ON ALGEBRAIC SPACES WITH AN ACTION OF $\mathbb{G}_m$

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Abstract. Let $Z$ be an algebraic space of finite type over a field, equipped with an action of the multiplicative group $\mathbb{G}_m$. In this situation we define and study a certain algebraic space equipped with an unramified morphism to $\mathbb{A}^1 \times Z \times Z$. (If $Z$ is affine and smooth this is just the closure of the graph of the action map $\mathbb{G}_m \times Z \to Z$.)

In articles joint with D. Gaitsgory we use this set-up to prove a new result in the geometric theory of automorphic forms and to give a new proof of a very important theorem of T. Braden.

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INTRODUCTION

0.1. The goals of this article. Algebraic varieties equipped with an action of the multiplicative group \( \mathbb{G}_m \) have been studied for quite a while (especially by A. Białynicki-Birula and his school); see, e.g., the works [Bi1, BS, Ju1, Ju2, Kon, Som].

This article has two goals.

The first one is to define the notion of attractor for an arbitrary algebraic space \( Z \) of finite type over a field \( k \) equipped with a \( \mathbb{G}_m \)-action and to prove the basic properties of attractors in this generality. The main difficulty is that if \( Z_{\text{red}} \) is not assumed to be a normal scheme then Sumihiro’s theorem is not applicable, so the \( \mathbb{G}_m \)-action on \( Z \otimes_k \bar{k} \) is not necessarily locally linear.\(^2\)

The second (and more important) goal is to provide the geometric background for the articles [DrGa1, DrGa2]. Namely, for any algebraic \( k \)-space \( Z \) of finite type acted on by \( \mathbb{G}_m \), we define a certain algebraic space \( \tilde{Z} \) of finite type over \( A^1 \times Z \times Z \) and study its properties. The space \( \tilde{Z} \) seems to be new even if \( Z \) is a separated smooth scheme (although it is somewhat similar to the space from [BS] Theorem 0.1.2 denoted there by \( Z \)). The space \( \tilde{Z} \) plays a crucial role in [DrGa2], where it is used to prove a new result in the geometric theory of automorphic forms. It also allows to give a new proof of a very important theorem of T. Braden, see [DrGa1]. In each of the articles [DrGa1, DrGa2] the space \( \tilde{Z} \) is used to construct the unit of a certain adjunction.

Now let us explain more details.

0.2. Attractors and repellers. Let \( k \) be any field, and let \( Z \) be a an algebraic \( k \)-space of finite type acted on by \( \mathbb{G}_m \). According to Theorem 1.4.2 and the easy Proposition 1.2.2, there exist algebraic spaces \( Z^0, Z^+, \) and \( Z^- \) of finite type over \( k \) representing the following functors:

\[
\begin{align*}
\text{Maps}(S, Z^0) &= \text{Maps}^\mathbb{G}_m(S, Z), \\
\text{Maps}(S, Z^+) &= \text{Maps}^\mathbb{G}_m(A^1 \times S, Z), \\
\text{Maps}(S, Z^-) &= \text{Maps}^\mathbb{G}_m(A^1_\mathbb{Z} \times S, Z),
\end{align*}
\]

\(^1\)We do not require \( Z \) to be either separated or normal. We include quasi-separatedness in the definition of algebraic space, but this is a very weak property (which holds automatically for schemes of finite type over \( k \)).

\(^2\)An action of \( \mathbb{G}_m \) on a scheme \( Z \) is said to be locally linear if \( Z \) can be covered by open affine subschemes preserved by the \( \mathbb{G}_m \)-action.
where \( S \) is a test scheme, \( \mathbb{A}^1 \) := \( \mathbb{P}^1 - \{ \infty \} \), and the \( \mathbb{G}_m \)-actions on \( \mathbb{A}^1 \) and \( \mathbb{A}^1_\perp \) are the usual ones. The space \( Z^0 \) (resp. \( Z^+ \) and \( Z^- \)) is called the space of \( \mathbb{G}_m \)-fixed points (resp. the attractor and repeller).

Let \( p^+ : Z^+ \to Z \) and \( q^+ : Z^+ \to Z^0 \) denote the maps corresponding to evaluating a \( \mathbb{G}_m \)-equivariant morphism \( \mathbb{A}^1 \times S \to Z \) at 1 \( \in \mathbb{A}^1 \) and 0 \( \in \mathbb{A}^1 \), respectively. One defines \( p^- : Z^- \to Z \) and \( q^- : Z^- \to Z^0 \) similarly. Let \( i^+ : Z^0 \to Z^+ \) (resp. \( i^- : Z^0 \to Z^- \)) denote the morphism induced by the projection \( \mathbb{A}^1 \times S \to S \) (resp. \( \mathbb{A}^1_\perp \times S \to S \)).

The morphisms \( p^\pm : Z^\pm \to Z \) are always unramified\(^3\) and if \( Z \) is separated they are monomorphisms (see Proposition \( \ref{1.4.11} \)). Of course, if \( Z \) is affine then so are \( Z^0 \) and \( Z^\pm \); moreover, in this case the morphisms \( p^\pm : Z^\pm \to Z \) are closed embeddings.

Let us also mention Proposition \( \ref{1.6.2} \) which says that the morphism
\[
\lambda := (i^+, i^-) : Z^0 \rightarrow Z^+ \times Z^- \tag{1.6.2}
\]
is an open embedding (and also a closed one). This fact is used in \[\text{DrGa1}\] to construct the co-unit of the adjunction in Braden’s theorem.

Of course, in the case where \( Z \) is a scheme equipped with a locally linear \( \mathbb{G}_m \)-action all above-mentioned results are well known (in a slightly different language).

0.3. The space \( \tilde{Z} \).

0.3.1. Hyperbolas. We now consider the following family of curves over \( \mathbb{A}^1 \), denoted by \( X \): as a scheme, \( X = \mathbb{A}^2 := \text{Spec} \ k[\tau_1, \tau_2] \), and the map \( X \to \mathbb{A}^1 \) is \( (\tau_1, \tau_2) \mapsto \tau_1 \tau_2 \). The fibers of this map are hyperbolas; the zero fiber is the coordinate cross, i.e., a degenerate hyperbola.

We let \( \mathbb{G}_m \) act on \( X \) hyperbolically:
\[
\lambda \cdot (\tau_1, \tau_2) := (\lambda \cdot \tau_1, \lambda^{-1} \cdot \tau_2).\]

0.3.2. The space \( \tilde{Z} \). According to Theorem \( \ref{2.2.2} \) there exists an algebraic space \( \tilde{Z} \) of finite type over \( \mathbb{A}^1 \) representing the following functor on the category of schemes over \( \mathbb{A}^1 \):
\[
\text{Maps}_{\mathbb{A}^1}(S, \tilde{Z}) := \text{Maps}^{\mathbb{G}_m}(X \times S, Z).\]

If \( Z \) is a scheme equipped with a locally linear \( \mathbb{G}_m \)-action then the existence of \( \tilde{Z} \) (as a scheme) is easy to prove, see Subsect. \( \ref{2.3} \).

In general, we prove representability of the above functor using M. Artin’s technique (see Section \( \ref{3} \). It would be nice if somebody finds a simpler and more constructive proof of representability.

0.3.3. The canonical morphism \( \tilde{p} : \tilde{Z} \rightarrow \mathbb{A}^1 \times Z \times Z \). Note that any section \( \sigma : \mathbb{A}^1 \rightarrow X \) of the morphism \( X \rightarrow \mathbb{A}^1 \) defines a map
\[
\sigma^* : \text{Maps}^{\mathbb{G}_m}(X \times S, Z) \rightarrow \text{Maps}(S, Z)
\]
and therefore a morphism \( \tilde{Z} \rightarrow Z \). Let \( \pi_1 : \tilde{Z} \rightarrow Z \) and \( \pi_2 : \tilde{Z} \rightarrow Z \) denote the morphisms corresponding to the sections
\[
t \mapsto (1, t) \in X \quad \text{and} \quad t \mapsto (t, 1) \in X.
\]

\(^3\)Using the map \( t \mapsto t^{-1} \), one can identify \( \mathbb{A}^1 \) with the scheme \( \mathbb{A}^1 \) equipped with the \( \mathbb{G}_m \)-action opposite to the usual one.

\(^4\)The definition of “unramified” is recalled in Subsect. \( \ref{0.5.5} \) below.
respectively. Now define

\[ \tilde{p} : \tilde{Z} \to \mathbb{A}^1 \times Z \times Z \]  

to be the morphism whose first component is the tautological projection \( \tilde{Z} \to \mathbb{A}^1 \), and the second and the third components are \( \pi_1 \) and \( \pi_2 \), respectively.

The morphism \( \tilde{p} : \tilde{Z} \to \mathbb{A}^1 \times Z \times Z \) is always unramified, and if \( Z \) is separated then \( \tilde{p} \) is a monomorphism (see Proposition 2.2.6). Moreover, if \( Z \) is affine then \( \tilde{p} \) is a closed embedding (see Proposition 2.3.3), so \( \tilde{p} \) identifies \( \tilde{Z} \) with a closed subscheme of \( \mathbb{A}^1 \times Z \times Z \).

0.3.4. The fibers of the morphism \( \tilde{Z} \to \mathbb{A}^1 \). Let \( \tilde{Z}_t \) denote the preimage of \( t \in \mathbb{A}^1 \) under the projection \( \tilde{Z} \to \mathbb{A}^1 \). Let \( \tilde{p}_t \) denote the corresponding map \( \tilde{Z}_t \to Z \times Z \).

By definition, \((\tilde{Z}_{t_1}, \tilde{p}_{t_1})\) identifies with \((Z_{t_1}, \Delta Z_{t_1})\). For any \( t \in \mathbb{A}^1 - \{0\} \), the pair \((\tilde{Z}_t, \tilde{p}_t)\) is the graph of the action of \( t \in \mathbb{G}_m \) on \( Z \). Moreover, the morphism \( \tilde{p} \) induces an isomorphism

\[ \tilde{Z} \stackrel{\sim}{\to} \Gamma, \quad \Gamma := \{(t, z_1, z_2) \mid t \cdot z_1 = z_2\}. \]

The space \( \tilde{Z}_0 \) identifies with \( Z^+ \times Z^- \) so that the morphism \( \tilde{p}_0 : \tilde{Z}_0 \to Z \times Z \) identifies with the composition

\[ Z^+ \times Z^- \to Z^+ \times Z^- \stackrel{p^+ \times p^-}{\to} Z \times Z. \]

The above-mentioned identification comes from the fact that the degenerate hyperbola \( X_0 \) is the union of the coordinate axes, one of which identifies with \( \mathbb{A}^1 \) and the other one with \( \mathbb{A}^1 - \{0\} \).

Thus \( \tilde{Z} \) provides an “interpolation” between the spaces \( \tilde{Z}_1 = Z \) and \( \tilde{Z}_0 = Z^+ \times Z^- \).

0.3.5. Smoothness. If \( Z \) is smooth then so is the morphism \( \tilde{Z} \to \mathbb{A}^1 \), see Proposition 2.2.4.

If \( Z \) is smooth and affine then by Proposition 2.3.3 the morphism \( \tilde{p} : \tilde{Z} \to \mathbb{A}^1 \times Z \times Z \) induces an isomorphism

\[ \tilde{Z} \stackrel{\sim}{\to} \tilde{\Gamma}, \]

where \( \Gamma \) is as in formula (0.2) and \( \tilde{\Gamma} \) is the scheme-theoretic closure of \( \Gamma \) in \( \mathbb{A}^1 \times Z \times Z \).

0.3.6. Remark. We prove that if the algebraic space \( Z \) is separated (resp. is a scheme) then so are all the algebraic spaces \( Z^0 \), \( Z^+ \), and \( Z \) (see Proposition 1.2.2, Corollary 1.4.3, and Proposition 2.2.6).

0.4. Organization of the paper. In Sect. 1 we define and study the space of \( \mathbb{G}_m \)-fixed points \( Z^0 \), the attractor \( Z^+ \), and the repeller \( Z^- \) corresponding to an algebraic \( k \)-space \( Z \) of finite type acted on by \( \mathbb{G}_m \).

In Sect. 2 we define and study the space \( \tilde{Z} \). A more detailed description of Section 2 can be found at the beginning of the section.

In Sect. 3 we prove some openness results. One of them is used in [DrGa1].

In Sections 4, 5 we prove Theorems 1.4.2 and Theorems 2.2.2 (the proofs are too long to be given in Sections 1 and 2).

In Appendix A we prove a very general Lemma 3.1.16.

In Appendix B we briefly recall some results on attractors due to A. Białynicki-Birula, J. Konarski, and A. J. Sommese.

0.5. Some conventions and recollections.
0.5.1. Maps and morphisms as synonyms. We often use the word “map” as a synonym of “morphism”. The space of morphisms between objects $X,Y$ of a category will usually be denoted by $\text{Maps}(X,Y)$.

0.5.2. General notion of $k$-space. Once and for all, we fix a field $k$ (of any characteristic). By a $k$-space (or simply space) we mean a contravariant functor $F$ from the category of $k$-schemes to that of sets which is a sheaf for the fpqc topology. Instead of considering all $k$-schemes as “test schemes”, it suffices to consider affine ones (any fpqc sheaf on the category of affine $k$-schemes uniquely extends to an fpqc sheaf on the category of all $k$-schemes). Instead of $F(\text{Spec } R)$ we write simply $F(R)$; in other words, we consider $F$ as a covariant functor on the category of $k$-algebras.

Note that for any $k$-scheme $S$ we have $F(S) = \text{Maps}(S,F)$, where Maps stands for the set of morphisms between $k$-spaces. Usually we prefer to write $\text{Maps}(S,F)$ rather than $F(S)$.

0.5.3. Algebraic $k$-spaces. General $k$-spaces will appear only as “intermediate” objects. For us, the really geometric objects are algebraic spaces. We will be using the definition of algebraic space from [LM] (which goes back to M. Artin).

Any quasi-separated $k$-scheme (in particular, any $k$-scheme of finite type) is an algebraic space. The reader may prefer to restrict his attention to schemes.

0.5.4. Monomorphisms. A morphism of $k$-spaces $f : X_1 \to X_2$ is said to be a monomorphism if the corresponding map

$$\text{Maps}(S, X_1) \to \text{Maps}(S, X_2)$$

is injective for any $k$-scheme $S$. In particular, this applies if $X_1$ and $X_2$ are algebraic spaces (e.g., schemes). It is known that a morphism of finite type between schemes (or algebraic spaces) is a monomorphism if and only if each of its geometric fibers is a reduced scheme with at most one point. It follows that a finite monomorphism is a closed embedding.

0.5.5. Unramified morphisms. According to Definition 17.3.1 from EGA IV-4, a morphism of schemes is said to be unramified if it is formally unramified and locally of finite presentation. The definition in [St] is slightly different: “locally of finite presentation” is replaced by “locally of finite type”. The difference is irrelevant for us because we will be dealing between morphisms between Noetherian schemes (or algebraic spaces).

Recall that a morphism $f$ is formally unramified if and only if the corresponding sheaf of relative differentials is zero. If $f$ has finite type this is equivalent to the geometric fibers of $f$ being finite and reduced.

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I am especially grateful to D. Gaitsgory. In fact, this article appeared as a part of a project joint with him (see [DrGa1, DrGa2]). Moreover, a part of the work on this article was done jointly with him (e.g., the formulation of Propositions 1.6.2 and 3.1.3 is due to D. Gaitsgory).

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5In particular, quasi-separatedness is included into the definition of algebraic space. Thus the quotient $A^1/Z$ (where the discrete group $Z$ acts by translations) is not an algebraic space.
1. Fixed points, attractors, and repellers

The main results of this section are Proposition 1.2.2, Theorem 1.4.2, Proposition 1.4.11, and Proposition 1.6.2 (the latter is used in [DrGa1] to construct the co-unit of the adjunction in Braden’s theorem). In the case of a scheme equipped with a locally linear \(G_m\)-action these results are well known (in a slightly different language).

We will be using the conventions of Subsect. 0.5 and especially those regarding the notions of \(k\)-space and algebraic \(k\)-space (see 0.5.2-0.5.3).

1.1. The space of \(G_m\)-equivariant maps. Let \(Y, Z\) be \(k\)-spaces equipped with an action of \(G_m\). Then we define a \(k\)-space \(\text{Maps}^{G_m}(Y, Z)\) as follows: for any \(k\)-scheme \(S\),

\[
(1.1) \quad \text{Maps}(S, \text{Maps}^{G_m}(Y, Z)) := \text{Maps}^{G_m}(Y \times S, Z)
\]

(the r.h.s. is clearly an fpqc sheaf with respect to \(S\)). The action of \(G_m\) on \(Z\) induces a \(G_m\)-action on \(\text{Maps}^{G_m}(Y, Z)\).

Note that even if \(Y\) and \(Z\) are schemes, the space \(\text{Maps}^{G_m}(Y, Z)\) does not have to be a scheme (or an algebraic space), in general.

1.2. The space of fixed points. Let \(Z\) be a \(k\)-space equipped with an action of \(G_m\). Then we set

\[
(1.2) \quad Z^0 := \text{Maps}^{G_m}((\text{Spec} \, k, Z)).
\]

Note that \(Z^0\) is a subspace of \(Z\) because \(\text{Maps}(S, Z^0) = \text{Maps}^{G_m}(S, Z)\) is a subset of \(\text{Maps}(S, Z)\).

Definition 1.2.1. \(Z^0\) is called the subspace of fixed points of \(Z\).

Proposition 1.2.2. If \(Z\) is an algebraic \(k\)-space (resp. scheme) of finite type then so is \(Z^0\). Moreover, the morphism \(Z^0 \to Z\) is a closed embedding.

This proposition is easy. The only surprise is that \(Z^0 \subset Z\) is closed even if \(Z\) is not separated. Idea of the proof: since \(G_m\) is connected \(Z^0 = Z_0\), where \(Z_0\) is the space of fixed points of the formal multiplicative group acting on \(Z\); on the other hand, \(Z_0\) is a closed subspace of \(Z\) (e.g., in the characteristic zero case \(Z_0\) is just the space of zeros of the vector field on \(Z\) corresponding to the \(G_m\)-action). The detailed proof is below.

Proof. It suffices to show that the morphism \(Z^0 \to Z\) is a closed embedding.

Let \(\mathfrak{G}\) be the space of stabilizers, i.e., an \(S\)-point of \(\mathfrak{G}\) is a pair \((z, g)\), where \(z \in Z(S)\) and \(g \in G_m(S)\) stabilizes \(z\). We have a monomorphism of group schemes over \(Z\)

\[
(1.3) \quad \varphi : \mathfrak{G} \to G_m \times Z.
\]

The corresponding morphism \(\hat{\varphi}\) of formal group schemes over \(Z\) is a closed embedding (because a finite monomorphism between schemes is a closed embedding). The image of \(\hat{\varphi}\) is a closed subspace of

\[
\lim_{\longrightarrow n} \text{Spec}A_n, \quad A_n := \mathcal{O}_Z[\lambda]/(\lambda - 1)^n.
\]

Let \(J_n \subset A_n\) be the corresponding sheaves of ideals. Set

\[
J_n := \text{Im}(J_n \otimes \mathcal{Hom}_\mathcal{O}_Z(A_n, \mathcal{O}_Z) \to \mathcal{O}_Z).
\]

\footnote{This is a slight abuse of language: if \(Z\) itself is not a scheme then \(\mathfrak{S}\) is a scheme only in the relative sense (i.e., \(\mathfrak{S} \times Z S\) is a scheme for any scheme \(S\) over \(Z\)).}
Each \( j_0 \) is an ideal in \( \mathcal{O}_Z \), and \( j_n \subset j_{n+1} \). Let \( Z_0 \subset Z \) be the closed subspace corresponding to the union of the ideals \( j_n \).

Let us prove that \( Z^0 = Z_0 \). It is clear that \( Z^0 \subset Z_0 \). It remains to show that the morphism \( \varphi_0 : \mathfrak{g} \times_Z Z_0 \to \mathbb{G}_m \times Z_0 \) induced by the map \( f \) is an isomorphism. Since \( \varphi \) is a monomorphism so is \( \varphi_0 \). On the other hand, \( \varphi_0 \) is etale by the definition of \( Z_0 \). So \( \varphi_0 \) is an open embedding. Since \( \mathbb{G}_m \) is connected this implies that \( \varphi_0 \) is an isomorphism. \( \square \)

**Example 1.2.3.** Suppose that \( Z \) is an affine scheme \( \text{Spec} \ A \). A \( \mathbb{G}_m \)-action on \( Z \) is the same as a \( \mathbb{Z} \)-grading on \( A \) (namely, the \( n \)-th component of \( A \) consists of functions \( f \in H^0(Z, \mathcal{O}_Z) \) such that \( f(\lambda \cdot z) = \lambda^n \cdot f(z) \)). It is easy to see that \( Z^0 = \text{Spec} \ A^0 \), where \( A^0 \) is the maximal graded quotient algebra of \( A \) concentrated in degree 0 (in other words, \( A^0 \) is the quotient of \( A \) by the ideal generated by homogeneous elements of non-zero degree).

**Lemma 1.2.4.** For any \( z \in Z^0 \) the tangent space \( T_z Z^0 \subset T_z Z \) equals \( (T_z Z)_{\mathbb{G}_m} \).

**Proof.** We can assume that the residue field of \( z \) equals \( k \) (otherwise do base change). Then compute \( T_z Z^0 \) in terms of morphisms \( \text{Spec} \ k[e]/(e^2) \to Z^0 \). \( \square \)

### 1.3. Attractors.

1.3.1. **The definition.** Let \( Z \) be a \( k \)-space equipped with an action of \( \mathbb{G}_m \). Then we set

\[
Z^+ := \text{Maps}^{\mathbb{G}_m}(\mathbb{A}^1, Z),
\]

where \( \mathbb{G}_m \) acts on \( \mathbb{A}^1 \) by dilations.

**Definition 1.3.2.** \( Z^+ \) is called the **attractor** of \( Z \).

Later we will prove (see Theorem 1.4.4) that if \( Z \) is an algebraic space of finite type then so is \( Z^+ \).

1.3.3. **Structures on \( Z^+ \).** (i) \( \mathbb{A}^1 \) is a monoid with respect to multiplication. The action of \( \mathbb{A}^1 \) on itself induces an \( \mathbb{A}^1 \)-action on \( Z^+ \), which extends the \( \mathbb{G}_m \)-action defined in Sect. 1.3.

(ii) Restricting a morphism \( \mathbb{A}^1 \times S \to Z \) to \( \{1\} \times S \) one gets a morphism \( S \to Z \). Thus we get a \( \mathbb{G}_m \)-equivariant morphism \( p^+ : Z^+ \to Z \).

Note that if \( Z \) is separated then \( p^+ : Z^+ \to Z \) is a monomorphism. To see this, it suffices to interpret \( p^+ \) as the composition

\[
\text{Maps}^{\mathbb{G}_m}(\mathbb{A}^1, Z) \to \text{Maps}^{\mathbb{G}_m}(\mathbb{G}_m, Z) = Z.
\]

Thus if \( Z \) is separated then \( p^+ \) identifies \( Z^+(k) \) with a subset of \( Z(k) \). It consists of those points \( z \in Z(k) \) for which the map \( \mathbb{G}_m \to Z \) defined by \( t \mapsto t \cdot z \) extends to a map \( f : \mathbb{A}^1 \to Z \); informally, the limit

\[
\lim_{t \to 0} t \cdot z
\]

should exist.

