The Banach space $H^1(X, d, \mu)$, II

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1 Introduction

In this paper we give the isomorphic classification of atomic $H^1(X,d,\mu)$, where $(X,d,\mu)$ is a space of homogeneous type, hereby completing a line of investigation opened by the word of Bernard Maurey [Ma1], [Ma2], [Ma3] and continued by Lennard Calrleson [C] and Przemyslaw Wojtaszczyk [Woj1], [Wpj2].

The resulting isomorphic representatives dyadic $H^1, (\sum H^1_n)_1^\infty$ and $l^1$; each isomorphic type is characterized by geometric properties of $(X,d,\mu)$.

Technically, the present paper deals with the existence of “Franklin-type” function on a space of homogeneous type $(X,d,\mu)$ and with the boundedness (in $H^1(X,d,\mu)$) of the orthogonal projection onto the span of this “Franklin-system”.

To this end a construction of S. Jaffard and Y. Meyer [J-M], together with methods of P. Wojtaszczyk [Woj2] have been adapted. (See Sections 3,4).

Applying Wojtaszczyk’s method one concludes that the “Franklin-system” in $H^1(X,d,\mu)$ is equivalent to special three valued martingale differences in some martingale $H^1$ space; (see Section 4). Using repeatedly Pełczyński’s decomposition method, this allows us to reduce in Section 5 the classification problem for atomic $H^1$ spaces to some of the other’s precious work on this subject:

- Classification of (the span of) special three valued martingale differences in martingale $H^1$ space [Mü2].

- Classification of the isomorphic type of martingale $H^1$ space [Mü1].
Atomic $H^1$ spaces are isomorphic to complemented subspaces of martingale $H^1$ [Mü3].

2 Preliminaries concerning spaces of homogeneous type

Definition and Notation

Let $d : X \times X \to \mathbb{R}^+$ be a quasi metric on a set $X$ and let $B(x, r) := \{ y \in X : d(x, y) < r \}$. Let $\mu$ be a non negative measure on $X$.

Let $A_1, A_2, K_1, K_2 (K_2 \leq 1 \leq K_1)$ be positive finite constants such that for each $x \in X$ and $r > 0$ the following relations hold:

1. $A_1 r \leq \mu(B(x, r))$ if $r \leq K_1 \mu(X)$
2. $B(x, r) = X$ if $r > K_1 \mu(X)$
3. $A_2 r \geq \mu(B(x, r))$ if $r \geq K_2 \mu(\{x\})$
4. $B(x, r) = \{x\}$ if $r < K_2 \mu(\{x\})$.

We assume moreover that there exists $\alpha > 0$ and $K_0 > 0$ such that for each $x, y, z \in X$

5. $|d(x, y) - d(y, z)| \leq K_0 r^{1-\alpha} d(x, y)^\alpha$

whenever $d(x, z) < r$ and $d(y, z) < r$.

Following Masias & Segovia [M-S1,2] a set $X$ equipped with a measure $\mu$ and a quasi metric $d$ satisfying (1) – (5) is called a “normal space of order $\alpha$”. The standard reference to these spaces is the article by Coifman & Weiss [C-W].
On \((X, d, \mu)\) certain analogues of dyadic intervals have been constructed by G. David and M. Christ; see [Dv] and [Ch].

**Theorem 1** There exists a collection \(\{Q^k_i \subset X : k \in \mathbb{Z}, i \in I_k\}\) and constants \(\delta \in (0, 1), \alpha_0 > 0, \eta > 0\) and \(C_1, C_2 < \infty\) such that:

\[
\mu(X \setminus \bigcup Q^k_i) = 0.
\]

If \(l \geq k\) then either \(Q^l_j \subset Q^k_i\) or \(Q^l_j \cap Q^k_i = 0\).

For each \((k, i)\) and \(l < k\) there exists a unique \(j\) such that \(Q^k_i \subset Q^l_j\).

\[
\text{diam } Q^k_i \leq C_1 \delta^k.
\]

Each \(Q^k_i\) contains a ball \(B(z^k_i, a_0, \delta^k)\).

\[
\mu\{x \in Q^k_i : d(x, X \setminus Q^k_i) \leq td^k\} \leq C_2 t^\eta \mu(Q^k_i) \text{ for each } k \in \mathbb{Z}, i \in I_k \text{ and } t \geq 0.
\]

As we shall see, the structure of this collection determines the isomorphic type of \(H^1(X, d, \mu)\). However we have to discard measure and diameter is too big.

**Lemma 2** In every normal space of order \(\alpha\) \((X, d, \mu)\) there exists \(L > 0\) depending on so that for every nonempty \(Q \subset X\) we have: \(\mu(Q)/\text{diam } Q > L\) implies \(Q\) consists of exactly one point.

**Proof.** Select \(M > 1\) so that \(2M > 1\). Then consider two cases.

**Case 1** Suppose there exists \(x_0 \in Q\) so that \(\mu(\{x_0\}) \geq M \text{diam } Q\) then \(\{x : d(x, x_0) < KM \text{ diam } Q\} = \{x_0\}\). Hence \(Q = \{x_0\}\).

**Case 2** For each \(x_0 \in Q\) we have \(\mu(\{x_0\}) \leq M \text{ diam } Q\); then as \(MK_2 > 1\)

\[
\begin{align*}
\mu(Q) & \leq \mu(B(x_0, K_2 M \text{ diam } Q) \\
& \leq A_2 MK_2 \text{ diam } Q.
\end{align*}
\]
Let now $L = A_2 MK_2$ then we have either $\mu(Q)/\text{diam } Q \leq L$ or $Q = \{x_0\}$ for some $x_0 \in X$.

Let $\mathcal{E} := \{Q^k_\alpha : k \in \mathbb{Z}, \alpha \in I_k\}$. And let $\mathcal{F}_n$ be the $\sigma$-Algebra generated the $n$-th generation of $\mathcal{E}$ and $\mathcal{F}_{n-1}$.

