NUMERICAL SOLUTION TO AN INVERSE PROBLEM ON A DETERMINATION OF PLACES AND CAPACITIES OF SOURCES IN THE HYPERBOLIC SYSTEMS

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Abstract. In the work we investigate the numerical solution to a class of inverse problems with respect to the system of differential equations of hyperbolic type. The specialties of considered problems are: 1) the impulse impacts are present in the system and it is necessary to determine the capacities and the place of their location; 2) the differential equations of the system are only related to boundary values, and arbitrarily; 3) because of the long duration of the object functioning, the exact values of the initial conditions are not known, but a set of possible values is given. The inverse problem under consideration is reduced to the problem of parametric optimal control without initial conditions with non-separated boundary conditions. For the solution it is proposed to use first-order optimization methods. The results of numerical experiments are given on the example of the inverse problem of fluid transportation in the pipeline networks of complex structure. The problem is to determine the locations and the volume of leakage of raw materials based on the results of additional observations of the state of the transportation process at internal points or at the ends of sections of the pipeline network.

1. Introduction. In this work we solve the inverse problem of definition as capacities, and unlike many other researches the locations of pointwise (external, internal) sources influencing the process (object) are determined. Process is described by the system of a large number of the differential equations of hyperbolic type of the second order connected among themselves only by boundary conditions. In the considered problem an accurate information about the initial conditions is missing, because of the long duration of the process functioning. In exchange the set of possible values of initial states is known. For the solution of the inverse problem there are additional observations on the process in isolated points.

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Examples of such problems are not difficult to cite from the field of ecology, geophysics, and underground hydrodynamics. In this paper, this problem has been studied in more detail in the framework of the problem of determining the volumes and places of leakage of raw materials in the fluid transport through a pipeline of a complex, loopback structure. Note that in order to solve the problem of monitoring the state of sections of pipelines, a lot of attention is paid to the creation of methods and technical devices for locating leaks [12], [20].

For underground and underwater sections of pipeline networks, for which a visual inspection and the use of such technical devices makes leak detection more complicated, one can use mathematical methods using the results of monitoring of flow modes at certain points of the pipeline network to determine the locations of leakages. Currently, there is an increased interest in such approaches, taking into account the increased level of development of numerical methods, computational tools and measuring technology [2], [3], [4], [9], [14], [19], [24], [30], [31].

This work differs from many others, which are mainly studied the solution to the problem of determining the location and volume of leakage either for steady modes of fluid flow in the pipelines of complex structure, or unsteady flow in the pipeline, consisting of only one linear section [21], [26], [27]. This work examines the numerical solution of the inverse problem of determining the locations and volumes of leakage at unsteady fluid flow in the pipeline network of complex loopback structure. The problem is described by the large number (equal to number of the section of pipeline network) of sub-systems of two partial differential equations of hyperbolic type with impulse impacts at the points of possible leakage in the sections of the pipeline network.

Another feature of this formulation of the problem is to assume the exact data on the initial state of the modes not available at the start of the monitoring, because of the long duration time of the transportation process. On the other hand it is not seem possible the operational measurements of the modes at all points of the pipeline network. In exchange there is information on a set of possible values of the modes on all sections of the pipeline on any initial instant and also from this point time there are results of measurements of the modes in particular points of a pipeline networks. The next feature of the problem lies in the boundary conditions, which are defined in the form of non-separated relations between modes (states) at the junctions of the pipeline network, defined by analogues of the first Kirchhoff's law and the continuity of the flow.

This formulation of the inverse problem coincides with the class of optimal control problems by objects with distributed parameters [13], [25], [29]. We suggest using the first order numerical methods for optimal control to solve this problem. The formulas for the components of the gradient of the minimizing target functional on identifying parameters are obtained. The results of numerical experiments to the solution of model inverse problems are given, with the presence of observations with different levels of measurement error in the input data.

2. General problem statement. Let us consider a system, consisting of \( M \) arcs arbitrarily joined by ends, represented in the form of an oriented graph. Each arc (edge) is an object with parameters, distributed over the length of the arc and time, the state of which is described by partial differential equations of hyperbolic type. We denote the set of nodes of the graph by \( I \), and the set of arcs (edges) \((k, s)\) with the length \( l^{(k,s)} \), with the beginning at the node \( k \in I \) and with the end at the node
s \in I \) denote by \( J = \{(k, s) : k, s \in I\}, |I| = N, |J| = M. \) Let \( J_i^+ = \{(j, i) : j \in I_i^+\}, J_i^- = \{(i, j) : j \in I_i^-\} \) are the sets of arcs respectively entering and leaving the node of \( i \), \( I_i^+ \) and \( I_i^- \) are the sets of nodes, connected with the \( i \)-th node, which are respectively the ends and the beginnings of the arcs from \( J_i \); \( J_i = J_i^+ \cup J_i^- \), \( I_i = I_i^+ \cup I_i^- \). We denote by \( |J_i^+| = |I_i^+| = n_i^+, |J_i^-| = |I_i^-| = n_i^-, n_i^++n_i^- = n_i \).

So, it’s known that
\[
\sum_{i \in I} n_i^+ = \sum_{i \in I} n_i^- = N, \quad \sum_{i \in I} n_i = 2M.
\]

In practical applications, as a rule, takes place \( n_i << N, i \in I \), i.e. the number of the nodes connected with any other node, are much less in comparison with common number of the nodes. Let the state of each arc is described by a differential equation of hyperbolic type of the following form \([15], [23], [26], [32]\

\[
\frac{\partial^2 u_{ks}(x, t)}{\partial t^2} = a_{ks} \frac{\partial^2 u_{ks}(x, t)}{\partial x^2} - b_{ks} \frac{\partial u_{ks}(x, t)}{\partial t} + q_{ks}(t) \delta(x - \xi_{ks}),
\]

\( x \in (0, l_{ks}), t \in [t_0, T]. \) (1)

Here \( u_{ks}(x, t) \) is the state of \((k, s)\)-th arc with the length \( l_{ks} \) at the point \( x \) at the time \( t \), \((k, s) \in J; a_{ks}, b_{ks} > 0 \) are given; the continuous functions \( q_{ks}(t) \) are determined by the capacity of internal or external sources concentrated at the points \( \xi_{ks} \in (0, l_{ks}) \) of some arcs. We denote a set of arcs containing concentrated sources by \( J^{ist} \subset J, N^{ist} = |J^{ist}| \), i.e.

\[
q_{ks}(t) \left\{ \begin{array}{ll}
\neq 0, & (k, s) \in J^{ist}, \\
= 0, & (k, s) \in J \setminus J^{ist}, 
\end{array} \right. 
\]

\( t \in [t_0, T]. \)

Remark 1. Consideration of a case of constancy of values of coefficients \( a_{ks}, b_{ks} \) is not principal and bound only to simplification of record of calculations provided in the following section.

The total number of equations of the form (1) is equal to \( M \) the number of arcs. The current states of the arcs are related to adjacent arcs only by \( 2M \) number non-separated boundary conditions:

\[
\sum_{k \in I_i^-} \left[ \alpha_{ik}^\nu u^{ik}(0, t) + \beta_{ik}^\nu \frac{\partial u^{ik}(0, t)}{\partial x} \right] +
\]
Here $\gamma_i^\nu$ is $\nu$-th characteristic of external influence in the $i$-th node. Such influences, as a rule are other than zero only in entrance and output nodes of system. The set of such nodes of the graph designate by $\mathcal{I}^f$, $N^f = |\mathcal{I}^f|$, and if $i \in \mathcal{I}^f$, then it means that, one of $n_i^+$ and $n_i^-$ is 1, other is 0. The set of internal nodes of the graph let designate by $\mathcal{I}^{int} = \mathcal{I} \backslash \mathcal{I}^f$, $N^{int} = |\mathcal{I}^{int}|$.

We note that the boundary conditions (2) have a significant specificity, consisting in the fact that they are non-separated (nonlocal) boundary conditions.

We will write down the expressions (2) in a matrix form more generally:

$$
A_1 u(0, t) + A_2 \partial u(0, t) - \gamma_1 = \gamma.
$$

Here we used the notations:

$$
\begin{align*}
u(0, t) &= (u^{1,1}(0, t), u^{1,2}(0, t), ..., u^{1,M}(0, t))^T, \\
u(l, t) &= (u^{1,1}(l, t), u^{1,2}(l, t), ..., u^{M,M}(l, t))^T, \\
\gamma &= (\gamma_1, ..., \gamma_{N^f})^T.
\end{align*}
$$

We assume $\nu^{i,j}(0, t) \equiv 0$, if the nodes $i$ and $j$ are not adjacent, and respectively, in matrices $A_1, A_2, B_1, B_2$ dimension $2M \times 4M$, made of coefficients $\alpha_{ks}, \beta_{ks}$, elements, corresponding non-existent edges are also 0. The components of the vector $\gamma \in R^{2M}$ corresponding to the conditions relative to the nodes in which there are no internal or external sources are equal to 0.

Let’s assume that the coefficients, participating in conditions (2) are such that, poorly filled augmented matrix $[A_1, A_2, B_1, B_2]$ have the rank equal to $2M$. Without loss of generality, let the rank of the augmented matrix $\tilde{A} = [A_1, A_2]$ is $2M$, i.e. $rang \, \tilde{A} = rang[A_1, A_2] = 2M$, $\hat{B} = [B_1, B_2]$.

It is also important to note the following feature of the problem under consideration, which one has to face in many practical applications.