(iii) Recall that \( Z^0 = \text{Maps}^{\mathbb{G}_m}(\text{Spec} \ k, Z) \). We equip \( \text{Spec} \ k \) and \( Z^0 \) with the trivial action of the multiplicative monoid \( \mathbb{A}^1 \).

The \( \mathbb{A}^1 \)-equivariant maps \( 0 : \text{Spec} \ k \to \mathbb{A}^1 \) and \( \mathbb{A}^1 \to \text{Spec} \ k \) induce \( \mathbb{A}^1 \)-equivariant maps \( q^+ : Z^+ \to Z^0 \) and \( i^+ : Z^0 \to Z^+ \) such that \( q^+ \circ i^+ = \text{id}_{Z^0} \) and the composition \( p^+ \circ i^+ \) is equal to the canonical embedding \( Z^0 \to Z \).

Note that if \( Z \) is separated then for \( z \in Z^+(k) \subset Z(k) \) the point \( q^+(z) \) is the limit (1.5).

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We define the tangent space by \( T_z Z := (T^*_z Z)^* \), where \( T^*_z Z \) is the fiber of \( \Omega^1_{Z/k} \) at \( z \). (The equality \( T^*_z Z = m_z/m_z^2 \) holds if the residue field of \( z \) is finite and separable over \( k \).)
1.3.4. **Affine case.** Suppose that $Z$ is affine, i.e., $Z = \text{Spec } A$, where $A$ is a $\mathbb{Z}$-graded commutative algebra. It is easy to see that in this case $Z^+$ is represented by the affine scheme $\text{Spec } A^+$, where $A^+$ is the maximal $\mathbb{Z}_+$-graded quotient algebra of $A$ (in other words, the quotient of $A$ by the ideal generated by all homogeneous elements of $A$ of strictly negative degrees).

By Example 1.2.3, $Z^0 = \text{Spec } A^0$, where $A^0$ is the maximal graded quotient algebra of $A$ (or equivalently, of $A^+$) concentrated in degree 0. Since $A^+$ is $\mathbb{Z}_+$-graded the algebra $A^0$ identifies with the 0-th graded component of $A^+$. Thus we get homomorphisms $A^0 \hookrightarrow A^+ \twoheadrightarrow A^0$. They correspond to the morphisms $Z^0 \leftarrow q^+ \leftarrow i^+ \to Z^0$.

1.4. **Results on attractors.**

1.4.1. **Representability of $Z^+$.**

**Theorem 1.4.2.** Let $Z$ be an algebraic $k$-space of finite type equipped with a $\mathbb{G}_m$-action. Then

(i) $Z^+$ is an algebraic $k$-space of finite type;

(ii) the morphism $q^+: Z^+ \to Z^0$ is affine.

The proof is given in Section 4. It yields a rather explicit description of the pair $(Z^+, q^+)$ in terms of the formal neighborhood of $Z^0 \subset Z$, see Theorem 4.3.1(ii) and Subsect. 4.4. Note that if $Z$ is affine Theorem 1.4.2 is clear from Subsect. 1.3.4, and this immediately implies the theorem in the case of schemes equipped with a locally linear $\mathbb{G}_m$-action, see Subsect. 1.4.4 below. This case is enough for most practical purposes, see Remark 1.4.6 below.

**Corollary 1.4.3.** (i) If $Z$ is a separated algebraic $k$-space of finite type then so is $Z^+$.

(ii) If $Z$ is a $k$-scheme of finite type then so is $Z^+$.

**Proof.** Follows from Theorem 1.4.2(ii) because by Proposition 1.2.2 $Z^0$ is a closed subspace of $Z$. □

1.4.4. **Proof of Theorem 1.4.2 for schemes with a locally linear $\mathbb{G}_m$-action.**

**Definition 1.4.5.** An action of $\mathbb{G}_m$ on a $k$-scheme $Z$ is said to be locally linear if $Z$ can be covered by open affine subschemes preserved by the $\mathbb{G}_m$-action.

**Remark 1.4.6.** If $k$ is algebraically closed and $Z_{\text{red}}$ is a normal separated scheme of finite type over $k$ then by a theorem of H. Sumihiro, any action of $\mathbb{G}_m$ on $Z$ is locally linear. (The proof of this theorem is contained in [Sum] and also in [KKMS, p.20-23] and [KKLV].)

For schemes equipped with a locally linear $\mathbb{G}_m$-action Theorem 1.4.2 is very easy: it follows from the affine case (which is clear from Subsect. 1.3.4) and the following lemma.

**Lemma 1.4.7.** Let $Z$ be a $k$-space equipped with a $\mathbb{G}_m$-action. Let $U \subset Z$ be a $\mathbb{G}_m$-stable open subspace. Then the subspace $U^+ \subset Z^+$ equals $(q^+)^{-1}(U^0)$, where $q^+$ is the natural morphism $Z^+ \to Z^0$.

**Proof.** For any test scheme $S$, we have to show that if $f: \mathbb{A}^1 \times S \to Z$ is a $\mathbb{G}_m$-equivariant morphism such that $\{0\} \times S \subset f^{-1}(U)$ then $f^{-1}(U) = \mathbb{A}^1 \times S$. This is clear because $f^{-1}(U) \subset \mathbb{A}^1 \times S$ is open and $\mathbb{G}_m$-stable. □

---

8We do not know if separateness is really necessary in Sumihiro’s theorem.
Lemma 1.4.9. Let $F$ be a $k$-space equipped with a $\mathbb{G}_m$-action. Let $F \subset Z$ be a $\mathbb{G}_m$-stable closed subspace. Then the subspace $F^+ \subset Z^+$ equals $(p^+)^{-1}(F)$, where $p^+$ is the natural morphism $Z^+ \to Z$.

Proof. An $S$-point of $(p^+)^{-1}(F)$ is a $\mathbb{G}_m$-equivariant morphism $f : \mathbb{A}^1 \times S \to Z$ such that $G_m \times S \subset f^{-1}(F)$. Since $f^{-1}(F)$ is closed in $\mathbb{A}^1 \times S$ this implies that $f^{-1}(F) = \mathbb{A}^1 \times S$, i.e., $f(\mathbb{A}^1 \times S) \subset F$.

1.4.10. The morphism $p^+ : Z^+ \to Z$.

Proposition 1.4.11. Let $Z$ be an algebraic $k$-space of finite type equipped with a $\mathbb{G}_m$-action.

(i) The morphism $p^+ : Z^+ \to Z$ is unramified (i.e., its geometric fibers are finite and reduced);

(ii) If $Z$ is separated then $p^+ : Z^+ \to Z$ is a monomorphism (i.e., each geometric fiber of $p^+ : Z^+ \to Z$ is reduced and has at most one point);

(iii) If $Z$ is proper then each geometric fiber of $p^+ : Z^+ \to Z$ is reduced and has exactly one point.

(iv) If $Z$ is an affine scheme then $p^+ : Z^+ \to Z$ is a closed embedding.

(v) The fiber of $p^+ : Z^+ \to Z$ over any geometric point of $Z^0 \subset Z$ has a single point (even if $Z$ is not separated).

(vi) Let $z \in Z^0$ and $\zeta := i^+(z) \in Z^+$, so $p^+ (\zeta) = z$. Then the map of tangent spaces $T_z Z^+ \to T_{\zeta} Z$ corresponding to $p^+ : Z^+ \to Z$ induces an isomorphism $T_z Z^+ \cong (T_{\zeta} Z)^+$, where $(T_{\zeta} Z)^+ \subset T_{\zeta} Z$ is the non-negative part with respect to the $\mathbb{G}_m$-action on $T_{\zeta} Z$. Moreover, the tangent maps $T_z Z^+ \to T_{\zeta} Z^+ \to T_z Z^0$ corresponding to the morphisms $i^+ : Z^0 \to Z^+$ and $q^+ : Z^+ \to Z^0$ identify with the canonical maps $(T_z Z)^0 \hookrightarrow (T_{\zeta} Z)^+ \to (T_{\zeta} Z)^0$.

Proof. Statement (iv) is clear from Subsect. 1.3.3. Statement (ii) was proved in Subsect. 1.3.3(ii). Statement (iii) follows from (ii) and the fact that any morphism from $\mathbb{A}^1 - \{0\}$ to a proper scheme extends to the whole $\mathbb{A}^1$.

Let us prove (i). We can assume that $k$ is algebraically closed. Then we have to check that for any $\zeta \in Z^+(k)$ the map of tangent spaces

\[ T_z Z^+ \to T_{p^+(\zeta)} Z \]

induced by $p^+ : Z^+ \to Z$ is injective. Let $f : \mathbb{A}^1 \to Z$ be the $\mathbb{G}_m$-equivariant morphism corresponding to $\zeta$. Then

\[ T_z Z^+ = \text{Hom}_{\mathbb{G}_m}(f^* \Omega^1_{\mathbb{A}^1}, O_{\mathbb{A}^1}), \]

and the map (1.6) assigns to a $\mathbb{G}_m$-equivariant morphism $\varphi : f^* \Omega^1_{\mathbb{A}^1} \to O_{\mathbb{A}^1}$ the corresponding map between fibers at $1 \in \mathbb{A}^1$. So the kernel of (1.6) consists of those $\varphi \in \text{Hom}_{\mathbb{G}_m}(f^* \Omega^1_{\mathbb{A}^1}, O_{\mathbb{A}^1})$ for which $\varphi|_{\mathbb{A}^1 - \{0\}} = 0$. This implies that $\varphi = 0$ because $O_{\mathbb{A}^1}$ has no nonzero sections supported at $0 \in \mathbb{A}^1$.

Let us deduce (vi) from formula (1.7). Since $\zeta := i^+(z)$ the morphism $f : \mathbb{A}^1 \to Z$ corresponding to $\zeta$ is constant, so $f^* \Omega^1_{\mathbb{A}^1} = T_z Z \otimes O_{\mathbb{A}^1}$. Thus formula (1.7) identifies $T_z Z^+$ with the space

\[ \text{Hom}_{\mathbb{G}_m}(T_z^* Z, k[t]) = \text{Hom}_{\mathbb{G}_m}((T_z^* Z)^+, k) = (T_z Z)^+. \]

To prove the lemma, note that by $\mathbb{G}_m$-equivariance, $f(t) = f(1)$ for $t \neq 0$. Now restricting $f$ to the Henselization of $\mathbb{A}^1$ at $0$ we see that $f$ is constant.

Let us prove (v). After base change, we can assume that $k$ is algebraically closed and the point in question is a $k$-point $z_0$. Any $k$-point of $Z$ is closed (because $Z$ is an algebraic $k$-space...
of finite type). So we can apply Lemma 1.4.9 to the \(G_m\)-stable closed subspace \(F = \{z_0\}\) and get \(p^{-1}(z_0) = F^+ \simeq \text{Spec } k\).

**Example 1.4.12.** Let \(Z\) be the projective line \(\mathbb{P}^1\) equipped with the usual action of \(G_m\). Then \(p^+ : Z^+ \to Z\) is the canonical morphism \(\mathbb{A}^1 \cup \{\infty\} \to \mathbb{P}^1\). In particular, \(p^+\) is *not* a locally closed embedding.

**Remark 1.4.13.** In the above example the restriction of \(p^+ : Z^+ \to Z\) to each connected component of \(Z^+\) is a locally closed embedding. This turns out to be true in a surprisingly large class of situations, but there are also important examples when this is false. More details can be found in Appendix [B].

**Remark 1.4.14.** It is easy to deduce from Proposition 1.4.11(i) that if the diagonal map \(Z \to Z \times Z\) is a locally closed embedding (e.g., if \(Z\) is a scheme) then the map

\[Z^+ \xrightarrow{(p^+, q^+)} Z \times Z^0\]

is a monomorphism.

**Proposition 1.4.15.** Let \(Z\) be an algebraic \(k\)-space of finite type equipped with a \(G_m\)-action. The morphism \(p^+ : Z^+ \to Z\) is an isomorphism if and only if the \(G_m\)-action on \(Z\) can be extended to an \(\mathbb{A}^1\)-action. In this case such extension is unique.

**Remark 1.4.16.** It is somewhat surprising that in this proposition \(Z\) does not have to be separated.

The “only if” part of Proposition 1.4.11 follows from the fact that the \(G_m\)-action on \(Z^+\) always extends to an \(\mathbb{A}^1\)-action, see Subsect. 1.3.3(i). The remaining parts of Proposition 1.4.11 immediately follow from the next lemma.

**Lemma 1.4.17.** Let \(Z\) be an algebraic \(k\)-space of finite type equipped with a \(\mathbb{A}^1\)-action. Equip \(Z^+\) with the \(\mathbb{A}^1\)-action from Subsect. 1.3.3(i). Then \(p^+ : Z^+ \to Z\) is an \(\mathbb{A}^1\)-equivariant isomorphism.

To prove the lemma, we will need the following

**Remark 1.4.18.** Let \(X \xrightarrow{i} Y \xrightarrow{\pi} X\) be morphisms of algebraic spaces such that \(\pi \circ i = \text{id}_X\) and \(\pi : Y \to X\) is unramified at each point of \(i(X)\), then \(i : X \to Y\) is an open embedding. Indeed, \(i\) is clearly a monomorphism, and it is also etale: to see this, look at the homomorphisms of Henselizations (or of completed local rings) induced by \(i : X \to Y\) and \(\pi : Y \to X\).

**Proof of Lemma 1.4.17.** The \(\mathbb{A}^1\)-action on \(Z\) defines an \(\mathbb{A}^1\)-equivariant morphism \(g : Z \to Z^+\) such that the composition of the maps

\[Z \xrightarrow{g} Z^+ \xrightarrow{p^+} Z\]

equals \(\text{id}_Z\). It remains to show that \(g\) is an isomorphism.

By Proposition 1.4.11(i), the morphism \(p^+ : Z^+ \to Z\) is unramified. So \(g\) is an open embedding by Remark 1.4.18. It remains to show that any point \(\zeta \in Z^+\) is contained in \(g(Z)\). Without loss of generality, we can assume that \(\zeta\) is a \(k\)-point (otherwise do base change). Set

\[U_\zeta := \{t \in \mathbb{A}^1 \mid t \cdot \zeta \in g(Z)\},\]

where \(t \cdot \zeta\) denotes the action of \(\mathbb{A}^1\) on \(Z^+\) from Subsect. 1.3.3(i). We have to show that \(1 \in U_\zeta\). Since \(U_\zeta\) is an open \(G_m\)-stable subset of \(\mathbb{A}^1\) it suffices to show that \(0 \in U_\zeta\). We claim that (1.8)

\[0 \cdot \zeta = g(q^+(\zeta)),\]
where \( q^+ : Z^+ \to Z^0 \) is the canonical morphism. Indeed, it is easy to check that
\[
q^+(0 \cdot \zeta) = q^+(\zeta), \quad q^+(g(q^+(\zeta))) = q^+(\zeta).
\]
Since \( q^+(\zeta) \in Z^0(k) \) the equality (1.3) follows from (1.0) and Proposition 1.4.11(v). \( \square \)

1.4.19. Smoothness. The following proposition is well known (at least, if \( Z \) is a scheme).

**Proposition 1.4.20.** Suppose that an algebraic \( k \)-space \( Z \) is smooth. Then \( Z^0 \) and \( Z^+ \) are smooth. Moreover, the morphism \( q^+ : Z^+ \to Z^0 \) is smooth.

**Proof.** We will only prove that \( q^+ \) is smooth. (Smoothness of \( Z^0 \) can be proved similarly, and smoothness of \( Z^+ \) follows.)

It suffices to check that \( q^+ \) is formally smooth. Let \( R \) be a \( k \)-algebra and \( R = R/I \), where \( I \subset R \) is an ideal with \( I^2 = 0 \). Let \( \bar{f} : \mathbb{G}_m^1 \to Z \) be a \( \mathbb{G}_m \)-equivariant morphism and let \( \bar{f}_0 : \text{Spec } \bar{R} \to Z^0 \) denote the restriction of \( \bar{f} \) to \( 0 \subset \mathbb{G}_m^1 \). Let \( \varphi : \text{Spec } R \to Z^0 \) be any morphism extending \( \bar{f}_0 \). We have to extend \( \bar{f} \) to a \( \mathbb{G}_m \)-equivariant morphism \( f : \mathbb{G}_m^1 \to Z \) so that \( f_0 = \varphi \).

Using smoothness of \( Z \), it is easy to show that there is a not-necessarily equivariant morphism \( f : \mathbb{G}_m^1 \to Z \) extending \( \bar{f} \) with \( f_0 = \varphi \). Then standard arguments show that the obstruction to existence of a \( \mathbb{G}_m \)-equivariant \( f \) with the required properties belongs to
\[
H^1(\mathbb{G}_m, M), \quad M := H^0(\mathbb{A}_m^1, f^*\Theta_Z \otimes \mathcal{O}_Z) \otimes_R I,
\]
where \( \Theta_Z \) is the tangent bundle of \( Z \) and \( \mathcal{O}_{\mathbb{A}_m^1} \) is the ideal of the zero section. But \( H^1 \) of \( \mathbb{G}_m \) with coefficients in any \( \mathbb{G}_m \)-module is zero. \( \square \)

1.5. **Repellers.** Set \( \mathbb{A}_m^- := \mathbb{P}^1 - \{\infty\} \); this is a monoid with respect to multiplication containing \( \mathbb{G}_m \) as a subgroup. One has an isomorphism of monoids
\[
(1.10) \quad \mathbb{A}_m^- \longrightarrow \mathbb{A}_m^1, \quad t \mapsto t^{-1}.
\]

Given a \( k \)-space equipped with a \( \mathbb{G}_m \)-action we set
\[
(1.11) \quad Z^- := \text{Maps}_{\mathbb{G}_m}(\mathbb{A}_m^1, Z).
\]

**Definition 1.5.1.** \( Z^- \) is called the **repeller** of \( Z \).

Just as in Subsect. 1.3.3 one defines an action of the monoid \( \mathbb{A}_m^- \) on \( Z^- \) extending the action of \( \mathbb{G}_m \), a \( \mathbb{G}_m \)-equivariant morphism \( p^- : Z^- \to Z \), and \( \mathbb{A}_m^- \)-equivariant morphisms \( q^- : Z^- \to Z^0 \) and \( i^- : Z^0 \to Z^- \) (where \( Z^0 \) is equipped with the trivial \( \mathbb{A}_m^- \)-action). One has \( q^- \circ i^- = \text{id}_{Z^0} \), and the composition \( p^- \circ i^- \) is equal to the canonical embedding \( Z^0 \hookrightarrow Z \).

Using the isomorphism (1.10), one can identify \( Z^- \) with the attractor for the inverse action of \( \mathbb{G}_m \) on \( Z \) (this identification is \( \mathbb{G}_m \)-anti-equivariant). Thus the results on attractors from Subsections 1.3.4 and 1.4 imply similar results for repellers.

In particular, if \( Z \) is the spectrum of a \( \mathbb{Z} \)-graded algebra \( A \) then \( Z^- \) canonically identifies with \( \text{Spec } A^- \), where \( A^- \) is the maximal \( \mathbb{Z}_- \)-graded quotient algebra of \( A \).

1.6. **Attractors and repellers.** In this subsection \( Z \) denotes an algebraic \( k \)-space of finite type equipped with a \( \mathbb{G}_m \)-action.

**Lemma 1.6.1.** The morphisms \( i^\pm : Z^0 \to Z^\pm \) are closed embeddings.

**Proof.** It suffices to consider \( i^+ \).

By Theorem 1.4.2(ii), the morphism \( q^+ : Z^+ \to Z^0 \) is separated. One has \( q^+ \circ i^+ = \text{id}_{Z^0} \). So \( i^+ \) is a closed embedding. \( \square \)

Now consider the fiber product \( Z^+ \times_Z Z^- \) formed using the maps \( p^\pm : Z^\pm \to Z \).
Proposition 1.6.2. The map
\[ j := (i^+, i^-) : Z^0 \to Z^+ \times Z^- \]
is both an open embedding and a closed one.

Remark 1.6.3. If \( Z \) is affine then \( j \) is an isomorphism (this immediately follows from the explicit description of \( Z^\pm \) in the affine case, see Subsections 1.3.4 and 1.5). In general, \( j \) is not necessarily an isomorphism. To see this, note that by (1.12) and (1.11), we have
\[ Z^+ \times Z^- = \text{Maps}^{G_m}(\mathbb{P}^1, Z) \]
(where \( \mathbb{P}^1 \) is equipped with the usual \( G_m \)-action), and a \( G_m \)-equivariant map \( \mathbb{P}^1 \to Z \) does not have to be constant, in general.

Proof of Proposition 1.6.2. Writing \( j \) as
\[ Z^0 = Z^0 \times Z^0 (i^+, i^-) Z^+ \times Z^- \]
and using Lemma 1.6.1 we see that \( j \) is a closed embedding.

Let us prove that \( j \) is an open embedding. Let \( \pi : Z^+ \times Z^- \to Z^0 \) denote the composition \( Z^+ \times Z^- \to Z^+ q^+ \to Z^0 \).

Then \( \pi \circ j = \text{id}_{Z^0} \). Now by Remark 1.4.18 it suffices to check that the tangent map
\[ T_{j(z)}(Z^+ \times Z^-) \to T_z Z^0 \]
corresponding to \( \pi \) is an isomorphism. Indeed, by Proposition 1.4.11(vi) and a similar statement for \( Z^- \), the map (1.14) is just the identity map \((T_z Z)^+ \cap (T_z Z)^- \to (T_z Z)^{G_m}\).

Remark 1.6.4. The fact that \( j \) is an open embedding can also be proved using (1.13) and the fact that every regular function on \( \mathbb{P}^1 \) is constant. (This type of argument is used in the proof of Proposition 3.1.13 below.)

Corollary 1.6.5. (i) If the map \( p^+ : Z^+ \to Z \) is an isomorphism then so are the maps \( Z^0 \xrightarrow{i^+} Z^- \xrightarrow{q^-} Z^0 \).
(ii) If the map \( p^- : Z^- \to Z \) is an isomorphism then so are the maps \( Z^0 \xrightarrow{i^-} Z^0 \xrightarrow{q^+} Z^0 \).

Proof. Let us prove (ii). By Proposition 1.6.2 the morphism \( i^+ : Z^0 \to Z^+ \) is an open embedding. It remains to show that any point \( \zeta \in Z^+ \) is contained in \( i^+(Z^0) \). Set
\[ U_\zeta := \{ t \in \mathbb{A}^1 \mid t \cdot \zeta \in i^+(Z^0) \} \]
We have to show that 1 \( \in U_\zeta \). But \( U_\zeta \) is an open \( G_m \)-stable subset of \( \mathbb{A}^1 \) containing 0, so \( U_\zeta = \mathbb{A}^1 \).

