q-Bernstein polynomials, q-Stirling numbers and q-Bernoulli polynomials

T. Kim

Division of General Education-Mathematics, Kwangwoon University, Seoul 139-701, Korea

Abstract: In this paper, we give new identities involving Phillips q-Bernstein polynomials and we derive some interesting properties of q-Bernstein polynomials associated with q-Stirling numbers and q-Bernoulli polynomials.

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1. Introduction

When one talks of q-extension, q is variously considered as an indeterminate, a complex number \( q \in \mathbb{C} \), or \( p \)-adic number \( q \in \mathbb{C}_p \). If \( q \in \mathbb{C} \), then we always assume that \( |q| < 1 \). If \( q \in \mathbb{C}_p \), we usually assume that \( |1 - q|_p < 1 \). Here, the symbol \( | \cdot |_p \) stands for the \( p \)-adic absolute value on \( \mathbb{C}_p \) with \( |p|_p \leq 1/p \).

For each \( x \), the q-basic numbers are defined by

\[
[x]_q = \frac{1 - q^x}{1 - q}, \quad \text{and} \quad [n]_q! = [n]_q[n - 1]_q \cdots [2]_q[1]_q, n \in \mathbb{N}, \quad (\text{see [1-17]}).
\]

Throughout this paper we assume that \( q \in \mathbb{C} \) with \( |q| < 1 \) and we use the notation of Gaussian binomial coefficient in the form

\[
\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} = \frac{[n]_q[n-1]_q \cdots [n-k+1]_q}{[k]_q!}, n, k \in \mathbb{N}.
\]

Note that

\[
\lim_{q \to 1} \binom{n}{k}_q = \binom{n}{k} = \frac{n(n-1) \cdots (n-k+1)}{k!}, \quad (\text{see [4-12]}).
\]

The Gaussian binomial coefficient satisfies the following recursion formula:

\[
\binom{n+1}{k}_q = \binom{n}{k-1}_q + q^k \binom{n}{k}_q = q^{n-k} \binom{n}{k-1}_q + \binom{n}{k}_q, \quad (\text{see [7, 8]}). \quad (1)
\]

The q-binomial formulae are known as

\[
(1 - b)_q^n = (b : q)_n = \prod_{i=1}^{n} (1 - bq^{i-1}) = \sum_{i=0}^{n} \binom{n}{i}_q q^{\binom{i}{2}} (-1)^i b^i, \quad (2)
\]

and

\[
\frac{1}{(1 - b)_q^n} = \frac{1}{(b : q)_n} = \prod_{i=1}^{n} (1 - bq^{i-1}) = \sum_{i=0}^{\infty} \binom{n + i - 1}{i}_q b^i, \quad (\text{see [7, 8]}).
\]
Now, we define the $q$-exponential function as follows:

$$
\lim_{n \to \infty} \frac{1}{(x : q)_n} = \lim_{n \to \infty} \sum_{k=0}^{\infty} \binom{n + k - 1}{k} x^k = \sum_{k=0}^{\infty} \frac{x^k (1 - q)^k}{[k]_q!} = e_q(x(1 - q)).
$$

A Bernoulli trial involves performing an experiment once and noting whether a particular event $A$ occurs. The outcome of Bernoulli trial is said to be “success” if $A$ occurs and a “failure” otherwise. Let $k$ be the number of successes in $n$ independent Bernoulli trials, the probabilities of $k$ are given by the binomial probability law:

$$
p_n(k) = \binom{n}{k} p^k (1 - p)^{n-k}, \text{ for } k = 0, 1, \ldots, n,
$$

where $p_n(k)$ is the probability of $k$ successes in $n$ trials. For example, a communication system transmit binary information over channel that introduces random bit errors with probability $\xi = 10^{-3}$. The transmitter transmits each information bit three times, and a decoder takes a majority vote of the received bits to decide on what the transmitted bit was. The receiver can correct a single error, but it will make the wrong decision if the channel introduces two or more errors. If we view each transmission as a Bernoulli trial in which a “success” corresponds to the introduction of an error, then the probability of two or more errors in three Bernoulli trial is

$$
p(k \geq 2) = \binom{3}{2}(0.001)^2(0.999) + \binom{3}{3}(0.001)^3 \approx 3(10^{-6}), \text{ see [18]}.\n$$

