Partial Balayage and a Generalization of the Divisible Sandpile Model

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Abstract

In recent work by L. Levine and Y. Peres, it was observed that three models for particle aggregation on the lattice—the divisible sandpile, rotor-router aggregation, and internal diffusion limited aggregation—share a common scaling limit as the lattice spacing tends to zero, if they are started with the same initial mass configuration. It is straightforward to observe that this scaling limit is precisely the same as the potential-theoretic operation of taking the partial balayage of this initial mass configuration to the Lebesgue measure. However, from the theory of the partial balayage operation it is clear that one may take the partial balayage of a mass configuration to a more general measure than the Lebesgue measure, which one cannot do for the three aggregation models described by Levine and Peres. In this paper we therefore generalize one of these models, the divisible sandpile model, in mainly a bounded setting, and show that a natural scaling limit of this generalization is given by a general partial balayage operation.

1 Introduction

In this section we review the results from L. Levine and Y. Peres [8, 9] regarding the divisible sandpile model (DS) and how its scaling limit is related to so-called partial balayage to unit density, Bal(\cdot, 1). Throughout, the dimension $d$ will be assumed to satisfy $d \geq 2$.

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1.1 Preliminaries and main result

Let $\mu : \mathbb{R}^d \to \mathbb{R}_+$ be a bounded and almost everywhere continuous function, with the property that $\{ x \in \mathbb{R}^d : \mu(x) \geq 1 \}$ is the closure of some open bounded set $\Omega$. Given a decreasing sequence $\{\xi_n\}_{n=1}^{\infty}$ of positive real numbers with limit zero as $n \to \infty$ we define the discretized mass configuration $\mu_n : \xi_n\mathbb{Z}^d \to \mathbb{R}_+$ on the scaled lattice $\xi_n\mathbb{Z}^d$ by

$$\mu_n(x) := \frac{1}{\xi_n^d} \int_{x^{\square}} \mu(y) dy,$$

where the symbol $x^{\square}$ denotes the closed cube in $\mathbb{R}^d$ of side length $\xi_n$ and midpoint $x$, i.e. the set

$$x^{\square} := x + \left[ -\frac{\xi_n}{2}, \frac{\xi_n}{2} \right]^d.$$

Since the volume of any such cube is $\xi_n^d$ we see from the above that the value of a discretization $\mu_n$ of $\mu$ at a point $x \in \xi_n$ is nothing but the mean value of $\mu$ in the set $x^{\square}$. We will also employ the notation that $x^{\varepsilon}$ is the closest lattice point to $x \in \mathbb{R}^d$ (i.e. if the lattice in question is $\xi\mathbb{Z}^d$, then $x^{\varepsilon} = (x + (\xi/2, \xi/2)^d) \cap (\xi\mathbb{Z}^d)$). Moreover, if $f$ is a function on $\mathbb{R}^d$ then $f^{\varepsilon}$ is defined as the restriction of $f$ to the underlying lattice (determined by the context), and, similarly, if $g$ is a lattice function on some lattice $\xi\mathbb{Z}^d$, then $g^{\square}$ is the extension of $g$ as a step function to $\mathbb{R}^d$ defined by $g^{\square}(x) := g(x^{\varepsilon})$.

We need to say a few words about convergence of sequences of sets relative to our sequence $\{\xi_n\}_{n=1}^{\infty}$ of decreasing lattice constants: a sequence of sets $\{A_n\}_{n=1}^{\infty}$, where $A_n \subset \xi_n\mathbb{Z}^d$, is said to converge to a set $D \subset \mathbb{R}^d$ if there for any given $\varepsilon > 0$ exists some integer $N$ such that we for all $n > N$ have

$$D_\varepsilon \cap \xi_n\mathbb{Z}^d \subset A_n \subset D^\varepsilon,$$

where $D_\varepsilon$ and $D^\varepsilon$ are subsets of $\mathbb{R}^d$, the inner and outer $\varepsilon$-neighbourhoods of $D$, respectively, defined by

$$D_\varepsilon := \{ x \in D : B(x, \varepsilon) \subset D \}$$

and

$$D^\varepsilon := \{ x \in \mathbb{R}^d : B(x, \varepsilon) \cap D \neq \emptyset \},$$

so that $D_\varepsilon \subset D \subset D^\varepsilon$; here $B(a, \rho)$ is the open ball in $\mathbb{R}^d$ of radius $\rho > 0$ centred at $a \in \mathbb{R}^d$. 

2
Having treated the necessary technicalities the divisible sandpile model on \( \xi \mathbb{Z}^d \) for some lattice constant \( \xi > 0 \) is now defined as follows: given a function \( \mu : \xi \mathbb{Z}^d \rightarrow \mathbb{R}_+ \), to be interpreted as our initial mass configuration, we pick to begin with any site \( x \in \xi \mathbb{Z}^d \) for which \( M := \mu(x) > 1 \)—we can think of \( \mu(x) \) to be the mass or number of (sand-)particles at \( x \) (ignoring the fact that we very much allow for non-integral number of particles), and hence that the site \( x \) is chosen in such a way that it has more than one particle. We now topple the site \( x \), by which we mean that we leave a unit mass at \( x \), and distribute the remaining mass of \( M - 1 \) uniformly amongst the \( 2^d \) neighbours \( y \) of \( x \); for sake of simplicity we will write \( y \sim x \) if \( y \) is a neighbour to \( x \). In essence, we alter the mass configuration \( \mu \) by replacing \( \mu(x) \) with \( 1 \), and \( \mu(y) \) with \( \mu(y) + (M - 1)/2^d \) for each \( y \sim x \) to obtain a new mass configuration \( \mu' : \xi \mathbb{Z}^d \rightarrow \mathbb{R}_+ \). We now do the previous steps again starting from \( \mu' \) instead of \( \mu \), and continue repeating this process over and over again until we reach (in the limit) a final mass configuration \( \nu : \xi \mathbb{Z}^d \rightarrow \mathbb{R}_+ \) which satisfies \( 0 \leq \nu \leq 1 \) everywhere. (That there even exists such a final mass configuration \( \nu \), not to mention the fact that this configuration actually also is independent of the particular choice of toppling sequence used, is highly non-trivial, but true under our assumptions on \( \mu \).) This process is what we call the (standard) divisible sandpile, and we call the final mass configuration \( \nu \) the (standard) divisible sandpile configuration of \( \mu \) (on \( \xi \mathbb{Z}^d \)).

A highly important function \( u \) called the odometer function can be defined for the divisible sandpile model: if \( \xi \mathbb{Z}^d \) is the lattice in question then \( u \) is the function defined by letting \( u(x) \) be \( \xi^2 \) times the total mass emitted from a lattice point \( x \in \xi \mathbb{Z}^d \) during the entire divisible sandpile process. Here the factor \( \xi^2 \) is to ensure the proper limiting behaviour when we later let \( \xi \to 0 \). If we study the algorithm for the divisible sandpile model in detail it becomes clear that any site \( x \in \xi \mathbb{Z}^d \) will, in the end, have emitted a total mass of \( u(x)/2d\xi^2 \) to each of its \( 2^d \) neighbours. But this reasoning also applies to the neighbouring sites of \( x \), hence each neighbour \( y \) will in total have sent mass of size \( u(y)/2d\xi^2 \) to \( x \). It follows that the net increase in mass at the site \( x \) will be the difference between the total mass received and the total mass emitted, i.e. precisely

\[
\sum_{y \sim x} \frac{u(y)}{2d\xi^2} - \sum_{y \sim x} \frac{u(x)}{2d\xi^2} = \sum_{y \sim x} \frac{u(y) - u(x)}{2d\xi^2} = \Delta u(x),
\]

where \( \Delta \) is the (for our purposes suitably renormalized) discrete Laplace operator. But this is only one way of calculating the net increase of mass at \( x \): with \( \mu \) the initial mass configuration and \( \nu \) the mass configuration we end
up with after the aggregation is completed as above, we evidently have

\[ \Delta u(x) = \nu(x) - \mu(x). \]  

The main goal of our study is to calculate the resulting set of fully occupied sites for the resulting divisible sandpile configuration, and for this we observe that the odometer function \( u \) can in fact be used to determine this set completely. The set of such fully occupied sites is of course the set \[ D := \{ x \in \xi \mathbb{Z}^d : \nu(x) = 1 \}. \]

If we consider any such \( x \in D \) we must either have that no toppling occurred at \( x \) at any stage during the course of the divisible sandpile algorithm, or that the site \( x \) did topple at least once. If \( x \) did not topple, then no mass has left \( x \), so \( x \) must either have had mass one during the entire course of the sandpile algorithm—if so then \( x \) must belong to the set \( \{ \mu \geq 1 \} \)—or must have received mass from one from its neighbouring points, i.e. must have a neighbour that did topple. On the other hand, if it in fact did perform a toppling at some stage during the course of the algorithm, then \( u(x) > 0 \). From these considerations we can conclude that, up to possibly a (in some sense negligible) set of boundary points, the set \( D \) of fully occupied sites is essentially

\[ \{ \mu \geq 1 \} \cup \{ u > 0 \}. \]

With this in mind, it is clear that we gain much information about the set \( D \) by finding the odometer function \( u \), and the approach we will take is to find \( u \) as the solution to the equation (1). Since \( \nu \) will, by construction, always satisfy \( \nu \leq 1 \), it is suitable to find a function \( \gamma \) that satisfies \( \Delta \gamma(x) = 1 - \mu(x) \), since if we then study the function \( s' := \gamma - u \) we see that

\[ \Delta s'(x) = \Delta(\gamma - u)(x) = \Delta \gamma(x) - \Delta u(x) \]
\[ = 1 - \mu(x) - \nu(x) + \mu(x) = 1 - \nu(x) \geq 0 \]

holds everywhere, i.e. \( s' \) is a subharmonic function on \( \xi \mathbb{Z}^d \). We note that \( s' \) satisfies \( s' \leq \gamma \), since \( u \geq 0 \) by definition. Moreover, if \( f \) is any other subharmonic function on \( \xi \mathbb{Z}^d \) satisfying \( f \leq \gamma \), then

\[ \Delta(f - \gamma + u)(x) = \Delta f(x) - 1 + \mu(x) + \nu(x) - \mu(x) \]
\[ = \Delta f(x) - 1 + \nu(x) = \Delta f(x) \geq 0 \]

if \( x \in D = \{ \nu = 1 \} \), and for \( x \) outside \( D \) we have \( u(x) = 0 \), hence

\[ f(x) - \gamma(x) + u(x) = f(x) - \gamma(x) \leq 0 \]
there. It follows that \( f - \gamma + u \) is a nonpositive function everywhere, i.e. that \( f \leq \gamma - u = s' \) on the whole of \( \xi \mathbb{Z}^d \). Thus, if we let \( s \) be the subharmonic function defined by

\[
s(x) := \sup\{ f(x) : f \text{ is subharmonic in } \xi \mathbb{Z}^d \text{ and } f \leq \gamma \}
\]  

(2)

it follows both that \( s \leq \gamma - u = s' \), but also \( s \geq \gamma - u \), since \( s' = \gamma - u \) is a competing function in the set defining \( s \) in (2). We can conclude that we in fact have

\[ u = \gamma - s \]

where \( s \) is given by (2).

