Time-consistent mean-variance reinsurance-investment problem with long-range dependent mortality rate

Ling Wang\textsuperscript{a}, Mei Choi Chiu\textsuperscript{b} and Hoi Ying Wong\textsuperscript{a}

\textsuperscript{a}Department of Statistics, The Chinese University of Hong Kong, Shatin, N.T., Hong Kong; \textsuperscript{b}Department of Mathematics & Information Technology, The Education University of Hong Kong, Tai Po, N.T., Hong Kong

\textbf{ABSTRACT}

This paper investigates the time-consistent mean-variance reinsurance-investment (RI) problem faced by life insurers. Inspired by recent findings that mortality rates exhibit long-range dependence (LRD), we examine the effect of LRD on RI strategies. We adopt the Volterra mortality model proposed in Wang et al. (2021). Volterra mortality model: actuarial valuation and risk management with long-range dependence. \textit{Insurance: Mathematics and Economics} \textbf{96}, 1–14 to incorporate LRD into the mortality rate process and describe insurance claims using a compound Poisson process with intensity represented by the stochastic mortality rate. Under the open-loop equilibrium mean-variance criterion, we derive explicit equilibrium RI controls and study the uniqueness of these controls in cases of constant and state-dependent risk aversion. We simultaneously resolve difficulties arising from unbounded non-Markovian parameters and sudden increases in the insurer’s wealth process. While the exiting literature suggests that LRD has a significant effect on longevity hedging, we find that reinsurance is a risk management strategy that is robust to LRD.

\textbf{ARTICLE HISTORY}

Received 17 July 2021
Accepted 9 June 2022

\textbf{KEYWORDS}

Mean-variance; time consistency; reinsurance investment; mortality model; long-range dependence

1. Introduction

Insurers can manage their risk exposure through reinsurance and enhance their profits by investing in financial markets. These market-related practices inspire studies on optimal reinsurance and investment (RI) strategies. Many RI strategies are developed using the mean-variance (MV) criterion, which is popular in the field of practical investment. For instance, Chen & Yam (2013) consider an optimal RI problem involving an insurer and the MV criterion under a regime-switching model. Shen & Zeng (2014) study an optimal MV-RI problem with a delay using the maximum principle approach. Yi et al. (2015) investigate robust optimal RI strategies under a benchmarking MV criterion.

As acknowledged in the literature, the optimal MV problem is complicated by time inconsistency, which prevents satisfaction of the dynamic programing principle. Specifically, a strategy that is optimal at the initial time point becomes suboptimal at a later time point. As time inconsistency is a universal property of problems with MV objectives, this potential weakness also appears in the corresponding RI problems. The first documentation of time inconsistency appears in Strotz (1955). Basak & Chabakauri (2010) propose to resolve problems complicated by time inconsistency by introducing an equilibrium feedback control framework based on the concept of sequential games. Björk et al. (2014) argue that the state-independent strategy provided in Basak & Chabakauri (2010) is economically unrealistic because the amount invested in the risky asset is independent of the investor’s
current wealth. They propose the concept of state-dependent risk aversion and establish an extended Hamilton–Jacobi–Bellman (HJB) framework for time inconsistency. However, Björk et al. (2017) show that it is technically difficult to study the uniqueness of equilibrium control within the HJB framework.

To simultaneously resolve time inconsistency and study uniqueness in MV problems, Hu et al. (2012) introduce an open-loop equilibrium control framework involving a system of forward–backward stochastic differential equations (FBSDEs). Hu et al. (2017b) further provide sufficient and necessary conditions for equilibrium control and rigorously prove uniqueness under the assumption of bounded parameters in the wealth process. Extensions of this approach to cases involving jumps, stochastic volatility, and constraints on frameworks with controls and robust controls can be found in Sun & Guo (2019), Yan & Wong (2019), Hu et al. (2017a) and Han et al. (2021), respectively.

The concept of time-consistent (TC) MV formulation extends to studies of RI strategies. Related studies based on diffusion approximations of the claims process include, but are not limited to, Li et al. (2015), Yi et al. (2015), Han et al. (2020), Wang & Siu (2020), Yan & Wong (2020), and the references therein. Studies that seriously consider jumps include Lin & Qian (2016), Zeng et al. (2016), Alia et al. (2016), Sun et al. (2020), Guan & Wang (2020), and the references therein. We follow the latter set of references and use jumps to describe the surplus process of the insurer and allow the intensity of the Poisson process, or the equivalent mortality rate, to be a stochastic process. The notable feature distinguishing this paper from prior studies is that the mortality rate follows a stochastic process with long-range dependence (LRD), which is also known as the long-memory property or persistence.

In a recent study using data from 16 countries, Yan et al. (2021) show empirically that mortality rate data exhibit LRD. Furthermore, Yan et al. (2020) empirically confirm LRD in a multivariate time series of multi-cohort mortality data. Delgado-Vences & Ornelas (2019) find strong empirical evidence for the existence of LRD in mortality data from an Italian population. Based on the empirical evidence, we investigate the impact of LRD on RI strategies, specifically the TCMV RI strategy. To the best of our knowledge, this paper is the first to consider the TC RI problem using a mortality rate with LRD.

Our investigation is based on the innovative Volterra mortality models (VMMs) proposed by Wang et al. (2021) for a mortality rate with LRD. Whereas the VMMs are tractable for actuarial valuation and longevity hedging, the non-Markovian and non-semimartingale features of the Volterra process generate subtle difficulties in deriving the open-loop equilibrium RI strategy. We also encounter jumps in the insurer’s wealth process. Therefore, the main contribution of this paper lies in its ability to overcome the aforementioned difficulties and derive unique explicit equilibrium controls for an RI problem under a VMM with both constant and state-dependent risk aversion TCMV objectives. Our work reinforces that the admissibility and uniqueness of equilibrium controls are non-trivial in a case involving an unbounded Volterra parameter that is simultaneously associated with the VMM and jumps. Specifically, the unbounded Volterra parameter requires additional efforts to establish the admissibility of the equilibrium strategies (Theorems 2.1 and 4.1) and the uniqueness of the equilibrium strategies (Theorems 3.3 and 4.2). For state-dependent risk aversion, we provide the first set of technical conditions (Assumptions 4.1–4.3) for ensuring the admissibility and uniqueness of the equilibrium RI policy.

In our open-loop equilibrium framework, the key mathematical challenge is the simultaneous encounter of the unbounded stochastic Volterra mortality rate and jumps in the state process. Although Sun & Guo (2019) extend their open-loop equilibrium framework to jump diffusion, their assumption regarding uniformly bounded parameters is too restrictive for our problem. Alia et al. (2016) study the open-loop RI problem under a jump diffusion model but require the Poisson intensity to be a bounded deterministic function. Yan & Wong (2020) study the open-loop RI problem with unbounded Markovian stochastic volatility, but their models do not include a jump term. We provide a necessary and sufficient condition for equilibrium control with jumps and unbounded Volterra parameters in the state process. We also carefully study the box constraint on the proportional reinsurance policy when the policy process falls in the interval [0, 1].
From an actuarial science perspective, we provide the first rigorous proof that the equilibrium reinsurance policy is independent of the historical mortality rate under the TCMV criterion with constant risk aversion even if the mortality rate exhibits LRD. Such a reinsurance strategy is a unique equilibrium policy. This is a rather strong result because it confirms that LRD does not affect the (unique equilibrium) reinsurance demand of TCMV investors with constant risk aversion. In contrast, Wang & Wong (2021) show that LRD has material effects on longevity pricing and hedging, implying that reinsurance is more robust to LRD in the mortality rate among TCMV investors with constant risk aversion. Furthermore, the unique equilibrium reinsurance strategy under constant risk aversion is a deterministic function, which enables the insurer to negotiate an implementable reinsurance contract with the reinsurer at the initial time. Hence, the reinsurance strategy is practical although reinsurance contracts are not traded in the market.

We also derive the first RI strategy for a TCMV insurer with state-dependent risk aversion. Our result is novel not only because it allows for the LRD mortality rate but also for the Markovian mortality rate. In fact, LRD has a mild effect on the RI strategy with state-dependent risk aversion. We show this by numerically comparing the LRD RI strategy to the Markovian strategy. In summary, the risk aversion even if the mortality rate exhibits LRD. Such an insurance strategy is a unique equilibrium policy independent of the historical mortality rate under the TCMV criterion with constant risk aversion.

The remainder of this paper is organized as follows. In Section 2, we describe our model of mortality with LRD and formulate an RI problem using the open-loop equilibrium control framework. In Section 3, we derive explicit forms for equilibrium strategies under constant risk aversion for the two cases with the positive constraint and the [0, 1] interval constraint on the reinsurance policy, respectively. In Section 4, we study the RI problem under state-dependent risk aversion. The admissibility of the equilibrium controls together with some technical conditions are provided. In Section 5, we use a numerical study to reveal the influence of the mortality rate with LRD on RI strategies. Our concluding remarks are presented in Section 6. Technical proofs are collected in the appendix.

2. Problem formulation

In a filtered complete probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, \mathbb{P})\), for any positive constant \(q\) and positive integer \(d\), let \(D\) be a nonempty subset of \(\mathbb{R}^d\). The following terms are defined:

\[ S^\mathcal{F}_t(t; D, \mathbb{P}): \text{the set of all } \{\mathcal{F}_s\}_{s \in [t,T]} \text{-adapted } D\text{-valued stochastic processes } X(\cdot) = \{X(s) : t \leq s \leq T\} \text{ with } \mathbb{E}[\sup_{t \leq s \leq T} |X(s)|^q] < +\infty; \]

\[ L^\mathcal{F}_t(t; D, \mathbb{P}): \text{the set of all essentially bounded } \{\mathcal{F}_s\}_{s \in [t,T]} \text{-adapted } D\text{-valued stochastic processes; } \]

\[ L^T \mathcal{F}_t(t; D, \mathbb{P}): \text{the set of all } \{\mathcal{F}_s\}_{s \in [t,T]} \text{-adapted } D\text{-valued stochastic processes with } \mathbb{E}[\int_t^T |X(s)|^q \, ds] < +\infty; \]

\[ H^{\mathcal{F}}_t(t, T; D, \mathbb{P}): \text{the set of all } \{\mathcal{F}_s\}_{s \in [t,T]} \text{-adapted } D\text{-valued stochastic processes } X(\cdot) = \{X(s) : t \leq s \leq T\} \text{ with } \mathbb{E}[(\int_t^T |X(s)|^2 \, ds)^{q/2}] < +\infty. \]

2.1. Mortality model

We use \(\hat{\lambda}_t\) to denote an intensity rate of a population. Insurance claims arrive according to a Poisson process with an intensity proportional to \(\hat{\lambda}(t)\). Yan et al. (2021) show that the mortality rates of different cohorts exhibit LRD, with an estimated Hurst parameter \(H\) of around 0.8; in this estimation, the Hurst parameter is restricted to the interval (0, 1). The mortality rate follows a Markovian process when \(H = 0.5\). Therefore, the empirical result reported by Yan et al. (2021) suggests significant LRD. Given the filtration \(\mathcal{F}_t\), the LRD feature means that the future state of the mortality rate depends on the entire historical path of the mortality rate observed in \([0, t]\). Let \(\mathbb{R}_+ = [0, \infty)\). To incorporate the LRD of the mortality rate, we adopt the VMM proposed in Wang et al. (2021), so that

\[ \hat{\lambda}_t = I(t) + \lambda_t, \]
where \( l(t) \geq 0 \) is a bounded deterministic function and \( \lambda \) follows a Volterra process. Specifically, \( \lambda_t \) follows a stochastic Volterra integral equation (SVIE):

\[
\lambda_t = \lambda_0 + \int_0^t K(t-s)(b_1 - a_1\lambda_s) \, ds + \int_0^t K(t-s)\sigma\sqrt{\lambda_s} \, dW_0(s),
\]

(2)

where \( \lambda_0, b_1, a_1, \) and \( \sigma \) are positive constants, \( W_0 \) represents a standard Brownian motion, and \( K \in L^2(\mathbb{R}_+, \mathbb{R}) \) is the Volterra kernel. When the kernel is set to the fractional kernel displayed in Table 1, then \( \int_0^T K(t-s) \, dW_0(s) \) is proportional to the classical fractional Brownian motion (fBM), a building block used in continuous-time models with LRD. The Volterra process is generally non-Markovian and non-semimartingale. Fortunately, the mortality model specified in (1) has an affine structure, as shown in Wang et al. (2021), and is therefore analytically tractable. In this paper, we make the following standard assumption regarding the kernel \( K(\cdot) \). Here, \( L^2_{\text{loc}}(\mathbb{R}_+, \mathbb{R}) = \{ f : \mathbb{R}_+ \rightarrow \mathbb{R} \text{ measurable satisfying } \int_A |f|^2 \, ds < +\infty, \forall A \subset \mathbb{R}_+, A \text{ compact} \} \).

**Assumption 2.1:** The kernel \( K \) in (2) is strictly positive and completely monotone on \((0, \infty)\), and there exist \( \chi \in (0, 2] \) and \( k > 0 \) such that

\[
\int_0^h K^2(t) \, dt + \int_0^T (K(t+h) - K(t))^2 \, dt \leq kh^\chi, \quad h > 0.
\]

**Lemma 2.1:** Under Assumption 2.1, the SVIE (2) admits a unique in law \( \mathbb{R}_+ \)-valued continuous weak solution for any initial condition \( \lambda_0 \in \mathbb{R}_+ \). (Theorem 6.1 in Abi Jaber et al. (2019).)

Table 1 offers some examples of kernel functions \( K \) that satisfy Assumption 2.1. Therefore, the fractional kernel satisfies our standing assumption. In fBM, the degree of LRD is reflected by the parameter \( \alpha \), which is related to the classical Hurst parameter \( H \) such that \( \alpha = H - 1/2 \). Table 1 also shows the resolvents corresponding to each kernel.

The **resolvent** or **resolvent of the second kind** corresponding to \( K \) shown in Table 1 is defined as the kernel \( R \), such that \( K \ast R = R \ast K = K - R \). The convolutions \( K \ast R \) and \( R \ast K \), where \( K \) is a measurable function on \( \mathbb{R}_+ \) and \( R \) is a measure on \( \mathbb{R}_+ \) of locally bounded variation, are defined by

\[
(K \ast R)(t) = \int_{[0,t]} K(t-s)R(ds), \quad (R \ast K)(t) = \int_{[0,t]} R(ds)K(t-s)
\]

for \( t > 0 \).

**Lemma 2.2 (Abi Jaber et al. 2019):** If \( \lambda \) follows the SVIE (2), then for any \( 0 \leq t \leq T \),

\[
\mathbb{E}[\lambda_T | \mathcal{F}_t] = \left( 1 - \int_0^T R_B(s) \, ds \right) \lambda_0 + b_1 \int_0^T E_B(T-s) \, ds + \int_0^t E_B(T-s)\sigma_\lambda \, dW_0(s),
\]

(3)

where \( B = -a_1, R_B \) is the resolvent of \(-KB\), and \( E_B = K - R_B \ast K \).
2.2. State process

Consider the classic risk process for an insurer’s surplus. When the insurer makes no reinsurance or investment, his surplus process can be described as follows:

$$I(t) = I_0 + \int_0^t (1 + \theta)k_1 \hat{\lambda}_t \mu_z \, ds - \sum_{i=1}^{N(t)} z_i,$$

where $I_0$ is the initial surplus and $(z_i)_{i=1}^\infty$ are independent and identically distributed (iid) positive random variables representing insurance claims. $N(t)$ is a stochastic Poisson process with intensity $k_1 \hat{\lambda}_t$, where $k_1 > 0$ is a constant. We assume that the insurance claims are independent of the mortality rate. $\tau_t = (1 + \theta)k_1 \mu_z \hat{\lambda}_t$ is the premium rate, with $\mathbb{E}[z] = \mu_z$ and $\theta > 0$ representing the safety loading of the insurer.

