Optimal potentials for hedging algorithms

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Abstract

We study a family of potential functions for online learning. We show that if the potential function has strictly positive derivatives of order 1-4 then the min-max optimal strategy for the adversary is Brownian motion. Using that fact we analyze different potential functions and show that the Normal-Hedge potential provides the tightest upper bounds on the cumulative regret of the top \( \epsilon \)-percentile.

1. Introduction

Online prediction with expert advise has been studied extensively over the years and the number of publications in the area is vast. For some starting points see Vovk (1990); Feder et al. (1992); Littlestone and Warmuth (1994); Cesa-Bianchi et al. (1997); Cesa-Bianchi and Lugosi (2006).

Here we focus on a simple variant of online prediction with expert advice called the decision-theoretic online learning game (DTOL) Freund and Schapire (1997), where we consider the signed version of the online game.

DTOL is a repeated zero sum game between a learner and an adversary. The adversary controls the losses of \( N \) experts, or (also referred to as actions), while the learner controls a distribution over the experts.

Iteration \( i = 1, \ldots, T \) of the game consists of the following steps:

1. The learner chooses a distribution \( P_i^j \) over the experts \( j \in \{1, \ldots, N\} \).

2. The adversary chooses an instantaneous loss for each of the \( N \) experts: \( l_i^j \in [-1, +1] \) for \( j \in \{1, \ldots, N\} \).

3. The learner incurs an expected loss defined to be \( \ell^i = \sum_{j=1}^{N} P_i^j l_i^j \).

A strategy for the learner (adversary) is a mapping from the history of choices made by both sides on previous iterations to the choice that is made on the current iteration.

We denote the cumulative loss of expert \( j \) on iteration \( k \) by \( L_j^k = \sum_{i=1}^{k} l_i^j \). Similarly, we denote the cumulative loss of the learner by \( L_L^k = \sum_{j=1}^{k} \ell^j \). The instantaneous regret of the learner relative to expert \( j \) on iteration \( i \) is defined as \( r_i^j = \ell^i - l_i^j \). The cumulative regret of the learner with respect to expert \( j \) for iterations \( i = 1, \ldots, k \) is \( R_j^k = \sum_{i=1}^{k} r_i^j \). We denote the vector of all \( N \) regrets at iteration \( k \) by \( R^k = \{R_j^k\}_{j=0}^{N} \).

The goal of the learner is to minimize the maximal regret at the end of the game \( R^*_T = \max_j R_j^T = L_L^T - \min_j L_j^T \). The goal of the adversary is to maximize the same quantity. We use \( L_L^T = \min_j L_j^T \) to denote the total loss of the best expert, and write \( R^*_T = L_L^T - L^*_T \). This paper describes strategies for the learner that provide upper bounds on \( R^*_T \) and strategies for the adversary that provide lower bounds on \( R^*_T \).
1.1. Some known Bounds

Zero-order bounds on the regret Freund and Schapire (1999) depend only on $N$ and $T$ and typically have the form

$$\max_j R^T_j < C\sqrt{T \ln N}$$

for some small constant $C$ (typically smaller than 2). These bounds can be extended to infinite sets of experts by defining the $\epsilon$-regret of the algorithm as the regret with respect to the best (smallest cumulative loss) $\epsilon$-percentile of the set of experts. This replaces the bound (1) with

$$\max_j R^T_j < C\sqrt{T \ln \frac{1}{\epsilon}}$$

Lower bounds have been proven that match these upper bounds up to a constant. These lower bounds typically rely on constructions in which the losses $l^i_j$ are chosen independently at random to be either +1 or −1 with equal probabilities.

Several algorithms with refined upper bounds on the regret have been studied. Of those, particularly relevant here is the second-order regret bound given in theorem 5 of Cesa-Bianchi et al. (2007):

$$\max_j R^T_j \leq 4\sqrt{V_T \ln N} + 2 \ln N + 1/2$$

Where

$$V_T = \sum_{i=1}^T \text{Var}_i = \sum_{j=1}^N P^j_i \left( l^i_j \right)^2 - \left( \sum_{j=1}^N P^j_i l^i_j \right)^2$$

A few things are worth noting. First, as $|l^i_j| \leq 1$, $\text{Var}_j \leq 1$ and therefore $V_T \leq T$. However $V_T/T$ can be arbitrarily small, in which case inequality (3) provides a tighter bound than inequality (1). Intuitively, we can say that $V_T$ replaces $T$ in the regret bound. This paper provides additional support for replacing $T$ with $V_T$ and provides tighter lower and upper bounds on the regret involving $V_T$.

1.2. Potential Functions

A common approach to designing online learning algorithms is to define a potential function. The potential function $\phi : \mathbb{R} \to \mathbb{R}$ is a positive, continuous and non-decreasing function of the regret. One popular potential function is the exponential function which has the form $\phi(R) = \exp(\eta R)$ where $\eta > 0$ is the learning rate. See Cesa-Bianchi and Lugosi (2006) for an extensive review of the use of potential functions for online learning.

A central quantity in the design and analysis of potential based algorithms is the average potential which we refer to here as the score. The score at time $k$ is defined as:

$$\Phi^k = \frac{1}{N} \sum_{j=1}^N \phi(R^k_j)$$

The fact that $\phi(R)$ is positive and non-decreasing implies an upper bound on the regret w.r.t. any expert. Suppose that the score at time $t$ is upper bounded by $A$ and that the learner suffers regret $B$ with respect to at least one expert: $B = \max_j (R^T_j)$, then $\phi(B) \leq N\Phi_T \leq NA$.

In most formulations of this technique, the potential function $\phi$ is fixed a-priori, and the learner’s strategy is designed based on this function. This raises a natural question:
**Question 1**  
Is there an optimal potential function $\phi$ for DTOL?

