CONFORMAL CONTINUATIONS AND WORMHOLE INSTABILITY IN SCALAR-TENSOR GRAVITY

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We study the stability of static, spherically symmetric, traversable wormholes existing due to conformal continuations in a class of scalar-tensor theories with zero scalar field potential (so that Fisher’s well-known scalar-vacuum solution holds in the Einstein conformal frame). Specific examples of such wormholes are those with nonminimally (e.g., conformally) coupled scalar fields. All boundary conditions for scalar and metric perturbations are taken into account. All such wormholes are shown to be unstable under spherically symmetric perturbations. The instability is proved analytically with the aid of the theory of self-adjoint operators in Hilbert space and is confirmed by a numerical computation.

1. Introduction

In our recent paper [1] we have considered spherically symmetric perturbations of wormhole solutions to the Einstein-massless scalar field equations which exist for scalar fields nonminimally coupled to gravity [2, 3]. The equations of motion were reduced to a single wave equation for the scalar field perturbation which in this case comprises the only dynamical degree of freedom. An analysis of this wave equation leads to the conclusion that such wormholes are unstable under spherically symmetric (monopole) perturbations, and this instability is of catastrophic nature since the increment of perturbation growth has no upper bound.

In this paper we continue this study and extend it in two respects. First, we discuss more general background configurations, namely, static, spherically symmetric wormholes which appear in arbitrary scalar-tensor theories (STT) admitting conformal continuations (CCs) [4]. The investigation is, however, restricted to massless fields for which Fisher’s well-known solution holds in the Einstein frame. Second, we examine the problem in more detail, including the behaviour of metric perturbations related to those of the scalar field. A physically meaningful metric perturbation of an initially regular configuration should be regular everywhere. This requirement turns out to impose an additional constraint on the scalar field perturbations, which makes the stability problem quite nontrivial. We finally prove that there exists at least a single growing mode of physically meaningful perturbations, i.e., such wormholes are indeed unstable. However, contrary to the conclusion of Ref. [1], the perturbation grows at a finite rate.

We thus find that gravitational instabilities, whose existence seems to be quite natural at surfaces where the gravitational coupling changes its sign (see, e.g., Ref. [5] for a discussion in a cosmological setting), still need much effort in their detailed study and even discovery.

The paper is organized as follows. Sec. 2 is a brief description of the background static configuration and its place among more general configurations of this kind, i.e., static, spherically symmetric wormhole solutions of a general class of scalar-tensor theories (STT) admitting conformal continuations (CCs) [4]. Sec. 3 discusses spherically symmetric perturbation equations and the corresponding gauge freedom. Sec. 4 is devoted to a stability investigation for wormholes, both analytical, using the theory of self-adjoint operators in Hilbert space, and numerical.

2. Conformal continuations and wormhole solutions of scalar-tensor theories

2.1. STT in Jordan and Einstein pictures

Consider the general (Bergmann-Wagoner-Nordtvedt) class of STT, where gravity is characterized by the metric $g_{\mu\nu}$ and the scalar field $\phi$; the action is

$$S = \int d^4x \sqrt{g} \left[ f(\phi) R[g] + h(\phi) g^{\mu\nu} \phi,\mu \phi,\nu - 2U(\phi) + 16\pi G L_m \right].$$

(1)

Here $R[g]$ is the scalar curvature, $f$, $h$ and $U$ are certain functions of $\phi$, varying from theory to theory, $L_m$ is the matter Lagrangian, and $G$ is the gravitational

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constant, not necessarily coinciding with its Newtonian value.

The action \( S \) is simplified by the well-known conformal mapping

\[ g_{\mu\nu} = \frac{\mathcal{M}_{\mu\nu}}{|f(\phi)|}, \]

accompanied by the scalar field transformation \( \phi \mapsto \psi \) such that

\[ \frac{d\psi}{d\phi} = \pm \sqrt{\frac{|f(\phi)|}{f(\phi)}} , \quad l(\phi) \overset{\text{def}}{=} f h + \frac{3}{2} \left( \frac{df}{d\phi} \right)^2. \]

In terms of \( \mathcal{M}_{\mu\nu} \) and \( \psi \), the action for \( U = L_m = 0 \), the case of massless scalar-vacuum fields to be considered here, takes the form

\[ S = \int d^4x \sqrt{\mathcal{G}} \left\{ R[g] + \left| \text{sign} l(\phi) \right| \psi_{,\mu} \psi_{,\nu} \right\} \]

up to a boundary term which does not affect the field equations. Here \( R[g] \) is the Ricci scalar obtained from \( \mathcal{M}_{\mu\nu} \).

The space-time \( M_J = \mathcal{M}[g] \) with the metric \( g_{\mu\nu} \) is referred to as the Jordan conformal frame (or picture), generally regarded as the physical frame in STT; the Einstein conformal frame \( M_E = \mathcal{M}[g] \) with the field \( \psi \) then plays an auxiliary role (see, however, discussions of the physical meaning of various conformal frames in [4, 5] and references therein). The action (1) corresponds to conventional general relativity if \( f > 0 \), and the normal sign of scalar kinetic energy is obtained for \( l(\phi) > 0 \). Scalar fields in anomalous STT, in which \( l(\phi) < 0 \), lead to a kinetic term in (1) with a “wrong” sign, are called phantom scalar fields. Such fields (with different potentials) are sometimes invoked in modern cosmological studies to describe dark energy.