Suppose that the insurer is allowed to purchase reinsurance or acquire new business. For any value of $t \in [0, T]$, the proportional reinsurance strategy is denoted by $a(t) \in [0, +\infty)$. $a(t) \in [0, 1]$ corresponds to proportional reinsurance coverage. Therefore, when an insurance claim occurs, the reinsurance company pays a $1 - a(t)$ fraction, while the insurer pays the remaining $a(t)$ fraction. Meanwhile, the reinsurance company charges the insurer at the rate of $\frac{1 + \eta}{1 + \theta} \tau_t (1 - a(t))$, where $\eta \geq \theta$ represents the safety loading of the reinsurer. When $a(t) > 1$, the insurer acquires new business. The surplus process of the insurer becomes

$$dI(t) = [(\theta - \eta)k_1 \hat{\lambda}_t \mu_z + (1 + \eta)k_1 \hat{\lambda}_t \mu_z a(t)] \, dt - a(t) \, d\sum_{i=1}^{N(t)} z_i.$$  

In practice, insurers also attempt to profit from the financial market. Consider a market consisting of risk-free and risky assets. The price of the risk-free asset, $B(t)$, is as follows:

$$dB(t) = r(t)B(t) \, dt,$$

where the interest rate $r > 0$ is a bounded deterministic function. The price of the risky asset, $S(t)$, is as follows:

$$dS(t) = S(t) [\mu(t) \, dt + \sigma(t) \, dW_1(t)],$$

where $\mu(t)$ and $\sigma(t) > 0$ are two bounded deterministic functions, and $W_1(t)$ represents the standard Brownian motion independent of $W_0(t)$.

Suppose that the insurer has an initial wealth of $X_0$. For $t \in [0, T]$, $\pi(t)$ is the amount of money invested in the risky asset and $X_t$ as the wealth process. The remaining amount of money, $X_t - \pi(t)$, is then invested in the risk-free asset. Hence, we obtain

$$dX_t = [r(t)X_t + (\mu(t) - r(t))\pi(t) + [(1 + \eta)\mu(t) + \theta - \eta]k_1 \hat{\lambda}_t \mu_z] \, dt$$

$$+ \pi(t)\sigma(t) \, dW_1(t) - a(t) \, d\sum_{i=1}^{N(t)} z_i.$$ 

Next, we simplify the notation by introducing a Poisson random measure. Denote the compound Poisson process by $\sum_{i=1}^{N(t)} z_i = \int_0^t \int_{\mathbb{R}_+} zN(ds, dz)$, where $\mathbb{R}_+ = [0, \infty)$ and $N(dt, dz)$ is a Poisson random measure in the space $[0, T] \times \mathbb{R}_+$. Assume that the positive random claim size $z$ has a probability density function $f(z)$ with finite expectation and moments. Here, $\mu_z = \int_{\mathbb{R}_+} z^2 f(z) \, dz$ and $\mathbb{E}[z^2] = \int_{\mathbb{R}_+} z^2 f(z) \, dz$. Thus, the Poisson random measure $N(dt, dz)$ has the compensator $\delta(dz)$ $dt \triangleq k_1 \hat{\lambda}_t f(z) \, dz \, dt$, where $\delta(dz) = k_1 \hat{\lambda}_t f(z) \, dz$. Define $F^2(t, T; \mathbb{R})$ as the set of all $\{F_s\}_{s \in [t, T]}$-predictable
processes \( X(\cdot, \cdot) : \Omega \times [t, T] \times \mathbb{R}_+ \rightarrow \mathbb{R} \), such that \( \mathbb{E}[\int_t^T \|X(s, \cdot)\|_{L^2}^2 \, ds] < \infty \). Here, \( \|X(s, z)\|_{L^2}^2 := \int_{\mathbb{R}_+} |X(s, z)|^2 \delta(dz) \). Let \( \tilde{N}(dt, dz) = N(dt, dz) - \delta(dz) \, dt \). The insurer’s wealth process becomes

\[
dX_t = [r(t)X_t + (\mu(t) - r(t))\pi(t) + \eta \hat{\lambda}_t \mu_z a(t) + (\theta - \eta) k_1 \hat{\lambda}_1 \mu_z] \, dt + \pi(t)\sigma(t) \, dW_1(t) - \int_{\mathbb{R}_+} a(t)zN(dt, dz).
\]

For convenience, we rewrite the wealth process as follows:

\[
dX_t = [r(t)X_t + v(t)^\top u(t) + c(t)] \, dt + \pi(t)\sigma(t) \, dW_1(t) - \int_{\mathbb{R}_+} a(t)zN(dt, dz),
\]

where \( u(t) = (\pi(t), a(t))^\top \) is the control pair; \( v = (v_1, v_2)^\top \) with \( v_1(t) = \mu(t) - r(t) \) and \( v_2(t) = \eta k_1 \hat{\lambda}_1 \mu_z \); and \( c(t) = (\theta - \eta) k_1 \hat{\lambda}_1 \mu_z \). The coefficients \( v_2(t) \) and \( c(t) \) are unbounded stochastic processes due to the unbounded parameter \( \hat{\lambda}_t \). Note that the existence of a constraint specifies that the control \( a(t) \) should always be nonnegative.

Traditional reinsurance problems assume a constant insurance claim intensity. However, the claim intensity can in fact be stochastic, especially for a long-term insurance contract. We investigate the influence of the LRD of the mortality rate on life reinsurance strategies and focus on life insurers in this paper. Thus, the intensity of insurance claims is proportional to the mortality rate.

### 2.3. Open-loop equilibrium framework

**Definition 2.1:** If the control \( u(\cdot) \in H^2_F(0, T; \mathbb{R}, \mathbb{P}) \times \bigcup_{q \geq 2} L^q_F(0, T; \mathbb{D}, \mathbb{P}) \) and Equation (6) admits a unique strong solution \( X \in S^2_F(0, T; \mathbb{R}, \mathbb{P}) \), then \( u \) is called an admissible control.

Two cases are considered in this paper.

- \( \mathbb{D} = \mathbb{R}_+ \). The reinsurance strategy is only required to be positive.
- \( \mathbb{D} = [0, 1] \). The reinsurance strategy is restricted to \([0, 1]\).

Although Volterra processes are unbounded in general, there is a bounded moment result for the continuous solution of the Volterra equation (2). As \( \hat{\lambda} \) and \( \lambda \) satisfy (1) where the deterministic function \( l(t) \) is bounded, we have the following Lemma 2.3 for \( \hat{\lambda} \).

**Lemma 2.3:** Let \( \lambda_t \) be the continuous solution of the SVIE (2). Suppose that kernel \( K \) satisfies Assumption 2.1; then, there exists a constant \( C > 0 \) such that

\[
\sup_{0 \leq t \leq T} \mathbb{E}[|\lambda_t|^q] < C.
\]

for any constant \( q \geq 2 \).

**Proof:** In (2), it is clear that

\[
|b_1 - a_1 \lambda| \vee |\sigma_2 \sqrt{\lambda}| \leq C_1(1 + |\lambda|),
\]

for a constant \( C_1 \). The result follows from Lemma 3.1 in Abi Jaber et al. (2019).

The insurer’s objective is to minimize

\[
J(t, x_t; u(\cdot)) = \frac{1}{2} \text{Var}_t(X_T) - (\phi_1 x_t + \phi_2) \mathbb{E}_t[X_T] = \frac{1}{2} (\mathbb{E}_t[X_T^2] - \mathbb{E}_t^2[X_T]) - (\phi_1 x_t + \phi_2) \mathbb{E}_t[X_T]
\]

(7)
by an admissible control $u$, where $x_t = X_t$, $E_t[\cdot] = E[\cdot|\mathcal{F}_t]$, and $\phi_1, \phi_2 \in \mathbb{R}_+$. If $\phi_1 = 0$ and $\phi_2 > 0$, the insurer has constant risk aversion; otherwise, if $\phi_1 > 0$, the insurer has state-dependent risk aversion.

Because of the unboundedness of the parameter $\hat{\lambda}$, the results in Sun & Guo (2019) cannot be applied directly to our problem. In our case, the admissibility of the control is highly nontrivial, especially for the state-dependent case. The following theorem and the proof detailed in Appendix A.1 are useful in this regard.

**Theorem 2.1:** If the control $u(\cdot) \in H^2_F(0, T; \mathbb{R}, \mathbb{P}) \times \bigcup_{q>2} L^q_F(0, T; D, \mathbb{P})$, then $X \in S^2_F(0, T; \mathbb{R}, \mathbb{P})$.

Theorem 2.1 asserts that the condition for the control space in Definition 2.1 alone is sufficient to ensure admissibility because the condition for the state process is immediate.

We use the open-loop equilibrium framework described in Hu et al. (2012, 2017a, 2017b). Given a control pair $(\pi^*(t), a^*(t)) \in H^2_F(0, T; \mathbb{R}, \mathbb{P}) \times \bigcup_{q>2} L^q_F(0, T; D, \mathbb{P})$, for any $t \in [0, T]$, $\epsilon > 0$, $\rho_1 \in H^2_F(t, T; \mathbb{R}, \mathbb{P})$, and $\rho_2 \in \bigcup_{q>2} L^q_F(t, T; D, \mathbb{P})$, define

$$\begin{align*}
\pi^{t, \epsilon, \rho_1}(s) &= \pi^*(s) + \rho_1(s)1_{s \in [t, t+\epsilon)}, \quad s \in [t, T], \\
a^{t, \epsilon, \rho_2}(s) &= a^*(s) + (\rho_2(s) - a^*(s))1_{s \in [t, t+\epsilon)}, \quad s \in [t, T].
\end{align*}$$

(8)

Under this construction, for any $t \in [0, T)$, we obtain $a^{t, \epsilon, \rho_2}(s) = \rho_2(s) \geq 0$, when $s \in [t, t+\epsilon]$, and $a^{t, \epsilon, \rho_2}(s) = a^*(s) \geq 0$, when $s \in [t + \epsilon, T)$. A similar setting is used in Hu et al. (2017a) and Yan & Wong (2020).

**Definition 2.2:** Let $(\pi^*(t), a^*(t)) \in H^2_F(0, T; \mathbb{R}, \mathbb{P}) \times \bigcup_{q>2} L^q_F(0, T; D, \mathbb{P})$ be a given control pair and $X^*$ be the corresponding state process. Then, the control pair $(\pi^*(t), a^*(t))$ is an equilibrium strategy for Problem (7) if, for any $t \in [0, T)$, $\rho_1 \in H^2_F(0, T; \mathbb{R}, \mathbb{P})$, and $\rho_2 \in \bigcup_{q>2} L^q_F(t, T; D, \mathbb{P})$, we have $X^* \in S^2_F(t, T; \mathbb{R}, \mathbb{P})$ and

$$\liminf_{\epsilon \downarrow 0} \frac{J(t, X^*_t; \pi^{t, \epsilon, \rho_1}, a^{t, \epsilon, \rho_2}) - J(t, X^*_t; \pi^*, a^*)}{\epsilon} \geq 0,$$

(9)

where $\pi^{t, \epsilon, \rho_1}, a^{t, \epsilon, \rho_2}$ is as defined in (8).

Note that we consider the constraint on the reinsurance control to be in line with Hu et al. (2017a). When the control $a^*$ is restricted to $D$, then $\rho_2$ is also restricted to $D$ in the definition.

The nature of the Volterra process prevents the use of a classic HJB framework in our problem. Inspired by Hu et al. (2012, 2017b) and Sun & Guo (2019), we adopt a BSDE approach and provide an equivalent condition to (9) as follows.

For any $t \in [0, T)$, define the adjoint process $(p^*(s; t), Z^*(s; t), Z^*_2(s, z; t)) \in L^2_F(t, T; \mathbb{R}, \mathbb{P}) \times H^2_F(t, T; \mathbb{R}, \mathbb{P}) \times H^2_F(t, T; \mathbb{R}, \mathbb{P})$ that satisfies the following BSDE:

$$\begin{align*}
\mathrm{d}p^*(s; t) &= -r(s)p^*(s; t) \mathrm{d}s + Z^*(s; t) \mathrm{d}W_s + \int_{\mathbb{R}_+} Z^*_2(s, z; t) \tilde{N}(\mathrm{d}s, \mathrm{d}z), \\
p^*(T; t) &= X^*_T - E_t[X^*_T] - (\phi_1 X^*_t + \phi_2),
\end{align*}$$

(10)

where $Z^*(s; t) = (Z^*_0(s; t), Z^*_1(s; t))^\top$, $W_s = (W_0(s), W_1(s))^\top$, and $X^*_t$ is the state process corresponding to the control $(\pi^*(t), a^*(t)) \in H^2_F(0, T; \mathbb{R}, \mathbb{P}) \times \bigcup_{q>2} L^q_F(0, T; D, \mathbb{P})$. The flow of BSDEs in (10) is constructed to perturb $J$ in (9) and thus obtain the leading order term of the $\liminf$ for a small value of $\epsilon$ when the terminal condition is set to match the objective function. Although Sun & Guo (2019) extend the perturbation in Hu et al. (2012, 2017b) to incorporate random jumps, in our case, the
unboundedness of the Volterra process creates additional difficulties. We prove the following theorem in Appendix A.2.

**Theorem 2.2:** For any $t \in [0, T]$, $\epsilon > 0$, $\rho_1 \in H^2_F(t, T; \mathbb{R}, \mathbb{P})$, and $\rho_2 \in \bigcup_{q > 2} L^q_F(t, T; \mathbb{D}, \mathbb{P})$, let $(\pi^*(t), a^*(t)) \in H^2_F(0, T; \mathbb{R}, \mathbb{P}) \times \bigcup_{q > 2} L^q_F(0, T; \mathbb{D}, \mathbb{P})$ be a given control pair and $\pi^{t, \epsilon, \rho_1}$ and $a^{t, \epsilon, \rho_2}$ be as defined in (8). Then,

\[
\int (t, X^*; \pi^{t, \epsilon, \rho_1}, a^{t, \epsilon, \rho_2}) - J(t, X^*_t; \pi^*, a^*) = \mathcal{E}_t \int_t^{t+\epsilon} \left[ (\Lambda(s; t), \rho(s)) + (\Theta(s) \rho(s), \rho(s)) \right] ds + o(\epsilon),
\]

where $\rho = (\rho_1, \rho_2 - a^*)^T$,

\[
\Lambda(s; t) = \left( v_1 p^*(s; t) + \sigma(s) Z^*_1(s; t), v_2 p^*(s; t) - \int \sigma(s) Z^*_2(s; z; t) \delta(dz) \right)^T,
\]

and $\Theta(s) = \frac{1}{2} \epsilon^T \sigma(s) \epsilon \begin{pmatrix} \epsilon^T \Sigma \epsilon & 0 \\ 0 & \lambda \end{pmatrix}$ with $(p^*(s; t), Z^*_1(s; t), Z^*_2(s; z; t))$ as defined in (10).

Theorem 2.2 asserts that the leading order term of $\lim \inf$ consists of two parts. The first part is a functional of $\Lambda$, and the second is a functional of $\Theta$. Then, it is sufficient to deduce an equilibrium control $u^*$ that makes both parts nonnegative. However, from this expression, it is clear that $(\Theta \rho, \rho) \geq 0$. Therefore, we present the following proposition.

**Proposition 2.1:** Under the same assumption applied to Theorem 2.2, $(\pi^*(t), a^*(t)) \in H^2_F(0, T; \mathbb{R}, \mathbb{P}) \times \bigcup_{q > 2} L^q_F(0, T; \mathbb{D}, \mathbb{P})$ is an equilibrium control if

\[
\lim \inf_{\epsilon \downarrow 0} \frac{1}{\epsilon} \int_t^{t+\epsilon} \mathcal{E}_s \left[ (\Lambda(s; t), \rho(s)) \right] ds \geq 0, \quad \text{a.s., } \forall t \in [0, T]
\]

for any $\rho_1 \in H^2_F(t, T; \mathbb{R}, \mathbb{P})$ and $\rho_2 \in \bigcup_{q > 2} L^q_F(t, T; \mathbb{D}, \mathbb{P})$, where $\Lambda(s; t)$ is defined as in (11), and $\rho = (\rho_1, \rho_2 - a^*)^T$.

