REALITY OF NON-FOCK SPINORS

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Abstract. The infinite-dimensional Clifford algebra has a maze of inequivalent irreducible unitary representations. Here we determine their type - real, complex or quaternionic. Some, related to the Fermi-Fock representations, do not admit any real or quaternionic structures. But there are many on $L^2$ of the circle that do and which seem to have analytic meaning.

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1. Introduction.

Let $H$ be a separable real prehilbert space and let $C(H)$ be the Clifford algebra of $H$, i.e., the quotient of the tensor algebra $\mathcal{T}(H)$ of $H$ by the ideal generated by the elements of the form

$$h \otimes h' + h' \otimes h + 2 < h, h' > \quad h, h' \in H.$$

Here we parametrize all the equivalence classes of representations of $C(H)$ on a separable real Hilbert space $U$, where $H \subset C(H)$ acts via skew-symmetric operators (“orthogonal”). $U$ is a space of (real) spinors. In infinite dimensions there is “a true maze” of inequivalent irreducible ones, in striking contrast to the finite case.

Choosing an orthonormal basis of $H$ and letting $J_k$ denote the action of $k^{\text{th}}$ element of the basis on $U$, the operators $J_1, J_2, \ldots$, are orthogonal complex structures on $U$ which anticommute with each other. We will often ignore $H$ altogether and regard a spinor structure on a real Hilbert space $U$ as a sequence of linear operators $J_1, \ldots, J_k, \ldots$, on it satisfying

$$||J_kv|| = ||v||, \quad J_k^2 = -I, \quad J_k J_l = -J_l J_k$$

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for all $k$ and all $l \neq k$.

The complex spinors, i.e., the unitary representations of $\mathbb{C} \otimes C(H)$, are the same as the representations of $C(H)$ on separable complex Hilbert spaces $V$ satisfying

$$||h \cdot v|| = ||h|| \cdot ||v||$$

for $h \in H \subset C(H)$ and $v \in V$. As we explain in §2, when $\dim H$ is even or infinite, $\mathbb{C} \otimes C(H)$ is the same as $\mathbb{C} \otimes T(H)$ modulo the so-called Canonical Anticommutations Relations of Quantum Field Theory. Gårding and Wightman’s parametrized the representations of the latter in [GW1]. Therefore, their parametrization yields a corresponding one of all the complex spinor structures up to unitary equivalence. We describe the result in detail in §2.

To parametrize the real representations it is then enough to determine the values of the Gårding-Wightman (or GW) parameters whose corresponding complex representation admits an invariant real structure, i.e., a $\mathbb{C}$-antilinear, norm-preserving operator $S$ such that

$$S^2 = I.$$

Then

$$U = \{v: Sv = v\}$$

is an invariant real form of $V$ which, by restriction, provides a real representation of $C(H)$. Since the complexification of a real representation of $C(H)$ is a representation of $\mathbb{C} \otimes C(H)$, one obtains a parametrization of the former.

Similar arguments yield those complex representations possessing a $\mathbb{C}$-antilinear, norm-preserving operator $Q$ such that

$$Q^2 = -I.$$

In the classical (finite dimensional) case, a complex representation of $C(H)$ is defined as of real, quaternionic or complex type, according to whether it admits an $S$, a $Q$, or neither, conditions that are mutually exclusive when the representation is irreducible. In the physics literature $S$ and $Q$ are called charge conjugation operators and the irreducible representations of real type Majorana spinors. By the way, we recover the classical result of Cartan and Dirac, namely that the unique irreducible complex representation of $C(\mathbb{R}^{2m})$ is of real type if and only if $m \equiv 0, 3$ modulo 4 and of quaternionic type otherwise. The geometric and physical significance of this in the finite dimensional case is well known (see, e.g., [C] and other chapters in the same book).

In infinite dimensions we find mazes of inequivalent irreducible spinors of each of the three types. Those corresponding to the Fermi-Fock representations of the Canonical Anticommuation Relations admit no real or quaternionic structures. But there are plenty of natural irreducible spinor structures of real and of quaternionic type on $L^2$ of the circle.

It is important to note that the final result is just a parametrization of all equivalence classes of unitary spinor structures of the three types. The questions of irreducibility and equivalence of the GW representations for the various values of the parameters are not completely resolved yet. Also, although every unitary representation of this algebra is completely reducible, the reduction is highly non-unique. These constitute insurmountable obstacles for proving most general statements about spinors in infinite dimensions using the GW parametrization. In a
way, one purpose of this paper is to exhibit one problem, namely the classification into types, for which these obstacles can be surmounted and has a neat answer.

As to other purposes, we mention some preliminary algebraic and analytic consequences.

First, and much like when \( \dim H = 1, 3, 7 \), such real representations are in correspondence with certain division algebras considered by Kaplansky \([K]\) as hypothetical infinite-dimensional analogs of the Octonions -although he himself was doubtful of their existence. Here is then a complete parametrization of such algebras. Of course, they are not commutative or associative, and they contain only one-sided inverses, but there are mazes of inequivalent ones. Regardless of their numerical status, their automorphism groups are reductive and come unitarily represented. By restriction, one obtains mazes of new irreducible unitary representations of the classical infinite dimensional Lie groups and algebras that appear as factors.

