On the boundary terms in Hardy’s inequalities for $W^{1,p}$ functions

Ahmed A. Abdelhakim

Mathematics Department, Faculty of Science, Assiut University, Assiut 71516 - Egypt
Email: ahmed.abdelhakim@aun.edu.eg

Abstract

With the help of a radially invariant vector field, we derive inequalities of the Hardy kind, with no boundary terms, for $W^{1,p}$ functions on bounded star domains. Our results are not obtainable from the classical inequalities for $W^{1,p}_0$ functions. Unlike in $W^{1,p}_0$, our inequalities admit maximizers that we describe explicitly.

Keywords: Hardy type inequalities, radially invariant vector field, Sobolev space, star domains, boundary terms

2010 MSC: Primary 35A23, Secondary 26D15, 46E35

1. Introduction

Let $1 \leq p < n$ and let $u \in C_0^\infty (\Omega)$ where $\Omega$ is a $C^1$ bounded domain in $\mathbb{R}^n$. Then

$$\frac{\|u\|_{L^p(|x|)}}{\|
abla u\|_{L^p(\Omega)}} \leq c_{p,n} \|\nabla u\|_{L^p(\Omega)}$$

where $c_{p,n} = p/ (n - p)$. The inequality (1) is a version of the wellknown Hardy’s inequality. It is interesting, obviously, if the origin belongs to $\Omega$. The constant $c_{p,n}$ in (1) is found to be optimal $[3]$, yet not attained in the corresponding Sobolev space $W^{1,p}_0(\Omega)$. This is a motivation to look for a remainder term. A substantial improvement of (1) when $p = 2$ was obtained by Brezis and Vazquez $[5]$ who proved the inequality

$$\left( \int_\Omega |\nabla u|^2 dx - \left( \frac{n - 2}{2} \right)^2 \int_\Omega \frac{u^2}{|x|^2} dx \right)^{\frac{1}{2}} \geq \frac{C(q,n)}{|\Omega|^{\frac{1}{q} - \frac{n+2}{2q}}} \left( \int_\Omega |u|^q dx \right)^{\frac{1}{2}},$$

(2)
1 < q < n/(n − 2), with the constant C(q, n) optimal when Ω is a ball centered at the origin and q = 2, but again, never achieved in H_0^1(Ω). Similar improvements for Hardy’s inequalities where a nonnegative correction term is introduced followed this result ([3], [6, 7], [13], [25]). But these mainly targeted versions of [11] that involve the distance from the boundary as opposed to the distance from the origin or treated the corresponding L^p cases. Filipas and Tertikas [12] optimized (2), in a certain sense, in terms of correction terms and showed that the best constants in their improvements cannot be achieved in H_0^1(Ω). Later, N. Ghoussoub, A. Moradifam [14] characterized radially symmetric potentials V and best constants c(V) for the Hardy inequality

\[ \int_{Ω} |∇u|^2 dx - \left( \frac{n-2}{2} \right)^2 \int_{Ω} \frac{u^2}{|x|^2} dx \geq c(V) \int_{Ω} V(|x|)|u|^2 dx, \quad u \in H_0^1(Ω). \]

These results were furthered in ([2], [8, 9, 10], [15], to name a few).

The unattainability of the aforementioned optimal constants persists regardless of the behaviour of W_0^{1,p} functions near the boundary. Notice that (11) does not hold for functions constant on Ω, and therefore not true for all u ∈ W^{1,p}(Ω). Any inequality of the Hardy type for u ∈ W^{1,p}(Ω) will certainly involve a boundary term.

The discussion above invokes a question: Does the inequality (11) hold true on a space larger than (or different from) W_0^{1,p}(Ω) on which the constant c_{p,n}, or a bigger one, is optimal and achieved?

Recently S. Machihara et al. [20] gave variants of (11) on the ball B(R) = \{x ∈ R^n : |x| < R\}, valid for H^1 functions, namely

\[ \int_{B(R)} \frac{|u(x) - u(R_x \frac{x}{|x|})|^2}{|x|^2} dx \leq \left( \frac{2}{n-2} \right)^2 \int_{B(R)} \frac{x}{|x|} \cdot ∇u \right|^2 dx, \quad n \geq 3, \quad (3) \]

\[ \int_{B(R)} \frac{|u(x) - u(R_x \frac{x}{|x|})|^2}{|x|^2} dx \leq 4 \int_{B(R)} \frac{x}{|x|} \cdot ∇u \right|^2 dx, \quad n = 2. \quad (4) \]

The novelty in inequalities (3) and (4) lies in obtaining inequalities of Hardy type on balls for a function u ∈ H^1(R^n) with no boundary terms. Observe the identity

\[ x \cdot ∇u \left( R_x \frac{x}{|x|} \right) = 0. \]
This idea is celebrated in \[16, 17\], \[21, 22, 23, 24\].

First we show that (3) can not be obtained from (1) directly, and neither can (4) be deduced from the inequality (18):

$$\int_{\Omega} \frac{|u(x)|^n}{|x|^n \left(\log \frac{M}{|x|}\right)} dx \leq \left(\frac{n}{n-1}\right)^n \int_{\Omega} \left| \frac{x}{|x|} \cdot \nabla u \right|^n dx, \quad M = \sup_{\Omega} |x|,$$

that holds for all $u \in W^{1,n}_0(\Omega)$, $n \geq 2$. Precisely, if $v(x) := u(x) - u(x/|x|)$ then $v$ is not necessarily in $H^1(B(1))$ whenever $u \in H^1(\mathbb{R}^n)$. In fact $u(x/|x|)$ is not always in $L^2(B(1))$ when $u \in L^2(\mathbb{R}^n)$. See Proposition 1.