In other words, it is sufficient for us to consider the nonnegativity of the functional of $\Lambda$, which involves $(p^*, Z^*, Z^*_2)$ as a solution of BSDE (10). In fact, $\Lambda$ can be expressed more explicitly as follows; the proof is given in Appendix A.3.

**Proposition 2.2:** Let $(\pi^*(t), a^*(t)) \in H^2_F(0, T; \mathbb{R}, \mathbb{P}) \times \bigcup_{q > 2} L^q_F(0, T; \mathbb{D}, \mathbb{P})$ be a given control pair, and $X^*$ be the corresponding state process. For any $t_1, t_2 \in [0, T]$, the unique solution of BSDE (10) satisfies $Z^*(s; t_1) = Z^*_2(s; t_2)$ and $Z^*_1(s; t_1) = Z^*_2(s; t_2)$ for a.e. $s \geq \max(t_1, t_2)$. Moreover, there exists a stochastic process $\Lambda_0$ valued in $\mathbb{R}^2$ and $\xi \in S^2_F(t, T; \mathbb{R}, \mathbb{P})$ such that

\[
\Lambda(s; t) = \Lambda_0(s) + \mathcal{E}_t^{T} \mathcal{R}(u) \mathcal{E}_t v(s) \xi(t).
\]

By substituting the expression of $\Lambda$ from Proposition 2.2 into Proposition 2.1, we obtain a sufficient and necessary condition for equilibrium control in the following theorem. This condition is useful for deducing an explicit solution to the corresponding equilibrium control.

**Theorem 2.3:** Let $(p^*(s; t), Z^*(s; t), Z^*_2(s; z; t)) \in L^2_F(t, T; \mathbb{R}, \mathbb{P}) \times H^2_F(t, T; \mathbb{R}^2, \mathbb{P}) \times F^2(t, T; \mathbb{R})$ be the unique solution of BSDE (10), then $(\pi^*(t), a^*(t)) \in H^2_F(0, T; \mathbb{R}, \mathbb{P}) \times \bigcup_{q > 2} L^q_F(0, T; \mathbb{D}, \mathbb{P})$ is an open-loop equilibrium control if and only if, for any $\rho_2 \in \bigcup_{q > 2} L^q_F(t, T; \mathbb{D}, \mathbb{P})$,

\[
\begin{cases}
\langle v_1(t) p^*(t; t) + \sigma(t) Z^*_1(t; t; t) \rangle = 0, & \text{a.e. } t \in [0, T], \ \text{a.s.,} \\
\langle v_2(t) p^*(t; t) - \int \sigma(s) Z^*_2(t, z; t) \delta(dz), \rho_2(t) - a^*(t) \rangle \geq 0 & \text{a.e. } t \in [0, T], \ \text{a.s.}
\end{cases}
\]
Although the proof of Theorem 2.3 is presented in Appendix A.4, we explain here the use of this theorem to derive an admissible equilibrium control \((\pi^*, a^*)\). In this case, we must first solve the flow of BSDEs in (10) for \((p^*, Z, Z^T)\) upon the existence of a solution. Here, Theorem 2.3 is useful for verifying a potential solution of \(p^*\). Once an appropriate solution is determined, Theorem 2.3 is also useful for extracting the corresponding equilibrium control \((\pi^*, a^*)\). Although this procedure is similar to others reported in the literature, we encounter the Volterra mortality rate when applying Theorem 2.3. Interestingly, this rate has no effect on the case of constant risk aversion \((\phi_1 = 0)\), such that the classical equilibrium control also works in the VMM. We further prove for the first time that such an equilibrium is unique. However, the proof under consideration is non-trivial, although the equilibrium control agrees with the classical equilibrium control without LRD. The state-dependent case \((\phi_1 > 0)\) presents an additional challenge.

3. Equilibrium strategy under constant risk aversion

Constant risk aversion refers to a setting where \(\phi_1 = 0\) and \(\phi_2 > 0\) in (10). Here, let \(\mathcal{D} = \mathbb{R}_+\). Note that the reinsurance strategy should obey the nonnegative constraint. When \(\phi_1 = 0\), the terminal condition in (10) does not involve \(X^*_T\); therefore, the derivation procedure is very similar to that discussed in Yan & Wong (2020). Fortunately, in the case with \(\phi_1 = 0\), the derivation procedure eliminates the Volterra mortality rate from the solution form considered below in (13). As the BSDE in (10) is linear, the solution is unique.

Inspired by Yan & Wong (2020), consider the solution form for \(p^*(s; t)\) as follows:

\[
p^*(s; t) = M_sX^*_s + \Gamma_s^{(2)} - \mathbb{E}_t[M_sX^*_s + \Gamma_s^{(2)} - \Phi_s],
\]

where \(M_s = e^{\int_t^s 2r(u)\,du}\), \(\Phi_s = -\phi_2 e^{\int_t^s r(u)\,du}\), and \((\Gamma_s^{(2)}, \gamma_s^{(2)})\) is the solution to the following BSDE:

\[
d\Gamma_s^{(2)} = -F_s^{(2)} \, ds + \gamma_s^{(2)} \, dW_s, \quad \Gamma_T^{(2)} = -\phi_2.
\]

Here, the driver \(F_s^{(2)}\) and \(\gamma_s^{(2)} = (\gamma_0^{(2)}(s), \gamma_1^{(2)}(s))^T\) remain to be determined. For a fixed \(t\), we apply the Itô lemma, with jumps at time point \(s\), to the solution form (13), and then compare the coefficients of the resulting SDE with those in (10). This enables us to identify \(F_s^{(2)}\).

After substituting the result obtained for \(p^*\) with an appropriate driver \(F_s^{(2)}\) into Theorem 2.3, we deduce that

\[
\pi^*(s) = -\frac{1}{M_s\sigma^2} (v_1 \Phi_s + \sigma \gamma_1^{(2)}(s)) = -\frac{(\mu(s) - r(s))\Phi_s}{M_s\sigma^2} - \frac{\gamma_1^{(2)}(s)}{M_s\sigma}, \quad \gamma_1^{(2)}(s) = \frac{-v_2 \Phi_s}{k_1\lambda M_s\mathbb{E}[z^2]} = -\frac{\eta \mu \Phi_s}{M_s\mathbb{E}[z^2]} > 0,
\]

where \((\Gamma^{(2)}, \gamma^{(2)})\) is the solution to the BSDE as follows:

\[
\left\{ \begin{array}{l}
d\Gamma_s^{(2)} = -\left\{ r(s)\Gamma_s^{(2)} - \frac{v_1}{\sigma} \gamma_1^{(2)}(s) - \frac{v_1^2}{\sigma^2} + \frac{v_2^2}{k_1\lambda M_s\mathbb{E}[z^2]} \right\} \Phi_s + M_s c(s) \, ds + \gamma_s^{(2)} \, dW_s, \\
\Gamma_T^{(2)} = -\phi_2.
\end{array} \right.
\]

As \(\hat{\lambda}\) is the only stochastic unbounded coefficient in BSDE (17), we consider the case in which \(\gamma_1^{(2)} = 0\) and obtain a BSDE with a unique solution.

**Proposition 3.1:** At a set \(\gamma_1^{(2)} = 0\), BSDE (17) admits a unique solution \((\Gamma^{(2)}, \gamma_0^{(2)}) \in S^q_{\mathcal{F}}(0, T; \mathbb{R}, \mathbb{P}) \times H^q_{\mathcal{F}}(0, T; \mathbb{R}, \mathbb{P})\) for any \(q > 2\).
Proof: Recall that $v_1$, $r$, and $\gamma$ are deterministic functions; 
\[ \frac{v_1^2}{k_1 s} = \frac{\eta^2 \mu^2}{\mathbb{E}[z^2]} k_1 \hat{\lambda}; \]  
and $c = (\theta - \eta) \mu z k_1 \hat{\lambda}$. According to Hölder’s inequality and Lemma 2.3, there exists a constant $C > 0$, such that 
\[ \mathbb{E} \left[ \left( \int_0^T |\hat{\lambda}_s| \, ds \right)^q \right] \leq C \int_0^T \mathbb{E} \left[ |\hat{\lambda}_s|^q \right] \, ds < \infty, \]
for any constant $q > 2$. The result follows according to Theorem 5.1 in El Karoui et al. (1997).

From (15) and (16), it is clear that $(\pi^*(t), a^*(t)) \in H_{\mathcal{F}}^2(0, T; \mathbb{R}, \mathbb{P}) \times \bigcup_{q > 2} L_\mathcal{F}^q(0, T; \mathbb{R}_+, \mathbb{P})$ when $\gamma_1^{(2)} = 0$. According to Theorem 2.1, we obtain the following equilibrium RI strategy.

**Theorem 3.1:** Let an explicit equilibrium control to Problem (7) with constant risk aversion $\phi_2 > 0$ be given as
\[ \pi^*(s) = \frac{\mu(s) - r(s)}{\sigma(s)^2} \phi_2 e^{-\int_s^T r(u) \, du}, \quad a^*(s) = \frac{\eta \mu z}{\mathbb{E}[z^2]} \phi_2 e^{-\int_s^T r(u) \, du} > 0. \]  

The equilibrium RI strategy in (18) is a pair of deterministic functions, which are the same as their Markovian counterparts in the literature. Theorem 3.1 shows that the LRD feature of the mortality model has no effect on the investment and reinsurance strategies under constant risk aversion. This is in sharp contrast to the longevity hedging results in Wang & Wong (2021), in which LRD significantly affects longevity hedging. In other words, reinsurance as an actuarial risk management tool is robust to the LRD property. Furthermore, this equilibrium control pair is unique, as proven in Section 3.2.

**Remark 3.1:** When constant risk aversion $\phi_2 = 0$, the insurer’s objective in Problem (7) is to minimize the variance. In this case, the equilibrium control in (18) becomes $\pi^* = a^* \equiv 0$. Therefore, the insurer does not invest in the risky asset, puts all of the money in the bank and transfers all of her risk to reinsurance ($a^* \equiv 0$ indicates that the reinsurer pays all insurance claims). In other words, if the insurer refuses to take a risk, a remedy is the cessation of business.

Although we obtain an equilibrium RI strategy in Theorem 3.1, it remains unclear whether this equilibrium is unique and whether the upper bound of the reinsurance strategy is 1. The boundedness condition eliminates the possibility of the insurer acquiring new (reinsurance) business.

### 3.1. The $[0, 1]$ interval constraint on reinsurance

If we impose $a(t) \in [0, 1]$ as a bounded constraint on the reinsurance strategy, then we set $\mathcal{D} = [0, 1]$ in Theorems 2.1 to 2.3 and reiterate the derivation procedure on such a constraint, using (10) and Theorem 2.3.

Consider the solution form for $\rho^*(s; t)$, as in (13). For a fixed $t$, we apply the Itô formula at $s$ and find that $\rho^*(t; t) = \Phi_t, Z_0^*(t; t) = \gamma_1^{(2)}(t), Z_1^*(t; t) = M_t \pi^*(t) \sigma(t) + \gamma_1^{(2)}(t),$ and $Z_2^*(t, z; t) = -M_t a^*(t) z$. Furthermore, we set $\gamma_1^{(2)}(t) = 0$. According to (12),
\[ \pi^*(t) = -\frac{1}{M_t \sigma^2} v_1 \Phi_t; \]
\[ \langle v_2(t) \Phi_t + M_t \mathbb{E}[z^2] k_1 \hat{\lambda}_t a^*(t), \rho_2(t) - a^*(t) \rangle \geq 0, \]
for any \( \rho_2 \in \bigcup_{q>2} L^q_\mathcal{F}(t, T; \mathcal{D}, \mathbb{P}) \). Hence, we deduce that
\[
a^\ast(t) = \operatorname{Proj}_\mathcal{D} \left( -\frac{\nu_2 \Phi_t}{k_1 \lambda_1 M_t \mathbb{E}[z^2]} \right) = \operatorname{Proj}_\mathcal{D} \left( -\frac{\eta \mu z \Phi_t}{M_t \mathbb{E}[z^2]} \right),
\]
where \( \operatorname{Proj}_\mathcal{D}(x) \) represents the projection of a point \( x \in \mathbb{R} \) onto \( \mathcal{D} \). We refer to Hiriart-Urruty & Lemarchal (2004) for detailed information about projections. Thus, \((\Gamma^{(2)}, \gamma_0^{(2)})\) becomes the solution to the BSDE.
\[
\begin{aligned}
\operatorname{d}\Gamma^{(2)}_s &= - \left\{ r(s) \Gamma^{(2)}_s - \frac{\nu_2}{\sigma^2} (s) + M_s \nu_2(s) \operatorname{Proj}_\mathcal{D} \left( -\frac{\eta \mu z \Phi_t}{M_t \mathbb{E}[z^2]} \right) + M_s c(s) \right\} \operatorname{d}s + \gamma_0^{(2)}(s) \operatorname{d}W_0(s), \\
\Gamma^{(2)}_T &= -\phi_2.
\end{aligned}
\]

**Proposition 3.2:** BSDE (20) admits a unique solution, \((\Gamma^{(2)}, \gamma_0^{(2)}) \in S^q_\mathcal{F}(0, T; \mathbb{R}, \mathbb{P}) \times H^q_\mathcal{F}(0, T; \mathbb{R}, \mathbb{P})\), for any \( q > 2 \).

**Proof:** Recall that \( \nu_2 = \eta \mu z k_1 \hat{\lambda}_1 \), and \( c = (\theta - \eta) \mu z k_1 \hat{\lambda}_1 \). Lemma 2.3 clearly demonstrates that \( \hat{\lambda}_1 \in H^q_\mathcal{F}(0, T; \mathbb{R}, \mathbb{P}) \) for any \( q > 2 \). The result follows according to Theorem 5.1 in El Karoui et al. (1997).

As both \( \pi^\ast \) and \( a^\ast \) are bounded, \((\pi^\ast(t), a^\ast(t)) \in H^2_\mathcal{F}(0, T; \mathbb{R}, \mathbb{P}) \times \bigcup_{q>2} L^q_\mathcal{F}(0, T; \mathcal{D}, \mathbb{P})\). The following theorem is immediate.

**Theorem 3.2:** An admissible equilibrium control to Problem (7), with constraint \([0, 1]\) on the reinsurance control and constant risk aversion \( \phi_2 > 0 \), is given by
\[
\pi^\ast(s) = \frac{\mu(s) - r(s)}{\sigma(s)^2} \phi_2 e^{-\int_s^T r(u) \operatorname{d}u}, \quad a^\ast(s) = \operatorname{Proj}_\mathcal{D} \left( \frac{\eta \mu z}{\mathbb{E}[z^2]} \phi_2 e^{-\int_s^T r(u) \operatorname{d}u} \right).
\]

The strategies in (21) are implemented in a straightforward manner. In fact, the RI strategy \((\pi^\ast(t), a^\ast(t))\) is exactly the same as that in Theorem 3.1 when \( a^\ast(t) \leq 1 \). Otherwise, if \( a^\ast(t) \leq 1 \) in Theorem 3.1, we obtain exactly the same \( \pi^\ast \) but must return \( a^\ast(t) \) to 1 to fulfill the upper bound. The uniqueness of this equilibrium strategy is demonstrated below.

### 3.2. Uniqueness of the equilibrium control

**Theorem 3.3:** Under Assumption 4.2, the control \( u^\ast = (\pi^\ast, a^\ast) \), given by (18), is the unique equilibrium control for the RI problem (7) when \( \phi_1 = 0 \).

**Proof:** Suppose that an alternative admissible equilibrium control pair, \((\pi, a)\), exists with the corresponding state process \( X \). By replacing \( X^\ast \) with \( X \), BSDE (10) admits a unique solution \((p(s,t), Z(s,t), \bar{Z}_2(s,z; t))\). This satisfies the condition in (12), where \( Z(s,t) = (Z_0(s,t), Z_1(s,t))^T \).