We have converted the problem of finding the odometer function \( u \), in particular finding the set \( \{ u > 0 \} = \{ \gamma > s \} \), into solving the obstacle problem (2), a problem that has a natural generalization to the continuous setting. Therefore, given some initial mass configuration \( \mu \) on \( \mathbb{R}^d \) (with appropriate assumptions on \( \mu \) to ensure existence, and so on) we define the obstacle \( \gamma_c : \mathbb{R}^d \to \mathbb{R} \) by

\[
\gamma_c(x) := -|x|^2 - N * \mu(x)
\]

where \( N(x) \) is the Newton kernel on \( \mathbb{R}^d \), proportional to \( \log |x|^{-1} \) in two dimensions and to \( |x|^{2-d} \) for \( d \geq 3 \), such that \( \Delta \gamma_c = \mu - 1 \). As in (2), we then define

\[
s_c(x) := \sup\{ f(x) : f \in \mathcal{CS}(\mathbb{R}^d) \text{ and } f \leq \gamma_c \},
\]

(3)

where \( \mathcal{CS}(S) \) denotes the set of functions continuous and subharmonic on some open set \( S \). Assuming we can find a solution \( s_c \) to (3), it can be seen that the set

\[
D := \{ x \in \mathbb{R}^d : \gamma(x) > s_c(x) \}
\]

(4)

will be the natural limit set, in the sense discussed above, to the sequence of sets \( \{ u_n > 0 \} \) where \( u_n \) is the \( n \)th odometer function for the divisible sandpile model for a sequence of decreasing positive lattice constants \( \{ \xi_n \}_{n=1}^\infty \) converging to zero.

We are now ready to state one of the main results from Levine’s thesis [8]:

**Theorem 1.1.** Let \( \{ \xi_n \}_{n=1}^\infty \) be a decreasing sequence of positive real numbers converging to zero, and let \( \{ \mu_n \}_{n=1}^\infty \) be a discretized mass configuration based on the sequence \( \{ \xi_n \}_{n=1}^\infty \) for some mass configuration \( \mu : \mathbb{R}^d \to \mathbb{R}_+ \) as above,
with $\Omega$ the open bounded set satisfying $\overline{\Omega} = \{\mu \geq 1\}$. Let $D_n$ be the domain of occupied sites from the standard divisible sandpile in the lattice $\xi_n \mathbb{Z}^d$ started from source density $\mu_n$. Then, as $n \to \infty$,

$$D_n \to D \cup \Omega,$$

where $D$ is the set given by (4).

We will later on observe that the obstacle problem in (3) is essentially the same obstacle problem as that occurring in the definition of the partial balayage operation $\text{Bal}(\cdot, m)$ of a mass configuration to the Lebesgue measure $m$—i.e. to density one, if we think of $m$ as a distribution—and so the limiting set in the above theorem is precisely

$$D \cup \Omega = \text{supp Bal}(\mu, m).$$

One consequence of the above is that if we let $\nu_n$ be the result of the standard divisible sandpile on $\xi_n \mathbb{Z}^d$ started from density $\mu_n$, but choose to interpret this resulting mass configuration as a measure on $\mathbb{R}^d$, i.e. with some abuse of notation we let

$$\nu_n = \xi_n^d \sum_{x \in \xi_n \mathbb{Z}^d} \nu_n(x) \cdot \delta_x,$$

where $\delta_x$ is the Dirac point mass measure at $x$, then $\nu_n \to \text{Bal}(\mu, m)$ in the sense of distributions as $n \to \infty$. This weak form of convergence is the approach we will take in the remainder of the paper.

## 2 Partial balayage

In this paper we are going to refer to two different variants of partial balayage: first a bounded version with Dirichlet boundary conditions, which we are going to relate to a bounded version of the generalized divisible sandpile algorithm, and also an unrestricted version when the dimension $d = 2$, which we in turn relate to a the possible limit of the generalized divisible sandpile in the setting where the confining radius grows infinitely large.

### 2.1 Bounded partial balayage

The bounded version of partial balayage was developed by B. Gustafsson and M. Sakai in [4], which we include here mainly for sake of completeness and for an overview of the minor adjustments to the notation we use in this
paper. For proofs we refer to [4], and for a good survey of partial balayage in general, see for instance [2].

Before we continue, we need to say a few words about our notation. If \( \mu \) is a signed Radon measure on \( \mathbb{R}^d \) with compact support, then we denote by \( U^\mu \) the Newtonian potential of \( \mu \). For greater compatibility with the analogous theory in the discrete setting, we use the (somewhat non-standard) normalization of the potential such that

\[
-\Delta U^\mu(x) = 2d \cdot \mu(x),
\]

which always holds in the sense of distributions (and pointwise wherever \( U^\mu \) is \( C^2 \)). Here \( \Delta \) is the usual Laplace operator

\[
\Delta = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2},
\]

with the natural generalization in terms of distributions.

**Definition 2.1.** Let \( \sigma = \sigma_+ - \sigma_- \) be a signed Radon measure on \( \mathbb{R}^d \) with compact support, and let \( R > 0 \). Define the set

\[
\mathcal{F}^{\sigma,R} := \{ V \in \mathcal{D}'(\mathbb{R}^d) : V \leq U^\sigma \text{ in } \mathbb{R}^d, \Delta V \geq 0 \text{ in } B(0,R) \},
\]

where \( \mathcal{D}'(\mathbb{R}^d) \) is the set of distributions in \( \mathbb{R}^d \).

**Theorem 2.2.** The set \( \mathcal{F}^{\sigma,R} \) in Definition 2.1 contains a largest element, \( V^\sigma \equiv V^{\sigma,R} := \sup \mathcal{F}^{\sigma,R} \). This \( V^\sigma \) satisfies the complementarity system

\[
\begin{align*}
V^\sigma &\leq U^\sigma \text{ in } \mathbb{R}^d, \\
\Delta V^\sigma &\geq 0 \text{ in } B(0,R), \\
V^\sigma &\equiv U^\sigma \text{ on } \mathbb{R}^d \setminus B(0,R), \\
-\Delta V^\sigma &= 0 \text{ in } \omega(\sigma) := \{ x \in B(0,R) : V^\sigma(x) < U^\sigma(x) \}.
\end{align*}
\]

It follows from the above that \( -\Delta V^\sigma \) is a signed Radon measure.

**Definition 2.3.** The partial balayage relative to the ball \( B(0,R) \), where \( R > 0 \), of a signed Radon measure \( \sigma = \sigma_+ - \sigma_- \) with compact support is defined to be the signed Radon measure

\[
\text{Bal}_R(\sigma,0) := -\frac{1}{2d} \Delta V^{\sigma,R},
\]

where \( V^{\sigma,R} \) is as in Theorem 2.2.
Remark 2.4. The modified Schwarz potential of the above problem is the function \( u \equiv u^{\sigma,R} := U^{\sigma} - V^{\sigma,R} \). In terms of the complementarity system in Theorem 2.2, \( u \) satisfies

\[
\begin{align*}
  u &\geq 0 \text{ in } \mathbb{R}^d, \\
  \Delta u &\geq 2d \cdot \sigma \text{ in } B(0,R), \\
  u &= 0 \text{ on } \mathbb{R}^d \setminus B(0,R), \\
  u &= 2d \cdot \sigma \text{ in } \omega(\sigma) := \{ x \in B(0,R) : u(x) > 0 \}.
\end{align*}
\]

For the partial balayage measure in terms of \( u \), we see from Definition 2.3 that \( \text{Bal}_{R}(\sigma,0) \) is given by

\[
\text{Bal}_{R}(\sigma,0) = -\frac{1}{2d} \Delta(U^{\sigma} - u) = \sigma + \frac{1}{2d} \Delta u.
\]

Remark 2.5. In [4], and several other articles, the partial balayage operation is often discussed in terms of \( \nu := \text{Bal}_{R}(\mu,\lambda) \), where \( \mu \) and \( \lambda \) are suitable (positive) measures. This resulting (also positive) measure \( \nu \) then satisfies \( \nu \leq \lambda \) in \( B(0,R) \), and \( U^{\nu} \) being equal to \( U^{\mu} \) (i.e. \( \nu \) and \( \mu \) are “graviequivalent”) outside of some \( a \text{ priori} \) unknown set \( \Omega \). At least in the finite energy setting, this \( \nu \) is the unique minimizer of the energy norm difference

\[
I[\mu - \nu] = \int U^{\mu} - d(\mu - \nu)
\]

over all \( \nu \) satisfying \( \nu \leq \lambda \) in \( B(0,R) \) (and, at least in two dimensions, with the extra condition that \( \nu \) has the same total mass as \( \mu \)).

In this paper, we will mostly focus on partial balayage measures of the form \( \text{Bal}_{R}(\cdot,0) \), as defined in Definition 2.3. At times when we need to refer to partial balayage measures of the form \( \text{Bal}_{R}(\cdot,\lambda) \) instead, we utilize a well-known translational invariance property of partial balayage (see [4]), in that, for suitable measures \( \mu, \lambda \) and \( \eta \) to ensure existence,

\[
\text{Bal}_{R}(\mu + \eta, \lambda + \eta) = \text{Bal}_{R}(\mu, \lambda) + \eta.
\]

In other words, when appropriate we simply think of \( \text{Bal}_{R}(\mu, \lambda) \) as the measure defined by

\[
\text{Bal}_{R}(\mu, \lambda) = \text{Bal}_{R}(\mu - \lambda, 0) + \lambda.
\]

2.2 Unrestricted partial balayage in the plane

In the plane it is known that Definition 2.3, under suitable assumptions on the signed measure \( \sigma \), can be generalized to allow for an infinite confining
radius. See [12] for details, and for recently developed connections between partial balayage measures and equilibrium measures in weighted potential theory [1,13].

**Definition 2.6.** Let $\sigma = \sigma_+ - \sigma_-$ be a signed Radon measure on $\mathbb{R}^2$ with compact support. Define the set

$$F^\sigma := \{ V \in D'(\mathbb{R}^d) : V \leq U^\sigma \text{ in } \mathbb{R}^2, \Delta V \geq 0 \text{ in } \mathbb{R}^2 \}.$$

**Theorem 2.7.** If $\sigma = \sigma_+ - \sigma_-$ is a signed Radon measure on $\mathbb{R}^2$ with compact support and negative total mass, with the property that $U^\sigma_-$ is a continuous function on $\mathbb{R}^2$, then $F^\sigma$ is non-empty and contains its largest element, $V^\sigma := \sup F^\sigma$. This $V^\sigma$ satisfies the complementarity system

$$V^\sigma \leq U^\sigma \text{ in } \mathbb{R}^2,$$
$$\Delta V^\sigma \geq 0 \text{ in } \mathbb{R}^2,$$
$$V^\sigma = U^\sigma \text{ in } \text{supp} \Delta V^\sigma \subset \text{supp} \sigma_-,$$
$$-\Delta V^\sigma = 0 \text{ in } \omega(\sigma) := \{x \in \mathbb{R}^2 : V^\sigma(x) < U^\sigma(x) \}.$$

**Definition 2.8.** The *(unrestricted) partial balayage* of a signed Radon measure $\sigma = \sigma_+ - \sigma_-$ with compact support, assumed to satisfy $U^\sigma_-$ continuous everywhere on $\mathbb{R}^2$, is defined to be the signed Radon measure

$$\text{Bal}(\sigma, 0) := -\frac{1}{2d} \Delta V^\sigma,$$

where $V^\sigma$ is as in Theorem 2.7.

### 3 Generalizing the divisible sandpile

As mentioned earlier, the scaling limit of the standard divisible sandpile obtained in L. Levine’s thesis [8] is related to taking partial balayage of a mass configuration to the Lebesgue measure $m$, i.e. $\text{Bal}_R(\mu, 1) \equiv \text{Bal}_R(\mu, m)$, where $R > 0$ is a large enough bounding radius.

However, as we saw in Section 2, there is mathematically no problem in calculating the partial balayage of a mass configuration relative to a more general measure than the Lebesgue measure, i.e. instead calculating $\text{Bal}_R(\mu, \lambda)$, where $\lambda$ is a measure that, in a sense, describes the maximal density that will be allowed for the final mass configuration. It is therefore a natural question to ask if the standard divisible sandpile model in [8] can be generalized to incorporate this measure $\lambda$, in such a way that the corresponding scaling limit of this modified particle aggregation model coincides with $\text{Bal}_R(\mu, \lambda)$.
In this section we shall see that this is, indeed, possible. With the translational invariance (5) in mind, we will, mainly for sake of simplicity in the formulation, actually develop a generalized sandpile model that converges to measures of the form $\text{Bal}_R(\cdot, 0)$ in the appropriate scaling limit. If desired, this can then readily be reformulated into a corresponding result in terms of $\text{Bal}_R(\mu, \lambda)$.