2. The space \( \tilde{Z} \)

We keep the conventions and notation of Subsect. 1.3.4 and Sect. 1. In particular, \( k \) is an arbitrary field, and \( Z \) denotes a \( k \)-space equipped with a \( G_m \)-action. The goal of this section is to construct and study a \( k \)-space \( \tilde{Z} \) equipped with a morphism \( \tilde{Z} \to \mathbb{A}^1 \times Z \times Z \) such that for \( t \in \mathbb{A}^1 - \{ 0 \} \) the fiber \( \tilde{Z}_t \) equals the graph of the map \( t : Z \to Z \) and the space \( \tilde{Z}_0 \) corresponding to \( t = 0 \) equals \( Z^+ \times Z^- \).
The organization of this section is as follows. In Subsect. 2.1 we define the space \( \overline{Z} \) and the main structures on it (e.g., the morphism \( \bar{\rho} : \overline{Z} \to \mathbb{A}^1 \times Z \times Z \) and the action of \( \mathbb{G}_m \) on \( \overline{Z} \)). In Subsect. 2.2 we formulate the main results on \( \overline{Z} \). The fact that the space \( \overline{Z} \) is algebraic is proved in Section 5; however, in the case of a scheme equipped with a locally linear \( \mathbb{G}_m \)-action the proof is much easier and is given in Subsect. 2.4. In Subsect. 2.3 (resp. 2.5) we prove additional properties of \( \overline{Z} \) in the case that \( Z \) is affine (resp. \( Z = \mathbb{P}^n \)).

2.1. The space \( \overline{Z} \): definition and structures.

2.1.1. A family of hyperbolas. Set \( X := \mathbb{A}^2 = \text{Spec} \ k[\tau_1, \tau_2] \). We will always equip \( X \) with the structure of a scheme over \( \mathbb{A}^1 \) defined by the map

\[
\mathbb{A}^2 \to \mathbb{A}^1, \quad (\tau_1, \tau_2) \mapsto \tau_1 \cdot \tau_2.
\]

Let \( X_t \) denote the fiber of \( X \) over \( t \in \mathbb{A}^1 \); in other words, \( X_t \subset \mathbb{A}^2 \) is the curve defined by the equation \( \tau_1 \tau_2 = t \). If \( t \neq 0 \) then \( X_t \) is a hyperbola, while \( X_0 \) is the “coordinate cross” \( \tau_1 \tau_2 = 0 \). One has \( X_0 = X_0^+ \cup X_0^- \), where

\[
(2.1) \quad X_0^+ := \{(\tau_1, \tau_2) \in \mathbb{A}^2 | \tau_2 = 0\}, \quad X_0^- := \{(\tau_1, \tau_2) \in \mathbb{A}^2 | \tau_1 = 0\}.
\]

2.1.2. The schemes \( X_S \). For any scheme \( S \) over \( \mathbb{A}^1 \) set

\[
(2.2) \quad X_S := X \times S \text{.}
\]

If \( S = \text{Spec} \ R \) we usually write \( X_R \) instead of \( X_S \).

2.1.3. The \( \mathbb{G}_m \)-action on \( X_S \). We equip \( X \) with the following hyperbolic \( \mathbb{G}_m \)-action:

\[
(2.3) \quad \lambda \cdot (\tau_1, \tau_2) := (\lambda \cdot \tau_1, \lambda^{-1} \cdot \tau_2).
\]

This action preserves the morphism \( \mathbb{A}^2 \to \mathbb{A}^1 \), so for any scheme \( S \) over \( \mathbb{A}^1 \) one gets an action of \( \mathbb{G}_m \) on \( X_S \).

Remark 2.1.4. If \( S \) is over \( \mathbb{A}^1 - \{0\} \) then \( X_S \) is \( \mathbb{G}_m \)-equivariantly isomorphic to \( \mathbb{G}_m \times S \). On the other hand, the “coordinate cross” \( X_0 \) has irreducible components \( X_0^+ \) (resp. \( X_0^- \)) such that \( X_0^+ \) (resp. \( X_0^- \)) is \( \mathbb{G}_m \)-equivariantly isomorphic to \( \mathbb{A}^1 \) (resp. to the scheme \( \mathbb{A}^1 \) defined in Subsect. 1.3).

2.1.5. The space \( \overline{Z} \). Given a \( k \)-space \( Z \) equipped with a \( \mathbb{G}_m \)-action, define \( \overline{Z} \) to be the following space over \( \mathbb{A}^1 \): for any scheme \( S \) over \( \mathbb{A}^1 \)

\[
\text{Maps}_{\mathbb{A}^1}(S, \overline{Z}) := \text{Maps}^{\mathbb{G}_m}(X_S, Z).
\]

In other words, for any \( k \)-scheme \( S \), an \( S \)-point of \( \overline{Z} \) is a pair consisting of a morphism \( S \to \mathbb{A}^1 \) and a \( \mathbb{G}_m \)-equivariant morphism \( X_S \to Z \).

Note that for any \( t \in \mathbb{A}^1(k) \) the fiber \( \overline{Z}_t \) has the following description:

\[
(2.4) \quad \overline{Z}_t = \text{Maps}^{\mathbb{G}_m}(X_t, Z).
\]

Remark 2.1.6. Later we will prove (see Theorem 2.2.2 and Proposition 2.2.3) that if \( Z \) is an algebraic \( k \)-space of finite type (resp. a \( k \)-scheme of finite type) then so is \( \overline{Z} \). For the spaces \( \overline{Z} \times (\mathbb{A}^1 - \{0\}) \) and \( \overline{Z}_0 := \overline{Z} \times \{0\} \) this follows from the easy Propositions 2.1.8 and 2.1.11 below (the latter has to be combined with Theorem 1.4.2).
2.1.7. The morphism $\tilde{p} : \tilde{Z} \to \mathbb{A}^1 \times Z \times Z$. Any section $\sigma : \mathbb{A}^1 \to X$ of the morphism $X \to \mathbb{A}^1$ defines a map $\sigma^* : \text{Maps}_{\mathbb{G}_m}^G(X_S, Z) \to \text{Maps}(S, Z)$ and therefore a morphism $\tilde{Z} \to Z$. Let $\pi_1 : \tilde{Z} \to Z$ and $\pi_2 : \tilde{Z} \to Z$ denote the morphisms corresponding to the sections
\begin{equation}
(2.9) \quad t \mapsto (1, t) \in X_t \quad \text{and} \quad t \mapsto (t, 1) \in X_t,
\end{equation}
respectively. Let
\begin{equation}
(2.8) \quad \tilde{p} : \tilde{Z} \to \mathbb{A}^1 \times Z \times Z
\end{equation}
denote the morphism whose first component is the tautological projection $\tilde{Z} \to \mathbb{A}^1$, and the second and the third components are $\pi_1$ and $\pi_2$, respectively.

For $t \in \mathbb{A}^1$ let
\begin{equation}
\tilde{p}_t : \tilde{Z}_t \to Z \times Z
\end{equation}
denote the morphism induced by (2.6) (as before, $\tilde{Z}_t$ stands for the fiber of $\tilde{Z}$ over $t$).

It is clear that $(\tilde{Z}_1, \tilde{p}_1)$ identifies with $(Z, \Delta_Z : Z \to Z \times Z)$. More generally, for $t \in \mathbb{A}^1 - \{0\}$ the pair $(\tilde{Z}_t, \tilde{p}_t)$ identifies with the graph of the map $Z \to Z$ given by the action of $t \in \mathbb{G}_m$.

Here is a slightly more precise statement.

**Proposition 2.1.8.** The morphism $\tilde{p}$ induces an isomorphism between
\begin{equation}
\mathbb{G}_m \times \tilde{Z} \subset \tilde{Z}
\end{equation}
and the graph of the action morphism $\mathbb{G}_m \times Z \to Z$.

**Remark 2.1.9.** Later we will show that if the $\mathbb{G}_m$-action on $Z$ extends to an $\mathbb{A}^1$-action and if $Z$ is an algebraic $k$-space of finite type then the whole space $\tilde{Z}$ identifies with the graph of the $\mathbb{A}^1$-action, see Proposition 2.1.12.

2.1.10. The space $\tilde{Z}_0$. Now let us construct a canonical isomorphism $\tilde{Z}_0 \sim \to Z^+ \times Z^-$. Recall that $\tilde{Z}_0 = \text{Maps}_{\mathbb{G}_m}^G(X_0, Z)$ and $X_0 = X_0^+ \cup X_0^-$, where $X_0^+$ and $X_0^-$ are defined by formula (2.1). One has $\mathbb{G}_m$-equivariant isomorphisms
\begin{equation}
(2.7) \quad \mathbb{A}^1 \sim \to X_0^+, \ s \mapsto (s, 0); \quad \mathbb{A}^1 \sim \to X_0^-, \ s \mapsto (0, s^{-1}).
\end{equation}
They define a morphism
\begin{equation}
\tilde{Z}_0 = \text{Maps}_{\mathbb{G}_m}^G(X_0, Z) \to \text{Maps}_{\mathbb{G}_m}^G(X_0^+, Z) \sim \to \text{Maps}_{\mathbb{G}_m}^G(\mathbb{A}^1, Z) = Z^+
\end{equation}
and a similar morphism $\tilde{Z}_0 \to Z^-$. 

**Proposition 2.1.11.** Assume that the $k$-space $Z$ is algebraic.
(i) The above morphisms $\tilde{Z}_0 \to Z^\pm$ induce an isomorphism
\begin{equation}
(2.8) \quad \tilde{Z}_0 \sim \to Z^+ \times Z^-,
\end{equation}
where the fiber product is taken with respect to the maps $q^\pm : Z^\pm \to Z^0$ from Subsections 1.3.3(iii) and 1.4.
(ii) The corresponding diagram
\begin{equation}
(2.9)
\begin{array}{ccc}
\tilde{Z}_0 & \overset{\tilde{p}_0}{\longrightarrow} & Z \times Z \\
\downarrow & & \downarrow \text{proj}^+ \times \text{proj}^- \\
Z^+ \times Z^- & \sim \to & Z^+ \times Z^-
\end{array}
\end{equation}
commutes.

Proof. It is easy to check that our morphisms \( \tilde{Z}_0 \to Z^\pm \) induce a morphism
\[
(2.10) \quad \tilde{Z}_0 \to Z^+ \times Z^-
\]
and that the corresponding diagram \( 2.10 \) commutes. To prove that the map \( 2.10 \) is an isomorphism, apply the following well known lemma for
\[Y = \mathbb{X}_0 \times S, \quad Y_1 = \mathbb{X}_0^+ \times S, \quad Y_2 = \mathbb{X}_0^- \times S,\]
where \( S \) is a test scheme. \( \square \)

Lemma 2.1.12. Let \( Y \) be a scheme and \( Y_1, Y_2 \subset Y \) closed subschemes whose scheme-theoretical union\( ^9 \) equals \( Y \). Then the square
\[
\begin{array}{ccc}
Y_1 \cap Y_2 & \longrightarrow & Y_1 \\
\downarrow & & \downarrow \\
Y_2 & \longrightarrow & Y
\end{array}
\]
is co-Cartesian in the category of algebraic spaces; that is, for any algebraic space \( Z \) the map
\[
(2.11) \quad \text{Maps}(Y, Z) \to \text{Maps}(Y_1, Z) \times_{\text{Maps}(Y_1 \cap Y_2, Z)} \text{Maps}(Y_2, Z)
\]
is bijective.

Proof. If \( Z \) is an affine scheme the map \( 2.11 \) is clearly bijective. Bijectivity of \( 2.11 \) for any scheme \( Z \) easily follows. For an arbitrary algebraic space \( Z \) bijectivity of \( 2.11 \) follows from the case where \( Z \) is an affine scheme and the following result \( \text{[SGA1, exp. IV, Theorem 4.7]} \): let \( \text{Et}^{\text{sep}}_{\text{fin}}(Y) \) be the category of separated etale schemes of finite type over \( Y \), then the functor
\[
\text{Et}^{\text{sep}}_{\text{fin}}(Y) \to \text{Et}^{\text{sep}}_{\text{fin}}(Y_1) \times_{\text{Et}^{\text{sep}}_{\text{fin}}(Y_1 \cap Y_2)} \text{Et}^{\text{sep}}_{\text{fin}}(Y_2)
\]
is an equivalence. \( \square \)

Proposition 2.1.13. (i) Let \( Y \subset Z \) be a \( \mathbb{G}_m \)-stable closed subspace. Then the diagram
\[
\begin{array}{ccc}
\tilde{Y} & \longrightarrow & \tilde{Z} \\
\tilde{p}_Y & & \tilde{p}_Z \\
\mathbb{A}^1 \times Y \times Y & \hookrightarrow & \mathbb{A}^1 \times Z \times Z
\end{array}
\]
is Cartesian. In particular, the morphism \( \tilde{Y} \to \tilde{Z} \) is a closed embedding.

(ii) Let \( Y \subset Z \) be a \( \mathbb{G}_m \)-stable open subspace. Then the above diagram identifies \( \tilde{Y} \) with an open subspace of the fiber product
\[
\tilde{Z} \times_{\mathbb{A}^1 \times Z \times Z} (\mathbb{A}^1 \times Y \times Y).
\]
In particular, the morphism \( \tilde{Y} \to \tilde{Z} \) is an open embedding.

\( ^9 \) By this we mean the supremum of \( Y_1 \) and \( Y_2 \) in the poset of closed subschemes.

\( ^{10} \) The separatedness assumption is harmless because any morphism from an affine scheme to an algebraic space \( Z \) is separated (even if \( Z \) itself is not separated).
Proof. (i) Let \( S \) be a scheme over \( \mathbb{A}^1 \) and \( f : X_S \to Z \) a \( \mathbb{G}_m \)-equivariant morphism. Formula (2.3) defines two sections of the map \( X_S \to S \). We have to show that if \( f \) maps these sections to \( Y \subset Z \) then \( f(X_S) \subset Y \). By \( \mathbb{G}_m \)-equivariance, we have \( f(X'_S) \subset Y \), where \( X' \) is the open subscheme \( \mathbb{A}^2 - \{0\} \subset \mathbb{A}^2 = X \) and \( X'_S := X' \times_{\mathbb{A}^1} S \). Since \( X'_S \) is schematically dense in \( X_S \), this implies that \( f(X_S) \subset Y \).

(ii) Just as before, we have a \( \mathbb{G}_m \)-equivariant morphism \( f : X_S \to Z \) such that \( f(X'_S) \subset Y \). The problem is now to show that the set

\[
\{ s \in S \mid X_s \subset f^{-1}(Y) \}
\]

is open in \( S \). Indeed, its complement equals \( \text{pr}_S(X_S - f^{-1}(Y)) \), where \( \text{pr}_S : X_S \to S \) is the projection. The set \( \text{pr}_S(X_S - f^{-1}(Y)) \) is closed in \( S \) because \( X_S - f^{-1}(Y) \) is a closed subset of \( X_S - X'_S \) and the morphism \( X_S - X'_S \to S \) is closed (in fact, it is a closed embedding). \( \square \)

2.1.14. Anti-action of \( \mathbb{A}^1 \times \mathbb{A}^1 \) on \( \widetilde{Z} \). The reader may prefer to skip the rest of Subsect. 2.1.16 for a while and proceed to Subsect. 2.2.

As usual, we consider \( \mathbb{A}^1 \) as a monoid with respect to multiplication. In this subsection we will define an “anti-action” of the monoid \( \mathbb{A}^2 = \mathbb{A}^1 \times \mathbb{A}^1 \) on \( \widetilde{Z} \) (the meaning of the word “anti-action” will become clear soon). In Subsect. 2.1.16 we will use it to define an action of \( \mathbb{G}_m^2 \) on \( \widetilde{Z} \).

The idea is as follows. Recall that \( X := \mathbb{A}^2 \), so the monoid \( \mathbb{A}^2 \) acts on \( X \). In particular, for any \( \lambda = (\lambda_1, \lambda_2) \in \mathbb{A}^2(k) \) the action of \( \mathbb{A}^2 \) on \( X \) defines a morphism \( X \to X \). For any \( t \in \mathbb{A}^1(k) \) it induces a \( \mathbb{G}_m \)-equivariant morphism \( X_t \to X_{t\lambda_1\lambda_2} \) (recall that \( X_t \) denotes the fiber over \( t \)). Since \( \widetilde{Z}_t := \text{Maps}_{\mathbb{G}_m}(X_t, Z) \) we get a morphism \( \widetilde{Z}_{t\lambda_1\lambda_2} \to \widetilde{Z}_t \). (Of course, if \( \lambda_1, \lambda_2 \neq 0 \) one can invert this morphism and get a morphism in the “expected” direction, i.e., \( \widetilde{Z}_t \to \widetilde{Z}_{t\lambda_1\lambda_2} \).

Same story if one works with \( S \)-points rather than \( k \)-points. Namely, suppose we have a \( k \)-scheme \( S \) and \( k \)-morphisms \( t : S \to \mathbb{A}^1 \) and \( \lambda_1, \lambda_2 : S \to \mathbb{A}^1 \). Let \( X_t \) (resp. \( \widetilde{Z}_t \)) denote the fiber product \( X \times_S (\mathbb{A}^1) \) (resp. \( \widetilde{Z} \times_S (\mathbb{A}^1) \)) with respect to \( t : S \to \mathbb{A}^1 \). The morphism

\[
X \times_S (\mathbb{A}^1) \to X \times_S (\mathbb{A}^1)
\]

maps \( X_t \subset X \times S \) to \( X_{t\lambda_1\lambda_2} \subset X \times S \), and the \( \mathbb{G}_m \)-equivariant morphism \( X_t \to X_{t\lambda_1\lambda_2} \) induces an \( S \)-morphism

\[
(2.12) \quad \phi_{\lambda_1, \lambda_2, t} : \widetilde{Z}_{t\lambda_1\lambda_2} \to \widetilde{Z}_t.
\]

The morphisms (2.12) have the following properties:

(i) compatibility with base change \( S' \to S \);
(ii) \( \phi_{1,1,t} \) equals the identity;
(iii) \( \phi_{\lambda_1\lambda'_1, \lambda_2\lambda'_2} = \phi_{\lambda_1, \lambda_2,t} \circ \phi_{\lambda'_1, \lambda'_2, t\lambda_1\lambda_2} \).

We use the word “anti-action” to denote this structure on the triple \( (\widetilde{Z}, \mathbb{A}^1, \widetilde{\mathbb{A}^1} \to \mathbb{A}^1) \).

Remark 2.1.15. An additional property of the above anti-action will be formulated later, see Subsect. 3.2.17.

Exercise. Describe the compositions

\[
\phi_{0,1,0} : \widetilde{Z}_0 \to \widetilde{Z}_1 \xrightarrow{\sim} Z, \quad \phi_{0,1,1} : \widetilde{Z}_0 \to \widetilde{Z}_1 \xrightarrow{\sim} Z, \quad \phi_{0,0,1} : \widetilde{Z}_0 \to \widetilde{Z}_1 \xrightarrow{\sim} Z
\]

and the idempotent endomorphisms \( \phi_{0,1,0}, \phi_{0,1,1}, \phi_{0,0,0} \in \text{End}(\widetilde{Z}_0) \) in terms of the isomorphism

\[
\widetilde{Z}_0 \xrightarrow{\sim} Z^+ \times \widetilde{Z}_0^-
\]

from Proposition 2.1.11.
2.1.16. Action of $G_m^2$ on $\tilde{Z}$. We equip $\mathbb{A}^1$ with the following action of $G_m^2$:

$$\lambda_1, \lambda_2 \ast t := \lambda_1^{-1} \lambda_2 t, \quad \lambda_i \in G_m, \ t \in \mathbb{A}^1.$$  

We lift it to an action of $G_m^2$ on $\tilde{Z}$ using the isomorphisms

$$\phi_{\lambda_1^{-1}, \lambda_2, t}^{-1} = \phi_{\lambda_1, \lambda_2^{-1}, \lambda_1^{-1} t \lambda_2} : \tilde{Z}_t \sim \tilde{Z}_{\lambda_1^{-1} t \lambda_2}$$

where $\phi$ is the morphism \ref{2.1.12}.

It is easy to check that the morphism $\tilde{p} : \tilde{Z} \to \mathbb{A}^1 \times Z \times Z$ and the isomorphism

$$\tilde{Z}_0 \sim \tilde{Z}^+ \times \tilde{Z}^-$$

from Proposition \ref{2.1.11} are $G_m^2$-equivariant. So all morphisms in diagram \ref{2.2} are $G_m^2$-equivariant.

Remark 2.1.17. In \cite{DrGa1} we make a different choice of signs in formulas \ref{2.1.13}--\ref{2.1.14}. Namely, the action of $G_m^2$ on $\mathbb{A}^1$ is defined there by $(\lambda_1, \lambda_2) \ast t := \lambda_1^{-1} \lambda_2^{-1} t$, and its lift to an action of $G_m^2$ on $\tilde{Z}$ is defined using the isomorphism $(\phi_{\lambda_1^{-1}, \lambda_2^{-1}, t})^{-1} : \tilde{Z}_t \sim \tilde{Z}_{\lambda_1^{-1} t \lambda_2^{-1}}$.

2.2. Main results on $\tilde{Z}$.

2.2.1. Formulation of the main results.

Theorem 2.2.2. Let $Z$ be any algebraic $k$-space of finite type equipped with a $G_m$-action. Then $\tilde{Z}$ is an algebraic $k$-space of finite type.

In full generality, the theorem will be proved in Section \ref{sec:5}. In Subsect. \ref{2.2.3} we will prove it in the case that $Z$ is a scheme equipped with a locally linear $G_m$-action (moreover, we will show that under these assumptions $\tilde{Z}$ is a scheme). This case is enough for most practical purposes.

From now on we assume that $Z$ is an algebraic $k$-space of finite type.

Proposition 2.2.3. (i) If $Z$ is separated then so is $\tilde{Z}$.

(ii) If $Z$ is a scheme then so is $\tilde{Z}$.

The proof will be given in Subsect. \ref{2.2.4} below.

Proposition 2.2.4. If $Z$ is smooth then the canonical morphism $\tilde{Z} \to \mathbb{A}^1$ is smooth.

Proof. It suffices to check formal smoothness. We proceed just as in the proof of Proposition \ref{1.4.20}. Let $R$ be a $k$-algebra equipped with a morphism $\text{Spec} \ R \to \mathbb{A}^1$. Let $\tilde{R} = R/I$, where $I \subset R$ is an ideal with $I^2 = 0$. Let $\tilde{f} \in \text{Maps}^G_m(\mathcal{X}_R, Z)$. We have to show that $\tilde{f}$ can be lifted to an element of $\text{Maps}^G_m(\mathcal{X}_{\tilde{R}}, Z)$. Since $\mathcal{X}_R$ is affine and $Z$ is smooth there is no obstruction to lifting $\tilde{f}$ to an element of $\text{Maps}(\mathcal{X}_R, Z)$. Then standard arguments show that the obstruction to existence of a $G_m$-equivariant lift is in $H^1(G_m, M)$, where $M := H^0(\mathcal{X}_R, \tilde{f}^*\Theta_Z) \otimes_R I$ and $\Theta_Z$ is the tangent bundle. But $H^1$ of $G_m$ with coefficients in any $G_m$-module is zero. \hfill $\Box$

2.2.5. Properties of $\tilde{p} : \tilde{Z} \to \mathbb{A}^1 \times Z \times Z$.

Proposition 2.2.6. (i) The morphism $\tilde{p} : \tilde{Z} \to \mathbb{A}^1 \times Z \times Z$ is unramified.

(ii) If $Z$ is separated then $\tilde{p}$ is a monomorphism.

