Properties of Trinomials of Height at least 2

V. Flammang and P. Voutier

Abstract

This paper is concerned with trinomials of the form \(z^n + az^m + b\), where \(0 < m < n\) are relatively prime integers and \(a\) and \(b\) are non-zero complex numbers. Typically (but not exclusively), \(a\) will be an integer with \(|a| \geq 2\), while \(b = \pm 1\). Our main results cover the irreducibility, Mahler measure and house of such trinomials.

1 Introduction

Trinomials have long been of interest to algebraists and number theorists.

For example, in 1907, O. Perron [Pe] proved that the trinomial \(z^n + az \pm 1\) is irreducible over \(\mathbb{Q}\) for \(|a| \geq 3\) and his results have been generalised since then. Still, our understanding of the irreducibility of such polynomials is weak. Theorem 1 below improves our knowledge when \(b = \pm 1\) and we have substantial evidence that the truth is captured in Conjecture 2 below.

The diophantine properties of trinomials are also of considerable importance. C.J. Smyth proved that if \(P\) is not a reciprocal polynomial, then the Mahler measure of \(P\) is bounded below by \(\theta_0 = 1.324717...\) the real zero of the trinomial \(z^3 - z - 1\). Furthermore, it is known that \(\theta_0\) is also the smallest Pisot number. The Mahler measure of \(P\) is also the smallest limit point of the Mahler measure of trinomials in \(\mathbb{Z}[z]\) and one of the smallest known limit points for general polynomials in \(\mathbb{Z}[z]\) [BM, Table 1, p. 412]. See Subsections 1.2 and 1.3 for our results on the Mahler measure of trinomials and, the related quantity, their house.

1.1 Irreducibility of trinomials \(z^n + az^m \pm 1 \in \mathbb{Z}[z]\)

We have the following result, generalising the result of Perron cited above.

**Theorem 1.** If \(a, m\) and \(n\) are integers satisfying \(n \geq 3, 0 < m < n, \gcd(m, n) = 1\) and \(|a| \geq n^2/3\), then \(x^n + ax^m \pm 1\) is irreducible over \(\mathbb{Q}\).

In fact, from calculations we have performed, it appears that the following much stronger conjecture is true. As in Perron’s result, we appear to have irreducibility once \(|a|\) is greater than a small absolute constant regardless of the degree of the middle term.

**Conjecture 2.** If \(a, m\) and \(n\) are integers satisfying \(n \geq 3, 0 < m < n, \gcd(m, n) = 1\) and \(|a| \geq 5\), then \(x^n + ax^m \pm 1\) is irreducible over \(\mathbb{Q}\).

Furthermore, there are only finitely many reducible polynomials of the form \(x^n + ax^m \pm 1\) with \(|a| = 3, 4\). They are \(x^3 \pm 3x^2 - 1, x^5 \pm 3x^4 - 1, x^{13} + 3x^4 - 1, x^{13} - 3x^4 + 1, x^{13} - 3x^6 - 1, x^{13} + 3x^6 + 1, x^{13} + 3x^7 \pm 1, x^{13} - 3x^9 \pm 1, x^{14} \pm 4x^5 - 1\) and \(x^{14} \pm 4x^9 - 1\).
Remark 1. Note that the condition that $\gcd(m, n) = 1$ in both Theorem 1 and Conjecture 2 is necessary. Examples from Bremner [Br] like $x^{33} + 67x^{11} + 1 = (x^3 + x + 1)(x^{30} - \cdots - 1)$ in his Theorem on pages 153–154 and $x^6 + (4\mu^4 - 4\mu)x^2 - 1 = (x^3 + 2\mu x^2 + 2\mu^2 x + 1)(x^3 - 2\mu x^2 + 2\mu^2 x - 1)$ for integers $\mu \neq 0, 1$ in his Postscript on page 154 demonstrate this.

1.2 Mahler Measure of trinomials $z^n + az^m + b \in \mathbb{C}[z]$

The Mahler measure of a polynomial $P(z) = a_0z^n + \cdots + a_n = a_0 \prod_{j=1}^{n} (z - \alpha_j) \in \mathbb{C}[z]$, with $a_0 \neq 0$, as defined by D. H. Lehmer [L] in 1933, is

$$M(P) = |a_0| \prod_{j=1}^{n} \max(1, |\alpha_j|).$$

Remark. Let $\alpha$ be a nonzero algebraic number. The Mahler measure of $\alpha$ is the Mahler measure of its minimal polynomial.

From a result of Kronecker, we know that, if $P$ is the minimal polynomial of an algebraic integer then $M(P) = 1$ and only if $P$ is a cyclotomic polynomial. D. H. Lehmer asked: does there exist a constant $c_0 > 0$ such that $M(P) > 1 + c_0$ for all $P$ not cyclotomic? The smallest known Mahler measure was found by D. H. Lehmer himself. It is the Mahler measure of the polynomial $z^{10} + z^9 + z^7 - z^6 - z^5 - z^4 - z^3 + z + 1$ for which the Mahler measure is 1.176280. . . .

