We prove that the binegativity is always positive for any two-qubit state. As a result, as suggested by the previous works, the asymptotic relative entropy of entanglement in two qubits does not exceed the Rains bound, and the PPT-entanglement cost for any two-qubit state is determined to be the logarithmic negativity of the state. Further, the proof reveals some geometrical characteristics of the entangled states, and shows that the partial transposition can give another separable approximation of the entangled state in two qubits.

Quantum entanglement plays an essential role in many quantum information tasks, and qualitative and quantitative understanding of the entanglement is one of the central topics in quantum information theory. An important mathematical operation in the theory of entanglement is the partial transposition \( \sigma^{T_B} \), where only the basis on one party, say Bob, is transposed. The states satisfying \( \sigma^{T_B} \geq 0 \) are called positive partial transposed (PPT) states, and all separable states are PPT states. Further, it has been shown that all PPT states in \( 2 \otimes 2 \) (two qubits) and \( 2 \otimes 3 \) are separable states.

Recently, Audenaert, Moor, Vollbrecht and Werner introduced an interesting and important mathematical operation, the binegativity \( |\sigma^{T_B}|_{T_B} \). They showed that, if \( |\sigma^{T_B}|_{T_B} \geq 0 \) holds, the asymptotic relative entropy of entanglement with respect to PPT states does not exceed the so-called Rains bound. Further they showed that \( |\sigma^{T_B}|_{T_B} \geq 0 \) holds for many classes of states and conjectured that it holds for any two-qubit state. Subsequently, Audenaert, Plenio, and Eisert showed that, if \( |\sigma^{T_B}|_{T_B} \geq 0 \) holds, the PPT-entanglement cost for the exact preparation is given by the logarithmic negativity. By this, they provided an operational meaning to the logarithmic negativity.

In this paper, we prove that \( |\sigma^{T_B}|_{T_B} \geq 0 \) indeed holds for any two-qubit state. The proof is geometrical in some sense, and reveals some geometrical characteristics of the entangled states. Further, it is found that the partial transposition can give another separable approximation of the entangled state in two qubits.

Before starting the proof of \( |\sigma^{T_B}|_{T_B} \geq 0 \), we briefly review several concepts necessary to the proof. The first is the entanglement witness \( 2 \otimes 2 \), which is the Hermitian operator \( W \) such that \( \text{Tr} W \rho \geq 0 \) for all separable states \( \rho \), and \( \text{Tr} W \sigma < 0 \) for some entangled states \( \sigma \). This is expressed such that \( W \) detects the entanglement of \( \sigma \). For \( \sigma^{T_B} \geq 0 \), \( (|\psi⟩⟨\psi|)^{T_B} \) is the entanglement witness where \( |\psi⟩ \) is the eigenstate of \( \sigma^{T_B} \) for a negative eigenvalue. Since the entanglement witness cannot be a positive operator, \( |\psi⟩ \) must always be entangled.

The concept of the entanglement witness is related to the existence of the hyper-plane separating the closed convex set of separable states and some entangled states. \( W \) itself plays the role of the normal vector of the hyper-plane. The concept can be applicable to another closed convex set of positive operators: if \( \text{Tr} \sigma |\phi⟩⟨\phi| < 0 \), it is certain that \( \sigma \) is not positive. In this paper, by analogy to the entanglement witness, we call \( |\phi⟩⟨\phi| \) detects the nonpositivity, and we say that \( |\phi⟩⟨\phi| \) detects the nonpositivity of \( \sigma \).