Exact static, spherically symmetric scalar-vacuum solutions of the theory (1) are well known [9, 10]. Among them, wormhole solutions are generic in the case of phantom scalar fields [11, 12]. Their stress-energy tensor \( T_{\mu\nu} \) manifestly violates the null energy condition (NEC) \( T_{\mu\nu}k^\mu k^\nu \geq 0 \), \( k_\mu k^\mu = 0 \), such violation being a necessary condition for wormhole existence [10], therefore wormhole solutions in their presence would have been naturally expected. One can note that, according to such solutions, both space-times \( M_J \) and \( M_E \) have wormhole properties, i.e., represent regular static traversable bridges between two flat asymptotics. The stability of such configurations has also been proved [11, 12] by a direct study of perturbation equations, though it seems quite strange for a field system with energy density unbounded from below. This question evidently deserves further investigation.

Our interest here will be in other wormhole solutions which appear in STT with \( l(\phi) > 0 \), in cases when the space-time manifold \( M_E \) is mapped, according to (2), to only a part of \( M_J \); this phenomenon was named conformal continuation (CC) [4, 13]. The Jordan space-time \( M_J \) is then globally regular, represents a wormhole, and it two non-intersecting parts map to two singular space-times \( M_E \) and \( M_E' \).

2.2. Fisher’s solution and its conformal continuations

The general static, spherically symmetric solution to the Einstein-scalar equations that follow from (1), was first found by Fisher [9] and was repeatedly rediscovered afterwards. Let us write it in the form suggested in [4], restricting ourselves to the “normal” case \( l > 0 \):

\[ \psi(u) = Cu + \psi_0, \]

\[ ds_E^2 = e^{2\gamma(u)} dt^2 - e^{2\alpha(u)} du^2 - e^{2\beta(u)} d\Omega^2 \]

\[ = e^{-2mu} du^2 - \frac{k^2 e^{2mu}}{\sin^2(ku)} \left[ \frac{k^2 du^2}{\sin^2(ku)} + d\Omega^2 \right] \]

where the subscript “E” stands for the Einstein frame; \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \) is the linear element on a unit sphere; \( C \) (scalar charge), \( m > 0 \) (mass in geometric units), \( k > 0 \) and \( \psi_0 \) are integration constants, of which the first three are related by

\[ k^2 = m^2 + \frac{1}{2} C^2. \]

Without loss of generality we put \( C > 0 \) and \( \psi_0 = 0 \). We are here using the harmonic radial coordinate \( u \in \mathbb{R}^+ \) in \( M_E[g] \), satisfying the coordinate condition \( \alpha = 2\beta + \gamma \).

Another convenient form of the solution is obtained in isotropic coordinates: with \( y = \tanh(\alpha ku/2) \), Eqs. (4), (6) are converted to

\[ \psi(y) = \frac{C}{k} \ln \left| \frac{1 + y}{1 - y} \right|, \]

\[ ds_E^2 = A(y) dt^2 - \frac{k^2(1 - y^2)^2}{y^2 A(y)} (dy^2 + y^2 d\Omega^2), \]

\[ A(y) = \left| \frac{1 - y}{1 + y} \right|^{2m/k}. \]

The solution is asymptotically flat at \( u \to 0 \) (\( y \to 0 \)), has no horizon when \( C \neq 0 \) and is singular at the centre \( u \to \infty, y \to 1 - 0, \psi \to \infty \). When the scalar field is “switched off” \( (C = 0, k = m) \), the Schwarzschild solution is recovered.

A feature of importance is the invariance of \( g_{\mu\nu} \) under the inversion \( y \mapsto 1/y \), noticed probably for the first time by Mitskievich [14]. Due to this invariance, the solution \( g_{\mu\nu} \) considered in the range \( y > 1 \) describes quite a similar configuration, but now \( y \to \infty \) is a flat asymptotic and \( y \to 1 + 0 \) is a singular centre. An attempt to unify the two ranges of \( y \), or, in other words, the two copies of Fisher’s solution, is meaningless due to the singularity at \( y = 1 \). We shall see that such a unification, leading to a wormhole, is achieved in \( M_J[g] \) where the singularity is smoothed out (in case \( C = \sqrt{6m} \)) owing to the conformal factor.

The corresponding Jordan-frame solution for any \( f(\phi) \) and \( h(\phi) \) such that \( l(\phi) > 0 \) are obtained from
If the function $f(\phi)$ is everywhere finite, $M_J$ has the same basic properties as $M_E$.

However, according to \textbf{11}, there is a class of STT in which some solutions produce structures of $M_J$ drastically different from that of $M_E$. Namely, let us use the $\phi$ field reparametrization freedom of the action \textbf{11} $[\phi = \phi(\phi_{\text{new}})]$ and fix the parametrization by putting in \textbf{11} $h(\phi) \equiv 1$. Then \textbf{11}, if the function $f(\phi)$ has a simple zero at some $\phi = \phi_0$, there is a subfamily of static, spherically symmetric solutions to the field equations admitting a CC. The latter means that a singular surface in $M_E$, corresponding to $\phi = \phi_0$, maps according to \textbf{2} to a regular surface $S_{\text{trans}}$ in $M_J$. Then $M$ can be continued in a regular manner through this surface, and the global properties of $M_J$ can be considerably richer than those of $M_E$; in the new region one can possibly find, e.g., a horizon or another spatial infinity. The above result was obtained in \textbf{4} for STT \textbf{11} in arbitrary dimensions and with arbitrary potentials $U(\phi)$.

