observed average sales for \(a\). In either case, fixing \(C_{aT}\) allows us to solve the optimization problem separately for each destination (given the independence of \(\eta_{aT}\) and \(\eta_{bT}\) imposed in Assumption 3(i)) rather than jointly, greatly reducing the computational burden of the optimization problem. Compared to the fully optimal counterfactual revenues without pre-allocation, our results below may thus be seen as lower bounds. But this only reinforces some of our conclusions. Also, we compare in Section 5.2 the revenues under uniform pricing with and without pre-allocation, and find that they are very similar. This suggests that the other counterfactual revenues would not change much either without pre-allocation.

**Bounds on counterfactual revenues.** In Appendix B, we show that under Assumption 3, given \(W_T\) and a pre-allocation \((C_{aT}, C_{bT})\), \(R^I_t\) has the following form (see Appendix B):

\[
R^I_t(W_T, C_{aT}, C_{bT}) = \sum_{d=a,b} \alpha^I_t(C_{dT}, \varepsilon, \lambda_{d0}) \exp\{X_{dT}' \beta_0 / \varepsilon\} g_0(W_T)^{1/\varepsilon}, \tag{9}
\]

where the non-random term \(\alpha^I_t(C_{dT}, \varepsilon, \lambda_{d0})\) satisfies a recursive formula.\(^{20}\) Then, using the bounds on \(g_0(W_T)\) in (4) and (8), we obtain the following bounds for \(R^I_t(W_T, C_{aT}, C_{bT})\):

\[
\left[ g^L_0(W_T)^{1/\varepsilon}, g^U_0(W_T)^{1/\varepsilon} \right] \times \sum_{d=a,b} \alpha^I_t(C_{dT}, \varepsilon, \lambda_{d0}) \exp\{X_{dT}' \beta_0 / \varepsilon\}. \tag{10}
\]

\(^{19}\)Such techniques are frequently used in quantity-based revenue management practices, especially in the presence of multiple resources, i.e., network revenue management. See Section 2 of van Ryzin and Talluri (2005).

\(^{20}\)The only exception is for revenues under uniform pricing with prices constrained to belong to the grid, for which the form is more complicated. Specifically, it corresponds to the maximum over the grid of the revenue displayed in (5). Nonetheless, we can still simply obtain bounds on these revenues using the monotonicity of the right-hand side of (5) with respect to \(g_0(W_T)\).
Bounds on $R^t$ then follow by averaging (10) over trains:

$$
\mathbb{E} \left[ \frac{g^{'\epsilon}_0(W_T)^{1/\epsilon}}{g^{'\epsilon}_0(W_T)^{1/\epsilon}} \right] \times \sum_{d=a,b} \alpha_t(C_{dT}, \varepsilon, \lambda_{d0}) \exp\{X'_d\theta/\varepsilon\} \right].
\end{equation}

We also consider below ratios of counterfactual revenues. Given what precedes, such ratios $r_0$ satisfy

$$
r_0 = \frac{\mathbb{E}[f_1(U_T)g_0(W_T)^{1/\epsilon}]}{\mathbb{E}[f_2(U_T)g_0(W_T)^{1/\epsilon}]}.
$$

for two identified, positive functions $f_1$ and $f_2$. Let $\mathcal{R}$ denote the identified set on $r_0$. Then one can show that $\mathcal{R}$ is an interval $[r, \overline{r}]$, where $\overline{r}$ and $r$ are defined as the unique solutions of

$$
\begin{align*}
\mathbb{E} \left[ g^{\epsilon}_0(W_T)^{1/\epsilon}(f_1(U_T) - r f_2(U_T)) + (g^{\epsilon}_0(W_T)^{1/\epsilon} - g^\epsilon_0(W_T)^{1/\epsilon})(f_1(U_T) - r f_2(U_T))_+ \right] &= 0, \\
\mathbb{E} \left[ g^{\epsilon}_0(W_T)^{1/\epsilon}(f_1(U_T) - \overline{r} f_2(U_T)) + (g^{\epsilon}_0(W_T)^{1/\epsilon} - g^\epsilon_0(W_T)^{1/\epsilon})(f_1(U_T) - \overline{r} f_2(U_T))_+ \right] &= 0.
\end{align*}
$$

### 4.3 Estimation and inference

We estimate $\theta_0 = (\varepsilon, \beta_0, \lambda_{a0}, \lambda_{b0})$ as follows. Let $Y_{jkT} = 1$ if seat $j$ in fare class $k$ for train $T$ is sold to $a$, $Y_{jkT} = 0$ otherwise. By (1), we have

$$
\Pr(Y_{jkT} = 1|\xi_{aT}, \xi_{bT}) = \Lambda \left( \ln(\xi_{bT}/\xi_{aT}) - \varepsilon \ln(p_{bkt}/p_{akt}) \right),
$$

and the $(Y_{jkT})_{j=1,\ldots,n_{kT}}$ (with $n_{kT} := n_{akt} + n_{bkt}$) are independent. Thus, we can estimate $\varepsilon$ and $\ln(\xi_{bT}/\xi_{aT})$ by maximizing the likelihood of a logit model including train fixed effects. Because the number of sales for each train is large (usually above 250), the bias related to the estimation of these fixed effects is expected to be negligible.

Second, under Assumption 3 (with the equality in (i) replaced by (2) to account for the bias related to the estimation of these fixed effects is expected to be negligible.

Then, we estimate $\beta_0$ by nonlinear least squares, replacing $\ln(\xi_{bT}/\xi_{aT})$ by its estimator. Finally, we estimate $(\lambda_{a0}, \lambda_{b0})$ by maximum likelihood on the sample $(\ln(\eta_{bT}/\eta_{aT}))_T$, with

$$
\ln(\eta_{bT}/\eta_{aT}) = \ln(\xi_{bT}/\xi_{aT}) \times \left[ \sum_{c\in b} \exp(X'_{cT}\beta_t) \right] - \ln \left[ \sum_{c\in a} \exp(X'_{cT}\beta_t) \right].
$$

25
Remark 4.3 We could directly estimate $\theta_0$ by maximum likelihood, as under Assumptions 1-3, the distribution of $(Y_{jKT})_{j=1,\ldots,n; k=1,\ldots,K}$ is fully parametric. We do not follow this path for numerical reasons. In fact, the corresponding estimator is much more complicated to compute, something turning out to be important when considering inference based on subsampling.

Next, we estimate the lower and upper bounds on $g_0(W_T)$ by the empirical counterparts of (4) and (8), where the conditional expectations $E(.)|W_T$ are replaced by empirical means (as $W_T$ is discrete here). We then estimate bounds on $R_I^l$ by the empirical counterpart of (11).

Turning to inference, we consider below statistical tests of whether $R_I^l$ improves upon the observed revenue $R_{obs}$. Specifically, we test for

$$H_0 : \Delta = R_I^l - R_{obs} \leq 0 \text{ vs } H_1 : \Delta > 0. \quad (12)$$

To this end, we consider critical regions of the form $\{\sqrt{N}\hat{\Delta}_l > \delta\}$ with $N$ the number of trains, $\delta$ specified below, $\hat{\Delta}_l := \hat{R}_{lower} - \hat{R}_{obs}$ and $\hat{R}_{lower}$ and $\hat{R}_{obs}$ estimators of the lower bound of $R_I^l$ and $R_{obs}$, respectively. By definition, $\Delta_l \leq \Delta$. Then, under $H_0$,

$$\Pr (\hat{\Delta}_l > \delta) = \Pr (\sqrt{N}(\hat{\Delta}_l - \Delta_l) > \delta - \sqrt{N}\Delta_l)$$
$$\quad \leq \Pr (\sqrt{N}(\hat{\Delta}_l - \Delta_l) > \delta - \sqrt{N}\Delta)$$
$$\quad \leq \Pr (\sqrt{N}(\hat{\Delta}_l - \Delta_l) > \delta). \quad (13)$$

Then, by using for $\delta$ a consistent estimator of the quantile of order $1 - \alpha$ of the asymptotic distribution of $\sqrt{N}(\hat{\Delta}_l - \Delta_l)$ ($q_{l,1-\alpha}$, say), we ensure control of the asymptotic level of our test. Because of the maximum in (4), the asymptotic distribution of $\sqrt{N}(\hat{\Delta}_l - \Delta_l)$ may not be Gaussian and the bootstrap may be invalid. We thus rely instead on subsampling (Politis et al., 1999) and estimate $q_{l,1-\alpha}$ by the $(1 - \alpha)$-quantile of the subsampling distribution of $\sqrt{N}(\hat{\Delta}_l - \Delta_l)$. We also use (13) and this subsampling distribution to compute an upper bound of the p-value corresponding to (12), which we will report in the next section.
5 Results

5.1 Demand estimation

We first consider the estimation of the price elasticity (−ε), the coefficients of destination-train specific effects (β₀), and the parameters of Gamma distribution (λₐ₀, λ₉₀). The variables we include in Wₜ are route dummies, time dummies for the year and month of the train, whether it occurs during the weekend, on public holidays, on school holidays and whether the departure time is during rush hour. Regarding the variables Xₜ₅ or, to be more precise, Xₜ₆ where c denotes a city (see our discussion around Equation (2)), we include travel time to c by train T, its square, city-specific effects Xₖ (namely, the population of the urban area of c and whether c is a regional capital) and all interactions Xₖ₇ × Wₜ₈ for all components Xₖ₇ and Wₜ₈ of the vectors Xₖ and Wₜ, respectively.

The estimates of price elasticities are displayed in the top panel of Table 3. In Column I (our baseline specification), we assume a constant price elasticity across routes and trains and obtain a price elasticity of −4.04. This result is larger (in absolute value) than those in the literature on the transportation industry. We refer for instance to the meta-analysis by Jevons et al. (2005) and the studies of Wardman (1997), Wardman (2006) and Wardman et al. (2007), which point to price elasticities in the range [−1.3; −2.2]. Unlike ours, most of the studies rely on aggregated data. This is likely to bias upwards price-elasticity estimates, a point that we illustrate in Appendix C by running regressions based on our data aggregated at different levels.
Table 3: Estimates of \((\varepsilon, \beta_0, \lambda_0)\)

|                        | Binomial model |            | Multinomial model |            |
|------------------------|----------------|------------|-------------------|------------|
|                        | I              | II         | III               | IV         |
| **Minus price elasticity** \((-\varepsilon)\) |                |            |                   |            |
| Constant               | 4.04 (0.22)    | 6.96 (0.57)| 4.04 (0.22)       | 6.96 (0.38)|
| Southwest              | \(-2.09\) (0.54)| \(-2.09\) (0.18) |            |            |
| Weekend/national holidays | \(-2.46\) (0.51)| \(-2.46\) (0.17) |            |            |
| Peak hour              | 0.37 (0.48)    | 0.37 (0.15) |                   |            |
| **Destination effects** |                |            |                   |            |
| Population (in M. inhabitants) | 2.23 (0.20)    | 2.24 (0.20)| 2.17 (0.19)       | 2.18 (0.19)|
| Regional capital       | 0.20 (0.19)    | 0.20 (0.19)| 0.17 (0.19)       | 0.17 (0.20)|
| Travel time by train (in hours) | \(-2.07\) (0.11)| \(-2.11\) (0.11)| \(-1.60\) (0.11)| \(-1.64\) (0.11)|
| Travel time by train, squared | 0.34 (0.06)    | 0.35 (0.06)| 0.28 (0.04)       | 0.29 (0.05)|
| **Gamma distributions** |                |            |                   |            |
| \(\lambda_{a0}\) (intermediate) | 3.63 (1.01)    | 3.63 (1.00)| 3.93 (1.07)       | 3.93 (1.06)|
| \(\lambda_{b0}\) (final)   | 2.62 (0.40)    | 2.62 (0.40)| 2.79 (0.41)       | 2.78 (0.41)|
| **Control for** \(X_d \times W_T\) | Yes | Yes | Yes | Yes |
| **\(R^2\) of the reg. of ln(\(\xi_b T / \xi_a T\))** | 0.502 | 0.506 | 0.521 | 0.525 |

*Notes:* The total number of trains is 2,909. In Columns I and II, with all fare classes, the total number of observations (fare classes \(\times\) trains) is 21,988. In Columns III and IV, the total number of observations (fare classes \(\times\) trains) is 34,908. Southwest correspond to the lines to Côte Basque, Toulouse and Perpignan. Standard errors (under parentheses) are calculated using the bootstrap with 500 re-sampled datasets.

The middle panel of Table 3 reports the estimates of the components of \(\beta_0\) corresponding to the travel time and city-specific effects. The effect of the population size and travel time by train are as expected. Larger cities lead to higher demand and a longer travel time by train leads to a lower demand for train tickets. The effect of
travel time may nonetheless be attenuated for long journeys, though the coefficient
of the square of travel time is not significant. The bottom panel of Table 3 reports
the estimates of the parameters \((\lambda_a, \lambda_b)\) of the gamma distribution. Intermediate
destinations are estimated to have larger uncertainty on demand \((V(\eta_{dT}) = \lambda_a)\) under
the gamma specification), though the difference between the two is not statistically
significant.

In Column II, we estimate the demand model by allowing price elasticity to vary
across routes and trains. We find that travelers of routes from Paris to the southwest
of France (namely, the routes to Côte basque, Toulouse and Perpignan) are less price-
sensitive to price than those of other routes. Travelers on weekend or national holidays
have a smaller price elasticity (in absolute value) than those on other days. On the
other hand, once controlling for weekend and national holidays, individuals traveling
during peak hours appear to have a similar elasticity to the others.

For several routes, there are actually multiple intermediate or final destinations. If
Assumptions 1 and 2 hold, Theorem 4.1 implies that the joint distribution of the
purchases for these multiple destinations, conditional on the total number of pur-
chasesthe train, is multinomial, rather than binomial in case the purchases for
the intermediate or final stops are aggregated. We re-estimate the demand models
corresponding to Columns I and II using a multinomial model. The results are dis-
played in Columns III and IV, respectively. The resulting price elasticities are almost
identical to those obtained before. The destination effects and estimates of \((\lambda_a, \lambda_b)\)
are also very similar.

5.2 Counterfactual revenues

We now turn to the counterfactual revenues under different pricing strategies, namely
uniform, stopping-time, and full dynamic pricing. As discussed in Section 4.2, for
counterfactual revenues \(R^I_r\) with \(r \in \{u, f, s\}\) and \(I \in \{c, i\}\), we simulate the revenue
with the optimally pre-allocated numbers of available seats for intermediate and final
stops; for \(R^I_r\) with \(r \in \{sM, sM+\}\) and \(I \in \{c, i\}\), i.e., stopping-time pricing strategy
with \(M\) (increasing) fares, we fix the pre-allocated number of available seats for
intermediate stop \(a\), \(C_{aT}\), to be equal to the average number of seats sold for \(a\) among
all the trains operated on the given route. We do this, rather than finding the optimal
value of $C_{aT}$, for computational reasons. Moreover, for the other pricing strategies ($r \not\in \{sM, sM+\}$), the revenues obtained this way secures at least 99% of the revenue based on the optimal pre-allocation, so we expect very little effect of considering this specific pre-allocation.

Table 4 summarizes the set estimates of counterfactual revenues averaging over all routes based on Column I in Table 3 – we discuss the results based on Column II in Section 5.3 below. We organize our discussion of the results along different themes.

**How does the actual strategy compare to counterfactual pricing strategies?**
Recall that by Assumption 4, the actual strategy is supposed to be better than any uniform pricing strategy under incomplete information and with prices constrained to belong to the price grid. The gains are however moderate: they range between 0% and 9.5%. Then, we already cannot exclude that the actual strategy performs actually worse than the same uniform pricing strategy but with unconstrained prices (see Scenario u.2). In any case, the gains would be at most 8.1%. When turning to dynamic pricing strategies, we observe a loss in observed revenue ranging between 9.2% and 16.7% relative to the optimal stopping-time pricing strategy (see Scenario s.5), and between 9.4% and 17.1% relative to the optimal dynamic pricing strategy (see Scenario f.1). Both losses are statistically significant at the 5% level, with the upper bound of p-values being 0.048 and 0.047, respectively.\textsuperscript{21} Under the most constrained dynamic pricing strategy, namely two fare classes and increasing prices, we still observe a loss in revenue ranging between 3.6% and 12.2%. When we consider the same constraints as in the actual pricing strategy, namely 12 fare classes and increasing prices, we estimate a loss in between 6.8% and 15.1%.\textsuperscript{22}

\textsuperscript{21}These p-values are obtained using a subampling procedure described in Section 4.3 and implemented by drawing 1,000 subsamples of size 300, with 50 trains from each of the 6 routes.

\textsuperscript{22}As discussed in Section 3, the gains are obtained by keeping competitors’ prices the same as in the observed scenario, i.e., a partial equilibrium. We can compute the decrease in competitors’ overall prices that offsets the gains. For instance, in the case with 12 increasing prices, we show that the decrease should be at least 6.8%. See Appendix A for more details.
Table 4: Revenues under counterfactual pricing strategies, average over lines

| Scenario                              | Point or set estimate (in K€)                  |
|---------------------------------------|-----------------------------------------------|
| **Observed pricing strategy**         | 12.21                                         |
| **Uniform pricing strategy**          |                                               |
| u.1 Incomplete information, constrained prices | [11.15, 12.21]                                 |
| u.2 Incomplete information, unconstrained prices | [11.29, 12.31]                                |
| u.3 Complete information, constrained prices | [12.56, 13.86]                                |
| u.4 Complete information, unconstrained prices | [13.23, 14.42]                                |
| **Stopping-time pricing strategy**    |                                               |
| s.1 Incomplete information, 2 increasing fares | [12.66, 13.90]                                |
| s.2 Incomplete information, 2 fares    | [12.85, 14.11]                                 |
| s.3 Incomplete information, 12 increasing fares | [13.10, 14.39]                               |
| s.4 Incomplete information, 12 fares   | [13.27, 14.57]                                 |
| s.5 Incomplete information             | [13.44, 14.66]                                 |
| s.6 Complete information, 2 increasing fares | [13.34, 14.54]                               |
| s.7 Complete information, 2 fares      | [13.41, 14.62]                                 |
| s.8 Complete information, 12 increasing fares | [13.39, 14.60]                               |
| s.9 Complete information, 12 fares     | [13.47, 14.69]                                 |
| s.10 Complete information              | [13.48, 14.70]                                 |
| **“Full” dynamic pricing strategy**   |                                               |
| f.1 Incomplete information             | [13.47, 14.68]                                 |
| f.2 Complete information               | [13.50, 14.72]                                 |

*Notes:* Estimated bounds on counterfactual revenues. With “constrained prices” (resp. “unconstrained prices”), optimization is conducted over the actual price grid (resp. over all positive real numbers).

Because of fixed pre-allocations for destinations $a$ and $b$, the revenues in Table 4 are just lower bounds on the true, optimal revenues, which still reinforces our conclusions above. To get a sense on the quantitative effect of these pre-allocations, we simulate counterfactual revenues under unconstrained uniform pricing without pre-allocating capacities among intermediate and final destinations. The corresponding formulas
are in Appendices E.3.1 and E.3.2, see the sections “uniform pricing” therein. In the complete information set-up, we obtain a set estimate of $[13.45, 14.68]$, corresponding to an increase in between 1.4% and 1.8% compared to Scenario u.4. In the incomplete information set-up, we obtain a higher gain of around 6%, with a set estimate of $[11.96, 14.68]$. This 6% might be the upper bound on possible gains from not imposing any pre-allocation, as one could expect that the effects of pre-allocation can be more easily mitigated with more flexible pricing strategies.

