A NETWORK FORMATION MODEL BASED ON SUBGRAPHS

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ABSTRACT. We develop a new class of random-graph models for the statistical estimation of network formation that allow for substantial correlation in links. Various subgraphs (e.g., links, triangles, cliques, and stars) are generated and their union results in a network. We provide estimation techniques for recovering the rates at which the underlying subgraphs were formed. We illustrate the models via a series of applications including testing for incentives to form cross-caste relationships in rural India, testing to see whether network structure is used to enforce risk-sharing, testing as to whether networks change in response to a community’s exposure to microcredit, and show that these models significantly outperform stochastic block models in matching observed network characteristics. We also establish asymptotic properties of the models and various estimators, which requires proving a new Central Limit Theorem for correlated random variables.

JEL Classification Codes: D85, C51, C01, Z13.

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This grew out of a paper: “Tractable and Consistent Random Graph Models,” (http://arxiv.org/abs/1210.7375), which we have now split into two pieces. This part contains the material on subgraph generation models and includes new results on identification, asymptotic normality, and estimation via minimum distance that were not part of the original paper. We thank Alberto Abadie, Isaiah Andrews, Emily Breza, Aureo de Paula, Paul Goldsmith-Pinkham, Bryan Graham, Han Hong, Guido Imbens, Michael Leung, Elena Manresa, Angelo Mele, Joe Romano, Elie Tamer, and the referees, as well as seminar participants, for helpful comments and suggestions. We thank Andres Drenik for valuable research assistance. Chandrasekhar is grateful for support from the NSF Graduate Research Fellowship Program and NSF grant SES-1156182. Jackson gratefuly acknowledges financial support from the NSF under grants SES-0961481 and SES-1155302 and from grant FA9550-12-1-0411 from the AFOSR and DARPA, and ARO MURI award No. W911NF-12-1-0509.

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1. Introduction

Networks of interactions impact many economic behaviors including insuring one’s self (e.g., Cai, deJanvry, and Sadoulet (2015)), participating in microfinance (e.g., Banerjee, Chandrasekhar, Duflo, and Jackson (2013)), educating one’s self (e.g., Calvo-Armengol, Patacchini, and Zenou (2009); Carrell, Sacerdote, and West (2013)), and engaging in criminal behavior (e.g., Glaeser, Sacerdote, and Scheinkman (1996); Patacchini and Zenou (2008)). Networks of interactions are also essential to understanding financial contagions (e.g., Gai and Kapadia (2010); Elliott, Golub, and Jackson (2014); Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015)), as well as world trade (e.g., Chaney (2016)), interstate war (e.g., Jackson and Nei (2015); Koenig, Rohner, Thoenig, and Zilibotti (2015)), and a host of other economic phenomena. As such, the structure that a network takes has profound consequences – changing the possibility of contagions, the decisions that people make, and the beliefs that people hold. In addition, to the direct interest in network formation, any analysis of peer effects or social learning must account for network endogeneity since peoples’ positions in networks are driven by their characteristics (e.g., see Goldsmith-Pinkham and Imbens (2013)). All of these applications make it essential to understand and be able to estimate network formation.

Moreover, networks are of interest in all such applications precisely because there are externalities – one agent’s behavior impacts the welfare and behaviors of others.\(^1\) This feature means that connections between pairs of agents are not independent, not only in determining behaviors but also in network formation. Thus, appropriate models of network formation must admit correlations in connections.

Despite the importance of network formation in such a wide range of social and economic settings, general, flexible, and tractable econometric models for the estimation of network formation are lacking. This stems from two challenges: the aforementioned dependence in connections and the fact that many studies involve one (large) network. Thus, one is often confronted with estimating a model of formation by taking advantage of the large number of connections, but having them all be dependent observations. Despite the dependence, it is possible that the many relationships in a network still provide rich enough information to consistently estimate the parameters of a network model and test of hypotheses from a single observed network, at least hypothetically. Here we develop a class of models that admit substantial correlations in links and also provide

\(^1\)For detailed discussion and references see Jackson, Rogers, and Zenou (2016) and Jackson (2017, forthcoming).
practical techniques of estimating the models, showing that they are easily estimable even if a researcher only has one network, as well as in cases with many networks.

Before discussing our approach, let us discuss some of the other approaches that are available.

1.1. Existing Models of Network Formation. The most basic models are what are known as ‘stochastic block models’, in which links may depend on node characteristics but are (conditionally) independent of each other. That approach requires correlation between links to be well-approximated by observables, and so is not suited for most applications beyond community detection. In particular, stochastic block models are not an option for estimation in most economic applications. In fact, in Section 4 we show that our models by incorporating correlation in links substantially outperform a stochastic block model in matching key network characteristics, even when the block model admits a rich set of covariates.

Given this void, a literature spanning several disciplines (sociology, statistics, economics, and computer science) turned to exponential random graph models – henceforth “ERGMs” – to meet these challenges. ERGMs admit link interdependencies and have become the workhorse models for estimating network formation. However, from the onset of the use of these models, people realized that the parameter estimates could be very unstable on all except very small networks. It turns out that these issues of instability are not simply a software issue: it has been shown that maximum likelihood and Bayesian estimators of the parameters are not be computationally feasible nor consistent for important classes of such models – effectively the ERGMs that include many link dependencies of interest – and so parameter estimates cannot always be trusted (nor can the standard errors). For details see Bhamidi, Bresler, and Sly (2008); Shalizi and Rinaldo (2012); Chandrasekhar and Jackson (2012).

A set of models that does allow for link dependencies and are estimable are those based on explicit link formation algorithms (e.g., Barabasi and Albert (1999); Jackson and Watts (2001); Jackson and Rogers (2007); Currarini, Jackson, and Pin (2009, 2010);

\[^2\] These grew from work on what were known as Markov models (e.g., Frank and Strauss (1986)) or \(p^*\) models (e.g., Wasserman and Pattison (1996)). An alternative approach is to work with regression models at the link (dyadic) level, but to allow for dependent error terms, as in the “MRQAP” approach (e.g., see Krackhardt (1988)). That approach, however, is not designed for identifying the incidence of particular patterns of network relationships that may be implied by various social or economic theories of the type that we wish to address here.

\[^3\] Recent work has made progress on both the speed of convergence of estimation algorithms as well as characterizing the asymptotic distribution of sufficient statistics in some classes of ERGMs that avoid extensive link dependencies (see e.g., Mele (2017a,b); Mele and Zhu (2017)).
Christakis, Fowler, Imbens, and Kalyanaraman (2010); Bramoullé, Currarini, Jackson, Pin, and Rogers (2012)). These models can be estimated since the algorithms are particular enough so that one can directly derive how parameters in the model translate into aggregate network statistics, such as the degree distribution or homophily levels. The advantage of such models is that a specific algorithm allows for estimation. The disadvantage is that the specificity of the algorithms also necessarily results in narrow models. Thus, these approaches are useful in some contexts, but they are not designed, nor intended, for general statistical testing of a wide variety of network formation models and hypotheses. For instance, such models under-perform with triadic closure (are links correlated across triples of nodes – so that if two people have a friend in common, are they more likely to be friends with each other than if link formation were independent).  

Another approach can be thought of as having roots in the spacial econometrics literature. In such models, nodes only link to other nodes that are close enough in some geographic or characteristic space, so that links between distant enough pairs of nodes are asymptotically (at a fast enough rate) independent (e.g., Boucher and Mouriﬁé (2012); Leung (2014)). This approach holds promise for some enormous networks – in which the graph can almost be decomposed into independent pieces.

A related but different approach views correlations as driven by unobserved heterogeneity (Chatterjee, Diaconis, and Sly, 2010), but has links be uncorrelated conditional on all (observed and unobserved) characteristics (as extended by Graham (2017)). Although there are still some challenges in taking such models to data, they have should be useful in settings that in which links are not formed in a correlated manner once one accounts for all characteristics.

Finally, there is a large literature on the theory of network formation from a strategic perspective (for references, see Jackson (2005, 2008))). Since the first writing of this paper, researchers have started to derive versions of such models that can be taken to data. One approach builds upon the relationship between certain classes of strategic network formation models and potential games (Mele (2017a); Badev (2013); Sheng  

4The Jackson and Rogers (2007) model does have a parameter that affects triadic closure, but in that model closure cannot be separated from the shape of the degree distribution - so it is best suited for growing random networks where new nodes are born over time.  

5The arguments therein have foundations in the mathematics literature on random geometric graphs (Penrose, 2003).  

6See also Charbonneau (2017) for related work in a panel data setting. Such models have been studied in the mathematics and statistics literatures (e.g., Holland and Leinhardt (1981); Park and Newman (2004); Chatterjee et al. (2010); Blitzstein and Diaconis (2011)) McCormick and Zheng (2015) merge both the insights from the unobserved heterogeneity model and the latent space distance model. Breza, Chandrasekhar, McCormick, and Pan (2017) evaluate its empirical performance.
Another derives restrictions on parameters of an observed network under the presumption that it is in equilibrium (pairwise stable) (De Paula, Richards-Shubik, and Tamer (2018)). Although the progress to date requires restrictions on how links can enter agent’s payoffs, they provide important first steps in deriving implications of the arsenal of strategic network formation models. Below, we also provide ways to incorporate strategic formation in SUGMs, thus in part bridging our approach here and the strategic formation approach.

1.2. Our Subgraph Model Approach. Our approach is quite distinct from all of the above, both in terms of the fundamentals of the approach (working with subgraphs as the basic building blocks) and the technicalities of allowing nontrivial conditional correlations (developing a new central limit theorem for non-trivially correlated random variables). Our contribution is to develop models of network formation that admit considerable interdependency, and have the presence of links be highly correlated – even across distances, but still prove consistency and asymptotic normality of the parameter estimates.

The paucity of flexible models that are computable and can be used across many applications for hypothesis testing and inference is what motivates our work here.

What we do is develop a new class of random-graph models for the statistical estimation of network formation that allow for substantial correlation in links. In these models, various subgraphs (e.g., links, triangles, cliques, and stars) are generated directly. For instance, students may form friendships with their roommate(s), members of a study group, teammates, band members, etc.; researchers may form collaborations on writing papers in pairs, or triples, or quadruples, etc. This results in links, and those links are then naturally correlated since they are formed in combinations. The union of all these subgraphs results in a network. The challenge to the researcher is that often only the final network is observed: a survey may ask people to list their friends and acquaintances, or links may be observed on a social platform, or emails or phone calls are observed, and so forth; but the original formation process is often not observed. The challenge that then arises in estimating how the network formed is that subgraphs may overlap and may also incidentally generate new subgraphs, and so the true rate of formation of the subgraphs cannot generally be inferred just by counting their presence in the resulting network.

\footnote{For a recent overview of the recent literature, see de Paula (2015).}
Although the basic ideas behind our models are very simple, we provide four different applications that illustrate how easily such models admit strategic network formation, general covariates, and generate rich network features.

Despite the fact that the formation can only be inferred, there are fairly simple conditions for identification, as different rates of generation for subgraphs lead to different observed network characteristics. Effectively, one can estimate the frequencies at which various subgraphs should appear in the final network based on their formation rate. So, we provide estimation techniques for recovering the frequencies at which the underlying subgraphs were formed from the observation of a single (large) network in addition to the case when the researcher observes many independent networks. We provide results on identification of the true underlying parameters governing subgraph formation from various statistics.

Beyond the identification issue, for the models to be useful in hypothesis testing and inference, we also need to provide results on the asymptotic distributions of the estimates. We provide results for two different approaches, depending on what sort of data the researcher has available. One approach is for researchers who observe many separate networks. Here asymptotics are straightforward since there are many independent observations. The other approach is for researchers who observe one network. There, the asymptotics are more technically challenging. In particular, existing central limit theorems from the spatial and time-series econometrics literatures do not apply to our setting, as we need to allow subgraphs to form on arbitrary groups of nodes, which then results in correlation patterns across all links in the network. In particular, the standard arguments exploiting strong mixing of random variables (e.g., Bolthausen (1982)) do not apply since there is no sense in which the random variables we are concerned with begin to become arbitrarily far from each other, and therefore essentially uncorrelated. Thus, we use a powerful lemma from Stein (1986) in order to prove a new central limit theorem for correlated random variables that provides for more general and permissive results than previously available for our setting. This establishes asymptotic normality for our estimators, and should be useful beyond our network setting. Our results may be of

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8This lemma and precursor work in Stein (1972) have been used to derive central limit theorems in two literatures: time-series/spatial statistics and dependency graphs. For instance the oft-used Bolthausen (1982) central limit theorem, crucial in time-series and spatial econometrics, uses a lemma from Stein (1972) to show asymptotic normality. In time-series and spatial econometrics, a non-exhaustive but illustrative list of papers using Bolthausen (1982) include Conley (1999), Jenish and Prucha (2009), Bester, Conley, and Hansen (2011), among others. However, the arguments of Stein (1972), and therefore Bolthausen (1982), do not apply to our setting in which we need to allow for much richer dependencies than are admitted in previous theorems.
independent interest as they have a connection to the study of central limit theorems for random variables described by dependency graphs (Baldi and Rinott (1989); Goldstein and Rinott (1996); Chen and Shao (2004)), though are less restrictive in the correlation structure of the random variables of interest. Finally, we also show that if the network is sparse enough then direct shares of subgraphs are consistent and asymptotically normal estimators – providing a very easy estimation technique for many network applications, as many social and economic networks are relatively sparse.

The paper is organized as follows. Section 2 presents the general SUGM framework and several applied examples to illustrate some uses of our approach. In Section 3, we show all SUGMs are identified and develop specific examples with various subgraph configurations, finite support covariates, and multiplexing. Section 4 we use SUGMs to study three empirical applications: incentives for forming risk-sharing links; demonstrating that low-dimensional SUGMs reflect empirical network structures more closely than high-dimensional conditional edge independent models – even those with unobserved node heterogeneity; and social norms restricting linking across social boundaries in public. We demonstrate consistency of our parameter estimates and asymptotic normality, both in situations in which one observes many networks as well as in which one only observes one large network, in Section 5. Section 6 presents a new and general central limit theorem utilized in the prior section. Section 7 concludes.

2. Model

2.1. Description. $n \geq 3$ is the number of nodes on which a network is formed. Nodes may have characteristics, such as age, profession, gender, race, caste, etc., that we denote by the vector $X_i$ for a generic $i \in \{1, \ldots, n\}$. In what follows we assume that the $X_i$ have finite support. As such nodes can be classified by a finite set of types.

We denote a network by $g$, the collection of subsets of $\{1, \ldots, n\}$ of size 2 that lists the edges or links that are present in its graph. So, $g = \{\{1, 3\}, \{2, 5\}\}$ indicates the network that has links between nodes 1 and 3 and between nodes 2 and 5. For notational ease, we simply write $g = \{13, 25\}$, and write $ij \in g$ to denote that link $ij$ is present in network $g$. In general our model easily accommodates directed graphs, and all of the definitions below extend directly, in which case instead of pairs of nodes, these would be ordered pairs so that $ij$ and $ji$ would differ. However, for ease of exposition, most of the examples and discussion refer to the undirected case.

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9 We conjecture that under sensible conditions our results hold for continuous covariates as well, though that remains beyond the scope of this paper.
\( \mathcal{G}^n \) denotes the set of all networks on \( n \) nodes.

In a subgraph generation model, henceforth SUGM, subgraphs are directly generated, and then the resulting network is the union of all of the links in all of the subgraphs. Degenerate examples of this are Erdos-Renyi random networks, and the generalization of that model, stochastic-block models, in which links are formed with probabilities based on nodes’ attributes. The more interesting classes of SUGMs include richer subgraphs, and hence involve dependencies in link formation. It might be that people of the same caste meet more frequently or are more likely to form a relationship when they do meet, as in a stochastic block model, but it could also be that groups of three (or more) meet and can decide whether to form a triangle, with the meeting probability and decision potentially driven by their castes and/or other characteristics. The model can then be described by a list of probabilities, one for each type of subgraph, where subgraphs can be based on the subgraph shape as well as the nodes’ characteristics.

SUGMs are formally defined as follows.

There are finitely many types of nonempty subgraphs, indexed by \( \ell \in \{1, \ldots, k\} \), on which the model is based – for instance in the links and triangles case \( \ell \in \{L, T\} \). The \( k \) subgraph types are denoted by \( (G_\ell)_{\ell \in \{1, \ldots, k\}} \), where each \( G_\ell \subset \mathcal{G}^n \) is a set of possible subgraphs on \( m_\ell \leq n \) nodes. Each subgraph in \( g' \in G_\ell \) is homomorphic to (a relabeling of) every other \( g'' \in G_\ell \). The definitions of the subgraph types can restrictions based on node characteristics, for instance, requiring that the characteristics \( X_i \) and \( X_{\pi(i)} \) be the same – for instance, \( G_\ell \) for some \( \ell \) could be the set of “triangles that involve one child and two adult nodes”. As an example, the set \( G_\ell \) for some \( \ell \) could be all stars with one central node and four other nodes, and another \( \ell \) could be all of the links that involve people of different castes, and so forth. These could also be directed subgraphs in the case of a directed network. A few examples are pictured in Figure 1.

The probability that various subgraphs form is described by a vector of parameters, denoted \( \beta \in \mathcal{B} \), where \( \mathcal{B} \) is (unless otherwise noted) a compact subset of some finite dimensional Euclidean space. For instance, \( \beta = (\beta_L, \beta_T) \in \mathcal{B} \subset [0, 1]^2 \) in a links and triangles example. In some applications, the parameters have the same dimension as the number of types of subgraphs, although this is not necessary. For example, the vector \( \beta \) may reflect preference parameters of agents who choose to form subgraphs based on their

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10 This definition does not admit isolates since we define subgraphs to be nonempty and connected, but isolates are easily admitted with notational complications, and are illustrated in some of our supplementary material and examples.

11 So, for any \( g', g'' \in G_\ell \), there exists a bijection \( \pi \) on \( \{1, \ldots, n\} \) such that \( ij \in g' \) if and only if \( \pi(i) \pi(j) \in g'' \).

12 We treat vectors as row or column vectors as is convenient in what follows.
A network $g$ on $n$ nodes is randomly formed as follows:

- Each of the possible subnetworks $g_\ell \in G_\ell$ is independently formed with probability $p_\ell(\beta)$ for each $\ell \in \{1, \ldots, k\}$. In many of our examples, each $p_\ell$ is synonymous with the parameter $\beta_\ell$. Therefore without loss of generality, in what follows use $\beta_\ell$ to denote the probability that subgraph of type $\ell$ forms, which again we emphasize can depend on discrete covariate compositions of nodes involved.
- The resulting network, $g$, is the union of all the links that appear in any of the generated subgraphs.

2.2. An Example with Node Characteristics.

To make things very clear, let us consider an example with discrete characteristics.

Suppose that nodes come in two colors: blue and red (for instance different genders, age groups, religions, etc., and clearly this extends directly to more than two colors). In our example of links and triangles, there are now three types of links: (blue, blue), (blue, red), (red, red); and four types of triangles (blue, blue, blue), (blue, blue, red), (blue, red, red), (red, red, red).

In this case, the set of possible $\ell$'s would be: $\{(\text{blue, blue}), (\text{blue, red}), (\text{red, red}), (\text{blue, blue, blue}), (\text{blue, blue, red}), (\text{blue, red, red}), (\text{red, red, red})\}$. 
Figure 2. Panel (A) shows all possible links and Panel (B) shows all possible triangles when a node has characteristic $X_i \in \{\text{red, blue}\}$.

So we have

\[ G_{(\text{blue,blue})} = \{ij : X_i = \text{blue}, X_j = \text{blue}\} \]

and

\[ G_{(\text{blue,blue,red})} = \{ijk : X_i = \text{blue}, X_j = \text{blue}, X_k = \text{red}\}, \]

and so forth, as depicted in Figure 2.

The parameters

\[ \{\beta_{(\text{blue,blue})}, \beta_{(\text{blue,red})}, \beta_{(\text{red,red})}, \beta_{(\text{blue,blue,blue})}, \beta_{(\text{blue,blue,red})}, \beta_{(\text{blue,red,red})}, \beta_{(\text{red,red,red})}\}, \]

could be the probabilities that the subgraphs in question form.

One could restrict or enrich the model by having simpler or more complex sets of parameters – for instance requiring that $\beta_{(\text{blue,blue})} = \beta_{(\text{red,red})}$, or by having preference parameters that govern the probabilities of various subgraphs forming, as we discuss below.\footnote{This example has a finite set of possible node characteristics. We conjecture that continuous node characteristics can be accommodated under appropriate assumptions, though this is beyond the scope of this paper. For instance, a model of interest could have $p_T(X_i, X_j, X_k; \beta)$ be the probability of a triangle governed by the (continuous) characteristics of the nodes in a logistic or probit formulation.}

2.3. Usefulness of this Framework. The SUGM perspective is useful for a number of purposes.

First, it can be used to explore incentives for linking in a reduced form way. There are many theories, including that of Coleman (1988), as well as game-theoretic models such as that studied by Jackson, Rodriguez-Barraquer, and Tan (2012), that suggest that triangles and other cliques play special roles in maintaining cooperation in favor exchange. In order to test such theories, we need a statistical model that allows us to test whether cliques appear significantly more often than being randomly generated by
links, and whether they appear in configurations that would be predicted by the game theory.

Second, simple SUGMs, even ones with just links and triangles, generate higher-order features of empirically observed social networks that link-based models (even those accounting for characteristics and geography) do not. It is important for a formation model to capture realistic features of empirical network data. For example, if a researcher observes only part of a network, having a reasonable model of network formation is important to interpolate over the missing data in a sensible way (e.g., see Chandrasekhar and Lewis (2013) or Breza, Chandrasekhar, McCormick, and Pan (2017)). Or, if one is interested in generating networks under a hypothetical policy, a model is only useful if it can generate networks that are likely to occur at a variety of parameter values.

Third, SUGMs can be used for structural estimation. There are parsimonious microfoundations – simple models of mutual consent or search – that give rise to SUGMs. Structural parameter estimates may have intrinsic value of their own in terms of welfare analyses, and also help with counterfactuals and policy evaluation.

To illustrate some of these potential uses of SUGMs, we provide several examples below. We then further develop and estimate SUGMs in the context of each of these examples in Section 4, and cover other examples in the appendix.

2.4. **Example 1: Do incentives for Risk Sharing Drive Network Formation?**

Our first example illustrates how SUGMs can be used to ask whether incentives for risk- and favor-sharing drive network formation.

2.4.1. *A model of mutual consent.* Consider a simple model in which individuals get utility from being in bilateral relationships (“links”) denoted by $L$, as well as trilateral relationships denoted by $T$. The value of a partner $j$ to $i$ in a bilateral relationship is given by $u_i^L$:

$$u_i^L (j) = \beta_L X_{ij} - \epsilon_{ij}$$

and the value of the trilateral set of relationships $jk$ to $i$ is given by $u_i^T$:

$$u_i^T (jk) = \beta_T X_{ijk} - \epsilon_{ijk}.$$ 

The value of the relationships depend on the finite support characteristics of the people involved (for instance, exhibiting homophily, etc.) via the $\beta_L X_{ij}$ and $\beta_T X_{ijk}$ terms. There are also idiosyncratic values to the relationships, $-\epsilon_{ij}$ and $-\epsilon_{ijk}$, which may capture personalities, compatibilities, etc. These are distributed according to some distributions $F_L$ and $F_T$ respectively.
Forming relationships requires mutual consent (e.g., as in the pairwise stability of Jackson and Wolinsky (1996)), so the net utility must be positive to all agents. The probability that a subgraph \( ij \) forms is

\[
p_L(X_{ij}, \beta_L) = F_L(\beta_L X_{ij}')^2
\]

and similarly the probability that subgraph \( ijk \) forms is

\[
p_T(X_{ijk}, \beta_T) = F_T(\beta_T X_{ijk}')^3.
\]

The squares and cubes come from the fact that links require two consents and trilateral relationships require three consents.

In this case by estimating the probabilities of subgraphs forming \((p_T(\cdot) \text{ and } p_L(\cdot))\), one can recover the marginal effects of changes in covariates on preferences for being in various configurations \((\beta_T \text{ and } \beta_L)\).

Without loss of generality we simply call the subgraph formation probabilities themselves \(\beta_{T,X_T}\) and \(\beta_{L,X_L}\) for pair and node covariate combination \(X_T\) and \(X_L\) respectively.

### 2.4.2. Incentives for Risk-Sharing.

Jackson, Rodriguez-Barraquer, and Tan (2012) show that whether or not a link is supported plays an important role in maintaining favor exchange. It characterizes renegotiation proof robust pairwise stable networks and shows that, in the homogenous parameter case all such networks are quilts, and in the inhomogenous parameter case every link must be supported. Of course, the model of Jackson, Rodriguez-Barraquer, and Tan (2012) is not meant to be taken literally and doesn’t lend itself to a simple econometric framework.

Consider a variation on the aforementioned mutual consent model wherein now there are multiple link types: favors and information, and for simplicity we do not consider the interaction of these links. We can use this to study the question raised by Jackson, Rodriguez-Barraquer, and Tan (2012). To make this simple assume there are no covariates, so all nodes are identical. Preferences are described by a random utility framework (McFadden, 1973). In this case the value of a link between \( i \) and \( j \) to \( i \) is given by

\[
u_{i}^{L,favor}(i) = \beta_{L,favor} - \epsilon_{ij,favor}, \quad u_{i}^{L,info}(i) = \beta_{L,info} - \epsilon_{ij,info}\]

and the value of a triangle is given by

\[
u_{i}^{T,favor}(jk) = \beta_{T,favor} - \epsilon_{ijk,favor}, \quad u_{i}^{T,info}(jk) = \beta_{T,info} - \epsilon_{ijk,info}.
\]
By the arguments of Jackson, Rodriguez-Barraquer, and Tan (2012), we expect that fraction of links that are supported should be higher in favor exchange than in information links. In the language of this model, one expects that \( \beta_{T,\text{favor}} > \beta_{T,\text{info}} \).

Given that triangles can be incidentally generated, one cannot test this simply by examining the ratio of supported links to unsupported ones. If \( \beta_{L,\text{info}} \) was very high, then it could be that there are many incidentally generated information triangles, and fewer links remain unsupported. By estimating a link and triangle SUGM, one can estimate the parameters and test this hypothesis, as we do in Section 4.2.

2.5. **Example 2: Matching Empirical Network Data.**

2.5.1. **SUGMs Match Non-Modeled Features of Observed Networks.** A challenge for network formation models has been to capture more than one or two observed features of social networks at a time.

For instance, many observed social networks are sparse but clustered, which motivates developing models that reflect this (Watts and Strogatz, 1998). They also have a variety of differing degree distributions ((Barabasi and Albert, 1999; Jackson and Rogers, 2007) and exhibit high levels of homophily (McPherson, Smith-Lovin, and Cook, 2001; Currrarini, Jackson, and Pin, 2009, 2010), which can lead to poverty traps and differences in employment between races (Calvo-Armengol and Jackson, 2007). There are also features such as the expansion properties of a network that are described by maximal eigenvalue of the adjacency matrix speaks to the speed of a diffusion process on the network (Bollobas (2001)). The depth of the max flow min cut speaks to several things such as consensus time in a social learning process Golub and Jackson (2012) as well as the degree of cooperation sustainable (Karlan, Mobius, Rosenblat, and Szeidl, 2009).

We show below that a simple links and triangles SUGM that only has four parameters (estimated from our data) captures a number of these features all within a simple model: average distance, the maximal eigenvalue, the cut (homophily), clustering, degrees, among other things) and does so better than a conditional edge independent model (a block model) with numerous parameters that can flexibly depend on a rich set of covariates even when allowing for unobserved heterogeneity for every node.

2.5.2. **Completing Network Data.** Network data can be expensive to collect, and so many data sets are only subsamples of a network. A simple way to proceed is to use such a sampled network to estimate the network formation parameters. This can then be applied to a predict the features that the network must exhibit on the full population.
As just mentioned, SUGMs are particularly well suited for this since they replicate many higher order features of network structure that are not captured by other methods.

2.6. Example 3: Links across Social Boundaries. Our final example shows how a SUGM can be used to investigate whether there are norms govern link-formation across different social groups. Identities can lead to strong social norms – prescriptions and proscriptions – concerning interactions across groups. For instance, in much of India there are strong forces that influence if and when individuals can form relationships across castes. Are people significantly more likely to form cross-caste relationships when those links are unsupported (without any friends in common) compared to when those links are supported with at least one friend in common (and thus have a witness to the relationship)? To answer this we need models that account for link dependencies, as cliques of three or more may exhibit greater adherence to a group norm prohibiting certain inter-caste relationships, while the norm may be circumvented in isolated bilateral relationships. We can test whether the relative frequency of triangles compared to links is higher when the relationships are all within caste than across caste.

