Regularity of Characteristic Exponents and Linear Response for Transfer Operator Cocycles

Julien Sedro¹, Hans Henrik Rugh²

¹ Laboratoire de Probabilités, Statistique et Modélisation (LPSM), Sorbonne Université, Université de Paris, 4 Place Jussieu, 75005 Paris, France. E-mail: sedro@lpsm.paris
² Institut de Mathématiques d’Orsay (IMO), Université Paris-Saclay, Site d’Orsay, 91405 Orsay Cedex, France. E-mail: hans.rugh@universite-paris-saclay.fr

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Abstract: We consider cocycles obtained by composing sequences of transfer operators with positive weights, associated with uniformly expanding maps (possibly having countably many branches) and depending upon parameters. Assuming $C^k$ regularity with respect to coordinates and parameters, we show that when the sequence is picked within a certain uniform family the top characteristic exponent and generator of top Oseledets space of the cocycle are $C^{k-1}$ in parameters. As applications, we obtain a linear response formula for the equivariant measure associated with random products of uniformly expanding maps, and we study the regularity of the Hausdorff dimension of a repeller associated with random compositions of one-dimensional cookie-cutters.

1. Introduction

The Oseledets–Lyapunov spectrum for transfer operators cocycles associated with random composition of maps plays a key role in the study of its ergodic properties, similar in many aspects to the one played by the spectrum of the transfer operator in the study of statistical properties of autonomous dynamical systems: this is the general philosophy arising from a series of papers generalizing Oseledets M.E.T to a cocycle of quasi-compact operators on a separable Banach space [25,26,31]. In a deterministic setting, a deep connection between stability of the statistical properties, such as statistical stability, linear and higher-order response, and stability of the transfer operator spectrum has been established: it then appears natural to wonder what remains of such a connection in the case of random composition of maps.

Indeed, the regularity with respect to parameters of the top characteristic exponent of operator cocycles has been thoroughly studied, mostly in analytical regularity or for random product of matrices: Starting with Ruelle’s seminal paper [43], in which he shows real-analyticity of the characteristic exponent for a random product of positive matrices (within a certain compact family contracting $\mathbb{R}^n_+$) to Le Page [40], establishing under less strict hypotheses, Hölder and smooth regularity in the case of an i.i.d product of matrices,
or Hennion’s paper [37], which gives sufficient conditions under which the characteristic exponent of an i.i.d product of matrices is differentiable. In an infinite-dimensional setting, Dubois [21] extended Ruelle’s result to more general cone-contractions and showed real-analyticity with respect to parameters of the top characteristic exponent for a sequence of real-analytical contracting operators. A further generalization to complex cone-contractions was given in [49]. One may also mention [16], where genericity of analyticity of Lyapunov exponents for random bounded linear operators is shown.

For transfer operator cocycles, the literature on characteristic exponents stability is more recent: in the context of finite regularity, we may include [20, 23, 24, 32], as well as the recent [14], which present a generalization of the Keller-Liverani approach to discrete spectrum stability [38] to the cocycle case. However, the results in [14] only gives continuity w.r.t parameters of the discrete spectrum, and do not give any explicit estimate on the modulus of continuity.

In the analytical setting, an investigation on characteristic exponent regularity w.r.t parameters was also recently used to establish a random analogue of the Nagaev-Guivarc’h method (see [18, 19]).

The problem we consider here concerns a family of cone-contracting transfer operators, specifically the problem of quenched (linear) response for a perturbed cocycle of expanding maps, i.e the regularity w.r.t parameters of the equivariant measure of this random dynamical system. This is a natural extension of the response problem for deterministic dynamics, which was investigated both for expanding maps [2, 4, 5, 7–10, 12, 28] and more general hyperbolic systems [5, 17, 33, 34, 45–47].

In the random setting, the annealed response (i.e. response for the invariant measure of the associated Markov chain) has also received extensive attention: besides the already mentioned [33], see the seminal [36] or the recent [1, 27, 29]. However, we emphasize that the above-mentioned works only cover the annealed case, and that to the authors knowledge, the present paper is the first time that quenched response is addressed. It is certainly of major interest in many applications, notably for climate science: see [13, 30, 42].

We note that even for a single deterministic system, in the context of finite regularity, the study of the regularity of the physical measure upon a parameter is a considerable task, as the map associating the system to its transfer operator is not even continuous in the usual operator norm. The methods to overcome this problem may be roughly divided in three categories: first, Ruelle’s original approach [45–47], relying on structural stability. This allows for changes of variables that, although not smooth themselves, depend smoothly on parameters. Second, various forms of quantitative stability results for fixed points of linear Markov operators, as in [1, 27–29]. Third, the so-called weak spectral perturbation theory, originally developed by Keller and Liverani [38], who managed under conditions of quasi-compactness to deduce Hölder continuity of the discrete part of the spectrum. Gouëzel and Liverani [33] then extended this to differentiability and higher-order regularity of the discrete spectrum, also establishing explicit formulae for the $n$-th derivative of this spectral data. However, this last approach relies heavily upon the study of the resolvent operator, which makes it unsuitable when considering non-autonomous dynamics, sequential or random compositions.

The approach we describe in this paper is suited both to deterministic dynamics and to random (non-autonomous) situations, including the quenched case. In the deterministic setting, our methods are easily adapted to the case of sequential dynamical systems (see Remark 1.4 below). This was considered, though not treated in full details, in Ruelle’s seminal paper [45]. As for the random case, our results cover the case of a possibly
non i.i.d random product chosen among an infinite, non-countable set, of uniformly expanding maps having an infinite number of branches, defined on any manifold that admits such maps. In the i.i.d case, we notice that our linear response result for the stationary measure of the skew-product associated to the random product (Theorem 4.21) implies a linear response for the annealed invariant measure of a random composition of uniformly expanding maps, whence parts of the results of [1].

Our approach will rely on two main ingredients: Birkhoff’s cone contraction theory to construct equivariant measures for random product of expanding maps and, to study their dependence on parameters, a new idea, first appearing in [50] in which a quantitative stability result for fixed point of a graded scheme of (possibly non-linear) operators was proven (we refer to the discussion at the beginning of Sect. 3 for more on graded schemes of operators). The main idea is that a fixed point of a contracting $C^k$ map, when viewed in $C^{k-1}$ may gain differentiability in parameters. We will apply this principle to Banach spaces which will consist of $C^k$-sections of a fiber bundle and show that the section becomes differentiable when viewed as a $C^{k-1}$ section: this implies the wanted regularity of characteristic exponents. Once one has become familiar with the general idea, the actual implementation is quite simple, although somewhat hampered by a complicated notation.

Our approach looks close in spirit to the ‘quantitative stability for fixed points’ family of results, but has two major differences: first, we may study fixed points of non-linear operators (notably fractional linear maps like in (2.2)). Second, we may study regularity of our fixed point beyond differentiability.

The paper is organized as follows: in Sect. 2, we consider a cone-contracting cocycle of linear operators and construct the associated fractional linear map (2.2). We construct its fixed point (Lemma 2.1), study some of its regularity properties (Lemma 2.3), and relate it to the top characteristic exponent of the linear operator cocycle (Theorem 2.2).

In Sect. 3, we consider families of such cone-contracting cocycle of linear operators, depending upon a parameter and acting on a scale of Banach spaces. In Theorem 3.2 we show that, under a suitable regularity assumption for the cocycle, both the fixed point map (seen in the right Banach space) and top characteristic exponent are smooth. As a by-product, we get an implicit-function like formula for the derivative of the fixed point map.

In Sect. 4, we present an application to the problem of linear response for the equivariant measure associated to a random product of uniformly expanding maps. Here, issues of uniformity with respect to various parameters are paramount and we give a detailed construction of the $c^{(k,\alpha)}$-structure on a general Riemann manifold. In Sect. 4.2, we construct a distance on a space of couples $(\bar{T}, g)$ of uniformly expanding maps and weights, satisfying some uniformity and regularity conditions. This allows us to establish strong Bochner measurability of the transfer operator cocycle. Based on a “parameter-extraction” approach, we also establish the regularity of the transfer operator cocycle with respect to parameters. In Sect. 4.3, we exhibit a family of Birkhoff cones that are uniformly contracted by the (weighted) transfer operator of a $c^\gamma$ uniformly expanding map. In Sects. 4.4 and 4.6, we establish a linear response formula for the equivariant (Theorem 4.18) and stationary (Theorem 4.21) measures of a random product of smooth maps.

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1 It should be noted that we only recover those results when one perturbs the maps themselves, which is not exactly the setup of [1], where also perturbations of the probability measure on the base are considered (see also [1, p.5, footnote 3]).

2 This step is quite classical, see e.g. [6]. For an occurrence of this idea in a context pertaining to linear response for non-autonomous systems, but somewhat different, see [51].
expanding maps. As an example, we show that the Hausdorff dimension of the repellor associated to one-dimensional cookie-cutters depends smoothly on the system.

Finally, in a series of appendices, we recall or prove various results useful to our purposes: in “Appendix A”, we recall various results in real cone contraction theory. In “Appendix B”, we have regrouped some material on (strong) Bochner measurability. In “Appendix C”, we prove some results in classical differential calculus, instrumental in our approach; in particular, we state and prove several “regularity extraction” theorems, central to the main sections of this paper. Finally, in “Appendix D”, we establish a generalization of the classical Leibniz principle to families of operators having some “loss of regularity” property.

We end this introduction by giving two fairly simple but non-trivial examples to which the results of the present paper apply. Below, let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space and let \(\tau : \Omega \to \Omega\) be an invertible \(\mathbb{P}\)-ergodic map (see however Remark 1.4).

**Example 1.1.** We equip \(\mathbb{R}^d\) with the standard Euclidean norm and we let \(\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d\) be the \(d \geq 1\) torus with the induced Riemannian metric. For \(\theta_0 < 1\) we consider the subset \(G(\theta_0) \subset M_d(\mathbb{Z})\) of invertible integer matrices \(M\) such that the \(\|M^{-1}\|_{L(\mathbb{R}^d; \mathbb{R}^d)} \leq \theta_0\). Finally let \(0 < \theta_1 < \frac{1}{\theta_0} - 1\).

Consider measurable maps \(\omega \in \Omega \mapsto M_\omega \in G(\theta_0)\) and \(\omega \in \Omega \mapsto \kappa_\omega \in C^r(\mathbb{R} \times \mathbb{T}^d; \mathbb{R})\) for \(r > 2\) for which \(\sup_\omega \|\partial_x \kappa_\omega(u, x)\|_{L(\mathbb{R}^d; \mathbb{R})} \leq \theta_1\) and \(\sup_\omega \|\kappa_\omega\|_{C^r} \leq K < +\infty\). We define the parametrized random expanding map: \(T_{\omega,u} : \mathbb{T}^d \to \mathbb{T}^d\), \(\omega \in \Omega\), \(u \in \mathbb{R}\) by

\[
T_{\omega,u}(x) := M_\omega x + \kappa_\omega(u, x) \mod \mathbb{Z}^d
\]

**Example 1.2.** Let \(\omega \mapsto \kappa_\omega \in C^r([0, 1]; [0, 1])\), \(r > 2\) be a measurable map for which \(\sup_\omega \|\kappa_\omega\|_{C^r} \leq K < +\infty\), \((x \mapsto \kappa_\omega(u, x)) \in \text{Diff}^r([0, 1])\) and \(\inf_x \partial_x \kappa_\omega(u, x) \geq \theta > \frac{1}{2}\) for all \(u \in \mathbb{R}\) and \(\omega \in \Omega\). We define in this case: \(T_{\omega,u} : D_{\omega,u} \to (0, 1)\), \(\omega \in \Omega\), \(u \in \mathbb{R}\) by

\[
T_{\omega,u}(x) := \frac{1}{\kappa_\omega(u, x)} - \left\lfloor \frac{1}{\kappa_\omega(u, x)} \right\rfloor,
\]

where the domain of definition is \(D_{\omega,u} = \{x \in (0, 1) : 1/\kappa_\omega(u, x) \notin \mathbb{N}\}\) (which is co-countable for fixed \(\omega, u\)). Note that when \(\kappa_\omega(u, x) \equiv x\), \(T_{\omega,u}\) is the standard Gauss map.

For either of the two examples above we consider the composition of random maps along orbits of \(\tau\):

\[
T_{\omega,u}^{(n)} := T_{\tau^{n-1}\omega,u} \circ \cdots \circ T_{\omega,u}.
\]

In the first example this is defined for all \(x \in \mathbb{T}^d\) and in the second example on a set of full Lebesgue measure in \((0, 1)\).

**Theorem 1.3.** For the above two examples we have with \(M = \mathbb{T}^d\) and \(M = (0, 1)\), respectively:

\[3\] By \(C^r\) we mean the closure of \(C^\infty\) functions in a space of \(C^r\) functions, cf. Definition 4.4.
For every $u \in \mathbb{R}$, the skew-product $F_u(\omega, x) := (\tau \omega, T_{\omega, u}(x))$ admits a unique stationary probability measure $\nu_u$ on $\Omega \times M$, whose decomposition along the marginal $\mathbb{P}$, $\nu_{\omega, u}$, has Lebesgue-density $h_{\omega, u} \in C^{r-1}(M)$. Furthermore, $\text{esssup}_{\omega \in \Omega} \| h_{\omega, u} \|_{C^{r-1}(M)} < +\infty$.

For any $1 \leq s < r - 2$, the map $u \in \mathbb{R} \mapsto h_u \in L^\infty(\Omega, C^s(M))$ is differentiable, and one has the following linear response formulae for any $\psi \in L^1(M)$:

$$\partial_u \left[ \int_M \psi h_{\omega, u} dm \right]_{u = u_0} = \sum_{n=0}^{\infty} \int_M \psi \circ T_{\tau - n, u_0}^{(n)} \partial_u \mathcal{L}_{\tau - (n+1), u_0} h_{\tau - (n+1), u_0} dm$$

(1.1)

$$\partial_u \left[ \int_\Omega \int_M \psi h_{\omega, u} dmd\mathbb{P} \right]_{u = u_0} = \sum_{n=0}^{\infty} \int_\Omega \int_M \psi \circ T_{\omega, u_0}^{(n)} \partial_u \mathcal{L}_{\omega, u_0} h_{\omega, u_0} dm d\mathbb{P}$$

(1.2)

Proofs for the above two examples are given in Sect. 4.5.

Remark 1.4. We have chosen to focus upon the linear response for an invertible ergodic random dynamical system. However, the method presented here allows for various other possibilities. We mention the following:

- In the article, by “quenched linear response” we mean a linear response when the random environment is fixed. This requires in the context of Formula (1.1), that $\tau$ is invertible since our construction of $h$ is computed along a (unique) backward orbit under $\tau$. It does not, however, rely upon the existence of a $\tau$-invariant probability measure on $\Omega$. Thus, modulo some obvious modifications, the formula also holds when taking $\Omega = \mathbb{Z}$ and letting $\tau$ be the shift. This corresponds to what is known as a sequential dynamical system.

- For our annealed formula, (1.2), measurability of $\tau$ and invariance of the measure $\mathbb{P}$ is necessary. In order to have unique characteristic exponents, ergodicity of $\tau$ is usually important but in the present situation it is not. This boils down to the characteristic exponent being identically zero in the above examples.

2. A Cone-Contracting Operator Cocycle

We will here give an abstract formulation of the above type of problems. We consider a Banach space $(E, \| \cdot \|)$ and a collection $M \subset L(E)$ of bounded linear operators which contracts uniformly a regular convex cone $C \subset E$. We make the following assumptions on $C$ and $M$ (see “Appendix A” for further details):

(H1) (outer regularity) There is $\ell \in E'$ of norm one and $K \in [1, +\infty)$ so that for every $\phi \in C$:

$$\langle \ell, \phi \rangle \geq \frac{1}{K} \| \phi \|$$

(H2) (inner regularity) There is $\rho \in (0, 1]$ so that the sub-cone

$$C(\rho) = \{ \phi \in C : B(\phi, \rho \| \phi \|) \subset C \}$$

is non-empty (in particular, $C$ has non-empty interior). We fix such a $\rho$ in the following. We define:

$$C_{\ell=1} = \{ \phi \in C : \langle \ell, \phi \rangle = 1 \} \quad \text{and} \quad C_{\ell=1}(\rho) = \{ \phi \in C(\rho) : \langle \ell, \phi \rangle = 1 \}.$$
(H3) (uniform contraction) We require that for every $L \in M$:

$$L(C^*) \subset C^*(\rho),$$

Here, we have written $C^* := C - \{0\}$ and $C^*(\rho) := C(\rho) - \{0\}$ for the punctured cones.

(H4) (uniform bounds) There is $1 \leq \vartheta < +\infty$ so that for every $L \in M$

$$\frac{1}{\vartheta} \leq \|L\| \leq \vartheta.$$

Such operators enjoy strong contraction properties for the projective Hilbert metric. In the following we will give a short summary of the consequences which will be needed below. As this is fairly standard we will be quite brief and refer to e.g. [3,49] for standard properties of cones and the above mentioned notions of regularity and to [21,43,48] for more details on the standard construction of the cone-contracting cocycles given below.

First, $C(\rho)$ has finite diameter in $C$. From Corollary A.5 one gets:

$$\Delta_1 = \text{diam}_{C^*}(C(\rho)^*) \leq 2 \log \left( 1 + \frac{2K}{\rho} \right).$$

This yields a strict Birkhoff contraction factor: $\eta = \tanh \Delta_1 \leq \frac{K}{K^*+\rho} < 1$. Also, the above four conditions imply (see Lemma A.9) that for any $L \in M$, $\phi \in C(\rho)$:

$$\frac{1}{\vartheta K} \|\phi\| \leq \langle \ell, L\phi \rangle \leq \vartheta \|\phi\| \text{ and } \frac{1}{\vartheta K} \langle \ell, \phi \rangle \leq \langle \ell, L\phi \rangle \leq \vartheta K \langle \ell, \phi \rangle. \quad (2.1)$$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\tau : \Omega \to \Omega$ a measure-preserving, invertible, and ergodic map. We will define an operator valued cocycle over $\Omega$ in the following way: let $X = L^\infty(\Omega, E)$ denote the set of uniformly bounded $\mathcal{F}$-Bochner measurable sections $\phi = (\phi_\omega)_{\omega \in \Omega} : \Omega \to E$ of the product bundle $\Omega \times E$. We write $\|\phi\|_X = \sup_{\omega \in \Omega} \|\phi_\omega\| < \infty$ for the norm: $(X, \| \cdot \|_X)$ is then a Banach space. We will also let $\mathcal{C} = C(\Omega) \subset X$ denote the bounded $\mathcal{F}$-Bochner measurable sections taking values in the cone $C$. We define in a similar way

$$\mathcal{C}(\rho) := \{ \phi \in X, \forall \omega \in \Omega, \phi_\omega \in C(\rho) \},$$

$$\mathcal{C}_{\ell=1} := \{ \phi \in X, \forall \omega \in \Omega, \phi_\omega \in C_{\ell=1}(\rho) \},$$

$$\mathcal{C}_{\ell=1}(\rho) := \{ \phi \in X, \forall \omega \in \Omega, \phi_\omega \in C_{\ell=1}(\rho) \}.$$

We write $\phi \in \mathcal{C}^*$ for a section that is nowhere vanishing, i.e. $\phi_\omega \in C^*(\rho)$ for every $\omega \in \Omega$.

We consider a family of operators of the form $\mathcal{L} = (\mathcal{L}_\omega)_{\omega \in \Omega} : \Omega \to M$, i.e. each $\mathcal{L}_\omega \in M$ is a uniform cone-contraction satisfying (H3) and (H4). We let this family act as a bundle map in the following way ($\phi \in X$ and $\omega \in \Omega$):

$$\mathcal{L} : X \to X, \quad (\mathcal{L}\phi)_\omega = \mathcal{L}_{\tau^{-1}\omega}\phi_{\tau^{-1}\omega}.$$

We will further assume that $\mathcal{L}$ is strongly Bochner measurable, i.e. it preserves Bochner measurability of sections (cf. “Appendix B”). The $n$-th iterate is given by:

$$(\mathcal{L}^n\phi)_\omega = \mathcal{L}_{\tau^{-1}\omega} \circ \cdots \circ \mathcal{L}_{\tau^{-n}\omega}\phi_{\tau^{-n}\omega}.$$
For $\ell \in E'$ satisfying (H1), we define the section of normalization factors:

$$
\Lambda : X \to L^\infty(\Omega; \mathbb{R}_+), \quad (\Lambda(\varphi))_\omega = \langle \ell, \mathcal{L}_{\tau^{-1}\omega}\phi_{\tau^{-1}\omega} \rangle.
$$

and the fractional linear map $\pi : \mathcal{G}^* \to \mathcal{G}_{\ell=1}(\rho)$, defined for $\omega \in \Omega$ by

$$
(\pi(\varphi))_\omega = \mathcal{L}_{\tau^{-1}\omega}\phi_{\tau^{-1}\omega} / \langle \ell, \mathcal{L}_{\tau^{-1}\omega}\phi_{\tau^{-1}\omega} \rangle.
$$

(2.2)

Here, as well as below, we will adopt the convention that a complex or real valued section over $\Omega$ acts upon $X$ by fiberwise multiplication. Similarly, a nowhere vanishing section has an inverse given by taking fiberwise inverse. The map (2.2) may then be written in the following compact way:

$$
\pi(\varphi) = \Lambda(\varphi)^{-1}\mathcal{L}(\varphi).
$$

We may then establish the following:

**Lemma 2.1.** The map (2.2) admits a unique fixed point section $f = \pi \circ f \in \mathcal{G}_{\ell=1}(\rho)$.

**Proof.** We see from Corollary A.8 that for $m \geq n \geq 1$, $\varphi, \psi \in \mathcal{G}^*$:

$$
\|\pi^n(\varphi) - \pi^m(\psi)\|_X \leq \frac{K}{2} \eta^{n-1} \Delta.
$$

In particular, $(\pi^n(\varphi))_n$ is Cauchy in $\mathcal{G}_{\ell=1}(\rho) \subset X$, whence converges to some $f = (f_\omega)_{\omega \in \Omega} \in \mathcal{G}_{\ell=1}(\rho)$, which is a fixed point of $\pi$. $\square$

Let us set $p = \Lambda(f) = \langle \ell, \mathcal{L}f \rangle : \Omega \to \mathbb{R}_+^+$, which by (2.1) verifies

$$
p_\omega = \langle \ell, \mathcal{L}_{\tau^{-1}\omega}f_{\tau^{-1}\omega} \rangle \in \left[ \frac{\rho}{\vartheta K}, \vartheta K \right].
$$

For fixed $\omega \in \Omega$ we define the characteristic exponent of the cocycle operator as follows:

$$
\chi_\omega = \limsup_n \frac{1}{n} \log \|\mathcal{L}^n\|_\omega
$$

(2.3)

We then have:

**Theorem 2.2.** Let $\mathcal{L} = (\mathcal{L}_\omega)_{\omega \in \Omega}$ be a strongly Bochner measurable cocycle consisting of uniform cone-contractions as defined above with

$\int_{\Omega} |\log \|\mathcal{L}_\omega\|| d\mathbb{P}(\omega) < +\infty$. Then

1. The real valued section $p = (p_\omega)_{\omega \in \Omega} = \langle \ell, \mathcal{L}f \rangle : \Omega \to \mathbb{R}_+^+$, is measurable and $\log p$ is $\mathbb{P}$-integrable.
2. The characteristic exponent (2.3) of the cocycle equals a.s.

$$
\chi(\mathcal{L}) = \mathbb{E}(\log p) = \int_{\Omega} \log p_\omega d\mathbb{P}(\omega),
$$

(2.4)
Proof. The first item is an easy consequence of (2.1).