Define
\[
\begin{aligned}
\tilde{p}(s,t) &= p(s,t) - \left( M_s X_s + \Gamma^{(2)}_s - \mathbb{E}_t[M_s X_s + \Gamma^{(2)}_s - \Phi_s] \right), \\
\tilde{Z}_0(s,t) &= Z_0(s,t) - \gamma_0^{(2)}(s), \quad \tilde{Z}_1(s,t) = Z_1(s,t) - M_s \pi(s) \sigma(s), \quad \tilde{Z}_2(s,z; t) = Z_2(s,z; t) + M_s a(s) z,
\end{aligned}
\]

using the definitions of \( M, \Gamma^{(2)}, \Phi, \) and \( \gamma_0^{(2)} \) provided in Section 3.
According to Proposition 3.1, \((\bar{p}(s; t), \bar{Z}(s; t), \bar{Z}_2(s, z; t)) \in L^2_\mathcal{F}(t, T; \mathbb{R}, \mathbb{P}) \times H^2_\mathcal{F}(t, T; \mathbb{R}^2, \mathbb{P}) \times F^2(t, T; \mathbb{R})\). Substituting this into (12), we obtain

\[
\begin{align*}
\left\{ \begin{array}{l}
\nu_1(t) \left[ \bar{p}(t; t) + \Phi_t \right] + \sigma(t) \left[ \bar{Z}_1(t; t) + M_t \pi(t) \sigma(t) \right] = 0, \\
\nu_2(t) \left[ \bar{p}(t; t) + \Phi_t \right] - k_1 \tilde{\lambda}_t \int_{\mathbb{R}_+} z \bar{Z}_2(t, z; t) - M_t a(t) z \, dz, \rho_2(t) - a(t) \right\} \geq 0
\end{array} \right.
\end{align*}
\tag{22}
\]

for any \(\rho_2 \in \bigcup_{q \geq 2} L^2_\mathcal{F}(t, T; \mathbb{R}^+, \mathbb{P})\). Therefore,

\[
\begin{align*}
\pi(t) &= -\frac{\nu_1}{M_t \sigma(t)^2} \Phi_t - \frac{1}{M_t \sigma(t)^2} \nu_1(t) \bar{p}(t; t) + \sigma(t) \bar{Z}_1(t; t) \\
&= \pi^*(t) - \frac{1}{M_t \sigma(t)^2} \nu_1(t) \bar{p}(t; t) + \sigma(t) \bar{Z}_1(t; t) \triangleq \pi^*(t) + D^\pi(t), \\
a(t) &= \frac{1}{k_1 \tilde{\lambda}_t M_t \mathbb{E}[z^2]} \left[ -\nu_2 \Phi_t - \left( \nu_2 \bar{p}(t; t) - k_1 \tilde{\lambda}_t \int_{\mathbb{R}_+} z \bar{Z}_2(t, z; t) \, dz \right) \right]^+ \\
&= a^*(t) - \frac{A^c_t}{k_1 \tilde{\lambda}_t M_t \mathbb{E}[z^2]} \left( \nu_2 \bar{p}(t; t) - k_1 \tilde{\lambda}_t \int_{\mathbb{R}_+} z \bar{Z}_2(t, z; t) \, dz \right) \triangleq a^*(t) + D^a_t(t),
\end{align*}
\]

where \(0 \leq A^c_t \leq 1\) is a bounded adapted process. To prove the uniqueness of the equilibrium control, we must demonstrate that \(D^\pi_t(t) \equiv D^a_t(t) \equiv 0\) as follows. Here,

\[
\begin{align*}
\bar{p}(s; t) &= \left\{ -r(s) \bar{p}(s; t) - \nu_1 D^\pi_t(s) M_s - \nu_2 D^a_t(s) M_s + \mathbb{E}_t[\nu_1 D^\pi_t(s) M_s + \nu_2 D^a_t(s) M_s] \right\} \, ds \\
&\quad + \bar{Z}(s; t) \, dW_s + \int_{\mathbb{R}_+} \bar{Z}_2(s, z; t) \, d\tilde{N}(ds, dz), \\
\bar{p}(T; t) &= 0, \quad s \in [t, T].
\end{align*}
\tag{23}
\]

As the interest rate \(r(\cdot)\) is a bounded deterministic function, we take \(r \equiv 0\) without loss of generality. Taking the conditional expectation on both sides of (23), we get \(\mathbb{E}_t[\bar{p}(s; t)] = 0\) for \(s \geq t\). Specifically, \(\bar{p}(t; t) = 0\). Hence, \(D^\pi_t(t) = -\bar{Z}_1(t; t) / M_t \sigma(t)\), and \(D^a_t(t) = A^c_t / M_t \mathbb{E}[z^2] \int_{\mathbb{R}_+} z \bar{Z}_2(t, z; t) \, dz\). Then,

\[
\begin{align*}
\nu_1 D^\pi_t(t) M_t &= -\frac{\mu(t) - t \nu_1(t)}{\sigma(t)^2} \bar{Z}_1(t; t), \\
\nu_1 D^a_t(t) M_t &= \frac{\nu_2 M_t \mathbb{E}[z^2]}{\sigma(t)^2} \int_{\mathbb{R}_+} z \bar{Z}_2(t, z; t) \, dz.
\end{align*}
\]

According to Proposition 2.2, we obtain \(\bar{Z}(s; t_1) = Z(s; t_2)\) and \(\bar{Z}_2(s, z; t_1) = \bar{Z}_2(s, z; t_2)\) for a.e. \(s \geq \max(t_1, t_2)\). We define \(\Delta_c(t) = -\theta_1(t) \bar{Z}_1(t; t) + \int_{\mathbb{R}_+} \theta_2(t, z) \bar{Z}_2(t, z; t) \, d\delta(dz)\) and \(\bar{p}(s; t) = \bar{p}(s; t) - \int_t^T \mathbb{E}_t[\Delta_c(u)] \, du\), where \(\theta_1(s) = \frac{\mu(s) - t \nu_1(s)}{\sigma(s)}\) and \(\theta_2(s, z) = \frac{\nu_2 M_t \mathbb{E}[z^2]}{\sigma(s)^2} z \geq 0\). Then,

\[
\begin{align*}
\bar{p}(s; t) &= -\nu_1 D^\pi_t(s) M_s - \nu_2 D^a_t(s) M_s \, ds + \bar{Z}(s; t) \, dW_s + \int_{\mathbb{R}_+} \bar{Z}_2(s, z; t) \, d\tilde{N}(ds, dz), \\
\bar{p}(T; t) &= 0, \quad s \in [t, T].
\end{align*}
\tag{24}
\]

We then introduce a new measure, \(\mathbb{Q}\), on \(\mathcal{F}_t\) by \(\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E}_t\), where

\[
\mathcal{E}_t = \exp \left\{ -\int_t^t \theta_1(s) \, dW_1(s) - \frac{1}{2} \int_0^t \theta_1(s)^2 \, ds \\
+ \int_0^t \int_{\mathbb{R}_+} \ln(1 + \theta_2(s, z)) \tilde{N}(ds, dz) + \int_0^t \int_{\mathbb{R}_+} [\ln(1 + \theta_2(s, z)) - \theta_2(s, z)] \delta(dz) \, ds \right\}.
\tag{25}
\]
Note that if $\theta_2(s, z) \geq 0$, then $\ln(1 + \theta_2(s, z)) \leq \theta_2(s, z)$. If $C_2 > k_1 \frac{\mu_\ast}{\|z\|^2}$ as in Assumption 4.2, then the Novikov condition,
\[
\mathbb{E} \left[ \exp \left( \frac{1}{2} \int_0^T \theta_1(s)^2 \, ds + \int_0^T \int \{ (1 + \theta_2(s, z)) \ln(1 + \theta_2(s, z)) - \theta_2(s, z) \} \delta(dz) \, ds \right) \right] < \infty,
\]
is satisfied. Using measure $\mathcal{Q}$, we obtain
\[
d\tilde{p}(s; t) = \tilde{Z}_0(s; t) \, dW_0(s) + \tilde{Z}_1(s; t) \, dW_1^\mathcal{Q}(s) + \int \tilde{Z}_2(s, z; t) \, d\widetilde{N}^\mathcal{Q}(ds, dz),
\]
where $W_1^\mathcal{Q}(s)$ and $\widetilde{N}^\mathcal{Q}(ds, dz)$ are the standard Brownian motion and compensated Poisson random measure under $\mathcal{Q}$, respectively.

For a set $q_0 \in (1, 2)$, and given that $z$ has finite moments, according to Lemma 2.3, there exists a constant $C$ such that
\[
\mathbb{E} \left[ \sup_{s \in [t, T]} \left| \int \mathbb{E}[\Delta_c(u)] \, du \right|^{q_0} \right] \leq C \mathbb{E} \left[ \left( \int \mathbb{E}[\Delta_c(u)] \, du \right)^{q_0} \right] \leq C \left\{ \mathbb{E} \left[ \int \tilde{Z}_1(s; s)^2 \, ds \right]^{q_0} \right\}^{\frac{1}{q_0}} + C \left\{ \left( \mathbb{E} \left[ \int \tilde{Z}_2(s, z; s)^2 \delta(dz) \, ds \right]^{\frac{q_0}{2}} \right)^{\frac{2}{q_0}} \right\} < \infty.
\]

Hence, $\mathbb{E}[\sup_{t \leq s \leq T} |\tilde{p}(s; t)|^{q_0}] < \infty$. For any $m \geq 1$, assume that $\mathbb{E}[\mathcal{E}_T^m] < \infty$; then, for the aforementioned $q_0, q_1 \in (1, q_0)$ exists such that
\[
\mathbb{E}^\mathcal{Q} \left[ \sup_{t \leq s \leq T} |\tilde{p}(s; t)| \right] \leq \left\{ \mathbb{E} \left[ \sup_{t \leq s \leq T} |\tilde{p}(s; t)|^{q_1} \right] \right\}^{\frac{1}{q_1}} \left\{ \mathbb{E} \left[ \mathcal{E}_T^{q_1 - 1} \right] \right\}^{\frac{q_1 - 1}{q_1}} < \infty.
\]

If $m \geq 2$, then $q_0$ and $q_1$ values exist such that the above boundedness holds. We analyze the condition $\mathbb{E}[\mathcal{E}_T^2] < \infty$, where $\mathcal{E}$ is given in (25). Note that $\theta_1$ is a bounded deterministic function and $\theta_2(s, z) \geq 0$. Therefore,
\[
\mathbb{E}[\mathcal{E}_T^2] \leq \mathbb{E} \left[ \exp \left( \int_0^T \int \ln(1 + \theta_2(s, z)) \tilde{N}(ds, dz) \right) \right] + \mathbb{E} \left[ \exp \left( \int_0^T \int \ln(1 + \theta_2(s, z))^2 \tilde{N}(ds, dz) \right) \right] \leq \mathbb{E} \left[ \exp \left( \sum_{i=1}^{N_T} \ln(1 + \theta_2(t_i, z_i))^2 \right) \right]
\]
for a constant $C$. Let $\mathbb{E}[(1 + \frac{\eta \mu_z}{\|z\|^2})^2] \triangleq A_0$ for a constant $A_0$. Then, we obtain
\[
\mathbb{E} \left[ \left( 1 + \frac{\eta \mu_z}{\|z\|^2} \right)^2 \right] = \mathbb{E} \left[ e^{(\ln A_0)N_T} \right] = \mathbb{E} \left[ e^{(\ln A_0)N_T} \right] \mathbb{E} \left[ (\tilde{\lambda}[0, T]) N_T \right]
\]
\[
= \mathbb{E} \left[ \exp \left( (A_0 - 1) \int_0^T k_1 \lambda_s \, ds \right) \right].
\]
As \( \mu_z^2 \leq \mathbb{E}[z^2], A_0 \leq 1 + \frac{2\eta\mu_z^2}{\mathbb{E}[z^2]} + \frac{\eta^2\mu_z^2}{\mathbb{E}[z^2]} \leq (1 + \eta)^2 \). There exists a constant value of \( C \), such that \( \mathbb{E}[\mathcal{E}_t^2] \leq C \mathbb{E}[\exp((2\eta + \eta^2) \int_0^T k_1\lambda_t ds)] \). If \( C_2 \geq k_1(2 + \eta)\eta \) as in Assumption 4.2, then \( \mathbb{E}[\mathcal{E}_T^2] < \infty \). Therefore, \( \tilde{p}(s; t) \) is a \( \mathbb{Q} \)-martingale, \( \tilde{p} = \tilde{Z} \equiv \tilde{Z}_2 \equiv 0 \), and \( D_1^c \equiv 0 \) and \( D_2^c \equiv 0 \). Following this, the equilibrium control \((\pi^*, a^*)\) derived by (18) is unique.

From the above proof, we recognize that \( C_2 \geq k_1(2 + \eta)\eta \) is sufficient to confirm the admissibility and uniqueness of equilibrium control (18) in a case with constant risk aversion.

We next study the uniqueness of the equilibrium strategy under the interval reinsurance constraint.

**Theorem 3.4:** Under Assumption 4.2, the strategy in (21) represents the unique equilibrium control to Problem (7), where \( \phi_1 = 0 \) and \( a \in [0, 1] \) is the constraint.

**Proof:** Suppose instead that there is an alternative admissible equilibrium control pair, \((\pi, a) \in H_2^2(0, T; \mathbb{R}, \mathbb{P}) \times \bigcup_{q>1} L^q_T(0, T; \mathbb{D}, \mathbb{P}), \) with the corresponding state process \( X \) and \( D = [0, 1] \). By replacing \( X^* \) with \( X, \) BSDE (10) admits a unique solution \((p(s; t), Z(s; t), Z_2(s, z; t))\). This satisfies the condition in (12), where \( Z(s; t) = (Z_0(s; t), Z_1(s; t))^T \). We define

\[
\tilde{p}(s; t) = p(s; t) - \left( M_sX_s + \Gamma_s^2 - E_t[M_sX_s + \Gamma_s^2 - \Phi_1] \right),
\]

\[
\tilde{Z}_0(s; t) = Z_0(s; t) - \gamma_0^2(s), \quad \tilde{Z}_1(s; t) = Z_1(s; t) - M_s\pi(s)\sigma(s),
\]

\[
\tilde{Z}_2(s, z; t) = Z_2(s, z; t) + M_s a(s)z,
\]

Using \( M, \Gamma, \Phi, \) and \( \gamma_0^2 \) as defined in Section 3.1. Substituting the above definitions into (12), we obtain

\[
\begin{aligned}
\begin{cases}
\nu_1(t) \left[ \tilde{p}(t; t) + \Phi_1 \right] + \sigma(t) \left[ \tilde{Z}_1(t; t) + M_t\pi(t)\sigma(t) \right] = 0, \\
\nu_2(t) \left[ \tilde{p}(t; t) + \Phi_1 \right] - k_1\lambda_t \int_{\mathbb{R}^+} z\tilde{f}(z) \left[ \tilde{Z}_2(t, z; t) - M_t a(t)z \right] dz, \rho_2(t) - a(t) \geq 0
\end{cases}
\end{aligned}
\]

for any \( \rho_2 \in \bigcup_{q>1} L^q_T(t, T; \mathbb{D}, \mathbb{P}). \) Therefore, \( \pi(t) = -\frac{\nu_1}{M_t\sigma(t)^2} \Phi_t - \frac{1}{M_t\sigma(t)^2} (\nu_1(t)\tilde{p}(t; t) + \sigma(t)\tilde{Z}_1(t; t)) \)

\[
= \pi^*(t) - \frac{1}{M_t\sigma(t)^2} (\nu_1(t)\tilde{p}(t; t) + \sigma(t)\tilde{Z}_1(t; t)) \triangleq \pi^*(t) + \tilde{D}_1(t),
\]

\[
a(t) = \text{Proj}_D \left\{ \frac{1}{k_1\lambda_t M_t\mathbb{E}[z^2]} \left[ -\nu_2 \Phi_t - \left( \nu_2 \tilde{p}(t; t) - k_1\lambda_t \int_{\mathbb{R}^+} z\tilde{f}(z) \tilde{Z}_2(t, z; t) dz \right) \right] \right\},
\]

\[
= \text{Proj}_D \left\{ -\frac{\eta_\mu_z \Phi_t}{M_t\mathbb{E}[z^2]} \right\} + \text{Proj}_D \left\{ -\frac{1}{k_1\lambda_t M_t\mathbb{E}[z^2]} \left( \nu_2 \tilde{p}(t; t) - k_1\lambda_t \int_{\mathbb{R}^+} z\tilde{f}(z) \tilde{Z}_2(t, z; t) dz \right) \right\},
\]

\[
= a^*(t) - \frac{\tilde{A}_t}{k_1\lambda_t M_t\mathbb{E}[z^2]} \left( \nu_2 \tilde{p}(t; t) - k_1\lambda_t \int_{\mathbb{R}^+} z\tilde{f}(z) \tilde{Z}_2(t, z; t) dz \right) \triangleq a^*(t) + \tilde{D}_2(t),
\]

where \( 0 \leq \tilde{A}_t \leq 1 \) is a bounded adapted process. By following a method similar to the proof of Theorem 3.3, we derive a unique control in (21) when \( C_2 \geq k_1(2 + \eta)\eta \) in Assumption 4.2.

