DENSITY OF STATES UNDER NON-LOCAL INTERACTIONS III.
N-PARTICLE BERNOULLI–ANDERSON MODEL

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ABSTRACT. Following [7, 8], we analyze regularity properties of single-site probability distributions of the random potential and of the Integrated Density of States (IDS) in the Anderson models with infinite-range interactions and arbitrary nontrivial probability distributions of the site potentials. In the present work, we study 2-particle Anderson Hamiltonians on a lattice and prove spectral and strong dynamical localization at low energies, with exponentially decaying eigenfunctions, for a class of site potentials featuring a power-law decay.

1. INTRODUCTION

This text is a follow-up of [7], where the reader can find the main motivations, a historical review, and a number of bibliographical references.

We consider a 2-particle lattice Anderson Hamiltonian

$$H(\omega) = H_0 + V(x, \omega) + U(x),$$

where $H$ is the kinetic energy operator, which we assume to be the standard second-order lattice Laplacian, $U(x)$ is the inter-particle interaction potential of the form

$$U(x) = \sum_{i<j} U^{(2)}(|x_i - x_j|),$$

with a compactly supported two-body interaction potential $U^{(2)}$, and $V(x, \omega)$ is the operator of multiplication by the random external potential energy of the form

$$V(x, \omega) = \sum_{j=1}^N V(x_j, \omega),$$

where $V : \mathbb{Z}^d \times \Omega \to \mathbb{R}$ is a random field on $\mathbb{Z}^d$ relative to a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with IID (independent and identically distributed) values.

Unlike all earlier mathematical works on localization in multi-particle models in presence of a random external potential, we do not assume any regularity of the random amplitudes of the site potentials. The prototypical example is given by the Bernoulli-Anderson model, but our techniques apply to arbitrary compactly supported probability measures not concentrated on a single point.

We always assume $d \geq 2$, since the analysis of one-dimensional models calls for more optimal, specifically one-dimensional techniques.

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As was shown in [11], an extension of the proof of localization for interactive 2-particle models to an arbitrary (but fixed from the beginning) number of particles \( N \) is conceptually not difficult, except for the proof of eigenvalue comparison estimates for norm-distant pairs of \( N \)-particle cubes. As the matter of fact, the transition from \( N = 2 \) to \( N \geq 3 \) aiming to prove localization estimates in the physically natural, norm-distance metric in the \( N \)-particle configuration space, requires new ideas and techniques. Such a program has been carried out in the general framework of the Multi-Scale Analysis (MSA) based on estimates in probability, as in the pioneering works on the single-particle MSA [18, 17, 14]; see [12, 6]. This task is yet to be performed in the context of the adaptation of the Fractional Moments Method to the \( N \)-particle models developed by Aizenman and Warzel [1] (for the lattice models) and by Fauser and Warzel [16] (in a Euclidean space).

Generally speaking, infinite-range particle-media interaction potentials make more difficult the localization analysis, and so do singular (e.g., Bernoulli) probability measures of the random amplitudes of the site potentials. Curiously enough, a combination of the two difficulties solves a thorny problem encountered in the eigenvalue comparison analysis of \( N \)-particle Anderson Hamiltonians, particularly in a continuum configuration space. In the present paper we study a "toy-model" with piecewise-constant ("staircase") site potentials, for which the proof of eigenvalue comparison estimates is simpler than for more realistic potentials, e.g., for \( u(r) = r^{-A}, A > d \). However, it was shown in [7] that satisfactory EV comparison estimates can be obtained for the realistic potentials, too. The bottom line is that neither the restriction \( N = 2 \) nor the use of the "staircase" site potentials \( u \) is crucial for the EV comparison estimates which, in turn, are vital for the efficient \( N \)-particle localization bounds, \( N \geq 2 \).

Let be given a function \( \mathbb{N} \ni r \mapsto u(r) \) the function

\[
Z^d \ni x \mapsto u(|x|)
\]

is absolutely summable. Then one can define a linear transformation \( U \), well-defined on any bounded function \( q : Z^d \to \mathbb{R} \):

\[
U : q \mapsto U[q], \quad \text{with} \quad U[q] : Z^d \to \mathbb{R},
\]

where, setting \( q_y = q(y) \), one has

\[
(U[q])(x) = \sum_{y \in Z^d} u(y - x)q_y.
\]

To clarify the main ideas of [5] and simplify some technical aspects, the interaction potential \( u : \mathbb{R}_+ \to \mathbb{R} \) is assumed to have the following form. Given a real number \( \kappa > 1 \), introduce a growing integer sequence

\[
\tau_k = \lfloor k^{\kappa} \rfloor, \quad k \in \mathbb{N},
\]

and let

\[
u(r) = \sum_{k=1}^{\infty} r_k^{-A} \mathbf{1}_{[\tau_k, \tau_{k+1})}(r),
\]

Making \( u(\cdot) \) piecewise constant will allow us to achieve, albeit in a somewhat artificial setting, an elementary derivation of infinite smoothness of the DoS from a similar property of single-site probability distributions of the potential \( V \). We refer to \( V \) as the cumulative...
potential in order to distinguish it from the interaction potential \( u \) (which is a functional characteristics of the model) and from the local potential amplitudes \( \{ q_y, y \in \mathbb{Z}^d \} \). The notation \( q_y \) will be used in formulae and arguments pertaining to general functional aspects of the model, while in the situation where the latter amplitudes are random we denote them by \( \omega_y \).

We always assume the amplitudes \( q_y \) and \( \omega_y \) to be uniformly bounded. In the case of random amplitudes, one should either to assume this a.s. (almost surely, i.e., with probability one) or to construct from the beginning a product measure on \([0, 1]^{\mathbb{Z}^d}\) rather than on \(\mathbb{R}^{\mathbb{Z}^d}\) and work with samples \( \omega \in [0, 1]^{\mathbb{Z}^d} \), which are thus automatically bounded. It is worth mentioning that boundedness is not crucial to most of the key properties established here, but results in a streamlined and more transparent presentation. On the other hand, as pointed out in [7], there are interesting models with unbounded amplitudes \( \omega \) such that
\[
\mathbb{E}\left[ (\omega - \mathbb{E}[\omega])^2 \right] < \infty.
\]
It is readily seen that single-site probability distribution of the cumulative potential \( V(x, \omega) \), necessarily compactly supported when \( \omega \) are uniformly bounded and the series (1.5) (with \( q_y \) replaced with \( \omega_y \)) converges absolutely, cannot have an analytic density, for it would be compactly supported and not identically zero, which is impossible. However, in some class of marginal measures of \( \omega \) with unbounded support, considered long ago by Wintner [24] in the framework of Fourier analysis of infinite convolutions of singular probability measures, the single-site density of \( V(\cdot, \omega) \) can be analytic on \( \mathbb{R} \).

We also always assume that \( \omega_y \) are IID.

2. Main Results

2.1. Infinite smoothness of single-site distributions.

**Theorem 2.1** (Cf. [8, Theorem 1]). Consider the potential \( u(r) \) of the form (1.7), with \( A > d \) and let \( d \geq 1 \). Then the characteristic functions of the random variables \( V(x, \omega) \) of the form (1.5) obey the upper bound
\[
|\varphi_{V_x}(t)| \leq \text{Const} e^{-c|t|^{d/A}}.
\]
Consequently, for any \( d \geq 1 \) the r.v. \( V_x \) have probability densities \( \rho_x \in C^\infty(\mathbb{R}) \).