The following properties of $\mathcal{F}_n$ are easily observed:

1. There exits $N_0 \in \mathbb{N}$, depending on $\delta$ (and the geometry of $(X,d,\mu)$) so that for every $Q \in \mathcal{E}$ the cardinality of $G_1(Q|\mathcal{E})$ is bounded by $N_0$.

2. There exists $L_0$, depending on $\delta$, (and the geometry of $(X,d,\mu)$) so that for every $Q \in \mathcal{E}$ and every $P \in G_1(Q|\mathcal{E})$ we have

$$\frac{\mu(P)}{\mu(Q)} \geq \frac{1}{L_0}.$$

3. Moreover for $Q \in \mathcal{E}$ we have $\frac{1}{C} \text{diam } Q \leq \mu(Q) \leq C \text{diam } Q$ where $C$ is as in Lemma 1.

The collection $\mathcal{E}$ has been linked to problems concerning the isomorphic structure of $H^1(X,d,\mu)$; see [Mü].

There we found finitely many sequences of increasing, pure by atomic $\alpha$-algebras

$$[\mathcal{F}_1]^\infty_{n=1}, \ldots, [\mathcal{F}_N]^\infty_{n=1}$$

so that $H^1(X,f,\mu)$ is isomorphic to a complemented subspace of the direct sum of the related martingale $H^1$-spaces, namely to

$$H^1([\mathcal{F}_1]^\infty_{n=1}) \oplus \ldots \oplus H^1([\mathcal{F}_N]^\infty_{n=1}).$$
Although not stated explicitly there, when combined with the result in [Mü] one observes the following implications:

1. If

\[ E = \{ t \in X : t \text{ lies in infinitely many } Q \in \mathcal{C} \} \]

satisfies \( \mu(E) = 0 \), then for each \( j \leq N \): \( H^1(\mathcal{F}_n^j \cup_{n=1}^\infty) \) is isomorphic to a complemented subspace of \( (\sum H_n^1)_\mu \).

2. If

\[
\sup_{Q \in \mathcal{E}} \sup_{P \subset Q, P \in \mathcal{E}} \frac{\mu(P)}{\mu(Q)} < \infty,
\]

then for \( 1 \leq j \leq N \) \( H^1(\mathcal{F}_n^j \cup_{n=1}^\infty) \) is isomorphic to a complemented subspace of \( l^1 \).

3 A smooth unconditional basic sequence in \( L^2(X, \mu) \)

Let \( Q \) be in \( G_n(X|\mathcal{E}) \) and let \( P_0, P_1, \ldots, P_N \) be an enumeration of \( G_1(Q|\mathcal{E}) \). By the above preliminary remarks, \( N \leq N_0 \), where \( N_0 \) is independent of \( Q \) and

\[
\inf_i \frac{\mu(P_i)}{\mu(Q)} > \frac{1}{C}
\]

\[
\frac{1}{C} \leq \inf_i \frac{\mu(P_i)}{\mu(P_0)} \leq \sup_i \frac{\mu(P_i)}{\mu(P_0)} \leq C
\]

where \( C \), depending on \( \delta \), is independent of \( Q \).

Let, for \( 1 < i \leq N \), the function \( h_{Q,i} \) satisfy the following conditions

1. \( \text{supp } h_{Q,i} \subset Q \)
2. \( h_{Q,i} \) is constant when restricted to one the sets \( P_j, 0 \leq j \leq N \).

3. There exists \( C > 0 \) (not depending on \( \delta \) or \( N \)) so that for \( \alpha_i \in \mathbb{R} \)
\[
\frac{1}{C} \left( \sum_{i=1}^{N} \alpha_i^2 \right) \leq \left\| \sum_{i=1}^{N} h_{Q,i} \alpha_i \right\|_{L^2(X,\mu)} \leq C \left( \sum_{i=1}^{N} \alpha_i^2 \right)^{1/2}.
\]

Using ideas related to the local Pełczyński decomposition, such a sytem was constructed by B. Maurey [Ma1].

As martingale differences an orthogonal in \( L^2(X,\mu) \) we get for \( f \in L^2(X,\mu) \) a uniquely determined sequence of coefficients \( \alpha_{Q,i}, Q \in \mathcal{E} \) so that
\[
f = \sum_{Q \in \mathcal{E}} \sum_{i \in I_Q} h_{Q,i} \alpha_{Q,i}
\]
and
\[
||f||_2 = \left( \sum_{Q \in \mathcal{E}} \left\| \sum_{i \in I_Q} h_{Q,i} \alpha_{Q,i} \right\|_2^2 \right)^{1/2} \sim \left( \sum_{Q \in \mathcal{E}} |Q| \sum_{i \in I_Q} \alpha_{Q,i}^2 \right)^{1/2}.
\]

In the other words \( h_{Q,i} \in \mathcal{E}, i \in I_Q \) forms an unconditional basis in \( L^2(X,\mu) \). Using smoth partition of unity we will modify \( h_{Q,i} \) to become a smoth unconditional basis for \( L^2(X,\mu) \).

For \( Q \in G_n(X|\mathcal{E}) \) we have
\[
C_2 \delta^n \leq \text{diam } Q < C_1 \delta^n.
\]

For \( \tau < 1/C_1500 \) we consider a partition of unity \( \psi_k^{(n)}, k = 1, \ldots, N_n \), so that:
\[
\text{diam} ( \text{ supp } \psi_k^{(n)} ) \leq \tau \delta^n \quad \text{Lip}_\beta(\psi_k^{(n)}) \leq (\tau \delta^n)^{-\beta} \quad \sum_{k=1}^{N_n} \psi_k^{(n)} = 1.
\]
See [M.-S.2] for a construction of such a partition of unity. We use it here to define the kernel

\[ K_n(x, y) := \sum_{k=1}^{N_n} \psi_k^{(n)}(x) \psi_k^{(n)}(y) \frac{1}{\|\psi_k^{(n)}\|_1} \]

and define

\[ \tilde{\varphi}_{Q,i}(x) := \int_X K_{n+1}(x, y) h_{Q,i}(y) d\mu(y) \]
\[ \varphi_{Q,i}(x) := \frac{\tilde{\varphi}_{Q,i}(x)}{\|\varphi_{Q,i}\|_2}. \]

