On strong duality in linear copositive programming

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Abstract
The paper is dedicated to the study of strong duality for a problem of linear copositive programming. Based on the recently introduced concept of the set of normalized immobile indices, an extended dual problem is deduced. The dual problem satisfies the strong duality relations and does not require any additional regularity assumptions such as constraint qualifications. The main difference with the previously obtained results consists in the fact that now the extended dual problem uses neither the immobile indices themselves nor the explicit information about the convex hull of these indices. The strong duality formulations presented in the paper for linear copositive problems have similar structure and properties as that proposed in the works by M. Ramana, L. Tuncel, and H. Wolkowicz, for semidefinite programming.

Keywords Linear Copositive Programming · Strong duality · Normalized immobile index set · Extended dual problem · Constraint Qualification · Semi-infinite Programming (SIP) · Semidefinite programming (SDP)

Mathematics Subject Classification 90C25 · 90C30 · 90C34

1 Introduction

Linear Copositive Programming problems can be considered as linear programs over the convex cone of so-called copositive matrices (i.e. matrices which are positive semi-defined on the non-negative orthant). Copositive problems form a special class of conic optimization problems and have many important applications, including $NP$-hard problems. For the references on applications of Copositive Programming see e.g. [5,9] and others.
Linear copositive problems are closely related to that of linear *Semi-Infinite Programming* (SIP) and *Semidefinite Programming* (SDP). Linear copositive and semidefinite problems are particular cases of SIP problems, but Linear Copositive Programming deals with more challenging and less studied problems than that of SDP. The literature on theory and methods of SIP, SDP and Linear Copositive Programming is very interesting and quite extensive, we refer the interested readers to [5,6,9,12,19,20,25,26], and the references therein.

Although the concepts of copositivity and complete positivity were originally formulated in 1952 in the paper by T. Motzkin [22], an active research in theory and methods of Linear Copositive Programming has begun only in the recent decades in papers of I. Bomze, M. Dür, E. de Klerk, and others (see [4,5,8]). Optimality conditions and related duality relationships are among the central topics of convex optimization and the importance of their research is well recognized (see e.g. [6,27], and the references therein). Optimality conditions are a crucial issue in the study of any optimization problem since they allow not only to test the optimality of a given feasible solution, but also to develop efficient numerical methods. As it is mentioned in [20], the duality plays a central role in detecting infeasibility, lower-bounding of the optimal objective value, as well as in design and analysis of iterative algorithms.

Often the studies on optimality conditions and duality for finite and (semi) infinite programming use certain regularity assumptions, so-called *constraint qualifications* (CQ). Such assumptions allow us to guarantee in some particular cases a strong (or zero-gap) duality which means that the optimal values of the primal and dual objective functions are equal and at least one of them is attained, hence the difference between these values (the duality gap) vanishes.

It is a known fact that in Linear Programming (LP), the strong duality is guaranteed without any CQ ([6]). The duality results for LP can be generalized to some particular classes of optimization problems. Several attempts were done to obtain CQ-free optimality and strong duality results for different classes of convex SIP problems (see e.g. [11,13,15]).

In [26,28], a CQ-free duality theory for general conic optimization problems was developed in terms of so-called *minimal cone*. Being quite universal, this theory has one disadvantage in terms of its application, namely, it is very abstract. In a number of publications, various explicit dual formulations were obtained by applying this theory to SDP problems (see e.g. [12,25,27,28]) and other optimization problems over symmetric (i.e., self-dual and homogeneous) cones (see [1,7,24]). As it was mentioned in [23,24], finding a broader family of conic problems for which such explicit dual formulations are possible, is an open problem.

Linear copositive problems belong to a wider and more complex class of linear conic problems than that of SDP, namely, to the class of optimization problems over cones of copositive and or completely positive matrices that are neither self-dual nor homogeneous (see [10]). The duality theory for these problems is not well studied yet. It is worth mentioning that almost all duality results and optimality conditions for Linear Copositive Programming are formulated under the Slater CQ ([2,4]).

In our papers [13,15,16], and others, we developed a new approach to optimality in SIP and SDP. This approach is based on the notion of *immobile indices* of constraints of an optimization problem, which refers to the indices of the constraints that are active for all feasible solutions. In [14,17], we have applied our approach to problems of Linear Copositive Programming and successfully obtained new explicit CQ-free optimality conditions and strong duality results. It is essential that to formulate our results, we used either the immobile indices ([17]), or the vertices of the convex hull of the *normalized immobile index set* ([14]).
In this paper, we further develop our approach to linear copositive problems and use it to obtain a new dual problem which we refer to here as the extended dual problem. As well as the regularized dual problem from [14], the extended dual one is constructed using the notion and properties of the normalized immobile indices, but in the formulation of this problem neither these indices nor the vertices of the convex hull of the corresponding index set are present. This permitted us to formulate the extended dual problem for the linear copositive problem in an explicit form and avoid the use of additional procedures for finding the immobile indices. The new extended dual problem satisfies the strong duality relations without any CQ.

The main worthiness of the obtained results consists in the fact that the new dual formulations for Linear Copositive Programming are akin to the dual problems proposed by M.V. Ramana, L. Tuncel and H. Wolkowicz in [25,26,28] for SDP: given a linear copositive problem, we construct its dual one using the same rules as it was done in [25,26,28] for SDP problems, but with the help of different dual cones. This relation not only confirms the already known deep connection between copositive and semidefinite problems but, taking into account the impact of the duality theory developed by M. Ramana et al. for SDP, permits one to expect that the duality results proposed in this paper, will be also very promising in Linear Copositive Programming.

It is worthwhile mentioning here that at present, with exception of [14], there are no explicit strong duality formulations without CQs for Linear Copositive Programming. All the results presented in the paper are original and cannot be obtained as a direct extension of any previous results.

The paper is organized as follows. Section 1 hosts Introduction. In Sect. 2, given a linear copositive problem, we formulate the corresponding normalized immobile index set and establish some new properties of this set. An extended dual problem is formulated in Sect. 3. We prove here that the strong duality property is satisfied. In Sect. 4, we compare the obtained duality results with that presented in [25,26,28] for SDP. It is shown that the compared dual formulations for Linear Copositive Programming and SDP are similar, being both CQ-free and providing strong duality. Although these dual formulations were obtained using different techniques, they almost coincide being applied to the class of linear SDP problems. The final Sect. 5 contains some conclusions.

2 Linear copositive programming problem

Here and in what follows, we use the following notations. Given an integer \( p > 1 \), \( \mathbb{R}_+^p \) denotes the set of all \( p \) vectors with non-negative components, \( S(p) \) denotes the space of real symmetric \( p \times p \) matrices, \( S_+(p) \) stays for the cone of symmetric positive semidefinite \( p \times p \) matrices, and \( \mathbb{COP}^p \) for the cone of symmetric copositive \( p \times p \) matrices

\[
\mathbb{COP}^p := \{ D \in S(p) : t^\top D t \geq 0 \ \forall t \in \mathbb{R}_+^p \}.
\]  

The space \( S(p) \) is considered here as a vector space with the trace inner product \( A \bullet B := \text{trace} \ (AB) \), for \( A, B \in S(p) \).

Consider a linear Copositive Programming problem in the form

\[
\min_x c^\top x \ \text{s.t.} \ A(x) \in \mathbb{COP}^p,
\]
where the decision variable is the $n$-vector $x = (x_1, \ldots, x_n)^\top$ and the constraints matrix function $A(x)$ is defined as

$$A(x) := \sum_{i=1}^n A_i x_i + A_0,$$

matrices $A_i \in S(p)$, $i = 0, 1, \ldots, n$, and vector $c \in \mathbb{R}^n$ are given.

Problem (2) can be rewritten as follows:

$$\min_x c^\top x \quad \text{s.t.} \quad t^\top A(x)t \geq 0 \quad \forall t \in \mathbb{R}_+^p. \quad (3)$$

It is well known that the copositive problem (3) is equivalent to the following convex SIP problem:

$$\min_x c^\top x \quad \text{s.t.} \quad t^\top A(x)t \geq 0 \quad \forall t \in T, \quad (4)$$

with a $p$-dimensional compact index set in the form of a simplex

$$T = \{ t \in \mathbb{R}_+^p : e^\top t = 1 \}, \quad (5)$$

where $e = (1, 1, \ldots, 1)^\top \in \mathbb{R}^p$, $t = (t_k, k \in P)^\top$, $P = \{1, 2, \ldots, p\}$.

Denote by $X$ the set of feasible solutions of problems (2)–(4),

$$X := \{ x \in \mathbb{R}^n : t^\top A(x)t \geq 0 \quad \forall t \in \mathbb{R}_+^p \} = \{ x \in \mathbb{R}^n : t^\top A(x)t \geq 0 \quad \forall t \in T \}.$$

Evidently, the set $X$ is convex.

According to the definition (see e.g. [17]), the constraints of the SIP problem (4) satisfy the Slater condition if

$$\exists \bar{x} \in \mathbb{R}^n \text{ such that } t^\top A(\bar{x})t > 0 \quad \forall t \in T, \quad (6)$$

and the constraints of the copositive problem (2) satisfy the Slater condition if

$$\exists \bar{x} \in \mathbb{R}^n : A(\bar{x}) \in \text{int} COP^p = \{ D \in S(p) : t^\top Dt > 0 \quad \forall t \in \mathbb{R}_+^p, \ t \neq 0 \}. \quad (7)$$

Here int $D$ denotes the interior of a set $D$.

Evidently, problem (2) (equivalently, problem (3)) satisfies the Slater condition (7) if and only if problem (4) satisfies condition (6).

