DISCRIMINANTS AND TORIC $K$–THEORY

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ABSTRACT. We discuss a categorical approach to the theory of discriminants in the combinatorial language introduced by Gelfand, Kapranov and Zelevinsky. Our point of view is inspired by homological mirror symmetry and provides $K$–theoretic evidence for a conjecture presented by Paul Aspinwall in a conference talk in Banff in March 2016 and later in a joint paper with Plesser and Wang.

1. Introduction.

In this note, we investigate a conjecture of Aspinwall, Plesser and Wang [APW]. Our calculations offer supporting evidence at the level of toric $K$–theory. We enhance the conjecture and propose a novel point of view on the multiplicities of discriminants which play an important implicit role in understanding the categorical aspects of the underlying toric birational geometry and the related web of spherical functors.

In the projective Calabi–Yau case, homological mirror symmetry [K1] is stated as an equivalence of the bounded derived category of coherent sheaves $D^b(X)$ on a smooth projective Calabi–Yau variety $X$ and the derived Fukaya category $DFuk(Y)$ of the mirror Calabi–Yau variety $Y$. Mirror symmetry and string theory considerations about $D$–brane moduli spaces also predict an identification between the global structure of locally trivial families of categories $D^b(X)$ and $DFuk(Y)$ over the Kähler parameters of $X$ and the complex parameters of $Y$, respectively. As a by–product of his general conjecture, Kontsevich [K2] conjectured that the action on cohomology of the group of self–equivalences of $D^b(X)$ matches the monodromy action on the cohomology of the mirror Calabi–Yau variety $Y$. In particular, for a smooth projective Calabi-Yau variety $X$, the action of of the spherical twist induced by the structure sheaf of $X$ is ”mirrored” by the monodromy action around a distinguished ”primary” component in the moduli space of complex structures of $Y$. While such a statement is implicit in the existing proofs of homological mirror symmetry as well as in the statement of the Strominger–Yau–Zaslow conjecture [SYZ] as identifications...
at large complex/radius limits, our work is concerned with a similar global identification “far” away from such special points in the moduli spaces.

In the present work, we consider the simplified case of a toric quasi-projective Calabi-Yau Deligne-Mumford stack viewed as a resolution of an affine toric Gorenstein singularity. This geometry is determined by a finite set of vectors $A \subset N = \mathbb{Z}^d$ contained in an integral hyperplane at distance 1 from the origin. The proposal by Aspinwall, Plesser and Wang expands Kontsevich’s identification to all the irreducible components of the principal $A$-determinant $E_A$. The polynomial $E_A$ has integer coefficients and generalizes the classical discriminant. It was introduced and studied by Gelfand, Kapranov and Zelevinsky [GKZ] and, among other remarkable features, has the property that its Newton polytope coincides with the secondary polytope $S(A)$ which is combinatorially determined by the starting configuration $A$. One can view the main result of our work as a first step towards the categorification of the principal $A$-determinant $E_A$.

The APW conjecture predicts that, for each non-empty face $\Gamma$ of the polytope $Q = \text{conv}(A)$, there exists a spherical functor $D_\Gamma \to D^b(X)$ corresponding to the component of the $A$-discriminant determined by $\Gamma$. For any stacky fan $\Sigma$ supported on the cone over the polytope $Q = \text{conv}(A)$, let $X = X_\Sigma$ denote the associated toric Deligne-Mumford stack as defined by Borisov, Chen and Smith [BCS]. In Section 3, we give a conjectural construction of the category $D_\Gamma$ as the bounded derived category of coherent sheaves $D^b(X_{\Sigma_\Gamma})$ of the toric DM stack $X_{\Sigma_\Gamma}$. The stacky fan $\Sigma_\Gamma$ is determined by $\Sigma$ and $\Gamma$ (cf. Definition 3.1) and the triangulated category $D_\Gamma$ is independent of the stacky fan $\Sigma$, but its $t$–structure and its image in $D^b(X)$ may change under a toric crepant birational transformation induced by a generalized flop of the stacky fan $\Sigma$. Intuitively, this change corresponds to some interesting wall–crossing phenomena in the Kähler parameter moduli space along the component of the $A$-discriminant determined by $\Gamma$. Moreover, for any edge $F$ of the secondary polytope, there exists a toric Deligne–Mumford stack $Z_F$ and a “wall–monodromy” spherical functor $D^b(Z_F) \to D^b(X)$ (cf. Definition 3.4 and Proposition 3.5). We state the main conjecture as follows.

**Conjecture 3.11.**

1) For any edge $F$ of the secondary polytope, the category $D^b(Z_F)$ admits a semiorthogonal decompositon consisting of $n_{\Gamma,F}$ components
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For each non-empty face $\Gamma$ of the polytope $Q = \text{conv}(A)$, for some explicitly defined algebraic multiplicities $n_{\Gamma,F}$ (cf. Definition 3.8).

2) For any non-empty face $\Gamma$ of the polytope $Q$, there exists a spherical functor $D^b(D_{\Gamma}) \to D^b(X)$ for any toric DM stack $X$ determined by a triangulation corresponding to a vertex of the secondary polytope.

When $n_{\Gamma,F} > 0$, the second part is a direct consequence of the first: the wall monodromy functors $D^b(Z_{F}) \to D^b(X)$ are spherical, so an unpublished result of Kuznetsov, and Halpern-Leistner–Shipman [HLS] implies that each component of the semiorthogonal decomposition determines a spherical functor.

The main result of this work is the following theorem.

**Theorem 3.7.** For any edge $F$ of the secondary polytope, the following equality holds:

$$\text{rk}(K_0(D^b(Z_F))) = \sum_{\Gamma \subset Q} n_{\Gamma,F} \cdot \text{rk}(K_0(D_{\Gamma})), $$

with the summation taken over all the non-empty faces $\Gamma$ of the polytope $Q$.

The theorem lends support to the conjecture by checking its consistency in terms of the ranks of the $K$-theory for the various toric Deligne-Mumford stacks that enter the geometrical picture and local multiplicities of discriminants and their intersecting components.

The importance of these multiplicities has been recognized in the work of Aspinwall, Plesser and Wang [APW], and their relevance is clearly explained in the more recent paper of Kite and Segal [KS]. Such multiplicities are also featured in earlier works [AHK], [HLS]. In the purely categorical sense, they are implicit in the theory of "windows" that began with the string theoretical work of Herbst, Hori and Page [HHP], and was made rigorous by Ballard, Favero, Katzarkov [BFK] and Halpern-Leistner [HL]. Our approach to determining multiplicities can be computationally involved but it is elementary algebraic (see Definition 3.8) and it uses the powerful GKZ toolbox for studying discriminants. Our definition avoids the well known subtleties related to the local topological picture of intersecting discriminants. A categorification of the classical braid factorization technique should offer
a truly conceptual picture of the expected semiorthogonal decompositions beyond the K–theoretic point of view. For a glimpse of how such a procedure might work, see [AHK].

