Random Defect Lines in Conformal Minimal Models

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Abstract

We analyze the effect of adding quenched disorder along a defect line in the 2D conformal minimal models using replicas. The disorder is realized by a random applied magnetic field in the Ising model, by fluctuations in the ferromagnetic bond coupling in the Tricritical Ising model and Tricritical Three-state Potts model (the $\phi_{12}$ operator), etc. We find that for the Ising model, the defect renormalizes to two decoupled half-planes without disorder, but that for all other models, the defect renormalizes to a disorder-dominated fixed point. Its critical properties are studied with an expansion in $\epsilon \propto 1/m$ for the $m^{\text{th}}$ Virasoro minimal model. The decay exponents $X_N = \frac{N}{2}(1 - \frac{2(3N-4)}{m^2}) + O(\frac{1}{m^3})$ of the $N^{\text{th}}$ moment of the two-point function of $\phi_{12}$ along the defect are obtained to 2-loop order, exhibiting multifractal behavior. This leads to a typical decay exponent $X_{12p} = \frac{1}{2}(1 + \frac{9}{(m+1)^2}) + O(\frac{1}{m^3})$. One-point functions are seen to have a non-self-averaging amplitude. The boundary entropy is larger than that of the pure system by order $1/m^3$.

As a byproduct of our calculations, we also obtain to 2-loop order the exponent $X_N = N(1 - \frac{27}{16}\pi^2(3N-4)(q-2)^2 + O(q-2)^3)$ of the $N^{\text{th}}$ moment of the energy operator in the $q$-state Potts model with bulk bond disorder.
1 Introduction

Conformal symmetry tends to emerge in pure (homogeneous and rotationally invariant) 2-D Statistical Mechanics models at their critical points. This high degree of symmetry severely constrains these theories, so that these critical points are well understood. Many models, including the Ising model, Tricritical Ising model, and Tricritical Three-state Potts model, are part of a class of conformal field theories known as Virasoro minimal models \([1, 2, 3]\). The Ising model \([1]\), Tricritical Ising model \([4, 5, 6]\), and Tricritical Three-state Potts model \([7, 8]\) have all been realized experimentally in adsorbed monolayer systems.

Because physical systems always have impurities, it is important to consider the effect of quenched disorder on the critical behavior of these theories. When disorder is added to the bulk, random fields are usually relevant, but random bonds may or may not be (see e.g. Ref.’s \([9, 10]\)). It is possible to show, for certain systems, that with the addition of disorder the system renormalizes into an infra-red fixed point. In fact, a rigorous theorem shows that when disorder in the order parameter is added to a system undergoing a first-order phase transition, the latent heat vanishes \([11]\), and a 2nd order transition can be expected.

However, one generally does not know a priori whether the new critical point is disorder-dominated. A number of studies have reported cases in which the addition of quenched disorder to a pure 2-D model at its critical point resulted in another pure (that is, non-random) critical model \([12, 13, 14, 15, 16, 17]\). Other studies found quenched disorder to result in new disorder-dominated fixed points \([18, 19, 20, 21, 22]\). One can often see that a critical point is disorder-dominated by showing that various universal quantities are not self-averaging (that measurements for a specific ‘typical’ sample may differ substantially from those on an average one). One particularly interesting manifestation of this is multifractal behavior, which occurs when an infinite hierarchy of independent scaling dimensions are associated with a single operator \([23, 24, 25]\). (See for example \([26]\) for a relevant discussion.)

All the above studies have focused on the effects of adding quenched disorder to the bulk of a system. However, it is also interesting to consider the case where the 2-D model has a defect along which impurities have clustered. In this paper we consider the effects of adding quenched disorder only along a defect line, in each of the minimal models; these models are labelled by an index \(m, m \geq 3\). Each \(m\) represents a different model: the Ising model \((m = 3)\), Tricritical Ising model \((m = 4)\), Tetracritical Ising model \((m = 5)\), Tricritical Three-state Potts model \((m = 6)\), etc. . . In Section 3 we introduce our defect model, adding quenched disorder in the coupling to \(\phi_{12}\) (an operator in the Kac Table \([1]\)) using replicas. The physical meaning of this disorder varies from model to model, representing a random magnetic field for the Ising model, but fluctuations in the ferromagnetic coupling (or chemical potential) in the Tricritical Ising model and Tricritical Three-state Potts model. We find that the Ising model renormalizes to a non-random model (consisting of decoupled half-planes with free boundary conditions and no random magnetic field), while all other models renormalize.
to disorder-dominated fixed points. Just as with bulk disorder, disorder on the
defect can result in either a pure or disorder-dominated fixed point.

In Section 3 we calculate the renormalization group equation of the strength
of the disorder $\Delta$, to 2-loop order by minimal subtraction \[21, 27\]. We find
that arbitrarily weak disorder grows, and flows to a new fixed point at which
$\Delta$ is of order $1/m$. This justifies a $1/m$ expansion, where critical quantities are
calculated perturbatively in $1/m$. The "boundary entropy" \[28\] at the random
fixed point is found to be $\mathcal{O}(1/m^3)$, and larger than the entropy of the pure
fixed point.

In Section 4 the same scheme is used to find the moments of correlation
functions of the operator $\phi_{12}$ along the defect. (Technical details associated
with the irreducible representations of the symmetric group \[29\] are delegated
to Appendices \[3\] and \[4\].) It is found that the moments fall off as a sum of power
laws, the dominant term decaying as

$$
< \phi_{12}(x_1)\phi_{12}(x_2) >^N \propto |x_1 - x_2|^{-2X_N}
$$

with

$$
X_N = \frac{N}{2} \left( 1 - \frac{1}{4}(3N - 4)\left( \frac{3}{m+1} \right)^2 + \mathcal{O}(\frac{3}{m+1})^3 \right)
$$

$X_N < N X_1$, so the operator $\phi_{12}$ exhibits multifractal behavior. This leads to a
typical decay exponent \[29\] :

$$
X_{\text{typical}} = \frac{1}{2} \left( 1 + \left( \frac{3}{m+1} \right)^2 + \mathcal{O}(\frac{3}{m+1})^3 \right)
$$

In Section 5 we calculate one-point functions of the same operator off the
defect line. The universal (normalized) amplitudes of the moments are found
to be non-self-averaging, whereas the power law is self-averaging.

In Section 6, the Ising model, which requires special considerations due to
the presence of an additional marginal (boundary) operator, is analyzed. It
is argued that when a random magnetic field is added along the defect line,
the system renormalizes to two decoupled half-plane Ising models with free
boundary conditions and no disorder.

We finally note that the manipulations needed to get the exponents in
Eq.(1.2) are similar to those needed to get the decay exponents of moments
of two-point functions of $\varepsilon$, the energy operator, for the $q$-state Potts model
with bulk disorder in the bond strength. This model has been analyzed else-
where by expanding about $q = 2$ (the Ising Model) \[21, 22, 29, 27\], and as a
byproduct of our calculations here, we find in Section 4, Eq.(4.11-4.13), that

$$
< \varepsilon(x_1)\varepsilon(x_2) >^N \propto |x_1 - x_2|^{-2\tilde{X}_N}
$$
where
\[
\tilde{X}_N = N \left( 1 - \frac{2}{9\pi^2} (3N - 4)(q - 2)^2 + \mathcal{O}(q - 2)^3 \right)
\] (1.5)

Results of numerical transfer matrix calculations for the random bond q-state Potts model can be found in [22].

### 2 The Defect Model

We start with a Virasoro minimal conformal field theory \([1, 2, 3]\) labelled by an integer \(m, m \geq 3\), and perturb it along a defect line. The perturbed action is

\[
S = S_m + \int_{-\infty}^{\infty} dx \ h(x) \phi_{12}(x, y = 0)
\] (2.1)

\(S_m\) is the action of the unperturbed conformal field theory. \(h(x)\) is a random coupling, is picked from a Gaussian probability distribution with zero mean and variance \(\Delta_0\), and is uncorrelated along the defect:

\[
\bar{h}(x) = 0, \quad \bar{h}(x)h(x') = 2\Delta_0 \delta(x - x')
\] (2.2)

The overbar indicates the disorder average. The operator \(\phi_{12}\) is located at position \((p, q) = (1, 2)\) in the Kac Table \([1]\), and exists for any minimal model. For the Ising model \((m = 3)\) it is the spin operator, while for the Tricritical Ising model \((m = 4)\) and Tricritical Three-State Potts model \((m = 6)\) it is the energy operator. The scaling dimensions \(2h_{pq}\) (twice the conformal weight) of operators \(\phi_{pq}\) in minimal models are known \([1]\) :

\[
2h_{pq} = \frac{[(m + 1)p - mq]^2 - 1}{2m(m + 1)},
\] (2.3)

which gives

\[
2h_{12} = \frac{1}{2} - \frac{3}{2(m + 1)}
\] (2.4)

The replicated \([30]\) effective action is

\[
S^\text{replica}_m = \sum_{\alpha=1}^n S_m^\alpha + \Delta_0 \int_{-\infty}^{\infty} dx \sum_{\alpha \neq \beta} \phi^{\alpha}_{12} \phi^{\beta}_{12}(x, y = 0),
\] (2.5)

where \(n\) is the number of replicas, and we take \(n \to 0\) at the end of the calculation.
We have ignored higher cumulants of the probability distribution of \( h(x) \), because power counting shows that they are irrelevant in the R.G. sense, for large \( m \), and we will be expanding about large \( m \).

