Revisiting Caianiello’s Maximal Acceleration

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Summary. — A quantum mechanical limit on the speed of orthogonality evolution justifies the last remaining assumption in Caianiello’s derivation of the maximal acceleration. The limit is perfectly compatible with the behaviour of superconductors of the first type.

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1. – Introduction

Following previous attempts [1], Caianiello showed in 1984 that Heisenberg’s uncertainty relations place an upper limit \( \mathcal{A}_m \) on the value that the acceleration of a particle can take along a worldline [2]. This limit, referred to as maximal acceleration (MA), is determined by the particle’s mass itself. It is distinct from the value that has been derived in some works [3, 4, 5] from the Planck mass \( m_P = \left( \frac{\hbar c}{G} \right)^{\frac{1}{2}} \) and is therefore a universal constant. With some modifications [6] and additions, Caianiello’s argument goes as follows.

If two observables \( \hat{f} \) and \( \hat{g} \) obey the commutation relation

\[
\left[ \hat{f}, \hat{g} \right] = -i\hbar \hat{\alpha},
\]

where \( \hat{\alpha} \) is a Hermitian operator, then their uncertainties

\[
\begin{align*}
(\Delta f)^2 &= \langle \Phi | \left( \hat{f} - \langle \hat{f} \rangle \right)^2 | \Phi \rangle \\
(\Delta g)^2 &= \langle \Phi | \left( \hat{g} - \langle \hat{g} \rangle \right)^2 | \Phi \rangle
\end{align*}
\]

also satisfy the inequality

\[
(\Delta f)^2 \cdot (\Delta g)^2 \geq \frac{\hbar^2}{4} \langle \Phi | \hat{\alpha} | \Phi \rangle^2,
\]

\[\Box\]
or

\[
\Delta f \cdot \Delta g \geq \frac{\hbar}{2} \left| \langle \Phi | \hat{\alpha} | \Phi \rangle \right|
\]

(4)

Using Dirac’s analogy between the classical Poisson bracket \( \{ f, g \} \) and the quantum commutator [7]

\[
\{ f, g \} \rightarrow \frac{1}{i\hbar} [\hat{f}, \hat{g}].
\]

(5)

one can take \( \hat{\alpha} = \{ f, g \} \). With this substitution, Eq. (4) yields the usual momentum-position commutation relations. If in particular \( \hat{f} = \hat{H} \), then Eq. (4) becomes

\[
[\hat{H}, \hat{g}] = -i\hbar \{ H, g \} \hat{1},
\]

(6)

Eq. (4) gives [7]

\[
\Delta E \cdot \Delta g \geq \frac{\hbar}{2} \left| \{ H, g \} \right|
\]

(7)

and

\[
\Delta E \cdot \Delta g \geq \frac{\hbar}{2} \left| \frac{dg}{dt} \right|
\]

(8)

when \( \frac{dg}{dt} = 0 \). Eqs. (7) and (8) are re-statements of Ehrenfest theorem. Criteria for its validity are discussed at length in the literature [8, 7, 9]. Eq. (8) implies that \( \Delta E = 0 \) when the quantum state of the system is an eigenstate of \( \hat{H} \). In this case \( \frac{dg}{dt} = 0 \).

If \( g \equiv v(t) \) is the (differentiable) velocity expectation value of a particle whose average energy is \( E \), then Eq. (8) gives

\[
\left| \frac{dv}{dt} \right| \leq \frac{2}{\hbar} \Delta E \cdot \Delta v(t).
\]

(9)

In general [10]

\[
\Delta v = ( < v^2 > - < v >^2 )^{\frac{1}{2}} \leq v_{\text{max}} \leq c.
\]

(10)

Caianiello’s additional assumption, \( \Delta E \leq E \), has so far remained unjustified. In fact, Heisenberg’s uncertainty relation

\[
\Delta E \cdot \Delta t \geq \hbar/2,
\]

