INTEGRABILITY OF CERTAIN DEFORMED NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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A systematic investigation of certain higher order or deformed soliton equations with (1 + 1) dimensions, from the point of complete integrability, is presented. Following the procedure of Ablowitz, Kaup, Newell and Segur (AKNS) we find that the deformed version of Nonlinear Schrödinger equation, Hirota equation and AKNS equation admit Lax pairs. We report that each of the identified deformed equations possesses the Painlevé property for partial differential equations and admits trilinear representation obtained by truncating the associated Painlevé expansions. Hence the above mentioned deformed equations are completely integrable.

Keywords: Integrable equations; nonlinear partial differential equations; soliton equations; deformed equations.

1. Introduction

The investigation of completely integrable higher order nonlinear partial differential equations (PDEs) with (1 + 1) dimensions admitting solitons has drawn considerable attention in recent years [6–11, 13, 15–18, 21]. For example by extending the Painlevé property for PDE [4, 19, 20], Karasu et al. [7] have recently identified a sixth order completely integrable nonlinear PDE

\[ u_{ttt} + 20u_{tx} + 40u_{ttx} + 120u_{ttxx} + u_{txxx} + 4u_{txxx} + 8u_{ttxxt} = 0 \]  

or

\[ (\partial_t^2 + 8u_t \partial_x + 4u_{xx})(u_{tt} + u_{txx} + 6u_t^2) = 0 \]  

or

\[ v_t + v_{xxx} + 12v_{xx} = g_v(t,x), \]  

\[ g_{xxx} + 8vg_v + 4v_g = 0, \]  

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can be written as

\[ u_t - u_{xxx} - 6u_{xx} = g(t, x), \]  
\[ g_{xx} + 4ug_x + 2ug = 0, \]

which can be viewed as a nonholonomic deformation of the Korteweg-deVries equation [9, 11]. It is appropriate to mention that Eq. (1.2) can also be derived through a different approach [13, 14, 18]. This suggests that one of the possibilities of finding a higher order but scalar completely integrable nonlinear PDEs with (1+1) dimensions possessing solitons is (in a sense) to finding a coupled or deformed equation (with integrability properties) consisting of (i) known nonlinear PDE with (1+1) dimensions possessing solitons with deformed variable and (ii) PDE in the deformed variable. Kundu has shown that Eq. (1.4) is (in a sense) to finding a coupled or deformed equation (with integrability properties) consisting of (i) known nonlinear PDE with (1+1) dimensions possessing solitons with deformed variable and (ii) PDE in the deformed variable. Kundu has shown that Eq. (1.4) admits a Lax pair [9]. Also Kupershmidt [8] has demonstrated that Eq. (1.4) admits a bi-Hamiltonian representation while Ramani et al. [13] have shown that it can be written in bilinear form. Recently we have shown that the coupled or deformed equations (1.4) possess other integrability structures such as the existence of infinitely many generalized symmetries, polynomial conserved quantities, nonlocal symmetries, master symmetries and a recursion operator [10, 16]. In this article we report that the deformed NLS, Hirota and AKNS equations with (1+1) dimensions, respectively, given by

\[ iu_t - u_{xx} - 2u^2u^* = g(x, t), \]  
\[ g_x = -2u\sqrt{c(t)^2 - gg^*}, \]  
\[ iu_t + |u|^2u_x + 2u^2 + g(x, t), \]  
\[ g_x = -2u\sqrt{c(t)^2 - gg^*}, \]  
\[ u_t = -u_{xx} + 2u^2v + \bar{g}(x, t), \]  
\[ v_x = v_{xx} - 2u^2v + h(x, t), \]  
\[ \bar{g}_x = 2u\sqrt{c(t)^2 + gh}, \]  
\[ h_x = 2u\sqrt{c(t)^2 + gh}, \]

where * denotes complex conjugate, \( \epsilon \) is a constant and \( c(t) \) is an arbitrary function admits Lax pair and possesses the Painlevé property for PDEs. Also we have shown, by truncating the associated Painlevé expansions, that each of them can be written in trilinear form. Thus the deformed NLS, Hirota and AKNS equations are completely integrable.

Note that on eliminating \( g(x, t), \bar{g}(x, t) \) and \( h(x, t) \) in the above coupled equations one can obtain higher order nonlinear PDEs. For example, the deformed Hirota equation (1.6) can be written as

\[
\left( \frac{\partial^2}{\partial x^2} - u_t \right) \left( iu_t + \epsilon |u|^2 u_x + \frac{u_{xx}}{2} + |u|^2 u + 2u^2 \right) \left( i(u_x^* + u^* u_x) + \frac{\partial}{\partial x} \left( |u|^2 u + u^2 u_x \right) \right) + i\epsilon \left( u_{xx} + u^* u_{xx} \right) + 6\epsilon \bar{u} u_x (u_x^* + u^* u_x) - \frac{1}{2} (u_{xx}^* - u^* u_{xx}) = 0
\]
which is a fifth order one. The plan of the article is as follows: In Sec. 2, we explain through AKNS procedure that how one can derive Lax matrices for deformed nonlinear PDEs with (1 + 1) dimensions in general and show that deformed NLS Eq. (1.5), Hirota (1.6) and AKNS equation Eq. (1.7) admit Lax pairs. In Sec. 3, we show explicitly that deformed NLS equation (1.5) possesses the Painlevé property. In Sec. 4, we show by truncating the Painlevé expansions that the deformed NLS admits trilinear representation. In Sec. 5, we give a brief summary of our results. In the Appendix, we show that both the deformed Hirota equation (1.6) and AKNS equation (1.7) possess the Painlevé property and admit trilinear forms.

2. Lax Pair of Nonlinear PDEs: AKNS Procedure

It is well known that the Lax pair of a scalar PDE, for example nonlinear evolution equation of the form

\[ u_t = F(u, u_x, u_{xx}, \ldots) \]  

(2.1)
can be constructed through AKNS procedure [1, 2, 12] in the following manner. Consider a linear system

\[ \psi_x = L\psi, \quad \psi_t = M\psi, \]  

(2.2)
or equivalently,

\[
\begin{bmatrix}
\psi_{1x}(\lambda) \\
\psi_{2x}(\lambda)
\end{bmatrix}
= \begin{bmatrix}
L_{11}(\lambda) & L_{12}(\lambda) \\
L_{21}(\lambda) & L_{22}(\lambda)
\end{bmatrix}
\begin{bmatrix}
\psi_1(\lambda) \\
\psi_2(\lambda)
\end{bmatrix},
\]

\[
\begin{bmatrix}
\psi_{1t}(\lambda) \\
\psi_{2t}(\lambda)
\end{bmatrix}
= \begin{bmatrix}
A(\lambda) & B(\lambda) \\
C(\lambda) & D(\lambda)
\end{bmatrix}
\begin{bmatrix}
\psi_1(\lambda) \\
\psi_2(\lambda)
\end{bmatrix},
\]

where \( \lambda \) is the spectral parameter and \( L_{ij}(\lambda), A(\lambda), B(\lambda), C(\lambda) \) and \( D(\lambda) \) are functions of dependent variable and their \( x \)-derivatives. The compatibility condition of the linear system (2.2) gives

\[ L_1 - M_2 + [L, M] = 0 \]  

(2.3)

which is usually referred to as Lax equation.

