Abstract

We study the canonical structure of the $SU(N)$ non-linear $\sigma$-model in a polynomial, first-order representation. The fundamental variables in this description are a non-Abelian vector field $L_{\mu}$ and a non-Abelian antisymmetric tensor field $\theta_{\mu\nu}$, which constrains $L_{\mu}$ to be a ‘pure gauge’ ($F_{\mu\nu}(L) = 0$) field. The second-class constraints that appear as a consequence of the first-order nature of the Lagrangian are solved, and the reduced phase-space variables explicitly found. We also treat the first-class constraints due to the gauge-invariance under transformations of the antisymmetric tensor field, constructing the corresponding most general gauge-invariant functionals, which are used to describe the dynamics of the physical degrees of freedom. We present these results in $1 + 1$, $2 + 1$ and $3 + 1$ dimensions, mentioning some properties of the $d+1$-dimensional case. We show that there is a kind of duality between this description of the non-linear $\sigma$-model and the massless Yang-Mills theory. This duality is further extended to more general first-class systems.

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Canonical Structure of the Non-Linear $\sigma$-Model in a Polynomial Formulation

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March 28, 2022
1 Introduction

One of the distinctive properties of the non-linear $\sigma$-model \cite{1}, is that its dynamical variables belong to a non-linear manifold \cite{2}, thus realizing the symmetry group in a non-linear fashion \cite{3}. Whence, either the Lagrangian becomes non-polynomial in terms of unconstrained variables, or it becomes polynomial but in variables which satisfy a non-linear constraint. It is often convenient to work in a polynomial or ‘linearized’ representation of the model. By this we mean an equivalent description where the symmetry is linearly realized, although the transformations now act on a different representation space. There is usually more than one way to construct such linearized representations. For example, in the $O(N)$ models, where the field is an $N$-component vector constrained to have constant modulus, a polynomial representation is constructed simply by introducing a Lagrange multiplier for that quadratic constraint. However, this simplicity is not present in general because the constraints required to define the manifold can be much more complex, like in the $SU(N)$ groups.

In references \cite{4, 5} a polynomial representation of the non-linear $\sigma$-model was introduced; let us briefly explain it for the $SU(N)$ model in d+1 dimensions.

The usual presentation \cite{8} of this model is in terms of an $SU(N)$ field $U(x)$, with Lagrangian density

$$L = \frac{1}{2} g^{d-1} tr(\partial_\mu U^\dagger \partial^\mu U) ,$$

\hspace{1cm} (1)

where $g$ is a coupling constant with dimensions of mass (the constant $f_\pi$ in its application to Chiral Perturbation Theory in 3+1 dimensions). The polynomial description \cite{5} of this model is constructed in terms of a non-Abelian ($SU(N)$) vector field $L_\mu$ plus a non-Abelian antisymmetric tensor field $\theta_{\mu\nu}$ with the Lagrangian

\footnote{To avoid the proliferation of indices, we frequently work in terms of the \textit{dual} of $\theta_{\mu\nu}$, which in 1 + 1 is a pseudo-scalar, in 2 + 1 a pseudovector, etcetera.}
density
\[ \mathcal{L} = \frac{1}{2} g^2 L_\mu \cdot L^\mu + g \theta_{\mu\nu} \cdot F^{\mu\nu}(L) \] (2)

where the fields \( L_\mu \) and \( \theta_{\mu\nu} \) are defined by their components in the basis of generators of the adjoint representation of the Lie algebra of SU\((N)\); i.e., \( L_\mu(x) \) is a vector with components \( L_\mu^a, a = 1, \cdots, N^2 - 1 \), and analogously for \( \theta_{\mu\nu} \). The components of \( F^{\mu\nu} \) in the same basis are: \( F^{a\mu\nu}(L) = \partial_\mu L_\nu^a - \partial_\nu L_\mu^a + g f_{abc} L_\mu^b L_\nu^c \). The dots mean SU\((N)\) scalar product, for example: \( L_\mu \cdot L^\mu = \sum_{a=1}^{N^2-1} L_\mu^a L^\mu_a \). The \( d \)-dependent exponents in the factors of \( g \) are chosen in order to make the fields have the appropriate canonical dimension for each \( d \).