2.2. Infinite smoothness of the DoS.

**Theorem 2.2** (Cf. [8, Theorem 2]). Fix a 2-particle cube \( B = B_L(u) \).

(A) There exists a \( \sigma \)-algebra \( \mathcal{B}_B \), an \( \mathcal{B}_A \)-measurable self-adjoint random operator \( \tilde{H}_B(\omega) \) acting in \( \ell^2(B) \), and a \( \mathcal{B}_B \)-independent real-valued r.v. \( \xi_B(\omega) \) such that
\[
H_B(\omega) = \tilde{H}_B(\omega) + \xi_B(\omega)1_B.
\]
(B) The characteristic function \( \varphi_{\xi_B} \) of \( \xi_B \) satisfies the decay bound
\[
|\varphi_{\xi_B}(t)| \lesssim e^{-|t|^{d/A}}.
\]
2.3. Wegner estimate. The next result is a development of [8, Theorem 3] for $N$-particle Hamiltonians.

**Theorem 2.3** ("Frozen bath" Wegner estimate). Fix the exponent $\varkappa > 1$ in the definition (1.6) of the sequence $r_k$, and consider a ball $B = B_L(u)$ and the Hamiltonian $H_B$. Let

$$\tau_{\varkappa} = \frac{\varkappa}{\varkappa - 1}. \quad (2.3)$$

Fix any $\tau > 2\tau_{\varkappa}$, consider a larger set $\overline{B} = B_{L^{\tau}}(u) \setminus B_L(u)$, and denote by $P_{\overline{B}} : \{\cdot\}$ the conditional probability given the $\sigma$-algebra $\mathcal{F}_{Z^d} \mid \overline{B}$. Then for any $\varepsilon \geq \varepsilon_L := L^{-A\tau}$

$$P_{\overline{B}} \{ \text{dist}(\Sigma_B, E) \lesssim \varepsilon_L \} \lesssim |B| L^{(A - \frac{d}{2})\tau_{\varkappa}} \varepsilon_L. \quad (2.4)$$

In particular, with $\tau > 2\tau_{\varkappa}$ one has for some $c > 0$

$$P_{\overline{B}} \{ \text{dist}(\Sigma_B, E) \leq \varepsilon_L \} \lesssim |B| L^{\frac{A + c \tau}{2}}. \quad (2.5)$$

**Corollary 1** (Stable Wegner estimate).

$$P \Bigg\{ \omega_{\overline{B}} : \inf_{\omega = (\omega_{\overline{B}}, \omega_B)} \text{dist}(\Sigma(H_B(\omega), E) \lesssim 2\varepsilon_L \Bigg\} \lesssim |B| L^{\frac{(A + c\tau)}{2}}. \quad (2.6)$$

**Remark 2.1.** It follows from the construction of the event in the LHS of (2.6) that it is measurable with respect to the $\sigma$-algebra generated by $\{V(y, \omega), y \in \Pi \overline{B}\}$. Consequently, any number of such events with pairwise distant centers are independent. This observation will be useful for the proof of Lemma 7.4.

**Theorem 2.4** (Eigenvalue comparison estimate). Fix the exponent $\varkappa > 1$ in the definition (1.6) of the sequence $r_k$, and consider a ball $B = B_L(u)$ and the Hamiltonian $H_B$. Then

$$\forall \varepsilon > 0 \quad P \{ \text{dist}(\Sigma(H_B), \Sigma(H_{B'})) \leq \varepsilon \} \leq C |B| L^{(A - \frac{d}{2})\tau_{\varkappa}} \varepsilon. \quad (2.7)$$

2.4. Localization. Below we denote by $\mathcal{B}_1(\mathbb{R})$ the set of all bounded Borel functions $\phi : \mathbb{R} \to \mathbb{C}$ with $\|\phi\|_{\infty} \leq 1$. As usual, $\langle x \rangle$ stands for $(|x|^2 + 1)^{1/2}$.

**Theorem 2.5** (Localization at low energy). Consider the potential $u(r)$ of the form (1.7), with $A > d > 1$, and let $b > d/2$. There exist an energy interval $I = [E_0, E_0 + \eta]$, $\eta > 0$, near the a.s. lower edge of spectrum $E_0$ of the random operator $H(\omega) = -\Delta + V(x, \omega) + U(x)$ such that

(A) with probability one, $H(\omega)$ has in $I$ pure point spectrum with square-summable eigenfunctions $\Psi(x, \omega)$ satisfying

$$|\Psi(x, \omega)| \leq C \psi(\omega) \langle x \rangle^{-b}; \quad (2.8)$$

(B) for any $x, y \in \mathbb{Z}^d$ and any connected subgraph $G \subseteq \mathbb{Z}^d$ containing $x$ and $y$ one has

$$\mathbb{E} \left[ \sup_{\phi \in \mathcal{B}_1(\mathbb{R})} \|1_x \mathcal{P}_I(\Omega_G(\omega)) \phi(\Omega_G(\omega)) 1_y \| \right] \leq \langle |x - y| \rangle^{-b}. \quad (2.9)$$
In the present paper, we focus mainly on the strong dynamical localization and privilege clarity of constructions and proofs, and for these reasons we use the fixed-energy MSA induction: it is substantially simpler than its variable-energy (energy-interval) counterpart initially developed in [11, 9] and streamlined by Klein and Nguyen [21, 22]. As it is well-known by now, the energy-interval MSA estimates, crucial to the proofs of spectral and dynamical localization, can actually be inferred from the fixed-energy variants without actually carrying out a separate energy-interval scale induction (the latter path has been employed by Germinet and Klein in [19] and subsequent papers). However, the direct derivations require some additional information and arguments, and the efficiency of the final estimates depends on the strength of the fixed-energy probabilistic bounds (cf. [15]) and on the specific form of the IAD (Independence At Distance) property featured by the model (cf. [2]).

Speaking of the method proposed by Elgart et al. [15], improving an older observation made by Martinelli and Scoppola [23], an exponential decay of the eigenfunctions in the localization interval of energies requires exponentially decaying probability bounds on unwanted events in the course of the fixed-energy analysis, and such bounds cannot be achieved today by the existing MSA techniques.

An alternative to the method of [15] was proposed in [4], and in the context of $N$-particle Anderson Hamiltonians in a Euclidean space it was used in [12] for the analysis of $N$-particle lattice models, and in [6] where a particular class of alloy potentials ("flat tiling" potentials) was studied. The specificity of the "flat tiling" alloys is that the sample space contains piecewise-constant functions whose plateaus can cover arbitrarily large cubes. A thorough analysis of the "staircase" potentials considered in the present paper shows that one can use a similar (and actually, even a slightly simpler) technique, and thus prove exponential spectral localization. I plan to provide the details in a forthcoming work.

The reader can also see that the main ingredients required for the energy-interval $N$-particle MSA induction are obtained in Sections 4 (eigenvalue concentration estimates for individual cubes) and 5 (eigenvalue comparison estimates for pairs of cubes). Therefore, a more tedious, direct proof of energy-interval estimates, leading to the exponential spectral localization, can also be obtained.

