Nonlocal Cauchy problems for wave equations and applications
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Abstract
In this paper, the existence, the uniqueness and estimates of solution to the integral Cauchy problem for linear and nonlinear abstract wave equations are proved. The equation includes a linear operator $A$ defined in a Banach space $E$, in which by choosing $E$ and $A$ we can obtain numerous classes of nonlocal initial value problems for wave equations which occur in a wide variety of physical systems.

Key Word: wave equations, Semigroups of operators, Hyperbolic-operator equations, cosine operator functions, operator valued $L^p$–Fourier multipliers

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1. Introduction

The subject of this paper is to study the existence, uniqueness and regularity properties to solution of the integral Cauchy problem (ICP) for the nonlinear abstract wave equation (NAWE)

$$u_{tt} - \Delta u + Au + F(u) = 0, \ x \in \mathbb{R}^n, \ t \in (0, \infty), \quad (1.1)$$

$$u(x,0) = \varphi(x) + \int_0^T \alpha(t)u(x,\sigma)\,d\sigma, \quad (1.2)$$

$$u_t(x,0) = \psi(x) + \int_0^T \beta(t)u_t(x,\sigma)\,d\sigma,$$

where $A$ is a linear operator in a Banach space $E$, $\alpha(s)$ and $\beta(s)$ are measurable functions on $(0,T)$, $u(x,t)$ denotes the $E$-valued unknown function, $f(u)$ is the given nonlinear function, $\varphi(x)$ and $\psi(x)$ are the given initial value functions, subscript $t$ indicates the partial derivative with respect to $t$, $n$ is the dimension of space variable $x$ and $\Delta$ denotes the Laplace operator in $\mathbb{R}^n$.

Remark 1.1. Note that, particularly, the conditions (1.2) can be expressed as the following multipoint nonlocal conditions

$$u(x,0) = \varphi(x) + \sum_{k=1}^l \alpha_k u(x,\lambda_k), \ u_t(0,x) = \psi(x) + \sum_{k=1}^m \beta_k u_t(x,\lambda_k), \quad (1.3)$$

where $l$ is a positive integer, $\alpha_k, \beta_k$ are complex numbers and $\lambda_k \in (0, \infty)$.
Since the Banach space $E$ is arbitrary and $A$ is a possible linear operator, by choosing $E$, $A$ and integral conditions, we can obtain numerous classes of nonlocal initial value problems for wave equations which occur in a wide variety of physical systems, particularly in the propagation of longitudinal deformation waves in an elastic rod, hydro-dynamical process in plasma, in materials science which describe spinodal decomposition, in the absence of mechanical stresses (see [29], [45], [48] and the references therein). If $F(u) = \lambda |u|^p u$, from (1.1) we get the ICP for the following NAWE

$$u_{tt} - \Delta u + Au + \lambda |u|^p u = 0, \quad x \in \mathbb{R}^n, \quad t \in (0, \infty),$$

(1.4)

where $p \in (1, \infty)$, $\lambda$ is a real number.

Let $N$, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of all natural, real and complex numbers, respectively. For $E = \mathbb{C}$, $\alpha_k = \beta_k = 0$ and $A = 0$ the problem (1.4) become as the following integral Cauchy problem

$$u_{tt} - \Delta u + \lambda |u|^{p-1} u = 0, \quad x \in \mathbb{R}^n, \quad t \in (0, \infty),$$

(1.5)

If we choose $E$ a concrete space, for example $E = L^2(\Omega)$, $A = L$, where $\Omega$ is a domain in $\mathbb{R}^m$ with sufficiently smooth boundary and $L$ is an elliptic operator in $L^2(\Omega)$, then from (1.1) – (1.2) we obtain the existence, uniqueness and the regularity properties of the mixed problem for linear wave equation

$$u_{tt} - \Delta u + Lu = F(x,t), \quad t \in [0,T], \quad x \in \mathbb{R}^n, \quad y \in \Omega,$$

and the following nonlinear wave equations (NWE) equation

$$u_{tt} - \Delta u + Lu + \lambda |u|^{p-1} u = 0, \quad t \in [0,T], \quad x \in \mathbb{R}^n, \quad y \in \Omega,$$

where $u = u(x,y,t)$.

Moreover, let we choose $E = L^2(0,1)$ and $A$ to be differential operator with generalized Wentzell-Robin boundary condition defined by

$$D(A) = \{u \in W^{2,2}(0,1), \quad B_j u = Au (j) = 0, \quad j = 0, 1\},$$

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\[ Au = au^{(2)} + bu^{(1)} \]  
where \( a = a(y) \) and \( b = b(y) \) are complex-valued functions. Then, from the main our theorem we get the existence, uniqueness and regularity properties of multipoint Wentzell-Robin type mixed problem for the linear wave equation

\[ u_{tt} - \Delta u + au_{yy} + bu_y = F(x, y, t), \]  

(1.7)

\[ u(x, y, 0) = \varphi(x, y) + \int_0^T \alpha(\sigma) u(x, y, \sigma) \, d\sigma, \]  

(1.8)

\[ u_t(x, y, 0) = \psi(x, y) + \int_0^T \beta(\sigma) u_t(x, y, \sigma) \, d\sigma, \]  

and for the following NWE

\[ u_{tt} - \Delta u + au_{yy} + bu_y + F(u) = 0, \]  

(1.10)

where

\[ u = u(x, y, t), \ x \in \mathbb{R}^n, \ y \in (0, 1), \ t \in [0, T]. \]

Note that, the regularity properties of Wentzell-Robin type boundary value problems (BVPs) for elliptic equations were studied e.g. in [12, 25] and the references therein. Moreover, if put \( E = l_2 \) and \( A \) choose as a infinite matrix \( [a_{mj}], m, j = 1, 2, \ldots, \infty \), then from our results we obtain the existence, uniqueness and regularity properties of integral Cauchy problem for infinity many system of linear wave equations

\[ (u_m)_{tt} - \Delta u_m + \sum_{j=1}^N a_{mj}u_j = F_j(x, t), \ x \in \mathbb{R}^n, \ t \in [0, T], \]  

(1.11)

\[ u_m(x, 0) = \varphi_m(x) + \int_0^T \alpha(\sigma) u_m(x, \sigma) \, d\sigma, \]  

and infinity many system of NWE equation

\[ (u_m)_{tt} - \Delta u_m + \sum_{j=1}^N a_{mj}u_j + F_m(u_1, u_2, \ldots u_N) = 0, \ x \in \mathbb{R}^n, \ t \in [0, T], \]  

(1.12)
where $a_{m,j}$ are complex numbers, $u_j = u_j(x,t)$.

The existence of solutions and regularity properties of Cauchy problem for NWE studied e.g in [5, 7], [14], [18], [20–24], [26], [30], [34], [42–46] and the references therein. In contrast to the mentioned above results we will study the regularity properties of the problem (1.1)–(1.2). Abstract differential equations studied e.g. in [1–3], [7–11], [13], [16–17], [19], [28], [32, 33], [35–39], [44] and [47]. The Cauchy problems for abstract hyperbolic equations were treated e.g. in [3], [7], [9], [11], [16], [32, 33].

The strategy is to express the wave equation as an integral equation, to treat in the nonlinearity as a small perturbation of the linear part of the equation, then use the contraction mapping theorem and utilize an estimate for solutions of the linearized version to obtain a priori estimates on $L^p$ norms of solutions.

Harmonic analysis, the operator theory, interpolation of Banach Spaces, embedding theorems in abstract Sobolev spaces are the main tools implemented to carry out the analysis.

Sometimes we use one and the same symbol $C$ (or $M$) without distinction in order to denote positive constants which may differ from each other even in a single context. When we want to specify the dependence of such a constant on a parameter, say $\alpha$, we write $C_\alpha$ (or $M_\alpha$).

In order to state our results precisely, we introduce some notations and some function spaces:

**Definitions and Background**

Let $E$ be a Banach space. $L^p(\Omega;E)$ denotes the space of strongly measurable $E$-valued functions that are defined on the measurable subset $\Omega \subset R^n$ with the norm

$$
\|f\|_{L^p} = \|f\|_{L^p(\Omega;E)} = \left( \int_\Omega \|f(x)\|^p_E \, dx \right)^{\frac{1}{p}}, 1 \leq p < \infty,
$$

$$
\|f\|_{L^\infty} = \text{ess sup}_{x \in \Omega} \|f(x)\|_E.
$$

The Banach space $E$ is called an UMD-space if the Hilbert operator

$$
(Hf)(x) = \lim_{\varepsilon \to 0} \int_{|x-y| > \varepsilon} \frac{f(y)}{x-y} \, dy
$$

is bounded in $L^p(R,E), p \in (1, \infty)$ (see. e.g. [4]). Any Hilbert space is a UMD space. Moreover, UMD spaces include e.g. $L^p$, $l_p$ spaces and Lorentz spaces $L_{pq}, p, q \in (1, \infty)$.

Here,

$$
S_\omega = \{ \lambda \in \mathbb{C}, |\arg \lambda| \leq \omega, 0 \leq \omega < \pi \},
$$

$$
S_{\omega,\kappa} = \{ \lambda \in S_\omega, |\lambda| > \kappa > 0 \}.
$$
A closed linear operator $A$ is said to be sectorial in a Banach space $E$ with bound $M > 0$ if $D(A)$ and $R(A)$ are dense on $E$, $N(A) = \{0\}$ and
$$\|(A + \lambda I)^{-1}\|_{B(E)} \leq M |\lambda|^{-1}$$
for any $\lambda \in S_\omega$, $0 \leq \omega < \pi$, where $I$ is the identity operator in $E$, $B(E)$ is the space of bounded linear operators in $E$; $D(A)$ and $R(A)$ denote domain and range of the operator $A$. It is known that (see e.g. [40, §1.15.1]) there exist fractional powers $A^\theta$ of a sectorial operator $A$. Let $E(A^\theta)$ denote the space $D(A^\theta)$ with the graphical norm
$$\|u\|_{E(A^\theta)} = \left(\|u\|^p + \|A^\theta u\|^p\right)^{\frac{1}{p}}, 1 \leq p < \infty, 0 < \theta < \infty.$$

A closed linear operator $A$ belong to $\sigma (M_0, \omega, E)$ (see [11], §11.2) if $D(A)$ is dense on $E$, the resolvent $(A - \lambda^2 I)^{-1}$ exists for $\Re \lambda > \omega$ and
$$\left\|(A - \lambda^2 I)^{-1}\right\|_{B(E)} \leq M_0 |\Re \lambda - \omega|^{-1}.$$

**Remark 1.2.** Let $0 \leq \gamma < 1$ It is known that if $A \in \sigma (M_0, \omega, E)$, then it is is a sectorial operator in $E$ and it is an infinitesimal generator of $C_0$ group of bounded linear operator $U_A(t)$ satisfying
$$\|U_A(t)\|_{B(E)} \leq M_0 e^{\omega |t|}, \quad t \in (-\infty, \infty),$$
$$\|A^\gamma U_A(t)\|_{B(E)} \leq M_0 |t|^{-\gamma}, \quad t \in (-\infty, \infty)$$
(see e.g. [33], [§ 1.6], Theorem 6.3).