Following [13,14], define the sets of immobile indices $T_{im}$ and $R_{im}$ in problems (4) and (3), respectively:

$$T_{im} := \{ t \in T : t^\top A(x)t = 0 \quad \forall x \in X \}$$

and

$$R_{im} := \{ t \in \mathbb{R}_+^p : t^\top A(x)t = 0 \quad \forall x \in X \}.$$

It is evident that the aforementioned sets are interrelated:

$$R_{im} = \{ t \in \mathbb{R}_+^p : t = \alpha \tau, \ \tau \in T_{im}, \ \alpha \geq 0 \} \text{ and } T_{im} = \{ t \in R_{im} : e^\top t = 1 \}.$$

From the latter relations, we conclude that the set $T_{im}$, the immobile index set for problem (4), can be considered as a normalized immobile index set for problem (3). In what follows, we will use mainly the set $T_{im}$, taking into account its relationship with the set $R_{im}$.

The following proposition is an evident corollary of Proposition 1 from [14].
Proposition 1 Given a linear copositive problem in the form (3), the Slater condition (6) is equivalent to the emptiness of the normalized immobile index set $T_{im}$.

It is evident that $T_{im} = \emptyset$ if and only if $R_{im} = \{0\}$.

Proposition 2 Given a linear copositive problem (3), let $\{\tau(i), i \in I\}$ be some set consisting of immobile indices of this problem. Then for any $x \in X$, the following inequalities hold:

$$A(x)\tau(i) \geq 0, \quad i \in I.$$  \hfill (8)

Proof It is evident that for all $x \in X$, any $t \in R_{im}\setminus\{0\}$ is an optimal solution of the lower level problem (see e.g. [6]) in the form

$$LLP(x) : \min t^\top A(x)t, \quad \text{s.t. } t \in \mathbb{R}^P_+.$$  

Taking into account the first order necessary optimality conditions for $\tau(i) = (\tau_k(i), k \in P) \in T_{im} \subset R_{im}, i \in I$, in the problem $LLP(x)$, one can conclude that for all $x \in X$, it holds

$$e_k^\top A(x)\tau(i) \begin{cases} = 0, & \text{if } \tau_k(i) > 0, \\ \geq 0, & \text{if } \tau_k(i) = 0, \end{cases} \quad k \in P, \quad i \in I,$$  \hfill (9)

where $e_k$ is the $k$-the vector of the canonic basis of the space $\mathbb{R}^P$.

It is evident that relations (9) imply inequalities (8).  \hfill $\square$

Proposition 3 Given an index set $\{\tau(i) \in T, i \in I\}$, the inequalities

$$A(x)\tau(i) \geq 0, \quad i \in I,$$  \hfill (10)

imply the inequalities

$$t^\top A(x)t \geq 0 \quad \forall t \in \text{conv}\{\tau(i), i \in I\}. \quad \hfill (11)$$

Here $\text{conv}S$ denotes the convex hull of a given set $S$.

Proof The proof of the proposition is evident.  \hfill $\square$

Let $\{\tau(i), i \in I\} \subset T_{im}$ be a nonempty subset of the set of normalized immobile indices in problem (3). For this subset and for any $e > 0$, denote

$$T(e) := T(e, \tau(i), i \in I) := \{t \in T : \rho(t, \text{conv} \{\tau(i), i \in I\}) \geq e\},$$  \hfill (12)

$$\hat{T}(e) := \hat{T}(e, \tau(i), i \in I) := \{t \in T : \rho(t, \text{conv} \{\tau(i), i \in I\}) \leq e\},$$  \hfill (13)

where $\rho(l, B) = \min_{\tau \in B} ||l - \tau||$ is the distance between a vector $l$ and a set $B$ associated with the norm $||a|| = \sqrt{a^\top a}$ in the vector space $\mathbb{R}^P$. Consider the sets

$$\mathcal{X} = \{x \in \mathbb{R}^n : A(x)\tau(i) \geq 0, i \in I\},$$

$$\mathcal{X}(e) = \{z \in \mathcal{X} : t^\top A(z)t \geq 0 \quad \forall t \in T(e)\}. \quad \hfill (14)$$

The following lemma is a generalization of Lemma 2 from [14].

Lemma 1 Given a copositive problem in the form (3), let $\{\tau(i), i \in I\}$ be a finite subset of the set of normalized immobile indices with $I \neq \emptyset$. Then there exists $e_0 > 0$ such that $\mathcal{X}(e_0) = X$, the set $\mathcal{X}(e)$ being defined in (14) with the set $T(e)$ as in (12).
Proof It follows from Proposition 2 that \( X \subset \mathcal{X}(\varepsilon) \) for all \( \varepsilon > 0 \). To finalize the proof, it is enough to show that there exists \( \varepsilon_0 > 0 \) such that \( \mathcal{X}(\varepsilon_0) \subset X \). Suppose the contrary. Then for each \( \varepsilon > 0 \) there exist \( z(\varepsilon) \in \mathcal{X}(\varepsilon) \) such that
\[
(t(\varepsilon))^\top A(z(\varepsilon))t(\varepsilon) < 0,
\quad \text{(15)}
\]
where \( t(\varepsilon) \) is an optimal solution of the problem
\[
\min_{t \in T} t^\top A(z(\varepsilon))t.
\quad \text{(16)}
\]
Since, by construction (see Proposition 3 and (14)), it holds
\[
t^\top A(z(\varepsilon))t \geq 0 \quad \text{forall } t \in T(\varepsilon) \cup \text{conv } \{\tau(i), \ i \in I\},
\]
then \( t(\varepsilon) \in \bar{T}(\varepsilon) \setminus \text{conv } \{\tau(i), \ i \in I\} \), for all \( \varepsilon > 0 \). Hence there exists a limit point \( t^* \) of the sequence \( t(\varepsilon) \) as \( \varepsilon \to 0 \) such that
\[
t^* = \lim_{\varepsilon \to 0} t(\varepsilon), \quad \lim_{\varepsilon \to 0} \varepsilon_s = 0, \quad \varepsilon_s > 0, \quad s = 1, 2, \ldots, \quad t^* \in \text{conv } \{\tau(i), \ i \in I\}.
\]

Given \( \varepsilon > 0 \), let us consider the vector \( l(\varepsilon) := t(\varepsilon) - t^* \). By construction, \( e^\top l(\varepsilon) = 0 \).
It is evident that there exists a sufficiently large \( s \in \mathbb{N} \) such that for \( k \in P \), the following conditions hold with \( \bar{\varepsilon} := \varepsilon_s \):
if \( t^*_k = 0 \), then \( l_k(\bar{\varepsilon}) = t_k(\bar{\varepsilon}) \geq 0 \) and if \( t_k(\bar{\varepsilon}) = 0 \), then \( t^*_k = 0 \) and \( l_k(\bar{\varepsilon}) = 0 \).

Consequently, the vector \( l := l(\bar{\varepsilon}) \) is a feasible direction for the indices \( t^* \) and \( t(\bar{\varepsilon}) \) in the set \( T \). Hence there exists \( \gamma_0 > 1 \) such that
\[
t^* + \gamma l = t^* + \gamma(t(\bar{\varepsilon}) - t^*) \geq 0, \quad e^\top (t^* + \gamma l) = 1 \quad \forall \gamma \in [0, \gamma_0].
\]

Define a quadratic function on \( \gamma, \ \gamma \in [0, \gamma_0] \):
\[
w(\gamma) := (t^* + \gamma l)^\top A(z(\bar{\varepsilon}))(t^* + \gamma l)
\]
\[
= (t^*)^\top A(z(\bar{\varepsilon}))t^* + 2\gamma l^\top A(z(\bar{\varepsilon}))t^* + \gamma^2 l^\top A(z(\bar{\varepsilon}))l = a\gamma^2 + 2by + c,
\]
where \( c := (t^*)^\top A(z(\bar{\varepsilon}))t^* \), \( b := l^\top A(z(\bar{\varepsilon}))t^* \), \( a := l^\top A(z(\bar{\varepsilon}))l \).

By construction, for \( \gamma^* := 1 \) we have \( w(\gamma^*) = (t^*)^\top A(z(\bar{\varepsilon}))t^* \) and it is the optimal value of the cost function of the problem (16) with \( \varepsilon = \bar{\varepsilon} \). Hence
\[
w(\gamma^*) = \min_{\gamma \in [0, \gamma_0]} w(\gamma) = \min_{\gamma \in [0, \gamma_0]} (a\gamma^2 + 2by + c).
\]

Since \( \gamma^* = 1 \in (0, \gamma_0) \) in the formula above, then \( 2a\gamma^* + 2b = 2a + 2b = 0 \).

Therefore \(-b = a\), which can be rewritten in the form
\[
- l^\top A(z(\bar{\varepsilon}))t^* = l^\top A(z(\bar{\varepsilon}))l,
\]
wherefrom we get
\[
(t(\bar{\varepsilon}))^\top A(z(\bar{\varepsilon}))t^* = (t(\bar{\varepsilon}))^\top A(z(\bar{\varepsilon}))t(\bar{\varepsilon}).
\quad \text{(17)}
\]

Since \( t^* \in \text{conv } \{\tau(i), \ i \in I\} \), then \( t^* = \sum_{i \in I} \beta_i \tau(i) \), \( \sum_{i \in I} \beta_i = 1 \), \( \beta_i \geq 0 \), \( i \in I \).

Consequently, taking into account that \( z(\bar{\varepsilon}) \in \mathcal{X}(\bar{\varepsilon}) \) and \( t(\bar{\varepsilon}) \geq 0 \), we have
\[
(t(\bar{\varepsilon}))^\top A(z(\bar{\varepsilon}))t^* = \sum_{i \in I} \beta_i (t(\bar{\varepsilon}))^\top A(z(\bar{\varepsilon}))\tau(i) \geq 0.
\]

But this inequality and inequality (15) contradict equality (17). The lemma is proved. \( \Box \)
It should be noticed that the lemma above can be considered as a generalization of Lemma 2 from [14] since it is formulated for an arbitrary subset \( \{ \tau(i), i \in I \} \) of \( T_{im} \), while in Lemma 2 from [14] we considered the fixed subset of \( T_{im} \), namely, the set of vertices of \( \text{conv} T_{im} \).