Secondly, there are interesting families of representations of \( C(H) \) on \( L^2(T) \) (or \( L^2(\mathbb{R}) \)) of real or quaternionic type which have some analytic content. For example, those of real type yield all manners of fitting the Hilbert transform

\[
\mathcal{H}f(x) = \int_{-\infty}^{\infty} \frac{f(y)}{x-y} \, dy
\]

into a sequence of mutually anticommuting real singular integral operators \( \mathcal{H}, \mathcal{H}_2, \mathcal{H}_3, \ldots \) of square \(-I\). The corresponding kernels are dyadic twistings of the Hilbert kernel and lead to analogs of the Cauchy kernel. In the quaternionic case, the charge conjugation operators are obtained by twisting real ones by Haar’s mother wavelet. Intriguing as they may be, we will not go into much detail about these issues here.

Instead, we will discuss two operators,

\[
D = \sum_{k=1}^{\infty} a_k \partial_k, \quad D' = \sum_{k=1}^{\infty} a_k^* \partial_k
\]

where \( a_k, a_k^* \), are the creation and anihilation operators associated to any spinor structure and the \( \partial_k \) are certain dyadic difference operators. Notably, for the standard Fermi-Fock representations they diverge off the vacuum. But for the spinor structures in \( L^2(T) \) that we discuss below, they have a dense domain and are mutually conjugate under any charge conjugation operator. We found remarkable that for one of these families, parametrized by infinite matrices of 0’s and 1’s, the associated operators \( D \) and \( D' \), which are far from self adjoint, can be diagonalized over \( \mathbb{Z} \): with integral eigenvalues and eigenfunctions that are polynomials with integral coefficients in the classical periodic Rademacher functions.

Any connection of all this with the real world must take into account that, as we prove below, a real or quaternionic structure requires that, in the standard statistical interpretation, changing all the occupied states to non-occupied and vice versa be a well defined operation. This may be an unlikely feature for fermions, but not necessarily for other systems of 0’s and 1’s. Indeed, the properties of \( D, D' \) and the higher Hilbert transforms \( \mathcal{H}_k \) seem more related to wave packings, splines and binary codes than to any particles or fields.

As this is a preliminary version, some proofs are only sketched, others are found in \([GKL]\) and a complete version will be ready shortly.

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2. Spinors as dyadic objects.

Let

\[ X = \mathbb{Z}_2^{\infty} \]

be the set of sequences \( x = (x_1, x_2, \ldots) \) of 0’s and 1’s, and \( \Delta \subset X \) the subset consisting of sequences with only finitely many 1’s. Then \( X \) is an abelian group under componentwise addition modulo 2 and \( \Delta \) is the subgroup generated by the sequences \( \delta^k \), where \( \delta^k \) is the Kronecker symbol. The product topology on \( X \) is compact and is generated by the sets

\[ X_k = \{x: x_k = 1\}, \quad X'_k = \{x: x_k = 0\}, \]

which, therefore, also generate the canonical \( \sigma \)-algebra of Borel sets in \( X \).

We will realize all the complex spinor structures on \( L^2 \) spaces of \( \mathbb{C} \)-valued functions on \( X \) or direct integrals thereof. As a motivation, let us realize the standard finite even-dimensional spinors in this manner. For each positive integer \( N \) consider the vector space

\[ V_N = \mathbb{C} \mathbb{Z}_2^N. \]

Then, clearly, \( \dim V_N = 2^N \) and the operators

\[
\begin{align*}
J_k f(x) &= -i(-1)^{x_1 + \cdots + x_{k-1}} f(x + \delta^k) \\
J'_k f(x) &= (-1)^{x_1 + \cdots + x_k} f(x + \delta^k)
\end{align*}
\]

where \( 1 \leq k \leq N \), \( x \in \mathbb{Z}_2^N \), addition is modulo 2 and the \( \delta^k \) is the standard basis of \( \mathbb{Z}_2^N \), define an irreducible complex representation of the Clifford algebra \( C(\mathbb{R}^{2N}) \) -the unique one modulo equivalence. In spite of its simplicity and of being implicit in the work of Friedrichs, Gårding, Wightman and von Neumann on the Anticommutation Relations, this description of even dimensional spinors does not seem to have been made explicit or exploited before.

The unitarity underlying the finite case is relative to the natural \( L^2 \) inner product in \( V_N \), which in turn is associated to the measure on \( \mathbb{Z}_2^N \) where each point has measure 1. But changing the measure to any equivalent one or changing the target space of the functions does not change the equivalence class of the representation.

This is no longer so when \( N = \infty \): in order to reach all equivalence classes one must allow for more general measures on the group \( X = \mathbb{Z}_2^{\infty} \) and replace \( \mathbb{C} \)-valued functions for sections of appropriate fiber spaces over \( X \). Two canonical, but very different measures on \( X \) that generalize the finite case are:

- \( \mu_X \), the Haar measure of \( X \).
- \( \mu_\Delta \), concentrated in the discrete set \( \Delta \), where \( \mu_\Delta(\{\delta\}) = 1 \). \( \mu_\Delta \) could be called the Fermi-Fock measure.

The first is invariant under all translations in \( X \) while the second is invariant only under those from \( \Delta \). It is \( \mu_\Delta \) that leads to the representations that appear most in QFT, however implicitly. It ignores all the points \( x \) with infinitely many \( x_i = 1 \), or occupied states, on the basis that the total number of particles -fermions in this case, must be finite. In any case, (2.1) define irreducible representations of \( C(H) \) on \( L^2(X, \mu_X) \) and \( L^2(X, \mu_\Delta) \), respectively, which we will prove to be inequivalent.
Recall that two measures $\lambda, \mu$ on a the same Borel algebra of sets can be said to be equivalent if there exists locally integrable functions, denoted by $d\lambda/d\mu$ and $d\mu/d\lambda$, such that for any measurable $A$, these Radon-Nykodim derivatives satisfy

$$
\lambda(A) = \int_A \frac{d\lambda}{d\mu} \, d\mu, \quad \mu(A) = \int_A \frac{d\mu}{d\lambda} \, d\lambda.
$$

$\mu$ is said to be quasi-invariant under a set of transformations $\{T\}$ of the underlying space, if the translated measures

$$
\mu_T(A) := \mu(T(A))
$$

are all equivalent to $\mu$. Now, consider triples

$$(\mu, V, C)$$

where

- $\mu$ is a positive Borel measure on $X$, quasi-invariant under $\Delta$.