Let $E_1$ and $E_2$ be two Banach spaces, $(E_1, E_2)_{\theta, p}$ denotes the interpolation spaces obtained from $\{E_1, E_2\}$ by $K$-method, where, $\theta \in (0, 1)$, $p \in [1, \infty]$ [40, §1.3.2].

The sectorial operator $A$ is said to be $R-$sectorial in a Banach space $E$ if the set $\{\xi(A + \xi)^{-1} : \xi \in S_\omega\}$, $0 \leq \omega < \pi$ is R-bounded (see e.g. [8]).

Let $E_0$ and $E$ be two Banach spaces and $E_0$ is continuously and densely embedded into $E$. Let $\Omega$ be a domain in $\mathbb{R}^n$ and $m$ is a positive integer. $W^{m,p}(\Omega; E_0, E)$ denotes the space of all functions $u \in L^p(\Omega; E_0)$ that have the generalized derivatives $\frac{\partial^m u}{\partial x_k} \in L^p(\Omega; E)$ for $1 \leq p \leq \infty$ with the norm
$$\|u\|_{W^{m,p}(\Omega; E_0, E)} = \|u\|_{L^p(\Omega; E_0)} + \sum_{k=1}^n \left\|\frac{\partial^m u}{\partial x_k}\right\|_{L^p(\Omega; E)} < \infty.$$

For $E_0 = E$ the space $W^{m,p}(\Omega; E_0, E)$ denotes by $W^{m,p}(\Omega; E)$. Here, $H^{s,p}(\mathbb{R}^n; E)$, $-\infty < s < \infty$ denotes the $E-$valued Sobolev space of order $s$ which is defined as:
$$H^{s,p} = H^{s,p}(\mathbb{R}^n; E) = (I - \Delta)^{-\frac{s}{2}} L^p(\mathbb{R}^n; E)$$
with the norm
$$\|u\|_{H^{s,p}} = \left\|(I - \Delta)^{\frac{s}{2}} u\right\|_{L^p(\mathbb{R}^n; E)} < \infty.$$

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It clear that \( H^{0,p} (\mathbb{R}^n; E) = L^p (\mathbb{R}^n; E) \). It is known that if \( E \) is a UMD space, then \( H^{m,p} (\mathbb{R}^n; E) = W^{m,p} (\mathbb{R}^n; E) \) for positive integer \( m \) (see e.g. [41, § 15]). \( H^{s,p} (\mathbb{R}^n; E_0, E) \) denote the Sobolev-Lions type space, i.e.,

\[
H^{s,p} (\mathbb{R}^n; E_0, E) = \{ u \in H^{s,p} (\mathbb{R}^n; E) \cap L^p (\mathbb{R}^n; E_0) \mid \| u \|_{H^{s,p}(\mathbb{R}^n; E_0)} < \infty \}.
\]

\( S (\mathbb{R}^n; E) \) denotes the Schwartz class, i.e., the space of \( E \)-valued rapidly decreasing smooth functions on \( \mathbb{R}^n \), equipped with its usual topology generated by seminorms. Here, \( S' (\mathbb{R}^n; E) \) denotes the space of all continuous linear operators \( L : S (\mathbb{R}^n; E) \to E \), equipped with the bounded convergence topology. Recall that \( S (\mathbb{R}^n; E) \) is norm dense in \( L^p (\mathbb{R}^n; E) \) when \( 1 \leq p < \infty \).

Let \( F \) denotes the Fourier transform. \( \Psi \in L^\infty (\mathbb{R}^n; B (E)) \) is called a multiplier from \( L^p (\mathbb{R}^n; E_1) \) to \( L^q (\mathbb{R}^n; E) \) if there exists a positive constant \( C \) such that

\[
\| F^{-1} \Psi (\xi) Fu \|_{L^q(\mathbb{R}^n; E)} \leq C \| u \|_{L^p(\mathbb{R}^n; E_1)} \text{ for all } u \in S (\mathbb{R}^n; E_1).
\]

We denote the set of all multipliers from \( L^p (\mathbb{R}^n; E) \) to \( L^q (\mathbb{R}^n; E) \) by \( M^q_p (E) \).

Let \( \Phi_h = \{ \psi_h \in M^q_p (E), h \in \sigma \} \) denote a collection of multipliers depending on the parameter \( h \).

We say that \( W_h \) is a uniform collection of multipliers if there exists a positive constant \( M \) independent of \( h \in \sigma \) such that

\[
\| F^{-1} \psi_h Fu \|_{L^q(\mathbb{R}^n; E)} \leq M \| u \|_{L^p(\mathbb{R}^n; E)}
\]

for all \( u \in S (\mathbb{R}^n; E) \) and \( h \in \sigma \).

**Definition 1.1.** Assume \( E \) is a Banach space and \( r \in [1, 2] \). Suppose there exists a positive constant \( C_0 = C_0 (r, E) \) so that

\[
\| Fu \|_{L^r(\mathbb{R}^n; E)} \leq C_0 \| Fu \|_{L^r(\mathbb{R}^n; E)}
\]

for \( \frac{1}{r} + \frac{1}{s} = 1 \) and each \( u \in S (\mathbb{R}^n; E) \). Then \( E \) is called Fourier type \( r \).

**Remark 1.3.** The simple estimate

\[
\| Ff (x) \|_E \leq \| f \|_{L^1(\mathbb{R}^n; E)}
\]

shows that each Banach space \( E \) has Fourier type 1. Bourgain [4] has shown that each \( B \)-convex Banach space (thus, in particular, each UMD space) has some non-trivial Fourier type \( p > 1 \).

In order to define \( E \)-valued Besov spaces we consider the dyadic-like subsets \( \{ J_k \}_{k=0}^\infty \), \( \{ I_k \}_{k=0}^\infty \) of \( \mathbb{R}^n \) and partition of unity \( \{ \varphi_k \}_{k=0}^\infty \). Let \( 1 \leq r, q \leq \infty \) and \( s \in \mathbb{R} \). The Besov space \( B^{s,r} (\mathbb{R}^n; E) \) is the space of all \( f \in S' (\mathbb{R}^n; E) \) with the norm

\[
\| f \|_{B^{s,r} (\mathbb{R}^n; E)} = \| \{ 2^{ks} (\varphi_k * f) \}_{k=0}^\infty \|_{L^r (L^q (\mathbb{R}^n; E))} = \]

\[= \]
\[ \left\{ \begin{array}{l}
\sum_{k=0}^{\infty} 2^{k r} \| \tilde{\varphi}_k \ast f \|_{L^q(R^n; E)} \leq \infty, \text{ if } 1 \leq r < \infty \\
\sup_{k \in \mathbb{N}_0} \sum_{k=0}^{\infty} 2^{k s} \| \tilde{\varphi}_k \ast f \|_{L^q(R^n; E)} < \infty, \text{ if } r = \infty
\end{array} \right. \]

\( B_{q,r}^s (R^n; E) \)-together with the above norm, is a Banach space. It can be shown that different choices of \( \{ \tilde{\varphi}_k \} \) lead to equivalent norms on \( B_{q,r}^s (R^n; E) \). Here, \( B_{q,r}^s (R^n; E_0) \) denotes the space \( L^q(R^n; E_0) \cap B_{q,r}^s (R^n; E) \) with the norm

\[ \| u \|_{B_{q,r}^s (R^n; E_0)} = \| u \|_{L^q(R^n; E_0)} + \| u \|_{B_{q,r}^s (R^n; E)} < \infty. \]

Let the operator \( A \) be a generator of a strongly continuous cosine operator function in a Banach space \( E \) defined by formula

\[ C(t) = \frac{1}{2} \left( e^{itA^\frac{1}{2}} + e^{-itA^\frac{1}{2}} \right) \]

(see [11, §11.2, 11.4], or [16], [33, 34]). Then, from the definition of sine operator-function \( S(t) \) we have

\[ S(t) u = \int_0^t C(\sigma) u d\sigma \]

and it follows that

\[ S(t) u = \frac{1}{2i} A^{-\frac{1}{2}} \left( e^{itA^\frac{1}{2}} - e^{-itA^\frac{1}{2}} \right). \]

**Lemma 1.1.** Let

\[ \left| 1 + \int_0^T \alpha(\sigma) \beta(\sigma) d\sigma \right| > \int_0^T (|\alpha(\sigma)| + |\beta(\sigma)|) d\sigma. \]

Then the operator \( O \) defined by

\[ O = \left[ 1 + \int_0^T \int_0^T \alpha(\sigma) \beta(\tau) d\sigma d\tau \right] I - \int_0^T (\alpha(s) + \beta(s)) C(s) ds \]

has an inverse \( O^{-1} \) and the following estimate is satisfied

\[ \| O^{-1} \|_{B(E)} \leq \left[ 1 + \int_0^T \alpha(s) \beta(s) ds \right]^{-1} \left[ \int_0^T (|\alpha(s)| + |\beta(s)|) ds \right]^{-1}. \]

The embedding theorems in vector valued spaces play a key role in the theory of DOEs. For estimating lower order derivatives we use following embedding theorem that is obtained from [37, Theorem 1]:

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Theorem A.1. Suppose the following conditions are satisfied:
(1) $E$ is a Banach space and $A$ is a sectorial operator in $E$;
(2) $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ is an $n$-tuples of nonnegative integer number and $s$ is
a positive number such that

$$\kappa = \frac{1}{s} \left[ |\alpha| + n \left( \frac{1}{p_1} - \frac{1}{p_2} \right) \right] \leq 1, 0 \leq \mu \leq 1 - \kappa, 1 \leq p_1 \leq p_2 \leq \infty; 0 < h \leq h_0,$$

where $h_0$ is a fixed positive number.