Without further constraints, this question is ill-defined. We take several steps to better define the question.

1. We allow the potential function to depend on the iteration number $i$, i.e. we study potential functions of the form $\phi(i, R)$. \(^1\)

2. We fix the number of iterations $T$. This assumption will later be removed.

3. We fix the final potential $\phi(T, \cdot)$. This function is required to have defined and strictly positive derivatives of degrees zero to four.\(^2\) This assumption is satisfied by most commonly used potential functions, including the exponential potential, the NormalHedge potential and polynomial potentials of degree higher than four.

This leads to a more specific question:

**Question 2**  
Given the length of the game $T$ and the final potential function $\phi(T, R)$, can we define the best potentials $\phi(i, R)$ for $0 \leq i < T$?

We give a qualified answer to this question. Rather than finding a single potential function, we associate an upper potential with learner strategy $P$ and a lower potential with adversarial strategy $Q$.

We start by fixing both $Q$ and $P$. As we show in Definition 4 and Lemma 5, for any pair of strategies $Q$ and $P$ and for any iteration $0 \leq i \leq T$ there exists a potential function $\phi_{Q,P}(i, \cdot)$ such that if we define the score $\Phi_{Q,P}(i, R^i)$ to be

$$
\Phi_{Q,P}(i, R^i) = \frac{1}{N} \sum_{j=1}^{N} \phi_{Q,P}(T - 1, R^i) \tag{5}
$$

then $\Phi_{Q,P}(0, R^0) = \cdots = \Phi_{Q,P}(T, R^T)$.

Using Equation 12 we define an upper potential and a lower potential for each iteration $0 \leq i \leq T$

$$
\phi^\uparrow_{P}(i, R^i) = \max_Q \phi_{Q,P}(i, R^i) \quad \text{and} \quad \phi^\downarrow_{Q}(i, R^i) = \min_P \phi_{Q,P}(i, R^i) \tag{6}
$$

Which are given given inductive definitions in Equations (13,14)

Clearly, for any $Q, P, i, R$ we have that $\phi^\uparrow_{P}(i, R) \leq \phi^\downarrow_{Q}(i, R)$. Our goal is to find a pair of strategies $Q, P$ such that $\phi^\uparrow_{P}(i, R) = \phi^\downarrow_{Q}(i, R)$. That would imply that $Q$ and $P$ are min-max optimal strategies.

It is unclear whether min-max strategies exist for standard DTOL. In order to close the apparent gap between the upper and lower potentials and identify the min-max strategies we expand the game. Here expansion means giving the adversary more choices while keeping the learner’s choices unchanged. As a consequence, any upper bounds guarantees that hold for the expanded game also hold for the standard DTOL (but the lower bounds might not).

We make two expansion steps. In the first, we allow the adversary to arbitrarily divide experts. In the second we we allow the adversary to choose the range of allowed instantaneous losses. Using both expansions we identify the min/max strategies and the min/max potential function thereby answering question 2.

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1. Potentials that depend on time were used for NormalHedge and related algorithms Chaudhuri et al. (2009); Luo and Schapire (2015)

2. The zeroth derivative is the function itself. Therefore positivity of derivatives 0 and 1 implies the standard assumption that $\phi(T, R)$ is positive and increasing.
The first expansion gives the adversary the ability to arbitrarily split experts. Intuitively, rather than associating a single loss with each expert, the adversary can arbitrarily divide the the expert into sub-experts and assign a different loss to each. This is similar to Chernov and Vovk (2010) where the number of experts is not known in advance.

We define the even split adversarial strategy \( Q^{1/2} \) as one which, at each iteration splits each current expert into two equal parts, one incurs loss 1 and the other loss -1. The result is that at iteration \( n \) we have \( 2^n \) experts, each of weight \( 2^{-n} \), each corresponding to a loss sequence in \( \{-1, +1\}^n \). The cumulative loss of the experts has a binomial distribution \( \mathbb{B}(n, 1) \)\(^3\). The symmetry of \( Q^{1/2} \) implies that the loss of the learner against this adversary is zero independently of the learner choices. Therefore the cumulative regret is also Binomially distributed according to \( \mathbb{B}(n, 1) \). The final score is equal to the expected value of the final potential \( \mathbb{E}_{R \sim \mathbb{B}(T, 1)} [\phi(T, R)] \). As we will show there is a simple strategy for the learner that guarantees an upper bound of \( \mathbb{E}_{R \sim \mathbb{B}(T, 2)} [\phi(T, R)] \). The upper and lower bounds are both based on binomial distribution. However, as the step sizes are \( 1 < 2 \), these strategies are not a min/max pair.

Intuitively, in order to achieve min/max optimality we need to make the step size of the adversary equal the step size of the learner. We next explain how the step size for the learner is reduced to the step size of the adversary.

We achieve that by taking the second expansion step. In this expansion we allow the adversary to set the step range in iteration \( i \) to \( [-s_i, +s_i] \) for any \( 0 \leq s_i \leq 1 \). In Section 5 we give a full description of the game. For now, suppose that the adversary chooses \( s_i = 1/m \) for some large natural number \( m \) and for all \( i \). Scaling \( Q^{1/2} \) to be \( \pm 1/m \) with equal probabilities we find that the lower score is equal to \( \mathbb{E}_{R \sim \mathbb{B}(T, 1/m)} [\phi(T, R)] \). It is not hard to see that if \( m \to \infty \) while \( T \) remains constant then the binomial distribution degenerates to the delta function. In order to avoid this degeneracy we set \( T = m^2 \). The binomial distribution \( \mathbb{B}(m^2, 1/m) \) has variance one for all \( m \) and converges to the standard normal as \( m \to \infty \). In Section 5 we show that in this limit the difference between the upper and lower potentials converges to zero.

