An Extension of Parikh’s Theorem beyond Idempotence

Michael Luttenberger and Maxmilian Schlund
{luttenbe,schlund}@model.in.tum.de

Tuesday 27th November, 2012

Abstract

The commutative ambiguity $\text{camb}_{G,X}$ of a context-free grammar $G$ with start symbol $X$ assigns to each Parikh vector $v$ the number of distinct leftmost derivations yielding a word with Parikh vector $v$. Based on the results on the generalization of Newton’s method to $\omega$-continuous semirings [EKL07b, EKL07a, EKL10], we show how to approximate $\text{camb}_{G,X}$ by means of rational formal power series, and give a lower bound on the convergence speed of these approximations. From the latter result we deduce that $\text{camb}_{G,X}$ itself is rational modulo the generalized idempotence identity $k = k + 1$ (for $k$ some positive integer), and, subsequently, that it can be represented as a weighted sum of linear sets. This extends Parikh’s well-known result that the commutative image of context-free languages is semilinear ($k = 1$).

Based on the well-known relationship between context-free grammars and algebraic systems over semirings [CS63, SS78, BR82, Kui97, Boz99], our results extend the work by Green et al. [GKT07] on the computation of the provenance of Datalog queries over commutative $\omega$-continuous semirings.

1 Introduction

Motivation Recently, Green et al. showed in [GKT07] that several questions regarding the provenance of an answer to a Datalog query \footnote{See e.g. [CGT89] for more details on Datalog.} reduce to computing the least solution of an algebraic system over a $\omega$-continuous commutative semiring. To illustrate the main idea, consider the following Datalog program that computes the transitive closure of a finite directed graph $G = (V, E)$:

\begin{align*}
\text{trans}(X, Y) : & \leftarrow \text{edge}(X, Y). \\
\text{trans}(X, Y) : & \leftarrow \text{trans}(X, Z), \text{trans}(Z, Y).
\end{align*}

Here, $X, Y, Z$ are variables ranging over the nodes $V$ of the graph, the interpretation of the (extensional) predicate $\text{edge}(X, Y)$ is given by the edge relation $E$ of $G$, while the interpretation of the (intensional) predicate $\text{trans}(X, Y)$ is implicitly given by the least Herbrand model, i.e. the transitive closure of $G$. In order to deduce which edges of $G$ give rise to a positive answer to the query $\text{?} \leftarrow \text{trans}(u, v)$, in [GKT07] the authors assign to each positive literal a unique identifier

\footnote{{This work was partially funded by DFG project “Polynomial Systeme über Semiringen: Grundlagen, Algorithmen, Anwendungen”}

\footnote{Institut für Informatik, Technische Universität München}
- for instance, let \( A = \{ e_{u,v} \mid (u, v) \in E \} \) and \( X = \{ X_{u,v} \mid u, v \in V \} \) – and then expands the above query into an abstract algebraic system in the formal parameters \( A \) and the variables \( X \):

\[
X_{u,w} = \begin{cases} 
  e_{u,w} + \sum_{v \in V} \frac{X_{u,v}X_{v,w}}{X_{u,v}} & \text{if } (u, w) \in E \\
  \text{otherwise}
\end{cases}
\]

In order to give a meaning to this system, the right-hand side is interpreted over some semiring \((S, +, \cdot, 0, 1)\), short \( S \), i.e. the abstract addition and multiplication are interpreted as the addition and multiplication in \( S \), and each formal parameter \( a \in A \) is interpreted as an element \( h(a) \in S \) by means of a valuation \( h : A \rightarrow S \). As is well-known [Ku97], each algebraic system has a least solution if \( S \) is \( \omega \)-continuous (see Section 2).

We demonstrate the connection between the Datalog program and the algebraic system by means of two examples. First, the transitive closure itself is essentially the least solution over the Boolean semiring \((\{0,1\}, \lor, \land, 0, 1)\) under the valuation \( h(e_{u,w}) = 1 \) for all \( e_{u,w} \in A \), i.e. the least solution assigns 1 to \( X_{u,w} \) if and only if \((u, w)\) is in the transitive closure. For a somewhat more interesting example, assume we want to analyze why an edge \((u, w)\) is included in the transitive closure. To this end, it suffices to represent a path by the set of its edges, and a set of paths by the set of corresponding sets of edges. This leads naturally to the semiring \((2^\mathbb{N}, \lor, \emptyset, \emptyset, \{\}\})\: a semiring element is a set of subsets of edge identifiers, two semiring elements \( s_1, s_2 \) are added by taking their union \( s_1 \lor s_2 \), while the (commutative) multiplication is defined by \( s_1 \lor s_2 = \{ a_1 \lor a_2 \mid a_1 \in s_1, a_2 \in s_2 \} \). Again, we obtain the answer to our question by computing the least solution of above system over this semiring under the valuation \( h(e_{u,w}) = \{ e_{u,w} \} \). For further examples, we refer the reader to [GKT07].

Note that in both examples, multiplication is commutative, and addition is idempotent. Naturally, the question arises over which commutative \( \omega \)-continuous semirings we can compute or, at least, approximate the least solution of an algebraic system. Of particular interest is the semiring of formal power series whose carrier is the set \( \mathbb{N}_\infty \langle \mathbb{N}^A \rangle \) of functions from Parikh vectors \( \mathbb{N}^A \) to the extended natural numbers \( \mathbb{N}_\infty = \mathbb{N} \cup \{ \infty \} \), as it is free in the following sense: every valuation \( h : A \rightarrow S \) into a concrete commutative \( \omega \)-continuous semiring induces a unique \( \omega \)-continuous homomorphism \( H : \mathbb{N}_\infty \langle \mathbb{N}^A \rangle \rightarrow S \) which maps the least solution over \( \mathbb{N}_\infty \langle \mathbb{N}^A \rangle \) to the least solution over \( S \) (we do not distinguish between \( h \) and \( H \) in the following). See e.g. [Boz99] [GKT07].

In general, a finite, explicit representation of the least solution \( s_X \mid X \in \mathcal{X} \) over \( \mathbb{N}_\infty \langle \mathbb{N}^A \rangle \) is not possible (see also Example 3.5). In [GKT07] the authors therefore present two algorithms All-Trees and Monomial-Coefficient for computing finitely representable information on this solution: All-Trees decides whether \( s_X : \mathbb{N}^A \rightarrow \mathbb{N}_\infty \) has only finite support and takes only finite values on its support, and can be used to evaluate Datalog over finite distributive lattices, a special case of commutative \( \omega \)-continuous semirings; Monomial-Coefficient computes the value of \( s_X \) for some Parikh vector \( v \in \mathbb{N}^A \). Both algorithms are based on the close relationship between algebraic systems and context-free grammars [CS63] [SS78] [Ku97] [ABB97] [Th67] [BR82] [Boz99] [EK07b] [EK07a] [EK08], and work by enumerating the derivation trees of the grammar associated with the algebraic system utilizing the pumping lemma for context-free languages in order to ensure termination. The associated context-free grammar \( G = (X, A, P) \) with terminals \( X \), alphabet \( A \), and productions \( P \) is obtained from the algebraic system by reinterpreting the right-hand sides of the algebraic system as rewriting rules for the variables. For instance, the algebraic system for computing the transitive closure translates to the grammar \( G \) defined by the rules

\[
X_{u,w} \rightarrow X_{u,v}X_{v,w} \quad \text{for all } u, v, w \in V, \quad \text{and } X_{u,w} \rightarrow e_{u,w} \quad \text{for all } (u, w) \in E.
\]
W.r.t. commutative $\omega$-continuous semirings, the grammar $G$ and the algebraic system are then connected by means of the commutative ambiguity $\text{camb}_{G,X} : \mathbb{N}^A \to \mathbb{N}^{\infty}$ which assigns to each Parikh vector $v \in \mathbb{N}^A$ the number of leftmost derivations w.r.t. $G$ with start symbol $X$ leading to a word with Parikh vector $v$: we have that $s_X = \text{camb}_{G,X}$ for all $X \in \mathcal{X}$, or short $s = \text{camb}_G$. See e.g. [CS63, Boz99, EKL07b].

**Contribution and related work** In this article, we study how to construct from a given context-free grammar a sequence $G[0], G[1], \ldots$ of nonexpansive context-free grammars $G[i]_X \leq \text{amb}_{G[i],X} (w)$ for all $w \in A^*$, Lemma 5.2, and, thus, also the commutative ambiguity. As $G[i]$ is nonexpansive, it is straightforward to show that $\text{camb}_{G[i],X}$ is rational in $\mathbb{N}^{\infty} (\mathbb{N}^A)_0$, and a rational expression representing $\text{camb}_{G[i],X}$ can easily be computed from $G[k]$. We then give a lower bound on the speed at which $\text{camb}_{G[i],X}$ converges to $\text{camb}_G$: letting $n$ be the number of variables of $G$, we show that for every positive integer $k$ and every $v \in \mathbb{N}^A$, if $\text{camb}_{G[n],X}(v) \neq \text{camb}_{G,X}(v)$, then at least $k \leq \text{camb}_{G[n],X}(v)$ (Theorem 4.2).

An immediate consequence of these results is an algorithm for evaluating Datalog queries over “collapsed” commutative semirings: call a $\omega$-continuous semiring $S$ collapsed at some positive integer $k$ if in $S$ the identity $k = k + 1$ holds I given a valuation $h : A \to S$ into a commutative $\omega$-continuous semiring collapsed at $k$, the least solution can be obtained by evaluating the corresponding rational expressions for $\text{camb}_{G[n],X}$ under the homomorphism induced by $h$.

In particular, this yields an algorithm for evaluating Datalog queries over the tropical semiring $(\mathbb{N}^{\infty}, \min, +, 0, \infty)$; this answers an open question of [GKT07]. We remark that in [EKL08] more efficient algorithms for the classes of star-distributive semirings, subsuming the tropical semiring, and of one-bounded semirings, subsuming finite distributive lattices, are presented.

Finally, we show that $\text{camb}_{G,X}$ can be represented modulo $k = k + 1$ as a finite sum $\gamma_1 1_{C_1} + \ldots + \gamma_k 1_{C_k}$ of weighted characteristic functions $1_C$ of linear sets $C \subseteq \mathbb{N}^A$ with weights $\gamma_i \in \{0, 1, \ldots, k\}$ (Theorem 5.2). This completes the extension of Parikh’s well-known theorem that the commutative image of a context-free grammar is a semilinear set ($k = 1$).