We have also dropped the terms with \( \alpha = \beta \) in Eq. (2.5). The terms with \( \alpha = \beta \) produce a non-random \( \phi_{1q} \) by the conformal fusion rules \[1\]. Upon renormalizing they will generate other non-random \( \phi_{1q} \) with \( q \geq 3 \). However, from Eq. (2.4) we can see that all these terms are irrelevant except when \( q = m = 3 \). So when \( m \neq 3 \) the perturbation in Eq. (2.5) is the only relevant one. The following analysis in Sections 3 - 5 will assume this, and will thus only hold for \( m \neq 3 \). This will not be too restrictive, since all quantities will be calculated by expansion in \( \epsilon = \frac{3}{m+1} \), and will thus be based on large \( m \). But if \( m = 3 \) (the Ising model), then \( \phi_{13} \) (the energy operator) is marginal, and we need to include the effects of a constant coupling to \( \phi_{13} \). This is done in Section 6.

3 Renormalization of the disorder strength \( \Delta \)

We calculate the renormalization group equation for \( \Delta \) to 2-loop order by minimal subtraction \[21, 27\]. To calculate the renormalization of \( \Delta_0 \), we want to know, given a microscopic disorder strength \( \Delta_0 \), what effective disorder strength \( \Delta(r) \) is seen on large length scales \( r \). In a region of size \( r \), we expand out \( \exp\{S_{\text{replica}}\} \) in powers of \( \Delta_0 \). \( \Delta_0 \) will come with disorder operators at various points in the region of size \( r \). \( p = 2, 3 \) or more disorder operators may look, using repeated operator product expansions, like a single disorder operator on larger length scales (or size \( r \)), and thus create a new effective disorder strength \( \Delta(r) \). This can be represented schematically as:

\[
\Delta_0 \int_{-\infty}^{\infty} dx \sum_{\alpha \neq \beta}^{n} \phi_{12}^{\alpha} \phi_{12}^{\beta}(x) + \frac{1}{2} \left( \Delta_0 \int_{-\infty}^{\infty} dx \sum_{\alpha \neq \beta}^{n} \phi_{12}^{\alpha} \phi_{12}^{\beta}(x) \right)^2 + \\
+ \frac{1}{6} \left( \Delta_0 \int_{-\infty}^{\infty} dx \sum_{\alpha \neq \beta}^{n} \phi_{12}^{\alpha} \phi_{12}^{\beta}(x) \right)^3 + \ldots \rightarrow \Delta \int_{-\infty}^{\infty} dx \sum_{\alpha \neq \beta}^{n} \phi_{12}^{\alpha} \phi_{12}^{\beta}(x) \quad (3.1)
\]

This analysis will of course generate numerous terms besides the disorder operator. However, as noted above, these terms are all irrelevant in the RG sense, so we will not calculate these terms.

For each power of \( \Delta_0 \), the integrals generated above are regulated at short distances by analytic continuation in \( \epsilon = \frac{3}{m+1} \), and at large distances by an

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1 For the lower values of \( m = 4 \) or 5, the \( 4^{\text{th}} \) order cumulants are respectively relevant and marginal, but the \( 2^{\text{nd}} \) order cumulant is more relevant, and presumably will be physically dominant. While the \( 4^{\text{th}} \) order cumulant is not irrelevant at the unperturbed fixed point for \( m = 4 \) or \( m = 5 \), it is irrelevant for large \( m \), and we expect it to be irrelevant for \( m = 4 \) or 5 at the new disordered fixed point. This would be similar to the Wilson-Fisher fixed point, where the \( \phi^6 \) operator becomes relevant at the Gaussian fixed point below three dimensions, whereas it is in fact irrelevant at the new fixed points obtained by an expansion about four dimensions.
infrared cutoff $r$. The calculation, which uses the method of [27], is done in Appendix A, where we obtain

$$\Delta(r) = r^\epsilon \Delta_0 + 4(n-2)\frac{r^{2\epsilon}}{\epsilon} \Delta_0^2 - 4(n-2) \left[ 2 - 4(n-2) \frac{1}{\epsilon} \right] \frac{r^{3\epsilon}}{\epsilon} \Delta_0^3 + O(\Delta_0^4)$$ (3.2)

When we calculate the beta function by taking a derivative with respect to $\log(r)$, we find that the poles in $\epsilon$ cancel, as they must for any physical quantity. The result is

$$\beta(\Delta) = \frac{d\Delta}{d(\log(r))} = \epsilon \Delta - 8\Delta^2 + 32\Delta^3 + O(\Delta^4)$$ (3.3)

where we have taken the replica limit $n \to 0$ (and have ceased writing the $r$ dependence of $\Delta(r)$). The RG flows take the unperturbed theory to a new infrared fixed point with disorder $\Delta_*$. Solving for $\Delta_*$ by putting $\beta(\Delta_*) = 0$ gives

$$\Delta_* = \frac{\epsilon}{8} + \frac{\epsilon^2}{16} + O(\epsilon^3), \quad (\epsilon = \frac{3}{m+1})$$ (3.4)

The new fixed point is at a distance of order $\epsilon$ from the unperturbed theory, and so we can calculate physical quantities by expanding in powers of $\epsilon$, which is small for minimal models with large index $m$. This is analogous to the Wilson-Fisher epsilon expansion in $d = 4-\epsilon$ dimensions. We will see below that the new fixed point is disorder-dominated. Because the analysis is based on expansion in powers of $\epsilon = \frac{3}{m+1}$, we really only show that there is a new disorder-dominated fixed point for large $m$ - but we expect no qualitative changes for lower $m$, such as $m = 4$. However, we again note that $m = 3$, the Ising model, is qualitatively different, and will be treated separately.

The ‘boundary entropy’ (the universal constant independent of the system size appearing in the disorder-averaged free energy) associated with the defect line, calculated as in Ref. [28], is found to be

$$\frac{\delta g}{n} = \frac{\pi^2 \epsilon^3}{96} + O(\epsilon^4).$$ (3.5)

Note that the entropy has increased from that of the unperturbed system (where it vanishes). This is to be contrasted with the case of a pure (non-random and unitary) system, where the entropy [28] is expected to only decrease upon renormalization.

4 Scaling Dimensions of Moments of $\phi_{12}$

We now look at $<\phi_{12}(x_1)\phi_{12}(x_2)>^N$, the disorder-averaged $N^{th}$ moment of the 2-point function for points $x_1$ and $x_2$ which lie both near the defect and far from each other (Figure 1). We want to see how these moments fall off at large distances. As formulated in replicas we have
\[
< \phi_{12}(x_1)\phi_{12}(x_2) >^N = \left( \prod_{\alpha=1}^{N} \phi_1^\alpha(x_1) \prod_{\beta=1}^{N} \phi_2^\beta(x_2) \right) ^N \tag{4.1}
\]

As explained in [29], we cannot simply calculate the dimension of \( \prod_{\alpha=1}^{N} \phi_1^\alpha \), because this operator is not multiplicatively renormalizable. Instead, it is a sum of independent scaling operators with different scaling dimensions. In the limit where the pure and disordered fixed points collide \((m = \infty)\), we have numerous operators with the same dimension. For example, looking at \( N = 2 \), \( \phi_1^1 \phi_2^2 \) and \( \phi_1^3 \phi_2^7 \) have the same scaling dimension and are equally good operators (note that the replica limit \( n \rightarrow 0 \) is not taken until after all calculations are completed). In general, for the \( N^{th} \) moment, and with \( n \) replicas, we have \( \binom{n}{N} \) operators with the same dimension at \( m = \infty \); because they all have the same scaling dimension at \( m = \infty \), any linear combination of them also has this scaling dimension at \( m = \infty \).

However, when we move to \( m < \infty \), only appropriate linear combinations of these operators will have well-defined scaling dimensions. The entity \( \prod_{\alpha=1}^{N} \phi_1^\alpha \) which appears in Eq.(4.1) will contain all of these scaling operators, so that the average \( < \phi_{12}(x_1)\phi_{12}(x_2) >^0 \) will decay as a mixture of power laws, and at large distances will be dominated by the scaling operator with the lowest dimension. We thus need to calculate the dimension of each of these scaling operators. The appropriate multiplicatively renormalizable operators transform in irreducible representations of the symmetric group [29]:

\[
\mathcal{O}_{NMn} = \sum_{1 \leq \alpha_i \leq n-M} (\phi_1^{\alpha_1} - \phi_2^{\alpha_2}) \ldots (\phi_1^{\alpha_{M-1}} - \phi_2^{\alpha_{M-1}}) \phi_1^{\alpha_M} \ldots \phi_2^{\alpha_N} \tag{4.2}
\]

where \( 0 \leq M \leq N \).

To calculate the scaling dimensions of \( \mathcal{O}_{NMn} \), we will add the term \( \Delta_{NMn,0} \int_{-\infty}^{\infty} dx \mathcal{O}_{NMn}(x) \) to the action. As in section 3, we calculate the renormalization of \( \Delta_{NMn,0} \) to \( \Delta_{NMn}(r) = Z_{NMn}(r)\Delta_{NMn,0} \) on length scales of size \( r \), by expanding the action in powers of \( \Delta_0 \):

\[
\begin{align*}
\Delta_{NMn,0} \left( \int_{-\infty}^{\infty} dx \mathcal{O}_{NMn}(x) \right) + \\
+ \Delta_{NMn,0} \Delta_0 \left( \int_{-\infty}^{\infty} dx \mathcal{O}_{NMn}(x) \right) \left( \int_{-\infty}^{\infty} dx' \sum_{\alpha \neq \beta}^n \phi_1^\alpha \phi_2^\beta (x') \right) + \\
+ \frac{1}{2} \Delta_{NMn,0} \Delta_0^2 \left( \int_{-\infty}^{\infty} dx \mathcal{O}_{NMn}(x) \right) \left( \int_{-\infty}^{\infty} dx' \sum_{\alpha \neq \beta}^n \phi_1^\alpha \phi_2^\beta (x') \right)^2 + \ldots \\
\rightarrow Z_{NMn}(r)\Delta_{NMn,0} \left( \int_{-\infty}^{\infty} dx \mathcal{O}_{NMn}(x) \right) \tag{4.3}
\end{align*}
\]
In Appendix B we check that to 2-loop order we don’t need to worry about mixing with other operators. $Z_{NMn}(r)$ is calculated with the same type of integrals as in the last section (they are again regulated at short distances by analytic continuation in $\epsilon$ and at large distances by an infrared cutoff $r$). However, technical combinatorial complexities arise in counting the number of contractions associated with various irreducible representations of the symmetric group. They are delegated to Appendices B and C. The result is

$$Z_{NMn}(r) = 1 + 2\tilde{b}_{NMn} \frac{r^\epsilon}{\epsilon} \Delta_0 - 4 \left( N(n - N) + (N - 1)\tilde{b}_{NMn} \right) \frac{r^{2\epsilon}}{\epsilon^2} \Delta_0^2$$

