An artificial game with equilibrium state of entangled strategy

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Abstract

Using the representation introduced in [1], an artificial game in quantum strategy space is proposed and studied. Although it has well-known classical correspondence, which has classical mixture strategy Nash Equilibrium states, the equilibrium state of this quantum game is an entangled strategy (operator) state of the two players. By discovering such behavior, it partially shows the independent meaning of the new representation. The idea of entanglement of strategies, instead of quantum states, is proposed, and in some sense, such entangled strategy state can be regarded as a cooperative behavior between game players.

Key Words: Game Theory, Quantum Game Theory, Entanglement

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Introduction — Recently, we proposed a new mathematical representation for Classical and Quantum Game Theory[1]. The idea is to define base vectors in strategy space and their inner product so as to form them as a Hilbert space. Then a system state is a vector in the direct product space of all single-player state spaces. A density matrix is used to describe a system state in this space. And then the payoff functions are reexpressed as hermitian operators acting on this system Hilbert space. Every player has a payoff matrix, which is a (1,1)-tensor no matter how many players and how many base strategies the game has. In that paper, many open questions were pointed out. Most of those questions can only be discussed in the new representation. If the conclusions of such questions shows some new phenomena, the special value of the new representation will partially be confirmed. One of such questions is the meaning of entangled strategies, which will happen when the direct product relation between system density matrix and single-player density matrix is destroyed. In this paper, we try to answer this question by one example. On the other hand, according to [2], a comment on quantum game, two questions should be answered for any quantum version of classical game. First, whether it is helpful to solve the original classical game; second, is a new truly quantum game, or still in the scope of classical game. In [3], we give a detailed answer of those two questions, while here, we try to provide a positive answer by this example.
The conclusion in this paper implies that from the viewpoint of classical game, besides an equivalent description, our new representation provides an applicable algorithm and even the process of the algorithm can be regarded as a reasonable evolutionary process, and it could be a way from static non-cooperative game to cooperative game; from the viewpoint of quantum game, the game in our representation is totally a new game, which could never be put into the framework of classical game, because a classical payoff matrix \( G^i \) is not enough to describe all information of a quantum game, which requires a much larger matrix \( H^i \).

The new representation — First, let’s shortly review the new representation. A game is defined as

\[
\Gamma = \left( \prod_{i}^{N} (\times S_{i,q}^i), \prod_{i}^{N} (\times S_{i,c}^i), \{H_i^i\} \right),
\]

in which \( S_{i,q}^i \) has base vectors \( \{\left| s_{\mu}^{i,q} \right>\} \), and \( S_{i}^c \) has base vectors \( \{\left| s_{\nu}^{i,c} \right>\} \). Usually the later is a subset of the former, but not necessary. A classical payoff function is defined on system base vectors such as \( H_{i,c}^i = \sum_{S} \left| S \right> H_{SS}^{i,c} \left< S \right| \), while a quantum payoff function is defined as \( H^i = \sum_{S S'} \left| S \right> H_{SS'}^{i} \left< S' \right| \), which has non-zero off-diagonal elements while \( H_{i,c}^i \) has only diagonal terms. \( H^i \) is hermitian and may have different forms for each player. A system state is defined as

\[
\rho_s = \prod_{i}^{N} \rho^i.
\]

The payoff of player \( i \) under a system state \( \rho^s \) is

\[
E^i (\rho^s) = Tr \left( \rho^s H^i \right).
\]

A reduced payoff matrix of player \( i \), which player \( i \) uses as a evaluation of all his own strategies under the fixed strategies of all other players, is defined as

\[
H_R^i = Tr_i (\rho^1 \cdots \rho^{i-1} \rho^{i+1} \cdots \rho^N H^i),
\]

where \( Tr_i (\cdot) \) means to do the trace in the space except the one of player \( i \). Then from equ(3), the payoff of player \( i \) also can be calculated by

\[
E^i (\rho^s) = Tr^i \left( \rho^i H_R^i \right),
\]

in which \( Tr^i (\cdot) \) is the trace in player \( i \)'s space. An equilibrium state is defined

\[
E^i (\rho_{eq}^s) \geq E^i (\rho_{eq}^1 \cdots \rho_{eq}^{i} \cdots \rho_{eq}^N), \forall i.
\]

This definition uses the same idea as Nash Equilibrium, but has independent meaning. First because the density matrix form allows more strategies than the traditional mixture strategy, and second, because \( \rho_{eq}^s \) might be an entangled strategy state, which destroys equ(2), while this is not allowed in both traditional classical and quantum
game. When such an entangled state is allowed, we need to adjust a little of equ(6), because at that case, $\rho_{eq}$ is not pre-defined. Here we try to define them as reduced density matrix, $E_i(\rho_{eq}^s) \geq E_i(Tr(\rho_{eq}^s) \cdot \rho^i), \forall i.$ (7)

It should be noticed that when equ(2) holds, the new definition equ(7) is equivalent with equ(6). A special case of the above definition is $E_i(\rho_{eq}^s) \geq E_i(\rho^s), \forall \rho^s, \forall i.$ (8)

Although is not always possible to find such a state $\rho_{eq}^s$, in this paper, we will ‘produce’ a game to make use of such states. Later on, we name such equilibrium as Global Equilibrium State (GES).

