Minimal extended seesaw and group symmetry realization of two-zero textures of neutrino mass matrices

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Abstract

We study the phenomenology of two-zero textures of $M^{4\times4}_\nu$ neutrino mass matrices in the minimal extended seesaw (MES) mechanism. $M^{4\times4}_\nu$ involves $3 \times 3$ Dirac neutrino mass matrix ($M_D$), $3 \times 3$ right-handed Majorana neutrino mass matrix ($M_R$) and $1 \times 3$ matrix ($M_S$) that couples the singlet field ‘$S$’ with the right-handed neutrinos. We consider the phenomenologically predictive cases $(5+3)$ and $(6+2)$ schemes for zero textures of $M_D$ and $M_R$ of $3 \times 3$ active sector of $M^{4\times4}_\nu$ along with admissible one or two-zero textures of $M_S$. Although there is a large number of combinations of $M_D$, $M_R$ and $M_S$ leading to the desired two-zero textures, $S_3$ group transformations between the different zero textures of $M_D$, $M_R$ and $M_S$ reduce them to a small number of basic combinations. In MES, $M^{4\times4}_\nu$ should be a matrix of rank 3, so we have 12 two-zero textures of neutrino mass matrices of rank 3 out of total 15 two-zero textures of neutrino mass matrices. In realization of the two-zero textures in $(5+3)$ scheme we have obtained a number of correlations among the neutrino mass matrix elements $m_{ij}$, while none of the two-zero textures could be realized in the $(6+2)$ scheme. The viability of each of the realizable textures is checked by plotting the scatter plots of their respective correlations under the current neutrino oscillation data. We have analysed the role played by the Dirac and Majorana CP phases for each of the textures. For constrained ranges of CP phases we also draw scatter plots for Jarlskog invariant ($J_C P$) and effective electron neutrino mass $|m_{\beta\beta}|$. In our study the viable textures are finally realized by $Z_8$ Abelian group symmetry by extending the Standard Model to include few scalar fields.

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1. Introduction

The neutrino physics has been enriched with the knowledge and understanding of massive neutrinos by both theoretical and experimental endeavours for the last couple of decades, but this field of research is still striving for solutions of some anomalous results, for example, anomalies in short-baseline neutrino experiments. These results are either to be confirmed or to be ruled out in ongoing or upcoming experiments dedicated for the purpose. In literature, there are excellent review works of neutrino physics [1] in which this issue has also been addressed and the present status of sterile neutrino searches has been nicely reviewed in a recent paper [2]. The problem arises as follows: for the first time, an unexpected excess amount of \( \bar{\nu}_e \)-like events in the decay process \( \mu^+ \to e^+ + \nu_e + \bar{\nu}_\mu \) was observed by the Liquid Scintillator Neutrino Detector (LSND) [3] in Fermi Laboratory, USA. In order to resolve the LSND anomaly, the Mini Booster Neutrino Experiment (MiniBooNE) was designed with the same oscillation frequency as that of the LSND experiment for detecting \( \nu_\mu \to \nu_e \) and \( \bar{\nu}_\mu \to \bar{\nu}_e \) signals [4]. MiniBooNE data have also corroborated the results of the LSND at the 4.7\( \sigma \) significance level.

The problem further deepens as about 15\% deficit of the measured to predicted neutrino induced signal rates in the GALLEX [5] and SAGE [6] solar experiments was observed with 3\( \sigma \) significance level. This can be explained as \( \nu_e \to \nu_e \) oscillations in (3 + 1) scheme of active-sterile neutrino mixings with [7] \( \Delta m_{41}^2 \approx O(1) \) eV\(^2\) and \( \sin^2 \theta_{14} \approx O(10^{-2}) \) to \( O(10^{-1}) \). Again, the reactor antineutrino anomaly stemmed from a 6\% deficit of the observed \( \bar{\nu}_e \) as compared to reactor antineutrino flux [8,9] of \(^{235}\)U, \(^{238}\)U, \(^{239}\)Pu and \(^{241}\)Pu with a significance of 3\( \sigma \). But the recent analysis of Daya Bay data has rejected the hypothesis of a constant \( \bar{\nu}_e \) flux as function of the \(^{239}\)Pu fission fraction and also the hypothesis of a constant \( \bar{\nu}_e \) energy spectrum, indicating that \(^{235}\)U may be responsible for the reactor antineutrino anomaly [10]. There are a number of ongoing or upcoming short-baseline reactor antineutrino experiments viz., NEOS [11], DANSS [12], STEREO [13], PROSPECT [14], Neutrino-4 [15], SoLid [16] planned for achieving conclusive results about these anomalies.

For analysis of the data sets from short base-line neutrino experiments, the parameter space of three neutrino paradigm has no room as it has been already exhausted with the solar, atmospheric and reactor neutrino oscillation data. So to resolve such anomalies, phenomenologically a sterile neutrino state of eV mass scale which does not take part in Standard Model (SM) weak interaction but can mix with the active neutrinos was hypothesised in Ref. [17]. On the other hand, there arises a conflicting situation in cosmological context as the existence of a light thermalised sterile neutrino state is strongly disfavoured by the PLANCK data [18]. But in a number of interesting works [19–22] and references therein, it has been argued for re-look on a particle physics model of active-sterile neutrino mixing as well as a consistent cosmological model.

It is a pertinent question on the number of light sterile states if one considers the hypothesis of sterile neutrinos being the answer to the anomalous results. In literature, 3+1, 3+2, 3+3 (all having stable sterile states) [23], and 3+1+Decay (having unstable sterile state) [24] models of active-sterile states have been investigated in the light of global fits of data of various reactor, radio-chemical and accelerator based neutrino experiments. The conclusions of the papers are that there is internal inconsistency in 3+1 model if one separates the dataset of appearance experiments from the dataset of disappearance experiments. Then the natural choice is 3+2 model. But there is no compelling improvement of results in 3+2 model and so is the case of 3+3. The 3+2 model and beyond are not economical as there is a proliferation of oscillation parameters. The paper [25] concludes that the improvement of results in 3+2 model is a statistical effect for more number of oscillation parameters, whereas the paper [24] claims that there is a spectacular
improvement in 3+1-decay model which relieves the internal tension in 3+1 model. As a first step, we are motivated to work with 3+1 models to address the issue of sterile neutrinos of \( \approx 1 \text{ eV} \) scale which will naturally be the simplest, minimal and more economical for extension of 3 active neutrino model.

The minimal extension of the existing 3-neutrino paradigm is the (3+1) scheme [26–29] to include one sterile neutrino and then one can calculate the probabilities of \( \nu_\mu \rightarrow \nu_e \) and \( \bar{\nu}_\mu \rightarrow \bar{\nu}_e \) to fix the LSND and MiniBooNE anomaly. The corresponding oscillation parameters are expected to be \( \Delta m^2_{41} \approx O(1) \text{ eV}^2 \) and \( \sin^2 \theta_{14} \approx \sin^2 \theta_{24} \leq O(10^{-2}) \) [2]. Similarly, the so-called gallium anomaly may be addressed as the consequence of \( \nu_e \rightarrow \nu_e \) oscillations in (3+1) framework with values of parameters \( \Delta m^2_{41} \approx O(1) \text{ eV}^2 \) and \( \sin^2 \theta_{14} \approx O(10^{-2}) \) to \( O(10^{-1}) \) [7]. Again the reactor antineutrino anomaly also points toward the existence of an eV scale sterile state of neutrino with the active-sterile flavor mixing parameter \( \sin^2 2\theta_{14} \approx O(10^{-1}) \) [30].

There may be some interesting phenomenological consequences of existence of one sterile neutrino species. For example, the disappearance oscillation probability of \( \bar{\nu}_e \rightarrow \bar{\nu}_e \) may depend upon the possible values of \( \theta_{13}, \Delta m^2_{41}, \Delta m^2_{42} \). This will be concern for JUNO [31] experiment which is designed primarily for probing the neutrino mass hierarchy i.e., normal or inverted. Again the phenomenology of appearance neutrino/antineutrino oscillations such as \( \nu_\mu \rightarrow \nu_e \) and \( \bar{\nu}_\mu \rightarrow \bar{\nu}_e \) in the presence of one or more sterile neutrinos is very interesting which includes new CP-violating effects and neutral current associated matter effects [32–34]. Moreover, the presence of such new degrees of freedom will modify the effective mass term of the \( \beta \) decays and that of 0ν2\( \beta \) decays.

The origin of an eV scale sterile neutrino in (3+1) model has been studied in the framework of an extended version of the canonical type-I seesaw mechanism known as Minimal Extended Seesaw (MES) mechanism [17,35,36] whose formalism is briefly discussed in Section 2. The MES mechanism involves 3 \( \times \) 3 Dirac neutrino mass matrix \( M_D \), 3 \( \times \) 3 right-handed Majorana neutrino mass matrix \( M_R \) and 1 \( \times \) 3 row matrix \( M_S \) that couples the sterile state to the three right-handed neutrinos.

In literature one of the interesting approaches for investigating the phenomenology of neutrino mass models is the study of texture zeros of neutrino mass matrices. Initially the texture zero models were developed to calculate the Cabibbo angle of quark flavor mixing for both two-family [37–39] and three-family [40,41] scheme. The zeros or vanishing elements in the fermion mass matrices lead to reducing the number of free parameters of the complicated mass matrices and thereby establishing testable relations between the physical quantities: masses, mixing angles, and CP phases. Texture zeros of a given fermion mass matrix originates from proper choice of the flavor basis that has been chosen for the mass matrix [1]. More number of zeros in the mass matrices imply more restrictions on the free parameters thereby enhancing the predictivity of the models. Zeros in \( M_\nu \) can be imposed directly by hand. From a more deeper theoretical front, zeros in \( M_\nu \) can also be realized via type-I seesaw mechanism which has been considered to be more compatible for understanding the smallness of neutrino mass [42–46]. Thus one is tempted to study the texture zeros of \( M_D \) and \( M_R \) which are ingredients of the seesaw formula than the study of texture zeros of \( M_\nu \) alone. This gives more insight into the problem. In addition, texture zeros of neutrino mass matrices indicate the underlying flavor symmetry. Zeros in arbitrary entries of the neutrino mass matrix \( M_\nu \) can also be implemented via suitable Abelian flavor symmetry group \( Z_n \) with an extended scalar sector [47,48]. This provides a reasonable theoretical justification to phenomenological assumptions to texture zeros. Texture zeros of neutrino mass models in 3-active neutrino scenario, and (3+1) scheme with one sterile state, have been extensively studied in literature [49–63]. There is detailed study of phenomenologically
allowed texture-zeros of neutrino mass matrix $M^4_{ν} \times 4$ in MES mechanism, assuming neutrinos to be Majorana fermions. With three active neutrinos, the light neutrino mass matrix $M^3_{ν} \times 3$ in the flavor basis can accommodate a maximum of two-zeros [64]. However, with one sterile neutrino i.e. (3+1) framework, one can have 1-4 possible zeros of $M^4_{ν} \times 4$ [61–63]. The MES neutrino mass matrix $M^4_{ν} \times 4$ is constructed with $3 \times 3$ Dirac neutrino mass matrix $M_D$, $3 \times 3$ right-handed Majorana neutrino mass matrix $M_R$ and $1 \times 3$ mass matrix $M_S$. Fundamentally the texture-zeros of $M^4_{ν} \times 4$ are results of the texture-zeros of $M_D$, $M_R$ and $M_S$ via MES mechanism. In view of this, our primary focus in this work is to study the texture-zeros of these fermion mass matrices in the basis where the charged lepton mass matrix, $M_l$ is diagonal and finally the viable zero textures shall be realized using Abelian flavor group symmetry $Z_n$ by extending the scalar sector of the Standard Model (SM) of particle physics.

In the 3-neutrino paradigm, out of 15 possible two-zero textures, only 7 such textures remain viable within experimental constraints [64]. However, the compatibility conditions in the 3+1 framework are different. Out of 45 possible two-zero textures of $M^4_{ν} \times 4$, only 15 textures have been found to be compatible with experimental constraints [61]. Also, these 15 experimentally allowed textures of $M^4_{ν} \times 4$ are the same as the 15 possible two-zero textures of $M^3_{ν} \times 3$ in three active neutrino scenario. The texture zeros of $M^3_{ν} \times 3$ in type-I seesaw mechanism are the result for transmission from the zeros of $M_D$ and $M_R$ [43–46]. In the three neutrino paradigm, the author of Ref. [46] explored the zero textures for $M_D$ and $M_R$ for predictive cases, wherein the total number of zeros of $M_D$ and $M_R$ should be 8, irrespective of the number and/or position of zeros in the light neutrino mass matrix $M_ν$. Considering the predictive cases, the authors of the paper [65] studied in details one-zero textures of $M^3_{ν} \times 3$ in MES mechanism, while two-zero textures could not be realized. Although the two-zero textures with [61] or without [64] a sterile neutrino are phenomenologically allowed, but none of them in MES can be realised not only in predictive case [65] but also in other cases [66].