The Lagrange multiplier \( \theta_{\mu\nu} \) imposes the constraint \( F^{\mu\nu}(L) = 0 \), which is equivalent \(^4\) to \( L_\mu = g^\mu U \partial_\mu U^\dagger \), where \( U \) is an element of SU\((N)\). When this is substituted back in (2), (1) is obtained \(^3\). This polynomial formulation could be thought of as a concrete Lagrangian realization of the Sugawara theory of currents \(^7\), where all the dynamics is defined by the currents, the energy-momentum tensor, and their algebra. Indeed, \( L_\mu \) corresponds to one of the conserved currents of the non-polynomial formulation, due to the invariance of \( \mathcal{L} \) under global (left) SU\((N)\) transformations of \( U(x) \). The energy-momentum tensor for (2) is indeed a function of \( L_\mu \) only:
\[ T^{\mu\nu} = g^2 (L_\mu \cdot L^\nu - \frac{1}{2} g^{\mu\nu} L^2) \] (3)

One can easily relate amplitudes with external legs of the field \( L_\mu \) to the corresponding pions’ scattering matrix elements, as shown in ref. \(^8\). It is also possible to relate off-shell Green’s functions of the field \( U \) to the ones of the field \( L_\mu \), although this relation is non-local. As

\[ L_\mu = g \frac{d}{dx} U \partial_\mu U^\dagger \Rightarrow D_\mu U = 0, \quad D_\mu = \partial_\mu + g \frac{d}{dx} L_\mu, \] (4)

\(^2\)For a complete derivation of the equivalence between the theories defined by (1) and (2) within the path integral framework, see ref. \(^5\).
then $U$ can be obtained at the point $x$ by parallel transporting its value at spatial infinity, which we fix to be equal to the unit matrix$^3$:

$$U(x) = \mathcal{P} \exp[-g^{\frac{2-d}{2}} \int_{C_x} dy^\mu L_\mu(y)], \quad (5)$$

where $\mathcal{P}$ is the path-ordering operator$^3$, and the line-integral in the exponent is along a curve $C_x$, a regular path starting at spatial infinity, and ending at $x$. The condition $F_{\mu\nu} = 0$ guarantees that $U$ is in fact invariant under deformations of $C_x$ which leave its endpoints unchanged. We can also construct products of two or more fields in a similar way, for example

$$U(x_2) U^{-1}(x_1) = \mathcal{P} \exp[-g^{\frac{2-d}{2}} \int_{C_{x_1 \rightarrow x_2}} dy^\mu L_\mu(y)], \quad (6)$$

where $C_{x_1 \rightarrow x_2}$ is a continuous path from $x_1$ to $x_2$. This shows how $U$-field correlation functions can in principle be calculated using Lagrangian (2); one has to evaluate, for example, the Wilson line (6) in the theory defined by (2).

The classical equations of motion for the Lagrangian (2) are

$$L^\nu(x) = \frac{1}{g} D_\mu \theta^{\mu\nu}(x),$$

$$F^{\mu\nu}(L) = 0. \quad (7)$$

Taking the covariant divergence on both sides of the first equation of motion, and using the second one, one gets

$$\partial \cdot L(x) = 0. \quad (8)$$

Inserting $L_\mu = g^{\frac{2-d}{2}} U \partial_\mu U^\dagger$ in (3), it yields the equations of motion for the usual non-polynomial Lagrangian (1). Note that the solutions of (7) will, in general, contain arbitrary functions of the time. If we know a solution, performing on it the transformation:

$$\theta_{\mu\nu}(x) \rightarrow \theta_{\mu\nu}(x) + \delta_\omega \theta_{\mu\nu}(x)$$

$$\delta_\omega \theta_{\mu\nu}(x) = D^\rho \omega_{\rho\mu\nu}(x), \quad (9)$$

$^3$We identify (as usual) all the points at spatial infinity.
where $\omega_{\rho\mu\nu}(x)$ is an arbitrary completely antisymmetric tensor field, will produce another solution, because $D_\rho \delta \omega^{\mu\nu}$ vanishes as a consequence of the Bianchi identity for $L_\mu$. Obviously $d$ must be larger than one in order to this transformation be well defined, since at least three different indices are needed to have a Bianchi identity. This degeneracy in the equations of motion is due to the gauge invariance of the action under the transformations (9).

