Determination of the prime bound of a graph

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December 11, 2013

Abstract

Given a graph \( G \), a subset \( M \) of \( V(G) \) is a module of \( G \) if for each \( v \in V(G) \setminus M \), \( v \) is adjacent to all the elements of \( M \) or to none of them. For instance, \( V(G) \), \( \emptyset \) and \( \{v\} \ (v \in V(G)) \) are modules of \( G \) called trivial. Given a graph \( G \), \( \omega_M(G) \) (respectively \( \alpha_M(G) \)) denotes the largest integer \( m \) such that there is a module \( M \) of \( G \) which is a clique (respectively a stable) set in \( G \) with \( |M| = m \). A graph \( G \) is prime if \( |V(G)| \geq 4 \) and if all its modules are trivial. The prime bound of \( G \) is the smallest integer \( p(G) \) such that there is a prime graph \( H \) with \( V(H) \supseteq V(G) \), \( H[V(G)] = G \) and \( |V(H) \setminus V(G)| = p(G) \). We establish the following. For every graph \( G \) such that \( \max(\alpha_M(G), \omega_M(G)) \geq 2 \) and \( \log_2(\max(\alpha_M(G), \omega_M(G))) \) is not an integer, \( p(G) = \lceil \log_2(\max(\alpha_M(G), \omega_M(G))) \rceil \). Then, we prove that for every graph \( G \) such that \( \max(\alpha_M(G), \omega_M(G)) = 2^k \) where \( k \geq 1 \), \( p(G) = k \) or \( k+1 \). Moreover \( p(G) = k+1 \) if and only if \( G \) or its complement admits \( 2^k \) isolated vertices. Lastly, we show that \( p(G) = 1 \) for every non prime graph \( G \) such that \( |V(G)| \geq 4 \) and \( \alpha_M(G) = \omega_M(G) = 1 \).

Mathematics Subject Classifications (2010): 05C70, 05C69

Key words: Module; prime graph; prime extension; prime bound; modular clique number; modular stability number

1 Introduction

A graph \( G = (V(G), E(G)) \) is constituted by a vertex set \( V(G) \) and an edge set \( E(G) \subseteq \binom{V(G)}{2} \). Given a set \( S, K_S = (S, \binom{S}{2}) \) is the complete graph on \( S \) whereas \( (S, \emptyset) \) is the empty graph. Let \( G \) be a graph. With each \( W \subseteq V(G) \) associate the subgraph \( G[W] = (W, \binom{W}{2} \cap E(G)) \) of \( G \) induced by \( W \). Given \( W \subseteq V(G) \), \( G[V(G) \setminus W] \) is also denoted by \( G-W \) and by \( G-w \) if \( W = \{w\} \). A

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graph $H$ is an \textit{extension} of $G$ if $V(H) \supseteq V(G)$ and $H[V(G)] = G$. Given $p \geq 0$, a $p$-extension of $G$ is an extension $H$ of $G$ such that $|V(H) \setminus V(G)| = p$. The \textit{complement} of $G$ is the graph $\overline{G} = (V(G), \left\{ \frac{V(G)}{2} \right\} \setminus E(G))$. A subset $W$ of $V(G)$ is a \textit{clique} (respectively a \textit{stable set}) in $G$ if $G[W]$ is complete (respectively empty). The largest cardinality of a clique (respectively a stable set) in $G$ is the clique number ($\alpha$) (respectively the stability number, $\omega$) of $G$, denoted by $\omega(G)$ (respectively $\alpha(G)$). Given $v \in V(G)$, the \textit{neighbourhood} $N_G(v)$ of $v$ in $G$ is the family $\{ w \in V(G) : \{ v, w \} \in E(G) \}$. We consider $N_G$ as the function from $V(G)$ to $2^{V(G)}$ defined by $v \mapsto N_G(v)$ for each $v \in V(G)$. A vertex $v$ of $G$ is isolated if $N_G(v) = \emptyset$. The number of isolated vertices of $G$ is denoted by $\iota(G)$.

We use the following notation. Let $G$ be a graph. For $v \neq w \in V(G)$,

$$(v, w)_G = \begin{cases} 0 & \text{if } \{ v, w \} \notin E(G), \\ 1 & \text{if } \{ v, w \} \in E(G). \end{cases}$$

Given $W \subseteq V(G)$, $v \in V(G) \setminus W$ and $i \in \{0, 1\}$, $(v, W)_G = i$ means $(v, w)_G = i$ for every $w \in W$. Given $W, W' \subseteq V(G)$, with $W \cap W' = \emptyset$, and $i \in \{0, 1\}$, $(W, W')_G = i$ means $(w, W')_G = i$ for every $w \in W$. Given $W \subseteq V(G)$ and $v \in V(G) \setminus W$, $v \leftrightarrow_G W$ means that there is $i \in \{0, 1\}$ such that $(v, W)_G = i$.

The negation is denoted by $v \not\leftrightarrow_G W$.

Given a graph $G$, a subset $M$ of $V(G)$ is a module of $G$ if for each $v \in V(G) \setminus M$, we have $v \leftrightarrow_G M$. For instance, $V(G)$, $\emptyset$ and $\{ v \} \ (v \in V(G))$ are modules of $G$ called trivial. Clearly, if $|V(G)| \leq 2$, then all the modules of $G$ are trivial. On the other hand, if $|V(G)| = 3$, then $G$ admits a nontrivial module. A graph $G$ is then said to be prime if $|V(G)| \geq 4$ and if all its modules are trivial. For instance, given $n \geq 4$, the path $\langle \{1, \ldots, n\}, \{\{p, q\} : |p - q| = 1\} \rangle$ is prime. Given a graph $G$, $G$ and $\overline{G}$ share the same modules. Thus $G$ is prime if and only if $\overline{G}$ is.

Given a set $S$ with $|S| \geq 2$, $K_S$ admits a prime $\lceil \log_2(|S| + 1) \rceil$-extension (see Sumner \cite{Sumner} Theorem 2.45 or Lemma \cite{Brignall} below). This is extended to any graph in \cite{Brignall} Theorem 3.7 and \cite{Brignall} Theorem 3.2 as follows.

\textbf{Theorem 1.} A graph $G$, with $|V(G)| \geq 2$, admits a prime $\lceil \log_2(|V(G)| + 1) \rceil$-extension.

Following Theorem \cite{Brignall} we introduce the notion of prime bound. Let $G$ be a graph. The \textit{prime bound} of $G$ is the smallest integer $p(G)$ such that $G$ admits a prime $p(G)$-extension. Observe that $p(G) = p(\overline{G})$ for every graph $G$. By Theorem \cite{Brignall} $p(G) \leq \lfloor \log_2(|V(G)| + 1) \rfloor$. By considering the clique number and the stability number, Brignall \cite{Brignall} Conjecture 3.8] conjectured the following.

\textbf{Conjecture 1.} For a graph $G$ with $|V(G)| \geq 2$,

\[ p(G) \leq \lfloor \log_2(\max(\alpha(G), \omega(G)) + 1) \rfloor. \]

We answer the conjecture positively by refining the notions of clique number and of stability number as follows. Given a graph $G$, the \textit{modular clique number}
of $G$ is the largest integer $\omega_M(G)$ such that there is a module $M$ of $G$ which is a clique in $G$ with $|M| = \omega_M(G)$. The modular stability number of $G$ is $\alpha_M(G) = \omega_M(G)$. The following lower bound is simply obtained.

**Lemma 1.** For every graph $G$ such that $\max(\alpha_M(G), \omega_M(G)) \geq 2$,

$$p(G) \geq \lceil \log_2(\max(\alpha_M(G), \omega_M(G))) \rceil.$$

Theorem 3.2 of [3] is proved by induction on the number of vertices. Using the main arguments of this proof, we improve Theorem 1 as follows.

**Theorem 2.** For every graph $G$ such that $\max(\alpha_M(G), \omega_M(G)) \geq 2$,

$$p(G) \leq \lceil \log_2(\max(\alpha_M(G), \omega_M(G)) + 1) \rceil.$$

The proof of Theorem 2 derives from an induction as well. A direct construction of a suitable extension is provided in [1, Theorem 2]. The following is an immediate consequence of Lemma 1 and Theorem 2.

**Corollary 1.** For every graph $G$ such that $\max(\alpha_M(G), \omega_M(G)) \geq 2$,

$$\lceil \log_2(\max(\alpha_M(G), \omega_M(G))) \rceil \leq p(G) \leq \lceil \log_2(\max(\alpha_M(G), \omega_M(G)) + 1) \rceil.$$

Let $G$ be graph such that $\max(\alpha_M(G), \omega_M(G)) \geq 2$. On the one hand, it follows from Corollary 1 that

$$\max(\alpha_M(G), \omega_M(G)) \notin \{2^k : k \geq 1\} \Rightarrow p(G) = \lceil \log_2(\max(\alpha_M(G), \omega_M(G))) \rceil.$$

On the other, if $\max(\alpha_M(G), \omega_M(G)) = 2^k$, where $k \geq 1$, then $p(G) = k$ or $k+1$. The next allows us to determine this.