3. Fourier Analysis of Probability Measures

3.1. The Main Lemma.

**Lemma 3.1** (Cf. [7, Lemma 4.1]). Let be given a family of IID r.v.

$$X_{n,k}(\omega), \ n \in \mathbb{N}, \ 1 \leq k \leq K_n, \ K_n \asymp n^{d-1},$$

and assume that their common characteristic function $\varphi_X(t) = \mathbb{E} \left[ e^{itX} \right]$ fulfills

$$\ln \left| \varphi_X(t) \right|^{-1} \geq C_X t^2, \ |t| \leq t_0.$$ (3.1)
Let
\[ S(\omega) = \sum_{n \geq 1} \sum_{k=1}^{K_n} a_n x_{n,k}(\omega), \quad a_n \asymp n^{-A}, \]
\[ S_{M,N}(\omega) = \sum_{n=M}^{N} \sum_{k=1}^{K_n} a_n x_{n,k}(\omega), \quad M \leq N. \]

The following holds true.

(A) There exists \( C = C(C_X, t_0, A, d) \in (0, +\infty) \) such that
\[ \forall t \in \mathbb{R} \quad |\varphi_S(t)| \leq Ce^{-|t|^{d/A}}. \]

(B) For any \( N \geq (1 + c)M \geq 1 \) with \( c > 0 \), and \( t \) with \( |t| \leq N^A \),
\[ \ln \left| \mathbb{E} \left[ e^{i t S_{M,N}(\omega)} \right] \right|^{-1} \geq M^{-2A + d} t^2. \]

(C) Let \( I_\varepsilon = [a, a + \varepsilon] \subset \mathbb{R} \). Then for any \( \varepsilon \geq N^{-A} \) one has
\[ \mathbb{P} \{ S_{M,N}(\omega) \in I_\varepsilon \} \lesssim M^{A - d/2} \varepsilon, \quad (3.2) \]

3.2. Thermal bath estimate for the cumulative potential. Here we recall some of the results obtained in [3].

Lemma 3.2. Consider a random field \( V(x, \omega) \) on \( \mathbb{Z}^d \) of the form
\[ V(x, \omega) = \sum_{y \in \mathbb{Z}^d} u(y-x) \omega_y, \]
where \( u \) is given by (1.7) and \( \{ \omega_x, x \in \mathbb{Z}^d \} \) are bounded IID r.v. with nonzero variance. Then the following holds true:

(A) The common characteristic function \( \varphi_V(\cdot) \) of the identically distributed r.v. \( V(x, \omega) \), \( x \in \mathbb{Z}^d \), obeys
\[ \forall t \in \mathbb{R} \quad |\varphi_V(t)| \leq Ce^{-|t|^{d/A}}. \quad (3.3) \]

(B) Consequently, the common probability distribution function \( F_V(\cdot) \) of the cumulative potential at sites \( x \in \mathbb{Z}^d \) has the derivative \( \rho_V \in \mathcal{C}(\mathbb{R}) \).

(C) Let \( v_* := \inf \text{supp} \rho_V \), then \( F_V(v_* + \lambda) = o(|\lambda|^\infty) \).

4. Infinite smoothness of the DoS and Wegner estimates

4.1. DoS in a thermal bath.

Proof of Theorem 2.2. Assertion (A). Fix a 2-particle cube \( B_L(u), u = (u_1, u_2) \), and consider the two possible situations.

Case (I) \(|u_1 - u_2| \leq 4L\).

Then \( \exists \tilde{u} \in \mathbb{Z}^d \) such that (see Fig. 1)
\[ \Pi B_L(u) = B_L(u_1) \cup B_L(u_2) \subset \tilde{B} := B_{10L}(\tilde{u}). \]
Here is shown the physical, single-particle space $Z \equiv \mathbb{Z}^d$ and not the multi-particle, product space. For each fixed $x \in \mathcal{X}_n$, $n \geq n_o(L, \varepsilon)$, the potential $\omega, \Pi^{(\varepsilon)}(x - \cdot)$ takes a constant value on an annulus $\mathcal{A}_n(x) = B_r(x) \setminus B_{r_n}(x)$ (the leftmost light-gray arc), hence on the entire cube $B_r(0) \subset \mathcal{A}_n(x)$. Therefore the sum of such potentials is a random constant on $B_r(0)$ with a smooth probability measure. The regularity of the latter can be assessed essentially in the same way as for the individual values of the cumulative potential $V(y, \omega)$, $y \in B_r(0)$. The remaining potentials $\omega, \Pi^{(\varepsilon)}(x - \cdot)$ (those which are non-constant on $B_r(0)$) can be rendered non-random by conditioning.

In this case, we can argue as in [8] and find an infinite set of sites $\mathcal{X}(\tilde{B})$ such that the random function on the lattice

$$(x, \omega) \mapsto \sum_{y \in \mathcal{X}(\tilde{B})} \omega_y u(|y - x|)$$

generates on $\tilde{B}$ (hence on both 1-particle projection cubes $B_r(u_i), i = 1, 2$) a random, constant in space potential, viz.

$$(x, \omega) \mapsto \sum_{y \in \mathcal{X}(\tilde{B})} \omega_y u(|y - \tilde{u}|) 1_{\tilde{B}}(x) = \xi(\omega) 1_{\tilde{B}}(x),$$

where the r.v. $\xi$ has an infinitely smooth probability measure.

For the 2-particle potential generated on $B$ this gives $2\xi(\omega) 1_{\Pi B}(x)$, since both projections of $B$ are affected by the same random constant.

Obviously, all the EVs of the Hamiltonian in $B_r(u_i)$ subject only to the above random potential admit a representation

$$E_i(\omega) = \lambda_i + \xi(\omega),$$

with non-random shifts $\lambda_i$, whence the infinite smoothness of their individual probability measures.

The total random potential induced on $B_r(u_i)$ is decomposed into the sum

$$V_B(x, \omega) = W_B(x, \omega) + 2\xi(\omega) 1_B(x),$$

where $W_B(x, \omega)$ is independent of $\xi(\omega)$ since it is generated by the random amplitudes not encountered in $\xi(\omega)$. Therefore, we can first condition on $W_B(x, \omega)$ and obtain an infinitely smooth probability measure for each random EV

$$\omega \mapsto E_i(\omega) = \lambda_i(\omega) + 2\xi(\omega)$$
with the shift $\lambda_i(\omega)$ rendered nonrandom by conditioning, and then switch $W_B(x, \omega)$ on, thus obtaining the a priori (unconditional) probability measure of $E_i(\cdot)$ as the convolution of the two independent random summands $\lambda_i(\omega)$ and $2\xi(\omega)$. The resulting convolution measure is at least as smooth as the one of $2\xi(\omega)$.

**Case (II)** $|u_1 - u_2| > 4L$.

In this case we can arrange an infinite sequence of scatterers’ subsets $\mathcal{X}_n$ which induce on each of the projection cubes $B_L(u_1)$ and $B_L(u_2)$ respective constant random fields, albeit with different values of the random constants (see Fig. 2),

$$\begin{align*}
B_L(u_1) &\ni x \mapsto a_x \xi_x(\omega)1_{B_L(u_1)}, \\
B_L(u_2) &\ni x \mapsto c_x a_x \xi_x(\omega)1_{B_L(u_2)},
\end{align*}$$

with $c_x \in [0, 1]$, and the resulting random potential induced by the scatterer at $x$ thus acts as a random scalar operator $(1 + c_x)a_x \xi_x(\omega)1_{B_L(u_1)}$. Since $(1 + c_x)a_x \approx a_x$, we conclude as in case (I) that the convolution of all admissible r.v. $(1 + c_x)a_x \xi_x$ has a $C^\infty(\mathbb{R})$-density.