It was also shown \textbf{4} that a wormhole was a generic type of a conformally continued Jordan-frame manifold. Before studying perturbations of such generic solutions (but with $U(\phi) \equiv 0$, so that we have Fisher’s solution in the Einstein picture), we discuss a specific example which makes evident the relations between Einstein and Jordan quantities.

2.3. Example: wormholes with a conformally coupled scalar field

A particular example of a CC is given by a free massless conformally coupled scalar field in GR. The latter is obtained when we put in \textbf{11}

$$f(\phi) = 1 - \phi^2/6, \quad h(\phi) \equiv 1. \quad U(\phi) = L_m = 0. \quad (10)$$

A transition sphere $S_{\text{trans}}$, if any, corresponds to $\phi^2=6$.

The transformation \textbf{13} now takes the form

$$\frac{d\psi}{d\phi} = \frac{1}{1 - \phi^2/6}. \quad (11)$$

We assume that spatial infinity, where $\psi \to 0$, corresponds in the Jordan space-time $M_J$ to $|\phi| < \sqrt{6}$, where $f(\phi) > 0$, so that the gravitational coupling has its normal sign. Then \textbf{11} gives

$$\psi = \sqrt{6} \tanh^{-1}(\phi/\sqrt{6}) + \psi_0, \quad \psi_0 = \text{const}. \quad (12)$$

Using \textbf{13} and \textbf{13}, it is now easy to write the metric in the Jordan picture.

A CC through the sphere $S_{\text{trans}}$ ($u = \infty$, $y = 1$, $\phi = \sqrt{6}$), which is singular in $M_E$, is obtained when the infinity of the conformal factor $1/f$ in \textbf{2} compensates the zero of both $g_{tt}$ and $g_{\theta\theta}$ simultaneously. This happens when, in accord with \textbf{7},

$$k = 2m = 2C/\sqrt{6}, \quad (13)$$

which selects a special subfamily among all solutions. We will restrict the consideration to this subfamily.

In terms of the isotropic coordinate $y$, the solution in the Jordan picture has the form \textbf{2}

$$ds^2 = \frac{(1+y y_0)^2}{1 - y_0^2} \left[ \frac{dt^2}{(1+y)^2} - \frac{m^2(1+y)^2}{y^4}(dy^2 + y^2 d\Omega^2) \right],$$

$$\phi(y) = \sqrt{6} \frac{y + y_0}{1 + y y_0}. \quad (14)$$

where $y_0 = \tanh(\psi_0/\sqrt{6})$. The range $0 < y < 1$, describing the whole manifold $M_E$ in the Fisher solution, corresponds to only a region $M_J^1$ of the manifold $M_J$ of the solution \textbf{14}. In all cases, $y = 0$ corresponds to a flat asymptotic, where $\phi \to \sqrt{6} y_0 < \sqrt{6}$. The global properties of the solution depend on the sign of $y_0$:

a) $y_0 < 0$. The solution is defined in the range $0 < y < 1/|y_0|$. At $y = 1/|y_0|$, there is a naked attracting central singularity: $g_{tt} \to 0$, $r^2 \to 0$, $\phi \to \infty$.

b) $y_0 = 0$, $\phi = \sqrt{6} y$, $y \in \mathbb{R}_+$. In this case it is helpful to pass to the conventional radial coordinate $r$, substituting $y = m/(r - m)$. The solution

$$ds^2 = (1 - m/r)^2 dt^2 - \frac{dr^2}{(1 - m/r)^2} - r^2 d\Omega^2,$$

$$\phi = \sqrt{6} m/(r - m) \quad (15)$$

is the well-known BH with a conformal scalar field \textbf{15}. The infinite value of $\phi$ at the horizon $r = m$ does not make the metric singular since, as is easily verified, the energy-momentum tensor remains finite there. This solution has been shown to be unstable under radial perturbations \textbf{17}.

c) $y_0 > 0$. This is the wormhole solution discussed in \textbf{11, 2} and, among other solutions, re-analyzed now. The solution is defined in the range $y \in \mathbb{R}_+$. At $y \to \infty$, we find another flat spatial infinity, where $\phi \to \sqrt{6}/y_0$, $r^2 \to \infty$ and $g_{tt}$ tends to a finite limit.

