1D Mott variable-range hopping with external field

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Abstract. Mott variable-range hopping is a fundamental mechanism for electron transport in disordered solids in the regime of strong Anderson localization. We give a brief description of this mechanism, recall some results concerning the behavior of the conductivity at low temperature and describe in more detail recent results (obtained in collaboration with N. Gantert and M. Salvi) concerning the one-dimensional Mott variable-range hopping under an external field\footnote{1}. 

Keywords: Random walk in random environment, Mott variable-range hopping, linear response, Einstein relation

1 Mott variable-range hopping

Mott variable range hopping is a mechanism of phonon-assisted electron transport taking place in amorphous solids (as doped semiconductors) in the regime of strong Anderson localization. It has been introduced by N.F. Mott in order to explain the anomalous non-Arrhenius decay of the conductivity at low temperature \cite{20,21,22,23,25}.

Let us consider a doped semiconductor, which is given by a semiconductor with randomly located foreign atoms (called impurities). We write $\xi := \{x_i\}$ for the set of impurity sites. For simplicity we treat spinless electrons. Then, due to Anderson localization, a generic conduction electron is described by a quantum wavefunction localized around some impurity site $x_i$, whose energy is denoted by $E_i$. This allows, at a first approximation, to think of the conduction electrons as classical particles which can lie only on the impurity sites, subject to the constraint of site exclusion (due to Pauli’s exclusion principle). As a consequence, a microscopic configuration is described by an element $\eta \in \{0, 1\}^\xi$, where $\eta_{x_i} = 1$ if and only if an electron is localized around the impurity site $x_i$. The dynamics is then described by an exclusion process, where the probability rate for a jump from $x_i$ to $x_j$ is given by (cf. \cite{1})

$$
\mathbb{I}(\eta_{x_i} = 1, \eta_{x_j} = 0) \exp\{-2\zeta|x_i - x_j| - \beta(E_j - E_i)\}.
$$

\footnote{1 If you want to cite this proceeding and in particular some result contained in it, please cite also the article where the result appeared, so that the contribution of my coworkers can be recognised. This can be important for bibliometric reasons.}
Above $\beta = 1/kT$ ($k$ being the Boltzmann’s constant and $T$ the absolute temperature), while $1/\zeta$ is the localization length. A physical analysis suggests that the energy marks $\{E_i\}$ can be modeled by i.i.d. random variables. In inorganic doped semiconductors, when the Fermi energy is set equal to zero, the common distribution $\nu$ of $E_i$ is of the form $c |E|^\alpha dE$ on some interval $[-A, A]$, where $c$ is the normalization constant and $\alpha$ is a nonnegative exponent.

At low temperature (i.e. large $\beta$) the form of the jump rates (1) suggests that long jumps can be facilitated when the energetic cost $\{E_j - E_i\}_+$ is small. This facilitation leads to an anomalous conductivity behavior for $d \geq 2$. Indeed, according to Mott’s law, in an isotropic medium the conductivity matrix $\sigma(\beta)$ can be approximated (at logarithmic scale) by $\exp(-c/\beta^{1+1/\alpha})\mathbb{I}$, where $\mathbb{I}$ denotes the identity matrix and $c$ is a suitable positive constant with negligible temperature-dependence.

The mathematical analysis of the above exclusion process presents several technical challenges and has been performed only when $\xi \equiv \mathbb{Z}^d$ (absence of geometric disorder) and with jumps restricted to nearest-neighbors (cf. [7,24]). This last assumption does not fit with the low temperature regime, where anomalous conductivity takes place.