**Theorem 3.** For every graph $G$ such that $\max(\alpha_M(G), \omega_M(G)) = 2^k$ where $k \geq 1$,

$$p(G) = k + 1 \text{ if and only if } \iota(G) = 2^k \text{ or } \overline{\iota(G)} = 2^k.$$

Lastly, we show that $p(G) = 1$ for every non prime graph $G$ such that $|V(G)| \geq 4$ and $\alpha_M(G) = \omega_M(G) = 1$ (see Proposition 8).

## 2 Preliminaries

Given a graph $G$, the family of the modules of $G$ is denoted by $\mathcal{M}(G)$. Furthermore set

$$\mathcal{M}_{\geq 2}(G) = \{M \in \mathcal{M}(G) : |M| \geq 2\}.$$

We begin with the well known properties of the modules of a graph (for example, see [5, Theorem 3.2, Lemma 3.9]).

**Proposition 1.** Let $G$ be a graph.

1. Given $W \subseteq V(G)$, $\{M \cap W : M \in \mathcal{M}(G)\} \subseteq \mathcal{M}(G[W])$. 


2. Given a module $M \in \mathcal{M}(G)$, $\mathcal{M}(G[M]) = \{ N \in \mathcal{M}(G) : N \subseteq M \}$.

3. Given $M, N \in \mathcal{M}(G)$ with $M \cap N = \emptyset$, there is $i \in \{0,1\}$ such that $(M,N)_G = i$.

Given a graph $G$, a partition $P$ of $V(G)$ is a modular partition of $G$ if $P \subseteq \mathcal{M}(G)$. Let $P$ be such a partition. Given $M \neq N \in P$, there is $i \in \{0,1\}$ such that $(M,N)_G = i$ by Proposition 1.3. This justifies the following definition. The quotient of $G$ by $P$ is the graph $G/P$ defined on $V(G/P) = P$ by $(M,N)_{G/P} = (M,N)_G$ for $M \neq N \in P$. We use the following properties of the quotient (for example, see [5, Theorems 4.1–4.3, Lemma 4.1]).

**Proposition 2.** Given a graph $G$, consider a modular partition $P$ of $G$.

1. Given $W \subseteq V(G)$, if $|W \cap X| = 1$ for each $X \in P$, then $G[W]$ and $G/P$ are isomorphic.

2. For every $M \in \mathcal{M}(G)$, $\{ X \in P : M \cap X \neq \emptyset \} \in \mathcal{M}(G/P)$.

3. For every $Q \in \mathcal{M}(G/P)$, $\cup Q \in \mathcal{M}(G)$.

The following strengthening of the notion of module is introduced to present the modular decomposition theorem (see Theorem 4 below). Given a graph $G$, a module $M$ of $G$ is said to be strong provided that for every $N \in \mathcal{M}(G)$, we have: if $M \cap N \neq \emptyset$, then $M \subseteq N$ or $N \subseteq M$. The family of the strong modules of $G$ is denoted by $\mathcal{S}(G)$. Furthermore set

$$\mathcal{S}_{\geq 2}(G) = \{ M \in \mathcal{S}(G) : |M| \geq 2 \}.$$ 

We recall the following well known properties of the strong modules of a graph (for example, see [5, Theorem 3.3]).

**Proposition 3.** Let $G$ be a graph. For every $M \in \mathcal{M}(G)$, $\mathcal{S}(G[M]) = \{ N \in \mathcal{S}(G) : N \subseteq M \} \cup \{ M \}$.

With each graph $G$, we associate the family $\Pi(G)$ of the maximal proper and nonempty strong modules of $G$ under inclusion. For convenience set

$$\Pi_1(G) = \{ M \in \Pi(G) : |M| = 1 \} \text{ and } \Pi_{\geq 2}(G) = \{ M \in \Pi(G) : |M| \geq 2 \}.$$ 

The modular decomposition theorem is stated as follows.

**Theorem 4** (Gallai [18, 19]). For a graph $G$ with $|V(G)| \geq 2$, the family $\Pi(G)$ realizes a modular partition of $G$. Moreover, the corresponding quotient $G/\Pi(G)$ is complete, empty or prime.

Let $G$ be a graph with $|V(G)| \geq 2$. As a direct consequence of the definition of a strong module, we obtain that the family $\mathcal{S}(G) \setminus \{ \emptyset \}$ endowed with inclusion is a tree called the modular decomposition tree [4] of $G$. Given $M \in \mathcal{S}_{\geq 2}(G)$, it follows from Proposition 3 that $\Pi(G[M]) \in \mathcal{S}(G)$. Furthermore, given $W \subseteq
Lemma 2. Let $S$ and $S'$ be disjoint sets such that $|S| \geq 2$ and $|S'| = \lceil \log_2(|S| + 1) \rceil$. There exists a prime graph $G$ defined on $V(G) = S \cup S'$ such that $S$ and $S'$ are stable sets in $G$.

Proof. If $|S| = 2$, then $|S'| = 2$ and we can choose a path on 4 vertices for $G$. Assume that $|S| \geq 3$. As $|S'| = \lceil \log_2(|S| + 1) \rceil$, $2^{|S'|-1} \leq |S|$ and hence $|S'| \leq |S|$. Thus there exists a bijection $\psi_{S'}$ from $S'$ onto $S'' \subseteq S$. Consider the injection $f_{S''} : S'' \to 2^{S'} \setminus \emptyset$ defined by $s'' \mapsto S' \setminus \{(\psi_{S'})^{-1}(s'')\}$. Since $|S'| = \lceil \log_2(|S| + 1) \rceil$, $|S| < 2^{|S'|}$ and there exists an injection $f_S$ from $S$ into $2^{S'} \setminus \emptyset$ such that $(f_S)_{S''} = f_{S''}$. Lastly, consider the graph $G$ defined on $V(G) = S \cup S'$ such that $S$ and $S'$ are stable sets in $G$ and $(N_G)_S = f_S$. We prove that $G$ is prime. If $|S| = 3$, then $|S'| = 2$ and $G$ is a path on 5 vertices which is prime. Assume that $|S| \geq 4$ and hence $|S'| \geq 3$. Let $M \in \mathcal{S}_{22}(G)$.

First, if $M \subseteq S$, then we would have $f_S(u) = f_S(v)$ for any $u \neq v \in M$. Thus $M \cap S' \neq \emptyset$.

Second, suppose that $M \subseteq S'$. Recall that for each $s \in S$, either $M \cap N_G(s) = \emptyset$ or $M \subseteq N_G(s)$. Given $u \in M$, consider the function $f : S \to 2^{(S' \setminus M) \cup \{u\}} \setminus \emptyset$ defined by

$$f(s) = \begin{cases} N_G(s) & \text{if } M \cap N_G(s) = \emptyset, \\ (N_G(s) \setminus M) \cup \{u\} & \text{if } M \subseteq N_G(s), \end{cases}$$

for every $s \in S$. Since $(N_G)_S$ is injective, $f$ is also and we would obtain that $|S| < 2^{|S'|}-1$. It follows that $M \cap S \neq \emptyset$.

Third, suppose that $S' \setminus M \neq \emptyset$. We have $(S \cap M, S' \setminus M)_G = (S' \cap M, S' \setminus M)_G = 0$. Given $s' \in S' \cap M$, $N_G(\psi_{S'}(s')) = S' \setminus \{s'\}$. In particular $S' \setminus M \subseteq N_G(\psi_{S'}(s'))$ and hence $\psi_{S'}(s') \in S \setminus M$. Furthermore $(\psi_{S'}(s'), S' \cap M)_G = (\psi_{S'}(s'), S \cap M)_G = 0$. Therefore $S' \cap M = \{s'\}$. Similarly, we prove that $|S' \setminus M| = 1$ which would imply that $|S'| = 2$. It follows that $S' \subseteq M$.

Lastly, suppose that $S \setminus M \neq \emptyset$. For each $s \in S \setminus M \neq \emptyset$, we would have $(s, S')_G = (s, S \cap M)_G = 0$ and hence $N_G(s) = \emptyset$. It follows that $S \subseteq M$ and $M = S \cup S'$.

\[ \Box \]
Lemma 3. Let \( C \) and \( S' \) be disjoint sets such that \( |C| \geq 2 \) and \( |S'| = \lceil \log_2(|C| + 1) \rceil \). There exists a prime graph \( G \) defined on \( V(G) = C \cup S' \) such that \( C \) is a clique and \( S' \) is a stable set in \( G \).

