Fractional Brownian Motion with Drift: Theory and Numerical Validation

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We study fractional Brownian motion of Hurst parameter \(H\) with both a linear and a non-linear drift. The latter appears naturally when applying non-linear variable transformations. Using a perturbative expansion around Brownian motion, we analytically give the first-order corrections to Brownian motion, resulting in a significant change in the probability distribution functions. Introducing an adaptive bisection algorithm, which is about 1000 times faster and more memory efficient than the standard Davies-Harte algorithm, we test these predictions for effective grid sizes of up to \(N_{	ext{eff}} = 2^{28} \approx 2.7 \times 10^8\) points. The agreement between theory and simulations is excellent, and by far exceeds in precision what can be obtained by scaling alone.

I. INTRODUCTION

Understanding the extreme-value statistics of random processes is important in a variety of contexts. Examples are records [1], e.g. in climate change [2], equivalent to depinning [3], in quantitative trading [4], or for earthquakes [5]. While much is known for Markov processes, and especially for Brownian motion [6–12], much less is known for correlated, i.e. non-Markovian processes, of which fractional Brownian motion (fBm) is the simplest scale-free version [13–20].

FBoM is important as it successfully models a variety of natural processes [21]: a tagged particle in single-file diffusion \((H = 0.25)\) [22, 23], the integrated current in diffusive transport \((H = 0.25)\) [24], polymer translocation through a narrow pore \((H \simeq 0.4)\) [25–27], anomalous diffusion [28], values of the log return of a stock \((H \simeq 0.6 \text{ to } 0.8)\) [14, 29–31], hydrology \((H \simeq 0.72 \text{ to } 0.87)\) [32], a tagged monomer in a polymer \((H = 0.25)\) [33], solar flare activity \((H \simeq 0.57 \text{ to } 0.86)\) [34], the price of electricity in a liberated market \((H \simeq 0.41)\) [35], telecommunication networks \((H \simeq 0.78 \text{ to } 0.86)\) [36], telomeres inside the nucleus of human cells \((H \simeq 0.18 \text{ to } 0.35)\) [37], or diffusion inside crowded fluids \((H \simeq 0.4)\) [38].

In this work, we consider the first-passage time for fBm with additional drift. Apart from a linear drift, a non-linear drift may appear, leading us to consider the process,

\[
\langle y_t \rangle = \langle e^{\nu t} \rangle = \exp \left( \langle z_t \rangle + \frac{1}{2} \langle z_t^2 \rangle \right) = \exp \left( \mu t + [\nu + 1] t^{2H} \right).
\]

The change of \(\nu\) to \(\nu + 1\) implies that non-linear drift is naturally generated. Note that the exponential transformation appears quite often, be it in the Black-Sholes theory of the stock market where the logarithm of the portfolio price is treated as a random walk [30, 39, 40], be it in non-linear surface growth of the Kardar-Parisi-Zhang universality class [41–43], where the transformation is known as the Cole-Hopf transformation [44, 45], or in the evaluation of the Pickands constant [46–53].

While for Brownian motion, equivalent to \(H = \frac{1}{2}\), many results can be obtained analytically [6–12], for fBm much less is known. Recently, we developed a framework [54] for a systematic expansion in

\[
\epsilon := H - \frac{1}{2}.
\]

It has since successfully been applied to obtain the distribution of the maximum and minimum of an fBm [55, 56], to fBm bridges [57], evaluation of the Pickands constant [47], the 2-sided exit problem [58] and the generalization of the three classical arcsine laws [59]. It is also known that the fractal dimension of the record set of an fBm is \(d_l = H\) [60].

This article is organized into four sections, the introduction, theory in section II, and numerics in section III, followed by conclusions in section IV.
II. THEORY

A. Scaling

For fractional Brownian motion

\[ x \sim t^H \iff t \sim x^{\frac{1}{H}}. \]  \hspace{1cm} (8)

Thus (without drift), any observable \( \mathcal{O}(x,t) \) can be written as

\[ \mathcal{O}(x,t) = x^{\dim(\mathcal{O})} f_\mathcal{O}(y), \quad y := \frac{x}{\sqrt{2t^H}}. \]  \hspace{1cm} (9)

The variable \( y \) is dimension free. In presence of drift, there are more possible combinations. Noting

\[ x \sim \mu t \iff \mu \sim \frac{x}{t} \sim x^{1-\frac{1}{H}} \sim t^{H-1}. \]  \hspace{1cm} (10)

Thus the combination \( u = \mu x^{\frac{1}{H}} \) is dimension free, as is \( \tilde{u} := u^{\frac{1}{H}} = \mu^\frac{1}{H} x \). The last relation reads

\[ x \sim \nu^{2H} \iff \nu \sim \frac{x}{t^{2H}} \sim \frac{1}{x} \sim \frac{1}{t^H}. \]  \hspace{1cm} (11)

Another scaling variable therefore is \( v = \nu x \). In conclusion, any observable \( \mathcal{O} \) can, in generalization of Eq. (9), be written as

\[ \mathcal{O}(x,t,\mu,\nu) = x^{\dim(\mathcal{O})} f_\mathcal{O}(y,u,v), \quad y := \frac{x}{\sqrt{2t^H}}, \quad u = \mu x^{\frac{1}{H}}, \quad v = \nu x. \]  \hspace{1cm} (12-15)

B. The path-integral of a fBm with drift

Consider a fBm with drift, as defined in Eq. (1). We can write the action as [54, 56]

\[ S[x] = \int_0^T dt \left[ \frac{\dot{x}^2}{4D_\varepsilon} - \frac{\varepsilon}{2} \int_{0 < t_1 < t_2 < T} \dot{x}_{t_1} \dot{x}_{t_2} \right]. \]  \hspace{1cm} (16)

A short-distance cutoff \(| t_1 - t_2 | > \tau \) in the last integral is implicit, reflected in the diffusion constant [56]

\[ D_\varepsilon = 2H \tau^{2H-1} = (1 + 2\varepsilon) \tau^{2\varepsilon} = (e\tau)^{2\varepsilon} + O(\varepsilon^2). \]  \hspace{1cm} (17)

Inserting the definition (1), we arrive after some algebra at

\[ S[z] = \int_0^T dt \left[ \frac{\dot{z}^2}{4D_\varepsilon} - \frac{\varepsilon}{2} \int_{0 < t_1 < t_2 < T} \dot{z}_{t_1} \dot{z}_{t_2} \right] + \int_0^T dt \frac{\varepsilon}{2} \dot{z}_t \left[ (\mu + \nu) \ln \left( \frac{t(T-t)}{\tau^2} \right) - 2\nu \ln \left( \frac{t}{\tau} \right) \right] - \frac{\varepsilon}{2} \int_0^T dt_1 \int_{t_1}^{t_2} dt_2 \dot{z}_{t_1} \dot{z}_{t_2} \left[ t_2 - t_1 \right] - \frac{\varepsilon}{2} \frac{z_T - z_0}{D_\varepsilon} + \frac{\mu T}{4} + \nu T \right] \right] - \frac{\varepsilon}{2} \left( \nu^2 - \mu^2 \right) \ln(T) + O(\varepsilon^2). \]  \hspace{1cm} (18)

Some checks are in order. In absence of boundaries, the exact free propagator reads

\[ P(0, z, T) = \frac{1}{2\sqrt{\pi T^H \varepsilon}} e^{-\frac{(z - \mu T - \nu T^2)^2}{4\varepsilon T^H}} \]  \hspace{1cm} (19)

Since the above formalism has variables \( \dot{z} \) only, the term \( \sim z^2 \) is given by the drift-free perturbation theory. We can further check that if we replace in the action \( \dot{z}(t) \) by its “classical trajectory”, i.e. \( \dot{z}(t) \to [z(T) - z(0)]/T \), then both the normalization and the drift term agree with the exact propagator.