Therefore, the equilibrium controls for the reinsurance policy situations with and without the boundedness constraint are unique in their respective settings. This finding is important because
it covers both Markovian models and non-Markovian VMM situations with respect to the mortality rate. The resulting strategies are Markovian and independent of the historical mortality rate for insurers with constant risk aversion. The result for the state-dependent case is rather different, as shown in the following section.

4. Equilibrium strategy under state-dependent risk aversion

Consider a case of state-dependent risk aversion when the reinsurance company offers a cheap reinsurance premium, where $\phi_2 = 0$ and $\phi_1 > 0$. In other words, $c \equiv 0$ or, equivalently, $\eta = \theta$. Here, let $\mathcal{D} = [0, \infty)$.

Consider the following solution form for BSDE (10):

$$p^*(s, t) = M_s X^*_s - \Gamma^{(1)}_t X^*_t - \mathbb{E}_t[M_s X^*_s],$$

where $\Gamma^{(1)}_t = \phi_1 \int_t^T r(s) \, ds$, $(M, U)$ is a solution to the BSDE, as follows:

$$dM_s = -F_s \, ds + U_s^\top \, dW_s, \quad M_T = 1,$$

where $U_s = (U_0(s), U_1(s))^\top$ represents a two-dimensional vector.

After suppressing the dependence of $s$, we repeat the procedure in a manner similar to that described in Section 3. We deduce that

$$\pi^*(t) = \frac{1}{M_t \sigma(t)} \left( \frac{\nu_1(t) \Gamma^{(1)}_t}{\sigma(t)} - U_1(t) \right) X^*_t,\quad a^*(t) = \frac{\nu_2(t)}{M_t \hat{\lambda}_s \mathbb{E}[z^2]} \Gamma^{(1)}_t X^*_t,$$

where $(M, U)$ is a solution to the BSDE:

$$\begin{cases} dM_s = \left\{ -2r(s)M_s + \left( \frac{\nu_1(s)}{\sigma(s)} - \frac{\nu_1(s) \Gamma^{(1)}_s}{M_s \sigma(s)} \right) U_1(s) + \frac{U_2^2(s)}{M_s} - \left( \frac{\nu_1(s)^2}{\sigma(s)^2} + \frac{\nu_2(s)^2}{k_1 \hat{\lambda}_s \mathbb{E}[z^2]} \right) \Gamma^{(1)}_s \right\} ds + U_s^\top \, dW_s, \\ M_T = 1. \end{cases}$$

As $\nu_1(s) = \mu(s) - r(s)$ is deterministic and $\nu_2(s) = \eta \mu z_k \hat{\lambda}_s$, $\hat{\lambda}_s$ is the only stochastic parameter among the coefficients of the above BSDE. Therefore, $U_1(\cdot) \equiv 0$ and we prove that the resulting BSDE admits a unique solution. Specifically,

$$\begin{cases} dM_s = \left\{ -2r(s)M_s - \left( \frac{\nu_1(s)^2}{\sigma(s)^2} + \frac{\nu_2(s)^2}{k_1 \hat{\lambda}_s \mathbb{E}[z^2]} \right) \Gamma^{(1)}_s \right\} ds + U_0(s) \, dW_0(s), \\ M_T = 1. \end{cases}$$

**Proposition 4.1:** BSDE (31) admits a unique solution $(M, U_0) \in \mathcal{S}_q^2(0, T; \mathbb{R}, \mathbb{P}) \times \mathcal{H}_q^q(0, T; \mathbb{R}, \mathbb{P})$ for any $q > 2$.

**Proof:** In this setting, $r$ and $\frac{\nu_1}{\sigma}$ are deterministic functions. Here, $\frac{\nu_2^2}{k_1 \hat{\lambda}_s \mathbb{E}[z^2]} = \eta^2 \mu^2 k_1 \hat{\lambda}$. According to Lemma 2.3, $\hat{\lambda} \in L_q^q(0, T; \mathbb{R}, \mathbb{P})$ for any $q > 2$. The result follows according to Theorem 5.1 in El Karoui et al. (1997).

**Proposition 4.2:** The explicit solution to BSDE (31) is given by

$$M_t = e^{\int_t^T 2r(s) \, ds} + \int_t^T e^{\int_t^u 2r(v) \, dv} \left( \frac{(\mu(s) - r(s))^2}{\sigma(s)^2} + \frac{\mu^2 \eta^2}{\mathbb{E}[z^2]} k_1 \mathbb{E}[\hat{\lambda}_s | \mathcal{F}_t] \right) \Gamma^{(1)}_s \, ds,$$
where
\[
\mathbb{E}[\hat{\lambda}_s | \mathcal{F}_t] = l(s) + \left(1 - \int_0^s R_B(u) \, du\right) \lambda_0 + \int_0^s E_B(s-u)(b_1 - a_1 \lambda_u) \, du \\
+ \int_0^t E_B(s-u)\sigma_\lambda \sqrt{\lambda_u} \, dW(0(u)); \tag{33}
\]

here, \( B = -a_1, R_B \) is the resolvent of \(-KB\), and \( E_B = K - R_B \ast K \). Furthermore, \( M_t \geq 1 \) for \( t \in [0, T] \).

**Proof:** From (31), we can easily see that
\[
M_t = \mathbb{E}\left[ e^{\int_t^T 2r(s) \, ds} + \int_t^T e^{\int_u^T 2r(u) \, du} \left( \frac{v_1(s)^2}{\sigma(s)^2} + \frac{v_2(s)^2}{k_1 \lambda_s \mathbb{E}[z^2]} \right) \Gamma_s^{(1)} \, ds \right] \\
= e^{\int_t^T 2r(s) \, ds} + \int_t^T e^{\int_u^T 2r(u) \, du} \left( \frac{(\mu(s) - r(s))^2}{\sigma(s)^2} + \frac{\mu_2^2 \eta^2}{k_1 \mathbb{E}[\lambda_s | \mathcal{F}_t]} \right) \Gamma_s^{(1)} \, ds.
\]

The result follows from Lemma 2.2.

As \( \hat{\lambda} \geq 0 \), the conditional expectation \( \mathbb{E}[\hat{\lambda}_s | \mathcal{F}_t] \geq 0 \) when \( 0 \leq t \leq s \leq T \). The interest rate \( r > 0 \). According to the representation of \( M_t \) in (32), we know that \( M_t \geq 1 \) for \( t \in [0, T] \). ■

Propositions 4.1 and 4.2 provide the unique explicit solution \((M, U)\) to BSDE (31). Combining it with Equation (30), we obtain an explicit expression of the equilibrium control. In the following, we prove its admissibility. The equilibrium control depends on \( X^* \) with state-dependent risk aversion. We also encounter a challenge posed by the unbounded Volterra process \( \hat{\lambda} \). We must make some additional assumptions to prove admissibility.

**Assumption 4.1:** A sufficiently large constant \( C_1 \) exists, such that
\[
\mathbb{E}\left[ \exp\left( C_1 \int_0^T U_0^2(s) \, ds \right) \right] < \infty.
\]

**Assumption 4.2:** A sufficiently large constant \( C_2 \) exists, such that
\[
\mathbb{E}\left[ \exp\left( C_2 \int_0^T \lambda_s \, ds \right) \right] < \infty.
\]

Assumptions 4.1 and 4.2 are similar to those made by Han & Wong (2019) and Yan & Wong (2019, 2020). These assumptions are regularities for unbounded parameters under Brownian filtration. However, we also encounter jumps. To enable the use of Itô’s calculus under a Poisson random measure, we must regulate the randomness of the claim size \( z \).

**Assumption 4.3:**
\[
\phi_1 \eta \mu_z \max\{z\} \leq \mathbb{E}[z^2].
\]

According to Assumption 4.3, the claim size \( z \) has an upper bound associated with the safety loading parameter \( \eta \), which is often a small value.
Theorem 4.1: As Assumptions 4.1–4.3 hold, there exists an admissible equilibrium control to Problem (7) under state-dependent risk aversion, as follows:

$$\pi^*(s) = \frac{\mu(s) - r(s)}{M_\sigma(s)^2} \phi_1 e^{\int_0^T (u_t) \, du} \, \mathcal{X}^*_s, \quad a^*(s) = \frac{\eta z}{M_\sigma(s)^2} \phi_1 e^{\int_0^T (u_t) \, du} \, \mathcal{X}^*_s > 0,$$

where $M$ is given in (??). Furthermore, $(\pi^*(s), a^*(s)) \in H^2_{\mathcal{F}}(0, T; \mathbb{R}, \mathbb{P}) \times \bigcup_{q > 2} L^q_{\mathcal{F}}(0, T; D, \mathbb{P})$, $X \in S^2_{\mathcal{F}}(0, T; \mathbb{R}^+ \cup \{0\}, \mathbb{P})$, and $a^*(s) \leq \frac{\mu(s)}{M_\sigma(s)^2} \phi_1 \mathcal{X}^*_s$.

Theorem 4.1 renders the explicit equilibrium control for a TCMV insurer with state-dependent risk aversion and ensures its admissibility. Note that the explicit form of $M$ given in Proposition 4.2 depends on the mortality rate $\hat{\lambda}$. As shown by (34), the LRD feature of the mortality rate affects the investment and reinsurance strategies under state-dependent risk aversion. In addition, the equilibrium strategies are proportional to the surplus level of the insurer.

Assumption 4.3 may be considered restrictive, although it incorporates the case of deterministic insurance claims. It is necessary to ensure the admissibility of the control. As the positive reinsurance strategy is proportional to the surplus process, $X_t$ should be a positive process. According to Assumption 4.3, Ito’s formula is applicable with respect to the Poisson random measure on $M_tX_t$, and $X_t$ is clearly positive in the proof of Theorem 4.1.

If Assumption 4.3 is violated for a practical model, we suggest setting the reinsurance and investment strategies to 0 once the surplus process reaches 0. In this way, the same form as (34) remains useful whenever $X_t$ is positive. Such a strategy pair is admissible and applicable in practice. The insurer will stop business once the surplus reaches 0.

We numerically examine the probability of having a negative surplus when Assumption 4.3 is violated in the next section. In fact, the probability is close to 0 in our numerical study.

We further analyze Assumption 4.1. As $U_0$ is given in (32), for any constant $C_1$,

$$\mathbb{E} \left[ \exp \left( C_1 \int_0^T U^2_0(s) \, ds \right) \right] = \mathbb{E} \left[ \exp \left( C_1 \int_0^T \left( \int_t^T e^{\int_0^r (u_t) \, du} \, du \right)^2 \mathcal{X}^*_t \, dt \right) \right],$$

where $E_B$ is defined as in Proposition 4.2. Hence, Assumption 4.1 holds when

$$C_2 \geq C_1 \sup_{0 \leq t \leq T} \left| \int_t^T e^{\int_0^r (u_t) \, du} \, du \mathcal{X}^*_t \right| k_1 E_B(s - t) \, \sigma \chi \, ds \right) \right|^2$$

in Assumption 4.2. Note that $\mu_z^2 \leq \mathbb{E}[z^2]$. According to the proof of Theorem 4.1, therefore, if $C_2$ in Assumption 4.2 satisfies

$$C_2 > \max \left\{ 125 \eta^4 \sigma \chi^2 \sup_{0 \leq t \leq T} \left| \int_t^T e^{\int_0^r (u_t) \, du} k_1 E_B(s - t) \, ds \right|^2, 18 \eta \phi_1 k_1 \right\},$$

it is sufficient to ensure the admissibility of the equilibrium control provided in (34).

4.1. Uniqueness of the equilibrium control

We further establish the uniqueness of the equilibrium control under state-dependent risk aversion. A detailed proof is provided in Appendix A.6.

Theorem 4.2: Suppose that Assumption 4.2 holds. Let $M$ be defined by (??). For the case involving state-dependent risk aversion, the control $u^*(s) = (\pi^*, a^*)$ given by (34) is the unique equilibrium control for the RI Problem (7) in which $\phi_1 > 0$ and $\phi_2 = 0$. 
By combining this theorem with the result of our analysis of admissibility, we discover that

$$C_2 \geq \max \left\{ k_1 (2 + \eta) \eta, 125 \eta^4 \sigma^2 \sup_{0 \leq t \leq T} \left| \int_t^T e^{\int_t^s 2r(u) du} k_1 E_B(s - t) ds \right|^2, 18 \eta \phi_1 k_1 \right\}$$ (35)

in Assumption 4.2 is sufficient to ensure both the admissibility and uniqueness of the equilibrium control in (34) under state-dependent risk aversion, where $E_B$ is defined as in Proposition 4.2. Although the value of 125 seems to be large, $\sigma_\lambda$ and the safety loading factor are usually very small. In other words, a large value of $C_2$ is not required in Assumption 4.2. Below, we give a sufficient condition under which Assumption 4.2 holds.

**Proposition 4.3:** For $\lambda$ defined in (2) and any positive constant $C_2$, if the condition $a_1^2 - 2C_2 \sigma_\lambda^2 > 0$ holds, then

$$\mathbb{E} \left[ \exp \left( C_2 \int_0^T \lambda_s ds \right) \right] < \infty.$$  

**Proof:** If $a_1^2 - 2C_2 \sigma_\lambda^2 > 0$, then according to Lemma A.1 in Han & Wong (2021), the Riccati equation,

$$\psi = \left( C_2 - a_1 \psi + \frac{1}{2} \sigma_\lambda^2 \psi^2 \right),$$ (36)

has a unique global continuous solution over $[0, T]$. According to Theorem 2.4 in Han & Wong (2021),

$$\mathbb{E} \left[ \exp \left( C_2 \int_0^T \lambda_s ds \right) \right] < \infty.$$  

Assumption 4.2 is hard to verify for a positive constant $C_2$. Proposition 4.3 provides a convenient sufficient condition for Assumption 4.2. For a small $\sigma_\lambda$, it further clarifies that the requirement for constant $C_2$ in (35) is actually not harsh.

5. Numerical study

To demonstrate the influence of the LRD of the mortality rate on the investment and reinsurance equilibrium strategies, we compare strategies under the LRD and Markovian mortality models. Using the VMM defined in (1), we can easily see that the model reduces to a Markovian mortality model,

$$d\lambda_t = (b_1 - a_1 \lambda_t) dt + \sigma_\lambda \sqrt{\lambda_t} dW_0,$$

when $K \equiv 1$. Hence, the VMM actually contains a Markovian mortality model as a special case. By setting $K \equiv 1$, our result can also be applied to a Markovian case. We thus compare the equilibrium strategies under the VMM with those under LRD and its Markovian counterparts by setting different values of $K$ and keeping the same values for other parameters.