How can we explain the suboptimality of the actual strategy, in particular compared to the optimal strategies under similar pricing constraints? First, the initial seat allocation planning determined by the CRS may sometimes be far away from the optimal allocation under complete information. Then, revenue managers may fail to adjust enough this initial allocation. Second, in our counterfactuals, we have considered that revenue managers knew the true $\epsilon$, the true effects of covariates, or the true $b_T(\cdot)$. This may not be the case in reality. In any case, our results emphasize the importance of not imposing strong optimality conditions on the supply side.

**Does it matter to have a fixed price grid?** We look at this question by comparing the revenues obtained under optimal uniform strategies with prices either chosen optimally on $[0, \infty)$ or only within the actual price grid of the train under consideration. The effect of the grid is higher in the complete information set-up, with a gain of an unconstrained optimization roughly ranging in between 4% and 5.3%. This is basically because demand is very high or very low for a few trains, in which case one would like to set a price above the maximal price, or below the minimal price of the grid. On the other hand, fixing the price grid has very small effects on revenues under incomplete information, with gains in between 0.8% and 1.3%.

**Does it pay to complexify pricing strategies?** The answer to that question very much depends on the information set-up. In the complete information case, the answer is basically “no”: the difference in revenue between uniform pricing with

\[^{23}\text{These are approximations obtained by dividing each lower bounds and each upper bounds. The exact bounds on the ratios are trickier to obtain because the revenues under uniform pricing and constrained prices do not take a simple form. The approximation we use works well on other ratios, for which we can compute the exact bounds.}\]
unconstrained prices and full dynamic pricing is only around 2.0%.\textsuperscript{24} This figure sharply contrasts with the 19.3% gain we estimate under incomplete information by comparing Scenarios f.1 and u.2.

Intuitively, dynamic pricing still helps in the complete information case because of the uncertainty on the demand process. But the possibility to adjust the pricing strategy as one learns about \((\xi_{aT}, \xi_{bT})\) (or, equivalently, \((\eta_{aT}, \eta_{bT})\)) in the incomplete information set-up plays a much more important role. To shed light on this point, we decompose the variance of the demand under the optimal uniform pricing in incomplete information into two parts:

\[
\mathbb{E}[V(D_{dT}(0,1,p_{dT}^u)|W_T)] = \mathbb{E}[V(D_{dT}(0,1,p_{dT}^u)|\xi_{dT})] + \mathbb{E}[\mathbb{E}[D_{dT}(0,1,p_{dT}^u)|\xi_{dT}]|W_T],
\]

where \(p_{dT}^u\) is the optimal price under uniform pricing for destination \(d\) and train \(T\). Even though they both involve \(g_0(W_T)\), one can show that the two terms in this decomposition are point identified. For intermediate and final destinations respectively, the variation of the demand process (the first term) only explains on average 1.3% and 0.9% of the total variance.

Now, even in the incomplete information set-up, one need not consider complex pricing strategies to obtain revenues close to the optimal ones. First, restricting to stopping-time pricing strategies incurs virtually no loss, compared to “full” dynamic pricing. By changing prices only when a purchase is observed, the firm can secure around 99.8% of the revenue gain from uniform pricing to dynamic pricing regimes (comparing here Scenarios s.5 and f.1). Considering pricing strategies with 12 fare classes, as in reality but with possibly decreasing prices, still yield revenues in between 98.7% and 99.1% of the revenues under full dynamic pricing (comparing here Scenarios s.4 and f.1).

**How fast does information accumulate?** First, the tiny difference between the gains of full dynamic pricing under complete and incomplete information shows that revenue management is an effective instrument for demand learning. By learning from consumers’ purchases in a Bayesian way, it can gradually pin down the uncertainty

\textsuperscript{24}This empirical finding is close to the existing simulation results in operational research. For example, Zhao and Zheng (2000) shows a similar improvement by between 2.4% and 7.3%.
on the overall demand. Pricing decision then takes this renewed information into account, improving total revenue. And actually, this demand learning can compensate almost all revenue loss due to ex ante uncertainty on demand. The difference in revenue under optimal uniform pricing between incomplete and complete information is around $2K\€$ (comparing Scenarios u.4 and u.2), while this difference decreases to around $0.03K\€$ only under optimal dynamic pricing (see Scenarios f.2 and f.1). This finding is in line with Lin (2006), who reports a similar near-optimality of demand learning in a simulation study.

The reason of this very modest loss compared to the complete information set-up is that information accumulates quickly. To illustrate this point, we simulate expected revenues under a class of intermediate stopping-time pricing strategies, where the firm is only allowed to dynamically price the first $x\%$ seats, turning to uniform pricing for the remaining seats. Thus, $x = 0$ and $x = 100$ correspond respectively to the optimal uniform and stopping-time pricing strategies. By quantifying the revenue gain from $x$ to $x + 1$, we can characterize how much can be marginally gained from being able to extract information on demand from additional purchases (1% of total seats) and optimally adjust its pricing. Figure 3 displays the lower bounds of the optimal revenues under these intermediate pricing strategies under complete (blue) and incomplete (red) information for $x = 1, \ldots, 100$.

Under incomplete information, demand learning is rather quick, as we can see from the important concavity of the red line. With just $x = 5$, the firm already achieves a revenue equal to the observed one; by learning from 50% of the seats, it obtains a revenue around 3% lower than that of the complete information. On the other hand, the blue line shows that the revenue gains under complete information are small. The incremental revenue from $x$ to $x + 1$ is almost constant and barely reaches 3€. This latter result could be expected, given that the difference between uniform pricing and the full stopping-time pricing is small under complete information. The striking difference in the pattern of marginal gain between complete and incomplete information settings is also in line with our previous findings: in terms of revenue

$^{25}$As the pricing strategies in Theorem 4.2, we show in Appendix B that one can partially identify the optimal revenues with intermediate stopping-time pricing strategies using the procedure described in Section 4.2.

$^{26}$We also simulate the upper bounds of these revenues. The obtained curve is very similar.
improvement by dynamic pricing, the effect of learning overall demand \((\xi_{aT}, \xi_{dT})\) is more remarkable than that of pinning down the uncertainty in demand process when \((\xi_{aT}, \xi_{dT})\) is fixed.

**Figure 3:** Revenues (lower bound) under intermediate pricing strategies

![Graph showing revenues under intermediate pricing strategies](image)

### 5.3 Tests and robustness checks

In this section, we first test the plausibility of Assumptions 1 and 2, on which the identification of \(\theta_0\) relies. Next, we relax the assumption of a time-invariant price elasticity. Then, we consider alternative parametric specifications. Finally, we explore the effect of specific routes on our results.

**Test of Assumptions 1 and 2.** These assumptions imply that the proportions \(n_{bkT}/(n_{akT} + n_{bkT})\) remain constant through fare classes \(k\) satisfying \(p_{bkT} = p_{akT}\). A convenient way to check this is to restrict ourselves to two routes, Paris-Marseille and Paris-Mulhouse, for which \(p_{bkT} = p_{akT}\) for all \(k \in \{1, ..., K\}\). By taking the first fare class as a reference, we simply regress \(n_{bkT}/(n_{akT} + n_{bkT})\) on the other 11 fare class dummies and train fixed effects. We then test whether the coefficients of the fare class dummies are equal to zero.

35
The results are presented in Table 5. As emphasized by the top panel, most coefficients are not significant, despite the large number of observations (453 and 499 for the two routes). For Paris-Marseille, the p-value of the joint test is larger than 0.05. For Paris-Mulhouse, the p-value is lower, but it appears that this result is mostly driven by the last fare classes (the joint test for nullity of the first 10 classes has a p-value of 15%). The coefficients of the last two fare classes are indeed positive and quite large for this route, indicating that there would be more “late purchasers” for Mulhouse than for Strasbourg.

Table 5: Test of the separability in Assumption 1

| Fare class | Paris-Marseille Coefficient estimates | Paris-Mulhouse Coefficient estimates |
|------------|--------------------------------------|-------------------------------------|
| 2          | -0.019                               | 0.003                               |
| 3          | -0.041***                            | -0.008                              |
| 4          | -0.019                               | -0.010                              |
| 5          | -0.004                               | -0.009                              |
| 6          | -0.005                               | -0.025                              |
| 7          | 0.002                                | -0.020                              |
| 8          | -0.003                               | 0.009                               |
| 9          | 0.033                                | 0.026                               |
| 10         | -0.003                               | 0.041                               |
| 11         | -0.03                                | 0.109***                            |
| 12         | -0.025                               | 0.168***                            |

Joint nullity test p-values

| 2-12       | 0.053 | 0.0004 |

Average ratio

| 0.589 | 0.249 |

Notes: Coefficient estimates of the regression of $n_{bkt}/(n_{akt} + n_{bkt})$ on train fixed effects and fare class dummies (fare class 1 being the reference).

To see whether this pattern could influence our results beyond this specific route, we re-estimate $\varepsilon$ using only the first 10 fare classes. We obtain a price elasticity of $-4.86$, which is thus somewhat higher in absolute value than the baseline estimate.
of -4.04 obtained with the 12 fare classes. We then recomputed the identified sets of counterfactual revenues for Scenarios u2, u4, s5, s10, f1 and f2 (as they are the simplest to compute). The optimal revenues are slightly higher but with differences never exceeding 3.3% on the lower bounds and 1.2% on the upper bounds.

**Time-varying price elasticities.** One could expect that consumers purchasing their tickets earlier would be more price elastic than those buying their tickets late. For instance, the latter could include more business travelers. If so, the assumption of a time-invariant price elasticity would be violated. We entertain this possibility by replacing $\varepsilon$ in Assumption 1.1 by $\varepsilon_{\text{early}}1\{k \leq S\} + \varepsilon_{\text{late}}1\{k > S\}$, for some threshold $S$, so that $V_{dT}$ is now a Poisson process with intensity

$$I_{dT}(t, p) = \xi_{dT}b_T(t) \left[\varepsilon_{\text{early}}p^{-1-\varepsilon_{\text{early}}}1\{t \leq t_S\} + \varepsilon_{\text{late}}p^{-1-\varepsilon_{\text{late}}}1\{t > t_S\}\right],$$

(14)

where $t_S \in (0, 1)$ is the closing time of fare class $S$ and is assumed to be a function of $W_T$. This last restriction holds for instance if the moment when fare class $S$ is closed is fixed initially depending on $W_T$, and not adjusted afterwards. Then, Equation (1) still holds, with $\varepsilon$ simply replaced by $\varepsilon_{\text{early}}1\{k \leq S\} + \varepsilon_{\text{late}}1\{k > S\}$. Thus, we can identify both elasticities provided there is enough variation in relative prices.

Results on demand are displayed in Table 6. We consider threshold values $S$ equal to 9, 10 and 11. In the three cases, “early purchasers” are estimated to be more price elastic than “late purchasers”, with the estimated price elasticity of the former being greater (in absolute value) than the baseline estimate in Table 3 (−4.04) but still close to its upper bound of the 95% confidence interval. Besides, we find that the estimates of destination-train-specific effect, $\beta_0$, and the parameters of the Gamma distribution, $\lambda_0$, are close to the baseline results.

Next, we assess the robustness of the time-invariance elasticity condition on counterfactual revenues. With two elasticities, it is unclear whether the lower and upper bounds on counterfactual revenues derived in (11) are still identified, but we can obtain bounds on the lower bound in (11). We refer to Appendix D for more details. We set below $S = 10$ and compute the aforementioned bounds on revenues with the optimal uniform, stopping-time and “full” dynamic pricing, under both complete and incomplete information.
Table 6: Binomial model of demand with \((\varepsilon_{\text{early}}, \varepsilon_{\text{late}})\)

| Specification      | S=9  | S=10 | S=11 |
|--------------------|------|------|------|
| Price elasticity   |      |      |      |
| \(\varepsilon_{\text{early}}\) | 4.76 | 4.55 | 4.53 |
| \(\varepsilon_{\text{late}}\)   | 3.01 | 3.17 | 2.88 |
| Destination effects|      |      |      |
| Population (in M. inhabitants) | 2.11 | 2.14 | 2.15 |
| Regional capital   | 0.25 | 0.24 | 0.24 |
| Travel time by train (in hours) | −1.90 | −1.94 | −1.93 |
| Travel time by train, squared | 0.33 | 0.33 | 0.33 |
| Gamma distributions|      |      |      |
| \(\lambda_{a0}\) (intermediate) | 3.63 | 3.63 | 3.63 |
| \(\lambda_{b0}\) (final)   | 2.62 | 2.62 | 2.62 |
| Control for \(X_d \times W_T\) | Yes | Yes | Yes |
| \(R^2\) of the reg. of \(\ln(\xi_{bT}/\xi_{aT})\) | 0.500 | 0.501 | 0.501 |

Notes: The total number of trains is 2,909 and the total number of observations (fare classes \(\times\) trains) is 21,988.

The results are summarized in Table 7. In all scenarios, the estimated lower bounds are close to those in Table 4, with differences of at most 500€, or 4% of the observed revenue. In particular, we still observe a loss of at least 6.5% in the observed revenue relative to the optimal stopping-time (s.5) and “full” dynamic pricing strategies (f.1), close to the numbers in Table 4. Second, the (relative) differences between the estimated lower bounds of different pricing strategies are quantitatively similar to those in Table 4. Overall, the findings in Tables 6 and 7 suggest that despite the difference in price elasticity of early and late purchasers, our results are robust to the assumption of a time-invariant price elasticity.

**Alternative parametric specifications.** We conduct two robustness checks. First, we simulate counterfactual revenues with a lognormal specification on \(\eta_{dT}\) instead of a gamma distribution (Assumption 3(ii)). The drawback of a lognormal specification is that it is not conjugate with the Poisson distribution. Then, the updated distribution of \(\eta_{dT}\) in the incomplete information set-up takes a complicated form, making it
very difficult to compute counterfactual scenarios. Nevertheless, this issue does not appear for uniform pricing and complete information. Table 8 shows the results for Scenarios u.2, u.4, s.10 and f.2. Even if the bounds are wider than in the baseline specification, the results are similar.

Second, we have focused so far on the demand model corresponding to Column I in Table 3. We did so because counterfactual revenues are harder to compute under the richer specification corresponding to Column II in the same table. Nevertheless, we were able to compute counterfactual revenues with this specification for a few scenarios. The results, presented in Table 9, are hardly affected.
Table 8: Counterfactual revenues with log-normally distributed $\eta_{dT}$

| Scenario | Point or set estimate (in K€) |
|----------|-------------------------------|
| **Observed pricing strategy** | 12.21 |
| **Uniform pricing strategy** | | |
| u.2 Incomplete inf., uncons. prices | Log-normal specification | Baseline |
| | [10.03, 12.30] | [11.29, 12.31] |
| u.4 Complete inf., uncons. prices | [12.67, 15.36] | [13.23, 14.42] |
| **Complete information** | | |
| s.10 Stopping-time pricing strategy | [12.90, 15.66] | [13.48, 14.70] |
| f.2 “Full” dynamic pricing strategy | [12.92, 15.68] | [13.50, 14.72] |

*Notes:* The baseline results correspond to those in Table 4. See the notes of that table for more details.

Table 9: Counterfactual revenues based on Column II in Table 3

| Scenario | Point or set estimate (in K€) |
|----------|-------------------------------|
| **Observed pricing strategy** | 12.21 |
| **Uniform pricing strategy** | | |
| u.1 incomplete information, constrained prices | Specification II | Baseline |
| | [11.51, 12.21] | [11.15, 12.21] |
| u.2 incomplete information, unconstrained prices | [11.77, 12.37] | [11.29, 12.31] |
| u.3 complete information, constrained prices | [12.85, 13.73] | [12.56, 13.86] |
| u.4 complete information, unconstrained prices | [13.63, 14.35] | [13.23, 14.42] |
| **Stopping-time pricing strategy** | | |
| s.5 incomplete information | [13.83, 14.57] | [13.44, 14.66] |
| s.10 complete information | [13.87, 14.60] | [13.48, 14.70] |
| **“Full” dynamic pricing strategy** | | |
| f.1 incomplete information | [13.86, 14.59] | [13.47, 14.68] |
| f.2 complete information | [13.89, 14.62] | [13.50, 14.72] |

*Notes:* The baseline results are those of Table 4. See the notes of that table for more details.
Effect of specific routes. Table 2 show that the routes to Marseille and Côte basque have unusually high and low loads, so one may worry that revenue management was very different for these lines. We resimulate the counterfactual revenues by excluding these two routes. The results are hardly affected, with changes in the bounds by at most 1% over all scenarios.

6 Conclusion

Though the framework we have developed is taylored to our application, several of our results could be applied to other set-ups. The insight that many counterfactual revenues only depend on price elasticity and total demand, and not on the precise timing of consumers’ arrival, is convenient when purchasing dates are unknown. Similarly, the censoring issue and the absence of exogenous variations in prices may often occur. Our identification strategy, combining exogenous variations in relative prices and moment inequalities based on basic rationality on consumer’s side and weak optimality conditions on the firm’s pricing strategy, could then be applied in such contexts. Our results also suggest that such moment inequalities may be quite informative in practice.
References

Abrate, G., G. Viglia, J. S. García, and S. Forgas-Coll (2016). Price competition within and between airlines and high-speed trains: the case of the milan—rome route. *Tourism Economics* 22(2), 311–323.

Aviv, Y. and A. Pazgal (2002). Pricing of short life-cycle products through active learning. Working paper.

Beria, P., S. Tolentino, A. Bertolin, and G. Filippini (2019). Long-distance rail prices in a competitive market. Evidence from head-on competition in Italy. *Journal of Rail Transport Planning & Management* 12, 100144.

Bitran, G. R. and S. V. Mondschein (1997). Periodic pricing of seasonal products in retailing. *Management Science* 43(1), 64–79.

Brémaud, P. (1981). *Point processes and queues: martingale dynamics*, Volume 50. Springer.

Brumelle, S. and J. McGill (1993). Airline seat allocation with multiple nested fare classes. *Operations Research* 41(1), 127–137.

Cho, S., G. Lee, J. Rust, and M. Yu (2018). Optimal dynamic hotel pricing. Working paper.

Cho, S., G. Lee, J. Rust, and M. Yu (2019). Semi-parametric instrument-free demand estimation: relaxing optimality and equilibrium assumptions. Working paper.

Delaplace, M. and F. Dobruszkes (2015). From low-cost airlines to low-cost high-speed rail? the french case. *Transport policy* 38, 73–85.

den Boer, A. V. (2015). Dynamic pricing and learning: historical origins, current research, and new directions. *Surveys in operations research and management science* 20(1), 1–18.

den Boer, A. V. and B. Zwart (2015). Dynamic pricing and learning with finite inventories. *Operations research* 63(4), 965–978.

D’Haultfoeuille, X. and R. Rathelot (2017). Measuring segregation on small units: A partial identification analysis. *Quantitative Economics* 8(1), 39–73.

Dressen, M. (2018). De la segmentation tarifaire au bas coût dans le ferroviaire français. essai d’analyse. *La nouvelle revue du travail* (12).

