PATHOLOGIES ON MORI FIBRE SPACES IN POSITIVE CHARACTERISTIC

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Abstract. We show that there exist Mori fibre spaces whose total spaces are klt but bases are not. We also construct Mori fibre spaces which have relatively non-trivial torsion line bundles.

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1. Introduction

The minimal model theory suggests a systematic approach to classify algebraic varieties. Given a variety $X$, the minimal model programme conjecture implies that $X$ is birational to either a minimal model or a

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Mori fibre space. The purpose of this paper is to find some phenomena on Mori fibre spaces which occur only in positive characteristic. Originally, one of the advantages to use Mori fibre spaces is to reduce some problems to their fibres and bases that are of lower dimensions. For instance, given a Mori fibre space $f : X \to S$ from a klt variety $X$ in characteristic zero, it is known that its base space $S$ is also klt (cf. [Amb05, Theorem 0.2], [Fuj99, Corollary 3.5]). Unfortunately, the same statement is no longer true in positive characteristic.

**Theorem 1.1.** Let $k$ be an algebraically closed field whose characteristic is two or three. Then there exists a projective $k$-morphism $f : V \to W$ of normal $k$-varieties that satisfies the following properties:

1. $V$ is a 4-dimensional $\mathbb{Q}$-factorial klt variety over $k$,
2. $W$ is a 3-dimensional normal $\mathbb{Q}$-factorial variety over $k$ which is not klt,
3. $f_*\mathcal{O}_V = \mathcal{O}_W$, $\rho(V/W) = 1$, $-K_V$ is $f$-ample,
4. any fibre of $f$ is an irreducible scheme of dimension one, and there is a non-empty open subset $W^0$ of $W$ such that the fibre $V \times_W \text{Spec} k(w)$ is isomorphic to $\mathbb{P}^1_{k(w)}$ for any point $w \in W$, where $k(w)$ is the residue field at $w$.

One of prominent properties of Mori fibre spaces in characteristic zero is that any relatively numerically trivial Cartier divisor is trivial (cf. [KMM87, Lemma 3-2-5(2)]). We construct an example in positive characteristic which violates this property.

**Theorem 1.2.** Let $k$ be an algebraically closed field whose characteristic $p$ is two or three. Then there exists a projective $k$-morphism $f : V \to W$ of normal $k$-varieties that satisfies the following properties:

1. $V$ is a 3-dimensional $\mathbb{Q}$-factorial klt variety over $k$,
2. $W$ is a smooth curve over $k$,
3. $f_*\mathcal{O}_V = \mathcal{O}_W$, $\rho(V/W) = 1$, $-K_V$ is $f$-ample, and
4. there is a Cartier divisor $D$ on $V$ such that $D \not\sim_f 0$ and $pD \sim_f 0$.

**Remark 1.3.** Since [KMM87, Lemma 3-2-5(2)] is a formal consequence of the relative Kawamata–Shokurov base point free theorem [KMM87, Theorem 3-1-1], the same statement as in [KMM87, Theorem 3-1-1] does not hold in positive characteristic.

1.1. **Construction of examples.** Let us overview how to construct the examples appearing in Theorem 1.1 and Theorem 1.2.
1.1.1. **Pathological surfaces over imperfect fields.** To find examples appearing in Theorem 1.1 and Theorem 1.2 we start with log del Pezzo surfaces over imperfect fields satisfying pathological properties as follows.

**Theorem 1.4.** Let \( k \) be an imperfect field whose characteristic \( p \) is two or three. Then there exists a \( k \)-morphism \( \rho : S \to C \) which satisfies the following properties.

1. \( S \) is a projective regular surface over \( k \) and there is an effective \( \mathbb{Q} \)-divisor \( \Delta_S \) such that \( (S, \Delta_S) \) is klt and \( -(K_S + \Delta_S) \) is ample,
2. \( C \) is a projective regular curve over \( k \) with \( K_C \sim 0 \),
3. \( \rho \) is a \( \mathbb{P}^1 \)-bundle, and
4. there is a Cartier divisor \( L \) on \( C \) such that \( L \not\sim 0 \) and \( pL \sim 0 \).

The surface \( S \) in Theorem 1.4 is a log Fano variety dominating a Calabi–Yau variety. Such an example does not exist in characteristic zero (cf. [PS09, Lemma 2.8], [FG12, Theorem 5.1]). For some related results in positive characteristic, we refer to [Eji].

Let us overview the construction of \( \rho : S \to C \) appearing in Theorem 1.4. We take a regular cubic curve \( C \) that is not smooth and has a \( k \)-rational point \( P \) around which \( C \) is smooth over \( k \). For example, if \( k \) is the function field of a curve over an algebraically closed field, then \( C \) is nothing but the generic fibre of a quasi-elliptic fibration equipped with a section. Since we have that \( H^1(C, \mathcal{O}_C(-P)) \neq 0 \) by Serre duality, a nonzero element \( \xi \) of \( H^1(C, \mathcal{O}_C(-P)) \) induces a locally free sheaf \( E \) of rank two. Then \( S \) is the \( \mathbb{P}^1 \)-bundle defined to be \( \mathbb{P}(E) \). In order to show that \( S \) is log del Pezzo, one of the essential facts is that we can find a purely inseparable field extension \( k \subset k' \) of degree \( p \) such that \( C \times_k k' \) is an integral but non-normal scheme and that its normalisation is isomorphic to \( \mathbb{P}_k^1 \). Since \( \varphi^*P \) is a \( k' \)-rational point, we have that

\[
H^1(\mathbb{P}^1_{k'}, \mathcal{O}_{\mathbb{P}^1_{k'}}(-\varphi^*P)) = H^1(\mathbb{P}^1_{k'}, \mathcal{O}_{\mathbb{P}^1_{k'}}(-1)) = 0.
\]

This implies that the pull-back \( \varphi^*\xi \) is zero. This property plays a crucial role in our construction. For more details, see Section 3.

1.1.2. **Proofs of the theorems.** Let us overview some of the ideas of the proofs of Theorem 1.1 and Theorem 1.2.

First let us treat the latter one: Theorem 1.2. This is a consequence of Theorem 1.4. Indeed, for an algebraically closed field \( k \) whose characteristic is two or three, it follows from Theorem 1.4 that we get a log del Pezzo surface \( (S, \Delta_S) \) over \( k(t) \) which has a non-trivial \( p \)-torsion. Then we can spread it out over some non-empty open subset \( W \) of \( \mathbb{A}^1_k \), i.e. there is a morphism \( V \to W \) with \( V \times_W \text{Spec} k(t) = S \). Although
this example does not satisfy the property $\rho(V/W) = 1$, we may assume it by replacing $S$ in advance with its appropriate birational contraction. For more details, see Subsection 4.1.

Second, let us overview the proof of Theorem 1.1. To this end, we first find a similar example over imperfect fields (cf. Theorem 4.4). The basic idea is taking cones of $\rho : S \to C$. However, there is no morphism between cones. What we will actually do is to take $\mathbb{P}^1$-bundles functorially for an ample divisor $M_C$ on $C$:

$$X := \mathbb{P}_S(\mathcal{O} \oplus \mathcal{O}(\rho^*M_C)) \to \mathbb{P}_C(\mathcal{O} \oplus \mathcal{O}(M_C)) =: W_0.$$ 

Let $W_0 \to W$ be the birational contraction of the section $C^-$ of $W_0 \to C$ with negative self-intersection number. Since $K_C \sim 0$, $W$ is not klt.

