Boson-fermion mapping and dynamical supersymmetry in fermion models

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Abstract

We show that a dynamical supersymmetry can appear in a purely fermionic system. This “supersymmetry without bosons” is constructed by application of a recently introduced boson-fermion Dyson mapping from a fermion space to a space comprised of collective bosons and ideal fermions. In some algebraic fermion models of nuclear structure, particular Hamiltonians may lead to collective spectra of even and odd nuclei that can be unified using the dynamical supersymmetry concept with Pauli correlations exactly taken into account.

I. INTRODUCTION

In its original inception supersymmetry pertains to a system of fermions and bosons exhibiting an invariance with respect to exchange between these two classes of particles. It may therefore be somewhat surprising to discover that a fermion system on its own can also exhibit dynamical supersymmetry. States with even and odd fermion numbers are then unified in a single representation of a supergroup. This is discussed in general below and demonstrated for a specific nuclear model.

States of even and odd nuclei are in principle eigenstates of the same Hamiltonian obtained for different particle numbers. Although there is no fundamental difference between even and odd nuclei from this point of view, their properties are, however, quite different. A unification of spectra of even and odd nuclei into a single framework is therefore a challenging possibility, with the prospect of unveiling a basic underlying symmetry.

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The notion of supersymmetry has in fact proved to be fruitful in nuclear structure physics \[1\]. Properties of some neighboring even and odd nuclei can namely be classified and understood in terms of an assumed supersymmetry within the framework of the interacting boson-fermion model (IBFM) \[2\]. Although this phenomenological supersymmetry does not necessarily imply dynamical supersymmetry on the microscopic level, the IBFM \[2\] does achieve a unification of even and odd states on the phenomenological level. Starting from a common boson-fermion Hamiltonian one finds that in some instances states of an even nucleus, described by many-boson wave functions, are linked by supersymmetry to states in a neighboring odd nucleus in which the odd fermion is treated explicitly. It is important to realize, however, that Pauli correlations between the odd particle and even core are not fully taken into account in the IBFM. In this sense the link between observed supersymmetry and an underlying microscopy is tenuous and no detailed microscopic understanding of the IBFM hypothesis in fact exists so far.

In this paper we investigate the link between dynamical supersymmetry and some fermion algebraic models. For this purpose we apply the recently developed Dyson boson-fermion mapping \[3\] which has the key property of mapping an even fermion system (comprised of collective pairs) to a purely boson system, and at the same time an odd fermion system (comprised of collective pairs and an odd fermion) to a system of collective bosons and a single ideal fermion. The construction is valid for any number of collective pairs and non-collective fermions treated individually, and preserves the Pauli principle exactly.

The paper is organized as follows. In Sections II and III we discuss respectively the general properties of dynamical supersymmetry as it is manifested in nuclear physics, and the boson-fermion mapping which we use to construct it. Sec. IV presents a simple example of the appearance of a dynamical supersymmetry in the SU(2) seniority model. Then, in Sec. V we describe a more realistic system with the algebraic structure of SO(8)\(\otimes\)SO(5) which also exhibits a dynamical supersymmetry similar to that appearing in the IBFM. Conclusions are drawn in Sec. VI. The present paper is an expanded version of a short unpublished communication \[4\].

**II. SUPERSYMMETRIC UNIFICATION OF STATES IN EVEN AND ODD NUCLEI: PHENOMENOLOGY VS MICROSCOPY**

The IBFM \[2\] generalizes the phenomenological interacting boson model (IBM) \[5\] to odd systems by coupling single fermion degrees of freedom to the IBM bosons, *assuming* these fermions to be kinematically independent from the bosons. The most general one-plus two-body IBFM Hamiltonian contains pure boson and pure fermion parts, as well as an interacting part, the only requirement being that boson and fermion numbers are conserved separately.

Even for the most restricted fermion subspaces this Hamiltonian still contains far too many parameters to serve as the basis for a purely phenomenological analysis. Microscopically inspired considerations can, on the one hand, serve as a guideline to effect a reduction to a few manageable terms \[2\], while dynamical bose-fermi symmetry and, eventually, dynamical supersymmetry can also serve to restrict the dynamics.

These dynamical symmetries, their consequences and eventual comparison with experimental data are discussed in detail in Refs. \[2,6–8\]. (See also Ref. \[9\] for a recent analysis...
where (extended) supersymmetry is successfully used to account for properties of negative parity states in $^{194}\text{Ir}$.) Here we only briefly recount the most salient aspects of, respectively, dynamical bose-fermi and supersymmetry on the phenomenological level in order to make it clear what we accomplish in our construction of a dynamical supersymmetry on the microscopic level.

Recall that dynamical symmetry refers to a Hamiltonian which can be written in terms of Casimir invariants of the algebras appearing in a single chain of associated subgroups only, whence an analytical expression can be found for the energy spectrum by exploiting a basis labelled according to irreps of the highest group in the chain [2,5]. This situation implies various relationships between strength parameters of the general Hamiltonian and thus represents a considerable restriction. From the phenomenological point of view one studies whether dynamical symmetries indeed appear among nuclear spectra; from the microscopic point of view one would endeavour to motivate a set of strength parameters conforming to a dynamical symmetry. An extensive literature exists for both types of study – see e.g. Ref. [2,5] for a presentation of phenomenological studies and Ref. [10] for a discussion of microscopic investigations.

The general form of the highest or first group in the subgroup chain pertaining to a bose-fermi symmetry of the IBFM type can be written as $G_B \otimes G_F$ and in the simplest version of the IBM with one $s$- and five $d$-bosons this takes the form $U(6) \otimes U(2\Omega)$ with $2\Omega$ the dimension of the fermion space. Dynamical symmetries for situations where the boson and fermion subchains are essentially independent are the least restrictive and not very interesting. However, when some of the algebras appearing in the two chains are the same or isomorphic, one has the possibility to couple them to a spinor algebra. When this happens beyond the level of the rotation group, the dynamical structure seems to be sufficiently restrictive to lead to interesting and manageable classification schemes which are also realised in some nuclei [2,6–8].

It is important to realize that a dynamical bose-fermi symmetry with a given set of strength parameters pertains to one specific odd nucleus, the number of bosons $N_B$ being fixed as $N_B = \frac{1}{2}(n - 1)$ with $n$ the number of valence nucleons.

A more ambitious attempt to implement dynamical symmetry is to consider $U(m_B) \otimes U(2\Omega)$, with $m_B$ the number of single boson states, as being embedded in the supergroup $U(m_B/2\Omega)$. This implies that one is considering the classification of nuclear states in a set of neighbouring even and odd nuclei within a single supersymmetric representation $|\mathcal{R}\rangle$, where $\mathcal{R} = N_B + \mathcal{N}$, with $\mathcal{N}$ the number of unpaired nucleons, and where the Hamiltonian parameters and all other parameters appearing in physical operators are the same for all the nuclei under consideration – clearly a severe dynamical restriction.

Note that the supersymmetric representation is specified not by the effective number of fermions, $2N_B + \mathcal{N}$, but by the number of bosons plus fermions, $N_B + \mathcal{N}$, which of course underlies the possibility to use the single supersymmetric representation for a set of adjacent nuclei. Although this type of dynamical supersymmetry is invariably found to be broken, there seems to be enough evidence [2,6–9] that it is a useful scheme for classification and calculation of nuclear properties for a number of nuclei.