A polynomial $P$ is reciprocal if $z^nP(1/z) = P(z)$ and an algebraic number is reciprocal if its minimal polynomial is reciprocal. As stated above, C.J. Smyth [S1] solved Lehmer’s problem when $P$ is not reciprocal. In the general case, the best known result is due to the second author [V], who proved that if $\alpha$ is an algebraic integer of degree $n \geq 2$ and not a root of unity, then

$$M(\alpha) \geq 1 + \frac{1}{4} \left( \frac{\log \log n}{\log n} \right)^3.$$

In 2013, the first author [F1] studied the Mahler measure of trinomials of height 1 and gave two criteria to identify those trinomials whose Mahler measure is less than 1.381356 . . . = $M(1 + z_1 + z_2)$. In this same work, she was able to prove a conjecture of Smyth on the Mahler measure of such trinomials for $n$ sufficiently large compared to $m$. Stankov [S] was interested in trinomials of the type $z^n - az - 1$ with $a \in (0, 2]$ and he presented the explicit expression by an integral of the limit of their Mahler measure when $n$ tends to $\infty$. We generalise this result to any trinomial of the form $z^n + az^k + b \in \mathbb{C}[z]$ in Theorem 3. In 2016, J-L. Verger-Gaugry [VG] studied the family of trinomials $z^n + z - 1$ for which he gave the asymptotic expansion of the Mahler measure as a function of $n$ only. Here, we prove the following results.

Theorem 3. Let $P(z) = z^n + az^m + b \in \mathbb{C}[z]$ with $ab \neq 0$.

(a) If $|a| - |b| \geq 1$, then $\lim_{n \to \infty} M(P) = |a|$.

(b) If $|b| - |a| \geq 1$, then $M(P) = |b|$.

(c) If $|a| + |b| \leq 1$, then $M(P) = 1$. 

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(d) If \(|a| - |b| < 1 < |a| + |b|\), then
\[
\lim_{n \to \infty} M(P) = \exp \left( \frac{1}{2\pi} \int_0^\gamma \ln \left( |a|^2 + 2|ab|\cos(t) + |b|^2 \right) dt \right),
\]
where we put \(\gamma = \arccos \left( \frac{1 - |a|^2 - |b|^2}{2|ab|} \right)\).

As a result of the following theorem, we have exact expressions for \(M(P)\) in all cases in Theorem 3 except (d).

**Theorem 4.** Let \(a\) and \(b \neq 0\) be fixed complex numbers with \(|a| - |b| \geq 1\). For all integers \(m\) and \(n\) with \(0 < m < n\) and \(\gcd(m, n) = 1\), we have
\[
\log M \left( z^n + az^m + b \right) = \log |a| - \sum_{k \geq 1} \frac{1}{km} (-1)^{kn} \left( \frac{k^n - 1}{km - 1} \right) \Re \left( b^{-km}(b/a)^{kn} \right). \tag{1}
\]

This behaviour is quite different from what happens when \(a, b = \pm 1\). In 2007, Duke [Duk] showed, for \(0 < m < n\) with \(\gcd(m, n) = 1\), that
\[
\log M \left( z^n + z^m + 1 \right) = \log M(x + y + 1) + \frac{c(n, m)}{n^2} + O \left( \frac{m}{n^3} \right),
\]
where \(c(n, m) = -\pi\sqrt{3}/6\) if 3 divides \(m + n\) and \(c(n, m) = \pi\sqrt{3}/18\) otherwise. The first author [F1] proved that similar results hold for the other trinomials with \(\pm 1\) as their coefficients and used them as mentioned above.

### 1.3 House of trinomials \(z^n + az^m \pm 1 \in \mathbb{Z}[z]\)

Let \(\alpha\) be a nonzero algebraic integer of degree \(n\), with conjugates \(\alpha_1 = \alpha, \ldots, \alpha_n\) and minimal polynomial, \(P\). The *house* of \(\alpha\) (and of \(P\)) is defined by:
\[
\overline{\alpha} = \max_{1 \leq i \leq n} |\alpha_i|.
\]

We have the inequality: \(\overline{\alpha} \geq M(\alpha)^{1/n}\). In 1965, A. Schinzel and H. Zassenhaus [SZ] conjectured that there exists a constant \(c > 0\) such that if \(\alpha\) is not a root of unity then \(\overline{\alpha} \geq 1 + c/n\). Thanks to the polynomial \(P(x) = x^n - 2\), we see that \(c \leq \log(2)\). In 1985, a result of C.J. Smyth [S1] led D. Boyd [B1] to conjecture that \(c\) should be equal to \(3/2 \log \theta_0\) where \(\theta_0 = 1.324717\ldots\), and that this value is attained too, via the polynomial \(x^3 + 2x^2 - 1\). The second author [V] proved that, if \(\alpha\) is an algebraic integer of degree \(n \geq 3\), not a root of unity, then
\[
\overline{\alpha} \geq 1 + \frac{1}{2n} \left( \log \log n / \log n \right)^3.
\]