For the second item, note that we have the identity $\mathcal{L}_{\tau^{-1} \omega} f_{\tau^{-1} \omega} = p_{\omega} f_{\omega}$. By the uniform bounds in (2.1), we see that (2.3) is equivalent to:

$$\chi_\omega = \limsup_{n \to \infty} \frac{1}{n} \log (\ell, (\mathcal{L}^n f)_\omega) = \limsup_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \log p_{\tau^{-k} \omega}.$$ 

We are thus reduced to consider Birkhoff averages of $\log p_\omega$. Since $\log p \in L^1(\mathbb{P})$, we conclude by Birkhoff’s theorem that a.e. the limsup is in fact a limit and (by ergodicity) a constant equal to the integral of $\log p$. □

By A.5 and A.8 we see that for $\phi \in \mathcal{C}_{\ell=1}(\rho)$ and $z \in X$ small enough:

$$\|\pi^n(\phi) - \pi^n(\phi + z)\|_\chi \leq \eta^{n-1} \frac{K}{\rho} (\|z\| + o(\|z\|)) \quad \text{(2.5)}.$$ 

This entails the following:

**Lemma 2.3.** There is $C = C(\rho, K, \vartheta) > 0$ so that for $\phi \in \mathcal{C}_{\ell=1}(\rho)$ and $z \in X$ with $\|z\| < \frac{\rho}{2\vartheta K}$:

$$\|\pi(\phi + z) - \pi(\phi) - Q(\phi)z\| \leq C\|z\|^2. \quad \text{(2.6)}$$

*Here, $Q(\phi)$ is the derivative of $\pi(\phi)$ which is given by the expression:

$$Q(\phi)z := \partial_\phi \pi(\phi)z = \Lambda(\phi)^{-1}(\mathcal{L}z - \Lambda(z)\pi(\phi)). \quad \text{(2.7)}$$

At the fixed point $f \in \mathcal{C}_{\ell=1}(\rho)$, the operator $1 - Q(f)$ is invertible and satisfies the bound:

$$\left\| (1 - Q(f))^{-1} \right\| \leq 1 + \frac{K}{\rho} \frac{1}{1 - \eta}.$$ 

Proof. In the following consider $\phi$ and $z$ as in the Lemma. From (2.1) we deduce that $\Lambda(\phi) : \Omega \to \left[\frac{\rho}{\vartheta K}, \frac{\vartheta K}{\rho}\right] \subset \mathbb{R}_+$, i.e. are uniformly bounded from above and below.

When $\|z\| < \frac{\rho}{2\vartheta K}$, we have $|\Lambda(z)| < \frac{\rho}{2\vartheta K} \leq \frac{1}{2} \Lambda(\phi)$ implying that

$$\Lambda(\phi + z) = \Lambda(\phi) + \Lambda(z) \geq \frac{\rho}{2\vartheta K}. \quad \text{(2.8)}$$

Thus, $\Lambda(\phi + z)$ is invertible in the Banach algebra $L^\infty(\Omega; \mathbb{R})$ through a Neumann series. and $\pi(\phi + z)$ is analytic in $z$ and uniformly bounded by $2\vartheta^2 K / \rho$. One has the identity:

$$\pi(\phi + z) - \pi(\phi) = \Lambda(\phi + z)^{-1}(\mathcal{L}z - \Lambda(z)\pi(\phi)) = (1 - \Lambda(\phi + z)^{-1}\Lambda(z))Q(\phi)z,$$

with $Q(\phi)$ as in (2.7). This implies, by (2.1), (2.5) and (2.8)

$$\|\pi(\phi + z) - \pi(\phi) - Q(\phi)z\| \leq \|\Lambda(\phi + z)^{-1}\Lambda(z)Q(\phi)z\| \leq \frac{2\vartheta K}{\rho} \|z\| \|z\| = \frac{4\vartheta^2 K^4}{\rho^2} \|z\|^2.$$ 

By (2.5) we get, at the fixed point $f$, the uniform bound $\|(Q(f))^n\| \leq \frac{K}{\rho} \eta^{n-1}, n \geq 1$ so by a Neumann series we obtain invertibility of $1 - Q(f)$ as well as the stated bound. □
Our goal in the following will be to consider a family of measurable cocycles of uniform cone-contractions depending on a parameter \( u \) (in a Banach space), and see how the regularity of the characteristic exponent and also the fixed point section w.r.t. \( u \) depends upon the regularity of the cocycle. We emphasize that since the work of Ruelle [43] on analytic matrix co-cycles, this is well-understood also when the cocycle is a family of transfer operators depending in an analytic way upon \( u \), see e.g. [21,43,48,49]. However, for transfer operators associated with compositions of maps with finite regularity, results are sparse and very incomplete due to an inherent 'loss of regularity'. It seems appropriate to first treat a slightly more abstract setup and then apply it to a cocycle of transfer operators. The readers who wish to have a concrete example in mind may want to consult Sect. 4 first.

3. Cone-Contracting Cocycles with Loss of Regularity

Let \( B \) be a Banach space and \( U \subset B \) an open and convex subset. An element \( u \in U \) will be considered as a parameter for our problem. Let \( r_0 > 0 \) and consider a family, indexed by \( r \in (0, r_0] \) of Banach spaces with associated regular cones as described in the previous section, i.e.

\[
S_r = (E_r, C_r, \rho_r, K_r, \partial_r, \ell_r), \quad r \in (0, r_0].
\]

We assume that the family is 'graded' in the sense that there are continuous linear injections \( j_{s,r} : E_r \hookrightarrow E_s \), for every \( 0 < s < r \leq r_0 \), satisfying uniform bounds with respect to \( s \) and \( r \), i.e there is some constant \( C > 0 \) independent from \( r \) and \( s \) such that \( \| j_{s,r} \|_{E_r \to E_s} \leq C \), and verifying a transitivity condition: \( \forall 0 < t < s < r \leq r_0: j_{t,s} j_{s,r} = j_{t,r}. \)

As a natural example, the reader may think of \( E_r \) as being \( C^r \)-functions on a manifold, the 'downgrading' as being the natural injection from \( C^r \) into \( C^s \) for \( 0 < s < r \leq r_0 \), and \( C_r \) to be a cone of nonnegative \( C^r \) functions of the form described in subsection 4.3.

For simplicity, we will assume that the injections preserve cones and the linear forms above, i.e. \( j_{s,r}(C_r) \subset C_s \) and \( \ell_r = \ell_s \circ j_{s,r} \). In this construction we also assume that all constants and norms are uniformly bounded on compact subsets of \( (0, r_0] \) (in our applications some constants diverge as \( r \to 0^+ \)).

Given a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) as above we construct Bochner-measurable sections for each system \( S_r \). In this way, we obtain spaces \( X_r = L^\infty(\Omega; E_r) \) as well as bounded cone fields \( C_r \equiv C_r(\Omega) = \{ \varphi : \Omega \to C_r \text{ Bochner measurable} \} \) and the associated slices \( C_r, \ell_{r,1} = C_r, \ell_{r,1}(\Omega) \) etc. as above. The injection \( j_{s,r} \) induces by pointwise action maps on Bochner-measurable sections: \( j_{s,r} : C_r(\Omega) \hookrightarrow C_s(\Omega) \) and \( j_{s,r} : X_r \hookrightarrow X_s \). This family will again be transitive.

For each system \( S_r, r \in (0, r_0] \) we assume given a family of cone-contracting cocycles, \( \mathcal{L}_{r,u} \) acting upon \( X_r \) and depending upon the parameter \( u \in U \). We assume that the cocycles commutes with our injections in the natural way: \( (0 < s < r \leq r_0): j_{s,r} \circ \mathcal{L}_{r,u} = \mathcal{L}_{s,u} \circ j_{s,r} \). When written out over fibers:

\[
\begin{align*}
\mathcal{L}_{r,u}(\phi_{r,\omega}) := \langle \ell_r, \mathcal{L}_{r,u}(\phi_{r,\omega}) \rangle \in \mathbb{R}, \quad \omega \in \Omega,
\end{align*}
\]

when acting upon \( \phi_{r, \omega} \in X_r \). We will often be using the normalizing fields obtained by acting with \( \ell_r \):

\[
\Lambda_{r,u,\omega}(\phi_{r,\omega}) := \langle \ell_r, \mathcal{L}_{r,u,\omega}(\phi_{r,\omega}) \rangle \in \mathbb{R}, \quad \omega \in \Omega.
\]
Omitting mentioning \( \omega \) explicitly, we write:

\[
\Lambda_{r,u}(\varphi_r) := \langle \ell_r, L_{r,u}(\varphi_r) \rangle \in L^\infty(\Omega).
\]

When \( \varphi_r \in \mathcal{C}_{r,\ell_r=1} \) the normalizing field \( \Lambda_{r,u}(\varphi_r) \) takes values in \([\varphi_r, \varnothing K]\), so is invertible in the Banach algebra \( L^\infty(\Omega) \). Therefore, one has

\[
\pi_{s,u}(j_{s,r}(\varphi_r)) = \frac{L_{s,u,\omega}(j_{s,r}\varphi_r,\omega)}{\langle \ell_s, L_{s,u,\omega}(j_{s,r}\varphi_r,\omega) \rangle} = j_{s,r}(\pi_{r,u}(\varphi_r)) = j_{s,r}\varphi_r, \omega).
\]

Thus, we also have \( \pi_{s,u} \circ j_{s,r} = j_{s,r} \circ \pi_{r,u} \).

A family of fields, \( \varphi = (\varphi_r)_{r \in (0,r_0]} \) with each \( \varphi_r \in X_r \), is said to be \textit{consistent} if for every \( 0 < s < r \leq r_0 \), \( \varphi_s = j_{s,r}\varphi_r \). Note, that given a consistent family one has for every \( 0 < s < r \leq r_0 \):

\[
\Lambda_{r,u}(\varphi_r(u)) = \langle \ell_s, j_{s,r}L_{r,u}\varphi_r(u) \rangle = \langle \ell_s, L_{s,u}j_{s,r}\varphi_r(u) \rangle = \Lambda_{s,u}(\varphi_s(u)).
\]

We sometimes write \( \Lambda_u(\varphi) \) for the common value in this case.

**Lemma 3.1.** There exists a unique parametrized consistent family \( f(u) = (f_r(u))_{r \in (0,r_0]} \), \( u \in U \) for which

\[
f_r(u) = \pi_{r,u}(f_r(u)) \in \mathcal{C}_{r,\ell_r=1}, \quad r \in (0,r_0], \quad u \in U.
\]

**Proof.** Pick a section \( s_{r_0} \in \mathcal{C}_{r_0,\ell_{r_0}=1} \) and set \( \varphi_s = j_{s,r_0}\varphi_{r_0} \in \mathcal{C}_{s,\ell_s=1}, \quad 0 < s \leq r_0 \) and iterate as in the previous section in each member of the family. We obtain fixed fields, \( f_r(u) = \pi_{r,u}(f_r(u)) \) and since \( j_{s,r}(\pi_{r,u})^n(\varphi_r,\omega) = (\pi_{s,u})^n(j_{s,r}\varphi_r,\omega) \) each iterate constitutes a consistent family and this relationship holds while taking the limits. Note, however, that the convergence may not be uniform over the scale \((0, r_0] \). \( \square \)

Up to now we haven’t assumed anything about the regularity of \( u \mapsto L_{r,u} \in L(X_r) \), whence of the fixed fields at the individual levels, \( r \in (0,r_0] \). In fact in the applications below, there is a priori no regularity of \( u \mapsto L_{r,u} \in L(X_r) \) w.r.t. \( u \), apart from being uniformly bounded. The idea in the following is that by ‘downgrading’ the target space within the family we recover regularity of the map \( u \mapsto j_{s,r}L_{r,u} = L_{s,u}j_{s,r} \in L(X_r, X_s), \quad 0 < s < r \leq r_0 \) which in turn we transform into regularity of the fixed field section \( u \mapsto f_s(u) \in X_s \). Here is the main theorem of this section:

**Theorem 3.2.** Let \( (f_r(u))_{r \in (0,r_0]} \), \( u \in U \) be the fixed point family from Lemma 3.1. Suppose that for every \( 0 < s < r \leq r_0 \) and \( t \in [0, r-s] \), the map

\[
u \in U \mapsto L_{s,u}j_{s,r} \in L(X_r, X_s)
\]

is \( C^t \). Then, the mappings taking \( u \in U \) to:

\[
f_s(u) \in X_s, \quad \Lambda_u(f) \in L^\infty(\Omega, \mathbb{R}_+) \quad \text{and} \quad \int_\Omega \log \Lambda_u(f) d\mathbb{P}(\omega)
\]

are all \( C^t \) for \( 0 < t < r_0 - s \). In particular, it follows that the last expression (the characteristic exponent) is \( C^{r_0-s} \) for any \( s > 0 \).

The proof will take the rest of Sect. 3 and is divided into the following steps taking us through our scale of Banach spaces \( X_r, r \in (0, r_0] \):

1. First, by contraction and transversality we show Hölder continuity of \( f_s \).
2. This continuity will then allow us to ‘bootstrap’ and show differentiability of \( u \mapsto f_s \) when \( 0 < s < r_0 - 1 \).
3. Finally, a recursive argument will yield higher order differentiability as well as the optimal Hölder exponent of the last derivative.

Step one, yielding Hölder-continuity, is done through

**Lemma 3.3.** Let \( 0 < s < r_0 \). Then \( u \in U \mapsto f_s(u) \in X_s \) is \( \alpha \)-Hölder continuous for \( \alpha = 1 \wedge (r_0 - s) \).

**Proof.** For \( n \geq 1 \) we make use of the following telescopic identity, putting the downgrading at the right place:

\[
j_{s,r_0}(\mathcal{L}^n_{r_0,u+h} - \mathcal{L}^n_{r_0,u}) = \sum_{k=0}^{n-1} \mathcal{L}^k_{s,u+h} \left( j_{s,r_0}(\mathcal{L}^n_{r_0,u+h} - \mathcal{L}^n_{r_0,u}) \right) \mathcal{L}^{n-k-1}_{r_0,u} \in L(X_{r_0}; X_s).
\]

The middle term is \( \alpha \)-Hölder from the assumptions in Theorem 3.2 and the other factors are bounded, so \( u \mapsto j_{s,r_0} \mathcal{L}^n_{r_0,u} \) is \( \alpha \)-Hölder. When \( \varphi_{r_0} \in \mathcal{C}_{r_0,\ell_{r_0}=1} \), the same is true for \( \Lambda_{u}^{(n)}(\varphi_{r_0}) = (\ell_{s}, j_{s,r_0} \mathcal{L}^n_{r_0,u} \varphi_{r_0}) \) and since the section is uniformly bounded from below, also \( u \in U \mapsto j_{s,r_0} \pi^n_{r_0,u} (\varphi_{r_0}) \) is \( \alpha \)-Hölder. For \( h \in B \) small (so that \( u + h \in U \)) we get by (A.5) and (A.8) in the appendix:

\[
\| f_s(u + h) - f_s(u) \|_{X_s} = \| \pi^n_{s,u+h} (f_s(u + h)) - \pi^n_{s,u} (f_s(u)) \|_{X_s}
\leq \| \pi^n_{s,u+h} (f_s(u + h)) - \pi^n_{s,u} (f_s(u + h)) \|_{X_s} + \| \pi^n_{s,u} (f_s(u + h)) - \pi^n_{s,u} (f_s(u)) \|_{X_s}
\leq \| j_{s,r_0} \pi^n_{r_0,u+h} (f_{r_0}(u)) - j_{s,r_0} \pi^n_{r_0,u} (f_{r_0}(u)) \|_{X_s} + \frac{K}{\rho} \eta^n \| f_s(u + h) - f_s(u) \|_{X_s}.
\]

Now, fix a value of \( n \) such that \( \frac{K}{\rho} \eta^n \leq \frac{1}{2} \) to obtain

\[
\| f_s(u + h) - f_s(u) \|_{X_s} \leq 2 \| j_{s,r_0} \pi^n_{r_0,u+h} (f_{r_0}(u)) - j_{s,r_0} \pi^n_{r_0,u} (f_{r_0}(u)) \|_{X_s}.
\]

And here the RHS is \( \alpha \)-Hölder continuous as we showed above. \( \square \)

### 3.1. Differentiability and beyond.

For the second step we will base our proof on arguments already given in [50]. For \( s \in (0, r_0) \), \( \varphi_s \in \mathcal{C}_{s,\ell_s=1} \) and \( z_s \in X_s \) we set:

\[
Q_s(u)(\varphi_s) \cdot z_s := \partial \varphi \pi_{s,u}(\varphi_s) \cdot z_s = \Lambda_u(\varphi_s)^{-1} \left( \mathcal{L}_{s,u} z_s - \Lambda u(z_s) \pi_{s,u}(\varphi_s) \right). \tag{3.1}
\]

The fact that this is well-defined follows from Lemma 2.3 where it is also shown that \( 1 - Q_s(u)(\varphi_s) \in L(X_s) \) is invertible with the bound \( \| (1 - Q_s(u))^{-1} \|_{L(X_s)} \leq 1 + \frac{K_s}{\rho_s} \frac{1 - \ell_s}{1 - \ell_{s_2}} \) and that for \( \| z_s \|_s < \rho_s/2\partial K \):

\[
\| \pi_{s,u}(\varphi_s + z_s) - \pi_{s,u}(\varphi_s) - Q_s(u) \cdot z_s \|_s \leq C(\| z_s \|_s^2).
\]

with \( C = C(\rho_s, \partial_s, K_s) < +\infty \). When downgrading the regularity we recover regularity as stated in the following

**Proposition 3.4.** Given \( 0 < s < r \leq r_0, 0 \leq t < r - s \) we have for \( \varphi_r \in \mathcal{C}_{r,\ell_r=1} \) that

\[
u \in U \mapsto j_{s,r} \pi_{r,u}(\varphi_r) \in X_s \quad \text{and} \quad u \in U \mapsto j_{s,r} Q_{r,u}(\varphi_r) \in L(X_r; X_s)
\]

are both \( C' \).
Proof. By hypothesis, the map \( u \in U \mapsto j_{s,r}^* \mathcal{L}_{r,u} \in L(X_r, X_z) \) is \( C^t \). By the uniform bounds \( u \in U \mapsto \Lambda_u(\varphi_r) := (\ell_s, j_{s,r}^* \mathcal{L}_{r,u} \varphi_r) \) is a \( C^t \) section over \( \Omega \) taking values in \( \left[ \frac{\partial}{\partial K}, \vartheta K \right] \subset (0, +\infty) \). So the same is true for its (fiber-wise) inverse \( (\Lambda_u(\varphi_r))^{-1} \) and then also for the fractional linear bundlemap \( \pi_{s,u}(j_{s,r}^*) = j_{s,r}^* \pi_{r,u} \varphi_r = \Lambda_u(\varphi_r)^{-1} j_{s,r}^* \mathcal{L}_{r,u} \varphi_r \). From the expression (3.1) for \( j_{s,r}^* Q_{s,u} = Q_{s,u} j_{s,r}^* \) we see that each term is \( C^t \) and the result follows from the Leibniz principle. \( \square \)

**Lemma 3.5.** Let \( 0 < s < r - 1 \leq r_0 - 1 \) and \( \varphi_r \in \mathcal{C}_{r,t}, t = 1 \). Then

\[
j_{s,r}^* (\pi_{r,u+h}(\varphi_r) - \pi_{r,u}(\varphi_r)) = P_{s,r,u}(\varphi_r) h + O_{s,r} \left( |h|^{1+\alpha} \right)
\]

(3.2)

with \( \alpha = (r - s - 1) \wedge 1 \) and

\[
P_{s,r,u}(\varphi_r) = \partial_u \left( j_{s,r}^* \pi_{r,u}(\varphi_r) \right) = \Lambda_u(\varphi_r)^{-1} \left( \partial_u (j_{s,r}^* \mathcal{L}_{r,u} \varphi_r) - (\partial_u \Lambda_u(\varphi_r)) \pi_{s,u}(\varphi_r) \right)
\]

(3.3)

Furthermore, the map \( u \in U \mapsto P_{s,r,u}(\varphi_r) \in L(\mathcal{B}, X_z) \) is \( C^t \) for \( t \in [0, r - s - 1) \).

Proof. From the hypothesis in Theorem 3.2 we have that \( j_{s,r}^* \mathcal{L}_{r,u} \in L(X_r, X_z) \) is \( C^{1+\alpha} \). So applying the MVT (recall that \( U \) is convex) we obtain:

\[
\| \mathcal{L}_{s,u+h} \varphi_s - \mathcal{L}_{s,u} \varphi_s - h \cdot \partial_u (j_{s,r}^* \mathcal{L}_{r,u}) \varphi_r \|_{X_s} \leq \sup_{0 \leq t \leq 1} |h|_\mathcal{B} \| \partial_u j_{s,r}^* (\mathcal{L}_{r,u+h} - \mathcal{L}_{r,u}) \| \| \varphi_r \|_{X_r} = O_{s,r} \left( |h|^{1+\alpha} \right) \| \varphi_r \|_{X_r}.
\]

Since \( \partial_u \Lambda_u(\varphi) = \partial_u (\Lambda_{s,u}(j_{s,r}^* \varphi_r)) \) we also have the expansion:

\[
\Lambda_{u+h}(\varphi) - \Lambda_u(\varphi) - h \cdot (\partial_u \Lambda_u(\varphi)) = O_{s,r} \left( |h|^{1+\alpha} \right).
\]

As \( \varphi \) is in the normalized cone, \( \Lambda_u(\varphi)^{-1} \) is bounded and (3.2) now follows from expanding the LHS using the two previous estimates, giving the wanted form for \( P_{s,r,u}(\varphi_r) \).

