A CLOSED ORIENTABLE 3-MANIFOLD WITH DISTINCT DISTANCE THREE GENUS TWO HEegaARD SPLITTINGS

JOHN BERGE

Abstract. We describe an example of a closed orientable 3-manifold $M$ with distinct distance three genus two Heegaard splittings. This demonstrates that the constructions of alternate genus two Heegaard splittings of closed orientable 3-manifolds described by Rubinstein and Scharlemann in their 1998 paper [RS], Genus Two Heegaard Splittings of Orientable 3-Manifolds, does not yield all alternate genus two splittings, and must therefore be augmented by the constructions described in [BS].

1. Introduction

The main result of this paper shows there are closed orientable 3-manifolds which have distinct distance three genus two Heegaard splittings. On the other hand, it is shown in [BS], which corrects an omission in the 1998 paper [RS] in which Rubinstein and Scharlemann applied their powerful method of sweepouts to the problem of classifying genus two Heegaard splittings of closed orientable 3-manifolds, that all of the constructions of potential alternate genus two Heegaard splittings described in [RS] yield splittings of distance no more than two. It follows that the constructions of [RS] must be augmented by those of [BS] in order to obtain all possible alternate genus two splittings of a closed orientable 3-manifold.

2. Preliminaries

A Heegaard splitting $(\Sigma; V, W)$, of genus $g$, of a closed orientable 3-manifold $M$ consists of a closed orientable surface $\Sigma$, of genus $g$, and two handlebodies $V$ and $W$, such that $\Sigma = \partial V = \partial W$, $V \cap W = \Sigma$, and $M = V \cup W$. A Heegaard splitting $(\Sigma; V, W)$ is reducible if there exists an essential separating simple closed curve in $\Sigma$ that bounds disks in both $V$ and $W$. A splitting $(\Sigma; V, W)$ is irreducible if it is not reducible. A set $v$ of pairwise disjoint disks in a handlebody $V$ is a complete set of cutting disks for $V$ if cutting $V$ open along the members of $v$ yields a 3-ball. If $v$ and $w$ are complete sets of cutting disks of $V$ and $W$ respectively, the set of simple closed curves $\partial v \cup \partial w$ in $\Sigma$ is a Heegaard diagram $(\Sigma; \partial v, \partial w)$ of $(\Sigma; V, W)$. The complexity $c(\Sigma; \partial v, \partial w)$ of $(\Sigma; \partial v, \partial w)$ is the number of points in $\partial v \cap \partial w$. (We always assume this number has been minimized by isotopies of $\partial v$ and $\partial w$ in $\Sigma$.)

The complete set of cutting disks $v$ of $V$ minimizes the complete set of cutting disks $w$ of $W$ if

$$c(\Sigma; \partial v, \partial w) \leq c(\Sigma; \partial v', \partial w),$$

for each complete set of cutting disks $v'$ of $V$.

The complete set of cutting disks $v$ of $V$ is a set of universal minimizers if for any complete sets of cutting disks $v'$ of $V$ and $w$ of $W$

$$c(\Sigma; \partial v, \partial w) \leq c(\Sigma; \partial v', \partial w).$$
A complete set of cutting disks $v$ of $V$ is a set of \textit{strict universal minimizers}, or set of SUMS, if for any complete sets of cutting disks $v'$ of $V$ and $w$ of $W$, with $v' \neq v$
\[ c(\Sigma; \partial v, \partial w) < c(\Sigma; \partial v', \partial w). \]

A Heegaard diagram $(\Sigma; \partial v, \partial w)$ is \textit{minimal} if
\[ c(\Sigma; \partial v, \partial w) \leq c(\Sigma; \partial v', \partial w') \]
whenever $v'$ and $w'$ are complete sets of cutting disks of $V$ and $W$ respectively. (Note that if the complete set of cutting disks $v$ of $V$ is a set of SUMS, and $(\Sigma; \partial v', \partial w)$ is minimal, then $v = v'$.)

Two Heegaard splittings $(\Sigma; V, W)$ and $(\Sigma'; V', W')$ of $M$ are \textit{homeomorphic} if there is a homeomorphism $h : M \rightarrow M$ such that $h(\Sigma) = \Sigma'$. Two Heegaard diagrams $(\Sigma; \partial v, \partial w)$ and $(\Sigma'; \partial v', \partial w')$ of $M$ are \textit{equivalent} if there is a homeomorphism $h : \Sigma \rightarrow \Sigma'$ such that $h(\partial v) = \partial v'$ and $h(\partial w) = \partial w'$. Note that if $(\Sigma; \partial v, \partial w)$ and $(\Sigma'; \partial v', \partial w')$ are equivalent Heegaard diagrams of $M$, then $(\Sigma; V, W)$ and $(\Sigma'; V', W')$ are equivalent Heegaard splittings of $M$.

Finally, two splittings $(\Sigma; V, W)$ and $(\Sigma'; V', W')$ of $M$ are \textit{isotopic} if there is an ambient isotopy of $M$ which carries $\Sigma$ to $\Sigma'$.

\textbf{Remark 2.1.} Methods for detecting the presence of a set of SUMS in one of the handlebodies of a genus two Heegaard splitting of a closed or orientable 3-manifold $M$, arguments that the existence of a set of SUMS is a generic condition among genus two Heegaard splittings, and applications of the existence of a detectable set of SUMS to the problem of determining all alternative genus two splittings of $M$ will appear elsewhere.

\textbf{Remark 2.2.} Note that we make extensive use of R-R diagrams. See [B1] for some background material on these.

\section{The forms of graphs underlying genus two Heegaard splittings.}

It will be helpful to have a list of the types of graphs which can underlie genus two Heegaard diagrams. Figure 1 displays the possible graphs. Lemmas 2.3 and 2.4 show these are the only possibilities. (We note a version of Lemma 2.3 appears in [HOT] where it is credited to [O].)

\textbf{Lemma 2.3.} If $W$ is a genus two handlebody with a complete set of cutting disks $\{D_S, D_T\}$, and $C$ is a set of disjoint essential simple closed curves in $\partial W$ such that each curve in $C$ has only essential intersections with $D_S$ and $D_T$, and no curve in $C$ is disjoint from both $D_S$ and $D_T$, then the Heegaard diagram of the curves in $C$ with respect to $\{D_S, D_T\}$ has a graph $G_C$ with the form of one of the three graphs in Figure 1.

\textbf{Proof.} It is easy to enumerate the possibilities here using the result of Lemma 2.3 which shows that all of the edges of $G_C$ which connect a vertex of $G_C$ corresponding to a given side of $D_S$ with a vertex of $G_C$ corresponding to a given side of $D_T$, must be parallel.

\textbf{Lemma 2.4.} Suppose the hypotheses of Lemma 2.3 hold. Then any two edges of $G_C$ connecting $S^+$ (resp $S^-$) to $T^+$ (resp $T^-$) must be parallel.

\textbf{Proof.} Suppose, to the contrary, that there are nonparallel edges in $G_C$ connecting, say, $S^+$ to $T^+$. Then it is easy to see that $G_C$ must have the form of one of the
three graphs in Figure 2 with $a > 0$ and $b > 0$. Next, if $v \in \{S^+, S^-, T^+, T^-\}$ is a
vertex of $G_C$, let $V(v)$ be the number of ends of edges of $G_C$ which meet $v$. Then,
if $G_C$ is a graph underlying a genus two Heegaard diagram, the equations $V(S^+) = V(S^-)$ and
$V(T^+) = V(T^-)$ must hold. However, one checks easily that these
equations do not hold in Figures 2a, 2b or 2c unless $a = b = 0$. □

**Figure 1.** If $W$ is a genus two handlebody with a complete set of
cutting disks $\{D_S, D_T\}$, and $C$ is a set of disjoint essential simple
closed curves in $\partial W$ such that each curve in $C$ has only essential
intersections with $D_S$ and $D_T$, and no curve in $C$ is disjoint from
both $D_S$ and $D_T$, then the Heegaard diagram of the curves in $C$
with respect to $\{D_S, D_T\}$ has a graph $G_C$ with the form of one of
these three graphs.

**Figure 2.** If $W$, $\{D_S, D_T\}$, and $C$ are as in Figure 1 then the
Heegaard diagram of the curves in $C$ with respect to $\{D_S, D_T\}$
does not have a graph $G_C$ with the form of one of the three graphs
in this figure unless $a = b = 0$.