Then the embedding $D^\alpha B^s_{p_1,r}(\mathbb{R}^n; E(A), E) \subset L^{p_2}(\mathbb{R}^n; E(A^{1 - \kappa - \mu}))$ is
continuous and for $u \in H^{s,p}(\mathbb{R}^n; E(A), E)$ the following uniform estimate holds

$$\|D^\alpha u\|_{L^{p_2}(\mathbb{R}^n; E(A^{1 - \kappa - \mu}))} \leq h^\mu \|u\|_{B^s_{p_1,r}(\mathbb{R}^n; E)} + h^{-(1 - \mu)} \|u\|_{L^{p_1}(\mathbb{R}^n; E)}.$$

By using [15, Theorem 4.3] we obtain:

Proposition A.1. Assume the Banach spaces $E_1, E_2$ have Fourier type $r \in [1, 2]$ and $$\Psi_h \in B_{r-1}^{\left( \frac{n}{p_1} + \frac{n}{p_2} - \frac{1}{r} \right)}(\mathbb{R}^n; B(E_1, E_2)).$$

Then $\Psi_h$ is a uniformly bounded collection of Fourier multiplier from $L^{p_1}(\mathbb{R}^n; E)$ to $L^{p_2}(\mathbb{R}^n; E)$ for $p_1 \leq p_2$ with $p_1, p_2 \in [1, \infty]$.

Proof. First, in a similar way as in [15, Theorem 4.3] we show that $\Psi_h$ is a
uniformly bounded collection of Fourier multiplier from $L^{p_1}(\mathbb{R}^n; E)$ to $L^{p_2}(\mathbb{R}^n; E)$.
Moreover, by Theorem A.1 we get that, for $s \geq n \left( \frac{1}{p_1} - \frac{1}{p_2} \right)$ the embedding $B_{p_1}^{s}(\mathbb{R}^n; E) \subset L^{p_2}(\mathbb{R}^n; E)$ is continuous. From these two fact we obtain the
conclusion.

The paper is organized as follows: In Section 1, some definitions and back-
ground are given. In Section 2, we obtain the existence of unique solution and
a priori estimates for solution of the linearized problem (1.1)-(1.2). In Section
3, we show the existence, uniqueness and estimates of strong solution of the
problem (1.1)-(1.2). In Section 4, the existence, uniqueness and a priori esti-
mates to solution of ICP for finite and infinite many system of wave equation is
derived.

2. Estimates for linearized equation

In this section, we make the necessary estimates for solutions of ICP for the
abstract linear wave equation

$$u_{tt} - \Delta u + Au = g(x,t), \quad x \in \mathbb{R}^n, \ t \in (0, \infty), \quad (2.1)$$

$$u(0,x) = \varphi(x) + \int_0^T \alpha(\sigma) u(\sigma, x) \, d\sigma, \quad (2.2)$$

$$u_t(0,x) = \psi(x) + \int_0^T \beta(\sigma) u_t(\sigma, x) \, d\sigma.$$
**Condition 2.1.** Assume:

\[ \begin{align*}
&\text{(1)} \quad 1 + \int_0^T \alpha(\sigma) \beta(\sigma) \, d\sigma > \int_0^T (|\alpha(\sigma)| + |\beta(\sigma)|) \, d\sigma; \\
&\text{(2)} \quad E \text{ is a Banach space of Fourier type } r \in [1, 2]; \\
&\text{(3)} \quad A \in \sigma(M_0, \omega, E) \text{ and } s > n \left( \frac{1}{p} + \frac{1}{p} \right) \text{ for } p \in [1, \infty).
\end{align*} \]

Let
\[ X_p = L^p(R^n; E), \quad Y^{s,p} = H^{s,p}(R^n; E), \quad Y^{s,p}_1(A) = H^{s,p}(R^n; E(A)) \cap L^1(R^n; E(A)), \quad Y^{s,p}_\infty(A) = H^{s,p} \cap L^\infty(R^n; E(A)). \]

First we need the following lemmas

**Lemma 2.1.** Suppose the Condition 2.1 hold. Moreover, \( \varphi, \psi \in Y^{s,p}_1(A) \). Then problem (2.1)–(2.2) has a unique generalized solution.

**Proof.** By using of the Fourier transform we get from (2.1)–(2.2):

\[ \hat{u}_{tt}(t, \xi) + A_\xi \hat{u}(t, \xi) = \hat{g}(t, \xi), \]

\[ \hat{u}(0, \xi) = \hat{\varphi}(\xi) + \int_0^T \alpha(\sigma) \hat{u}(\xi, \sigma) \, d\sigma, \tag{2.3} \]

\[ \hat{u}_t(0, \xi) = \hat{\psi}(\xi) + \int_0^T \beta(\sigma) \hat{u}(\sigma, \xi) \, d\sigma, \quad \xi \in R^n, \quad t \in (0, T), \]

where \( \hat{u}(\xi, t) \) is a Fourier transform of \( u(x, t) \) with respect to \( x \), where

\[ A_\xi = A + |\xi|^2, \quad \xi \in R^n. \]

Consider the problem

\[ \hat{u}_{tt}(t, \xi) + A_\xi \hat{u}(t, \xi) = \hat{g}(t, \xi), \tag{2.4} \]

\[ \hat{u}(\xi, 0) = u_0(\xi), \quad \hat{u}_t(\xi, 0) = u_1(\xi), \quad \xi \in R^n, \quad t \in [0, T], \]

where \( u_0(\xi) \in D(A) \) and \( u_1(\xi) \in D(A^+) \) for \( \xi \in R^n \). By virtue of [11, §11.2, 11.4] we obtain that \( A_\xi \) is a generator of a strongly continuous cosine operator function and problem (2.4) has a unique solution for all \( \xi \in R^n \), moreover, the solution can be written as

\[ \hat{u}(\xi, t) = C(\xi, t, A) u_0(\xi) + S(\xi, t, A) u_1(\xi) + \]

\[ \int_0^T S(\xi, t - \tau, A) \hat{g}(\xi, \tau) \, d\tau, \quad t \in (0, T), \tag{2.5} \]
where \( C(t, \xi, A) \) is a cosine and \( S(t, \xi, A) \) is a sine operator-functions (see e.g. [11]) with generator of \( A\xi \), i.e.

\[
C(t, \xi, A) = \frac{1}{2} \left( e^{itA\xi} + e^{-itA\xi} \right), \quad S(t, \xi, A) = \frac{1}{2i} A^{-\frac{1}{2}} \left( e^{itA\xi} - e^{-itA\xi} \right).
\]

Using formula (2.5) and nonlocal boundary condition

\[
u_0(\xi) = \hat{\varphi}(\xi) + \int_0^T \alpha(\sigma) \hat{\dot{u}}(\xi, \sigma) \, d\sigma,
\]

we get

\[
u_0(\xi) = \hat{\varphi}(\xi) + \int_0^T \alpha(\sigma) [C(\xi, \sigma, A) \nu_0(\xi) + S(\xi, \sigma, A) \nu_1(\xi)] \, d\sigma +
\]

\[
\int_0^T \int_0^\sigma S(\xi, \sigma - \tau, A) \hat{g}(\xi, \tau) \, d\tau \, d\sigma, \quad \tau \in (0, T).
\]

Then,

\[
\begin{bmatrix}
I - \int_0^T \alpha(\sigma) C(\xi, \sigma, A) \, d\sigma
\end{bmatrix} \nu_0(\xi) = \begin{bmatrix}
\int_0^T \alpha(\sigma) S(\xi, \sigma, A) \, d\sigma
\end{bmatrix} \nu_1(\xi) =
\]

\[
\int_0^T \int_0^\sigma \alpha(\sigma) S(\xi, \sigma - \tau, A) \hat{g}(\xi, \tau) \, d\tau \, d\sigma + \hat{\varphi}(\xi).
\]

Differentiating both sides of formula (2.5) we obtain

\[
\hat{u}_t(\xi, t) = -AS(\xi, t, A) \nu_0(\xi) + C(\xi, t, A) \nu_1(\xi) +
\]

\[
\int_0^t C(\xi, t - \tau, A) \hat{g}(\xi, \tau) \, d\tau, \quad t \in (0, \infty).
\]

Using this formula and integral condition

\[
u_1(\xi) = \hat{\psi}(\xi) + \int_0^T \beta(\sigma) \hat{u}_t(\xi, \sigma) \, d\sigma
\]

we obtain

\[
u_1(\xi) = \hat{\psi}(\xi) + \int_0^T \beta(\sigma) [-AS(\xi, \sigma, A) \nu_0(\xi) + C(\xi, \sigma, A) \nu_1(\xi)] \, d\sigma +
\]
\[
\int_0^T \int_0^\sigma C(\xi, \sigma - \tau, A) \hat{g}(\xi, \tau) \, d\tau \, d\sigma. 
\] 

(2.6)

Thus,
\[
\int_0^T \beta(\sigma) AS(\xi, \sigma, A) \, d\sigma u_0(\xi) + \left[ I - \int_0^T \beta(\sigma) C(\xi, \sigma, A) \, d\sigma \right] u_1(\xi) = \\
\int_0^T \int_0^\sigma \beta(\sigma) C(\xi, \sigma - \tau, A) \hat{g}(\xi, \tau) \, d\tau \, d\sigma + \hat{\psi}(\xi). 
\] 

(2.7)

Now, we consider the system of equations (2.6) and (2.7) in \( u_0(\xi) \) and \( u_1(\xi) \).