What concerns us mostly in the remainder of this paper is the question whether dynamical supersymmetry, as summarized above, can be compatible with the Pauli principle or, alternatively, if dynamical supersymmetry can be an exact property of a fermion sys-
tem. From the point of view that there are important Pauli corrections to the lowest order association between collective fermion pairs and IBM bosons \([\text{[1]}]\), one might anticipate a negative answer to this question. Nevertheless, we show that the implementation of appropriate boson-fermion mappings indeed reveals instances where this compatibility holds. These mappings, introduced and refined in Refs. \([\text{[1,3]}]\), and discussed and exploited below, introduce an equivalence between a system of interacting fermions on the one hand, and on the other a system of interacting bosons and fermions which are \textit{by construction kinematically independent}. Although no guarantee in itself, this clearly fulfills the minimum requirement for a dynamical supersymmetry to exist, namely to have the appropriate degrees of freedom. Moreover, as elaborated below, one can indeed find instances where dynamical supersymmetry emerges as an exact classification scheme of fermion states. This is simply another, but equivalent, classification to whatever classification scheme may have been adopted on the fermion level. What the boson-fermion mapping accomplishes is to make the inherent supersymmetric nature transparent in these instances, although they should also be identifiable on a purely algebraic level in terms of relationships between standard and supersymmetric representations.

Apart from providing a concrete link between fermion dynamics and dynamical supersymmetry, the use of boson-fermion mappings also allows one to \textit{construct} various transition operators appropriate to the boson-fermion description. This is in contrast to the phenomenological situation where one is obliged to truncate an infinite series of combinations of boson and fermion operators with phenomenological parameters and terms only restricted by their tensor and particle number changing properties \([\text{[2]}]\). It should be emphasized that the choice of these transition operators in phenomenological models such as the IBM or IBFM is \textit{not} dictated by the Hamiltonian parameters in general, specifically also in the case of dynamical symmetry or supersymmetry. (See also Ref. \([\text{[12]}]\) for a discussion of this point.) In the present context we show e.g. how the single fermion transfer operator is uniquely specified in the boson-fermion description by the appropriate mapping and how it compares with phenomenological results.

Finally, our work provides the possibility of a direct link between phenomenological models which introduce dynamical supersymmetry in an inherently fermion problem, such as is in the IBFM, and fermion models which exploit a similar structure by considering both vector and spinor representations of the the overall symmetry group, such as in the very recent fermion dynamical symmetry model (FDSM) analysis \([\text{[13]}]\). Our formalism is thus tailor made to investigate the similarities and differences between the two approaches.

**III. DYSON BOSON-FERMION MAPPING OF THE COLLECTIVE ALGEBRA**

In order to keep the paper as self-contained as possible, we briefly retrace the basic steps in the derivation of the Dyson boson-fermion mapping \([\text{[3]}]\) used in subsequent sections. We start from a fermion collective pair algebra comprised of collective fermion pair creation operators \(A^j = \frac{1}{2} \chi_{\mu \nu} a^\mu a^\nu\), pair annihilation operators \(A_i\), and their commutators \([A_i, A^j]\). The notation exploits a summation convention in which upper (lower) indices denote creation (annihilation) operators, as in \(a^\mu = a^\mu_+\) and \(A^j = A^j_+\). We also assume \([\text{[1]}]\) that the collective fermion operators obey the algebraic closure relations \([A_i, A^j] = e_{ik}^{jl} A^l\), i.e., operators \(A^j, A_i\), and \([A_i, A^j]\) are assumed to form a collective spectrum generating algebra \([\text{[15]}]\). These
relations guarantee an exact decoupling of the even collective space $|\Psi_{\text{even}}\rangle = A^i A^j \ldots A^k |0\rangle$ from all other even fermion states. Similarly, by adding an odd fermion, one obtains an exactly decoupled odd collective space $|\Psi_{\text{odd}}\rangle = a^\mu A^i A^j \ldots A^k |0\rangle$.

In a boson-fermion description an even state $|\Psi_{\text{even}}\rangle$ should be represented by an ideal space state $|\Psi_{\text{even}}\rangle$, say, which contains bosons only, with the odd ideal space states $|\Psi_{\text{odd}}\rangle$ containing an additional fermion $a^\mu$. Following traditional terminology this is also referred to as an ideal fermion $a^\mu$. By definition it commutes with all boson operators, $[a^\mu, B_i] = [a^\mu, B^i] = 0$. The required representation in the ideal space can be achieved by constructing an appropriate boson-fermion mapping (see the Appendix),

$$
A^i \leftrightarrow R^i - A^i = B^i [A_i, A^j] - \frac{1}{2} \epsilon^{ijk} B^j B^k B_i, \quad (1a)
$$

$$
[A_i, A^j] \leftrightarrow [A_i, A^j] - \epsilon^{ijk} B^k B_i, \quad (1b)
$$

$$
A_j \leftrightarrow B_j, \quad (1c)
$$

$$
a^\nu \leftrightarrow X^{-1} (a^\nu + B^i [A_i, a^\nu]) X, \quad (1d)
$$

$$
a^\nu \leftrightarrow X^{-1} \alpha^\nu X, \quad (1e)
$$

where $A^i = \frac{1}{2} \chi_{i\mu\nu} a^\mu a^\nu$ are collective pairs of ideal fermions. The similarity transformation $X$ has the explicit form

$$
X = \sum_{n=0}^{\infty} \left( \frac{1}{C_F - \tilde{C}_F} A^i B_i \right)^n, \quad (2)
$$

where $C_F = F^k F_k$ is an operator which leaves invariant the ideal-fermion core subalgebra composed of operators $[A_i, A^j]$, i.e., $[C_F, [A_i, A^j]] = 0$. Here we use the notation of a deferred-action operator $\tilde{C}_F$ which should be evaluated at the position indicated by "\text{\_\_}". Equivalently, one can write Eq. (2) by using multiple sums over eigenstates of $C_F$, in which case $1/(C_F - \tilde{C}_F)$ can be replaced by typical energy denominators.

We can now discuss the structure of the images of single fermion operators, Eqs. (1d) and (1e). An explicit general evaluation of these images is difficult because the operators $a^\nu$ and $\alpha^\nu$ do not commute with the ideal-fermion invariant operator $C_F$ and one has to consider branching rules of the collective algebra. Before presenting solutions in particular cases we can, however, analyze some general properties of these images.

We note that the similarity transformation (2) does not change any state in which there are no bosons $B^j$, and therefore the single-fermion states are mapped onto single ideal fermion states, $a^\nu |0\rangle \leftrightarrow a^\nu |0\rangle$. Since the images of pair creation operators (1a) do not change the ideal fermion number, collective odd states $|\Psi_{\text{odd}}\rangle$ are mapped onto ideal states with one ideal fermion only.