If there was a divisorial contraction whose exceptional locus is the pull-back of $C^-$, then the resulted variety would be what we are looking for. Although we can not hope this, we will get close to this situation by running a suitable minimal model programme. To this end, we first construct a minimal model programme after taking a purely inseparable cover of $X$, and descends it to $X$ after that. For more details, see Subsection 4.2.

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2. Preliminaries

2.1. Notation. In this subsection, we summarise notation used in this paper.

- We will freely use the notation and terminology in [Har77] and [Kol13].
- For a scheme $X$, its *reduced structure* $X_{\text{red}}$ is the reduced closed subscheme of $X$ such that the induced morphism $X_{\text{red}} \to X$ is surjective.
- For an integral scheme $X$, we define the *function field* $K(X)$ of $X$ to be $\mathcal{O}_{X,\xi}$ for the generic point $\xi$ of $X$.
- For a field $k$, we say $X$ is a *variety over $k$* or a *$k$-variety* if $X$ is an integral scheme that is separated and of finite type over $k$. We say $X$ is a *curve over $k$* or a *$k$-curve* (resp. a *surface over $k$* or a *$k$-surface*, resp. a *threefold over $k$*) if $X$ is a $k$-variety of dimension one (resp. two, resp. three).
- We say that two schemes $X$ and $Y$ over a field $k$ are *$k$-isomorphic* if there exists an isomorphism $\theta : X \to Y$ of schemes such
that both $\theta$ and $\theta^{-1}$ commute with the structure morphisms: $X \to \text{Spec } k$ and $Y \to \text{Spec } k$.

**Definition 2.1.** Let $k$ be a field.

(1) Let $C$ be a proper curve over $k$. Let $M$ be an invertible sheaf on $C$. It is well-known that

$$
\chi(C, mM) = \dim_k(H^0(C, mM)) - \dim_k(H^1(C, mM)) \in \mathbb{Z}[m]
$$

and that the degree of this polynomial is at most one (cf. [Kle66, Ch I, Section 1, Theorem in page 295]). We define the degree of $L$ over $k$, denoted by $\deg_k M$ or $\deg M$, to be the coefficient of $m$.

(2) Let $X$ be a separated scheme of finite type over $k$, let $L$ be an invertible sheaf on $X$, and let $C \hookrightarrow X$ be a closed immersion over $k$ from a proper $k$-curve $C$. We define the intersection number over $k$, denoted by $L \cdot_k C$ or $L \cdot C$, to be $\deg_k (L|_C)$.

2.2. Some properties extensible from the generic fibre. In this subsection, we summarise some properties extensible from the generic fibre: Lemma 2.2. Also, we give a criterion to be of generically relative Picard number one (Lemma 2.5). To this end, we establish two auxiliary lemmas: Lemma 2.3 and Lemma 2.4.

**Lemma 2.2.** Let $k$ be a field. Let $f : X \to Y$ be a projective $k$-morphism of normal $k$-varieties with $f_* \mathcal{O}_X = \mathcal{O}_Y$.

(1) Assume that $k$ is algebraically closed. Then the generic fibre $X_{K(Y)}$ is $\mathbb{Q}$-factorial if and only if there is a non-empty open subset $Y'$ of $Y$ such that $X \times_Y Y'$ is $\mathbb{Q}$-factorial.

(2) Let $\Delta$ be an effective $\mathbb{Q}$-divisor such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Assume that there is a log resolution of $(X_{K(Y)}, \Delta|_{X_{K(Y)}})$. Then $(X_{K(Y)}, \Delta|_{X_{K(Y)}})$ is klt (resp. log canonical) if and only if there is a non-empty open subset $Y'$ of $Y$ such that $(X \times_Y Y', \Delta|_{X \times_Y Y'})$ is klt (resp. log canonical).

**Proof.** The assertion (1) holds by [BGS, the third Theorem in Introduction]. The assertion (2) follows from a straightforward argument. □

**Lemma 2.3.** Let $k$ be a field. Let $f : X \to Y$ be a projective $k$-morphism of normal $k$-varieties. Assume that the generic fibre $X_{K(Y)}$ is $K(Y)$-isomorphic to $\mathbb{P}^n_{K(Y)}$ for some non-negative integer $n$. Then there exists a non-empty open subset $Y'$ of $Y$ such that the fibre $X_y$ is $k(y)$-isomorphic to $\mathbb{P}^n_{k(y)}$ for any point $y \in Y'$.

**Proof.** Replacing $Y$ by a non-empty open subset, we may assume that the following properties hold:
(1) $f$ is a smooth morphism and $f_*\mathcal{O}_X = \mathcal{O}_Y$,
(2) $-K_X$ is $f$-ample,
(3) the tangent bundle $T_{X_y}$ is ample for any $y \in Y$ (cf. [Laz04, Proposition 6.1.9]), and
(4) there is a section of $f$, i.e. there exists a closed immersion $j : Y_1 \to X$ such that the composite morphism $Y_1 \to X \to Y$ is an isomorphism.

Fix $y \in Y$ and let $X_{\overline{k(y)}}$ be the base change of the fibre $X_y$ to its algebraic closure $\overline{k(y)}$. Since $X_{\overline{k(y)}}$ is a smooth projective variety whose tangent bundle $T_{X_{\overline{k(y)}}}$ is ample, we have that $X_{\overline{k(y)}}$ is $\overline{k(y)}$-isomorphic to $\mathbb{P}^n_{\overline{k(y)}}$ by [Mor79, Theorem 8]. This implies that $X_y$ is a Severi–Brauer variety. Since $X_y$ has a $\overline{k(y)}$-rational point by (4), we have that $X_y$ is $\overline{k(y)}$-isomorphic to $\mathbb{P}^n_{\overline{k(y)}}$ by [GS06, Theorem 5.1.3].

**Lemma 2.4.** Let $k$ be a field of characteristic $p > 0$. Consider a commutative diagram of projective $k$-morphisms of normal $k$-varieties

$$
\begin{array}{ccc}
X' & \xrightarrow{\alpha} & X \\
\\
\downarrow{f'} & & \downarrow{f} \\
Y' & \xrightarrow{\beta} & Y,
\end{array}
$$

where $\alpha$ and $\beta$ are finite universal homeomorphisms. Then $\rho(X/Y) = \rho(X'/Y').$

**Proof.** Since there is a positive integer $e$ such that the $e$-th iterated absolute Frobenius morphisms $F^e_X : X \to X$ and $F^e_Y : Y \to Y$ factor through $\alpha$ and $\beta$ respectively:

$$
F^e_X : X \xrightarrow{\tilde{\alpha}} X' \xrightarrow{\alpha} X, \quad F^e_Y : Y \xrightarrow{\tilde{\beta}} Y' \xrightarrow{\beta} Y,
$$

we have that $\rho(X/Y) \leq \rho(X'/Y')$. The opposite inequality follows from the fact that the $e$-th iterated absolute Frobenius morphisms $F^e_{X'} : X' \to X'$ and $F^e_{Y'} : Y' \to Y'$ factor through $\tilde{\alpha}$ and $\tilde{\beta}$, respectively. $
$

**Lemma 2.5.** Let $k$ be a field of characteristic $p > 0$. Let $f : X \to Y$ be a projective $k$-morphism of normal $k$-varieties with $f_*\mathcal{O}_X = \mathcal{O}_Y$.

Assume that there exists a finite universal homeomorphism $\varphi : \mathbb{P}^n_L \to X_{K(Y)}$ over $K(Y)$ for some finite purely inseparable extension $K(Y) \subset L$. Then there exists a non-empty open subset $Y'$ of $Y$ such that $\rho(X'/Y') = 1$ for $X' := X \times_Y Y'$. 