Let us specify Eq. (18) to the two cases of interest: For a fBm with linear drift as given in Eq. (1) with \( \nu = 0 \), we have

\[ S^{ld}[z] = \int_0^T dt \left[ \frac{\dot{z}^2}{4D_\varepsilon} - \frac{\mu}{2} \left( z_T - z_0 \right) + \frac{T-1}{4} \nu^2 \right] \]  \hspace{1cm} (19)

For a fBm with non-linear drift as given in Eq. (1) with \( \mu = 0 \), we have

\[ S^{nl}[z] = \int_0^T dt \left[ \frac{\dot{z}^2}{4D_\varepsilon} - \frac{\nu}{2} \left( z_T - z_0 \right) + \frac{T+1}{4} \nu^2 \right] \]  \hspace{1cm} (20)

Note the appearance of the diffusion constant in the “bias” (Girsanov) term \( z_T - z_0 \) for a linear drift, and its absence for a non-linear drift.

In the following, we define

\[ \alpha := \mu - \nu, \quad \beta := \mu + \nu, \]  \hspace{1cm} (22)

\[ \mu = \frac{\alpha + \beta}{2}, \quad \nu = \frac{\beta - \alpha}{2}. \]  \hspace{1cm} (23)

This comes with the drift terms grouped as

\[ S^\alpha[z] := \frac{\varepsilon \alpha}{2} \int_0^T dt \dot{z}_t \ln \left( \frac{t}{\tau} \right), \]  \hspace{1cm} (24)

\[ S^\beta[z] := \frac{\varepsilon \beta}{2} \int_0^T dt \dot{z}_t \ln \left( \frac{T-t}{\tau} \right). \]  \hspace{1cm} (25)

Perturbation theory is done in terms of (24) and (25), as well as the non-local interaction

\[ S^1[z] = -\frac{\varepsilon}{2} \int_T dt_1 \int_{t_1}^{t_2} dt_2 \dot{z}_{t_1} \dot{z}_{t_2}. \]  \hspace{1cm} (26)
C. Brownian motion with absorbing boundaries

With the normalizations introduced in Eq. (1), Brownian motion from \( x_0 \) to \( x \) in time \( t \) satisfies the equation
\[
\partial_t P_+(x_0, x, t) = \frac{\partial^2}{\partial x^2} P_+(x_0, x, t) - \mu \partial_x P_+(x_0, x, t). 
\]
We have for an absorbing boundary at \( x = 0 \)
\[
P^\mu_+(x_0, x, t) = \frac{1}{\sqrt{4\pi t}} \left( e^{-(x-x_0)^2/4t} - e^{-(x+x_0)^2/4t} \right) \times e^{\frac{\mu}{2} (x-x_0)^2}. 
\]
It satisfies the b.c. in time
\[
P_+(x, x', t = 0) = \delta(x - x'). 
\]
This yields
\[
\tilde{\tilde{f}}(s) := \mathcal{L} \tilde{f}(t) := \mathcal{L}_{t \rightarrow s} [f(t)] = \int_0^{\infty} dt \, e^{-st} f(t). 
\]
The probability to survive at time \( t \) is after Laplace transform
\[
\tilde{P}_\text{survive}^\mu(m, s) = \int_0^\infty dx \, \tilde{P}_\text{survive}^\mu(m, x, s) = \frac{1 - e^{-m \sqrt{s + \frac{\mu^2}{4}}}}{s}. 
\]
Cutoff dependence (at order \( e \) logarithmically) on the UV cutoff \( \tau \). The result is
\[
P^\mu_\text{survive}(m, t) = 1 - \frac{1}{2} e^{-m|\mu|} \text{erfc} \left( \frac{m - t |\mu|}{2 \sqrt{t}} \right) - \frac{1}{2} e^{-m|\mu|} \text{erfc} \left( \frac{t |\mu| + m}{2 \sqrt{t}} \right). 
\]
Taking the limit of \( t \to \infty \) yields
\[
P^\mu_\text{escape}(m) = \lim_{t \to \infty} P^\mu_\text{survive}(m, t) = 1 - e^{-m|\mu| - \frac{m}{2}} - e^{-m|\mu| - \frac{m}{2}} = 1 - P_\text{abs}(m),
\]
leading back to \( P_\text{abs}(m) \) defined in Eq. (36).

D. Diagrammatic rules

We start by calculating the Green function \( G(m, t) \), i.e. \( 1/|x_0| \) times the probability density to go from \( m \) to \( x_0 \) at time \( t \) for the first time (see Eq. (33) for Brownian motion). It depends (at order \( e \) logarithmically) on the UV cutoff \( \tau \) in time and \( x_0 \), and as a consequence is not normalised; it is noted \( G(m, t) \). We reserve the symbol \( P \) for probabilities, and \( \bar{P} \) for probability densities. As we discuss in section II L, \( G(m, t) \) is proportional to the probability density of first-passage times.

We now use the perturbation expansion established in Ref. [54–57]; we refer to [56, 61] for an introduction. To summarise, each of the perturbations (24)-(26) is inserted into the path integral with absorbing boundaries at \( x = 0 \), to evaluate the Green function from \( m \) to \( x_0 \). In order to eliminate integrations over the times appearing in Eqs. (24)-(26), the calculation is performed in the Laplace-conjugate variable \( s \), defined by Eq. (30).

The next step is to use (Eq. (31) of [56]),
\[
\frac{1}{\tau_2 - \tau_1} = \int_{y>0} e^{-y(\tau_2 - \tau_1)}. 
\]
The integral over times necessitates a cutoff \( \tau \) at small times, which can be replaced by a cutoff \( \Lambda \) for large \( y \) (Eq. (A3) of [56]). Their relation is
\[
\int_0^T dt \int_0^\Lambda e^{-yt} dy = \ln(T\Lambda) + \gamma_E + O(e^{-T\Lambda})
\]
\[
\ln \left( \frac{T}{\tau} \right) = \int_{\tau}^{T} \frac{1}{t} dt. 
\]
This implies the choice
\[
\Lambda = e^{-\gamma_E / \tau}. 
\]
The variable \( y \) on the r.h.s. of Eq. (40) can be interpreted as a shift in the Laplace variable \( s \) associated to the time difference \( \tau_2 - \tau_1 \), i.e.

\[
s \to s + y
\]

for all propagators between times \( \tau_1 \) and time \( \tau_2 \). For an example see the first diagram in Eq. (49) below.

Last, while the insertion of the position \( x_t \) at time \( t \) with \( 0 < t < T \) leads to a factor of \( x \) in the corresponding propagators,

\[
\langle x_t \rangle_{x_0=a,x_T=b} \to \int_x P_+(a, x, t) x \partial_x P_+(x, b, T-t),
\]

the insertion of \( \dot{x}_t \) yields (Eq. (A1) of \([56]\))

\[
\langle \dot{x}_t \rangle_{x_0=a,x_T=b} \to 2 \int_x P_+(a, x, t) \partial_x P_+(x, b, T-t).
\]

\section{Diagrams to be evaluated}

There are three diagrams, presented on figure 1. They give the first order in \( \epsilon \) for \( G \),

\[
\mathcal{G}(m, T) := \exp \left( \frac{m}{2} \left( \frac{\mu}{D_\epsilon} + \nu \right) - \frac{T}{4} (\mu T^\epsilon + \nu T^\epsilon)^2 \right) \times \{ G_0(m, T) + \epsilon \left[ G_1(m, T) - \alpha G_\alpha(m, T) - \beta G_\beta(m, T) \right] \}.
\]