(4.4)

where we have defined

$$\tilde{b}_{NMn} \equiv 2 \left( (N - M)n - N^2 + M(M - 1) \right)$$

(4.5)

$\tilde{b}_{NMn}$ is a solely combinatorial factor which arises from counting the number of replica contractions associated with $O_{NMn}$. We then use Eq.(3.2) to rewrite the series in $\Delta_0$ as a series in $\Delta$, yielding $\gamma_{NMn}(\Delta)$.

$$\gamma_{NMn}(\Delta) = \frac{d(\log(Z_{NMn}(r)))}{d(\log(r))} = 2\tilde{b}_{NMn}\Delta - 8(N(n - N) + (N - 1)\tilde{b}_{NMn})\Delta^2 + O(\Delta^3)$$

(4.6)

Note that the poles in $\epsilon$ again cancel. To get the scaling dimensions of $O_{NMn}$ at the disordered fixed point, we take $n \to 0$ and $\Delta \to \Delta_*$, getting

$$\gamma_{NM}(\Delta_*) = \frac{\epsilon}{4} \tilde{b}_{NM0} + \frac{\epsilon^2}{8} (N^2 - (N - 2)\tilde{b}_{NM0}) + O(\epsilon^3)$$

(4.7)

In the unperturbed theory, for all $M$, $<O_{NMn}(x_1)O_{NMn}(x_2)>_N$ will fall off as $|x_1 - x_2|^{4Nh_{12}}$. But with the defect it will fall off as $|x_1 - x_2|^{-2X_{NM}}$, where $2X_{NM} = 4Nh_{12} - 2\gamma_{NM}(\Delta_*)$. Because $\prod_{i=1}^N \phi_{12}^{x_i}$ is a linear combination of the scaling operators $O_{NMn}$, the moment $<\phi_{12}(x_1)\phi_{12}(x_2)>_N$ will be a sum of terms decaying with powers $X_{NM}$. For $|x_1 - x_2|$ large, this will be dominated by the smallest power, and it is easy to see that this is $X_N = X_{NN}$, giving our main result:

$$<\phi_{12}(x_1)\phi_{12}(x_2)>_N \propto |x_1 - x_2|^{-2X_N}$$

(4.8)

$$X_N = \frac{N}{2} \left( 1 - \frac{\epsilon^2}{4}(3N - 4) + O(\epsilon^3) \right)$$

(4.9)

$X_N < NX_1$, so the system is not self-averaging. Instead, we have an infinite number of independent scaling dimensions all associated with the single operator...
Note that $X_{N_1}/N_1 > X_{N_2}/N_2$ for $N_2 > N_1$, as required by convexity. The result above also yields the typical exponent:

$$X_{\text{typical}} = \frac{1}{2}(1 + \epsilon^2 + \mathcal{O}(\epsilon^3))$$  \hspace{1cm} (4.10)

The combinatorial problems encountered here are the same as those for the $q$-state Potts model with disorder in the bulk ferromagnetic couplings. The specific integrals are different but have already been done in [27]. The only new difficulty is that to two loop order, the operators $O_{NM_n}$ are no longer always multiplicatively renormalizable, but instead mix with other descendant operators. However, luckily, the leading and subleading operators, $O_{N,N_1}$ and $O_{N,N_1-1,n}$, remain multiplicatively renormalizable (see Appendix B for details). So the combinatorics in Appendices B and C also give the 2-loop result for the decay exponent of the $N^{th}$ moment of the two-point function of the energy operator $\epsilon$, in a Potts model with bulk bond disorder. We have:

$$\langle \epsilon(x_1)\epsilon(x_2) \rangle_N \propto |x_1 - x_2|^{-2\tilde{X}_N}$$  \hspace{1cm} (4.11)

$$\tilde{X}_N = N \left( 1 - \frac{2}{9\pi^2} (3N - 4)(q - 2)^2 + \mathcal{O}(q - 2)^3 \right)$$  \hspace{1cm} (4.12)

This also gives us the typical decay exponent:

$$\tilde{X}_{\text{typical}} = 1 + \frac{8}{9\pi^2} (q - 2)^2 + \mathcal{O}(q - 2)^3$$  \hspace{1cm} (4.13)

5 Moments of the One-point Function of $\phi_{12}$

We now calculate the disorder-averaged moments of the one-point function of $\phi_{12}$, evaluated at a distance $y$ from the defect line (Figure 2). The method is similar to that in [31]. We no longer need to worry about the various irreducible representations of the symmetric group, because the one point function of $O_{NM_n}$ vanishes by symmetry except when $M = 0$. The disorder-averaged odd moments of $\phi_{12}$ vanish because the disorder-averaged system is symmetric under $\phi_{12} \rightarrow -\phi_{12}$. The even moments are calculated perturbatively:

$$\langle \phi_{12}(x = 0, y)^{2N} \rangle = \left\langle \prod_{\alpha=1}^{2N} \phi_{12}^\alpha(0, y) \right\rangle$$

$$= \left\langle \left[ \prod_{\alpha=1}^{2N} \phi_{12}^\alpha(0, y) \right] e^{\Delta_0 \int_{-\infty}^{\infty} dx \sum_{\alpha, \beta} \phi_{12}^\alpha \phi_{12}^\beta} \right\rangle_{0, \text{cutoff a}}$$

$$= \frac{1}{y^{-(1-\epsilon)N}} \sum_{i=0}^{\infty} \frac{1}{i!} (\Delta_0 y^\epsilon)^i \left\langle \left[ \prod_{\alpha=1}^{2N} \phi_{12}^\alpha(0, 1) \right] \left[ \prod_{j=1}^{i} \int_{-\infty}^{\infty} dx \right] \right\rangle$$
\[
\left[ \prod_{j=1}^{i} \left( \sum_{\beta_j \neq \gamma_j} \phi_{12}^{\beta_j}(x_j,0)\phi_{12}^{\gamma_j}(x_j,0) \right) \right] \left[ \prod_{(k<l)}^{i} \theta \left( |x_i - x_j| - \frac{a}{y} \right) \right] \right) \equiv y^{-(1-\epsilon)N} \sum_{i=M}^{\infty} \frac{1}{i!} (\Delta_0 y^\epsilon)^i I^{(N)}_i \left( \frac{a}{y} \right) \tag{5.1}
\]

In the 2nd line we have introduced a cutoff \( a \) to regulate the short-range divergences. In the 3rd line we have expanded out the effect of the defect perturbatively in \( \Delta_0 \), and used the conformal symmetry to rescale each expectation value by \( y \). In the lowest order term, \( I^{(N)}_N \), to get a nonzero expectation value, the 2N operators on the defect must lie in different replicas – so nothing special happens when two defect terms get close together, and we can drop the cutoff \( a \).

\[
I^{(N)}_N \left( \frac{a}{y} \right) = (2N)! \left[ \int_{-\infty}^{\infty} \frac{dx}{(x^2 + 1)^{1-\epsilon}} \right]^N = (2N)! \pi^N (1 + O(\frac{a}{y}, \epsilon)) \tag{5.2}
\]

We can now use the operator product expansion (OPE)

\[
\left[ \sum_{\alpha \neq \beta}^{n} \phi_{12}^\alpha(x+\delta,y=0)\phi_{12}^\beta(x+\delta,y=0) \right] \left[ \sum_{\alpha' \neq \beta'}^{n} \phi_{12}^{\alpha'}(x,y=0)\phi_{12}^{\beta'}(x,y=0) \right] \rightarrow 4(n-2) \delta^{1-\epsilon} \sum_{\alpha \neq \beta}^{n} \phi_{12}^\alpha(x,y=0)\phi_{12}^\beta(x,y=0) \quad \text{for} \quad \delta \to 0 \tag{5.3}
\]

to get the leading poles in \( \epsilon \) for the higher-order terms, by taking derivatives of \( I^{(N)}_i \left( \frac{a}{y} \right) \) with respect to \( \frac{a}{y} \). When we take derivatives of the step function \( \theta( |x_i - x_j| - \frac{a}{y} ) \), we bring two operators close together, and so can use the OPE. We get

\[
\frac{\partial I^{(N)}_i \left( \frac{a}{y} \right)}{\partial (a/y)} = -4i(i-1)(n-2)\left( \frac{a}{y} \right)^{-1+\epsilon} I^{(N)}_{i-1} \left( \frac{a}{y} \right) \quad \text{for} \quad i > N \tag{5.4}
\]

We already have \( I^{(N)}_N \), so we can now repeatedly integrate to get \( I^{(N)}_i \) for \( i > N \). \( I^{(N)}_i \) must be finite as \( \epsilon \to 0 \) for \( a \neq 0 \), because with a finite cutoff \( a \), there are no ultraviolet singularities in the integrals for \( I^{(N)}_i \) – this requirement determines all the constants of integration. So, for example, the leading term in \( I^{(N)}_{N+1} \left( \frac{a}{y} \right) \) is

\[
I^{(N)}_{N+1} \left( \frac{a}{y} \right) = -4N(N+1)(n-2)I^{(N)}_N(0) \left[ \frac{1}{\epsilon} \left( \frac{a}{y} \right)^\epsilon + \text{constant} \right] (1 + O(\frac{a}{y})) \tag{5.5}
\]

\[
= -4N(N+1)(n-2)I^{(N)}_N(0) \left( \frac{(\frac{a}{y})^\epsilon - 1}{\epsilon} \right) (1 + O(\frac{a}{y})) \tag{5.6}
\]