* By construction we obtain at once the following properties of \( \varphi_{Q,i} \):

\[ \text{supp} \ varphi_{Q,i} \subset \{ z \in X, \ \text{dist}(\text{supp} h_{Q,i}, z) \leq \tau \delta^n \} \]
\[ \text{Lip}_\beta(\varphi_{Q,i}) \leq \left( \frac{\mu(Q)}{\tau \delta} \right)^\beta \left( \frac{\mu(Q)}{\delta} \right)^{-1/2} \]
\[ \int_X \varphi_{Q,i} d\mu = 0. \]

And for \( Q \in \mathcal{E} \) fixed we obtain

\[ \frac{1}{C} \left( \sum_i \alpha_i^2 \right)^{1/2} \leq \left\| \sum_i \alpha_i \varphi_{Q,i} \right\|_{L^2(X, \mu)} \leq \left( \sum_i \alpha_i^2 \right)^{1/2} C \]

where \( C \) is independent of \( Q \) or \( \tau \) and depends only on the geometry of \( (X, d, \mu) \).

Moreover we have the following theorem.

**Theorem 3** Let \( E_1 = \bigcup_{n=1}^{\infty} G_{2n}(X|\mathcal{E}) \) then for

\[ f = \sum_{Q \in E_1} \sum_{i \in I_Q} \alpha_{Q,i} \varphi_{Q,i} \]

we have

\[ \left( \sum_Q \sum_{i \in I_Q} \alpha_{Q,i}^2 \right)^{1/2} \leq \|f\|_2 \leq \left( \sum_Q \sum_{i \in I_Q} \alpha_{Q,i}^2 \right)^{1/2}. \]
Proof. Suppose \( \text{supp} \varphi_{Q,i} \cap \text{supp} \varphi_{Q,i} \neq 0 \) then w.l.o.g. assume that

\[
\text{diam ( supp } \varphi_{Q,i} \text{)} \leq \text{diam ( supp } \varphi_{P,j} \text{)}.
\]

Let \( z \) be a fixed point in \( \text{supp} \varphi_{Q,i} \), then

\[
\int_X \varphi_{Q,i} \varphi_{P,j} d\mu = \int_X \varphi_{Q,i}(\varphi_{p,j} - \varphi_{p,j}(z)) d\mu \leq ||\varphi_{Q,i}||_{1} \sup_{x \in Q} |\varphi_{p,j}(x) - \varphi_{p,j}(z)| \leq |\delta^{1/2} \mu(Q)^{1/2}(\text{Lip}_\beta \varphi_{p,j}) \text{diam ( supp } \varphi_{Q})^\beta|
\]

\[
\leq \frac{\mu(Q)^{1/2+\beta}}{\mu(P)^{1/2+\beta}} \frac{1}{(\tau \delta)^{\beta}}.
\]

Then given \( Q \), consider \( P \in G_n(Q|E_1) \) then \( \frac{\mu(P)}{\mu(Q)} \leq C \delta^{2n} \) and \( G_n(Q|E) \) contains at most \( C \delta^{-2n} \) elements.

From these observations we see (using e.g. the argument in [U, Lemma 3.3]) that there exists \( C \) (not depending on \( \delta \) or \( \tau \)) so that

\[
C ||f||_2^2 + \frac{\delta^{2\beta}}{(\tau \delta)^\beta} \sum \sum \alpha_{Q,i}^2 \geq \frac{1}{C} \sum \sum \alpha_{Q,i}^2
\]

and

\[
||f||_2^2 < 2C \sum \sum \alpha_{Q,i}^2.
\]

Now choosing \( \delta \) so small that \( (\frac{\delta}{\tau})^\beta < \frac{1}{C^2} \) we obtain the result.

4 A smooth biorthogonal sequence in \( L^2(X, \mu) \)

Let \( G_n := G_n(X|E) \). Fix \( K \gg 1 \). Using Lemma 9 from [Mü3] we split \( G_n \) into \( P_{n,1}, \ldots, P_{n,l} \) so that for \( P, Q \in P_{n,j} \) we have \( \text{dist}(P, Q) \geq K \mu\{\mu(P), \mu(Q)\} \) and \( l \) depends only on \( K \) and the geometry of \( (X, d, \mu) \).
Now fix $m \in \mathbb{N} \setminus \{1\}$ $0 < s \leq m$ and $j \leq l$. Then let $\mathcal{F} := \bigcup_{k=0}^{\infty} \mathcal{P}_{mk+s,j} \cup \{X\}$.

Next fix $i_0 \leq N$ and for $Q \in \mathcal{F}$ let

$$\varphi_Q := \varphi_{Q,i_0}$$
$$\varphi_X := 1_X.$$

Observe now that for $P,Q \in \mathcal{P}_{mk+s,j}$

$$\text{supp } \varphi_Q \cap \text{supp } \varphi_P \neq \emptyset$$

and for each $P \in \mathcal{P}_{mk+s,j}$ and $r \in \mathbb{N}$ there exists at most one $Q \in \mathcal{P}_{m(k-r)+s,j}$ so that

$$\text{supp } \varphi_Q \cap \text{supp } \varphi_P = \emptyset.$$

Moreover by Theorem 3 the Gram matrix

$$G := \left( \int \varphi_Q \varphi_P d\mu \right)_{Q,P \in \mathcal{F}}$$

is invertible (and positive definite).

The Gram-matrix is used to construct a biorthogonal system from the $\varphi_Q - s$.