The determinant of this system is
\[
D(\xi) = \begin{vmatrix}
\alpha_{11}(\xi) & \alpha_{12}(\xi) \\
\alpha_{21}(\xi) & \alpha_{22}(\xi)
\end{vmatrix},
\]

where
\[
\alpha_{11}(\xi) = I - \int_0^T \alpha(\sigma) C(\xi, \sigma, A) \, d\sigma, \quad \alpha_{12}(\xi) = -\int_0^T \alpha(\sigma) S(\xi, \sigma, A) \, d\sigma,
\]
\[
\alpha_{21}(\xi) = \int_0^T \beta(\sigma) AS(\xi, \sigma, A) \, d\sigma, \quad \alpha_{22}(\xi) = I - \int_0^T \beta(\sigma) S(\xi, \sigma, A) \, d\sigma.
\]

Then by using the properties
\[
[C(\sigma, A) C(\tau, A) + AS(\sigma, A) S(\tau, A)] = I
\]

of sine and cosine operator function \([11, \S11.2, 11.4]\) we obtain
\[
D(\xi) = I - \int_0^T [\alpha(\sigma) + \beta(\sigma)] C(\sigma) \, d\sigma + \\
\int_0^T \int_0^\tau \alpha(\sigma) \beta(\tau) \left[ C(\xi, \sigma, A) C(\xi, \tau, A) + AS(\xi, \sigma, A) S(\xi, \tau A) \right] \, d\sigma d\tau = \\
I - \int_0^T [\alpha(\sigma) + \beta(\sigma)] C(\sigma) \, d\sigma + \int_0^T \int_0^\tau \alpha(\sigma) \beta(\tau) \, d\sigma d\tau = O(\xi).
\]
Solving the system (2.6) – (2.7), we get

\[ u_0 (\xi) = O^{-1} (\xi) \left\{ \left[ I - \int_0^T \beta (\sigma) C (\xi, \sigma, A) d\sigma \right] f_1 + \int_0^T \alpha (\sigma) S (\xi, \sigma, A) d\sigma f_2 \right\} , \]

\[ u_1 (\xi) = O^{-1} (\xi) \left\{ \left[ I - \int_0^T \alpha (\sigma) C (\xi, \sigma, A) d\sigma \right] f_2 - \int_0^T [\beta (\sigma) A_\xi S (\xi, \sigma, A) d\sigma] f_1 \right\} , \]

where

\[ f_1 = \int_0^T \int_0^\sigma \alpha (\sigma) S (\xi, \sigma - \tau, A) \hat{g} (\xi, \tau) d\tau d\sigma + \hat{\phi} (\xi) , \]

\[ f_2 = \int_0^T \int_0^\sigma \beta (\sigma) C (\xi, \sigma - \tau, A) \hat{g} (\tau, \xi) d\tau d\sigma + \hat{\psi} (\xi) . \]

From (2.5), (2.8) and (2.9) we get that, the solution of the problem (2.4) can be expressed as

\[ \hat{u} (t, \xi) = O^{-1} (\xi) \left\{ C (\xi, t, A) \left[ \left( I - \int_0^T \beta (\sigma) C (\xi, \sigma, A) d\sigma \right) f_1 + \int_0^T \alpha (\sigma) S (\xi, \sigma, A) d\sigma f_2 \right] + S (t, \xi, A) \left[ \left( I - \int_0^T \alpha (\sigma) C (\xi, \sigma, A) d\sigma \right) f_2 - \int_0^T \beta (\sigma) A_\xi S (\xi, \sigma, A) d\sigma f_1 \right] \right\} + \int_0^t S (t - \tau, \xi, A) \hat{g} (\tau, \xi) d\tau , \quad t \in (0, T) . \]

Taking into account (2.10) we obtain from (2.11) that there is a generalized solution of (2.1) – (2.2) given by

\[ u (x, t) = S_1 (t, A) \varphi (x) + S_2 (t, A) \psi (x) + \Phi (x, t) , \]

where \( S_1 (t, A) \) and \( S_2 (t, A) \) are linear operator functions in \( E \) defined by

\[ S_1 (t, A) \varphi = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \{ e^{ix\xi} O^{-1} (\xi) \]

\[ \left[ C (t, \xi, A) \left( I - \int_0^T \beta (\sigma) C (\xi, \sigma, A) d\sigma \right) - A_\xi S (\xi, \sigma, A) \right] d\sigma \hat{\varphi} (\xi) d\xi , \]

\[ S_2 (t, A) \psi = \int_{\mathbb{R}^n} \{ e^{ix\xi} O^{-1} (\xi) \]

\[ \left[ C (t, \xi, A) \left( I - \int_0^T \alpha (\sigma) C (\xi, \sigma, A) d\sigma \right) \right] d\sigma \hat{\psi} (\xi) d\xi . \]
\[
S_2(t, A) \psi = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \left\{ e^{ix\xi} O^{-1}(\xi) C(t, \xi, A) \right\} d\xi 
\]

\[
\int_{0}^{T} \left[ \alpha(\sigma) S(\xi, \sigma, A) + S(\xi, \sigma, A) \left( I - \int_{0}^{T} \alpha(\sigma) C(\xi, \sigma, A) d\sigma \right) \right] \hat{\psi}(\xi) d\xi
\]

\[
\left( I - \int_{0}^{T} \beta(\sigma) C(\xi, \sigma, A) - A_\xi S(\xi, \sigma, A) d\sigma \right) \hat{\phi}(\xi) \right\} d\xi,
\]

\[
\Phi(x, t) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} O^{-1}(\xi) e^{ix\xi} \left\{ \int_{0}^{t} S(\xi, t - \tau, A) \hat{g}(\xi, \tau) d\tau + \right\}
\]

\[
\left[ C(\xi, t, A) \left( I - \int_{0}^{T} \beta(\sigma) C(\xi, \sigma, A) d\sigma \right) + \right.
\]

\[
S(\xi, t, A) \int_{0}^{T} \beta(\sigma) A_\xi S(\xi, \sigma, A) d\sigma \right] g_1(\xi) + C(\xi, t, A) \int_{0}^{T} \alpha(\sigma) S(\xi, \sigma, A) d\sigma + \]

\[
S(\xi, t, A) \left( I - \int_{0}^{T} \alpha(\sigma) C(\xi, \sigma, A) d\sigma \right) g_2(\xi) \right\} d\xi,
\]

here

\[
g_1(\xi) = \int_{0}^{T} \int_{0}^{\sigma} \alpha(\sigma) S(\xi, \sigma - \tau, A) \hat{g}(\xi, \tau) d\tau d\sigma, \quad (2.14)
\]

\[
g_2(\xi) = \int_{0}^{T} \int_{0}^{\sigma} \beta(\sigma) C(\xi, \sigma - \tau, A) \hat{g}(\xi, \tau) d\tau d\sigma.
\]

**Lemma 2.2.** Suppose the Condition 2.1 hold. Let \(0 \leq \gamma < \frac{1}{2}\) and \(\varphi \in Y_{1,p}^\gamma(A)\). Then the following uniform estimate holds

\[
\|A_\gamma S_1(t, A) \varphi\|_{X_\infty} \leq C_1 \left[ \|A \varphi\|_{Y_{\gamma,p}} + \|A \varphi\|_{X_1} \right]. \quad (2.15)
\]

**Proof.** Let \(N \in \mathbb{N}\) and

\[
\Pi_N = \{ \xi : \xi \in \mathbb{R}^n, \ |\xi| \leq N \}, \quad \Pi'_N = \{ \xi : \xi \in \mathbb{R}^n, \ |\xi| \geq N \}.
\]

It is clear to see that
\[ \|A^\gamma S_1(t, A) \varphi\|_{X_\infty} = \|A^\gamma F^{-1} C(t, A) \hat{\varphi}\|_{X_\infty} \leq \]

\[ \|F^{-1} A^\gamma C(\xi, t, A) \hat{\varphi}(\xi)\|_{L^\infty(\Pi_N; E)} + \|F^{-1} A^\gamma C(\xi, t, A) \hat{\varphi}(\xi)\|_{L^\infty(\Pi_N; E)} \leq \]

\[ \|F^{-1} A^\gamma C(\xi, t, A) \hat{\varphi}(\xi)\|_{L^\infty(\Pi_N; E)} + \]

\[ \left(1 + |\xi|^2\right)^{-\frac{\gamma}{2}} A^{-(1-\gamma)} C(\xi, t, A) \left(1 + |\xi|^2\right)^{-\frac{\gamma}{2}} A^{-(1-\gamma)} C(\xi, t, A) - \]

\[ it \xi_k \left(1 + |\xi|^2\right)^{-\frac{\gamma}{2}} A^{-(1-\gamma)} S(\xi, t, A). \]

By differentiating again we obtain the same type operator-functions. By applying [11, Theorem 2.1] and replacing \( A \) by \( A_\xi = A + |\xi|^2 \) for \( \xi \in \Pi_N \) we obtain the uniform estimate

\[ \|C(\xi, t, A)\|_{B(E)} \leq M_\xi |\omega|^t, \quad \|S(\xi, t, A)\|_{B(E)} \leq M_\xi |\omega|^t, \]

where

\[ M_\xi = M \left(2 + \lambda^2 \right) |\xi|^2 - |\lambda^2|^{-1} \right) \leq 2M \text{ for all } \lambda \in (0, \infty). \]

Moreover, by resolvent properties of sectorial operator \( A \) we have

\[ \left\| A_\xi^{-(1-\gamma)} \right\| \leq C |\xi|^{-(1-\gamma)}. \]

Hence, by above estimates of \( C(\xi, t, A), S(\xi, t, A) \) and resolvent properties of sectorial operator \( A \) we obtain

\[ \left(1 + |\xi|^2\right)^{-\frac{\gamma}{2}} A_\xi^{-(1-\gamma)} C(\xi, t, A), \quad \left(1 + |\xi|^2\right)^{-\frac{\gamma}{2}} A_\xi^{-(1-\gamma)} C(\xi, t, A) \in \]

\[ W^{m,r}(R^n; B(E)) \subset B_{r,\lambda_{1/2}}(R^n; B(E)) \]

for \( m \geq s > n \left(\frac{1}{r} + \frac{1}{p}\right) \) and \( t \in [0, T]. \) By Proposition A1 from (2.18) we get

\[ \left(1 + |\xi|^2\right)^{-\frac{\gamma}{2}} A_\xi^{-(1-\gamma)} C(\xi, t, A) \in M_p^\infty(E), \]
uniformly in \( t \in [0, T] \), i.e. we have the following estimates

\[
\left\| F^{-1} A^\gamma C (\xi, t, A) \hat{\varphi} (\xi) \right\|_{L^\infty (\Pi_N; E)} \leq M_3 \, \| A\varphi \|_{Y^{\gamma,p}},
\]

\[
\left\| F^{-1} A^\gamma S (\xi, t, A) \hat{\varphi} (\xi) \right\|_{L^\infty (\Pi_N; E)} \leq M_3 \, \| A\varphi \|_{Y^{\gamma,p}}.
\]

Then by Minkowski’s inequality and from (2.16) and (2.17) we obtain (2.15).