10
\[ -4N(N+1)(n-2)I_N^{(N)}(0) \log(\frac{a}{y}) \]  
(5.5)

(Note that this does diverge as \( a \to 0 \).) More generally, by repeatedly integrating Eq. (5.4) and using the requirement that all terms be finite as \( \epsilon \to 0 \) for \( a \neq 0 \), we find

\[
I_i^{(N)}(\frac{a}{y}) = \frac{i! (i-1)!}{(i-N)! (N-1)!} \left( -4(n-2) \log(\frac{a}{y}) \right)^{i-N} I_N^{(N)}(0), \quad i \geq N
\]  
(5.6)

Now taking the leading divergences of each term in Eq.(5.1) gives

\[
\langle \phi_{12}(y) \rangle^{2N} = \frac{\Delta_0 y^{-(1+\epsilon)N} I_N^{(N)}(0)}{N! (N-1)!} \sum_{i=N}^{\infty} \frac{(i-1)!}{(i-N)!} (-4(n-2) \log(\frac{a}{y}) y^i \Delta_0)^{i-N} \]  
(5.7)

In the front we have extracted a constant and the expected power-law dependence \( y^{-(1-\epsilon)N} = y^{-2h_{12}N} \). The remaining terms are written as a function \( F[a, y, \Delta_0] \) of the large distance \( y \) and the microscopic variables \( a \) and \( \Delta_0 \). However, we know from the Callen-Symanzik equation that this can be rewritten in terms of a single scaling function dependent only on the renormalized coupling for length scales of order \( y \), \( \Delta(\log(\frac{y}{a})) \) \[31\]. Explicitly,

\[
F[a, y, \Delta_0] = F[ae^{\ell}, y, \Delta(\ell)] = F[y, y, \Delta(\log(\frac{y}{a}))] = G[\Delta(\log(\frac{y}{a}))]
\]  
(5.8)

We rewrite \( \Delta_0 \) in terms of \( \Delta(\log(\frac{y}{a})) \) by integrating Eq.(3.3) to 1st order, getting

\[
\Delta_0 y^{\epsilon} = \Delta(\log(\frac{y}{a})) + 8 \log(\frac{y}{a}) (\Delta(\log(\frac{y}{a})))^2 + \mathcal{O}(\Delta(\log(\frac{y}{a})))^3 + \ldots
\]  
(5.9)

Substituting this into Eq.(5.7), we indeed get that the leading divergences at all orders of perturbation theory sum up to give a function which depends only on \( \Delta(\log(\frac{y}{a})) \):

\[
\frac{(2N)!}{N!} \left( \frac{\pi \Delta(\log(\frac{y}{a}))}{y^{2h_{12}}} \right)^N, \quad \langle \phi_{12}(y) \rangle^{2N+1} = 0
\]  
(5.10)

We see that the amplitudes of one-point functions are not self-averaging:

\[
\langle \phi_{12}(y) \rangle^{2N} \neq [\langle \phi_{12}(y) \rangle^{2N}]^N.
\]  
That is, the average of the \( N \text{th} \) power is different than the \( N \text{th} \) power of the average. The amplitude ratios are universal properties of the random defect fixed point.

Also note that

\[
\frac{1}{N_1} \log \left[ \langle \phi_{12}(y) \rangle^{2N_1} \right] < \frac{1}{N_2} \log \left[ \langle \phi_{12}(y) \rangle^{2N_2} \right] \text{ for } N_1 < N_2,
\]  
as required by convexity.
6 Ising Model

We now look at the defect introduced in Eq. (2.1) for the special case of the Ising model. As noted at the end of Section 2, the analysis in the three sections above fails when \( m = 3 \), because along with the perturbation in Eq. (2.5), we generate the marginal non-random operator \( \phi_{13} \). The minimal model with \( m = 3 \) corresponds to the Ising model, and in this model \( \phi_{12} \) is the spin operator (\( \sigma \)) and \( \phi_{13} \) is the energy operator (\( \epsilon \)). So in this case our defect consists of a random applied magnetic field and a non-random energy operator. Our action is

\[
S = S_{\text{Ising}} + \int_{-\infty}^{\infty} dx \left[ \Delta \sum_{\alpha \neq \beta} \sigma_{\alpha}(x,0)\sigma_{\beta}(x,0) + \lambda \sum_{\alpha=1}^{n} \epsilon_{\alpha}(x,0) \right] \tag{6.1}
\]

In the lattice formulation of the Ising model, perturbing with the energy operator is the same as changing the bond strength. Defects where the bond strength is changed along a single line have been studied and solved exactly by Bariev and McCoy et. al. \[32, 33\]. They dealt with cases where the bond strength was changed only in the bonds perpendicular to the defect (the ladder geometry – Figure 3), or only in the bonds parallel to the defect (the chain geometry – Figure 4). Our defect, for the Ising model, is thus a perturbation with a random magnetic field, of these exactly solved defects. The scaling dimension of the spin operator along the defect is \( x_{\sigma} = \frac{1}{2} g(\lambda)^2 \), where

\[
g(\lambda) = \frac{2}{\pi} \tan^{-1} \left( \frac{\sqrt{2} - 1}{\tanh(K_c + \lambda)} \right) \quad \text{for the ladder geometry,} \tag{6.2}
\]

and

\[
g(\lambda) = \frac{1}{\pi} \cos^{-1} (\tanh(2\lambda)) \quad \text{for the chain geometry} \tag{6.3}
\]

Here we use the results of Bariev and McCoy et. al. \[32, 33\], and along the defect have changed the bond coupling from \( K_c = \log\left(\frac{1}{2}(1 + \sqrt{2})\right) \) to \( K_c + \lambda \). This is not quite correct, because \( \lambda \) is the coefficient of the energy operator in the continuum formulation, while \( K = K_c + \lambda \) is the coupling in the lattice formulation. However, taking this difference into account will only give a \( \lambda \)-dependent rescaling of our renormalization group flows, which will not affect the qualitative results. Some subtleties regarding the branch of the arctangent in Eq.(6.2) for antiferromagnetic ladder couplings are dealt with in Appendix D.

We get the OPE coefficient \( b_{\sigma \sigma \epsilon} = -g(\lambda) \) from \[34, 35\] :

\[
\sigma(x - \frac{\delta}{2})\sigma(x + \frac{\delta}{2}) = \delta^{-g^2} \left[ 1 + \delta g \epsilon + \ldots \right] \tag{6.4}
\]

We can now get the renormalization group equations to 1-loop order solely from the OPE’s \[21, 31\] :
\[\frac{d\Delta}{d\ell} = (1 - g(\lambda)^2)\Delta - 8\Delta^2 \quad (6.5)\]

\[\frac{d\lambda}{d\ell} = -4g(\lambda)\Delta^2 \quad (6.6)\]

The flows for the ladder and chain cases are shown in Figures 5 and 6. The flows for the ladder case show that perturbations about the point \(\Delta = \lambda = 0\) (the defect-free point) eventually flow to the point with \((\Delta, \lambda) = (0, -K_c)\). This value of \(\lambda\) corresponds to vanishing bond strength along the ladder. So the renormalization group flow takes us to a point with two decoupled half-plane Ising models and no random magnetic field.

We can check our result by looking at the flows around the decoupled point. At this point the spins can be represented by free fermions, and we can calculate the renormalization group equations about this point. These results agree with the flows around the decoupled point obtained from Eq.(6.5) and Eq.(6.6). In making the comparison, it is important to note that in the replica formalism, disorder in the magnetic field corresponds to an operator \(\sum_{\alpha=1}^{\beta-1} \sigma_\alpha(x,0)\sigma_\beta(x,0)\), but in Eq.(6.1), \(\Delta\) is the coefficient of this operator with the \(\alpha = \beta\) terms removed. This means that \(\Delta\) is not really the strength of the random magnetic field, but a linear combination of the strength of the random magnetic field and the bond strength \((\lambda)\). So, in figure 5, adding a random magnetic field to the decoupled point, \((\Delta, \lambda) = (0, -K_c)\), moves us to a point with \(\Delta > 0\) and \(\lambda > -K_c\), and the renormalization group flows from this new point eventually go back to the original decoupled point.

The flows in the chain case take perturbations about the defect-free point to \((\Delta, \lambda) = (0, -\infty)\). Again, the random magnetic field has vanished at the new fixed point. To interpret this value of \(\lambda\), we look at the spin operator. The dimension of the spin operator at \(\lambda = -\infty\) is given by Eq.(6.3) as \(x_\sigma = \frac{1}{2}g(-\infty)^2 = \frac{1}{2}\), which is the same as the dimension of the spin operator along the edge of an Ising model with free boundary conditions. The possible boundary conditions of the Ising model have been completely classified \[36\] and the only boundary condition where the spin operator has dimension \(1/2\) is the free boundary condition. This makes sense physically, because \(\lambda = -\infty\) corresponds to an infinitely antiferromagnetic coupling that produces alternating spins along the defect, which upon coarse-graining gives net magnetization zero everywhere along the line. We conclude that in the chain case we also flow to two decoupled Ising models with free boundary conditions.

If is not hard to see that adding higher cumulant terms of the magnetic field to Eq.(6.1) will not change the qualitative results of our 1-loop renormalization group calculations.