We are now motivated to revisit the two-zero textures in MES $M^4_{ν} \times 4$ obtained from the full neutrino mass matrix $M^7_{ν} \times 7$ subject to implementing the seesaw formula once. In this work, we adopt the conjecture that $3 \times 3$ active sector of this MES $M^4_{ν} \times 4$ in Eq. (3) also prefers the predictive scenario as they do in case of 3-active neutrino scenario. Phenomenology does not allow zero texture in sterile sector, i.e., no zero in the fourth row or column of $M^4_{ν} \times 4$ [61]. There are three possibilities in predictive scenario: (4+4) scheme, (5+3) scheme and (6+2) scheme, where the digits in the pairs represents the number of zeros of $M_D$ and $M_R$ respectively. Two-zero textures in $M^3_{ν} \times 4$ for (4+4) scheme within MES mechanism have already been studied in the paper [67]. In this paper we plan to explore the (5+3) and (6+2) schemes for realization of two-zero textures of MES $M^4_{ν} \times 4$ along with one and/or two-zero textures of $M_S$. It is to be noted that three-zero and four-zero texture of $M^4_{ν} \times 4$ can never be realized in the MES mechanism, as $M_D$ and $M_R$ should remain as non-singular in MES.

The MES $4 \times 4$ neutrino mass matrix is of rank 3, thus one of the mass eigenvalues should be massless and hence one has to consider one of the active neutrinos massless as vanishing sterile neutrino mass becomes trivial. So we have two mass patterns: normal hierarchy (NH) and inverted hierarchy (IH). 12 out of 15 allowed two-zero textures in $M^4_{ν} \times 4$ are of rank 3. We shall study all the 12 two-zero textures under the predictive scenario of (5+3) and (6+2) scheme. It is noted that none of the 12 two-zero textures can be realized within the (6+2) scheme. However, 9 out of 12 textures are found to be realizable within (5+3) scheme. We shall study all possible texture zeros of $M_D$ and $M_R$ and their $S_3$ transformations in the (5+3) scheme and check the consistency of the correlations of the textures by plotting scatter plots with the current neutrino oscillation data. We also study the Dirac and Majorana CP phases dependence of the
coveted textures. In addition, we study the dependence of effective electron neutrino mass $|m_{\beta \beta}|$ on the lightest neutrino mass $m_{\text{lightest}}$ for the considered constrained range of Majorana CP phases for each texture. Also, we present scatter plots for Jarlskog invariant $J_{C P}$ for each texture considering constrained range of Dirac CP phases.

The paper is organized as follows: In Section 2 we present a brief discussion on MES mechanism. Section 3 includes a brief review on the two-zero textures of $M_{3\times 3}$. In its corresponding subsection 3.1, five-zero textures of $M_{D}$ and three-zero textures of $M_{R}$ along with zero textures of $M_{S}$ is being presented. Also, $S_3$ permutation of fermion mass matrices under MES mechanism is presented in Sec. 3.1. In Section 4 we present the realization of the two-zero textures. In Section 5 we check the viability of the textures under recent neutrino oscillation data for both unconstrained and constrained CP phases. Also, scatter plots for $|m_{\beta \beta}|$ and $J_{C P}$ for constrained CP phases are presented in this section. Symmetry realization of the viable textures are presented in Section 6. Finally, we conclude in Section 7 followed by the Appendix that contains the expressions for all ten elements $m_{ij}$, ($i,j = e, \mu, \tau, s$) of the symmetric light neutrino mass matrix $M_{v}^{4\times 4}$.

2. Minimal Extended Seesaw mechanism

The Minimal Extended Seesaw (MES) mechanism is an extension of the canonical type-I seesaw mechanism with an additional gauge singlet chiral field ‘S’ [17]. Thus, the fermion sector of the SM gets extended with four additional particles - the three right-handed neutrinos and one chiral field ‘S’. Within this scenario, an eV scale sterile neutrino naturally appears without needing to insert any tiny Yukawa coupling or mass scales. The Lagrangian representing the mass term for neutrinos takes the form

$$-\mathcal{L}_m = \bar{\nu}_L M_D \nu_R + \bar{\nu}_R^c M_S \nu_R + \frac{1}{2} \bar{\nu}_R^c M_R \nu_R + h.c. \quad (1)$$

Here $M_{D}, M_{R}$ are $3 \times 3$ Dirac and right-handed Majorana mass matrices respectively. As the extension is made with only one extra gauge singlet field ‘S’, therefore $M_{S}$ which couples ‘S’ with the right-handed neutrinos $\nu_R$’s is a $(1 \times 3)$ row matrix. In the basis $(\nu_L, \nu_R^c, S^c)$, the full neutrino mass matrix is a $7 \times 7$ matrix of the form:

$$M_{v}^{7\times 7} = \begin{pmatrix} 0 & M_D & 0 \\ M_D^T & M_R & M_S^T \\ 0 & M_S & 0 \end{pmatrix}. \quad (2)$$

Considering the hierarchy $M_R \gg M_S > M_D$ and implementing type-I seesaw formula, the right-handed neutrinos get decoupled at low energy scales. The process of the block diagonalization of Eq. (2) leads to the effective neutrino mass matrix in the basis $(\nu_L, S^c)$ as

$$M_{v}^{4\times 4} = -\begin{pmatrix} M_D M_R^{-1} M_D^T \\ M_S (M_R^{-1})^T M_D^T \\ M_D M_R^{-1} M_S^T \\ M_S M_R^{-1} M_S^T \end{pmatrix}. \quad (3)$$

The square matrix in Eq. (3) contains four light eigenstates corresponding to three active neutrinos and one sterile neutrino [35]. However, the determinant of the mass matrix $M_{v}^{4\times 4}$ is zero with the condition of $M_D$ and $M_R$ being non-singular. Thus the mass matrix $M_{v}^{4\times 4}$ is a matrix of rank 3. This implies that at least one of the active neutrino mass states remains as massless, while the sterile neutrino mass is

$$m_s \simeq -M_S M_R^{-1} M_S^T. \quad (4)$$
In Ref. [17] the authors explained that similar to type-I seesaw formula, in MES also \( M_D \) and \( M_R \) can assume masses around the electroweak scale \( \approx 100 \text{ GeV} \) and the grand unification scale \( \approx 10^{14} \text{ GeV} \) respectively. Furthermore, there is no bare mass term for the gauge singlet chiral field, \( S \), being involved in \( M_S \) term of the Lagrangian in Eq. (1) which originates from certain Yukawa interactions with right-handed neutrinos and a SM singlet scalar. Hence the mass scale of \( M_S \) is not constrained. For an illustration, assuming \( M_D \approx 10^2 \text{ GeV} \), \( M_S \approx 5 \times 10^2 \text{ GeV} \) and \( M_R \approx 2 \times 10^{14} \text{ GeV} \), one gets \( m_\nu \approx 0.05 \text{ eV} \) and a sterile neutrino of mass \( m_s \approx 1.3 \text{ eV} \).

Assuming the charged lepton mass matrix to be diagonal, in the flavor basis the\( 4 \times 4 \) Majorana neutrino mass matrix can be expressed as

\[
M^4_{\nu} = V M^\text{diag}_\nu V^T = \begin{pmatrix}
  m_{ee} & m_{e\mu} & m_{e\tau} & m_{e\tau} \\
  m_{e\mu} & m_{\mu\mu} & m_{\mu\tau} & m_{\mu\tau} \\
  m_{e\tau} & m_{\mu\tau} & m_{\tau\tau} & m_{\tau\tau} \\
  m_{e\tau} & m_{\mu\tau} & m_{\tau\tau} & m_{\tau\tau}
\end{pmatrix},
\]

where \( V \) corresponds to the \( 4 \times 4 \) PMNS lepton mixing matrix and \( M^\text{diag}_\nu = \text{diag}(m_1, m_2, m_3, m_4) \). The mass matrix elements of Eq. (5) are presented in the appendix section.

Considering the parametrization of the \( 4 \times 4 \) PMNS lepton mixing matrix as [68]

\[
V = U P,
\]

where

\[
U = (R_{34} \tilde{R}_{24} \tilde{R}_{14})(R_{23} \tilde{R}_{13}) R_{12},
\]

\[
P = \text{diag}(1, e^{-i\alpha/2}, e^{-i(\beta/2-\delta_{13})}, e^{-i(\gamma/2-\delta_{14})}).
\]

Here \( \alpha, \beta, \gamma \) are the Majorana phases and \( R_{ij}/\tilde{R}_{ij} \) are the rotation matrices in the \( ij \) flavor space with \( \delta_{13}, \delta_{14}, \delta_{24} \) as the Dirac CP phases, e.g.,

\[
R_{34} = \begin{pmatrix}
  1 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & c_{34} & s_{34} \\
  0 & 0 & -s_{34} & c_{34}
\end{pmatrix}, \quad \tilde{R}_{24} = \begin{pmatrix}
  1 & 0 & 0 & 0 \\
  0 & c_{24} & 0 & s_{24} e^{-i\delta_{24}} \\
  0 & 0 & 1 & 0 \\
  0 & -s_{24} e^{i\delta_{24}} & 0 & c_{24}
\end{pmatrix}.
\]

\( c_{ij} = \cos \theta_{ij}, s_{ij} = \sin \theta_{ij} \).

The current experimental data on the oscillation parameters for 3-neutrino as well as for (3+1)-picture are given in Table 1.

### 3. Two-zero textures of \( M^4_{\nu} \)

In MES \( M^4_{\nu} \) matrices are required to be of rank 3 as one of the mass eigenvalue value of active neutrinos is to vanish. There are 12 possible two-zero textures of \( M^4_{\nu} \) which are of rank 3 presented in Table 2. In this work we have conjectured that the predictive case is the a priori condition for the total number of zeros of \( M_D \) and \( M_R \) being 8 in the active sector of \( M^4_{\nu} \) as they follow in the type-I seesaw without sterile neutrino. There are three possibilities (a) 4+4, (b) 5+3 and (c) 6+2 for predictive scenario. The (4+4) predictive scheme was studied in the paper [67]. Now we are motivated to study (5+3) and (6+2) schemes of predictive cases. In the paper [61] authors found the following: (i) the textures \( A_1 \) and \( A_2 \) belong to class A, which allow only the normal hierarchical mass patterns, (ii) the textures \( D_1, D_2 \) of class D allow both NH and IH mass orderings, (iii) the textures \( B_3, B_4 \) of class B and \( F_1, F_2, F_3 \) of class F favours all the three
Table 1
Best fit and 3σ values of active ν oscillation parameters [69] and the current constraints on sterile neutrino parameters [70–72]. Here \( R_\odot \) is the solar to atmospheric mass squared difference ratio.

| Parameter | Best fit | 3σ range |
|-----------|----------|-----------|
| \( \Delta m_{12}^2 [10^{-5} \text{eV}^2] \) | 7.37 | 6.93–7.97 |
| \( \Delta m_{23}^2 [10^{-3} \text{eV}^2] \) (NH) | 2.50 | 2.37–2.63 |
| \( \Delta m_{13}^2 [10^{-3} \text{eV}^2] \) (IH) | 2.46 | 2.33–2.60 |
| \( \sin^2 \theta_{12} / 10^{-1} \) | 2.97 | 2.50–3.54 |
| \( \sin^2 \theta_{13} / 10^{-2} \) (NH) | 2.14 | 1.85–2.46 |
| \( \sin^2 \theta_{13} / 10^{-2} \) (IH) | 2.18 | 1.86–2.48 |
| \( \sin^2 \theta_{23} / 10^{-1} \) (NH) | 4.37 | 3.79–6.16 |
| \( \sin^2 \theta_{23} / 10^{-1} \) (IH) | 5.69 | 3.83–6.37 |
| \( \delta_{13} / \pi \) (NH) | 1.35 | 0–2 |
| \( \delta_{13} / \pi \) (IH) | 1.32 | 0–2 |
| \( R_\odot \) (NH) | 0.0295 | 0.0263–0.0336 |
| \( R_\odot \) (IH) | 0.0299 | 0.0266–0.0342 |
| \( \Delta m_{S N D}^2 (\Delta m_{31}^2 \text{ or } \Delta m_{41}^2) \text{ eV}^2 \) | 1.63 | 0.87–2.04 |
| \( |V_{e4}|^2 \) | 0.027 | 0.012–0.047 |
| \( |V_{\mu e}|^2 \) | 0.013 | 0.005–0.03 |
| \( |V_{\mu e}|^2 \) | $< 0.16$ |

Table 2
Viable two-zero textures [61] of rank 3. Here ‘X’ indicates the elements with non-zero entries.

| \( A_1 \) | \( A_2 \) | \( B_3 \) | \( B_4 \) |
|-----------|----------|----------|----------|
| \( \begin{array}{cccc} 0 & 0 & X & X \\ X & X & X & X \\ X & X & X & X \\ X & X & X & X \end{array} \) | \( \begin{array}{cccc} 0 & X & 0 & X \\ X & X & X & X \\ X & X & X & X \\ X & X & X & X \end{array} \) | \( \begin{array}{cccc} X & 0 & X & X \\ X & X & X & X \\ X & X & X & X \\ X & X & X & X \end{array} \) | \( \begin{array}{cccc} X & X & 0 & X \\ X & X & X & X \\ X & X & X & X \\ X & X & X & X \end{array} \) |

mass patterns: normal hierarchy (NH), inverted hierarchy (IH) and quasi degenerate (QD). As one of the mass eigenvalues of the matrix \( M_\nu^{4 \times 4} \) in MES is massless, so the NH and IH mass patterns of the textures are allowed in MES but the QD textures are not allowed. In advance we note from our work that the textures \( C, E_1, E_2 \) can not be realized under (5+3) scheme and also no one of the textures can be realized under (6+2) scheme in the context of MES mechanism.