The artificial game — Now we define our 2-player game on base vector set $\{|B\rangle, |S\rangle\}$, which means Box and Show respectively. We use them as base vectors for both classical and quantum game. So the base vectors of system space are $\{|BB\rangle, |BS\rangle, |SB\rangle, |SS\rangle\}$. An arbitrary system state can be $\rho^s = \rho_{\mu\nu} |\mu\rangle \langle \nu|$, (9) where $\mu, \nu$ is anyone of the base vectors. Now we just write down the artificial payoff matrix, $H^1 = \begin{bmatrix} \epsilon_1 & 0 & 0 & \epsilon_1 \\ 0 & \epsilon_2 & \epsilon_2 & 0 \\ 0 & \epsilon_2 & \epsilon_2 & 0 \\ \epsilon_1 & 0 & 0 & \epsilon_1 \end{bmatrix} = H^2$, (10)

which does not come from a real quantum game at this stage. But now we will try to figure out a real quantum game with similar situation.

The manipulative definition of a quantum game, in the traditional framework which uses the concepts of strategy according to [4, 3], has been reexpress in [5] as

$$\Gamma^{q,o} = \left( \rho^q_0 \in \mathbb{H}^q, \prod_{i=1}^N \otimes \mathbb{H}^i, \mathcal{L}, \{P^i\} \right).$$ (11)

Where $\rho^q_0$ is the initial state of a quantum object, $\mathbb{H}^i$ is player $i$’s strategy space, $\mathcal{L}$ is a mapping from $\prod_{i=1}^N \otimes \mathbb{H}^i$ to $\mathbb{H}^q$, the quantum object’s operator space, and $P^i$ is the payoff scale for player $i$. Now in this game, the quantum object is still chosen as spin, which has base state vectors $|U\rangle$ and $|D\rangle$. In matrix form, we denote them as $(1, 0)^T$ and $(0, 1)^T$. We choose

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$ (12)

as our pure strategies are also the base vectors in player $i$’s Hilbert space of quantum strategy. So a general quantum strategy (operator) has the form,

$$A = xB + yS = \begin{bmatrix} x & y \\ y & x \end{bmatrix}.$$ (13)
If we also require it’s a unitary operator, then $A^\dagger A = I$, then $x$ and $y$ are not independent. The general form is

$$U = \cos \theta B + i \sin \theta S = \begin{bmatrix} \cos \theta & i \sin \theta \\ i \sin \theta & \cos \theta \end{bmatrix} \quad (14)$$

The initial state of the quantum object is chosen as

$$\rho^0_0 = \begin{bmatrix} 3/4 & \sqrt{3}/4 \\ \sqrt{3}/4 & 1/4 \end{bmatrix} \quad (15)$$

Mapping $\mathcal{L}$ is just the product $U = U^2 U^1$. Payoff scale matrix are set as

$$P^1 = \frac{1}{2} \begin{bmatrix} 3\epsilon_1 - \epsilon_2 & 0 \\ 0 & 3\epsilon_2 - \epsilon_1 \end{bmatrix} = P^2 \quad (16)$$

Then in this framework, the payoff is defined as

$$E^i = Tr \left( P^i \mathcal{L} \rho^0_0 (\mathcal{L})^\dagger \right) = Tr \left( P^i U^2 U^1 \rho^0_0 (U^1)^\dagger (U^2)^\dagger \right). \quad (17)$$

This is quite similar with the quantum penny flip game[3], which uses different $\rho^0_0$ and $P^i$. Using the transformation procedure proposed in [1], we will need other two quantum pure strategies as base vectors, $\sigma_y, \sigma_z$. So we will get a $16 \times 16$-matrix as our whole payoff matrix. And the sub-matrix related with $|B\rangle$ and $|S\rangle$ is just the payoff matrix of our artificial game. So this game should be investigated in the larger space with the whole payoff matrix, and just because of this, we call our game here as an artificial game. Since the difficulty to deal with the whole payoff matrix in the larger space, we wish the discussion of this artificial game still can give us most information. Of course, if necessary, we can turn to analyze the game with the whole payoff matrix. However, in this paper, we think , a clear picture is more important than the results of a full game.