To investigate Mott variable range hopping at low temperature, in the regime of low impurity density some effective models have been proposed. One is given by the random Miller-Abrahams resistor network [19]. Another effective model is the following (cf. [9]): one approximates the localized electrons by classical non–interacting (independent) particles moving according to random walks with jump probability rate given by the transition rate (1) multiplied by a suitable factor which keeps trace of the exclusion principle. To be more precise, given $\gamma \in \mathbb{R}$, we call $\mu_\gamma$ the product probability measure on $\{0, 1\}^\xi$ with $\mu_\gamma(\eta_{x_i}) = e^{-\beta(E_i - \gamma)} (1 + e^{-\beta(E_i - \gamma)})$. Then it is simple to check that $\mu_\gamma$ is a reversible distribution for the exclusion process. In the independent particles approximation, the probability rate for a jump from $x_i$ to $x_j$ is given by

$$
\mu_\gamma(\eta_{x_i} = 1, \eta_{x_j} = 0) \exp\{-2\zeta|x_i - x_j| - \beta\{E_j - E_i\}_+\}.
$$

(2)

It is simple to check that at low temperature, i.e. large $\beta$, (2) is well approximated by (cf. [11])

$$
\exp\{-2\zeta|x_i - x_j| - \beta/2(|E_i - \gamma| + |E_j - \gamma| + |E_i - E_j|)\}.
$$

(3)

In what follows, without loss of generality, we shift the energy so that the Fermi energy $\gamma$ equals zero, we take $2\zeta = 1$ and we replace $\beta/2$ by $\beta$. Since the above random walks are independent, we can restrict to the analysis of a single random walk, which we call Mott random walk.

2 Mott random walk and bounds on the diffusion matrix

We give the formal definition of Mott random walk as random walk in a random environment. The environment is given by a marked simple point process $\omega =$
\{(x_i, E_i)\} where \(\{x_i\} \subset \mathbb{R}^d\) and the (energy) marks are i.i.d. random variables with common distribution \(\nu\) having support on some finite interval \([-A, A]\). Given a realization of the environment \(\omega\), Mott random walk is the continuous-time random walk \(X_t^{\omega}\) with state space \(\{x_i\}\) and probability rate for a jump from \(x_i\) to \(x_j \neq x_i\) given by
\[
r_{x_i, x_j}(\omega) = \exp\{-(|x_i - x_j| - \beta(|E_i| + |E_j| + |E_i - E_j|))\}. \tag{4}
\]
As already mentioned, one expects for \(d \geq 2\) that the contribution to the transport of long jumps dominates as \(\beta \to \infty\). In [3] a quenched invariance principle for Mott random walk has been proved. Calling \(D(\beta)\) the diffusion matrix of the limiting Brownian motion, in [8, 9] bounds in agreement with Mott’s law have been obtained for the diffusion matrix. More precisely, under very general conditions on the isotropic environment and taking \(\nu\) of the form \(c|E|^{\alpha}dE\) for some \(\alpha \geq 0\) and on some interval \([-A, A]\), it has been proved that for suitable \(\beta\)-independent positive constant \(c_1, c_2, \kappa_1, \kappa_2\) it holds
\[
c_1 \exp\{-\kappa_1 \beta^{\frac{\alpha+1}{\alpha+2}}\} \mathbb{I} \leq D(\beta) \leq c_2 \exp\{-\kappa_2 \beta^{\frac{\alpha+1}{\alpha+2}}\} \mathbb{I}. \tag{5}
\]
We point out that for a genuinely nearest-neighbor random walk \(D(\beta)\) would have an Arrhenius decay, i.e. \(D(\beta) \approx e^{-c\beta}I\), thus implying that (5) is determined by long jumps.

In dimension \(d = 1\) long jumps do not dominate. Indeed, the following has been derived for \(d = 1\) in [2] when \(\omega\) is a renewal marked simple point process. We label the points in increasing order, i.e. \(x_i < x_{i+1}\), take \(x_0 = 0\) and assume that \(E[x_1^2] < \infty\). Then the following holds [2]:
(i) If \(E[e^{x_1}] < \infty\), then a quenched invariance principle holds and the diffusion coefficient satisfies
\[
c_1 \exp\{-\kappa_1 \beta\} \leq D(\beta) \leq c_2 \exp\{-\kappa_2 \beta\}, \tag{6}
\]
for \(\beta\)-independent positive constants \(c_1, c_2, \kappa_1, \kappa_2\);
(ii) If \(E[e^{x_1}] = \infty\), then the random walk is subdiffusive. More precisely, an annealed invariance principle holds with zero diffusion coefficient.