### 2.2. The distance of a Heegaard splitting.

Suppose $\Sigma$ is a Heegaard surface in a closed orientable 3-manifold $M$ in which
$\Sigma$ bounds handlebodies $H$ and $H'$. The Heegaard splitting of $M$ by $\Sigma$ is a splitting
of *distance* $n$ if there is a sequence $c_0, \ldots, c_n$ of essential simple closed curves in $\Sigma$
such that:

1. $c_0$ bounds a disk in $H$;
2. $c_n$ bounds a disk in $H'$;
3. if $n > 0$, $c_i$ and $c_{i+1}$ are disjoint for $0 \leq i < n$;
(4) \( n \) is the smallest nonnegative integer such that (1), (2) and (3) hold.

A Heegaard splitting of distance 0 is reducible. A splitting of distance 1 is weakly reducible. Any Heegaard splitting of a reducible manifold is reducible. A Heegaard splitting of distance at most 2 has the disjoint curve property or DCP \([TH]\); i.e. there is an essential nonseparating simple closed curve in \( \Sigma \) which is disjoint from nonseparating properly embedded disks in both \( H \) and \( H' \). Any Heegaard splitting of a toroidal 3-manifold has the DCP \([He, TH]\). A weakly reducible genus two Heegaard splitting is also reducible, so an irreducible Heegaard splitting of genus two has distance at least two \([TH]\).

Remark 2.5. If a closed orientable 3-manifold has a Heegaard splitting with distance at least three, then it is irreducible, atoroidal, and it is not a Seifert manifold by Hempel \([He]\), so by Perelman’s proof of Thurston’s Geometrization Conjecture, the manifold is hyperbolic.

3. Alternate genus two Heegaard splittings and (SF,PP) pairs

Suppose \( H \) is a genus two handlebody, and \( \alpha \) is a nonseparating simple closed curve in \( \partial H \). The curve \( \alpha \) is Seifert Fiber or SF in \( H \) if attaching a 2-handle to \( H \) along \( \alpha \) yields an orientable Seifert fibered space over the disk \( D^2 \) with 2 exceptional fibers. A nonseparating simple closed curve \( \beta \) in \( \partial H \) is primitive in \( H \) if there exists a disk \( D \) in \( H \) such that \( |\beta \cap D| = 1 \). Equivalently \( \beta \) is conjugate to a free generator of \( \pi_1(H) \). The curve \( \beta \) is a proper power or PP in \( H \) if \( \beta \) is disjoint from a separating disk in \( H \), \( \beta \) does not bound a disk in \( H \), and \( \beta \) is not primitive in \( H \). A pair of disjoint nonseparating simple closed curves \( (\alpha, \beta) \) in the boundary of a genus two handlebody \( H \) is a \( (\text{Seifert Fiber, Proper Power}) \) pair, or \( (\text{SF,PP}) \) pair if attaching a 2-handle to \( H \) along \( \alpha \) yields an orientable Seifert fibered space over the disk \( D^2 \) with 2 exceptional fibers, and \( \beta \) is a proper power of a free generator of \( \pi_1(H) \).

Remark 3.1. Suppose \( H \) is a genus two handlebody. Due to the work of Zieschang and others, nonseparating simple closed curves in \( \partial H \) which are SF curves are completely understood. (See the expository paper \([Z2]\) of Zieschang and its excellent bibliography.) Using this classification, it is not hard to show that if \( \alpha \) is SF in \( \partial H \), then there exists a nonseparating curve \( \beta \) disjoint from \( \alpha \) such that \( (\alpha, \beta) \) is a \( (\text{SF,PP}) \) pair in \( \partial H \), and the pair \( (\alpha, \beta) \) has an R-R diagram with the form of Figure 3a or 3b.

The following theorem explains our interest in \( (\text{SF,PP}) \) pairs.

Theorem 3.2. Suppose \( \Sigma \) is a genus two Heegaard surface bounding handlebodies \( H \) and \( H' \) in a closed orientable 3-manifold \( M \). If \( \alpha \) and \( \beta \) are disjoint nonseparating simple closed curves in \( \Sigma \) such that \( (\alpha, \beta) \) is a \( (\text{SF,PP}) \) pair in \( H \) and a \( (\text{PP,SF}) \) pair in \( H' \), then \( M \) has an alternative genus two Heegaard surface \( \Sigma' \), which is not obviously isotopic to \( \Sigma \).

Proof. Let \( N_\alpha \) and \( N_\beta \) be disjoint regular neighborhoods of \( \alpha \) and \( \beta \) respectively in \( \Sigma \). Since \( \beta \) is a proper power in \( H \), the boundary components of \( N_\beta \) bound an essential separating annulus \( A_\beta \) in \( H \), and cutting \( H \) open along \( A_\beta \) cuts \( H \) into a genus two handlebody \( H_\beta \) and a solid torus \( V_\beta \). Similarly, since \( \alpha \) is a proper power in \( H' \), the boundary components of \( N_\alpha \) bound an essential separating annulus \( A_\alpha \).
If $(\alpha, \beta)$ is a $(SF, PP)$ pair on the boundary of a genus two handlebody $H$, then $\alpha$ and $\beta$ have an R-R diagram on $\partial H$ with the form of Figure 3a or the form of Figure 3b. Here, $|P|, |S| > 1$, $a, b > 0$, and $\gcd(a, b) = 1$.

Then $N_\alpha$ lies in $\partial H_\beta$, and $\alpha$ is primitive in $H_\beta$. And similarly, $N_\beta$ lies in $\partial H'_\alpha$, and $\beta$ is primitive in $H'_\alpha$. To see that $\alpha$ is primitive in $H_\beta$, let $H[\alpha]$ denote the manifold obtained by adding a 2-handle to $H$ along $\alpha$. By hypothesis, $H[\alpha]$ is Seifert fibered over $D^2$ with two exceptional fibers. The annulus $A_\beta$ is an essential separating annulus in $H[\alpha]$, which must be vertical in the Seifert fibration of $H[\alpha]$. This implies that $H_\beta[\alpha]$ is Seifert fibered over the disk $D^2$ with one exceptional fiber. So $H_\beta[\alpha]$ is a solid torus. This can occur only if $\alpha$ is primitive in $H_\beta$. Similarly, $\beta$ is primitive in $H'_\alpha$.

Returning to the main argument, $N_\alpha$ lies in $\partial V'_\alpha$, and $N_\beta$ lies in $\partial V_\beta$. It follows that $H'_\alpha \cup_{N_\beta} V_\beta$ and $H_\beta \cup_{N_\alpha} V'_\alpha$ are each genus two handlebodies. Their common boundary is then an alternative genus two Heegaard surface $\Sigma'$ for $M$. □

Remark 3.3. The type of alternative genus two Heegaard splittings described in Theorem 3.2 which arise from $(\alpha, \beta)$ pairs in genus two Heegaard surfaces that are $(SF, PP)$ pairs in one of the handlebodies bounded by the surface and $(PP, SF)$ pairs in the other handlebody bounded by the surface, are exactly those which were overlooked in the classification [RS].
4. An example of distinct distance three genus two splittings

**Theorem 4.1.** The R-R diagrams in Figures 4 and 5 represent distinct genus two Heegaard splittings, each of distance 3, of a closed orientable 3-manifold $M$.

**Proof.** Corollary 4.6 shows the Heegaard splittings described by the R-R diagrams in Figures 4 and 5 are each distance three splittings, while Section 5 shows the splittings are both splittings of the same closed orientable 3-manifold $M$. Then Proposition 5.9 finishes the proof by showing the splittings of $M$ described by the R-R diagrams in Figures 4 and 5 are not homeomorphic.

□
Proofs of the results leading to Theorem 4.1 occupy the bulk of the remainder of the paper. However, we start with two preliminary subsections. The first of these, Subsection 4.1, explains the R-R diagrams used in the remainder of the paper, such as Figures 4 and 5. The second subsection, Subsection 4.2, describes certain rectangles in the Heegaard surfaces of Figures 4 and 5, which are later used to show that sets of SUMS exist.

4.1. A word about the R-R diagrams in Figures 4 and 5

The R-R diagrams appearing in this and following sections differ slightly from those described in [B1], and those appearing in previous sections. This subsection aims to explain these diagrams.

Suppose Σ is a genus two Heegaard surface bounding handlebodies $H$ and $H'$ in a closed orientable 3-manifold $M$, $\{D_A, D_B\}$ and $\{D_X, D_Y\}$ are complete sets of cutting disks of $H$ and $H'$ respectively, and $\Gamma$ is an essential separating simple closed curve in $\Sigma$ disjoint from $\partial D_A$ and $\partial D_B$. (Thus $\Gamma$ bounds a disk in $H$ which separates $D_A$ and $D_B$.) In addition, suppose $\partial D_X$ and $\partial D_Y$ have only essential intersections with $\partial D_A$, $\partial D_B$ and $\Gamma$, and both $D_X \cap \Gamma$ and $D_Y \cap \Gamma$ are nonempty. Finally, let $A$ be a regular neighborhood of $\Gamma$ in $\Sigma$ chosen so that $A$ is disjoint from $\partial D_A$ and $\partial D_B$, and let $F_A$ and $F_B$ be the two once-punctured tori components of $\Sigma - \text{int}(A)$ with $F_A$ and $F_B$ labeled so that $\partial D_A \subset F_A$ and $\partial D_B \subset F_B$.

