Asymptotic stability of small solitary waves for nonlinear Schrödinger equations with electromagnetic potential in $\mathbb{R}^3$

Eva Koo *
evahk@math.ubc.ca

Abstract

We consider the nonlinear magnetic Schrödinger equation for $u : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{C}$,

$$iu_t = (i\nabla + A)^2 u + Vu + g(u), u(x,0) = u_0(x),$$

where $A : \mathbb{R}^3 \to \mathbb{R}^3$ is the magnetic potential, $V : \mathbb{R}^3 \to \mathbb{R}$ is the electric potential, and $g = \pm |u|^2 u$ is the nonlinear term. We show that under suitable assumptions on the electric and magnetic potentials, if the initial data is small enough in $H^1$, then the solution of the above equation decomposes uniquely into a standing wave part, which converges as $t \to \infty$ and a dispersive part, which scatters.

1 Introduction

Consider the nonlinear Schrödinger equation with magnetic and electric potentials for $\psi(x,t) : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{C}$,

$$\begin{cases} i\partial_t \psi = (\Delta + 2iA \cdot \nabla + i(\nabla \cdot A) + V)\psi + g(\psi) \\ \psi(x,0) = \psi_0(x) \in H^1(\mathbb{R}^3) \end{cases}$$

where

$$g(\psi) = \pm |\psi|^2 \psi.$$
Here, \( A(x) = (A_1(x), A_2(x), A_3(x)) : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) and \( V(x) : \mathbb{R}^3 \rightarrow \mathbb{R} \). Equation (1) can be equivalently written as
\[
 i \partial_t \psi = (i \nabla + A)^2 \psi + V \psi + g(\psi) \tag{3}
\]
by replacing \( V \) with \( V - |A|^2 \). Here, \( A(x) = (A_1(x), A_2(x), A_3(x)) \) is the magnetic potential (also known as the vector potential), and \( V(x) \) is the electric potential (also known as the scalar potential). In this paper, we consider potentials \( A(x) \) and \( V(x) \) which decay to 0 as \( |x| \rightarrow \infty \).

Equation (1) describes a charged quantum particle subject to external electric and magnetic fields, and a self-interaction (nonlinearity). Such nonlinear Schrödinger equations find numerous physical applications, for example, in Bose-Einstein condensates and nonlinear optics.

Just as for linear Schrödinger equations
\[
 i \partial_t \psi = (-\Delta + 2iA \cdot \nabla + i(\nabla \cdot A) + V) \psi, \tag{4}
\]
an important role is played by standing wave solutions (or bound states)
\[
 \psi(x, t) = e^{iEt} Q(x) \tag{5}
\]
of (1). Existence of standing waves to equation (1) for certain electrical and magnetic potentials was first proved in [1].

Here we consider small solutions of the form (5) which bifurcate from zero along an eigenvalue of the linear Hamiltonian operator
\[
 H = -\Delta + 2iA \cdot \nabla + i(\nabla \cdot A) + V. \tag{6}
\]
Physical intuition suggests that the ground-state standing wave (the one corresponding to the lowest eigenvalue \( E \)) should remain stable when the self-interaction (nonlinearity) is turned on, and indeed should become asymptotically stable (that is, nearby solutions should relax to the ground state by radiating excess energy to infinity – see below for a more precise statement). When only one bound state is present, this was first proved in [7] for scalar potentials \( (A \equiv 0) \) and well-localized perturbations of the ground state. Later works addressed the more complicated situation of multiple bound states (e.g. [11], [8]). For merely energy-space (i.e. \( H^1(\mathbb{R}^3) \)) perturbations of the ground state, asymptotic stability was proved in [5], again for scalar potential \( (A \equiv 0) \). The main goal of the
The present paper is to prove asymptotic stability of the ground state, in the energy space, and in the additional presence of the magnetic field.

**Remark 1.** Our argument should also go through for nonlinearities $g(\psi) = \pm|\psi|^{p-1}\psi$ for $1 + 4/3 \leq p < 5$. For concreteness, we will work with $g(\psi) = \pm|\psi|^2\psi$.

In order to ensure the operator $H$ is self-adjoint, we make the following assumption,

**Assumption 1.** *(Self-adjointness assumption)* We assume that each component of $A$ is a real-valued function in $L^q + L^\infty$ for some $q > 3$ that $\nabla \cdot A \in L^2 + L^\infty$, and that $V$ is a real-valued function in $L^2 + L^\infty$.

Then by Theorem X.22 of [13], the operator $H$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^3)$.

**Assumption 2.** *(Spectral assumption)* We assume that $H$ supports only one eigenvalue $e_0 < 0$ which is nondegenerate. We also assume 0 is not a resonance of $H$ (see e.g. [4] for the definition of resonance).

We need the following assumption to show the existence and exponential decay of the nonlinear bound states.

**Assumption 3.** *(Assumptions for existence and exponential decay of nonlinear bound states)* We assume

$$\|A\|_{L^q + L^\infty(|x| > R)} + \|V\|_{L^2 + L^\infty(|x| > R)} \to 0 \text{ as } R \to \infty$$

for some $q > 3$.

Under the above assumptions, we have the following lemma on the existence and decay of nonlinear bound states. Let $\phi_0 > 0$ be the positive, $L^2$-normalized eigenfunction corresponding to the eigenvalue $e_0$ of $H$.

**Lemma 1.** *(Existence and decay of nonlinear bound states)* For each sufficiently small $z \in \mathbb{C}$, there is a corresponding eigenfunction $Q[z] \in H^2$ solving the nonlinear eigenvalue problem

$$HQ + g(Q) = EQ$$

with the corresponding eigenvalue $E[z] = e_0 + o(z)$ and $Q[z] = z\phi_0 + q(z)$ with

$$q(z) = o(z^2), \quad DQ[z] = (1,i)\phi_0 + o(z) \quad \text{and} \quad D^2Q[z] = o(1) \quad \text{in } H^2$$
where we denote
\[ DQ[z] = (D_1Q[z], D_2Q[z]) = \left( \frac{\partial}{\partial z_1}Q[z], \frac{\partial}{\partial z_2}Q[z] \right), \quad \text{and} \quad z = z_1 + iz_2. \] (10)

Furthermore, \( Q \) has exponential decay in the sense that
\[ e^{\beta|x|}Q \in H^1 \cap L^\infty \] (11)
for some \( \beta > 0 \) (independent of \( z \)).

Next, we need assumptions on \( A \) and \( V \) which ensure our linear Schrödinger evolution obeys some dispersive estimates. For \( f, g \in L^2(\mathbb{R}^3, \mathbb{C}) \), define the real inner product \( \langle f, g \rangle \) by
\[ \langle f, g \rangle = \Re(\int_{\mathbb{R}^3} \bar{f}g dx). \] (12)
Denote \( \langle x \rangle = (1 + |x|^2)^{\frac{1}{2}} \) and fix \( \sigma > 4 \). Let \( P_c \) be the projection onto the continuous spectral subspace of \( H \). Following [4], we have:

**Assumption 4. (Strichartz estimates assumption)** We assume that for all \( x, \xi \in \mathbb{R}^3 \),
\[ |A(x)| + \langle x \rangle|V(x)| \lesssim \langle x \rangle^{-1-\epsilon}, \] (13)
\[ \langle x \rangle^{1+\epsilon'} A(x) \in \dot{W}^{\frac{7}{6},6}(\mathbb{R}^3), \] (14)
and
\[ A \in C^0(\mathbb{R}^3) \] (15)
for some \( \epsilon > 0 \) and all sufficiently small \( \epsilon' \in (0, \epsilon) \).

Define the space-time norm
\[ \|\psi\|_X = \|\langle x \rangle^{-\sigma}\psi\|_{L_t^2H_x^1} + \|\psi\|_{L_t^3W_x^{1,6}} + \|\psi\|_{L_t^\infty H_x^1}. \] (16)

We can now state the main result, which says that all \( H^1 \)-small solutions converge to a solitary wave (nonlinear bound state) as \( t \to \infty \):

**Theorem 1. (Asymptotic stability of small solitary waves)** Let assumptions [4] hold. For \( 0 \leq t < \infty \), every solution \( \psi \) of equation (1) with initial data \( \psi_0 \) sufficiently small in \( H^1 \) can be uniquely decomposed as
\[ \psi(t) = Q[z(t)] + \eta(t), \] (17)
with differentiable $z(t) \in \mathbb{C}$ and $\eta(t) \in H^1$ satisfying $\langle i\eta, D_1 Q[z] \rangle = 0$, $\langle i\eta, D_2 Q[z] \rangle = 0$ and
\[
\|\eta\|_X \lesssim \|\psi_0\|_{H^1}, \quad \|\dot{z} + iE[z]z\|_{L^1_t} \lesssim \|\psi_0\|_{H^1}^2. \tag{18}
\]

Furthermore, as $t \to \infty$,
\[
z(t) \exp(i \int_0^t E[z(s)] ds) \to z_+, \quad E(z(t)) \to E(z_+) \tag{19}
\]
for some $z_+ \in \mathbb{C}$ and
\[
\|\eta(t) - e^{-itH} \eta_+\|_{H^1} \to 0 \tag{20}
\]
for some $\eta_+ \in H^1_x \cap \text{Range}(P_c)$.

For comparison, consider the nonlinear Schrödinger equation with just a scalar potential $V$,
\[
i\partial_t \psi = (-\Delta + V)\psi + g(\psi) \tag{21}
\]
for the same nonlinearity $g$ as above, which is a special case of equation (1) with $A = 0$. The corresponding asymptotic stability result for (21) was obtained in dimension three in [5], in dimension one in [9] and in dimension two in [10, 6]. Our approach for equation (1) will be similar to that in [5], which uses the Strichartz estimates
\[
\|e^{it(\Delta-V)} P_c \phi\|_{\tilde{X}} \lesssim \|\phi\|_{H^1} \tag{22}
\]
and
\[
\|\int_{-\infty}^t e^{i(t-s)(\Delta-V)} P_c F(s) ds\|_{\tilde{X}} \lesssim \|F\|_{L^1_t L^{1,6}} \tag{23}
\]
where $\tilde{X} = L^\infty_t H^1 \cap L^2_t W^{1,6} \cap L^2_t L^{6,2}$, which are known to hold for a class of scalar potentials $V$. Our approach will use the Strichartz estimates for $H$ from [4]. However, the proof of [4] of the inhomogeneous Strichartz estimates
\[
\|\int_{-\infty}^t e^{i(t-s)H} P_c F(s) ds\|_{L^q_t L^p_x} \lesssim \|F\|_{L^\infty_t L^{q'}_x} \tag{24}
\]
for $H = -\Delta + 2iA \cdot \nabla + i(\nabla \cdot A) + V$ uses a lemma from [2] which does not hold for the endpoint case $(q, p) = (2, 6)$ or $(\tilde{q}, \tilde{p}) = (2, 6)$. To overcome the lack of endpoint Strichartz estimates, we will use estimates in weighted spaces, as in [9] and [10].

Section 2 is devoted to the proofs of the various linear dispersive estimates needed for the asymptotic stability argument. In addition to the estimates taken from [4], we need to establish estimates in weighted Sobolev spaces, which require some work. We will prove the following theorem.
Theorem 2. We say that \((p, q)\) is Strichartz admissible if
\[
\frac{2}{q} + \frac{3}{p} = \frac{3}{2} \quad \text{with} \quad 2 \leq p < 6.
\]
(25)

If \((q, p)\) and \((\tilde{p}, \tilde{q})\) are Strichartz admissible, then
\[
\| \int_0^t e^{i(t-s)H} P_c F(s) ds \|_{L^q_t W^{1,p}_x} + \| \langle x \rangle^{-\sigma} \int_0^t e^{i(t-s)H} P_c F(s) ds \|_{L^2_t H^1_x} \leq \min(\| \langle x \rangle^\sigma F \|_{L^q_t H^1_x}, \| F \|_{L_{q'}^q W^{1,p'}_x}).
\]
(26)

The asymptotic stability theorem is proved in section 3. Finally, the existence and decay of nonlinear bound states (Lemma 1) is given in an appendix.

2 Linear estimates

The following lemmas 2 and 3 are from [1]:

Lemma 2. (Non-endpoint Strichartz estimates) Under assumptions 4 and 2, if \((p, q)\) and \((\tilde{p}, \tilde{q})\) are Strichartz admissible, we have
\[
\| e^{itH} P_c f \|_{L^q_t L^p_x} \lesssim \| f \|_{L^2(\mathbb{R}^3)}
\]
(28)
and
\[
\| \int_{-\infty}^t e^{i(t-s)H} P_c F(x) ds \|_{L^q_t L^p_x} \lesssim \| F \|_{L^\tilde{q'}_{\tilde{p}'} L^\tilde{p}_{\tilde{q}'}_x}.
\]
(29)

Notice that the above does not include the \(L^2_t\)-norm. Fix \(\sigma > 4\).

