A CLASSIFICATION OF PAIRS OF DISJOINT NONPARALLEL PRIMITIVES IN THE BOUNDARY OF A GENUS TWO HANDLEBODY

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Abstract. Embeddings of pairs of disjoint nonparallel primitive simple closed curves in the boundary of a genus two handlebody are classified. Briefly, two disjoint primitives either lie on opposite ends of a product $F \times I$, or they lie on opposite ends of a kind of “twisted” product $F \times e \times I$, where $F$ is a once-punctured torus.

If one of the curves is a proper power of a primitive, the situation is simpler. Either the curves lie on opposite sides of a separating disk in the handlebody, or they bound a nonseparating essential annulus in the handlebody.

1. INTRODUCTION

Suppose $H$ is a genus two handlebody. A simple closed curve $\alpha$ in $\partial H$ is primitive in $H$ if there exists a disk $D$ in $H$ such that $|\alpha \cap D| = 1$. Equivalently $\alpha$ is conjugate to a free generator of $\pi_1(H)$. A pair of disjoint properly embedded disks in $H$ is a complete set of cutting disks of $H$ if cutting $H$ open along the pair of disks yields a 3-ball. A pair of disjoint simple closed curves $(\alpha, \beta)$ in $\partial H$ is a primitive pair if both $\alpha$ and $\beta$ are primitive in $H$. (Note that a “pair of primitives” is not generally a “primitive pair”.) A pair of nonseparating simple closed curves $(\alpha, \beta)$ in $\partial H$ is separated in $H$ if there exists a separating disk $D$ embedded in $H$ such that $\alpha$ and $\beta$ lie on opposite sides of $\partial D$ in $\partial H$.

2. PRELIMINARIES

This section recalls some of the basic properties of genus two Heegaard diagrams, their underlying graphs, and genus two R-R diagrams which will be helpful.

2.1. GENUS TWO HEegaRD DIAGRAMS AND THEIR UNDERLYING GRAPHS. Suppose $\alpha$ and $\beta$ are disjoint nonparallel simple closed curves in the boundary of a genus two handlebody $H$, neither $\alpha$ nor $\beta$ bound disks in $H$, and $\{D_A, D_B\}$ is a complete set of cutting disks of $H$. Cutting $H$ open along $D_A$ and $D_B$ cuts $\alpha$ and $\beta$ into sets of arcs $E(\alpha)$ and $E(\beta)$ respectively, and cuts $H$ into a 3-ball $W$. Then $\partial W$ contains disks $D^+_A$, $D^-_A$, $D^+_B$, and $D^-_B$ such that gluing $D^+_A$ to $D^-_A$ and $D^+_B$ to $D^-_B$ reconstitutes $\alpha$, $\beta$, and $H$.

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The sets of arcs $E(\alpha)$ and $E(\beta)$ form the edges of Heegaard diagrams in $\partial W$ with “fat”, i.e., disk rather than point, vertices $D^+_A$, $D^-_A$, $D^+_B$, and $D^-_B$. Let $HD_\alpha$, $HD_\beta$, and $HD_{\alpha,\beta}$ be the Heegaard diagrams in $\partial W$ whose edges are the arcs of $E(\alpha)$, $E(\beta)$, and $E(\alpha) \cup E(\beta)$ respectively.

If one ignores how $D^+_A$ and $D^+_B$ are identified with $D^-_A$ and $D^-_B$ to reconstitute $H$, the sets of arcs $E(\alpha)$, $E(\beta)$, and $E(\alpha) \cup E(\beta)$ also form the edges of graphs in $\partial W$ with vertices $D^+_A$, $D^-_A$, $D^+_B$, and $D^-_B$. Let $G_\alpha$, $G_\beta$, and $G_{\alpha,\beta}$ denote the graphs in $\partial W$ whose edges are the arcs of $E(\alpha)$, $E(\beta)$, and $E(\alpha) \cup E(\beta)$ respectively. Then $G_\alpha$ is the graph underlying $HD_\alpha$, and $G_\beta$ is the graph underlying $HD_\beta$ etc.. Note that these graphs are not just abstract graphs, since they inherit specific embeddings in the 2-sphere $S^2 \cong \partial W$ from the Heegaard diagrams which they underlie.

**Remark 2.1.** The notation $HD_\alpha$, $HD_\beta$, and $HD_{\alpha,\beta}$, does not specify the set of cutting disks $\{D_A, D_B\}$. However, this shouldn’t lead to ambiguity because context will make it clear which set of cutting disks of $H$ is playing the role of $\{D_A, D_B\}$.

**Remark 2.2.** In figures of $G_\alpha$, $G_\beta$, or $G_{\alpha,\beta}$, the disks $D^+_A$, $D^-_A$, $D^+_B$, and $D^-_B$ in $\partial W$ are smashed to points denoted by $A^+$, $A^-$, $B^+$, and $B^-$ respectively.

### 2.2. Genus two R-R diagrams.

R-R diagrams are a type of planar diagram related to Heegaard diagrams. These diagrams were originally introduced by Osborne and Stevens in [OS74]. They are particularly useful for describing embeddings of simple closed curves in the boundary of a handlebody so that the embedded curves represent certain conjugacy classes in $\pi_1$ of the handlebody.

Here is a description of the basics of genus two R-R diagrams, which is all we need. Suppose $\Sigma$ is a closed orientable surface of genus two obtained by capping off the two boundary components of an annulus $A$ with a pair of once-punctured tori $F_A$ and $F_B$, so that $\Sigma = A \cup F_A \cup F_B$, $A \cap F_A = \partial F_A$, and $A \cap F_B = \partial F_B$. Following Zieschang [ZS8], the separating simple closed curves $\partial F_A$ and $\partial F_B$ in $\Sigma$ are belt curves, and $F_A$ and $F_B$ are handles.