The whole manifold $M_J$ can be represented as the union

$$M_J = M_{J1} \cup S_{\text{trans}} \cup M_{J2} \quad (16)$$

where the region $M_{J1}$ ($y < 1$) is, according to \textbf{2}, in one-to-one correspondence with the whole manifold $M_E$ of Fisher’s solution \textbf{5, 6}. The “antigravitational” ($f(\phi) < 0$) region $M_{J2}$ ($y > 1$) is in a similar correspondence with another “copy” of Fisher’s solution, $M_E'$ \textbf{7}, where, instead of \textbf{12},

$$\psi = \sqrt{6} \coth^{-1}(\phi/\sqrt{6}) + \psi_0', \quad \psi_0' = \text{const.} \quad (17)$$

The metric $\bar{g}_{\mu\nu}$ of this second Einstein-frame manifold $M_E'$ should also be regularized by the factor $1/f$ on $S_{\text{trans}}$, hence the integration constants in it should satisfy the condition \textbf{12}. Moreover, one can verify that, to provide a smooth transition in the Jordan-frame metric $g_{\mu\nu}$ through $S_{\text{trans}}$, all the constants $k$, $h$, $C$ and $\psi_0$ should coincide in $M_E$ and $M_E'$. The latter statement
is proved using the coordinate $y$ which is common on both sides of $S_{\text{trans}}$.

This example well illustrates the general properties of conformally continued solutions \[4\]. Namely, in the region beyond $S_{\text{trans}}$, there can be a singularity due to $l(\phi) = 0$, as happens in the above case a) at $y = 1/|y_0| > 1$. If there is no such singularity, we obtain a wormhole. Case b), with a horizon, is exceptional, inherent only to the field \[10\] in four dimensions. Thus, for a more general action discussed in \[1, 18\], with the coupling constant $\xi > 0$, in case $\xi > 1/6$ all solutions exhibiting a CC describe wormholes, whereas for $\xi < 1/6$ everything depends on an integration constant similar to $y_0$ in the above example, and we may have either a wormhole or a naked singularity.

The stability analysis developed below covers wormhole solutions obtained in the theory \[11\] under the conditions

$$f(\phi) = 1 - \xi \phi^2, \quad h(\phi) \equiv 1, \quad U(\phi) = L_m = 0, \quad (18)$$

with the coupling constant $\xi > 0$, in case $\xi > 1/6$ all solutions exhibiting a CC describe wormholes, whereas for $\xi < 1/6$ everything depends on an integration constant similar to $y_0$ in the above example, and we may have either a wormhole or a naked singularity.

The stability analysis developed below covers wormhole solutions obtained in the theory \[11\] under the conditions

$$h \equiv 1, \quad U = L_m = 0, \quad l(\phi) > 0, \quad (19)$$

with an arbitrary function $f(\phi)$, having a simple zero at some $\phi = \phi_0$. In other words, we consider massless scalar fields in a general non-phantom STT, for which wormhole solutions exist due to a CC.

3. Spherically symmetric perturbations and gauge freedom

Consider small spherically symmetric (monopole) perturbations of any static, spherically symmetric solution of the theory \[11, 19\]. The only dynamical degree of freedom is evidently related to the scalar field due to the generalized Birkhoff theorem \[19\]: if we take a time-independent scalar field, the equations of motion automatically lead to a static solution.

We will use, for simplicity, the Einstein conformal frame, since the perturbation equations in $M_1$, being equivalent to those in $M_\Gamma$, look much more complicated, and it is even hard to decouple different components of the Einstein equations. However, the boundary conditions that select physically meaningful perturbation should be formulated for variables specified in $M_1$ and only then converted to Einstein-frame quantities.

In the Einstein picture, the equations of motion are the Einstein-scalar field equations due to \[11\]

$$\nabla_\alpha \nabla^\alpha \psi = 0, \quad (20)$$

$$R^\nu_\mu = -\psi_{,\nu} \psi^{,\nu}. \quad (21)$$

We now write the metric in $M_\Gamma$ in the form

$$ds_\Gamma^2 = e^{2\gamma} dt^2 - e^{2\alpha} dx^2 - e^{2\beta} d\Omega^2, \quad (22)$$

where the functions $\alpha, \beta, \gamma$ as well as the scalar field $\psi$ are split into a static background part and a small (linear) time-dependent perturbation:

$$\alpha = \alpha(u) + \delta \alpha(u, t)$$

where $u$ is a radial coordinate, and similarly for $\beta, \gamma$ and $\psi$. Now, in addition to the freedom of choosing the radial coordinate $u$, we have an additional freedom of specifying the frame of reference in perturbed spacetime, called gauge freedom. The latter makes it possible to specify (by hand) some linear relation between the perturbations. Certain care is needed to ensure that the resulting perturbation will not be a “pure gauge”, i.e., will not be removable by coordinate transformations.

Consider an infinitesimal coordinate transformation of the static metric \[22\] preserving its spherical symmetry, i.e.,

$$x^\mu_{\text{new}} = x^\mu_{\text{old}} + \zeta^\mu, \quad \zeta^\mu = (\bar{\eta}, \bar{\xi}, \bar{0}, \bar{0}), \quad (23)$$

where the time dependence of the perturbations $\bar{\eta}, \bar{\xi}$ is separated: $\bar{\eta} = \eta(u) e^{\Omega t}$ and $\bar{\xi} = \xi(u) e^{\Omega t}$. To preserve the diagonal form of the metric, we should put $\Omega \xi = \eta' e^{2\gamma - 2\alpha}$ (where $\alpha$ and $\gamma$ are unperturbed), so that all perturbations are expressed in terms of $\eta(u)$. For the metric functions and the scalar field $\phi$ we obtain (omitting the factor $e^{\Omega t}$)

$$\delta \alpha = \frac{1}{\Omega} e^{2\gamma - 2\alpha} [\eta'' + \eta'(2\gamma' - \alpha')],$$

$$\delta \beta = \frac{\beta'}{\Omega} e^{2\gamma - 2\alpha},$$

$$\delta \gamma = \frac{1}{\Omega} (\Omega^2 \gamma + \gamma' \eta' e^{2\gamma - 2\alpha}],$$

$$\delta \phi = \frac{\phi'}{\Omega} e^{2\gamma - 2\alpha}, \quad (24)$$

where the prime denotes $d/du$.