3 An extended dual problem for Linear Copositive Programming

In this section, we will discuss some dual formulations of the linear copositive problem (2).

Given an arbitrary cone \( K \in S(p) \), the corresponding dual cone \( K^* \) is defined as

\[
K^* := \{ A \in S(p) : A \bullet D \geq 0 \quad \forall D \in K \}.
\]

It is known that the cone of symmetric positive semidefinite matrices \( S^+(p) \) is self-dual, i.e. \( S^+_+(p) = S^+(p) \) but the cone of symmetric copositive matrices \( \mathcal{COP}p \) defined in (1), is not.

It can be shown (see e.g. [3]) that given the cone \( \mathcal{COP}p \), its dual cone \( \mathcal{CP}p \) is the cone of so-called completely positive matrices:

\[
\mathcal{CP}p := \text{conv}\{tt^T : t \in \mathbb{R}_+^p\},
\]

and \( \mathcal{CP}p \subset \mathcal{COP}p \).

Then the (standard) Lagrangian dual problem for (2) can be written as follows [2]:

\[
\max_{U} (-U \bullet A_0), \quad \text{s.t. } U \bullet A_i = c_i, \; i = 1, 2, \ldots, n; \quad U \in \mathcal{CP}p. \tag{18}
\]

It is well known (see [2] Theorem 3.1) that if the constraints of problem (2) satisfy the Slater condition, then there is no gap between the optimal values of problems (2) and (18).

If the constraints of problem (2) do not satisfy the Slater condition, then the positive gap is possible (see an example at the end of this section).

The aim of this section is to formulate an extended dual problem for problem (2) and show that there is no duality gap for this pair of dual problems.

For a given finite integer \( m_0 \geq 0 \), consider the following problem:

\[
\max_{U} - (U + W_{m_0}) \bullet A_0, \quad \text{s.t. } (U + W_{m_0}) \bullet A_j = c_j, \; j = 1, 2, \ldots, n; \quad U \in \mathcal{CP}p, \quad W_0 = \mathbb{0}_p, \quad \left( U_m W_m \right) \in \mathcal{CP}^{2p}, \; m = 1, \ldots, m_0. \tag{19}
\]

where \( U_m \in S(p) \), \( D_m \in S(p) \), \( W_m \in \mathbb{R}^{p \times p} \), \( m = 1, \ldots, m_0 \), and \( \mathbb{0}_p \) stays for the \( p \times p \) null matrix.

Notice that in the case \( m_0 = 0 \), we consider that the index set \( \{1, \ldots, m_0\} \) is empty and the constraints

\[
(U_m + W_{m-1}) \bullet A_j = 0, \; j = 0, 1, \ldots, n; \quad \left( U_m W_m \right) \in \mathcal{CP}^{2p}, \; m = 1, \ldots, m_0,
\]

are absent in problem (19). Hence, for \( m_0 = 0 \), problem (19) takes the form (18):

\[
\max_{U} - U \bullet A_0, \quad \text{s.t. } U \bullet A_j = c_j, \; j = 1, 2, \ldots, n; \quad U \in \mathcal{CP}p.
\]
Lemma 2 [Weak duality] Let $x \in X$ be a feasible solution of the primal linear copositive problem (2) and

$$ (U_m, W_m, D_m, m = 1, ..., m_0; \ U) $$

be a feasible solution of problem (19). Then the following inequality holds:

$$ c^\top x \geq -(U + W_{m_0}) \cdot A_0. $$

(21)

Proof For $m = 1, ..., m_0$, it follows from the condition $\left( U_m W_m W_m^\top D_m \right) \in CP^{2p}$, that there exists a matrix $B_m$ with non-negative elements in the form

$$ B_m = \begin{pmatrix} V_m \\ L_m \end{pmatrix} \in \mathbb{R}^{2p \times k(m)}, $$

such that

$$ \begin{pmatrix} U_m & W_m \\ W_m^\top & D_m \end{pmatrix} = B_m B_m^\top = \begin{pmatrix} V_m & L_m \\ V_m^\top & L_m^\top \end{pmatrix}. $$

The matrix $B_m$ above is composed by the blocks containing some matrices

$$ V_m = (\tau^m(i), i \in I_m), \ L_m = (\lambda^m(i), i \in I_m). $$

(22)

where $\tau^m(i) \in \mathbb{R}_+, \lambda^m(i) \in \mathbb{R}_+, i \in I_m, k(m) := |I_m|$. Here and in what follows, $|I|$ denotes the number of elements in a set $I$.

Hence, for $m = 1, ..., m_0$, the matrices $U_m, W_m, D_m$ in (19) admit representations

$$ U_m = V_m V_m^\top, \ W_m = V_m L_m^\top, \ D_m = L_m L_m^\top. $$

(23)

Consider the first group of constraints of the dual problem (19):

$$ U_1 \cdot A_j = 0, \ j = 0, 1, ..., n. $$

Due to (22) and (23), these constraints can be rewritten in the form

$$ \sum_{i \in I_1} (\tau^1(i))^\top A_j \tau^1(i) = 0, \ j = 0, 1, ..., n. $$

(24)

It follows from (24) that for any $x \in \mathbb{R}^n$, we have

$$ \sum_{i \in I_1} (\tau^1(i))^\top A(x)(\tau^1(i)) = 0. $$

(25)

Since the inequalities

$$ t^\top A(x)t \geq 0 \ \forall t \in \mathbb{R}_+^P, \ \forall x \in X, $$

(26)

should be fulfilled, equality (25) implies $(\tau^1(i))^\top A(x)\tau^1(i) = 0, i \in I_1, \forall x \in X$.

Thus, one can conclude that $\tau^1(i) \in T_{m_1}, i \in I_1$, and, consequently (see Proposition 2), it holds

$$ A(x)\tau^1(i) \geq 0, \ i \in I_1, \forall x \in X. $$

(27)

Suppose that for some $m \geq 1$, it was shown that

$$ A(x)\tau^m(i) \geq 0, \ i \in I_m, \forall x \in X. $$

(28)
Due to (22) and (23), the constraints \((U_{m+1} + W_m) \bullet A_j = 0, \ j = 0, 1, \ldots, n,\) of problem (19) can be rewritten as follows:

\[
\sum_{i \in I_{m+1}} (\tau^{m+1}(i))^\top A_j \tau^{m+1}(i) + \sum_{i \in I_m} (\lambda^m(i))^\top A_j \tau^m(i) = 0, \ j = 0, 1, \ldots, n.
\]

It follows from the latter equalities that for any \(x \in \mathbb{R}^n,\) we have

\[
\sum_{i \in I_{m+1}} (\tau^{m+1}(i))^\top A(x) \tau^{m+1}(i) + \sum_{i \in I_m} (\lambda^m(i))^\top A(x) \tau^m(i) = 0.
\]  \(\text{(29)}\)

By the hypothesis above, inequalities (28) are satisfied. Then, taking into account that \(\lambda^m(i) \in \mathbb{R}^+_n, \ i \in I_m,\) and for any \(x \in X,\) inequalities (26) hold, we conclude from (29) that for any \(x \in X,\) the following equalities are valid:

\[
(\tau^{m+1}(i))^\top A(x) \tau^{m+1}(i) = 0, \ i \in I_{m+1}, \ (\lambda^m(i))^\top A(x) \tau^m(i) = 0, \ i \in I_m.
\]

Hence, \(\tau^{m+1}(i) \in T_{im}, \ i \in I_{m+1},\) and, according to Proposition 2, it holds

\[
A(x) \tau^{m+1}(i) \geq 0, \ i \in I_{m+1}, \ \forall x \in X.
\]

Replace \(m\) by \(m + 1\) and repeat the considerations for all \(m < m_0.\)

Let \(m = m_0.\) In this case, relations (28) have the form

\[
A(x) \tau^{m_0}(i) \geq 0, \ i \in I_{m_0}, \ \forall x \in X,
\]  \(\text{(30)}\)

and for \(U = \sum_{i \in I} \tau(i) \tau(i)^\top, \ \tau(i) \in \mathbb{R}^+_n, \ i \in I,\) the constraints

\[
(U + W_{m_0}) \bullet A_j = c_j, \ j = 1, \ldots, n,
\]

of problem (19) can be represented as follows:

\[
\sum_{i \in I} (\tau(i))^\top A_j \tau(i) + \sum_{i \in I_{m_0}} (\lambda^{m_0}(i))^\top A_j \tau^{m_0}(i) = c_j, \ j = 1, \ldots, n.
\]  \(\text{(31)}\)

Then, evidently,

\[
\sum_{j=1}^n c_j x_j = \sum_{i \in I} (\tau(i))^\top A(x) \tau(i) + \sum_{i \in I_{m_0}} (\lambda^{m_0}(i))^\top A(x) \tau^{m_0}(i)
\]

\[
-\left(\sum_{i \in I} (\tau(i))^\top A_0 \tau(i) + \sum_{i \in I_{m_0}} (\lambda^{m_0}(i))^\top A_0 \tau^{m_0}(i)\right)
\]

\[
= \sum_{i \in I} (\tau(i))^\top A(x) \tau(i) + \sum_{i \in I_{m_0}} (\lambda^{m_0}(i))^\top A(x) \tau^{m_0}(i) - (U + W_{m_0}) \bullet A_0.
\]  \(\text{(32)}\)

From (26) and (30), we conclude that

\[
(\tau(i))^\top A(x) \tau(i) \geq 0, \ i \in I; \ (\lambda^{m_0}(i))^\top A(x) \tau^{m_0}(i) \geq 0, \ i \in I_{m_0}, \ \forall x \in X.
\]

These inequalities together with equality (32) imply (21). The lemma is proved.