Dubé, J.-P. and S. Misra (2023). Personalized pricing and consumer welfare. *Journal of Political Economy* 131(1), 131–189.
Feng, Y. and G. Gallego (1995). Optimal starting times for end-of-season sales and optimal stopping times for promotional fares. *Management Science* 41(8), 1371–1391.

Feng, Y. and B. Xiao (2000). Optimal policies of yield management with multiple predetermined prices. *Operations Research* 48(2), 332–343.

Gallego, G. and G. Van Ryzin (1994). Optimal dynamic pricing of inventories with stochastic demand over finite horizons. *Management science* 40(8), 999–1020.

Huang, Y., P. B. Ellickson, and M. J. Lovett (2022). Learning to set prices. *Journal of Marketing Research* 59, 411–434.

Jevons, D., A. Meaney, N. Robins, J. Dargay, J. Preston, P. Goodwin, and M. Wardman (2005). How do passengers respond to change? Working paper, Transport Studies Unit and Oxford University Centre for the Environment.

Lazarev, J. (2013). The welfare effects of intertemporal price discrimination: An empirical analysis of airline pricing in U.S. monopoly markets. Working paper.

Lee, A. (1990). *Airline Reservations Forecasting: Probabilistic and Statistical Models of the Booking Process*. Ph. D. thesis, Flight Transportation Laboratory, Massachusetts Institute of Technology, Cambridge, MA.

Lin, K.-Y. (2006). Dynamic pricing with real-time demand learning. *European Journal of Operational Research* 174(1), 522–538.

Littlewood, K. (1972). Forecasting and control of passenger bookings. *Airline Group International Federation of Operational Research Societies Proceedings, 1972* 12, 95–117.

Mariton, H. (2008). La politique tarifaire de la SNCF. Technical report, Assemblée nationale. Commission des Finances, de l’Economie générale et du Plan.

McAfee, R. P. and V. te Velde (2008). Dynamic pricing with constant demand elasticity. *Production and Operations Management* 17(4), 432–438.

Phillips, R. L. (2021). *Pricing and revenue optimization*. Stanford university press.

Politis, D. N., J. P. Romano, and M. Wolf (1999). *Subsampling*. Springer Science & Business Media.

Sauvant, J. (2002). Le transport ferroviaire de voyageurs en france : enfin un bien normal? Technical report, Note de synthèse du SES.

Stefanescu, C. (2012). Multivariate demand: Modeling and estimation from censored sales. Working paper.
Swan, W. (1990). Revenue management forecasting biases. Working paper, Boeing Commercial Aircraft, Seattle.

Talluri, K. and G. van Ryzin (2005). *The theory and practice of revenue management*. Springer Verlag.

van Ryzin, G. J. and K. T. Talluri (2005). An introduction to revenue management. In *Emerging Theory, Methods, and Applications*, pp. 142–194. Informs.

Von Martens, T. and A. Hilbert (2011). Customer-value-based revenue management. *Journal of Revenue and Pricing Management* 10(1), 87–98.

Wardman, M. (1997). Inter-urban rail demand elasticities and competition in Great Britain: Evidence from direct demand models. *Transportation Research* 33E(1), 15–28.

Wardman, M. (2006). Demand for rail travel and the effects of external factors. *Transportation Research* 42E, 129–148.

Wardman, M., W. Lythgoe, and G. Whelan (2007). Rail passenger demand forecasting: cross-sectional models revisited. *Research in Transportation Economics* 20(3), 119–152.

Williams, K. R. (2022). The welfare effects of dynamic pricing: Evidence from airline markets. *Econometrica* 90(2), 831–858.

Zhao, W. and Y.-S. Zheng (2000). Optimal dynamic pricing for perishable assets with nonhomogeneous demand. *Management science* 46(3), 375–388.
A Microfoundations of the demand model

This section aims at showing that Assumption 1 may be derived from a model of consumer’s choice. First, we suppose that the number of travelers who search at time $t \in [0, 1]$ for traveling to destination $d$ (e.g., from Paris to Marseille) at time 1 follows a non-homogeneous Poisson process with rate $\kappa_d T \lambda_T$. These consumers then choose between $m$ modes (iDTGV, airline etc.), with mode 1 corresponding to iDTGV. We assume that traveler $i$ searching at time $t$ for route $r = (d, T)$ has the following indirect utility when choosing mode $m$:

$$U_{itrm} = v_{itr} - \alpha \ln p_{trm} - c_{itrm},$$

where $v_{itr}$ is $i$’s expected utility at time $t$ to take route $r$ (e.g., their utility of traveling from Paris to Toulouse to attend a rugby match in Toulouse), $p_{trm}$ is the price at time $t$ of mode $m$ for route $r$, and $c_{itrm}$ is $i$’s expected cost incurred when choosing mode $m$ that is not captured by price (e.g., their traveling time to Gare Montparnasse in Paris). The probability of purchasing an iDTGV ticket at time $t$ for route $r$ is then:

$$s_{tr1} = \operatorname{Pr} (U_{itr1} \geq U_{itrm}, \forall m \geq 2) = \operatorname{Pr} \left( C_{itrm} \geq \frac{p_{tr1}^\alpha}{p_{trm}^\alpha} C_{itr1}, \forall m \geq 2 \right), \quad (15)$$

where $C_{itrm} = \exp(c_{itrm})$. Suppose that the $\{C_{itrm}\}_{m=1}^m$ are independent and follow Pareto distributions with parameters $(A_{itrm}, \delta_{Tm})$, respectively, so that $\operatorname{Pr} (C_{itrm} > c) = (1 - \frac{A_{itrm}}{c})^\delta_{Tm}$. The parameter $A_{itrm}$ can be interpreted as the minimal cost of taking mode $m$ for route $r$ among travelers who search at $t$. Suppose also that $A_{itrm}/A_{itr1} = a_{rm}$ does not depend on $t$, i.e., the minimal cost of taking mode $m$ relative to iDTGV does not depend on the time of search (the cost of traveling to airport relative to train station). Then, as long as $a_{rm} \geq p_{trm}^\alpha/p_{tr1}^\alpha$ for all $m = 2, ..., m$, we obtain:

$$s_{tr1} = \mathbb{E} \left[ \operatorname{Pr} \left( C_{itrm} \geq \frac{p_{tr1}^\alpha}{p_{trm}^\alpha} C_{itr1}, \forall m \geq 2 \mid C_{itr1} \right) \right]$$

$$= \prod_{m=2}^m \left( \frac{p_{trm}^\alpha A_{itrm}}{A_{itr1}} \right) \mathbb{E} \left[ \frac{-\alpha \sum_{m=2}^m \delta_{Tm} C_{itr1}^{-1} \sum_{m=2}^m \delta_{Tm}}{p_{tr1} \sum_{m=2}^m \delta_{Tm}} \right]$$

$$= \frac{\delta_{T1}}{\sum_{m=1}^m \delta_{Tm}} \prod_{m=2}^m \left( \frac{p_{trm}^\alpha A_{itr1}}{A_{itrm}} \right) \frac{\delta_{Tm}}{p_{tr1}} \frac{-\alpha \sum_{m=2}^m \delta_{Tm}}{p_{tr1} \sum_{m=2}^m \delta_{Tm}}$$

$$= \frac{\delta_{T1}}{\sum_{m=1}^m \delta_{Tm}} \prod_{m=2}^m \left( \frac{p_{trm}^\alpha a_{rm}}{p_{tr1}} \right) \frac{\delta_{Tm}}{p_{tr1}} \frac{-\alpha \sum_{m=2}^m \delta_{Tm}}{p_{tr1} \sum_{m=2}^m \delta_{Tm}}. \quad (16)$$
Then, given \((p_{trm}, a_{rm})\) with price \(p_{tr1}\) is

\[
I_{dT}(t, p) = -\kappa_{dT} \lambda_{T1} \frac{\partial s_{tr1}}{\partial p_{tr1}} = \kappa_{dT} \lambda_{T1} \alpha \delta_{T1} \prod_{m=2}^{m} (p_{trm}^{\alpha} a_{rm})^{\delta_{Tm}} p^{-1 - \alpha \sum_{m=2}^{m} \delta_{Tm}}.
\] (17)

To obtain our demand model from (17), we finally assume that \(p_{trm} = p_{Tm} b_{dT}\), a separability restriction in the pricing dynamics of mode \(m\). Then, we obtain:

\[
I_{dT}(t, p) = \frac{\delta_{T1} \kappa_{dT} b_{dT}^{\alpha} \sum_{m=2}^{m} \delta_{Tm}}{\sum_{m=2}^{m} \delta_{Tm}} \prod_{m=2}^{m} a_{dT}^{\delta_{Tm}} \int_{0}^{1} \lambda_{Tu} \prod_{m=2}^{m} p_{uTm}^{\alpha \delta_{Tm}} du
\]

\[
\times \frac{\lambda_{T1} \prod_{m=2}^{m} p_{Tm}^{\alpha \delta_{Tm}}}{\int_{0}^{1} \lambda_{Tu} \prod_{m=2}^{m} p_{uTm}^{\alpha \delta_{Tm}} du} \left( \frac{\alpha \sum_{m=2}^{m} \delta_{Tm}}{\varepsilon} \right) p^{-1 - \alpha \sum_{m=2}^{m} \delta_{Tm}}.
\] (18)

The price-elasticity \(\varepsilon\) is related to both travelers’ disutility of price (\(\alpha\)) and non-price costs for alternative modes of transportation, through the term \(\sum_{m=2}^{m} \delta_{Tm}\). Note that Equation (18) allows \(\varepsilon\) to depend on \(T\). The term \(b_{T}(t)\) captures both overall demand through the term \(\lambda_{T1}\) and competition effect, through the term \(\prod_{m=2}^{m} p_{Tm}^{\alpha \delta_{Tm}}\).

Combining (18) and revenue formula (9), we can compute the compensating variation in prices of competing transportation modes, \(p_{Tm}\), that offsets the loss due to the observed revenue management relative to counterfactual pricing strategies. Suppose that \(p_{Tm}\) decreases to \((1 - \Delta)p_{Tm}\) for all \(m = 2, ..., m\) and \(t \in [0, 1]\), i.e., an overall relative decrease in competitors’ prices by \(\Delta\). According to (18), \(\xi_{dT}\) will decrease to \((1 - \Delta)^{\alpha} \sum_{m=2}^{m} \delta_{Tm} \xi_{dT} = (1 - \Delta)^{\varepsilon} \xi_{dT}\). Then, because of (9), we obtain that \(R_{r}\) decreases by \(1 - [(1 - \Delta)^{\varepsilon}]^{1/\varepsilon} = \Delta\). For instance, consider Scenario s.3 in Table 4 and its lower bound 13.10K\(\xi\). To offset the gain of the counterfactual scenario relative to observed scenario relative to this counterfactual one, competitors’ prices need to decrease by \(1 - 12.21/13.10 = 6.8\%\) on average.

**B Expressions for the counterfactual revenues**

In this appendix, we list the formulas for the counterfactual revenues. The proofs of these formulas are given in Appendix E.3. The formulas are given conditional on
\((X_{aT}, X_{bT}, W_T)\) and for simplicity, we assume here that \(C_{aT}\) and \(C_{bT}\) are constant; if not, the results should just be seen conditional on \((C_{aT}, C_{bT})\). Due to the independence between \(\eta_{aT}\) and \(\eta_{bT}\) in Assumption 3(i) and the pre-allocation of capacity, we can separably simulate the counterfactual revenue for each destination, and sum them up to obtain the revenue for train \(T\). Consequently, we focus on pricing for destination \(d\) served by train \(T\) to simplify the exposition.

We both consider arbitrary distributions for \(\xi_{dT}\) and the gamma distribution in Assumption 3. Hereafter, in addition to the scenarios described in Section 3, we consider intermediate pricing strategies described in Section 5.2, see Figure 3 and the discussion above it. The corresponding revenues are denoted by \(R^I_{ik}\), with \(I \in \{c, i\}\) and where \(K \in [0, 100]\) indexes the proportion of seats that are dynamically priced.

Under Assumption 3(i), revenue formulas under both complete and incomplete information have the following form: for \(I \in \{c, i\}\) and \(r \in \{u, f, s, sM, sM+\}\),

\[ R^I_r = \alpha^I_r(\varepsilon, C_{dT}, f) \exp\{X'_{dT}/\varepsilon\} g_0^{1/\varepsilon}(W_T), \]

where \(f(.)\) denotes the distribution of \(\eta_{dT}\). In each scenario, we will display the result for a general \(f(.)\), and then apply it to the case where \(f\) is the density of a gamma distribution \(\Gamma(\lambda_{d0}, 1)\). To simplify the formulas and the proofs, we only specify \(\alpha^I_r(\varepsilon, C_{dT}, f)\) in each case. Finally, \(D(q)\) denotes hereafter a random variable satisfying \(D(q) \sim \mathcal{P}(q)\).

### B.1 Complete information

**Uniform pricing** \(\alpha^c_u = \max_{q > 0} \left\{ q^{-\frac{1}{\varepsilon}} \mathbb{E}[D(q) \land C_{dT}] \right\} \mathbb{E}\left[\eta_{dT}^{1/\varepsilon}\right].\)

**Full-dynamic pricing** \(\alpha^c_f = \alpha^c_{C_{dT}, f} \mathbb{E}\left[\eta_{dT}^{1/\varepsilon}\right], \) where \(\alpha^c_{0, f} = 0\) and for all \(k \geq 1, \)

\( \alpha^c_{k, f} = (\alpha^c_{k, f} - \alpha^c_{k-1, f})^{1-\varepsilon} (1 - 1/\varepsilon)^{\varepsilon-1}. \)

**Stopping-time pricing** \(\alpha^c_s = \alpha^c_{C_{dT}, s} \mathbb{E}\left[\eta_{dT}^{1/\varepsilon}\right], \) where \(\alpha^c_{0, s} = 0\) and for all \(k \geq 1, \)

\[ \alpha^c_{k, s} = \max_{q > 0} \left\{ q^{-\frac{1}{\varepsilon}} (1 - e^{-q}) + \alpha^c_{k-1, s} \int_0^1 q e^{-sq}(1 - s)^{1/\varepsilon} ds \right\}. \]
Stopping-time pricing with $M$ fares

$$R_{sM}^c = \alpha_{C_{dt}, sM}^c \mathbb{E} \left[ \eta_{dT}^{1/\varepsilon} \right],$$
where $\alpha_{C_{dt}, sM}^c = \max_{q > 0} \alpha_{C_{dt}, M}(q)$, $\alpha_k(q) = q^{-1/2} \mathbb{E}[D(q) \land k]$ and for all $k \in \{1, ..., C_{dt}\}$,

$$\alpha_{k,m}(q) = \max \left\{ \int_0^1 q e^{-qz} \left[ q^{-1/2} + \alpha_{k-1,m \land (k-1)}(q(1-z))(1-z)^{1/2} \right] dz, \max_{q > 0} \int_0^1 q e^{-qz} \left[ q^{-1/2} + \alpha_{k-1,m-1}(q(1-z))(1-z)^{1/2} \right] dz \right\}.$$  

Stopping-time pricing with $M$ increasing fares

$$\alpha_{sM+}^c = \alpha_{C_{dt}, sM+}^{c+} \mathbb{E} \left[ \eta_{dT}^{1/\varepsilon} \right],$$
where $\alpha_{C_{dt}, sM+}^{c+} = \max_{q > 0} \alpha_{C_{dt}, M}^{c+}(q)$ with $\alpha_{0,0}(q) = \alpha_{0,0}(q)$ and

$$\alpha_{k,m}^{c+}(q) = \max \left\{ q \int_0^1 e^{-qz} \left[ q^{-1/2} + \alpha_{k-1,m \land (k-1)}^{c+}(q(1-z))(1-z)^{1/2} \right] dz, \max_{q' \in [0,q]} \int_0^1 e^{-q'z} \left[ q'^{-1/2} + \alpha_{k-1,m-1}^{c+}(q'(1-z))(1-z)^{1/2} \right] dz \right\}.$$  

Intermediate-$K$ stopping-time pricing

$$\alpha_{C_{dt}, iK}^c \mathbb{E} \left[ \eta_{dT}^{1/\varepsilon} \right],$$
where $\alpha_{C_{dt}, (1-K)\%}^{c+}$ is

$$\alpha_{k,iK}^c = \max_{q > 0} \left\{ q^{-1/2} (1-e^{-q}) + \alpha_{k-1,iK}^c \int_0^1 q e^{-qs} (1-s)^{1/2} ds \right\}.$$  

Under Assumption 3(ii), $\eta_{dT} \sim \Gamma(\lambda_{d0}, 1)$. We have $\mathbb{E} \left[ \eta_{dT}^{1/\varepsilon} \right] = \Gamma(\lambda_{d0} + 1/\varepsilon)/\Gamma(\lambda_{d0})$.

### B.2 Incomplete Information

Hereafter, we denote by $g_{\lambda,\mu}$ the density of the $\Gamma(\lambda, \mu)$ distribution.