3.1 Bounded divisible sandpile for signed mass configurations on a fixed lattice

Let $\sigma : \xi \mathbb{Z}^d \rightarrow \mathbb{R}$ be a bounded function on the lattice $\xi \mathbb{Z}^d$ for some lattice constant $\xi > 0$; this function will be our generalization of the initial mass configuration. We shall always assume that $\sigma$ has compact support

$$\text{supp } \sigma := (\text{supp } \sigma_+) \cup (\text{supp } \sigma_-),$$

where $\sigma_+ = \max(\sigma, 0)$ and $\sigma_- = -\min(\sigma, 0)$; a bounded lattice function of compact support will for sake of brevity be called a generalized mass configuration. We are only going to be interested in admissible generalized mass configurations, by which we mean

$$\sum_{x \in \xi \mathbb{Z}^d} \sigma_-(x) \geq \sum_{x \in \xi \mathbb{Z}^d} \sigma_+(x). \quad (6)$$

Much like we in the standard divisible sandpile model ended up with a mass configuration satisfying $\nu \leq 1$ everywhere, we will, in our generalized divisible sandpile, in the end obtain a generalized mass configuration $\nu$ satisfying $\nu \leq 0$ everywhere. Since we want the total mass of our mass configuration to remain the same throughout this process, so that $\sum_{x \in \xi \mathbb{Z}^d} \nu(x) = \sum_{x \in \xi \mathbb{Z}^d} \sigma(x)$, this explains requirement (6), as we then have

$$\sum_{x \in \xi \mathbb{Z}^d} \sigma(x) = \sum_{x \in \mathbb{Z}^d} \nu(x) \leq 0.$$

The main way we will generalize the divisible sandpile model is by generalizing the toppling step described in Section 1.1 for the standard divisible sandpile. In the standard model, at every site $x$ where the mass $M$ exceeds one, we redefine our mass configuration locally around $x$, leaving a mass of one at $x$ and spreading the remaining mass $M - 1$ equally amongst the nearest neighbours of $x$. We here essentially do more or less the same, with the difference that we instead look for sites where $\sigma$ is positive (i.e. violating the desired property of the mass configuration being $\leq 0$ everywhere). Thus,
for every site $x$ in our lattice where we have $\sigma_+(x) > 0$ we modify our mass configuration around $x$, leaving no mass at all at $x$ (so that $\nu \leq 0$ at least is satisfied at $x$ for our new mass configuration $\nu$), and relocate the remaining mass $\sigma_+(x)$ equally amongst the $2d$ neighbouring sites of $x$.

To formalize the above we do the following: consider $x \in \xi \mathbb{Z}^d$ arbitrary but for the moment fixed, let $\eta : \xi \mathbb{Z}^d \to \mathbb{R}$ be some generalized mass configuration and define toppling of $\eta$ at the site $x$ to be the mass configuration $T_x \eta$ defined by

$$T_x \eta(y) := \eta(y) + \eta_+(x) \xi^2 \cdot \Delta \delta_x(y),$$

where $\delta_x$ is the (discrete) delta function at $x$, and $\Delta$ is the (for our purposes suitably normalized) discrete Laplace operator defined by

$$\Delta f(y) = \frac{1}{2d\xi^2} \sum_{y' \sim y} (f(y') - f(y)),$$

where $y' \sim y$ means $y' \in \xi \mathbb{Z}^d$ is one of the $2d$ neighbouring points of distance $\xi$ from $y$ in $\xi \mathbb{Z}^d$. If $x$ happens to be a lattice point for which $\eta(x) \leq 0$ holds, then clearly $\eta_+(x) = 0$, hence $T_x \eta(y) = \eta(y)$ holds for every $y \in \xi \mathbb{Z}^d$, as desired. If we on the other hand happen to have $\eta(x) = \eta_+(x) > 0$ then we get a contribution from the second term in (7) and need to calculate $\Delta \delta_x(y)$ to determine what $T_x \eta(y)$ is. From (8) we obtain

$$\Delta \delta_x(y) = \frac{1}{2d\xi^2} \sum_{y' \sim y} (\delta_x(y') - \delta_x(y)),$$

and see that this function obtains different values depending on how close $y$ is to $x$. If $y = x$, then $\delta_x(y) = 1$ and $\delta_x(y') = 0$ for every $y' \sim y = x$, from which it follows that

$$\Delta \delta_x(x) = \frac{1}{2d\xi^2} \sum_{y' \sim y} (0 - 1) = -\frac{1}{2d\xi^2} \sum_{y' \sim y} 1 = -\frac{2d}{2d\xi^2} = -\frac{1}{\xi^2}.$$ 

If $y$ instead is a neighbouring point of $x$, then $x$ is a neighbouring point of $y$ (naturally), so $\delta_x(y')$ will be zero for every $y' \sim y$ except for when $y' = x$. Clearly we then also have $\delta_x(y) = 0$ as $y \neq x$, so we in this case instead obtain

$$\Delta \delta_x(y) = \frac{1}{2d\xi^2} \sum_{y' \sim y} (\delta_x(y') - 0) = \frac{1}{2d\xi^2}.$$
Finally, if $y$ is neither equal to $x$ nor a neighbouring point of $x$, then $\delta_x(y')$ is zero for every $y' \sim y$ and evidently also $\delta_x(y) = 0$, yielding $\Delta \delta_x(y) = 0$. We summarize these cases into

$$\Delta \delta_x(y) = \begin{cases} -\frac{1}{\xi^2} & \text{if } y = x, \\ \frac{1}{2d\xi^2} & \text{if } y \sim x, \\ 0 & \text{otherwise.} \end{cases}$$

This yields that we obtain

$$T_x \eta(y) = \begin{cases} \eta(x) + \eta_+(x)\xi^2 \cdot (-\frac{1}{\xi^2}) & \text{if } y = x, \\ \eta(y) + \eta_+(x)\xi^2 \cdot \frac{1}{2d\xi^2} & \text{if } y \sim x, \\ \eta(y) & \text{otherwise;} \end{cases}$$

$$= \begin{cases} -\eta_-(x) & \text{if } y = x, \\ \eta(y) + \frac{\eta_+(x)}{2d} & \text{if } y \sim x, \\ \eta(y) & \text{otherwise.} \end{cases}$$

We see that this way of defining the toppling agrees precisely with how we want to modify the mass configuration if $x$ is a site where the mass configuration has a violating positive mass.

Naturally, the site $x$ need not be the only site in $\xi \mathbb{Z}^d$ where the initial mass configuration possibly is in violation of the desired nonpositivity, and we also note that as we perform the above toppling at $x$ we could in fact turn some of the neighbouring points of $x$ into violating points if we add too much mass to these points during the toppling process. To ensure that we in the end obtain a mass configuration $\nu$ which satisfies $\nu \leq 0$ everywhere, and not only at specific points, we therefore need to do this toppling procedure over all violating points and repeat when necessary. To avoid problems with mass possibly escaping to infinity, we will in this section treat a bounded generalization of the divisible sandpile, i.e. fix some $R > 0$ and restrict our study for the moment to the set $\tilde{B}_R := B(0, R) \cap \xi \mathbb{Z}^d$, where $B(a, r) \subset \mathbb{R}^d$ is the open ball centred at $a \in \mathbb{R}^d$ of radius $r > 0$; we choose $R$ large enough so that $\tilde{B}_R$ contains the support of our initial mass configuration. Now fix a sequence $x_1, x_2, \ldots$ of points of $\tilde{B}_R$ with the property that if $x \in \tilde{B}_R$ is arbitrary, then there are infinitely many points in the sequence $x_1, x_2, \ldots$
for which $x_k = x$; we call such a sequence an \textit{infinitely covering sequence} (of $\hat{B}_R$). For $k \geq 1$ we define the mass configuration $\sigma_k^R \equiv \sigma_k$ to be the mass configuration obtained from $\sigma$ after successive toppling of the sites $x_1, x_2, \ldots, x_k$, i.e. we let

$$\sigma_k(y) := T_{x_k}T_{x_{k-1}} \ldots T_{x_2}T_{x_1}\sigma(y).$$

Also, for each $k \geq 1$ we define the $k$th odometer function $u_k : \xi \mathbb{Z}^d \rightarrow \mathbb{R}_+$ to be $\xi^2$ times the total mass emitted from the site $x$ after toppling the sites $x_1, x_2, \ldots, x_k$. These odometer functions are, as already seen in the introduction, highly useful when studying what happens to the mass configuration as $k$ tends to infinity.

For any subset $S \subset \xi \mathbb{Z}^d$ of the lattice, we will by $\partial S$ denote the outer boundary of $S$, defined by

$$\partial S := \{ y \in S : \text{there exists } y' \in S \text{ with } y \sim y' \};$$

note that we by definition always have $S \cap \partial S = \emptyset$. Our first main result is the following:

\textbf{Proposition 3.1.} Let $\sigma : \xi \mathbb{Z}^d \rightarrow \mathbb{R}$ be a generalized mass configuration, let $R > 0$ be such that $\text{supp } \sigma \subset \hat{B}_R := B(0, R) \cap \xi \mathbb{Z}^d$ and let $x_1, x_2, x_3, \ldots$ be an infinitely covering sequence of $\hat{B}_R$. For each $k \geq 1$ let $\sigma_k$ be the generalized mass configuration obtained from $\sigma$ after toppling the $k$ points $x_1, \ldots, x_k$, and let $u_k$ be the corresponding odometer function.

Then there exists a generalized mass configuration $\nu$ on $\xi \mathbb{Z}^d$ and a function $u : \xi \mathbb{Z}^d \rightarrow \mathbb{R}_+$ such that $\sigma_k(x) \rightarrow \nu(x)$ and $u_k(x) \not\rightarrow u(x)$ for every $x \in \xi \mathbb{Z}^d$ as $k \rightarrow \infty$. Moreover, $\nu = \nu_+ - \nu_-$ has the structure $\text{supp } \nu_+ \subset \partial \hat{B}_R$ and $\text{supp } \nu_- \subset \text{supp } \sigma_-$, so that $\nu \geq 0$ on $\partial \hat{B}_R$ and $\nu \leq 0$ on $\hat{B}_R$.

\textbf{Note:} The proof of the above proposition is essentially identical to the proof of the analogous statement for the standard divisible sandpile, as given in Lemma 3.1 in [9], with only minor adjustments for change in notation and the restriction that our infinitely covering sequence now is a subset of $\hat{B}_R$ instead of $\xi \mathbb{Z}^d$ as in [9]; we include it here for completeness.