Proof. After shrinking $Y$, we can find a commutative diagram of projective $k$-morphisms of normal $k$-varieties

$$
\begin{array}{ccc}
X' & \xrightarrow{\alpha} & X \\
\downarrow f' & & \downarrow f \\
Y' & \xrightarrow{\beta} & Y,
\end{array}
$$

where $\alpha$ and $\beta$ are finite universal homeomorphisms, $K(Y') = L$ and the generic fibre of $f'$ is $K(Y')$-isomorphic to $\mathbb{P}^n_{K(Y')}$. Then the assertion follows from Lemma 2.3 and Lemma 2.4.

2.3. Varieties of Fano type. In this subsection, we recall the definition of varieties of Fano type and one of basic properties (Lemma 2.7).

Definition 2.6. Let $k$ be a field. We say a projective normal $k$-variety $X$ is of Fano type if there is an effective $\mathbb{Q}$-divisor $\Delta$ such that $(X, \Delta)$ is klt and $-(K_X + \Delta)$ is ample. In this case, we say $(X, \Delta)$ is log Fano. We say $(X, \Delta)$ is log del Pezzo if $X$ is log Fano and $\dim X = 2$.

Lemma 2.7. Let $k$ be a field of characteristic $p > 0$. Let $X$ and $Y$ be projective normal varieties over $k$. Assume that a rational map $f : X \dashrightarrow Y$ over $k$ satisfies one of the following properties.

(1) $f$ is a birational morphism.
(2) $f$ is a birational map which is an isomorphism in codimension one.

If $X$ is of Fano type, $[k:k^p] < \infty$ and $\dim X \leq 3$, then $Y$ is of Fano type.

Proof. For both the cases, we can apply the same argument as in [Bir16, Lemma 2.4]. However, we give a proof only for the case (1) since our setting differs from [Bir16, Lemma 2.4]. Thanks to the assumptions $[k:k^p] < \infty$ and $\dim X \leq 3$, we may freely use log resolutions by [CP08, CP09].

Since $X$ is of Fano type, we can find an effective $\mathbb{Q}$-divisor $\Delta$ on $X$ such that $(X, \Delta)$ is klt and $-(K_X + \Delta)$ is ample. By taking a log resolution of $(X, \Delta)$, we can find an effective $\mathbb{Q}$-divisor $A_X$ such that $-(K_X + \Delta) \sim_{\mathbb{Q}} A_X$ and $(X, \Delta + A_X)$ is klt. Taking the push-forward by $f$, we have that

$$-(K_Y + f_*\Delta + f_*A_X) \sim_{\mathbb{Q}} 0.$$

We have that $(Y, f_*\Delta + f_*A_X)$ is klt. Since $f_*A_X$ is big and log resolutions exist, $Y$ is of Fano type.
Remark 2.8. The assumptions $|k : k^p| < \infty$ and $\dim X \leq 3$ in Lemma 2.7 is used only to assure the existence of log resolutions \cite{CP08, CP09}.

2.4. Jacobian criterion for regularity. For a later use, we summarise results for regularity of some explicit varieties that follow from Jacobian criterion.

Lemma 2.9. Let $k$ be a field of characteristic $p > 0$. Take elements $s, t \in k \setminus k^p$. Then the following hold.

(1) If $p = 2$, then $\text{Spec } k[x, y]/(x^2 + ty^2)$ is regular outside the origin $(0, 0)$.
(2) If $p = 2$, then $k[x, y]/(tx^2 + 1)$ is regular.
(3) If $p = 2$ and $[k(s^{1/2}, t^{1/2}) : k] = 4$, then $k[x, y]/(sx^2 + ty^2 + 1)$ is regular.
(4) If $p = 2$, then $\text{Proj } k[x, y, z]/(y^2z + x^3 + sxz^2)$ is regular.
(5) If $p = 3$, then $\text{Proj } k[x, y, z]/(y^2z + x^3 + sz^3)$ is regular.

Proof. Since all the proofs are quite similar, we only prove (1). We consider the following open subset of $\text{Spec } k[x, y]/(x^2 + ty^2)$:

$$D(y) = \text{Spec } k[x, y, z]/(x^2 + ty^2, zy + 1) \simeq \text{Spec } k[x, y, y^{-1}]/(x^2y^{-2} + t).$$

We can find an $F_2$-derivation $D_1$ of $k[x, y, z]$ with $D_1(t) = 1$ by $t \not\in k^2$ and \cite{Mat89} Theorem 26.5. We have that the ring $k[x, y, z]/(x^2 + ty^2, zy + 1)$ is regular by applying the Jacobian criterion \cite{Gro66} Proposition 22.6.7(iii) for $k_0 := F_2$, $B = k[x, y, z]$, $\mathfrak{q}$ is a prime ideal of $B$ containing $(x^2 + ty^2, zy + 1)$, $f_1 := x^2 + ty^2$, and the $F_2$-derivation $D_1$ defined above. It follows from the same argument that also the open subset $D(x)$ of $\text{Spec } k[x, y]/(x^2 + ty^2)$ is regular. To summarise, $\text{Spec } k[x, y]/(x^2 + ty^2)$ is regular outside the origin $(0, 0)$.

\section{2.5. Slc-ness of conics.} The purpose of this subsection is to show that any plane conic curve is semi log canonical (Lemma 2.11). We start with a typical case of characteristic two.

Lemma 2.10. Let $k$ be a field of characteristic two with an element $t \in k$ such that $t \not\in k^2$. Let

$$Z := \text{Spec } k[x, y]/(x^2 + ty^2).$$

Then $Z$ is a semi log canonical curve.

Proof. By Lemma 2.9(1), $Z$ is regular outside the origin. By using the assumption: $t \not\in k^2$, we can directly check that $Z$ has a node at the origin (cf. \cite{Kol13} 1.41). It follows from \cite{Tan16} Proposition 3.6 that $Z$ is semi log canonical.
Lemma 2.11. Let $k$ be a field and let $Z = \text{Spec } k[x, y]/(f)$, where $Z$ is reduced and $f$ is a polynomial of degree two. Then $Z$ is semi log canonical.

Proof. We assume that the characteristic of $k$ is two, since otherwise the problem is easier. We may assume that $k$ is separably closed.

We can write

$$f = a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + a_{00}$$

where $a_{ij} \in k$.

Assume that $a_{11} \neq 0$. Since $k$ is separably closed, we can write $a_{20}x^2 + a_{11}xy + a_{02}y^2 = l_1l_2$ for some homogeneous polynomials $l_1$ and $l_2$ of degree one with $(l_1, l_2) = (x, y)$. Then it is easy to check that $Z$ is semi log canonical. Thus we may assume that $a_{11} = 0$.

Assume that $a_{10} \neq 0$. Taking a suitable linear transform, we may assume that $a_{00} = 0$. We now have $f = a_{20}x^2 + a_{02}y^2 + a_{10}x + a_{01}y$. Since $a_{10} \neq 0$, we have that $Z \simeq k[x, y]/(x + bx^2 + cy^2)$ for some $b, c \in k$. It follows from the Jacobian criterion for smoothness that $Z$ is smooth.

Thus we may assume that $a_{11} = a_{10} = a_{01} = 0$. If $a_{00} = 0$, then we may assume that $a_{20} = 1$ by the symmetry, i.e. $f = x^2 + a_{02}y^2$. Since $Z$ is reduced, we have that $a_{02} \notin k^2$. Thus $Z$ is semi log canonical by Lemma 2.10.