The explicit form of the Lax matrices can be derived in the following way, that is, for a given suitable matrix \( L \) the matrix \( M \) can be derived by expanding its entries as a polynomial in the spectral parameter \( \lambda \) or \( \psi \) satisfying Eq. (2.3). Let us fix the entries of matrix \( L \) as

\[ L_{11} = i\lambda, \quad L_{12} = iq, \quad L_{21} = ir, \quad L_{22} = -i\lambda \]

or equivalently,

\[ \psi_{1x} = i\lambda\psi_1 + iq\psi_2 \]  

(2.4)

\[ \psi_{2x} = ir\psi_1 - i\lambda\psi_2. \]  

(2.5)
Proceeding further from Eq. (2.3) we find

\[ D = -A, \]
\[ A_x = i q C - i r B, \]
\[ i q_t = B_2 - 2i \lambda B + 2q A, \]
\[ i r_t = C_2 + 2\lambda C - 2i r A. \]

Expanding \( A(\lambda), B(\lambda) \) and \( C(\lambda) \) as polynomials in \( \frac{\lambda^j}{j!}, j = -3, -2, -1, 0, 1, 2 \) and then equating different powers of \( \lambda \) in the above equations to zero yields the following:

\[ A(\lambda) = a_0 \lambda^3 + a_2 \lambda^2 + \left( \frac{a_3 q r}{2} + a_1 \right) \lambda + \left( \frac{i a_1}{4} r q_x - q r_x \right) - \frac{a_2 q r}{2} + a_0 \right) + \frac{A_1}{\lambda} + \frac{A_2}{\lambda^2}, \]

\[ B(\lambda) = a_0 \lambda^2 + \left( \frac{i a_1}{2} q r_x + a_2 \right) \lambda + \left( \frac{i a_1}{4} q r_x + q r_x \right) - \frac{i a_2 q r}{2} + a_1 \right) + \frac{B_1}{\lambda} + \frac{B_2}{\lambda^2}, \]

\[ C(\lambda) = a_0 \lambda + \left( \frac{a_1 q r_x}{2} + a_2 \right) \lambda + \left( \frac{a_1 q r_x}{4} (r_x r + q r_x^2) + \frac{i a_2 q r_x}{2} + a_1 \right) + \frac{C_1}{\lambda} + \frac{C_2}{\lambda^2}, \]

in addition with

\[ i q_t + \frac{a_1}{4} (q_{xx} + 6 q r q_x) + \frac{i a_2}{2} q_{xx} + i a_2 q r_x^2 - 2 i a q - a_1 q_x = -2i B_1, \]

\[ i r_t + \frac{a_1}{4} (r_{xx} + 6 q q r_x) - \frac{i a_2}{2} r_{xx} - i a_2 q r_x^2 + 2 i a q - a_1 r_x = 2i C_1, \]

\[ A_{1x} = i q C_1 - i r B_1, \quad B_{2x} = 2i (B_2 - q A_1), \quad C_{1x} = -2i (C_2 - r A_1), \]

\[ A_{2x} = i q C_2 - i r B_2, \quad B_{2x} + 2i q A_2 = 0, \quad C_{2x} - 2i r A_2 = 0, \]

where \( a_0, a_1, a_2 \) are integration constants. We would like to mention that the Lax matrices of well known soliton equations such as Korteweg–de Vries equation (KdV), modified KdV, etc can be derived by choosing \( A_i, B_i \) and \( C_i, i = 1, 2 \) as zero. This suggests that by choosing nonzero expressions for \( A_i, B_i \) and \( C_i, i = 1, 2 \) satisfying (2.13)–(2.16) one can derive Lax matrices for higher order or coupled or deformed nonlinear PDEs. Some of the identified higher order or deformed PDEs are as follows:

(i) **Deformed NLS Equation**: PDEs (2.13)–(2.16) reduce into

\[ i u_t - u_{xx} - 2u^2 u^* = g(x,t) \]

and its conjugate for the choice

\[ a_0 = a_1 = a_2 = 0, \quad a_2 = 2i, \quad q = u, \quad r = u^*, \quad B_1 = \frac{i q}{2}, \quad C_1 = \frac{i q^*}{2}. \]
Proceeding further with the above expressions for $B_1$ and $C_1$ we find that (2.15)–(2.16) reduce into
\begin{align*}
b_x &= i(u g^* - u^* g), \\ g_x &= -2iub
\end{align*}
where
\begin{align*}
A_1 &= \frac{ib}{2}, \\ A_2 &= 0, \\ B_2 &= 0, \\ C_2 &= 0
\end{align*}
and so the Lax matrices $L$ and $M$ become
\begin{align*}
L &= \begin{pmatrix}
i\lambda & iu \\
ing & -i\lambda
\end{pmatrix}, \\
M &= \begin{pmatrix}2i\lambda^2 - iu u^* & 2i\lambda u + u_x \\
2i\lambda u^* - u_x^* & -2i\lambda^2 + iu u^*
\end{pmatrix} + \frac{i}{2\lambda} \begin{pmatrix}b & g \\
g^* & -b
\end{pmatrix}.
\end{align*}
Note that Eqs. (2.17)–(2.18) become the deformed NLS equation (1.5) when $b(x, t) = \sqrt{c(t)^2 - gg^*}$.