The classical correspondence and its solution — First, we study this game in classical strategy space, which has the general state as

$$\rho^c = (p^1_b \langle B | B \rangle + p^1_s \langle S | S \rangle) (p^2_b \langle B | B \rangle + p^2_s \langle S | S \rangle), \quad (18)$$

or in matrix form

$$\rho^c = \begin{bmatrix} p^1_b p^2_b & 0 & 0 & 0 \\ 0 & p^1_s p^2_s & 0 & 0 \\ 0 & 0 & p^1_b p^2_b & 0 \\ 0 & 0 & 0 & p^1_s p^2_s \end{bmatrix} \quad (19)$$

For such diagonal density matrix, according to the trace operator in equ(3), only the diagonal term of payoff matrix will effect the payoff value. So payoff matrix of the classical correspondence of this artificial game is

$$H^1 = \begin{bmatrix} \epsilon_1 & 0 & 0 & 0 \\ 0 & \epsilon_2 & 0 & 0 \\ 0 & 0 & \epsilon_2 & 0 \\ 0 & 0 & 0 & \epsilon_1 \end{bmatrix} = H^2, \quad \text{or traditionally,} \quad G = \begin{bmatrix} \epsilon_1, \epsilon_1 & \epsilon_2, \epsilon_2 \\ \epsilon_2, \epsilon_2 & \epsilon_1, \epsilon_1 \end{bmatrix}. \quad (20)$$
They are similar with Battle of the Sexes ($\epsilon_1 > \epsilon_2$) and Hawk-Dove ($\epsilon_1 < \epsilon_2$). The solution for a general mixture strategy can be solved by the pseudo-dynamical way introduced by [1]. The general reduced payoff matrix are

\[
H^1_R = \begin{bmatrix}
p_b^2 \epsilon_1 + p_s^2 \epsilon_2 & 0 \\
0 & p_b^2 \epsilon_2 + p_s^2 \epsilon_1
\end{bmatrix}
\quad \text{and} \quad
H^2_R = \begin{bmatrix}
p_b^1 \epsilon_1 + p_s^1 \epsilon_2 & 0 \\
0 & p_b^1 \epsilon_2 + p_s^1 \epsilon_1
\end{bmatrix}.
\]

So the iteration equation given by the pseudo-dynamical equation is

\[
p_b = \frac{1}{1 + e^{\gamma \delta (1 - 2p_b (3 - i))}},
\]

in which $\delta = \epsilon_1 - \epsilon_2$. The fixed points of this iteration is shown in fig(1).

**Solution in quantum pure strategy space** — If a general unitary operator as equ(14) can be used as the strategy of player 1 and player 2, what the equilibrium state? In order to simplify our discussion, unlike in classical game we deal with both cases of $\epsilon_1 > \epsilon_2$ and $\epsilon_1 < \epsilon_2$, here we only focus on the former case. A general state of player $i$ is $U^i (\theta^i)$ in equ(14). Then the reduced payoff matrix when player 2 chooses $|U^2 (\theta^2)|$ is

\[
H^1_R = Tr_1 (\rho^2 H^1) = \begin{bmatrix}
\epsilon_1 \cos^2 \theta^2 + \epsilon_2 \sin^2 \theta^2 & i (\epsilon_1 - \epsilon_2) \cos \theta^2 \sin \theta^2 \\
i (\epsilon_2 - \epsilon_1) \cos \theta^2 \sin \theta^2 & \epsilon_2 \cos^2 \theta^2 + \epsilon_1 \sin^2 \theta^2
\end{bmatrix}.
\]

When $\epsilon_1 > \epsilon_2$ the eigen-state with maximum eigenvalue is

\[
E^1 = \epsilon_1, |\epsilon_1\rangle = (\cos \theta^2, -i \sin \theta^2)^T
\]

Compared with equ(14), we know

\[
\theta^1 = -\theta^2.
\]

On the other hand, if we solve the inverse question that the solution of player 2 when player 1’s strategy is fixed at $U^1 (\theta^1)$, we can get

\[
\theta^2 = -\theta^1.
\]

The combination of equ(25) and equ(26), the equilibrium state of this game in quantum strategy space is

\[
(U^1, U^2) = (U^1 (\theta), U^2 (-\theta)), \forall \theta \quad \text{and} \quad (E^1, E^2) = (\epsilon_1, \epsilon_1).
\]