The second concept relates to the state representation based on the local filtering. According to Refs. \( 2 \) and \( 3 \), all states in two qubits can be transformed by local filtering of full rank into either the Bell diagonal states or the states of the form

\[
\sigma = \frac{1}{N} \sum_{i=0}^{3} p_i (A \otimes B) |e_i⟩⟨e_i| (A^\dagger \otimes B^\dagger)
\]

or

\[
\sigma = \frac{1}{N} (A \otimes B) \sigma_c (A^\dagger \otimes B^\dagger),
\]

where \( N \) is the normalization, \( \sum_i p_i = 1 \), and \( |e_i⟩ \) is the set of orthogonal Bell basis. Without loss of generality, we can fix \( |e_i⟩ = \{|\psi^-⟩, |\psi^+⟩, |\phi^-⟩, |\phi^+⟩\} \) \( 2 \), \( 3 \), \( 4 \), \( 5 \), \( 6 \), \( 7 \), \( 8 \), \( 9 \), \( 10 \), \( 11 \), \( 12 \), where \( |\psi^±⟩ = (|0⟩ \pm |1⟩)/\sqrt{2} \) and \( |\phi^±⟩ = (|00⟩ \pm |11⟩)/\sqrt{2} \). In fact, we can assume that \( p_0 \) is largest and \( p_3 \) is smallest among \( p_0, p_1, p_2, p_3 \). Further, since \( A \) and \( B \) are full rank, we can put \( \det A = \det B = 1 \) without loss of generality. This leads to a convenient relation of \( A^\dagger \tilde{A} = B^\dagger \tilde{B} = I \) \( 2 \), \( 3 \), \( 4 \), \( 5 \), \( 6 \), \( 7 \), \( 8 \), \( 9 \), \( 10 \), \( 11 \), \( 12 \), where the tilde operation is defined as \( \tilde{\sigma} \equiv \sigma_2 A^* \sigma_2 \) for local operators and \( |\tilde{\psi}⟩ \equiv (\sigma_2 \otimes \sigma_2) |\psi⟩ \) for states. In this paper, these forms of the state representation [Eqs. \( 2 \) or \( 3 \)] are called normal form.

Then, let us start the proof of \( |\sigma^{T_B}|_{T_B} \geq 0 \). It is trivial when \( \sigma \) is a PPT state, since \( |\sigma^{T_B}|_{T_B} = \sigma \geq 0 \). Therefore,
we restrict ourselves to the case that \( \sigma \) is entangled. The partial transposition of \( \sigma \) can be written as

\[
\sigma^{T_B} = P - \lambda |\psi\rangle\langle\psi|,
\]
where \( |\psi\rangle \) is the (normalized) eigenstate for the negative eigenvalue of \(-\lambda\) (there is only one negative eigenvalue for entangled states in two qubits [14]). The remainder \( P \) is the (unnormalized) positive part \( (P \geq 0) \), which is orthogonal to \( |\psi\rangle \), and hence \( P|\psi\rangle = 0 \). Then, \( \sigma \) and \( |\sigma^{T_B}\rangle^{T_B} \) are

\[
|\sigma^{T_B}\rangle^{T_B} = P + \lambda |\psi\rangle\langle\psi|^{T_B},
\]

Here, it is worth discussing the geometrical meaning of the problem. Let us consider the space of all Hermitian operators. The set of quantum states which are positive operators is a subset of the whole (we do not care about the normalization explicitly). Further, let us consider PPT operators, which are those such that its partial transposed operators are positive (a PPT operator can be either positive or nonpositive). These two sets are schematically shown in Fig. 1 (this figure is essentially the same as Fig. 1 in Ref. [13]). Hereafter, we call these sets positive ball and PPT ball, although the actual shape is not a spherical ball [12]. The intersection of the two balls corresponds to the set of PPT states (separable states). Therefore, entangled state \( \sigma \) is located in the positive ball outside the intersection. Since \( |\sigma^{T_B}\rangle^{T_B} \) is a PPT operator \( (|\sigma^{T_B}\rangle^{T_B})^{T_B} = |\sigma^{T_{B}}\rangle^{T_{B}} \geq 0 \), it is contained in the PPT ball. Then, the geometrical meaning to prove \( |\sigma^{T_B}\rangle^{T_B} \geq 0 \) is to prove that \( |\sigma^{T_{B}}\rangle^{T_{B}} \) is always located in the intersection.