The two-fermion states are mapped as $a^\mu a^\nu |0\rangle \leftrightarrow (a^\mu a^\nu + \chi_{i\mu\nu} B^i - \frac{1}{C_F} \chi_{i\mu\nu} A^i) |0\rangle$ and in general contain the non-collective pair of ideal fermions $a^\mu a^\nu |0\rangle$. However, when the collective pair $A_i$ is formed by summing the pairs $a^\mu a^\nu$ with collective amplitudes $\frac{1}{2} \chi_{i\mu\nu}$, the ideal non-collective pairs above recombine (note that $C_F A^i |0\rangle = g A^i |0\rangle$) and only the boson state $g B^i |0\rangle$ remains. Again, since the images (1a) conserve the ideal fermion number, the same recombination mechanism is also valid for any even state.

Similarly as in the even case, spurious states may appear in spectra of mapped operators when diagonalized in the complete ideal boson-fermion space. However, they do not
IV. THE SU(2) SENIORITY MODEL AND DYNAMICAL SUPERSYMMETRY

The textbook SU(2) seniority model \[18\] is very often used in group theory models to illustrate the key aspects of the methods used. Although nothing new can be said about the model itself, its simplicity and intuitiveness allows for a very suitable test ground where advanced approaches can be explained in simple terms. In this Section we apply to this model our boson-fermion mapping and discuss the resulting dynamical supersymmetry.

The SU(2) model is defined by considering in a single-\(j\) shell the monopole pair creation operator
\[
S^+ = \sqrt{\frac{\Omega}{2}}(a_j^+a_j^+)^{(0)},
\]
with \(\Omega = j + \frac{1}{2}\). It fulfills the commutation relation
\[
[S, S^+] = \Omega - n,
\]
where \(n\) is the fermion number operator in the single-\(j\) shell. The SU(2) algebra can be generalized to describe odd systems by constructing the superalgebra generated by the operators \(S^+, S, \Omega-n, a_{jm}\), and \(a_{jm}^+\). The relevant commutation relation is
\[
[a_{jm}^+, S] = -\tilde{a}_{jm},
\]
with \(\tilde{a}_{jm} = (-1)^{j-m}a_{j,-m}\), while the single-fermion operators obey the standard anticommutation relations. Clearly this is a rather trivial superalgebra as the elements of the odd sector (single fermion operators) anti-commute only to the identity. Alternatively, by considering the commutator of single-fermion operators, the set of bi- and single-fermion operators may of course also be viewed as generators of a standard (orthogonal) algebra.

In the single-\(j\) shell we consider the pairing Hamiltonian
\[
H = -GS^+S,
\]
which has the energy spectrum \[18\]
\[
E = -\frac{1}{4}G(n-v)(2\Omega - n - v + 2),
\]
with the seniority quantum number \(v\) denoting the number of fermions not coupled to angular momentum zero. This Hamiltonian describes both the even and odd systems, and the spectra in both cases are given by the same expression \[4\] with \(v\) even or odd, respectively.

We can apply to this model the Dyson boson-fermion mapping described in Sec. \[III\] to find an equivalent description in the boson-fermion space. For the SU(2) algebra, the operator \(C_F\) depends on the ideal fermion number operator \(\mathcal{N} = \sum_m a_m^+a_m\), and one may derive an explicit form of the similarity transformation \[3\]. As we have...
\[ C_F - \hat{C}_F = \frac{1}{2}(N - \hat{N})(\Omega + 1 - \frac{1}{2}(N + \hat{N})) , \] (8)
the transformation (2) is
\[ X = \frac{(\Omega - \frac{1}{2}(N + \hat{N}))!}{(\Omega - \hat{N})!} \exp \left[ S^\dagger B \right] . \] (9)

Specializing from the general case to SU(2), we have introduced here the ideal fermion operators \( \alpha^\dagger_{jm} \) and \( \alpha_{jm} \), which commute with the ideal boson operators \( B^\dagger \) and \( B \), and the ideal fermion pair operators \( S^\dagger \) and \( S \) obtained from \( S^+ \) and \( S \) by replacing \( a \) by \( \alpha \). The general Dyson boson-fermion mapping (4) is then obtained for the SU(2) case in the form
\[ S^+ \longleftrightarrow \Omega B^\dagger - B^\dagger B - B^\dagger \hat{N} = B^\dagger (\Omega - N_B - \hat{N}) = B^\dagger (\Omega - \aleph) , \] (10a)
\[ S \longleftrightarrow B , \] (10b)
\[ n \longleftrightarrow 2B^\dagger B + \hat{N} = 2N_B + \hat{N} = \aleph + N_B , \] (10c)
\[ a^+_{jm} \longleftrightarrow \alpha^\dagger_{jm} \frac{\Omega - \aleph}{\Omega - \hat{N}} + B^\dagger \hat{\alpha}_{jm} - S^\dagger \hat{\alpha}_{jm} \frac{\Omega - \aleph}{(\Omega - \hat{N})(\Omega - \hat{N} + 1)} , \] (10d)
\[ a_{jm} \longleftrightarrow \alpha_{jm} + \hat{\alpha}^\dagger_{jm} B \frac{1}{\Omega - \hat{N}} + S^\dagger \alpha_{jm} B \frac{1}{(\Omega - \hat{N})(\Omega - \hat{N} + 1)} , \] (10e)
where \( \aleph = N_B + \hat{N} \). We see that the single fermion images (10c) and (10d) are finite and contain terms changing the ideal fermion number by one only. Furthermore, they preserve exactly the anti-commutation relations on the full ideal space, i.e., as operator identities. This is guaranteed by our construction method, and can be verified by explicit calculation. The preservation of the commutation and anticommutation relations ensures the exact preservation of the Pauli exclusion principle, once the original fermion problem is mapped into the boson-fermion space.

Clearly, the mapping (10) transforms the 2-body Hamiltonian (3) into a 1-plus-2-body boson-fermion Hamiltonian of the form
\[ H_{BF} = -G N_B (\Omega - N_B + 1 - \hat{N}) . \] (11)

Hamiltonians (3) and (11) have exactly the same spectrum (7). Hamiltonian (11) can also be expressed in a form which stresses its dependence on the total number of bosons and fermions \( \aleph \), i.e.,
\[ H_{BF} = -G (\aleph - \hat{N})(\Omega + 1 - \aleph) . \] (12)

Note also that the boson-fermion interaction term in (11), which reads \( G N_B \hat{N} \), can be expressed in terms of the odd generators, \( O^\dagger_m = a^\dagger_{jm} B \) and \( O_m = B^\dagger \alpha_{jm} \), of the U(1/2\Omega) superalgebra. Since the boson and ideal fermion number operators can be linked to even generators (see also expression (15) and its discussion below), this identification makes it possible to write the Hamiltonian (11) in yet another form in terms of both even generators and supergenerators of U(1/2\Omega):
\[ H_{BF} = -G \left( N_B (\Omega - N_B + 1) + \hat{N} - \sum_m O^\dagger_m O_m \right) . \] (13)
After the mapping states with a given fermion number \( n \) are classified according to the number of bosons \( N_B \) and the number of ideal fermions \( N \), these numbers still being related to the fermion number \( n \) through \( n = 2N_B + N \). The number of ideal fermions now corresponds to the seniority quantum number, because all fermion pairs coupled to zero angular momentum are mapped onto bosons. This latter identification follows by construction, as any attempt to create an ideal fermion pair of angular momentum zero by applying (10d) successively (with appropriate coupling), results in the creation of the boson \( B \). As long as we do not overfill the \( j \) level by too many particles, i.e., restrict the ideal space to physical states with \( 2N_B + N ≤ 2Ω \), the fermion space results are exactly reproduced in the boson-fermion space.