Another example is discussed in Appendix D based on Banerjee, Chandrasekhar, Duflo, and Jackson (2016). We describe how our framework can be used to study how social network structure may change in response to exposure to a credit market.

2.7. Links and Triangles as Our Leading Example. The bulk of our discussion has focused on (and will continue to focus on) a variety of link and triangle SUGMs, though other subgraphs can be included and are covered by our general results. Our exposition focuses on links and triangles for two main reasons: first, this case is simple to understand and illustrates all of the main points since it exhibits correlated links and incidental generation, second, the link and triangle model already matches most all of the moments that a researcher is generally interested and thus should be sufficient for most research projects.

We leave any further specification of the particulars of a model to the researcher as it will depend on their context and the phenomenon being modeled. In practice if a key concern is modeling sparse networks that are clustered, a links and triangle model, generally with covariates, will do the heavy lifting. But if there are other the types of subgraphs that are hypothesized to arise in some particular context, then that model can be constructed and estimated in the ways outlined in our approach and some of our general results.
The selection of subgraphs in a data-driven manner may also be possible, but formalizing that is left as the subject of future research. For instance, one could have a list of subgraphs as a possible basis for the SUGM with only a sparse set of them actually forming the true SUGM; allowing the data to tell the researcher which to include.

3. Identification and Estimation

3.1. The Challenge. The researcher’s goal is to use the observed data to recover the parameters of interest, for example, the \((\beta_L, \beta_T)\) in a SUGM of links and triangles. If the researcher observed the links and triangles that were formed directly, then estimation would be straightforward. Indeed, in some instances a researcher might have information on all the various groups a given individual is involved in: for instance in the case of a co-authorship network, the researcher may observe all the papers a researcher has written and thus observes papers with two authors, three authors, and so forth. Instead, for instance, it may be that there are groups of three people who commonly share favors and risks together – who really form a triangle, but the researcher only has information from a survey asking which pairs of agents are ‘friends’ based a survey (as in networks derived from the Add Health data set as in Currairini et al. (2009)), or who borrows from whom and who lends kerosene and rice to whom and other bilateral nominations (as in our Indian village data Banerjee et al. (2013)), or from observing that they are friends on a social platform (as in Facebook network data as in Bailey et al. (2016)), or from observing that two people phone each other or remit payments to each other (as in many Phone data sets Blumenstock et al. (2011)).

Thus, the general problem is that the formation of the subgraphs is not directly observed, and so must be inferred in order to estimate the parameters of interest. From the perspective of the researcher, the observed network \(g\) is a projection \emph{directly generated} links and triangles. For example, if \(g_{ij}g_{jk}g_{ik} = 1\), is it the case that \(ijk\) formed as a triangle, or that \(ij, jk\) and \(ik\) formed as links, or that \(ij\) and \(jk\) formed as links and \(ik\) formed as part of a different triangle \(ikm\), or some combination of these or other combinations? Figure 3 provides an illustration.

This presents a challenge for estimating a parameter related to triangle formation since some of the observed triangles were “\emph{directly generated}” in the formation process, and others were “\emph{incidentally generated};” and similarly, it presents a challenge to estimating a parameter for link formation since some truly generated links end up as parts of triangles. It could also be that the link 12 formed directly or as part of the triangle.
A NETWORK FORMATION MODEL BASED ON SUBGRAPHS

Figure 3. The network that is formed and eventually observed is shown in panel D. The process comes from forming triangles independently with probability $\beta_T$ as in (B) in red; and also forming links, in grey, independently with probability $\beta_L$ as in (C). New links are dashed while links that overlap with some link also formed in a triangle are in solid and bold. We see that there is both (i) overlap as some links coincide with links already in triangles, as well as (ii) extra triangles that were generated ‘incidentally’. Given that we only observe the resulting network in panel D, we need to infer the formation of the different subgraphs carefully and not simply by directly counting observed links and triangles.

124 or both. This would not be observed either, and so we face several challenges in estimating the number of directly generated links.\textsuperscript{14}

Despite this challenge, the model parameters can generally be identified, as we prove below. For instance, as the probabilities of links and triangles vary so do the properties of the expected networks. To understand how, note that as one varies $(\beta_L, \beta_T)$, the relative rates of overall observed links and triangles change, as do the number of triangles that overlap with each other. One can calculate the relative rates at which incidental links and triangles are expected to be generated, and there is an invertible relationship between observed counts of links and triangles, and the underlying rates at which they

\textsuperscript{14}One could view this as an example of measurement error with correlation: which parts in the resulting observed graph are direct versus incidental is unobserved. The observed graph, which is a projection, and provides a count of observable subgraphs of various types, could be viewed as a mismeasurement of the list of subgraphs directly generated by the SUGM process.
were expected to be directly formed. Moreover, this can be done via estimators that are easily computed. In addition, as we show later, these have nice consistency as well as asymptotic normality properties despite the fact that the links are all correlated, under the appropriate additional relative frequency assumptions among subgraphs as described in Section 5. Thus, beyond the identification problem, we also prove a new central limit theorem for correlated random variables that could be correlated in ways that do not satisfy standard mixing assumptions used in time series or spatial econometrics.

3.2. Identification. To keep the discussion uncluttered suppose that \( p_\ell(\beta) = \beta_\ell \), and we consider SUGMs with a list of \( k \) subgraphs. We remind the reader that here a subgraph can be a configuration (e.g., links, triangles, stars) where nodes are marked by finite support covariates; and also covers multiplexing, as described before.

Let \( S(g) = (S_1(g), \ldots, S_K(g)) \) be a vector of statistics of the graph used to identify the parameters of interest, where \( K \geq k \).

In terms of estimation, writing out an analytic expression for the probability of observing \( S(g) \) as a function of \( \beta \) in the general case can be challenging (we offer these functions for links and triangles, and links and stars, below). One way to estimate the parameter is via a minimum distance estimator.

To estimate \( \beta \) define the objective function

\[
\hat{Q}(\beta) = (S(g) - E_\beta [S(g)]) (S(g) - E_\beta [S(g)])'
\]

and then estimate of the parameters, denoted \( \hat{\beta} \), solves the equation:

\[ (3.1) \quad E_{\hat{\beta}}[S(g)] = S(g). \]

The notation here suggests a setting in which a researcher observes a single large network, as is often the case. In some case, as we discuss in Sections 5.2 and 5, a researcher has \( R \) networks. In that case \( g = (g_1, \ldots, g_R) \) and \( S(g) = \frac{1}{R} \sum_r S(g_r) \) is the empirical average. Our argument for identification presented here does not depend on the data frame (single large network or many independent networks) because what is crucial is whether \( \beta \neq \beta' \) implies \( E_\beta [S(g)] \neq E_{\beta'} [S(g)] \).

For identification we need to show that

\[ \beta \neq \beta' \implies E_\beta [S(g)] \neq E_{\beta'} [S(g)]. \]

We now show that given a collection of subgraphs that comprise a SUGM, there is a natural set of statistics based on counting subgraphs that identifies the parameters.
In particular, order subgraph types so that the number of links\textsuperscript{15} a subgraph of type $\ell$ is nondecreasing in $\ell$. Let $S_{\ell}(g)$ be the fraction of subgraph of type $G_{\ell}$ that are present out of the total possible.

So, for instance, links are labeled with $\ell$s that come before triangles, which come before stars with four links, which come before cliques of four nodes, etc. If there are subgraphs with the same number of links (e.g., lines of three links, and triangles) it does not matter which of those two appears first. Similarly, if we have different types of triangles depending upon characteristics of the nodes (e.g., triangles with all blue nodes, triangles with two blue and one red node, etc.), then it does not matter which relative order those appear in.

The reason for this ordering is that is a logic to the way that incidental generation works: a single isolated link cannot be generated by any combination of larger subgraphs. Generally, no isolated subgraph of type $\ell$ can be generated by subgraphs of types appearing later in the sequence. This fact will be useful in proving some of the results below.

Let $N_{\ell}(g)$\textsubscript{sub-only} be the number of instances where there is a situation in which there could exist a particular instance of subnetwork $g_{\ell} \in G_{\ell}$, then in $g \cup g_{\ell}$ the instance $g_{\ell}$ could only have been generated by subgraphs of type $\ell' \leq \ell$. Let $S_{\ell}(g)$\textsubscript{sub-only} be the share of instances out of $N_{\ell}(g)$\textsubscript{sub-only} in which the subgraph $g_{\ell}$ is actually present in $g$.

For example, for links and triangles, $N_{L}(g)$\textsubscript{sub-only} is the number of pairs of nodes that have no common neighbors in $g$, and $S_{L}(g)$\textsubscript{sub-only} fraction of times that there is a link between those two nodes. For $\ell = k$, $S_{k}(g)$\textsubscript{sub-only} = $S_{k}(g)$, since there are no larger subnetworks.

Consider the \textit{"subgraph-only counts"}:

$$S_{\ell}(g) = (S_{1}(g)\textsubscript{sub-only}, ..., S_{k-1}(g)\textsubscript{sub-only}, S_{k}(g)\textsubscript{sub-only}),$$

where for $k$ note that $S_{k}(g)$\textsubscript{sub-only} = $S_{k}(g)$.

In some realized networks, $N_{k}(g)$\textsubscript{sub-only} = 0 – for instance when a network is nearly complete, and then $S_{k}(g)$\textsubscript{sub-only} = 0 is not well defined. Generally observed networks are far from being complete, and $N_{k}(g)$\textsubscript{sub-only} > 0 for each $k$, denoted $N(g)$\textsubscript{sub-only} > 0. We condition on this for the identification results on sub-only counts.

To run the minimum distance estimation with sub-only counts, then one needs to work with the conditional probability space, $E \left[\mid N(g)\textsubscript{sub-only} > 0\right]$, which works fine as we show in the proof of Proposition 3 and will fit most applications.

\textsuperscript{15}In the case of multiplexing, count the total number of relationships present in the subgraph.
Proposition 1. [Identification via Subgraph Counts] Every SUGM is identified. That is, for any collection of subgraphs \((G_\ell)_{\ell \in \{1, \ldots, k\}}\) on \(n\) nodes, the vector of subgraph-only counts, \(S(g)_{\text{sub-only}}\), satisfies:

\[
\beta \neq \beta' \implies E_\beta \left[ S(g)_{\text{sub-only}} | N(g)_{\text{sub-only}} > 0 \right] \neq E_{\beta'} \left[ S(g)_{\text{sub-only}} | N(g)_{\text{sub-only}} > 0 \right].
\]

Recalling the general definition of the SUGM, note that this means that for any SUGM, even one comprised of subgraphs that could have nodes with varying (discrete) covariates and allowing for multiplexing, we have identification.

We now elaborate on two pedagogical examples, to make this result clear and to provide some other identification techniques that can also be useful. The first example concerns links and triangles. We look at this model in three ways: without covariates, with covariates, and with multiplexing (where edges are not binary but can take on assorted values). The second example is a model of links and Z-stars.

3.3. Links and Triangles.

3.3.1. Without Covariates. We begin with the simplest case where there are no covariates. It is useful to define by \(UP(g)\) the set of pairs of nodes that have no common neighbors in \(g\) (\(UP\) for “unsupported pairs of nodes”).

Here

\[
S_L(g)_{\text{sub-only}} := \frac{\sum_{i < j, (ij) \notin UP(g)} g_{ij}}{|UP(g)|},
\]

as the share of links in the subnetwork excluding all existing triangles, and

\[
S_T(g)_{\text{sub-only}} = S_T(g) = \frac{\sum_{i < j < k} g_{ij}g_{jk}g_{ik}}{\binom{n}{3}}.
\]

For the case of links and triangles, note that the expectation of \(S_L(g)_{\text{sub-only}}\) is the expected fraction of links present among pairs of nodes that do not have any common neighbors. This is \(\beta_L\), and so identifies that parameter. Thus, we identify the link parameter from this statistic. If the link parameter has not changed, then only the triangle parameter has changed and then it follows directly that the expectation of \(S_T(g)\) must change. The proof for the general case is a direct extension of this logic.

Corollary 1. A SUGM of links and triangles is identified with moments \(S(g) = (S_L(g)_{\text{sub-only}}, S_T(g))\) for any \(\beta = (\beta_L, \beta_T) \in (0, 1)^2\). That is, if \((\beta'_L, \beta'_T) \neq (\beta_L, \beta_T)\) then \(E_{\beta'} \left[ S(g) | N(g)_{\text{sub-only}} > 0 \right] \neq E_\beta \left[ S(g) | N(g)_{\text{sub-only}} > 0 \right].\)
3.3.2. Links and Triangles With Covariates. We describe how identification works with
covariates, again taking the support of \( X \) to be finite.

In this case, for every pair of node types \( a, b \) note that links cannot be generated by
other links. Thus, all of the sub only counts are direct to calculate for links:

\[
S_{L,ab}(g)^{\text{sub-only}} = \frac{\sum_{i<j,(X_i,X_j)\in\{(a,b),(b,a)\}} g_{ij} \left(1 - \max_{h\neq i,j} [g_{ih}g_{jh}]\right)}{\sum_{i<j,(X_i,X_j)\in\{(a,b),(b,a)\}} \left(1 - \max_{h\neq i,j} [g_{ih}g_{jh}]\right)}.
\]

The expression \( 1 - \max_{h\neq i,j} [g_{ih}g_{jh}] \) is 1 if and only if the pair of nodes have no common
neighbors, and so if there were to be a link between \( i, j \) it could only come directly.

Triangles can be incidentally generated by other triangles, so here we have to be exact
in following the sub-only definitions for triangles. To illustrate the definitions, let us
consider the case in which \( X_i \in \{\text{blue},\text{red}\} \). So, the triangles are \( (\text{blue,blue,blue}) \),
\( (\text{blue,blue,red}) \), \( (\text{blue,red,red}) \), \( (\text{red,red,red}) \), and let us follow this order in ordering
our triangles. Then,

\[
S_{T,(\text{blue,blue,blue})}(g)^{\text{sub-only}} = \frac{\sum_{i<j<k,X_i,X_j,X_k=\text{blue}} g_{ij}g_{jk}g_{ik} \left(1 - \max_{h\neq i,j,k} [g_{ih}g_{jh}g_{kh}]\right)}{\sum_{i<j<k,X_i,X_j,X_k=\text{blue}} \left(1 - \max_{h\neq i,j,k} [g_{ih}g_{jh}g_{kh}]\right)}.
\]

So, for the all-blue triangles, our base set of nodes on which we count our share of
triangles that are present is out of triples of blues which if connected could only have
been generated by all-blue links or triangles. In this case, that means that no pair of
the three blues can have a red neighbor in common (here noting that the only incidental
generation involving other types of triangles would have to be a \( (\text{blue,blue,red}) \) triangle
generating one of the blue-blue edges in the triangle here).

Next, then for the \( (\text{blue,blue,red}) \) case, the triples of nodes that we consider are
those for neither of the blue,red pairs has a red neighbor in common. But now it is
acceptable for the two blues to have a red neighbor in common since that is the case in
consideration; and similarly it is acceptable for the blue,red pair to have a blue neighbor
in common for the same reason.

Then for the \( (\text{blue,red,red}) \) case, we can consider all triples of such nodes for which
the two reds don’t have a red neighbor in common.

Finally, for the all \( \text{red} \) case, we consider all such triples since that is the last subgraph
in our sequence.

**Corollary 2.** A SUGM of links and triangles with covariates is identified by moments

\[
S(g)^{\text{sub-only}} = \left(S_{L,ab}(g)^{\text{sub-only}}, S_{T,abc}(g)^{\text{sub-only}}\right)_{a\leq b\leq c}
\]

for any \( \beta = \left((\beta_{L,ab})_{a\leq b\leq c}, (\beta_{T,abc})_{a\leq b\leq c}\right) \).

That is, if \( \beta \neq \beta' \) then \( E^{g'}\left[ S(g)^{\text{sub-only}} | N(g)^{\text{sub-only}} > 0 \right] \neq E^{\beta}\left[ S(g)^{\text{sub-only}} | N(g)^{\text{sub-only}} > 0 \right] \).
3.3.3. Multiplexing. In this example we show that the same results from above hold even if the network is a multigraph. For example, if households can have information-sharing links, risk-sharing links, and/or social links, there could be 8 different sorts of observations that could occur between two nodes, as they might have no relation, exactly one relation (three different types), exactly two relations (three different ways), or all three relations.

In saving notation, let us ignore triangles and consider a situation in which we have info and favor relationships. So, $ij$ could have an info link, a favor link, neither, or both. Here, our $g$ is extended to track two networks $g^{\text{info}}$ and $g^{\text{favor}}$.

There are many reasons to believe that combinations of information and favor links would not be independent, and so there are three parameters of interest $\beta_{\text{info}}, \beta_{\text{favor}}$ and $\beta_{\text{info-favor}}$, where it could be that info and favor links happen to form independently, or they might be generated as a pair. So, even without triangles, this already has correlated observations and incidental generation.

Nonetheless, we can still think of this in terms of subgraphs. Here, the count statistics are easy to see. One would count info links that have no favor links out of pairs of nodes that have no favor links, and one would do the same in reverse, and then count the overall fraction in which both are present.

![Figure 4](image-url)

**Figure 4.** Panel (A) shows all possible links and Panel (B) shows all possible triangles when a edges can either be blue, red or both (double line, purple).
Extending this to triangles is then straightforward. One would first examine links in the order we described. Next, one would examine triangles consisting of just one relationship on each edge, then triangles in which some edge has two relationships, and so forth growing upward in the number of links.

For example, in terms of Figure 4, one such ordering would start at the bottom left and proceed to the right, then go up and order triangles in the each row from left to right and then keep going upward. At each step, the counting would follow exactly the procedures described above for links and then similarly in defining triples of nodes for which if a triangle in question were present it could not have been generated incidentally by an set of subgraphs including a triangle further to the right or above it in the figure.

**Corollary 3.** A SUGM of links and triangles with multiplexing is identified by moments $S(g)^{\text{sub-only}}$. That is, if $\beta \neq \beta'$ then

$$E_{\beta'} [S(g)^{\text{sub-only}} | N(g)^{\text{sub-only}} > 0] \neq E_{\beta} [S(g)^{\text{sub-only}} | N(g)^{\text{sub-only}} > 0].$$

**3.3.4. Identification With Straight Link and Triangle Counts.** For a links and triangles SUGM, identification can be also be achieved using only $S(g) = (S_L(g), S_T(g))$ instead of the sub-only counts, which is slightly easier to code for use in empirical work (and the moment $S_L(g)$ will generally have more observations than $S_L(g)^{\text{sub-only}}$). We now show this. This result on identification is much harder to prove and does not directly generalize.

To understand the source of identification here, consider Figure 5. Each configuration involves two triangles, but the graph in Panel B with only five links is relatively more easily incidentally formed than the one in Panel A. Thus, by looking at the combination of how many triangles and how likely links there are, we can sort out relative rates of the two parameters.

**Proposition 2.** A SUGM of links and triangles is identified with moments $S(g) = (S_L(g), S_T(g))$ for any $\beta = (\beta_L, \beta_T) \in (0,1)^2$. That is, if $(\beta'_L, \beta'_T) \neq (\beta_L, \beta_T)$ then $E_{\beta'} [S(g)] \neq E_{\beta} [S(g)]$.

Let us outline the basic ideas behind the proof, with the full proof appearing in the appendix.

Let $\bar{q}_L$ denote the probability that a link forms conditional upon exactly one particular triangle that it could be a part of not forming. For instance, for nodes $ij$ it is the
probability that \( ij \) is formed either as a link or as part of a triangle that is not triangle \( hij \) for some other node \( h \). In this case:

\[
E_{\beta_L,\beta_T} [S_L(g), S_T(g)] = [\beta_T + (1 - \beta_T)\tilde{q}_L, \beta_T + (1 - \beta_T)(\tilde{q}_L)^3].
\]

For instance, note that the term \( \beta_T + (1 - \beta_T)(\tilde{q}_L)^3 \) is the probability that a triangle forms, either directly \( \beta_T \), or does not form directly \( 1 - \beta_T \) but then each of the links then forms on its own \( (\tilde{q}_L)^3 \).\(^{16}\) The term for the links is similar as it could form if some particular triangle forms, or else if that triangle does not form then it forms with probability \( \tilde{q}_L \). Although there are more direct ways to write the probability of a link forming, this particular expression is useful in the proof since it is easy to compare it to and this distinguish it from the triangle expression, as they are identical except for the exponent. This is very helpful in showing how different parameters lead to different rates of formation of links and triangles since we can isolate the difference via the \( \tilde{q}_L \) versus \( (\tilde{q}_L)^3 \) expressions.

Analogs of this proposition extend to cases with covariates and multiplexing, simply with more complicated extensions of (3.2) accounting for the specific types of triangles or links needed to incidentally generate any given link or part of a triangle.

### 3.4. Links and Stars

Our second example is a model where a link forms with probability \( \beta_L \) and a star (with some number of links \( z > 1 \)) with probability \( \beta_{\text{Star}} \). We consider the case without covariates or multiplexing for simplicity.

Here, when we count links in the sub-only, we have to count them out of all pairs of nodes that if they did have a link would not be involved in a star. Thus, these are pairs of nodes \( ij \) for which both \( i \) and \( j \) have degree not counting \( g_{ij} \) that is less than \( z - 1 \).

\(^{16}\)Conditional upon the triangle not forming directly, the links are then independent.
Let $d_i(g \setminus j) = \sum_{h \neq j, i} g_{ih}$ be the degree of $i$ in $g$ excluding node $j$. So,

$$S_L(g)_{sub-only} = \frac{\sum_{i<j, d_i(g \setminus j) < z-1, d_i(g \setminus j) < z-1} g_{ij}}{\sum_{i<j, d_i(g \setminus j) < z-1, d_i(g \setminus j) < z-1} 1},$$

and $S_{star}(g)_{sub-only} = S_{star}(g) = \frac{\sum_{i:d_i(g) \geq z} \binom{d_i(g)}{z}}{n^{n-1}}$ as the share of $z$-stars in the network.

**Corollary 4.** A SUGM of links and $z$-stars is identified by $S(g)_{sub-only} = (S_L(g)_{sub-only}, S_{star}(g))$. That is, if $\beta' \neq \beta$, then $E_{\beta'} \left[ S_{sub-only}(g) | N(g)^{sub-only}_L > 0 \right] \neq E_{\beta} \left[ S_{sub-only}(g) | N(g)^{sub-only}_L > 0 \right]$. 

![Observed Network and 5-stars highlighted in red, centers denoted](image)

**Figure 6.** An example of a link and 5-star model on $n = 11$ nodes. Here there are two 5-stars in the network, and either or both could have been generated incidentally by links, and also they generate incidental links. The link parameter is identified from the links not in the stars (the black nodes in panel (B)) counted relative to pairs of nodes which if connected would still not be part of a star. Once the link parameter is identified, then the star parameter is identified by the share of stars present, at that expected share is a function of the link and star parameters with an already identified link parameter.

There are also other moments that can be used to identify parameters from a links and $z$-stars SUGM as we discuss in Appendix E.

4. Applications

We now apply our model to study the four examples from Section 2 to illustrate the kinds of questions SUGMs can be used to addressed. Though slightly unorthodox, we defer the presentation of the asymptotic framework and the demonstration that the estimator for the parameters is consistent and asymptotically normally distributed in Section 5, after showcasing several applications of SUGMs.

4.1. Data. We use the Banerjee, Chandrasekhar, Duflo, and Jackson (2013, 2014) data consisting of a variety of social and economic networks from 75 Indian villages as well as detailed demographic background.\(^{17}\) Having 75 villages worth of data allows us to show

\(^{17}\)See Banerjee, Chandrasekhar, Duflo, and Jackson (2013) for more information about the data.
not only how the model scales with the number of nodes, but also with the number of networks observed.

The networks have households as nodes. There are an average of 220 households per village. We surveyed adults, asking them about a variety of their daily interactions, as well as their demographics (caste, education, profession, religion, family size, wealth variables, voting and ration cards, self-help group participation, savings behavior, etc.). We have network data from 89.14 percent of the 16,476 households based on interviews with 65 percent of all adults between the ages of 18 and 55. As we study the undirected, unweighted network described below, this means that we observe 98.8% of the potential links between pairs.\textsuperscript{18} We have data concerning twelve types of interactions: (1) whose houses he or she visits, (2) who visits his or her house, (3) his or her relatives in the village, (4) non-relatives who socialize with him or her, (5) who gives him or her medical help, (6) from whom he or she borrows money, (7) to whom he or she lends money, (8) from whom he or she borrows material goods (e.g., kerosene, rice), (9) to whom he or she lends material goods, (10) from whom he or she gets important advice, (11) to whom he or she gives advice, (12) with whom he or she goes to pray (e.g., at a temple, church or mosque).

The answers are aggregated to the household level, but one can also work with the individual-level networks to get very similar results as those presented below. How a link is defined varies based on the application. We use undirected,\textsuperscript{19} unweighted networks that may allow for multiplexing.

For much of what follows, we work with the borrowing and lending of material goods (questions 8 and 9, with any positive answer indicating a link being present) that we call “favor” links, and the exchange of advice (questions 10 and 11, with any positive answer indicating a link being present) that we call “info” links.

4.2. Example 1 (cont.): Do incentives for risk sharing drive network formation? Continuing Example 1 from Section 2.4, we test whether supported relationships are significantly more likely to appear in favor exchange than informational links. The (joint) hypothesis that we are testing is that exchanging material goods is more costly

\textsuperscript{18}This is a new wave of data relative to our original microfinance study that includes more surveys. Note that $1 - (1 - 0.8914)^2 = 0.988$.

\textsuperscript{19}Some links are not reciprocated, but that is true at similar rates for the questions regarding relatives as compared to the other questions, and so much of the failure of reciprocation may simply be measurement error rather than true one-way relationships. For our purposes here, which are purely to illustrate the ability of the models to work with data, this distinction is inconsequential.
and/or happens less frequently for agents, and so requires more incentives and supporting enforcement than exchanging information which is less costly and/or more frequent.

To keep the illustration in this first example clear, we abstract from covariates. We illustrate the incorporation of covariates in the examples below.

Thus, from Section 2.4, we know that $p_{T,favor} = F(\beta_{T,favor})^2$ and $p_{L,favor} = F(\beta_{L,favor})^2$, and similarly for information. Without loss of generality, we renormalize things so that $F(\beta) = \beta$ in each case, as the precise scaling of the parameters is irrelevant to the overall question. Thus, the test of whether $\frac{\beta_{T,favor}}{\beta_{L,favor}} > \frac{\beta_{T,info}}{\beta_{L,info}}$ corresponds to $\frac{p_{T,favor}/p_{L,favor}^{3/2}}{p_{T,info}/p_{L,info}^{3/2}} > 1$.

This test takes into account that there are more consents for a group than a pair (the 3/2), and is also robust to information links simply being more or less valuable, as it adjusts by relative link prevalence.

Note that the $p_T$ and $p_L$ are not the realized frequencies of links and triangles, as those involve incidentally formed instances, so we need to work from our identification theorem to infer these probabilities.

| Table 1. Parameter estimates by network type |
|---------------------------------------------|
|         | Information | Favors |
| $\hat{p}_L$ | 0.0123      | 0.0088 |
| (0.0005) | (0.00003)   | (0.00002) |
| $\hat{p}_T$ | 0.0002      | 0.0002 |
| (0.00002) | (0.00002)   | (0.00002) |

Notes: Standard errors computed by nonparametric bootstrapping with replacement 75 networks.

We estimate the four parameters in question under the assumption that all villages in our sample are independent network generated from the same common parameter values. Table 1 presents the parameter estimates and standard errors.

We reject the hypothesis that there is no difference in the support of favor relationships compared to information relationships ($p = 0.0146$).20 We conclude that the data are consistent with the theory that incentives for favor exchange matters in network formation in these data.

While in the above we have taken all villages to be generated from a common parameter, we can also push this further by estimating $\hat{p}_L$ and $\hat{p}_T$ separately for each village, allowing for underlying heterogeneity in the parameters across villages, but with large

---

20Specifically, the $p$-value is computed for a test of the null hypothesis $p_{T,favor}/p_{T,info}^{3/2} = p_{L,favor}/p_{L,info}^{3/2}$, where the parameters are held to be common across all villages in the sample. See Section 5.1 for a justification.
numbers of nodes, we can consistently estimate the parameters per network. We see the results in Figure 7, though standard errors are omitted for visual clarity. We see that for most villages, the favor over info ratios are higher for triangles compared to links.