Thus we may assume that $a_{00} \neq 0$. Replacing $f$ with $f/a_{00}$ we get

$$f = a_{20}x^2 + a_{02}y^2 + 1.$$ 

Since $f$ is reduced, we may assume that $a_{20} \notin k^2$, hence $[k(a_{20}^{1/2}) : k] = 2$. This implies that $[k(a_{20}^{1/2}, a_{02}^{1/2}) : k]$ is equal to either 4 or 2. If $[k(a_{20}, a_{02}^{1/2}) : k] = 4$, then $Z$ is regular by Lemma 2.9.

Thus we may assume that $[k(a_{20}^{1/2}, a_{02}^{1/2}) : k] = 2$. We have that $a_{02} \in k^2(a_{20}) = k^2 \oplus k^2a_{20}$, hence $a_{02} = b^2 + c^2a_{20}$ for some $b, c \in k$. We get

$$f = a_{20}x^2 + a_{02}y^2 + 1 = a_{20}(x + cy)^2 + (by + 1)^2.$$ 

After replacing $x + cy$ by $x$, we can write

$$f = a_{20}x^2 + (by + 1)^2.$$ 

If $b = 0$, then $Z$ is regular by Lemma 2.9. If $b \neq 0$, then we get $k[x, y]/(f) \simeq k[x, y]/(a_{20}x^2 + y^2)$. It follows from Lemma 2.10 that $Z$ is semi log canonical. \qed
3. Pathological surfaces over imperfect fields

3.1. Construction in a general setting. In this subsection, we give a criterion to find a log Fano variety dominating a Calabi–Yau variety (Proposition 3.5). Although our construction is analogous to a standard one over an algebraically closed field (cf. [Muk13]), we give details of them because our setting is more general and our base field is not necessarily algebraically closed.

Notation 3.1. Let \( k \) be a field of characteristic \( p > 0 \). Assume that there exist a \( k \)-morphism \( \varphi : C' \to C \) of regular projective \( k \)-varieties and a Cartier divisor \( D \) on \( C \) which satisfy the following properties.

1. \( \varphi \) is a finite universal homeomorphism of degree \( p \).
2. There is a nonzero element \( \xi \in H^1(C, \mathcal{O}_C(D)) \) whose pull-back \( \varphi^*(\xi) \in H^1(C', \mathcal{O}_{C'}(\varphi^*D)) \) is zero.

The element \( \xi \) induces a locally free sheaf \( E \) of rank two on \( C \) equipped with the following exact sequence that does not split:

\[
0 \to \mathcal{O}_C(D) \xrightarrow{\alpha} E \xrightarrow{\beta} \mathcal{O}_C \to 0.
\]

By our assumption (2), the pull-back of this sequence to \( C' \) splits: 
\( \varphi^*E \simeq \mathcal{O}_{C'} \oplus \mathcal{O}_{C'}(\varphi^*D) \). We set

\[
S := \mathbb{P}_C(E), \quad S' := \mathbb{P}_{C'}(\varphi^*E)
\]

and obtain a cartesian diagram:

\[
\begin{array}{ccc}
S' & \xrightarrow{\psi} & S \\
\downarrow{\rho'} & & \downarrow{\rho} \\
C' & \xrightarrow{\varphi} & C.
\end{array}
\]

The surjection \( \beta : E \to \mathcal{O}_C \) in (3.1.1) induces a section \( C_1 \) of \( \rho \). We set \( C'_1 := \psi^*C_1 \) which is a section of \( \rho' \). We have another section \( C'_2 \) of \( \rho' \) corresponding to the surjection:

\[
\varphi^*E \simeq \mathcal{O}_{C'} \oplus \mathcal{O}_{C'}(\varphi^*D) \to \mathcal{O}_{C'}(\varphi^*D),
\]

where the latter homomorphism is the natural projection. We set \( C_2 \) to be the reduced closed subscheme of \( X \) which is set-theoretically equal to \( \psi(C'_2) \).

Lemma 3.2. \( C_2 \) is an integral scheme and the induced morphism \( \rho_{C_2} : C_2 \to C \) is a finite universal homeomorphism of degree \( p \).

Proof. Since \( \psi \) is a universal homeomorphism, we have that \( C_2 \) is an integral scheme. Since the induced composite morphism

\[
C'_2 \xrightarrow{\psi} C_2 \xrightarrow{\rho_{C_2}} C
\]

is a finite universal homeomorphism of degree \( p \), we conclude that \( C_2 \) is an integral scheme and the induced morphism \( \rho_{C_2} : C_2 \to C \) is a finite universal homeomorphism of degree \( p \).
is a finite universal homeomorphism of degree $p$, the degree of the latter morphism $\rho_{C_2} : C_2 \to C$ is equal to either 1 or $p$. It suffices to show that the latter case actually happens.

Assuming that $\rho_{C_2} : C_2 \to C$ is of degree one i.e. birational, let us derive a contradiction. Since $\rho_{C_2} : C_2 \to C$ is a finite birational morphism and $C$ is normal, it follows that $\rho_{C_2}$ is an isomorphism. Thus $C_2$ is a section of $\rho$, hence it is corresponding to a surjective $O_C$-module homomorphism

$$\gamma : E \to O_C(\tilde{D})$$

for some Cartier divisor $\tilde{D}$ on $C$. Since $C_2' = C_2 \times_S S'$, we have that the pull-back $\varphi^*(\gamma \circ \alpha)$ is an isomorphism. It follows from the faithfully flatness of $\varphi$ that $\gamma \circ \alpha$ is an isomorphism. This implies that the sequence (3.1.1) splits, which is a contradiction.

\[\square\]

**Lemma 3.3.** The following hold:

1. $O_{S'}(C_2')|_{C_2} \simeq O_{C'}(\varphi^*D)$.
2. $O_{S'}(C_1')|_{C_1} \simeq O_{C'}(-\varphi^*D)$.

**Proof.** We can apply the same argument as in [Har77, Ch. V, Proposition 2.6]. \[\square\]

**Lemma 3.4.** The following $\mathbb{Q}$-linear equivalence holds:

$$-K_S \overset{\mathbb{Q}}{\sim} \frac{2}{p} C_2 + \rho^*(-D - K_C).$$

**Proof.** Since the induced morphism $\rho_{C_2} : C_2 \to C$ is of degree $p$ by Lemma 3.2, we have that

(3.4.1) \[
K_{S/C} + \frac{2}{p} C_2 \overset{\mathbb{Q}}{\sim} \rho^*(L)
\]

for some $\mathbb{Q}$-divisor $L$ on $C$. Taking the pull-back $\psi^*$ of (3.4.1), we get

(3.4.2) \[
K_{S'/C'} + \frac{2}{p} \psi^*C_2 \overset{\mathbb{Q}}{\sim} \rho'^*\varphi^*(L).
\]

Since $K_{S'} + C'_1 + C'_2 \sim \rho'^*K_{C'}$, we have that

(3.4.3) \[
-C'_1 - C'_2 + \frac{2}{p} \psi^*C_2 \overset{\mathbb{Q}}{\sim} \rho'^*\varphi^*(L).
\]

Restricting (3.4.3) to $C'_1$, we have that $\varphi^*D \overset{\mathbb{Q}}{\sim} -C'_1|_{C_1'} \overset{\mathbb{Q}}{\sim} \varphi^*(L)$ by Lemma 3.3 if we identify $C$ with $C_2$. Since the absolute Frobenius morphism $C \to C$ factors through $\varphi : C' \to C$, we get the $\mathbb{Q}$-linear equivalence

(3.4.4) \[
D \overset{\mathbb{Q}}{\sim} L.
\]
Substituting (3.4.4) for (3.4.1), we get
\[-K_S \sim_{\mathbb{Q}} 2 \frac{p}{C_2} + \rho^*(-L - K_C) \sim_{\mathbb{Q}} 2 \frac{p}{C_2} + \rho^*(-D - K_C),\]
as desired. □

**Proposition 3.5.** If \(-D - K_C\) is ample and \((S, \frac{2}{p}C_2)\) is log canonical, then there exists an effective \(\mathbb{Q}\)-divisor \(\Delta\) on \(S\) such that \((S, \Delta)\) is klt and \(-(K_S + \Delta)\) is ample.