In this note, we do not perform any mirror monodromy calculations for the associated GKZ $D$–module. A true homological mirror symmetry consistency check of the APW conjecture would require such computations, but we leave this discussion for future work. However, by simply combining the results of this work with the analytic monodromy calculations from [H], [BH2], we can check the APW predictions in some cases under certain transversality assumptions, see Corollary 4.2.

It is important to note that the principal A-determinant is determined by the characteristic cycle of the GKZ $D$–module. We expect that the "stacky" nature of the toric geometrical context will require the use of the better behaved GKZ system [BH] and an appropriately adapted version of the topological mirror map.

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2. Review of toric geometry and $A$-discriminants

2.1. Toric Geometry. We briefly review the definition of a toric Deligne-Mumford stack as developed in [BCS] (see also [JT]). A stacky fan $\Sigma$ is defined as the data $(\Sigma, \{v_i\})$ where $\Sigma$ is a simplicial fan in $\bar{N} = N \otimes \mathbb{Z} \mathbb{Q}$, and $\{v_i\} 1 \leq i \leq n$, is a collection of elements in the finitely generated group $N$. We assume that the rays of the simplicial fan $\Sigma$ are generated by the possibly non-minimal non-zero integral elements $\bar{v}_i$.

The set $\{v_i\}$ defines a map $\alpha : \mathbb{Z}^n \to N$ with finite cokernel. If we dualize, we get an exact sequence

$$0 \to K \to G \to (\mathbb{C}^\times)^n \to (\mathbb{C}^\times)^d \to 1,$$

defining the Gale dual of $\alpha$ (see [BCS]), with $K$ a finite abelian group. We apply the functor Hom$(\cdot, \mathbb{C}^\times)$ to get the exact sequence

$$0 \to K \to G \to (\mathbb{C}^\times)^n \to (\mathbb{C}^\times)^d \to 1,$$
where $G$ is the algebraic group $\text{Hom}(N', \mathbb{C}^\times)$.

Consider the subset $Z$ of $\mathbb{C}^n$ that consists of all the points $z = (z_1, \ldots, z_n)$ such that the set of $v_i$ for the zero coordinates of $z$ is contained in a cone of $\Sigma$. Then the toric DM stack $X_\Sigma$ that corresponds to the stacky fan $\Sigma$ is defined as the stack quotient $[Z/G]$ where $Z$ and $G$ are endowed with the natural reduced scheme structures. It has been shown in [BCS] that $X_\Sigma$ is a Deligne-Mumford stack whose moduli space is the simplicial toric variety $X_\Sigma$. The category of coherent sheaves on $[Z/G]$ is equivalent to that of $G$-linearized coherent sheaves on $Z$, see [V, Example 7.21].

To a cone $\sigma$ in $\Sigma$, one can associate a closed substack $X_\sigma$ of $X_\Sigma$ by looking at the quotient of $N$ by the subgroup $N_\sigma$ spanned by $v_i$ with $\bar{v}_i \in \sigma$. The cones of the new fan $\Sigma/\sigma$ are obtained as images in $N/N_\sigma \otimes \mathbb{R}$ of the cones in the star of $\sigma$ in $\Sigma$, consisting of those cones in $\Sigma$ containing $\sigma$ as a subcone. The elements $\bar{v}_i$ for the fan $\Sigma/\sigma$ are images of those $v_i$ that belong to cones in the star of $\sigma$ but are not in $\sigma$. The quotient $N/N_\sigma$ may have torsion, even in the case of a torsionfree $N$. Therefore, the closed substack $X_\sigma$ may not be reduced even though $X_\Sigma$ is.

2.2. Combinatorics of $A$-sets. In this section, we review some concepts discussed in chapter 7 of the book [GKZ]. A finite subset $A = \{v_1, \ldots, v_n\}$ of the integral lattice $N = \mathbb{Z}^d$ is called an $A$-set if the set $A$ generates the lattice $N$ as an abelian group, and there exists a linear function $h : N \to \mathbb{Z}$ such that $h(v_i) = 1$, for all $i, 1 \leq i \leq n$.

For the rest of the paper, we will assume that the set $A = \{v_1, \ldots, v_n\} \subset N = \mathbb{Z}^d$ is an $A$-set. However, see Remark 3.12 below.

We will use the notations

$$Q := \text{conv}(A) \subset \mathbb{R}^{d-1},$$
$$K := \sum_{1 \leq i \leq n} \mathbb{R}_{\geq 0}v_i = \mathbb{R}_{\geq 0}Q \subset N \otimes \mathbb{R} = \mathbb{R}^d,$$
$$S := \sum_{1 \leq i \leq n} \mathbb{Z}_{\geq 0}v_i = \mathbb{Z}_{\geq 0}A \subset N = \mathbb{Z}^d.$$
K. The notion of a regular triangulation will be explained below. There is a natural crepant birational morphism $\pi : X_\Sigma \to Y$ from the induced smooth toric Deligne-Mumford stack $X_\Sigma$ to the toric affine Gorenstein singularity $Y$.

A triangulation of the set $A = \{v_1, \ldots, v_n\}$ is a triangulation of the polytope $Q$ such that all vertices are among the the points $v_1, \ldots, v_n$. A marked polytope $(P, B)$ consists of the convex polytope $P$ and a finite subset $B$ of $P$ such that $P = \text{Conv}(B)$.

**Definition 2.1.** [GKZ, Definition 7.2.1] Given the set $A \subset N = \mathbb{Z}^d$, a polyhedral subdivision $\mathcal{P}$ of $(Q, A)$ is a family of $(d-1)$-dimensional marked polytopes $(Q_i, A_i), i = 1, \ldots, l$, $A_i \subset A$, such that any intersection $Q_i \cap Q_j$ is a face (possible empty) of both $Q_i$ and $Q_j$,

$$A_i \cap (Q_i \cap Q_j) = A_j \cap (Q_i \cap Q_j),$$

and the union of all $Q_i$ coincides with $Q$.