Lemma 3. (Weighted homogeneous \(L^2_t\) estimates) Under assumptions 4 and 2, we have
\[
\| \langle x \rangle^{-\sigma} e^{-itH} f \|_{L^2_t L^2_x} \lesssim \| f \|_{L^2_x},
\]
(30)
and
\[
\sup_{\lambda \geq 0} (\lambda) \| \langle x \rangle^{-\sigma} (H - (\lambda^2 + i0))^{-1} \langle x \rangle^{-\sigma} \|_{L^2 \to L^2} \lesssim 1.
\]
(31)

The weighted resolvent estimate of lemma 3 implies weighted inhomogeneous estimates for the linear evolution:

Lemma 4. (Weighted \(L^2_t\) inhomogeneous estimates) Under the assumptions of lemma 3
\[
\| \langle x \rangle^{-\sigma} \int_0^t e^{i(t-s)H} P_c \langle x \rangle^{-\sigma} F(s) ds \|_{L^2_t L^2_x} \lesssim \| F \|_{L^2_t L^2_x}.
\]
(32)
Proof. For simplicity we may restrict to times $t \geq 0$. By Plancherel, we have

$$\|\chi_{\{t\geq 0\}}\langle x \rangle^{-\sigma} \int_0^t e^{i(t-s)(H-\i e)} P_c \langle x \rangle^{-\sigma} F(s) ds \|_{L_x^2}$$

(33)

Next, change the order of the $ds$ and $dt$ integral and use that

$$\int_s^\infty dt e^{it(H-\tau-\i e)} P_c \langle x \rangle^{-\sigma} F(s)$$

(35)

$$= \frac{1}{i} (H - \tau - \i e)^{-1} e^{it(H-\tau-\i e)} \bigg|_{t=s} P_c \langle x \rangle^{-\sigma} F(s)$$

(36)

$$= \frac{-1}{i} (H - \tau - \i e)^{-1} e^{is(H-\tau-\i e)} P_c \langle x \rangle^{-\sigma} F(s),$$

(37)

we get

$$\|\chi_{\{t\geq 0\}}\langle x \rangle^{-\sigma} \int_0^t e^{i(t-s)(H-\i e)} P_c \langle x \rangle^{-\sigma} F(s) ds \|_{L_x^2}$$

(38)

$$= \|\langle x \rangle^{-\sigma} \int_0^\infty dse^{-\i s(H-\i e)} (H - \tau - \i e)^{-1} e^{\i s(H-\tau-\i e)} P_c \langle x \rangle^{-\sigma} F(s) \|_{L_x^2}$$

(39)

$$= \|\langle x \rangle^{-\sigma} (H - \tau - \i e)^{-1} P_c \langle x \rangle^{-\sigma} \int_0^\infty dse^{-\i s\tau} F(s) \|_{L_x^2}.$$  

(40)

If we take the $L_x^2$-norm of both sides, we get

$$\|\langle x \rangle^{-\sigma} \int_0^t e^{i(t-s)(H+\i e)} P_c \langle x \rangle^{-\sigma} F(s) ds \|_{L_x^2 L_x^2}$$

(42)

$$\lesssim \|\langle x \rangle^{-\sigma} (H - \tau + \i e)^{-1} P_c \langle x \rangle^{-\sigma} \int_0^\infty dse^{-\i s\tau} F(s) \|_{L_x^2 L_x^2}$$

(43)

$$\lesssim \sup_{\tau} \|\langle x \rangle^{-\sigma} (H - \tau + \i e)^{-1} P_c \langle x \rangle^{-\sigma} \|_{L^2 \to L^2} \| \int_0^\infty dse^{-\i s\tau} F(s) \|_{L_x^2 L_x^2}$$

(44)

$$\lesssim \|F\|_{L_x^2 L_x^2} \text{ by Plancherel and Lemma 3}$$

(45)

Now sending $\epsilon$ to 0, we have

$$\|\langle x \rangle^{-\sigma} \int_0^t e^{i(t-s)H} P_c \langle x \rangle^{-\sigma} F(s) ds \|_{L_x^2 L_x^2} \lesssim \|F\|_{L_x^2 L_x^2}$$

(46)

as needed.

\[ \square \]
Lemma 5. (Mixed Strichartz weighted estimates) Let $(q,p)$ and $(\tilde{p},\tilde{q})$ be Strichartz admissible. Then

$$\| \int_0^t e^{i(t-s)H} P_c F(s)ds \|_{L_t^q L_x^p} + \| \langle x \rangle^{-\sigma} \int_0^t e^{i(t-s)H} P_c F(s)ds \|_{L_t^\sigma L_x^\sigma} \leq \min(\| \langle x \rangle^\sigma F \|_{L_t^q L_x^p}, \| F \|_{L_t^\sigma L_x^\sigma}).$$

Proof. First,

$$\| \int_0^\infty e^{-isH} P_c F(s)ds \|_{L_x^2}^2 = ( \int_0^\infty e^{-isH} P_c F(s)ds, \int_0^\infty e^{-itH} P_c F(s)dt ).$$

Moving the integrals through the inner product and rearranging the terms, we get

$$\| \int_0^\infty e^{-isH} P_c F(s)ds \|_{L_x^2}^2 = \int_0^\infty ds (P_c F(s), \int_0^\infty e^{-i(t-s)H} P_c F(s)dt )$$

$$= \int_0^\infty ds (\langle x \rangle^\sigma P_c F(s), \langle x \rangle^{-\sigma} \int_0^\infty e^{-i(t-s)H} P_c F(s)dt )$$

by Hölder inequality

$$\leq \| \langle x \rangle^\sigma P_c F(s) \|_{L_x^2} \| \langle x \rangle^{-\sigma} \int_0^\infty e^{-i(t-s)H} P_c F(s)dt \|_{L_x^2}$$

and by lemma 4

$$\lesssim \| \langle x \rangle^\sigma P_c F(s) \|_{L_x^2}^2.$$ (56)

Hence,

$$\| \int_0^t e^{i(t-s)H} P_c F(s)ds \|_{L_t^q L_x^p} = \| e^{itH} \int_0^\infty e^{-isH} P_c F(s)ds \|_{L_t^q L_x^p}$$

$$\lesssim \| \int_0^\infty e^{-isH} P_c F(s)ds \|_{L_x^2} \text{ by lemma 2}$$

$$\lesssim \| \langle x \rangle^\sigma F(s) \|_{L_x^2}.$$ (59)

Now, by a lemma of Christ-Kiselev (see [2]), we have

$$\| \int_0^t e^{i(t-s)H} P_c F(s)ds \|_{L_t^q L_x^p} \lesssim \| \langle x \rangle^\sigma F(s) \|_{L_x^2}.$$ (60)

Next, let $\langle x \rangle^\sigma g(x,t) \in L_t^q L_x^p$. Then

$$\int_0^\infty (\langle x \rangle^\sigma g(x,t), \langle x \rangle^{-\sigma} \int_0^\infty e^{i(t-s)H} P_c F(s)ds dt$$

$$= \int_0^\infty \langle g(x,t), \int_0^\infty e^{i(t-s)H} P_c F(s)ds dt \rangle$$

(62)
Moving the integrals through the inner product and rearranging the terms, we get

\[
\int_0^\infty (\langle x \rangle^\sigma g(x,t), \langle x \rangle^{-\sigma} \int_0^\infty e^{i(t-s)H} P_c F(s) ds) dt
\]

(64)

\[
= \int_0^\infty ds \left( \int_0^\infty e^{i(s-t)H} P_c g(x,t) dt, F(s) \right)
\]

by Hölder inequality

(65)

\[
\leq \| \int_0^\infty e^{i(s-t)H} P_c g(x,t) dt \|_{L_q^2 L_p^2} \| F(s) \|_{L_q^\prime L_{p'}^\prime}
\]

(66)

\[
\lesssim \| \langle x \rangle^\sigma g \|_{L_q^2 L_p^2} \| F(s) \|_{L_q^\prime L_{p'}^\prime}
\]

(67)

Hence,

\[
\| \langle x \rangle^{-\sigma} \int_0^\infty e^{i(t-s)H} P_c F(s) ds \|_{L_q^2 L_p^2} \lesssim \| F(s) \|_{L_q^\prime L_{p'}^\prime}.
\]

(68)

Again, by the lemma of Christ-Kiselev, we have

\[
\| \langle x \rangle^{-\sigma} \int_0^t e^{i(t-s)H} P_c F(s) ds \|_{L_q^2 L_p^2} \lesssim \| F(s) \|_{L_q^\prime L_{p'}^\prime}.
\]

(69)

Now by lemma 2 and lemma 4, we have shown lemma 5.

\[\square\]

**Lemma 6.** (Derivative Strichartz estimates) Let \( p \geq 2 \) and let

\[
H_1 = H + K = -\Delta + 2iA \cdot \nabla + i(\nabla \cdot A) + V + K
\]

(70)

for a sufficiently large number \( K \). Then \( H_1 \) is a positive operator on \( L^p \), and

\[
\| \phi \|_{W^{1,p}} \sim \| H_1^{\frac{1}{2}} \phi \|_{L^p}.
\]

(71)

From this, it follows that

\[
\| e^{-itH} f \|_{L_q^1 W_{1,p}^1} \lesssim \| f \|_{H_1^{\frac{1}{2}}}
\]

(72)

and

\[
\| \int_0^t e^{i(t-s)H} P_c F(s) ds \|_{L_q^1 W_{1,p}^1} \lesssim \| F \|_{L_q^\prime W_{1,p'}^1},
\]

(73)

for Strichartz admissible \( (q,p) \) and \( (\tilde{p}, \tilde{q}) \).

**Proof.** We would like to first show

\[
\| \phi \|_{W^{1,p}} \sim \| H_1^{\frac{1}{2}} \phi \|_{L^p} \quad \text{for} \quad \phi \in W^{1,p}.
\]

(74)
Clearly \( \| \phi \|_{W^0, p} = \| \phi \|_{L^p} = \| H_1^0 \phi \|_{L^p} \). We will show in the appendix that for \( K \) large enough, \( H_1 \) is a positive operator on \( L^p \), and

\[
\| \phi \|_{W^2, p} \sim \| H_1 \phi \|_{L^p}. \tag{76}
\]

By Theorem 1 of [3], there exist positive numbers \( \epsilon \) and \( C \), such that \( H_1^t \) is a bounded operator on \( L^p \) for \( -\epsilon \leq t \leq \epsilon \) and \( \| H_1^t \| \leq C \). Therefore the hypothesis of section 1.15.3 of [12] holds and we have that

\[
[D(H_1), D(H_0^0)]_{\frac{1}{2}} = D(H_1^{\frac{1}{2}}). \tag{77}
\]

Using that \( D(H_1) = W^{2, p} \), \( D(H_0^0) = L^p \) and \([W^{2, p}, L^p]_{\frac{1}{2}} = W^{1, p} \), we find that

\[
D(H_1^{\frac{1}{2}}) = W^{1, p}. \tag{78}
\]

Now by section 1.15.2 of [12], \( H_1^{\frac{1}{2}} \) is an isomorphic mapping from \( D(H_1^{\frac{1}{2}}) = W^{1, p} \) onto \( L^p \). Therefore, we have

\[
\| \phi \|_{W^{1, p}} \sim \| H_1^{\frac{1}{2}} \phi \|_{L^p}. \tag{79}
\]

Finally,

\[
\left\| \int_0^t e^{i(t-s)H} P_c F(s)ds \right\|_{L^q W^{1, p}} \sim \left\| \int_0^t e^{i(t-s)H} P_c F(s)ds \right\|_{W^{2, p}} \| L^q \|_{L^q} \tag{80}
\]

\[
\sim \left\| H_1^{\frac{1}{2}} \int_0^t e^{i(t-s)H} P_c F(s)ds \right\|_{L^q} \| L^q \|_{L^q} \tag{81}
\]

\[
= \left\| \int_0^t e^{i(t-s)H} P_c H_1^{\frac{1}{2}} F(s)ds \right\|_{L^q} \| L^q \|_{L^q} \tag{82}
\]

\[
\lesssim \left\| H_1^{\frac{1}{2}} F \right\|_{L^{q', q'}} \| L^{q', q'} \|_{L^q} \tag{83}
\]

\[
\sim \| F \|_{L^{q', q'}} \| W^{1, q'} \| \tag{84}
\]

For \( s \in \mathbb{R} \), denote the norm \( \| \phi \|_{(x)^s L^2} \) by

\[
\| \phi \|_{(x)^s L^2} = \| (x)^{-s} \phi \|_{L^2} \tag{85}
\]

and the norm \( \| \phi \|_{(x)^s H^1} \) by

\[
\| \phi \|_{(x)^s H^1} = \| \phi \|_{(x)^s L^2} + \| \nabla \phi \|_{(x)^s L^2}. \tag{86}
\]

Next we need derivative version of the weighted estimates of Lemma [4] - this is given in Lemma [9] below. First, we need two preparatory lemmas.
Lemma 7. For $t > 0$, let $A_t(x) = \frac{1}{\sqrt{t}} A(\frac{x}{\sqrt{t}})$ and $V_t(x) = \frac{1}{t} V(\frac{x}{\sqrt{t}})$. Let
\[ \tilde{H} = -\Delta + 2i A_t \cdot \nabla + i(\nabla \cdot A_t) + V_t + \frac{1}{t} K + 1. \] (87)
Then there exists $T > 0$ such that $\sup_{t>T} \| \tilde{H}^{-1} \|_{L^2 \rightarrow H^2} < \infty$.

