Abstract. We consider a one-frequency, quasi-periodic, block Jacobi operator, whose blocks are generic matrix-valued analytic functions. We establish Anderson localization for this type of operator under the assumption that the coupling constant is large enough but independent of the frequency. This generalizes a result of J. Bourgain and S. Jitomirskaya on localization for band lattice, quasi-periodic Schrödinger operators.

1. Introduction and statement

An integer lattice quasi-periodic Schrödinger operator is an operator $H_\lambda(x)$ on $l^2(\mathbb{Z}) \ni \psi = \{\psi_n\}_{n \in \mathbb{Z}}$, defined by

$$[H_\lambda(x) \psi]_n := -(\psi_{n+1} + \psi_{n-1} - 2\psi_n) + \lambda f(x + n\omega) \psi_n,$$

where $\lambda \neq 0$ is a coupling constant, $x \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$ is a phase parameter that introduces some randomness into the system, $f: \mathbb{T} \to \mathbb{R}$ is a (bounded) potential function, and $\omega \in \mathbb{T}$ is a fixed irrational frequency.

Note that $H_\lambda(x)$ is a bounded, self-adjoint operator. Moreover, due to the ergodicity of the system, the spectral properties of the family of operators $\{H_\lambda(x): x \in \mathbb{T}\}$ are independent of $x$ almost surely.

In this paper we study a more general Schrödinger-like operator on a band integer lattice (which in some sense may be regarded as an approximation of a higher dimensional lattice). Before we define such operators, let us introduce some notations and terminology.

All throughout, if $m \in \mathbb{N}$ and if $M: \mathbb{T} \to \text{Mat}_m(\mathbb{R})$ is any matrix-valued function, we denote by $M^\top(x)$ the transpose of $M(x)$.

We say that $M(x)$ has no constant eigenvalues if for any $w \in \mathbb{C}$, we have $\det[M(x) - w I] \neq 0$ as a function of $x$.

Furthermore, given a frequency $\omega \in \mathbb{T}$, for all $n \in \mathbb{Z}$ we denote by $M_n(x)$ the quasi-periodic matrix-valued function

$$M_n(x) := M(x + n\omega).$$

All such matrix-valued functions $M$ will be assumed real analytic (meaning that their entries are real analytic functions).
Let us then denote by $C^\omega_{r}(T, \text{Mat}_m(\mathbb{R}))$ the space of all analytic functions $M: T \to \text{Mat}_m(\mathbb{R})$ having a holomorphic, continuous up to the boundary extension to $A_r := \{ z \in \mathbb{C} : 1 - r < |z| < 1 + r \}$, the annulus of width $2r$ around the torus $T$. We endow this space with the uniform norm $\|M\|_r := \sup_{z \in A_r} \|M(z)\|$.

Let $l \in \mathbb{N}$ be the width of the band lattice, fix an irrational frequency $\omega \in T$ and let $W, R, F \in C^\omega_{r}(T, \text{Mat}_l(\mathbb{R}))$. Assume that for all phases $x \in T$, $R(x)$ and $F(x) \in \text{Sym}_l(\mathbb{R})$, i.e they are symmetric matrices.

A quasi-periodic block Jacobi operator is an operator $H = H_\lambda(x)$ acting on $l^2(\mathbb{Z} \times \{1, \ldots, l\}, \mathbb{R}) \cong l^2(\mathbb{Z}, \mathbb{R}^l)$ by

$[H_\lambda(x) \vec{\psi}]_n := -(W_{n+1}(x) \vec{\psi}_{n+1} + W_n^\top(x) \vec{\psi}_{n-1} + R_n(x) \vec{\psi}_n + \lambda F_n(x) \vec{\psi}_n)$,

where $\vec{\psi} = \{\vec{\psi}_n\}_{n \in \mathbb{Z}} \in l^2(\mathbb{Z}, \mathbb{R}^l)$ is any state, and as before, $x \in T$ is a phase and $\lambda \neq 0$ is a coupling constant.

This model contains all quasi-periodic, finite range hopping Schrödinger operators on integer or band integer lattices. The hopping term is given by the “weighted” Laplacian:

$[\Delta_W(x) \vec{\psi}]_n := W_{n+1}(x) \vec{\psi}_{n+1} + W_n^\top(x) \vec{\psi}_{n-1} + R_n(x) \vec{\psi}_n,$

where the hopping amplitude is encoded by the quasi-periodic matrix-valued functions $W_n(x)$ and $R_n(x)$.

The potential is the quasi-periodic matrix-valued function $\lambda F_n(x)$.

Denote $V(x) := \lambda F(x) + R(x)$. Since $H = H_\lambda(x)$ acts on $l^2(\mathbb{Z}, \mathbb{R}^l)$, this operator can be represented in matrix form as an infinite, tri-diagonal block matrix, whose building blocks are matrices in $\text{Mat}(l, \mathbb{R})$:

$H = \begin{bmatrix}
\cdot & \cdot & \cdot & 0 & \cdot \\
-\lambda & V_n & W_{n+1} & 0 & \\
0 & -W_{n+1}^\top & V_{n+1} & -W_{n+2} & \\
\cdot & 0 & \cdot & \cdot & \cdot \\
\end{bmatrix}.$

\footnote{We warn the reader that we identify the torus $T = \mathbb{R}/\mathbb{Z}$ (an additive group) with the unit circle $S^1 \subset \mathbb{C}$ (a multiplicative group) via the map $x + \mathbb{Z} \mapsto e(x) := e^{2\pi i x}$, but we maintain the additive notation, e.g. we write $x + \omega$ instead of $e(x)e(\omega)$.}
**Definition 1.** We say that an operator satisfies Anderson localization if it has pure point spectrum with exponentially decaying eigenfunctions.

We may now formulate the main result of this paper.

**Theorem 1.** Consider the quasi-periodic block Jacobi operator $H_\lambda(x)$ defined in (1.1), and assume that

\[
\text{det}[W(x)] \neq 0, \quad (1.2a) \\
F(x) \text{ has no constant eigenvalues.} \quad (1.2b)
\]

There is $\lambda_0 = \lambda_0(W,R,F,l) < \infty$ so that if $|\lambda| > \lambda_0$ and $x \in \mathbb{T}$, then for almost every frequency $\omega \in \mathbb{T}$, the operator $H_\lambda(x)$ satisfies Anderson localization.

**Remark 1.** This theorem generalizes a result of J. Bourgain and S. Jitomirskaya [5]. The Anderson localization result proven in [5] corresponds to the special case $W(x) \equiv I_l \in \text{Mat}_l(\mathbb{R})$, $R(x) \equiv R := \begin{bmatrix} \ldots & \ldots & 0 & \ldots \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \ldots & 0 & \ldots & \ldots \end{bmatrix} \in \text{Mat}_l(\mathbb{R})$ and $F(x) \equiv \text{diag } [f_1(x), \ldots, f_l(x)]$, where the diagonal entries of $F$ are non-constant functions $f_j \in C^\omega(\mathbb{T}, \mathbb{R})$.

**Remark 2.** P. Duarte and S. Klein have shown (see Theorem 3.1 in [7]) that the sets of matrix valued analytic functions $W \in C^\omega(\mathbb{T}, \text{Mat}_m(\mathbb{R}))$ and respectively $F \in C^\omega(\mathbb{T}, \text{Sym}_l(\mathbb{R}))$ satisfying the assumptions (1.2) of the theorem are generic in a strong sense, namely they are open, dense and prevalent in their respective spaces.

Furthermore, under these same assumptions on the data, and for every frequency $\omega$, it was shown (see Theorem 2.3 in [7]) that all non-negative Lyapunov exponents of the operator $H_\lambda(x)$ have a lower bound of the type $\log |\lambda| - \mathcal{O}(1)$.

Recently, in the same setting and under the same assumptions, P. Duarte and S. Klein have shown (see Corollary 6.1 in [9]) that if the frequency $\omega$ is Diophantine, then the integrated density of states of the operator $H_\lambda(x)$ is weak-H"older continuous.

Given a finite set $N \subset \mathbb{Z}$, denote by $H_N$ the finite-volume restriction of $H$ to $l^2(N, \mathbb{R}^l) \simeq l^2(N \times \{1, \ldots, l\}, \mathbb{R})$. When $N = \{1, \ldots, N\}$ for
some $N \geq 1$, we use the shorthand notation $H_N$ instead. Thus

$$H_N = \begin{bmatrix} V_1 & -W_2 & 0 \\ -W_2^\top & V_2 & \ddots & \ddots \\ \vdots & \vdots & \ddots & -W_N \\ 0 & -W_N^\top & V_N \end{bmatrix}$$

is an $N \times N$ block matrix-valued function. Each block is an $l \times l$ analytic matrix. Hence $H_N(x)$ may also be regarded as an $Nl \times Nl$ matrix.