A function $f : Q \to \mathbb{R}$ is said to be $\mathcal{P}$–linear for the polyhedral subdivision $\mathcal{P} = (Q_i, A_i)$ if $f$ is continuous and the restriction of $f$ to each $Q_i$ is affine-linear. A function $\psi : A \to \mathbb{R}$ is said to be $\mathcal{P}$–linearizable if there exists a $\mathcal{P}$–affine function $g_\psi : Q \to \mathbb{R}$ such that $g_\psi(v) = \psi(v)$ for any $v \in \bigcup A_i$. A continuous function $g : Q \to \mathbb{R}$ is concave, if for any $x, y \in Q$, we have $g(tx+(1-t)y) \geq tg(x)+(1-t)g(y)$, $0 \leq t \leq 1$. For any polyhedral subdivision $\mathcal{P} = (Q_i, A_i)$, let $C(\mathcal{P})$ denote the cone of $\mathcal{P}$–linearizable functions $\psi : A \to \mathbb{R}$ such that the associated function $g_\psi : Q \to \mathbb{R}$ is concave and $g_\psi(v) \geq \psi(v)$ for all $v \in A \setminus \cup A_i$. We say that the subdivision $\mathcal{P}$ is regular if the cone $C(\mathcal{P})$ has relative non-empty interior inside the linear space of $\mathcal{P}$–linearizable functions.

For two polyhedral subdivisions $\mathcal{P} = (Q_i, A_i)$ and $\mathcal{P}' = (Q'_j, A'_j)$, we shall say that $\mathcal{P}$ is a refinement $\mathcal{P}'$ if, for each $j$, the collection of $(Q_i, A_i)$ such that $Q_i \subset Q_j$, forms a polyhedral subdivision of $(Q'_j, A'_j)$. The set of polyhedral subdivisions is then naturally a partially ordered set (poset) with respect to refinement. Triangulations are minimal elements of this poset. The maximal element is the polyhedral subdivision $(Q = \text{conv}(A), A)$.

The cones $C(\mathcal{T})$ for all the regular triangulations of $(Q, A)$ together with all the faces of these cones form a complete fan $F(A)$ in $\mathbb{R}^A = \mathbb{R}^n$ called the secondary fan. We choose a translation invariant volume form $\text{vol}$ in $\mathbb{R}^{n-1}$ (or in $\mathbb{R}^n$) such that elementary simplex in the lattice $N = \mathbb{Z}^n \subset \mathbb{R}^n$ has volume 1. The characteristic function of the
triangulation $T$ is the function $\phi_T : A \to \mathbb{R}$ defined by

$$\phi_T(v) := \sum_{v \in \text{Vert}(\sigma)} \text{vol}(\sigma),$$

where the summation is taken over all the maximal simplices of the triangulations $T$ for which $v$ is vertex. We set $\phi(v) = 0$ if $v$ is not a vertex of simplex in $T$. The secondary polytope $S(A)$ is the convex hull in $\mathbb{R}^A = \mathbb{R}^n$ of the functions $\phi_T$ for all the triangulations $T$ of $(Q,A)$.

Theorem 7.1.7 in [GKZ] shows that the secondary polytope is $n-d$ dimensional and its vertices are the characteristics functions $\phi_T$ for all triangulations $T$ of $(Q,A)$. Moreover, the secondary fan $F(A)$ is the normal fan of the secondary polytope $S(A)$. We identify $\mathbb{R}^A = \mathbb{R}^n$ with its dual in the canonical way.

Consider two regular triangulations such that the corresponding vertices in the secondary polytope are joined by an edge. The two triangulations differ by a modification along a circuit. A circuit in $A$ is a minimal dependent subset $\{v_i, i \in I\}$ with $I \subset \{1, \ldots, n\}$. In particular any circuit determines an integral relation of the form

$$\sum_{i \in I_+} l_i v_i + \sum_{i \in I_-} l_i v_i = 0,$$

with $I = I_+ \cup I_-$, where the two subsets $I_+ := \{i : l_i > 0\}$ and $I_- := \{i : l_i < 0\}$ are uniquely defined by the circuit up to replacing $I_+$ by $I_-$. We will assume that the relation (1) is primitive, i.e. that the integers $l_i$ have no common prime factor. Given a circuit, one can write a (possibly) non-primitive relation by choosing $|l| := \text{vol}(\text{conv}(v_i, i \in I \setminus i))$.

When there is no danger of confusion, we may call the index subset $I$ a circuit. The minimizing condition in the definition of a circuit implies that the sets $\text{conv}(v_i, i \in I_+)$ and $\text{conv}(v_i, i \in I_-)$ intersect in their common interior point. The polytope determined by $v_i, i \in I$, admits exactly two triangulations $T_+(I)$ and $T_-(I)$ defined by the simplices $\text{conv}(\{v_j, j \in I \setminus i\}$, for $i \in I_+$, respectively $i \in I_-$.

Suppose that the regular triangulations $T$ and $T'$ of $A$ are obtained from each other by a modification along a circuit $I$. We say that a subset $J \subset A \setminus I$, is separating for $T$ and $T'$ if, for some $i \in I$, the set of $v_i$, with $i \in (I \setminus i) \cup J$ is the set of vertices of a simplex (of maximal dimension) of $T_i$. It turns out that a separating set $J$ has the property that the sets $(I \setminus i) \cup J$, for $i \in I_+$, determine simplices in $T$ and the sets $(I \setminus i) \cup J$, for $i \in I_-$, determine simplices in $T'$. 
Proposition 2.2. [GKZ, Prop 7.2.12] The polyhedral subdivision $\mathcal{F} = \mathcal{F}(\mathcal{T}, \mathcal{T}')$ corresponding to the edge of the secondary polytope joining the vertices determined by $\mathcal{T}$ and $\mathcal{T}'$ consists of the simplices $(\text{conv}(K), K)$ which $\mathcal{T}$ and $\mathcal{T}'$ have in common and the polyhedra $(\text{conv}(I \cup J), I \cup J)$ for all separating subsets $J \subset A \setminus I$.

2.3. Principal $A$-Determinants. Define $\nabla_A$ as the Zariski closure in $\mathbb{C}^n$ of the set of polynomials $f = \sum_{1 \leq j \leq n} a_i x^{v_i}$ in $\mathbb{C}[x_1, \ldots, x_d]$ such that there exists some $y \in (\mathbb{C}^\times)^n$ with the property that $f = 0$ is singular at $y$. By definition, the discriminant $\Delta_A \in \mathbb{Z}[a_1, \ldots, a_n]$ is the irreducible polynomial (defined up to a sign) whose zero set is given by the union of the irreducible codimension 1 components of $\nabla_A$. For the case $\text{codim} \nabla_A > 1$, one sets $\nabla_A = 1$.

