On Some Results of Topological Groupoid

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Abstract. Our main interest in this study is to look for groupoid space and topological groupoid. In this paper, we give some results of groupoid space and topological groupoid, which are properties of source proper topological groupoid (SP-groupoid). In the end we introduced remarks, propositions and theorems.

1. Introduction
The notion of groupoid as a generalization of that of group was found by Brandt [4] as a result of his work on the arithmetic of quaternary quadratic forms, generalizing the work of Gauss [3] for case of binary quadratic forms. This concept of groupoid now can be seen as a significant extension of the range of discourse, allowing for a more flexible and powerful approach to symmetry [2]. The notion of groupoid action was introduce by Ehresmann (1959) [6] generalizing the group action in his work on the space of all isomorphisms between the fibers of a fiber bundle. On the other hand, Jean louis Tu [4] call the continuous action of topological groupoid \( G \) on a topological space \( E \) proper.

The present paper consists of two sections. In section one we introduce some definitions and propositions which we need in our work to give the definitions and theorems about categories, functor, fiber product, proper mapping, groupoid space and topological groupoid. In section two, we introduce the basic the construction of source proper topological groupoid SP-groupoid, propositions and theorems. These results are novel in the present time (to the best of our knowledge).

2. Groupoid Space and Topological Groupoid

Definition (1-1) [1]:
A groupoid is a pair of set \((G, B)\) on which are given:
1. Two surjections \(\alpha, \beta: G \rightarrow B\) called the source and the target map respectively.
2. An injection \(w: B \rightarrow G\) called the unit map with \(w \circ \beta = \beta \circ w = \beta \circ w = I_B\) where \(I_B: B \rightarrow B\) is the identity map.
3. A law of partial composition \(\gamma\) in \(G\) defined as a law of composition on the set \(G \times G = \{(g, g') \in G \times G, \alpha(g) = \beta(g')\}\) “fiber product of \(\alpha\) and \(\beta\) over \(B\)” such that:
   (a) \(\gamma(g, \gamma(g_1, g_2)) = \gamma(\gamma(g_1, g_2), g_2), \forall (g, g_1, g_2) \in G \times G\).
   (b) \(\alpha(\gamma(g_1, g_2)) = \alpha(g_2), \beta(\gamma(g_1, g_2)) = \beta(g_1)\) for each \((g_1, g_2) \in G \times G\).
   (c) \(\gamma(g, w(\alpha(g))) = g\) and \(\gamma(w(\beta(g)), g) = g\), for all \(g \in G\).
4. A bijection $\sigma: G \to G$ called the inversion of $G$ satisfying:
   (a) $\alpha(\sigma(g)) = \beta(g)$, $\beta(\sigma(g)) = \alpha(g)$, for all $g \in G$.
   (b) $\gamma(\sigma(g), g) = w(\alpha(g))$, $\gamma(g, \sigma(g)) = w(\beta(g))$, for all $g \in G$.
We write $(g) = g^{-1}$, called the inverse element of $g \in G$ and $w(x) = \bar{x}$ called the unit element in $G$
associated to the element $x \in B$. Also, we write $\gamma(g, \dot{g}) = g\dot{g}$. $G$ is called the groupoid and $B$ is
called the base. Also we say that $G$ is a groupoid on $B$.

Example (1-2):
(1) Each non empty set $B$ is a groupoid of base $B$ where $w = I_B$. This groupoid is called null groupoid.
(2) $GL(n, R)$ of base $R$ is groupoid.

Definition (1-3) [5]:
Let $f$ be a mapping of a topological space $x$ into topological space $Y$. Then $f$ is said to be proper map
if $f$ is continuous and the mapping $f \times I_Z: X \times Z \to Y \times Z$ is closed for every topological spac.

Proposition (1-4) [6]:
Let $f$ and $g$ be continuous maps then:
(i) If $f$ and $g$ are proper then $gof$ is proper.
(ii) If $gof$ is proper and $f$ is surjective then $g$ is proper.
(iii) If $gof$ is proper and $g$ is injective then $f$ is proper.
(iv) If $gof$ is proper and $y$ is a Hausdorff space then $f$ is proper map.

Definition (1-5) [3]:
A topological groupoid is a groupoid $(G, B)$ together with topologies on $G$ and $B$ such that the maps
$\alpha, \beta, w, \gamma$ and $\sigma$ are continuous where $G \ast G$. Has the subspace topology from $G \times G$. A topological
subgroupoid is subgroupoid $(H, A)$ with the subspace topology from $(G, B)$.