- $V = \{V_x\}_{x \in X}$ is a family of complex Hilbert spaces a.e. invariant under translations by $\Delta$ and such that the function

  $$
  \nu(x) = \dim V_x
  $$

  is measurable.

- $C = \{c_k(x) : k \in \mathbb{Z}_+, x \in X\}$ is a family of unitary operators

  $$
  c_k(x) : V_x \to V_{x+\delta_k} = V_x
  $$

  depending measurably on $x$ and satisfying

  $$
  c_k(x) = c_k(x + \delta^k)
  $$

  (2.2)

  $$
  c_k(x)c_l(x + \delta^k) = c_l(x)c_k(x + \delta^l)
  $$

  for all $\delta \in \Delta$ and almost all $x \in X$.

We will often write $(\mu, \nu, C)$ instead of $(\mu, V, C)$, in view of the fact that changing $V$ unitarily will yield equivalent representations. Given such triple, consider the Hilbert space

$$
V = V(\mu, \nu, C) = \int_X \oplus V_x \, d\mu(x).
$$

For example, when $V_x = \mathbb{C}$ for all $x$,

$$
V(\mu, 1, C) = L^2(X, \mu),
$$

the ordinary $L^2$ space of $\mathbb{C}$-valued functions. Finally, define operators on $V$ by

$$
J_k f(x) = -i(-1)^{x_1 + \cdots + x_{k-1}} \, c_k(x) \left( \sqrt{\frac{d\mu(x + \delta^k)}{d\mu(x)}} f(x + \delta^k) \right)
$$

(2.3)

$$
J'_k f(x) = (-1)^{x_1 + \cdots + x_k} \, c_k(x) \left( \sqrt{\frac{d\mu(x + \delta^k)}{d\mu(x)}} f(x + \delta^k) \right)
$$

where an $f \in V$ is regarded as an assignment $x \mapsto f(x) \in V_x$ and all $+$ are modulo 2.
Theorem 2.4. The operators $J_1, J_1', J_2, J_2', \ldots$ are mutually anticommuting orthogonal complex structures and, therefore, define a (complex) spinor structure on $V$. Conversely, every spinor structure on a separable Hilbert space is unitarily equivalent to some $V(\mu, \nu, C)$.

For the proof, one observes that the Clifford commutation relations for the $J$’s translate into the Fermi commutation relations for the operators

$$a_k = \frac{1}{2} (iJ_k - J_k'),$$

which, according to [GW1], are themselves parametrized up to equivalence by the triples $(\mu, \nu, C)$. The Fermi-Fock representation corresponds to the triple $(\mu_{\Delta}, 1, \{1\})$. Von Neumann’s first examples of non-Fock representations, were infinite tensor products, which in our notation are the $V(\mu_X, 1, \{C\})$, where $\mu_X$ is the Haar measure on $X$ and

$$C_k^{\otimes} (x) = \omega_k^{(-1)^x_k},$$

the $\omega_k$ being arbitrary complex numbers of absolute value 1. In particular,

$$V(\mu_X, 1, \{1\}).$$

is one such.

3. Real and Quaternionic structures.

If $U$ is a real module over $C(H)$, then $C \otimes U$ is a complex module over $C \otimes C(H)$, which comes with the $C(H)$-invariant decomposition

$$C \otimes U = U \oplus \mathbb{R} iU.$$

$U$ is an invariant real form of $C \otimes U$. Conversely, any complex module over $C(H)$ with an invariant real form determines a real module over $C(H)$. Hence, by determining all the invariant real forms of the Gårding-Wightman modules we will be parametrizing all the real representations of $C(H)$ up to orthogonal equivalence.

The first problem is equivalent to determining all the charge-conjugation operators of the representations $V(\mu, \nu, C)$, i.e., the $\mathbb{C}$-antilinear operators $S : V \to V$ which commute with the action of $C(H)$ and such that

$$S^2 = 1, \quad ||Sf|| = ||f||.$$

The invariant real form in question is then $\{v \in V : Sv = v\}$.

Let

$$x \mapsto \bar{x}$$

be the involution of $X$ which changes all 0’s to 1’s and viceversa. Modulo 2, $\bar{x} = x + 1$ where $1_k = 1 \forall k$. We have induced involutions on subsets of $X$ and on functions and measures on $X$:

$$\bar{A} = \{\bar{x} : x \in A\}, \quad \bar{f}(x) = f(\bar{x}), \quad \bar{\mu}(A) = \mu(\bar{A}).$$
Theorem 3.2. $V(\mu, \nu, C)$ admits an invariant real form if and only if the measures $\mu$ and $\hat{\mu}$ are equivalent, $\hat{\nu}(x) = \nu(x)$ for almost all $x \in X$ and there exist a measurable family of operators

$$r(x) : V_x \to V_{\hat{x}} \cong V_x$$

which are $\mathbb{C}$-antilinear, preserve the norm and satisfy

$$(3.3) \quad r(x)r(\hat{x}) = 1$$

$$r(x)c_k(\hat{x}) = (-1)^k c_k(x)r(x + \delta^k)$$

for all $k \in \mathbb{N}$ and almost all $x \in X$.