This gauge-invariance makes the quantization of the model interesting, and it will allow us to discuss some properties of the non-linear $\sigma$-model from the (unusual) point of view of gauge systems. The Hamiltonian formulation of the model possesses a rich structure, since there are second-class constraints ($\mathcal{L}$ is first-order), first-class constraints (for $d > 1$), and moreover they are reducible for $d > 2$.

The structure of the paper is as follows: In section 2 we discuss the Hamiltonian formulation of the $1+1$, $2+1$ and $3+1$ models, following the Dirac algorithm [6]. In section 3 we construct the general gauge invariant functionals for the transformations generated by the first-class constraints found in section 2, and in section 4 we apply the Dirac’s brackets method to the second-class system formed by the first-class constraints plus some canonical gauge-fixing conditions. In section 5 we present our conclusions.

In Appendix we discuss a duality relationship between first-class systems, which generalizes a property we discuss for the $2+1$-dimensional model.

2 Hamiltonian formalism and constraints

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This kind of symmetry also appears when considering the dynamics of a two-form gauge field, see for example references [4].
2.1 1 + 1 dimensions

From Section 1, the polynomial Lagrangian in 1 + 1 dimensions becomes

\[ \mathcal{L} = \frac{1}{2} g^2 L_\mu L^\mu + \frac{1}{2} g \theta \epsilon_{\mu \nu} F^{\mu \nu}(L) \]  

(10)

where \( \theta \) is a pseudoscalar field. It is evident that there is no gauge symmetry in this case. Thus there will not be first-class constraints in the Hamiltonian formulation. However, there are second-class constraints, because \( \mathcal{L} \) is of first-order in the derivatives. This property will also appear in higher dimensions, so we will only discuss it in some detail for this case. To start with, we rewrite (10) in a more explicit form

\[ \mathcal{L} = \frac{1}{2} g^2 L_0^a L_0^a - \frac{1}{2} g^2 L_1^a L_1^a + g \theta^a \partial_0 L_1^a - g \theta^a \partial_1 L_0^a + g^2 \theta^a f^{abc} L_0^b L_1^c. \]  

(11)

Next we define the canonical momenta, where the primary constraints appear:

\[ \pi_0^a(x) \equiv \frac{\partial \mathcal{L}}{\partial (\partial_0 L_0^a)} \approx 0 \]

\[ \pi_1^a \equiv \frac{\partial \mathcal{L}}{\partial (\partial_1 L_1^a)} \approx g \theta^a(x) \]

\[ \pi_0^\theta(x) \equiv \frac{\partial \mathcal{L}}{\partial (\partial_0 \theta^a)} \approx 0 \]  

(12)

and the canonical Hamiltonian becomes

\[ H = \int dx \left[ \frac{1}{2} g^2 L_0^a L_0^a + \frac{1}{2} g^2 L_1^a L_1^a - g L_0^a (D_1 \theta)^a \right], \]  

(13)

where

\[ (D_1 \theta)^a = \partial_1 \theta^a + g f^{abc} L_1^b \theta^c. \]  

(14)

The ‘total’ Hamiltonian is constructed as usual, adding to (13) a Lagrange multiplier term for each of the primary constraints (12). Following the Dirac’s algorithm one more constraint is obtained:

\[ g L_0^a(x) \approx -(D_1 \theta)^a(x) \]  

(15)
and the Lagrange multipliers become fully determined. The full set of (primary plus secondary) constraints is second-class, and its particular form allows us to eliminate the canonical pairs of $L^a_0$ and $\theta^a$, thus effectively eliminating the associated degrees of freedom. The Dirac bracket becomes equal to the Poisson bracket for the remaining degrees of freedom. The resulting Hamiltonian is

$$H = \int dx \left[ \frac{1}{2g^2} D_1 \pi_1 \cdot D_1 \pi_1 + \frac{1}{2} g^2 L_1 \cdot L_1 \right],$$

with canonical brackets between the $L^a_1$'s and their momenta $\pi^a_1$. Thus these two variables become symplectic coordinates on the reduced phase-space, or constraint surface.