Next, let $S$, $C_A$, and $C_B$ be the sets of arcs $(\partial D_X \cup \partial D_Y) \cap A$, $(\partial D_X \cup \partial D_Y) \cap F_A$, and $(\partial D_X \cup \partial D_Y) \cap F_B$ respectively. Then R-R diagrams like Figures 4 and 5 display the annulus $A$, minus a point at infinity together with the arcs of $S$ embedded in the plane $\mathbb{R}^2$ as the closure of the complement of a pair of disjoint hexagons $H_A$ and $H_B$.

This results in the identification of the boundaries $\partial F_A$ and $\partial F_B$ of the once-punctured tori $F_A$ and $F_B$ with $\partial H_A$ and $\partial H_B$ respectively. Let $G \in \{A, B\}$, and
let $p$ and $q$ be two points of $(\partial D_X \cup \partial D_Y) \cap \partial F_G$. Then the identification of $\partial F_G$ with $\partial H_G$ has the following properties:

- The points $p$ and $q$ lie in the same face of $H_G$ if and only if $p$ and $q$ are endpoints of connections $\delta_p$ and $\delta_q$ respectively in $C_G$ such that $\delta_p$ and $\delta_q$ are properly isotopic in $F_G$ under an isotopy that carries $p$ to $q$.

- The points $p$ and $q$ lie in opposite faces of $H_G$ if and only if $p$ and $q$ are endpoints of connections $\delta_p$ and $\delta_q$ respectively in $C_G$ such that $\delta_p$ and $\delta_q$ are properly isotopic in $F_G$ under an isotopy that does not carry $p$ to $q$.

- Suppose $p$ and $q$ lie in opposite faces of $H_G$, and let $f_p$ and $f_q$ be the faces of $H_G$ containing $p$ and $q$ respectively. Then $|S \cap f_p| = |S \cap f_q| = n$, for some nonnegative integer $n$, and there exists a unique set $\Delta$ of $n$ disjoint properly embedded arcs in $H_G$ such that each member of $\Delta$ connects a point of $S \cap f_p$ to a point of $S \cap f_q$. Then $p$ and $q$ are endpoints of a connection in $C_G$ if and only if $p$ and $q$ are connected by an arc in $\Delta$.

Once the proper isotopy classes of the connections $C_G$ in $F_G$ have been determined as above, the only remaining problem is to specify the isotopy class of the simple closed curve $\partial D_G$ in $F_G$. We do this by putting a set of three integer labels next to three consecutive faces of $H_G$: so that one member of each pair of opposite faces of $H_G$ is labeled, and we interpret these integers as algebraic intersection numbers of oriented connections with an oriented simple closed curve $\partial D_G$. This is enough to completely specify the isotopy class of $\partial D_G$ in $F_G$. We note that, a priori, the three consecutive labels can be any 3-triple of integers of the form $(m, m + n, n)$ with $\gcd(m, n) = 1$; so that only two labels would suffice. However, three labels are often convenient.

Finally, suppose $f_A$ and $f_B$ are faces of $H_A$ and $H_B$ respectively, and let $S' \subset S$ be the set of arcs in $S$ which connect points in $f_A$ to points in $f_B$. Then the arcs in $S$ are parallel in $A$, and in order to reduce the number of arcs which are displayed in an R-R diagram like Figure 4 or 5, we often group the arcs of $S'$ together with brackets, which are then connected by a single arc in $A$.

(Figure 7 illustrates most of the points mentioned above.)

4.2. Rectangles in $\Sigma$.

The proofs that each of the Heegaard splittings described by the R-R diagrams of Figures 3 and 5 is a distance 3 splitting, and that the splittings of Figures 4 and 5 are not homeomorphic, depends on the existence of certain rectangles in the genus two Heegaard surfaces of these diagrams. This subsection describes the rectangles we need.

Suppose $\Sigma$ is a genus two Heegaard surface in a closed orientable 3-manifold $M$ such that $\Sigma$ bounds genus two handlebodies $H$ and $H'$, $\{D_A, D_B\}$ and $\{D_X, D_Y\}$ are complete sets of cutting disks of $H$ and $H'$ respectively, and $\Gamma$ is an essential separating curve in $\Sigma$ which bounds a disk in $H$ separating $D_A$ and $D_B$.

Assuming, as we may, that $\partial D_X$ and $\partial D_Y$ have only essential intersections with $\partial D_A$, $\partial D_B$ and $\Gamma$, suppose $\Gamma$ intersects both $\partial D_X$ and $\partial D_Y$. Then the curves $\Gamma$, $\partial D_X$ and $\partial D_Y$, cut $\Sigma$ into sets of faces, which are either four-sided, i.e. rectangles, or have more than four sides. Let $R$ denote the set of rectangles cut from $\Sigma$ by $\Gamma$, $\partial D_X$ and $\partial D_Y$. We are interested in four subsets of $R$, which we denote by $R_{ax}$, $R_{ay}$, $R_{bx}$, and $R_{by}$. The meaning of the subscripts of these sets is as follows: The first letter of the 2-letter subscript $pq$ of the subset $R_{pq}$ of $R$ is $a$ (resp $b$) if
Figure 7. Figures illustrating how the identification of $\partial F_A$ with the boundary of a hexagon in Figure 7a, together with the set of three integers placed near three consecutive faces of the hexagon in Figure 7a, encodes the embedding of the simple closed curve $\partial D_A$ and the connections of $(\partial D_X \cup \partial D_Y) \cap F_A$ in $F_A$ shown in Figure 7c. Here Figure 7c shows the once-punctured torus $F_A$ cut open along a pair of essential arcs parallel to connections in $(\partial D_X \cup \partial D_Y) \cap F_A$.

And Figure 7b shows how points on opposite faces of the hexagon in Figure 7a are joined by connections in $F_A$.

each rectangle in $R_{pq}$ lies on the same side of $\Gamma$ in $\Sigma$ as $\partial D_A$ (resp $\partial D_B$). The second letter of the 2-letter subscript $pq$ of the subset $R_{pq}$ of $R$ is $x$ (resp $y$) if each rectangle in $R_{pq}$ has two subarcs of $\partial D_X$ (resp $\partial D_Y$) in its boundary. In addition, each rectangle $R_{pq}$ in $R_{pq}$ intersects $\partial D_P$ in a number of essential arcs. In each such case, let $|R_{pq}|$ be the number of essential arcs in $R_{pq} \cap \partial D_P$. (It is possible that an $R_{pq}$ is empty.)

Finally, we mention that a rectangle $R_{pq}$ with $p \in \{a, b\}$, $q \in \{x, y\}$, and $|R_{pq}| = e$ exists in the Heegaard surface of the R-R diagram in Figure 4 (resp Figure 5), if and only if $\partial D_Q$ intersects the face of the $P$-hexagon in Figure 4 (resp Figure 5), with label $e$ in two adjacent points.

**Theorem 4.2.** Suppose that for each $R_{pq} \in \{R_{ax}, R_{ay}, R_{bx}, R_{by}\}$ there exist rectangles $R_{pq1} \in R_{pq}$ and $R_{pq2} \in R_{pq}$ such that $|R_{pq1}| - 1 > |R_{pq2}| > 1$. Then the set of cutting disks $\{D_A, D_B\}$ of $H$ is a set of SUMS, and the Heegaard splitting has no disjoint curves.

**Proof.** Suppose $C_1$ and $C_2$ are a pair of disjoint nonseparating simple closed curves in $\Sigma$ such that $C_1$ bounds a disk in $H'$. We may assume $C_1$ and $C_2$ have only essential intersections with $\partial D_X$, $\partial D_Y$, $\partial D_A$, $\partial D_B$, and $\Gamma$.

Consider the curve $C_1$. One possibility is that $C_1 \cap (\partial D_X \cup \partial D_Y)$ is nonempty. Suppose this is the case. Then Lemma 4.3 shows the graph $G(D_X, D_Y \mid C_1)$ of the Heegaard diagram of $C_1$ with respect to $D_X$ and $D_Y$ has the form of Figure 11 with $b > 0$, and with the pairs of vertices $\{S^+, S^-\}$ and $\{T^+, T^-\}$ of Figure 11 replaced by $\{X^+, X^-\}$ and $\{Y^+, Y^-\}$.