To show that this limit is the min/max, we need to show that, in fact, the optimal choice for the adversary is to set \( m \) arbitrarily high. If the adversary prefers a finite \( m \), then this is not the min/max. This is where the strictly positive fourth derivative is important. In Section 5 we showed that, if the final potential function is in \( \mathcal{P}^4 \) then the lower potential strictly increases with \( m \). Establishing the min/max properties of the limit \( m \to \infty \).

One benefit of working with the Weiner process is that the backward equations that define the potential transform, in the limit, into the Kolmogorov Backwards equations (KBE) The potential function \( \phi^\uparrow(t, R) = \phi^\downarrow(t, R) = \phi(t, R) \) is the solution of the Kolmogorov backwards equation (KBE) with the boundary condition defined by \( \phi(T, R) \). We have thus answered Question 2 in the affirmative.

Having identified the min/max strategies for a given final potential function \( \phi(T, R) \) we next turn to choosing \( \phi(T, R) \). We say that a parametrized potential function \( \phi(\theta_T, R) \) is compatible with KBE if the solution to KBE with boundary condition \( \phi(T, R) = \phi(\theta_T, R) \) can be written in the form \( \phi(t, R) = \phi(\theta_t, R) \). In Section 6 we identify two KBE compatible functions, one corresponds to the exponential potential, the other to the NormalHedge potential. An additional benefit of characterizing the solution as a solution of KBE is that the solution can be extended to \( t > T \). This removes the need to know \( T \) a priori and yields an any time algorithm.

Finally, we come back to Question 1. Intuitively, we want \( \phi(T, R) \) (properly normalized) to increase as fast as possible, without causing the score to increase with \( t \). As we have identified the worst case adversary

\[^3\) We use \( \mathbb{B}(n, s) \) to denote the binomial distribution that assigns probability \( (n)!2^{-n} \) to the point \( (n-2j)s \) for \( j = 0, \ldots, n \).
to be Brownian motion, we can ask what is the fastest increasing potential for which Brownian motion will not increase the score. In section 7 we show that an appropriate limit of NormalHedge has this distinction.

2. Preliminaries

The integer time game takes place on the set \((i, R) \in \{0, 1, \ldots, T\} \times \mathbb{R}\), where \(i\) corresponds to the iteration number, and \(R\) corresponds to the (cumulative) regret.

As in the standard DTOL setting, an expert corresponds to a sequence of cumulative regrets. However, unlike the standard DTOL, the number of experts is allowed to be infinite. In DTOL, the state of the learning process is defined by the regret of each of the \(N\) experts: \(\langle R_1(i), \ldots, R_N(i) \rangle\). In order to represent the regret of a potentially uncountable set of experts, we define the state as a distribution over possible regret values. Thus the state of the game on iteration \(i\) is a distribution (probability measure) over the real line, denoted \(\Psi(i)\). We denote by \(R \sim \Psi(i)\) a random regret \(R\) that is chosen according to the distribution corresponding to the state at iteration \(i\). Given a measurable function \(f : \mathbb{R} \to \mathbb{R}\) we define \(D_{R \sim \Psi(i)}[f(R)]\) to be the distribution of \(f(R)\) when \(R \sim \Psi(i)\). We similarly define the expected value of \(f(R)\) by \(E_{R \sim \Psi(i)}[f(R)]\), and the probability of an event \(e\) defined using \(f(R)\) by \(P_{R \sim \Psi(i)}[e(f(R))]\).

The initial state \(\Psi(0)\) is a point mass at \(R = 0\). The state \(\Psi(t)\) is defined by \(\Psi(t_i - 1)\) and the choices made by the two players as described in the next section.

The final potential function \(\phi(T, \cdot)\) is predefined and known to both sides. The final score is defined to be

\[
\Phi(T) = E_{R \sim \Psi(T)}[\phi(T, R)] \quad (7)
\]

The goal of the learner is to minimize \(\Phi(T)\) and the goal of the adversary is to maximize it.

We assume that the final potential is strictly positive of degree \(k\), which is defined as follows:

**Definition 1 (Strict Positivity of degree \(k\))** A function \(f : \mathbb{R} \to \mathbb{R}\) is strictly positive of degree \(k\), denoted \(f \in \mathcal{P}^k\), if the derivatives of orders 0 to \(k\): \(f(x), \frac{d}{dx}f(x), \ldots, \frac{d^k}{dx^k}f(x)\) exist and are strictly positive.

A simple lemma connects an upper bound on any score function in \(\mathcal{P}^1\) with a bound on the regret.

**Lemma 2** Let \(\phi(T, R) \in \mathcal{P}^1\), \(\Phi(T) \leq U\) and \(\epsilon = P_{R \sim \Psi(T)}[R > R']\) be the probability of the set of experts with respect to which the regret is larger than \(R'\). Then

then the regret of the algorithm relative to the top \(\epsilon\) of the experts is upper bounded by

\[
\phi(T, R) \leq U/\epsilon
\]

3. Integer time game

We start with a setup in which time and iteration number are the same, i.e. \(t_i = i\). In this section we suppress \(t_i\) and instead use the iteration number \(i = 0, 1, 2, \ldots, T\).

We define the state of the game on iteration \(i\): \(\Psi(i)\) as the distribution of regret over the experts. Note that experts are allowed to be uncountably infinite. In particular the adversary can assign to the experts with regret \(x\) at iteration \(t\) an arbitrary distribution of losses in the range \([-1, +1]\).

The game is defined by three parameters:

- \(T\) : The number of iterations
- \(\Psi(0) = \delta(0)\) is the initial state of the game which is a point mass distribution at 0.
• \( \phi(T, R) \): The function that is in \( \mathcal{P}^2 \).