### 2.2 2 + 1 dimensions

The polynomial Lagrangian in this case becomes

$$\mathcal{L} = \frac{1}{2} g^2 L_\mu \cdot L^\mu + \frac{1}{2} g \theta_\mu \cdot \epsilon^{\mu\nu\lambda} F_{\nu\lambda}(L).$$

(17)

The constraint algorithm\footnote{The full details of the application of the Dirac algorithm to this system will be presented elsewhere.} produces the $2 + 1$ analogous of the second-class constraints we showed in Section 1, allowing us to eliminate the 0-component of $L_\mu$ and all the components of $\theta_\mu$. However, there will remain a set of first-class constraints

$$G^a(x) = \frac{1}{2} \epsilon_{jk} F^{a}_{jk} \approx 0,$$

with the first-class Hamiltonian

$$H = \int d^2x \left[ \frac{1}{2g^2} D_j \pi_j \cdot D_k \pi_k + \frac{1}{2} g^2 L_j \cdot L_j \right].$$

(19)

They satisfy the algebra

$$\{G^a(x), G^b(y)\} = 0$$

$$\{H, G^a(x)\} = V^{ab} G^b(x)$$

$$V^{ab} \equiv g^{-\frac{3}{2}} f^{acb} (D_j \pi_j)^c(x).$$

(20)
Now we show in what sense we can relate the massless Yang-Mills theory to the non-linear $\sigma$-model in this formulation. The $SU(N)$ Yang-Mills theory is defined by the Lagrangian

$$\mathcal{L}_{YM} = -\frac{1}{4} F_{\mu\nu}(L) \cdot F^{\mu\nu}(L),$$

which in the temporal gauge gives rise to the canonical Hamiltonian

$$H = \int d^2 x \left[\frac{1}{2} \pi_j \cdot \pi_j + \frac{1}{4} F_{jk} \cdot F_{jk}\right],$$

and the first-class constraints (‘Gauss’ laws’):

$$H_a(x) = (D_j \pi_j)_a(x) \approx 0,$$

which satisfy the $SU(N)$ algebra

$$\{H_a(x), H_b(y)\} = \delta(x - y) f_{abc} H_c(x).$$

Note that (22) can be rewritten as

$$H = \int d^2 x \left[\frac{1}{2} \pi_j \cdot \pi_j + \frac{1}{2} G \cdot G\right],$$

where the $G_a$’s are the ones defined in (18). We then see that the first-class systems corresponding to the Yang-Mills model and the non-linear $\sigma$-model can be related by: 1) Interchanging the constraints:

$$H_a(x) \leftrightarrow G_a(x),$$

and 2) Interchanging $L_j$ by $\frac{1}{g^2} \pi_j$ in the non-derivative terms in the Hamiltonian. The generalization of this mapping is constructed in Appendix.

### 2.3 $3 + 1$ dimensions

After eliminating the second-class constraints, one obtains a first-class Hamiltonian which looks exactly like the one of the $2 + 1$-dimensional case:

$$H = \int d^3 x \left[\frac{1}{2g^2} D_j \pi_j \cdot D_k \pi_k + \frac{1}{2} g^2 L_j \cdot L_j\right],$$
and the set of first-class constraints

\[ G^a_j(x) = \frac{1}{2} \varepsilon_{jkl} F^a_{kl}(x) \approx 0 . \]  

(28)

Although the system seems to be the obvious generalization of the 2+1-dimensional one, there is an essential difference: The constraints (28) are not all independent, but verify the Bianchi identity:

\[ (D_j G_j)^a(x) = 0 , \ \forall a . \]  

(29)

This implies that the set of constraints is reducible, containing only two independent functions. The counting of degrees of freedom then gives 1 for the number of physical dynamical variables \((1 = 3 - 2)\). We mention that the elimination of the second-class constraints applies in a similar way to the general \(d\)-dimensional case, and that the Hamiltonian and constraints are the obvious generalizations of (27) and (28), respectively. Due to the existence of the Bianchi identity in general, the number of independent constraints in an arbitrary dimension is just enough to kill \(d - 1\) out of the \(d\) degrees of freedom in \(H\), leading to only one physical variable, as it should be for a model which describes the dynamics of a scalar field.