Further, let us pay attention to the geometrical location of \( P^{T_B} \): it is located on the middle of the line connecting \( \sigma \) and \( |\sigma^{T_B}\rangle^{T_B} \) since \( P^{T_B} = \sigma/2 + |\sigma^{T_B}\rangle^{T_B}/2 \). In addition, \( P^{T_B} \) must be located on the edge of the PPT ball, since \( P^{T_B} \) itself is a PPT operator and its partial transposition is rank deficient \( (P|\psi\rangle = 0) \). However, so that \( |\sigma^{T_B}\rangle^{T_B} \) is located in the intersection, it is geometrically obvious that \( P^{T_B} \) must be located on the edge of the intersection (thick solid curve in Fig. 1). This corresponds to \( P^{T_B} > 0 \), which is indeed necessary for \( |\sigma^{T_B}\rangle^{T_B} \geq 0 \) because \( |\sigma^{T_B}\rangle^{T_B} = 2P^{T_B} - \sigma \).

Then, the outline of the proof is as follows: We first prove a lemma which simplifies the proof of \( |\sigma^{T_{B}}\rangle^{T_{B}} \geq 0 \). Second, we prove that \( P^{T_B} > 0 \) (positive definite) whenever a given state \( \sigma \) is entangled. Finally, we search for an operator \( X \) in the intersection such that \( |\sigma^{T_B}\rangle^{T_B} \) is located on the line connecting \( P^{T_B} \) and \( X \) (see Fig. 1). As a result, it is found that \( |\sigma^{T_{B}}\rangle^{T_{B}} \) can be always written as a convex sum of two positive operators \( (P^{T_B} \) and \( X \)), which can complete the proof.

The key point of the proof for \( |\sigma^{T_{B}}\rangle^{T_{B}} \geq 0 \) is to represent \( P \) (not \( \sigma \)) in the normal form mentioned before. A lemma we first prove is concerned with the state representation of \( P \).

**Lemma 1.** Let \( \sigma \) be any entangled state in two qubits and write down as \( \sigma^{T_B} = P - \lambda |\psi\rangle\langle\psi| \) where \( P \geq 0 \) and \( P|\psi\rangle = 0 \). If \( P \) is rank 3, \( P \) is always represented in the normal form of Eq. (4). If the rank of \( P \) is less than 3, there always exist \( \sigma' \) in the vicinity of \( \sigma \) such that \( \sigma^{T_{B}} = P' - \lambda |\psi\rangle\langle\psi| \) where \( P'|\psi\rangle = 0 \), \( P' \) is rank 3, and \( \sigma' \) is represented in the normal form of Eq. (4).

**Proof:** In the case that \( P \) is rank 3, let us assume that \( P \) is represented in the normal form of Eq. (5) as \( P = \frac{1}{M} \sum_{i=0}^{3} p_i (A_i \otimes B_i) |\psi_i\rangle\langle\psi_i| (A_i \otimes B_i) \) where \( \sigma_i \) is given by Eq. (1). Since \( P \) is assumed to be rank 3 and \( A \otimes B \) is full rank, \( \sigma_i \) must be rank 3. By using \( A_i A = B_i B = I \), it can be easily checked that only the state of \( |\psi_i\rangle \) is \( \frac{1}{\sqrt{M}} (A \otimes B) |01\rangle \) satisfies \( P|\psi\rangle = 0 \) where \( M \) is the normalization. However, this state is a product state which contradicts that \( |\psi_i\rangle \) must be an entangled state in order that \( (|\psi_i\rangle\langle\psi_i|^{T_B} \) is an entanglement witness detecting the entanglement of \( \sigma \) (entanglement witness cannot be a positive operator as mentioned before). Therefore, \( P \) of rank 3 must be represented in the normal form of Eq. (2) as