Second, with the help of a vector field $f$ satisfying a particular boundary value problem, we extend inequalities (3) and (4) to bounded $C^1$ star-shaped with respect to the origin domains. This is easy for inequalities like (1) obtainable from their analogues on balls:

Since $c_{p,n}$ is independent of $\Omega$, then a standard proof \[3\] implies

$$\int_{B(R)} \frac{|u|^p}{|x|^p} dx \leq c_{p,n}^p \int_{B(R)} |\nabla u|^p dx, \quad u \in H^1_0(B(R)).$$

(6)

Set $R = \max_{x \in \Omega} |x|$ and simply extend $u \in H^1_0(\Omega)$ to

$$\bar{u}(x) := \begin{cases} u(x), & x \in \Omega; \\ 0, & \text{otherwise}. \end{cases}$$

Then $\bar{u} \in H^1_0(B(\max_{x \in \Omega} |x|))$ and (1) follows from (6). This extension by zero argument \[18\], Remark 1.3 does not work for inequalities (3) and (4).

The term $u(Rx/|x|)$ that grants $v$ zero “trace” on $\partial B(R)$ becomes idle and the field $x/|x|$ needs to be replaced by another radially invariant field that will definitely depend on the domain. As a result, spherical coordinates used in \[20\] can be no longer helpful. As with (3) and (4), by Proposition 1 our inequalities are unobtainable from (1) when $1 \leq p < n$, $n \geq 3$ or from (5) in the critical case $p = n \geq 2$.

Finally, sharpness and equality are discussed using a method different from Ioku et al.’s in \[16\], \[17\]. The case $p = 1$ missing in \[17\] is also recovered.

We actually identify maximizers for our inequalities in $W^{1,p}$. We show how exactly the existence of these maximizers depends on both $f$ and the domain.
2. Main results

**Proposition 1.** Let $n \geq 2$, $1 \leq p < n$, and assume $u \in L^p(\mathbb{R}^n)$. Then $v(x) := u(x) - u(x/|x|)$ needs not belong to $L^p(B(1))$. Moreover, $u \in W^{1,p}(\mathbb{R}^n)$, does not guarantee $v \in W^{1,p}(B(1))$.

**Proof.** Fix $x_0 \in \mathbb{R}^n$ with $|x_0| = 1$. Let $0 \leq \gamma \leq 1$ denote a cutoff function with $\gamma(x) = 1$ for $\frac{3}{4} < |x| < \frac{5}{4}$ and $\gamma = 0$ for $|x| < \frac{1}{2}$ and $|x| > \frac{3}{2}$. Let

$$u(x) := \frac{\gamma(x)}{|x - x_0|^\alpha}, \quad \alpha > 0.$$  

We have

$$\int_{\mathbb{R}^n} |u|^p \, dx \lesssim \int_{\frac{1}{2} \leq |x| \leq \frac{3}{2}} \frac{dx}{|x - x_0|^{p\alpha}} \lesssim \int_{|x| \leq 1} \frac{dx}{|x|^{p\alpha}}.$$  

Thus $u \in L^p(B(1))$ if $\alpha < \frac{n-1}{p}$. Whereas, using spherical coordinates,

$$\int_{B(1)} \left| u \left( \frac{x}{|x|} \right) \right|^p \, dx = \int_{\frac{1}{2} \leq |x| \leq 1} \frac{dx}{|x - x_0|^{p\alpha}} \approx \int_{|\omega| = 1} \frac{d\omega}{|\omega - x_0|^{p\alpha}}$$  

which converges iff $\alpha < \frac{n-1}{p}$. This excludes $v$ from $L^p(B(1))$ for $\alpha \geq \frac{n-1}{p}$.

Also

$$\int_{\mathbb{R}^n} |\nabla u|^p \, dx \lesssim \int_{\frac{1}{4} \leq |x| \leq \frac{1}{2}} \frac{|\nabla \gamma|^p}{|x - x_0|^{p\alpha}} \, dx + \int_{\frac{1}{2} \leq |x| \leq \frac{3}{2}} \frac{|\gamma|^p}{|x - x_0|^{p(\alpha + 1)}} \, dx \lesssim 1 + \int_{|x| \leq 1} \frac{dx}{|x|^{p(\alpha + 1)}},$$  

whence $u \in W^{1,p}(\mathbb{R}^n)$ whenever $\alpha < \frac{n-p}{p}$. Meanwhile

$$\nabla \left( u \left( \frac{x}{|x|} \right) \right) = \frac{1}{|x|} (\nabla u) \left( \frac{x}{|x|} \right) - \frac{x \cdot (\nabla u) \left( \frac{x}{|x|} \right)}{|x|^3} x$$  

$$= \frac{1}{|x|} \left( (\nabla u) \left( \frac{x}{|x|} \right) - \frac{x}{|x|} \cdot (\nabla u) \left( \frac{x}{|x|} \right) \frac{x}{|x|} \right)$$  

$$= \frac{1}{r} \left[ (\nabla u)(\omega) - (\hat{r} \cdot (\nabla u)(\omega)) \hat{r} \right] = \frac{1}{r} \nabla_{\omega} u(\omega)$$.
in spherical coordinates. Consequently

\[
\int_{B(1)} \left| \nabla u \left( \frac{x}{|x|} \right) \right|^p \, dx \approx \int_{|\omega|=1} |\nabla_{\omega} u(\omega)|^p \, d\omega = \int_{|\omega|=1} \frac{d\omega}{|\omega - x_0|^{p(\alpha+1)}}.
\]

Therefore \( v \notin W^{1,p}(B(1)) \) for any \( \alpha \geq (n - p - 1)/p \).

