Anomalous transport and relaxation in classical one-dimensional models

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Abstract. After reviewing the main features of anomalous energy transport in 1D systems, we report simulations performed with chains of noisy anharmonic oscillators. The stochastic terms are added in such a way to conserve total energy and momentum, thus keeping the basic hydrodynamic features of these models. The addition of this “conservative noise” allows to obtain a more efficient estimate of the power-law divergence of heat conductivity \( \kappa(L) \sim L^\alpha \) in the limit of small noise and large system size \( L \). By comparing the numerical results with rigorous predictions obtained for the harmonic chain, we show how finite–size and –time effects can be effectively controlled. For low noise amplitudes, the \( \alpha \) values are close to \( 1/3 \) for asymmetric potentials and to \( 0.4 \) for symmetric ones. These results support the previously conjectured two-universality-classes scenario.

1 Introduction

Many-body systems constrained in reduced spatial dimensions (1 and 2D) display unusual relaxation and transport properties. Following familiar arguments of equilibrium statistical mechanics, this should be traced back to the predominant role of fluctuations that give rise to, e.g., long–range order etc. In the context of non–equilibrium processes, the presence of anomalies is signalled by the appearance of long–time tails in the correlation functions of the relevant currents [1,2] leading to ill–defined transport coefficients i.e. to the breakdown of standard hydrodynamics. The case of one–dimensional models is perhaps the most striking. For the sake of an example, we mention the divergence of viscosity in cellular automata fluids [3], anomalous diffusion in single-file systems [4], and the enhancement of vibrational energy transmission in polymers [5] or individual carbon nanotubes [6].

Within this general context, one of the issues that attracted a renewed interest in the last decade is the problem of anomalous heat conduction in one-dimensional models (for a review see [7]). The interest in such models originates from the need of constructing a minimal, nonperturbative theory of nonequilibrium stationary states and the quest for a rigorous microscopic foundation of phenomenological relations (the Fourier’s law in this case). This motivation generated a vast literature, especially in the mathematical physics community [8]. Moreover, many of the peculiarities of 1D models turned out to be of interest by themselves, as examples of highly correlated, and thus complex, behaviour. Those latter features are shortly reviewed in the first part of this paper. In the second part, the usefulness of an additional stochastic noise [9] is discussed with reference to some open questions.

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The class of models we will consider consists of a set of \( N \) classical point-like particles with masses \( m_n \) and positions \( x_n \) ordered along a line. Interactions are restricted only to nearest-neighbour pairs and the dynamics is ruled by

\[
m_n \ddot{x}_n = -F_n + F_{n-1}; \quad F_n = -V'(x_{n+1} - x_n),
\]

where \( V'(y) \) is a shorthand notation for the first derivative of the the interparticle potential \( V \) with respect to the argument. Boundary conditions for the first and last particle will be specified in the various cases. The two most intensively studied systems are the Fermi–Pasta–Ulam (FPU) model (with equal masses \( m_n = m \)) \cite{10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26},

\[
V(y) = \begin{cases} \frac{g_2}{2} (y-a)^2 + \frac{g_3}{3} (y-a)^3 + \frac{g_4}{4} (y-a)^4, & \text{for } y = 0 \\ 0 & \text{otherwise} \end{cases}
\]

and the diatomic Hard-Point Gas (HPG), where \( m_n = m/rm \) for even (odd) \( n \), with the interaction potential \cite{14,15,16,17}.

\[
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\]

We clarify from the beginning that we will always deal with genuine nonintegrable dynamics. For the FPU, this means working with high enough energies/temperatures to avoid all the difficulties induced by quasi-integrability and the associated slow relaxation to equilibrium \cite{18}. For the HPG this requires fixing a mass-ratio \( r \) not too close to unity.