These results continue the study of Newton’s method over $\omega$-continuous semirings presented in [EKL07b, EKL07a, EKL10]. There it was shown that Newton’s method, as known from calculus, also applies to the setting of algebraic systems over $\omega$-continuous semirings, and converges always to the least solution at least as fast as (and many times much faster than) the standard fixed-point iteration. Although it is shown in [EKL07a, EKL10] that Newton’s method is well-defined on any $\omega$-continuous semiring, the definition does not yield an effective way of applying Newton’s method as it requires the user to supply at each iteration a semiring element which represents a certain difference. Only for special cases it is stated how to compute those differences, but a general construction is missing in these articles.

The grammars $G[i]$ defined in Definition 3.1 address this shortcoming. Their construction is based on the notion of “tree dimension” introduced in [EKL07b] to characterize the structure of terms evaluated by Newton’s method, where it was shown that the $k$-th Newton approximation of

---

2A context-free grammar is nonexpansive if every variable $X$ derives only sentential forms containing $X$ at most once [CS63].
3Where $k$ denotes the term $1 + \ldots + 1$ consisting of the corresponding number of 1s. For instance, any $\omega$-continuous idempotent semiring is “collapsed” at 1. See also [BE09] for a much more general discussion of these semirings.
4$C \subseteq \mathbb{N}^A$ is linear if $C = \{v_0 + \sum_{i=1}^s \lambda_i v_i \mid \lambda_1, \ldots, \lambda_s \in \mathbb{N}\}$ for vectors $v_0, \ldots, v_s \in \mathbb{N}^A$. 

3
the least solution of an algebraic system corresponds exactly to the derivation trees of dimension at most $k$ generated by the context-free grammar associated with the system. This allows us to explicitly define a grammar, resp. equation system, which captures exactly the update computed by Newton’s method within a single step. That is, we may define the difference of two consecutive Newton approximations over any $\omega$-continuous semiring by constructing a grammar which generates exactly the derivation trees of $G$ of dimension exactly $k$. By taking the sum of all these updates, we obtain the grammar, $G^{[k]}$ which generates exactly the derivation trees of $G$ of dimension at most $k$. Hence, if the least solution of (the equation system associated with) $G^{[k-1]}$ is known, we only need to solve the equation system corresponding to the derivation trees of dimension exactly $k$. We remark that this construction does not require multiplication to be commutative; it is merely a partition of the regular tree language of derivation trees of $G$.

If multiplication is commutative, $\text{camb}_{G^{[k]}}$ represents the $k$-th Newton approximation over any commutative $\omega$-continuous semiring. Similarly, the bound on the speed at which $\text{camb}_{G^{[k]}}$ converges to $\text{camb}_G$ given in Theorem 1.2 generalizes the result of [EKŁ07b] on the convergence of Newton’s method over idempotent commutative $\omega$-continuous semirings.

If multiplication is not commutative, we may not represent the least solution of $G^{[k]}$ as regular expressions, but only as regular tree expressions with the particular property that tree substitution only occurs at a unique leaf. It might be worthwhile to study if there are interesting (distributive) abstract interpretations whose widening operator can take advantage of this representation.

Structure of the paper In Section 2 we recall the most fundamental definitions, in particular the definition of the dimension of a tree. We then show in Section 3 how to unfold a given context-free grammar $G$ into a new context-free grammar $G^{[k]}$ that generates exactly those derivation trees of $G$ that are of dimension at most $k$ and, thus, represents exactly the $k$-th Newton approximation. We show that the commutative ambiguity of each grammar $G^{[k]}$ is rational over $\mathbb{N}_\infty(\langle \mathbb{N}^A \rangle)$. In Section 4 we give a lower bound on the speed at which the ambiguity of $G^{[k]}$ converges to that of $G$. We use this result in Section 5 to obtain from a rational expression for $\text{camb}_{G^{[k]}}$ a semilinear representation of $\text{camb}_G$ modulo the generalized idempotence assumption of $k = k + 1$, thereby completing the extension of Parikh’s theorem from $k = 1$ to arbitrary $k$.

All proofs can be found in the appendix.

2 Preliminaries

The power set of a set $M$ is denoted by $2^M$. For $k \in \mathbb{N}$, set $[k] := \{1, 2, \ldots, k\}$ with $[0] = \emptyset$. The natural numbers extended by a greatest element $\infty$, and the natural numbers “collapsed” at a given positive integer $k$ are denoted by $\mathbb{N}_\infty$, and $\mathbb{N}_k = \{0, 1, \ldots, k\}$, respectively. For $a \in \mathbb{N}_\infty$, set $a + \infty = \infty$, $\emptyset \cdot \infty = 0$, and $a \cdot \infty = \infty$ if $a \neq 0$. Addition and multiplication are defined on $\mathbb{N}_k$ by identifying $k$ with $\infty$.

The set of words over the (finite) alphabet $A$ is denoted by $A^*$ with $\varepsilon = (\emptyset)$ the empty word. The length of a word $w \in A^*$ is denoted by $|w|$. The Parikh map is $c : A^* \to \mathbb{N}^A : w \mapsto (c_a(w) | a \in A)$ where $c_a(w)$ denotes the number of occurrences of $a$ in $w$.

Let $\Sigma$ be finite ranked set (signature) where $\Sigma_r$ denotes the subset of $\Sigma$ consisting of exactly those symbols having arity $r$. Then $T_\Sigma$ denotes the set of $\Sigma$-terms where we use Polish notation so that $T_\Sigma \subseteq \Sigma^*$. When $t \in T_\Sigma$, we denote by $t = \sigma t_1 \ldots t_r$ that $\sigma \in \Sigma_r$ and $t_1, \ldots, t_r \in T_\Sigma$ are
the uniquely determined subterms; for inductive definitions, we set \( t = \sigma t_1 \ldots t_r = \sigma \) if \( r = 0 \). \( T_\Sigma \) is canonically identified with the set of finite, \( \Sigma \)-labeled, rooted trees: the rooted tree underlying \( t = \sigma t_1 \ldots t_r \) has as nodes the set \( V_t = \{ \varepsilon \} \cup \{ i\pi \mid i \in [r], \pi \in V_{t_i} \} \) with \( \varepsilon \) the root, and the edges \( E_t := \{(\pi, \pi i) \mid \pi i \in V_t \} \) pointing away from the root. The label \( \text{lbl}_t(i) \) of a node in \( V_t \) is then defined inductively by \( \text{lbl}_t(\varepsilon) = \sigma \) and \( \text{lbl}_t(i\pi) = \text{lbl}_t(\pi) \) for \( t = \sigma t_1 \ldots t_r \). The height \( \text{hgt}(t) \) of a tree \( t = \sigma t_1 \ldots t_r \) is defined to be 0 if \( r = 0 \), and otherwise by \( \text{hgt}(t) = \max_{i \in [r]} \text{hgt}(t_i) \).

Analogously, define the subtree \( t|_\pi \) of \( t \) rooted at \( \pi \), and the tree \( t|_{t/\pi} \) obtained by substituting the tree \( t' \) for \( t|_\pi \) inside of \( t \).

**Definition 2.1.**

The dimension \( \text{dim}(t) \) of \( t = \sigma t_1 \ldots t_r \in T_\Sigma \) is defined to be \( \text{dim}(t) = 0 \) if \( r = 0 \); otherwise let \( d = \max_{i \in [r]} \text{dim}(t_i) \), and set \( \text{dim}(t) = d \) if there is a unique child \( i \in [r] \) of dimension \( d \), else set \( \text{dim}(t) = d + 1 \).

From the definition it easily follows that \( \text{dim}(t) \) is the height of the greatest perfect binary tree that can be obtained from the rooted tree \((V_t, E_t)\) via edge contractions. Thus, \( \text{dim}(t) \) is bounded from above by \( \text{hgt}(t) \).

**Example 2.2.**

Assume \( \Sigma = \{a, b\} \) with \( a \in \Sigma_2 \) and \( b \in \Sigma_0 \). Then \( aabbaabb \in T_\Sigma \) is identified with the tree:

```
  \varepsilon: a
    1: a
  11: b  12: b
    21: a  22: b
  211: b  212: b
```

For instance, the node 212 is labeled by \( b \). Computing the dimension bottom-up, we obtain \( \text{dim}(t|_{211}) = 1, \text{dim}(t|_{212}) = 1, \text{dim}(t|_{11}) = 1, \) and \( \text{dim}(t) = 2 \).

The tree dimension \( \text{dim}(t) \) is also known as Horton-Strahler number [Hor45, Sh52], or the register number \( \text{rs58} \) [FFV79, DK95], and is closely related to the pathwidth \( \text{pw}(T) \) of the tree \( T = (V_t, E_t) \) underlying \( t \): it can be shown that \( \text{pw}(T) - 1 \leq \text{dim}(t) \leq 2\text{pw}(T) + 1 \).

**Semirings** We recall the basic results on semirings (see e.g. to [Knu97, DK09]). A semiring \((S, +, \cdot, 0, 1)\) consists of a commutative additive monoid \((S, +, 0)\) and a multiplicative monoid \((S, \cdot, 1)\) where multiplication distributes over addition from both left and right, and multiplication by 0 always evaluates to 0. We simply write \( S \) for \((S, +, \cdot, 0, 1)\) if the signature is clear from the context. \( S \) is commutative if its multiplication is commutative. \( S \) is naturally ordered if the relation \( a \sqsubseteq b \) defined by \( a \sqsubseteq b :\Leftrightarrow \exists d \in S: a + d = b \) is a partial order on \( S \); then 0 is the least element.

A partial order \((P, \leq)\) is \( \omega \)-continuous if for every monotonically increasing sequence \((\omega \text{-chain}) \) \((a_i)_{i \in \mathbb{N}}, i.e. a_i \leq a_{i+1} \) for all \( i \in \mathbb{N} \), the supremum \( \sup_{i \in \mathbb{N}} a_i \) exists in \( (P, \leq) \); a function \( f: (P, \leq) \to (P, \leq) \) is called \( \omega \)-continuous if for every \( \omega \)-chain \((a_i)_{i \in \mathbb{N}} \) we have \( f(\sup_{i \in \mathbb{N}} a_i) = \sup_{i \in \mathbb{N}} f(a_i) \).

