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Citation for published version:
Brochier, A & Jordan, D 2017, 'Fourier transform for quantum D-modules via the punctured torus mapping class group' Quantum Topology, vol 8, no. 2, pp. 361-379. DOI: 10.4171/QT/92

Digital Object Identifier (DOI):
10.4171/QT/92

Link:
Link to publication record in Edinburgh Research Explorer

Document Version:
Peer reviewed version

Published In:
Quantum Topology

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FOURIER TRANSFORM FOR QUANTUM D-MODULES VIA THE PUNCTURED TORUS MAPPING CLASS GROUP

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Abstract. We construct a certain cross product of two copies of the braided dual $\tilde{H}$ of a quasitriangular Hopf algebra $H$, which we call the elliptic double $E_H$, and which we use to construct representations of the punctured elliptic braid group extending the well-known representations of the planar braid group attached to $H$. We show that the elliptic double is the universal source of such representations. We recover the representations of the punctured torus braid group obtained in [Jo], and hence construct a homomorphism to the Heisenberg double $D_H$, which is an isomorphism if $H$ is factorizable.

The universal property of $E_H$ endows it with an action by algebra automorphisms of the mapping class group $\tilde{SL}_2(\mathbb{Z})$ of the punctured torus. One such automorphism we call the quantum Fourier transform; we show that when $H = U_q(\mathfrak{g})$, the quantum Fourier transform degenerates to the classical Fourier transform on $D(\mathfrak{g})$ as $q \to 1$.

1. Introduction

Let $(H, \mathcal{R})$ be a quasi-triangular Hopf algebra, and let $\tilde{H}$ denote the braided dual – also known as the reflection equation algebra – of $H$ [DKM, DM2, DM1, Ma]. This is the restricted dual vector space $H^\circ$, but the multiplication is twisted from the standard one by the $\mathcal{R}$-matrix (see Section 2 for details).

Let $\{e_i\}$ and $\{e^i\}$ denote dual bases of $H$ and $\tilde{H}$, respectively. Then the canonical element $X = \sum e^i \otimes e_i \in \tilde{H} \otimes H$ is known to satisfy the following relation in $\tilde{H} \otimes H^{\otimes 2}$:

$$X^{0,12} := (\text{id} \otimes \Delta)(X) = (\mathcal{R}^{1,2})^{-1} X^{0,2} \mathcal{R}^{1,2} X^{0,1} \tag{1.1}$$

Here, $\tilde{H}$ has index 0 in the tensor product, and $\Delta$ denotes the coproduct of $H$.

There is a canonical action of the planar braid group $B_n(\mathbb{R}^2)$ on the $n$th tensor power of any $H$-module $V$. Given modules $M$ for $\tilde{H}$ and $V$ for $H$, equation (1.1) allows one to define a similarly canonical action of the punctured planar braid group $B_n(\mathbb{R}^2 \setminus \text{disc})$ on $M \otimes V^{\otimes n}$, and moreover to show that $\tilde{H}$ is universal for this action. We have:

Theorem 1.1 (DKM, Prop 10). Let $B$ be an algebra, and suppose that $X_B \in B \otimes H$ satisfies relation (1.1). Then there is a unique homomorphism $\phi_B : \tilde{H} \to B$ such that $(\phi_B \otimes \text{id})(X) = X_B$.

The main goal of this paper is to define elliptic analogs of the reflection equation algebra. The punctured elliptic braid group $B_n(T^2 \setminus \text{disc})$ is the free product of two copies of $B_n(\mathbb{R}^2 \setminus \text{disc})$, modulo certain relations. In Section 3 we construct an algebra $E_H$ as a certain crossed product of two copies of $\tilde{H}$, mimicking the cross relations of $B_n(T^2 \setminus \text{disc})$. We define canonical elements $X, Y \in E_H \otimes H$ by

$$X = \sum (e^i \otimes 1) \otimes e_i, \quad Y = \sum (1 \otimes e^i) \otimes e_i,$$

and characterize the cross relations on $E_H$ as follows:
Theorem 1.2. The cross relations of $E_H$ are equivalent to the following commutation relation for $X, Y, R$:

$$X^{0.1}R^{2.1}Y^{0.2} = R^{2.1}Y^{0.2}R^{1.2}X^{0.1}R^{2.1}$$ (1.2)

We prove the following elliptic analog of Theorem 1.1:

Theorem 1.3. Let $B$ be an algebra, and $X_B, Y_B \in B \otimes H$ satisfying (1.1) individually, and (1.2) together. Then there exists a unique algebra homomorphism

$$\phi_B : E_H \rightarrow B$$

such that $X_B = (\phi_B \otimes \text{id})(X)$ and $Y_B = (\phi_B \otimes \text{id})(Y)$. Explicitly, $\phi_B$ is given by

$$\phi_B(x \otimes 1) = (\text{id} \otimes x)(X_B) \quad \phi_B(1 \otimes x) = (\text{id} \otimes x)(Y_B).$$

Equation (1.2) can be used to define representations of $B_n(T^2 \text{disc})$ in the same way as (1.1) is used for $B_n(\mathbb{R}^2 \text{disc})$; see Theorem 4.3. Recall that $B_n(T^2 \text{disc})$ carries a natural action of the punctured torus mapping class group, which is isomorphic to a certain central extension $\hat{SL}_2(\mathbb{Z})$ of $SL_2(\mathbb{Z})$. In the case $H$ is a ribbon Hopf algebra, we show that this extends to an action of $\hat{SL}_2(\mathbb{Z})$ on $E_H$.