Also, \( P_{\mu \tau} \) symmetry [61] exists between the textures \( A_1 – A_2; B_3 – B_4; D_1 – D_2 \) and \( F_2 – F_3 \) according to Eq. (10). However, such a symmetry does not exist for the texture \( F_1 \).

\[
B_4 = P_{\mu \tau}^T B_3 P_{\mu \tau},
\]  
(10)
where
\[ P_{\mu\tau} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \] (11)

### 3.1. (5+3) scheme and $S_3$ permutation group

There are a number of combinations of $M_D$, $M_R$ and $M_S$ for which the mass matrices $M^{4\times4}_v$ in Eq. (3) remain invariant under $S_3$ transformations that lead to the same correlations. These transformations reduce the voluminous work of dealing with a large number of possible combinations of $M_D$, $M_R$ and $M_S$ under the $(5+3)$ scheme to only a few basic combinations. The following $S_3$ transformations of $M_D$, $M_R$ and $M_S$ \footnote{The $M_v$, $M_R$ and $M^T_SM_S$ are symmetric and hence they entail permutations of both rows and columns.} keep $M^{4\times4}_v$ invariant:

\[ M_D \rightarrow M_DZ, \quad M_R \rightarrow Z^T M_R Z, \quad M_S \rightarrow M_S Z \] (12)

Here $Z$ represents the six elements of the $S_3$ permutation group
\[ Z \in S_3 = (A, A^2, B, AB, BA, ABA) \] (13)

where
\[ A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \] (14)

3-zero textures of $M_R$:

The right-handed Majorana mass matrix $M_R$ is symmetric with six independent entries and so we have $^6C_3 = 20$ possible 3-zero textures, out of which only 14 are non-singular (Table 3). According to Eq. (12) $S_3$ group permutations enable us to work with only four basic 3-zero textures of $M_R$: $M^{(1)}_R$, $M^{(7)}_R$, $M^{(9)}_R$ and $M^{(10)}_R$. However, it has been observed that all the elements of the inverse of $M^{(1)}_R$ are non-zero and as a result, no zero in $M^{4\times4}_v$ can be generated via MES mechanism. So $M^{(1)}_R$ is redundant. Thus, only 13 three-zero textures of $M_R$ shall be considered for investigation and three basic textures: $M^{(7)}_R$, $M^{(9)}_R$ and $M^{(10)}_R$.

\[ M^{(7)}_R = \begin{pmatrix} A & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & F \end{pmatrix}, \quad M^{(9)}_R = \begin{pmatrix} A & B & 0 \\ B & 0 & E \\ 0 & E & 0 \end{pmatrix}, \quad M^{(10)}_R = \begin{pmatrix} A & B & 0 \\ B & 0 & 0 \\ 0 & 0 & F \end{pmatrix}. \] (15)

5-zero textures of $M_D$:

As the Dirac neutrino mass matrices are non-symmetric with all 9 elements being independent, there might be $^9C_5 = 126$ possible 5-zero textures. However, as the MES mechanism demands $M_D$ to be non-singular, 90 such textures of $M_D$ which have either row zero, column zero or block zero are not useful for being singular. The remaining 36 non-singular textures are
viable. Again we find that 26 textures of $M_D$ (Table 4) out of aforesaid 36 textures play the role in the basic combinations with $M_R$ in Eq. (15) and $M_S$ (Eq. (16), (17)) to reproduce the desired two-zero textures of $M_v^{4 \times 4}$. All other combinations can be obtained via $S_3$ transformations according to Eq. (12).

Zero textures of $M_S$:

The $1 \times 3$ row matrix $M_S = (s_1 \ s_2 \ s_3)$ can have two possible zero textures:

(1) One-zero textures:
$$M_S^{(1)} = \begin{pmatrix} 0 & s_2 & s_3 \end{pmatrix}, \quad M_S^{(2)} = \begin{pmatrix} s_1 & 0 & s_3 \end{pmatrix}, \quad M_S^{(3)} = \begin{pmatrix} s_1 & s_2 & 0 \end{pmatrix}.$$  

(2) Two-zero textures:
$$M_S^{(4)} = \begin{pmatrix} s_1 & 0 & 0 \end{pmatrix}, \quad M_S^{(5)} = \begin{pmatrix} 0 & s_2 & 0 \end{pmatrix}, \quad M_S^{(6)} = \begin{pmatrix} 0 & 0 & s_3 \end{pmatrix}.$$  

4. Realization of two-zero textures

We have 36 five-zero textures of $M_D$, 13 three-zero textures of $M_R$ and 6 one and two-zero textures of $M_S$ under $(5+3)$ scheme mentioned in the previous section to realize the two-zero textures of $M_v^{4 \times 4}$ in MES. Although there exist 2808 number of possible choices for the combinations of $M_D, M_R$ and $M_S$ in the framework of MES, but 252 number of possible combinations are effective for realization of our desired two-zero textures of $M_v^{4 \times 4}$. Again the complexity of the texture study of these 252 combinations can further be reduced by $S_3$ transformations to only 42 basic combinations for each of which there are five $S_3$ transformations. The zeros of $M_D, M_R$ and $M_S$ propagate to $M_v^{4 \times 4}$ through MES formula in Eq. (3) which acquires a form of two-zero texture. In this process we also obtain the correlations among some of the matrix elements $m_{ij}$
with \( i, j = e, \mu, \tau, s \) of \( M^4 \times 4 \) as results of the functional relations among the parameters of \( M_D \), \( M_R \) and \( M_S \). Again the question of viability of a two-zero texture is addressed by the consistency check of these correlations under the current neutrino data. This analysis shall follow in the next section. Now we present three representative cases out of 42 basic combinations for understanding of the problem:

**Case I:** The following basic combination of

\[
M_R = M^{(9)}_R, \quad M_D = M^{(11)}_D, \quad M_S = M^{(6)}_S,
\]

in Eq. (15), Table 4 and Eq. (17) respectively used in Eq. (3) leads to the form of \( M^4 \times 4 \) as

\[
M^4 \times 4 = \begin{pmatrix}
0 & (b-a)B & (b-a)A & (b-a)A \\
0 & E & E & E \\
(b-a)A & E & E & E \\
(b-a)A & E & E & E
\end{pmatrix},
\]
Table 5
S₃ symmetric textures of the basic combination in Eq. (18).

| Case | M_D | M_R   | M_S   |
|------|-----|-------|-------|
| (a)  | \(\begin{pmatrix} a & b & 0 \\ d & 0 & 0 \\ 0 & 0 & l \end{pmatrix}\) | \(M_R^{(12)}\) | \(M_S^{(6)}\) |
| (b)  | \(\begin{pmatrix} 0 & b & c \\ 0 & e & 0 \\ g & 0 & 0 \end{pmatrix}\) | \(M_R^{(3)}\) | \(M_S^{(4)}\) |
| (c)  | \(\begin{pmatrix} 0 & b & c \\ 0 & 0 & f \\ g & 0 & 0 \end{pmatrix}\) | \(M_R^{(2)}\) | \(M_S^{(5)}\) |
| (d)  | \(\begin{pmatrix} a & 0 & c \\ d & 0 & 0 \\ 0 & h & 0 \end{pmatrix}\) | \(M_R^{(11)}\) | \(M_S^{(4)}\) |
| (e)  | \(\begin{pmatrix} a & 0 & c \\ 0 & 0 & f \\ 0 & h & 0 \end{pmatrix}\) | \(M_R^{(5)}\) | \(M_S^{(4)}\) |

This is the texture \(B_3\) in Table 2 and reproduces the following correlation

\[
\frac{m_{\tau\tau}}{m_{es}} = \frac{m_{\mu\tau}}{m_{\mu s}} = \frac{m_{\tau s}}{m_{\tau s}} = \frac{m_{\tau s}}{m_{s s}} = \frac{m_{\tau s}}{m_{s s}} \sqrt{\frac{m_{\tau \tau}}{m_{s s}}} \tag{20}
\]

According to Eq. (12) \(S_3\) transformations of the basic combination in Eq. (18) give a number of cases which generate textures \(B_3\) with the same correlations as in Eq. (20). These are presented in Table 5.

Case II: Another basic combination of

\[
M_R = M_R^{(9)}, \quad M_D = M_D^{(21)}, \quad M_S = M_S^{(6)} \tag{21}
\]

in Eq. (15), Table 4 and Eq. (17) respectively applied in Eq. (3) produces the following \(M_v^{4\times4}\)

\[
M_v^{4\times4} = \begin{pmatrix}
\frac{c^2 B^2}{A E^2} & \frac{c e}{E} & \frac{(hA-gB)c}{A E} & \frac{c s_3 A E^2}{B^2} \\
\frac{c e}{E} & 0 & 0 & \frac{e s_3}{E} \\
\frac{(hA-gB)c}{A E} & 0 & \frac{g^2}{A} & \frac{(hA-gB)s_3}{A E} \\
\frac{c s_3 A E^2}{B^2} & \frac{e s_3}{E} & \frac{(hA-gB)s_3}{A E} & \frac{s_3 B^2}{A E^2}
\end{pmatrix} . \tag{22}
\]

This is of the texture \(D_1\) in Table 2 that leads to the following correlation

\[
\frac{m_{ee}}{m_{es}} = \frac{m_{es}}{m_{ss}} = \frac{m_{\tau\tau}}{m_{\tau s}} = \frac{m_{e\mu}}{m_{\mu s}} = \sqrt{\frac{m_{ee}}{m_{ss}}} \tag{23}
\]

In this case also there exist another five combinations of \(M_D, M_R, M_S\) which are \(S_3\) symmetric to Eq. (21) giving the same correlation as in Eq. (23) (Table 6).

Case III: The basic combination of

\[
M_R^{(9)}, M_D^{(5)}, M_S^{(6)} \tag{24}
\]
Table 6
S\(_3\) symmetric textures of the basic combination in Eq. (21).

| Case | \( MD \) | \( MR \) | \( MS \) |
|------|--------|--------|--------|
| (a)  | \[
\begin{pmatrix}
 0 & 0 & c \\
 0 & 0 & d \\
 0 & 0 & g \\
\end{pmatrix}
\] | \( M_{R}^{(12)} \) | \( M_{S}^{(6)} \) |
| (b)  | \[
\begin{pmatrix}
a & 0 & 0 \\
0 & e & 0 \\
0 & h & t \\
\end{pmatrix}
\] | \( M_{R}^{(3)} \) | \( M_{S}^{(4)} \) |
| (c)  | \[
\begin{pmatrix}
a & 0 & 0 \\
0 & 0 & f \\
0 & h & l \\
\end{pmatrix}
\] | \( M_{R}^{(2)} \) | \( M_{S}^{(5)} \) |
| (d)  | \[
\begin{pmatrix}
0 & b & 0 \\
0 & 0 & d \\
0 & 0 & f \\
\end{pmatrix}
\] | \( M_{R}^{(11)} \) | \( M_{S}^{(4)} \) |
| (e)  | \[
\begin{pmatrix}
a & 0 & 0 \\
0 & 0 & f \\
0 & 0 & g \\
\end{pmatrix}
\] | \( M_{R}^{(5)} \) | \( M_{S}^{(4)} \) |

Table 7
S\(_3\) symmetric textures of the basic combination in Eq. (24).

| Case | \( MD \) | \( MR \) | \( MS \) |
|------|--------|--------|--------|
| (a)  | \[
\begin{pmatrix}
0 & b & 0 \\
0 & e & f \\
h & 0 & 0 \\
\end{pmatrix}
\] | \( M_{R}^{(12)} \) | \( M_{S}^{(6)} \) |
| (b)  | \[
\begin{pmatrix}
0 & 0 & c \\
0 & 0 & d \\
0 & 0 & f \\
0 & 0 & h \\
\end{pmatrix}
\] | \( M_{R}^{(3)} \) | \( M_{S}^{(4)} \) |
| (c)  | \[
\begin{pmatrix}
0 & b & 0 \\
0 & d & e \\
0 & 0 & 0 \\
0 & 0 & i \\
\end{pmatrix}
\] | \( M_{R}^{(2)} \) | \( M_{S}^{(5)} \) |
| (d)  | \[
\begin{pmatrix}
0 & 0 & c \\
0 & 0 & d \\
0 & 0 & e \\
g & 0 & 0 \\
\end{pmatrix}
\] | \( M_{R}^{(11)} \) | \( M_{S}^{(4)} \) |
| (e)  | \[
\begin{pmatrix}
a & 0 & 0 \\
0 & d & e \\
0 & 0 & 0 \\
0 & 0 & h \\
\end{pmatrix}
\] | \( M_{R}^{(5)} \) | \( M_{S}^{(4)} \) |

in Eq. (15), Table 4 and Eq. (17) respectively used in Eq. (3) leads to the texture \( B_4 \) and it reproduces two correlations:

\[
\frac{m_{el}}{m_{\mu s}} = \frac{m_{ee}}{m_{es}} \tag{25}
\]

\[
m_{ee}m_{ss} = m_{es}^2 \tag{26}
\]

However, there exist another five combinations (Table 7) which are \( S_3 \) symmetric to Eq. (24) giving the same correlations as in Eq. (25) and (26).
Table 8

Basic combination required for realization of the allowed two-zero textures under (5+3) scheme and their respective correlations.