4.3. Example 2 (cont.): Matching Features of Empirical Network Data. Revisiting the example from Section 2.5, we compare a standard ‘stochastic block’ model that estimates linking probabilities based on node characteristics – here caste and geography – to a SUGM based on links and triangles. We also compare SUGMs to an extension of this stochastic block model which includes the rich set of covariates but in addition adds node-level unobserved heterogeneity parameters.

The idea is to compare how well each of these models replicates various features of empirically observed networks, including many characteristics that are not directly in the model such as clustering, the size of the giant component, average path length, and various eigenvalue properties of the adjacency matrices (the largest eigenvalue, and an eigenvalue measure of homophily.

Beyond average degree and clustering (which turn out to be well-captured by links and triangles), we are interested whether a very basic SUGM does a good job of replicating observed networks in terms of characteristics other than those that involve link and triangle counts. We look at the first eigenvalue of the adjacency matrix, which is a measure of diffusiveness of a network under a percolation process (e.g., Bollobás, Borgs, Chayes, and Riordan (2010); Jackson (2008)). This is intimately related to the expansiveness of
the network – namely, for any subset of nodes the number of links leaving the subset relative to the number of links within the subset. We are also interested in the second eigenvalue of the stochasticized adjacency matrix.\textsuperscript{21} This is a quantity that is key in local average learning processes and modulates the time to consensus (DeMarzo, Vayanos, and Zwiebel (2003); Golub and Jackson (2012)), but is also closely related to homophily (Golub and Jackson (2012)) and is labeled as such in the table below. Additionally, we look at the fraction of nodes that belong to the giant component of the network, as well as the number of isolates, as empirical networks are often not completely connected. Finally, we also consider average path length (in the largest component).

Again, we present the results for favor and info networks. These networks are reasonably connected (with more than ninety percent of the nodes being in a giant component) and yet also reasonably sparse for small networks.

Our procedure is as follows. For every village, we estimate four network formation models. One network formation model is a link-based model (stochastic block model) in which the probabilities can depend on geographic distance, caste, the number of rooms households have, number of beds, quality of electricity provision, quality of latrines, household ownership status, and squared differences in non-binary variables. The probabilities are estimated using logistic regression and the model has 12 parameters. The next is the model of Graham (2017). This is the same formulation of the preceding model, but adds unobserved heterogeneity in the form of node-fixed effects:

\[
P(g_{ij} = 1 | X_{ij}) = \Lambda (\alpha_i + \alpha_j + \beta' X_{ij})
\]

where \( \Lambda (\cdot) \) is the logit link function and \( X_{ij} \) is the aforementioned vector of demographic characteristics and polynomials therein. This model has \( n+12 \) parameters per network.\textsuperscript{22}

The other models are very low-dimension SUGMs. One is the basic SUGM with links and triangles. Pairs of household are categorized as either being “close” or “far,” where “close” refers to pairs of nodes that are of the same caste and “far” to those that differ in caste. Similarly, we categorize triangles as being “close” if all nodes are of the same caste and “far” otherwise. Thus, we allow for four parameters, close and far link parameters and close and far triangle parameters. The other model is a slightly richer SUGM in which we allow some nodes to be isolates, meaning there are five parameters. Neither includes any other demographic covariates nor unobserved heterogeneity.

\textsuperscript{21}The stochasticized adjacency matrix \( T \) is defined as \( T_{ij} = \frac{g_{ij}}{\sum_k g_{ik}} \), where either \( g_{ii} = 1 \), or \( g_{ik} > 0 \) for some \( k \neq i \), as this captures the set of people to whom \( i \) listens.

\textsuperscript{22}Consistency of all \( \alpha_i \) in addition to \( \beta \) has been proven for a dense sequence of graphs (e.g., Graham (2017)).
To make the strongest point, we compare these very stark SUGMs that use only caste variables to account for homophily, to very rich covariate dependent (block) models that can incorporate a large set of covariates – including much richer demographics that are usually available to a researcher as well as node-level fixed effects. We show that even though we have considerably more information on the nodes, such as geographic distance and demographic characteristics, and we do not make use of this information for the SUGMs they recreate networks much more accurately than a link-based model that does take advantage of a rich set of node characteristics. Adding over 12 parameters to the block model to flexibly control for demographic attributes, or even \( n+12 \) parameters with unobserved heterogeneity, does not come close to doing as well as the simple SUGMs. Moreover, since the specification developed here makes use of considerably richer data than those used in the two candidate SUGM models, it suggests that by decomposing a network into a tapestry of random structures (triangles, links, and even isolates), considerable value is added in modeling higher order features of networks in a parsimonious way.

We estimate parameters for the village network for each model and then generate random network from each model based on the estimated parameters. We do 100 such simulations for each of the 75 village and for each of the models. We then compare the true network characteristics with those from the simulations.

Table 2 presents the results and Figure 8 presents the estimated network statistics under the various models as well as the true values, village-by-village. Both of the SUGMs match the various features of the networks substantially better than the conditional edge independent models (with and without node fixed effects). Including isolates in the SUGM further improves the fits not only for isolates, but also for fraction in the giant component and the maximum eigenvalue. This suggests that there are more isolated households in a village for a reason outside of randomness in network formation.

The most obvious thing to note is that the link-based models do extremely poorly when it comes to matching clustering while the SUGM does much better, and here adding unobserved dimensions to generate unconditional link correlations (e.g., clustering) does worse than a SUGM that allows correlated link formation directly. More interestingly, conditioning on the triangles in the SUGM is enough to deliver better matches on all of the other dimensions, and the difference on homophily is perhaps most interesting, since one would imagine that the block models could get that right given that they include many covariates. This tells us that triangles and correlation between links play a subtle but important role in homophily – something that is better picked up by a SUGM
Table 2. Network Properties

|                  | Truth | Links/Triangles SUGM | Links/Tri/Isolates SUGM | Covariates (Block Model) | Covariates + Unobserved Heterogeneity (Latent Block Model) |
|------------------|-------|---------------------|------------------------|-------------------------|----------------------------------------------------------|
|                  | (1)   | (2)                 | (3)                    | (4)                     | (5)                                                      |
| Degree           | 8.096 | 8.076               | 8.042                  | 8.815                   | 9.621                                                    |
|                  | (0.261) | (0.263)             | (0.255)                | (0.311)                 | (0.354)                                                  |
| Clustering       | 0.220 | 0.159               | 0.147                  | 0.051                   | 0.075                                                    |
|                  | (0.006) | (0.003)             | (0.003)                | (0.003)                 | (0.005)                                                  |
| Isolates         | 10.972 | 3.503               | 13.787                 | 0.499                   | 0.873                                                    |
|                  | (0.841) | (0.408)             | (0.998)                | (0.092)                 | (0.157)                                                  |
| % in Giant       | 0.950 | 0.984               | 0.938                  | 0.998                   | 0.996                                                    |
| Maximal Eigenvalue | 11.914 | 10.453             | 10.816                 | 10.374                  | 12.583                                                   |
|                  | (0.374) | (0.301)             | (0.300)                | (0.321)                 | (0.430)                                                  |
| Homophily        | 0.887 | 0.815               | 0.804                  | 0.686                   | 0.680                                                    |
|                  | (0.007) | (0.009)             | (0.009)                | (0.010)                 | (0.010)                                                  |
| Average Path Length | 3.027 | 2.957               | 2.871                  | 2.758                   | 2.641                                                    |
|                  | (0.048) | (0.043)             | (0.040)                | (0.040)                 | (0.037)                                                  |

Notes: Average value of various network statistics for the information and favor networks across 75 villages are shown in Column 1. Columns 2-5 present the average values across the 75 villages with 100 simulations per village generated from the estimated parameter value for each model specified. Column 2 presents links and triangles in a four-parameter SUGM with covariates where links and triangle probability can vary by binary classification: whether the pairs or triples are "close" or "far", as described in the text. Column 3 adds isolates to the previous SUGM to constitute a five-parameter SUGM. Column 4 consists of a twelve-parameter conditional edge independent model that includes flexible controls for continuous covariates, as described in the text. Column 5 adds to this unobserved heterogeneity, by including fixed effects for each of node $i$ and node $j$, when determining whether the pair $ij$ are linked, which adds $n$-more parameters per network of size $n$. Standard errors for the means in parentheses.

than an independent link model even when that model includes rich demographics and unobserved heterogeneity.

That SUGMs do a much better job at recreating a multitude of features of observed network structures that standard link-based models, especially with rich demographic information, is important, and suggests that there is substantial value added of modeling the formation of triangles and isolates. Knowing that our model is better able to capture the realistic correlation of links within observed networks should make us more confident in trusting the results of some other empirical applications. For example, when we look at links across social boundaries, we can be comfortable that to a first order, thinking about a SUGM with links and triangles across and within caste groups can do a good job of matching patterns in the data, and thus tracing them back to model parameters.
Figure 8. Each panel shows the expected values of six network statistics question generated across 100 simulations at the estimated parameter value for each of the 75 village networks. Panel A presents results for the information network and panel B for the favors network.
4.4. Example 3 (cont.): Links across Social Boundaries. Our final example looks at the propensities to link across caste. Individuals have identities that can lead to strong social norms – prescriptions and proscriptions – about interactions across groups. For instance, in much of India there are strong forces that influence if and when individuals form relationships across castes. Are people significantly more likely to form cross-caste relationships when those links are unsupported (without any friends in common and therefore the relationship is less public) compared to when those links are supported with at least one friend in common? To answer this we need models that account for link dependencies, as cliques of three or more may dictate greater adherence to a group norm prohibiting certain inter-caste relationships, while the norm may be circumvented in bilateral relationships.

We link two households if members of either engaged in favor exchange with each other: that is, they borrowed or lent goods such as kerosene, rice or oil in times of need. We work with two caste categories: the first consists of people in scheduled castes and scheduled tribes and the second consists of those people in any other caste (Munshi and Rosenzweig, 2006). Scheduled castes and scheduled tribes are those defined by the Indian government as being disadvantaged. This is a fundamental distinction over which the strongest cultural forces are likely to focus. Additional norms are at work with finer caste or subcaste distinctions, but those norms are more varied depending on the particular castes in question while this provides a clear barrier.

As a simple model to address this issue, consider a process in which individuals may meet in pairs or triples and then decide whether to form a given link or triangle. The link is formed if and only if both individuals prefer to form the link, and a triangle is formed if and only if all three individuals prefer to form it. This minimally complicates an independent-link model enough to require modeling link interdependencies.

In particular, there are probabilities, denoted $\pi_L(diff), \pi_L(same)$, that a given link has an opportunity to form (i.e., the pair meets and can choose to form the relationship) that depend on the pair of individuals being of different castes or of the same caste, respectively. Similarly, there are probabilities, denoted $\pi_T(diff), \pi_T(same)$, that a given triangle has an opportunity to form (that the three people involved meet and can choose to form the relationship) that depend on the triple of individuals being of all the same castes or two of the same and one of a different caste.
As noted above, individual \( i \)'s utility of having a relationship with \( j \) can be influenced by whether they share caste and is given by

\[
    u_i(ij) = \alpha_{0,L} + \beta_{0,L} \text{SameCaste}_{ij} + \delta_{0,L} X'_{ij} - \epsilon_{L,ij},
\]

where \( \text{SameCaste}_{ij} \) is a dummy for whether both individuals are members of the same caste, \( X_{ij} \) is a vector of covariates depending on \( X_i \) and \( X_j \) (saturated and discrete, including same-caste interactions). For expositional simplicity here, we set \( \delta_L = 0 \). The outside option is normalized to zero, so \( P_L(\text{same}) \) is the probability that an individual desires to form a link with an individual of the same caste group, and \( P_L(\text{diff}) \) is the probability that an individual desires to form a link with an individual of a different caste group.

The crucial point is that \( i \) can have returns that depend on being in a multilateral relationship with \( j \) and \( k \) – that is conceptually distinct from having these two bilateral relationships – and this can be given by

\[
    u_i(ijk) = \alpha_{0,T} + \beta_{0,T} \text{SameCaste}_{ijk} + \delta_{0,T} X'_{ijk} - \epsilon_{T,i,jk},
\]

where \( \text{SameCaste}_{ijk} \) is a dummy for whether all three individuals are members of the same caste, \( X_{ijk} \) is a vector of covariates depending on \( X_i \), \( X_j \), and \( X_k \). Again for expositional simplicity, we set \( \delta_{0,T} = 0 \). Correspondingly, \( P_T(\text{same}) \) is the probability that an individual desires to form a triangle when all individuals are of the same caste group, and \( P_T(\text{diff}) \) is the probability that an individual desires to form a triangle when it consists of people from both caste groups.\(^{23}\)

The hypothesis that we explore is that \( P_T(\text{diff})/P_T(\text{same}) < P_L(\text{diff})/P_L(\text{same}) \) so that people are more reluctant to involve themselves in cross-caste relationships when those are “public” in the sense that other individuals observe those relationships; with a null hypothesis that they are equal \( P_T(\text{diff})/P_T(\text{same}) = P_L(\text{diff})/P_L(\text{same}) \).

Note that the probability that a “same” link forms is

\[
    p_L(\text{same}) = P_L(\text{same})^2 \pi_L(\text{same})
\]

as it requires both agents to agree, and the probability that a “different” link forms is

\[
    p_L(\text{diff}) = P_L(\text{diff})^2 \pi_L(\text{diff}).
\]

\(^{23}\)This is a simplified model for illustration, but one can clearly consider preferences conditional on any string of covariates. This extends a model such as that of Curcurarini, Jackson, and Pin (2009, 2010) to allow for additional link dependencies. We could also be interested in higher order relationships.
Analogously for triangles we have

\[ p_T(\text{same}) = P_T(\text{same})^3 \pi_T(\text{same}) \quad \text{and} \quad p_T(\text{diff}) = P_T(\text{diff})^3 \pi_T(\text{diff}), \]

where the cubic captures the fact that it takes three agreements to form the triangle. The difference in the exponents reflects that it is more difficult to get a triangle to form than a link. Hence, to perform a proper test, we have to adjust for the exponents as otherwise we would just uncover a natural bias due to the exponent that would end up favoring cross-caste links.

One challenge in identifying a preference bias is that it could be confounded by the meeting bias. Thus, we first model the meeting process more explicitly and show that we still have identification as the meeting bias makes triangles relatively more likely to be cross-caste than links. Thus, our test is conservative in the sense that if we find cross-caste links relatively more likely, that is evidence for a (strong) preference bias.

Consider a meeting process where people spend a fraction \( f \) of their time mixing in the community that is predominantly of their own types and a fraction \( 1 - f \) of their time mixing in the other caste’s community. Then at any given snapshot in time, a community would have \( f \) of its own types present and \( 1 - f \) of the other type present, as depicted in Figure 9. (Variations on this sort of biased meeting process appear in Currarini, Jackson, and Pin (2009, 2010); Bramoullé, Currarini, Jackson, Pin, and Rogers (2012).)

\[ \text{Community A} \quad \text{Community B} \]

\[ \text{(A) Individuals all on own-community side of river} \]

\[ \text{Community A} \quad \text{Community B} \]

\[ \text{(B) Fraction} \quad \frac{f}{1} = \frac{1}{4} \text{ mixed across communities} \]

**Figure 9.** Geographically driven meeting process where agents spend 3/4 of their time in their own community.

**Lemma 1.** A sufficient condition for \( \frac{P_T(\text{diff})}{P_T(\text{same})} < \frac{P_L(\text{diff})}{P_L(\text{same})} \) is that \( \frac{p_T(\text{diff})}{p_T(\text{same})} < \left( \frac{p_L(\text{diff})}{p_L(\text{same})} \right)^{3/2} \).

The proof appears in Appendix A, but follows from straightforward calculations.
Given Lemma 1, we can test our hypothesis directly from a SUGM that compares relative link and triangle counts (we can also include isolated nodes, but those do not impact this hypothesis). In particular, we only need examine whether \( \frac{p_T(\text{diff})}{p_T(\text{same})} < \left( \frac{p_L(\text{diff})}{p_L(\text{same})} \right)^{3/2} \).

**Table 3.** Parameter estimates by network type

|          | \( \hat{p}_L(\text{same}) \) | \( \hat{p}_T(\text{same}) \) | \( \hat{p}_L(\text{diff}) \) | \( \hat{p}_T(\text{diff}) \) |
|----------|-----------------|-----------------|-----------------|-----------------|
| Information | 0.0167 | 0.0004 | 0.0064 | 0.00003 |
| (0.0007) | (0.00004) | (0.0004) | (0.00000) |
| Favors | 0.0125 | 0.0004 | 0.0042 | 0.00002 |
| (0.0005) | (0.00005) | (0.0003) | (0.00001) |

Notes: Standard errors computed by nonparametric bootstrapping with replacement 75 networks.

Table 3 presents the parameter estimates, again where we assume that all 75 networks are independent draws from the same distribution, and a formal test rejects the null \( \frac{p_T(\text{diff})}{p_T(\text{same})} = \left( \frac{p_L(\text{diff})}{p_L(\text{same})} \right)^{3/2} \) with \( p < 0.001 \) for each network type.\(^{24}\)

Finally, figure 10 shows the results when we allow the parameter estimates to vary by village. For the bulk of villages, cross-caste relationships relative to within-caste relationships are more frequent as isolated links compared to being embedded in triangles, for both information and favor networks.

\(^{24}\)This is from doing a nonparametric bootstrap with replacement over the 75 villages for 10000 repetitions.
5. Asymptotics

We now discuss the asymptotic properties of estimators of SUGMs. We provide conditions under which the estimators are consistent and describe their asymptotic distributions.

There are two natural perspectives to take. The first holds the number of nodes, \( n \), fixed, and allows the number of different realizations of networks \( R \) to tend to infinity. In this case estimation and inference is straightforward, as there are a growing number of independent realizations of the model and standard techniques apply. Notice that most of the results in the previous section were done under this frame.

The second perspective holds the number of networks observed \( R \) fixed, usually at \( R = 1 \), and then lets the number of nodes grow: \( n \to \infty \). This is the more challenging perspective as the observations of various parts of a network are not independent.

Since in a typical SUGM the links may all be correlated, we also prove a new central limit theorem for correlated random variables that do not satisfy the standard mixing conditions used in time series and spatial econometrics. This result and technique should be of interest beyond network models.

5.1. Many networks case. We do not go into much detail on this first perspective because it follows standard statistical arguments. One has a collection of \( R \) networks, each drawn independently according to a SUGM with the same parameter \( \beta^0 \). We hold \( n \) fixed in this thought experiment. Then the estimator of the parameters is consistent and asymptotically normally distributed. For notational simplicity we omit covariates.

Let us define a moment function for network \( g_r \), \( r \in \{1, \ldots, R\} \), as

\[
h (g_r; \beta) := S (g_r) - E_{\beta} [S (g_r)].
\]

Also define

\[
H (\beta^0) := E \left[ \nabla_{\beta} h (g_r; \beta^0) \right] \quad \text{and} \quad \Sigma := E \left[ h (g_r; \beta^0) h (g_r; \beta^0)' \right].
\]

Proposition 3. Consider a SUGM model and parameter estimated by GMM (using conditional moments) as specified in Proposition 1; with identity-matrix weighting and for which the rows of \( H \) are linearly independent. Then

\[
\hat{\beta} \xrightarrow{P} \beta^0 \quad \text{and} \quad \sqrt{R} \left( \hat{\beta} - \beta^0 \right) \rightsquigarrow N (0, V)
\]

where \( V = (HH')^{-1} H\Sigma H' (HH')^{-1} \).
This result follows by verifying the conditions for consistency and asymptotic normality with independent observations (each network), such as those in (Newey and McFadden, 1994).

5.2. Single large network case. In many cases of interest the researcher has data from one large network. Therefore we consider the case of large $n$ asymptotics where the researcher observes a single large network (so $R = 1$). This case is considerably more challenging as it involves only correlated observations.

Network data tend to be sparse and clustered: with few links relative to the potential number of links and where one’s neighbors tend to be linked to each other with much higher than an independent probability (e.g., see the background in Newman (2003); Jackson (2008)). The clustering is the challenging aspect of the asymptotics since subgraphs we observe are not only the directly generated subgraphs of various types but also include incidentally generated features. Thus, we have to provide new techniques for our asymptotic results. We provide two angles, one providing a new central limit theorem for correlated random variables and the other taking advantage of sparseness.

5.2.1. Sequences of random networks. To describe how parameter estimates behave as a function of the number of nodes $n$, it is useful to consider a sequence of distributions governed by parameters indexed by $n$ and study the asymptotic behavior of estimators of parameters along the sequence. This approach is standard in the random graphs literature (e.g., see the classic book of Bollobas (2001)). Research on social networks has long observed that parameters need to adjust with the number of nodes. For example, friendship networks among a small set of agents (say 50 or 100) and large set of agents (thousands or much more) often have comparable average degrees. As a concrete example, consider friendships among high school students in the U.S. based on the Add Health data set (e.g., see (Currarini, Jackson, and Pin, 2009, 2010)). There are some high schools with only 30 students and others with around 3000 students. The average degree is ranges between 6 and 8 over the high schools, the link probability shrinks dramatically with $n$: from roughly $6/30$ to roughly $8/3000$. Thus, irrespective of the size of their school, students have numbers of friends of roughly the same order of magnitude, and so the frequency of friendship formation must decrease with $n$.

To address this, models of network formation generally tune the density as a function of $n$. For instance an example of a sparse sequence is one in which $P(g_{ij} = 1) = \delta n^{-1}$ so

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25 See Chandrasekhar (2015) for examples networks of varying size ranging from village network data in sub-saharan Africa or India to university dorm friendship network data which all exhibit somewhat comparable number of links per node.
that $E[d_i] = \delta$ and an example of a dense sequence is one in which $P(g_{ij} = 1) = \delta$ so that $E[d_i] = \delta \cdot (n - 1)$. In standard random graph theory, for instance the case of Erdos-Renyi random graphs, researchers prove asymptotic theorems that apply to a sequence $p^n$ of link probabilities, as a function of a growing number of nodes $n$ (again, see Bollobas (2001)). Here, to prove asymptotic theorems we sequence the parameters $\beta^n$. Allowing parameters to change in $n$ complicates identification and consistency/asymptotic normality arguments, but is necessary in order to have a useful theory.

There are many things that could drive the rates of subgraph formation as a function of $n$. Generally humans have time constraints that limit meeting rates and relationship formation. For instance at a conference with thirty people, a person might have an opportunity to talk with most of the other attendees, but at a conference with ten thousand people, the person might only have an opportunity to talk with a hundred or so of the others. For example, agents may apply effort to meet at a “town square” and pairs meet at some rates, triples at some rate, and so on, and then by mutual consent choose to link or not, which gives a foundation to interpret the value of a type of subgraph given a meeting as well as the meeting rate (see Banerjee et al. (2016)).

Thus, we consider a sequence of SUGMs with subgraphs $(G_1, ..., G_k)$ that form on $n$ nodes that are generated with probabilities $\beta^n = (\beta_1^n, ..., \beta_k^n)$. The superscript on the $\beta^n$ indicates the dependence on $n$ to allow for meeting and subgraph formation rates to vary along the sequence. It is convenient to express these in a form

$$\beta^n_{\ell} = \frac{b_\ell}{n^{h_\ell}}$$

for some $b_\ell > 0$ and $h_\ell > 0$, as this allows us to directly see how the parameter varies with $n$. So this is a general way of encoding the rates that could come from meeting, time budgets, or any other story that gives rise to sparse sequences.

We are still interested in estimation $\beta^n$, and the $h_\ell$ parameters just help in stating the theorems and are not necessarily of interest on their own but could be estimated if one also models things like meeting rates and preferences. In the main body of the paper we state results about estimation of the parameters $b_\ell$, (which presumes the researcher knows, estimates, or makes an assumption about $n^{h_\ell}$), while in the appendix we prove (stronger) results about the estimation of the $\beta_\ell$s.

Assumptions on such rates are ubiquitous in this literature. For example, assuming probabilities that do not change in $n$ implicitly assumes a sequence of dense graphs. Putting structure on, say, how nodes meet or what opportunities they have to interact can also be mapped onto assumptions about $h_\ell$s, as can assumptions that degrees are proportional to some function of $\log(n)$, for instance (e.g., see Bollobas (2001)).
5.2.2. Identification, Consistency and Asymptotic Normality. There are two issues that need to be dealt with to derive results on the asymptotic properties of estimators. One we have already dealt with, which is to ensure that as \( n \to \infty \), different enough values of the parameter \( b^n \) and \( b^n' \) must lead to correspondingly different expectations of the moment functions, in the sense of identifiable uniqueness. Otherwise, loosely speaking, the differences in counts of subgraphs will be insufficient to accurately estimate the parameters. As we show in the proofs of the results below, this follows from our identification results above.

The second issue is that deriving a limiting distribution for \( \beta^n \) or \( b^n \) requires deriving a limiting distribution for \( S^n(g) \). With a single graph, a central limit theorem for \( S^n(g) \) does not exist, because there are potential correlations in all the links. For instance, any two links can be part of the same clique of 4, and so in any model with cliques of 4 nodes, all links are correlated. Even with just links and triangles all adjacent links are correlated since they could be part of the same triangle, and any two adjacent triangles are correlated since they share a common link.

For instance, in the case of links and triangles, we would like to show that

\[
\frac{S^n_L(g) - E_{\beta^n_0}[S_L(g)]}{\sigma^n_L} \xrightarrow{d} \mathcal{N}(0, 1), \quad \text{and} \quad \frac{S^n_T(g) - E_{\beta^n_0}[S_T(g)]}{\sigma^n_T} \xrightarrow{d} \mathcal{N}(0, 1),
\]

and jointly as well, where \((\sigma^n_L)^2 := \text{var}(S^n_L(g))\) and \((\sigma^n_T)^2 := \text{var}(S^n_T(g))\). Since

\[
S^n_L(g) = \frac{\sum_{i<j} g_{ij}}{\binom{n}{2}} \quad \text{and} \quad S^n_T(g) = \frac{\sum_{i<j<k} g_{ij} g_{ik} g_{jk}}{\binom{n}{3}}
\]

and \(g^n_{ij}\) and \(g^n_{ik}\) are correlated for any \(k\), \(S^n_L\) involves correlated random variables, and since any two triples in \(S^n_T\) that involve a common link are correlated, we need to carefully check that we can still prove a central limit theorem and that the correlation will not cause problems in the limit.

To accomplish this, we prove a new central limit theorem for a class of generally correlated random variables, and then apply it to prove our SUGM results. Such a new central limit theorem appears in Section 6 and Appendix B.

Relying on our new central limit theorem, we prove the following result for the model with links and triangles. This approach can be extended, of course, to other SUGM models, but the actual proof is case-by-case.

It is useful to define the variance-covariance matrix of the moments

\[
V_n = \begin{pmatrix}
\text{var}(n^{h_L} S_L) & \text{cov}(n^{h_L} S_L, n^{h_T} S_T) \\
\text{cov}(n^{h_L} S_L, n^{h_T} S_T) & \text{var}(n^{h_T} S_T)
\end{pmatrix}.
\]
With this defined we can state our result.

**Proposition 4.** Consider a links and triangles SUGM with associated parameters \( \beta_{0,L}, \beta_{0,T} = \left( \frac{b_{0,L}}{n^{h_L}}, \frac{b_{0,T}}{n^{h_T}} \right) \) with \( 0 \leq D < b_{0,L}, b_{0,T} < D \) such that

\[
h_L \in (1/2, 2) \text{ and } h_T \in [h_L + 1, \min[3, 3h_L]),
\]

excluding the case in which \( h_T = h_L + 1 \) and \( h_L > 1 \).

Consider the estimator \( \tilde{b} \) using moments \( S = (S_L(g), S_T(g)) \). Then \( |\tilde{b}^n - b| \xrightarrow{P} 0 \) and \( V_n^{-1/2} (\tilde{b}^n - b) \xrightarrow{D} \mathcal{N}(0, I) \).

The proposition establishes that not only do the parameter estimates converge to their true values, but that they are approximately normally distributed around the true parameters for large networks.

The proposition requires some rates on the size of the \( \beta^n \)'s. If the parameters are too tiny (converge to 0 much too rapidly), then the networks will be nearly empty even in the limit and there will not be enough data to generate a central limit theorem. Conversely, if the network is extremely dense then it will be nearly complete and there will also not be enough variation to generate a central limit theorem. The proposition states that if the parameters are not too extreme then the estimates of \( \beta^n \)'s are consistent and asymptotically normally distributed as the central limit theorem holds. Proposition 4 is proven as a corollary to Proposition B.3, which appears in Appendix B.3.3.

