Bosonization and the generalized Mandelstam operators

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Abstract

The generalized massive Thirring model (GMT) with \( N_f \) (=number of positive roots of \( su(n) \)) fermion species is bosonized in the context of the functional integral and operator formulations and shown to be equivalent to a generalized sine-Gordon model (GSG) with \( N_f \) interacting soliton species. The generalized Mandelstam-Halpern soliton operators are constructed and the fermion-boson mapping is established through a set of generalized bosonization rules in a quotient positive definite Hilbert space of states. Each fermion species is mapped to its corresponding soliton in the spirit of particle/soliton duality of Abelian bosonization. The examples of \( su(3) \) and \( su(4) \) are presented.

1 Introduction

The transformation of Fermi fields into Bose fields, called \textit{bosonization}, provided in the last years a powerful tool to obtain non-perturbative information in two-dimensional field theories [1]. The Abelian and non-Abelian bosonizations have been derived in [2] and [3], respectively. In the non-abelian developments the appearance of solitons in the bosonized model, which generalizes the sine-Gordon solitons, to our knowledge has not been fully explored. The interacting multi-flavor massive fermions deserves a consideration in the spirit of the particle/soliton duality of the Abelian bosonization.

In this context, an important question is related to the multi-flavor extension of the well known massive Thirring (MT) and sine-Gordon relationship (SG)[2]. In [4, 5] it has been shown that the generalized massive Thirring model (GMT) is equivalent to the generalized sine-Gordon model (GSG) at the classical level; in particular, the mappings between spinor bilinears of the GMT theory and exponentials of the GSG fields were established on shell and the various soliton/particle correspondences were uncovered.

In [6] the bosonization of the GMT model has been performed following a hybrid of the operator and functional formalisms. This approach introduces a redundant Bose field algebra containing some un-physical degrees of freedom [7]. The redundant Bose fields constitute a set of pairwise massless fields quantized with opposite metrics. In the GMT cases, under consideration here, these features are reproduced according to an affine \( su(n) \) Lie algebraic construction.

A positive definite Hilbert space of states \( \mathcal{H} \) is identified as a quotient space in the Hilbert space hierarchy emerging in the bosonization process. One has that each GMT fermion is bosonized in terms of a Mandelstam “soliton” operator and a spurious exponential field with zero scale dimension, this spurious field behaves as an identity in the Hilbert space \( \mathcal{H} \) and,
so, has no physical effects. Afterwards, a set of generalized bosonization rules are established
mapping the GMT fermion bilinears into the corresponding operators composed of the GSG boson fields.

2 Functional integral and operator approaches

The two-dimensional massive Thirring model with current-current interactions of $N_f$ (Dirac) fermion species is defined by the Lagrangian density

$$\frac{1}{k'} \mathcal{L}_{GMT}[\psi^j, \overline{\psi}^j] = \sum_{j=1}^{N_f} \left\{ i \overline{\psi}^j \gamma^\mu \partial_\mu \psi^j - m^j \overline{\psi}^j \psi^j \right\} - \frac{1}{4} \sum_{k,l=1}^{N_f} \left[ \hat{G}_{kl} J^\mu_k J^\mu_l \right],$$

(2.1)

where the $m^j$'s are the mass parameters, the overall coupling $k'$ has been introduced for later purposes, the currents are defined by $J^\mu_j = \overline{\psi}^j \gamma^\mu \psi^j$, and the coupling constant parameters are represented by a non-degenerate $N_f \times N_f$ symmetric matrix $\hat{G} = \hat{g} \hat{G} \hat{g}$, $\hat{g}_{ij} = g \delta_{ij}$, $\hat{G}_{jk} = \hat{G}_{kj}$.

The GMT model (2.1) is related to the weak coupling sector of the $su(n)$ ATM theory in the classical treatment of Refs. [4, 5]. We shall consider the special cases of $su(n)$ ($n = 3, 4$).

In the $n = 3$ case the currents at the quantum level must satisfy

$$J^\mu_3 = \hat{\delta}_1 J^\mu_1 + \hat{\delta}_2 J^\mu_2,$$

(2.2)

where the $\hat{\delta}_1, 2$ are some parameters related to the couplings $\hat{G}_{kl}$. Similarly, in the $n = 4$ case the currents at the quantum level satisfy

$$J^\mu_4 = \hat{\sigma}_{41} J^\mu_1 + \hat{\sigma}_{42} J^\mu_2, \quad J^\mu_5 = \hat{\sigma}_{31} J^\mu_1 + \hat{\sigma}_{53} J^\mu_3, \quad J^\mu_6 = \hat{\sigma}_{62} J^\mu_2 + \hat{\sigma}_{63} J^\mu_3,$$

(2.3)

where the $\hat{\sigma}_{ia}$'s (i=4,5,6; a=1,2,3) are related to the couplings $\hat{G}_{kl}$.