The transition from \( \Psi(i) \) to \( \Psi(i+1) \) is defined by the choices made by the adversary and the learner.

1. The learner chooses weights. Formally, \( P(i, \cdot) \) is a density over \( \mathbb{R} \):

2. The adversary chooses the losses of the experts. Formally this is a mapping from \( \mathbb{R} \) to distributions over \([-1, +1] \): \( Q(i) : \mathbb{R} \to \Delta[-1, +1] \). We use \( l \sim Q(i, R) \) to denote the distribution over the instantaneous loss associated with iteration \( i \) and regret \( R \).

3. The aggregate loss (also called “the loss of the master”) is calculated:

\[
\ell_P(i) = \mathbb{E}_{R \sim \Psi(i)} \left[ P(i, R) \mathbb{E}_{l \sim Q(i, R)} [l] \right] \tag{8}
\]

We define the bias at \( (i, R) \) to be \( B(i, R) \equiv \mathbb{E}_{l \sim Q(i, R)} [l] \) which allows us to rewrite Eqn (8) as

\[
\ell_P(i) = \mathbb{E}_{R \sim \Psi(i)} \left[ P(i, R) B(i, R) \right] \tag{9}
\]

Note that \( B(i, R) \) is in \([-1, 1] \) and that \( \ell_P(i) \) is the mean of \( P(i, R) B(i, \cdot) \). Note also that \(-2 \leq y - \ell_P(i) \leq 2 \) corresponds to the instantaneous regret.

4. The state is updated.

\[
\Psi(i + 1) = \mathcal{D}_{R \sim \Psi(i), l \sim Q(i, R)} \left[ R + l - \ell_P(i) \right] \tag{10}
\]

Which is the distribution of \( R + l - \ell_P(i) \) where \( R \) is the cumulative regret at iteration \( i \), whose distribution is to \( \Psi(i) \), \( l \) is the instantaneous loss chosen according to the adversarial distribution \( Q(i, R) \) and \( \ell_P(i) \) is the average loss as defined above.

The final score is the mean of the potential according to the final state, as given in Equation (7). The goal of the learner is to minimize the final score and the goal of the adversary is to maximize it. Equations (15, 17) define simple strategies for the adversary and the learner. For these strategies we prove our main result regarding the integer game.

**Theorem 3**

*There exists a strategy for the adversary such that for any strategy of the learner,*

\[
\Phi(T) \geq \mathbb{E}_{R \sim \mathbb{B}(T, 1)} \left[ \phi(T, R) \right]
\]

*There exists a strategy for the learner such that for any strategy of the adversary,*

\[
\Phi(T) \leq \mathbb{E}_{R \sim \mathbb{B}(T, 2)} \left[ \phi(T, R) \right]
\]

\( \mathbb{B}(n, s) \) is defined in Footnote 3, and the proof is given in Appendix A. The potential This proof is based on the concept of upper and lower potentials which is our main tool in this paper and is explained in the following section.
4. Potentials

The definition of an integer time game specifies the The final potential \( \phi(T, R) \) is set in the definition of a game. In this section we show a natural way to extend the definition to all game iterations.

**Definition 4 (potential backwards recursion)**  Let \( T \) be the number of iterations, \( \phi(T, R) \) be the final potential, \( P \) be a learner strategy and \( Q \) be an adversarial strategy. We define the intermediate potential functions of \( i = T - 1, T - 2, \ldots, 0 \) using the following backwards recursion:

\[
\phi_{P,Q}(i, R) = \mathbb{E}_{l \sim Q(i,R)} \left[ \phi_{P,Q}(i + 1, R + l - \ell_P(i)) \right]
\]

(11)

Where \( \ell_P(i) \) is defined in Eqn (8).

The following lemma guarantees that, for this definition of the potential function, the score function does not change from iteration to iteration.

**Lemma 5**  Define the score at iteration \( i \) to be

\[
\Phi_{P,Q}(i) = \mathbb{E}_{R \sim \Psi(i)} \left[ \phi_{P,Q}(i + 1, R) \right]
\]

(12)

Then

\[
\phi_{P,Q}(0, 0) = \Phi_{P,Q}(0) = \Phi_{P,Q}(1) = \cdots = \Phi_{P,Q}(T)
\]

The proof is given in Appendix B.

Definition 4 and Lemma 5 correspond to a fixed pair of strategies. Using those, there is a natural way to define an upper potential \( \phi_P^\uparrow \), we characterizes the least upper bound on the potential that is guaranteed by the learner strategy \( P \).

\[
\phi_P^\uparrow(i, R) = \begin{cases} 
\phi(T, R) & \text{if } i = T \\
\sup_Q \mathbb{E}_{l \sim Q(i,R)} \left[ \phi_P^\uparrow(i + 1, R + l - \ell_P(i)) \right] & \text{if } 0 \leq i < T
\end{cases}
\]

(13)

Similarly, the lower potential \( \phi_Q^\downarrow \) characterizes the highest lower bound on the potential guaranteed by the adversarial strategy \( Q \).

\[
\phi_Q^\downarrow(i, R) = \begin{cases} 
\phi(T, R) & \text{if } i = T \\
\inf_P \mathbb{E}_{l \sim Q(i,R)} \left[ \phi_Q^\downarrow(i + 1, R + l - \ell_P(i)) \right] & \text{if } 0 \leq i < T
\end{cases}
\]

(14)

4.1. Strategies for the integer time game

We assign the adversary the even split strategy, described in the introduction. Formally, even split is defined as

\[
Q^{1/2}(i, R) = \begin{cases} 
-1 & \text{w.p. } 1/2 \\
+1 & \text{w.p. } 1/2
\end{cases}
\]

(15)