(ii) Deformed Hirota Equation

The PDEs (2.13)–(2.14) reduce into
\begin{align*}
iu_t + i\epsilon (u^3 u_x + 6|u|^2 u_x) + u_{xx} + 2|u|^2 u &= g(x, t) \quad (2.19)
\end{align*}
and its conjugate for the choice
\begin{align*}
a_0 &= a_1 = 0, \\ a_2 &= -i, \\ a_3 &= 4i\epsilon, \\ q &= u, \\ r &= u^*, \\ B_1 &= \frac{iq}{2}, \\ C_1 &= \frac{iq^*}{2}.
\end{align*}
Proceeding further with the restrictions given above we find that (2.15)–(2.16) reduce into
\begin{align*}
g_x &= -2iub, \\ b_x &= i(u g^* - u^* g)
\end{align*}
where
\begin{align*}
A_1 &= \frac{ib}{2}, \\ A_2 &= 0, \\ B_2 &= 0, \\ C_2 &= 0.
\end{align*}
Note that the deformed Hirota equation (1.6) can be obtained by choosing $b(x, t) = \sqrt{c(t)^2 - gg^*}$ in (2.19)–(2.20) and admits Lax pair with Lax matrices $L$ and $M$ as
\begin{align*}
L &= \begin{pmatrix}
i\lambda & iu \\
ing & -i\lambda
\end{pmatrix}, \\
M &= \begin{pmatrix}4i\epsilon\lambda^3 - i\lambda^2 + \frac{iuu^*}{2} - 2i\epsilon\lambda uu^* + i\epsilon u u_x u_x \\
+ i(u u_x^* - u^* u_x) & 4i\epsilon\lambda^2 u + 2i\epsilon\lambda u_x - i\lambda u - i\epsilon u u_x \\
4i\epsilon\lambda^2 u^* - 2i\epsilon\lambda u_{x}^* - i\lambda u^* \\
+ i(u u_x^* - u^* u_x) & -4i\epsilon\lambda^2 + i\lambda^2 - \frac{iuu^*}{2} + 2i\epsilon\lambda uu^*
\end{pmatrix} + \frac{i}{2\lambda} \begin{pmatrix}b & g \\
g^* & -b
\end{pmatrix},
\end{align*}
satisfying the Lax equation (2.3).
Deformed AKNS Equation

\begin{align}
\frac{u_t}{u_{xx}} &= -u_{xx} + 2u^2v + \tilde{g}, \\
v_t &= v_{xx} - 2v^3u + h
\end{align}

for the choice \( a_0 = a_1 = a_3 = 0, \ a_2 = 2, \ q = u, \ r = v, \ B_1 = \frac{-\tilde{g}}{2}, \ C_1 = \frac{-h}{2} \).

Proceeding further with the restrictions given above we find that (2.15)–(2.16) reduce into

\begin{align}
\tilde{g}_x &= 2ub, \\
h_x &= 2vb, \\
b_x &= (ub + v\tilde{g})
\end{align}

where

\begin{align}
A_1 &= -\frac{ib}{2}, \quad A_2 = 0, \quad B_2 = 0, \quad C_2 = 0.
\end{align}

Note that the deformed AKNS equation (1.7) can be obtained by choosing \( b(x, t) = \sqrt{c(t)^2 + \tilde{g}^2} \) in (2.23) and admits Lax pair with Lax matrices

\begin{align}
L &= \begin{pmatrix}
i\lambda & iu \\
-iw & -i\lambda
\end{pmatrix}, \\
M &= \begin{pmatrix}
2\lambda^2 + uv & 2\lambda u - iu_x \\
-2\lambda v - iv_x & -2\lambda^2 - uv
\end{pmatrix} + \frac{i}{2\lambda} \begin{pmatrix}
-b & -\tilde{g} \\
-h & b
\end{pmatrix},
\end{align}

satisfying the Lax equation (2.3).

3. Painlevé Analysis of Deformed NLS Equation

We wish to report that each of the above identified deformed equations possesses the Painlevé property for PDEs [4, 19, 20]. In this section, we present the computational details for the deformed NLS equation while the details for the deformed Hirota and AKNS equation are given in the Appendix. We would like to mention that deformed NLS (1.5) can be written as

\begin{align}
&\begin{align}
&i u_t - u_{xx} - 2u^2u^* = g(x, t), \\
&b_x = i(au^* - a^*g), \quad g_x = -2ibu.
\end{align}
\end{align}

In order to extend the Painlevé analysis for PDEs we write the above equation as

\begin{align}
&\begin{align}
&U_t - V_{xx} - 2U^2V - 2V^3 = H, \\
&V_t + U_{xx} + 2U^3 + 2UV^2 = -G, \\
&G_x = 2VB, \\
&H_x = -2UB, \\
&B_x = 2UH - 2VG,
\end{align}
\end{align}

where \( u = U + iV, g = G + iH \). Obviously \( b = B \) is a real valued function.
It is well known that the Painlevé analysis for PDEs consists essentially of three steps [19]: (i) determination of the leading-order behavior of the Laurent series solution, (ii) identifying the resonances at which the arbitrary functions enter into the series and (iii) verifying that sufficient number of arbitrary functions exist without the introduction of movable critical singularity manifolds.

Let us assume that the leading order behavior of the solutions of (3.2) be

\[ U(x, t) \approx U_0 \phi^{\alpha_1}, \quad V(x, t) \approx V_0 \phi^{\alpha_2}, \quad G(x, t) \approx G_0 \phi^{\alpha_3}, \]

\[ H(x, t) \approx H_0 \phi^{\alpha_4}, \quad B(x, t) \approx B_0 \phi^{\alpha_5}, \]

where \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \) and \( \alpha_5 \) are negative integers to be determined and \( U_0, V_0, G_0, H_0, \) and \( B_0 \) are functions of \( (x, t) \) and \( \phi(x, t) \) is the singularity manifold. Substituting (3.3) in (3.2) and equating the most dominant terms we find

\[ \alpha_1 = \alpha_2 = -1, \quad \alpha_3 = \alpha_4 = \alpha_5 = -2 \]

and

\[ U_0^2 + V_0^2 = -\phi_x^2 \text{(repeated twice)} \Rightarrow \text{either } U_0 \text{ or } V_0 \text{ is arbitrary} \]

\[ G_0 = \frac{-V_0 B_0}{\phi_x}, \quad H_0 = \frac{U_0 B_0}{\phi_x}, \quad B_0(x, t) - \text{arbitrary}. \]