\(\Box\)

**Lemma 3** [Strong duality] Let problem (2) have an optimal solution. Then there exist a number \(0 \leq m_0 < \infty\) and a feasible solution

\[
\left(U^0_m, \ W^0_m, D^0_m, \ m = 1, \ldots, m_0; \ U^0\right)
\]  \(\text{(33)}\)
of problem (19) such that for any optimal solution $x^0$ of problem (2), it holds
\begin{equation}
    c^\top x^0 = -(U^0 + W^0_{m_0}) \bullet A_0. \tag{34}
\end{equation}

**Proof** To prove the lemma, we will algorithmically construct the number $m_0$ and the matrices (33).

Iteration # 0. Consider the following SIP problem:
\begin{equation}
    \min_{(x, \mu)} \mu, \quad \text{s.t.} \quad t^\top A(x)t + \mu \geq 0, \ t \in T, \tag{35}
\end{equation}
with the set $T$ defined in (5). If there exists a feasible solution $(\tilde{x}, \tilde{\mu})$ of this problem with $\tilde{\mu} < 0$, then set $m_0 := 0$ and GO TO the Final step.

Otherwise for any $x \in X$, the vector $(x, \mu^0 = 0)$ is an optimal solution of problem (35). It should be noticed that in problem (35), the index set $T$ is a compact set, and the constraints of this problem satisfy the Slater condition. Hence, based of Propositions 5.105 and 5.107 from [6], we conclude that there exist indices and numbers
\begin{equation}
    \tau(i) \in T, \quad \gamma(i) > 0, \ i \in I_1, \ |I_1| \leq n + 1,
\end{equation}
such that
\begin{equation}
    \sum_{i \in I_1} \gamma(i)(\tau(i))^\top A_j \tau(i) = 0, \ j = 0, 1, ..., n; \quad \sum_{i \in I_1} \gamma(i) = 1. \tag{36}
\end{equation}
It follows from (36) that the set $I_1$ is nonempty and $\tau(i) \in T_{im}, \ i \in I_1$. In fact, relations $\gamma(i) > 0, \ i \in I_1, \ \sum_{i \in I_1} \gamma(i) = 1$ imply that $I_1 \neq \emptyset$. Moreover, from the first group of equalities in (36), it follows
\begin{equation}
    x_j \sum_{i \in I_1} \gamma(i)(\tau(i))^\top A_j \tau(i) = 0 \ \forall x_j \in \mathbb{R}, \ j = 1, ..., n;
\end{equation}
\begin{equation}
    \sum_{i \in I_1} \gamma(i)(\tau(i))^\top A_0 \tau(i) = 0.
\end{equation}
Let us sum up all these equalities. As a result we get
\begin{equation}
    \sum_{i \in I_1} \gamma(i)(\tau(i))^\top A(x) \tau(i) = 0 \ \forall x \in \mathbb{R}^n. \tag{37}
\end{equation}
Since $\gamma(i) > 0, \ i \in I_1$ and inequalities (26) should hold true, then it follows from (37) that
\begin{equation}
    (\tau(i))^\top A(x) \tau(i) = 0 \ \forall x \in X \ \forall i \in I_1.
\end{equation}
By definition, this means $\tau(i) \in T_{im}, \ i \in I_1$.

Let us set
\begin{equation}
    \beta_1(i) := \sqrt{\gamma(i)}, \ i \in I_1, \ V_1^0 := (\tau(0)(i) := \beta_1(i) \tau(i), \ i \in I_1), \ U_1^0 := V_1^0 (V_1^0)^\top.
\end{equation}
Then equalities (36) take the form
\begin{equation}
    U_1^0 \bullet A_j = 0, \ j = 0, 1, ..., n. \tag{38}
\end{equation}
Denote $T_1 := \text{conv} \{\tau(i), \ i \in I_1\}$ and proceed to the next iteration.

Iteration # 1. Consider the problem
\begin{equation}
    \min_{(x, \mu)} \mu, \quad \text{s.t.} \quad A(x) \tau(i) \geq 0, \ i \in I_1, \ t^\top A(x)t + \mu \geq 0, \ t \in \{t \in T : \rho(t, T_1) \geq \varepsilon_1\}. \tag{39}
\end{equation}
where $\varepsilon_1 > 0$ is such a number that the set of feasible solutions of problem (39) with $\mu = 0$ coincides with the set $X$ of feasible solutions of problem (2). According to Lemma 1, such $\varepsilon_1 > 0$ exists.

If there exists a feasible solution $(\hat{x}, \hat{\mu})$ of problem (39) with $\hat{\mu} < 0$, then STOP and GO TO the Final step with $m_0 := 1$.

Otherwise, $(x, \mu^0 = 0)$ with any $x \in X$ is an optimal solution of problem (39). In the SIP problem (39), the index set $\{t \in T : \rho(t, T_1) \geq \varepsilon_1\}$ is compact, and the constraints satisfy the following Slater type condition:

$$\exists(\hat{x}, \hat{\mu}) : A(\hat{x})\tau(i) \geq 0, \ i \in I_1, \ i^T A(\hat{x}) t + \hat{\mu} > 0, \ t \in \{t \in T : \rho(t, T_1) \geq \varepsilon_1\}.$$ 

Hence (see Theorem 1 and its Corollary from [18]), there exist indices and numbers

$$\tau(i) \in \{t \in T : \rho(t, T_1) \geq \varepsilon_1\}, \ \gamma(i) > 0, \ i \in \Delta I_1, \ \vert \Delta I_1 \vert \leq n + 1,$$

and vectors $\lambda^1(i) \in \mathbb{R}^p_+, \ i \in I_1$, such that

$$\sum_{i \in \Delta I_1} \gamma(i)(\tau(i))^T A_j \tau(i) + \sum_{i \in I_1} (\lambda^1(i))^T A_j \tau(i) = 0, \ j = 0, 1, ..., n;$$

$$\sum_{i \in \Delta I_1} \gamma(i) = 1. \tag{42}$$

It follows from (42) that $\Delta I_1 \neq \emptyset$.

Let us show that for all $x \in X$, it holds

$$(\tau(i))^T A(x)\gamma(i) = 0, \ i \in \Delta I_1; \ (\lambda^1(i))^T A(x)\gamma(i) = 0, \ i \in I_1. \tag{43}$$

In fact, from (41) it follows

$$x_j \left( \sum_{i \in \Delta I_1} \gamma(i)(\tau(i))^T A_j \tau(i) + \sum_{i \in I_1} (\lambda^1(i))^T A_j \tau(i) \right) = 0, \ j = 1, ..., n;$$

$$\sum_{i \in \Delta I_1} \gamma(i)(\tau(i))^T A_0 \tau(i) + \sum_{i \in I_1} (\lambda^1(i))^T A_j \tau(i) = 0.$$ 

Let us sum up these equalities. As a result, we get

$$\sum_{i \in \Delta I_1} \gamma(i)(\tau(i))^T A(x)\gamma(i) + \sum_{i \in I_1} (\lambda^1(i))^T A(x)\gamma(i) = 0 \ \forall x \in \mathbb{R}^n. \tag{44}$$

We have proved above that $\tau(i) \in T_{im}, \ i \in I_1$. Then, from Proposition 2 it follows

$$A(x)\gamma(i) \geq 0, \ i \in I_1, \ \forall x \in X.$$ 

These inequalities and conditions $\lambda^1(i) \in \mathbb{R}^p_+, \ i \in I_1$, imply the inequalities

$$(\lambda^1(i))^T A(x)\gamma(i) \geq 0, \ i \in I_1, \ \forall x \in X. \tag{45}$$

Equalities (43) are the consequence of inequalities (26), (45) and equalities (44). From (43), it follows $\tau(i) \in T_{im}, \ i \in \Delta I_1$, and from (36) and (41), we get

$$\sum_{i \in I_2} \gamma(i)(\tau(i))^T A_j \tau(i) + \sum_{i \in I_1} (\lambda^1(i))^T A_j \tau(i) = 0, \ j = 0, 1, ..., n, \tag{46}$$

where $I_2 := I_1 \cup \Delta I_1$.

To prove the finiteness of the algorithm described in this lemma, we will use a specific procedure that modifies the index set $\{\tau(i), i \in \Delta I\}$ using the index set $\{\tau(i), i \in I\}$,
where $I$ and $\Delta I$ are the sets which we have at the current iteration $#m$ of the algorithm: $I := I_m$, $\Delta I := \Delta I_m$.

The main aim of the procedure is as follows: having fixed the index set $\{\tau(i), i \in I\}$, (found at the previous iteration $#(m - 1)$), we replace the indices $\tau(i) \in T_{im}, i \in \Delta I$, (found at the current iteration $#m$) by the new ones $\tilde{\tau}(i) \in T_{im}, i \in \Delta I$, so as to satisfy the inclusion

$$\{k \in P : \tau_k(i) = 0\} \subset \{k \in P : \tilde{\tau}_k(i) = 0\}$$

for all $i \in \Delta I$, and meet the relations

$$\{k \in P : \tau_k(i) > 0\} \cap \{k \in P : \tilde{\tau}_k(s) = 0\} \neq \emptyset \ \forall \ i \in I, \ \forall \ s \in \Delta I.$$

Moreover, we want that the indices from the fixed index set $\{\tau(i), i \in I\}$, together with new ones $\tilde{\tau}(i) \in T_{im}, i \in \Delta I$, satisfy relations similar to (46). For this purpose we will modify some coefficients $\gamma(i), i \in I \cup \Delta I$, and $\lambda(i), i \in I$.