### 3 Gauge invariant functionals

Gauge invariant functionals\(^6\) are important from both the classical and quantum mechanical points of view. Classically, a complete set of gauge invariant functionals and their equations of motion completely determines the dynamics of the observable, i.e., physical degrees of freedom. In Quantum Mechanics, Dirac’s method for first-class constraints defines the ‘physical’ subspace of the complete Hilbert space as the one whose state vectors are annihilated by the first-class constraints, i.e., the gauge-

\(^6\)We assume the denomination ‘gauge-invariant’ to mean on-shell gauge invariance, i.e., the gauge-invariant functionals are invariant on the constraint surface.
invariant ones. In the Schrödinger representation, the physical subspace consists of gauge invariant functionals of the fields.

To construct the gauge invariant functionals, we make use of the concept of gauge invariant projection, defined as follows: Let \( I = I[\pi, L] \) be an arbitrary functional of the phase-space fields. Then its gauge-invariant projection \( \mathcal{P}(I)[\pi, L] \) is defined by:

\[
\mathcal{P}(I)[\pi, L] = \frac{1}{\mathcal{N}} \int \mathcal{D}\omega \, I[\pi^\omega, L],
\]

(30)

where \( \pi^\omega \) is the gauge-transformed of \( \pi \) by the gauge group element \( \omega(x) \) (for example, in \( 2 + 1 \) dimensions, \( \pi^\omega_j = \pi_j + \epsilon_{jk}D_k\omega \)) and the functional integration is over all the possible configurations for \( \omega \). The normalization factor \( \mathcal{N} \) is just the volume of the gauge group: \( \mathcal{N} = \int \mathcal{D}\omega \). It is then easy to see that the gauge invariant projection of an arbitrary functional is indeed gauge invariant:

\[
\mathcal{P}(I)[\pi^\omega, L] = \mathcal{P}(I)[\pi, L],
\]

(31)

and that \( \mathcal{P} \) is a linear projection operator:

\[
\mathcal{P}(\lambda_1 I_1 + \lambda_2 I_2) = \lambda_1 \mathcal{P}(I_1) + \lambda_2 \mathcal{P}(I_2),
\]

\[
\mathcal{P}^2 = \mathcal{P}, \quad \forall I.
\]

(32)

A functional \( F \) is gauge invariant if and only if \( \mathcal{P}(F) = F \). This can be shown to be equivalent to saying that \( F \) belongs to the image of \( \mathcal{P} \). We then construct the most general gauge-invariant functional by applying \( \mathcal{P} \) to an arbitrary functional.

In \( 2 + 1 \) dimensions, we further decompose the momentum as

\[
\pi_j(x) = D_j\alpha(x) + \epsilon_{jk}D_k\beta(x),
\]

(33)

(where \( \alpha \) and \( \beta \) are scalar and pseudoscalar, respectively) to show that

\[
\mathcal{P}(I)[\pi, L] = \mathcal{P}(I)[\alpha, \beta, L] = \frac{1}{\mathcal{N}} \int \mathcal{D}\omega \, I[D_j\alpha + \epsilon_{jk}D_k(\beta + \omega), L]
\]

\[
= I[\alpha, 0, L]
\]

(34)
where the last line was obtained by performing the shift $\omega \to \omega - \beta$. (34) shows that any gauge invariant functional is independent of $\beta$; the reciprocal is immediate. The conclusion can be put as follows: The general gauge invariant functional depends arbitrarily on $L_j$, an on $\pi$ only through the combination $D_j \pi_j$.