The results that emerged from a long series of works can be summarized as follows. All studied models of the form (1) display **anomalous transport and relaxation features**, meaning by this that (at least) one of the following phenomena has been reported:

- The finite-size heat conductivity \( \kappa(L) \) diverges as \( L^\alpha \) in the limit of a large system size \( L \rightarrow \infty \) \cite{13}. This means that this transport coefficient is ill-defined in the thermodynamic limit, i.e. Fourier’s law does not hold;
- The equilibrium correlator of the energy current displays a nonintegrable power-law decay, \( \langle J(t)J(0) \rangle \propto t^{-(1-\delta)} \) \( (0 \leq \delta < 1) \) for long times \( t \rightarrow \infty \) \cite{19}. Accordingly, the Green-Kubo formula yields an infinite value of the conductivity;
- Energy perturbations propagate superdiffusively \cite{20}: a local perturbation of the energy broadens and its variance \( \sigma^2(t) \) grows in time as \( t^\beta \) with \( \beta > 1 \);
- Relaxation of spontaneous fluctuations is fast (i.e. superexponential) \cite{21,22}: at variance with standard hydrodynamics, the typical decay rate in time of fluctuations, \( \tau(q) \), is found to scale as \( \tau(q) \sim q^{-z} \) (with \( z < 2 \)).

Altogether, these features indicate that the kinetics of energy carriers is so correlated that they are able to propagate faster with respect to the the standard (diffusive) case. In view of this common physical origin, it is expected that the exponents describing such process will be related to each other by some “hyperscaling relations”. Their value should be ultimately dictated by the dynamical scaling of the underlying dynamics. Moreover, we may at least hope that they are largely independent of the microscopic details, thus allowing for a classification of anomalous behaviour in terms of “universality classes”.

Numerical studies \cite{7} indicate that anomalies occur generically in 1 and 2D, whenever momentum is conserved. This is connected to the existence of long-wavelength (Goldstone) modes (an acoustic phonon branch in the linear spectrum of (1)) that are very weakly damped. Indeed, it is sufficient to add external (e.g. substrate) forces to make the anomaly disappear. This is precisely the case of the ding-a-ling model \cite{23} and of other models in the same class, like the Frenkel–Kontorova \cite{24,25} or the nonlinear Klein-Gordon chains \cite{26}. The only remarkable exception is the coupled rotor chain \cite{27,28} where, however, different mechanisms are at work.\footnote{For the sake of nomenclature, it is useful to mention two particular cases: the quadratic plus cubic \( (g_4 = 0) \) and quadratic plus quartic \( (g_3 = 0) \) potentials that, for historical reasons, are referred to as the “\( \alpha \)-FPU” and “\( \beta \)-FPU” models, respectively. In the former one, sufficiently small coupling constant \( g_3 \) and/or energies must be considered to avoid runaway instabilities.}
Recently introduced stochastic models provide some mathematically rigorous results about the importance of momentum conservation. A random exchange of momentum between neighbor particles is added to the Hamiltonian deterministic dynamics. These exchanges may conserve only the energy \([29]\), or also the total momentum \([9]\). While in the first case conductivity remains finite and Fourier law is proved for harmonic interaction \([29]\), when total momentum is conserved it diverges as \(L^{1/2}\) \([9]\). In the anharmonic case, conductivity is much harder to compute or estimate also with this noise. So we performed some numerical simulations on the quartic and cubic FPU models with momentum-energy conserving noise.

In the first part of the paper we will review in more detail the anomalous features mentioned above. The second part is devoted to some preliminary numerical studies of the stochastic models.

1.1 Diverging finite-size conductivity

A natural way to simulate a heat conduction experiment consists in putting the system in contact with two heat reservoirs operating at different temperatures \(T_+\) and \(T_-\). Several models for the thermostats have been proposed based on both deterministic and stochastic algorithms \([7]\). Regardless of the actual thermostating scheme, after a transient, an off-equilibrium stationary state sets in, with a net heat current flowing through. The thermal conductivity \(\kappa\) is then estimated as the ratio between the time-averaged flux \(\overline{j}\) and the overall temperature gradient \((T_+ - T_-)/L\), where \(L = aN\) is the chain length (\(a\) denoting the lattice spacing). In this manner, \(\kappa\) should be considered as an effective transport coefficient, accounting for both boundary and bulk scattering mechanisms. The average \(\overline{j}\) can be estimated in several equivalent ways, depending on the employed thermostating scheme. One possibility is to directly measure the energy exchanges with the two baths. A more general definition (thermostat-independent) consists in averaging

\[
\dot{j}_n = \frac{1}{2}(\dot{x}_{n+1} + \dot{x}_n) F(x_{n+1} - x_n),
\]

which is obtained from the continuity equation for the energy density

\[
e_n = \frac{1}{2} m_n \dot{x}_n^2 + \frac{1}{2} [V(x_{n+1} - x_n) + V(x_n - x_{n-1})]
\]

As a result of many independent simulations performed with the above-described methods, it is now established that \(\kappa \propto N^\alpha\). It is also remarkable that the same type of behaviour has been observed for a realistic model of a single–walled carbon nanotube \([6]\). This type of molecular dynamics simulations, that involves complicated three-body interactions for carbon atoms, confirm that toy models like ours can indeed capture some general features.