**Proof.** Since \(-D - K_C\) is ample, so is \(-(K_S + \left(\frac{2}{p} - \epsilon\right)C_2)\) \(\sim_{\mathbb{Q}} \epsilon C_2 + \pi^*(-D - K_C)\) for some rational number \(\epsilon\) with \(0 < \epsilon < \frac{2}{p}\), where the \(\mathbb{Q}\)-linear equivalence follows from Lemma 3.4. Set \(\Delta := \left(\frac{2}{p} - \epsilon\right)C_2\). Since \(S\) is regular and \((S, \frac{2}{p}C_2)\) is log canonical, we have that \((S, \Delta)\) is klt. □

### 3.2. Non-smooth K-trivial curves.

We summarise the properties of \(K\)-trivial curves, which we will need later.

**Proposition 3.6.** Let \(k\) be an imperfect field whose characteristic \(p\) is two or three. Then there exists a projective regular curve \(C\) over \(k\) which satisfies the following properties.

1. \(K_C \sim 0\),
2. the number of the \(k\)-rational points of \(C\) is at least three,
3. there is a purely inseparable field extension \(k \subset k'\) of degree \(p\) such that \(C \times_k k'\) is an integral scheme which has a unique non-regular point \(Q\),
4. the normalisation \(C'\) of \(C \times_k k'\) is \(k'\)-isomorphic to \(\mathbb{P}^1_{k'}\), and
5. there is a Cartier divisor \(L\) on \(C\) such that \(L \not\sim 0\) and \(pL \sim 0\).

**Proof.** Since \(k\) is not perfect, we can find an element \(t \in k\) with \(t \not\in k^p\).

First we treat the case where \(p = 2\). Consider the following equation, which is taken from [Ito94, Table 1 in page 243]:
\[C := \text{Proj} k[x, y, z]/(y^2z + x^3 + (t^3 + t)zx^2).\]
We have that \(C\) is regular by Lemma 2.9. By the adjunction formula, (1) holds. The assertion (2) holds since all of \([0 : 1 : 0]\), \([0 : 0 : 1]\) and \([t + 1 : t^2 + 1 : 1]\) are \(k\)-rational points on \(C\). Let \(k' := k(\sqrt{t^3 + t})\). The Jacobian criterion for smoothness implies that \(C \times_k k' \setminus Q\) is smooth over \(k'\), where \(Q := [\sqrt{t^3 + t} : 0 : 1]\). We can check that for the open set
\[\text{Spec} k'[x, y]/(y^2 + x^3 + (t^3 + t)x)\]
of $C \times_k k'$, its normalisation is isomorphic to $\mathbb{A}^1_{k'}$. Indeed, the integral closure of $k'[x, y]/(y^2 + x^3 + (t^3 + t)x) = k[\overline{x}, \overline{y}]$ is equal to $k'[\overline{y}/(\sqrt[3]{t^3 + t})_x]$, where $\overline{x}$ and $\overline{y}$ are the images of $x$ and $y$, respectively. Thus (3) and (4) hold. The assertion (5) holds by setting $L := P_1 - P_2$, where $P_1$ and $P_2$ are $k$-rational points around which $C$ is smooth.

Second we assume that $p = 3$. Let

$$C := \text{Proj } k[x, y, z]/(y^2z + x^3 + t^2z^3).$$

All of $[0 : 1 : 0], [0 : t : 1]$ and $[0 : -t : 1]$ are $k$-rational points on $C$. By Lemma 2.9, $C$ is regular. We omit the remaining proof since it is similar to but easier than the one for the case where $p = 2$. □

### 3.3. Log del Pezzo surfaces.

**Notation 3.7.** Let $k$ be an imperfect field whose characteristic $p$ is two or three. Let $C$ be a projective regular curve over $k$ as in Proposition 3.6. By Proposition 3.6(3)(4), there is a purely inseparable field extension $k \subset k'$ of degree $p$ such that $C \times_k k'$ is integral and its normalisation $C'$ of $C \times_k k'$ is $k'$-isomorphic to $\mathbb{P}^1_{k'}$:

$$\varphi : C' \to C \times_k k' \to C.$$

In particular, $\varphi$ is a finite universal homeomorphism of degree $p$. By Proposition 3.6(2)(3)(4), we can find a $k$-rational point $P$ around which $C$ is smooth over $k$. We set $D := -P$ and let $\xi \in H^0(C, \mathcal{O}_C(D))$ be a nonzero element whose existence guaranteed by Serre duality. Since $C' \simeq \mathbb{P}^1_{k'}$ and $P' := \varphi^*P$ is a $k'$-rational point, we have that $H^1(C', \mathcal{O}_{C'}(\varphi^*D)) = 0$ by Serre duality. Therefore, we can apply the construction as in Subsection 3.1 (cf. Notation 3.1). Then we obtain a cartesian diagram of regular projective $k$-varieties:

$$\begin{array}{ccc}
S' & \xrightarrow{\psi} & S \\
\downarrow \varphi' & & \downarrow \varphi \\
C' & \xrightarrow{\varphi} & C.
\end{array}$$

**Lemma 3.8.** We use Notation 3.7. Then the following hold.

1. If $p = 2$, then $C_2$ is $k$-isomorphic to a conic curve in $\mathbb{P}^2_k$ or $\mathbb{P}^2_{k'}$.
2. If $p = 3$, then $C_2$ is $k'$-isomorphic to $\mathbb{P}^1_{k'}$.

**Proof.** We have that

$$(K_{X/C} + C_2) \cdot_k C_2 = \deg_k(\omega_{C_2}) - \deg_k((\pi|_{C_2})^*\omega_C) = \deg_k(\omega_{C_2})$$

and

$$\psi^*(K_{X/C} + C_2) \cdot_k C_2' = (K_{X'/C'} + pC_2') \cdot_k C_2' = (p - 1)C_2' \cdot_k C_2' = (p - 1)\deg_k(\varphi^*D) = -p(p - 1).$$
Since $\psi|_{C'_2} : C'_2 \to C_2$ is birational, it follows that
\[ \deg_k(\omega_{C'_2}) = (K_{X/C} + C_2) \cdot k C_2 = \psi^*(K_{X/C} + C_2) \cdot k C'_2 = -p(p - 1). \]
By [Kol13] Lemma 10.6, we have that $\deg_K(\omega_{C_2}) = -2$ for $K = H^0(C_2, \mathcal{O}_{C_2})$. The natural morphisms $C' \simeq C'_2 \to C_2 \to C$ induce field extensions
\[ k = H^0(C, \mathcal{O}_C) \subset H^0(C_2, \mathcal{O}_{C_2}) \subset H^0(C'_2, \mathcal{O}_{C'_2}) = k', \]
which implies that $H^0(C_2, \mathcal{O}_{C_2})$ is either $k$ or $k'$. Thus the assertion (1) follows from [Kol13] Lemma 10.6.

We show (2). Since $p = 3$, we get $\deg_k(\omega_{C_2}) = -6$. Thus we have that $k' = H^0(C_2, \mathcal{O}_{C_2})$ and $\deg_{k'}(\omega_{C_2}) = -2$. Since $\psi|_{C'_2} : C'_2 \to C_2$ is birational and $\deg_{k'}(\omega_{C_2}) = \deg_{k'}(\omega_{C_2})$, $\psi|_{C'_2}$ is an isomorphism, hence (2) hold.