This result is generalizable to $3 + 1$ dimensions. $F$ is shown to depend only on $D_j \pi_j$ and $L_j$, by using the same argument as in the $2 + 1$ case. The decomposition of $\pi_j$ is now

$$\pi_j(x) = D_j \alpha(x) + \epsilon_{jkl} D_k \beta_l(x), \quad (35)$$

and the $\beta$ dependence is removed as before by a shift in $\omega$. The only difference appears in the actual construction of the projection operator, which appears to be ill-defined at first sight. This is so because the gauge transformations in $d = 3$:

$$\pi_j^\omega = \pi_j + \epsilon_{jkl} D_k \omega_l, \quad (36)$$

are invariant under $\omega_j(x) \to \omega_j(x) + D_j \lambda(x)$, for any $\lambda$. This produces an infinite factor when one integrates over $\omega$ in the definition (30) of $P(I)$. Of course, this factor is also present in $N$, but to explicitly cancel them on needs to ‘fix the gauge’ for the integration over $\omega$. A convenient way to do that is by using the Faddeev-Popov trick, which gives the ‘gauge fixed’ projector

$$P(I)[\pi, L] = \frac{1}{\mathcal{D}\omega \det M_f[\omega] \delta[f(\omega)]} \int \mathcal{D}\omega \det M_f[\omega] \delta[f(\omega)] I[D_j \alpha + \epsilon_{jkl} D_k \omega_l], \quad (37)$$

where $M_f[\omega] = \frac{\delta}{\delta \omega^\lambda} f(\omega^\lambda)$. We have seen that the gauge invariant functionals depend on $D_j \pi_j$ and $L_j$ (the result is indeed true in any number of dimensions). However, there is still a degree of redundancy in this description because one is interested only in gauge invariant functions on-shell, i.e., on the surface $F_{jk}(L) = 0$. Thus we do not need the full $L_j$, but only its restriction to the constraint surface. As it was shown in ref. [8], it is possible to solve that kind of equation using a perturbative
approach. The main result we need to recall is that perturbative expansion allows one to express $L_j$ as a function of the scalar $\partial_j L_j$ only. Then we obtain a more symmetrical description in terms of the gauge-invariant, scalar variables:

$$(D_j \pi_j)^a(x), \ (D_j L_j)^a(x) = \partial_j L_j^a(x).$$

Their equations of motion link each other:

$$\frac{\partial}{\partial t}(D_j \pi_j) = -g^2 \partial_j L_j$$
$$\frac{\partial}{\partial t}(\partial_j L_j) = -g^{-2} \partial_j D_j(D_k \pi_k)$$
$$F_{jk}(L) \approx 0,$$

(39)

(where we have included the constraints). They imply the second order equations

$$(\partial_t^2 - \partial_j D_j) D_k \pi_k = 0$$
$$(\partial_t^2 - \partial_j D_j) \partial_k L_k = 0,$$

(40)

which show the scalar particle nature of the (only) physical degree of freedom. Let us consider in more detail the issue of static solutions in 3 + 1 dimensions. In this situation, (39) reduces to

$$\partial_j L_j = 0, \ F_{jk} \approx 0$$
$$\partial_j D_j(D_k \pi_k) = 0.$$

(41)

The first two equations in (41) are equivalent to

$$L_j = U \partial_j U^\dagger, \ \partial_j L_j = 0.$$

(42)

They are exactly the set of equations one gets when considering the Gribov problem [12] (for the Yang-Mills theory) in the Coulomb gauge, on the orbit of the trivial configuration ($L_j = 0$). It is well known that there are more solutions than just the
trivial one, in particular, one obtains the ‘fermionic’ configurations of the Skyrme model\cite{9}, which verify

$$n = -\frac{1}{24\pi^2} \int d^3x \epsilon_{jkl} \text{tr}(L_j L_k L_l) = \pm \frac{1}{2}. \quad (43)$$