For finding the powers at which the arbitrary functions enter into the series solution, we substitute

\[ U(x, t) \approx U_0 \phi^{-1} + U_j \phi^{j-1}, \quad V(x, t) \approx V_0 \phi^{-1} + V_j \phi^{j-1}, \quad G(x, t) \approx G_0 \phi^{-2} + G_j \phi^{j-2}, \]

\[ H(x, t) \approx H_0 \phi^{-2} + H_j \phi^{j-2}, \quad B(x, t) \approx B_0 \phi^{-2} + B_j \phi^{j-2} \]

into the leading order terms of (3.2), and equating the lowest-order terms to zero we obtain a system of five equations linear in \((U_j, V_j, G_j, H_j, B_j)\). In matrix form it may be conveniently written as

\[
\begin{pmatrix}
4U_0 V_0 \\
4V_0^2 + (j^2 - 3j)\phi_x^2 \\
4U_0 \phi_x \\
0 \\
0 \\
0 \\
2B_0 \\
(2 - j)\phi_x \\
0 \\
2V_0 \\
0 \\
-2B_0 \\
0 \\
0 \\
2H_0 \\
-2G_0 \\
-2V_0 \\
2U_0 \\
(2 - j)\phi_x \\
0 \\
0
\end{pmatrix}
\begin{pmatrix}
U_1 \\
V_1 \\
G_1 \\
H_1 \\
B_1
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}
\]

(3.7)

Upon evaluation, Eq. (3.7) yields the following resonance values

\[ j = -1, 0, 0, 2, 3, 4, 4. \]

Obviously, the resonance value at \(-1\) represents the arbitrariness of the singularity manifold \( \phi(x, t) = 0 \).
To compute the arbitrary functions at the resonance values we now substitute the following Laurent series expansions, 

\[
U(x, t) = \sum_{j=0}^{4} U_j \phi^{j-1}, \quad V(x, t) = \sum_{j=0}^{4} V_j \phi^{j-1}, \quad G(x, t) = \sum_{j=0}^{4} G_j \phi^{j-2},
\]

\[
H(x, t) = \sum_{j=0}^{4} H_j \phi^{j-2}, \quad B(x, t) = \sum_{j=0}^{4} B_j \phi^{j-2}
\]

(3.9)

into (3.2). From the leading-order analysis it is clear that \( B_0(x, t) \) and either \( U_0(x, t) \) or \( V_0(x, t) \) are arbitrary corresponding to the resonance values 0 and 0. Equating the coefficients of \( \phi^{-2}, \phi^{-2}, \phi^{-2}, \phi^{-2} \) in (3.2) to zero, we obtain

\[
\begin{pmatrix}
4U_0V_0 & 4V_0^2 - 2\phi_x^2 & 0 & 0 & 0 \\
4V_0^2 - 2\phi_x^2 & 4U_0V_0 & 0 & 0 & 0 \\
0 & 2B_0 & \phi_x & 0 & 2V_0 \\
-2B_0 & 0 & 0 & \phi_x & -2U_0 \\
2H_0 & -2G_0 & -2V_0 & 2U_0 & \phi_x
\end{pmatrix}
\begin{pmatrix}
U_1 \\
V_1 \\
G_1 \\
H_1 \\
B_1
\end{pmatrix}
= \begin{pmatrix}
-U_0\phi_t + V_0\phi_{xx} + 2(V_0)\phi_x - H_0 \\
V_0\phi_t + U_0\phi_{xx} + 2(U_0)\phi_x - G_0 \\
G_{0x} \\
H_{0x} \\
B_{0x}
\end{pmatrix}.
\]

(3.10)

Solving the above equation (3.10), we obtain

\[
U_1(x, t) = \frac{1}{2\phi_x^2} (U_0\phi_{xx}\phi_x - V_0\phi_{xx}\phi_x - 2U_0\phi_x^2 - V_0B_0),
\]

\[
V_1(x, t) = \frac{1}{2\phi_x^2} (V_0\phi_{xx}\phi_x + U_0\phi_{xx}\phi_x - 2V_0\phi_x^2 + U_0B_0),
\]

\[
G_1(x, t) = \frac{1}{\phi_x^2} (-U_0B_0^2 + B_0V_0\phi_x^2 + V_0B_0\phi_x^2 - 2V_0B_0\phi_x\phi_{xx} - U_0B_0\phi_x\phi_x),
\]

\[
H_1(x, t) = -\frac{1}{\phi_x^2} (V_0B_0^2 + B_0U_0\phi_x^2 + U_0B_0\phi_x^2 - 2U_0B_0\phi_x\phi_{xx} + V_0B_0\phi_x\phi_x),
\]

\[
B_1(x, t) = \frac{1}{\phi_x^2} (B_{0x}\phi_x - B_0\phi_{xx})
\]

so that

\[
U_0U_1 + V_0V_1 = \frac{\phi_{xx}}{2}, \quad V_0G_1 - U_0H_1 = B_1\phi_x.
\]
Similarly, by equating the coefficients of $(\phi^{-1}, \phi^{-1}, \phi^{-1}, \phi^{-1})$ to zero, we obtain
\begin{align}
4U_0V_0U_2 + (4V_0^2 - 2\phi_x^2)V_2 &= U_{0x} - V_{0xx}, \\
-2V_1\phi_x - 2(U_1^2 + V_1^2)V_0 &= -H_1, \quad (3.11a) \\
(4U_0^2 - 2\phi_x^2)U_2 + 4U_0V_0V_2 &= -(V_0 + U_{0xx} + 2U_1\phi_{xx}) \\
+2(U_1^2 + V_1^2)U_0 + G_1 + 2V_0^2U_2, \quad (3.11b) \\
2V_0B_2 + 4V_2B_0 &= -2V_1B_1 + G_{1x}, \quad (3.11c) \\
2U_0B_2 + 2U_2B_0 &= -2U_1B_1 - H_{1x}, \quad (3.11d)
\end{align}

Solving Eqs. (3.11a) and (3.11b) we obtain
\begin{align}
U_2 &= \frac{1}{6\phi_x}[(2U_0V_0(-U_{0x} + V_{0xx} + 2V_1\phi_{xx} + H_1) - 2V_0^2(V_0 + U_{0xx} + 2U_1\phi_{xx} + G_1) \\
&+ \phi_x^2(V_0 + U_{0xx} + 2U_1\phi_{xx} + G_1 + 2(U_1^2 + V_1^2)U_0)] \\
V_2 &= \frac{1}{6\phi_x}[(2U_0V_0(V_0 + U_{0xx} + 2U_1\phi_{xx} + G_1) + 2U_0^2(U_0 - V_{0xx} - 2V_1\phi_{xx} - H_1) \\
- \phi_x^2(U_{0x} - V_{0xx} - 2V_1\phi_{xx} - H_1 - 2(U_1^2 + V_1^2)V_0)].
\end{align}

From Eqs. (3.11c) and (3.11d), we find
\begin{align}
B_2 &= \frac{1}{2V_0^2}(2V_2B_0 + 4V_2B_1 + G_{1x}) = -\frac{1}{2V_0^2}(2U_2B_0 + 2U_1B_1 + H_{1x}).
\end{align}

From Eq. (3.11e) we infer that either $G_2(x, t)$ or $H_2(x, t)$ is arbitrary corresponding to the resonance at $2$.