In 1991, E.M. Matveev [M] proved that, if \(\alpha\) is an algebraic integer of degree \(n \geq 2\), not a root of unity, then \(\overline{\alpha} \geq \exp \left( \log (n + 0.5) / n^2 \right)\). Until Dimitov’s very recent result (see below), the best-known asymptotic result was given by A. Dubickas [Dub]:
\[
\overline{\alpha} > 1 + \frac{1}{n} \left( 64/\pi^2 - \epsilon \right) \left( \log \log n / \log n \right)^3 \quad \text{for} \ n > n_0(\epsilon).
\]
In 2007, G. Rhin and Q. Wu [RW2] verified the conjecture of Schinzel and Zassenhaus with the constant of Boyd up to degree 28. They also established that, if \( \alpha \) is an algebraic integer of degree \( n \geq 4 \), not a root of unity, then 
\[
\rho_n \geq \exp\left(3 \log(3n/3)/n^2\right)
\]
for \( n \leq 12 \), and 
\[
\rho_n \geq \exp\left(3 \log(n/2)/n^2\right)
\]
for \( n \geq 13 \). It appears that the result of [RW2] improves Matveev for \( n \geq 6 \). Very recently (Dec. 2019), Dimitrov [Di] has proven the Schinzel-Zassenhaus conjecture with \( c = \log(2)/4 \). His Theorem 1 states that when \( P(z) \in \mathbb{Z}[z] \) is a non-cyclotomic monic polynomial of degree \( n \) that is irreducible over \( \mathbb{Q} \) and \( \alpha \) is a zero of \( P(z) \), then 
\[
\alpha \geq 2^{1/(4n)} = 1 + \frac{\log(2)}{4n} + O\left(\frac{1}{n^2}\right).
\]

For trinomials, let \( n \geq 2 \) and \( \theta_n \) be the unique real zero in \((0,1)\) of the trinomial \( z^n + z - 1 \). By his method of asymptotic expansions of the zeros mentioned above, J-L. Verger-Gaugry [VG] obtained a direct proof of the conjecture of Schinzel-Zassenhaus for \( \theta_n^{-1} \), proving that 
\[
\left|\theta_n^{-1}\right| > 1 + \frac{(\log n)\left(1 - \frac{\log\log n}{\log n}\right)}{n}.
\]

Here we focus on trinomials of the form \( z^n + az^m \pm 1 \), where \( 0 < m < n \) are relatively prime integers and \( a \) is a positive real number. Upon replacing \( z \) by \( -z \), we find that all such polynomials are of the form 
\[
R_{n,m,a}(z) = z^n - az^m + 1 \quad \text{with} \quad a > 0, \ m \text{ odd and } n \text{ even};
\]
\[
S_{n,m,a}(z) = z^n + az^m - 1 \quad \text{with} \quad a > 0 \text{ and } n \text{ odd}; \text{ or}
\]
\[
T_{n,m,a}(z) = z^n - az^m - 1 \quad \text{with} \quad a > 0.
\]

**Theorem 5.** For any positive real number \( a \geq 2 \) and relatively prime positive integers \( 0 < m < n \) with \( m \) odd and \( n \) even, \( r_{n,m,a}(z) \) has a real root \( r_{n,m,a}^{(1)} \) and 
\[
r_{n,m,a}^{(1)} \geq 1 + \frac{\log(a - 1)}{n - m}.
\]
Therefore, if we restrict \( a \) to be an integer with \( a \geq 2 \), we have 
\[
|R_{n,m,a}| \geq 1 + \frac{\log(a - 1)}{n - m}.
\]

**Theorem 6.** For any positive real number \( a \geq 2 \) and relatively prime positive integers \( 0 < m < n \) with \( m \) even and \( n \) odd, \( S_{n,m,a}(z) \) has a real root \( s_{n,m,a}^{(3)} \) and 
\[
\left|s_{n,m,a}^{(3)}\right| \geq 1 + \frac{\log(a - 1)}{n - m}.
\]
If we restrict \( a \) to be an integer with \( a \geq 2 \), we have 
\[
|S_{n,m,a}| \geq 1 + \frac{\log(a - 1)}{n - m}.
\]

Unfortunately, when \( m \) and \( n \) are both odd, the single real zero of \( S_{n,m,a} \) is between 0 and 1, while \( |S_{n,m,a}| > 1 \), so we are unable to obtain a non-trivial lower bound for \( |S_{n,m,a}| \) in this case.
Theorem 7. For any positive real number $a \geq 2$ and relatively prime positive integers $0 < m < n$, $T_{n,m,a}(z)$ has a real root $t^{(1)}_{n,m,a} > 1$ and

$$t^{(1)}_{n,m,a} > 1 + \frac{\log(a)}{n-m}.$$ 

Therefore, if $a$ is restricted to be an integer with $a \geq 2$, we have

$$|T_{n,m,a}| > 1 + \frac{\log(a)}{n-m}.$$ 

An algebraic integer $\alpha$ of degree $n$ is extremal if $\min$ is the minimum of the houses of the algebraic integers of degree $n$. Denote by $m(n)$ this minimum. From Smyth’s example, $P_n(x) = x^{3n} + x^{2n} - 1$, given above (again, see [B1, S1]), it is known that $m(n) \leq \theta_0^{3/(2n)} = \lfloor \theta_0 \rfloor$. Here we have

**Corollary 8.** (a) For all positive integers $a$, $m$ and $n$ with $a \geq 2$, gcd$(m,n) = 1$, $m$ odd and $n$ even, if $R_{n,m,a}$ is irreducible over $\mathbb{Q}$, then its zeros are not extremal.

(b) For all positive integers $a$, $m$ and $n$ with $a \geq 2$, gcd$(m,n) = 1$, $m$ even and $n$ odd, if $S_{n,m,a}$ is irreducible over $\mathbb{Q}$, then its zeros are not extremal.

(c) For all positive integers $a$, $m$ and $n$ with $a \geq 2$ and gcd$(m,n) = 1$, if $T_{n,m,a}$ is irreducible over $\mathbb{Q}$, then its zeros are not extremal.