\[
P = \frac{1}{N} \sum_{i=0}^{3} p_i (A_i \otimes B_i) |\psi_i\rangle\langle\psi_i| (A_i \otimes B_i),
\]

where \( p_3 = 0 \) in order that \( P \) is rank 3 since \( p_3 \) is smallest among \( p_i \). Further, by fixing the Bell basis as \( |\psi_i\rangle = \{|\psi_i^-, \psi_i^+, \phi_i^-, \phi_i^+\} \), it is found that only the state of

\[
|\psi\rangle = \frac{1}{\sqrt{M}} (A \otimes B) |\phi^+\rangle
\]

satisfies \( P|\psi\rangle = 0 \).

In the case that the rank of \( P \) is less than 3, there always exist \( P' \) of rank 3 in the vicinity of \( P \) such that \( P'|\psi\rangle = 0 \) (for example, if \( P \) is rank 2, using \( |\psi^+\rangle \) orthogonal to \( |\psi\rangle \) and satisfying \( P'|\psi^+\rangle = 0 \), let \( P' = P + \epsilon |\psi^+\rangle\langle\psi^+| \) with \( \epsilon \) being an infinitesimally small positive value). This \( P' \) must be represented in the normal form of Eq. (2) for the same reason discussed above \((|\psi_i\rangle\langle\psi_i|^{T_B} \) is entangled). Since \( P' \) is in the vicinity of \( P \), \( \sigma' = P' - \lambda |\psi\rangle\langle\psi|^{T_B} \) is also in the vicinity of \( \sigma \).

It should be noted that the rank of \( P \) will be shown to be 3, and the possibility of the second case in Lemma 1 will be denied (see Corollary 1 below).
The next task for the proof of $|\sigma^{TB}|_{TB} \geq 0$ is to prove $P^{TB} > 0$ whenever $\sigma$ is entangled. To this end, it suffices to show that, if we assume $P^{TB} \neq 0$, $\sigma = P^{TB} - \lambda |\psi\rangle \langle \psi|^{TB}$ cannot be any entangled state for $\lambda > 0$ and for $|\psi\rangle$ satisfying $P|\psi\rangle = 0$. According to Lemma 1, so that $\sigma$ is an entangled state, $P$ (or $P'$ in the close vicinity of $P$) must be written as Eq. (6) at least. For those $P$ (or $P'$) of rank 3, Eq. (6) is only the state satisfying $P|\psi\rangle = 0$ (or $P'|\psi\rangle = 0$). Therefore, in the following, we shall show that, if $P^{TB} \neq 0$, $\sigma = P^{TB} - \lambda |\psi\rangle \langle \psi|^{TB}$ cannot be positive (and hence cannot be an entangled state) for every $P$ of Eq. (6) and for $|\psi\rangle$ of Eq. (6). By this, when the rank of $P$ is less than 3, since $\sigma' = P^{TB} - \lambda |\psi\rangle \langle \psi|^{TB}$ cannot be positive as well, $\sigma$ in the close vicinity of $\sigma'$ cannot be positive.

The partial transposition of Eq. (6) is calculated as

$$P^{TB} = \frac{1}{N} \sum_{i=0}^{3} p_i (A \otimes B^*)|e_i\rangle\langle e_i|^{TB} (A^\dagger \otimes B^T)$$

$$= \frac{1}{2N} \sum_{i=0}^{3} (1 - 2p_{3-i}) (A \otimes B^*)|e_i\rangle\langle e_i|(A^\dagger \otimes B^T)$$

where we fixed $|e_i\rangle$ as $\{ |\psi^-\rangle, |\psi^+\rangle, |\phi^-\rangle, |\phi^+\rangle \}$ and used

$$(|\psi^\pm\rangle|\psi^\pm\rangle)^{TB} = \frac{1}{2} (|\psi^-\rangle\langle \psi^-| + |\psi^+\rangle\langle \psi^+|$$

$$(|\phi^\pm\rangle|\phi^\pm\rangle)^{TB} = \frac{1}{2} (|\phi^-\rangle\langle \phi^-| + |\phi^+\rangle\langle \phi^+|$$