Now, equating the coefficients of $(\phi^0, \phi^0, \phi^0, \phi^0, \phi^0)$ in (3.2) to zero, we obtain the following equations
\begin{align}
4U_0U_3 + 4V_0V_3 &= \frac{1}{V_0}[-4(U_1V_0 + U_0V_1)U_2 - 8V_0V_1V_2 - 3\phi_{xx}V_2 - V_{1xx} - H_2 + U_{1t} \\
&+ U_2\phi_x - 2V_1\phi_x - 2(U_1^2 + V_1^2)V_1], \quad (3.12a) \\
4U_0U_3 + 4V_0V_3 &= \frac{1}{V_0}[(U_1V_0 + U_0V_1)V_2 + 3\phi_{xx}U_2 + U_{1xx} + G_2 + V_{1t} + V_2\phi_x \\
&+ 2U_2\phi_x + 2(U_1^2 + V_1^2)U_1], \quad (3.12b) \\
2V_0B_3 + 2B_0V_3 - \phi_2G_3 &= -2V_1B_2 - 2V_2B_1 + G_{2x}. \quad (3.12c)
\end{align}
4. Trilinear Representation of Deformed NLS Equation

To investigate whether or not the deformed NLS equation admits bilinear or trilinear representation, first we introduce a set of new dependent variables, that is,

\[ g \rightarrow l_x, \quad b \rightarrow k_x, \quad l_x = \frac{\partial l}{\partial x}, \quad k_x = \frac{\partial k}{\partial x} \]

and so the deformed NLS equation (3.1) becomes

\[
\begin{align*}
iv & - uu_x - 2u^2u^* = l_x, \\
kk & = i(u l_x - u^* k_x), \\
lk & = -2 i u k_x.
\end{align*}
\] (4.1)

The associated Painlevé expansions of Eq. (4.1) truncated up to constant term read

\[
\begin{align*}
U &= \frac{U_0}{\phi} + U_1, \\
V &= \frac{V_0}{\phi} + V_1, \\
l_R &= \frac{G_0}{\phi} + G_1, \\
l_I &= \frac{H_0}{\phi} + H_1, \\
k &= \frac{B_0}{\phi} + B_1.
\end{align*}
\] (4.2)

where \( l = l_R + il_I \). Without loss of generality, we consider the vacuum solution

\[ U_1 = V_1 = G_1 = H_1 = B_1 = 0. \]

Note that Eq. (4.2) can be rewritten as

\[
\begin{align*}
u &= \frac{R}{S}, \\
u^* &= \frac{R^*}{S}, \\
l &= \frac{Q}{S}, \\
l^* &= \frac{Q^*}{S} \quad \text{and} \quad k = \frac{T}{S}
\end{align*}
\] (4.3)

with \( S(x,t) \) and \( T(x,t) \) real, and \( Q(x,t) \) and \( R(x,t) \) are complex valued functions. Substituting (4.3) into (4.1) and rearranging the terms we find that Eq. (4.1) can be written into a trilinear form. They are

\[
\begin{align*}
(iD_t - D_R^2)R \cdot S &= D_Q Q \cdot S, \\
D_R^2 S \cdot S &= 2R R^*.
\end{align*}
\] (4.4a, 4.4b)
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\begin{align}
S(D_t^2 Q &- S) - Q(D_x^2 S - S) = -2iR(D_x T - S), \quad (4.4c) \\
S(D_t^2 T &- T(D_x^2 S - S) = i(R(D_x Q - S) - R'(D_x Q - S)), \quad (4.4d)
\end{align}

where \( D \) is the Hirota operator defined by [5]

\[ D^n_f g = (\partial/\partial t - \partial/\partial x^n)^m(\partial/\partial x^n - \partial/\partial x')^m f(x, t)g(x', t') \quad | x = x', t = t'. \]

Expanding the functions \( Q(x, t), R(x, t), S(x, t) \) and \( T(x, t) \) as power series in \( \epsilon \) that is

\[ Q = \sum_{n=1}^{\infty} Q_{2n-1} \epsilon^{2n-1}, \quad R = \sum_{n=1}^{\infty} R_{2n-1} \epsilon^{2n-1}, \]

\[ S = 1 + \sum_{n=1}^{\infty} S_{2n} \epsilon^{2n}, \quad T = T_0 + \sum_{n=1}^{\infty} T_{2n} \epsilon^{2n} \]

and using them in (4.1), one can construct the N-soliton solutions in the usual way. To obtain one-soliton solution, we consider

\[ Q = iQ_1, \quad R = \epsilon R_1, \quad S = 1 + \epsilon^2 S_2, \quad T = T_0 + \epsilon^2 T_2, \quad (4.5) \]

where \( \epsilon \) is an arbitrary small parameter. Substituting Eq. (4.5) in Eq. (4.4) and then equating different powers of \( \epsilon \) to zero yields an over determined system of linear PDEs. Solving them consistently yields

\[ R_1 = e^{\eta_1}, \quad S_2 = e^{\eta_1 + S_2}, \quad Q_1 = \frac{-2}{p_1^2} c(t) e^{\eta_1}, \]

\[ T_0 = c(t) x, \quad T_2 = \frac{(p_1 x - 4)}{4p_1^2} c(t) e^{\eta_1 + \eta_2} \]

where

\[ \eta_1 = p_1 x - i p_2 t - \frac{2}{p_1} \int c(t) dt + p_2, \quad e^A = \frac{1}{4p_1^2} \]

\( p_1 \) and \( p_2 \) are real constants and \( c(t) \) is an arbitrary function and so the one-soliton solution of Eq. (4.1) reads

\[ a(x, t) = \frac{R}{S} = \frac{e^{\eta_1}}{1 + e^{\eta_1 + \eta_2 + A}}, \]

\[ b(x, t) = \frac{Q}{S} = \frac{-2c(t) e^{\eta_1}}{p_1^2 (1 + e^{\eta_1 + \eta_2 + A})}, \]

\[ c(x, t) = \frac{T}{S} = \frac{c(t)(4p_1^2 x + (p_1 x - 4)e^{\eta_1 + \eta_2})}{4p_1^2 (1 + e^{\eta_1 + \eta_2 + A})} \]

The above form of one-soliton solution implies that the speed of the solution no longer has explicit relation with the amplitude as in the case of NLS equation without deformation.
5. Summary

A systematic investigation of certain higher order or deformed soliton equations with \((1+1)\) dimensions, from the point of complete integrability, is presented. Following the procedure of Ablowitz, Kaup, Newell and Segur (AKNS) we find that the deformed version of Nonlinear Schrödinger equation, Hirota equation and AKNS equation admit Lax pairs. It is shown that each of identified deformed equation possesses the Painlevé property for PDEs. Also we have shown by truncating the associated Painlevé expansions that each of the deformed equations admits trilinear representation. Hence they are completely integrable. The analysis shows that the deformation may cause variations in the speed of the soliton solutions.