Proof. There exists a bijection \( \psi_{S'} \) from \( S' \) onto \( S'' \subseteq C \). Consider the injection \( f_{S''} : S'' \rightarrow 2^{S'} \setminus \{S'\} \) defined by \( s'' \mapsto (\psi_{S'}^{-1}(s'')) \). Let \( f_S \) be any injection from \( S \) into \( 2^{S'} \setminus \{S'\} \) such that \( (f_S)_S = f_{S''} \). Lastly, consider the graph \( G \) defined on \( V(G) = C \cup S' \) such that \( C \) is a clique in \( G \), \( S' \) is a stable set in \( G \) and \( N_G(c) \cap S' = f_S(c) \) for each \( c \in C \). We prove that \( G \) is prime. Let \( M \in \mathcal{M}_{22}(G) \). As in the proof of Lemma 2, we have \( M \cap C \neq \emptyset \) and \( M \cap S' \neq \emptyset \).

Now, suppose that \( S' \cap M \neq \emptyset \). We have \( (C \cap M, S' \setminus M)_G = (S' \cap M, S' \setminus M)_G = 0 \). Given \( t' \in S' \setminus M \), \( N_G(\psi_{S'}(t')) \cap S' = \{t'\} \). Thus \( \psi_{S'}(t') \in C \setminus M \). But \( (\psi_{S'}(t'), S' \cap M)_G = (\psi_{S'}(t'), C \cap M)_G = 1 \) which contradicts \( N_G(\psi_{S'}(t')) \cap S' = \{t'\} \). It follows that \( S' \subseteq M \).

Lastly, suppose that \( C \cap M \neq \emptyset \). For each \( c \in C \setminus M \neq \emptyset \), we would have \( (c, S')_G = (c, C \cap M)_G = 1 \) and hence \( N_G(c) \cap S' = S' \). It follows that \( S \subseteq M \) and \( M = S' \cup S' \).

The question of prime extensions of a prime graph is not detailed enough in [3]. For instance, the number of prime 1-extensions of a prime graph given in [3] is not correct. Moreover, Corollary 2 below is used without a precise proof.

Lemma 4. Let \( G \) be a prime graph \( G \). Given a \( a \notin V(G) \), there exist

\[ 2^{|V(G)|} - 2|V(G)| - 2 \]

distinct prime extensions of \( G \) to \( V(G) \cup \{a\} \).

Proof. Consider any graph \( H \) defined on \( V(H) = V(G) \cup \{a\} \) such that \( H[V(G)] = G \). We prove that \( H \) is not prime if and only if

\[ N_H(a) \in \{\emptyset, V(G)\} \cup \{N_G(v) : v \in V(G)\} \cup \{N_G(v) \cup \{v\} : v \in V(G)\}. \]

To begin, assume that \( N_H(a) \in \{\emptyset, V(G)\} \cup \{N_G(v) : v \in V(G)\} \cup \{N_G(v) \cup \{v\} : v \in V(G)\} \). If \( N_H(a) = \emptyset \) or \( V(G) \), then \( V(G) \) is a nontrivial module of \( H \). If there is \( v \in V(G) \) such that \( N_H(a) \setminus \{v\} = N_G(v) \), then \( \{a, v\} \) is a nontrivial module of \( H \).

Conversely, assume that \( H \) admits a nontrivial module \( M \). By Proposition 1, \( M \setminus \{a\} \in \mathcal{M}(G) \). As \( G \) is prime and as \( M \setminus \{a\} \neq \emptyset \) and \( M \not
subseteq V(H) \), either \( |M \setminus \{a\}| = 1 \) or \( M = V(G) \). In the second instance, \( N_H(a) = \emptyset \) or \( V(G) \). In the first, there is \( v \in V(G) \) such that \( M = \{a, v\} \). Thus \( N_H(a) = N_G(v) \) or \( N_G(v) \cup \{v\} \).

To conclude, observe that

\[ |\{\emptyset, V(G)\} \cup \{N_G(v) : v \in V(G)\} \cup \{N_G(v) \cup \{v\} : v \in V(G)\}| = 2 + 2|V(G)| \]

because \( G \) is prime. \( \Diamond \)
Corollary 2. Let $G$ be a prime graph $G$. For any $a \neq b \notin V(G)$, there exists a prime extension $H$ of $G$ to $V(G) \cup \{a, b\}$ such that $(a, b)_H = 0$.

Proof. Since $|V(G)| \geq 4$, $2^{|V(G)|} - 2|V(G)| - 2 \geq 2$. Consequently there is an extension $H$ of $G$ to $V(G) \cup \{a, b\}$ such that $(a, b)_H = 0$, $N_H(a) \neq N_H(b)$ and

$$N_H(a), N_H(b) \notin \emptyset, V(G) \cup \{N_G(v) : v \in V(G)\} \cup \{N_G(v) \cup \{v\} : v \in V(G)\}.$$

By the proof of Lemma 4, $H - a$ and $H - b$ are prime. We show that $H$ is prime also. Let $M \in M_{22}(H)$. By Proposition 11, $M \setminus \{a\} \notin M(H - a)$. As $H - a$ is prime and $M \setminus \{a\} \neq \emptyset$, either $|M \setminus \{a\}| = 1$ or $M \setminus \{a\} = V(H) \setminus \{a\}$. In the first, there is $v \in V(G) \setminus \{b\}$ such that $M = \{a, v\}$. If $v = b$, then $N_H(a) = N_H(b)$. If $v \in V(G)$, then $\{a, v\}$ would be a nontrivial module of $H - b$. Consequently $M \setminus \{a\} = V(H) \setminus \{a\}$. Since $H - b$ is prime, $a \leftrightarrow_H V(G)$ and hence $a \in M$. Thus $M = V(H)$.

4 Proof of Theorem 2

Let $G$ be a graph with $|V(G)| \geq 2$. By Theorem 3.2, there exists a prime extension $H$ of $G$ such that

$$2 \leq |V(H) \setminus V(G)| \leq \log_2(|V(G)| + 1)$$

and

$$V(H) \setminus V(G)$$

is a stable set in $H$.

We can consider the smallest integer $q(G)$ such that $q(G) \geq 2$ and $G$ admits a prime $q(G)$-extension $H$ such that $V(H) \setminus V(G)$ is a stable set in $H$.

The results below, from Proposition 4 to Corollary 4, are suggested by the proof of Theorem 3.2.

We introduce a basic construction. Consider a graph $G$ and a modular partition $P$ of $G$ such that $P \subseteq S(G)$ and $P \cap S_2(G) \neq \emptyset$. Let $X \in P \cap S_2(G)$ such that

$$q(G[X]) = \max\{(q(G[Y]) : Y \in P \cap S_2(G))\}.$$

Consider a set $S$ such that $S \cap V(G) = \emptyset$ and $|S| = q(G[X])$. There exists a prime $q(G[X])$-extension $H_X$ of $G[X]$ to $X \cup S$ such that $S$ is a stable set in $H_X$. Since $X$ is not a module of $H_X$, there is $s_X \in S$ such that $s_X \leftrightarrow_G X$. Furthermore, if there is $v \in S$ such that $\{v, X\}_{H_X} = 0$, then $V(H_X) \setminus \{v\}$ would be a nontrivial module of $H_X$. Thus $\{v \in S : v \leftrightarrow_{H_X} X\} = \{v \in S : (v, X)_{H_X} = 1\}$. As $S$ is a stable set in $H_X$, $\{v \in S : (v, X)_{H_X} = 1\}$ is a module of $G$. It follows that

$$\{v \in S : v \leftrightarrow_{H_X} X\} = \{v \in S : (v, X)_{H_X} = 1\}$$

and

$$|\{v \in S : v \leftrightarrow_{H_X} X\}| \leq 1$$

and

$$s_X \in S \setminus \{v \in S : v \leftrightarrow_{H_X} X\}.$$
Now, for each $Y \in (P \cap S_2(G)) \setminus \{X\}$, there is a prime $q(G[Y])$-extension $H_Y$ of $G[Y]$ to $Y \cup S_Y$ such that $\{v \in S : v \leftrightarrow_{H_X} X\} \subseteq S_Y \subseteq S$ and $S_Y$ is a stable set in $H_Y$. Consider the extension $H$ of $G$ to $V(G) \cup S$ satisfying

- for each $Y \in P \cap S_2(G)$, $H[Y \cup S_Y] = H_Y$;
- for each $v \in V(G)$ such that $\{v\} \in P$, $(v, S \setminus \{S_X\})_H = 0$ and $(v, S_X)_H = 1$.

**Proposition 4.** Given a graph $G$, consider a modular partition $P$ of $G$ such that $P \subseteq S(G)$ and $P \cap S_2(G) \neq \emptyset$. If the corresponding extension $H$ is not prime, then all the nontrivial modules of $G$ are included in $\{v \in V(G) : \{v\} \in P\}$.

**Proof.** Let $M$ be a nontrivial module of $H$. By Proposition 1.1, $M \cap (X \cup S) \in M(H[X \cup S])$. Since $H[X \cup S]$ is prime, we have $M \supseteq (X \cup S) \cap \{v \in V(G) : \{v\} \subseteq P\}$ or $M \cap (X \cup S) = \emptyset$.