Discovering all solutions of a quantum game in quantum mixture strategy space is not a trivial problem, but will not be a topic of this paper, because here, we just want to compare these solutions with the entangled strategy solution in next section, not a general way to calculate all the solutions. Anyway, for this game, since it has many pure strategy NEs, any mixture combination of all the pure NEs will be mixture NE.
Figure 1: In figure(a), self-mapping function $p^1_b = \frac{1}{1+e^{\beta \delta (1-2p^1_b)}}$ for different $\beta$ are plotted. For such a mapping, different signs of $\delta$ corresponds to the same function. In figure(b), iteration process $\left(p^1_b = \frac{1}{1+e^{\beta \delta (1-2p^1_b)}}\right)$ for different $\beta$ are plotted. From figure(a), we know that for $p^1_b$, 0.5 is always the unstable fixed point no matter whether $\delta$ is positive or not and there are other two stable fixed points depending on $\beta$. In traditional situation, under infinite resolution level ($\beta = \infty$), the fixed points are 0 and 1 depending on initial value. From figure(b), much detailed information can be extracted. In figure(b-1), when $\delta > 0$, $(0,0)$ and $(1,1)$ are the stable fixed points, while in figure(b-2), when $\delta < 0$, they are $(0,1)$ and $(1,0)$. So we can say, besides $(0.5,0.5)$, when $\delta > 0$, we have two NEs, but when $\delta < 0$, we have other two NEs. They are just the NEs from traditional analysis, and here they are given by our pseudo-dynamics process. Furthermore, thinking about the iteration process, step by step, in some sense, this process can be regarded as an evolution. So even the pseudo-dynamics process will have a good meaning besides as expected it does end at the reasonable states — NEs if they are stable. So will it also be a way to Evolutionary Game from Static Game?
Entangled quantum game and the GES — Now we try to solve the equilibrium state in the most wider strategy space, system strategy space, or sometimes, entangled strategy space, in which a general state can be equ(9). As in [1], we named it as Entangled Quantum Game because the strategy space permits a state without direct product relation equ(2). In fact, because in this game, both classical and quantum game use the same base vectors, it also can be named as Entangled Classical Game according to the rule we proposed in [1]. The payoff matrix of equ(10) have Global Equilibrium State (GES). The payoff matrix can be rewritten as

$$H_1 = H_2 = \epsilon_1 (|BB\rangle + |SS\rangle) \langle BS| + \epsilon_2 (|BS\rangle + |SB\rangle) \langle SS|$$  \hspace{1cm} (28)

So the eigen-state with maximum eigenvalue is $|BB\rangle + |SS\rangle$ when $\epsilon_1$ is bigger and is $|BS\rangle + |SB\rangle$ when $\epsilon_2$ is bigger. And it’s easy to know they are GES when $\epsilon_1$ or $\epsilon_2$ is larger respectively. So the equilibrium state is

$$\rho_{ges} = |BB\rangle \langle BB| + |BS\rangle \langle BS| + |SB\rangle \langle SB|$$ \hspace{1cm} (29)

Both of these states are entangled states between the players. This implies that an entangled strategy can win over both quantum and classical players. And even more, since it’s GES,

$$E^i (\rho_{ges}) > E^i (Tr^1 (\rho_{ges}) Tr^2 (\rho_{ges}))$$ \hspace{1cm} (30)

This means when we destroy the correlation between player 1 and player 2, both players get less payoff. In some sense, this implies that those two players should negotiate and reach agreement. This is the topic of Cooperative Game. So, could we generally say, if an entangled state in system space has the property that

$$E^i (\rho^S) > E^i (Tr^1 (\rho^S) Tr^2 (\rho^S))$$ \hspace{1cm} (31)

it will imply a cooperative behavior?

Discussion — From the results above, we know solution in quantum strategy space includes solution in classical strategy space as special case, while solution in entangled strategy space beats quantum solution. At the first sight of the equilibrium entangled strategy, one might regard it as a natural result of the classical correspondence. Because, in the case of $\epsilon_1 > \epsilon_2$, in some sense, the entangled equilibrium means they choose to stay together $(\frac{1}{2} (|BB\rangle + |SS\rangle) \langle BS| + |SB\rangle \langle SS|)$. But the NEs of the classical correspondence, equ(20), are $|BB\rangle \langle BB|, |SS\rangle \langle SS|$, and $\frac{1}{2} |BB\rangle \langle BS| + \frac{1}{2} |BS\rangle \langle SS|$. They are different with our GES. So it’s totally a new phenomena in entangled game. However, although it seems correct theoretically, how to experimentally entangle two operators, not the usual meaning as entanglement of quantum objects? Another question is if there is no GES in the game, how to find the equilibrium state defined in equ(7). In [1], we proposed a pseudo-dynamical iteration process on the basis of Kinetics Equation in Statistical Mechanics, and use it to calculate the equilibrium state of classical game. It seems work well, although we are still pursuing a general proof. But
still, we have no applicable algorithm for quantum game. Is it possible to generalize this approach into quantum game? At last, the same payoff matrix of player 1 and player 2 makes the classical correspondence of our artificial game not very like a battle, so it’s exactly a quantized version of Battle of the Sexes and Hawk-Dove Game. This is also one of the reasons that we call this game and artificial game. Anyway, according to the manipulative definition of the quantum game $\Gamma^{q,o}$ in equ(11), it still can be realized by quantum object and operators.

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