Given an energy $E \in \mathbb{R}$, the corresponding Dirichlet determinant $\det [H_N(x) - EI_{Nl}]$ and the minors of the matrix $H_N(x) - EI_{Nl}$ will play a crucial role in our analysis.

We derive upper bounds on the minors, uniformly in the phase (see Section 2), and lower bounds for the Dirichlet determinant, on average (see Section 3). From these bounds we derive, using Cramer’s rule, estimates on the entries of the Green’s function $G_N(x; E) := (H_N(x) - EI_{Nl})^{-1}$, which in turn lead to the proof of localization (see Section 4).

The main challenge of this paper, and the place where our argument differs most from the one in [5], is deriving the average lower bounds. To that end, we use the method introduced in P. Duarte and S. Klein [7] for obtaining lower bounds on Lyapunov exponents of quasi-periodic cocycles.

We note that this method depends upon Hardy’s convexity theorem for subharmonic functions, and thus it is a one variable argument. Therefore, the several variables model (where the potential and weight functions are defined on $\mathbb{T}^d$ with $d > 1$, and $\omega \in \mathbb{T}^d$ is a translation vector) requires a different set of tools, and it will be considered in a separate project.

2. AN UPPER BOUND ON MINORS OF DIRICHLET MATRICES

To ease the presentation, given a block matrix $g$, we use roman letters for the indices of its block-matrix entries, and greek letters for the indices of its scalar entries. More precisely, we write

$$g = (g_{\gamma,\gamma'})_{1 \leq \gamma,\gamma' \leq Nl} \in \text{Mat}_{Nl}(\mathbb{C}),$$

which is identified with the block matrix

$$g = (g_{n,n'})_{1 \leq n,n' \leq N} \in \text{Mat}_N(\mathbb{R}), \text{ where } R = \text{Mat}_l(\mathbb{C}).$$
Moreover, given $\gamma \in \{1,\ldots, N\}$, let $n(\gamma) \in \{1,\ldots, n\}$ such that
\[ \gamma = l \cdot n(\gamma) + r \quad \text{with} \quad -l \leq r < 0. \]
Hence given a pair of indices $(\gamma, \gamma')$ with $1 \leq \gamma, \gamma' \leq N$, the scalar entry $g_{\gamma,\gamma'}$ belongs to the $l \times l$-block $g_{n(\gamma), n(\gamma')}$. 

2.1. Combinatorial preliminaries. Some of the terminology and the intuition in what follows are related to the lemma of Gessel-Viennot-Lindström (see Chapter 31 in [2] or Chapter 5 in [1]), although we do not directly use this beautiful result relating paths in a graph and determinants.

Given a matrix $g = (g_{\gamma,\gamma'})_{1 \leq \gamma, \gamma' \leq m} \in \text{Mat}_m(\mathbb{C})$, it is useful to represent it as a directed bipartite graph as follows. The vertices of the graph are precisely the rows $R_1, \ldots, R_m$ and the columns $C_1, \ldots, C_m$ of the matrix, while for every pair $(\gamma, \gamma')$ of indices $\gamma, \gamma' \in \{1, \ldots, m\}$, there is an edge (or path) $R_{\gamma} \rightarrow C_{\gamma'}$ with weight (or cost) $g_{\gamma,\gamma'}$.

A path system is any collection $\Gamma$ of paths
\[ R_{\gamma_1} \rightarrow C_{\gamma'_1}, \ldots, R_{\gamma_k} \rightarrow C_{\gamma'_k}. \]
We will call cost (or weight) of a path system $\Gamma$ the product of the corresponding weights:
\[ c(\Gamma) := g_{\gamma_1,\gamma'_1} \cdots g_{\gamma_k,\gamma'_k}. \]
A subset $M \subset \{1, \ldots, m\}$ and a permutation $\sigma \in \sigma_M$ determine the following path system
\[ \Gamma_\sigma := \{ R_{\gamma} \rightarrow C_{\sigma(\gamma)} : \gamma \in M \}. \]
Note that $\Gamma_\sigma$ is a vertex-disjoint path system with no isolated vertices from $\{R_{\gamma} : \gamma \in M\}$ to $\{C_{\gamma} : \gamma \in M\}$. Moreover, its cost is
\[ c(\Gamma_\sigma) = \prod_{\gamma \in M} g_{\gamma,\sigma(\gamma)}. \]
Any subset $M \subset \{1, \ldots, m\}$ is considered ordered with the natural order. We denote by $\neg M$ its complement in $\{1, \ldots, m\}$, again, ordered with the natural order. If $M = \{\alpha\}$ is a singleton, we denote $\neg\{\alpha\}$ by $\neg\alpha$.

If $M, M' \subset \{1, \ldots, m\}$ are two non-empty sets of indices, and if $g \in \text{Mat}_m(\mathbb{C})$, we denote by
\[ g_{M, M'} = (g_{\gamma,\gamma'})_{\gamma \in M, \gamma' \in M'} \]
the corresponding reduced matrix.
In particular, if $\alpha, \alpha' \in \{1, \ldots, m\}$, then $g_{-\alpha,-\alpha'}$ represents the matrix obtained from $g$ by removing row $\alpha$ and column $\alpha'$, so $\det [g_{-\alpha,-\alpha'}]$ is the $(\alpha, \alpha')$-minor of $g$.

Let $\alpha, \alpha' \in \{1, \ldots, m\}$ with $\alpha \neq \alpha'$. Any tuple $(\gamma_1, \ldots, \gamma_s)$ with $\gamma_1 = \alpha'$, $\gamma_s = \alpha$ and $\gamma_i \in \{1, \ldots, m\}$ all distinct indices determines the path system

$$\Gamma = \{(R_{\alpha}^\prime =) R_{\gamma_1} \to C_{\gamma_2}, R_{\gamma_2} \to C_{\gamma_3}, \ldots, R_{\gamma_{s-1}} \to C_{\gamma_s} (= C_{\alpha})\}.$$ 

We refer to any such path system simply as “a path from $R_{\alpha}^\prime$ to $C_{\alpha}$” and identify it with the tuple that determines it, hence $\Gamma = (\gamma_1, \ldots, \gamma_s)$. Moreover, we denote by $P_{\alpha' \to \alpha}$ the set of all such paths from $R_{\alpha'}$ to $C_{\alpha}$.

Given $\Gamma = (\gamma_1, \ldots, \gamma_s) \in P_{\alpha' \to \alpha}$, denote by $|\Gamma| = s$ its length and note that its cost is

$$c(\Gamma) = g_{\gamma_1, \gamma_2} g_{\gamma_2, \gamma_3} \cdots g_{\gamma_{s-1}, \gamma_s}.$$

By abuse of notation, $-\Gamma$ will denote the complement of the set $\{\gamma_1, \ldots, \gamma_s\}$ in $\{1, \ldots, m\}$.

The following lemma shows how to express a minor of a matrix in terms of the costs of such paths. It was proven by J. Bourgain and S. Jitomirskaya (see Lemma 10 in [6]). We reformulate it here using the terminology introduced above and then present a more detailed proof.

**Lemma 1.** Let $g \in \text{Mat}_m(\mathbb{C})$ and let $\alpha, \alpha' \in \{1, \ldots, m\}$ with $\alpha \neq \alpha'$. Then

$$\det [g_{-\alpha,-\alpha'}] = (-1)^{\alpha + \alpha'} \sum_{\Gamma \in P_{\alpha' \to \alpha}} (-1)^{|\Gamma|+1} c(\Gamma) \det [g_{-\Gamma,-\Gamma}].$$

In other words, the $(\alpha, \alpha')$-minor of $g$ can be expressed as a signed sum over paths from $R_{\alpha'}$ to $C_{\alpha}$ of the product between the cost of the path and the determinant of the reduced matrix obtained by removing the rows and columns corresponding to vertices in the path.

**Proof.** Let $\hat{g} \in \text{Mat}_m(\mathbb{C})$ be the matrix obtained from $g$ by replacing its $(\alpha, \alpha')$-entry with 1, and all other entries on row $\alpha$ or column $\alpha'$ with 0 (while keeping all other entries the same).