Let there be a static configuration with $\beta' \neq 0$ and $\phi' \neq 0$. Then, if we have nontrivial time-dependent perturbations under the gauge condition $\delta \beta = 0$ (or $\delta \phi = 0$), Eqs. \[24\] immediately lead to $\eta' = 0$, which means that our perturbation cannot be caused by a transformation like \[23\], i.e., is physical. The same is true for any gauge of the form $f_1 \delta \beta + f_2 \delta \phi = 0$ where $f_1$ and $f_2$ are any fixed functions of $u$, provided $f_1 \beta' + f_2 \phi' \neq 0$. The reason is that $\beta$ and $\phi$ are scalars with respect to coordinate transformations of the 2-surfaces $(x^0, x^1)$. Thus, choosing such gauges, we can be sure that the perturbations to be studied will be physical. For other gauges, involving $\delta \alpha$ and/or $\delta \gamma$, an additional inspection will be required.

A more general approach to the problem of gauge in perturbation theory for spherically symmetric spacetimes can be found in Ref. \[20\]; though, in the present case, our explicit treatment seems more transparent.

4. Stability analysis

4.1. The problem

We have considered our set of linear perturbation equations using two different systems of analytical computation, Maple and Mathematica, which made it possible to compare the results and to avoid errors.
We use the Einstein conformal frame, in which the equations are much simpler, and the gauge
\[ \delta \psi = 0 \] (25)
which is manifestly physical (see Sec. 3) and, in addition, transforms to \( \delta \phi = 0 \) in the Jordan frame. Moreover, according to Eq. (20), we have the following relation between the metric perturbations:
\[ \delta \alpha = 2 \delta \beta + \delta \gamma. \] (26)

Two independent components of the Einstein equations for perturbations in the gauge may be written as
\[ e^{2\beta} R_1^0 = 2 [\delta \beta' - \beta' (\delta \beta + \delta \gamma)] - \gamma \delta \beta = 0, \]
\[ e^{2\beta} R_2^0 = 2 \beta'' (2 \delta \beta + \delta \gamma) - 2 e^{2\beta + 2\gamma} \delta \beta + e^{4\beta} \delta \beta - \delta \beta'' = 0 \] (27)

Here primes denote derivatives with respect to \( u \), the harmonic radial coordinate in the Einstein frame, \( \alpha, \beta \) and \( \gamma \) describe the background configuration and satisfy the static field equations. We separate the variables using the substitution
\[ \delta(r, t) = \delta(u) e^{iut}, \] (28)
where \( \delta \) is a perturbation of any variable in our problem. After substitution of into \( \delta \gamma \) is expressed from the first equation, and then the second equation takes the form ( \( e^{iu} \) is omitted)
\[ \delta \beta'' - \Omega^2 \delta \beta = s(u) + F(u) \delta \beta' + G(u) \delta \beta = 0, \] (29)
where \( s, F, G \) are functions of \( u \) obtained from the metric:
\[ F(u) = -2 \beta'' / \beta', \]
\[ G(u) = -2 \beta'' + 2 \beta'' \gamma / \beta' + 2 e^{2\beta + 2\gamma}, \]
\[ s(u) = e^\beta. \] (30)

A few words about the boundary conditions. At spatial infinity the choice is evident: \( \delta \beta \to 0 \). At the transition sphere \( S_{\text{trans}} \) \( \delta \beta \) should be finite, as well as its first two derivatives in \( u \). This is necessary for the metric perturbations in the Jordan picture to be finite and smooth at \( S_{\text{trans}} \), which is easily checked using the transformation and expressions in terms of the invariant length in the Jordan frame.

As usual, we perform a transition from to a Schrödinger-like form of the perturbation equation:
\[ d^2 y / dx^2 + [E - V(x)] y(x) = 0, \] (31)
where the subscript \( x \) denotes \( d/dx \) and \( E = -m^2 \Omega^2 \). The notations are chosen in such a way that the potential \( V(x) \) and the “energy” \( E \) are dimensionless. The asymptotic forms of \( V(x) \) are
\[ V(x) \approx -2 / x^3 \quad (x \to \infty, \ \text{spatial asymptotic}), \]
\[ V(x) \approx -1 / (4x^2) \quad (x \to 0, \ \text{the sphere } S_{\text{trans}}). \] (34)

Thus we have a quadratic potential well at \( S_{\text{trans}} \), which is placed at \( x = 0 \) by choosing the proper value of the arbitrary constant in the definition of \( x \) in Eq. 32.

The boundary condition at spatial infinity \( (u \to 0, \ x \to \infty) \) is \( y \to 0 \) while the asymptotic form of any solution of 31 with \( E < 0 \) at large \( |x| \) is
\[ y \approx C_1 e^{-m \Omega |x|} + C_2 e^{m \Omega |x|}, \quad C_{1,2} = \text{const.} \] (35)

An admissible solution is the one with \( C_1 = 0 \), with only a decaying exponential.