In general it is not possible, as in this simple case, to make, for a given fixed particle number, an a priori selection of simple ideal space states which will span the physical subspace. However, as elaborated in Ref. [14], one can exploit the simplicity of the ideal space basis for the diagonalization of the mapped Hamiltonian. The physical eigenstates do not mix with the unphysical states and can be identified after the diagonalization. (See Ref. [19] for a recent application of this procedure.)

In the original fermion space of the SU(2) seniority model, the Hamiltonian eigenstates are classified according to the representations of the subgroup chain

\[
SO(4Ω+1) \supset SU(2) \otimes Sp(2Ω) \supset U(1) \otimes SO(3),
\]

while in the boson-fermion space they are classified by the quantum numbers derived from the chain

\[
U(1/2Ω) \supset U_B(1) \otimes U_F(2Ω) \supset U_B(1) \otimes U_F(1) \otimes Sp_F(2Ω) \supset U_B(1) \otimes U_F(1) \otimes SO_F(3).
\]

In the second chain the supergroup \( U(1/2Ω) \) is generated by \( B^\dagger B, α_{jm}^\dagger α_{jm'}, α_{jm}^\dagger B, \) and \( B^\dagger α_{jm}, \) with the first two operators belonging to the even sector and the remaining two to the odd sector of the superalgebra, respectively. The appearance of \( U(1/2Ω) \) in the chain is clearly suggested by the form (13) and also dictates that the same Hamiltonian parameters in (11) are used for both even and odd states, if the Hamiltonian should be viewed as a phenomenological boson-fermion Hamiltonian. From the mapping point of view there is of course no real choice in this matter, as the original fermion Hamiltonian pertains to any number of particles. Nevertheless, it is interesting to note already here how the phenomenological extension to superalgebras à la IBFM may be suggested from a microscopic point of view.

In the dynamical supersymmetry concept, those states of the system with the same number of ideal particles \( N \) belong to a single supersymmetric representation \( |N⟩ \) of the supergroup, here \( U(1/2Ω) \). (Note that \( N = N_B + N \) differs from the fermion number \( n \) and its ideal space image \( |10d⟩, n ←→ 2N_B + N \).) Starting from, say, the state with no fermions \( |N_B⟩ \), the other states of the multiplet are obtained by successive application of supergenerators \( α_{jm}^\dagger B \).

Although one can embed all states of the SU(2) model in a sequence of representations of the supergroup \( U(1/2Ω) \), this does not really provide other interesting physical consequences. This is so because in the chain (13) the supergroup \( U(1/2Ω) \) is immediately split into the boson and fermion sectors which remain separate down to the bottom of the chain.
As discussed in Sec. II, potentially interesting situations which are restrictive from the supersymmetry point of view, occur if at a certain level of subgroups one may find the same subgroups in the boson and fermion sectors, and combine them together into a given boson-fermion subgroup. Such a situation arises in a fermion model with a richer structure than the present SU(2) case and is discussed in the next section. In terms of algebras appearing in the chain (14), we here only find a trivial example of the boson subalgebra $U_B(1)$, generated by the boson number operator $N_B$, and the fermion subalgebra $U_F(1)$, generated by the fermion number operator $N$, which can be combined into the boson-fermion subalgebra $U_{BF}(1)$, generated by $\hat{\mathbb{N}}=N_B+N$.

At the same time, even the simplest example of the SU(2) seniority model discussed here provides interesting consequences for the structure of the single-fermion transfer operators and the spectroscopic factors. After the mapping, the single-fermion operators acquire terms which are responsible for the Pauli correlations between the even core and the odd particle. In the phenomenological supersymmetric models these terms are postulated together with some arbitrary numerical constants, whereas in the supersymmetric picture derived from the boson-fermion mapping they are fixed by the mapping procedure itself. For example, the image of the fermion annihilation operator (10e) is a combination of the ideal fermion annihilation operator with two corrective terms. The first one corresponds to an annihilation of the same ideal fermion accompanied by an attempt to replace a boson by a collective ideal fermion pair. This term ensures that after the annihilation the remaining fermions still obey the correct statistics. The second term corresponds to the standard annihilation of the boson replaced by an appropriate fermion which has been the partner of the annihilated fermion in a collective pair. Of course, the relative weights of these corrective terms depend on how many ideal fermions have already been present before the annihilation.

To evaluate and understand different aspects of the original fermion and the mapped boson-fermion description, it is instructive to calculate the spectroscopic factors for the simplest states in the SU(2) model. In the original fermion space this can be done by using the commutation relations (11) and (12), and we arrive at the result

$$\langle n+1, v=1, j|\alpha_j^+|n, v=0\rangle = -\langle n, v=0|\bar{\alpha}_j|n+1, v=1, j\rangle = -\sqrt{2\Omega-n}.$$ (16)

To get this relation one has to calculate the normalization factors of the fermion states.

Evaluating matrix elements (16) in the mapped boson-fermion space is much simpler. In particular, one can work with the ideal boson-fermion eigenstates of the Hamiltonian (11) corresponding to the states with $N=0$ and 1, e.g., $|B^{N_B}\rangle$, and $|B^{N_B}, \alpha_j^+\rangle$. Using Eqs. (10d) and (10e) we directly read off the results

$$\langle B^{N_B}, \alpha_j^+|\alpha_j^+|B^{N_B}\rangle = -\sqrt{\frac{2}{\Omega}(\Omega-N_B)},$$ (17a)

$$-\langle B^{N_B}|\bar{\alpha}_j|B^{N_B}, \alpha_j^+\rangle = -\sqrt{2\Omega},$$ (17b)

where $(a_j^+|BF$ and $(\bar{\alpha}_j)|BF$ are the boson-fermion images of $a_j^+$ and $\bar{\alpha}_j$, respectively. In this calculation, only simple boson normalization factors enter. After the hermitization (see Ref. [20] and references therein) and identification of $2N_B = n$, matrix elements (17) reproduce the correct value of $-\sqrt{2\Omega-n}$.

It is probably worthwhile to conclude this section by carefully distinguishing two superalgebraic structures we identified in two different contexts. Firstly, the algebraic structure
obtained by extending a collective bifermion algebra with single fermion operators leads to a trivial superalgebra structure. Nevertheless, this identification plays an important role in the construction of boson-fermion images developed in Refs. [3,11]. Secondly, after mapping of the fermion Hamiltonian, it becomes possible to analyse the resulting equivalent boson-fermion Hamiltonian in terms of a subgroup chain with a supergroup as first member, and classify states of neighbouring even and odd systems according to a single representation of that supergroup. This analysis relies on the identification of the mapped Hamiltonian as being constructed from the generators of subalgebras of a non-trivial superalgebra, generally different from the one appearing on the original fermion level.