For a first contradiction, suppose that $M \supseteq (X \cup S)$. Let $v \in V$ such that $\{v\} \in P$. As $v \leftrightarrow_{H_Y} S$, $v \in M$. Thus $\{v \in V(G) : \{v\} \in P\} \subseteq M$. Let $Y \in P \cap S_2(G)$.

By Proposition 1.1, $M \cap (Y \cup S_Y) \in M(H[Y \cup S_Y])$. Since $H[Y \cup S_Y]$ is prime and since $S_Y \subseteq M \cap (Y \cup S_Y)$, $Y \subseteq M$. Therefore $\bigcup(P \cap S_2(G)) \subseteq M$ and we would have $M = V(H)$.

For a second contradiction, suppose that $|M \cap (X \cup S)| = 1$. Consider $v \in S \cup X$ such that $M \cap (X \cup S) = \{v\}$. Suppose that $v \in X$. We have $M \subseteq V(G)$ and $M \in M(G)$ by Proposition 1.1. As $X \subseteq S(G)$ and $v \in X \cap M$, $X \subseteq M$ or $X \subseteq M$. In both cases, we would have $|M \cap (X \cup S)| \geq 2$. Suppose that $v \in S$. There is $Y \in P \setminus \{X\}$ such that $Y \cap M \neq \emptyset$. Let $y \in Y \cap M$. Since $y \leftrightarrow_{G} X$, $y \leftrightarrow_{H_Y} X$ and hence $v \neq S_X$. If $Y \in P \cap S_2(G)$, then $v \in S_Y$ and $M \cap (Y \cup S_Y)$ would be a nontrivial module of $H[Y \cup S_Y]$. If $Y = \{y\}$, then $(y, S_X)_H = 1$. Thus $(v, S_X)_H = 1$ and $S$ would not be a stable set in $H$.

It follows that $M \cap (X \cup S) = \emptyset$. By Proposition 1.1, $M \in M(G)$. Let $Y \in P \cap S_2(G) \setminus \{X\}$. Suppose for a contradiction that $Y \cap M \neq \emptyset$. As $Y \subseteq S(G)$, $Y \subseteq M$ or $Y \subseteq M$. In both cases, $M \cap (Y \cup S_Y)$ would be a nontrivial module of $H[Y \cup S_Y]$. It follows that $Y \cap M = \emptyset$. Therefore $M \subseteq \{v \in V(G) : \{v\} \in P\}$. ◇

**Corollary 3.** Given a graph $G$ such that $G / \Pi(G)$ is prime, we have

$$q(G) \leq \begin{cases} 2 & \text{if } \Pi_2(G) = \emptyset \\ \max\{2, [\log_2(\Pi_1(G))] + 1\} & \text{if } \Pi_2(G) \neq \emptyset. \end{cases}$$

**Proof.** If $G$ is prime, then $q(G) \leq 2$ by Corollary 2 and hence $q(G) = 2$. Assume that $G$ is not prime, that is, $\Pi_2(G) \neq \emptyset$. Let $H$ be the extension of $G$ associated with $\Pi(G)$. Suppose that $H$ admits a nontrivial module $M$. By Proposition 4, $\{\{u\} : u \in M\} \subseteq \Pi_1(G)$. Thus $M \in M(G)$ by Proposition 1.1. By Proposition 2, $\{\{u\} : u \in M\}$ would be a nontrivial module of $G / \Pi(G)$. ◇

**Proposition 5.** Given a graph $G$ such that $G / \Pi(G)$ is complete or empty, we have

$$q(G) \leq \max\{2, [\log_2(\Pi_1(G))] + 1\} \quad \text{or} \quad q(G) \leq \max\{q(G[X]) : X \in \Pi_2(G)\}.$$
Proof. Assume that $G/\Pi(G)$ is empty. If $\Pi(G) = \Pi_1(G)$, then $G$ is empty by Proposition 1, and it suffices to apply Lemma 2. Assume that $\Pi_{2\geq}(G) \neq \emptyset$ and set

$$W_2 = \bigcup \Pi_{2\geq}(G).$$

Let $H$ be the extension of $G$ associated with $\Pi(G)$. Recall that $V(H) = V(G) \cup S, V(G) \cap S = \emptyset$ and $|S| = q(G[X])$ where $X \in \Pi_{2\geq}(G)$ such that $q(G[X]) = \max(\{q(G[Y]) : Y \in \Pi_{2\geq}(G)\})$. Moreover $H[X \cup S]$ is prime.

If $|\Pi_1(G)| \leq 1$, then $H$ is prime by Proposition 4 so that $q(G) \leq \max(\{q(G[Y]) : Y \in \Pi_{2\geq}(G)\})$. Assume that $|\Pi_1(G)| \geq 2$ and set

$$W_1 = V(G) \setminus W_2.$$

By Lemma 2, there exists a prime extension $H_1$ of $G[S_1]$ to $W_1 \cup S_1$ such that $|S_1| = \lceil \log_2(|W_1| + 1) \rceil$ and $S_1$ is stable in $H_1$. As $G/\Pi(G)$ is empty, $\Pi_{2\geq}(G) \in \mathcal{M}(G/\Pi(G))$. By Proposition 3, $W_2 \in \mathcal{M}(G)$. Thus $\Pi_{2\geq}(G) \subseteq S(G[W_2])$ by Proposition 3. It follows from Proposition 4 that $H[W_2 \cup S]$ is prime. We construct suitable extensions of $G$ according to whether $|S_1| \leq |S|$ or not.

To begin, assume that $|S_1| \leq |S|$. We can assume that

$$\{v \in S : v \not\in H[X \cup S] \cap X \subseteq S_1 \subseteq S$$

and we consider an extension $H'$ of $H_1$ and $H[W_2 \cup S]$ to $V(G) \cup S$. We show that $H'$ is prime. Let $M \in \mathcal{M}_{2\geq}(H')$. By Proposition 1, $M \cap (W_2 \cup S) \in \mathcal{M}(H[W_2 \cup S])$. Since $H[W_2 \cup S]$ is prime, $M \cap (W_2 \cup S) = \emptyset$ or $M \supseteq (W_2 \cup S)$.

• Suppose for a contradiction that $M \cap (W_2 \cup S) \neq \emptyset$. By Proposition 1, $M$ would be a nontrivial module of $H_1$.

• Suppose for a contradiction that $|M \cap (W_2 \cup S)| = 1$ and consider $w \in W_2 \cup S$ such that $M \cap (W_2 \cup S) = \{w\}$. First, suppose that $w \in W_2$ and consider $Y \in \Pi_{2\geq}(G)$ such that $w \in Y$. By Proposition 1, $M \in \mathcal{M}(G)$. As $Y \in S(G)$ and $w \in X \cap M, X \leq M$ or $M \leq X$. In both cases, we would have $|M \cap (W_2 \cup S)| \geq 2$. Second, suppose that $w \in S$ and consider $v \in W_1 \cap M$. Since $v \not\in H[X \cap S]$ and hence $w \in S_1$. It follows from Proposition 1 that $M$ would be a nontrivial module of $H_1$.

Consequently $M \supseteq (W_2 \cup S)$. By Proposition 1, $M \cap (W_1 \cup S_1) \in \mathcal{M}(H_1)$. As $H_1$ is prime and $M \cap (W_1 \cup S_1) \supseteq S_1, M \cap (W_1 \cup S_1) = (W_1 \cup S_1)$ so that $M = V(H')$.

Now, assume that $|S_1| > |S|$. We can assume that $S \not\subseteq S_1$ and we consider the unique extension $H''$ of $H_1$ and $H[W_2 \cup S]$ to $V(G) \cup S_1$ such that

$$(W_2, S_1 \setminus S)_{H''} = 0. \quad (1)$$

We show that $H''$ is prime. Let $M \in \mathcal{M}_{2\geq}(H'')$. We obtain $M \cap (W_1 \cup S_1) = \emptyset, |M \cap (W_1 \cup S_1)| = 1$ or $M \supseteq (W_1 \cup S_1)$. If $M \cap (W_1 \cup S_1) = \emptyset$, then $M$ would be a nontrivial module of $H[W_2 \cup S]$.  

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Suppose for a contradiction that $|M\cap(W_1\cup S)_1| = 1$ and consider $w \in W_1\cup S_1$ such that $M\cap(W_1\cup S_1) = \{w\}$. There is $v \in W_2\cap M$. Let $Y \in \Pi_{22}(G)$ such that $v \in Y$.

- Suppose that $w \in W_1$. By Proposition 1.1, $M \in \mathcal{M}(G)$. Since $Y \in \mathcal{S}(G)$ and since $Y\cap M \neq \varnothing$ and $w \in M \setminus Y$, $Y \subseteq M$. It follows from Proposition 1.1 that $M \cap (W_2\cup S)$ would be a nontrivial module of $H[W_2\cup S]$.