At the other asymptotic, \( x \to 0 \), the condition that follows from the above continuity requirements reads \( y / \sqrt{x} < \infty \) whereas the solution of 31 behaves as
\[ y \approx \sqrt{x} (C_3 + C_4 \ln x). \] (36)

It follows that we must select the solution with \( C_4 = 0 \). In other words, our problem is to find out whether there is a solution to the boundary-value problem for Eq. 31 such that \( y \to 0 \) as \( x \to \infty, \ y / \sqrt{x} < \infty \) as \( x \to 0 \) and \( E < 0 \). In the remainder of the section we solve this problem.

4.2. Summary of the solution

We begin with a proof of the fact that the Hamiltonian operator \( H \) related to Eq. 31 is self-adjoint and is bounded from below. To this end, we use an auxiliary operator \( T \) which has the same singularity at \( S_{\text{trans}} \) as \( H \). The one-sided boundedness indicates that the real part of the increment \( \Omega \) cannot be infinite. A further comparison of \( T \) and \( H \) shows that the continuous parts of their spectra coincide and lie in the non-negative part of the real number axis. So, if there are any solutions of our boundary value problem with \( E < 0 \), they belong to a discrete spectrum.

To prove the existence of a solution with \( E < 0 \) we use the well-known fact from quantum mechanics (its more general form is called the minimax principle) that the lower bound \( \mu_0 \) of the spectrum of an operator \( A \) is the infimum of the functional
\[ (\psi, A\psi), \] (37)
where the parentheses denote the scalar product (defined a bit later), the infimum is taken on the set of functions \( \psi \) which lay in the definition domain of \( T \), and the norm \( \| \psi \| = 1 \). Thus the value \( (\psi, A\psi) \) for any specified function \( \psi \) is an upper estimate for \( \mu_0 \), and if it is negative, then \( \mu_0 < 0 \). Functions which may closely resemble the unknown function that realizes the...
infinite can give values of the functional \[ E \] closest to \( \mu_0 \). We guess such a function, which shows that the ground state of \( H \) lies below zero. This function is a ground state of a certain operator which is similar to \( H \) but simpler.

### 4.3. The solution

Consider the auxiliary differential equation

\[
- \frac{d^2}{dx^2} y(x) - \frac{1}{4x^2} y(x) = E y(x)
\]

and investigate the question of self-adjointness of the Schrödinger operator

\[
T y(x) \equiv - \frac{d^2}{dx^2} y(x) - \frac{1}{4x^2} y(x)
\]

on the subset \( D(T) \) of real Hilbert space \( L_2([0, \infty)) \) such that, for \( y(x) \in D(T) \), (a) our boundary conditions (BCs) hold (so that, e.g., \( |y|/\sqrt{x} < \infty \) as \( x \to 0 \)) and (b) \( T y(x) \in L_2 \). The space \( L_2([0, \infty)) \) is a Hilbert space with an inner (scalar) product defined as the Lebesgue integral

\[
(\varphi, \psi) = \int_0^\infty \varphi^* \psi \, dx,
\]

where the star stands for complex conjugation. \( D(T) \) is dense in \( L_2 \) since \( C_0^\infty(0, \infty) \subset D(T) \), where \( C_0^\infty(0, \infty) \) is the subset of functions in \( C^\infty(0, \infty) \) with a compact support separated from 0. It is a dense subset in \( L_2 \).

One can show that the operator \( T \) defined in this way is symmetric and therefore closable. Obviously, the BCs of our Hilbert space are homogenous. The Schrödinger equation \( \frac{d^2}{dx^2} y(x) - \frac{1}{4x^2} y(x) = E y(x) \) related to the operator \( T \) has the solution

\[
c_1 \sqrt{x} K_0(\sqrt{-E} x) + c_2 \sqrt{x} I_0(\sqrt{-E} x),
\]

where \( E \) is the “energy” corresponding to \( -m^2 \Omega^2 \) of our problem, so to prove the instability we should show that there are “quantum states” with \( E < 0 \); \( K_0 \) and \( I_0 \) are the zero-order modified Bessel functions of the first kind. Neither of these functions, nor any linear combination, satisfy our BCs. This means that the operator \( T - EI \), \( E < 0 \) has a bounded inverse operator \( (T - EI)^{-1} \) with a definition domain dense in \( L_2 \). The existence of a reverse operator follows from the well-known alternative: under given homogenous BCs, either the differential equation \( L[y] = g(x) \) has a uniquely defined solution \( y(x) \), or the homogeneous equation \( L[y] = 0 \) has a non-zero solution. In our case,

\[
L[y] \equiv - \frac{d^2}{dx^2} y(x) - \frac{1}{4x^2} y(x) - E y(x).
\]

The boundedness and the density property of the definition domain of \( (T - EI)^{-1} \) in \( L_2 \) follow from studying the properties of solutions to the equation \( L[y] = g(x) \) with nonzero \( g(x) \in L_2 \). The existence of \( (T - EI)^{-1} \), \( E < 0 \) means that the domain \( (-\infty, 0) \subset \rho(T) \), \( \rho(T) \) being the resolvent set of \( T \).