2 Proof of Theorem 1

To prove Theorem 1, we use the following result of Schinzel.

**Lemma 9** (Schinzel). For positive integers $0 < m < n$, put $m_1 = m / \text{gcd}(m,n)$ and $n_1 = n / \text{gcd}(m,n)$.

Let $a, b, c \in \mathbb{Z}\setminus\{0\}$, gcd$(a,b,c) = 1$. If $ax^n + bx^m + c$ is reducible, then at least one of the following four conditions is satisfied:

(a) $|b| \leq |a|^{m_1} |c|^{n_1-m_1} + 1$;

(b) $|b| \leq \frac{2m_1(n_1-m_1)}{\log(2m_1(n_1-m_1))}|a|^{m_1} |c|^{-m_1} \min\{|a|, |c|\} = 1$ and $\sqrt[n_1]{\text{min}\{|a|, |c|\}} \in \mathbb{Z}$ for some prime $p|n_1$;

(c) for some $q$ |gcd$(m,n)$, $q$ a prime or $q = 1$, $\sqrt[q]{|a|} \in \mathbb{Z}$, $\sqrt[q]{|c|} \in \mathbb{Z}$ and if $q = 2$, then $(-1)^{m_1} ac > 0$, while if $q = 4$, then $ac > 0$ and $n_1 \equiv 0 \mod 2$;

(d) $|a| \equiv 0 \mod 2$ and either $\sqrt[4]{|a|} \in \mathbb{Z}$, $\sqrt[4]{|c|} \in \mathbb{Z}$ or $\sqrt[4]{|a|} \in \mathbb{Z}$, $\sqrt[4]{|c|} \in \mathbb{Z}$.

**Proof.** This is Theorem 9 on pages 12–13 of [Sch].

**Proof.** To prove Theorem 1, we suppose that $x^n + ax^m + 1$ is reducible.

We apply Lemma 9 with $a = 1$, $c = \pm 1$ and $b$ equal to our $a$ here.

Condition (a) implies that $|a| \leq 2$.

Since $m_1(n_1-m_1) \leq n_1^2/4$ for all $0 \leq m_1 \leq n_1$, condition (b) implies that $|a| < n^2/2$.

A quick calculation for small relatively prime values of $m$ and $n$ shows that we must have $|a| < 0.321n^2$ for $n \geq 3$.

Since gcd$(m,n) = 1$, conditions (c) and (d) cannot hold.

\[ \square \]
3 Proof of Theorem 3

The proof requires two preliminary results.

Lemma 10. If \( Q \) is a polynomial with complex coefficients, then

\[
\lim_{n \to \infty} M(z^n + Q(z)) = \exp \left( \frac{1}{2\pi} \int_0^{2\pi} \log \max \left( 1, |Q(e^{it})| \right) dt \right).
\]

Proof. D. Boyd [B2, Appendix 3] proved that \( \lim_{n \to \infty} M(F(z, z^n)) = M(F(z, w)) \) if \( F \) is a polynomial. Thus we have \( \lim_{n \to \infty} M(z^n + Q(z)) = M(w + Q(z)) \). Now we apply Jensen’s formula \([J]\) with respect to the variable \( w \) to obtain the lemma (also see \([B2, \text{equation (21)}]\)). \( \square \)

Corollary 11.

\[
\lim_{n \to \infty} M(z^n + az^m + b) = \exp \left( \frac{1}{2\pi} \int_0^{2\pi} \log \max \left( 1, |ae^{it} + b| \right) dt \right).
\]

Proof. Lemma 10 gives \( \lim_{n \to \infty} M(z^n + az^m + b) = \exp \left( \frac{1}{2\pi} \int_0^{2\pi} \log \max \left( 1, |ae^{itm} + b| \right) dt \right) \). Furthermore,

\[
\int_0^{2\pi} \log \max \left( 1, |ae^{itm} + b| \right) dt = \frac{1}{m} \sum_{\ell=0}^{m-1} \int_0^{2\pi} \log \max \left( 1, |ae^{it} + b| \right) dt
\]

proving the corollary. \( \square \)

We are now able to prove Theorem 3. Let \( P(z) = z^n + az^m + b \) be a trinomial with \( a, b \in \mathbb{C} \). We will need this preliminary calculation: if \( a = |a|e^{iu} \) and \( b = |b|e^{iv} \) then \( |ae^{it} + b|^2 = |a|e^{(t+u-v)} + |b|^2 = |a|^2 + 2|a||b|\cos(t') + |b|^2 \) where \( t' = t + u - v \).

(a) \( |a| - |b| \geq 1 \):

We have \( |ae^{it} + b|^2 \geq (|a| - |b|)^2 \geq 1 \). Therefore, \( \max(1, |ae^{it} + b|) = |ae^{it} + b| \), so Corollary 11 gives

\[
\lim_{n \to \infty} M(z^n + az^m + b) = M(az + b) = |a| \max(1, |b|/|a|) = \max(|a|, |b|).
\]

(b) \( |b| - |a| \geq 1 \):

We use Rouché’s Theorem (again, see Corollary on page 153 \([Ah]\)). We want to prove that all the zeros of \( z^nP(1/z) = bz^n + az^{-m} + 1 \) have absolute value at most 1.