We point out that the above 1d results hold also for a larger class of jump rates and energy distributions \(\nu\) [2]. Moreover, we stress that the above bounds [5] and [6] refer to the diffusion matrix \(D(\beta)\) and not to the conductivity matrix \(\sigma(\beta)\). On the other hand, believing in the Einstein relation (which states that \(\sigma(\beta) = \beta D(\beta)\)), the above bounds would extend to \(\sigma(\beta)\), hence one would recover lower and upper bounds on the conductivity matrix in agreement with the physical Mott law for \(d \geq 2\) and with the Arrhenius-type decay for \(d = 1\).

The rigorous derivation of the Einstein relation for Markov processes in random environment is in general a difficult task and has been the object of much investigation also in the last years (cf. e.g. [10,11,12,13,14,15,16,17,18]). In what follows, we will concentrate on the effect of perturbing 1d Mott random walk by an external field and on the validity of the Einstein relation. Hopefully, progresses on the Einstein relation for Mott random walk in higher dimension will be obtained in the future.
3 Biased 1d Mott random walk

We take $d = 1$ and label points $\{x_i\}$ in increasing order with the convention that $x_0 := 0$ (in particular, we assume the origin to be an impurity site). It is convenient to define $Z_i$ as the interpoint distance $Z_i := x_{i+1} - x_i$.

![Diagram of points, energy marks, and interpoint distances.](image)

**Fig. 1.** Points $x_i$, energy marks $E_i$, and interpoint distances $Z_i$.

We make the following assumptions:

(A1) The random sequence $(Z_k, E_k)_{k \in \mathbb{Z}}$ is stationary and ergodic w.r.t. shifts;

(A2) $\mathbb{E}[Z_0]$ is finite;

(A3) $\mathbb{P}(\omega = \tau_{\omega} \ell)$ is zero for all $\ell \in \mathbb{Z} \setminus \{0\}$;

(A4) There exists some constant $d > 0$ satisfying $\mathbb{P}(Z_0 \geq d) = 1$.

Note that we do not restrict to the physically relevant energy mark distributions $\nu$ which are of the form $c |E|^{\alpha}dE$ on some interval $[-A,A]$.

Given $\lambda \in [0,1)$ we consider the biased generalized Mott random walk $X^{\omega,\lambda}_t$ on $\{x_i\}$ with jump probability rates given by

$$r^{\lambda}_{x_i,x_j}(\omega) = \exp \{ -|x_i - x_j| + \lambda(x_j - x_i) - u(E_i, E_j) \} , \quad x_i \neq x_j , \quad (7)$$

and starting at the origin. Above, $u$ is a given bounded and symmetric function.

Note that we do not restrict to $\lambda$. The special form (4) is relevant when studying the regime $\beta \to \infty$, on the other hand here we are interested in the system at a fixed temperature (which is included in the function $u$) under the effect of an external field.

In the rest, it is convenient to set $r^{\lambda}_{x,x}(\omega) \equiv 0$. We also point out that one can easily prove that the random walk $X^{\omega,\lambda}_t$ is well defined since $\lambda \in (0,1)$.

The following result, obtained in [5], concerns the ballistic/sub-ballistic regime:

**Proposition 1** [5] For $\mathbb{P}$–a.a. $\omega$ the random walk $X^{\omega,\lambda}_t$ is transient to the right, i.e. $\lim_{t \to \infty} X^{\omega,\lambda}_t = +\infty$ a.s.