We say that \( S \) is \( \omega \)-continuous if \((S, \sqsubseteq)\) is \( \omega \)-continuous, and addition and multiplication are both \( \omega \)-continuous in every argument. In any \( \omega \)-continuous semiring finite summation \( \sum \) can
be extended to countable sequences and families by means of \( \sum_{i \in \mathbb{N}} a_i := \sup_{k \in \mathbb{N}} \sum_{i \in [k]} a_i \). The Kleene star \( *: S \to S \) is defined by \( a^* := \sum_{i \in \mathbb{N}} a^i \).

If not stated otherwise, we always assume that \( \mathbb{N}_\infty \) carries the semiring structure \((\mathbb{N}_\infty, +, 0, 1)\) with addition and multiplication as stated above so that \( 1^* = \infty \). For any \( \omega \)-continuous semiring \( S \) there is exactly one \( \omega \)-continuous homomorphism \( h \) from \( \mathbb{N}_\infty \) to \( S \) as \( h(0) = 0, h(1) = 1 \), and \( h(\infty) = h(1^*) = 1^* \) have to hold; we therefore embed \( \mathbb{N}_\infty \) into \( S \) by means of this unique homomorphism.

For a commutative semiring \((S, +, 0, 1)\), and a finitely decomposable\(^5\) monoid \((M, \circ, e)\) we recall the definition of the semiring \( S\langle M \rangle \) of formal power series. Its carrier is the set of total functions from \( M \) to \( S \). For \( s \in S\langle M \rangle \) denote by \((s, m)\) the value of \( s \) at \( m \in M \). Then addition on \( S \) is extended pointwise to \( S\langle M \rangle \), while multiplication is defined by means of the generalized Cauchy product, i.e.:

\[
(s + t, m) = (s, m) + (t, m) \quad \text{and} \quad (s \cdot t, m) = \sum_{u, v \in M: u \cdot v = m} (s, u) \cdot (t, v).
\]

That is, we treat \( s \in S\langle M \rangle \) as a (formal) power series \( \sum_{m \in M} (s, m) m \) with \((s, m)\) the coefficient of the monomial \( m \). If the support \( \text{supp}(s) = \{ m \in M \mid (s, m) \neq 0 \} \) is finite, \( s \) is called a (formal) polynomial. The subset of polynomials is denoted by \( S\langle M \rangle \). The semiring \( S \) and the monoid \( M \) are canonically embedded into \( S\langle M \rangle \) by means of the monomorphisms \( h_S: S \to S\langle M \rangle; s \mapsto s e \) and \( h_M: M \to S\langle M \rangle; m \mapsto 1 m \), respectively. W.r.t. these definitions \( S\langle M \rangle \) and \( S\langle M \rangle \) become semirings with neutral elements \( 0 = h_M(0) = 1 = h_M(1) = h_M(e) \); if \( S \) is \( \omega \)-continuous, then so is \( S\langle M \rangle \), and the Kleene star is defined everywhere on \( S\langle M \rangle \). For instance, \( S\langle M \rangle \) is \( \omega \)-continuous for \( S \) either \( \mathbb{N}_\infty \) or \( \mathbb{N}_k \), and \( M \) either \( \mathbb{A}^* \) or \( \mathbb{N}^A \); but \( \mathbb{N}\langle \mathbb{A}^* \rangle \) and \( \mathbb{N}\langle \mathbb{N}^A \rangle \) are not. Not that \( \mathbb{N}_\infty\langle \mathbb{A}^* \rangle \) is free in the following sense: let \( \langle S, +, \cdot, 0_S, 1_S \rangle \) be some \( \omega \)-continuous semiring; then every valuation \( h: \mathbb{A} \to S \) extends uniquely to a \( \omega \)-continuous homomorphism \( h: \mathbb{N}_\infty\langle \mathbb{A}^* \rangle \to S \) defined by \( h(a) = \sum_{x \in \mathbb{A}} (s, a) h(a) \). Similarly, \( \mathbb{N}_\infty\langle \mathbb{N}^A \rangle \) is a representation of the free commutative \( \omega \)-continuous semiring generated by \( \mathbb{A} \), and, thus, isomorphic to \( \mathbb{N}_\infty\langle \mathbb{A}^* \rangle \) modulo commutativity.

Let \( S \) be commutative and \( \omega \)-continuous so that the Kleene star is defined for every power series in \( S\langle M \rangle \). A power series \( s \in S\langle M \rangle \) is called rational, if it can be constructed from the elements of \( S \) and \( M \) by means of the rational operations addition, multiplication, and Kleene star, i.e. if either \( r \in S \), or \( r \in M \), or \( r = (r_1 + r_2) \), or \( r = r_1 \cdot r_2 \), or \( r = r_1^* \) for \( r_1, r_2 \) rational in \( S\langle M \rangle \). A rational expression (over \( M \) with weights in \( S \)) is any term constructed from elements of \( S \) and \( M \), and the rational operations. For every rational series \( r \) in \( S\langle M \rangle \) there is a rational expression \( \rho \) which evaluates to \( r \) over \( S\langle M \rangle \). By our assumption that \( S \) is \( \omega \)-continuous, also every rational expression evaluates to a rational series \( r \) over \( S\langle M \rangle \). Note that \( \omega \)-continuous homomorphisms preserve rationality.

**Context-free grammars** A context-free grammar \( G = (X, A, P) \) consists of variables \( X \), an alphabet \( A \), and rules \( P \subseteq X \times (A \cup X)^* \). By \( (G, X) \) we denote the grammar \( G \) with start symbol \( X \in X \). For a rule \( (P, \gamma) \in P \) we also write \( X \rightarrow_G \gamma \) or simply \( X \rightarrow \gamma \) if \( G \) is apparent from the context. \( \Rightarrow_G \) denotes the binary relation on \((A \cup X)^* \) induced by the rules \( P \), i.e., if \( X \rightarrow_G w \), then \( a X \beta \Rightarrow_G aw \beta \) for all \( a, \beta \in (A \cup X)^* \). The (reflexive) transitive closure of \( \Rightarrow_G \) is denoted by \( \Rightarrow_G^* \). The language generated by \( (G, X) \) is \( L(G, X) = \{ w \in A^* \mid X \Rightarrow_G^* w \} \).

---

\(^5\)A monoid \((M, \circ, e)\) is finitely decomposable if for every \( m \in M \) there exists only finitely many pairs \((u, v) \in M^2 \) that \( u \circ v = m \). This ensures that the Cauchy product is also well-defined over semirings \( S \) which are not \( \omega \)-continuous.
Let $\Sigma_G$ denote the set $\{\sigma_{X,\gamma} \mid X \rightarrow_G \gamma\}$ and define the arity of $\sigma_{X,\gamma}$ to be the number of variables occurring in $\gamma$. Define the new context-free grammar $G_T$ with alphabet $\Sigma_G$ by setting $X \rightarrow G_T \sigma_{X,\gamma} X_1 \ldots X_r$ for $\gamma = \gamma_0 X_1 \gamma_1 \ldots \gamma_{r-1} X_r \gamma_r$. Then $T_{G,X} := L(G_T, X) \subseteq T_{\Sigma_G}$ is called the set of $(G, X)$-trees (or simply $X$-trees if $G$ is apparent from the context) and $T_{G,X}$ “yields” $L(G, X)$ in the sense of [Tha67, BR82, Boz99, EKL07]. The word represented by a tree $t \in T_{G,X}$ is called its yield $Y(t)$ and is inductively defined by $Y(t) = u_0 Y(t_1) u_1 \ldots u_{r-1} Y(t_r) u_r$ for $t = \sigma_{X,\gamma} t_1 \ldots t_r$ and $\gamma = u_0 X_1 u_1 \ldots u_{r-1} X_r u_r$. We then have $L(G, X) = \{Y(t) \mid t \in T_{G,X}\}$, and

$$amb_{G,X}(w) = |\{t \in T_{G,X} \mid Y(t) = w\}|$$

and

$$camb_{G,X}(v) = |\{t \in T_{G,X} \mid c(Y(t)) = v\}|$$

where $amb_{G,X} \in N_\infty \langle \langle A^* \rangle \rangle$, $camb_{G,X} \in N_\infty \langle \langle N^A \rangle \rangle$, and $L(G, X) = \text{supp}(amb_{G,X}) \in N_1 \langle \langle A^* \rangle \rangle$.

The dimension of a derivation tree is closely related to the index of a derivation.

**Definition 2.3** (see e.g. [GS68]).
The index of a derivation is the maximal number of variables occurring in any sentential form of the derivation.

**Definition 2.4.**
For $G$ a context-free grammar and $t \in T_{\Sigma_G}$, let $\minidx(t)$ be the minimum index taken over all derivations associated with $t$.

**Lemma 2.5** ([EKL07], [EKL07b]).
Let $G$ be a context-free grammar and $r_{\text{max}}$ the maximal arity of a symbol in $\Sigma_G$. Then: $\dim(t) < \minidx(t) \leq \dim(t) \cdot (r_{\text{max}} - 1) + 1$.

**Example 2.6.**
Consider $G$ defined by the productions:

$$X \rightarrow YaYaY \quad Y \rightarrow X \quad Y \rightarrow b.$$

Then $\Sigma_G = \{\sigma_{X,XX}, \sigma_{X,Y}, \sigma_{Y,a}\}$. The leftmost derivation

$$X \Rightarrow YaYaY \Rightarrow XaYaY \Rightarrow YaYaYaYaY \Rightarrow^+ bababab$$

has index 5, and corresponds to the derivation tree

$$t = \sigma_{X,YaYaY} \sigma_{Y,X} \sigma_{X,Y,YaYaY} \sigma_{Y,b} \sigma_{Y,b} \sigma_{Y,b} \sigma_{Y,b} \sigma_{Y,b}$$

depicted as

$$\varepsilon: \sigma_{X,YaYaY}$$

$$1: \sigma_{Y,X}$$

$$2: \sigma_{Y,b}$$

$$3: \sigma_{Y,b}$$

$$11: \sigma_{X,YaYaY}$$

$$111: \sigma_{Y,b}$$

$$112: \sigma_{Y,b}$$

$$113: \sigma_{Y,b}$$

This tree has dimension 1. A derivation of minimal index first processes the subtree $t|_2$ and $t|_3$ leading to an index of 3.
3 Unfolding

In this section, we describe how to unfold a given context-free grammar $G = (\Sigma, A, P)$ into a new context-free grammar $G^{[k]}$ which generates exactly the trees of dimension at most $k$ (Definition 3.1 and Lemma 3.2). Hence, $\text{amb}_{G^{[k]}} \leq \text{amb}_{G}$. By construction, $G^{[k]}$ is nonexpansive, i.e. every variable $X$ can only be derived into sentential forms in which $X$ occurs at most once [GS68, Ynt67]. From this, it easily follows that the commutative ambiguity $\text{camb}_{G^{[k]}}$ is a rational power series in $N_{\infty}([N^k])$ (Lemma ??).