The cases under consideration $n = 3, 4$ correspond to $N_f = 3, 6$, respectively; however, most of the construction below is valid for $N_f > 6$. In the hybrid approach the Thirring fields are written in terms of the “generalized” Mandelstam “soliton” $\Psi^j(x)$ and $\sigma^j$ fields

$$\psi^j(x) = \Psi^j(x) \sigma^j, \quad j = 1, 2, 3, ..., N_f;$$

(2.4)

where

$$\Psi^j(x) = \left( \frac{\mu}{2\pi} \right)^{1/2} K^j e^{-i\pi \gamma_5/4} e^{-i \left( \frac{\mu}{4} \gamma_5 \Phi^j(x) + \frac{\mu}{2} \int_{x_1}^{x_2} \Phi^j(x^0, z) dz \right)};$$

(2.5)

$$\sigma^j = e^{\frac{i}{2} \left( m_j - \sqrt{4 \pi} \xi_j \right)};$$

(2.6)

$$= e^{-\frac{i}{2} a_j l_k}.$$

(2.7)

In (2.5) the factor $K^j$ makes the fields anti-commute for different flavors [8].

The Lagrangian in terms of purely bosonic fields becomes

$$\frac{1}{k'} \mathcal{L}'_{eff} = \sum_{j,k=1}^{N_f} \left\{ C_{jk} \partial_\mu \Phi_j \partial^\mu \Phi_k + E_{jk} \partial_\mu \xi_j \partial^\mu \xi_k + F_{jk} \partial_\mu \eta_j \partial^\mu \eta_k \right\} + \sum_{j=1}^{3} M^j \cos(\Phi^j),$$

(2.8)
where $C_{jk}, D_{jk}, E_{jk}$ and $M^j$ are some parameters such that the fields $ξ_j$ and $η_j$ are quantized with opposite metric.

Notice that each $Ψ^j$ is written in terms of a non-local expression of the corresponding bosonic field $Φ^j$ and the appearance of the couplings $β_j$ in (2.5) in the same form as in the standard sine-Gordon construction of the Thirring fermions [2]; so, one can refer the fermions $Ψ^j(x)$ as generalized SG Mandelstam-Halpern soliton operators.

2.1 The $su(3)$ case

The generalized massive Thirring model (GTM) (2.1) with three fermion species, satisfying the currents constraint (2.2), bosonizes to the generalized sine-Gordon model (GSG) (2.8) with three boson fields satisfying the linear constraint $β_3 Φ_3 = β_1 Φ_1 δ_1 + δ_2 β_2 Φ_2$, by means of the “generalized” bosonization rules

\[
i \bar{ψ}^j γ^μ \partial_μ ψ^j = \frac{1}{2} (1 - ρ_j) (\partial_μ Φ^j)^2, \quad j = 1, 2, 3; \quad (2.9)
\]

\[
m_j \bar{ψ}^j ψ^j = M_j \cos (β_j Φ^j), \quad β_j^2 = \frac{4π}{1 + \frac{g_j^2}{4G_{im}}}; \quad (2.10)
\]

\[
\bar{ψ}^j γ^μ ψ^j = -\frac{β_j}{2π} e^{μν} ∂_ν Φ_j, \quad (2.11)
\]

where $ρ_j$ can be written in terms of $g_j$ and the correlation functions on the right hand sides must be understood to be computed in a positive definite quotient Hilbert space of states $H \sim \frac{π^2}{πα}$ [6].

2.2 The $su(4)$ case

In the $su(4)$ case [9] one has the constraints $Φ_4 = σ_{41} Φ_1 + σ_{42} Φ_2$, $Φ_5 = σ_{51} Φ_1 + σ_{53} Φ_3$, $Φ_6 = σ_{62} Φ_2 + σ_{63} Φ_3$, with $σ_{ij}$ being some parameters. The bosonization rules become

\[
i \bar{ψ}^j γ^μ \partial_μ ψ^j = -\frac{β_j}{π} \bar{M}_i^+ (g_i)^2 \bar{M}_i \quad i = 1, 2, 3, ..., 6; \quad (2.12)
\]

\[
i \bar{ψ}^j γ^μ ψ^j = \frac{1}{2} (1 - \hat{ρ}_j) (\partial_μ Φ^j)^2, \quad j = 1, 2, 3, ..., 6; \quad (2.13)
\]

\[
m_j \bar{ψ}^j ψ^j = M_j \cos (β_j Φ^j). \quad (2.14)
\]

The parameters $M_i^+, β_j$ and $\hat{ρ}_j$ can be expressed in terms of $g_i, G_{ij}$. The $z_i$'s are regularization parameters.

3 Discussions

Using the mixture of the functional integral and operator formalisms we have considered the bosonization of the multiflavour GMT model with $N_f$ [=number of positive roots of $su(n)$] species. The sets of free bosonic fields $(ξ_j, η_j)$ are quantized with opposite metrics...
and their contributions are essential in order to define the correct Hilbert space of states and the relevant fermion-boson mappings. One must emphasize that the classical properties of the ATM model [4, 5, 10] motivated the various insights considered in the bosonization procedure of the GMT model performed in this work. The form of the quantum GSG model (2.8) is similar to its classical counterparts in [4, 5], except for the field renormalizations and the relevant quantum corrections to the coupling constants. The bosonization results presented in this talk can be useful to study particle/soliton duality and confinement in multi-fermion two-dimensional models, extending the results of [11, 12].

Acknowledgments

The author thanks the organizers of the V-SILAFAE and the hospitality of IMPA, CBPF and ICET-UFMT. This work was supported by FAPEMAT-CNPq.

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