On the first iteration of the algorithm, we apply the procedure to the data set

$$\{\tau(i), \gamma(i), i \in \Delta I_1; \tau(i), \lambda^1(i), \gamma(i), i \in I_1\}.$$  \hspace{1cm} \text{(47)}

**Procedure DAM** (Data Modification).

The Procedure starts with an initial data set

$$\{\tau(i), \gamma(i), i \in \Delta I; \tau(i), \lambda(i), \gamma(i), i \in I\}$$  \hspace{1cm} \text{(48)}

such that

$$\tau(i) \in T_{im}, \tau(i) \not\in \text{conv}\{\tau(j), j \in I\}, \ \gamma(i) > 0, \ i \in \Delta I;$$

$$\tau(i) \in T_{im}, \lambda(i) \in \mathbb{R}_+^p, \ \gamma(i) > 0, \ i \in I,$$

and

$$\sum_{i \in \Delta I \cup I} \gamma(i)(\tau(i))^\top A_j \tau(i) + \sum_{i \in I} (\lambda(i))^\top A_j \tau(i) = 0, \ j = 0, 1, ..., n.$$  \hspace{1cm} \text{(49)}

Set $P_+(i) := \{k \in P : \tau_k(i) > 0\}, i \in \Delta I \cup I$.

If

$$P_+(i) \cap (P \setminus P_+(s)) \neq \emptyset \ \forall s \in \Delta I, \ \forall i \in I,$$  \hspace{1cm} \text{(50)}

then STOP. The Procedure **DAM** is complete.

If (50) is not satisfied, then find $s_0 \in \Delta I$ and $i_0 \in I$ such that

$$P_+(i_0) \subset P_+(s_0).$$  \hspace{1cm} \text{(51)}

Let us find index $\hat{\tau}(s_0)$ that has the form $\hat{\tau}(s_0) = \tau(s_0) - \theta \tau(i_0)$ and satisfies the relations

$$\hat{\tau}(s_0) \geq 0, \ ||\{k \in P : \hat{\tau}_k(s_0) = 0\}|| \geq ||\{k \in P : \tau_k(s_0) = 0\}|| + 1.$$  \hspace{1cm} \text{(52)}

For this purpose, set $\theta := \min_{k \in P} \theta_k > 0$, where

$$\theta_k := \begin{cases} \infty, & \text{if } k \in P \setminus P_+(i_0), \\ \frac{\tau_k(s_0)}{\tau_k(i_0)}, & \text{if } k \in P_+(i_0). \end{cases}$$

It is easy to verify that the constructed index $\hat{\tau}(s_0)$ satisfies (52).

Let us show that $\theta < 1$. 
Suppose the contrary: \( \theta \geq 1 \). Hence \( \theta_k \geq 1 \ \forall k \in P_+(i_0) \), and consequently, \( \tau_k(s_0) \geq \tau_k(i_0) > 0 \), \( k \in P_+(i_0) \). Notice that since

\[
1 = \mathbf{e}^\top \tau(i_0) = \sum_{k \in P_+(i_0)} \tau_k(i_0) \leq \sum_{k \in P_+(i_0)} \tau_k(s_0) \leq \sum_{k \in P_+(s_0)} \tau_k(s_0) = \mathbf{e}^\top \tau(s_0) = 1,
\]

we conclude that

\[
\sum_{k \in P_+(i_0)} \tau_k(s_0) = 1, \quad \sum_{k \in P_+(i_0)} \tau_k(i_0) = 1, \quad \text{and} \quad \tau_k(s_0) \geq \tau_k(i_0) > 0 \ \forall k \in P_+(i_0).
\]

It follows from the latter conditions that \( \tau(s_0) = \tau(i_0) \) which contradicts the assumption \( \tau(s_0) \not\in \text{conv}\{\tau(i), i \in I\} \). The contradiction proves the inequality \( \theta < 1 \). Since, by construction, \( \theta \) is strictly positive, then the double inequality \( 0 < \theta < 1 \) is valid.

Now we replace the index \( \tilde{\tau}(s_0) \geq 0 \) by the normalize one

\[
\tilde{\tau}(s_0) := \frac{\tau(s_0)}{\mathbf{e}^\top \tau(s_0)} = \left( \tau(s_0) - \theta \tau(i_0) \right) / (1 - \theta) \geq 0.
\]

Let us show that \( \tilde{\tau}(s_0) \in T_{im} \). Since, by assumption, \( \tau(s_0) \in T_{im} \), then the following relations (see \( 9 \)) hold true:

\[
e_k^\top A(x)\tau(s_0) = 0, \quad k \in P_+(s_0); \quad e_k^\top A(x)\tau(s_0) \geq 0, \quad k \in P \setminus P_+(s_0) \ \forall x \in X.
\]

These relations together with \( 51 \) imply the equalities

\[
(\tau(i_0))^\top A(x)\tau(s_0) = 0 \ \forall x \in X.
\]

From the latter equalities and the equalities

\[
(\tau(i_0))^\top A(x)\tau(i_0) = 0, \quad (\tau(s_0))^\top A(x)\tau(s_0) = 0 \ \forall x \in X,
\]

it follows:

\[
(1 - \theta)^2(\tilde{\tau}(s_0))^\top A(x)\tilde{\tau}(s_0) = (\tau(s_0))^\top A(x)\tau(s_0) - 2\theta(\tau(i_0))^\top A(x)\tau(s_0) + \theta^2(\tau(i_0))^\top A(x)\tau(i_0) = 0 \ \forall x \in X,
\]

and therefore \( \tilde{\tau}(s_0) \in T_{im} \).

In the data set \( 48 \), let us perform the following replacements:

\[
\tau(s_0) \rightarrow \tilde{\tau}(s_0) = (\tau(s_0) - \theta \tau(i_0)) / (1 - \theta) \geq 0, \quad e^\top \tilde{\tau}(s_0) = 1,
\]

\[
\tilde{\tau}(s_0) \not\in \text{conv}\{\tau(i), i \in I\};
\]

\[
\lambda(i_0) \rightarrow \tilde{\lambda}(i_0) = \lambda(i_0) + 2\gamma(s_0)\theta(1 - \theta)\tilde{\tau}(s_0) \geq 0;
\]

\[
\gamma(i_0) \rightarrow \tilde{\gamma}(i_0) = \gamma(i_0) + \gamma(s_0)\theta^2 > 0;
\]

\[
\gamma(s_0) \rightarrow \tilde{\gamma}(s_0) = \gamma(s_0)(1 - \theta)^2 > 0.
\]

All other data remain unchanged.

It is easy to verify that relations \( 49 \) hold true with the modified data set.

For the modified data set, check condition \( 50 \). If it is satisfied, then STOP; the procedure is complete. If \( 50 \) is not satisfied, then find new indices \( s_0 \in \Delta I \) and \( i_0 \in I \) such that inclusion \( 51 \) is valid and repeat the steps described above.

The Procedure \textbf{DAM} is completely described.

\( \bigcirc \) Springer
Let us go back to the proof of the lemma. Recall that we are performing the Iteration #1 of the algorithm. Having applied the Procedure DAM to the data set (47), one obtains a new (modified) data set in the same form (47) such that

- the indices \( \tau(i), i \in I_1 \), are the same as in the initial data set (i.e., the procedure left these indices unchanged);
- the modified indices \( \tau(i), i \in \Delta I_1 \), are the immobile ones in problem (3);
- for the modified indices \( \tau(i) \) and numbers \( \gamma(i), i \in \Delta I_1 \), relations (40) are fulfilled;
- for the modified vectors \( \lambda^1(i) \) and numbers \( \gamma(i), i \in I_1 \), it holds \( \lambda^1(i) \in \mathbb{R}^p_+ \), \( \gamma(i) > 0, i \in I_1 \);
- for the modified data set (47), relations (50) with \( \Delta I = \Delta I_1, I = I_1 \) and (46) are satisfied.

Using the new data (obtained as the result of applying the Procedure DAM to the initial data set (47)), denote:

\[
\beta_2(i) := \sqrt[\gamma(i)]{\tau(i)}, \quad i \in I_2, \quad V_0^0 := (\beta_2(i) \tau(i), i \in I_2), \quad L_1^0 := (\lambda^1(i)/\beta_1(i), i \in I_1),
\]

\[
U_2^0 := V_2^0 V_0^0, \quad W_1^0 := L_1^0 V_0^0, \quad D_1^0 := L_1^0 L_1^0.
\]

Then relations (46) can be written as follows:

\[
(U_2^0 + W_1^0) \bullet A_j = 0, \quad j = 0, 1, \ldots, n.
\]  

(53)

GO TO the next iteration.

**Iteration # m, m \geq 2.** By the beginning of this iteration, the numbers \( \beta_m(i) > 0, i \in I_m \), as well as the indices, vectors and numbers

\[
\tau(i) \in T_{im}, \quad \gamma(i) > 0, \quad i \in I_m = I_{m-1} \cup \Delta I_{m-1}, \quad \lambda^{m-1}(i) \in \mathbb{R}^p_+, \quad i \in I_{m-1},
\]

are found such that

- relations (50) with \( \Delta I := \Delta I_{m-1} \neq \emptyset, I := I_{m-1} \) hold;
- the following equalities are satisfied:

\[
\sum_{i \in I_m} \gamma(i) \tau(i)^\top A_j \tau(i) + \sum_{i \in I_{m-1}} (\lambda^{m-1}(i))^\top A_j \tau(i) = 0, \quad j = 0, 1, \ldots, n.
\]  

(54)

Using these data, matrix

\[
V_m^0 = (\beta_m(i) \tau(i), i \in I_m)
\]

with \( \beta_m(i) > 0, i \in I_m \),

(55)

was constructed.

Denote \( T_m := \text{conv}\{\tau(i), i \in I_m\} \) and consider the problem

\[
\min_{(x, \mu) \in \mathbb{R}^{n+1}} \mu,
\]

s.t. \( A(x) \tau(i) \geq 0, i \in I_m, t^\top A(x) + \mu \geq 0, \quad t \in \{t \in T : \rho(t, T_m) \geq \varepsilon_m\}, \]

(56)

where \( \varepsilon_m > 0 \) is such a number that the feasible set of problem (56) with \( \mu = 0 \) coincides with the feasible set \( X \) of problem (2). According to Lemma 1, such \( \varepsilon_m \) exists.