Since $g_{-\alpha,-\alpha'} \in \text{Mat}_{m-1}(\mathbb{C})$ is the matrix obtained from $g$ by removing row $\alpha$ and column $\alpha'$, we have

$$\det [g_{-\alpha,-\alpha'}] = (-1)^{\alpha + \alpha'} \det [\hat{g}].$$

Using the permutations description of the determinant we have:

$$\det [\hat{g}] = \sum_{\sigma \in \sigma_m} \text{sign}(\sigma) \prod_{\gamma=1}^m \hat{g}_{\gamma, \sigma(\gamma)}.$$
In the language of graphs introduced earlier and applied to the matrix \( \hat{g} \) and to its corresponding directed bipartite graph, we have

\[
\det [\hat{g}] = \sum_{\sigma \in \sigma_m} \sign(\sigma) \, \epsilon(\Gamma_\sigma),
\]

or in other words, the determinant of \( \hat{g} \) is the signed sum over all vertex-disjoint path systems (from the rows \( R_1, \ldots, R_m \) to the columns \( C_1, \ldots, C_m \) of \( \hat{g} \)) of the costs of these path systems.

It turns out that if \( \sigma \in \sigma_m \) with \( \sigma(\alpha) \neq \alpha' \), then the product

\[
\epsilon(\Gamma) = \prod_{\gamma=1}^{m} \hat{g}_{\gamma, \sigma(\gamma)} = 0,
\]

since it contains the term \( \hat{g}_{\alpha, \sigma(\alpha)} \), which is zero (precisely because \( \sigma(\alpha) \neq \alpha' \)).

Therefore, the cost of paths \( \Gamma_\alpha \) corresponding to such permutations is zero, hence

\[
\det [\hat{g}] = \sum_{\sigma \in \sigma_m \atop \sigma(\alpha) = \alpha'} \sign(\sigma) \, \epsilon(\Gamma_\sigma). \quad (2.1)
\]

Every permutation can be written in a unique way as a product of cycles. Since \( \sigma(\alpha) = \alpha' \), there is a unique cycle \( \tau \) in this representation of \( \sigma \) that contains \( \alpha \) and \( \alpha' \).

We may write this cycle as \( \tau = (\gamma_1, \ldots, \gamma_s) \) with \( \gamma_1 = \alpha' \), \( \gamma_s = \alpha \) and \( \gamma_i \) all distinct. We then have the representation

\[
\sigma = \tau \cdot \tau^\perp,
\]

where \( \tau^\perp \) is a permutation of the set \( \{1, \ldots, m\} \setminus \{\gamma_1, \ldots, \gamma_s\} \) (unless this set is empty, in which case \( \tau^\perp \) is the identity permutation).

Note that \( \sign(\sigma) = \sign(\tau) \cdot \sign(\tau^\perp) \) and \( \sign(\tau) = (-1)^{s+1} \).

Also note that the cycle \( \tau \) determines the path (the reader should not mind the abuse of notation) \( \Gamma = (\gamma_1, \ldots, \gamma_s) \in \mathcal{P}_{\alpha' \rightarrow \alpha} \), and vice-versa.

We can now write

\[
\epsilon(\Gamma_\sigma) = \prod_{\gamma=1}^{m} \hat{g}_{\gamma, \sigma(\gamma)}
\]

\[
= \hat{g}_{\gamma_1, \sigma(\gamma_1)} \cdot \cdots \cdot \hat{g}_{\gamma_{s-1}, \sigma(\gamma_{s-1})} \cdot \hat{g}_{\gamma_s, \sigma(\gamma_s)} \cdot \prod_{\gamma \in -\Gamma} \hat{g}_{\gamma, \sigma(\gamma)}
\]

\[
= g_{\gamma_1, \gamma_2} \cdot g_{\gamma_2, \gamma_3} \cdot \cdots \cdot g_{\gamma_{s-1}, \gamma_s} \cdot \hat{g}_{\alpha, \alpha'} \cdot \prod_{\gamma \in -\Gamma} g_{\gamma, \tau^\perp(\gamma)}
\]

\[
= \epsilon(\Gamma) \cdot 1 \cdot \prod_{\gamma \in -\Gamma} g_{\gamma, \tau^\perp(\gamma)} = \epsilon(\Gamma) \cdot \prod_{\gamma \in -\Gamma} g_{\gamma, \tau^\perp(\gamma)}.
\]
We rearrange the sum in (2.1) as
\[
\det [\hat{g}] = \sum_{\Gamma \in \mathcal{P}} (\alpha' \rightarrow \alpha) (\alpha \rightarrow \alpha') \left( \prod_{\gamma \in \Gamma} g_{\gamma,\sigma_\gamma} \right)
\]
\[= \sum_{\Gamma \in \mathcal{P}} (\alpha' \rightarrow \alpha) (\alpha \rightarrow \alpha') \det [g_{\Gamma - \Gamma}]. \]

\[\square\]

### 2.2. Statement and derivation of the upper bound.

We fix any coupling constant \(\lambda\) and restrict the set of energies \(E\) to a compact interval, say \(|E| \lesssim |\lambda|\). If \(N \in \mathbb{N}\) and \(1 \leq \alpha, \alpha' \leq Nl\), we denote by \(\mu_{N,(\alpha,\alpha')}(x;E)\) the \((\alpha,\alpha')\)-minor of the Dirichlet matrix \(H_N(x) - E I_{Nl}\). That is,
\[
\mu_{N,(\alpha,\alpha')}(x;E) := \det [(H_N(x) - E I_{Nl})_{\alpha,\alpha'}].
\]

**Proposition 2.** There is \(C = C(W,F,R,l) < \infty\) such that for all \(N \in \mathbb{N}\), for all indices \(1 \leq \alpha, \alpha' \leq Nl\) and for all \(x \in \mathbb{T}\) we have the following uniform upper bound on the corresponding minor:
\[
\frac{1}{Nl} \log |\mu_{N,(\alpha,\alpha')}(x;E)| \leq \left(1 - \frac{|n(\alpha) - n(\alpha')|}{Nl}\right) \log |\lambda| + C.
\]

**Proof.** Recall that
\[
H_N - E I = \begin{bmatrix}
V_1 - E I & -W_2 & 0 \\
-W_2^\top & V_2 - E I & \ddots & \ddots \\
\ddots & \ddots & \ddots & -W_N \\
0 & -W_N^\top & V_N - E I
\end{bmatrix}
\]
where \(V = \lambda F + R\). We may assume without loss of generality that \(|\lambda| \geq 1\).

Let \(C = C(W,F,R) < \infty\) be such that the immediately off-diagonal blocks have the bound \(\|W\|_\infty < C\) and the diagonal blocks have the bound \(\|V - EI\|_\infty < C|\lambda|\).

We assume that \(\alpha \neq \alpha'\), otherwise the statement is trivial.

We fix all parameters (i.e. \(N, E, x, \alpha, \alpha'\)) and let
\[
g := H_N(x) - E I \in \text{Mat}_{Nl}(\mathbb{C}), \text{ so that}
\]
µN,α,α′(x,E) = det [g¬α,¬α′] = (-1)^{α+α′} \sum_{Γ \in P_{α′→α}} (-1)^{|Γ|+1} c(Γ) \det [g_{Γ,-Γ}] \tag{2.2}

where the last equality is due to Lemma 1.

We estimate the determinant of the matrix $g_{Γ,-Γ} \in \text{Mat}_{N-|Γ|}(C)$ simply by using Hadamard’s inequality:

$$|\det [g_{Γ,-Γ}]| \leq (C \lambda)^{Nl-|Γ|}.$$ \tag{2.3}

Combining (2.2) and (2.3) we have

$$|\det [g_{¬α,¬α′}]| \leq \sum_{Γ \in P_{α′→α}} |c(Γ)| (C \lambda)^{Nl-|Γ|}. \tag{2.4}

In order to estimate the cost $c(Γ)$ of a path $Γ \in P_{α′→α}$, we need a more careful analysis.

Let $1 \leq γ, γ′ \leq NL$.

(i) If $n(γ) = n(γ′)$ then $g_{γ,γ′}$ belongs to a diagonal block. Thus

$$|g_{γ,γ′}| \leq C |\lambda| \text{ (high cost)}.$$

(ii) If $|n(γ) - n(γ′)| = 1$ then $g_{γ,γ′}$ belongs to an immediately off-diagonal block. Thus

$$|g_{γ,γ′}| \leq C \text{ (bounded cost)}.$$

(iii) If $|n(γ) - n(γ′)| \geq 2$ then $g_{γ,γ′}$ does not belong to a tridiagonal block. Thus

$$|g_{γ,γ′}| = 0 \text{ (no cost)}.$$

Note also that if $|n(γ) - n(γ′)| \leq 1$ then $|γ′ - γ| < 2l$.