Suppose first that \( |b| - |a| > 1 \). Putting \( f(z) = bz^n \) and \( g(z) = az^{-m} + 1 \), we see that \( |g(z)| \leq |a| + 1 < |b| = |f(z)| \) on the unit circle, since \( |b| - |a| > 1 \). Hence \( P(z) = f(z) + g(z) \) and \( f(z) \) have the same number of zeros inside the unit circle. Since \( f(z) \) has \( n \) such zeros, so does \( P(z) \). Hence all the zeros of \( P \) have absolute value at least 1 and \( M(P) = |b| \).

Now suppose that \( |b| - |a| = 1 \) and let \( \varepsilon > 0 \). We now consider \( g(z) \) and \( f(z) \) on the circle of radius \( 1 + \varepsilon \) centred at 0. Now \( |g(z)| \leq |a|(1 + \varepsilon)^{-m} + 1 = |a| + |a|\varepsilon_1 + 1 \) and \( |f(z)| = b(1 + \varepsilon)^n = |b| + |b|\varepsilon_2 = |a| + 1 + |a|\varepsilon_2 + \varepsilon_2 \), for some \( 0 < \varepsilon_1 < \varepsilon_2 \). So \( f(z) - |g(z)| \geq |a|\varepsilon_2 + \varepsilon_2 - |a|\varepsilon_1 > 0 \).
Hence within any circle of radius $1 + \varepsilon$ centred at 0, $P(z) = f(z) + g(z)$ and $f(z)$ have the same number of zeros. Since $f(z)$ has $n$ such zeros, so does $P(z)$. Taking the limit as $\varepsilon \to 0$, we see that all the zeros of $P(z)$ have absolute value at most 1. Hence $M(P) = |b|$.

(c) $|a| + |b| \leq 1$:
We use Rouché’s Theorem here too and proceed in the same way as above. We put $f(z) = z^n + b$ and $g(z) = az^m$, and consider the cases $|a| + |b| < 1$ and $|a| + |b| = 1$ separately.

(d) $|a - b| < 1 < |a + b|:
We have $|ae^{it} + b| > 1$ iff $t \in (0, \gamma)$ where $\gamma = \arccos \left( \frac{1 - |a|^2 - |b|^2}{2|ab|} \right)$. From Corollary 11, we deduce

$$\lim_{n \to \infty} M(z^n + az^m + b) = \exp \left( \frac{1}{2\pi} \int_0^\gamma \log(|a|^2 + 2|ab| \cos(t) + |b|^2) \, dt \right).$$

4 Proof of Theorem 4

We first need a lemma to show that we can interchange an integral and a sum that arise in our proof.

**Lemma 12.** Let $a$ and $b \neq 0$ be complex numbers with $|a| - |b| \geq 1$ and let $m$ and $n$ be integers satisfying $0 < m < n$. Then

$$\int_0^{2\pi} \sum_{k \geq 1} \frac{1}{k} \left( \frac{e^{int}}{-ae^{int} - b} \right)^k \, dt = \sum_{k \geq 1} \int_0^{2\pi} \frac{1}{k} \left( \frac{e^{int}}{-ae^{int} - b} \right)^k \, dt.$$  

**Proof.** According to Fubini’s Theorem, this relationship holds if

$$\int_0^{2\pi} \sum_{k \geq 1} \left| \frac{1}{k} \left( \frac{e^{int}}{-ae^{int} - b} \right)^k \right| \, dt < \infty \quad \text{or} \quad \sum_{k \geq 1} \int_0^{2\pi} \left| \frac{1}{k} \left( \frac{e^{int}}{-ae^{int} - b} \right)^k \right| \, dt < \infty.$$

We will use the first condition here:

$$\int_0^{2\pi} \sum_{k \geq 1} \frac{1}{k} \left( \frac{1}{ae^{int} + b} \right)^k \, dt.$$  

For $|a| - |b| > 1$, then $|ae^{int} + b| > 1$, so $\sum_{k \geq 1} \frac{1}{k} \left( \frac{1}{ae^{int} + b} \right)^k$ converges and can be bounded from above independently of $t$. Since the integral is over a bounded set, it is finite and the lemma holds.

For $|a| - |b| = 1$, we use the fact that $\sum_{k \geq 1} z^k/k = -\log(1 - z)$ if $|z| \leq 1$ and $z \neq 1$. So we need to show that

$$\int_0^{2\pi} \log \left( 1 - \frac{1}{ae^{int} + b} \right) \, dt < \infty.$$  

The only place where $|ae^{int} + b| = 1$ is where $ae^{int} = (|b| + 1)(-b)/|b|$. Note that there are $m$ such values of $t$ satisfying $0 \leq t < 2\pi$. Elsewhere $|ae^{int} + b| > 1$, so away from these $m$ values of $t$, the integral is bounded. We only need to show that in all neighbourhoods of each of
these \( m \) values of \( t \), the integral is also bounded. It suffices to consider only one such value of \( t \), \( t_0 \), and only neighbourhoods on one side of such a \( t \). That is to show that

\[
\int_{t_0}^{t_0 + \varepsilon} \log \left( 1 - \left| \frac{1}{ae^{it} + b} \right| \right) dt < \infty,
\]

for \( \varepsilon > 0 \).

Suppose that \( a \) and \( b \) are both real numbers. Here \( |ae^{it} + b| = \sqrt{a^2 + b^2 + 2ab \cos(t)} \).