- Suppose that $w \in S_1$. By Proposition 1.1, $M \cap (W_2\cup S) \in \mathcal{M}(H[W_2\cup S])$. As $H[W_2\cup S]$ is prime and as $v \in M \cap W_2$ and $M \cap S \subseteq \{w\}$, $M \cap (W_2\cup S) = \{v\}$ and hence $w \in S_1 \setminus S$. For every $u \in W_2 \setminus \{v\}$, we have $(u,v)_G = (u,w)_{H''} = 0$ by (1). Since $(v,W_1)_G = 0$, we would have $N_G(v) = \varnothing$ and hence $\{v\} \in \Pi_1(G)$.

It follows that $M \supseteq (W_1\cup S_1)$. By Proposition 1.1, $M \cap (W_2\cup S) \in \mathcal{M}(H[W_2\cup S])$. As $H[W_2\cup S]$ is prime and $M \cap (W_2\cup S) \supseteq S$, $M \cap (W_2\cup S) = (W_2\cup S)$ so that $M = V(H'')$.

Finally, observe that when $G/\Pi(G)$ is complete, we can proceed as previously by replacing (1) by $(W_2, S_1 \setminus S)_H'' = 1$.

The next result follows from Corollary 3 and Proposition 5 by climbing the modular decomposition tree from bottom to top.

**Corollary 4.** Given a graph $G$, if there is $X \in \mathcal{S}_{22}(G)$ such that $\lambda_G(X) \in \{\Box,\underline{\Box}\}$ and $|\Pi_1(G[X])| \geq 2$, then

$$q(G) \leq \max(\{\lfloor \log_2(|\Pi_1(G[Y])| + 1)\} : Y \in \mathcal{S}_{22}(G), \lambda_G(Y) \in \{\Box,\underline{\Box}\}).$$

Given Corollary 4, Theorem 2 follows from the next transcription in terms of the modular decomposition tree. Let $G$ be a graph. Denote by $\mathcal{M}(G)$ the family of the maximal elements of $\mathcal{M}_{22}(G)$ under inclusion which are cliques or stable sets in $G$.

**Proposition 6.** Given a graph $G$ such that $\max(\alpha_M(G),\omega_M(G)) \geq 2$,

$$M \in \mathcal{M}(G) \iff \begin{cases} M \in \mathcal{M}_{22}(G) \\ \lambda_G(M) \in \{\Box,\underline{\Box}\} \\ M = \{v \in M : \{v\} \in \Pi(G[M])\}. \end{cases}$$

**Proof.** To begin, consider $M \in \mathcal{M}(G)$ and assume that $M$ is a stable set in $G$. By Proposition 1.1, $M \in \mathcal{M}(G[M])$. Set

$$Q = \{X \in \Pi_1(G[M]) : X \cap M \neq \varnothing\}.$$ 

By definition of $M$, $|Q| \geq 2$ and hence $M = \bigcup Q$ because $Q \subseteq \mathcal{S}(G[M])$. Furthermore, $Q \subseteq \mathcal{S}(G[M])$ by Proposition 3. As all the strong modules of an empty graph are trivial, we obtain $|X| = 1$ for each $X \in Q$, that is,

$$M \subseteq \{v \in M : \{v\} \in \Pi(G[M])\}. $$

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By Proposition \(\text{22}\), \(Q \in \mathcal{M}(G[\overline{M}] / \Pi(G[\overline{M}]))\). For a contradiction, suppose that \(\lambda_G(\overline{M}) = \sqcup\). Since \(Q \in \mathcal{M}_{22}(G[\overline{M}] / \Pi(G[\overline{M}]))\), \(Q = \Pi(G[\overline{M}])\) and hence \(M = \overline{M}\). As \(|X| = 1\) for each \(X \in Q\), \(G[\overline{M}] / \Pi(G[\overline{M}])\) and \(G[\overline{M}]\) are isomorphic by Proposition \(\text{21}\). It would follow that \(G[M]\) is prime. Consequently \(\lambda_G(\overline{M}) \in \{\square, \blacksquare\}\). Given \(v \neq w \in M\), we have \(\{\{v\}, \{w\}\}_{G[\overline{M}] / \Pi(G[\overline{M}])} = (v, w) = 0\). Thus

\[
\lambda_G(\overline{M}) = \square.
\]

Since \(\lambda_G(\overline{M}) = \square\), \(\Pi_1(G[\overline{M}]) \in \mathcal{M}(G[\overline{M}] / \Pi(G[\overline{M}]))\). By Proposition \(\text{23}\), \(\bigcup \Pi_1(G[\overline{M}]) \in \mathcal{M}(G[\overline{M}] / \Pi(G[\overline{M}]))\) and hence \(\bigcup \Pi_1(G[\overline{M}]) \in \mathcal{M}(G)\) by Proposition \(\text{12}\). Given \(v \neq w \in \bigcup \Pi_1(G[\overline{M}])\), we have \(\{v, w\}_G = \{\{v\}, \{w\}\}_{G[\overline{M}] / \Pi(G[\overline{M}])} = 0\). Therefore \(\bigcup \Pi_1(G[\overline{M}])\) is a stable set of \(G\). As \(M \subseteq \bigcup \Pi_1(G[\overline{M}])\), \(M = \bigcup \Pi_1(G[\overline{M}])\) by maximality of \(M\). It follows that

\[
M = \{v \in \overline{M} : \{v\} \in \Pi(G[\overline{M}])\}.
\]

Conversely, consider \(M \in \mathcal{M}_{22}(G)\) such that \(\lambda_G(\overline{M}) = \square\) and \(M = \{v \in \overline{M} : \{v\} \in \Pi(G[\overline{M}])\}\). As \(\lambda_G(\overline{M}) = \square\), \(\Pi_1(G[\overline{M}]) \in \mathcal{M}(G[\overline{M}] / \Pi(G[\overline{M}]))\). By Proposition \(\text{23}\), \(M = \bigcup \Pi_1(G[\overline{M}]) \in \mathcal{M}(G[\overline{M}] / \Pi(G[\overline{M}]))\) and hence \(M \in \mathcal{M}(G)\) by Proposition \(\text{12}\). Since \((v, w)_G = \{\{v\}, \{w\}\}_{G[\overline{M}] / \Pi(G[\overline{M}])} = 0\) for \(v \neq w \in M\), \(M\) is a stable set in \(G\). There is \(N \in \mathcal{M}(G)\) such that \(N \supseteq M\). As \(M\) is a stable set in \(G\), \(N\) is as well. By what precedes, \(N = \{v \in \overline{N} : \{v\} \in \Pi(G[\overline{N}])\}\). We have \(\overline{M} \subseteq \overline{N}\) because \(M \subseteq N\). Furthermore \(\overline{M} \in \mathcal{S}(G[\overline{N}])\) by Proposition \(\text{3}\). Given \(v \in M\), we obtain \(\{v\} \notin \overline{M} \subseteq \overline{N}\). Since \(\{v\} \in \Pi(G[\overline{N}]), \overline{M} = \overline{N}\). Therefore \(M = N\) because \(M = \{v \in \overline{M} : \{v\} \in \Pi(G[\overline{M}])\}\) and \(N = \{v \in \overline{N} : \{v\} \in \Pi(G[\overline{N}])\}\).

Let \(G\) be a graph such that \(\max(\alpha_M(G), \omega_M(G)) \geq 2\). Consider \(M \in \mathcal{M}(G)\). By Proposition \(\text{8}\), \(\lambda_G(\overline{M}) \in \{\square, \blacksquare\}\) and \(|\Pi_1(G[\overline{M}])| = |M| \geq 2\). By Corollary \(\text{3}\)

\[
p(G) \leq q(G) \leq \max(\{\left\lfloor \log_2(\Pi_1(G[\overline{Y}]) + 1)\right\rfloor : \overline{Y} \in \mathcal{S}_{22}(G), \lambda_G(\overline{Y}) \in \{\square, \blacksquare\})
\]

We have also

\[
\max(\{\left\lfloor \log_2(\Pi_1(G[\overline{Y}]) + 1)\right\rfloor : \overline{Y} \in \mathcal{S}_{22}(G), \lambda_G(\overline{Y}) \in \{\square, \blacksquare\})
\]

\[
\leq \max(\{\left\lfloor \log_2(|M| + 1)\right\rfloor : M \in \mathcal{M}(G))\}
\]

\[
= \left\lfloor \log_2(\max(\alpha_M(G), \omega_M(G)) + 1)\right\rfloor.
\]

Consequently

\[
p(G) \leq \left\lfloor \log_2(\max(\alpha_M(G), \omega_M(G)) + 1)\right\rfloor. \quad \text{(Theorem \text{2})}
\]

To obtain Corollary \(\text{11}\) we prove Lemma \(\text{5}\).