Sketch of proof: let

$$(Tf)(x) = \sqrt{\frac{d\hat{\mu}(x)}{d\mu(x)}} f(\hat{x}).$$

If $S$ is an invariant real structure the the product $TS$ must commute with the operators $a_k a_k^*$ and $a_k^* a_k$ for all $k$. This is a commuting set of projections for which $V = \int_X V_x \, d\mu(x)$ is the spectral decomposition. From this one can deduce that $TS$ must act pointwise on each $V_x$. This is our $r(x)$.

All this applies to the finite case as well. If $\dim H = 2m$, $\mu$ and $\hat{\mu}$ are equivalent for any $\mu$. From (3.3) one deduces

$$r(1) = (-1)^{m(m+1)/2} r(0).$$

Assuming, as we may, that $r(0)$ is the standard conjugation on $\mathbb{C}$, we see that $V$ splits over $\mathbb{R}$ if and only if $m(m+1)/2$ is an even integer, i.e., for

$$m \equiv 0, 3 \pmod{4}$$

as is well known.

Assume now that $V$ is infinite dimensional and separable. The axiom of choice implies that there are always plenty of solutions $r(x)$ to the equations (3.3) but most of them -and often all, are non-measurable. Indeed, the latter turns out to be the case in the following two cases.

Corollary 3.4. If $\mu$ is discrete and $V$ is irreducible over $\mathbb{C}$, then it is irreducible over $\mathbb{R}$. In particular, this is the case for the Fermi-Fock representations.

Corollary 3.5. The tensor product representations $V(\mu_X, 1, C^\otimes)$ are irreducible over $\mathbb{R}$.

The proofs of these results involve arguments of ergodicity.

Next, we describe a standard form for representations of real type for which the operator-valued function $r(x)$ is constant, namely

$$r(x) v = \bar{v}.$$
with respect to a fixed choice of real form in each integrand $V_x$. A maze of examples will come out of it.

Let

$$U = \int_X^\oplus U_x \, d\mu(x)$$

be a direct integral of real Hilbert spaces satisfying

$$U_{x+\delta} = U_x, \quad U_{\bar{x}} = U_x$$

for all $\delta \in \Delta$ and almost all $x \in X$. With $V = \mathbb{C} \otimes U$ and $V_x = \mathbb{C} \otimes U_x$,

$$V = \int_X^\oplus V_x \, d\mu(x).$$

Clearly, $V_{x+\delta} = V_x = V_{\bar{x}}$, $U_x$ is a real form of $V_x$ and $U$ one of $V$. Denote by $-\bar{}$ the corresponding conjugations. If $f \in V$ and $A \in \text{End}(V)$ set

$$\bar{f}(x) := \overline{f(x)}, \quad \bar{A}(f) := \overline{A(f)}.$$  

$U$ is not the invariant real form we are looking for - this would be incompatible with the anticommutation relations. Instead we have

**Theorem 3.7.** If $\hat{\mu}$ is equivalent to $\mu$, $\hat{\nu} = \nu$ and the operators $c_k$ satisfy

$$\overline{c_k(x)} = (-1)^k c_k(\bar{x}),$$

then

$$V^\mathbb{R} = \{ f \in V : \bar{f}(x) = \sqrt{\frac{d\hat{\mu}(x)}{d\mu(x)}} f(\bar{x}) \}$$

is an invariant real form of $V = V(\mu, \nu, \mathbb{C})$.

**Theorem 3.8.** Any unitary representation of $C(H)$ with an invariant real structure is unitarily equivalent to one in standard form.

Perhaps the simplest infinite-dimensional Majorana spinors are given by $V(\mu_X, 1, \mathbb{C})$ with $\mu_X$ being the Haar measure of $X$ and the $c_k$ given by the dyadic Rademacher functions

$$c_{2\ell}(x) = 1$$
$$c_{4\ell+1}(x) = (-1)^{4\ell+3}$$
$$c_{4\ell+3}(x) = (-1)^{4\ell+1}.$$  

**Theorem 3.9.** With these $c_k$, $V(\mu_X, 1, \mathbb{C})$ is irreducible over $\mathbb{C}$, but the real form

$$L^2(X)^\mathbb{R} = \{ f \in L^2(X) : f(\bar{x}) = \overline{f(x)} \}$$

is an invariant real subspace. The real representation on $L^2(X)^\mathbb{R}$ so obtained, is irreducible over $\mathbb{R}$ and does not arise from any representation of $\mathbb{C} \otimes C(H)$ by restriction of the scalars.

The quaternionic case is treated similarly, although in the irreducible case the operator-valued function $q(x)$ cannot be taken to be constant and does not arise from any quaternionic structure in each $V_x$, like real ones do. It is not obvious a priori that $V(\mu, \nu, \mathbb{C})$ can support any quaternionic structure when $\nu(x) = 1$. 

Theorem 3.10. \( V(\mu, \nu, C) \) admits an invariant quaternionic structure if and only if \( \mu \) and \( \tilde{\mu} \) are equivalent, \( \tilde{\nu}(x) = \nu(x) \) for almost all \( x \in X \) and there exist a measurable family of operators

\[
q(x): V_x \to V_{\tilde{x}} \cong V_x
\]

which are \( \mathbb{C} \)-antilinear, preserve the norm and satisfy

\[
\begin{align*}
q(x)q(\tilde{x}) &= -1, \\
q(x)c_k(\tilde{x}) &= (-1)^k c_k(x)q(x + \delta^k)
\end{align*}
\]

for all \( k \in \mathbb{N} \) and almost all \( x \in X \).

