Regularization of Kerr-NUT spacetimes and their thermodynamical quantities

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In the context of the teleparallel equivalent of general relativity (TEGR) theory, continuous calculations of the total energy and momentum for Kerr-NUT spacetimes using three different methods, the gravitational energy-momentum, the Riemannian connection 1-form, \( \tilde{\Gamma}^{\beta}_{\alpha} \) and the Euclidean continuation method, have been achieved. Many local Lorentz transformations, that play the role of regularizing tool, are given to get the commonly known form of energy and momentum. We calculate the thermodynamic quantities of Kerr-NUT spacetime. We also investigate the first law of thermodynamics and quantum statistical relation.

1. Introduction

For several decades, four dimensional solutions of Einstein field equations have been widely inquiry in gravity community. Taub-NUT metric\(^1\) is an analytic solution of the vacuum Einstein equations. When the metric is expressed in Schwarzschild-like coordinates, one has coordinate singularity that occurs at certain values of radial coordinate where \( g_{rr} \) component becomes infinity and corresponding to bifurcate Killing horizons. The Taub-NUT spacetime is participate in many recent studies of general relativity (GR). Hawking has suggested that the Euclidean geometry of Taub-NUT metric could lead to gravitational analogue of the Yang-Mills instanton.\(^2\) In that case the Einstein equations are met with zero cosmological constant and the manifold is \( R_4 \) with the limit that is a twisted three-sphere possessing a distorted metric. The Kaluza-Klein monopole has been obtained by embedding the Taub-NUT gravitational instanton into five dimensional Kaluza-Klein theory.\(^3,4\)

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∗PACS numbers: 04.20.Cv, 04.20.Fy, 04.50.-h
Keywords: gravitation, teleparallel gravity, energy-momentum, Weitzenböck connection, regularization teleparallelism

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Egyptian Relativity Group (ERG) URL: http://www.erg.eg.net
The non-Abelian goals space duals of the Taub-NUT spacetime have been also examined in terms of the local isometry group $SU(2) \times U(1)$. The Taub-NUT spacetime has been shown to be associated with $SU(2)$ through T-duality. Carter has demonstrated that the Hamilton-Jacobi equation for the geodesics in the Taub-NUT metric separates in specific coordinate systems. The gravitomagnetic monopole source effects have been also examined in the Taub-NUT spacetime. In the Kerr-Taub-NUT-de Sitter metrics, separability of the Hamiltonian-Jacobi equation has been studied in higher dimensions. Recently, a rotating Schwarzschild black hole has been studied to investigate effective potentials for null and timelike geodesics of particles and hydrodynamics associated with general relativistic Euler equation for the steady state axisymmetric fluid. Recently, calculations of energy, spatial momentum and angular momentum for some specific Kerr-NUT spacetimes have been done.

Hawking’s ground-breaking work on black hole evaporation and information loss is based on the idea that a pair of particles is created just inside the event horizon and from this pair the positive energy particle tunnels out of the hole and appears as Hawking radiation. The negative energy particle tunnels inwards and results in decrease of the mass of the black hole. The energy of the particle changes sign as it crosses the horizon. These particles follow trajectories which cannot be explained classically. This process of particle evaporation from black holes is thus a phenomenon of quantum tunneling of particles through the event horizon. This contributes to the change in mass, angular momentum and charge of the black hole, which change its thermodynamics as well. The particle travels in time so that the action becomes complex and the dynamics of the outgoing particle is governed by the imaginary part of this action. This action has been calculated using the radial null geodesics or the so-called Hamilton-Jacobi method for various spacetimes. Using the method of the radial null geodesics Hawking radiation as a tunneling process has been analyzed for the Kerr and Kerr-Newman black holes. They have done the analysis using transformed forms of these metrics and they do not consider the question of quantum corrections at all. The Hamilton-Jacobi method has been used to calculate quantum corrections to the Hawking temperature and the Bekenstein-Hawking area law for the Schwarzschild, anti-de Sitter Schwarzschild and Kerr black holes.

At present, teleparallel theory seems to be popular again, and there is a trend of analyzing the basic solutions of general relativity (GR) with teleparallel theory and comparing the results. The TEGR is a viable alternative geometrical description of Einstein’s GR written in terms of the tetrad field. It is considered as an essential part of generalized non-Riemannian theories such as the Poincaré gauge theory or metric-affine gravity. The physics relevant to geometry may be related to the teleparallel description of gravity. Within the context of metric-affine gravity, a stationary axially symmetric exact solution of the vacuum field equations is obtained for a specific gravitational Lagrangian by using prolongation techniques (see [18] and references therein). A relation between spinor Lagrangian and teleparallel theory is established. In the framework of the TEGR it has been possible to address the long-standing problem of defining energy, momentum and angular momentum of the gravitational field. The tetrad field seems to be a suitable field quantity to address this problem, because it yields the gravitational field and at the same time establishes a class of reference frames in spacetime. Moreover, there are simple and clear indications that the gravitational energy-momentum defined in the context of the TEGR provides a unified picture of the concept of mass-energy in special and general relativity. It is the object of the current research to extend the previous work by including other Kerr-NUT spacetimes to show if the previous procedure works correctly or not? Also the physical quantities related to
these tetrads, i.e., entropy, Hawking temperature are calculated.*

In §2, the first Kerr-NUT spacetime is given and calculation of its energy, spatial and angular momentum have been achieved using the definition of the gravitational energy-momentum, which is coordinate independent, and common know results are obtained. In §3, the second Kerr-NUT spacetime is provided, that is linked to the first tetrad through a local Lorentz transformation, and the conserved quantities, energy, spatial and angular momentums are calculated and a finite result is obtained. In §4, it is shown by explicate calculation that the energy of the third Kerr-NUT spacetime using the definition of the gravitational energy-momentum is always divergent! To make the calculation clearer we use the Riemannian connection 1-form, $\tilde{\Gamma}^{\beta}_{\alpha}$ and repetition of the calculation of energy has been done and same divergent value is obtained. Therefore, a new local Lorentz transformation is employed in §5. When this transformation is applied to the Kerr-NUT spacetime, third one, and repetition of the calculation of energy either using the gravitational energy-momentum or the Riemannian connection 1-form, $\Gamma^{\beta}_{\alpha}$ one can get a finite and acceptable result. In §6, another Kerr-NUT spacetime is given, fourth one, calculation of energy using the two definitions has been achieved and a divergent result is presented. In §7, using another local Lorentz transformation an acceptable and finite result is obtained. In §8, we use the Euclidean continuation method to calculate the energy, entropy, temperature. Also we show the consistient of the first law of thermodynamics for Kerr-NUT spacetime and also show that the quantum statistical relation holds. §9 is devoted to main result and discussion. In §10, many local Lorentz transformations, that can be viewed as regularizing tools for the calculations of energy and momentum, are provided.

### 2. First Kerr-NUT spacetime

The covariant form of the first Kerr-NUT tetrad field having axial symmetry in spherical coordinates ($t, r, \theta, \phi$), can be written as

$$
(h^\alpha_i)_1 = \begin{pmatrix}
F_1 & 0 & 0 & F_2 \\
0 & F_3 & 0 & 0 \\
0 & 0 & F_4 & 0 \\
0 & 0 & 0 & -F_5 \sin \theta
\end{pmatrix},
$$

where $F_i, i = 1 \cdots 5$ are functions of $r$ and $\theta$ having the form

$$
F_1 = \sqrt{\frac{D}{C}}, \quad F_2 = \frac{F}{\sqrt{CD}}, \quad F_3 = \sqrt{\frac{C}{D_1}}, \quad F_4 = \sqrt{C}, \quad F_5 = \sqrt{\frac{CD_1}{D}},
$$

$$
C = r^2 + (L + a \cos \theta)^2, \quad D = r^2 - 2Mr + a^2 \cos^2 \theta - L^2, \quad D_1 = r^2 - 2Mr + a^2 - L^2, \quad F = -2 \left( aL^2 + Mr \right) \sin^2 \theta + D_1 L \cos \theta
$$

*Many basic equations used in this calculation are reviewed explicitly in11), and references therein. Therefore, in this work repetitions will be omitted.
where $M$, $a$ and $L$ are the mass, the rotation and the NUT parameters respectively.\textsuperscript{22)} We consider a non asymptotically flat spacetime in this paper, and impose the boundary condition that for $r \to \infty$ and $L \to 0$ the tetrad (1) approaches the tetrad of Minkowski spacetime, in Cartesian coordinate, i.e., $O_{\alpha\beta} = \text{diag}(+1, -1, -1, -1)$. The metric tensor $g_{ij} \overset{\text{def}}{=} O_{\mu\nu} h^{\mu i} h^{\nu j}$ associated with the tetrad field (1) has the form

$$
\text{ds}^2 = F_1^2 \text{dt}^2 - F_3^2 \text{dr}^2 - F_4^2 \text{d}\theta^2 - \left( F_3^2 \sin^2 \theta - F_2^2 \right) \text{d}\phi^2 - F_1 F_2 \text{dtd}\phi,
$$

which is the Kerr-NUT spacetime written in Boyer-Lindquist coordinates.\textsuperscript{11)}