Finally, from the hypothesis in 3.2, \( \partial_u (j_{s,r}^* \mathcal{L}_{r,u}) \varphi_r \) is \( C^t \) for \( t \in [0, r - s - 1) \), whence also \( \partial_u \Lambda_u(\varphi) \) and the claim follows. \( \square \)

**Theorem 3.6** (cf. Theorem 1 in [50]). Let \( 0 < s < r - 1 \). Then \( u \in U \mapsto \mathfrak{f}_s(u) \in X_s \) is differentiable. The derivative is given by:

\[
D_u \mathfrak{f}_s(u) = (1 - Q_{s,u}(\mathfrak{f}_s(u)))^{-1} P_{s,r_0,u}(\mathfrak{f}_{r_0}(u)) \in L(\mathcal{B}, X_z),
\]

(3.4)

Proof. Fix \( u \) in \( U \) and set \( \varphi_t = \mathfrak{f}_t(u) \) (all \( 0 < t \leq r_0 \)). For \( h \in \mathcal{B} \) small (in particular, \( u + h \in U \)) we set \( z_t = z_t(h) = \mathfrak{f}_t(u + h) - \mathfrak{f}_t(u) \). From the fixed point property, consistency and the above estimates we deduce:

\[
z_s = \pi_{s,u+h}(\varphi_s + z_s) - \pi_{s,u}(\varphi_s)
\]

\[
= \pi_{s,u+h}(\varphi_s + z_s) - \pi_{s,u+h}(\varphi_s) + j_{s,r_0} \left( \pi_{r_0,u+h}(\varphi_{r_0}) - \pi_{r_0,u}(\varphi_{r_0}) \right)
\]

\[
= Q_{s,u+h}(\varphi_s) z_s + O_s(\|z_s\|^2) + P_{s,r_0,u}(\varphi_{r_0}) h + O_{s,r_0}(|h|^{1+\alpha})
\]

By Lemma 3.3 and since \( s < r_0 - 1 \) we already know that \( z_s = O_s(|h|) \) so the term \( O_s(\|z_s\|^2) \) can be neglected. We still need to convert \( Q_{s,u+h} \) into \( Q_{s,u} \). For \( s + \alpha < t < r_0 - 1 \) we have \( z_t(h) = O_t(|h|) \) and then

\[
(Q_{s,u+h}(\varphi_s) - Q_{s,u}(\varphi_s)) z_s = j_{s,t} (Q_{t,u+h}(\varphi_t) - Q_{t,u}(\varphi_t)) z_t(h)
\]

\[
= O_{s,t}(|h|^\alpha) \times O_t(|h|) = O_{s,t}(|h|^{1+\alpha}),
\]
where we have inserted a downgrading operator to exploit $\alpha$-Hölder continuity from the second term (Proposition 3.4) and the Lipschitz continuity shown in (3.3) at the fixed point $\varphi_z$ for the first term. Thus, $z_{s} = Q_{s,u}(\varphi_{z})z_{s} + P_{s,r_{0},u}(\varphi_{r_{0}})h + O_{s,r_{0}}(|h|^{1+\alpha})$ and using invertibility of $1 - Q_{s,u}$ we get:

$$z_{s} = (1 - Q_{s,u}(\varphi_{z}))^{-1}(P_{s,r_{0},u}(\varphi_{r_{0}})h) + O_{s,r_{0}}(|h|^{1+\alpha}).$$

This shows differentiability of $f_{s}(u)$ with the stated formula for the derivative. □

For higher order regularity, we will bootstrap through the formula (3.4). Showing that the RHS has some regularity means that $f_{s}(u)$ has the same regularity plus one. Obtaining a $C^{1,\alpha}$-regularity this way is not very hard, using equicontinuity and telescoping as above. But already for $C^{2}$ the algebra starts getting quite involved. To treat the general case, we have developed what to us appears to be a new algebraic differentiation tool: A graded Leibniz principle. This allows us to use a simple inductive argument. A detailed description of the notation involved and the relevant technical lemmas have been relegated to “Appendix D”. Our inductive hypothesis is the following:

**Hypothesis 3.7.** For $\gamma \in (0, r_{0}]$ we let $\mathcal{H}(\gamma)$ denote the following property: $\forall 0 < s < r_{0}$ and $0 \leq t < \gamma \wedge (r_{0} - s)$, the map: $u \mapsto f_{s}(u) = j_{s,r_{0}}f_{r_{0}}(u) \in X_{s}$ is $C^{t}$.

What we have shown above is that $\mathcal{H}(1)$ holds and that in addition when $0 < s < r_{0} - 1$ then $u \mapsto f_{s}(u)$ is differentiable with a derivative given by formula (3.4). For $0 < s < r_{0} - 1$, by equivariance, the expression $P_{s,u}(f(u)) = P_{s,r,u}(f_{r}(u))$ is independent of $r \in (s + 1, r_{0}]$.

**Lemma 3.8.** Under hypothesis $\mathcal{H}(\gamma)$ with $\gamma \in (1, r_{0}]$:  

1. The map: $u \mapsto P_{s,u}(f(u)) \in L(B; X_{s})$ is $C^{t}$ for $0 \leq t < \gamma \wedge (r_{0} - s - 1)$.
2. The map: $u \mapsto (1 - Q_{s,u}(f_{s}(u)))^{-1}j_{s,r} \in L(X_{s}; X_{s})$ is $C^{t}$ for $0 \leq t < \gamma \wedge (r - s)$.

The proof of this lemma is given at the end of “Appendix D”.

**Proof of Theorem 3.2.** Write $r_{0} = k_{0} + \alpha_{0}$ with $k_{0} \in \mathbb{N}$ and $\alpha_{0} \in (0, 1]$. By the above $\mathcal{H}(\alpha_{0})$ is satisfied. We will show that when $k \in \{0, 1, \ldots, k_{0} - 1\}$ then hypothesis $\mathcal{H}(k + \alpha_{0})$ implies $\mathcal{H}(k + 1 + \alpha_{0})$. This will prove that $\mathcal{H}(r_{0})$ holds which is precisely the conclusion of the theorem.

Thus, assume that we have shown that $\mathcal{H}(k + \alpha_{0})$ holds for some $k \in \{0, 1, \ldots, k_{0} - 1\}$. Let $\epsilon > 0$ be small, $r = r_{0} - 1 - \epsilon$ and $0 < s < r$. Setting $R_{s,u} := \left(1 - Q_{s,u}(f_{s}(u))\right)^{-1} \in L(X_{s})$ we may write

$$D_{u}f_{s}(u) = R_{s,u}(f_{s}(u))j_{s,r}P_{r,u}(f(u)) \in L(B, X_{s}).$$

Here, in terms of “Appendix D”, $M_{s,r}^{1}(u) = R_{s,u}(f_{s}(u))j_{s,r}P_{r,u}(f(u))$ defines an equivariant $(\gamma, 0)$-regular family of operators and $N_{r}(u) := j_{s,r}P_{r,u}(f(u))$ defines a left-equivariant $(\gamma - 1, 1)$-regular family. By the graded Leibniz principle, Proposition D.5, the above product is $(\gamma - 1, 1)$-regular so in particular $u \mapsto D_{u}f_{s}(u)$ is $C^{t}$ for $0 \leq t < \gamma \wedge (r - s - 1)$. Therefore, $u \mapsto f_{s}(u)$ is $C^{t}$ for $0 \leq t < (\gamma \wedge (r - s)) + 1 = (\gamma + 1) \wedge (r_{0} - s - \epsilon)$. We may here let $\epsilon$ go to zero and conclude that $\mathcal{H}(\gamma + 1)$ holds. As already mentioned this implies $\mathcal{H}(r_{0})$, thus concluding the proof. □
4. Dynamical Applications

We illustrate the above results through applications to dynamical systems. Let \((M, g)\) be an \(n\)-dimensional \(\mathcal{C}^\infty\) connected Riemannian manifold without boundaries and of bounded diameter. Also let \(\lambda > 1\) be a fixed expansion constant. Further geometric conditions on an atlas for \(M\) will be given below. We will work with parametrized (Bochner)-measurable families of \(\mathcal{C}^r\) uniformly \(\lambda\)-expanding maps on \(M\) and a continuous scale of Hölder spaces \((\mathcal{C}^s(M))_{0 < r \leq r_0}\) for some \(r_0 > 1\). We consider parameters taking values in an open non-empty convex subset \(U\) of a Banach space \(B\). Our strategy will be to exhibit a regular Birkhoff cone in every \(\mathcal{C}^s\) space which is contracted in a strict and uniform way by the transfer operators associated with the expanding maps. This will allow us to construct the fixed point \(f_y \in L^\infty(\Omega, C^s(M))\), \(u \in U\) of a transfer operator cocycle, and then to study regularity with respect to \(u \in U\) of the stationary measure and the top characteristic exponent with the tools of the previous sections.

When \(M\) is a torus or an interval (as in our examples in the introduction) the tangent bundle is trivializable, which simplifies the discussion when dealing with expanding measure and the top characteristic exponent with the tools of the previous sections.

We illustrate the above results through applications to dynamical systems. Let

Hypothesis 4.1. There exists a \(\mathcal{C}^\infty\)-atlas \(\mathcal{A}_M = \{(V_j, \eta_j) : j \in \mathcal{J}\}\) for \((M, g)\) with the following properties:

1. The atlas consists of simply connected charts and is uniformly \(\mathcal{C}^\infty\). By ‘uniformly’ we mean that there is a sequence \((c_k)_{k \geq 1}\) of positive reals so that every transition map between overlapping charts (thus a map between open sets in Banach spaces):

\[
\eta_j \circ \eta_m^{-1} : \eta_m(V_j \cap V_m) \to \eta_j(V_j \cap V_m) \quad (\subset \mathbb{R}^n), \quad j, m \in \mathcal{J}
\]

has a \(\mathcal{C}^k\) norm bounded by \(c_k\).
2. There is \(\delta_0 > 0\) so that \(\forall y \in M\) the set \(\mathcal{J}(y) := \{j \in \mathcal{J} : B(y, \delta_0) \subset V_j\}\) is non-empty.
3. Each chart \(V_j = \eta_j(V_j), j \in \mathcal{J}\) is convex in \(\mathbb{R}^n\).
4. There is \(L < +\infty\) so that \(\forall j \in \mathcal{J}, x, x' \in V_j\), \(x_j = \eta_j(x), x'_j = \eta_j(x'):\)

\[
\frac{1}{L} \|x_j - x'_j\|_{\mathbb{R}^n} \leq d_M(x, x') \leq L\|x_j - x'_j\|_{\mathbb{R}^n}.
\]

In the following we consider a fixed atlas verifying the above conditions and we fix a value for the Lebesgue number \(\delta_0 > 0\). We set \(\delta_1 = \delta_0/3\).

Remark 4.2. 1. Our functional analysis will be built with respect to this fixed atlas, but our main results are independent of the choices made (though constants in estimates may change). We also note that the existence of an atlas verifying the above is automatic when \(M\) is a compact \(\mathcal{C}^\infty\) manifold without boundary.

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4 The manifold could also be modeled over any Hilbert space. Finite dimension is however used when dealing with physical measures.

5 As shown by Gromov in [35] any expanding self-map of a compact manifold is topologically conjugate to an infra-nil-endomorphism.
2. The convexity condition 4.1.4.1 on the $V_j$’s may be relaxed, but to find a good geometric condition is complicated, notably for the result C.4 in the Appendix to hold. We refer to e.g. De La Llave and Obaya in [15, section 6] for sufficient conditions in this direction. We have here opted for the simplest hypothesis.

As mentioned above $U \subset \mathcal{B}$ is a non-empty open convex subset (our parameter space) in a Banach space. Then $U \times M$ inherits a manifold structure with an atlas consisting of charts $(U \times V_j, \hat{\eta}_j)_{j \in \mathcal{J}}$ in which

$$\hat{\eta}_j : (u, y) \in U \times V_j \mapsto (u, \eta_j(y)) \in U \times V_j.$$ 

Throughout this section we equip $\mathbb{R}^n$ with the Euclidean metric and $B \times \mathbb{R}^n$ with the norm $\|(u, x)\| = |u|_B \lor |x|_{\mathbb{R}^n}$ with “$\lor$” meaning max.

Let $k \geq 0$ be an integer and $\alpha \in (0, 1]$. Let $Z$ be a Banach space and for $j \in \mathcal{J}$ let $\phi_j : U \times V_j \to Z$ be a $C^k$ map (thus from an open convex subset of a Banach space into a Banach space). We then define (Fréchet) derivatives of $\phi_j$ in the usual way and for $0 \leq q \leq k$ the associated semi-norm of $\phi_j$:

$$p^{(q)}(\phi_j) = \sup\{|D^q \phi_j (u, x)| : u \in U, x \in V_j\}$$

Recall that $\delta_1 = \delta_0/3$ (Hypothesis 4.1.4.1). We will define a $\delta_1$-local Hölder-continuity of $D^k \phi_j$ in the chart $U \times V_j$. For $\xi = (u, x_j), \xi' = (u', x'_j) \in U \times V_j$ we set:

$$d_j(\xi, \xi') = |u - u'|_B \lor |x_j - x'_j|_{\mathbb{R}^n}.$$ 

We say that $D^k \phi_j$ is $(\alpha; \delta_1)$-Hölder continuous if there is $C < +\infty$ so that: $\forall \ \xi, \xi' \in U \times V_j$ with $d_j(\xi, \xi') \leq \delta_1$:

$$\|D^k \phi_j(\xi) - D^k \phi_j(\xi')\| \leq C \ d_j(\xi, \xi')^{\alpha}. \quad (4.1)$$

We let $h^{(k, \alpha)}(\phi_j) := h^{(k, \alpha)}_{\delta_1, d_j}(\phi_j)$ be the smallest constant $C$ for which (4.1) holds and define the norm:

$$\|\phi_j\|_{(k, \alpha)} := \|\phi_j\|^{\delta_1, d_j}_{(k, \alpha)} = \max_{0 \leq q \leq k} p^{(q)}(\phi_j) \lor h^{(k, \alpha)}_{\delta_1, d_j}(\phi_j). \quad (4.2)$$

As shown in Lemma 4.14 making another choice of $\delta_1$ and the metric $d_j$, if equivalent to the above, will lead to an equivalent $\|\cdot\|_{(k, \alpha)}^{\delta_1, d_j}$-norm. Unless explicitly stated otherwise $\delta_1$ and the metric $d_j$ are as above and will be omitted in the notation for the Banach spaces and the norms.

**Definition 4.3.** We let $C^{(k, \alpha)}(U \times V_j; Z)$ denote the space of functions $\phi_j : U \times V_j \to Z$ for which the norm (4.2) is finite. $(C^{(k, \alpha)}(U \times V_j; Z), \|\cdot\|_{(k, \alpha)})$ is a Banach space.

A function on the parametrized manifold, $\Phi : U \times M \to Z$, induces a collection $(\phi_j)_{j \in \mathcal{J}}$ of partial maps on charts: for each $j \in \mathcal{J}$, $\phi_j := \Phi \circ (\hat{\eta}_j)^{-1} : U \times V_j \to Z$. This leads to the following:
**Definition 4.4.** The \((k, \alpha)\)-norm of \(\Phi : U \times M \rightarrow Z\) is defined as:

\[
\| \Phi \|_{(k, \alpha)} = \sup_{j \in \mathcal{J}} \| \phi_j \|_{(k, \alpha)}.
\]

In the following, \(C^{(k, \alpha)}(U \times M; Z)\) denotes the Banach space of functions \(\Phi\) for which the above norm is finite, and \(c^{(k, \alpha)}(U \times M; Z)\) denotes the closure of \(C^\infty\) functions in \(C^{(k, \alpha)}(U \times M; Z)\).

\(C^{(k, \alpha)}(M; Z)\) and \(c^{(k, \alpha)}(M; Z)\) are defined in the same way (omitting \(U\) in the construction).

**Remark 4.5.** The above construction yields the following description of \(C^{(k, \alpha)}(U \times M)\) (and similarly for the other spaces in the above definition): A family \((\phi_j)_{j \in \mathcal{J}}\) with each \(\phi_j \in C^{(k, \alpha)}(U \times V_j)\) corresponds to a (unique) element \(\Phi \in C^{(k, \alpha)}(U \times M)\) if and only if we have the following identification:

\[
\forall j, m \in \mathcal{J}, \ u \in U, \ y \in V_j \cap V_m : \phi_j(u, \eta_j(y)) = \phi_m(u, \eta_m(y)).
\]

Denoting this identification by \(\sim\), we may write:

\[
C^{(k, \alpha)}(U \times M) \simeq \left( \prod_{j \in \mathcal{J}} C^{(k, \alpha)}(U \times V_j) \right) / \sim.
\]

We also want to define a local \(C^{(k, \alpha)}\)-structure (and a \(c^{(k, \alpha)}\)-structure) for parametrized locally defined maps of \(M\) into itself which satisfies a uniform Lipschitz condition. Again we shall do this in terms of our fixed atlas.

More precisely, for each \(j \in \mathcal{J}\) consider maps of the type \(\vartheta : U \times V_j \rightarrow M\) that verifies the following uniform Lipschitz-condition:

\[
d_M(\vartheta(u, y), \vartheta(u', y')) \leq |u - u'|_B + \frac{1}{\lambda} d_M(y, y') \quad (4.3)
\]

for all \((u, y), (u', y') \in U \times V_j\). Thus, the map is \(1/\lambda\)-contracting in the manifold variable and 1-Lipschitz in the \(u\)-variable. More generally, it suffices to assume that the map is locally Lipschitz in the \(u\)-variable. But by restricting to a smaller subset of \(U\) (our results are only local with respect to \(U\)) and scaling the norm in \(B\) we may always reduce to the above case. Given two maps \(\vartheta, \widehat{\vartheta} : U \times V_j \rightarrow M\) we define their uniform distance to be:

\[
d_0(\vartheta, \widehat{\vartheta}) = \sup_{\xi \in U \times V_j} d_M(\vartheta(\xi), \widehat{\vartheta}(\xi)).
\]

The following lemma will allow us to compare in a finer way uniformly close maps:

**Lemma 4.6.** For \(j \in \mathcal{J}\) let \(\vartheta, \widehat{\vartheta} : U \times V_j \rightarrow M\) verify (4.3) and \(d_0(\vartheta, \widehat{\vartheta}) \leq \delta_1\). Then for any \(\xi_0 = (u_0, y_0) \in U \times V_j\) and \(m \in \mathcal{J}(\vartheta(\xi_0)) \cup \mathcal{J}(\widehat{\vartheta}(\xi_0))\) we have:

\[
\vartheta, \widehat{\vartheta} : B_B \times V_j(\xi_0, \delta_1) \rightarrow B_M(\vartheta(\xi_0), \delta_0) \cap B_M(\widehat{\vartheta}(\xi_0), \delta_0) \subset V_m.
\]

**Proof.** \(\vartheta, \widehat{\vartheta}\) are 2-Lipschitz and \(d_0(\vartheta, \widehat{\vartheta}) \leq \delta_1\). So the result follows from \(0 < \delta_1 < \delta_0/3\) and Hypothesis 4.1 on the atlas. \(\Box\)
Thus, when \( \vartheta, \hat{\vartheta} \) are uniformly close, we may go to the same local coordinates in the source and in the image. Let us write \( \psi_j := \vartheta \circ (\tilde{\eta}_j)^{-1} : U \times V_j \to M \) and \( \hat{\psi}_j := \hat{\vartheta} \circ (\tilde{\eta}_j)^{-1} : U \times V_j \to M \) when expressing the above two maps in local coordinates in the source (but not the image).

Given \( \xi_0 = (u_0, y_0) \in U \times V_j \) (thus in local coordinates) let us write
\[
B_j(\xi_0) = \{(u, y) \in U \times V_j : d_j((u, y), (u_0, y_0)) < \delta_1\}
\]
for the product ball in charts. For any \( m \in \mathcal{J}(\psi_j(\xi_0)) \cup \mathcal{J}(\hat{\psi}_j(\xi_0)) \) we may go to local coordinates also in the image if we restrict the domain to \( B_j(\xi_0) \) and define \( \psi_m, j = (\eta_m \circ \psi_j)|B_j(\xi_0) \) and \( \hat{\psi}_m, j = (\eta_m \circ \hat{\psi}_j)|B_j(\xi_0) \). Then \( \psi_m, j, \hat{\psi}_m, j : B_j(u_0, y_0) \to V_m \subset \mathbb{R}^n \) are local maps between (the same) convex subsets of normed vector spaces, whence may be compared.

Given \( \vartheta, \hat{\vartheta} : U \times V_j \to M \), or equivalently, \( \psi_j, \hat{\psi}_j : U \times V_j \to M \) for which \( d_0(\psi, \hat{\psi}) \leq \delta_1 \) we define their \((k, \alpha)\)-distance:
\[
d_j^{(k, \alpha)}(\psi_j, \hat{\psi}_j) := \sup_{\xi \in U \times V_j} \sup_{m \in \mathcal{J}(\psi_j(\xi)) \cup \mathcal{J}(\hat{\psi}_j(\xi))} \| \psi_m, j - \hat{\psi}_m, j \|_{C^{(k, \alpha)}(B_j(\xi); V_m)},
\]
with local maps defined as in the previous paragraph. When \((k, \alpha) = (0, 1)\) that’s it. For \( k \geq 1 \), however, we also introduce the following ‘gauge’ on \( \vartheta \) (or equivalently, \( \psi_j \)):
\[
P_j^{(k, \alpha)}(\psi_j) := \sup_{\xi \in U \times V_j} \sup_{m \in \mathcal{J}(\psi_j(\xi))} \| D\psi_m, j \|_{C^{(k-1, \alpha)}(B_j(\xi); V_m)}.
\]
We denote by \( C^{(k, \alpha)}(U \times V_j; M), d_j^{(k, \alpha)} \) the metric space of maps verifying (4.3) and of finite gauge in the case \( k \geq 1 \). Again \( c^{(k, \alpha)}(U \times V_j; M) \) is the subspace of maps which may be uniformly approximated by \( C^\infty \) functions.

### 4.1. Uniformly expanding maps and associated weights

In order to specify regularity of an expanding map, e.g. for the examples in the introduction, it is convenient to do so for its inverse branches rather than the map itself. The following definition may look like an overkill but it allows for countably many branches and the presence of (certain types of) singularities.

**Definition 4.7.** Let \( \mathcal{A} \) be a finite or countable index set. For each \( j \in \mathcal{J} \) let \( \Psi_j = \{ \vartheta_i^j : i \in \mathcal{A} \} \) be a family of maps \( \vartheta_i^j : U \times V_j \to M, i \in \mathcal{A} \) satisfying (4.3).

1. We say that two such families \( \Psi_j \) and \( \Psi_m \) are compatible if \( \forall u \in U, y \in V_j \cap V_m \), there exist \( \sigma \in \text{Aut}(\mathcal{A}) \) and an open neighborhood \( W \subset U \times (V_j \cap V_m) \) of \((u, y)\) such that
\[
\forall i \in \mathcal{A} : \vartheta_i^j = \vartheta_m^{\sigma(i)} \text{ on } W.
\]
2. Let the collection \( \Psi = (\Psi_j)_{j \in \mathcal{J}} \) consist of compatible families. For any given \((u, y) \in U \times M\) there is \( j \in \mathcal{J} \) so that \( y \in V_j \). By compatibility the set \( D(u, y) := \{ \vartheta_i^j(u, y) : i \in \mathcal{A} \} \) is then independent of the choice of \( j \). The collection is said to be separating if for every \( u \in U\):
(a) \( D(u, y) \cap D(u, y') = \emptyset \) whenever \( y \neq y' \in M \) and
(b) The map \( i \in \mathcal{A} \mapsto \vartheta_i^j(u, y) \in M \) is injective for every \( j \in \mathcal{J} \) and \( y \in \mathcal{V}_j \).

For fixed \( u \in U \) the definition is designed so that \( G_u = \bigcup_{y \in M} D(u, y) \times \{y\} \subset M \times M \) is the graph of a map \( T_u \) over some domain.

Consider the two natural projections \( \pi_1 : G_u \rightarrow D_u := \bigcup_{y \in M} D(u, y) \) and \( \pi_2 : G_u \rightarrow M \). By separability, the first is injective so indeed \( G_u \) is the graph of the map \( T_u = \pi_2 \circ \pi_1^{-1} : D_u \rightarrow M \), defined on the domain \( D_u \). By construction, it is a covering map of \( M \) of degree \( \text{Card} \mathcal{A} \) which is locally \( \lambda \)-expanding. For each \( j \in \mathcal{J} \) the family \( \psi_j^i : U \times V_j \rightarrow M \), \( i \in \mathcal{A} \) is precisely the (parametrized) \( \frac{1}{\lambda} \)-contracting inverse branches of \( T_u \) on the chart \( \mathcal{V}_j \).

Consider a compatible and separating collection of maps \( \Psi \) and the associated dynamical system \( T_u : D_u \rightarrow M \) from the above construction. As before we write \( \psi_j^i := \vartheta_j^i \circ (\hat{\pi}_j)^{-1} : U \times V_j \rightarrow M \) with the source expressed in local coordinates. For \( k \geq 1 \) we now make a \((k, \alpha)\)-regularity assumption on the branches: We require that there is a constant \( k_T < +\infty \) so that our gauges, cf. (4.5), are all uniformly bounded:

\[
\forall i \in \mathcal{A}, \ j \in \mathcal{J} : p_j^{(k, \alpha)}(\psi_j^i) \leq k_T.
\] (4.6)

We will also associate a weight (not to be confused with the metric tensor) \( g_u : D_u \rightarrow \mathbb{R} \) defined on the domain of \( T_u \). The most convenient way to specify regularity of the weight \( g \) is through its composition with inverse branches. Again we do so using the charts in our fixed atlas. So with \( j \in \mathcal{J} \) and \( \psi_j^i : U \times V_j \rightarrow M \) being a local inverse branch we define \( g_j^i(u, y) := g(u, \psi_j^i(u, y)) \) for \( (u, y) \in U \times V_j \).