**Theorem 1** [5] Fix $\lambda \in (0,1)$.

(i) If $\mathbb{E}[e^{(1-\lambda)Z_0}] < \infty$ and $u$ is continuous, then for $\mathbb{P}$–a.a. $\omega$ the following limit exists

$$v_X(\lambda) := \lim_{t \to \infty} \frac{X^{\omega,\lambda}_t}{t} \quad a.s.$$

and moreover it is deterministic, finite and strictly positive.
(ii) If \( \mathbb{E}[e^{-(1+\lambda)Z_{-1}+(1-\lambda)Z_0}] = \infty \), then for \( \mathbb{P} \)-a.a. \( \omega \) it holds

\[
v_X(\lambda) := \lim_{t \to \infty} \frac{X_t^\xi,\lambda}{t} = 0 \quad \text{a.s.}
\]

As discussed in [5], the condition \( \mathbb{E}[e^{-(1-\lambda)Z_0}] = \infty \) does not imply that \( v_X(\lambda) = 0 \). On the other hand, if \( (Z_k)_{k \in \mathbb{Z}} \) are i.i.d. (or even if \( Z_k, Z_{k+1} \) are independent for every \( k \)) and \( u \) is continuous, then the above two cases (i) and (ii) in Theorem 1 are exhaustive and one concludes that \( \mathbb{E}[e^{-(1-\lambda)Z_0}] < \infty \) if and only if \( v_X(\lambda) = 0 \).

The above Theorem 1 extends also to the jump process associated to \( X_t^{\omega,\lambda} \), i.e. to the discrete time random walk \( Y_n^{\omega,\lambda} \) with probability \( p_{x_i,x_k}^\lambda(\omega) \) of a jump from \( x_i \) to \( x_k \neq x_i \) given by

\[
p_{x_i,x_k}^\lambda(\omega) = \frac{r_{x_i,x_k}^\lambda(\omega)}{\sum_k r_{x_i,x_k}^\lambda(\omega)}.
\]

In particular, if \( \mathbb{E}[e^{-(1-\lambda)Z_0}] < \infty \) and \( u \) is continuous then the random walk \( Y_n^{\omega,\lambda} \) is ballistic (\( v_Y(\lambda) > 0 \)), while if \( \mathbb{E}[e^{-(1+\lambda)Z_{-1}+(1-\lambda)Z_0}] = \infty \) then the random walk \( Y_n^{\omega,\lambda} \) is sub-ballistic (i.e. \( v_Y(\lambda) = 0 \)). We point out that indeed the result has been proved in [5] first for the random walk \( Y_n^{\omega,\lambda} \) and then extended to the continuous time case by a random time change argument.

We write \( \tau_x \omega \) for the environment \( \omega \) translated by \( x \in \mathbb{R} \), more precisely we set \( \tau_x \omega := \{(x_j-x,E_j)\} \) if \( \omega = \{(x_j,E_j)\} \). We recall that the environment viewed from the walker \( Y_n^{\omega,\lambda} \) is given by the discrete time Markov chain \( (\tau_{Y_n^{\omega,\lambda}} \omega)_{n \geq 0} \). This is the crucial object to analyze in the ballistic regime. The following result concerning the environment viewed from the walker \( Y_n^{\omega,\lambda} \) is indeed at the basis of the derivation of Theorem 1 (ii) as well as the starting point for the analysis of the Einstein relation.

**Theorem 2** [56] Fix \( \lambda \in (0,1) \). Suppose that \( \mathbb{E}[e^{-(1-\lambda)Z_0}] < \infty \) and that \( u \) is continuous. Then the environment viewed from the walker \( Y_n^{\omega,\lambda} \) admits an invariant and ergodic distribution \( Q_\lambda \) mutually absolutely continuous w.r.t. \( \mathbb{P} \). Moreover, it holds

\[
v_Y(\lambda) = Q_\lambda[\varphi_\lambda] \quad \text{and} \quad v_X(\lambda) = \frac{v_Y(\lambda)}{Q_\lambda[1/(\sum_k r_{0,x_i}^\lambda)]},
\]

where \( \varphi_\lambda \) denotes the local drift, i.e. \( \varphi_\lambda(\omega) := \sum_i x_i p_{0,x_i}^0(\omega) \).