Observe that if $\{S^+, S^-\} = \{X^+, X^-\}$ and $\{T^+, T^-\} = \{Y^+, Y^-\}$ in Figure 11, then $C_1$ intersects every rectangle in $R_{ax} \cup R_{by}$ in an essential arc. On the other hand, if $\{S^+, S^-\} = \{Y^+, Y^-\}$ and $\{T^+, T^-\} = \{X^+, X^-\}$ in Figure 11, then $C_1$ intersects every rectangle in $R_{ax} \cup R_{bx}$ in an essential arc.
The remaining possibility is that \( C_1 \) is disjoint from \( \partial D_X \cup \partial D_Y \). In this case, either (2) or (3) of Lemma 4.3 applies. If \( C_1 \) is isotopic to \( \partial D_X \), then \( C_1 \) intersects every rectangle in \( R_{ax} \cup R_{bx} \) in an essential arc. If \( C_1 \) is isotopic to \( \partial D_Y \), then \( C_1 \) intersects every rectangle in \( R_{ay} \cup R_{by} \) in an essential arc. If \( C_1 \) is not isotopic to \( \partial D_X \) or \( \partial D_Y \), then, since \( C_1 \) is nonseparating in \( \Sigma \), it separates \( X^+ \) from \( X^- \) and \( Y^+ \) from \( Y^- \) in the graph \( G(D_X, D_Y \mid C_1) \) of the Heegaard diagram of \( C_1 \) with respect to \( D_X \) and \( D_Y \). So, in this case, \( C_1 \) intersects every rectangle in \( R_{ay} \cup R_{by} \), and every rectangle in \( R_{ax} \cup R_{bx} \) in an essential arc.

Next we turn attention to the \( H \) side of \( \Sigma \). Here the simple closed curve \( \Gamma \) cuts \( \Sigma \) into two once-punctured tori, \( F^+ \) and \( F^- \) such that \( \partial D_A \subset F_A \subset F^+ \), \( \partial D_B \subset F_B \subset F^- \), and \( \partial F^+_A = \partial F^-_B = \Gamma \).

Let \( P \) be either \( A \) or \( B \), and consider the once-punctured torus \( F^+_P \). We have just observed that either \( C_1 \) intersects every rectangle in \( R_{ax} \cup R_{bx} \) in an essential arc, or \( C_1 \) intersects every rectangle in \( R_{ay} \cup R_{by} \) in an essential arc. If \( C_1 \) intersects every rectangle in \( R_{ax} \cup R_{bx} \) in an essential arc, let \( q = x \); otherwise, let \( q = y \). In either case, the hypothesis of Theorem 4.2 guarantees there exists a pair of rectangles \( R_{pq_1} \in R_{pq} \) and \( R_{pq_2} \in R_{pq} \) such that \( |R_{pq_1}| - 1 > |R_{pq_2}| > 1 \). Let \( m = |R_{pq_1}| \), and let \( n = |R_{pq_2}| \). Then \( m - 1 > n > 1 \), and the configuration of \( R_{pq_1} \), \( R_{pq_2} \) and \( \partial D_P \) in \( F^+_P \) must be homeomorphic to that shown in Figure 8. Next, consider the set of connections \( C_1 \cap F^+_P \) and observe that, since \( C_1 \) intersects both \( R_{pq_1} \) and \( R_{pq_2} \) in essential arcs, there exist connections \( \omega_1 \) and \( \omega_2 \) in \( C_1 \cap F^+_P \) such that \( \omega_1 \subset R_{pq_1} \) and \( \omega_2 \subset R_{pq_2} \).

It is time to consider \( C_2 \). Note first that, because \( C_1 \) and \( C_2 \) are disjoint, and both \( C_1 \cap F^+_A \) and \( C_1 \cap F^-_B \) contain nonisotopic pairs of connections, \( C_2 \) can not lie completely in \( F^+_A \) or \( F^-_B \). So \( C_2 \cap F^+_P \) is a set of connections in \( F^+_P \). It follows that if \( \delta \) is any connection in \( C_2 \cap F^+_P \), then \( \delta \) is properly isotopic to \( F^+_P \) to one of the four connections \( \delta_1, \delta_2, \delta_3, \delta_4 \) shown in Figure 8. Then, since \( m - 1 > n > 1, \delta_1, \delta_2, \delta_3, \delta_4 \) intersect \( \partial D_P \) respectively \( m > 3, n > 1, m + n > 4 \) and \( m - n > 1 \) times. In particular, each connection \( \delta \in C_2 \cap F^+_P \) satisfies \( |\delta \cap \partial D_P| \geq 2 \).

**Claim 1.** There are no disjoint curves in \( \Sigma \).

**Claim 2.** The set of cutting disks \( \{D_A, D_B\} \) of \( H \) is a set of SUMS.

**Proof of Claim 1.** Since \( C_2 \) is an arbitrary nonseparating simple closed curve in \( \Sigma \) disjoint from a disk in \( H' \), it is enough to show that \( C_2 \) has essential intersections with every cutting disk of \( H \). Lemma 4.3 shows that if \( C_2 \) is disjoint from a disk in \( H \), then the graph \( G(D_A, D_B \mid C_2) \) of the Heegaard diagram of \( C_2 \) with respect to \( D_A \) and \( D_B \) either has no edges connecting \( A^+ \) to \( A^- \), or it has no edges connecting \( B^+ \) to \( B^- \).

But, this is not the case. Since \( C_2 \) does not lie completely in \( F^+_A \), or completely in \( F^-_B \), there is a connection \( \delta_A \in C_2 \cap F^+_A \) such that \( |\delta_A \cap \partial D_A| \geq 2 \), and there is a connection \( \delta_B \in C_2 \cap F^-_B \) such that \( |\delta_B \cap \partial D_B| \geq 2 \). This implies there exist edges in \( G(D_A, D_B \mid C_2) \) connecting \( A^+ \) to \( A^- \), and there exist edges in \( G(D_A, D_B \mid C_2) \) connecting \( B^+ \) to \( B^- \). It follows that \( C_2 \) is not disjoint from a disk in \( H \), and so there are no disjoint curves in \( \Sigma \). This proves Claim 1.

**Proof of Claim 2.** Now suppose that in addition to being disjoint from \( C_1 \), \( C_2 \) bounds a cutting disk of \( H' \). Then we may assume \( C_1 \) and \( C_2 \) bound an arbitrary complete set of cutting disks of \( H' \). And then, by Lemma 4.3, the complete set of
cutting disks \( \{D_A, D_B\} \) of \( H \) will be a set of SUMS provided we can show that the graph \( G(D_A, D_B \mid C_1, C_2) \) of the Heegaard diagram of \( C_1 \) and \( C_2 \) with respect to \( D_A \) and \( D_B \) has the form of Figure 11 with \( c > a + b > 0 \) and \( d > a + b > 0 \). We proceed to do this.

It was shown in the proof of Claim 1 that \( G(D_A, D_B \mid C_2) \) has edges connecting \( A^+ \) to \( A^- \) and edges connecting \( B^+ \) to \( B^- \). So, since \( G(D_A, D_B \mid C_2) \) is a subgraph of \( G(D_A, D_B \mid C_1, C_2) \), the graph \( G(D_A, D_B \mid C_1, C_2) \) also has edges connecting \( A^+ \) to \( A^- \) and edges connecting \( B^+ \) to \( B^- \). This implies \( G(D_A, D_B \mid C_1, C_2) \) has the form of Figure 11.

It remains to establish that \( c > a + b > 0 \) and \( d > a + b > 0 \) in Figure 11. To see this, observe that \( a + b \) in Figure 11 is equal to the number of connections in \( (C_1 \cup C_2) \cap F_P^+ \). Also observe that any connection in \( C_1 \cap F_P^+ \) is properly isotopic in \( F_P^+ \) to one of the four connections \( \delta_1, \delta_2, \delta_3, \delta_4 \) shown in Figure 8. Therefore, since \( m - 1 > n > 1, \delta_1, \delta_2, \delta_3 \) and \( \delta_4 \) intersect \( \partial D_P \) respectively \( m > 3, n > 1, m + n > 4 \) and \( m - n > 1 \) times. In particular, each connection \( \delta \in C_1 \cap F_P^+ \) satisfies \( |\delta \cap \partial D_P| \geq 2 \). Since each connection \( \delta \in C_2 \cap F_P^+ \) also satisfies \( |\delta \cap \partial D_P| \geq 2 \), we have \( c > a + b > 0 \) and \( d > a + b > 0 \). However, there is also a connection \( \omega_1 \in C_1 \cap F_P^+ \) with \( |\omega_1 \cap \partial D_P| > 2 \). This implies \( c > a + b > 0 \) and \( d > a + b > 0 \), which is what we need. This proves Claim 2.

And also completes the proof of Theorem 4.2.