When $H = U_q(g)$, we produce degenerations of $E_H$ to the algebras of differential operators on $G$ and, upon further degeneration, on $g$. Recall that the algebra of differential operators on an algebraic group $G$ can be constructed as a semi-direct product

$$D(G) = U(g) \ltimes O(G),$$

where the action of $U(g)$ on $O(G)$ is induced by that of $g$ on $G$ by left invariant differential operators. This construction can be extended to any Hopf algebra and is known as the Heisenberg double $S^2$. This is a semi-direct product $D_H = H \ltimes H^o$, where $H$ acts on its dual by the right coregular action.

In [10], canonical functors are constructed from the category of modules over the Heisenberg double of a quasi-triangular Hopf algebra to the category of modules over the (unpunctured) torus braid group. This relies upon an alternate construction – due to Varagnolo-Vasserot [VV] – of the Heisenberg double of a quasi-triangular Hopf algebra, which uses the braided dual $\hat{H}$ in place of $H^o$. This presentation for the Heisenberg double also yields an isomorphism with the handle algebras $S_{1,1}$ of [ACS] (see Remark 5.3).

Lifting the constructions of [10] to the unpunctured torus braid group, they can easily be re-interpreted as producing canonical elements $X$ and $Y$ in $D_H \otimes H$, satisfying equations (1.1) and (1.2). Hence, Theorem 1.3 yields a unique homomorphism $\Phi : E_H \rightarrow D_H$, compatible with the representations of the $B_n(T^2 \text{disc})$ on both sides. The map $\Phi$ is an isomorphism if, and only if, $H$ is factorizable. Since the quantum group $U_q(g)$ is factorizable, we may identify the elliptic double $E_{U_q(g)}$ with the algebra $D_q(G) := D_{U_q(g)}$ of quantum differential operators on $G$.

In particular we obtain an $\hat{SL}_2(\mathbb{Z})$ action on $D_q(G)$ by the above considerations. One such automorphism of $D_q(G)$ we call the quantum Fourier transform; its classical limit upon an appropriate degeneration is the classical Fourier transform on the Weyl algebra $D(g)$. We expect that our quantum Fourier transform for $D_q(G)$ will be compatible with that on the braided dual of $U_q(g)$ defined in [LM], realizing the braided dual as an $\hat{SL}_2(\mathbb{Z})$-equivariant $D_q(G)$-module. Studying this category of $\hat{SL}_2(\mathbb{Z})$-equivariant $D_q(G)$-modules more generally is an interesting direction of future research.
Acknowledgments. This paper is a companion to work in progress with D. Ben-Zvi [BZBJ], in which we generalize the elliptic double construction to arbitrary genus, and to any braided tensor category, using the language of topological field theory. We are grateful to D. Ben-Zvi, and to all three authors of [CEE], for their many helpful discussions and encouragement, and to P. Roche for bringing the article [AGS] to our attention.

2. The braided dual and its relatives

Let \((H, \mathcal{R})\) be a quasi-triangular Hopf algebra, and denote by:

- \(H^e = H^{\text{coop}} \otimes H\) where \(H^{\text{coop}}\) is \(H\) with opposite comultiplication
- \(H^{[2]}\) the Hopf algebra which is \(H \otimes H\) as an algebra, and with coproduct given by

\[
\tilde{\Delta}(x \otimes y) = (\mathcal{R}^{2,3})^{-1}(\tau^{2,3} \circ \Delta(x \otimes y))\mathcal{R}^{2,3}
\]

where \(\tau(a \otimes b) = b \otimes a\). Recall that the twist \(H^F\) of \(H\) by an invertible element \(F \in H \otimes H\) is the Hopf algebra with the same multiplication, and with coproduct given by

\[
\Delta^F(x) = F^{-1}\Delta(x)F.
\]

In order for \(H^F\) to be co-associative, \(F\) must satisfy two conditions:

\[
F^{1,2}F^{3,2} = F^{1,3}F^{2,3}, \quad (\epsilon \otimes \text{id})(F) = (\text{id} \otimes \epsilon)(F) = 1.
\]

Two twists \(F, F'\) are equivalent if there exists an invertible element \(x \in H\), such that \(\epsilon(x) = 1\) and \(F' = \Delta(x)F(x^{-1} \otimes x^{-1})\).

The following is standard (see [Dr2]):

Proposition 2.1. A twist induces a tensor equivalence \(H\text{-mod} \rightarrow H^F\text{-mod}\). Equivalent twists leads to isomorphic tensor functors.

It is easily checked that \(F = \mathcal{R}^{1,3}\mathcal{R}^{1,4} \in (H^e)^{\otimes 2}\) is a twist, and that

\[H^{[2],\text{coop}} = (H^e)^F = H^E\]

Let \(D\) be the “double braiding” \(\mathcal{R}^{2,1}\mathcal{R}^{1,2}\). Since \(D\Delta(x) = \Delta(x)D\) for all \(x\), we have:

\[H^D = H\]

as Hopf algebras. Similarly, \(H^{[2],\text{coop}}\) is in fact equal to \((H^e)^{F(D^{1,3})^k}\) for any \(k \in \mathbb{Z}\), with \(F\) as above.