Now we are going to calculate the energy content of the tetrad field (1) using (2). The non-vanishing components of the tensor $\Sigma^{abc}$ needed to the calculation of energy have the form

$$
\Sigma^{(0)01} \approx -\frac{\sin \theta}{2r^2} \left( 2r^3 - 2Mr + 2a^2 M \cos^2 \theta + ra^2 \sin^2 \theta + 2(2M - r)aL \cos \theta - 2rL^2 \right) + O \left( \frac{1}{r^3} \right),
$$

$$
\Sigma^{(3)01} \approx -\frac{r^2 L \cos \theta + Mra \sin^2 \theta - 2aL \cos^2 \theta (L + a \cos \theta)}{2r^2} + O \left( \frac{1}{r^3} \right).
$$

Using Eq. (4), the energy associated with spacetime (1) takes the form

$$
P^{(0)} = E = -\oint_{S \to \infty} dS_k \Pi^{(0)k} = -\frac{1}{4\pi} \oint_{S \to \infty} dS_k \sqrt{-g} h^{(0)}_{\mu} \left( \Sigma^{\mu0k} - \Sigma^{k0\mu} \right) M = 0, a = 0, L = 0
$$

$$
\approx M + \frac{L^2}{r} + O \left( \frac{1}{r^2} \right).
$$

It follows from Eq. (5) that energy of the Kerr-NUT spacetime is finite and physically acceptable. Using Eq. (4) one can get the spatial momentum in the form

$$
P_1 = -\oint_{S \to \infty} dS_k \Pi^{(1)k} = -\frac{1}{4\pi} \oint_{S \to \infty} dS_k \epsilon \Sigma^{(1)0k} = 0, \text{ by same method } P_2 = 0, \ P_3 \approx \left( \frac{1}{r} \right).
$$

Now turn our attention to calculate the angular-momentum. The non vanishing components of the angular-momentum are given by

$$
M^{(0)1}(r, \theta, \phi) \approx \frac{\sin \theta}{4\pi r} \left( L^2 + rM + aL \cos \theta - 2r^2 \right) + O \left( \frac{1}{r^2} \right),
$$

$$
M^{(0)2}(r, \theta, \phi) \approx -\frac{1}{16\pi r} \left( 2r^2 \cos \theta - 2L a(2 - 3 \cos^2 \theta) - a^2 \cos \theta [2 - 3 \cos \theta] \right) + O \left( \frac{1}{r^2} \right),
$$

$$
M^{(1)3}(r, \theta, \phi) \approx -\frac{Ma \sin^2 \theta}{4\pi r} + O \left( \frac{1}{r^2} \right), \ M^{(2)3}(r, \theta, \phi) \approx \frac{[ML - rL + 2Ma \cos \theta] \sin \theta}{4\pi r} + O \left( \frac{1}{r^2} \right).
$$

Using Eq. (7) we get

$$
L^{(0)1} = \int_0^\pi \int_0^{2\pi} \int_0^\infty d\theta d\phi dr \left[ M^{(0)1} \right] = 0,
$$

where $* \Sigma^{abc} \overset{\text{def}}{=} \frac{1}{4} (T^{abc} + T^{bac} - T^{cab}) + \frac{1}{2} (O^{a}T^{b} - O^{ab}T^{c})$ where $T^{abc}$ and $T^{a}$ are the torsion tensor and the basic vector field.\textsuperscript{11)}
which is a consistence results. By the same method one can obtain
\[
L^{(0)(2)} = L^{(1)(3)} = L^{(0)(3)} = L^{(1)(2)} = L^{(2)(3)} = 0.
\] (9)

Eq. (9) shows that the angular momentum associated with tetrad (1) not right and this may
be linked to the definition used in the calculation.\(^{11}\)

3. Second Kerr-NUT spacetime

The covariant form of the second Kerr-NUT tetrad field having axial symmetry in spherical
coordinates, can be written as
\[
(h^\alpha_i)_2 = \begin{pmatrix}
F_1 & 0 & 0 & F_2 \\
0 & \sin \theta \cos \phi & \cos \theta \cos \phi & F_3 \\
0 & \sin \theta \sin \phi & \cos \theta \sin \phi & F_4 \\
0 & \cos \theta & -\sin \theta & F_5
\end{pmatrix}
\]
where \(F_i, \ i = 1 \cdots 5\) are defined by Eq. (13). (10)

Tetrad (10) has the same associated metric of tetrad (1), i.e., Kerr-NUT spacetime given
by Eq. (3). Tetrad (10) is related to tetrad (1) through the relation \((h^\alpha_i)_2 = (\Lambda^\alpha_\gamma)(h^\gamma_i)_1\),
where \((\Lambda^\alpha_\gamma)\) is the local Lorentz transformation given by
\[
(\Lambda^\alpha_\gamma) \triangleq \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \sin \theta \cos \phi & \cos \theta \cos \phi & -\sin \phi \\
0 & \sin \theta \sin \phi & \cos \theta \sin \phi & \cos \phi \\
0 & \cos \theta & -\sin \theta & 0
\end{pmatrix}. \] (11)

Following the same technics used in §2 to calculate the energy associated with tetrad (10),
we finally get the non-vanishing components of \(\Sigma^{a01}\) up to order \(O \left(\frac{1}{r^3}\right)\) as
\[
\Sigma^{(0)01} \approx -\frac{\sin \theta}{2r} \left(2r^3 - 2Mr + 2a^2 M(3 \cos^2 \theta - 1) + ra^2 \sin^2 \theta + 2(2M - r)aL \cos \theta - 2rL^2\right) + O \left(\frac{1}{r^3}\right),
\]
\[
\Sigma^{(1)01} \approx \frac{Ma \sin^2 \theta \cos \phi}{2r} + O \left(\frac{1}{r^2}\right), \quad \Sigma^{(2)01} \approx \frac{Ma \sin^2 \theta \sin \phi}{2r} + O \left(\frac{1}{r^2}\right). \] (12)

Using Eq. (12) to calculate energy one can get the same value of first tetrad as given by
Eq. (5). Also the form of spatial momentum components are the same as that of Eq. (6).
Finally, the angular momentum linked to tetrad (10) has the form (9).
4. Third Kerr-NUT spacetime

The covariant form of the third Kerr-NUT tetrad can be written as

\[
(h_\alpha^i)_3 = \begin{pmatrix}
-\sqrt{\frac{D_1}{C}} & 0 & 0 & \sqrt{\frac{D_1}{C}} (a \sin^2 \theta - 2L \cos \theta) \\
0 & -r \sin \theta & C_1 & 0 \\
a \sin \theta & 0 & 0 & -\sin \theta (r^2 + a^2 + L^2) \\
0 & \frac{C_1}{\sqrt{D_1}} & -r \sin \theta & 0
\end{pmatrix},
\]

(13)

where \(C_1 = \sqrt{C - r^2 \sin^2 \theta}\). Tetrad (13) has the same associated metric of tetrad (1). Tetrad (13) is related to tetrad (1) through the relation

\[
(h_\alpha^i)_3 = (\Lambda_1^{\alpha \gamma}) (h_\gamma^i)_1
\]

where \((\Lambda_1^{\alpha \gamma})\) is another local Lorentz transformation given by

\[
(\Lambda_1^{\alpha \gamma}) \overset{\text{def.}}{=} \begin{pmatrix}
\sqrt{\frac{D_1}{D}} & 0 & a \sin \theta & 0 \\
0 & r \sin \theta & \frac{C_1}{\sqrt{C}} & 0 \\
a \sin \theta & 0 & \frac{C_1}{\sqrt{D}} & -r \sin \theta \\
0 & \frac{D_1}{\sqrt{D}} & 0 & 0
\end{pmatrix},
\]

(14)

Following the same technics used in §2 and §3 to calculate energy one can get the non-vanishing components of \(\Sigma^{a01}\) asymptotically as

\[
\Sigma^{(0)01} \approx -\frac{\sin \theta}{4r \cos^2 \theta} \left(6r^2 \cos^2 \theta - 7a^2 \cos^2 \theta \sin^2 \theta + 4Mr \cos^2 \theta + L^2(1 + 3 \cos^2 \theta) + 12aL \cos^3 \theta \right) + O \left(\frac{1}{r^2}\right),
\]

\[
\Sigma^{(2)01} \approx \frac{[Ma - 3ar] \sin^2 \theta + 4L \cos \theta |r - M|}{4r} + O \left(\frac{1}{r^2}\right).
\]

(15)

Using Eq. (15) we finally obtain

\[
P^{(0)} = E = -\oint_{S \rightarrow \infty} dS_k \Pi^{(0)k} = -\frac{1}{4\pi} \oint_{S \rightarrow \infty} dS_k \sqrt{-g} h^{(0)} \Sigma^{00k} \approx \infty!
\]

(16)

It follows from Eq. (16) that the energy of the Kerr-NUT spacetime is divergent and this is not an acceptable.