Proof. Take $t \geq 1$. For $\phi \in L^2$, let $h = \tilde{H}^{-1} \phi$. Then
\[ \| \phi \|_2^2 = \left( (-\Delta + 2i A_t \cdot \nabla + i(\nabla \cdot A_t) + V_t + \frac{1}{t} K + 1)h, \right. \] (88)
\[ \left. (-\Delta + 2i A_t \cdot \nabla + i(\nabla \cdot A_t) + V_t + \frac{1}{t} K + 1)h \right) \]
\[ = \| \Delta h \|_2^2 + \| h \|_2^2 + \| A_t \cdot \nabla h \|_2^2 + 2\| \nabla h \|_2^2 + \| h \|_2^2 + F \] (90)
\[ \geq \| \Delta h \|_2^2 + \| h \|_2^2 + F \] (91)
where $F$ denotes the rest of the terms, and recall that $q > 3$. We would like to show that every term in $F$ is bounded by $\| h \|_{H_2}^2$. Here,
\[ |F| \leq 2\| (\Delta h)(A_t \cdot \nabla h) \|_1 + 2\| (\Delta h)(\nabla \cdot A_t + V_t + \frac{1}{t} K)h \|_1 \] (93)
\[ + 2\| [A_t(\nabla \cdot A_t + V_t + \frac{1}{t} K)] \cdot (\nabla h)h \|_1 \] (94)
\[ + 2\| A_t \cdot (\nabla h)h \|_1 + 2\| (A_t + V_t + \frac{1}{t} K)^2 h^2 \|_1 \] (95)
Here,
\[ \| (\Delta h)(A_t \cdot \nabla h) \|_1 \lesssim \frac{1}{\sqrt{t}} \| \Delta h \|_2 \| (\frac{A_t}{\sqrt{t}}) \|_{L^\infty + L^q}(\| \nabla h \|_2 + \| \nabla h \|_{\frac{2q}{q-2}}) \] (96)
where $\frac{2q}{q-2} < 6$ (97)
\[ \lesssim \frac{1}{\sqrt{t}} \| \Delta h \|_2 \| (\frac{A_t}{\sqrt{t}}) \|_{L^\infty + L^q}(\| \nabla h \|_2 + \| \Delta h \|_2^\frac{3}{2} \| \nabla h \|_2^\frac{3q-3}{2}) \] (98)
\[ \lesssim \frac{t^{-\frac{(q-3)}{2q}}}{2} \| \Delta h \|_2 \| (\frac{A_t}{\sqrt{t}}) \|_{L^\infty + L^q}(\| \nabla h \|_2 + \| \Delta h \|_2^\frac{3}{2} \| \nabla h \|_2^\frac{3q-3}{2q}), \] (99)
\[ \| (\Delta h)(\nabla \cdot A_t) + V_t + \frac{1}{t} K)h \|_1 \] (100)
\[ \lesssim \frac{1}{t} \| \Delta h \|_2 \| (\nabla \cdot A_t) \|_{L^\infty + L^2} + \| V_t \|_{L^\infty + L^2} + K(\| h \|_2 + \| h \|_\infty) \] (101)
\[ \lesssim t^{-\frac{q}{2}} \| \Delta h \|_2 (\| \nabla A_t \|_{L^\infty + L^2} + \| V \|_{L^\infty + L^2} + K)(\| h \|_2 + \| h \|_2^\frac{1}{2} \| \Delta h \|_2^\frac{1}{2} ). \] (102)
Similar bounds hold for the other terms of $F$. We conclude that
\[ \|\phi\|_2^2 \geq (1 + o(1)) \|h\|_{H^2}^2 \text{ as } t \to \infty. \] (105)
Hence, for all $t$ large enough, we have
\[ \|h\|_{H^2}^2 \lesssim \|\phi\|_2^2. \] (106)

**Lemma 8.** Let $H_1$ be as in lemma 6. For $\phi \in L^2$ and $t > 0$, we have
\[ \|\nabla(H_1 + t)^{-1}\phi\|_{L^2} \lesssim (1 + t)^{-\frac{1}{2}} \|\phi\|_{L^2}. \] (107)

*Proof.* For $\phi \in L^2$, let $\psi = (H_1 + t)^{-1}\phi$. For $t$ bounded away from zero, define $\hat{\psi}$ by $\psi(x) = \frac{1}{t}\hat{\psi}(\sqrt{t}x)$. Then $\Delta \psi(x) = \Delta \hat{\psi}(\sqrt{t}x)$, $\nabla \psi(x) = \frac{1}{\sqrt{t}}\nabla \hat{\psi}(\sqrt{t}x)$ and $V(x)\psi(x) = \frac{1}{t}V(x)\hat{\psi}(\sqrt{t}x)$ and
\[ (\tilde{H}\hat{\psi})(\sqrt{t}x) = \phi(x). \] (108)
Replacing $x$ by $\frac{x}{\sqrt{t}}$ and inverting $\tilde{H}$, we get
\[ \hat{\psi}(x) = \tilde{H}^{-1}\phi\left(\frac{x}{\sqrt{t}}\right). \] (109)
Hence,
\[ \psi(x) = \frac{1}{t}[\tilde{H}^{-1}\phi\left(\frac{\cdot}{\sqrt{t}}\right)](\sqrt{t}x) \] (110)
and
\[ \nabla \psi(x) = \frac{1}{\sqrt{t}}[\nabla(\tilde{H})^{-1}\phi\left(\frac{\cdot}{\sqrt{t}}\right)](\sqrt{t}x). \] (111)
By Lemma 4 $\|\tilde{H}^{-1}\|_{L^2 \to L^2}$ is uniformly bounded for $t \geq T$. Therefore,
\[ \|\nabla \psi(x)\|_2 = \left\| \frac{1}{\sqrt{t}}[\nabla(\tilde{H})^{-1}\phi\left(\frac{\cdot}{\sqrt{t}}\right)](\sqrt{t}x) \right\|_2 \]
\[ = t^{-\frac{3}{4}}\|\nabla \tilde{H}^{-1}\phi\left(\frac{\cdot}{\sqrt{t}}\right)\|_2 \]
\[ \lesssim t^{-\frac{3}{4}}\|\nabla \tilde{H}^{-1}\|_{L^2 \to L^2}\|\phi\left(\frac{\cdot}{\sqrt{t}}\right)\|_2 \]
\[ = t^{-\frac{1}{2}}\|\phi\|_2. \] (112)
Therefore, for $t \geq T$,
\[ \|\nabla(H_1 + t)^{-1}\phi\|_2 \lesssim t^{-\frac{1}{2}}\|\phi\|_2 \] (116)
and the lemma follows. \[ \square \]
Lemma 9. \textit{(Derivative weighted estimates)} Let $H_1$ be as in lemma 6. We have
\[ \|\phi\|_{(x)^sH^1} \sim \|H_1^{1/2}\phi\|_{(x)^sL^2} \text{ for } s \in \mathbb{R}. \] (117)

From this, it follows that
\[ \|\langle x\rangle^{-\sigma} \int_0^t e^{i(t-s)H}P_sF(s)ds\|_{L^2_xL^2_t} \lesssim \|\langle x\rangle^\sigma F\|_{L^2_xL^2_t}. \] (118)

\textbf{Proof.} Since \[ \|f\|_{(x)^sH^1} = \|\langle x\rangle^{-s}f\|_{L^2} + \|\nabla\langle x\rangle^{-s}f\|_{L^2}, \] to show the lemma, it suffices to show
\[ \|\langle x\rangle^{-s}H_1^{-1/2}\langle x\rangle^s\|_{L^2 \to L^2} < \infty \] (119)
and
\[ \|\nabla\langle x\rangle^{-s}H_1^{-1/2}\langle x\rangle^s\|_{L^2 \to L^2} < \infty. \] (120)

The second bound above is the harder of the two. We will show the second bound and the first one follows by a similar argument. First,
\[ \nabla\langle x\rangle^{-s}H_1^{-1/2}\langle x\rangle^s\phi = \nabla H_1^{-1/2}\phi + \nabla\langle x\rangle^{-s}[H_1^{-1/2}, \langle x\rangle^s]\phi \] (121)

Now $\nabla H_1^{-1/2}$ is bounded from $L^2$ to $L^2$ since $H_1^{-1/2}$ maps from $L^2$ to $H^1$ while $\nabla$ maps from $H^1$ to $L^2$.

For the second term, we use $H_1^{-1/2} = \int_0^\infty \frac{dt}{\sqrt{t}}(H_1 + t)^{-1}$ and $[(H_1 + t)^{-1}, \langle x\rangle^s] = (H_1 + t)^{-1}[H_1 + t, \langle x\rangle^s](H_1 + t)^{-1}$ to get
\[ \nabla\langle x\rangle^{-s}[H_1^{-1/2}, \langle x\rangle^s] = \nabla\langle x\rangle^{-s} \int_0^\infty \frac{dt}{\sqrt{t}}(H_1 + t)^{-1}[H_1 + t, \langle x\rangle^s](H_1 + t)^{-1} \] (122)

Recall that
\[ H_1 = -\Delta + 2iA \cdot \nabla + i(\nabla \cdot A) + V + K, \] (123)
so
\[ [H_1 + t, \langle x\rangle^s] = (-\Delta\langle x\rangle^s) - 2(\nabla\langle x\rangle^s) \cdot \nabla + 2iA \cdot (\nabla\langle x\rangle^s). \] (124)

Let $g(x) = (-\Delta\langle x\rangle^s) + 2iA \cdot (\nabla\langle x\rangle^s)$ and $h(x) = -2(\nabla\langle x\rangle^s)$. Then
\[ \nabla\langle x\rangle^{-s}[H_1^{-1/2}, \langle x\rangle^s] = \nabla\langle x\rangle^{-s} \int_0^\infty \frac{dt}{\sqrt{t}}(H_1 + t)^{-1}(g(x) + h(x) \cdot \nabla)(H_1 + t)^{-1}. \] (125)
Since $g(x) \lesssim \langle x \rangle^{s-1}$, we rewrite the $g(x)$-part of the above as

\begin{align}
\nabla \langle x \rangle^{-s} & \int_0^\infty \frac{dt}{\sqrt{t}} (H_1 + t)^{-1} g(x)(H_1 + t)^{-1} \\
= \nabla & \int_0^\infty \frac{dt}{\sqrt{t}} \langle x \rangle^{-s} g(x)(H_1 + t)^{-1}(H_1 + t)^{-1} \\
+ \nabla & \langle x \rangle^{-s} \int_0^\infty \frac{dt}{\sqrt{t}} (H_1 + t)^{-1}[H_1 + t, g(x)](H_1 + t)^{-1}(H_1 + t)^{-1} 
\end{align}

(126)

(127)

(128)

The first part of the above sum is bounded. For the second part, writing $[H_1 + t, g(x)] = \tilde{g}(x) + \tilde{h}(x) \cdot \nabla$ as before, we can iterate the above process until $\tilde{g}(x) \lesssim 1$. Since $h(x) \lesssim \langle x \rangle^{s-1}$, so by the similar argument, we have

\begin{align}
\nabla \langle x \rangle^{-s} & \int_0^\infty \frac{dt}{\sqrt{t}} (H_1 + t)^{-1} h(x) \cdot \nabla (H_1 + t)^{-1} \\
= \nabla & \int_0^\infty \frac{dt}{\sqrt{t}} \langle x \rangle^{-s} h(x)(H_1 + t)^{-1}\nabla (H_1 + t)^{-1} \\
+ \nabla & \langle x \rangle^{-s} \int_0^\infty \frac{dt}{\sqrt{t}} (H_1 + t)^{-1}[H_1 + t, h(x)](H_1 + t)^{-1}\nabla (H_1 + t)^{-1} 
\end{align}

(129)

(130)

(131)

As before, the first part of the above sum is bounded. For the second part, $[H_1 + t, g(x)] = \tilde{g}(x) + \tilde{h}(x) \cdot \nabla$ as before, we can iterate the above process until $\tilde{h}(x) \lesssim 1$. As a result, it suffices to consider

$$
\int_0^\infty \frac{dt}{\sqrt{t}} ((H_1 + t)^{-1})^m 
$$

(132)

and

$$
\int_0^\infty \frac{dt}{\sqrt{t}} ((H_1 + t)^{-1}\nabla (H_1 + t)^{-1})^m 
$$

(133)

for $m \geq 1$. Now by lemma 8, both of the expressions above are bounded in $L^2$.  

\[\square\]

Now, to prove theorem 2, apply lemma 6 and 9 to lemma 5, we get the result.