So far, we have represented the perturbation in the energy operator as changing the bond strength either in the vertical or the horizontal direction, but not in both. More generally, we should represent the perturbation as changing bond strengths in both directions. However, analogously to the ladder and chain cases, we expect that a more isotropic treatment would only give
a different monotonically decreasing function $g(\lambda)$, and that the point with $(\Delta, \lambda) = (0, g^{-1}(1))$ would still be a stable fixed point with a large basin of attraction (including the defect-free model at $\Delta = \lambda = 0$). And by the classification of Ising model boundary states in [36], the point with $(\Delta, \lambda) = (0, g^{-1}(1))$ will always correspond to two decoupled half-plane Ising models with free boundary conditions and no random magnetic field.

7 Conclusions

We have found a new universality class of disordered defect lines. The defect lines exist in various two-dimensional Statistical Mechanical models, such as the Tricritical Ising model and Tricritical 3-state Potts model. The large-distance behavior of these defect lines has been shown to be disorder-dominated. Two-point functions along the defect exhibit multifractal behavior, and universal (normalized) amplitudes of one-point functions are non-self-averaging. We have also argued that when a random magnetic field is applied along a single line of the Ising model, it causes the two sides of the defect to decouple, and to turn into two half-plane Ising models with free boundary conditions and no disorder.

Results for the defect line in the $m^{th}$ Virasoro minimal model were obtained by a $1/m$ expansion. However, the physically most interesting models are at low $m$. For example, $m = 4$ corresponds to the Tricritical Ising model, with a random bond strength (or a random chemical potential) along a line. Our calculations show that this results in disorder-dominated long-distance behavior. It would be interesting to understand this model in a more fundamental and non-perturbative fashion. The random boundary/defect fixed points that we have found in this paper are, besides the bulk random bond $q$-state Potts models, a rare case where detailed analytic information about random critical behavior is available. In particular, it would be most interesting to compare our results for the random defect lines in minimal models with future numerical results, such as those obtained by Jacobsen and Cardy for the bulk random Potts models [22].

As a byproduct of our calculations for the defect line, we have also obtained multifractal energy-energy correlations in the bulk random bond $q$-state Potts model.

Finally, we comment on the R.G. analysis performed under the assumption of broken replica symmetry. Such a calculation was done in [22] for the $q$-state Potts model with bulk disorder in the bond strength, where it was found that the replica symmetric disordered fixed point is unstable to a new fixed point with broken replica symmetry. However, numerical tests show that the Potts model with random bonds is best described by the replica symmetric fixed point [38, 39]. An identical analysis to that of [34] for the random defect problem that is the subject of the present paper shows that, again, the fixed point considered in this paper is unstable, and flows to a new stable fixed point with broken replica symmetry. Thus, our random defect problem may provide further insights into the significance of the replica broken fixed point.
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A  Renormalization of $\Delta$

We calculate the renormalization of $\Delta$ to second order. The structure of the calculations closely parallels that in [27]. We get the coefficients in

$$\Delta(r) = r^\epsilon \left( \Delta_0 + A_2(r, \epsilon) \Delta_0^2 + (A_{31}(r, \epsilon) + A_{32}(r, \epsilon) + A_{33}(r, \epsilon)) \Delta_0^3 + \ldots \right)$$  \hspace{1cm} (A.1)

where, as explained below, the different terms come from the different types of contractions at each order of perturbation theory. All integrals are regularized at short distances by analytic continuation in $\epsilon \equiv \frac{3}{m+1}$ and at large distances by an infrared cutoff $r$.

A.1 First Order

To lowest order we bring two perturbation terms together

$$\int_{|x_1-x_2|<r} dx_1 dx_2 \sum_{\alpha \neq \beta} \phi_{12}^\alpha(x_1) \phi_{12}^\beta(x_1) \sum_{\gamma \neq \delta} \phi_{12}^\gamma(x_2) \phi_{12}^\delta(x_2)$$  \hspace{1cm} (A.2)

We get the same perturbation back again (and thus a contribution to $\Delta$) when $\beta = \gamma$ and $\alpha \neq \delta$. This gives us the $A_2$ term in Eq. (A.1)

$$A_2 = 2(n-2) \int_{|x_1-x_2|<r} dx_2 < \phi_{12}(x_1) \phi_{12}(x_2) >_0 = 4(n-2) \frac{r^\epsilon}{\epsilon}$$  \hspace{1cm} (A.3)

The 0 subscript on the correlator indicates that it is calculated in the defect-free theory.

A.2 Second Order

To second order we have

$$\int_{|x_1-x_2|<r} dx_1 dx_2 dx_3 \sum_{\alpha \neq \beta} \phi_{12}^\alpha(x_1) \phi_{12}^\beta(x_1) \sum_{\gamma \neq \delta} \phi_{12}^\gamma(x_2) \phi_{12}^\delta(x_2)$$  \hspace{1cm} (A.4)

This gives us several possible contractions. We get one possible contraction when $\beta = \gamma$, $\delta = \mu$, $\alpha \neq \delta$, $\beta \neq \nu$, $\alpha \neq \nu$:

$$A_{31} = 4(n-2)(n-3)\mathcal{I}_1,$$  \hspace{1cm} (A.5)
where we have defined

$$I_1 \equiv \int_{|x_2-x_1|<r} \int_{|x_3-x_1|<r} dx_2 \int_{|x_3-x_1|<r} dx_3 \phi_{12}(x_1)\phi_{12}(x_2) >0 < \phi_{12}(x_2)\phi_{12}(x_3) >0$$

$$= \int_{|x_2-x_1|<r} \int_{|x_3-x_1|<r} dx_2 \int_{|x_3-x_1|<r} dx_3 |x_1-x_2|^{-1+\epsilon} |x_2-x_3|^{-1+\epsilon}$$

$$= 2r^{2\epsilon} \int_0^1 dy |y|^{-1+2\epsilon} \int_{|z|<1/y} dz |z|^{-1+\epsilon} |1-z|^{-1+\epsilon} \tag{A.6}$$

We have transformed to coordinates $y = \frac{x_2-x_1}{r}$ and $z = \frac{x_3-x_1}{x_2-x_1}$. We can now extend the integral over $z$ to go from $-\infty$ to $\infty$, since this will only change $I_1$ to $O(\epsilon^0)$. This gives us two integrals which we can do exactly:

$$I_1 = 2r^{2\epsilon} \left( \frac{1}{2\epsilon} \right) \left( \frac{2^{1-2\epsilon}}{\sqrt{\pi}} \right) (1 + \cos \pi \epsilon) \Gamma(\frac{1}{2} - \epsilon) + O(\epsilon^0)$$

$$= \frac{4r^{2\epsilon}}{\epsilon^2} + O(\epsilon^0) \tag{A.7}$$

Note that in the integrals we have picked out one point $x_1$ as special, and integrated over $x_2$ and $x_3$, but in the combinatorial factor $4(n-2)(n-3)$ we have treated all points symmetrically (i.e., have treated $\alpha = \delta$, $\beta = \mu$ the same as $\beta = \gamma$, $\delta = \mu$). This is permissible to this order in perturbation theory, since we are only concerned with the poles in $\epsilon$, which result from short range divergences and are the same for any permutation of the contractions. We can see this explicitly by considering a different arrangement of the same contractions:

$$I'_1 \equiv \int_{|x_2-x_1|<r} \int_{|x_3-x_1|<r} dx_2 \int_{|x_3-x_1|<r} dx_3 \phi_{12}(x_1)\phi_{12}(x_2) >0 < \phi_{12}(x_1)\phi_{12}(x_3) >0$$

$$= 4 \frac{r^{2\epsilon}}{\epsilon^2} = I_1 + O(\epsilon^0) \tag{A.8}$$

We get a second possible set of contractions when $\alpha = \gamma = \mu$ and $\delta = \nu \neq \beta$:

$$A_{32} = 4(n-2)I_2 \tag{A.9}$$

where we have defined

$$I_2 = \int_{|x_2-x_1|<r} \int_{|x_3-x_1|<r} dx_2 \int_{|x_3-x_1|<r} dx_3 \left[ < \phi_{12}(x_1)\phi_{12}(x_2)\phi_{12}(x_3)\phi_{12}(\infty) >0 \times \right.$$

$$\left. < \phi_{12}(x_2)\phi_{12}(x_3) >0 - ( < \phi_{12}(x_2)\phi_{12}(x_3) >0)^2 \right] \tag{A.10}$$
The four-point function gives the coefficient of three \( \phi_{12} \)'s projecting to a single \( \phi_{12} \). We have subtracted off \(< \phi_{12}(x_2)\phi_{12}(x_3) >_0 \)^2 – this term corresponds to the contribution from two perturbation terms (disorder operators) getting close, and does not affect the dimension of the operator at \( x_1 \). This term is only a contribution to the free energy, so subtracting it off simply corresponds to normalizing the correlation functions. We get the four-point function from the Coulomb gas formalism \[40, 41\]:

\[
< \phi_{12}(0)\phi_{12}(1)\phi_{12}(z)\phi_{12}(\infty) >_0 = |z|^{-1+\epsilon} |1-z|^{1-\epsilon/3} \left[ \frac{3k_1}{4} |z|^{2-\epsilon/3} F(1 - \frac{\epsilon}{3}, 2 - \epsilon; 2 - \frac{2\epsilon}{3}; z) |^2 + |F(1 - \frac{\epsilon}{3}, 1 - \frac{\epsilon}{3}, 1 - \frac{2\epsilon}{3}; z) |^2 \right] \tag{A.11}
\]

where we have defined

\[
k_1 \equiv -\frac{4(\Gamma(\frac{2\epsilon}{3}))^2 \Gamma(2 - \epsilon) \Gamma(1 - \frac{\epsilon}{3})}{3 \Gamma(\frac{\epsilon}{3}) \Gamma(-1 + \epsilon) \Gamma(2 - \frac{2\epsilon}{3})} = 1 + \mathcal{O}(\epsilon) \tag{A.12}
\]

and \( F \) is the hypergeometric function. If we take the limit as \( \epsilon \to 0 \) at fixed \( z \) for the four-point function, we get

\[
< \phi_{12}(0)\phi_{12}(1)\phi_{12}(z)\phi_{12}(\infty) >_0 \xrightarrow{\epsilon \to 0} |\text{sign}(z)\text{sign}(z-1) + \frac{\text{sign}(z)}{|z-1|} - \frac{\text{sign}(z-1)}{|z|}| \tag{A.13}
\]