We will assume that each \( g_j^i \in c^{(k, \alpha)}(U \times V_j; \mathbb{R}) \) and that when \( k \geq 1 \) there is a constant \( k_g < +\infty \) so that the derivative verifies the uniform bound:

\[
\|D(g_j^i)\|_{(k-1, \alpha)} \leq k_g, \ i \in \mathcal{A}, \ j \in \mathcal{J}.
\] (4.7)

Finally, we require that there is a uniform constant \( 1 \leq k_\Sigma < +\infty \) so that for \( (u, y) \in U \times M \):

\[
\frac{1}{k_\Sigma} \leq \sum_{x:T_u(x)=y} e^{g(u,x)} \leq k_\Sigma.
\] (4.8)

**Remark 4.8.** 1. Below we shall make use of the following observation: Let \( \xi_0 = (u_0, y_0) \in U \times V_j \). Each \( g_j^i \) is by hypothesis \( k_g \)-Lipschitz and if \( \xi = (u, y) \in B_j(\xi) \) then \( d_j(\xi, \xi_0) \leq \delta_1 \). Thus,

\[
\sum_{i \in \mathcal{A}} \sup_{B_j(\xi_0)} e^{g_j^i} \leq k_\Sigma e^{\delta_1 k_g}.
\] (4.9)

2. Finally, note that when \( \mathcal{A} \) is infinite, (4.8) implies that \( g \) itself can not be uniformly bounded. This is why we (have to) use the derivative of \( g_j^i \) in the regularity condition (4.7).
4.2. Pairs of maps and weights. We fix in the following a real value of \( r_0 > 1 \). We set \( r_0 = (k_0, \alpha_0) \) with \( (k_0, \alpha_0) \in \mathbb{N} \times (0, 1] \) and \( \delta_1 > 0 \) as in the previous section. We also choose arbitrary but fixed values of the associated constants \( k_T, k_g, k_{\Sigma} \) as described in (4.6), (4.7) and (4.8). We consider the collection \( \mathcal{P} = \mathcal{P}_{k_T, k_g, k_{\Sigma}} \) of \( e^{(k_0,\alpha_0)} \) pairs \( (T, g) \) with \( T \) being a parametrized family of \( \lambda \)-expanding covering maps and \( g \) is an associated weight as constructed above, for which the above regularity conditions are verified with the afore-mentioned constants.

We define a \((k, \alpha)\)-distance \( d_{\mathcal{P}} \) between pairs \((T, g), (\hat{T}, \hat{g}) \in \mathcal{P}\) in the following way: First, if the two covering maps \( T, \hat{T} \) have different degrees then the distance is infinity.

If they have the same degree we let \( \mathcal{A} \) be a common index set and consider for each \( j \in J \) the two sets of inverse branches (expressed in local coordinates) \( \psi_j^i : U \times V_j \to M \) and \( \hat{\psi}_j^i : U \times V_j \to M \), \( i \in \mathcal{A} \). We will compare these up to a (simultaneous) permutation of branches and weights. Let \( P \) be the subset of permutations \( \sigma \in \text{Aut}(\mathcal{A}) \) for which \( d_0(\psi_j^i, \hat{\psi}_j^i) \leq \delta_1 \) for all \( i \in \mathcal{A} \).

For \( \sigma \in P \) we set:

\[
d_{j,\sigma} = \sup_{i \in \mathcal{A}} \left\{ d_{j}^{(k,\alpha)}(\psi_j^i, \hat{\psi}_j^i) + \| \psi_j^i - \hat{\psi}_j^i \|_{(k,\alpha)} \right\}
\]

and

\[
d_{\mathcal{P}}((T, g), (\hat{T}, \hat{g})) = \sup_{j \in J} \inf_{\sigma \in P} d_{j,\sigma}.
\] (4.10)

If \( P \) is empty the distance is infinity. This provides a suitable notion of distance on our collection of pairs in \( \mathcal{P} \).

**Lemma 4.9.** Let \( 0 < s \leq r_0 \) and write \( s = (k, \alpha) \) with \( s = k + \alpha \), \( k \in \mathbb{N} \) and \( \alpha \in (0, 1] \). To every pair \( (T, g) \in \mathcal{P} \) there is a well-defined bounded linear parametrized transfer operator \( \mathcal{L}_{T,g} \in L(c^s(M); c^s(U \times M)) \) which for \( \phi \in c^s(M) \) is given by the expression:

\[
(\mathcal{L}_{T,g}\phi)(u, y) = \sum_{x \in D_{\psi_j^i}(u) = y} e^{g(u,x)} \phi(x), \quad (u, y) \in U \times M.
\]

**Proof.** For \( j \in J \) and \( y \in V_j \) we may rewrite the expression as a sum over inverse branches using local coordinates on \( V_j \) (recall that \( g_j^i(u, y) = g(u, \psi_j^i(u, y)) \)):

\[
(\mathcal{L}_{T,g,j}\phi)(u, y) = \sum_{i \in \mathcal{A}} e^{g_j^i(u,y)} \phi(\psi_j^i(u, y)), \quad (u, y) \in U \times V_j.
\]

Let \( (u_0, y_j) \in U \times V_j \) and define as in (4.4) \( B_j = B_j(u_0, y_j) \subset U \times V_j \). It suffices to show that the \( C^s \) norm of \( \mathcal{L}_{T,g}\phi \) is uniformly bounded when restricted to \( B_j \) for arbitrary \( (u_0, y_j) \in U \times V_j \). This may be done going to local coordinates. For each \( i \in \mathcal{A} \), let \( x^i = \psi_j^i(u_0, y_j) \) and pick \( m_i \in J(x_i) \) (cf. Hypothesis 4.1). The above sum then becomes

\[
(\mathcal{L}_{T,g,j}\phi)(u, y) = \sum_{i \in \mathcal{A}} e^{g_j^i(u,y)} \phi_{m_i} \circ \psi_{m_i,j}(u, y),
\] (4.11)

\[ ^{6} \] \( d_{\mathcal{P}} \) does not necessarily verify the triangle inequality. One could turn it into a real metric by a Kobayashi-like construction, but this does not seem to be of any particular use in our context.
with \( \phi_{m_i} = \phi \circ \eta_{m_i}^{-1} \), \( \psi_{m_i,j}^i : B_j \to V_{m_i} \) being the induced maps, now between open sets in Banach spaces. Using Proposition C.1 we see that all terms in the above sum are \( c(\delta_1) \). From the estimates in that proposition and (4.9) we get the following uniform bound:

\[
\|\hat{L}_{T,g,}\phi\|_s \leq \sum_{i \in A} C_s \left| e^{g_j^i} \right|_{C^0(B_j)} \|\phi_{m_i}\|_s (1 + k_g)^s (1 + k_T)^s \\
\leq C_s (1 + k_g)^s (1 + k_T)^s \delta_1 \|\phi\|_s, \tag{4.12}
\]

where \( C_s \) is a constant depending only upon \( s \) and the constants related to the manifold and the choice of norms. (and not to the specific choice of \( (T, g) \)). The above bound is first for each \( B_j \), but then extends to all of \( U \times M \) by definition of our \( c(\delta_1) \) structure.

We then have the following result:

**Lemma 4.10.** With the notation as in Lemma 4.9 above, for any fixed \( \phi \in c(\delta_1) \), the map \( (T, g) \in \mathcal{P} \mapsto \hat{L}_{T,g}\phi \in c(\delta_1) \) is uniformly continuous.

**Proof.** Let \( \epsilon > 0 \) and \( \hat{\phi} \in c(\delta_1) \). First, by the uniform bounds on the operator (4.12) and since \( \phi \) is the \( C^s \)-limit of smooth functions, we may find \( \phi \in C^{s+1}(M, \mathbb{C}) \) so that

\[
\|\hat{L}_{T,g}\phi - \hat{L}_{\tilde{T},g}\phi\|_s \leq \epsilon/2 \text{ for every } (T, g) \in \mathcal{P}. \]

We fix this \( \phi \).

Let \( \Delta = (0, \delta_1) \subset (0, 1) \) and consider two pairs \( (T, g), (\tilde{T}, \tilde{g}) \in \mathcal{P} = \mathcal{P}_{k_T, k_g, k_\Sigma} \) of distance \( d_P((T, g), (\tilde{T}, \tilde{g})) < \Delta \). Let \( j \in J \) and for \( (u_0, y_j) \in U \times V_j \) set \( B_j = B_j(u_0, y_j) \) as before. By definition of \( d_P \) there is a common index set \( A \) and a permutation \( \sigma \) such that for every \( i \in \mathcal{A} \) there is \( m_i \in J \) and local inverses: \( \psi_{m_i,j}^i, \psi_{m_i,j}^\sigma: B_j \to V_{m_i}, \) such that \( d_j^{(k,\alpha)}(\psi_{m_i,j}^i, \psi_{m_i,j}^\sigma) < \Delta \) and \( \|g_j^i - \tilde{g}_j^\sigma\|_{(k,\alpha), j} < \Delta \). We may then write:

\[
\hat{L}_{T,g,j}\phi - \hat{L}_{\tilde{T},g,j,\phi} = \sum_{i \in A} (e^{g_j^i} - e^{g_j^{\sigma(i)}})\phi_{m_i} \circ \psi_{m_i,j}^i \\
+ \sum_{i \in A} e^{g_j^{\sigma(i)}}(\phi_{m_i} \circ \psi_{m_i,j}^i - \phi_{m_i} \circ \psi_{m_i,j}^{\sigma(i)}). \tag{13.14}
\]

Taking the \( C^s \) norm and using Proposition C.1 the first sum in (13.14) is bounded by

\[
\sum_{i \in A} C_{1,s} C_{4,s} |e^{1 + g_j^i}|_0 (1 + k_g)^s \Delta \ C_{3,s} (1 + k_g)^s \|\phi\|_s \leq C_1 \|\phi\|_{s+1} \Delta,
\]

with \( C_1 = C_1(s, k_T, k_g, k_\Sigma, \delta_1) \) depending only upon the constants. For the second term we note that by Lemma 4.6 the images of \( \psi_{m_i,j}^i \) and \( \psi_{m_i,j}^{\sigma(i)} \) are contained in the same convex set \( V_{m_i} \). Corollary C.4 then applies and implies:

\[
\sum_{i \in A} C_{2,s} |e^{g_j^{\sigma(i)}}|_0 (1 + k_g)^s \ C_{s,1} \|\phi\|_{s+1} \Delta \leq C_2 \|\phi\|_{s+1} \Delta,
\]

again with \( C_2 \) depending only upon constants. Choosing \( \Delta \) small enough each term can be made smaller than \( \epsilon/4 \), thus concluding the proof. \( \square \)
Remark 4.11. How small we have to take $\Delta$ depends in an essential way upon $\hat{\phi}$ through the norm $\|\phi\|_{s+1}$ of the approximant. The theorem is in general false if $c^\xi$ is replaced by $C^\xi$. To see what may go wrong consider the following explicit example: For $s = (k, \alpha)$ with $k \in \mathbb{N}$ and $0 < \alpha < 1$, the function $\phi(x) = |x|^{k+\alpha}$ belongs to $\phi \in C^\xi([-1, 1])$. However, the map $u \in (-1, 1) \mapsto \phi\left(\frac{u+\xi}{2}\right) \in C^\xi([-1, 1])$ is no-where continuous. We leave the verification of this fact to the interested reader.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, endowed with a measure-preserving, invertible and ergodic map $\tau : \Omega \to \Omega$. Let $\omega \in \Omega \mapsto (T_\omega, g_\omega) \in \mathcal{P} = (\mathcal{P}_{kT, k\Sigma, dP})$ be a Bochner measurable family of pairs.

By this we mean that for every $\Delta > 0$ there is a countable measurable partition $(\Omega_i)_{i \in \mathbb{N}}$ of $\Omega$ such that the diameter in $\mathcal{P}$ for the $d_P$ distance (4.10) of each set $\{(T_\omega, g_\omega) : \omega \in \Omega_i\}$ is smaller than $\Delta$. In particular, the degree of $T_\omega$ is constant on each $\Omega_i$.

For $\omega \in \Omega$ and $u \in U$ we write $T_{\omega,u}(x) := T_\omega(u, x), x \in D_{\omega,u}$. A random product of expanding maps above $(\Omega, \tau)$ is then given by

$$T_{\omega,u}(n) := T_{\tau^{-n} \omega, u} \circ \cdots \circ T_{\omega, u} : D_{\omega,u} \to M$$

with domain of definition given by $D_{\omega,u}(n) = T_{\omega,u}^{-1} \circ \cdots \circ T_{\omega,u}^{-1}(M)$.

We associate to $\omega \in \Omega$ the transfer operator $\hat{L}_\omega := L_{T_{\omega,u}, g_{\omega,u}} \in L(c^\xi(M); c^\xi(U \times M))$, $0 < s \leq r_0$ and parametrized by $u \in U$. For fixed $\phi \in c^\xi(M)$ we see by Lemma 4.10 that the map $\omega \in \Omega \mapsto \hat{L}_\omega \phi \in c^\xi(U \times M)$ as a composition of a continuous and a Bochner measurable map is Bochner measurable. Thus, the map $\omega \mapsto \hat{L}_\omega \in L(c^\xi(M); c^\xi(U \times M))$ is strongly Bochner measurable (in the sense of Definition B.1).

Let $X_s(M) = L^\infty(\Omega; c^\xi(M))$ and $X_s(U \times M) = L^\infty(\Omega; c^\xi(U \times M))$ denote the $\Omega$-Bochner measurable sections of $c^\xi$-functions on $M$, respectively $U \times M$. The operator $\hat{L}_{T_{\tau}, g} : c^\xi(M) \to c^\xi(U \times M)$ has norm bounded by $C_r$, a constant that only depends upon the constants of $\mathcal{P}$ and the manifold. Our uniform choice of constants implies that $\hat{L}_{\tau}$ induces a well-defined, bounded operator $\hat{L}_s : X_s(M) \to X_s(U \times M)$ by declaring that for $\phi = (\phi_\omega)_{\omega \in \Omega} \in X_s(M)$:

$$(\hat{L}_s \phi)_\omega(u, y) = (\hat{L}_{\tau^{-1} \omega} \phi_{\tau^{-1} \omega})(y), \quad \omega \in \Omega, u \in U, y \in M.$$ 

This time we also make explicit in the notation the dependency upon $s \in (0, r_0]$. We may now extract the variable $u$ as a parameter to obtain a parametrized operator co-cycle over $\Omega$:

$$u \in U \mapsto \mathcal{L}_{s,u} \in L(X_s(M)), \quad (\mathcal{L}_{s,u} \phi)_\omega(y) = (\hat{L}_s \phi)_\omega(u, y), \quad \omega \in \Omega, \quad \phi \in X_s(M).$$

The key-point of this section is that if we pre- or post-compose with a down-grading operator we recover regularity from the above parameter extraction:

**Theorem 4.12.** For any $0 < s < r \leq r_0$ and $t \in [0, r - s)$, the map $u \in U \mapsto j_{s,r} \mathcal{L}_{s,u} = \mathcal{L}_{s,u} j_{s,r} : X_r(M) \to X_s(M)$ is $C^t$.

**Proof.** The core of the proof is Proposition C.8 in the appendix, except that it only deals with convex subset in Banach spaces. In view of the description in remark 4.5 of our $c^\xi(U \times M)$-structure and the parameter extraction Theorem C.2 we have, however, the following inclusions:

$$c^\xi(U \times M) \hookrightarrow \prod_{j \in \mathcal{J}} c^\xi(U \times V_j) \hookrightarrow \prod_{j \in \mathcal{J}} C^{r-s}(U; c^\xi(V_j)) \simeq C^{r-s}(U; \prod_{j \in \mathcal{J}} c^\xi(V_j)).$$
The identification in Remark 4.5 carries over to the last expression and shows that \( c^\mathcal{L}(U \times M) \leftrightarrow C^{r-s}(U; c^\mathcal{L}(M)) \). With similar identifications for \( X_r(M) = L^\infty(\Omega; c^\mathcal{L}(M)) \) and using Proposition C.8 we have \( X_r(U \times M) \leftrightarrow C^{r-s}(U; X_s(M)) \). Thus, \( \mathcal{L} : X_r(M) \to X_r(U \times M) \leftrightarrow C^{r-s}(U; X_s(M)) \) and we infer that \( u \in U \to J_{x,r}^{s,r,u} : X_r(M) \to X_s(M) \) is \( C' \) for \( 0 \leq t < r - s \) as wanted. □

The above result is precisely the regularity-hypothesis needed in Theorem 3.2. What is lacking in order to apply that theorem is to show that our operator cocycle is contracting a cone-field in a uniform way. This will be the subject of the next section.

4.3. A family of cones adapted to expanding systems. Our setup is such that we are able to use cone-families that do not depend upon the parameter \( u \in U \) or the random variable \( \omega \in \Omega \). So in the following, and to simplify notation, we will omit any reference to \( u \) and \( \omega \).

Let \( r > 0 \) and write as usual \( r = (k, \alpha) \) with \( k \in \mathbb{N} \) and \( \alpha \in (0, 1] \). Recall that a real-valued function \( \phi \in c^\mathcal{L}(M; \mathbb{R}) \) induces \( c^\mathcal{L} \) maps on our charts \( \phi_j = \phi \circ \eta_j^{-1} : V_j \to \mathbb{R} \), \( j \in \mathcal{J} \). We will construct a family of cones adapted to this \( c^\mathcal{L} \)-structure. Recall that when \( \xi, \xi' \in V_j \) we let \( d^\mathcal{L}_j(x, x') = d_M(\xi, \xi') \) denote the induced Riemannian distance between \( x = \eta_j(\xi), x' = \eta_j(\xi') \in V_j \). Also let \( g^j_\alpha \) be the induced Riemannian metric tensor in the chart \( V_j \). The length of \( v \in \mathbb{R}^n \simeq T_x V_j \) for this metric is:

\[
\|v\|_{g^j_\alpha} = \sqrt{g^j_\alpha(v, v)}.
\]

By Hypothesis 4.1 this norm is \( L \)-equivalent to the Euclidean norm on \( \mathbb{R}^n \).

Given a \( k \)-linear form \( A \) on \( \mathbb{R}^n \) and a base point \( x \in V_j \) we define its \( g \)-norm at \( x \) to be:

\[
\|A\|_{g^j} = \sup \left\{ |A(v_1, \ldots, v_k)| : \forall 1 \leq i \leq k : v_i \in \mathbb{R}^n, \|v_i\|_{g^j} \leq 1 \right\}.
\]

With this definition we have in particular:

\[
\frac{1}{L^q} \|D^q \phi_j(x)\| \leq \|D^q \phi_j(x)\|_{g^j} \leq L^q \|D^q \phi_j(x)\| \quad (4.15)
\]

Given a vector of strictly positive numbers \( a := (a_1, \ldots, a_k, a_{k+1}) \in (\mathbb{R}_+^k)^{k+1} \), we consider the convex cone \( C_a \) of non-negative functions \( \phi \in c^\mathcal{L}(M) \), \( \phi \geq 0 \) satisfying the following conditions:

\[
\begin{align*}
\text{(C1)} & \quad \text{For } j \in \mathcal{J}, q \in \{1, \ldots, k\} \text{ and } x \in V_j: \|D^q \phi_j(x)\|_{g^j} \leq a_q \phi_j(x). \\
\text{(C2)} & \quad \text{For } j \in \mathcal{J} \text{ and } x, x' \in V_j \text{ with } 0 < d^g_j(x, x') \leq \delta_1: \\
& \quad \frac{\|D^k \phi_j(x) - D^k \phi_j(x')\|_{g^j}}{d^g_j(x, x')^\alpha} \leq a_{k,\alpha} \phi_j(x).
\end{align*}
\]

Remark 4.13. Similar cones were considered in [6], but the treatment in the case of a Hölder condition on derivatives was somewhat incomplete.
The semi-norms used to define the cone $C_a$ are tailored to prove contraction properties for the Hilbert-metric. We also have to prove regularity of the cone and for that we need the following lemma to connect norms in the Banach spaces and the semi-norms used in the above definition. For use elsewhere in the article it is slightly more general than needed here.

**Lemma 4.14.** Let $K, L \in [1, +\infty)$ and let $W \subset X$ be an open convex set in a Banach space. Let $d_1$ be the norm-metric on $W$ and let $d_2$ be another metric which is $L$-equivalent with $d_1$, i.e. $\frac{1}{L}d_1 \leq d_2 \leq Ld_1$ as functions on $W \times W$. Also let $\delta_1, \delta_2 > 0$ be $K$-equivalent constants. Then $E = C_{\delta_1, d_1}(W; Z) = C_{\delta_2, d_2}(W; Z)$ (with $Z$ any Banach space) and for any $\phi \in E$ the Hölder-exponents $h_{\delta_1, d_1}^{(k, \alpha)}(\phi)$ and $h_{\delta_2, d_2}^{(k, \alpha)}(\phi)$ are $[LK]^{1-\alpha}L^\alpha$-equivalent.

**Proof.** It is enough to show the lemma for $k = 0$.

Let $\xi, \xi' \in W$ with $\|\xi - \xi'\| \leq \delta_1$. Let $N \geq 1$ be an integer. The points $\xi_\ell = \xi + \frac{\ell}{N}(\xi' - \xi)$, $0 \leq \ell \leq N$, belong to $W$ (convex). So

$$d_2(\xi_\ell, \xi_{\ell+1}) \leq \frac{L}{N}\|\xi - \xi'\| \leq \frac{L\delta_1}{N} \leq \frac{LK}{N}\delta_2.$$

When $N = [LK]$ then $d_2(\xi_\ell, \xi_{\ell+1}) \leq \delta_2$ so by the $\delta_2, d_2$-Hölder estimate:

$$|\phi(\xi_\ell) - \phi(\xi_{\ell+1})| \leq C_2d_2(\xi_\ell, \xi_{\ell+1})^\alpha \leq C_2 \left(\frac{L}{N}\right)^\alpha\|\xi - \xi'\|^\alpha.$$

Finally, $|\phi(\xi) - \phi(\xi')| \leq C_2N^{1-\alpha}L^\alpha\|\xi - \xi'\|^\alpha \leq C_2[\frac{LK}{N}]^{1-\alpha}L^\alpha d_1(\xi, \xi')^\alpha$, implying one inequality. Interchanging $d_1$ and $d_2$ turns out to yield a bound with the same constant (even though the problem is not symmetric). $\Box$

The above lemma shows that Hölder constants evaluated for the Euclidean metric and for the induced manifold metric are equivalent within uniform constants. It follows that our usual chart-norm (4.2) is equivalent to: $\|\phi\|_g := \max_{0 \leq q \leq k} \sup_x \|D^q\phi_j(x)\|_{g_j} \vee h_{\delta_1, d_1}^{(k, \alpha)}(\phi_j)$. For the latter norm it is clear from the construction that $C_a$ is a closed, convex cone but this is then also the case in our usual chart-norm. $\Box$

**Lemma 4.15.** For every $a$ there is a constant $R > 0$ such that for $\phi \in C_a$:

$$\phi(x) \leq R\phi(y), \quad \forall x, y \in M.$$

**Proof.** Consider $k \geq 1$ and $j \in J$. By Hypothesis 4.1 on the atlas we have $\|D\phi_j(x)\|_{g_j} \leq a_1\phi_j(x)$ for all $x \in V_j$. So if a geodesic lies in $V_j$ and is joining $y, y' \in V_j$ we have

$$\phi(y') \leq e^{a_1d_M(y, y')}\phi(y).$$

Since the $d_M$-distance is realized by geodesics, we deduce that for every $y, y' \in M$: $\phi(y') \leq e^{a_1D\phi(y)}$ with $D$ being the diameter of $M$.