If $S$ is a set of pairwise disjoint simple closed curves in $\Sigma$, then, after isotopy, we may assume each curve $\zeta \in S$ is either disjoint from $\partial F_A \cup \partial F_B$, or $\zeta$ is cut by its intersections with $\partial F_A \cup \partial F_B$ into arcs, each properly embedded and essential in one of $A$, $F_A$, $F_B$. A properly embedded essential arc in $F_A$ or $F_B$ is a **connection**.

Two connections in $F_A$ or $F_B$ are **parallel** if they are isotopic in $F_A$ or $F_B$ via an isotopy keeping their endpoints in $\partial F_A$ or $\partial F_B$. A collection of pairwise disjoint connections on a given handle can be partitioned into **bands** of pairwise parallel connections. Since each handle is a once-punctured torus, there can be at most three nonparallel bands of connections on a given handle.

Note that sets of pairwise nonparallel connections in a once-punctured torus are unique up to homeomorphism. In particular, if $F$ and $F'$ are once-punctured tori, $\Delta = \{\delta_1, \cdots, \delta_k\}$, is a set of pairwise nonparallel connections in $F$, and $\Delta' = \{\delta'_1, \cdots, \delta'_l\}$, $1 \leq i \leq 3$ is a set of pairwise nonparallel connections in $F'$, then there is a homeomorphism $h: F \to F'$ which takes $\Delta$ to $\Delta'$.

**Remark 2.3.** In practice, it is often inconvenient to have curves in $S$ that lie completely in $F_A$ or $F_B$. This situation can be avoided by relaxing the supposition that each curve in $S$ has only essential intersections with the belt curves $\partial F_A$ and $\partial F_B$. Then, if a curve $\zeta \in S$ lies completely in $F_A$ or $F_B$, say $F_B$, $\zeta$ can be isotoped in $\Sigma$ so that $\zeta \cap F_B$ consists of one properly embedded essential arc, while $\zeta \cap A$ is
an inessential arc in $A$, isotopic in $A$ into $\partial F_B$, keeping its endpoints fixed. (In the R-R diagrams of this paper, the curve $\beta$ is always displayed in this manner.)

Some simplifications can be made at this point without losing any information about the embedding of the curves of $\mathcal{S}$ in $\Sigma$. For example, suppose $F$ is either $F_A$ or $F_B$, and let $\mathcal{S}_A$ be the set of arcs in which curves of $\mathcal{S}$ intersect $A$. Then each set of parallel connections on $F$ can be merged into a single connection. (This also merges some endpoints of arcs in $\mathcal{S}_A$ meeting $\partial F$.)

After such mergers, $F$ carries at most 3 pairwise nonparallel connections. Continuing, after each set of parallel connections on $F_A$ and $F_B$ has been merged, additional mergers of sets of properly embedded parallel subarcs of $\mathcal{S}_A$ can also be made; although now whenever, say, $n$ parallel arcs are merged into one, this needs to be recorded by placing the integer $n$ near the single arc resulting from the merger.

Merging parallel connections in $F_A$ and $F_B$ turns the set of pairwise disjoint simple closed curves in $\mathcal{S}$ into a graph $\mathcal{G}$ in $\Sigma$ whose vertices are the endpoints of the remaining connections in $F_A$ and $F_B$. Clearly $\mathcal{G}$ and its embedding in $\Sigma$ completely encodes the embedding of the curves of $\mathcal{S}$ in $\Sigma$.

Understanding $\mathcal{G}$ and the curves of $\mathcal{S}$ it represents is complicated by the fact that $\mathcal{G}$ is usually nonplanar. However, $\mathcal{G}$ always has a nice immersion $I(\mathcal{G})$ in the plane $\mathbb{R}^2$, which we now describe. It is this immersion of $\mathcal{G}$ in $\mathbb{R}^2$ which becomes an R-R diagram of the curves of $\mathcal{S}$ in $\Sigma$.

To produce $I(\mathcal{G})$, first remove a small disk $D$, disjoint from $\mathcal{G}$, from the interior of $\mathcal{A}$. Then embed $\mathcal{A} - D$ in $\mathbb{R}^2$ so that $\partial F_A$ and $\partial F_B$ bound disjoint round disks, say $\mathcal{F}_A$ and $\mathcal{F}_B$ respectively, in $\mathbb{R}^2$.

Next, note that if $\delta$ and $\delta'$ are nonparallel connections on a handle $F_X$, with $X \in \{A, B\}$, the endpoints of $\delta$ separate the endpoints of $\delta'$ in the belt curve $\partial F_X$. It follows that, if $u$ and $v$ in $\partial F_X$ are the endpoints of a connection $\delta$ in $F_X$, we may assume $u$ and $v$ bound a diameter $d_\delta$ of the disk $F_X$, and then $\delta$ can be embedded in $F_X$ as the diameter $d_\delta$ of $F_X$. This results in each round disk $F_X$ containing 0, 1, 2, or 3 diameters passing through its center $X$, with each diameter an image of a connection in $F_X$, where the number of such diameters depends upon whether $F_X$ originally contained respectively 0, 1, 2, or 3 bands of parallel connections.