Let $C[0,1]$ denote the set of continuous function on $[0,1]$. For $f \in C[0,1]$, Bernstein introduced the following well known linear operator in [2]:

$$
B_n(f|x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1 - x)^{n-k} = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) B_{k,n}(x).
$$

Here $B_n(f|x)$ is called the Bernstein operator of order $n$ for $f$. For $k, n \in \mathbb{Z}_+$, the Bernstein polynomials of degree $n$ is defined by

$$
B_{k,n}(x) = \binom{n}{k} x^k (1 - x)^{n-k}.
$$

By the definition of Bernstein polynomials, we can see that Bernstein basis is the probability mass function of binomial distribution. Based on the $q$-integers Phillips introduced the $q$-analogue of well known Bernstein polynomials (see [15, 16]). For $f \in C[0,1]$, Phillips introduced the $q$-extension of $B_n(f|x)$ as follows:

$$
B_{n,q}(f | x) = \sum_{k=0}^{n} B_{k,n}(x, q) f\left(\frac{[k]_q}{[n]_q}\right) = \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right) \binom{n}{k}_q x^k (1 - x)^{n-k}_q.
$$

Here $B_{n,q}(f | x)$ is called the $q$-Bernstein operator of order $n$ for $f$. For $k, n \in \mathbb{Z}_+$, the $q$-Bernstein polynomial of degree $n$ is defined by

$$
B_{k,n}(x, q) = \binom{n}{k}_q x^k (1 - x)^{n-k}_q, x \in [0,1].
$$
For example, $B_{0,1}(x, q) = 1 - x$, $B_{1,1}(x, q) = x$, and $B_{0,2}(x, q) = 1 - [2]_q x + qx^2, \cdots$. Also $B_{k,n}(x, q) = 0$ for $k > n$, because $\binom{n}{k}_q = 0$. The $q$-binomial distribution associated with the $q$-boson oscillator are introduced in [19, 20]. For $n, k \in \mathbb{Z}_+$, its probabilities are given by

$$p(X = k) = \binom{n}{k}_q x^k (1 - x)^{n-k}, \text{ where } x \in [0, 1].$$

This distributions are studied by several authors and has applications in physics as well as in approximation theory due to the $q$-Bernstein polynomials and the $q$-Bernstein operators (see [16, 19, 20]). From the definition of $q$-Bernstein polynomials, we note that the $q$-Bernstein basis is the probability mass function of $q$-binomial distribution. Recently, several authors have studied the analogs of Bernstein polynomials (see [1, 2, 5, 8, 9, 10, 15, 16, 17]). In [5], Gupta-Kim-Choi-Kim gave the generating function of Phillips $q$-Bernstein polynomials as follows:

$$\frac{x^k t^k}{[k]_q!} e_q((1 - x)_q t) = \sum_{n=0}^{\infty} \frac{(1 - x)_q^n t^n}{[n]_q!} = \sum_{n=k}^{\infty} \binom{n}{k}_q x^k (1 - x)^{n-k} \frac{t^n}{[n]_q!}.$$

Because $B_{k,0}(x, q) = B_{k,1}(x, q) = B_{k,2}(x, q) = \cdots = B_{k,k-1}(x, q) = 0$, we obtain the generating function for $B_{k,n}(x, q)$ as follows:

$$F_q^{(k)}(t, x) = \frac{x^k t^k}{[k]_q!} e_q((1 - x)_q t) = \sum_{n=0}^{\infty} B_{k,n}(x, q) \frac{t^n}{[n]_q!}, \text{ see [5]},$$

where $n, k \in \mathbb{Z}_+$ and $x \in [0, 1]$.

Notice that

$$B_{k,n}(x, q) = \begin{cases} \binom{n}{k}_q x^k (1 - x)^{n-k}, & \text{if } n \geq k \\ 0, & \text{if } n < k, \end{cases}$$

for $n, k \in \mathbb{Z}_+$ (see [5, 15, 16]).

In this paper we study the generating function of Phillips $q$-Bernstein polynomial and give some identities on the Phillips $q$-Bernstein polynomials. From the generating function of $q$-Bernstein polynomial, we derive recurrence relation and derivative of the Phillips $q$-Bernstein polynomials. Finally, we investigate some interesting properties of $q$-Bernstein polynomials related to $q$-Stirling numbers and $q$-Bernoulli polynomials.