Definition (1-6) [2]:
A morphism of topological groupoid is morphism of groupoids $(f, f_0): (G, B) \to (\hat{G}, \hat{B})$ such that $f$
and $f_0$ are continuous. An isomorphism of topological groupoid is a morphism of topological groupoids such that $f: G \to \hat{G}$ is a homeomorphism.

Remark (1-7) [1]:
We denote by $TG$ the category whose objects are topological groupoids and whose morphisms are continuous groupoid morphisms.

Example (1-8):
Any trivial groupoid $E \times \Gamma \times E$ is a topological groupoid where $\Gamma$ is a topological group and $E$ is a
topological space.

Definition (1-9) [7]:
Let $(G, B)$ be topological groupoid space, $\pi: E \to B$ be a continuous map and let $G \ast E$ denote the fiber
product of $\alpha$ and $\pi$ over $B$. A left action of $G$ on $(E, \pi, B)$ is a continuous map $\varphi^*: G \ast E \to E$ such that:
(i) $\pi(\varphi^*(g, z)) = \beta(g)$ for each $(g, z) \in G \ast E$.
(ii) $\varphi^*(w(\pi(z)), z) = z$ for each $Z \in E$.
(iii) $\varphi^*(g, \varphi^*(g, z)) = \varphi^*(\gamma(g, \dot{g})), z)$ for each $(g, \dot{g}) \in G \ast E$.
The bundle $(E, \pi, B)$ together with action $\varphi^*$ is called groupoid space and is denote by $G -$space, in
similar way, we can define right action of $G$ on $(E, \pi, B)$.
3. Source Proper Topological groupoid (SP-groupoid):

Definition (2-1) [3]:
A topological groupoid \((G,B)\) is called source proper groupoid "sp-groupoid" if:

1. The source map \(\alpha: G \rightarrow G\) is a proper.
2. The base space \(B\) is a Housdorff.

Example (2-2):
Every compact topological groupoid \(G\) on a Hausdorff space \(B\) is a Sp-groupoid, since the source map \(\alpha: G \rightarrow B\) is proper map.

Proposition (2-3) [3]:
Let \((G,B)\) be an sp-groupoid then:

1. The inversion map \(\sigma: G \rightarrow G\) is proper.
2. The target map \(\beta: G \rightarrow B\) is proper.
3. The unit map \(w: B \rightarrow G\) is proper.

Proposition (2-4):
Let \((f,f_0):(G,B) \rightarrow (G',B')\) and \((g,g_0):(G',B') \rightarrow (G'',B'')\) be morphism of Sp-groupoid such that \(gof\) is a proper map then the base map \(g_0f_0: B \rightarrow B''\) is a proper map.

Proof:
Consider the following commutative diagram in topological groupoid.

\[
\begin{array}{ccc}
B & \xrightarrow{g_0f_0} & B'' \\
\downarrow w & & \downarrow w'' \\
w(B) & \xrightarrow{(gof)^*} & w''(B'')
\end{array}
\]

which \(w\) and \(w''\) are both proper Proposition (2-3) and \((gof)^*\) is proper map since \(w(B)\) is known subspace in \(G\) and \(w(g_0f_0)\) is proper map since \(w_0(g_0,f_0) = (gof)^*\) and \((gof)^* w\) is proper map proposition(1-4). Hence \((g_0,f_0)\) is proper map, since \(w''\) is injective continuous proposition (1-4).

Definition (2-5) [2]:
Let \((f_1,f'_1):(G_1,B_1) \rightarrow (G'_1,B'_1)\) and \((f_2,f'_2):(G_2,B_2) \rightarrow (G'_2,B'_2)\) each are proper maps, then the direct sum \((f_1 \oplus f_2,f'_1 \oplus f'_2):(G_1 \oplus G_2,B_1 \oplus B_2) \rightarrow (G'_1 \oplus G'_2,B'_1 \oplus B'_2)\) is proper map.

Proposition (2-6):
Let \((f,f_0):(G,B) \rightarrow (G',B')\) and \((g,g_0):(G',B') \rightarrow (G'',B'')\) be morphisms of topological groupoid, if \((gof)\) is an injective proper map and \((G'',B'')\) is Sp-groupoid then \((G,B)\) is Sp-groupoid.