Let \(H^e\) be the restricted Hopf algebra dual of \(H\). It has a natural \(H\)-bimodule structure, hence a \(H^e\) left module structure given by:

\[(x \otimes y) \triangleright f := f(S^{-1}(x) \cdot y)\]

where \(S\) is the antipode of \(H\) and we use the fact that \(S^{-1}\) is a Hopf algebra isomorphism \(H^{\text{coop}} \rightarrow H_{\text{op}}\). It turns \(H^o\) into an algebra in \(H^e\text{-mod}\).

Remark 2.2. We use the inverse of the antipode rather than the antipode itself because it is convenient to consider the canonical element as an invariant element of \(H^e \otimes H\), the image of \(1 \in \mathbb{C}\) under the evaluation map \(k \rightarrow H^e \otimes H\), which means that \(H^e\) really denotes the left dual of \(H\) in the rigid monoidal category of \(H\)-modules. This is slightly different from the convention used in [DKM, da] but it allows us to label tensor factors from left to right.
**Definition 2.3.** The kth twisted braided dual $\hat{H}_k$ is the algebra image of $H^o$ via the tensor functor $H^o$-mod $\rightarrow H^{[2],\text{coop}}$-mod given by the twist $F(D^{1,3})^k$. Explicitly, this is $H^o$ as a vector space, with multiplication given by

$$x \cdot y = m(R^{1,3}R^{1,4}(D^{1,3})^k \triangleright (x \otimes y))$$

where $m$ is the multiplication of $H^o$. This is an algebra in the category of $H^{[2],\text{coop}}$-module with the same action as above, namely

$$(x \otimes y) \triangleright f = (u \mapsto f(S^{-1}(x)uy)).$$

Let $X$ be the canonical element of $\hat{H} \otimes H$, that is the image of 1 under the coevaluation map $k \rightarrow \hat{H} \otimes H$. If $e_i$ is a basis of $H$ and $e^i$ the dual basis of $\hat{H} \cong H^o$, then $X = \sum e^i \otimes e_i$. If $H$ is infinite dimensional then $X$ lives in an appropriate completion of the tensor product.

**Proposition 2.4.** The element $X$ satisfies:

$$X^{0,12} = D^k(R^{1,2})^{-1}X^{0,2}R^{1,2}X^{0,1}. \tag{2.1}$$

This implies that $X$ satisfies the reflection equation

$$R^{2,1}X^{0,2}R^{1,2}X^{0,1} = X^{0,1}R^{2,1}X^{0,2}R^{1,2}.$$

The braided dual is in fact universal for this property in the following sense:

**Proposition 2.5.** Let $B$ be an algebra and $X_B \in B \otimes H$ satisfying equation (2.1) for some $k \in \mathbb{Z}$. Then there exists a unique algebra morphism

$$\phi_B : \hat{H}_k \rightarrow B$$

such that $(\phi_B \otimes \text{id})(X) = X_B$. Explicitly, $\phi_B$ is given by

$$H^o \cong \hat{H} \ni f \mapsto (f \otimes \text{id})(X).$$

Propositions 2.4 and 2.5 are proved in [DKM] in the case $k = 0$. The general proof is similar. Note that the fact that these axioms all lead to the same reflection equation, regardless of the value of $k$, essentially follows from the fact that the left hand side of (2.1) is invariant under conjugation by $D$.

Let $u = m((S \otimes \text{id})(R^{2,1}))$ where $m$ is the multiplication of $H$. Then $\nu = uS(u)$ is central and satisfies

$$\Delta(\nu) = D^{-2}(\nu \otimes \nu)$$

implying that

$$D^{k-2} = \Delta(\nu)D^k(\nu^{-1} \otimes \nu^{-1})$$

meaning that $D^{k-2}$ and $D^k$ are equivalent. Therefore, they lead to isomorphic tensor functors, from which follows the following:

**Proposition 2.6.** For any $k \in \mathbb{Z}$, the algebras $\hat{H}_k$ and $\hat{H}_{k+2}$ are isomorphic.

Therefore, it is enough to consider $\hat{H}_0$ and $\hat{H}_1$. Moreover, if $H$ is a ribbon Hopf algebra, then by definition $\nu$ admits a central square root implying by a similar argument:

**Proposition 2.7.** If $H$ is a ribbon Hopf algebra then all the $\hat{H}_k$ are isomorphic.

**Remark 2.8.** The algebra $\hat{H}_0$ is usually called the reflection dual, the braided dual or the reflection equation algebra in the literature.

**Remark 2.9.** For any $k$, equation (2.1) plays the same role in the reflection equation, as the hexagon axiom in the Yang-Baxter equation, encoding some kind of compatibility with the tensor product of $H$-modules. Topologically, it corresponds to a "strand doubling" operation for the additional generator of the braid group of the punctured plane. Formally, such an operation depends on the choice of a framing, while a ribbon element removes the dependence on the framing.
3. The elliptic double

Let \( T \) denote the following element in \((H^{[2],\text{coop}})^{\otimes 2}\), which we identify as a vector space with \( H^{\otimes 4} \):

\[
T = (R^{3,2})^{-1}(R^{3,1})^{-1}(R^{4,2})^{-1}R^{1,4}.
\]

**Proposition 3.1.** The element \( T \) satisfies the hexagon axioms

\[
(id \otimes \Delta)(H^{[2],\text{coop}})T = T_{1,3}T_{1,2},
\]

\[
(\Delta(H^{[2],\text{coop}}) \otimes id)T = T_{1,3}T_{2,3}
\]

in \((H^{[2],\text{coop}})^{\otimes 3}\).