Finally, we need a lemma from [5] for the projection operator $P_c$ onto the continuous spectral subspace.

**Lemma 10.** (Continuous spectral subspace comparison) Let the continuous spectral subspace $H_c[z]$ be defined as

$$
H_c[z] = \{ \eta \in L^2 | \langle i\eta, D_1Q[z] \rangle = \langle i\eta, D_2Q[z] \rangle = 0 \}.
$$

(134)
Then there exists $\delta > 0$ such that for each $z \in \mathbb{C}$ with $|z| \leq \delta$, there is a bijective operator $R[z] : \text{Ran } P_c \to \mathcal{H}_c[z]$ satisfying

$$P_c|\mathcal{H}_c[z]| = (R[z])^{-1}. \quad (135)$$

Moreover, $R[z] - I$ is compact and continuous in $z$ in the operator norm on any space $Y$ satisfying $H^2 \cap W^{1,1} \subset Y \subset H^{-2} + L^\infty$.

The proof of lemma 10 is given in lemma 2.2 of [5]. We will use lemma 10 with $Y = L^2$.

### 3 Proof of the main theorem

Lemma 11 gives the following corollary which will form part of the main theorem.

**Lemma 11.** (Best decomposition) There exists $\delta > 0$ such that any $\psi \in H^1$ satisfying $\|\psi\|_{H^1} \leq \delta$ can be uniquely decomposed as

$$\psi = Q[z] + \eta \quad (136)$$

where $z \in \mathbb{C}$, $\eta \in H^1$, $\langle i\eta, D_1 Q[z] \rangle = \langle i\eta, D_2 Q[z] \rangle = 0$, and $|z| + \|\eta\|_{H^1} \lesssim \|\psi\|_{H^1}$.

The proof of lemma 11 is essentially an application of the implicit function theorem on the equation $B(z) = 0$ with

$$B(z) = (B_1(z), B_2(z)), \quad B_j = \langle i(\psi - Q[z]), D_j Q[z] \rangle \quad \text{for } j = 1, 2. \quad (137)$$

Details can be found in lemma 2.3 of [5].

Now, we prove theorem 1.

**Proof.** Substitute

$$\psi(t) = Q[z(t)] + \eta(t) \quad (138)$$

into equation (11) to get

$$i(DQ \dot{z} + \partial_t \eta) = HQ + H\eta + g(Q + \eta) \quad (139)$$
where for $w \in \mathbb{C}$, we denote $DQ[z]w = D_1Q[z]\Re w + D_2Q[z]\Im w$. Since $HQ + g(Q) = EQ$ and $DQ[z]iz = iQ[z]$ (since $Q[e^{ia}z] = e^{ia}Q[z]$ for $\alpha \in \mathbb{R}$), we have

$$i\partial_t \eta = H\eta - iDQ\dot{z} + EQ - g(Q) + g(Q + \eta)$$  \hspace{1cm} (140)

$$= H\eta - iDQ(\dot{z} + iEz) - g(Q) + g(Q + \eta).$$  \hspace{1cm} (141)

We can write this as

$$i\partial_t \eta = H\eta + F$$  \hspace{1cm} (142)

where

$$F = g(Q + \eta) - g(Q) - iDQ(\dot{z} + iEz).$$  \hspace{1cm} (143)

In integral form,

$$\eta(t) = e^{-itH}(\eta(0) - i \int_0^t e^{isH} F(s) ds).$$  \hspace{1cm} (144)

Let $\eta_c = P_c \eta$. Then

$$\eta_c = e^{-itH} P_c \eta(0) - i \int_0^t e^{i(t-s)H} P_c F(s) ds.$$  \hspace{1cm} (145)

Then for fixed $\sigma > 4$, since $\eta = \Re[z] \eta_c$, we have

$$\|\eta\|_X \lesssim \|\eta_c\|_X$$  \hspace{1cm} (146)

$$\lesssim \|\eta(0)\|_{H^1_x} + \| \int_0^t e^{-i(s-t)H} P_c(F(s) - 2Q|\eta|^2 - \bar{Q}\eta^2 - |\eta|^2 \eta) ds \|_X$$  \hspace{1cm} (147)

$$+ \| \int_0^t e^{-i(s-t)H} P_c(2Q|\eta|^2 + \bar{Q}\eta^2 + |\eta|^2 \eta) ds \|_X$$  \hspace{1cm} (148)

$$\lesssim \|\eta(0)\|_{H^1_x} + \| \int_0^t e^{-i(s-t)H} P_c(F(s) - 2Q|\eta|^2 - \bar{Q}\eta^2 - |\eta|^2 \eta) ds \|_X$$  \hspace{1cm} (149)

$$+ \|Q\eta^2\|_{L_t^4W_x^{1, \frac{18}{13}}} + \|\eta\|^3_{L_t^6W_x^{1, \frac{18}{13}}}.$$  \hspace{1cm} (150)

(151)

For $\|Q\eta^2\|_{L_t^{\frac{4}{3}}W_x^{1, \frac{18}{13}}}$, we have

$$\|Q\eta^2\|_{L_t^{\frac{4}{3}}W_x^{1, \frac{18}{13}}} = \|Q\eta^2\|_{L_t^{\frac{4}{3}}L_x^{\frac{18}{13}}} + \|\nabla(Q\eta^2)\|_{L_t^{\frac{4}{3}}L_x^{\frac{18}{13}}}$$  \hspace{1cm} (152)

$$\lesssim \|(|Q| + |\nabla Q|)\eta^2\|_{L_t^{\frac{4}{3}}L_x^{\frac{18}{13}}} + \|Q\eta \nabla \eta\|_{L_t^{\frac{4}{3}}L_x^{\frac{18}{13}}}$$  \hspace{1cm} (153)

$$\lesssim \|Q\|_{L_t^{\infty}W_x^{1,6}} \|\eta\|^2_{L_t^{\frac{3}{2}}L_x^{\frac{18}{13}}} + \|Q\|_{L_t^{\infty}L_x^6} \|\eta\|_{L_t^{\frac{3}{2}}L_x^{\frac{18}{13}}} \|\nabla \eta\|_{L_t^{\frac{3}{2}}L_x^{\frac{18}{13}}}$$  \hspace{1cm} (154)

$$\lesssim \|Q\|_{W_x^{1,6}} \|\eta\|^3_X.$$  \hspace{1cm} (155)
For $\|\eta^3\|_{L^2_t W^{1,\frac{11}{18}}_x}$, we have

$$\|\eta^3\|_{L^2_t W^{1,\frac{11}{18}}_x} = \|\eta^3\|_{L^2_t L^{\frac{11}{3}}_x} + \|\nabla \eta^3\|_{L^2_t L^{\frac{11}{3}}_x} \quad (156)$$

$$\lesssim \|\eta^3\|_{L^2_t L^{\frac{11}{3}}_x} + \|\eta^2 \nabla \eta\|_{L^2_t L^{\frac{11}{3}}_x} \quad (157)$$

$$\leq \|\eta^2\|_{L^2_t L^\frac{3}{2}_x} \|\eta\|_{L^2_t L^{\frac{3}{2}}_x} + \|\eta^2\|_{L^2_t L^\frac{3}{2}_x} \|\nabla \eta\|_{L^2_t L^{\frac{11}{3}}_x} \quad (158)$$

$$\leq \|\eta^2\|_{L^2_t L^\frac{3}{2}_x} \|\eta\|_{L^2_t L^{\frac{3}{2}}_x} \|\nabla \eta\|_{L^2_t L^{\frac{11}{3}}_x} \quad (159)$$

Now, using $\|\eta\|_{L^2_t} \lesssim \|\nabla \eta\|_{L^\infty_t L^2_x}$, we get

$$\|\eta\|_{L^2_t L^\frac{3}{2}_x} \lesssim \|\nabla \eta\|_{L^\infty_t L^2_x} \|\eta\|_{L^2_t L^{\frac{3}{2}}_x} \quad (160)$$

So

$$\|\eta^3\|_{L^2_t W^{1,\frac{11}{18}}_x} \lesssim \|\nabla \eta\|_{L^\infty_t L^2_x} \|\eta\|_{L^2_t L^{\frac{3}{2}}_x} \|\nabla \eta\|_{L^2_t L^{\frac{11}{3}}_x} \quad (161)$$

Together we have

$$\|\eta\|_{X} \lesssim \|\eta(0)\|_{H^1_x} + \int_0^t e^{-i(s-t)H} P_c(F(s) - 2Q|\eta|^2 - \tilde{Q} \eta^2 - |\eta|^2 \eta) ds \|_X \quad (162)$$

$$+ \|Q\|_{W^{1,6}_x} \|\eta\|_{X} + \|\eta\|_{X}^3 \quad (163)$$

$$\lesssim \|\eta(0)\|_{H^1_x} + \|\langle F(s) - 2Q|\eta|^2 - \tilde{Q} \eta^2 - |\eta|^2 \eta\rangle\|_{L^2_t(x) H^1_x} \quad (164)$$

$$+ \|\eta\|_{X} + \|\eta\|_{X}^3 \quad (165)$$

Next, for $g(\psi) = |\psi|^2 \psi$,

$$\|\langle F - 2Q|\eta|^2 - \tilde{Q} \eta^2 - |\eta|^2 \eta\rangle\|_{L^2_t(x) H^1_x} \quad (166)$$

$$= \|Q^2 \partial_{\chi} + 2|Q|^2 \eta - i DQ(\partial_{\chi} + i Ez)\|_{L^2_t(x) H^1_x} \quad (167)$$

$$\lesssim \|\langle x \rangle 2^\sigma Q^2\|_{W^{1,\infty}_x} \|\eta\|_{L^2_t(x) H^1_x} + \|DQ\|_{L^2(x) H^1_x} \|\dot{z} + i Ez\|_{L^2_t} \quad (168)$$

Next, we would like to bound $\langle \dot{z} + i Ez \rangle$. Recall that we imposed

$$\langle in_{\chi} \frac{\partial}{\partial z_1} Q[z]\rangle = 0 \quad \text{and} \quad \langle in_{\chi} \frac{\partial}{\partial z_2} Q[z]\rangle = 0 \quad (169)$$

through Lemma [11]. By Gauge covariance of $Q$, we have

$$Q[e^{i\alpha}z] = e^{i\alpha}Q[z]. \quad (170)$$
So for $z = z_1 + iz_2$,

$$Q[z] = e^{i\alpha} \tilde{Q} |z|^2$$

where $\alpha = \tan^{-1}\left(\frac{z_2}{z_1}\right)$. (171)

Here $\tilde{Q} : \mathbb{R}^+ \to \mathbb{R}$. So

$$\partial_{z_1} Q = \partial_{z_1} (e^{i\alpha}) \tilde{Q} + 2z_1 e^{i\alpha} \tilde{Q}' = e^{i\alpha} i (\partial_{z_1} \alpha) \tilde{Q} + 2z_1 e^{i\alpha} \tilde{Q}' = i(\partial_{z_1} \alpha) Q + 2z_1 e^{i\alpha} \tilde{Q}'$$

and

$$\partial_{z_2} Q = \partial_{z_2} (e^{i\alpha}) \tilde{Q} + 2z_2 e^{i\alpha} \tilde{Q}' = e^{i\alpha} i (\partial_{z_2} \alpha) \tilde{Q} + 2z_2 e^{i\alpha} \tilde{Q}' = i(\partial_{z_2} \alpha) Q + 2z_2 e^{i\alpha} \tilde{Q}'.$$ (172)

So

$$0 = \langle i\eta, -z_2 \partial_{z_1} Q + z_1 \partial_{z_2} Q \rangle = \langle \eta, -z_2 (\partial_{z_1} \alpha) Q + z_1 (\partial_{z_2} \alpha) Q \rangle$$

$$= (-z_2 (\partial_{z_1} \alpha) + z_1 (\partial_{z_2} \alpha)) \langle \eta, Q \rangle = \langle \eta, Q \rangle. (174)$$

Now differentiate $\langle i\eta, \partial_{z_1} Q[z] \rangle = 0$ and $\langle i\eta, \partial_{z_2} Q[z] \rangle = 0$ with respect to $t$ and substitute $i \partial_t \eta = H\eta + F$, we get

$$0 = \langle i\partial_t \eta, \frac{\partial}{\partial z_j} Q[z] \rangle + \langle i\eta, D \frac{\partial}{\partial z_j} Q \hat{z} \rangle$$

$$= \langle H\eta + F, \frac{\partial}{\partial z_j} Q[z] \rangle + \langle i\eta, D \frac{\partial}{\partial z_j} Q \hat{z} \rangle$$

$$= \langle (H\eta + g(Q + \eta) - g(Q) - iDQ(\hat{z} + iEz)), \frac{\partial}{\partial z_j} Q[z] \rangle + \langle i\eta, D \frac{\partial}{\partial z_j} Q \hat{z} \rangle.$$ (176)