It is easy to see that, when \( Q^{1/2} \) is used, the expected loss \( \ell_P = 0 \) regardless of \( P \). In other words, the learner has no influence on the lower potential which is simply:

\[
\phi_Q^\downarrow(i - 1, R) = \frac{\phi_Q^\downarrow(i, R + 1) + \phi_Q^\downarrow(i, R - 1)}{2}
\]

(16)

4 To bootstrap the recursion we set \( \phi_{P,Q}(T, R) = \phi(T, R) \)
The learner strategy:

\[ P^d(i - 1, R) = \frac{1}{Z} \frac{\phi(i, R + 2) - \phi(i, R - 2)}{2} \]  

(17)

Where \( Z \) is a normalization factor

\[ Z = E_{R \sim \Psi(i)} \left[ \frac{\phi(i, R + 2) - \phi(i, R - 2)}{2} \right] \]

guarantees the upper potential

\[ \phi^+_{pd}(i - 1, R) = \frac{\phi^+_{pd}(i, R + 2) + \phi^+_{pd}(i, R - 2)}{2} \]  

(18)

These strategies satisfy Theorem 3, the proof is given in Appendix A.

We find that the lower bound corresponds to an unbiased random walk with step size \( \pm 1 \). The upper bound also corresponds to a an unbiased random walk with step size \( \pm (1 + c) \). The natural setting in the natural game is \( c = 1 \), which means that there is a significant difference between the upper and lower bounds. As we show in the next section, this gap converges to zero in the continuous time setting, and the upper and lower bounds match, making the strategies for both sides min-max optimal.

Note also that the adversarial strategy the aggregate loss \( \ell(t) \) is always zero, regardless of the strategy of the learner, and state progression is independent of the learner’s choices.

5. Discrete time game

We now describe a game that allows the adversary to choose the size of the step on each iteration. Differently from the introduction in which we assumed all step sizes are equal.

In this game we use \( i = 1, 2, 3, \ldots \) as the iteration index. We use \( t_i \) to indicate a sequence of real-valued time points. \( t_0 = 0 \) and \( t_{i+1} = t_i + s_i^2 \) we assume there exists a finite \( n \) such that \( t_n = T \).

We need to put a bound on \( B(t_i, R) \) because without a bound the bias can be \( s_i \) and the cumulative bias after unit time would be \( s_i/s_i^2 = 1/s_i \) which goes to infinity as \( s_i \to 0 \). Infinite biases make that total loss undefined.

We will later give some particular potential functions for which no a-priori knowledge of the termination condition is needed. The associated bounds will hold for any iteration of the game.

On iteration \( i = 1, 2, \ldots \)

1. If \( t_{i-1} = T \) the game terminates.
2. The adversary chooses a step size \( 0 < s_i \leq 1 \), which advances time by \( t_i = t_{i-1} + s_i^2 \).
3. Given \( s_i \), the learner chooses a distribution \( P(i) \) over \( \mathbb{R} \).
4. The adversary chooses a mapping from \( \mathbb{R} \) to distributions over \([ -s_i, +s_i ]\): \( Q : \mathbb{R}^2 \to \Delta[-s_i, +s_i] \) such that \( B(t_i, R) \leq E_{y \sim Q(t_i, R)} [y] \leq cs_i^2 \) for all \( R \in \mathbb{R} \).
5. The aggregate loss is calculated:

\[ \ell(t_i) = E_{R \sim \Psi(t_i)} [P(t_i, R)B(t_i, R)] \]  

(19)
6. The state is updated. The expectation below is over distributions. and the notation \( G \oplus R \) means that distribution \( G \) over the reals is shifted by the amount defined by the scalar \( R \):

\[
\Psi(t_i) = \mathbb{E}_{R \sim \Psi(t_{i-1})} [Q(t_i)(R) \oplus (R - \ell(t_i))] 
\]

When \( t_i = T \) the game is terminated, and the final value is calculated:

\[
\Phi(T) = \mathbb{E}_{R \sim \Psi(T)} [\phi(T, R)]
\]

In the discrete time game the adversary has an additional choice, the choice of \( s_i \). Thus the adversary’s strategy includes that choice. There are two constraints on this choice: \( s_i \geq 0 \) and \( \sum_{i=1}^{n} s_i^2 = T \). Note that even that by setting \( s_i \) arbitrarily small, the adversary can make the number of steps - \( n \) - arbitrarily large. We will therefore not identify a single adversarial strategy but instead consider the supremum over an infinite sequence of strategies.

We use \( N(0, \sigma) \) to denote the normal distribution with mean 0 and std \( \sigma \).

**Theorem 6**

let \( A = \mathbb{E}_{R \sim N(0, \sqrt{T})} [\phi(T, R)] \)

- For any \( \epsilon > 0 \) there exists a strategy for the adversary such that for any strategy of the learner 
  \( \Phi(T) \geq A - \epsilon \)

- There exists a strategy for the learner that guarantees, against any adversary \( \Phi(T) \leq A \).

**5.1. The adversary prefers smaller steps**

As noted before, if the adversary chooses \( s_i = 1 \) for all \( i \) the game reduces the the integer time game. The question is whether the adversary would prefer to stick with \( s_i = 1 \) or instead prefer to use \( s_i < 1 \). In this section we give a surprising answer to this question – the adversary always prefers a smaller value of \( s_i \) to a larger one. This leads to a preference for \( s_i \to 0 \), as it turns out, this limit is well defined and corresponds to Brownian motion, also known as Wiener process.