Lemma 4.3. Suppose \( W \) is a genus two handlebody with a complete set of cutting disks \( \{D_S, D_T\}, C_1 \) and \( C_2 \) are a pair of disjoint nonseparating simple closed curves in \( \partial W \) such that \( C_1 \) and \( C_2 \) have only essential intersections with \( D_S \) and \( D_T \), and \( C_1 \) bounds a disk in \( W \). Let \( G(D_S, D_T \mid C_1) \) and \( G(D_S, D_T \mid C_2) \) be the graphs of the Heegaard diagrams of \( C_1 \) and \( C_2 \) respectively with respect to \( D_S \) and \( D_T \). Then either:

1. \( G(D_S, D_T \mid C_1) \) has the form of Figure 11: with \( b > 0 \);
2. \( C_1 \) is isotopic to \( \partial D_S \) or \( \partial D_T \);
3. \( C_1 \) is a bandsram of \( \partial D_S \) with \( \partial D_T \), so \( C_1 \) appears as a simple closed curve which separates vertex \( S^+ \) from vertex \( S^- \) and vertex \( T^+ \) from vertex \( T^- \) in \( G(D_S, D_T \mid C_1) \).

In any case, either there are no edges in \( G(D_S, D_T \mid C_2) \) connecting \( S^+ \) to \( S^- \), or there are no edges in \( G(D_S, D_T \mid C_2) \) connecting \( T^+ \) to \( T^- \).

Proof. Let \( D_C \) be the disk which \( C_1 \) bounds in \( W \), and suppose \( C_1 \) has essential intersections with \( \partial D_S \cup \partial D_T \). Then \( D_C \cap (D_S \cup D_T) \) is nonempty, and we may assume \( D_C \cap (D_S \cup D_T) \) consists only of arcs of intersection. So there is an arc of intersection in \( D_C \cap (D_S \cup D_T) \) which cuts off an outermost subdisk \( D \) of \( D_C \). Let \( \omega \) be the arc \( \partial D \cap C_1 \). Then \( \omega \) has both of its endpoints at the same vertex of \( G(D_S, D_T \mid C_1) \). It follows that, in this case, \( G(D_S, D_T \mid C_1) \) has the form of Figure 11 with \( b > 0 \). On the other hand, if \( C_1 \) does not have essential intersections with \( \partial D_S \cup \partial D_T \), it is clear that (2) or (3) holds.

Turning to the form of \( G(D_S, D_T \mid C_2) \), we see that if \( C_1 \) has essential intersections with \( \partial D_S \cup \partial D_T \), and \( \omega \) has both endpoints at \( S^+ \) or \( S^- \) (resp \( T^+ \) or \( T^- \)), then there are no edges in \( G(D_S, D_T \mid C_2) \) connecting \( T^+ \) to \( T^- \) (resp \( S^+ \) to \( S^- \)).

On the other hand, if \( C_1 \) does not have essential intersections with \( \partial D_S \cup \partial D_T \), so (2) or (3) holds, then again, either there are no edges in \( G(D_S, D_T \mid C_2) \) connecting \( S^+ \) to \( S^- \), or there are no edges in \( G(D_S, D_T \mid C_2) \) connecting \( T^+ \) to \( T^- \).
Lemma 4.4. Suppose $W$ is a genus two handlebody with a complete set of cutting disks $\{D_S, D_T\}$, and $\{C_1, C_2\}$ is a pair of disjoint essential simple closed curves in $\partial W$ such that the graph $G(D_S, D_T | C_1, C_2)$ of the Heegaard diagram of $C_1$ and $C_2$ with respect to $D_S$ and $D_T$ has the form of Figure 1, with $c > a + b > 0$ and $d > a + b > 0$. Then the set of cutting disks $\{D_S, D_T\}$ of $W$ is the one and only complete set of cutting disks of $W$ intersecting $C_1 \cup C_2$ minimally.

Proof. Suppose, to the contrary, that there exists a complete set of cutting disks $\{D_1, D_2\}$ of $W$, with $\{D_1, D_2\}$ not isotopic to $\{D_S, D_T\}$ in $W$, such that the complexity of the Heegaard diagram $D(D_1, D_2 | C_1, C_2)$ is less than or equal to the complexity of $D(D_S, D_T | C_1, C_2)$. Suppose furthermore, as we may, that among such complete sets of cutting disks of $H$, $\{D_S, D_T\}$ and $\{D_1, D_2\}$ intersect minimally.

If $\{D_S, D_T\}$ and $\{D_1, D_2\}$ are disjoint, then one of $\{D_1, D_2\}$, say $D_1$, is a bandsum of $D_S$ and $D_T$ in $W$, and

$$|D_1 \cap (C_1 \cup C_2)| \leq \max\{|D_A \cap (C_1 \cup C_2)|, |D_B \cap (C_1 \cup C_2)|\}.$$

However, because $c > a + b > 0$ and $d > a + b > 0$ in the graph $G(D_S, D_T | C_1, C_2)$, this is impossible. So $\{D_S, D_T\}$ and $\{D_1, D_2\}$ must have essential intersections.

We may assume disks in $\{D_S, D_T\}$ intersect disks in $\{D_1, D_2\}$ only in arcs. So some disks in $\{D_S, D_T, D_1, D_2\}$ contain outermost subdisks cut off by outermost arcs of intersection of disks in $\{D_S, D_T\}$ with disks in $\{D_1, D_2\}$. Among the set of outermost subdisks of the disks in $\{D_S, D_T, D_1, D_2\}$, let $D$ be one that intersects $C_1 \cup C_2$ minimally.

Note that if $D$ were a subdisk of $D_S$ or $D_T$, then $D$ could be used to perform surgery on one of $D_1$, $D_2$, leading to a contradiction of the assumed minimality properties of $\{D_1, D_2\}$ vis $\{D_S, D_T\}$. So $D$ must be a subdisk of $D_1$ or $D_2$. But then $D$ could be used to perform surgery on one of $\{D_S, D_T\}$, say $D_S$, yielding a cutting disk $D'_S$ of $W$, disjoint from $D_S$ and $D_T$, i.e. a bandsum of $D_S$ and $D_T$, such that

$$|D'_S \cap (C_1 \cup C_2)| \leq |D_S \cap (C_1 \cup C_2)|.$$

However, as before, since $c > a + b > 0$ and $d > a + b > 0$ in the graph $G(D_S, D_T | C_1, C_2)$, this is impossible.

Corollary 4.5. The complete set of cutting disks $\{D_A, D_B\}$ of the handlebody $H$ in the Heegaard splittings described by the R-R diagrams in Figures 4 and 5 is a set of SUMS.

Proof. First, recall, as mentioned before, that a rectangle $R_{pq}$ with $p \in \{a, b\}$, $q \in \{x, y\}$, and $|R_{pq}| = e$ exists in the Heegaard surface of the R-R diagram in Figure 4 (resp Figure 5), if and only if $\partial D_Q$ intersects the face of the $P$-hexagon in Figure 4 (resp Figure 5), with label $e$ in two adjacent points.

Then examination of Figures 4 and 5 shows that, in each case, there are four pairs of rectangles in the Heegaard surface which satisfy the hypothesis of Theorem 4.2. It follows that, in each case, the set of cutting disks $\{D_A, D_B\}$ of the underlying handlebody $H$ is a set of SUMS.

Corollary 4.6. The Heegaard splittings in the R-R diagrams of Figures 4 and 5 are distance three splittings.
Proof. As, in Corollary 4.5, examination of Figures 4 and 9 shows that, in each case, there are four pairs of rectangles in the Heegaard surface which satisfy the hypothesis of Theorem 4.2. It follows that the Heegaard splittings described by the R-R diagrams in Figures 4 and 9 do not have the DCP, and so they are splittings of distance at least three.

On the other hand, Figures 9 and 6 show that, in each splitting, there exist disjoint nonseparating simple closed curves $\alpha$ and $\beta$ in the Heegaard surface which are $(SF, PP)$ pairs. This implies these splittings have distance at most three. Hence each of these splittings is a distance three splitting. □

Figure 8. Here $F_P^+$ is one of the two once-punctured tori in $\Sigma$ with boundary $\Gamma$. The figure shows $F_P^+$ cut open along a pair of properly embedded arcs parallel to edges of rectangles $R_{pq1}$ and $R_{pq2}$ in $\mathcal{R}_{pq}$ where $|R_{pq1}| = |R_{pq1} \cap \partial D_P| = m$, $|R_{pq2}| = |R_{pq2} \cap \partial D_P| = n$, and $m - 1 > n > 1$. (It is always the case that $\gcd(m, n) = 1$).

5. Deriving an R-R Diagram of the Second Splitting of $M$ from the R-R Diagram of the First Splitting of $M$

In order to obtain an R-R diagram of the second splitting of $M$ from the first splitting of $M$, we carry out the following three steps.