For any non-empty face $\Gamma$ of the polytope $Q$, let $i(\Gamma)$ denote the index of the sublattice $\mathbb{Z}(A \cap \Gamma)$ inside the lattice $N \cap \mathbb{R}\Gamma$,

$$i(\Gamma) := [N \cap \mathbb{R}\Gamma : \mathbb{Z}(A \cap \Gamma)].$$

Recall that $S = \mathbb{Z}_{\geq 0}A$ is the semigroup generated by $A$. Furthermore, let $S/\Gamma$ denote the image semigroup of $S$ in the quotient free group $N_\mathbb{R} / \mathbb{R}\Gamma$, with $N_\mathbb{R} = N \otimes \mathbb{Z} \mathbb{R}$. The semigroup $S/\Gamma$ generates a pointed cone in $N_\mathbb{R} / \mathbb{R}\Gamma$, so we can define

$$u(\Gamma) := \text{vol}(\text{conv}(S/\Gamma) \setminus \text{conv}(S/\Gamma \setminus \{0\})),$$

where the volume form on $N_\mathbb{R} / \mathbb{R}\Gamma$ is induced from the standard volume form on $N$.

Definition 2.3. [GKZ, Theorem 10.1.2] The principal $A$-determinant $E_A$ is the polynomial in $\mathbb{Z}[a_1, \ldots, a_n]$ defined as

$$E_A := \prod_{\Gamma} (\Delta_A \cap \Gamma)^{u(\Gamma) \cdot i(\Gamma)},$$

where the product is taken over all the non-empty faces of the polytope $Q = \text{conv}(A)$.

Note that we have taken the liberty of not using the definition given in [GKZ]. Of course, the above definition is equivalent to the one given in that book. For the faces $\Gamma$ such that $\text{codim} \nabla_A > 1$, the corresponding factor in the product is equal to 1. This implies that the product is taken over the vertices and the faces $\Gamma$ such that $A \cap \Gamma$ is not simplicial and there is no proper subface $\Gamma'$ of $\Gamma$ such that the linear relations among the elements of $A \cap \Gamma$ are linear relations among the elements of $A \cap \Gamma'$.

The principal $A$–determinant has the following remarkable property.
Theorem 2.4. [GKZ, Theorem 10.1.4] For a given set \( A \), the Newton polytope of \( E_A \) coincides with the secondary polytope \( S(A) \). In particular, the vertices of the Newton polytope of \( E_A \) are given by the characteristic functions \( \phi_T \) for all the regular triangulations \( T \) of \( (Q, A) \).

3. K-theory and discriminant intersection multiplicities

We will now compute the dimensions of the Grothendieck rings of some toric stacks involved in the conjecture discussed in this paper. Let \( \Gamma \) be a non-empty face of the polytope \( Q = \text{conv}(A) \).

Definition 3.1. The stacky fan \( \Sigma_\Gamma = (\Sigma_\Gamma, \{\pi(v_i)\}_{i \in I}) \), with \( \pi : N \to N/\mathbb{Z}(A \cap \Gamma) \) the canonical map, is defined by a simplicial fan \( \Sigma_\Gamma \) in the real linear space \( (N/\mathbb{Z}(A \cap \Gamma)) \otimes \mathbb{R} = N_{\mathbb{R}}/\mathbb{R}\Gamma \) with rays generated by the images of the vectors \( v_i \) for \( i \in I \). The set \( I \) consists of all indices \( i \) such that the image of \( v_i \) in \( N_{\mathbb{R}}/\mathbb{R}\Gamma \) is contained in the closure of the set

\[
\text{conv}(S/\Gamma) \setminus \text{conv}(S/\Gamma \setminus \{0\}),
\]

where \( S/\Gamma \) is the image semigroup \( S = \mathbb{Z}A \) in \( N_{\mathbb{R}}/\mathbb{R}\Gamma \).

It is important to note that the set \( A \) and the nonempty face \( \Gamma \) uniquely determine the vectors \( v_i, i \in I \), but the cones of the simplicial fan \( \Sigma_\Gamma \) are not uniquely determined. However, the following statement holds.

Proposition 3.2. For any two choices of stacky fans \( \Sigma_\Gamma \) and \( \Sigma'_\Gamma \) as above, the bounded derived categories of coherent categories \( D^b(X_{\Sigma_\Gamma}) \) and \( D^b(X_{\Sigma'_\Gamma}) \) are equivalent.

Proof. The stacky fans \( \Sigma_\Gamma \) and \( \Sigma'_\Gamma \) are connected by a finite sequence of toric birational flops. Each of them induces an equivalence of the bounded derived categories of coherent sheaves on the corresponding DM stacks [BO], [Ka], so the result follows. \( \square \)

Moreover, for any choice of the stacky fan \( \Sigma_\Gamma \), we can prove the following result.

Proposition 3.3. The rank of the Grothendieck group of the DM stack \( X_{\Sigma_\Gamma} \) is equal to the product \( u(\Gamma) \cdot i(\Gamma) \).

Proof. Since \( N \) is a lattice, the quotient group \( N/N \cap \mathbb{R}\Gamma \) is torsion free. This implies that the torsion part of the quotient \( N/\mathbb{Z}(A \cap \Gamma) \) is isomorphic to the torsion part of the quotient \( N \cap \mathbb{R}\Gamma/\mathbb{Z}(A \cap \Gamma) \). The order of the torsion part is given by the index \( i(\Gamma) \). The discussion in section 6 of [BH1], as well as the work of Jiang-Tseng [JT], show that the rank of \( K_0(X_{\Sigma_\Gamma}) \) is equal to \( K_0(X_{\Sigma_\Gamma}) \cdot i(\Gamma) \).
In order to finish the proof, we have to show that the dimension of $K_0(X_{\Sigma})$ is equal to $u(\Gamma)$. Formula $3$ implies that $u(\Gamma)$ is equal to the sum of the volumes of the (simplicial) maximal dimensional cones in the fan $\Sigma_{\Gamma}$. The result follows from [DKK, Proposition 3.20] which shows that the desired rank is equal to the sum of the volumes of the maximal dimensional cones in the fan $\Sigma_{\Gamma}$. $\square$

We now consider two triangulations $\mathcal{T}$ and $\mathcal{T}'$ corresponding to vertices of the secondary polytope which are joined by an edge $F$. According to Proposition $2.2$, there exists a circuit $I = \{v_i\}, i \in I$, associated to this edge. Let $\mathbb{Z}I := \sum_{i \in I} \mathbb{Z}v_i$ be the sublattice generated in $\mathbb{N}$ by the vectors of the circuit. Recall that the difference between the simplices in $\mathcal{T}$ and $\mathcal{T}'$ is determined by the separating sets $J \subset A \setminus I$ (cf. Proposition $2.2$). Note that for any separating set $J$, the cone determined by the polytope $\text{conv}(J)$ is simplicial. Let $K \subset A$ denote the subset of elements in $A$ that belong to some separating set associated to the edge $F$.

