Measuring the orbital angular momentum of light beams by using a single intensity distribution

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Abstract. Here, we modify and study a method for measuring the orbital angular momentum (OAM) based on angle-averaged intensity in a single transverse plane. We also simulate propagation of a superposition of two Laguerre-Gaussian modes after passing through a diffuser with a random phase. Despite the distortions caused by the diffuser, the modified method allows OAM measurement with an average error less than 10%. The main advantages of the described technique is that (i) simultaneously with the full OAM the OAM spectrum is obtained and (ii) no any optical elements are needed.

1. Introduction

Orbital angular momentum (OAM) is studied for a long time since laser beams with the OAM have numerous practical uses, among which the most prominent are optical communications and micromanipulation [1, 2]. Modern reviews [3, 4] and monographs [5] in this area have recently been published. For the conventional rotationally symmetric optical vortices, the OAM per photon coincides with the vortex topological charge, but it is not the case for asymmetric laser beams. Thus, developing simple and high-efficiency techniques for measuring the OAM of such laser beams has been a topical issue.

There are many tools to measure the OAM of paraxial beams, including diffractive optical elements, transformation optics, metasurfaces, interferograms, triangular apertures, annular diffraction gratings, and cylindrical or astigmatic lenses. Not all these techniques allow measuring fractional OAM values, whereas an arbitrary OAM can be obtained by measuring the intensity with a cylindrical lens [6–8] and computing intensity moments [7–10].

In [11], we proposed a technique to obtain the OAM by registering only one intensity distribution and without using any optical elements. The only limitation is that the laser beam should be approximately a finite superposition of Laguerre-Gauss (LG) modes $(n, 0)$. However, the beam in [11] was supposed undistorted and we tested the method in the initial plane, without any propagation in free space. Besides, the method relies on the angle-averaged intensity on several rings and choosing these rings can affect the obtained OAM value.

In this work, we modified the method from [11]. To eliminate the dependence of the measured OAM value on the radii of circles where the intensity is angle-averaged, we use a much larger number of circles and instead of solving a system of linear equations we use the least squares method. In
addition, we simulate propagation of a laser beam passed through a diffuser with a random phase, which imitates propagation in a medium with random inhomogeneities. We show that despite the distortions caused by the diffuser, the method allows OAM measurement with an error less than 10%.

2. General theory of measuring the OAM by using a single intensity distribution

Here we briefly describe the OAM measurement method proposed in [11]. As in [9], we assume that the complex amplitude of a laser beam in a transverse plane is given by

\[ E(r, \varphi, z) = \sum_{n=-N}^{N} C_n \exp\left(\mathbf{i}n\varphi\right)\Psi_n(r, z), \] (1)

where \((r, \varphi)\) are polar coordinates, \(C_n\) are weight coefficients, \(\Psi_n(r, z)\) are some rotationally symmetric functions. Although the finite sum in equation (1) is not suitable for approximating an arbitrary function, many practical laser beams are described by equation (1). The intensity of the field in equation (1) reads as

\[ I(r, \varphi, z) = \left| E(r, \varphi, z) \right|^2 = \sum_{n=-N}^{N} \sum_{m=-N}^{N} C_n C_m^* \exp\left[\mathbf{i}(n-m)\varphi\right] \Psi_n(r, z) \Psi_m^*(r, z), \] (2)

where the asterisk means complex conjugation. Integration of equation (2) over the variable \(\varphi\) yields:

\[ \bar{I}(r, z) = \int_0^{2\pi} I(r, \varphi, z) d\varphi = 2\pi \sum_{n=-N}^{N} \left| C_n \right|^2 \left| \Psi_n(r, z) \right|^2 . \] (3)