Therefore, in order to estimate the cost of a path $Γ \in P_{α′→α}$ more precisely, we have to take into account the time spent by the path in the diagonal and immediately off-diagonal blocks.

Indeed, if a path $Γ = (γ_1, γ_2, \ldots, γ_s) \in P_{α′→α}$ ventures off the tridiagonal blocks, that is, if $|n(γ_{i+1}) - n(γ_i)| \geq 2$ for some $1 \leq i \leq s - 1$, then $g_{γ_i,γ_{i+1}} = 0$, so $c(Γ) = 0$.

We may then restrict ourselves to the collection $P_{α′→α}^*$ of paths $Γ = (γ_1, γ_2, \ldots, γ_s) \in P_{α′→α}$ with $|n(γ_{i+1}) - n(γ_i)| \leq 1$ for all $1 \leq i \leq s - 1$.

Define for such a path

$$b = b(Γ) := \text{card} \{1 \leq i \leq s - 1 : |n(γ_{i+1}) - n(γ_i)| = 1\}$$

to be the number of steps of bounded cost taken by $Γ$. 
Since \( \gamma_i \in \{1, \ldots, Nl\} \) are all distinct, \( s \leq Nl \), so \( b(\Gamma) < Nl \). Also,
\[
|n(\alpha) - n(\alpha')| = |n(\gamma_s) - n(\gamma_1)| \leq \sum_{i=1}^{s-1} |n(\gamma_{i+1}) - n(\gamma_i)| = b(\Gamma).
\]
Then for every path \( \Gamma \in \mathcal{P}_{\alpha' \rightarrow \alpha}^* \)
we have
\[
|n(\alpha) - n(\alpha')| \leq b(\Gamma) \leq Nl,
\]
so its cost has the bound
\[
|c(\Gamma)| = s - \sum_{i=1}^{s-1} |g_{\gamma_i, \gamma_{i+1}}| = b(\Gamma) \leq Nl, \quad \text{(2.5)}
\]
Furthermore, for any path \( \Gamma = (\gamma_1, \gamma_2, \ldots, \gamma_s) \in \mathcal{P}_{\alpha' \rightarrow \alpha}^* \)
we have
\[
|\gamma_{i+1} - \gamma_i| < 2l \quad \text{for all indices } 1 \leq i \leq s - 1.
\]
Since \( s \leq Nl \), it follows
that there are at most \((4l)^Nl\) paths in \( \mathcal{P}_{\alpha' \rightarrow \alpha}^* \).
Therefore, using (2.4) and summing over all paths \( \Gamma \in \mathcal{P}_{\alpha' \rightarrow \alpha}^* \)
with the same number \( b = b(\Gamma) \) of steps of bounded cost, we have
\[
|\det [g_{-\alpha, -\alpha'}]| \leq \sum_{b=|n(\alpha) - n(\alpha')|}^{Nl} \sum_{\Gamma \in \mathcal{P}_{\alpha' \rightarrow \alpha}^* \atop b(\Gamma) = b} |c(\Gamma)| (C |\lambda|)^{s-b(\Gamma)} = C^{(|\Gamma| - b(\Gamma))} (4l)^{Nl} \quad \text{(3.1)}
\]
which completes the proof, after taking logarithms.

3. A lower bound on Dirichlet determinants

**Proposition 3.** There are finite constants \( \lambda_0 = \lambda_0(W, F, R) \) and \( C = C(W, F, R) \)
such that for all \( \omega \in \mathbb{R} \), \( N \in \mathbb{N} \) and \( \lambda \) with \( |\lambda| \geq \lambda_0 \) we have
\[
\int_T \frac{1}{Nl} \log |\det [H_N(x) - E I_{Nl}]| \, dx \geq \log |\lambda| - C
\]
for all energies \( E \) in a compact interval (say for \( |E| \leq |\lambda| \)).

**Proof.** The matrix-valued functions \( W \) and \( V = \lambda F + R \) are holomorphic on \( A_r \)
and continuous up to the boundary; in particular, for some constant \( C < \infty \), \( \|W\|_r \leq C \) and \( \|V\|_r \leq C |\lambda| \). It follows that the
block matrix-valued function $H_N - E I$ is also holomorphic on $A_r$ and
considering only energies $E$ with $|E| \leq C |\lambda|$, $\|H_N - E I\|_r \leq 2C |\lambda|$.
Let
$$u(z) := \frac{1}{N!} \log |\det [H_N(z) - E I]|.$$  
Then $u(z)$ is subharmonic on $A_r$ and using Hadamard’s inequality to estimate the determinant, it satisfies the upper bound
$$u(z) \leq \log |\lambda| + O(1) \quad \text{for all } z \in A_r. \quad (3.2)$$
We will derive a lower bound for $u(z)$ on a circle $|z| = 1 + y_0$, where $0 < y_0 \ll r$.
Write
$$H_N(z) - E I = \lambda D_N(z) + B_N(z), \quad (3.3)$$
where
$$D_N(z) = \text{diag} \left[ F_1(z) - \frac{E}{\lambda} I, \ldots, F_l(z) - \frac{E}{\lambda} \right]$$
is a block diagonal matrix and
$$B_N = \begin{bmatrix} R_1 & -W_2 & \cdots & \cdots & \cdots & -W_N \\ -W_2^T & R_2 & \cdots & \cdots & \cdots & -W_N^T \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -W_N^T & R_N \end{bmatrix}$$is the remaining block tridiagonal matrix.
Note that
$$\|B_N\|_r \leq C \quad \text{and} \quad \|D_N\|_r \leq C.$$We assume that $F$ has no constant eigenvalues: for any $t \in \mathbb{R}$ we have
$\det [F(x) - t I] \neq 0$ (as a function of $x \in \mathbb{T}$). So given any $0 < \delta \ll r$, there is $y_0 \sim \delta$ and there is $\epsilon_0 = \epsilon_0(\delta, r, F) > 0$ such that
$$|\det [F(z) - t I]| \geq \epsilon_0$$for all $z$ with $|z| = 1 + y_0$ and for all $t$ with $|t| \leq C$.
This follows from Corollary 4.6 in [7]. We present here an alternative, self-contained argument.
Let $f_t(z) := \det [F(z) - t I]$. Since $f_t(z)$ is holomorphic and non-identically zero on $A_r$, it has a finite number of zeros in any compact set, in particular in the annulus $A := \{ z : 1 + \frac{\delta}{2} \leq |z| \leq 1 + 2\delta \}$.
Therefore, there are circles $|z| = 1 + y$ with $\frac{\delta}{2} \leq y \leq 2\delta$ on which $f_t(z)$ has no zeros, so
$$\epsilon(t) := \sup_{\frac{\delta}{2} \leq y \leq \delta} \inf_{z : |z| = 1 + y} |f_t(z)| > 0.$$
We show below that the map $\mathbb{R} \ni t \mapsto \varepsilon(t)$ is lower semicontinuous. This will imply that it has a positive lower bound when $t$ is restricted to a compact interval, thus establishing (3.4).

To prove the lower semicontinuity of $\varepsilon(t)$, fix $t$ and assume that

$$\varepsilon(t) > \varepsilon_0^t.$$

(Since $l$ is fixed, the power $l$ in $\varepsilon_0^t$ is just for aesthetic reasons related to the calculations that follow). Then there is $y \in \left[\frac{\delta}{2}, 2\delta\right]$ such that on the circle $|z| = 1 + y$ we have

$$|\det [F(z) - t I]| = |f_{t_0}(z)| \geq \varepsilon_0^t.$$

Hence on $|z| = 1 + y$, the matrix $F(z) - t I$ is invertible and by Cramer’s formula,

$$\| (F(z) - t I)^{-1} \| \leq \frac{\| \text{adj} [F(z) - t I] \|}{\| \det [F(z) - t I] \|} \leq \frac{C^t}{\varepsilon_0^t} = (C \varepsilon_0^{-1})^t. \quad (3.5)$$

Furthermore, for any $t'$ we have

$$f_{t'}(z) = \det [F(z) - t' I] = \det [F(z) - tI + (t - t') I] = \det [F(z) - tI] \det [I + (t - t') (F(z) - tI)^{-1}] \quad (3.6)$$

To estimate from below the second determinant on the right hand side above, we use the following simple fact.