Appendix: Painlevé Analysis of Deformed Hirota and AKNS Equations

A. Painlevé analysis of deformed Hirota equation

The deformed Hirota equation (1.6) can be written as

\[
\begin{align*}
    u_t + i\epsilon (u_{3x} + 6|u|^2 u_x) + 2u_x + |u|^2 u &= g(x,t) \quad \text{(A.1a)} \\
    b_x &= i(ug^* - u^*g), \quad g_x = -2ibu. \quad \text{(A.1b)}
\end{align*}
\]

In order to extend the Painlevé analysis for PDEs we rewrite the above equations as

\[
\begin{align*}
    U_t + \frac{V_{xx}}{2} + (U^2 + V^2) + 6\epsilon(U^2 + V^2)U_x + \epsilon U_{xxx} &= H, \quad \text{(A.2a)} \\
    V_t - \frac{U_{xx}}{2} - (U^2 + V^2) - 6\epsilon(U^2 + V^2)V_x + \epsilon V_{xxx} &= -G, \quad \text{(A.2b)} \\
    G_x &= 2VB, \quad \text{(A.2c)} \\
    H_x &= -2UB, \quad \text{(A.2d)} \\
    B_x &= 2UH - 2VG. \quad \text{(A.2e)}
\end{align*}
\]

where \(u(x,t) = U(x,t) + iV(x,t), \quad g(x,t) = G(x,t) + iH(x,t)\). Obviously \(B(x,t)\) is a real valued function. From the leading order behavior we obtain the following:

\[
\begin{align*}
    U(x,t) &\approx U_0\phi^{-1}, \quad V(x,t) \approx V_0\phi^{-1}, \quad G(x,t) \approx G_0\phi^{-2}, \quad H(x,t) \approx H_0\phi^{-2}, \quad B(x,t) \approx B_0\phi^{-2} \\
\end{align*}
\]

and

\[
\begin{align*}
    U_0^2 + V_0^2 &= -\phi^2 \quad \text{(repeated twice)} \quad G_0 = \frac{-V_0B_0}{\phi x}, \quad H_0 = \frac{U_0B_0}{\phi x}, \quad B_0 - \text{arbitrary.} \quad \text{(A.4)}
\end{align*}
\]

Similarly from the resonance analysis we obtain the following resonance values

\[
\begin{align*}
    j &= -1, 0, 0, 1, 2, 3, 4, 5.
\end{align*}
\]

and the resonance value at \(-1\) represents the arbitrariness of the singularity manifold \(\phi(x,t) = 0\), while the resonance 0, 0 are associated with the arbitrariness of \(U_0\) or \(V_0\) and \(B_0\).
To compute the arbitrary functions associated with the obtained resonance values we now introduce the following series expansions,

\[
U(x, t) = \sum_{j=0}^{5} U_j \phi^{j-1}, \quad V(x, t) = \sum_{j=0}^{5} V_j \phi^{j-1}, \quad G(x, t) = \sum_{j=0}^{5} G_j \phi^{j-2}, \\
H(x, t) = \sum_{j=0}^{5} H_j \phi^{j-2}, \quad B(x, t) = \sum_{j=0}^{5} B_j \phi^{j-2}
\]  

(A.5)

into Eq. (A.2). Now collecting the coefficients of \((\phi^{-3}, \phi^{-3}, \phi^{-2}, \phi^{-2}, \phi^{-2})\) in (A.2) to zero, we obtain

\[
\begin{pmatrix}
2\epsilon U_0 \phi_x & 2\epsilon V_0 \phi_x & 0 & 0 & 0 \\
2\epsilon U_0 \phi_x & 2\epsilon V_0 \phi_x & 0 & 0 & 0 \\
0 & 2B_0 & \phi_x & 0 & 2V_0 \\
2B_0 & 0 & 0 & -\phi_x & 2U_0 \\
-2H_0 & 2G_0 & 2V_0 & -2U_0 & -\phi_x
\end{pmatrix}
\begin{pmatrix}
U_1 \\
V_1 \\
G_1 \\
H_1 \\
B_1
\end{pmatrix}
= 
\begin{pmatrix}
\epsilon \phi_x \phi_{xx} \\
\epsilon \phi_x \phi_{xx} \\
G_{1xx} \\
H_{1xx} \\
B_{1xx}
\end{pmatrix}.
\]  

(A.6)

From Eqs. (A.6) we conclude that either \(U_1(x, t)\) or \(V_1(x, t)\) is arbitrary corresponding to the resonance value at 1.

Proceeding as before, we find that the functions \(G_2\) or \(H_2\), \(U_3\) or \(V_3\), \(U_4\) or \(V_4\), \(B_4\) and \(U_5\) or \(V_5\) are arbitrary corresponding to the resonance values at 2, 3, 4, 5. Thus the general solution of deformed Hirota (A.2) possesses the required number namely nine arbitrary functions without the introduction of movable critical manifolds. Thus, the deformed Hirota equation (A.2) possesses the Painlevé property for PDEs.

**B. Trilinear representation of deformed Hirota equation**

To investigate whether or not the deformed Hirota equation admits bilinear or trilinear representation, first we introduce a set of new dependent variables, that is, \(g \rightarrow l_x, \quad b \rightarrow k_x\) and so the deformed Hirota equation (A.1) becomes

\[
iu_t + i\epsilon(u_{3x} + 6|u|^2u_x) + \frac{u_{xx}}{2} + |u|^2u = l_x(x, t) \\
k_{xx} = i(u_x^2 - u^2 l_x)\]  

(B.1)

The associated Painlevé expansions of Eq. (B.1) truncated up to constant term reads

\[
U = \frac{U_0}{\phi} + U_1, \quad V = \frac{V_0}{\phi} + V_1, \quad l_R = \frac{G_0}{\phi} + G_1, \quad l_I = \frac{H_0}{\phi} + H_1, \quad k = \frac{B_0}{\phi} + B_1.
\]  

(B.2)