\section{Order \( \epsilon \), first diagram \( \tilde{G}_1 \)}

The Laplace transform of the first diagram is obtained from the one without drift, as represented by the first diagram of figure 1. (The global factor of \( 2 = 2^2/2 \) comes from a factor of 2 for each insertion of \( \dot{x} \), and the 1/2 from the action.)

\[
\tilde{G}_1(m, s) = \frac{2}{x_0} \int_0^\Lambda dy \int_{x_1>0} \int_{x_2>0} \tilde{P}_+(m, x_1, s) \partial_{x_1} \tilde{P}_+(x_1, x_2, s + y) \partial_{x_2} \tilde{P}_+(x_2, x_0, s)
\]

\[
= 2 \int_0^\Lambda dy \frac{e^{-m\sqrt{s}} (my - 2\sqrt{s} + y) + \sqrt{s} + ye^{-m\sqrt{s}y}}{2y^2}
\]

\[
= e^{m\sqrt{s}} (m\sqrt{s} + 1) \text{Ei}(-2m\sqrt{s}) + m\sqrt{s} e^{-m\sqrt{s}} \left[ \ln \left( \frac{m^2}{2\pi} \right) - 1 \right] - e^{-m\sqrt{s}} (m\sqrt{s} + 1) \ln (m\sqrt{s}).
\]
Note that we can also work with a subtracted version, for which \( \tilde{G}_1(m, s)/\tilde{G}_0(m, s) \) vanishes at \( m = 0 \), equivalent to a global change in normalization. This is

\[
\tilde{G}_1^a(m, s) = e^{m\sqrt{s}} (m\sqrt{s} + 1) \mathrm{Ei} (-2m\sqrt{s}) + e^{-m\sqrt{s}} \left[ m\sqrt{s} \left( \ln \left( \frac{m}{2\sqrt{s} \tau} \right) - 1 \right) - \ln (2m\sqrt{s}) - \gamma_E \right].
\] (50)

For the inverse Laplace transform we find using appendix C of Ref. [57]

\[
G_1(m, t) = \frac{me^{-\frac{m^2}{2t}}}{2\sqrt{\pi} t^{3/2}} + \frac{me^{-\frac{m^2}{4\sqrt{\pi} t^{5/2}}}}{4\sqrt{\pi} t^{5/2}} \left[ (m^2 - 4t) \ln (\Lambda m^2) + (2\gamma_E - 1)m^2 + 2t \ln \left( \frac{t}{\Lambda x_0^4} \right) + 2(3 - 7\gamma_E)t \right] = G_0(m, t) \left[ \mathcal{I} \left( \frac{m}{\sqrt{2t}} \right) + 2 \left( \frac{m^2}{4t} - 1 \right) \ln (\Lambda m^2) + \frac{(2\gamma_E - 1)m^2 + 2(3 - 7\gamma_E)t}{2t} + \ln \left( \frac{t}{\Lambda x_0^4} \right) \right].
\] (51)

The special function \( \mathcal{I} \) appearing in this expression was introduced in Ref. [54], Eq. (B53)

\[
\mathcal{I}(z) = \frac{z^2}{6} \left[ 2 F_2 \left( 1, 1; \frac{5}{2}, 3; \frac{z^2}{2} \right) + \pi(1 - z^2) \text{erf} \left( \frac{z}{\sqrt{2}} \right) \right] - 3z^2 + \sqrt{2\pi} e^{\frac{z^2}{2}} z + 2.
\] (52)

Using the definition (42) of \( \Lambda \), Eq. (51) and introducing the variable

\[
z := \frac{m}{\sqrt{2t}},
\] (53)

\( G_0(m, t), G_1(m, t) \) and its subtracted version \( \tilde{G}_1^a(m, t) \) can be written more compactly as

\[
t G_0(m, t) = e^{-\tau^2 z} \sqrt{2\pi},
\] (54)

\[
G_1(m, t) = G_0(m, t) \left\{ \mathcal{I}(z) + (z^2 - 1) \left[ \ln \left( \frac{t}{\tau} \right) - 1 \right] + \ln \left( \frac{\tau^2}{4x_0^4 z^4} \right) + z^2 \left[ \ln \left( 2z^2 + \gamma_E \right) - 4\gamma_E + 2 \right] \right\},
\] (55)

\[
G_1^a(m, t) = G_0(m, t) \left\{ \mathcal{I}(z) - \ln \left( \frac{4\tau^2 z^4}{\tau} \right) + z^2 \left[ \ln \left( \frac{2t z^2}{\tau} \right) + \gamma_E - 1 \right] - 2\gamma_E - 1 \right\}.
\] (56)

Note that there is a global prefactor of \( 1/t \), and a logarithmic dependence on \( t, \tau \) and \( x_0 \).

### G. Order \( \epsilon \), second diagram \( \tilde{G}_\alpha \)

To study perturbations with \( S_\alpha \) defined in Eq. (24), we represent the logarithm as

\[
\ln \left( \frac{t}{\tau} \right) = \int_0^\infty \frac{dy}{y} \left[ e^{-\tau y} - e^{-ty} \right].
\] (57)

This yields

\[
\tilde{G}_\alpha(m, s) = \lim_{x_0 \to 0} \frac{1}{x_0} \int_{x_0}^{\Lambda / \rho} \frac{dy}{y} \int_{x_0}^{1} \left[ \tilde{P}(m, x_1, s) e^{-\gamma s} - \tilde{P}(m, x_1, s) e^{-s y} \right] y \partial_{x_1} \tilde{P}(x_1, x_0, s)
\]

\[
= \int_{0}^{\Lambda / \rho} \frac{dy}{\sqrt{y^2}} \left[ \frac{e^{-m\sqrt{s}}}{\sqrt{y^2}} - \frac{e^{-m\sqrt{s}}}{\sqrt{s y^2}} - \frac{me^{-m\sqrt{s}}}{2} \right] 2y
\]

\[
= \frac{1}{4} me^{-m\sqrt{s}} \left[ 2e^{m\sqrt{s}} \text{Ei} (-2m\sqrt{s}) + \ln \left( \frac{4\tau^2 z^4}{m^2} \right) + 2 \right] + \mathcal{O}(\Lambda^{-1}).
\] (58)

We checked that the \( y \) integrand is convergent, at least as \( 1/y^2 \) for large \( y \), and has a finite limit for \( y \to 0 \); thus neither \( x_0 \) nor \( \Lambda \) are necessary as UV cutoffs, and the \( y \)-integral is finite. The \( \tau \)-dependence stems from the \( \ln(t/\tau) \) of the perturbation term.

Doing the inverse Laplace transform using appendix C of [57], we get with \( z \) defined in Eq. (53)

\[
\sqrt{t} G_\alpha(m, t) = \frac{e^{-\tau^2 z^2} \mathcal{I}(z) - 2}{2\sqrt{\pi}(1 - z^2)} + z \text{erfc} \left( \frac{z}{\sqrt{2}} \right) - \frac{e^{-\tau^2 z^2} \left[ \ln \left( \frac{2t z^2}{\tau} \right) + \gamma_E - 1 \right]}{2\sqrt{\pi}}.
\] (59)
Note that there is no pole at $z = 1$. Indeed, for $z \to 1$ one obtains
\[
-2F_2(1, 1; \frac{5}{2}, 3; \frac{1}{2}) - 4F_2(1, 1; \frac{3}{2}, 2; \frac{1}{2}) + 2\sqrt{2\pi} \left( \text{erfc} \left( \frac{1}{\sqrt{2}} \right) - 3 \right) + 4\pi \text{erfi} \left( \frac{1}{\sqrt{2}} \right) - 4\ln \left( \frac{2}{\tau} \right) - 4\gamma_E + 22 \over 8\sqrt{\pi}.
\]  

(60)

\[ \text{H. Order } \epsilon, \text{ third diagram } \tilde{G}_\beta \]

