Bosonization of vertex operators for Zn symmetric Belavin model and its correlation functions

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\[ a \] CCAST (World Laboratory), P.O.Box 8730, Beijing 100080, China
\[ b \] Institute of Modern Physics, Northwest University, Xian 710069, China

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Abstract

Based on the bosonization of vertex operators for \( A_n^{(1)} \) face model by Asai, Jimbo, Miwa and Pugai, using vertex-face correspondence we obtain vertex operators for Zn symmetric Belavin model, which are constructed by deformed boson oscillators. The correlation functions are also obtained.

1 Introduction

The Bosonization of vertex operators for solvable models is very powerful for studying their correlation functions\(^{[1-3]}\). These vertex operators realize the Zamolodichikov-Faddeev algebras\(^{[4-7]}\) with the R-matrices of the models. From the bosonized vertex operators, one can obtain multi-point correlation functions as explicit integrals.

Recently, the studies of q-deformed Virasoro algebra\(^{[8,21,22]}\) and its vertex operators make it clear to understand the correspondence between the unitary minimal conformal models\(^{[26]}\) and the ABF models\(^{[11]}\), this thing has been considered as a mysterious one for a long time. The bosonization for q-deformed Virasoro algebra and its vertex operators also make it possible to calculate the correlation functions for ABF models. The bosonization for q-deformed W algebra\(^{[21,22]}\) and its vertex operations\(^{[10,21]}\) would promote to investigate the \( A_n^{(1)} \) RSOS models\(^{[15]}\). Naively, the q-deformed W algebra (including q-deformed Virasoro algebra) would play an important role in the elliptic face models. How about the elliptic vertex
model—eight vertex model and Zn Belavin model. Jimbo et al obtained the difference equations for eight-vertex operators using the method of the corner transfer matrix (CTM) and “physical picture”, moreover, give the spontaneous polarization of eight-vertex model\cite{20}. Through the same method, Quano obtained the difference equations for Zn Belavin model and the corresponding spontaneous polarization\cite{19}. However, to calculate the general correlation functions for the elliptic vertex model practically is very complicated and still is an open problem\cite{24,27}.

After the works for trigonomatric models\cite{1-3} several important works for elliptic models has been done\cite{8-10}. Lukyanov et al give the bosonization of vertex operators for ABF model. Miwa give the corresponding bosonized boundary operators\cite{9}. Recently, Asai, Jimbo et al obtain the bosonized vertex operators\cite{12} for $A_n^{(1)}$ face model\cite{12}. These work will greatly promote the study of solvable models of elliptic type. Based on their work, using vertex-face correspondence\cite{13-15}, we obtain bosonized vertex operators for Zn symmetric Belavin model\cite{16,17}.

In the following, we first find an intertwiner in section 2.3, which intertwines the Belavin R-matrix and the Boltzmann weight of $A_n^{(1)}$ face model in Ref.\cite{10}. We then review the vertex operators in Ref.\cite{8-10} and their exchange relations in section 4. Combining their vertex operators and the intertwiners, we finally obtain the bosonization for vertex operators and give the correlation functions of Zn symmetric Belavin model in section 5.

\section{Vertex-face correspondence}

Given an integer $n$ ($2 \leq n$), $\tau, w \in \mathbb{C}$ and $\text{Im} \tau > 0$, we can construct Zn symmetric Belavin R-matrix\cite{16,17}. Define $n \times n$ matrices $g, h, I_\alpha$ where

\begin{align*}
g_{jk} &= \omega^j \delta_{jk}, \quad h_{jk} = \delta_{j+1,k}, \quad \omega = \exp\left(\frac{2i\pi}{n}\right) \\
I_\alpha &= I_{(\alpha_1, \alpha_2)} = g^{\alpha_2} h^{\alpha_1}, \quad (\alpha_1, \alpha_2) \in Z_n^2
\end{align*}

Define $I_n^{(j)} = I \otimes I \otimes ... \otimes I_\alpha \otimes I \otimes ... \otimes I$, where $I_\alpha$ is at the $j^{th}$ site, $I$ is the $n \times n$ unit matrix, and

\begin{align*}
W_\alpha(z, \tau) &= \frac{1}{n} \theta \left[ \frac{1}{2} + \frac{\alpha_2}{n} \right] (z + \frac{w}{n}, \tau) / \theta \left[ \frac{1}{2} + \frac{\alpha_1}{n} \right] (\frac{w}{n}, \tau) \\
\theta \left[ \begin{array}{c} a \\ b \end{array} \right] (z, \tau) &= \sum_{m \in \mathbb{Z}} \exp\{i\pi(m+a)(m+a)\tau + 2(z+b)\} \\
\sigma_0(z, \tau) &= \theta \left[ \begin{array}{c} 1 \\ \tau \end{array} \right] (z, \tau)
\end{align*}