We point that for \( \lambda = 0 \) the probability distribution \( Q_0 \) defined as

\[
dQ_0 = \frac{\sum_k r_{0,x_k}^\lambda}{\mathbb{E}[\sum_k r_{0,x_k}^\lambda]} d\mathbb{P}
\]

is indeed reversible for the environment viewed from the walker \( Y_n^{\omega,\lambda=0} \), and that \( v_Y(0) = v_X(0) = 0 \). In what follows, when \( \lambda = 0 \) we will often drop \( \lambda \) from the notation (in particular, we will write simply \( r_{x_i,x_j}(\omega), p_{x_i,x_k}(\omega), X_t^{\omega}, Y_n^{\omega} \)).
The proof of Theorem 2 takes inspiration from the paper [4]. The main technical difficulty comes from the presence of arbitrarily long jumps, which does not allow to use standard techniques based on regeneration times. Following the method developed in [4] we have considered, for each positive integer $\rho$, the random walk obtained from $Y_n^{\omega,\lambda}$ by suppressing jumps between sites $x_i, x_j$ with $|i - j| > \rho$ [5]. For this $\rho$-indexed random walk, under the same hypothesis of Theorem 2, we have proved that there exists a distribution $Q^{(\rho)}$ which is invariant and ergodic for the associated environment viewed from the walker, and that $Q^{(\rho)}$ is mutually absolutely continuous w.r.t. $P$. In particular, the methods developed in [4] provide a probabilistic representation of the Radon–Nykodim derivative $dQ^{(\rho)} / dP$, which (together with a suitable analysis based on potential theory) allows to prove that $Q^{(\rho)}$ weakly converges to $Q$, and that $Q$ has indeed the nice properties stated in Theorem 2.

4 Linear response and Einstein relation for the biased 1d Mott random walk

Let us assume again (A1), (A2), (A3), (A4) as in the previous section. The probabilistic representation of the Radon–Nykodim derivative $dQ^{(\rho)} / dP$ mentioned in the above section is the starting point for the derivation of estimates on the Radon–Nykodim derivative $dQ / dQ_0$:

**Proposition 2** [6] Suppose that for some $p \geq 2$ it holds $E[e^{pZ_0}] < +\infty$. Fix $\lambda_0 \in (0, 1)$. Then

$$\sup_{\lambda \in [0, \lambda_0]} \left\| \frac{dQ_\lambda}{dQ_0} \right\|_{L^p(Q_0)} < \infty.$$  

The above proposition allows to prove the continuity of the expected value $Q_\lambda(f)$ of suitable functions $f$:

**Theorem 3** [6] Suppose that $E[e^{pZ_0}] < \infty$ for some $p \geq 2$ and let $q$ be the conjugate exponent of $p$, i.e. $q$ satisfies $\frac{1}{p} + \frac{1}{q} = 1$. Then, for any $f \in L^q(Q_0)$ and $\lambda \in [0, 1)$, it holds that $f \in L^1(Q_\lambda)$ and the map

$$[0, 1) \ni \lambda \mapsto Q_\lambda(f) \in \mathbb{R}$$

is continuous.