Using again the integral representation (57), the third diagram is read off from Fig. 1 as
\[
\tilde{G}_\beta(m, s) = \lim_{x_0 \to 0} \frac{1}{x_0} \int_0^\Lambda \frac{dy}{y} \int_{x_0}^{1/y} \tilde{P}_+(m, x_1, s) \partial_{x_1} \left[ \tilde{P}_+(x_1, x_0, s) e^{-\tau y} - \tilde{P}_+(x_1, x_0, s + y) \right]
\]
\[
= \int_0^\infty \frac{dy}{\sqrt{y + 4m^{\sqrt{\pi}}}} - \frac{\sqrt{y + 4m^{\sqrt{\pi}}}}{\sqrt{y}} - \frac{m e^{-m^{\sqrt{\pi}}}}{2y}
\]
\[
= \frac{e^{-m^{\sqrt{\pi}}}}{4\sqrt{s}} \left[ 2 - \ln \left( \frac{m^2}{4s^2} \right) + \ln (4m^2s) + 2\gamma_E \right] - \frac{e^{m^{\sqrt{\pi}}}}{2\sqrt{s}} \left( m^{\sqrt{\pi}} + 1 \right) \text{Ei} \left( -2m\sqrt{s} \right).\]

(61)

We checked that the $y$ integrand is convergent, as it decays at least as $1/y^{3/2}$ for large $y$, and has a finite limit for $y \to 0$, thus $x_0$ or $\Lambda$ are not necessary as an UV cutoff, and the $y$-integral is finite.

\[ \text{I. Combinations} \]

Let us remind that the result for $G_0(z)$ is given in Eq. (54), while $G_1(z)$ and $G_2(z)$ are given in Eqs. (55) and (56). Let us now turn to the corrections for drift. While $G_\alpha$ and $G_\beta$ are the appropriate functions for the calculations, we finally need the corrections for linear drift $\mu$ and non-linear drift $\nu$.

Demanding that
\[
\alpha G_\alpha + \beta G_\beta = \mu G_\mu + \nu G_\nu,
\]
and using Eqs. (22) and (23) yields
\[
\sqrt{t}G_\mu(m, t) = \sqrt{t}[G_\alpha(m, t) + G_\beta(m, t)]
\]
\[
= -\frac{e^{-\frac{z^2}{2}} (z^2 + 1) \text{Ei}(z) - 2}{2\sqrt{\pi} (z^2 - 1)} + \frac{\sqrt{2} z \text{erfc} (\frac{z}{\sqrt{2}})}{z^2 - 1}
\]
\[
- \frac{e^{-\frac{z^2}{2}} z^2 \left( \ln \left( \frac{2^2 z^2}{e^2} \right) + \gamma_E - 2 \right)}{2\sqrt{\pi}}
\]
\[
\sqrt{t}G_\nu(m, t) = \sqrt{t}[G_\beta(m, t) - G_\alpha(m, t)]
\]
\[
= \frac{e^{-\frac{z^2}{2}} \left[ \text{Ei}(z) - 2 \right]}{2\sqrt{\pi}} + \frac{e^{-\frac{z^2}{2}} z^2 \left( \ln (2z^2) + \gamma_E \right)}{2\sqrt{\pi}}.
\]

(64)

\[ \text{J. Scaling and corrections from the diffusion constant, final result} \]

The natural scaling variable for fBm is not $z$, but
\[
y := \frac{m}{\sqrt{2\mu}}.
\]

(68)

Consider
\[
e^{-\frac{z^2}{2}} \frac{y}{\sqrt{2\pi}} \left[ 1 + \left( z^2 - 1 \right) \epsilon \ln(t) \right] + O(\epsilon^2).
\]

(69)

There is also a correction to the diffusion constant,
\[
D_\epsilon \simeq (\epsilon \tau)^2 \nu.
\]

(70)
So what we measure is

\[ G(m, t) = \mathcal{G}(m, t D_e) \]

\[ = \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi t D_e}} \times \exp \left( \frac{m}{2} \left[ \frac{\mu}{D_e} + \nu \right] \right) \]

\[ \times \exp \left( -\frac{D_e t}{4} \left[ \mu^2 (D_e t)^{-2} + \nu^2 (D_e t)^2 \right] \right) \]

\[ \times \exp \left( \epsilon \left[ \frac{G_1(m, t) - \mu G_\mu(m, t) - \nu G_\nu(m, t)}{G_0(m, t)} \right] \right) \]

\[ - (z^2 - 1) \ln(t) \right) \]

\[ = \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi t}} \times \]

\[ \times \exp \left( -\frac{\mu m^{1-2e/H}}{2} y^{2e} - \frac{\nu m}{2} y^e - \frac{t}{4} [\mu t^{-e} + \nu t^e] \right) \]

\[ \times \exp \left( \epsilon \left[ \bar{F}_1(y) - \mu m \bar{F}_\mu(y) - \nu m \bar{F}_\nu(y) \right] \right) . \tag{71} \]

This leads to

\[ tG(m, t) = \frac{y^{\frac{3}{2e}}}{\sqrt{2\pi}} \times \]

\[ \times \exp \left( -\frac{y^2}{2} + \epsilon \bar{F}_1(y) - \mu m^{\frac{1}{2e}} y^{2e} \left[ \frac{1}{2} + \epsilon F_\mu(y) \right] \right) \]

\[ \times \exp \left( -\nu m y^{2e} \left[ \frac{1}{2} + \epsilon F_\nu(y) \right] - \frac{m^2}{8y^2} \left[ \mu \left( \frac{2y^2}{m^2} \right) + \nu \right]^2 \right) . \]

Note that our \( \epsilon \) expansion is restricted to the first order in \( \epsilon \). Therefore, in expressions like

\[ \frac{1}{H} - 1 = 1 - 4\epsilon + O(\epsilon^2) , \quad 1 - \frac{1}{2H} = 2\epsilon + O(\epsilon^2) , \tag{73} \]

we have no means to distinguish between left- and right-hand side. Some choices are given by scaling, as the prefactor of \( y^{\frac{3}{2e}} \), or seem natural, others are educated guesses.

Eq. (72) can be rewritten as a probability measure in the variable \( y \), given distance \( m \) from the absorbing boundary for the starting point. Using that

\[ \frac{dt}{\tau} = \frac{1}{H} \frac{dy}{y} , \tag{74} \]

this yields

\[ \mathcal{P}(y|m, \mu, \nu) = \mathcal{P}_>(y|m, \mu, \nu) + P_{\text{escape}}(m, \mu, \nu) \delta(y) . \tag{75} \]

The function \( \mathcal{P}_>(y|m, \mu, \nu) \) is

\[ \mathcal{P}_>(y|m, \mu, \nu) = \frac{y^{\frac{3}{2e}}}{\sqrt{2\pi H}} \times \]

\[ \exp \left( -\frac{y^2}{2} + \epsilon \left[ F_1(y) - 2(\gamma_E + ln 2) \right] \right) \]

\[ - \mu m^{\frac{1}{2e}} y^{2e} \left[ \frac{1}{2} + \epsilon F_\mu(y) \right] \]

\[ - \nu m y^{2e} \left[ \frac{1}{2} + \epsilon F_\nu(y) \right] - \frac{m^2}{8y^2} \left[ \mu \left( \frac{2y^2}{m^2} \right) + \nu \right]^2 . \]

We have added a constant next to \( F_1(y) \), which we choose with \( F_1(0) = 0 \), in order to ensure that to leading order in \( \epsilon \) the expression is normalised to 1.