Zn symmetric Belavin R-matrix is

\begin{equation}
R_{jk}(z, \tau) = \frac{\sigma_0(w, \tau)}{\sigma_0(z + w, \tau)} \sum_{\alpha \in Z_n^2} W_\alpha(z, \tau) I_n^{(j)} (I_\alpha^{-1})^{(k)}
\end{equation}
which satisfies Yang-Baxter equation (YBE)

\[ R_{12}(z_1 - z_2, \tau)R_{13}(z_1 - z_3, \tau)R_{23}(z_2 - z_3, \tau) = R_{23}(z_2 - z_3, \tau)R_{13}(z_1 - z_3, \tau)R_{12}(z_1 - z_2, \tau) \]  

(2)

Given a n-vector \(a \in \mathbb{Z}^n\), we define Boltzmann weight of \(A(1)_{n-1}\) face model[12], which can be written in the vertex form \(W(a|z, \tau)^{\mu\nu}_{\mu\nu}\), where the non-zero elements are

\[
W(a|z, \tau)^{\mu\mu}_{\mu\mu} = 1 \\
W(a|z, \tau)^{\mu\nu}_{\mu\nu} = \frac{\sigma_0(z + w, \tau) \sigma_0(z + a_{\mu\nu}w, \tau)}{\sigma_0(w, \tau) \sigma_0(a_{\mu\nu}w, \tau)}, \quad \mu \neq \nu
\]

(3)

(4)

where \(a_{\mu\nu}\) is defined by \(a = (a_1, ..., a_n)\) by

\[
\overline{a}_{\mu} = a_{\mu} - \frac{1}{n} \sum_{i=1}^{n} a_{l} + w_{\mu} \tag{6}
\]

\[
\overline{a}_{\mu\nu} = \overline{a}_{\mu} - \overline{a}_{\nu} = a_{\mu} - a_{\nu} + w_{\mu} - w_{\nu} \tag{7}
\]

\(\{w_j\}\) is a set of generic complex numbers specified by the face model under investigation.

The intertwiners of vertex-face correspondence are [14,15] n-column vectors \(\varphi_{\mu,a}(z, \tau)\) whose k-th component is

\[
\varphi^{(k)}_{\mu,a}(z, \tau) = \theta^{(j)}(z + nw(\overline{a}_{\mu} + 1 - \frac{1}{n}), \tau)
\]

\[
\theta^{(j)}(z, \tau) = \theta \left[ \frac{1}{\tau} - \frac{j}{n} \right] (z, n\tau)
\]

The vertex-face correspondence is

\[
R(z_1 - z_2, \tau)\varphi_{\mu,a+e_{\mu}}(z_1, \tau) \otimes \varphi_{\nu,a}(z_2, \tau) = \sum_{\mu' \nu'} W(a|z_1 - z_2, \tau)^{\mu\nu}_{\mu'\nu'} \varphi_{\mu',a+e_{\mu'}}(z_1, \tau) \otimes \varphi_{\nu',a}(z_2, \tau) \tag{8}
\]

where \(e_{\mu} = (0, 0, ..., 1, 0, ..., 0)\) and “1” is at the \(\mu^{th}\) site. We can introduce n-row vectors \(\tilde{\varphi}_{\mu,a}(z, \tau)\) such that

\[
\sum_{k} \varphi^{(k)}_{\mu,a}(z, \tau) \tilde{\varphi}^{(k)}_{\mu,a}(z, \tau) = \delta_{\mu\nu} \tag{9}
\]

Thus we have

\[
\sum_{\mu} \varphi_{\mu,a}(z, \tau) \tilde{\varphi}_{\mu,a}(z, \tau) = I \tag{10}
\]

Notice \(\varphi_{\mu,a}(z, \tau)\) is a function of \(a, \mu, k, z, n, \tau, w, \{w_j\}\). One can show the vertex-face correspondence by \(\tilde{\varphi}\)

\[
\varphi_{\mu,a}(z_1, \tau) \otimes \varphi_{\nu,a+e_{\mu}}(z_2, \tau) R(z_1 - z_2, \tau) = \sum_{\mu' \nu'} W(a|z_1 - z_2, \tau)^{\mu'\nu'}_{\mu\nu'} \tilde{\varphi}_{\mu',a+e_{\mu'}}(z_1, \tau) \otimes \tilde{\varphi}_{\nu',a}(z_2, \tau) \tag{11}
\]
3 Modular transformation of Boltzmann weight