We now go back to \( \mathcal{I}_2 \), which we can calculate by using the symmetries \( x_2 \to -x_2 \) and \( x_2 \leftrightarrow x_3 \) to cut the integration region down by a fourth, and then transforming to new coordinates \( y = \frac{x_2 - x_1}{r} \) and \( z = \frac{x_3 - x_1}{x_2 - x_1} \). The integral now exactly factorizes into two one-dimensional integrals:

\[
\mathcal{I}_2 = 4\pi^2 \int_0^1 dy \ y^{-1+2\epsilon} \int_{-1}^1 dz \left[ < \phi_{12}(0)\phi_{12}(1)\phi_{12}(z)\phi_{12}(\infty) >_0 \times < \phi_{12}(1)\phi_{12}(z) >_0 - (< \phi_{12}(1)\phi_{12}(z) >_0)^2 \right] \tag{A.14}
\]

The \( y \)-integral gives \( \frac{1}{2\pi} \), while the \( z \)-integral can be rewritten as

\[
\int_{-1}^1 dz \left[ < \phi_{12}(0)\phi_{12}(1)\phi_{12}(z)\phi_{12}(\infty) >_0 < \phi_{12}(1)\phi_{12}(z) >_0 - < \phi_{12}(0)\phi_{12}(z) >_0 - (< \phi_{12}(1)\phi_{12}(z) >_0)^2 \right] + \int_{-1}^1 dz < \phi_{12}(0)\phi_{12}(z) >_0 \tag{A.15}
\]

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The second integral is exactly $\frac{2}{\epsilon}$. It is straightforward to use Eq. (A.11) to check that as $z \to 1$, the four-point function goes to $|z - 1|^{-1+\epsilon} + O(z - 1)$. So the first integral converges everywhere, and we can get its value to $O(\epsilon^0)$ by simply replacing $\epsilon$ with zero inside the integral:

$$
\int_{-1}^{1} dz \left[ \frac{1}{|z - 1|} \text{sign}(z) \text{sign}(z - 1) + \frac{\text{sign}(z)}{|z - 1|} - \frac{1}{|z|} - \frac{1}{|z - 1|^2} \right] + O(\epsilon) = -1 + O(\epsilon) \tag{A.16}
$$

Putting this all together gives

$$
I_2 = 4 \pi^2 \left( \frac{1}{2\epsilon} \right) \left[ \frac{2}{\epsilon} - 1 + O(\epsilon) \right] = -2 \pi^2 \left( \frac{1}{\epsilon} - \frac{2}{\epsilon^2} \right) \tag{A.17}
$$

Again, as with $A_{31}$, other possible permutations of this contraction like

$$
I'_2 = \int dx_2 \int dx_3 <\phi_{12}(x_1)\phi_{12}(x_2)\phi_{12}(x_3)\phi_{12}(\infty)>_0 \quad <\phi_{12}(x_1)\phi_{12}(x_2)>_0 \tag{A.18}
$$

give the same result up to terms of $O(\epsilon^0)$.

Finally, the last possible contraction in Eq. (A.4) comes from $\alpha = \gamma = \mu$ and $\beta = \delta = \nu$. This is

$$
A_{33} = \frac{4}{3} \int dx_2 \int dx_3 \left[ (\phi_{12}(x_1)\phi_{12}(x_2)\phi_{12}(x_3)\phi_{12}(\infty)>_0)^2 - (\phi_{12}(x_2)\phi_{12}(x_3)>_0)^2 \right] = 16 \pi^2 \int_0^1 dy \int_{-1}^{1} dz \left[ (\phi_{12}(0)\phi_{12}(1)\phi_{12}(z)\phi_{12}(\infty)>_0)^2 - (\phi_{12}(1)\phi_{12}(z)>_0)^2 \right] \tag{A.19}
$$

The $y$-integral gives $\frac{2}{\epsilon}$. The $z$-integral is evaluated similarly to Eq. (A.15). We rewrite it as

$$
\int_{-1}^{1} dz \left[ (\phi_{12}(0)\phi_{12}(1)\phi_{12}(z)\phi_{12}(\infty)>_0)^2 \right]
$$

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\[
- (\langle \phi_{12}(0) \phi_{12}(z) \rangle_0^2 - (\langle \phi_{12}(1) \phi_{12}(z) \rangle_0^2 - 1) + \\
\int_{-1}^{1} dz \left[ 1 + (\langle \phi_{12}(0) \phi_{12}(z) \rangle_0^2 \right]
\]

(A.20)

The second integral can be evaluated exactly, and is \( O(\epsilon) \). The first integral is nowhere divergent, so we can get its value to \( O(\epsilon^0) \) by taking \( \epsilon \) to 0 inside the integral. Then squaring Eq.(A.13) gives

\[
(\langle \phi_{12}(0) \phi_{12}(1) \phi_{12}(z) \phi_{12}(\infty) \rangle_0^2 \rightarrow 0 \left( \text{sign}(z) \text{sign}(z - 1) + \frac{\text{sign}(z)}{|z - 1|} - \frac{\text{sign}(z - 1)}{|z|} \right)^2 \\
= 1 + \frac{1}{z^2} + \frac{1}{(z - 1)^2}
\]

(A.21)

So both \( z \)-integrals are \( O(\epsilon) \), and \( A_{33} \) is \( O(\epsilon^0) \). Since \( A_{33} \) has no pole in \( \epsilon \), it can be dropped to this order. Combining all these results gives Eq.(3.2).
B  Renormalization of $\Delta_{NMn}$

To get the dimensions of the moments of $\phi_{12}$, we need the coefficients of $\Delta_0$ and $\Delta_2^0$ in $Z_{NMn}(r)$, where $\Delta_{NMn}(r) = Z_{NMn}(r)\Delta_{NMn,0}$. The integrals in this section will be the same as the integrals in appendix A. The only difference will be in the combinatorial factors which precede them. In this section we will use $\langle N, M, n |$ and $| N, M, n \rangle$ to represent the bra and ket forms of the operator $O_{NMn}$ defined in Eq.(4.2). However, before we start, we should make sure that no problems occur with mixing, either for the defect line considered in this paper, or for the parallel calculation in the q-state Potts model with bulk disorder in the bond strength.

B.1  Mixing With Other Operators

The combinations $O_{NMn}$ of Eq.(4.2) diagonalize the operators of the form $\prod \phi_{12}$. However, we also need to check that these operators don’t mix with other operators that have the same scaling dimensions at $m = \infty$. We only need to consider operators of the form $\phi_{1q}$, because these operators form a closed subalgebra in the minimal models. To 2 loops, only 4 $\phi_{12}$ operators can be affected by the perturbation, and $4 \dim(\phi_{12}) = 2$ at $m = \infty$. It easy to see that the only operator, or descendent of an operator, or combination of operators, in the $\phi_{1q}$ subalgebra with dimension 2, is $\phi_{13}$ (therefore, to this order, no mixing occurs with derivative operators). $\phi_{13}$ does in fact mix with $(\phi_{12})^4$, and so $\phi_{13}(\phi_{12})^{N-4}$ mixes with $(\phi_{12})^N$. However, the 2-loop overlap integral is

$$\int_{|x_2-x_1|<r} dx_2 \int_{|x_3-x_1|<r} dx_3 \langle \phi_{12}(x_1)\phi_{12}(x_2)\phi_{13}(\infty) \rangle_0 >0$$

This has no poles in $\epsilon$, so we would only need to take this mixing into account to do 3-loop calculations.

Mixing is potentially more of a problem when we use these calculations to get moments of the energy operator in the q-state Potts model with bulk bond disorder – this model is analyzed by expanding about the Ising Model [27, 29], so we need to look for mixing with other operators that have the same scaling dimension at $q = 2$. Denoting the energy operator in the $\alpha$ replica by $\varepsilon^\alpha$, we see that $\varepsilon^\alpha\varepsilon^\beta\varepsilon^\gamma$ and $\nabla^2\varepsilon^\alpha$ have the same dimension at $q = 2$, and that in general we can replace any two $\varepsilon$’s with a $\nabla^2$ without changing the dimension. These operators mix to two loops, with overlap integrals such as

\begin{align*}
\int_{|x_2-x_1|<r} dx_2 \int_{|x_3-x_1|<r} dx_3 < \phi_{12}(x_1)\phi_{12}(x_2) >0 \\
< \phi_{12}(x_1)\phi_{12}(x_3) >0 < \phi_{12}(x_1)\phi_{12}(x_3) >0^2 (B.1)
\end{align*}
\[ \int d^2 x_2 \int d^2 x_3 \varepsilon^\alpha \varepsilon^\beta \varepsilon^\gamma (0) \varepsilon^\alpha \varepsilon^\beta (x_2) \varepsilon^\alpha \varepsilon^\gamma (x_3) \nabla^2 \varepsilon^\alpha (\infty) \]

\[ = \int d^2 x_2 d^2 x_3 < \varepsilon (0) \varepsilon (x_2) \varepsilon (x_3) \nabla^2 \varepsilon (\infty) >_0 \]

\[ < \varepsilon (0) \varepsilon (x_2) >_0 < \varepsilon (0) \varepsilon (x_3) >_0 \quad (B.2) \]