(1) Obtain a geometric 4-generator, 4-relator presentation $\mathcal{P}$ of $\pi_1(M)$ from the diagram of the first splitting of $M$ in Figure 4.

(2) Reduce the presentation $\mathcal{P}$ of step (1) to a 2-generator, 2-relator geometric presentation which has minimal length under automorphisms.

(3) Produce an R-R diagram realizing the presentation $\mathcal{P}$ obtained in step (2).

Figure 5 shows the R-R diagram of the original splitting of $M$ in Figure 4 with four simple closed curves $\alpha$, $\alpha_1$, $\beta$ and $\beta_1$ added to the diagram so that $\alpha$, $\alpha_1$, $\beta$ and $\beta_1$ represent $A^5B^3$, $B^5A^2B^2A^2$, $B^5$, and $B^2$ respectively in $\pi_1(H)$, while they represent $X^3$, $X^2$, $Y^3x^3Y^2$, and $x^5y^3$ respectively in $\pi_1(H')$. Thus $(\alpha, \beta)$ is a $(SF, PP)$ pair in $H$, and a $(PP, SF)$ pair in $H'$. 

\[ \delta_2 \rightarrow \omega_2 \rightarrow \omega_1 \rightarrow \delta_1 \rightarrow \delta_4 \rightarrow m \rightarrow n \]

\[ \delta_3 \rightarrow \partial D_P \rightarrow m \rightarrow n \]

\[ F_P^+ \rightarrow R_{pq1} \rightarrow R_{pq2} \]

\[ \beta_1 \rightarrow \beta \rightarrow \alpha_1 \rightarrow \alpha \]
Figure 9. The R-R diagram of the first splitting of $M$ in Figure 4 with four curves $\alpha$, $\alpha^\perp$, $\beta$ and $\beta^\perp$ added to the diagram. Here $\alpha$, $\alpha^\perp$, $\beta$ and $\beta^\perp$ represent $A^5B^5$, $b^5A^2B^2A^2$, $B^3$, and $B^2$ respectively in $\pi_1(H)$, while they represent $X^3$, $X^2$, $Y^5x^3Y^2$, and $x^5y^3$ respectively in $\pi_1(H')$. So $(\alpha, \beta)$ is a $(SF, PP)$ pair in $H$, and a $(PP, SF)$ pair in $H'$. (Note that we adopt the space-saving convention of using pairs of uppercase and lowercase letters to denote generators and their inverses in free groups and the relators of presentations. So if $x$ is a generator of a free group, then $X = x^{-1}$.)

The curve $\beta^\perp$ has been chosen so that $\beta^\perp$ is disjoint from the separating curve $\Gamma$ in $\Sigma$, and so that $\beta$ and $\beta^\perp$ intersect transversely in a single point $q$. Then, in particular, $\beta$ and $\beta^\perp$ lie completely on the $B$-handle of Figure 9.

The curve $\alpha^\perp$ has been chosen so that the pair $(\alpha, \alpha^\perp)$ has properties with respect to the handlebody $H'$ analogous to those enjoyed by the pair $(\beta, \beta^\perp)$ with respect to $H$. That is: $\alpha$ and $\alpha^\perp$ intersect transversely once in a single point $p$, and they are both disjoint from a separating curve $\Gamma'$ in $\Sigma$ such that $\Gamma'$ bounds a disk in $\partial H'$ separating the cutting disks $D_X$ and $D_Y$ of $H'$. (Note $\Gamma'$ is not shown in Figure 9).

5.1. Obtaining a genus four splitting of $M$.

The separating disk in $H$ which $\Gamma$ bounds cuts $H$ into two solid tori $V_A$ and $V_B$, with meridional disks $D_A$ and $D_B$ respectively. Similarly, the separating curve $\Gamma'$, cuts $H'$ into two solid tori $V_X$ and $V_Y$, with meridional disks $D_X$ and $D_Y$ respectively.

Let $v_B$ be a regular neighborhood in $V_B$ of a core of $V_B$. Then $V_B \setminus \text{int}(v_B)$ is homeomorphic to $\partial V_B \times I$. And if $D_q$ is a small disk in $\partial V_B$ containing the point $q = \beta \cap \beta^\perp$, then $v_B$ can be attached to $H'$ by the one-handle $D_q \times I$.

In similar fashion, let $v_X$ be a regular neighborhood in $V_X$ of a core of $V_X$. Then $V_X \setminus \text{int}(v_X)$ is homeomorphic to $\partial V_X \times I$. And if $D_p$ is a small disk in $\partial V_X$ containing the point $p = \alpha \cap \alpha^\perp$, then $v_X$ can be attached to $H$ by the one-handle
$D_p \times I$. These two changes transform $H$ and $H'$ into genus four handlebodies $H_4$ and $H'_4$, giving a genus four Heegaard splitting of $M$.

5.2. Locating complete sets of cutting disks of $H_4$ and $H'_4$.

The next step is to locate complete sets of cutting disks of $H_4$ and $H'_4$. First, note that $D_X \cap v_X$ is a meridional disk of $v_X$. Next, note that $\beta \times I$ and $\beta^\perp \times I$ are annuli in $\partial V_B \times I$, which become cutting disks $D_\beta$ and $D_{\beta^\perp}$ of $H_4$ when the one handle $D_q \times I$ is attached to $H'$. Also note that $D_A$ is still a cutting disk of $H_4$. It follows that $H_4$ has a complete set of cutting disks comprised of $D_A$, $D_\beta$, $D_{\beta^\perp}$, and $D_X \cap v_X$. Similarly, $H'_4$ has a complete set of cutting disks comprised of $D_Y$, $D_\alpha$, $D_{\alpha^\perp}$, and $D_B \cap v_B$.

5.3. Obtaining an initial geometric presentation of $\pi_1(M)$.

If we take generators $A$, $C$, $D$, and $E$ of $\pi_1(H_4)$ which are dual in $H_4$ to $D_A$, $D_\beta$, $D_{\beta^\perp}$, and $D_X \cap v_X$ respectively, then $\pi_1(M)$ has the geometric presentation

\[(5.1) \quad \mathcal{P} = \langle A, C, D, E \mid \partial D_\alpha, \partial D_{\alpha^\perp}, \partial(D_B \cap v_B), \partial D_Y \rangle.\]

The next step is to express the abstract relators of (5.1) as cyclic words in the generators $A$, $C$, $D$, and $E$ of $\mathcal{P}$.

To obtain the cyclic word which $\partial D_\alpha$ represents in $\pi_1(H_4)$, start at the point $p = \alpha \cap \alpha^\perp$ and proceed around $\alpha$ recording the oriented intersections of $\alpha$ with $\beta$, $\beta^\perp$, and $\partial D_A$, while ignoring intersections of $\alpha$ with $\partial D_X$ until returning to $p$. Then starting at $p$, retrace $\alpha$ in the opposite direction and record only the oriented intersections of $\alpha$ with $\partial D_X$ until returning to $p$. This yields $\partial D_\alpha = A^5 D c^3$.

The cyclic word which $\partial D_{\alpha^\perp}$ represents in $\pi_1(H_4)$ can be obtained in the same way as that of $\partial D_\alpha$. This yields $\partial D_{\alpha^\perp} = dA^2 c A^2 c^2$. (This works for $\partial D_\alpha$ and $\partial D_{\alpha^\perp}$ because the annulus in $D_X$ bounded by $\partial D_X$ and $\partial(D_X \cap v_X)$ lies in $H_4$.)

It is fairly easy to find the cyclic word which $\partial(D_B \cap v_B)$ represents in $\pi_1(H_4)$ because $\partial(D_B \cap v_B)$ and $\partial D_B$ bound an annulus in $D_B$ which lies in $H_4$, and the curves $\beta$ and $\beta^\perp$ form a basis for a once-punctured torus in $\partial H_4$ in which $\partial D_B$ lies. It follows that the cyclic word which $\partial(D_B \cap v_B)$ represents in $\pi_1(H_4)$ can be obtained by traversing $\partial D_B$ in $\Sigma$ while recording the oriented intersections of $\partial D_B$ with $\beta$ and $\beta^\perp$. This yields $\partial(D_B \cap v_B) = DC^2 DC^3$.

Finally, it is straightforward to obtain the cyclic word which $\partial D_Y$ represents in $\pi_1(H_4)$ by proceeding around $\partial D_Y$ in Figure 9 while recording the oriented intersections of $\alpha$ with $\beta$, $\beta^\perp$ and $\partial D_A$ in terms of the new set of generators $A$, $C$ and $D$ of $\pi_1(H_4)$. This yields $\partial D_Y = A^2 D c(A^2 D c A^2 c)^2$.