Also note that the immersion $I(\mathcal{G})$ still encodes the embedding of the curves in $\mathcal{S}$ in $\Sigma$ up to homeomorphism. This follows from the aforementioned fact that if $\Delta = \{\delta_1, \ldots, \delta_i\}$ and $\Delta' = \{\delta'_1, \ldots, \delta'_i\}$, 1 ≤ $i$ ≤ 3 are each sets of pairwise nonparallel connections in a once-punctured torus $F$, then there is a homeomorphism of $F$ which takes $\Delta$ to $\Delta'$.

The partition of $\Sigma$ into $\mathcal{A}$, $F_A$ and $F_B$ makes it easy to describe infinite families of parametrized embeddings of the curves of $\mathcal{S}$ in the boundary of a genus two handlebody. To do this, consider $\Sigma$ as 2-sided, with sides $\Sigma^+$ and $\Sigma^-$, and suppose the curves of $\mathcal{S}$ lie in $\Sigma^+$. Then gluing a pair of disks $D_A$ and $D_B$ to $\Sigma^-$ so that $\partial D_A$ is glued to a nonseparating simple closed curve in $F_A^{-1}$, $\partial D_B$ is glued to a nonseparating simple closed curve in $F_B^-$, and the resulting 2-sphere boundary component is capped off with a 3-ball, makes $\Sigma$ the boundary of a genus two handlebody $H$.

Continuing, note that if $\delta$ and $\delta'$ are two nonparallel oriented connections in a once-punctured torus $F$, then the isotopy class of an oriented nonseparating simple closed curve $\gamma$ in $F$ is determined by its algebraic intersection numbers with $\delta$ and
This makes it possible to parametrize the isotopy classes of attaching curves of \( \partial D_A \) and \( \partial D_B \) in \( F_A \) and \( F_B \) by adding integer labels to the endpoints of the diameters of \( F_A \) and \( F_B \) which represent connections in \( F_A \) and \( F_B \). (There are minor restrictions on the values of these parameters; all related to similar restrictions on meridional and longitudinal coordinates of simple closed curves in \( H_1(\partial V) \), where \( V \) is a solid torus. Figure 1 illustrates these.)

\[ \text{Figure 1. There is a simple closed curve in the once-punctured torus } F_A \text{ which intersects the bands of connections in Figures 1a, 1b, and 1c with the indicated intersection numbers if and only if: A) } \gcd(p, r) = 1 \text{ in Figure 1a. B) } \gcd(p, r) = 1 \text{ and } q = p + r \text{ in Figure 1b. C) } p \in \mathbb{Z} \text{ in Figure 1c. Figure 1d shows a variant labeling, useful when we wish to think of } F_A \text{ as carrying a meridional and longitudinal pair of simple closed curves, say } m \text{ and } l, \text{ meeting transversely at a single point. In this case, if } \delta \text{ is a connection with label } (p, q) \text{ in } F_A, \text{ so } [\delta] = p[l] + q[m] \text{ in } H_1(F_A, \partial F_A), \text{ then } ps - rq = \pm 1. \]

\[ \text{Remark 2.4. In practice, given a set of curves } S \text{ in the boundary of a genus two handlebody } H, \text{ we usually reverse the order in which the partition of } \partial H \text{ into } A, F_A, \text{ and } F_B \text{ and a complete set of cutting disks } \{D_A, D_B\} \text{ are chosen by choosing a desired set of cutting disks } \{D_A, D_B\} \text{ first, and then choosing an appropriate compatible partition of } \partial H \text{ into } A, F_A, \text{ and } F_B \text{ with } \partial D_A \subset F_A \text{ and } \partial D_B \subset F_B. \]

3. The classification

**Theorem 3.1.** Suppose \( H \) is a genus two handlebody, and \( \alpha \) and \( \beta \) are a pair of disjoint nonparallel simple closed curves in \( \partial H \) such that both \( \alpha \) and \( \beta \) are primitive in \( H \). Then \( \alpha \) and \( \beta \) have an R-R diagram with the form of Figure 2, 3 or 4.

**Proof.** Recall that if \( \gamma \) is a primitive simple closed curve in the boundary of a genus two handlebody \( H \), then there is a unique cutting disk of \( H \) (up to isotopy) disjoint from \( \gamma \), while there are an infinite number of cutting disks of \( H \) that intersect \( \gamma \) transversely exactly once. Then given \( (\alpha, \beta) \), let \( \{D_A, D_B\} \) be a complete set of cutting disks of \( H \) such that \( |\beta \cap \partial D_A| = 0 \), and such that \( \partial D_B \) intersects \( \alpha \) minimally subject to \( |\beta \cap \partial D_B| = 1 \). Now the goal is to show that if \( |\alpha \cap \partial D_B| = 0 \), \( |\alpha \cap \partial D_B| = 1 \), or \( |\alpha \cap \partial D_B| > 1 \), then \( \alpha \) and \( \beta \) have an R-R diagram with the form of Figure 2, 3 or 4 respectively.

Consider the first possibility \( |\alpha \cap \partial D_B| = 0 \). In this case, since \( \alpha \) is primitive in \( H \), we must have \( |\alpha \cap \partial D_A| = 1 \), and then the pair of disjoint primitives \( (\alpha, \beta) \)
is also a pair of primitives in $\pi_1(H)$. It follows that the pair $(\alpha, \beta)$ has an R-R diagram with the form of Figure 2.