If \( a \) and \( b \) have different signs, then we need to consider what happens near \( t_0 = 0 \). We want a lower bound for \( 1 - \frac{1}{\sqrt{a^2 + b^2 + 2ab \cos(t)}} \) of the form \( ct \) where \( c > 0 \) for \( t \) in some interval whose left endpoint is \( t_0 = 0 \). There are two cases to consider: (i) \( a > 0 \), \( b < 0 \) and \( a = -b + 1 \); or (ii) \( a < 0 \), \( b > 0 \) and \( a = -b - 1 \). In the first case we consider the value of the function

\[
1 - \frac{1}{\sqrt{a^2 + b^2 + 2ab \cos(t)}} = 1 - \frac{1}{\sqrt{2b^2 - 2b + 1 + 2(1-b)\cos(t)}}
\]

at \( t = 0 \) (where it has the value \( 0 \)) and \( t = \pi/2 \) (where it has the value \( 1 - \frac{1}{\sqrt{2b^2 - 2b + 1}} \)). This function is also convex for \( t \in [0, \pi/2] \) (this follows from the fact that \( 2b^2 - 2b + 1 + 2(1-b)\cos(t) \) is non-negative and increasing in this interval, approaching \( b^2 + (b-1)^2 \) as \( t \to \pi/2 \), and that \( 1 - 1/\sqrt{x} \) is a convex function). So we have

\[
1 - \frac{1}{\sqrt{a^2 + b^2 + 2ab \cos(t)}} \geq \frac{2}{\pi} \left( 1 - \frac{1}{\sqrt{2b^2 - 2b + 1}} \right) t,
\]

for \( 0 \leq t \leq \pi/2 \). That is, we can take \( c \) above to be \( \frac{2}{\pi} \left( 1 - \frac{1}{\sqrt{2b^2 - 2b + 1}} \right) \).

Now \( \int_0^\varepsilon \log(ct) \, dt = \varepsilon \log(c\varepsilon) - \varepsilon \), so

\[
\lim_{\varepsilon \to 0} \int_0^\varepsilon \log(ct) \, dt = \lim_{\varepsilon \to 0} \varepsilon \log(c\varepsilon) - \varepsilon = \lim_{\varepsilon \to 0} \log(c\varepsilon) = \lim_{\varepsilon \to 0} \frac{1}{1/\varepsilon} = -\lim_{\varepsilon \to 0} \varepsilon/c = 0,
\]

by L'Hôpital's rule. Thus

\[
\int_{t_0}^{t_0 + \varepsilon} \log \left( 1 - \left| \frac{1}{ae^{it} + b} \right| \right) dt < \infty,
\]

and so, as argued above,

\[
\int_0^{2\pi} \log \left( 1 - \left| \frac{1}{ae^{imt} + b} \right| \right) dt < \infty.
\]

The other cases, namely when \( a < 0 \), \( b > 0 \) and \( a = -b - 1 \); and when \( a \) and \( b \) have the same sign are treated in the very same way, proving the lemma when \( a, b \in \mathbb{R} \).

Furthermore, a more complicated version of this same argument can be used to prove the lemma for any \( a, b \in \mathbb{C} \) satisfying the conditions of the lemma. \( \square \)

The idea of the proof of Theorem 4 is the same as the one used by the first author in the study of trinomials of height 1 (see [F1]).

Using Jensen's formula, we put

\[
\lambda_{n,m,a} = \log M(z^n + az^m + b) = \frac{1}{2\pi} \int_0^{2\pi} \log |e^{imt} + ae^{imt} + b| \, dt.
\]
Note that $\lambda_{n,m,a} = \log M \left( -z^n - az^m - b \right)$ holds, so we work with $-z^n - az^m - b$ here. This change simplifies slightly what follows.

Using Jensen’s formula again and since $|a| > |b|$ (from our hypothesis that $|a| - |b| \geq 1$), we have

$$\frac{1}{2\pi} \int_0^{2\pi} \log \left| -ae^{int} - b \right| \, dt = \log M \left( -az^m - b \right) = \log |a|.$$ 

Thus

$$\lambda_{n,m,a} - \log |a| = \frac{1}{2\pi} \int_0^{2\pi} \log \left| -e^{int} - ae^{int} - b \right| - \log \left| -ae^{int} - b \right| \, dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \log \left| 1 - \frac{e^{int}}{-ae^{int} - b} \right| \, dt.$$ 

Writing any $0 \neq x \in \mathbb{C}$ as $x = re^{i\theta}$ where $r > 0$ and $|\theta| \leq \pi$, we have $\log(x) = i\theta + \log(r)$ so $|x| = \Re(\log(x))$ -- we have written this using the principal value of the logarithm function, but it holds for any branch of the logarithm function. Therefore,

$$\lambda_{n,m,a} - \log |a| = \frac{1}{2\pi} \Re \int_0^{2\pi} \log \left( 1 - \frac{e^{int}}{-ae^{int} - b} \right) \, dt.$$ 

Applying Lemma 12 and the series expansion

$$\log \left( 1 - \frac{e^{int}}{-ae^{int} - b} \right) = -\sum_{k \geq 1} \frac{1}{k} \left( \frac{e^{int}}{-ae^{int} - b} \right)^k,$$

we obtain

$$\lambda_{n,m,a} - \log |a| = -\frac{1}{2\pi} \sum_{k \geq 1} \frac{1}{k} \Re \int_0^{2\pi} e^{int} \left( -ae^{int} - b \right)^{-k} \, dt. \tag{2}$$

From $z = e^{it}$, we have $dz = ie^{it}dt$, so

$$I_k = -i \int_{|z|=1} z^{kn-1} (-az^m - b)^{-k} \, dz.$$ 

We evaluate this integral using the Residue theorem:

$$I_k = -i(2\pi i) \sum_{r; r^m = -b/a} \text{res} \left( z^{kn-1} (-az^m - b)^{-k}; r \right),$$

where the sum is over all $m$-th roots of $-b/a$ and $\text{res} \left( f(z); z_0 \right)$ is the residue of a function $f(z)$ at $z = z_0$.