Proof. Theorem 2.2.2 implies that properties (i) and (ii) can be checked fiberwise. By Proposition 2.1.8 the map $\tilde{p} : \tilde{Z} \to \mathbb{A}^1 \times Z \times Z$ is a monomorphism over $\mathbb{A}^1 - \{0\} \subset \mathbb{A}^1$. It remains to
consider the morphism \( \tilde{p}_0 : \tilde{Z}_0 \to Z \times Z \). By Proposition 2.1.11 this is equivalent to considering the composition

\[
Z^+ \times Z^- \to Z^+ \times Z^- \xrightarrow{p^+ \times p^-} Z \times Z.
\]

By Proposition 1.4.11(i-ii), this composition is unramified, and if \( Z \) is separated then it is a monomorphism. □

Remark 2.2.7. If \( Z \) is affine then \( \tilde{p} : \tilde{Z} \to \mathbb{A}^1 \times Z \times Z \) is a closed embedding, see Proposition 2.3.3 below.

Remark 2.2.8. Suppose that \( Z \) admits a \( \mathbb{G}_m \)-equivariant locally closed embedding into a projective space \( \mathbb{P}(V) \), where \( \mathbb{G}_m \) acts linearly on \( V \). We claim that in this case the morphism \( \tilde{p} : \tilde{Z} \to \mathbb{A}^1 \times Z \times Z \) is a locally closed embedding\(^{11}\) By Proposition 2.1.13 it suffices to check this if \( Z = \mathbb{P}(V) \). This will be done in Subsect. 2.5 below.

Remark 2.2.9. If \( Z \) is the projective line \( \mathbb{P}^1 \) equipped with the usual \( \mathbb{G}_m \)-action then the map \( \tilde{p} : \tilde{Z} \to \mathbb{A}^1 \times Z \times Z \) is not a closed embedding (because the scheme \( \tilde{Z}_0 = Z^+ \times Z^- \) is not proper).

Remark 2.2.10. Let \( Z \) be the curve obtained from \( \mathbb{P}^1 \) by gluing 0 with \( \infty \). Equip \( Z \) with the \( \mathbb{G}_m \)-action induced by the usual action on \( \mathbb{P}^1 \). Then \( \tilde{p} : \tilde{Z} \to \mathbb{A}^1 \times Z \times Z \) is not a locally closed embedding. In fact, already \( \tilde{p}_0 : \tilde{Z}_0 \to Z \times Z \) is not a locally closed embedding (because the maps \( p^\pm : Z^\pm \to Z \) are not).

2.2.11. Description of \( \tilde{Z} \) if the \( \mathbb{G}_m \)-action on \( Z \) extends to an action of \( \mathbb{A}^1 \) or \( \mathbb{A}^1 \). Recall that by Proposition 1.4.11 a \( \mathbb{G}_m \)-action on \( Z \) has at most one extension to an action of the multiplicative monoid \( \mathbb{A}^1 \) and such extension exists if and only if the morphism \( p^+ : Z^+ \to Z \) is an isomorphism. Of course, this remains true if \( \mathbb{A}^1 \) is replaced by the monoid \( \mathbb{A}^1 \) defined in Subsect. 1.5 and \( p^+ \) is replaced by \( p^- : Z^- \to Z \).

Proposition 2.2.12. Suppose that a \( \mathbb{G}_m \)-action on \( Z \) extends to an \( \mathbb{A}^1 \)-action. Then

(i) the morphism \( \tilde{p} : \tilde{Z} \to \mathbb{A}^1 \times Z \times Z \) is a monomorphism, which identifies \( \tilde{Z} \) with the graph of the \( \mathbb{A}^1 \)-action on \( Z \); in particular, the composition

\[
(2.15) \quad \tilde{Z} \xrightarrow{\tilde{p}} \mathbb{A}^1 \times Z \times Z \xrightarrow{\alpha} \mathbb{A}^1 \times Z \times \text{Spec } k = \mathbb{A}^1 \times Z
\]

is an isomorphism;

(ii) the inverse of (2.15) is the morphism

\[
(2.16) \quad \mathbb{A}^1 \times Z \to \tilde{Z}
\]

corresponding to the \( \mathbb{G}_m \)-equivariant map \( \mathbb{X} \times Z \to Z \) defined by

\[
(\tau_1, \tau_2, z) \mapsto \tau_1 \cdot z, \quad (\tau_1, \tau_2) \in \mathbb{X}, \ z \in Z.
\]

Proof. Let \( \alpha : \tilde{Z} \to \mathbb{A}^1 \times Z \) denote the composition (2.15) and \( \beta : \mathbb{A}^1 \times Z \to \tilde{Z} \) the morphism (2.10). It is easy to see that \( \alpha \circ \beta = \text{id} \). The problem is to show that \( \beta \circ \alpha = \text{id} \). To do this, it suffices to prove that \( \alpha \) is a monomorphism. But being a monomorphism is a fiberwise condition, so it suffices to show that \( \beta \) induces an isomorphism between fibers over any \( t \in \mathbb{A}^1 \). For \( t \neq 0 \) this follows from Proposition 2.1.8. If \( t = 0 \) then by Proposition 2.1.11 the morphism in question is the composition

\[
Z^+ \times Z^- \xrightarrow{Z^+ \times Z^- \xrightarrow{p^+}} Z.
\]

\(^{11}\)Note that the map \( p^\pm : Z^\pm \to Z \) is typically not a locally closed embedding, see Example 1.4.12.
By Proposition 1.4.15, $p^+$ is an isomorphism. So the projection $q^- : Z^{-} \to Z^{0}$ is also an isomorphism by Corollary 1.6.5(i). □

The above proposition formally implies the following one.

**Proposition 2.2.13.** Suppose that a $\mathbb{G}_m$-action on $Z$ extends to an action of the monoid $\mathbb{A}_1^1$. Then

(i) the morphism $\tilde{p} : \tilde{Z} \to \mathbb{A}_1^1 \times Z \times Z$ is a monomorphism, which identifies $\tilde{Z}$ with

$$\{(t, z_1, z_2) \in \mathbb{A}_1^1 \times Z \times Z \mid z_1 = t^{-1} \cdot z_2\};$$

in particular, the composition

$$\tilde{Z} \xrightarrow{\tilde{p}} \mathbb{A}_1^1 \times Z \times Z \to \mathbb{A}_1^1 \times \text{Spec} \, k \times Z = \mathbb{A}_1^1 \times Z$$

is an isomorphism;

(ii) the inverse of (2.17) is the morphism

$$\mathbb{A}_1^1 \times Z \to \tilde{Z}$$

corresponding to the $\mathbb{G}_m$-equivariant map $X \times Z \to Z$ defined by

$$(\tau_1, \tau_2, z) \mapsto \tau_2^{-1} \cdot z, \quad (\tau_1, \tau_2) \in X, \quad z \in Z.$$

2.2.14. **Proof of Proposition 2.2.3.** If $\tilde{Z}$ is separated then $\tilde{p} : \tilde{Z} \to \mathbb{A}_1^1 \times Z \times Z$ is a monomorphism by Proposition 2.2.6(ii). Any monomorphism is separated. Proposition 2.2.3(i) follows.

To prove Proposition 2.2.3(ii), we need the following well known fact.

**Proposition 2.2.15.** A separated quasi-finite morphism between algebraic spaces is quasi-affine. In particular, it is schematic. For the proof, see [LM, Theorem A.2] or [Kn, ch. II, Theorem 6.15].

**Corollary 2.2.16.** A monomorphism between algebraic $k$-spaces of finite type is schematic.

Now it is easy to prove Proposition 2.2.3(ii) under an additional assumption that $\tilde{Z}$ is separated: indeed, in this case Proposition 2.2.6(ii) allows to apply Corollary 2.2.16 to the morphism $\tilde{p} : \tilde{Z} \to \mathbb{A}_1^1 \times Z \times Z$.

To prove Proposition 2.2.3(ii) in general, we will apply Corollary 2.2.16 to a more complicated morphism $\tilde{p}'$ constructed below.

Recall that $X := \mathbb{A}_2^2 = \text{Spec} \, k[\tau_1, \tau_2]$ and that $X$ is equipped with the structure of a scheme over $\mathbb{A}_1^1$ defined by the map $(\tau_1, \tau_2) \mapsto \tau_1 \tau_2$. Let $\mathbb{B} \subset X$ be the line defined by the equation $\tau_1 = \tau_2$, then $\mathbb{B}$ is finite and flat over $\mathbb{A}_1^1$. So for any open subscheme $U \subset Z$ there is an algebraic space $U$ over $\mathbb{A}_1^1$ such that for any scheme $S$ over $\mathbb{A}_1^1$

$$\text{Maps}_{\mathbb{A}_1^1}(S, U) := \text{Maps}(\mathbb{B} \times S, U);$$

moreover, if $U$ is affine then $U$ is scheme. The canonical morphism $U \to Z$ is an open embedding.

The embedding $\mathbb{B} \hookrightarrow X$ induces a morphism $\alpha : \tilde{Z} \to \tilde{X}$ over $\mathbb{A}_1^1$. Combining it with $\tilde{p} : \tilde{Z} \to \mathbb{A}_1^1 \times Z \times Z$ one gets a morphism

$$\tilde{p}' : \tilde{Z} \to \mathbb{Z} \times (\mathbb{A}_1^1 \times Z \times Z) = \mathbb{Z} \times Z \times Z.$$

**Lemma 2.2.17.** Suppose that $Z$ is a scheme (or more generally, an algebraic space such that the diagonal map $Z \to Z \times Z$ is a locally closed embedding). Then the map $\tilde{p}' : \tilde{Z} \to \mathbb{Z} \times Z \times Z$ is a monomorphism.

**Proof.** Follows from Remark 1.4.14 combined with Propositions 2.1.8 and 2.1.11. □
Proof of Proposition 2.2.3(ii). Let $Z$ be a scheme. Then $\tilde{Z} \times (\mathbb{A}^1 - \{0\})$ is a scheme by Proposition 2.1.8. So to prove that $\tilde{Z}$ is a scheme, it suffices to show that for any point $\zeta \in \tilde{Z}_0$ there exists an open subscheme $V \subset \tilde{Z}$ containing $\zeta$. Let $z \in Z$ be the image of $\zeta$ under the composition

$$\tilde{Z}_0 \xrightarrow{\tilde{p}} Z^+ \times Z^- \supseteq Z^0 \hookrightarrow Z.$$

Let $U \subset Z$ be an open affine containing $z$. Then the open subspace $\mathcal{O} \subset Z$ is a scheme. Define an open subscheme $\tilde{V} \subset \tilde{Z}$ by

$$V := (\tilde{p})^{-1}(U \times Z \times Z).$$

By Lemma 2.2.17 and Corollary 2.2.16 $V$ is a scheme. It is clear that $\zeta \in V$. □

2.3. The case where $Z$ is affine.

2.3.1. The scheme $X_Z$. Let $R$ be an algebra over $k[t]$, so $S := \text{Spec } R$ is a scheme over $\mathbb{A}^1$. In this situation the scheme $X_Z := X \times S$ introduced in Subsect. 2.1.2 will be denoted by $X_R$. It has the following explicit description:

$$(2.19) \quad X_R := \text{Spec } A_R, \quad \text{ where } A_R := R[\tau_1, \tau_2]/(\tau_1 \tau_2 - t).$$

It is clear that $A_R$ is a free $R$-module with basis $e_n$, $n \in \mathbb{Z}$, where

$$(2.20) \quad e_n = \tau_1^n \quad \text{ for } n \geq 0, \quad e_n = \tau_2^{-n} \quad \text{ for } n \leq 0.$$

The $G_m$-action on $X_R$ defines a $\mathbb{Z}$-grading on $A_R$. The element $e_n$ defined by (2.20) has degree $n$ with respect to this grading.

2.3.2. The space $\tilde{Z}$ in the case that $Z$ is affine. Recall that $\tilde{Z}$ is the space over $\mathbb{A}^1$ such that

$$(2.21) \quad \text{Maps}_{\mathbb{A}^1}(\text{Spec } R, \tilde{Z}) := \text{Maps}^{G_m}(X_R, Z)$$

for any algebra $R$ over $k[t]$.

Proposition 2.3.3. Assume that $Z$ is affine. Then the morphism $\tilde{p} : \tilde{Z} \to \mathbb{A}^1 \times Z \times Z$ is a closed embedding. In particular, $\tilde{Z}$ is an affine $k$-scheme of finite type.

Proof. If $Z$ is a closed subscheme of an affine scheme $Z'$ and the proposition holds for $Z'$ then it holds for $Z$ by Proposition 2.1.13(i). So we are reduced to the case that $Z$ is a finite-dimensional vector space equipped with a linear $G_m$-action.

If the proposition holds for affines schemes $Z_1$ and $Z_2$ then it holds for $Z_1 \times Z_2$. So we are reduced to the case of $Z = \mathbb{A}^1$ and $\lambda \in G_m$ acts on $\mathbb{A}^1$ as multiplication by $\lambda^n$, $n \in \mathbb{Z}$.

In this case it is straightforward to compute $\tilde{Z}$ and $\tilde{p}$ using (2.21), (2.11), and the definition of $\tilde{p}$ from Subsect. 2.1.7. In particular, one checks that $\tilde{p}$ identifies $\tilde{Z}$ with the closed subscheme of $\mathbb{A}^1 \times Z \times Z$ defined by the equation $x_2 = t^n x_1$ if $n \geq 0$ and by the equation $x_1 = t^{-n} x_2$ if $n \leq 0$ (here $t, x_1, x_2$ are the coordinates on $\mathbb{A}^1 \times Z \times Z = \mathbb{A}^3$). □

As before, assume that $Z$ is affine. Then by Proposition 2.3.3 the morphism $\tilde{p}$ identifies $\tilde{Z}$ with the closed subscheme $\tilde{p}(\tilde{Z}) \subset \mathbb{A}^1 \times Z \times Z$. By Proposition 2.1.8 the intersection of $\tilde{p}(\tilde{Z})$ with the open subscheme

$$G_m \times Z \times Z \subset \mathbb{A}^1 \times Z \times Z$$

is equal to the graph of the action map $G_m \times Z \to Z$. Hence, $\tilde{Z}$ contains the closure of the graph in $\mathbb{A}^1 \times Z \times Z$. In general, this containment is not an equality. However, one has the following

[12] E.g., take $Z$ to be the hypersurface in $\mathbb{A}^{2n}$ defined by the equation $x_1 y_1 + \ldots + x_n y_n = 0$ and define the $G_m$-action by $x_i = \lambda x_i$, $y_i = \lambda^{-1} y_i$. 

2.3.5. Explicit description of $\bar{Z}$ in the case that $Z$ is affine. This subsection can be skipped by the reader.

Define a map $\mu : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}_+$ by

$$\mu(n_1, n_2) := \frac{(|n_1| + |n_2| - |n_1 + n_2|)}{2}.$$  

(2.22)

So if $n_1, n_2$ are nonzero and have opposite signs then $\mu(n_1, n_2) = \min(|n_1|, |n_2|)$; otherwise one has $\mu(n_1, n_2) = 0$.

**Proposition 2.3.6.** If $Z$ is the spectrum of a $\mathbb{Z}$-graded $k$-algebra $B$ then $\bar{Z} = \text{Spec} \, \bar{B}$, where $\bar{B}$ is the $k[t]$-algebra with generators

$$[b], \quad b \in B_n, \quad n \in \mathbb{Z},$$

and defining relations

$$[b_1 \cdot b_2] = \mu(n_1, n_2) \cdot [b_1] \cdot [b_2], \quad b_1 \in B_{n_1}, b_2 \in B_{n_2}, \quad n_1, n_2 \in \mathbb{Z},$$

$$[\lambda_1 b_1 + \lambda_2 b_2] = \lambda_1 [b_1] + \lambda_2 [b_2], \quad \lambda_i \in k, b_i \in B_n, n \in \mathbb{Z}.$$  

(2.21)

Proof. By (2.21) and (2.19), for any $k[t]$-algebra $R$, a morphism of $k[t]$-algebras $\bar{B} \to R$ is the same as a morphism of graded $k$-algebras $\varphi : B \to A_R$. Our $A_R$ is a free $R$-module whose basis is formed by elements $e_n$ defined by (2.20). Let $B_n$ denote the $n$-th graded component of $B$, then for $b \in B_n$ one has $\varphi(b) = \varphi_n(b) e_n$, where $\varphi_n : B_n \to R$ is some $k$-linear map. It is easy to check that

$$e_{n_1} e_{n_2} = t^{\mu(n_1, n_2)} e_{n_1 + n_2},$$

so the condition $\varphi(b_1 b_2) = \varphi(b_1) \varphi(b_2)$ can be rewritten as

$$\varphi_{n_1 + n_2}(b_1 b_2) = t^{\mu(n_1, n_2)} \varphi_{n_1}(b_1) \varphi_{n_2}(b_2), \quad b_1 \in B_{n_1}, b_2 \in B_{n_2}, \quad n_1, n_2 \in \mathbb{Z}.$$  

The proposition follows.  

2.4. **Proof of Theorem 2.2.2 in the case of a locally linear $G_m$-action.** Let $Z$ be a $k$-scheme of finite type equipped with a $G_m$-action. Suppose that the action is locally linear, i.e., $Z$ can be covered by open affine $G_m$-stable subschemes $U_i$. Let us show that under this assumption $\bar{Z}$ is a $k$-scheme of finite type.

By Proposition 2.3.3 each $U_i$ is an affine $k$-scheme of finite type. By Proposition 2.1.13(ii), for each $i$ the canonical morphism $\bar{U}_i \to \bar{Z}$ is an open embedding. It remains to show that $\bar{Z}$ is covered by the open subschemes $\bar{U}_i$.

It suffices to check that for each $t \in \mathbb{A}^1$ the fiber $\bar{Z}_t$ is covered by the open subschemes $(\bar{U}_i)_t$. For $t \neq 0$ this is clear from Proposition 2.1.8. It remains to consider the case $t = 0$.

By Proposition 2.1.11 $\bar{Z}_0 = Z^+ \times Z^-$. So a point of $\bar{Z}_0$ is a pair $(z^+, z^-) \in Z^+ \times Z^-$ such that $q^+(z^+) = q^-(z^-)$. The point $q^+(z^+) = q^-(z^-)$ is contained in some $U_i$. By Lemma 1.4.7 we have $z^+, z^- \in U_i$. So our point $(z^+, z^-) \in \bar{Z}_0$ belongs to $(\bar{U}_i)_0$.  

□
2.5. **The morphism** $\tilde{p} : \tilde{Z} \to \mathbb{A}^1 \times Z \times Z$ **in the case** $Z = \mathbb{P}^n$. In this subsection (which can be skipped by the reader) we prove the following statement promised in Remark 2.2.8.

**Proposition 2.5.1.** Let $Z$ be a projective space $\mathbb{P}^n$ equipped with an arbitrary $\mathbb{G}_m$-action. Then the morphism $\tilde{p} : \tilde{Z} \to \mathbb{A}^1 \times Z \times Z$ is a locally closed embedding.

**Proof.** For a suitable coordinate system in $\mathbb{P}^n$, the $\mathbb{G}_m$-action is given by

$$\lambda \ast (z_0 : \ldots : z_n) = (\lambda^{m_0} z_0 : \ldots : \lambda^{m_n} z_n), \quad \lambda \in \mathbb{G}_m.$$ 

Let $U_i \subset Z = \mathbb{P}^n$ denote the open subset defined by the condition $z_i \neq 0$. It is affine, so by Proposition 2.3.3 the canonical morphism $\tilde{U}_i \to \mathbb{A}^1 \times U_i \times U_i$ is a closed embedding. Thus to finish the proof of the proposition, it suffices to show that $\tilde{p}^{-1}(\mathbb{A}^1 \times U_i \times U_i) = \tilde{U}_i$. By Proposition 2.1.8, $\tilde{p}^{-1}(\mathbb{G}_m \times U_i \times U_i) = \mathbb{G}_m \times \tilde{U}_i$. So it remains to prove that the morphism $\tilde{p}_0 : \tilde{Z}_0 \to Z \times Z$ has the following property: $(\tilde{p}_0)^{-1}(U_i \times U_i) = (\tilde{U}_i)_0$. Identifying $\tilde{Z}_0$ with $Z^+ \times Z^-$ and using Lemma 1.4.7 we see that it remains to prove the following lemma. \hfill $\square$

**Lemma 2.5.2.** Let $z^+, z^- \in \mathbb{P}^n$. Suppose that

$$\lim_{\lambda \to 0} \lambda \ast z^+ = \lim_{\lambda \to \infty} \lambda \ast z^- = \zeta .$$

If $z^+, z^- \in U_i$ then $\zeta \in U_i$.

**Proof.** Write $z^+ = (z^+_0 : \ldots : z^+_n)$, $z^- = (z^-_0 : \ldots : z^-_n)$, $\zeta = (\zeta_0 : \ldots : \zeta_n)$. We have $z^+_i \neq 0$, and the problem is to show that $\zeta_i \neq 0$.

Suppose that $\zeta_i = 0$. Choose $j$ so that $\zeta_j \neq 0$. Then $z^+_j \neq 0$ and

$$\lim_{\lambda \to 0} \lambda^{m_i - m_j} (z_i / z_j) = \zeta_i / \zeta_j = 0, \quad \lim_{\lambda \to \infty} \lambda^{m_i - m_j} (z_i / z_j) = \zeta_i / \zeta_j = 0 .$$

This means that $m_i > m_j$ and $m_i < m_j$ at the same time, which is impossible. \hfill $\square$

3. **Some openness results**

In this section $Z$ denotes an algebraic $k$-space of finite type equipped with a $\mathbb{G}_m$-action.

The main results are Propositions 3.1.3 and 3.2.14. They say that certain morphisms involving $\tilde{Z}$ are open embeddings.

Proposition 3.1.3 is used in DrGa1 in a crucial way. The method of its proof is used in Subsect. 3.2 to describe the $n$-fold fiber product $\tilde{Z} \times_Z \cdots \times_Z \tilde{Z}$ formed using the two projections $\tilde{Z} \to Z$.

3.1. **The fiber products** $Z^- \times_Z \tilde{Z}$ and $\tilde{Z} \times_Z Z^+$. The constructions and results of this subsection are used in DrGa1 (in the verification of the adjunction properties).

3.1.1. **Definition of the fiber products.** In Subsect. 2.1.7 we defined morphisms $\pi_1, \pi_2 : \tilde{Z} \to Z$. We will study the fiber product

$$(3.1) \quad Z^- \times_Z \tilde{Z},$$

formed using $\pi_1 : \tilde{Z} \to Z$ and the fiber product

$$(3.2) \quad \tilde{Z} \times_Z Z^+,$$

formed using $\pi_2 : \tilde{Z} \to Z$. Note that both fiber products are spaces over $\mathbb{A}^1$ (because $\tilde{Z}$ is).
3.1.2. **Formulation of the result.** Consider the composition

\[ A^1 \times Z^+ \to \tilde{Z}^+ = \tilde{Z}^+ \times \tilde{Z} \to \tilde{Z} \times Z^+, \]

where the first arrow is the morphism (2.16) for the space \( Z^+ \) and the second arrow comes from the morphism \( p^+ : Z^+ \to Z \). Consider also the similar composition

\[ A^1 \times Z^- \to \tilde{Z}^- = \tilde{Z}^- \times \tilde{Z} \to \tilde{Z} \times Z^-, \]

where the first arrow is the morphism (2.18) for the space \( Z^- \). In [DrGa1] we use the following result.

**Proposition 3.1.3.** The compositions (3.3) and (3.4) are open embeddings.

Note that unlike the situation of Proposition 1.6.2, these embeddings are usually not closed.