**Lemma 2.3.** Suppose the Condition 2.1 hold. Let \( 0 \leq \gamma < \frac{1}{2} \) and \( \psi \in Y^{\gamma,p} (A) \). Then the uniform estimate holds

\[
\| A^\gamma S_2 (t, A) \psi \|_{X_\infty} \leq M_5 \left( \| A\psi \|_{Y^{\gamma,p}} + \| A\psi \|_{X_1} \right).
\]

**Proof.** It is clear that

\[
\| A^\gamma S_2 (t, A) \psi \|_{X_\infty} = \left\| F^{-1} A^\gamma S (t, A) \hat{\psi} \right\|_{X_\infty} \leq \]

\[
\left\| F^{-1} A^\gamma C (\xi, t, A) \hat{\psi} (\xi) \right\|_{L^\infty (\Pi_N; E)} + \left\| F^{-1} A^\gamma C (\xi, t, A) \hat{\psi} (\xi) \right\|_{L^\infty (\Pi_N; E)} \leq \]

\[
\left\| F^{-1} S (\xi, t, A) A^{-\gamma} \hat{\psi} (\xi) \right\|_{L^\infty (\Pi_N; E)} + (2.21)
\]

\[
\left\| F^{-1} \left( 1 + |\xi|^2 \right)^{-\frac{\gamma}{2}} A^{-\gamma} S (\xi, t, A) \left( 1 + |\xi|^2 \right)^{\frac{\gamma}{2}} A\hat{\psi} (\xi) \right\|_{L^\infty (\Pi_N; E)}.
\]

Then by (2.19) and by resolvent properties of sectorial operator \( A \) we obtain

\[
\left\| F^{-1} A^\gamma S (\xi, t, A) \hat{\psi} (\xi) \right\|_{L^\infty (\Pi_N; E)} \leq M_4 \, \| A^\gamma \psi \|_{X_1},
\]

\[
\left\| F^{-1} A^\gamma S (\xi, t, A) \hat{\psi} (\xi) \right\|_{L^\infty (\Pi_N; E)} \leq M_5 \, \| A\psi \|_{Y^{\gamma,p}}.
\]

Hence, from (2.21) and (2.22) we obtain (2.20).

**Lemma 2.4.** Suppose the Condition 2.1 hold. Let \( 0 \leq \gamma < \frac{1}{2} \) and \( g (., t) \in Y^{\gamma,p} \) for \( t \in [0, T] \). Then we have the following uniform estimate

\[
M_6 \left( \| g (., t) \|_{Y^{\gamma,p}} + \| g (., t) \|_{X_1} \right) d\tau.
\]

\[
\int_0^t A^\gamma S_2 (x, t - \tau, A) g (x, \tau) d\tau \leq (2.23)
\]
Proof. By reasoning as it has been made in the above, we get
\[
\left\| \frac{F^{-1}}{t} \int_0^t A^\gamma S(\xi, t - \tau, A) \hat{g}(\xi, \tau) \, d\tau \right\|_{X_{\infty}} \leq (2.24)
\]
\[
\left\| \int_0^t A^\gamma S(\xi, t - \tau, A) \hat{g}(\xi, \tau) \, d\tau \right\|_{L^\infty(I_{\infty}; E)} + \left\| \int_0^t (1 + |\xi|^2)^{-\frac{\gamma}{2}} A^\gamma S(\xi, t - \tau, A) (1 + |\xi|^2)^{\frac{\gamma}{2}} \hat{g}(\xi, \tau) \, d\tau \right\|_{L^\infty(I_{\infty}; E)}.
\]

In view of smoothness $C(\xi, t, A)$ with respect to $\xi \in \mathbb{R}^n$, resolvent properties of sectorial operator $A$ and by Remark 1.2 we get
\[
(1 + |\xi|^2)^{-\frac{\gamma}{2}} \int_0^t A^\gamma S(\xi, t - \tau, A) \, d\tau \in B^n_{q,1}(\mathbb{R}^n; B(E))
\]
for $s > n \left( \frac{1}{p} + \frac{1}{p} \right)$ for all $t \in [0, T]$.

Then by Proposition A1 we get that, the operator-valued functions
\[
\left( 1 + |\xi|^2 \right)^{-\frac{\gamma}{2}} \int_0^t A^\gamma S(\xi, t - \tau, A) \, d\tau \in M^\infty_{p}(E)
\]
uniformly in $t \in [0, T]$. Hence, (2.24) implies (2.23).

**Theorem 2.1.** Assume the Condition 2.1 hold and $0 \leq \gamma < \frac{1}{2}$. Moreover, $\varphi \in Y_{1}^{s,p}(A)$, $\psi \in Y_{1}^{s,p}(A)$ and $g(., t) \in Y_{1}^{s,p}$ for $t \in [0, T]$. Then problem (2.1) – (2.2) has a unique solution $u(x, t) \in C^2([0, T]; Y_{\infty}^{s,p}(A))$ and the following estimate holds
\[
\|A^\gamma u\|_{X_{\infty}} + \|A^\gamma u_t\|_{X_{\infty}} \leq C \left\{ \|A\varphi\|_{Y_{\infty}^{s,p}} + \|A\varphi\|_{X_{1}} + \right\}
\]
\[
\|A\psi\|_{Y_{\infty}^{s,p}} + \|A\psi\|_{X_{1}} + \int_0^t (\|\Delta g(., \tau)\|_{Y_{\infty}^{s,p}} + \|\Delta g(., \tau)\|_{X_{1}}) \, d\tau \right\}
\]
uniformly in $t \in [0, T]$.

**Proof.** From Lemma 2.1 we obtain that the problem (2.1) – (2.2) has a solution $u$. From (2.12) and from the estimates (2.15), (2.20), (2.23) for $0 \leq \gamma < \frac{1}{2}$ we get the estimate
\[
\|A^\gamma u(., t)\|_{X_{\infty}} \leq C \left\{ \|A\varphi\|_{Y_{\infty}^{s,p}} + \|A\varphi\|_{X_{1}} + \|A\psi\|_{Y_{\infty}^{s,p}} + \|A\psi\|_{X_{1}} + \right\}
\]
By differentiating, in view of (2.5) we get from (2.12) the estimate of type (2.26) for \( u_t \), when \( 0 \leq \gamma < \frac{1}{2} \). From this we obtain the estimate (2.25).

**Theorem 2.2.** Assume the Condition 2.1 hold and \( 0 \leq \gamma < \frac{1}{2} \). Moreover, \( \varphi \in Y^{s,p}_1 (A) \), \( \psi \in Y^{s,p}_1 (A) \) and \( g (., t) \in Y^{s,p} \) for \( t \in (0, \infty) \). Then the problem (2.1) - (2.2) has a unique solution \( u (x, t) \in C^2 ([0, T]; Y^{s,p} (A)) \) and the following uniform estimate holds

\[
\| \Delta^\gamma u (., t) \|_{Y^{s,p}} + \| \Delta^\gamma u_t (., t) \|_{Y^{s,p}} \leq \frac{2}{\pi} \int_0^t \| \Delta g (., \tau) \|_{Y^{s,p}} d\tau.
\]

**Proof.** From (2.5) we have the following estimate

\[
\left( \left\| F^{-1} \left( 1 + |\xi|^2 \right)^{\frac{s}{2}} \hat{u} \right\|_{X_p} + \left\| F^{-1} \left( 1 + |\xi|^2 \right)^{\frac{s}{2}} \hat{u}_t \right\|_{X_p} \right) \leq C \left( \left\| A \varphi \right\|_{Y_{s,p}} + \left\| A \psi \right\|_{Y_{s,p}} + \int_0^t \left\| \Delta g (., \tau) \right\|_{Y_{s,p}} d\tau \right).
\]

Then by Proposition A.1 from (2.18) we get

\[
\left( 1 + |\xi|^2 \right)^{-\frac{s}{2}} A_\xi^{-(1-\gamma)} C (\xi, t, A), \quad \left( 1 + |\xi|^2 \right)^{-\frac{s}{2}} A_\xi^{-(1-\gamma)} C (\xi, t, A) \in B^p_{r,1} (R^n; B (E)).
\]

Then by Proposition A.1 from (2.18) we get

\[
\left( 1 + |\xi|^2 \right)^{-\frac{s}{2}} A_\xi^{-(1-\gamma)} C (\xi, t, A) \in M^p_E (E), \quad \left( 1 + |\xi|^2 \right)^{-\frac{s}{2}} A_\xi^{-(1-\gamma)} S (\xi, t, A) \in M^p_E (E),
\]

for all \( t \). So, the relations (2.29) by using the Minkowski’s inequality for integrals implies (2.27).
3. Initial value problem for nonlinear equation

In this section, we will show the local existence and uniqueness of solution to Cauchy problem (1.1) – (1.2).

For the study of the nonlinear problem (1.1) – (1.2) we need the following lemma from [31].

Lemma 3.1 (Abstract Nirenberg’s inequality). Let \( E \) be a UMD space. Assume that \( u \in L^p (\mathbb{R}^n ; E) \), \( D^m u \in L^q (\mathbb{R}^n ; E) \), \( p, q \in (1, \infty) \). Then for \( i \) with \( 0 \leq i \leq m \), \( m > \frac{n}{q} \) we have

\[
\left\| D^i u \right\|_r \leq C \left\| u \right\|_p^{1-\mu} \sum_{k=1}^{n} \left\| D^m_k u \right\|_q^m,
\]

where

\[
\frac{1}{r} = \frac{i}{m} + \mu \left( \frac{1}{q} - \frac{m}{n} \right) + (1-\mu) \frac{1}{p}, \quad \frac{i}{m} \leq \mu \leq 1.
\]

Note that, for \( E = \mathbb{C} \) the lemma considered by L. Nirenberg [31].

Using the chain rule of the composite function, from Lemma 3.1 we obtain the following result

Lemma 3.2. Let \( E \) be an UMD space. Assume that \( u \in W^{s,p} (\mathbb{R}^n ; E) \cap L^\infty (\mathbb{R}^n ; E) \), and \( F(u) \) possesses continuous derivatives up to order \( s \geq 1 \). Then \( F(u) - F(0) \in W^{m,p} (\Omega ; E) \) and

\[
\left\| F(u) - F(0) \right\|_p \leq \left\| F^{(1)}(u) \right\|_\infty \left\| u \right\|_p,
\]

\[
\left\| D^k F(u) \right\|_p \leq C_0 \sum_{j=1}^{k} \left\| F^{(j)}(u) \right\|_\infty \left\| u \right\|_p^{j-1} \left\| D^k u \right\|_p, \quad 1 \leq k \leq s,
\]

where \( C_0 \geq 1 \) is a constant and \( k \) is an integer number.