Since the Boltzmann weight in Eq.(3)-Eq.(5) is not the same as that in Ref.[10], we need to rescale the Zn Belavin R-matrix and specify the parameters $z\tau$, so that we could directly use the beautiful results in Ref.[10]. Let us restrict the parameter $w$ : $\text{Im} w > 0$ and set

$$ x = e^{i\pi w}, \quad |x| < 1 \quad (\text{Im} w > 0), \quad n + 2 \leq r $$

$$ [v] = e^{i\pi \frac{w}{r_w}} \sigma_0(wv, rw) = \text{const} \times x^{\frac{r}{2} - v} \Theta_{z,\tau}(x^{2v}) $$

$$ \Theta_q(z) = (z, q)(q^{-1}, q) = \prod_{n=0}^{\infty} (1 - q^n) $$

Notice the modular transformation for theta function $\sigma_0(z, \tau)$

$$ \sigma_0(z, -\frac{1}{\tau}) = \text{const} \times e^{i\pi \frac{z}{2} \tau} \sigma_0(z, \tau) $$

we have

$$ [v] = \text{const} \sigma_0(\frac{wv}{r_w}, -\frac{1}{r_w}) $$

Let Zn symmetric Belavin R-matrix be rescaled, and the parameter $z\tau$ be specified as follows

$$ R(v, -\frac{1}{r_w}) = r_1(-v) \frac{\sigma_0(\frac{1}{r_w}, -\frac{1}{r_w})}{\sigma_0(\frac{1}{r_w}, -\frac{1}{r_w})} \sum_a W_\alpha(v, -\frac{1}{r_w}) I_\alpha \otimes I_a^{-1} $$

$$ W_\alpha(v, -\frac{1}{r_w}) = \frac{\sigma_0(\frac{1}{r_w}, -\frac{1}{r_w})}{n_\alpha \sigma_0(\frac{1}{r_w}, -\frac{1}{r_w})} $$

$$ r_1(v) = x^{2v} \frac{1}{n_\alpha} \frac{g_1(-v)}{g_1(v)}, \quad g(v) = \frac{x^{2v+2n-2} \{x^{2v+2n-2} \}}{x^{2v+2n} \{x^{2v+2n} \}} $$

$$ \{ z \} = (z; x^{2r}, x^{2n}), \quad (z, q_1, q_2, ..., q_m) = \prod_{\{n_j\}=0}^{\infty} (1 - zq_1^{n_j} q_2^{n_j} ... q_m^{n_j}) $$

we also specify the intertwiners $\varphi$ and $\tilde{\varphi}$ as follows

$$ \varphi^{(k)}_{\mu, a}(v, -\frac{1}{r_w}) = g(k) \left( v + n(\pi_\mu + 1 - \frac{1}{n}) \frac{1}{r}, -\frac{1}{r_w} \right) $$

$$ \sum_k \tilde{\varphi}^{(k)}_{\mu, a}(v, -\frac{1}{r_w}) \varphi^{(k)}_{\nu, a}(v, -\frac{1}{r_w}) = \delta_{\mu \nu} $$

The vertex-face correspondence becomes

$$ \tilde{\varphi}_{\mu, a}(v_1, -\frac{1}{r_w}) \otimes \tilde{\varphi}_{\nu, a + \epsilon_\mu}(v_2, -\frac{1}{r_w}) R(v_1 - v_2, -\frac{1}{r_w}) $$

$$ = \sum_{\mu', \nu'} W'(a) v_1 - v_2, -\frac{1}{r_w} \tilde{\varphi}_{\mu', a + \epsilon_\mu}(v_1, -\frac{1}{r_w}) \otimes \tilde{\varphi}_{\nu', a}(v_2, -\frac{1}{r_w}) $$
using the Eq.(11), the non-zero element of \( W'(a|v, -\frac{1}{r^w})[^\mu,^\nu] \) can be written

\[
W'(a|v, -\frac{1}{r^w})[^\mu,^\nu] = r_1(-v) \tag{17}
\]

\[
W'(a|v, -\frac{1}{r^w})[^\mu,^\nu] = r_1(-v) \frac{|v| [a_{\mu\nu} - 1]}{|v| + 1} [a_{\mu\nu}] \tag{18}
\]

\[
W'(a|v, -\frac{1}{r^w})[^\mu,^\nu] = r_1(-v) \frac{|v + a_{\mu\nu}|}{|v| + 1} [a_{\mu\nu}] \tag{19}
\]