Theorem 4  

a) The coefficients $(a_{P,Q})_{P,Q \in \mathcal{F}}$ of the matrix $G^{-1/2}$ satisfy the estimates

$$|a_{P,Q}| \leq C \min \left\{ \frac{\mu(P) \mu(Q)}{\mu(Q)}, \frac{\mu(P)}{\mu(Q)} \right\}^{1/2-\alpha} \left( 1 + \frac{\text{dist}(P,Q)}{3 \max\{\mu(P),\mu(Q)\}} \right)^{-1-\alpha}$$

where $0 < \alpha < \beta/2$.

b) The functions

$$f_Q := \sum_{P \in \mathcal{F}} a_{P,Q} \varphi_P, \quad Q \in \mathcal{F}$$

form an orthonormal system in $L^2(X,\mu)$, the closed span of which coincides with the closed span of $\{\varphi_Q : Q \in \mathcal{F}\}$. 

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Proof. Part b) is a well known algebraic identity, so we shall concentrate on the Proof of part a):

Recall first that for \( Q \cap P \neq 0 \) we have the estimate

\[
\int \varphi_Q \varphi_P d\mu \leq \left( \frac{1}{\tau_0} \right)^{\beta} \min \left\{ \frac{\mu(Q)}{\mu(P)}, \frac{\mu(P)}{\mu(Q)} \right\}^{1/2+\beta}.
\]

Combining this with \( \text{supp} \varphi_Q \subseteq \{ z \in X : d(z, Q) \leq \alpha \} \) we obtain in particular

\[
(3.1) \int \varphi_Q \varphi_P d\mu \leq \min \left\{ \frac{\mu(P)}{\mu(Q)}, \frac{\mu(Q)}{\mu(P)} \right\}^{1/2+\beta} \left( 1 + \frac{\text{dist}(P, Q)}{\max\{\mu(P), \mu(Q)\}} \right)^{-1-\beta}.
\]

Moreover if \( Q, P \in \mathcal{F}, Q \neq P \) and \( \int \varphi_Q \varphi_P \neq 0 \) then necessarily

\[
\min \left\{ \frac{\mu(P)}{\mu(Q)}, \frac{\mu(Q)}{\mu(P)} \right\} \leq \delta^m.
\]

(To obtain this conclusion we introduced the splitting of \( G_n \) into \( \mathcal{P}_{n,j} \).)

Using this information, the proof of Lemma 3.3 in [U] gives that for \( a := (a_P)_{P \in \mathcal{F}}, a_P \in \mathbb{R} \) the following norm estimate for the matrix \( \text{Id} - G \) holds:

\[
\left\| (\text{Id} - G)a \right\|_{l^2} \leq \left( \frac{\delta^m}{(\delta \tau)^{\beta}} \right) C_2 \|a\|_{l^2}.
\]

(\( C_2 \) is a universal constant.) Now put \( R = \text{Id} - G \). Observe that \( G^{-1/2} \) can be developed in a power series of \( R \), indeed:

\[
G^{-1/2} = \sum_{k=0}^{\infty} C_k R^k
\]

where \( C_k = o(k^{-1/2}) \).

Clearly the coefficients \( R(P, Q), P, Q \in \mathcal{F} \) of \( R \) satisfy the estimates (3.1).

Now by a result of Frazier-Jawerth, estimates of this form are stable under the formation of products. More precisely by [F-J, Theorem 9.1] for \( 0 < \gamma < \beta \)
there exists $C_1 > 1$ so that for each $k \in \mathbb{N}$ the coefficients $R^{(k)}(P, Q)$ of $R^k$ satisfy

$$R^{(k)}(P, Q) \leq \left( \frac{1}{\tau \beta} \right)^\beta C_1 \min \left\{ \frac{\mu(P)}{\mu(Q)}, \frac{\mu(Q)}{\mu(P)} \right\}^{1/2+\gamma} \left( 1 + \frac{\text{dist}(P, Q)}{\max\{\mu(P), \mu(Q)\}} \right)^{-1-\gamma}.$$

On the other hand we trivially have

$$R^{(k)}(P, Q) \leq ||R^k||_{l^2} \leq \left[ \frac{\delta m C_2}{(\tau \delta)^\beta} \right]^k.$$

Fix now $P, Q \in \mathcal{F}$ and let

$$\sigma(P, Q) := \left\{ \min \frac{\mu(P)}{\mu(Q)}, \frac{\mu(Q)}{\mu(P)} \right\}^{1/2+\gamma} \left( 1 + \frac{\text{dist}(P, Q)}{\max\{\mu(P), \mu(Q)\}} \right)^{-1-\gamma}.$$

Next consider the number $k_0 = k_0(P, Q)$ which is defined by

$$k_0 := \left[ \frac{\gamma \log \sigma(P, Q)}{2 \log(C_1/(\delta \tau))} \right].$$

We assume that $k_0$ is integer. At this point we make a suitable choice for $m$. Namely we choose $m$ so that

$$\frac{\log(C_2 \delta^m/((\delta \tau)^\beta))}{\log((\tau \delta)^\beta C_1)} \geq 1 + \frac{\gamma}{2}.$$ 

(Observe that $m$ is of course not depending on $P, Q$.)

We then have the numerical estimates:

$$\sum_{k=0}^{k_0} \left( \frac{C_1}{(\delta \tau)^\beta} \right)^k \leq \sigma(P, Q)^{-\gamma/2} C'$$

$$\sum_{k=k_0+1}^{\infty} \left( \frac{\delta^m}{(\tau \delta)^\beta C_2} \right)^k \leq \sigma(P, Q)^{1+\gamma/2} C'.$$

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Hence the following estimates hold for $P,Q \in \mathcal{F}$

$$\sum_{k=0}^{k_0} R^{(k)}(P,Q) \leq \sigma(P,Q)^{1+\gamma/2} C$$

$$\sum_{k=k_0+1}^{\infty} R^{(k)}(P,Q) \leq \sigma(P,Q)^{1+\gamma/2} C.$$ 

Summing up we have for the coefficients of $G^{-1/2}$ the following estimate

$$G^{-1/2}(P,Q) \leq C \min \left\{ \frac{\mu(P)}{\mu(Q)}, \frac{\mu(Q)}{\mu(P)} \right\}^{1/2+\gamma/2} \left( 1 + \frac{\text{dist}(P,Q)}{\max\{\mu(P), \mu(Q)\}} \right)^{-1-\gamma/2} C.$$ 

**Remark.** The above proof merges arguments of Franzier & Jawerth [F,J] and Uchiyama [U] with those of Jaffard & Meyer [J-M] to conclude that – in the language if Franzier and Jawerth – $G^{-1/2}$ is an almost diagonal matrix.