The upcoming lemma provides an implemental estimate of weak derivatives.

**Lemma 2.** \(\square\) Let \( 1 \leq p \leq \infty \) and \( g \in W^{1,p}(\Omega \to \mathbb{R}) \). Then \( |x \cdot \nabla g| \leq |x \cdot \nabla u| \).

**Theorem 3.** Let \( n \geq 3 \), \( 1 \leq p < n \), and let \( u \in W^{1,p}(\Omega) \) where \( \Omega \) is a bounded \( C^1 \) domain star-shaped with respect to the origin in \( \mathbb{R}^n \). Suppose \( f \in C^1(\Omega \to \mathbb{R}^n) \cap L^\infty(\Omega \to \mathbb{R}^n) \) solves the boundary value problem

\[
\begin{cases}
(x \cdot \nabla) f(x) = 0, & x \in \Omega_0; \\
 f(x) = x, & x \in \partial \Omega
\end{cases}
\]

where \( \Omega_0 \) denotes \( \Omega \setminus \{0\} \). Then

\[
\int_{\Omega} \frac{|u - u \circ f|^p}{|x|^p} \, dx \leq \left( \frac{p}{n - p} \right)^p \int_{\Omega} \frac{x}{|x|} \cdot \nabla u \bigg| \frac{x}{|x|} \, dx. \tag{8}
\]

**Proof.** The proof is standard. By density \(\square\), we may argue assuming \( u \in C^1(\overline{\Omega}) \). Let \( F(x) := |u(x) - u(f(x))|^p/|x|^p \). Then, for \( x \neq 0 \), we have

\[
F(x) = -|u(x) - u(f(x))|^p \nabla \left( \frac{1}{|x|} \right) \cdot \frac{x}{|x|^{p-1}} = -\nabla \cdot (F(x) x) + (n - (p - 1)) F(x) + pF^{\frac{p-1}{p}}(x) \frac{x}{|x|} \cdot \nabla|u(x) - u(f(x))|,
\]

which we rewrite as

\[
F(x) = \frac{1}{n - p} \nabla \cdot (F(x) x) - \frac{p}{n - p} F^{\frac{p-1}{p}}(x) \frac{x}{|x|} \cdot \nabla|u(x) - u(f(x))|. \tag{9}
\]

Now, choose \( \epsilon > 0 \) such that \( \Omega \supset B(\epsilon) \). Then

\[
I(\epsilon) := \int_{B(\epsilon)} F(x) \, dx \lesssim \|u - u \circ f\|_{L^\infty(\Omega)}^p \epsilon^{n-p}. \tag{10}
\]
The estimate (10) helps isolate the singularity so that, if $\nu$ is the outward pointing normal on $\Omega$, then the divergence theorem yields
\[
J(\epsilon) := \int_{\Omega \setminus B(\epsilon)} \nabla \cdot (F(x) \, x) \, dx = -\int_{|x| = \epsilon} |u(x) - u(f(x))|^p |x|^{p-1} \, dS(x),
\]
as assumption (7) ensures
\[
\int_{\partial \Omega} F(x) \, x \cdot \nu(x) dS(x) = 0.
\]
And from (11) follows the estimate
\[
|J(\epsilon)| \lesssim \|u - u \circ f\|_{L^\infty(\Omega)}^p \epsilon^{n-p}.
\]
In addition, applying Hölder’s inequality
\[
|K(\epsilon)| := \left| \int_{\Omega \setminus B(\epsilon)} F^{\frac{p-1}{p}}(x) \frac{x}{|x|} \cdot \nabla |u(x) - u(f(x))| \, dx \right|
\leq \left( \int_{\Omega \setminus B(\epsilon)} F(x) \, dx \right)^{1 - \frac{1}{p}} \left( \int_{\Omega \setminus B(\epsilon)} \left| \frac{x}{|x|} \cdot \nabla |u(x) - u(f(x))| \right|^p \, dx \right)^{\frac{1}{p}}.
\]
But Lemma 2 affirms the pointwise estimate
\[
|x \cdot \nabla |u(x) - u(f(x))|| \leq |x \cdot \nabla (u(x) - u(f(x)))|, \quad x \in \Omega_0.
\]
Also, from (7) follows
\[
x \cdot \nabla u(f(x)) = (\nabla u) (f(x)) \cdot (x \cdot \nabla) f = 0, \quad x \in \Omega_0.
\]
Returning with (14) and (15) to (13) implies
\[
|K(\epsilon)| \leq \left( \int_{\Omega \setminus B(\epsilon)} F(x) \, dx \right)^{1 - \frac{1}{p}} \left( \int_{\Omega \setminus B(\epsilon)} \left| \frac{x}{|x|} \cdot \nabla u \right|^p \, dx \right)^{\frac{1}{p}}.
\]
Integrating (9) over $\Omega$ we get
\[
\int_{\Omega} F(x) \, dx = I(\epsilon) + \frac{1}{n-p} J(\epsilon) - \frac{p}{n-p} K(\epsilon).
\]
Finally, by (10), $\lim_{\epsilon \to 0^+} I(\epsilon) = 0$. Similarly $\lim_{\epsilon \to 0^+} J(\epsilon) = 0$ from (12). And since $|\nabla u|^p \in L^1(\Omega)$, then, using the dominated convergence theorem in (16), we deduce (8) from (17).
Remark 1. Star-shapedness of the domain $\Omega$ with respect to the origin is necessary for a nontrivial vector field $f$ to satisfy (7).