Recall that $F = g(Q + \eta) - g(Q) - iDQ(\hat{z} + iEz)$. Therefore, we have

$$0 = \langle H\eta + g(Q + \eta) - g(Q) - iDQ(\hat{z} + iEz), \frac{\partial}{\partial z_j} Q[z] \rangle + \langle i\eta, D \frac{\partial}{\partial z_j} Q \hat{z} \rangle$$

$$= \langle (H\eta + \frac{\partial}{\partial \epsilon} g(Q + \epsilon\eta)|_{\epsilon=0}) + \eta, D \frac{\partial}{\partial z_j} Q \hat{z} \rangle$$

$$-iDQ(\hat{z} + iEz), \frac{\partial}{\partial z_j} Q[z] \rangle + \langle i\eta, D \frac{\partial}{\partial z_j} Q \hat{z} \rangle$$

$$= \langle (H\eta + \frac{\partial}{\partial \epsilon} g(Q + \epsilon\eta)|_{\epsilon=0}) + \eta, D \frac{\partial}{\partial z_j} Q \hat{z} \rangle.$$ (180)
From the above, we get that
\[
\langle (g(Q + \eta) - g(Q) - \partial_0^\ell g(Q + \epsilon \eta)), \frac{\partial}{\partial z_j} Q[z] \rangle \tag{183}
\]
\[
= \langle -iDQ(\dot{z} + iEz), \frac{\partial}{\partial z_j} Q[z] \rangle \tag{184}
\]
\[
+ \langle (H\eta + \partial_0^\ell g(Q + \epsilon \eta)), \frac{\partial}{\partial z_j} Q[z] \rangle \tag{185}
\]
\[
+ \langle i\eta, D \frac{\partial}{\partial z_j} Q \dot{z} \rangle. \tag{186}
\]

Let \( \mathcal{H}\eta = H\eta + \partial_0^\ell g(Q + \epsilon \eta) \). By the symmetry of \( \mathcal{H} \) and differentiating equation (8) by \( z_j \), we have
\[
\langle \mathcal{H}\eta, \frac{\partial}{\partial z_j} Q \rangle = \langle \eta, \mathcal{H} \frac{\partial}{\partial z_j} Q \rangle = \langle \eta, E \frac{\partial}{\partial z_j} Q \rangle + (\frac{\partial}{\partial z_j} E) \langle \eta, Q \rangle \tag{188}
\]
\[
= \langle \eta, E \frac{\partial}{\partial z_j} Q \rangle = \langle i\eta, iE \frac{\partial}{\partial z_j} Q \rangle \tag{189}
\]
\[
= \langle i\eta, E \frac{\partial}{\partial z_j} DQiz \rangle \tag{190}
\]
using \( \langle \eta, Q \rangle = 0 \) and \( DQ[z]iz = iQ[z] \). So
\[
\langle (g(Q + \eta) - g(Q) - \partial_0^\ell g(Q + \epsilon \eta)), \frac{\partial}{\partial z_j} Q[z] \rangle \tag{191}
\]
\[
= \langle -iDQ(\dot{z} + iEz), \frac{\partial}{\partial z_j} Q[z] \rangle + \langle i\eta, E \frac{\partial}{\partial z_j} DQiz \rangle + \langle i\eta, D \frac{\partial}{\partial z_j} Q \dot{z} \rangle \tag{192}
\]
\[
= \langle -iDQ(\dot{z} + iEz), \frac{\partial}{\partial z_j} Q[z] \rangle + \langle i\eta, (D \frac{\partial}{\partial z_j} Q)(\dot{z} + iEz) \rangle \tag{193}
\]
For \( g(\psi) = |\psi|^2 \psi \),
\[
\partial_\ell^0 g(Q + \epsilon \eta) = Q^2 \bar{\eta} + 2|Q|^2 \eta. \tag{195}
\]
Therefore,
\[
g(Q + \eta) - g(Q) - \partial_\ell^0 g(Q + \epsilon \eta) = |Q + \eta|^2(Q + \eta) - |Q|^2 Q - Q^2 \bar{\eta} - 2|Q|^2 \eta \tag{196}
\]
\[
= 2Q|\eta|^2 + \bar{Q}\eta^2 + |\eta|^2 \eta \tag{197}
\]
Since
\[ \langle \frac{\partial}{\partial z_j} Q, i \frac{\partial}{\partial z_k} Q \rangle = j - k + o(1), \]
we have that
\[ |\dot{z} + iE z| \lesssim |\langle 2Q|\eta|^2 + Q\eta^2 + |\eta|^2\eta, DQ \rangle| (1 + \|\eta\|_{L^2}). \]
Therefore,
\[ \|\dot{z} + iE z\|_{L^2_t} \lesssim \|\langle 2Q|\eta|^2 + Q\eta^2 + |\eta|^2\eta, DQ \rangle\|_{L^2_t} (1 + \|\eta\|_{L^\infty_t L^2_x}) \]
(202)
\[ \lesssim (\|QDQ|\eta|^2\|_{L^2_x L^2_t} + \|DQ|\eta|^2\eta\|_{L^2_x L^2_t}) (1 + \|\eta\|_{L^\infty_t L^2_x}) \]
(203)
\[ \lesssim (\|QDQ\|_{L^\infty_x L^2_x} \|\eta\|_{L^2_x L^2_t} + \|DQ\|_{L^\infty_x L^2_t} \|\eta\|_{L^2_x L^2_t}^3) (1 + \|\eta\|_{L^\infty_t L^2_x}) \]
(204)
\[ \lesssim (\|QDQ\|_{L^\infty_x L^2_x} \|\eta\|_{L^2_x L^2_t}^2 + \|DQ\|_{L^\infty_x L^2_t} \|\eta\|_{L^2_x L^2_t}^3) (1 + \|\eta\|_{L^\infty_t L^2_x}) \]
(205)
\[ \lesssim \|\eta\|_X^2 + \|\eta\|_X^4 \]
(207)

For \(\|\eta\|_{L^4_x L^4_t}\), we used
\[ \|\eta\|_{L^4_x} \lesssim \|\nabla \eta\|_{L^2_x}^{\frac{3}{4}} \|\eta\|_{L^2_x}^{\frac{13}{4}}. \]
(208)

For \(\|\eta\|_{L^6_x L^6_t}\), we used
\[ \|\eta\|_{L^6_x} \lesssim \|\nabla \eta\|_{L^2_x}^{\frac{7}{5}} \|\eta\|_{L^2_x}^{\frac{12}{5}}. \]
(209)

Putting the preceding estimates together we have
\[ \|\eta\|_X \lesssim \|\eta(0)\|_{H^1} + \|\langle x \rangle^{2\sigma} Q^2\|_{L^\infty_x} \|\eta\|_X + \|\eta\|_X^2 + \|\eta\|_X^4, \]
(210)
and since \(\|\langle x \rangle^{2\sigma} Q^2\|_{L^\infty_x} \ll 1\),
\[ \|\eta\|_X \leq C[\|\eta(0)\|_{H^1} + \|\eta\|_X^2 + \|\eta\|_X^4] \]
(211)
for some constant \(C \geq 1\).

Now, let \(X_T\) be the norm defined by
\[ \|\psi\|_{X_T} = \|\langle x \rangle^{-\sigma} \psi\|_{L^2_t([0,T],H^1_x)} + \|\psi\|_{L^2_t([0,T],W^{1,4}_{x,\sigma})} + \|\psi\|_{L^\infty_t([0,T],H^1_x)} \]
(212)
Fix the initial condition \(\|\psi(0)\|_X\) to be small enough so that
\[ \|\eta(0)\|_{H^1} \leq \frac{1}{20C^2}. \]
(213)
Let
\[ T_1 = \sup\{T > 0 : \|\eta\|_{X_T} \leq \frac{1}{10C} \} > 0. \] (214)

Then for \( 0 \leq T \leq T_1 \),
\[ \|\eta\|_{X_T} \leq \frac{1}{20C} + \frac{1}{10^2C} + \frac{1}{10^4C^3} \leq \frac{1}{15C}, \] (215)
showing that \( T_1 = \infty \).

Next, we would like to bound \( \|\hat{z} + iEZ\|_{L^1_t} \). We have
\[ \|\hat{z} + iEZ\|_{L^1_t} \lesssim (\|QDQ\|_{L^1_t} + \|DQ\|_{L^1_t}) \] (216)
\[ \lesssim (\|QDQ\|_{L^1_t} + \|DQ\|_{L^1_t}) (1 + \|\eta\|_{L^2}) \] (217)
\[ \lesssim (\|QDQ\|_{L^1_t} + \|DQ\|_{L^1_t}) (1 + \|\eta\|_{L^2}) \] (218)
\[ \lesssim (\|QDQ\|_{L^1_t} + \|DQ\|_{L^1_t}) (1 + \|\eta\|_{L^2}) \] (219)
Here, the factor \( \|\langle x \rangle^{-\sigma}\eta^3\|_{L^1_t} \) can be bounded by
\[ \|\langle x \rangle^{-\sigma}\eta^3\|_{L^1_t} \lesssim \|\langle x \rangle^{-\sigma}\eta\|_{L^2_t} \|\eta\|_{L^4_t} \lesssim \|\langle x \rangle^{-\sigma}\eta\|_{L^2_t} \|\eta\|_{L^4_t} \lesssim \|\langle x \rangle^{-\sigma}\eta\|_{L^2_t} \] (220)
\[ \|\langle x \rangle^{-\sigma}\eta\|_{L^2_t} \lesssim \|\eta\|_{L^2_t}. \] (221)

Putting everything together, we have
\[ \|\hat{z} + iEZ\|_{L^1_t} \lesssim \|\eta\|_{L^2_t} + \|\eta\|_{L^4_t}. \] (222)

Therefore, \( |\partial_t (e^{i\int_0^t E(s)ds} z(t))| = |\hat{z} + iEZ| \in L^1_t \). This means that \( \lim_{t \to \infty} e^{i\int_0^t E(s)ds} z(t) \) exists. Since \( |e^{i\int_0^t E(s)ds} z(t)| = |z|, \lim_{t \to \infty} |z(t)| \) exists. Furthermore, \( E \) is continuous and \( E(z) = E(|z|), \lim_{t \to \infty} E(z(t)) \) exists.

Finally, let \( H = -\Delta + 2iA \cdot \nabla + i(\nabla \cdot A) + V \). So
\[ \eta_c(t) = e^{itH}(\eta_c(0) - i \int_0^t e^{-isH} P_c F(s) ds). \] (223)

By Strichartz estimates as above, we have
\[ \| \int_S^T e^{-isH} P_c F(s) ds \|_{H^1} \lesssim \| F \|_X \to 0 \] (224)
as \( T > S \to \infty \). Therefore, \( \int_0^\infty e^{-isH} P_c F ds \) converges in \( H^1 \), and
\[ \lim_{t \to \infty} e^{-itH} \eta_c(t) = \eta_c(0) - i \int_0^\infty e^{-isH} P_c F(s) ds : =: \eta_+ \] (225)
for some $\eta_+ \in H^1$. From this, we get that $\eta_c(t)$ converges to 0 weakly in $H_1$. Now, by compactness of $R[z(t)] - I$, we have that $\eta_d(t) := \eta(t) - \eta_c(t) = (R[z(t)] - I)\eta_c(t)$ converges to 0 strongly in $H^1$. Therefore

$$\|\eta(t) - e^{itH} \eta_+\|_{H^1} \to 0.$$  \hfill (226)

\[ \square \]

### A Nonlinear bound states

The following is the proof for Lemma 1, the existence and exponential decay of nonlinear bound states.

**Proof of existence of nonlinear bound states:**

For each small $z \in \mathbb{C}$, we look for a solution

$$Q = z\phi_0 + q \quad \text{and} \quad E = e_0 + e'$$  \hfill (227)

of

$$(-\Delta + 2i A \cdot \nabla + i(\nabla \cdot A) + V)Q + g(Q) = EQ$$  \hfill (228)

with $(\phi_0, q) = 0$ and $e' \in \mathbb{R}$ small.