V. SO(8) AND SO(8)⊗SO(5) MAPPING, AND DYNAMICAL SUPERSYMMETRY

In this section we discuss the appearance of dynamical supersymmetry in a more complicated model, namely, in the Ginocchio SO(8) model [21]. This model is defined by collective pairs

\[
F^\pm_{JM} = \sqrt{\frac{2}{k}} \sum_{j_1 j_2} \frac{J_1 J_2 j_1 j_2}{k} \left\{ \begin{array}{ccc} j_1 & j_2 & J \\ i & i & k \end{array} \right\} (a^+_{j_1} a^+_{j_2})^{(J)}_M, \quad (18a)
\]

\[
P_{JM} = -\sqrt{\frac{2\Omega}{k}} \sum_{j_1 j_2} (-1)^{J_1 + k + j_1} \frac{J_1 J_2}{k} \left\{ \begin{array}{ccc} j_1 & j_2 & J \\ i & i & k \end{array} \right\} (a_{j_1} a_{j_2})^{(J)}_M, \quad (18b)
\]

with \( i = \frac{3}{2} \) and \( k \) integer. In (18a) only \( S^+ \ (J=0) \) and \( D^+ \ (J=2) \) pairs are allowed, while in (18b) \( J \) takes values 0, 1, 2, and 3.

In order to generalize the SO(8) algebra to a superalgebra, we include the \( 2\Omega=(2i+1)(2k+1) \) creation and annihilation operators \( a^\dagger_{jm} \) and \( a_{jm} \), where \( j=|k-i|, \ldots, k+i \). A boson-fermion mapping of this algebra was derived in Ref. [3]. In this case the series defining the similarity transformation

\[
X = \sum_{k=0}^{\infty} \frac{1}{C_F - \tilde{C}_F} F^\dagger F^{(k)}, \quad (19)
\]

cannot be explicitly summed up because the denominator

\[
C_F - \tilde{C}_F = \frac{1}{\Omega} \left[ \frac{1}{2} (N - \hat{N})(\Omega + 6) - \frac{1}{2} (N + \hat{N}) + \tilde{C}_{2Spin} - C_{2Spin} \right], \quad (20)
\]

contains the quadratic Casimir operator of the Spin(6) group which is not expressible in terms of number operators. Here \( N \) is the ideal fermion number operator \( N = \sum_{jm} a^\dagger_{jm} \alpha_{jm} \) and \( C_{2Spin} = \frac{1}{4} (P_1 \cdot P_1 + P_2 \cdot P_2 + P_3 \cdot P_3) \).

The similarity transformation \( X \) can nevertheless be applied with the result that the transformed mapping then reads

\[
F^\pm_{JM} \leftrightarrow B^\dagger_{JM} - \frac{2}{k^2} \sum_{j_1 j_2 j_3 j'} \hat{J}_1 \hat{J}_2 \hat{J}_3 \hat{J}' \left\{ \begin{array}{ccc} i & i & J \\ i & i & J_3 \\ J_2 & J_1 & J' \end{array} \right\} (B^\dagger_{J_1 J_2} (J') B_{J_1})^{(J)}_M
\]
\[ +\frac{2}{k^2} (-1)^{J_2} \hat{J}_1 \hat{J}_2 \left\{ \begin{array}{ccc} J_1 & J_2 & J \\ i & i & i \end{array} \right\} \left( \hat{B}_1 \hat{P}_M \right)^{(J)} \]

\[ F_{JM} \leftrightarrow B_{JM}, \quad (21a) \]

\[ P_{JM} \leftrightarrow \frac{2\sqrt{2\Omega}}{k} (-1)^{j+2i} \hat{J}_1 \hat{J}_2 \left\{ \begin{array}{ccc} J_1 & J_2 & J \\ i & i & i \end{array} \right\} \left( \hat{B}_1 \hat{P}_M \right)^{(J)} + \mathcal{P}_{JM}, \quad (21c) \]

for the collective pair operators while for the single-fermion operators we obtain

\[ a_{jm}^+ \leftrightarrow \alpha_{jm}^+ + B_J^+ \cdot [\mathcal{F}_J, \alpha_{jm}^+] - \frac{1}{C_F - C_F} \mathcal{F}_J^\dagger \cdot \hat{B}_J - B_J^+ \cdot [\mathcal{F}_J, \alpha_{jm}^+] \]

\[ +B_J^+ \cdot [\mathcal{F}_J, \alpha_{jm}^+] \frac{1}{C_F - C_F} \mathcal{F}_J^\dagger \cdot \hat{B}_J \sim \ldots \]

\[ a_{jm} \leftrightarrow \alpha_{jm} - \frac{1}{C_F - C_F} \mathcal{F}_J^\dagger \cdot \hat{B}_J \sim \alpha_{jm} + \alpha_{jm} \frac{1}{C_F - C_F} \mathcal{F}_J^\dagger \cdot \hat{B}_J \sim \ldots \]. \quad (22a, 22b)\]

The ideal fermion pair operators \( \mathcal{F}_J \) and \( \mathcal{P}_J \) are given by Eqs. (18a) and (18b), respectively, with the fermion operators \( \alpha_{jm} \) replaced by ideal fermion operators \( \alpha_{jm} \). The dots \( \ldots \) refer to a class of higher order terms which will not contribute to matrix elements between physical states. (See Ref. [3] for further discussion.)

A condensate of collective fermion pairs have now been mapped onto a condensate of \( s^+ \equiv B_0^+ \) and \( d^+ \equiv B_2^+ \) bosons, and a condensate of collective fermion pairs with one additional odd fermion, onto a condensate of \( s^+ \) and \( d^+ \) bosons with one additional ideal fermion. This situation is quite reminiscent of the phenomenological IBFM. The six bosons above span the symmetric representation of \( U(6) \). The size of the fermion sector depends on \( k \). Let us first discuss the situation with \( k=0 \) (\( j=3/2 \)). From Eq. (21b) we observe that the boson-fermion images of the multipole operators are

\[ P_{1M} \leftrightarrow 2\sqrt{2} \left[ (d^+ \bar{d})_M^{(1)} - \frac{1}{\sqrt{2}} (\alpha_{j\bar{a}_j})_M^{(1)} \right], \quad (23a) \]

\[ P_{2M} \leftrightarrow 2 \left[ (d^+ s + s^+ \bar{d})_M^{(2)} - (\alpha_{j\bar{a}_j})_M^{(2)} \right], \quad (23b) \]

\[ P_{3M} \leftrightarrow -2\sqrt{2} \left[ (d^+ \bar{d})_M^{(3)} + \frac{1}{\sqrt{2}} (\alpha_{j\bar{a}_j})_M^{(3)} \right]. \quad (23c) \]

In the boson-fermion multipole operators (23) we recognize immediately the generators of the Spin_{BF}(6) group applied in the phenomenological IBFM boson-fermion dynamical symmetry \( U_B \otimes U_{BF}(4) \). Similarly, from Eqs. (22) we identify single nucleon transfer operators in the form used in the above phenomenological dynamical symmetry, provided that we limit ourselves to \( n = 0,1 \) states only and neglect the \( B^+B \) type terms:

\[ a_{jm}^+ \leftrightarrow \alpha_{jm}^+ + \frac{1}{\sqrt{2}} \left[ s^+ \bar{a}_{jm} + \sqrt{5} (d^+ \bar{a}_{jm})_M \right]. \quad (24) \]

Clearly, once we construct a fermion dynamical symmetry Hamiltonian from the Casimir operators of the subgroups appearing in the chain

\[ \text{SO}(8) \supset \text{Spin}(6) \supset \text{Spin}(5) \supset \text{Spin}(3), \quad (25) \]
its boson-fermion image would be just the IBFM $U_B \otimes U_F(4)$ dynamical symmetry Hamiltonian. Since the original SO(8) Hamiltonian does not distinguish even and odd systems, the mapped Hamiltonian must also be the same for both even and odd system. Therefore, as already argued in the SU(2) case, the classification scheme in the boson-fermion space should be extended by embedding the boson-fermion dynamical symmetry into the U(6/4) dynamical supersymmetry. However, there is a difference between the mapped boson-fermion system and the phenomenological IBFM U(6/4) supersymmetry. In the SO(8) model considered here all particles occupy the same $j=3/2$ level while in the IBFM it is assumed that the bosons occupy the whole valence shell with only the fermion restricted to $j=3/2$.