**Assertion (B).** The claim follows easily from the Main Lemma 3.1; we only need to identify the key ingredients of the latter:

$$\begin{align*}
\mathcal{X}_n := \left\{ x \in \mathbb{Z}^d : \text{dist}(x, \Lambda_L) \in [r_n, r_n + 1) \right\}, &\quad K_n := |\mathcal{X}_n|, \\
\{ \omega, x \in \mathcal{X}_n \} \leftrightarrow \left\{ X_{n,k}, k = 1, \ldots, K_n \right\}, \\
M := L, &\quad N = +\infty, \\
S_{M,N}(\omega) = \sum_{n=M}^{\infty} \sum_{k=1}^{K_n} a_n X_{n,k} \equiv \sum_{x: |x| \geq L} u(|x|) \omega_x.
\end{align*}$$

4.2. **Wegner estimates.** Aiming to the applications to Anderson localization, we now have to operate with a restricted, annular "bath" of finite size, the complement of which is
"frozen". This is necessary for obtaining a satisfactory replacement for the IAD property very valuable in the short-range interaction models.

**Proof of Theorem 2.3.** The required bound follows from assertion (C) of Lemma 3.1. Identification of the principal ingredients of Lemma 3.1 is as follows:

\[ X_n := \left\{ x \in \mathbb{Z}^d : |x| = n \right\}, \quad K_n := |X_n|, \]

\[ \{ \omega_{x, k} \} \leftrightarrow \{ X_n, k, k = 1, \ldots, K_{n} \} \]

\[ M := L^{\tau}, \quad \tau > \tau_e, \]

\[ N = L^\tau, \quad \tau > \tau_e, \]

\[ S_{M,N}(\omega) = \sum_{n=M}^{N} \sum_{k=1}^{K_n} a_n X_{n,k} \equiv \sum_{x:|x|\in[L,R_L]} u(|x|) \omega_{x} \]

Proceeding as in Theorem 2.2, we obtain the representation

\[ H_B(\omega) = \tilde{H}_B(\omega) + \xi_B(\omega) 1_B, \tag{4.1} \]

where the random operator \( \tilde{H}_B(\omega) \) is independent of the r.v. \( \xi_B(\omega) \). By Lemma 3.1, \( \xi_B \) fulfills, for any interval \( I \) of length

\[ \varepsilon_L = N^{-A} \equiv L^{-A\tau}, \tag{4.2} \]

the concentration estimate (cf. (3.2))

\[ \mathbb{P}\{ \xi_B(\omega) \in I_{\varepsilon_L} \} \lesssim M^{A^{-\frac{d}{2}}} \varepsilon_L \equiv C L(A^{-\frac{d}{2}}) \varepsilon_L \lesssim L^{-(A^{-\frac{d}{2}} + (A^\tau) - A^\tau)}. \tag{4.3} \]

In particular, with \( \tau \geq 2\tau_e \) we have

\[ \mathbb{P}\{ \xi_B(\omega) \in I_{\varepsilon_L} \} \lesssim L^{-\frac{1}{4}} \varepsilon_L^{-\frac{d}{4}}. \tag{4.4} \]

This proves the EVC estimate (2.6), since \( H_B(\omega) \) acts in the Hilbert space \( \ell^2(B) \) of finite dimension \( |B| \).

**5. Eigenvalue Comparison Bound. Proof of Theorem 2.4**

Consider two 2-particle cubes, \( B' = B_L(u^1) \) and \( B'' = B_L(u^2) \), of radius \( L \), with \( |u^1 - u^2| > \hat{C}L \). Introduce the lattice subsets

\[ X_n := \left\{ x \in \mathbb{Z}^d : \text{dist}(x,B_L(0)) \in [r_n,r_{n+1}] \right\}. \]

and the spherical layers \( A_r = \{ x : |x| \in [r,r+1) \} \), \( r \in \mathbb{N} \). We will have to work again with \( r_n \leq r < r_{n+1} \), where \( n \) suits the conditions \( r_{n+1} - r_n \geq CL \) (cf. [7, Eqn (6.3)–(6.4)], thus with \( r_n \gg L \). Elementary geometric arguments show that if \( \text{dist}(B_L(u^1),B_L(u^2)) \geq CL \), with a sufficiently large \( C > 0 \), then there exist constants \( C_1, C_2, c > 0 \) depending on the dimension \( d \) with the following properties. Exchanging if necessary \( u^1 \leftrightarrow u^2 \) and then
For $y \in B_L(u^1_1)$ one has $\overline{\pi}^{(\infty)}(x - y) = r_n^{-A}$, while for $y \in B_L(u^1_2)$, $\overline{\pi}^{(\infty)}$ jumps to the next plateau: $u^{(2)}(x - y) = r_n^{-A}$. The separation sphere between the two plateaus is indicated by the thick black circle. This sphere depends of course upon its centre $x$, but for all $x \in X$ obeying (5.1) with a suitable $c > 0$, the separation does occur with the same the radii $r_{n+1}$, $r_{n+1} - CL$.

$u^1_1 \leftrightarrow u^1_2$ (cf. Fig. 3), one can find an infinite sequence of lattice subsets $X_n \subset \mathbb{Z}^d$, $n \geq n_0$ such that for all $x \in X_n$, with

$$
|x - u^1_n| = r \in [r_{n+1} - C_1L, r_{n+1} - C_2L]
$$

we have for all $x \in A_r$, with some $n_1 \geq n + 1$,

$$
\overline{\pi}^{(\infty)}(x - y)1_{B_L(u^1_1)}(y) = r_n^{-A}1_{B_L(u^1_1)}(y),
$$

while for some $n_2, n_3 \geq n + n(C)$, where $n(C) \to +\infty$ as $\hat{C} \to +\infty$,

$$
\overline{\pi}^{(\infty)}(x - y)1_{B_L(u^1_2)}(y) = r_n^{-A}1_{B_L(u^1_2)}(y),
$$

Fix a measurable labeling of the eigenvalues of $H = H_B(\omega)$ and $H'' = H_{B'}(\omega)$, in increasing order: $\{\lambda''_{u,a} \in [1, |B'|] \}$ and, respectively, $\{\lambda''_{b,b} \in [1, |B''|] \}$. Denote

$$
\xi_{a,b}(\omega) = \lambda''_{u,a}(\omega) - \lambda''_{b,b}(\omega)
$$

and for the rest of the argument, fix some pair of indices $(a, b) \in [1, |B'|] \times [1, |B''|]$. Denote $X' = \bigcup_{n \geq n_0}X_n$, and decompose $\omega = (\omega_X, \omega_{X'})$. From this point on, $\omega_{X'}$ will be fixed, so the probabilistic estimates will be made with respect to the conditional probability $P\{ \cdot \mid \overline{\delta}_{X'}^{(\infty)} \}$. Writing

$$
V(y, \omega) = W(\omega_X) + W(\omega_X)(y), \quad W(\omega_X)(y) = \sum_{n \geq n_0} \sum_{x \in X_n} \omega_x \overline{\pi}^{(\infty)}(x - y),
$$

one obtains by straightforward calculations that

$$
\xi_{a,b}(\omega) = c_{a,b}(\omega_{X'}) + \eta_{a,b}(\omega_{X'}),
$$
where $\eta_{a,b}$ has a $C^\infty$-density $p_{a,b}(\cdot)$ with
\[ \|p_{a,b}(\cdot)\|_\infty \leq L^c, \quad c_x = \frac{x}{x-1}. \]

For example, $c_x \leq 2$ for $x \geq 2$.