1.2 Long-time tails

In the spirit of linear–response theory, transport coefficients can be computed from equilibrium fluctuations of the associated currents. More precisely, by introducing the total heat flux

\[
J = \sum_n j_n,
\]

the Green-Kubo formalism tells us that heat conductivity is determined from the expression

\[
\kappa = \frac{1}{k_B T^2} \lim_{t \to \infty} \lim_{N \to \infty} \frac{1}{N} \int_0^t dt' \langle J(t')J(0) \rangle
\]

where the average is performed in a suitable equilibrium ensemble, e.g. microcanonical with zero total momentum.
A condition for the formula (6) to give a well-defined conductivity is that the time integral is convergent. This is clearly not the case when the current correlator vanishes as \( \langle J(t)J(0) \rangle \propto t^{-\delta} \) with \( 0 \leq \delta < 1 \). Here, the integral diverges as \( t^{\delta} \) and we may thus define a finite-size conductivity \( \kappa(L) \) by truncating the time integral in the above equation to \( t \approx L/c \), where \( c \) is the sound velocity. Consistency with the above definition thus implies \( \alpha = \delta \). The available data are consistent with this expectation, thus providing independent method for estimating the exponent \( \alpha \).

For later purposes, we mention that, by means of the Wiener–Khintchine theorem, one can equivalently extract \( \delta \) from the low-frequency behaviour of the spectrum of current fluctuations

\[
S(\omega) = \int d\omega \langle J(t)J(0) \rangle e^{i\omega t}
\]

that displays a low-frequency singularity of the form \( S(\omega) \propto \omega^{-\delta} \). From the practical point of view, this turns out to be the most accurate numerical strategy as divergence rates are better estimated than vanishing ones.

1.3 Diffusion of perturbations

Consider the infinite system at equilibrium with an energy \( e_0 \) per particle and average momentum 0, and perturb it by increasing the energy of a subset of adjacent particles by some preassigned amount \( \Delta e \). Let us denote by \( e(x,t) \) the energy profile evolving from such a perturbed initial condition (for simplicity we identify \( x \) with the average particle location \( n\ell \)). We then ask how the perturbation \( f(x,t) = \langle e(x,t) - e_0 \rangle \) behaves in time and space \([30]\), where the angular brackets denote an ensemble average over independent trajectories. Because of energy conservation, \( \sum_n f(n\ell,t) = \Delta e \) at all times, so that \( f \) can be interpreted as a probability density (provided that it is also positive-defined and normalized).

At sufficiently long times and for large \( x \), one expects \( f(x,t) \) to scale as

\[
f(x,t) = t^{-\gamma} G(x/t^{\gamma})
\]

for some probability distribution \( G \) and a parameter \( 0 \leq \gamma \leq 1 \).

The case \( \gamma = 1/2 \) corresponds to normal diffusion and to a normal conductivity. On the other hand, \( \gamma = 1 \) corresponds to a ballistic motion and to a linear divergence of the conductivity.

Consequently an \( \alpha \) strictly contained between 0 and 1 implies a superdiffusive behaviour of the macroscopic evolution of an energy perturbation. It is an open problem to determine the dynamical nature of this superdiffusion. In \([31]\) has been proposed that \textit{Lévy walks} \([32]\) may describe these dynamics.