To determine the values of \((2N+1)\) expansion coefficients \(\left| C_n \right|^2\), we need \((2N+1)\) equations, which can be obtained by measuring the intensity \(I(r, \varphi, z)\) and integrating it over \((2N+1)\) circles of radii \(r_m\):

\[ \bar{I}_m = \sum_{n=-N}^{N} M_m x_n, m = -N, -(N-1), \ldots, (N-1), N , \] (4)

where \(\bar{I}_m = \bar{I}(r_m, z)\) is the angle-averaged intensity on the radii \(r_m\), \(M_m = \left| \Psi_n(r, z) \right|^2\) are values of squared modules of the basis functions \(\Psi_n(r, z)\), and \(x_n = \left| C_n \right|^2\) are the sought-for squared modules of the expansion coefficients in equation (1). Solving the system (4) gives the squares modules of all coefficients in equation (1). Note that for the system (4) to have an unambiguous solution, the radial basis functions in equation (1) need to meet the following condition at \(n \neq 0\):

\[ \Psi_n(r, z) \neq \Psi_m(r, z), n \in [-N,N]. \]

Axial OAM component \(J_z\) and the total field energy \(W\) are given by the well-known relations [12]:

\[ J_z = \Im \left\{ \int_0^{2\pi} \int_0^{\infty} E^*(r, \varphi, z) \frac{\partial E(r, \varphi, z)}{\partial \varphi} rdrd\varphi \right\} , \] (5)

\[ W = \int_0^{2\pi} \int_0^{\infty} E^*(r, \varphi, z) E(r, \varphi, z) rdrd\varphi . \] (6)

Substituting (1) into (5) and (6) yields the normalized total OAM of the beam (1):

\[ \frac{J_z}{W} = \left[ \sum_{n=-N}^{N} n \left| C_n \right|^2 I_n \right]^{-1} \begin{bmatrix} \sum_{n=-N}^{N} \left| C_n \right|^2 I_n \end{bmatrix}, \]

where \(I_n = \int_0^{\infty} \left| \Psi_n(r, z) \right|^2 rdr\) is the power of each constituent beam in the superposition (1) and the squared modules of the coefficients of equation (1) are obtained from equation (4): \(x_n = \left| C_n \right|^2\), with the powers \(I_n\) of all constituent beams calculated using the known expansion basis \(\Psi_n(r, z)\).
3. Least squares modification

The coefficients \( C_n \) for equation (7) are obtained by solving the linear system (4). However, this system itself depends on the chosen radii \( r_m \), where the intensity is angle-averaged. To make the method more stable, a larger number of radii can be used, but in this case the system (4) becomes overdefined. Therefore, instead of solving it, it is possible to minimize the errors between the right and left parts. So, if we use \( P \) radii \( (P > 2N + 1) \), then the following function should be minimized:

\[
\Delta = \left( \sum_{n=-N}^{N} \left( \sum_{p=1}^{P} M_{p\mu} x_n - \tilde{I}_p \right) \right)^2 \rightarrow \min
\]

Differentiating this function over \( x_l \) \((l = -N, ..., N)\) yields the following system of linear equations:

\[
\sum_{n=-N}^{N} \left( \sum_{p=1}^{P} M_{p\mu} M_{p\nu} \right) x_n = \sum_{p=1}^{P} \tilde{I}_p M_{p\nu}, \quad l = -N,...,N,
\]

from which the coefficients \( |C_n|^2 \) can be defined.

4. Fractional OAM of superposition of LG modes

Using equation (7), we can tailor a light field with the desired OAM value. Here we choose a superposition of LG modes with orders \((0,2)\) and \((0,4)\), where coefficients \( C_2 \) and \( C_4 \) are such that the normalized OAM equals 3.0 and 3.5 respectively:

\[
E(r,\varphi,z=0) = \sqrt{2} \frac{C_2}{\sqrt{\lambda w_0}} \exp\left(2i\varphi \left( \frac{r \sqrt{2}}{w_0} \right)^2 \right) + \sqrt{2} \frac{C_4}{\sqrt{12\pi w_0}} \exp\left(4i\varphi \left( \frac{r \sqrt{2}}{w_0} \right)^4 \right) \exp\left( - \frac{r^2}{w_0^2} \right),
\]

where \( w_0 \) is the waist radius. If \( C_2 = 1, C_4 = 1 \), we have \( J_z/W = 3 \). If \( C_2 = 1, C_4 = 3^{\frac{1}{2}} \) then \( J_z/W = 3.5 \).