It can be found that our Boltzmann weight \( W'(a| - v, -\frac{1}{r^w})[^\mu,^\nu] \) is the same as the Boltzmann weight \( W \left( a + \tau_\mu + \bar{\tau}_\nu, a + \tau_\mu, |v \right) \) in the Ref.[10]

\[
W'(a| - v, -\frac{1}{r^w})[^\mu,^\nu] = W \left( a + \tau_\mu + \bar{\tau}_\nu, a + \tau_\mu, |v \right) \tag{20}
\]

\[
W'(a| - v, -\frac{1}{r^w})[^\mu,^\nu] = W \left( a + \tau_\mu + \bar{\tau}_\nu, a + \tau_\mu, |v \right) \tag{21}
\]

\[
W'(a| - v, -\frac{1}{r^w})[^\mu,^\nu] = W \left( a + \tau_\mu + \bar{\tau}_\nu, a + \tau_\mu, |v \right) \tag{22}
\]

4 Vertex operators in \( A_{n-1} \) face model

We review Asai,Jimbo,Miwa and Pugai’s bosonization of vertex operator for \( A_{n-1} \) face model[10].

According to Ref.[8-10],introduce bosonic oscillators \( \beta^m_j \) ( \( 1 \leq j \leq n - 1, m \in Z \{0\} \)) which satisfy

\[
[\beta^m_j, \beta^m_k] = m \frac{(n-1)m}{nm} \delta_{m+m',0} \tag{23}
\]

\[
[\beta^m_j, \beta^m_k] = -m x^{(j-k)mn} \frac{m}{nm} \delta_{m+m',0}, \quad j \neq k \tag{24}
\]

\[
[a]_x = \frac{x^a - x^{-a}}{x - x^{-1}}, \quad x = e^{\pi w}
\]

Define \( \beta^m_n = -x^{2mn} \sum_{j=1}^{n-1} x^{-2jm} \beta^m_j \).

Introduce zero modes \( p_\mu , q_\mu \) ( \( \mu = 1, ..., n \)) , such that \( [ip_\mu, q_\nu] = \delta_{\mu,\nu} \).Consider orthonormal bases \( \{e_\mu\} , \mu = 1, ..., n < e_\mu, e_\nu > = \delta_{\mu,\nu} \), and

\[
\bar{e}_\mu = e_\mu - \frac{1}{n} \sum_k c_k
\]

Define

\[
Q_{\tau_\mu} = q_\mu , \quad R_{\tau_\mu} = p_\mu - \frac{1}{n} \sum_k p_k
\]

\[
\text{5}
\]
One have

$$[i P_{\mu}, Q_{\nu}] = \delta_{\mu \nu} - \frac{1}{n} = \langle \tau_{\mu}, \tau_{\nu} \rangle$$

Let the vacuum $|0\rangle$ be such that

$$\beta_{m}^{j} |0\rangle = p_{\mu} |0\rangle = 0, \quad \text{for } m > 0$$

and that

$$|l, k\rangle = e^{i \sqrt{\frac{r}{r-1}} Q_{l} - i \sqrt{\frac{r}{r-1}} Q_{k}} |0\rangle$$

where $l = \sum_{j=1}^{L} l_{j} e_{j}$, $k = \sum_{j=1}^{L} k_{j} e_{j}$, $Q_{l} = \sum_{j} k_{j} g_{j}$, $Q_{l} = \sum_{j} l_{j} g_{j}$, and let

$$\gamma = \gamma_{j} e_{j}, \quad \beta = \sum_{j} \beta_{j} e_{j}, \quad P_{k} = \sum_{j} k_{j} P_{j}$$

we have

$$[i P_{\gamma}, Q_{\beta}] = \langle \tau_{\gamma}, \beta \rangle = \langle \gamma, \beta \rangle$$

where $\tau = \sum_{j} \gamma_{j} \tau_{j} - \sum_{j} \beta_{j} \tau_{j}$. The Fock space $F_{l,k} = C[(\beta_{-1}, \beta_{-2}, \ldots)_{1 \leq j \leq n} | l, k \rangle$ with

$$\beta_{m}^{j} |l, k\rangle = 0, \quad (m > 0)$$

$$P_{\gamma} |l, k\rangle = \langle \tau_{l}, \sqrt{\frac{r}{r-1}} l - \sqrt{\frac{r-1}{r}} k \rangle$$