\textbf{Proof.} It is evident from the definition of the toppling procedure that the $k$th mass configuration $\sigma_k$ can only be nonzero in $\hat{B}_R$ (the set covered by the sites at which we perform toppling) and possibly also on the boundary of $\hat{B}_R$, so for every $k$ we have $\sigma_k(x) = 0$ if $|x| \geq R + 2$. We define the $k$th quadratic weight $Q_k$ through

$$Q_k := \sum_{x \in \xi \mathbb{Z}^d} \sigma_k(x)|x|^2. \quad (9)$$
(Here $\sigma_0 = \sigma$.) On one hand, this immediately yields
\[
Q_k = \sum_{x \in \xi \mathbb{Z}^d} ((\sigma_k)_+(x) - (\sigma_k)_-(x))|x|^2 \leq \sum_{x \in \xi \mathbb{Z}^d} (\sigma_k)_+(x)|x|^2.
\tag{10}
\]

We now claim that for every $k \geq 1$ we have
\[
\sum_{x \in \xi \mathbb{Z}^d} (\sigma_k)_+(x) \leq \sum_{x \in \xi \mathbb{Z}^d} (\sigma_{k-1})_+(x),
\tag{11}
\]
which then, by iteration and the fact that $\sigma_k(x) = 0$ for all $|x| \geq R + 2$, leads to the inequality
\[
Q_k \leq (R + 2)^2 M_+ \quad \text{for all } k \geq 0,
\tag{12}
\]
where $M_+$ is the total mass of the non-negative part of the initial mass configuration $\sigma$:
\[
M_+ := \sum_{x \in \xi \mathbb{Z}^d} \sigma_+(x).
\]

To prove (11), we first observe that if $(\sigma_{k-1})_+(x_k) = 0$ there is nothing to prove, since $\sigma_k(x) = T_x \sigma_{k-1}(x) = \sigma_{k-1}(x)$ then holds for every $x$. For now we therefore assume $(\sigma_{k-1})_+(x_k) > 0$. This implies that $(\sigma_k)_+(x_k) = 0$, $(\sigma_k)_+(x) = (\sigma_{k-1})_+(x)$ for all $x \neq x_k$ with $x \not\sim x_k$, and for every $x \sim x_k$ the inequality
\[
(\sigma_k)_+(x) \leq (\sigma_{k-1})_+(x) + \frac{(\sigma_{k-1})_+(x_k)}{2d}
\tag{13}
\]
holds. The left hand side of (11) then becomes
\[
\sum_{x \in \xi \mathbb{Z}^d} (\sigma_k)_+(x) = \sum_{x \sim x_k} (\sigma_k)_+(x) + \sum_{x \neq x_k, x \not\sim x_k} (\sigma_k)_+(x)
\leq \sum_{x \sim x_k} (\sigma_{k-1})_+(x) + \frac{(\sigma_{k-1})_+(x_k)}{2d} + \sum_{x \neq x_k, x \not\sim x_k} (\sigma_{k-1})_+(x)
= (\sigma_{k-1})_+(x_k) + \sum_{x \neq x_k} (\sigma_{k-1})_+(x) = \sum_{x \in \xi \mathbb{Z}^d} (\sigma_{k-1})_+(x),
\tag{14}
\]
as desired. Since the total mass of $\sigma_k$ is equal to the total mass of $\sigma_{k-1}$ by construction, inequality (11) immediately implies that we also have
\[
\sum_{x \in \xi \mathbb{Z}^d} (\sigma_k)_-(x) \leq \sum_{x \in \xi \mathbb{Z}^d} (\sigma_{k-1})_-(x)
\tag{15}
\]
for each \( k \geq 1 \), which, in a similar manner, in turn implies the lower bound

\[
Q_k \geq -(R + 2)^2 M_\text{--} \text{ for all } k \geq 0,
\]

where \( M_\text{--} \) is the total mass of the non-positive part of the initial mass configuration \( \sigma \):

\[
M_\text{--} := \sum_{x \in \xi \mathbb{Z}^d} \sigma_-(x).
\]

We have thus established the following bounds on \( Q_k \) for each \( k \geq 0 \):

\[
-(R + 2)^2 M_\text{--} \leq Q_k \leq (R + 2)^2 M_+.
\]

(16)

From (9) it follows for \( k \geq 1 \) that

\[
Q_k - Q_{k-1} = \sum_{x \in \xi \mathbb{Z}^d} (\sigma_k(x) - \sigma_{k-1}(x)) |x|^2,
\]

and from the definition of \( \sigma_k \) as \( \sigma_k = T_{x_k} \sigma_{k-1} \) one obtains slightly different but related results depending on if the \( x \in \xi \mathbb{Z}^d \) is equal to the toppling point \( x_k \), is merely adjacent to \( x_k \), or neither of these: for \( x = x_k \) a trivial calculation shows that \( \sigma_k(x_k) - \sigma_{k-1}(x_k) = -(\sigma_{k-1})_+(x_k) \), if instead \( x \sim x_k \) then \( \sigma_k(x) - \sigma_{k-1}(x) = \frac{1}{2d} (\sigma_{k-1})_+(x_k) \) holds, and if \( x \) is neither equal to nor adjacent to \( x_k \) then \( \sigma_k(x) = \sigma_{k-1}(x) \). Inserting these results into (18) yields

\[
Q_k - Q_{k-1} = -(\sigma_{k-1})_+(x_k) |x_k|^2 + \frac{1}{2d} (\sigma_{k-1})_+(x_k) \sum_{x \sim x_k} |x|^2
\]

\[
= (\sigma_{k-1})_+(x_k) \cdot \frac{1}{2d} \sum_{x \sim x_k} (|x|^2 - |x_k|^2) \\
\overset{=\xi^2 (\Delta |x|^2)(x_k) = \xi^2}{=} \xi^2 \cdot (\sigma_{k-1})_+(x_k).
\]

This in turn implies that

\[
Q_k = Q_0 + \xi^2 \cdot \sum_{j=1}^{k} (\sigma_{j-1})_+(x_j).
\]

(19)

Now consider the \( k \)th odometer function \( u_k \): the value of \( u_k(x) \) is defined as \( \xi^2 \) times the total mass emitted from \( x \) during the \( k \) first applications of the toppling procedure, therefore we can write the value of \( u_k \) at \( x \) as

\[
u_k(x) = \xi^2 \sum_{1 \leq j \leq k: x_j = x} (\sigma_{j-1})_+(x).
\]
If we now sum $u_k(x)$ over all $x \in \xi \mathbb{Z}^d$, keeping in mind that $u_k(x)$ will be zero for every $x$ outside $\hat{B}_R$ and that our sequence $x_1, x_2, \ldots$ is an infinitely covering sequence of $\hat{B}_R$, then we obtain

$$\sum_{x \in \xi \mathbb{Z}^d} u_k(x) = \xi^2 \sum_{x \in \xi \mathbb{Z}^d} \sum_{1 \leq j \leq k: x_j = x} (\sigma_{j-1})_+(x) = \xi^2 \sum_{j=1}^k (\sigma_{j-1})_+(x_j).$$

(20)

Combining this last result with (19) and our previously established bounds $Q_k \leq (R + 2)^2 M_+$ and $-Q_0 \leq (R + 2)^2 M_-$, we get

$$\sum_{x \in \xi \mathbb{Z}^d} u_k(x) \leq (R + 2)^2 M,$$

(21)

where $M := M_+ + M_-$. As the right side of (21) is independent of $k$, and $u_k(x)$ clearly is an increasing function of $k$ for each fixed $x \in \xi \mathbb{Z}^d$, it follows that for any fixed $x \in \xi \mathbb{Z}^d$ the sequence $\{u_k(x)\}_{k=1}^\infty$ is increasing and bounded from above, hence convergent. We define the *odometer function* $u$ to be this limit: for any $x \in \mathbb{Z}^d$ let

$$u(x) := \lim_{k \to \infty} u_k(x).$$

(22)

Now, if $y \sim x$ then it is clear from how we defined the toppling that $x$ after $k$ toppling steps has received a contribution of mass of size $\frac{1}{2d\xi^2} u_k(y)$ from $y$. Since this holds for each neighbouring point of $x$, it is clear that $x$ in total has received a mass of size $\frac{1}{2d\xi^2} \sum_{y \sim x} u_k(y)$ after the $k$th toppling step. But during these steps we may also have performed toppling at $x$ itself, so to calculate the net difference in mass at $x$ at the $k$th step from our initial mass configuration at $x$ we need to subtract the mass emitted from $x$ up to this point, i.e. $u_k(x)/\xi^2$, from the total mass received. Hence we see that

$$\sigma_k(x) = \sigma(x) + \frac{1}{2d\xi^2} \sum_{y \sim x} (u_k(y) - u_k(x)) = \sigma(x) + \Delta u_k(x).$$

(23)

However, we just showed that $u_k$ had a well-defined limit as $k$ tends to infinity, and so relation (23) shows that also $\sigma_k$ has a limit, namely

$$\nu := \sigma + \Delta u.$$

(24)

Finally, the proposed structure of $\nu = \nu_+ + \nu_-$ with $\text{supp} \nu_+ \subset \partial \hat{B}_R$ and $\text{supp} \nu_- \subset \text{supp} \sigma_-$ is now evident: for any $x \in \hat{B}_R$ we have for infinitely many values of $k$ that $\sigma_k(x) \leq 0$ holds true (namely whenever we just toppled at $x$), hence the same inequality must hold for the limiting mass configuration that we now know exists, i.e. $\nu(x) \leq 0$ for all $x \in \hat{B}_R$. Iteration
of the estimate \((\sigma_k)_-(x) \leq (\sigma_{k-1})_-(x)\) for any \(x \in \xi \mathbb{Z}^d\) and \(k \geq 1\) implies that \((\sigma_k)_-(x) \leq \sigma_-(x)\), which in the limit \(k \to \infty\) becomes \(\nu_- (x) \leq \sigma_- (x)\), establishing \(\text{supp} \, \nu_- \subset \text{supp} \, \sigma_-\). Finally, the fact that we only perform toppling in the set \(\hat{B}_R\) implies that \(\nu\) in principle only can be non-zero on the set \(\hat{B}_R \cup \{x : x \sim y \text{ where } y \in \hat{B}_R\} = \hat{B}_R \cup \partial \hat{B}_R\). However, since we already know that \(\nu\) is non-positive on \(\hat{B}_R\), it follows, as desired, that \(\text{supp} \, \nu_+ \subset \partial \hat{B}_R\).

Proposition 3.1 has an inherent problem in that the limiting mass configuration \(\nu\) seemingly may depend on the choice of infinitely covering sequence of \(\hat{B}_R\), but this is in fact not the case. To see this, we will establish a characterization of the odometer function \(u\), and hence of the limiting mass configuration \(\nu\) via \(\nu = \sigma + \Delta u\), that does not depend on the choice of infinitely covering sequence; this characterization will also in fact be our link to the partial balayage operation in the continuous setting discussed later on in the paper.

To begin with, we need to define a discrete analogue of the potential function in continuous potential theory. For any given function \(\mu : \xi \mathbb{Z}^d \to \mathbb{R}\), assumed to have compact (i.e. finite) support, we define the (discrete) potential \(U^\mu_\xi\) (or simply \(U^\mu\) whenever it is clear which lattice we are referring to) of \(\mu\) via

\[
U^\mu_\xi (x) := \xi^d \sum_{y \in \xi \mathbb{Z}^d} g_\xi (x, y) \mu (y).
\]

Here \(g_\xi (\cdot, \cdot)\) is the discrete Green’s function on the underlying lattice, defined for \(x, y \in \xi \mathbb{Z}^d\) by

\[
g_\xi (x, y) := \begin{cases} 
\frac{2}{\pi} \log \xi - \gamma_0 \left( \frac{x}{\xi}, \frac{y}{\xi} \right) & \text{if } d = 2, \\
\frac{1}{\xi^{d-2}} \gamma_1 \left( \frac{x}{\xi}, \frac{y}{\xi} \right) & \text{if } d \geq 3,
\end{cases}
\]

where

\[
\gamma_0 (x, y) = \lim_{n \to \infty} (\mathbb{E}_x | \{k \leq n : X_k = x\} | - \mathbb{E}_x | \{k \leq n : X_k = y\} |)
\]

is the (recurrent) potential kernel for simple random walk on \(\mathbb{Z}^2\), and \(\gamma_1 (x, y)\) is the Green’s function for simple random walk on \(\mathbb{Z}^d\) for \(d \geq 3\),

\[
\gamma_1 (x, y) = \mathbb{E}_x | \{k : X_k = y\} |.
\]

Here \(\mathbb{E}_x\) denotes expectation with the simple random walk started at the lattice site \(x\); see [6, 7] for details on these Green’s functions. The above definitions imply in particular that

\[
-\Delta_1 g_\xi (x, y) = \frac{1}{\xi^d} \delta_x (y) = -\Delta_2 g_\xi (x, y),
\]

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17
where \( \delta_{x,y} \) is the Kronecker delta, and \( \Delta_j \) is the discrete Laplace operator acting on the \( j \)th variable. As an immediate and important consequence, it follows that

\[
-\Delta U^\mu_\xi (x) = \xi^d \sum_{y \in \xi \mathbb{Z}^d} (\Delta_1 g_\xi(x,y)) \mu(y) = \sum_{y \in \xi \mathbb{Z}^d} \delta_{x,y} \mu(y) = \mu(x),
\]

just as in the continuous setting. In a similar manner, we can via an easy calculation moreover see that for any function \( v : \xi \mathbb{Z}^d \to \mathbb{R} \) having finite support we have

\[
-\Delta U^v_\xi (x) = v(x).
\]

That \( u \) indeed is independent of the choice of infinitely covering sequence now follows from the following proposition:

**Proposition 3.2.** Let \( \sigma \) and \( R > 0 \) be as in Proposition 3.1, and let \( \nu \) and \( u \) denote the corresponding limit functions relative to toppling of some infinitely covering sequence \( x_1, x_2, \ldots \) of \( \hat{B}_R \).