Further, $P^{TB} \neq 0$ corresponds to $p_0 \geq 1/2$ in Eq. (9), since $1 - 2p_i$ is smallest among $1 - 2p_i$ ($P^{TB}$ is positive semidefinite for $p_0 = 1/2$, and has a negative eigenvalue for $p_0 > 1/2$). By introducing the state of

$$|\phi\rangle = \frac{1}{\sqrt{2}} (\hat{A} \otimes \hat{B}^*)|\phi^+\rangle,$$

where $L$ is the normalization, it is found that

$$\langle \phi | \sigma | \phi \rangle = \langle \phi | P^{TB} | \phi \rangle - \lambda \langle \phi | (|\psi\rangle \langle \psi|)^{TB} | \phi \rangle$$

$$= \frac{1 - 2p_0}{2NL} - \lambda \text{Tr}(C \otimes I) V(C^\dagger \otimes I) P^+$$

where $V = 2(|\phi^+\rangle \langle \phi^+|)^{TB}$ is the flip operator, and $C \equiv H_1 H_2$ is the product of two positive definite operators $H_1 \equiv A^\dagger A$ and $H_2 \equiv (B^\dagger B^*) = B^2 B^*$. In the third equality, we used $(A \otimes B)V(A^\dagger \otimes B^T) = (B A^\dagger) V(B A^\dagger)$. Using $\det H_1 = \det H_2 = \det C = 1$, it can be shown that $\text{Tr} CC^* \geq 2$. As a result, it is found that $|\phi\rangle | | \phi > 0$ for $\lambda > 0$, and $\sigma$ cannot be positive where it is assumed that $P^{TB} \neq 0$. In this way, $\langle \phi | \phi \rangle$ works as a witness operator detecting the nonpositivity of $\sigma$. Then, the following theorem was proven.

**Theorem 1.** For any two-qubit state $\sigma$, the positive part $(P)$ of $\sigma^{TB}$ is a PPT state. Further, if $\sigma$ is entangled, the partial transposition of the positive part $(P^{TB})$ is full rank.

It has been shown that the partial transposition of any separable state of rank 2 is also rank 2 \cite{18}, and it is obvious that the partial transposition of any pure separable state is also a pure separable state. Therefore, the fact that $P^{TB}$ is separable and full rank implies that $P$, which is rank deficient, must be rank 3, and we obtain the following corollary.

**Corollary 1.** For any entangled two-qubit state $\sigma$, the positive part $(P)$ of $\sigma^{TB}$ is rank 3.

Theorem 1 states that, in some sense, the partial transposition in two qubits can also give separable approximations of the entangled states as well as the best separable approximation \cite{18}, the closest disentangled state in the relative entropy measure \cite{19}, and so on. Every entangled state in two qubits can be decomposed into the separable approximation \cite{18} $P^{TB}$ (normalized) and the deviation from it $[\{ |\psi\rangle \langle \psi|^{TB} \}^2$.