As a more realistic situation, consider $k=2$, corresponding to $j=1/2$, 3/2, 5/2, and 7/2. In the nuclear shell model this corresponds to the 3s$_{1/2}$, 2d$_{3/2}$, 2d$_{5/2}$, and 1g$_{7/2}$ orbitals located between the $N=50$ and 82 nuclear magic numbers. In the IBFM, a related supersymmetry with the same single-particle content, is U(6/20), realized in the Au-Pt isotopes [23]. The group reduction chain is $U_B(6) \otimes U_F(20) \supset SO_B(6) \otimes SU_F(4) \supset Spin_{BF}(6) \supset Spin_{BF}(5) \supset \tilde{Spin}(3)$. A Hamiltonian chosen as a linear combination of quadratic Casimir operators appearing in this chain yields the analytical U(6/20) IBFM energy formula [23]

$$E = A \sigma (\sigma + 4) + \tilde{A} [\sigma_1 (\sigma_1 + 4) + \sigma_2 (\sigma_2 + 2) + \sigma_3^2]$$

$$+ B [\tau_1 (\tau_1 + 3) + \tau_2 (\tau_2 + 1)] + CJ (J + 1).$$

In the simple extension of the SO(8) Ginocchio model to odd systems the inactive angular momentum $k$ of the odd fermion gives rise to an unrealistic degeneracy. One can, however, lift this degeneracy by adding to the SO(8) algebra multipole operators corresponding to an interchange of the active and inactive angular momenta $k$ and $i$. Suppose we add two such operators $\tilde{P}_J$ for $J=1$ and 3,

$$\tilde{P}_{JM} = -\sqrt{2} \Omega \sum_{j_1 j_2} (-1)^{J+i+k+j_2} \hat{j}_1 \hat{j}_2 i \left\{ \begin{array}{ccc} j_1 & j_2 & J \\ k & k & i \end{array} \right\} (a_{j_1}^{+} a_{j_2})_{M}^{(J)},$$

with $k = 2$ and $i = \frac{3}{2}$. These operators form an SO(5) algebra and commute with the SO(8) generators (18). The resulting algebraic structure is therefore SO(8)$\otimes$SO(5). Hence, we can form a new dynamical symmetry subgroup chain by combining the SO(5) generators with the Spin(5) generators to obtain an additional group denoted by $\tilde{Spin}(5)$. The fermion group reduction chain is then

$$SO(8) \otimes SO(5) \supset Spin(6) \otimes SO(5) \supset Spin(5) \otimes SO(5) \supset \tilde{Spin}(5) \supset \tilde{Spin}(3),$$

with $\tilde{Spin}(5)$ generated by

$$G_3 = P_3 - 2 \sqrt{\frac{2}{5}} \tilde{P}_3,$$

$$G_1 = P_1 + 2 \sqrt{\frac{2}{5}} \tilde{P}_1,$$

and $\tilde{Spin}(3)$ by $G_1$, with $P_J$ the original SO(8) multipole operators (18). We are now in a position to write a Hamiltonian corresponding to the fermion dynamical symmetry (28) which is composed of quadratic Casimir operators of the groups appearing in the chain:

$$H = AC_{2\text{Spin}(6)} + BC_{2\tilde{\text{Spin}}(5)} + \tilde{B}C_{2\tilde{\text{Spin}}(5)} + CC_{2\tilde{\text{Spin}}(3)},$$

(30)
In the odd case, when the system is formed by \( N \sigma \tau \) where (comprised of \( N \) from the reduction pattern \( N, N-1 \), characterizing the representations of the subgroups in the dynamical symmetry chain, namely

\[
\text{The corresponding spectrum is obtained in a basis classified by quantum numbers characterizing the representations of the subgroups in the dynamical symmetry chain, namely}
\]

\[
E = A[\sigma_1(\sigma_1 + 4) + \sigma_2(\sigma_2 + 2) + \sigma_3^2] + B[\tau_1(\tau_1 + 3) + \tau_2(\tau_2 + 1)] + \tilde{B}[\tilde{\tau}_1(\tilde{\tau}_1 + 3) + \tilde{\tau}_2(\tilde{\tau}_2 + 1)] + CJ(J + 1),
\]

where \((\tau_1, \tau_2)\) are the Spin(5) irreps and \((\tilde{\tau}_1, \tilde{\tau}_2)\) are the irreps of \( \tilde{\text{Spin}}(5) \). For an even system comprised of \( N \) fermion pairs, the quantum numbers are: \((\sigma_1, \sigma_2, \sigma_3) = (\sigma, 0, 0)\), with \( \sigma = N, N-2, N-4, \ldots 1 \), or \( 0 \), \((\tau_1, \tau_2) = (\tau, 0)\), with \( \tau = 0, 1, \ldots \sigma \). One also has \((\tilde{\tau}_1, \tilde{\tau}_2) = (\tau_1, \tau_2)\). In the odd case, when the system is formed by \( N \) fermion pairs and a single odd fermion, the branching rules are \((\sigma_1, \sigma_2, \sigma_3) = (\sigma \pm \frac{1}{2}, \frac{1}{2}, \frac{1}{2})\), with \( \sigma = N, N-2, N-4, \ldots 1 \), or \( 0 \), \((\tau_1, \tau_2) = (\tau_1, \frac{1}{2})\), with \( \sigma_1 \geq \tau_1 \geq \frac{1}{2} \). Eventually, the values of \((\tilde{\tau}_1, \tilde{\tau}_2)\), \( \tilde{\tau}_1 \geq \frac{1}{2} \) are deduced from the reduction pattern

\[
(\tau_1, \frac{1}{2}) \otimes (1, 0) = (\tau_1 + 1, \frac{1}{2}) + (\tau_1, \frac{1}{2}) + (\tau_1 - 1, \frac{1}{2}) + (\tau_1, \frac{3}{2}) .
\]

In table [I] we present the angular momentum \( J \) content of the lowest Spin(5) representations.