The numerical simulation is done by the Fresnel transform. The simulation parameters are: the wavelength, \( \lambda = 532 \text{ nm} \), the waist radius, \( w_0 = 1 \text{ mm} \), computation domain: \(-R \leq x, y \leq R, (R = 5 \text{ mm})\), and the distance between the grid points, \( \Delta x = \Delta y = 20 \mu \text{m} \). Figure 1 depicts distributions of intensity [figure 1(a, c)] and phase [figure 1(b, d)] in the initial plane \( z=0\), for superposition of the LG modes with the normalized OAM equal to 3 [figure 1(a, b)] and 3.5 [figure 1(c, d)].

![Figure 1](image-url)

**Figure 1.** Intensity (a,c) and phase (b,d) distributions in the initial plane \( z=0\) of a superposition of LG modes, whose normalized OAM equals 3 (a,b) and 3.5 (c,d).

For the beams in figure 1, OAM was also obtained by equations (4), (7), assuming that the beam contained modes with the topological charges \( m = 1, ..., 5 \). The radii \( r_m \) were chosen in the form \( m \Delta r \) so that 80% of the beam energy was within a radius of 5\( \Delta r \) (\( \Delta r = 0.34 w_0 \) in figure 1(a,b) and \( \Delta r = 0.36 w_0 \) in figure 1(c,d)).

For the beam in figure 1(a,b) the superposition (1) was calculated for the coefficients \( C_1 = 0; C_2 = 1; C_3 = 0; C_4 = 1; C_5 = 0 \). By solving equation (4), the squared modules of the coefficients were found to
be $|C_1|^2 = 0.001; |C_2|^2 = 0.98; |C_3|^2 = 0.05; |C_4|^2 = 0.94; C_5 = 0.03$. With the theoretical value of the normalized OAM being $J/W = 3.00$, the value calculated from equation (7) was 3.01.

For the beam in figure 1(c, d) the superposition (1) was calculated for the coefficients $C_1 = 0; C_2 = 1; C_3 = 0; C_4 = 3^{1/2}; C_5 = 0$. With the radii $r_m$ calculated the same way, the solution of equation (4) gave the squared modules of the coefficients $|C_1|^2 = -0.002; |C_2|^2 = 1.03; |C_3|^2 = -0.09; |C_4|^2 = 3.12$; and $|C_5|^2 = -0.06$. With the theoretical value of the beam normalized OAM being $J/W = 3.50$, equation (7) gives a value of 3.51. Thus, the error in determining the OAM using the numerical solution of the system in equation (4) is less than 1%.

5. Experimental determining the OAM of a superposition of LG modes

Figure 2 shows an experimental setup for determining the OAM by using equations (4) and (7). Collimated and expanded beam from a solid state laser was incident onto a $1920 \times 1080$ spatial light modulator SLM HOLOEYE PLUTO VIS (each pixel of $8 \mu m$). The phase mask on the modulator was obtained by encoding the complex amplitudes distribution of a superposition of the LG modes with the Gaussian beam waist radius of $w_0 = 190 \mu m$. The reflected and modulated laser beam was spatially filtered by lenses $L_2 (f_2 = 500 \text{ mm})$ and $L_3 (f_3 = 150 \text{ mm})$ and a diaphragm $D$, which blocked the unmodulated light. Intensity distributions of the generated superpositions of the LG beams were recorded by the camera in the focal plane of the lens $L_2$ (the plane conjugate to the plane of the modulator display $z = 0$).

Figure 2. Experimental setup for determination of OAM of superpositions of the LG modes: Laser is a solid state laser ($\lambda = 532 \text{ nm}$), MO is a microobjective (8×, $NA = 0.3$), PH is a pinhole (the hole diameter is $40 \mu m$), $L_1$, $L_2$ and $L_3$ are spherical lenses ($f_1 = 350 \text{ mm}, f_2 = 500 \text{ mm}, f_3 = 150 \text{ mm}$), SLM is a spatial light modulator HOLOEYE PLUTO-VIS, $M$ is a mirror, $D$ is a diaphragm, Cam is a camera ToupCam U3CMOS08500KPA.