**Remark 3.1.4.** By Propositions 2.2.12 and 2.2.13 the maps \( A^1 \times Z^+ \to \tilde{Z}^+ \) and \( A^1 \times Z^- \to \tilde{Z}^- \) are isomorphisms, so Proposition 3.1.3 means that the morphisms

\[ \tilde{Z}^+ \to \tilde{Z} \times Z^+ , \quad \tilde{Z}^- \to \tilde{Z} \times Z^- \]

are open embeddings.

**Remark 3.1.5.** Using (1.13), it is easy to see that if every \( G_m \)-equivariant map \( \mathbb{P}^1 \otimes \bar{k} \to Z \otimes \bar{k} \) is constant then the maps (3.3) and (3.4) are surjective. In this case they are isomorphisms by Proposition 3.1.3.

3.1.6. **Plan.** We will interpret the fiber products (3.3) and (3.4) as spaces of \( G_m \)-equivariant maps. More precisely, we will define schemes \( X^- \) and \( X^+ \) over \( A^1 \) equipped with \( G_m \)-action, such that for any scheme \( S \) over \( A^1 \) one has natural bijections

\[ \text{Maps}_{A^1}(S, \tilde{Z} \times Z^+) \sim \text{Maps}_{A^1}(X_S^+, Z), \quad X_S^+ := X^+ \times S \]

(3.5)

\[ \text{Maps}_{A^1}(S, \tilde{Z} \times Z^-) \sim \text{Maps}_{A^1}(X_S^-, Z), \quad X_S^- := X^- \times S. \]

(3.6)

Then we will give a simple description of \( X^\pm \). We will see that after reformulating Proposition 3.1.3 in terms of \( X^\pm \) it becomes almost obvious.

3.1.7. **Definition of \( X^\pm \).** We will define \( X^\pm \) so that the bijections (3.5) - (3.6) are tautological.

We have

\[ \text{Maps}_{A^1}(S, \tilde{Z}) = \text{Maps}_{A^1}(X_S, Z), \]

(3.7)

\[ \text{Maps}_{A^1}(S, A^1 \times Z^+) = \text{Maps}_{A^1}(S \times A^1, Z), \]

(3.8)

\[ \text{Maps}_{A^1}(S, A^1 \times Z^-) = \text{Maps}_{A^1}(S \times A^1, Z), \]

\[ \text{Maps}_{A^1}(S, A^1 \times Z) = \text{Maps}_{A^1}(S \times G_m, Z). \]

Recall that the maps \( \pi_1, \pi_2 : \tilde{Z} \to Z \) used in Subsect. 3.1.1 come from the two sections of the morphism \( \mathbb{X} \to A^1 \) that are given by formula (2.5), namely, \( \pi_1 \) corresponds to the section \( t \mapsto (1, t) \) and \( \pi_2 \) to the section \( t \mapsto (t, 1) \). These two sections define two \( G_m \)-equivariant maps \( A^1 \times G_m \to \mathbb{X} \), where the \( G_m \)-action on \( \mathbb{X} \) is defined by (2.3). Namely, the section \( t \mapsto (t, 1) \) defines the map

\[ A^1 \times G_m \to \mathbb{X}, \quad (t, \lambda) \mapsto (\lambda t, \lambda^{-1}), \]

(3.9)
and the section \( t \mapsto (1, t) \) defines the map
\[
A^1 \times \mathbb{G}_m \to X, \quad (t, \lambda) \mapsto (\lambda, \lambda^{-1}t).
\]
Note that both maps are open embeddings.

**Definition 3.1.8.** (i) \( X^+ \) is the push-out of the diagram of open embeddings
\[
A^1 \times A^1 \hookrightarrow A^1 \times \mathbb{G}_m \hookrightarrow X
\]
in which the right arrow is the map \((3.9)\).
(ii) \( X^- \) is the push-out of the diagram of open embeddings
\[
A^1 \times A^1_\perp \hookrightarrow A^1 \times \mathbb{G}_m \hookrightarrow X
\]
in which the right arrow is the map \((3.10)\).

Both \((3.11)\) and \((3.11)\) are diagrams in the category of schemes over \(A^1\) equipped with a \(\mathbb{G}_m\)-action over \(A^1\) (in the case of \(A^1 \times A^1\) the structure of scheme over \(A^1\) is given by the first projection \(A^1 \times A^1 \to A^1\)). So \(X^+\) and \(X^-\) are also in this category.\(^{13}\)

The bijections \((3.10)\) and \((3.11)\) are clear.

**3.1.9. Description of \(X^\pm\).** We claim that both schemes \(X^+\) and \(X^-\) are isomorphic to the blow-up of \(A^2\) at a point. Here is a more precise statement, whose verification is straightforward.

**Lemma 3.1.10.** (i) The morphisms
\[
A^1 \times A^1 \to A^1 \times A^1, \quad (t, \lambda) \mapsto (t, \lambda t)
\]
are compatible via diagram \((3.11)\). The corresponding morphism \(\sigma^+: X^+ \to A^1 \times A^1\) is a blow-up at the point \((0, 0) \in A^1 \times A^1\).

(ii) The morphisms
\[
A^1 \times A^1_\perp \to A^1 \times A^1_\perp, \quad (t, \lambda) \mapsto (t, \lambda t^{-1})
\]
are compatible via diagram \((3.12)\). The corresponding morphism \(\sigma^-: X^- \to A^1 \times A^1_\perp\) is a blow-up at the point \((0, \infty) \in A^1 \times A^1_\perp\).

(iii) Both \(\sigma^+\) and \(\sigma^-\) are \(\mathbb{G}_m\)-equivariant morphisms of schemes over \(A^1\).

**3.1.11. The canonical morphisms** \(A^1 \times Z^+ \to Z \times Z^+\) and \(A^1 \times Z^- \to Z^- \times Z\). For any scheme \(S\) over \(A^1\), the morphisms \(\sigma^\pm\) from Lemma 3.1.10 (i-ii) induce morphisms
\[
\sigma^+: S^+_S \to S \times A^1, \quad \sigma^-: S^-_S \to S \times A^1_\perp.
\]

By \((3.10)-(3.11)\) and \((3.11)-(3.12)\), these morphisms induce canonical maps
\[
(\sigma^+)^*: \text{Maps}_{A^1}(S, A^1 \times Z^+) \to \text{Maps}_{A^1}(S, Z \times Z^+),
\]
\[
(\sigma^-)^*: \text{Maps}_{A^1}(S, A^1 \times Z^-) \to \text{Maps}_{A^1}(S, Z^- \times Z),
\]
which are natural in \(S\). These maps define canonical morphisms
\[
(\sigma^+)^*: A^1 \times Z^+ \to Z \times Z^+, \quad (3.13)
\]
\[
(\sigma^-)^*: A^1 \times Z^- \to Z^- \times Z.
\]

\(^{13}\)Moreover, one can define an action of the torus \(G_m^2\) on each of the diagrams \((3.11)\) and \((3.12)\) so that they become diagrams in the category of toric varieties (a.k.a. toric embeddings); then \(X^+\) and \(X^-\) are also in this category. The above \(G_m\)-action is a part of the \(G_m^2\)-action.
Lemma 3.1.12. The morphisms \((3.3)\) and \((3.4)\) are equal, respectively, to \((3.13)\) and \((3.14)\).

We skip the verification of the lemma, which is straightforward.

The lemma implies that Proposition 3.1.3 is equivalent to the following one.

Proposition 3.1.13. The morphisms \((3.13)\) and \((3.14)\) are open embeddings.

We will prove the part of Proposition 3.1.13 about \((\sigma^{-})^{*}\). We will use the following property of the morphism \(\sigma^{-}: X^{-} \to \mathbb{A}^{1} \times \mathbb{A}^{-}_{1}\).

Lemma 3.1.14. Let \(S\) be a spectrum of an Artinian local ring equipped with a morphism \(S \to \mathbb{A}^{1}\). Then the morphism \(\sigma^{-}_{S}: X^{-}_{S} \to S \times \mathbb{A}^{1}_{-}\) has the following property: the map \(\mathcal{O}_{S \times \mathbb{A}^{1}_{-}} \to (\sigma^{-}_{S})_{*}\mathcal{O}_{X^{-}_{S}}\) is an isomorphism (here \((\sigma^{-}_{S})_{*}\) denotes the naive direct image rather than the derived one).

Proof. If \(S\) is a spectrum of a field the statement is clear from the explicit description of \(\sigma^{-}\) given in Lemma 3.1.10(ii). The case of a general Artinian local ring follows by devissage (one uses flatness of \(X^{-}_{S}\) and \(S \times \mathbb{A}^{1}_{-}\) over \(S\)). \(\square\)

Remark 3.1.15. It is easy to prove Lemma 3.1.14 for any scheme \(S\) over \(\mathbb{A}^{1}\) and for the derived direct image \(R(\sigma^{-}_{S})_{*}\) instead of the naive one. However, the above minimalistic formulation of Lemma 3.1.14 will allow us to skip the proof of Lemma 3.2.14(i) (because it is identical to that of Proposition 3.1.13).

We will also use the following general lemma, which is proved in Appendix A.

Lemma 3.1.16. Let \(A, B, Z\) be algebraic \(k\)-spaces and \(f: A \to B\) a surjective morphism with \(f_{*}\mathcal{O}_{A} = \mathcal{O}_{B}\) (here \(\mathcal{O}_{A}, \mathcal{O}_{B}\) are sheaves on the etale sites \(A_{et}, B_{et}\) and \(f_{*}\) is understood in the non-derived sense). Then

(i) the map \(\text{Maps}(B, Z) \to \text{Maps}(A, Z)\) induced by \(f\) is injective;
(ii) if \(B_{0} \subset B\) is a closed subspace containing \(B_{\text{red}}\) and \(A_{0} = f^{-1}(B_{0})\) then the diagram

\[
\begin{array}{ccc}
\text{Maps}(B, Z) & \longrightarrow & \text{Maps}(A, Z) \\
\downarrow & & \downarrow \\
\text{Maps}(B_{0}, Z) & \longrightarrow & \text{Maps}(A_{0}, Z)
\end{array}
\]

induced by \(f\) is Cartesian.

Now let us prove the part of Proposition 3.1.13 about the morphism \((\sigma^{-})^{*}\). To prove that \((\sigma^{-})^{*}\) is an open embedding, it suffices to show that it is etale and induces an injective map of field-valued points. This amounts to checking the following:

(a) if \(S\) is a spectrum of a field equipped with a morphism \(S \to \mathbb{A}^{1}\) then the map
\[
(\sigma^{-}_{S})^{*}: \text{Maps}^{\mathbb{G}_{m}}(S \times \mathbb{A}^{-}_{1}, Z) \to \text{Maps}^{\mathbb{G}_{m}}(X^{-}_{S}, Z)
\]

is injective;
(b) let \(S\) be a spectrum of an Artinian local ring equipped with a morphism \(S \to \mathbb{A}^{1}\) (so \(S_{\text{red}}\) is a spectrum of a field), then the diagram

\[
\begin{array}{ccc}
\text{Maps}^{\mathbb{G}_{m}}(S \times \mathbb{A}^{-}_{1}, Z) & \longrightarrow & \text{Maps}^{\mathbb{G}_{m}}(X^{-}_{S}, Z) \\
\downarrow & & \downarrow \\
\text{Maps}^{\mathbb{G}_{m}}(S_{\text{red}} \times \mathbb{A}^{-}_{1}, Z) & \longrightarrow & \text{Maps}^{\mathbb{G}_{m}}(X^{-}_{S_{\text{red}}}, Z)
\end{array}
\]
is Cartesian.

To prove this, it suffices to apply Lemma 3.1.16 for $A = X \setminus S$, $B = S \times A^1$, $f = \sigma_S^-$ and also for $A = \mathbb{G}_m \times X \setminus S$, $B = \mathbb{G}_m \times S \times A^1$, $f = \text{id}_{\mathbb{G}_m} \times \sigma_S^-$. Lemma 3.1.16 is applicable by Lemma 3.1.14.

3.2. The fiber product $\tilde{Z} \times Z \ldots \times Z \tilde{Z}$. The material of this subsection is not used in the rest of the article. But we think it is interesting on its own.

3.2.1. Plan. In Subsect. 3.1 we described the fiber products $Z^- \times \tilde{Z}$ and $\tilde{Z} \times Z^+$ and constructed open embeddings

$$(3.15) \quad A^1 \times Z^- \hookrightarrow Z^- \times \tilde{Z}, \quad A^1 \times Z^+ \hookrightarrow \tilde{Z} \times Z^+.$$  

Similarly to the above fiber products, one defines $\tilde{Z} \times \tilde{Z}$ and, more generally, the $n$-fold fiber product

$$(3.16) \quad \tilde{Z}_n := \tilde{Z} \times \ldots \times \tilde{Z}$$

using the projections $\pi_1, \pi_2 : \tilde{Z} \to Z$.

We will describe $\tilde{Z}_n$ as a space of maps and construct an open embedding

$$(3.17) \quad \tilde{Z} \times \mathbb{A}^n \hookrightarrow \tilde{Z}_n,$$

which is an isomorphism if $Z$ is affine. The morphism $\mathbb{A}^n \to \mathbb{A}^1$ implicit in formula (3.17) is the multiplication map

$$(3.18) \quad (t_1, \ldots, t_n) \mapsto t_1 \cdot \ldots \cdot t_n.$$  

It will be clear that the embeddings (3.15) can be obtained by base change from the embedding (3.17) for $n = 2$.

The strategy will be similar to the one used in Subsect. 3.1. The role of the blow-up of $\mathbb{A}^2$ (see Subsect. 3.1.9) will be played by a certain “very small” resolution of singularities of the scheme

$$(3.19) \quad X_{\mathbb{A}^n} := X \times \mathbb{A}^n_{\mathbb{A}^1};$$

here the fiber product is defined using the map (3.18), so it is, in fact, the hypersurface

$$t_1 \cdot \ldots \cdot t_n = uv.$$  

The above-mentioned small resolution is well known for $n = 2$.

3.2.2. $\tilde{Z}_n$ as a space of maps. Let $\mathcal{C}_n$ denote the category of spaces over $\mathbb{A}^n$ equipped with a $\mathbb{G}_m$-action over $\mathbb{A}^n$. For instance, $X$ and $\tilde{Z}$ are objects of $\mathcal{C}_1$, and the space $\tilde{Z}_n$ defined by (3.16) is an object of $\mathcal{C}_n$ (because $\tilde{Z} \in \mathcal{C}_1$).

Now we will define a scheme $X_n \in \mathcal{C}_n$ such that for any scheme $S$ over $\mathbb{A}^n$ one has

$$(3.20) \quad \text{Maps}_{\mathbb{A}^n}(S, \tilde{Z}_n) = \text{Maps}_{\mathbb{A}^1}(X_n, Z),$$

where $(X_n)_S := X_n \times_{\mathbb{A}^n} S$. (For instance, if $n = 1$ then $X_n = X$.)

First, set

$$U_r := \mathbb{A}^{r-1} \times X \times \mathbb{A}^{n-r}, \quad 1 \leq r \leq n.$$  

Note that $U_r \in \mathcal{C}_n$ because $X \in \mathcal{C}_1$. In $\mathcal{C}_1$ we have two open embeddings $\mathbb{G}_m \times \mathbb{A}^1 \to X$ defined by (3.9), (3.10). Multiplying them by $\mathbb{A}^{r-1}$ on the left and $\mathbb{A}^{n-r}$ on the right one gets two open
embeddings $G_m \times \mathbb{A}^n \hookrightarrow U_r$ in the category $\mathcal{C}_n$. Let $\alpha_r : G_m \times \mathbb{A}^n \hookrightarrow U_r$ be the embedding corresponding to (3.19) and $\beta_r : G_m \times \mathbb{A}^n \rightarrow U_r$ the one corresponding to (3.10).

Now define $X_n \in \mathcal{C}_n$ to be the colimit (a.k.a. inductive limit) of the diagram

$$
(U_1 \xrightarrow{\alpha_1} U_2 \xrightarrow{\beta_2} \cdots) \xrightarrow{\alpha_1 \beta_2 \cdots} G_m \times \mathbb{A}^n
$$

Then the bijection (3.20) is tautological.

**Lemma 3.2.3.** (i) The canonical morphisms

$$U_r = \mathbb{A}^{r-1} \times X \times \mathbb{A}^{n-r} \rightarrow X_n, \quad 1 \leq r \leq n$$

are open embeddings, and their images cover $X_n$.

(ii) $X_n$ is a smooth scheme over $k$ of dimension $n+1$, which is flat over $\mathbb{A}^n$.

**Proof.** Statement (i) is proved by induction. Statement (ii) follows. \qed

**Remark 3.2.4.** It is clear that the fiber of $X_n$ over each field-valued point of $\mathbb{A}^n$ is a curve. (It is obtained by gluing hyperbolas. Such gluing is non-tautological only if some of these hyperbolas are degenerate.)

**Remark 3.2.5.** Here is a more precise version of the previous remark. For $m \geq 0$ let $C_m$ denote the following curve: take $m+1$ copies of $\mathbb{P}^1$, denoted by $(\mathbb{P}^1)_i$, $0 \leq i \leq m$; then for all $i < m$ glue $0 \in (\mathbb{P}^1)_i$ with $\infty \in (\mathbb{P}^1)_{i+1}$ and finally, remove $\infty \in (\mathbb{P}^1)_0$ and $0 \in (\mathbb{P}^1)_m$. It is easy to see that the fiber of $X_n$ over each field-valued point of $\mathbb{A}^n$ is isomorphic to $C_m$ for some $m$, $0 \leq m \leq n$.

3.2.6. The locally closed embedding $X_n \hookrightarrow \mathbb{A}^n \times (\mathbb{P}^1)^{n+1}$. We will first construct a quasi-projective scheme $X'_n \in \mathcal{C}_n$: more precisely, $X'_n$ will be a locally closed subscheme of the product $\mathbb{A}^n \times (\mathbb{P}^1)^{n+1}$. Then we will construct a $\mathcal{C}_n$-isomorphism $X_n \sim X'_n$.

Points of $\mathbb{P}^1$ will be denoted by $(p : q)$. We equip $\mathbb{P}^1$ with the usual action of $G_m$, i.e., $\lambda \in G_m$ takes $(p : q)$ to $(\lambda p : q)$.

**Convention.** Let $\xi, \xi' \in \mathbb{P}^1$, $\xi = (p : q)$, $\xi' = (p' : q')$, $t \in \mathbb{A}^1$. Then

$$t \cdot \xi = \xi'$$

as a shorthand for $tpq' = p'q$.

(Thus $0 \cdot \infty = \xi'$ for any $\xi' \in \mathbb{P}^1$.)

We equip $(\mathbb{P}^1)^{n+1}$ with the diagonal action of $G_m$. So $\mathbb{A}^n \times (\mathbb{P}^1)^{n+1}$ is an object of $\mathcal{C}_n$.

Points of $\mathbb{A}^n$ will be denoted by $(t_1, \ldots, t_n)$. Points of $(\mathbb{P}^1)^{n+1}$ will be denoted by $(\xi_0, \ldots, \xi_n)$, where $\xi_i \in \mathbb{P}^1$.

**Definition 3.2.7.** $X'_n \subset \mathbb{A}^n \times (\mathbb{P}^1)^{n+1}$ is the locally closed subscheme defined by the inequalities

$$\xi_0 \neq \infty, \quad \xi_n \neq 0$$

and the equations

$$\xi_{i-1} = t_i \cdot \xi_i, \quad 1 \leq i \leq n;$$

here the equations are understood according to Convention (3.22).
The subscheme $X'_r \subset A^n \times (P^1)^{n+1}$ is $G_m$-stable, so $X'_r \in \mathcal{C}_n$.

Let $U'_r \subset X'_n$ denote the $G_m$-stable open subscheme defined by the inequalities

\[(3.25)\]
\[\xi_{r-1} \neq \infty, \quad \xi_r \neq 0.\]

Note that the inequalities (3.25) follow from (3.24) and (3.26).

**Lemma 3.2.8.** (i) The open subschemes $U'_r$ cover $X_n$.

(ii) If $r_1 \leq r_2 \leq r_3$ then $U'_{r_1} \cap U'_{r_3} \subset U'_{r_2}$.

**Proof.** (i) Let $(\xi_0, \ldots, \xi_n) \in X'$. Let $r$ be the minimal number such that $\xi_r \neq 0$. Then $\xi_{r-1} = 0 \neq \infty$, so $(\xi_0, \ldots, \xi_n) \in U'_r$.

(ii) Let $(\xi_0, \ldots, \xi_n) \in U'_{r_1} \cap U'_{r_2}$. Since $\xi_{r_3-1} \neq \infty$ and $\xi_{r_1} \neq 0$ the equations (3.24) imply that $\xi_{r_2-1} \neq \infty$ and $\xi_r \neq 0$. \qed

**Corollary 3.2.9.** $X'_n$ is the colimit of the diagram

\[(3.26)\]

Now we will construct a $\mathcal{C}_n$-isomorphism between diagrams (3.21) and (3.26).

Recall that the scheme $U'_r$ from diagram (3.21) equals $A^{r-1} \times A^{n-r}$, and the coordinates on $X$ are denoted by $\tau_1, \tau_2$.

**Lemma 3.2.10.** (i) Formulas $\tau_1 = \xi_{r-1}, \tau_2 = \xi_r^{-1}$ define a $\mathcal{C}_n$-isomorphism $U'_r \simto U_r$. Its inverse is given by

\[\xi_i = t_{i+1} \cdots t_{i-1} \cdot \tau_1\quad \text{for } i < r, \quad \xi_i = (\tau_2 \cdot t_{r+1} \cdots t_i)^{-1}\quad \text{for } i \geq r.\]

(ii) There exists an isomorphism between diagrams (3.26) and (3.21) inducing the above isomorphism $U'_r \simto U_r$ for each $r \in \{1, \ldots, n\}$.

The proof is straightforward. Let us just say that the composition

\[U'_r \cap U'_{r+1} \simto G_m \times A^n \to G_m\]

given by $\xi_r$ (note that the values of $\xi_r$ on $U'_r \cap U'_{r+1}$ are in $P^1 - \{0, \infty\} = G_m$).

Finally, Corollary 3.2.9 tells us that the isomorphism between diagrams (3.26) and (3.21) constructed in Lemma 3.2.10 induces a $\mathcal{C}_n$-isomorphism $X'_n \simto X_n$. We will always identify $X'_n$ with $X_n$ using this isomorphism.

3.2.11. *The map $X_n \to X_{A^n}$.* Recall that $X_{A^n} := X \times A^n$; equivalently, $X_{A^n}$ is the hypersurface

\[(3.27)\]
\[t_1 \cdots t_n = uv.\]

The equations (3.24) imply that $\xi_0 \cdot \xi_n^{-1} = t_1 \cdots t_n$, so we have a morphism

\[(3.28)\]
\[X_n = X'_n \to X_{A^n}\]

defined by

\[u = \xi_0, \quad v = \xi_n^{-1}.\]

**Lemma 3.2.12.** The morphism (3.28) is projective and small.
Proof. The morphism (3.28) is projective because it is a composition
\[ X'_n \hookrightarrow X_{\mathbb{A}^n} \times (\mathbb{P}^1)^{n-1} \to X_{\mathbb{A}^n} \]
in which the first arrow is a closed embedding and the second one is the projection. By Remark 3.2.4, the fibers of the morphism (3.28) have dimension \( \leq 1 \) (the fibers that have more than one point are chains of projective lines). Finally, it is easy to check that the map (3.29) is an isomorphism over \( X_{\mathbb{A}^n} \) if \( F \subset X_{\mathbb{A}^n} \) is a closed subset of codimension 3; namely, a point \( (t_1, \ldots, t_n, u, v) \in X_{\mathbb{A}^n} \) is in \( F \) if and only if \( u = v = 0 \) and \( t_i = 0 \) for more than one \( i \).