Without loss of generality, we consider the vacuum solution

\[
U_1 = V_1 = G_1 = H_1 = B_1 = 0.
\]
Note that Eq. (B.2) can be rewritten as

\[ u = \frac{R}{S}, \quad u^* = \frac{R^*}{S}, \quad l = \frac{Q}{S}, \quad l^* = \frac{Q^*}{S} \quad \text{and} \quad k = \frac{T}{S} \quad (B.3) \]

with \( S(x, t) \) and \( T(x, t) \) real, and \( Q(x, t) \) and \( R(x, t) \) are complex functions. Substituting (B.3) into (B.1) and rearranging the terms we find that Eq. (B.1) can be written into a tilinear form. They are

\[
\left( iD_t + \frac{1}{2}D_x^2 + \imath \epsilon D^3_x \right) R \cdot S = D_x Q \cdot S,
\]

\[
D_x^2 S \cdot S = 2RR^*,
\]

\[
S(D_x^2 Q \cdot S) - Q(D_x^2 S \cdot S) = -2iR(D_x T \cdot S),
\]

\[
S(D_x^2 T \cdot S) - T(D_x^2 S \cdot S) = i(R(D_x Q^* \cdot S) - R^*(D_x Q \cdot S)).
\]  

Hence we obtain the one-soliton solution of Eq. (B.1) as

\[
u(x, t) \approx \frac{R}{S} e^{\eta_1}, \quad l(x, t) \approx \frac{Q}{S} e^{\eta_1}, \quad l^*(x, t) \approx \frac{Q^*}{S} e^{\eta_1}, \quad k(x, t) \approx \frac{T}{S} e^{\eta_1},
\]

where

\[
\eta_1 = \frac{p_1 x + (\imath p_2 t + \epsilon)}{2} - \epsilon p_1^2 t^2 + \int c(t) dt^2, \quad \epsilon^2 = \frac{1}{4p^2_1},
\]

and \( c(t) \) is an arbitrary function.

C. Painlevé analysis of deformed AKNS equation (1.7)

The deformed AKNS equation (1.7) can be written as

\[
u_t = -u_{xx} + 2u^2 v + \dot{g}, \quad (C.1a)
\]

\[
v_t = -v_{xx} - 2v^2 u + h, \quad (C.1b)
\]

\[
\dot{g}_t = 2ab, \quad (C.1c)
\]

\[
h_x = 2\epsilon b, \quad (C.1d)
\]

\[
b_x = uh + vg, \quad (C.1e)
\]

From the leading order behavior we obtain the following:

\[
u(x, t) \approx u_0 \phi^{-1}, \quad v(x, t) \approx v_0 \phi^{-1}, \quad \dot{g}(x, t) \approx \dot{g}_0 \phi^{-2},
\]

\[
h(x, t) \approx h_0 \phi^{-2}, \quad b(x, t) \approx b_0 \phi^{-2} \quad (C.2)
\]
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and

\[ u_0 v_0 = \phi_x^2, \quad \tilde{g}_0 = \frac{b_0 u_0}{v_0}, \quad b_0 = \frac{\tilde{g}_0 v_0}{\phi_x} \]  

(C.3)

Similarly from the resonance analysis, we obtain the following resonance values

\[ j = -1, 0, 2, 3, 4. \]

Obviously, the resonance value at \(-1\) represents the arbitrariness of the singularity manifold \(\phi(x, t) = 0\), while the resonance 0 are associated with the arbitrariness of \(u_0\) or \(v_0\) and \(\tilde{g}_0\) or \(b_0\).

To compute the arbitrary functions associated with the obtained resonance values we now introduce the following series expansions,

\[ u(x, t) = \sum_{j=0}^{4} u_j \phi^{j-1}, \quad v(x, t) = \sum_{j=0}^{4} v_j \phi^{j-1}, \quad \tilde{g}(x, t) = \sum_{j=0}^{4} \tilde{g}_j \phi^{j-2}, \]

\[ h(x, t) = \sum_{j=0}^{4} h_j \phi^{j-2}, \quad b(x, t) = \sum_{j=0}^{4} b_j \phi^{j-2} \]

(C.4)

into Eq. (C.1). Equating the coefficients of \((\phi^{-2}, \phi^{-1}, \phi^{-1}, \phi^{-2}, \phi^{-2})\) in (C.1) to zero, we obtain

\[
\begin{pmatrix}
4\phi_x^2 & 2\phi_x^2 & 0 & 0 & 0 \\
2\phi_x^2 & 4\phi_x^2 & 0 & 0 & 0 \\
2\phi_x & 0 & \phi_x & 0 & 2u_0 \\
0 & 2\phi_x & 0 & \phi_x & 2v_0 \\
h_0 & \tilde{g}_0 & v_0 & u_0 & \phi_x
\end{pmatrix}
\begin{pmatrix}
u_1 \\
v_1 \\
\tilde{g}_1 \\
h_1 \\
h_1
\end{pmatrix} =
\begin{pmatrix}
-4u_0\phi_x - u_0\phi_{xx} - 2(u_0\phi_x)\phi_x - \tilde{g}_0 \\
v_0\phi_x - v_0\phi_{xx} - 2(v_0\phi_x)\phi_x + h_0 \\
\tilde{g}_0 \\
h_0 \\
h_0
\end{pmatrix}.
\]

(C.5)

Solving Eq. (C.5), we obtain the explicit values for \(u_1, v_1, \tilde{g}_1, h_1\) and \(b_1\). Proceeding as before, we find that \(\tilde{g}_2\) or \(b_2\), \(u_3\) or \(v_3\), \(u_4\) or \(v_4\) and \(h_4\) are arbitrary corresponding to the resonance values at \(j = 2, 3, 4, 4\). Thus the general solution of deformed AKNS (C.1) possesses the required number namely seven arbitrary functions with out the introduction of movable critical manifolds. Thus, the deformed AKNS equation (C.1) possesses the Painlevé property for PDEs.