Putting these pieces together yields the presentation

\[(5.2) \quad \mathcal{P} = \langle A, C, D, E \mid A^5 D c^3, dA^2 c A^2 c^2, DC^2 DC^3, A^7 D c(A^7 D c A^2 c)^2 \rangle.\]

5.4. Reducing $\mathcal{P}$ to a 2-generator, 2-relator geometric presentation.

Suppose $(\Sigma; V, W)$ is a Heegaard splitting, with $\Sigma$ bounding handlebodies $V$ and $W$. Recall that the splitting $(\Sigma; V, W)$ has a trivial handle if there exist cutting disks $D_v$ of $V$ and $D_w$ of $W$ such that $D_v$ and $D_w$ intersect transversely in a single point.

In order to obtain the second genus two splitting of $M$ from the genus four splitting by $H_4$ and $H'_4$, we want to find two independent trivial handles, one involving $D_\alpha$, and one involving $D_\beta$. Since $D_{\beta^\perp}$ is a disk in $H_4$, which meets $D_\alpha$ transversely at a single point, $D_\alpha$ and $D_{\beta^\perp}$ can be used as the first trivial handle.
To eliminate the \((D_\alpha, D_{\beta^+})\) trivial handle, the cutting disks of \(H'_4\), other than \(D_\alpha\), need to have their intersections with \(D_{\beta^+}\) removed by forming bandsums of these disks with \(D_\alpha\) along arcs of \(\partial D_{\beta^+}\). Algebraically, this amounts to changing presentation \((5.2)\) by solving \(\partial D_\alpha = A^5 B De^3 = 1\) for \(D\), from which \(D = a^5 E^3\), then replacing all occurrences of \(D\) in the other three relators of \((5.2)\) with \(a^5 E^3\), and then dropping the generator \(D\) and the relator \(A^5 B De^3\) from \((5.2)\). This changes \((5.2)\) into the geometric presentation\((5.3)\)

\[
\mathcal{P} = \langle A, C, E \mid A^7 c A^2 e^5, a^5 E^3 C^2 a^5 E^3 C^3, A^2 E^3 c(A^2 E^3 c A^7 c A^2 c) \rangle.
\]

Topologically, eliminating the trivial handle \((D_\alpha, D_{\beta^+})\), has destabilized the genus four Heegaard splitting of \(M\) by \(H_4\) and \(H'_4\) by turning it into a genus three Heegaard splitting of \(M\) by genus three handlebodies \(H_3\) and \(H'_3\). Then, in this genus three splitting, \(D\) and the cutting disk of \(H'_3\) whose boundary represents the first relator \(A^7 c A^2 e^5\) of \((5.3)\) form a trivial handle.

Algebraically, eliminating this trivial handle amounts to solving \(A^7 c A^2 e^5 = 1\) in \((5.3)\) for \(C\), from which \(C = A^2 e^5 A^7\) and \(c = a^7 E^5 a^2\), then replacing \(C\) and \(c\) in the other two relators of \((5.3)\) with \(A^2 e^5 A^7\) and \(a^7 E^5 a^2\) respectively, and then dropping the generator \(C\) and the relator \(A^7 c A^2 e^5\) from \((5.3)\). This turns \((5.3)\) into the geometric presentation\((5.4)\)

\[
\mathcal{P} = \langle A, E \mid A^9 e (A^2 E^4 A^2 e^5 A^9 e^5)^2, E^8 a^7 (E^8 a^7 E^5 a^2 E^5 a^7)^2 \rangle.
\]

Finally, taking the inverse of the first relator, and replacing \(E\) with \(A\) and \(a\) with \(B\) in \((5.4)\), turns \((5.4)\) into\((5.5)\)

\[
\mathcal{P} = \langle A, B \mid A^8 B^7 (A^8 B^7 A^5 B^2 A^5 B^7)^2, A^5 B^9 (A^5 B^9 A^5 B^2 a^3 B^2)^2 \rangle.
\]

5.5. Obtaining an \(R-R\) diagram of a realization of \(\mathcal{P}\).

At this point, we have a genus two Heegaard splitting of \(M\) by handlebodies \(H_2\) and \(H'_2\), which was obtained by eliminating a trivial handle from the genus three splitting of \(M\) by \(H_3\) and \(H'_3\). Next, for simplicity, we drop the subscripts from \(H_2\) and \(H'_2\). So we have a genus two Heegaard surface \(\Sigma\) bounding handlebodies \(H\) and \(H'\). Then there exist complete sets \(\{D'_A, D'_B\}\) and \(\{D_X, D_Y\}\) of cutting disks of \(H\) and \(H'\) respectively such that the Heegaard diagram \(D(D'_A, D'_B \mid \partial D_X + \partial D_Y)\) of \(\partial D_X\) and \(\partial D_Y\) with respect to \(D'_A\) and \(D'_B\) realizes \((5.5)\).

Note that, in general, there is nothing unique about Heegaard diagrams realizing \((5.5)\) since diagrams such as \(D(D'_A, D'_B \mid \partial D_X + \partial D_Y)\) can be modified by Dehn twists of \(\partial D_X\) and \(\partial D_Y\) about simple closed curves which bound disks in \(H\). The following claim provides a way to deal with this minor annoyance.

Claim 3. There exists a complete set of cutting disks \(\{D_A, D_B\}\) of \(H\) such that the Heegaard diagram \(D(D_A, D_B \mid \partial D_X + \partial D_Y)\) has both minimal complexity and realizes presentation \((5.5)\).

Proof of Claim 3. Lemma 5.6 shows \((5.6)\) has minimal length under automorphisms of the free group \(F(A, B)\), and it also shows \(\{A, B\}\) is the only basis of \(F(A, B)\) in which \((5.6)\) has minimal length. It then follows from the main result of [Z] that \(H\) has a unique set of cutting disks \(\{D_A, D_B\}\) such that \(D(D_A, D_B \mid \partial D_X + \partial D_Y)\) has both minimal complexity and realizes \((5.5)\). \(\square\)

Claim 4. The graph \(G(D_A, D_B \mid \partial D_X + \partial D_Y)\) underlying \(D(D_A, D_B \mid \partial D_X + \partial D_Y)\) has the form of Figure 5.1, with \(c, d > 0\).
Proof of Claim 4] By Lemma 2.3, $G(D_A, D_B | \partial D_X, \partial D_Y)$ has the form of one of the three graphs in Figure 4. Since $D(D_A, D_B | \partial D_X, \partial D_Y)$ has minimal complexity, $G(D_A, D_B | \partial D_X, \partial D_Y)$ does not have the form of Figure 4. And, because both generators appear in the relators of (5.5) with exponents having absolute value greater than one, $G(D_A, D_B | \partial D_X, \partial D_Y)$ does not have the form of Figure 4. It follows $G(D_A, D_B | \partial D_X, \partial D_Y)$ has the claimed form. 

Finally, we can produce an R-R diagram $D$ with $D(D_A, D_B | \partial D_X, \partial D_Y)$ as its underlying Heegaard diagram. R-R diagrams with $D(D_A, D_B | \partial D_X, \partial D_Y)$ as their underlying Heegaard diagram are parametrized by the isotopy class of an essential separating simple closed curve $\Gamma$ in the Heegaard surface $\Sigma$ which is disjoint from $D_A$ and $D_B$. Since $G(D_A, D_B | \partial D_X, \partial D_Y)$ has the form of Figure 4, it is natural to take $\Gamma$ to be the unique separating simple closed curve in $\Sigma$, which is disjoint from $\partial D_A$ and $\partial D_B$, and is also disjoint from any edge of $D(D_A, D_B | \partial D_X, \partial D_Y)$ connecting $D_A^+ \to D_A^-$ or $D_B^+ \to D_B^-$. Thus $\Gamma$ becomes the essential separating simple closed curve in $\Sigma$, disjoint from $\partial D_A$ and $\partial D_B$, which intersects $\partial D_X$ and $\partial D_Y$ minimally.

Then in order to determine the form of $D$ we need to:

1. Determine the labels for each face of the hexagons representing the A-handle and B-handle of $D$;
2. Determine how the cyclic orders of the labeling of the faces of the A-handle hexagon and the B-handle hexagon need to be coordinated;
3. Determine how many connections lie in each isotopy class of connections on each handle of $D$;
4. Determine how to connect endpoints of connections on the A-handle with endpoints of connections on the B-handle of $D$.