(11)

that follows from [9] by writing \( \Delta t = \Delta v/|dv/dt| \), seems to imply that, given a fixed average energy \( E \), a state can be constructed with arbitrarily large \( \Delta E \), contrary to Caianiello’s assumption. This conclusion is erroneous. The correct interpretation of (11) is that a quantum state with spread in energy \( \Delta E \) takes a time \( \Delta t \geq \frac{\hbar}{2\Delta E} \) to evolve to a distinguishable (orthogonal) state. This evolution time has a lower bound. Margolus
and Levitin have in fact shown \[11\] that the evolution time of a quantum system with fixed average energy $E$ must satisfy the more stringent limit

\[
\Delta t \geq \frac{\hbar}{2E},
\]

which determines a *maximum speed of orthogonality evolution* \[12\]. Their argument is simple. If at $t = 0$ an arbitrary state of a quantum system is written as a superposition of energy eigenstates $|\psi(0)\rangle = \sum_n c_n |E_n\rangle$, then at time $t$ the state has evolved to $|\psi(t)\rangle = \sum_n c_n e^{-iE_n\pi t/\hbar} |E_n\rangle$. The shortest time after which $|\psi(0)\rangle$ and $|\psi(t)\rangle$ are distinguishable is given by the orthogonality condition

\[
<\psi(0)|\psi(t)> = \sum_n |c_n|^2 e^{-iE_n\pi t/\hbar} = 0.
\]

The factor $\pi$ in \[13\] has been introduced because \[11\] requires that the energy distribution oscillate in time with a period at least $\frac{\hbar}{\Delta E}$. On using the inequality $\cos(x) \geq 1 - \frac{1}{2}(x + \sin(x))$, which is valid for $x \geq 0$, and equating to zero both real and imaginary parts of \[13\], Margolus and Levitin arrive at the equation

\[
Re(<\psi(0)|\psi(t)> \geq \sum_n |c_n|^2 \cos \left( \frac{E_n\pi \Delta t}{\hbar} \right) \geq 1 - \frac{E\Delta t}{\hbar},
\]

from which \[12\] follows. Obviously, both limits \[11\] and \[12\] can be achieved only for $\Delta E = E$, while spreads $\Delta E > E$, that would make $\Delta t$ smaller, are precluded by \[12\]. This effectively restricts $\Delta E$ to values $\Delta E \leq E$, as conjectured by Caianiello. One can now derive an upper limit on the value of the proper acceleration. In fact, in the instantaneous rest frame of the particle, where the acceleration is largest \[6\], $E = mc^2$ and \[9\] gives

\[
|\frac{dv}{dt}| \leq 2\frac{mc^3}{\hbar} \equiv A_m.
\]

It also follows that in the rest frame of the particle, where $\frac{d^2x^\mu}{ds^2} = 0$, the absolute value of the proper acceleration is \[16\]

\[
\left( \frac{d^2x^\mu}{ds^2} \frac{d^2x^\nu}{ds^2} \right)^{\frac{1}{2}} = \left( \frac{1}{c^2} \frac{d^2\mathbf{v}}{dt^2} \right)^{\frac{1}{2}} \leq A_m.
\]

Eq. \[16\] is a Lorentz invariant. The validity of \[16\] under Lorentz transformations is therefore assured.

Result \[12\] can also be used to extend \[15\] to include the average length of the acceleration $<\mathbf{a}>$. If, in fact, $v(t)$ is differentiable, then fluctuations about its mean are given by

\[
\Delta v \equiv v - <v> \simeq \left( \frac{dv}{dt} \right)_0 \Delta t + \left( \frac{d^2v}{dt^2} \right)_0 (\Delta t)^2 + \ldots.
\]
Eq. (17) reduces to \( \Delta v \approx |\frac{dv}{dt}| \Delta t = < a > \Delta t \) for sufficiently small values of \( \Delta t \), or when \( |\frac{dv}{dt}| \) remains constant over \( \Delta t \). Eq. (12) then yields

\[
< a > \leq \frac{2eE}{\hbar}
\]

and again (15) follows.