Due to the above non physical result and to make the picture clearer another method will be use to calculate the energy of tetrad (13) to show if the divergent result will continue or not?
The coframe of tetrad (13), is described by the components:

\[
\begin{align*}
\vartheta_1^0 &= -\sqrt{\frac{D_1}{C}} \left( [a \cos^2 \theta + 2L \cos \theta]d\phi + dt \right), \\
\vartheta_1^1 &= -\frac{r \sin \theta dr - \sqrt{D_1} C_1 d\theta}{\sqrt{D_1}}, \\
\vartheta_1^2 &= -\frac{\sin \theta}{\sqrt{C}} \left( [r^2 + L^2 + a^2]d\phi - adt \right), \\
\vartheta_1^3 &= -\frac{C_1 dr + r \sin \theta \sqrt{D_1} d\theta}{\sqrt{D_1}}.
\end{align*}
\]

(17)

Taking coframe (17), as well as the Riemannian connection \( \tilde{\Gamma}_\alpha^\beta \) and substitute into the translational momentum, \( \tilde{H}_\alpha \), we finally get

\[
\tilde{H}_0 \cong -\frac{\sin \theta}{8r \pi \cos^2 \theta} \left[ 2r^2 \cos^2 \theta + 5a^2 \cos^2 \theta - L^2 \sin^2 \theta - 8aL \cos^3 \theta - 5a^2 \cos^4 \theta \right] d\theta \wedge d\phi + \cdots + O \left( \frac{1}{r^2} \right).
\]

(18)

Using Eq. (18) to compute the total energy at a fixed time in the 3-space with a spatial 2-dimensional boundary surface \( \partial S = \{ r = R, \theta, \phi \} \) we finally obtain

\[
\tilde{E} = \int_{\partial S} \tilde{H}_0 = \infty!
\]

(19)

which is identical with Eq. (16). Eqs. (16) and (19) show that the two definitions, the gravitational energy-momentum and the translational momentum give equal form of energy which is not acceptable. This means that tetrad (13) is inconvenient one and should multiply by some appropriate local Lorentz transformation to bypass the above inconsistence result.

5. On the choice of the frame

Let us consider the local Lorentz transformation described by the matrix

\[
\left( \Lambda_2^\alpha \right)_{\beta} = \begin{pmatrix} H_1 & H_2 & H_3 & H_4 \\ K_1 \sin \theta \cos \phi & K_2 \sin \theta \cos \phi & K_3 \cos \theta \cos \phi & K_4 \sin \phi \sin \theta \\ L_1 \sin \theta \sin \phi & L_2 \sin \theta \sin \phi & L_3 \cos \theta \sin \phi & L_4 \cos \phi \sin \theta \\ N_1 \cos \theta & N_2 \cos \theta & N_3 \sin \theta & N_4 \cos \theta \end{pmatrix},
\]

(20)

where \( H_i, K_i, L_i \) and \( N_i, i = 1 \cdots 4 \) are defined as:

\[
\begin{align*}
H_1 &= -\frac{r^2 - Mr + a^2 + aL \cos \theta}{\sqrt{CD_1}}, & H_2 &= -\frac{(Mr + L^2 + aL \cos \theta)r \sin \theta}{C \sqrt{D_1}}, & H_3 &= -\frac{a \sin \theta}{\sqrt{C}}, \\
H_4 &= -\frac{(Mr + L^2 + aL \cos \theta)c_1}{C \sqrt{D_1}}, & K_1 &= -\frac{(Mr + L^2 + aL \cos \theta - a^2) \cos \phi + a \sin \phi}{\cos \phi \sqrt{CD_1}}.
\end{align*}
\]

\( ^*\Theta^\alpha = h^\alpha dx^\alpha \). We use the relativistic units in this calculation. \( \tilde{H}_\alpha = \frac{1}{2c} \tilde{\Gamma}_\beta \gamma \eta_{\alpha \beta \gamma}, \quad \Gamma_\alpha^\beta \equiv \tilde{\Gamma}_\alpha^\beta - K_\alpha^\beta \)

with \( \tilde{\Gamma}_\alpha^\beta \) is the purely Riemannian connection and \( K^\mu \nu \) is the contorsion 1-form

\( \cdots \) means terms which are multiply by \( d\theta \wedge dr, d\theta \wedge dt, dr \wedge d\phi \) etc.
\( K_2 = -\frac{r[(r^2 - Mr + aL \cos \theta) \cos \phi + ar_1 \sin \phi] \sin^2 \theta - \alpha \cos \theta \mathcal{C}_1 \sqrt{\mathcal{D}_1}}{\cos \phi \sin \theta \sqrt{\mathcal{D}_1 \mathcal{C}}}, \)

\( K_3 = -\frac{\beta}{\cos \theta \cos \phi \sqrt{\mathcal{C}}}, \)

\( K_4 = -\frac{C_1[(r^2 + aL \cos \theta - Mr) \cos \phi + ar_1 \sin \phi] + r \alpha \cos \theta \sqrt{\mathcal{D}_1}}{\sin \phi \sqrt{\mathcal{D}_1 \mathcal{C}}}, \)

\( L_1 = -\frac{(Mr + L^2 + aL \cos \theta - a^2) \sin \phi - ar_1 \cos \phi}{\sqrt{\mathcal{C} \mathcal{D}_1} \sin \phi}, \)

\( L_2 = -\frac{r[(r^2 - Mr + aL \cos \theta) \sin \phi - ar_1 \cos \phi] \sin^2 \theta - \beta \cos \theta \mathcal{C}_1 \sqrt{\mathcal{D}_1}}{\sin \phi \sin \theta \sqrt{\mathcal{D}_1 \mathcal{C}}}, \)

\( L_3 = \frac{\alpha}{\cos \theta \sin \phi \sqrt{\mathcal{C}}}, \)

\( L_4 = -\frac{C_1[(r^2 + aL \cos \theta - Mr) \sin \phi - ar_1 \cos \phi] + r \beta \cos \theta \sqrt{\mathcal{D}_1}}{\cos \phi \sqrt{\mathcal{D}_1 \mathcal{C}}}, \)

\( N_1 = -\frac{(Mr + L^2 + aL \cos \theta)}{\sqrt{\mathcal{C} \mathcal{D}_1}}, \)

\( N_2 = -\frac{\sin \theta[r \cos \theta(r^2 + a^2 - Mr + aL \cos \theta) + r \mathcal{C}_1 \sqrt{\mathcal{D}_1}]}{\cos \theta \sqrt{\mathcal{D}_1}} \),

\( N_3 = 0 \)

\( N_4 = -\frac{\cos \theta(r^2 + a^2 - Mr + aL \cos \theta) \mathcal{C}_1 - rr_1 \sqrt{\mathcal{D}_1} \sin^2 \theta}{\cos \theta \sqrt{\mathcal{D}_1}} \),

where \( \alpha, \beta, \) and \( r_1 \) are defined by

\( \alpha \overset{\text{def}}{=} r_1 \cos \phi + a \sin \phi, \quad \beta \overset{\text{def}}{=} r_1 \sin \phi - a \cos \phi, \quad r_1 \overset{\text{def}}{=} \sqrt{r^2 + L(L + 2a \cos \theta)}. \)

From Eqs. (13) and (20) we construct the new tetrad

\( (h^\alpha)_{\gamma} = \left( \Lambda_2 \alpha \gamma \right) (h^\gamma)_{\delta}. \)

Using Eq. (23) to calculate the non-vanishing components needed to the calculations of energy, we finally get

\[ \Sigma^{(0)01} \cong \frac{\sin \theta(Mr^2 + rL^2 - raL \cos \theta)}{r^2}, \]

\[ \Sigma^{(1)01} = \Sigma^{(2)01} = \Sigma^{(3)01} \cong -\frac{\sin \theta(r^3 - 2Mr^2 - 2rL^2 - 2raL \cos \theta)}{r^2}. \]