Define for $j=1, \ldots, n-1$

the simple root $\alpha_{j} = e_{j} - e_{j+1}$, the basic weight $\omega_{j} = \sum_{k=1}^{j} \tau_{k}$

$$\xi_{j}(v) = e^{i \sqrt{\frac{r}{r-1}} (Q_{a_{j}} - i P_{a_{j}} 2e_{lnx})} e^{\sum_{m \neq 0} \frac{1}{m} \left( \beta_{m}^{j} - \beta_{m+1}^{j} \right) x_{j}^{m} \beta_{m}^{j}}$$

$$\eta_{j}(v) = e^{-i \sqrt{\frac{r}{r-1}} (Q_{a_{j}} - i P_{a_{j}} 2e_{lnx})} e^{-\sum_{m \neq 0} \frac{1}{m} \left( \beta_{m}^{j} x_{j}^{m} \beta_{m+1}^{j} \right)}$$

Introduce vertex operators

$$\phi_{\mu}(v) = \int d(x^{2 v_{j}}) \prod_{j=1}^{\mu-1} \eta_{j}(v) \xi_{j}(v) \eta_{j+1}(v) \xi_{j+1}(v) f(v_{j} - v_{j-1}, \hat{\pi}_{j})$$

$$\hat{\pi}_{\mu} = \sqrt{r(r-1)} P_{\tau_{\mu}} + w_{\mu} \quad f(v, y) = \frac{|v + \frac{1}{2} - y|}{|v - \frac{1}{2}|}$$

where $\{ w_{j} \}$ is a set complex number defined in Eq.(6) and $\hat{\pi}_{\mu}$ is a set of operators. Here we set $v_{0} = v$ and take the integration contours to be simple closed curves around the origin satisfying

$$|x^{2 v_{j+1}}| < |x^{2 v_{j}}| < |x^{-1} x^{2 v_{j+1}}| \quad (j = 1, \ldots, \mu - 1)$$
Following Ref.[10], one can verify

$$\phi_\mu(v_1)\phi_\nu(v_2) = \sum_{\mu'\nu'} \phi_{\mu'}(v_2)\phi_{\nu'}(v_1)W'(\hat{\pi}|v_2 - v_1, -\frac{1}{r_{\mu\nu'}})$$

Namely,

$$\phi_\mu(-v_1)\phi_\nu(-v_2) = \sum_{\mu'\nu'} \phi_{\mu'}(v_2)\phi_{\nu'}(v_1)W'(\hat{\pi}|v_1 - v_2, -\frac{1}{r_{\mu\nu'}})$$

Now the Boltzmann weight $W'(\hat{\pi}|v, -\frac{1}{r_{\mu\nu'}})$ be some functions like Eq.(17)-Eq.(19) with $a_{\mu\nu}$ replaced by operator $\hat{\pi}_{\mu\nu}$. Thus it does not commutate with vertex operator $\phi_\mu(v)$ and the exchange relations Eq.(27) should be written in that order.

5 vertex operators for Zn symmetric Belavin model and its correlation functions

In “physical picture” of lattice models[1,3,15,20], the vertex operators for Zn symmetric Belavin model can be realized by a half column transfer matrix, and these vertex operators realize the Zamolodchikov-Fadeev algebra with the Zn symmetric R-matrix as its construction coefficient[19,20,29].

$$Z^j(z_1)Z^k(z_2) = \sum_{j'k'} Z^{j'}(z_2)Z^{k'}(z_1)R_{j'k'}^{jk}(z_1 - z_2, \tau)$$

where $Z^j(z)$ are some operators acting on the eigenvalue vectors spaces $H^i$ of the Hamiltonian $D^{(i)}$ of CTM, where

$$e^{i\pi w D^j}Z^j(z)e^{-i\pi w D^j} = Z^j(nw + z)$$

The correlation functions are expressed in terms of the trace of vertex operators in the spaces $H^i$. For the detail, we refer readers to the Ref.[3].

Our main idea is to realize these vertex operators and the $D^{(i)}$ in a direct sum of Fock space $L_i = \bigoplus_{(m_i)\in z F_{\nu}, \sum_{j=1}^{n-1} m_j \alpha_j, \Lambda_i} (\alpha_i)$ is the basic weight of Lie algebra $A_{n-1}$ (resp. simple root of $A_{n-1}$) This representation is expect to be reducible. In fact, the $H^i$ can be consider as the same as the Fock space $L_i$ from the chacter [18] (see Eq.(39))

$$tr_{H^i}(x^n D^{(i)}) = tr_{L_i}(x^n D^{(i)})$$

A similar bosonization produce was applied for caculations of conformal blocks in the conformal field theory (CTF)[28], lattice correlation functions for XXZ model[3] and ABF model[8,9].
Define \( \hat{a}_\mu = -\sqrt{\frac{1}{r_w}} p_\mu + w_\mu + \frac{1}{r_w} < e_\mu, l > \) \( \hat{a}_{\mu\nu} = \hat{a}_\mu - \hat{a}_\nu \), we have