**Proof.** This is straightforward computation with the Yang-Baxter equation. The computation is depicted in braids in Figure 1.

![Braid diagram](https://via.placeholder.com/150)

**Figure 1.** A braid diagram proof of \((id \otimes \Delta)(T) = T_{1,3}T_{1,2} \). □

Since \( \tilde{H}_k \) is a \( H^{[2],\text{coop}} \)-module algebra, one can make the following definition:

**Definition 3.2.** The \( k \)th elliptic double \( E^{(k)}_H \) of \( H \) is the braided tensor square of \( \tilde{H}_k \) with respect to \( T \). Explicitly, it is \( \tilde{H}_k^{\otimes 2} \) as a vector space, \( \tilde{H}_k \otimes 1 \) and \( 1 \otimes \tilde{H}_k \) are subalgebras and the cross relations are given by

\[
(1 \otimes g)(f \otimes 1) = T \triangleright (f \otimes g).
\]

The fact that \( E^{(k)}_H \) is indeed an associative algebra follows from the hexagon axioms. Choose a basis \((e_i)_{i \in I}\) of \( H \) and define \( X, Y \in E^{(k)}_H \otimes H \) by

\[
X = \sum e_i \otimes 1 \otimes e_i, \quad Y = \sum 1 \otimes e_i \otimes e_i,
\]

where we use the vector space identification \( E^{(k)}_H \cong \tilde{H}^{\otimes 2} \). The main result of this section is the following:

**Theorem 3.3.** The cross relations of \( E^{(k)}_H \) are equivalent to the commutation relation for \( X, Y, R \):

\[
X^{0,1}R^{2,1}Y^{0,2} = R^{2,1}Y^{0,2}R^{1,2}X^{0,1}R^{2,1}.
\] (3.1)
Therefore all commutations relation can be gathered into a “matrix” equation
\[ Y^{0.2}X^{0.1} = Y^{0.1}X^{0.2} \]

where \( T \) acts on the \( E_H^{(k)} \) (i.e. 0th) component. We recall the following identities:
\[ R^{-1} = (S \otimes \text{id})(R) = (\text{id} \otimes S^{-1})(R). \] (3.3)

Applying \( S^{-1} \) to the first factor of the relation \((S \otimes \text{id})(R)R = 1\), setting \( R = \sum r_1 \otimes r_2 = \sum r_1' \otimes r_2' \) using apostrophes to distinguish between copies of \( R \) – one has the following useful identity (note the order of the terms):
\[ \sum S^{-1}(r_1)r_1' \otimes r_2' = 1. \] (3.4)

Then equation (3.2) reads, in coordinates:
\[
\begin{align*}
(1 \otimes e^j)(e^i \otimes 1) & \otimes e_i \otimes e_j \\
& = ((2r_1' \otimes r_2' \otimes r_1') \otimes S(r_1) \otimes S(r_2)) \otimes e^i \otimes e^j \otimes e_i \otimes e_j.
\end{align*}
\] (3.5)

The left \( H^{[2]} \) action on \( \hat{H}_k \) is by definition dual to the right \( H^{[2]} \) action on \( H \), therefore:
\[
\sum ((x \otimes y) \otimes e^i) \otimes e_i = \sum e^i \otimes S^{-1}(x)e_ig
\]

Using this, equation (3.5) can be rewritten:
\[
(1 \otimes e^j)(e^i \otimes 1) \otimes e_i \otimes e_j = e^i \otimes e^j \otimes S^{-1}(r_1') \otimes S^{-1}(r_2) \otimes r_1'r_2' \otimes r_1'r_2' \text{S}(r_1') \text{S}(r_2').
\]

Then, using the \( R \)-matrix relations (3.3) and (3.4) to move elements from the right hand side to the left hand side (and reassigning apostrophes for the sake of clarity) we obtain:
\[
((1 \otimes e^j)(e^i \otimes 1)) \otimes r_2r_1' \otimes e_i \otimes e_j \otimes r_1r_2' = e^i \otimes e^j \otimes e_i \otimes e_j \otimes r_1r_2
\]
which is exactly (1.2).

\[ \square \]

Remark 3.4. The relations of Theorem 3.3 should be compared with those of the graph algebra \( S_{1.1} \) of AGS.

Equation (1.2) is a defining relation for \( E_H^{(k)} \), in the following sense:

**Corollary 3.5.** Let \( B \) be an algebra, and \( X_B, Y_B \in B \otimes H \) satisfying both the axiom (2.1) and equation (1.2) (with \( X \) and \( Y \) replaced by \( X_B \) and \( Y_B \)). Then there exists a unique algebra morphism
\[
\phi_B : E_H^{(k)} \rightarrow B
\]
such that \( X_B = (\phi_B \otimes \text{id})(X) \) and \( Y_B = (\phi_B \otimes \text{id})(Y) \). Explicitly, \( \phi_B \) is given by
\[
\phi_B(1 \otimes 1) = (\text{id} \otimes x)(X_B) \quad \phi_B(1 \otimes x) = (\text{id} \otimes x)(Y_B).
\]
4. Braid group and mapping class group actions

In this section we construct representations of the punctured torus braid group from \( E_H^{(k)} \). First, we have:

**Definition 4.1.** The punctured elliptic braid group \( B_n(T^2 \setminus \text{disc}) \) is the fundamental group of the configuration space of \( n \) points in \( T^2 \setminus \text{disc} \).