Theorem 4. Consider $\Omega$ and $f$ of Theorem 3. Let $M := \sup_{\Omega} |x|$ and assume $u \in W^{1,n}(\Omega)$ with $n \geq 2$. Then
\[
\int_{\Omega} \frac{|u - u \circ f|^n}{|x|^n \left(\log \frac{M}{|x|}\right)^n} \, dx \leq \left(\frac{n}{n-1}\right)^n \int_{\Omega} \frac{|x|}{|x|^n \left(\log \frac{M}{|x|}\right)^{n-1}} \cdot \nabla u \left|\nabla u \right|^n \, dx. \tag{18}
\]

Proof. Invoking the density argument [1], we may take $u \in C^1(\bar{\Omega})$. Since, for $x \in \Omega_0$,
\[(n - 1) \frac{|u(x) - u(f(x))|^n}{|x|^n \left(\log \frac{M}{|x|}\right)^n} = |u(x) - u(f(x))|^n \nabla \cdot \left(\frac{x}{|x|^n \left(\log \frac{M}{|x|}\right)^{n-1}} \right) = \nabla \cdot \left(\frac{|u(x) - u(f(x))|^n}{|x|^n \left(\log \frac{M}{|x|}\right)^{n-1}} x \right) + \frac{n}{|x|^{n-1} \left(\log \frac{M}{|x|}\right)^{n-1}} x \cdot \nabla |u(x) - u(f(x))|.
\]

The rest resembles the proof of Theorem 3 once the components (19)-(21) below are recognized:
\[
\int_{B(\epsilon)} \frac{|u(x) - u(f(x))|^n}{|x|^n \left(\log \frac{M}{|x|}\right)^n} \, dx \lesssim \frac{\|u - u \circ f\|_{L^\infty(\Omega)}}{\left(\log \frac{M}{\epsilon}\right)^{n-1}} \to 0, \text{ as } \epsilon \to 0^+. \tag{19}
\]
\[
\int_{\partial \Omega} \frac{|u(x) - u(f(x))|^n}{|x|^n \left(\log \frac{M}{|x|}\right)^{n-1}} (x \cdot \nu) \, dx = 0, \tag{20}
\]
\[
-\int_{\Omega \setminus B(\epsilon)} \nabla \cdot \left(\frac{|u(x) - u(f(x))|^n}{|x|^n \left(\log \frac{M}{|x|}\right)^{n-1}} x \right) \, dx
\]
\[
= \int_{|x|=\epsilon} \frac{|u(x) - u(f(x))|^n}{|x|^{n-1} \left(\log \frac{M}{|x|}\right)^{n-1}} dS(x) \lesssim \frac{\|u - u \circ f\|_{L^\infty(\Omega)}}{\left(\log \frac{M}{\epsilon}\right)^{n-1}} \to 0, \text{ as } \epsilon \to 0^+, \tag{21}
\]
an implication of the divergence theorem.

\[ \square \]

3. Applications

1- On ellipsoids

Let \( a := (a_i) \in \mathbb{R}^n_+ \) and let \( \| \cdot \|_a \) denote the norm \( \| x \|_a := (\sum_{i=1}^{n} a_i^2 x_i^2)^{\frac{1}{2}} \)
\( x = (x_i) \in \mathbb{R}^n \). Consider the open ellipsoid \( E_a := \{ x : |x^2| < 1 \} \). The essentially bounded field \( f_a(x) := x/|x| \) is smooth on \( E_a \setminus \{0\} \) and satisfies (7) with \( \Omega = E_a \). Therefore, if \( u \in W^{1,p}(E_a) \), \( 1 \leq p < n \), \( n \geq 3 \),

\[ \int_{E_a} \frac{|u - u \circ f_a|^p}{|x|^p} \, dx \leq \left( \frac{p}{n-p} \right)^p \int_{E_a} \frac{|x| \cdot \nabla u|^p}{|x|} \, dx \]

by Theorem 3. Moreover, applying Theorem 4 if \( u \in W^{1,n}(E_a) \) with \( n \geq 2 \) then

\[ \int_{E_a} \frac{|u - u \circ f_a|^n}{|x|^n \left( \log \frac{M_a}{|x|} \right)^n} \, dx \leq \left( \frac{n}{n-1} \right)^n \int_{E_a} \frac{|x| \cdot \nabla u|^n}{|x|} \, dx \]

where \( M_a := \sup_{x \in E_a} |x| = 1/\min_i a_i \).

2- On star-shaped domains

Let the \( C^1 \) domain \( C \) be such that \( 0 \in C \) and \( C \) is star-shaped w.r.t. \( 0 \). Suppose \( \partial C \) has the representation \( r = r(\omega) \) in spherical coordinates. Then \( C \) and

\[ f(x) := r \left( \frac{x}{|x|} \right) \frac{x}{|x|} \]

fulfill all prerequisites of both Theorems 3 and 4.

4. Sharpness and maximizers

Back to Theorems 3 and 4. If \( u \in W^{1,p}(\Omega) \cap C(\bar{\Omega}) \) and \( f \) is such that \( u \circ f \in W^{1,p}(\Omega) \), then \( u - u \circ f \in W^{1,p}_0(\Omega) \). Hence, by (15), the sharpness of the inequalities (1) and (5) then assure the optimality of the constants in (8) and (18), respectively. Equality in (8) is clear for constant \( u \), for which both sides vanish. The same applies to (18). In fact if \( u \) is radially invariant in \( \Omega \) then \( u(x) = u(f(x)) \) and \( x \cdot \nabla u = 0 \), in which case both sides of (8) and
are zero.