We first give an informal description of the notation used in the definition of $G^{[k]}$: given the bound $k$ on the maximal dimension we split every variable $X \in \Sigma$ of $G$ into the variables $X^{(d)}$ and $X^{[d]}$, where $d \in \{0, 1, \ldots, k\}$, with the intended meaning that $X^{(d)}$ resp. $X^{[d]}$ generates all $G_X$-trees of dimension exactly resp. at most $d$; a variable $X^{[d]}$ can only be rewritten to $X^{(d')}^\prime$ for some $d' \leq d$, i.e. nondeterministically the dimension of the tree to be generated from $X^{[d]}$ has to be chosen; the rules rewriting the variable $X^{(d)}$ are derived from the rules $X \to \gamma$ by replacing each variable $Y$ occurring in $\gamma$ by either $Y^{(d')}$ or $Y^{[d']}$ for some $d' \leq d$ in such a way that, inductively, it is guaranteed that every $X$-tree of dimension exactly $d$ is generated exactly once. In particular, as for each $X$-tree $t = \sigma t_1 \ldots t_r$ there is at most one $i \in [r]$ with $\dim(t) = \dim(t_i)$, the grammar $G^{[k]}$ is nonexpansive.

**Definition 3.1.**
Let $G$ be a context-free grammar $G = (\Sigma, A, P)$, and let $k$ be a fixed natural number. Set $X^{[k]} := \{X^{[d]}, X^{(d)} \mid X \in \Sigma, 0 \leq d \leq k\}$. The grammar $G^{[k]} = (X^{[k]}, A, P^{[k]})$ consists then of exactly the following rules:

- $X^{[d]} \to X^{(e)}$ for every $d \in [k] \cup \{0\}$, and every $e \in [d] \cup \{0\}$.
- If $X \to_G u_0$, then $X^{(0)} \to_{G^{[k]}} u_0$.
- If $X \to_G u_0 X_1 u_1$, then $X^{(d)} \to_{G^{[k]}} u_0 X^{(d)}_1 u_1$ for every $d \in [k] \cup \{0\}$.
- If $X \to_G u_0 X_1 u_1 \ldots u_{r-1} X_r u_r$ with $r > 1$:
  - For every $d \in [k]$, and every $j \in [r]$:
    
    Set $Z_j := X^{(d)}_j$ and $Z_i := X^{[d-1]}_i$ if $i \neq j$ for all $i \in [r] - \{j\}$. Then:
    
    $X^{(d)} \to_{G^{[k]}} u_0 Z_1 u_1 \ldots u_{r-1} Z_r u_r$.
  - For every $d \in [k]$, and every $J \subseteq [r]$ with $|J| \geq 2$:
    
    Set $Z_i := X^{(d-1)}_i$ if $i \in J$ and $Z_i := X^{[d-2]}_i$ if $i \notin J$. If all $Z_i$ are defined, i.e., $d \geq 2$ if $r > 2$, then:
    
    $X^{(d)} \to_{G^{[k]}} u_0 Z_0 u_1 \ldots u_{r-1} Z_{r-1} u_r$.

As the sets of variables of $G$ and $G^{[k]}$ are disjoint, in the following, we simply write $\text{amb}_X$ for $\text{amb}_{G,X}$, $\text{amb}_{X^{[d]}}$ for $\text{amb}_{G^{[k]},X^{[d]}}$, $X$-tree for $(G, X)$-tree, and so on.

**Lemma 3.2.**
Every $X^{(d)}$-tree resp. $X^{[d]}$-tree has dimension exactly resp. at most $d$. There is a yield-preserving bijection between the $X^{(d)}$-trees resp. $X^{[d]}$-trees and the $X$-trees of dimension exactly resp. at most $d$. 

8
Hence, by induction each of these subtrees has dimension less than \( l \). All nodes of subtrees each labeled by at most \( l \) distinct variables \( X \) for which there is at least one node of \( t \). Induction on \( l(t) \). Let \( t \) be number of distinct variables \( Y \) for which there is at least one node of \( t \) which is labeled by a rule rewriting \( Y \). Obviously, \( l(t) \leq |X| \). Induction on \( l(t) \) shows that every derivation tree \( t \) satisfying this property has dimension less than \( l(t) \). For \( l(t) = 1 \) a tree with this property cannot contain any nodes of arity two or more. Hence, its dimension is trivially zero. For \( l(t) > 1 \) given such an \( X \)-tree \( t = \sigma t_1 \ldots t_r \) we can find a simple path \( \pi \) leading from the root of \( t \) to a leaf which visits all nodes of \( t \) which are labeled by a rule rewriting \( X \). Removing \( \pi \) from \( t \) we obtain a forest of subtrees each labeled by at most \( l(t) - 1 \) distinct variables, and each still having above property. Hence, by induction each of these subtrees has dimension less than \( l(t) - 1 \), and, thus, \( t \) has dimension less than \( l(t) \).

We illustrate the construction in the following example.

**Example 3.5.**

Let \( G \) be defined by the productions

\[
X \to aX XXXXX | bXXXXX | c.
\]

The abstract algebraic system associated with this grammar is

\[
X = aX^6 + bX^5 + c.
\]

Using the valuation \( h(a) = 1/6 \), \( h(b) = 1/2 \), \( h(c) = 1/3 \), we interpret this abstract system as the concrete system

\[
X = 1/6X^6 + 1/2X^5 + 1/3
\]

over the \( \omega \)-continuous semiring \( \langle [0, \infty], +, \cdot, 0, 1 \rangle \) of nonnegative reals extended by a greatest element \( \infty \) with addition and multiplication extended as in the case of \( \mathbb{N} \). The least solution \( \mu \)
of this system, i.e. the least nonnegative root of $1/6X^6 + 1/2X^5 - X + 1/3$, can be shown to be neither rational nor expressible using radicals. We may approximate $\mu$ by evaluating $camb_{X[i]}$ under $h$. Up to commutativity, the grammar $G[k]$ corresponds to the following algebraic system:

$$
X^{(0)} = c \quad X^{[0]} = c
$$

$$
\vdots
$$

$$
X^{(k)} = \left( \binom{6}{1} a X^{[k-1]} + \binom{5}{1} b X^{[k-1]} \right) X^{(k)} + \sum_{j=2}^{6} \binom{6}{j} a X^{[k-2]} X^{[k-2]} + \sum_{j=2}^{5} \binom{5}{j} b X^{[k-2]} X^{[k-2]}. \right.
$$

From this, rational expressions for $camb_{X[i]}$ can easily be obtained:

$$
camb_{X^{(0)}} = c \quad camb_{X^{[0]}} = c
$$

$$
camb_{X^{(1)}} = (6ac^5 + 5bc^4)^* (ac^6 + bc^5) \quad camb_{X^{[1]}} = camb_{X^{(1)}} + camb_{X^{[0]}}
$$

$$
\vdots
$$

$$
camb_{X^{(k)}} = \left( \binom{6}{1} a camb_{X^{[k-1]}}^5 + \binom{5}{1} b camb_{X^{[k-1]}}^4 \right)^* \quad camb_{X^{[k]}} = camb_{X^{(k)}} + camb_{X^{[k-1]}}
$$

$$
+ \sum_{j=2}^{6} \binom{6}{j} a camb_{X^{[k-2]}}^6 - \sum_{j=2}^{5} \binom{5}{j} b camb_{X^{[k-2]}}^5 - \sum_{j=2}^{5} \binom{5}{j} b camb_{X^{[k-2]}}^{5-j}. \right.
$$

Evaluating the first three expressions for $camb_{X[i]}$ under $h$ we obtain the following approximations of $\mu$:

$$
h(camb_{G[i],X^{[0]}}) = 1/3
$$

$$
h(camb_{G[i],X^{[1]}}) = 1/3 + (6^{-1}3^{-6} + 2^{-1}3^{-5})(1 - 6 \cdot 6^{-1}3^{-5} - 5 \cdot 2^{-1}3^{-4})^{-1}
$$

$$
h(camb_{G[i],X^{[2]}}) = \frac{1417}{4221} \approx 0.335702 \approx 0.335704
$$

It can be shown that $h(camb_{X[i]})$ is exactly the $k$-th approximation obtained by applying Newton’s method to $1/6X^6 + 1/2X^5 - X + 1/3$ starting at $X = 0$.

\section{Speed of Convergence}

For this section, let $n$ denote the number of variables of the context-free grammar $G$. In [EKL07b] it was shown that, if $camb_{X[n]}(v) < camb_X(v)$, then $1 \leq camb_{X[n]}(v)$, i.e. $supp(camb_{X[n]}) = supp(camb_X)$. As $camb_{X[n]}$ is rational, this lower bound yields an alternative proof that $c(L(G,X))$ is a regular language. In this section we extend this result to a lower bound on the speed at which $camb_{X[i]}$ converges to $camb_X$ for $k \to \infty$.

By $l(t)$ we denote the number of variables occurring in a derivation tree $t$. The following lemma was proven in [EKL07b].

\textbf{Lemma 4.1.}

For every $X$-tree $t$ there is a Parikh-equivalent tree $\tilde{t}$ of dimension at most $l(t)$.
By similar arguments as before we can derive an even stronger convergence-theorem:

**Theorem 4.2.**
Let \( n \) be the number of variables of \( G \). Then for all \( k \geq 0 \) and \( v \in \mathbb{N}^A \): \( \text{camb}_{G^{n+k}}(v) \geq \min(\text{camb}_X(v), 2^k) \).