| Texture | $M_D$, $M_R$, $M_S$ | Correlations |
|---------|---------------------|--------------|
| $A_1(i)$ | (1), (10), (1) | $m_{es}(m_{\mu\tau \mu} - m_{\mu\tau \tau}) = m_{ee}(m_{\tau\mu \tau} - m_{\tau\mu \mu})$ |
| (ii) | (2), (9), (6) | (a) $m_{\mu\tau \mu} = m_{\mu\tau \tau}$, (b) $m_{\mu\tau \mu}m_{\mu\tau \tau} = m_{\mu\tau \mu}m_{\mu\tau \tau}$ |
| (iii) | (3), (9), (6) | $m_{es} = m_{\mu\tau \mu} = m_{\mu\tau \tau}$ |
| (iv) | (4), (9), (6) | (a) $m_{es} = m_{\mu\tau \mu}$, (b) $m_{\mu\tau \mu}m_{\mu\tau \tau} = m_{\mu\tau \mu}m_{\mu\tau \tau}$ |
| (v) | (5), (10), (5) | $m_{es} = m_{\mu\tau \mu}$ |
| $A_2(i)$ | (6), (10), (1) | $m_{es}(m_{\mu\tau \mu} - m_{\mu\tau \tau}) = m_{ee}(m_{\tau\mu \tau} - m_{\tau\mu \mu})$ |
| (ii) | (8), (9), (6) | (a) $m_{\mu\tau \mu} = m_{\mu\tau \tau}$, (b) $m_{\mu\tau \mu}m_{\mu\tau \tau} = m_{\mu\tau \mu}m_{\mu\tau \tau}$ |
| (iii) | (7), (10), (5) | $m_{es} = m_{\mu\tau \mu} = m_{\mu\tau \tau}$ |
| (iv) | (9), (9), (6) | (a) $m_{es} = m_{\mu\tau \mu}$, (b) $m_{\mu\tau \mu}m_{\mu\tau \tau} = m_{\mu\tau \mu}m_{\mu\tau \tau}$ |
| (v) | (10), (9), (6) | $m_{es} = m_{\mu\tau \mu}$ |
| $B_3(i)$ | (11), (9), (6) | $m_{es}(m_{\mu\tau \mu} - m_{\mu\tau \tau}) = m_{ee}(m_{\tau\mu \tau} - m_{\tau\mu \mu})$ |
| (ii) | (12), (10), (5) | (a) $m_{\mu\tau \mu} = m_{\mu\tau \tau}$, (b) $m_{\mu\tau \mu}m_{\mu\tau \tau} = m_{\mu\tau \mu}m_{\mu\tau \tau}$ |
| (iii) | (7), (9), (6) | $m_{es} = m_{\mu\tau \mu} = m_{\mu\tau \tau}$ |
| (iv) | (13), (9), (6) | (a) $m_{es} = m_{\mu\tau \mu}$, (b) $m_{ee}m_{\tau\tau} = m_{ee}m_{\tau\tau}$ |
| (v) | (14), (10), (1) | $m_{es}(m_{\mu\tau \mu} - m_{\mu\tau \tau}) = m_{ee}(m_{\tau\mu \tau} - m_{ee}m_{\tau\tau})$ |
| $B_4(i)$ | (15), (9), (6) | (a) $m_{\mu\tau \mu} = m_{\mu\tau \tau}$, (b) $m_{ee}m_{\tau\tau} = m_{ee}m_{\tau\tau}$ |
| (ii) | (16), (10), (5) | (a) $m_{\mu\tau \mu} = m_{\mu\tau \tau}$, (b) $m_{ee}m_{\tau\tau} = m_{ee}m_{\tau\tau}$ |
| (iii) | (5), (9), (6) | (a) $m_{es} = m_{\mu\tau \mu}$, (b) $m_{ee}m_{\tau\tau} = m_{ee}m_{\tau\tau}$ |
| (iv) | (17), (9), (6) | (a) $m_{es} = m_{\mu\tau \mu}$, (b) $m_{ee}m_{\tau\tau} = m_{ee}m_{\tau\tau}$ |
| (v) | (18), (10), (1) | $m_{es}(m_{\mu\tau \mu} - m_{\mu\tau \tau}) = m_{ee}(m_{\tau\mu \tau} - m_{\tau\mu \mu})$ |
| $D_1(i)$ | (19), (10), (1) | $m_{es} = m_{\mu\tau \mu}$ |
| (ii) | (20), (9), (6) | (a) $m_{\mu\tau \mu} = m_{\mu\tau \tau}$, (b) $m_{\mu\tau \mu}m_{\mu\tau \tau} = m_{\mu\tau \mu}m_{\mu\tau \tau}$ |
| (iii) | (2), (10), (5) | (a) $m_{\mu\tau \mu} = m_{\mu\tau \tau}$, (b) $m_{\mu\tau \mu}m_{\mu\tau \tau} = m_{\mu\tau \mu}m_{\mu\tau \tau}$ |
| (iv) | (21), (9), (6) | (a) $m_{es} = m_{\mu\tau \mu}$, (b) $m_{ee}m_{\tau\tau} = m_{ee}m_{\tau\tau}$ |
| (v) | (16), (9), (6) | (a) $m_{es} = m_{\mu\tau \mu}$, (b) $m_{ee}m_{\tau\tau} = m_{ee}m_{\tau\tau}$ |
| $D_2(i)$ | (23), (10), (1) | $m_{es} = m_{\mu\tau \mu}$ |
| (ii) | (22), (9), (6) | (a) $m_{\mu\tau \mu} = m_{\mu\tau \tau}$, (b) $m_{\mu\tau \mu}m_{\mu\tau \tau} = m_{\mu\tau \mu}m_{\mu\tau \tau}$ |
| (iii) | (8), (10), (5) | (a) $m_{\mu\tau \mu} = m_{\mu\tau \tau}$, (b) $m_{\mu\tau \mu}m_{\mu\tau \tau} = m_{\mu\tau \mu}m_{\mu\tau \tau}$ |
| (iv) | (24), (9), (6) | (a) $m_{es} = m_{\mu\tau \mu}$, (b) $m_{ee}m_{\tau\tau} = m_{ee}m_{\tau\tau}$ |
| (v) | (12), (9), (6) | (a) $m_{es} = m_{\mu\tau \mu}$, (b) $m_{ee}m_{\tau\tau} = m_{ee}m_{\tau\tau}$ |
| $F_1(i)$ | (24), (10), (2) | $m_{ee}m_{\tau\tau} = m_{es}^2$ |
| (ii) | (21), (10), (2) | $m_{ee}m_{\tau\tau} = m_{es}^2$ |

(continued on next page)

The remaining 39 basic combinations of $M_D$, $M_R$ and $M_S$ along with their correlations are presented in Table 8. In our analysis we have found that some combinations are having multiple correlations.
5. Viability of the textures

In our analysis we calculate the mass matrix elements $m_{ij}$, $i, j = (e, \mu, \tau$ and $s$) using the current neutrino data (Table 1) in the expressions given in the Appendix. The range of Dirac and the Majorana CP phases is considered to be $(0 - 2\pi)$. The viability of a texture shall be examined under the following two conditions:

(i) The left hand side (lhs) and the right hand side (rhs) of the correlation(s) of a given texture are plotted against $\sin \theta_{34}$ in the range $(0 - 0.4)$. If there happens a reasonable overlapping of the plots of lhs and rhs, then the texture is considered as an allowed texture within the overlapping range of $\sin \theta_{34}$. We also study the viability of the textures when the CP phases are constrained to some random ranges of values within $(0 - 2\pi)$. In analysis, we categorise them as (a) CP phase dependent textures and (b) CP phase independent textures. It is to be noted that in case, a texture is having more than one correlation and at least one of them is CP phase dependent, then the texture belongs to CP phase dependent category.

(ii) The effective electron neutrino mass $|m_{\beta\beta}|$ which indicates the rate of neutrinoless double beta decay. We calculate $|m_{\beta\beta}|$ for each texture using the data in Table 1 in the following Eq. (27):

$$|m_{\beta\beta}| = |m_1 c_{12}^2 c_{13}^2 c_{14}^2 + m_2 c_{12}^2 c_{13}^2 s_{12}^2 e^{-i\alpha} + m_3 c_{14}^2 s_{13}^2 e^{-i\beta} + m_4 s_{14}^2 e^{-i\gamma}|.$$  \hspace{1cm} (27)

For NH we put $m_1 = 0$ and for IH $m_3 = 0$. Again three Majorana CP violating phases $\alpha, \beta, \gamma$ are not constrained by experimental limits i.e., they may take values between 0 to $2\pi$. There are a number of experiments [73–76] which are looking for more accurate limit on $|m_{\beta\beta}|$. The most constraint upper bound has been set to $|m_{\beta\beta}| < 0.06 - 0.165$ eV at 90% C.L. by the KamLAND ZEN Collaboration [76].

Now naturally a texture is considered to be viable if both the above conditions are satisfied simultaneously. The procedure of our analysis is the following: in the first place, we examine the textures under the condition (i) i.e., the consistency of the correlation(s) of a texture and categorize it into CP phase dependent or independent. If consistency is not found for a texture, then such texture is ruled out for not being viable. In the second place, we calculate $|m_{\beta\beta}|$ for the viable textures only under the condition (i). Again a viable texture under condition (i) may be ruled out if $|m_{\beta\beta}|$ for the texture is not within the experimental constraints.
In addition, we also consider to determine the Jarlskog invariant \( J_{CP} \) of a texture. For (3+1) scheme, Jarlskog invariant \( J_{CP}^{4v} = \text{Im}(U_{e1}U_{\mu2}U_{e2}^\ast U_{\mu1}^\ast) \) according to the parametrization in Eq. (6) takes the form

\[
J_{CP}^{4v} = J_{CP}^{3v} c_{14}^2 c_{24}^2 + s_{12} s_{14} c_{24} c_{23} c_{14}^2 c_{13}^2 c_{12} s_{12} \sin(\delta_{14} - \delta_{24}),
\]

(28)

where

\[
J_{CP}^{3v} = s_{12} s_{23} s_{13} c_{12} c_{23} c_{13}^2 \sin \delta_{13}.
\]

(29)

The ranges of Dirac CP phases \((\delta_{13}, \delta_{14}, \delta_{24})\) for calculating \( J_{CP}^{4v} \) are considered the same for which a texture becomes viable. The textures which are not consistent with the viability conditions (i) and (ii), will not be considered for calculating \( J_{CP}^{4v} \).

5.1. CP phase dependent textures

There are two categories of CP phase dependent textures because of their phenomenology illustrated below.

Category-I: This class of textures shows overlapping of scatter plots only for partial range of \( \sin \theta_{34} \) when CP phases are not constrained to any segmented range. In a systematic analysis we have found that there exist some ranges of CP phases for which overlapping of scatter plots of such textures always disappears within the range of \( \sin \theta_{34} = (0 - 0.4) \). For example, one such case is presented here:

We pick up the case of the texture \( B_3(i) \) in Table 8. Firstly we calculate the matrix elements \( m_{ij} \) with \( i, j = (e, \mu, \tau \text{ and } s) \) given in the Appendix using 3\( \sigma \) range of the parameters of neutrino data in Table 1 and taking unconstrained CP phases in the range \((0 - 2\pi)\). We present the NH case for the texture \( B_3(i) \) for which \( m_1 = 0 \) is set for calculation of the matrix elements. Each of 5 expressions in the correlations for \( B_3(i) \) in Eq. (20) separated by equality sign is plotted against \( \sin \theta_{34} \) with its range \((0 - 0.4)\). Fig. 1 shows these correlation plots.

It is evident from Fig. 1 that the correlations in Eq. (20) are consistent only for \( \sin \theta_{34} > 0.1 \) for unconstrained CP phases. For values of \( \sin \theta_{34} < 0.1 \) the overlapping of the plots for \( m_{\tau\tau} \) and \( \sqrt{m_{\tau\tau}} \) disappears and the texture is not allowed below that value of \( \sin \theta_{34} \). This predicts a lower bound on \( \sin \theta_{34} \) to be 0.1 for this texture.
We play textures \( P \). Kumar, M. Patgiri / Nuclear Physics B 957 (2020) 115082

\(\tau\tau\)

If CP phases are constrained to the ranges: \( \delta_{13} = \delta_{14} = (45^0 - 90^0) \), \( \delta_{24} = (180^0 - 225^0) \), \( \alpha = (135^0 - 180^0) \), \( \beta = \gamma = (0 - 45^0) \) (Texture \( B_3(i) \), NH),

\[
\begin{align*}
\text{Fig. 2. Scatter plot for Eq. (20) against } \sin \theta_{34} \text{ for constrained ranges of CP phases: } & \delta_{13} = \delta_{14} = (45^0 - 90^0), \delta_{24} = (180^0 - 225^0), \\
& \alpha = (135^0 - 180^0), \beta = \gamma = (0 - 45^0) \text{ (Texture } B_3(i) \text{, NH).}
\end{align*}
\]

\(\mu\tau\)

The textures belonging to this category have the special features that their scatter plots always have reasonable overlapping in part of the range of \( \sin \theta_{34} \) when CP phases are randomly constrained to some range within \( 0 - 2\pi \).

We present the case of \( D_1(iv) \) (NH) of this category. Following the similar procedure of Category-I, we have plotted the Fig. 3. Plot (a) and Plot (b) are the correlation plots of Eq. (23) for unconstrained and constrained CP phases respectively. The figure shows that the texture is allowed for all ranges of \( \sin \theta_{34} = (0.0 - 0.4) \) for unconstrained CP phases. However, on constraining the phases to their respective ranges: \( \delta_{13} = (45^0 - 90^0) \), \( \delta_{14} = (180^0 - 225^0) \), \( \delta_{24} = (270^0 - 315^0) \), \( \alpha = (270^0 - 315^0) \), \( \beta = (90^0 - 135^0) \), \( \gamma = (330^0 - 360^0) \), the texture is allowed for the range \( (0.06 - 0.40) \) of \( \sin \theta_{34} \). On surveying the correlation with different ranges of CP phases, we find that the texture is always allowed at least for some values of \( \sin \theta_{34} \), unlike the Category-I, and this is true even when CP phases are made to vanish. This clearly argues for different phenomenology of Category-II from Category-I.