Eq. (12) is relevant to quantum geometry \([14, 15, 16]\), the entire subject of maximal acceleration \([17]\) and the field of computation \([11]\). This does not exhaust its usefulness. Its predictions and those of (9) are compared, in the example below, with the behaviour of a well known class of quantum systems.

2. – "Maximal Acceleration" in Type-I Superconductors

The static behavior of superconductors of the first kind is adequately described by London’s theory \([18]\). The fields and currents involved are weak and vary slowly in space. The equations of motion of the superelectrons are in this case \([19]\)

\[
\frac{D\vec{v}}{Dt} = \frac{e}{m} \left[ \vec{E} + \left( \frac{\vec{v}}{c} \times \vec{B} \right) \right] = \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v}.
\]

(19)

On applying (8) to (19), one finds

\[
\sqrt{\left( \frac{1}{2} \nabla \cdot \vec{v}^2 - \vec{v} \times \left( \nabla \times \vec{v} \right) \right)^2} \leq \frac{2}{\hbar} \Delta E \cdot \Delta \vec{v},
\]

(20)

and again

\[
\sqrt{\frac{1}{4} \left( \nabla \cdot \vec{v}^2 \right)^2 + \frac{e}{mc} \epsilon_{ijk} (\nabla^i \vec{v}^2) v^j B^k + \left( \frac{e}{mc} \right)^2 \left[ v^2 B^2 - (v_i B_i)^2 \right]} \leq \frac{2}{\hbar} \Delta E \cdot \Delta \vec{v},
\]

(21)

where use has been made of London’s equation

\[
\nabla \times \vec{v} = -\frac{e}{mc} \vec{B},
\]

(22)

and \( \epsilon_{ijk} \) is the Levi-Civita tensor. Static conditions, \( \frac{\partial \vec{v}}{\partial t} = 0 \), make (20) and (21) simpler. Eq. (19) can be used to express the acceleration in terms of the quantities \( \vec{E} \) and \( \vec{B} \) that are of more direct experimental and theoretical interest for this class of superconductors. London’s theory in the linear case predicts that \( \vec{E} = 0 \) in the superconductor. Eqs. (20) and (21) can therefore be used to calculate an upper limit on \( \vec{E} \) in the nonlinear version of London’s theory. It is also useful, for the sake of numerical comparisons, to apply (21) to the case of a sphere of radius \( R \) in an external magnetic field of magnitude \( B_0 \) parallel to the polar axis. This problem has an obvious symmetry and can be solved exactly. The
exact solutions of London’s equations for \( r \leq R \) are well-known \(^{20}\) and are reported here for completeness. They are

\[
B_r = \frac{4\pi}{\beta^2 c} r \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta j_\varphi),
\]

\[
B_\theta = \frac{4\pi}{\beta^2 c} r \frac{\partial}{\partial r} (r j_\varphi),
\]

\[
j_\varphi = n e v_\varphi = \frac{A}{r^2} (\sinh \beta r - \beta r \cosh \beta r) \sin \theta,
\]

where \( A = -\frac{c}{4\pi} \frac{3B_0 R}{\sinh \beta r} \), \( n \) is the density of superelectrons and \( \beta = \left( \frac{4\pi n e^2}{mc^2} \right)^{\frac{1}{2}} \) represents the reciprocal of the penetration length. From (19) and (20) one obtains

\[
| E_r | \leq \frac{v_\varphi B_\theta}{c} + \sqrt{\left( \frac{2mc}{\sqrt{\hbar}} \right)^2 (\Delta E)^2 (\Delta v)^2 - \left( \frac{v_\varphi B_r}{c} \right)^2}.
\]