Let us investigate all candidates for radially varying maximizers of (8). As intuited from Proposition 1, the existence of a maximizer in \( W^{1,p}(\Omega) \) depends on \( f \) and the geometry of \( \Omega \). Now, equality in (8) requires equality in (13) which necessitates the existence of a \( \lambda > 0 \) such that

\[
F^\pm(x) = \lambda \left| \frac{x}{|x|} \cdot \nabla |u(x) - u(f(x))| \right| = \lambda \left| \frac{x}{|x|} \cdot \nabla u(x) \right|, \quad p > 1, \tag{22}
\]

for almost every \( x \in \Gamma \) with \( \Gamma := \{ x \in \Omega_0 : u(x) - u(f(x)) \neq 0 \} \). Comparing (22) against (8) with an equality implies

\[
\lambda = \frac{p}{n - p},
\]

and (22) becomes

\[
x \cdot \nabla w(x) = \pm \frac{n - p}{p} w(x), \quad w(x) = u(x) - u(f(x)), \quad \text{a.e. } \Gamma \tag{23}
\]

or equivalently

\[
x \cdot \nabla \left( |x|^{\frac{n-p}{p}} w(x) \right) = 0 \quad \text{a.e. } \Gamma.
\]

Thus a maximizer of (8), if exists, has to satisfy

\[
u(x) = u(f(x)) + |x|^{\frac{n-p}{p}} \psi(x)
\]

for some \( \psi \) that solves

\[
x \cdot \nabla \psi = 0 \quad (\text{equivalently } \frac{\partial}{\partial |x|} \psi = 0) \quad \text{on } \Gamma. \tag{24}
\]

By this radial invariance of \( \psi \) in \( \Omega \), we can write \( \psi(x) = \psi(x/|x|) \), after extending it radially, wherever necessary, to the unit ball. Moreover, with a sufficiently small \( \eta > 0 \), one can fit \( B(\eta) \) inside \( \Omega \). So if \( u(x) = u(f(x)) + |x|^{-\frac{n-p}{p}} \psi(x) \) then both sides of (8) dominate

\[
\int_{B(\eta)} \frac{|\psi(x)|^n}{|x|^n} \, dx = \int_{S^{n-1}} |\psi(\omega)|^n \, d\omega \int_0^\eta \frac{1}{r} \, dr = +\infty
\]
unless \( \psi = 0 \). Therefore, for \( u \) to maximize (8), it must satisfy

\[
u(x) = u(f(x)) + |x|^{\frac{n-p}{p}} \psi(x) \quad \text{a.e. } \Omega, \tag{25}\]

assuming \( u - u \circ f \neq 0 \) a.e. \( \Omega \).

Recall that (8) lacks nontrivial maximizers when \( u - u \circ f \in W^{1,p}_0(\Omega) \). To avoid that, take an \( \psi \in C(\bar{\Omega}_0) \) that satisfies (24). If \( |x|^{\frac{n-p}{p}} \psi \in W^{1,p}(\Omega) \) then \( u - u \circ f \in W^{1,p}(\Omega) \cap C(\Omega_0) \). However \( u(x) - u(f(x)) \neq 0 \) for every \( x \in \partial \Omega \), for otherwise \( \psi \) vanishes identically on \( \partial \Omega \) which, by (24), implies \( \psi \) vanishes on \( \bar{\Omega}_0 \). So assume \( \psi \in C(\bar{\Omega}_0) \) henceforth.

A nontrivial maximizer of (8) cannot be in \( C_{\text{rad}}(\bar{\Omega}_0) \), the space of continuous in the radial direction functions on \( \bar{\Omega}_0 \). Indeed, since \( f \in C(\bar{\Omega}_0) \) and \( f(x) = x \) on \( \partial \Omega \), then for a function \( u \in C_{\text{rad}}(\bar{\Omega}_0) \) that verifies (25), we would have \( u(x) - u(f(x)) = 0 \) on \( \partial \Omega \). Let

\[
\phi(x) := \begin{cases} 
|f(x)|^{\frac{n-p}{p}} \psi(x), & \text{in } \Omega_0; \\
0, & \text{on } \partial \Omega.
\end{cases}
\]

Then \( \phi \in C(\Omega_0) \), and is evidently radially invariant in \( \Omega_0 \). Since \( f \neq 0 \) in \( \Omega_0 \), then \( \phi \notin C_{\text{rad}}(\bar{\Omega}_0) \). Define

\[
\xi(x) := |x|^{\frac{n-p}{p}} \psi(x) + \phi(x), \quad x \in \bar{\Omega}_0. \tag{26}
\]

Notice that either \( \xi \notin C_{\text{rad}}(\bar{\Omega}_0) \), or else \( \xi \) is identically zero. Since

\[
\psi(f(x)) = \psi(x), \quad \phi(f(x)) = 0
\]

for every \( x \in \Omega_0 \), then

\[
\xi(f(x)) = |f(x)|^{\frac{n-p}{p}} \psi(x) = \phi(x) \quad \text{in } \Omega_0
\]

and \( \xi \) satisfies (25). (See Figure A).

\[\text{Figure A: } f(x) = f(p) = p, \quad \xi(f(x)) = \xi(p) = |p|^{\frac{n-p}{p}} \psi(p).\]
Thus, (26) provides a maximizer of (8) for all \(1 \leq p < n\) as long as \(f\) and \(\psi\) are such that \(\psi \in L^\infty(\Omega), |x|^{\frac{n-p}{p}} \psi, |f|^{\frac{n-p}{p}} \psi\) are in \(W^{1,p}(\Omega)\). For these maximizers, both sides of (8) equal

\[
\int_\Omega \frac{|\psi|^p}{|x|^{2p-n}} \, dx \leq \|\psi\|_{L^\infty(\Omega)}^p \int_\Omega \frac{1}{|x|^{2p-n}} \, dx < \infty
\]

for all \(1 \leq p < n\). Interestingly, since \(\xi \notin C_{rad}(\bar{\Omega}_0)\) then \(\xi - \xi \circ f\) cannot be a \(W^{1,1}(\Omega)\) function. This is because \(\xi - \xi \circ f \in C(\Omega_0) \setminus C(\bar{\Omega}_0)\) and, although \(\xi(x) - \xi(f(x)) = 0, \forall x \in \partial \Omega\), its trace \(|x|^{\frac{n-p}{p}} \psi \neq 0\) on \(\partial \Omega\). See Example II.