Some trajectories escape, which we count as absorption time \( t = \infty \), equivalent to \( y = 0 \), resulting into the contribution proportional to \( \delta(y) \) in Eq. (75), with amplitude

\[ P_{\text{escape}}(m, \mu, \nu) = 1 - P_{\text{abs}}(m, \mu, \nu) , \tag{77} \]

where

\[ P_{\text{abs}}(m, \mu, \nu) := \int_{0}^{\infty} dy \mathcal{P}(y|m, \mu, \nu) . \tag{78} \]

It is evaluated in the next section, see Eqs. (105)-(107).

The special functions are

\[ \bar{F}_1(y) = \frac{G_1(y)}{G_0(y)} - (y^2 - 1) \left[ \ln(t/\tau) - 1 \right] + 4 \ln y \tag{79} \]

\[ = \mathcal{I}(y) + \ln \left[ \frac{\tau^2}{4\pi y^2} \right] + y^2 \left( \ln \left( 2y^2 \right) + \gamma_E \right) - 4\gamma_E + 2 . \]

Choosing normalizations s.t. \( \bar{F}_1(0) = 0 \), this gives

\[ \bar{F}_1(y) = \bar{F}_1(y) - \bar{F}_1(0) \]

\[ = \mathcal{I}(y) + y^2 \left( \ln \left( 2y^2 \right) + \gamma_E \right) - 2 . \tag{80} \]

Its asymptotic expansions for small and large \( y \) are

\[ \tilde{F}_1(y) = 2\sqrt{2\pi} y + y^2 \left( \ln \left( 2y^2 \right) + \gamma_E - 3 \right) - \frac{1}{3} \sqrt{2\pi} y^3 \]

\[ + \frac{y^4}{6} - \frac{1}{30} \sqrt{\frac{\pi}{2}} y^5 + \frac{y^6}{90} - \frac{1}{420} \sqrt{\frac{\pi}{2}} y^7 + \frac{y^8}{1260} \]

\[ - \sqrt{x^{10}} + \frac{y^{10}}{6048} + O(y^{11}) , \tag{81} \]

\[ \tilde{F}_1(y) = \ln(y^2/2) + 1 - \psi \left( \frac{1}{2} \right) + \frac{1}{2y^2} - \frac{1}{2y^4} + \frac{5}{4y^5} \]

\[ - \frac{21}{4y^6} + \frac{63}{2y^7} + O(y^{-11}) . \tag{82} \]

Note that Eq. (80) is equivalent to Eqs. (55) in [54], and (56) in [56].

The second function is for the drift proportional to \( \mu \),

\[ \tilde{F}_\mu(y) := \frac{G_\mu(m, t)}{mG_0(m, t)} + \partial_t \left[ \frac{m^{4e}}{2D_e y^{2e}} \right]_{e=0} . \tag{83} \]
It is evaluated as

\[
\mathcal{F}_\nu(y) = \frac{(y^2 + 1) [I(y) - 2]}{2y^2(1 - y^2)} + \frac{\sqrt{2\pi} e^{\frac{y^2}{2}} \text{erfc} \left( \frac{y}{\sqrt{2}} \right)}{y (y^2 - 1)} + \frac{1}{2} \ln(2 - \gamma_E). \tag{84}
\]

Its asymptotic expansions are given by

\[
\mathcal{F}_\nu(y) = \frac{1}{2} \left[ 1 - \gamma_E + \ln(2) \right] + \frac{1}{3} \sqrt{2\pi} y - \frac{y^2}{4} + \frac{1}{15} \sqrt{\frac{\pi}{2}} y^3 - \frac{y^4}{36} + \frac{1}{140} \sqrt{\frac{\pi}{2}} y^5 - \frac{y^6}{360} + \frac{\sqrt{\frac{\pi}{2}} y^7}{1512} - \frac{y^8}{4200} + \frac{\sqrt{\frac{\pi}{2}} y^9}{19008} + \frac{y^{10}}{56700} + O(y^{11}), \tag{85}
\]

\[
\mathcal{F}_\nu(y) = \ln(2y) + \frac{\ln(2y^2) + \gamma_E - 1}{2y^2} + \frac{3}{4y^2} - \frac{5}{4y^6} + \frac{35}{8y^8} - \frac{189}{8y^{10}} + O(y^{-11}). \tag{86}
\]

The third function is for the drift proportional to \( \nu \),

\[
\mathcal{F}_\nu(y) := \frac{G_\nu(y)}{G_0(y)}m - \ln(y). \tag{87}
\]

It is evaluated as

\[
\mathcal{F}_\nu(y) = \frac{I(y) - 2}{2y^2} + \frac{\ln(2) + \gamma_E}{2}. \tag{88}
\]

Its asymptotic expansions read

\[
\mathcal{F}_\nu(y) = \frac{\sqrt{2\pi}}{y} + \frac{-3 + \gamma_E + \ln(2)}{2} - \frac{1}{3} \sqrt{\frac{\pi}{2}} y + \frac{y^2}{12} - \frac{1}{60} \sqrt{\frac{\pi}{2}} y^3 + \frac{y^4}{180} - \frac{1}{840} \sqrt{\frac{\pi}{2}} y^5 + \frac{y^6}{2520} - \frac{\sqrt{\frac{\pi}{2}} y^7}{12096} + \frac{y^8}{37800} - \sqrt{\frac{\pi}{2}} y^9 + \frac{y^{10}}{623700} + O(y^{11}) \tag{89}
\]

\[
\mathcal{F}_\nu(y) = \ln(y) + \frac{2 \ln(y) + \gamma_E + 1 + \ln(2)}{2y^2} + \frac{1}{4y^4} - \frac{1}{4y^6} + \frac{5}{8y^8} - \frac{21}{8y^{10}} + O(y^{-11}) \tag{90}
\]

Note that we added some strangely looking factors into the result (87). The factor \( m \times m^{-\frac{\nu}{2}} = m^{-\nu - 1} \) accounts for the dimension of the diffusion constant, \( m/Dc \sim m^{-2\nu} \), and takes out the term \( \ln(m) \) from \( \mathcal{F}_\nu(y) \). We moved out also a remaining term \( \sim \ln y \).

Using Eq. (77) for small \( y \), there is a problem when \( \nu \nu < 0 \), since then the combination (second-to-last term in the exponential)

\[
-\epsilon \nu m y^{2\nu} \left[ \frac{1}{2} + \epsilon \mathcal{F}_\nu(y) \right]_{y \to 0} - \epsilon \nu m \sqrt{2\pi} y^{2\nu - 1} \approx -2\nu \sqrt{\pi \epsilon t^H}. \tag{91}
\]

diverges (at least for \( \frac{1}{2} < H < \frac{1}{2} \)), which is amplified since it appears inside the exponential. We propose to use the following Padé variant, which seems to work well numerically,

\[
\left[ \frac{1}{2} + \epsilon \mathcal{F}_\nu(y) \right]_{\epsilon < 0, \nu > 0} \xrightarrow{y \to 0} \frac{1}{2 - 4\epsilon \mathcal{F}_\nu(y)}. \tag{92}
\]

While \( \mathcal{F}_\nu(y) \) diverges for small \( y \), this is at leading order nothing but a normalization factor depending on \( \nu t^H \).