Once a particular solution of (43) is obtained, it can be inserted in the last equation of (41) to get an equation for $\pi$. Note that the momenta should then satisfy

$$D_j \pi_j = f_0, \quad (44)$$

where $f_0$ is a zero mode of the operator $\partial_j D_j$ (of course, the trivial solution $\pi_j = 0$ is included). For each Gribov solution $L_j$, there will be a non-trivial zero mode for this operator, and then a non-zero solution for the momenta. These solutions can be compared to the static solutions of the usual non-polynomial formulation. To do that we must regain the field $L_0$, which was eliminated by using the second-class constraints. That is very simple, since in fact $L_0$ is equal to a constant times $D_j \pi_j$, and then (44) implies

$$L_0 = f_0. \quad (45)$$

So the family of static solutions in the polynomial version seems to be larger than in the usual treatment. Indeed, as $L_0 = U \partial_0 U^\dagger$, a non-zero $L_0$ implies that there is a time-dependence for $U$. Note, however, that such configurations contribute to the energy in an amount:

$$E(L_0) = \frac{1}{2} g^2 \int d^3 x |f_0|^2, \quad (46)$$

which is proportional to the norm of the zero mode, and then the minimum energy will correspond to the trivial configuration $L_0 = 0$. A simple example of a configuration with $L_0 \neq 0$ is:

$$\tilde{L}_j(x, t) = \exp(iht)L_j(x)\exp(-iht)$$

$$\tilde{L}_0(t) = h, \quad (47)$$

\footnote{The stabilizing term can be added without changing the canonical structure of the model.}
where \( h \) is a hermitian (constant) traceless matrix, and \( L_j(x) \) satisfies (42). Thus for (47), \( E(L_0) = \frac{1}{2}g^2tr(h^2) \int d^3x \), which is divergent for infinite volume.

## 4 Dirac’s brackets method

As an alternative to the previous approach, we apply here the ‘Dirac’s brackets method’ to the treatment of the first-class constraints in the 2 + 1 model (it can however be straightforwardly generalized to the \( d + 1 \) model). It consists in constructing the Dirac’s brackets for the set of \textit{second-class} constraints containing all the original first-class constraints plus a suitable set of gauge fixing conditions. We choose the canonical gauge fixing functions:

\[
\chi^a(x) = \pi^a_2(x) = 0 .
\]

The basic ingredient to calculate the Dirac’s brackets is the Poisson bracket between \( \chi^a \) and \( G^a(x) \): \( \{\chi^a(x), G^b(y)\} = (D_1)^{ab}\delta(x - y) \). From this it follows that the only non-trivial Dirac’s brackets between canonical variables are

\[
\begin{align*}
\{L^a_1(x), \pi^b_1(y)\}_D &= \delta_{ab}\delta(x - y) , \\
\{L^a_2(x), \pi^b_1(y)\}_D &= \langle x, a \mid D^{-1}_1D_2 \mid y, b \rangle .
\end{align*}
\]

The second one is a complicated non-local function. It is more convenient to take advantage of the results of the previous section to work with \( L_j \) and \( D_1\pi_1 \). Then the Dirac’s brackets become local

\[
\begin{align*}
\{L^a_1(x), (D_1\pi_1)^b(y)\}_D &= -D_1^{ab}\delta(x - y) \\
\{L^a_2(x), (D_1\pi_1)^b(y)\}_D &= -D_2^{ab}\delta(x - y) .
\end{align*}
\]

## 5 Conclusions

The polynomial formulation (2) has an interesting canonical structure. Some of its properties are:
The system has second-class constraints which can be solved explicitly for some coordinates in terms of the others. This leaves the canonical pairs associated to the spatial components of a non-Abelian vector field only.

For \( d > 1 \) there remain first-class constraints which form an Abelian algebra. They, and the first-class Hamiltonian have essentially the same structure in any number of dimensions. However, for \( d > 2 \), the constraints are reducible. The number of independent constraints is just enough to leave only one physical degree of freedom.

These first-class systems can be regarded as ‘duals’ of the Yang-Mills model in the temporal gauge, in the sense that the constraints in one of the systems are non-trivial gauge invariant functions in the other. This duality can be generalized to a greater class of first-class systems.

We also constructed the most general gauge invariant functional explicitly. Note that the Gauss-law constraints of the dual Yang-Mills system appear here as (non-trivial) gauge invariant objects, verifying the general property discussed in the Appendix.