As a consequence $f_Q$ is centered around $Q$. More precisely we have the following pointwise estimate.

**Lemma 5** There exists $C = C(\delta) \sim \log \delta$ so that

1. for $x \in X$

   $$|f_Q(x)| \leq C \left( 1 + \frac{\text{dist}(x,Q)}{\mu(Q)} \right)^{-1-\alpha/2} \frac{1}{\mu(Q)^{1/2}}$$

2. for $x,y \in X$ with $d(x,y) \leq \mu(Q)$

   $$|f_Q(x) - f_Q(y)| \leq C \left( 1 + \frac{\text{dist}(x,Q)}{\mu(Q)} \right)^{-1-\alpha/2-\beta} \frac{d(x,y)^{\beta}}{\mu(Q)^{1/2+\beta}}.$$
Proof. Given the estimates of $G^{-1/2}$ the proof is quite standard, and only the argument for part 1) will be outlined. Fix $x \in X$ then clearly for some $C > 1$

$$|f_Q(x)| \leq C \sum_{\{P : \text{dist}(x,P) \leq C\mu(P)\}} \min\left\{\frac{\mu(P)}{\mu(Q)}, \frac{\mu(Q)}{\mu(P)}\right\}^{1/2+\alpha}$$

$$\times \frac{1}{\mu(P)^{1/2}} \left(1 + \frac{\text{dist}(P,Q)}{\max\{\mu(P)\mu(Q)\}}\right)^{-1-\alpha}.$$ 

\[\square\]

Let now $K \in \mathbb{N}$ be such that

$$K\mu(Q) \leq \text{dist}(x,Q) \leq 2K\mu(Q),$$

and split the above sum into three, by dividing the index set:

$$\{P : \text{dist}(x,P) < C\mu(P)\} = A \cup B \cup D$$

where

$$A := \{P : \text{dist}(x,P) \leq C\mu(P) \text{ and } \mu(P) > K\mu(Q)\}$$

$$B := \{P : \text{dist}(x,P) \leq C\mu(P) \text{ and } \mu(Q) \leq \mu(P) \leq K\mu(Q)\}$$

$$D := \{P : \text{dist}(x,P) \leq C\mu(P) \text{ and } \mu(P) \leq \mu(Q)\}.$$ 

Case 1

$$\sum_A \left\{\frac{\mu(Q)}{\mu(P)}\right\}^{1/2+\alpha} \left(1 + \frac{\text{dist}(P,Q)}{\mu(P)}\right)^{-1-\alpha} \frac{1}{\mu(P)^{1/2}}$$

$$\leq \sum_{\mu(P)>K\mu(Q)} \left\{\frac{\mu(Q)}{\mu(P)}\right\}^{1/2+\alpha} \frac{1}{\mu(P)^{1/2}}$$

$$\leq \frac{\mu(Q)^{\alpha+1/2}}{(K\mu(Q))^{1+\alpha}} = \frac{1}{\mu(Q)^{1/2}K^{1+\alpha}}.$$ 

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\[
\leq \frac{1}{\mu(Q)^{1/2}} \left(1 + \frac{\text{dist}(x,Q)}{\mu(Q)}\right)^{-1-\alpha}
\]

**Case 2** Let \(\mu(P) = \mu(Q)\delta^{-k}\) and \(\delta^{-k_1} = K\). Then

\[
\sum_{\mathcal{B}} \left\{ \frac{\mu(Q)}{\mu(P)} \right\}^{1/2+\alpha} \left(1 + \frac{\text{dist}(P,Q)}{\mu(P)}\right)^{-1-\alpha} \frac{1}{\mu(P)^{1/2}}
\]

\[
\leq \sum_{k=0}^{k_1} \delta^{k(1/2+\alpha)} \frac{\mu(Q)^{\delta^k}}{k} \frac{1}{\mu(Q)^{1/2}}
\]

\[
\leq \mu(Q)^{1+\alpha} \mu(Q)^{-1/2} \frac{k_1}{K^{1+\alpha}}
\]

\[
\leq \mu(Q)^{-1/2} \left|\log \delta\right| \log K \frac{1}{K^{1+\alpha}}
\]

\[
\leq c_\alpha (\log \delta) \mu(Q)^{-1/2} \left(1 + \frac{\text{dist}(x,Q)}{\mu(Q)}\right)^{-1-\alpha/2}
\]

**Case 3**

\[
\sum_{\mathcal{D}} \left\{ \frac{\mu(Q)}{\mu(P)} \right\}^{1/2+\alpha} \left(1 + \frac{d(P,Q)}{\mu(Q)}\right)^{-1-\alpha} \frac{1}{\mu(P)^{1/2}} \leq \begin{cases} \frac{1}{\mu(Q)^{1/2}} & \text{if dist}(x,Q) \leq C\mu(Q) \\ 0 & \text{otherwise.} \end{cases}
\]

5 **Bounded Projections in** \(H^1(X,d,\mu)\)

First we determine the norm of \(f_Q, Q \in \mathcal{F}\) in \(H^1(X,d,\mu)\). Given the decay of \(f_Q\) and the fact that \(\int_X f_Q d\mu = 0\) it is natural to use molecules as in [Woj].