All three functions \( \mathcal{F}_1(y) \), \( \mathcal{F}_\mu(y) \) and \( \mathcal{F}_\nu(y) \) are measured in section III, see figures 6, 7, and 9.
K. Absorption probability

From Eq. (46), we obtain, $P_{\text{abs}}(m, \alpha, \beta)$

$$P_{\text{abs}}(m, \alpha, \beta) = \int_0^\infty dt G(m, tD_\epsilon)$$

$$= \int_0^\infty dt \exp \left( -\frac{m}{2D_\epsilon} + \nu \right) G_0(m, tD_\epsilon) + \epsilon \int_0^\infty dt \exp \left( -\frac{m}{2\beta} - \frac{t}{4} |\beta|^2 \right) \times$$

$$\times \left\{ \int_0^\infty dt \exp \left( -\frac{t}{4} |\mu|^2 \right) G_0(m, tD_\epsilon) + \epsilon \left[ \tilde{G}_1(m, s) - \alpha \tilde{G}_\alpha(m, s) - \beta \tilde{G}_\beta(m, s) \right] \sqrt{\tau = |\beta|/2} \right\}$$

$$+ ... \quad (93)$$

Here $\tilde{G}_1(m, s)$ is given by Eq. (49), $\tilde{G}_\alpha(m, s)$ by Eq. (58), and $\tilde{G}_\beta(m, s)$ by Eq. (61). We still need the integral

$$\int_0^\infty dt \exp \left( -\frac{t}{4} |\mu|^2 \right) G_0(m, tD_\epsilon) = \epsilon^{-|\mu|m/(2\sqrt{D_\epsilon})} + \frac{\alpha \beta}{2} \epsilon G_3(m, \beta) \quad (94)$$

$$G_3(m, \beta) = \int_0^\infty dt e^{-\frac{|\beta|^2}{4} t \ln(t)} |G_0(m, t)|. \quad (95)$$

The last expression can be calculated as

$$G_3(m, \beta) := \int_0^\infty dt e^{-\frac{|\beta|^2}{4} t \ln(t)} G_0(m, t)$$

$$= \partial_k |_{k=0} \int_0^\infty dt e^{-\frac{|\beta|^2}{4} t^1 + K} |G_0(m, t)|$$

$$= \partial_k |_{k=0} \sqrt{\frac{\pi}{|\beta|}} K^{-\frac{1}{2}} \left( |\beta|m \right)$$

$$= - \frac{m^{3/2} \partial_k |_{k=0} K^{-\frac{1}{2}} \left( |\beta|m \right)}{\sqrt{\pi |\beta|}} + \frac{mc}{|\beta|}$$

$$= - \frac{me^{-m|\beta|/4 \ln(m)}}{|\beta|} + \frac{mc}{|\beta|}$$

$$= \frac{m}{|\beta|} [\gamma_E - 2 \ln(1|\beta|) - \gamma_E + 1/2] m^2 \ln(m) + \gamma_E + 2] + O(m^3). \quad (96)$$

With the above formulas, Eq. (93) is rewritten as

$$P_{\text{abs}}(m, \alpha, \beta) = e^{-m(\beta + |\beta|)/2} \left\{ 1 + \epsilon e^{m|\beta|/2} \times \right.$$}

$$\times \left[ \frac{\alpha \beta}{2} G_3(m, \beta) + \frac{\alpha + \beta + |\beta|}{2} m (1 + \ln \tau) e^{-m|\beta|/2} \right.$$}

$$+ \tilde{G}_1(m, s) - \alpha \tilde{G}_\alpha(m, s) - \beta \tilde{G}_\beta(m, s) \right\} \sqrt{\tau = |\beta|}$$

$$+ O(\epsilon^2) \right\}. \quad (97)$$

We note the exact relations

$$\tilde{G}_1(m, s) + 2\sqrt{\tau} \tilde{G}_\beta(m, s) = 0 \quad (98)$$

$$G_3(m, |\beta|) + 2 \tilde{G}_\alpha(m, s)$$

$$- |\beta| (1 + \ln \tau) e^{-m|\beta|/4 \ln(m)} + O(\epsilon^2) \quad (99)$$

Let us analyse $P_{\text{abs}}$ separately for $\beta < 0$ and $\beta > 0$, starting with the former. Using both cancelations in Eqs. (98) and (99), we find

$$P_{\text{abs}}(\alpha, \beta < 0) = 1 + O(\epsilon^2). \quad (100)$$

Thus there is no change in normalisation for a drift towards the absorbing boundary. For $\beta > 0$, we find again with the use of Eqs. (98) and (99)

$$P_{\text{abs}}(\alpha, \beta > 0) = e^{-m\beta} \times$$

$$\times \left\{ 1 + \epsilon \left( (\alpha + \beta) m (1 + \ln \tau) + 2 e^{\beta \ln(m)} \left( \tilde{G}_1(m, s) - \alpha \tilde{G}_\alpha(m, s) \right) \sqrt{\tau = |\beta|} \right) + \gamma_E \right\}. \quad (101)$$

For what follows, we note regularity of the combination

$$\text{Ei}(-x) - \ln(x) - \gamma_E = -x - \frac{x^2}{4} - \frac{x^3}{18} - \frac{x^4}{96} - \frac{x^5}{600} + O(x^6). \quad (102)$$

We can write Eq. (101) as

$$P_{\text{abs}}(m, \alpha, \beta) = e^{-m\beta} \times$$

$$\times \left\{ 1 + \epsilon \left( (\alpha + \beta) m (1 + \ln \tau) \right) \right.$$

$$\left. - \alpha m (2 \ln(\beta) + \gamma_E) + \beta m (2 \ln(m) + \gamma_E) \right\} + O(\epsilon^2) \right\}$$

$$= e^{-m\beta} \times$$

$$\times \left\{ 1 + \epsilon \left[ 2(\beta - \alpha) \ln(\beta) - \gamma_E (\alpha + 3\beta) - 2\beta + 4\beta \ln(m) \right] \right.$$

$$\left. + O(\epsilon^2) + O(m^2 \epsilon) \right\}. \quad (103)$$
As the asymptotic expansion in the last line shows, a common
resummation is possible, passing to variables $\mu$ and $\nu$

\[
P_{\text{abs}}(m, \mu, \nu) = \exp \left( -m^{\frac{1}{H}} \mu \left[ 1 + 2(1 - \gamma_E)\epsilon \right] \right.
\]
\[
- m^{\frac{1}{H}} \mu (\mu + \nu)^{\frac{1}{H} - 2} \left[ 1 + 2(1 - 2\gamma_E)\epsilon \right] \left. \right) + O(\epsilon^2) + O(m^2\epsilon).
\]  

Note that the (inverse) powers of $H$ were chosen s.t. the resulting object is scale invariant. Expanding in $\epsilon$ leads back to Eq. (103). One finally arrives at

\[
P_{\text{abs}}^{(0)}(m, \mu, \nu) = \exp \left( -m^{\frac{1}{H}} \left\{ \mu \left[ 1 + 2(1 - \gamma_E)\epsilon \right] + \nu (\mu + \nu)^{\frac{1}{H} - 2} \left[ 1 + 2(1 - 2\gamma_E)\epsilon \right] \right\} \right.
\]
\[
+ \epsilon \left\{ 2(m\nu + 1) \left[ e^{m(\mu + \nu)} \text{Ei}( - m(\mu + \nu) ) - \ln (m(\mu + \nu)) - \gamma_E \right] - 2m(\mu + \nu) \left[ \ln (m(\mu + \nu)) + \gamma_E - 1 \right] \right\} \left. \right) + O(\epsilon^2).
\]  

In order that this formula be invariant under $m \to \lambda m$, $\mu \to \lambda^{-1} \frac{\mu}{\mu}$ and $\nu \to \lambda^{-1} \nu$, we can either replace $m\mu$ by $m\mu^{\frac{\mu}{\mu}}$, or $m^{\frac{1}{H}} - 1 \mu$. The first version is

\[
P_{\text{abs}}^{(1)}(m, \mu, \nu) = \exp \left( -m^{\frac{1}{H}} \left\{ \mu \left[ 1 + 2(1 - \gamma_E)\epsilon \right] + \nu (\mu^{\frac{1}{H}} + \nu)^{\frac{1}{H} - 2} \left[ 1 + 2(1 - 2\gamma_E)\epsilon \right] \right\} \right.
\]
\[
+ \epsilon \left\{ 2(m\nu + 1) \left[ e^{m(\mu^{\frac{1}{H}} + \nu)} \text{Ei}( - m(\mu^{\frac{1}{H}} + \nu) ) - \ln (m(\mu^{\frac{1}{H}} + \nu)) - \gamma_E \right] - 2m(\mu^{\frac{1}{H}} + \nu) \left[ \ln (m(\mu^{\frac{1}{H}} + \nu)) + \gamma_E - 1 \right] \right\} \left. \right) + O(\epsilon^2).
\]  