**Proof of Lemma** \(\text{4}\). Let \(G\) be a graph such that \(\max(\alpha_M(G), \omega_M(G)) \geq 2\). There exists \(S \in \mathcal{M}(G)\) such that \(|S| = \max(\alpha_M(G), \omega_M(G))\) and \(S\) is a clique or a
Lemma 5. Given a graph $G$, if $\iota(G) \neq \emptyset$ or $\iota(\overline{G}) \neq \emptyset$, then

$$p(G) \geq \lceil \log_2(\max(\alpha_M(G), \omega_M(G))) \rceil,$$

Proof. By interchanging $G$ and $\overline{G}$, assume that $\iota(G) \geq \iota(\overline{G})$. Given $p < \lceil \log_2(\iota(G) + 1) \rceil$, consider any $p$-extension $H$ of $G$. We have $2^{V(H) \setminus V(G)} < |S|$ so that the function $S \to 2^{V(H) \setminus V(G)}$, defined by $s \mapsto N_H(s) \cap (V(H) \setminus V(G))$, is not injective. There are $s \neq t \in S$ such that $v \leftrightarrow_H \{s, t\}$ for every $v \in V(H) \setminus V(G)$. As $S$ is a module of $G$, we have $v \leftrightarrow_H \{s, t\}$ for every $v \in V(G) \setminus S$. Since $S$ is a stable set in $G$, $\{s, t\}$ is a nontrivial module of $H$. ♦

When a graph or its complement admits isolated vertices, we obtain the following.

Lemma 5. Given a graph $G$, if $\iota(G) \neq \emptyset$ or $\iota(\overline{G}) \neq \emptyset$, then

$$p(G) \geq \lceil \log_2(\max(\iota(G), \iota(\overline{G})) + 1) \rceil.$$
Proposition 7. For every graph $G$ such that $\max(\alpha_M(G), \omega_M(G)) = 2$,

$$p(G) = 2 \text{ if and only if } \iota(G) = 2 \text{ or } \iota(\overline{G}) = 2.$$ 

Proof. It follows from Lemma 1 and Theorem 2 that $p(G) = 1$ or 2. To begin, assume that $\iota(G) = 2$ or $\iota(\overline{G}) = 2$. By Lemma 5, $p(G) \geq 2$ and hence $p(G) = 2$. Conversely, assume that $p(G) = 2$. Let $a \notin V(G)$. As $\max(\alpha_M(G), \omega_M(G)) = 2$, $|N| = 2$ for each $N \in M(G)$. Let $N_0 \in M(G)$. For $N \in \mathbb{P}(G)$, we have $G[N]$ is prime. By Lemma 3, $G[N]$ admits a prime extension $H_N$ defined on $N \cup \{a\}$. We consider any 1-extension $H$ of $G$ to $V(G) \cup \{a\}$ satisfying the following.

1. For each $N \in M(G)$, $a \leftrightarrow_H N$.
2. For each $N \in \mathbb{P}(G)$, $H[N \cup \{a\}] = H_N$.
3. Let $v \in I(G)$. There is $i \in \{0, 1\}$ such that $(v, N_0)_G = i$. We require that $(v, a)_H \neq i$.

To begin, we prove that $S_{22}(G) \cap M(H) = \emptyset$. Given $M \in S_{22}(G)$, we have to verify that $a \leftrightarrow_H M$. Let $N$ be a minimal element under inclusion of $\{N' \in S_{22}(G) : N' \subseteq M\}$. By Proposition 1, $\Pi(G[N']) \subseteq S(G)$. By minimality of $N$, $\Pi(G[N]) = \Pi_1(G[N])$ so that $G[N]$ and $G[N]/\Pi(G[N])$ are isomorphic by Proposition 1. We distinguish the following two cases.

- Assume that $\lambda_G(N) = \emptyset$. We obtain that $G[N]$ is prime, that is, $N \in \mathbb{P}(G)$.
  As $H[N \cup \{a\}]$ is prime, $a \leftrightarrow_H N$.

- Assume that $\lambda_G(N) \in \{\square, \blacksquare\}$. By Proposition 5, $N \in M(G)$. Thus $|N| = 2$ and $a \leftrightarrow_H N$ by definition of $H$.

In both cases, $a \leftrightarrow_H N$ and hence $a \leftrightarrow_H M$.

Now we prove that $M_{22}(G) \cap M(H) = \emptyset$. Let $M \in M_{22}(G)$. Since $S_{22}(G) \cap M(H) = \emptyset$, assume that $M \notin S_{22}(G)$. Set $Q = \{X \in \Pi(G[\overline{M}]) : X \cap M \neq \emptyset\}$.

By Proposition 1, $M \in M(G[\overline{M}])$. By definition of $\overline{M}$, $|Q| \geq 2$. Thus $M = \bigcup Q$ because $\Pi(G[\overline{M}]) \subseteq S(G[\overline{M}])$. Furthermore $Q \notin \Pi(G[\overline{M}])$ because $M \notin S_{22}(G)$. By Proposition 2, $Q \in M(G[\overline{M}]/\Pi(G[\overline{M}]))$. As $2 \leq |Q| < |\Pi(G[\overline{M}])|$, $\lambda_G(M) \in \{\square, \blacksquare\}$. If there is $X \in Q \cap S_{22}(G)$, then $a \leftrightarrow_H X$ by what precedes and hence $a \leftrightarrow_H M$. Assume that $Q \notin \Pi_1(G[\overline{M}])$. We obtain that $M$ is a clique or a stable set in $G$. Since $\max(\alpha_M(G), \omega_M(G)) = 2$, $M \in M(G)$ and $a \leftrightarrow_H M$ by definition of $H$.

As $p(G) = 2$, $H$ admits a nontrivial module $M_H$. We have $a \in M_H$ because $M_{22}(G) \cap M(H) = \emptyset$.

First, we show that $N \subseteq M_H$ for each $N \in \mathbb{P}(G)$. By Proposition 1, $M_H \cap (N \cup \{a\}) \in M(H[N \cup \{a\}])$. Since $H[N \cup \{a\}]$ is prime and $a \in M_H \cap (N \cup \{a\})$, we obtain either $(M_H \setminus \{a\}) \cap N = \emptyset$ or $N \subseteq M_H \setminus \{a\}$. Suppose for a contradiction that $(M_H \setminus \{a\}) \cap N = \emptyset$. By Proposition 1, $M_H \setminus \{a\} \in M(G)$.

There is $i \in \{0, 1\}$ such that $(M_H \setminus \{a\}) \cap N = i$ by Proposition 3. Therefore $(a, N)_H = i$ which contradicts the fact that $H[N \cup \{a\}]$ is prime. It follows that $N \subseteq M_H$.

$$\bigcup \mathbb{P}(G) \subseteq M_H.$$ \hfill (2)
Thus we show that $N \cap M_H \neq \emptyset$ for each $N \in \mathcal{M}(G)$. Otherwise consider $N \in \mathcal{M}(G)$ such that $N \cap M_H = \emptyset$. There is $i \in \{0,1\}$ such that $(M_H \setminus \{a\},N)_G = i$. Thus $(a,N)_H = i$ which contradicts $a \leftrightarrow_H N$. Therefore

$$N \cap M_H \neq \emptyset \quad \text{for each} \quad N \in \mathcal{M}(G).$$

(3)

Third, let $v \in I(G)$. By (5), $N_0 \cap M_H \neq \emptyset$. Since $(v,N_0 \cap M_H)_G \neq (v,a)_H$, $v \in M_H$. Hence

$I(G) \subseteq M_H$.

(4)

By (2) and (4),

$$V(G) \setminus M_H \subseteq \mathcal{M}(G).$$

(5)

To conclude, consider $v \in V(H) \setminus M_H$. By (5), there is $N_v \in \mathcal{M}(G)$ such that $v \in N_v$. By interchanging $G$ and $\overline{G}$, assume that $N_v$ is a stable set in $G$. Since $v \leftrightarrow_H M_H$ and $(v,N_v \cap M_H)_G = 0$, we obtain $(v,M_H)_H = 0$. Let $N \in \mathcal{M}(G) \setminus \{N_v\}$. By Corollary 5, $N \cap N_v = \emptyset$. As $N \cap M_H \neq \emptyset$ by (3), we have $(v,N \cap M_H)_G = 0$ and hence $(v,N)_G = 0$. It follows that $N_G(v) = \emptyset$. Therefore $(N_v,V(G) \setminus N_v)_G = 0$ because $N_v \in \mathcal{M}(G)$. Since $N_v$ is a stable set in $G$, we obtain $N_v \subseteq \{u \in V(G) : N_G(u) = \emptyset\}$. Clearly $\{u \in V(G) : N_G(u) = \emptyset\}$ is a stable set in $G$. Thus $i(G) \leq \max(\alpha_M(G),\omega_M(G)) = 2$. Consequently $N_v = \{u \in V(G) : N_G(u) = \emptyset\}$.