To arrive at an equivalent boson-fermion description we apply the boson-fermion mapping \([21]\) and \([22]\), whereas the new SO(5) algebra is simply mapped from the original fermion space to the ideal fermion space by replacing the operators \( a_j \) by operators \( \alpha_j \).

\[
\tilde{P}_{JM} \leftrightarrow -\sqrt{2\bar{N}} \sum_{j_1j_2} (-1)^{J+i+k+j_2} \frac{j_1j_2}{j_1} \begin{pmatrix} j_1 & j_2 & J \\ i & k & i \end{pmatrix} (\alpha_{j_1} \alpha_{j_2})^{(J)}_M .
\]

Obviously, the boson-fermion images of the generators of \( \tilde{\text{Spin}}(5) \) are obtained by combining the images of Spin(5) and SO(5), as in expressions \([23]\). It can be verified that, up to a normalization factor, these are just the generators of the subgroup Spin\(_{\text{BF}}\)(5) of the IBFM \( U(6/20) \) \([24]\) given explicitly in Ref. \([24]\). Consequently, at the boson-fermion level we have now the following group chain

\[
\text{U}_B(6) \otimes \text{U}_F(20) \supset \text{U}_B(6) \otimes \text{U}_F(4) \otimes \text{U}_F(5) \supset \text{SO}_B(6) \otimes \text{SU}_F(4) \otimes \text{SO}_F(5) \\
\supset \text{Spin}_{\text{BF}}(6) \otimes \text{SO}_F(5) \supset \text{Spin}_{\text{BF}}(5) \otimes \text{SO}_F(5) \supset \text{Spin}_{\text{BF}}(5) \supset \text{Spin}_{\text{BF}}(3) .
\]

The groups Spin\(_{\text{BF}}\)(l) correspond to Spin(l) and are generated by the images \([21c]\), while the groups \( \tilde{\text{Spin}}_{\text{BF}}\)(l) correspond to \( \tilde{\text{Spin}}(l) \) and their generators are obtained as outlined above.

It is interesting to note that the construction \([27]\) and its implementation in the group chain \([28]\) lead to a Hamiltonian \([20]\) with spectrum \([32]\) which is the same as that used by Jolie \textit{et al} \([24]\), but motivated from a different point of view. The construction in Ref. \([24]\)
introduces an *ad hoc* coupling between a pseudo-orbital angular momentum $2$ and pseudo spin $3/2$ to obtain the appropriate set of single-particle orbits for their analysis of the $A=130$ mass region which is furthermore based on the introduction of phenomenological $s$- and $d$-bosons to establish a $U(6/20)$ supersymmetry framework. Our construction therefore implies that the analysis of Ref. [24] can be linked to and equivalently carried out in an $SO(8) \otimes SO(5)$ framework, which becomes even more transparent after we introduce below the boson-fermion image of the Hamiltonian (30).

To obtain the boson-fermion image of the Hamiltonian (30), all the generators are simply replaced by their boson-fermion images. The eigenenergy expression remains the same as in (32). Also the representation reduction pattern remains the same as in the original fermion system, provided the boson number satisfies $N_B = N$ and the number of ideal fermions is either 0, or 1. Consequently, for the present model application, there is a one-to-one correspondence between fermion states and states in the boson-plus-one-fermion space, so that spurious states do not appear. Furthermore, following the same reasoning as in Sec. [14] since the fermion interaction strengths pertain to even and odd systems alike, boson-fermion eigenstates of the mapped Hamiltonian are contained in a single supersymmetric representation of $U(6/20)$.

As a simple application of the supersymmetric $SO(8) \otimes SO(5)$ energy formula (32), we compare the corresponding even and odd spectra in Fig. 1. The Hamiltonian parameters $B = 35$ keV, $\tilde{B} = 12.63$ keV, and $C = 18$ keV, are chosen so that the even part coincides with the one of Ref. [23]. In the odd spectrum we then find more low-lying $J = (\frac{3}{2})^+$ states than one gets from the phenomenological IBFM expression (26). The dotted lines connect states belonging to the same $\tilde{\text{Spin}}(5)$ representations. We observe that these representations are ordered with increasing energy as $(\tilde{\tau}_1, \tilde{\tau}_2) = (\frac{1}{2}, \frac{1}{2}), (\frac{3}{2}, \frac{1}{2}), (\frac{3}{2}, \frac{3}{2}), (\frac{5}{2}, \frac{3}{2}), (\frac{5}{2}, \frac{1}{2})$. On the other hand, the phenomenological IBFM $U(6/20)$ supersymmetry (26) gives, approximately, the ordering of $\text{Spin}_{BF}(5)$ representations as follows: $(\tau_1, \tau_2) = (\frac{1}{2}, \frac{1}{2}), (\frac{3}{2}, \frac{1}{2}), (\frac{3}{2}, \frac{3}{2}), (\frac{5}{2}, \frac{3}{2}), (\frac{5}{2}, \frac{1}{2})$. A different ordering of representations also changes the selection rules of the allowed transitions. For example, the observed $(\frac{5}{2})^+_2 \rightarrow (\frac{3}{2})^+_1$ transition in $^{195}$Au [29] is forbidden in $U(6/20)$ (26) as it violates the $\Delta \tau_1 = 0, \pm 1$ selection rule. In the scheme presented here the $(\frac{5}{2})^+_2$ state belongs to the $(\frac{3}{2}, \frac{1}{2})$ representation and the above transition would be allowed. However, it is at the moment too speculative to claim that the dynamical supersymmetry discussed here may be realized in the Pt-Au region and allow such selection rules to be tested. It is interesting, though, to note that the discussed $SO(8) \otimes SO(5)$ scheme, which we introduced in Ref. [4], was also used recently in Ref. [13] and appears to have experimental evidence in the Xe-Ba region. See e.g. Fig. 3 of Ref. [13] with more experimental low-lying low spin states in the odd systems than seems typical for phenomenological supersymmetric spectra derived from (26).

We have discussed the cases of one or zero unpaired fermions only. However, it may be shown that a one-to-one correspondence between the fermion states and the boson-fermion states holds when more unpaired particles are considered as well, provided that the ideal fermions do not form collective $SO(8)$ pairs and, moreover, provided that the condition $2N_B + n \leq \Omega$ is satisfied.
VI. CONCLUSION

In summary, we have applied the recently constructed generalized Dyson boson-fermion mapping of collective algebras to the seniority SU(2) and SO(8)⊗SO(5) models. The mapping gives finite non-hermitian boson-fermion images of collective pairs and single fermion operators expressed in terms of ideal boson and fermion annihilation and creation operators. In these models, and for the specific interactions chosen, we have revealed a dynamical supersymmetric structure analogous to what is found in the interacting boson-fermion model, but with full recognition of the Pauli principle.

The findings discussed in this paper are based on the construction of a Hamiltonian from the generators of the fermion spectrum generating algebra which defines a class of dynamical symmetries [15]. The Hamiltonian itself is not invariant with respect to the symmetry group, but its eigenstates can be classified by the group and subgroup representations. At this level the supersymmetric structure is not readily visible, and even and odd states simply belong to different representations of the fermion group. However, we showed that an equivalent description which does reveal a dynamical supersymmetry can be obtained after a boson-fermion mapping.