**Theorem 6** There exists \(C = C(\delta,\alpha)\) and \(\varepsilon > 0\) so that for each \(Q \in \mathcal{F}\) we have

\[
\left(\int f_Q^2 \frac{d\mu}{\mu(Q)}\right) \left(\int f_Q^2(x) d(x,Q)^{1-\varepsilon} \frac{d\mu}{\mu(Q)}\right)^{1/\varepsilon} \leq C(\delta,\alpha).
\]

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PROOF. First we have clearly
\[ \int f_Q^2 \frac{d\mu}{\mu(Q)} = \frac{1}{\mu(Q)}. \]
Let \( Q_n := \{ x \in X, \mu(Q)(2^n - 1) \leq d(x, Q) \leq (2^{n+1} - 1)\mu(Q) \} \), then
\[ \int f_Q^2(x)d(x, Q)^{1+\varepsilon}d\mu = \sum_{n=0}^{\infty} \int_{Q_n} f_Q^2(x)d(x, Q)^{1+\varepsilon}d\mu. \]
Let us first consider the case \( n \geq 1 \):
\[ \int_{Q_n} f_Q^2(x)d(x, Q)^{1+\varepsilon}d\mu \leq C(\delta) \int_{Q_n} \left( 1 + \frac{d(x, Q)}{\mu(Q)} \right)^{-2-\alpha} (d(x, Q)^{1+\varepsilon} \frac{d\mu}{\mu(Q)^{1/2}} \leq C(\delta)2^{n(-2-\alpha)} \mu(Q)^{-1} (2^{n+1} \mu(Q))^{1+\varepsilon}2^{n+1} \mu(Q) \leq C(\delta) \mu(Q)^{1+\varepsilon}4 \cdot 2^{n(-\alpha+\varepsilon)}. \]
And for \( n = 0 \) we have
\[ \int_{Q_0} f_Q^2(x)d(x, Q)^{1+\varepsilon}d\mu \leq \mu \frac{1}{\mu(Q)} \int_{Q_0} d(x, Q)^{1+\varepsilon}d\mu \leq \mu(Q)^{1+\varepsilon}. \]
Summing up we obtain
\[ \int_X f_Q^2(x)d(x, Q)^{1+\varepsilon}d\mu \leq C(\delta) \frac{1}{1 - 2^{-\alpha+\varepsilon}} \mu(Q) \]
and finally
\[ \left( \int f_Q^2 \frac{d\mu}{\mu(Q)} \right)^{1/\varepsilon} \left( \int f_Q^2(x)d(x, Q)^{1+\varepsilon} \frac{d\mu}{\mu(Q)} \right)^{1/\varepsilon} \leq \left( C(\delta) \frac{1}{1 - 2^{-\alpha+\varepsilon}} \right)^{1/\varepsilon}. \]
Choosing \( \varepsilon = \alpha/2 \) gives the required estimate.
We shall show next that \( \text{span} \{ f_Q : Q \in \mathcal{F} \} \) is a complemented subspace of \( H^1(X, d, \mu) \) and that \( \{ f_Q : Q \in \mathcal{F} \} \) is equivalent to a martingale difference sequence in \( H^1([\mathcal{F}_n]) \) (where \( \mathcal{F}_n \) was defined in Section 1).
The operator $Pf = \sum_{Q \in \mathcal{F}} (f|_{f_Q})f_Q$ is clearly a projection, i.e., satisfies $P^2 = P$.

Theorem together with the smoothness and localization properties of $f_Q$ will be used to show that $P$ defines a bounded projection on $H^1(X, d, \mu)$.

**Theorem 7** There exists $C > 0$ so that for $f \in H^1(X, d, \mu)$

$$
\left\| \sum_{Q \in \mathcal{F}} (f|_{f_Q})f_Q \right\|_{H^1(X, d, \mu)} \leq \|f\|_{H^1(X, d, \mu)}.
$$

**Remark.** The following proof not new! It is a simple modification of the proof in [Woj Theorem], and is included here just for sake of completeness.

**Proof.** It is enough to consider atoms in $(X, d, \mu)$: Let $a : X \to \mathbb{R}$ be supported on a ball $B$ so that $\int ad\mu = 0$, $\|a\|_{\infty} \leq \mu(B)^{-1}$ and $\mu(B) \leq C \text{ diam } B$. Then decompose $\mathcal{F} = E \cup F \cup G$ where

$$
E = \{Q \in \mathcal{F} : \mu(Q) \geq \mu(B)\}
$$

$$
F = \{Q \in \mathcal{F} : \mu(Q) \geq \mu(B) \text{ and } \text{dist}(P, Q) \leq L\mu(Q)\}
$$

$$
G = \{Q \in \mathcal{F} : \mu(Q) \geq \mu(B) \text{ and } \text{dist}(P, Q) \geq L\mu(Q)\}.
$$

**Case 1** By the triangle inequality we have: using Theorem 6:

$$
\left\| \sum_{Q \in E} (a|_{f_Q})f_Q \right\|_{H^1} \leq \sum_{Q \in E} |(a|_{f_Q})| \cdot \|f_Q\|_{H^1}
$$

$$
\leq \sum_{Q \in E} \mu(Q)^{1/2} \frac{\text{diam } B^\beta}{\mu(Q)^{1/2+\beta}} \left\{ 1 + \frac{d(B, Q)}{\mu(Q)} \right\}^{-1-\alpha/2-\beta}
$$
\[
\sum_{\mu(Q) > \mu(B)} \left\{ \sum_{k=1}^{\mu(Q)-1} k^{-1-\alpha/2-\beta} \right\} \text{diam } B^3 \mu(Q)^{-3} \\
\leq \frac{1}{\alpha/2 + \beta} \left\{ \sum_{\mu(Q) > \mu(B)} \mu(Q)^{-\beta} \right\} \text{diam } (B) \\
\leq \frac{1}{\alpha/2 + \beta} \mu(B)^{-\beta} \text{diam } (B) \leq \text{const.}
\]

**Case 2** Again by triangle inequality and Theorem 6:

\[
\left\| \sum_{Q \in G} (a|f_Q)f_Q \right\|_{H^1} \leq \sum_{Q \in G} \int_B |f_Q| d\mu(B)^{-1}\mu(Q)^{1/2} \\
\leq C \sum_{Q \in G} \mu(Q)^{-1/2} \left( 1 + \frac{\text{dist}(B,Q)}{\mu(Q)} \right)^{1-\alpha/2} \mu(Q)^{1/2} \\
\leq C \sum_{\mu(Q) \leq \mu(B)} \left( 1 + \frac{\mu(B)}{\mu(Q)} \right)^{-\alpha/2} \\
\leq C \sum_{\mu(Q) \leq \mu(B)} \left( \frac{\mu(Q)}{\mu(B)} \right)^{\alpha/2} \leq C \text{const.}
\]