Proof of Theorem 3. Consider a graph $G$ such that $\max(\alpha_M(G),\omega_M(G)) = 2^k$ where $k \geq 1$. It follows from Lemma 1 and Theorem 2 that $p(G) = k$ or $k + 1$. To begin, assume that $i(G) = 2^k$ or $i(G) = 2^k$. By Lemma 2, $p(G) \geq k + 1$ and hence $p(G) = k + 1$.

Conversely, assume that $p(G) = k + 1$. If $k = 1$, then it suffices to apply Proposition 7. Assume that $k \geq 2$. For convenience set

$$\mathcal{M}_{\text{max}}(G) = \{N \in \mathcal{M}(G) : |N| = \max(\alpha_M(G),\omega_M(G))\}.$$

With each $N \in \mathcal{M}_{\text{max}}(G)$ associate $w_N \in N$. Set $W = \{w_N : N \in \mathcal{M}_{\text{max}}(G)\}$.

We prove that $\max(\alpha_M(G-W),\omega_M(G-W)) = 2^k - 1$. Let $N \in \mathcal{M}_{\text{max}}(G)$. By Corollary 1, the elements of $\mathcal{M}_{\text{max}}(G)$ are pairwise disjoint. Thus $N \setminus W = N \setminus \{w_N\}$ is a clique or a stable set in $G-W$. Furthermore $N \setminus \{w_N\} \in \mathcal{M}(G-W)$. Therefore $2^k - 1 = |N \setminus \{w_N\}| \leq \max(\alpha_M(G-W),\omega_M(G-W))$. Now consider $N' \in \mathcal{M}_{\text{max}}(G-W)$. We show that $N' \in \mathcal{M}(G)$. We have to verify that for each $N \in \mathcal{M}_{\text{max}}(G)$, $w_N \leftrightarrow_G N'$. Let $N \in \mathcal{M}_{\text{max}}(G)$. First, assume that there is $v \in (N \setminus \{w_N\}) \setminus N'$. We have $v \leftrightarrow_G N'$. As $N$ is a clique or a stable set in $G$, $\{v,w_N\} \in \mathcal{M}(G[N])$. By Proposition 1, $\{v,w_N\} \in \mathcal{M}(G)$. Thus $w_N \leftrightarrow_G N'$. Second, assume that $N \subseteq \{w_N\} \subseteq N'$. Clearly $w_N \leftrightarrow_G N'$ when $N \setminus \{w_N\} = N'$. Assume that $N' \setminus (N \setminus \{w_N\}) \neq \emptyset$. By interchanging $G$ and $\overline{G}$, assume that $N'$ is a clique in $G-W$. As $N \setminus \{w_N\} \subseteq N'$ and $|N \setminus \{w_N\}| \geq 2$, we obtain that $N$ is a clique in $G$. Since $(N \setminus \{w_N\},N' \setminus N)_G = 1$ and since $N \in \mathcal{M}(G)$, we have $(w_N,N' \setminus N)_G = 1$. Furthermore $(w_N,N \setminus \{w_N\})_G = 1$ because $N$ is a clique in $G$. Therefore $(w_N,N')_G = 1$. Consequently $N' \in \mathcal{M}(G)$.

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\( \mathcal{M}(G) \). As \( N' \) is a clique in \( G \), there is \( M \in \mathcal{M}(G) \) such that \( M \supseteq N' \). If \( M \not\subseteq \mathcal{M}_{\text{max}}(G) \), then \( |N'| \leq |M| > \max(\alpha_M(G), \omega_M(G)) \). If \( M \in \mathcal{M}_{\text{max}}(G) \), then \( N' \subseteq M \setminus \{ w_M \} \) and hence \( |N'| < |M| = \max(\alpha_M(G), \omega_M(G)) \). In both cases, we have \( |N'| = \max(\alpha_M(G - W), \omega_M(G - W)) < \max(\alpha_M(G), \omega_M(G)) \). It follows that \( \max(\alpha_M(G - W), \omega_M(G - W)) = 2^k - 1 \).

By Lemma \( \Box \) and Theorem \( \Box \), \( p(G - W) = k \) and hence there exists a prime \( k \)-extension \( H' \) of \( G - W \). We extend \( H' \) to \( V(H') \cup W \) as follows. Let \( N \in \mathcal{M}_{\text{max}}(G) \). Consider the function \( f_N : N \setminus \{ w_N \} \rightarrow 2^{V(H') \cup V(G - W)} \) defined by \( v \mapsto N_H(v) \setminus V(G - W) \) for \( v \in N \setminus \{ w_N \} \). Since \( H' \) is prime, \( f_N \) is injective. As \( |N \setminus \{ w_N \}| = 2^k - 1 \) and \( |2^{V(H') \cup V(G - W)}| = 2^k \), there is a unique \( X_N \subseteq V(H') \cup V(G - W) \) such that \( f_N(v) = X_N \) for every \( v \in N \setminus \{ w_N \} \). Let \( H \) be the extension of \( H' \) to \( V(H') \cup W \) such that \( N_H(w_N) \cap (V(H') \cup V(G - W)) = X_N \) for each \( N \in \mathcal{M}_{\text{max}}(G) \). As \( p(G) = k + 1 \), \( H \) is not prime. Consider a nontrivial module \( M_H \) of \( H \).

Observe the following. Given \( N \neq N' \in \mathcal{M}_{\text{max}}(G) \),

\[
N \cap M_H \neq \emptyset \quad \text{and} \quad N' \setminus M_H \neq \emptyset \quad \implies \quad M_H \not\supseteq V(H'). \tag{6}
\]

Indeed, by Proposition \( \Box \), \( M_H \cap V(G) \in \mathcal{M}(G) \). Since \( \bar{N} \), \( \bar{N}' \in S(G) \) and since \((M_H \cap V(G)) \cap \bar{N} \not\subseteq \emptyset \) and \((M_H \cap V(G)) \cap \bar{N}' \not\subseteq \emptyset \), \( M_H \cap V(G) \) is comparable to \( \bar{N} \) and \( \bar{N}' \) under inclusion. Suppose for a contradiction that \( M_H \cap V(G) \not\subseteq \bar{N} \) and \( M_H \cap V(G) \not\subseteq \bar{N}' \). It follows that \( N \cap \bar{N} \neq \emptyset \) and \( N \cap \bar{N}' \neq \emptyset \). As \( \bar{N}', \bar{N} \in S(G) \), \( \bar{N}' \not\subseteq N \) or \( N \not\subseteq \bar{N}' \). In the first instance, it follows from Proposition \( \Box \) that \( \bar{N}' \) would be a nontrivial strong module of \( G[N] \) which contradicts the fact that \( N \) is a clique or a stable set in \( G \). Thus \( N \not\subseteq N' \) and hence \( \bar{N} \not\subseteq \bar{N}' \). Similarly \( N' \not\subseteq \bar{N} \) and \( \bar{N}' \not\subseteq \bar{N} \). Therefore \( \bar{N} = \bar{N}' \) and it would follow from Proposition \( \Box \) that \( N = N' \). Consequently \( \bar{N} \not\subseteq (M_H \cap V(G)) \) or \( \bar{N}' \not\subseteq (M_H \cap V(G)) \). For instance, assume that \( \bar{N} \not\subseteq (M_H \cap V(G)) \). By Proposition \( \Box \), \( M_H \cap V(H') = \mathcal{M}(H') \). Furthermore \((M_H \cap V(H')) \supseteq (N \setminus W) \) and \( N \setminus W = N \setminus \{ w_N \} \) by Corollary \( \Box \). Since \( H' \) is prime, we have \( V(H') \subseteq M_H \). It follows that \( \Box \) holds.

As \( H' \) is prime and \( M_H \cap V(H') \in \mathcal{M}(H') \), we have either \( |M_H \cap V(H')| \leq 1 \) or \( M_H \supseteq V(H') \). For a contradiction, suppose that \( |M_H \cap V(H')| \leq 1 \). There is \( N \in \mathcal{M}_{\text{max}}(G) \) such that \( w_N \in M_H \). It follows from \( \Box \) that

\[
N' \cap M_H = \emptyset \text{ for each } N' \in \mathcal{M}_{\text{max}}(G) \setminus \{ N \}. \tag{7}
\]

Thus \( M_H \cap W = \{ w_N \} \) and there is \( v \in V(H') \) such that \( M_H \cap V(H') = \{ v \} \). Clearly \( M_H = \{ v, w_N \} \) and we distinguish the following two cases to obtain a contradiction.

- Suppose that \( v \in V(G - W) \). By Proposition \( \Box \), \( \{ v, w_N \} \in \mathcal{M}(G) \). Therefore there is \( N' \in \mathcal{M}_{\text{max}}(G) \) such that \( N' \supseteq \{ v, w_N \} \). By \( \Box \), \( N = N' \) and we would obtain \( N_H(w_N) \cap (V(H') \setminus V(G - W)) = f_N(v) \).