From Eq. (24) one can obtain the energy of Eq. (23) in the from

\[ P^{(0)} = E = -\frac{1}{4\pi} \int_{S \rightarrow \infty} dS_k \Pi^{(0)k} = \frac{1}{4\pi} \int_{S \rightarrow \infty} dS_k \sqrt{-g} (h^{(0)}_{\mu}(\Sigma_{\mu0k} - \Sigma_{\mu0k}^{M=0,a=0,L=0})) \]

\[ \cong M + \frac{L^2}{r} - \frac{L^2 M}{r^2} + O\left(\frac{1}{r^2}\right). \]

Eq. (25) is a satisfactory results. Using Eq. (24) one can get the spatial momentum related to tetrad (23) to has the same form of Eq. (6).\footnote{Eq. (23) is an exact solution to the equation of motion of TEGR. This case is studied intensively by Hayashi and Shirafuji (cf., Ref. 13) Eqs. (7. 2)−(7. 11) and references therein).}

\footnote{\( \Sigma^{(a)bc} = (h^{\alpha})_{\delta} \Sigma^{\delta bc} \). Terms like \( aM L, M a^2, M a^2 L, \) etc. \( \cdots \) are neglected in these calculations.}

\footnote{We introduce \( \Sigma_{\mu0k}^{M=0,a=0,L=0} \) to remove the divergence appearers from term like \( r \). It is worth to mention that we cannot use the expression \( \Sigma_{\mu0k}^{r \rightarrow \infty} \) because the spacetime we use is not asymptotically flat.}

\footnote{Terms like \( M^2, L^3, L^3 M, M^2 a, \cdots \) etc. are neglected in this calculations.}
Now turn our attention to calculate the angular-momentum related to tetrad (23). The non vanishing components needed to compute the angular-momentum up to order $O \left( \frac{1}{r^2} \right)$ are given by

\[
M^{(0)(1)}(r, \theta, \phi) \approx \frac{-1}{8\pi r} \left( \{2r^2 + 2aL \cos^3 \theta + L^2 \cos^2 \theta - 2[L^2 + rM + a^2] \sin^2 \theta - 4aL \sin^2 \theta \cos \theta \} \cos \phi \\
+ ar \sin^2 \theta \sin \phi \right) + O \left( \frac{1}{r^2} \right),
\]

\[
M^{(0)(2)}(r, \theta, \phi) \approx \frac{-1}{8\pi r} \left( \{2r^2 + 2aL \cos^3 \theta + L^2 \cos^2 \theta - 2[L^2 + rM + a^2] \sin^2 \theta - 4aL \sin^2 \theta \cos \theta \} \sin \phi, \\
+ ar \sin^2 \theta \cos \phi \right) + O \left( \frac{1}{r^2} \right),
\]

\[
M^{(0)(3)}(r, \theta, \phi) \approx \frac{\sin \theta(6aL \cos^2 \theta + 2Mr \cos \theta + 3L^2 \cos \theta - 2aL)}{8\pi r} + O \left( \frac{1}{r^2} \right),
\]

\[
M^{(1)(2)}(r, \theta, \phi) \approx \frac{\sin \theta(3aM \cos^2 \theta + 2ML \cos \theta - 2rL \cos \theta - aM)}{8\pi r} + O \left( \frac{1}{r^2} \right),
\]

\[
M^{(1)(3)}(r, \theta, \phi) \approx \frac{- \sin^2 \theta([3aM \cos \theta - 2rL + 2LM] \sin \phi + aL \cos \phi)}{8\pi r} + O \left( \frac{1}{r^2} \right),
\]

\[
M^{(2)(3)}(r, \theta, \phi) \approx \frac{\sin^2 \theta([3aM \cos \theta - 2rL + 2LM] \cos \phi - aL \sin \phi)}{8\pi r} + O \left( \frac{1}{r^2} \right).
\]

Using Eq. (26) we obtain the components of the angular momentum linked to tetrad (23) same as that provided by Eqs. (8) and (9).

We show by explicit calculations that the energy-momentum tensor, that is a coordinate independent, gives inconsistent result of the angular momentum when applied to the tetrad field given by Eq. (23)!

Using the translational momentum employed in the previous section to show if tetrad (23) to continue providing the consistence physical results of energy and spatial momentum or not. The coframe related to tetrad (23) is described by the components:

\[
\psi_4^0 = \frac{1}{\mathcal{D}_1 C} \left\{ \mathcal{D}_1[r^2 - Mr + a \cos \theta(L + a \cos \theta)]dt + C[Mr + L(L + a \cos \theta)]dr \\
+ \mathcal{D}_1[a^2L \cos^3 \theta - arM \cos^2 \theta + aL^2 \cos^2 \theta + 2r^2L \cos \theta - 2rML \cos \theta + a^2L \cos \theta + arM + aL^2]d\phi \right\},
\]

\[
\psi_4^1 = \frac{1}{\mathcal{D}_1 C} \left\{ \mathcal{D}_1[Mr + L(L + a \cos \theta)] \sin \theta \cos \phi dt + C[\{r^2 - Mr + aL \cos \theta\} \cos \phi + ar_1 \sin \phi] \sin \theta dr \\
+ \mathcal{C} D_1 \alpha \cos \theta d\theta + \mathcal{D}_1[r_1 \sin \phi C - \cos \phi \{ar^2 + 2aL^2 - 3aL^2 \cos^2 \theta - 2L^3 \cos \theta - a^2L^3 \cos^3 \theta \\
+ 3a^2L \cos \theta + a^3 \cos^2 \theta - arM \cos^2 \theta - 2rML \cos \theta + raM]) \sin \theta d\phi \right\},
\]

\[
\psi_4^2 = \frac{1}{\mathcal{D}_1 C} \left\{ \mathcal{D}_1[Mr + L(L + a \cos \theta)] \sin \theta \sin \phi dt + C[\{r^2 - Mr + aL \cos \theta\} \sin \phi - ar_1 \cos \phi] \sin \theta dr \\
+ \mathcal{D}_1 \beta \cos \theta d\theta - \mathcal{D}_1[r_1 \cos \phi C + \sin \phi \{ar^2 + 2aL^2 - 3aL^2 \cos^2 \theta - 2L^3 \cos \theta - a^2L^3 \cos^3 \theta \\
+ 3a^2L \cos \theta + a^3 \cos^2 \theta - arM \cos^2 \theta - 2rML \cos \theta + raM]) \sin \theta d\phi \right\},
\]
\[
\dot{\psi}_4^3 = \frac{1}{\mathcal{D}_1 C} \left\{ \mathcal{D}_1 [Mr + L(L + a \cos \theta)] \cos \theta dt + C \{r^2 - Mr + aL \cos \theta\} \cos \theta dr \
- CD_1 r_1 \sin \theta d\theta + \mathcal{D}_1 [2L \cos \theta - a \sin^2 \theta][Mr + L(L + a \cos \theta)] \cos \theta d\phi \right\}.
\]

(27)

If we take coframe (27), as well as the trivial Weitzenböck connection, i.e., \( \Gamma^\alpha_{\beta \gamma} = 0 \) we finally reach the temporal component of the translation momentum in the form

\[
\tilde{H}_0 \approx -\frac{\sin \theta}{8r^2 \pi} \left[ 3a^2 M \cos^2 \theta + 6aLM \cos \theta - 2arL \cos \theta + 2ML^2 - 2rL^2 - a^2 M + 2r^3 - 2r^2 M \right] d\theta \wedge d\phi + \ldots + O \left( \frac{1}{r^3} \right).
\]

(28)

Computing the total energy up to order \( O \left( \frac{1}{r^2} \right) \) at a fixed time in the 3-space with a spatial 2-dimensional boundary surface \( \partial S = \{ r = R, \theta, \phi \} \) we obtain

\[
\hat{E} = \int_{\partial S} \left( \hat{H}_0 - \left\{ \hat{H}_0 \right\}_{M=0,a=0,L=0} \right) \approx M + \frac{L^2}{R} - \frac{L^2 M}{R^2} + O \left( \frac{1}{R^3} \right),
\]

(29)

which is the energy of Kerr black hole when \( L = 0 \).\(^{22)\}

The necessary components needed to calculate the spatial momentum \( \tilde{H}_\alpha, \ \hat{\alpha} = 1, 2, 3 \) have the following components

\[
\tilde{H}_1 = \frac{2 \cos \phi \sin^2 \theta [3aL \cos \theta + 2L^2 + 2Mr + 4M^2]}{r} d\theta \wedge d\phi + \ldots,
\]

\[
\tilde{H}_2 = \frac{2 \sin \phi \sin^2 \theta [3aL \cos \theta + 2L^2 + 2Mr + 4M^2]}{r} d\theta \wedge d\phi + \ldots
\]

\[
\tilde{H}_3 = \frac{2 \sin \theta (3aL \cos \theta + 2L^2 + Mr \cos \theta - aL + 4M^2 \cos \theta - aL)}{r} d\theta \wedge d\phi + \ldots.
\]

(30)

Using Eq. (30) we finally get the spatial momentum in the form

\[
P_1 = P_2 = P_3 \approx O \left( \frac{1}{R^2} \right).
\]

(31)

6. Fourth Kerr-NUT spacetime

\*We introduce \( \tilde{H}_0 \) to remove the divergence appearers from term like \( r \). It is worth to mention that we cannot use the expression \( \tilde{H}_0 \) because the spacetime we use is not asymptotically flat.