The alternative second version is

\[
P_{\text{abs}}^{(2)}(m, \mu, \nu) = \exp \left( -m^{\frac{1}{H}} \left\{ \mu \left[ 1 + 2(1 - \gamma_E)\epsilon \right] + \nu (\mu^{\frac{1}{H}} + \nu)^{\frac{1}{H} - 2} \left[ 1 + 2(1 - 2\gamma_E)\epsilon \right] \right\} \right.
\]
\[
+ \epsilon \left\{ 2(m\nu + 1) \left[ e^{m^{\frac{1}{H}} - 1 \mu + m\nu} \text{Ei}( - m^{\frac{1}{H}} - 1 \mu - m\nu ) - \ln (m^{\frac{1}{H}} - 1 \mu + m\nu) - \gamma_E \right] - \left( m^{\frac{1}{H}} - 1 \mu + m\nu) \left[ \ln (m^{\frac{1}{H}} - 1 \mu + m\nu) + \gamma_E - 1 \right] \right\} \left. \right) + O(\epsilon^2).
\]  

From the appearance of fractal powers of $m$ and $\nu$ in Eq. (104), we suspect that both power series in $m^{\mu^{\frac{1}{H}}}$ and $m^{\frac{1}{H} - 1} \mu$ might appear. While numerical simulations could decide which version is a better approximation, only higher-order calculations would be able to settle the question.

L. Relation between the full propagator, first-passage times,
and the distribution of the maximum

(i) In Ref. [54] was calculated $P_+(m, t)$ the normalised probability density to be at $m$, given $t$, when starting at $x_0$ close to 0.

(ii) Here we consider the probability density to be absorbed at time $t$ when starting at $m$. This is a first-passage time, with distribution $P_{\text{first}}(m, t)$.

(iii) Third, let the process start at 0, and consider the distribution of the max $m$, given a total time $t$, $P_{\text{max}}(m, t)$, denoted by $P_{\text{max}}^T(m)$ (with $t = T$) in Ref. [56].

All three objects have a scaling form depending on the same
The factors of $H$ and $\sqrt{2}$ where chosen for later convenience. These objects are related. Denote $P_{\text{surv}}(m, t)$ the probability to start at $x = 0$, and to survive in presence of an absorbing boundary at $m$ up to time $t$. Note that $P_{\text{surv}}(m, t)$ is a probability, whereas $P_{\text{first}}(m, t)$, $P_{+}(m, t)$, and $P_{\text{max}}(m, t)$ are densities, the first two in $t$, the latter in $m$. Then

$$P_{+}(m, t) = P_{\text{first}}(m, t) = -\partial_t P_{\text{surv}}(m, t) ,$$

$$P_{\text{max}}(m, t) = \partial_m P_{\text{surv}}(m, t) .$$

Since $P_{\text{surv}}(m, t)$ is a probability, it is scale free, and scaling implies that

$$P_{\text{surv}}(m, t) = R_{\text{surv}}\left( y = \frac{m}{\sqrt{2H}} \right) .$$

Putting together Eqs. (111), (112) and (113) proves Eqs. (108) to (110), with

$$R_{\text{first}}(y) = R_{+}(y) = y R'_{\text{surv}}(y) ,$$

$$R_{\text{max}}(y) = R'_{\text{surv}}(y) .$$

The scaling functions appearing are almost the same, differing by (innocent looking) factors of $t$ and $H$ and a (non-innocent looking) factor of $y$. However, when changing to the measure in $y$, all of them become identical, i.e.

$$P_{\text{first}}(m, t) dt = \frac{H}{t} R_{\text{first}}(y) dy ,$$

$$= R'_{\text{surv}}(y) dy \quad (116)$$

$$P_{+}(m, t) dt = \frac{H}{t} R_{+}(y) dy = \frac{R_{+}(y)}{y} dy ,$$

$$= R'_{\text{surv}}(y) dy \quad (117)$$

$$P_{\text{max}}(m, t) dm = \frac{1}{\sqrt{2H}} R_{\text{max}}(y) dy = \frac{R_{\text{max}}(y)}{y} dy ,$$

$$= R'_{\text{surv}}(y) dy . \quad (118)$$

No prefactor remains at the end, since integration over $y$ must yield 1, or equivalently $R_{\text{surv}}(\infty) = 1$, $R_{\text{surv}}(0) = 0$.

### III. NUMERICS

#### A. Simulation protocol

Fractional Brownian motion can be simulated with the classical Davis-Harte (DH) algorithm [18, 62], whose algorithmic complexity (execution time) scales with system size $N$ as $N \ln N$. Here we use the adaptive bisection algorithm introduced in Refs. [63, 64], and explained below. For $H = 1/3$ its measured algorithmic complexity grows as $(\ln N)^3$, making it about 5000 times faster, and 10000 times less memory consuming than DH for an effective grid size of $N = 2^{12}$.

To measure the functions $F_{\nu}$, $F_{\mu}$ and $F_{\mu}$, which all depend on $y$ only, we

(i) generate a (drift free) fBm $x_t$ with $x_0 = 0$, of length $N$; the latter corresponds to a time $T = 1$.

(ii) add the drift terms to yield $z_t = x_t + \mu t + \nu t^{2H}$.
The third order correction can be extracted as
\[ F_3^\epsilon(y|m) := \frac{1}{2\epsilon^2} \left[ F_1^\epsilon(y|m) + F_1^{\epsilon^2}(y|m) - 2F_1(y|m) \right] + \mathcal{O}(\epsilon). \] (125)

For the remaining functions \( F_\mu \) and \( F_\nu \), we can employ similar formulas; we have to decide how to subtract numerically or analytically the denominator in
\[ F_\mu^\epsilon(y|m, \mu) := -\frac{1}{\epsilon} \ln \left( \frac{\mathcal{P}(y|m, \mu, \nu = 0)}{\mathcal{P}(y|m, \mu = \nu = 0)} \right) \times \frac{y^{-2\epsilon}}{\mu m \pi^{-1}} \]
\[ + \frac{1}{2} + \frac{\mu}{4} \left( \frac{m}{2} \right)^{\frac{\nu}{2}} - \frac{\nu}{\pi}, \] (126)
\[ F_\nu^\epsilon(y|m, \nu) := -\frac{1}{\epsilon} \ln \left( \frac{\mathcal{P}(y|m, \mu = 0, \nu)}{\mathcal{P}(y|m, \mu = 0, 0)} \right) \times \frac{y^{-2\epsilon}}{\nu m} \]
\[ + \frac{1}{2} + \frac{\nu m}{8} y^{-\epsilon-2}. \] (127)

We can also work symmetrically
\[ F_\mu^\epsilon(y|m) := -\frac{1}{\epsilon} \ln \left( \frac{\mathcal{P}(y|m, \mu, \nu = 0)}{\mathcal{P}(y|m, -\mu, 0)} \right) \frac{y^{-2\epsilon}}{2\mu m \pi^{-1}} + \frac{1}{2}. \] (128)
\[ F_\nu^\epsilon(y|m) := -\frac{1}{\epsilon} \ln \left( \frac{\mathcal{P}(y|m, \mu = 0, \nu)\mathcal{P}(y|m, \mu = 0, -\nu)}{\mathcal{P}(y|m, \mu = 0, 0)} \right) \frac{y^{-2\epsilon}}{2\nu m} + \frac{1}{2}. \] (129)

Finally, a more precise estimate of the theoretical curves is given by symmetrizing results for the same \(|\epsilon|\), using the analogue of Eq. (122).