For \( E = \mathbb{C} \) the lemma coincide with the corresponding inequality in [22]. Let

\[
X = L^p (\mathbb{R}^n ; E), \ Y = W^{s,p} (\mathbb{R}^n ; E(A)), \ E_0 = (X,Y)_{\frac{p}{p}}.
\]

Remark 3.1. By using J. Lions-I. Petree result (see e.g [27] or [40, § 1.8.]) we obtain that the map \( u \to u(t_0) \), \( t_0 \in [0,T] \) is continuous from \( W^{s,p} (0,T; X,Y) \) onto \( E_0 \) and there is a constant \( C_1 \) such that

\[
\left\| u(t_0) \right\|_{E_0} \leq C_1 \left\| u \right\|_{W^{s,p}(0,T;X,Y)}, \quad 1 \leq p \leq \infty.
\]

Let we define the space \( Y(T) = C([0,T]; Y^{s,p}_\infty(A)) \) equipped with the norm defined by

\[
\left\| u \right\|_{Y(T)} = \max_{t \in [0,T]} \left\| u \right\|_{Y^{s,p}} + \max_{t \in [0,T]} \left\| u \right\|_{L^\infty(\mathbb{R}^n ; E(A))}, \quad u \in Y(T).
\]

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It is easy to see that $Y(T)$ is a Banach space. For $\varphi \in Y_{s,p}^\infty (A)$ and $\psi \in Y_{s,p}^\infty (A)$ let 

$$M = \|A\varphi\|_{Y_{s,p}} + \|A\varphi\|_{X^\infty} + \|A\psi\|_{Y_{s,p}} + \|A\psi\|_{X^\infty}.$$ 

**Definition 3.1.** For any $T > 0$ if $\varphi \in Y_{s,p}^\infty (A)$, $\psi \in Y_{s,p}^\infty (A)$ and $u \in C ([0,T] ; Y_{s,p}^\infty (A))$ satisfies the equation (1.1)−(1.2) then $u(x,t)$ is called the continuous solution or the strong solution of the problem (1.1)−(1.2). If $T < \infty$, then $u(x,t)$ is called the local strong solution of the problem (1.1) − (1.2). If $T = \infty$, then $u(x,t)$ is called the global strong solution of the problem (1.1)−(1.2).

**Condition 3.1.** Assume:

1. $E$ is a UMD space and $A \in \sigma (C_0, \omega, E)$;
2. $\varphi \in Y_{s,p}^\infty (A)$, $\psi \in Y_{s,p}^\infty (A)$ and $1 \leq p < \infty$ for $s > n \frac{1}{r} + \frac{1}{p}$;
3. the function $u \to F(u) : R^n \times [0,T] \times E_0 \to E$ is a measurable in $(x,t) \in R^n \times [0,T]$ for $u \in E_0$;
4. $F(x,t,.,.)$ is continuous in $u \in E_0$ for $x \in R^n$, $t \in [0,T]$ and $f(u) \in C^{(1)} (E_0; E)$.

Main aim of this section is to prove the following result:

**Theorem 3.1.** Assume the Condition 3.1 are satisfied. Then problem (1.1)−(2.2) has a unique local strange solution $u \in C^{(2)} ([0, T_0] ; Y_{s,p}^\infty (A))$, where $T_0$ is a maximal time interval that is appropriately small relative to $M$.

Moreover, if

$$\sup_{t \in [0,T_0]} \left( \|u\|_{Y_{s,p}} + \|Au\|_{X^\infty} + \|Au_t\|_{Y_{s,p}} + \|Au_t\|_{X^\infty} \right) < \infty \quad (3.0)$$

then $T_0 = \infty$.

**Proof.** First, we are going to prove the existence and the uniqueness of the local continuous solution of the problem (1.1)−(1.2) by contraction mapping principle. Suppose that $u \in C^{(2)} ([0,T] ; Y_{s,p}^\infty (A))$ is a strong solution of the problem (1.1)−(1.2). Consider a map $G$ on $Y(T)$ such that $G(u)$ is the solution of the Cauchy problem

$$G_{tt}(u) - \Delta G_{tt}(u) + AG(u) = F(G(u)), \ x \in R^n, \ t \in (0,T), \quad (3.1)$$

$$G(u)(x,0) = \varphi(x) + \int_0^T \alpha(\sigma) G(u)(x,\sigma) \, d\sigma,$$

$$G(u)_t(x,0) = \psi(x) + \int_0^T \beta(\sigma) G(u)_t(x,\sigma) \, d\sigma,$$
From Lemma 3.2 we know that \( F(u) \in L^p(0,T;Y_s^p) \) for any \( T > 0 \). Thus, in view of Remark 1.3 and by Theorem 2.1, problem (3.1) has a unique solution which can be written as
\[
G(u)(t,x) = S_1(t,A) \varphi(x) + S_2(t,A) \psi(x) + \tilde{\Phi}(t,x),
\]
where \( S_1(t,A) \), \( S_2(t,A) \) are defined by (2.13) and
\[
\tilde{\Phi}(t,x) = (2\pi)^{-1} \int_{\mathbb{R}^n} O^{-1}(\xi) e^{ix\xi} \left\{ \int_0^t S(t-\tau,\xi,A) \hat{F}(u)(\tau,\xi) d\tau + \right. \]
\[
\left. \begin{bmatrix} C(t,\xi,A) \left( I - \int_0^T \beta(\sigma) C(\sigma,\xi,A) d\sigma \right) + \\ S(t,\xi,A) \int_0^T \beta(\sigma) A_\xi S(\sigma,\xi,A) d\sigma \end{bmatrix} g_1(\xi) + \begin{bmatrix} C(t,\xi,A) \left( I - \int_0^T \alpha(\sigma) C(\sigma,\xi,A) d\sigma \right) g_2(\xi) \end{bmatrix} \right\} d\xi.
\]
Here,
\[
\tilde{g}_1(\xi) = \int_0^\sigma \int_0^\tau \alpha(\sigma) S(\sigma-\tau,\xi,A) \hat{F}(u)(\tau,\xi) d\tau d\sigma, \quad (3.4)
\]
\[
\tilde{g}_2(\xi) = \int_0^\sigma \int_0^\tau \beta(\sigma) C(\sigma-\tau,\xi,A) \hat{F}(u)(\tau,\xi) d\tau d\sigma.
\]
For the sake of convenience, we assume that \( F(0) = 0 \). Otherwise, we can replace \( F(u) \) with \( F(u) - F(0) \). Hence, from Lemma 3.2 we have \( F(u) \in Y_s^p \) iff \( F \in C^{(k)}(R;E) \) for \( k \geq s > 1 \). Consider the operator in \( Y(T) \) defined as
\[
Gu = S_1(t,A) \varphi(x) + S_2(t,A) \psi(x) + \tilde{\Phi}(t,x).
\]
From Lemma 3.2 we get that the operator \( G \) is well defined for \( F \in C(R;E) \). Moreover, from Lemma 3.2 it is easy to see that the map \( G \) is well defined for \( F \in C^{(k)}(X_0;E) \) for \( k \geq s > 1 \) and \( k \in \mathbb{N} \). We put
\[
Q(M;T) = \left\{ u \mid u \in Y(T), \|u\|_{Y(T)} \leq M + 1 \right\}.
\]
Let us prove that the map \( G \) has a unique fixed point in \( Q(M;T) \). For this aim, it is sufficient to show that the operator \( G \) maps \( Q(M;T) \) into \( Q(M;T) \).
and $G : Q ( M ; T ) \rightarrow Q ( M ; T )$ is strictly contractive if $T$ is appropriately small relative to $M$. Consider the function $\tilde{f} ( \xi ) : [ 0, \infty ) \rightarrow [ 0, \infty )$ defined by

$$\tilde{f} ( \xi ) = \max \left\{ \left\| F^{(1)} ( x ) \right\|_E, \ldots, \left\| F^{(k)} ( x ) \right\|_E \right\} ; \xi \geq 0.$$ 

It is clear to see that the function $\tilde{f} ( \xi )$ is continuous and nondecreasing on $[ 0, \infty )$. From Lemma 3.2 for $1 \leq k \leq s$ we have

$$\left\| F ( u ) \right\|_{Y^{k,s}} \leq \left\| F^{(1)} ( u ) \right\|_\infty \| u \|_p + C_0 \sum_{j=1}^{k} \left\| F^{(j)} ( u ) \right\|_\infty \left\| u \right\|_\infty^{j-1} \left\| D^k u \right\|_p \leq 2C_0 \tilde{f} ( M + 1 ) ( M + 1 ) \| u \|_{Y^{s,p}} . \quad (3.6)$$

By using the Theorem 2.1 we obtain from (3.5)

$$\left\| A^s G ( u ) \right\|_{X \infty} \leq \| A \varphi \|_{X \infty} + \left\| A^\frac{s}{2} \psi \right\|_{X \infty} + \int_0^t \left\| \Delta F ( u ( \tau ) ) \right\|_E , \quad (3.7)$$

$$\left\| A^s G ( u ) \right\|_{Y^{2,s}} \leq \left\| A \varphi \right\|_{Y^{2,s}} + \left\| A^\frac{s}{2} \psi \right\|_{Y^{2,s}} + \int_0^t \left\| \Delta F ( u ( \tau ) ) \right\|_{Y^{2,p}} d \tau . \quad (3.8)$$

Thus, from (3.6) – (3.8) and Lemma 3.2 we get

$$\left\| A^s G ( u ) \right\|_{Y(T)} \leq M + T ( M + 1 ) \left[ 1 + 2C_0 ( M + 1 ) \tilde{f} ( M + 1 ) \right] .$$

If $T$ satisfies

$$T \leq \left\{ ( M + 1 ) \left[ 1 + 2C_0 ( M + 1 ) \tilde{f} ( M + 1 ) \right] \right\}^{-1} \left\| G u \right\|_{Y(T)} \leq M + 1 . \quad (3.9)$$

then therefore, if (3.9) holds, then $G$ maps $Q ( M ; T )$ into $Q ( M ; T )$. Now, we are going to prove that the map $G$ is strictly contractive. Assume $T > 0$ and $u_1, u_2 \in Q ( M ; T )$ given. We get

$$G ( u_1 ) - G ( u_2 ) = (2\pi)^{-\frac{n}{2}} \int_{R^n} \int_0^t S ( t - \tau, \xi, A ) \left[ \hat{F} ( u_1 ) ( \xi, \tau ) - \hat{F} ( u_2 ) ( \xi, \tau ) \right] d \tau d \xi .$$