Turning to the second possibility, suppose $|\alpha \cap \partial D_B| = 1$. In this case, let $C$ be a separating simple closed curve in $\partial H$, disjoint from $\partial D_A$, $\partial D_B$, and $\beta$, such that $C$ separates $\partial D_A$ and $\partial D_B$. Then let $A$ be a small annular regular neighborhood of $C$ in $\partial H$, chosen so that $A$ is also disjoint from $\partial D_A$, $\partial D_B$, and $\beta$. Then $\partial H - \text{int}(A)$ is the union of two once-punctured tori $F_A$ and $F_B$, with $\partial D_A \subset F_A$ and $\partial D_B \subset F_B$, and we may suppose that $\alpha$ has only essential intersections with $A$, $F_A$ and $F_B$, as well as $\partial D_A$ and $\partial D_B$. This partition of $\partial H$ into $A \cup F_A \cup F_B$ provides a natural framework of the sort described in Subsection 2.2, which leads to an R-R diagram describing how $\alpha$, $\beta$, $\partial D_A$, and $\partial D_B$, are configured in $\partial H$.

In this case, this is quite easy. Since $\alpha$ and $\beta$ are disjoint, while $|\alpha \cap \partial D_B| = 1$, $\alpha \cap F_B$ must consist of a single connection. Then $\alpha \cap F_A$ must also consist of exactly one connection, which can be any connection in $F_A$. It follows that the pair $(\alpha, \beta)$ has an R-R diagram with the form of Figure 3.

Finally, consider the last possibility $|\alpha \cap \partial D_B| > 1$. In this case, Lemma 3.2 lays the foundation for the analysis by showing that the graph $G_{\alpha, \beta}$ underlying the Heegaard diagram $HD_{\alpha, \beta}$ has the form of Figure 5.

The same partition of $\partial H$ into the union of $\alpha$, $F_A$, and $F_B$, can be used to obtain an R-R diagram of $\alpha$ and $\beta$ on $\partial H$ as was used in the previous case. The R-R diagram will differ of course, because now $\alpha$ has more than one connection in each of $F_A$, $F_B$. We can determine what these connections of $\alpha$ can be by looking at the cyclic word which $\alpha$ represents in $\pi_1(H)$.

Theorem 3.3 below, which is the main result of [CMZ81], shows that, if $\alpha$ is primitive in $H$, then the cyclic word which $\alpha$ represents in $\pi_1(H)$ must have a particular form, and this provides what we need. To use Theorem 3.3 let $A$ and $B$ be generators of $\pi_1(H)$ chosen so that $A$ and $B$ are represented by simple closed curves in $H$ dual to $D_A$ and $D_B$ respectively. Then, since $\alpha$ is primitive in $H$, and $G_{\alpha, \beta}$ has the form of Figure 5 with $|\alpha \cap \partial D_B| > 1$, Theorem 3.3 implies $\alpha$ represents a cyclic word in $\pi_1(H)$ of the form $w = A^{m_1}B \ldots A^{m_j}B$, with $\{m_1, \ldots, m_j\} = \{e, e + 1\}$ and $j > 1$. In addition, since $j > 1$ and $\alpha$ is a primitive rather than a proper power of a primitive in $H$, both $e$ and $e + 1$ must appear as exponents of $A$ in $W$.

It follows that the A-handle of an R-R diagram $D$ describing the embedding of $\alpha$ and $\beta$ in $\partial H$ must have exactly two nonparallel types of connections bearing labels $p$ and $p + \epsilon$, where $\epsilon = \pm 1$, and $\{p, p + \epsilon\} = \{e, e + 1\}$.

Next, let $a = |A^p B|$ and $b = |A^{p+\epsilon} B|$ be the number of subwords of the form $A^p B$ and $A^{p+\epsilon} B$ respectively in $w$. Then $a + b = |\alpha \cap \partial D_B| > 1$. And, since $c \geq a + b$ in Figure 5, the set of exponents $\{m_1, \ldots, m_j\}$ of $A$ in $w$ must satisfy $\{m_1, \ldots, m_j\} = \{e, e + 1\}$ with $e > 1$. So $\min\{p, p + \epsilon\} > 1$ in $D$. Finally, since $\{A^p B, A^{p+\epsilon} B\}$ is a set of free generators of $\pi_1(H)$, $\gcd(a, b) = 1$.

Then, putting this all together, it follows readily that the pair $(\alpha, \beta)$ has an R-R diagram with the form of Figure 4.

Lemma 3.2. Suppose $\alpha$ and $\beta$ are disjoint nonparallel primitive simple closed curves in the boundary of a genus two handlebody $H$, and $\{D_A, D_B\}$ is a complete set of cutting disks of $H$ such that $|\beta \cap \partial D_A| = 0$, $|\beta \cap \partial D_B| = 1$, and $|\alpha \cap \partial D_B| > 1$, where $|\alpha \cap \partial D_B|$ is minimal subject to $|\beta \cap \partial D_B| = 1$. 

□
Then the graph $G_{\alpha,\beta}$ underlying the Heegaard diagram $HD_{\alpha,\beta}$ of $\alpha$ and $\beta$ with respect to $\{D_A, D_B\}$ has the form of Figure 5 with $c \geq a + b > 1$.

Proof. Since $\alpha$ is primitive in $H$, there is a cutting disk $D$ of $H$ disjoint from $\alpha$. This implies the subgraph $G_\alpha$ of $G_{\alpha,\beta}$ is either not connected, or has a cut vertex. We will show that the only way either of these alternatives can hold is if $G_{\alpha,\beta}$ has the form of Figure 5.

