Knots from wall–anti-wall annihilations with stretched strings

Muneto Nitta\textsuperscript{1}

\textsuperscript{1}Department of Physics, and Research and Education Center for Natural Sciences, Keio University, Hiyoshi 4-1-1, Yokohama, Kanagawa 223-8521, Japan

(Dated: May 1, 2014)

Abstract

A pair of a domain wall and an anti-domain wall is unstable to decay. We show that when a vortex-string is stretched between the walls, there remains a knot soliton (Hopfion) after the pair annihilation.
I. INTRODUCTION

The Faddeev-Skyrme (FS) model is an $O(3)$ model with a fourth derivative term. As a skyrmion is a stable soliton in the Skyrme model \cite{1}, the FS model admits a knot soliton, which is a stable soliton with a non-trivial Hopf number $\pi_3(S^2) \simeq \mathbb{Z} \cite{2}$. Such a soliton with the minimum Hopf charge is also called a Hopfion. A knot soliton, which is mathematically interesting in itself, is also considered as a model of a glue ball. Although several studies have been conducted on knot solitons, they have, thus far, focused most on soliton stability, construction of solutions, and interaction between multiple solutions, among other topics. However, thus far, there has been no discussion on how knot solitons are created. In contrast, a mechanism has already been proposed for the generation of topological defects: the Kibble-Zurek mechanism, by which topological defects are inevitably generated during a phase transition accompanied with spontaneous symmetry breaking.

In this paper, we propose a mechanism of the creation of knot solitons. The key concept is the identification of a Hopfion as a closed and twisted baby skyrmie string, \textit{i.e.} a “twisted baby skyrmie ring”. A baby skyrmion \cite{3} is a lump soliton in the $\mathbb{C}P^1$ model with a four-derivative term in 2+1 dimensions. In 3+1 dimensions, it is a string-like soliton. When one makes a closed baby skyrmie string, one can twist the phase modulus. Such a twisted baby skyrmie ring is nothing but a Hopfion \cite{4-6}. The original baby skyrmie model admits the unique vacuum \cite{3}, whereas the new baby skyrmie model, proposed in a subsequent study \cite{7}, admits two discrete vacua. Therefore, the new model admits a domain wall solution interpolating between these two vacua \cite{8,9}. We consider a pair of a domain wall and an anti-domain wall. Such a configuration is clearly unstable to decay. However, it does not have to end up uniformly with the vacuum state. We first show that there can remain (anti-)baby skyrmions in $d = 2 + 1$ dimensions and baby skyrmie rings in $d = 3 + 1$ dimensions. Untwisted baby skyrmie rings are unstable to decay. However, when vortex-strings are stretched between the wall–anti-wall pair, there can appear twisted baby skyrmie rings, or Hopfions.
II. FADDEEV-SKYRME MODEL

The Lagrangian of the FS model is expressed as

\[ \mathcal{L} = \frac{1}{2} \partial_{\mu} \mathbf{n} \cdot \partial^{\mu} \mathbf{n} - \mathcal{L}_{4}(\mathbf{n}) - V(\mathbf{n}), \quad \mathbf{n} \cdot \mathbf{n} = 1 \]  

with \( \mathbf{n} = (n_1, n_2, n_3) \); the fourth derivative FS term is expressed as

\[ \mathcal{L}_{4}(\mathbf{n}) = \kappa [\mathbf{n} \cdot (\partial_{\mu} \mathbf{n} \times \partial_{\nu} \mathbf{n})]^{2} = \kappa (\partial_{\mu} \mathbf{n} \times \partial_{\nu} \mathbf{n})^{2}, \]  

in which we have used \( \mathbf{n} \cdot \partial \mathbf{n} = 0 \). The original FS model has no potential, i.e., \( V = 0 \) \[2\].

The original baby skyrme model has been proposed to have the potential \( V = m^{2}(1 - n_{3}) \), which admits the unique vacuum \( n_{3} = 1 \) \[3\]. On the other hand, the new (or planar) baby skyrme model has been proposed with the potential \( V = m^{2}(1 - n_{3}^{2}) = m^{2}(1 - n_{3})(1 + n_{3}) \) with two vacua \( n_{3} = \pm 1 \) \[7\]. The latter admits both a baby skyrmion \[7\] and a domain wall interpolating between the two vacua \[8, 9\].