Using the mean value theorem, we obtain

$$\hat{F} ( u_1 ) - \hat{F} ( u_2 ) = \hat{F}^{(1)} ( u_2 + \eta_1 ( u_1 - u_2 ) ) ( u_1 - u_2 ) ,$$

$$D_{\xi} \left[ \hat{F} ( u_1 ) - \hat{F} ( u_2 ) \right] = \hat{F}^{(2)} ( u_2 + \eta_2 ( u_1 - u_2 ) ) ( u_1 - u_2 ) D_{\xi} u_1 +$$
\[ \hat{F}^{(1)}(u_2) (D_\xi u_1 - D_\xi u_2), \]
\[ D_\xi^2 \left[ \hat{F}(u_1) - \hat{F}(u_2) \right] = \hat{F}^{(3)}(u_2 + \eta_3 (u_1 - u_2)) (u_1 - u_2) (D_\xi u_1)^2 + \]
\[ \hat{F}^{(2)}(u_2) (D_\xi u_1 - D_\xi u_2) (D_\xi u_1 + D_\xi u_2) + \]
\[ \hat{F}^{(2)}(u_2 + \eta_4 (u_1 - u_2)) (u_1 - u_2) D_\xi^2 u_1 + \hat{F}^{(1)}(u_2) (D_\xi^2 u_1 - D_\xi^2 u_2), \]
where \(0 < \eta_i < 1, i = 1, 2, 3, 4\). Thus using Holder’s and Nirenberg’s inequality, we have
\[ \left\| \hat{F}(u_1) - \hat{F}(u_2) \right\|_{X_\infty} \leq \bar{f}(M + 1) \|u_1 - u_2\|_{X_\infty}, \quad (3.10) \]
\[ \left\| \hat{F}(u_1) - \hat{F}(u_2) \right\|_{X_p} \leq \bar{f}(M + 1) \|u_1 - u_2\|_{X_p}, \quad (3.11) \]
\[ \left\| D_\xi \left[ \hat{F}(u_1) - \hat{F}(u_2) \right] \right\|_{X_p} \leq (M + 1) \bar{f}(M + 1) \|u_1 - u_2\|_{X_\infty} + \quad (3.12) \]
\[ \bar{f}(M + 1) \left\| \hat{F}(u_1) - \hat{F}(u_2) \right\|_{X_p}, \]
\[ \left\| D_\xi^2 \left[ \hat{F}(u_1) - \hat{F}(u_2) \right] \right\|_{X_p} \leq (M + 1) \bar{f}(M + 1) \|u_1 - u_2\|_{X_\infty} \|u_1 - u_2\|_{X_p} D_\xi^2 u_1 \|
\[ + \bar{f}(M + 1) \|u_1 - u_2\|_{X_\infty} \|D_\xi^2 u_1\|_{X_p} + \bar{f}(M + 1) \|D_\xi^2 (u_1 - u_2)\|_{X_p} \leq \]
\[ C^2 \bar{f}(M + 1) \|u_1 - u_2\|_{X_\infty} \|u_1 - u_2\|_{X_p} D_\xi^2 u_1 \|
\[ + \bar{f}(M + 1) \|u_1 - u_2\|_{X_\infty} \|D_\xi^2 (u_1 - u_2)\|_{X_p} \leq 3C^2 (M + 1)^2 \bar{f}(M + 1) \|u_1 - u_2\|_{X_\infty} + 2C^2 (M + 1) \bar{f}(M + 1) \|D_\xi^2 (u_1 - u_2)\|_{X_p}, \]
where \(C\) is the constant in Lemma 3.1. In a similar way for \(1 \leq k \leq s\) we obtain
\[ \left\| D_\xi^k \left[ \hat{F}(u_1) - \hat{F}(u_2) \right] \right\|_{X_p} \leq \quad (3.13) \]
\[ M_0 \bar{f}(M + 1) \|u_1 - u_2\|_{X_\infty} + M_1 \bar{f}(M + 1) \|D_\xi^k (u_1 - u_2)\|_{X_p}. \]
From (3.10) – (3.13), using Minkowski’s inequality for integrals, Fourier multiplier theorems for operator-valued functions in \(X_p\) and Young’s inequality, we obtain
\[ \|G(u_1) - G(u_2)\|_{Y(T)} \leq (2\pi)^{-\frac{1}{2}} \int_0^T \int_{\mathbb{R}^n} \|u_1 - u_2\|_{X_{\infty}} \|u_1 - u_2\|_{Y_{2,p}} \, d\xi \, dt + \]
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\[
\int_0^t \| F(u_1) - F(u_2) \|_{X_\infty} \, d\tau + (2\pi)^{-\frac{1}{2}} \int_0^t \int_{R^n} \| F(u_1) - F(u_2) \|_{Y_{2,p}} \, d\xi d\tau \leq T \left[ 1 + C_1 (M + 1)^2 \bar{f} (M + 1) \right] \| u_1 - u_2 \|_{Y(T)},
\]
where \( C_1 \) is a constant. If \( T \) satisfies (3.9) and the following inequality
\[
T \leq \frac{1}{2} \left[ 1 + C_1 (M + 1)^2 \bar{f} (M + 1) \right]^{-1},
\]
then
\[
\| Gu_1 - Gu_2 \|_{Y(T)} \leq \frac{1}{2} \| u_1 - u_2 \|_{Y(T)}.
\]
That is, \( G \) is a constructive map. By contraction mapping principle, we know that \( G(u) \) has a fixed point \( u(x, t) \in Q (M; T) \) that is a solution of the problem (1.1) – (1.2). From (3.2) we get that \( u \) is a solution of (3.5). Let us show that this solution is a unique in \( Y (T) \). Let \( u_1, u_2 \in Y (T) \) are two solution of the problem (1.1) – (1.2). Then
\[
u_1 - u_2 = \]
\[
(2\pi)^{-\frac{1}{2}} \int_{R^n}^t \int_0^t \mathcal{S}(t - \tau, \xi, A) \left[ \hat{F}(u_1)(\xi, \tau) - \hat{F}(u_2)(\xi, \tau) \right] \, d\tau d\xi.
\]
By the definition of the space \( Y (T) \), we can assume that
\[
\| u_1 \|_{X_{\infty}} \leq C_1 (T), \quad \| u_1 \|_{X_{\infty}} \leq C_1 (T).
\]
Hence, by Lemmas 3.1, 3.2, Theorem 2.1 and Minkowski’s inequality for integrals we obtain from (3.15)
\[
\| u_1 - u_2 \|_{Y_{s,p}} \leq C_2 (T) \int_0^t \| u_1 - u_2 \|_{Y_{s,p}} \, d\tau.
\]
From (3.16) and Gronwall’s inequality, we have \( \| u_1 - u_2 \|_{Y_{2,p}} = 0 \), i.e. problem (1.1) – (1.2) has a unique solution which belongs to \( Y (T) \). That is, we obtain the first part of the assertion. Now, let \( [0, T_0) \) be the maximal time interval of existence for \( u \in Y (T_0) \). It remains only to show that if (3.0) is satisfied, then \( T_0 = \infty \). Assume contrary that, (3.0) holds and \( T_0 < \infty \). For \( T' \in [0, T_0) \), we consider the following integral equation
\[
u (x, t) = S_1 (t, A) u (x, T) + S_2 (t, A) u_t (x, T) + \hat{\Phi} (t, x).
\]
By virtue of (3.5), for \( T' > T \) we have
\[
\sup_{T \in [0, T_0)} \| A u (., T') \|_{Y_{s,p}} + \| A u (., T') \|_{X_{\infty}} + \| A u_t (., T') \|_{Y_{s,p}} +
\]
\[ \| Au_t (\cdot, T') \|_{X^\infty} < \infty, \]

where
\[
\begin{align*}
u (x, T') &= \varphi (x) + \int_{T'}^T \alpha (\sigma) u (x, \sigma) \, d\sigma, \\
u_t (x, T') &= \psi (x) + \int_{T'}^T \beta (\sigma) u_t (x, \sigma) \, d\sigma.
\end{align*}
\]

By reasoning as a first part of theorem and by contraction mapping principle, there is a \( T^* \in (0, T_0) \) such that for each \( T' \in [0, T_0) \), the equation (3.17) has a unique solution \( v \in Y (T^*) \). The estimates (3.9) and (3.14) imply that \( T^* \) can be selected independently of \( T' \in [0, T_0) \). Set \( T' = T_0 - \frac{T^*}{2} \) and define
\[
\tilde{u} (x, t) = \begin{cases}
u (x, t), & t \in \left[ 0, T' \right] \\
v (x, t - T'), & t \in \left[ T', T_0 + \frac{T^*}{2} \right].
\end{cases}
\]

By construction \( \tilde{u} (x, t) \) is a solution of the problem (1.1)-(1.2) on \( \left[ T', T_0 + \frac{T^*}{2} \right] \) and in view of local uniqueness, \( \tilde{u} (x, t) \) extends \( u \). This is against to the maximality of \( [0, T_0) \), i.e we obtain \( T_0 = \infty \).

4. The nonlocal Cauchy problem for the finite and infinite many system of wave equation

Consider first, the linear problem (1.11). Let (see [40, § 1.18])
\[
l_q (N) = \left\{ u = \{ u_j \}, \ j = 1, 2, ... N, \| u \|_{l_q(N)} = \left( \sum_{j=1}^{N} |u_j|^q \right)^{\frac{1}{q}} < \infty \right\},
\]
and
\[
l_q^\sigma (N) = \left\{ \| u \|_{l_q^\sigma(N)} = \left( \sum_{j=1}^{N} 2^{\sigma j} |u_j|^q \right)^{\frac{1}{q}} < \infty \right\}, \ \sigma > 0.
\]

Here,
\[
X_{pq} = L^p (R^n; l_q), \ Y^{s,p,q} = H^{s,p} (R^n; l_q), \ Y^{s,p,q,s} = H^{s,p} (R^n; l_q^\sigma) \cap L^1 (R^n; l_q),
\]
\[
Y^{s,p,q,s} = H^{s,p} (R^n; l_q^\sigma) \cap L^1 (R^n; l_q), \ Y^{s,p,q,s} = H^{s,p} (R^n; l_q) \cap L^\infty (R^n; l_q),
\]
\[
Y^{s,p,q,s} = H^{s,p} (R^n; l_q^\sigma, l_q).
\]

Condition 4.1. Assume:
\[1 + \int_0^T \alpha (\sigma) \beta (\sigma) d\sigma > \int_0^T (|\alpha (\sigma)| + |\beta (\sigma)|) d\sigma;\]

(2) \( s > n \left( \frac{1}{p} + \frac{1}{q} \right) \) for \( r \in [1, 2] \), \( p \in [1, \infty] \).