Without entering into the details of the proof (which can be found in [6]) we give some comments on the derivation of Theorem 3 from Proposition 2. To this aim, we take for simplicity $p = 2$. Hence we are supposing that $E[e^{2Z_0}] < \infty$ and that $f \in L^2(Q_0)$, and we want to prove that $f \in L^1(Q_\lambda)$ and that the function in (8) is continuous. For simplicity, let us restrict to its continuity at $\lambda = 0$. The fact that $f \in L^1(Q_\lambda)$ follows by writing $Q_\lambda(f) = Q_0(\frac{dQ_\lambda}{dQ_0} f)$ and then by applying Schwarz inequality and Proposition 2. The proof of the continuity at $\lambda = 0$
is more involved. We recall that by Kakutani’s theorem balls are compact for the $L^2(Q_0)$–weak topology. Hence, due to Proposition 2, the family of Radon–Nykodim derivatives $\frac{dQ_\lambda}{dQ_0}$, $\lambda \in [0, \lambda_0]$, is relatively compact for the $L^2(Q_0)$–weak topology. In [8] we then prove that any limit point of this family is given by $\mathds{1}$. As a byproduct of the representation $Q_\lambda(f) = Q_0(\frac{dQ_\lambda}{dQ_0}f)$ and of the weak convergence $\frac{dQ_\lambda}{dQ_0} \rightarrow \mathds{1}$, we get the continuity of (8) at $\lambda = 0$.

We now move to the study of $\partial_{\lambda = 0}Q_\lambda(f)$. To this aim we introduce the operator $L_0 : L^2(Q_0) \rightarrow L^2(Q_0)$ as

$$L_0 f(\omega) = \sum_k p_{0,x_k}(\omega)[f(\tau_{x_k}\omega) - f(\omega)] , \quad f \in L^2(Q_0).$$

We recall that a function $f$ belongs to $L^2(Q_0) \cap H_{-1}$ if there exists $C > 0$ such that

$$|\langle f, g \rangle| \leq C \langle g, -\mathbb{L}_0 g \rangle^{1/2} \quad \forall g \in L^2(Q_0).$$

Above $\langle \cdot, \cdot \rangle$ is the scalar product in $L^2(Q_0)$. Due to the theory developed by Kipnis and Varadhan [12], for any $f \in L^2(Q_0) \cap H_{-1}$, we have the weak convergence

$$\frac{1}{\sqrt{n}} \left( \sum_{j=0}^{n-1} f(\omega_j), \sum_{j=0}^{n-1} \varphi(\omega_j) \right) \rightharpoonup_{n \rightarrow \infty} (N^f, N^\varphi)$$

for a suitable 2d gaussian vector $(N^f, N^\varphi)$. Above $\varphi$ denotes the local drift $\varphi_\lambda$ with $\lambda = 0$ (cf. Theorem 2) and $\omega_j = \tau_{y_j}\omega$, i.e. $(\omega_n)_n$ represents the environment viewed from the walker $Y_n$.

Finally, we need another ingredient coming from the theory of square integrable forms in order to present our next theorem. We consider the space $\Omega \times \mathbb{Z}$ endowed with the measure $M$ defined by

$$M(v) = Q_0 \left[ \sum_{k \in \mathbb{Z}} p_{0,x_k} v(\cdot, k) \right] , \quad \forall v : \Omega \times \mathbb{Z} \rightarrow \mathbb{R} \quad \text{Borel, bounded}.$$

A generic Borel function $v : \Omega \times \mathbb{Z} \rightarrow \mathbb{R}$ will be called a form. $L^2(M)$ is known as the space of square integrable forms. Given a function $g = g(\omega)$ we define

$$\nabla g(\omega, k) := g(\tau_{x_k}\omega) - g(\omega).$$

If $g \in L^2(Q_0)$, then $\nabla g \in L^2(M)$. The closure in $M$ of the subspace $\{\nabla g : g \in L^2(Q_0)\}$ is the set of the so called potential forms (its orthogonal subspace is given by the so called solenoidal forms).