It is known [10] that, if the process (1) acts rather long, the impact of values of initial conditions on the state of the processes occurring on the arcs weakens over time, because of the friction, determined by the second terms in the right-hand sides of the equations (1). Therefore, when the process is observed for a long time, there exists such $\tau$, $\tau > t_0$ that, only boundary conditions impact on the state of the processes in the time interval $[t_0, T]$ for $t > \tau$. The value $\tau$ is determined by the parameters of the process $a_{ks}, b_{ks}$ [10]. This specificity is important for the considered problem from a practical point of view, i.e. the measurements of the states for all arcs of the system as a rule, technically cannot be realized in all points of the phase variable at any moment of time. In practice, measurements of the states of the objects distributed in the space are carried out at the ends and/or in some specific internal points of the arcs.

Therefore, we will assume that, at some initial instant $t_0$ the initial conditions for the process (1)

$$
u^{ks}(x, t_0) = \varphi_{0}^{ks}(x), \quad \partial \nu^{ks}(x, t_0) = \varphi_{1}^{ks}(x)
$$

are not defined precisely, but a set of possible values $U^0$ is given:

$$
U^0 = \{(\varphi_0^{ks}(x), \varphi_1^{ks}(x)) : (k, s) \in J \}
$$
The initial conditions belonging to $U^0$ can be defined in a parametric form:

$$
\varphi^{ks}(x, \gamma) = (\varphi_0^{ks}(x; \gamma), \varphi_1^{ks}(x; \gamma)), \quad (k, s) \in J, \quad \gamma \in \Gamma \subseteq \mathbb{R}^n,
$$

(4)

$\Gamma$ is a given set of admissible values of the parameters, the function $\varphi(x, \gamma)$ is determined precisely to (1) - (3):

for all initial conditions from $U^0$ with the probabilities of getting the given functions or parameters that define them:

$$
U^0 = \{\varphi_0^{ks}(x; \gamma^j), \varphi_1^{ks}(x; \gamma^j) : \ j = 1, ..., N_{U^0}, (k, s) \in J\}
$$

(5)

with the probabilities of getting the $j$-th value:

$$
p_j = P(\gamma = \gamma^j), \quad j = 1, ..., N_{U^0}.
$$

We assume that, the functions and parameters involved in the initial-boundary value problem (1) - (3) with nonlocal conditions are such that, it has a unique solution at given values of the source capacities $q_i(t)$, $t \in [0, T]$ and their locations $\xi^{ks}_i \in [0, l^{ks}]$ for all initial conditions from $U^0$.

We will demand from the functions $u^{ks}(x, t)$ defining the solution of a problem (1)-(3):

- on variables $x, t$ on all arcs $(k, s) \in J$ ,
- twice continuously differentiability on arcs, in which there are no concentrated sources, i.e. for $(k, s) \in J \setminus J^{ist}$, 
- on the arcs containing the concentrated sources, functions $u^{ks}(x, t)$ have to be continuously differentiable and almost everywhere are twice continuously differentiable on variables $x, t$, besides the points $x = \xi^{ks} \in (0, l^{ks})$, $(k, s) \in J^{ist}$.

Remark 2. Many processes can be described by the initial boundary value problems (1)-(5) in practice. For example, vibrations of multi-link metal structures exposed to external shocks, the unsteady flow of hydrocarbon raw material in pipeline networks of a complex structure, which have external sources (inflows or outflows). Similar problems also arise in complex acoustic devices. For each of these applications, the function $u^{ks}(x, t)$ is determined by the state parameter of the studied process at the point $x$ of $(k, s)$-th link at time $t$. Particularly, in the inverse problem considered in Section 4 for determining the locations and volumes of fluid leaks in a complex pipeline system the derivative with respect to $x$ of the function $u^{ks}(x, t)$ determines pressure and the derivative with respect to $t$ characterizes the flow amount at point $x$ of the $(k, s)$-th segment of the pipeline at time $t$.

Note that a lot of work has been devoted to the existence and uniqueness of the solution of systems of hyperbolic equations with nonlocal conditions, but unfortunately the constructive conditions have not yet been obtained. Therefore it is necessary to be content with the results of numerical studies, based on calculations conducted for each specific practical class of problems.

Now we formulate the basic problem considered in this paper, which belongs to a class of inverse problems or to the problems of parametric identification of systems with distributed parameters [2], [3], [4], [13], [25], [29].

We suppose the capacities $q^{ks}(t)$ and the locations $\xi^{ks}$ of the sources at the arcs of the given set $J^{ist} \subseteq J$ are unknown and they need to be determined using additional information on the state of the processes, given below. Let the measurements of the process state are made in some points $\eta^{ks} \in (0, l^{ks})$ of the arcs (edges) $(k, s) \in J^{obser}$ of the graph. We designate the locations of these points by $\eta^{ks} \in (0, l^{ks})$, 

and the set of arcs in which observations are carried out we designate by $J_{\text{observe}}$, $J_{\text{observe}} \subset J$:

$$
\left. u^{ks}(\eta^{ks}, t), \frac{\partial u^{ks}(\eta^{ks}, t)}{\partial x} \right| = \left. \left( V_0^{ks}(t), V_1^{ks}(t) \right) \right|.
$$

We’ll assume the restrictions on the identifying functions and parameters, proceeding from the meaning of the problem considered:

$$
0 \leq \xi^{ks} \leq l^{ks}, \quad q \leq q^{ks}(t) \leq \bar{q}, \quad t \in [t_0, T], (k, s) \in J^{loss},
$$

where $q, \bar{q}$ are the given values.

We reduce the formulated inverse problem to the optimization problem. We use the conditions (6) to form the next minimized functional of the quadratic deviation $J$ and the set of arcs in which observations are carried out we designate by

$$
\Phi(q, \xi) = \int_\Gamma \Phi(q, \xi; \gamma) p_T(\gamma) d\gamma,
$$

$$
\Phi(q, \xi; \gamma) = \sum_{(k, s) \in J_{\text{observe}}} \int_T^T \left[ u^{ks}(\eta^{ks}, t; \gamma, q, \xi) - V_0^{ks}(t) \right]^2 + \left[ \frac{\partial u^{ks}(\eta^{ks}, t; \gamma, q, \xi)}{\partial x} - V_1^{ks}(t) \right]^2 dt + \mathcal{R}(\xi, q),
$$

$$
\mathcal{R}(\xi, q) = \varepsilon_1 \sum_{(k, s) \in J_{\text{obs}}} \| q^{ks}(t) - \hat{q}^{ks}(t) \|^2 + \varepsilon_2 \sum_{(k, s) \in J_{\text{obs}}} \left[ \xi^{ks} - \hat{\xi}^{ks} \right]^2.
$$

where $u^{ks}(\eta^{ks}, t; \gamma, q, \xi)$ are the calculated values of the process state at the moment of time $t$ at the observed points as a result of the solution of the boundary value problem for admissible values of the parameters $\gamma \in \Gamma$ in the initial conditions $\varphi(x, \gamma)$ and for given admissible places and capacities of the sources $(\xi, q(t))$; $[\tau, T]$ is the time interval of observation on the process whose states do not depend on the initial conditions at $t = t_0; \xi^{ks}, q^{ks}(t) \in R^n, \varepsilon_1, \varepsilon_2$ are the regularization parameters.

Since the initial conditions determined at the instant of time $t_0$ do not influence the process in the time interval $[\tau, T]$, an exact knowledge of the initial value $t_0$ and of the initial functions $\varphi_0^{ks}(x; \gamma), \varphi_1^{ks}(x; \gamma), (k, s) \in J$ for the value of the functional (8) is not principal.

Particularly, in the case of a finite set of admissible initial conditions in the form of (5) with a uniform distribution of the parameter $\gamma \in \Gamma$, the functional (8) takes the form:

$$
\bar{\Phi}(q, \xi) = \frac{1}{N_{\text{obs}}} \sum_{j=1}^{N_{\text{obs}}} \int_T^T \left[ u^{ks}(\eta^{ks}, t; q, \xi, \gamma_j) - V_0^{ks}(t) \right]^2 + \left[ \frac{\partial u^{ks}(\eta^{ks}, t; q, \xi)}{\partial x} - V_1^{ks}(t) \right]^2 dt + \mathcal{R}(\xi, q).
$$

3. **Numerical solution to the problem.** Formulated problem (1)–(5), (8) or (1)–(6), (8) on the one hand belongs to the class of inverse problems or problems of parametric identification, on the other hand it can be attributed to the problem of parametric optimal control of object with distributed parameters, described by a hyperbolic system [2], [3], [4], [13], [17], [23], [29].
DETERMINATION OF PLACES AND CAPACITIES OF SOURCES

Its main difference from the previously discussed statements is the necessity to determine the locations of the sources \( \xi \), in addition to their capacities \( q(t) \). It is clear that in the general case the optimized functional is not convex, because of the nonlinear presence of optimized parameters in the differential equations, so it can be multiextreme. To solve such problems, as a rule, either a “good” initial condition is required for iterative minimization methods, or is not required its multiple solution from different initial points. In this paper, as illustrated in test problems, the solution of the optimization problem was carried out multiply, from different initial points for the iterative process.