**Proposition 4.2.** The group \( B_n(T^2 \setminus \text{disc}) \) is generated by \( X_1, \ldots, X_n, Y_1, \ldots, Y_n, \sigma_1, \ldots, \sigma_{n-1} \) with relations:

- the \( X_i \)'s (resp. \( Y_i \)'s) pairwise commute,
- the planar braid relation for the \( \sigma_i \)'s,
- the following cross relations:
  
  \[ X_{i+1} = \sigma_i X_i \sigma_i \quad Y_{i+1} = \sigma_i Y_i \sigma_i \]  

  \[ X_1 Y_2 = Y_2 X_1 \sigma_i^2 \]  

  \[ (4.1) \quad (4.2) \]

The results of the previous section easily imply:

**Theorem 4.3.** There exists unique group morphisms

\[ \phi : B_n(T^2 \setminus \text{disc}) \rightarrow (E_H^{(k)} \otimes H^\otimes n) \rtimes S_n \]

given by

\[ X_i \mapsto X_i^{0,1}, \quad Y_i \mapsto Y_i^{0,1}, \quad \sigma_i \mapsto (i, i + 1)R^{i, i+1}. \]

**Proof.** The two first set of cross relations can obviously be taken as a definition of \( X_i, Y_i \) for \( i > 1 \). That these operators pairwise commute follows from the reflection equation and the Yang-Baxter equation. The last cross relation is nothing but the defining equation \( (1.2) \) of \( E_H^{(k)} \).

\[ \square \]

Let \( \widetilde{SL_2(\mathbb{Z})} \) denote the group generated by \( A, B, Z \) with relations:

\[ A^4 = (AB)^3 = Z, \quad (A^2, B) = 1. \]

Clearly, \( Z \) is central, so this is a central extension,

\[ 1 \rightarrow \mathbb{Z} \rightarrow \widetilde{SL_2(\mathbb{Z})} \rightarrow SL_2(\mathbb{Z}) \rightarrow 1. \]

**Proposition 4.4.** The group \( \widetilde{SL_2(\mathbb{Z})} \) acts on \( B_n(T^2 \setminus \text{disc}) \) in the following way:

\[ A \cdot \sigma_i = \sigma_i \quad B \cdot \sigma_i = \sigma_i \]

\[ A \cdot X_i = Y_i \quad A \cdot Y_i = Y_i X_i^{-1} Y_i^{-1} \]

\[ B \cdot X_i = X_i \quad B \cdot Y_i = Y_i X_i^{-1}. \]

**Proposition 4.5.** Let \( B \) be an algebra and \( (X_B, Y_B) \in B \otimes H \) satisfying equation \( (1.2) \) and axioms \( (2.1) \) with \( k = 1 \). Then, so does \((X_B, Y_B X_B^{-1})\) and \((Y_B, Y_B X_B^{-1} Y_B^{-1})\).

**Proof.** Equation \( (1.2) \) is exactly one of the defining relation of \( B_{1,n} \) so that it is satisfied follows from the previous proposition. So we just have to check that \( Y_B X_B^{-1} \) and \( Y_B X_B^{-1} Y_B^{-1} \) satisfies \( (2.1) \) with \( k = 1 \). This is a direct computation:

\[ (Y_B X_B^{-1})^{0,12} = \mathcal{R}^{2,1} Y_B^{0,2} \mathcal{R}^{1,2} Y_B^{0,1} (X_B^{0,1})^{-1} (\mathcal{R}^{1,2})^{-1} (X_B^{0,2})^{-1} (\mathcal{R}^{2,1})^{-1} \]

\[ = \mathcal{R}^{2,1} Y_B^{0,2} \mathcal{R}^{1,2} Y_B^{0,1} (X_B^{0,2})^{-1} (\mathcal{R}^{2,1})^{-1} (X_B^{0,1})^{-1} (\mathcal{R}^{1,2})^{-1} \mathcal{R}^{2,1} (\mathcal{R}^{2,1})^{-1} \]

\[ = \mathcal{R}^{2,1} Y_B^{0,2} (X_B^{0,2})^{-1} \mathcal{R}^{2,1} Y_B^{0,1} (X_B^{0,1})^{-1}, \]

where at lines 2 and 3 we use the reflection equation and the elliptic commutation relation respectively. The second part is proved by doing the exact same computation replacing \( Y_B \) by \( Y_B X_B^{-1} \) and \( X_B \) by \( Y_B \).

\[ \square \]
Corollary 4.6. There is an action of $\widetilde{SL}_2(\mathbb{Z})$ on $E_H^{(1)}$, uniquely determined by its action on canonical elements $X, Y$ as follows:

\[ A \cdot X = Y, \quad A \cdot Y = YX^{-1}Y^{-1}, \]
\[ B \cdot X = X, \quad B \cdot Y = YX^{-1}. \]

Moreover, the action is compatible with the $\widetilde{SL}_2(\mathbb{Z})$-action on $B_n(T^2\setminus \text{disc})$.

