Calculation of the circular plates’ stability in stresses

Yu Ya Tyukalov
Polytechnic Institute, Vyatka State University (PI of VSU), 36, Moskovskaya str.,
Kirov 610000, Russia

E-mail: yutvgu@mail.ru

Abstract. The method is proposed for solving the plates stability problems by the finite element method based on piecewise constant approximations of moments. The solution was obtained on the basis of the principles of minimum additional energy and the possible displacements. To ensure the moment fields equilibrium, the equilibrium algebraic equations of grid nodes are compiled using the possible displacements principle. Such equilibrium equations are written as a system of linear homogeneous algebraic equations. Using the Lagrange multipliers method the equilibrium algebraic equations are including to the functional. The proposed method ensures the critical stress convergence to the exact value from below, which provides reserve of the plate stability.

1. Introduction
The round plates elastic stability analysis is using when a significant number of structural elements, especially thin plates, are designing. To ensure the margin of structures stability, it is important to determine the lower critical load limit. The solutions of various structures stability problems by the finite element method in displacements had get widely spread [1-2]. Such solutions are using the various functionals and types of the displacement fields approximations. This solutions allow to consider the features of structural elements deformations [3-15]. Alternative variational principles, in particular, the principle of additional energy minimum, are also used to solve various problems of the elasticity theory [16-22]. In [17-19,21,22], piecewise constant approximations of the moment fields were used to calculate bent plates by the finite element method. It is shown that in this case, when the finite element mesh is grinding, the displacements are going to exact values from above. Thus, it can be assumed that using of such approximations to analyse of the plates’ stability will allow us to obtain the critical load lower boundary.

This work aim is to develop calculating method of the thin round plates’ stability, which is based on piecewise constant approximations of the moments’ fields.

2. Methods
The stability problem solution of thin plates corresponding to Kirchhoff’s theory can be constructed using the functional of additional energy [1]:

$$\Pi^c = \frac{1}{2} \frac{12}{E\cdot t} \int_{\Omega} \left( M_x^2 + M_y^2 - 2\nu M_x M_y + 2(1+\nu)M_w^2 \right) \mathrm{d}\Omega.$$  (1)
$E$ is the elastic modulus of the plate material; $t$ is the plate thickness; $\nu$ - Poisson's ratio; $M_x$ is bending moment directed along the $X$ axis; $M_y$ is bending moment directed along the $Y$ axis; $M_{xy}$ is torque. We write the functional (1) on matrix form:

$$II^c = \frac{1}{2} \int_{\Omega} M_{x}^T E^{-1} M_{x} \, d\Omega, \quad M_{\Omega} = \begin{pmatrix} M_x \\ M_y \\ M_{xy} \end{pmatrix}, \quad E^{-1} = \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix}. \quad (2)$$

We represent the subject area as a set of triangular finite elements and introduce the notations $M_{x,i}, M_{y,i}, M_{xy,i}$ for the nodal moments and $M_k$ for the nodal moments vector of finite element. In the finite element region, we will approximate the moment fields by using piecewise constant functions [17-19]. To simplify the expressions, we introduce auxiliary unit step functions $\psi_i(x,y)$ and the corresponding diagonal matrices $\psi_i$, as well as the approximation matrix of the moments $Z_k$ at the region of triangular finite element.

$$\psi_i(x,y) = \begin{cases} 1, & (x,y) \in \Omega_i \\ 0, & (x,y) \notin \Omega_i \end{cases}, \quad \psi_i = \begin{bmatrix} \psi_{i1} & \psi_{i2} & \psi_{i3} \end{bmatrix}, \quad Z_k = \begin{bmatrix} \psi_{1k} & \psi_{2k} & \psi_{3k} \end{bmatrix}. \quad (3)$$

Then $M_{\Omega} = Z_k M_k$ and functional (2) will have the following form:

$$II^c_k = \frac{1}{2} \int_{\omega_k} M_{x}^T \big( Z_k^T E^{-1} Z_k \big) M_{x} \, d\Omega. \quad (4)$$

We introduce the notation $D_k$ for the local flexibility matrix of a finite element.

$$D_k = \int_{\omega_k} Z_k^T E^{-1} Z_k \, d\Omega. \quad (5)$$

Note that the matrix elements are calculated analytically and has the block-diagonal form. Expressions of matrix elements $D_k$ can be found in [17-19]. From the flexibility matrices of finite elements, the global flexibility matrix $D$ is formed. Also, from the vectors $M_i$ the global vector $M$ is formed. Using the introduced notation, we obtain the following expression of functional (2):

$$II^c = \frac{1}{2} M^T D M. \quad (6)$$

It is important to note that the matrix $D$ is block-diagonal. The block is the square matrix measuring 3 by 3. Therefore, the matrix $D$ is analytically invertible. This case greatly simplifies the construction of the problem solution.