First, note that if there are no edges of $HD_\alpha$ connecting $D_A^+$ to either $D_B^+$ or $D_B^-$, then $|\alpha \cap \partial D_B| \leq 1$, contrary to hypothesis. So there must be edges of $G_\alpha$ connecting $A^+$ to either $B^+$ or $B^-$. However, if there are edges of $G_\alpha$ connecting $A^+$ to both $B^+$ and $B^-$, then $G_\alpha$ is connected and has no cut vertex. So, up to swapping $A^+$ and $A^-$ or $B^+$ and $B^-$, we may assume $A^+$ is connected to $B^+$, and $A^+$ is not connected to $B^-$ in $G_\alpha$.

Continuing, if edges of $HD_\alpha$ only connect $D_A^+$ to $D_B^-$, and $D_A^-$ to $D_B^+$, then the bandsum of $D_A$ and $D_B$ along one of these edges is a cutting disk $D'_B$ of $H$ such that $\{D_A, D'_B\}$ is a complete set of cutting disks of $H$, $|\beta \cap \partial D'_B| = 1$, and $|\alpha \cap \partial D_B| < |\alpha \cap \partial D'_B|$. This contradicts the assumed minimality of $|\alpha \cap \partial D_B|$.

It follows that there are edges of $G_\alpha$ connecting $A^+$ to $A^-$. And then, since $G_\alpha$ must have a cut vertex, no edges of $G_\alpha$ connect $B^+$ to $B^-$.

Finally, if $c < a + b$ in Figure 5 then, as before, the bandsum of $D_A$ and $D_B$ along an edge of $HD_\alpha$ connecting $D_A^+$ to $D_B^-$ is a cutting disk $D'_B$ of $H$ such that $\{D_A, D'_B\}$ is a complete set of cutting disks of $H$, $|\beta \cap \partial D'_B| = 1$, and $|\alpha \cap \partial D_B| < |\alpha \cap \partial D'_B|$. This again contradicts the assumed minimality of $|\alpha \cap \partial D_B|$. It follows that $G_{\alpha,\beta}$ has the form shown in Figure 5. \qed

3.1. Recognizing primitives. The following result of Cohen, Metzler, and Zimmermann makes it possible to determine if a cyclically reduced word in a free group of rank two is primitive.

**Theorem 3.3** (CMZ81). Suppose a cyclic conjugate of

$$w = A^{m_1}B^{n_1} \cdots A^{m_j}B^{n_j}$$

is a member of a basis of $F(A, B)$, where $j \geq 1$ and each indicated exponent is nonzero. Then, after perhaps replacing $A$ by $A^{-1}$ or $B$ by $B^{-1}$, there exists $e > 0$ such that:

$$m_1 = \cdots = m_j = 1, \quad \text{and} \quad \{n_1, \ldots, n_j\} = \{e, e + 1\},$$

or

$$\{m_1, \ldots, m_j\} = \{e, e + 1\}, \quad \text{and} \quad n_1 = \cdots = n_j = 1.$$

Note that if $w$ in $F(A, B)$ has the form $w = AB^{n_1} \cdots AB^{n_j}$, say, with $j \geq 1$ and $\{n_1, \ldots, n_j\} = \{e, e + 1\}$, then the automorphism $A \mapsto AB^{-e}$ of $F(A, B)$ reduces the length of $w$, so repeated applications of such automorphisms can be used to determine if a given word $w$ in $F(A, B)$ is a primitive.

3.2. Ends of (twisted) products.

**Definition 3.4.** Suppose $F$ is a once-punctured torus. Then $F \times I$ is a genus two handlebody $H$, and the surfaces $F \times 0$ and $F \times 1$ in $\partial H$ are the ends of the product $F \times I$. Suppose $\delta$ is a nonseparating simple closed curve lying in $F \times 0$ or $F \times 1$. If $\delta$ is pushed into the interior of $F \times I$ by an isotopy and Dehn surgery is performed on $\delta$, the result is another genus two handlebody $H'$. Then $H'$ is a
Figure 2. If $H$ is a genus two handlebody, $(\alpha, \beta)$ is a pair of disjoint nonparallel primitives in $\partial H$, and \{\(D_A, D_B\)\} is a complete set of cutting disks of $H$ such that $|\beta \cap \partial D_A| = 0$, $|\beta \cap \partial D_B| = 1$, and $|\alpha \cap \partial D_B| = 0$, then $\alpha$ and $\beta$ have an R-R diagram with the form of this figure. Here $(\alpha, \beta)$ represents $(A, B)$ in $\pi_1(H)$.

Figure 3. If $H$ is a genus two handlebody, $(\alpha, \beta)$ is a pair of disjoint nonparallel primitives in $\partial H$, and \{\(D_A, D_B\)\} is a complete set of cutting disks of $H$ such that $|\beta \cap \partial D_A| = 0$, $|\beta \cap \partial D_B| = 1$, and $|\alpha \cap \partial D_B| = 1$, then $\alpha$ and $\beta$ have an R-R diagram with the form shown in this figure. Here the parameters $p$ and $q$ are intersection numbers of the connection $\alpha \cap F_A$ with $\partial D_A$ and a simple closed longitudinal curve $l$ on $F_A$ with $|l \cap \partial D_A| = 1$. Then $p \in \mathbb{Z}$, $\gcd(p, q) = 1$, and $(\alpha, \beta)$ represents $(A^p, B)$ in $\pi_1(H)$.

twisted product $F \times I$, and the two surfaces $F \times 0$ and $F \times 1$ in $\partial H'$ are the ends of the twisted product $F \times I$.