We use the numerical methods of optimal control based on the first order iterative procedure of optimization for instance, the method of projection of the gradient [22] to solve the problem

\[
\left( \frac{\xi}{q(t)} \right)^{i+1} = \text{Pr} \left[ \left( \frac{\xi}{q(t)} \right)^i - \alpha^i \text{grad} \, \Im((\xi, q(t))^i) \right], \quad i = 0, 1, ..., (10)
\]

where \( \text{Pr}(\cdot) \) is the projection operator of the vector into a set, defined by constraints (7), \( \alpha^i > 0 \) is a step of one dimensional minimization:

\[
\alpha^i = \arg \min_{\alpha \geq 0} \Im \left[ \text{Pr} \left( \left( \frac{\xi}{q(t)} \right)^i - \alpha \text{grad} \, \Im((\xi, q(t))^i) \right) \right].
\]

We obtain the formulas for the gradient of the target functional (8) with respect to the identified parameters \((\xi, q(t))\) to use the procedure (10). The components of the gradient vector will be received independently of each other considering their mutual independence. So, we assume that there is a source with unknown values of capacities \( q^k_s(t) \) and location \( \xi^k_s \in [0; l^k_s] \) at any one given arc (edge) \((k, s) \in J^\text{out} \) of the graph, but for other sources these data exist. We use the method of variation of optimized parameters to obtain the formulas for the components of the gradient for the identified parameters.

Taking into account the mutual independence of the admissible initial conditions in \( U^0 \), it is clear that the following formulas take place for the component of the gradient:

\[
\text{grad} \Phi(q, \xi) = \text{grad} \int_\Gamma \Phi(q, \xi; \gamma)\rho_\Gamma(\gamma)d\gamma = \int_\Gamma \text{grad} \Phi(q, \xi; \gamma)\rho_\Gamma(\gamma)d\gamma.
\]

Therefore it is enough to receive the formulas for \( \text{grad}_{q^k_s} \Phi(q, \xi) \) and \( \text{grad}_{\xi^k_s} \Phi(q, \xi) \) at any arbitraries admissible initial conditions or the parameters corresponding to them. In the \((k, s)\) -th equation of the system (1) we introduce the notation:

\[
v(x, t) = q^k_s(t)\delta(x - \xi^k_s), \quad (11)
\]

The function \( v(x, t) \) will be called an identified influence. Let \( v = v(x, t) \) and \( \tilde{v} = \tilde{v}(x, t) \) are two admissible influences, to which correspond \( u^{k_s}(x, t; v) \) and \( \tilde{u}^{k_s}(x, t; \tilde{v}) \) the solutions to initial-boundary value problem (1)–(3) for any concretely selected admissible initial conditions \((\varphi^k_0(x), \varphi^k_1(x))\), \((k, s) \in J \). Let \( \Delta v = \Delta v(x, t) = \tilde{v}(x, t) - v(x, t), \Delta u^{k_s}(x, t) = \tilde{u}^{k_s}(x, t; \tilde{v}) - u^{k_s}(x, t; v) \). It is reduced from (1)–(3) that, the function \( \Delta u^{k_s}(x, t) \) is the solution to the following initial-boundary-value
problem with nonlocal boundary conditions:
\[
\frac{\partial^2 \Delta u^{ks}(x,t)}{\partial t^2} = a^{ks} \frac{\partial^2 \Delta u^{ks}(x,t)}{\partial x^2} - b^{ks} \frac{\partial \Delta u^{ks}(x,t)}{\partial t} + \Delta v(x,t), \\
\text{where } x \in (0,l^{ks}), t \in [t_0,T], (k,s) \in J^{ist}
\]  
(12)

\[
\frac{\partial^2 \Delta u^{ks}(x,t)}{\partial t^2} = a^{ks} \frac{\partial^2 \Delta u^{ks}(x,t)}{\partial x^2} - b^{ks} \frac{\partial \Delta u^{ks}(x,t)}{\partial t}, \\
\text{where } x \in (0,l^{ks}), t \in [t_0,T], (k,s) \in I / J^{ist},
\]  
(13)

\[
\sum_{k \in I_i^{−}} \left[ a^{\nu} \Delta u^{k,i}(0,t) + \beta^{\nu}_{ki} \frac{\partial \Delta u^{k,i}(0,t)}{\partial x} \right] + \\
+ \sum_{k \in I_i^{−}} \left[ a^{\nu} \Delta u^{k,i}(l^{ik},t) + \beta^{\nu}_{ik} \frac{\partial \Delta u^{k,i}(l^{ik},t)}{\partial x} \right] = 0, \nu = \Gamma, \eta, i \in I,
\]  
(14)

\[
\Delta u^{ks}(x,t_0) = 0, \quad \frac{\partial \Delta u^{ks}(x,t_0)}{\partial t} = 0, \quad s \in I_i^{−}, k \in I. 
\]  
(15)

The relations (14) in general form can be written according to (3) in the matrix form:
\[
A_1 \Delta u(0,t) + A_2 \frac{\partial \Delta u(0,t)}{\partial x} + B_1 \Delta u(l,t) + B_2 \frac{\partial \Delta u(l,t)}{\partial x} = 0,
\]  
(16)

Then according to the made assumption, a rank of an augmented matrix \([A_1, A_2 B_1, B_2]\) is equal to 2M. Multiplying both parts (16) on \(\bar{A}^{-1} = [A_1, A_2]^{-1}\) and designating \(\bar{B} = [-B_1 - B_2]\), we have:
\[
\begin{bmatrix} \Delta u(0,t) \\ \frac{\partial \Delta u(0,t)}{\partial x} \end{bmatrix} = \bar{A}^{-1} \bar{B} \begin{bmatrix} \Delta u(l,t) \\ \frac{\partial \Delta u(l,t)}{\partial x} \end{bmatrix}
\]  
(17)

The increment of the functional (8) can be written in the following form, taking into account the designations accepted above and for the function \(\psi^{ks}(x,t)\) arbitrarily almost everywhere twice continuously differentiable on the arguments at \(x \in (0,l^{ks}), t \in (t_0,T), \) we have:
\[
\Delta \Phi(v) = \Phi(\tilde{v}) - \Phi(v) = \\
= \sum_{(k,s) \in J_{\text{observe}}} 2 \int_{t_0}^{T} \left[ u^{ks}(\eta^{ks},t;\gamma,v) - V_0^{ks}(t) \right] \Delta u^{ks}(\eta^{ks},t;\gamma,v) + \\
+ \left[ \frac{\partial u^{ks}(\eta^{ks},t;\gamma,v)}{\partial x} - V_1^{ks}(t) \right] \Delta u^{ks}(\eta^{ks},t;\gamma,v) dt + + \Re(\Delta v) + \\
+ o(||\Delta u(x,t)||^2) + \sum_{(k,s) \in J_{\text{observe}}} o(||\Delta u^{ks}(\eta^{ks},t;\gamma,v)||^2) d\gamma + \\
+ \sum_{(k,s) \in J_{\text{observe}}} \int_{t_0}^{T} \psi^{ks}(x,t) \left( - \frac{\partial^2 \Delta u^{ks}(x,t)}{\partial t^2} + a^{ks} \frac{\partial^2 \Delta u^{ks}(x,t)}{\partial x^2} - \\
b^{ks} \frac{\partial \Delta u^{ks}(x,t)}{\partial t} + \Delta v(x,t) \right),
\]  
(18)

Here through \(|| \cdot ||\) we designate any norm corresponding to the considered space of functions. Let carry out integration by parts on the last term of the integrals over
the spatial and temporal variables,

\[
\int_{t_0}^{T} \int_{0}^{k,s} \psi_{k,s}(x,t) \left( \frac{\partial^2 \Delta u_{k,s}(x,t)}{\partial t^2} + a_{k,s} \frac{\partial^2 \Delta u_{k,s}(x,t)}{\partial x^2} - b_{k,s} \frac{\partial \Delta u_{k,s}(x,t)}{\partial t} + \Delta v(x,t) \right) = \\
- \int_{t_0}^{T} \int_{0}^{k,s} \psi_{k,s}(x,t) \frac{\partial \Delta u_{k,s}(x,t)}{\partial t} \big|_{t_0}^{T} dx + \int_{t_0}^{T} \frac{\partial}{\partial t} \left( \int_{0}^{k,s} \psi_{k,s}(x,t) \frac{\partial \Delta u_{k,s}(x,t)}{\partial x} \big|_{0}^{k,s} dt \right) - \\
- a_{k,s} \int_{t_0}^{T} \psi_{k,s}(x,t) \frac{\partial \psi_{k,s}(x,t)}{\partial x} \big|_{0}^{k,s} dt + a_{k,s} \int_{t_0}^{T} \psi_{k,s}(x,t) \frac{\partial \Delta u_{k,s}(x,t)}{\partial x^2} dx dt - \\
- b_{k,s} \int_{t_0}^{T} \psi_{k,s}(x,t) \Delta u_{k,s}(x,t) \big|_{0}^{k,s} dt + b_{k,s} \int_{t_0}^{T} \frac{\partial \psi_{k,s}(x,t)}{\partial x} dx dt
\]

We introduce the matrix \( \hat{A} = (a_{k,s})_{k,s=1}^{m} \) assuming that its elements \( a_{k,s} = 0 \), if there is no corresponding edge (arc) \((k,s) \notin J\). Using (17), we will obtain:

\[
\sum_{(k,s) \in J} a_{k,s} \left( \int_{t_0}^{T} \int_{0}^{k,s} \Delta u_{k,s}(x,t) \frac{\partial \psi_{k,s}(x,t)}{\partial x} \big|_{0}^{k,s} dt - \int_{t_0}^{T} \Delta u_{k,s}(x,t) \frac{\partial \psi_{k,s}(x,t)}{\partial x} \big|_{0}^{k,s} dt \right) = \\
= \sum_{(k,s) \in J} a_{k,s} \left( \int_{t_0}^{T} \int_{0}^{k,s} \left( \frac{\partial \Delta u_{k,s}(x,t)}{\partial x} \big|_{0}^{k,s} \Delta u_{k,s}(x,t) \big|_{0}^{k,s} dt - \frac{\partial \psi_{k,s}(0,t)}{\partial x} \Delta u_{k,s}(0,t) \big|_{0}^{k,s} dt \right) dt = \\
= \hat{A} \left[ \frac{\partial \psi(0,t)}{\partial x}, -\psi(0,t) \right] \left[ \frac{\Delta u(0,t)}{\partial \Delta u(0,t)} \right] + \hat{A} \left[ \psi(l,t), -\frac{\partial \psi(l,t)}{\partial x} \right] \left[ \frac{\Delta u(l,t)}{\partial \Delta u(l,t)} \right] = \\
= \hat{A} \left[ \frac{\partial \psi(0,t)}{\partial x}, -\psi(0,t) \right] A^{-1} B \left[ \frac{\Delta u(0,t)}{\partial \Delta u(0,t)} \right] + \hat{A} \left[ -\frac{\partial \psi(l,t)}{\partial x}, \psi(l,t) \right] \left[ \frac{\Delta u(l,t)}{\partial \Delta u(l,t)} \right] = \\
= \left[ \hat{A} \left[ \frac{\partial \psi(0,t)}{\partial x}, -\psi(0,t) \right] A^{-1} B + \hat{A} \left[ \psi(l,t), -\frac{\partial \psi(l,t)}{\partial x} \right] \right] \left[ \frac{\Delta u(l,t)}{\partial \Delta u(l,t)} \right]
\]