**Proof.** Assume there is a \( v \in \mathbb{N}^A \) with \( \text{camb}_{X^{n+k}}(v) < \text{camb}_X(v) \), i.e. we have some \( X \)-tree \( t \) of dimension at least \( n + k + 1 \) with \( c(Y(t)) = v \). We show that \( t \) witnesses the existence of at least \( 2^k \) distinct \( X \)-trees of dimension at most \( n + k + 1 \) with a yield that is Parikh-equivalent to \( t \).

We will prove the following stronger statement which implies the statement of the theorem: If \( \dim(t) \geq l(t) + k + 1 \) then there exist at least \( 2^k \) Parikh-equivalent trees of dimension at most \( l(t) + k \).

We prove the claim by induction on \( |V(t)| \), the number of nodes of \( t \). If \( |V(t)| = 1 \), then \( \dim(t) = 0 \) whereas \( l(t) + k + 1 = k + 2 > 0 \), so the claim trivially holds. Observe that if \( t \) has a subtree of dimension at least \( l(t) + k + 1 \) we can apply the induction hypothesis to every such subtree and thus obtain altogether at least \( 2^k \) Parikh-equivalent trees of dimension lower than \( \dim(t) \).

Therefore we can restrict ourselves to the case where \( \dim(t) = l(t) + k + 1 \) and all subtrees have dimension at most \( l(t) + k \). Note that in this case \( t \) must have (at least) two subtrees \( t_1, t_2 \) of dimension exactly \( l(t) + k \). We distinguish two cases:

- **Case** \( l(t_1) < l(t) \) or \( l(t_2) < l(t) \): Suppose w.l.o.g. \( l(t_1) < l(t) \). Apply the induction hypothesis to \( t_1 \), since \( \dim(t_1) = l(t_1) + k \geq l(t) + k + 1 \) and obtain at least \( 2^k \) Parikh-equivalent trees of dimension at most \( l(t_1) + k \). Then we apply Lemma 4.1 to every other subtree of \( t \) to obtain at least \( 2^k \) different trees \( l \) of dimension at most \( l(t) + k \).

- **Case** \( l(t_1) = l(t_2) = l(t) \): (This is the only case that requires actual work) Since \( t_1 \) has dimension \( l(t) + k \) it contains a perfect binary tree of height \( l(t) + k \) as a minor. The set of nodes of this minor on level \( k \) define \( 2^k \) (independent) subtrees of \( t_1 \). Each of these \( 2^k \) subtrees has height at least \( l(t) \), thus by the Pigeonhole principle contains a path with two variables repeating. We reallocate any subset of these \( 2^k \) pump-trees to \( t_2 \) which is possible since \( l(t_2) = l(t) = l(t_1) \). This changes the subtrees \( t_1, t_2 \) into \( t_1, t_2 \). Each of these \( 2^k \) choices produces a different tree \( l \)—the trees differ in the subtree \( t_1 \). As in the previous case we now apply Lemma 4.1 to every subtree of \( t \) except \( t_1 \) thereby reducing the dimension of \( t \) to at most \( \dim(t_1) = l(t) + k \) thus obtaining at least \( 2^k \) different Parikh-equivalent trees of dimension at most \( \dim(t_1) = l(t) + k \).

\( \square \)

We state some straightforward consequences of Theorem 4.2 based on the generalization of context-free grammars to algebraic systems. We say that a \( \omega \)-continuous semiring \( S \) is collapsed at some positive integer \( k \) if in \( S \) the identity \( k = k + 1 \) holds. For instance, the semirings \( \mathbb{N}_k \langle \mathbb{A}^* \rangle \) and \( \mathbb{N}_k \langle \mathbb{N}^A \rangle \) are collapsed at \( k \). For \( k = 1 \), the semiring is idempotent.

**Corollary 4.3.**
\( \text{camb}_{X^{n+k}} = \text{camb}_X \) over \( \mathbb{N}_k \langle \mathbb{N}^A \rangle \), and \( \text{camb}_X \) is rational in \( \mathbb{N}_k \langle \mathbb{N}^A \rangle \).

**Corollary 4.4.**
The least solution of an algebraic system with associated context-free grammar \( G \) and valuation \( h \) over a commutative \( \omega \)-continuous semiring \( S \) collapsed at \( k \) is \( (h(\text{camb}_{X^{n+k}}) \mid X \in \mathcal{X}) \).
By the results of [EKL07], the latter corollary is equivalent to saying that Newton’s method reaches the least solution of an algebraic system in $n$ variables over a commutative $\omega$-continuous semiring collapsed at $k$ after at most $n + \log \log k$ steps.

## 5 Semilinearity

In the following, let $k$ denote a fixed positive integer. By Corollary 4.3 we know that $\text{camb}_G$ is rational modulo $k = k + 1$. In this section, we give a semilinear characterization also of $\text{camb}_G$. We identify in the following a word $w \in A^*$ with its Parikh vector $c(w) \in \mathbb{N}^A$.

In the idempotent setting ($k = 1$), see e.g. [Pi73, KS86, HK99, AEI01], the identities (i) $(x^*)^* = x^*$, (ii) $(x + y)^* = x^*y^*$, and (iii) $(xy^*)^* = 1 + xx^*y^*$ can be used to transform any regular expression into a regular expression in “semilinear normal form” $\sum_{i=1}^r w_{i,0} w_{i,1} \cdots w_{i,l_i}$ with $w_{i,j} \in A^*$. It is not hard to deduce the following identities over $\mathbb{N}_k \langle \langle \mathbb{N}^A \rangle \rangle$ where $x^{<r}$ abbreviates the sum $\sum_{i=0}^{r-1} x^i$ and $\text{supp}(x)$ is identified with its characteristic function:

**Lemma 5.1.**
The following identities hold over $\mathbb{N}_k \langle \langle \mathbb{N}^A \rangle \rangle$:

1. $kx = k \text{supp}(x)$
2. $(\gamma x)^* = (\gamma x)^{<[\log, k]} + kx^{[\log, k]}x^*$
3. $(x^*)^* = kx^*$
4. $(x + y)^* = (x + y)^{<k} + x^k x^* + y^k y^* + kxy(x + y)^{\max(k-2,0)} x^*y^*$
5. $(xy^*)^* = 1 + xy^* + x^2 x^* + x^2 y \sum_{0 \leq m, j < k-2} (x^{2+m+j}) x^m y^j + kx^2 y(x^{\max(k-2,0)} + y^{\max(k-2,0)}) x^*y^*$

for $\gamma$ any integer greater than one.

Consider a rational series $\tau \in \mathbb{N}_k \langle \langle \mathbb{N}^A \rangle \rangle$ represented by the rational expression $\rho$. The above identities, where (13), (14), (15) generalizes (i), (ii), (iii), respectively, allow one to reduce the star height of $\rho$ to at most one by distributing the Kleene stars over sums $(\rho_1 + \rho_2)^*$ and products $(\rho_1 \rho_2)^*$ – in the latter case if $\rho_1 \rho_2 \not\in A^*$ – yielding a rational expression $\rho'$ of the form

$$\rho' = \sum_{i=1}^s \gamma_i w_{i,0} w_{i,1} \cdots w_{i,l_i}, \quad (w_{i,j} \in A^*, \gamma_i \in \mathbb{N}_k).$$

which still represents $\tau$ over $\mathbb{N}_k \langle \langle \mathbb{N}^A \rangle \rangle$. By (11) we know that, if $\gamma_{i,0} = k$, we may replace $w_{i,0} w_{i,1} \cdots w_{i,l_i}$ by its support which is a linear set in $\mathbb{N}^A$. This can be generalized to $k > 1$.

**Theorem 5.2.**
Every rational $\tau \in \mathbb{N}_k \langle \langle \mathbb{N}^A \rangle \rangle$ can be represented as a finite sum of weighted linear sets, i.e.

$$\tau = \sum_{i \in s} \gamma_i \text{supp}(w_{i,0} w_{i,1} \cdots w_{i,l_i}) \quad \text{with } w_{i,j} \in A^* \text{ and } \gamma_i \in \mathbb{N}_k.$$

**Example 5.3.**
The rational expression $\rho = (a + 2b)^*$ represents the series $\sum_{i,j \in \mathbb{N}} 2^i a^i b^j$ in $\mathbb{N}_\infty \langle \langle \mathbb{N}^A \rangle \rangle$. Computing
over $N_2\langle\langle N^A\rangle\rangle$ we may transform $\rho$ as follows:

\[
\begin{align*}
(a + 2b)^* &= (a + 2b)^2 + a^2a^* + (2b)^2(2b)^* + 2a(2b)a^*(2b)^* \\
\varepsilon + a + 2b + a^2a^* + 2b^2b^* + 2aba^*b^* &= a^* + 2(bb^* + aba^*b^*) \\
&= a^* + 2(bb^*a^*) \\
&= 1\text{ supp}(bb^*a^*) + 2\text{ supp}(bb^*a^*).
\end{align*}
\]

**Corollary 5.4.**

For every $k \in \mathbb{N}_\infty$ we can construct a formula of Presburger arithmetic that represents the set $\{v \in N^A \mid \text{camb}_{G,X}(v) = k\}$.

### 6 Acknowledgment

The authors would like to thank Volker Diekert for his help with a first version of Theorem 5.2, Rupak Majumdar for his pointer to [GKT07], and Javier Esparza and Andreas Gaiser for many helpful discussions.

### References

[ABB97] J. M. Autebert, J. Berstel, and L. Boasson. *Handbook of Formal Languages*, volume 1, chapter 3: Context-Free Languages and Pushdown Automata, pages 111 – 174. Springer, 1997.

[AEI01] L. Aceto, Z. Ésik, and A. Ingólfsdóttir. A fully equational proof of Parikh’s theorem. *RAIRO, Theoretical Informatics and Applications*, 36:200–2, 2001.

[BÉ09] Stephen L. Bloom and Zoltán Ésik. Axiomatizing rational power series over natural numbers. *Inf. Comput.*, 207(7):793–811, 2009.

[Boz99] S. Bozapalidis. Equational elements in additive algebras. *Theory Comput. Syst.*, 32(1):1–33, 1999.

[BR82] J. Berstel and C. Reutenauer. Recognizable formal power series on trees. *Theor. Comput. Sci.*, 18:115–148, 1982.

[CGT89] S. Ceri, G. Gottlob, and L. Tanca. What you always wanted to know about datalog (and never dared to ask). *IEEE Trans. Knowl. Data Eng.*, 1(1):146–166, 1989.