Proof. It follows from Proposition 4.5 together with the universal property stated in Corollary 3.5. □

5. Relation with the Heisenberg double and quantum Fourier transform

Since $\tilde{H}_0$ is a $H^{[2],\text{coop}}$-module algebra, one can form the semi-direct product $\tilde{H} \rtimes H^{[2],\text{coop}}$. It is easily checked that $H \otimes 1 < H^{[2],\text{coop}}$ is a coideal subalgebra, hence the following definition makes sense:

Definition 5.1. The Heisenberg double $D_H$ is the subalgebra $\tilde{H}_0 \rtimes (H \otimes 1)$.

Remark 5.2. The standard definition of the Heisenberg double involves $H^e$ and the usual dual, instead of $H^{[2]}$ and the braided dual. However, it is shown in [VV] that these two algebras are isomorphic.

Clearly, the double braiding $R^{2,1}R^{1,2}$ satisfies axiom (2.1) with $k = 0$. This is a manifestation of the embedding of the cylinder braid group on $n$ strands into the ordinary braid group on $n + 1$ strands. We have:

Theorem 5.3. [Jo] The canonical element $X \in D_H \otimes H$ together with the image of the double braiding under the inclusion $H \otimes H \to D_H \otimes H$ satisfy the commutation relation (1.2).

Corollary 5.4. There exists a canonical map from the elliptic double to the Heisenberg double.

By construction, this map is the identity on the first $\tilde{H}_0$ component and defined on the second component by the factorization map,

\[ \phi : \tilde{H}_0 \to H, \quad f \mapsto (f \otimes \text{id})(R^{2,1}R^{1,2}). \]

Definition 5.5. A quasi-triangular Hopf algebra is called factorizable if $\phi$ is injective.

Let $I_H$ be the image of $\phi$ and let $D'_H$ be the subalgebra $\tilde{H} \rtimes (I_H \otimes 1)$ of $D_H$.

Theorem 5.6. If $H$ is a factorizable Hopf algebra, then $D'_H$ is isomorphic as an algebra to $E_H^{(0)}$.

Let $G$ be a reductive algebraic group, $\mathfrak{g}$ its Lie algebra and $U = U_q(\mathfrak{g})$ the corresponding quantum group. Recall (see e.g. [CP, Chap. 9]) that this is a quasi-triangular Hopf algebra over $\mathbb{C}(q)$ for $q$ a variable which, roughly, specialize to the enveloping algebra of $\mathfrak{g}$ at $q = 1$. Denote by $U' = U_q(\mathfrak{g})'$ its ad-locally finite part.

Theorem 5.7 ([BS, RSTS]). $U$ is a factorizable ribbon Hopf algebra, and the image of the factorization map $(U^*) \to U$ is $U'$.

1This is not quite true since the R-matrix does not belongs to $U_q(\mathfrak{g})^{\otimes 2}$ but only to a certain completion of it, but it is still enough for our purpose.
Let $D_q(G)$ be the subalgebra $\hat{U} \rtimes U'$ of the Heisenberg double of $U$. It is a deformation of the algebra of differential operators on $G$. Thanks to the above theorem, $D_q(G)$ is isomorphic to $E_d^{(1)}$ which is itself isomorphic to $E_r^{(1)}$. Altogether this yields the action of $SL_2(\mathbb{Z})$ on $D_q(G)$.

6. Relation to classical Fourier transform

In this section we show how the Weyl algebra of $\mathfrak{g}$ and the classical Fourier transform can be obtained both directly as the elliptic double of a certain Hopf algebra and via an appropriate degeneration of the elliptic double of the corresponding quantum group. Let $U_\hbar(\mathfrak{g})$ be the “formal” version of the quantum group. This a topological quasi-triangular Hopf algebra over $\mathbb{C}[\hbar]$, where $\hbar$ is a formal variable, deforming the enveloping algebra of $\mathfrak{g}$ and whose definition can be found, e.g., in [CP, Chap. 6]. Since directly taking the classical (i.e. $\hbar = 0$) limit of the elliptic commutation relation gives the commutative algebra $S(\mathfrak{g})^{\otimes 2}$ we will have to consider a slightly more complicated degeneration.

Let $S(\mathfrak{g})$ denote the symmetric algebra on $\mathfrak{g}$, equipped with its standard co-product $\Delta(X) = X \otimes 1 + 1 \otimes X$ for $X \in \mathfrak{g}$, making it a commutative, cocommutative Hopf algebra. Let $r \in \mathfrak{g}^{\otimes 2}$ denote the quasi-classical limit of the R-matrix of $U_\hbar(\mathfrak{g})$, i.e.:

$$R = 1 + \hbar r + O(\hbar^2).$$

Then, in a straightforward way, the completion of the symmetric algebra $(\hat{S}(\mathfrak{g}), R_0 = \exp(r))$ is a quasi-triangular, factorizable Hopf algebra\(^2\). Let $t = r + r^{2,1} \in S^2(\mathfrak{g})^\#$ and let $C$ denote the corresponding Casimir element, i.e. $C = m(t)$ where $m$ is the multiplication of $S(\mathfrak{g})$. Then $\nu_0 = \exp(-C/2)$ is a ribbon element. Since $R_0 \not\in S(\mathfrak{g})^{\otimes 2}$, $S(\mathfrak{g})$ is not strictly speaking a ribbon Hopf algebra, but the construction of the elliptic double is still well defined in this situation.