Taking into account the arbitrary functions \( \psi(x,t) = \psi_{k,s}(x,t), (k,s) \in J \), obtain the following conjugate problem:

\[
\frac{\partial^2 \psi_{k,s}(x,t)}{\partial t^2} = \begin{cases} 
    a_{k,s} \frac{\partial^2 \psi_{k,s}(x,t)}{\partial x^2} + b_{k,s} \frac{\partial \psi_{k,s}(x,t)}{\partial t}, & t_0 \leq t < \tau, \\
    a_{k,s} \frac{\partial^2 \psi_{k,s}(x,t)}{\partial x^2} + b_{k,s} \frac{\partial \psi_{k,s}(x,t)}{\partial t} + 2 \left[ u_{k,s}(x,t,v) \right. & \\
    - V_{k,s}(t) \bigg] + \left[ \frac{\partial \psi_{k,s}(x,t,v)}{\partial x} - V_{k,s}(t) \right] \delta(x - t_{k,s}), & \tau \leq t \leq T, \\
    a_{k,s} \frac{\partial^2 \psi_{k,s}(x,t)}{\partial x^2} + b_{k,s} \frac{\partial \psi_{k,s}(x,t)}{\partial t}, & \tau \leq t \leq T, \quad (k,s) \notin J_{\text{observe}}, \\
\end{cases}
\]
\[
\psi^{ks}(x,T) = 0, \quad \frac{\partial \psi^{ks}(x,T)}{\partial t} = 0,
\]
\[
\left( \frac{\partial \psi(0,t)}{\partial x}, -\psi(0,t) \right) \tilde{A}^{-1} \tilde{B} + \left( \psi(l,t), -\frac{\partial \psi(l,t)}{\partial x} \right) = 0
\]  

(20)

Note that if the point of observation \( \eta^{ks} \) is on the boundary of the arc \((k,s)\), then the third term on the right hand (19) will be in the boundary condition (20) of corresponding arc as a term.

Using the evaluation

\[
\int_{t_0}^{T} \int_{l}^{l^{ks}} |\Delta u^{ks}(x,t,v)|^2 dxdt \leq C \int_{t_0}^{T} \int_{l}^{l^{ks}} |\Delta v(x,t)|^2 dxdt, \quad (k,s) \in J,
\]

obtained in [20] for the case, when control functions belong to a class of measurable functions \((C > 0)\) is a constant not depending on the value of \(\Delta u\), for the increment of the functional \(\Phi(v)\) we have:

\[
\Delta \Phi(v) = \Phi(\tilde{v}) - \Phi(v) = \int_{\Gamma} \left[ \sum_{(k,s) \in J^{\text{obs}}} \int_{t_0}^{T} \int_{l}^{l^{ks}} \psi^{ks}(x,t;\gamma,v)\Delta v dxdt + \Re(\Delta v) + o(\Delta v(x,t)) \right] \rho_{\Gamma}(\gamma)d\gamma.
\]

(21)

Hence, at the expense of the increment \(\Delta q^{ks}(t)\) of an argument \(q^{ks}(t)\), considering (11), (21), we have the following formula for components of functional gradient at the value of capacity \(q^{ks}(t)\) in a point \(\xi^{ks} \in [0,l^{ks}]\):

\[
\text{grad}_{q^{ks}} \Phi(\xi,q) = \int_{\Gamma} \sum_{(k,s) \in J^{\text{obs}}} \int_{t_0}^{T} \{ \psi^{ks}(\xi^{ks},t;\gamma) + 2\varepsilon_1(q^{ks}(t) - \hat{q}^{ks}(t)) \} \rho_{\Gamma}(\gamma)d\gamma, t \in [t_0,T],
\]

(22)

To obtain the formula for \(\text{grad}_{\xi^{ks}} \Phi(\xi,q)\), in (13) we use the following function \(\delta_{\varepsilon}(x)\) instead of Dirac function \(\delta(x)\):

\[
\delta_{\varepsilon}(x) = \begin{cases} 
1/\varepsilon, & x \in [-\varepsilon/2,\varepsilon/2], \\
0, & x \notin [-\varepsilon/2,\varepsilon/2],
\end{cases}
\]

where \(\varepsilon > 0\) is a small enough, and instead of \(v(x,t)\) in (11) let consider the function \(v_{\varepsilon}(x,t) = q^{ks}(t)\delta_{\varepsilon}(x - \xi^{ks})\). It is evident, that at \(\varepsilon\) tending to zero, the function \(\delta_{\varepsilon}(x)\) approaches the generalized Dirac function \(\delta(x)\), and \(v_{\varepsilon}(x,t)\) to \(v(x,t)\) [37].

If the argument \(\xi^{ks}\) of the function \(v(x,t)\) in (11) receives an increment \(\Delta \xi^{ks} > \varepsilon/2\) it is equivalent to what \(v(x,t)\) will receive the following increment

\[
\Delta v_{\varepsilon}(x,t) = \begin{cases} 
-q^{ks}(t), & x \in [\xi^{ks} - \frac{\varepsilon}{2}, \xi^{ks} - \frac{\varepsilon}{2}], \\
+q^{ks}(t), & x \in [\xi^{ks} + \Delta \xi^{ks} - \frac{\varepsilon}{2}, \xi^{ks} + \Delta \xi^{ks} + \frac{\varepsilon}{2}], \\
0, & x \in [0, \xi^{ks} - \frac{\varepsilon}{2}] \cup (\xi^{ks} + \frac{\varepsilon}{2}, \xi^{ks} + \Delta \xi^{ks} + \frac{\varepsilon}{2}) \\
+\Delta \xi^{ks} - \frac{\varepsilon}{2}, & x \in (\xi^{ks} + \Delta \xi^{ks} + \frac{\varepsilon}{2}, l^{ks}].
\end{cases}
\]
Using (21), we have:

\[
\Delta \Phi(\xi, q) = \frac{1}{\varepsilon} \int_0^T \sum_{(k,s) \in J_{\text{observation}}} \int_{t_0}^T \int \psi^{ks}(x,t) \delta(x - \xi^{ks}) dx dt + \\
+ 2\varepsilon_2 (\xi^{ks} - \hat{\xi}^{ks}) \Delta \xi^{ks} + o(\Delta \xi^{ks}) \rho_T(\gamma) d\gamma = \\
= \int_0^T \sum_{(k,s) \in J_{\text{observation}}} \int_{t_0}^T \frac{q^{ks}(t)}{\varepsilon} (\psi^{ks}(x + \varepsilon/2, t) - \psi^{ks}(x - \varepsilon/2, t)) |_{x = \xi^{ks}} \Delta \xi^{ks} dx dt + \\
+ 2\varepsilon_2 (\xi^{ks} - \hat{\xi}^{ks}) \Delta \xi^{ks} + o(\Delta \xi^{ks}) \rho_T(\gamma) d\gamma.
\]

Taking account this, dividing both sides by \(\Delta \xi^{ks}\) and passing to the limit at \(\Delta \xi^{ks} \to 0, \varepsilon \to 0\), we have the following formula:

\[
\text{grad}_{\xi^{ks}} \Phi(\xi, q(t)) = \\
= \int_0^T \sum_{(k,s) \in J_{\text{observation}}} \left\{ \int_{t_0}^T q^{ks}(t)(\psi^{ks}(x,t))'_x |_{x = \xi^{ks}} dt + 2\varepsilon_2 (\xi^{ks} - \hat{\xi}^{ks}) \rho_T(\gamma) d\gamma \right\}. \tag{23}
\]

Using the calculations similar given above, the formula (22) can be received also for cases \(0 < \Delta \xi^{ks} < \varepsilon/2\) and \(\Delta \xi^{ks} < 0\) [6].

From conditions (20) it is visible that the conjugate problem (19)–(20), as well as the direct problem, has nonlocal non-separated boundary conditions and for its numerical solution it is necessary to use the methods, based on special sweep schemes of the boundary conditions [7], [5], [11].

The direct (1)–(4) and adjoint (19)–(20) initial–boundary value problems have the following features. The number of systems of hyperbolic partial differential equations is equal to the number of segments and in real problems, can reach several tens and even hundreds. All systems are mutually independent and are related only via nonseparated conditions involving the boundary values of the unknown state functions of corresponding segments sharing network vertices.