[CS63] N. Chomsky and M.P. Schützenberger. *Computer Programming and Formal Systems*, chapter The Algebraic Theory of Context-Free Languages, pages 118 – 161. North Holland, 1963.

[DK95] L. Devroye and P. Kruszewski. A note on the Horton-Strahler number for random trees. *Inf. Process. Lett.*, 56(2):95–99, 1995.

[DK09] M. Droste and W. Kuich. *Handbook of Weighted Automata*, volume 1, chapter 1: Semirings and formal power series, pages 3 – 27. Springer, 2009.
[EGKL11] J. Esparza, P. Ganty, S. Kiefer, and M. Luttenberger. Parikh’s theorem: A simple and direct automaton construction. Inf. Process. Lett., 111(12):614–619, 2011.

[EKL07a] J. Esparza, S. Kiefer, and M. Luttenberger. An extension of Newton’s method to $\omega$-continuous semirings. In DLT, pages 157–168, 2007.

[EKL07b] J. Esparza, S. Kiefer, and M. Luttenberger. On fixed point equations over commutative semirings. In STACS, pages 296–307, 2007.

[EKL08] J. Esparza, S. Kiefer, and M. Luttenberger. Derivation tree analysis for accelerated fixed-point computation. In DLT, pages 301–313, 2008.

[EKL10] J. Esparza, S. Kiefer, and M. Luttenberger. Newtonian program analysis. J. ACM, 57(6):33, 2010.

[Ers58] A. P. Ershov. On programming of arithmetic operations. Commun. ACM, 1(8):3–9, 1958.

[FFV79] P. Flajolet, J. Françon, and J. Vuillemin. Towards analysing sequences of operations for dynamic data structures (preliminary version). In FOCS, pages 183–195, 1979.

[GKT07] T. J. Green, G. Karvounarakis, and V. Tannen. Provenance semirings. In PODS, pages 31–40, 2007.

[GS68] S. Ginsburg and E. Spanier. Derivation-bounded languages. Journal of Computer and System Sciences, 2:228–250, 1968.

[HK99] M. W. Hopkins and D. Kozen. Parikh’s theorem in commutative Kleene algebra. In Logic in Computer Science, pages 394–401, 1999.

[Hor45] R. E. Horton. Erosional development of streams and their drainage basins: hydrophysical approach to quantitative morphology. Geological Society of America Bulletin, 56(3):275–370, 1945.

[KS86] Werner Kuich and Arto Salomaa. Semirings, Automata, Languages, volume 5 of Monographs in Theoretical Computer Science. An EATCS Series. Springer, 1986.

[Kui97] W. Kuich. Handbook of Formal Languages, volume 1, chapter 9: Semirings and Formal Power Series: Their Relevance to Formal Languages and Automata, pages 609–677. Springer, 1997.

[Pil73] D. L. Pilling. Commutative regular equations and Parikh’s theorem. Journal of the London Mathematical Society, pages 663–666, 1973.

[RS83] N. Robertson and P. D. Seymour. Graph minors. i. excluding a forest. J. Comb. Theory, Ser. B, 35(1):39–61, 1983.

[SS78] A. Salomaa and M. Soittola. Automata-theoretic aspects of formal power series. Texts and monographs in computer science. Springer, 1978.

[Str52] A. N. Strahler. Hypsometric (area-altitude) analysis of erosional topology. Geological Society of America Bulletin, 63(11):1117–1142, 1952.

[Tha67] J. W. Thatcher. Characterizing derivation trees of context-free grammars through a generalization of finite automata theory. J. Comput. Syst. Sci., 1(4):317–322, 1967.

[Ynt67] M.K. Yntema. Inclusion relations among families of context-free languages. Information and Control, 10:572–597, 1967.
A Missing proofs

Proof of Lemma 3.2

Let \( t \) be a derivation tree of dimension \( \dim(t) = d \). Then \( t = \sigma t_1 \ldots t_r \) has at most one child \( t_c (c \in [r]) \) with \( \dim(t) = \dim(t_c) \) by definition of \( \dim \). Hence, there is a unique maximal path \( v_0 \ldots v_l \) starting in \( v_1 = \varepsilon \) such that (i) \( \dim(t_1) = \dim(t_{v_1}) \) and (ii) either \( v_l \) is a leaf of \( t \) or every proper subtree of \( v_l \) has dimension less than \( d \). Let \( \text{dlen}(t) = l \) denote the length of this unique path. Further, we use \( \text{dchar}(t) = \{ (i, \dim(t'_i)) \mid i \in [r'] \text{ for } t|_{v_i} = \sigma t'_i \ldots t'_{r'} \} \) to remember the dimensions of the children of \( t|_{v_1} \). (\( \text{dchar}(t) = \emptyset \) if \( v_l \) is a leaf of \( t \).)

We first construct a mapping \( \hat{\cdot} \) from the derivation trees of \( G[k] \) to the derivation trees of \( G \) of dimension at most \( d \) and exactly \( d \), respectively:

- If \( t = \sigma X[\alpha],X[\sigma] t_1 \), then \( \hat{t} := \hat{t}_1 \).
- If \( t = \sigma X[\alpha],u_0 Z_1 u_1 \ldots u_{r-1} Z_r u_r t_1 \ldots t_r \), then \( \hat{t} := \sigma X[\alpha],u_0 X_1 u_1 \ldots u_{r-1} X_r u_r \hat{t}_1 \ldots \hat{t}_r \) where \( X_i \in \mathcal{X} \) is the variable from which \( Z_i \in \mathcal{X}[k] \) was derived.

Informally, \( \hat{\cdot} \) contracts edges induced by rules \( X[d] \to X[e] \) which choose a concrete dimension \( e \leq d \), and then forgets the superscripts. By definition, the rules of \( G[k] \) which rewrite the variable \( X[d] \) are obtained from the rules of \( G \) which rewrite the variable \( X \) by only adding superscripts. Hence, \( \hat{\cdot} \) maps any \( X[d]-\text{tree} \) and any \( X[d]-\text{tree} \) to a \( X \)-tree while preserving its yield (\( Y(t) = Y(\hat{t}) \)). Further, as the edges induced by the rules \( X[d] \to X[e] \) do not influence the tree dimension, we also have \( \dim(t) = \dim(\hat{t}) \) and \( \text{dchar}(t) = \text{dchar}(\hat{t}) \). We also have \( \text{dlen}(t) \geq \text{dlen}(\hat{t}) \) as contracting the edges induced by \( X[d] \to X[e] \) can only reduce \( \text{dlen}(\cdot) \).

We claim that \( \hat{\cdot} \) maps the set of \( X[d]-\text{trees} (X[d]) \) one-to-one onto the set of \( X \)-trees of dimension at most \( d \) (exactly \( d \)). We proceed by induction on \( d \). Let \( d = 0 \).

- Consider a \( X(0) \)-tree \( t \). The only rules rewriting \( X(0) \) are of the form \( X(0) \to u Y(0) v \) (for \( u, v \in A^* \) and \( Y \in \mathcal{X} \)). For these rules, forgetting the superscript is an injective operation. Hence, \( \hat{\cdot} \) is injective on the set of \( X(0) \)-trees. Obviously, \( t \) is also a chain, and, thus, \( 0 = \dim(t) = \dim(\hat{t}) \). (In fact, \( \text{dlen}(t) = \text{dlen}(\hat{t}) \).)

Consider now a \( X(0) \)-tree \( t \). By definition of \( G[k] \), \( X[0] \) can only be rewritten to \( X(0) \). So \( t = \sigma X[\alpha],X[\sigma] t_1 \) for \( t_1 \) a \( X(0) \)-tree, and \( \hat{t} = t_1 \). Again, \( 0 = \dim(t) = \dim(\hat{t}) \).

- Let \( t \) be a \( X(d) \)-tree for \( d > 0 \) where \( t = \sigma X[\alpha],u_0 Z_1 u_1 \ldots u_{r-1} Z_r u_r t_1 \ldots t_r \) for some \( r > 0 \) where there is a rule \( X = u_0 X_1 u_1 \ldots u_{r-1} X_r u_r \) in \( G \) (\( X_i \in \mathcal{X}, \ u_i \in A^* \)) such that for all \( i \in [r] \) either \( Z_i \in \{ X[d], X[d-1] \} \) or \( Z_i \in \{ X_i(d-1), X_i(d-2) \} \).

Assume first that \( t \) has no \( Y[d]-\text{subtree} \) for any \( Y \in \mathcal{X} \), i.e. \( t \) is a \( X(d) \)-tree of minimal height. Then \( \sigma = \sigma X[\alpha],u_0 Z_1 \ldots Z_r u_r \), where \( Z_i = X_i(d-1) \) or, if \( d \geq 2 \), \( Z_i = X_i(d-2) \) for some \( X_i \in \mathcal{X} \) such that \( X = u_0 X_1 \ldots X_r u_r \in G \). Inductively, we already know that \( \dim(t') = e \) (\( \dim(t') \leq e \)) for every \( X(e) \)-tree (\( X[e]-\text{tree} \)) and all \( e < d \). Hence, \( \dim(t) = \dim(d) = d \) and \( \text{dlen}(t) = \text{dlen}(\hat{t}) = 0 \).

Thus, assume that \( t \) contains a \( Y(d)-\text{subtree} \) for some \( Y \in \mathcal{X} \). By construction, there occurs at most one \( "(d)\)-variable\), i.e. a variable of \( \{ Y(d) \mid Y \in \mathcal{X} \} \), in the right-hand side \( \gamma \) of every rule \( X(d) \to \gamma \). By construction, there is a unique \( j \in [r] \) such that \( Z_j = X_j(d) \), while \( Z_i = X_i(d-1) \) for all \( i \in [r] \setminus \{ j \} \). Then the \( X(d)-\text{tree} t_j \) has height less
than \( t \), so by induction on the height of \( X^{(d)} \)-trees, we have \( \dim(t_j) = \dim(\hat{t}_j) = d \) and \( \dlen(t_j) = \dlen(\hat{t}_j) \). By induction on \( d \), we also know that \( \dim(t_i) < d \) \( (i \in [r] - \{j\}) \). Hence, \( \dim(t) = d \) and \( \dlen(t) = \dlen(t_j) + 1 \). As the edge to \( t_j \) is not contracted by \( \hat{\cdot} \), also \( \dlen(\hat{t}) = \dlen(t_j) + 1 = \dlen(\hat{t}_j) \).