#### Uniform pricing

$$\alpha_{u}^i = \max_{q > 0} \left\{ \int_{R^+} q^{-1/2} \mathbb{E}[D(qz) \land C_{dt}] f(z) dz \right\}.$$  

Under Assumption 3(ii), $\alpha_{u}^i = \max_{q > 0} \left\{ \int_{R^+} q^{-1/2} \mathbb{E}[D(qz) \land C_{dt}] g_{\lambda,0,1}(z) dz \right\}.$

#### Full-dynamic pricing

Under Assumption 3, $\alpha_{f}^i = \alpha_{C_{dt}, f}(\lambda_{d0})$, where $\alpha_{0,f}(\lambda) = 0$ for any $\lambda > 0$ and for all $k \in \{1, ..., C_{dt}\}$,

$$\alpha_{k,f}^i(\lambda) = \lambda \left( 1 - \frac{1}{\varepsilon} \right)^{\varepsilon-1} \left[ -\alpha_{k-1,f}^i(\lambda + 1) + (1 + \frac{1}{\lambda \varepsilon}) \alpha_{k,f}^i(\lambda) \right]^{1-\varepsilon}.$$  

48
Stopping-time pricing $\alpha_s^i = \alpha_{C_{dT},s}^i(f)$, where $\alpha_{0,s}^i(f) = 0$ and for any $k \in \{1, \ldots, C_{dT}\}$,
\[
\alpha_{k,s}^i(f) = \max_{q > 0} q \int_0^1 \left[ q^{-1/\varepsilon} + (1 - u)^{1/\varepsilon} \alpha_{k-1,s}^i(T(f; qu)) \right] \left[ \int_0^\infty z e^{-quz} f(z) dz \right] du
\]
and $T(f; q)$ is a transformation of density function $f$ defined in Lemma E.2 below.
Under Assumption 3(ii), $\alpha_s^i = \alpha_{C_{dT},s}(\lambda_{d0})$, where $\alpha_{0,s}^i(\lambda) = 0$ for $\lambda > 0$, and for all $k \in \{1, \ldots, C_{dT}\}$,
\[
\alpha_{k,s}^i(\lambda) = \max_{q > 0} q \int_0^1 \frac{\lambda}{(1 + qs)^{\lambda+1}} \left[ q^{-1/\varepsilon} + \left( \frac{1 - s}{1 + qs} \right)^{1/\varepsilon} \alpha_{k-1,s}^i(\lambda + 1) \right] ds.
\]

**Stopping-time pricing with $M$ fares** $\alpha_{s,M}^i(M, f) = \alpha_{s,M}^i(f)$ where $\alpha_{s,M}^i(f) = \max_{q > 0} c_{C_{dT},M}(q, f)$ and for all $k$, $c_{k,0}(q, f) = q^{-1/\varepsilon} \int \mathbb{E}[D(qz) \wedge k] f(z) dz$ and
\[
c_{k,m}(q, f) = \max \left\{ q \int_0^1 \int ze^{-quz} f(z) dz \left[ q^{-1/\varepsilon} + c_{k-1,m\wedge(k-1)}(q(1 - u), T(f; qu)) \right. \right.
\]
\[
(1 - u)^{1/\varepsilon} \left. \left. \right] du, \max_{q' > 0} q' \int_0^1 \int ze^{-quz} f(z) dz \left[ q'^{-1/\varepsilon} + c_{k-1,m-1}(q'(1 - u), T(f; q'u)) \right. \right.
\]
\[
(1 - u)^{1/\varepsilon} \left. \left. \right] du \right\}
\]
for any $m \in \{1, \ldots, k\}$, $T$ being the same transform as in the case of stopping-time pricing. Further under Assumption 3(ii), $\alpha_{s,M}^i(M, \lambda_{d0}) = \alpha_{s,M}^i(\lambda_0)$, where $\alpha_{s,M}^i(\lambda) = \max_{q > 0} c_{C_{dT},M}(q, \lambda)$ with, for all $k$, $c_{k,0}(q, \lambda) = q^{-1/\varepsilon} \int \mathbb{E}[D(qz) \wedge k] g_{\lambda,1}(z) dz$ and for all $k \in \{1, \ldots, C_{dT}\}$ and all $m \in \{1, \ldots, k\}$,
\[
c_{k,m}(q, \lambda) = \max \left\{ q \int_0^1 \frac{\lambda}{(1 + qu)^{\lambda+1}} \left[ q^{-1/\varepsilon} + c_{k-1,m\wedge(k-1)} \left( q(1 - u), \frac{1 - u}{1 + qu} \right) \right. \right.
\]
\[
\left. \left. \left( \lambda + 1 \right) \left( \frac{1 - u}{1 + qu} \right)^{1/\varepsilon} \right] du, \max_{q' > 0} q' \int_0^1 \frac{\lambda}{(1 + qu')^{\lambda+1}} \left[ q'^{-1/\varepsilon} + c_{k-1,m-1} \left( q'(1 - u), \frac{1 - u}{1 + qu'} \right) \right. \right.
\]
\[
\left. \left. \left( \lambda + 1 \right) \left( \frac{1 - u}{1 + qu'} \right)^{1/\varepsilon} \right] du \right\}.
\]

**Stopping-time pricing with $M$ increasing fares** $\alpha_{s,M+}^i(M, f) = \alpha_{C_{dT},s,M+}^i(f)$, where $\alpha_{C_{dT},s,M+}(f) = \max_{q > 0} c_{C_{dT},M}(q, f)$ with, for any $k \in \{0, \ldots, C_{dT}\}$, $c_{k,0}(q, f) = c_{k,0}(q, f)$ and for any $m \geq 1$,
\[
c_{k,m}^+(q, f) = \max \left\{ q \int_0^1 \int ze^{-quz} f(z) dz \left[ q^{-1/\varepsilon} + c_{k-1,m\wedge(k-1)}^+(q(1 - u), T(f; qu)) \right. \right.
\]
\[
(1 - u)^{1/\varepsilon} \left. \left. \right] du, \max_{q' \in [0,q]} q' \int_0^1 \int ze^{-quz} f(z) dz \left[ q'^{-1/\varepsilon} + c_{k-1,m-1}^+(q'(1 - u), T(f; q'u)) \right. \right.
\]
\[
(1 - u)^{1/\varepsilon} \left. \left. \right] du \right\}
\]
49
Under Assumption 3(ii), we have $R_{sM+}(M, \lambda_0) = \alpha_{sM+}(\lambda_0)$, where $\alpha_{sM+}(\lambda) = \max_{q>0} c_{C_{at},M}^+(q, \lambda)$ with $c_{k,m}^+(q, \lambda) = c_{k,m}^+(q, g_{\lambda,1})$ as defined above. Further, we have the following simplifications:

$$c_{k,m}^+(q, \lambda) = \max \left\{ q \int_0^1 \frac{\lambda}{1 + qu} \left[ q^{-\frac{1}{\epsilon}} + c_{k-1,m}^+(\lambda - 1) \left( \frac{q(1-u)}{1 + qu}, \lambda + 1 \right) \left( \frac{1 - u}{1 + qu} \right)^{\frac{1}{\epsilon}} \right] du \right\},$$

$$\max_{q' \in (0, q]} q' \int_0^1 \frac{\lambda}{1 + q'u} \left[ q'^{-\frac{1}{\epsilon}} + c_{k-1,m-1}^+(\lambda - 1) \left( \frac{q'(1-u)}{1 + q'u}, \lambda + 1 \right) \left( \frac{1 - u}{1 + q'u} \right)^{\frac{1}{\epsilon}} \right] du \right\}.$$  

**Intermediate-$K$ stopping-time pricing** \( \alpha_{K}^i = \alpha_{C_{at},iK}^i(f) \), where \( \alpha_{C_{at}(1-K\%)iK}^i(f) = \max_{q>0} \left\{ \int_{R^+} q^{-\frac{1}{\epsilon}} E[D(qz) \wedge (C_{at}(1-K\%)) f(z) dz] \right\} \) and for any \( k > C_{at}(1-K\%) \),

$$\alpha_{k,iK}^i(f) = \max_{q>0} q \int_0^1 \left[ q^{-1/\epsilon} + (1-u)^{\frac{1}{\epsilon}} \alpha_{k-1,iK}^i(T(f; qu)) \right] \int_0^\infty ze^{-quz} f(z) dz du.$$  

Under Assumption 3(ii), \( \alpha_s^i = \alpha_{C_{at},iK}^i(\lambda_0) \), where

$$\alpha_{C_{at}(1-K\%)iK}^i(\lambda) = \max_{q>0} \left\{ \int_{R^+} q^{-\frac{1}{\epsilon}} E[D(qz) \wedge (C_{at}(1-K\%)) g_{\lambda,1} z) dz] \right\}$$

for \( \lambda > 0 \), and for all \( k > C_{at}(1-K\%) \),

$$\alpha_{k,iK}^i(\lambda) = \max_{q>0} \left\{ q \int_0^1 \frac{\lambda}{1 + qs} \left[ q^{-\frac{1}{\epsilon}} + \left( \frac{1 - s}{1 + qs} \right)^{\frac{1}{\epsilon}} \alpha_{k-1,iK}^i(\lambda + 1) \right] ds \right\}.$$  

### C Demand Estimation with Aggregated Data

The difference between our results and those from studies relying on aggregated data comes precisely from the fact that we dispose of micro-level data. The approach based on aggregate data is likely to bias upwards the price-elasticity estimates. Average prices are endogenous, since the weights associated to each price or, equivalently, to each fare class, is fully driven by the demand. Basically, trains in high demand are likely to have a few number of seats available at a low price, resulting in a higher average price. To illustrate this point, we aggregate our micro data and estimate the corresponding price elasticities. For instance, we propose to aggregate data over fare classes at the train level, and thus to consider an average price for every train. Then we regress the logarithm of total purchases on the logarithm of this average price.
We first aggregate the data at the train and destination level. Let $Q_{dT}$ be the total quantity of tickets purchased for destination $d$ in the train $T$, $Q_{dT} = \sum_{k=1}^{K} n_{dkT}$. The corresponding average price $\overline{p}_{dT}$ is given by:

$$\overline{p}_{dT} = \frac{\sum_{k=1}^{K} n_{dkT}p_{dkT}}{\sum_{k=1}^{K} n_{dkT}}.$$

We then consider a constant elasticity demand model with train fixed effects:

$$\ln(Q_{dT}) = -\epsilon \ln(\overline{p}_T) + \delta_T + \xi_d + \nu_{dT},$$

where $\xi_d$ accounts for a destination-specific component.

We then aggregate further our data at the train level, by considering $Q_T = Q_{aT} + Q_{bT}$ and defining the corresponding average price:

$$\overline{p}_T = \frac{\sum_{d \in \{a, b\}} \sum_{k=1}^{K} n_{dk}p_{dkT}}{\sum_{d \in \{a, b\}} \sum_{k=1}^{K} n_{dk}}.$$

We consider a similar model as (19), except that at that level of aggregation, we cannot include train and destination fixed effects. Instead, we include day of departure and route fixed effects:

$$\ln(Q_T) = -\epsilon \ln(\overline{p}_T) + \delta_t(T) + \xi_r(T) + \nu_T,$$

where $t(T)$ and $r(T)$ denote the day of departure and the route of train $T$. Finally, the most aggregated approach consists in aggregating these demands at a weekly or monthly level, either by train route or at the national level.

Results are given in Table 10. The first line presents the price elasticity estimate for the less disaggregated specification. Strikingly, the estimate ($-1.02$) is already much higher than ours. It is close to the estimate of $-0.70$ obtained by Sauvant (2002) on SNCF aggregated data. By aggregating further at the train level, we exacerbate the bias and obtain already a positive coefficient ($0.15$). Aggregating further at the week or at the month level increases further the coefficient, up to $1.14$. Using data aggregated at the national level leads to somewhat lower coefficients, but still positive ones ($0.14$ and $0.56$ for weekly and monthly data, respectively).
Table 10: Estimated price elasticities with aggregated data

| Model                                      | Price elasticity |
|--------------------------------------------|------------------|
| Train and destination level (Equation (19))| -1.02 (0.24)     |
| Train level (Equation (20))               | 0.15 (0.03)      |
| Week \times route level                   | 0.29 (0.12)      |
| Month \times route level                  | 1.14 (0.40)      |
| Week level (France)                       | 0.14 (0.09)      |
| Month (France)                            | 0.56 (0.33)      |

Notes: We refer to the text for a detailed explanation of each model.

D Bounding counterfactual revenues with time-varying elasticity

In this appendix, we give details on the partial identification of the lower bound in (11) with a time-varying elasticity, namely when the intensity of $V_{dT}$ satisfies (14). We first show how to partially identify $g_0$ in this context and then turn to counterfactual revenues.

Demand estimation: $g_0(.)$. To partially identify $g_0(.)$, we still rely on consumers’ rationality and weak optimality condition described in Section 4.2. Because $\varepsilon_{\text{early}} \neq \varepsilon_{\text{late}}$, the resulting moment inequalities differ from those under the assumption of constant price elasticity. The inequalities originating from consumers’ rationality are modified as follows. First, for $k \leq S$ (recalling that $t_S$ is a function of $W_T$),

$$E \left[ \sum_{j=K}^{n_{dT}} n_{dT} - C_T \wedge D \left( \exp(X_{dT} \beta_0) \eta_{dT} \left[ p_{dT}^{-\varepsilon_{\text{early}}} g_{0,\text{early}}(W_T) + p_{d(S+1)T}^{-\varepsilon_{\text{late}}} g_{0,\text{late}}(W_T) \right] \right) \bigg| W_T \right] \leq 0, \quad (21)$$

and for $k > S$,

$$E \left[ \sum_{j=K}^{n_{dT}} n_{dT} - C_T \wedge D \left( \exp(X_{dT} \beta_0) \eta_{dT} p_{dT}^{-\varepsilon_{\text{late}}} \left[ g_{0,\text{early}}(W_T) + g_{0,\text{late}}(W_T) \right] \right) \bigg| W_T \right] \leq 0, \quad (22)$$

52
where \( g_0^{early}(W_T) = g_0(W_T) \int_0^{t^s} b_T(t)dt \) and \( g_0^{late}(W_T) = g_0(W_T) \int_{t^s}^1 b_T(t)dt \). Note that the left-hand side of (21) is strictly decreasing in \( \bar{g}(W_t) := g_0^{early}(W_T) + p_{dST} g_0^{late}(W_T) \). Then, given \( d, k \leq S \), and \( W_T \), we obtain a lower bound on \( \bar{g}(W_t) \).

Similarly, we obtain a lower bound on \( g_0^{early}(W_T) + g_0^{late}(W_T) \) from (22) given \( d, k \leq S \), and \( W_T \). As a result, for a given \( W_T \), we obtain 24 linear constraints on \((g_0^{early}(W_T), g_0^{late}(W_T))\).

Turning to the weak optimality condition, we have

\[
\mathbb{E}[R_T(p_a, p_b)|W_T] = \max_{C_{aT}+C_{bT}=C_T} \left\{ \sum_{d=a,b} p_d \int_0^\infty \mathbb{E} \left[ D \left( \exp \{ X_{dT}^\prime \beta_0 \} \right) \left( p_d^{-\varepsilon_{early}} g_0^{early}(W_T) \right. \right. \\
\left. \left. + p_d^{-\varepsilon_{late}} g_0^{late}(W_T) \right) z \right] \wedge C_{dT} \right\} g_{\lambda_{d0,1}}(z) dz \}
\]

Let \( R(g_0^{early}(W_T), g_0^{late}(W_T); \varepsilon_{early}, \varepsilon_{late}; \beta_0, \lambda_0) := \max_{k=1,\ldots,K} \mathbb{E}[R_T(p_{dkT}, p_{bkT})|W_T] \). Then, we obtain:

\[
R(g_0^{early}(W_T), g_0^{late}(W_T); \varepsilon_{early}, \varepsilon_{late}; \beta_0, \lambda_0) \leq \mathbb{E}\left[R_T^{obs}|W_t\right]. \quad (23)
\]

Unlike the moment inequalities built on the consumers’ rationality, the weak optimality condition does not deliver a linear constraint on \((g_0^{early}(W_T), g_0^{late}(W_T))\) because of the maximization over \( k \). The estimation of the identified set of the counterfactual revenue is complicated by this non-linearity. To circumvent this numerical challenge, we exploit a looser weak optimality inequality, namely

\[
R(g_0^{early}, g_0^{late}; \varepsilon_{early}, \varepsilon_{late}; \beta_0, \lambda_0) \geq R(g_0^{early}, g_0^{late}; \varepsilon_{early}, \varepsilon_{early}, \beta_0, \lambda_0).
\]

This inequality holds because \( \varepsilon_{early} < \varepsilon_{late} \) and \( p_{dkT} \geq 1 \) for all \( T, d \in \{a, b\} \) and \( k \in \{1, \ldots, 12\} \). Then, (23) implies

\[
R(g_0^{early}(W_T), g_0^{late}(W_T); \varepsilon_{early}, \varepsilon_{early}, \beta_0, \lambda_0) \leq \mathbb{E}\left[R_T^{obs}|W_t\right].
\]

In other words, the actual revenue management yields a higher revenue than the optimal uniform pricing strategy with a price chosen from the price grid and early purchasers’ price elasticity. This looser inequality leads to an upper bound on \( g_0^{early}(W_T) + g_0^{late}(W_T) \) and delivers a linear constraint.

To summarize, for a given \( W_T \), consumers’ rationality and the loosened version of the weak optimality condition imply linear constraints on \((g_0^{early}(W_T), g_0^{late}(W_T))\). The computation of the estimator of the identified set of counterfactual revenues then reduces to an optimization problem with linear constraints.
**Counterfactual revenues, uniform pricing.** We give details for the optimal uniform pricing under complete information. The derivation for the case under limited information is similar. Given \( W_T \) and capacity \( C_{dT} \) for destination \( d \), the expected revenue \( R_{udT}^c \) of the optimal uniform pricing strategy under complete information for destination \( d \) in train \( T \) is

\[
R_{udT}^c(\varepsilon_{\text{early}}, \varepsilon_{\text{late}}, g_0^\text{early}, g_0^\text{late}, C_{dT}) = \int_0^\infty \max_{p_d > 0} \left\{ p_d \mathbb{E}\left[ D\left( \exp\{X_d' \beta_0\} \left( p_d^{-\varepsilon_{\text{early}}} g_0^\text{early}(W_T) + p_d^{-\varepsilon_{\text{late}}} g_0^\text{late}(W_T) \right) \right) \right] \right\} g_{\lambda_{d_0}}(w) dw,
\]

and the expected optimal revenue at the train level is:

\[
R_{uT}^c(\varepsilon_{\text{early}}, \varepsilon_{\text{late}}, g_0^\text{early}, g_0^\text{late}, C_T) = \max_{C_{aT} + C_{bT} = C_T} \sum_{d=a,b} R_{udT}^c(\varepsilon_{\text{early}}, \varepsilon_{\text{late}}, g_0^\text{early}, g_0^\text{late}, C_{dT}).
\]

Denote by \( \mathcal{G} \) the set of linear constraints on \((g_0^\text{early}, g_0^\text{late})\) derived in the previous section. Then, the lower bound of the set estimate of \( R_{uT}^c(\varepsilon_{\text{early}}, \varepsilon_{\text{late}}, g_0^\text{early}, g_0^\text{late}, C_T) \) is expressed as:

\[
R_{uT}^c(\varepsilon_{\text{early}}, \varepsilon_{\text{late}}, C_T) = \min_{(g_0^\text{early}, g_0^\text{late}) \in \mathcal{G}} \max_{C_{aT} + C_{bT} = C_T} \sum_{d=a,b} R_{udT}^c(\varepsilon_{\text{early}}, \varepsilon_{\text{late}}, g_0^\text{early}, g_0^\text{late}, C_{dT}).
\]

Solving this program exactly is difficult. Rather, we obtain lower and upper bounds on \( R_{uT}^c(\varepsilon_{\text{early}}, \varepsilon_{\text{late}}, C_T) \). First, note that using Jensen’s inequality, we have

\[
p_d^{-\varepsilon_{\text{early}}} g_0^\text{early} + p_d^{-\varepsilon_{\text{late}}} g_0^\text{late} \geq (g_0^\text{early} + g_0^\text{late}) p_d^{-\varepsilon(g_0^\text{early}, g_0^\text{late})},
\]

where \( \varepsilon(g_0^\text{early}, g_0^\text{late}) = \frac{g_0^\text{early} + g_0^\text{late}}{g_0^\text{early} + g_0^\text{late}} \). Then,

\[
R_{udT}^c(\varepsilon_{\text{early}}, \varepsilon_{\text{late}}, g_0^\text{early}, g_0^\text{late}, C_{dT}) \geq R_{udT}^c(\varepsilon(g_0^\text{early}, g_0^\text{late}), \varepsilon(g_0^\text{early}, g_0^\text{late}), g_0^\text{early}, g_0^\text{late}, C_{dT}),
\]

with \( R_{udT}^c(\varepsilon(g_0^\text{early}, g_0^\text{late}), \varepsilon(g_0^\text{early}, g_0^\text{late}), g_0^\text{early}, g_0^\text{late}, C_{dT}) \) the expected revenue of the optimal uniform pricing under constant price elasticity \( \varepsilon(g_0^\text{early}, g_0^\text{late}) \). Then,

\[
R_{uT}^c(\varepsilon_{\text{early}}, \varepsilon_{\text{late}}, C_T) \geq \min_{(g_0^\text{early}, g_0^\text{late}) \in \mathcal{G}} \max_{C_{aT} + C_{bT} = C_T} \sum_{d=a,b} R_{udT}^c(\varepsilon(g_0^\text{early}, g_0^\text{late}), \varepsilon(g_0^\text{early}, g_0^\text{late}), g_0^\text{early}, g_0^\text{late}, C_{dT}).
\]