5.2.3. Direct Estimation with Sparse Networks. While the results presented to this point provide general methods of estimating SUGMs, they involve calculations that can be circumvented in many cases of interest. In particular, many observed networks are sparse. For instance, even looking within limited settings, typical researchers have at most hundreds of co-authors even though there are tens of thousands of researchers in any field. Generally, average degrees of nodes are of small order compared to the number of nodes, and this means that very simple direct counting methods can be accurate in estimating SUGMs in many applications - one can directly estimate parameters from observed counts. We give precise definitions of sparsity that ensure that the direct count estimates are accurate estimators for parameters in such sparse domains; and these restrictions apply in many settings of interest.

We call these estimators *direct estimators*, and denote them by \( \tilde{\beta} \) and \( \tilde{b} \) to distinguish them from the previous estimators, \( \hat{\beta} \) and \( \hat{b} \). The core idea is that when subgraph...
formation is sufficiently sparse, it is rare for a smaller subgraphs to incidentally generate larger ones. So, starting by counting the frequency of larger subgraphs (e.g., triangles in this case), then we can directly and accurately estimate the parameter that drives their formation. Next, after removing the triangles (since they always incidentally generate links), we then can count the relative frequency of links on the remaining pairs of nodes, which consistently estimates link formation.

Note that even though the parameters are estimated based on direct counts of subgraphs, there is still an important logic that needs to be imposed on how subgraphs are counted – for example, only estimating the frequency of links once we have removed the triangles. The ordering in which we do our counting is important since even in a sparse network larger subgraphs can still incidentally generate smaller subgraphs, but smaller ones will rarely incidentally generate larger ones.

The following proposition covers the case of links and triangles, and follows as a corollary to Propositions C.1 and C.2, that provide stronger convergence results (for $\beta$s rather than the $b$s and for general SUGMs), in the appendix. Proposition C.2 in fact shows that any sequence of SUGMs satisfying some basic subgraph frequency conditions has such a result, so this includes the case with finite support covariates and general subgraph combinations. We present only the links and triangles case here for parallelism with Proposition 4.

It is useful to define the variance-covariance matrix of the moments

$$V_n = \begin{pmatrix} \text{var}(n^{h_L}S_U) & \text{cov}(n^{h_L}S_U, n^{h_T}S_T) \\ \text{cov}(n^{h_L}S_U, n^{h_T}S_T) & \text{var}(n^{h_T}S_T) \end{pmatrix}.$$ 

With these defined we can state our result.

**Proposition 5** (Consistency and Asymptotic Normality of Direct Estimators of Sparse Link and Triangle SUGMs). Consider a links and triangles SUGM with associated parameters $\beta^n_{0,L}, \beta^n_{0,T} = \left(\frac{b_{0,L}}{n^{h_L}}, \frac{b_{0,T}}{n^{h_T}}\right)$ with $0 \leq D < b_{0,L}, b_{0,T} < D$ such that

- $h_L \in (2/3, 2)$ and $h_T \in [2, \min[3, 3h_L])$.

Consider the direct estimator $\tilde{b}$ using $S = (S_U(g), S_T(g))$. Then

$$|\tilde{b} - b| \overset{p}{\rightarrow} 0 \text{ and } V_n^{-1/2} \left(\tilde{b} - b\right) \overset{d}{\sim} \mathcal{N}(0, I).$$

Proposition 5 states that growing and relatively sparse SUGMs are consistently estimable via a very simple estimation technique that is easily computable. We illustrate the consistency of the direct estimator in Appendix B.3.4 with a simulation, and show how it is consistent for low parameter values, but then as parameter values grow one
must use minimum distance to get fully consistent estimates. The illustration is for 500 nodes – and as \( n \) grows there is a larger range of degrees that are admitted, as we know from our results that degree that can grow at any rate that is less than \( n^{1/3} \) and still satisfy the sparsity conditions for consistency.

The proof of the proposition involves showing that under the counting convention the fraction of incidentally generated remaining subnetworks vanishes for each \( \ell \), and the observed counts of subnetworks converge to the truly generated ones. And then, by a standard limiting argument applied to the truly generated subgraphs (which are independent), the appropriately normalized vector of subgraph counts are asymptotically normally distributed (with an approximately independent distribution).

6. A Central Limit Theorem for Correlated Random Variables

We now state a new central limit theorem that applies for a variety of settings in which all variables may be correlated (well-beyond network settings), but in which the total amount of covariance is bounded.

Many existing central limit theorems that allow for correlated random variables do not apply to our setting as they require a spatial/ordered lattice structure (e.g., Bolthausen (1982). In the typical logic of central limit theorems based on strong mixing arguments in the spatial and time series literature, random variables are embedded in some space where there are “close” and “far” random variables and the further they are, the less correlated they are. Some researchers working on network formation (e.g., Boucher and Mourifié (2012); Leung (2014)) exploit these spatial techniques by embedding nodes in some space so that only “nearby” nodes can link and “distant” nodes cannot link in order to satisfy mixing conditions and apply a central limit theorem like Bolthausen (1982). As \( n \to \infty \) most nodes get further and further apart and therefore essentially never link. The reason that this is unsatisfying for our purposes is that such a strategy imposes a specific structure on the adjacency matrix: it has to be nearly block-diagonal. To see this, consider the simple case where nodes live on a line. Then in the adjacency matrix, only nodes within some limited distance to the left or right of any given node tend to be linked. While this may be fine for certain contexts, it is not an adequate description of a village network where there is no natural space on which some households in a village should be considered, ex ante, to be infinitely far apart (or students in a dorm or university should be considered, ex ante, to be infinitely unlikely to link to each other).

Our proof technique builds on a foundation developed by Stein (1986). That result (and precursor work in Stein (1972)) have been used before to derive central limit
theorems in two literatures: time-series/spatial statistics and dependency graphs. For instance the oft-used Bolthausen (1982) central limit theorem, crucial in time-series and spatial econometrics, uses a lemma from Stein (1972) to show asymptotic normality. The basic idea is to arrange the data in some manner to identify "close" and "far" random variables and establish conditions on mixing as a function of these distances. In this setting, a non-exhaustive but illustrative list of econometrics papers include Conley (1999), Jenish and Prucha (2009), Bester, Conley, and Hansen (2011), among numerous others.

The literature on dependency graphs has not been explored as exhaustively in econometrics. In such previous approaches, collections of random variables are represented on a graph (the graph here is a manner of representing correlation between random variables, not the target SUGM that we are modeling) where a link between two indices mean that they are correlated and no link means they are independent. Our insight can be thought of as a generalization of the dependency graph literature: what we show is that if the overall covariances satisfy some bounds, then one can still prove a CLT no matter how that dependency is arranged (even with a complete dependency graph). This differs from much the previous literature which assumes that many variables have zero correlation. The normalized sum is then shown to be asymptotically normally distributed provided that the dependency graph is sufficiently sparse (Baldi and Rinott (1989); Goldstein and Rinott (1996); Chen and Shao (2004)). However, those results overly-restrictive, for our purposes, conditions on how various random variables at different nodes in the dependency graph can be correlated. For instance, we want models where in principle all random variables which represent links in the SUGM can be ex ante correlated, and in overlapping ways. Even the few previous results allowing for high- and low-correlation dependency sets are far too stringent to apply to our setting (Ross, 2011; Goldstein and Rinott, 1996; Chen and Shao, 2004). We work with weaker conditions that allow us to work with SUGMs, and are stated based on bounds on sums of covariances, differently from conditions in the previous literature.

We require some new notation.

Let \( \{X^N_\alpha : \alpha \in \Lambda^N \} \) be an array of random variables taking on values in \([0, 1]\). Here \( \alpha \in \Lambda^N \) is the set of labels, and the index is such that \( |\Lambda^N| = N \). For instance, in our SUGM settings the \( X_\alpha \) may be an indicator of the appearance of some particular subgraph, such as a link or triangle, and \( \alpha \) would track the pairs of nodes involved in a potential link \((ij)\) or triples of nodes in a triangle \((ijk)\), and \( N \) captures the \( \binom{n}{2} \) possible
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links or \( \binom{n}{3} \) possible triangles. That is, when considering link counts \( \alpha \) would track pairs of nodes involved in links and when considering triangles \( \alpha \) would track triples.

Let

\[
S^N := \sum_{\alpha \in \Lambda^N} \left( X^N_{\alpha} - E \left[ X^N_{\alpha} \right] \right).
\]

We provide conditions under which a normalized statistic

\[
\bar{S}^N := \frac{S^N}{a_N^{1/2}} \sim \mathcal{N}(0, 1),
\]

where the normalizer, \( a_N \), is a measure of the variance of \( S^N \).

6.1. Dependency neighborhoods. For each \( \alpha, N \), we partition the index set of other random variables, \( \Lambda^N \), into two pieces. In particular, we define a set, called a dependency neighborhood, for each \( \alpha, N \):

\[
\Delta (\alpha, N) \subset \Lambda^N \text{ such that } \alpha \in \Delta (\alpha, N).
\]

The conditions for \( \eta \in \Delta (\alpha, N) \) are precisely defined below. We need \( \Lambda^N \) to be partitioned into \( \Delta (\alpha, N) \) sets for each \( \alpha \) in a specific manner to satisfy a few sufficient conditions.

Basically, the set \( \Delta (\alpha, N) \) includes the \( X_\eta \)'s that have relatively “high” correlation with \( X_\alpha \), and generally its complement includes the \( X_\eta \)'s that have relatively “low” correlation with \( X_\alpha \). There is substantial freedom in defining these sets, but an easy rule to applying them to (non-sparse) SUGMs is to set the \( \Delta (\alpha, N) \) sets to include the other subgraphs with which the subgraph \( \alpha \) shares some edges and could have potentially been incidentally generated.\(^{29}\)

We show that under conditions on the relative correlations inside and outside of the dependency neighborhoods, a central limit theorem applies.

6.2. The Central Limit Theorem. We consider sequences such that the distribution over the \( X_\alpha \)s is symmetric (although not necessarily exchangeable).\(^{30}\)

\(^{29}\)In the sparse case, one can set \( \Delta (\alpha, N) = \alpha \), as in Corollary 6.1.

\(^{30}\)Symmetry is the requirement that for any \( \alpha \) and \( \alpha' \), there exists a permutation of labels that maps \( \alpha \) to \( \alpha' \) and leaves the distribution unchanged. For instance, the marginal distribution of any link \( ij \) is similar to the distribution of any other link \( kl \), and we can find a permutation of labels for which the joint distributions over this link and all other links are the same. This does not, however, mean that exchangeability holds, as the joint distribution of \( ij \) and \( jk \), is not the same as the distribution over \( ij \) and \( rs \). Note also that this does not preclude allowing for node characteristics, as those will be encoded into the specification of \( \alpha \). An example that violates symmetry would be that links involving person 1 are twice as likely as links involving person 2 (excluding link 12).
Let
\[ a_N := \sum_{\alpha, \eta \in \Delta(\alpha, N)} \text{cov} (X_\alpha, X_\eta) , \]
be the total sum of covariances across all the pairs of variables in each other’s dependency neighborhoods, and recall
\[ \overline{S}^N = \frac{S^N}{a_N^{1/2}} \]
is the normalized statistic.

In what follows, we maintain that \( a_N \to \infty \), as otherwise there is insufficient variation to obtain a central limit theorem.

The following are the key conditions for the theorem:

\[ \sum_{\alpha, \eta, \gamma \in \Delta(\alpha, N)} \text{E} [X_\alpha X_\eta X_\gamma] = o \left( a_N^{3/2} \right) , \]

\[ \sum_{\alpha, \alpha', \eta \in \Delta(\alpha, N), \eta' \in \Delta(\alpha', N)} \text{cov} ((X_\alpha - \mu)(X_\eta - \mu), (X_{\alpha'} - \mu)(X_{\eta'} - \mu)) = o \left( a_N^2 \right) , \]

\[ \sum_{\alpha, \eta \notin \Delta(\alpha, N)} \text{cov} (X_\alpha, X_\eta) = o (a_N) , \]

\[ \text{E}[ (X_\alpha - \mu)(X_\eta - \mu) | X_\eta] \geq 0 \text{ for every } \alpha, \eta \notin \Delta(\alpha, N). \]

Even though \( \text{E}[ (X_\alpha - \mu)(X_\eta - \mu) | X_\eta] \geq 0 \) in most applications (as subgraphs either incidentally generate each other or don’t overlap at all, but do not tend to interact negatively\(^{31}\)), we can do without the condition – it is used to provide a simpler statement of (6.3).\(^{32}\)

Condition (6.3) is an intuitive one that states that covariances between subgraphs outside of each other’s dependency sets have a lower order of covariance than within the dependency sets. Essentially, this just captures that dependency sets are properly defined and therefore the variance of the sum is captured by the sum of variances and covariances within dependency sets.

\(^{31}\)For an example for where subgraphs interact negatively, consider the following alternative model. Each person wants to have exactly one friend. In that case, links involving some node are negatively correlated with each other or don’t overlap at all, but do not tend to interact negatively.

\(^{32}\)Below, we prove a stronger version of the theorem with a combined version of (6.3) and (6.4) that only requires that
\[ \sum_{\alpha, \eta \notin \Delta(\alpha, N)} \text{E}[ (X_\alpha - \mu)(X_\eta - \mu) \cdot \text{sign} (\text{E}[ (X_\alpha - \mu)|X_\eta](X_\eta - \mu))] = o(a_N) ; \]
without requiring the nonnegative conditional covariance, \( \text{E}[ (X_\alpha - \mu)(X_\eta - \mu)|X_\eta] \geq 0 \) for every \( \alpha, \eta \notin \Delta(\alpha, N). \) See (B.1).
Conditions (6.1) and (6.2) are conditions that limit the extent to which there are dependencies between more than two subgraphs at a time, requiring that these be of lower order than interactions between two at a time. As we show below, these are satisfied by basic examples of SUGMs. Some such conditions are clearly needed since excessive dependence leads to a failure of a central limit theorem. These extend the literature sufficiently to cover our SUGMs, which were not covered before, except for degenerate cases.

**Theorem 6.1.** If (6.1)-(6.4) are satisfied, then \( S^N \Rightarrow N(0,1) \).

It is useful to consider the special case in which \( \Delta(\alpha, N) = \{ \alpha \} \), which still extends and nests many standard central limit theorems. This corollary is particularly useful when we get to the case of sparse networks, where incidental networks are unlikely and the correlation between different subgraphs becomes small.

**Corollary 6.1.** If \( E[ (X_\alpha - \mu)(X_\eta - \mu) | X_\eta ] \geq 0 \) for every \( \eta \neq \alpha \), and\(^{33}\)

(i) \( \text{var}(X_\alpha) \geq \mu^2 N^{-1/3+\varepsilon} \) for some \( \varepsilon > 0 \) and large enough \( N \),
(ii) \( \sum_{\alpha \neq \eta} \text{cov}((X_\alpha - \mu)^2, (X_\eta - \mu)^2) = o \left( N^2 \text{var} (X_\alpha)^2 \right) \), and
(iii) \( \sum_{\alpha \neq \eta} \text{cov}(X_\alpha, X_\eta) = o (N \text{var} (X_\alpha)) \),

then \( S^N \Rightarrow N(0,1) \).

Moreover, if the \( X_\alpha \)s are Bernoulli random variables and have \( E[X_\alpha] \rightarrow 0 \), then (ii) is implied by (iii).

Note that (ii) is often satisfied whenever (iii) is, and so this is an easy corollary that is based on two intuitive conditions: the variance of the variable in question cannot vanish too quickly (as there needs to be enough variation/information about the variables to get convergence), and the covariance between variables cannot be too large. An application of this corollary is given below, and the proof of the ‘Moreover’ statement appears there.

We outline the steps of the proof of Theorem 6.1, which applies some key techniques pioneered by Stein (1972, 1986) (see also Bolthausen (1982); Baldi and Rinott (1989); Ross (2011)) to a more general structure than have been analyzed before. Here we provide the high-level outline and a detailed proof appears in the appendix.

An application of a lemma from Stein (1986), allows us to prove the theorem by showing that

\[
(6.5) \quad \left| E \left[ S f \left( S \right) \right] - E \left[ f' \left( S \right) \right] \right|
\]

\(^{33}\)Condition (i) can be weakened to \( N^{-1/3} \mu^2 = o (\text{var} (X_\alpha)) \), as shown in the appendix.
tends to zero.\footnote{The key observation of \textcite{stein1986} is that if a random variable satisfies \( \mathbb{E}[f'(Y) - Yf(Y)] = 0 \) for every \( f(\cdot) \) that is continuous and continuously differentiable, then it must have a standard normal distribution.}

In working with (6.5) it is useful to break it into pieces which include both the dependency neighborhoods and the rest; and so it is useful to define the (normalized) sum over the terms not in the dependency neighborhood:

\[
\bar{S}_\alpha := \sum_{\eta \notin \Delta(\alpha, N)} (X_\eta - \mu)/a_N^{1/2}.
\]

In order to show that (6.5) tends to 0, our first step is to develop an expression for the \( \mathbb{E}[\bar{S} f(\bar{S})] \) term:

\[
\mathbb{E}[\bar{S} f(\bar{S})] = \mathbb{E} \left[ \frac{1}{a_N^{1/2}} \sum_{\alpha} (X_\alpha - \mu) \cdot f(\bar{S}) \right] \quad \text{(by definition)}
= \mathbb{E} \left[ \frac{1}{a_N^{1/2}} \sum_{\alpha} (X_\alpha - \mu) \left( f(\bar{S}) - f(\bar{S}_\alpha) \right) \right] + \mathbb{E} \left[ \frac{1}{a_N^{1/2}} \sum_{\alpha} (X_\alpha - \mu) \cdot f(\bar{S}_\alpha) \right].
\]

The second line contains two pieces which we analyze separately. We show that the second piece vanishes, largely based on condition (6.3) and several lemmas that appear in Appendix A. We bound the first part of the expression via a Taylor expansion. We add and subtract a term containing \( f(\bar{S}_\alpha) \), which is useful for a Taylor expansion (below). This generates an extra term, which in the usual dependency graph literature is assumed to be zero, but is not in our case. We allow for modest but nontrivial amounts of correlation in these terms and still establish the result. So we have

\[
\mathbb{E}[\bar{S} f(\bar{S})] = \mathbb{E} \left[ \frac{1}{a_N^{1/2}} \sum_{\alpha} (X_\alpha - \mu) \left( f(\bar{S}) - f(\bar{S}_\alpha) \right) \right] + \mathbb{E} \left[ \frac{1}{a_N^{1/2}} \sum_{\alpha} (X_\alpha - \mu) \cdot f(\bar{S}_\alpha) \right].
\]

The second part of the proof uses the expression from above to rewrite our crucial expression as

\[
\left| \mathbb{E}[\bar{S} f(\bar{S})] - \mathbb{E}[f'(\bar{S})] \right| \leq \mathbb{E} \left[ \frac{1}{a_N^{1/2}} \sum_{\alpha} (X_\alpha - \mu) \left( f(\bar{S}) - f(\bar{S}_\alpha) \right) \right] \quad \text{bounded by 2nd order remainder}
+ \mathbb{E} \left[ \frac{1}{a_N^{1/2}} \sum_{\alpha} (X_\alpha - \mu) \left( \bar{S} - \bar{S}_\alpha \right) f'(\bar{S}) \right] + o(1).
\]
Both (6.1) and (6.2) are then used to show that these two terms go to zero as $N \to \infty$. Specifically, the first term, which is bounded by a second order remainder term, involves expectations over products of triples and therefore is controlled by (6.1). The second term can be factored to show that it involves the terms of the form in (6.2). These sorts of terms arise in the dependency graph literature as well as the spatial statistics literature as well, but our bounds on them are looser and more permissive, which turns out to be critical for our applications.

7. Conclusion

We have provided a new class of network models – SUGMs – that are designed for the practical statistical estimation of social and economic networks, especially as the edges in such networks tend to be correlated. These models substantially outperform standard stochastic block models involving a rich set of observables (even those including unobserved node-level heterogeneity) in generating networks that match observed network characteristics. We have shown that a broad class of these models are well-identified and can be easily estimated by minimum distance, or even more basic and direct techniques in the case of sparse networks. In order to establish asymptotic properties of our estimators, we have proven a new central limit theorem for correlated random variables (that appears in the appendix). We illustrated the power of these models via several applications.

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Watts, D. and S. Strogatz (1998): “Collective dynamics of small-world networks,” *Nature*, 393, 440–442. 2.5.1
Proof of Proposition 1. Order subgraph types so that the number of links a subgraph of type \( \ell \) is nondecreasing in \( \ell \).

Let \( \ell^* \) be the smallest \( \ell \) for which \( \beta_\ell \neq \beta'_\ell \).

Note that \( E_\beta \left[ S_{\ell^*} (g) | N(g) > 0 \right] \) depends only on \( \beta_\ell \) for \( \ell \leq \ell^* \) since this expectation considers only instances of the subgraph that do not intersect with larger \( \ell \) and also compare them to possible instances in which if the graph were present it would not have been incidentally generated by any larger subgraph. Note also that \( E_\beta \left[ S_{\ell^*} (g) | N(g) > 0 \right] \) is nondecreasing in \( \beta_\ell \) for \( \ell < \ell^* \) and strictly increasing in \( \beta_{\ell^*} \). This follows the presence of any such subgraph can be written as the max of an indicator that the subgraph was generated directly, and an indicator that it was generated incidentally by some combination of smaller-indexed subgraphs and adjacent subgraphs of its own type. The likelihood of the first indicator is exactly \( \beta_{\ell^*} \) and so strictly increases in \( \beta_{\ell^*} \), and the second likelihood is nondecreasing in \( \beta_\ell \) for \( \ell \leq \ell^* \), and so overall the probability is nondecreasing in \( \beta_\ell \) for \( \ell < \ell^* \) and strictly increasing in \( \beta_{\ell^*} \).

Thus, since \( \beta_\ell = \beta'_\ell \) for all \( \ell < \ell^* \) and \( \beta_{\ell^*} \neq \beta'_{\ell^*} \), it follows directly that

\[
E_\beta \left[ S_{\ell^*} (g) | N(g) > 0 \right] \neq E_{\beta'} \left[ S_{\ell^*} (g) | N(g) > 0 \right]
\]

completing the proof. \( \blacksquare \)

Proof of Proposition 2. First, note that \( 1 - \left(1 - \beta_T^n \right)^2 \) is the probability that some link is formed as part of at least one triangle out of \( x \) possible triangles that could have it as an edge (independently of whether it also forms directly).

Next, note that the probability that a link forms conditional on some particular triangle that it could be a part of \( \text{not forming} \) is

\[
\tilde{q}_L = \beta_L + (1 - \beta_L) (1 - (1 - \beta_T)^{n-3}) \tag{A.1}
\]

Given this, note that the probability that a link forms can be written as

\[
q_L = \beta_T + (1 - \beta_T) \tilde{q}_L \tag{A.2}
\]

noting that a link could form as part of a triangle that it is part of, or else form conditional upon that triangle not forming.

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\( ^{35} \)That is, consider a given pair of nodes \( i,j \) and a third node \( k \). Consider the probability that link \( ij \) is formed conditional on triangle \( ijk \) not forming directly as a triangle.
We can write the probability of some triangle forming as

\[ q_T = \beta_T + (1 - \beta_T)\tilde{q}_L^3, \]

where the first expression \( \beta_T \) is the probability that the triangle is directly generated, and then the second expression \( (1 - \beta_T)\tilde{q}_L^3 \) is the probability that it was not generated directly, but instead all three of the edges formed on their own (which happen independently, conditional on the triangle not forming, which has probability \( \tilde{q}_L^3 \)).

The result follows from Lemma A.1, with \( x_1 = \beta_L, x_2 = \beta_T, q_L = a_1(x), q_T = a_2(x) \) and \( \tilde{q}_L = f(x) \).

**Lemma A.1.** Let \( x = (x_1, x_2) \in (0, 1)^2 \) and \( a(x) = (a_1(x), a_2(x)) \) be two real-valued functions

\[
\begin{align*}
  a_1(x) &= x_2 + (1 - x_2) f(x) \\
  a_2(x) &= x_2 + (1 - x_2) f(x)^3,
\end{align*}
\]

with

\[
f(x) = x_1 + (1 - x_1) \left[ 1 - (1 - x_2)^N \right] = 1 - (1 - x_1) (1 - x_2)^N
\]

for some integer \( N \geq 0 \).

Then

\[ x \neq x' \implies a(x) \neq a(x'). \]

**Proof.** Suppose the contrary. Then

\[
\begin{align*}
  x'_2 + (1 - x'_2) f(x') &= x_2 + (1 - x_2) f(x) \\
  x'_2 + (1 - x'_2) f(x')^3 &= x_2 + (1 - x_2) f(x)^3,
\end{align*}
\]

First, note that if \( x'_2 = x_2 \), then since these are both less than one, the first equation above implies that \( f(x') = f(x) \). However, that is not possible since \( f \) is increasing in \( x_1 \) and \( x'_1 \neq x_1 \) - recalling that \( x' \neq x \) and \( x'_2 = x_2 \). Thus, \( x'_2 \neq x_2 \), and so without loss of generality consider the case in which \( x'_2 < x_2 \). This implies that both

\[ f(x') = bf(x) + c \]

and

\[ f(x')^3 = bf(x)^3 + c, \]

where \( b = \frac{1 - x_2}{1 - x'_2} \in (0, 1) \) and \( c = \frac{x_2 - x'_2}{1 - x'_2} \in (0, 1) \), and \( b + c = 1 \).
This implies that
\[
bf(x)^3 + 1 - b = (bf(x) + 1 - b)^3.
\]
This as an equation of the form
\[
by^3 + 1 - b = (by + 1 - b)^3
\]
where \(b \in (0, 1)\) and \(y \in (0, 1)\). Note that the left hand side is larger when \(y = 0\) and the two are equal when \(y = 1\), and that the derivative of the difference is
\[
3by^2 - 3b(by + 1 - b)^2 = 3b \left[ y^2 - (by + 1 - b)^2 \right] < 0
\]
so the difference is decreasing over the entire interval, and hits 0 at the end of the interval. Therefore the difference is always positive in \((0, 1)\) and so there is no solution. Thus, our supposition was incorrect, establishing the lemma.

**Lemma A.2.** Any event (in the discrete \(\sigma\)-algebra generated by all possible realizations of all subgraphs) associated with any SUGM has a probability that is an analytic function (and so it is in \(C^\infty\)), and has derivatives and cross partials at all levels being uniformly continuous and bounded on the whole parameter space of \([0, 1]^k\).

**Proof.** An ‘outcome’ is a specification of exactly which subgraphs form and which do not - so a complete specification of what happens. Any event then corresponds to a set of outcomes, and so its probability is a sum of probabilities of the outcomes.

Each outcome’s probability is of the form
\[
\prod_{\ell} \beta_{\ell}^{z_{\ell}} (1 - \beta_{\ell})^{m_{\ell} - z_{\ell}}
\]
where \(z_{\ell}\) indicates how many subgraphs of type \(\ell\) are present in the outcome. Since each of these functions is analytic (and hence in \(C^\infty\)), all of the derivatives and partials, cross partials, etc., are continuous and bounded on \([0, 1]^k\) and hence uniformly continuous on \([0, 1]^k\). Any event is then a finite sum of analytic functions and so the result follows directly.

**Proof of Proposition 3.**

First we check consistency by the conditions of Theorem 2.6 of Newey and McFadden (1994). Here each observation is an independently drawn network.\(^{36}\) For the conditional distribution, we work with the subgraph model, but then condition on \(g_s\) for which

\[^{36}\]The mapping between our notation and theirs is that our \(\beta\) translate to their \(\theta\), our \(g_s\) are their \(z_s\), and our \(S_{\text{sub-only}}\) is their \(g\).
$N_{\text{sub-only}}(g) > 0$. This works in terms of having a pdf $f(g|\beta)$ that draws $g$ according to $\beta$, but then conditions on $N_{\text{sub-only}}(g) > 0$. This is simply restricting the probability space to $N_{\text{sub-only}}(g) > 0$ and renormalizing.

To check that their Theorem 2.6 applies, we check each condition. For (i): let $\hat{W}$ be the identity matrix and then observe that Proposition 1 implies that $\mathbb{E}_{\beta_0} \left[ S(g_r) | N_{\text{sub-only}}(g) > 0 \right] - \mathbb{E}_{\beta} \left[ S(g_r) | N_{\text{sub-only}}(g) > 0 \right] = 0$ only if $\beta = \beta_0$. For (ii), we have assumed that the parameter space $\mathcal{B}$ is compact. (iii) follows from the fact that $\mathbb{E}_{\beta} [S(g_r) | N_{\text{sub-only}}(g) > 0]$ is continuous at each $\beta$ with probability one since it composes continuous functions of parameter entries. Finally (iv) follows from the fact that since every $S_{\ell}^{\text{sub-only}}$ are shares, they are strictly less than 1 and therefore $\mathbb{E} \left[ \sup_{\beta} \| S_{\ell}^{\text{sub-only}}(g_r; \beta) \| | N_{\text{sub-only}}(g) > 0 \right] \leq 1 < \infty$.