Let $D(\mathfrak{g})$ be the algebra of differential operators on $\mathfrak{g}$, i.e. the Weyl algebra. As a vector space it is $S(\mathfrak{g}^*)^{\otimes 2}$, the two copies of $S(\mathfrak{g}^*)$ are subalgebras and the cross relations are:

$$\forall f,g \in \mathfrak{g}^*, [f \otimes 1, 1 \otimes g] = (f,g),$$

where $(\cdot, \cdot)$ is the pairing on $\mathfrak{g}^*$ induced by $t$. The first result of this section is:

**Proposition 6.1.** The 0th elliptic double of $(S(\mathfrak{g}), R_0)$ is isomorphic to the Weyl algebra $D(\mathfrak{g})$ and the action of the generator $A$ of $SL_2(\mathbb{Z})$ coincides with the classical Fourier transform. That is, on generators $(f,g) \in \mathfrak{g}^* \times \mathfrak{g}^* \subset D(\mathfrak{g})$, we have,

$$A(f,g) = (-g, f).$$

**Proof.** Let $x, y$ denote two copies of the canonical element in $\mathfrak{g}^* \otimes \mathfrak{g}$. The restricted dual of $S(\mathfrak{g})$ is $S(\mathfrak{g}^*)$ and the corresponding canonical element is $X = \exp(x)$. Since $S(\mathfrak{g})$ is commutative, equation (2.4) reduces to the standard relation,

$$(id \otimes \Delta)(X) = X^{0,1} X^{0,2},$$

hence the braided dual and the restricted dual coincide. Likewise, the defining equation of the elliptic double reduces to:

$$(X^{0,1}, X^{0,2}) = R_0^{1,2} R_0^{1,2},$$

where $(a,b) = ab a^{-1} b^{-1}$ and $Y = \exp(y)$. Since

$$[x^{0,1}, y^{1,2}] = [y^{0,2}, t^{1,2}] = 0,$$

this equation is equivalent to:

$$[x^{0,1}, y^{0,2}] = t^{1,2}.$$

\(^2\)Here the tensor product is the topological one, i.e. $\hat{S}(\mathfrak{g})^{\otimes 2} := \hat{S}(\mathfrak{g} \times \mathfrak{g})$
Applying $f$ and $g$ to the first and second components, respectively, of the above equation gives the defining relations (2) of $D(\mathfrak{g})$.

Since $(S(\mathfrak{g}), R_0)$ is ribbon, $E^{(0)}_{S(\mathfrak{g})}$ is isomorphic to $E^{(1)}_{S(\mathfrak{g})}$. Pulling back the action of the $A$ generator of $\widehat{SL}_2(\mathbb{Z})$ through this isomorphism, we find:

$$x \mapsto y \quad \quad y \mapsto Y^{-1}(-x + (1 \otimes C))Y$$

It is easily seen that the cross relations of $D(\mathfrak{g})$ implies

$$Y^{-1}xY = x + (1 \otimes C).$$

Hence $A$ map $x$ to $y$ and $y$ to $-x$. □

Let $U_{h^2}(\mathfrak{g})$ be the $\mathbb{C}[[h]]$-Hopf algebra obtained by formally replacing $\hbar$ by $h^2$ in the definition of the product, the coproduct and the R-matrix of $U_h(\mathfrak{g})$. Denote by $\delta_n$ the map $(\text{id} - \epsilon)^{\otimes n} \circ \Delta^n$ where $\epsilon$ is the counit of $U_{h^2}(\mathfrak{g})$. Denote by $\widehat{U}$ the quantum formal series Hopf algebra (QFSHA) attached to $U_{h^2}(\mathfrak{g})$, i.e. the sub-algebra

$$\widehat{U} = \{ x \in U_{h^2}(\mathfrak{g}), \delta_n(x) \in h^n U_{h^2}(\mathfrak{g}), \forall n \geq 0 \}$$

It is known [Dr1, Ga] that $\widehat{U}$ is a flat deformation of $\widehat{S}(\mathfrak{g})$. Hence, choose a $\mathbb{C}[[h]]$-module identification

$$\psi : \widehat{U} \rightarrow \widehat{S}(\mathfrak{g})[[h]]$$

which is the identity modulo $h$, and let $U \subset \widehat{U}$ be the preimage under $\psi$ of $S(\mathfrak{g})[[h]]$.

**Proposition 6.2.** The following holds:

(a) $U$ is a Hopf algebra.

(b) We have canonical bialgebra isomorphisms:

$$\widehat{U}/(h) \cong \widehat{S}(\mathfrak{g}), \quad U/(h) \cong S(\mathfrak{g}).$$

(c) The R-matrix of $U_{h^2}(\mathfrak{g})$ belongs to $\widehat{U}^{\otimes 2}$ and its image in $\widehat{S}(\mathfrak{g})^{\otimes 2}$ is $R_0$.