Note that \( R_{udT}^c(\varepsilon(g_0^\text{early}, g_0^\text{late}), \varepsilon(g_0^\text{early}, g_0^\text{late}), g_0^\text{early}, g_0^\text{late}, C_{dT}) \) is a function of \( g_0^\text{early} + g_0^\text{late} \) and \( \varepsilon(g_0^\text{early}, g_0^\text{late}) \). Moreover, for \( d = a, b \) and any \( C_{dT} \), it is increasing with respect to \( g_0^\text{early} + g_0^\text{late} \) and decreasing with respect to \( \varepsilon(g_0^\text{early}, g_0^\text{late}) \). Then

\[
\max_{C_{aT} + C_{bT} = C_T} \sum_{d=a,b} R_{udT}^c(\varepsilon(g_0^\text{early}, g_0^\text{late}), \varepsilon(g_0^\text{early}, g_0^\text{late}), g_0^\text{early}, g_0^\text{late}, C_{dT})
\]

54
is also increasing with respect to \(g_0^{\text{early}} + g_0^{\text{late}}\) and decreasing with respect to \(\varepsilon(g_0^{\text{early}}, g_0^{\text{late}})\). As a result, its minimization with respect to \((g_0^{\text{early}}, g_0^{\text{late}}) \in \mathcal{G}\) is achieved either when \(g_0^{\text{early}} + g_0^{\text{late}}\) is minimized or \(\varepsilon(g_0^{\text{early}}, g_0^{\text{late}})\) is maximized. Denote the minimizer of \(g_0^{\text{early}} + g_0^{\text{late}}\) by \(g_0^*\) and the maximizer of \(\varepsilon(g_0^{\text{early}}, g_0^{\text{late}})\) by \(\bar{\varepsilon}\). Then,

\[
\begin{align*}
\min_{(g_0^{\text{early}}, g_0^{\text{late}}) \in \mathcal{G}} \max_{g \in \mathcal{G}} & \sum_{d=a,b} R_{udT}(\varepsilon(g_0^{\text{early}}, g_0^{\text{late}}), \varepsilon(g_0^{\text{early}}, g_0^{\text{late}}); g_0^{\text{early}}, g_0^{\text{late}}, C_{dT}) \\
= \min \left\{ \varepsilon(g_0^{\text{early}}, g_0^{\text{late}}) \in \mathcal{G} : \min_{g \in \mathcal{G}} \max_{g \in \mathcal{G}} \sum_{d=a,b} R_{udT}(\varepsilon(g_0^{\text{early}}, g_0^{\text{late}}), \varepsilon(g_0^{\text{early}}, g_0^{\text{late}}); g_0^{\text{early}}, g_0^{\text{late}}, C_{dT}) \right\}.
\end{align*}
\]

To perform the first minimization, it suffices to compute the upper bound of \(\varepsilon(g_0^{\text{early}}, g_0^{\text{late}})\) in \(\mathcal{G}\) subject to \(g_0^{\text{early}} + g_0^{\text{late}} = g_0^*\), and compute the corresponding optimal revenue. Similarly, to perform the second minimization, it suffices to compute the lower bound of \(g_0^{\text{early}} + g_0^{\text{late}}\) in \(\mathcal{G}\) subject to \(\varepsilon(g_0^{\text{early}}, g_0^{\text{late}}) = \bar{\varepsilon}\), and compute the corresponding optimal revenue with constant price elasticity \(\bar{\varepsilon}\). The minimum of the two simulated revenues then bounds \(R_{udT}^c(\varepsilon_{\text{early}}, \varepsilon_{\text{late}}, C_T)\) from below. Second, note that

\[
R_{udT}^c(\varepsilon_{\text{early}}, \varepsilon_{\text{late}}, g_0^{\text{early}}, C_{dT}) \leq \min_{(g_0^{\text{early}}, g_0^{\text{late}}) \in \mathcal{G}} \max_{g_0^{\text{early}}, g_0^{\text{late}}} \sum_{d=a,b} R_{udT}(\varepsilon_{\text{early}}, \varepsilon_{\text{late}}, g_0^{\text{early}}, g_0^{\text{late}}, C_{dT})
\]

The revenue \(R_{udT}^c(\varepsilon_{\text{early}}, \varepsilon_{\text{late}}, g_0^{\text{early}}, 0, C_{dT})\) is the expected revenue of the optimal uniform pricing under constant price elasticity \(\varepsilon_{\text{early}}\) and only depends on \(g_0^{\text{early}}\). Then it suffices to compute the lower bound of \(g_0^{\text{early}}\) in \(\mathcal{G}\) \(\cap \{ (g_0^{\text{early}}, 0) : g_0^{\text{early}} \in \mathbb{R}^+ \}\) and simulate the corresponding optimal revenue with constant price elasticity \(\varepsilon_{\text{early}}\) to bound \(R_{udT}^c(\varepsilon_{\text{early}}, \varepsilon_{\text{late}}, C_T)\) from above.

**Counterfactual revenues, dynamic pricing.** Given an allocation \((C_{dT}, C_{bT})\) between destinations \(a\) and \(b\) served by train \(T\), we can consider the optimal dynamic pricing (stopping-time or the full one) separably for either destination. Take the full dynamic pricing under complete information as example. Using the notations in Appendix B, the Bellman equation is written as: for \(t < t_S\),

\[
V^*_k(t) = \partial_t B_T(1 - t, 1) \frac{\varepsilon_T}{\varepsilon_{\text{late}} + 1} \varepsilon_{\text{late}} \left[ V^*_k(t) - V^*_{k-1}(t) \right]^{1 - \varepsilon_{\text{late}}}, \text{ with } V^*_k(0) = 0 \forall k \geq 0, \quad (24)
\]
and for $t \geq t_S$,

$$V_{k'}^*(t) = \partial_1 B_T(1 - t, 1) \frac{\xi_{dT}}{\varepsilon_{\text{early}} - 1} \left(1 - \frac{1}{\varepsilon_{\text{early}}}\right)^{\varepsilon_{\text{early}}} \left[V_k^*(t) - V_{k-1}^*(t)\right]^{1-\varepsilon_{\text{early}}}, \quad (25)$$

where the initial conditions $V_k^*(t_S)$ are defined by (24). We are interested in $V_k^*(1)$. If the elasticity in (24) were $\varepsilon_{\text{early}}$, we would then obtain the same explicit solution in Appendix B. However, because $\varepsilon_{\text{late}} < \varepsilon_{\text{early}}$, the initial conditions for (25) at $t = t_S$ are no longer the same. Such jumps in the initial conditions introduce substantial challenges in identifying and deriving explicit formula for $V_k^*(1)$.

We propose lower and upper bounds for $V_k^*(1)$. To obtain the lower bound, we consider a preallocation of $C_{dT}$ into $C_{dT\text{early}}$ and $C_{dT\text{late}}$, each corresponding to the capacity during early and late periods for destination $d$, respectively. Any seat allocated to the early period cannot be re-used during the late one. Intuitively, this preallocation is suboptimal because any unsold seats during the early period could have been sold later. Therefore, the optimal dynamic pricing with such preallocation constraints provides a lower bound for the unconstrained one. Technically, this constrained dynamic pricing corresponds to setting $V_k^*(t_S) = 0$ in (25) for any $k > 0$. Then, we can derive the revenues before and after $t_S$ by using the formulas in Appendix B, and explicitly express the total revenue as a nonlinear function of $(g_{\text{early}}^0, g_{\text{late}}^0)$.

Denote by $V_k^*(t; \varepsilon_{\text{early}})$ the value function corresponding to the Bellman equation with constant elasticity $\varepsilon_{\text{early}}$ for $t \in [0, 1]$. To obtain the upper bound, we use the following property:

$$V_k^*(t) - V_{k-1}^*(t) > V_k^*(t; \varepsilon_{\text{early}}) - V_{k-1}^*(t; \varepsilon_{\text{early}}) \quad \text{for} \quad t > t_S,$$

i.e., the opportunity cost of selling a seat during the early period is greater when consumers in the late period are less price elastic. Using this property and formulas in Appendix B, we have:

$$V_k^*(1) - V_k^*(t_S) < V_k^*(1; \varepsilon_{\text{early}}) - V_k^*(t_S; \varepsilon_{\text{early}}) = \alpha_{k,f} \xi_{dT}^{1/\varepsilon_{\text{early}}} [(g_{\text{early}}^0 + g_{\text{late}}^0)^{1/\varepsilon_{\text{early}}} - (g_{\text{early}}^0)^{1/\varepsilon_{\text{early}}}]$$

Then, using (24), we obtain the upper bound for $V_k^*(1)$:

$$V_k^*(1) < \alpha_{k,f} \xi_{dT}^{1/\varepsilon_{\text{early}}} [(g_{\text{early}}^0 + g_{\text{late}}^0)^{1/\varepsilon_{\text{early}}} - (g_{\text{early}}^0)^{1/\varepsilon_{\text{early}}}] + \xi_{dT}^{1/\varepsilon_{\text{late}}} (g_{\text{late}}^0)^{1/\varepsilon_{\text{late}}}.$$

We can then minimize the lower/upper bound with respect to $(g_{\text{early}}^0, g_{\text{late}}^0)$ to obtain the lower/upper bound for the lower bound of the counterfactual revenue.
E  Proofs

E.1  Theorem 4.1

Hereafter, $\ell$ denotes the dimension of $X_{dT}$, $\text{Supp}(A)$ denotes the support of any random variable $A$. and $\Delta X_T := X_{bT} - X_{aT}$. We first state Assumption 5 appearing in the statement of the theorem.

**Assumption 5**  There exists a component $X_{djT}$ of $X_{dT}$ such that $\text{Supp}(\Delta X_{jT} | \Delta X_{-jT}) = \mathbb{R}$, where $X_{d-jT}$ is the vector stacking all the components of $X_{dT}$ except $X_{djT}$. Also, there exists $x_{-j}^1, x_{-j}^2, ..., x_{-j}^\ell \in \text{Supp}(X_{b-jT} - X_{a-jT})$ such that the matrix

$$M := \begin{pmatrix} x_{-j}^1' - x_{-j}^2' \\ \vdots \\ x_{-j}^1' - x_{-j}^\ell' \end{pmatrix}$$

is nonsingular. Finally, $\sup \text{Supp} \left( \sum_{k=1}^{K} n_{akT} + n_{bkT} \right) \geq K + 1$.

The most restrictive condition is $\text{Supp}(\Delta X_{jT} | \Delta X_{-jT}) = \mathbb{R}$. However, the proof below reveals that identification is not achieved at infinity. The large support condition simply ensures that we can produce the compensating variations used in the proof. Also, the nonparametric identification of the distribution of $\eta_{bT}/\eta_{aT}$ can be obtained without large support, using the analyticity of the density of the logistic distribution. The last condition ($\sup \text{Supp} \left( \sum_{k=1}^{K} n_{akT} + n_{bkT} \right) \geq K + 1$) easily holds in our application. It shows that our result does not require a large number of units of the perishable good (namely, tickets for a given train in our application) to be applicable.

1. First, note that because the realization of $\tau_k$ is determined by the Poisson process before $\tau_k$ and is independent of $D_{dT}(\tau_k, \tau_{k+1}; p_{dT})$ for $d \in \{a, b\}$, it suffices to show (1) if $\tau_k$ is replaced by any fixed number that we suppose equal to 0 without loss of generality. To ease the exposition, we often omit the index $T$ hereafter and define $\mu_d = \xi_d p_{dk}^{e}$ and $\rho = \mu_a/ (\mu_a + \mu_b)$. We also introduce $D_{d, \tau_n} = D_{dT}(0, \tau_n; p_{dT})$ for $d \in \{a, b\}$, $D_{\tau_n} = D_{a, \tau_n} + D_{b, \tau_n}$ and $\tau_n = \inf \{t > 0 : D_t \geq n \} \land 1$. We will show that for all $n \geq 1$,

$$D_{a, \tau_n} | D_{\tau_n}, b_T(\cdot), \xi_a, \xi_b \sim \text{Binomial} \left( D_{\tau_n}, \rho \right).$$

(26)
Given the previous discussion and because the right-hand side of (26) does not depend on \( b_T(\cdot) \), (1) will follow from (26).

To prove (26), we introduce, for any \( n \geq 1 \), the hitting times \( \sigma_n = \inf\{ t \in [0, 1] : D_t \geq n \} \), with \( \sigma_n = 2 \) if \( D_1 < n \). Let us also fix \( t \in (0, 1) \) and let us partition the interval \( I = [t, 1] \) into \( m \) intervals \( I_1, \ldots, I_m \) of equal length \( \Delta t = (1 - t)/m \). Finally, for all \( c \leq n \), let
\[
q_{c,n;k} = \Pr[D_{a,\sigma_n} = c | D_{\sigma_n} = n, \sigma_n \in I_k].
\]

By Lemma E.1, there exists \((c_l, c_r)\), independent of \( k \) and \( m \), such that for all \( k = 1, \ldots, m \),
\[
-c_l(1 + n)\Delta t \leq q_{c,n;k} - \left(\frac{n}{c}\right)\rho^c(1 - \rho)^{n-c} \leq c_r\Delta t.
\]

Moreover, we have
\[
\Pr[D_{a,\sigma_n} = c | D_{\sigma_n} = n, \sigma_n \in I] = \frac{\sum_{k=1}^m \Pr[D_{a,\sigma_n} = c, D_{\sigma_n} = n, \sigma_n \in I_k]}{\sum_{k=1}^m \Pr[D_{\sigma_n} = n, \sigma_n \in I_k]} 
\in \left[ \min_{k=1,\ldots,m} q_{c,n;k}, \max_{k=1,\ldots,m} q_{c,n;k} \right].
\]

Consequently,
\[
-c_l(1 + n)\Delta t \leq \Pr[D_{a,\sigma_n} = c | D_{\sigma_n} = n, \sigma_n \in I] - \left(\frac{n}{c}\right)\rho^c(1 - \rho)^{n-c} \leq c_r\Delta t.
\]

By letting \( m \to \infty \) and then let \( t \to 0 \), we obtain
\[
\Pr[D_{a,\sigma_n} = c | D_{\sigma_n} = n, \sigma_n \leq 1] = \left(\frac{n}{c}\right)\rho^c(1 - \rho)^{n-c}.
\]

Now, because \( D_{\tau_n} = n \) if and only if \( \sigma_n \leq 1 \), we obtain (26) in this case. Further, because \( D_{\tau_n} = n' < n \) if and only if \( D_1 = n' \) and \( \sigma_n = 2 \), we have
\[
\Pr[D_{a,\tau_n} = c | D_{\tau_n} = n'] = \Pr[D_{a,1} = c | D_1 = n', \sigma_n = 2] 
= \Pr[D_{a,1} = c | D_1 = n'] 
= \left(\frac{n'}{c}\right)\rho^c(1 - \rho)^{n'-c}.
\]

Thus, (26) also holds when \( D_{\tau_n} = n', n' < n \). The result follows.

2. Consider two fare classes \( k, k' \) such that \( p_{ak}/p_{bk} \neq p_{ak'}/p_{bk'} \) (hereafter, we implicitly reason conditional on prices). Fix \( x \in \mathbb{R} \) and let
\[
\bar{x} = x - \frac{\varepsilon}{\beta_{0j}} \ln \left(\frac{p_{ak} p_{bk'}}{p_{bk} p_{ak'}}\right),
\]

58
Then, \( x\beta_{0j} - \varepsilon \ln(p_{ak}/p_{bk}) = \bar{x}\beta_{0j} - \varepsilon \ln(p_{ak}/p_{bk}) \). In turn, given the index structure,

\[
\Pr(n_{bkT} = n_{|n_{akT} + n_{bkT} = n, \Delta X_{-jT} = x})
= \Pr(n_{bk'T} = n_{|n_{ak'T} + n_{bk'T} = n, \Delta X_{-jT} = x})
\]

Conversely, there is a single solution \( \bar{x} \) to this equation, given by (29). Hence, \( \bar{x} \) and thus \( \beta_{0j} \) are identified. Similarly, for any two \( x_{-j} \neq \bar{x}_{-j} \) in the support of \( \Delta X_{-jT} \),

\[
\Pr(n_{akT} = n_{|n_{akT} + n_{bkT} = n, \Delta X_{-jT} = x_{-j}, \Delta X_{jT} = x})
= \Pr(n_{akT} = n_{|n_{akT} + n_{bkT} = n, \Delta X_{-jT} = \bar{x}_{-j}, \Delta X_{jT} = \bar{x})
\]

if and only if \( x\beta_{0j} + x'_{-j}\beta_{0-j} = \bar{x}\beta_{0j} + \bar{x}'_{-j}\beta_{0-j} \). By considering \( x^1_{-j}, \ldots, x^\ell_{-j} \) as in Assumption 5, we obtain \( M\beta_{0j} = y \) for some identified vector \( y \). Since \( M \) is nonsingular, \( \beta_{0-j} \) is identified.

We now show the nonparametric identification of the cumulative distribution function (cdf) \( F \) of \( \ln(\eta_{aT}/\eta_{bT}) \). Since \( \sup \text{Supp} \left( \sum_{k=1}^{K} n_{akT} + n_{bkT} \right) \geq K + 1 \), there exists a fare class \( k \) for which \( \sup \text{Supp}(n_{akT} + n_{bkT}) \geq 2 \). Fix \( n \geq 2 \). The distribution of \( n_{bkT}|n_{akT} + n_{bkT} = n, \Delta X_{jT} = x \) is a binomial mixture, with mixture distribution \( G_x \), say. Then (see, e.g. D’Haultfœuille and Rathelot, 2017), the first \( n \) moments of \( G_x \) are identified. In particular, we identify \( \int_0^1 p(1-p)dG_x(p) \). Now, given the structure of the problem,

\[
\int_0^1 p(1-p)dG_x(p) = \int \Lambda'(x\beta_0 - u)dF(u),
\]

with \( \Lambda' = \Lambda(1 - \Lambda) \) the density of the logistic distribution. By varying \( x_j \) over \( \mathbb{R} \), we thus identify the distribution of \( U + V \), where \( U \) and \( V \) are independent, \( U \) is logistic and \( V \sim F \). Taking the Fourier transform, we thus identify \( \Psi_U \times \Psi_V \), where \( \Psi_U \) and \( \Psi_V \) (resp. \( \Psi_V \)) denotes the characteristic function of \( U \) (resp. \( V \)). Since \( \Psi_U(t) = \pi t/\sinh(\pi t) \neq 0 \), \( \Psi_V \) is identified. Hence, \( F \) is identified as well.

Finally, under Assumption 3(ii), \( \eta_{aT}/\eta_{aT} \) follows a beta prime distribution with parameters \((\lambda_{b0}, \lambda_{a0})\). Because this beta prime distribution is identified by what precedes, so are \((\lambda_{b0}, \lambda_{a0})\).