For $k = 0$, the condition (C2) yields: $\phi(x') \leq (1 + a_{0, \alpha}d_1^\alpha(x, x'))^\alpha\phi(x)$ whenever $d_1^\alpha(x, x') \leq \delta_1$. Given arbitrary $y, y' \in M$ we consider a geodesic between the two points and chop it into $N$ pieces with $N = \lceil\frac{D}{\delta_1}\rceil + 1$. Each piece has length at most $\delta_1$. For endpoints of such a piece we may thus apply the previous estimate showing the wanted bound with $R = (1 + a_{0, \alpha}\delta_1)^N$. $\Box$
Lemma 4.16. Every cone $C_a$ is inner and outer regular. Furthermore, if $\sigma \in (0, 1)$ then
\[
\text{diam}_{C_a}(C_{\sigma a})^* \leq 2 \log \left( \frac{1 + \sigma}{1 - \sigma} \right) + 2 \log R.
\]

Proof. The constant function $1 \in C_a$. Let $\phi = 1 + \delta \psi \in c^\ell(M)$ with $\|\delta \psi\|_r \leq c$. Then by the equivalence of seminorms (4.15), we have for $j \in J, x \in V_j, 1 \leq q \leq k$:
\[
\|D^q \phi(x)\|_{g^*_q} \leq L q c \leq a_q (1 - c) \leq a_q \phi(x)
\]
provided $c \leq \frac{a_q}{L^q + a_q}$.

Similarly for the Hölder-condition using Lemma 4.14. Thus, $B_{c^\ell}(1, \rho) \subset C_a$ whenever $0 < \rho < \min_{1 \leq q \leq k} \frac{a_q}{1 + a_q} \wedge \frac{d_{k, \alpha}}{1 + a_q}$, showing inner regularity.

For outer regularity, one may take $\ell$ to be any probability measure on $M$, e.g. evaluation at a point. Then for $\phi \in C_a$ we have by the previous lemma $\phi(x) \leq R \ell(\phi)$ for all $x \in M$, whence
\[
\|\phi\|_r \leq K \left( \max_{1 \leq q \leq k} \frac{a_q}{1 + a_q} \vee \frac{d_{k, \alpha}}{1 + a_q} \right) \sup_x \phi(x) \leq R \left( \max_{1 \leq q \leq k} \frac{a_q}{1 + a_q} \vee \frac{d_{k, \alpha}}{1 + a_q} \right) \langle \ell, \phi \rangle
\]
again with a constant $K$ depending upon the equivalence-constants of semi-norms. This shows outer regularity.

In order to show that the inclusion is uniformly bounded we will describe a general (and fairly standard) way to obtain such bounds. Note that in both (C1) and (C2) the left hand-side of the two inequalities is a semi-norm on $c^\ell(M)$. So (C1) and (C2) may be formulated as a (in fact, uncountable) collection of elementary conditions of the form:
\[
p(\phi) \leq a \phi(x) \quad (4.16)
\]
where $a$ is one of the cone-constants, $x \in M$ and $p$ is a semi-norm on $c^\ell(M)$.

Now consider $\phi_1, \phi_2 \in C_{\sigma a}^*$. In order to calculate the projective Hilbert-distance of the two functions in $C_a$ we need to calculate the infimum of $t, t' > 0$ so that $t \phi_1 - \phi_2, t' \phi_2 - \phi_1 \in C_a$. However, when $\phi_1, \phi_2$ verify the elementary condition (4.16) with $\sigma a$ instead of $a$ we have the following sequence of inequalities:
\[
p(t \phi_1 - \phi_2) \leq tp(\phi_1) + p(\phi_2)
\]
\[
\leq \sigma a (t \phi_1(x) + \phi_2(x))
\]
\[
\leq a (t \phi_1(x) - \phi_2(x)) \quad (4.17)
\]
where equality in (4.17) is satisfied as soon as we set:
\[
t = \frac{1 + \sigma}{1 - \sigma} \sup_{x \in M} \frac{\phi_2(x)}{\phi_1(x)}.
\]

Here, the semi-norm has completely disappeared in the last condition. With a similar calculation for $t'$ we get for the Hilbert-distance:
\[
\exp(d_{C_a}(\phi_1, \phi_2)) \leq tt' = \left( \frac{1 + \sigma}{1 - \sigma} \right)^2 \sup_{x, x' \in M} \frac{\phi_2(x)}{\phi_1(x)} \cdot \frac{\phi_1(x')}{\phi_2(x')} \leq \left( \frac{1 + \sigma}{1 - \sigma} R \right)^2.
\]
Taking log we get the wanted estimate. □

We claim that the (weighted) transfer operator is a strict and uniform contraction of the cones $C_a$, in the following sense:
Theorem 4.17 ([6] Lemma 3.2). Let \( r > 0 \) and \( r = (k, \alpha) \) as above. If \( k = 0 \) take \( \sigma \in \left( \frac{1}{\lambda}, 1 \right) \) and if \( k \geq 1 \) take instead \( \sigma \in \left( \frac{1}{\lambda}, 1 \right) \) with \( \lambda > 1 \) being the expansion constant for our expanding maps. There exists a vector \( a = (a_1, ..., a_k, a_{k, \alpha}) \in (\mathbb{R}^*_+)^{k+1} \) depending only upon the constants of \( P = \mathcal{P}^{(r)}_{k_T, k_g, k_\Sigma} \) and the manifold such that for every pair \((T, g) \in P\) the transfer operator \( \mathcal{L}_{T, g} \in L(c^\infty(M, \mathbb{R})) \), defined for \( \phi \in c^\infty(M, \mathbb{R}) \) by

\[
\mathcal{L}_{T, g}\phi(x) = \sum_{y, Ty = x} e^{s(y)}\phi(y), \quad x \in M,
\]

satisfies: \( \mathcal{L}_{T, g}(C_\alpha) \subset C_\sigma \).

In particular, by the previous lemma, \( \mathcal{L}_{T, g} \) is a uniform contraction of \( C_\alpha \) for the Hilbert metric. Furthermore, there exist \( \rho > 0 \), depending only on the constants of the problem such that for every \( \phi \in (C_\alpha)^* \)

\[
B(\mathcal{L}_{T, g}\phi, \rho \| \mathcal{L}_{T, g}\phi \|) \subset C_\alpha.
\]

Proof. (Our proof follows the same lines as in the proof of [6] Lemma 3.2. Here, we provide details of the uniform bounds also in the presence of a Hölder condition on the top-most derivative). Write \( L := \mathcal{L}_{T, g} \). For the first statement, take a chart \( V_j, j \in J \).

The transfer operator takes the form as in (4.11) (where as above we omit the \( u \)-variable):

\[
(\mathcal{L}_j\phi)(y) = \sum_{i \in A} e^{g_j^i(y)}\phi_{m_i} \circ \psi_{m_i, j}^i(y), \quad y \in V_j
\]

When \( \phi \in C_\alpha \) we will determine suitable values of \( a_1, ..., a_k \) by induction starting with \( a_1 \).

Assume that for \( 1 \leq q < k \) we have determined \( a_1, ..., a_{q-1} \). We then get by the Faà di Bruno formula and algebra for the \( q' \)-th derivative:

\[
D^q(\mathcal{L}_j\phi)(y) = \sum_{i \in A} e^{g_j^i} \left[ D^q \phi_{m_i} \circ \psi_{m_i, j}^i(y). (D\psi_{m_i, j}^i, ..., D\psi_{m_i, j}^i) + R_{q-1}(y) \right]
\]

where \( R_{q-1} \) is linear in \( \phi, ..., D^{q-1}\phi \) and multinomial in derivatives up to order \( q \) of \( \psi_{m_i, j}^i \) and \( g_j^i \). Note that the only term with a \( q' \)-th derivative of \( \phi \) has the form \( D^q \phi \circ \psi(y).(D\psi(y), ..., D\psi(y)) \). Since the \( \psi \)'s are \( 1/\lambda \)-contractions in the Riemannian metric, the \( \| \cdot \|_{g_j^i} \)-norm of this term is bounded by \( \| D^q \phi \circ \psi(y) \|_{g_j^i} \). Inserting in the above expansion, using the induction hypothesis and the telescopic principle \((C.1)\) we get:

\[
\| D^q \mathcal{L}_j\phi(y) \|_{g_j^i} \leq \left( \frac{a_q}{\lambda^q} + \sum_{\ell=1}^{q-1} a_\ell p_q, \ell(k_T, k_g) + p_q, 0(k_T, k_g) \right) \mathcal{L}_j\phi(y), \quad y \in V_j,
\]

where \( p_q, \ell, 0 \leq \ell \leq q - 1 \) are polynomials in the constants (except for \( k_\Sigma \)) of the problem. Given \( \frac{1}{\lambda} < \sigma < 1 \) and having chosen \( a_1, ..., a_{q-1} \) it then suffices to choose \( a_q \) large enough to ensure that

\[
\| D^q \mathcal{L}_j\phi(y) \|_{g_j^i} \leq \sigma a_q \mathcal{L}_j\phi(y), \quad y \in V_j.
\]
For the Hölder-condition on the $k$’th derivative consider $y, y' \in V_j$, $0 < d_j^g(y, y') \leq \delta_1$. In order to achieve cone-contraction for the Hölder condition we notice that in the difference $D^k L_j \phi(y') - D^k L_j \phi(y)$ there is only one term which involves the Hölder bound for the $k$’th derivative of $\phi$. It takes the following form (we may treat the other terms using telescoping):

$$e^{g_j^i(y)} (D^g \phi \circ \psi(y') - D^g \phi \circ \psi(y)) (D \psi(y), \ldots, D \psi(y)).$$

Now, this being taken at $y$, the $\| \cdot \|_{g_j^i}$-norm is bounded by $e^{g_j^i(y)} (\frac{a_k, \alpha}{\lambda^{k+\alpha}} d_j^g(y, y')^\alpha \phi \circ \psi(y)$ and we obtain as in the previous calculation:

$$\frac{\| D^k L_j \phi(y') - D^k L_j \phi(y) \|_{g_j^i}}{d_j^g(y, y')^\alpha} \leq \left( \frac{a_k, \alpha}{\lambda^{k+\alpha}} + \sum_{\ell=1}^k a_{\ell} p_{k, \ell}(k_T, k_g) + p_{k, 0}(k_T, k_g) \right) L_j \phi(y), \ y \in V_j,$n

This time, however, we have to distinguish if $k \geq 1$ in which case we may choose $\sigma$ as above, and the case $k = 0$ for which we must choose $\sigma \in (\frac{1}{\lambda^r}, 1)$ in order to assure that

$$\frac{\| D^g L_j \phi(y') - D^g L_j \phi(y) \|_{g_j^i}}{d_j^g(y, y')^\alpha} \leq \sigma a_q L_j \phi(y), \ y \neq y' \in V_j.$$

This proves the uniform cone-contraction. The last statement follows from Lemma A.10.

4.4. Random physical measure and quenched linear response. We assume here that the metric tensor is uniformly $C^\infty$ (cf. Hypothesis 4.1 for the meaning of being uniform) in all charts and write $m$ for the associated volume form. In a local chart $V_j, j \in J$, we have: $dm_x = \rho_j(x) d^n x := \sqrt{\det(g_j^i)} d^n x$, $x \in V_j$ with $\rho_j$ being uniformly $C^\infty$ and uniformly bounded away from zero.

Going back to the setting of a random product (4.14) of $c^r$ uniformly expanding maps (section 4.1), we consider $(T_{\omega, g}, g_{\omega})_{\omega \in \Omega} \in \mathcal{P}_{k_T, k_g, k_\Sigma}$ where the exponential of the weight is defined as the metric derivative of the map: $e^{g_{\omega}} := 1/|\det g(\partial_x T_{\omega})|$ (beware of the two different uses of the letter $g$). In order to get physical measures we must further assume that the domain $D_{\omega, u}$ for each $T_{\omega, u} = T_{\omega}(u, \cdot)$ has full measure in $M$.

The regularity condition on $g_{\omega}$ implicitly impose a regularity condition on $T_{\omega}$ (or rather its inverse branches). Omitting $\omega$ in the notation, let $j \in J$ and $\psi_j^i : U \times V_j \to M$, $i \in A$ be a local inverse branch. Take a point $\xi_0 = (u_0, y_0) \in V_j$ and pick $m_i \in J(\psi_j^i(\xi_0))$. Restricting to $B(\xi_0) \subset U \times V_j$ we get in local coordinates:

$$g_{\omega} := \log \frac{\rho_j(u, y)}{\rho_{m_i}(\psi_j^i(u, y))} - \log \left| \det \partial_x \psi_j^i(u, y) \right|, \ (u, y) \in B(\xi_0).$$

This should be uniformly bounded in $C^{r-1}$ norm. Here, as the $\rho$’s are uniformly $C^\infty$ and uniformly bounded from above and below, the first term is bounded as soon as $\psi_j^i$
is. For the second term, however, the bound (4.7) imposes the auxiliary condition that there should be $k'_g < +\infty$ so that

$$\|D \log |\operatorname{det} \frac{\partial}{\partial y} \psi_i^j|\|_{(r-1,\alpha)} \leq k'_g, \quad \forall i \in A, j \in J.$$  \tag{4.18}

By change of variables (and the hypothesis that $D\omega, u$ has full measure in $M$):

$$\int_M 1 \, dm_x = \int_M S_{\omega,u}(y) \, dm_y$$

where $S(y) = S_{\omega,u}(y) := \sum_{x:T_{\omega,u}=y} e^{g(u,x)}$. It follows that $S$ must take values both above and below one (if it is not identically one). Now, by condition (4.18) ratios of $S(y)/S(y')$ are uniformly bounded from above and below, whence we see that condition (4.8) is automatically verified.

Given the above family of maps and weights we may thus associate as in the previous section the transfer operator cocycle $\hat{L}u$, acting on $X_{r-1}(M, C)$. Then for each $u \in U$, $\hat{L}u$ is a cone-contracting cocycle (in the sense of Sect. 2) of the cone $C_a$ for some well-chosen $a$: this follows straightforwardly from Theorem 4.17.

In this setting, Theorem 2.1 apply, so we may construct $f_u \in X_{r-1}(M, C)$, fixed point of $\pi_u$. The coincidence of the outer regularity form of $C_a$ and the left eigenvector of $L_{\omega,u}$ ensures that

- For almost every $\omega \in \Omega$, every $u \in U$,

$$p_{\omega,u} = \langle \ell, L_{\omega,u} f_{\omega,u} \rangle = \int_M L_{\omega,u} f_{\omega,u} \, dm = \int_M f_{\omega,u} \, dm = 1.$$  

In particular, the fixed point $f_u$ of $\pi_u$ is also a fixed point for $\hat{L}u$.

- The top characteristic exponent of this transfer operator cocycle is zero: by virtue of (2.4), one has

$$\chi_u = \mathbb{E}[\log(p_u)] = 0.$$  

By Theorem 4.12, we may apply Theorem 3.2: for any $0 \leq s < r - 1$ the map $f : U \to X_s(M, \mathbb{C})$ is $C^t$ for any $0 < t < r - 1 - s$. For $\psi \in L^1(M)$ we then set

$$\nu_{\omega,u}[\psi] = \int_M \psi \, f_{\omega,u} \, dm,$$  \tag{4.19}

which defines an absolutely continuous equivariant probability measure for $T_{\omega,u}$, i.e. for any $\omega \in \Omega$, any $u \in U$,

$$\nu_{\omega,u}[\psi \circ T_{\omega,u}] = \nu_{\tau_{\omega,u}}[\psi].$$

$\nu_u$ is called the random a.c.i.m.\footnote{This is a somewhat particular case. In applications, often the linear form used for the outer regularity of $C_a$ and the left eigenvector of $L_{\omega,u}$ do not coincide.}

We may now formulate the main result of this section: a linear response formula for the random a.c.i.m associated to a random product of expanding maps:

\footnote{A.c.i.m stands for absolutely continuous invariant measure. Depending on the context, it is also called the random S.R.B measure (for Sinai-Ruelle-Bowen), or the random physical measure.}
Theorem 4.18. Let \((T_{\omega,u})_{\omega \in \Omega, u \in U} \in \mathcal{P}_{k_T,k_x,k_\Sigma}\) be a random family of uniformly expanding maps, and let \(L_u\) be the transfer operator cocycle it generates above \((\Omega, \tau)\), acting on \(X_{r-1}(M, \mathbb{C})\). Let \(v_u\) be the random a.c.i.m introduced in (4.19).

For every observable \(\psi \in L^1(M)\), the map \(u \in U \mapsto v_u[\psi] \in L^\infty(\Omega, \mathbb{C})\) is differentiable at \(u = u_0\), for every \(u_0 \in U\) with

\[
D_u \left[ \int_M \psi \, d\nu_{\omega,u} \right]_{u=u_0} = \sum_{n=0}^\infty \int_M \psi \circ T_{\tau^{-n}\omega,u_0}^{(n)} P_{\tau^{-n}\omega,u_0} f_{\tau^{-n}\omega,u_0} \, dm
\]

(4.20)

For the proof of this result, we will use an estimate on the speed of convergence of the random product \(L_{\omega,u}^{(n)}\) towards its equivariant line, analogous to the spectral gap estimate for a deterministic cone contraction.

Lemma 4.19. Let \((T_{\omega,u})_{\omega \in \Omega, u \in U} \in \mathcal{P}_{k_T,k_x,k_\Sigma}\) be a random family of expanding maps, with dilation constants all bounded from below by some \(\lambda > 1\).

Let \(L_u\) be the associated transfer operator cocycle above \((\Omega, \tau, \mathbb{P})\), and let \(f_u \in X_{r-1}(M, \mathbb{C})\) be its fixed point. Then for every integer \(0 < s \leq r - 1\) and every \(\phi \in X_s(M, \mathbb{C})\):

\[
\|L_{\omega,u}^{(n)} \phi \omega - f_{\tau^{-n}\omega,u} \int_M \phi \omega dm\|_{C^s} \leq C \eta^{n-1} \|\phi\|_{C^s}
\]

(4.21)

where \(C\) depends only on the cone \(C_a\) and \(\eta < 1\).

Proof. Let \(n \geq 1\). We write \((A.3)\), at \(\phi, \ f_{\tau^{-n}\omega}\), for some Bochner-measurable, essentially bounded family \((\phi\omega)_{\omega \in \Omega} \in C_a \subset C^s(M)\), to get

\[
\left\|L_{\tau^{-n}\omega,u}^{(n)} f_{\tau^{-n}\omega} - \frac{\int_M \phi \omega dm \alpha_n - \int_M \phi \omega dm}{f_{\tau^{-n}\omega,u} \int_M \phi \omega dm} \right\|_{C^s} \leq K \Delta \|f\|_{C^s}\]

with \(K\) the sectional aperture of \(C_a\), \(\Delta = \text{diam}_{C_a} C_a < \infty\), \(0 < \sigma < 1\) and \(a\) given by Theorem 4.17 and \(\eta = \tanh \left( \frac{\Delta}{2} \right) < 1\). One obtains

\[
\left\|f_{\omega,u} - \frac{L_{\tau^{-n}\omega,u}^{(n)} \phi \omega}{\int_M \phi dm} \right\|_{C^s} \leq K \Delta \|f\|_{C^s}\]

which yields, with the “change of variables” \(\omega \leftrightarrow \tau^{-n}\omega\) and once multiplied by \(0 < \int_M \phi dm \leq \|\phi\|_{C^s}\), (4.21) \(\Box\)

In the general case take \(\phi \in C^s(M)\) of norm one. As shown in Lemma 4.16 the cone \(C_a\) is inner regular. In fact, as shown in the proof of that Lemma there is \(\rho = \rho_a > 0\) so that \(B(1, \rho_a) \subset C_a\) (with \(1\) being the constant function). Then \(\phi = \frac{1}{\Sigma_\rho}(1 + \rho \phi) - \frac{1}{\Sigma_\rho}(1 - \rho \phi)\) shows that we have a decomposition \(\phi = \phi_1 - \phi_2\), with \((\phi_1, \phi_2) \in (C_a)^2\) and \(\|\phi_1\|_{C^s} + \|\phi_2\|_{C^s} \leq \frac{2}{\rho} \|\phi\|_{C^s}\). Thus,

\[
\left\|f_{\tau^{-n}\omega,u} \int_M \phi \omega dm - L_{\omega,u}^{(n)} \phi \omega \right\|_{C^s} = \left\|f_{\tau^{-n}\omega,u} \int_M \phi \omega dm - L_{\omega,u}^{(n)} \phi \omega \right\|_{C^s} \leq \frac{K \Delta}{\rho} \eta^{n-1} \|\phi\|_{C^s}\]
which conclude the proof in the general case \( \phi \in c^s(M) \). \( \square \)

**Remark 4.20.** • From estimate (4.21), we may draw the following conclusion: if \( (\phi_\omega)_{\omega \in \Omega} \in c^s(M) \) is such that \( \int_M \phi_\omega dm = 0 \), then for any \( n \geq 1 \),

\[
\| L^{(n)}_\tau^{-n} \phi_\omega \|_{C^s} \leq \frac{K \Delta}{\rho} \eta^{n-1} \| \phi_\omega \|_{C^s}.
\]

In particular, the limit \( \sum_{n=0}^{\infty} L^{(n)}_\tau^{-n} \phi_\omega \) is well defined whenever \( \int_M \phi_\omega dm = 0 \).

• Estimate (4.21) has far reaching consequences: in particular, it can be used to establish exponential decay of random correlations, in the same way as one establishes exponential decay of correlations in the deterministic case. We refer to [6, Thm B] for more details on this.

• It is possible to establish uniform exponential decay of random correlations for general weighted cocycles of transfer operators \((L_{\tau, g_\omega})_{\omega \in \Omega}\): see [21, Theorem 3.1].