1.4 Relaxation of spontaneous fluctuations

The evolution of a fluctuation of wavenumber \( q \) excited at \( t = 0 \) is described by its correlation functions \( G(q,t) \). For 1D models like \([1]\) they are defined by considering the relative displacements \( u_n = x_n - n\ell \) and defining the collective coordinates

\[
Q(q) = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} u_n \exp(-iqn) = \frac{2\pi k}{Na}
\]

for \( k \) being an integer comprised between \(-N/2 + 1\) and \( N/2 \). The normalized correlator \( \tilde{G}(q,t) = (Q^*(q,t)Q(q,0))/\langle|Q(q)|^2\rangle \), is thus proportional to the density–density correlator, which is routinely used in condensed–matter physics \([33]\).
For sufficiently small wavenumbers $q$, the Fourier transform of $G$, the dynamical structure factor $S(q, \omega)$, usually displays sharp peaks at finite frequency. The associated linewidths are a measure of the fluctuation’s inverse lifetime. Simulations\textsuperscript{15,22} indicate that these lifetimes scale as $q^{-z}$ with $z = 1.50 - 1.67$. As explained above, one may think of this as a superdiffusive process, intermediate between standard diffusive and ballistic propagation.

This property is supported by theoretical results obtained in the framework of Mode-Coupling Theory (MCT). This approach has been traditionally invoked to estimate long-time tails of fluids\textsuperscript{1} and to describe the glass transition\textsuperscript{36}. Basically, it amounts to writing a set of approximate equations for $G(q, t)$ that must be solved self-consistently. For the problem at hand, the simplest version of the theory amounts to consider the equations \textsuperscript{37,38}

$$
\dot{G}(q, t) + \varepsilon \int_0^t \Gamma(q, t - s) \dot{G}(q, s) \, ds + \omega^2(q) G(q, t) = 0 \quad ,
$$

where the memory kernel $\Gamma(q, t)$ is proportional to $\langle \mathcal{F}(q, t) \mathcal{F}(q, 0) \rangle$, with $\mathcal{F}(q)$ being the nonlinear part of the fluctuating force between particles. Equations (11) must be solved with the initial conditions $G(q, 0) = 1$ and $\dot{G}(q, 0) = 0$. Equations (11) are derived within the well-known Mori–Zwanzig projection approach\textsuperscript{34}.

The mode–coupling approach basically amounts to replacing the exact memory function $\Gamma$ with an approximate one, where higher–orders correlators are written in terms of their bare values, obtained with an approximate one, where higher–orders correlators are written in terms of

$$
\Gamma(q, t) = \omega^2(q) \frac{2\pi}{N} \sum_{p+p' - q = 0, \pm \pi} G(p, t) G(p', t) \quad ,
$$

(11)

Here $p$ and $p'$ range over the whole Brillouin zone (from $-\pi$ to $\pi$ in our units). This yields a closed system of nonlinear integro–differential equations. Both the coupling constant $\varepsilon$ and the frequency $\omega(q)$ are temperature-dependent input parameters, which should be computed independently by numerical simulations or approximate analytical estimates\textsuperscript{37,38}. For the aims of the present work, we may restrict ourselves to considering their bare values, obtained in the harmonic approximation. In the adopted dimensionless units they read $\varepsilon = 3g_3^2 k_B T/2\pi$ and $\omega(q) = 2|\sin \frac{q}{2}|$. Of course, the actual renormalized values are needed for a quantitative comparison with specific models.

The long-time behaviour of $G$ can be determined by looking for a solution of the form

$$
G(q, t) = C(q, t)e^{i\omega(q)t} + c.c. \quad ,
$$

(12)

with $\dot{C} \ll \omega G$. It can thus be shown\textsuperscript{39,40} that, for small $q$-values and long times $C(q, t) = g(\sqrt{t}q^{3/2})$ i.e. $z = 3/2$ in agreement with the above mentioned numerics. Furthermore, in the limit $\sqrt{t}q^{3/2} \to 0$ one can explicitly evaluate the functional form of $g$, thus obtaining

$$
C(q, t) = \frac{1}{2} \exp \left( -Dtq^2 t^{\frac{1}{2}} \right) \quad ,
$$

(13)

where $D$ is a suitable constant of order unity. Correlation display a “compressed exponential” behaviour\textsuperscript{41} in this time range. This also means that the lineshapes of the structure factors $S(q, \omega)$ are non-Lorenzial but rather display a faster power-law decay $(\omega - \omega_{\text{max}})^{-7/3}$ around their maximum.

Upon inserting this scaling result into the definition of the heat flux, one eventually concludes that the conductivity $\kappa$ diverges with a rate $\alpha = 1/3$.