Figure 3 shows the generated intensity distribution for the superposition of the LG modes from the figure 1(a, b), i.e. for the beam in figure 3, the coefficients from equation (1) are equal to $C_1 = 0; C_2 = 1; C_3 = 0; C_4 = 1; C_5 = 0$. Using equations (4) and (7) with $m = 1, ..., 5$, OAM of such a beam was calculated. Solution of equation (4) gives the values $C_1 = -0.31i; C_2 = 0.95; C_3 = -0.71i; C_4 = 1.00; C_5 = -0.21i$. Calculation by equation (7) using the experimentally obtained intensity distribution gives the value 3.17. The standard error thus equals 5.6%.

Figure 3. Experimentally generated intensity distribution of a superposition of LG modes $\text{LG}_{0,2} + \text{LG}_{0,4}$ registered by the camera. Image size is $1500 \times 1500 \mu m$.

6. Numerical simulation of propagation in random medium

In this section, we study numerically how propagation of a superposed beam (10) in random medium affects the beam transverse shape and its OAM defined by the above described methods. Instead of the
random medium, we used a diffuser with a random phase distribution $\Psi(x, y)$ placed in the initial plane. The diffuser has the correlation function described by the Gaussian exponential function:

$$\langle \exp(i\Psi(x, y) - i\Psi(x', y')) \rangle = \exp\left[-\frac{(x-x')^2 + (y-y')^2}{\sigma^2}\right],$$

(11)

where $\sigma$ is the correlation radius of the homogeneous phase diffuser, and angular brackets in equation (11) mean the ensemble averaging over statistically similar phase diffusers.

Figure 4 depicts phase distributions from figure 1 but distorted by the diffuser [figure 4(a) for the normalized OAM of 3.00 and figure 4(d) for the normalized OAM of 3.50], as well as the intensity and phase distributions in the transverse plane on propagation. The correlation radius of the diffuser is $\sigma = 50 \mu m$, while its phase varies from $-\pi$ to $\pi$. Propagation distance is $z = 6 m$ (it equals approximately the Rayleigh range). As seen in figures 4 (b,e) the intensity pattern consists of two light spots rotated by $45^\circ$ counterclockwise, similarly to the case when the initial phase is undistorted. However, the spots itself have distorted shapes.

Figure 4. Propagation of a superposition of LG modes with the normalized OAM of 3.00 (a-c) and 3.50 (d-f): initial phase distributions distorted by a phase diffuser (a,d), as well as intensity (b,e) and phase (c,f) distributions upon propagation over a distance $z = 6 m$. Red rings show the radii $r_m$ over which the intensity is angle-averaged.

Using the intensity patterns from figures 4(b,e), the OAM was calculated by equations (4), (7). For figure 4(b), the normalized OAM is 3.17 (theoretical value is 3.00), whereas for figure 4(e), the normalized OAM is 3.64 (theoretical value is 3.50). Thus, the error is 6% in the first case and 4% in the second.

To compare the two methods for obtaining the superposition coefficients (by solving either the linear system (4) or linear system (9)), we performed 20 tests for both superpositions with the OAM of 3.50 and 3.00. For the system (9), we used intensity distributions angle-averaged on 20 different radii (instead of 5 radii for the system (4)). For the superposition with the OAM of 3.50, the average error is 3.3% and 2.4% when the coefficients are obtained by solving respectively the linear system (4) and linear system (9). For the superposition with the OAM of 3.00, these average errors are respectively 8.2% and 6.7%. It should be noted here that both methods are significantly more accurate then reconstructing the phase distribution and calculating the OAM by direct trapezoidal evaluation of the integrals (5), (6), which gives the average error of 12%.
7. Conclusion
In this work, we have modified and studied a method for measuring the OAM by single intensity distribution described in [11]. In the modified method, the number of rings with the angle-averaged intensity can be much larger than the number of modes in a laser beam, thus making the method much less dependent on choosing the rings radii. Mathematically, the method is based on solving an overdefined system of linear equations by the least squares method. In addition, we have simulated propagation of a superposition of two LG modes after passing through a diffuser with a random phase. Despite the distortions caused by the diffuser, it is shown that the method allows OAM measurement with an average error less than 10%: the obtained values are 2.4% for the beam with the OAM of 3.50 and 6.7% for the beam with the OAM of 3.00. The main advantages of the described technique is that (i) simultaneously with the full OAM the OAM spectrum is obtained (squared modules of the expansion coefficients in equation (1)) and (ii) no any optical elements are needed.

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