Table 10 shows the allowed ranges of \( \sin \theta_{34} \) for the textures belonging to this Category-II with and without constraining CP phases. Their respective constrained ranges of CP phases are presented in Table 11. There is a lower limit of \( \sin \theta_{34} \) for many textures observed in case of unconstrained CP phases.

Analysis under \( J_{CP} \) and \( |m_{\beta\beta}| \)

Now we check the compatibility of the textures which are allowed in scatter plots in context of the values of \( |m_{\beta\beta}| \) under the same conditions of CP phases as imposed in case of the scatter plots. We also calculate the value of Jarlskog invariant \( J_{CP} \) for each case considering the constrained range of Dirac CP phases.
Table 9

Category-I: textures not allowed for the ranges of CP phases.

| Texture          | Constrained CP phases                                                                 |
|------------------|----------------------------------------------------------------------------------------|
| $A_1(iii), (v)$  | $\delta_{13} = \delta_{14} = (45^0 - 90^0)$, $\delta_{24} = (180^0 - 225^0)$, $\alpha = (135^0 - 180^0)$, $\beta = \gamma = (0 - 45^0)$ |
| $A_1(iv)(a)$     | $\delta_{13} = (180^0 - 225^0)$, $\delta_{14} = (0 - 30^0)$, $\alpha = (0 - 90^0)$, $\beta = (315^0 - 360^0)$, $\gamma = (270^0 - 315^0)$ |
| $B_3(i), (ii)$   | $\delta_{13} = (45^0 - 90^0)$, $\delta_{14} = (90^0 - 135^0)$, $\delta_{24} = (0 - 45^0)$, $\alpha = (270^0 - 315^0)$, $\beta = \gamma = (0 - 30^0)$ |
| $B_3(iii)(a)$    | $\delta_{13} = (0 - 90^0)$, $\delta_{14} = (90^0 - 130^0)$, $\delta_{24} = (0 - 180^0)$, $\alpha = (0 - 30^0)$, $\beta = (0 - 180^0)$, $\gamma = (0 - 90^0)$ |
| $B_3(iv)(a)$     | $\delta_{13} = (180^0 - 225^0)$, $\delta_{14} = (225^0 - 270^0)$, $\delta_{24} = (180^0 - 210^0)$, $\alpha = \gamma = (180^0 - 225^0)$, $\beta = (225^0 - 270^0)$ |
| $B_3(v)$         | $\delta_{13} = (0 - 90^0)$, $\delta_{14} = (90^0 - 180^0)$, $\delta_{24} = (45^0 - 90^0)$, $\alpha = (0 - 45^0)$, $\beta = \gamma = (0 - 30^0)$ |
| $B_4(iv)(a)$     | $\delta_{13} = (0 - 90^0)$, $\delta_{14} = (90^0 - 180^0)$, $\delta_{24} = (45^0 - 90^0)$, $\alpha = (0 - 45^0)$, $\beta = \gamma = (0 - 30^0)$ |
| $D_1(ii)(a)$     | $\delta_{13} = (0 - 45^0)$, $\delta_{14} = \delta_{24} = (0 - 30^0)$, $\alpha = (270^0 - 315^0)$, $\beta = (130^0 - 180^0)$, $\gamma = (225^0 - 270^0)$ |
| $D_1(iii)$       | $\delta_{13} = (45^0 - 90^0)$, $\delta_{14} = (0 - 30^0)$, $\delta_{24} = (315^0 - 360^0)$, $\alpha = (0 - 45^0)$, $\beta = (0 - 30^0)$, $\gamma = (315^0 - 360^0)$ |
| $D_1(iv)$        | $\delta_{13} = \delta_{24} = \text{unconstrained}$, $\delta_{14} = (0 - 270^0)$, $\alpha = (0 - 90^0)$, $\beta = (0 - 90^0)$, $\gamma = (0 - 30^0)$ |
| $D_2(v)$         | $\delta_{13} = (45^0 - 90^0)$, $\delta_{14} = \delta_{24} = (0 - 45^0)$, $\alpha = \gamma = (180^0 - 225^0)$, $\beta = (0 - 45^0)$ |
| $F_3(iii)$       | $\delta_{13} = (45^0 - 90^0)$, $\delta_{14} = (315^0 - 360^0)$, $\delta_{24} = \alpha = (0 - 30^0)$, $\alpha = (0 - 30^0)$, $\beta = (90^0 - 135^0)$, $\gamma = (270^0 - 315^0)$ |

Fig. 3. Scatter plot for Eq. (23) against $\sin \theta_{14}$ for unconstrained (Plot (a)) and constrained (Plot (b)) ranges of CP phases: $\delta_{13} = (45^0 - 90^0)$, $\delta_{14} = (180^0 - 225^0)$, $\delta_{24} = (270^0 - 315^0)$, $\alpha = (270^0 - 315^0)$, $\beta = (90^0 - 135^0)$, $\gamma = (330^0 - 360^0)$ (Texture $D_1(iv)$, NH).
Table 10
Table shows the allowed range of $\sin \theta_{34}$ for unconstrained and constrained CP phases for textures under Category (II). Here ‘All’ represents that the texture allows all values of $\sin \theta_{34} = (0 - 0.4)$.

| Texture       | Range of $\sin \theta_{34}$ for unconstrained CP | Range of $\sin \theta_{34}$ for constrained CP |
|---------------|--------------------------------------------------|-----------------------------------------------|
| $A_1(i) (NH)$ | $(0.04 - 0.4)$                                   | $(0.08 - 0.4)$                                |
| $A_1(ii) (NH)$| $> 0.02$                                          | $(0.02 - 0.1)$                                |
| $A_2(ii) (NH)$| $> 0.08$                                          | $> 0.1$                                       |
| $A_2(ii) (b) (NH)$ | $> 0.08$                                     | $> 0.12$                                      |
| $A_2(iii), (v) (NH)$ | $> 0.04$                                    | $> 0.08$                                      |
| $A_2(iv) (NH)$ | $> 0.04$                                          | $> 0.14$                                      |
| $B_3(iii) (a) (NH)$ | *All*                                      | $> 0.2$                                       |
| $B_3(v) (NH)$ | *All*                                             | $> 0.06$                                      |
| $B_3(iv) (a) (NH)$ | $> 0.02$                                     | $< 0.06$                                      |
| $B_3(iv) (a) (IH)$ | $> 0.02$                                     | $< 0.18$                                      |
| $B_3(v) (NH)$ | *All*                                             | $< 0.14$                                      |
| $B_3(v) (IH)$ | *All*                                             | $< 0.06$                                      |
| $D_1(ii) (NH)$ | $> 0.02$                                          | $> 0.1$                                       |
| $D_1(iii) (NH)$ | *All*                                        | $> 0.04$                                      |
| $D_1(iv) (NH)$ | *All*                                             | $> 0.06$                                      |
| $D_1(v) (a) (NH)$ | $> 0.02$                                     | $> 0.06$                                      |
| $D_1(v) (a) (IH)$ | $> 0.06$                                     | $> 0.24$                                      |
| $F_1(iii) (NH)$ | $> 0.04$                                          | $> 0.1$                                       |
| $F_1(iii) (IH)$ | $> 0.04$                                          | $> 0.32$                                      |
| $F_1(iv) (IH)$ | $> 0.04$                                          | $(0.02 - 0.06)$                               |
| $F_2(iv) (NH)$ | $> 0.1$                                            | $> 0.14$                                      |
| $F_2(iv) (IH)$ | $> 0.04$                                          | $(0.04 - 0.06)$                               |

Table 11
Constrained ranges of CP phases for textures under Category-(II). The third column represents the value of $|m_{\beta\beta}|$ and $J_{CP}$ for constrained ranges of Majorana CP phases ($\alpha, \beta, \gamma$) and Dirac CP phases ($\delta_{13}, \delta_{14}, \delta_{24}$) respectively. For the textures $A_1, A_2$, the element $m_{ee} = 0$. This is represented as ‘−’ for $|m_{\beta\beta}|$ in the table.

| Texture       | Constrained ranges of CP phases | $|m_{\beta\beta}|$ (eV) & $J_{CP}$          |
|---------------|---------------------------------|---------------------------------|
| $A_1(i) (NH)$ | $\alpha = \gamma = (225^0 - 270^0), \beta = (135^0 - 180^0)$ | $|m_{\beta\beta}| = -$          |
| $A_1(ii) (a)$ (NH) | $\delta_{13} = (45^0 - 90^0), \delta_{14} = (180^0 - 225^0), \delta_{24} = (0 - 45^0)$ | $J_{CP} = (0 - 0.02)$          |
| $A_2(ii) (a)$ (NH) | $\alpha = (0 - 45^0), \beta = (0 - 30^0), \gamma = (120^0 - 160^0)$ | $|m_{\beta\beta}| = -$          |
| $A_2(ii) (b)$ (NH) | $\delta_{13} = (135^0 - 180^0), \delta_{14} = (325^0 - 360^0), \delta_{24} = (90^0 - 135^0)$ | $(0 - 0.02)$                   |
| $A_2(iii), (v)$ (NH) | $\alpha = (225^0 - 270^0), \beta = (180^0 - 225^0), \gamma = (315^0 - 360^0)$ | $|m_{\beta\beta}| = -$          |
| $A_2(iv) (a)$ (NH) | $\delta_{13} = (135^0 - 180^0), \delta_{14} = (225^0 - 270^0), \delta_{24} = (270^0 - 315^0)$ | $(0 - 0.02)$                   |
| $A_2(iv) (a)$ (NH) | $\alpha = (315^0 - 360^0), \beta = (0 - 30^0)$ | $|m_{\beta\beta}| = -$          |
| $A_2(iv) (a)$ (NH) | $\delta_{13} = (45^0 - 90^0), \delta_{14} = (180^0 - 225^0), \delta_{24} = (180^0 - 225^0)$ | $(0.02 - 0.04)$                |
| $A_2(iv) (a)$ (NH) | $\alpha = (135^0 - 180^0), \beta = (0 - 30^0), \gamma = (315^0 - 360^0)$ | $|m_{\beta\beta}| = -$          |
| $A_2(iv) (a)$ (NH) | $\delta_{13} = (180^0 - 225^0), \delta_{14} = \delta_{24} = (0 - 30^0)$ | $(0 - 0.02)$                   |
Here we consider $D_1(i v)$ (NH) as a representative case for illustration. We show the variation of $|m_{\beta\beta}|$ and $J_{CP}$ (Fig. 5) for the constrained ranges of Majorana CP phases $\alpha = (270^0 - 315^0)$, $\beta = (90^0 - 135^0)$, $\gamma = (330^0 - 360^0)$ and Dirac CP phases $\delta_{13} = (45^0 - 90^0)$, $\delta_{14} = (180^0 - 225^0)$, $\delta_{24} = (270^0 - 315^0)$ respectively. For unconstrained CP phases, we obtain $|m_{\beta\beta}| \approx (0.01 - 0.06)$ eV for NH and $\approx (10^{-3} - 0.1)$ eV for IH (Fig. 4). In Fig. 5 (left plot), we study the dependence of $|m_{\beta\beta}|$ on the lightest neutrino mass for constrained range of