3.3. Type I and Type II pairs of disjoint primitives.

Definition 3.5. Suppose $(\alpha, \beta)$ is a pair of disjoint nonparallel primitive simple closed curves in the boundary of a genus two handlebody $H$. The pair $(\alpha, \beta)$ is a Type I pair if there is a cutting disk $D$ of $H$ such that $|\alpha \cap \partial D| = 1$ and $|\beta \cap \partial D| = 1$. The pair $(\alpha, \beta)$ is a Type II pair if there is a once-punctured torus $F$ and a homeomorphism $h: H \to F \times I$ such that $h(\alpha) \subset F \times 1$ and $h(\beta) \subset F \times 0$. (Somewhat loosely, Type I pairs are twisted pairs, while Type II pairs are untwisted pairs.)

The remaining results of this section show that pairs of disjoint nonparallel primitives on the boundary of a genus two handlebody either lie on disjoint ends
Figure 4. If $H$ is a genus two handlebody, $(\alpha, \beta)$ is a pair of disjoint nonparallel primitives in $\partial H$, and $\{D_A, D_B\}$ is a complete set of cutting disks of $H$ with $|\beta \cap \partial D_A| = 0$, $|\beta \cap \partial D_B| = 1$, and $|\alpha \cap \partial D_B| = s > 1$, with $s$ minimal subject to $|\beta \cap \partial D_B| = 1$, then $\alpha$ and $\beta$ have an R-R diagram with the form shown in this figure with parameters which satisfy $\gcd(a, b) = 1$, $a + b > 1$, $\epsilon = \pm 1$, and $\min\{p, p + \epsilon\} > 1$.

Figure 5. Lemma 3.2 shows that if $|\alpha \cap \partial D_B| > 1$ in Theorem 3.1, then, up to swapping $A^+$ and $A^-$ or $B^+$ and $B^-$, the graph $G_{\alpha, \beta}$ of the Heegaard diagram $HD_{\alpha, \beta}$ has the form shown here with $c \geq a + b$ and $a + b = |\alpha \cap \partial D_B|$.

We begin with the following lemma which characterizes $(\alpha, \beta)$ pairs which lie on disjoint ends of a product, $F \times I$, where $F$ is a once-punctured torus.

Lemma 3.6. Suppose $H$ is a genus two handlebody, $\alpha$ and $\beta$ are two disjoint nonparallel simple closed curves in $\partial H$, each of which is primitive in $H$, and $A$ and $B$ are a pair of free generators of $\pi_1(H)$. Then there is a once-punctured torus $F$ and a homeomorphism $h: H \to F \times I$ such that $h(\alpha) \subset F \times 1$ and $h(\beta) \subset F \times 0$ if and only if there exists a simple closed curve $\Gamma$ in $\partial H$ separating $\alpha$ and $\beta$ such that $\Gamma$ represents the cyclic word $ABA^{-1}B^{-1}$ or its inverse in $\pi_1(H)$. 

of an ordinary product $F \times I$, or on disjoint ends of a twisted product $F_\sim \times I$, where $F$ is a once-punctured torus.
Figure 6. This figure shows the pair of disjoint primitives $(\alpha, \beta)$ of Figure 2 separated by a simple closed curve $\Gamma$ such that $\Gamma$ represents $ABA^{-1}B^{-1}$ in $\pi_1(H)$. (The existence of $\Gamma$ shows that in addition to being separated and a Type I pair, $(\alpha, \beta)$ is also a Type II pair.)

Figure 7. This figure and Lemma 3.6 show that, if $(\alpha, \beta)$ is a pair of disjoint primitives in $\partial H$ with an R-R diagram of the form shown in Figure 4, then there exists a curve $\Gamma$ in $\partial H$, separating $\alpha$ and $\beta$, and a once-punctured torus $F$, such that $H$ is homeomorphic to $F \times I$, under a homeomorphism which takes $\partial F \times I$ to a regular neighborhood of $\Gamma$ in $\partial H$. This follows from Lemma 3.6, since $\Gamma$ represents $A^\epsilon B^{-1}A^{-\epsilon}B$ in $\pi_1(H)$ with $\epsilon = \pm 1$.

Proof. The proof follows directly from the well-known fact, see Proposition 5.1 of [LS77], that any automorphism of the free group of rank two $F(A, B)$ carries the cyclic word represented by the commutator $ABA^{-1}B^{-1}$ onto itself or its inverse.

Suppose there is a once-punctured torus $F$ and a homeomorphism $h: H \to F \times I$ such that $h(\alpha) \subset F \times 1$ and $h(\beta) \subset F \times 0$. Let $X$ and $Y$ be a pair of free generators of $\pi_1(F)$. Then the simple closed curve $\Gamma = h^{-1}(\partial F \times \frac{1}{2})$ in $\partial H$ separates $\alpha$ and $\beta$ and $\Gamma$ represents a cyclic word in $\pi_1(H)$ equal to the commutator $h^{-1}(XYX^{-1}Y^{-1})$ or its inverse in $\pi_1(H)$. 
Figure 8. This R-R diagram shows that if $\alpha$ and $\beta$ have an R-R diagram with the form of Figure 3 and $\Gamma$ is a simple closed curve in $\partial H$, separating $\alpha$ and $\beta$, such that $\Gamma$ represents a cyclic word in $\pi_1(H)$ with no more than four syllables, then $\alpha$, $\beta$, and $\Gamma$ have an R-R diagram with the form of this figure. Here the pairs of parameters $(p, q)$ and $(r, s)$ are intersection numbers of connections on the A-handle of this R-R diagram with $\partial D_A$ and a simple closed longitudinal curve $l$ on the A-handle. (By twisting $l$ around $\partial D_A$ when $p \neq 0$, we may assume $|r| < |p|$.) Note $|l \cap \partial D_A| = 1$ if and only if $ps - rq = \pm 1$. Then, by Lemma 3.6, $(\alpha, \beta)$ is a Type II pair if and only if $r = \pm 1$, and this occurs if and only if $q = \pm (ps + 1)$.