**Definition 3.4.** The stacky fan $\Sigma_F = (\Sigma_F, \{\pi(v_i)\}_{i \in K})$, with $\pi : \mathbb{N} \to \mathbb{N}/\mathbb{Z}I$ the canonical map, is defined by the fan $\Sigma_F$ in the real linear space $\mathbb{N}_R/\mathbb{R}I$ whose maximal cones are images under the map $\pi$ of cones over $\text{conv}(J)$, for some separating set $J$.

Denote by $Z_F$ the toric DM stack defined by the stacky fan $\Sigma_F$. For the general theory of spherical functors, see [AL].

**Proposition 3.5.** There exists a “wall monodromy” spherical functor $D^b(Z_F) \to D^b(X)$ where $X$ is the toric DM stack induced by either one of the triangulations $\mathcal{T}$ or $\mathcal{T}'$.

*Proof.* The EZ-twist argument in [EZ] is mostly formal and can be adapted to the toric DM stacky context. The result also follows from the work of Halpern-Leistner-Shipman [HLS]. In the physics literature, the spherical twist induced by this spherical functor is known as the “wall monodromy”. $\square$

**Proposition 3.6.** The rank of the Grothendieck group of the DM stack $Z_F$ is equal to the sum

$$\sum_{J \text{- sep}} [N : \mathbb{Z}(I \cup J)]$$

over all the separating sets $J$.

*Proof.* Since $I \cup J$ is a simplex, and $\mathbb{Z}I \cap \mathbb{Z}J = (0)$, we can write a non-canonical splitting for the torsion group $N/\mathbb{Z}(I \cup J)$ as a direct sum of
finite torsion groups $\mathbb{Z}^k/\mathbb{Z}I \oplus \mathbb{Z}^{d-k}/\mathbb{Z}J$. Note that the index $[\mathbb{Z}^{d-k}/\mathbb{Z}J]$ is the volume of the cone generated by the separating set $J$ in the fan $\Sigma_I$ in $N_{\mathbb{R}}/\mathbb{R}I$ associated to the toric DM stack $Z_F$. As in the proof of Proposition 3.3 we conclude that the rank of the Grothendieck group of the toric DM stack is indeed the sum $\sum_J [N : \mathbb{Z}(I \cup J)]$ over all the separating sets $J$.

The main result of this note is the following theorem.

**Theorem 3.7.** For any edge $F = [\phi_T, \phi_{T'}]$ of the secondary polytope, between the triangulations $T$ and $T'$. the following equality holds:

$$\text{rk}(K_0(D^h(Z_F))) = \sum_{\Gamma \subseteq Q} n_{\Gamma,F} \cdot \text{rk}(K_0(D_\Gamma)),$$

for some non-negative integers $n_{\Gamma,F}$, and the summation is taken over all the non-empty faces $\Gamma$ of the polytope $Q$.

The precise combinatorial definition of the non-negative integers $n_{\Gamma,F}$ will be given below in Definition 3.8. It arises naturally as a by-product of the proof of the theorem and it is close in spirit to the string theoretical analysis in [AGM]. Geometrically, the index $n_{\Gamma,F}$ coincides in most cases with the intersection multiplicity of the discriminant component $\nabla_{\Gamma,F}$ with the rational curve induced by edge $F$ in the DM toric stack defined by the secondary fan. This point of view has been discussed in [HLS], [KS].

**Proof.** We will prove the result by giving two interpretations to the coefficient restriction of the principal $A$-determinant $E_A$ to the edge $F$ of the secondary polytope. By [GKZ, Prop 6.1.3], the coefficient restriction of any Laurent polynomial $P(x_1, \ldots, x_n)$ to any face $F$ of its Newton polytope $S(A)$ has the leading term of the polynomial in $t$

$$t \mapsto P(t^{\psi(x_1, \ldots, x_n)})$$

equal to $t^{\psi(F)} P_F(x_1, \ldots, x_n)$, where $\psi$ is a linear support function for the face $F$, and $P_F$ is the coefficient restriction of $P$ to the face $F$. Recall that $\psi : \mathbb{R}^A \to \mathbb{R}$ is a linear support function for the face $F$ if $F$ is the maximal face the secondary polytope $S(A)$ where $\psi$ attains its maximum value.

Let $P = (Q_i, A_i)$ be a polyhedral subdivision of the marked polytope $(Q, A)$. Theorem 10.1.12' in [GKZ] shows that, up to multiplication by a constant, the coefficient restriction of $E_A$ to the edge $F(P)$ of the secondary polytope equals

$$\prod_i (E_{A_i})^{[N : \mathbb{Z}A_i]}.$$
In particular, if $\mathcal{F}$ is the polyhedral subdivision corresponding to the circuit $I$ and the edge $[\phi_T, \phi_T']$ for two triangulations $\mathcal{T}, \mathcal{T}'$ (see Proposition 2.2), then, up to multiplication by a constant, $E_A$ is equal to

$$\prod_{i \in K} (E_A)_{[N:ZA_i]} \cdot \prod_{J - \text{sep}} (E_{I \cup J})_{[N:Z(I \cup J)]},$$

where $(\text{conv}(A_i), A_i)_{i \in K}$ is the family of simplices that $\mathcal{T}$ and $\mathcal{T}'$ have in common, and the second product is taken over all the separating sets $J$ such that $J \cup I$ has maximal dimension. Note that each such $J \cup I$ contains as a unique circuit the circuit $I$. In the terminology of [GKZ, pg 309], the set $J \cup I$ is weakly dependent. As such, the argument on [GKZ, pg 309] shows that, up to a multiplication by a constant, $E_{I \cup J}$ is in fact equal to the the discriminant $\Delta_{I \cup J}$, which by [GKZ, Prop 9.1.8] is a non-zero scalar multiple of the polynomial

$$(6) \quad \Delta_I := \left( \prod_{i \in I_+} l_i^i \right) \prod_{i \in I_-} a_i^{-l_i} - \left( \prod_{i \in I_-} l_i^{-l_i} \right) \prod_{i \in I_+} a_i^l_i,$$

where $\sum_{i \in I_+ \cup I_-} l_i v_i = 0$ is a primitive integer relation (unique up to sign) associated to the circuit $I$. Note [GKZ, Prop 9.1.8] assumes that the circuit generates the ambient lattice so the choice $|l_i| := \text{vol}(\text{conv}(v_i, i \in I \setminus i))$, would work in that case. We do not make that assumption.

