The Quadratic Formula Made Hard
or
A Less Radical Approach to Solving Equations

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Introduction

It appears that, along with many of my friends and colleagues, I had been brainwashed by the great and tragic lives of Abel and Galois to believe that no general formulas are possible for roots of equations higher than quartic. This seemed to be confirmed by the brilliant and arduous solution of the general quintic by Hermite. Yet, below we find a formula giving a root to any algebraic equation of degree 2-5 and any reduced equation (see below) of higher degree. This algorithm, which must have been familiar to Lagrange, resulted when I was working on a paper on the asymptotics of hypergeometric functions where Gauss’ multiplication formula for the gamma function is used to reduce certain infinite series, and by a happy accident my copy of Whittaker and Watson opened at p. 133.

The Formula

Without loss of generality it is sufficient to find at least one root to the reduced equation

\[ x^N - x + t = 0 \quad (N = 2, 3, 4 \ldots). \]  

(1)

Letting \( x = \zeta^{-1/(N-1)} \), we easily find that (1) becomes

\[ \zeta = e^{2\pi i} + t\phi(\zeta) \]  

(2)

where

\[ \phi(\zeta) = \zeta^{N/(N-1)}. \]  

(3)

Lagrange’s theorem states that for any function \( f \) analytic in a neighborhood of a root of (2)

\[ f(\zeta) = f(e^{2\pi i}) + \sum_{n=1}^{\infty} \frac{t^n}{n!} \frac{d^{n-1}}{da^{n-1}}[f'(a)|\phi(a)|^{n}]_{a=e^{2\pi i}}. \]  

(4)
We now simply let \( f(\zeta) = \zeta^{-1/(N-1)} \), carry out the elementary differentiations (noting that \( D_kx^p = \Gamma(p + 1)x^{p-k}/\Gamma(p - k + 1) \)) and we come up with the root

\[
x_1 = \exp[-2\pi i/(N-1)] - \frac{t}{N-1} \sum_{n=0}^{\infty} \frac{(te^{2\pi i/(N-1)})^n \Gamma(N+1)}{\Gamma(n+2) \Gamma(N) \Gamma(n+1)}.
\] (5)

(N-2 further roots are found by replacing \( \exp(2\pi i/(N-1)) \) by the other N-1-st roots of unity, and the remaining root from the relation \( \sum x_j = \delta_{N,2} \)). By the use of Gauss’ multiplication theorem, the infinite series can be broken up into a (finite) sum of hypergeometric functions.

\[
x_1 = \omega^{-1} - \frac{t}{(N-1)^2} \sqrt{\frac{N}{2\pi(N-1)}} \sum_{q=0}^{N-2} \left( \frac{\omega t}{N-1} \right)^{qN/(N-1)} \frac{\Pi_{k=0}^{N-1} \Gamma(Nq/(N-1)+1+k)}{\Gamma(q/(N-1)+1) \Pi_{k=0}^{N-2} \Gamma(Nq/(N-1)+2)} \times
\]

\[
\begin{array}{c}
\frac{qN/(N-1) + 1}{N} \ldots \frac{qN/(N-1) + N}{N}, 1; \\
\int N+1 F_N \left[ \frac{qN/(N-1) + 1}{N} \ldots \frac{qN/(N-1) + N}{N}, 1; \frac{q + 2}{N-1} \ldots \frac{q + N}{N-1}, \frac{q}{N-1} + 1; \left( \frac{t\omega}{N-1} \right)^N N \right],
\end{array}
\]

where \( \omega = \exp(2\pi i/(N-1)) \). In practice, \( N+1 F_N \) will always be reducible to at least \( N+1 F_{N-1} \). Hence the root is a sum of at most \( N-1 \) hypergeometric functions.

The one technical point is that the convergence of these series requires that \( t \) be “sufficiently small”, but this can be overcome by certain hypergeometric identities tantamount to analytic continuation.