If $g \in \text{Mat}_m(\mathbb{R})$ is an invertible matrix and if we denote by $s_1(g) \geq \ldots \geq s_m(g) > 0$ its singular values, then $\|g^{-1}\| = s_m(g)^{-1}$, so

$$|\det [g]| = s_1(g) \ldots s_m(g) \geq \|g^{-1}\|^{-m}.$$

Now if $g \in \text{Mat}_m(\mathbb{R})$ and $\|g\| < 1$, then $I + g$ is invertible and $\|(I + g)^{-1}\| \leq (1 - \|g\|)^{-1}$. By the previous considerations applied to $I + g$ we conclude

$$|\det [I + g]| \geq (1 - \|g\|)^m. \quad (3.7)$$

Applying (3.7) to $g = (t - t') (F(z) - tI)^{-1}$ and using (3.5) (which also ensures that $\|g\| < 1$ if $|t - t'| \ll 1$), we have:

$$|\det [I + (t - t') (F(z) - tI)^{-1}]| \geq \left(1 - |t - t'| (C \varepsilon_0^{-1})^t\right)$$

$$= 1 - \mathcal{O}(|t - t'|).$$

Combined with (3.6), this implies that on $|z| = 1 + y$ we have

$$|f_{t'}(z)| \geq \varepsilon_0^t \left(1 - \mathcal{O}(|t - t'|)\right).$$

Hence

$$\varepsilon(t') \geq \inf_{z: |z| = 1 + y} |f_{t'}(z)| \geq \varepsilon_0^t \left(1 - \mathcal{O}(|t - t'|)\right),$$
which proves the lower semicontinuity of the function $\varepsilon$.

Thus the lower bound (3.4) holds, and it implies that on the circle $|z| = 1 + y_0$ the matrix $F(z) - tI$ is invertible and as in (3.5) we have
$$\left\| (F(z) - tI)^{-1} \right\| \leq (C \epsilon_0^{-1})^t.$$

Since the complexified dynamics leaves the circle $|z| = 1 + y_0$ invariant, all the blocks $F_j(z) - tI = F(z + j\omega) - tI$ are invertible and their inverses have the same bound as above there.

Therefore, on the circle $|z| = 1 + y_0$, for all $E$ with $|E| \leq C |\lambda|$, the block matrix $D_N(z) = \text{diag} \left( F_1(z) - E \frac{\lambda}{X} I, \ldots, F_l(z) - E \frac{\lambda}{X} I \right)$ is invertible as well, and since
$$D_N^{-1}(z) = \text{diag} \left[ \left( F_1(z) - E \frac{\lambda}{X} I \right)^{-1}, \ldots, \left( F_l(z) - E \frac{\lambda}{X} I \right)^{-1} \right],$$
we conclude that
$$\| D_N^{-1}(z) \| \leq (C \epsilon_0^{-1})^t. \quad (3.8)$$

Furthermore, using (3.4) again,
$$|\det [D_N(z)]| = \prod_{j=1}^{N} \left| \det \left[ F_j(z) - E \frac{\lambda}{X} I \right] \right| \geq \epsilon_0^N. \quad (3.9)$$

Going back to (3.3), we can now write for $|z| = 1 + y_0$
$$H_N(z) - E I = \lambda D_N(z) \left[ I + \lambda^{-1} D_N^{-1}(z) B_N(z) \right],$$
so
$$|\det [H_N(z) - E I]| = |\lambda|^N |\det [D_N(z)]| \left| \det \left[ I + \lambda^{-1} D_N^{-1}(z) B_N(z) \right] \right|. \quad (3.10)$$

As before, we estimate from below the second determinant on the right hand side of (3.10) by applying (3.7) to $g := \lambda^{-1} D_N^{-1}(z) B_N(z) \in \text{Mat}_{Nl}(\mathbb{R})$. Using (3.8), on the circle $|z| = 1 + y_0$ we have
$$\left\| \lambda^{-1} D_N^{-1}(z) B_N(z) \right\| \leq \frac{(C \epsilon_0^{-1})^t C}{|\lambda|} \leq \frac{1}{2},$$
provided $|\lambda|$ is large enough. Then
$$\left| \det \left[ I + \lambda^{-1} D_N^{-1}(z) B_N(z) \right] \right| \geq 2^{-Nl}. \quad (3.11)$$

Combining (3.10), (3.9) and (3.11) we conclude that on $|z| = 1 + y_0$ we have
$$u(z) = \frac{1}{Nl} \log |\det [H_N(z) - E I]| \geq \log |\lambda| + \log \frac{\epsilon_0}{2}. \quad (3.12)$$
We gather up below the estimates (3.2) and (3.12) on the subharmonic function \( u(z) \), defined on the annulus \( \mathcal{A}_r \).

\[
\begin{align*}
  u(z) &\leq \log |\lambda| + \mathcal{O}(1) \quad \text{on } |z| = 1 + r \\
  u(z) &\geq \log |\lambda| - \mathcal{O}(1) \quad \text{on } |z| = 1 + y_0.
\end{align*}
\]

By Hardy’s convexity theorem (see for instance [10]), the mean of a subharmonic function on an annulus is radially log-convex. More precisely, the function

\[
(1 - r, 1 + r) \ni s \mapsto \int_{|z| = s} u(z) \frac{dz}{2\pi}
\]

is convex as a function of \( \log s \).

Since \( 1 < 1 + y_0 < 1 + r \), letting \( \alpha := \frac{\log(1 + y_0)}{\log(1 + r)} \in (0, 1) \) so that

\[
\log(1 + y_0) = (1 - \alpha) \log(1) + \alpha \log(1 + r),
\]

we then have

\[
\int_{|z| = 1 + y_0} u(z) \frac{dz}{2\pi} \leq (1 - \alpha) \int_{|z| = 1} u(z) \frac{dz}{2\pi} + \alpha \int_{|z| = 1 + r} u(z) \frac{dz}{2\pi}.
\]

Combined with the lower bound on \( |z| = 1 + y_0 \) and the upper bound on \( |z| = 1 + r \), this implies a lower bound for the mean on \( |z| = 1 \):

\[
(1 - \alpha) \int_T u(x) \, dx = (1 - \alpha) \int_{|z| = 1} u(z) \frac{dz}{2\pi} \\
\geq \log |\lambda| - \mathcal{O}(1) - \alpha (\log |\lambda| + \mathcal{O}(1)) = (1 - \alpha) \log |\lambda| - \mathcal{O}(1),
\]

hence

\[
\int_T u(x) \, dx \geq \log |\lambda| - \mathcal{O}(1),
\]

which concludes the proof of this proposition. \( \square \)

4. Green’s functions estimates and the proof of localization

Let \( \mathcal{N} = [a, b] = \{a, a + 1, \ldots, b - 1, b\} \subset \mathbb{Z} \) be an interval of integers and let \( |\mathcal{N}| = b - a + 1 \) be its length. Consider the corresponding Green’s function

\[
G_N(x) = G_N(x; E) := (H_N(x) - E I)^{-1},
\]

defined whenever \( H_N(x) - E I \) is invertible.

We may regard \( G_N(x; E) \) as a \( |\mathcal{N}| \times |\mathcal{N}| \) block matrix, whose blocks are \( l \times l \) matrices over \( \mathbb{R} \). In this case we denote its entries by \( G_N,(n,n') x; E) \), where \( n, n' \in \mathcal{N} \) (as before, we use roman letters for the indices of the block entries of such a matrix).
The Green’s function may also be regarded as a $|N| \times |N|$ matrix with real entries, which we denote by $G_{N,(a,a')}(x; E)$, where $(a-1)l+1 \leq \alpha, \alpha' \leq bl$.

Furthermore, given a function $\vec{\psi} : \mathbb{Z} \to \mathbb{R}^l$, its finite volume restriction is

$$\vec{\psi}_N := \begin{bmatrix} \vec{\psi}_a \\ \vdots \\ \vec{\psi}_b \end{bmatrix} \in (\mathbb{R}^l)^{|N|} \simeq \mathbb{R}^{|N| l}.$$ 

When the integers interval $N = [1, N]$ for some $N \geq 1$, we use the shorthand notations $G_N(x; E)$ and $\vec{\psi}_N$ for $G_{N,(x,a)}(x; E)$ and $\vec{\psi}_N$ respectively.

Note that for any $j \in \mathbb{Z}$ we have $G_N(x+j\omega; E) = G_{N+j}(x; E)$. Hence for any interval $N = [a,b] \subset \mathbb{Z}$,

$$G_N(x; E) = G_{N|}(x + (a - 1)\omega; E).$$

Recall also the function considered in the previous section

$$u_N(x) = u_N(x; E) := \frac{1}{N^l} \log |\det [H_N(x) - E I]|.$$ 

4.1. Green’s functions estimates. A crucial ingredient in the proof of Anderson localization for the operator $H_\lambda(x)$ is the exponential decay of the off-diagonal entries of the Green’s function $G_N(x)$. These estimates are obtained using a certain concentration of measure inequality for the function $u_N(x)$ as well as the upper and lower bounds in Proposition 2 and Proposition 3.