Proof:
Consider the following commutative diagram in topological groupoid.
\[ G \xrightarrow{f} G' \xrightarrow{g} G'' \]

\[ \alpha \quad \downarrow \quad \alpha' \quad \downarrow \quad \alpha'' \]

\[ G \xrightarrow{f_0} G' \xrightarrow{g_0} G'' \]

i.e \( \alpha'of = f_0\alpha \)
\( \alpha''og = g_0\alpha' \)
\( \alpha''o(gof) = (g_0, f_0)\alpha \)
\( (gof) \) and \( \alpha \) are proper map \((G'', B'')\) is Sp- groupoid and then
\( \alpha''o(gof) \) is proper map .proposition (1-4) and then \((g_0, f_0)\alpha \) is proper.
Hence \( \alpha \) is proper map since \((g_0, f_0)\) is injective continuous proposition (1-4).

Proposition (2-7):
Let \((f, f_0): (G, B) \rightarrow (G', B') \) and \((g, g_0): (G', B') \rightarrow (G'', B'') \) be morphisms of topological groupoid
, if \((gof): G \rightarrow \tilde{G}\) is surjective proper map and \((G, B)\) is Sp-groupoid then \((G'', B'')\) is Sp- groupoid.

**Proof:**
Consider the following commutative diagram in topological groupoid.

\[ G \xrightarrow{f} G' \xrightarrow{g} G'' \]

\[ \alpha \quad \downarrow \quad \alpha' \quad \downarrow \quad \alpha'' \]

\[ G \xrightarrow{f_0} G' \xrightarrow{g_0} G'' \]

i.e \( \alpha'of = f_0\alpha \)
\( \alpha''og = g_0\alpha' \)
\( \alpha''o(gof) = (g_0, f_0)\alpha \)
\( (gof) \) and \( \alpha \) are proper map \((G, B)\) is Sp-groupoid and then \(g_0, f_0\) is proper map proposition (2-3)
Hence \( \alpha \) is proper map then \( \alpha'' \) is proper since \((gof)\) is surjective continuous proposition(1-4).

Theorem (2-8):
Let \((f_1, f'_1): (G, B) \rightarrow (G_1, B_1) \)
\((f_2, f'_2): (G_1, B_1) \rightarrow (G_2, B_2) \)
\[
\vdots
\]
and \((f_n, f'_n): (G_n, B_n) \rightarrow (G_{n+1}, B_{n+1}) \)
be morphisms of topological groupoid , if \(f_n \circ \ldots \circ f_2 \circ f_1\) is an injective proper map and \((G_n, B_n)\) is Sp-groupoid then \((G, B)\) is Sp-groupoid.

**Proof:**
Consider the following commutative diagram in topological groupoid

\[ G \xrightarrow{f_1} G_1 \xrightarrow{f_2} G_2 \rightarrow \ldots \rightarrow G_n \]

\[ \alpha \quad \downarrow \quad \alpha_1 \quad \downarrow \quad \alpha_2 \quad \downarrow \quad \alpha_n \]

\[ B \xrightarrow{f'_1} B_1 \xrightarrow{f'_2} B_2 \rightarrow \ldots \rightarrow B_n \]
\[ \alpha_n \circ (f_n \circ \ldots \circ f_2 \circ f_1) = (f_n' \circ \ldots \circ f_2' \circ f_1') \circ \alpha \]
\[ (f_n, \ldots, f_2, f_1) \text{ and } \alpha_n \text{ are proper map.} \]
\[ (G_n, B_n) \text{ is Sp-groupoid and then } \alpha_n \circ (f_n \circ \ldots \circ f_2 \circ f_1) \]
\[ \text{is proper map proposition(1-4). And then } (f_n' \circ \ldots \circ f_2' \circ f_1') \circ \alpha \]
\[ \text{is proper .since } \alpha \text{ is proper map} \]
\[ \text{(proposition(2-5)).} \]

**Theorem (2-9):**
Let \( (f_1, f_1'): (G, B) \rightarrow (G_1, B_1) \)
\[ (f_2, f_2'): (G_1, B_1) \rightarrow (G_2, B_2); \]
\[ \vdots \]
and \( (f_n, f_n'): (G_{n-1}, B_{n-1}) \rightarrow (G_n, B_n) \)
\[ \text{be morphisms of topological groupoid ,then } f_n \circ \ldots \circ f_2 \circ f_1: G \rightarrow G_n \]
\[ \text{is surjective proper map and } (G, B) \]
\[ \text{is Sp-groupoid.} \]