Conversely, suppose there exists a simple closed curve $\Gamma$ in $\partial H$ separating $\alpha$ and $\beta$, and a pair of free generators $A$ and $B$ of $\pi_1(H)$ such that $\Gamma$ represents the cyclic word $ABA^{-1}B^{-1}$ or its inverse in $\pi_1(H)$. Then there exists a complete set of cutting disks $\{D_A, D_B\}$ of $H$ such that $|\partial D_A \cap \Gamma| = |\partial D_B \cap \Gamma| = 2$. It follows readily that there exists a once-punctured torus $F$, together with a pair of nonparallel connections $\delta_A$, $\delta_B$ in $F$, and a homeomorphism $h: H \to F \times I$ such that $D_A = h^{-1}(\delta_A \times I)$, $D_B = h^{-1}(\delta_B \times I)$, and $\Gamma = h^{-1}(\partial F \times \frac{1}{2})$. So, in particular, either $h(\alpha) \subset F \times 1$ and $h(\beta) \subset F \times 0$, or $h(\beta) \subset F \times 1$ and $h(\alpha) \subset F \times 0$. □

Corollary 3.7. If $H$ is a genus two handlebody and $(\alpha, \beta)$ is a pair of disjoint primitives in $\partial H$ with an R-R diagram of the form of Figure 4, then $\alpha$ and $\beta$ lie on disjoint ends of a product $F \times I$, where $F$ is a once-punctured torus.

Proof. Figure 7 shows that, if $\alpha$ and $\beta$ have an R-R diagram with the form of Figure 4 then there exists a simple closed $\Gamma$ in $\partial H$ separating $\alpha$ and $\beta$ such that $\Gamma$ represents $A^\epsilon B^{-1}A^{-\epsilon}B$ in $\pi_1(H)$ with $\epsilon = \pm 1$. Then Lemma 3.6 implies the claim. □

Lemma 3.8. Suppose $H$ is a genus two handlebody and $(\alpha, \beta)$ is a pair of disjoint primitives in $\partial H$ with an R-R diagram of the form of Figure 3.

(1) If $r = 0$, $\alpha$ and $\beta$ are separated and also have an R-R diagram with the form of Figure 2.

(2) If $|r| = 1$, then $\alpha$ and $\beta$ lie on disjoint ends of a product $F \times I$, where $F$ is a once-punctured torus.
(3) If $|r| > 1$, then $\alpha$ and $\beta$ lie on disjoint ends of a twisted product $F \times I$, where $F$ is a once-punctured torus.

Proof. Figure 8 shows that, if $\alpha$ and $\beta$ have an R-R diagram with the form of Figure 4, then there exists a simple closed $\Gamma$ in $\partial H$ separating $\alpha$ and $\beta$ such that $\Gamma$ represents $A^r B^{-1} A^{-r} B$ in $\pi_1(H)$. If $r = 0$, $\Gamma$ bounds a separating disk in $H$. So $\alpha$ and $\beta$ are separated. Otherwise, if $|r| = 1$, Lemma 3.6 shows $\alpha$ and $\beta$ lie on disjoint ends of a product $F \times I$.

This leaves the case $|r| > 1$. In this case, there is a nonseparating simple closed curve $\lambda$ in $F_A$ such that $|\Gamma \cap \lambda| = 2$. Then there is a Dehn surgery on a core curve of the $A$-handle of $H$ which turns $H$ into another genus two handlebody $H'$ in which $\lambda$ bounds a cutting disk and, by Lemma 3.6, $\alpha$ and $\beta$ lie on disjoint ends of a product $F \times I$. From this it is easy to see that $\alpha$ and $\beta$ lie on disjoint ends of a twisted product $F \times I$. $\square$

This next result characterizes the cyclic words in $\pi_1(H)$ which are represented by the separating simple closed curves $\Gamma$ in Figures 7 and 8.

**Proposition 3.9.** Suppose $H$ is a genus two handlebody with a pair of disjoint nonparallel simple closed curves $\alpha$ and $\beta$ in $\partial H$ such that both $\alpha$ and $\beta$ are primitive in $H$. Then there is a simple closed curve $\Gamma$ in $\partial H$ separating $\alpha$ and $\beta$, and a complete set of cutting disks $\{D_A, D_B\}$ of $H$, such that either $|\Gamma \cap \partial D_A| = 2$, or $|\Gamma \cap \partial D_B| = 2$. In particular, up to replacing $A$ with $A^{-1}$, $B$ with $B^{-1}$, or perhaps exchanging $A$ and $B$, there is an integer $n$ such that $\Gamma$ represents the cyclic word $A^n B A^{-n} B^{-1}$ in $\pi_1(H)$.

Proof. Examination of the curve $\Gamma$ in Figures 6, 7, and 8 shows that, in each case, $\Gamma$ separates $\alpha$ and $\beta$ and $|\Gamma \cap \partial D_B| = 2$. It follows that $\Gamma$ represents a cyclic word in $\pi_1(H)$ of the claimed form. $\square$

Finally, the following theorem restates the classification in terms of Type I and Type II pairs.

**Theorem 3.10.** If $(\alpha, \beta)$ is a pair of disjoint nonparallel primitive simple closed curves in the boundary of a genus two handlebody $H$, then $(\alpha, \beta)$ is either a Type I or Type II pair. In particular:

1. $(\alpha, \beta)$ is a Type I pair if $\alpha$ and $\beta$ have an R-R diagram with the form of Figure 4.