All of these items can be determined by scanning the relators of $P$. For example, suppose $G \in \{A, B\}$. Then, since $\Gamma$ is disjoint from edges of $D(D_A, D_B | \partial D_X, \partial D_Y)$ connecting $D_A^+ \to D_A^-$ or $D_B^+ \to D_B^-$, $G$ must appear in the relators of $P$ with exponents having at most three absolute values, say $e_1, e_2$ and $e_3$, with $e_2 = e_1 + e_3$, and then the labels of the faces of the $G$-hexagon must be in clockwise cyclic order either $(e_1, e_2, e_3, -e_1, -e_2, -e_3)$ or $(e_3, e_2, e_1, -e_3, -e_2, -e_1)$.

Next, scanning the relators of $P$ shows $A$ appears with exponents having absolute values 3, 8 and 5, while $B$ appears with exponents having absolute values 2, 9 and 7. Up to orientation reversing homeomorphism of $D$, the cyclic order of the labeling of the faces of one of the handles of $D$ may be chosen arbitrarily. So we may label the faces of the A-handle of $D$ with $(5, 8, 3, -5, -8, -3)$ in clockwise cyclic order. Then, with the cyclic order of the labels of the A-handle specified, it is easy to see that, because $(A^8 B^7)^{\pm 1}$, $(B^7 A^8)^{\pm 1}$ and $(A^5 B^2)^{\pm 1}$ appear in $P$, the faces of the B-hexagon of $D$ must be labeled in clockwise cyclic order $(7, 9, 2, -7, -9, -2)$ if $D$ is to realize $P$.

Next, let $|G^{\pm e}|$, for $G \in \{A, B\}$, denote the total number of appearances of $G$ in the relators of $P$ with exponent having absolute value $e$. Then $|A^{\pm 5}| = 9$, $|A^{\pm 8}| = 3$, $|A^{\pm 3}| = 2$, $|B^{\pm 7}| = 5$, $|B^{\pm 9}| = 3$, $|B^{\pm 2}| = 6$, and clearly these values determine the number of connections in each isotopy class of connections on the two handles of $D$.

It remains to determine how edges of $D$ connect endpoints of connections on the A-hexagon of $D$ with endpoints of connections on the B-hexagon of $D$. In general,
each 2-syllable subword of the relators of \( \mathcal{P} \) of the form \((A^m B^n)^{\pm 1}\) with \( m \in \{\pm 5, \pm 8, \pm 3\} \) and \( n \in \{\pm 7, \pm 9, \pm 2\} \) corresponds to an edge of \( \mathcal{D} \) which connects an endpoint of a connection in the face of the A-hexagon of \( \mathcal{D} \) with label \(-m\) to an endpoint of a connection in the face of the B-hexagon of \( \mathcal{D} \) with label \(n\). In particular, since \(|A^{\pm 5}| = 9\), there must be a set \( S \) of 9 edges in \( \mathcal{D} \) connecting the 9 endpoints of connections in the face of the A-hexagon of \( \mathcal{D} \) with label \(-5\) to 9 consecutive endpoints of connections in the boundary of the B-hexagon of \( \mathcal{D} \).

One can see where these 9 consecutive endpoints of edges in \( S \) are located in the boundary of the B-hexagon in the following way. For each \( e \in \{\pm 7, \pm 9, \pm 2\} \), let \(|(A^5 B^e)^{\pm 1}|\) denote the total number of appearances of two syllable subwords in the relators of \( \mathcal{P} \) of the form \((A^5 B^e)^{\pm 1}\). A scan of the relators of \( \mathcal{P} \) shows the only nonzero values in this set are: \(|(A^5 B^7)^{\pm 1}| = 2\), \(|(A^5 B^9)^{\pm 1}| = 3\) and \(|(A^5 B^2)^{\pm 1}| = 4\). This implies the edges of \( S \) must appear in \( \mathcal{D} \) so that \( S \) is the disjoint union of subsets of 2, 3 and 4 edges which meet the faces of the B-hexagon of \( \mathcal{D} \) with labels 7, 9 and 2 respectively. It is easy to see there is only one way to do this. And this, in turn, implies that how edges of \( \mathcal{D} \) connect endpoints of connections on the A-hexagon of \( \mathcal{D} \) with endpoints of connections on the B-hexagon of \( \mathcal{D} \) is completely determined.

The resulting R-R diagram \( \mathcal{D} \) appears in Figure 5. One checks easily that \( \mathcal{D} \) realizes (5.5). And, by construction, it is an R-R diagram of a genus two Heegaard splitting of \( M \).

**Lemma 5.6.** Suppose \( \mathcal{P} = \langle A, B | R_1, R_2 \rangle \) is a two-generator, two-relator presentation in which both \( R_1 \) and \( R_2 \) are cyclically reduced cyclic words. If there is an automorphism of the free group \( F(A, B) \) generated by \( A \) and \( B \) which reduces the length of \( \mathcal{P} \), then one of the four Whitehead automorphisms

\[
(5.7) \quad (A, B) \mapsto (AB^{\pm 1}, B) \\
(5.8) \quad (A, B) \mapsto (A, BA^{\pm 1})
\]

will also reduce the length of \( \mathcal{P} \).

**Proof.** This is a well-known result of Whitehead. See [W] or [LS]. \( \square \)

### 5.6. The two genus two splittings of \( M \) are not homeomorphic.

**Proposition 5.9.** The genus two Heegaard splittings of \( M \) determined by the R-R diagrams of Figures 4 and 5 are not homeomorphic.

**Proof.** Consider how the simple closed curves \( \partial D_P \), for \( P \in \{A, B\} \) intersect the bands of connections \( F_P \cap (\partial D_X \cup \partial D_Y) \) in \( F_P \) in Figures 4 and Figure 5. It is not hard to see that, in both Figures 4 and 5, the graph \( G(D_X, D_Y | \partial D_A, \partial D_B) \) of the Heegaard diagram \( D(D_X, D_Y | \partial D_A, \partial D_B) \) of \( \partial D_A \) and \( \partial D_B \) with respect to \( D_X \) and \( D_Y \) has the form of Figure 4, with \( c > a + b > 0 \) and \( d > a + b > 0 \). It follows from Lemma 4.4 that, in both diagrams, the set of cutting disks \( \{D_X, D_Y\} \) of \( H \) is the unique complete set of cutting disks of \( H \) intersecting \( \partial D_A \cup \partial D_B \) minimally.

Next, since Corollary 4.5 shows that, in both Figure 4 and Figure 5, \( \{D_A, D_B\} \) is a set of SUMS, it follows that, in both Figures 4 and Figure 5, the set of simple closed curves \( \{\partial D_A, \partial D_B, \partial D_X, \partial D_Y\} \) form the unique minimal complexity Heegaard diagram carried by the splitting surface \( \Sigma \). However, in the Heegaard surface \( \Sigma \) of Figure 4, \( |(\partial D_A \cup \partial D_B) \cap (\partial D_X \cup \partial D_Y)| = 121 \), while in the Heegaard surface \( \Sigma \)
of Figure 3 $|\partial D_A \cup \partial D_B) \cap (\partial D_X \cup \partial D_Y)| = 149$. Since these minimal complexities differ, the Heegaard surfaces of $M$ in Figures 4 and 5 are not homeomorphic. □

References

[B1] J. Berge, A classification of pairs of disjoint nonparallel primitives in the boundary of a genus two handlebody, [arXiv:0910.3028]

[BS] J. Berge and M. Scharlemann, Multiple Genus 2 Heegaard Splittings: A Missed Case, [arXiv:0910.3921]

[He] J. Hempel, 3-manifolds as viewed from the curve complex, Topology 40 (2001) 631–657.

[HOT] Homma, T., Ochiai, M. and Takahashi, M., An Algorithm for Recognizing $S^3$ in 3-Manifolds with Heegaard Splittings of Genus Two, Osaka J. Math. 17 (1980), 625–648.

[LS] R. Lyndon and P. Schupp, Combinatorial group theory, Springer Verlag, Berlin, 1977.

[O] M. Ochiai, Heegaard-Diagrams and Whitehead-Graphs, Math. Sem. Notes of Kobe Univ. 7 (1979), 573–590.

[RS] H. Rubinstein and M. Scharlemann, Genus two Heegaard splittings of orientable 3-manifolds, in Proceedings of the 1998 Kirbyfest, Geometry and Topology Monographs 2 (1999) 489–553.

[Th] A. Thompson, The disjoint curve property and genus 2 manifolds, Topology Appl. 97 (1999) 273–279.

[W] J. H. C. Whitehead, On Equivalent Sets of Elements in a Free Group, Ann. of Math. 37 (1936), 782–800.

[Z1] Zieschang, H., On Simple Systems of Paths on Complete Pretzels Amer. Math. Soc. Transl. (2) (92), (1970), 127–137. Transl. of Mat. Sb. (66) 108 (1965), 230–239.

[Z2] H. Zieschang, On Heegaard Diagrams of 3-Manifolds, Asterisque, 163–164 (1988), 147–280.

E-mail address: jberge@charter.net