\[
\hat{a}_\mu e^{-i\sqrt{\frac{1}{r_w}} q_\mu} = e^{-i\sqrt{\frac{1}{r_w}} q_\nu}(\hat{a}_\mu + \delta_{\mu\nu})
\]

\[
\hat{\pi}_{\mu\nu} F_{l,k} = (r < e_\mu - e_\nu, l - k) + (r < e_\mu - e_\nu, k) + w_\mu - w_\nu) F_{l,k}
\]

\[
\hat{a}_{\mu\nu} F_{l,k} = (r < e_\mu - e_\nu, k) + w_\mu - w_\nu) F_{l,k}
\]

(31)

From above equation and \([v + r] = -[v]\), we can derive

\[
W'(\hat{\pi}|v, -\frac{1}{r_w} \mu\nu|_{F_{l,k}} = W'(\hat{a}|v, -\frac{1}{r_w} \mu\nu|_{F_{l,k}})
\]

(32)

From Eq.(11) and Eq.(27), we can construct the bosonization for the vertex operators of Zn symmetric Belavin model which satisfy relation Eq(28) with specifying the parameters \( z, \tau \). Define

\[
\Phi^{(j)}(v) = \sum_{\mu} \phi_\mu(-v) \varphi_{\mu,\tilde{\alpha}}(v + \delta, -\frac{1}{r_w})
\]

(33)

where \( \delta \) is a generic parameter. Notice Eq.(27) , Eq.(33) and the vertex-face correspondence Eq(11), we have

\[
\Phi^{(i)}(v_1)\Phi^{(j)}(v_2)|_{F_{l,k}} = \sum_{\mu\nu} \phi_\mu(-v_1) \varphi_{\nu,\tilde{\alpha}}(v_1 + \delta, -\frac{1}{r_w}) \phi_\nu(-v_2) \varphi_{\nu,\tilde{\alpha}}(v_2 + \delta, -\frac{1}{r_w})|_{F_{l,k}}
\]

\[
= \sum_{\mu\nu} \phi_\mu(-v_1) \phi_\nu(-v_2) \varphi_{\nu,\tilde{\alpha}}(v_1 + \delta, -\frac{1}{r_w}) \varphi_{\nu,\tilde{\alpha}}(v_2 + \delta, -\frac{1}{r_w})|_{F_{l,k}}
\]

\[
= \sum_{\mu\nu} \varphi_{\mu,\tilde{\alpha}}(v_1 + \delta, -\frac{1}{r_w}) \varphi_{\nu,\tilde{\alpha}}(v_2 + \delta, -\frac{1}{r_w})|_{F_{l,k}}
\]

(34)

Namely ,we have the bosonization for vertex operator of Zn symmetric Belavin model

\[
\Phi^{(i)}(v_1)\Phi^{(j)}(v_2)|_{L_i} = \sum_{i'j'} \Phi^{(i')}(|v_2|)\Phi^{(j')}|v_1)R^{ij}_{ij'}(v_1 - v_2, -\frac{1}{r_w})|_{L_i}
\]

We also can give the bosonization for the Hamiltonian of CTM \( D^{(i)} \) in Fock space

\[
D = \sum_{m=1}^{\infty} \sum_{j=1}^{n-1} \frac{[rm]x}{(r-1)m} \Omega^j_m S^j_m + \frac{1}{2} \sum_{j=1}^{n-1} P_{\omega_j} P_{\alpha_j}
\]

(35)

\[
\Omega^j_m = \sum_{k=1}^{j} \beta^{(2k-j-1)m}_{-m} \beta^k_{-m} \quad S^j_m = x^{-jm}(\beta^j_m - \beta^{j+1}_m)
\]

\[
D|_{L_i} = D^{(i)}|_{L_i}
\]

\[
x^n \Phi^{(j)}(v)x^{-nD} = \Phi^{(j)}(v + nw)
\]

(36)
We can also define the dual vertex operators $\Phi^\ast_j(v)$ through the skew-symmetric fusion of n-1s $\Phi(v)$ \cite{10}

$$\Phi^\ast_j(v) = \sum_{\mu} \phi^{\ast (n-1)}_{\mu}(v - \frac{n}{2}) A_{\mu}^{-1} \phi_{\mu,\hat{\eta} - \hat{\mu}}(v, -\frac{1}{rW})$$

$$A_{\mu} = (-1)^{n-1} \frac{x}{(x^2; x^{2r})} \prod_{k=1} (x^{2r} - x^{2(r-k)})^{\hat{\mu}} = 1^{n}[1 + \hat{\pi}_{k\mu}]$$

where we set $v_n = v$ and $|x^2^{x^{2r-1}}| < |x^{2r}| < |x^{-1}x^{2r-1}|$ ($j = \mu, \ldots, n - 1$)