One can therefore consider the 0th elliptic double of $U$. A direct consequence of the above proposition is then:

**Corollary 6.3.** The algebra $E_U$ is a flat deformation of the Weyl algebra $D(\mathfrak{g})$, and the $\widehat{SL}_2(\mathbb{Z})$-action on $E_U$ degenerates to the $\widehat{SL}_2(\mathbb{Z})$-action on $D(\mathfrak{g})$. In particular, the quantum Fourier transform degenerates to the classical one.

**Proof of Prop. 6.2.** All of this can be checked explicitly. A more conceptual argument is as follows: recall that $(\mathfrak{g}, \mu, \delta, r)$ is a quasi-triangular Lie bialgebra, where we denote by $\mu$ its bracket and by $\delta$ its co-bracket. The quantum group $U_{h^2}(\mathfrak{g})$ is obtained by applying an Etingof–Kazhdan quantization functor [EK] to the $\mathbb{C}[[h]]$-quasi-triangular Lie bialgebra $(\mathfrak{g}[[h]], \mu, h^2\delta, h^2r)$. On the other hand, $\widehat{U}$ is the quasi-triangular Hopf algebra obtained by applying the same functor to the quasi-triangular Lie bialgebra $(\mathfrak{g}[[h]], \mu, h\delta, \hbar r)$. The QFSHA construction is the lift of the inclusion,

$$(\mathfrak{g}[[h]], h\mu, h\delta, r) \rightarrow (\mathfrak{g}[[h]], \mu, h^2\delta, h^2r),$$

given by $x \mapsto h x$ (since $r \in \mathfrak{g}^{\otimes 2}$, its image is indeed $h^2r$).

One can show that the product, the coproduct and the antipode on $\widehat{U}$ restrict to a well-defined Hopf algebra structure on $U$. By construction, the reduction modulo $\hbar$ of $\widehat{U}$ is the quantization of the $C$-quasi-triangular Lie bialgebra,

$$(\mathfrak{g}[[h]], h\mu, h\delta, r)/(\hbar) \cong (\mathfrak{g}, 0, 0, r),$$

which is easily seen to be $(\widehat{S}(\mathfrak{g}), R_0)$.
REFERENCES

[AGS] A. Alekseev, H. Grosse, V. Schomerus. Combinatorial quantization of the hamiltonian chern-simons theory ii. *Communications in Mathematical Physics* (1996), 174(3):561–604.

[BS] P. Baumann, F. Schmitt. Classification of bicovariant differential calculi on quantum groups (a representation-theoretic approach). *Communications in Mathematical Physics* (1998), 194(1):71–86.

[BZBJ] D. Ben-Zvi, A. Brochier, D. Jordan. Quantum D-modules and quantum geometric Langlands. (in preparation).

[CEE] D. Calaque, B. Enriquez, P. Etingof. Universal KZB equations: The elliptic case. In Y. Tschinkel, Y. Zarhin, editors, *Algebra, Arithmetic, and Geometry*, vol. 269 of *Progress in Mathematics*, pp. 165–266 (Birkhäuser Boston, 2009).

[CP] V. Chari, A. Pressley. *A guide to quantum groups* (Cambridge University Press, Cambridge, 1994).

[DM] J. Donin, P. P. Kulish, A. I. Mudrov. On a universal solution to the reflection equation. *Lett. Math. Phys.* (2003). 63(3):179–194.

[DM2] J. Donin, A. Mudrov. Reflection equation- and FRT-type algebras. *Czechoslovak J. Phys.* (2002). 52(11):1201–1206. Quantum groups and integrable systems (Prague, 2002).

[Dr1] V. G. Drinfeld. Quantum groups. In *Proc. Int. Cong. Math. (Berkeley, Calif., 1986)*, vol. 1 (Amer. Math. Soc., Providence, RI, 1987) pp. 798–820.

[Dr2] V. G. Drinfeld. Quasi-Hopf algebras. *Leningrad Math. J.* (1990). 1(6):1419–1457.

[EK] P. Etingof, D. Kazhdan. Quantization of Lie bialgebras. I. *Selecta Math. (N.S.)* (1996). 2(1):1–41.

[Ga] F. Gavarini. The quantum duality principle. *Ann. Inst. Fourier (Grenoble)* (2002). 52(3):809–834.

[Jo] D. Jordan. Quantum D-modules, elliptic braid groups, and double affine hecke algebras. *International Mathematics Research Notices* (2009). 2009:2081–2105(25).

[LM] V. Lyubashenko, S. Majid. Braided groups and quantum fourier transform. *Journal of Algebra* (1994). 166(3):506 – 528.

[Ma] S. Majid. *Foundations of quantum group theory* (Cambridge University Press, Cambridge, 1995).

[RSTS] N. Y. Reshetikhin, M. A. Semenov-Tian-Shansky. Quantum R-matrices and factorization problems. *Journal of Geometry and Physics* (1988). 5(4):533–550.

[STS] M. A. Semenov-Tian-Shansky. Poisson Lie groups, quantum duality principle, and the quantum double. In *Mathematical aspects of conformal and topological field theories and quantum groups* (South Hadley, MA, 1992), vol. 175 of *Contemp. Math.*, pp. 219–248 (Amer. Math. Soc., Providence, RI, 1994).

[VV] M. Varagnolo, E. Vasserot. Double affine Hecke algebras at roots of unity. *Represent. Theory* (2010). 14:510–600.

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