| Texture | Constrained ranges of CP phases | $|m_{\beta\beta}|$ (eV) & $J_{CP}$ |
|---------|---------------------------------|------------------|
| $B_3(i i i)(a)$ (NH) | $\alpha = (315^0 - 360^0), \beta = \gamma = (330^0 - 360^0)$ | (0.02 - 0.07) |
| $B_3(v)$ (NH) | $\alpha = (180^0 - 225^0), \beta = \gamma = (315^0 - 360^0)$ | (0.01 - 0.06) |
| $B_4(iv)(a)$ (NH) | $\alpha = (180^0 - 225^0), \beta = \gamma = (315^0 - 360^0)$ | (0.01 - 0.06) |
| $B_4(iv)(a)$ (IH) | $\alpha = (180^0 - 225^0), \beta = (225^0 - 270^0)$ | (0.02 - 0.07) |
| $B_4(v)$ (NH) | $\alpha = (0 - 45^0), \beta = \gamma = (0 - 30^0)$ | (0.06 - 0.1) |
| $B_4(v)$ (IH) | $\alpha = (90^0 - 180^0)$, $\beta = \gamma = (0 - 30^0)$ | (0.02 - 0.07) |
| $D_1(ii)(a)$ (NH) | $\alpha = (315^0 - 360^0), \beta = \gamma = (330^0 - 360^0)$ | (0.01 - 0.06) |
| $D_1(i i i)$ (NH) | $\alpha = (270^0 - 315^0)$, $\beta = (330^0 - 360^0)$, $\gamma = (315^0 - 360^0)$ | (0.05 - 0.06) |
| $D_1(iv)$ (NH) | $\alpha = (45^0 - 90^0)$, $\beta = (90^0 - 135^0)$, $\gamma = (330^0 - 360^0)$ | (0.01 - 0.06) |
| $D_1(v)(a)$ (NH) | $\alpha = (135^0 - 180^0), \beta = (0 - 30^0)$, $\gamma = (225^0 - 315^0)$ | (0.01 - 0.06) |
| $D_1(v)(a)$ (IH) | $\alpha = (90^0 - 135^0), \beta = (0 - 30^0)$, $\gamma = (225^0 - 315^0)$ | (0.01 - 0.06) |
| $F_1(i i i)$ (NH) | $\alpha = (135^0 - 180^0), \beta = (0 - 30^0)$, $\gamma = (225^0 - 315^0)$ | (0.01 - 0.06) |
| $F_1(i i i)$ (IH) | $\alpha = (45^0 - 90^0), \beta = (90^0 - 135^0)$, $\gamma = (330^0 - 360^0)$ | (0.05 - 0.06) |
| $F_1(iv)$ (NH) | $\alpha = (270^0 - 315^0), \beta = (180^0 - 225^0)$, $\gamma = (315^0 - 360^0)$ | (0.02 - 0.07) |
| $F_1(iv)$ (IH) | $\alpha = (135^0 - 180^0), \beta = (0 - 30^0)$ | (0.02 - 0.07) |
| $F_2(iv)$ (NH) | $\alpha = (180^0 - 225^0), \beta = (0 - 30^0), \gamma = (0 - 45^0)$ | (0.02 - 0.07) |
| $F_2(iv)$ (IH) | $\alpha = (45^0 - 90^0), \beta = (0 - 30^0)$ | (0.02 - 0.07) |
Majorana CP phases \( (\alpha = 270^0 - 315^0), \beta = 90^0 - 135^0), \gamma = 330^0 - 360^0) \). Accordingly we present the parameter space of \(|m_{\beta\beta}|\) with respect to the lightest neutrino mass \( (m_{\text{lightest}}) \) in left plot of Fig. 5 for normal hierarchy (NH) pattern. As demanded by MES mechanism, for NH \( m_1 = 0 \). In our analysis, we have plotted \(|m_{\beta\beta}|\) against \( m_2 \) for NH pattern. Similarly for cases with inverted hierarchy \(|m_{\beta\beta}|\) is obtained against \( m_1 \) (as \( m_2 > m_1 \) and \( m_3 = 0 \) according to MES mechanism). From Fig. 5 it is evident that for the constrained range of \( \alpha, \beta, \gamma \), the texture predicts \(|m_{\beta\beta}| \approx (0.01 - 0.06) \text{ eV}\) which lies below the upper bound of \(|m_{\beta\beta}| < 0.061 - 0.165 \) [76]. Hence, we consider this texture with the above constrained CP phases to be an allowed texture. Again, we find that for unconstrained Dirac CP phases, the Jarlskog invariant \( J_{CP} \) which provides a measure of the Dirac-type CP violation is \( J_{CP} \approx (0 - 0.05) \) for both NH and IH and the maximal value of \( J_{CP} \approx 0.05 \) is for \( \delta_{13} = 90^0 \) and \( 270^0 \). From Fig. 5 (right plot) it is seen that for constrained range of Dirac CP phases \( \delta_{13} = (45^0 - 90^0), \delta_{14} = (180^0 - 225^0), \delta_{24} = (270^0 - 315^0), J_{CP} = (0.01 - 0.03) \). The range of \(|m_{\beta\beta}|\) and \( J_{CP} \) for constrained ranges of CP phases for textures under Category-II are presented in Table 11.

5.2. CP phase independent textures

The textures \( A_2(i) \) and \( B_3(iii) \) among all the textures remain viable for the whole range of \( \sin \theta_{34} = (0.0 - 0.4) \) whether CP phases are unconstrained or constrained to different ranges, or
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Fig. 6. Scatter plots of texture $B_4(iii)$ for Eq. (25) (left plot) and Eq. (26) (right plot) against $\sin \theta_{34}$ for unconstrained CP phases. \(m_{\mu\mu} = \frac{m_{\mu\mu}}{m_{ee}}\) (left plot) and \(m_{ee} = m_{e\mu} = m_{\mu\mu} = \frac{m_{\mu\mu}}{m_{ee}}\) (right plot).

Fig. 7. Scatter plot of texture $B_4(iii)$ for Eq. (25) against $\sin \theta_{34}$ for constrained ranges of CP phases. Plot (a) is for: $\delta_{13} = \beta = (45^0 - 90^0)$, $\gamma < 30^0$, $\delta_{14} = (90^0 - 130^0)$, $\delta_{24} = (180^0 - 270^0)$, $\alpha = (270^0 - 360^0)$. Plot (b) for: $\delta_{13} = \gamma = (135^0 - 180^0)$, $\beta = (45^0 - 90^0)$, $\delta_{14} = (180^0 - 225^0)$, $\delta_{24} = (0 - 45^0)$, $\alpha = (180^0 - 270^0)$. Plot (c) for: $\delta_{13} = \gamma = (135^0 - 180^0)$, $\beta = (225^0 - 270^0)$, $\delta_{14} = (180^0 - 225^0)$, $\delta_{24} = (0 - 45^0)$, $\alpha = (0 - 30^0)$. Plot (d) for: $\delta_{13} = (0 - 30^0)$, $\gamma = (225^0 - 270^0)$, $\beta = (135^0 - 180^0)$, $\delta_{14} = (225^0 - 270^0)$, $\delta_{24} = (45^0 - 90^0)$, $\alpha = (90^0 - 135^0)$. \(m_{\mu\mu} = \frac{m_{\mu\mu}}{m_{ee}}\). \(m_{ee} = m_{e\mu} = m_{\mu\mu} = \frac{m_{\mu\mu}}{m_{ee}}\).

even CP phases enforced to be zero. For illustration, we take the texture $B_4(iii)$. The texture has two collateral correlations Eq. (25) and Eq. (26). The Fig. 6 represents the scatter plot for Eq. (25) (left plot) and for Eq. (26) (right plot) for unconstrained CP phases. Again the Fig. 7 and 8 show the scatter plots for different chosen ranges of CP phases for Eq. (25) and (26) respectively. It is observed that the overlapping of the plots for the left-hand side and right-hand side retains even for the choice of different ranges of CP phases, thereby showing no dependence of CP phase change. Thus it is a CP phase independent texture. Similar phenomenology of CP phase independence is shown by the texture $A_2(i)$. 
Fig. 8. Scatter plot of texture $B_4(iii)$ for Eq. (26) against $\sin\theta_{34}$ for constrained ranges of CP phases: Plot (a) is for: $\delta_{13} = \beta = (45^0 - 90^0)$, $\gamma < 30^0$, $\delta_{14} = (90^0 - 130^0)$, $\delta_{24} = (180^0 - 270^0)$, $\alpha = (270^0 - 360^0)$. Plot (b) for: $\delta_{13} = \gamma = (135^0 - 180^0)$, $\beta = (45^0 - 90^0)$, $\delta_{14} = (180^0 - 225^0)$, $\delta_{24} = (0 - 45^0)$, $\alpha = (180^0 - 270^0)$. Plot (c) for: $\delta_{13} = \gamma = (135^0 - 180^0)$, $\beta = (225^0 - 270^0)$, $\delta_{14} = (180^0 - 225^0)$, $\delta_{24} = (0 - 45^0)$, $\alpha = (0 - 30^0)$. Plot (d) for: $\delta_{13} = (0 - 30^0)$, $\gamma = (225^0 - 270^0)$, $\beta = (135^0 - 180^0)$, $\delta_{14} = (225^0 - 270^0)$, $\delta_{24} = (45^0 - 90^0)$, $\alpha = (90^0 - 135^0)$. $m_{ee}m_{ss}$, $m_{e2}^2$.

Table 12
Table shows the value of $|m_{\beta\beta}|$ and $J_{CP}$ obtained for different constrained ranges of CP phases considered in Fig. 7 and 8.

| Constrained ranges of CP phases | $|m_{\beta\beta}|$ (eV) & $J_{CP}$ |
|---------------------------------|------------------------|
| $\alpha = (270^0 - 360^0)$, $\beta = (45^0 - 90^0)$, $\gamma < 30^0$ | $|m_{\beta\beta}| = (0.015 - 0.07)$ eV, $J_{CP} = (0.01 - 0.03)$ |
| $\delta_{13} = (45^0 - 90^0)$, $\delta_{14} = (90^0 - 130^0)$, $\delta_{24} = (180^0 - 270^0)$ | |
| $\alpha = (180^0 - 270^0)$, $\beta = (45^0 - 90^0)$, $\gamma = (135^0 - 180^0)$ | $(0.01 - 0.06)$ |
| $\delta_{13} = (135^0 - 180^0)$, $\delta_{14} = (180^0 - 225^0)$, $\delta_{24} = (0 - 45^0)$ | $(0 - 0.03)$ |
| $\alpha = (0 - 30^0)$, $\beta = (225^0 - 270^0)$, $\gamma = (135^0 - 180^0)$ | $(0.01 - 0.06)$ |
| $\delta_{13} = (135^0 - 180^0)$, $\delta_{14} = (180^0 - 225^0)$, $\delta_{24} = (0 - 45^0)$ | $(0 - 0.03)$ |
| $\alpha = (90 - 135^0)$, $\beta = (135^0 - 180^0)$, $\gamma = (225^0 - 270^0)$ | $(0.01 - 0.06)$ |
| $\delta_{13} = (0 - 30^0)$, $\delta_{14} = (225^0 - 270^0)$, $\delta_{24} = (45^0 - 90^0)$ | $(0 - 0.02)$ |

We also check the range of $|m_{\beta\beta}|$ for different constrained ranges of the Majorana CP phases and found that for all the cases, the range of $|m_{\beta\beta}|$ lies within the experimental upper bound $|m_{\beta\beta}| < 0.06 - 0.165$ [76]. Similarly we also calculate the range of $J_{CP}$ for each case of constrained Dirac CP phases. In Table 12 we have presented the values of $|m_{\beta\beta}|$ and $J_{CP}$ for the different ranges of constrained CP phases considered in the plots of Fig. 7 and 8.
6. Symmetry realization

For every set of the fermion mass matrices with texture zeros in arbitrary entries, there corresponds to a scalar sector such that the texture zeros can always be realized by the Abelian flavor symmetries \[47,48\]. In the paper \[48\] the author discussed two methods for symmetry realization of texture zeros in the lepton sector in the framework of the seesaw mechanism. Now we adopt the method-2 of the paper for symmetry realization of the textures in our present context of MES mechanism. In doing so, the charged lepton mass matrix \(M_l\) shall be considered to be diagonal. In our systematic study, we find that \(Z_8\) is successful for generating all the viable textures. \(Z_8\) consists of the group elements

\[
(1, \omega, \omega^2, \omega^3, \omega^4, \omega^5, \omega^6, \omega^7) \]

where \(\omega = e^{i\pi/8}\) is the generator of the group. We show the symmetry realization for texture \(B_3(i)\) as a representative case. The leptonic fields are considered to transform under \(Z_8\) as

\[
\begin{align*}
\bar{D}_{el} &\rightarrow \omega^7 \bar{D}_{el}, & e_R &\rightarrow \omega^3 e_R, & \nu_{eR} &\rightarrow \omega \nu_{eR} , \\
\bar{D}_{\mu L} &\rightarrow \omega \bar{D}_{\mu L}, & \mu_R &\rightarrow \omega^2 \mu_R, & \nu_{\mu R} &\rightarrow \omega^7 \nu_{\mu R}, \\
\bar{D}_{\tau L} &\rightarrow \omega^2 \bar{D}_{\tau L}, & \tau_R &\rightarrow \omega^6 \tau_R, & \nu_{\tau R} &\rightarrow \omega^3 \nu_{\tau R},
\end{align*}
\]

(30)

Here, \(\bar{D}_{jl}, l_R\) and \(\nu_{kr}\) represent the \(SU(2)_L\) doublets, the RH \(SU(2)_L\) singlets and the RH neutrino singlets respectively. The bilinears \(\bar{D}_{jl} l_R, \bar{D}_{jl} \nu_{kr}, \nu_{kr}^T C^{-1} \nu_{jr}\) relevant for \(M_l, M_D\) and \(M_R\) respectively transform as

\[
\begin{align*}
\bar{D}_{kl} l_R &= \begin{pmatrix} \omega^2 & \omega & \omega^5 \\ \omega^4 & \omega^3 & \omega^7 \\ \omega^5 & \omega^4 & 1 \end{pmatrix}, & \bar{D}_{kl} \nu_{jr} &= \begin{pmatrix} 1 & \omega^6 & \omega^2 \\ \omega^2 & 1 & \omega^4 \\ \omega^6 & \omega^4 & \omega^6 \end{pmatrix}, \\
\nu_{kr}^T C^{-1} \nu_{jr} &= \begin{pmatrix} \omega^2 & 1 & \omega^4 \\ 1 & \omega^6 & \omega^2 \\ \omega^4 & \omega^2 & \omega^6 \end{pmatrix}.
\end{align*}
\]

(31)

We consider three \(SU(2)_L\) doublet Higgs (\(\Phi_1, \Phi_2, \Phi_3\)) transforming under \(Z_8\) as

\[
\Phi_1 \rightarrow \Phi_1, \quad \Phi_2 \rightarrow \omega^6 \Phi_2, \quad \Phi_3 \rightarrow \omega^5 \Phi_3.
\]

(32)

The \(Z_8\) invariant Yukawa Lagrangian becomes

\[
-\mathcal{L} = Y_{11}^l \bar{D}_{el} \Phi_2 e_R + Y_{22}^l \bar{D}_{\mu L} \Phi_3 \mu_R + Y_{33}^l \bar{D}_{\tau L} \Phi_1 \tau_R + Y_{11}^D \bar{D}_{el} \Phi_1 \nu e_R + Y_{12}^D \bar{D}_{el} \Phi_2 \nu_{\mu R} + Y_{22}^D \bar{D}_{\mu L} \Phi_1 \nu_{\mu R} + Y_{33}^D \bar{D}_{\tau L} \Phi_3 \nu_{\tau R} + h.c.
\]

(33)

After acquiring a non-zero vacuum expectation value \(<\phi_0>\neq 0\) by the Higgs fields, \(M_l\) and \(M_D\) take the form

\[
M_l = \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix}, \quad M_D = \begin{pmatrix} a & b & 0 \\ 0 & e & 0 \\ 0 & 0 & l \end{pmatrix}.
\]

(34)

We consider a scalar singlet \(\chi\) transforming under \(Z_8\) as

\[
\chi \rightarrow \omega^6 \chi.
\]

(35)
which leads to the following form of $M_R$

$$M_R = \begin{pmatrix} A & B & 0 \\ B & 0 & E \\ 0 & E & 0 \end{pmatrix}.$$  \hfill (36)

We also consider transformation of the singlet chiral field ‘S’, so as to prevent bare mass term of the form $\bar{S}^T S$.

$$S \rightarrow \omega^5 S.$$  \hfill (37)

Scalar singlet $\lambda_1$ transforming as

$$\lambda_1 \rightarrow \lambda_1$$  \hfill (38)

leads to the following form of $M_S$

$$M_S = \begin{pmatrix} 0 & 0 & s_3 \end{pmatrix},$$  \hfill (39)

which are the zero textures of the mass matrices in Eq. (18) for texture $B_3$.