**Case 3** Here we show that \( \sum_{Q \in F} (a|f_Q)f_Q \) is a molecule.

Consider first

\[
\int_B \left\| \sum_{Q \in F} (a|f_Q)f_Q \right\|^2 d(x,x_B)^{1+\epsilon} \leq C \mu(B)^{1+\epsilon} \|a\|_2^2 \leq c\mu(B)^\epsilon.
\]

Then we consider

\[
\int_{X \setminus B} \left\| \sum_{Q \in F} (a|f_Q)f_Q \right\|^2 d(x,x_B)^{1+\epsilon} d\mu(x) \\
\leq C \|a\|_2^2 \sum_{Q \in F} \mu(Q)^{-1} \int_{X \setminus B} \left( 1 + \frac{d(x,Q)}{\mu(Q)} \right)^{-2-\alpha} d(x,x_B)^{1+\epsilon} d\mu \\
\leq C \mu(B)^{-1} \sum_{\mu(Q) \leq \mu(B)} \frac{\mu(B)}{\mu(Q)} \int_{X \setminus B} \left( \frac{d(x,x_B)}{\mu(Q)} \right)^{-2-1} d(x,x_B)^{1+\epsilon} d\mu
\]
\[
\leq C \sum_{\mu(Q) \leq \mu(B)} \mu(Q)^\alpha \int_{X \setminus B} d(x, x_B)^{-1+\varepsilon-\alpha} d\mu
\]
\[
\leq C \left\{ \sum_{\mu(Q) \leq \mu(B)} \mu(Q)^\alpha \mu(B)^\alpha \right\} \mu(B)^\varepsilon.
\]

Summing up we have for \(\varepsilon < \alpha\):
\[
\left( \int_X \left\| \sum_{Q \in F} |a| f_Q f_Q \right\|^2 d(x, x_B)^{1+\varepsilon} \right)^{1/\varepsilon} \leq C \mu(B)
\]
and
\[
\int_X \left\| \sum_{Q \in F} |a| f_Q f_Q \right\|^2 d\mu \leq ||a||^2 \leq C \mu(B)^{-1}.
\]
Multiplying the above estimates one sees that \(\sum_{Q \in F} |a| f_Q f_Q\) is indeed a molecule.

In Section 1, using successive, generations of \(\varepsilon\), an increasing sequence of \(\alpha\)-algebra, \((\mathcal{F}_n)_{n=1}^\infty\) has been defined.

In Section 2, we defined on unconditional basis \(\{h_{Q,i}, Q \in \varepsilon, i \in I_Q\}\) for \(L^2(X, \mu)\). As recorded in [Ma2] this system forms an unconditional basis in the martingale \(H^1([\mathcal{F}_n])\) space.

We fix now \(i_0 \leq N\) as in Section 3 and let
\[
h_Q = h_{Q,i_0}, \quad Q \in \mathcal{F}.
\]
The family \(\{h_Q : Q \in \mathcal{F}\}\) forms a three valued martingale difference sequence with respect to the filtration \([\mathcal{F}_n]_{n=1}^\infty\) satisfying the following condition:
\[
\text{supp } h_Q \cap \text{ supp } h_P \neq \emptyset
\]
implies
\[ \text{supp } h_Q \subseteq \text{supp } h_P \text{ or } \text{supp } h_P \subseteq \text{supp } h_Q. \]

We will show next, that \( \{f_Q, Q \in \mathcal{F}\} \) in \( H^1(X, d, \mu) \) is equivalent to \( \{h_Q : Q \in \mathcal{F}\} \) in \( H^1([\mathcal{F}_n]) \).

Let \( Y \) be the closed linear span of \( \{f_Q : Q \in \mathcal{F}\} \) equipped with the norm inherited by \( H^1(X, d, \mu) \), then we have:

**Theorem 8**

\[
T : Y \to H^1([\mathcal{F}_n]) \\
f_Q \to h_Q
\]

extends to a bounded operator.

**Proof.** Let \( f \in Y \) implies clearly \( f \in H^1(X, d, \mu) \). Hence there exist atoms \( a_i \), and \( \lambda_i \in \mathbb{R} \) so that

\[
f = \sum \lambda_i a_i \text{ and } \sum |\lambda_i| \leq C\|f\|_{H^1}.
\]

Moreover

\[
f = Pf = \sum \lambda_i Pa_i
\]

and

\[
\|Pa_i\|_{H^1(x,d,\mu)} \leq C|a_i|_{H^1(x,d,\mu)}.
\]

So it remains to show that there exists \( C > 0 \) so that for any atom \( a \) on \( (X, d, \mu) \) we have \( ||TPa||_{H^1([\mathcal{F}_n])} \leq C. \)

To estimate

\[
TPa = \sum_{Q \in \mathcal{F}} (a|f_Q)h_Q
\]

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in $H^1([F_n])$ we observe that

$$||h_Q||_{H^1([F_n])} \leq C\mu(Q)^{1/2},$$

split $\mathcal{F}$ into $E \cup F \cup G$ as in the proof of Theorem (7) and argue exactly as P. Wojtaszczyk in [Woj2, Theorem 5].

Let $Z$ be closure of the linear span of $\{h_Q : Q \in \mathcal{F}\}$ in $H^1([F_n])$, equipped with the norm inherited by $H^1([F_n])$. By [Ma1, Theorem 2] $\{h_{Q,i}, Q \in \varepsilon, i \in I_Q\}$ is an unconditional basis in $H^1([F_n])$ the natural restriction operator

$$Q : H^1([F_n]) \rightarrow H^1([F_n])$$

$$\sum_{Q \in \varepsilon} \sum_{i \in I_Q} \alpha_{Q,i} h_{Q,i} \rightarrow \sum_{Q \in \mathcal{F}} a_{Q,i_0} h_{Q,i_0}$$

is a bounded projection.