**Proof:**
Consider the following commutative diagram in topological groupoid
\[
\begin{array}{ccc}
G & \xrightarrow{f_1} & G_1 \xrightarrow{f_2} G_2 \xrightarrow{f_3} \cdots \xrightarrow{f_n} G_n \\
\downarrow & & \downarrow \alpha_1 & & \downarrow \alpha_2 & & \downarrow \alpha_3 & & \cdots & & \downarrow \alpha_n \\
B & \xrightarrow{f_1'} & B_1 & \xrightarrow{f_2'} B_2 & \xrightarrow{f_3'} B_3 & \cdots & \xrightarrow{f_n'} B_n \\
\end{array}
\]
\[ \text{i.e } \alpha_n \circ (f_n \circ \ldots \circ f_2 \circ f_1) = (f_n' \circ \ldots \circ f_2' \circ f_1') \circ \alpha \]
\[ (f_n' \circ \ldots \circ f_2' \circ f_1') \text{ and } \alpha \text{ are proper map. Since } (G, B) \text{ is Sp-groupoid} \]
\[ (f_n' \circ \ldots \circ f_2' \circ f_1') \text{ is proper} \]
\[ \text{proposition (1-4). Hence } \alpha_{n-1} \text{ is proper map then } \alpha_n \text{ is proper since} \]
\[ (f_n \circ \ldots \circ f_2 \circ f_1) \text{ is surjective continuous} \]
\[ \text{proposition (2-6).} \]

**Theorem (2-10):**
Let \( (f_1, f_1'): (G^2, B^1) \rightarrow (G^2, B^2) \)
\[ (f_2, f_2'): (G^2, B^2) \rightarrow (G^3, B^3); \]
\[ \vdots \]
and \( (f_n, f_n'): (G^{n-1}, B^{n-1}) \rightarrow (G^n, B^n) \)
\[ \text{be morphisms of topological groupoid , if } (f_n, o \circ f_2 \circ f_1) \text{ is an injective proper map and } (G^n, B^n) \]
\[ \text{is Sp-groupoid then } (G, B) \text{ is Sp-groupoid.} \]

**Proof:**
Consider the following commutative diagram in topological groupoid
\[
\begin{array}{ccc}
G & \xrightarrow{f_1} G \times G \xrightarrow{f_2} G \times G \xrightarrow{f_3} \cdots \xrightarrow{f_n} G \\
\downarrow & & \downarrow \alpha_2 & & \downarrow \alpha_3 & & \cdots & & \downarrow \alpha_n \\
B & \xrightarrow{f_1'} B \times B \xrightarrow{f_2'} B \times B \xrightarrow{f_3'} B \cdots \xrightarrow{f_n'} B \\
\end{array}
\]
\[ \text{i.e } \alpha_n \circ (f_n \circ \ldots \circ f_2 \circ f_1) = (f_n' \circ \ldots \circ f_2' \circ f_1') \circ \alpha \]
\[ (f_n' \circ \ldots \circ f_2' \circ f_1') \text{ and } \alpha \text{ are proper map. Since } (G, B) \text{ is Sp-groupoid} \]
\[ (f_n' \circ \ldots \circ f_2' \circ f_1') \text{ is proper} \]
\[ \text{proposition (1-4). Hence } \alpha_{n-1} \text{ is proper map then } \alpha_n \text{ is proper since} \]
\[ (f_n \circ \ldots \circ f_2 \circ f_1) \text{ is surjective continuous} \]
\[ \text{proposition (2-6).} \]
Proof:
(i.e \((f_0 o \ldots o f_2 o f_1)\) and \(\alpha_n\) are proper maps, \((G^n, B^n)\) is Sp-groupoid and then \(\alpha_n o(f_0 o \ldots o f_2 o f_1)\) is proper map (Proposition (1-4)), and then \((f_n o \ldots o f_2 o f_1)\) is proper. Hence \(\alpha\) is proper map.

\textbf{Theorem (2-11):}
Let \((f_1, f'_1):(G, B) \rightarrow (G^2, B^2)\)
\((f_2, f'_2):(G^2, B^2) \rightarrow (G^3, B^3)\)
\vdots
and \((f_n, f'_n):(G^{n-1}, B^{n-1}) \rightarrow (G^n, B^n)\)
be morphisms of topological groupoid, then \((f_n o \ldots o f_2 o f_1):G \rightarrow G^n\) is surjective proper map and \((G^n, B^n)\) is Sp-groupoid.