\( ^{22)} \)
The covariant form of the fourth Kerr-NUT tetrad can be written as

\[
(h^\alpha_i)_4 = \begin{pmatrix} -\sqrt{\frac{D_1}{C}} & 0 & 0 & \sqrt{\frac{D_1}{C}}(a \sin^2 \theta - 2L \cos \theta) \\ -a \sin \theta \sin \phi \sqrt{\frac{1}{C}} & -r \sin \theta \cos \phi \sqrt{\frac{1}{D_1}} & C_1 \cos \phi & (r^2 + a^2 + L^2) \sin \theta \sin \phi \sqrt{\frac{1}{C}} \\ a \sin \theta \cos \phi \sqrt{\frac{1}{C}} & -r \sin \theta \sin \phi \sqrt{\frac{1}{D_1}} & C_1 \sin \phi & -(r^2 + a^2 + L^2) \sin \theta \cos \phi \sqrt{\frac{1}{C}} \\ 0 & -\frac{C_1}{\sqrt{D_1}} & -r \sin \theta & 0 \end{pmatrix}.
\]

(32)

Tetrad (32) has the same associated metric of tetrad (1). Tetrad (32) is related to tetrad (1) through the relation \((h^\alpha_i)_5 = (\Lambda_3^\alpha_\gamma)(h^\gamma_i)_1\) where \((\Lambda_3^\alpha_\gamma)\) is the local Lorentz transformation given by

\[
(\Lambda_3^\alpha_\gamma) \overset{\text{def.}}{=} \begin{pmatrix} \sqrt{\frac{D_1}{D}} & -a \sin \theta \sin \phi \sqrt{\frac{1}{D}} & a \sin \theta \cos \phi \sqrt{\frac{1}{D}} & 0 \\ -a \sin \theta \sin \phi \sqrt{\frac{1}{D}} & -\frac{L_5}{\sqrt{CD}} & -\frac{L_6 \cos \phi \sin \phi}{\sqrt{CD}} & -\frac{L_7 \cos \phi \sin \theta}{\sqrt{C}} \\ a \sin \theta \cos \phi \sqrt{\frac{1}{D}} & -\frac{L_6 \sin \phi \cos \phi}{\sqrt{CD}} & -\frac{L_8}{\sqrt{CD}} & -\frac{L_7 \sin \phi \sin \theta}{\sqrt{C}} \\ 0 & -\frac{L_7 \cos \phi \sin \theta}{\sqrt{C}} & -\frac{L_7 \sin \phi \sin \theta}{\sqrt{C}} & \frac{C_1 \cos \theta + r \sin^2 \theta}{\sqrt{C}} \end{pmatrix},
\]

where

\[
L_5 = \sqrt{D}[r \sin^2 \theta - C_1 \cos \theta] \cos^2 \phi - \sqrt{CD_1} \sin^2 \phi, \quad L_6 = -\sqrt{D}[r \sin^2 \theta - C_1 \cos \theta] + \sqrt{CD_1},
\]

\[
L_7 = (C_1 + r \cos \theta), \quad L_8 = \sqrt{D}[r \sin^2 \theta - C_1 \cos \theta] \sin^2 \phi - \sqrt{CD_1} \cos^2 \phi.
\]

(34)

Following the same technics used in the previous sections to calculate energy, we finally get the non-vanishing components of \(\Sigma^{01}\) asymptotically as

\[
\Sigma^{(0)01} \approx \frac{-\sin \theta}{4r \cos^2 \theta} \left(6r^2 \cos^2 \theta - 7a^2 \cos^2 \theta \sin^2 \theta + 4Mr \cos^2 \theta + L^2(1 + 3 \cos^2 \theta) + 12aL \cos^2 \theta \right) + O \left(\frac{1}{r^2}\right),
\]

\[
\Sigma^{(2)01} \approx \frac{[Ma - 3ar] \sin^2 \theta + 4L \cos \theta[r - M]}{4r} + O \left(\frac{1}{r^2}\right).
\]

Using Eq. (35), we finally obtain the energy of tetrad (32) in the form

\[
P^{(0)} = E = -\oint_{S \rightarrow \infty} dS_k \Pi^{(0)k} = -\frac{1}{4\pi} \oint_{S \rightarrow \infty} dS_k \sqrt{-g} h^{(0)}_{\mu} \Sigma^{\mu 0k} \approx \infty!
\]

Using the translational momentum to recalculate the energy of tetrad (32) whose coframe is described by the components:

\[
\vartheta_1^0 = -\sqrt{\frac{D_1}{C}} \left((a \cos^2 \theta + 2L \cos \theta) d\phi + dt\right), \quad \vartheta_1^1 = -\frac{(r \sin \theta dr - \sqrt{D_1} \sqrt{C - r^2 \cos^2 \theta} \theta d\theta)}{\sqrt{D_1}}.
\]
\[ \psi_1^2 = -\frac{\sin \theta}{\sqrt{C}} \left( [r^2 + L^2 + a^2] d\phi - adt \right), \quad \psi_1^3 = -\frac{\left( \sqrt{C} - r^2 \cos^2 \theta dr + r \sin \theta \sqrt{D_1} d\theta \right)}{\sqrt{D_1}}. \] (37)

Taking coframe (37), as well as the Riemannian connection \( \tilde{\Gamma}_\alpha^\beta \) and substitute into the translational momentum, \( \tilde{H}_\alpha \), we finally get

\[ \tilde{H}_0 \approx \frac{-\sin \theta}{8\pi \cos^2 \theta} \left[ 2r^2 \cos^2 \theta + 5a^2 \cos^2 \theta - L^2 \sin^2 \theta - 8aL \cos^3 \theta - 5a^2 \cos^4 \theta \right] d\theta \wedge d\phi + \cdots + O \left( \frac{1}{r^2} \right). \] (38)

Using Eq. (38) to compute the total energy at a fixed time in the 3-space with a spatial 2-dimensional boundary surface \( \partial S = \{ r = R, \theta, \phi \} \) we finally obtain

\[ \tilde{E} = \int_{\partial S} \tilde{H}_0 = \infty! \] (39)
which is identical with Eq. (36).

7. On the choice of the frame

Let us consider the local Lorentz transformation described by the matrix

\[ (\Lambda_4^\alpha_\beta) = \begin{pmatrix} H_1 & H_5 & H_6 & H_4 \\ K_1 \sin \theta \cos \phi & K_5 \sin \theta \cos \phi & K_6 \cos \theta \cos \phi & K_4 \sin \phi \sin \theta \\ L_1 \sin \sin \phi & L_5 \sin \sin \phi & L_6 \cos \theta \sin \phi & L_4 \cos \phi \sin \theta \\ N_1 \cos \theta & N_5 \cos \theta & N_6 \sin \theta & N_7 \cos \theta \end{pmatrix}, \] (40)

where \( H_i, K_i, L_i \) and \( N_i, i = 5, 6 \) and \( N_7 \) are defined as:

\[ H_5 = -\frac{\sin \theta[(Mr + L^2 + aL \cos \theta)r \cos \phi - a \sin \phi \sqrt{CD_1}]}{C \sqrt{D_1}}, \]
\[ H_6 = -\frac{\sin \theta[(Mr + L^2 + aL \cos \theta)r \sin \phi + a \cos \phi \sqrt{CD_1}]}{C \sqrt{D_1}}, \]
\[ K_5 = -\frac{1}{\cos \phi \sin \theta \ C \sqrt{D_1}} \left\{ \left( r_1 \sqrt{D_1} C + r [r^2 + La \cos \theta - rM] \sin^2 \theta - r_1 C_1 \cos \theta \sqrt{D_1} \right) \cos \phi \right. \]
\[ + [rr_1 a \sin^2 \theta + a \sqrt{CD_1} - a \ C_1 \cos \theta \sqrt{D_1}] \cos \phi \sin \phi - r_1 \sqrt{CD_1} \right\}, \]
\[ K_6 = -\frac{1}{\cos \phi \cos \theta \ C \sqrt{D_1}} \left\{ \left( r_1 \sqrt{D_1} C + r [r^2 + La \cos \theta - rM] \sin^2 \theta - r_1 \cos \theta \ C_1 \sqrt{D_1} \right) \cos \phi \sin \phi \right. \]
\[ - [rr_1 a \sin^2 \theta + a \sqrt{CD_1} - a \ C_1 \cos \theta \sqrt{D_1}] \cos^2 \phi - a \ C_1 \cos \theta \sqrt{D_1} + arr_1 \sin^2 \theta \}, \]
\[ L_5 = \frac{-1}{\cos \phi \sin \theta \sqrt{D_1}} \left\{ (r_1 \sqrt{D_1} C + r [r^2 + La \cos \theta - r M] \sin^2 \theta - r_1 C_1 \cos \theta \sqrt{D_1}) \cos \phi \sin \phi \\
- [r r_1 a \sin^2 \theta + a \sqrt{C D_1} - a C_1 \cos \theta \sqrt{D_1}] \cos^2 \phi + a \sqrt{C D_1} \right\}, \]

\[ L_6 = \frac{1}{\cos \phi \cos \theta \sqrt{D_1}} \left\{ (r_1 \sqrt{D_1} C + r [r^2 + La \cos \theta - r M] \sin^2 \theta - r_1 C_1 \cos \theta \sqrt{D_1}) \cos \phi + r_1 C_1 \cos \theta \sqrt{D_1} - r [r^2 + a L \cos \theta - r M] \sin^2 \theta \right\}, \]

\[ N_5 = \frac{-\sin \theta \cos \phi [r \cos \theta (r^2 + a^2 - M r + a L \cos \theta) + r_1 C_1 \sqrt{D_1}]}{C \sqrt{D_1}}, \]

\[ N_6 = \frac{-\sin \phi [r \cos \theta (r^2 + a^2 - M r + a L \cos \theta) + r_1 C_1 \sqrt{D_1}]}{C \sqrt{D_1}}. \]

\[ N_7 = -\frac{[\cos \theta (r^2 + a^2 - M r + a L \cos \theta) C_1 + r r_1 \sin^2 \theta \sqrt{D_1}]}{C \sqrt{D_1}}. \]

From Eqs. (32) and (40) one can get

\[ (h^\alpha_i)_6 = \left( \Lambda_4^\alpha \right) (h^\gamma_i)_5. \]  \hspace{1cm} (42)

Using Eq. (42) to calculate the non-vanishing components needed to the calculations of energy, we finally get

\[ \Sigma^{(0)01} \cong -\frac{\sin \theta}{r} \left( r^2 - M r - L^2 - L a \cos \theta \right) + O \left( \frac{1}{r^2} \right), \]

\[ \Sigma^{(1)01} = \Sigma^{(2)01} = \Sigma^{(3)01} = 0. \]  \hspace{1cm} (43)

Using Eq. (43) one can finally get

\[ P^{(0)} = E = -\int_{S \to \infty} dS_k \Pi^{(0)k} = -\frac{1}{4 \pi} \int_{S \to \infty} dS_k \sqrt{-g} h^{(0)}_{\mu} \left( \Sigma^{\mu 0k} - \Sigma^{\mu 0k}_M = 0, a = 0, L = 0 \right) \]

\[ \cong M + \frac{L^2}{r} - \frac{L^2 M}{r^2} + O \left( \frac{1}{r^3} \right), \]  \hspace{1cm} (44)

which is a satisfactory results.\(^{22}\) Also from Eq. (43) one can get the spatial momentum in the form

\[ P_1 = -\int_{S \to \infty} dS_k \Pi^{(1)k} = -\frac{1}{4 \pi} \int_{S \to \infty} dS_k \sqrt{-g} h^{(1)}_{\mu} \Sigma^{\mu 0k} = 0, \]

by same method \(P_2 = P_3 = 0.\)  \hspace{1cm} (45)

Now turn our attention to the calculation of angular-momentum. The non vanishing components of the angular-momentum are given asymptotically up to order \(O \left( \frac{1}{r} \right)\) by

\[ M^{(0)(1)}(r, \theta, \phi) \cong \frac{(2 r \cos \phi - 2 M \cos \phi \sin^2 \theta + a \sin \phi (1 + \cos^2 \theta)}{8 \pi} + O \left( \frac{1}{r} \right), \]
\[
M^{(0)(2)}(r, \theta, \phi) \approx \frac{(2r \sin \phi - 2M \sin \phi \sin^2 \theta - a \cos \phi(1 + \cos^2 \theta)}{8\pi} + O\left(\frac{1}{r}\right),
\]
\[
M^{(0)(3)}(r, \theta, \phi) \approx \frac{M \sin \theta \cos \theta}{4\pi} + O\left(\frac{1}{r}\right),
\]
\[
M^{(1)(2)}(r, \theta, \phi) \approx -\frac{L \sin \theta \cos \theta}{4\pi} + O\left(\frac{1}{r}\right),
\]
\[
M^{(1)(3)}(r, \theta, \phi) \approx -\frac{L \sin^2 \theta \sin \phi}{8\pi r} + O\left(\frac{1}{r}\right),
\]
\[
M^{(2)(3)}(r, \theta, \phi) \approx -\frac{L \sin^2 \theta \cos \phi}{8\pi r} + O\left(\frac{1}{r}\right).
\]

Using Eq. (46) one get the angular momentum components in the form
\[
L^{(0)(1)} = \int_0^\pi \int_0^{2\pi} \int_0^\infty d\theta d\phi dr \left[M^{(0)(1)}\right] = 0,
\]
which is a consistence results. By the same method we finally obtain
\[
L^{(0)(2)} = L^{(0)(3)} = L^{(1)(2)} = L^{(1)(3)} = L^{(2)(3)} = 0.
\]

We show by explicit calculations that the energy-momentum tensor which is a coordinate independent gives a consistent result of the angular momentum when applied to the tetrad field given by Eq. (12)!

\section{8. Thermal properties of Kerr-NUT Spacetime}

In GR, thermodynamical quantities are calculated using Euclidean continuation of metric.\cite{23} However, in TEGR these quantities are calculated through the divergence term which appears in the Lagrangian\cite{24}. This term has no effect on the field equation.

Hawking and Gibbons\cite{23} discussed the thermal properties of the Schwarzschild solution, for which the line-element takes the positive-definite standard form
\[
ds^2 = + \left(1 - \frac{2M}{r}\right) d\tau^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2,
\]
after Euclidean continuation of the time variable, \(t = -i\tau\). By using the transformation \(x = 4M(1 - 2M/r)^{1/2}\), the line-element squared becomes
\[
ds^2 = + \left(\frac{x}{4M}\right)^2 d\tau^2 + \left(\frac{r^2}{4M^2}\right)^2 dx^2 + r^2 d\Omega^2,
\]
which shows that \(\tau\) can be regarded as an angular variable with period \(8\pi M\). Now the Euclidean section of the Schwarzschild solution is the region defined by \(8\pi M \geq \tau \geq 0\) and
where the metric is positive definite, asymptotically flat, and non-singular. They calculated the Euclidean action, \( \hat{I} \), of GR from the surface term as follows:

\[
\hat{I} = 4\pi M^2 = \frac{\beta^2}{16\pi},
\]

(51)

where \( \beta = 8\pi M = T^{-1} \) with \( T \) being interpreted as the absolute temperature of the Schwarzschild black hole.

For a canonical ensemble the energy is given by

\[
E = \frac{\sum_n E_n e^{-\beta E_n}}{\sum_n e^{-\beta E_n}} = -\frac{\partial}{\partial \beta} \log Z,
\]

(52)

where \( E_n \) is the energy in the \( n \)th state, and \( Z \) is the partition function, which is in the tree approximation related to the Euclidean action of the classical solution by

\[
\hat{I} = -\log Z.
\]

(53)

Using of (51) and (53) in (52) gives

\[
E = \frac{\beta}{8\pi} = M.
\]

(54)

They also calculated the entropy of the Schwarzschild black hole to obtain

\[
S = -\sum_n P_n \log P_n = \beta E + \log Z = 4\pi M^2 = \frac{1}{4} A,
\]

(55)

where \( P_n = Z^{-1} e^{-\beta E_n} \), and \( A \) is the area of the event horizon of the Schwarzschild black hole.