**Theorem 3.9.** We use Notation 3.7. Then there exists an effective $\mathbb{Q}$-divisor $\Delta_S$ on $S$ such that $(S, \Delta_S)$ is klt and $-(K_S + \Delta_S)$ is ample.

**Proof.** By Proposition 3.5, it suffices to show that $-D - K_C$ is ample and $(S, \frac{2}{3}C_2)$ is log canonical. The ampleness of $-D - K_C$ follows from $D = -\tilde{P}$ and $K_C \sim 0$. If $p = 3$, then it follows from Lemma 3.8 that $(S, \frac{2}{3}C_2)$ is log canonical. If $p = 2$, then $C_2$ is semi log canonical by Lemma 2.11 and Lemma 3.8. Therefore, $(S, C_2)$ is log canonical by inversion of adjunction (cf. [Tanb] Theorem 5.1). \qed

**Theorem 3.10.** Let $k$ be an imperfect field whose characteristic $p$ is two or three. Then there exists a projective $\mathbb{Q}$-factorial klt surface $T$ over $k$ with $k = H^0(T, \mathcal{O}_T)$ which satisfies the following properties.

1. $-K_T$ is ample,
2. $\rho(T) = 1$,
3. there is a Cartier divisor $M$ such that $M \not= 0$ and $pM \sim 0$, and
4. there exists a finite universal homeomorphism $\mathbb{P}_{k'}^2 \to T$, where $k \subset k'$ is a purely inseparable extension of degree $p$.

**Proof.** We use Notation 3.7. There is a $\mathbb{P}^1$-bundle structure $\rho : S \to C$. Since $S' = \mathbb{P}_{\mathbb{P}_{k'}}(\mathcal{O} \oplus \mathcal{O}(1))$, we have the blow-down $f' : S' \to \mathbb{P}_{k'}^2 =: T'$ contracting $C'_2$. Thus, we get a commutative diagram
\[
\begin{array}{ccc}
S' & \xrightarrow{\psi} & S \\
\downarrow f' & & \downarrow f \\
T' & \xrightarrow{\psi_T} & T,
\end{array}
\]
where $\psi_T$ is a finite universal homeomorphism of degree $p$ and $f$ is the birational morphism to a projective normal surface $T$ satisfying
Thus (4) holds. Since \( f \) is corresponding to \( (K_S + \Delta_S) \)-negative extremal ray, we have that \( (T, f_*\Delta_S) \) is klt and \( T \) is \( \mathbb{Q} \)-factorial (cf. [Tanb, Theorem 4.3]). In particular, \( T \) is klt. By [Tanb, Theorem 4.3], the assertions (1) and (2) holds. We get (3) by Proposition 3.6(5) and [Tanb, Theorem 4.3].

\[ \square \]

### 4. Pathological Mori fibre spaces

#### 4.1. Mori fibre spaces with non-trivial torsion divisors.

*Proof of Theorem 1.2.* By Theorem 3.10, there exist a projective \( \mathbb{Q} \)-factorial klt surface \( T \) over \( k(t) \) with \( H^0(T, \mathcal{O}_T) = k(t) \) and a Cartier divisor \( M \) on \( T \) which satisfy the properties (1)–(4) in Theorem 3.10. We can find a projective morphism \( f : V \to W \) and a Cartier divisor \( D \) on \( V \), where \( W \) is a non-empty open subset \( W \) of \( \text{Spec} \, k[t] \), \( V \times W \) Spec \( k(t) = T \), and \( D|_T = M \). In particular, the property (2) in the statement holds. After possibly shrinking \( W \), we may assume that \( V \) is normal, \( f_*\mathcal{O}_V = \mathcal{O}_W \), \( pD \sim 0 \), and \( -K_V \) is an \( f \)-ample \( \mathbb{Q} \)-Cartier \( \mathbb{Q} \)-divisor. Since \( D|_T = M \), the property (4) in the statement holds. The property (1) (resp. (3)) in the statement holds by Lemma 2.2 (resp. Lemma 2.5). 

\[ \square \]

#### 4.2. Mori fibre spaces with non-klt bases.

The main purpose of this subsection is to show Theorem 4.4, since it directly implies one of our main results: Theorem 1.1. In (4.2.1), we summarise notation. In (4.2.2) and (4.2.3), we run a suitable minimal model programme which will be needed in the proof of Theorem 4.4. In (4.2.4), we prove Theorem 4.3 and Theorem 1.1.

### 4.2.1. Setup.

We use Notation 3.7. Assume that \([k : k^p] < \infty \). Let \( M_C := \mathcal{O}_C(P) \), where \( P \) is a \( k \)-rational point on \( C \) around which \( C \) is smooth over \( k \). Let \( M_S := \varphi^*M_C, M_{C'} := \varphi^*M_C, \) and \( M_{S'} := \psi^*\varphi^*M_C \).

We set

\[ X := \mathbb{P}(\mathcal{O}_S \oplus M_S), \quad R := \mathbb{P}(\mathcal{O}_C \oplus M_C) \]

\[ X' := \mathbb{P}(\mathcal{O}_{S'} \oplus M_{S'}), \quad R' := \mathbb{P}(\mathcal{O}_{C'} \oplus M_{C'}) \]

and obtain a cartesian diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{\rho_X} & R \\
\downarrow{\pi} & & \downarrow{\pi_R} \\
S & \xrightarrow{\rho} & C,
\end{array}
\]
whose base change by \((-) \times_C C'\) can be written by
\[
\begin{array}{ccc}
X' & \xrightarrow{\rho'_X} & R' \\
\downarrow \pi' & & \downarrow \pi'_R \\
S' & \xrightarrow{\rho'_1} & C'.
\end{array}
\]
Let \(C^\pm\) be the sections of \(\pi_R\) corresponding to the direct sum decomposition \(\mathcal{O}_C \oplus M_C\) such that \(\mathcal{O}_R(C^\pm)|_{C^\pm} = \pm M_C\) if we identify \(C\) with \(C^\pm\). We set \(S^\pm, C'^\pm\) and \(S'^\pm\) to be the pull-backs of \(C^\pm\) to \(X, R'\) and \(X'\), respectively.

Since \(R' \simeq \mathbb{P}_{k'}^2(\mathcal{O} \oplus \mathcal{O}(1))\), we have that \(C'^-\) is a \((-1)\)-curve on \(R'\), i.e. \(K_{R'} \cdot C'^- = C'^- \cdot k' = -1\). Let
\[
\theta' : R' \to \mathbb{P}_{k'}^2(= Q')
\]
be the blow-down contracting \(C'^-\).

Corresponding to \(\theta'\), we can find a birational morphism \(\theta : R \to Q\) to a projective surface \(Q\) such that \(\theta_* \mathcal{O}_R = \mathcal{O}_Q, \text{Ex}(\theta) = C^-\). Indeed, for a positive integer \(e\) such that the \(e\)-th iterated absolute Frobenius morphism \(F^e : R' \to R' =: \mathcal{O}^{(p^e)}\) factors through the induced morphism \(R \to R'\), we define \(Q\) to be the normalisation of \(\mathcal{O}^{(p^e)}\) in \(K(R)\), where \(\theta^{(p^e)} : R^{(p^e)} \to Q^{(p^e)}\) is defined to be the same morphism as \(\theta\).