Take again $f \in H_{-1} \cap L^2(Q_0)$ and, given $\varepsilon > 0$, define $g_\varepsilon^f \in L^2(Q_0)$ as the unique solution of the equation

$$(\varepsilon - \mathbb{L}_0)g_\varepsilon^f = f .$$

As discussed with more details in [8], as $\varepsilon$ goes to zero the family of potential forms $\nabla g_\varepsilon^f$ converges in $L^2(M)$ to a potential form $h^f$:

$$h^f = \lim_{\varepsilon \rightarrow 0} \nabla g_\varepsilon^f \quad \text{in } L^2(M).$$
Theorem 4  Suppose $\mathbb{E}(e^{pZ_0}) < \infty$ for some $p > 2$. Then, for any $f \in H_{-1} \cap L^2(\mathbb{Q}_0)$, $\partial_{\lambda=0}Q_\lambda(f)$ exists. Moreover the following two probabilistic representations hold with $h = h^f$ (see (11)):

$$
\partial_{\lambda=0}Q_\lambda(f) = \begin{cases} 
Q_0[\sum_{k \in \mathbb{Z}} p_{0,x_k}(x_k - \varphi) h(\cdot, k)] \\
-Cov(N^f, N^\varphi)
\end{cases}
$$

We point out that a covariance representation of $\partial_{\lambda=0}Q_\lambda(f)$ as the second one in (12) appears also in (11) and (18).

We give some comments on the derivation of Theorem 4 up to the first representation in (12). Fix $f \in H_{-1} \cap L^2(\mathbb{Q}_0)$, thus implying that $Q_0(f) = 0$. Given $\varepsilon > 0$ take $g_{\varepsilon} \in L^2(\mathbb{Q}_0)$ as the unique solution of the equation $(\varepsilon - L_0)g_{\varepsilon} = f$, i.e. $g_{\varepsilon} = g^f_\varepsilon$ with $g^f_\varepsilon$ as in (10). Then we can write

$$
\frac{Q_\lambda(f) - Q_0(f)}{\lambda} = \frac{Q_\lambda(f)}{\lambda} = \frac{\varepsilon Q_\lambda(g_{\varepsilon})}{\lambda} - \frac{Q_\lambda(L_0 g_{\varepsilon})}{\lambda}.
$$

The idea is to take first the limit $\varepsilon \to 0$, afterwards the limit $\lambda \to 0$. By the results of Kipnis and Varadhan [12] we have that $\varepsilon Q_\lambda(g_{\varepsilon})$ is negligible as $\varepsilon \to 0$. Hence we have $\partial_{\lambda=0}Q_\lambda(f) = -\lim_{\lambda \to 0} \frac{Q_\lambda(L_0 g_{\varepsilon})}{\lambda}$ if the latter exists. On the other hand we have the following identity and approximations for $\varepsilon, \lambda$ small:

$$
-\frac{Q_\lambda(L_0 g_{\varepsilon})}{\lambda} = Q_\lambda\left[\frac{L_{\lambda} - L_0}{\lambda}g_{\varepsilon}\right] = Q_\lambda\left[\sum_{k \in \mathbb{Z}} \frac{\lambda}{\lambda} p^\lambda_{0,x_k} - p_{0,x_k}\right] g_{\varepsilon}(\tau_{x_k} \cdot) - g_{\varepsilon}
\approx Q_\lambda\left[\sum_{k \in \mathbb{Z}} \partial_{\lambda=0}p^\lambda_{0,x_k} h(\cdot, k)\right] \approx Q_0\left[\sum_{k \in \mathbb{Z}} \partial_{\lambda=0}p_{0,x_k} h(\cdot, k)\right]
$$

$$
= Q_0\left[\sum_{k \in \mathbb{Z}} p_{0,x_k}(x_k - \varphi) h(\cdot, k)\right],
$$

where $L_{\lambda} f(\omega) = \sum_{\omega} p^\lambda_{0,x_k}(\omega) [f(\tau_{x_k} \omega) - f(\omega)]$ and $h = h^f$ (see (11)). Roughly, the first identity in (14) follows from the stationarity of $Q_\lambda$ for the environment viewed from $Y_{\omega,\lambda}$, the first approximation in the second line follows from (11), the second approximation in the second line follows from Theorem 3 the identity in the third line follows from the equality $\partial_{\lambda=0}p^\lambda_{0,x_k} = p_{0,x_k}(x_k - \varphi)$.