From Theorem 2.1 we obtain

**Theorem 4.1.** Assume the Condition 4.1 hold and \( 0 \leq \gamma < \frac{1}{2} \). Moreover, \( \varphi \in Y_{1,0}^{s,p,q,\sigma} \), \( \psi \in Y_{1,1}^{s,p,q,\sigma} \) and \( g (., t) \in Y_{1}^{s,p,q} \) for \( t \in [0, T] \). Then problem (1.11) has a unique solution \( u (x, t) \in C^2 ([0, T]; Y_{s,p,q,\sigma}) \) and the following estimate holds

\[
\| A^\gamma u (., t) \|_{X_{\infty,q}} + \| A^\gamma u_t (., t) \|_{X_{\infty,q}} \leq C \left\{ \| A\varphi \|_{Y_{1,0}^{s,p,q,\sigma}} + \| A\varphi \|_{Y_{1,1}^{s,p,q,\sigma}} + \int_0^t \left( \| \Delta g (., \tau) \|_{Y_{s,p,q}} + \| \Delta g (., \tau) \|_{X_{1,q}} \right) d\tau \right\}
\]

uniformly in \( t \in [0, T] \).

Consider now, the integral Cauchy problem for the following nonlinear system

\[
(u_m)_{tt} - \Delta (u_m)_{tt} + \sum_{j=1}^N a_{mj} u_j (x, t) = F_m (u), \ x \in \mathbb{R}^n, \ t \in (0, T), \quad (4.1)
\]

\[
u_m (x, 0) = \varphi (x) + \int_0^T \alpha (\sigma) u_m (x, \sigma) d\sigma, \quad (4.2)
\]

\[
(u_m)_t (x, 0) = \psi (x) + \int_0^T \beta (\sigma) u_t (x, \sigma) d\sigma, \quad (4.3)
\]

where \( u = (u_1, u_2, ..., u_N) \), \( a_{mj} \) are complex numbers, \( \varphi_m (x) \) and \( \psi_m (x) \) are data functions. Here,

\[
l_q = l_q (N) = \left\{ u = \{u_j\}, \ j = 1, 2, ..., N, \|u\|_{l_q (N)} = \left( \sum_{j=1}^N |u_j|^q \right)^{\frac{1}{q}} < \infty \right\},
\]

(see [44, § 1.18]). Let \( A \) be the operator in \( l_q (N) \) defined by

\[
A = [a_{mj}], \ a_{mj} = g_m 2^{\nu j}, \ m, j = 1, 2, ..., N, \quad D (A) = l_q^p (N) =
\]
\[
\left\{ u = \{u_j\}, \ j = 1, 2, \ldots, N, \|u\|_{l_q(N)} = \left( \sum_{j=1}^{N} 2^{qj} u_j^q \right)^{\frac{1}{q}} < \infty \right\}.
\]

From Theorem 3.1 we obtain the following result

**Theorem 4.1.** Assume:

1. The Condition 4.1 hold and \(0 \leq \gamma < \frac{1}{q}\).
2. The function \(u \to F(u) : R^n \times [0, T) \times E_{0q} \to l_q\) is a measurable function in \((x, t) \in R^n \times [0, T)\) for \(u \in E_{0q}\).
3. The function \(u \to F(u) : R^n \times [0, T) \times E_{0q} \to l_q\) is a measurable function in \((x, t) \in R^n \times [0, T)\) for \(u \in E_{0q}\).
4. Moreover, \(F(u) \in C^{1}(E_{0q}; l_q)\).

Then problem (4.1) - (4.2) has a unique local strange solution \(u \in C^{2}(0, T_0); Y_{s,p,q,0})\), where \(T_0\) is a maximal time interval that is appropriately small relative to \(M\).

Moreover, if

\[
\sup_{t \in [0, T_0]} \left( \|Au\|_{Y_{s,p,q}} + \|Au\|_{X_{\infty,p,q}} + \|Au_t\|_{Y_{s,p,q}} + \|Au_t\|_{X_{\infty,p,q}} \right) < \infty \quad (4.4)
\]

then \(T_0 = \infty\).

**Proof.** By virtue of [43], \(l_q(N)\) is a Fourier type space. It is easy to see that the operator \(A\) is positive in \(l_q(N)\). Moreover, by interpolation theory of Banach spaces [40, § 1.3], we have

\[
E_{0q} = \left( W^{2,p}(R^n; l_q), L^p(R^n; l_q) \right) = B_{p,q}^{2(1 - \frac{1}{p})} \left( R^n; l_q^{(1 - \frac{1}{p})}, l_q \right).
\]

By using the properties of spaces \(Y_{s,p,q}, Y_{\infty,p,q,0}, E_{0q}\), we get that all conditions of Theorem 3.1 are hold, i.e., we obtain the conclusion.

5. The Wentzell-Robin type mixed problem for wave equations

Consider at first, the linear problem (1.7) - (1.9). Here,

\[
X_{p2} = L^p(R^n; L^2(0, 1)), Y_{s,p,2} = H^{s,p}(R^n; L^2(0, 1)), Y_{s,2} = H^{s,p}(R^n; L^2(0, 1)) \cap L^\infty(R^n; L^2(0, 1)), Y_{\infty,2} = H^{s,p}(R^n; L^2(0, 1)) \cap L^\infty(R^n; L^2(0, 1)), Y_{s,2} = H^{s,p}(R^n; H^2(0, 1), L^2(0, 1)).
\]

From Theorem 2.1 we obtain the following result

**Theorem 5.1.** Suppose the the following conditions are satisfied:

1. The Condition 4.1 is hold and \(0 \leq \gamma < \frac{1}{q}\).
(2) $a$ is positive, $b$ is a real-valued functions on $(0, 1)$. Moreover, $a(\cdot) \in C(0, 1)$ and
\[
\exp \left( - \int b(t) a^{-1}(t) \, dt \right) \in L_1(0, 1);
\]
(3) $\varphi, \psi \in Y^{s,p}$ and $g(\cdot, t) \in Y^{s,p}$ for $t \in [0, T]$.

Then the problem $(1.7) - (1.9)$ has a unique solution $u(x, t) \in C^2([0, T]; Y^{s,p,2})$ and the following estimate holds
\[
\|A^2 u\|_{X_{\infty},2} + \|A^2 u_t\|_{X_{\infty},2} \leq C \left\{ \|A\varphi\|_{Y^{s,p,2}} + \|A\varphi\|_{X_{1,2}} + \int_0^T \left( \|\Delta g(\cdot, \tau)\|_{Y^{s,p,2}} + \|\Delta g(\cdot, \tau)\|_{X_{1,2}} \right) d\tau \right\}
\]
uniformly in $t \in [0, T]$.

**Proof.** Let $H = L^2(0, 1)$ and $A$ is a operator defined by $(1.6)$. Then the problem $(1.7) - (1.9)$ can be rewritten as the problem $(2.1) - (2.2)$. By virtue of $[12, 25]$ the operator $A$ generates analytic semigroup in $L^2(0, 1)$. Hence, by virtue of $(1)-(3)$ all conditions of Theorem 2.1 are satisfied. Then Theorem 2.1 implies the assertion.

Consider now, the nonlinear problem
\[
u_{tt} - \Delta u + au_{yy} + bu_y + F(u) = 0, \quad (5.2)
\]
\[
u(x, y, 0) = \varphi(x, y) + \int_0^T \alpha(\sigma) u(x, y, \sigma) \, d\sigma, \quad (5.3)
\]
\[
u_t(x, y, 0) = \psi(x, y) + \int_0^T \beta(\sigma) u_t(x, y, \sigma) \, d\sigma,
\]
\[
a(j) u_{yy}(x, j, t) + b(j) u_y(x, j, t) = 0, \quad j = 0, 1 \text{ for all } t \in [0, T]. \quad (5.4)
\]

Let
\[
E_{0,2} = (X_{p2}, Y^{s,p,2})_{1,p,2}.
\]
Now, we obtain the following result:

**Theorem 5.2.** Suppose the the following conditions are satisfied:
1. \(a\) is positive, \(b\) is a real-valued functions on \((0, 1)\). Moreover, \(a(\cdot) \in C(0, 1)\) and
\[
\exp \left( - \int_{\frac{1}{2}}^x b(t) a^{-1}(t) \, dt \right) \in L_1(0, 1);
\]
2. the Condition 4.1 is hold and \(0 \leq \gamma < 1\);
3. (4) and (5) assumptions of the Conditions 3.1 are satisfied for \(E = L^2(0, 1)\);
4. the function \(u \to F(u) : \mathbb{R}^n \times [0, T] \times E_{02} \to L^2(0, 1)\) is a measurable in \((x, t) \in \mathbb{R}^n \times [0, T]\) for \(u \in E_{02}\); moreover \(f(u) \in C^{(1)}(E_{02}; L^2(0, 1))\).
5. the function \(u \to F(u)\) is continuous in \(u \in E_{02}\) for \(x, t \in \mathbb{R}^n \times [0, T]\); moreover \(f(u) \in C^{(1)}(E_{02}; L^2(0, 1))\).

Then problem (5.2) – (5.4) has a unique local strange solution
\[
u \in C(2) \left( [0, T_0); Y_{s,p,2}^{\infty,2} \right),
\]
where \(T_0\) is a maximal time interval that is appropriately small relative to \(M\). Moreover, if
\[
\sup_{t \in [0, T_0]} \left( \|Au\|_{Y_{s,p,2}^{\infty,2}} + \|Au\|_{X_{s,p,2}^\infty} + \|Au_t\|_{Y_{s,p,2}^{\infty,2}} + \|Au_t\|_{X_{s,p,2}^\infty} \right) < \infty \quad (5.5)
\]
then \(T_0 = \infty\).

**Proof.** By virtue of [43], \(L^2(0, 1)\) is a Fourier type space. It is easy to see that the operator \(A\) is positive in \(L^2(0, 1)\). Moreover, by interpolation theory of Banach spaces [40, § 1.3], we have
\[
E_{02} = \left( W^{s,p} \left( \mathbb{R}^n; H^2(0, 1), L^2(0, 1) \right), L^p \left( \mathbb{R}^n; L^2(0, 1) \right) \right) \frac{1}{2}, 2 =
\]
\[
B^{s(1 - \frac{1}{p})}_{p,2} \left( \mathbb{R}^n; H^{s(1 - \frac{1}{p})}, L^2(0, 1) \right).
\]
By using the properties of spaces \(Y_{s,p,2}^{\infty,2}, Y_{s,p,2}^{\infty,2}, E_{02}\) we get that all conditions of Theorem 3.1 are hold, i.e., we obtain the conclusion.

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