Let \(q := \theta(\text{Ex}(\theta))\) and \(q' := \theta'(\text{Ex}(\theta'))\). In the proof of Theorem 4.4 we will run an \(S^-\)-MMP over \(Q\)
\[
X =: X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2,
\]
consisting of two steps: \(f_0\) is a flip and \(f_1\) is a divisorial contraction. To this end, we construct the corresponding \(S'^-\)-MMP in (4.2.2) and (4.2.3).

4.2.2. The first step: flip. We use the same notation as in (4.2.1). Let \(H_{X'}\) be an ample Cartier divisor on \(X'\) and we define \(\lambda' := \sup\{\lambda \in \mathbb{R}_{\geq 0} \mid H_{X'} + \lambda S'^- \text{ is nef}\}\). Set
\[
L' := H_{X'} + \lambda' S'^-.
\]
Since \(\mathcal{O}_{X'}(S'^-)|_{S'^-} = -M_{S'}\), we have that \(\lambda'\) is a positive rational number and \(L'|_{S'^-}\) is semi-ample. It follows from Keel’s theorem ([Kee99 Proposition 1.6]) that \(L'\) is semi-ample. Let
\[
g' : X' \to Z'
\]
be the birational contraction with \(g'_* \mathcal{O}_{X'} = \mathcal{O}_{Z'}\) induced by \(L'\). We have that \(\text{Ex}(g')\) is equal to the \((-1)\)-curve \(\Gamma'\) on \(S'^-\). In particular, \(g'\) is a small birational morphism.
We construct a flip of \( g' \). Let 
\[
h' : Y' \to X'
\]
be the blowup along \( \Gamma' \). We have that \( E' := \text{Ex}(h') \) is isomorphic to 
\[
\mathbb{P}_{\Gamma'}(N_{\Gamma'/X'})\text{, where } N_{\Gamma'/X'} \text{ is the normal bundle, hence it is an extension of } N_{S^-/X'|\Gamma'} \text{ and } N_{\Gamma'/S^-}. \]
Since 
\[
S'^- \cdot \Gamma' = -1, \quad (\Gamma' \text{ in } S'^-) \cdot \Gamma' \text{ in } S'^-) = -1,
\]
the locally free sheaf \( N_{\Gamma'/X'} \) is corresponding to an extension class \( \alpha \in \text{Ext}^1_{\mathbb{P}_k}(\mathcal{O}(-1), \mathcal{O}(-1)) = 0. \) Therefore, we get \( N_{\Gamma'/X'} \simeq \mathcal{O}_{\Gamma'}(-1) \oplus \mathcal{O}_{\Gamma'}(-1). \) It follows that \( E' \simeq \mathbb{P}^1_k \times_{k'} \mathbb{P}^1_{k'} \). Let \( T' \) be the proper transform of \( S'^- \) on \( Y' \).

**Lemma 4.1.** The following hold.

1. \( K_{X'} \cdot \Gamma' = 0. \)
2. \( \mathcal{O}_{Y'}(E')|_{E'} \simeq \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, -1) \) if we identify \( E' \) with \( \mathbb{P}^1_{k'} \times_{k'} \mathbb{P}^1_{k'} \).
3. \( \mathcal{O}_{Y'}(T')|_{E'} \simeq \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0, 1) \) if we identify \( E' \) and \( E' \to \Gamma' \) with \( \mathbb{P}^1_{k'} \times_{k'} \mathbb{P}^1_{k'} \) and its first projection, respectively.

**Proof.** The assertion (1) follows from \( S'^- \cdot \Gamma' \) \( \Gamma' = -1 \) and 
\[
(K_{X'} + S'^-) \cdot \Gamma' = K_{S'^-} \cdot \Gamma' = -1,
\]
where the latter equation follows from the fact that \( \Gamma' \) is a \((-1\)-curve).

We show (2). Since \( K_{X'} = h^*K_{X'} + E' \), we have that 
\[
\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-2, -2) \simeq \mathcal{O}_{Y'}(K_{Y'} + E')|_{E'} = \mathcal{O}_{Y'}(h^*K_{Y'} + 2E')|_{E'} \simeq \mathcal{O}_{Y'}(2E')|_{E'},
\]
where the last isomorphism holds because (1) implies \( K_{X'}|_{\Gamma'} \sim 0. \) Thus (2) holds.

The assertion (3) follows from \( h^*S'^- = T' + E' \), (2) and \( \mathcal{O}_X(S'^-)|_{\Gamma'} \simeq \mathcal{O}_{\mathbb{P}^1}(-1). \)

**Lemma 4.2.** Let \( k_1 \) be a field. Let \( \zeta : Y_1 \to Z_1 \) be a birational \( k_1 \)-morphism of projective normal \( k_1 \)-varieties. For any \( \mathbb{Q} \)-Cartier \( \mathbb{Q} \)-divisor \( N \) on \( Y_1 \) and an ample \( \mathbb{Q} \)-Cartier \( \mathbb{Q} \)-divisor \( H \) on \( Z_1 \), there exists a positive integer \( m \) such that \( (N + m\zeta^*H)|_S \) is big for any integral closed subscheme \( S \) on \( Y_1 \) such that \( S \not\in \text{Ex}(\zeta) \). In particular, if \( N \) is \( \zeta \)-nef and \( \zeta(\text{Ex}(\zeta)) \) is one point, then \( N + m\zeta^*H \) is nef.

**Proof.** Let 
\[
I := \{ S \mid S \text{ is an integral closed subscheme of } Y_1 \text{ such that } S \not\in \text{Ex}(\zeta) \}.
\]
If \( S \in I \), the induced morphism \( S \to \zeta(S) \) is birational. Therefore, for any \( S \in I \), there is \( n_S \in \mathbb{Z}_{>0} \) such that \( (N + n_S\zeta^*H)|_S \) is big. Let \( n_1 := n_{Y_1} \). By Kodaira’s lemma, we may write \( N + n_1\zeta^*H = A + D \) where \( A \) is an ample \( \mathbb{Q} \)-Cartier \( \mathbb{Q} \)-divisor and \( D \) is an effective \( \mathbb{Q} \)-divisor.
on $Y_1$. Let $D = \sum e_j D_j$ be the decomposition into the irreducible components and we set $n_2 := \max_{D_j \in I}\{n_1, n_{D_j}\}$. For any $D_j \in I$, let $D_j^N$ be its normalisation and we again apply Kodaira’s lemma to $(N + n_2 f^* H)|_{D_j^N}$. Repeating the same procedure finitely many times, we can find $n \in \mathbb{Z}_{>0}$ such that $(N + n f^* H)|_S$ is big for every $S \in I$. \qed

For an ample Cartier divisor $H_{Z'}$ on $Z'$ and a sufficiently large integer $m$, we set

$$M' := T' + mh^* g^* H_{Z'}.$$  

We have that $M'$ is nef and big by Lemma 4.1(3) and Lemma 4.2. It follows from Lemma 4.3 that $\mathcal{E}(M') = E'$, where we refer to [Kee99, Definition 0.1] for the definition of $\mathcal{E}(M')$. By [Kee99, Theorem 0.2], we get the birational morphism

$$h'_1 : Y' \to X'_1$$  

with $(h'_1)_* \mathcal{O}_{Y'} = \mathcal{O}_{X'_1}$ induced by $M'$. By our construction, we have that $X'_1$ is $\mathbb{Q}$-factorial, $\rho(X'_1) = \rho(X'_0) = 3$ (cf. [CT, Lemma 2.1]), $Y' \to Z'$ factors through $h'_1$, the fibre of the induced morphism $X'_1 \to Q'$ over $q'$ is set-theoretically equal to $S'_1 \cup \Gamma_1$, and $\Gamma_1 \not\subseteq S'_1$, where $\Gamma_1 := h'_1(E')$ and $S'_1$ is the proper transform of $S'$. In particular, $S'_1$ is ample over $Z'$, hence $X'_1 \to Z'$ is a flip of $X' \to Z'$.