Considering in a similar way Eq. \( (37) \) with \( E > 0 \), one can show that \( [0, \infty) \) is a continuous spectrum.

Thus we have shown that \( T \) is a closed symmetric operator which contains real numbers in its resolvent set. It satisfies the conditions of the second corollary of Theorem X.1 in [21]. If the resolvent set of a closed symmetric operator contains at least one real number, then this operator is self-adjoint. The self-adjointness of this operator was also mentioned in passing in Ref. [22].

The proved properties of \( T \) make it possible to use the wealth of results obtained in the theory of self-adjoint operators. In particular, we use the following two theorems:

**Theorem 1** (Rellich [22]). Let \( A \) be a self-adjoint operator on \( D(A) \) and \( B \) a symmetric operator on \( D(B) \), so that \( D(B) \supset D(A) \) and

\[
||B\psi|| \leq a||\psi|| + b||A\psi||,
\]

\( b < 1 \). Then the operator \( A + B \) is self-adjoint and \( D(A + B) = D(A) \).

**Theorem 2** (Kato [22]). Let the conditions of Theorem 1 hold and \( A \) be bounded from below (or from above, or from both sides), then \( A+B \) is bounded from below (or from above, or from both sides), but not necessarily with the same bound (bounds).

Considering \( T \) as \( A \) in these theorems, we can rewrite \( 31 \) as

\[
T y(x) + \tilde{V}(x)y(x) = E y(x),
\]

where

\[
\tilde{V}(x) = V(x) + 1/4x^2.
\]

Since \( \tilde{V} \) is bounded (this is true since \( \tilde{V}(x) \to 0 \) as \( x \to \infty \), and \( \tilde{V}(x) \) is bounded everywhere), the conditions of Theorem 1 are fulfilled, and the operator

\[
H \equiv -d^2/dx^2 + V(x)y(x)
\]

connected with Eq. \( 31 \) is self-adjoint on \( D(T) \).

Using the spectral theorem for unbounded operators [21], we prove that \( T \) is non-negative and consequently is bounded from below. Therefore the operator \( H \) is bounded from below too (Theorem 2).

We now wish to show that the continuous spectrum of \( H \) coincides with the continuous spectrum of \( T \). We use the following theorem:

**Theorem 3** [21]. Let \( A \) be a self-adjoint operator and \( C \) a symmetric operator such that \( C(A^n + i)^{-1} \), \( n \in \mathbb{N} \) is a compact operator. Then, if \( B = A + C \) is self-adjoint on \( D(A) \), \( \sigma_{\text{ess}}(A) = \sigma_{\text{ess}}(B) \).

\[\text{We denote: } \sigma_{\text{disc}} = \text{discrete spectrum}, \sigma_{\text{cont}} = \text{continuous spectrum}, \sigma_{\text{ess}} = \text{essential spectrum} \]
The compactness of the operator $\tilde{V}(T+i)^{-1}$ follows from its integral representation:

$$
(\tilde{V}(T+i)^{-1}f)(x) = \int_0^\infty dy K(x, y) f(y),
$$

where $G(x, y)$ is the Green function, core of the integral operator $(T+i)^{-1}$. As follows from the asymptotic properties of $\tilde{V}(x)$ and $G(x, y), K(x, y) \in L_2([0, \infty) \times [0, \infty))$. So, $\tilde{V}(T+i)^{-1}$ is a Hilbert-Schmidt operator and hence is compact \[21\]. The conditions of Theorem 3 are fulfilled, and $\sigma_{\text{ess}}(H) = \sigma_{\text{ess}}(T) = [0, \infty)$. Since $\sigma = \sigma_{\text{ess}} \cup \sigma_{\text{disc}} [21]$, the remaining part of $\sigma(H)$ belongs to the discrete spectrum.

Eq. (29) may also be expressed in a non-Schrödinger form using the new variable $w$ instead of $u$:

$$
u = \frac{1}{2}\ln \sqrt{1/w+1}$$

and further converted to a normal form:

$$
\left\{ 16E + \frac{11+32E}{2w} - \frac{64}{3(1+4w)} + \frac{1}{4w^2} + \frac{1}{4(1+w)^2} - \frac{1}{6(1+w)} - \frac{32}{(1+w)^2} \right\} n(w) + \frac{d^2n(w)}{dw^2} = 0,
$$

where

$$
\begin{align*}
n(w) &= \delta \beta \exp\left(-\int \frac{(2w-1)dw}{2w(1+w)(1+4w)}\right). \\
\end{align*}
$$