Corollary 3.11. If \( \mu \) is discrete and \( V \) is irreducible over \( \mathbb{C} \), then \( V(\mu, \nu, C) \) admits no quaternionic structure. In particular, the Fermi-Fock representations are of complex type.

Corollary 3.12. The tensor product representations \( V(\mu_X, 1, C^\otimes) \) are all of complex type. In particular, this is so for \( V(\mu_X, 1, \{1\}) \).

Finally, we give a standard form for spinors of quaternionic type, for which the operator-valued function \( q \) will be

\[
q(x)v = (-1)^{x_1} \bar{v}.
\]

The bar indicates that we are in the context of (3.6), where \( V \) comes with a (non-invariant) real form \( U \). With this understood we have

Theorem 3.13. If \( \tilde{\mu} \) is equivalent to \( \mu \), \( \tilde{\nu}(x) = \nu(x) \) and the operators \( c_k \) satisfy

\[
\begin{align*}
\overline{c_1(x)} &= c_1(\tilde{x}), \\
\overline{c_k(x)} &= (-1)^k c_k(\tilde{x}),
\end{align*}
\]

\( \forall k \geq 2 \) and almost all \( x \in X \), then

\[
Qf(x) = (-1)^{x_1} \sqrt{\frac{d\mu(\tilde{x})}{d\mu(x)}} f(\tilde{x})
\]

is an invariant quaternionic structure in \( V(\mu, \nu, C) \).

Theorem 3.14. Any unitary representation of \( C(H) \) with an invariant quaternionic structure is unitarily equivalent to a standard one.

The simplest irreducible infinite-dimensional spinors of quaternionic type are realized in \( L^2(X, \mu_X) \) as \( V(\mu_X, 1, C) \), with

\[
\begin{align*}
c_{2\ell}(x) &= 1 = c_1(x) \\
c_{2\ell+1}(x) &= (-1)^{x_{2\ell+1}} (\ell \geq 1) \\
c_{2\ell+2}(x) &= (-1)^{x_{2\ell+1}} (\ell \geq 1)
\end{align*}
\]
The following dyadic representations give many examples of real and quaternionic spinors with special properties. Recall that the Walsh functions, as functions on $X$, are defined by
\[
\phi_\alpha(x) = (-1)^{\sum \alpha_k x_k}
\]
for $\alpha \in \Delta$, which are precisely the characters of $X$. Setting
\[
\sigma^k = \delta^1 + \cdots + \delta^k
\]
then
\[
\phi_{\sigma^k}(x) = (-1)^{x_1 + \cdots + x_k}
\]
which appear as multipliers in the definition of the operators $J, J'$.

Let $\Gamma$ denote the set of infinite symmetric matrices $\gamma$ of 0’s and 1’s, with only finitely many 1’s in each row or column and none along the diagonal. We regard each row $\gamma^k$ as a point in $\Delta$. $\Gamma$ contains the disjoint subsets
\[
\Gamma_1 = \{ \gamma \in \Gamma : \sum_j \gamma^k_j \equiv k \ \forall k \}
\]
\[
\Gamma_{-1} = \{ \gamma \in \Gamma : \sum_j \gamma^1_j \equiv 0, \sum_j \gamma^k_j \equiv k \ \forall k \geq 2 \}
\]
where the congruences are modulo 2. In other words, $\Gamma_1$ consists of the matrices where the number of 1’s in a row has the same parity as the position of that row, while for $\Gamma_{-1}$ the condition is the same except for the first row, for which it is reversed.

In what follows we will take $\mu = \mu_X$, the Haar measure on $X$ and $\nu = 1$, so that $V = L^2(X, \mathbb{C})$.

**Theorem 3.15.** For any $\gamma \in \Gamma$, the multiplier operators
\[
c_k(x) = \phi_{\gamma^k}(x)
\]
satisfy (2.2) and, therefore,
\[
J_k f(x) = -i \phi_{\sigma^{k-1}+\gamma^k}(x)f(x+\delta^k)
\]
\[
J'_k f(x) = \phi_{\sigma^k+\gamma^k}(x)f(x+\delta^k)
\]
define a spinor representation. This representation is irreducible over $\mathbb{C}$. If $\gamma \in \Gamma_1$ (respectively, $\gamma \in \Gamma_{-1}$) then the representation is of real (respectively, quaternionic) type, in its respective standard form. Those of real type have
\[
L^2(X)^R = \{ f \in L^2(X) : \overline{f(x)} = f(\bar{x}) \}
\]
as the invariant real form.

On the Walsh basis,
\[
J_k \phi_\alpha = -i(-1)^{\alpha_k} \phi_{\alpha+\gamma^k+\sigma^{k-1}}
\]
\[
J'_k \phi_\alpha = (-1)^{\alpha_k} \phi_{\alpha+\gamma^k+\sigma^k}
\]
4. $L^2(\mathbb{T})$ as a space of spinors.

Any representation $V(\mu_X, 1, \mathbb{C})$ where $\mu_X$ is the Haar measure and $\nu = 1$, can be realized on the standard $L^2(\mathbb{T})$ of complex-valued functions on the circle.