Now the claim follows by counting the number of pairs $(a,b)$, which is $O(L^{2d})$. \hfill \square

6. ILS estimates at low energies via "thin tails"

**Theorem 6.1** (Stable ILS estimate). Fix any $L_0 > 1$ and consider the Hamiltonian $H_{B_{L_0}(u)}(\omega)$ with an arbitrary $u \in \mathbb{Z}^d$. Assume that the interaction potential is positive and decays as $u(r) = r^{-A}$, $A > d$, and introduce a larger ball $B^+ = B_{cL_0}(u)$ and the sigma-algebras

- $\mathcal{F}_B^+$ generated by all scatterers’ amplitudes $\omega_y$ with $y \in B^+$,
- $\mathcal{F}_B^\perp$ generated by all scatterers’ amplitudes $\omega_y$ with $y \in \mathbb{Z}^d \setminus B^+$.

Then for any $\theta \in (0,1)$ there exists some $C_\theta > 0$ such that
\[
\mathbb{P}\left\{ \mathbb{E}_{\omega_{IB}(u)}^{A} : \inf_{\omega=(\omega_{IB}(u),\omega_{IB}(u^\perp))} E_0^A(\omega) \leq L_0^{-\theta} \right\} \leq e^{-C_\theta L_0^d}, \tag{6.1} \]

Consequently, for any nontrivial amplitudes $\omega_x$ of the site potentials $x \mapsto u(C - x)$, for any $b > 0$ there exists a nontrivial interval $I_x = [0,E_x]$ and $L_0$ large enough such that one has
\[
\mathbb{P}\left\{ \mathbb{E}_{\omega_{IB}(u)}^{A} : \inf_{\omega=(\omega_{IB}(u),\omega_{IB}(u^\perp))} E_0^A(\omega) \leq E_x \right\} \leq L_0^{-b}. \tag{6.2} \]

**Proof.** Due to the assumed positivity of the interaction potential $U(2)$, we have with $B = B' \times B''$
\[
H_{B_{L}(u)}(\omega) \geq H_{B_{L}(u_1)}(\omega) \otimes 1_{B_{L}(u_2)} + 1_{B_{L}(u_1)} \otimes H_{B_{L}(u_2)}(\omega). \tag{6.2} \]

It has been noticed already in earlier works on multi-particle Anderson Hamiltonians (with regular probability distribution of the amplitudes of the site potentials) that the ILS estimate for $N$-particle Hamiltonians follow directly for their 1-particle counterparts (projection Hamiltonians). The derivation itself does not rely on the regularity properties of the disorder distribution, so the claim actually follows from its 1-particle variant established in [7]. \hfill \square

7. Proof of localization

7.1. Deterministic analysis. We adapt the strategy from [20].

Working with a Hamiltonian $H_{B_{L}(u)} = -\Delta_{B_{L}(u)} + gV$ in a given cube $B_{L}(u)$, it will be necessary to know the values of the amplitudes $\omega_y$ with $y$ in a larger cube $B_{R_L}(u) \supset B_{L}(u)$, where the specific choice of $R_L$ depends upon the decay rate $r \mapsto r^{-A}$ of the interaction potential $u(r)$, along with some other parameters of the model and of the desired rate of decay of EFCs to be proved. Below we set $R_L = L^\tau$, $\tau > 1$.

**Definition 7.1.** A 2-particle cube $B_{L}(u) = B_{L}(u_1) \times B_{L}(u_2)$ is called non-interactive (NI) if $|u_1 - u_2| > 4L$, and partially interactive (PI), otherwise.
Definition 7.2. Let be given a cube \( B = B_L(u) \). A configuration \( q \in \Omega_{Zd} \) is called

1. \((E, \delta, B)\)-non-singular iff the resolvent \( G_B(E) \) of the operator
   \[
   H_{B_L(u)} = -\Delta_{B_L(u)} + U[q]|_{B_L(u)}
   \]
   (cf. the definition of \( U[q] \) in (1.5)) is well-defined and satisfies
   \[
   \max_{x \in B_{E^2}(u)} \max_{y \in \partial B_{E^2}(u)} \|G_B(x, y; E)\| \leq \delta ;
   \]
   (7.1)

2. \((E, \varepsilon, B)\)-non-resonant iff
   \[
   \text{dist}(\Sigma(H_B), E) \geq \varepsilon .
   \]
   (7.2)

When the condition (7.1) (resp., (7.2)) is violated, \( q \) will be called \((E, \varepsilon)\)-singular (resp., \((E, \varepsilon)\)-resonant). We will be using obvious shortcuts \((E, \varepsilon)\)-NS, \((E, \delta)\)-S, \((E, \varepsilon)\)-NR, \((E, \varepsilon)\)-R.

Definition 7.3. Let be given a cube \( B = B_L(x) \) and a real number \( \tau > 1 \). Denote \( \overline{B} = B_{L^\tau}(x) \). A configuration \( q_{\overline{B}} \in \Omega_{\overline{B}} \) is called

1. \((E, \varepsilon)\)-SNS (strongly non-singular in \( B \), or stable non-singular) iff for any configuration of amplitudes \( q_{\Omega_{\overline{B}}} \in \Omega_{\overline{B}} \): the extension of \( q_{\Omega_{\overline{B}}} \) to the entire lattice, \( q = (q_{\Omega_{\overline{B}}} \cdot q_{\Omega_{\overline{B}}}) \) is \((E, \varepsilon)\)-NS in \( B \);

2. \((E, \varepsilon)\)-SNR (strongly NR, or stable NR) iff for the zero-configuration \( \Omega_{\overline{B}} \in \Omega_{\overline{B}}: q_{\Omega_{\overline{B}}} \equiv 0 \) the function \( V_B = U[q_{\overline{B}} + q_{\Omega_{\overline{B}}}]|_{\overline{B}} = U[q_{\overline{B}}]|_{\overline{B}} \) is \((E, \varepsilon)\)-CNR.

3. \((E, \delta, K)\)-strongly-good (\((E, \delta, K)\)-S-good) in \( B \) iff for the configuration \( \Omega_{\overline{B}} \) the cube \( B \) contains no collection of \( K \) or more PI cubes \( B_{L_k}(u') \), with pairwise \( L^\tau \)-distant centers \( u' \), none of which is \((E, \delta)\)-SNS.

In subsection 7.2 we work in the situation where the potential \( V : \mathbb{Z}^d \to \mathbb{R} \) is fixed, and perform a deterministic analysis of finite-volume Hamiltonians. It will be convenient to use a slightly abusive but fairly traditional terminology and attribute the non-singularity and non-resonance properties to various cubes \( B \) rather than to a configuration \( q \) or a cumulative potential \( V = U[q] \), which will be fixed anyway. Therefore, we will refer, for example, to \((E, \varepsilon)\)-NS balls instead of \((E, \varepsilon, B)\)-NS configurations \( q \).