It has been observed that, symmetry realization of the other five $S_3$ symmetric textures (Table 5) of the basic combination in Eq. (18) follows an interesting pattern. For instance, considering the textures in Case (b) of Table 5 obtained by transforming the basic combination Eq. (18) by the element “B” of the $S_3$ group, where

$$B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$  \hfill (40)

There exists an interchange of the first and third column of the matrix “B”. Following the similar pattern for symmetry realization of the textures of Case (b) (Table 5), and interchanging only the $Z_8$ transformation of the right-handed neutrino singlets ($\nu_{e_R} \leftrightarrow \nu_{\tau_R}$) of the basic combination in Eq. (30), that is

$$\nu_{e_R} \rightarrow \omega^3 \nu_{e_R}, \quad \nu_{\tau_R} \rightarrow \omega \nu_{\tau_R},$$  \hfill (41)

meanwhile keeping the transformation of all the other fields, that is, $\nu_{\mu_R}, \bar{D}_{jl}, l_R, \Phi, S, \chi$ and $\lambda$ same as that of basic combination, we arrive at the following set of matrices

$$M_D = \begin{pmatrix} 0 & b & c \\ 0 & e & 0 \\ g & 0 & 0 \end{pmatrix}, \quad M_R = M_R^{(3)}, \quad M_S^{(4)} = \begin{pmatrix} s_1 & 0 & 0 \end{pmatrix}.$$  \hfill (42)

Similarly, symmetry realization of all the other combinations (Table 5) can be obtained by simply interchanging the transformation of the RH neutrino singlets of the basic combination, according to the interchange of the columns of the elements of the $S_3$ group via which the combinations are obtained. A similar study was performed in Ref. [65].

In Table 13, we present the symmetry realization of all the basic combinations of $M_D, M_R$ and $M_S$. The basic combinations for each texture involves only three basic forms of the right-handed Majorana mass matrix $M_R = M_R^{(7)}, M_R^{(9)}, M_R^{(10)}$. For those textures which are realized via $M_R^{(9)}$, we keep the $Z_8$ transformation of the RH neutrino singlets $\nu_{k_R}$ to be the same as in Eq. (30). The transformation for the scalar singlet $\chi$, therefore, remains the same as in Eq. (35).

For textures obtained via $M_R^{(10)}$, we consider the transformation of the RH neutrino singlets and scalar singlet $\chi$ as:
Table 13

$Z_3$ symmetry realization of all the basic cases.

| Texture | $\tilde{D}_{\omega L}, \tilde{D}_{\mu L}, \tilde{D}_{\tau L}$ | $\Phi, \mu, \tau, \tau_R$ | $S'$ | $\lambda'$ |
|---------|-------------------------------------------------|--------------------------|-----|---------|
| $A_1(i)$ | $\omega^1$, $\omega^2$, $\omega^3$ | $\omega, \omega^5$, $\omega^7$ | $1, \omega, \omega^7$ | $\omega^2$ | $\omega, \omega^5$ |
| $A_1(ii)$ | $\omega^4, \omega^5$ | $\omega^4, \omega^5$ | $1, \omega^3, \omega^6$ | $\omega^5$ | $1$ |
| $A_1(iii)$ | $\omega^7, \omega^5$ | $1, \omega^7$ | $1, \omega^3, \omega^7$ | $\omega^2$ | $\omega^7$ |
| $A_1(iv)$ | $\omega^3, \omega^7$ | $\omega^2, \omega^7$ | $1, \omega^3, \omega^7$ | $\omega^2$ | $\omega^7$ |
| $A_2(i)$ | $\omega^3, \omega^5, \omega^3$ | $\omega^3, \omega^4$ | $1, \omega, \omega^7$ | $\omega^2$ | $\omega, \omega^5$ |
| $A_2(ii)$ | $\omega^4, \omega^5$ | $\omega^4, \omega^6$ | $1, \omega^3, \omega^6$ | $\omega^5$ | $1$ |
| $A_2(iii)$ | $\omega^7, \omega^5$ | $\omega^7, 1, \omega^5$ | $1, \omega^3, \omega^7$ | $\omega^2$ | $\omega^7$ |
| $A_2(iv)$ | $\omega^3, \omega^7, \omega^6$ | $\omega^4, \omega^3, \omega^7$ | $1, \omega^3, \omega^7$ | $\omega^2$ | $\omega^7$ |
| $A_2(v)$ | $\omega^4, \omega^7, \omega^6$ | $\omega^4, \omega^7, \omega^6$ | $1, \omega^3, \omega^7$ | $\omega^2$ | $\omega^7$ |
| $B_3(i)$ | $\omega^3, \omega^2, \omega^2$ | $\omega^3, \omega^2, \omega^6$ | $1, \omega^6, \omega^5$ | $\omega^5$ | $1$ |
| $B_3(ii)$ | $\omega^4, \omega^2, \omega^2$ | $\omega^4, \omega^2, \omega^4$ | $1, \omega^4, \omega^3$ | $\omega^2$ | $\omega^7$ |
| $B_3(iii)$ | $\omega^5, \omega^2, \omega^2$ | $\omega^5, \omega^2, \omega^5$ | $1, \omega^5, \omega^5$ | $\omega^7$ | $1$ |
| $B_3(iv)$ | $\omega^6, \omega^2, \omega^2$ | $\omega^6, \omega^2, \omega^6$ | $1, \omega^6, \omega^6$ | $\omega^7$ | $1$ |
| $B_3(v)$ | $\omega^7, \omega^2, \omega^2$ | $\omega^7, \omega^2, \omega^7$ | $1, \omega^7, \omega^7$ | $\omega^7$ | $1$ |
| $D_1(i)$ | $\omega^3, \omega^5, 1$ | $\omega^6, 1$ | $1, \omega^6, \omega^5$ | $\omega^2$ | $\omega, \omega^7$ |
| $D_1(ii)$ | $\omega^4, \omega^5, 1$ | $\omega^7, \omega^4, \omega^2$ | $1, \omega^7, \omega^5$ | $\omega^5$ | $1$ |
| $D_1(iii)$ | $\omega^5, \omega^5, 1$ | $\omega^5, \omega^5, 1$ | $1, \omega^5, \omega^5$ | $\omega^5$ | $1$ |
| $D_1(iv)$ | $\omega^6, \omega^5, 1$ | $\omega^6, \omega^6, 1$ | $1, \omega^6, \omega^6$ | $\omega^5$ | $1$ |
| $D_1(v)$ | $\omega^7, \omega^5, 1$ | $\omega^7, \omega^7, 1$ | $1, \omega^7, \omega^7$ | $\omega^5$ | $1$ |
| $D_2(i)$ | $\omega^3, \omega^7, \omega^2$ | $\omega^3, \omega^3, \omega^6$ | $1, \omega, \omega^6$ | $\omega^5$ | $1$ |
| $D_2(ii)$ | $\omega^4, \omega^7, \omega^2$ | $\omega^4, \omega^3, \omega^3$ | $1, \omega^4, \omega^3$ | $\omega^5$ | $1$ |
| $D_2(iii)$ | $\omega^5, \omega^7, \omega^2$ | $\omega^5, \omega^3, \omega^3$ | $1, \omega^5, \omega^3$ | $\omega^5$ | $1$ |
| $D_2(iv)$ | $\omega^6, \omega^7, \omega^2$ | $\omega^6, \omega^3, \omega^3$ | $1, \omega^6, \omega^3$ | $\omega^5$ | $1$ |
| $D_2(v)$ | $\omega^7, \omega^7, \omega^2$ | $\omega^7, \omega^3, \omega^3$ | $1, \omega^7, \omega^3$ | $\omega^5$ | $1$ |
| $F_1(i)$ | $\omega^3, \omega^7, \omega^2$ | $\omega^3, \omega^3, \omega^6$ | $1, \omega, \omega^6$ | $\omega^5$ | $1$ |
| $F_1(ii)$ | $\omega^4, \omega^7, \omega^2$ | $\omega^4, \omega^3, \omega^3$ | $1, \omega^4, \omega^3$ | $\omega^5$ | $1$ |
| $F_1(iii)$ | $\omega^5, \omega^7, \omega^2$ | $\omega^5, \omega^3, \omega^3$ | $1, \omega^5, \omega^3$ | $\omega^5$ | $1$ |
| $F_1(iv)$ | $\omega^6, \omega^7, \omega^2$ | $\omega^6, \omega^3, \omega^3$ | $1, \omega^6, \omega^3$ | $\omega^5$ | $1$ |
| $F_2(i)$ | $\omega^3, \omega^3, \omega^2$ | $\omega^3, \omega^3, \omega^6$ | $1, \omega, \omega^6$ | $\omega^5$ | $1$ |
| $F_2(ii)$ | $\omega^4, \omega^3, \omega^2$ | $\omega^4, \omega^3, \omega^3$ | $1, \omega^4, \omega^3$ | $\omega^5$ | $1$ |
| $F_2(iii)$ | $\omega^5, \omega^3, \omega^2$ | $\omega^5, \omega^3, \omega^3$ | $1, \omega^5, \omega^3$ | $\omega^5$ | $1$ |
| $F_2(iv)$ | $\omega^6, \omega^3, \omega^2$ | $\omega^6, \omega^3, \omega^3$ | $1, \omega^6, \omega^3$ | $\omega^5$ | $1$ |
| $F_3(i)$ | $\omega^3, \omega^3, \omega^3$ | $\omega^3, \omega^3, \omega^4$ | $1, \omega, \omega^6$ | $\omega^5$ | $1$ |
| $F_3(ii)$ | $\omega^4, \omega^3, \omega^3$ | $\omega^4, \omega^3, \omega^4$ | $1, \omega, \omega^6$ | $\omega^5$ | $1$ |
| $F_3(iii)$ | $\omega^5, \omega^3, \omega^3$ | $\omega^5, \omega^3, \omega^4$ | $1, \omega, \omega^6$ | $\omega^5$ | $1$ |
| $F_3(iv)$ | $\omega^6, \omega^3, \omega^3$ | $\omega^6, \omega^3, \omega^4$ | $1, \omega, \omega^6$ | $\omega^5$ | $1$ |
\[ v_{eR} \rightarrow \omega v_{eR}, \quad v_{\mu R} \rightarrow \omega^7 v_{\mu R}, \quad v_{\tau R} \rightarrow \omega^5 v_{\tau R}, \quad \chi \rightarrow \omega^6 \chi \]  

(43)

Textures that are realized by the diagonal RH Majorana mass matrix \( M_R^{(7)} \), we consider the transformations of \( v_{kR}, \chi \) as:

\[ v_{eR} \rightarrow v_{eR}, \quad v_{\mu R} \rightarrow \omega^4 v_{\mu R}, \quad v_{\tau R} \rightarrow \omega v_{\tau R}, \quad \chi \rightarrow \omega^6 \chi \]  

(44)

\( Z_8 \) transformations of the left-handed \( SU(2)_L \) doublets \( \tilde{D}_{ijL} \), right-handed \( SU(2)_L \) singlets \( l_R \), Higgs doublets \( \phi \), singlet field ‘S’ and scalar singlets \( \lambda \) of all the basic cases for each texture are presented in Table 13.

7. Conclusion

We have considered the MES (3+1) model of active-sterile neutrino oscillations with one eV mass scale sterile neutrino for resolving the anomalous results of various reactor, radio-chemical and accelerator based neutrino experiments. We have discussed briefly the minimal extended seesaw mechanism that imposes the condition on \( M_v^{4 \times 4} \) to be of rank 3 along with both \( M_D \) and \( M_R \) to be invertible. Then we have undertaken the study of texture zeros of the fermion mass matrices that reduces the complexity of the free parameters and leads to the testable correlations among the elements of neutrino mass matrices. Again texture zeros indicate the underlying symmetry of the models. In our work, we have taken 15 phenomenologically viable two-zero textures of \( M_v^{4 \times 4} \) [61] under consideration, of which 12 two-zero textures are found to be of rank 3 for MES realization: \( A_1, A_2, B_3, B_4, C, D_1, D_2, E_1, E_2, F_1, F_2, F_3 \). For realization of these two-zero textures, the predictive case i.e., the sum of zeros of \( M_D \) and \( M_R \) being 8 in the active sector of \( M_v^{4 \times 4} \) have been implemented. In this context, we have considered (5+3) and (6+2) predictive cases of zeros in \( M_D \) and \( M_R \) along with the admissible texture zeros of \( M_S \). In our analysis, we have found that out of 12 viable two-zero textures of rank 3, only 9 textures \( (A_1, A_2, B_3, B_4, D_1, D_2, F_1, F_2, F_3) \) could be realized in the (5+3) scheme while none in (6+2) scheme could be succeeded.