In this section, we use the fractional $K(t) = \frac{\alpha-1}{t^{\alpha}}$ for the VMM such that the Hurst parameter $H = \alpha - \frac{1}{2}$. The VMM reflects the LRD feature for $\alpha > 1$ and the Markovian feature for $\alpha = 1$. We focus on a population whose members are all 50 years old at time $t = 0$. To reflect the effect of LRD, the insurer is assumed to have access to historical mortality rate data for this population beginning at age 30. Let $T = 3$ years. Next, we simulate a sample path of the mortality rate from age 30 to 53, as shown in Figure 1(a). Yan et al. (2021) show empirically that the value of the Hurst parameter in the mortality rate is around 0.8. We therefore set $\alpha = 1.33$ in our numerical study. For simplicity, we
set \( l(t) \equiv 0 \). The values of the other parameters are as follows: \( b_1 = 0.15, a_1 = 0.5, \lambda_0 = 0.18, \) and \( \sigma_\lambda = 0.1 \).

Based on this mortality rate path, we compare the reinsurance and investment strategies under the VMM and Markovian mortality model. In the constant risk aversion case, Equation (18) reveals that the equilibrium strategies are the same under both models. Hence, the LRD feature of the mortality rate does not affect the equilibrium RI strategy under constant risk aversion. Under state-dependent risk aversion, the difference in equilibrium strategies deduced using (34) depends on the process \( M_t \). In the expression of \( M_t \), only \( \mathbb{E}[\hat{\lambda}_s | \mathcal{F}_t] \) for \( s \geq t \) differs between the two mortality models. The value of \( \mathbb{E}[\hat{\lambda}_s | \mathcal{F}_t] \) is calculated using (33) under the VMM, and \( \mathbb{E}[\hat{\lambda}_s | \mathcal{F}_t] = l(s) + \lambda_t e^{-a_1(s-t)} + \frac{b_1}{a_1} (1 - e^{-a_1(s-t)}) \) under the Markovian mortality model. According to (33), the historical mortality rate enables an adjustment to the value of \( \mathbb{E}[\hat{\lambda}_s | \mathcal{F}_t] \). Under the Markovian mortality model, no adjustment is made, and the value of \( \mathbb{E}[\hat{\lambda}_s | \mathcal{F}_t] \) only depends on the current mortality rate. Based on this observation, we numerically compare the equilibrium strategies under the two models. The following parameter values are used: \( k_1 = 10, r \equiv 0.05, \eta \equiv 0.2, \mu = 0.07, \) and \( \sigma = 0.2 \). The path of the risky asset \( S \) on \([0, T]\) is simulated, as shown in Figure 1(b). For insurance claims, we simulate the claim sizes by assuming that \( z \) follows an exponential distribution with mean 1, i.e. \( z \sim \text{Exp}(1) \). Thus, \( \mu_z = 1 \) and \( \mathbb{E}[z^2] = 2 \). Note that the exponential claim size violates Assumption 4.3. This makes our numerical study in a weaker theoretical condition and enables us to demonstrate that the derived strategy pair is useful even when Assumption 4.3 is violated.

We set an initial wealth value of \( X_0 = 10 \) at \( t = 0 \) and let \( \phi_1 = 1 \). Then, we calculate the equilibrium strategies under the VMM and Markovian mortality model according to (34). Figures 2 and 3 plot the strategies and wealth processes under the two models. Figure 2 shows that the LRD feature slightly influences both the investment and reinsurance strategies compared with the Markovian mortality model. Figure 3 further confirms that the LRD feature has a mild effect on the performance of the equilibrium RI strategy. The study in Wang & Wong (2021) shows that the LRD property has a significant impact on the equilibrium longevity hedging strategy. The implication is that reinsurance is more robust to LRD than longevity hedging is.

We then analyze the sensitivity of LRD for different levels of \( \phi_1 \). We vary the risk aversion coefficient from 0.6 to 1, and then plot the percentage differences between the VMM and Markovian

**Figure 1.** Sample paths of the mortality rate and risky asset.

![Figure 1](image-url)
mortality model in the reinsurance strategies and wealth processes in Figure 4. We reveal that the percentage difference increases with the risk aversion coefficient $\phi_1$.

Note that the exponentially distributed claim size $z$ violates Assumption 4.3. However, as long as the wealth level is positive, as shown in Figure 3, the equilibrium RI strategy pair in (34) is still implementable. We simulate 500 sample paths of mortality rates and asset prices, and plot the wealth process under the equilibrium RI strategy in Figure 5. All simulated paths are positive, indicating that the probability of negative wealth is extremely low for a reasonable set of parameter values.

To examine the general impact of the LRD feature of the mortality rate on equilibrium strategies, we plot the sample means and variances for both strategies under LRD and Markovian mortality models by varying $\phi_1$ from 0 to 1. The efficient frontiers in Figure 6 further reveal that reinsurance is robust to the LRD property. Given that the mortality rate exhibits an LRD property, Figure 6 suggests that
correctly identifying the Hurst parameter $H$ slightly improves the performance of both reinsurance and investment strategies.

In the following, we investigate the sensitivity of the impact of LRD to the Hurst parameter. Let $\phi_1 = 1$. We first set $\alpha = 1.33$ ($H = 0.83$) and simulate 500 sample paths. Corresponding to each path, we calculate the values of the equilibrium reinsurance strategies at the initial time point ($t = 0$) under LRD and Markovian mortality models. The histogram of the percentage differences in equilibrium reinsurance strategies at the initial point is shown in Figure 7(a). We repeat 500 simulation runs for $\alpha = 1.10$ ($H = 0.60$) by keeping other parameters unchanged. Figure 7(b) provides the histograms of the percentage differences between the initial values of the equilibrium reinsurance strategies under the two mortality models. The effect of the LRD feature is smaller when $H = 0.60$ than when $H = 0.83$.  

Figure 4. Percentage differences in (a) reinsurance strategies and (b) wealth levels under the two models with different values of $\phi_1$.

Figure 5. Simulated wealth paths.
6. Conclusion

Studies by Wang et al. (2021) and Wang & Wong (2021) show that including the LRD feature in a mortality rate has a significant impact on longevity hedging. As a life risk management tool, reinsurance is a popular alternative to the longevity hedging approach. We show that the LRD feature in the mortality rate has a limited effect on equilibrium reinsurance strategies. In other words, risk management with reinsurance is robust to LRD. Specifically, under constant risk aversion, the equilibrium control under the LRD and Markovian mortality models coincides with each other and is unique. Under state-dependent risk aversion, LRD has a slight influence, which increases as risk aversion increases. This paper presents a mathematical solution to the TCMV RI problem for a mortality rate that includes LRD. We derive explicit open-loop equilibrium strategies for both constant and state-dependent risk.
aversion cases. By using unbounded parameters and imposing some technical conditions, we also prove the admissibility and uniqueness of the equilibrium controls in these two cases.

**Disclosure statement**

No potential conflict of interest was reported by the author(s).

**Funding**

This work was supported by Research Grants Council, University Grants Committee [RMG8601495].

**ORCID**

Mei Choi Chiu  [http://orcid.org/0000-0003-2799-8360](http://orcid.org/0000-0003-2799-8360)

Hoi Ying Wong  [http://orcid.org/0000-0001-9743-1832](http://orcid.org/0000-0001-9743-1832)

**References**

Abi Jaber E., Larsson M. & Pulido S. (2019). Affine Volterra processes. The Annals of Applied Probability 29(5), 3155–3200.

Alia I., Chighoub F. & Sohail A. (2016). A characterization of equilibrium strategies in continuous-time mean-variance problems for insurers. Insurance: Mathematics and Economics 68, 212–223.

Basak S. & Chabakauri G. (2010). Dynamic mean-variance asset allocation. The Review of Financial Studies 23(8), 2970–3016.

Björk T., Khapko M. & Murgoci A. (2017). On time-inconsistent stochastic control in continuous time. Finance and Stochastics 21(2), 331–360.

Björk T., Murgoci A. & Zhou X. Y. (2014). Mean-variance portfolio optimization with state-dependent risk aversion. Mathematical Finance 24(1), 1–24.

Chen P. & Yam S. C. P. (2013). Optimal proportional reinsurance and investment with regime-switching for mean-variance insurers. Insurance: Mathematics and Economics 53(3), 871–883.

Delgado-Vences F. & Ornelas A. (2019). Modelling Italian mortality rates with a geometric-type fractional Ornstein-Uhlenbeck process. *arXiv preprint arXiv:1901.00795*.

El Karoui N., Peng S. & Quenez M. C. (1997). Backward stochastic differential equations in finance. Mathematical Finance 7(1), 1–71.

Guan G. & Wang X. (2020). Time-consistent reinsurance and investment strategies for an AAI under smooth ambiguity utility. Scandinavian Actuarial Journal 2020(8), 677–699.

Han X., Liang Z. & Young V. R. (2020). Optimal reinsurance to minimize the probability of drawdown under the mean-variance premium principle. Scandinavian Actuarial Journal 2020(10), 879–903.

Han B., Pun C. S. & Wong H. Y. (2021). Robust state-dependent mean-variance portfolio selection: a closed-loop approach. Finance and Stochastics 25, 529–561.

Han B. & Wong H. Y. (2019). Time-consistent mean-variance reinsurance-investment problems under unbounded random parameters: BSDE and uniqueness. Available at SSRN 3182387.

Han B. & Wong H. Y. (2021). Mean-variance portfolio selection under Volterra Heston model. Applied Mathematics & Optimization 84, 683–710.

Hiriart-Urruty J. B. & Lemarchal C. (2004). Fundamentals of convex analysis. Springer Science & Business Media, Berlin, Heidelberg.

Hu Y., Huang J. & Li X. (2017a). Equilibrium for time-inconsistent stochastic linear–quadratic control under constraint. *arXiv preprint arXiv:1703.09415*.

Hu Y., Jin H. & Zhou X. Y. (2012). Time-inconsistent stochastic linear–quadratic control. SIAM Journal on Control and Optimization 50(3), 1548–1572.

Hu Y., Jin H. & Zhou X. Y. (2017b). Time-inconsistent stochastic linear-quadratic control: characterization and uniqueness of equilibrium. SIAM Journal on Control and Optimization 55(2), 1261–1279.

Li D., Rong X. & Zhao H. (2015). Time-consistent reinsurance-investment strategy for a mean-variance insurer under stochastic interest rate model and inflation risk. Insurance: Mathematics and Economics 64, 28–44.

Lin X. & Qian Y. (2016). Time-consistent mean-variance reinsurance-investment strategy for insurers under CEV model. Scandinavian Actuarial Journal 2016(7), 646–671.

Øksendal B. K. & Sulem A. (2019). Applied Stochastic Control of Jump Diffusions, 3rd ed. Springer Berlin, Heidelberg.

Shen Y. & Zeng Y. (2014). Optimal investment-reinsurance with delay for mean-variance insurers: a maximum principle approach. Insurance: Mathematics and Economics 57, 1–12.
Appendix. Proofs

A.1 Proof of admissibility in Theorem 2.1

Proof: From (6), we have

\[ X_t = e^{\int_0^t r(u) \, du} X_0 + \int_0^t e^{\int_0^u r(v) \, dv} \left( \nu(s)^\top u(s) + c(s) \right) \, ds \]

\[ + \int_0^t e^{\int_0^u r(v) \, dv} \pi(s) \sigma(s) \, dW_1(s) - \int_0^t \int_{\mathbb{R}_+} e^{\int_0^u r(v) \, dv} a(s) z \, \tilde{N}(ds, dz) \]

Then, by applying the Cauchy–Schwarz inequality,

\[ \sup_{0 \leq t \leq T} |X_t|^2 \leq 4 \sup_{0 \leq t \leq T} \left[ \left( e^{\int_0^t r(u) \, du} X_0 \right)^2 + \left( \int_0^t e^{\int_0^u r(v) \, dv} \left( \nu(s)^\top u(s) + c(s) \right) \, ds \right)^2 \right] \]

\[ + \left( \int_0^t e^{\int_0^u r(v) \, dv} \pi(s) \sigma(s) \, dW_1(s) \right)^2 + \left( \int_0^t \int_{\mathbb{R}_+} e^{\int_0^u r(v) \, dv} a(s) z \, \tilde{N}(ds, dz) \right)^2 \] \]

As \( r(\cdot) \) is a deterministic function, there exists a positive constant \( C \) such that

\[ \sup_{0 \leq t \leq T} \left( \int_0^t e^{\int_0^u r(v) \, dv} \left( \nu(s)^\top u(s) + c(s) \right) \, ds \right)^2 \leq C \sup_{0 \leq t \leq T} \left[ \int_0^t |\nu_1(s)\pi(s)| + |\nu_2(s)a(s)| + |c(s)| \, ds \right]^2 \]

\[ \leq C \left( \int_0^T |\nu_1(s)\pi(s)| \, ds + \int_0^T |\nu_2(s)a(s)| \, ds + \int_0^T |c(s)| \, ds \right)^2 \]

\[ \leq C \left[ \left( \int_0^T |\pi(s)| \, ds \right)^2 + \left( \int_0^T |\nu_2(s)a(s)| \, ds \right)^2 + \left( \int_0^T |c(s)| \, ds \right)^2 \right] \]

As \( \pi(s) \sigma(s) \in L^2_{\mathbb{F}}(0, T; \mathbb{R}, \mathbb{F}), \int_0^t e^{\int_0^u r(v) \, dv} \pi(s) \sigma(s) \, dW_1(s) \) is a martingale. As \( a(s) \in \bigcup_{q=2}^\infty L^q_{\mathbb{F}}(0, T; \mathcal{D}, \mathbb{F}) \), for any \( q \geq 2 \),

\[ \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}} a(s)^2 z^2 \delta(dz) \, ds \right] \leq k_1 \mathbb{E}[z^2] \left( \mathbb{E} \left[ \int_0^T a(s)^q \, ds \right]^2 \right)^{\frac{1}{q}} \left( \mathbb{E} \left[ \int_0^T |\lambda s|^{\frac{q}{2}} \, ds \right] \right)^{1-\frac{2}{q}} < \infty. \]
According to the BDG inequality, there exists a positive constant $C$ where
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \left( \int_0^t e^{r(u)} du \right)^2 \right] \leq CE \left[ \int_0^T \pi(s)^2 ds \right],
\]
and
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \left( \int_0^t \int_{\mathbb{R}_+} e^{r(u)} du a(s) z \tilde{N}(ds, dz) \right)^2 \right] \leq CE \left[ \int_0^T \lambda_\alpha(s)^2 ds \right]
\]
for a positive constant $C$. Combining the above results, by Hölder’s inequality and Lemma 2.3, for any constant $q > 2$, there exists a constant $C > 0$ such that
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |X_t|^2 \right] \leq C \left\{ 1 + \mathbb{E} \left[ \int_0^T \pi(s)^2 ds \right] + \mathbb{E} \left[ \left( \int_0^T \lambda_\alpha(s) ds \right)^2 \right] + \mathbb{E} \left[ \int_0^T \lambda_\alpha(s)^2 ds \right] \right\} \leq C \left\{ 1 + \mathbb{E} \left[ \int_0^T \pi(s)^2 ds \right] + \left( \mathbb{E} \left[ \int_0^T a(s)^q ds \right] \right)^{\frac{2}{q}} \left( \mathbb{E} \left[ \int_0^T \lambda_\alpha(s)^{\frac{2q}{2-q}} ds \right] \right)^{\frac{2-q}{q}} \right\} < \infty.
\]

### A.2 Proof of Theorem 2.2

**Proof:** Let $X^t(\cdot)$ be the state process corresponding to $u^{t e, \rho'}(s) = (\pi^{t e, \rho_1}(s), a^{t e, \rho_2}(s))^T$. Using the standard perturbation approach,

\[
X^t_s = X^t_t + Y^t_s + Z^t_s, \quad s \in [t, T]
\]

where $Y^t_s$ and $Z^t_s$ are the respective solutions to the following SDEs:

\[
\begin{align*}
\text{d}Y^t_s &= (\pi^t Y^t_s) ds + \sigma \rho_1 1_{[t, t+\epsilon)}(s) dW_1(s) - \int_{\mathbb{R}_+} z(\rho_2 - a^*) 1_{[t, t+\epsilon)}(s) \tilde{N}(ds, dz), \\
Y^t_t &= 0, \quad s \in [t, T], \\
\text{d}Z^t_s &= \left\{ r(s)Z^t_s + \nu^T \rho_1 1_{[t, t+\epsilon)}(s) \right\} ds, \quad Z^t(t) = 0, \quad s \in [t, T].
\end{align*}
\]