Such problems can be solved, for example, by applying the numerical approach based on the grid method and can be described as follows. A grid domain is introduced on each network segment by partitioning the interval \([0, l^{ks}]\) into points \(x_i = x_{i-1} + h_x, i = 1, ..., L_x^{ks}, x_a = 0, (k,s) \in J\), with a given step \(h_x\). The time interval \([0, T]\) \(L_t = \frac{T}{h_t}\) is divided by the points \(t_j = j h_t, j = 0, ..., L_t\) into \(L_t\) intervals with a given step \(h_t\). The values \(h_x, h_t\) are determined by the required accuracy of the solution to the initial–boundary value problem.

Defining \(U_{ij}^{ks}, i = 0, ..., L_x^{ks}, j = 0, ..., L_t, (k,s) \in J\) let use the implicit scheme [28] for approximation of systems of differential equations for all sections \((k,s)\).

We obtain a system of linear algebraic equations at each time layer \(j, j = 0, ..., L_t\), which we write in the form [5]:

\[
\bar{A}^{ks}_{ij} \bar{U}^{ks}_{ij} = \hat{B}^{ks}_{ij}, \quad i = 1, ..., L_x^{ks}, \tag{24}
\]

here \(\bar{A}^{ks}_{ij}\) – is the three-diagonal matrix of dimension \(M \times M\) and \(\hat{B}^{ks}_{ij}\) – is a \(M\)-dimensional vector; they are both determined by the difference approximation scheme. On introducing \(M(L_x^{ks} + 1)\)–dimensional vectors \(U_j^{ks} = (U_0^{ks}, ..., U_{L_x^{ks}}^{ks})^T\), \((k,s) \in J\) and \(M\)–dimensional vectors \(U_{0j} = \{u_0^{ks} : (k,s) \in J\}, U_{L_x+j} = \{u_{L_x+j}^{ks} : (k,s) \in J\}:\)
\( (k, s) \in J \), the system (24) is written as:

\[
A_{j}^{ks} u_{j}^{ks} = B_{j}^{ks},
\]

and the nonseparated conditions (2) are written as:

\[
\Re_{1} U_{0j} + \Re_{2} U_{Lxj} = \Re_{3}.
\]

For each \( j \), \( A_{j}^{ks} \) are three-dimensional matrices, while \( B_{j}^{ks} \) is a \( L_{x}^{ks} \)-dimensional vector.

For each \( j, j = 1, \ldots, L, 2M \) boundary values of grid functions for adjacent segments are used in (26).

To obtain separated conditions of the form

\[
\tilde{\Re}_{1} U_{0j} = \tilde{\Re}_{3}
\]

or

\[
\tilde{\Re}_{2} U_{Lxj} = \tilde{\Re}_{3}
\]

instead of (26), a sweep scheme for transfer of conditions to the left or right is proposed in [5]. The boundary values of a segment are transferred using only one system of LAE corresponding to this segment, whereas the boundary values of the other segments are not used. As a result, the transfer process can be parallelized with respect to segments and conditions on these segments.

The initial–boundary value problems (1)–(4) and (19)–(20) can also be solved by applying a numerical algorithm based on the method of lines (see [7]). Let us define the straight lines \( t = t_{j}, t_{j} = j h_{t}, j = 0, \ldots, L_{t}, h_{t} = T / L_{t} \) and introduce the notation \( u_{j}^{ks}(x) = u_{j}^{ks}(x, t_{j}) \).

Approximating the time derivatives in differential equations (1) on each line \( t = t_{j}, j = 0, \ldots, L_{t} \), we obtain a system consisting of systems of ordinary differential equations:

\[
\frac{du_{j}^{ks}(x)}{dx} = A_{j}^{ks} u_{j}^{ks}(x) + B_{j}^{ks}, x \in (0, l_{x}), (k, s) \in J.
\]

These systems are related only by boundary conditions, which are written as:

\[
\Re_{1j} u_{j}(0) + \Re_{2j} u_{j}(l_{x}) = \Re_{3j}.
\]

Here \( \Re_{1j}, \Re_{2j} \) are given \( 2M \times 2M \) matrices determined by the adjacency matrix of the network and \( \Re_{3j} \) is a \( 2M \)-dimensional vector:

\[
u_{j}(0) = \{ u_{j}^{ks}(0) : (k, s) \in J \}, u_{j}(l_{x}) = \{ u_{j}^{ks}(l_{x}) : (k, s) \in J \}.
\]

A differential scheme for the transfer of conditions (28) to one of the ends (e.g., to the left end) was proposed in [7] for deriving a 2Mth-order SLAE of the form:

\[
\tilde{\Re}_{1j} u_{j}(0) = \tilde{\Re}_{3j}.
\]

The boundary condition for the unknown function \( u^{ks}(x) \) in (28) is transferred from one end to the other by using only the corresponding \( (k, s) \)-th system (27). After solving the \( 2M \)-th-order SLAE (29) and determining \( u_{j}(0) \), the Cauchy problems for each system (27) are solved regardless of one another.

Note that the transfer of conditions and the solution of Cauchy problems for each segment at each time level \( j, j = 1, \ldots, L_{t} \), is well parallelized.

A more detailed exposition of the above schemes for solving initial–boundary value problems, results of numerous computer experiments, and their analysis can be found in [7],[5].
Let us describe the algorithm of the procedure (10) for the gradient projection method:

- step 1. Set the initial admissible values for the locations and capacities of the identified sources \((\xi, q(t))_0\).
- step 2. Solve the initial-boundary value problem (1)-(4) and determine the values for pressure and flow amount in all nodes (ends of sections) of the network.
- step 3. Calculate the value of the functional (8) using the observed values (6).
- step 4. Solve the adjoint problem (19)-(20).
- step 5. Calculate the values of the components of the functional gradient by the formulas (22), (23).
- step 6. Calculate new values of \((\xi, q(t))_1\) according to the formula (10), using the method of the golden section for calculating the step of one-dimensional minimization of \(\xi_0\).
- step 7. Check whether the obtained value is optimal, if yes, the algorithm is completed, otherwise go to step 2. (The stopping criteria for the procedure (10) can be, for example, a small step size \(\alpha_i^0\) or a small change of value of the functional on two subsequent iterations).

Remark 3. If the sections on which leakages were occurred are not known beforehand, then it is necessary to solve the aforementioned problem for various variants of the groups of arcs, suspected for the presence of sources. It is clear that the case under which the minimal value of the functional (8) has been obtained corresponds to the solution to the considered problem. To carrying out such search on the graph is a laborious task in the case of a large number of arcs with sources or a very large number of arcs \(M\). In practice, as a rule, arcs suspicious for the presence of a possible source, with sufficient plausibility is appointed by experts on the basis of experience or any other considerations.

It is clear that if it is obtained \(|q^{ks}(t)| \leq \varepsilon, t \in [\tau, T]\) (\(\varepsilon\) is a sufficiently small number) as a result of the solution to the problem (1)–(8) for sufficiently small value of the functional, it means that there isn’t a concentrated source in the arc \((k, s)\) and as a whole there are not concentrated sources in the system. As a rule corollary of the irregular purpose of an arc for definition on it of a source when it is not there, is the big in size of minimum value of a functional, or if the received value \(\xi^{ks}\) such that, \(0 \leq \xi^{ks} \leq \varepsilon\) or \(l^{ks} - \varepsilon \leq \xi^{ks} \leq l^{ks}\).

4. Inverse problem on determination the places and volumes of leakages in the pipelines networks. Within the common statement of the inverse problem (1)-(8) we will consider a problem of finding of places of leakages and volume of fluid at its transportation in a pipeline network of complex structure.

Let’s assume that the mode of a dropping fluid flow in network is isothermal, unsteady, laminar [15], [16]. Let’s present a pipeline network in the form of the oriented graph with \(M\) sections and \(N\) nodes and use the designations accepted in item 2.

The state of process of fluid flow on each arc (edge, the line section) can be described by the equation of a form (1). But the following system of differential equations of hyperbolic type is most often used [1], [15], [16], [17], [18], [26], [32], [33]:
- \frac{\partial P^{ks}(x,t)}{\partial x} = \frac{\rho}{S^{ks}} \frac{\partial Q^{ks}(x,t)}{\partial t} + 2a^{ks} \frac{\rho}{S^{ks}} Q^{ks}(x,t),
- \frac{\partial P^{ks}(x,t)}{\partial t} = \begin{cases} \frac{c^2 \rho}{S^{ks}} \frac{\partial Q^{ks}(x,t)}{\partial x} + \frac{c^2 \rho}{S^{ks}} q^{ks}(t) \delta(x - \xi^{ks}), & x \in (0, l^{ks}), (k,s) \in J^{ist}, \\ \frac{c^2 \rho}{S^{ks}} \frac{\partial Q^{ks}(x,t)}{\partial x}, & x \in (0, l^{ks}), (k,s) \in J \setminus J^{ist}. \end{cases} \tag{30}

Here: \( t \in (t_0, T], s \in I_k^+, k \in I; P^{ks}(x,t), Q^{ks}(x,t) \) is a pressure and flow amount at the moment of time \( t \) at the point \( x \in (0, l^{ks}) \) in the section \((k,s)\) of the pipeline network; \( c \) is the sound velocity in the medium; \( S^{ks} \) is the area of an internal cross-section of the section \((k,s)\) of the pipeline network; \( \delta^{ks} \) is the coefficient of dissipation in the section \((k,s)\) (it can be assumed that the kinematic viscosity \( \gamma \) does not depend on the pressure in the case of laminar mode) and \( 2a^{ks} = 32\gamma/(\mu^{ks})^2 = const \). The values of \( Q^{ks}(x,t) \) can be positive or negative. The positive or negative value of \( Q^{ks}(x,t) \) means that an actual flow in the section \((k,s)\) is directed from the node \( k \) to the node \( s \) or is directed from \( s \) to \( k \), respectively. Each section obviously is an inflow or outflow for some nodes and the following conditions are satisfied for \( k_i \in I, k_j \in I_k^i \) because of \( l^{k_i,k_i} = l^{k_j,k_j} \):

\[
Q^{k_i,k_j}(x,t) = -Q^{k_j,k_i}(l^{k_i,k_j} - x,t),
P^{k_i,k_j}(x,t) = P^{k_j,k_i}(l^{k_i,k_j} - x,t), \quad x \in (0, l^{k_i,k_j}).
\]

The system (30) consists of \( 2M \) equations and includes \( M \) subsystems of two equations on all sections of the network, and only one subsystem corresponds to each section. Taking into account that each section is an outflow for some nodes of the network, then the fluid flow process on the network can be written for sections of outflows instead of (30) (the indices will be get values \( s \in I_k^+, k \in I \) in (30) for this case) or in a mixed form.