It is known that the \( O(3) \) sigma model is equivalent to the \( CP^{1} \) model. To demonstrate this, we use the relations \( \mathbf{n} = \Phi^{\dagger} \sigma \Phi \) and \( \Phi^{T} = (1, u)/\sqrt{1 + |u|^{2}} \), where \( u \) is the projective coordinate of \( CP^{1} \). The Lagrangian \[11\] with the new baby skyrme potential can then be rewritten as

\[ \mathcal{L} = \frac{\partial_{\mu} u^{*} \partial^{\mu} u - m^{2}|u|^{2}}{(1 + |u|^{2})^{2}} - 2\kappa \frac{\partial_{\mu} u^{*} \partial^{\mu} u - |\partial_{\mu} u^{*} \partial^{\mu} u|^{2}}{(1 + |u|^{2})^{4}}. \]

Here, the two discrete vacua \( n_{3} = +1 \) and \( n_{3} = -1 \) are mapped to \( u = \infty \) and \( u = 0 \), respectively.

III. BABY SKYRMIONS FROM WALL-ANTI-WALL ANNIHILATION

Let us first construct a domain wall solution \[8, 9\]. For the existence of a domain wall, the FS term is not essential. We simply consider a small coupling \( \kappa \). The model reduces to the massive \( CP^{1} \) model, which can be made supersymmetric by adding fermions \[10\]. However, supersymmetry is not essential in our study.

Then, a domain wall interpolating the two vacua \( u = 0 \) and \( u = \infty \) can be obtained as \[10\]

\[ u_{w} = e^{\mp m(x^{1} - x_{0}^{1}) + i\varphi} \]  

(4)
FIG. 1: (a,b) Single domain wall and (c,d) a pair of a domain wall and an anti-domain wall in the $\mathbb{C}P^1$ model. (a) The wall is perpendicular to the $x^1$-axis. The arrows denote points in the $\mathbb{C}P^1$. (b) The $\mathbb{C}P^1$ target space. The north and south poles are denoted by $\odot$ and $\oplus$, respectively. The path connecting them represents the map from the path in (a) along the $x^1$-axis in real space from $x^1 \rightarrow -\infty$ to $x^1 \rightarrow +\infty$. The path in the $\mathbb{C}P^1$ target space passes through one point on the equator, which is represented by $\leftarrow$ in (a) in this example. In general, the $U(1)$ zero mode is localized on the wall as a phase modulus. (c) The phases of the wall and anti-wall are opposite to each other. (d) The path represents the map from the wall and anti-wall configuration.

with the width $1/m$, where $\mp$ represents a wall and an anti-wall. Here, $x^1_0$ and $\varphi$ are real constants representing the position and phase of the (anti-)domain wall. The domain wall is mapped to a large circle starting from the north pole, denoted by $\odot$, and ending at the south pole, denoted by $\oplus$, in the $\mathbb{C}P^1$ target space.

We consider a pair of a wall at $x^1 = x^1_1$ and an anti-wall at $x^1 = x^1_2$. An approximate solution valid at large distance, $x^1_2 - x^1_1 \gg m^{-1}$, is obtained as

$$u_{w-aw} = e^{-m(x^1-x^1_1)+i\varphi_1} + e^{+m(x^1-x^1_1)+i\varphi_2}.$$  

(5)

Here, the phases $\varphi_1$ and $\varphi_2$ of the wall and anti-wall are considered to be opposite to each other, $\varphi_1 = \varphi_2 + \pi$, as in Fig. 1(c). The configuration is mapped to a loop in the $\mathbb{C}P^1$ target space, as in Fig. 1(d). This configuration is unstable as it should end up with the vacuum state $\odot$. In the decaying process, the loop is unwound from the south pole in the target space. The unwinding of the loop can be achieved in two topologically inequivalent processes, schematically shown in Fig. 2(c) and (f).

In real space, at first, a bridge connecting the wall and anti-wall is created, as in Fig. 2(a) and (d). Here, there exist two possibilities of the spin structure along the bridge, corre-
FIG. 2: Decaying processes of the wall and anti-wall. (a,d) A bridge is created between the wall and the anti-wall. In this process, there are two possibilities of the $CP^1$ structure along the bridge. (b,e) The upper and lower regions are connected by breaking the bridge. (c,f) Accordingly, the loop in the $CP^1$ target space is unwound in two ways.

sponding to the two inequivalent ways of the unwinding processes: along the bridge in the $x^1$-direction, the spin rotates (a) anti-clockwise or (b) clockwise on the equator of the $CP^1$ target space. Let us label these two kinds of bridges by $\uparrow$ and $\downarrow$. Second, the bridge is broken into two pieces, as in Fig. 2(b) and (e) between which the vacuum state $\odot$ is filled between them. Let us again label these two kinds of holes by $\uparrow$ and $\downarrow$. In either case, the two regions separated by the domain walls are connected through a hole created by the decay of the walls. Once created, these holes grow with reducing the wall energy.