Let $H_0 = -\Delta + 2i A \cdot \nabla + i(\nabla \cdot A) + V - e_0$. If we substitute $Q = z\phi_0 + q$ and $E = e_0 + e'$ into equation (228), we get

$$H_0 q + g(z\phi_0 + q) = e'(z\phi_0) + e'q.$$  \hfill (229)

Projecting equation (229) on the $\phi_0$ and $\phi_0^\perp$ directions, we get

$$e'z = (\phi_0, g(z\phi_0 + q))$$  \hfill (230)

and

$$H_0 q = -P_c g(z\phi_0 + q) + e'q.$$  \hfill (231)

Now, let

$$K = \{(q, e') \in H^2_+ \times \mathbb{R} ||q||_{H^2} \leq |z|^2, |e'| \leq |z|\}$$  \hfill (232)

for sufficiently small $z \in \mathbb{C}$ where $H^2_+ = \{q \in H^2 | (q, \phi_0) = 0\}$. Also, define the map $M : (q_0, e'_0) \mapsto (q_1, e'_1)$ by

$$g_0 := g(z\phi_0 + q_0),$$  \hfill (233)
and
\[ q_1 := H_0^{-1}(-P_cq_0 + e'_0 q_0). \] (235)

Now if \((q_0, e'_0) \in K\), we have
\[ |z e'_1| = |(\phi_0, g_0)| = |(\phi_0, g\phi_0 + q_0)| = |(\phi_0, z\phi_0 + q_0)| \lesssim O(z^3) \] (236)
and
\[ \|q_1\|_{H^2} \lesssim \|P_c q_0 + e'_0 q_0\|_{L^2} \lesssim \|g_0\|_{L^2} + \|e'_0\|_{H^2} \lesssim O(z^3). \] (237)
Therefore, \( |e'_1| \lesssim O(z^2) \) and \( \|q_1\|_{H^2} \lesssim O(z^3) \). This shows that \( M \) maps \( K \) into \( K \) for sufficiently small \( z \).

Next, we would like to show that \( M \) is a contraction mapping. Let \((a_1, b_1) := M(q_0, e'_0)\) and \((a_2, b_2) := M(q_1, e'_1)\) with \( g_j = g(\phi_0 + q_j) \) for \( j = 0, 1 \). Then
\[
|z(b_2 - b_1)| = |(\phi_0, g_0 - g_1)| \\
= |(\phi_0, g(\phi_0 + q_0) - g(\phi_0 + q_1))| \\
= |(\phi_0, z(\phi_0 + q_0) - z(\phi_0 + q_1)| \\
\lesssim \int \phi_0(|z|^2 \phi_0^2 + |q_0|^2 + |q_1|^2)q_0 - q_1 \lesssim |z|^2 \|q_0 - q_1\|_{L^2}. \] (241)

As \( a_i = H_0^{-1}(-P_c q_{i-1} + e'_{i-1} q_{i-1}) \) for \( i = 1, 2 \) and \( \|H_0^{-1}\|_{L^2 \rightarrow H^2} \leq \infty \), we have
\[
\|a_1 - a_2\|_{H^2} \lesssim \|P_c(q_1 - g_0) + e'_0q_0 - e'_1q_1\|_{L^2} \\
\lesssim \|g_1 - g_0\|_{L^2} + \|e'_0 - e'_1\|_{q_0\|_{L^2}} + \|e'_1\|_{q_0 - q_1\|_{L^2}.} \] (243)

Since
\[
\|g_1 - g_0\|_{L^2} = \|g(\phi_0 + q_1) - g(\phi_0 + q_0)\|_{L^2} \\
\lesssim |z|^2 \|\phi_0^2(q_1 - q_2)\|_{L^2} + |z|\|\phi_0(q_1^2 - q_2^2)\|_{L^2} + \|q_1^3 - q_2^3\|_{L^2} \\
\lesssim |z|^2 \|\phi_0^2\|_{L^2}\|q_1 - q_2\|_{L^5} + |z|\|\phi_0\|_{L^5}\|q_1 + q_2\|_{L^6}\|q_1 - q_2\|_{L^5} \\
+ \|(|q_1|^2 + |q_1 q_2| + |q_2|^2)\|_{L^4}\|q_1 - q_2\|_{L^4}, \] (247)

together, we have
\[
\|a_1 - a_2\|_{H^2} \lesssim |z|\|q_1 - q_2\|_{H^2} + |z|^2 |e'_0 - e'_1|. \] (248)
Hence, $M$ is a contraction mapping for $z$ sufficiently small. Now by the contraction mapping theorem, there exists a unique fixed point $(q,e')$ satisfying $\|q\|_{H^2} = O(z^3)$ and $|e'| = O(z^2)$ as $z \to 0$. The statements about derivatives of $Q$ and $E$ with respect to $z$ follow by differentiating (229) with respect to $z$ and applying the contraction mapping principle again.

Proof of exponential decay:

**Lemma 12.** For $\epsilon > 0$, define the exponential weight function $\chi_R$ by

$$
\chi_{R,\epsilon} = \begin{cases} 
  e^{\epsilon(|x|-R)} - 1 & \text{if } R < |x| \leq 2R, \\
  e^{\epsilon(3R-|x|)} - 1 & \text{if } 2R < |x| < 3R, \\
  0 & \text{else} 
\end{cases}
$$

(249)

Suppose for $\epsilon > 0$ small enough, $f \in H^1$ satisfies

$$
\|\chi_{R,\epsilon} f\|_{H^1} \leq C
$$

(250)

for some constant $C$ independent of $R$, then

$$
e^{-|x|} f \in H^1
$$

(251)

for some $\epsilon' > 0$.

**Proof.** For $R > 0$, $\|\chi_{R,\epsilon} f\|_{H^1} \leq C$ implies that

$$
\|(e^{\epsilon(|x|-R)} - 1)f\|_{H^1[\frac{3}{2}R,2R]} \leq C.
$$

(252)

Since $f \in H^1$,

$$
\|e^{\epsilon(|x|-R)} f\|_{H^1[\frac{3}{2}R,2R]} \leq C + \|f\|_{H^1} \leq C'.
$$

(253)

$e^{\frac{1}{2}\epsilon R} \leq e^{\epsilon(|x|-R)}$ for $|x| \in [\frac{3}{2}R,2R]$, so

$$
\|e^{\frac{1}{2}\epsilon R} f\|_{H^1[\frac{3}{2}R,2R]} \leq C'.
$$

(254)

So

$$
\|e^{\left(\frac{1}{2}(\frac{1}{2}\epsilon))}\right)2R f\|_{H^1[\frac{3}{2}R,2R]} \leq C'.
$$

(255)

Let $\epsilon' = (\frac{1}{2}(\frac{1}{2}\epsilon))$. Using $e^{\epsilon'2R} \geq e^{\epsilon'|x|}$ for $|x| \in [\frac{3}{2}R,2R]$, we get that

$$
\|e^{\epsilon'|x|} f\|_{H^1[\frac{3}{2}R,2R]} \leq C'.
$$

(256)
for some constant $C'$ independent of $R$.

Let $\epsilon'' = \frac{1}{2} \epsilon'$. Then
\[
\|e^{\epsilon'' |x|f}\|_{H^1(|x|>1)}^2 = \sum_{k=0}^{\infty} \|e^{\epsilon'' |x|f}\|_{H^1[\frac{2^{2k}}{3^k}, \frac{2^{2(k+1)}}{3^k+1}]}^2.
\]

Now, for each $k$, since $e^{\epsilon'} = e^{\epsilon''} e^{\epsilon'}$, taking $R = \frac{2^{2k+1}}{3^k+1}$ in (256), we have
\[
C' \geq \|e^{\epsilon' |x|f}\|_{H^1[\frac{2^{2k}}{3^k}, \frac{2^{2(k+1)}}{3^k+1}]}.
\]

This means that,
\[
\|e^{\epsilon'' |x|f}\|_{H^1[\frac{2^{2k}}{3^k}, \frac{2^{2(k+1)}}{3^k+1}]} \leq C' e^{-\epsilon'' \frac{2k}{3^k}}
\]

Therefore,
\[
\|e^{\epsilon'' |x|f}\|_{H^1(|x|>1)}^2 = \sum_{k=0}^{\infty} \|e^{\epsilon'' |x|f}\|_{H^1[\frac{2^{2k}}{3^k}, \frac{2^{2(k+1)}}{3^k+1}]}^2
\leq C'^2 \sum_{k=0}^{\infty} e^{-\epsilon'' \frac{2k+1}{3^k}}
\leq \infty
\]

By Lemma 12 to show that $\|e^{\alpha |x|Q}\|_{H^1} < \infty$ for some $\alpha > 0$, it suffices to show that $\|\chi_{R,\epsilon} Q\|_{H^1} \leq C$ for some constant $C$ independent of $R$. Here, $\chi_{R,\epsilon}$ is the exponential weight function as in Lemma 12.

Consider the bilinear form
\[
E(\psi, \phi) = (\nabla \psi, \nabla \phi) + i \int (2 \bar{\psi} A \cdot \nabla \phi + \bar{\psi} (\nabla \cdot A) \phi) dx + \int V \bar{\psi} \phi dx
\]
associated to the magnetic Schrödinger operator $-\Delta + 2iA \cdot \nabla + i(\nabla \cdot A) + V$. Then
\[
E(\psi, \psi) = (\nabla \psi, \nabla \psi) + i \int (2 \bar{\psi} A \cdot \nabla \psi + \bar{\psi} (\nabla \cdot A) \psi) dx + \int V \bar{\psi} \psi dx
\]
\[
= (\nabla \psi, \nabla \psi) + 2i \int \bar{\psi} A \cdot \nabla \psi dx + \int V \bar{\psi} \psi dx
\]
Set
\[ b := \lim_{R \to \infty} \inf \{ E(\phi, \phi) | \phi \in H^1, \| \phi \|_2 = 1, \phi(x) = 0 \text{ for } |x| < R \}. \tag{268} \]

We will show that \( b \geq 0 \) by contradiction. Suppose \( b < 0 \). Then there exists a sequence \( \phi_{R_j} \in H^1 \) with \( R_j \to \infty \), satisfying \( \| \phi_{R_j} \|_2 = 1, \phi_{R_j}(x) = 0 \) for \( |x| < R_j \), and \( E(\phi_{R_j}, \phi_{R_j}) < \delta \) for some fixed \( \delta < 0 \).

Suppose \( V \in L^\infty \), then
\[
\int V \phi_{R_j}^* \phi_{R_j} \, dx \leq \| V \|_\infty \| \phi_{R_j} \|_2^2 = \| V \|_\infty. \tag{269} \]

Suppose \( V \in L^2 \), then
\[
\int V \phi_{R_j}^* \phi_{R_j} \, dx \leq \| V \|_2 \| \phi_{R_j} \|_2^2 \tag{270} \]
\[
\leq \| V \|_2^2 \| \phi_{R_j} \|_2^2 \| \nabla \phi_{R_j} \|_2^2 \tag{271} \]
\[
\leq \tilde{\delta} \left( \| \nabla \phi_{R_j} \|_2^2 \right)^{\frac{3}{2}} + \frac{1}{\tilde{\delta}} \left( \| V \|_2 \| \phi_{R_j} \|_2 \right)^4 \tag{272} \]
\[
= \tilde{\delta} \| \nabla \phi_{R_j} \|_2^2 + \frac{1}{\tilde{\delta}} \| V \|_2^4. \tag{273} \]

Hence,
\[
\int V \phi_{R_j}^* \phi_{R_j} \, dx \lesssim \tilde{\delta} \| \nabla \phi_{R_j} \|_2^2 + \frac{1}{\tilde{\delta}} \| V \|_{L^\infty + L^2} \] \text{ where } \tilde{\delta} \text{ is sufficiently small}. \tag{274} \]

Similarly, suppose \( A \in L^\infty \), then
\[
|\langle \phi_{R_j}, A \cdot \nabla \phi_{R_j} \rangle| \leq \| A \|_\infty \| \phi_{R_j} \|_2 \| \nabla \phi_{R_j} \|_2 = \| A \|_\infty \| \nabla \phi_{R_j} \|_2. \tag{275} \]

On the other hand, suppose \( A \in L^{(3+\tilde{\epsilon})} \), then
\[
|\langle \phi_{R_j}, A \cdot \nabla \phi_{R_j} \rangle| \leq \| A \|_{(3+\tilde{\epsilon})} \| \phi_{R_j} \|_{2^{\frac{3+\tilde{\epsilon}}{2}}} \| \nabla \phi_{R_j} \|_2 \tag{276} \]
\[
\lesssim \| A \|_{(3+\tilde{\epsilon})} \| \phi_{R_j} \|_{2^{\frac{3+\tilde{\epsilon}}{2}}} \| \nabla \phi_{R_j} \|_2 \tag{277} \]
\[
= \| A \|_{(3+\tilde{\epsilon})} \| \nabla \phi_{R_j} \|_2^{\frac{3}{2} \frac{3+\tilde{\epsilon}}{2(3+\tilde{\epsilon})}}. \tag{278} \]

Hence,
\[
|\langle \phi_{R_j}, A \cdot \nabla \phi_{R_j} \rangle| \lesssim \| A \|_{L^{(3+\tilde{\epsilon})} + L^\infty} \left( \| \nabla \phi_{R_j} \|_2 \| \nabla \phi_{R_j} \|_2^{\frac{3(1+\tilde{\epsilon})}{2(3+\tilde{\epsilon})}} \right), \tag{279} \]
in which \( \frac{3}{2} - \frac{3(1+\tilde{\epsilon})}{2(3+\tilde{\epsilon})} \) is strictly less than 2 for \( \tilde{\epsilon} > 0 \).
Since \( \text{supp}(\phi_{R_j}) \subset \{|x| \geq R_j\} \), by the assumption \( \|V_-(L^{2}+L^{\infty})(|x|>R_j) \to 0 \) and \( \|A\|_{(L^{2}+L^{\infty})(|x|>R_j)} \to 0 \), \( \int V_-|\phi_{R_j}|^2 \, dx \) and the negative part of \( \Im \int \overline{\phi_{R_j}} A \cdot \nabla \phi_{R_j} \) converge to 0. Hence, the negative part of the energy converges to 0, a contradiction. Thus \( b \geq 0 \).