For a gas of fermions in thermal equilibrium \( \Delta E \sim \frac{3}{5} \mu, \Delta v \sim \frac{3}{5} \sqrt{\frac{\epsilon_F}{2m}} \) and the chemical potential behaves as \( \mu \approx \epsilon_F - \frac{1}{2} \) \( \frac{kT}{1.28} \). For \( T \) close to the transition temperatures of type-I superconductors. The reality of (26) requires that \( \Delta E \geq \mu_R B_r \), where \( \mu_B = \frac{\hbar}{2mc} \) is the Bohr magneton, or that \( \frac{3}{5} \epsilon_F \geq \mu_B B_r \). This condition is certainly satisfied for values of \( B_r \leq B_c \), where \( B_c \) is the critical value of the magnetic field applied to the superconductor. From (26) one also obtains

\[
| E_r | \leq \frac{3}{2m} \left( \frac{\epsilon_F}{2} \right)^{\frac{3}{8}} \left[ \frac{B_\theta}{c} + \sqrt{\left( \frac{3\epsilon_F}{\mu_B} \right)^2 - \left( \frac{B_r}{c} \right)^2} \right].
\]

More restrictive values for \( \Delta E \) and \( \Delta v \) can be obtained from \( B_c \). The highest value of the velocity of the superelectrons must, in fact, be compatible with \( B_c \) itself, lest the superconductor revert to the normal state. This value is approximately a factor \( 10^3 \) smaller than that obtained by statistical analysis. The upper value \( v_0 \) of \( v_\varphi \) is at the surface. From \( \Delta E \leq \frac{1}{2} m v_0^2, \Delta v \leq v_0 \) and (26) one finds that at the equator, where \( B_r = 0 \), \( E_r \) satisfies the inequality

\[
| E_r | \leq \frac{v_0}{c} \left( | B_\theta | + \frac{v_0^2}{2\mu_B} \right).
\]

For a sphere of radius \( R = 1 \text{ cm} \) one finds \( v_0 \approx 4.4 \times 10^4 \text{ cm/s} \) and \( E_r \leq 69 \text{ N/C} \). If no magnetic field is present, then (28) gives \( E_r \leq 4.2 \text{ N/C} \). On the other hand London’s equation gives

\[
E_r = \frac{m}{2e} \frac{\partial v_\varphi^2}{\partial r} \approx 0.32 \text{ N/C}.
\]
The experimental work of Bok and Klein agrees with (29). The MA limits and are therefore consistent with and its experimental verification.