2 Universality

The crucial question at this point is: how universal are the above defined exponents? The renormalization group argument by Narayan and Ramaswamy\textsuperscript{42}, predicts $\alpha = (2-d)/(2+d)$.\textsuperscript{5}
Following the arguments exposed above, this implies that in 1D the values of the exponents are

$$\alpha = \delta = \frac{1}{3}, \quad \beta = \frac{4}{3}, \quad z = \gamma^{-1} = \frac{3}{2}. \quad (14)$$

According to this approach, any possible additional term in the noisy Navier-Stokes equation yields irrelevant corrections in the renormalization procedure [42].

On the other hand, there now exists a rigorous result, proving that $\alpha = 1/2$ [9] in a chain of harmonic oscillators subject to an energy and momentum–conserving noise. Moreover, the application of kinetic theories to the $\beta$-FPU model [43,44,45] predict $\alpha = 2/5$, while a modified version of the MCT adapted to this specific case gives instead $\alpha = 1/2$ [46,47].

The validation of these theoretical results by numerical simulations is, to some extent, challenging. Generally speaking, the available numerical estimates of $\alpha$ and $\delta$ range between 0.25 and 0.44 [7,48]. The existence of crossovers among different scaling regimes has been observed [50]. However, even in the most favorable cases of computationally efficient models as the HPG, finite–size corrections to scaling are sizeable. As a matter of fact, $\alpha$–values as diverse as 0.33 [17] and 0.25 [49] for comparable parameter choices have been reported. On the other hand, a numerically convincing confirmation of the $\alpha = 1/3$ prediction comes from the diffusion of perturbations [20]. The most compelling deviations from the values given in (14) have been reported for the $\beta$-FPU model [48], where a better agreement with the predictions of the kinetic theory has been observed.

A reasonable argument that can be invoked to delimit the $\alpha = 1/3$ universality class appears to be the symmetry of the interaction potential with respect to the equilibrium position. In fact, systematically larger $\alpha$–values have been reported only for symmetric potentials such as the $\beta$-FPU model or, more recently, for a modified version of the HPG model [35], where the interaction can be tuned to yield symmetric fluctuations of the force. Moreover, the rigorous result $\alpha = 1/2$ [9] indeed concerns a model whose potential (harmonic) is symmetric and where the stochastic updating is symmetric as well. In the framework of MCT, one can understand that the symmetry of the fluctuations implies that the quadratic kernel (11) should be replaced by a cubic one [40], thus yielding different values of the exponents (see [49] for a thermodynamic interpretation of this difference).

Whether symmetry is the only necessary ingredient to identify the asymptotic behaviour is however not fully proven. As mentioned above, difficulties manifestly arise e.g. for the $\beta$-FPU model where it is not yet completely assessed which of the predicted exponents one should expect. In this respect some further reconsideration of MCT and kinetic theories may be of help. Actually, there are even some controversial results about the dynamical scaling exponent $z$ in the supposed broad $\alpha = 1/3$ class [47]. It is thus clear that further rigorous results on simple models would be definitely welcome.

### 3 Stochastic models with energy and momentum conservation

Recently, some of the authors proposed a new class of models for analyzing the anomalous properties of heat conduction for a system of oscillators [9]. The Hamiltonian dynamics is perturbed by a stochastic noise, which acts only on the momenta, while conserving total energy and momentum. In particular, such an approach allows to compute explicitly the heat flux correlation function in the harmonic case, i.e. model (2) with $g_3 = g_4 = 0$. For the 1D model it is found that the current correlator vanishes as $\langle J(t)J(0) \rangle \sim t^{-1/2}$, thus implying that $\kappa \sim L^{1/2}$. The average of the current correlator is performed over the equilibrium measure, which is the uniform measure over the hypersurface of constant total energy and momentum. As usual in this microcanonical measure, the total momentum is set to zero. There are various ways for translating such stochastic dynamics into a suitable algorithm. The numerical simulations hereafter reported have been performed by making the system evolve over a finite number $n$ of steps (each of duration $h$) and, then, by updating the momenta of $n_c$ triplets of nearest-neighbor oscillators, whose position are randomly chosen with a uniform probability density over the chain sites. Each “collision” in the momentum-subspace is performed by extracting
from the uniform distribution over \([0, 2\pi]\) a random angle which rotates the three momenta in such a way to maintain fixed their sum and the sum of their squares. The resulting configuration is the initial condition for a new deterministic trajectory lasting again over \(n\) steps, and so forth. Because of the conservation laws, the microcanonical measure (uniform on the energy surface) is still invariant and it can be proven to be ergodic.