The concentration of measure bounds require an arithmetic assumption on the frequency $\omega$. If $t > 0$ let $DC_t$ be the set of frequencies $\omega \in \mathbb{T}$ satisfying the following Diophantine condition:

$$\|k\omega\| \geq \frac{t}{|k|^2} \quad \text{for all } k \in \mathbb{Z} \setminus \{0\},$$

where for any real number $x$ we write $\|x\| := \min_{k \in \mathbb{Z}} |x - k|$.

Note that $\bigcup_{t>0} DC_t$ is a set of full measure.

Let us formulate the Green’s functions estimates needed in the proof of localization.

**Definition 2.** Let $N = [a, b] \subset \mathbb{Z}$ be an interval and let $(x, \omega, E) \in \mathbb{T} \times \mathbb{T} \times \mathbb{R}$. We say that $G_N(x; E)$ is a **good** Green’s function if for all $n, n' \in N$

$$\|G_{N,(n,n')}(x; E)\| \leq e^{-(|n-n'| - |N|) \log |\lambda|}.$$ 

(4.2)
Note that $G_{N,(n,n')} (x; E)$ is an $l \times l$ block, and $\|G_{N,(n,n')} (x; E)\|$ refers to any of the equivalent norms on $\text{Mat}_l(\mathbb{K})$. Clearly (4.2) is equivalent to
\[
|G_{N,(\alpha,\alpha')} (x; E)| \lesssim e^{- \frac{|n(\alpha) - n(\alpha')| - |N|}{50}} \log |\lambda| (4.3)
\]
for all $(a - 1)l + 1 \leq \alpha, \alpha' \leq bl$.

We now describe the setting under which the concentration of measure and consequently the Green’s functions estimates will hold.

Fix $t > 0$, $\omega \in DC_t$ and $\delta > 0$ small enough, say $\delta = \frac{1}{100l}$.

Let $n_0$ be a large enough integer depending on $t$ and on $l$ and let $n_1 \gg n_0$.

Let $\lambda_0$ be a large enough, finite constant depending on the data (e.g. on $W, F, R, l$); $\lambda_0$ is essentially the constant from Proposition 3, but we might slightly increase it so that, for instance, $\log |\lambda_0|$ is much larger than the other constants (denoted by $C$) from Propositions 2 and 3.

Fix any coupling constant $\lambda$ with $|\lambda| \geq \lambda_0$ and drop it from notations.

The energy $E$ will be such that $|E| \lesssim |\lambda|$.

With this setup, we have the following result.

**Proposition 4.** There are absolute constants $a > 0$ and $p \in \mathbb{N}$ such that if $M \geq n_0$ and $N \geq n_1$ then the following hold.

(i) Let
\[
\mathcal{B}_N^M = \mathcal{B}_N^M (\omega, E) := \left\{ x \in \mathbb{T} : \frac{1}{M} \sum_{j=0}^{M-1} u_N (x + j\omega) \leq (1 - \delta) \log |\lambda| \right\}. (4.4)
\]

Then
\[
\text{meas} (\mathcal{B}_N^M (\omega, E)) < e^{- M^a}.
\]

(ii) For every $x \notin \mathcal{B}_N^M (\omega, E)$ there is $0 \leq j < M$ such that $G_N (x + j\omega; E)$ is a good Green’s function.

(iii) For every $x \in \mathbb{T}$ there are integers $0 \leq n < N^p$ such that $G_N (x + n\omega; E)$ is a good Green’s function. In fact,
\[
\# \{ 0 \leq n < N^p : G_N (x + n\omega; E) \text{ is not a good Green’s function} \} \ll N^p.
\]

**Proof.** (i) Recall from the proof of Proposition 3 that the function $u(x) = u_N (x)$ has a subharmonic extension
\[
u(z) = u_N (z) = \frac{1}{Nl} \log |\det [H_N (z) - E I]|.
\]

From (3.2), this function has the upper bound
\[
u(z) \leq \log |\lambda| + O(1) \quad \text{for all } z \in \mathcal{A}_r.
\]
While it is not bounded from below, we showed in Proposition 3 that on average, it has the bound
\[ \int_{\mathbb{T}} u(x) \, dx \geq \log |\lambda| - O(1). \]

Therefore, there is \( x_0 \in \mathbb{T} \) such that
\[ u(x_0) \geq \log |\lambda| - O(1). \]

This is all we need for the quantitative version of Birkhoff’s ergodic theorem (showing that the averages of \( u(x) \) along the orbits of the torus translation by a Diophantine frequency concentrate near the mean of \( u \)) to be applicable. See for instance Subsection 6.2.3 in [8] (specifically, Lemma 6.6) for more details. Therefore, say by Theorem 6.5 in [8], we have that if \( \omega \in DC \) and \( M \geq t^{-2} \), then for some absolute constant \( a > 0 \) we have
\[
\text{meas}\left\{ x \in \mathbb{T} : \left| \frac{1}{M} \sum_{j=0}^{M-1} u(x + j \omega) - \int_{\mathbb{T}} u(x) \, dx \right| > SM^{-a} \right\} < e^{-M^a},
\]
where \( S \) is of the order of the above bounds on \( u \), hence \( S = O(\log |\lambda|) \).

If \( x \) does not belong to the set above, then using again the lower bound on the mean of \( u \) derived in Proposition 3 we have
\[
\frac{1}{M} \sum_{j=0}^{M-1} u(x + j \omega) \geq \int_{\mathbb{T}} u(x) \, dx - S M^{-a} \geq \log |\lambda| - O(1) - C \log |\lambda| M^{-a} > (1 - \delta) \log |\lambda|,
\]
provided \( M > l^{1/a} \) and \(|\lambda| \) is large enough.

(ii) By Cramer’s rule, for all indices \( 1 \leq \alpha, \alpha' \leq Nl \) we have
\[
G_{N,(\alpha,\alpha')}(x) = ((H_N(x) - EI)^{-1})_{(\alpha,\alpha')} = \frac{\mu_{N,(\alpha,\alpha')}(x)}{\det [H_N(x) - EI]}.
\]

Taking logarithms,
\[
\frac{1}{N} \log |G_{N,(\alpha,\alpha')}(x)| = \frac{1}{N} \log |\mu_{N,(\alpha,\alpha')}(x)| - \frac{1}{N} \log |\det [H_N(x) - EI]| = \frac{1}{N} \log |\mu_{N,(\alpha,\alpha')}(x)| - l u_N(x).
\]

By Proposition 2 for all \( x \in \mathbb{T} \) we have the upper bound
\[
\frac{1}{N} \log |\mu_{N,(\alpha,\alpha')}(x)| \leq l \log |\lambda| - \frac{n(\alpha) - n(\alpha')}{{N}} \log |\lambda| + Cl.
\]
If \( x \notin B_M \), then by item (i) we have

\[
\frac{1}{M} \sum_{j=0}^{M-1} u_N(x + j\omega) > (1 - \delta) \log |\lambda|.
\]

Then for some \( 0 \leq j < M \) we must have

\[
u_N(x + j\omega) > (1 - \delta) \log |\lambda|.
\]

We conclude that

\[
\frac{1}{N} \log |G_{N, (\alpha, \alpha')} (x + j\omega)| \leq l \log |\lambda| - \left| \frac{n(\alpha) - n(\alpha')}{N} \right| \log |\lambda| + Cl
\]

\[
- l \log |\lambda| + \delta l \log |\lambda|
\]

\[
< - \left( \frac{|n(\alpha) - n(\alpha'|}{N} - \frac{1}{50} \right) \log |\lambda|.
\]

This implies (4.3) for \( N = [1, N] \), so \( G_N(x + j\omega; E) \) is a good Green’s function.

(iii) In order to prove this last statement, it is enough to show that there is \( p < \infty \) such that for every \( x \in \mathbb{T} \), there is \( 0 \leq k < N^p \) with \( x + k\omega \notin B_M \), where \( M \) is an integer with \( n_0 \leq M \ll N \).

By the pointwise ergodic theorem, any typical orbit \( \{x + k\omega: k \geq 0\} \) will of course visit any set of positive measure. The point is to obtain a precise, quantitative version of this statement. To that extent, the fact that \( \omega \) is Diophantine and the algebraic structure of the set \( B_M \) are crucial.