Now let use apply Hawking and Gibbons procedure to Eqs (1) and (13) that reproduce Kerr-NUT spacetime. The Euclidean action is given by

\[
\hat{I} = -\frac{1}{2\kappa} \int \sqrt{g} (R - 2T^\mu;_\mu) d^4x = \frac{1}{\kappa} \int \sqrt{g} T^\mu;_\mu d^4x.
\]

(56)

where \( R \) is the Riemann-Christoffel scalar curvature, which is vanishing for Kerr-NUT spacetime, and \( T^\mu \) is the basic vector field of the torsion.

Eq. (1) after using Euclidean continuation, \( t = -\sqrt{-1}\tau, \quad L = \sqrt{-1}L \) and \( a = \sqrt{-1}a \), takes the following form

\[
(h^{\alpha}_i)_{1E} = \begin{pmatrix}
\mathcal{F}_{1E} & 0 & 0 & \mathcal{F}_{2E} \\
0 & \mathcal{F}_{3E} & 0 & 0 \\
0 & 0 & \mathcal{F}_{4E} & 0 \\
0 & 0 & 0 & -\mathcal{F}_{5E} \sin \theta
\end{pmatrix},
\]

(57)
where $F_{iE}, i = 1 \cdots 5$, and $E$ refers to Euclidean continuation having the form

$$
F_{1E} = \sqrt{\frac{D_E}{C_E}}, \quad F_{2E} = \frac{F_E}{a \sqrt{C_E D_E}}, \quad F_{3E} = \sqrt{\frac{C_E}{D_1E}}, \quad F_{4E} = \sqrt{C_E}, \quad F_{5E} = \sqrt{\frac{C_E D_1E}{D_E}},
$$

where 

$$
C_E = r^2 - (L + a \cos \theta)^2, \quad D_E = r^2 - 2Mr + 2a^2 - a^2 \cos^2 \theta - L^2, \quad D_{1E} = r^2 - 2Mr + a^2 - L^2, \quad D_3E = r^2 - 2Mr + 2a^2 - a^2 \cos^2 \theta - L^2, \quad D_{2E} = r^2 - 2Mr + a^2 - L^2,
$$

$F_E = -aD_{1E} \cos \theta (a \cos \theta + 2L) + a^2 D_{1E} - a^2 L^2 - a^2 \sin^2 \theta (r^2 - a^2)$.

The Euclidean line element squared of the above tetrad takes the form

$$
ds^2 = F_{1E}^2 d\tau^2 + F_{3E}^2 dr^2 + F_{4E}^2 d\theta^2 + (F_{5E}^2 \sin^2 \theta + F_{2E}^2) d\phi^2 + F_{1E} F_{2E} dt d\phi,
$$

which is Euclidean Kerr-NUT. Using Eq. (57) we get the following

$$
h = \sin \theta \sqrt{\frac{[r^2 - a^2 \cos^2 \theta]^2 - 4aLC_E \cos \theta + L^2 (L^2 - 2[r^2 + a^2 \cos^2 \theta])}{[r^2 - a^2 \cos^2 \theta]^2 - 4aLC_E \cos \theta + L^2 (L^2 - 2[r^2 + a^2 \cos^2 \theta])}},
$$

$$
T^1 = \frac{r D_{1E} + (r - M) C_E}{[r^2 - a^2 \cos^2 \theta]^2 - 4aLC_E \cos \theta + L^2 (L^2 - 2[r^2 + a^2 \cos^2 \theta])},
$$

where $h = \sqrt{g}$ is the determinant. There is another non-vanishing components of $T^\mu$, i.e., $T^2$ but this is not necessary in our computations. The volume integral (56) of tetrad (57) is calculated to give

$$
\hat{I} = \frac{1}{\kappa} \int \sqrt{g} (T^\mu_{\nu \mu}) d^4 x \approx \pi (r_+^2 + 3L^2),
$$

where $r_+$ is the outer event horizon located at the largest positive root of $g^{rr} = 0$, i.e.,

$$(r_+^2 - 2Mr - a^2 + L^2) = 0 \Rightarrow r_+ = M + \sqrt{M^2 + a^2 - L^2}.\quad (61)$$

Using (61) in (52) we get the energy of tetrad (57) using the Euclidean continuation method to have the form

$$
E \approx M + \frac{L^2}{r_+}, \quad which \ is \ acceptable \ result \ [11].
$$

Using Eqs. (61) and (62) in (55) we get the entropy of tetrad (57) to have the form

$$
S = \pi (r_+^2 + 3L^2).
$$

The Hawking temperature of tetrad (57) has the form

$$
T \approx \frac{1}{4\pi r_+}.
$$

Using Eqs. (62), (63) and (64) we get

$$
dM = T dS.
$$

Eq. (65) shows that the first law of thermodynamic is satisfied. From Eqs. (61), (62), (63) and (64) we find the following relation

$$
E - TS \approx T \hat{I},
$$

where $\hat{I}$ is the Euclidean integral of the tetrad.
is satisfied. Eq. (66) shows that quantum statistical relation holds.

Now turn us to Eq. (13) which after using Euclidean continuation takes the following form

\[
(h^\alpha)_{3E} = \begin{pmatrix}
-\sqrt{\frac{D_{1E}}{C_E}} & 0 & 0 & \frac{C_{2E}}{a D_E} \sqrt{\frac{D_{1E}}{C_E}} \\
0 & -r \sin \theta & \sqrt{D_{1E}} C_{1E} & 0 \\
-\frac{a \sin \theta}{\sqrt{C_E}} & 0 & 0 & -\sin \theta (r^2 - a^2) \\
0 & -\frac{C_{1E}}{\sqrt{D_{1E}}} & -r \sin \theta & 0
\end{pmatrix},
\]

where \( C_{1E} \) and \( C_{2E} \) are defined as

\[
C_{1E} = \sqrt{C_E - r^2 \sin^2 \theta}, \quad C_{2E} = -a D_E \cos \theta (a \cos \theta + 2L) + (a^2 - L^2) D_{1E} - a^2 \sin^2 \theta (L^2 - a^2).
\]

Using Eq. (67), the asymptotic form of the basic vector up to \( \frac{1}{r^4} \) has the form

\[
T^1 \approx \frac{(6r^2 - 8Mr + 2a^2 - M^2 + 5L^2) \cos^2 \theta + L^2 + 18aL \cos^3 \theta + 10a^2 \cos^4 \theta}{r^3 \cos^2 \theta},
\]

The volume integral (56) of tetrad (67) is calculated to give

\[\hat{I} = \infty.\]

Therefore, we need to use local Lorentz transformation to multiply it by Eq. (67) to get finite physics result. This local Lorentz transformation has the form (76) below. Multiply Eq. (67) by Eq. (76) we get

\[
= (h^\alpha)_{3E\text{Regularized}} \begin{pmatrix}
-\sqrt{\frac{D_E}{C_E}} & 0 & 0 & \frac{1}{D_E C_E} \sqrt{\frac{C_{3E}}{D_E}} \\
0 & \frac{\cos \phi \sin \theta \sqrt{C_E}}{\sqrt{D_{1E}}} & \frac{\cos \phi \cos \theta \sqrt{C_E}}{\sqrt{D_{1E}}} & \sin \phi \sin \theta \frac{\sqrt{C_{3E}}}{\sqrt{D_E}} \\
0 & \frac{\sin \phi \sin \theta \sqrt{C_E}}{\sqrt{D_{1E}}} & \frac{\sin \phi \cos \theta \sqrt{C_E}}{\sqrt{D_{1E}}} & \cos \phi \sin \theta \frac{\sqrt{C_{3E}}}{\sqrt{D_E}} \\
0 & \frac{\cos \theta \sqrt{C_E}}{\sqrt{D_{1E}}} & -\sin \theta \sqrt{C_E} & 0
\end{pmatrix}
\]

where \( C_{3E} \) takes the form

\[
C_{3E} = -a D_{1E} \cos \theta (a \cos \theta + 2L) + (r^2 - L^2) D_{1E}.
\]
Using Eq. (71), the asymptotic form of the basic vector up to \( \left( \frac{1}{r} \right) \) has the form

\[
T^1 \cong \frac{\sin \theta (4r^2 - 6Mr + 3a^2 + a^2 \cos^2 \theta + 4aL \cos \theta)}{r},
\]

(73)

The volume integral (56) of tetrad (71) is calculated to give

\[
\hat{I} = \frac{1}{\kappa} \int \sqrt{g} (T^\mu_{\mu}) d^4x \cong \pi(r_+^2 + 3L^2),
\]

(74)

Using (74) in (52) we get the energy of tetrad (71) using the Euclidean continuation method to has the form (62). Also the entropy, temperature, the first law of thermodynamics and quantum statistical relation have the form of Eqs. (63), (64), (65) and (66).