Iterating this approach, maximizers of (18), if exist, verifiably take the form

\[
u(x) = \nu(f(x)) + \psi(x) \left(\log \frac{M}{|x|}\right)^{\frac{n-1}{n}} \text{ a.e. } \Omega \tag{27}
\]

with an \(\psi\) that satisfies (24), and will be momentarily further determined.

Observe that we excluded the functions \(\nu(x) = \nu(f(x)) + \psi(x) \left(\log \frac{M}{|x|}\right)^{\frac{n-1}{n}}\) with which both sides in (18) diverge. Indeed, for \(0 < \eta < M\) small enough, both sides of (18) dominate

\[
\int_{B(\eta)} \frac{|\psi(x)|^n}{|x|^n \log \frac{M}{|x|}} = \int_{S^{n-1}} |\psi(\omega)|^n \int_0^\eta \frac{1}{r \log \frac{M}{r}} \, dr \, d\omega = +\infty.
\]

With \(\nu\) in (27), both sides of (18) equal

\[
I_\Omega := \int_\Omega \frac{|\psi|^n}{|x|^n \left(\log \frac{M}{|x|}\right)^{2n-1}} \, dx.
\]

Whether \(I_\Omega\) converges is a little tricky. Let’s find \(\psi\) that makes \(I_\Omega\) converge. Such \(\psi\) undoubtedly depends on \(\partial \Omega\) where the logarithmic singularity lies. The radial invariance of \(\psi\), and the radial symmetry of its factor suggest writing \(I_\Omega\) in spherical coordinates. So, let \(\partial \Omega\) be given by \(r = r(\omega)\). Since \(\Omega\) is \(C^1\) then \(r(\omega) \in C^1(S^{n-1})\). And since \(0 \in \Omega\) then there exists \(r_{0,\Omega} > 0\) such that \(r(\omega) \geq r_{0,\Omega}\) on \(S^{n-1}\). Therefore

\[
I_\Omega = \int_{S^{n-1}} \int_0^{r(\omega)} \frac{|\psi(\omega)|^n}{r \left(\log \frac{M}{r(\omega)}\right)^{2n-1}} \, dr \, d\omega = \frac{1}{2n-2} \int_{S^{n-1}} \frac{|\psi(\omega)|^n}{\left(\log \frac{M}{r(\omega)}\right)^{2n-2}} \, d\omega.
\]
Let $\psi \in C(\bar{\Omega}_0)$. If $\Lambda := \{x \in \partial \Omega : |x| = M\}$ has positive Lebesgue surface measure, $S(\Lambda)$, and $\psi \neq 0$ on $\Lambda$, then $I_\Omega$ is ill-defined. Suppose that $\Lambda$ is, in addition, simply connected. Then, for $I_\Omega$ to be well-defined, $\psi$ must vanish in the cone with apex at 0 and base on $\Lambda$. Obviously $\Lambda = \partial \Omega$ iff $\Omega = B(M)$. Thus, the inequality \[ \text{on balls admits no maximizers.} \]

Assume that $S(\Lambda) = 0$. We need $|\psi|^n$ to decrease at least as fast as the now radially invariant logarithmic factor $(\log (M/r(\omega)))^{(2n-2)}$ when $r(\omega) \to M$. Luckily, if $g(t) := (1-t)^{n\alpha} / (\log (1/t))^{2n-2}$, with $\alpha \geq (2n-2)/n$, $0 < t < 1$, then $g \in C([0, 1]) \cap L^\infty ([b, 1])$, for any fixed $b > 0$. Also if $h(x) := r(x)/M$, $x \in S^{n-1}$, then $h \in C(S^{n-1})$, and consequently $g \circ h \in C(S^{n-1})$. This urges the choice $\psi(x) = (M/r(|x|))^{\alpha}$, $x \in \Omega_0$, for which $I_\Omega$ converges. Suppose (28)

$$
\eta(x) := \begin{cases} 
(M - r \left( \frac{x}{|x|} \right) )^\alpha \left( \log \frac{M}{|x|} \right)^{-\frac{n-1}{n}}, & \text{in } \Omega_0; \\
0, & \text{on } \partial \Omega.
\end{cases}
$$

Then $\eta(f(x)) = 0$ for every $x \in \bar{\Omega}_0$ and we get

$$
\eta(x) - \eta(f(x)) = \left( M - r \left( \frac{x}{|x|} \right) \right)^\alpha \left( \log \frac{M}{|x|} \right)^{-\frac{n-1}{n}}, \quad x \in \Omega_0.
$$

Hence, if $\partial \Omega$ has the representation $r = r(x/|x|)$ with $r(x/|x|) < M$, $S$-a.e on $\partial \Omega$, then $\eta$ in (28) is a maximizer of (18) for all $n \geq 2$, provided we choose $\alpha \geq (2n-2)/n$ large enough that $\eta \in W^{1,n}(\Omega)$. Observe here that $\eta - \eta \circ f \in C(\Omega_0) \setminus C_{\text{rad}}(\Omega_0)$, with its trace $(M - r(x/|x|))^{\alpha} (\log M/|x|)^{-n/\alpha}$ > 0, $S$-a.e $\partial \Omega$. Thus $\eta - \eta \circ f \notin W^{1,1}_0(\Omega)$ where (18) has no maximizers. See Example 2.