So there exists \( \delta(R) \) with \( \delta(R) \to b \geq 0 \) as \( R \to \infty \), such that for any \( \phi \in H^1 \) satisfying \( \phi(x) = 0 \) for \( |x| < R \), we have

\[
\mathcal{E}(\phi, \phi) \geq \delta(R) \|\phi\|^2.
\]  
(280)

For \( \phi \in H^1 \), we have

\[
\delta(R) \|\chi_{R \phi}\|^2 \leq \mathcal{E}(\chi_{R \phi}, \chi_{R \phi}) \leq (\nabla \chi_{R \phi}, \nabla \chi_{R \phi}) - 2\Im(\int \overline{\chi_{R \phi}} A \cdot \nabla \chi_{R \phi} \, dx) + \int V \chi_{R \phi} \chi_{R \phi} \, dx. \]  
(281)

If we expand the factor \( \nabla \chi_{R \phi} \), we get that

\[
(\nabla \chi_{R \phi}, \nabla \chi_{R \phi}) = (\phi \nabla \chi_{R \phi}, \phi \nabla \chi_{R \phi}) + 2(\phi \nabla \chi_{R \phi}, \chi_{R \phi} \nabla \phi) + (\chi_{R \phi} \nabla \phi, \chi_{R \phi} \nabla \phi) \]  
(282)

and since \( \Im(\int |\phi|^2 A \cdot \chi_{R \phi}^2 \nabla \chi_{R \phi}) = 0 \)

\[
-2\Im(\int \overline{\chi_{R \phi}} A \cdot \nabla \chi_{R \phi} \, dx) = -2\Im(\int \chi_{R \phi}^2 \overline{A} \cdot \nabla \phi) - 2\Im(\int |\phi|^2 A \cdot \chi_{R \phi}^2 \nabla \chi_{R \phi}) = -2\Im(\int \chi_{R \phi}^2 \overline{A} \cdot \nabla \phi). \]  
(283)

Since

\[
2(\phi \nabla \chi_{R \phi}, \chi_{R \phi} \nabla \phi) + (\chi_{R \phi} \nabla \phi, \chi_{R \phi} \nabla \phi) - 2\Im(\int \chi_{R \phi}^2 \overline{\phi} A \cdot \nabla \phi) + \int V \chi_{R \phi} \chi_{R \phi} \, dx \]  
(284)

is nothing but \( \mathcal{E}(\chi_{R \phi}^2, \phi) \), we have

\[
\delta(R) \|\chi_{R \phi}\|^2 \leq \mathcal{E}(\chi_{R \phi}^2, \phi) + \|\phi \nabla \chi_{R \phi}\|^2 \leq (\chi_{R \phi}, H_0 \phi) + c_0 \|\chi_{R \phi}\|^2 + \|\phi \nabla \chi_{R \phi}\|^2 \]  
(285)

where \( H_0 = -\Delta + i(A \cdot \nabla + \nabla \cdot A) + V - e_0 \).

From direct calculation, we see that for \( R > 0 \),

\[
|\nabla \chi_{R \phi}| \lesssim \epsilon(\chi_{R \phi} + 1), \]  
(286)

so

\[
\|\phi \nabla \chi_{R \phi}\|^2 \lesssim \epsilon^2 \|\phi(\chi_{R \phi} + 1)\|^2. \]  
(287)
Putting everything together, we have

$$\delta(R)\|\chi_R \phi\|_2^2 \lesssim (\chi_R^2 \phi, H_0 \phi) + (e_0 + \epsilon^2)\|\chi_R \phi\|_2^2 + \epsilon^2\|\phi\|_2^2. \quad (291)$$

Since $e_0 < 0$ and $\lim_{R \to \infty} \delta(R) \geq 0$, for $\epsilon$ small enough and $R$ sufficiently large, $\delta(R) - e_0 - \epsilon^2$ is positive and bounded away from zero. Therefore, we have

$$\|\chi_R \phi\|_2^2 \lesssim (\chi_R^2 \phi, H_0 \phi) + \epsilon^2\|\phi\|_2^2. \quad (292)$$

Next,

$$\|\chi_R \nabla \phi\|_2^2 \leq \|\nabla (\chi_R \phi)\|_2^2 + \|\phi \nabla \chi_R\|_2^2 \lesssim \mathcal{E}(\chi_R \phi, \chi_R \phi) + 2\mathfrak{I}(\int \chi_R \phi A \cdot \nabla \chi_R \phi dx) - \int V \chi_R \phi \chi_R \phi dx \quad (294)$$

$$+ \epsilon^2\|\phi\|_2^2 \quad (295)$$

$$\mathfrak{I}(\int \chi_R \phi A \cdot \nabla \chi_R \phi dx) \leq \|A\|_{L^\infty(|x| \geq R)} \|\chi_R \phi\|_{L^2} \|\nabla (\chi_R \phi)\|_{L^2} \quad (297)$$

and

$$\mathfrak{I}(\int \chi_R \phi A \cdot \nabla \chi_R \phi dx) \leq \|A\|_{L^3(|x| \geq R)} \|\chi_R \phi\|_{L^6} \|\nabla (\chi_R \phi)\|_{L^2} \quad (298)$$

$$\leq \|A\|_{L^3(|x| \geq R)} \|\chi_R \phi\|_{H^1} \|\nabla (\chi_R \phi)\|_{L^2}, \quad (299)$$

we have that

$$\mathfrak{I}(\int \chi_R \phi A \cdot \nabla \chi_R \phi dx) \leq \|A\|_{(L^\infty + L^3) (|x| \geq R)} \|\chi_R \phi\|_{H^1} \|\nabla (\chi_R \phi)\|_{L^2} \quad (300)$$

$$\leq \|A\|_{(L^\infty + L^3) (|x| \geq R)} \|\chi_R \phi\|_{H^1}^2. \quad (301)$$

Therefore,

$$\|\chi_R \nabla \phi\|_2^2 \lesssim \mathcal{E}(\chi_R \phi, \chi_R \phi) + \|A\|_{(L^\infty + L^3) (|x| \geq R)} \|\chi_R \phi\|_{H^1}^2 + \|\chi_R \phi\|_2^2 + \epsilon^2\|\phi\|_2^2. \quad (302)$$

Now using $\mathcal{E}(\chi_R \phi, \chi_R \phi) = (\chi_R^2 \phi, H_0 \phi) + e_0 \|\chi_R \phi\|_2^2$ and $\|\chi_R \phi\|_2^2 \lesssim (\chi_R^2 \phi, H_0 \phi) + \epsilon^2\|\phi\|_2^2$, we have that

$$\|\chi_R \nabla \phi\|_2^2 \lesssim (\chi_R^2 \phi, H_0 \phi) + \|A\|_{(L^\infty + L^3) (|x| \geq R)} \|\chi_R \phi\|_{H^1}^2 + \epsilon^2\|\phi\|_2^2. \quad (303)$$
Since
\[ \| \nabla (\chi \phi) \|_{L^2} = \| \phi \nabla \chi \|_{L^2} + \| \chi \phi \nabla \|_{L^2} \lesssim \epsilon \| \phi (\chi R + 1) \|_{L^2} + \| \chi \phi \nabla \|_{L^2}, \] (304)
putting everything together, we have that
\[ \| \chi R \phi \|_{H^1}^2 \lesssim (\chi R \phi, \chi R H_0 \phi) + \epsilon^2 \| \phi \|_2^2 + \| A \|_{(L^\infty + L^3)(|x| \geq R)} \| \chi R \phi \|_{H^1}, \] (305)
so for \( R \) sufficiently large,
\[ \| \chi R \phi \|_{H^1}^2 \lesssim (\chi R \phi, \chi R H_0 \phi) + \epsilon^2 \| \phi \|_2^2. \] (306)
If we let \( \phi = \phi_0 \) and use that \( H_0 \phi_0 = 0 \), we have
\[ \| \chi R \phi_0 \|_{H^1}^2 \lesssim \| \phi_0 \|_2^2 = 1. \] (307)
Next, let \( \phi = q \). Using that \( H_0 q = -P_c g(z \phi_0 + q) + e'q \), we get
\[ \| \chi R q \|_{H^1}^2 \lesssim (\chi R q, \chi R H_0 q) + \epsilon^2 \| q \|_2^2 \] (308)
\[ \lesssim (\chi R q, \chi R (-P_c g(z \phi_0 + q) + e'q)) + \epsilon^2 \| q \|_2^2 \] (309)
\[ \lesssim \| \chi R^2 q g(z \phi_0 + q) \|_1 + e' \| \chi R q \|_2^2 + \epsilon^2 \| q \|_2^2. \] (310)
As \( g(z) = |z|^2 z \), we have
\[ \| \chi R^2 q g(z \phi_0 + q) \|_1 \] (311)
\[ \lesssim |z|^3 \| \chi R^2 q \phi_0^3 \|_2 + |z|^2 \| \chi R^2 q^2 \phi_0^2 \|_1 + |z| \| \chi R^2 q^3 \phi_0 \|_1 + \| \chi R^2 q^4 \|_1 \] (312)
\[ \lesssim |z|^3 \| \chi R^2 \phi_0^3 \|_2 \| q \|_2 + |z|^2 \| \chi R^2 \phi_0^2 \|_2 \| q^2 \|_2 + |z| \| \chi R^2 \phi_0 \|_2 \| \chi R q^3 \|_2 \] (313)
\[ + \| \chi R^2 q^4 \|_2 \| q^2 \|_2 \] (314)
\[ \leq o(z^2). \] (315)
Hence,
\[ \| \chi R q \|_{H^1}^2 \leq o(z^2) \] (316)
by (307) and \( \| q \|_{H^2} = o(z^2) \).
Next if we substitute \( \phi = Dq \), and use that
\[ H_0 Dq = -P_c Dg(z \phi_0 + q) + q De' + e'Dq, \] (317)
we get

$$\|\chi_RDq\|_{H^1}^2 \leq (\chi_RDq, \chi_RH_0Dq) + \epsilon^2\|Dq\|_2^2$$  \hspace{1cm} (318)

$$\leq (\chi_RDq, \chi_R(-(PcDg(z\phi_0 + q) + qDe' + e'Dq)) + \epsilon^2\|Dq\|_2^2$$  \hspace{1cm} (319)

$$\leq \|\chi_R^2DqDg(z\phi_0 + q)\|_1 + \|\chi_R^2DqDqDe'\|_1 + e'\|\chi_RDq\|_2^2 + \epsilon^2\|q\|_2^2.$$  \hspace{1cm} (320)

Here, the first term $\|\chi_R^2DqDg(z\phi_0 + q)\|_1$ is bounded by

$$\|\chi_R^2DqDg(z\phi_0 + q)\|_1 \leq \|\chi_R^2Dq\phi_0z\phi_0 + q\|_1^2$$  \hspace{1cm} (321)

$$\leq |z|^2\|\chi_R^2Dq\phi_0\|_1^2 + |z|^2\|\chi_R^2Dq\phi_0^2q\|_1^2 + \|\chi_R^2\phi_0q\|_1^2$$  \hspace{1cm} (322)

$$\leq |z|^2\|\chi_R^2Dq\phi_0\|_1^2 + |z|^2\|\chi_R^2Dq\phi_0^2q\|_1^2 + \|\chi_R^2\phi_0q\|_1^2$$  \hspace{1cm} (323)

$$\leq |z|^2\|\chi_R^2Dq\|_{H^1}\|\chi_R\phi_0\|_{H^1}\|\phi_0\|_{H^1}^2 + |z|\|Dq\|_{H^1}\|Q\|_{H^1}\|\chi_R\phi_0\|_{H^1}^2$$  \hspace{1cm} (324)

$$\leq o(z^2).$$  \hspace{1cm} (325)

and the second term $\|\chi_R^2DqDqDe'\|_1$ is bounded by

$$\|\chi_R^2DqDqDe'\|_1 \leq \|\chi_RDq\|_{H^1}\|\chi_Rq\|_{H^1}\|De'\|_{H^1}$$  \hspace{1cm} (326)

$$\leq o(z^2).$$  \hspace{1cm} (327)

Therefore,

$$\|\chi_RDq\|_{H^1}^2 \leq o(z^2).$$  \hspace{1cm} (328)

Hence, by Lemma 12 and $Q = z\phi_0 + q$, we have $\|e^{|x|}Q\|_{H^1} \leq \infty$ and $\|e^{|x|}DQ\|_{H^1} \leq \infty$ for some $\beta > 0$.