Then \( \nu = \sigma + \Delta u \) and \( u = U^\sigma - v \), where

\[
v(x) := \sup \{ f(x) : \Delta f \geq 0 \text{ in } \hat{B}_R, \ f \leq U^\sigma \text{ in } \xi \mathbb{Z}^d \}.
\]

**Proof.** We know that \( \nu = \sigma + \Delta u \), hence \( \Delta u = \nu - \sigma \). Let \( v' := U^\sigma - u \). We immediately obtain

\[
-\Delta v' = -\Delta U^\sigma + \Delta u = \sigma + \nu - \sigma = \nu = \nu_+ - \nu_-.
\]

Since \( \nu \) is \( \nu = -\nu_- \leq 0 \) in \( \hat{B}_R \), it follows that \( \Delta v' \geq 0 \) in \( \hat{B}_R \). Moreover, as \( u(x) \) is \( \xi^2 \) times the total mass emitted from a site \( x \in \xi \mathbb{Z}^d \) it is clear that \( u \geq 0 \) holds everywhere in \( \mathbb{Z}^d \), and so \( v' = U^\sigma - u \leq U^\sigma \). We conclude that \( v' \) is a competing function in the definition of \( v \) in (25), which shows that \( v \geq v' \) holds everywhere in \( \xi \mathbb{Z}^d \).

For the converse inequality, let us study the difference \( v - v' \). First of all, we observe that (25) implies that also the solution \( v \) to the obstacle problem will satisfy \( \Delta v \geq 0 \) in \( \hat{B}_R \). Indeed, let \( f \) be any function satisfying both \( \Delta f \geq 0 \) in \( \hat{B}_R \) and \( f \leq U^\sigma \) in \( \xi \mathbb{Z}^d \). That \( f \) is subharmonic in \( \hat{B}_R \) means that

\[
f(x) \leq \frac{1}{2d} \sum_{y \sim x} f(y)
\]

holds for every \( x \in \hat{B}_R \). Using the inequality \( v \geq f \) on the right hand side implies

\[
f(x) \leq \frac{1}{2d} \sum_{y \sim x} v(y),
\]

18
and taking supremum over all such functions \( f \) on the left hand side shows that \( \Delta v \geq 0 \) must hold everywhere in \( \hat{B}_R \).

Now, we have
\[
\Delta(v - v') = \Delta v - \Delta v' = \Delta v + \nu.
\]
For every \( x \) belonging to the set \( D := \{ y \in \hat{B}_R : \nu(y) = 0 \} \) it is thus clear that \( \Delta(v - v')(x) = \Delta v(x) \geq 0 \), since we just established that \( v \) is subharmonic in \( \hat{B}_R \). On the other hand, for every \( x \in \hat{B}_R \setminus D \) we must have \( \nu(x) < 0 \), which evidently implies that \( x \) must be a site that, during the toppling process, never emitted any mass, i.e. a site where \( u(x) = 0 \). Since we only do toppling at the sites belonging to \( \hat{B}_R \), it is moreover clear that \( u(x) = 0 \) for every \( x \notin D \). But for any such \( x \) we then obtain
\[
(v - v')(x) = v(x) - U_\sigma(x) + u(x) = v(x) - U_\sigma(x) \leq 0.
\]
Hence \( v - v' \) is a function that is subharmonic on \( D \) and satisfies \( v - v' \leq 0 \) outside \( D \), and so the maximum principle implies that \( v - v' \leq 0 \) in fact must hold everywhere on \( \xi \mathbb{Z}^d \), i.e. \( v \leq v' \) holds everywhere. We can now finally conclude that \( v = v' \), hence \( u = U_\sigma - v \) as stated.

As seen in the two previous propositions, we obtain for each \( R > 0 \) a well-defined generalized mass configuration \( \nu = \nu_+ - \nu_- \) as long as the support of \( \sigma \) belongs to \( \hat{B}_R \). For sake of simplicity, we introduce the following notation:

**Definition 3.3.** Let \( \sigma : \xi \mathbb{Z}^d \to \mathbb{R} \) be a generalized mass configuration and let \( R > 0 \) be such that \( \text{supp} \sigma \subset \hat{B}_R \). We call the resulting generalized mass configuration \( \nu \) in Propositions 3.1 and 3.2 the *generalized divisible sandpile configuration of \( \sigma \) in \( \hat{B}_R \)*, and denote this configuration \( \text{GDS}_R(\sigma) \equiv \text{GDS}_{\xi R}^\xi(\sigma) := \nu \) (as a function on \( \xi \mathbb{Z}^d \)).

**Remark 3.4.** In the previous definition \( \text{GDS}_{\xi R}^\xi(\sigma) \) is a function defined on the same lattice \( \xi \mathbb{Z}^d \) as \( \sigma \). However, we can in a natural way interpret \( \text{GDS}_{\xi R}^\xi(\sigma) \) as a (signed) *measure on \( \mathbb{R}^d \)* (with some slight abuse of notation):
\[
\text{GDS}_{\xi R}^\xi(\sigma) = \xi^d \sum_{x \in \xi \mathbb{Z}^d} \text{GDS}_{\xi R}^\xi(\sigma)(x) \delta_x,
\]
where \( \delta_x \) is the Dirac measure at \( x \). That this is well-defined follows from the fact that \( \text{GDS}_{\xi R}^\xi(\sigma)(x) \) is bounded, and zero except for finitely many \( x \in \xi \mathbb{Z}^d \).
3.2 GDS and energy minimization

There is a rather natural interpretation of the algorithm for the generalized divisible sandpile as that minimizing a certain energy. In the continuous setting, the energy of a measure $\mu : \mathbb{R}^d \to \mathbb{R}$ is often defined as

$$I[\mu] = \int_{\mathbb{R}^d} U^\mu \, d\mu.$$ 

Following this, we define in the discrete setting the energy $E[\eta]$ of a mass configuration $\eta : \mathbb{Z}^d \to \mathbb{Z}$ using

$$E[\eta] := \xi^d \sum_{y \in \mathbb{Z}^d} U^\eta_y(y) \eta(y) = \xi^{2d} \sum_{x,y \in \mathbb{Z}^d} g_\xi(x,y) \eta(x) \eta(y).$$

For later use, we also define the mutual energy $E[\sigma, \kappa]$ between two mass configurations $\eta$ and $\kappa$ defined on the same lattice $\mathbb{Z}^d$ as

$$E[\eta, \kappa] := \xi^d \sum_{y \in \mathbb{Z}^d} U^\eta_y(y) \kappa(y).$$

Note that $E[\eta, \kappa] = E[\kappa, \eta]$ and $E[\eta] = E[\eta, \eta]$.

In a rather straightforward way, we can explicitly calculate how the energy behaves when we perform a toppling in the algorithm for the generalized divisible sandpile. Let $\eta, x_1, x_2, \ldots$ and $\eta = T_{x_k} \eta$ be as in Proposition 3.1, let $E_k = E[\eta_k]$, and study the difference $E_k - E_{k-1}$ in energy between two mass configurations that only differ in that we have toppled in precisely one point (the point $x_k$). We write

$$E_k - E_{k-1} = \xi^{2d} \sum_{x,y \in \mathbb{Z}^d} g_\xi(x,y) d(x,y),$$

where we let $d(x,y) := \sigma_k(x) \sigma_k(y) - \sigma_{k-1}(x) \sigma_{k-1}(y)$. If the mass of $\sigma_{k-1}$ at the point $x_k$ where we want to topple satisfies $\sigma_{k-1}(x_k) \leq 0$, then the mass configuration is unchanged, i.e. $\sigma_k = \sigma_{k-1}$ everywhere, hence $d(x,y) = 0$ for all $x, y \in \mathbb{Z}^d$ and $E_k = E_{k-1}$. Assume therefore that $(\sigma_{k-1})_+(x_k) > 0$, so that $\sigma_{k-1}$ and $\sigma_k$ are not equal everywhere. In that case, the double sum in (26) can be split into nine different terms, depending on if $x$ (and similarly for $y$) is either equal to the toppling point $x_k$, is a neighbour of $x_k$, or belongs to
the set $S_k := \xi \mathbb{Z}^d \setminus (\{x_k\} \cup \{z : z \sim x_k\})$. We get

$$
\mathcal{E}_k - \mathcal{E}_{k-1} = \xi^{2d} \left[ g_\xi(x_k, x_k) d(x_k, x_k) + \sum_{y \sim x_k} g_\xi(x_k, y) d(x_k, y) 
+ \sum_{y \in S_k} g_\xi(x_k, y) d(x_k, y) + \sum_{x \sim x_k} g_\xi(x, x_k) d(x, x_k) 
+ \sum_{x \sim x_k \ y \sim x_k} g_\xi(x, y) d(x, y) + \sum_{x \sim x_k \ y \sim x_k} g_\xi(x, y) d(x, y) 
+ \sum_{x \in S_k \ y \sim x_k} g_\xi(x, y) d(x, y) + \sum_{x \in S_k \ y \sim x_k} g_\xi(x, y) d(x, y) 
\right].
$$

Since both $g_\xi(\cdot, \cdot)$ and $d(\cdot, \cdot)$ are symmetric functions in their respective arguments, the above can be reduced to

$$
\mathcal{E}_k - \mathcal{E}_{k-1} = \xi^{2d} \left[ g_\xi(x_k, x_k) d(x_k, x_k) + 2 \sum_{x \sim x_k} g_\xi(x, x_k) d(x, x_k) 
+ 2 \sum_{x \sim x_k} g_\xi(x, x_k) d(x, x_k) + \sum_{x \sim x_k \ y \sim x_k} g_\xi(x, y) d(x, y) 
+ \sum_{x \sim x_k \ y \sim x_k} g_\xi(x, y) d(x, y) + \sum_{x \sim x_k \ y \sim x_k} g_\xi(x, y) d(x, y) 
\right],
$$

which shows that we only have to calculate the combination $g_\xi(x, y) d(x, y)$ for the six different cases appearing in this expression:

- $x = x_k$, $y = y_k$: since we topple at $x_k$ we have $\sigma_k(x_k) = 0$, thus $d(x_k, x_k) = -\sigma_{k-1}(x_k)^2$,
- $x \sim x_k$, $y = x_k$: $d(x, x_k) = -\sigma_{k-1}(x_k)\sigma_{k-1}(x)$,
- $x \in S_k$, $y = x_k$: $d(x, x_k) = -\sigma_{k-1}(x_k)\sigma_{k-1}(x)$,
- $x \sim x_k$, $y \sim x_k$: $d(x, y) = \frac{\sigma_{k-1}(x_k)}{2d} \cdot \left( \sigma_{k-1}(x) + \sigma_{k-1}(y) + \frac{\sigma_{k-1}(x_k)}{2d} \right)$,
- $x \sim x_k$, $y \in S_k$: $d(x, y) = \frac{\sigma_{k-1}(x_k)\sigma_{k-1}(y)}{2d}$,
- $x \in S_k$, $y \in S_k$: $d(x, y) = 0$. 