Consider a sequence of adversarial strategies \( S_k \) indexed by \( k = 0, 1, 2, \ldots \). The adversarial strategy \( S_k \) is corresponds to always choosing \( s_i = 2^{-k} \), and repeating \( Q_{\pm 2^{-k}}^{1/2} \) for \( T^{2^{2k}} \) iterations. This corresponds to the distribution created by a random walk with \( T^{2^{2k}} \) time steps, each step equal to \( +2^{-k} \) or \( -2^{-k} \) with probabilities \( 1/2, 1/2 \). Note that in order to preserve the variance, halving the step size requires increasing the number of iterations by a factor of four.

Let \( \phi(S_k, t, R) \) be the value associated with adversarial strategy \( S_k \), time \( t \) (divisible by \( 2^{-2k} \)) and location \( R \). We are ready to state our main theorem.

**Theorem 7** *If the final value function has a strictly positive fourth derivative:*

\[
\frac{d^4}{dR^4} \phi(T, R) > 0, \forall R
\]

*then for any integer \( \kappa > 0 \) and any \( 0 \leq t \leq T \), such that \( t \) is divisible by \( 2^{-2k} \) and any \( R \),

\[
\phi(S_{k+1}, t, R)) > \phi(S_k, t, R)
\]
Proofs for Theorem 7 and Lemma 8 are given in Appendix C. Before proving the theorem, we describe it’s consequence for the online learning problem. We can restrict Theorem 7 for the case \( t = 0, R = 0 \) in which case we get an increasing sequence:

\[
\phi(S_1, 0, 0) < \phi(S_2, 0, 0) < \cdots < \phi(S_k, 0, 0) <
\]

The limit of the strategies \( S_k \) as \( k \to \infty \) is the well studied Brownian or Wiener process. The backwards recursion that defines the value function is the celebrated Backwards Kolmogorov Equation with zero drift and unit variance

\[
\frac{\partial}{\partial t} \phi(t, R) + \frac{1}{2} \frac{\partial^2}{\partial R^2} \phi(t, R) = 0 \quad (20)
\]

Given a final value function with a strictly positive fourth derivative we can use Equation (20) to compute the value function for all \( 0 \leq t \leq T \). We will do so in the next section.

We now go back to proving Theorem 7. The core of the proof is a lemma which compares, essentially, the value recursion when taking one step of size 1 to four steps of size 1/2.

Consider the adversarial strategies \( S_k \) and \( S_{k+1} \) at a particular time point \( 0 \leq t \leq T \) such that \( t \) is divisible by \( \Delta t = 2^{-2k} \) and at a particular location \( R \). Let \( t' = t + \Delta t \), and fix a value function for time \( t' \), \( \phi(t', R) \) and compare between two values at \( R, t \). The first value denoted \( \phi_k(t, R) \) corresponds to \( S_k \), and consists of a single random step of \( \pm 2^{-k} \). The other value \( \phi_{k+1}(t, R) \) corresponds to \( S_{k+1} \) and consists of four random steps of size \( \pm 1/2 \).

**Lemma 8** If \( \phi(t', R) \) is, as a function of \( R \) continuous, strictly convex and with a strictly positive fourth derivative. Then

- \( \phi_k(t, R) < \phi_{k+1}(t, R) \)
- Both \( \phi_k(t, R) \) and \( \phi_{k+1}(t, R) \) are continuous, strictly convex and with a strictly positive fourth derivative.

### 5.2. Strategies for the Learner in the discrete time game

The strategies we propose for the learner in the continuous time game are an adaptation of the strategies \( P_1^d, P_2^d \) to the case where \( s_i < 1 \).

We start with the high-level idea. Consider iteration \( i \) of the continuous time game. We know that the adversary prefers \( s_i \) to be as small as possible. On the other hand, the adversary has to choose some \( s_i > 0 \). This means that the adversary always plays sub-optimally. Based on \( s_i \), the learner makes a choice and the adversary makes a choice. As a result the current state \( \Psi(t_{i-1}) \) is transformed to \( \Psi(t_i) \). To choose it’s strategy, the learner needs to assign value possible states \( \Psi(t_i) \). How can she do that? By assuming that in the future the adversary will play optimally, i.e. setting \( s_i \) arbitrarily small. While the adversary cannot be optimal, it can get arbitrarily close to optimal, which is Brownian motion.

Solving the backwards Kolmogorov equation with the boundary condition \( \phi(T, R) \) yields \( \phi(t, R) \) for any \( R \in \mathbb{R} \) and \( t \in [0, T] \). We now explain how using this potential function we derive strategies for the learner.

Note that the learner chooses a distribution after the adversary set the value of \( s_i \). The discrete time version of \( P_1^d \) (Eqn 17) is

\[
P^{1d}(t_{i-1}, R) = \frac{1}{Z^{1d}} \frac{\phi(t_i, R + s_{i-1} + cs^2_{i-1}) - \phi(t_i, R - s_{i-1} - cs^2_{i-1})}{2}
\]

where \( Z^{1d} = E_{R \sim \Psi(t_i)} \left[ \frac{\phi(t_i, R + s_{i-1} + cs^2_{i-1}) - \phi(t_i, R - s_{i-1} - cs^2_{i-1})}{2} \right] \)
Next, we consider the discrete time version of $P^2$: (Eqn ??)

$$P^{2d}(t_{i-1}, R) = \frac{1}{Z^{2d}} \frac{\partial}{\partial r} \bigg|_{r=R} \phi(t_{i-1} + s_{i-1}^2, r)$$

(22)

where $Z^{2d} = E_{R \sim \Psi(t_i)} \left[ \frac{\partial}{\partial r} \bigg|_{r=R} \phi(t_{i-1} + s_{i-1}^2, r) \right]$

6. Two KBE compatible potential functions

The potential functions, $\phi(t, R)$ is a solution of PDE (20):

$$\frac{\partial}{\partial t} \phi(t, R) + \frac{1}{2} \frac{\partial^2}{\partial r^2} \phi(t, R) = 0$$

(23)

under a boundary condition $\phi(T, R) = \phi(T, R)$, which we assume is in $\mathcal{P}^4$

So far, we assumed that the game horizon $T$ is known in advance. We now show two value functions where knowledge of the horizon is not required. Specifically, we call a value function $\phi(t, R)$ self consistent if it is defined for all $t > 0$ and if for any $0 < t < T$, setting $\phi(T, R)$ as the final potential and solving for the Kolmogorov Backward Equation yields $\phi(t, R)$ regardless of the time horizon $T$.