Now observe that $z^{kn-1} (-az^m - b)^{-k}$ has one more pole when we consider this function over the Riemann sphere. It has a pole at $\infty$ too. Furthermore, the sum of the residues of a function over the Riemann sphere equals 0 [Ca, equation (2.7), p. 94], so

$$I_k = -2\pi \text{res} \left( z^{kn-1} (-az^m - b)^{-k}; \infty \right), \tag{3}$$

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Lemma 13. Let $$a \geq 2$$ be a real number, $$m$$ and $$n$$ are positive relatively prime integers with $$0 < m < n$$.

(a) Suppose also that $$m$$ odd and $$n$$ even. $$R_{n,m,a}(z)$$ has two real zeros, $$r_{n,m,a}^{(1)}$$ and $$r_{n,m,a}^{(2)}$$ satisfying $$r_{n,m,a}^{(1)} > 1$$ if $$a > 2$$, $$r_{n,m,a}^{(1)} = 1$$ if $$a = 2$$, and $$0 < r_{n,m,a}^{(2)} < 1$$.

5 Proofs of Theorems 5–7 and Corollary 8

We start with a lemma about the location of the real zeros of the trinomials we are considering.

It is also known [Ca, top of page 92] that

$$\text{res} \left( f(z); \infty \right) = - \text{res} \left( \frac{1}{z^2} f \left( \frac{1}{z} \right); 0 \right),$$

so we compute this quantity now. With $$f(z) = z^{kn-1} (-az^m - b)^{-k}$$, we have

$$\frac{1}{z^2} f \left( \frac{1}{z} \right) = \frac{z^{-1-kn+km}}{(-a - bz^m)^k}.$$

The negative binomial series $$(-a - bz^m)^{-k}$$ equals

$$\sum_{i=0}^{\infty} \binom{-k}{i} (-b)^i z^{im} (-a)^{-k-i} = (-1)^k \sum_{i=0}^{\infty} \binom{-k}{i} b^i z^{im} a^{-k-i}.$$

To get the residue, we need the coefficient for $$z^{kn-km}$$ term in this sum. I.e., $$i = k(n-m)/m$$. If $$m \nmid kn$$, then there is no such coefficient and the residue is 0. Otherwise, the coefficient of the $$z^{kn-km}$$ term is

$$\binom{-k}{k(n-m)/m} b^{k(n-m)/m} a^{-k-k(n-m)/m} = (-1)^k \binom{k(n-m)/m}{k(n-m)/m} \binom{kn/m-1}{kn/m-k} b^{kn/m} a^{-kn/m},$$

since $$\binom{-r}{s} = (-1)^s \binom{r+s-1}{s}$$ for positive integers $$r$$ and $$s$$. In fact, recall that $$\gcd(m, n) = 1$$, so $$m|kn$$ if and only if $$m|k$$. Therefore, from this expression and $$\binom{kn/m-1}{kn/m-k} = \binom{kn/m-1}{kn/m-k}$$, we obtain

$$\text{res} \left( z^{kn-1} (-az^m - b)^{-k}; \infty \right) = \begin{cases} 
(-1)^{kn/m} \binom{kn/m-1}{k-1} b^{kn/m} a^{-kn/m} & \text{if } m|k, \\
0 & \text{otherwise}.
\end{cases}$$

Applying this expression to (3) and then (2), we obtain

$$\lambda_{n,m,a} - \log |a| = - \sum_{k \geq 1} \frac{1}{km} (-1)^{kn} \binom{kn/m-1}{k-1} \text{Re} \left( b^{-km}(b/a)^{kn} \right),$$

which is equation (1).
(b) Suppose also that \( n \) is odd. If \( m \) is odd, then \( S_{n,m,a}(z) \) has one real zero. If \( m \) is even, then \( S_{n,m,a}(z) \) has three real zeros. There is always one real zero, \( 0 < s_{n,m,a}^{(1)} < 1 \). If \( m \) is even, then there is another real zero, \(-1 < s_{n,m,a}^{(2)} < 0 \) and a third real zero, \( s_{n,m,a}^{(3)} \). If \( a > 2 \), then \( s_{n,m,a}^{(3)} < -1 \). If \( a = 2 \), then \( s_{n,m,a}^{(3)} = -1 \).

(c) If \( n \) is odd, then \( T_{n,m,a}(z) \) has three real zeros if \( m \) is odd and one real zero if \( m \) is even. There is always one zero, \( t_{n,m,a}^{(1)} > 1 \). If \( m \) is odd, then there is another real zero, \(-1 < t_{n,m,a}^{(2)} < 0 \) and a third real zero, \( t_{n,m,a}^{(3)} \leq -1 \).