**Proof of theorem 4.18.** For any \( 0 \leq s < r - 1 \), it follows from Theorem 3.2 that the map \( u \in U \mapsto f_u \in X_{s-1}(M) \) is differentiable with

\[
D_u f_u = [1 - Q_u(f_u)]^{-1} P_u(f_u),
\]

(4.22)

with \( Q_u(f_u) \), \( P_u(f_u) \) coming (3.1) and (3.3). In this particular case, one has

\[
(Q_{\omega, u})_\omega = L_{\tau^{-1} \omega, u} - \left( \int_M dm \right) f_{\omega, u}
\]

\[
[P_u]_\omega(f_{\omega, u}) = (\partial_u L_{\tau^{-1} \omega, u}) f_{\tau^{-1} \omega, u} - \left( \int_M (\partial_u L_{\tau^{-1} \omega, u}) f_{\tau^{-1} \omega, u} dm \right) f_{\omega, u}.
\]

The normalization \( \int_M f_u dm = 1 \) implies that \( \int_M D_u f_u dm = 0 \). Similarly, \( \int_M \hat{L}_u f_u dm = 1 \) yields that

\[
0 = D_u \left( \int_M L_{\omega, u} f_{\omega, u} \right) = \int_M \partial_u L_{\omega, u} f_{\omega, u} dm + \int_M L_{\omega, u} D_u f_{\omega, u} dm
\]

\[
= \int_M \partial_u L_{\omega, u} f_{\omega, u} dm.
\]

Hence, one obtains \( Q_{\omega, u} D_u f_{\omega, u} = L_{\tau^{-1} \omega, u} f_{\tau^{-1} \omega, u} \) and \( (P_u)_\omega(f_{\omega, u}) = \partial_u L_{\tau^{-1} \omega, u} f_{\tau^{-1} \omega, u} \). \( \square \)

A straightforward computation shows that

\[
(Q^n_u)_\omega = L^{(n)}_\tau^{-n \omega, u} - \left( \int_M dm \right) f_{\omega, u}
\]

and thus by (4.21), for any \( 0 \leq s < r - 1 \), the sum \( \sum_{n=0}^{\infty} Q^n_u \) converges in \( X_s(M, \mathbb{C}) \). Hence for any \( \phi \in X_s(M, \mathbb{C}) \), one may define a bounded operator via

\[
\left( [1 - Q_u]^{-1} \phi \right)_\omega = \sum_{n=0}^{\infty} (Q^n_u)_\omega \phi_{\tau^{-n} \omega}.
\]
Let $u_0 \in U$. From remark 4.20 and the previous discussion, one sees that (4.22) can be rewritten as

$$D_u f_{\omega,u_0} = \sum_{n=0}^{\infty} L_{\tau^{-n}\omega,u_0}^{(n)} \partial_u L_{\tau^{-(n+1)}\omega,u_0} f_{\tau^{-(n+1)}\omega,u_0},$$

which is valid in $X_{\tau^{-1}}(M, \mathbb{C})$. This last equation yields

$$D_u v_{\omega,u_0}[\psi] = \int_M \psi D_u f_{\omega,u_0} dm = \int_M \sum_{n=0}^{\infty} \psi L_{\tau^{-n}\omega,u_0}^{(n)} \partial_u L_{\tau^{-(n+1)}\omega,u_0} f_{\tau^{-(n+1)}\omega,u_0} dm,$$

which yields (4.20) by using boundedness of the integral on $C^s(M)$ and the duality property of the transfer operator $\int_M \psi L_{\omega,u_0} \phi dm = \int_M \psi \circ T_{\omega,u_0} \phi dm$. □

Now that we have established the regularity of the map $u \in U \mapsto \int_M \psi d v_{\omega,u}$ as an element of $L^\infty(\Omega, \mathbb{C})$, we obtain for free the regularity of $u \in U \mapsto R_{\psi}(u) := \mathbb{E}(\int_M \psi d v_{\omega,u})$, i.e an annealed version of theorem 4.18:

**Theorem 4.21.** Let $(T_{\omega,u})_{\omega\in\Omega,u\in U} \in \mathcal{P}_{k_T,k_{\kappa},k_{\Sigma}}$ be a random family of uniformly expanding maps, let $(\mathcal{Z}_u)_{u\in U}$ be the transfer operator cocycle it generates above $(\Omega, \tau)$, acting on $X_{\tau^{-1}}(M, \mathbb{C})$, and $v_u$ be the associated random a.c.i.m.

For every observable $\psi \in L^1(M)$, the map $u \in U \mapsto \mathbb{E}[\int_M \psi d v_u]$ is differentiable at $u = u_0$, for every $u_0 \in U$ with

$$D_u \left[ \mathbb{E}[\int_M \psi d v_u] \right]_{u=u_0} = \sum_{n=0}^{\infty} \int_\Omega \int_M \psi \circ T_{\omega,u_0}^{(n)} \partial_u L_{\tau^{-1}\omega,u_0} f_{\tau^{-1}\omega,u_0} dm d \mathbb{P}.$$

(4.23)

4.5. **Proof of Theorem 1.3 (for Examples 1.1 and 1.2).** Consider the type of map used in Example 1.1 in the introduction of the form: $T_u(x) = Mx + \kappa(u,x) \mod \mathbb{Z}^d$, where for simplicity we have omitted reference to $\omega \in \Omega$ in the notation. We want to show that for a measurable collection of such maps we may apply Theorem 4.18 and Theorem 4.21. For this we need to show that $T_u$ is uniformly $\lambda$-expanding and that the map together with the natural weight verify the uniform bounds in section 4.

For the atlas in Hypothesis 4.1 we may take as charts e.g. a collection of $4^d$ cubes $\mathcal{V}_1, \ldots, \mathcal{V}_{4^d}$ of side-length $\frac{1}{2}$ placed in a uniform way on $\mathbb{T}^d$. Then any $\delta_0 \in (0, 1/4)$ is a Lebesgue number and we may identify lengths on charts by the lengths in corresponding cubes $V_j$. The metric in each chart is the standard Euclidean metric. From the conditions in the introduction on the matrix $M$ and the function $\kappa$ we note that for any $v \in \mathbb{R}^d$ we have $|\partial_i T_u(x)v| \geq \lambda|v|$ with $\lambda = \frac{1}{\delta_0} - \theta_1 > 1$, so $T_u$ is indeed uniformly $\lambda$-expanding with $\lambda > 1$.

Let $n = |\det(M)| \geq 2$. The map $x \in \mathbb{T}^d \mapsto Mx \in \mathbb{T}^d$ is an $n$-fold covering map of the torus. By a deformation and continuity argument the same holds for our map $T_u$. For every $1 \leq j \leq 4^d$ we thus have $n$ local inverses of $T_u$: $\psi_j^i : \mathcal{V}_j \rightarrow \mathbb{T}^d$, $i \in \mathcal{A} = \{1, \ldots, n\}$. By the above all local inverses of $T_u$ are $\frac{1}{\lambda}$-Lipschitz contracting.
Applying the implicit function theorem to $T_u(\psi^i_j(u, y)) = y$ and using the specific form for $T_u$ we get for each inverse branch:

$$\partial_y \psi^i_j(u, y) = \left( M + \partial_x \kappa(u, \psi^i_j(u, y)) \right)^{-1}$$

$$\partial_u \psi^i_j(u, y) = -\left( M + \partial_x \kappa(u, \psi^i_j(u, y)) \right)^{-1} \partial_u \kappa(u, \psi^i_j(u, y)).$$

From these we obtain Faà di Bruno like algebraic formulae. Every $q$-th order derivative of $\psi$ will be given in terms of derivatives up to order $q$ of $\kappa$ and up to order $q - 1$ of $\psi$ itself. Since $\|(M + \partial_x \kappa)^{-1}\| \leq \frac{1}{x}$ it follows by a recursive argument that if $\kappa$ is uniformly bounded in $C^{r_0}$-norm then so is every $\psi$. This shows that our inverse branches verify condition $(4.6)$.

For condition $(4.7)$, the weight for the natural measure is given by $g(x) = -\log |\det DT(x)|$ which when composed with an inverse branch on $V_j$ takes the form:

$$g^i_j(u, y) = -\log \left| \det \frac{\partial \psi^i_j(u, y)}{\partial u} \right|$$

$$= -\log n - \log \det \left( 1 + M^{-1} \partial_x \kappa \right). \quad (4.24)$$

An eigenvalue of $M^{-1} \partial_x \kappa$ never exceeds $\theta_0 \theta_1 < 1$ in absolute value. Therefore,

$$0 < (1 - \theta_0 \theta_1)^d \leq \det \left( 1 + M^{-1} \partial_x \kappa \right) \leq (1 + \theta_0 \theta_1)^d. \quad (4.25)$$

As the constant $n$ disappears when taking derivatives, we conclude that $|Dg^i_j|$ is uniformly bounded. Repeating the arguments from above we see that $\|D(g^i_j)|_{(k-1, \alpha)}$ is uniformly bounded as well, proving $(4.7)$.

The upper and lower bounds sum of weights, condition $(4.8)$, follows from formula $(4.24)$ for $g^i_j$ and the bound $(4.25)$, yielding the uniform bound:

$$\frac{1}{(1 + \theta_0 \theta_1)^d} \leq \sum_{x: T_u(x) = y} e^{g(u, x)} \leq \frac{1}{(1 - \theta_0 \theta_1)^d}.$$

We are thus in position to apply Theorems 4.18 and 4.21, which achieves our goal for this example.

For our Example 1.2, the manifold consists of one chart $V = (0, 1)$. We consider auxiliary functions of the type $\kappa \in C^{r_0}(\mathbb{R} \times [0, 1])$ with $\|\kappa\|^{r_0} \leq K$, $\partial_x \kappa(u, x) \geq \theta > \frac{1}{2}$ and $\kappa(u, 0) = 0$, $\kappa(u, 1) = 1$. The parametrized expanding maps $T_u: D_u \rightarrow V = (0, 1)$ given by $T_u = 1/\kappa - [1/\kappa]$, is defined whenever $1/\kappa$ is not an integer, so that $D_u$ is $V$ minus a countable set. For fixed $u$ there are countably many inverse branches, which we may index by $i \in \mathbb{N}$ the corresponding integer value of $\frac{1}{i}$ for that branch.

We thus have: $T_u(\psi^i(u, y)) = \frac{1}{\kappa(u, \psi^i(u, y))} - i = y$, $i \in A = \mathbb{N}$. Since $\partial_x T_u = -\partial_x \kappa/\kappa^2$ and $\partial_u T_u = -\partial_u \kappa/\kappa^2$ we get for each inverse branch ($i \in \mathbb{N}$) the two identities:

$$\partial_y \psi^i(u, y) = \frac{-1}{(i + y)^2 \partial_x \kappa(u, \psi^i(u, y))} \quad \text{and} \quad \partial_u \psi^i(u, y) = -\frac{\partial_u \kappa(u, \psi^i(u, y))}{\partial_x \kappa(u, \psi^i(u, y))}.$$

As in the previous example the first condition $(4.6)$ is clearly verified.
For the weight condition note that $\partial_x \kappa$ takes values in $[\frac{1}{\lambda}, 2]$ implying that $\log(\partial_x \kappa(u, \psi^i_j(y)))$ is $C^{r_0 - 1}$ bounded, and similarly the derivative of $\log(i + y)$ is $C^{r_0 - 2}$ bounded.

By summability of $\left(\frac{1}{t^i}\right)_{i \geq 1}$ (4.8) also holds.

The only lacking element is the uniform contraction (since $\partial_y \psi(u, y)$ need not be smaller than one). For this we introduce the metric in the chart $(0, 1)$: $g(x) = \frac{1}{1 + x}$, or equivalently given by the line element $ds = \frac{1}{1 + x} |dx|$. Our atlas verifies Hypothesis (4.1) and for the metric derivative we have:

$$\|\partial_y \psi^i(u, y)\|_{g_\psi} = \frac{1 + y}{1 + \psi^i(u, y)} \left|\partial_y \psi^i(u, y)\right|.$$  

From $\partial_x \kappa \geq \theta$ we get $1 - \kappa = \int_x^1 \partial_x \kappa \, dx \geq \theta (1 - x)$ or $\kappa = \frac{1}{1 + x} \leq \theta x + (1 - \theta)$. Then as $\theta > 1/2$ and $x = \psi^i(u, y) \in [0, 1]$:

$$\|\partial_y \psi^i(u, y)\|_{g_\psi} \leq \frac{1 + y}{1 + x} \times \frac{1}{\theta(i + y)^2} \leq \frac{1}{1 + x} \frac{1}{\theta} \frac{1}{i + y} \leq \frac{\theta x + 1 - \theta}{\theta x + \theta} \leq \frac{1}{2\theta} < 1.$$  

We may then apply our theorems as wanted. Note that for $\theta \in (0, \frac{1}{2})$ we may find $\kappa$ satisfying all other conditions and for which the associated $T_u$ has an attractive fixed point.  

\[\square\]

4.6. Application: Hausdorff dimension of repellers for 1D expanding maps. In this section we are interested in the random product of one-dimensional maps, with uniform dilation but not necessarily defined everywhere. More precisely, we are interested in the following class of systems:

**Definition 4.22.** Let $I_1, \ldots, I_N \subset [0, 1]$ be disjoint intervals, and $r \geq 2$. A $C^r$ map $T : I_1 \cup \cdots \cup I_N \to [0, 1]$ is called a **cookie-cutter** if it satisfies the following conditions:

- There exists some $\lambda > 1$ such that $\inf |T'| \geq \lambda$
- For each $i \in \{1, \ldots, n\}$, $T(I_i) = [0, 1]$

If $T$ is a cookie-cutter, we introduce its **repeller**,  

$$\Lambda := \{x \in I_1 \cup \cdots \cup I_N, T^n(x) \text{ is well-defined for all } n\} = \bigcap_{i=1}^{\infty} T^{-i}([0, 1])$$

We will denote by $CC^r([0, 1])$ the set of all $C^r$ cookie-cutters.

In other words, a cookie-cutter is a one-dimensional expanding map with full branches. It is a well-known fact that the repeller associated to such a map is a Cantor set. We now define perturbed cookie-cutters, in the following way:

**Definition 4.23.** Let $U$ be an open subset of some Banach space $B$, and let $\psi_i : U \times [0, 1] \to (0, 1), i \in \{1, \ldots, N\}$ be $C^r$ maps such that

- For every $i \in \{1, \ldots, N\}$, $\|\partial_x \psi_i\|_\infty \leq 1/\lambda < 1$
- For every $i \in \{1, \ldots, N\}$, every $u \in U$, the intervals $I_{i,u} = \psi_i(u_i([0, 1]))$ are pairwise disjoint.
This data defines a cookie-cutter map $T_u$ on $I_{1,u}, \ldots, I_{N,u}$, by $T_u = \psi_{i,u}^{-1}$ on $I_{i,u}$. We call it a perturbed cookie-cutter.

The question we want to study is the following: if one were to choose at each step a random cookie-cutter, and then perturb it in the sense of definition 4.23, does the Hausdorff dimension of the repeller change in a smooth way?

To answer that question, we will use a random version of Bowen formula, which connects the transfer operator cocycle’s top characteristic exponent and the Hausdorff dimension of the associated (random) repeller (cf. theorem 4.24).

More precisely, one considers a random product $T_{\omega,u}^{(n)} := T_{\tau^{n-1}\omega,u} \circ \cdots \circ T_{\omega,u}$ of perturbed cookie-cutters, i.e we assume that each $T_{\omega,u} \in P_{kT,k_g,k_\Sigma}$ is a perturbed cookie-cutter in the sense of Definition 4.23.

Associated to this random product is a random repeller, defined by

$$\Lambda_{\omega,u} := \bigcap_{i=1}^{\infty} \left( T_{\omega,u}^{(i)} \right)^{-1} ([0, 1]).$$

Given $t \geq 0$, we also define the transfer operator $L_{\omega,t,u}$ by

$$L_{\omega,t,u} \phi(x) := \sum_{T_{\omega,u}y = x} \frac{1}{|T_{\omega,u}'(y)|^t} \phi(y)$$

It follows from Theorem 4.17 that $L_{\tau,t,u}$ is a cone-contracting cocycle in the sense of Sect. 2, for $C_a$.

It is also clear from the definition that $L_{\omega,t,u}$ depends analytically on $t \geq 0$ (up to considering a small complex extension of $t$), so that it follows from [49, Theorem 10.2] that for every fixed $u_0 \in U$, the map $t \mapsto \chi_{t,u_0}$ is analytic.

We also introduce the following quantities:

$$M_n(\omega,t,u) := \sup_{y \in \Lambda_{\omega,t,u}} L_{\omega,t,u}^{(n)} 1(y)$$

$$m_n(\omega,t,u) := \inf_{y \in \Lambda_{\omega,t,u}} L_{\omega,t,u}^{(n)} 1(y)$$

where $\Lambda_{\omega,t,u} := \bigcap_{i=1}^{\infty} \left( T_{\omega,u}^{(i)} \right)^{-1} ([0, 1])$, and finally we let

$$-\infty \leq P(\omega,t,u) := \lim \inf \frac{1}{n} \log(m_n(\omega,t,u)) \leq \overline{P}(\omega,t,u)$$

$$:= \lim \sup \frac{1}{n} \log(M_n(\omega,t,u)) \leq +\infty$$

Those last quantities exists by super-multiplicativity (resp. sub-multiplicativity) and Kingman’s ergodic theorem, and are $P$-almost surely constant by ergodicity of $\tau$.

One can show that those quantities almost surely agree, their common value being $\chi_{t,u}$ the top characteristic exponent of the random product, and that $t \geq 0 \mapsto \chi_{t,u} + t \log(\lambda) \in \mathbb{R}$ is decreasing (see [48, Lemma 3.5 and Theorem 4.4]).

Furthermore, this decreasing map admits a unique zero that coincide with the (a.s) Hausdorff dimension of the random repeller $\Lambda_{\omega,u}$ (see [48, Theorem 4.4 and 5.3]).
Theorem 4.24 ([48] Theorem 5.3). Let $\tau$ be an invertible and ergodic map of $(\Omega, \mathbb{P})$. Let $(T_\omega)_{\omega \in \Omega}$ be a random product of cookie-cutters, such that $\mathbb{E} \left[ \log \| T_\omega' \|_\infty \right] < \infty$.

Then $\mathbb{P}$-almost surely the Hausdorff dimension of the random repeller $\Lambda_\omega$ is given by the unique zero $z(T)$ of the top characteristic exponent $\chi_t$ of the transfer operator cocycle $\hat{L}_t$.

For a proof, we refer to [48, 4-5]. The question is now the dependence of that zero on the parameter $u$:

Theorem 4.25. Let $(T_{\omega,u})_{\omega \in \Omega, u \in U}$ be a random product of perturbed cookie-cutters.

Then the Hausdorff dimension of the random repeller defined by (4.26) is $C^s$ with respect to $u \in U$.

Proof. Theorem 4.24 entails that the almost-sure Hausdorff dimension of $\Lambda_{\omega,u}$ is given by some $z(u)$ such that $\chi_{z(u)}, u = 0$.

From Theorems 4.24 and 3.2, one has that

- For every $u \in U$, $\chi_{z(u)}, u = 0$
- The map $(t, u) \mapsto \chi_t, u$ is $C^{r-2}$.
- $\partial_t \chi(t, u) \leq -\log(\lambda) < 0$

Hence the result follows from the implicit function theorem. \hfill \Box

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A Real cone contraction theory

As the literature is somewhat disparate on the subject, we provide here a catalog of more or less standard results on cone contractions. Most of these results already occur in similar forms in e.g. [11, 21, 41]. In a few cases, we have added (short) proofs where appropriate. In the following, $(E, \| \cdot \|)$ will denote a real Banach space.

Definition A.1. Let $C \subset E$. We say that $C$ is a proper closed convex cone (for short, a Birkhoff cone) if

- $\mathbb{R}^+_C = C$, i.e $C$ is stable by multiplication with a positive scalar.
- $C$ is a closed and convex subset of $E$.
- $C \cap (-C) = \{ 0 \}$ (the cone is proper).

Definition A.2. Let $C \subset E$ be a Birkhoff cone. We say that $C$ is

1. inner regular iff $C$ has non-empty interior. Equivalently, there is $\rho > 0$ so that:
\[ C(\rho) = \{ x \in C : B_E(x, \rho \| x \|) \subset C \} \]

is non-trivial (contains other points than the origin).
2. outer regular if there is $\ell \in E'$, $\| \ell \| = 1$ and $K < +\infty$ such that for every $u \in C$:
\[ \frac{1}{K} \| u \| \leq \langle \ell, u \rangle \leq \| u \|. \quad (A.1) \]
We will say that $C$ is **regular** if it is both inner and outer regular. We define:

$$C_{\ell=1}(\rho) = \{ u \in C(\rho) : \langle \ell, u \rangle = 1 \}.$$  

**Definition A.3.** Let $C \subset E$ be a Birkhoff cone, and let $x, y \in C^*$. We define the projective Hilbert metric:

$$\delta(x, y) = \inf \{ t > 0 , \ t x - y \in C \}$$

$d_C(x, y) = \log(\delta(x, y)\delta(y, x)) \in [0, +\infty]$.

Hilbert’s original (but equivalent) definition was through cross-ratios which may be formulated as follows: When $x, y \in C^*$ are non-colinear, it is always possible to normalize $x$ and $y$ so that $I(x, y) = \{ t \in \mathbb{R} : (1 + t)x + (1 - t)y \in C \} = [t_1, t_2]$ is a bounded (closed) interval, with $t_1 \leq -1 < 1 \leq t_2$. One then has for their projective distance, relative to the cone:

$$d_C(x, y) = \log \frac{t_2 + 1}{t_2 - 1} \frac{t_1 - 1}{t_1 + 1} \in [0, +\infty] \quad (A.2)$$

**Lemma A.4.** Let $C \subset E$ be a regular Birkhoff cone, and let $d_C$ be the associated Hilbert metric. Suppose that $B(x_1, r_1) \subset C$ and $B(x_2, r_2) \subset C$ for some $r_1, r_2 > 0$. Then

$$d_C(x_1, x_2) \leq \log \left( 1 + \frac{\|x_1 - x_2\|}{r_1} \right) + \log \left( 1 + \frac{\|x_1 - x_2\|}{r_2} \right) \leq \left( \frac{1}{r_1} + \frac{1}{r_2} \right) \|x_1 - x_2\|.$$

**Proof.** Inclusion of the two balls imply that $I(x_1, x_2) \supseteq [-1 - \frac{2r_1}{\|x_1 - x_2\|}, +1 + \frac{2r_2}{\|x_1 - x_2\|}]$ which together with (A.2) yields the result. $\Box$

**Corollary A.5.** For $x, y \in C_{\ell=1}(\rho)$: $d_C(x, y) \leq 2 \log \left( 1 + \frac{\|x - y\|}{\rho} \right) \leq \frac{2}{\rho} \|x - y\|$. When $x \in C_{\ell=1}(\rho)$ and $\|u\| < \rho$ we have: $d_C(x, x+u) \leq \frac{2\|u\|}{\rho - \|u\|} = \frac{2}{\rho} \|u\| + o(\|u\|)$.

**Lemma A.6** ([21], Appendix A). Let $C \subset E$ be an outer regular Birkhoff cone with $\ell \in E'$ as above. Then for all $x, y \in C^*$:

$$\left\| \frac{x}{\langle \ell, x \rangle} - \frac{y}{\langle \ell, y \rangle} \right\| \leq \frac{K}{2} d_C(x, y)$$

**Theorem A.7** (Birkhoff’s theorem, [11]). Let $C \subset E$ be a Birkhoff cone and let $L \in L(E)$ be a contraction of $C^*$, i.e. such that $L(C^*) \subset C^*$. Setting $\Delta = \text{diam}_C(L(C^*)) \in [0, +\infty]$ we have for $x, y \in C^*$:

$$d_C(Lx, Ly) \leq \left( \tanh \frac{\Delta}{4} \right) d_C(x, y)$$

**Corollary A.8.** Let $C$ be a regular cone and $n \in \mathbb{N}$. Let $(L_i)_{1 \leq i \leq n} \in L(E)$ be cone contractions, i.e $L_i(C^*) \subset C^*$ for any $1 \leq i \leq n$ and set $\Delta_i = \text{diam}_C(L_i(C^*)) \in [0, +\infty]$. Then for all $x, y \in C^*$:

$$\left\| \frac{L_n \ldots L_1 x}{\langle \ell, L_n \ldots L_1 \ell \rangle} - \frac{L_n \ldots L_1 y}{\langle \ell, L_n \ldots L_1 \ell \rangle} \right\| \leq \frac{K}{2} \left( \prod_{i=1}^{n} \tanh \frac{\Delta_i}{4} \right) d_C(x, y) \quad (A.3)$$
Lemma A.9. Let $C$ be a regular cone with associated $\ell$, $K < +\infty$ and $\rho > 0$ as in Definition A.2. Let $L \in L(E)$ with $L(C) \subset C$. Then for every $x \in C(\rho)^*$:

$$\frac{\rho}{K} \|L\| \|x\| \leq \langle \ell, Lx \rangle \leq \|L\| \|x\| \text{ and } \frac{\rho}{K} \|L\| \leq \frac{\langle \ell, Lx \rangle}{\langle \ell, x \rangle} \leq K \|L\|.$$  

Proof. Suppose $x \in C(\rho)$, $\|x\| = 1$ and consider $\|u\| \leq 1$. Then $x \pm \rho u \in C$ so also $L(x \pm \rho u) \in C$. By outer regularity: $\langle \ell, L(x \pm \rho u) \rangle \geq \frac{1}{K} \|L(x \pm \rho u)\|$. Therefore,

$$2\rho \|Lu\| \leq \|L(x + \rho u)\| + \|L(x - \rho u)\| \leq K \langle \ell, L(x + \rho u) + L(x - \rho u) \rangle = 2K \langle \ell, Lx \rangle$$

Thus, $\frac{\rho}{K} \|L\| \leq \langle \ell, Lx \rangle$ from which we deduce the left-most inequality. The rest follows from (A.1). □

Lemma A.10. Let $C$ be a regular Birkhoff cone with associated linear functional $\ell \in E'$ and $K < +\infty$ as in Definition A.2. Let $C_1 \subset C$ be a subcone of finite diameter, $\Delta = \text{diam}_C(C_1^+) < +\infty$. Suppose that there is $x \in C_1$, $\langle \ell, x \rangle = 1$ with $B(x, r) \subset C_1$.