The above steps are indeed rigorously proved in [6]. Since, as already observed, $\partial_{\lambda=0}Q_\lambda(f) = -\lim_{\lambda \to 0} \frac{Q_\lambda(L_0 g_{\varepsilon})}{\lambda}$, the content of Theorem 4 up to the first representation in (12) follows from (14).

We conclude this section with the Einstein relation. To this aim we denote by $D_X$ the diffusion coefficient associated to $X^\tau_\nu$ and by $D_Y$ the diffusion coefficient associated to $Y_{\nu,\lambda}$. $D_X$ is the variance of the Brownian motion to which $X^\tau_\nu$ converges under diffusive rescaling, and a similar definition holds for $D_Y$.

Theorem 5  The following holds:

(i) If $\mathbb{E}[e^{2Z_0}] < \infty$, then $v_Y(\lambda)$ and $v_X(\lambda)$ are continuous functions of $\lambda$;
(ii) If \( E[e^{\varphi Z_0}] < \infty \) for some \( p > 2 \), then the Einstein relation is fulfilled, i.e.
\[
\partial_{\lambda=0} v_Y(\lambda) = D_Y \quad \text{and} \quad \partial_{\lambda=0} v_X(\lambda) = D_X.
\] (15)

If we make explicit the temperature dependence in the jump rates (7) we would have
\[
r_{\lambda,x_i,x_j}(\omega) = \exp \{-|x_i - x_j| + \lambda \beta(x_j - x_i) - \beta u(E_i, E_j)\},
\]
where \( \lambda \) is the strength of the external field. Then Einstein relation (15) takes the more familiar (from a physical viewpoint) form
\[
\partial_{\lambda=0} v_Y(\lambda, \beta) = \beta D_Y(\beta) \quad \text{and} \quad \partial_{\lambda=0} v_X(\lambda, \beta) = \beta D_X(\beta).
\]

We conclude by giving some ideas behind the proof of the Einstein relation for \( v_Y(\lambda) \) in Theorem 5. We do not fix the details, but only the main arguments. By applying Theorem 4 with \( f := \phi \) and using the first representation in the r.h.s. of (12), one can express \( \partial_{\lambda=0} Q_{\lambda}[\varphi] \) as a function of \( h = h^\varphi \):
\[
\partial_{\lambda=0} Q_{\lambda}[\varphi] = Q_0 \left[ \sum_{k \in \mathbb{Z}} p_{0,x_k}(x_k - \varphi) h(\cdot, k) \right].
\]
Recall that \( v_Y(\lambda) = Q_{\lambda}[\varphi \lambda] \) (cf. Theorem 2) and that \( v_Y(0) = 0 \). In [6] we have proved the following approximations and identities (for the last identity see Section 9 in [6]):
\[
\frac{v_Y(\lambda) - v_Y(0)}{\lambda} = \frac{v_{\lambda}[\varphi \lambda]}{\lambda} = Q_{\lambda}[\varphi \lambda - \varphi] + Q_{\lambda}[\varphi - 0]_{\lambda} \\
\approx Q_0 \left[ \partial_{\lambda=0} \varphi \lambda + \partial_{\lambda=0} Q_{\lambda}[\varphi] \right] \\
= Q_0 \left[ \partial_{\lambda=0} \varphi \lambda + Q_0 \left[ \sum_{k \in \mathbb{Z}} p_{0,x_k}(x_k - \varphi) h(\cdot, k) \right] \right] \\
= Q_0 \left[ \sum_{k \in \mathbb{Z}} p_{0,x_k}(x_k - \varphi)(x_k + h(\cdot, k)) \right] = D_Y.
\]

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