Indeed, we can identify each $V_\mu$ with $\mathbb{C}$, so that $V = L^2(X)$ and now use the dyadic expansions to identify $X$ with the interval $(0, 1)$, or with $\mathbb{T}$ -except for a set of measure zero. The Haar measure on $X$ becomes the Lebesgue measure on $(0, 1)$, and the Haar measure on $\mathbb{T}$. This is in spite of the fact that translations in $X$ do not correspond to rigid rotations of $\mathbb{T}$ and that, as a topological space, $X$ is homeomorphic, not to $\mathbb{T}$, but to the Cantor set. The operation $x \mapsto \overline{x}$ in $X$ correspond to $y \mapsto 1 - y$ in $(0, 1)$ which, on $\mathbb{T} \subset \mathbb{C}$, becomes ordinary complex conjugation. In particular, for a standard representation $s$ of real type with $\nu = 1$, the $c_k$’s are functions from $\mathbb{T}$ to itself satisfying

$$c_k(t) = (-1)^k c_k(\overline{t})$$

and the $C(H)$-invariant real form is

$$L^2(\mathbb{T})^R = \{ f \in L^2(\mathbb{T}) : \overline{f(t)} = f(\overline{t}) \}$$

Via $X \approx \mathbb{T}$ the functions $\phi_\alpha(x)$ become the classical periodic Walsh functions $w_n(t)$, where $n = 0, 1, 2, ...$ corresponds to $\alpha \in \Delta$ via the dyadic expansion

$$n = \sum_{k=0}^{\infty} \alpha_{k+1} 2^k$$

Now recall the classical Hilbert transform

$$\mathcal{H} f(s) = \int_{-\infty}^{\infty} \frac{f(u)}{s - u} du$$

When transported to the circle coordinatized by $-\pi \leq \theta \leq \pi$, it becomes

$$\mathcal{H} f(\theta) = \int_{-\pi}^{\pi} \cot(\xi/2) f(\theta - \xi) d\xi.$$ 

It is a complex structure on $L^2(\mathbb{T})$, i.e., $\mathcal{H}^2 = -I$, which evidently preserves the ordinary real form of real-valued functions

$$L^2(\mathbb{T})_\mathbb{R} = \{ f \in L^2(\mathbb{T}) : \overline{f(t)} = f(t) \}.$$ 

If we set

$$\mathcal{H}' f(s) = i \int_{-\infty}^{\infty} \frac{f(u)}{s + u} du,$$

then $\mathcal{H}'^2 = -I$ and

$$\mathcal{H} \mathcal{H}' = -\mathcal{H}' \mathcal{H}.$$ 

Since any two unitary complex or quaternionic structures on a Hilbert space are mutually conjugate by a unitary transformation, every spinor structure on $L^2(\mathbb{T})$ can be assumed to start with $J_1 = \mathcal{H}$ and $J'_1 = \mathcal{H}'$. Indeed, one can adjust the unitary conjugation in such a way that all the remaining generators are also singular integral operators on $\mathbb{R}$

$$J_\ell f(s) = \int_{-\infty}^{\infty} K_\ell(s, u) f(u) du$$

and similarly for the $J'_\ell$. Of course, the kernels $K_\ell, K'_\ell$, are not of convolution type. They are dyadic twistings of the Hilbert kernel, which, in turn, lead to analogs of the Cauchy kernel.
5. Dyadic Difference Operators.

For any vector-valued function $f$ on $X$ define

$$\partial_k f(x) := \phi_\delta^k (x)(f(x + \delta^k) - f(x))$$

where, as usual, addition is modulo 2. These are natural difference operators in two ways. Firstly, they are natural partial derivatives in $X = \mathbb{Z}_2^\infty$ once we fix the motion from 0 to 1 (resp., from 0 to 1) as positive (resp., negative). Secondly, if we identify $X$ with the interval $(0, 1)$ so that $x \in X$ corresponds to $t \in (0, 1)$, then the ordinary derivative on $(0, 1)$ is

$$f'(t) = \lim_{k \to \infty} 2^k \partial_k f(x).$$

This follows by taking incremental quotients of the form

$$\frac{f(t + \frac{(-1)^t}{2^k}) - f(t)}{\frac{(-1)^t}{2^k}} = 2^k (-1)^t (f(t + \frac{(-1)^t}{2^k}) - f(t))$$

and noting that the translation $x \mapsto x + \delta^k$ in $X$, corresponds in $(0, 1)$ to $t \mapsto t + (-1)^t 2^{-k}$. Equivalently,

$$\frac{d}{dt} = \sum_{k=0}^{\infty} 2^k (2\partial_{k+1} - \partial_k).$$

This suggests a few obvious deformations of the derivative operator, starting with

$$\sum_{k=0}^{\infty} z^k (2\partial_{k+1} - \partial_k),$$

$z \in \mathbb{C}$. Another, related to the subject at hand, is obtained by expressing the operators $\partial_k$ in terms of the the $J_k$ and $J'_k$ of the special spinor structure $V(\mu_X, 1, 1)$, then replacing the operators $c_k = 1$ by arbitrary ones. The resulting “twisted derivative” is, for any spinor structure $V(\mu_X, 1, \mathbb{C})$,

$$\frac{d}{dc} = \lim_{k \to \infty} i2^k (\phi_\sigma^k \partial_k + J_k J'_k).$$

In this article we will concentrate instead on the operators $\sum_k J_k \partial_k$ and $\sum_k J_k^* \partial_k$, or, better yet,

$$D = \sum_{k=0}^{\infty} a_k \partial_k \quad D' = \sum_{k=0}^{\infty} a_k^* \partial_k$$

associated to any representation $V(\mu, \nu, \mathbb{C})$. We will not attempt to motivate them a priori. They are of course linear wherever defined and anihilate constants, but their resemblance to Dirac operators does not go very far because the $\partial_k$ do not commute with the spinor representation. But the following observations makes them worth of some attention.