Then for every $y \in C_1$,

$$B_E \left( y, \frac{1}{K} re^{-\Delta} \|y\| \right) \subset C.$$  

Proof. Pick $x, y \in C_{\ell=1}$ and set $u_t = \frac{1}{2}((1+t)y + (1-t)x)$. Then $\{t \in \mathbb{R} : u_t \in C\} = [t_1, t_2]$ is a bounded (closed) interval (definition A.3) with $t_1 \leq -1 < 1 \leq t_2$. Then $\Delta \geq d_C(x, y) \geq \log \frac{t_2+1}{t_2-1} = \log \frac{t_2+1}{t_2-1} \geq e^{-\Delta}$. For $|u| < 1$ we have $x + ru \in C$. Also $u_{t_2} \in C$ so the following convex combination is also in $C$:

$$\frac{(t_2 - 1)(x + ru) + 2u_{t_2}}{t_2 + 1} = y + r \frac{t_2 - 1}{t_2 + 1} u \in C.$$  

Here $\frac{1}{K} \|y\| \leq \langle \ell, y \rangle = 1$ so $B(y, \rho \|y\|) \subset C$ with $\rho = \frac{1}{K} re^{-\Delta}$. □

### B Bochner and Strong Measurability

In our setup we need measurability of quantities related to sections and operators. It is close to standard Bochner measurability and strong measurability in the sense of e.g. [31, Appendix A] but not quite the same so we bring here a brief account of the notions we use.

In this appendix $(X, \| \cdot \|_X)$ denotes a Banach space and $(\Omega, \mathcal{F})$ a non-empty space equipped with a $\sigma$-algebra.

Definition B.1. 1. A map $\phi : \Omega \to X$ is said to be $\sigma$-simple if it has a countable image and is measurable (with respect to $\mathcal{F}$). We write $S_{\sigma}(\Omega, X)$ for the set of such functions.

2. A map $\psi : \Omega \to X$ is said to be Bochner-measurable if it may be written as the uniform limit of a sequence of $\sigma$-simple functions. We write $L^\infty(\Omega; X)$ for the set of Bochner measurable maps that are uniformly bounded. It is a Banach space under the uniform norm.
3. Let $A \subset X$ be a non-empty set. We say that $\mathcal{L} : \Omega \to L(X)$ is strongly Bochner measurable on $A$ provided that $\omega \in \Omega \mapsto \mathcal{L}(\omega)a \in X$ is Bochner-measurable for every $a \in A$.

4. We say that $\mathcal{L}$ in the previous definition is measurably bounded provided there is a measurable function $\rho : \Omega \mapsto [0, +\infty)$ so that $\|\mathcal{L}(\omega)\|_{L(X)} \leq \rho(\omega)$ for every $\omega \in \Omega$.

$\mathcal{L}$ is of course bounded if $\sup_{\omega \in \Omega} \|\mathcal{L}(\omega)\|_{L(X)} < +\infty$.

**Remark B.2.** A pointwise limit of measurable functions is measurable [39, VI, § 1, M7], so the above definition a Bochner-measurable function is equivalent to saying that $\psi$ is measurable with image having a countable dense subset.

**Proposition B.3.** Let $\phi : \Omega \to A \subset X$ be Bochner measurable with values in $A$ and let $\mathcal{L} : \Omega \to L(X)$ be measurably bounded and strongly Bochner measurable on $A$. Then

$$\omega \in \Omega \mapsto \mathcal{L}(\omega)\phi(\omega) \in X$$

is also Bochner measurable.

**Proof.** Let $\rho$ be as in the last part of Definition B.1. For $m = 0, 1, \ldots$ we set $\Omega_m = \rho^{-1}(\{m, m + 1\})$ which provides a measurable partition of $\Omega$. Let $\epsilon > 0$. By Bochner measurability, and for each $m \geq 0$ we may find sequences $x_{m,k} \in A \subset X$ and $E_{m,k} \in \mathcal{F}$, $k \geq 1$ so that $(E_{m,k})_{k \geq 1}$ form a measurable partition and $|\phi(\omega) - x_{m,k}| \leq \frac{\epsilon}{2^{m+1}}$. By strong measurability on $A$ we have for each $m \geq 0, k \geq 1$ sequences $y_{m,k,\ell} \in X$ and $F_{m,k,\ell} \in \mathcal{F}$, $\ell \geq 1$ so that $(F_{m,k,\ell})_{\ell \geq 1}$ forms a measurable partition and $|\mathcal{L}(\omega)x_{m,k} - y_{m,k,\ell}| \leq \epsilon/2$ for every $\omega \in F_{m,k,\ell}$. Then for $\omega \in G_{m,k,\ell} = \Omega_m \cap E_{m,k} \cap F_{m,k,\ell}$ we have

$$|\mathcal{L}(\omega)\phi(\omega) - y_{m,k,\ell}| \leq |\mathcal{L}(\omega)(\phi(\omega) - x_{m,k})| + |\mathcal{L}(\omega)x_{m,k} - y_{m,k,\ell}| \leq \epsilon.$$

The $G_{m,k,\ell}$-collection gives a measurable partition of $\Omega$ and the conclusion follows. \qed

**C Differential Calculus**

We provide a listing of some of the more or less standard results in differential calculus which we are using. Let $U$ and $V$ denote open convex sets in Banach spaces $B_U$ and $B_V$, respectively, and let $Z$ be a fixed Banach space. We write $|\phi|_0$ (or $\|\phi\|_0$) to denote a uniform norm on the relevant domain of definition.

Let $r > 0$ and $\delta_1 \in (0, 1]$ be fixed number in this section. We write $r = (k, \alpha)$, with $k \in \mathbb{N}_0$ and $\alpha \in (0, 1]$ such that $r = k + \alpha$. $C^r$-norms between open convex subsets of Banach spaces are defined in the standard way. By $C^s(U; Z)$ we understand the closure of $C^s(U; Z)$ in $C^r(U; Z)$ with $s > r$ (the closure is independent of the choice of $s$).

Given $A \in L(E_1 \times \cdots \times E_n; Z)$, a multilinear form with $E_1, \ldots, E_n$ being Banach spaces, there is a natural isomorphism obtained by singling out the $j$’th Banach space, yielding: $A_j \in L(E_1 \times \cdots \times \hat{E}_j \cdots \times E_n; L(E_j; Z))$. In many places in this article we make use of the telescopic principle which asserts that

$$\|A(M_1, \ldots, M_n) - A(N_1, \ldots, N_n)\|$$

$$\leq \sum_{j=1}^n \|A_j(M_1, \ldots, M_{j-1}, N_{j+1}, \ldots, N_n)\| \|M_j - N_j\|. \quad (C.1)$$
**Proposition C.1.** Let \( \phi_1, \phi_2 \in c^2(V; \mathbb{C}) \), \( \phi \in c^2(V; Z) \) and \( \psi \in c^1(U; V) \), \( t = r \lor 1 \). Then there are constants \( C_{r,1}, C_{r,2}, C_{r,3}, C_{4,r} \) depending only upon \( r \) and the chosen norms so that:

1. \( \phi_1 \phi_2 \in c^2(V; \mathbb{C}) \) and \( \|\phi_1 \phi_2\|_r \leq C_{1,r} \|\phi_1\|_r \|\phi_2\|_r \).
2. \( e^{\phi_1} \in c^2(V; \mathbb{C}) \) and \( \|e^{\phi_1}\|_r \leq C_{2,r} |e^{\phi_1}|_0 (1 + \|\phi_1\|_r)^r \).
3. \( \phi \circ \psi \in c^2(U; Z) \) and \( \|\phi \circ \psi\|_r \leq C_{3,r} \|\phi\|_r (1 + \|D\psi\|_t-1)^r \).

and also:

4. \( \|e^{\phi_1} - e^{\phi_2}\|_r \leq C_{4,r} (|e^{\phi_1}|_0 \lor |e^{\phi_2}|_0) \|\phi_1 - \phi_2\|_r (1 + \|\phi_1\|_r \lor \|\phi_2\|_r)^r \).

**Proof.** 1. For \( C^k \)-functions with \( k \) an integer, this is standard. When \( r = (k, \alpha) \) with \( \alpha \in (0, 1] \) we have \( D^k(\phi_1 \phi_2) = (D^k \phi_1) \phi_2 + \phi_1 (D^k \phi_2) + R_{k-1} \), where \( R_{k-1} \) is a bilinear form in \( \phi_1 \) and \( \phi_2 \) involving derivatives of order at most \( k-1 \). The 1-local Hölder constant may then be estimated using the MVT on the last term and the above-mentioned telescopic principle to obtain: \( h^r_{\delta_1}(D^k(\phi_1 \phi_2)) \leq h^r_{\delta_1}(D^k \phi_1)\|\phi_2\|_0 + \|\phi_1\|_0 h^r_{\delta_1}(D^k \phi_2) + \|D R_{k-1}\|_0 \).

The last term is bounded by the \( C^k \) norm of \( \phi_1 \phi_2 \).

2. Write \( D^q(e^{\phi_1}) = e^{\phi_1} (D^q \phi_1 + D^q 1) \), \( 0 \leq q \leq k \) and develop. For the Hölder constant for \( D^k(e^{\phi_1}) \) we use the same argument as above.

3. The so-called Faà di Bruno formula gives a combinatorial expression for \( D^k(\phi \circ \psi) \). Exhibiting the \( k \)’th order derivative one has:

\[
D^k(\phi \circ \psi) = D^k \phi \circ \psi, (D\psi, \ldots, D\psi) + D\phi \circ \psi, D^k \psi + R_{k-1}.
\]

Again \( R_{k-1} \) involves only derivatives up to order \( k - 1 \) and may be treated using the MVT. For the Hölder constant of the \( k \)’th derivative, let \( K = \|D\psi\|_0 \). If \( y, y' \in U \) are at a distance at most \( \delta_1 \), then the \( \psi \)-image of this path has length (in \( V \)) at most \( h_{\delta_1} \). Lemma 4.14 then shows that \( h_{\delta_1}(D^k \phi \circ \psi) \leq [K]^{1-\alpha} h_{\delta_1}(D^k \phi) \).

4. For the last inequality, the MVT yields \( |e^{\phi_1} - e^{\phi_2}|_0 \leq L(|e^{\phi_1}|_0 \lor |e^{\phi_2}|_0) \|\phi_1 - \phi_2\|_0 \). Developing \( D^q(e^{\phi_1} - e^{\phi_2}) \) and using the telescopic principle yields the wanted bound. The above calculations were done for \( C^r \) functions. But the uniform bounds implies that the result easily carries over to \( c^2 \) functions as well. \( \square \)

### C.1 Regularity when extracting a parameter

In this section we will show how the regularity of a function of two variables behaves when extracting one variable as a parameter. The notation is as above. We have the following smoothness result when extracting a variable as a parameter:

**Theorem C.2.** We equip the product \( B_U \times B_V \) with the max norm. Let \( r, s, t \geq 0 \) with \( t = r - s > 0 \). We have the following canonical continuous injections:

\[
\phi \in C^r(U \times V; Z) \hookrightarrow \widehat{\phi} \in C^s(U; C^t(V; Z)) \quad \text{(C.2)}
\]

\[
\phi \in c^r(U \times V; Z) \hookrightarrow \hat{\phi} \in c^s(U; c^t(V; Z)) \quad \text{(C.3)}
\]

under the natural identification: \( \widehat{\phi}_u(x) := \phi(u, x) \), \( u \in U, x \in V \).
Proof. Consider the first inclusion. Denote \( Y_s = C^s(V; Z) \) and let \( \phi \in C^r(U \times V; Z) \). Our claim is that \( \hat{\phi}_u \in Y_s \) and that the map \( u \in U \mapsto \hat{\phi}_u \in Y_s \) is \( C^r \) with \( t = r - s > 0 \).

Case 1: We first show this for \( 0 \leq s < r \leq 1 \). Here \( \| \phi \|_r = |\phi|_0 \vee h^r_{\delta_1}(\phi) \) is simply the local \( r \)-Hölder norm. For fixed \( u \in U \), clearly \( |\hat{\phi}_u|_0 \leq |\phi|_0 \) and \( h^r_{\delta_1}(\hat{\phi}_u) \leq h^r_{\delta_1}(\hat{\phi}_u) \leq h^r_{\delta_1}(\phi) \) (since \( 0 < \delta_1 \leq 1 \)), so \( \hat{\phi}_u \in Y_s \) with \( \| \hat{\phi}_u \|_s \leq \| \phi \|_r \). To check the regularity w.r.t. \( u \) pick \( u_1, u_0 \in U \) with \( 0 < |u_1 - u_0| \leq \delta_1 \) and set \( \Delta = \hat{\phi}_{u_1} - \hat{\phi}_{u_0} \in Y_s \). We have \( |\Delta|_0 \leq h^r_{\delta_1}(\phi)|u_1 - u_0|^r \).

To check Hölder regularity with respect to \( x \) consider \( x_0, x_1 \in V \) with \( 0 < |x_1 - x_0| \leq \delta_1 \). Then \( \Delta(x_1) - \Delta(x_0) = \phi(u_1, x_1) - \phi(u_1, x_0) + \phi(u_0, x_1) - \phi(u_0, x_0) \) may be estimated using Hölder-regularity either with respect to \( u \) or to \( x \). This yields (the middle term is maximal when \( |x_1 - x_0| = |u_1 - u_0| ):

\[
\frac{|\Delta(x_1) - \Delta(x_0)|}{|x_1 - x_0|^s} \leq 2h^r_{\delta_1}(\phi) \frac{|x_1 - x_0|^r \wedge |u_1 - u_0|^r}{|x_1 - x_0|^s} \leq 2h^r_{\delta_1}(\phi)|u_1 - u_0|^{r-s}.
\]

Thus, \( h_s(\Delta) \leq 2h^r_{\delta_1}(\phi)|u_1 - u_0|^{r-s} \) from which: \( \| \hat{\phi}_{u_1} - \hat{\phi}_{u_0} \|_Y_s \leq 2 \| \phi \|_r |u_1 - u_0|^{r-s} \).

So \( \hat{\phi} \in C^{(0,r-s)}(U; Y_s) \) with \( \| \hat{\phi} \|_{r-s} \leq 2 \| \phi \|_r \).

Case 2: Consider now when \( 0 \leq s \leq 1 < r \leq 1 + s \). Again for fixed \( u \): \( \hat{\phi}_u \in Y_s \) with \( \| \hat{\phi}_u \|_s \leq \| \phi \|_r \). With \( \Delta \) as above we have \( |\Delta|_0 \leq |D\phi|_0 |u_1 - u_0| \leq \| \phi \|_r |u_1 - u_0|^{r-s} \).

Let \( u(\tau) = \tau u_1 + (1 - \tau)u_0 \), \( 0 \leq \tau \leq 1 \) be the segment joining \( u(0) = u_0 \) and \( u(1) = u_1 \) (included in \( U \) by convexity). Applying the MVT, one gets:

\[
|\Delta(x_1) - \Delta(x_0)| \leq \int_0^1 \left| \frac{d}{d\tau}(\hat{\phi}_{u_\tau}(x_1) - \hat{\phi}_{u_\tau}(x_0)) \right| d\tau \\
\leq \int_0^1 |D\hat{\phi}_{u_\tau}(x_1) - D\hat{\phi}_{u_\tau}(x_0)| |u'()|d\tau \\
\leq h^{r-1}_{\delta_1}(\phi)|x_1 - x_0|^{r-1}|u_1 - u_0|.
\]

Interchanging the roles of \( x_1, x_0 \) and \( u_1, u_0 \) we also have:

\[
|\Delta(x_1) - \Delta(x_0)| \leq h^{r-1}_{\delta_1}(\phi)|u_1 - u_0|^{r-1}|x_1 - x_0|,
\]

and then (again the middle term is maximal for \( |u_1 - u_0| = |x_1 - x_0| ):

\[
\frac{|\Delta(x_1) - \Delta(x_0)|}{|x_1 - x_0|^s} \leq h^{r-1}_{\delta_1}(\phi) \frac{|x_1 - x_0|^r \wedge |u_1 - u_0|^r}{|x_1 - x_0|^s} \leq h^{r-1}_{\delta_1}(\phi)|u_1 - u_0|^{r-s} \leq \| \phi \|_r |u_1 - u_0|^{r-s}.
\]

It follows that \( \| \hat{\phi}_{u_1} - \hat{\phi}_{u_0} \|_Y_s \leq \| \phi \|_r |u_1 - u_0|^{r-s} \) and \( \hat{\phi} \in C^{(0,r-s)}(U; Y_s) \) with \( \| \hat{\phi} \|_{r-s} \leq \| \phi \|_r \). Note that when \( r - s = 1 \) the conclusion is that \( \phi \in C^{(1,s)}(U \times V; Z) \leftrightarrow \hat{\phi} \in C^{(0,1)}(U; C^s(V; Z)) \),

i.e. in general, \( u \mapsto \hat{\phi}_u \) need not be differentiable in this case, but it is Lipschitz continuous.

Higher order regularity can be reduced to the above two cases.

To see this let \( 0 \leq \beta < \alpha \leq 1 \) and \( r = k + m + \alpha, s = m + \beta \) with \( k, m \in \mathbb{N}_0 \). For \( \phi \in C^r(U \times V; Z) \) and fixed \( u \in U \) it is clear that \( \hat{\phi}_u \in C^{(m,\beta)}(V; Z) \). For the regularity of the map \( u \mapsto \hat{\phi}_u \) we leave intermediate derivatives to the reader and consider only
the highest order. Note that the first case treated above yields the following injection and identification (using natural isomorphisms for the linear maps involved):

\[ \partial^k \partial^m \phi \in C^{(0,\alpha)}(U \times V; L(B^k_U \times B^m_V; Z)) \]
\[ \iff C^{(0,\alpha-\beta)}(U; C^{(0,\beta)}(V; L(B^k_U \times B^m_V; Z))) \]
\[ \simeq C^{(0,\alpha-\beta)}(U; L(B^k_U; C^{(0,\beta)}(V; L(B^m_V; Z)))). \]

with \( W = C^{(0,\beta)}(V; L(B^m_V; Z)) \). We observe that the latter precisely gives the identification with \( \partial^m(\partial^k \phi_u) \) and implies that \( \partial^k \phi_u \in C^{(0,\alpha-\beta)}(U; L(B^k_U; C^s(V; Z))) \) whence that \( \phi_u \in C^t(U; C^s(V; Z)) \) with \( t = (k, \alpha - \beta) \) as we wished to show.

In the case \( 0 < \alpha \leq \beta \leq 1 \) and \( r = k + m + 1 + \alpha, s = m + \beta \) we consider again \( \partial^k \partial^m \phi \) which reduces the necessary injection to the second case treated above. In either case the norm increases at most by a factor of 2.

For the second injection, if \( \phi \) is the \( C^\infty \) limit of smooth functions then the induced function \( \phi_u \) is the \( C^s \) limit of smooth functions. The norm-estimates carry over from before. □

**Remark C.3.** For non-compact \( V \) and for integer values \( n \geq k \geq 0 \), there is in general no natural injection of \( C^n(U \times V; \mathbb{R}) \) into \( C^{n-k}(U; C^k(V; \mathbb{R})) \). An easy (counter) example is for \( n = k = 0, U = V = (0, 1) \), where you may consider e.g. \( \phi(u, x) = \sin(u/x) \).

**Corollary C.4.** Assume that \( W \subset B_W \) is an open convex subset of a Banach space. Let \( r > 1 \) and \( \beta \in (0, 1) \). Let \( \psi_0, \psi_1 \in C^r(V; W) \) and \( \phi \in C^{r+\beta}(W; Z) \). We write \( M = \|\psi_0\|_r \vee \|\psi_1\|_r \vee 1 \). Suppose that \( \|\psi_0 - \psi_1\|_r \leq \delta_1 \). Then we have

\[ \|\phi \circ \psi_1 - \phi \circ \psi_0\|_r \leq C_{r,\beta} \|\phi\|_{r+\beta} M^r \|\psi_0 - \psi_1\|_r^\beta \]

with \( C_{r,\beta} \) a constant that depends only upon \( r, \beta \) and the choice of norms.

**Proof.** We first show this under the additional assumption that \( \psi_0, \psi_1 \in C^{r+\beta} \). Let \( a = \|\psi_0 - \psi_1\|_r \). We may assume that \( a > 0 \) or else there is nothing to show. We define for \( (t, y) \in [0, a] \times V \) the following linear interpolation between \( \psi_0 \) and \( \psi_1 \) (allowed since \( W \) is convex):

\[ \widehat{\psi}(t, y) = \frac{t}{a} \psi_1(y) + \left(1 - \frac{t}{a}\right) \psi_0(y) = \psi_0(y) + \frac{t}{a} (\psi_1(y) - \psi_0(y)). \]

One has \( \widehat{\psi} \in C^{r+\beta}([0, a] \times V; W) \).\(^9\) Note that \( \partial_1 \widehat{\psi} = \frac{1}{a} (\psi_1 - \psi_0) \) has \( (r+\beta-1) \)-norm not greater than 1. It follows that \( \|\widehat{\psi}\|_{r+\beta} \leq \|\psi_0\|_{r+\beta} \vee \|\psi_1\|_{r+\beta} \vee 1 \). It follows from Proposition C.1 that \( F = \phi \circ \widehat{\psi} \) has \( (r+\beta) \)-norm bounded by \( K = C_{r,\beta} \|\phi\|_{r+\beta} M^r \).