It is clear that if to include the function \( u^{ks}(x,t), x \in (0, l^{ks}) \), such that

\[
\frac{\partial u^{ks}(x,t)}{\partial x} = -\frac{S^{ks}}{c^2 \rho} P^{ks}(x,t), \quad \frac{\partial u^{ks}(x,t)}{\partial t} = Q^{ks}(x,t), \tag{31}
\]

then the system (30) is reduced to the form of (1).

The \( 2M \) boundary conditions must be specified for the system (30). We’ll use the conditions of material balance for internal nodes of the network

\[
\sum_{s \in I_k^+} Q^{ks}(l^{ks},t) - \sum_{s \in I_k^-} Q^{ks}(0,t) = \tilde{q}^{k}(t), \quad k \in I^{int}, \quad t \in [t_0, T], \tag{32}
\]

and the flow continuity conditions:

\[
P^{k_i,k}(l^{k_i,k},t) = P^{k_j,k}(0,t), \quad k_i \in I_k^+, k_j \in I_k^-, k \in I^{int}, \quad t \in [t_0, T], \tag{33}
\]

where \( \tilde{q}^{k}(t) \) is a given external inflow (\( \tilde{q}^{k}(t) > 0 \)) or outflow (\( \tilde{q}^{k}(t) < 0 \)) for internal node \( k \).

The relations (32) include \( N^{int} \) conditions. In the relations (33) the number of independent conditions is equal to \( N_k - 1 \) for each internal \( k \) th node from the set \( I^{int} \). Internal sections of the network are related with two boundary conditions in (33), but the total number of all conditions is \( N_f \), one end of sections which
belong to $I_f$. So the total number of boundary conditions in (32), (33) is $N^{int} + \sum_{k \in I^{int}} (N_k - 1) = 2M - N_f$.

We’ll give the value of the pressure (the set of which we denote by $I_p^f \subset I^f$) or flow amount (the set of which we denote by $I_q^f \subset I^f$) at each node from $I^f$, which is not an internal node of the network. Then the following $N_f$ conditions will be added to the conditions (32), (33):

$$
\begin{align*}
\begin{cases}
P^n(t) = P^{ns}(0, t) = \tilde{P}^n(t), & s \in I_n^+, \hspace{1cm} n \in I_p^f, \hspace{1cm} t \in [t_0, T], \\
P^n(t) = P^{sn}(I_n^+, s) = P^n(t), & s \in I_n^-, \hspace{1cm} n \in I_p^f, \hspace{1cm} t \in [t_0, T],
\end{cases}
\end{align*}
$$

and it is important that conditions were met: $I^f = I_q^f \cup I_p^f$, $I_q^f \cap I_p^f = \emptyset$, $I_p^f \neq \emptyset$.

Here $\tilde{Q}^n(t)$, $\tilde{P}^n(t)$, $m \in I_q^f$, $n \in I_p^f$ are given functions, defining the modes of sources functioning.

So, we assume the initial conditions for the process (30) are not precisely defined at the initial time $t_0$, but the initial conditions belonging to a set $U^0$ can be defined in parametrical a form

$$
\varphi^{ks}(x, \gamma) = (Q^{ks}(x, t_0; \gamma), P^{ks}(x, t_0; \gamma)), \hspace{1cm} (k, s) \in J, \hspace{1cm} \gamma \in \Gamma \in R^n
$$

with the corresponding density functions of distributions $\rho_T(\gamma)$. The set of admissible initial conditions can be set by a finite number $N^{int}$ of the given functions or parameters defining them.

Thus, to calculate the fluid flow mode $(P^{ks}(x, t), Q^{ks}(x, t))$ in all sections $(k, s) \in J$, of the hydraulic network for given locations and volumes of leaks it’s necessary to solve the system of $2M$ number differential equations (30) with $2M$ number any initial conditions at $t = t_0$, satisfying (36) and $2M$ boundary conditions (32)–(33).

At the same time the obtained solution to the problem for $t > \tau$ will be correspond to real-time current modes.

We suppose the volume $q^{ks}(t)$ and the locations $x^{ks}$ of leakages at predetermined $N^{loss}$ sections of the set $J^{loss} \subset J$ are unknown and they need to be determined using additional information on flow mode.

To determine the unknown locations and volumes of leakages $(x^{ks}, q^{ks}(t)), (k, s) \in J^{loss}$ let assume that, there are the results of observations on the pressure and/or flow amount at certain points of the various sections of the network, which quantity exceeds $2N^{loss}$, the number of identified parameters.

It is natural to assume that sought locations of leaks do not coincide with the points of measurements carried out. In order not to load the formulas by new indices and symbols, let’s assume the additional measurement points are again from $I^f$ the entering points and the leaving points of the pipeline network. Moreover, if the measurements on pressure are used for the nodes from the set $I_p^f$, but measurements on flow amount for the nodes from $I_q^f$ in the boundary conditions (33), (34), then additional information will be the results of measurements of pressure at some points of subsets $I_{qp}^f$ of the set $I_q^f$, i.e. $I_{qp}^f \subset I_q^f$, and measurements of flow amount at the points of some subsets $I_{pq}^f$ of the set $I_p^f$, i.e. $I_{pq}^f \subset I_p^f$:

$$
\begin{align*}
\begin{cases}
P^n(t) = P^{ns}(0, t) = P^{ns}_{mes}(t), & s \in I_n^+, \hspace{1cm} n \in I_p^f, \hspace{1cm} t \in [t_0, T], \\
P^n(t) = P^{ns}(I_n^+, s) = P_{mes}^{sn}(t), & s \in I_n^-, \hspace{1cm} n \in I_p^f, \hspace{1cm} t \in [t_0, T],
\end{cases}
\end{align*}
$$
\[
\begin{align*}
Q^m(t) &= Q^{\text{mes}}(0, t) = Q^{\text{mes}}_x(t), \quad s \in I^+_m, \text{ if } I^-_m = \emptyset, \\
Q^m(t) &= Q^{\text{mes}}(t^m, t) = Q^{\text{mes}}_x(t), \quad s \in I^-_m, \text{ if } I^+_m = \emptyset, \quad m \in I_p \subset I_p'.
\end{align*}
\]

The considered inverse problem consists of finding the locations of leaks
\[
\xi = (\xi^{(k_1, k_1)}, ..., \xi^{(k_2, k_2)}), \xi^{(k_i, k_j)} \in (0; \ell^{k_i, k_j})
\]
and the corresponding values of raw material losses
\[
g(t) = (q^{k_1, k_1}(t), ..., q^{k_2, k_2}(t)), \quad (k_i, k_j) \subset J^{\text{loss}}, \quad s = 1, ..., L
\]
for \(t \in [t_0, T]\) by using the given mathematical model and additional observed data (37), (38).

We’ll consider these conditions as additional and use to form the following minimizing functional of quadratic deviation on average over all possible initial conditions from the set:

We will consider these conditions as additional information which are used for creation of the discrepancy functional depending on initial conditions from a set \(\Gamma\):

\[
\Phi(\xi, q) =
\]

\[
= \int_{\Gamma} \left\{ \sum_{m \in I_p} \int_{t_0}^T |Q^m(t; \xi, q(t), \gamma) - Q^{\text{mes}}_x(t)|^2 dt + \Re(\xi, q) \right\} \rho_{\Gamma}(\gamma) d\gamma \to \min,
\]

where \(Q^m(t; \xi, q(t), \gamma), m \in I_p'\) are the calculated values of flow amount in the observed points by solving the boundary value problem with any possible initial conditions \(\psi^{ks}(x, \gamma), \gamma \in \Gamma\) and for given admissible locations and volumes of leaks \((\xi, q(t)); [\tau, T]\) is the time interval of observation on the process, modes of which do not depend on the initial conditions.