Several holes are created during the entire decaying process. Let us focus a pair of two nearest holes. One can find a ring of a domain wall between the holes, as shown in Fig. 3. Here, since there exist two kinds of holes ($\uparrow$ and $\downarrow$), there exist four possibilities of the rings, (a) $\uparrow\downarrow$, (b) $\downarrow\uparrow$, (c) $\uparrow\uparrow$, and (d) $\downarrow\downarrow$ in Fig. 3. Clearly, the rings of types (c) of (d) can decay and end up with the vacuum state $\odot$. However, the decay of the rings of types (a) and (b) is topologically forbidden, because of a nontrivial winding of the spin along the rings.
FIG. 3: Stable and unstable rings. (a,b) Stable rings. The phase winds once (the winding number is ±1) along the rings. Total configurations are baby skyrmions with a non-trivial element, ±1, of the second homotopy group, \( \pi_2 \). (c,d) Unstable rings. The phase does not wind (the winding number is 0) along the rings. They decay into ground state (up pseudo-spin).

What are these topologically protected rings of a domain wall? They are nothing but baby skyrmions. The solutions can be written as 
\[
(\omega \equiv x^1 + ix^2) \quad u = u_0 = \lambda/(\omega - \omega_0) \quad \text{and} \quad u = \bar{u}_0
\]
for a baby skyrmion and an anti-baby skyrmion, respectively, where \( \omega_0 \in \mathbb{C} \) represents the positions of the lump. Here, \( \lambda \in \mathbb{C}^* \) where the size \(|\lambda|\) is fixed by the balance between the potential term and the FS term, while \( U(1) \) orientation \( \text{arg} \lambda \) is a modulus of the baby skyrmion. In fact, these configurations can be shown to have a nontrivial winding in the second homotopy group \( \pi_2(\mathbb{C}P^1) \simeq \mathbb{Z} \); (a) and (b) belong to, respectively, +1 and −1 of \( \pi_2(\mathbb{C}P^1) \).

In \( d = 3 + 1 \) dimensions, domain walls have two spatial dimensions in their world-volume. When a domain wall-pair decay occurs, there appear two-dimensional holes, which are labeled as ↓ or ↑ in Fig. 2 (b) and (e), respectively. Along the boundary of these two kinds of holes, there appear baby skyrme strings, which generally create closed baby skyrme strings, i.e., baby skyrme rings, as in Fig. 4. This process can be numerically verified \[11\]. These rings are unstable to decay into the fundamental excitations in the end.

IV. CREATING KNOTS

Here, we discuss the approach to stabilizing baby skyrme rings. To this end, we place a baby skyrme string stretching between the domain wall and the anti-domain wall, as in Fig. 5.
FIG. 4: An untwisted baby srkyme-ring.

FIG. 5: (a) A pair of a domain wall and an anti-domain wall stretched by a string (vortex). The domain walls are perpendicular to the $x^3$-axis, and a baby skyrme string along the $x^3$-axis is stretched between the domain walls. The arrows denote points in the $\mathbb{C}P^1$ target space. (b) Loops in the wall-vortex systems. While the loop A yields an untwisted baby skyrme ring in Fig. 4, the loop B (C) yields a baby skyrme ring twisted once (twice) with the Hopf charge one (two).

First, we consider $k$ baby skyrme strings ending on a domain wall. Such a configuration was constructed in [12, 13] in the absence of the FS term ($z \equiv x^2 + ix^3$)

$$u_{w-v} = e^{\pm m(x^1-x^1_0)+iv_0} Z(z), \quad Z(z) = \sum_{i=1}^{k} \frac{\lambda_i}{z - z_i}. \quad (6)$$

This solution precisely coincides with a BIon [14] in the Dirac-Born-Infeld action on a D2-
FIG. 6: A knot created after a wall–anti-wall pair annihilation. (a) The torus divides the regions of $\odot$ and $\otimes$. The vertical section of the torus in the $x^1$-$x^2$ plane is a pair of a baby skyrmion and an anti-baby skyrmion. While they rotate along the $x^3$-axis, their phases are twisted and connected to each other at the $\pi$ rotation. This configuration is different from that of a closed and untwisted baby skyrmion string in Fig. 4. (b) The preimages of $\uparrow$ and $\downarrow$ make a link.

brane, and so, this was referred to as a “D-brane soliton” [12]. The wall surface in the above solution is logarithmically bent. We can also place the strings on the both sides of the wall [13]

$$Z(z) = \prod_{j=1}^{k_+} (z - z_j^+)/\prod_{i=1}^{k_-} (z - z_i^-),$$

(7)

where $z_j^\pm$ and $k_\pm$ denote the positions and numbers of strings extending to $x^1 \to \pm \infty$, respectively. If the numbers of strings coincide on both sides, i.e., $k_1 = k_2$, the wall surface is asymptotically flat.