**Examples**

\( N=2 \)

\[
x^2 - x + t = 0
\]

Here we have

\[
x_1 = 1 - t \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(n+1)} \frac{\Gamma(2n + 1)}{\Gamma(n+2)}.
\] (6)

However, by Gauss’ formula

\[
\Gamma(2n + 1) = 4^n(1/2)_n(1)_n \quad ((n)_k = \Gamma(n + k)/\Gamma(n))
\] (7)

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so

\[ x_1 = 1 - t \, _2F_1(1/2, 1; 2; 4t) \]  \quad (8)

Since

\[ _2F_1(1/2, 1; 2; z) = \frac{2}{z} \left\{ \begin{array}{ll}
1 - \sqrt{1 - z} & |z| \leq 1 \\
1 - i \sqrt{z - 1} & |z| > 1
\end{array} \right. \]  \quad (8)

we reproduce the quadratic formula. Note that the second root comes from \( x_1 + x_2 = 1 \).

**N=3**

\[ x^3 - x + t = 0 \]

By separating the sum in (5) into sums over the even and odd values of \( n \) we obtain

\[
x_1 = -1 + \frac{t}{2} \sum_{n=0}^{\infty} \frac{\Gamma(3n + 1) t^{2n}}{\Gamma(n + 1) \Gamma(2n + 2)} + \frac{t^2}{2} \sum_{n=0}^{\infty} \frac{\Gamma(3n + 5/2) t^{2n}}{\Gamma(n + 3/2) \Gamma(2n + 3)}. \]  \quad (9)

By breaking up the gamma functions of multiple argument by using Gauss’ multiplication theorem, the sums are easily identified as hypergeometric series:

\[
x_1 = -1 - \frac{t}{2} \, _2F_1(1/3, 2/3; 3/2; 27t^2/4) + \frac{3t^2}{8} \, _3F_2(5/6, 7/6, 1; 3/2, 2; 27t^2/4). \]  \quad (10)

However, from A.P. Prudnikov et.al, Integrals and Series, Vol.3[Gordon and Breach, 1990]) we find

\[ _2F_1(1/3, 2/3; 3/2; z) = \frac{3}{\sqrt{z}} \sin(\frac{1}{3} \sin^{-1} \sqrt{z}) \]

\[ _3F_2(5/6, 7/6, 1; 3/2, 2; z) = \frac{18}{z} \left[ \cos(\frac{1}{3} \sin^{-1} \sqrt{z}) - 1 \right], \]  \quad (11)

and we therefore have the three roots

\[
x_1 = -\frac{1}{\sqrt{3}} \sin\left[ \frac{1}{3} \sin^{-1} (t\sqrt{27}/2) \right] - \cos\left[ \frac{1}{3} \sin^{-1} (t\sqrt{27}/2) \right] \]

\[
x_2 = -\frac{1}{\sqrt{3}} \sin\left[ \frac{1}{3} \sin^{-1} (t\sqrt{27}/2) \right] + \cos\left[ \frac{1}{3} \sin^{-1} (t\sqrt{27}/2) \right]
\]
\[ x_3 = \frac{2}{\sqrt{3}} \sin\left[ \frac{1}{3} \sin^{-1}(t\sqrt{27}/2) \right]. \]  

Once again, for \( t > 2/\sqrt{27} \) equation (10) must be analytically continued to obtain the correct form of (12). This amounts to writing \( \sin^{-1} z = \frac{\pi}{2} - i\ln(z + \sqrt{z^2 - 1}) \).

**Conclusion**

For \( N=2,3,4 \) Eq.(5) is definitely not preferable to the standard formulas, but for \( N=5 \), e.g. we get the root

\[ x = t \ _4F_3 \left[ \begin{array}{cc} 1/5, 2/5, 3/5, 4/5; & 3125t^4/256 \\ 1/2, 3/4, 5/4; & \end{array} \right] \]  

in an elementary fashion with considerably less difficulty than by following the procedure in Davis’ book [Introduction to Nonlinear Ordinary Differential Equations (Dover)]. It might also be pointed out that the above procedure carries over in a trivial way to the trinomial equation

\[ y^N - ay^{N-1} + a = 0 \]  

where \( y = 1/x, \ a = 1/t \). Numerically, these formulas are not of much use since solutions can be obtained with the push of a button on many pocket calculators, but formulas such as (13) should have numerous entertaining uses, such as summing the odd hypergeometric series.

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