\textit{Proof:}
Consider the following commutative diagram in topological groupoid
\[
\begin{array}{ccccccc}
G & \xrightarrow{f_1} & G \times G & \xrightarrow{f_2} & G \times G \times G & \xrightarrow{f_3} & \ldots & \xrightarrow{n \text{ times}} & G \\
\downarrow{\alpha} & & \downarrow{\alpha_2} & & \downarrow{\alpha_3} & & \downarrow{\alpha_3} & & \downarrow{n \text{ times}} \\
B & \xrightarrow{f'_1} & B \times B & \xrightarrow{f'_2} & B \times B \times B & \xrightarrow{f'_3} & \ldots & \xrightarrow{n \text{ times}} & B
\end{array}
\]
i.e \(\alpha_n o(f_0 o \ldots o f_2 o f_1) = (f'_0 o \ldots o f'_2 o f'_1) o \alpha\)
\((f'_0 o \ldots o f'_2 o f'_1)\) and \(\alpha\) are proper maps since \((G, B)\) is Sp-groupoid and \((f'_0 o \ldots o f'_2 o f'_1)\) is proper proposition (1-4). Hence \(\alpha_{n-1}\) is proper map then \(\alpha_n\) is proper since \((f_n o \ldots o f_2 o f_1)\) is surjective continuous (Theorem (2-8)).

\textbf{Proposition (2-12):}
Let \((f_1, f'_1):(G_1, B_1) \rightarrow (G'_1, B'_1)\) and \((f_2, f'_2):(G_2, B_2) \rightarrow (G'_2, B'_2)\) be morphisms of Sp-groupoid such that \(f_1 \oplus g_1\) is a proper map then the \(f'_1 \oplus f'_2: B_1 \oplus B_2 \rightarrow B'_1 \oplus B'_2\) is proper map.

\textit{Proof:}
Consider the following commutative diagram in topological groupoid[TG].
\[
\begin{array}{ccc}
B_1 \oplus B_2 & \xrightarrow{f_1 \oplus f_2} & B'_1 \oplus B'_2 \\
\downarrow{w} & & \downarrow{\hat{w}} \\
w(B_1 \oplus B_2) & \xrightarrow{(f_1 \oplus f_2)^*} & w'(B'_1 \oplus B'_2)
\end{array}
\]
In which \(w\) and \(\hat{w}\) are both proper proposition (2-3) and \((f_1 + f_2)^*\) is proper map since \(w(B_1 \oplus B_2)\) is closed subspace in \((G_1 \oplus G_2)\) and \((f_1 \oplus f_2)\) is proper map since
\[ w' o (f_1 \oplus f_2) = (f_1 \oplus f_2)' o w \] and \( (f_1 \oplus f_2)' o w \) is proper map since \( \dot{w} \) is injective continuous.

**Theorem (2-11):**
Let \((f_1 \oplus f_2, f_1' \oplus f_2') = (G, B) \rightarrow (G', B')\), \((G, B)\) are Sp-groupoid then \( f_1 \oplus f_2: G_1 \oplus G_2 = G \rightarrow B_1 \oplus B_2 = B \) and \( f_1' \oplus f_2': G_1' \oplus G_2' = G \rightarrow B_1' \oplus B_2' = B' \) are proper maps.

**Proof:**
The map \( f_1 \oplus f_2 \) is bijective continuous map and its inverse is \( (f_1 \oplus f_2)^{-1}(g) = (f_1 \oplus f_2)(g)^{-1} \), \( \forall g \in G \) which is continuous so \( f_1 \oplus f_2 \) is homomorphism and then \( f_1 \oplus f_2 \) is proper map, and \( f_1' \oplus f_2' \) is proper map.

**References**

[1] Brown, R. 1989 Symmetry, Groupoids and Higher dimensional analogons, computer Math. Application, Vol.17, No(1-3), pp.49-57.
[2] Chiriac. L. 2009 On topological groupoid and multiple Indentities, (BADS), number 1(59), pp.67-78.
[3] Gursuy, M., H. 2005 Topological group- groupoids and their Covering, Indian Pure and appl. Math., Vol.36(a), pp.493-502.
[4] Hamza, S., H. 2007 Certain types of groupoids spaces, A thesis AL-Mustansiriyah University, Collage of Sciences, Department of Mathemetics.
[5] Mackenzie, K. 1987 Lie groupoids and Lie algebroids in differential Geometry, London Math. Soc. Lecture note series Cambridge University press.
[6] Meyer .R. 2012 On the Category of groupoids, GAP Anogia.
[7] Moerdijk, I. 2002 Orbifolds as Groupoids an Introduction, Madison W., pp. 205-222.
[8] Tu, J.L. 2004 Non-Hansdorff Groupoid , proper actions and K-theory, Document a math-a, pp.565-597.