Assume now that \( \hat{t} = \hat{t}' \) for two \( X^{(d)} \)-trees \( t, t' \). Then \( \dim(t) = \dim(t') = \dim(\hat{t}) \), \( \dlen(t) = \dlen(t') = \dlen(\hat{t}) \), and \( \dchar(t) = \dchar(t') = \dchar(\hat{t}) \). Let \( t = \sigma_{X, u_0 X_1 u_1 \ldots u_{r-1} X u_r} \). Then necessarily, \( t = \sigma_{X^{(d)}, u_0 Z_1 \ldots Z_j u_r} \) and \( t' = \sigma_{X^{(d)}, u_0 Z_1' \ldots Z_j' u_r} \) with either \( Z_i \in \{X_i^{(d)}, X_i^{[d-1]} \} \) or \( Z_i \in \{X_i^{(d-1)}, X_i^{[d-2]} \} \), and, analogously, for all \( Z_i' \). As \( \hat{\cdot} \) only forgets superscripts and removes \( \sigma_{X^{(d)}, X^{(s)}} \).

If \( \dlen(\hat{t}) = 0 \), then \( t, t', \hat{t} \) have only subtrees of dimension at most \( d-1 \). By definition of \( G^{[k]} \), it follows that \( Z_i, Z_i' \in \{X_i^{(d-1)}, X_i^{[d-2]} \} \). By induction, we know that only \( (d-1) \)-variables can generate trees of dimension \( d-1 \), hence, necessarily \( Z_i = Z_i' = X_i^{(d-1)} \) for all children \( i \in [r] \) of \( t \) which have dimension exactly \( d-1 \), while \( Z_i = Z_i' = X_i^{[d-2]} \) for all remaining children. Again by induction, we know that \( \hat{\cdot} \) is injective on sets of \( Y^{(d-2)} \)-trees and \( Y^{(d-1)} \)-trees, respectively. Hence, \( t = t' \).

Finally, assume \( \dlen(\hat{t}) > 0 \). Then \( \hat{t} \) has a unique child \( t_j \) of dimension \( d \), while \( \dim(t_j) < d \) for \( j \in [r] - \{i\} \). Consequently, \( Z_j = Z_j' = X_j^{(d)} \) and \( Z_j = Z_j' = X_j^{[d-1]} \) for \( j \in [r] - \{i\} \) by definition of \( G^{[k]} \). By induction on \( d \) and \( \dlen(t) \), we may assume that \( \hat{\cdot} \) is injective on the subtrees of \( t \) and \( t' \), hence, \( t = t' \) follows.

It remains to show that for any \( X \)-tree \( t' \) of dimension exactly \( d \) (at most \( d \)), there is a \( X^{(d)} \)-tree \( (X^{(d)} \)-tree) \( t \) such that \( \hat{t} = t' \). To this end, we define an operator \( \tilde{\cdot} \) which maps a \( X \)-tree of dimension exactly \( d \) to a \( X^{(d)} \)-tree by, essentially, introducing the superscripts into a symbol \( \sigma_{X, u_0 X_1 u_1 \ldots u_{r-1} X u_r} \) as required by the dimensions of the subtrees \( t_1, \ldots, t_r \):

Let \( t = \sigma_{X, u_0 X_1 u_1 \ldots u_{r-1} X u_r} t_1 \ldots t_r \) with \( d = \dim(t) \) and \( d_i = \dim(t_i) \), then

\[
\hat{t} := \sigma_{X^{(d)}, X^{(s)}} \sigma_{X^{(d)}, u_0 Z_1 u_1 \ldots Z_j u_r} t_1' \ldots t_r',
\]

where \( Z_i, t_i' \) are defined as follows:

- If \( d > \max_{i \in [r]} d_i \), then let \( J = \{i \in [r] \mid d_i = d-1\} \) and set \( Z_i := X_i^{(d-1)} \) and \( t'_i := \hat{t}_i \) if \( i \in J \), and \( Z_i := X_i^{[d-2]} \) and \( t'_i := \sigma_{X^{[d-2]}}, X^{(s)}, \hat{t}_i \) otherwise.

- If \( d = \max_{i \in [r]} d_i \), then there is a unique \( j \in [r] \) such that \( d_j = d \). Set \( Z_j = X_j^{(d)} \) and \( t'_j := \hat{t}_j \). For the remaining \( i \in [r] - \{j\} \), set \( Z_i := X_i^{[d-1]} \) and \( t'_i := \sigma_{X_i^{[d-1]}}, X^{(s)}, \hat{t}_i \).

It is straightforward to check that \( \hat{t} \) is indeed a \( X^{(d)} \)-tree for \( \dim(t) = d \), and that \( \hat{t} = t \). Obviously, \( \tilde{\cdot} \) is injective. Finally, for every \( d' \geq d \) there is exactly one rule \( X^{[d']} \rightarrow X^{(d)} \). Hence, \( \sigma_{X^{[d']}}, X^{(s)}, \hat{t} \)
is, by definition of \( G^{[k]} \), the unique \( X^{(d)} \)-tree which is mapped by \( \hat{\cdot} \) back to \( t \).

**Proof of Lemma 5.1**

The proofs are straightforward, and essentially only require to unroll and cut off the power series underlying the Kleene star using the \( \omega \)-continuity of the Kleene star and the assumption that
\[ k = k + 1. \] We several times make use of the trivial bound \((\binom{a}{b}) \geq a\) for \(0 < b < a\). on the binomial coefficient.

(11) \(kx = k \text{supp}(x)\) is obviously true modulo \(k = k + 1\).

(12) \((\gamma x)^* = (\gamma x)^{[\log_k x]} + k \cdot x^{[\log_k x]} x^*

This follows from the \(\omega\)-continuity of the star \((\gamma x)^* = \sum_{n \in \mathbb{N}} (\gamma x)^n\) and the first identity.

(13) \((x^*)^* = kx^*

Choose any \(w \in \text{supp}((x^*)^*)\). Then \(w\) can be factorized into \(w = u_1 \ldots u_l\) with \(u_i \in \text{supp}(x^*)\), i.e., \(w \in \text{supp}((x^*)^l)\). Obviously, we then can also find a factorization of \(w\) into \(l + i\) words for any \(i > 0\) as we may add an arbitrary number of neutral elements into this factorization. Hence, \(w \in \text{supp}((x^*)^{l+1})\) for all \(i \geq 0\). So, the coefficient of \(w\) in \((x^*)^*\) is \(\infty = k\) modulo \(k = k + 1\).

(14) \((x + y)^* = (x + y)^{ < k} + x^k x^* + y^k y^* + kxy(x + y)^{ \max(k-2,0)} x^* y^*

Proof:

\[
\begin{align*}
(x + y)^* &= (x + y)^{ < k} + \sum_{n \geq k} (x + y)^n \\
(xy = yx) &= (x + y)^{ < k} + \sum_{n \geq k} \sum_{j=0}^{n} (\binom{n}{j}) x^j y^{n-j} \\
&= (x + y)^{ < k} + \sum_{n \geq k} x^n y^n + \sum_{j=1}^{n-1} (\binom{n}{j}) x^j y^{n-j} \\
&= (x + y)^{ < k} + x^k x^* + y^k y^* + \sum_{n \geq k} \sum_{j=1}^{n-1} (\binom{n}{j}) x^j y^{n-j} \\
(j = i + 1, n = m + 2) &= (x + y)^{ < k} + x^k x^* + y^k y^* \\
&\quad + \sum_{m \geq \max(k-2,0)} \sum_{i=0}^{m} \binom{m+2}{i+1} x^{i+1} y^{m-i+1} \\
((m+2) \geq k, (11)) &= (x + y)^{ < k} + x^k x^* + y^k y^* \\
&\quad + kxy \sum_{m \geq \max(k-2,0)} \sum_{i=0}^{m} \binom{m}{i} x^i y^{m-i} \\
&= (x + y)^{ < k} + x^k x^* + y^k y^* \\
&\quad + kxy(x + y)^{ \max(k-2,0)} (x + y)^* \\
((ii) \text{ supp}(x + y)^*) = \text{supp}((x^* y^*), (11)) &= (x + y)^{ < k} + x^k x^* + y^k y^* \\
&\quad + kxy(x + y)^{ \max(k-2,0)} x^* y^* \\

(15) (xy)^* &= 1 + xy^* + x^2 x^* + x^2 y \sum_{0 \leq m, j < k-2} \binom{2+m+j}{1+j} x^m y^j \\
&\quad + kx^2 y x^{\max(k-2,0)} x^* y^* + kx^2 y x^{\max(k-2,0)} y^* \]
Proof:

\[(xy^*) = \sum_{n \in \mathbb{N}} x^n (y^*)^n\]
\[= 1 + xy^* + \sum_{n \geq 2} x^n \sum_{l \geq 0} \binom{n+l-1}{l} y^l\]
\[= 1 + xy^* + x^2 y + \sum_{n \geq 2, l \geq 1} \binom{n+l-1}{l} x^n y^l\]

\[\text{If } (n = m + 2, l = j + 1, xy = yx)\]
\[= 1 + xy^* + x^2 y + x^2 y \sum_{m \geq 0, j \geq 0} \binom{2+m+j}{1+j} x^m y^j\]

\[\text{If } (k = k + 1)\]
\[= 1 + xy^* + x^2 y + k x^2 y \sum_{m \geq k-2, j \geq k-2} \binom{1+m+j}{1+j} x^m y^j\]

\[(I)\]
\[= 1 + xy^* + x^2 y + k x^2 y \sum_{m \geq 0, j \geq 0} \binom{0+m+j}{1+j} x^m y^j\]

\[\text{Proof of Theorem 5.2}\]

We identify a word \(w \in A^*\) with its Parikh vector \(c(w) \in \mathbb{N}^A\). We show that, if \(\text{supp}(w_1, \ldots, w_k) \neq w_1^* \cdots w_k^* \in \mathbb{N}_k\langle \mathbb{N}^A \rangle\), then we can split the linear term in a finite sum of weighted linear terms where in each linear term with weight less than \(k\) the number of Kleene stars is strictly less than \(l\). Then the result follows inductively.