- Suppose that \( v \in V(H') \setminus V(G - W) \). There is \( i \in \{ 0, 1 \} \) such that \( (w_N, N \setminus \{ w_N \})_{G} = i \). We obtain \( (v, N \setminus \{ w_N \})_{G} = i \) because \( \{ v, w_N \} \in \mathcal{M}(H) \).

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Since \( f_N \) is injective, the function \( g_N : N \setminus \{ w_N \} \to 2^{(V(H') \setminus V(G-W)) \setminus \{ v \}} \), defined by \( g_N(u) = f_N(u) \setminus \{ v \} \) for \( u \in N \setminus \{ w_N \} \), is injective as well. We would obtain \( 2^k - 1 \leq 2^k - 1 \).

Consequently \( V(H') \subseteq M_H \). As \( M_H \) is a nontrivial module of \( H \), there exists \( N \in \mathcal{M}_{\text{max}}(G) \) such that \( w_N \notin M_H \). By interchanging \( G \) and \( G^\alpha \), assume that \( N \) is a stable set in \( G \). We have \( (w_N, N \setminus \{ w_N \})_G = 0 \) and hence \( (w_N, V(H'))_H = 0 \).

In particular \( (w_N, V(G - W))_G = 0 \). Given \( N' \in \mathcal{M}_{\text{max}}(G) \setminus \{ N \} \), we obtain \( (w_N, N' \setminus \{ w_N \})_G = 0 \). Since \( N' \in \mathcal{M}(G) \), \( (w_N, w_N)_G = 0 \). It follows that \( N_G(w_N) = \emptyset \). As at the end of the proof of Proposition \( \dagger \), we conclude by \( N = \{ u \in V(G) : N_G(u) = \emptyset \} \).

\( \diamond \)

Lastly, we examine the graphs \( G \) such that \( \alpha_M(G) = \omega_M(G) = 1 \). For these, \( \mathcal{M}(G) = \emptyset \). Thus either \( |V(G)| \leq 1 \) or \(|V(G)| \geq 4 \) and \( G \) is not prime.

**Proposition 8.** For every non prime graph \( G \) such that \(|V(G)| \geq 4 \) and \( \alpha_M(G) = \omega_M(G) = 1 \), we have \( p(G) = 1 \).

**Proof.** Consider a minimal element \( N_{\text{min}} \) of \( \mathcal{S}_2(G) \). By Proposition \( \mathbb{3} \), \( \Pi(G[N_{\text{min}}]) \subseteq \mathcal{S}(G) \). By minimality of \( N_{\text{min}} \), \( \Pi(G[N_{\text{min}}]) = \Pi_1(G[N_{\text{min}}]) \). Thus \( G[N_{\text{min}}] \) and \( G[N_{\text{min}}]/\Pi(G[N_{\text{min}}]) \) are isomorphic by Proposition \( \mathbb{2} \).

If \( \lambda_G(N_{\text{min}}) \in \{ \Box, \blacksquare \} \), then \( N_{\text{min}} \) is a clique or a stable set in \( G \) and there would be \( N \in \mathcal{M}(G) \) such that \( N \geq N_{\text{min}} \). Therefore \( \lambda_G(N_{\text{min}}) = \cup \) and \( N_{\text{min}} \in \mathbb{P}(G) \).

Let \( a \notin V(G) \). For each \( N \in \mathbb{P}(G) \), \( G[N] \) is prime. By Lemma \( \mathbb{4} \), \( G[N] \) admits a prime 1-extension \( H_N \) to \( N \cup \{ a \} \). We consider the 1-extension \( H \) of \( G \) to \( V(G) \cup \{ a \} \) satisfying the following.

1. For each \( N \in \mathbb{P}(G) \), \( H[N \cup \{ a \}] = H_N \).
2. Let \( v \in I(G) \). There is \( i \in \{ 0, 1 \} \) such that \( (v, N_{\text{min}})_G = i \). We require that \( (v, a)_H \neq i \).

We proceed as in the proof of Proposition \( \mathbb{7} \) to show that \( \mathcal{M}_{\mathbb{2}}(G) \cap \mathcal{M}(H) = \emptyset \). To begin, we prove that \( \mathcal{S}_{\mathbb{2}}(G) \cap \mathcal{M}(H) = \emptyset \). Given \( M \in \mathcal{S}_{\mathbb{2}}(G) \), we have to verify that \( a \leftrightarrow_H M \). Let \( N \) be a minimal element under inclusion of \( \{ N' \in \mathcal{S}_{\mathbb{2}}(G) : N' \subseteq M \} \). We obtain that \( \Pi(G[N]) = \Pi_1(G[N]) \) so that \( G[N] \) and \( G[N]/\Pi(G[N]) \) are isomorphic by Proposition \( \mathbb{2} \).

If \( \lambda_G(N) \in \{ \Box, \blacksquare \} \), then \( N \) is a clique or a stable set in \( G \) and there would be \( N' \in \mathcal{M}(G) \) such that \( N' \geq N \). Thus \( \lambda_G(N) = \cup \). We obtain that \( G[N] \) is prime, that is, \( N \in \mathbb{P}(G) \).

Since \( H[N \cup \{ a \}] \) is prime, \( a \leftrightarrow_H N \) and hence \( a \leftrightarrow_H M \).

Now we prove that \( \mathcal{M}_{\mathbb{2}}(G) \cap \mathcal{M}(H) = \emptyset \). Let \( M \in \mathcal{M}_{\mathbb{2}}(G) \). Since \( \mathcal{S}_{\mathbb{2}}(G) \cap \mathcal{M}(H) = \emptyset \), assume that \( M \notin \mathcal{S}_{\mathbb{2}}(G) \). Set \( Q = \{ X \in \Pi(G[M]) : X \cap M \neq \emptyset \} \). We obtain that \( M = \cup Q \), \( |Q| \geq 2 \) and \( \lambda_G(M) \in \{ \Box, \blacksquare \} \). If \( |\Pi_1(G[M])| = 2 \), then we would have \( \{ v \in M : \{ v \} \in \Pi(G[M]) \} \in \mathcal{M}(G) \) by Proposition \( \mathbb{6} \). Consequently \( |\Pi_1(G[M])| \leq 1 \) and there is \( X \in Q \cap \Pi_{\mathbb{2}}(G[M]) \). By what precedes \( a \leftrightarrow_H X \) and hence \( a \leftrightarrow_H M \).

Lastly, we establish that \( H \) is prime. Let \( M_H \in \mathcal{M}_{\mathbb{2}}(H) \). As previously shown, \( a \in M \). We show that \( N \in M_H \) for each \( N \in \mathbb{P}(G) \). By Proposition \( \mathbb{1} \),
$M_H \cap (N \cup \{a\}) \in \mathcal{M}(H[H \cup \{a\}]).$ Since $H[H \cup \{a\}]$ is prime and $a \in M_H \cap (N \cup \{a\})$, we obtain either $(M_H \setminus \{a\}) \cap N = \emptyset$ or $N \subseteq M_H \setminus \{a\}$. Suppose for a contradiction that $(M_H \setminus \{a\}) \cap N = \emptyset$. By Proposition 1.1, $M_H \setminus \{a\} \in \mathcal{M}(G)$. There is $i \in \{0,1\}$ such that $(M_H \setminus \{a\}, N)_G = i$ by Proposition 1.3. Therefore $(a, N)_H = i$ which contradicts the fact that $H[H \cup \{a\}]$ is prime. It follows that $N \subseteq M_H$ for each $N \in \mathcal{P}(G)$. In particular $N_{\text{min}} \subseteq M$. Let $v \in I(G)$. As $(v, N_{\text{min}})_G \neq (v, a)_H$, $v \in M_H$. Consequently $M_H = V(H)$. \hfill \diamond

References

[1] A. Boussaïri, P. Ille, Prime bound of a graph, 2011, http://arxiv.org/abs/1110.2935v1.

[2] R. Brignall, Simplicity in relational structures and its application to permutation classes, Ph.D. Thesis, University of St Andrews, 2007.

[3] R. Brignall, N. Ruškuc and V. Vatter, Simple extensions of combinatorial structures, Mathematika (2011) 57, 193–214.

[4] R. McConnell, F. de Montgolfier, Linear-time modular decomposition of directed graphs, Discrete Appl. Math. 145 (2005), 198–209.

[5] A. Ehrenfeucht, T. Harju, G. Rozenberg, The Theory of 2-Structures, A Framework for Decomposition and Transformation of Graphs, World Scientific, Singapore, 1999.

[6] T. Gallai, Transitiv orientierbare Graphen, Acta Math. Acad. Sci. Hungar. 18 (1967), 25–66.

[7] F. Maffray, M. Preissmann, “A translation of Tibor Gallai’s paper: Transitiv orientierbare Graphen,” Perfect Graphs J.L. Ramirez-Alfonsin and B.A. Reed, (Editors), Wiley, New York (2001), pp. 25–66.

[8] D.P. Sumner, Indecomposable graphs, Ph.D. Thesis, University of Massachusetts, 1971.