The algorithmic procedure introduces a new control parameter in the dynamics, namely the ratio between the number of collisions per the unit time and the number of particles \(N\).

In order to test the effectiveness of the algorithm, we have first performed numerical simulations of the stochastic dynamics of the harmonic chain, as we can make a comparison with the rigorous results \([9]\). In the left panel of figure 1 we show the power spectrum defined in (7) for different values of \(N\), while maintaining fixed the ratio \(n_c/N\approx 0.1\). All curves align to the same power–law behavior in a range of small values of \(\omega\) of about 3 decades. In particular, the spectrum is found to diverge as \(S(\omega) \sim \omega^{-\delta}\) with \(\delta\) slightly larger than 1/2. Moreover, finite size effects are clearly visible in the spectrum corresponding to the smallest chain length (\(N = 512\)). In the right panel of figure 1 we have kept \(N = 2048\), while we have considered different values of \(n_c\). For decreasing values of \(n_c\), we see that \(\delta\) remains close to 1/2, in agreement with the rigorous prediction reported in \([9]\).

![Fig. 1. Power spectra of the heat flux for the stochastic harmonic chain at energy density \(e = 10\). Units along the vertical axis are arbitrary. Left panel: data for different values of \(N\), while maintaining fixed the ratio \(n_c/N\). Right panel: \(N = 2048\), and different values of \(n_c\). Each curve is averaged over about 500 independent trajectories. Here and in the following figures, the time interval between collisions is 10 \(h\) with integration time step \(h = 0.01\). The thin dashed line corresponds to the law \(\omega^{-1/2}\).](image)

Relying on these results, we have also studied the power spectra of the stochastic FPU models \([2]\) with \(g_2 = g_3 = g_4 = 1\) and \(g_2 = g_4 = 1\) and \(g_3 = 0\) (\(\beta\)-FPU). For both models, we report the power spectrum at fixed \(N\) for different \(n_c\) values (see Figs. \([2, 3]\)). Analogously to what observed for the harmonic case, we see that upon increasing \(n_c\), the scaling region widens towards higher frequencies, thus allowing for a more accurate analysis than in the purely deterministic case. Moreover, in both cases the effective divergence exponent \(\delta\) appears to increase with \(n_c\), approaching 1/2. More specifically, for the FPU with cubic and quartic nonlinearities (Fig. \([2]\)), \(\delta\) increases from \(\approx 0.35\) for \(n_c = 10\) to \(\approx 0.48\) for \(n_c = 200\). Analogously, for the \(\beta\)-FPU case (Fig. \([3]\)), we find that \(\delta\) increases from \(\approx 0.41\), to \(\approx 0.47\).

Based on the existence of a few universality classes, a continuous dependence of the exponent \(\delta\) on the number \(n_c\) of collisions is quite unlikely. However, one should recognize that the frequency scaling range is as wide as three decades and the dependence of \(\delta\) on \(\omega\) is pretty weak. Altogether, the fact that \(\delta\) is close to 1/2 for the larger \(n_c\) values suggests that the scaling behaviour predicted in \([9]\) applies to a larger class of systems and enforces the hypothesis of a second distinct universality class. Moreover, it is interesting to notice that in both anharmonic models, the divergence of thermal conductivity increases with the strength of the noise. This means that, quite surprisingly, noise contributes to slowing down the decay of energy current correlations.
Fig. 2. Power spectrum of the heat flux for the stochastic FPU model (cubic and quartic nonlinearities) with energy density $e = 10$, $N = 2048$ and different number $n_c$ of colliding triplets. Units along the vertical axis are arbitrary. Each curve is averaged over about 800 independent trajectories. The thin dashed line corresponds to the law $\omega^{-1/2}$.

Fig. 3. Power spectrum of the heat flux for the stochastic $\beta$-FPU model (quartic nonlinearity) with energy density $e = 10$, $N = 2048$ and different number $n_c$ of colliding triplets. Units along the vertical axis are arbitrary. Each curve is averaged over about 800 independent trajectories. The thin dashed line corresponds to the law $\omega^{-1/2}$.

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