Moreover given any atom $a$ in the martingale $H^1([F_n])$ space then $Qa$ is again an atom in $H^1([F_n])$.

(In Section 1 we remarked that the filtration $[F_n]_{n=1}^{\infty}$ is regular (see [G, p 96]) and therefore an atom in $H^1([F_n])$ is simply a function $a : X \rightarrow \mathbb{R}$ for which is supported in an atom $Q$ of $F_n$ so that $||a||_{\infty} \leq \mu(Q)^{-1}C$ and $\int a d\mu = 0$.) Now we have the following

**Theorem 9**

$$S : Z \rightarrow H^1(X, d, \mu)$$

$$h_Q \rightarrow f_Q$$

defines a bounded operator.
PROOF. Let $f \in Z$. Then there exists a sequence of atoms $a_i$ for $H^1([\mathcal{F}_n])$ and $\lambda_i \in \mathbb{R}$ so that

$$f = \sum \lambda_i a_i$$

and

$$\sum_{i=1}^{\infty} |\lambda_i| \leq C\|f\|_{H^1([\mathcal{F}_n])}.$$  

As

$$f = Qf = \sum_{i=1}^{\infty} \lambda_i Qa_i,$$

we have: that for any $f \in Z$ there exists a sequence of atoms $q_i$: in $H^1([\mathcal{F}_n])$, $\lambda_i \in \mathbb{R}$ and $q_i \in Z$ (sic!) satisfying

$$f = \sum_{i=1}^{\infty} \lambda_i q_i \quad \text{and} \quad \sum_{i=1}^{\infty} |\lambda_i| \leq C\|f\|_{H^1([\mathcal{F}_n])}.$$  

It is therefore enough to consider atoms $q$ of the form

$$q = \sum_{Q \in \mathcal{F}} \alpha_Q h_Q$$

and to show that

$$\|Sq\|_{H^1(X,d,\mu)} = \left\| \sum_{Q \in \mathcal{F}} \alpha_Q f_Q \right\|_{H^1(X,d,\mu)}$$

is bounded by an absolute constant independent of $q$.

As moreover $\{h_Q : Q \in \mathcal{F}\}$ is biorthogonal it remains to show that there exists $C > 0$ so that for any atom $q \in Z$

$$\left\| \sum_{Q \in \mathcal{F}} (q|h_Q)f_Q \right\|_{H^1(X,d,\mu)} \leq C.$$  

To do so we just follow the argument in [Woj Theorem 5] again.
6 Dénouement

In this paragraph we will give a solution to the classification problem of atomic $H^1(X, d, \mu)$ spaces:

In addition to the material developed in Sections 1 — 4 we will use the following ingredients:

- The isomorphic classification of martingal $H^1$-spaces generated by an increasing sequence of purely atomic $\sigma$-algebras.

- The isomorphic classification three-valued martingale difference sequences in martingale $H^1$ spaces.

- $H^1(X, d, \mu)$ is isomorphic to a complemented subspace of martingale $H^1$ space.

**Theorem 10** If $H^1(X, d, \mu)$ is infinite dimensional, it is isomorphic to one of the following spaces: $H^1(\delta), (\sum H^1_n)^\mu, l^1$.

**Proof.**

1. **The Case $H^1(\delta)$**

Let $E = \{t \in X : t \text{ lies in infinitely many elements of } \varepsilon\}$. Suppose $\mu(E) > 0$. Then there exists a subcollection $\mathcal{F} \subset \varepsilon$ as constructed in Section 3 so that

$$F := \{t \in X : t \text{ lies in infinitely many elements of } \mathcal{F}\}$$

satisfies $\mu(F) > 0$. 

By [Mu2], \( \text{span}\{h_Q : Q \in \mathcal{F}\} \) equipped with the norm of \( H^1([\mathcal{F}_n]) \) is then isomorphic to \( H^1(\delta) \). Hence by Section 4
\[
H^1(\delta) \overset{C}{\hookrightarrow} H^1(X,d,\mu).
\]
On the other hand by the results in [Mü3] and [Ma3]
\[
H^1(X,d,\mu) \overset{C}{\hookrightarrow} H^1(\delta).
\]
So the Pelczyński decomposition method gives that \( H^1(\delta) \) is isomorphic to \( H^1(X,d,\mu) \).

2. The Case \( (\sum H^1_n)_{\mu} \)

Suppose that \( \mu(E) = 0 \) and \( \sup_{Q \in \varepsilon} \sum_{P \subset Q, P \in \varepsilon} \mu(P)/\mu(Q) = \infty \). Then there exists a subcollection \( \mathcal{F} \subset \varepsilon \) constructed as in Section 3 so that \( \mu(F) = 0 \) and
\[
\sup_{Q \in \mathcal{F}} \sum_{P \subset Q, P \in \mathcal{F}} \mu(P)/\mu(Q) = \infty.
\]
By the result of [Mu2] \( \text{span}\{h_Q : Q \in \mathcal{F}\} \) is then isomorphic to \( (\sum |H^1_n|_{\mu}) \). Hence by Section 4
\[
(\sum H^1_n)_{\mu} \overset{C}{\hookrightarrow} H^1(X,d,\mu).
\]
On the other hand by [Mu2] \( \mu(E) = 0 \) implies
\[
H^1(X,d,\mu) \overset{C}{\hookrightarrow} (\sum H^1_n)_{\mu}.
\]
So the Pelczyński decomposition method gives that \( (\sum H^1_n)_{\mu} \) is isomorphic to \( H^1(X,d,\mu) \).
3. The Case $l^1$

Suppose
\[ \sup_{Q \in \varepsilon} \sum_{P \subseteq Q, P \in \varepsilon} \frac{\mu(P)}{\mu(Q)} < \infty \]

then by [Mü1] and [Mü3]
\[ H^1(X, d, u) \overset{\mathcal{C}}{\rightarrow} l^1. \]

By a theorem of Pełczyński a complemented subspace of $l^1$ is either finite dimensional or isomorphic to $l^1$. 

\[ \blacksquare \]
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