Next we check asymptotic normality by the conditions of Theorem 3.4 of Newey and McFadden (1994). We meet the conditions of Theorem 2.6 already and have $\beta_0$ in the interior of the compact parameter space, so (i) is met. We see (iii) holds since by definition the subgraph counts are fractions between 0 and 1. Both (ii) (that the empirical moment function is continuously differentiable in a neighborhood of the true parameter) and (iv) (that the gradient of the moment function is continuous at the true parameter and that it satisfies a ULLN) follow from Lemma A.2 (noting that this still holds when restricting the support to $N_{\text{sub-only}}(g) > 0$ since that simply renormalizes the expectation on a restricted set of graphs). Analytic functions are $C^\infty$, so there are arbitrarily many derivatives. Finally, for (v), that $HH'$ is non-singular follows from the assumption of linear independence of rows of $H$.

Proof of Lemma 1. Having two randomly picked nodes bump into each other within a community, there is a $f^2 + (1-f)^2$ probability of the nodes being of the same type, and a $1 - (f^2 + (1-f)^2)$ probability of them being of different types.\(^{37}\) Thus, the relative meeting frequency of different type links compared same type links is

$$\frac{\pi_L(\text{diff})}{\pi_L(\text{same})} = \frac{1 - (f^2 + (1-f)^2)}{f^2 + (1-f)^2}.$$  

For triangles, picking three individuals out of the community at any point in time would lead to a $f^3 + (1-f)^3$ probability that all three are of the same type, and $1 - (f^2 + (1-f)^2)$

\(^{37}\)To keep things simple, we consider equal-sized groups, but the argument extends with some adjustments to asymmetric sizes.
of them being of mixed types, and so

$$\frac{\pi_T(\text{diff})}{\pi_T(\text{same})} = \frac{1 - (f^3 + (1 - f)^3)}{f^3 + (1 - f)^3}.$$  

It follows directly that for $f \in (0, 1)$:

(A.4) \[ \frac{\pi_T(\text{same})}{\pi_T(\text{diff})} < \frac{\pi_L(\text{same})}{\pi_L(\text{diff})}. \]

So different type triangles are more likely to have opportunities to form under this random mixing model than different type links. In particular, note that

$$\frac{P_T(\text{diff})}{P_T(\text{same})} < \frac{P_L(\text{diff})}{P_L(\text{same})} \text{ if and only if } \left( \frac{p_T(\text{diff})}{p_T(\text{same})} \frac{\pi_T(\text{same})}{\pi_T(\text{diff})} \right)^{1/3} < \left( \frac{p_L(\text{diff})}{p_L(\text{same})} \frac{\pi_L(\text{same})}{\pi_L(\text{diff})} \right)^{1/2}.$$  

In summary, given (A.4), a sufficient condition for $\frac{P_T(\text{diff})}{P_T(\text{same})} < \frac{P_L(\text{diff})}{P_L(\text{same})}$ is that

$$\left( \frac{p_T(\text{diff})}{p_T(\text{same})} \right)^{3/2} < \left( \frac{p_L(\text{diff})}{p_L(\text{same})} \right)^{3/2}$$

which completes the argument. ■
B.1. **Stein’s Lemma.**

Our proof uses a lemma from Stein (1986). We review it here, both to be self-contained and also to explain why this approach to proving asymptotic normality is useful and distinct from other approaches in the networks literature.

The key observation of Stein (1986) is that if a random variable satisfies

$$\mathbb{E}[f'(Y) - Y f(Y)] = 0$$

for every $f(\cdot)$ that is continuous and continuously differentiable, then it must have a standard normal distribution.

This observation leads to a useful lemma, that allows one to characterize the Kolmogorov distance between a random variable $Y$ and a standard normally distributed $Z$, denoted $d_K(Y, Z)$. We can bound this from above by (a constant times) the Wasserstein distance, $d_W(Y, Z)$, which itself is bounded by the below expression.

**Lemma B.1** (Stein (1986); Ross (2011)). If $Y$ is a random variable and $Z$ has the standard normal distribution, then

$$d_W(Y, Z) \leq \sup\{ f : ||f||, ||f'|| \leq 2, ||f''|| \leq \sqrt{2/\pi} \} |\mathbb{E}[f'(Y) - Y f(Y)]|.$$

Further

$$d_K(Y, Z) \leq (2/\pi)^{1/4} (d_W(Y, Z))^{1/2}.$$ 

By this lemma, if we show that a normalized sum of random variables satisfies

$$\sup\{ f : ||f||, ||f'|| \leq 2, ||f''|| \leq \sqrt{2/\pi} \} |\mathbb{E}[f'(\mathcal{X}^N) - \mathcal{X}^N f(\mathcal{X}^N)]| \to 0,$$

then $d_W(\mathcal{X}^N, Z) \to 0$, and so it must be asymptotically normally distributed.

Arguments based on Stein (1986), and his precursor work, Stein (1972), have been used to derive central limit theorems in two literatures: spatial statistics and dependency graphs. For example, for sequences of $\alpha$-mixing random variables in the spatial statistics literature, the oft-used Bolthausen (1982) central limit theorem uses a lemma from Stein (1972) to show asymptotic normality. In the spatial statistics literature, a standard structure would have $\Lambda^N \subset \mathbb{Z}^d$, and this set is thought of as growing outwards with $N$. Then as $\Lambda^N$ expands, the $\alpha$’s that are added along the sequence are increasingly far
apart, and correlations vanish with distance so that $X_\alpha$ is only correlated with $X_\eta$ that are close to $\alpha$ in the lattice. This does not work for our purposes because our setting has no such spatial structure, and we wish to allow any nodes to link to each other in our networks.

The dependency graph approach is closer to our approach and eliminates such spatial structure. Instead, collections of random variables are represented on a graph, where a link between two indices mean that they are correlated and no link means they are independent. The normalized sum is then asymptotically normally distributed provided that the dependency graph is sufficiently sparse (Baldi and Rinott (1989); Goldstein and Rinott (1996); Chen and Shao (2004)). However, previous results in that literature place overly-restrictive conditions on how various $X_\alpha$s can be correlated across $\alpha$. For instance, we want models where in principle all links can be ex ante correlated, and in overlapping ways. Even previous results allowing for high- and low-correlation dependency sets are far too stringent to apply to our setting (Ross, 2011; Goldstein and Rinott, 1996; Chen and Shao, 2004).

Our approach can be thought of as extending the dependency neighborhoods approach. In principle every $X_\alpha$ and $X_\eta$ can be correlated, but we separate those into more highly and less highly correlated sets, and we still obtain a central limit theorem under fairly weak conditions that bound the total and relative correlations in these sets. For instance, in our context, the highly correlated sets for our example with triangles would be triangles that share a node or an edge, and the less correlated sets would be triangles that have no nodes in common.

B.2. Proofs of Theorem 6.1 and Corollary 6.1. The following lemmas are useful in the proof of Theorem 6.1.

**Lemma B.2.** A solution to $\max_h E[Z h(Y)]$ s.t. $|h| \leq 1$ (where $h$ is measurable) is $h(Y) = \text{sign}(E[Z|Y])$, where we break ties, setting $\text{sign}(E[Z|Y]) = 1$ when $E[Z|Y] = 0$.

**Proof.** This can be seen from direct calculation:

$$E[Z h(Y)] = \int_Y E[Z|Y] h(Y) dP(Y)$$

Maximizing $E[Z|Y] h(Y)$ pointwise when $|h| \leq 1$ is achieved by setting $h(Y) = \text{sign}(E[Z|Y])$, and it is clearly ok to break ties by setting $\text{sign}(E[Z|Y]) = 1$ when $E[Z|Y] = 0$, as that makes no difference in the integral. ■
**Lemma B.3.** $E[XYh(Y)]$ when $h(\cdot)$ is measurable and bounded by $\sqrt{\frac{2}{\pi}}$ satisfies

$$E[XYh(Y)] \leq \sqrt{\frac{2}{\pi}} E[XY \cdot \text{sign}(E[X|Y])].$$

**Proof.** This follows from Lemma B.2, setting $Z = XY$. ■

**Proof of Theorem 6.1.** By Lemma B.1, it is sufficient to show that the appropriate sequence of random variables $S^N$ satisfies

$$\sup_{\{f: \|f\|, \|f''\| \leq \sqrt{\frac{2}{\pi}}\}} |E[f'(S^N) - S^N f(S^N)]| \to 0.$$

Recall

$$a_N = \sum_{\alpha, \eta \in \Delta(\alpha, N)} \text{cov} (X_\alpha, X_\eta),$$

and

$$S^N = S^N / a_N^{1/2}.$$  

Also let the size of the dependency set be given by $M(N) = |\Delta(\alpha, N)|$.

Then define the following average covariances:

$$c_1^N = \frac{\sum_{\eta \in \Delta(\alpha, N)} \text{cov} (X_\alpha, X_\eta)}{M(N)},  \quad c_2^N = \frac{\sum_{\eta \notin \Delta(\alpha, N)} \text{cov} (X_\alpha, X_\eta)}{N - M(N)}.$$

Note that $a_N = NM(N)c_1^N$.

For ease of notation, we omit the superscript $Ns$ below.

Let

$$S_\alpha := \sum_{\eta \notin \Delta(\alpha, N)} (X_\eta - \mu)$$

and $S_\alpha := S_\alpha / a_1^{1/2}$.

Observe that

$$E[Sf(S)] = E\left[\frac{1}{a^{1/2}} \sum_\alpha (X_\alpha - \mu) \cdot f(S)\right]$$

$$= E\left[\frac{1}{a_{1/2}} \sum_\alpha (X_\alpha - \mu) \left(f(S) - f(S_\alpha)\right)\right] + E\left[\frac{1}{a_{1/2}} \sum_\alpha (X_\alpha - \mu) \cdot f(S_\alpha)\right].$$

Now we follow the steps of the proof that were outlined in Section 5, where we described the steps of the proof.

Recall that the first step is to show that

$$\left|E\left[\frac{1}{a_{1/2}} \sum_\alpha (X_\alpha - \mu) \cdot f(S_\alpha)\right]\right| = o(1),$$

by employing condition (6.3).
In order to do this, we can expand the term to

\[
\left| E \left[ \frac{1}{a_N^{1/2}} \sum_{\alpha \in \Lambda} (X_\alpha - \mu) \cdot f \left( \overline{S}_\alpha \right) \right] \right| = \left| E \left[ \frac{1}{a_N^{1/2}} \sum_{\alpha \in \Lambda} (X_\alpha - \mu) \cdot f \left( \frac{1}{a_N^{1/2}} \sum_{\eta \notin \Delta(\alpha, N)} (X_\eta - \mu) \right) \right] \right| \leq E \left[ \frac{1}{a_N^{1/2}} \sum_{\alpha \in \Lambda} (X_\alpha - \mu) \cdot f \left( 0 \right) \right] = 0 \text{ since } E[X_\alpha - \mu] = 0.
\]

\[
+ \left| E \left[ \frac{1}{a_N^{1/2}} \sum_{\alpha \in \Lambda} (X_\alpha - \mu) \cdot \left( \frac{1}{a_N^{1/2}} \sum_{\eta \notin \Delta(\alpha, N)} (X_\eta - \mu) \right) \cdot f' \left( \overline{S}_\alpha \right) \right] \right|
\]

where \( \overline{S}_\alpha \) is an intermediate value between \( \overline{S}_\alpha \) and 0.

To bound the second term, we apply Lemma B.3 to conclude that

\[
\left| \frac{E \left[ \sum_{\alpha \in \Lambda; \eta \notin \Delta(\alpha, N)} (X_\alpha - \mu) (X_\eta - \mu) f' \left( \overline{S}_\alpha \right) \right]}{a_N} \right| \leq \sqrt{\frac{2}{\pi}} \left| \frac{E \left[ \sum_{\alpha \in \Lambda; \eta \notin \Delta(\alpha, N)} (X_\alpha - \mu) (X_\eta - \mu) \cdot \text{sign} \left( E \left[ (X_\alpha - \mu) (X_\eta - \mu) \mid (X_\eta - \mu) \right] \right) \right]}{a_N} \right|.
\]

Thus, it is sufficient that

(B.1)

\[
E \left[ \sum_{\alpha \in \Lambda; \eta \notin \Delta(\alpha, N)} (X_\alpha - \mu) (X_\eta - \mu) \cdot \text{sign} \left( E \left[ (X_\alpha - \mu) (X_\eta - \mu) \mid (X_\eta - \mu) \right] \right) \right] = o(a_N)
\]

to ensure that

\[
\left| \frac{E \left[ \sum_{\alpha \in \Lambda; \eta \notin \Delta(\alpha, N)} (X_\alpha - \mu) \cdot (X_\eta - \mu) \cdot f' \left( \overline{S}_\alpha \right) \right]}{a_N} \right| = o(1).
\]

Note that (B.1) is weaker than (6.3), since if \( E \left[ (X_\alpha - \mu) (X_\eta - \mu) \mid (X_\eta - \mu) \right] \geq 0 \), then (using our tie-breaking convention from the lemma)

\[
\text{sign} \left( E \left[ (X_\alpha - \mu) (X_\eta - \mu) \mid (X_\eta - \mu) \right] \right) = 1,
\]

and so (B.1) becomes

\[
E \left[ \sum_{\alpha \in \Lambda; \eta \notin \Delta(\alpha, N)} (X_\alpha - \mu) (X_\eta - \mu) \right] = o(a_N),
\]

which is ensured by (6.3).
Next, the second step of the proof, as we outlined in Section 5, is to apply a similar reasoning as in Ross (2011) with an \( o(1) \) adjustment (from the first step above), to write

\[
\left| E \left[ f'(\bar{S}) - \bar{S}f'(\bar{S}) \right] \right| \leq E \left[ \frac{1}{a^{1/2}} \sum_{\alpha} (X_\alpha - \mu) (f(\bar{S}) - f(\bar{S}_\alpha) - (\bar{S} - \bar{S}_\alpha)f'(\bar{S})) \right] \\
+ \left| E \left[ f'(\bar{S}) \left( 1 - \frac{1}{a^{1/2}} \sum_{\alpha} (X_\alpha - \mu) (\bar{S} - \bar{S}_\alpha) \right) \right] \right| + o(1),
\]

and then to show that the right hand side of this expression goes to 0.

By a Taylor series approximation and given the bound on the derivatives of \( f \), it follows that

\[
\left| E \left[ f'(\bar{S}) - \bar{S}f'(\bar{S}) \right] \right| \leq \frac{\|f''\|}{2a^{1/2}} \sum_{\alpha} E \left[ |X_\alpha - \mu| (\bar{S} - \bar{S}_\alpha)^2 \right] \\
+ \left| E \left[ f'(\bar{S}) \left( 1 - \frac{1}{a^{1/2}} \sum_{\alpha} (X_\alpha - \mu) (\bar{S} - \bar{S}_\alpha) \right) \right] \right| + o(1).
\]

Let us denote the first two terms on the right hand side as \( A_1 \) and \( A_2 \) respectively. We bound each, and show that each is \( o(1) \), which then completes the proof.

\[
A_1 = \frac{\|f''\|}{2a^{3/2}} \sum_{\alpha} E \left[ |X_\alpha - \mu| \left( \sum_{\eta \in \Delta(\alpha, N)} (X_\eta - \mu) \right)^2 \right] \\
\leq \frac{\|f''\|}{2a^{3/2}} \sum_{\alpha} E \left[ X_\alpha \left( \sum_{\eta \in \Delta(\alpha, N)} (X_\eta - \mu) \right)^2 \right] \\
+ \frac{\|f''\|}{2a^{3/2}} \sum_{\alpha} E \left[ \mu \left( \sum_{\eta \in \Delta(\alpha, N)} (X_\eta - \mu) \right)^2 \right] \\
\leq \frac{\|f''\|}{2a^{3/2}} \sum_{\alpha, \eta \in \Delta(\alpha, N), \gamma \in \Delta(\alpha, N)} E \left[ X_\alpha (X_\eta - \mu) (X_\gamma - \mu) \right] \\
+ \frac{\|f''\|}{2a^{3/2}} \sum_{\alpha, \eta \in \Delta(\alpha, N), \gamma \in \Delta(\alpha, N)} E \left[ \mu (X_\eta - \mu) (X_\gamma - \mu) \right] \\
= \frac{\|f''\|}{2a^{3/2}} \sum_{\alpha, \eta \in \Delta(\alpha, N), \gamma \in \Delta(\alpha, N)} \left( E \left[ X_\alpha X_\eta X_\gamma \right] - \mu E \left[ X_\alpha, X_\eta \right] - \mu E \left[ X_\alpha, X_\gamma \right] + \mu^2 E \left[ X_\alpha \right] \right) \\
+ \frac{\|f''\|}{2a^{3/2}} \sum_{\alpha, \eta \in \Delta(\alpha, N), \gamma \in \Delta(\alpha, N)} \mu \text{cov} \left[ X_\eta, X_\gamma \right] \\
= \frac{\|f''\|}{2a^{3/2}} \sum_{\alpha, \eta \in \Delta(\alpha, N), \gamma \in \Delta(\alpha, N)} \left( E \left[ X_\alpha X_\eta X_\gamma \right] - \mu \left( \text{cov} \left[ X_\alpha, X_\eta \right] + \mu^2 \right) - \mu \left( \text{cov} \left[ X_\alpha, X_\gamma \right] + \mu^2 \right) + \mu^3 \right)
\]
where the last step follows from (6.1).

Similarly

\[ A_2 = \left| E \left[ f'(S) \left( 1 - \frac{1}{a^{1/2}} \sum_{\alpha} (X_\alpha - \mu) (S - S_\alpha) \right) \right] \right| \]

\[ = \frac{1}{a} \left| E \left[ f'(S) \left( a - \sum_{\alpha, \eta \in \Delta(\alpha, N)} (X_\alpha - \mu) (X_\eta - \mu) \right) \right] \right| \]

\[ \leq \frac{||f'||}{a} E \left( \sum_{\alpha, \beta \in \Delta(\alpha, N)} (X_\alpha - \mu) (X_\beta - \mu) - E [(X_\alpha - \mu) (X_\beta - \mu)] \right) \]

\[ \leq \frac{\sqrt{2}}{a \sqrt{\pi}} \left( \text{var} \left[ \sum_{\alpha, \beta \in \Delta(\alpha, N)} (X_\alpha - \mu) (X_\eta - \mu) \right] \right)^{1/2} \]

\[ = \frac{\sqrt{2}}{a \sqrt{\pi}} \left( \sum_{\alpha, \alpha', \eta \in \Delta(\alpha, N), \eta' \in \Delta(\alpha', N)} \text{cov} ((X_\alpha - \mu)(X_\eta - \mu), (X_{\alpha'} - \mu)(X_{\eta'} - \mu)) \right)^{1/2}, \]
where the last inequality follows by the Cauchy-Schwarz Inequality. Note that the final expression is $o(1)$ by (6.2).

**Proof of Corollary 6.1.** We apply Theorem 6.1 to the case in which $\Delta(\alpha, N) = \{\alpha\}$.

In this case, note that (6.1) becomes

$$\sum_{\alpha} E \left[ X_{\alpha}^3 \right] = o \left( \left( \sum_{\alpha} \text{var} \left( X_{\alpha} \right) \right)^{3/2} \right).$$

which is satisfied if

$$N \mu^3 = o \left( N^{3/2} \text{var} \left( X_{\alpha} \right)^{3/2} \right).$$

or

$$N^{-1/3} \mu^2 = o \left( \text{var} \left( X_{\alpha} \right) \right).$$

which is implied by (i).

Next, (6.2) becomes

$$\sum_{\alpha, \alpha'} \text{cov} \left( (X_{\alpha} - \mu)^2, (X_{\alpha'} - \mu)^2 \right) = o \left( N^2 \text{Var} \left( X_{\alpha} \right)^2 \right).$$

Note that the terms on the left-hand-side where $\alpha = \alpha'$ are equal to $N \text{var} \left( X_{\alpha} \right)^2$, and so are inconsequential to satisfying the equation, and thus it becomes

$$\sum_{\alpha \neq \alpha'} \text{cov} \left( (X_{\alpha} - \mu)^2, (X_{\alpha'} - \mu)^2 \right) = o \left( N^2 \text{var} \left( X_{\alpha} \right)^2 \right),$$

which is condition (ii).

Finally, (6.3) becomes

$$\sum_{\alpha \neq \alpha'} \text{cov} \left( X_{\alpha}, X'_{\alpha} \right) = o \left( N \text{var} \left( X_{\alpha} \right) \right),$$

which is condition (iii).

**B.3. Links and Triangles SUGMs.** Let the moment (normalized) be

$$\tilde{M} (\beta) = R_n S (g) - E_\beta [R_n S (g)],$$

where $R_n = \text{diag} \left\{ n^{h_L}, n^{h_T} \right\}$ properly normalizes the moments. So, for example, for links we have

$$\tilde{M}_L^n (\beta) = \frac{n^{h_L}}{\binom{n}{2}} \sum_{i<j} \left\{ g_{ij} - E_\beta g_{ij} \right\}$$
\[ \frac{n^h L}{(1 - 1)} \sum_{i < j} g_{ij} - q L (\beta) n^h L. \]

The objective function is
\[ \hat{Q}_n (g, \beta) := \hat{M}^n (\beta)' \hat{M}^n (\beta). \]

And we will need
\[ \bar{Q}_n (\beta) = E \left[ \hat{M}^n (\beta) \right]' E \left[ \hat{M}^n (\beta) \right] \]
which is the non-stochastic analogue.

**B.3.1. Identification.** We now prove identification for sequences of parameters, in the strong sense of identifiable uniqueness in the sense of Lemma 3.1 of Pötscher and Prucha (1997): the parameters \( \beta^n_0 \) are identifiably unique in the sense that for any \( \varepsilon > 0 \)
\[ \liminf_{n \to 0} \inf_{\beta \in B} : \delta (\beta, \beta^n_0) > \varepsilon \]
\[ \| Q_n (\beta) - Q_n (\beta^n_0) \| > 0. \]

We prove this now.

To do so, and in what follows, we use a strong metric for distances between parameters (\( \delta (\beta, \beta^n_0) \) in the above) so that they will be accurately estimated even if the network is sparse so that the parameters are small. We set\(^{38}\)
\[ \delta(x, y) := \max_{\ell} \left[ \frac{|x_{\ell} - y_{\ell}|}{\max(|x_{\ell}|, |y_{\ell}|)} \right], \]
then the requirement becomes
\[ \max_{\ell} \frac{\left| \hat{\beta}^n_{\ell} - \beta^n_{0, \ell} \right|}{\max(\left| \hat{\beta}^n_{\ell} \right|, \left| \beta^n_{0, \ell} \right|)} \xrightarrow{P} 0. \]

This requires that \( \hat{\beta}^n_{\ell} \) and \( \beta^n_{0, \ell} \) be proportional to each other far enough along the sequence. Thus, if \( \beta^n_0 \) approaches 0, saying that \( \hat{\beta}^n_{\ell} \) is a good estimate of it under this metric also requires that \( \hat{\beta}^n_{\ell} \) approach 0 at the same rate, which is a much stronger conclusion than just requiring that the two parameters converge in the usual Euclidean metric.\(^{39}\)

\(^{38}\)We take \( 0/0 = 0. \)

\(^{39}\)To see why this is better than a weaker metric, consider the degenerate estimator of \( \hat{\beta}^n_{\ell} = 0. \) In this case \( \delta(0, \beta^n_{0, \ell}) = \frac{n^h |0 - b^n_{0, \ell}|}{b_{0, \ell}} = \frac{|0 - b_{0, \ell}|}{b_{0, \ell}} = 1 \) which does not tend to zero - so the \( \delta \) metric tells us that this is not a good estimator, but if we just worked with standard distance as our metric, then \( |0 - \beta^n_{0, \ell}| \to 0. \)
Note that if $\delta(\beta, \beta^n) \overset{P}{\to} 0$ then $|\beta - \beta^n| \overset{P}{\to} 0$, and so the results in the paper follow as corollaries to the results below. To see this observe

$$\delta(\beta, \beta^n) = \max_{\ell} \frac{|\beta_\ell - \beta^n_\ell|}{\max(|\beta_\ell|, |\beta^n_\ell|)} = \max_{\ell} \frac{n^{h_\ell} |\beta_\ell - \beta^n_\ell|}{\max(|\beta_\ell|, |\beta^n_\ell|)}$$

$$= \max_{\ell} \frac{|b_\ell - b^n_\ell|}{\max(|b_\ell|, |b^n_\ell|)} \geq \max_{\ell} \frac{|b_\ell - b^n_\ell|}{D}.$$  

since by assumption $b_\ell$ lives in a compact set with maximum $D$. Since $\delta(\beta, \beta^n) \overset{P}{\to} 0$ then so must $\max_{\ell} \frac{|b_\ell - b^n_\ell|}{D}$, proving the result.

**Proposition B.1.** Consider a links and triangles SUGM with associated parameters $\beta^n_{0,L}, \beta^n_{0,T} = \begin{pmatrix} b^n_{0,L} / h^n_L, b^n_{0,T} / h^n_T \end{pmatrix}$ with $h_L > 1/2$ and $h_T \in [h_L + 1, 3h_L]$, then $\beta^n_{0,L}, \beta^n_{0,T}$ are identifiably unique.

**Proof of Proposition B.1.**

Write

$$\beta^n = \begin{pmatrix} b^n_L / n^{h_L}, b^n_T / n^{h_T} \end{pmatrix}, \quad \beta^n_0 = \begin{pmatrix} b^n_{0,L} / n^{h_L}, b^n_{0,T} / n^{h_T} \end{pmatrix},$$

where $b^n_L, b^n_T, b^n_{0,L}, b^n_{0,T}$ lie in $[D, D]$.

Let $r^n_L = 1/n^{h_L}$ and $r^n_T = 1/n^{h_T}$.

First, note that $1 - (1 - \beta^n_T)^x$ is the probability that some link is formed as part of at least one triangle out of $x$ possible triangles that could have it as an edge (independently of whether it also forms directly).

Next, note that the probability that a link forms conditional on some particular triangle that it could be a part of not forming is

$$(B.3) \quad q^n_L = \beta^n_L + (1 - \beta^n_L) \left(1 - (1 - \beta^n_T)^{n-3}\right).$$

So, we can write the probability of some triangle forming as

$$(B.4) \quad q^n_T := E_{\beta^n_L, \beta^n_T} [S_T(g)] = \beta^n_T + (1 - \beta^n_T) (q^n_L)^3,$$

---

40 We allow the constants to depend on $n$ to capture that some applications have both rates and constants that adjust with scale, and we may want to fit across data of networks of varying sizes. But this is largely semantic, as estimating any particular network has only one $b$, and one can ignore the superscripts on the $b$s if one likes.

41 That is, consider a given pair of nodes $i, j$ and a third node $k$. Consider the probability that link $ij$ is formed conditional on triangle $ijk$ not forming directly as a triangle.
where the first expression $\beta^n_T$ is the probability that the triangle is directly generated, and then the second expression $(1 - \beta^n_T)(q^n_T)^3$ is the probability that it was not generated directly, but instead all three of the edges formed on their own (which happen independently, conditional on the triangle not forming, which has probability $(q^n_T)^3$).

It is useful to note that since $\beta^n_T = o(1)$, $(1 - \beta^n_T) \to 1$ and since $h_T > 1$, $|(1 - \beta^n_T)^{n-3} = (1 - \frac{b_T n}{n^2 + r})| \to 0$. Thus,\footnote{We use Bachmann-Landau notation so $f(n) = \Theta(g(n))$ means that $f$ is bounded above and below asymptotically by $g$. That is, $\exists k_1 > 0, \exists k_2 > 0, \exists n_0$ such that $\forall n > n_0$, $k_1 g(n) \leq f(n) \leq k_2 g(n)$.}

$$\tilde{q}^n_L = \Theta \left( \frac{1}{n^{h_L}} + \frac{1}{n^{h_T-1}} \right) = \Theta \left( \frac{1}{n^{h_L}} \right)$$

where the second equality follows since $h_T \geq h_L + 1$.