If there exists a feasible solution (\( \tilde{x}, \tilde{\mu} \)) of problem (56) with \( \tilde{\mu} < 0 \), then STOP and GO TO the Final step with \( m_0 := m \).

Otherwise for any \( x \in X \), vector \( (x, \mu^0 = 0) \) is an optimal solution of problem (56).

Since in problem (56) the index set \( \{t \in T : \rho(t, T_m) \geq \varepsilon_m\} \) is compact, and the constraints satisfy the Slater type condition, then the optimality of \( (x, \mu^0 = 0) \) provides that there exist indices and numbers

\[
\tau(i) \in \mathbb{R}^p_+, \quad e^\top \tau(i) = 1, \quad \tau(i) \notin T_m; \quad \gamma(i) > 0, \quad i \in \Delta I_m, |\Delta I_m| \leq n + 1,
\]  

(57)
and vectors
\[ \hat{\lambda}^m(i) \in \mathbb{R}_+^p, \ i \in I_m, \] (58)
that satisfy the following equalities:
\[ \sum_{i \in \Delta I_m} \gamma(i) (\tau(i))^\top A_j \tau(i) + \sum_{i \in I_m} (\hat{\lambda}^m(i))^\top A_j \tau(i) = 0, \ j = 0, 1, ..., n; \]
\[ \sum_{i \in \Delta I_m} \gamma(i) = 1. \] (59)

By analogy with how this was done at Iteration # 1, we can show that equalities (59) imply the equalities
\[ (\tau(i))^\top A(x) \tau(i) = 0, \ i \in \Delta I_m; \ (\hat{\lambda}^m(i))^\top A(x) \tau(i) = 0, \ i \in I_m, \ \forall x \in X, \]
and, therefore, \( \tau(i) \in T_{im}, \ i \in \Delta I_m. \) Based on (54) and (59), one can conclude that
\[ \sum_{i \in I_{m+1}} \gamma(i) (\tau(i))^\top A_j \tau(i) + \sum_{i \in I_m} (\hat{\lambda}^m(i))^\top A_j \tau(i) = 0, \ j = 0, 1, ..., n, \] (60)
where \( I_{m+1} := I_m \cup \Delta I_m, \) and the vectors \( \lambda^m(i) \in \mathbb{R}_+^p, \ i \in I_m, \) are constructed as follows:
\[ \lambda^m(i) = \begin{cases} 
\lambda^{m-1}(i) + \hat{\lambda}^m(i), & i \in I_{m-1}, \\
\hat{\lambda}^m(i), & i \in \Delta I_{m-1} = I_m \setminus I_{m-1}. 
\end{cases} \]

Having applied the Procedure DAM to the data set
\[ \{\tau(i), \gamma(i), i \in \Delta I_m; \ \tau(i), \lambda^m(i), \gamma(i), i \in I_m\}, \] (61)
one will get the modified data set (in the same form) such that

- the indices \( \tau(i), i \in I_m, \) are the same as in the initial data set (these indices are not changed by the Procedure DAM);
- the modified indices \( \tau(i), i \in \Delta I_m, \) are the immobile ones in problem (3);
- for the modified indices \( \tau(i) \) and numbers \( \gamma(i), i \in \Delta I_m, \) relations (57) are fulfilled;
- for the modified vectors \( \lambda^m(i) \) and numbers \( \gamma(i), i \in I_m, \) it holds
\[ \lambda^m(i) \in \mathbb{R}_+^p, \ \gamma(i) > 0, \ i \in I_m; \]

- the modified data (61) satisfy relations (50) with \( \Delta I = \Delta I_m, \ I = I_m, \) and relations (60).

Using these new data, let us set
\[ \beta_{m+1}(i) := \sqrt{\gamma(i)}, \ i \in I_{m+1}; \ V^0_{m+1} := (\beta_{m+1}(i) \tau(i), i \in I_{m+1}), \]
\[ L^0_m := (\lambda^m(i) / \beta_m(i), i \in I_m), \ U^0_{m+1} := V^0_{m+1} (V^0_{m+1})^\top, \]
\[ W^0_m := L^0_m (V^0_m)^\top, \ D^0_m := L^0_m (L^0_m)^\top, \]
where matrix \( V^0_m \) was defined at the previous iteration according to (55). Then relations (60) can be written in the form:
\[ (U^0_{m+1} + W^0_m) \bullet A_j = 0, \ j = 0, 1, ..., n. \] (62)

Perform the next Iteration # \((m + 1)\).

Final step. It will be proved in Lemma 4 (see below) that the algorithm consists of a finite number of iterations.

Hence, for some \( 0 \leq m_0 < \infty, \) one of the following situations will arise:
a) $m_0 = 0$ and for problem (35) there exists a feasible solution $(\bar{x}, \bar{\mu})$ with $\bar{\mu} < 0$;

b) $m_0 > 0$ and for problem (56) with $m = m_0$ there exists a feasible solution $(\bar{x}, \bar{\mu})$ with $\bar{\mu} < 0$.

In situation a) the constraints of the original problem (2) satisfy the Slater condition. Hence, according to the well-known optimality conditions (e.g. see the KKT condition (4) in [2]), if $x^0$ is an optimal solution of problem (2), then there exists a matrix $U^0 \in \mathbb{C} \mathbb{P}^p$ such that

$$U^0 \bullet A_j = c_j, \quad j = 1, 2, \ldots, n; \quad U^0 \bullet A(x^0) = 0.$$ 

It follows from the relations above that $U^0$ is a feasible solution of the dual problem (20) and equality (34) holds.

Consider situation b): $m_0 > 0$. By the beginning of the final step, the matrices $U_x^0, W_x^0, D_x^0$, $s = 1, \ldots, m_0 - 1$; $W_0 = \mathbb{C} \mathbb{P}_p$, $U_m^0 = V_m^0 (V_m^0)\top$, the immobile indices $\tau(i)$, and the numbers $\beta_m(i) > 0$, $i \in I_m^0$, have been computed.

Consider the problem

$$\min_x c^\top x,$$

s.t. $A(x)\tau(i) \geq 0$, $i \in I_m^0$, $t^\top A(x)t \geq 0$, $t \in \{t \in T : \rho(t, T_m^0) \geq \varepsilon_m^0\}$,

where $\varepsilon_m^0 > 0$ is the number used when problem (56) with $m = m_0$ was formulated. The way the number $\varepsilon_m^0$ has been chosen guarantees that the feasible set of problem (63) coincides with the feasible set of problem (2).

Problem (63) satisfies a Slater type condition since, by construction,

$$A(\bar{x})\tau(i) \geq 0, \quad i \in I_m^0, \quad t^\top A(\bar{x})t \geq -\bar{\mu} > 0, \quad t \in \{t \in T : \rho(t, T_m^0) \geq \varepsilon_m^0\},$$

and $\{t \in T : \rho(t, T_m^0) \geq \varepsilon_m^0\}$ is a compact set.

Let $x^0$ be an optimal solution of problem (2). Then vector $x^0$ is optimal in problem (63) as well. Hence, there exist indices, numbers and vectors

$$\tau(i) \in \mathbb{R}^+_p, \quad e^\top \tau(i) = 1, \quad \tau(i) \notin T_m^0, \quad \gamma(i) > 0, \quad i \in I; \quad \lambda_m(i) \in \mathbb{R}^+_p, \quad i \in I_m^0,$$

such that

$$\sum_{i \in I} \gamma(i)(\tau(i))^\top A_j \tau(i) + \sum_{i \in I_m^0} (\lambda_m(i))^\top A_j \tau(i) = c_j, \quad j = 1, \ldots, n,$$

$$\tau(i))^\top A(x^0) \tau(i) = 0, \quad i \in I; \quad (\lambda_m(i))^\top A(x^0) \tau(i) = 0, \quad i \in I_m^0.$$ 

Let us set

$$V^0 := ((\tau(i))/\gamma(i), \quad i \in I), \quad L_m^0 := (\lambda_m(i)/\beta_m(i), \quad i \in I_m^0),$$

$$U^0 := V^0 (V^0)^\top, \quad W_m^0 := L_m^0 (V_m^0)^\top, \quad D_m^0 := L_m^0 (U_m^0)^\top.$$ 

Then relations (64) take the form

$$(U^0 + W_m^0) \bullet A_j = c_j, \quad j = 1, \ldots, n.$$ 

It follows from (38), (53), (62), and (66) that the constructed set of matrices (33) is a feasible solution of problem (19).

It was shown above that for any feasible solution $x \in X$ of problem (2) and any feasible solution (20) of problem (19), the inequality (21) holds. From the equalities (32) and (65), it follows that the feasible solution $x^0 \in X$ of the primal problem (2) and the feasible solution
(33) of problem (19) constructed above, turn the inequality (21) into equality. The lemma is proved.

**Lemma 4** The algorithm, described in the proof of Lemma 3, is finite (i.e. it stops after a finite number of iterations).

**Proof** If the algorithm has stopped on the Iteration # 0 or the Iteration # 1, the lemma is proved.

Otherwise, let us consider an Iteration # $m$ of the algorithm for some $m \geq 2$. At the beginning of this iteration, we have the set of indices $\tau(i) \in \mathbb{R}_+^P$, $i \in I_m$, where

$$I_m = I_{m-1} \cup \Delta I_{m-1} = \Delta I_0 \cup \Delta I_1 \cup \ldots \cup \Delta I_{m-1}$$

and

$$\Delta I_0 := I_1, \ \Delta I_s \neq \emptyset, s = 0, \ldots, m - 1.$$  

As before, denote $P_+(i) := \{k \in P : \tau_k(i) > 0\}$, $i \in \Delta I_s$, $s = 0, 1, \ldots, m - 1$.

Let $i_s$ be an index from the set $\Delta I_s$: $i_s \in \Delta I_s$, $s = 0, 1, \ldots, m - 1$.