Following Eq(c.20) in the Ref.[10] and Eq(10), we have the following invertibility

$$\Phi^{(i)}(v)\Phi_j^\ast(v)vert_{L_i} = c_n^{-1} \delta_i^j \times id\rvert_{L_i}$$

(37)

Thus the correlation function for Zn symmetric Belavin model can be described by the following trace functions

$$F^{(i)}(v_1, \ldots, v_N)_{i_1, \ldots, i_N} = \frac{tr_{L_i}(x^D \Phi^{(i)}(v_1) \ldots \Phi^{(N)}(v_N) \Phi^\ast_{i_1}(v_N) \ldots \Phi^\ast_{i_N}(v_N) \ldots)_{tr_{L_i}(x^N)}}{tr_{L_i}(x^D)}$$

(38)

Using the cyclic properties of a matrix trace and the relations Eq(34) and Eq(36), it is easy to derive the difference equations which the correlation functions should be satisfied. The difference equations for eight-vertex model (n=2) was given by Jimbo et al \cite{20} from the “physical picture”, and the difference equation for Zn symmetric Belavin model (n is generic integer, $2 \leq n$) was given by Quano \cite{19} also through the “physicial picture”. The correlation functions for Zn symmetric Belavin model in Eq(38) can be expressed explicitly in terms of integrals after carrying out the trace functions of Eq(38). Therefore, the trace functions of Eq(38) is the solution to difference equations with elliptic R-matrix as its construction coefficient. To directly solve the general difference equations of elliptic type is still an open problem \cite{24,27}.

Fortunately, the bosonization method give a system way to solve these difference equations.

In the simple case for N=1, the trace functions will give the charcter of Z-grade spaces $L_i$

$$tr_{L_i}(x^D) = \sum_{\{m_i\} \in Z} x^{\frac{1}{2} \sum_{k=1}^{n-1} m_j <\omega_k \sqrt{\frac{r-1}{r\omega_k}} - \sqrt{\frac{r-1}{r\omega_k}} (X_k + \sum_{j=1}^{n-1} m_j \alpha_j)} <\alpha_k \sqrt{\frac{r-1}{r\alpha_k}} - \sqrt{\frac{r-1}{r\alpha_k}} (X_k + \sum_{j=1}^{n-1} m_j \alpha_j)}$$

(39)

**Remark:** Actually, the above charcter of space $L_i$ are the same as that of level one integrable represenation of q-deformed affine algebra $(U_q(sl(n))$, of course, it is also equal to that of level one representation of affine algebra $(A^{(1)}_{n-1})$ \cite{18}. Thus the space $L_i$ would be some level one representation of some elliptic deformation of affine algebra, which are not known but many phenomena suggest that it would be existed. We expect
to find this elliptic deformation of affine algebra which would play a role of the symmetric algebra of the elliptic vertex model.

For generic N, one will encounter the following trace functions

\[
tr_{L_i}(x^n e^{\sum_{m=1}^{\infty} \sum_{j=1}^{n-1} A_{\alpha j} \beta_{m-j} e^{\sum_{m=1}^{\infty} \sum_{j=1}^{n-1} B_{\gamma j} \beta_{m-j}} f \rho})
\]

(40)

Since the operator \(e^{Q_\alpha}\) would shift a Fock sector to another different sector unless \(Q_\alpha = 0\), the term of \(e^{Q_\alpha}\) in general no-zero trace functions should be equal to 1. We can calculate the contributions in the trace for tensor components (i) oscillators modes and (ii) the zero mode separately. The trace over oscillator part can be carry out by using the Clavelli-Shapiro technique[25]. More explicitly, let us introduce other oscillators \(\beta_j^i\) \((j = 1, ..., n - 1)\) which commutate with the old ones \(\beta_j^m\). Define the following operators acting in the tensor product of Fock space of \(\beta_j^i\) and that of \(\beta_j^m\)

\[
b_j^m = \frac{\beta_j^m \otimes 1}{1 - x^{2nm}} + 1 \otimes \beta_{-j}^m , \ m > 0
\]

and

\[
b_j^m = \beta_j^m \otimes 1 + \frac{1 \otimes \beta_{-j}^m}{x^{2nm} - 1} , \ m < 0
\]