In accordance with the Castigliano principle, the moment fields must satisfy the equilibrium equations and static boundary conditions. To ensure the equilibrium of the moment fields, we will compose the grid nodes equilibrium equations using the possible displacements principle[17-22]. Such equilibrium equations for finite element grid nodes are written as the system of linear homogeneous algebraic equations:

$$C_i^M M_{i,eq} = 0, \quad i \in \Xi. \quad (7)$$
$M_{i,eq}$ is the vector of nodal moments included in the equilibrium equation of node $i$. The equilibrium equation of the node will include unknown moments of nodes that belong to finite elements that are adjoining to node $i$. $C_i$ is the vector of coefficients at unknowns entering into the equilibrium equation. In [17], the algorithm and necessary formulas for obtaining equilibrium equations (7) are presented. When we solving stability problems, at nodes lying on some boundaries of the domain, the moment $M_{n,i}$ (8) normal to the boundary or the moment directed along the boundary $M_{s,i}$ (9), or both, can be equal to zero.

$$M_{n,i} = M_{x,i} \cos^2 \alpha_i + M_{y,i} \sin^2 \alpha_i - 2M_{xy,i} \sin \alpha_i \cos \alpha_i = 0.$$  

$$M_{s,i} = (M_{x,i} + M_{y,i}) \sin \alpha_i \cos \alpha_i + M_{xy,i} \left( \cos^2 \alpha_i - \sin^2 \alpha_i \right) = 0.$$  

$\alpha_i$ is the angle between the tangent to the boundary at node $i$ and the global axis $X$. Using the Lagrange multipliers method, we add the equilibrium equations (7) and the static boundary conditions (8-9) to the functional (6):

$$\Pi^c = \frac{1}{2} M^T DM + \sum_{i \in \Omega_1} w_i C^T_i M_{i,eq} + \sum_{i \in \Omega_{n}} \phi_{i,n} M_{n,i} + \sum_{i \in \Omega_{s}} \phi_{i,s} M_{s,i}. $$

$$w_i$$ is displacement of node $i$; $\phi_{i,n}$, $\phi_{i,s}$ are the Lagrange multipliers, which are additional nodal unknowns. Multipliers have the dimension of rotation angles of unit length. $\Omega_{s}$ is set of nodes, that the vertical displacements are not equal to zero. $\Omega_{n}^-$, $\Omega_{s}^-$ are sets of nodes on the border, where the corresponding moments are equal to zero. Expression (10) can be represented in the simpler matrix form:

$$\Pi^c = \frac{1}{2} M^T DM + w^T L M.$$  

$L$ is the equilibrium matrix of the nodes for the entire system. The number of this matrix rows is equal to the sum of unknown nodal displacements and number of static boundary conditions (8-9). The vector $w$ includes unknown nodal displacements and Lagrange multipliers $\phi_{i,n}$, $\phi_{i,s}$.

As is known [1], when we solve the problem of stability of plates, it is necessary consider additional tensile-compression deformations arising when the plate is bending:

$$\varepsilon_{x,y} = \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2, \varepsilon_{y,x} = \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2, \varepsilon_{xy} = \frac{\partial^2 w}{\partial x \partial y}.$$  

Deformations (12) correspond to the work of internal stresses $\sigma_x, \sigma_y, \sigma_{xy}$ acting in the median plane of the plate. The vertical displacements function of the finite element is represented using triangular coordinates:

$$w(x,y) = \frac{1}{2} A \left( a_i + b_i x + c_i y \right), \quad a_i = x_{i+1} y_{i+2} - x_{i+2} y_{i+1}, \quad b_i = y_{i+1} - y_{i+2}, \quad c_i = x_{i+2} - x_{i+1}.$$  

$x_{i}, y_{i}$ are the finite element nodes coordinates in the global coordinate system; $\bar{w}_i$ is the node vertical displacement; $A$ is the finite element area. The stresses work is expressed by the integral:
\[ U'_{*} = \int_{A} (\sigma_{x} e_{x,y} + \sigma_{y} e_{y,x} + \sigma_{xy} e_{x,y})dA \]  

(14)

We introduce the notations for vectors:

\[ w_k = \begin{pmatrix} \bar{w}_1 \\ \bar{w}_2 \\ \bar{w}_3 \end{pmatrix}, \quad N_{k,x} = \frac{1}{2A} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \quad N_{k,y} = \frac{1}{2A} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}. \]  

(15)

Using (15) we will obtain the following matrix expressions for the deformations (12):

\[ e_{x,y} = \frac{1}{2} w_k^T \left( N_{k,x} N_{k,x}^T \right) w_k, \quad e_{y,x} = \frac{1}{2} w_k^T \left( N_{k,y} N_{k,y}^T \right) w_k, \quad e_{x,y} = w_k^T \left( N_{k,x} N_{k,y}^T \right) w_k. \]  

(16)

\[ H_{k,x} = \int_{A} \sigma_{x} N_{k,x} N_{k,x}^T dA = \frac{\sigma_{x}}{4A} \begin{pmatrix} b_1^2 & h b_2 & b_1 b_3 \\ h b_2 & b_2^2 & b_2 b_3 \\ b_1 b_3 & b_2 b_3 & b_3^2 \end{pmatrix}. \]  