21
Inserting this into the above and simplifying, once more also using the symmetric property of $g_\xi(\cdot,\cdot)$, we obtain

\[
\mathcal{E}_k - \mathcal{E}_{k-1} = \xi^{2d} \sigma_{k-1}(x_k) \left[ -g_\xi(x_k, x_k) \sigma_{k-1}(x_k) - 2 \sum_{x \neq x_k} g_\xi(x, x_k) \sigma_{k-1}(x) \right.
\]

\[
+ 2 \frac{1}{2d} \sum_{x \neq x_k} \sum_{y \sim x_k} g_\xi(x, y) \sigma_{k-1}(x) + \frac{1}{4d^2} \sum_{x \sim x_k} \sum_{y \sim x_k} g_\xi(x, y) \sigma_{k-1}(x_k)
\]

Two of the four terms vanish, since they can be combined in the following manner:

\[
- \sum_{x \neq x_k} g_\xi(x, x_k) \sigma_{k-1}(x) + \frac{1}{2d} \sum_{x \neq x_k} \sum_{y \sim x_k} g_\xi(x, x_k) \sigma_{k-1}(x)
\]

\[
= \sum_{x \neq x_k} \sigma_{k-1}(x) \left[ -g_\xi(x, x_k) + \frac{1}{2d} \sum_{y \sim x_k} g_\xi(x, y) \right]
\]

\[
= \sum_{x \neq x_k} \sigma_{k-1}(x) \left[ \frac{1}{2d} \sum_{y \sim x_k} (g_\xi(x, y) - g_\xi(x, x_k)) \right]
\]

\[
= \xi^{2-d} \sum_{x \neq x_k} \sigma_{k-1}(x) \delta_{x_k}(x) = 0.
\]

As for the two remaining terms in $\mathcal{E}_k - \mathcal{E}_{k-1}$, we see in a similar way that

\[
-g_\xi(x_k, x_k) + \frac{1}{4d^2} \sum_{x \sim x_k} \sum_{y \sim x_k} g_\xi(x, y)
\]

\[
= \frac{1}{2d} \sum_{x \sim x_k} \left[ -g_\xi(x_k, x_k) + g_\xi(x, x_k) - g_\xi(x, x_k) + \frac{1}{2d} \sum_{y \sim x_k} g_\xi(x, y) \right]
\]

\[
= \xi^2 \Delta_1 g_\xi(x_k, x_k) + \frac{1}{2d} \sum_{x \sim x_k} \frac{1}{2d} \sum_{y \sim x_k} (g_\xi(x, y) - g_\xi(x, x_k))
\]

\[
= -\xi^{2-d} \delta_{x_k}(x_k) - \xi^{2-d} \frac{1}{2d} \sum_{x \sim x_k} \delta_{x_k}(x) = -\xi^{2-d},
\]

from which it immediately finally follows that

\[
\mathcal{E}_k - \mathcal{E}_{k-1} = -\xi^{2d} \sigma_{k-1}(x_k)^2 \xi^{2-d} = -\xi^{d+2} (\sigma_{k-1})_+(x_k)^2,
\]

22
i.e. whenever it happens that $(\sigma_{k-1})_+(x_k)$ is positive at toppling step $k$, then the energy strictly decreases. The total energy after $k$ steps is

$$E_k = E_0 - \xi^{d+2} \sum_{j=1}^{k} (\sigma_{j-1})_+(x_j)^2.$$  (27)

Comparing this with (20) immediately shows that $E_k$ has a finite limit as $k \to \infty$.

Now, consider the problem of finding a mass configuration $\tilde{\nu}$ with the properties $\tilde{\nu} \leq 0$ in $\hat{B}_R$ and with the same total mass as $\sigma$, that minimizes energy of the difference between $\sigma$ and $\tilde{\nu}$, i.e. that solves the problem

$$\min E[\sigma - \nu, \nu - \tilde{\nu}] : \tilde{\nu} \leq 0 \text{ in } \hat{B}_R \text{ and } \sum_{y \in \xi \mathbb{Z}^d} \tilde{\nu}(y) = \sum_{y \in \xi \mathbb{Z}^d} \sigma(y).$$

We claim that $\nu := \text{GDS}_R^\xi(\sigma)$ is the (unique) solution to this problem. By usual Hilbert space theory arguments, it suffices to show that

$$E[\sigma - \nu, \nu - \tilde{\nu}] \geq 0$$

holds for all $\tilde{\nu}$ with $\tilde{\nu} \leq 0$ in $\hat{B}_R$ and $\sum_{y \in \xi \mathbb{Z}^d} \tilde{\nu}(y) = \sum_{y \in \xi \mathbb{Z}^d} \sigma(y)$. To begin with, we have

$$E[\sigma - \nu, \nu - \tilde{\nu}] = \xi^d \sum_{y \in \xi \mathbb{Z}^d} (U^\sigma(y) - U^\nu(y))(\nu(y) - \tilde{\nu}(y)).$$

By the definition of $\nu = \sigma + \Delta u$, where $u$ is the limiting odometer function, it follows that $U^\nu = U^\sigma - u$, i.e. the first factor in the sum above is precisely $U^\sigma - U^\nu = u$. It follows that we may reduce the set over which we sum to the set of points where $u$ is non-zero, i.e. $\{y \in \xi \mathbb{Z}^d : u(y) > 0\}$ (which is a subset of $\hat{B}_R$). However, if $u(y) > 0$ then some mass must have been emitted from $y$ in the construction of $\text{GDS}_R^\xi(\sigma)$, thus $\nu(y) = 0$ must hold. We then obtain

$$E[\sigma - \nu, \nu - \tilde{\nu}] = \xi^d \sum_{y : u(y) > 0} u(y)(-\tilde{\nu}(y)) \geq 0,$$

since both $u$ and $-\tilde{\nu}$ are non-negative. We can in fact calculate an explicit expression for the minimizing energy by studying $E[\sigma - \sigma_k]$ and letting $k \to \infty$. For the difference $E[\sigma - \sigma_k] - E[\sigma - \sigma_{k-1}]$ between two successive steps in the algorithm we get

$$E[\sigma - \sigma_k] - E[\sigma - \sigma_{k-1}] = E[\sigma_k] - E[\sigma_{k-1}] - 2E[\sigma, \sigma_k - \sigma_{k-1}].$$
We already know that $E[\sigma_k] - E[\sigma_{k-1}] = -\xi^{d+2}(\sigma_{k-1})_+(x_k)^2$. For the last term, we get

$$E[\sigma, \sigma_k - \sigma_{k-1}] = E[\sigma_k - \sigma_{k-1}, \sigma] = \xi d \sum_{x \in \xi \mathbb{Z}^d} U^{\sigma_k - \sigma_{k-1}}(x) \sigma(x),$$

and, utilizing that $(\sigma_k - \sigma_{k-1})(x) = (\sigma_{k-1})_+(x_k)\xi^2 \Delta \delta_{x_k}(x)$, hence

$$U^{\sigma_k - \sigma_{k-1}}(x) = (\sigma_{k-1})_+(x_k)\xi^2 \Delta U \delta_{x_k}(x) = -(\sigma_{k-1})_+(x_k)\xi^2 \delta_{x_k}(x),$$

it follows that

$$E[\sigma, \sigma_k - \sigma_{k-1}] = -\xi^{d+2}(\sigma_{k-1})_+(x_k)\sigma(x_k).$$

We can summarize the above to draw the conclusion

$$E[\sigma - \sigma_k] - E[\sigma - \sigma_{k-1}] = -\xi^{d+2}(\sigma_{k-1})_+(x_k)^2 + 2\xi^{d+2}(\sigma_{k-1})_+(x_k)\sigma(x_k) = \xi^{d+2}(\sigma_{k-1})_+(x_k)(2\sigma(x_k) - (\sigma_{k-1})_+(x_k)),$$

hence

$$E[\sigma - \sigma_k] = \sum_{j=1}^k (E[\sigma - \sigma_j] - E[\sigma - \sigma_{j-1}]) = \xi^{d+2} \sum_{j=1}^k (\sigma_{j-1})_+(x_j)(2\sigma(x_j) - (\sigma_{j-1})_+(x_j)).$$

From this it follows that the minimizing energy is precisely

$$E[\sigma - \nu] = \xi^{d+2} \sum_{j=1}^\infty (\sigma_{j-1})_+(x_j)(2\sigma(x_j) - (\sigma_{j-1})_+(x_j)). \quad (28)$$

Note that this is convergent, as the factor $2\sigma(x_j) - (\sigma_{j-1})_+(x_j)$ is bounded and the sum $\xi^2 \sum_{j=1}^\infty (\sigma_{j-1})_+(x_j)$ is by (20) precisely equal to

$$\sum_{x \in \xi \mathbb{Z}^d} u(x) = \sum_{x \in B_R} u(x) < \infty.$$
3.3 A natural scaling limit of the bounded GDS

As mentioned in the introduction, we are interested in studying all of the above in the natural scaling limit, i.e. as the lattice spacing tends to zero. For this reason, we simply fix a sequence of positive real numbers \( \{\xi_n\}_{n=1}^{\infty} \), which is assumed to be monotonically decreasing and with limit zero as \( n \) tends to infinity. Our initial generalized mass configuration \( \sigma \) is now assumed to be a bounded function defined on \( \mathbb{R}^d \) instead of some lattice, and for each lattice constant \( \xi_n \) we now discretize \( \sigma \) in precisely the same way as in Section 1.1, i.e. we define for each \( n = 1, 2, \ldots \) the function \( \sigma_n : \xi_n \mathbb{Z}^d \to \mathbb{R} \) via

\[
\sigma_n(x) := \frac{1}{\xi_n^d} \int_{x} \sigma(y) \, dy.
\]

(29)

For each \( n \) we thus obtain a generalized mass configuration on a lattice, can perform the generalized divisible sandpile algorithm on each such configuration, and hence will obtain a sequence of generalized mass configurations \( \{\text{GDS}_{\xi_n}^R(\sigma_n)\}_{n=1}^{\infty} \) (for some \( R \) chosen in a suitable manner). Note that the discretization above comes at a (slight) price: in general we do not necessarily have \( (\sigma_n)_+ = (\sigma)_+ \) or \( (\sigma_n)_- = (\sigma)_- \), only in the limit \( n \to \infty \).

We claim the following:

**Theorem 3.5.** Let \( \sigma : \mathbb{R}^d \to \mathbb{R} \) be a bounded and almost everywhere continuous function with compact support for which \( \int_{\mathbb{R}^d} \sigma(x) \, dx < 0 \), let \( \{\xi_n\}_{n=1}^{\infty} \) be a sequence of positive decreasing lattice constants such that \( \xi_n \searrow 0 \) as \( n \to \infty \), and for each \( n = 1, 2, \ldots \) let \( \sigma_n : \xi_n \mathbb{Z}^d \to \mathbb{R} \) be the discretization of \( \sigma \) relative to \( \xi_n \mathbb{Z}^d \) as in (29). Assume \( R > 0 \) is such that \( \text{supp} \sigma \subset B(0, R) \) and \( \text{supp} \sigma_n \subset B(0, R) \) for all \( n \). Then, in the sense of distributions,

\[
\text{GDS}_{\xi_n}^R(\sigma_n) \to \text{Bal}_R(\sigma, 0) \text{ as } n \to \infty.
\]

(30)

To prove this theorem we need a few lemmas.

**Lemma 3.6.** Let \( U_{\xi_n}^\sigma \equiv U_{\xi_n}^{\sigma_n} \) be the discrete potential of \( \sigma_n \), defined on \( \xi_n \mathbb{Z}^d \), let \( (U_{\xi_n}^\sigma)^\square \) be its extension to \( \mathbb{R}^d \) as a step function and let \( U^\sigma \) be the potential of the measure \( \sigma(x) \, dx \). Then \( (U_{\xi_n}^\sigma)^\square \to U^\sigma \) uniformly on compact subsets of \( \mathbb{R}^d \) as \( n \to \infty \).