**A key lemma**  The proof of Point 1 crucially relies on the following lemma, which we prove below (the notation we use are introduced in the proof of Theorem 4.1).
Lemma E.1 Suppose that Assumption 1 holds. Then, there exists \( c_l \) and \( c_r \), independent of \( k \) and \( m \), such that for all \( k = 1, \ldots, m \),

\[
-c_l(1+n)\Delta t \leq q_{c,n;k} - \binom{n}{c} \rho^c(1-\rho)^{n-c} \leq c_r \Delta t.
\]

Proof: First, observe that \( \{\sigma_n \in I_k\} = \{D_{L+(k-1)\Delta t} < n, \ D_{L+k\Delta t} \geq n\} \). Then

\[
\Pr[D_{a,\sigma_n} = c, D_{\sigma_n} = n, \sigma_n \in I_k] = \Pr[D_{a,\sigma_n} = c, D_{\sigma_n} = n, D_{L+(k-1)\Delta t} < n, \ D_{L+k\Delta t} \geq n]
\]

\[
= \Pr[D_{a,\sigma_n} = c, D_{\sigma_n} = n, D_{L+(k-1)\Delta t} = n-1, \ D_{L+k\Delta t} \geq n]
\]

\[
+ \Pr[D_{a,\sigma_n} = c, D_{\sigma_n} = n, D_{L+(k-1)\Delta t} < n-1, \ D_{L+k\Delta t} \geq n]
\]

(31)

We first show that the second term in (31) is negligible, as being of order \((\Delta t)^2\). Simple algebra shows that if \( U \sim \mathcal{P}(\mu) \), then \( \Pr(U \geq 2) \leq \mu^2 \). Hence,

\[
\Pr[D_{a,\sigma_n} = c, D_{\sigma_n} = n, D_{L+(k-1)\Delta t} < n-1, \ D_{L+k\Delta t} \geq n]
\]

\[
\leq \Pr[D_{L+k\Delta t} - D_{L+(k-1)\Delta t} \geq 2]
\]

\[
\leq \left[ (\mu_a + \mu_b) \int_{L+(k-1)\Delta t}^{L+k\Delta t} b_s ds \right]^2
\]

\[
\leq \left[ (\mu_a + \mu_b) \bar{b} \Delta t \right]^2,
\]

where \( \bar{b} = \sup_{t \in [0,1]} b(t) \). Now, the first term in (31) satisfies:

\[
\Pr[D_{a,\sigma_n} = c, D_{\sigma_n} = n, D_{L+(k-1)\Delta t} = n-1, \ D_{L+k\Delta t} \geq n]
\]

\[
= \Pr[D_{a,\sigma_n} = n, D_{\sigma_n} = n, D_{L+(k-1)\Delta t} = n-1, \ D_{L+k\Delta t} = n]
\]

\[
+ \Pr[D_{a,\sigma_n} = c, D_{\sigma_n} = n, D_{L+(k-1)\Delta t} = n-1, \ D_{L+k\Delta t} > n],
\]

where the second term can be similarly controlled as above:

\[
\Pr[D_{a,\sigma_n} = c, D_{\sigma_n} = n, D_{L+(k-1)\Delta t} = n-1, \ D_{L+k\Delta t} > n] \leq \Pr[D_{L+k\Delta t} - D_{L+(k-1)\Delta t} \geq 2]
\]

\[
\leq \left[ (\mu_a + \mu_b) \bar{b} \Delta t \right]^2.
\]

As a consequence,

\[
\Pr[D_{a,\sigma_n} = c, D_{\sigma_n} = n, D_{L+(k-1)\Delta t} = n-1, \ D_{L+k\Delta t} = n]
\]

\[
< \Pr[D_{a,\sigma_n} = c, D_{\sigma_n} = n, \sigma_n \in I_k]
\]

\[
\leq \Pr[D_{a,\sigma_n} = c, D_{\sigma_n} = n, D_{L+(k-1)\Delta t} = n-1, \ D_{L+k\Delta t} = n]
\]

\[
+ 2[ (\mu_a + \mu_b) \bar{b} ]^2 (\Delta t)^2.
\]

(32)
Now, we have

\[
\Pr[D_{a,\sigma} = c, D_{\sigma} = n, D_{t+\Delta t} = n - 1, D_{t+k\Delta t} = n]
= \Pr[D_{a,\sigma} = c, D_{t+k\Delta t} = n, D_{t+(k-1)\Delta t} = n - 1]
= \Pr[D_{a,\sigma} = c, D_{t+k\Delta t} = n, D_{t+(k-1)\Delta t} = n - 1, D_{a,t+\Delta t} = c] + \Pr[D_{a,\sigma} = c, D_{t+k\Delta t} = n, D_{t+(k-1)\Delta t} = n - 1, D_{a,t+\Delta t} = c - 1]
= \Pr[D_{a,\sigma} = c, D_{t+k\Delta t} = n, D_{t+(k-1)\Delta t} = n - 1, D_{a,t+\Delta t} = c - 1] + \Pr[D_{a,\sigma} = c, D_{t+k\Delta t} = n, D_{t+(k-1)\Delta t} = n - 1, D_{a,t+\Delta t} = c]. \quad (33)
\]

Now, by independence between \((D_{a,t})_{t \geq 0}\) and \((D_{b,t})_{t \geq 0}\), and independence between \(D_{d,t+s} - D_{d,t}\) and \(D_{d,t}\) for all \(s > 0\) and \(d \in \{a, b\}\),

\[
\Pr[D_{a,t+k\Delta t} = c, D_{t+k\Delta t} = n, D_{t+(k-1)\Delta t} = n - 1, D_{a,t+\Delta t} = c - 1] = \Pr[D_{a,t+(k-1)\Delta t} = c - 1] \Pr[D_{a,t+k\Delta t} = c | D_{a,t+(k-1)\Delta t} = c - 1] \times \Pr[D_{b,t+k\Delta t} = n - c | D_{b,t+(k-1)\Delta t} = n - c]
= \frac{\mu_a^c \mu_b^{n-c} (\int_0^{t+k\Delta t} b_s ds)^{n-1}}{(n-c)!(c-1)!} \exp \left\{ - (\mu_a + \mu_b) \int_0^{t+k\Delta t} b_s ds \right\} \int_{t+(k-1)\Delta t}^{t+k\Delta t} b_s ds.
\]

Similarly,

\[
\Pr[D_{a,t+k\Delta t} = c, D_{t+k\Delta t} = n, D_{t+(k-1)\Delta t} = n - 1, D_{a,t+\Delta t} = c] = \frac{\mu_a^c \mu_b^{n-c} (\int_0^{t+k\Delta t} b_s ds)^{n-1}}{(n-c)!c!} \exp \left\{ - (\mu_a + \mu_b) \int_0^{t+k\Delta t} b_s ds \right\} \int_{t+(k-1)\Delta t}^{t+k\Delta t} b_s ds.
\]

By plugging the last two equalities into (33), we obtain

\[
\Pr[D_{a,\sigma} = c, D_{t+k\Delta t} = n, D_{t+(k-1)\Delta t} = n - 1] = \frac{n \mu_a^c \mu_b^{n-c} (\int_0^{t+k\Delta t} b_s ds)^{n-1}}{(n-c)!c!} \exp \left\{ - (\mu_a + \mu_b) \int_0^{t+k\Delta t} b_s ds \right\} \left( \mu_a + \mu_b \right) \int_{t+(k-1)\Delta t}^{t+k\Delta t} b_s ds.
\]
By combining (34), (35), and (27), we obtain the following inequalities:

\[\frac{n\mu^c_b n^{-c} (\int_0^{t+(k-1)\Delta t} b_s ds)^{n-1}}{(n-c)!c!} \exp \{-(\mu_a + \mu_b) \int_0^{t+k\Delta t} b_s ds\} (\mu_a + \mu_b) \int_0^{t+k\Delta t} b_s ds\]

\[< P(D_{a,n} = c, D_{\sigma_n} = n, \sigma_n \in I_k) \]

\[\leq \frac{n\mu^c_b n^{-c} (\int_0^{t+(k-1)\Delta t} b_s ds)^{n-1}}{(n-c)!c!} \exp \{-(\mu_a + \mu_b) \int_0^{t+k\Delta t} b_s ds\} (\mu_a + \mu_b) \int_0^{t+k\Delta t} b_s ds\]

\[+ 2[\mu_a + \mu_b \tilde{b}]^2 (\Delta t)^2. \] (34)

By summing (34) over \(c = 0, 1, \ldots, n\), we obtain

\[\frac{n(\mu_a + \mu_b)^n (\int_0^{t+(k-1)\Delta t} b_s ds)^{n-1}}{n!} \exp \{-(\mu_a + \mu_b) \int_0^{t+k\Delta t} b_s ds\} (\mu_a + \mu_b) \int_0^{t+k\Delta t} b_s ds\]

\[< P(D_{a,n} = n, \sigma_n \in I_k) \]

\[\leq \frac{n(\mu_a + \mu_b)^n (\int_0^{t+(k-1)\Delta t} b_s ds)^{n-1}}{n!} \exp \{-(\mu_a + \mu_b) \int_0^{t+k\Delta t} b_s ds\} (\mu_a + \mu_b) \int_0^{t+k\Delta t} b_s ds\]

\[+ 2(n+1)[(\mu_a + \mu_b \tilde{b})^2 (\Delta t)^2. \] (35)

By combining (34), (35), and (27), we obtain the following inequalities:

\[-c_{l,k}(1+n)(\Delta t)^2 \leq q_{c,n;k} - \binom{n}{c} \rho^c (1-\rho)^{n-c} \leq c_{r,k}(\Delta t)^2, \]

where

\[c_{r,k} = \frac{2(n+1)[(\mu_a + \mu_b \tilde{b})^2}{n(\mu_a + \mu_b)^n (\int_0^{t+(k-1)\Delta t} b_s ds)^{n-1}} \exp \{-(\mu_a + \mu_b) \int_0^{t+k\Delta t} b_s ds\} (\mu_a + \mu_b) \int_0^{t+k\Delta t} b_s ds\]

\[c_{l,k} = c_{r,k} \binom{n}{c} \rho^c (1-\rho)^{n-c}. \]

Finally, note that \(c_{r,k}(\Delta t) \leq c_r\) where

\[c_r = \frac{2(n+1)[(\mu_a + \mu_b \tilde{b})^2}{n(\mu_a + \mu_b)^n (\int_0^{t+(k-1)\Delta t} b_s ds)^{n-1}} \exp \{-(\mu_a + \mu_b) \int_0^{t+k\Delta t} b_s ds\} (\mu_a + \mu_b) \inf_{s \in I} b_s\]

Moreover, \(c_r\) does not depend on \(k\) and \(m\). Finally, defining \(c_l = c_r \binom{n}{c} \rho^c (1-\rho)^{n-c}, \)
\(c_l\) does not depend on \(k\) and \(m\) either, and (30) holds for all \(k = 1, \ldots, m\).
E.2 Theorem 4.2

The idea is to define a new time variable and new Poisson processes such that (i) $b_T(\cdot)$ does not appear anymore in this new set-up; (ii) the corresponding counterfactual revenues are equal to the initial ones. The result then follows. First, fix $(I, r) \in \{c, i\} \times \{u, f, s, sM, sM+\}$ and let $B_T(t) = \int_0^t b_T(u)du$. By assumption, $B_T(\cdot)$ is continuous and strictly increasing. We now define our new set-up, using tildes. First, let $(\tilde{X}_{aT}, \tilde{X}_{bT}, \tilde{W}_T) = (X_{aT}, X_{bT}, W_T)$, $\tilde{t} = B_T(t)$ and for $d \in \{a, b\}$, $t \in [0, 1]$, $A \subset [0, \infty)$ and $p > 0$, let us define $\tilde{V}_{dT}$ as

$$\tilde{V}_{dT}(\{0, \tilde{t}\}, A) = V_{dT}(\{0, \tilde{t}\}, A).$$

By Assumption 1 and definition of $\tilde{t}$, $\tilde{V}_{dT}$ is a Poisson process with intensity $(t, p) \mapsto \xi_{dT} \in p^{-\varepsilon - 1}$. In other words, $\tilde{\xi}_{dT} = \xi_{dT}$, $\varepsilon = \varepsilon$ and $\tilde{b}_T(t) = 1$. In particular, $\tilde{V}_{dT}$ does not depend on $b_T(\cdot)$.

We now prove that $\tilde{R}^I_t$, the optimal revenue associated with $(\tilde{V}_{aT}, \tilde{V}_{bT})$, satisfies $\tilde{R}^I_t = R^I_t$. Let us consider a pricing strategy $(p^I_{r,a}(\cdot), p^I_{r,b}(\cdot))$ associated with the processes $(V_{aT}, V_{bT})$, satisfying the constraints associated with $r$ and $I$ and leading, on expectation, to the optimal revenue $R^I_t$. As feasible pricing strategies, $p^I_{r,a}(t)$ and $p^I_{r,b}(t)$ only depend on purchases up to $t$, on $\varepsilon$, $b_T(\cdot)$ and on $(\xi_{aT}, \xi_{bT})$ (if $I = c$) or on $\tilde{f}_{\xi_{aT}, \xi_{bT}|X_{aT}, X_{bT}, W_T}$ (if $I = i$). Now, let us define, for $d \in \{a, b\}$ and $t \in [0, 1]$,

$$\tilde{p}^I_{r,d}(\tilde{t}) = p^I_{r,d}(t).$$

By construction, $\tilde{p}^I_{r,d}(\tilde{t})$ only depends on purchases (associated with $\tilde{V}_{dT}$) up to $\tilde{t}$, on $\varepsilon$, $b_T(\cdot)$ and on $(\tilde{\xi}_{aT}, \tilde{\xi}_{bT}) = (\xi_{aT}, \xi_{bT})$ (if $I = c$) or on $\tilde{f}_{\xi_{aT}, \xi_{bT}|X_{aT}, X_{bT}, W_T} = f_{\xi_{aT}, \xi_{bT}|X_{aT}, X_{bT}, W_T}$ (if $I = i$). Also, as $p^I_{r,d}$, it satisfies all the constraints associated with $r$, since no constraints are related to time. Hence, $(\tilde{p}^I_{r,a}, \tilde{p}^I_{r,b})$ is a feasible pricing strategy, up to one point: it depends on $b_T(\cdot)$. This dependence is nevertheless useless, since the Poisson processes $(\tilde{V}_{aT}, \tilde{V}_{bT})$ do not depend on $b_T(\cdot)$. Hence, the expected revenue $\tilde{R}^I_t$, say, associated with $(\tilde{p}^I_{r,a}, \tilde{p}^I_{r,b})$ satisfies $\tilde{R}^I_t \leq \tilde{R}^I_t$. Also, by construction, we obtain with $(\tilde{p}^I_{r,a}, \tilde{p}^I_{r,b})$, associated with $(\tilde{V}_{aT}, \tilde{V}_{bT})$, the same purchases at the same

---

There may be no such pricing strategy, but only sequences of pricing strategies with corresponding expected revenue tending to $R^I$. If so, we just replace $p^I(\cdot)$ by the corresponding sequence in the rest of the proof.
prices as with the pricing strategy \((p_{r,a}, p_{r,b})\) associated to \((V_{aT}, V_{bT})\). In other words, \(\tilde{R}_r = R^l_r\). Therefore, \(R^l_r \leq \tilde{R}_r\).

Finally, consider a pricing strategy \((\tilde{p}_{r,a}^l(\cdot), \tilde{p}_{r,b}^l(\cdot))\) associated with \((\tilde{V}_{aT}, \tilde{V}_{bT})\), satisfying the constraints associated with \(r\) and \(I\) and leading, on expectation, to the optimal revenue \(\tilde{R}_r\). Then let \(p_{r,a}^l(t) = \tilde{p}_{r,a}^l(i)\). By construction, \(p_{r,a}^l(t)\) only depends on purchases (associated with \(V_{dT}\)) up to \(t\), on \(\varepsilon\), \(b_T(\cdot)\) (through \(\tilde{t} = B_T(t)\)) and on \((\xi_{aT}, \xi_{bT})\) (if \(I = c\)) or on \(f_{\xi_{aT}:\xi_{bT}|X_{aT},X_{bT},W_T}(\xi_{aT},\xi_{bT})\) (if \(I = i\)). Thus, it is a feasible pricing strategy. Reasoning as above, this yields \(\tilde{R}_r \leq R^l_r\). Hence, \(\tilde{R}_r = R^l_r\) and the result follows.

### E.3 Formulas for counterfactual revenues

#### E.3.1 Complete information

**Uniform pricing** Given \(\xi_{dT}\), the revenue under uniform prices \(p_d\) is

\[
R^c_u(p_d; \xi_{dT}) = \mathbb{E}[p_d D_{dT}(0, \tau_{C_{dT}} \wedge 1; p_d) | \xi_{dT}],
\]

where \(\tau_C = \inf\{t : D_{dT}(0, t; p_d) \geq C_{dT}\}\) is the stopping time of selling out all \(C_{dT}\) seats. Then,

\[
R^c_u(\xi_{dT}) = \max_{p > 0} R^c_u(p, \xi_{dT}) = \max_{p > 0} p \mathbb{E}[D(\xi_{dT} p^{-\varepsilon}) \wedge C_{dT} | \xi_{dT}].
\]

We obtain the result by defining \(q = \xi_{dT} p^{-\varepsilon}\) and integrate over \(\xi_{dT}\).