5. Examples

Example 1. Let $n \geq 3$, and define on the ellipsoid $E_\alpha$ in Section 3 the functions $\psi(x) = x_1/|x|_\alpha$ and

$$
\phi(x) := \begin{cases} 
|f_a(x)|^{\frac{n-2}{p}} \psi(x) = |x|/|x|_\alpha^{\frac{n-2}{p}} \psi(x), & \text{in } E_\alpha \setminus \{0\}; \\
0, & \text{on } \partial E_\alpha.
\end{cases}
$$

Since $\phi, \psi \in L^\infty(E_\alpha)$ and are both weakly differentiable, and since

$$
\max \left\{ \left| \nabla \frac{|x|}{|x|_\alpha} \right|, \left| \nabla \frac{x_1}{|x|_\alpha} \right|, \left| \nabla \frac{|x|^{\frac{n-2}{p}}}{|x|_\alpha} \right| \right\} \lesssim_a \frac{1}{|x|}, \quad x \neq 0,
$$
then $\xi(x) = |x|^\frac{n-p}{p}\psi + \phi$ is in $W^{1,p}(E_a)$ for all $1 \leq p < n$. We note that $\psi \in C^\infty(\overline{E_a} \setminus \{0\})$ and $\xi \notin C_{rad}(\overline{E_a} \setminus \{0\})$. We also find

$$\xi(f_a(x)) = |f_a(x)|^\frac{n-p}{p}\psi(x) = \phi(x), \quad \text{in } E_a \setminus \{0\}. \tag{1}$$

Consequently

$$\xi(x) - \xi(f_a(x)) = |x|^\frac{n-p}{p}\psi(x), \quad \text{in } E_a \setminus \{0\}. \tag{2}$$

Moreover

$$\int_{E_a} \frac{|\xi - \xi \circ f_a|^p}{|x|^p} \, dx = \left(\frac{p}{n-p}\right)^p \int_{E_a} \left|\frac{x}{|x|}\cdot\nabla\xi\right|^p \, dx = \int_{E_a} \frac{|x|^{n-p}|\psi|^p}{|x|^p} \, dx \leq \frac{1}{a_1^p} \int_{E_a} \frac{dx}{|x|^{2p-n}} < \infty \tag{3}$$

for all $1 \leq p < n$.

**Example 2.** Again, consider the ellipsoid $E_a$. Using spherical coordinates, $\partial E_a$ has the representation

$$r = r(\omega) = |\omega_a|^{-1}, \quad \omega_a := (a_1\omega_1, \ldots, a_n\omega_n).$$

Losing no generality, let $a_1 = \min_i a_i$, $a_2 = \max_i a_i$ so that $M_a = 1/a_1$ and

$$1 \leq \frac{M_a}{r(\omega)} \leq M_a a_2, \quad \text{for every } \omega \in S^{n-1}. \tag{4}$$

Assume that $a_1 < a_i$ for every $2 \leq i \leq n$. Then $r(\omega) = M_a$ precisely at the two points $p_\pm = (\pm 1/a_1, 0, \ldots, 0)$. Fix $\alpha \geq (2n-2)/n$, and let

$$\psi(x) = \begin{cases} \left(M_a - r\left(\frac{x}{|x|}\right)\right)^\alpha, & \text{in } E_a \setminus \{0\}; \\ 0, & \text{on } \partial E_a. \end{cases} \tag{5}$$

Then $\psi \in C^\infty(\overline{E_a} \setminus \{0\}) \cap L^\infty(E_a)$. Computations show that

$$\eta(x) = \psi(x) \left(\log \frac{M_a}{|x|}\right)^{-\frac{n-1}{n}}$$

13
satisfies
\[ \int_{E_a} \frac{|\eta - \eta \circ f_a|^n}{|x|^n \left( \log \frac{M_a}{|x|} \right)^n} \, dx = \left( \frac{n}{n-1} \right)^n \int_{E_a} \frac{|x|}{|x|^n} \cdot \nabla \eta \cdot n \, dx = I_{E_a} \]

where
\[ I_{E_a} = \int_{E_a} \frac{|\psi|^n}{|x|^n \left( \log \frac{M_a}{|x|} \right)^n} \, dx = \frac{M_a^n}{2n-2} \int_{S^{n-1}} \frac{(1 - r(\omega))^{n\alpha}}{\left( \log \frac{M_a}{r(\omega)} \right)^{2n-2}} \, d\omega, \]

which converges since \( r(\omega) \geq a^{-1} > 0 \) uniformly, and its integrand is the composite of the continuous bounded functions \( \omega \mapsto r(\omega)/M_a = 1/(M_a|\omega_a|) \), and
\[ x \mapsto \begin{cases} (1 - x)^{\alpha n}/(\log (1/x))^{2n-2}, & 0 < x < 1; \\ 0, & x = 1. \end{cases} \]

Evidently \( \eta \in L^\infty(E_a) \) when \( \alpha \geq (n - 1)/n \). Let us check \( \nabla \eta \in L^n(E_a) \) for all \( n \geq 2 \). For all \( x \in E_a \setminus \{0\} \), we have
\[ |\nabla \psi| = \alpha \left( M_a - r \left( \frac{x}{|x|} \right) \right)^{\alpha - 1} \left| \nabla \frac{|x|}{|x|^n} \right| \lesssim_{\alpha,n} \frac{\psi^{\alpha - 1}}{|x|}. \]