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REFERENCES

[1] E.R. Caianiello, Lett. Nuovo Cimento 32(1981)65.
[2] E.R. Caianiello, Lett. Nuovo Cimento 41(1984)370. See also E. R. Caianiello, Rivista Nuovo Cimento 15, No. 4 (1992).
[3] H.E. Brandt, Lett. Nuovo Cimento 38 (1983)522 ; 39(1984)192; Found. Phys. Lett. 11(1998)265.
[4] E.R. Caianiello, M. Gasperini, G. Scarpetta, Class. Quantum Grav. 8(1981)659.
[5] M. Gasperini in Advances in Theoretical Physics, edited by E. R. Caianiello (World Scientific, Singapore, 1991), p. 77.
[6] W.R. Wood, G. Papini, Y.Q. Cai, Nuovo Cimento 104 B (1989)361 and (errata corrig) 727; G. Papini, Mathematica Japonica 41(1995)81.
[7] L.D. Landau and E.M. Lifshitz, Quantum Mechanics, third edition (Pergamon Press, New York, 1977), pp.27 and 49.
[8] Albert Messiah, Quantum Mechanics (North-Holland, Amsterdam, 1961), Vol.I, Chs. 4.10 and VIII.13.
[9] Roger Balian, From Microphysics to Macrophysics. Methods and Applications of Statistical Physics, Vol. I (Springer-Verlag, Berlin, 1991).
[10] C.S. Sharma, S. Srinranakathan, Lett. Nuovo Cimento 44(1985)275 .
[11] Norman Margolus, Lev B. Levitin, Physica D 120(1998)188.
[12] See also Jacob D. Bekenstein, Phys. Rev. Lett. 46(1981)623; J. Anandan, Y. Aharonov, Phys. Rev. Lett. 65(1990)1697.
[13] G. Stephenson and C.W. Kilmister, Special Relativity for Physicists (Longmans, London, 1965).
[14] E.R. Caianiello, Lett. Nuovo Cimento 25(1979)225; 27(1980)89; Il Nuovo Cimento B59(1980)350; E.R. Caianiello, G. Marmo, and G. Scarpetta, Il Nuovo Cimento A 86(1985)337; E.R. Caianiello, La Rivista del Nuovo Cimento 15 n.4 (1992) and references therein.
[15] G. Papini and W.R. Wood, Phys. Lett. A170(1992)409; W.R. Wood and G. Papini, Phys. Rev. D45(1992)3617; Found. Phys. Lett. 6(1993)409.
[16] E.R. Caianiello, A. Feoli, M. Gasperini, G. Scarpetta, Int. J. Theor. Phys. 29(1990)131. E.R. Caianiello, M. Gasperini, G. Scarpetta, Class. Quantum Grav. 8(1991)659; M. Gasperini, in “Advances in Theoretical Physics” ed. E.R. Caianiello, (World Scientific, Singapore, 1991), p. 77. A. Feoli, Nucl. Phys. B396(1993)261. E.R. Caianiello, M. Gasperini, and G. Scarpetta, Il Nuovo Cimento B105(1990)259. V. Bozza, G. Lambiase, G. Papini, G. Scarpetta, Phys. Lett. A279(2001)163. G. Papini, A. Feoli, and G. Scarpetta, Phys. Lett. A202(1995)50. A. Feoli, G. Lambiase, G. Papini, and G. Scarpetta, Il Nuovo Cimento 112B(1997)913. G. Lambiase, G. Papini, and G. Scarpetta, Il Nuovo Cimento 114B(1999)189. See also S. Kuwata, Il Nuovo Cimento 111B(1996)893. G. Lambiase, G. Papini, and G. Scarpetta, Phys. Lett. 244A(1998)349. C.X. Chen, G. Lambiase, G. Mobed, G. Papini, and G. Scarpetta, Il Nuovo Cimento B114(1999)1135. C.X. Chen, G. Lambiase, G. Mobed, G. Papini, and G. Scarpetta, Il Nuovo Cimento B114(1999)199. A. Feoli, G. Lambiase, G. Papini, G. Scarpetta, Phys. Lett. A263(1999)147. S. Capozziello, A. Feoli, G. Lambiase, G. Papini, G. Scarpetta, Phys. Lett. A268(2000)247. V. Bozza, A. Feoli, G. Papini, G. Scarpetta, Phys. Lett. A271(2000)35.
[17] A. Das, J. Math. Phys. 21(1980)1506;
  M. Gasperini, Astrophys. Space Sci. 138(1987)387;
  M. Toller, Nuovo Cimento B102(1988)261; Int. J. Theor. Phys. 29(1990)963; Phys. Lett. B256(1991)215;
  B. Mashhoon, Physics Letters A143(1990)176;
  V. de Sabbata, C. Sivaram, Astrophys. Space Sci. 176(1991)145; Spin and Torsion in gravitation, World Scientific, Singapore, (1994);
  D.F. Falla, P.T. Landsberg, Il Nuovo Cimento B 106(1991) 69;
  A.K. Pati, Il Nuovo Cimento B 107(1992)895; Europhys. Lett. 18(1992)285. C.W. Misner, K.S. Thorne, J. A. Wheeler, Gravitation, W.H. Freeman and Company, S. Francisco, 1973.
[18] P.G. De Gennes, Superfluidity of Metals and Alloys (W.A. Benjamin, New York, 1966).
[19] D.R. Tilley and J. Tilley, Superfluidity and Superconductivity, third edition (Adam Hilger, Bristol, 1990).
[20] F. London, Superfluids, Vol. I (Dover Publications, New York, 1961).
[21] J. Bok and J. Klein, Phys. Rev. Lett. 20(1968)660.