### B. The adaptive bisection algorithm

When simulating fractional Brownian motion, there are unavoidable numerical errors. Suppose the total time is \( T = 1 \), and we discretize with \( N \) points. This leads to an effective time discretisation of \( \delta t = 1/N \). Since \( \delta x \sim \delta t^H \), the effective resolution in \( x \) scales as
\[ \delta x \simeq N^{-H}. \] (130)

To achieve the moderate resolution of \( \delta x = 10^{-3} \), at \( H = 0.67 \) we need \( N \approx 2^{15} \) points, while for \( H = 0.33 \) this grows to \( N \approx 2^{31} \), resulting in more than 1000 longer run times when using the standard Davis-Harte (DH) algorithm [18, 62].

The source of the problem, and its resolution, can be inferred from Fig. 4, where we marked the first-passage time of the continuous fBm as \( \tau_\infty \). In the rather course initial resolution (red), the first-passage time \( \tau_\infty \) is largely overestimated, as it is for the 4 times finer grid (in green), and marked as \( \tau_4 \). Working with a fine grid is computationally costly as the complexity can not be smaller than \( \mathcal{O}(N) \), as each of the \( N \) points has to be generated.

The Davis-Harte (DH) algorithm achieves this with complexity \( \mathcal{O}(N) = \mathcal{O}(N \ln N) \). Given that all points are correlated, this is actually quite good. The only way to do better,
is to not generate all gridpoints, but only those “which matter”, i.e. those close to the barrier. This is the starting idea for the Adaptive Bisection Algorithm we introduced in references [63, 64]. There we start from a coarse grid with $2^{10}$ points, which is then iteratively refined at midpoints between already existing points, if the probability that these midpoints can pass the barrier is larger than a control parameter $c$. Our benchmarks for $H = 0.33$ show that one can bound the error rate, i.e. failure to identify the correct first-passage time on the largest testable grid of size $N_{\text{eff}} = 2^{24}$ by $10^{-6}$, which means that in a high-quality simulation of $10^{5}$ samples about 10 samples exhibit a larger first-passage time. This should be well within the inherent noise of a Monte Carlo simulation, which scales with the square root of the number of samples. As we show in Ref. [63], the algorithmic complexity of the adaptive bisection algorithm is (at least in practice) $C^{\text{ABSEC}}(N_{\text{eff}}) = O(\ln N_{\text{eff}})^3).$ The idea behind this scaling is that a finite number of points is needed per generation of refinement, thus the total number of points to be generated scales logarithmically with the effective grid size $N_{\text{eff}}$...
FIG. 8: Left: first-passage-time density plotted with overlapping bins as in Fig. 5 for various values of \( H \) and \( \nu \) compared to the theory given in Eq. (77). Right: Ratio of simulation and theoretical values.

FIG. 9: Left: Numerical estimate of \( \mathcal{F}_\nu \), using Eq. (129). The black curve is the theoretical prediction (88). The colored curves are simulation results using Eq. (129). The cyan and olive curves are the symmetrised results using the equivalent of Eq. (122) for \( H = 0.4/0.6 \) (cyan) and \( H = 0.33/0.67 \) (olive). The former one is the best numerical estimate of the theory, and very close to the latter. The larger inset shows the estimated second-order corrections, analogous to Eqs. (123)-(124). There seem to be non-negligible corrections of order three. Using the analogue of Eq. (125) these are estimated in the inset of the right figure. An almost perfect data collapse can be obtained as \( \epsilon \mathcal{F}_\nu(y) \approx \mathcal{F}_\nu(y) e + (2y^{-2} - 4y^{-1} - 6 + y)e^2 + (3y - 20)e^3 \), see right figure (main plot). Since extrapolation problems mentioned around Eq. (92) become important for small \( y \), this estimate is intended as a fit only, to show that the scatter on the left plot is consistent with higher-order corrections.

(i.e. the size of the grid, if all midpoints were generated). The inverse correlation matrices needed to sample the midpoints are constructed recursively, which necessitates a number of operations scaling as the square of their size. Repeating this procedure until all points are sampled then scales cubicly with \( \ln(N_{\text{eff}}) \), thus the scaling given above.

Below, we measure the three scaling functions \( \mathcal{F}_1 \), \( \mathcal{F}_\mu \), and \( \mathcal{F}_\nu \) for \( H = 0.33 \) (initial grid size \( 2^9 \)) and final gridsize \( 2^{12+G} \), with \( g = 8, G = 18 \), \( H = 0.4 \) (\( g = 10, G = 14 \)), \( H = 0.6 \) (\( g = 8, G = 8 \)), and \( H = 0.67 \) (\( g = 8, G = 6 \)). Thanks to the adaptive bisection algorithm, which is about 1000 times faster for \( H = 0.33 \) at the used resolution \( N_{\text{eff}} = 2^{28} \), we can maintain a resolution in \( x \) of \( 10^{-3} \), with about 25 million samples at \( H = 0.33 \), \( H = 0.6 \) and \( H = 0.67 \), and twice as much for \( H = 0.4 \). As we will see below, this allows us to precisely validate our analytical predictions.

C. Simulation results

We show simulation results on Figs. 5 to 9. First, on figure 5 (left), we present results for the first-passage probability \( P(y|m, \mu, \nu = 0) \). The numerical results (in color) are compared to the predictions from Eq. (77). One sees that the-
ory and simulations are in good quantitative agreement. This comparison is made more precise by plotting the ratio between simulation and theory on the right of Fig. 5.

The function $\mathcal{F}_1(y)$ is extracted on Fig. 6. The theoretical result (80) agrees very well with numerical simulations for all $H$. Using the symmetrized form (122) with $H = 0.4/0.6$ shows a particularly good agreement. It allows us to extract the subleading correction via Eqs. (123) and (124). This is shown in the inset of Fig. 6; again the symmetrized estimate is the most precise.

Using the data presented on Fig. 5, Fig. 7 shows the order-$\epsilon$ correction $\mathcal{F}_{\mu}$ extracted via Eq. (128). The symmetrized estimate is rather close to the analytical result. The inset estimates the subleading correction.

The results for non-linear drift $\nu$ are presented on Fig. 8, starting with the probability distribution $P(y|m)$ (left), followed by the ratio between simulation and theory on the right. The agreement is again good. From these data is extracted the function $\mathcal{F}_\nu(y)$ defined in Eq. (88), see Fig. 9. Note that $\mathcal{F}_\nu(y)$ is much larger than $\mathcal{F}_\mu(y)$ (Fig. 7), and diverges for small $y$. The two subleading corrections to $\mathcal{F}_\nu(y)$ are not negligible and estimated as well, allowing us to collapse all measured estimates on the theoretical curve.

In summary, we have measured all scaling functions with good to excellent precision, ensuring that the analytical results are correct.

IV. CONCLUSION

In this article, we gave analytical results for fractional Brownian motion, both with a linear and a non-linear drift. Thanks to a novel simulation algorithm, we were able to verify the analytical predictions with grid sizes up to $N = 2^{28}$, leading to a precise validation of our results.

Our predictions to first order in $H - 1/2$ are precise, and many samples of very large systems are needed to see statistically significant deviations. We therefore hope that our formulas will find application in the analysis of data, as e.g. the stock market.

Another interesting question is how a trajectory depends on its history, i.e. prior knowledge of the process. We obtained analytical results also in this case, and will come back with its numerical validation in future work.

Our study can be generalised in other directions, as e.g. making the variance a stochastic process, as in [65] or in the rough-volatility model of Ref. [66], which both use fBm in their modelling.

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