2. $(\alpha, \beta)$ is a Type II pair if $\alpha$ and $\beta$ have an R-R diagram with the form of Figure 4.

3. $(\alpha, \beta)$ is both a Type I pair and a Type II pair if and only if $\alpha$ and $\beta$ are separated in $H$, or $\alpha$ and $\beta$ have an R-R diagram with the form of Figure 3 in which $\alpha$ wraps around the $A$-handle of the R-R diagram $p$ times longitudinally and $q$ times meridionally with $q = ps \pm 1$ for some integer $s$. (See Figure 8.)

Proof. This follows from the other results of this section. Details omitted. $\square$

4. **Pairs in which $\beta$ is a proper power and $\alpha$ is a primitive or proper power**

A nonseparating simple closed curve $\beta$ in the boundary of a genus two handlebody $H$ is a proper power if $\beta$ is disjoint from an essential separating disk in $H$, $\beta$
does not bound a disk in $H$, and $\beta$ is not primitive in $H$. With the classification of $\alpha, \beta$ pairs in which both $\alpha$ and $\beta$ are primitives finished, it seems natural to generalize slightly to the situation in which one or both of $\alpha, \beta$ are proper powers of primitives.

Here, as promised in the abstract, the situation is simpler and completely described by the following theorem.

**Theorem 4.1.** Suppose $H$ is a genus two handlebody, and $\alpha$ and $\beta$ are two disjoint nonparallel simple closed curves in $\partial H$ such that $\beta$ is a proper power in $H$ and $\alpha$ is primitive or a proper power in $H$. Then either $\alpha$ and $\beta$ are separated in $H$, or $\alpha$ and $\beta$ bound a nonseparating annulus in $H$.

**Proof.** Suppose $\{D_A, D_B\}$ is a complete set of cutting disks of $H$ with the property that $|\beta \cap (\delta D_A \cup \delta D_B)|$ is minimal, and also $|\alpha \cap (\delta D_A \cup \delta D_B)|$ is as small as possible among the complete sets of cutting disks of $H$ minimizing $|\beta \cap (\delta D_A \cup \delta D_B)|$.

Then, since $\beta$ is a proper power in $H$, one of $D_A, D_B$, say $D_A$, is disjoint from $\beta$. And then $|\beta \cap \delta D_B| = s$ with $s > 1$.

**Claim 1.** $\alpha$ intersects only one of $\partial D_A$, $\partial D_B$.

**Proof of Claim 1.** We use the notation of Subsection 2.1. Then, since $\alpha$ is disjoint from a disk in $H$, $G_\alpha$ is either not connected, or it has a cut vertex. And, of course, the graph $G_\beta$ is not connected, since all $s$ of its edges connect $D_A^+$ to $D_B^-$.

Now the proof of Claim 1 breaks into two cases depending upon whether there are nonparallel edges in $G_\beta$.

**Case 1:** There are edges of $G_\beta$ which are not parallel.

In this case, the vertices $D_A^+$ and $D_A^-$ of $G_\beta$ lie in different faces of $G_\beta$. Since $\alpha$ and $\beta$ are disjoint, this implies there are no edges of $G_\alpha$ connecting $D_A^+$ to $D_A^-$ in $G_\alpha$. It follows that, up to exchanging $D_A^+$ and $D_A^-$ or $D_B^+$ and $D_B^-$, $G_\alpha$ has the form of Figure 6.

If $a = 0$ in Figure 6 the claim holds. So suppose $a > 0$ in Figure 6. Then the band sum of $D_A$ and $D_B$, along a subarc of $a$ representing one of the $a$ edges of $G_\alpha$ connecting $D_A^+$ to $D_B^-$, is a disk $D'_B$ of $H$ such that $\{D_A, D'_B\}$ is a complete set of cutting disks of $H$, $|\beta \cap \delta D'_B| = s$, and $|\alpha \cap \delta D'_B| < |\alpha \cap \delta D_B|$, contrary to hypothesis. It follows that $a = 0$ in Figure 6 and the claim holds in this case.

**Case 2:** Any two edges of $G_\beta$ are parallel.

In this case, since any two edges of $G_\beta$ are parallel, there is a once-punctured torus $F$ in $\partial H$ which contains $\partial D_B$ and $\beta$. If $\alpha$ and $\partial F$ are disjoint, then $\alpha$ and $\partial D_B$ are disjoint and the claim holds. So suppose the set of connections $F \cap \alpha$ is nonempty, and $\delta$ is a connection in $F \cap \alpha$. Then, since $\alpha$ and $\beta$ are disjoint, $|\delta \cap \partial D_B| = s > 1$. This implies the graph $G_\alpha$ also has the form of Figure 6 in this case. Then the argument of Case 1 shows the claim also holds here. This finishes the proof of Claim 1.

To finish the proof of Theorem 4.1 observe that if $\alpha$ only intersects $\partial D_A$, then clearly $\alpha$ and $\beta$ are separated in $H$. On the other hand, if $\alpha$ only intersects $\partial D_B$, then both $\alpha$ and $\beta$ are disjoint from $D_A$, and so $\alpha$ and $\beta$ bound an annulus $A$ in the solid torus obtained by cutting $H$ open along $D_A$. Finally, $A$ must be nonseparating in $H$ since $\alpha$ and $\beta$ are not parallel in $\partial H$. \(\Box\)
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Figure 9. The graph $G_\alpha$ of the proof of Theorem 4.1.

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