We consider two solutions to the PDE, the exponential potential and the NormalHedge potential. We give the form of the potential function that satisfies Kolmogorov Equation 20, and derive the regret bound corresponding to it.

The exponential potential function which corresponds to exponential weights algorithm corresponds to the following equation

$$\phi_{\text{exp}}(R, t) = e^{\sqrt{2\eta R - \eta^2 t}}$$

Where $\eta > 0$ is the learning rate parameter.

Given $\epsilon$ we choose $\eta = \sqrt{\frac{\ln(1/\epsilon)}{t}}$ we get the regret bound that holds for any $t > 0$

$$R_\epsilon \leq \sqrt{2t \ln \frac{1}{\epsilon}}$$

(24)

Note that the algorithm depends on the choice of $\epsilon$, in other words, the bound does not hold for all values of $\epsilon$ at the same time.

The NormalHedge value is

$$\phi_{\text{NH}}(R, t) = \begin{cases} \frac{1}{\sqrt{t+\nu}} \exp \left( \frac{R^2}{2(t+\nu)} \right) & \text{if } R \geq 0 \\ \frac{1}{\sqrt{t+\nu}} & \text{if } R < 0 \end{cases}$$

(25)

Where $\nu > 0$ is a small constant. The function $\phi_{\text{NH}}(R, t)$, restricted to $R \geq 0$ is in $\mathcal{P}^4$ and is a constant for $R \leq 0$.

The regret bound we get is:

$$R_\epsilon \leq \sqrt{(t + \nu) \left( \ln(t + \nu) + 2 \ln \frac{1}{\epsilon} \right)}$$

(26)

This bound is slightly larger than the bound for exponential weights, however, the NormalHedge bound holds simultaneously for all $\epsilon > 0$ and the algorithm requires no tuning.
7. NormalHedge yields the fastest increasing potential

Up to this point, we considered any continuous value function with strictly positive derivatives 1-4. We characterized the min-max strategies for any such function. It is time to ask whether value functions can be compared and whether there is a “best” value function. In this section we give an informal argument that NormalHedge is the best function. We hope this argument can be formalized.

We make two observations. First, the min-max strategy for the adversary does not depend on the potential function! (as long as it has strictly positive derivatives). That strategy corresponds to the Brownian process.

Second, the argument used to show that the regret relative to $\epsilon$-expert of the expert is based on two arguments

- The average value function does not increase with time.
- The (final) value function increases rapidly as a function of $R$

The first item is true by construction. The second argument suggests the following partial order on value functions. Let $\phi_1(t, R), \phi_2(t, R)$ be two value functions such that

$$\lim_{{R \to \infty}} \frac{\phi_1(t, R)}{\phi_2(t, R)} = \infty$$

then $\phi_1$ dominates $\phi_2$, which we denote by, $\phi_1 > \phi_2$.

On the other hand, if the value function increases too quickly, then, when playing against Brownian motion, the average value will increase without bound. Recall that the distribution of the Brownian process at time $t$ is the standard normal with mean 0 and variance $t$. The question becomes what is the fastest the value function can grow, as a function of $R$ and still have a finite expected value with respect to the normal distribution.

The answer seems to be NormalHedge (Eqn. 25). More precisely, if $\epsilon > 0$, the mean value is finite, but if $\epsilon = 0$ the mean value becomes infinite.

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**Appendix A. Proof of Theorem 3**

**Proof**

1. By symmetry adversarial strategy (15) guarantees that the aggregate loss (9) is zero regardless of the choice of the learner: $\ell(i) = 0$. Therefor the state update (10) is equivalent to the symmetric random walk:

$$\Psi(i) = \frac{1}{2}((\Psi(i - 1) \oplus 1) + (\Psi(i - 1) \ominus 1))$$

Which in turn implies that if the adversary plays $Q^*$ and the learner plays an arbitrary strategy $P$

$$\phi^+(i - 1, R) = \frac{1}{2}(\phi(i, R - 1) + \phi(i, R + 1))$$

(27)

As this adversarial strategy is oblivious to the strategy, it guarantees that the average value at iteration $i$ is *equal* to the average of the lower value at iteration $i - 1$.

2. Plugging learner’s strategy (17) into equation (9) we find that

$$\ell(i - 1) = \frac{1}{Z_{i-1}}E_{R \sim \Psi(i-1)} [(\phi(i, R + 1 + c) - \phi(i, R - 1 - c))B(i - 1, R)]$$

(28)

Consider the average value at iteration $i - 1$ when the learner’s strategy is $P^*$ and the adversarial strategy is arbitrary $Q$:

$$\Phi_{P^*,Q}(i - 1, R) = E_{R \sim \Psi(i-1)} [E_{y \sim Q(i-1)}(R) \phi(i, R + y - \ell(i - 1))]$$

(29)
As \( \phi(i, \cdot) \) is convex and as \((y - \ell(i - 1)) \in [-1 - c, 1 + c], \)

\[
\phi(i, R + y) \leq \frac{\phi(i, R + 1 + c) + \phi(i, R - 1 - c)}{2} + (y - \ell_P(i)) \frac{\phi(i, R + 1 + c) - \phi(i, R - 1 - c)}{2}
\]  

(30)