For the standard Fermi-Fock representation the domains of $D$ and $D'$ consist of the constants alone - they diverge elsewhere. However, if $\mu_X$ is the Haar measure on $X \approx \mathbb{R}$, we have
Theorem 5.1. For any representation $V(\mu_X, 1, \mathcal{C})$, the domains of $D$ and $D'$ contain the algebraic span of the Walsh and the Fourier basis and, therefore, are dense in $L^2(X)$.

In terms of the Walsh basis, the matrices of $D$ and $D'$ involve only 0 and ±1 and are not symmetric. However, they appear to be always diagonalizable. Here we shall concentrate on what actually happens for the dyadic representations of §3, where the diagonalization can be done over $\mathbb{Z}$.

Let $\gamma \in \Gamma$ be an infinite symmetric matrix of 0's and 1's such that all the diagonal elements and almost all elements in each row $\gamma^k$ are zero. The corresponding spinor representation on $L^2(X, \mu_X)$ is

$$J_k f(x) = -i\phi_{\sigma^{k-1}+\gamma^k}(x)f(x+\delta^k)$$

$$J'_k f(x) = \phi_{\sigma^k+\gamma^k}(x)f(x+\delta^k)$$

where $\phi_\alpha$ are the Walsh functions. The set $W_\mathbb{Z}$ of integral linear combinations of Walsh functions defines an integral structure in $L^2(X) \cong L^2(\mathbb{T})$.

Theorem 5.2. For any matrix $\gamma \in \Gamma$, the operators $D$ and $D'$ associated to the representation $V(\mu_X, 1, \{\phi_{\gamma^k}\})$ can be diagonalized over $\mathbb{Z}$: with integral eigenvalues and eigenvectors in $W_\mathbb{Z}$.

There is an algorithm involving only the matrix $\gamma$ to obtain all the eigenvalues and eigenvectors of $D$ and $D'$. It goes roughly as follows. For any positive integer $n$ let $W_n$ be the set of functions $X \to \mathbb{C}$ that depend only on the first $n$ components of $x$. Fix $\gamma \in \Gamma$ and define a sequence of integers $0 < N_1 < N_2 < ...$ by

$$N_k := \max\{k, \min\{m : \gamma^1, ..., \gamma^k \in W_m\}\}.$$ 

Then

$$0 \subset W_{N_1} \subset W_{N_2} \subset ...$$

is a filtration of the space of functions on $X$ by finite-dimensional subspaces invariant under both $D$ and $D'$. Consider now the more general operators

$$D_{[k, \lambda]} := D - \lambda \phi_{\sigma^{N_k}} I$$

with $k \in \mathbb{N}$ and $\lambda \in \mathbb{C}$. Then, one can obtain the eigenvalues and eigenvectors of $D_{[k, \lambda]}$ in $W_{N_k}$, recursively from the eigenvalues and eigenvectors of $D_{[k-1, \mu]}$ in $W_{N_{k-1}}$ for all $\mu$.

In general, $D$ and $D'$ are very different operators. For example, the domain may be 0 for one and dense for the other and there is no relation with the adjoints either. The following result essentially characterizes the standard spinor representations of real type.

Theorem 5.3. If $V(\mu_X, 1, \mathcal{C})$ is standard and of real type, then

$$D' = TDT^{-1}$$

where

$$T f(r) = \sqrt{\frac{d\mu(\tilde{r})}{d\mu(r)}} f(\tilde{r}).$$
Finally, either operator determines the representation. For example, for any \( V(\mu_X, 1, C) \), one has

\[
-2a_k f = \phi_{\delta^k} D(\phi_{\delta^k} f) - Df \\
-2a^*_k f = \phi_{\delta^k} D'(\phi_{\delta^k} f) - D'f.
\]

\textbf{6. Kaplansky’s Division Algebras.}

The real finite-dimensional division algebras -associative or not- occur only in dimensions 1,2,4 and 8. If we require them to have a multiplicative identity and be \textit{normed} relative to a fixed inner product, namely, to satisfy

\[
||ab|| = ||a|| ||b||,
\]

one obtains just the usual algebras of real, complex, quaternionic and octonionic numbers.

In [Ka], Kaplansky proved that in infinite dimensions there were no real normed division algebras, i.e. no strict analog of the numbers above. Of course, there are many division algebras -even associative and commutative ones (e.g., \( \mathbb{R}[X] \)), as well as many normed algebras (since \( V \otimes V \cong V \)), but none will satisfy both conditions simultaneously.

After noticing that weakening “division” to, say, “left-division”, did not introduce any new algebras in finite dimensions, Kaplansky comments on his attempts to prove that the same was true in the infinite case. But counterexamples were given in [Cu],[R].

Now we can describe all such structures, that is, all bilinear operations on a real separable Hilbert space such that \( ||ab|| = ||a|| ||b|| \) and such that for every \( a \neq 0 \) there exists \( a^{-1} \) satisfying \( a^{-1}(ab) = b \). Indeed,

\textbf{Theorem 6.1.} \textit{The left-division real normed algebras of countable dimension are parametrized up to equivalence by the triples \((\mu, \nu, C)\) of Theorem 3.7.}

Explicitly: the product in such algebra \( A \) can be linearly modified so as to have a left-identity 1. If \( H \) is identified with the orthogonal complement of 1 in \( A \), so that

\[
A = H \oplus \mathbb{R}1,
\]

then left-multiplication on \( A \) by elements of \( H \) satisfies the relation

\[
(h_1 h_2 + h_2 h_1)a = -2 < h_1, h_2 > a
\]

and, therefore \( A \) becomes an orthogonal \( C(H) \)-module, corresponding to some triple \((\mu, \nu, C)\) satisfying the conditions of 3.7. Conversely, given an orthogonal \( C(H) \)-module \( A \) and any identification \( A \cong H \oplus \mathbb{R} \), the product

\[
(h + c) \ast a := ha + ca
\]

where \( a \mapsto ha \) is the Clifford action, satisfies the desired properties.