In MES realization of 9 two-zero textures in (5+3) scheme, we have obtained some correlations among the elements, \( m_{ij} \) of \( M_v^{4 \times 4} \). If the scatter plots of the lhs and rhs of each such correlation of a texture against \( \sin \theta_{34} \) with current neutrino data have a reasonable overlapping, we have hypothesised them to be viable models. Accordingly \( m_{ij} \) have been calculated with 3\( \sigma \) range of the current neutrino oscillation data using formulae given in Appendix. The scatter plots have been drawn under two conditions (i) keeping the Dirac and Majorana CP phases unconstrained \((0 – 360^\circ)\) and (ii) constraining the CP phases to certain ranges. It has been observed that there are a number of textures whose viabilities get affected when CP phases are constrained to certain ranges, while for a number of textures the phenomenology remains unchanged when CP phases are constrained to different segments of values or even when CP phases are made to be zero. Accordingly we have classified the textures under two categories (i) CP phase dependent textures and (ii) CP phase independent textures.

In our study we have seen that all 9 textures exhibit reasonable overlappings for some ranges of \( \sin \theta_{34} \) when CP phases are unconstrained. However, for some selective ranges of CP phases, some textures are not allowed within \( \sin \theta_{34} = (0 – 0.4) \). For example, the phenomenology of the texture \( B_3(i) \) is represented by the scatter plots for the correlations as shown in Fig. 1 and 2. It has been observed that CP phases play a fair role in determining the viability of the texture. The textures which exhibit similar phenomenology have been presented in Table 9 with their few ranges of CP phases for which the textures become invalid.
In case of the textures like $D_1(i\nu)$ of which the scatter plots are in (Fig. 3), it has been observed that for unconstrained CP phases, the texture is viable within the complete range of $\sin \theta_{34} = (0 - 0.4)$, while on constraining the CP phases, the allowed range of $\sin \theta_{34}$ have been squeezed to $(0.06 - 0.4)$. The texture has been found to be viable at least for some range of $\sin \theta_{34}$ whatever choice of the ranges of the CP phases. Textures showing similar phenomenology have been listed in Table 10 with their respective allowed range of $\sin \theta_{34}$ under unconstrained and constrained CP phases. The constrained ranges of CP phases for each texture have been presented in Table 11.

It has been observed that the textures: $A_2(i)$ and $B_4(iii)$ remain unaffected whether CP phases are unconstrained or constrained to different ranges. On surveying these textures for different segments of the six CP phases, it has been observed that the correlations are allowed for all values of $\sin \theta_{34}$. These textures are insensitive to variation of CP phases and are therefore categorised as CP phase independent textures. As a representative case realization of texture $B_4(iii)$ have been presented in section 5.2. The CP phase independent nature of the texture has been depicted in the scatter plots in Fig. 6 for unconstrained CP phases and Fig. 7 and Fig. 8 for different ranges of constrained CP phases.

For each of the textures, we have calculated the effective neutrino mass $|m_{\beta\beta}|$ for the considered constrained range of Majorana CP phases. For all the textures with respective constrained range of Majorana CP phases, $|m_{\beta\beta}|$ has been found to lie within experimental constraints. In addition, we have also calculated the Jarlskog invariant $J_{CP}$ for each texture with their corresponding constrained ranges of Dirac CP phases. In the $(3+1)$ scheme, we have observed that, $J_{CP} \neq 0$ for $\delta_{13} = 0$ which differs from the case of three neutrino scenario.

Also, it has been observed that there exist $S_3$ transformations of a given combination of $M_D, M_R, M_S$ leading to a particular two-zero textures of $M^\nu_{\nu} 4 \times 4$ which give the same correlations. This enables us to study only few basic combinations of $M_D, M_R$ and $M_S$ instead of all possible five-zero textures of $M_D$ and three-zero textures of $M_R$. As a representative case Table 5 shows the combinations of $M_D, M_R$ and $M_S$ which are obtained via $S_3$ transformation from the basic combination in Eq. (18) for texture $B_3(i)$. Basic combinations of each of the textures have been listed in Table 8.

The viable textures have been finally realized by means of $Z_8$ Abelian flavor symmetry group. With the charged lepton mass matrix $M_l$ as diagonal and with 5 zeros in $M_D$ we have required three Higgs doublet $\Phi_1, \Phi_2, \Phi_3$, one of which is the SM Higgs ($\Phi_1$) which transforms trivially under $Z_8$. To realize the three-zero texture of $M_R$, one scalar singlet $\chi$ is required. For one-zero texture of $M_S$, two additional scalar singlets ($\lambda_1, \lambda_2$) are needed, while one singlet $\lambda_1$ for the two-zero texture of $M_S$. Also, we have considered $Z_8$ transformation of the singlet chiral field ‘$S$’, in order to prevent bare mass term of ‘$S$’ as demanded by MES mechanism. We have presented the symmetry realization of texture $B_3(i)$ as a representative case. We have also demonstrated the symmetry realization of the Case (b) (Table 5) to show that all the $S_3$ symmetric textures also follow $S_3$ transformations of the fields of the basic combinations.

In this work we have succeeded to obtain viable two-zero textures of $(3+1)$ MES neutrino mass matrices $M^\nu$ in $(5+3)$ predictive scheme of $M_D$ and $M_R$, and one/two zero in $M_S$ and their $Z_8$ symmetry realization. There are further scope for study of these textures from other aspects like unstable sterile state i.e., $(3+1+$ decay) model.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.
Appendix A. Light neutrino mass matrix elements

\[ m_{ee} = c_{12}^2 c_{14}^2 m_1 + e^{-i\alpha} c_{13}^2 c_{14}^2 m_2 s_{12}^2 + e^{-i\beta} c_{14}^2 m_3 s_{13}^2 + e^{-i\gamma} m_4 s_{14}^2, \]
\[ m_{\mu\tau} = e^{i(\delta_{14} - \delta_{23}) - i(\gamma)} c_{14} m_4 s_{14} s_{24} + c_{12} c_{13} c_{14} m_1 (-c_{23} c_{24} s_{12} - e^{i\delta_{13}} c_{12} c_{24} s_{13} s_{23}) - e^{i(\delta_{14} - \delta_{24})} c_{13} c_{14} m_2 s_{12} (c_{12} c_{23} c_{24} - e^{i\delta_{13}} c_{24} s_{13} s_{23}) - e^{i(\delta_{14} - \delta_{24})} c_{13} s_{12} s_{14} s_{24} + e^{-i(\beta + \delta_{24})} (c_{12} c_{23} c_{24} c_{34} s_{13} s_{14} s_{24} + e^{i\delta_{13}} c_{12} c_{23} c_{24} c_{34} s_{12} s_{13} s_{23} - e^{i\delta_{14}} s_{12} s_{13} s_{14} s_{24})), \]
\[ m_{\mu\tau} = e^{-i(\gamma - 2\delta_{14})} c_{14} c_{24} m_4 s_{14} s_{24} + c_{12} c_{13} c_{14} m_1 (-e^{i\delta_{13}} c_{12} c_{23} c_{34} s_{13} s_{14} + c_{34} s_{12} s_{23}) - e^{i\delta_{14}} c_{12} c_{23} c_{24} c_{34} s_{14} + e^{i\delta_{24}} (c_{23} s_{12} + e^{i\delta_{13}} c_{12} s_{13} s_{23} s_{24} s_{34}) - e^{-i\alpha} c_{13} c_{14} m_2 s_{12} (e^{i\delta_{13}} c_{23} c_{34} s_{12} s_{13} + c_{12} c_{34} s_{23}) + e^{i\delta_{13}} c_{12} c_{24} s_{14} s_{24} + e^{i\delta_{24}} (c_{12} c_{23} - e^{i\delta_{13}} s_{12} s_{13} s_{23} s_{24} s_{34}) + e^{-i\beta} c_{14} m_3 s_{13} \times (-e^{i\delta_{14}} c_{24} s_{13} s_{14} s_{24} + e^{i\delta_{13}} c_{13} (c_{23} c_{34} - e^{i\delta_{24}} s_{23} s_{24} s_{34})), \]
\[ m_{ee} = e^{-i(\gamma - 2\delta_{14})} - e^{i(\gamma - 2\delta_{14})} c_{14} c_{24} c_{34} m_4 s_{14} + e^{-i\beta} c_{14} m_3 s_{13} (-e^{i\delta_{14}} c_{24} s_{14} s_{13} s_{14} - e^{i\delta_{13}} c_{12} c_{23} c_{34} s_{14} + e^{i\delta_{24}} c_{12} c_{24} c_{34} s_{24} + c_{12} c_{34} s_{23} + c_{12} c_{23} c_{24} c_{34} s_{23} s_{24} s_{34} - e^{i\delta_{14}} c_{12} c_{24} c_{34} s_{13} s_{14} s_{24})), \]
\[ m_{\mu\tau} = e^{-i(\gamma - 2\delta_{14})} - e^{i(\gamma - 2\delta_{14})} c_{14} c_{24} c_{34} m_4 s_{14} + e^{-i\beta} c_{14} m_3 s_{13} (-e^{i\delta_{14}} c_{24} s_{14} s_{13} s_{14} - e^{i\delta_{13}} c_{12} c_{23} c_{34} s_{14} + e^{i\delta_{24}} c_{12} c_{24} c_{34} s_{24} + c_{12} c_{34} s_{23} + c_{12} c_{23} c_{24} c_{34} s_{23} s_{24} s_{34} - e^{i\delta_{14}} c_{12} c_{24} c_{34} s_{13} s_{14} s_{24})), \]
\[ m_{\mu\tau} = e^{-i(\gamma - 2\delta_{14})} - e^{i(\gamma - 2\delta_{14})} c_{14} c_{24} c_{34} m_4 s_{14} + e^{-i\beta} c_{14} m_3 s_{13} (-e^{i\delta_{14}} c_{24} s_{14} s_{13} s_{14} - e^{i\delta_{13}} c_{12} c_{23} c_{34} s_{14} + e^{i\delta_{24}} c_{12} c_{24} c_{34} s_{24} + c_{12} c_{34} s_{23} + c_{12} c_{23} c_{24} c_{34} s_{23} s_{24} s_{34} - e^{i\delta_{14}} c_{12} c_{24} c_{34} s_{13} s_{14} s_{24})), \]
\[ m_{\mu\tau} = e^{-i(\gamma - 2\delta_{14})} - e^{i(\gamma - 2\delta_{14})} c_{14} c_{24} c_{34} m_4 s_{14} + e^{-i\beta} c_{14} m_3 s_{13} (-e^{i\delta_{14}} c_{24} s_{14} s_{13} s_{14} - e^{i\delta_{13}} c_{12} c_{23} c_{34} s_{14} + e^{i\delta_{24}} c_{12} c_{24} c_{34} s_{24} + c_{12} c_{34} s_{23} + c_{12} c_{23} c_{24} c_{34} s_{23} s_{24} s_{34} - e^{i\delta_{14}} c_{12} c_{24} c_{34} s_{13} s_{14} s_{24})), \]
\[ m_{\tau\tau} = e^{-i(\gamma - 2\delta_{14})} c_{14}^2 c_{24} \delta_{34}^2 + m_1 (-e^{i\delta_{14}} c_{12} c_{23} c_{34} s_{14} + c_{34} s_{12} s_{23} - e^{i\delta_{14}} c_{12} c_{13} c_{24} s_{14} s_{34} + e^{i\delta_{24}} c_{12} s_{12} + e^{i\delta_{13}} c_{12} s_{13} s_{23} s_{24} s_{34})^2 + e^{-i\alpha} m_2 (e^{i\delta_{13}} c_{12} c_{34} s_{12} s_{13} + c_{12} c_{34} s_{23} + e^{i\delta_{14}} c_{13} c_{24} s_{14} s_{34} + e^{i\delta_{14}} c_{12} s_{13} s_{23} s_{24} s_{34})^2 + e^{-i\beta} m_3 \]

\[ m_{\tau s} = e^{-i(\gamma - 2\delta_{14})} c_{14}^2 c_{24} c_{34} s_{14} s_{34} + m_1 (-e^{i\delta_{14}} c_{12} c_{23} c_{34} s_{13} + c_{34} s_{12} s_{23} - e^{i\delta_{24}} c_{12} s_{12} + e^{i\delta_{13}} c_{12} s_{13} s_{23} s_{24} s_{34} + e^{i\delta_{12}} c_{13} c_{24} s_{14} s_{34} + e^{i\delta_{24}} c_{12} s_{13} s_{23} s_{24} s_{34})^2 + e^{-i\alpha} m_2 (e^{i\delta_{13}} c_{12} c_{34} s_{12} s_{13} + c_{12} c_{34} s_{23} + e^{i\delta_{14}} c_{13} c_{24} s_{14} s_{34} + e^{i\delta_{12}} c_{13} c_{24} c_{34} s_{13} s_{23} s_{24} s_{34})^2 + e^{-i\beta} m_3 (e^{i\delta_{13}} c_{13} c_{24} s_{13} s_{34} + e^{i\delta_{14}} c_{13} c_{14} s_{23} s_{24} s_{34})^2. \]

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