Next, note that the probability that a link forms is

$$q^n_L := E_{\beta^n_T, \beta^n_L} [S_L(g)] = \beta^n_L + (1 - \beta^n_L) \left( (1 - (1 - \beta^n_T)^{n-2} \right), \quad (B.5)$$

where the first expression $\beta^n_L$ is the probability that the link is directly generated, and then the second expression $(1 - \beta^n_L) (1 - (1 - \beta^n_T)^{n-2})$ is the probability that it was not generated directly, but instead appeared as an edge in some triangle (and there are $n-2$ such possible triangles).

It is also useful to write this in a very different way:

$$q^n_L := E_{\beta^n_T, \beta^n_L} [S_L(g)] = \beta^n_T + (1 - \beta^n_T)\tilde{q}^n_L, \quad (B.6)$$

noting that a link could form as part of a triangle that it is part of, or else form conditional upon that triangle not forming.

The following derivative expressions are useful:

$$\frac{\partial \tilde{q}^n_L}{\partial \beta^n_L} = (1 - \beta^n_T)^{-3}, \quad \frac{\partial \tilde{q}^n_L}{\partial \beta^n_T} = (n - 3)(1 - \beta^n_L)(1 - \beta^n_T)^{-2}. \quad (B.7)$$

$$\frac{\partial q^n_L}{\partial \beta^n_L} = (1 - \beta^n_T)^{-2}.$$ 

$$\frac{\partial q^n_L}{\partial \beta^n_T} = 3(1 - \beta^n_L)(\tilde{q}^n_T)^2 \frac{\partial \tilde{q}^n_T}{\partial \beta^n_L} = 3(\tilde{q}^n_T)(1 - \beta^n_T)^{-2}.$$ 

$$\frac{\partial q^n_L}{\partial \beta^n_T} = 1 - \tilde{q}^n_T + (1 - \beta^n_T) \frac{\partial \tilde{q}^n_T}{\partial \beta^n_T} = 1 + (n - 3)(1 - \beta^n_L)(1 - \beta^n_T)^{-1}.$$ 

$$\frac{\partial q^n_L}{\partial \beta^n_T} = 1 - (\tilde{q}^n_T)^3 + 3(1 - \beta^n_L)(\tilde{q}^n_T)^2 \frac{\partial \tilde{q}^n_T}{\partial \beta^n_T} = 1 - (\tilde{q}^n_T)^3 + 3(\tilde{q}^n_T)^2(n - 3)(1 - \beta^n_L)(1 - \beta^n_T)^{-1}.$$
Given that $\beta_n^L = o(1)$ (since $h_L > 0$), $\beta_n^T = o(1/n)$ (since $h_T > 1$), and $q^L_n = \Theta \left( \frac{1}{n^{h_L}} \right)$ the above expressions imply that:

(B.8) \[ \frac{\partial q^L_n}{\partial \beta_n^L} = 1 - o(1), \]

(B.9) \[ \frac{\partial q^T_n}{\partial \beta_n^L} = \Theta \left( \frac{1}{n^{2h_L}} \right), \]

(B.10) \[ \frac{\partial q^L_n}{\partial \beta_n^T} = n - 2 - o(1), \]

(B.11) \[ \frac{\partial q^T_n}{\partial \beta_n^T} = \Theta \left( \max[1, n^{1-2h_L}] \right). \]

Note that (B.8)-(B.11) hold for any parameters $h_L > 0$ and $3h_L > h_T \geq h_L + 1$ - and thus uniformly for any $\beta^n$ in a compact set of such $h_L, h_T$, and thus as long as we restrict attention to $\beta^n$ in that compact set, we have the same order derivatives and so then we approximate:

(B.12) \[ \frac{E_{\beta^n} [S_L(g)] - E_{\beta_0^n} [S_L(g)]}{r_n^L} \approx n^{h_L} \left[ \frac{b_n^L - b_{L0}^n}{n^{h_L}} + (n - 2) \frac{b_T^n - b_{T0}^n}{n^{h_T}} \right] \]

\[ \approx b_L^n - b_{L0}^n + (b_T^n - b_{T0}^n)\Theta(n^{h_L+1-h_T}), \]

and

(B.13) \[ \frac{E_{\beta^n} [S_T(g)] - E_{\beta_0^n} [S_T(g)]}{r_n^T} \approx n^{h_T} \left[ \frac{b_L^n - b_{L0}^n}{n^{h_L}} \Theta(1/n^{2h_L}) + \frac{b_T^n - b_{T0}^n}{n^{h_T}} \Theta \left( \max[1, n^{1-2h_L}] \right) \right] \]

\[ \approx (b_L^n - b_{L0}^n)\Theta(n^{h_T-3h_L}) + (b_T^n - b_{T0}^n)\Theta \left( \max[1, n^{1-2h_L}] \right). \]

To establish identifiable uniqueness (given the additive separability of $\bar{Q}^n(\beta)$ across $U,T$) it is sufficient to argue that for any $\varepsilon > 0$ there exists $\phi > 0$ such that for large enough $n$, if $\delta((\beta_L^n, \beta_T^n), (\beta_{L0}^n, \beta_{T0}^n)) > \varepsilon$, then at least one of the following inequalities holds:

(B.14) \[ \left| \frac{E_{\beta^n} [S_L(g)] - E_{\beta_0^n} [S_L(g)]}{r_n^L} \right| > \phi \]
or
\begin{equation}
\left| \frac{E_{\beta^n} [S_T(g)] - E_{\beta^n_0} [S_T(g)]}{r^n_T} \right| > \phi.
\end{equation}

Note that \( \delta((\beta^n_L, \beta^n_T), (\beta^n_0, L, \beta^n_L, \beta^n_T)) > \varepsilon \) translates into \(|b^n_L - b^n_0| > c\varepsilon \) and/or \(|b^n_T - b^n_0| > c\varepsilon \) for some \( c > 0 \). If the second inequality holds, then by (B.13) it follows that (B.15) holds. If (B.15) does not hold for any \( \phi \), then by (B.13) it must be that \(|b^n_L - b^n_0| > c\varepsilon \) and/or \(|b^n_T - b^n_0| < \delta^n \) for a sequence \( \delta^n \rightarrow 0 \). In that case, noting that since \( h_T \geq h_L - 1 \) (and so the second term of (B.12) is of order at most 1 times \( \delta^n \) while the first term is at least \( c\varepsilon \) in magnitude), then by (B.12) it follows that (B.14) holds. ■

B.3.2. Consistency.

Proposition B.2. Consider a links and triangles SUGM with \( 2 > h_L > 1/2 \) and \( h_T \leq h_L + 1, \min\{3h_L, 3\} \). Then \( \delta(\beta^n_0, \beta^n_0) \rightarrow P \rightarrow 0 \).

Proof of Proposition B.2. The proof follows from checking the condition of Lemma 3.1 of Pötscher and Prucha (1997) (see also Jenish and Prucha (2009)). Clearly \( \mathcal{B} \) is compact, the weighting function is the identity matrix so it is positive semi-definite, and the moment function is continuous in \( \beta \). Identifiable uniqueness was demonstrated in Proposition B.1. Uniform convergence of the objective function comes from checking a Lipschitz condition.

The first thing to observe is that this just requires showing
\[
\sup_{\beta} \left| \hat{M}^n (\beta) - E \hat{M}^n (\beta) \right| = o_p (1).
\]

This is because
\[
\sup_{\beta} \left| \hat{Q}_n (g, \beta) - \hat{Q}_n (\beta) \right| \leq \sup_{\beta} \left| \hat{M}^n (\beta)' \hat{M}^n (\beta) - E \left[ \hat{M}^n (\beta) \right]' E \left[ \hat{M}^n (\beta) \right] \right|
\]
\[
\leq \sup_{\beta} \left| \left\{ \hat{M}^n (\beta) - E \left[ \hat{M}^n (\beta) \right] \right\}' \hat{M}^n (\beta) \right|
\]
\[
+ \sup_{\beta} \left| E \left[ \hat{M}^n (\beta) \right]' \left\{ \hat{M}^n (\beta) - E \left[ \hat{M}^n (\beta) \right] \right\} \right|
\]
\[
\leq 2K \cdot \sup_{\beta} \left| \hat{M}^n (\beta) - E \left[ \hat{M}^n (\beta) \right] \right|
\]

for a constant \( K \), recalling we have assumed \( D_L < b_L < \bar{D}_L \) and \( D_T < b_T < \bar{D}_T \).

So, we show that \( \sup_{\beta} \left| \hat{M}^n (\beta) - E \hat{M}^n (\beta) \right| = o_p (1) \). It is enough to show pointwise convergence, which is clear by inspection, and stochastic equicontinuity.
Stochastic equicontinuity requires that for any $\epsilon > 0$, there exists $\eta > 0$ such that
\[
\limsup_n P \left\{ \sup_{\beta, \beta' \in \delta(\beta, \beta') < \eta} \left| \hat{M}^n (\beta) - \hat{M}^n (\beta') \right| > \epsilon \right\} < \epsilon.
\]
A sufficient condition is a Lipschitz condition: for every $\beta, \beta'$,
\[
\left| \hat{M} (\beta) - \hat{M} (\beta') \right| = O_p (1) \cdot \delta (\beta, \beta') .
\]
We now show this condition. Recall that $\Delta = h_T - h_L$. It is also useful to note (see the proof of Proposition B.1) that
\[
|q^n_L (\beta) - q^n_L (\beta')| \leq (1 + o (1)) |\beta_L - \beta'_L| + \Theta (n) |\beta_T - \beta'_T|
\]
and
\[
|q^n_T (\beta) - q^n_T (\beta')| \leq \Theta \left( n^{-2h_L} \right) |\beta_L - \beta'_L| + \Theta (1) |\beta_T - \beta'_T| .
\]
Returning to the moments computation:
\[
\left| \hat{M} (\beta) - \hat{M} (\beta') \right| \leq n^{h_L} |q_L (\beta) - q_L (\beta')| + n^{h_T} |q_T (\beta) - q_T (\beta')|
\]
\[
\leq (1 + o (1)) |\beta_L - \beta'_L| n^{h_L} + \Theta (n) |\beta_T - \beta'_T| n^{h_L}
\]
\[
+ \Theta \left( n^{-2h_L} \right) |\beta_L - \beta'_L| n^{h_T} + \Theta (1) |\beta_T - \beta'_T| n^{h_T}
\]
\[
\leq \Theta (1) \delta_L (\beta_L, \beta'_L) + \Theta \left( n^{1-\Delta} \right) \delta_T (\beta_T, \beta'_T)
\]
\[
+ \Theta \left( n^{h_T-3h_L} \right) \delta_L (\beta_L, \beta'_L) + \Theta (1) \delta_T (\beta_T, \beta'_T)
\]
\[
\leq \Theta (1) \delta (\beta, \beta')
\]
since $\Delta > 1$ and $h_T < 3h_L$, which proves the result.

B.3.3. Asymptotic Normality of the SUGM Estimators.

We prove a stronger version of Proposition 4. Hence we prove some results with (B.2) as a metric, and the previous results follow as corollaries. As we show, our estimation will be consistent even in this strong metric. If the parameters are all positive, then this ensures a strong form of convergence. And we note that if one of the true parameters is 0 then this is very strong and requires that the actual parameter estimate is 0 with a probability going to 1. To see why this will actually hold in many models, consider the case of the links and triangles model. If the true link parameter is 0, then all links will be in triangles. Parameter estimates of 0 for links and the fraction of triangles actually observed for triangles (adjusted for incidentals from triangles) minimizes the objective
function and gives a 0, so will be the actual estimates (as the model is identified and has a unique minimizer as we have shown). Thus, the link parameter estimate will be 0 with probability 1, and so consistency will hold. If instead, the true triangle parameter was 0, then the only triangles would be incidentals, and with a probability growing in the number of nodes (under our assumptions below) the identification equations will again have a unique corner maximizer which would have 0 for the triangle parameter.

It is useful to define the variance-covariance matrix of the moments and a rate matrix

$$V_n = \begin{pmatrix} \text{var}(n^{h_L} S_L) & \text{cov}(n^{h_L} S_L, n^{h_T} S_T) \\ \text{cov}(n^{h_L} S_L, n^{h_T} S_T) & \text{var}(n^{h_T} S_T) \end{pmatrix} \quad \text{and} \quad R_n = \begin{pmatrix} n^{h_L} & 0 \\ 0 & n^{h_T} \end{pmatrix}.$$   

**Proposition B.3.** Consider a links and triangles SUGM with associated parameters $\beta^{n,0}_L, \beta^{n,0}_T = (b^{n,0}_L/n^{h_L}, b^{n,0}_T/n^{h_T})$ with $0 \leq \overline{D} < b^{n,0}_L, b^{n,0}_T < \overline{D}$ such that

$$h_L \in \left(\frac{1}{2}, 2\right) \quad \text{and} \quad h_T \in [h_L + 1, \min\{3, 3h_L\}],$$

excluding the case in which $h_T = h_L + 1$ and $h_L > 1$.\(^{43}\) Consider the GMM estimator $\hat{\beta}$ using moments $S = (S_L(g), S_T(g))$. Then

$$\delta(\hat{\beta}^{n}_n, \beta^{n}_0) \xrightarrow{P} 0$$

and\(^{44}\)

$$V_{n}^{-1/2} R_n \left( \hat{\beta}^{n}_n - \beta^{n}_0 \right) \Rightarrow \mathcal{N}(0, I),$$

**Proof of Proposition B.3.**

The proof of the result follows the outline of standard results on asymptotic normality of parameter estimates (e.g., Newey and McFadden (1994)).

It is convenient to normalize things via a change of variables via the diagonal normalizing matrix $R_n = \text{diag}\{n^{h_L}, n^{h_T}\}$ to a parameter vector $b := R_n \beta$, so that the magnitude of the parameter vector does not change with $n$. Observe that $\delta(\hat{\beta}, \beta^{n}_0) \xrightarrow{P} 0$ if and only if $\hat{b} \xrightarrow{P} b_0$, and consistency in the $\delta$-metric holds by Proposition B.2.

It is also useful to then define the expected and empirical moment functions in terms of this rescaled parameter

$$\tilde{M}_L (b) = \left[ \frac{n^{h_L}}{n^{m_L}} \sum_{i < j} g_{ij} - q_{L} (b_L, b_T) \right]$$

\(^{43}\)We conjecture that the theorem works for this measure zero case as well, though we have not been able to show it using our current proof technique. Simulations suggest that the result should hold in this region as well.

\(^{44}\)The expression for $V_n$ is different when $h_T = h_L + 1$, and is given in the proof of the proposition.
and
\[ \hat{M}_T(b) = \left[ \frac{n^{ht}}{n^{mt}} \sum_{i<j<k} g_{ij}g_{jk}g_{ik} - \bar{q}_T(b_L, b_T) \right] \]
where \( \bar{q}_L(b) = n^{ht}q_L \) and \( \bar{q}_T(b) = n^{ht}q_T \) are the normalized expectations given parameter \((b_L, b_T) = R_n\beta\).

Let \( \Delta = h_T - h_L \). We treat two separate cases, \( \Delta > 1 \) and \( \Delta = 1 \). The second case allows links to generate triangles at a similar rate as triangles, and so is a more complex case to treat, and so each step of the argument involves different arguments for the two cases.

From the first order condition of GMM estimation, we take a mean value expansion around the true normalized parameter \( b_0 \) by applying the mean-value theorem, and then solve for \( b - b_0 \).\(^{45}\) Note that the mean value \( b \) is evaluated component by component in the matrix \( \nabla \hat{M}(\hat{b}) \). This abuse of notation is standard (e.g., Newey and McFadden (1994)).

\[ R_n (\hat{\beta} - \beta_0) = (\hat{b} - b_0) = - \left[ \nabla \hat{M}(\hat{b})' \nabla \hat{M}(\hat{b}) \right]^{-1} \nabla \hat{M}(\hat{b})' \hat{M}(b_0) . \]

Below we will show that for \( \Delta > 1 \)
\[ - \left[ \nabla \hat{M}(\hat{b})' \nabla \hat{M}(\hat{b}) \right]^{-1} \nabla \hat{M}(\hat{b})' \xrightarrow{p} I \]
and by Lemma B.4, for
\[ V_n = \begin{pmatrix} \text{var}(n^{ht}S_L) & \text{cov}(n^{ht}S_L, n^{ht}S_T) \\ \text{cov}(n^{ht}S_L, n^{ht}S_T) & \text{var}(n^{ht}S_T) \end{pmatrix} \]
it follows that
\[ V_n^{-1/2}\hat{M}(b_0) \xrightarrow{d} N(0, I) . \]
Therefore by Slutsky’s theorem, it follows that
\[ V_n^{-1/2}R_n (\hat{\beta} - \beta_0) \xrightarrow{d} N(0, I) . \]

Thus, to complete the proof for the case of \( \Delta > 1 \), it suffices to show that
\[ - \left[ \nabla \hat{M}(\hat{b})' \nabla \hat{M}(\hat{b}) \right]^{-1} \nabla \hat{M}(\hat{b})' \xrightarrow{p} I \]

\(^{45}\) This is valid because \( b_0 \) is assumed to lie in the interior of \( B \), a compact set, which then implies the sequence of \( B^n \) under consideration.
For the case of $\Delta = 1$ we will end up with a different expression for the limit of
\[- \left( \nabla \hat{M} \left( \bar{b} \right) \right) \nabla \hat{M} \left( \bar{b} \right) \] and so will have a different covariance and normalization.

To find the limit of these gradient terms, we need to compute $\nabla q$, where we define
\[
\bar{q}_L \left( b \right) := n^{h_L} \left[ q_L \left( \beta \right) \right] \\
= n^{h_L} \left[ \beta_L + (1 - \beta_L) \left[ 1 - (1 - \beta_T)^{n-2} \right] \right] \\
= b_L + \left( n^{h_L} - b_L \right) \left( n - 2 \right) \cdot \frac{b_T}{n^{h_T}} \\
= b_L + \frac{b_T}{n^{h_T-h_L-1}} + o \left( 1 \right).
\]

Similarly
\[
\bar{q}_T \left( b \right) := n^{h_T} \left[ q_T \left( \beta \right) \right] \\
= n^{h_T} \left[ \beta_T + (1 - \beta_T) \left[ \beta_L + (1 - \beta_T) \left[ 1 - (1 - \beta_T)^{n-2} \right] \right] \right] \\
= b_T + \left( n^{h_T} - b_T \right) \left( \frac{b_L}{n^{h_L}} + \frac{b_T}{n^{h_T-1}} - \frac{b_T}{n^{h_T+h_L-1}} b_L \right) \\
= b_T + \left\{ \frac{b_L}{n^{h_L-h_T/3}} + \frac{b_T}{n^{h_T/3-1}} - \frac{b_T}{n^{h_T+h_L-1-h_T/3}} b_L \right\}.
\]

Note that the third term will always be of lesser order, so
\[
x = O \left( \left\{ \frac{b_L}{n^{h_L-h_T/3}} + \frac{b_T}{n^{h_T/3-1}} \right\} \right).
\]

Also notice
\[
h_L - h_T/3 < 2h_T/3 - 1 \iff \Delta > 1.
\]

Thus, if $\Delta > 1$ only the first term in $x$ matters, while if $\Delta = 1$ then the two terms are of the same order.

Finally it will be useful to write
\[
\nabla \bar{q}_L = \left( 1 + o \left( 1 \right) \right) n^{1-\Delta} + o \left( n^{1-\Delta} \right)
\]
and
\[
\nabla \bar{q}_T = \left( 3x^2 \left( \frac{1}{n^{h_L-h_T/3}} (1 + o(1)) \right) \right) + 3x^2 \left( \frac{1}{n^{h_T/3-1}} (1 + o(1)) \right) \right).
Consider the case where $\Delta > 1$. Then
\[
\nabla q_L = \begin{pmatrix} 1 + o(1) \\ n^{1-\Delta} + o(n^{1-\Delta}) \end{pmatrix} = \begin{pmatrix} 1 + o(1) \\ o(1) \end{pmatrix}
\]
and
\[
\nabla q_T = \begin{pmatrix} o(1) \\ 1 + o(1) \end{pmatrix},
\]

since $3h_L > h_T$ and $h_T > \frac{3}{2}$.

Now consider the case where $\Delta = 1$. Again
\[
\nabla q_T = \begin{pmatrix} o(1) \\ 1 + o(1) \end{pmatrix}
\]
but in this case
\[
\nabla q_L = \begin{pmatrix} 1 + o(1) \\ n^{1-\Delta} + o(n^{1-\Delta}) \end{pmatrix} = \begin{pmatrix} 1 + o(1) \\ 1 + o(1) \end{pmatrix}.
\]

Notice that $\varphi(b)$ is a continuously differentiable function of $b \in B$, where $B$ is compact, and $\nabla_b \varphi(b)$ has a bounded derivative. This allows us to write
\[
\nabla \hat{M}(b) = \nabla \hat{M}(b_0) + o_p(1) = -\nabla \tilde{q}(b_0) + o_p(1).
\]

We explicitly compute the inverse of $\nabla(q(b))'\nabla(q(b))$ below, which exists.

If $\Delta > 1$ we can write
\[
-\nabla \hat{M}(b) = \begin{pmatrix} \frac{\partial \varphi_L}{\partial q_L} & \frac{\partial \varphi_T}{\partial q_L} \\ \frac{\partial \varphi_L}{\partial q_T} & \frac{\partial \varphi_T}{\partial q_T} \end{pmatrix} = \begin{pmatrix} 1 + o(1) & o(1) \\ o(1) & 1 + o(1) \end{pmatrix}
\]
and if $\Delta = 1$ we can write
\[
-\nabla \hat{M}(b) = \begin{pmatrix} \frac{\partial \varphi_L}{\partial q_L} & \frac{\partial \varphi_T}{\partial q_L} \\ \frac{\partial \varphi_L}{\partial q_T} & \frac{\partial \varphi_T}{\partial q_T} \end{pmatrix} = \begin{pmatrix} 1 + o(1) & 1 + o(1) \\ o(1) & 1 + o(1) \end{pmatrix}.
\]

We can also compute
\[
\nabla \hat{M}(b)'\nabla \hat{M}(b) = \begin{pmatrix} \frac{\partial \varphi_L}{\partial q_L} & \frac{\partial \varphi_T}{\partial q_L} \\ \frac{\partial \varphi_L}{\partial q_T} & \frac{\partial \varphi_T}{\partial q_T} \end{pmatrix} \begin{pmatrix} \frac{\partial \varphi_L}{\partial q_L} & \frac{\partial \varphi_T}{\partial q_L} \\ \frac{\partial \varphi_L}{\partial q_T} & \frac{\partial \varphi_T}{\partial q_T} \end{pmatrix}
\]
\[
= \begin{pmatrix} \left(\frac{\partial \varphi_L}{\partial q_L}\right)^2 + \left(\frac{\partial \varphi_T}{\partial q_L}\right)^2 & \frac{\partial \varphi_L}{\partial q_L} \frac{\partial \varphi_T}{\partial q_L} + \frac{\partial \varphi_L}{\partial q_T} \frac{\partial \varphi_T}{\partial q_T} \\ \frac{\partial \varphi_L}{\partial q_L} \frac{\partial \varphi_T}{\partial q_L} + \frac{\partial \varphi_L}{\partial q_T} \frac{\partial \varphi_T}{\partial q_T} & \left(\frac{\partial \varphi_L}{\partial q_T}\right)^2 + \left(\frac{\partial \varphi_T}{\partial q_T}\right)^2 \end{pmatrix}
\]
and so the inverse is
\[
\left[ \nabla \tilde{M} (b)' \nabla \tilde{M} (b) \right]^{-1} = \frac{1}{\det \left[ \nabla \tilde{M} (b)' \nabla \tilde{M} (b) \right]} \left( \left( \frac{\partial q_L}{\partial b_T} \right)^2 + \left( \frac{\partial q_T}{\partial b_T} \right)^2 \right) - \left[ \frac{\partial q_T}{\partial b_T} \frac{\partial q_L}{\partial b_T} + \frac{\partial q_T}{\partial b_L} \frac{\partial q_L}{\partial b_L} \right]^2.
\]

The determinant is given by
\[
\det \left[ \nabla \tilde{M} (b)' \nabla \tilde{M} (b) \right] = \left( \frac{\partial q_L}{\partial b_T} \right)^2 + \left( \frac{\partial q_T}{\partial b_T} \right)^2 - \left( \frac{\partial q_T}{\partial b_T} \frac{\partial q_L}{\partial b_T} \right)^2.
\]

If \( \Delta > 1 \) then the determinant is \( 1 + o(1) \). If \( \Delta = 1 \) it is the same.

So the inverse is, if \( \Delta > 1 \),
\[
\left[ \nabla \tilde{M} (b)' \nabla \tilde{M} (b) \right]^{-1} = \frac{1}{1 + o(1)} \begin{pmatrix} 1 + o(1) & o(1) \\ o(1) & 1 + o(1) \end{pmatrix}.
\]

We can compute the final object in the case \( \Delta > 1 \) as
\[
- \left[ \nabla \tilde{M} (b_0)' \nabla \tilde{M} (b_0) \right]^{-1} \nabla \tilde{M} (b_0)' = \frac{1}{1 + o(1)} \begin{pmatrix} 1 + o(1) & o(1) \\ o(1) & 1 + o(1) \end{pmatrix} \begin{pmatrix} 1 + o(1) & o(1) \\ o(1) & 1 + o(1) \end{pmatrix} \rightarrow I,
\]

which completes the argument for the case of \( \Delta > 1 \).

Meanwhile if \( \Delta = 1 \) then the inverse is
\[
\left[ \nabla \tilde{M} (b)' \nabla \tilde{M} (b) \right]^{-1} = \frac{1}{1 + o(1)} \begin{pmatrix} 2 + o(1) & -1 + o(1) \\ -1 + o(1) & 1 + o(1) \end{pmatrix}.
\]

Therefore,
\[
- \left[ \nabla \tilde{M} (b_0)' \nabla \tilde{M} (b_0) \right]^{-1} \nabla \tilde{M} (b_0)' = \frac{1}{1 + o(1)} \begin{pmatrix} 2 + o(1) & -1 + o(1) \\ -1 + o(1) & 1 + o(1) \end{pmatrix} \begin{pmatrix} 1 + o(1) & o(1) \\ o(1) & 1 + o(1) \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.
\]
Now consider the case with \( \Delta = 1 \). In this case since \( n^{2-h_L} = n^{3-h_T} \), it follows from our calculations above that

\[
\sqrt{n^{2-h_L}} R_n (\hat{\beta} - \beta_0) = \sqrt{n^{2-h_L}} \left( \begin{array}{cc} 1 & -1 \\ 0 & 1 \end{array} \right) \left( \begin{array}{c} \frac{n^{h_L}}{2} \sum g_{ij} - \overline{q}_L (b_0) \\ \frac{n^{h_T}}{3} \sum g_{ijk} - \overline{q}_T (b_0) \end{array} \right)
\]

\[
= \sqrt{n^{2-h_L}} \left( \frac{n^{h_L}}{2} \sum g_{ij} - \overline{q}_L (b_0) \right) - \left( \frac{n^{h_T}}{3} \sum g_{ijk} - \overline{q}_T (b_0) \right),
\]

which still jointly converge to a mean zero random variable, but with a different variance-covariance matrix:

\[
V_n = \left( \begin{array}{cc}
\text{var}(n^{h_L} S_L - n^{h_T} S_T) & \text{cov}(n^{h_L} S_L - n^{h_T} S_T, n^{h_T} S_T) \\
\text{cov}(n^{h_L} S_L - n^{h_T} S_T, n^{h_T} S_T) & \text{var}(n^{h_T} S_T)
\end{array} \right)
\]

for the \( \Delta = 1 \) case.

**Lemma B.4.** Consider a links and triangles SUGM with associated parameters \( \beta_0^{n,L}, \beta_0^{n,T} = \left( \frac{b_{0,L}}{n^{h_L}}, \frac{b_{0,T}}{n^{h_T}} \right) \) such that

\[
2 > h_L > 1/2 \quad \text{and} \quad \min[3, 5h_L - 1] > h_T > h_L + 1
\]
or

\[
2 > h_L > 1 \quad \text{and} \quad 3 > h_T > 2.
\]

Then the model satisfies the conditions of Theorem 6.1, so

\[
V_n^{-1/2} \left( \begin{array}{c}
\frac{n^{h_L}}{m_L} \sum_{i<j} \{ g_{ij} - \text{E}g_{ij} \} \\
\frac{n^{h_T}}{m_T} \sum_{i<j<k} \{ g_{ijk} - \text{E}g_{ijk}g_{ijk} \}
\end{array} \right) \sim \mathcal{N} (0, I)
\]

where

\[
V_n = \left( \begin{array}{cc}
\text{var}(n^{h_L} S_L) & \text{cov}(n^{h_L} S_L, n^{h_T} S_T) \\
\text{cov}(n^{h_L} S_L, n^{h_T} S_T) & \text{var}(n^{h_T} S_T)
\end{array} \right).
\]

**Proof of Lemma B.4.**

Our proof uses Theorem 6.1 directly to establish that the conditions hold for the statistic \( S_L(g) \), and then we use Corollary 6.1 to establish that the conditions hold for the statistic \( S_T(g) \). Then we can apply the theorem dimension by dimension and because the off-diagonals of the covariance matrix are vanishing, the vector is jointly normal.\(^{46}\)

\(^{46}\)Alternatively we could use the Cramér-Wold theorem and demonstrate convergence in distribution for all fixed linear combinations in \( \mathbb{R}^2 \), though it follows in this case.
First, note that our maintained assumption that $a_N \to \infty$ requires that $h_L < 2$ and $h_T < 3$.