With regard to $\kappa$, the above solution should be modified; however, we consider a regime with a small $\kappa$ so that the modifications to the above solution can be neglected.

Now let us consider the strings stretched between domain walls in Fig. 5. An approximate solution for a wall with the phase $\varphi_1$ at $x^1 = x^1_1$ and an anti-wall with the phase $\varphi_2$ at $x^1 = x^1_2$ is obtained from Eq. (5) as

$$u_{w-\nu-aw} = (e^{-m(x^1-x^1_1)+i\varphi_1} + e^{+m(x^1-x^1_2)+i\varphi_2})Z(z),$$

(8)

with $Z(z)$ in Eq. (6) for $k$ stretched strings, or Eq. (7) for $k_2$ stretched strings and $k_1$ strings attached from outside. In our case, we take $\varphi_1 = \varphi_2 + \pi$. 

8
As in the case without a stretched string, the configuration itself is unstable to decay, and baby skyrme rings are created. A ring is not twisted if it does not enclose the stretched strings, as the loop A in Fig. 5(b). However, if a ring encloses $n$ stretched strings, as the loops B and C in Fig. 5(b), it is twisted $n$ times. A baby skyrme ring twisted once is shown in Fig. 6(a). The vertical section of the torus in the $x^1$-$x^2$-plane is a pair of a baby skyrmion and an anti-baby skyrmion. Moreover, the presence of the stretched string implies that the phase winds anti-clockwise along the loops, as is indicated by the arrows on the top and bottom of the torus in Fig. 6(a). When the baby skyrmions in the pair rotate along the $x^1$-axis, their phases are twisted and connected to each other at the $\pi$ rotation. This configuration is nothing but a Hopfion, which is stable as discussed in [6]. One can calculate the Hopf charge, but there is a simpler way to confirm that the configuration is a Hopfion.

A preimage of a point on $\mathbb{C}P^1$ is a loop in real space. When two loops of the preimages of two arbitrary points on $\mathbb{C}P^1$ have a linking number $n$, the configuration has a Hopf charge $n$. In Fig. 6(b), we plot the preimages of $\uparrow$ and $\downarrow$, which are linked with a linking number one. Thus, we obtain a knot soliton with a Hopf charge of one (a Hopfion). Similarly, a created ring enclosing $n$ stretched strings yields a knot soliton with a Hopf charge of $n$.

In summary, when a pair of a domain wall and an anti-domain wall annihilates in the FS model with the potential term, (anti-)baby skyrmions are created in $d = 2+1$ dimensions. In $d = 3+1$ dimensions, there appear baby skyrme-strings. When a string is stretched between the wall pair, the baby skyrme-ring that encircles it becomes a twisted baby skyrme ring, which is a Hopfion. This mechanism is the first proposal for creating knot solitons.

Acknowledgements

This work is supported in part by Grant-in Aid for Scientific Research (No. 23740198) and by the “Topological Quantum Phenomena” Grant-in Aid for Scientific Research on Innovative Areas (No. 23103515) from the Ministry of Education, Culture, Sports, Science and Technology (MEXT) of Japan.

[1] T. H. R. Skyrme, Proc. Roy. Soc. Lond. A 260, 127 (1961); Nucl. Phys. 31, 556 (1962).
[2] L. D. Faddeev and A. J. Niemi, Nature 387, 58 (1997).
[3] B. M. A. Piette, B. J. Schroers and W. J. Zakrzewski, Z. Phys. C 65, 165 (1995); Nucl. Phys.
[4] H. J. de Vega, Phys. Rev. D 18, 2945 (1978).
[5] A. Kundu and Y. P. Rybakov, J. Phys. A A 15, 269 (1982).
[6] J. Gladikowski and M. Hellmud, Phys. Rev. D 56, 5194 (1997).
[7] T. Weidig, Nonlinearity 12, 1489-1503 (1999).
[8] A. E. Kudryavtsev, B. M. A. Piette and W. J. Zakrzewski, Nonlinearity 11, 783 (1998).
[9] D. Harland and R. S. Ward, Phys. Rev. D 77, 045009 (2008).
[10] E. R. C. Abraham and P. K. Townsend, Phys. Lett. B 291, 85 (1992); Phys. Lett. B 295, 225 (1992).
[11] H. Takeuchi, K. Kasamatsu, M. Nitta and M. Tsubota, J. Low Temp. Phys. 162, 243 (2011).
[12] J. P. Gauntlett, R. Portugues, D. Tong and P. K. Townsend, Phys. Rev. D 63, 085002 (2001).
[13] Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, Rev. D 71, 065018 (2005).
[14] C. G. Callan and J. M. Maldacena, Nucl. Phys. B 513, 198 (1998); G. W. Gibbons, Nucl. Phys. B 514, 603 (1998).