According to the BDG inequality, there exists a positive constant $C$, such that

\[
\mathbb{E}_t \left[ \sup_{s \in [t, T]} (Y^t_s)^2 \right] = \mathbb{E}_t \left[ \sup_{s \in [t, T]} \left( \int_t^s e^{r(u)} du \sigma \rho_1 1_{[t, t+\epsilon)}(s) dW_1(s) - \int_t^s e^{r(u)} du e^{r(u)} du z(\rho_2 - a^*) 1_{[t, t+\epsilon)}(s) \tilde{N}(ds, dz) \right)^2 \right] \leq C E_t \left[ \int_t^{t+\epsilon} \rho_1^2(s) + (\rho_2(s) - a^*(s))^2 \lambda_\alpha \right] ds.
\]

As $\rho_1 \in H^2_q(t, T; \mathbb{R}, \mathbb{P}), \rho_2 \in \bigcup_{q > 2} L^q(t, T; \mathbb{D}, \mathbb{P}),$ for any $q > 2$, we have

\[
\int_t^T \mathbb{E}_t[\rho_1^2(s) + (\rho_2(s) - a^*(s))^2 \lambda_\alpha] ds \leq \int_t^T \mathbb{E}_t[\rho_1^2(s)] ds + \left( \int_t^T \mathbb{E}_t[(\rho_2(s) - a^*(s))^q] ds \right)^{\frac{2}{q}} \left( \int_t^T \mathbb{E}_t[\lambda_\alpha(s)^{\frac{2q}{2-q}}] ds \right)^{\frac{2-q}{q}} < \infty.
\]

Thus, $\rho_1^2(s) + (\rho_2(s) - a^*(s))^2 \lambda_\alpha \in L^2(t, T; \mathbb{R}, \mathbb{P}).$ By the Lebesgue differentiation theorem,

\[
\lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \int_t^{t+\epsilon} \rho_1^2(s) + (\rho_2(s) - a^*(s))^2 \lambda_\alpha ds = \rho_1^2(t) + (\rho_2(t) - a^*(t))^2 \lambda_\alpha(t).
\]
Note that \( \rho_{1}^{2}(t) + (\rho_{2}(t) - a^{*}(t))^{2} \) is a known constant given \( \mathcal{F}_{t} \). Thus,

\[
\mathbb{E}_{t} \left[ \sup_{s \in [t,T]} (Y_{s}^{*})^{2} \right] = O(\epsilon).
\]

In addition, we have \( \mathbb{E}_{t}[Y_{s}^{*}] = 0 \). By the SDE of \( Z_{s}^{*}, Z_{t}^{*} \) admits the expression \( \int_{t}^{s} e^{\int_{\tau}^{t} r(u) \, du} v^{T} \mathbf{1}_{[t,t+\epsilon]}(v) \, dv \). There exists a positive constant \( C \) such that

\[
\mathbb{E}_{t} \left[ \sup_{s \in [t,T]} (Z_{s}^{*})^{2} \right] = \mathbb{E}_{t} \left[ \sup_{s \in [t,T]} \left( \int_{t}^{s} e^{\int_{\tau}^{t} r(u) \, du} v^{T} \mathbf{1}_{[t,t+\epsilon]}(v) \, dv \right)^{2} \right] \\
\leq C \mathbb{E}_{t} \left[ \left( \int_{t}^{t+\epsilon} |v^{T} \rho| \, dv \right)^{2} \right] = C \mathbb{E}_{t} \left[ \int_{t}^{t+\epsilon} \int_{t}^{t+\epsilon} \left| v(s)^{T} \rho(s) \right| \left| v(v)^{T} \rho(v) \right| \, ds \, dv \right]. \tag{A1}
\]

Similar to the analysis of \( Y_{s}^{*} \), we have \( \mathbb{E}_{t}[\int_{t}^{T} \int_{t}^{T} |v(s)^{T} \rho(s)| |v(v)^{T} \rho(v)| \, ds \, dv] < \infty \). Thus,

\[
\lim_{\epsilon \downarrow 0} \frac{1}{\epsilon^{2}} \mathbb{E}_{t} \left[ \int_{t}^{t+\epsilon} \int_{t}^{t+\epsilon} |v(s)^{T} \rho(s)| |v(v)^{T} \rho(v)| \, ds \, dv \right] = (v(t)^{T} \rho(t))^{2}.
\]

We obtain

\[
\mathbb{E}_{t} \left[ \sup_{s \in [t,T]} (Z_{s}^{*})^{2} \right] = O(\epsilon^{2}).
\]

By suppressing \( s \) and \( t \), we have

\[
J(t,X_{t}^{*}; \pi^{t,e},u^{t,e},\rho_{1}^{t,e},\rho_{2}^{t,e}) - J(t,X_{t}^{*}; \pi^{*},a^{*})
\]

\[
= \mathbb{E}_{t} \left[ \left( \frac{1}{2} (X_{t}^{*} + X_{t}^{+}) + \frac{1}{2} \mathbb{E}_{t}[X_{t}^{*} + X_{t}^{+}] + \phi_{1} X_{t}^{*} + \phi_{2} X_{t}^{+} - X_{t}^{+} \right) \right] \\
= \frac{1}{2} \mathbb{E}_{t} \left[ (X_{t}^{*} - X_{t}^{+}) + \frac{1}{2} \mathbb{E}_{t}[X_{t}^{*} - X_{t}^{+}] \right] \\
= \mathbb{E}_{t} \left[ \left( \mathbb{E}_{t}[X_{t}^{*} + X_{t}^{+}] + \phi_{1} X_{t}^{*} + \phi_{2} X_{t}^{+} - X_{t}^{+} \right) \right] \\
= \mathbb{E}_{t} \left[ \left( \mathbb{E}_{t}[X_{t}^{*} + X_{t}^{+}] + \frac{1}{2} \mathbb{E}_{t}[X_{t}^{*} - X_{t}^{+}] \right) \right] \\
= o(\epsilon). \tag{A2}
\]

By applying Itô’s formula to \( \langle p^{*}(s,t), Y_{s}^{*} + Z_{s}^{*} \rangle \),

\[
\mathbb{E}_{t} \left[ \langle p^{*}(s,t), Y_{s}^{*} + Z_{s}^{*} \rangle \right]
\]

\[
= \mathbb{E}_{t} \left[ \int_{t}^{t+\epsilon} \left( \langle p^{*}(s,t) \nu(s), \rho(s) \rangle + \sigma(s) Z_{s}^{*}(s,t) \rho_{1}(s) - \frac{1}{2} \int_{R_{+}} z Z_{s}^{*}(s,z,t) (\rho_{2}(s) - a^{*}(s)) \delta(dz) \, ds \right) \right]. \tag{A3}
\]

By applying Itô’s formula to \( e^{\int_{t}^{T} r(u) \, du} \langle Y_{s}^{*} + Z_{s}^{*}, Y_{s}^{*} + Z_{s}^{*} \rangle \), we have

\[
\mathbb{E}_{t} \left[ \langle Y_{s}^{*} + Z_{s}^{*}, Y_{s}^{*} + Z_{s}^{*} \rangle \right]
\]

\[
= \mathbb{E}_{t} \left[ \int_{t}^{t+\epsilon} \left( e^{\int_{\tau}^{t} r(u) \, du} \sigma^{2}(s)^{2} \rho_{1}^{2}(s) + e^{\int_{\tau}^{t} r(u) \, du} \int_{R_{+}} z^{2} \delta(dz) (\rho_{2}(s) - a^{*}(s))^{2} \right) \, ds \right] + o(\epsilon) \tag{A4}
\]

By substituting (A3) and (A4) into (A2), the result in Theorem 2.2 follows.

### A.3 Proof of Proposition 2.2

**Proof:** Set \( \mathbf{p}(s,t) = e^{-\int_{t}^{s} r(u) \, du} p^{*}(s,t) + \mathbb{E}_{t}[X_{s}^{*}] + \phi_{1} X_{s}^{*} + \phi_{2} \), \( \tilde{Z}(s,t) = e^{-\int_{t}^{s} r(u) \, du} Z^{*}(s,t) \), and \( \tilde{Z}_{2}(s,\cdot,t) = e^{-\int_{t}^{s} r(u) \, du} Z_{2}^{*}(s,\cdot,t) \). According to Itô’s lemma, we obtain

\[
\mathcal{d} \mathbf{p}(s,t) = \tilde{Z}(s,t) \mathcal{d} W_{s} + \int_{R_{+}} \tilde{Z}_{2}(s,z,t) \mathcal{N}(ds,dz), \quad \mathbf{p}(T,t) = X_{T}^{*}. \tag{A5}
\]

Note that neither the coefficients nor the terminal condition of the above equation depend on \( t \). (A5) can be regarded as a BSDE on the entire time interval \([0,T]\). For \( s \in [0,T] \), denote the solution of (A5) by
According to Proposition 2.1, \(\tilde{p}(s, \tilde{z}(s), \tilde{Z}_2(s, \cdot)) \in S^2_T(t; T; \mathbb{R}, \mathbb{P}) \times L^2_T(t; T; \mathbb{R}^2, \mathbb{P}) \times F^2(t; T; \mathbb{R})\). Given the uniqueness of the solution, for any \(t \in [0, T]\), \(\tilde{p}(s, t), \tilde{Z}(s, t), \tilde{Z}_2(s, t) = (\tilde{p}(s), \tilde{z}(s), \tilde{Z}_2(s, \cdot))\). Then, the first claim in this lemma follows.

Using the definition of \(\tilde{p}(s, t)\), we obtain

\[
p^*(s; t) = e^{\tilde{I}^{t}_{T} r(u) du} \tilde{p}(s) - e^{\tilde{I}^{t}_{T} r(u) du} (E_t[\xi^T + \phi_1 X_t^s + \phi_2]) = e^{\tilde{I}^{t}_{T} r(u) du} \tilde{p}(s) + e^{\tilde{I}^{t}_{T} r(u) du} \xi(t),
\]

where \(\xi(t) = -(E_t[X_t^s] + \phi_1 X_t^s + \phi_2)\), and \(\xi(\cdot) \in S^2_T(t; T; \mathbb{R}, \mathbb{P})\). Then,

\[
\Lambda(s; t) = \Lambda_0(s) + e^{\tilde{I}^{t}_{T} r(u) du} \nu(s) \xi(t),
\]

where \(\Lambda_0(s) = \nu(s) e^{\tilde{I}^{t}_{T} r(u) du} \tilde{p}(s) + e^{\tilde{I}^{t}_{T} r(u) du} (\sigma(s) \tilde{Z}_1(s), k_1 \lambda_s, \int_{R_+} \varphi(z) \tilde{Z}_2(s, z) dz)^T\).

**A.4 Proof of Theorem 2.3**

**Proof:** First, to prove sufficiency, recall the representation \(\Lambda(s; t) = \Lambda_0(s) + e^{\tilde{I}^{t}_{T} r(u) du} \nu(s) \xi(t)\) from Proposition 2.2. We still set \(\rho = (\rho_1, \rho_2 - a^*)^T\). Then,

\[
\frac{1}{\epsilon} \int_t^{t+\epsilon} E_t[(\Lambda(s; t), \rho(s))] ds = \frac{1}{\epsilon} \int_t^{t+\epsilon} E_t[\Lambda(s; t)] ds
\]

\[
= \frac{1}{\epsilon} \int_t^{t+\epsilon} E_t \left[ e^{\tilde{I}^{t}_{T} r(u) du} \nu(s) \xi(t) \right] ds.
\]

Recall the expression of \(\nu, \rho \in H^2_T(t; T; \mathbb{R}, \mathbb{P}) \times \bigcup_{q>2} L^q_T(t; T; \mathbb{R}, \mathbb{P})\), and \(\xi \in S^2_T(t; T; \mathbb{R}, \mathbb{P})\). By Lemma 2.3 and Hölder’s inequality, for any constant \(q > 2\), we have

\[
\int_t^T E_t \left[ e^{\tilde{I}^{t}_{T} r(u) du} \nu(s) \xi(t) \right] ds \leq C \left( \int_t^T E_t[[\nu(s), \rho(s)]^2] ds \right)^\frac{1}{2} \left( \int_t^T E_t[[\xi(t) - \xi(s)]^2] ds \right)^\frac{1}{2}
\]

\[
\leq C \left( \int_t^T E_t[\rho_1^2] ds + \left( \int_t^T E_t[[\nu_2(s)]^\frac{2q}{q-2}] ds \right)^{\frac{q-2}{q}} \left( \int_t^T E_t[[\rho_2(s) - a^*(s)]^q] ds \right)^{\frac{q}{q}} \right)^\frac{1}{2}
\]

\[
\times \left( \int_t^T E_t[[\xi(t) - \xi(s)]^2] ds \right)^\frac{1}{2} < \infty.
\]

for a positive constant \(C\). Hence, \(e^{\tilde{I}^{t}_{T} r(u) du} \nu(s) \xi(t) - \xi(s) \in L^2_T(t; T; \mathbb{R}, \mathbb{P})\). By the Lebesgue differentiation theorem,

\[
\liminf_{\epsilon \to 0} \frac{1}{\epsilon} \int_t^{t+\epsilon} E_t \left[ e^{\tilde{I}^{t}_{T} r(u) du} \nu(s) \xi(t) \right] ds = E_t \left[ e^{\tilde{I}^{t}_{T} r(u) du} \nu(t) \xi(t) \right] = 0.
\]

Hence, we obtain

\[
\liminf_{\epsilon \to 0} \frac{1}{\epsilon} \int_t^{t+\epsilon} E_t[(\Lambda(s; t), \rho(s))] ds - \frac{1}{\epsilon} \int_t^{t+\epsilon} E_t[(\Lambda(s; s), \rho(s))] ds = 0. \quad (A6)
\]

If the condition in (12) is satisfied, then

\[
\liminf_{\epsilon \to 0} \frac{1}{\epsilon} \int_t^{t+\epsilon} E_t[(\Lambda(s; t), \rho(s))] ds = \liminf_{\epsilon \to 0} \frac{1}{\epsilon} \int_t^{t+\epsilon} E_t[(\Lambda(s; s), \rho(s))] ds \geq 0.
\]

According to Proposition 2.1, \((\pi^*, a^*)\) is an open-loop equilibrium control.