7.2. Scaling scheme. Fix \( N \ni d \geq 1, A > d \) and the interaction potential \( u(r) \left( \sim r^{-A} \right) \) of the form (1.7). Further, fix an arbitrary number \( b > d \), which will represents the desired polynomial decay rate of the key probabilities in the MSA induction, and let

\[
\alpha > \tau > \frac{b}{A - d}, \quad N \ni S > \frac{b\alpha}{b - \alpha d}, \quad L_{k+1} = \left\lfloor L_k^\alpha \right\rfloor, \quad k \geq 0,
\]

(7.3)

with \( L_0 \) large enough, to be specified on the as-needed basis. A direct analog of the well-known deterministic statement [14, Lemma 4.2] is the following statement adapted to long-range interactions essentially as in [20]. Denote

\[
m_k := \left(1 + L_k^{-1/8}\right) m, \quad \varepsilon_k := 4L_k^{-(A-\frac{d}{2})\tau}, \quad \delta_k := e^{-m_k L_k}.
\]

(7.4)
Lemma 7.1 (Conditions for strong non-singularity). Consider a cube $B = B_{L_{k+1}}(u)$, $k \geq 0$, and suppose that

(i) $B$ is $(E, \varepsilon_k)$-SNR;

(ii) all non-interactive cubes $B_{L_k}(x) \subset B$ are $(E, \delta_k)$-SNS;

(iii) $B$ is $(E, \delta_k, K)$-S-good for some $K \in \mathbb{N}$.

There exists $L_*(K) \in \mathbb{N}$ such that if, in addition, $L_0 \geq L_*(K)$, then $B$ is $(E, m)$-SNS.

Proof. Derivation of the NS property can be done essentially in the same way as in [14] and in numerous subsequent papers, with minor adaptations. See for example [6, proof of Lemma 7] where the singular balls are also supposed to be pairwise $L_k^2$-distant, $\tau > 1$.

To show that the strong (stable) non-singularity property also holds true, one can use induction on scales $L_k$. We have to show that the NS property of the larger cube $B_{L_{k+1}}(u)$ is stable with respect to arbitrary fluctuations of the random amplitudes $\omega_y$ with $y \notin \Pi B_{L_{k+1}}(u)$. According to what has just been said in the previous paragraph, it suffices to check the stability of the properties

(i') $B_{L_{k+1}}(u)$ is $(E, \varepsilon_k)$-NR,

(ii') $B_{L_{k+1}}(u)$ contains no collection of cubes $\{B_{L_k}(x^i) : 1 \leq i \leq S + 1\}$, with pairwise $2L_k^2$-distant centers, neither of which is $(E, m)$-NS,

under the hypotheses (i)–(ii).

There is nothing to prove for the stability of (i'), as it is asserted by (i).

On the scale $L_0$ the non-singularity is derived from non-resonance, with a comfortable gap between an energy $E$ and the spectrum in the cube of radius $L_0$, which provides the base of induction. Evidently, given any cube $B_{L_j}(x) \subset B_{L_{k+1}}(u)$ one has

$$\forall j = 0, \ldots, k \quad B_{L_{j+1}}^x(u) \subset B_{L_j}^x(x).$$

In other words, stability encoded in the SNS or SNR properties of smaller balls $B_{L_j}(x) \subset B_{L_{k+1}}(u)$ is stronger than what is required for the stability w.r.t. fluctuations $\omega_y$ outside a much larger cube $B_{L_{k+1}}^y(u)$. We conclude that the claim follows indeed from the the hypotheses (i)–(ii). \hfill \Box

7.3. Conditions for non-singularity of NI cubes.

Lemma 7.2. Consider a NI cube $B = B_{L_k}^{(N)}(u) = B' \times B''$, and the respective reduced Hamiltonians $H' = H_{B'}$ and $H'' = H_{B''}$. Assume that $A$ is $(E, 2\varepsilon_k)$-SNR and, in addition,

- $\forall \lambda' \in \Sigma(H')$ the cube $B''$ is $(E - \lambda', \delta_k)$-SNS, and
- $\forall \lambda'' \in \Sigma(H'')$ the cube $B'$ is $(E - \lambda'', \delta_k)$-SNS.

Then $B$ is $(E, \delta_k)$-SNS.

The proof of this deterministic statement is similar to that of its counterpart from [10] and subsequent papers on $N$-particle localization with short-range site potentials, except for the stability aspect. Since the non-singularity of the projection cubes is assumed to be stable (SNS), so is the resulting non-singularity of the 2-particle cube $B_{L_k}(u)$. 

7.4. **Probabilistic analysis.** The following statement is merely an adaptation of Corollary 2.6, stated in a form suitable for the fixed-energy scaling analysis.

**Lemma 7.3** (Probability of SNR-cubes). *For all \(k \in \mathbb{N}\) and SI cubes \(B_{L_k}(u)\) one has*

\[
P\left\{ B_{L_k}(u) \text{ is } (E, \varepsilon_k)\text{-SNR} \right\} \geq 1 - L_k^{-\frac{(A+\tau)p}{2}}. \tag{7.5}
\]

It follows from Definition 7.3 that any event of the form

\[
A\left( B_L(x), E, m \right) = \left\{ V_q(\cdot; \omega) \big| B_L(x) \text{ is (E, m)-SNS} \right\}
\]

is measurable w.r.t. the sigma-algebra \(\mathcal{F}_{B_L(x)}\).

The next estimate of the probability of occurrence of multiple singular 2-particle cubes \(B_i \equiv B_{L_k}(u^{(i)})\), \(1 \leq i \leq S_{k+1}\), is quite similar to its single-particle counterpart, since it treats the case of distant PI cubes, each located — by definition — “closely enough” to the diagonal, so that their full projections \(\Pi B_i = B_{L_k}(u^{(i)}) \cup B_{L_k}(u^{(i)})\) are pairwise distant, essentially as in the single-particle case. The main technical difference is that now we have to control the fluctuations of the locally constant (due to the staircase nature of \(u\)) random potential on the entire projections \(\Pi B_i\), while in the 1-particle systems one has \(\Pi B_{L_k}(u) \equiv B_{L_k}(u)\). In fact, a necessary adaptation was already made in the eigenvalue concentration estimate given in Section 4.