W.l.o.g. we may assume that each \(w_i \neq \varepsilon\), i.e. \(c(w_i) \neq 0\), as \(\varepsilon^* = \infty = k\). Denote by \(M \in \mathbb{N}^{A \times I}\) the matrix whose \(i\)-th row is given by \(c(w_i)\) (w.r.t. some chosen order on \(A\)), and let \(\lambda = (\lambda_1, \ldots, \lambda_l) \in \mathbb{N}^l\). Then the coefficient \(c_v := (w_1^* \cdots w_k^* v)\) is exactly the number of solutions over \(\mathbb{N}^l\) of the linear equation \(v = \lambda M\). If the set \(\{c(w_1), c(w_2), \ldots, c(w_l)\}\) is linearly independent, then trivially \(c_v \leq 1\) and we are done.

Assume thus that the set \(\{c(w_1), c(w_2), \ldots, c(w_l)\}\) is linearly dependent, i.e. there is some kernel vector \(n = (n_1, \ldots, n_l) \in \mathbb{Z}^l \setminus \{0\}\). Let \(I_+ = \{i \in [l] \mid n_i > 0\}\), \(I_- = \{i \in [l] \mid n_i < 0\}\), and \(I_0 = \{i \in [l] \mid n_i = 0\}\). As all components of \(M\) are nonnegative, \(n\) necessarily has a positive and a negative component, i.e. \(I_+ \neq \emptyset \neq I_-\). Let \(\|n\|_\infty := \max_{i \in [l]} |n_i|\) and \(C := \|n\|_\infty \cdot (k-1)\).

Consider now any \(\lambda = (\lambda_1, \ldots, \lambda_l) \in \mathbb{N}^l\) with \(\lambda_i > C\) for all \(i \in I_+\). Then also \(\lambda - in \in \mathbb{N}^l\) for \(i = 0, \ldots, k-1\) and trivially \(v = \lambda M = (\lambda - in) M\) which implies that \(c_v \geq k\). If \(\lambda_i > C\) for all \(i \in I_-\), consider analogously \(\lambda + in\). For \(I \subseteq \{I_+, I_-\}\) we split the series \(\prod_{i \in I} w_i^*\) into series \(s_I\) and \(t_I\) defined by

\[s_I := \prod_{i \in I} (w_i^c u_i^*)\]
\[t_I := \sum_{\emptyset \neq J \subseteq I} \prod_{i \in J} w_i^{< C} \prod_{i \in I \setminus J} (w_i^c u_i^*)\]

18
As discussed above, all positive coefficients of \( s = \prod_{i \in I} (w_i^+ w_i^-) \) (for \( I \in \{I_+, I_-\} \)) are greater than or equal to \( k \). Hence \( s_I = k \supp(s_I) \) over \( \mathbb{N}_k \langle \mathbb{N}^A \rangle \).

\[
\begin{align*}
& w^*_1 w^*_2 \ldots w^*_I \\
= & \prod_{i \in I_0} w^*_i (s_{I_+} + t_+)(s_{I_-} + t_-) \\
= & \prod_{i \in I_0} w^*_i (k s_{I_+} + t_+)(k s_{I_-} + t_-) \\
= & \prod_{i \in I_0} w^*_i \left( t_+ t_- + k (t_{I_+} s_{I_-} + t_{I_-} s_{I_+} + s_{I_+} s_{I_-}) \right) \\
= & \prod_{i \in I_0} w^*_i \left( t_+ t_- + k (t_{I_+} s_{I_-} + t_{I_-} s_{I_+} + 2 s_{I_+} s_{I_-}) \right) \\
= & \prod_{i \in I_0} w^*_i \left( t_+ t_- + k \left( \prod_{i \in I_0} w^*_i \right) \right) \\
= & k \left( \prod_{i \in I_0} w^*_i \right) \\
\end{align*}
\]

It remains to consider the second summand which can be written as a finite sum of products of which each contains at most \( |I| - (J_+ \cup J_-) \leq l - 2 \) Kleene stars:

\[
t_{I_+} t_{I_-} \prod_{i \in I_0} w^*_i = \sum_{\emptyset \neq J_+ \subseteq I_+, \emptyset \neq J_- \subseteq I_-} \prod_{i \in (I_+ - J_+) \cup (I_- - J_-)} w^C_i \prod_{i \in [I] - (J_+ \cup J_-)} w^*_i.
\]

**Proof of Corollary 5.4**

As \( c(L(G, X)) = \supp(\text{camb}_{G, X}) = \{ v \in \mathbb{N}^A \mid \text{camb}_{G, X}(v) > 0 \} \) is semilinear by Parikh’s theorem, it is effectively representable by a formula of Presburger arithmetic, and so is its complement (\( k = 0 \)).

Assume thus \( 1 \leq k < \infty \) and let \( K = k + 1 \). Then we may compute from \( \text{camb}_{X^{[n]}} \) a weighted semilinear representation of \( \text{camb}_{X} \) modulo \( K = K + 1 \):

\[
\text{camb}_{X} = \sum_{i=1}^{r} \gamma_i \supp(v_{i,0} v_{i,1}^* \ldots v_{i,l_i}^*) \text{ with } \gamma_i \in \mathbb{N}_K \text{ and } v_{i,j} \in \mathbb{N}^A.
\]

From each term \( \supp(v_{i,0} v_{i,1}^* \ldots v_{i,l_i}^*) \) we can construct an equivalent Presburger formula \( F_i \).

Then \( \text{camb}_{X}(v) = k \) if and only if

\[
v \models \exists y_1, \ldots, y_r: \forall i = 1 \ldots r: \gamma_i y_i = k \land (F_i(v) \rightarrow y_i = 1 \land \neg F_i(v) \rightarrow y_i = 0).
\]

Finally, let \( k = \infty \). As for any \( v \in \mathbb{N}^A \) there are only finitely many \( w \in \mathbb{A}^* \) with \( c(w) = v \), we have \( \text{camb}_{G, X}(v) = \infty \) if and only if there is a \( w \in \mathbb{A}^* \) with \( c(w) = v \) and \( \text{amb}_{G, X}(w) = \infty \). We
therefore construct from $G = (\mathcal{X}, A, P)$ a context-free grammar $G' = (\mathcal{X}', A, P')$ with $\mathcal{X} \subseteq \mathcal{X}'$ such that $L(G', X) = \{ w \in A^* \mid \text{amb}_{G,X}(w) = \infty \}$. Then $\{ v \in \mathbb{N}^A \mid \text{amb}_{G,X}(v) = \infty \} = c(L(G', X))$ and is a semilinear set by Parikh’s theorem where the corresponding Presburger formula is again effectively constructible.

We discuss the construction of $G'$ for the sake of completeness: we have $\text{amb}_{G,X}(w) = \infty$ if and only if there are infinitely many $X$-trees $t$ with $Y(t) = w$. In particular, for every $h \in \mathbb{N}$ we can find a $X$-tree $t$ of height at least $h$ with $Y(t)$, as there are only finitely many $X$-trees of bounded height. For instance, choose $h \geq (|w| + 1)|X|$ and consider a maximal path $v_0 \ldots v_h$ from the root of such a $t$ to a leaf. For all $i = 0 \ldots h$ assume $t|_{v_i}$ is a $X_i$-tree ($X = X_0$). This path then corresponds to a derivation of the form

$$X = X_0 \Rightarrow^+ u_0 X_1 v_0 \Rightarrow^+ \ldots \Rightarrow^+ u_0 \ldots u_{h-1} X_h v_h \ldots v_0 \Rightarrow u_1 \ldots u_{h-1} u_h v_h v_{h-1} \ldots v_1 = w$$

for suitable $u_i, v_i \in A^*$. In the sequence $X_0, X_1, \ldots, X_h$ color $X_i$ black if $|u_i| = 0$; otherwise color $X_i$ red. Then there are at most $|w|$ red variables in this sequence. In particular, there is a subsequence $X_i, X_{i+1}, \ldots, X_{i+|X|}$ consisting of $1 + |X|$ consecutive black variables, as otherwise $h + 1 \leq (|w| + 1)|X|$. Hence, the derivation contains a cyclic derivation $Y \Rightarrow^+ Y$.

Therefore compute the set $\mathcal{X}_C = \{ X \in \mathcal{X} \mid X \Rightarrow^+_G X \}$ of cyclic variables as usual, and define $G'$ such that a derivation can only terminate in a word if the derivation visits at least one cyclic variable:

- Set $\mathcal{X}' = \{ X, X' \mid X \in \mathcal{X} \}$ with the intended meaning that an unprimed variable still has to be derived into a sentential form containing at least one cyclic variable $Y \in \mathcal{X}_C$.

- Construct $P'$ as follows:
  - If $X \Rightarrow_G u_0$ for $u_0 \in A^*$, then $X' \Rightarrow_{G'} u_0$.
  - If $X \Rightarrow_G u_0 X_1 u_1 X_2 u_2 \ldots u_r X_r u_r$ for $r > 0$ and $u_i \in A^*$, then
    $$X' \Rightarrow_{G'} u_0 X'_1 u_1 X'_2 u_2 \ldots u_{r-1} X'_r u_r$$
    and
    $$X \Rightarrow_{G'} u_0 X_1 u_1 X'_2 u_2 \ldots u_{r-1} X'_r u_r$$
    $$X \Rightarrow_{G'} u_0 X'_1 u_1 X_2 u_2 \ldots u_{r-1} X'_r u_r$$
    $$\cdot$$
    $$X \Rightarrow_{G'} u_0 X'_1 u_1 X'_2 u_2 \ldots u_{r-1} X'_r u_r$$
  - If $X \in \mathcal{X}_C$, then $X \Rightarrow_{G'} X'$.

By construction, an unprimed variable $Y$ can only be rewritten to a sentential form containing exactly one unprimed variable, except $Y$ is cyclic in $G$, in which case the rule $Y \Rightarrow_{G'} Y'$ can also be applied.

Then $w \in L(G', X)$ if and only if there is a derivation $X \Rightarrow^+_G uYv \Rightarrow^+_G uY'v \Rightarrow^+_G w$, as only primed variables can be rewritten to terminal words. By construction, this is equivalent to $X \Rightarrow^+_G uYv \Rightarrow^+_G w$ and $Y \in \mathcal{X}_C$, which in turn is equivalent to $\text{amb}_{G,X}(w) = \infty$. 

20