Further, it is important to discuss the geometrical meaning of $|\phi\rangle$ of Eq. (6). In the case of $p_0 = 1/2$, $P^{TB}$ of Eq. (6) becomes positive semidefinite (rank 3) and it can be seen that $|\phi\rangle$ satisfies $P^{TB} |\phi\rangle = 0$. Geometrically, $P^{TB}$ of rank 3 is just located at the crossing point of two edges as shown in Fig. 4. The hyper-plane corresponding to the entanglement witness of $|\langle \psi \rangle \langle \psi |^{TB}$ is in contact with the PPT ball at the crossing point, and $P^{TB} - \lambda |\langle \psi \rangle \langle \psi |^{TB}$ is located in the direction perpendicular to this hyper-plane (since the entanglement witness plays the role of the normal vector). In addition, the witness for the nonpositivity $|\phi\rangle | \phi \rangle$, which is the eigenstate of $P^{TB}$ for a zero-eigenvalue, also specifies a hyper-plane which is in contact with the positive ball at the crossing point. What we showed in the proof of Theorem 1 is that these two hyper-planes always cross with shallow angles so that the nonpositivity of $|\langle \psi \rangle \langle \psi |^{TB}$ is always detected by the hyper-plane specified by $|\phi\rangle | \phi \rangle$. The inner product of two normal vectors of the hyper-planes corresponds to $\text{Tr}(\langle \psi \rangle \langle \psi |^{TB} | \phi \rangle | \phi \rangle$, which was shown always to be positive.

The remaining task for the proof of $|\sigma^{TB}|_{TB} \geq 0$ is to search for a positive operator $X$. According to Lemma
1 and Corollary 1, $P$ and $P_{TB}$ are represented by Eqs. \textbf{[4]} and \textbf{[5]}, respectively, when $\sigma$ is an entangled state. Further, according to $P_{TB} > 0$ (Theorem 1), $p_0 < 1/2$ (and $p_3 = 0$ since $P$ is rank 3). So that $\sigma = P_{TB} - \lambda |\psi\rangle\langle\psi|_{TB} \geq 0$, the range of $\lambda$ is limited, and an upper bound of $\lambda$ must be found. It is slightly surprising that the hyper-plane of $|\phi\rangle\langle\phi|$ also plays a crucial role for this purpose. Using $|\phi\rangle$ of Eq. \textbf{[11]} and $|\phi\rangle |\sigma\rangle |\phi\rangle$ of Eq. \textbf{[12]} (but $p_1 < 1/2$ here), the condition of $|\phi\rangle |\sigma\rangle |\phi\rangle \geq 0$ leads to $\lambda \leq \lambda_0 \equiv (1 - 2p_0)M/N$ (we again used $\text{TrCC}^* \geq 2$). Then, we define the operator $X$ as

$$X \equiv P_{TB} + \lambda_0 |\psi\rangle\langle\psi|_{TB},$$  \hspace{1cm} (12)

whose geometrical location is shown in Fig. \textbf{[8]} This $X$ is always positive as shown below. Let us introduce

$$X' = 2N(\tilde{A}^\dagger \otimes \tilde{B}^T)X(\tilde{A} \otimes \tilde{B}^*)$$

$$= 2 \sum_{i=0}^{2} (p_0 - p_{3-i}) |e_i\rangle \langle e_i|$$

$$+ (1 - 2p_0) [I \otimes I + (H_1 \otimes H_2)V(H_1 \otimes H_2)]$$

$$= \sum_{i=0}^{2} (p_0 - p_{3-i}) |e_i\rangle \langle e_i|$$

$$+ (1 - 2p_0)(C \otimes I)[\tilde{C}^\dagger \tilde{C} \otimes I + V](C^\dagger \otimes I).$$  \hspace{1cm} (13)

Since $A$ and $B$ are full rank, $X \geq 0$ if and only if $X' \geq 0$. The first term of $X'$ is positive since $p_0$ is largest. According to Ref. \textbf{[1]}, for a given $R \geq 0$, if $|\xi\rangle$ belongs to the range of $R$ and $\kappa \leq \frac{\text{Tr}(R)}{2}$, then $R - \kappa |\xi\rangle\langle\xi|$ $\geq 0$. Since $\text{det}(\tilde{C}^\dagger \tilde{C}) = 1$, the eigenvalues of $\tilde{C}^\dagger \tilde{C}$ are written as $\{t_1, t_2\}$, and we obtain

$$|\psi\rangle \langle|\psi|\rangle_{TB}^{-1}$$

and $\tilde{C}^\dagger \tilde{C} \otimes I + V = (\tilde{C}^\dagger \tilde{C} + 1) \otimes I - 2|\psi\rangle\langle|\psi|\rangle_{TB}^{-1}$ $\geq 0$. As a result, since $p_0 < 1/2$, the second term of $X'$ is also positive and $X'$ is found to be positive. Since $0 < \lambda \leq \lambda_0$, $|\sigma_{TB}|_{TB}$ can be always written as a convex sum of two positive operators ($X$ and $P_{TB}$), and the following theorem was proven.