Without pre-allocation of capacities among intermediate and final destinations, given \((p_a, p_b), (\xi_{aT}, \xi_{bT})\), and using the first statement of Theorem 4.1, we have:

\[
R^c_u(p_a, p_b, \xi_{aT}, \xi_{bT}) = \mathbb{E} \left[ \mathbb{E} \left[ p_a D(\xi_{aT} p_a^{-\varepsilon}) + p_b D(\xi_{bT} p_b^{-\varepsilon}) \left| D(\xi_{aT} p_a^{-\varepsilon} \xi_{bT} p_b^{-\varepsilon}) \wedge C_T \right. \right] \right] \\
= \frac{\xi_{aT} p_a^{1-\varepsilon} + \xi_{bT} p_b^{1-\varepsilon}}{\xi_{aT} p_a^{-\varepsilon} + \xi_{bT} p_b^{-\varepsilon}} \mathbb{E} \left[ D(\xi_{aT} p_a^{-\varepsilon} + \xi_{bT} p_b^{-\varepsilon}) \wedge C_T \right].
\]

Then, the optimal revenue under uniform pricing without pre-allocation is achieved when \(p_a = p_b\) and therefore:

\[
R^c_u(\xi_{aT}, \xi_{bT}) = \max_{p > 0} p \mathbb{E} \left[ D((\xi_{aT} + \xi_{bT}) p^{-\varepsilon}) \wedge C_T \right]
\]

64
**Full dynamic pricing.** Denote by $V_k(t,p_d)$ the expected revenue when there remains $k$ vacant seats before the departure and the current seat is priced at $p_d$ at time $1-t$. From $1-t$ to $1-t+\Delta t$, the probability of selling one seat is $b_T(1-t)\xi_{dT}p_d^\varepsilon \Delta t + o(\Delta t)$ and generates $p_d$ revenue if one seat is sold. With probability $o(\Delta t)$, more than one seats are sold. Then, following Gallego and Van Ryzin (1994) (Section 2.2.1 on page 1004), we have:

$$V_k^*(t) = \max_{p_d>0} \left\{ b_T(1-t)\xi_{dT}p_d^\varepsilon \Delta t \left(p_d + V_{k-1}(t-\Delta t)\right) + \left[1 - b_T(1-t)\xi_{dT}p_d^\varepsilon \Delta t\right]V_k^*(t-\Delta t) + o(\Delta t)\right\}. \quad (36)$$

Letting $\Delta t \to 0$, this equation shows that $V_k^*$ is continuous. Further, by considering $(V_k^*(t) - V_k^*(t-\Delta t))/\Delta t$ and letting $\Delta t \to 0$, we obtain that $V_k^*$ is differentiable, with\(^{28}\)

$$V_k''(t) = \max_{p_d>0} b_T(1-t)\xi_{dT}p_d^\varepsilon \left[p_d + V_{k-1}^*(t) - V_k^*(t)\right] \quad (37)$$

with boundary conditions $V_k^*(0) = 0$ for any $k = 1, ..., C_{dT}$ and $V^*(t,0) = 0$ for any $t \in [0, 1]$. As a consequence, the optimal price $p_{tk}^*$ can be obtained from the first-order condition of the right-hand side of (37):

$$p_{tk}^* = \frac{-\varepsilon}{\varepsilon - 1} \left[V_k^*(t) - V_{k-1}^*(t)\right]. \quad (38)$$

By plugging $p_{tk}^*$ into (37) and using $B_T(t, 1) = \int_t^1 b_T(s)ds$ (where we let $B_T(t, t') := \left[\int_t^1 b_T(s)ds\right]$), we obtain:

$$V_k''(t) = \partial_j B_T(1-t, 1) \frac{\xi_{dT}}{\varepsilon - 1} \left(1 - \frac{1}{\varepsilon}\right)^\varepsilon \left[V_k^*(t) - V_{k-1}^*(t)\right]^{1-\varepsilon}, \quad (39)$$

where $\partial_j B_T$ denotes the derivative of $B_T$ with respect to its $j$-th argument. We now prove by induction on $k$ that

$$V_k^*(t) = \alpha_{k,f}^c [\xi_{dT}B_T(1-t, 1)]^{\frac{1}{\varepsilon}} \quad (40)$$

for all $k \in \{0, ..., C_{dT}\}$, with $\alpha_f^c(0) = 0$ and $\alpha_{k,f}^c = (\alpha_{k,f}^c - \alpha_{k-1,f}^c)^{1-\varepsilon} \left(1 - \frac{1}{\varepsilon}\right)^{1-\varepsilon}$. The result holds for $k = 0$ since $V_0^*(t) = 0$. Next, suppose that (40) holds for $k-1 \geq 0$ and let us show that the result holds for $k$. By plugging this solution for $k-1$ into the differential equation (37), we obtain:

$$V_k''(t) = \partial_j B_T(1-t, 1) \frac{\xi_{dT}}{\varepsilon - 1} \left(1 - \frac{1}{\varepsilon}\right)^\varepsilon \left[V_k^*(t) - \alpha_{k-1,f}^c [\xi_{dT}B_T(1-t, 1)]^{\frac{1}{\varepsilon}}\right]^{1-\varepsilon}, \quad (41)$$

\(^{28}\)For conditions that enable to interchange $\lim_{\Delta t \to 0}$ and max, we refer to Brémaud (1981) for details.
with $V_k^*(0) = 0$. We can check that $V_k^*(t) = \alpha_{k,f}^\epsilon \xi_{dT}B_T(1 - t, 1)^{1/\epsilon}$ is a solution to (41). To show uniqueness, let $\phi(v, z) = \frac{1}{\epsilon - 1} \left(1 - \frac{1}{\epsilon}\right) \left[v - \alpha_{k-1,f}^\epsilon z^{1/\epsilon}\right]^{1-\epsilon}$. Consider the diffeomorphism $z(t) = \xi_{dT}B_T(1 - t, 1)$ and define $\tilde{V}_k^*(z) = V_k^*(t(z))$. Then, (41) can be written as

$$\tilde{V}_k^{*\epsilon}(z) = \phi(\tilde{V}_k^*(z), z),$$

with $\tilde{V}_k^*(0) = 0$. It is enough to prove that $\tilde{V}_k^*$ is the unique solution of (42) and we prove this by contradiction. Suppose that there is another differentiable solution $\tilde{V}_k(\cdot)$ different from $\tilde{V}_k^*(z) = \alpha_{k,f}^\epsilon z^{1/\epsilon}$. Without loss of generality, $\tilde{V}_k(z_0) > \tilde{V}_k^*(z_0)$ for some $z_0 > 0$. Because $\tilde{V}_k(0) = \tilde{V}_k^*(0) = 0$, then $z_m = \sup\{z \leq z_0 : \tilde{V}_k(z_0) \leq \tilde{V}_k^*(z_0)\}$ exists and $z_m < z_0$. Moreover, $\tilde{V}_k(z_m) = \tilde{V}_k^*(z_m)$. Then, (42) implies the contradiction

$$0 < \tilde{V}_k(z_0) - \tilde{V}_k^*(z_0) = \int_{z_m}^{z_0} [\phi(\tilde{V}_k(z), z) - \phi(\tilde{V}_k^*(z), z)] dz \leq 0,$$

where the second inequality follows from the fact that $\phi$ is a decreasing function of $z$ and $\tilde{V}_k(s) > \tilde{V}_k^*(s)$ for all $s \in (z_m, z_0)$. Finally, we conclude that $\tilde{V}_k^*(\cdot)$ is the unique solution. Hence, the result holds for $k$, and (40) holds. By taking $t = 1, k = C_{dT}$ and integrating over $\xi_{dT}$, we obtain the formula in Section B.

**Stopping-time pricing.** Denote by $V_k(t, p_d)$ the expected optimal revenue at time $1 - t$ when pricing the next seat at $p_d$ and with $k$ remaining seats. In this scenario, prices do not change until the next seat is sold. Define $\tau_{1-t, p_d} = \inf\{s > 0 : D_T(1 - t, 1 - t + s; p_d) \geq 1\}$. Then,

$$\Pr[\tau_{1-t, p_d} > s] = \Pr[D(1 - t, 1 - t + s; p_d) = 0] = \exp\{-B_T(1 - t, 1 - t + s)\xi_d p_d^{-\epsilon}\},$$

and the density of $\tau_{1-t, p_d}$ is

$$f_{\tau_{1-t, p_d}}(s) = \xi_d p_d^{-\epsilon} \partial_2 B_T(1 - t, 1 - t + s) e^{-B_T(1-t,1-t+s)\xi_d p_d^{-\epsilon}}.$$ (43)

Then, the Bellman equation is

$$V_k(t, p_d) = \mathbb{E}\left[\mathbf{1}_{\tau_{1-t, p_d} < t} \left(p_d + V_{k-1}^*(t - \tau_{1-t, p})\right)\right]$$

$$= \int_0^t f_{\tau_{1-t, p_d}}(s) \left(p_d + V_{k-1}^*(t - s)\right) ds$$

$$= \int_0^t \xi_d p_d^{-\epsilon} \partial_2 B_T(1 - t, 1 - t + s) e^{-B_T(1-t,1-t+s)\xi_d p_d^{-\epsilon}} \times \left(p_d + V_{k-1}^*(t - s)\right) ds.$$

(44)
Let \( V_k^*(t) = \max_{p > 0} V_k(t, p) \). We now show by induction that

\[
V_k^*(t) = \alpha_k^c \xi_{\Delta T} B_T(1-t, 1)^{\frac{1}{2}},
\]

where \( \alpha_{0,s}^c = 0 \) and

\[
\alpha_k^c = \max_{q > 0} \left\{ q^{-\frac{1}{2}}(1 - e^{-q}) + \alpha_{k-1,s}^c \int_0^1 q e^{-sq}(1 - s)^{\frac{1}{2}}ds \right\}.
\]

The result holds for \( k = 0 \) since \( V_0^*(1-t) = 0 \). Now, suppose that (45) is true for \( k - 1 \geq 0 \). By using the change of variable \( z = B_T(1-t, 1-t+s)/B_T(1-t, 1) \) and applying (45) for \( V_{k-1}^*(t) \) in Equation (44), we get

\[
V_k(t, p) = \int_0^1 \xi_{\Delta T} B_T(1-t, 1)p - e^{-B_T(1-t, 1)\xi_{\Delta T}p - z} \left( p + [\xi_{\Delta T} B_T(1-t, 1)(1-z)]^{\frac{1}{2}} \alpha_{k-1,s}^c \right) dz
\]

\[
= \int_0^1 \xi_{\Delta T} B_T(1-t, 1)^{\frac{1}{2}} \left( q^{-\frac{1}{2}}(1 - e^{-q}) + \alpha_{k-1,s}^c \int_0^1 q e^{-qz}(1 - z)^{\frac{1}{2}}dz \right),
\]

where \( q = \xi_{\Delta T} B_T(1-t, 1)p - e^{-z} \). As a consequence,

\[
V_k^*(t) = \max_{p > 0} V_k(t, p)
\]

\[
= \int_0^1 \xi_{\Delta T} B_T(1-t, 1)^{\frac{1}{2}} \max_{q > 0} \left\{ q^{-\frac{1}{2}}(1 - e^{-q}) + \alpha_{k-1,s}^c \int_0^1 q e^{-qz}(1 - z)^{\frac{1}{2}}dz \right\}
\]

\[
= \alpha_k^c \xi_{\Delta T} B_T(1-t, 1)^{\frac{1}{2}},
\]

and (45) is true for \( k \). Thus, (45) holds for all \( k \in \{0, ..., C_{\Delta T}\} \). Finally, by taking \( t = 1, k = C_{\Delta T} \) and the expectation with respect to \( \xi_{\Delta T} | (X_{\Delta T}, W_T) \), we obtain the expression in Appendix B.

**Stopping-time pricing with \( M \) fares.** Denote by \( V_k(0; t, p, m) \) (resp. \( V_k(1; t, p, m) \)) the expected revenue of the firm at time \( 1-t \), with a current price \( p \), a remaining capacity \( k \) and a remaining number of fares \( m \), if it decides to keep the same price \( p \) (resp. to choose a new price). Then, we have the following Bellman equations:

\[
\begin{align*}
&V_k(1; t, p, m) = \max_{p' > 0} \int_0^t f_{\tau_{1-t,p}}(s) \left[ p' + V_{k-1}^*(t-s, p', m-1) \right] ds, \\
&V_k(0; t, p, m) = \int_0^t f_{\tau_{1-t,p}}(s) \left[ p + V_{k-1}^*(t-s, p, m) \right] ds, \\
&V_k^*(t, p, m) = \max_{d \in \{0,1\}} V_k(d; t, p, m),
\end{align*}
\]

(46)
with initial conditions $V_0^*(t,p,m) = 0$. We show by induction on $k$ that for all $(k,m) \in \{0, ..., C_{dT}\} \times \mathbb{N}$,

$$V_k^*(t,p,m) = \alpha_{k,m}(q(t,p)) \left[ \xi_{dT}B_T(1 - t, 1) \right]^\frac{1}{2},$$  \hspace{1cm} (47)

where $q(t,p) = p^{-\varepsilon}\xi_{dT}B_T(1 - t, 1)$, $\alpha_{k,0}(q) = q^{-\frac{1}{2}} \mathbb{E}[D(q) \land k]$ and for $m \geq 1$,

$$\alpha_{k,m}(q) = \max \left\{ q \int_0^1 e^{-qz}\left[q^{-\frac{1}{2}} + \alpha_{k-1,m\land(k-1)}(q(1 - z))(1 - z)^{\frac{1}{2}}\right] dz, \right. \hspace{1cm} \left. \max_{q' > 0} q' \int_0^1 e^{-q'z}\left[q'^{-\frac{1}{2}} + \alpha_{k-1,m-1}(q'(1 - z))(1 - z)^{\frac{1}{2}}\right] dz \right\}. $$

Because for any $m \geq k$ and $d \in \{0, 1\}$, we have $V_k(d; t, p, m) = V_k(d; t, p, k)$, it suffices to prove the result for $m \leq k$. The result holds for $k = m = 0$ since $V_0^*(t, p, m) = 0$. Now, suppose that (47) holds for $k - 1 \geq 0$ and all $m \leq k - 1$. If $m = 0$, the price cannot be changed anymore, so $V_k^*(t, p, m)$ is simply the revenue with price $p$ from $1 - t$ to 1, and (47) holds.

If $m \geq 1$, we have, by Equations (43), (46), the change of variable $z = B_T(1 - t, 1 - t + s)/B_T(1 - t, 1)$ and the induction hypothesis,

$$V_k(0; t, p, m)$$

$$= \int_0^t f_{\tau_{1-t,s}}(s) \left[ p + V_{k-1}^*(t - s, p, m \land (k - 1)) \right] ds$$

$$= \int_0^t \xi_{dT} p^{-\varepsilon} \partial_2 B_T(1 - t, 1 - t + s)e^{-\xi_{dT} p^{-\varepsilon} B_T(1 - t, 1 - t + s)}$$

$$\left[ p + \alpha_{k-1,m\land(k-1)}(\xi_{dT} B_T(1 - t + s, 1)p^{-\varepsilon})[\xi_{dT} B_T(1 - t + s, 1)]^\frac{1}{2} \right] ds$$

$$= \int_0^1 \xi_{dT} p^{-\varepsilon} B_T(1 - t, 1)e^{-\xi_{dT} p^{-\varepsilon} B_T(1 - t, 1)z}$$

$$\left[ p + \alpha_{k-1,m\land(k-1)}(\xi_{dT} B_T(1 - t, 1)p^{-\varepsilon}(1 - z))[\xi_{dT} B_T(1 - t, 1)]^\frac{1}{2}(1 - z)^\frac{1}{2} \right] dz$$

$$= [\xi_{dT} B_T(1 - t, 1)]^\frac{1}{2} \int_0^1 q(t,p)e^{-q(t,p)z}\left[q(t,p)^{-\frac{1}{2}} + \alpha_{k-1,m\land(k-1)}(q(t,p)(1 - z))(1 - z)^\frac{1}{2}\right] dz,$$  \hspace{1cm} (48)

With the same reasoning, we also obtain

$$V_k(1; t, p, m)$$

$$= \max_{q' > 0} \int_0^t f_{\tau_{1-t,s'}}(s) \left[ p' + V_{k-1}^*(t - s, p', m - 1) \right] ds$$

$$= [\xi_{dT} B_T(1 - t, 1)]^\frac{1}{2} \max_{q > 0} \int_0^1 q e^{-qz}\left[q^{-\frac{1}{2}} + \alpha_{k-1,m-1}(q(1 - z))(1 - z)^\frac{1}{2}\right] dz.$$
Then,
\[ V^*_k(t, p, m) = \max_{d \in \{0, 1\}} V_k(d, t, p, m) \]
\[ = \alpha_{k,m}(q(t, p)) [\xi_d T B_T(1 - t, 1)]^{\frac{1}{2}} , \]
Thus, (47) holds for \( k \), and hence for all \( k \in \{0, ..., C_{dT}\} \). By setting \( t = 0 \) and optimizing \( V^*_k(t, p, m) \) over \( p \) (or equivalently over \( q(t, p) \)) and taking the expectation with respect to \( \xi_d T |(X_{dT}, W_T) \), we obtain the desired expression in Appendix B.

**Stopping-time pricing with \( M \) increasing prices.** The reasoning is very similar to the previous case. The only change in (46) is in the formula of \( V_k(1; t, p, m) \): the maximization is now over \( p' \geq p \) rather than \( p' \geq 0 \), since the new price has to be higher than the current one. Then, following a similar strategy by induction, we get

\[ V^*_k(t, p, m) = \alpha^+_k,0(q(t, p)) [\xi_d T B_T(1 - t, 1)]^{\frac{1}{2}} , \]
where \( \alpha^+_k,0(q) = \alpha_{k,0}(q) \) and

\[ \alpha^+_k,m(q) = \max \left\{ q \int_0^1 e^{-qz} \left[ q^{\frac{-1}{2}} + \alpha^+_{k-1,m\wedge(k-1)}(q(1 - z))(1 - z)^{\frac{1}{2}} \right] dz, \right. \]
\[ \left. \max_{q' \in [0,a]} q' \int_0^1 e^{-q'z} \left[ q'^{\frac{-1}{2}} + \alpha^+_{k-1,m-1}(q'(1 - z))(1 - z)^{\frac{1}{2}} \right] dz. \right\} \]

We obtain the result by taking \( t = 0, k = C_{dT} \) and defining \( \alpha^+_{C_{dT},sM+} = \max_{q > 0} \alpha^+_C(q) \).

**Intermediate-\( K \) stopping-time pricing** The proof is the same as that for (45) except for the initial value because the firm must apply uniform pricing whenever there remain \( C_{dT}(1 - K\%) \) seats. Thus, the Bellman equation and the updating of the constants \( \alpha^+_{i,K,k} \) take the same form as under the stopping-time pricing strategy in (45) for \( k \geq C_{dT}(1 - K\%) \). The initial value becomes \( \alpha^+_{i,K,C_{dT}(1-K\%)} \), which comes from the optimal uniform pricing with \( C_{dT}(1 - K\%) \) seats.

**E.3.2 Incomplete information**

Denote \( Y_{dT} = \exp\{X'_{dT}\beta_0\} g_0(W_T) \). Denote the density function of \( \xi_{dT} \) by \( f \). Under Assumption 3(ii), \( f \) is a gamma density \( \Gamma(\lambda_{d0}, Y_{dT}^{-1}) \).
Uniform pricing. We have:

\[ R^i_u = \max_{p>0} R^i_u(p; \varepsilon, f) \]

\[ = \max_{p>0} p \int_{z>0} \mathbb{E}[D(p^{-\varepsilon} z) \land C_{dT}] f(z) dz. \]

By the change of variable \( q = Y_{dT} p^{-\varepsilon} \), we obtain the desired formula.

Now, given \((p_a, p_b)\) and \((C_{aT}, C_{bT})\), the total revenue generated by train \( T \) is:

\[ \mathbb{E} [R_T(p_a, p_b; C_{aT}, C_{bT})|W_T] \]

\[ = \sum_{d=a,b} p_d \int_{z>0} \mathbb{E} \left[ D(p_d^{-\varepsilon} \exp\{X_{dT}^{\ast}\} g_0(W_T) z) \land C_{dT} \right] g_{\lambda_0,1}(z) dz, \quad (49) \]

where \( g_{\lambda_0,1}(z) \) is the density of a \( \Gamma(\lambda_0, 1) \). Then, we obtain (5) by maximizing the revenue in (49) over all possible allocations \((C_{aT}, C_{bT})\) subject to \( C_{aT} + C_{bT} = C_T \).

Without pre-allocation of capacities among intermediate and final destinations, we obtain:

\[ \mathbb{E} [R_T(p_a, p_b; C_T)|W_T] = \mathbb{E} \left[ \sum_{d=a,b} p_d D_{dT} | (D_{aT} + D_{bT}) \land C_T, z_a, z_b \right] \]

\[ = \mathbb{E} \left[ \left( \sum_{d=a,b} p_d^{-\varepsilon} \exp\{X_{dT}^{\ast}\} g_0(W_T) z_d \right) \land C_T \right] \frac{\sum_{d=a,b} p_d^{-\varepsilon} \exp\{X_{dT}^{\ast}\} g_0(W_T) z_d}{\sum_{d=a,b} p_d^{-\varepsilon} \exp\{X_{dT}^{\ast}\} g_0(W_T) z_d} \]

where \( z_a \) and \( z_b \) follows \( \Gamma(\lambda_0, 1) \) and \( \Gamma(\lambda_0, 1) \), respectively, and are independent.

Then, the optimal revenue is obtained by maximizing \( \mathbb{E} [R_T(p_a, p_b; C_T)|W_T] \) over \((p_a, p_b) \in \mathbb{R}_+^2\).