By our parameter-extraction theorem C.2 we deduce that \( t \in [0, a] \mapsto (F(t, \cdot) \in C^r(U; Z)) \) is \( C^{(0,\beta)} \) with at most twice the indicated bound for the norm. But then the Hölder bound implies:

\[ \|\phi \circ \psi_1 - \phi \circ \psi_0\|_r = \|F(a, \cdot) - F(0, \cdot)\|_r \leq 2K|a - 0|^\beta = 2K \|\psi_0 - \psi_1\|_r^\beta \]

\( [0, a] \times V \) is not open, but the construction of \( C^r \) functions works equally well on a space obtained by intersecting an open convex set with a closed half-space.
as we wanted to show. Returning to the general case, consider the telescopic form for the derivative:

\[ D(\phi \circ \psi_1) - D(\phi \circ \psi_0) = (D\phi \circ \psi_1 - D\phi \circ \psi_0) . D\psi_1 + (D\phi \circ \psi_0) . (D\psi_1 - D\psi_0) \]

Here, \( D\phi \in C^{r+\beta-1} \) and \( \psi_1 \in C^r \subset C^{r+\beta-1} \) (since \( \beta \leq 1 \)). The first part then applies (with \( r + \beta - 1 \) instead of \( r + \beta \)) and shows that \( \| D\phi \circ \psi_1 - D\phi \circ \psi_0 \|_{r-1} \leq C_{r-1,\beta} \| D\phi \|_{r+\beta-1} \| \psi_1 - \psi_0 \|_{\beta} \). The last term trivially verifies the same type of bound. From this we deduce the result for the difference without derivatives. \( \square \)

**Proposition C.5.** Let us consider \( r = k + \alpha \) with \( k \in \mathbb{N} \) and \( 0 < \alpha \leq 1 \), and \( \underline{r} = (k, \alpha), B, X, Y \) three Banach spaces and \( U \subset B \) an open subset. Under the natural identifications, we have the following injections:

\[ L(X, C^\alpha(U, Y)) \hookrightarrow C^\alpha(U, L(X, Y)) \]

**Proof.** We present a proof by induction on \( k \in \mathbb{N} \).

For \( \underline{r} = (0, \alpha) \), we assume that we have an operator \( \mathcal{L}_u : X \to Y \), satisfying: there is a \( C > 0 \), such that for any \( \phi \in X \), any \( u \neq v, \ u, v \in U \),

\[ \mathcal{L}_u \phi \in C^{0,\alpha}(U, Y) \]

\[ \| \mathcal{L}_u \phi \|_Y \leq C \| \phi \|_X \]

\[ \| \mathcal{L}_u \phi - \mathcal{L}_v \phi \|_Y \leq C \| \phi \|_X \| u - v \|_{\mathcal{B}}^\alpha \]

Define \( \hat{\mathcal{L}} \) as the map \( u \in U \mapsto \mathcal{L}_u \in L(X, Y) \). Then it is easy to see that under the previous assumptions, \( \hat{\mathcal{L}} \) is a \( C^\alpha \) map

Let us assume that the wanted injection is established at rank \( k - 1 \), and consider \( r = k + \alpha \), an operator \( \mathcal{L}_u : X \to Y \), with \( \mathcal{L}_u \phi \in C^\alpha(U, Y) \) and \( \| \mathcal{L}_u \phi \|_{C^r(U, Y)} \leq C \| \phi \|_X \).

For any \( \phi \in X \), we may consider the partial derivative (w.r.t \( u \)) of \( \mathcal{L}_u \phi, \partial_u (\mathcal{L}_u \phi) \in L(B, Y) \), which is, by assumption, a \( C^{k-1,\alpha} \) map w.r.t \( u \), with \( \| \partial_u \mathcal{L}_u \phi \|_{C^{k-1,\alpha}} \leq C \| \phi \|_X \). But by induction hypothesis, this means that the map \( \hat{\mathcal{L}} : U \to L(X, Y) \) admits a derivative which is \( C^{k-1,\alpha} \), i.e that \( \hat{\mathcal{L}} \) is \( C^{k,\alpha} \). \( \square \)

**C.2 Bochner measurable smooth sections.** Let \((\Omega, \mathcal{F})\) be a measurable space. In this section there is no measure involved. We write

\[ X_r(U; Z) := X_{(k,\alpha)}(U; Z) = L^\infty(\Omega; c^{(k,\alpha)}(U; Z)) \]

for a Bochner measurable map from \( \Omega \) to the Banach space \( Y = c^{(k,\alpha)}(U; Z) \), with \( r = k + \alpha > 0, \alpha \in (0, 1], k \in \mathbb{N}_0 \). The norm of \( \phi \in X_r(U; Z) \) is the uniform norm:

\[ \| \phi \|_{X_r} = \sup_{\omega \in \Omega} \| \phi_\omega \|_r \].

Operations in the following proposition is understood to take place fiber-wise, e.g. for fixed \( \omega \in \Omega \), \( (\phi \circ \psi)_\omega := \phi_\omega \circ \psi_\omega \). We have:

**Proposition C.6.** Let \( \phi, \phi \in X_{r}(V; \mathbb{C}), \phi \in X_{r}(V; Z) \) and \( \psi \in X_{s}(U; V) \), \( s = r \vee 1 \). Then there are constants \( C_{r,1}, C_{r,2}, C_{r,3} \) depending only upon \( r \) and the chosen norms so that:

1. \( \phi \circ \phi \in X_{r}(V; \mathbb{C}) \) and \( \| \phi \circ \phi \|_{X_r} \leq C_{1,r} \| \phi \|_{X_r} \| \phi \|_{X_r} \).
2. \( e^{\phi} \in X_{r}(V; \mathbb{C}) \) and \( \| e^{\phi} \|_{X_r} \leq C_{2,r} \| e^{\phi} \|_{X_{r-1}} \).
3. \( \phi \circ \psi \in X_{r}(U; Z) \) and \( \| \phi \circ \psi \|_{X_r} \leq C_{3,r} \| \phi \|_{X_r} \| \psi \|_{X_{s-1}} \).

\( (r > 1) \).
Proof. As operations are fiber-wise we clearly have the stated bounds on the norms. The only issue is Bochner-measurability. We show this for the first case: Let $M > \|\varphi_1\|_{X_r} \vee \|\varphi_2\|_{X_r}$. Given $\epsilon > 0$ we may find a countable measurable partition $(\Omega_m)_{m \in \mathbb{N}}$ so that for every $m \in \mathbb{N}$, $\omega, \omega' \in \Omega_m$, $i = 1, 2$ we have: $\|(\varphi_i)_\omega - (\varphi_i)_{\omega'}\|_r \leq \frac{\epsilon}{2C_{1,r}M}$. Then by the telescopic principle and the above bounds:

$$\|(\varphi_1\varphi_2)_\omega - (\varphi_1\varphi_2)_{\omega'}\|_r \leq 2C_{1,r} \cdot \frac{\epsilon}{2C_{1,r}M} \leq \epsilon,$$

implying that $\varphi_1\varphi_2$ is Bochner measurable in the sense of definition B.1. The other two statements follow in the same way. □

Lemma C.7. One has the following injections:

- $\varphi \in L^\infty(\Omega; C^r(V; Z)) \hookrightarrow \widehat{\varphi} \in C^r(V; L^\infty(\Omega; Z))$;
- $\varphi \in L^\infty(\Omega; c^r(V; Z)) \hookrightarrow \widehat{\varphi} \in c^r(V; L^\infty(\Omega; Z))$,

under the natural fiber-wise identification:

$$\widehat{\varphi}(v)(\omega) := \varphi(\omega)(v), \quad \omega \in \Omega, v \in V.$$

Proof. Given $\epsilon > 0$ we find a countable measurable partition $(\Omega_i)_{i \in \mathbb{N}}$ so that for every $i \in \mathbb{N}$, $\omega, \omega' \in \Omega_i$, we have: $\|(\varphi)_\omega - (\varphi)_{\omega'}\|_r \leq \epsilon$. Pick also for each $i \geq 1$: $\omega_i \in \Omega_i$ and set $f_i = \varphi(\omega_i)$. Define:

$$\widehat{f}(v)(\omega) = \widehat{f}(v)(\omega) := \sum_i 1_{\Omega_i}(\omega) f_i(v).$$

Then $f$ is a $\sigma$-simple $\epsilon$-uniform approximation to $\varphi$. Clearly, $\widehat{f}$ takes values in $Y = L^\infty(\Omega, Z)$. Also $\partial^q_v \widehat{f}(v) \in L(B^q_v; Y)$ and $\|\partial^q_v \widehat{f} - \partial^q_v f\| \leq \epsilon$ for $0 \leq q \leq k$ and similarly for the $\alpha$-Hölder estimate for the $k$'th derivative. Thus $\widehat{\varphi} \in c^r(V; L^\infty(\Omega; Z))$ and it has the same norm as $\varphi$. □

We conclude this section with a key ingredient for our applications section:

Proposition C.8. With the notation as in Theorem C.2 and this section, we have the following injection of norm at most 2:

$$\varphi \in X_r(U \times V; Z) \hookrightarrow \widehat{\varphi} \in C^{r-s}(U; X_s(V; Z)),$$

under the natural fiber-wise identification:

$$\widehat{\varphi}(u)(\omega)(v) := \varphi(\omega)(u, v), \quad \omega \in \Omega, u \in U, v \in V.$$

Proof. Combining C.2 and C.7 we have the following injections:

$$X_r(U \times V; Z) = L^\infty(\Omega; c^r(U \times V; Z)) \hookrightarrow L^\infty(\Omega; C^{r-s}(U; c^\xi(V; Z))) \hookrightarrow C^{r-s}(U; L^\infty(\Omega; c^\xi(V; Z))) = C^{r-s}(U; X_s(V; Z)).$$
D Graded Differential Calculus

An essential ingredient in differential calculus is the Leibniz principle: When e.g. \( f, g \) are \( C^r(\mathbb{R}) \)-functions for \( r \geq 1 \) (\( r \) not necessarily an integer) then so is their product and one has a formula for the derivative of the product \( (f \cdot g)' = f' \cdot g + f \cdot g' \). The derivative is then \( C^{r-1} \) and one may iterate the derivation formula when \( r \geq 2 \). The aim here is to develop a similar theory for graded differential calculus, in particular the Leibniz principle, when \( f, g \) are replaced by linear operators depending on a parameter \( u \) but where regularity with respect to the parameter only appears when downgrading the codomain (the image space) or upgrading the domain within a certain scale of Banach spaces. The upshot of this appendix is to show that the resulting regularity when performing algebraic operations on graded differential operators is as good as one could possibly hope for. In particular, we prove Lemma 3.8.

We will stick to the notation of Sect. 3. More precisely, let \( X = (X_t)_{t \in (0, r_0]} \) denote the scale of Banach spaces. By this we mean a parametrized family of Banach spaces coming with a family of bounded linear (downgrading) operators \( j_s, r \in L(X_r; X_s), 0 < s \leq r \leq r_0 \). We assume that each operator is injective and has dense image and that the collection satisfies the transitivity condition: \( j_{s, s} = \text{Id} \) and \( j_{s, c, j_{c, r}} = j_{s, r} \) whenever \( 0 < s \leq c \leq r \leq r_0 \).

**Example D.1.** An instructive example to have in mind is \( X_t = C^t(S^1), t \in (0, r_0] \) with \( j_{s, r} : C^r(S^1) \rightarrow C^s(S^1) \) being the natural embedding for \( 0 < s \leq r \leq r_0 \).

We let \( B \) denote a Banach space and let \( U \subset B \) be a non-empty open convex subset.

**Definition D.2.** Let \( n \in \mathbb{N}_0 = \{0, 1, \ldots\}, \gamma > 0 \) with \( \gamma + n \leq r_0 \). We associate to the integer \( n \) the following set of ordered pairs: \( I_n = \{(s, r) \in (0, r_0]^2 : s + n \leq r\} \). Consider a family \( \mathcal{M} \) of bounded linear operators \( M_{s, r}(u) \in L(B^n; L(X_r, X_s)), (s, r) \in I_n \) and parametrized by \( u \in U \). We say that the family \( \mathcal{M} \) is \((j, n)\)-equivariant and \((\gamma, n)\)-regular provided that:

1. For every \((s, r),(s', r') \in I_n\) with \( s < s', r < r' \) and \( u \in U \): \( j_{s, s'} M_{s', r'}(u) = M_{s, r}(u) j_{r, r'} \).
2. The map \( u \in U \mapsto M_{s, r}(u) \in L(B^n; L(X_r; X_s)) \) is \( C^t \) for all \((s, r) \in I_n \) and \( 0 \leq t < \gamma \wedge (r - s - n) \).

Keeping the same notation as in the previous definition we define:

**Definition D.3.** Consider a family \( \mathcal{N} \) of functions \( N_s(u) \in L(B^n; X_s), 0 \leq s < r_0 - n \), parametrized by \( u \in U \). We say that \( \mathcal{N} \) is left-equivariant and \((\gamma, n)\)-regular provided that for all \( 0 \leq s < r < r_0 - n, u \in U \): \( N_s(u) = j_{s, r} N_r(u) \), and the map \( u \in U \mapsto N_s(u) \in L(B^n; X_s) \) is \( C^t \) for all \( 0 \leq t < \gamma \wedge (r_0 - n - s) \).

**Lemma D.4.** Let \( \mathcal{M} \) be an equivariant \((\gamma, n)\)-regular family with \( \gamma > 1 \). We define the derived family \( \partial_u \mathcal{M} \) given by: \( \partial_u M_{s, r}(u) \in L(B; L(B^n; L(X_r, X_s))) \equiv L(B^{n+1}; L(X_s, X_r)) \) for all \((s, r) \in I_{n+1} \). This derived family is equivariant and \((\gamma - 1, n + 1)\)-regular. Conversely, suppose that \( \mathcal{M} \) is \((1 + \alpha, n)\)-regular with \( \alpha > 0 \) and that derived family \( \partial_u \mathcal{M} \) is \((\gamma', n + 1)\)-regular, then setting \( \gamma = \alpha \vee \gamma' + 1 \), we have that \( \mathcal{M} \) is \((\gamma, n)\)-regular. A similar statement holds for a left-equivariant family \( \mathcal{N} \).

**Proof.** The first statement is obvious from definitions. For the second, we may assume \( \gamma' > \alpha \) or else it is trivial. Suppose that \( \mathcal{M} \) is \((1 + \alpha, n)\)-regular with \( \alpha > 0 \) and let \( \partial_u \mathcal{M} \) be the derived family. If \( 0 < s < r \leq r_0 \) with \( r - s \leq r_0 - n \) and \( 1 < t < t_s =
$$(\gamma' + 1) \wedge (r - s - n).$$ Then $$u \mapsto \partial_u M_{s,r}(u)$$ is $$C^{t-1}$$ and consequently $$u \mapsto M_{s,r}(u)$$ is $$C^t$$ as we wanted to show.

The main reason for introducing equivariant, $$(\gamma, n)$$-regular families comes from the stability under products: □

**Proposition D.5.** Let $$\mathcal{M}^1$$ and $$\mathcal{M}^2$$ be two families of $$j$$-equivariant, $$(\gamma_1, n_1)$$-regular, respectively $$(\gamma_2, n_2)$$-regular; operators. Suppose that $$n = n_1 + n_2 < r_0$$ and set $$\gamma = \gamma_1 \wedge \gamma_2 \wedge (r_0 - n) > 0$$. Then there is a well-defined product family $$\mathcal{M} = \mathcal{M}^1 \star \mathcal{M}^2$$ obtained by declaring for $$(s, r) \in I_n$$:

$$M_{s,r}(u) := M^1_{s,c}(u) M^2_{c,r}(u) \in L(B^n \times B^{n_2}; L(X_r; X_s)) \simeq L(B^n; L(X_r; X_s))$$

with $$c$$ being any number in the non-empty interval $$(s + n_1, r - n_2)$$. This product family is equivariant and $$(\gamma, n)$$-regular. Similarly, with $$\mathcal{M}^1$$ as above and $$\mathcal{N}^2$$ a left-equivariant and $$(\gamma_2, n_2)$$-regular family, the product $$N_s(u) := M^1_{s,c} N^2_{c}(u)$$ defines a family $$\mathcal{N} = \mathcal{M}^1 \star \mathcal{N}^2$$ which is left-equivariant and $$(\gamma, n)$$-regular.

**Proof.** If $$c < c'$$ are two numbers in the above interval, then we have by equivariance (all operators being well-defined):

$$M^1_{s,c}(u) M^2_{c',r}(u) = M^1_{s,c}(u) \left[ j_{c,c'} M^2_{c',r}(u) \right] = M^1_{s,c}(u) M^2_{c',r}(u),$$

showing that the product does not depend upon the choice of $$c$$. Let $$(s, r) \in I_n$$ and $$t_* = \gamma \wedge (r - s - n) > 0$$. Regularity will be shown by induction in $$\gamma$$. First, assume that $$\gamma \in (0, 1]$$. Then, in particular, $$t_* \leq 1$$. We will show that $$u \mapsto M_{s,r}(u) \in L(X_r; X_s)$$ is $$t$$-Hölder for every $$0 < t < t_*$$ and set $$s' := s + n_1 + \epsilon/3 < r' := r - n_2 - \epsilon/3$$. Then we have by equivariance:

$$M_{s,r}(u) = M^1_{s,s'}(u) j_{s',r} M^2_{r',r}(u).$$

Note that $$r' - s - n_1 = r - s' - n_2 = r - s - n - \epsilon/3$$. Also let $$c = (s' + r')/2$$. When $$u, u + h \in U$$, using a telescopic sum, equivariance and Hölder continuity we have the following identity:

$$M_{s,r}(u + h) - M_{s,r}(u) = M^1_{s,s'}(u + h) j_{s',r} M^2_{r',r}(u + h) - M^1_{s,s'}(u) j_{s',r} M^2_{r',r}(u)$$

(D.1)

$$= \left( M^1_{s,s'}(u + h) - M^1_{s,s'}(u) \right) M^2_{r',r}(u)$$

$$\quad + \left( M^1_{s,c}(u + h) - M^1_{s,c}(u) \right) \left( M^2_{c,r}(u + h) - M^2_{c,r}(u) \right)$$

$$\quad + M^1_{s,c}(u) \left( M^2_{c',r}(u + h) - M^2_{c',r}(u) \right)$$

$$\quad = O(h^{t_* - \epsilon/3}) O(1) + O(h^{t_* - \epsilon/3}) O(h^{t_* - \epsilon/3}) + O(1) O(h^{t_* - \epsilon/3})$$

$$\quad = O(h^{t_* - 2\epsilon/3}).$$

(D.2)

We may here let $$\epsilon \to 0$$ and obtain the claim for this case.

For higher order regularity, let $$k \geq 1$$ be an integer and suppose that the proposition has been proven whenever $$0 < \gamma \leq k$$. For $$k = 1$$ this was done above. So assume now that $$\gamma \in (k, k + 1)$$. For $$(s, r) \in I_n$$ we set $$t_* = \gamma \wedge (r - s - n)$$. We may assume
that $t_n \in (k, k + 1]$ as well (or else there is nothing to show). We write $t_n = k + \alpha$
with $\alpha \in (0, 1]$. We want to show that $u \mapsto M_{s,r}(u)$ is $C^t$ for all $0 < t < t_s$. Set
$s' = s + n_1 + \alpha/3$ and $r' = r - n_2 - \alpha/3$. Then as before $r'^- s' = r - s - n - 2\alpha/3 > k + \alpha/3$.
Since $k \geq 1$ we obtain derived families $\partial_u M^1$ and $\partial_u M^2$ as described in Lemma D.4.
We have e.g. $M^1_{s,s'}(u) j_{s',r'} = M^1_{s,r'}(u)$ so by the MVT we get:

$$
\left| M^1_{s,r'}(u + h) - M^1_{s,r'}(u) - h \cdot \partial_u M^1_{s,r'}(u) \right|_{L(X_i;X_r)} 
\leq |h| \sup_{\tau \in [0,1]} \left| \partial_u M^1_{s,r'}(u + \tau h) - \partial_u M^1_{s,r'}(u) \right|_{L(X_i;X_r)} 
= O(h^1)O(h^{\alpha/3}) = O(h^{1+\alpha/3}).
$$

With a similar expansion for $M^2$ and using Hölder estimates for the middle term we expand (D.2) to get:

$$
M_{s,r}(u + h) - M_{s,r}(u) = (h \partial_u M^1_{s,r'})(u) M^2_{r',r}(u) + M^1_{s,s'}(u) (h \partial_u M^2_{s',r})(u) + O(h^{1+\alpha/3}),
$$

showing that $M_{s,r}$ is differentiable with derivative

$$
\partial_u M_{s,r}(u) = \partial_u M^1_{s,r'}(u) M^2_{r',r}(u) + M^1_{s,s'}(u) \partial_u M^2_{s',r}(u).
$$

Now, in this expression we may again use equivariance to write

$$
\partial_u M_{s,r}(u) = \partial_u M_{s,c_1}(u) M^2_{c_1,c_2}(u) + M^1_{s,c_2}(u) \partial_u M^2_{c_2,c_3}(u),
$$

with $c_1 \in (s + n_1 + 1, r - n_2)$ and $c_2 \in (s + n_1, r - 1 - n_2)$. The first term is the product of
two j-equivariant families that are $(\gamma - 1, n_1 + 1)$ and $(\gamma, n_2)$ regular, respectively. Since
$(\gamma - 1) \wedge (r - s - n - 1) = t_n - 1 \leq k$ we may apply the induction hypothesis on this
term to conclude that this first product is $(\gamma - 1, n + 1)$-regular. Similarly for the second
term. Thus $u \mapsto \partial_u M_{s,r}(u)$ is $C^t$ for every $t < \gamma - 1$, whence $u \mapsto M_{s,r}(u)$ is $C^t$ for
every $t < \gamma$ as we wanted to show (see Lemma D.4). The proof in the left-equivariant
case follows the same path. □

**Lemma D.6.** Let $(Q_{s,r}(u))$ be an equivariant, $(\gamma, 0)$-regular family with the additional
property that $1 - Q_{s}(u)$ is invertible for all $0 < s \leq r_0$ and having a uniformly bounded
inverse $R_{s}(u) = (1 - Q_{s}(u))^{-1}$ when $\epsilon < s \leq r_0$ for any $\epsilon > 0$. Then the family of
operators $R_{s,r}(u) = j_{s,r} R_{r}(u)$, $0 < s < r \leq r_0$ is again equivariant and $(\gamma, 0)$-regular.

**Proof.** This boils down to the resolvent identity combined with equivariance. We have e.g. for $u, u + h \in U$:

$$
R_{s,r}(u + h) - R_{s,r}(u) = R_{s}(u + h) \left( Q_{s,r}(u + h) - Q_{s,r}(u) \right) R_{r}(u).
$$

Hölder-continuity then follows using regularity of the middle term. When $t_n = \gamma \wedge
(r - s - 1) > 1$ we may again develop the middle term and conclude that $R_{s,r}(u)$ is
differentiable with derivative:

$$
\partial_u R_{s,r}(u) = R_{s}(u) \left( \partial_u Q_{s,r}(u) \right) R_{r}(u) \in L(B; L(X_r; X_s)).
$$

Here we have a product of 3 operators being $(t_n, 0)$, $(t_n - 1, 1)$ and $(t_n, 0)$-regular,
respectively. The product is then itself $(t_n - 1, 1)$-regular and therefore $R_{s,r}(u)$ is $(t_n, 0)$-regular as we wanted to show. □
Proof of Lemma 3.8. First note that in Theorem 3.2 the collection of operators $L_{s,r}(u) := \mathcal{L}_{s,u} J_{s,r}$ with $0 < s \leq r \leq r_0$, forms an equivariant family $\mathcal{L}$ of $(r_0, 0)$-regular operators over $(0, r_0]$. The derived family $\left( \partial_u(\mathcal{L}_{s,u} J_{s,r_0}) \right)_{s \in (0, r_0]}$ is then $(r_0 - 1, 1)$-regular.

Under Hypothesis $\mathcal{H}(\gamma)$ the family of fixed fields $(f_s(u))_{s \in (0, r_0]}$ is left-equivariant and $(\gamma, 0)$-regular. From Proposition D.5 it follows that the family of products $\left( \partial_u(\mathcal{L}_{s,u} J_{s,r_0}) f_{r_0}(u) \right)_{s \in (0, r_0]}$ is $(\gamma \land (r_0 - 1), 1)$-regular. This is in fact the principal term in the definition 3.3 of $P_{r,r_0,u}(f_{r_0}(u))$ which is therefore also $(\gamma \land (r_0 - 1), 1)$-regular: this shows the first claim in Lemma 3.8. In a similar way, using Proposition 3.4 and $\mathcal{H}(\gamma)$ we see that $M_{s,r}(u) = Q_{s,u}(f_s(u)) J_{s,r}$ is equivariant and $(\gamma, 0)$-regular. This implies the second claim in the Lemma. □

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