**Lemma 7.4** (Probability of a bad PI-cluster). *Assume that \(A > 2Nd + 3\gamma\) with \(\gamma > 0\), and \(\tau > 2 + \frac{A}{Nd}\). Set \(s = (A - Nd)\tau - (A + Nd + 1)\), \(\sigma = \gamma(Nd)^{-1}\) and \(\alpha = (1 + \sigma)\tau\). Then*

\[
s - \alpha Nd > \gamma \tau > 0. \tag{7.6}
\]

*Further, let \(K \in \mathbb{N}\) satisfy \(\frac{2M(1+\sigma)s}{\tau} < K < L_k^{\alpha-\tau}, M \geq 1\). Then for \(L_0\) large enough*

\[
\sup_{u \in \mathbb{Z}^d} P\left\{ B_{L_{k+1}}(u) \text{ is not } (E, \varepsilon, \tau, K)\text{-S-good} \right\} \leq p_{k+1} \leq L_{k+1}^{-M_0}. \tag{7.7}
\]

*Proof.* (7.6) follows from the assumptions on \(A\) and \(\tau\) by a simple calculation:

\[
s - \alpha Nd = (A - Nd)\tau - (A + Nd + 1) - (1 + \sigma)Nd\tau
\]

\[
= (\gamma + (\gamma - \sigma Nd))\tau - (A + 2Nd)
\]

\[
> \gamma \tau + (\gamma \tau - (A + 2Nd)) > \gamma \tau.
\]

Further, by Remark 2.1, any collection of events \(E_i = \left\{ B_{L_k}(u^i) \text{ is not } (E, m)\text{-SNS} \right\}\) with pairwise \(L_k\)-distant centers \(u^i\) is independent, hence

\[
P\left\{ \bigcap_{i=1}^{K} E_i \right\} = \prod_{i=1}^{K} P\left\{ E_i \right\} \leq p_k^K.
\]

By (7.6) we have \(\frac{s - \alpha Nd}{\alpha} > \frac{\tau}{1 + \sigma}\). Thus with \(K > 2Ms(1 + \sigma)\gamma^{-1}\), we have for the maximal number \(S(\omega)\) of pairwise \(L_k\)-distant singular PI cubes \(B_{L_k}(x^i)\) inside \(B_{L_{k+1}}(u)\):

\[
P\left\{ S(\omega) \geq K \right\} \leq L_{k+1}^{K\alpha} p_k^K \leq L_{k+1}^{-K}\left(\frac{s}{\alpha} - Nd\right) \leq L_{k+1}^{-M_0}. \tag{7.8}
\]
The claim is proved. □

It is to be stressed that the positive integer $M$ (hence the exponent $M\delta$ of the length scale $L_{k+1}$ in (7.7)) can be made arbitrarily large by taking $L_0$ large enough.

The next statement relies on the 1-particle localization results for the staircase potentials (cf. [8]).

**Lemma 7.5.** If $L_0$ is large enough, then for any $E \in \mathbb{R}$ and any NI cube $B_{L_k}(u)$ one has

$$\mathbb{P}\left\{ B_{L_k}(u) \text{ is } (E,m)\text{-S} \right\} \leq L_{k+1}^{-\frac{2b}{4}}. \quad (7.9)$$

Consequently, for any $x$

$$\mathbb{P}\left\{ B_{L_{k+1}}(x) \text{ contains a NI } (E,\delta)\text{-S cube } B_{L_k}(u) \right\} \leq \frac{1}{4}L_{k+1}^{-b}. \quad (7.10)$$

The proof is similar to that of [12, Lemma 3.4].

**Lemma 7.6** (Scaling of probabilities). Consider a Hamiltonian $H$ where, as in Lemma 7.4, the site potential $u(r) = r^{-A}, A > 2Nd + 3\gamma, \gamma > 0$. Let $s = (A - Nd)\tau - (A + Nd + 1)$, and assume that the scale growth exponent (cf. (7.3)) has the form $\alpha = (1 + \sigma)\tau$ with $\sigma = \gamma/(Nd)$. Fix an arbitrarily large $b > 0$ and assume that

$$\sup_{u \in \mathbb{Z}^{Nd}} \mathbb{P}\left\{ B_{L_k}(u) \text{ is not } (E,m)\text{-SNS} \right\} \leq p_k \leq L_{k+1}^{-b}.$$  

Furthermore, let the cluster cardinality parameter $K$ in the definition of $(E,\delta,K,\tau)$-good cubes satisfy $K > \frac{2M(1+\sigma)}{\gamma}$, where $N \ni M > \frac{2b}{s}$. Then

$$\sup_{u \in \mathbb{Z}^{Nd}} \mathbb{P}\left\{ B_{L_{k+1}}(u) \text{ is not } (E,m)\text{-SNS} \right\} \leq p_{k+1} \leq L_{k+1}^{-b}.$$  

**Proof:** By Lemma 7.1, if $B_{L_{k+1}}(u)$ is not $(E,m)$-SNS, then

(i) either it is not $(E,\varepsilon_{k+1})$-SCNR,

(ii) or it contains a $(E,m)$-singular, non-interactive cube $B_{L_k}(y)$,

(iii) or it is not $(E,\delta_k,\tau,K)$-S-good.

- The probability of the event (i) is assessed with the help of the Wegner-type estimate from Theorem 2.3, relying on the disorder in the cubes $B_{L_{k+1}}(u)$ with $B_{L_k}(u) \subseteq B_{L_{k+1}}(u)$. The largest of these cubes, $B_{L_{k+1}}(u)$, is surrounded by a belt of width $L_{k+1}^{\tau}$ where the random amplitudes are not fixed hence can contribute to the Wegner estimate with $\varepsilon = L_{k+1}^{-A\tau}$, hence the same is true for all of these balls: we have (cf. (2.3) and (2.6))

$$\mathbb{P}\left\{ B_{L_{k+1}}(u) \text{ is not } (E,\varepsilon_R)\text{-SNR} \right\} \leq L_{k+1}^{-\frac{1}{2}A\tau}. \quad (7.11)$$

Since $\tau > 2(b+1)/A$, the RHS of (7.11) is bounded by $\frac{1}{2}L_{k+1}^{-b-1}$ with $b' > b$. The total number of such balls is $Y_{k+1} = L_{k+1}^{\alpha-1} = \frac{1}{2}L_{k+1}^{1-\alpha-1}$, with $1 - \alpha^{-1} < 1$, whence

$$\mathbb{P}\left\{ B_{L_{k+1}}(u) \text{ is not } (E,\varepsilon_R)\text{-CNR} \right\} \leq L_{k+1}^{-(b'+1)} \leq \frac{1}{4}L_{k+1}^{-b}.$$  

- Next, by Lemma 7.5, the probability of (ii) is upper-bounded by $\frac{1}{4}L_{k+1}^{-b}$. 

To assess the probability of \( (iii) \), recall that by Lemma 7.4

\[
P\{S_{k+1} > K\} \leq \frac{1}{4} L_{k+1}^{-M_s},
\]

whence

\[
P\{B_{L_{k+1}}(u) \text{ is not } (E, m)\text{-SNS} \} \leq \frac{1}{2} L_{k+1}^{-b} + \frac{1}{2} L_{k+1}^{-b} = L_{k+1}^{-b}.
\]

Collecting the above three estimates, the claim follows. \( \Box \)

By induction on \( k \), we come to the conclusion of the fixed-energy MSA under a polynomially decaying interaction.

**Theorem 7.1.** Suppose that the ILS estimate

\[
\sup_{u \in \mathbb{Z}^d} P\{B_{L_0}(u) \text{ is not } (E, m)\text{-SNS}\} \leq L_0^{-b} \tag{7.12}
\]

holds for some \( L_0 \) large enough, uniformly in \( E \in I_* \subset \mathbb{R} \). Then for all \( k \geq 0 \) and all \( E \in I \)

\[
\sup_{u \in \mathbb{Z}^d} P\{B_{L_k}(u) \text{ is not } (E, m)\text{-SNS}\} \leq L_k^{-b}. \tag{7.13}
\]

The required ILS estimate is established in Section 6. This concludes the fixed-energy MSA induction.