This means at the combinations \( \mathcal{O}_{NMn} \) of Eq.(4.2) (with \( \phi_{12} \) replaced by \( \varepsilon \)) are no longer generally multiplicatively renormalizable, but mix with derivatives of \( \mathcal{O}_{N-2,M,n} \). Luckily, however, no problem with mixing occurs for the operators \( \mathcal{O}_{N,N,n} \) and \( \mathcal{O}_{N,N-1,n} \), which we know from the one-loop calculations to provide the leading and subleading decay exponents. No problem with mixing occurs for \( \mathcal{O}_{N,N,n} \) because the overlap

\[ \sum_{\alpha \neq \beta} \varepsilon^\alpha \varepsilon^\beta \sum_{\gamma \neq \delta} \varepsilon^\gamma \varepsilon^\delta |N, N, n > \rightarrow \nabla^2 |N - 2, M, n > \quad (B.3) \]

is 0 for all \( M \), where the \( \nabla^2 \) is understood as acting on only one of the \( \varepsilon \)'s. It is not hard to see this by counting the connected contractions of

\[ < N, N, n | \sum_{\alpha \neq \beta} \varepsilon^\alpha \varepsilon^\beta \sum_{\gamma \neq \delta} \varepsilon^\gamma \varepsilon^\delta |N - 2, M, n > \quad (B.4) \]

If we expand \(|N - 2, M, n >\) into monomials, each monomial term gives a contraction of 0 — this is because after contracting all the \( \varepsilon \)'s in the monomial term, we will be left with at least two antisymmetric terms from the \(< N, N, n >\), and thus positive and negative contractions will cancel. So the contractions give a total of 0 (even before the replica limit \( n \rightarrow 0 \) is taken). This is true even if the \(|N - 2, M, n >\) has a \( \nabla^2 \) on it, and even if \(< N, N, n >\) is replaced with \(< N, N - 1, n >\). So mixing is not a problem for the \( \mathcal{O}_{N,N,n} \) and \( \mathcal{O}_{N,N-1,n} \) operators in the q-state Potts model with bulk bond disorder.

**B.2 First Order**

Returning to our defect model, the 1st order correction comes from

\[ \left( \sum_{\alpha \neq \beta} \phi_{12}^\alpha \phi_{12}^\beta \right) |N, M, n > \rightarrow \tilde{b}_{NMn} |N, M, n > \quad (B.5) \]

where we contract one \( \phi_{12} \) operator in the \( \sum_{\alpha \neq \beta} \phi_{12}^\alpha \phi_{12}^\beta \) with one \( \phi_{12} \) operator in the \(|N, M, n >\). The integral is the same integral done in Eq.(A.3), giving \( 2^{c''} \). We can get the combinatorial factor \( \tilde{b}_{NMn} \) by contracting with \( \langle N, M, n \rangle \) on both sides:

\[ 22 \]
\[ \hat{b}_{NMn} = \frac{\langle N, M, n | \sum_{\alpha \neq \beta}^{n} \phi^{\alpha}_{12} \phi^{\beta}_{12} | N, M, n \rangle}{\langle N, M, n | M, N, n \rangle} \quad (B.6) \]

This combinatorial factor is found in subappendix C.2. Leaving our result in terms of \( \hat{b}_{NMn} \), the order \( \Delta_0 \) term in \( Z_{NMn} \) is

\[ 2\hat{b}_{NMn} \left( \frac{r^4}{e} \right) \Delta_0 \quad (B.7) \]

## B.3 Second Order

To second order, we need to look at the number of ways that we can take

\[ < N, M, n | \mu \sum_{\alpha \neq \beta}^{n} \phi^{\alpha}_{12} \phi^{\beta}_{12} \sum_{\gamma \neq \delta}^{n} \phi^{\gamma}_{12} \phi^{\delta}_{12} \rightarrow < N, M, n | \nu \quad (B.8) \]

The index \( \mu \) in \( < N, M, n | \mu \) is a label used to make discussion easier, and does not signify an independent variable. The \( \mu \) simply indicates that for discussing contractions, the dummy indices \( \alpha_i \) in Eq.(4.2) will all be called \( \mu_i \) for \( i = 1, 2 \ldots n \).

There are now a number of possible contractions to consider. The one with \( \mu_i = \alpha = \gamma = \nu_j \) and \( \mu_k = \beta = \delta = \nu_l \) (for some \( i, j, k, l \)) gives an integral over a squared four-point function. We did this integral for \( A_{33} \) in appendix A and found that it had no singularities. So we can ignore this term.

We now consider the contraction with \( \mu_i = \alpha = \gamma = \nu_j \) and \( \beta = \delta \), which gives a combinatorial factor that we call \( Q^{(2)}_{NMn} \). It is to be understood that no replica indices other than the specified ones are equal to each other. In both cases, we get the same integral as in Eq.(A.10) (or Eq.(A.18)), which was found to be \( I_2 = -2r^{22}(\frac{1}{2} - \frac{3}{2}e) \).

To get the combinatorial factor for \( Q^{(2)}_{NMn} \), we look at

\[ \mu < N, M, n | \sum_{\alpha \neq \beta}^{n} \phi^{\alpha}_{12} \phi^{\beta}_{12} \sum_{\gamma \neq \delta}^{n} \phi^{\gamma}_{12} \phi^{\delta}_{12} | N, M, n > \quad (B.9) \]

and only allow the contractions described above. For the contractions allowed in \( Q^{(2)}_{NMn} \), the replica indices that appear in a monomial term must be the same in the bra and ket sides. The \( \alpha = \gamma \) replica index can be the same as any of the these \( N \) monomial terms. The \( \beta = \delta \) replica index must be something different, so has \( (n - N) \) choices. We also get an overall factor of 4 for the different pairs of terms that we could have chosen to contract. So \( Q^{(2)}_{NMn} = 4N(n - N) \).

\( Q^{(3)}_{NMn} \) can be calculated similarly. The \( \alpha \) index contracts to the left and the \( \delta \) index contracts to the right – this gives a combinatorial factor \( < N, M, n | \sum_{\alpha \neq \beta}^{n} \phi^{\alpha}_{12} \phi^{\beta}_{12} | N, M, n > = \hat{b}_{NMn} < N, M, n | N, M, n > \).

The \( \beta = \gamma \) index can then be any of the \( (N - 1) \) replica indices in the right (or left) \( | N, M, n > \).
(excluding the two which are equal to \( \alpha \) or \( \beta \)). And there is again an overall factor of 4, so \( Q_{NMn}^{(3)} = 4(N - 1) \tilde{b}_{NMn} \).

Now consider the contraction with \( \mu_i = \alpha, \beta = \gamma \) and \( \delta = \nu_j \), which gives a combinatorial factor \( Q_{NMn}^{(4)} \), and the contraction with \( \mu_i = \alpha, \nu_i = \beta, \nu_j = \delta \), which gives a combinatorial factor \( Q_{NMn}^{(5)} \). Both of these terms give the same integral as in Eq.(A.6) (or Eq.(A.8)), and thus a factor of \( I_1 = 4 \frac{r^2}{\epsilon} \).

\( Q_{NMn}^{(4)} \) can be evaluated in the same manner as \( Q_{NMn}^{(3)} \), giving \( Q_{NMn}^{(4)} = 4(n - N - 1) \tilde{b}_{NMn} \). The term \( Q_{NMn}^{(5)} \) is more complicated and is found in subsection C.3 to be \( Q_{NMn}^{(5)} = -4N(n - N) + 2(n - 2) \tilde{b}_{NMn} + (\tilde{b}_{NMn})^2 \). Putting this all together, we get the result

\[
Z_{NMn} = 1 + 2 \tilde{b}_{NMn} \frac{r^2}{\epsilon} \Delta_0 + \frac{1}{2} (Q_{NMn}^{(2)} + Q_{NMn}^{(3)}) (-2 \frac{r^2}{\epsilon} + 4 \frac{r^2}{\epsilon^2}) + \frac{1}{2} (Q_{NMn}^{(4)} + Q_{NMn}^{(5)}) (4 \frac{r^2}{\epsilon^2}) + O(\Delta_0^3)
\]

\[
= 1 + 2 \tilde{b}_{NMn} \frac{r^2}{\epsilon} \Delta_0 - 4 \left( N(n - N) + (N - 1) \tilde{b}_{NMn} \right) \frac{r^2}{\epsilon} \Delta_0^2
\]

\[
+ \left( 2(\tilde{b}_{NMn})^2 + 4(n - 2) \tilde{b}_{NMn} \right) \frac{r^2}{\epsilon^2} \Delta_0^2 + O(\Delta_0^3)
\]  

(B.10)
C Combinatorial Factors

In this appendix we calculate the combinatorial factors used in appendix B. We want the number of possible contractions between the irreducible representations of $S_N$, given by $\mathcal{O}_{N,M,n}$ in Eq. (4.2), and copies of the disorder operator. The trivial spatial dependence is suppressed in the equations below.