For any $k, 2 \leq k \leq m$, and any $s, 0 \leq s \leq k - 2$, by construction, it holds

$$i_s \in I_{k-1} = \Delta I_0 \cup \Delta I_1 \cup \ldots \cup \Delta I_{k-2}, \ i_{k-1} \in \Delta I_{k-1},$$

and the relations (50) are fulfilled with $\Delta I = \Delta I_{k-1}$ and $I = I_{k-1}$. Hence

$$P_+(i_s) \cap (P \setminus P_+(i_{k-1})) \neq \emptyset, \ s = 0, 1, \ldots, k - 2, \ k = 2, \ldots, m,$$

wherefrom we conclude

$$P_+(i_s) \subset P_+(i_{k-1}), \ s = 0, 1, \ldots, k - 2, \ k = 2, \ldots, m. \quad (67)$$

Consequently, all the sets $P_+(i_s), s = 0, 1, \ldots, m - 1$, are different. Taking into account that on each Iteration # $m$ it holds $\Delta I_s \neq \emptyset, s = 0, 1, \ldots, m - 1$, one can conclude that the number $m_0$ of the iterations fulfilled by the algorithm, cannot be greater than some finite number $m_*$, where $m_*$ is the maximal number of all different subsets of the set $P$ satisfying (67). The lemma is proved. 

**Remark 1** The main contribution of the algorithm used in the proof of Lemma 3, consists in the justification of the existence of a finite number $m_0$ and the corresponding feasible solution (33) of problem (19) for which equality (34) is satisfied.

It is worthwhile mentioning that it was not the aim of this paper to find a “good” estimate of the minimal value of the number $m_0$.

Notice also that it is possible that someone can offer other (perhaps more complex) procedures for finding the finite sets of matrices (33) satisfying the constraints of problem (19) and the equality (34). Some of such procedures may provide a better (smaller than $m_*$) estimate of the number $m_0$.

**Remark 2** In the case when the immobile indices are isolated points in the set $T$, the set of immobile indices is finite: $T_{im} = \{r^*(j), j \in J_s\}$, $|J_s| < \infty$ (see Proposition 2.5 in [15]). Then on each Iteration # $m$ of the algorithm, it holds

$$\Delta I_m \neq \emptyset, \ \{\tau(i), i \in \Delta I_m\} \subset \{r^*(j), j \in J_s\},$$

$$\{\tau(i), i \in \Delta I_k\} \cap \{\tau(i), i \in \Delta I_s\} = \emptyset \ \forall k = 1, \ldots, m \ \forall s = 1, \ldots, m; \ k \neq s,$$

and relations (50) are satisfied. Hence is this case one does not need to use the Procedure DAM and has $m_0 \leq |J_s|$.
The main result of the paper can be formulated in the form of the following theorem which is a consequence of Lemmas 2 and 3.

**Theorem 1** There exists a finite \( m_0 \geq 0 \) such that problem (19) is dual to the original linear copositive problem (2) and the strong duality relations are satisfied, i.e. if the primal problem (2) admits an optimal solution \( x^0 \), then the dual problem also has an optimal solution in the form (33) and equality (34) holds.

**Remark 3** In our recent paper [14], we have suggested another strong dual formulation for Linear Copositive Programming. This formulation was based on the knowledge of the extremal points of the set \( \operatorname{conv} T_m \). In the present paper, the extended dual problem for the linear copositive problem (2) is also obtained using the concept and the properties of the normalized immobile index set, but in its final formulation, we do not use neither the elements of this set (the immobile indices), nor the extremal points of its convex hull.

To illustrate the relationship between the original copositive problem (2), its Lagrangian dual problem (18) and the extended dual (19), let us consider an example from [17] which is a slight modification of the Example 2.2 from [27].

Consider a linear copositive problem in the form (2) with the following data:

\[
\begin{align*}
A_0 &= \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & A_1 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & A_2 &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix},
\end{align*}
\]

where \( a > 0 \). It was shown in [17] that the vector \( x^0 = (-1, 0)^\top \) is an optimal solution of this problem and the optimal value of the cost function is equal to \( c^\top x^0 = 0 \).

Let us consider the corresponding Lagrangian dual problem (18),

\[
\begin{align*}
\max_U (-U \cdot A_0), & \quad \text{s.t. } U \cdot A_1 = 0, \quad U \cdot A_2 = -1, \\
& \quad \text{with } U := \sum_{s \in S} t(s)(t(s))^\top, \quad t(s) \in \mathbb{R}^3_+,
\end{align*}
\]

for some finite index set \( S : |S| < \infty \). For data (68), this problem takes the form

\[
\begin{align*}
\max_{k=1, 2, 3} (-a \sum_{s \in S} t_k^2(s)),
\end{align*}
\]

It follows from the constraints of the dual problem above that for any dual feasible solution it holds \( t_2(s) = 0, s \in S, \) and \( \sum_{s \in S} t_k^2(s) = 1 \). Hence, the optimal value of the cost function is equal to \( -a \sum_{s \in S} t_k^2(s) = -a < 0 \). Consequently, using the Lagrangian dual problem (69), we get the positive duality gap: \( c^\top x^0 - (-a \sum_{s \in S} t_1^2(s)) = a > 0 \).

Now, for the same primal copositive problem (2) with data (68), consider the extended dual problem (19). It was shown in [17] that in the primal problem under consideration, there is a unique immobile index \( x^* = (0, 0, 1)^\top \). Hence, according to Remark 2 we may set \( m_0 = 1 \) in the dual problem (19).

Denote \( V_1 = (0, 0, 1)^\top, \) \( L_1 = (0, 1, 0)^\top, \) and set

\[
\begin{align*}
W_0 := \mathbb{O}_3, & \quad U_1 := V_1 V_1^\top, & \quad W_1 := V_1 L_1^\top, & \quad D_1 := L_1 L_1^\top, & \quad U := \mathbb{O}_3.
\end{align*}
\]
It is easy to check that
\[ U_1 \cdot A_j = 0, \quad j = 0, 1, 2; \quad (U + W_1) \cdot A_j = c_j, \quad j = 1, 2; \quad (U + W_1) \cdot A_0 = 0, \]
\[ U \in \mathcal{CP}^3, \quad \left( \begin{array}{c|c} U_1 & W_1 \\ \hline W_1^T & D_1 \end{array} \right) = \left( \begin{array}{c|c} V_1 & V_1 \\ \hline L_1 & L_1 \end{array} \right)^\top \in \mathcal{CP}^6. \]

It follows from the latter relations that the constructed set of matrices \((U, W_1, D_1; U)\) is a feasible solution of problem (19) with \(m_0 = 1\) and the corresponding value of the dual cost function is equal to zero. Taking into account the weak duality inequality and the equality \(c^\top x^0 = 0\), we conclude that the feasible solution (70) is optimal for the extended dual problem (19) with \(m_0 = 1\). Consequently we have shown that the extended dual problem has an optimal solution and there is no positive duality gap between the original primal problem and its extended dual.

At the end of this section, we would like to note that as far as we know, with the exception of the paper [14] mentioned above, all previously published optimal conditions and duality results for Linear Copositive Programming are formulated under the Slater condition. In our research, we do not suppose that the Slater condition is satisfied. All this demonstrates the importance and novelty of the results of the paper.

4 A comparison of the obtained duality results for the linear copositive problem with dual formulations for the linear SDP

The main goal of this section is to show that for the given linear copositive problem (2), its extended dual problem (19) is constructed here using the same rules which were used in [25,26,28] to build a dual problem for SDP but with the help of different dual cones.

Consider a linear SDP problem
\[ \min_x c^\top x \quad \text{s.t. } A(x) \in S_+(p). \tag{71} \]

Following [26], let us adduce the M. Ramana et al.’ extended dual for this problem:
\[ \max \quad -(\tilde{U} + \tilde{W}_{m_0}) \cdot A_0, \quad \text{s.t. } (\tilde{U}_m + \tilde{W}_{m-1}) \cdot A_j = 0, \quad j = 0, 1, \ldots, n, \quad m = 1, \ldots, m_0, \]
\[ (\text{ED-R}) : \quad (\tilde{U} + \tilde{W}_{m_0}) \cdot A_j = c_j, \quad j = 1, 2, \ldots, n; \quad \tilde{U} \in S_+(p), \quad \tilde{W}_0 = \mathbb{0}_p, \]
\[ \left( \begin{array}{c|c} \tilde{U}_m & \tilde{W}_m \\ \hline \tilde{W}_m^\top & I \end{array} \right) \in S_+(2p), \quad m = 1, \ldots, m_0. \tag{72} \]

It is easy to notice that the new dual problem (19) obtained in this paper for problem (2), has a similar structure and properties as the dual problem (ED-R) for SDP problem (71).

Nevertheless, it is worth mentioning that these dual problems were obtained using different approaches: the dual problem (19) was formulated and its properties were established using (implicitly) the concept of the immobile indices while the dual SDP problem (ED-R) (referred in [26] as the regularized dual problem (DRP)) was derived using the notion of the minimal cone which was described there as the output of a special procedure.

To show that there exists more closed relationship between copositive and SDP dual formulations, we will formulate for the SDP problem (71), a new dual problem which is a little different from the formulation (ED-R).

For this purpose, we apply to the SDP problem (71), the approach developed hereby for Linear Copositive Programming. Having repeated the process of building the dual problem
as it was described in Sect. 3, one can obtain the extended dual to problem (71) in the form

$$\begin{align*}
&\max - (U + W_{m_0}) \cdot A_0, \\
&\text{s.t. } (U_m + W_{m-1}) \cdot A_j = 0, \ j = 0, 1, ..., n, \ m = 1, ..., m_0. \\
&\text{(ED): } (U + W_{m_0}) \cdot A_j = c_j, \ j = 1, 2, ..., n, \ U \in S_+(p), \ W_0 = \emptyset, \\
&\quad \quad \quad \quad (U_m W_m W_m^\top D_m) \in S_+(2p), \ m = 1, ..., m_0. \quad (73)
\end{align*}$$

The only difference in formulations (ED) and (ED-R) consists of the right lower blocks of the matrices (72) and (73). Let us show that these problems are equivalent.