Second, we prove necessity. If \(a^* = (\pi^*, a^*)^T\) is an open-loop equilibrium control, then according to Definition 2.2 and the variational equation in Theorem 2.2,

\[
\liminf_{\epsilon \to 0} \frac{1}{\epsilon} \int_t^{t+\epsilon} E_t \left[ (\Lambda(s; t), \rho(s)) + (\Theta(s) \rho(s), \rho(s)) \right] ds \geq 0,
\]

where \(\rho\) is defined as in Theorem 2.2 and Proposition 2.1. Let \(\rho_2 - a^* = 0\); then, the first condition in (12) is a direct
result of Theorem 3.1 in Sun & Guo (2019). For the second condition in (12), let $\rho_1 = 0$. According to Theorem 2.2,
\[
\liminf_{\epsilon \downarrow 0} \int_t^{t + \epsilon} \mathbb{E}_t \left[ \left( v_2(s)p^*(s; t) - \int_{\mathbb{R}_+} z k_1 \hat{\lambda}_c f(z) Z_2^*(s, z; t) \, dz \right) \tilde{\rho}_2(s) \right] \nonumber \\
+ \frac{1}{2} e^{t \int_0^T \mathbb{E}_t \left[ \left( v_2(s)p^*(s; t) - \int_{\mathbb{R}_+} z k_1 \hat{\lambda}_c f(z) Z_2^*(s, z; t) \, dz \right) \text{sgn}(\tilde{\rho}_2(s)) \right] ds} \geq 0,
\]
where $\tilde{\rho}_2 = \rho_2 - a^* \in \bigcup_{q > 2} L^q(0, T; \mathbb{R}, \mathbb{P})$. Thus,
\[
\liminf_{\epsilon \downarrow 0} \int_t^{t + \epsilon} \mathbb{E}_t \left[ \left( v_2(s)p^*(s; t) - \int_{\mathbb{R}_+} z k_1 \hat{\lambda}_c f(z) Z_2^*(s, z; t) \, dz \right) \text{sgn}(\tilde{\rho}_2(s)) \right] ds \geq 0.
\]
According to Lemma 2.3 and Hölder’s inequality, for any constant $q \in (1, 2)$,
\[
\mathbb{E}_t \left[ \int_0^T (\hat{\lambda}_c p^*(s; t))^q \, ds \right] \leq \left\{ \mathbb{E}_t \left[ \int_0^T p^*(s; t)^2 \, ds \right] \right\}^{\frac{q}{2}} \left\{ \mathbb{E}_t \left[ \int_0^T \frac{1}{\hat{\lambda}_c} \, ds \right] \right\}^{1 - \frac{q}{2}} < \infty.
\]
Similarly, $\mathbb{E}_t [\int_0^T (\hat{\lambda}_c \tilde{\rho}_2(s))^2 \, ds] < \infty$. Thus, according to Lemma 3.5 in Hu et al. (2017a),
\[
\left( v_2(t)p^*(t; t) - \int_{\mathbb{R}_+} z k_1 \hat{\lambda}_c f(z) Z_2^*(t, z; t) \, dz \right) \tilde{\rho}_2(t) + \frac{1}{2} e^{t \int_0^T \mathbb{E}_t \left[ \left( v_2(s)p^*(s; t) - \int_{\mathbb{R}_+} z k_1 \hat{\lambda}_c f(z) Z_2^*(s, z; t) \, dz \right) \text{sgn}(\tilde{\rho}_2(s)) \right] ds} \geq 0.
\]
For any $\theta \in (0, 1]$, we use the same method as in Hu et al. (2017a) to obtain
\[
\left( v_2(t)p^*(t; t) - \int_{\mathbb{R}_+} z k_1 \hat{\lambda}_c f(z) Z_2^*(t, z; t) \, dz \right) \tilde{\rho}_2(t) + \frac{1}{2} e^{t \int_0^T \mathbb{E}_t \left[ \left( v_2(s)p^*(s; t) - \int_{\mathbb{R}_+} z k_1 \hat{\lambda}_c f(z) Z_2^*(s, z; t) \, dz \right) \text{sgn}(\tilde{\rho}_2(s)) \right] ds} \rho_2(t)^2 \geq 0.
\]
Let $\theta \to 0^+$; we thus obtain
\[
\left( v_2(t)p^*(t; t) - \int_{\mathbb{R}_+} z k_1 \hat{\lambda}_c f(z) Z_2^*(t, z; t) \, dz \right) (\rho_2(t) - \alpha^*(t)) \geq 0.
\]

\section{A.5 Proof of Theorem 4.1}

\textbf{Proof:} We need only prove the admissibility of the equilibrium control in (34). Define $\pi^*(t) = \pi_1(t)X^*_t$ and $a^*(t) = \pi_2(t)X^*_t$, where $\pi_1(t) = \frac{\pi^{(1)}(t)}{\mu'(\sigma_1(t))\Gamma^{(1)}_1}$, and $\pi_2(t) = \frac{\pi^{(1)}(t)}{\mu'\hat{\lambda}_c}\Gamma^{(1)}_1$. Note that $0 < \frac{\Gamma^{(1)}_1}{\hat{\lambda}_c} \leq \phi_1$. According to Assumption 4.3, we obtain $0 \leq \pi_2(t)z \leq 1$ for $t \in [0, T]$. By substituting $\pi^*(t)$ and $a^*(t)$ into (6) and applying Itô’s formula, we obtain
\[
\begin{align*}
\frac{d(\ln M^*_t)}{M^*_t} &= \left( -r(t) - \frac{1}{2} \sigma(t)^2 \pi_1(t)^2 - \frac{U^2_0(t)}{2M^*_t} \right) dt + \int_{\mathbb{R}_+} \left[ \ln \left( 1 - \pi_2(t) z + \pi_2(t) z \right) \delta(dz) \right] dW_1(t) \\
&\quad + \frac{U^0(t)}{M_t} dW_0(t) + \sigma(t)\pi_1(t) dW_1(t) + \int_{\mathbb{R}_+} \left[ \ln \left( 1 - \pi_2(t) z + \pi_2(t) z \right) \tilde{N}(dt, dz) \right] dW_1(t).
\end{align*}
\]
Under the standard Brownian motion $W$, $\frac{d(\ln M^*_t)}{M_t} = \exp(\int_0^t h_s \, dW_s - \frac{1}{2} \int_0^t h_s^2 \, ds)$. Therefore,
\[
X^*_t = e^{\int_0^t r(u) \, du} \frac{X_0 M_0}{M_t} \mathbb{E}_t \left[ \exp \left( U_0 \frac{M}{M_t} \cdot W_0 \right) \right] \mathbb{E}_t (\sigma_1 \cdot W_1) \exp \left\{ \int_0^t \int_{\mathbb{R}_+} \left[ \ln \left( 1 - \pi_2(s) z + \pi_2(s) z \right) \delta(dz) \right] dW_1(s) + \int_0^t \int_{\mathbb{R}_+} \ln \left( 1 - \pi_2(s) z + \pi_2(s) z \right) \tilde{N}(ds, dz) \right\}.
\]
Note that
\[
\begin{align*}
\int_0^t \int_{\mathbb{R}_+} \ln \left( 1 - \pi_2(s) z \right) \delta(dz) \, ds + \int_0^t \int_{\mathbb{R}_+} \ln \left( 1 - \pi_2(s) z \right) \tilde{N}(ds, dz) \\
= \sum_{i=0}^{N(t)} \ln \left( 1 - \pi_2(t_i) z_i \right) \leq 0.
\end{align*}
\]
Hence,

\[
0 < \exp \left\{ \int_0^t \int_{\mathbb{R}_+} \ln (1 - \xi_2(t)z) \, \delta(dz) \, dt + \int_0^t \int_{\mathbb{R}_+} \ln (1 - \xi_2(t)z) \, \tilde{N}(dr, dz) \right\} \leq 1.
\]

As \(M_0\) is bounded, for any \(m > 1\), we obtain

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |X^*|^m \right] \leq C \mathbb{E} \left[ \sup_{0 \leq t \leq T} \mathcal{E}_t^{2m} \left( \frac{U_0}{M} \cdot W_0 \right) \mathcal{E}_t^{2m} (\sigma \xi_1 \cdot W_1) \exp \left( \int_0^t \int_{\mathbb{R}_+} 2m \xi_2(t)z \delta(dz) \, dt \right) \right]
\]

\[
\leq C \left( \mathbb{E} \left[ \sup_{0 \leq t \leq T} \mathcal{E}_t^{4m} \left( \frac{U_0}{M} \cdot W_0 \right) \right] \mathbb{E} \left[ \sup_{0 \leq t \leq T} \mathcal{E}_t^{8m} (\sigma \xi_1 \cdot W_1) \right] \right)^{\frac{1}{2}} \left( \mathbb{E} \left[ \sup_{0 \leq t \leq T} \exp \left( \int_0^t \int_{\mathbb{R}_+} 8m \xi_2(t)z \delta(dz) \, dt \right) \right] \right)^{\frac{1}{2}} < \infty.
\]

As \(\sigma \xi_1\) is a bounded deterministic function, we have \(\mathbb{E}[\sup_{0 \leq t \leq T} \mathcal{E}_t^{8m} (\sigma \xi_1 \cdot W_1)] < \infty\). Setting \(C_2 \geq 8m \frac{\eta \mu_2 \psi(k)}{E[z]}\) in Assumption 4.2,

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \exp \left( \int_0^t \int_{\mathbb{R}_+} 8m \xi_2(t)z \delta(dz) \, dt \right) \right] = \mathbb{E} \left[ \exp \left( \int_0^T \int_{\mathbb{R}^2} \frac{8m \eta \mu_2 \psi(k)}{M_i \mathbb{E}[z^2]} k_1 \lambda_i \, dz \, dt \right) \right] < \infty.
\]

Hence, we obtain \(X^* \in S^m_T(0, T; \mathbb{R}_+, \mathbb{F})\) for any \(m > 1\), and \((\pi^*(t), a^*(t)) \in H^2_T(0, T; \mathbb{R}, \mathbb{F}) \times F^2(t, T; \mathbb{R})\), where \(Z(s; t) = (Z_0(s; t), Z_1(s; t))^T\). This satisfies the condition in (12). Then, we define

\[
\tilde{p}(s; t) = p(s; t) - \left( M_i X_i - \Gamma_0 \right) X_i - \mathbb{E} [M_i X_i]
\]

\[
\tilde{Z}_0(s; t) = Z_0(s; t) - X_i U_0(s), \quad \tilde{Z}_1(s; t) = Z_1(s; t) - M_i \pi(s) \sigma(s),
\]

\[
\tilde{Z}_2(s, z; t) = Z_2(s, z; t) + M_i a(s) z,
\]

where \(M, U_0\), and \(\Gamma_0\) are defined in Section 4.

Clearly, \((\tilde{p}(s; t), \tilde{Z}(s; t), \tilde{Z}_{2}(s, z; t)) \in L^2_T(t, T; \mathbb{R}, \mathbb{F}) \times H^2_T(t, T; \mathbb{R}^2, \mathbb{F}) \times F^2(t, T; \mathbb{R})\). Similar to the proof of Theorem 3.3, we obtain

\[
\begin{align*}
(\nu_1(t) \left[ \tilde{p}(t; t) - \Gamma_0 \right] X_i) + \sigma(t) \left[ \tilde{Z}_1(t; t) + M_i \pi(t) \sigma(t) \right] &= 0, \\
(\nu_2(t) \left[ \tilde{p}(t; t) - \Gamma_0 \right] X_i) - \int_{\mathbb{R}_+} zk_1 \lambda_i f(z) \left[ \tilde{Z}_2(t, z; t) - M_i a(t) z \right] \, dz, \rho_2(t) - a(t) &\geq 0,
\end{align*}
\]

for any \(\rho_2 \in \bigcup_{q \geq 2} L^q_T(t, T; \mathbb{R}_+, \mathbb{F})\). Then,

\[\text{A.6 Proof of Theorem 4.2}\]

\[\text{Proof:}\] Suppose that there is another equilibrium control \(u(\cdot) = (\pi(\cdot), a(\cdot))\) with the corresponding state process \(X\). Then, when \(X^*\) is replaced by \(X\), BSDE (10) admits a unique solution \((\tilde{p}(s; t), Z(s; t), \tilde{Z}_2(s, z; t)) \in L^2_T(t, T; \mathbb{R}, \mathbb{F}) \times H^2_T(t, T; \mathbb{R}^2, \mathbb{F}) \times F^2(t, T; \mathbb{R})\), where \(Z(s; t) = (Z_0(s; t), Z_1(s; t))^T\). This satisfies the condition in (12). Then, we define

\[
\tilde{p}(s; t) = p(s; t) - \left( M_i X_i - \Gamma_0 \right) X_i - \mathbb{E} [M_i X_i]
\]

\[
\tilde{Z}_0(s; t) = Z_0(s; t) - X_i U_0(s), \quad \tilde{Z}_1(s; t) = Z_1(s; t) - M_i \pi(s) \sigma(s),
\]

\[
\tilde{Z}_2(s, z; t) = Z_2(s, z; t) + M_i a(s) z,
\]

where \(M, U_0\), and \(\Gamma_0\) are defined in Section 4.

Clearly, \((\tilde{p}(s; t), \tilde{Z}(s; t), \tilde{Z}_{2}(s, z; t)) \in L^2_T(t, T; \mathbb{R}, \mathbb{F}) \times H^2_T(t, T; \mathbb{R}^2, \mathbb{F}) \times F^2(t, T; \mathbb{R})\). Similar to the proof of Theorem 3.3, we obtain

\[
\begin{align*}
(\nu_1(t) \left[ \tilde{p}(t; t) - \Gamma_0 \right] X_i) + \sigma(t) \left[ \tilde{Z}_1(t; t) + M_i \pi(t) \sigma(t) \right] &= 0, \\
(\nu_2(t) \left[ \tilde{p}(t; t) - \Gamma_0 \right] X_i) - \int_{\mathbb{R}_+} zk_1 \lambda_i f(z) \left[ \tilde{Z}_2(t, z; t) - M_i a(t) z \right] \, dz, \rho_2(t) - a(t) &\geq 0,
\end{align*}
\]

for any \(\rho_2 \in \bigcup_{q \geq 2} L^q_T(t, T; \mathbb{R}_+, \mathbb{F})\). Then,
\[
\pi(t) = \frac{v_1}{M_1\sigma(t)} \Gamma_t^{(1)} X_t - \frac{1}{M_1\sigma(t)^2}(v_1(t)\bar{\bar{p}}(t; t) + \sigma(t)\bar{z}_1(t; t))
\]
\[
= \pi^*(t) - \frac{1}{M_1\sigma(t)^2}(v_1(t)\bar{\bar{p}}(t; t) + \sigma(t)\bar{z}_1(t; t)) \equiv \pi^*(t) + D_d^1(t),
\]
\[
a(t) = \frac{1}{k_1\lambda_t M_2 \mathbb{E}[\nu^2]} \left[ v_2 \Gamma_t^{(1)} X_t - \left( v_2 \bar{p}(t; t) - k_1 \lambda_t \int_{\mathbb{R}^+} zf(z)\bar{z}_2(t, z; t) \, dz \right) \right]^{+}
\]
\[
= a^*(t) - \frac{A_d^t}{k_1\lambda_t M_2 \mathbb{E}[\nu^2]} \left( v_2 \bar{p}(t; t) - k_1 \lambda_t \int_{\mathbb{R}^+} zf(z)\bar{z}_2(t, z; t) \, dz \right) \equiv a^*(t) + D_d^2(t),
\]
where \(0 \leq A_d^t \leq 1\) is a bounded adapted process. Next, we show \(D_d^1(t) \equiv D_d^2(t) \equiv 0\) to prove the uniqueness of the equilibrium control. We obtain
\[
d\bar{p}(s; t) = \left\{ -r(s)\bar{\bar{p}}(s; t) - v_1 D_d^1(s) M_1 + v_2 D_d^2(s) M_2 \right\} ds
\]
\[
+ \bar{z}(s; t)^\top dW(s) + \int_{\mathbb{R}^+} \bar{z}_2(s, z; t) \, d\tilde{N}(ds, dz), \quad s \in [t, T],
\]
\[
\bar{p}(T; t) = 0.
\]
As the interest rate \(r(\cdot)\) is a bounded deterministic function, we take \(r \equiv 0\) without loss of generality. By taking the conditional expectation on both sides of (A8), we obtain \(\mathbb{E}_s[\bar{\bar{p}}(s; t)] = 0\) at \(s \geq t\). Particularly, \(\bar{\bar{p}}(t; t) = 0\). Hence, \(D_d^1(t) = -\frac{\bar{z}_1(t)}{M\sigma(t)}\) and \(D_d^2(t) = - \frac{A^t}{M_2 \mathbb{E}[\nu^2]} \int_{\mathbb{R}^+} zf(z)\bar{z}_2(t, z; t) \, dz\). Then, we have \(v_1(t)D_d^1(t) M_1 = -\frac{\mu(t) - r(t)}{\sigma(t)} \bar{z}_1(t; t)\) and \(v_2(t)D_d^2(t) M_1 = \frac{\mu(t) - r(t)}{\sigma(t)} \int_{\mathbb{R}^+} z\bar{z}_2(t, z; t) \, d\delta(dz)\).

Following the same method used in the proof of Theorem 3.3, we obtain \(\bar{z} \equiv \bar{z}_2 \equiv 0\). Thus, \(D_d^1 \equiv D_d^2 \equiv 0\). As a result, \((\pi^*, a^*)\) given by (34) is the unique open-loop equilibrium control when \(C_2 \geq k_1(2 + \eta)\eta\) in Assumption 4.2. ■