**Acknowledgements**

C. D. F. was supported by an European Community Postdoctoral Fellowship. T. M. was supported in part by the Daiwa Anglo-Japanese Foundation. We also would like to express our acknowledgement to Dr. I. J. R. Aitchison for his kind hospitality.
Appendix: A duality transformation for first-class systems

The kind of ‘duality’ that exists between the massless Yang-Mills theory and the polynomial version of the non-linear $\sigma$-model is a particular case of a more general concept, which we define in this Appendix. Let us consider a constrained dynamical system defined on a phase-space of coordinates $q_j, p_j$, $j = 1 \cdots N$, with first-class Hamiltonian $H$ and a complete set of irreducible first-class constraints $G_a \approx 0$, $a = 1, \cdots, N$. We assume that the first-class constraints satisfy the closed algebra

$$\{G_a, G_b\} = g_{abc}(q, p) G_c$$

and regarding the Hamiltonian, we impose on it the requirement of having the structure

$$H = \frac{1}{2} F_a(q, p) F_a(q, p)$$

where the functions $F_a(q, p)$, $a = 1, \cdots, N$ verify the relations

$$\{F_a, F_b\} = f_{abc}(q, p) F_c$$

$$\{G_a, F_b\} = \lambda_{abc}(q, p) F_c$$

and $\lambda$ is completely antisymmetric with respect to the last two indices. This implies that the Poisson bracket of $H$ and each of the $G_a$’s will be strongly equal to zero, what is stronger than what we need in a general first-class system. Indeed, Equations (52) and (53) select among all the possible first-class systems the class which admit a duality of the kind we are going to define.

The associated dual first-class system is defined on the same phase-space, and its Hamiltonian and constraints (denoted with a tilde) are defined by

$$\tilde{H} = \frac{1}{2} G_a G_a$$

$$\tilde{G}_a = F_a \approx 0$$

16
where $F_a$ and $G_a$ are the ones introduced in (51), (52) and (53). We then verify that the new system is also first-class, since

\[
\{ \tilde{G}_a, \tilde{G}_b \} = \tilde{g}_{abc}(q,p) \tilde{G}_c \\
\{ \tilde{G}_a, \tilde{H} \} = V_{ab}(q,p) \tilde{G}_b ,
\]

(55)

where:

\[
\tilde{g}_{abc}(q,p) = f_{abc}(q,p) \\
V_{ab}(q,p) = \lambda_{abc}(q,p) G_c(q,p) .
\]

(56)

Thus evidently this mapping leaves the first-class nature of the system invariant. Note however, that the irreducibility of the new constraints is by no means guaranteed. That will depend upon the particular form of the $F_a$’s. An interesting property of the new system is that, because of (53),

\[
\{ \tilde{G}_a, \tilde{G}_b \} \approx 0 \quad \forall a, b ,
\]

(57)

which proves that the $G_a$’s constitute a set of $M$ independent gauge invariant functions, which is a very helpful property when one wants to study the classical or quantal dynamics of the system.

Note that the transformation we defined is not necessarily involutive; to guarantee that we would need a completely antisymmetric $\lambda_{abc}$ in Equation (53).

The Hamiltonian (52) resembles the one of the Yang-Mills system, except for the absence of the term quadratic in the canonical momenta. We did not include this, neither the corresponding one in the dual, to keep the discussion as general as possible. As they are gauge invariant by themselves, their presence or not do not alter the essence of the discussion.

This duality transformation can be interpreted as transforming a gauge-invariant theory into another. The unphysical gauge variables of the first theory come to be
the true physical degrees of freedom of the second one, which are actually defined on the fibers generated by the gauge group of the first model (note that the second theory is not invariant under the gauge group of the first one). Hence it follows naturally that the first-class constraints which generate the gauge transformations of the first theory are the dynamical variables which describe motions (‘translations’) along the gauge group fiber.

This duality transformation may be useful in cases where the number of physical degrees of freedom is the same in both the original and dual models, as for the Yang-Mills and non-linear sigma-model in 2+1 dimensions. Here the physical excitations are massless scalar fields for both models. Although it is very difficult to write down an explicit dynamics for the physical degree of freedom of the Yang-Mills theory, this mapping could make it easier, since the identification of physical variables is much simpler in the non-linear sigma-model.
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