Full dynamic pricing. Define \( V_k(t, p, f) \) as the expected revenue at time \( 1 - t \) when there remains \( k \) vacant seats before the departure, the current seat is priced at \( p \) and the density of \( \xi_{dT} \), given the current information, is \( f \). Let also \( V_k^{\ast}(t, f) = \max_{p>0} V_k(t, p, f) \). When \( \eta_T \sim \Gamma(\lambda, \mu) \), we use respectively \( V_k(t, p, \lambda, \mu) \) and \( V_k^{\ast}(t, \lambda, \mu) \) instead of \( V_k(t, p, g_{\lambda, \mu}) \) and \( V_k^{\ast}(t, g_{\lambda, \mu}) \).

Between \( 1 - t \) and \( 1 - t + \Delta t \), if one seat is sold, which occurs with probability \( \xi_{dT} p^{-\varepsilon} \partial_1 B_T(1 - t, 1) \Delta t + o(\Delta t) \), the posterior cdf of \( \xi_{dT} \), \( F_1(\xi; \Delta t) \) satisfies

\[ F_1(\xi; \Delta t) \propto [p^{-\varepsilon} \partial_1 B_T(1 - t, 1) \xi \Delta t + o(\Delta t)] \xi^{\lambda-1} e^{-\mu \xi}, \]

and the corresponding density is

\[ f_1(\xi; \Delta t) = \xi^{\lambda} e^{-\mu \xi} \frac{\mu^{\lambda+1}}{\Gamma(\lambda+1)} + o(\Delta t). \]
As $\Delta t \to 0$, the posterior density converges to $g_{\lambda+1,\mu}$. If the seat is not sold between $1 - t$ and $1 - t + \Delta t$, then the posterior cdf of $\eta_T$ is
\[
F_0(\xi; \Delta t) \propto \xi^{\lambda-1} \exp(-\mu(t, \Delta t, p)\xi),
\]
where $\mu(t, \Delta t, p) = \mu + p^{-\varepsilon}B_T(1 - t, 1 - t + \Delta t)$. Therefore, the posterior density is $g_{\lambda,\mu}(t, \Delta t, p)$. Then, the Bellman equation can be written as:
\[
V_k(t, p, \lambda, \mu) = \int \left\{ [p^{-\varepsilon}\xi \partial_1 B_T(1 - t, 1)\Delta t + o(\Delta t)] \times [p + V_{k-1}^*(t - \Delta t, f_1(; \Delta t))] \right. \\
\left. + \left[ 1 - p^{-\varepsilon}\xi \partial_1 B_T(1 - t, 1)\Delta t - o(\Delta t) \right] \times V_k^*(t - \Delta t, \lambda, \mu(p, t, \Delta)) \right\} g_{\lambda,\mu}(\xi) d\xi.
\]
Then, using $V_k^*(t, \lambda, \mu) = \max_{p>0} V_k(t, p, \lambda, \mu)$ and letting $\Delta t \to 0$, we obtain:\textsuperscript{29}
\[
\partial_1 V_k^*(t, \lambda, \mu) \\
= \max_{p>0} \int \left\{ p^{-\varepsilon}\xi \partial_1 B_T(1 - t, 1) \left[ p + V_{k-1}^*(t, \lambda + 1, \mu) - V_k^*(t, \lambda, \mu) \right] \right. \\
\left. + \lim_{\Delta t \to 0} \frac{V_k^*(t - \Delta t, \lambda, \mu(t, \Delta t, p)) - V_k^*(t - \Delta t, \lambda, \mu)}{\Delta t} \right\} g_{\lambda,\mu}(\xi) d\xi \\
= \partial_1 B_T(1 - t, 1) \max_{p>0} \int \left\{ p^{-\varepsilon}\xi \left[ p + V_{k-1}^*(t, \lambda + 1, \mu) - V_k^*(t, \lambda, \mu) \right] + \partial_3 V_k^*(t, \lambda, \mu)p^{-\varepsilon} \right\} g_{\lambda,\mu}(\xi) d\xi \\
= \partial_1 B_T(1 - t, 1) \max_{p>0} \left\{ p^{-\varepsilon}\frac{\lambda}{\mu} \left[ p + V_{k-1}^*(t, \lambda + 1, \mu) - V_k^*(t, \lambda, \mu) \right] + \partial_3 V_k^*(t, \lambda, \mu)p^{-\varepsilon} \right\}.
\]
Solving for the optimal price, we then obtain:
\[
\partial_1 V_k^*(t, \lambda, \mu) = \left[ \frac{\varepsilon}{\varepsilon - 1} \right]^{-\varepsilon} \frac{\lambda}{\mu(\varepsilon - 1)} \partial_1 B_T(1 - t, 1) \\
\times \left[ -V_{k-1}^*(t, \lambda + 1, \mu) + V_k^*(t, \lambda, \mu) - \frac{\mu}{\lambda} \partial_3 V_k^*(t, \lambda, \mu) \right]^{1-\varepsilon}.
\]
Letting $z(t) = B_T(1 - t, 1)$ and $V^*(z(t), \lambda, \mu) = V^*(t, \lambda, \mu)$, we obtain:
\[
\partial_1 V_k^*(z, \lambda, \mu) = \left[ \frac{\varepsilon}{\varepsilon - 1} \right]^{-\varepsilon} \frac{\lambda}{\mu(\varepsilon - 1)} \left[ -V_{k-1}^*(z, \lambda + 1, \mu) + V_k^*(z, \lambda, \mu) \right. \\
\left. - \frac{\mu}{\lambda} \partial_3 V_k^*(z, \lambda, \mu) \right]^{1-\varepsilon}.
\]
\textsuperscript{29}For conditions that enable to interchange $\lim_{\Delta t \to 0}$ and $\max$, we refer to Brémaud (1981) for details.
We prove by induction on $k$ that for all $k \in \{0, ..., C_{dT}\}$,
\[
\bar{V}_k^*(z, \lambda, \mu) = \left(\frac{z}{\mu}\right)^{\frac{1}{\varepsilon}} \alpha_{k,f}^*(\lambda),
\] (51)
where $\alpha_{i}^j(0, \lambda) = 0$ and for $k \geq 1$,
\[
\alpha_{k,f}^i(\lambda) = \lambda \left(1 - \frac{1}{\varepsilon}\right)^{\varepsilon-1} \left[-\alpha_{k-1,f}^i(\lambda + 1) + \left(1 + \frac{1}{\lambda\varepsilon}\right)\alpha_{k,f}^i(\lambda)\right]^{1-\varepsilon}.
\]
The result holds for $k = 0$ since $V_0^*(z, \lambda, \mu) = 0$. Suppose that (51) holds for $k - 1$. Then, (50) and the induction hypothesis yield
\[
\partial_t \bar{V}_k^*(z, \lambda, \mu) = \left[\frac{\varepsilon}{\varepsilon - 1}\right]^{1-\varepsilon} \frac{\lambda}{\mu(\varepsilon - 1)} \left[-\left(\frac{z}{\mu}\right)^{\frac{1}{\varepsilon}} \alpha_{k-1,f}(\lambda + 1) + \bar{V}_k^*(z, \lambda, \mu)\right]
- \frac{\mu}{\lambda} \partial_t \bar{V}_k^*(z, \lambda, \mu) \right]^{1-\varepsilon}.
\] (52)
The function $(z, \lambda, \mu) \mapsto \alpha_{k,f}^i(\lambda)(z/\mu)^{1/\varepsilon}$ is a solution to (52). We now show that $\bar{V}_k^*(z, \lambda, \mu)$ is equal to this solution. First, note that $V_k^*(t, \lambda, \mu)$ remains unchanged if the distribution of $B_T(t, t')\xi$ remains unchanged. Now,
\[
B_T(t, t')\xi = (B_T(t, t')/\delta) \times (\delta\xi),
\]
with $\delta\xi \sim \Gamma(\lambda, \mu/\delta)$. Hence, $V_k^*(t, \lambda, \mu)$ remains unchanged if we replace $\mu$ by $\mu/\delta$ and $z(t)$ by $z(t)/\delta$. Given the definition of $\bar{V}_k^*(z, \lambda, \mu)$, this implies $\bar{V}_k^*(z/\delta, \lambda, \mu/\delta) = \bar{V}_k^*(z, \lambda, \mu)$ for all $\delta > 0$. Then, to prove the induction step, we only need to show that $V(x) := V_k^*(x, \lambda, 1)$ satisfies $V(x) = \alpha_{k,f}^i(\lambda)x^{1/\varepsilon}$. By Equation (52),
\[
V'(x) = \left[\frac{\varepsilon}{\varepsilon - 1}\right]^{1-\varepsilon} \frac{\lambda}{\varepsilon - 1} \left[-x^{\frac{1}{\varepsilon}}\alpha_{k-1,f}(\lambda + 1) + V(x) + \frac{x}{\lambda}V'(x)\right]^{1-\varepsilon},
\] (53)
with initial condition $V(0) = 0$. Suppose that (53) has two distinct solutions $V_1, V_2$ and let $x_0$ be such that $V_1(x_0) \neq V_2(x_0)$, say $V_1(x_0) > V_2(x_0)$. Define $x_m = \sup\{x \leq x_0 : V_1(x) \leq V_2(x)\}$. Because $V_1(0) = V_2(0)$ and $V_1(x_0) > V_2(x_0)$, we have $0 \leq x_m < x_0$ and $V_1(x) > V_2(x)$ for $x \in (x_m, x_0]$. Moreover, because both solutions are continuous, $V_1(x_m) = V_2(x_m)$. According to (53), because $\varepsilon > 1$, as long as $V_1(x) > V_2(x)$, we have $V_1'(x) < V_2'(x)$. Then,
\[
V_1(x_0) - V_2(x_0) = \int_{x_m}^{x_0} [V_1'(x) - V_2'(x)] dx < 0,
\]
which contradicts $V_1(x_0) > V_2(x_0)$. Hence, $V(x) = \alpha_{k,f}^i(\lambda)x^{1/\varepsilon}$, and the induction step holds. Thus, (51) is satisfied for $k \in \{0, ..., C_{dT}\}$. Finally, we obtain the result in Appendix B by taking $t = 0$ and $k = C_{dT}$. 72
**Stopping-time pricing**  The difference from the stopping-time pricing under complete information is that the firm updates in a Bayesian way its belief on the distribution of $\xi_{t\tau}$. Even if the firm continuously updates its belief, only moments where a sale occurs matter, since this is the time where it can decide to change its prices. Thus, starting at time $1 - t$, we can focus on time $1 - t + \tau_{t,p}$. The next lemma characterizes the corresponding posterior distribution of $\xi_{t\tau}$.

**Lemma E.2** Suppose that the density function of $\xi_{t\tau}$ at time $1 - t$ is $f$ and the firm prices the next seat at $p$. Then, the posterior distribution of $\xi_{t\tau}|_{t_1 - t,p} = s$ is $T(f; q(t, s, p))$, with $q(t, s, p) = p^{-\varepsilon}B_T(1 - t, 1 - t + s)$ and

$$T(f; u)(z) = \frac{ze^{-uz}f(z)}{\int ze^{-uz}f(z)dz}.$$  

$\xi p^{\varepsilon}_d \partial_2 B_T(1 - t, 1 - t + s)e^{-B_T(1-t,1-t+s)\xi p^{\varepsilon}_d}$

**Proof:** As Equation (43) shows, given $\xi_{t\tau} = z$, the density function of $\tau_{t_1 - t,p}$ is

$$f_{\tau_{t_1 - t,p}\xi_{t\tau}}(s|z) = p^{-\varepsilon}z\partial_2 B_T(1 - t, 1 - t + s)e^{-q(t,s,p)}.$$  

Then, the joint distribution of $(\tau_{t_1 - t,p}, \xi_{t\tau})$ is

$$f_{\tau_{t_1 - t,p},\xi_{t\tau}}(s, z) = p^{-\varepsilon}z\partial_2 B_T(1 - t, 1 - t + s)e^{-q(t,s,p)}f(z)$$

The result follows.

Now, using the same notation as in the full dynamic pricing case above and the same arguments as in proof of (44), we have

$$V_k(t, p, f) = \int_0^t f_{\tau_{t_1 - t,p}}(s)[p + V_{k-1}(t - s; T(f; q(t, s, p)))]ds.$$  

and

$$V_k^*(t, f) = \max_{p > 0} \int_0^t f_{\tau_{t_1 - t,p}}(s) [p + V_{k-1}^*(t - s; T(f; q(t, s, p)))]ds.$$  

(55)

We now prove by induction on $k$ that for all $k \in \{0, ..., C_{t\tau}\}$,

$$V_k^*(t, f) = [B_T(1 - t, 1)]^{\frac{1}{2}} \alpha^{i}_k(f).$$  

(56)

where $\alpha^{i}_0(0, f) = 0$ and for all $k \in \{1, .., C_{t\tau}\}$,

$$\alpha^{i}_{k,s}(f) = \max_{q > 0} q \int_0^1 [q^{-1/\varepsilon} + (1 - u)^{1/2}\alpha^{i}_{k-1,s}(T(f; qu))] \int_0^\infty ze^{-quat}f(z)dzdu.$$  

(57)
The result holds for $k = 0$ since $V_0^*(t; f) = 0$. Suppose that it holds for $k - 1 \geq 0$. First, by (54), we have

$$f_{\tau_1, \tau, B_T}(s, z) = \int_0^\infty p^{-\varepsilon} z \partial_2 B_T(1 - t, 1 - t + s)e^{-q(t,s,p)z} f(z)dz.$$  \hspace{1cm} (57)

Using (55), we obtain

$$V_k^*(t, f) = \max_{p > 0} \int_0^t f_{\tau_1, \tau, B_T}(s) \left\{ p + [B_T(1 - t + s, 1)]^{\frac{1}{2}} \times \alpha_k^{i_{-1, s}}(T(f; q(t, s, p))) \right\} ds$$

$$= \max_{p > 0} \int_0^1 \left[ \int_0^\infty q(t, p)ze^{-q(t,p)uz} f(z)dz \right]$$

$$\times \left\{ p + [B_T(1 - t, 1)(1 - u)]^{\frac{1}{2}} \alpha_k^{i_{-1, s}}(T(f; q(t, p)u)) \right\} du$$

$$= [B_T(1 - t, 1)]^{\frac{1}{2}} \max_{q > 0} q \int_0^1 \left[ \int_0^\infty ze^{-quz} f(z)dz \right]$$

$$\times \left[ q^{-1/\varepsilon} + (1 - u)^{\frac{1}{2}} \alpha_k^{i_{-1, s}}(T(f; qu)) \right] du.$$  \hspace{1cm} (58)

The second equality follows using the change of variable $u = B_T(1-t, 1-t+s)/B_T(1-t, 1)$ and the third by the change of variable $q = q(t, p)$. Hence, the induction step holds, and (56) is satisfied for all $k \in \{0, ..., C_{dT}\}$. We obtain the desired expression by taking $t = 0$ and $k = C_{dT}$.

If Assumption 3(ii) further holds, we obtain by Lemma E.2 that if $f = g_{\lambda, \mu}$, then $T(f; u) = g_{\lambda + 1, \mu + u}$. Let $V_k(t, p; \lambda, \mu)$ and $V_k^*(t; \lambda, \mu)$ be defined as in the full dynamic pricing case. Then, by the same induction as above, we have, for all $k \in \{0, ..., C_{dT}\}$,

$$V_k^*(t; \lambda, \mu) = \left[ \frac{B_T(1 - t, 1)}{\mu} \right]^{\frac{1}{2}} \alpha_k^{i_{k, s}}(\lambda),$$

where $\alpha_k^{i_{k, s}}(0, \lambda) = 0$ for $\lambda > 0$, and

$$\alpha_k^{i_{k, s}}(\lambda) = \max_{q > 0} q \int_0^1 \frac{\lambda}{(1 + qs)^{\lambda + 1}} \left[ q^{-\frac{1}{\varepsilon}} + \left( \frac{1 - s}{1 + qs} \right)^{\frac{1}{2}} \alpha_k^{i_{-1, s}}(\lambda + 1) \right] ds.$$  \hspace{1cm} (59)

The result follows by taking $t = 0$, $\mu = Y_{dT}^{-1}$, and $k = C_{dT}$, we obtain the desired expression.

**Stopping-time pricing with $M$ fares.** As in the complete information case, let $V_k(0; t, p, m)$ (resp. $V_k(1; t, p, m, f)$) denote the optimal revenue at time $1 - t$, with a
current price $p$, a remaining capacity $k$, a remaining number of fares $m$ and a density of $f$ for $\xi_{dT}$ (conditional on the current information) if the firm decides to keep the same price (resp. to change it). Then, as (46), we have:

$$V_k(0; t, p, m, f) = \int_0^t f_{\tau_{t-s},p}(s) \left[ p + V^*_k(t-s, p, m, T(f; q(t, s, p))) \right] ds,$$

$$V_k(1; t, p, m, f) = \max_{p' > 0} \int_0^t f_{\tau_{t-s},p'}(s) \left[ p' + V^*_k(t-s, p', m-1, T(f; q(t, s, p))) \right] ds,$$

$$V^*_k(t, p, m, f) = \max_{d \in \{0, 1\}} V_k(d; t, p, m, f),$$

with the initial conditions $V^*_0(t, p, m, f) = 0$. We prove by induction on $k$ that for all $(k, m) \in \{0, \ldots, C_{dT}\} \times \mathbb{N},$

$$V^*_k(t, p, m, f) = c_{k,m}(q(t, p), f) \left[ B_T(1-t, 1) \right]^\frac{1}{2},$$

where $c_{k,0}(q, f) = q^{-\frac{1}{2}} \int \mathbb{E}[D(qz) \land k] f(z) dz$ and

$$c_{k,m}(q, f) = \max \left\{ q \int_0^1 \int z e^{-qz} f(z) dz \left[ q^{-1/\varepsilon} + c_{k-1,m\wedge(k-1)}(q(1-u), T(f; qu)) \right] (1-u)^{\frac{1}{2}} du, \left. \max_{q' > 0} q' \int_0^1 \int z e^{-q'zu} f(z) dz \left[ q'^{-1/\varepsilon} + c_{k-1,m-1}(q'(1-u), T(f; q'u)) (1-u)^{\frac{1}{2}} \right] du \right\}.$$
obtain

\[ V_k(0; t, p, m, f) = \left[ \int_0^t \left( \int_0^\infty (\xi_a + \xi_b) p^{-\varepsilon} \partial_2 B_T(1 - t, 1 - t + s) e^{-q(t,s)p} f(z) dz \right) \right] ds + \int_0^t \left[ \int_0^\infty q(t, p) e^{-q(t,p)u} f(z) dz \right] \left[ p + c_{k-1, m\wedge(k-1)}(q(t, p), T(f; q(t, p))) [B_T(1 - t + s, 1)]^{1/\varepsilon} \right] ds \]

By the same reasoning and the change of variable \( q = q(t, p) \),

\[ V_k(1; t, p, m, f) = [B_T(1 - t, 1)]^{1/\varepsilon} \max_{d \in \{0, 1\}} V_k(d; t, p, m, f) \]

This concludes the induction step, proving that (60) holds for all \( k \in \{0, ..., C_d T\} \).

**Stopping-time pricing with \( M \) increasing fares**  The proof follows by making the same changes as those made in the complete information set-up.

**Intermediate-\( K \) stopping-time pricing**  The proof follows by making the same changes as those made in the complete information set-up.