In fact, let \((\tilde{U}_m, \tilde{W}_m, m = 1, ..., m_0, \tilde{U})\) be a feasible solution of problem (ED-R). It is evident that the set

$$(U_m = \tilde{U}_m, W_m = \tilde{W}_m, D_m = I, \ m = 1, ..., m_0, \ U = \tilde{U})$$

is a feasible solution of problem (ED) with the same value of the cost function.

Now let us show that for any feasible solution

$$(U_m, W_m, D_m, m = 1, ..., m_0, U) \quad (74)$$

of problem (ED) there exists a feasible solution

$$(\tilde{U}_m, \tilde{W}_m, m = 1, ..., m_0, \tilde{U}) \quad (75)$$

of problem (ED-R) with the same value of the cost function.

Notice that for \(m = 1, ..., m_0\), from the inclusion (73) it follows that there exists a matrix

$$B_m = \begin{pmatrix} V_m \\ L_m \end{pmatrix} \in \mathbb{R}^{2p \times k(m)} \text{ with } V_m \in \mathbb{R}^{p \times k(m)} \text{ and } L_m \in \mathbb{R}^{p \times k(m)},$$

such that

$$\begin{pmatrix} U_m \\ W_m \\ W_m^\top D_m \end{pmatrix} = B_m B_m^\top = \begin{pmatrix} V_m \\ L_m \end{pmatrix} \begin{pmatrix} V_m^\top \\ L_m^\top \end{pmatrix}.$$

Hence, for \(m = 1, ..., m_0\), matrices \(U_m, W_m, D_m\) admit representations

$$U_m = V_m V_m^\top, \ W_m = V_m L_m^\top, \ D_m = L_m L_m^\top,$$

with some matrices \(V_m\) and \(L_m\).

Let (74) be a feasible solution of problem (ED). Set

$$\tilde{U} := U, \ \tilde{W}_{m_0} := W_{m_0}, \ \rho(m_0) := \max\{1, \mu_{\text{max}}(L_{m_0}^\top L_{m_0})\},$$

$$\tilde{W}_{m_0} := \rho(m_0)U_{m_0}, \ \tilde{W}_{m_0-1} := \rho(m_0)W_{m_0-1}.$$

Here \(\mu_{\text{max}}(Q)\) denotes the maximal eigenvalue of a matrix \(Q \in \mathbb{R}^{p \times p}\).

It is easy to check that, by construction, we have

$$(\tilde{U} + \tilde{W}_{m_0}) \cdot A_j = c_j, \ j = 1, 2, ..., n; \quad (\tilde{U}_{m_0} + \tilde{W}_{m_0-1}) \cdot A_j = 0, \ j = 0, 1, ..., n.$$

Let us show that

$$\tilde{U}_{m_0} - \tilde{W}_{m_0} \tilde{W}_{m_0}^\top \in S_+(p), \quad (76)$$

or equivalently,

$$t^\top V_{m_0}(\rho(m_0)I - \tilde{L}(m_0)) V_{m_0}^\top t \geq 0 \ \forall t \in \mathbb{R}^p,$$
or
\[ \tau^\top (\rho(m_0)I - \tilde{L}(m_0)) \tau \geq 0 \quad \forall \tau \in \{ \tau \in \mathbb{R}^p : \tau = V^\top_{m_0} t, \ t \in \mathbb{R}^p \} \subset \mathbb{R}^p, \]
where \( \tilde{L}(m_0) := L^\top_{m_0} L_{m_0} \).

It is known (see [21], p. 230) that for any real symmetric matrix \( Q \in S(p) \), the inequality \( t^\top Q t \leq \mu_{max}(Q) t^\top t \) is satisfied for any \( t \in \mathbb{R}^p \). Hence
\[ t^\top (\rho(m_0)I - \tilde{L}(m_0)) \tau = \rho(m_0) t^\top \tau - t^\top \tilde{L}(m_0) \tau \geq (\rho(m_0) - \mu_{max}(\tilde{L}(m_0))) t^\top \tau \geq 0 \quad \forall \tau \in \mathbb{R}^p. \]

Inclusion (76) is proved.

Suppose that for some \( m \leq m_0 \) we have constructed matrices
\( \tilde{U}, \tilde{U}_{m_0}, ..., \tilde{U}_{m}, \tilde{W}_{m_0}, ..., \tilde{W}_m, \tilde{W}_{m-1} \),
and a number \( \rho(m) > 0 \) such that \( \tilde{W}_{m-1} = \rho(m)W_{m-1} \) and the following relations hold:
\( (\tilde{U} + \tilde{W}_{m_0}) \bullet A_j = c_j, \ j = 1, 2, ..., n; \)
\( (\tilde{U}_s + \tilde{W}_{s-1}) \bullet A_j = 0, \ j = 0, 1, ..., n, \ \tilde{U}_s - \tilde{W}_s \tilde{W}_s^\top \in S_+(p), \ s = m_0, m_0 - 1, ..., m. \)

Let us set
\[ \rho(m - 1) := \max \{ 1, \rho^2(m) \mu_{max}(L^\top_{m-1} L_{m-1}) \}, \]
\[ \tilde{U}_{m-1} := \rho(m - 1)U_{m-1}, \ \tilde{W}_{m-2} := \rho(m - 1)W_{m-2}. \]

Applying the rules described above for the cases where \( m = m_0, m_0 - 1, ..., 2 \), we can construct matrices \( \tilde{U}, \tilde{U}_{m_0}, ..., \tilde{U}_2, \tilde{W}_{m_0}, ..., \tilde{W}_2, \tilde{W}_1 \), and the number \( \rho(2) > 0 \) such that \( \tilde{W}_1 = \rho(2)W_1 \).

Set \( \rho(1) := \max \{ 1, \rho^2(2) \mu_{max}(L^\top_1 L_1) \}, \tilde{U}_1 := \rho(1)U_1 \). One can check that the matrices constructed above, form a feasible solution (75) of problem (ED-R) and it holds
\( (\tilde{U} + \tilde{W}_{m_0}) \bullet A_0 = (U + W_{m_0}) \bullet A_0. \)

Hence, for the SDP problem (71), we have shown that the dual problem in the form (ED) is a slight modification of the known dual problem (ED-R).

Now, let us compare two pairs of primal and dual problems:

(\( \alpha \)) the linear copositive problem (2) and its dual one (19), and
(\( \beta \)) the SDP problem (71) and its dual one (ED).

One can see that these pairs of dual problems are constructed in the spaces \( S(p) \) and \( S(2p) \) using the same rules, but their constraints are defined with the help of different dual cones:

- in the pair of problems (2) and (19), the cone \( \mathcal{COP}^p \) is used to formulate the constraints of the primal copositive problem and the dual cones \( \mathcal{CP}^p \) and \( \mathcal{CP}^{2p} \) are used to formulate the constraints of the dual one;
- in the pair of SDP problems (71) and (ED), the cone \( S_+(p) \) is used to formulate the constraints of the primal SDP problem and the dual cones \( S_+^*(p) = S_+(p) \) and \( S_+^*(2p) = S_+(2p) \) are used for the dual formulation.

This similarity points to a deep relationship between these two classes of conic problems, Linear Copositive Programming and SDP. At the same time, it is worth mentioning that copositive problems are more complex and less studied when compared with that of SDP.
Remark 4 When comparing the complexity of the procedures described above for constructing the pairs of dual problems in SDP and Linear Copositive Programming, notice the following.

- For SDP problems, one has an estimate $m_0 \leq \min\{n, p\}$ of the number $m_0$. This estimate can be found using the fact that the set of immobile indices for an SDP problem is a subspace of $\mathbb{R}^p$ and the properties of positive semidefinite matrices are well-studied.
- For linear copositive problems, determining a good estimate of $m_0$ is a much more challenging task as the set of immobile indices is a union of a finite number of convex cones in $\mathbb{R}^p$. Notice that the cone of copositive matrices and its dual cone (the cone of completely positive matrices) are not so well studied (there are many open questions here [5,9]).
- The cones of copositive and completely positive matrices are neither self-dual nor homogeneous (see [10]).

As it was noticed above, finding a good estimate of the number $m_0$ for copositive problems was not our purpose here. We plan to devote a special paper to this issue.

5 Conclusions and future work

The main contribution of the paper consists in developing a new approach to dual formulations in Linear Copositive Programming. This approach permitted us to formulate a new extended dual problem in explicit form and to close the duality gap between the optimal values of the copositive problem and its extended dual without any CQs or other additional assumptions.

To the best of our knowledge, with the exception of our previous papers [14,17], in Linear Copositive Programming, there are no other known explicit strong dual formulations that do not require CQs.

In [14,17], the dual problems were formulated based on the explicit knowledge of the immobile index set. The advantage of the dual results presented in this paper if compare with the results mentioned above, consists in the fact that now there is no need to find explicitly either the elements of the normalized immobile index set or the extremal points of its convex hull. For linear copositive problems, the dual formulation obtained in the paper is original and different from that published before.

The new dual formulation for Linear Copositive Programming is similar to the dual formulation for SDP problem proposed by M.Ramana et al. in [26]. This similarity and the fact that the duality results obtained in this paper (i) do not use CQs, (ii) have explicit formulation, and (iii) are strong, motivate us to study other applications of the developed approach based on the notion of the immobile indices.

In our future work, we are going to find a better estimate of the number $m_0$ that is essential for our dual formulation. To obtain this estimate, it will be necessary to study new properties of the extended dual problem and its feasible set. We plan also to apply the results of the paper for other classes of copositive problems with the aim to develop new explicit optimality conditions.

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