$S_L$ case:
In this case, $\alpha$ refers to a generic link $ij$.

We first apply the theorem letting $\Delta \left( \alpha, \left(\begin{array}{c} n \\ 2 \end{array}\right) \right) = \{ \eta : \eta \cap \alpha \neq \emptyset \}$, so $\Delta(ij, N) = \{ij\} \cup \{ik|k \neq i,j\} \cup \{jk|k \neq i,j\}$. After that we will apply the corollary. Each covers different parameter intervals.

Applying Theorem 6.1: So, for the application of the theorem, $\Delta$ includes the link $ij$ itself, as well as adjacent links, $ik$ and $jk$, for $k \neq i,j$. We show that the conditions for Theorem 6.1 hold.

(6.3) is obvious from the definition of $\Delta \left( ij, \left(\begin{array}{c} n \\ 2 \end{array}\right) \right)$, because if $ij$ and $kl$ do not share nodes then no triangle (nor any link) could generate both, and so they are independent and the left hand side term is then 0.

Next we verify (6.1). Consider the product

$$E |X_\alpha X_\eta X_\gamma|$$

for $\eta, \gamma \in \Delta \left( \alpha, N \right)$. There are several cases to consider. The first three are cases in which $\alpha, \eta, \gamma$ are distinct and the last two handle cases in which two or more of these are the same link. In each case the letters $i,j,k,l$ are distinct generic nodes, and $\alpha$ could be any one of the links

[1]: $ij, jk, il$ (a line) - there are order $n^4$ of these.
[2]: $ij, ik, il$ (a star) - there are order $n^4$ of these.
[3]: $ij, jk, ik$ (a triangle) - there are order $n^3$ of these.
[4]: $ij, ij, jk$ or $ij, jk, jk$, two of the links repeat - there are order $n^3$ of these.
[5]: $ij, ij, ij$ all of the links repeat - there are order $n^2$ of these.

Note that

$$E |X_\alpha X_\eta X_\gamma| = P (X_\alpha X_\eta X_\gamma = 1),$$

which we now bound in each case.

From the proof of Proposition B.1, recall that $q_L^n$ is the probability of a link forming in the graph which can be due to a link forming directly or as a part of a triangle, and $\overline{q}_L^n$ is the probability of a link forming if a particular triangle that it could be part of does not form. Let $\overline{q}_L'$ denote the probability that a link forms conditional on two triangles that it could be part of not forming.
It is useful to suppress the $n$ subscripts unless explicitly needed. Then loose upper bounds on the probabilities of the various structures are

- for $[1]$: $\beta_{0,T}^2 + 2(1 - \beta_{0,T})\beta_{0,T}q_L + (1 - \beta_{0,T})^2q_L^2 \leq \beta_{0,T}^2 + 2\beta_{0,T}q_L + q_L^2$,
- for $[2]$: $\beta_{0,T}^3 + 3(1 - \beta_{0,T})\beta_{0,T}^2 + 3(1 - \beta_{0,T})^2\beta_{0,T}q_L + (1 - \beta_{0,T})^3(q_L^2)^3 \leq 4\beta_{0,T}^3 + 3\beta_{0,T}q_L + q_L^3$,
- for $[3]$: $\beta_{0,T} + (1 - \beta_{0,T})(q_L')^3 \leq \beta_{0,T} + q_L^2$,
- for $[4]$: $\beta_{0,T} + (1 - \beta_{0,T})(q_L')^2 \leq \beta_{0,T} + q_L^2$
- for $[5]$: $q_L$

Thus, from the numbers and bounds on probabilities, it follows that

\[
\text{(B.16)} \quad \mathbb{E}|X_\alpha X_\eta X_\gamma| \leq \Theta(n^4(\beta_{0,T}^2 + \beta_{0,T}q_L + q_L^2) + n^3(\beta_{0,T} + q_L^2) + n^2q_L).
\]

Next, note that straightforward calculations show that for $k \neq i$

\[
\text{(B.17)} \quad \text{cov}(X_{ij}, X_{jk}) = \beta_{0,T}(1 - \beta_{0,T})(1 - \tilde{q}_L)^2 \approx \beta_{0,T}.
\]

It then follows directly that

\[
a_N = \Theta\left(n^2q_L + n^3\beta_{0,T}\right).
\]

Now we can compare our expression for $\mathbb{E}|X_\alpha X_\eta X_\gamma|$ from (B.16) to

\[
o(a_N^{3/2}) = o\left(n^3q_L^{3/2} + n^{9/2}\beta_{0,T}^{3/2}\right),
\]

to check the condition.

This imposes a number of constraints (omitting the ones that are obviously satisfied, such as $n^4\beta_{0,T}^2 + n^3q_L^2 = o(n^3q_L^{3/2} + n^{9/2}\beta_{0,T}^{3/2})$):

\[
\text{(B.18)} \quad n^4\beta_{0,T}q_L + n^4q_L^3 + n^3\beta_{0,T} + n^2q_L = o(n^3q_L^{3/2} + n^{9/2}\beta_{0,T}^{3/2}),
\]

Noting, that as in the proof of Proposition B.1, (working there with $\tilde{q}_L$, which is of the same order)

\[
q_L^3 = \Theta\left(\frac{1}{n^{h_L}} + \frac{1}{n^{h_T-1}}\right).
\]

it follows that

\[
o\left(n^3q_L^{3/2} + n^{9/2}\beta_{0,T}^{3/2}\right) = o\left(n^{3-(3/2)h_L} + n^{9/2-(3/2)h_T}\right).
\]

Thus, (B.18) becomes

\[
\max[4 - h_T - \min[h_L, h_T - 1], 4 - 3\min[h_L, h_T - 1], 3 - h_T, 2 - \min[h_L, h_T - 1]] < \max[3 - (3/2)h_L, 9/2 - (3/2)h_T],
\]
We break this into two cases: If $h_L \geq h_T - 1$ then this is satisfied whenever $h_T \in (5/3, 3)$. If $h_L < h_T - 1$ then this is satisfied whenever $h_L \in (2/3, 2)$ and $h_T > \max[(3/2)h_L, h_L + 1]$.

Next, we turn to (6.2). We compute terms of the form
\[
\text{cov} \left( (g_{ij} - q_L) (g_{jk} - q_L), (g_{rs} - q_L) (g_{st} - q_L) \right).
\]

since $\eta \in \Delta (\alpha, N)$ and $\eta' \in \Delta (\alpha', N)$, where here we allow for the cases that $k = i$ and $r = t$. Iterating on the expectations, one can show that
\[
\text{cov} \left( (g_{ij} - q_L) (g_{jk} - q_L), (g_{rs} - q_L) (g_{st} - q_L) \right) \leq E (g_{ij} g_{jk} g_{rs} g_{st}).
\]

It is easy to see that if $\{i, j, k\} \cap \{r, s, t\} = \emptyset$, then the covariance is zero since the events are independent. Thus, we are summing over cases in which the intersection is nonempty. The cases with intersection of two or more nodes are handled as we already did above, noting that the condition is less restrictive here (note that $a_N > 1$, so $a_N^2 > a_N^{3/2}$).

So, we restrict attention to the cases in which there is only one node of intersection. In this case the intersection could come from (1) $s = j$, so we are looking at two two-stars that are joined at the center, (2) $i = r$ so we are looking at a line, or (3) $s = i$, where the center of one star is attached to the leaf of the other. These exhaust all configurations up to a relabeling. Consider the event that $g_{ij} g_{jk} g_{rs} g_{st} = 1$ and say we are in case 1. This has the highest probability relative to the other two cases, so we can construct a crude bound using this. This probability is of order no more than:
\[
\beta_0^3 + \beta_0^2 q_L + \beta_0 q_L^3 + q_L^4
\]
So, we need to check that
\[
n^5(\beta_0^3 + \beta_0^2 q_L + \beta_0 q_L^3 + q_L^4) = o \left( n^4 q_L^2 + n^6 \beta_0^2 \right).
\]

It is sufficient that
\[
n \beta_0 q_L^3 = o \left( q_L^2 + n^2 \beta_0^2 \right), \quad n \beta_0 q_L^2 = o \left( q_L^2 + n^2 \beta_0^2 \right), \quad n q_L^3 = o \left( q_L^2 + n^2 \beta_0^2 \right)
\]

These conditions become:
\[
\max[1 - 2h_T - \min[h_L, h_T - 1], 1 - h_T - 2 \min[h_L, h_T - 1], 1 - 3h_L, 4 - 3h_T] < \max[-2h_L, 2 - 2h_T]
\]

We break this into two cases: If $h_L \geq h_T - 1$ then this is satisfied whenever
\[
h_L + 1 \geq h_T, \quad h_T \in (2, 1/2 + (3/2)h_L).
\]

If $h_L < h_T - 1$ then this is satisfied whenever
\[
h_L > 1, \quad h_T > \max[(2/3)h_L + 4/3, h_L + 1].
\]
Note that the first case requires that $h_L > 1$. Also, then $h_L + 1 < \frac{1}{2} + \frac{3}{2}h_L$, and so the first case becomes $h_L > 1$ and $h_T \in (2, h_L + 1)$. Next, note, that when $h_L > 1$, $(2/3)h_L + 4/3 < h_L + 1$ and so the second case becomes $h_L > 1$, $h_T > h_L + 1$.

Thus, overall the conditions are thus that $h_L \in (1, 2)$ and $h_T \in (2, 3)$.

Applying Corollary 6.1: Next, let us analyze the conditions imposed by the corollary. Condition (i) is straightforward, since $\var(X) = \mu(1 - \mu)$ for a Bernoulli random variable.

Condition (iii) is verified by checking that (noting our expression for $\cov(X_{ij}, X_{jk})$ above and that $\var(X) = \mu(1 - \mu) \approx q_L$):

$$\sum_{\alpha \neq \eta} \cov(X_{\alpha}, X_{\eta}) \approx n^3 \beta_T = o(n^2 q_L)$$

which is satisfied whenever $h_L < h_T - 1$.

It is also easy to check that (ii) holds whenever (iii) does. In particular, note that

$$\sum_{\alpha \neq \eta} \cov((X_{\alpha} - \mu)^2, (X_{\eta} - \mu)^2) = \mathbb{E}[(X_{\alpha}^2 - 2X_{\alpha}\mu + \mu^2)(X_{\eta}^2 - 2X_{\eta}\mu + \mu^2)]$$

and, since for a Bernoulli random variable $X_{\alpha}^2 = X_{\alpha}$, this becomes

$$\sum_{\alpha \neq \eta} \cov((X_{\alpha} - \mu)^2, (X_{\eta} - \mu)^2) = \mathbb{E}[(X_{\alpha}(1 - 2\mu) + \mu^2)(X_{\eta}(1 - 2\mu) + \mu^2)]$$

$$= \mathbb{E}[X_{\alpha}X_{\eta} - \mu^2] + O(\mu^3) \approx \cov(X_{\alpha}, X_{\eta}).$$

Thus, (ii) is satisfied whenever (iii) is (and note that $a_N \geq 1$ so $a_N^2 \geq a_N$).

Therefore, again, re-summarizing all of the conditions together, to get asymptotic normality for the links statistics is sufficient that either

$$2 > h_L > 0 \quad \text{and} \quad 3 > h_T > h_L + 1$$

or

$$2 > h_L > 1 \quad \text{and} \quad 3 > h_T > 2.$$  

**ST case:** Set $\Delta(ij, (n/2)) = \{ijk\}$ and we apply Corollary 6.1.

Condition (i) follows from the fact that since $\mu \leq 1/2$ for large enough $N$, $\var(X_{\alpha}) = \mu(1 - \mu) \geq \mu^2 \geq \mu^2 N^{-1/3+\varepsilon}$.

Condition (ii) is implied by condition (iii), below, just as argued above.

So we check condition (iii):

$$\sum_{\alpha \neq \eta} \cov(X_{\alpha}, X_{\eta}) = o(N \cdot \var(X_{\alpha})).$$
To see this first observe that (again, noting that $X_{\alpha}$ is Bernoulli):

$$\text{var}(X_{\alpha}) = q_T(1 - q_T) = \beta_{0,T}(1 + o(1)),$$

which follows from the proof of Proposition B.1.

We compute $\text{cov}(X_{\alpha}, X_{\eta})$ for various cases of $\alpha, \eta$ as a function of how many nodes the two triangles have in their intersection:

- $|\alpha \cap \eta| = 0$: $\text{cov}(X_{\alpha}, X_{\eta}) = 0$ by independence.
- $|\alpha \cap \eta| = 1$: $\text{cov}(X_{\alpha}, X_{\eta}) = O(\beta_{0,T} \cdot q_L^4)$. There are $O(n^5)$ of these.
  - This is because we need at least one link from each triangle to have formed together and not have already formed independently, which can happen only if the joint node is part of a triangle, and neither of the triangles formed directly. This gives us $\beta_{0,T} \tilde{q}_L^4 \leq \beta_{0,T} q_L^4$.

- $|\alpha \cap \eta| = 2$: $\text{cov}(X_{\alpha}, X_{\eta}) = O(\beta_{0,T} q_L^2 + q_L^5)$.
  - This is because we need the common link from each triangle to have formed together and not have already formed independently in both cases, which can happen only if exactly one of the triangles formed directly and the other did not, or else neither triangle to have formed and all of the links to have formed. This is of order $\beta_{0,T} \tilde{q}_L^2 + (\tilde{q}_L^5) \leq \beta_{0,T} q_L^2 + q_L^5$.

There are $O(n^4)$ of these.

Thus, it must be that

$$n^5 \beta_{0,T} q_L^4 + n^4(\beta_{0,T} q_L^2 + q_L^5) = o\left(n^3 \beta_{0,T}\right).$$

These conditions become $h_L > 1/2$ and $h_T < 5h_L - 1$.

If we put all of the conditions together from both the links and the triangles, we end up with either

$$2 > h_L > 1/2 \quad \text{and} \quad \min[3, 5h_L - 1] > h_T > h_L + 1$$

or

$$2 > h_L > 1 \quad \text{and} \quad 3 > h_T > 2.$$

which completes the proof.

B.3.4. Simulations.

We illustrate the consistency of the estimators of links and triangles example via some simulations.
We set \( n = 500 \) and run 500 simulations of generating a network under the SUGM and then calculating the estimates for each of 20 parameter values for \((\beta^n_{0,L}, \beta^n_{0,T})\):

\[
\beta^n_{0,L} = \frac{b_0}{n^{1.1}} \quad \text{and} \quad \beta^n_{0,T} = \frac{3b_0}{n^{2.1}}
\]

with \( b_0 \in \{.5, 1, ..., 9.5, 20\} \). This generates networks with expected degrees ranging between 2 and 50 and for parsimony guarantees that on average the number of links generated directly and by triangles are similar.

Figure 11 presents the results. We see that for the sparse case the direct estimator and GMM do well, but as density increases only the GMM estimator remains accurate. With larger \( n \), the sparse estimator does well for wider ranges of parameters.
Appendix C. Direct Estimation of Sparse Networks

We use the following convention in ordering the counting of statistics.

C.1. A Convention for Counting Subgraphs in Sparse Networks.

Consider a SUGM and order the classes of the subgraphs, \( G_1, \ldots, G_\ell, \ldots, G_k \), from ‘largest’ to ‘smallest’. In particular, we choose the ordering of \( 1, \ldots, k \) so that a subgraph in \( G_\ell \) cannot be a subnetwork of the subnetworks in \( G_{\ell'} \) for \( k \geq \ell' > \ell \geq 1 \):

\[
g_\ell \in G_\ell \text{ and } g_{\ell'} \in G_{\ell'} \implies g_\ell \not\subset g_{\ell'}.
\]

There exists at least one such ordering - for instance, any ordering in which subgraphs with more links are counted before subgraphs with fewer links. In an example with links, 2-stars and triangles: triangles precede 2-stars which precede links. Note that this is a partial order: for instance, a ‘three link line’ \( ij, jk, kl \) is neither a subgraph nor a supergraph of a ‘3-star’ \( ij, ik, il \), which is also a three link subgraph on four nodes. It is irrelevant in which order subgraphs with the same number of links are counted.

So, we count subgraphs in this order, and after having removed links associated with all of the subgraphs already counted, denoted \( \tilde{S}_n^\ell \):

\[
\tilde{S}_n^\ell (g) = |\{ g_\ell \in G_\ell : g_\ell \subset g \text{ and } g_\ell \not\subset g_{\ell'} \text{ for any } g_{\ell'} \in G_{\ell'} \text{ such that } g_{\ell'} \subset g \text{ for some } \ell'' < \ell \}|.
\]

C.2. Direct Parameter Estimation.

To define the direct parameter estimates, \( \tilde{\beta}_s \), from the counts, we then need to divide by the number of possible subgraphs that could exist on the remaining pairs of nodes after having removed the larger subgraphs. In particular, let \( \tilde{r}_n^\ell (g) \) denote the number of potential remaining subgraphs of type \( \ell \) exist after removing all those of types \( \ell'' < \ell \):

\[
\tilde{r}_n^\ell (g) = |\{ g_\ell \in G_\ell : g_\ell \not\subset g \text{ for any } g_{\ell'} \in G_{\ell'} \text{ such that } g_{\ell'} \subset g \text{ for some } \ell'' < \ell \}|.
\]

In our links and triangles example, then \( \tilde{r}_n^3 = \binom{n}{3} \) and \( \tilde{r}_n^2 = \binom{n}{2} - L(\tilde{S}_T(g)) \) where \( L(\tilde{S}_T(g)) \) is the number of links that are part of triangles in \( g \). Typically, in sparse networks, the adjustments of the denominators to account for the deletion of links already in larger subgraphs will be inconsequential (see the proof of Proposition

\footnote{Note in terms of the notation here, counting in order from ‘largest’ to ‘smallest’ subnetworks means that we count things from smallest to largest index \( \ell \): so the specification of how we ordered labels moves in the opposite direction of the size of the subgraphs.}

\footnote{This ignores the fact that not all of these could be formed without incidentally generating more larger networks. For instance, with links and triangles on 4 nodes, if we remove triangle 123, we are left with links 14, 24, 34, and at most one of those could form without incidentally generating another triangle. This bias will disappear in the sparse case with large number of links.}
For instance, there are relatively few triangles relative to what could be present and not many links will be lost to triangles.\footnote{This does not mean that we can simply do away with our ordered-counting convention entirely - as the presence of directly formed links could still be of a similar order as the presence of links in triangles, it is just that both are relatively rare. So, the adjustments in the numerator associated with counting subgraphs are essential, while the ones in the denominator to track how many could have been present are not essential, but improve small-sample accuracy.}

The direct estimator $\tilde{\beta}^n$ is then

$$\tilde{\beta}^n_\ell = \frac{\tilde{S}^n_\ell(g)}{\tilde{r}^n_\ell(g)}.$$\hspace{1cm}(C.1)

For example, in a links and triangles model, direct estimators are

$$\left(\tilde{\beta}_T, \tilde{\beta}_L\right) = \left(\frac{\# \text{ of triangles}}{\# \text{ of triples of nodes}}, \frac{\# \text{ of links not in triangles}}{\# \text{ of pairs of nodes that are not already together in some triangle}}\right).$$

As we prove, under a sparsity condition these direct estimators are consistent estimates of the true parameters, and they are asymptotically Normally distributed.

As an illustration, consider Figure 12 in which links and triangles are formed on 41 nodes. There are 9 truly generated triangles, but 10 observed overall. So, the frequency of triangles, $\tilde{S}^n_T(g)$, is overestimated by using 10 instead of 9. The true frequency was $9/10660$ but is estimated as $10/10660$.

With respect to links, there were actually 25 truly directly generated, but one becomes part of an incidentally generated triangle and two others overlap on existing triangles, and so $\tilde{S}^n_L(g)$ becomes 22 instead. Here we count them just out of the $820 - 30 = 790$ remaining pairs of nodes that are not in triangles, so we estimate $22/790$ while the true frequency was $25/820$.

C.3. Generating Classes.

To define sparsity, we have to track how many ways a potential subnetwork $g' \in G^\ell_\ell$ could be incidentally generated, many of the ways being equivalent up to relabelings. For instance, many different combinations of triangles and edges could incidentally generate a triangle $g' = \{12, 23, 31\}$. However, notice that there are only eight ways in which it can be done if we ignore the labels of the nodes outside of $g'$: link 12 could be generated either by a triangle or link, and same for links 23 and 31, leading to $2^3 = 8$ ways in which this could happen.

We first provide a precise specification of what it means to be incidentally generated. We say that a subgraph $g' \in G_\ell$ for some $\ell$ can be incidentally generated by the subgraphs $\{g^j\}_{j \in J}$, indexed by $J$, if $g' \subset \cup_{j \in J} g^j$.\footnote{This does not mean that we can simply do away with our ordered-counting convention entirely - as the presence of directly formed links could still be of a similar order as the presence of links in triangles, it is just that both are relatively rare. So, the adjustments in the numerator associated with counting subgraphs are essential, while the ones in the denominator to track how many could have been present are not essential, but improve small-sample accuracy.}
Consider any potential subgraph $g' \in G^n_\ell$ that can be incidentally generated by a set of subnetworks $\{g^j\}_{j \in J}$ with associated indices $\ell_j$ and also by another set $\{g'^{j'}\}_{j' \in J'}$. We say that $\{g^j\}_{j \in J}$ and $\{g'^{j'}\}_{j' \in J'}$ are equivalent generators of $g'$ if there exists a bijection $\pi$ from $J$ to $J'$ such that $\ell_j = \ell_{\pi(j)}$ and $|g_j \cap g'| = |g_{\pi(j)} \cap g'|$. So the equivalent generating sets have the same configurations in terms of numbers and types of subgraphs, and in terms of how many nodes each of those subgraphs intersects the given network.

So, for instance a triangle 123, could be incidentally generated by links 12, 23, and triangle 134; and an equivalent generator is links 12, 23, and triangle 135, and another is links 23, 13; and triangle 128, and so forth.

Given this equivalence relation, ignoring the specific labels of subgraphs we can define generating classes for any type of subgraph $G_\ell$. We just keep track of the number and
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So, each generating class $\mathcal{C}$ of some $G^n_\ell$ is a list $\mathcal{C} = (\ell_1, c_1, \ldots, \ell_C, c_C)$ consisting of a list of types of subgraphs used for the incidental generation and how many nodes each has intersecting with the given incidentally generated subgraph. Thus, $\mathcal{C} = (\ell_1, c_1, \ldots, \ell_C, c_C)$ is such that there $\exists g' \in G^n_\ell$ generated by some $\{g^j\}_{j \in J}$ for which $|J| = C$ and for each $j$: $g^j \in G^n_{\ell_j}$ and $c_j = |g^j \cap g'|$.

We order generating classes so that the indices are ordered: $\ell_j \leq \ell_{j+1}$, and lexicographically $c_j \leq c_{j+1}$ whenever $\ell_j = \ell_{j+1}$. This ensures that we avoid counting the same class twice.\(^{50}\)

We only need to work with a small set of generating classes, so we restrict attention to the following:

- generating classes that are minimal: in the above $J$ there cannot be $j'$ such that $g' \subseteq \bigcup_{j \in J, j \neq j'} g^j$, and
- generating classes that only involve smaller subgraphs: $\ell_j \geq \ell$ for all $j \in J$.

The second condition states that we can ignore many generating classes because of our counting convention: when counting any given subgraph type, we only have to worry about incidental generation by the remaining (weakly smaller) subgraphs.

So, for a links and triangles example, where $G^n = (G^n_T, G^n_L)$ are triangles and links, there are four generating classes of a triangle: a triangle could be incidentally generated by three other triangles, two triangles and one link, two links and one triangle, or three links.\(^{51}\) Under the last condition above, there are no generating classes for links to worry about, since they cannot be incidentally generated by themselves and we only count them after removing all triangles.

C.4. Relative Sparsity.

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50 However, a generating class of two links and a triangle is a different generating class than one link and two triangles - this numbering just avoids the double counting of two links and a triangle separately from a triangle and two links.

51 Here, then we would represent a generating class of two triangles and a link as $(T, 2; T, 2; L, 2)$, where this indicates that two triangles were involved and each intersected the subgraph in question in two nodes and then $L, 2$ indicates that a link was involved intersecting the subgraph in two nodes.
Consider a set of (ordered) subgraphs $G^n = (G^n_1, \ldots, G^n_k)$ and any $\ell \in \{1, \ldots, k\}$ and any generating class of some $\ell, C = (\ell_1, c_1; \ldots; \ell_C, c_C)$. Let $M_C = (\sum_{j=1}^{C} c_j) - m_\ell$.

For example, in forming a triangle from any combination of triangles and links, each $c_j = 2$ and so $M_C = 6 - 3 = 3$.

We say that a sequence SUGMs with associated (ordered) subgraphs $G^n = (G^n_1, \ldots, G^n_k)$ and parameters $\beta^n_\ell$ is relatively sparse if and for each $\ell$:

$$\beta^n_{0,\ell} \to 0,$$

and for each associated generating class $C$ with associated $(\ell_j, c_j)_{j=1,\ldots,C}$:

$$\frac{\prod_{j=1}^{C} E_{\beta^n_0}(S^n_{\ell_j}(g))}{n^{MC} E_{\beta^n_0}(S^n_\ell(g))} \to 0,$$

and

$$\frac{\prod_{j=1}^{C} E_{\beta^n_0}(S^n_{\ell_j}(g))}{n^{MC} E_{\beta^n_0}(S^n_{\ell_j'}(g))} \to 0,$$

for each $j' \in 1, \ldots, C$. Note that this condition applies to the actual frequencies of subgraphs $S^n_\ell$ - a condition on primitives - rather than the directly counted subgraphs $\tilde{S}_\ell^n$, although the condition turns out to be equivalent when it holds. This is a condition that provides us with the necessary bounds the relative frequency with which subgraphs are incidentally generated (the numerator) compared to directly generated (the denominator). It applies in two ways: one is that new graphs are not being incidentally generated at too fast a rate, and secondly, that given subgraphs are not disappearing into incidentally generated larger subgraphs at too fast a rate. Although notationally complex, they are easily checked with links and triangles, for instance if $\beta_L = b_L/n$ and $\beta_T = b_T/n^2$, or for many other values.$^{53}$

C.5. Estimation of Sparse Models.

$^{52}$Note that $M_C \geq 1$ since $C \geq 2$ and some set of $c_j$ nodes intersects with at least one other set of $c_j'$ nodes for some $j' \neq j$ (noting that the incidentally generated subgraph is not a collection of disconnected links). Recall that under the ordering, lower-indexed subgraphs cannot be generated as a subset of some single higher-indexed one.

$^{53}$Again, we emphasize that many empirical applications have degrees that are fairly constant with network size, and $\beta_L = b_L/n$ and $\beta_T = b_T/n^2$ covers a case in which nodes have expected degree roughly $b_L + b_T/2$ irrespective of how large the network grows.
We give precise definitions of sparsity that ensure that the direct count estimates are accurate estimators for parameters in such sparse domains; and these restrictions apply in many settings of interest.

We call these estimators direct estimators, and denote them by $\tilde{\beta}$ to distinguish them from the GMM estimators, $\hat{\beta}$, though obviously these too correspond to a set of moments. The core idea is that when subgraph formation is sufficiently sparse, it is rare for a smaller subgraphs to incidentally generate larger ones. So, starting by counting the frequency of larger subgraphs (e.g., triangles in this case), then we can directly and accurately estimate the parameter that drives their formation. Next, after removing the triangles (since they always incidentally generate links), we then can count the relative frequency of links on the remaining pairs of nodes, which consistently estimates link formation.

Note that even though the parameters are estimated based on direct counts of subgraphs, there is still an important logic that needs to be imposed on how subgraphs are counted - for example, only estimating the frequency of links once we have removed the triangles. The ordering in which we do our counting is important since even in a sparse network larger subgraphs can still incidentally generate smaller subgraphs, but smaller ones will rarely incidentally generate larger ones.