(17)

\[ H_{k,y} = \int_{A} \sigma_{y} N_{k,y} N_{k,y}^T dA = \frac{\sigma_{y}}{4A} \begin{pmatrix} c_1^2 & c_1 c_2 & c_1 c_3 \\ c_1 c_2 & c_2^2 & c_2 c_3 \\ c_1 c_3 & c_2 c_3 & c_3^2 \end{pmatrix}. \]  

(18)

\[ H_{k,y} = \int_{A} \sigma_{xy} N_{k,y} N_{k,x}^T dA = \frac{\sigma_{xy}}{4A} \begin{pmatrix} b_1 c_1 & b_1 c_2 & b_1 c_3 \\ b_2 c_1 & b_2 c_2 & b_2 c_3 \\ b_3 c_1 & b_3 c_2 & b_3 c_3 \end{pmatrix}. \]  

(19)

Calculating the derivative of the additional energy (14), we will obtain:

\[ H_k = \frac{dU'_{*}}{dw_k} = (H_{k,x} + H_{k,y} + H'_{k,y}) w_k, \]  

\[ H'_{k,y} = \frac{\sigma_{xy}}{4A} \begin{pmatrix} 2 h c_1 & h c_2 + h c_1 & h c_3 + h c_1 \\ h c_1 + b_2 c_1 & 2 h c_2 & b_2 c_1 + b_2 c_2 \\ b_3 c_1 + b_3 c_1 & b_3 c_2 + b_3 c_2 & 2 h c_3 \end{pmatrix}. \]  

(20)

From the geometrical matrices of finite elements \( H_k \), the global matrix \( H \) is formed for the entire system, and from vectors \( w_k \) the global vector \( w \) of nodal displacements is formed. Summing up (11) and (14), we obtain the following functional for solving the plate stability problem:

\[ II^c = \frac{1}{2} M^T DM + w^T LM + \frac{\lambda^c}{2} w^T H w. \]  

(21)

We will obtain the resolving equations if the derivatives of (21) with respect to the vectors \( M \) and \( w \) will be equal to zero:

\[ DM + L^T w = 0, \]  

\[ LM + \lambda^c H w = 0. \]  

(22)

Expressing the vector \( M \) from the first equation and substituting it into the second equation, we obtain:
10. \[ T - \lambda_c \lambda w = 0. \] (23)

Using (23), we obtain the algebraic equations system for determining the critical parameter \( \lambda_c \): \[ K w = \lambda_c H w. \] \[ K = LD^3 L'. \] (24)

As noted above, the matrix \( D \) has the block-diagonal form and is easily reversible analytically. Therefore, the matrix \( K \) has a tape structure of nonzero elements, which significantly reduces the computational cost of the equations system solving. To determine the critical parameter value, we will apply the reverse iteration method:

\[
\begin{align*}
\text{set the vector } w_i, i = 0; \\
\text{do } i = i + 1; \\
K w_i = H w_{i-1}; \\
w_{\max} = \max \left| w_{i,j} \right|, j = 1..n; \\
\lambda_{cr,i} = \frac{1}{w_{\max}}; \\
w_i = \frac{1}{w_{\max}} w_i; \\
w_{i} = \frac{w_i + w_{i-1}}{2}; \\
\text{end, if } \left| \lambda_{cr,i} - \lambda_{cr,i-1} \right| \leq \varepsilon.
\end{align*}
\] (25)

In (25) \( \varepsilon \) determines the critical parameter calculation accuracy.

3. Results and discussion

To assess the accuracy and convergence of the proposed method, for various finite element grids, calculations the round plate stability, which is clamped along the contour, were performed (Figure 1). The action of uniform radial compression was considered.

![Figure 1. Finite element grids for half a plate.](image)

The following plate parameters were adopted: \( R = 3 \text{ m}, E = 10000 \text{ kN} / \text{m}^2, \mu = 0.3, t = 0.6 \text{ m} \). The calculation results are shown in Table 1.

The exact, analytically obtained value of the critical compression stress is determined by the following formula:

\[ q_{cr} = \frac{1.49 \pi^2 E t^2}{12 R^2 (1 - \mu^2)}. \] (26)
Table 1. Value of critical stress.

| Grid   | $q_\text{cr}$, kN/m² |
|--------|----------------------|
| Grid a | 259.13               |
| Grid b | 283.62               |
| Grid c | 318.97               |
| Exact value | 323.20            |

The calculation results demonstrate the convergence of the obtained critical stress value to the exact value from below, which provides a margin of the plate stability. The critical stress value obtained with the finest grid differs from the exact value by about one percent, which indicates good the proposed method accuracy.

4. Conclusions

The algorithm is proposed for solving the problems of plate stability by the finite element method based on piecewise constant approximations of moments. The solution was obtained on the basis of the principle of minimum additional energy. The principle of possible displacements is used to obtain algebraic equilibrium equations of the nodes of finite elements mesh. The proposed method ensures the convergence of the calculated critical stress value to the exact value from below, which provides a margin of plate stability.

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