First recall that \( \text{meas}(B_M) < e^{-Ma} \). Furthermore, the inequality (4.4) defining the set \( B_M \) may be rewritten as

\[
\prod_{j=0}^{M-1} \det [H_N(x + j\omega) - EI] \leq |\lambda|^{(1-\delta)NMl}.
\]

(4.5)

\( H_N(x) - EI \) may be regarded as an \( Nl \times Nl \) matrix-valued analytic function. Let \( f(x) \) be any of its entries. Expand \( f(x) \) into its Fourier series

\[
f(x) = \sum_{m \in \mathbb{Z}} \hat{f}_m e^{2\pi imx}
\]

and consider the truncation

\[
f_1(x) := \sum_{|m| \leq N^2} \hat{f}_m e^{2\pi imx},
\]

so that \( \|f - f_1\|_\infty \lesssim e^{-N^2} \).
Therefore, substituting each entry of $H_N(x) - EI$ by its truncation will produce negligible errors (i.e. super-exponentially small) in (4.3).

Hence we may assume that (4.5) is in fact a trigonometric polynomial inequality of degree $\leq N^2 \cdot Nl \cdot M = lN^3M$. Further truncating the power series for cos and sin, we conclude that the set $\mathcal{B}_N^M$ described by (4.5) may be regarded as a semi-algebraic set of degree at most $O(N^4M)$.

Put $M = N^{1/2}$. We have a set $\mathcal{B} = \mathcal{B}_N^M$ which is contained in a semi-algebraic set of degree at most $O(N^5)$, whose measure is comparable to that of $\mathcal{B}$, hence it is sub-exponentially (in $N$) small.

Then if $N_1 := N^p$, where $p$ is a large enough absolute constant, by Corollary 9.7 in J. Bourgain’s monograph [3], for every $x \in \mathbb{T}$,

$$\#\{k = 0, \ldots, N_1 : x + k\omega \in \mathcal{B}\} \ll N_1.$$ 

Thus there are (plenty of) integers $0 \leq k \leq N^p$ for which $x + k\omega \notin \mathcal{B}$, which completes the proof. \hfill \Box

**Remark 3.** In the proof of localization we will apply item (ii) in Proposition 4 with $M$ being a small fraction of $N$, say $M = \frac{1}{100}N$. Then the statement may be reformulated as follows.

There is an absolute constant $a > 0$ and for every large enough integer $N$ there is a set of phases $\mathcal{B}_N = \mathcal{B}_N(\omega, E)$ with $\text{meas}(\mathcal{B}_N(\omega, E)) < e^{-N^a}$ such that if $x \notin \mathcal{B}_N(\omega, E)$ then $G_N(x; E)$ is a good Green’s function for some

$$N \in \left\{[1, N] + j : 0 \leq j < \frac{1}{100}N\right\}.$$ 

That is, $G_N(x; E)$ is a good Green’s function where $N$ is the interval $[1, N]$ or a relatively small displacement thereof.

With this reformulation, we have the analogue of Proposition 7.19 in J. Bourgain’s monograph [3]. That proposition forms the basis for the proof of localization for quasi-periodic Schrödinger operators described in Chapter 10 of the monograph.

Finally, it is clear that similar statements to those in items (ii) and (iii) of Proposition 4 also hold when replacing $G_{[1,N]}$ by $G_{[-N,N]}$.

### 4.2. The sketch of the proof of localization.

The proof of localization consists of a parameter elimination argument that uses the Green’s functions estimates in Proposition 4 and semi-algebraic sets considerations. The argument is analogous to the one sketched in [3], which in turn references the method introduced by J. Bourgain and M. Goldstein in [4] to establish non-perturbative Anderson localization for quasi-periodic Schrödinger operators on the integer lattice. We present
below a sketch of the argument and indicate the point where it differs from \[1\].

Bourgain-Goldstein’s method uses the Sch’nol-Simon theorem\[2\] to establish localization, it is enough to show that every extended state (i.e. generalized eigenvector) is exponentially decaying.

More precisely, if for any \(\vec{\psi} = (\vec{\psi}_n)_{n \in \mathbb{Z}} \subseteq \mathbb{R}^l\) and \(E \in \mathbb{R}\) such that

\[
H(x) \vec{\psi} = E \vec{\psi}, \quad \|\vec{\psi}_0\|_2 = 1 \quad \text{and} \quad \|\vec{\psi}_n\|_2 \lesssim (1 + |n|)^2 \quad \text{for all} \ n \in \mathbb{Z}, \ (4.6)
\]

one shows that necessarily for some \(c > 0\)

\[
\|\vec{\psi}_n\|_2 \leq e^{-cn} \quad \text{as} \ |n| \rightarrow \infty, \ (4.7)
\]

then \(H(x)\) satisfies Anderson localization.

Consider \(x \in \mathbb{T}, \vec{\psi} = (\vec{\psi}_n)_{n \in \mathbb{Z}}\) and \(E \in \mathbb{R}\) such that (4.6) holds. Clearly \(H(x) \vec{\psi} = E \vec{\psi}\) is equivalent to

\[-W_{n+1}(x) \vec{\psi}_{n+1} - W_n^T(x) \vec{\psi}_{n-1} + (V_n(x) - E I) \vec{\psi}_n = 0 \quad \text{for all} \ n \in \mathbb{Z},\]

which implies that for every interval \(N = [a, b] \subset \mathbb{Z},\)

\[
(H_N(x) - E I) \vec{\psi}_N = \begin{bmatrix}
W_a^T(x) \vec{\psi}_{a-1} \\
0 \\
\vdots \\
0 \\
W_{b+1}(x) \vec{\psi}_{b+1}
\end{bmatrix} = \begin{bmatrix}
W_a^T(x) \vec{\psi}_{a-1} \\
0 \\
\vdots \\
0 \\
\vec{0}
\end{bmatrix} + \begin{bmatrix}
0 \\
\vec{0}
\end{bmatrix},
\]

where \(\vec{0}\) is the null vector in \(\mathbb{R}^l\).

From here we derive that

\[
\vec{\psi}_N = G_N(x; E) \begin{bmatrix}
W_a^T(x) \vec{\psi}_{a-1} \\
0 \\
\vdots \\
\vec{0}
\end{bmatrix} + G_N(x; E) \begin{bmatrix}
\vec{0} \\
\vdots \\
\vec{0}
\end{bmatrix}.
\]

It follows that for \(j \in N = [a, b]\) we have the following identity:

\[
\vec{\psi}_j = G_{N,(j,a)}(x; E) W_a^T(x) \vec{\psi}_{a-1} + G_{N,(j,b)}(x; E) W_{b+1}(x) \vec{\psi}_{b+1}. \ (4.8)
\]

This formula makes it apparent how exponential decay of off-diagonal terms of the Green’s function can lead to exponential decay of an extended state. It is more complicated than its counterpart in \[1\] because

\[2\]This result holds in our more general setting, for instance as a consequence of the generalization of Sch’nol’s theorem obtained in \[11\]. See also Theorem 4.1 in \[12\].
its terms are $l \times l$ blocks rather than numbers, and there are extra factors related to the weights $W(x)$. However, as the weights are bounded, the relevant estimates are derived in a similar manner, as it can be seen in the two lemmas below.

Given two integers $j, k$, we write $j \lesssim k$ when $j \leq Ck$ for some appropriate absolute constant $C > 0$, while $j \asymp k$ means that $j \lesssim k$ and $k \lesssim j$.

Let us say that an integer $j$ is well inside an interval $N = [a, b] \subset \mathbb{N}$ if it is far away from its endpoints, for instance if $2a \leq j \leq b$, so that

$$|j - a| = j - a \gtrsim j \quad \text{and} \quad |j - b| = b - j \gtrsim j.$$ 

**Lemma 5.** Let $N \in \mathbb{N}$ and let $N = [a, b] \subset \mathbb{N}$ with $a \asymp N$ and $b \asymp N$. For instance $N$ could be $[\frac{1}{2}N, 2N]$. Consider $x \in \mathbb{T}$, $\vec{\psi} = (\vec{\psi}_n)_{n \in \mathbb{Z}} \subset \mathbb{R}^l$ and $E \in \mathbb{R}$ such that (4.6) holds. If $G_N(x; E)$ is a good Green’s function and if $j$ is well inside $N$ then

$$\|\vec{\psi}_j\|_2 \lesssim e^{-cj},$$

where $c = O(\log |\lambda|)$.