D. Trilinear representation of deformed AKNS equation

To investigate whether or not the deformed AKNS equation admits bilinear or trilinear representation, first we introduce a set of new dependent variables, that is,

\[ u \rightarrow iu, \quad v \rightarrow iv, \quad \tilde{g} \rightarrow i\tilde{g}, \quad h \rightarrow im, \quad b \rightarrow k. \]

and so the deformed AKNS equation (D.1) becomes

\[ u_t + u_{xx} + 2u^2v = l, \]

\[ v_t - v_{xx} - 2v^2u = m, \]

where

\[ l = \sum_{j=0}^{4} l_j \phi^{j-2}, \quad m = \sum_{j=0}^{4} m_j \phi^{j-2}. \]
\[ l_{xx} = 2uk_x, \]
\[ m_{xx} = 2vk_x \]
\[ k_{xx} = -(um_x + vl_x). \]  
\( \text{(D.1)} \)

The associated Painlevé expansions of Eq. (D.1) truncated up to constant term reads
\[ u = u_0 + u_1, \quad v = v_0 + v_1, \quad l = \tilde{g}_0 + \tilde{g}_1, \quad m = h_0 + h_1, \quad k = b_0 + b_1. \]  
\( \text{(D.2)} \)

Without loss of generality, we consider the vacuum solution
\[ u_1 = v_1 = \tilde{g}_1 = h_1 = b_1 = 0. \]

Note that Eq. (D.2) can be rewritten as
\[ u = \frac{R}{S}, \quad v = \frac{P}{S}, \quad l = \frac{Q}{S}, \quad m = \frac{M}{S} \text{ and } k = \frac{T}{S} \]  
\( \text{(D.3)} \)

where \( P(x, t), Q(x, t), R(x, t), S(x, t), M(x, t) \) and \( T(x, t) \) are real. Substituting and rearranging the terms we find that Eq. (D.1) can be written into a trilinear form. They are
\[ (D_t + D_x^2)R \cdot S = D_x Q \cdot S, \]
\[ (D_t - D_x^2)P \cdot S = D_x M \cdot S, \]
\[ D_x^2 S \cdot S = 2RP, \]
\[ S(D_x^2 Q \cdot S) - Q(D_x^2 S \cdot S) = 2R(D_x T \cdot S), \]
\[ S(D_x^2 M \cdot S) - M(D_x^2 S \cdot S) = 2P(D_x T \cdot S), \]
\[ S(D_x^2 T \cdot S) - T(D_x^2 S \cdot S) = -(R(D_x M \cdot S) + P(D_x Q \cdot S)). \]  
\( \text{(D.4)} \)

Hence we obtain the one-soliton solution of Eq. (D.1) as
\[ u(x, t) = \frac{R}{S} e^{\eta_1}, \]
\[ v(x, t) = \frac{P}{S} e^{\eta_0}, \]
\[ l(x, t) = \frac{Q}{S} \frac{2(x(t)e^{\eta_0}}{p_1(1 + e^{\eta_1 + \eta_2 + \delta})}, \]
\[ m(x, t) = \frac{M}{S} \frac{2(x(t)e^{\eta_0}}{p_1(1 + e^{\eta_1 + \eta_2 + \delta})}, \]
\[ k(x, t) = \frac{T}{S} \frac{e(x(t)(4p_1 x^2 + (p_1 x - 4)e^{\eta_1 + \eta_2})}{4p_1(1 + e^{\eta_1 + \eta_2 + \delta})}. \]
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where

\[ \eta_1 = p_1 x - p_1^2 t + \frac{2}{p_1} \int c(t) dt + p_2, \]

\[ \eta_2 = p_1 x + p_1^2 t + \frac{2}{p_1} \int c(t) dt + p_2, \]

\[ e^A = \frac{1}{4p_1^2}, \]

and \( c(t) \) is an arbitrary function.

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References

[1] M. J. Ablowitz, D. J. Kaup, A. C. Newell and H. Segur, The inverse scattering transform — Fourier analysis for nonlinear problems, Stud. Appl. Math. 53 (1974) 249.

[2] M. J. Ablowitz and P. A. Clarkson, Solitons, Nonlinear Evolution Equations and Inverse Scattering (Cambridge University Press, Cambridge, 1991).

[3] M. J. Ablowitz, A. Ramani and H. Segur, A connection between nonlinear evolution equations and ordinary differential equations of P-type I, J. Math. Phys. 21 (1980) 715.

[4] R. Conte and M. Musette, The Painlevé Handbook (Springer, Dordrecht, 2008).

[5] R. Hirota, The Direct Method in Soliton Theory (Cambridge University Press, Cambridge, 2004).

[6] J. Ji, J.-B. Zhang and D.-J. Zhang, Soliton solutions for a negative order AKNS equation hierarchy, Commun. Theor. Phys. 52 (2009) 395.

[7] A. Karasu-Kalkanli, A. Karasu, A. Sakovich, S. Sakovich and R. Turhan, A new integrable generalization of the Korteweg-de-Vries equation, J. Math. Phys. 49 (2008) 073516.

[8] B. A. Kupershmidt, KdV6: An integrable system, Phys. Lett. A 372 (2008) 2634.

[9] A. Kundu, Exact accelerating solitons in Nonholonomic deformation of the KdV equation with two fold integrable hierarchy, J. Phys. A 41 (2008) 495201.

[10] A. Kundu, R. Sahadevan and L. Nalinidevi, Nonholonomic deformation of KdV and mKdV equations and their symmetries, hierarchies and integrability, J. Phys. A 42 (2009) 115213.

[11] A. Kundu, Nonlinearizing linear equations to integrable systems including new hierarchies with nonholonomic deformations, J. Math. Phys. 50 (2009) 102702.

[12] M. Lakshmanan and S. Rajasekar, Nonlinear Dynamics: Integrability, Chaos and Patterns (Springer, Berlin, 2003).

[13] R. Lin, Y. Zeng and W. X. Ma, Solving the KdV hierarchy with self-consistent sources by inverse scattering method, Physica A 291 (2001) 287.

[14] V. M. McLaughlin, Integration method of the Korteweg-de-Vries equation with self-consistent source, Phys. Lett. A 133 (1988) 493.

[15] A. Ramani, B. Grammaticos and R. Willox, Bilinearization and solutions of the KdV6 equations, Anal. Appl. 6 (2008) 401.

[16] R. Sahadevan and L. Nalinidevi, Similarity reduction, Nonlocal and Master symmetries of Sixth order Korteweg-de-Vries equation, J. Math. Phys. 42 (2000) 053505.

[17] Y. Shao and Y. Zeng, The solutions of the NLS equations with self-consistent sources, J. Phys. A 38 (2005) 2441.
[18] J. P. Wang, Extension of integrable equations, *J. Phys. A: Math. Theor.* **42** (2009) 362004.

[19] J. Weiss, M. Tabor and G. Carnevale, The Painlevé property for partial differential equation, *J. Math. Phys.* **24** (1983) 522.

[20] J. Weiss, The Painlevé property for partial differential equations, Backlund transformation, Lax pairs and the Schwarzian derivatives, *J. Math. Phys.* **24** (1983) 1405.

[21] Y. Yao and Y. Zeng, The bi-Hamiltonian structure and new solutions of KdV6 equation, *Lett. Math. Phys.* **86** (2008) 193.