Thus
\[ \int_{E_a} |\nabla \eta|^n \, dx \lesssim_{\alpha,n} \int_{E_a} \frac{\psi^n}{|x|^n \left( \log \frac{M_a}{|x|} \right)^n} \, dx + \int_{E_a} \frac{\psi^{\alpha(n-1)}}{\alpha} \, dx \]
\[ = \frac{M_a^n}{2n-2} \int_{S^{n-1}} \frac{(1 - r(\omega))^{n\alpha}}{\left( \log \frac{M_a}{r(\omega)} \right)^{2n-2}} \, d\omega + \frac{M_a^{\alpha(n-1)}}{2n-2} \int_{S^{n-1}} \frac{(1 - r(\omega))^{\alpha(n-1)}}{\left( \log \frac{M_a}{r(\omega)} \right)^{n-2}} \, d\omega < \infty \]
for all \( \alpha \geq n/2 \). Noteworthily, \( \eta \in C(E_a) \setminus C_{rad}(\Omega_0) \), and \( \eta - \eta \circ f = \eta \) has positive trace. Hence \( \eta - \eta \circ f \notin W^{1,1}(E_a) \).

6. Acknowledgement

The author is grateful to Craig Cowan at the university of Manitoba for his valuable comments on the counterexample that proves Proposition 1.
References

[1] R. A. Adams, Sobolev Spaces, Academic Press, New York, 1975.

[2] A. Alvino, R. Volpicelli, B. Volzone, On Hardy inequalities with a remainder term, Ric. Mat. 59 (2010) 265 - 280.

[3] A. Balinsky, W. D. Evans, R. T. Lewis, The analysis and geometry of hardy’s inequality. Springer, New York 2015.

[4] G. Barbatis, S. Filippas, A. Tertikas A unified approach to improved $L^p$ Hardy inequalities with best constants, Trans. Amer. Math. Soc., 356 (6) (2004), 2169 - 2196.

[5] H. Brezis and J. L. Vázquez, Blowup solutions of some nonlinear elliptic problems, Revista Mat. Univ. Complutense Madrid 10 (1997), 443 - 469.

[6] H. Brezis and M. Marcus, Hardy’s inequality revisited, Ann. Scuola. Norm. Sup. Pisa, 25 (1997), 217 - 237.

[7] H. Brezis, M. Marcus, and I. Shafrir, Extremal functions for Hardy’s inequality with weight, J. Funct. Anal. 171 (2000), 177 - 191.

[8] C. Cowan, Optimal Hardy inequalities for general elliptic operators with improvements, Commun. Pure Appl. Anal. 9 (2010), no. 1, 109 - 140.

[9] S. Cuomo, A. Perrotta, On best constants in Hardy inequalities with a remainder term, Nonlinear Analysis 74 (2011) 5784 - 5792.

[10] B. Devyver, M. Fraasb, Y. Pinchover, Optimal hardy weight for second-order elliptic operator: An answer to a problem of Agmon, J. Funct. Anal., 266, (2014), No. 7, 4422 - 4489.

[11] L. C. Evans and R. F. Gariepy, Measure theory and fine properties of functions, CRC Press, New York, 1992.

[12] S. Filippas, and A. Tertikas, Optimizing improved Hardy inequalities , J. Funct. Anal. 192 (2002), 186 - 233.
[13] F. Gazzola, H.-C. Grunau and E. Mitidieri, Hardy inequalities with optimal constants and remainder terms, Trans. Amer. Math. Soc. 356 (2004), 2149 - 2168.

[14] N. Ghoussoub, A. Moradifam, On the best possible remaining term in the Hardy inequality, Proc. Natl. Acad. Sci. USA, 105 (37) (2008), 13746 - 13751.

[15] N. Ghoussoub, A. Moradifam, Functional inequalities: New perspectives and new applications, Math. Surveys Monogr., vol. 187, American Mathematical Society, Providence, RI, 2013.

[16] N. Ioku, M. Ishiwata and T. Ozawa, Sharp remainder of a critical Hardy inequality, Archiv der Mathematik, 106 (2016), 65 - 71.

[17] N. Ioku, M. Ishiwata and T. Ozawa, Hardy type inequalities in $L^p$ with sharp remainders, Journal of Inequalities and Applications, (2017) 2017: 5.

[18] A scale invariant form of a critical Hardy inequality, International Mathematics Research Notices, Volume 2015, Issue 18 (2015), 8830 - 8846.

[19] E. H. Lieb and M. Loss, Analysis, 2nd ed., Graduate Studies in Mathematics, vol. 14, American Mathematical Society, Providence, RI, 2001.

[20] S. Machihara, T. Ozawa and H. Wadade, Hardy type inequalities on balls, Tohoku Mathematical Journal, Volume 65, (2013), No. 3, 321 - 330.

[21] S. Machihara, T. Ozawa, H. Wadade, Remarks on the Rellich inequality, Mathematische Zeitschrift, 286 (2017), 34, 1367 - 1373.

[22] S. Machihara, T. Ozawa and H. Wadade, Remarks on the Hardy type inequalities with remainder terms in the framework of equalities, arXiv: 1611.03580 [math.AP].

[23] S. Machihara, T. Ozawa and H. Wadade, Scaling invariant Hardy inequalities of multiple logarithmic type on the whole space, Journal of Inequalities and Applications (2015) 2015: 281.

[24] M. Ruzhansky and D. Suragan, Critical Hardy inequalities, arXiv:1602.04809 [math.AP].

16
[25] J. L. Vazquez, E. Zuazua, The Hardy inequality and the asymptotic behaviour of the heat equation with an inverse square potential, J. Funct. Anal. 173, (2000), 103 - 153.