3.2.13. The open embedding \( \tilde{Z} \times_{\mathbb{A}^1} \mathbb{A}^n \hookrightarrow \tilde{X}_n \). By (3.20), the map (3.28) induces a morphism
\[ (\tilde{Z} \times_{\mathbb{A}^1} \mathbb{A}^n) \to \tilde{X}_n. \]

Lemma 3.2.14. (i) This morphism is an open embedding.
(ii) If every \( \mathbb{G}_m \)-equivariant map \( \mathbb{P}^1 \otimes_k \bar{k} \to Z \otimes_k \bar{k} \) is constant then the map (3.29) is an isomorphism.

Proof. Statement (i) is proved just as Proposition 3.1.13 (to prove an analog of Lemma 3.1.14, use Remark 3.2.4 and the flatness statement from Lemma 3.2.3 (ii)).

Statement (ii) is a consequence of (i) and the following corollary of Remark 3.2.6: any fiber of the morphism (3.28) is either a point or a chain of projective lines each of which is equipped with the standard \( \mathbb{G}_m \)-action.

3.2.15. Toric action. Recall that \( X_{\mathbb{A}^n} \) is the variety of solutions to the equation (3.27). Let \( T \subset X_{\mathbb{A}^n} \) denote the set of those solutions all of whose coordinates are nonzero. This is a group with respect to multiplication. The torus \( T \) acts on \( X_{\mathbb{A}^n} \) by multiplication; in fact, \( X_{\mathbb{A}^n} \) is a toric variety with respect to \( T \). There is a unique structure of toric variety on \( X_n \) (with the same torus \( T \)) such that the map (3.28) is a morphism of toric varieties. Therefore one can describe \( X_n, X_{\mathbb{A}^n} \), and the map (3.28) using the language of fans, see [KKMS, Ch. I].

Note that the action of \( \mathbb{G}_m \) on \( X_n \) and \( X_{\mathbb{A}^n} \) considered above is a part of the \( T \)-action.

3.2.16. Relation with the anti-action from Subsect. 2.1.14. The morphism (3.29) can be expressed in terms of the anti-action from Subsect. 2.1.13. Let us explain this for \( n = 2 \).

Given a \( k \)-scheme \( S \) and morphisms \( t_1, t_2 : S \to \mathbb{A}^1 \), we have a commutative diagram
\[ \begin{array}{ccc}
\tilde{Z}_{t_1t_2} & \xrightarrow{\phi_{t_1,t_2}} & \tilde{Z}_{t_2} \\
\phi_{t_2,t_1} \downarrow & & \downarrow \phi_{t_1,t_2} \\
\tilde{Z}_{t_1} & \xrightarrow{\phi_{t_1,1}} & \mathbb{A}^1 \\
\end{array} \]
whose arrows are given by the anti-action of \( \mathbb{A}^2 \); see formula (2.12). It is easy to check that the morphism \( \phi_{t_1,t_2,1} : \tilde{Z}_{t_2} \to \tilde{Z}_{t_1} = \mathbb{A}^1 \) comes from the morphism \( \pi_1 : \tilde{Z} \to Z \) defined in Subsect. 2.1.14 and the morphism \( \phi_{t_1,1,1} : \tilde{Z}_{t_1} \to \tilde{Z}_1 = Z \) comes from \( \pi_2 \). So diagram (3.30) defines a morphism \( \tilde{Z}_{t_1t_2} \to \tilde{Z} \times \tilde{Z} \). As \( t_1 \) and \( t_2 \) vary, we get a morphism \( \tilde{Z} \times \mathbb{A}^2 \to \tilde{Z} \times \tilde{Z} \). It is straightforward to check that it is equal to the morphism (3.29) for \( n = 2 \).

3.2.17. Remark. By virtue of Subsect. 3.2.16 one can interpret Proposition 3.2.14 as a property of the anti-action from Subsect. 2.1.14. Since the map (3.29) involves a number \( n \), we have, in fact, a sequence of properties for \( n = 2, 3, 4, \ldots \). However, it is easy to see that the property for \( n = 2 \) implies the rest.
4. Proof of Theorem 1.4.2

4.1. Plan. Let $Z$ be an algebraic $k$-space of finite type equipped with a $\mathbb{G}_m$-action. We have to prove that $Z^+$ is an algebraic space and that $q^+ : Z^+ \to Z^0$ is an affine morphism of finite type. To this end, we will decompose the morphism $q^+ : Z^+ \to Z^0$ as

$$Z^+ \to Z^+_\infty \to Z^0,$$

where $Z^+_\infty$ is defined in Subsect. 4.2 below. Then we will prove that the morphism $Z^+ \to Z^+_\infty$ is, in fact, an isomorphism and the morphism $Z^+_\infty \to Z^0$ is an affine morphism of finite type.

4.2. The space $Z^+_\infty$. For $n \in \mathbb{Z}_+$ let $(\mathbb{A}^1)_n \subset \mathbb{A}^1$ denote the $n$-th infinitesimal neighborhood of $0 \in \mathbb{A}^1$, i.e., $(\mathbb{A}^1)_n := \text{Spec } k[[t]]/(t^{n+1})$. Set $Z^+_n := \text{Maps}^{\mathbb{G}_m}((\mathbb{A}^1)_n, Z)$. Note that $Z^+_0 = Z^0$. The spaces $Z^+_n$ form a projective system.

**Definition 4.2.1.** $Z^+_\infty := \lim_n Z^+_n$. Equivalently, $Z^+_\infty = \text{Maps}^{\mathbb{G}_m}(\hat{\mathbb{A}}^1, Z)$, where $\hat{\mathbb{A}}^1$ is the formal completion of $\mathbb{A}^1$ at $0 \in \mathbb{A}^1$, i.e., $\mathbb{A}^1 := \lim_n (\mathbb{A}^1)_n$.

The multiplicative monoid $\mathbb{A}^1$ acts on itself by multiplication, and this action preserves the subschemes $(\mathbb{A}^1)_n \subset \mathbb{A}^1$. So $\mathbb{A}^1$ acts on the spaces $Z^+_n$ and $Z^+_\infty$.

4.3. A theorem which implies Theorem 1.4.2 The embeddings

$$\text{Spec } k = \{0\} \hookrightarrow \hat{\mathbb{A}}^1 \hookrightarrow \mathbb{A}^1$$

induce morphisms

$$\text{Maps}^{\mathbb{G}_m}(\mathbb{A}^1, Z) \to \text{Maps}^{\mathbb{G}_m}(\hat{\mathbb{A}}^1, Z) \to \text{Maps}^{\mathbb{G}_m}(\text{Spec } k, Z)$$

or equivalently,

$$(4.1) \quad Z^+ \to Z^+_\infty \to Z^0.$$

The composition in (4.1) equals $q^+ : Z^+ \to Z^+_\infty$. So Theorem 1.4.2 follows from the next one.

**Theorem 4.3.1.** (i) The morphism $Z^+ \to Z^+_\infty$ is an isomorphism.

(ii) $Z^+_\infty$ is an algebraic space. Moreover, the morphism $Z^+_\infty \to Z^0$ is affine and of finite type.

The easier statement (ii) will be proved in the next subsection. Statement (i) of the theorem will be proved in Subsections 4.5-4.7.

4.4. Proof of Theorem 4.3.1 (ii). We will first construct a finitely generated $\mathbb{Z}_+$-graded quasi-coherent $\mathcal{O}_{Z^0}$-algebra $\mathcal{A}^+$ (see Definition 4.3.2). Then we will construct an isomorphism $Z^+ \cong \text{Spec } \mathcal{A}^+$ of spaces over $Z^0$. Thus we will get an explicit description of $Z^+$ in the spirit of Subsect. 1.3.1.

Let $J_n$ denote the sheaf of $n$-jets of functions on $Z$. In particular,

$$(4.2) \quad J_0 = \mathcal{O}_Z, \quad J_1 = \mathcal{O}_Z \oplus \Omega^1_Z.$$

Let $J_n^0$ denote the pullback of $J_n$ to $Z^0$.

Each $J_n$ is a coherent $\mathcal{O}_Z$-algebra. So $J_n^0$ is a coherent $\mathcal{O}_{Z^0}$-algebra. In addition, the $\mathcal{O}_{Z^0}$-algebra is $\mathbb{Z}_+$-graded: the grading corresponds to the $\mathbb{G}_m$-action on $J_n^0$. The epimorphism

$$J_n^0 \twoheadrightarrow J_0^0 = \mathcal{O}_{Z^0}$$
will be called *augmentation*. If \(0 \leq m \leq n\) then \(J_n^m\) identifies with the quotient of \(J_n^0\) by the \((m + 1)\)-th power of the augmentation ideal \(\ker(J_n^0 \to \mathcal{O}_{Z^n})\).

Let \(A^+_n\) denote the quotient of \(J_n^0\) by the ideal generated by the degree 0 component of \(\ker(J_n^0 \to \mathcal{O}_{Z^n})\) and by the components of negative degrees of \(J_n^0\). Clearly \(A^+_n\) is a \(\mathbb{Z}_+\)-graded coherent \(\mathcal{O}_{Z^n}\)-algebra whose degree 0 component equals \(\mathcal{O}_{Z^n}\).

**Lemma 4.4.1.** If \(0 \leq m \leq n\) then \(A^+_n\) identifies with the quotient of \(A^+_m\) by the \((m + 1)\)-th power of the augmentation ideal \(\ker(A^+_m \to \mathcal{O}_{Z^n})\).

**Proof.** Follows from a similar property of the algebras \(J_n^m\). \(\square\)

Lemma 4.4.1 implies that \(\ker(A^+_m \to A^+_m)\) is concentrated in degrees \(> m\); in other words, the degree \(m\) component of \(A^+_m\) does not depend on \(n\) if \(n \geq m\).

**Definition 4.4.2.** \(A^+\) is the projective limit of the \(\mathcal{O}_{Z^n}\)-algebras \(A^+_n\). In other words, \(A^+\) is the \(\mathbb{Z}_+\)-graded quasi-coherent \(\mathcal{O}_{Z^n}\)-algebra whose degree \(m\) component is the degree \(m\) component of \(A^+_n\), where \(n \geq m\).

**Lemma 4.4.3.** (i) The \(\mathcal{O}_{Z^n}\)-algebra \(A^+\) is finitely generated.

(ii) Suppose that the pullback of \(\Omega_1^Z\) to \(Z^0\) is concentrated in degrees \(\leq n\) with respect to the \(\mathbb{Z}\)-grading corresponding to the \(\mathbb{G}_m\)-action. Then the \(\mathcal{O}_{Z^n}\)-algebra \(A^+\) is generated by its graded components of degrees \(\leq n\).

**Proof.** Statement (i) follows from Lemma 4.4.1 and the fact that \(A^+\) is coherent. Statement (ii) follows from Lemma 4.4.1 and the following description of \(A^+_1\), which is an immediate corollary of \(\Gamma_2\): \(A^+_1 = \mathcal{O}_{Z^n} \oplus M\), where \(M\) is the strictly positive part of the pullback of \(\Omega_1^Z\) to \(Z^0\). \(\square\)

Theorem 1.4.2(ii) immediately follows from Lemma 4.4.3(i) and the next proposition.

**Proposition 4.4.4.** \(Z^+\) is canonically isomorphic to \(\text{Spec} A^+\) as a space over \(Z^0\).

**Proof.** Let \(S = \text{Spec} R\) be an affine test scheme. Fix a morphism \(\varphi : S \to Z^0\). Set

\[A^+_R := H^0(S, \varphi^* A^+).\]

We have to construct a canonical bijection

\[\text{Maps}_{\mathcal{O}_{Z^n}}(S, Z^+) \xrightarrow{\sim} \text{Hom}_R(A^+_R, R),\]

where \(\text{Maps}_{\mathcal{O}_{Z^n}}\) stands for the set of morphisms of spaces over \(Z^0\) and \(\text{Hom}_R\) stands for the set of \(R\)-algebra homomorphisms.

Define \(\Phi : S \to Z \times S\) by \(\Phi := (\varphi, \text{id}_S)\). By definition, elements of \(\text{Maps}_{\mathcal{O}_{Z^n}}(S, Z^+)\) correspond to \(\mathbb{G}_m\)-equivariant \(S\)-morphisms \(f : \mathbb{A}^1 \times S \to Z \times S\) whose restriction to \(\{0\} \times S\) equals \(\Phi : S \to Z \times S\). Note that \(f(\mathbb{A}^1_n \times S)\) is contained in the \(n\)-th infinitesimal neighborhood of the subspace \(\Phi(S) \subset Z \times S\). This neighborhood equals \(\text{Spec} J^R_n\), where \(J^R_n := H^0(S, \varphi^* J^0_n)\) (this follows from the definition of the jet sheaves).

So elements of the l.h.s. of (4.3) correspond to homomorphisms of augmented topological algebras

\[\lim_{\rightarrow n} J^R_n \to R[[t]]\]

compatible with the \(\mathbb{Z}\)-gradings (here \(t \in R[[t]]\) has degree 1 and the augmentation \(R[[t]] \to R\) is the “constant term” map). Such a homomorphism has to kill all elements of negative degrees.

---

\[\text{If } Z \text{ is not a scheme then it may happen that the monomorphism } \Phi \text{ is not a locally closed embedding. But the notion of the } n\text{-th infinitesimal neighborhood still makes sense in this situation. E.g., one can use Definition 16.1.2 from EGA IV (if the algebraic spaces involved are equipped with the etale topology).}\]
and all degree 0 elements of the augmentation ideal of $J^n_R$. Thus elements of the l.h.s. of (4.3) correspond to graded $R$-algebra homomorphisms $A^n_R \to R[t]$.

Finally, graded $R$-algebra homomorphisms $A^n_R \to R[t]$ are in bijection with elements of the r.h.s of (4.3): to a graded homomorphism $A^n_R \to R[t]$ one associates its composition with $ev_1 : R[t] \to R$, where $ev_1$ is evaluation at $t = 1$.

\[ \square \]

4.5. Proof of Theorem 4.3.1(i) modulo Lemma 4.5.3 Our goal is to prove that the morphism $Z^+ \to Z^+_\infty$ is an isomorphism.

Lemma 4.5.1. The morphism $Z^+ \to Z^+_\infty$ is a monomorphism.

Proof. We have to prove the injectivity of the map

\[ \text{Maps}^{G_m}(S \times \mathbb{A}^1, Z) \to \text{Maps}^{G_m}(S, Z^+_\infty) \]

for any $k$-scheme $S$. Since $Z^+_\infty$ has finite type over $k$ we can assume that $S$ is Noetherian (and moreover, has finite type over $k$).

Let $f_1, f_2 : S \times \mathbb{A}^1 \to Z$ be $G_m$-equivariant morphisms having the same restriction to the formal neighborhood of $S \times \{0\} \subset S \times \mathbb{A}^1$. We have to prove that $f_1 = f_2$. Let $E$ denote the equalizer of $f_1, f_2$, i.e., the preimage of the diagonal with respect to $(f_1, f_2) : S \times \mathbb{A}^1 \to Z \times Z$. Clearly $E$ is a scheme of finite presentation over $S$ equipped with an $\mathbb{A}^1$-action and an $\mathbb{A}^1$-equivariant monomorphism $\nu : E \hookrightarrow S \times \mathbb{A}^1$. Moreover, the “sub”scheme $E = S \times \mathbb{A}^1$ contains the formal neighborhood of $S \times \{0\} \subset S \times \mathbb{A}^1$. So $\nu$ is etale at $S \times \{0\} \subset E$. Let $E'$ be the maximal open subscheme of $E$ such that $\nu|_{E'}$ is etale. Then $\nu|_{E'}$ is an open embedding. So $E'$ is an open subscheme of $S \times \mathbb{A}^1$ containing $S \times \{0\}$ and stable with respect to the $\mathbb{A}^1$-action. Therefore $E' = S \times \mathbb{A}^1$, $E = S \times \mathbb{A}^1$, and $f_1 = f_2$.

Remark 4.5.2. It is clear that the space $Z^+$ is locally of finite presentation (i.e., the corresponding functor $\{k$-algebras$\} \to \{\text{sets}\}$ commutes with filtering inductive limits).

Lemma 4.5.3. Let $R$ be a complete local Noetherian $k$-algebra. Then the map $Z^+(R) \to Z^+\infty(R)$ is bijective.

Let us assume this lemma for now; it will be proved in Subsect. 4.7.

Proof of Theorem 4.3.1(i). We have to show that the morphism $Z^+ \to Z^+\infty$ is an isomorphism. By Lemma 4.5.1 it is a monomorphism, so it remains to show that for any point $z \in Z^+_\infty$ the morphism $Z^+ \to Z^+\infty$ admits a section over some etale neighborhood of $z$. By Remark 4.5.2 we can replace “etale neighborhood” by “Henselization”. By Artin approximation [Ar, Theorem 1.10] and Remark 4.5.2 one can replace “Henselization” by “spectrum of the completed local ring of $z$”. It remains to use Lemma 4.5.3.

\[ \square \]

Remark 4.5.4. If $Z$ is separated the proofs of Theorems 4.3.1 and 4.3.1(i) can be simplified (in particular, Artin approximation is unnecessary). Namely, if $Z$ is separated it is easy to prove directly that the canonical morphism $Z^+ \to Z^+\times Z$ is a closed embedding. Combining this with Theorem 4.3.1(ii), one immediately gets Theorem 4.3.2 and Theorem 4.3.1(i) follows from Lemma 4.5.3 in the particular case of Artinian local $k$-algebras.

\[ \text{The quotation marks are due to the fact that $\nu$ is not necessarily a locally closed embedding. Of course, $\nu$ is a locally closed embedding if $Z$ is separated or if $Z$ is a scheme.} \]
4.6. **A descent theorem of Moret-Bailly.** To prove Lemma [1.5.3](#) we need the following result from [MB](#).

**Theorem 4.6.1.** Let \( S \) be a k-scheme and \( Y \subseteq S \) a closed subscheme whose defining ideal in \( \mathcal{O}_S \) is finitely generated. Let \( S' \) be a scheme flat and affine over \( S \) such that the map \( S' \times_S Y \rightarrow Y \) is an isomorphism. Set \( U := S - Y \), \( U' := U \times_S S' \). Then for any algebraic k-space \( Z \) the map

\[
Z(S) \xrightarrow{\sim} Z(S') \times_{Z(U')} Z(U)
\]

is bijective.

This is Theorem 1.2 from [MB](#). If \( Z \) is a scheme the proof is easy (see [MB](#)); more generally, there is an easy proof if the diagonal map \( Z \rightarrow Z \times Z \) is a locally closed embedding. In the general cast, the proof from [MB](#) uses Proposition 4.2 from [FR](#), which says that in the situation of Theorem 4.6.1 the functor

\[
QC(S) \xrightarrow{\sim} QC(S') \times_{QC(U')} QC(U)
\]

is an equivalence; here QC stands for the category of quasi-coherent \( \mathcal{O} \)-modules.

**Remark 4.6.2.** As far as we understand, the scheme \( S' \) is required in [MB](#) to be affine over \( S \) only to simplify the exposition.

**Remark 4.6.3.** For any Noetherian ring \( A \), Theorem [1.6.1](#) is applicable in the following situation:

\[
S = \text{Spec } A[t], \quad Y = \text{Spec } A[t]/(t) \subset S, \quad U = \text{Spec } A[t, t^{-1}],
\]

\[
S' = \text{Spec } A[[t]], \quad U' = \text{Spec } A((t)).
\]

4.7. **Proof of Lemma 4.5.3**. By Lemma [4.5.1](#) we only have to prove that the map \( Z^+(R) \rightarrow Z^+_\infty(R) \) is surjective.

The coordinate on \( \mathbb{A}^1_R \) will be denoted by \( t \). So \( \mathbb{A}^1_R = \text{Spec } R[[t]] \), and the formal completion of \( \mathbb{A}^1_R \) along 0 is \( \text{Spf } R[[t]] \), where \( R[[t]] \) is equipped with the \( t \)-adic topology.

By definition, an element of \( Z^+_\infty(R) \) is a \( \mathbb{G}_m \)-equivariant morphism \( \tilde{f} : \text{Spf } R[[t]] \rightarrow Z \). Using the fact that \( R \) and \( R[[t]] \) are Henselian one easily checks that

\[
\text{Maps}(\text{Spf } R[[t]], Z) = \text{Maps}(\text{Spec } R[[t]], Z),
\]

so we can also consider \( \tilde{f} \) as a morphism \( \text{Spec } R[[t]] \rightarrow Z \). Let \( \hat{f}' : \text{Spec } R((t)) \rightarrow Z \) denote the restriction of \( \tilde{f} : \text{Spec } R[[t]] \rightarrow Z \) to \( \text{Spec } R((t)) \). The problem is to extend \( \hat{f}' \) to a \( \mathbb{G}_m \)-equivariant morphism \( f : \mathbb{A}^1_R = \text{Spec } R[t] \rightarrow Z \). By Theorem [4.6.1](#) and Remark [4.6.3](#) this problem is equivalent to extending \( \hat{f}' \) to a \( \mathbb{G}_m \)-equivariant morphism \( f' : (\mathbb{G}_m)_R = \text{Spec } R[t, t^{-1}] \rightarrow Z \).

Specifying \( f' \) is the same as specifying its restriction to \( \{1\} \subset (\mathbb{G}_m)_R \); denote it by \( \hat{z} \in Z(R) \).

The requirement that \( f'|_{\text{Spec } R((t))} = \hat{f}' \) translates into the following condition:

\[
i_t(z) = \hat{z},
\]

where \( i : Z(R) \rightarrow Z(R((t))) \) is induced by the embedding \( R \hookrightarrow R((t)) \) and \( \hat{z} : \text{Spec } R((t)) \rightarrow Z \) is the composition

\[
\text{Spec } R((t)) \xrightarrow{(t^{-1}, f')} \mathbb{G}_m \times Z \rightarrow Z
\]

(the second morphism is the action map).
Lemma 4.7.2. Since
Assume that unique point of \( \text{Spec} \ R \) is Artinian. Then so is \( R((t)) \). Let \( z_0 \in Z \) denote the image of the unique point of \( \text{Spec} R((t)) \) and \( O_{Z, z_0} \) the corresponding Henselian local ring. Since \( R((t)) \) is Henselian the morphism \( \tilde{z} : \text{Spec} R((t)) \to Z \) factors through \( \text{Spec} O_{Z, z_0} \). So \( \tilde{z} \) defines a homomorphism \( \varphi : O_{Z, z_0} \to R((t)) \), and the problem is to show that \( \varphi(O_{Z, z_0}) \subset R \). Indeed, if \( p \in R((t)) \) belongs to \( \varphi(O_{Z, z_0}) \) then by \( \ref{4.6.1} \), \( p \) satisfies the identity \( p(\lambda t) = p(t) \), so \( p \in R \).