Consider a linear form $\psi : \mathbb{R}^A \to \mathbb{R}$ lying in the interior of cone $C(\mathcal{F})$ (the normal cone to the edge $F$) of the secondary fan as defined in section 2.2. This means that the edge $F$ is the supporting face of $\psi$. In particular, $\psi$ is equal to a constant $\psi(F)$ along $F$, and $\psi(v_i) < \psi(F)$ when $v_i$ is not in $F$. It follows that the coefficient of the leading term in $t$ of the Laurent polynomial $t \mapsto E_A(\sum_i t^{\psi} a_i x^{v_i})$ is, up to multiplication by a constant and a monomial in the variables $a_i$, the product

$$(7) \quad \prod_{J - \text{sep}} (E_{I \cup J})_{[N:Z(I \cup J)]} = (\Delta_I)_{\sum_{J - \text{sep}} [N:Z(I \cup J)]}$$

where the sum and product in the formula above are over all the separating sets $J$, the discriminant polynomial $\Delta_I$ is defined by (6).

On the other hand, according to Definition 2.3 the principal $A$-determinant $E_A$ is the product

$$E_A := \prod_{\Gamma} ((\Delta_{A \cap \Gamma})^{\alpha(\Gamma) - \epsilon(\Gamma)}),$$

where the product is taken over all the non-empty faces of the polytope $Q = \text{conv}(A)$. Due to the properties of Minkovski sums for Newton polytopes, the linear form $\psi$ attains its maximum value on the Newton
polytope of $\Delta_{A\cap \Gamma}$ at either a vertex or an edge parallel to $F$. This means that, up to multiplication by a constant and a monomial in in the variables $a_i$, the product the coefficient of the leading term in $t$ of the Laurent polynomial $t \mapsto \Delta_{A\cap \Gamma}(\sum_i t^{\psi_i} a_i x^{v_i})$ is a power of $\Delta_I$.

**Definition 3.8.** The multiplicity $n_{\Gamma,F}$ is the non-negative integer defined by the property that, up to multiplication by a constant and a monomial in the variables $a_i$, the coefficient of the leading term in $t$ of the Laurent polynomial $t \mapsto \Delta_{A\cap \Gamma}(\sum_i t^{\psi_i} a_i x^{v_i})$ is equal to $(\Delta_I)^{n_{\Gamma,F}}$.

The theorem follows as a direct consequence of the results of Propositions 3.3 and 3.6.

**Remark 3.9.** Note that $n_{\Gamma,F} = 0$, when $\Gamma$ is a vertex of $Q$. Furthermore, the definition shows that, if the circuit $I$ determined by the edge $F$ is not contained in the face $\Gamma$, then $n_{\Gamma,F} = 0$.

**Remark 3.10.** In practice, calculating the multiplicities $n_{\Gamma,F}$ can be computationally daunting. We will provide some examples in the next section. On the other hand, this definition avoids the well known difficulties related to the local topological picture of intersecting discriminants. Moreover, our definition does not use the Horn uniformization (see Kapranov [Kap]) of the $A$–discriminant which is a useful tool but only provides a birational parametrization of the discriminant.

The previous theorem adds to the body of evidence supporting the conjecture of Aspinwall-Plesser-Wang [APW]. The importance of multiplicities and other aspects of this conjecture have been discussed by Kite–Segal [KS].

**Conjecture 3.11.**

1) For any edge $F$ of the secondary polytope, the category $D^b(Z_F)$ admits a semiorthogonal decompositon consisting of $n_{\Gamma,F}$ components $D^b(X_{\Sigma_{\Gamma}})$ for each non-empty face $\Gamma$ of the polytope $Q = \text{conv}(A)$.

2) For any non-empty face $\Gamma$ of the polytope $Q$, there exists a spherical functor $D^b(D_\Gamma) \to D^b(X)$ for any toric DM stack $X$ determined by a triangulation corresponding to a vertex of the secondary polytope.

**Remark 3.12.** One of the hypotheses in the definition of an $A$-set is that the set $A$ generates the lattice $N = \mathbb{Z}^d$. This hypothesis guarantees that the multiplicity of the "primary" component of the discriminant associated to the face $\Gamma = Q$ has $u(\Gamma) = i(\Gamma) = 1$ and $Z_\Gamma$ is a point Spec $\mathbb{C}$. This assumption is likely not essential. We could simply ask that $A$ linearly generates the vector space $N_\mathbb{R} = \mathbb{R}^d$. The adapted results of this note should continue to hold, although the multiplicities involved
gain the extra factor $[N :ZA]$. On the other hand, the technical details in [GKZ] are obtained for $A$-sets in this more restricted context, so one would have to work out and adapt the calculations to cover the more general situation.

4. Applications

Example 4.1. Recall that for a general set $A$ in the lattice $N = \mathbb{Z}^d$, the rational polyhedral cone $K$ generated by $A$ defines a Gorenstein affine toric variety $Y = \text{Spec} \mathbb{C}[K^\vee \cap N^\vee]$. For any stacky resolution $\pi : X_\Sigma \to Y$ induced by the regular triangulation of the polytope $Q$, the structure of the exceptional set is quite complicated. It may contain irreducible components of different dimensions. Consider an edge of the secondary polytope and let $I = I_+ \cup I_-$ be the associated circuit in $A$. We assume that $I$ is maximal dimension, $\dim I = \dim C$. Let $X_+$ and $X_-$ denote the corresponding toric stacky resolutions of $Y$ that differ by a flop. In this case $Z_F$ is the stacky point $[\text{Spec} \mathbb{C}/\mathbb{Z}_n]$ where $n$ is the index of the sublattice generated by the circuit $I$ in $N, n = [N : \mathbb{Z}I]$. The sets $I_+$ and $I_-$ generate cones that determine the exceptional loci $E_+ \subset X_+$, and $E_- \subset X_-$, and, by Proposition 3.5, we have spherical functors $\mathbb{D}^b(\text{Spec} \mathbb{C}/\mathbb{Z}_n) \to \mathbb{D}^b(E_+) \to \mathbb{D}^b(X_+)$. The category $\mathbb{D}^b([\text{Spec} \mathbb{C}/\mathbb{Z}_n])$ splits into $n$ orthogonal copies of $\mathbb{D}^b(\text{Spec} \mathbb{C})$, so we get the spherical functors predicted by Kontsevich’s conjecture described in the introduction. Theorem 3.7 implies that $n_{F,Q} = n$, where $F$ is the edge of the secondary polytope associated to the flop. This multiplicity statement is not obvious if one attempts a direct calculation. In the transversal case $n_{F,Q} = 1$, the analytic monodromy calculations from [H], [BH2] for the classical GKZ $D$–module are valid, so we get that:

Corollary 4.2. The wall monodromy spherical functor induced by the edge $F$ is the spherical functor $\mathbb{D}^b(\text{Spec} \mathbb{C}) \to \mathbb{D}^b(E_F) \to \mathbb{D}^b(X)$, where $E_F \subset X$ is the exceptional locus corresponding to the circuit $I$. The $K$–theory action of this functor matches the analytical continuation functor along a corresponding loop $\gamma_F$ (see [BH2]).

A similar argument expands the range of cases where Kontsevich’s conjecture is true beyond the the cases considered in [H]. We hope to return to this issue in future work.

Example 4.3. Let $X$ be the resolution of the $A_3$ singularity, with $v_0 = (1,0), v_1 = (1,1), v_2 = (1,2), v_3 = (1,3), v_4 = (1,4)$. As it is well known, the secondary polytope is combinatorially equivalent to a cube in $\mathbb{R}^3$. 
The principal $A$–determinant is
\[
E_A = a_0 a_4 \Delta_Q = a_0 a_4 (256 a_0^3 a_4^3 - 192 a_0^2 a_1 a_3 a_4^2 - 128 a_0^2 a_2 a_4^2 \\
+ 144 a_0^2 a_2 a_3 a_4 - 27 a_0^2 a_4^4 + 144 a_0 a_1^2 a_2 a_4^2 - 6 a_0 a_1^2 a_3 a_4^2 - 80 a_0 a_1 a_2^2 a_3 a_4 \\
+ 18 a_0 a_1 a_2 a_3^3 + 16 a_0 a_2^3 a_4 - 4 a_0 a_3^3 a_4^2 - 27 a_1^4 a_4^2 \\
+ 18 a_1^3 a_2 a_3 a_4 - 4 a_1^3 a_2^3 a_4 + a_1^2 a_2^2 a_4^2).
\]

It is clear that in this case $Z_Q$ is the point $\text{Spec } \mathbb{C}$. Let $X$ denote the DM stack $[\mathbb{C}^2/\mathbb{Z}_4]$ whose stacky fan has one cone and the rays generated by the vectors $v_0, v_4$. Consider the edge $F_1$ corresponding to the birational transformation $X \leftrightarrow X_1$, where $X_1$ is the toric DM stack with cones determined by the pairs $v_0, v_1$ and $v_1, v_4$. The associated polyhedral subdivision is $(\text{conv} \{0, 4\}, \{0, 1, 4\})$, and the circuit relation $I$ is $3v_0 - 4v_1 + v_4 = 0$. The discriminant $\Delta_I$ is $256 a_0^3 a_4 - 27 a_1^4$, and the leading term with respect to the edge $F_1$ in the quartic discriminant $\Delta_Q$ is
\[256 a_0^3 a_4^3 - 27 a_1^4 a_4^2 = a_4^2 \cdot \Delta_I.\]

This means that $n_{Q,F_1} = 1$, which is consistent with the fact that $Z_{F_1} = \text{Spec } \mathbb{C}$. The associated spherical functor is
\[D^b(\text{Spec } \mathbb{C}) \to D^b([\mathbb{C}^2/\mathbb{Z}_4]).\]

Let $F_2$ denote the edge corresponding to the birational transformation $X \leftrightarrow X_2$, where $X_2$ is the toric DM stack with cones determined by the pairs $v_0, v_2$ and $v_2, v_4$. The associated polyhedral subdivision is $(\text{conv} \{0, 4\}, \{0, 2, 4\})$, and the circuit relation $I$ is $v_0 - 2v_2 + v_4 = 0$. The discriminant $\Delta_I$ is $4a_0 a_1 - a_2^2$, and the leading term with respect to the edge $F_2$ in the quartic discriminant $\Delta_Q$ is
\[256 a_0^3 a_4^3 - 128 a_0^2 a_2 a_4^2 + 16 a_0 a_4^4 a_4 = 16 a_0 a_4 \cdot \Delta_I^2.\]

This means that $n_{Q,F_2} = 2$ which is consistent with the fact that $Z_{F_2}$ is the stacky point $[\text{Spec } \mathbb{C}/\mathbb{Z}_2]$. The derived category $D^b([\text{Spec } \mathbb{C}/\mathbb{Z}_2])$ splits into two copies of $D^b(\text{Spec } \mathbb{C})$ and the associated wall monodromy spherical functor is
\[D^b([\text{Spec } \mathbb{C}/\mathbb{Z}_2]) \to D^b([\mathbb{C}^2/\mathbb{Z}_4]).\]

Example 4.4. Let $X_1$ be the well known quasi–projective Calabi-Yau toric variety defined as the total space of the canonical bundle of $\mathbb{P}^2$. The toric structure is given by the vectors $v_0 = (0, 0, 1), v_1 = (1, 0, 1), v_2 = (0, 1, 1)$ and $v_3 = (-1, -1, 1)$. The lattice of linear relations is one dimensional and generated by the vector $(-3, 1, 1, 1)$, and the secondary polytope is the segment $F = [(3, 2, 2, 2), (0, 3, 3, 3)].$
There are two toric birational models in this case, namely $X_1$ and the stacky quotient $X_2 = [\mathbb{C}^3/\mathbb{Z}_3]$. The principal $A$–determinant is $E_A = a_1^2a_2^2a_3^2(a_0^3 - 27a_1a_2a_3)$. The multiplicity of the discriminant components corresponding to $v_1, v_2, v_3$ are all equal to 2, and $n_{F,Q} = 1$. The spherical twist of $D^b(X_1)$ is determined by the spherical object $\mathcal{O}_{F_2} \subset X_1$.

**Example 4.5.** Let’s consider the case the Calabi-Yau toric variety $X_1$ is the total space of the canonical bundle of the Hirzebruch surface $F_2$. The toric structure is given by the vectors $v_0 = (0, 0, 1), v_1 = (1, 0, 1), v_2 = (0, 1, 1), v_3 = (-1, 2, 1)$ and $v_3 = (0, -1, 1)$ with the obvious cones. The lattice of relations is two dimensional and generated by the vectors $(-2, 0, 1, 0, 1)$ and $(0, 1, -2, 1, 0)$. There are four stacky toric birational models for $X_1$.