For the proof of Lemma 3.6 we refer to the proofs of Lemma 2.16 (i) and Lemma 2.22 in [10] which, although there stated with slightly different assumptions than the ones in this paper, go through in our setting as well, with more or less only notational changes.
Lemma 3.7. Let \( u_n \) be the limiting odometer function for the generalized divisible sandpile on \( \xi_n \mathbb{Z}^d \) from mass configuration \( \sigma_n \), and let \( u = U^\sigma - V^\sigma \) be the modified Schwarz potential of \( \text{Bal}_R(\sigma, 0) \) as in Remark 2.4, with \( \sigma \) and \( \sigma_n \) as in Theorem 3.5. Then for every \( x \in \mathbb{R}^d \), \((u_n)^\square(x) \to u(x)\) pointwise as \( n \to \infty \).

Proof. Let us first restrict the problem slightly. We know that the function \( u \) is zero on the complement of \( B(0, R) \), and for each \( n \) we also know that the odometer function \( u_n \) is zero outside of the set \( \hat{B}_R(n) := B(0, R) \cap (\xi_n \mathbb{Z}^d) \). For any \( x \notin B(0, R) \) it therefore follows that for all \( n \) large enough we have \((u_n)^\square(x) = u_n(x) = 0 = u(x)\). The set we have to study in detail is thus \( B(0, R) \). The slightly more challenging part of the proof is thus the convergence for \( x \) in the set \( B(0, R) \).

We mainly repeat the arguments made in the proof of Lemma 3.8 in [10], with a few modifications due to the fact that we here work in a slightly different setting, being bounded to the set \( B(0, R) \). As a first step, we use that \( u = U^\sigma - V^\sigma, u_n = U_n^\sigma - v_n \) along with the convergence \((U_n^\sigma)^\square \to U^\sigma \) from Lemma 3.6 to conclude that it suffices to show that \((v_n)^\square(x) \to V^\sigma(x)\) for all \( x \in B(0, R) \), where

\[
V^\sigma(x) = \sup\{ f(x) : f \in \mathcal{CS}(B(0, R)), f \leq U^\sigma \text{ in } \mathbb{R}^d \},
\]

the set \( \mathcal{CS}(B(0, R)) \) is the set of functions on \( \mathbb{R}^d \) that are continuous and subharmonic on \( B(0, R) \), and

\[
v_n(x) := \sup\{ f(x) : \Delta f \geq 0 \text{ in } \hat{B}_R(n), f \leq U_n^\sigma \text{ in } \xi_n \mathbb{Z}^d \}.
\]

The method we will employ will in essence be to construct help functions that are comparable to \( V^\sigma, R \) and \( v_n \), respectively, but have discrete or continuous analogues that are competing functions in the obstacle problems (31) and (32), thereby allowing us to conclude both \( V^\sigma, R(x) \leq (v_n)^\square(x) \) and \((v_n)^\square(x) \leq V^\sigma, R(x)\) for \( n \) large enough.

Let \( \varepsilon > 0 \) be arbitrary but fixed. We want to show that

\[
(v_n)^\square(x) \geq V^\sigma(x)
\]

holds for all \( n \) large enough and all \( x \in B(0, R) \). For any \( h > 0 \) let \( \tilde{V}^\sigma := J_h V^\sigma \) be the mollification of \( V^\sigma \) (for instance as in [5, Section 3.5]):

\[
\tilde{V}^\sigma(x) := J_h V^\sigma(x) = \frac{1}{h^d} \int_{\mathbb{R}^d} V^\sigma(y) m \left( \frac{x - y}{h} \right) dy,
\]

where \( m(y) = C \exp(-1/(1 - |y|^2)) \) if \( |y| < 1 \) and zero otherwise, with \( C \) such that \( \int m(y) dy = 1 \). By taking \( h \) small enough we obtain \( |V^\sigma - \tilde{V}^\sigma| < \varepsilon \)
on $B(0, R - h)$, in particular $\tilde{V}^\sigma(x) + \varepsilon > V^\sigma(x)$ for all $x \in B(0, R - h)$. We will construct our helper function from the discretization $(\tilde{V}^\sigma)^\sigma$ of $\tilde{V}^\sigma$, and need to relate the discrete Laplacian of $(\tilde{V}^\sigma)^\sigma$ to the continuous Laplacian of $\tilde{V}^\sigma$ (which is well-defined since $\tilde{V}^\sigma$ is infinitely differentiable). In general, a straightforward calculation (for instance in [10, Lemma 2.20]) shows that if $f \in C^\infty(D)$ on some open set $D \subset \mathbb{R}^d$, $A$ is a bound for the third derivative of $f$ in $D$, and $x \in D \cap \xi\mathbb{Z}^d$ with $B(x, \xi) \subset D$, then

$$|\Delta f(x) - 2d \Delta f_\sigma(x)| \leq \frac{A d}{3} \xi.$$ 

For any fixed value of $n$, note that we can always choose $h > 0$ small enough so that the set $\hat{B}_R^{(n)}$ is contained in $B(0, R - h)$. Let $A$ be a bound for the third partial derivatives of $\tilde{V}^\sigma$ in $B(0, R - h)$, and let $\phi_n : \xi_n\mathbb{Z}^d \to \mathbb{R}$ be defined by

$$\phi_n(x) := (\tilde{V}^\sigma)^\sigma(x) + \frac{A \xi_n |x|^2}{6}.$$ 

It follows that for all $x \in \hat{B}_R^{(n)}$, $\Delta \phi_n(x) \geq \Delta \tilde{V}^\sigma(x)/2d$. However, $V^\sigma$ is subharmonic in $B(0, R)$, hence $V^\sigma$ is subharmonic in $B(0, R - h)$, and thus $\phi_n$ is (discrete) subharmonic in $\hat{B}_R^{(n)}$. If $n$ is large enough then the term $A \xi_n |x|^2/6$ is strictly less than $\varepsilon$ in $\hat{B}_R^{(n)}$, from which it follows that

$$\phi_n(x) < (V^\sigma)^\sigma(x) + 2\varepsilon$$

for all $x \in \hat{B}_R^{(n)}$. Since $V^\sigma$ is bounded from above by $U^\sigma$, and we again use the property $|(U_n^{\sigma(n)})^\square - U^\sigma| < \varepsilon$ for all $n$ large enough by Lemma 3.6, we obtain

$$\phi_n(x) - 3\varepsilon \leq U_n^{\sigma(n)}(x)$$

for all $x \in \hat{B}_R^{(n)}$. Now define $\Phi_n : \xi_n\mathbb{Z}^d \to \mathbb{R}$ via

$$\Phi_n(x) := \begin{cases} \phi_n(x) - 3\varepsilon & \text{if } x \in \hat{B}_R^{(n)}, \\ U_n^{\sigma(n)}(x) & \text{otherwise.} \end{cases}$$

It follows that $\Phi_n$ is a lattice function that is subharmonic on $\hat{B}_R^{(n)}$ and satisfies $\Phi_n \leq U_n^{\sigma(n)}$ everywhere on $\xi_n\mathbb{Z}^d$. The function $\Phi_n$ is thus a competing element in the obstacle problem (32), hence $\Phi_n \leq v_n$ holds everywhere on $\xi_n\mathbb{Z}^d$. For any $x \in B(0, R)$ we can now conclude that

$$(v_n)^\square(x) = v_n(x^\square) \geq \Phi_n(x^\square) \geq (\tilde{V}^\sigma)^\sigma(x^\sigma) - 3\varepsilon$$

$$> V^\sigma(x^\sigma) - 4\varepsilon > V^\sigma(x) - 5\varepsilon,$$

(33)
where we in the last equality used that for all \( n \) large enough we have \( |(V^\sigma) - V^\sigma| < \varepsilon \).

For the converse result, we again repeat the techniques in the proof of Lemma 3.8 in [10]: let \( R' > R \) and introduce the function \( \psi_n : \xi_n \mathbb{Z}^d \to \mathbb{R} \) defined by

\[
\psi_n(x) := -\Delta(v_n \chi_{B_{R'}})(x).
\]

On the one hand we have \( U_{\psi_n}^\sigma(x) = v_n(x)\chi_{B_{R'}}(x) \) \( (= v_n(x) \text{ for } x \in \hat{B}_{R'} \) ). It can be shown (see the proof of Lemma 3.7 in [10], the same methods apply here) that we have a similar property for \( x \in B(0, R) \) if we try to take the continuous potential of the function \( \psi_n \) considered as a measure on \( \mathbb{R}^d \) (in the sense that \( d\psi_n(x) = (\psi_n)^\square(x) \, dm(x) \), where \( m \) is the Lebesgue measure on \( \mathbb{R}^d \): for any \( \varepsilon > 0 \) we have \( |(\psi_n)^\square(x) - U(\psi_n)^\square(x)| < \varepsilon \) for all \( x \in B(0, R) \) if \( n \) is large enough. Assuming \( n \) is also large enough for \( |(U_{\psi_n}^\sigma)^\square(x) - U^\sigma(x)| < \varepsilon \) to hold for all \( x \in B(0, R) \), we obtain

\[
U(\psi_n)^\square(x) < (\psi_n)^\square(x) + \varepsilon \leq (U_{\psi_n}^\sigma)^\square(x) + \varepsilon < U^\sigma(x) + 2\varepsilon.
\]

Let \( \Psi_n : \mathbb{R}^d \to \mathbb{R} \) be defined by

\[
\Psi_n(x) := \begin{cases} 
U(\psi_n)^\square(x) - 2\varepsilon & \text{if } x \in B(0, R), \\
U^\sigma(x) & \text{otherwise.}
\end{cases}
\]

The function \( \Psi_n \) is then subharmonic and continuous in \( B(0, R) \), since \( (\psi_n)^\square \) is non-positive there and bounded. By the above, we clearly also have \( \Psi_n \leq U^\sigma \) everywhere in \( \mathbb{R}^d \). It immediately follows that \( \Psi_n \) is a competing function in the obstacle problem (31), hence satisfies \( \Psi_n(x) \leq U^\sigma(x) \) everywhere on \( \mathbb{R}^d \). In particular, for \( x \in B(0, R) \) this implies

\[
(\psi_n)^\square(x) < U(\psi_n)^\square(x) + \varepsilon = \Psi_n(x) + 3\varepsilon \leq U^\sigma(x) + 3\varepsilon. \tag{34}
\]

Finally, combining (33) and (34), we can conclude that if \( \varepsilon' > 0 \) is arbitrary and \( x \in B(0, R) \) then there exists \( N \) such that we for all \( n > N \) have both \( (\psi_n)^\square(x) > V^\sigma(x) - \varepsilon' \) and \( (\psi_n)^\square(x) < V^\sigma(x) + \varepsilon' \), i.e. precisely

\[
|((\psi_n)^\square(x) - V^\sigma(x)| < \varepsilon',
\]

which completes the proof. \( \square \)

**Lemma 3.8.** Let \( \{g_n\}_{n=1}^{\infty} \) be a sequence of functions with \( g_n : \xi_n \mathbb{Z}^d \to \mathbb{R} \), for some fixed lattice constants \( \xi_n \) satisfying \( \xi_n \searrow 0 \) as \( n \to \infty \), and assume that \( (g_n)^\square(x) \) converges to \( g(x) \) for every \( x \in \mathbb{R}^d \) for some function \( g \in L^1_{\text{loc}}(\mathbb{R}^d) \). Then \( (\nabla g_n)^\square \to \Delta g/2d \) in the sense of distributions as \( n \to \infty \).
Proof. Let $\varphi \in C_0^\infty(\mathbb{R}^d)$ be an arbitrary test function. We obtain

$$\langle (\Delta g_n)\Box, \varphi \rangle = \int_{\mathbb{R}^d} (\Delta g_n)(x) \varphi(x) \, dx = \sum_{y \in \xi_n Z^d} (\Delta g_n)(y) \varphi(x) \, dx$$

$$= \frac{1}{2d} \xi_n^2 \sum_{y \in \xi_n Z^d} \sum_{z \sim y} (g_n(z) - g_n(y)) \int_{y\Box} \varphi(x) \, dx$$

$$= \frac{1}{2d} \xi_n^2 \sum_{y \in \xi_n Z^d} \sum_{k=1}^d (g_n(y + \xi_n e_k) - 2g_n(y) + g_n(y - \xi_n e_k)) \int_{y\Box} \varphi(x) \, dx$$

Utilizing that we sum over the entire lattice $\xi_n Z^d$ we can rewrite this last expression as

$$\frac{1}{2d} \xi_n^2 \sum_{y \in \xi_n Z^d} g_n(y) \int_{y\Box} \sum_{k=1}^d \frac{(\varphi(x + \xi_n e_k) - 2\varphi(x) + \varphi(x - \xi_n e_k))}{\xi_n^2} \, dx$$

$$= \frac{1}{2d} \int_{\mathbb{R}^d} g_n(x) \sum_{k=1}^d \frac{\varphi(x + \xi_n e_k) - 2\varphi(x) + \varphi(x - \xi_n e_k)}{\xi_n^2} \, dx$$

$$\to \frac{1}{2d} \int_{\mathbb{R}^d} g(x) \sum_{k=1}^d \frac{\partial^2 \varphi}{\partial x_k^2} \, dx = \left\langle \frac{1}{2d} \Delta g, \varphi \right\rangle \text{ as } n \to \infty,$$

where the last convergence follows from the dominated convergence theorem.