**Theorem 2.** $|\sigma_{TB}|_{TB} \geq 0$ for any two-qubit state $\sigma$.

Finally, we briefly discuss the case in the higher dimensional systems. It has been already mentioned that $|\sigma_{TB}|_{TB} \geq 0$ does not hold in general, and the states violating $|\sigma_{TB}|_{TB} \geq 0$ have been named binegative states. In order to obtain some insights into how the binegative states emerge, we numerically generated the random binegative states of full rank in two qudits, and confirmed that $P_{TB} \equiv \sigma/2 + |\sigma_{TB}|_{TB}/2$ is not positive in general. This implies that the necessary condition corresponding to Theorem 1 is already violated in the higher dimensional systems (binegative states satisfying Theorem 1 also exist). This seems to imply that the two hyper-planes at the crossing point (like those shown in Fig. \textbf{[8]} sometimes cross with steep angles (it was shown to be always shallow in two qubits).

It will be important to clarify the geometry of the state space more, which might lead to a geometrical understanding of the quantum information tasks.

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[9] First convert the given Bell diagonal state by a suitable local unitary into the canonical Hilbert-Schmidt form where basis set is $|\varepsilon_i\rangle = \{|\psi^-\rangle, |\psi^+\rangle, |\phi^+\rangle, |\phi^+\rangle\}$ \textbf{[10]}. After that, if $p_k$ is largest, convert $|\varepsilon_k\rangle$ to $|\psi^-\rangle$ by a unilaterial $\pi$ rotation \textbf{[11]}. Finally, if $p_t$ is smallest, convert $|\varepsilon_k\rangle$ to $|\phi^+\rangle$ by a bilateral $\pi/2$ rotation \textbf{[12]}, where $|\psi^-\rangle$ remains unchanged. These transformations can be absorbed in $A$ and $B$.

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[16] For $C = H_1 H_2 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ and $\det C = \alpha \delta - \beta \gamma = 1$,

\[
\text{Tr} C C^* = |\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 + \alpha \delta^* + \beta \gamma^* + \alpha^* \delta + \beta^* \gamma + \beta^* \gamma^* + \alpha^* \delta^* + \beta \gamma + \beta^* \gamma^* + (\alpha - \delta^*)^2 + (\beta - \gamma^*)^2 + 2.
\]

Then, putting

\[
H_k = \begin{pmatrix} a_k & b_k + i c_k \\ b_k - i c_k & d_k \end{pmatrix}
\]

where $a_k, d_k > 0$, and $b_k$ and $c_k$ are real, $\text{Tr} C C^* = (a_1 a_2 - d_1 d_2)^2 + 4(a_1 b_2 + d_2 b_1)(a_2 b_1 + d_1 b_2) + 2 = (a^2 - d^2)^2 + (a + d)^2 b^2 - (a - d)^2 b^2 + 2$ where $a = \sqrt{a_1 a_2}, d = \sqrt{d_1 d_2}$ and $b = \left(\frac{a_1 d_2 - a_2 d_1}{a_1 d_2 + a_2 d_1}\right)^{1/4}$. Since $\det H_k = 1$ and $c_k$ must be real, $b_k^2 \leq a_k d_k - 1$. Then, $b^2 \leq 4ad$ and $\text{Tr} C C^* \geq (a - d)^2 + (a + d)^2 b^2 + 2 \geq 2$.

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