\textbf{Examples:} The Fermi-Fock representations yield the examples of [Cu],[R]. Letting instead \( \mu_X \) be the Haar measure on \( X \) and

\[
\gamma_k(x) = 1, \quad \gamma_k(x) = (-1)^{\frac{k(k+1)}{2}}, \quad \gamma_k(x) = (-1)^{\frac{k(k+3)}{2}}
\]
yields an essentially inequivalent algebra. The corresponding $C(H)$-module is irreducible over $\mathbb{C}$ but splits over $\mathbb{R}$ as a sum of two copies of a real irreducible module $U$. The normed algebra constructed from $U$ is therefore the simplest infinite-dimensional analog of the Octonions -if there is to be one.

For the dyadic spinors introduced in §3 one can describe the resulting algebras purely in dyadic terms. Let then $\gamma \in \Gamma$. For any non-negative integers $k, m$, let

$$N_\gamma(k, m) = \sum_{j=0}^{k-2} (m_j + \gamma_{j+1}^{k-1} + 1)2^j + \sum_{j=k-1}^{\infty} (m_j + \gamma_{j+1}^{k-1})2^j$$

$$N'_\gamma(k, m) = \sum_{j=0}^{k-2} (m_j + \gamma_{j+1}^k + 1)2^j + \sum_{j=k-1}^{\infty} (m_j + \gamma_{j+1}^k)2^j$$

where

$$k = \sum_{j \geq 0} k_j2^j, \quad m = \sum_{j \geq 0} m_j2^j$$

$(k_j, m_j \in \{0, 1\})$ are the dyadic expansions of $k$ and $m$ and the sums in parenthesis are modulo 2.

**Theorem 6.2.** Let $\gamma$ be an infinite matrix zeroes and ones with finitely many ones in each row and such that

$$\gamma_l^k = \gamma_k^l, \quad \gamma_k^k = 0$$

for all $k, l$. On a real vector space $V$ with basis

$$w_0, w'_0, w_1, w'_1, w_2, w'_2, ...$$

define a linear $\star_\gamma : A \otimes A \to A$ by

$$w_0 \star w_m = w_m$$
$$w_0 \star w'_m = w'_m$$
$$w_k \star w_m = (-1)^{m_k-1} w_{N_\gamma(k, m)} (k \geq 1)$$
$$w_k \star w'_m = (-1)^{m_k-1} w'_{N_\gamma(k, m)} (k \geq 1)$$
$$w'_k \star w_m = -i(-1)^{m_k-1} w_{N'_\gamma(k, m)} (k \geq 0)$$
$$w'_k \star w'_m = -i(-1)^{m_k-1} w'_{N'_\gamma(k, m)} (k \geq 0)$$

for all $m \geq 0$. Then $(V, \star)$ has no zero divisors and every non-zero element is a left-unit. Furthermore, under the inner product defined by declaring $\{w_n, w'_n\}$ to be orthonormal, $\star$ is a composition of the corresponding quadratic forms, i.e.,

$$||a \star b|| = ||a|| \cdot ||b||$$

The algebra of the theorem corresponds to the spinor representation $V(\mu_X, 1, \{\gamma^k\})$. For example take $\gamma = 0$. The corresponding product $\star$ is antilinear in the first slot, linear in the second and satisfies

$||||$
$$w_k \star w_m = (-1)^{m_{k-1}} w_m (k-1)$$
for $k \geq 2$, where $N(k, m) = m^{(k-1)}$ is the number obtained from $m$ by changing its first $k-1$ dyadic coefficients $m_0, ..., m_{k-2}$.

More interesting than the products themselves may be their automorphism groups of various kinds. In particular, $\text{Pin}(\infty)$, the Banach Lie group generated by the elements of unit length in $H$ under the Clifford product, comes with a natural unitary spin representation $v \mapsto J_1...J_r v$, once we fix one for $C(H)$. It satisfies

$$J_h (J_k \cdot v) = -J_{r_h(k)} \cdot J_h (v)$$
where $r_h : H \rightarrow H$ denotes the reflection with respect to the hyperplane $h^\perp$. Therefore, $\text{Pin}(\infty)$ acts by orthogonal transformations in $H$. Let it act trivially on the factor $\mathbb{R}1$ of $H \oplus \mathbb{R}1$. If now $V$ is any spin representation, an identification $V = H \oplus \mathbb{R}1$ yields two actions of $\text{Pin}(\infty)$ on $V$, $v \mapsto B_g v$ and $v \mapsto \Sigma_g v$, which satisfy

$$\Sigma_g (u\star v) = B_g u \star \Sigma_g v.$$ 

In this way we obtain an inclusion

$$1 \rightarrow \text{Pin}(\infty) \rightarrow \mathcal{G} = \{(g_1,g_2) \in U(H) \times U(V) : g_2 (u \star v) = g_1 u \star g_2 v\}$$
for any real representation of $C(H)$. $\mathcal{G}$ is reductive and comes with a unitary representation. Its specific structure depends very much on the equivalence class of the spinor representation but, at least in the examples treated here, their semisimple part is a classical infinite-dimensional group. Hence, by restriction, one obtains many irreducible unitary representations of the latter. Most are “new” and not of highest weight type.

For an inspiring discussion of the groups of symmetries associated to the Octonions, see [B].

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