Step 1. Assume that \( R \) is Artinian. Then so is \( R((t)) \). Let \( z_0 \in Z \) denote the image of the unique point of \( \text{Spec} R((t)) \) and \( O_{Z, z_0} \) the corresponding Henselian local ring. Since \( R((t)) \) is Henselian the morphism \( \tilde{z} : \text{Spec} R((t)) \to Z \) factors through \( \text{Spec} O_{Z, z_0} \). So \( \tilde{z} \) defines a homomorphism \( \varphi : O_{Z, z_0} \to R((t)) \), and the problem is to show that \( \varphi(O_{Z, z_0}) \subset R \). Indeed, if \( p \in R((t)) \) belongs to \( \varphi(O_{Z, z_0}) \) then by \( \ref{4.6.1} \), \( p \) satisfies the identity \( p(\lambda t) = p(t) \), so \( p \in R \).

Step 2. Now drop the Artinian assumption. Let \( m \subset R \) be the maximal ideal. Set \( R_n := R/m^n \). Let \( z_n \in Z(R_n((t))) \) be the image of \( \tilde{z} \). By Step 1, \( z_n \) comes from a unique \( z_n \in Z(R_n) \). Since \( R \) is a complete local ring the sequence \( z_n \) defines a point \( z \in Z(R) \), i.e., a morphism \( z : \text{Spec} R((t)) \to Z \). We have to prove that the composition \( \text{Spec} R((t)) \to \text{Spec} R \xrightarrow{\beta} Z \) equals \( \tilde{z} : \text{Spec} R((t)) \to Z \). Just as in the proof of Lemma \( \ref{4.5.1} \), let \( E \) denote the equalizer of the two morphisms \( \text{Spec} R((t)) \to Z \); this is a \( \mathbb{G}_m \)-stable “sub”scheme of \( \text{Spec} R((t)) \) containing \( \text{Spec} R_n((t)) \) for each \( n \in \mathbb{N} \). Just as in the proof of Lemma \( \ref{4.6.1} \), this implies that \( E \) contains a \( \mathbb{G}_m \)-stable subscheme \( E' \) open in \( \text{Spec} R((t)) \) and containing \( \text{Spec}(R/m)((t)) \). Let us show\( ^{16} \) that such \( E' \) has to be equal to \( \text{Spec} R((t)) \).

Choose a closed subscheme \( F \subset \text{Spec} R((t)) \) whose complement equals \( E' \) and let \( I \subset R((t)) \) be the corresponding ideal.

\[ \text{Lemma 4.7.2.} \]
\[(i) \quad I + m((t)) = R((t)). \]
\[(ii) \quad \text{Let } I' \subset R((t)) \text{ be the ideal of all formal series } \sum r_i t^i, r_i \in R, \text{ such that the series } \sum r_i \lambda^i t^i \in R[\lambda, \lambda^{-1}][(t)] \text{ belongs to } I \cdot R[\lambda, \lambda^{-1}][(t)]. \text{ Then } I \text{ is contained in the radical of } I'. \]

\[ \text{Proof.} \quad \text{The open subset } E' = (\text{Spec } R((t))) - F \text{ contains } \text{Spec}(R/m)((t)), \text{ so } F \cap \text{Spec}(R/m)((t)) = \emptyset. \text{ This translates into (i). The fact that } E' \text{ is } \mathbb{G}_m \text{-stable translates into (ii).} \]

It remains to show that any ideal \( I \subset R((t)) \) with properties (i)-(ii) from the lemma is the unit ideal. Since \( I \) is contained in the radical of \( I' \) property (i) implies that \( I' + m((t)) = R((t)) \), so \( I' \) contains an element of the form \( \sum r_i t^i \), where \( r_i \in R \) and
\[ r_0 \in 1 + m. \]

By the definition of \( I' \), one has an equality of the form
\[ \sum r_i \lambda^i t^i = \sum g_j h_j, \quad g_j \in R[\lambda, \lambda^{-1}][(t)], \quad h_j \in I. \]

Equating the coefficients of \( \lambda^0 \) in this equality, we see that \( r_0 \in I \). On the other hand, \( r_0 \) is invertible by \( \ref{4.6.1} \). So \( I \) is the unit ideal, and we are done.

\[ ^{16} \text{This step is unnecessary if } Z \text{ is separated: indeed, in this case } Z \text{ is a closed subscheme of } \text{Spec } R((t)) \text{ containing } \text{Spec } R_n((t)) \text{ for all } n, \text{ so } E = \text{Spec } R((t)) \text{ and we are done.} \]
5. Proof of Theorem 2.2.2

In this section we prove Theorem 2.2.2 which says that for any algebraic $k$-space of finite type equipped with a $\mathbb{G}_m$-action, the space $\tilde{Z}$ defined in Subsect. 2.1.5 is an algebraic $k$-space of finite type.

We will use M. Artin’s technique for proving representability. In particular, in Subsect. 5.4 we use his Approximation Theorem to prove existence of a scheme equipped with a surjective étale morphism to $\tilde{Z}$. (Unfortunately, such proof of existence is not really constructive.)

We will be using the notation $X$ and $X_S$ introduced in Subsections 2.1.1-2.1.2. Recall that $X := \mathbb{A}^2 = \text{Spec } k[\tau_1, \tau_2]$ and for any scheme $S$ over $\mathbb{A}^1$ we set $X_S := X \times S$, where $X$ is mapped to $\mathbb{A}^1$ by $(\tau_1, \tau_2) \mapsto \tau_1 \cdot \tau_2$.

5.1. Plan. We will use the canonical morphism $\tilde{p} : \tilde{Z} \to \mathbb{A}^1 \times Z \times Z$, see Subsect. 2.1.7. As explained in Subsect. 2.2.3 representability of $\tilde{Z}$ would immediately imply that $\tilde{p}$ is unramified.

To prove representability of $\tilde{Z}$, we will first prove some properties of $\tilde{p}$, which are weaker than being representable and unramified. Namely, in Subsect. 5.2 we prove that the diagonal morphism

$$\Delta : \tilde{Z} \to \tilde{Z} \times_{\mathbb{A}^1 \times Z \times Z} \tilde{Z}$$

is an open embedding (in particular, it is representable). This immediately implies that the morphism $\tilde{p} : \tilde{Z} \to \mathbb{A}^1 \times Z \times Z$ is formally unramified. Then we prove another property of $\tilde{p}$ (see Proposition 5.3.2) and deduce from it Proposition 5.3.8 which is a strong form of pro-representability. Proposition 5.3.8 implies “openness of formal etaleness” for morphisms from schemes to $\tilde{Z}$ (see Corollary 5.3.10). After that, it remains to check effective pro-representability, see Subsections 5.4-5.5.

Finally, in Subsect. 5.6 (which is not used in the rest of the article) we give a reasonable “upper bound” for the conormal sheaf of $\tilde{Z}$ with respect to the unramified morphism $\tilde{p} : \tilde{Z} \to \mathbb{A}^1 \times Z \times Z$.

This bound is closely related to the proof of Proposition 5.3.2.

5.2. The diagonal morphism.

Proposition 5.2.1. The diagonal morphism (5.1) is an open embedding.

Let us prove the proposition. We have to show that for any scheme $S$ and any morphisms $\varphi_1, \varphi_2 : S \to \tilde{Z}$ giving rise to the same morphism $h : S \to \mathbb{A}^1 \times Z \times Z$, the equalizer $\text{Eq}(\varphi_1, \varphi_2)$ is representable by an open subscheme of $S$. Let $f_1, f_2 : X_S \to Z$ be the $\mathbb{G}_m$-equivariant morphisms corresponding to $\varphi_1, \varphi_2$ and let $E := \text{Eq}(f_1, f_2)$ be their equalizer. Then $E$ is a scheme of finite presentation over $X_S$ equipped with a monomorphism $E \hookrightarrow X_S$. Moreover, since $\varphi_1$ and $\varphi_2$ correspond to the same morphism $h : S \to \mathbb{A}^1 \times Z \times Z$ we have $E \supset X'_S$, where $X'$ is the open subscheme $\mathbb{A}^2 - \{0\} \subset \mathbb{A}^2 = X$ and $X'_S := X' \times_{\mathbb{A}^1} S$. Now it remains to prove the following lemma.

\footnote{Instead of M. Artin’s technique one could use the one from [Mur] (which does not rely on Artin’s Approximation Theorem). This would not make the proof of representability more constructive.}
Lemma 5.2.2. Let $S$ be a scheme over $\mathbb{A}^1$. Let $E$ be a scheme of finite presentation over $X_S$ such that the map $E \to X_S$ is a monomorphism. Assume that the morphism $X_S \to X_S$ factors through $E$.

Let $U$ be the set of all $s \in S$ such that the corresponding morphism $E_s \to X_s$ is an isomorphism (here $E_s$ and $X_s$ are the fibers of $E$ and $X_S$ over $s$). Then

(i) the subset $U \subset S$ is open;
(ii) the map $E \times_S U \to X_U$ is an isomorphism.

Remark 5.2.3. If the monomorphism $E \to X_S$ is a closed embedding then Lemma 5.2.2 is obvious; moreover, in this case $U = S$. So if $Z$ is separated then Proposition 5.2.1 is obvious; moreover, in this case the map (5.1) is an isomorphism (i.e., $\tilde{p} : \tilde{Z} \to \mathbb{A}^1 \times Z \times Z$ is a monomorphism).

Proof. We proceed in 3 steps.

Step 1. Assume that $S$ is Artinian. Then statement (i) is tautological, and the morphism $E \to X_S$ is a closed embedding. Let $J \subset O_{X_S}$ be the ideal corresponding to $E \subset X_S$. Since $E \supset X_S$ the restriction of $J$ to $X'_S$ is zero. This easily implies that $J = 0$. So $X_S = X_S$, which proves statement (ii).

Step 2. Assume that $S$ is Noetherian. Let $\tilde{E} \subset E$ be the biggest open subscheme such that the morphism $\tilde{E} \to X_S$ is etale. Then $\tilde{E}$ is an open subscheme of $X_S$ containing $X'_S$. Applying the result of Step 1 to Artinian closed subschemes of $S$, we see that $X_s \subset \tilde{E}$ for any $s \in U$. This allows to replace $E$ by $\tilde{E}$; in other words, we can assume that the morphism $E \to X_S$ is an open embedding. Then statements (i) and (ii) are clear because $X_S - E$ is a closed subset of $X_S - X'_S$ and the morphism $X_S - X'_S \to S$ is closed (in fact, it is a closed embedding).

Step 3. Since $E$ is of finite presentation we can remove the Noetherian assumption. \hfill \Box

Thus we have proved Proposition 5.2.1. Before formulating some corollaries of it, let us make an obvious remark.

Remark 5.2.4. It is clear that the space $\tilde{Z}$ is locally of finite presentation (i.e., the corresponding functor $\{k\text{-algebras}\} \to \{\text{sets}\}$ commutes with filtering inductive limits).

Corollary 5.2.5. Let $S$ be a $k$-scheme. Then

(i) any morphism $S \to \tilde{Z}$ is representable;
(ii) if $S$ is locally of finite presentation over $k$ then any morphism $S \to \tilde{Z}$ is locally of finite presentation.

Proof. It suffices to show that the diagonal morphism $\tilde{Z} \to \tilde{Z} \times \tilde{Z}$ is representable and locally of finite presentation. Both properties follow from Proposition 5.2.1 (The second property also follows from Remark 5.2.1). \hfill \Box

Corollary 5.2.6. The morphism $\tilde{p} : \tilde{Z} \to \mathbb{A}^1 \times Z \times Z$ is formally unramified. In other words, for any commutative diagram

\[
\begin{array}{c}
S_0 \xrightarrow{c} S \\
\downarrow \quad \downarrow \\
\tilde{Z} \xrightarrow{} \mathbb{A}^1 \times Z \times Z
\end{array}
\]

(5.2)
where $S$ is a scheme and $S_0$ is a closed subscheme defined by a nilpotent ideal, there exists at most one way to complete (5.2) to a commutative diagram

\[
\begin{array}{ccc}
S_0 & \longrightarrow & S \\
\downarrow & & \downarrow \\
Z & \longrightarrow & \mathbb{A}^1 \times Z \times Z
\end{array}
\]

**Proof.** Follows from Proposition 5.2.1. \qed

**Remark 5.2.7.** In Corollary 5.2.6 the condition “$S_0$ is defined by a nilpotent ideal” can be replaced by a weaker condition $S_0 \supset S_{\text{red}}$. This follows from Remark 5.2.4.

5.3. **Constructing formal neighborhoods.**

5.3.1. *The property of $\tilde{p} : \tilde{Z} \rightarrow \mathbb{A}^1 \times Z \times Z$ to be proved.* Fix a commutative diagram (5.2). Say that a morphism of schemes $T \rightarrow S$ is *liftable* (with respect to this diagram) if there exists a morphism $T \rightarrow \tilde{Z}$ such that the corresponding diagram

\[
\begin{array}{ccc}
T \times S_0 & \longrightarrow & T \\
\downarrow & & \downarrow \\
Z & \longrightarrow & \mathbb{A}^1 \times Z \times Z
\end{array}
\]

commutes (note that such a morphism $T \rightarrow Z$ is unique by Corollary 5.2.6). Let us explain that the vertical arrows of (5.4) are obtained by composing the vertical arrows of (5.2) with the morphisms $T \rightarrow S$ and $T \times S S_0 \rightarrow S_0$.

**Proposition 5.3.2.** For any commutative diagram (5.2) the corresponding functor

\[T \mapsto \{\text{liftable morphisms } T \rightarrow S\}\]

is representable by a closed subscheme $S' \subset S$.

The proof of Proposition 5.3.2 will use the following lemma, which is very abstract ($\tilde{Z}$ and $\mathbb{A}^1 \times Z \times Z$ can be replaced by any spaces or functors).

**Lemma 5.3.3.** It suffices to prove Proposition 5.3.2 if $n(S_0, S) \leq 2$. Here $n(S_0, S)$ is the nilpotence degree of the ideal of the closed subscheme $S_0 \subset S$.

**Proof.** Proceed by induction on $n(S_0, S)$. If $n(S_0, S) > 2$ we can choose a closed subscheme $S' \subset S$ containing $S_0$ so that $n(S_0, S') < n(S_0, S)$ and $n(S', S) \leq 2$. Applying Proposition 5.3.2 to the embedding $S_0 \hookrightarrow S'$ we get a closed subscheme $S' \hookrightarrow S$ and a commutative diagram

\[
\begin{array}{ccc}
S_0' & \longrightarrow & S' \\
\downarrow & & \downarrow \\
\tilde{Z} & \longrightarrow & \mathbb{A}^1 \times Z \times Z
\end{array}
\]

such that for any liftable morphism $f : T \rightarrow S$ one has $T \times_S S' = T \times_S S'$. Then for any liftable $f : T \rightarrow S$ one has

\[n(T \times_S S', T) = n(T \times_S S', T) \leq n(S', S) \leq 2,
\]

so $f : T \rightarrow S$ factors through the first infinitesimal neighborhood of $S'$ in $S$. Replacing $S$ by this neighborhood we can assume that $n(S', S) \leq 2$. Now it remains to apply Proposition 5.3.2 to the embedding $S' \hookrightarrow S$. \qed
The proof of Proposition 5.3.2 given below is straightforward; the elementary Lemma 5.3.3 is its heart.

5.3.4. Proof of Proposition 5.3.3 By Lemma 5.3.3 we can assume that \( J^2 = 0 \), where \( J \subset \mathcal{O}_S \) is the ideal of \( S_0 \).

Recall that for any scheme \( S \) over \( \mathbb{A}^1 \), an \( \mathbb{A}^1 \)-equivariant morphism \( S \to Z \) is the same as a \( \mathbb{G}_m \)-equivariant morphism \( \mathbb{X}_S \to Z \), where \( \mathbb{X}_S := \mathbb{X} \times_{\mathbb{A}^1} S \). We can think of an \( \mathbb{A}^1 \)-morphism \( S \to \mathbb{A}^1 \times Z \times Z \) as a \( \mathbb{G}_m \)-equivariant morphism \( \mathbb{Y}_S \to Z \), where \( \mathbb{Y}_S := \mathbb{Y} \times_{\mathbb{A}^1} S \) and \( \mathbb{Y} := \mathbb{A}^1 \times (\mathbb{G}_m \sqcup \mathbb{G}_m) \). The morphism \( \tilde{p} : \tilde{Z} \to \mathbb{A}^1 \times Z \times Z \) comes from the \( \mathbb{G}_m \)-equivariant morphism \( \nu : \mathbb{Y} \to \mathbb{X} \) whose restriction to the first copy of \( \mathbb{A}^1 \times \mathbb{G}_m \) is given by

\[
(t, \lambda) \mapsto (\lambda, \lambda^{-1} \cdot t)
\]

and whose restriction to the second copy of \( \mathbb{A}^1 \times \mathbb{G}_m \) is given by

\[
(t, \lambda) \mapsto (\lambda \cdot t, \lambda^{-1}).
\]

(Note that both restrictions are open embeddings.)

So a diagram (5.2) corresponds to the following data:

(i) a scheme \( S \) over \( \mathbb{A}^1 \) and a closed subscheme \( S_0 \subset S \) defined by an ideal \( J \subset \mathcal{O}_S \) such that \( J^2 = 0 \);

(ii) a \( \mathbb{G}_m \)-equivariant morphism \( f_0 : \mathbb{X}_{S_0} \to Z \);

(iii) a \( \mathbb{G}_m \)-equivariant morphism \( f : \mathbb{Y}_S \to \mathbb{Z} \) whose restriction to \( \mathbb{Y}_{S_0} \) is equal to the composition of \( \nu_{S_0} : \mathbb{Y}_{S_0} \to \mathbb{X}_{S_0} \) and \( f_0 : \mathbb{X}_{S_0} \to \mathbb{Z} \).

Clearly (iii) is equivalent to the following datum:

(iii') a lift of the composition \( f_0 \mathcal{O}_Z \to \mathcal{O}_{\mathbb{X}_{S_0}} \to (\nu_{S_0})_* \mathcal{O}_{\mathbb{Y}_{S_0}} \) to a \( \mathbb{G}_m \)-equivariant morphism

\[
f_0 \mathcal{O}_Z \to (\nu_{S_0})_* \mathcal{O}_{\mathbb{Y}_{S_0}}.
\]

Here each algebraic space is equipped with the etale topology, and \( f_0 \) denotes the pullback with respect to \((f_0)_\text{et} : (\mathbb{X}_{S_0})_{\text{et}} = (\mathbb{X}_S)_{\text{et}} \to \mathbb{Z}_\text{et} \).

We can rewrite (iii') as follows:

(iii'') a lift of the morphism \( f_0 \mathcal{O}_Z \to \mathcal{O}_{\mathbb{X}_{S_0}} \) to a \( \mathbb{G}_m \)-equivariant morphism

\[
f_0 \mathcal{O}_Z \to \mathcal{O}_{\mathbb{X}_{S_0}} \times (\nu_{S})_* \mathcal{O}_{\mathbb{Y}_{S}},
\]

Extending diagram (5.2) to diagram (5.3) is equivalent to lifting the map (5.5) further to a morphism \( f_0 \mathcal{O}_Z \to \mathcal{O}_{\mathbb{X}_S} \). By Corollary 5.2.6 there is at most one such lift. This also follows from the first part of the next lemma.

Lemma 5.3.5. (a) The morphism \( \mathcal{O}_{\mathbb{X}_S} \to (\nu_{S})_* \mathcal{O}_{\mathbb{Y}_S} \) is injective.

(b) Set \( \mathcal{F}_S := \text{Coker}(\mathcal{O}_{\mathbb{X}_S} \to (\nu_{S})_* \mathcal{O}_{\mathbb{Y}_S}) \) and let \( \text{pr}_S : \mathbb{X}_S \to S \) denote the projection. Then \( (\text{pr}_S)_* \mathcal{F}_S \) is a free \( \mathcal{O}_S \)-module (of countable rank).

Proof. It suffices to consider the case where the morphism \( S \to \mathbb{A}^1 \) is an isomorphism. In this case we have to check that the map

\[
k[\tau_1, \tau_2] \to k[\tau_1, \tau_2, \tau_1^{-1} \times k[\tau_1, \tau_2, \tau_2^{-1}]
\]

is injective and its cokernel is a free module over \( k[\tau_1 \tau_2] \subset k[\tau_1, \tau_2] \). Injectivity is clear. The cokernel identifies via the map \((u, v) \mapsto u - v \) with

\[
k[\tau_1, \tau_2, \tau_1^{-1}] + k[\tau_1, \tau_2, \tau_2^{-1}] \subset k[\tau_1, \tau_2, \tau_1^{-1}, \tau_2^{-1}],
\]

which is a module over \( k[\tau_1 \tau_2] \) freely generated by the elements 1 and \( \tau_i^{-n} \), where \( n \in \mathbb{N} \) and \( i = 1, 2 \).
End of the proof of Proposition 5.3.2. Let \( F_S \) and \( \text{pr}_S \) be as in Lemma 5.3.5(b). The obstruction to solving our lifting problem is a morphism \( f_0^* \Omega_Z \rightarrow F_S \otimes (\text{pr}_S)^* \mathcal{J} \), which is a derivation with respect to the ring homomorphism \( f_0^* \Omega_Z \rightarrow \mathcal{O}_{X_{S_0}} \) (here we use that \( T^2 = 0 \)). We can rewrite this obstruction as a morphism of coherent \( \mathcal{O}_{X_{S_0}} \)-modules \( f_0^* \Omega_Z^1 \rightarrow F_S \otimes (\text{pr}_S)^* \mathcal{J} \) and then (using the fact that \( X_S \) is affine over \( S \)) as a morphism of quasi-coherent \( \mathcal{O}_{S_0} \)-modules in a neighborhood of \( S \) in which each \( S \) has the property from Proposition 5.3.6.

Now let us explain how to construct the closed subscheme \( S \subset S \) from Proposition 5.3.2. By Lemma 5.3.7(b), \( (\text{pr}_S)_* F_S \) is a free \( \mathcal{O}_S \)-module. After choosing a basis in it, we can think of the morphism \( \Omega_S^1 \) as an (infinite) collection of morphisms \( (\text{pr}_S)_* f_0^* \Omega_Z^1 \rightarrow F_S \otimes (\text{pr}_S)^* \mathcal{J} \). Let \( \mathcal{J}_1 \subset \mathcal{J} \) be the submodule (or equivalently, the ideal) generated by their images. Finally, let \( S \subset S \) be the closed subscheme corresponding to \( \mathcal{J}_1 \subset \mathcal{O}_S \). It is easy to see that \( S \) has the property from Proposition 5.3.2. □

5.3.6. Constructing formal neighborhoods.

Lemma 5.3.7. Let \( S_0 \) be a k-scheme of finite type. The following properties of a morphism \( \varphi : S_0 \rightarrow \tilde{Z} \) are equivalent:

(i) \( \varphi \) is formally unramified;
(ii) \( \varphi \) is unramified;
(iii) the composition

\[
S_0 \xrightarrow{\varphi} \tilde{Z} \xrightarrow{\tilde{p}} \mathbb{A}^1 \times Z \times Z
\]

is unramified.