If the set of admissible initial conditions is finite in the form of (5) with a uniform distribution, then the functional (39) takes the following form:

\[
\Phi(\xi, q) =
\]

\[
= \frac{1}{N_{\gamma}} \sum_{j=1}^{N_{\gamma}} \left( \sum_{m \in I_p} \int_{t_0}^T |Q^m(t; \xi, q(t), \gamma^j) - Q^{\text{mes}}_x(t)|^2 dt + \Re(\xi, q) \right) \to \min,
\]

For components of a gradient of a functional (39) it is possible to use formulas (18)-(19), (21), (22). Considering designations (31), we can receive the following formulas:

\[
\text{grad}_{\xi^k} \Phi(\xi, q) = \int_{\Gamma} \left\{ c^2 \frac{\rho}{S_{ks}} \psi^{ks}(\xi^{ks}, t; \gamma) + 2 \varepsilon_1 (q^{ks}(t) - \dot{q}^{ks}(t)) \right\} \rho_{\Gamma}(\gamma) d\gamma, \quad t \in [t_0, T],
\]

\[
\text{grad}_{x^k} \Phi(\xi, q(t)) = \int_{\Gamma} \left\{ c^2 \frac{\rho}{S_{ks}} \int_{t_0}^T q^{ks}(t)(\psi^{ks}(x, t; \gamma))'_{x=x=\xi^{ks}} dt + 2 \varepsilon_2 (\xi^{ks} - \ddot{\xi}^{ks}) \right\} \rho_{\Gamma}(\gamma) d\gamma.
\]

Here the functions \( \psi^{ks}(x, t) = \varphi^{ks}(x, t; \hat{Q}, \hat{P}) \) are the solutions to the conjugate problem:

\[
\begin{align*}
-\frac{\partial \varphi^{ks}(x, t)}{\partial x} &= \frac{\partial \varphi^{ks}(x, t)}{\partial t}, \quad x \in (0, \ell^{ks}), \quad s \in I^+_k, \quad k \in I, \quad t \in [t_0, T], \\
-\frac{\partial \varphi^{ks}(x, t)}{\partial t} &= c^2 \frac{\partial \varphi^{ks}(x, t)}{\partial x} - 2a^{ks}\varphi^{ks}(x, t),
\end{align*}
\]
The following especially constructed test problem for oil pipeline network consisting of 5 nodes. The results of numerical experiments. Problem 1 on the pumping stations set in the four ends of the pipeline, which determined by \( q \), \( \rho \), \( \gamma \). We assume the oil with a kinematic viscosity \( \nu \), density \( \rho \) and the pressure get the following values: \( \nu = 920 \text{ kg/m}^2 \text{ sec} \) and \( \rho = 0.8 \text{ kg/m}^3 \) is transported via network for a long time. Let \( 2a = 0.017 \) and the sound speed in the medium is \( c = 1200 \text{ m/sec} \).

Let there exists a leakage at the point \( \xi = 30 \) km in the segment (1,2) of the network and the volume of leakage is defined by a function \( q(1,2) = 50 - 10e^{3at} \text{ m}^3/\text{hour} \), \( \alpha = -0.0001 \). The pressure was observed during \( T = 1800 \text{ sec} \) on the pumping stations set in the four ends of the pipeline, which determined by the values of the following functions:

\[
\begin{align*}
\hat{P}^1(t) &= 2 \cdot 10^6 + 3 \cdot 10^5e^{3at} \text{ Pa}, \\
\hat{P}^3(t) &= 1.9 \cdot 10^6 - 7.2 \cdot 10^5e^{3at} \text{ Pa}, \\
\hat{P}^4(t) &= 1.8 \cdot 10^6 - 6.7 \cdot 10^5e^{3at} \text{ Pa}, \\
\hat{P}^6(t) &= 1.6 \cdot 10^6 + 8.6 \cdot 10^5e^{3at} \text{ Pa} t \in [0, T].
\end{align*}
\]

5. The results of numerical experiments. Problem 1 We consider the following especially constructed test problem for oil pipeline network consisting of 5 nodes as shown in figure 1. Here \( N = 6, M = 5 \), \( I^f = \{1, 3, 4, 6\} \), \( N^f = 4 \), \( N^{int} = 2 \), external inflows and outflows do not exist inside the network. All segments are of diameter \( d = 530 \text{ (mm)} \) and are of various lengths:

\[
\begin{align*}
l^{(1,2)} &= 100 \text{ km}, l^{(5,2)} = 30 \text{ km}, l^{(3,2)} = 70 \text{ km}, l^{(5,4)} = 100 \text{ km}, l^{(5,6)} = 60 \text{ km}.
\end{align*}
\]

We assume the oil with a kinematic viscosity \( \gamma = 1.5 \cdot 10^{-4} \text{ (m}^2/\text{ sec)} \) and density \( \rho = 920 \text{ (kg/m}^3) \) is transported via network for a long time. Let \( 2a = 0.017 \) and the sound speed in the medium is \( c = 1200 \text{ (m/sec)} \).

It was found by numerical experiments that the solution of the initial-boundary-value problem (30) – (36) for a given hydraulic network at \( t > \tau \approx 480 \text{ sec} \) doesn’t already depend on the choice of initial conditions given at \( t_0 \) 0, but is determined only by the boundary conditions (50).

It will be assumed known, that there was a steady regime in all pipelines at initial moment of time \( t_0 = 0 \), i.e. \( N_{i_{t=0}} = 1 \) in the functional (39), and the flow amount and the pressure get the following values:

\[
\begin{align*}
\hat{Q}^{(1,2)}_0(x) &= 300 \text{ m}^3/\text{hour}, & \hat{Q}^{(5,2)}_0(x) &= 200 \text{ m}^3/\text{hour}, \\
\hat{Q}^{(3,2)}_0(x) &= 100 \text{ m}^3/\text{hour}, & \hat{Q}^{(5,4)}_0(x) &= 120 \text{ m}^3/\text{hour}, \\
\hat{Q}^{(5,6)}_0(x) &= 80 \text{ m}^3/\text{hour}, & \\
\hat{P}^{(1,2)}_0(x) &= 2.3 \cdot 10^6 - 5.8955x \text{ Pa}, & \hat{P}^{(5,2)}_0(x) &= 1.7 \cdot 10^6 - 1.174x \text{ Pa}, \\
\hat{P}^{(3,2)}_0(x) &= 1.8 \cdot 10^6 - 1.67x \text{ Pa}, & \hat{P}^{(5,4)}_0(x) &= 1.8 \cdot 10^6 - 2.358x \text{ Pa}, \\
\hat{P}^{(5,6)}_0(x) &= 1.8 \cdot 10^6 - 0.944x \text{ Pa}.
\end{align*}
\]
First, we determine the values of the observed flow regimes in the hydraulic network. We solve the initial-boundary-value problem \((30), (50)-(52)\) numerically by using these data and define the values of the functions of flow amount \(Q^m(t), m \in \tilde{I}_q\), at the ends of the segment. Further we use these values to generate the target functional \((40)\), the location and volume of leakage \(\xi^{(1,2)}\) ‘‘forget’’.

We use the scheme proposed in the work [5], to calculate the values of the regimes of fluid flow over the network, i.e. to solve an initial-boundary-value problem \((30), (50)-(52)\) with non-separated boundary conditions numerically, and step of temporal variable is equal to \(h_t = 100\) sec, but step of spatial variable is \(h_x = 100m\). \(N_t = T/h_t\) \(N_x = l^k/s/h_x\), \((k, s) \in I\) are the numbers of steps for corresponding variables (these values were determined at the results of a specially conducted experiments for defining the effective values of these parameters).

Different schemes [1], [10], [6] were analyzed to approximate the \(\delta\)–Dirac function, and calculations were made using the method proposed by the authors of the work [6] by using Gaussian type of function.

The results shown in the table 1 are of the solution of an identification problem to determine the locations and leak amount on a predetermined section \((1,2)\). Five various sets of values of identified parameters \((\xi^{(1,2)}, q^{(1,2)}(t))_0\) are used as initial values for the iterative process \((10)\).

Here are given the results of the problem under the assumptions that the observed values of flow amount on the ends of the network are the values of measurement error, respectively, 0.5% and 1%. We add the values randomly generated numbers

| \(\eta\) | \(\xi_0^{(1,2)}\) | 60 | 20 | 90 | 10 | 45,685 |
|---|---|---|---|---|---|---|
| \(q_0^{(1,2)}(t)\) | \(90 - 10e^{3\beta t}\) | \(20 - 10e^{3\beta t}\) | \(30 - 10e^{3\beta t}\) | \(66 + 20e^{3\beta t}\) | \(66 + 20e^{3\beta t}\) |
| \(\Phi_0\) | 76.104 | 16.704 | 11.664 | 42.48 | 57.636 |
| \(\Phi\) | 5.73 \times 10^{-7} | 1.26 \times 10^{-7} | 3.19 \times 10^{-6} | 1.85 \times 10^{-6} | 7.43 \times 10^{-7} |
| \(N_{iter}\) | 6 | 5 | 16 | 14 | 8 |
| \(\delta\xi^{(1,2)}\) | 0.00009 | 0.00006 | 0.0003 | 0.0002 | 0.0006 |
| \(\delta\eta^{(1,2)}\) | 0.0003 | 0.0006 | 0.0003 | 0.0002 | 0.0003 |

| \(\eta\) | \(\xi_0^{(1,2)}\) | 60 | 20 | 90 | 10 | 45,685 |
|---|---|---|---|---|---|---|
| \(q_0^{(1,2)}(t)\) | \(90 - 10e^{3\beta t}\) | \(20 - 10e^{3\beta t}\) | \(30 - 10e^{3\beta t}\) | \(66 + 20e^{3\beta t}\) | \(66 + 20e^{3\beta t}\) |
| \(\Phi_0\) | 76.352 | 16.924 | 11.832 | 43.744 | 57.413 |
| \(\Phi\) | 0.023 | 0.014 | 0.024 | 0.017 | 0.020 |
| \(N_{iter}\) | 6 | 5 | 14 | 12 | 7 |
| \(\delta\xi^{(1,2)}\) | -0.005 | 0.011 | 0.002 | -0.011 | -0.006 |
| \(\delta\eta^{(1,2)}\) | 0.043 | 0.030 | 0.049 | 0.041 | 0.042 |

| \(\eta\) | \(\xi_0^{(1,2)}\) | 60 | 20 | 90 | 10 | 45,685 |
|---|---|---|---|---|---|---|
| \(q_0^{(1,2)}(t)\) | \(90 - 10e^{3\beta t}\) | \(20 - 10e^{3\beta t}\) | \(30 - 10e^{3\beta t}\) | \(66 + 20e^{3\beta t}\) | \(66 + 20e^{3\beta t}\) |
| \(\Phi_0\) | 77.119 | 16.924 | 11.832 | 43.744 | 57.413 |
| \(\Phi\) | 0.067 | 0.071 | 0.073 | 0.062 | 0.065 |
| \(N_{iter}\) | 6 | 5 | 14 | 12 | 7 |
| \(\delta\xi^{(1,2)}\) | -0.049 | -0.020 | -0.002 | 0.020 | -0.005 |
| \(\delta\eta^{(1,2)}\) | 0.109 | 0.092 | 0.107 | 0.094 | 0.093 |
\eta \chi_i Q^{k,s}(t_i) to the values $Q^{k,s}(t_i) = Q^{k,s}(t_i; \xi, q)$, $t_i = ih_i$, $(k, s) \in J^{\text{observ}}$, $i = 1, \ldots, N_t$ obtained by solving the direct problem to generate observations for the identification problem. Here $\chi_i$ is a random value, uniformly distributed in the interval $[-1, 1]$, $i = 1, \ldots, N_t$. $\eta$ gets the values, equal to 0, 0.005 and 0.01, which correspond to the levels of noise in the measurements of flow amount at the nodes from $J^{\text{observ}}$. accordingly in 0% (without noise), 0.5% and 1% of the measured value.