Whereas the Hamiltonian in the SU(2) model was constructed from the pair operators which are mapped in a non-hermitian way (but nevertheless yields a hermitian image for the pairing Hamiltonian), in the SO(8) model application the Hamiltonian consisted of multipole operators with manifestly hermitian boson-fermion images. Apparently, a dynamical supersymmetric construction like the one applied to the SO(8) model could also be extended to any fermion algebraic model based on a pseudo-spin and pseudo-orbital recoupling, as e.g. the SD Sp(6) [21] and SDG SO(12) [19] models. In such cases the unification of the collective and non-collective algebras can always be achieved on the Spin(3) level. In the SO(8) example the corresponding group was denoted by \( \tilde{\text{Spin}}(3) \). The interesting question, however, is if one can find a unification of some of the groups relevant to a given model at some higher level, as in the case of \( \tilde{\text{Spin}}(5) \) for the SO(8) example. This would be analogous to a dynamical supersymmetry, or a boson-fermion dynamical symmetry in the usual IBFM sense, where the unification is usually considered to be formed at the highest level possible in the group chain.

Obviously, a fermion algebraic model puts stronger restrictions on the possible extensions to odd systems. In any case, should an algebraic model be applied to describe even-even nuclei, like e.g. the fermion dynamical symmetry model (FDSM) [27], its extension to the description of odd nuclei should also be investigated. As we have shown in this paper, the identification of dynamical symmetries (the usual starting point of such analyses) can sometimes be incorporated into a framework of dynamical supersymmetries after the boson-fermion mapping, thus unveiling a possibly richer dynamical symmetry structure than is immediately visible on the original fermion level. It is of interest to enquire which other “hidden” non-trivial dynamical supersymmetries exist in algebraic fermion models and to understand the relationship between representations on the fermion level and those on the boson-fermion level from a purely algebraic point of view, without necessarily relying on the boson-fermion type mapping considered here.

Let us finally remark that the dynamical supersymmetry discussed in the present context presupposes that all ideal space states enter the analysis, as in the case of the SO(8) and
SO(8)⊗SO(5) models. If some of these states do turn out to be spurious in other cases, as they most likely will, one will have a situation which is the equivalent of what is termed the “dynamical Pauli effect” in the FDSM [27, 29]. An incomplete supersymmetric spectrum will result, but the classification and description of the remaining physical states in terms of (ideal) bosons and fermions will still be perfectly in order.

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APPENDIX:

In order to prove that the boson-fermion mapping (1) preserves the commutation and anticommutation relations from the original fermion space we proceed in two steps. First we use the boson-fermion mapping,

\[ A^j \leftrightarrow R^j \equiv A^j + B^i [A_i, A^j] - \frac{1}{2} c_{ikl} B^i B^k B_l, \]  

(A1a)

\[ [A_i, A^j] \leftrightarrow [A_i, A^j] - c_{ikl} B^i B^k B_l, \]  

(A1b)

\[ A_j \leftrightarrow B_j, \]  

(A1c)

\[ a^\nu \leftrightarrow \alpha^\nu + B^i [A_i, \alpha^\nu], \]  

(A1d)

\[ a_\nu \leftrightarrow \alpha_\nu, \]  

(A1e)

derived in Ref. [11] using supercoherent states. This mapping preserves exactly the commutation relations of the collective algebra, as well as the commutation relations between single-fermion and pair operators, and also the anticommutation relations between single-fermion operators. However, a repeated application of the image of the pair creation operator (A1a) fails to produce an ideal even space spanned by bosons only. (See also Ref. [3].)

Second, to prove that the similarity transformation, transforming the mapping (A1) into the desired mapping (1), is given by Eq. (2), we proceed as follows. We begin by noticing that \( X^{-1} B_j X = B_j \), because \( X \) explicitly commutes with the boson annihilation operator \( B_j \), accounting for (1d). Next we show that \( X^{-1} R^j X = R^j - A^j \) by proving the identity

\[ [X, R^j - A^j] = A^j X. \]  

(A2)

This is achieved by splitting \( X \) into an infinite sum of terms which individually increase the number of ideal fermions by 0, 2, 4, ..., i.e., \( X = \sum_{n=0}^{\infty} X_n \). As \( R^j - A^j \) must conserve the number of ideal fermions, from Eq. (A2) we find the recurrence relation \( [X_{n+1}, R^j - A^j] = A^j X_n \) which allows all \( X_n \) to be determined, if \( X_0 \) is known.
We are free to choose \( X_0 \) as an arbitrary operator which commutes with \( R^j - A^j \), and Eq. (2) reflects the simplest choice \( X_0 = 1 \). Then the term \( X_1 \) is a solution of the recurrence relation for \( n = 0 \). Indeed we have

\[
\left( \frac{1}{C_F - \tilde{C}_F} A^i B_i, R^j - A^j \right) = \frac{1}{C_F - \tilde{C}_F} \left( A^i B_i, R^j - A^j \right) = \frac{1}{C_F - \tilde{C}_F} (C_F A^j - A^j C_F) = A^j, \tag{A3}
\]

where the first equality results from the fact that \( C_F \) commutes with \( R^j - A^j \) and the second one from the explicit calculation of the commutator. Similarly, we prove by induction that the \( X_{n+1} \) term of Eq. (2) is a solution of the recurrence relation provided the same is true for \( X_n \):

\[
[X_{n+1}, R^j - A^j] = \frac{1}{C_F - \tilde{C}_F} \left( A^i B_i X_n, R^j - A^j \right) = \frac{1}{C_F - \tilde{C}_F} \left( A^i B_i A^j X_{n-1} + (C_F A^j - A^j C_F) X_n \right) = \frac{1}{C_F - \tilde{C}_F} \left( A^j \frac{1}{C_F - \tilde{C}_F} (1 + \frac{\tilde{C}_F - C_F}{C_F - \tilde{C}_F}) A^i B_i X_{n-1} \right) = A^j X_n \tag{A4}
\]

where the operator \( \tilde{C}_F \) is in turn evaluated at "\( \tilde{\} \)."

Finally, to see that Eq. (1b) is invariant under the transformation (2) we use the facts that \( [C_F, [A_i, A^j]] = 0 \) and \( [A^m B_m, [A_i, A^j] - c^{ij}_{lk} B^k B_l] = 0 \) which follow from the algebraic closure and the Jacobi identities.
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FIGURES

FIG. 1. The lower part of an SO(8)⊗SO(5) spectrum. In the odd part (right), the (1/2,1/2) and (3/2,1/2) irreps of Spin(5) are shown only. The dotted lines connect states belonging to the same Spin(5) irrep.
TABLE I. Angular momentum content of the lowest Spin(5) representations is presented. In the upper (lower) part the representations relevant to the even (odd) system are shown, respectively.

| \((\tau_1, \tau_2)\) | \(J\)          |
|----------------------|-------------|
| (0, 0)               | 0           |
| (1, 0)               | 2           |
| (2, 0)               | 4, 2        |
| (3, 0)               | 6, 4, 3, 0  |
| (4, 0)               | 8, 6, 5, 4, 2 |
| \((\frac{1}{2}, \frac{3}{2})\) | \(\frac{3}{2}\) |
| \((\frac{3}{2}, \frac{5}{2})\) | \(\frac{7}{2}, \frac{7}{2}, \frac{1}{2}\) |
| \((\frac{5}{2}, \frac{7}{2})\) | \(\frac{11}{2}, \frac{9}{2}, \frac{7}{2}, \frac{5}{2}, \frac{3}{2}\) |
| \((\frac{3}{2}, \frac{5}{2})\) | \(\frac{9}{2}, \frac{7}{2}, \frac{5}{2}, \frac{3}{2}\) |
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