C.1 Normalization

First, we calculate the normalization of $|N,M,n>$

\[ A[N,M,n] \equiv \langle N,M,n | N,M,n \rangle = \sum_{\alpha_i \neq \alpha_j} \sum_{1 \leq \alpha_i \leq (n-M)} \sum_{1 \leq \beta_i \leq (n-M)} \langle \phi_{12}^{\alpha_1} - \phi_{12}^{\alpha} \cdots \phi_{12}^{\alpha_{M-1}} \phi_{12}^{\alpha + 1} \cdots \phi_{12}^{\alpha_n} | \phi_{12}^{\beta_1} - \phi_{12}^{\beta} \cdots \phi_{12}^{\beta_{M-1}} \phi_{12}^{\beta + 1} \cdots \phi_{12}^{\beta_n} \rangle \] (C.1)

If we expand out the two terms ($\phi_{12}^{\alpha_1} - \phi_{12}^{\alpha}$) and ($\phi_{12}^{\beta_1} - \phi_{12}^{\beta}$) into monomials, the cross terms are 0, and the remaining terms are of the same form $A[\cdot]$ as before, but with different values of $M$, $N$ and $n$. We get

\[ A[N,M,n] = A[N,M-1,n-1] + (n-M-N+1)^2 A[N-1,M-1,n-1] \] (C.2)

We can calculate $A[N,M,n]$ explicitly when $M = 0$:

\[ A[N,0,n] = \sum_{\alpha_i \neq \alpha_j} \sum_{\beta_i \neq \beta_j} \langle \phi_{12}^{\alpha_1} \phi_{12}^{\alpha_2} \cdots \phi_{12}^{\alpha_n} | \phi_{12}^{\beta_1} \phi_{12}^{\beta_2} \cdots \phi_{12}^{\beta_n} \rangle = N! \prod_{i=1}^{N} (n-N+i) \] (C.3)

Here the $N!$ comes from the number of ways to contract the left side with the right side, and the factors of $n-N+i$ come from the different ways to pick the $\alpha_i$ once the contractions have been done. Given the value of $A[N,M,n]$ when $M = 0$, and the recursion relation above, we can show by induction that the general solution is

\[ A[N,M,n] = (N-M)! \prod_{i=1}^{N} (n-M-N+i) \prod_{i=1}^{M} (n-2M+1+i) \] (C.4)
C.2 OPE – 1st order term

Here we want to calculate

\[ B[N, M, n] \equiv N, M, n \mid \sum_{\alpha \neq \beta} \phi_{12}^\alpha \phi_{12}^\beta \mid N, M, n > \quad (C.5) \]

Note that three-point functions of \( \phi_{12} \) are 0 by the \( \phi_{12} \rightarrow -\phi_{12} \) duality symmetry. As before, we expand out the \( (\phi_{12}^{\alpha M} - \phi_{12}^{n-(M-1)}) \) and \( (\phi_{12}^{\beta M} - \phi_{12}^{n-(M-1)}) \) terms into monomials. The direct terms again give contractions of the form \( B[...] \), but we now have a cross term in which either \( \alpha \) or \( \beta \) must be equal to \( n - (M - 1) \). We get

\[
\frac{1}{2} B[N, M, n] = \frac{1}{2} B[N, M - 1, n - 1] \\
+ (n - M - N + 1)^2 \frac{1}{2} B[N - 1, M - 1, n - 1] \\
- 2(n - M - N + 1) \sum_{1 \leq \alpha, \beta \leq (n - M)} \sum_{\beta_i \neq \beta} \sum_{\alpha = 1}^{\beta_i} \mid (\phi_{12}^{\alpha M} - \phi_{12}^{n-M}) \phi_{12}^{\alpha M} \phi_{12}^{\beta M+1} \phi_{12}^{\alpha N} \mid \phi_{12}^{\beta_i} \mid \\
- 2(n - M - N + 1) \sum_{\beta_i \neq \beta} \sum_{\alpha = 1}^{\beta_i} \mid (\phi_{12}^{\beta_1} - \phi_{12}^{\beta_2}) \phi_{12}^{\beta M-1} \phi_{12}^{n-M-2} \phi_{12}^{\beta M+1} \phi_{12}^{\beta N} \mid \phi_{12}^{\beta_i} > \quad (C.6)
\]

The contraction in the last term looks just like the contraction \( A[N, M - 1, n - 1] \), with \( \alpha \) taking the place of \( \beta_M \). The only difference between the sum above and \( A[N, M - 1, n - 1] \), is that the sum over \( \alpha \) above includes terms with \( (n - M + 2) \leq \alpha \leq n \), and terms where \( \alpha \) is the same as some other term \( \beta_i \), for \( 1 \leq i \leq (M - 1) \). However, these two extra contributions are equal and opposite, being equal to \( \pm (M - 1)(n - M - N + 1)A[N - 1, M - 2, n - 2] \). So they cancel, and we have

\[
\frac{1}{2} B[N, M, n] = \frac{1}{2} B[N, M - 1, n - 1] + \\
(n - M - N + 1)^2 \frac{1}{2} B[N - 1, M - 1, n - 1] - \\
2(n - M - N + 1)A[N, M - 1, n - 1] \quad (C.7)
\]

As with \( A[N, 0, n] \), it is easy to calculate \( B[N, 0, n] \):

\[
B[N, 0, n] = 2N(N!) \prod_{i=0}^{N} [(n - N + i)] = 2N(n - N)A[N, 0, n] \quad (C.8)
\]

Given the recursion relation for \( B[N, M, n] \), the initial condition \( B[N, 0, n] \), and the result for \( A[N, M, n] \) in the previous section, we can show by induction that
\[ \tilde{b}_{NMn} \equiv \frac{B[N, M, n]}{A[N, M, n]} = 2((N - M)n - N^2 + M(M - 1)) \quad (C.9) \]

### C.3 2nd order term

We want to calculate

\[ C[M, N, n] \equiv 4 < N, M, n \mid \sum_{\alpha \neq \beta} \phi_{12}^{\alpha} \phi_{12}^{\beta} \sum_{\gamma \neq \delta} \phi_{12}^{\gamma} \phi_{12}^{\delta} \mid N, M, n > \quad (C.10) \]

where we require that \( \alpha \) and \( \gamma \) contract to the left, the \( \beta \) and \( \delta \) contract to the right, and \( \alpha, \beta, \gamma, \delta \) are all distinct from one another (other possible directions of contractions give the factor of 4 in front). This combinatorial factor arises in the contraction for \( Q^{(5)}_{NMn} \) in subappendix B.3. As in the previous subsection, expanding out the \( (\phi_{12}^{\alpha} - \phi_{12}^{n-(M-1)}) \) and \( (\phi_{12}^{\beta} - \phi_{12}^{n-(M-1)}) \) terms into monomials gives back two terms of the form \( C[\ldots] \), and a more complicated cross-term. The cross term almost has the same form as \( B[N, M - 1, n - 1] \), but we also get some extra terms because the sums over \( \alpha, \beta, \gamma, \delta \) are different than we would have in a \( B[\ldots] \) term. Counting all the ways in which our cross term differs from \( B[N, M - 1, n - 1] \) is tedious but straightforward, and we get

\[ \frac{1}{2} B[N, M - 1, n - 1] - (M - 1)(n - M - N + 1)^2 A[N - 1, M - 2, n - 2] \]
\[ - (M - 1)A[N, M - 2, n - 2] - (n - M - N)A[N, M - 1, n - 1] \]
\[ = \frac{1}{2} B[N, M - 1, n - 1] - (n - N - 1)A[N, M - 1, n - 1] \quad (C.11) \]

The recursion relation is

\[ \frac{1}{4} C[N, M, n] = \frac{1}{4} C[N, M - 1, n - 1] + \]
\[ (n - M - N + 1)^2 \frac{1}{4} C[N - 1, M - 1, n - 1] - 4(n - M - N + 1) \]
\[ \left\{ \frac{1}{2} B[N, M - 1, n - 1] - (n - N - 1)A[N, M - 1, n - 1] \right\} \quad (C.12) \]

Combined with the initial condition,

\[ C[N, 0, n] = 4N(N - 1)(n - N)(n - N - 1)A[N, 0, n], \quad (C.13) \]

we can find the value of \( C[N, M, n] \) for all \( M \) and \( N \) by induction. The result is

\[ Q^{(5)}_{NMn} \equiv \frac{C[N, M, n]}{A[N, M, n]} = -4N(n - N) - 2(n - 2)\tilde{b}_{NMn} + \left( \tilde{b}_{NMn} \right)^2 \quad (C.14) \]

where \( \tilde{b}_{NMn} \) was defined in Eq. (C.9).
D Ising Ladder Defect for $K < 0$

The branch of the arctangent used in Eq.(5.2) for antiferromagnetic ($K = K_c + \lambda < 0$) ladder couplings requires some explanation. We take the value of the arctangent to be in $(0, \frac{\pi}{2})$ if its argument is positive (i.e. $K > 0$) and to be in $(\frac{\pi}{2}, \pi)$ if its argument is negative (i.e. $K < 0$). This makes the slope of $g(\lambda)$ continuous through $K = 0$. On the other hand, the results of [32, 33] have a slope discontinuity at $K = 0$, and have $g$ symmetric under $K \to -K$. This slope discontinuity results from a level crossing in the lowest scaling dimension for operators on the boundary [36]. If we let $\sigma_t$ be the spin on one side of the defect, and $\sigma_b$ the spin on the other side, we see that the operator with the lowest scaling dimension changes from $\sigma_t + \sigma_b$ to $\sigma_t - \sigma_b$ as $K$ goes through 0, so that while the dimension of each operator changes smoothly through $K = 0$, the dimension of the lowest scaling operator does not. However, if we take our random applied magnetic field to not vary across the defect, it couples to $\sigma_t + \sigma_b$ only, and we want the lowest scaling dimension for $K > 0$, but the 2nd lowest scaling dimension for $K < 0$. We can get these from [36], thus justifying the branches of arctangent chosen above.

Note that if we had used the other branch of the arctangent, corresponding to a magnetic field uncorrelated across the defect, the flow picture in the ladder case would have been symmetric under $K \to -K$, and the entire $(\Delta, \lambda)$ plane would have flowed into the decoupled point.
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Figure Captions

Fig. 1. A two-point correlation function for operators lying along the defect.

Fig. 2. A one-point function for an operator in the bulk.

Fig. 3. The ladder defect in the Ising model.

Fig. 4. The chain defect in the Ising model.

Fig. 5. Renormalization Group flows for the ladder defect in the Ising model. $\Delta$ is the strength of the disordered magnetic field along the defect line, and $\lambda$ is the bond strength along the line.

Fig. 6. Renormalization Group flows for the chain defect in the Ising model.