Next, we would like to show $\|e^{|x|}Q\|_{L^\infty} \leq \infty$ by bounding $\|\Delta(e^{|x|}Q)\|_{L^\frac{2}{1}+}. \hspace{1cm} (329)$ Since $\|\Delta(e^{|x|}Q)\|_{L^\infty(|x| \leq 1)} < \infty$ already holds, it remains to show $\|\Delta(e^{|x|}Q)\|_{L^\frac{2}{1}+(|x| > 1)} < \infty$. Let $\gamma = \frac{\beta}{4}$. Using the equation for $Q$, we get

$$\|\Delta(e^{|x|}Q)\|_{L^\frac{2}{1}+(|x| > 1)} \leq |(\Delta e^{|x|})Q|_{L^\frac{2}{1}+(|x| > 1)} + |(\nabla e^{|x|})\cdot(\nabla Q)|_{L^\frac{2}{1}+(|x| > 1)}$$  \hspace{1cm} (330)

$$+ |e^{|x|}A \cdot \nabla Q|_{L^\frac{2}{1}+(|x| > 1)} + |e^{|x|}[(\nabla \cdot A) + V]Q|_{L^\frac{2}{1}+(|x| > 1)}$$  \hspace{1cm} (332)

$$+ |e^{|x|}g(Q)|_{L^\frac{2}{1}+(|x| > 1)} + |e^{|x|}EQ|_{L^\frac{2}{1}+(|x| > 1)}.$$  \hspace{1cm} (333)
Let $f$ and $g$ be such that $\Delta e^{\gamma|x|} = f(x)e^{\gamma|x|}$ and $\nabla e^{\gamma|x|} = g(x)e^{\gamma|x|}$. We can bound the first two terms loosely by

$$
\| (\Delta e^{\gamma|x|}) Q \|_{L^{3+}(|x|>1)} \lesssim \| e^{-\frac{H}{2}} e^{\gamma|x|} f(x) \|_{L^{6+}(|x|>1)} \| e^{\beta|x|} Q \|_{L^2}
$$

and

$$
\| (\nabla e^{\gamma|x|}) \cdot (\nabla Q) \|_{L^{3+}(|x|>1)} \lesssim \| e^{-\frac{1}{3} \beta|x|} g(x) \|_{L^{6+}(|x|>1)} \| e^{\frac{2}{3} \beta|x|} (\nabla Q) \|_{L^2}
$$

$$
\lesssim \| e^{\frac{2}{3} \beta|x|} Q \|_{H^1} + \| e^{-\frac{1}{3} \beta|x|} g(x) \|_{L^{\infty}(|x|>1)} \| e^{\beta|x|} Q \|_{L^2}.
$$

Using similar ways, we can also bound $\| e^{\gamma|x|} g(Q) \|_{L^{3+}(|x|>1)}$ and $\| e^{\gamma|x|} E Q \|_{L^{3+}(|x|>1)}$.

Next, for $\| e^{\gamma|x|} A \cdot \nabla Q \|_{L^{3+}(|x|>1)}$, we have

$$
\| e^{\gamma|x|} A \cdot \nabla Q \|_{L^{3+}(|x|>1)} \leq \| A \|_{L^{3+} + L^{\infty}} \| e^{\frac{1}{3} \beta|x|} \nabla Q \|_{L^3(|x|>1)} + \| e^{\frac{2}{3} \beta|x|} \nabla Q \|_{L^2(|x|>1)} \lesssim \| e^{\frac{2}{3} \beta|x|} e^{\beta|x|} (\nabla Q) \|_{L^3} + \| e^{e^{2\beta|x|}} \nabla Q \|_{L^2}.
$$

We already shown above that $\| e^{\beta|x|} \nabla Q \|_{L^2} < \infty$. To bound $\| e^{\frac{2}{3} \beta|x|} e^{\beta|x|} (\nabla Q) \|_{L^3}$, let $h = e^{\beta|x|} (\nabla Q)$ and from above, we know that $h \in L^2$. Now, consider the set

$$
M = \{ x | (e^{-\frac{2}{3} \beta|x|} |h|) \beta > |h|^2 \} = \{ x | |h| > e^{|2\beta|x|} \}.
$$

Clearly,

$$
\| e^{\frac{2}{3} \beta|x|} e^{\beta|x|} (\nabla Q) \|_{L^3(M^c)} = \| e^{\frac{2}{3} \beta|x|} h \|_{L^3(M^c)} \leq \| |h|^{\beta} \|_{L^3} = \| h \|_{L^2}^{\frac{2}{3}} < \infty.
$$

On the other hand, inside $M$, $|e^{\beta|x|} (\nabla Q)| > e^{2\beta|x|}$ and hence, $|\nabla Q| > e^{\beta|x|}$. Then

$$
\| e^{\frac{2}{3} \beta|x|} e^{\beta|x|} (\nabla Q) \|_{L^3(M)} \leq \| e^{\frac{2}{3} \beta|x|} |\nabla Q|^2 \|_{L^3}
$$

$$
\leq \||\nabla Q|^2\|_{L^3} \lesssim |\nabla Q|_{L^6}^3 \lesssim |\nabla Q|_{H^1}^3.
$$

Hence, we have

$$
\| \Delta (e^{\gamma|x|} Q) \|_{L^{\frac{3}{2}}} \leq \infty.
$$

By Sobolev embedding, we have

$$
\| e^{\gamma|x|} \|_{L^{\infty}} \leq \infty.
$$
B Proof of $\|\phi\|_{W^{2,p}} \sim \|H_1\phi\|_{L^p}$

Recall that $H_1 = H + K = -\Delta + i(2A \cdot \nabla + \nabla \cdot A) + V + K$. Let

$$W = i\nabla \cdot A + V + K.$$  \hfill (351)

Then

$$\|H_1\phi\|_{L^p} \leq \|\Delta \phi\|_{L^p} + \|W\phi\|_{L^p} + 2\|A \cdot \nabla \phi\|_{L^p} \lesssim \|\phi\|_{W^{2,p}}.$$  \hfill (352)

In the above, we bounded $\|A \cdot \nabla \phi\|_{L^p}$ by $\|A\|_{L^\infty} \|\nabla \phi\|_{L^p}$. Next,

$$\|\phi\|_{W^{2,p}}^2 \lesssim \|\Delta \phi\|^2_{L^p} + \|\phi\|^2_{L^p}$$  \hfill (353)

$$= \|(-H_1 + W + i2A \cdot \nabla)\phi\|^2_{L^p} + \|\phi\|^2_{L^p}$$  \hfill (354)

$$\lesssim \|H_1\phi\|^2_{L^p} + \|W\phi\|^2_{L^p} + 2\|A \cdot \nabla \phi\|^2_{L^p} + \|\phi\|^2_{L^p}$$  \hfill (355)

$$\lesssim \|H_1\phi\|^2_{L^p} + \|W\|^2_{L^\infty} \|\phi\|^2_{L^p} + 2\|\nabla A\|^2_{L^\infty} \|\phi\|^2_{L^p} + \|\phi\|^2_{L^p}$$  \hfill (356)

$$\lesssim \|H_1\phi\|^2_{L^p} + \|\phi\|^2_{L^p}.$$  \hfill (357)

To bound $\|\phi\|^2_{L^p}$, consider

$$(|\phi|^{p-2}\phi, H_1\phi) = (|\phi|^{p-2}\phi, -\Delta \phi) + \int W|\phi|^p + \int |\phi|^{p-2}\phi A \cdot \nabla \bar{\phi}. \hfill (358)$$

Taking real parts on both sides, we get

$$\Re(|\phi|^{p-2}\phi, H_1\phi) - 2\Re \int |\phi|^{p-2}\phi A \cdot \nabla \bar{\phi} - \Re(|\phi|^{p-2}\phi, -\Delta \phi) \geq \int W|\phi|^p \geq C||\phi||_{L^p}^p \hfill (359)$$

by choosing $K \geq ||V||_{L^\infty} + ||\nabla \cdot A||_{L^\infty} + C + 1$ where $C$ is a large constant that will be used below. Using that

$$\int |\phi|^{p-2}|
abla \phi|^2 + (p-2) \int |\phi|^{p-4} |\Re((\phi) \nabla \phi)|^2 = \Re \int \nabla(|\phi|^{p-2}\phi) \cdot \nabla \phi$$  \hfill (360)

$$= \Re(|\phi|^{p-2}\phi, -\Delta \phi) \geq 0,$$  \hfill (361)

we get

$$C||\phi||_{L^p}^p \lesssim \|H_1\phi\|_{L^p}||\phi||_{L^p}^{p-1}||L^{p-1} + \|A\|_{L^\infty}||\phi||_{L^p}^{p-2} \nabla \bar{\phi}||_{L^1}$$  \hfill (362)

$$- \int |\phi|^{p-2}|
abla \phi|^2 - (p-2) \int |\phi|^{p-4} |\Re((\phi) \nabla \phi)|^2 \hfill (363)$$

$$\lesssim \|H_1\phi\|_{L^p}||\phi||_{L^p}^{p-1} + \|A\|_{L^\infty}||\phi||_{L^p}^{p-2} \nabla \phi||_{L^1}^2 \hfill (364)$$

$$\lesssim \|H_1\phi\|_{L^p}||\phi||_{L^p}^{p-1} + 2\|\phi||_{L^p}^{p-2} \nabla \phi||_{L^2} + \frac{1}{e^2} \|A\|_{L^\infty}||\phi||_{L^p}^{p-2} \nabla \phi||_{L^2}$$  \hfill (365)

$$- \|\phi||^{p-2} \nabla \phi||_{L^2}.$$  \hfill (366)
Now if we choose $\epsilon$ small enough, we have
\[
C\|\phi\|_{L^p}^{p-1} \lesssim \|H_1 \phi\|_{L^p} \|\phi\|_{L^p}^{p-1} + \frac{1}{\epsilon^2} \|A\|_\infty \|\phi\|_{L^p}^p.
\] (367)

Dividing by $\|\phi\|_{L^p}^{p-1}$, we have
\[
C\|\phi\|_{L^p} \lesssim \|H_1 \phi\|_{L^p} + \frac{1}{\epsilon^2} \|A\|_\infty \|\phi\|_{L^p}.
\] (368)

Finally, if we choose $C$ large enough and put everything together, we have
\[
\|\phi\|_{W^{2,p}} \lesssim \|H_1 \phi\|_{L^p}.
\] (369)

**Acknowledgement**

The author would like to thank Stephen Gustafson and Tai-Peng Tsai for many helpful discussions.

**References**

[1] M. Esteban, P.L. Lions, *Stationary solutions of nonlinear Schrödinger equations with an external magnetic field*, PDE and Calculus of Variations, in Honor of E. De Giorgi, Birkhuser, 1990.

[2] M. Christ, A. Kiselev, *Maximal functions associated to filtrations*, J. Fun. Anal. 179 (2001), 409-425.

[3] M. G. Cowling, *Harmonic analysis on semigroups*, Ann. of Math. (2) 117 (1983), no. 2, 267-283.

[4] M. Burak Erdogan, Michael Goldberg, Wilhelm Schlag, *Strichartz and smoothing estimates for Schrödinger operators with almost critical magnetic potentials in three and higher dimensions*, Forum Math. 21, no. 4, 687 - 722.

[5] Stephen Gustafson, Kenji Nakanishi, Tai-Peng Tsai, *Asymptotic stability and completeness in the energy space for nonlinear Schrödinger equation with small solitary waves*, IMRN 2004, no. 66, 3559-3584.
[6] E. Kirr, A. Zarnescu, *On the asymptotic stability of bound states in 2D cubic Schrödinger equation*, Comm. Math. Phys. 272 (2007), no. 2, 443-468.

[7] A. Soffer, M.I. Weinstein, *Multichannel nonlinear scattering for nonintegrable equations*, Comm. Math. Phys. 133 (1990), no. 1, 119-146.

[8] A. Soffer, M.I. Weinstein, *Selection of the ground state in the nonlinear Schrödinger equation*, Rev. Math. Phys. 16, no. 8 (2004), 977-1071.

[9] Tetsu Mizumachi, *Asymptotic stability of small solitons to 1D NLS with potential*, J. Math. Kyoto Univ. 48, no. 3 (2008), 471-497.

[10] Tetsu Mizumachi, *Asymptotic stability of small solitons for 2D Nonlinear Schrödinger equations with potential*, J. Math. Kyoto Univ. 47, no. 3 (2007), 599-620.

[11] T.-P. Tsai, H.-T.Yau, *Asymptotic dynamics of nonlinear Schrödinger equations: resonance-dominated and dispersive-dominated solutions*, Comm. Pure Appl. Math. 55 (2002), no. 2, 153-216.

[12] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, 2nd ed., Johann Ambrosius Barth, Heidelberg, 1995.

[13] M. Reed, B. Simon, *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness*, Academic Press, New York, 1975