Combining the equations (28) and (29) we find that

\[
\Phi_{P\ast, Q}(i - 1, R) = \mathbb{E}_{R \sim \Psi(i - 1)} \left[ \mathbb{E}_{y \sim Q(i - 1)(R)} \left[ \phi(i, R + y - \ell(i - 1)) \right] \right] \
\leq \mathbb{E}_{R \sim \Psi(i - 1)} \left[ \frac{\phi(i, R + 1 + c) + \phi(i, R - 1 - c)}{2} \right] 
\]  

(31)

\[
+ \mathbb{E}_{R \sim \Psi(i - 1)} \left[ \mathbb{E}_{y \sim Q(i - 1)(R)} \left[ (y - \ell(i - 1)) \frac{\phi(i, R + 1 + c) - \phi(i, R - 1 - c)}{2} \right] \right]
\]  

(32)

The final step is to show that the term (33) is equal to zero. As \( \ell(i - 1) \) is a constant with respect to \( R \) and \( y \) the term (33) can be written as:

\[
\mathbb{E}_{R \sim \Psi(i - 1)} \left[ \mathbb{E}_{y \sim Q(i - 1)(R)} \left[ y - \ell(i - 1) \right] \frac{\phi(i, R + 1 + c) - \phi(i, R - 1 - c)}{2} \right]
\]  

(34)

\[
= \mathbb{E}_{R \sim \Psi(i - 1)} \left[ B(i - 1, R) \frac{\phi(i, R + 1 + c) - \phi(i, R - 1 - c)}{2} \right] 
\]  

(35)

\[
- \ell_P(i) \mathbb{E}_{R \sim \Psi(i - 1)} \left[ \frac{\phi(i, R + 1 + c) - \phi(i, R - 1 - c)}{2} \right]
\]  

(36)

\[
= 0
\]  

(37)

Appendix B. Proof of Lemma 5

Proof We prove the lemma by showing that \( \Phi_{P, Q}(i) = \Phi_{P, Q}(i + 1) \) for all \( i \in \{T - 1, T - 2, \ldots, 0\} \)

\[
\Phi_{P, Q}(i + 1) = \mathbb{E}_{R \sim \Psi(i + 1)} \left[ \phi_{P, Q}(i + 1, R) \right] = \mathbb{E}_{R \sim \Psi(i), l \sim Q(i, R)} \left[ \phi_{P, Q}(i + 1, R + l - \ell_P(i)) \right] 
\]  

(38)

\[
= \mathbb{E}_{R \sim \Psi(i)} \left[ \mathbb{E}_{l \sim Q(i, R)} \left[ \phi_{P, Q}(i + 1, R + l - \ell_P(i)) \right] \right] 
\]  

(39)

\[
= \mathbb{E}_{R \sim \Psi(i)} \left[ \phi_{P, Q}(i, R) \right] 
\]  

(40)

\[
= \Phi_{P, Q}(i)
\]  

(41)

Appendix C. Proofs of Lemma 8 and Theorem 7

Proof of Lemma 8 Recall the notations \( \Delta t = 2^{-2k} \) \( l' = t + \Delta t \) and \( s = 2^{-k} \). We can write out explicit expressions for the two values:
• For strategy $S_0$ the value is
\[
\phi_k(t, R) = \frac{\phi(t', R + s) + \phi(t', R - s)}{2}
\]

• For strategy $S_1$ the value is
\[
\phi_{k+1}(t, R) = \frac{1}{16} \left( \phi(t', R + 2s) + 4\phi(t', R + s) + 6\phi(t', R) + 4\phi(t', R - s) + \phi(t', R - 2s) \right)
\]

We want to show that $\phi_1(T - 1, R) > \phi_0(T - 1, R)$ for all $R$, in other words we want to characterize the properties of $\phi(T, R)$ the would guarantee that
\[
\phi_1(t, R) - \phi_0(t, R) = \frac{1}{16} \left( \phi(t', R + 2) - 4\phi(t', R + 1) + 6\phi(t', R) - 4\phi(t', R - 1) + \phi(t', R - 2) \right) > 0 \tag{43}
\]

Inequalities of this form have been studied extensively under the name “divided differences” Popoviciu (1965); Butt et al. (2016); de Boor (2005). A function $\phi(T, R)$ that satisfies inequality 43 is said to be $4$’th order convex (see details in Butt et al. (2016)).

$n$-convex functions have a very simple characterization:

**Theorem 9** Let $f$ be a function with is differentiable up to order $n$, and let $f^{(n)}$ denote the $n$’th derivative, then $f$ is $n$-convex ($n$-strictly convex) if and only if $f^{(n)} \geq 0$ ($f^{(n)} > 0$).

We conclude that if $\phi(t', R)$ has a strictly positive fourth derivative then $\phi_{k+1}(t, R) > \phi_k(t, R)$ for all $R$, proving the first part of the lemma.

The second part of the lemma follows from the fact that both $\phi_{k+1}(t, R)$ and $\phi_k(t, R)$ are convex combinations of $\phi(t, R)$ and therefore retain their continuity and convexity properties. \hfill \blacksquare

**Proof** of Theorem 7

The proof is by double induction over $k$ and over $t$. For a fixed $k$ we take a finite backward induction over $t = T - 2^{-2k}, T - 2 \times 2^{-2k}, T - 3 \times 2^{-2k}, \ldots, 0$. Our inductive claims are that $\phi_{k+1}(t, R) > \phi_k(t, R)$ and $\phi_{k+1}(t, R), \phi_k(t, R)$ are continuous, strongly convex and have a strongly positive fourth derivative. That these claims carry over from $t = T - i \times 2^{-2k}$ to $t = T - (i + 1) \times 2^{-2k}$ follows directly from Lemma 8.

The theorem follows by forward induction on $k$. \hfill \blacksquare