The discriminant picture is obtained by studying the singularities of the Laurent polynomial $\sum a_i x^{v_i}$. Besides the vertices corresponding to $v_1, v_3$ and $v_4$ there are two other faces that contain circuits. The discriminant component determined by the face $\Gamma$ generated by the vectors $v_1, v_2$ and $v_3$ is

$$f_\Gamma = a_2^2 - 4a_1a_3.$$  

For the maximal dimensional face $Q$ we obtain the discriminant

$$f_Q = a_0^4 - 8a_0^2a_2a_4 + 16a_2^2a_4^2 - 64a_1a_3a_4^2.$$  

The principal $A$–determinant is

$$E_A = a_1^2a_2^2a_4^2 \cdot f_\Gamma \cdot f_Q.$$
Figure 2. The "mirror" complex discriminant moduli space.

If we choose the coordinates adapted to the large complex structure point
\[ x := \frac{a_2 a_4}{a_0^2}, \quad y = \frac{a_1 a_3}{a_2^2} \]
the we obtain that the two relevant discriminant components are given by
\[ y = \frac{1}{4}(1 - \frac{1}{4x})^2 \quad \text{and} \quad y = \frac{1}{4}. \]

Figure 1 is suggestive and intuitively very useful and appeared for the first time in the paper by Dave Morrison [DRM] on the 3-fold octic in \( \mathbb{P}^4(1, 1, 1, 2, 2) \) which is a higher dimensional generalization of our example.

The analysis of the neighborhood of the point \( A \) is completely analogous to the work [AHK], so we will not discuss here. By analyzing the moduli space picture, we see that a rational curve \( x = \text{constant} \) infinitesimally close to the \( y \)-axis corresponds to a transition from \( X_1 \), the total space of the canonical bundle of \( F_2 \) to the stacky version \( X_4 \) of the weighted projective space \( \mathbb{P}(2, 1, 1) \). This curve intersects \( E_A = 0 \) in the neighborhood of the point \( B \). The circuit \( I \) associated to the corresponding edge of the secondary quadrilateral is \( v_1 - 2v_2 + v_3 = 0 \). One of the associated EZ-spherical functor is induced by the diagram
\[
\begin{array}{ccc}
E & \hookrightarrow & X_1 \\
\downarrow q & & \\
Z & = & \mathbb{A}^1
\end{array}
\]
The birational transformation corresponding a rational curve $x = k$ ($k$ a very large constant) corresponds to a transition from $X_2$ to $X_3$ which is the stacky quotient $[\mathbb{C}^3/\mathbb{Z}_4]$. This curve intersects $E_A = 0$ in the neighborhood of the point $C$. One of the associated spherical functors is induced by the diagram

$$
E = [\mathbb{A}^1/\mathbb{Z}_2] \times \mathbb{P}^1 \longrightarrow X_4 = [\mathbb{C}^3/\mathbb{Z}_4]
$$

$$
\psi \downarrow \downarrow \downarrow
$$

$$
Z = [\mathbb{A}^1/\mathbb{Z}_2]
$$

It is interesting to note that the circuit $I$ associated to this transition and determined by corresponding edge of the secondary polytope is also $v_1 - 2v_2 + v_3 = 0$.

Recall that in determining multiplicities, the circuit discriminant $\Delta_I = 4a_1a_3 - a_2^2$ played a crucial role. For the transition $X_1 \leftrightarrow X_4$, we can consider the linear form $\psi$ in the cone $C(F_{14})$ whose supporting face is the edge $F_{14}$ given by

$$
\psi(v_0) = u, \psi(v_1) = \psi(v_2) = \psi(v_3) = 1, \psi(v_4) = 0,
$$

with $u > 1/2$. A similar linear form $\psi$ will work for the transition $X_2 \leftrightarrow X_3$, with $u < 1/2$ in that case. The monomials in $f_Q a_0^4, a_0^2a_2a_4, a_2^2a_4^2, a_1a_3a_4^2$ have the weights $4u, 2u + 1, 2, 2$, respectively. The leading term for the edge $F_{14}$ is $a_0^4$. The highest weight terms for the edge $F_{23}$ are $a_2^2a_4^2$ and $a_1a_3a_4^2$, so in this case the leading term for $f_Q$ is

$$
16a_2^2a_4^2 - 64a_1a_3a_4^2 = -16a_4^2 \cdot \Delta_I.
$$

We conclude that $n_{Q,F_{14}} = 0$ and $n_{Q,F_{23}} = 1$, as Figure 2 indicates. In this simple example, we do know the true parametrizations of the components, so the weight calculation is not really needed. However, in the general case, only the birational Horn parametrizations are available, so the weight calculation becomes necessary.

5. Final comments and some speculations

The proposed constructions of spherical functors appearing in this paper seem to be related of the notion of co-sheaf over the topological space of the cone $K = \sum \mathbb{R}v_i$, in the sense of Bressler-Lunts [BL], section 6.7.1.

Since the defined spherical functors can be thought as categorical maps corresponding to the Bressler-Lunts strata, it is an intriguing question whether the whole structure can be expressed in the language
of schobers introduced by Kapranov and Schechtman [KaS]. This abstract realization would be consistent with the fact that the proposed $EZ$–spherical functors map “stalks” between various strata of a Whitney stratification of the characteristic cycle of the GKZ hypergeometric system. Moreover, we expect that there are analogues of the wall monodromy functors corresponding to all the toric strata in the stacky moduli space compactification given by the secondary fan, together with the corresponding semi-orthogonal decompositions with the categories $D_{\Gamma}$ as components. Such a general picture would require a refinement of our Minkovski sum argument in the proof of our multiplicity result. The stalks in this case would be bounded derived categories of coherent sheaves of the form $D_{\Gamma}$, indexed by the faces $\Gamma$. In particular, if $\Gamma = \emptyset$, the “generic stalk” is the bounded derived category of coherent sheaves on the ambient toric DM stack.

Certain wall crossing phenomena were studied in the language of spherical pairs and schobers by Donovan [Do] and, from a mirror symmetric point of view, by Nadler [N]. In a combinatorial context similar to ours, Špenko and Van den Bergh [SvB] produced a categorification of the GKZ system in the quasi-symmetric toric case. More work is needed in order to relate these points of view and make our speculation rigorous.

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