### 8. Derivation of Spectral and Dynamical Localization

#### 8.1. Energy-interval estimates.

Proposition 8.1, proved in [4], provides an alternative to an earlier method developed by Elgart et al. [15], and Proposition 8.3 is essentially a reformulation of an argument by Germinet and Klein (cf. [19, proof of Theorem 3.8]) which substantially simplified the derivation of strong dynamical localization from the energy-interval MSA bounds, compared to [13].

Introduce the following notation: given a cube \( B_L(z) \) and \( E \in \mathbb{R} \),

\[
F_{x,L}(E) := \left| B_L(z) \right| \max_{|y-z| \leq L} \left| G_{B_L}(z, y; E) \right|
\]

with the convention that \( \left| G_{B_L}(z, y; E) \right| = +\infty \) if \( E \) is in the spectrum of \( H_{B_L(z)} \). Further, for a pair of balls \( B_L(x), B_L(y) \) set

\[
F_{x,y,L}(E) := \max \left[ F_{x,L}(E), F_{y,L}(E) \right].
\]

The fixed-energy MSA in an interval \( E \in I \subset \mathbb{R} \) provides probabilistic bounds on the functional \( F_{1,L}(E) \) of the operator \( H_{B_L(x)}(\omega) \); as a rule, they are easier to obtain that those on \( \sup_{E \in I} F_{x,y,L}(E) \) (referred to as energy-interval bounds). Martinelli and Scoppola [23] were apparently the first to notice a relation between the two kinds of bounds, and used it to prove a.s. absence of a.c. spectrum for Anderson Hamiltonians obeying suitable fixed-energy bounds on fast decay of their Green functions. Elgart, Tautenhahn and Veselić [15] improved the Martinelli–Scoppola technique, so that energy-interval bounds implying spectral and dynamical localization could be derived from the outcome of the fixed-energy MSA.
Proposition 8.1 (Cf. [4, Theorem 4]). Let be given a random ensemble of operators \( H(\omega) \) acting in a finite-dimensional Hilbert space \( \mathcal{H}, \dim \mathcal{H} = D \), two subspaces \( \mathcal{H}', \mathcal{H}'' \subset \mathcal{H} \) with their respective orthonormal bases \( \{ \phi_i, 1 \leq i \leq D' \} \) and \( \{ \psi_j, 1 \leq j \leq D'' \} \), an interval \( I \subset \mathbb{R} \) and real numbers \( a, p > 0 \) such that for all \( E \in I \) the function
\[
\mathcal{M} : (E, \omega) \mapsto \max_i \max_j \| \Pi_\phi G(E, \omega) \Pi \psi_j \|
\]
(with \( \Pi_\phi \equiv |\phi \rangle \langle \phi| \)) satisfies
\[
\mathbb{P}\{ \mathcal{M}(E, \omega) \geq a \} \leq p. \tag{8.1}
\]
Then the following holds true:

(A) For any \( b > p \) there exists an event \( B(b) \subset \Omega \) such that \( \mathbb{P}\{ B(b) \} \leq b^{-1} p \) and for any \( \omega \not\in B(b) \) the random set of energies
\[
\mathcal{E}(a, \omega) := \{ E \in I : \mathcal{M}(E, \omega) \geq a \}
\]
is covered by \( K < 3n'n''N \leq 3N^4 \) intervals \( J_k = [E^-_k, E'^+_k] \) of total length \( \sum_k |J_k| \leq b \).

(B) The random endpoints \( E^k_\pm \) depend upon \( H \) in such a way that, for a one-parameter family \( A(t) := H(\omega) + t \mathbf{1} \), the endpoints \( E^k_\pm(t) \) for the operators \( A(t) \) (replacing \( H \)) have the form
\[
E^k_\pm(t) = E^k_\pm(0) + t, \quad t \in \mathbb{R}. \tag{8.2}
\]

For our purposes, it suffices to set \( b = p^{1/2} \) in assertion (A).

The next statement is an adaptation of [6, Theorem 6].

Theorem 8.2. Consider two 2-particle cubes \( B' = B_L(x), B'' = B_L(y) \), and introduce the functions
\[
\mathcal{M}_x(E, \omega) = \max_{z \in \partial B_L(x)} |G_{B_L(x)}(x, z; E)|, \\
\mathcal{M}_y(E, \omega) = \max_{z \in \partial B_L(y)} |G_{B_L(y)}(y, z; E)|. \tag{8.3}
\]

Then
\[
\mathbb{P}\{ \exists E \in I : \min [\mathcal{M}_x(E, \omega), \mathcal{M}_y(E, \omega)] \geq a \} \lesssim |B_L(x)||B_L(y)|L^{4p} p^{1/2}. \tag{8.4}
\]
The proof follows essentially the same path as in [4, Proof of Theorem 5], and is even slightly simpler, for it uses a representation
\[
\mathbf{V}(z, \omega) \mathbf{1}_{B_L(x)} = \xi(\omega) \mathbf{1}_{B_L(x)} + \mathbf{V}'(z, \omega) \mathbf{1}_{B_L(x)}, \\
\mathbf{V}(z, \omega) \mathbf{1}_{B_L(y)} = c \xi(\omega) \mathbf{1}_{B_L(y)} + \mathbf{V}''(z, \omega) \mathbf{1}_{B_L(y)}, \tag{8.5}
\]
where \( \mathbf{V}'(z, \omega) \) and \( \mathbf{V}''(z, \omega) \) are measurable with respect to some sub-sigma-algebra \( \mathcal{B} \subset \mathcal{F}, \) while \( \xi(\cdot) \) is independent of \( \mathcal{B} \) and has the continuity modulus \( \varepsilon_x \). Such a representation was obtained in Section 5. In [6] one had to assess first the (random) continuity modulus of the conditional sample mean of the random potential in a finite cube.
8.2. Decay of eigenfunction correlators. Given an interval \( I \subset \mathbb{R} \), denote by \( \mathcal{B}_1(I) \) the set of bounded Borel functions \( \phi : \mathbb{R} \to \mathbb{C} \) with \( \text{supp} \phi \subset I \) and \( \| \phi \|_{\infty} \leq 1 \).

**Proposition 8.3** (Cf. [5, Theorem 3], [19]). Assume that the following bound holds for some \( \varepsilon > 0 \), \( h_L > 0 \), \( L \in \mathbb{N} \) and a pair of balls \( B_L(x), B_L(y) \) with \( |x - y| \geq 2L + 1 \):

\[
\mathbb{P}\left\{ \sup_{E \in I} \min_{z \in \{x, y\}} M_z(E, \omega) > \varepsilon \right\} \leq h_L. \tag{8.6}
\]

Then for any cube \( B \supset (B_{L+1}(x) \cup B_{L+1}(y)) \)

\[
\mathbb{E}\left[ \sup_{\phi \in \mathcal{B}_1(I)} \left| \langle 1_x, \phi(H_B) 1_y \rangle \right| \right] \leq 4\varepsilon + h_L. \tag{8.7}
\]

**Proof of assertion (B), Theorem 2.5.** The validity of the condition (8.6) with \( h_L = L^{-\frac{1}{2b}+C(A)} \) follows from Theorem 8.2. Since \( b > 0 \) can be made arbitrarily large by taking \( L_0 \) and the auxiliary parameter \( \tau \) large enough, the assertion (B) of Theorem 2.5 follows. \( \square \)

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