9. Discussion and conclusion

The main results of this study are as follows:

- We have calculated the energy content of many tetrad fields, who have Kerr-NUT spacetime metric, using the gravitational energy-momentum tensor and the Riemannian connection 1-form. The first tetrad gave common known form of energy and spatial momentum, however, the components of the angular momentum are not in agreement with the known result.\(^{28}\) May we claim that the definition of the angular momentum\(^ {29}\) is not working properly.

- The second tetrad show that is linked to the first tetrad through a local Lorentz transformation show a consistence results of energy and spatial momentum but suffers from the same defect of the first one.

- The third tetrad is shown to be inconvenient one because the energy associated with it is divergent. Therefore, a local Lorentz transformation is multiplied by the third tetrad, the new tetrad still remains a solution to the field equation of TEGR. Calculations of energy and spatial are rerun and satisfactory results are obtained. Yet, the calculation of angular momentum still yields abnormal result.

- It became an important issue in the TEGR to accompany with inconvenient tetrad, i.e., tetrad that has not gave physical acceptable results of energy and spatial momentum, local Lorentz transformations satisfy \((\Lambda^\alpha_{\gamma}) (\Lambda^\gamma_{\beta}) = \delta^\alpha_{\beta}\). These transformations played key role in eliminating inertia, which participate the physical quantities, from the inconvenient tetrad and bring its relevant physics in an acceptable form.

- In this study several local Lorentz transformations are given. These transformations can be used as a regularizing tool to tetrads (13) and (32).

- We calculate the energy of the first tetrad using the Euclidean continuation method. We show that the result of energy is acceptable and consistent with the present result and with the previous result.\(^ {11}\) The thermodynamic quantities associated with this tetrad are calculated and the result are in consistent with the previous.\(^ {27}\) We also show that the first law
of thermodynamic is satisfied and the quantum statistical relation holds.

- Repeated the same calculations done for tetrad (57) to tetrad field (67) we obtain a divergent results. This anomalous results is consistence with the previous two methods, the gravitational energy-momentum tensor and the Riemannian connection 1-form. This ensures that such kind of tetrad field must be multiply with some appropriate local Lorentz transformation to get a finite result.

- Using one of the local Lorentz transformation given in the Appendix, i.e., Eq. (76) and multiply it to tetrad (67) and repeat the same calculation done for tetrad (57) by using the Euclidean continuation method we get the well known results and same thermodynamic quantities that are consistent with what obtained before.27

- Still calculations of angular momentum using the definition contained in have critical problems at least as the current study indicated and also as shown in.11) It is unclear how to tackle such a problem. We may claim that one way to solve such a problem is to calculate the conserved charges, using their definition within the TEGR, related to the spacetime used in this study.

10. Appendix

There are many local Lorentz transformations that can be employed to do the same job done by Eqs. (20) and (40). Among these ones are the following:

First local Lorentz transformation

\[
\left( \Lambda_5^{\alpha \gamma} \right)_{\text{def.}} = \begin{pmatrix}
\frac{\sqrt{D_1}}{D} & 0 & a \sin \theta \\
-a \sin \theta \sin \phi & \cos \phi (C_1 \cos \theta - r \sin^2 \theta) & -\sqrt{\frac{D_1}{D}} \sin \phi \\
-a \sin \theta \cos \phi & \sin \phi (C_1 \cos \theta - r \sin^2 \theta) & -\sqrt{\frac{D_1}{D}} \cos \phi \\
0 & -\sin \theta (C_1 + r \cos \theta) & 0 \\
\end{pmatrix}
\]

Second local Lorentz transformation

\[
\left( \Lambda_6^{\alpha \gamma} \right)_{\text{def.}} = \begin{pmatrix}
\frac{\sqrt{D_1}}{D} & -a \sin \theta \sin \phi & a \sin \theta \cos \phi & 0 \\
0 & r \sin \theta \cos \phi & r \sin \theta \sin \phi & \frac{C_1}{\sqrt{C}} \\
0 & \frac{C_1}{\sqrt{C}} \cos \phi & \frac{C_1}{\sqrt{C}} \sin \phi & -r \sin \theta \\
a \sin \theta \sqrt{\frac{D_1}{D}} & -\sqrt{\frac{D_1}{D}} \sin \phi & \sqrt{\frac{D_1}{D}} \cos \phi & 0 \\
\end{pmatrix}
\]
Besides the above two local Lorentz transformation also Eqs. (14) and (33) serve as regularization.

References

[1] A. Taub, *Ann. Math.* **53** (1951), 472; E.T. Newman, L. Tamburino, and T. Unti, *J. Math. Phys.* **4** (1963), 915.

[2] S. W. Hawking, *Phys. Lett.* A **60** (1977), 81.

[3] N. S. Manton, *Phys. Lett.* B **110**, (1985) 54.

[4] M. F. Atiyah and N. Hitchin, *Phys. Lett.* A **107** (1985), 21.

[5] S. Hewson, *Class. Quantum Grav.* **13** (1996), 1739.

[6] T. C. Kraan and P. van Baal, *Phys. Lett.* B **428** (1998), 268.

[7] B. Carter, *Comm. Math. Phys.* **10** (1968), 280.

[8] D. Bini, C. Cherubini, M. de Mattia and R.T. Jantzen, *Gen. Rel. Grav.* **35** (2003), 2249; D. Bini, C. Cherubini, R.T. Jantzen and B. Mashhoon, *Class. Quantum Grav.* **20** (2003), 457; D. Bini, C. Cherubini, R.T. Jantzen and B. Mashhoon, *Phys. Rev.* D **67** (2003), 084013.

[9] Z. W. Chong, G.W. Gibbons, H. Lü and C.N. Pope, *Phys. Lett.* B **609** (2005), 124.

[10] S. T. Hong and S.W. Kim, *J.Korean Phys.Soc.* **49** (2006), S748.

[11] G. G. L. Nashed, *Prog. Theor. Phys.* **27** No. 3 (2012), 561.

[12] S. W. Hawking, *Nature* **248** (1989), 30; S.W. Hawking, *Commun. Math. Phys.* **43** (1975); 199; Erratum ibid. **46** (1976), 206.

[13] M.K. Parikh and F. Wilczek, *Phys. Rev. Lett.* **85** (2000), 5042; M.K. Parikh, *Gen. Rel. Grav.* **36** (2004), 2419.

[14] R. Banerjee and B.R. Majhi *JHEP* **06** (2008), 095.

[15] Q. Q. Jiang, S. Q. Wu and X. Cai,*Phys. Rev.* D **73** (2006), 064003.

[16] K. Hayashi, *Phys. Lett.* **69B**, 441 (1977); K. Hayashi and T. Shirafuji. *Phys. Rev.* **D19**, 3524 (1979); *Phys. Rev.* **D24**, 3312 (1981); M. Blagojević and M. Vasil’č *Class. Quant. Grav.* **5** (1988), 1241; T. Kawai, *Phys. Rev.* D **62** (2000), 104014, T. Kawai, K. Shibata and I. Tanaka, *Prog. Theor. Phys.* **104** (2000) 505.

[17] F.W. Hehl, J.D. MacCrea, E.W. Mielke, Y. Ne’eman, *Phys. Rep.* **258** (1995), 1.

[18] P. Baekler, F.W. Hehl, *Int. J. Mod. Phys.* D **15** (2006), 635.

[19] R.S. Tung, J.M. Nester, *Phys. Rev.* D **60** (1999), 021501.

[20] J.W. Maluf, F.F. Faria, K.H. Castello-Branco, *Class. Quantum Grav.* **20** (2003), 4683.
[21] J.W. Maluf, M.V.O. Veiga, J.F. da Rocha-Neto, *Gen. Relat. Grav.* **39** (2007), 227.

[22] M. Ahmed and S. M. Hossain 1995 *Prog. Theor. Phys.* **93** (1995) 901.

[23] G. W. Gibbons and S. W. Hawking *Phys. Rev.* **D15** (1977), 2752.

[24] T. Shirafuji, G.G.L. Nashed and K. Hayashi *Prog. Theor. Phys.* **95** (1996), 665.

[25] S. W. Hawking *Phys. Rev.* **D14** (1976), 2460.

[26] S. W. Hawking in *"General Relativity" 1979*, S. W. Hawking and W. Israel, eds. (Cambridge Univ. Press, Cambridge).

[27] X. Ge, Y. Shen *Class. Quant. Grav.* **20** (2003), 3593

[28] Y. N. Obukhov, and G. G. Rubilar, *Phys. Rev* **D73** (2006), 124017 .

[29] J. W. Maluf, J. F. da Rocha-neto, T. M. L. Toribio and K. H. Castello-Branco, *Phys. Rev.* **D65** (2002), 124001.