A similar statement also holds for $N \subset \mathbb{Z}_-$.

**Proof.** Using (4.8) we have

$$\|\vec{\psi}_j\|_2 \leq \|G_N,(j,a)(x; E)\| \|W_a^\top(x)\| \|\vec{\psi}_{a-1}\|_2$$

$$+ \|G_N,(j,b)(x; E)\| \|W_{b+1}(x)\| \|\vec{\psi}_{b+1}\|_2.$$ 

Since $G_N(x; E)$ is a good Green’s function and $j$ is well inside $N$, its $(j, a)$ and $(j, b)$ entries decay exponentially fast in $j$.

Moreover, since $\vec{\psi}$ is a generalized eigenvector, $\|\vec{\psi}_{a-1}\|_2 \lesssim a^2 \lesssim j^2$, and similarly for $\vec{\psi}_{b+1}$.

Recall that for any integer $k$, $W_k(x) = W(x + k\omega)$ and $W(x)$ is bounded.

The conclusion then follows. \hfill \Box

**Lemma 6.** Let $x \in \mathbb{T}$, $\vec{\psi} = (\vec{\psi}_n)_{n \in \mathbb{Z}} \subset \mathbb{R}^l$ and $E \in \mathbb{R}$ such that (4.6) holds and let $N_0 \in \mathbb{N}$. If for some $c > 0$

$$\|\vec{\psi}_{N_0}\|_2 \leq \eta \quad \text{and} \quad \|\vec{\psi}_{-N_0}\|_2 \leq \eta,$$

then

$$\text{dist} \left( E, \text{spectra} \ H_{(-N_0,N_0)}(x) \right) \lesssim \eta.$$
Proof. Using (4.8) with $N = (-N_0, N_0)$ and $j = 0$ we have:

\[
1 = \|\vec{\psi}_0\|_2 \leq \|G_{N, (0, -N_0+1)}(x; E)\| \|W\|_\infty \|\vec{\psi}_{-N_0}\|_2 \\
+ \|G_{N, (0, N_0-1)}(x; E)\| \|W\|_\infty \|\vec{\psi}_{N_0}\|_2 \\
\lesssim \|G_N(x; E)\| \eta.
\]

Then

\[
\text{dist} (E, \text{spectrum } H_{(-N_0, N_0)}(x)) = \| (H_N(x) - E I)^{-1} \|^{-1} \lesssim \|G_N(x; E)\|^{-1} \lesssim \eta,
\]

which proves the lemma. \qed

Fix any $x_0 \in \mathbb{T}$ and let $N \in \mathbb{N}$ be any large enough scale. Consider a much larger scale $N' = NC$.

The idea of the proof of localization for the operator $H(x_0)$ is to pave\(^3\) an interval $N' \supseteq \left[\frac{1}{2} N', 2 N' \right]$ of length $|N'| \approx N'$ by a collection $\{N_n\}_n$ of intervals of length $|N_n| = N$, where $G_{N_n}(x_0; E)$ are all good Green’s functions.

An application of the resolvent identity (see Lemma 10.33 in [3] or Section 15, Step 3 in [4]) implies that $G_{N'}(x_0; E)$ is also a good Green’s function.

Since $N'$ is well inside the interval $N'$, Lemma 5 implies $\|\vec{\psi}_{N'}\|_2 \leq e^{-cN'}$, where $c = O(\log |\lambda|)$.

This argument applies to any large enough $N$, and it may be replicated on the negative side $\mathbb{Z}_-$. Hence it establishes exponential decay of the extended state $\vec{\psi}$ and thus Anderson localization.

It remains to explain the paving procedure of a large interval by small intervals whose corresponding Green’s functions are good. It is enough to show that for all $n$ with $\sqrt{N'} \leq n \leq 2N'$, there is $0 \leq j_n \leq \frac{1}{100} N$ such that

\[
G_{[1,N]+j_n}(x_0 + n\omega; E)
\]

is a good Green’s function for all $E$. \hspace{1cm} (4.9)

Indeed, if we denote $N_n := [1, N] + j_n + n$, then

\[
G_{N_n}(x_0; E) = G_{[1,N]+j_n}(x_0 + n\omega; E)
\]

is a good Green’s function. Moreover,

\[
\bigcup \{N_n : \sqrt{N'} \leq n \leq 2N' \} \supseteq \left[\frac{1}{2} N', 2N' \right],
\]

and consecutive intervals in this collection overlap by a lot (i.e. their intersection has length $\gtrsim N$). Hence the paving property holds.

\(^3\)Paving $N'$ by the intervals $\{N_n\}_n$ means that every point $k \in N'$ is well inside some interval $N_n$ in the collection.
In order to establish (4.9), by item (ii) of Proposition 4 and by Remark 3, it is enough to have that

\[ x_0 + n\omega \notin \bigcup_E B_N(\omega, E) \quad \text{for all} \quad \sqrt{N'} \leq n \leq 2N'. \]  

(4.10)

Ensuring that such a stretch of the orbit of \( x_0 \) under the translation by the frequency \( \omega \) avoids the corresponding exceptional set above, will require the elimination of a set \( \Omega \) of frequencies. The problem is the dependence of these exceptional sets on the eigenvalue \( E \). We explain below how to eliminate this dependence on \( E \) (for full details, see Section 4 in [4]).

By Proposition 4 item (iii), there are plenty of integers \( N_1 \leq n \leq N_p \) such that

\[ G_{[1,N]+n}(x_0; E) = G_N(x_0 + n\omega; E) \]  

is a good Green’s function. Since there are plenty of integers \( N_1 \leq j \leq N_p \) well inside \([1,N]+n\), by Lemma 5, \( \| \vec{\psi}_j \|_2 \lesssim e^{-c j} \), where \( c = O(\log |\lambda|) \).

Repeating this argument on \( \mathbb{Z}_- \), we can ensure that there is an integer \( N_0 \) with \( N_1 \leq N_0 \leq N_p \) such that both estimates

\[ \| \vec{\psi}_{N_0} \|_2 \leq e^{-c N_0} \quad \text{and} \quad \| \vec{\psi}_{-N_0} \|_2 \leq e^{-c N_0} \]

hold (it would be enough if they held for some \( N_0 \) and \( -N_0' \), where \( N_0' \approx N_0 \)).

Applying Lemma 6, we conclude the following. If \( E \) is a generalized eigenvalue for \( H(x_0) \), then for every large enough scale \( N \), there is \( N_0 \) with \( N_1 \leq N_0 \leq N_p \) such that \( E \) is an almost eigenvalue of the finite volume operator \( H_{(-N_0,N_0)}(x_0) \):

\[ \text{dist} (E, \text{spectrum } H_{(-N_0,N_0)}(x_0)) \lesssim e^{-c N_0} < e^{-c N_1}. \]

We may then replace the condition (4.10) by

\[ x_0 + n\omega \notin S_N(\omega) \quad \text{for all} \quad \sqrt{N'} \leq n \leq 2N', \]  

where \( \Omega := \bigcup \{ \mathcal{B}_N(\omega, E) : E \in \text{spectrum } H_{(-N_0,N_0)}(x_0), N_0 \leq N_p \} \).

This is a union of at most \( N^{p+1} \) many sets of measure \( < e^{-N^a} \), so \( \text{meas} (S_N(\omega)) < e^{-N^{a'}} \). Consider the set

\[ S_N := \{ (\omega, x) : \omega \text{ Diophantine and } x \in S_N(\omega) \}. \]

As in the proof of item (iii) of Proposition 4, \( S_N \) may be regarded as a semi-algebraic set of sub-exponentially small measure in \( N \) but polynomial complexity (we would have to restrict the Diophantine condition (4.1) to integers \( j < N \)). Lemma 6.1 in [4] then provides an...
estimate on the measure of the set
\[ \Omega_N := \{ \omega \in \mathbb{T} : (\omega, x_0 + n\omega) \in S_N \text{ for some } n \sim N' \} , \]
namely
\[ \text{meas} (\Omega_N) < (N')^{-1/2} = N^{-C/2} . \]

Since \( \sum_N N^{-C/2} < \infty \), by Borel-Cantelli, the set \( \Omega := \bigcap_k \bigcup_{N \geq k} \Omega_N \) has measure zero. We conclude that if \( \omega \) is Diophantine and \( \omega \notin \Omega \), then \( \omega \notin \Omega_N \) for all large enough \( N \). This ensures that (4.11) holds, and hence it justifies the paving procedure.

The interested reader may find all the remaining details in [4] as well as in Chapter 10 of J. Bourgain’s monograph [3].

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