Proof. By Corollary 5.2.5, we have (i) ⇔ (ii). Since \( S_0 \) is of finite type property (iii) is equivalent to the composition \( S_0 \xrightarrow{\varphi} \tilde{Z} \xrightarrow{\tilde{p}} \mathbb{A}^1 \times Z \times Z \) being formally unramified. The latter is equivalent to (i) by Lemma 5.2.6. □

Let \( S_0 \) and \( \varphi : S_0 \rightarrow \tilde{Z} \) be as in Lemma 5.3.7. Define the formal neighborhood of \( S_0 \) with respect to \( \varphi : S_0 \rightarrow \tilde{Z} \) to be the following contravariant functor \{ affine k-schemes \} → \{ sets \}:

\[
T \mapsto \text{Maps}(T, \tilde{Z}) \times \text{Maps}(T_{\text{red}}, S_0).
\]

Proposition 5.3.8. Let \( S_0 \) and \( \varphi : S_0 \rightarrow \tilde{Z} \) be as in Lemma 5.3.7. Let \( S_\infty \) denote the formal neighborhood of \( S_0 \) with respect to \( \varphi : S_0 \rightarrow \tilde{Z} \). Let \( S_\infty' \) denote the formal neighborhood of \( S_0 \) with respect to \( \tilde{p} \circ \varphi : S_0 \rightarrow \mathbb{A}^1 \times Z \times Z \). Then

(i) the morphism \( S_0 \rightarrow S_\infty \) is a closed embedding;
(ii) \( S_\infty \) can be represented as an inductive limit of a diagram

\[
S_0 \hookrightarrow S_1 \hookrightarrow S_2 \hookrightarrow \ldots
\]

in which each \( S_n \) is a k-scheme of finite type, the morphisms \( S_n \rightarrow S_{n+1} \) are closed embeddings, and for each \( N \geq n \) the \( n \)-th infinitesimal neighborhood of \( S_0 \) in \( S_N \) equals \( S_n \).

Proof. Let \( S_\infty' \) denote the \( n \)-th infinitesimal neighborhood of \( S_0 \) in \( S_\infty \). Clearly \( S_\infty' \) is a k-scheme of finite type. Set

\[
S_n := S_\infty' \times S_\infty.
\]

\(^{18}\) Of course, this morphism is also \( G_m \)-equivariant and commutes with the action of the algebra \( (\text{pr}_{S_0})_* \mathcal{O}_{X_{S_0}} \).
It remains to prove that for every \( n \in \mathbb{N} \) the morphism \( S_n \to S'_n \) is a closed embedding. To this end, consider the diagram

\[
\begin{array}{ccc}
S_0 & \to & S'_n \\
\downarrow & & \downarrow \\
\tilde{Z} & \to & \mathbb{A}^1 \times Z \times Z
\end{array}
\]

of type (5.2). Applying Proposition 5.3.2 to this diagram, one gets a closed subscheme of \( S'_n \).

It is easy to check that this closed subscheme equals \( S_n \) (use Corollary 5.2.6 and Remark 5.2.7).

5.3.9. An openness lemma.

Corollary 5.3.10. Let \( S_0 \) be a \( \mathbb{k} \)-scheme of finite type and \( \varphi : S_0 \to \tilde{Z} \) a morphism. Let \( s \in S_0 \) be a closed point. Suppose that

(i) the morphism \( s \to \tilde{Z} \) is a monomorphism (so the formal neighborhood of \( s \) in \( \tilde{Z} \) is well-defined);

(ii) \( \varphi \) induces an isomorphism between the formal neighborhoods of \( s \) in \( S_0 \) and \( \tilde{Z} \).

Then there is an open subscheme \( U \subset S_0 \) containing \( s \) such that the restriction of \( \varphi \) to \( U \) is etale.

Proof. Since \( \tilde{p} : \tilde{Z} \to \mathbb{A}^1 \times Z \times Z \) is formally unramified, condition (ii) implies that the morphism \( \tilde{p} \circ \varphi : S_0 \to \mathbb{A}^1 \times Z \times Z \) is unramified at \( s \). So after shrinking \( S \) we can assume that the morphism \( \tilde{p} \circ \varphi \) is unramified. Then Proposition 5.3.8 is applicable.

Let \( S_1 \) be as in Proposition 5.3.8(ii). Let \( I \subset \mathcal{O}_{S_1} \) be the ideal of the closed subscheme \( S_0 \subset S_1 \) and let \( F \subset S_0 \) be the support of the coherent \( \mathcal{O}_{S_0} \)-module \( I \). Let us check that the open subscheme \( U := S_0 - F \) has the required properties.

Condition (ii) implies that \( s \in U \). After replacing \( S \) by \( U \) we get \( I = 0 \), i.e., \( S_1 = S_0 \). This implies that \( S_\infty = S_0 \), i.e., \( \varphi \) is etale.

5.4. Proof of Theorem 2.2.2 modulo Proposition 5.4.1. Given a \( \mathbb{k}[t] \)-algebra \( R \), set

\[
\tilde{Z}(R) := \text{Maps}_{\mathbb{A}^1}(\text{Spec } R, \tilde{Z}).
\]

Proposition 5.4.1. Let \( R \) be a complete Noetherian \( \mathbb{k}[t] \)-algebra and \( m \subset R \) the maximal ideal. Then the map

\[
\tilde{Z}(R) \to \lim_{\longrightarrow} \tilde{Z}(R/m^n)
\]

is bijective.

The proposition will be proved in Subsect. 5.5. Now let us deduce Theorem 2.2.2 from Proposition 5.4.1 and the results of Subsections 5.2-5.3.

We already know that \( \tilde{Z} \times_{\mathbb{A}^1} (\mathbb{A}^1 \setminus \{0\}) \) and \( \tilde{Z}_0 := \tilde{Z} \times_{\mathbb{A}^1} \{0\} \) are algebraic \( \mathbb{k} \)-spaces of finite type (see Remark 2.1.6). So to prove Theorem 2.2.2 it suffices to check that \( \tilde{Z} \) is an algebraic \( \mathbb{k} \)-space locally of finite type.

Let \( \tilde{z}_0 \) be a spectrum of a finite extension of \( \mathbb{k} \) equipped with a monomorphism \( \varphi : \tilde{z}_0 \to \tilde{Z} \). Our goal is to construct a \( \mathbb{k} \)-scheme \( S \) of finite type equipped with an etale morphism \( S \to \tilde{Z} \) whose fiber over \( \tilde{z}_0 \) is non-empty.

Applying Proposition 5.3.8 to the monomorphism \( \varphi : \tilde{z}_0 \to \tilde{Z} \), we see that the formal neighborhood of \( \tilde{z}_0 \) in \( \tilde{Z} \) equals \( \text{Spf } A \) for some complete Noetherian local ring \( A \). Applying Proposition 5.4.1 we upgrade this pro-representability result to effective pro-representability;
in other words, we get a morphism $\text{Spec } A \to \tilde{Z}$ extending the morphism $\text{Spf } A \to \tilde{Z}$. Using Artin approximation \cite[Theorem 1.10]{AN} and the fact that $\tilde{Z}$ is locally of finite presentation, we get a $k$-scheme $S'$ of finite type equipped with a closed point $s_0 \in S'$ and a morphism $(S', s_0) \to (\tilde{Z}, \tilde{s}_0)$ inducing an isomorphism between the formal completions. By Corollary \cite[5.3.10]{AN} $s_0$ has a Zariski neighborhood $S \subset S'$ such that the morphism $S \to \tilde{Z}$ is etale. Thus we have proved Theorem \cite[2.2.2]{AN} modulo Proposition \cite[5.4.1]{AN}.

5.5. **Proof of Proposition 5.4.1** The proof below is parallel to that of Lemma \cite[4.5.3]{AN}.

If $t$ is invertible in $R$ the statement is clear because $\tilde{Z} \times_{\mathbb{A}^1} (\mathbb{A}^1 \setminus \{0\}) \simeq \mathbb{G}_m \times Z$ is an algebraic space. So from now we will assume that $t \in m$.

Set $A_R := R/(\tau_1, \tau_2 - t)$. Recall that $\tilde{Z}(R) := \text{Maps}_{G^m}(\mathbb{X}_R, Z)$, $\mathbb{X}_R := \text{Spec } A_R$.

We will use the following notation:

\[
\mathbb{X}_R[t^{-1}] := \text{Spec } A_R[t^{-1}]; \quad \mathbb{X}_R[\tau_i^{-1}] := \text{Spec } A_R[\tau_i^{-1}], \quad i = 1, 2;
\]

\[
\hat{A}_R := R[\tau_1, \tau_2]/(\tau_1 \tau_2 - t); \quad \hat{\mathbb{X}}_R := \text{Spec } \hat{A}_R;
\]

\[
\hat{\mathbb{X}}_R[t^{-1}] := \text{Spec } \hat{A}_R[t^{-1}]; \quad \hat{\mathbb{X}}_R[\tau_i^{-1}] := \text{Spec } \hat{A}_R[\tau_i^{-1}], \quad i = 1, 2.
\]

Applying Moret-Bailly’s Theorem \cite[4.6.1]{AN} for $S = \mathbb{X}_R$, $S' = \hat{\mathbb{X}}_R$, $Y = \text{Spec } R/(t) = \text{Spec } R/(\tau_1, \tau_2, \tau_1 \tau_2 - t) \subset \mathbb{X}_R$ and then applying Zariski descent to the covering $\mathbb{X}_R - Y = \mathbb{X}_R[\tau_1^{-1}] \cup \mathbb{X}_R[\tau_2^{-1}]$ one gets an exact sequence

\[
\tilde{Z}(R) \to F(R) \to G_1(R) \times G_2(R) \times \text{Maps}_{G^m}(\mathbb{X}_R[t^{-1}], Z),
\]

where

\[
F(R) := \text{Maps}_{G^m}(\mathbb{X}_R, Z) \times \text{Maps}_{G^m}(\mathbb{X}_R[\tau_1^{-1}], Z) \times \text{Maps}_{G^m}(\mathbb{X}_R[\tau_2^{-1}], Z),
\]

\[
G_i(R) := \text{Maps}_{G^m}(\hat{\mathbb{X}}_R[\tau_i^{-1}], Z), \quad i = 1, 2.
\]

**Lemma 5.5.1.** The sequence

\[
\tilde{Z}(R) \to F(R) \to G_1(R) \times G_2(R)
\]

is still exact.

**Proof.** It suffices to show that the map

\[
\text{Maps}_{G^m}(\mathbb{X}_R[t^{-1}], Z) \to \text{Maps}_{G^m}(\hat{\mathbb{X}}_R[t^{-1}], Z)
\]

is injective.

Let $f_1, f_2 : \mathbb{X}_R[t^{-1}] \to Z$ be $G_m$-equivariant morphisms. Since

\[
\mathbb{X}_R[t^{-1}] \simeq G_m \times \text{Spec } R[t^{-1}]
\]

the equalizer $E := \text{Eq}(f_1, f_2)$ equals $E_0 \times \mathbb{X}_R[t^{-1}]$ for some scheme $E_0$ equipped with a monomorphism $\mu : E_0 \to \text{Spec } R[t^{-1}]$. 


Now suppose that \( f_1 \) and \( f_2 \) have equal images in Maps\( ^{G_m}(\mathcal{X}_R[t^{-1}], Z) \). Then \( \mu \) becomes an isomorphism after base change with respect to the morphism \( \pi : \mathcal{X}_R[t^{-1}] \to \text{Spec} \ R[t^{-1}] \).

But \( \pi \) is faithfully flat (because \( \mathcal{X}_R \) is faithfully flat over Spec \( R \)). So \( \mu \) is an isomorphism and therefore \( f_1 = f_2 \).

We have to prove that the map \( (5.8) \) is bijective. Let \( F \) and \( G_i \) be as in \( (5.9)-(5.10) \). It is easy to see that the map

\[
F(R) \to \lim_{\to n} F(R/m^n)
\]

is bijective. So by Lemma \( 5.5.1 \) it remains to show that the map

\[
G_i(R) \to \lim_{\to n} G_i(R/m^n)
\]

is injective for \( i = 1, 2 \). Let us prove this for \( i = 1 \). We will proceed as at Step 2 of the proof of Lemma \( 4.5.3 \) (see Subsect. 4.7).

Suppose that \( f_1, f_2 \in G_1(R) \) have equal images in \( G_1(R/m^n) \) for each \( n \in \mathbb{N} \). Let \( E \) denote the equalizer of the \( G_m \)-equivariant morphisms \( f_1, f_2 : \mathcal{X}_R[1^{-1}] \to Z \). Just as at Step 2 of the proof of Lemma \( 4.5.3 \) we see that \( E \) contains an open \( G_m \)-stable subscheme \( E' \subset \mathcal{X}_R[1^{-1}] \) such that \( E' \supset \mathcal{X}_R/m[1^{-1}] \). It remains to show that such \( E' \) has to be equal to \( \mathcal{X}_R[1^{-1}] \).

Choose a closed subscheme \( F \subset \mathcal{X}_R \) whose complement equals \( E' \) and let

\[
I \subset \mathcal{A}_R := R[[\tau_1, \tau_2]]/(\tau_1 \tau_2 - t)
\]

be the corresponding ideal. The inclusion \( E' \supset \mathcal{X}_R/m[1^{-1}] \) and the fact that \( E' \) is \( G_m \)-stable translate into the following properties of \( I \).

**Lemma 5.5.2.** (i) The image of \( I \) in \( \mathcal{A}_R/m \) contains \( 1, \mathcal{X}_N \) for some \( N \in \mathbb{Z}_+ \).

(ii) Let \( \varphi : \mathcal{A}_R \to \mathcal{A}_R[1, \lambda, -1] \) be the continuous \( R \)-algebra homomorphism such that

\[
\tau_1 \mapsto \lambda \tau_1, \quad \tau_2 \mapsto \lambda^{-1} \tau_2;
\]

then \( I \) is contained in the radical of the ideal \( I' := \varphi^{-1}(1, \mathcal{A}_R[1, \lambda, -1]) \).

It remains to show that any ideal \( I \subset \mathcal{A}_R \) with properties (i)-(ii) from the lemma contains a power of \( \tau_1 \). Since \( I \) is contained in the radical of \( I' \) property (i) implies that \( I' \) contains an element of the form

\[
\sum_{i=0}^{\infty} r_i \tau_1^i + \sum_{i<0} r_i \lambda^i \tau_2^{-i}, \quad r_i \in R
\]

for some \( N \in \mathbb{Z}_+ \). By the definition of \( I' \), one has an equality of the form

\[
\sum_{i=0}^{\infty} r_i \lambda^i \tau_1^i + \sum_{i<0} r_i \lambda^i \tau_2^{-i} = \sum_{j=1}^{n} g_j h_j, \quad g_j \in \mathcal{A}_R[1, \lambda, -1], \quad h_j \in I.
\]

Equating the coefficients of \( \lambda^N \) in this equality, we see that \( r_N \tau_1^N \in I \). On the other hand, \( r_N \) is invertible by \( (5.11) \). So \( I \) contains \( \tau_1^N \), and we are done.
5.6. Virtual conormal sheaf of \( \tilde{Z} \) with respect to \( \mathbb{A}^1 \times Z \times Z \). Let \( N \) denote the conormal sheaf of \( \tilde{Z} \) with respect to the unramified morphism \( \tilde{p} : \mathbb{A}^1 \times Z \times Z \). We are going to define another coherent sheaf \( N' \) on \( \tilde{Z} \) such that \( N \) is canonically a quotient of \( N' \). One could call \( N' \) the virtual conormal sheaf.

Here is the definition of \( N' \): for any affine scheme \( S \) equipped with a morphism \( \varphi : S \to \tilde{Z} \)

\[
H^0(S, \varphi^*N') := H^0(X_S, f^*\Omega^1_{\tilde{Z}} \otimes \omega_{X_S/S})^{G_m},
\]

where \( f : X_S \to Z \) is the \( G_m \)-equivariant morphism corresponding to \( \varphi \) and \( \omega_{X_S/S} \) is the relative dualizing sheaf.

The following facts are not used in the rest of the article; we formulate them for completeness.

First, the “obstruction theory” from the proof of Proposition 5.3.2 yields a canonical epimorphism \( N' \to N \) (we leave the definition to the reader).

Second, let \( \tilde{Z}_{der} \) denote the derived version of \( \tilde{Z} \) (to define it, replace the space \( \text{Maps}^{G_m} \) from the definition of \( \tilde{Z} \) by its derived version). Then \( N' \) is, in fact, the “conormal sheaf” of \( \tilde{Z}_{der} \) with respect to \( \mathbb{A}^1 \times Z \times Z \); more precisely, \( N' \) is the \((-1)\)-st cohomology sheaf of the relative cotangent complex of \( \tilde{Z}_{der} \) over \( \mathbb{A}^1 \times Z \times Z \).

Appendix A. Proof of Lemma 5.1.10

**Lemma A.0.1.** Let \( A \) and \( B \) be algebraic \( k \)-spaces and \( f : A \to B \) a surjective morphism with \( f_*\mathcal{O}_A = \mathcal{O}_B \) (here \( \mathcal{O}_A, \mathcal{O}_B \) are sheaves on the etale sites \( A_{et}, B_{et} \) and \( f_* \) is understood in the non-derived sense). Suppose that \( f : A \to B \) factors as

\[
A \to B' \to B,
\]

where \( i : B' \to B \) is a monomorphism of \( \mathbf{m} \)-spaces. Then \( i \) is an isomorphism (i.e., \( B' = B \)).

**Proof.** It suffices to show that for every \( b \in B \) there exists an etale morphism \( (B_1, b_1) \to (B, b) \) such that \( i \) becomes an isomorphism after base change to \( B_1 \).

Note that since \( f \) is surjective so is \( i : B' \to B \). In other words, \( B' \) and \( B \) have the same field-valued points. In particular, \( b \in B' \).

The monomorphism \( i : B' \to B \) has finite type, so after etale base change \( (B_1, b_1) \to (B, b) \), we can assume that there is an open subspace \( \overline{B} \subset B' \) which is closed in \( B \) (and therefore closed in \( B' \)). Set \( A := (f')^{-1}(\overline{B}) \), then \( \overline{A} \subset A \) is both open and closed. Let \( 1_{\overline{A}} \in H^0(A, \mathcal{O}_A) \) denote the characteristic function of \( \overline{A} \). Since \( f_*\mathcal{O}_A = \mathcal{O}_B \) the map \( H^0(B, \mathcal{O}_B) \to H^0(A, \mathcal{O}_A) \) is an isomorphism. So \( 1_{\overline{A}} \) comes from an idempotent element of \( H^0(B, \mathcal{O}_B) \). After shrinking \( B \), we can assume that this element equals 1. This means that \( 1_{\overline{A}} = 1 \), \( \overline{A} = A \), \( \overline{B} = B' \), and \( i : B' \to B \) is a closed embedding.

Let \( \mathfrak{J}_{B'} \subset \mathcal{O}_B \) be the ideal of the closed subspace \( B' \subset B \). Then \( \mathfrak{J}_{B'} \subset \text{Ker}(\mathcal{O}_B \to f_*\mathcal{O}_A) = 0 \). So \( B' = B \).

Now let us prove Lemma 5.1.10. It says the following:

**Lemma A.0.2.** Let \( A, B, Z \) be algebraic \( k \)-spaces and \( f : A \to B \) a surjective morphism with \( f_*\mathcal{O}_A = \mathcal{O}_B \). Then

(i) the map \( \text{Maps}(B, Z) \to \text{Maps}(A, Z) \) induced by \( f \) is injective;

\[\text{E.g., see Lemma 37.17.2 from [51] (whose “tag” is 04HI).}\]
(ii) if \( B_0 \subset B \) is a closed subspace containing \( B_{\text{red}} \) and \( A_0 = f^{-1}(B_0) \) then the diagram
\[
\begin{array}{ccc}
\text{Maps}(B, Z) & \longrightarrow & \text{Maps}(A, Z) \\
\downarrow & & \downarrow \\
\text{Maps}(B_0, Z) & \longrightarrow & \text{Maps}(A_0, Z)
\end{array}
\]
induced by \( f \) is Cartesian.

Proof. (i) Let \( g_1, g_2 : B \to Z \) be morphisms such that \( g_1 \circ f = g_2 \circ f \). Let \( i : B' \hookrightarrow B \) denote the equalizer of \( g_1 \) and \( g_2 \) (i.e., the preimage of the diagonal with respect to \((g_1, g_2) : B \to Z \times Z\)). By Lemma A.0.1 i is an isomorphism. So \( g_1 = g_2 \).

(ii) Suppose that \( g_0 \in \text{Maps}(B_0, Z) \) and \( h \in \text{Maps}(A, Z) \) have the same image in \( \text{Maps}(A_0, Z) \). We have to extend \( g_0 : B_0 \to Z \) to a morphism \( g : B \to Z \) whose composition with \( f : A \to B \) equals \( h \).

We have a canonical \( k \)-algebra homomorphism
\[(A.1)\]
\[
g_0^* O_Z \rightarrow O_{B_0},
\]
where \( g_0^* \) denotes the sheaf-theoretical pullback with respect to \((g_0)_{\text{et}} : (B_0)_{\text{et}} \to Z_{\text{et}}\). Note that \( B_{\text{et}} = (B_0)_{\text{et}} \). So extending \( g_0 \) to a morphism \( g : B \to Z \) is equivalent to lifting the map \[(A.1)\]
to a \( k \)-algebra homomorphism \( \varphi : g_0^* O_Z \rightarrow O_B \). Define \( \varphi \) to be the composition
\[
g_0^* O_Z \rightarrow f^* O_A \rightarrow O_B,
\]
where the first arrow comes from the homomorphism \( f g_0^* O_Z = h^* O_Z \rightarrow O_A \) corresponding to \( h : A \to Z \).

\[\square\]

Appendix B. Some results of Bialynicki-Birula, Konarski, and Sommese

Recall that if \( Z \) is separated then \( p^+ : Z^+ \to Z \) is a monomorphism. But already if \( Z \) is the projective line equipped with the standard \( \mathbb{G}_m \)-action, the morphism \( p^+ : Z^+ \to Z \) is not a locally closed embedding.

Theorem B.0.3. Let \( Z \) be a separated scheme over an algebraically closed field \( k \) equipped with a \( \mathbb{G}_m \)-action. Then each of the following conditions ensures that the restriction of \( p^+ : Z^+ \to Z \) to each connected component \( Z \) of \( Z^+ \) is a locally closed embedding:

(i) \( Z \) is smooth;
(ii) \( Z \) is normal and quasi-projective;
(iii) \( Z \) admits a \( \mathbb{G}_m \)-equivariant locally closed embedding into a projective space \( \mathbb{P}(V) \), where \( \mathbb{G}_m \) acts linearly on \( V \).

Case (i) is due to A. Bialynicki-Birula [Bia].

Case (ii) immediately follows from the easy case \( Z = \mathbb{P}(V) \). Case (ii) turns out to be a particular case of (iii) because by Theorem 1 from [Sum], if \( Z \) is normal and quasi-projective then it admits a \( \mathbb{G}_m \)-equivariant locally closed embedding into a projective space.

In case (i) the condition that \( Z \) is a scheme (rather than an algebraic space) is essential, as shown by A. J. Sommese [Som]. In case (ii) the quasi-projectivity condition is essential, as shown by J. Konarski [Kon] using a method developed by J. Jurkiewicz [Ju1, Ju2]. In this example \( Z \) is a 3-dimensional toric variety which is proper but not projective; it is constructed by drawing a certain picture on a 2-sphere, see the last page of [Kon].

\[20\]

Using the \( k^1 \)-action on \( Z^+ \), it is easy to see that each connected component of \( Z^+ \) is the preimage of a connected component of \( Z^0 \) with respect to the map \( q^+ : Z^+ \to Z^0 \).
In case (ii) normality is clearly essential (to see this, take $Z$ to be the curve obtained from $\mathbb{P}^1$ by gluing 0 with $\infty$).

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