If \( n \) is even, then \( T_{n,m,a}(z) \) has two real zeros. One, \( t_{n,m,a}^{(1)} > 1 \) and the other, \(-1 < t_{n,m,a}^{(2)}. \)

Proof. We prove only part (c), as the proofs of parts (a) and (b) are identical.

Using Descartes’ Sign Rule, we find there is one positive real zero, since there is one sign change among the coefficients of \( T_{n,m,a}(z) \). Since \( T_{n,m,a}(1) = -a \) and \( \lim_{z \to +\infty} T_{n,m,a}(z) = +\infty \), this zero is strictly greater than 1.

Applying Descartes’ Sign Rule to \( T_{n,m,a}(-z) = (-1)^n z^n + (-1)^{m-1} a z^m - 1 \), there are two negative zeros if \( m \) and \( n \) are both odd; no negative zeros if \( n \) is odd and \( m \) is even; and one negative zero if \( n \) is even (in which case, \( m \) is odd).

We have \( T_{n,m,a}(-1) = (-1)^n - (-1)^{m-1} a - 1 \) equals \( (-1)^{m-1} a = a \) if \( n \) is even (recalling that \( m \) is odd in this case) and \( (-1)^{m-1} a - 2 \) if \( n \) is odd. For \( n \) even and since \( T_{n,m,a}(0) = -1 \), it follows that the unique negative real zero is strictly between -1 and 0.

Lastly, for \( n \) odd and \( m \) odd, we have \( T_{n,m,a}(0) = -1 \) and \( T_{n,m,a}(-1) = a - 2 \). So if \( a > 2 \) and since \( \lim_{z \to -\infty} T_{n,m,a}(z) = -\infty \), there must be one in \((-1, 0)\) and another in \((-\infty, -1)\). For \( a = 2 \), there is a zero at \( z = -1 \) and the second negative zero turns out to be less than -1 if \( m > n/2 \) or in \((-1, 0)\) if \( m < n/2 \).

5.1 Proof of Theorem 5

When \( n \) is even and \( m \) is odd, from Lemma 13(a), \( r_{n,m,a}^{(1)} \) is the unique real zero of the trinomial \( R_{n,m,a} \) which satisfies \( r_{n,m,a}^{(1)} \geq 1 \) and put \( r_{n,m,a}^{(1)} = 1 + t \) with \( t \geq 0 \). From the expression for \( R_{n,m,a} \), we see that \((1 + t)^a - a(1 + t)^m + 1 = 0 \), so \((1 + t)^a - a(1 + t)^m + 1 + (1 + t)^m \geq 1 \), i.e., \((1 + t)^a - (a - 1)(1 + t)^m \geq 0 \). Thus

\[(1 + t)^a \geq (a - 1)(1 + t)^m.\]

Let \( t_0 \) be the largest real number such that

\[(1 + t_0)^a = (a - 1)(1 + t_0)^m,\]

then \( t \geq t_0 \) and \( t_0 = \exp(\log(a - 1)/(n - m)) - 1 \). Using the Taylor expansion of \( \exp(x) \), \( t_0 \geq \log(a - 1)/(n - m) \) and the theorem follows.

5.2 Proof of Theorem 6

When \( n \) is odd and \( m \) is even, recall from Lemma 13(b) that \( s_{n,m,a}^{(3)} \) is the unique real zero of the trinomial \( S_{n,m,a} \) which satisfies \( s_{n,m,a}^{(3)} \leq -1 \) and put \( s_{n,m,a}^{(3)} = -(1 + t) \) with \( t \geq 0 \). From the
expression for $S_{n,m,a}$, we see that $(1 + t)^n - a(1 + t)^m + 1 = 0$, so as above

$$(1 + t)^n \geq (a - 1)(1 + t)^m$$

and the proof follows as above too.

### 5.3 Proof of Theorem 7

From Lemma 13(c), $t_{n,m,a}^{(1)}$ is the unique real zero of $T_{n,m,a}$ which satisfies $t_{n,m,a}^{(1)} > 1$ and put $t_{n,m,a}^{(1)} = 1 + t$ with $t > 0$. From the expression for $T_{n,m,a}$, we know that

$$(1 + t)^n > a(1 + t)^m.$$  

If

$$(1 + t_0)^n = a(1 + t_0)^m,$$

then $t > t_0$ and $t_0 = \exp(\log(a)/(n - m)) - 1$. Using the Taylor expansion of $\exp(x)$, $t_0 > \log(a)/(n - m)$ and our result follows.

### 5.4 Proof of Corollary 8

We prove only part (c) as the proof for the other two parts is identical.

It is clear that $m(n) \leq 2^{1/n}$. We have $T_{n,m,a}((2^{1/n}) = 1 - a2^{m/n} < 0$, since $a \geq 2$. Since $T_{n,m,a}(z) > 0$ for all $z > \theta_{m,n,a}$, we deduce $t_{n,m,a}^{(1)} > 2^{1/n}$, which implies the non-extremality of $t_{n,m,a}^{(1)}$.

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V. Flammang:
UMR CNRS 7502. IECL, Université de Lorraine, site de Metz,
Département de Mathématiques, UFR MIM,
Ile du Saulcy, CS 50128. 57045 METZ cedex 01. FRANCE
E-mail address: valerie.flammang@univ-lorraine.fr

P. Voutier:
London, UK
E-mail address: Paul.Voutier@gmail.com