4.2.3. The second step: divisorial contraction. We use the same notation as in (4.2.1) and (4.2.2).

Lemma 4.3. The following hold.

(1) The normalisation of $S'_1$ is a universal homeomorphism.
(2) $-S'_1|_{S'_1}$ is ample.

Proof. We show (1). Let $S'_{Y'}$, be the proper transform of $S'$ on $Y'$. Note that $\tilde{h}_1 : S'_{Y'} \to S'$ and the exceptional locus of $\tilde{h}_1 : S'_{Y'} \to S'_1$ is equal to $\Gamma'_{Y'}$, where $\Gamma'_{Y'} := (\tilde{h}_1)^{-1}(\Gamma')$. Since $\Gamma'_{Y'} \simeq \Gamma' \simeq \mathbb{P}_{k'}$, we have that $\tilde{h}_1(\Gamma'_{Y'})$ is a $k'$-rational point and the induced morphism $\Gamma'_{Y'} \to \tilde{h}_1(\Gamma'_{Y'})$ is the same as the structure morphism $\mathbb{P}_{k'}^1 \to \text{Spec } k'$. In particular, any fibre of $\tilde{h}_1 : S'_{Y'} \to S'_1$ is geometrically connected, hence (1) holds.

We show (2). Take a curve $B'$ on $Q'$ passing through $q'$ and $|B'|$ is base point free. Then the inverse image $D'$ to $X'_1$ can be written by

$$D' = a S'_1 + F'$$  

where $a > 0$ and $F'$ is a nonzero effective $\mathbb{Q}$-divisor with $S'_1 \not\subseteq \text{Supp } F'$. Take a general curve $G'$ on $S'_1$. Since $D' \cdot G' = 0$ and $F' \cdot G' > 0$, we have that $S'_1 \cdot G' < 0$. Thus (2) holds by $\rho(S'_1) = 1$. \qed
Let $H_{X'_1}$ be an ample Cartier divisor on $X'_1$ and we define $\nu'$ by $\nu' := \sup \{ \nu \in \mathbb{R}_{\geq 0} \mid H_{X'_1} + \nu S'_1 \text{ is nef} \}$. We have that $\nu'$ is a positive rational number and let

$$N' := H_{X'_1} + \nu S'_1.$$ 

Since we can find a positive integer $m$ such that $\mathcal{O}_{X'_1}(mN')|_{S'_1} \cong \mathcal{O}_{S'_1}$ by Lemma [4.3], we have that $N'$ is semi-ample by Keel’s theorem ([Kee99, Proposition 1.6]). Let

$$f'_1 : X'_1 \to X'_2$$

be the birational morphism induced by $N'$ with $(f_1)_* \mathcal{O}_{X'_1} = \mathcal{O}_{X'_2}$. We also get $\alpha' : X'_2 \to Q'$ and $\rho(X'_2/Q') = 1$.

4.2.4. Proof of Theorem 1.1.

Theorem 4.4. Let $k$ be an imperfect field whose characteristic $p$ is two or three. If $[k : k^p] < \infty$, then there exists a $k$-morphism $\alpha : X_2 \to Q$ of projective normal $k$-varieties, with $\alpha_* \mathcal{O}_{X_2} = \mathcal{O}_Q$ and $H^0(Q, \mathcal{O}_Q) = k$, that satisfies the following properties.

1. $X_2$ is a $Q$-factorial threefold of Fano type,
2. $Q$ is a projective $Q$-factorial log canonical surface which is not klt,
3. any fibre of $\alpha$ is geometrically irreducible of dimension one, and general fibre of $\alpha$ is $\mathbb{P}^1$, and
4. $\rho(X_2/Q) = 1$.

Proof. We use the same notation as in (4.2.1), (4.2.2) and (4.2.3). We get the rational maps

$$X =: X_0 \to X_1 \to X_2 \to Q$$

corresponding to

$$X' =: X'_0 \to X'_1 \to X'_2 \to Q',$$

where $X_0 \to X_1 \to X_2$ is an $S^-$-MMP over $Q$. Indeed, for a positive integer $e$ such that the $e$-th iterated absolute Frobenius morphism $F^e : X' \to X' := X'^{(p^e)}$ factors through the induced morphism $X' \to X$, we define $X_i$ to be the normalisation of $X'^{(p^e)}$ in $K(X)$, where $X'^{(p^e)} \to X_i^{(p^e)}$ is the same birational map as $X' \to X_i$. Since $X'_0$ is of Fano type, $f_0$ is small and $f_1$ is birational, we have that also $X_2$ is of Fano type by Lemma [2.7]. Thus (1) holds. Since $Q' \cong \mathbb{P}^2_{k'}$ is $Q$-factorial, so is $Q$ (cf. [Tana, Lemma 2.5]). By [Koll13, Lemma 3.1], $Q$ is not klt but log canonical. Thus (2) holds. The assertion (3) follows from the construction, because the fibre of $X' \to Q'$ over $q'$ is
an image of \( \text{Ex}(h') \simeq \mathbb{P}^1_{k'} \times_k \mathbb{P}^1_{k'} \) and hence geometrically irreducible. The assertion (4) holds by \( \rho(X'_2/Q') = 1 \) and Lemma 2.4.

**Proof of Theorem 1.1.** We apply Theorem 4.4 for a field \( k(t) \). Then there exists a \( k(t) \)-morphism \( \alpha : X_2 \to Q \) of projective normal \( k(t) \)-varieties, with \( \alpha_* \mathcal{O}_{X_2} = \mathcal{O}_Q \) and \( H^0(Q, \mathcal{O}_Q) = k(t) \), satisfying the properties (1)–(4) in Theorem 4.4. We can find projective \( k \)-morphisms

\[
V \xrightarrow{\varphi} W \xrightarrow{\eta} T
\]

of normal \( k \)-varieties such that \( T \) is a non-empty open subset of Spec \( k[t] \) and \( f \times_T \text{Spec} \ k(t) = \alpha \). After possibly shrinking \( T \), we may assume that

- \( V \) and \( W \) are \( \mathbb{Q} \)-factorial by Lemma 2.2,
- \( V \) is klt by Lemma 2.2,
- \( W \) is not klt by Lemma 2.2, and
- \( f_* \mathcal{O}_V = \mathcal{O}_W \).

We set \( W_1 \) to be the subset of \( W \) consisting of the points \( w \in W \) such that \( V_w \) is geometrically irreducible and of dimension one. By [Gro66, 9.5.5 and 9.7.7], \( W_1 \) is a constructible subset of \( W \).

**Claim.** There exists an open subset \( W_2 \) of \( W \) such that \( W_\eta \subset W_2 \subset W_1 \), where \( \eta \) is the generic point of \( T \).

**Proof of Claim.** By Theorem 4.4(3), we have that \( W_\eta \subset W_1 \). This inclusion implies that \( \eta \not\in g(W_1^c) \), hence the constructible subset \( g(W_1^c) \) of a curve \( T \) is a proper closed subset of \( T \), where let \( B^c := A \setminus B \) for any subset \( B \) of a set \( A \). Thus the inclusions \( W_\eta \subset W_2 \subset W_1 \) hold for \( W_2 := g^{-1}(g(W_1^c)^c) \). It completes the proof of Claim.

After replacing \( W \) by \( W_2 \), we may assume that the fibre \( V_w \) over any point \( w \in W \) is geometrically irreducible and of dimension one. In particular, \( \rho(V/W) = 1 \). It completes the proof of Theorem 1.1. □

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