We use the following notations in the table 1: $\tilde{\chi}$ is the initial value of the functional, $\tilde{\varphi}$ is the computed optimal value of the functional, $N_{\text{iter}}$ is the number of iterations (of one dimensional minimization), in the conjugate gradient projection method, $\delta \xi^{(1,2)} = (\xi^{(1,2)} - \xi^{(1,2)})/\xi^{(1,2)}$, $\delta q^{(1,2)} = \max_{t \in [t_0, T]} |q^{(1,2)}(t) - q^{(1,2)}(t)|/|q^{(1,2)}(t)|$ are the relative errors, for the leak location and amount, respectively.

The increment of the accuracy of measurements has the most significant affect on the accuracy of determination of the leak amounts, as one can see from Table 1, but generally the error in the obtained values of the identifiable parameters are the same as the error in measurements. Suppose the state of the regimes for all segments aren’t specified accurately at $t = 0$ when the monitoring of boundary conditions starts, but it’s known from technological considerations that the flow amount and the pressure in the segments are defined on a finite set, obtained from (53) uniformly splitting the range of values by $\nu$ points with step $h_{\nu} = 20/\nu$. We define the set of admissible initial conditions from the assumption that, there was steady regime in all segments of the network at $t_0 = 0$. The flow amount and the pressure had the following values:

\begin{align*}
Q^{(1,2)} &= \{Q^{(1,2)}_0(x) : 300 + nh_{\nu}\}; \\
Q^{(5,2)} &= \{Q^{(5,2)}_0(x) : 200 + nh_{\nu}\}; \\
Q^{(3,2)} &= \{Q^{(3,2)}_0(x) : 100 + nh_{\nu}\}; \\
Q^{(5,4)} &= \{Q^{(5,4)}_0(x) : 120 + nh_{\nu}\}; \\
Q^{(5,6)} &= \{Q^{(5,6)}_0(x) : 80 + nh_{\nu}\}; \\
Q^{(5,2)} &= \{Q^{(5,2)}_0(x) : 2.3 \cdot 10^{-6} - 5.8955x + 2 \cdot 10^4nh_{\nu}\}; \\
P^{(1,2)} &= \{P^{(1,2)}_0(x) : 1.7 \cdot 10^6 - 1.174x + 2 \cdot 10^4nh_{\nu}\}; \\
P^{(3,2)} &= \{P^{(3,2)}_0(x) : 1.8 \cdot 10^6 - 1.67x + 2 \cdot 10^4nh_{\nu}\}; \\
P^{(5,4)} &= \{P^{(5,4)}_0(x) : 1.8 \cdot 10^6 - 2.358x + 2 \cdot 10^4nh_{\nu}\}; \\
P^{(5,6)} &= \{P^{(5,6)}_0(x) : 1.8 \cdot 10^6 - 0.944x + 2 \cdot 10^4nh_{\nu}\},
\end{align*}

$x \in [0, l]$, $n = 1, 2, \ldots, \nu$.
Table 2. The results of the solution to the problem 1 regarding leakage in the section (1,2) for different $\nu$ and for different initial states $(\xi^{(1,2)}, q^{(1,2)}(t))_i^0, i = 1,\ldots, 5$.

| $\nu$ | $\xi_0^{(1,2)}$ | $q_0^{(1,2)}(t)$ | $\Phi_0$ | $\Phi$ | $\tilde{\Phi}$ | $\delta\xi^{(1,2)}$ | $\delta q^{(1,2)}$ | $\tilde{\Phi}$ |
|-------|-----------------|-----------------|--------|-------|-------------|---------------|---------------|--------|
| 3     | 155.248         | 22.323          | 22.323 | 84.303 | 118.809     | 0.004         | 0.0001        | 0.0007 | 0.0005 |
| 5     | 155.390         | 33.608          | 22.443 | 84.086 | 118.854     | 0.001         | 0.0003        | 0.0005 | 0.0001 |

Table 2 presents the results obtained by minimizing the functional for various initial values of the identified parameters $(\xi, q(t))$ and for various numbers $\nu$ of initial conditions of form (37). Figure 2 shows the plots of the exact leakage function and the computed leak amounts when solving a considered problem. Table 3 presents the results of numerical experiments obtained in solving the problem, without noise in the measurements, but with the leak at the point $\xi^{(5,6)} = 15$ km in the segment (5,6) and with the leak amount specified by the function $q^{(5,6)}(t) = 10 + 10e^{3\alpha t}$ $m^3/hour$. 

![Figure 2](image-url)

**Figure 2.** Plots of the exact $(q_{loss} = q^{(1,2)}(t) = 50 - 10e^{0.0003t} m^3/hour$ and obtained leak volume functions $q^{(1,2)}(t)$ for five various initial approximations $(\xi^{(1,2)}, q^{(1,2)}(t))_i^0, i = 1,\ldots, 5$. for the iteration process (10) in problem 1.
Problem 2. The network shown in Fig. 1 is known to have a single leak, but the leak-containing segment is not known. To construct test flow rate measurements at the terminal points of the pipeline, the leak was assumed to be on the segment \((3, 2)\) at the point and the leak amount was specified by the function \(\xi\). As in Problem 1, solving the initial-boundary value problem produced flow rates at the terminal points of the oil pipeline, which were used to form the cost functional. The leak-containing segment was determined simultaneously with the leak locations and amounts by solving Problem 1 for each of the segments, assuming that it has a leak. As expected, the smallest value of the functional was obtained in the problem solved for the segment \((3, 2)\), which indeed contained a leak and the cost functional value was considerably smaller than in the case when the inverse problem was solved on the other suspected segments. The numerical results are given in Table 4. Figure 3 shows the exact and computed leak amounts. The latter were obtained by solving Problem 1, assuming that a leak occurred on various network segments.

6. Conclusion. In article the solution of the inverse problem of definition of locations and capacities of the external and internal sources influencing on multi edge system is investigated. Each edge of system is a subsystem with distributed parameters, described by a differential equation of hyperbolic type. An example of such system is the pipeline transport network of the complex structure. Flow of fluid on each its section is described by the hyperbolic equation. In internal points or on the

| \(\xi_0^{(5,6)}\) | \(q_0^{(5,6)}(t)\) | \(\xi^{(5,6)}\) | \(\Phi_0\) | \(\Phi\) | \(N_{iter}\) |
|---|---|---|---|---|---|
| 25 | \(5 + 10e^{3\alpha t}\) | 15.219 | 2.556 | \(3.55 \cdot 10^{-5}\) | 165 |
| 10 | \(30 + 10e^{3\alpha t}\) | 14.781 | 44.615 | \(3.55 \cdot 10^{-5}\) | 204 |
| 22.56 | \(15 + 10e^{3\alpha t}\) | 15.222 | 3.467 | \(3.59 \cdot 10^{-5}\) | 197 |
Table 4. Results obtained by solving Problem 2

| $\xi^0$ | $q^0(t)$ | Segment indices ($k, s$) | $\xi^{ks}$ | $\Im_0$ | $\Im$ | Number of iterations |
|---------|----------|--------------------------|------------|---------|------|---------------------|
| 25      | $15 + 10e^{-0.0003t}$ | (1, 2) | 5.991 | 43.272 | 39.614 | 6 |
|         |          | (5, 2) | 2.279 | 139.536 | 60.386 | 58 |
| 5       | $5 + 10e^{-0.0003t}$  | (3, 2) | 29.974 | 10.188 | $1.22 \times 10^{-5}$ | 12 |
|         |          | (5, 4) | 2.244 | 54.185 | 37.958 | 79 |
|         |          | (5, 6) | 2.995 | 72.396 | 20.621 | 185 |
|         |          | (1, 2) | 5.986 | 46.836 | 39.614 | 5 |
|         |          | (5, 2) | 2.305 | 84.132 | 60.408 | 5 |
|         |          | (3, 2) | 29.935 | 32.148 | $2.44 \times 10^{-5}$ | 12 |
|         |          | (5, 4) | 1.898 | 38.628 | 37.728 | 150 |
|         |          | (5, 6) | 2.947 | 21.786 | 20.621 | 72 |

ends of arcs the discrepancy functional is constructed by the results of additional measurements of a state of process. The formulas for the gradient of the minimized functional are obtained in the work.

Results of numerical experiments are given on the example of problems for definitions of places of leakages on the oil pipeline.

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