Now the trace of some bosonic operator \(O(\beta_j^m)\) can be expressed in terms of the vacuum expectation value \(\langle 0|O(b_j^m)|0 \rangle\). Namely,

\[
tr(x^n O(\beta_j^m)) = \frac{\langle 0|O(b_j^m)|0 \rangle}{(x^{2n}; x^{2n})}
\]

(41)

We denote \(\langle 0|O(b_j^m)|0 \rangle\) by \(\langle \langle O(\beta_j^m) \rangle \rangle\) (we choose the same symbol as that of the Lukaynov’s in the Ref.[8]). Due to the Wick theorem, the expectation value of a product of exponential operators is factorized into the two point functions

\[
\langle \langle \eta_1(v_1) \eta_1(v_2) \rangle \rangle = \langle \langle \eta_{n-1}(v_1) \eta_{n-1}(v_2) \rangle \rangle C_1^2 G_1(v_2 - v_1)
\]

(42)

\[
\langle \langle \eta_1(v_1) \eta_{n-1}(v_2) \rangle \rangle = C_1^2 G_{n-1}(v_2 - v_1)
\]

(43)

\[
\langle \langle \eta_1(v_1) \xi_1(v_2) \rangle \rangle = C_1 C_2 S(v_2 - v_1)
\]

(44)

\[
\langle \langle \xi_1(v_1) \xi_{j+1}(v_2) \rangle \rangle = C_2^2 S(v_2 - v_1)
\]

(45)

\[
\langle \langle \xi_1(v_1) \xi_j(v_2) \rangle \rangle = C_2^2 T(v_2 - v_1)
\]

(46)

\[
\langle \langle \eta_{n-1}(v_1) \eta_{n-1}(v_2) \rangle \rangle = C_1 C_2 S(v_2 - v_1)
\]

(47)

\[
\{z\} = (z; x^{2r}, x^{2n}, x^{2n}, x^{2n}) \quad \rho(\eta_j) = C_1 = \frac{\{x^{2r+2n}\}'(x^{2r+4n-2})'}{\{x^{2r+2n}\}'(x^{4n})'}
\]

\[
\rho(\xi_j) = C_2 = (x^{2n}, x^{2n}) \frac{\{x^{2r+2n}\}}{\{x^{2r-2n+2}\}}
\]

\[
G_1(v) = \frac{\{x^{2r+2n}\}'(x^{2r+2n-2+2v})'(x^{2r+2n-2v})'(x^{2r+4n-2-2v})'}{\{x^{2r+2n}\}'(x^{x^{2r+2n}})'(x^{2r+2n-2})'(x^{4n-2})'}
\]
\[ G_{n-1}(v) = \frac{\{x^{n+2v}\}' \{x^{2r+n+2v}\}' \{x^{3n-2v}\}' \{x^{2r+3n-2v}\}'}{\{x^{2r+n-2+2v}\}' \{x^{2+n+2v}\}' \{x^{2r+3n-2-2v}\}' \{x^{3n+2-2v}\}'} \]

\[ S(v) = \frac{\{x^{2r-1+2v}\}' \{x^{2r-1+2n-2v}\}'}{\{x^{1+2v}\}' \{x^{1+2n-2v}\}'} \]

\[ T(v) = (x^{2v}; x^{2n})(2^n-2v; x^{2n}) \frac{\{x^{2+2v}\}' \{x^{2+2n-2v}\}'}{\{x^{2r-2+2v}\}' \{x^{2r-2+2n-2v}\}'} \]

For all other combinations of \( \eta_1(v) , \eta_{n-1}(v) , \xi_j(v) \), we have \( <XY> = \rho(X)\rho(Y) \) and \( \rho(\eta_j) = C_1 \) \( \rho(\xi_j) = C_2 \).

### 6 Discussions

In this paper we construct the bosonic realization of vertex operators for \( Z_n \) symmetric Belavin model in the Fock spaces \( \bigoplus_{i=1}^{n-1} L_i \). In fact, they are the intertwiner operators among spaces \( L_i \) and \( L_i \) would be the level one representation space of some deformed affine algebra (should not be \( q \)-deformed affine algebra). Moreover, we guess that this deformed algebra would be some elliptic deformed affine algebra: for the case of eight-vertex model (\( n=2 \)) it would be the elliptic algebra \( A_{q,p}(\hat{sl}_2) \) \([23]\); for the case of \( 2 < n \), it would be some elliptic algebra which is generalization of \( A_{q,p}(\hat{sl}_2) \) for high rank \( n \). We will present the results of this kind algebra in the further paper.

We only consider the type I vertex operators\([3]\). We can further construct the bosonization for type II vertex operators of \( Z_n \) symmetric Belavin model .

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