The following proposition covers the case of links and triangles. Proposition C.2 provides the result for general SUGMs and appears below.

It is useful to define the variance-covariance matrix of the moments and a rate matrix

$$V_n = \begin{pmatrix}
\text{var}(n^{h_L} S_U) & \text{cov}(n^{h_L} S_U, n^{h_T} S_T) \\
\text{cov}(n^{h_L} S_U, n^{h_T} S_T) & \text{var}(n^{h_T} S_T)
\end{pmatrix} \quad \text{and} \quad R_n = \begin{pmatrix} n^{h_L} & 0 \\
0 & n^{h_T} \end{pmatrix}.$$  

With these defined we can state our result.

**Proposition C.1** (Consistency and Asymptotic Normality of Direct Estimators of Sparse Link and Triangle SUGMs). Consider a links and triangles SUGM with associated parameters $\beta^n_{0,L}, \beta^n_{0,T} = \left( \frac{b_{0,L}}{n^{h_L}}, \frac{b_{0,T}}{n^{h_T}} \right)$ with $0 \leq D < b_{0,L}, b_{0,T} < \overline{D}$ such that

$h_L \in (2/3, 2)$ and $h_T \in [2, \min[3, 3h_L])$.

Consider the direct estimator $\tilde{\beta}$ using $S = (S_U(g), S_T(g))$. Then

$$\delta (\tilde{\beta}^n, \beta^n_0) \xrightarrow{p} 0$$

and

$$V_n^{-1/2} R_n \left( \tilde{\beta}^n - \beta^n_0 \right) \sim \mathcal{N} (0, I),$$
Proposition C.1 states that growing and relatively sparse SUGMs are consistently estimable via a very simple estimation technique that is easily computable. We illustrate the consistency of the direct estimator in Appendix B.3.4, and show how it is consistent for low parameter values, but then as parameter values grow one must use GMM to get fully consistent estimates. The illustration is for 500 nodes – and as $n$ grows there is a larger range of degrees that are admitted, as we know from our results that degree that can grow at any rate that is less than $n^{1/3}$ and still satisfy the sparsity conditions for consistency.

The proof of the proposition involves showing that under the counting convention the fraction of incidentally generated remaining subnetworks vanishes for each $\ell$, and the observed counts of subnetworks converge to the truly generated ones. And then, by a standard limiting argument applied to the truly generated subgraphs (which are independent), the appropriately normalized vector of subgraph counts are asymptotically normally distributed (with an approximately independent distribution).

It follows from the next result on more general subgraphs in sparse networks.

In order to have $\delta \left( \tilde{\beta}_n^{\ell}, \beta_{0,\ell}^n \right) \xrightarrow{P} 0$, beyond the network being relatively sparse, it must also be that the potential number of observations of a particular kind of subgraph grows as $n$ grows. For instance, if nodes have different characteristics (say some demographics), and we are counting triangles and links by node types, then it also has to be that the number of nodes that have each demographic grows as $n$ grows. If there were never more than 5 nodes with some demographic, then we cannot get an accurate estimate of link formation among those nodes.

We say a SUGM is growing if the probability that $\tilde{S}_n^{\ell}(g) \to \infty$ for each $\ell$ goes to 1.

**Proposition C.2** (Consistency and Asymptotic Normality of Direct Estimators of Sparse SUGMs). Consider a sequence of growing and relatively sparse SUGMs with associated subgraph statistics $\tilde{S}_n = (\tilde{S}_1^n, \ldots, \tilde{S}_k^n)$ and parameters $\beta_0^n = (\beta_{0,1}^n, \ldots, \beta_{0,k}^n)$. Consider the direct estimator $\tilde{\beta}$ described above. Then

1. $\delta(\tilde{\beta}_n, \beta_{0,n}) \xrightarrow{P} 0$
2. $\Sigma^{-1/2}(\tilde{\beta}_n - \beta_{0,n}) \sim N(0, I)$ where $\Sigma_{\ell,\ell} = \frac{\beta_{0,\ell}^{n-1-\beta_{0,\ell}^n}}{\kappa_{\ell}(n_{\ell})}$ and the off-diagonals are all 0.

Proposition C.2 states that growing and relatively sparse SUGMs are consistently estimable via a very simple estimation technique that is easily computable.

The proof of the proposition involves showing that, under the growing and sparsity conditions, the fraction of incidentally generated subnetworks vanishes for each $\ell$, and
the observed counts of subnetworks converge to the truly generated ones. And then, by a standard limiting argument applied to the truly generated subgraphs (which are independent), the appropriately normalized vector of subgraph counts are asymptotically normally distributed (with an approximately independent distribution).

**Proof of Proposition C.1.** This is a corollary of Proposition C.2. First, note that our growing condition and our vanishing fraction condition imply that $2 > h_L > 0$ and $3 > h_T > 0$. Next, let us examine the conditions that triangles are not incidentally generated. First, the condition that links don’t generate triangles requires that $3h_L > h_T$. Next, the condition that triangles don’t generate triangles requires $h_T > \frac{3}{2}$ (the condition implies that $\beta_T > (1 - (1 - \beta_T)^{-2})^3$ which implies $\beta_T/3 > n\beta_T$ or $h_T/3 < h_T - 1$ ). Mixtures of links and triangles not generating triangles are implied by the combination of the two above. Finally, we have a condition that a significant fraction of links don’t disappear into triangles is implied by $h_T \geq 2$ (and a way to see this directly is that degree from triangles is order $n^2/n^{h_T}$, which must be order constant or below). So, combining these conditions we get the claimed intervals. ■

**Proof of Proposition C.2.**

When obvious, we omit superscript $n$’s to simplify notation, but they are implicit.

Let $D_{\ell}(g)$ denote the set of all links which are deleted due to counting $\ell' < \ell$:

$$D_{\ell}(g) = \{ij : 
ij \in g', g' \subset g, g' \in G_{\ell'}, \ell' < \ell\}.$$  

For instance, $D_L(g)$ is the set of links that are members of triangles that appear in $g$ in the links and triangles SUGM, and therefore are not considered when counting links.

Note that under our relative sparsity condition, overall link presence vanishes\(^{54}\), and so it then easily follows that $E[|D_{\ell}(g)|] = o_p(n^2)$ for any $\ell$. It follows that,

\begin{equation}
\widetilde{\beta}_{\ell} = \frac{\tilde{S}_{\ell}(g)}{\kappa_{\ell}(n_{\ell})} - D_{\ell}(g) = \frac{\tilde{S}_{\ell}(g) - S_{\ell'}(g)}{\kappa_{\ell}(n_{\ell})}(1 + o_p(1)) \tag{C.2}
\end{equation}

$$= \left(\frac{S_{\ell'} \text{true}}{\kappa_{\ell}(n_{\ell})} + \frac{\bar{S}_{\ell'} \text{true} - S_{\ell'} \text{true}}{\kappa_{\ell}(n_{\ell})} + \frac{\tilde{S}_{\ell}(g) - S_{\ell'} \text{true}}{\kappa_{\ell}(n_{\ell})}\right)(1 + o_p(1)),$$

\(^{54}\)This can be checked from the condition directly, or simply by noting that if it did not, then the probability of any finite subgraph forming simply from link generation (and hence incidentally from whatever subgraphs form a particular link) would not vanish which would contradict parts of what is proven below.
where $S^{true}_\ell$ is the number of truly generated such subgraphs (unobserved) on the whole network, and $\tilde{S}^{true}_\ell$ is the number of truly generated such subgraphs (unobserved) on the networks that the after removing the links in $D_\ell(g)$, and $\binom{n}{m_\ell}$ counts the number of ways to pick $m_\ell$ nodes out of $n$, and $\kappa_\ell$ is the (finite number) of relabelings to count different subgraphs of type $\ell$ on a given set of $m_\ell$ nodes.

So, we show below that $|\tilde{S}^{true}_\ell - S^{true}_\ell| = o_p(S^{true}_\ell)$ and $|\tilde{S}_\ell(g) - S^{true}_\ell| = o_p(S^{true}_\ell)$; which then also implies that $\tilde{S}_\ell(g) - \tilde{S}^{true}_\ell = o_p(S^{true}_\ell)$. Together with (C.2), these tell us that

$$\tilde{\beta}_\ell = \left( \frac{S^{true}_\ell}{\kappa_\ell \binom{n}{m_\ell}} \right) (1 + o_p(1)).$$

Given that the network is growing that $S_\ell(g)$ has a binomial distribution with parameter $\beta_{0,\ell}$, parts (1) and (2) follow directly.

To see this, let us define an estimator $\tilde{\beta}^{true}_\ell$ based on the $S^{true}_\ell$:

$$\tilde{\beta}^{true}_\ell = \left( \frac{S^{true}_\ell}{\kappa_\ell \binom{n}{m_\ell}} \right),$$

noting that this is a theoretical construct since $S^{true}_\ell$ is unobserved as noted above.

We know that

$$\Sigma^{-1/2}(\tilde{\beta}^{true}_\ell - \beta^{n}_{0,\ell}) \sim N(0, I)$$

where $\Sigma_{\ell,\ell} = \frac{\beta^{n}_{0,\ell}(1-\beta^{n}_{0,\ell})}{\kappa_\ell \binom{n}{m_\ell}}$ and the off-diagonals are all 0. Then, since we have shown that $\tilde{\beta} = \tilde{\beta}^{true}_\ell H$, where $H$ is a diagonal matrix with $H_{\ell\ell} = 1 + \varepsilon_\ell$, with $\varepsilon_\ell = o_p(1)$, (2) then follows.

So, to complete the proof we show that $|\tilde{S}^{true}_\ell - S^{true}_\ell| = o_p(S^{true}_\ell)$ and $|\tilde{S}_\ell(g) - S^{true}_\ell| = o_p(S^{true}_\ell)$.

To establish these claims, we establish two facts. One is that the probability that some observed subgraph of type $\ell$ was incidentally generated (by subgraphs that are no larger than it in the ordering) is $o_p(1)$. This establishes that $|\tilde{S}_\ell(g) - S^{true}_\ell| = o_p(S^{true}_\ell)$. The other is that a truly formed subgraph of type $\ell$ becomes part of an incidentally generated subgraph of type $\ell' < \ell$ is $o_p(1)$. This establishes that $|\tilde{S}^{true}_\ell - S^{true}_\ell| = o_p(S^{true}_\ell)$.

Let $z^{n}_\ell$ denote the probability that any given $g' \in G^{n}_\ell$ is incidentally generated. We now show that $z^{n}_\ell/\beta^{n}_{0,\ell} = o(1)$, which establishes the first claim.

Consider $g_\ell \in G^{n}_\ell$ and a (minimal, ordered) generating subclass $\mathcal{C} = (\ell_j, c_j)_{j \in J}$, and for which $\ell_j \geq \ell$ fr all $j$.

\footnote{For example, note that $\kappa_\ell = 1$ for a triangle but for a $K$-star it is $K$ since each star is different when a different member of the $K$ nodes is the center.}
We show that the probability \( z^n_\ell \) that it is generated by this subclass goes to zero relative to \( \beta^n_{0,\ell} \), and since there are at most \( M_\ell \leq k^{m_\ell} \) such generating classes, this implies that \( z^n_\ell / \beta^n_{0,\ell} \to 0 \).

Consider a subnetwork in \( G^n_\ell \). The probability of getting at least one such network that has the \( c_j \) nodes out of the \( m_\ell \) in \( g_\ell \) is no more than

\[
\kappa_\ell \left( \frac{n}{m_\ell} - c_j \right) \beta^n_{0,\ell_j} \leq \kappa_\ell_j n^{m_\ell_j - c_j} \beta^n_{0,\ell_j}.
\]

Then, we can bound the desired ratio by

\[
\frac{z^n_\ell \beta^n_{0,\ell}}{\beta^n_{0,\ell_j}} \leq \frac{\prod_{j \in J} n^{m_\ell_j - c_j} \kappa_\ell_j \beta^n_{0,\ell_j}}{\beta^n_{0,\ell}} \leq \frac{\prod_{j \in J} n^{m_\ell_j} \kappa_\ell_j \beta^n_{0,\ell_j}}{\sum_j c_j \beta^n_{0,\ell}} \leq 0.
\]

where the penultimate step follows from the fact that \( M_C = \sum_{j \in J} c_j - m_\ell \) and \( M_C \geq 1 \) (since \( |J| \geq 2 \) and some \( c_j \) intersects with at least one other set of \( c_{j'} \) for some \( j' \neq j \), as the subgraph is not just isolated pairs of links) and the final step follows from the sparseness condition

\[
\frac{\prod_{j \in J} E[S^n_j]}{n^{M_C} E[S^n_\ell]} \to 0
\]

since the numerator of the final expression is of the order \( \Pi_{j \in 1, \ldots, C} E[S^n_j] \) while the denominator is of the order \( n^{M_C} E[S^n_\ell] \).

The second claim follows from a similar calculation. It is sufficient to show that the probability that some subgraph of type \( \ell_j' \) becomes part of a subgraph of type \( \ell < \ell_j' \) (where \( j' \in J \) is part of a generating class of some \( \ell < \ell_j' \)), compared to the likelihood of the formation of a subgraph of type \( \ell_j' \), is of vanishing order. Again, as there are a finite number of larger subgraphs, and a finite number of generating classes, it is sufficient to show this for a generic \( \ell < \ell_j' \) and generic generating class. In the following, the numerator is on the order of the expected number of incidentally formed subgraphs of type \( \ell \) from this type of generating class, while the denominator is the expected number of the subgraphs of type \( \ell \).

\[
\frac{\kappa_\ell \left( \frac{n}{m_\ell} \right) \prod_{j \in J} n^{m_\ell_j - c_j} \kappa_\ell_j \beta^n_{0,\ell_j}}{\kappa_{\ell_j'} \left( \frac{n}{m_{j'}} \right) \beta^n_{0,\ell_{j'}}}
\]
$$= \Theta \left( \frac{\prod_{j \in J} n^{m_{ij}} \beta_{0,ij}}{n^\sum_{j=1,\ldots,C} c_j - m_{ij} + m_{ij}'} \beta_{0,ij}' \right) \to 0.$$  

This convergence to 0 follows from the second part of the sparsity condition, which implies that

$$\prod_{j \in 1,\ldots,C} E_{\beta_0^n \left( S^{n}_{ij} \left( g \right) \right)} \to 0,$$

for each $j' \in 1,\ldots,C$. $\blacksquare$
In some cases, we are interested in testing how networks change in response to some change in the setting. As an example, we wish to know how the introduction of microfinance changes network structure, and not only how this affects people who end up with formal loans, but also among those who do not.

Banerjee, Chandrasekhar, Duflo, and Jackson (2016) examine the introduction of microfinance to 43 out of 75 villages - and based on network surveys before and after the microfinance introduction. This allows one to do a diff-n-diff analysis of the networks. SUGMs are particularly relevant since we wish to estimate not only how this affects links, but also how it affects triangles (supported relationships) and other features of the network. Moreover, we need to see how this varies with demographics. In particular, households vary in their appetite and eligibility for microfinance. We can distinguish (at least) two types of households in a village: those that are likely to join microfinance if available and those that are not. A SUGM-style dynamic matching model helps us identify the effects that microfinance has on the overall social network structure in a society and break that down by relationship type and household characteristics.

By estimating changes in SUGMs in villages where microfinance is introduced and comparing those changes to villages in which microfinance was not introduced, we can test these hypotheses.

To do this, let us first describe a stylized simplified version of the model in which people search for or invest in relationships. One can think about this as people putting in costly effort to meet other people in a “town square” and the odds of meeting people depend on one’s effort and others’ efforts. Consider a game where agents have $k$ tasks, each of which require a clique of $m_\ell$ nodes. For simplicity, assume that the value of being in a group does not depend on others’ observables or your own. An agent searches with effort $e$ for partners of each type of subgraph, and for simplicity we can think of this picking the contribution to the probability of the subgraph forming directly and again assume that this is additive for simplicity. So if $i, j, k$ put in some efforts, the probability of them forming a group is $e_i + e_j + e_k$ in this example. (In Banerjee, Chandrasekhar, Duflo, and Jackson (2016) this is more complicated and exhibits complementarities.) Assume there is a convex cost of effort and that high enough effort becomes infinitely costly.
Then the expected utility is

\[ U_i(e) = \sum_{\ell=1}^{k} \sum_{i_1, \ldots, i_{m-1}} v_\ell \left( \sum_{r \in \{i_1, \ldots, i_{m-1}\}} e_r \right) - \frac{1}{2} c \sum_{\ell} \sum_{i_1, \ldots, i_{m-1}} e_{\ell,i,i}^2. \]

and therefore

\[ v_\ell \left( \frac{n}{m_{\ell-1}} \right) = \left( \frac{n}{m_{\ell-1}} \right) e_{\ell,i} \implies e_{\ell,i}^* = v_\ell. \]

By symmetry then

\[ p_\ell = \frac{1}{m_{\ell}} \cdot \frac{v_\ell}{c}. \]

Therefore the fundamental parameter to be estimated is the ratio of the marginal benefit to the marginal cost, which we can define as \( \beta_\ell := p_\ell. \)

This is of course a stylized model. Banerjee et al. (2016) consider a more elaborate model when thinking about how to model network response to microfinance exposure. There we model the payoff to a link or a triangle as \( v_{L,\theta_i,\theta_j} \) and \( v_{T,\theta_i,\theta_j,\theta_k} \) where \( \theta_i \) is a node’s type: whether the node is likely (\( \theta_i = H \)) or not (\( \theta_i = L \)) to join microfinance if it were available in the village. \( H \) types are those, for instance, who have a female of eligible age in the household because that is a necessary condition to be eligible. Agents spend effort to meet in the “town square” where the meeting probabilities are complementary in the efforts applied, as described above.

If microfinance generates crowd-out of direct links by raising the value of autarky relative to the value of maintaining a link, then \( v_{L,H,\theta_j} \) and \( v_{T,H,\theta_j,\theta_k} \) should decline in microfinance villages for every \( \theta_i = H \) type. Ceteris paribus, equilibrium efforts in maintaining or searching for partners of even \( L \)-types may decline in this case, leading to an externality wherein links between all types decline. This is precisely what we observe. In fact, we observe that \( LL \) links and \( LLL \) triads — subgroups involving no agents directly likely to receive microcredit if it were offered — experience the greatest declines in financial links. This model also allows for structural estimation of parameters, which we do.
Note that $Z$-stars are somewhat thorny as adding one link from the center of a $Z$-star to another node now results in $Z$ incidental $Z$-stars. Nonetheless, we can still identify them from counts of simple statistics.

It is useful to work with a parameter

$$\beta_{0,Z}^n = 1 - (1 - \beta_{0,Z}^n)^2 \left( \frac{n-2}{Z-1} \right)$$

which is the probability that a link $ij$ forms incidentally via on of $2 \left( \frac{n-2}{Z-1} \right)$ possible $Z$-stars of which $ij$ could potentially be a part.

Again, we say that a sequence of SUGMs is well-balanced if the probability that a link is formed directly or formed as part of some $Z$-star are of the same order: $\beta_{0,L, Z}^n = \Theta(\beta_{0,L}^n)$.

**Proposition E.1.**

- A SUGM based on links and $Z$-stars is identified by $S_L, S_d S_{d-1}$, where $S_d$ is the fraction of nodes that have degree $d$, for any choice of $0 < d < Z$. That is, if $\beta'_L, \beta'_Z \neq \beta_L, \beta_Z$ then

$$\left( E_{\beta'_L, \beta'_Z} [S_L(g)], \frac{E_{\beta'_L, \beta'_Z} [S_d(g)]}{E_{\beta'_L, \beta'_Z} [S_{d-1}(g)]} \right) \neq \left( E_{\beta_L, \beta_Z} [S_L(g)], \frac{E_{\beta_L, \beta_Z} [S_d(g)]}{E_{\beta_L, \beta_Z} [S_{d-1}(g)]} \right).$$

- Moreover, if a sequence of SUGMs is well-balanced and $\beta_{0,L}^n, \beta_{0,Z}^n$ are bounded away from 1, then $\beta_{0,L}^n, \beta_{0,Z}^n$ are identifiably unique.\(^{56}\)

In Proposition E.1 the identification is based on a comparison between the relative frequency of two degrees: $d$ and $d - 1$, for any $0 < d < Z$. For many choices of $\beta$, identification can be achieved with just $E_{\beta_L, \beta_Z} [S_d(g)]$ rather than $E_{\beta_L, \beta_Z} [S_{d-1}(g)]$, but to get identification across all parameter values requires comparison of two degrees.

**Proof of Proposition E.1.**

First, note that

\[(E.1) \quad q_L := E_{\beta_L, \beta_Z} [S_L(g)] = \beta_L + (1 - \beta_L) \left( 1 - (1 - \beta_Z)^2 \left( \frac{n-2}{Z-1} \right) \right),\]

\(^{56}\)Here the $r_{L,Z}$ are both set to be the order of $\beta_{0,L}^n$, and the identifiable uniqueness is achieved via

$$\left( \frac{E_{\beta_L, \beta_Z} [S_L(g)]}{r_{L,Z}}, \frac{E_{\beta_L, \beta_Z} [S_d(g)]}{r_{L,Z}} \right).$$
where the $(1 - \beta Z)^{2(n-2)}$ represents the probability that none of the $2^{\binom{n-2}{Z-1}}$ possible $Z$-stars, of which $ij$ could potentially be a part, form.

The expected fraction of nodes having degree $d < Z$ is

$$(E.2) \quad q_d := \mathbb{E}_{\beta_L, \beta_Z} [S_d(g)] = \binom{n-1}{d} \pi^d (1 - \pi)^{n-1-d} (1 - \beta Z)^{(n-1)},$$

where

$$\pi = \beta_L + (1 - \beta_L) \left( 1 - (1 - \beta Z)^{\binom{n-2}{Z-1}} \right)$$

is the probability that some link $ij$ forms without it being part of a $Z$-star centered at $i$.

We show the first part of proposition by showing that if

$$(E.3) \quad \left( q_L, \frac{q_d}{q_d-1} \right) = \left( \mathbb{E}_{\beta'_L, \beta'_Z} [S_L(g)], \frac{\mathbb{E}_{\beta'_L, \beta'_Z} [S_d(g)]}{\mathbb{E}_{\beta'_L, \beta'_Z} [S_{d-1}(g)]} \right) = \left( \mathbb{E}_{\beta_L, \beta_Z} [S_L(g)], \frac{\mathbb{E}_{\beta_L, \beta_Z} [S_d(g)]}{\mathbb{E}_{\beta_L, \beta_Z} [S_{d-1}(g)]} \right)$$

then $\beta_L, \beta_Z = \beta'_L, \beta'_Z$.

If $\beta_L, \beta_Z = (0, 0)$ then $q_L = 0$ which implies directly from (E.1) that $\beta'_L, \beta'_Z = (0, 0)$, and so that case is straightforward. So, we concentrate on the case in which $\beta_L, \beta_Z \neq (0, 0)$, which then implies that $q_L > 0$. Similarly, we consider the case in which $\beta'_L < 1, \beta'_Z < 1$ (as otherwise the network is complete), which implies that $q_L < 1$ and then that $\beta'_L < 1, \beta'_Z < 1$, which then also implies that $0 < q_d < 1$.

By (E.1):

$$q_L = \beta_L + (1 - \beta_L) \left( 1 - (1 - \beta Z)^{\binom{n-2}{Z-1}} \right).$$

which implies that

$$(1 - \beta Z)^{\binom{n-2}{Z-1}} = \left( \frac{1 - q_L}{1 - \beta_L} \right)^{1/2}.$$

This implies that

$$q_d = \binom{n-1}{d} \pi(\beta_L)^d (1 - \pi(\beta_L))^{n-1-d} \left( \frac{1 - q_L}{1 - \beta_L} \right)^{(n-1)/(2Z)},$$

where

$$\pi(\beta_L) = 1 - (1 - \beta L)^{1/2} (1 - q_L)^{1/2}.$$

This implies that

$$(E.4) \quad \frac{q_d}{q_d-1} = \frac{\pi(\beta_L)}{(1 - \pi(\beta_L))}.$$
Thus, repeating the argument for $\beta'$ and then employing (E.3):

$$
\frac{1 - (1 - \beta_L)^{1/2} (1 - q_L^{n})^{1/2}}{(1 - \beta_L)^{1/2} (1 - q_L^{n})^{1/2}} = \frac{1 - (1 - \beta'_L)^{1/2} (1 - q_L^{n})^{1/2}}{(1 - \beta'_L)^{1/2} (1 - q_L^{n})^{1/2}}.
$$

Thus

$$
\frac{1}{(1 - \beta_L)^{1/2}} = \frac{1}{(1 - \beta'_L)^{1/2}},
$$

which implies that $\beta_L = \beta'_L$. This then implies that $\beta_Z = \beta'_Z$ via (E.1), and so we have established the result.

To establish identifiable uniqueness we argue that for any $\varepsilon > 0$ there exists $\phi > 0$ such that for large enough $n$, if $\delta((\beta^n_L, \beta^n_Z), (\beta^n_{0,L}, \beta^n_{0,Z})) > \varepsilon$, then at least one of the following inequalities holds:

$$
\left| \frac{E_{\beta^n} [S_L(g)] - E_{\beta^n_0} [S_L(g)]}{r^n} \right| > \phi
$$

or

$$
\left| \frac{E_{\beta^n} [S_d(g)] - E_{\beta^n_0} [S_d(g)]}{r^n E_{\beta^n} [S_{d-1}(g)]} \right| > \phi.
$$

So, suppose the contrary: there exists a subsequence of $(\beta^n_L, \beta^n_Z), (\beta^n_{0,L}, \beta^n_{0,Z})$ and $\phi^n \to 0$ such that for every $n$ $\delta((\beta^n_L, \beta^n_Z), (\beta^n_{0,L}, \beta^n_{0,Z})) > \varepsilon$ and yet

$$
\left| \frac{E_{\beta^n} [S_L(g)] - E_{\beta^n_0} [S_L(g)]}{r^n} \right| \leq \phi^n
$$

and

$$
\left| \frac{E_{\beta^n} [S_d(g)] - E_{\beta^n_0} [S_d(g)]}{r^n E_{\beta^n} [S_{d-1}(g)]} \right| \leq \phi^n.
$$

By (E.4), it follows that

$$
q^n_{d} / q^n_{d-1} = 1 - (1 - \beta^n_L)^{1/2} (1 - q^n_L)^{1/2}
$$

and

$$
q^n_{0,d} / q^n_{0,d-1} = 1 - (1 - \beta^n_{0,L})^{1/2} (1 - q^n_{0,L})^{1/2}.
$$

Suppose that $|q^n_L - q^n_{0,L}| \leq r^n \phi^n$. For any $\gamma > 0$, if

$$
\frac{|\beta^n_L - \beta^n_{0,L}|}{\max(\beta^n_L, \beta^n_{0,L})} > \gamma
$$

then

$$
|\beta^n_L - \beta^n_{0,L}| > c_1 r^n \gamma,
$$

We use the obvious notation for $q^n_d$ and $q^n_{0,d}$, and so forth following (E.1) and (E.2), to indicate the expected counts associated with the parameters $\beta^n$ and $\beta^n_0$. 

\[57\]
and so it follows that for small enough $\phi^n$, $|q^n_d/q^n_{d-1} - q^n_{0,d}/q^n_{0,d-1}| > \phi^n$, which is a contradiction. Thus, it must also be that there is a sequence $\gamma^n \to 0$, and a further subsequence of our parameters such that

$$|\beta^n_L - \beta^n_{0,L}| \leq \gamma^n$$

along the subsequence.

Along the subsequence it must then be that

$$\frac{|\beta^n_Z - \beta^n_{0,Z}|}{\max(\beta^n_Z, \beta^n_{0,Z})} > \varepsilon,$$

and so

$$|\beta^n_Z - \beta^n_{0,Z}| > c_1 r^n \varepsilon.$$

But now, from (E.1)

$$q^n_L = \beta^n_L + (1 - \beta^n_L)\beta^n_Z$$

and

$$q^n_{0,L} = \beta^n_{0,L} + (1 - \beta^n_{0,L})\beta^n_{0,Z}.$$ 

Then given that $\beta^n_{0,L}, \beta^n_{0,Z}$ are bounded away from 1, and that $|\beta^n_L - \beta^n_{0,L}| \leq \gamma^n \to 0$, while $|\beta^n_Z - \beta^n_{0,Z}| > c_1 r^n \varepsilon$ implies the difference in the two equations above cannot tend to 0, which is a contradiction. Thus, our supposition was incorrect. ■