Proof of Theorem 3.5. We know that $\text{GDS}_R^\xi(\sigma_n) = \sigma_n + \Delta u_n$, so that

$$\text{GDS}_R^\xi(\sigma_n) = (\text{GDS}_R^\xi(\sigma_n))\Box = (\sigma_n)\Box + (\Delta u_n)\Box.$$ 

Since $\{\sigma_n\}_{n=1}^\infty$ is assumed to be a discretization of $\sigma$, we have $(\sigma_n)\Box \to \sigma$ as $n \to \infty$. Lemma 3.8 combined with Lemma 3.7 yields $(\Delta u_n)\Box \to \Delta u/2d$. It follows that

$$\text{GDS}_R^\xi(\sigma_n) = (\sigma_n)\Box + (\Delta u_n)\Box \to \sigma + \frac{\Delta u}{2d} = \text{Bal}_R(\sigma, 0),$$

as desired.

3.4 Boundary properties for large confining radii

Given the recent development in [12] of partial balayage in an unrestricted setting (at least in the plane) described in Section 2.2, one might expect
there to be a result similar to Theorem 3.5 if we attempt to study the limit $R \to \infty$. For instance, it is rather easy to show that there is a sort of invariance in the choice of the confining radius in the generalized divisible sandpile, in the sense that successive applications of $GDS_\rho(\cdot)$ operators for, say, first $\rho = R_1$ and then $\rho = R_2$ for some $0 < R_1 < R_2$, yields the same result as if we would have used $\rho = R_2$ from the start. In view of Theorem 3.5 this is natural, since it is known that a similar iterative property holds for $Bal_\rho(\cdot, 0)$ [4, Theorem 2.2 (iii)].

**Proposition 3.9.** If $\sigma : \xi \mathbb{Z}^d \to \mathbb{R}$ is a generalized mass configuration and $R_1 > 0$ is such that $\text{supp} \sigma \subset \hat{B}_{R_1}$, and $R_2 > R_1 + \xi$ is arbitrary, then

$$GDS_{R_2}(GDS_{R_1}(\sigma)) = GDS_{R_2}(\sigma).$$

**Proof.** For sake of simplicity, we define the three mass configurations $\nu_1, \nu_2$ and $\tilde{\nu}$ via

$$\nu_1 := GDS_{R_1}(\sigma) = -\Delta v_1,$$

$$\nu_2 := GDS_{R_2}(\sigma) = -\Delta v_2,$$

$$\tilde{\nu} := GDS_{R_2}(\nu_1) = -\Delta \tilde{v},$$

where $v_1$, $v_2$ and $\tilde{v}$ are, by (25), the solutions to the obstacle problems

$$v_1(x) = \sup \{ f(x) : \Delta f \geq 0 \text{ in } \hat{B}_{R_1}, f \leq U^\sigma \text{ in } \xi \mathbb{Z}^d \},$$

$$v_2(x) = \sup \{ f(x) : \Delta f \geq 0 \text{ in } \hat{B}_{R_2}, f \leq U^\sigma \text{ in } \xi \mathbb{Z}^d \},$$

$$\tilde{v}(x) = \sup \{ f(x) : \Delta f \geq 0 \text{ in } \hat{B}_{R_2}, f \leq U^{\nu_1} \text{ in } \xi \mathbb{Z}^d \}.$$

Note that $\tilde{\nu} = GDS_{R_2}(\nu_1)$ is well-defined since $\nu_1$ is a mass configuration of negative total mass satisfying $\text{supp} \nu_1 \subset (\partial \hat{B}_{R_1} \cup \hat{B}_{R_1}) \subset \hat{B}_{R_2}$ by our assumption on $R_2$. We claim that the solutions $\tilde{v}$ and $v_2$ above are in fact equal, from which $\tilde{\nu} = \nu_2$ clearly will follow, proving the proposition.

First of all, by the definitions of $\tilde{\nu}$ and $v_1$ we see that

$$\tilde{v} \leq U^{\nu_1} = U^{-\Delta v_1} = v_1 \leq U^\sigma$$

holds throughout $\xi \mathbb{Z}^d$. Moreover, as was seen in the proof of Proposition 3.2, it is clear that $\Delta \tilde{v} \geq 0$ holds in $\hat{B}_{R_2}$. Combining this, we see that $\tilde{v}$ is a competing function in the definition of $v_2$, from which it follows that $\tilde{v} \leq v_2$ holds everywhere.

For the contrary, we know similarly that $v_2 \leq U^\sigma$ holds everywhere and that $\Delta v_2 \geq 0$ holds in $\hat{B}_{R_2}$. But $\hat{B}_{R_2} \supset \hat{B}_{R_1}$ by assumption, hence $v_2$ is a competing function in the definition of $v_1$, yielding $v_2 \leq v_1$ everywhere. Since
\[ v_1 = U^{-\Delta v_1} = U^{\nu_1} \]
it thus follows that \( v_2 \leq U^{\nu_1} \), and so we can conclude that \( v_2 \) is a competing function in the definition of \( \tilde{v} \), finally yielding \( v_2 \leq \tilde{v} \) everywhere in \( \xi \mathbb{Z}^d \), and we are done.

With Proposition 3.9 in mind, we can also observe that the total mass of \((\text{GDS}_{R_2}(\sigma))_+\), i.e. the total mass of \(\text{GDS}_{R_2}(\sigma)\) that resides on \(\partial \hat{B}_{R_2}\), in fact always must be \textit{strictly} less than the total mass residing on \(\partial \hat{B}_{R_1}\) for \(\text{GDS}_{R_1}(\sigma)\). (When calculating \(\text{GDS}_{R_2}(\sigma) = \text{GDS}_{R_2}(\text{GDS}_{R_1}(\sigma))\) we must topple all the points on \(\partial \hat{B}_{R_1}\), with the consequence that at least a fraction of that mass has to move inwards into the region where \(\text{GDS}_{R_1}(\sigma)\) is negative, thereby annihilating and resulting in that the positive part of \(\text{GDS}_{R_2}(\sigma)\) must have strictly less total mass than the positive part of \(\text{GDS}_{R_1}(\sigma)\).) One might therefore guess that this boundary mass would vanish if we keep increasing the confining radius \(R\), i.e. let \(R \to \infty\). However, there does not seem to be any reason for such a result to hold in general, at least not for dimensions \(d \geq 3\). In the upcoming paper [3] the example \(\text{Bal}_{R}(\sigma, 0)\) is treated in detail, where \(\sigma\) is the measure

\[
\sigma = t\eta - \chi_{B(0, \rho)},
\]
and \(t > 0\) is a parameter, \(\eta\) is the hypersurface measure on the unit sphere \(\partial B(0, 1)\), and \(\chi_{B(0, \rho)}\) is interpreted as the characteristic function of the set \(B(0, \rho)\) times the Lebesgue measure in \(\mathbb{R}^d\) (or, equivalently, the restriction of the Lebesgue measure to \(B(0, \rho)\), extended with zero outside of \(B(0, \rho)\)). The two radii \(\rho\) and \(R\) appearing in the problem are assumed to satisfy \(0 < \rho < 1 < R\). If \(t\) and \(\rho\) are chosen suitably then \(\text{Bal}_{R}(\sigma, 0)\) exists, and by the radial symmetry of the problem it is possible to explicitly calculate the part of \(\text{Bal}_{R}(\sigma, 0)\) that is supported on the boundary \(\partial B(0, R)\), i.e. the positive part of \(\text{Bal}_{R}(\sigma, 0)\). In particular, if we write \(\nu := \text{Bal}_{R}(\sigma, 0) = \nu_+^{(R)} - \nu_-^{(R)}\), so that \(\text{supp} \nu_+^{(R)} \subset \partial B(0, R)\), then the quantity \(M_R := \nu_+^{(R)}(\mathbb{R}^d)\), i.e. the total mass residing at the boundary \(\partial B(0, R)\), has a limit

\[
\lim_{R \to \infty} M_R = M_{\infty} := \frac{d-2}{d} \cdot |S^{d-1}| t,
\]
where \(|S^{d-1}|\) is the surface area of the unit sphere in \(\mathbb{R}^d\). By this example it therefore seems rather likely that any attempt of finding an unbounded version of Theorem 3.5 would be rather futile. Also, it is noteworthy that if the dimension \(d\) is very large then in the above example nearly \textit{all} of the total mass of \(\sigma_+\) would be relocated out to the boundary \(\partial B(0, R)\) by \(\text{Bal}_{R}(\cdot, 0)\) for large \(R\), suggesting that the boundary has a rather important impact on the problem for \(d \geq 3\).
In dimension $d = 2$ however, the situation seems slightly different. For the above example it turns out that the total mass $M_R$ on the boundary tends to zero as $R \to \infty$. In fact, in [3] it is shown that the boundary mass vanishes in general in dimension $d = 2$. As already mentioned in Section 2.2, in [12] it is shown that one can define a partial balayage operation $\text{Bal}(\sigma, 0)$ (see Definition 2.8), in some sense corresponding to letting $R = \infty$ in Definition 2.3. As seen in Theorem 2.7, the assumptions on the signed measure $\sigma$ for this partial balayage measure to exist do not need to be very harsh—negative total mass and (for instance) continuity of the potential of the negative part of $\sigma$ are sufficient. It thus seems rather likely that there exists a limit of the generalized divisible sandpile model for $d = 2$ as the confining radius grows infinitely large. We have unfortunately been unable to prove such a result, and will have to settle with a conjecture:

**Conjecture 3.10.** Let $\sigma$ be a generalized mass configuration on $\xi \mathbb{Z}^2$ with finite support. Then $M_R \to 0$ as $R \to \infty$, where $M_R$ is the boundary mass of $\text{GDS}_R(\sigma)$:

$$M_R := \sum_{x \in \xi \mathbb{Z}^2} (\text{GDS}_R(\sigma))_+(x).$$

As a final remark, we note that one way of interpreting such a result—if it holds—is that the confining radius $R > 0$ that we introduced to ensure convergence of the generalized model in a sense is unnecessary in dimension $d = 2$. On the other hand, based on the above example the confining radius seems required in dimensions $d \geq 3$. Given the recently developed strong connections between the standard divisible sandpile and the so-called internal diffusion limited aggregation (IDLA) model for particle aggregation, which uses simple random walks as a means to relocate excess mass, it does not seem too unlikely that the apparent difference in behaviour between $d = 2$ and $d \geq 3$ for the generalized divisible sandpile may have something to do with the result by G. Pólya [11] that the simple random walk is recurrent in dimension $d = 2$ and transient if $d \geq 3$.

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