We cannot solve this equation, but its truncated version

$$
\frac{d^2n(w)}{dw^2} + \left\{ 16E + \frac{11+32E}{2w} + \frac{1}{4w^2} \right\} n(w) = 0
$$

has the solution:

$$
n(w) = c_1 M\left(\frac{(11+32E)}{16\sqrt{-E}}, 0, 8\sqrt{-E}\right) + c_2 W\left(\frac{(11+32E)}{16\sqrt{-E}}, 0, 8\sqrt{-E}\right),
$$

where $M$ and $W$ are Whittaker functions. If the first argument, usually denoted as the index $k$, takes on the values $k = 1/2, 3/2, \ldots, n+1/2, n \in \{0, N\}$, then $M(k, 0, w) = W(k, 0, w)$, and the corresponding quantity $E$ has the values

$$
E_k = -\frac{11}{32} - \frac{k^2}{8} \left(1 - \sqrt{1 + \frac{11}{2k^2}}\right),
$$

which satisfy the equation

$$
\frac{(11+32E_k)}{16\sqrt{-E_k}} = k.
$$

Hence $E$ takes a discrete set of values in the interval $[-3/8 + \sqrt{3}/32, 0]$, and they lie in $\sigma_{\text{disc}}$ of the Schrödinger operator related to Eq. (30). The latter operator has an infinite discrete spectrum with the limiting point 0. Any other eigenvalue $E$ gives a solution related to the essential spectrum, or the resolvent set of the latter operator. The solutions obtained make it possible to apply the minimax principle ([21], theorem XIII.1). The theorem is applicable because $H$ is self-adjoint and bounded from below.

We use the solution with $k = 1/2$ as a trial function to find an upper estimate of the lower bound of $\sigma(H)$:

$$
\mu_0 = (\psi(E_{1/2}, x), H \psi(E_{1/2}, x)),
$$

where the parentheses denote the scalar product on $L_2$, i.e., Lebesgue integration, $\psi(E_{1/2}, x)$ is a normalized function obtained from $W(1/2, 0, 8\sqrt{-E_{1/2}})$ by the substitution which transforms Eq. (42) into \[31\]. Explicit integration in $x$ is impossible because we cannot represent $w(x)$ in elementary functions, but we can integrate in $w$ after necessary substitutions. It is convenient to represent $H$ as a sum of two Schrödinger operators, where the first, $H'$, corresponds to Eq. (50) and the second, $V'$, corresponds to the remaining part of the operator. We obtain

$$
\begin{align*}
\mu_0 &= (\psi(E_{1/2}, x), H' + V' \psi(E_{1/2}, x)) \\
&= E_{1/2} + (\psi(E_{1/2}, x), V' \psi(E_{1/2}, x)) \\
&= E_{1/2} + \frac{\int W(1/2, 0, 8\sqrt{-E_{1/2}})U(w)dw}{\int W(1/2, 0, 8\sqrt{-E_{1/2}})U(w)dw},
\end{align*}
$$

where

$$
U(w) = \frac{64}{3(1+4w)} - \frac{1}{4(1+w)^2} + \frac{1}{6(1+w)} + \frac{32}{(1+w)^2},
$$

and $r(w) = 16(1+1/w)$. Integration gives $\mu_0 \simeq -0.039$. Since the essential spectrum begins from 0, according to the minimax principle, $\mu_0$ is an upper estimate of the ground state which is thus located below zero.

Thus we have proved the existence of negative eigenvalues of the operator $H$ under physically justified BCS, and therefore there are exponentially growing solutions (at least one) of our perturbation equation \[20\]. We have also solved our boundary-value problem \[20\] numerically, applying the Fortran procedure SLEIG and using the coordinate transformation \[17\]. We found a single discrete eigenvalue at $-0.048$, which fits our estimate of $\mu_0$. The corresponding problem for Eq. (45) is not suitable for using SLEIG because Fortran does not "understand" the BC $y/\sqrt{t} < \infty$.

Recall that we have been working in the Einstein frame, so that the coordinates used cover only half of the wormhole, so that we should use two copies of this patch and verify whether the metric perturbations remain really smooth at the transition sphere $S_{\text{trans}}$. Computation of the corresponding metric perturbations ($\delta \beta$, \[31\])
\( \delta \gamma \) at the point where \( f(\phi) = 0 \) shows that their first derivatives in \( l \) (where \( l \) is the Gaussian radial coordinate in the Jordan picture, such that \( g_{tt} = -1 \)) are zero, while the second derivatives are finite, so that the linearized gravity equations are there meaningful and hold. We conclude that the metric perturbations found are physical, and the wormholes under consideration are unstable under small spherically symmetric perturbations. The decay rate depends on the value of \( m \) in (5), since \( E = -m^2\Omega^2 \). The wormhole radius (understood, for simplicity, as \( \sqrt{-g_{22}} \) at the transition sphere rather than throat radius which is smaller but generically of the same order) is proportional to \( m \) and also depends on the integration constant in the solution of (3). Let us discuss the special case of conformal coupling. According to (14), the wormhole radius is \( r_{wh} = 2m\sqrt{(1 + y_0)/(1 - y_0)} \). If we assume \( y_0 \ll 1 \), then both \( r_{wh} \) and the throat radius are approximately equal to \( 2m \). The characteristic time of decay, \( \tau = 1/\Omega \), is proportional to \( m \) (which has the dimension of length):

\[
\tau \simeq m/\sqrt{\mu_0} \simeq 5m. \quad (57)
\]

For a wormhole radius of the order of a typical stellar size \( \sim 10^6 \) km, the time \( \tau \) is a few seconds (slightly greater than the time needed for a light signal to cover the stellar diameter). If \( y_0 \) increases under fixed wormhole radius, then \( m \) decreases, so \( \tau \) decreases too. We see that, for all wormholes with a fixed radius, \( \tau \leq 5m \).

Similar estimates can be obtained for other STT characterized by different \( f(\phi) \).

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