2. $q$-Bernstein polynomials, $q$-Stirling numbers and $q$-Bernoulli polynomials

Let

$$F_q^{(k)}(t, x) = \sum_{n=0}^{\infty} B_{k,n}(x, q) \frac{t^n}{[n]_q!}.$$

From (5) and (3), we note that
\[ F_q^{(k)}(t, x) = \sum_{n=0}^{\infty} \binom{n}{k} x^n (1-x)^{n-k} t^n \frac{t^n}{[n]_q!}, \]
\[ = \sum_{n=0}^{\infty} \binom{n+k}{k} x^n (1-x)^{n-k} t^n \frac{t^n}{[n+k]_q!}, \]
\[ = \frac{x^k t^k}{[k]_q!} \sum_{n=0}^{\infty} \frac{(1-x)^n t^n}{[n]_q!}, \]
\[ = \frac{x^k t^k}{[k]_q!} e_q((1-x)_q), \]
where \( n, k \in \mathbb{Z}_+ \) and \( x \in [0, 1] \).

Note that
\[ \lim_{q \to 1} F_q^{(k)}(t, x) = \frac{x^k t^k}{k!} e^{(1-x)t} = \sum_{n=0}^{\infty} B_{k,n}(x) \frac{t^n}{n!}, \]
where \( B_{k,n}(x) \) are the Bernstein polynomial of degree \( n \).

The \( q \)-derivative \( D_q f \) of function \( f \) is defined by
\[ (D_q f)(x) = \frac{df(x)}{d_q x} = \frac{f(x) - f(qx)}{(1-q)x}, \quad \text{(see [6]).} \tag{7} \]

From (7), we note that
\[ D_q(fg)(x) = g(x) D_q f(x) + f(qx) D_q g(x), \quad \text{(see [6]).} \tag{8} \]

The \( q \)-Bernstein operator is given by
\[ B_{n,q}(f \mid x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} q^n \frac{t^n}{n!}, \]
\[ = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \frac{t^n}{n!}, \]
\[ = \frac{x^k t^k}{[k]_q!} e_q((1-x)_q), \]
where \( x \in [0, 1] \) and \( n, k \in \mathbb{Z}_+ \).

For \( f \in C[0, 1] \), we have
\[ B_{n,q}(f \mid x) = \sum_{k=0}^{n} f \left( \frac{[k]_q}{[n]_q} \right) B_{k,n}(x, q) \]
\[ = \sum_{k=0}^{n} f \left( \frac{[k]_q}{[n]_q} \right) \binom{n}{k} x^k (1-x)^{n-k} q^n \frac{t^n}{n!}, \]
\[ = \sum_{k=0}^{n} f \left( \frac{[k]_q}{[n]_q} \right) x^k \binom{n}{k} \sum_{j=0}^{n-k} \binom{n-k}{j} (-1)^j q^j (1)_q^j x^j. \]
It is easy to show that
\[
\binom{n}{k}_q \binom{n-k}{j}_q = \binom{n}{k+j}_q \binom{k+j}{k}_q.
\]

Let \( k + j = m \). Then we have
\[
\binom{n}{k}_q \binom{n-k}{j}_q = \binom{n}{m}_q \binom{m}{k}_q.
\]  
(10)

By (9) and (10), we easily get
\[
B_{n,q}(f | x) = \sum_{m=0}^{n} \binom{n}{m}_q x^m \sum_{k=0}^{m} \binom{m}{k}_q q^{m-k} (-1)^{m-k} f \left( \frac{\lfloor k \rfloor}{\lfloor m \rfloor} \right).
\]  
(11)

Therefore, we obtain the following proposition.

**Proposition 1.** For \( f \in C[0,1] \) and \( n \in \mathbb{Z}_+ \), we have
\[
B_{n,q}(f | x) = \sum_{m=0}^{n} \binom{n}{m}_q x^m \sum_{k=0}^{m} \binom{m}{k}_q q^{m-k} (-1)^{m-k} f \left( \frac{\lfloor k \rfloor}{\lfloor m \rfloor} \right).
\]  
(11)

It is well known that the second kind Stirling numbers are defined by
\[
\frac{(e^t - 1)^k}{k!} = \frac{1}{k!} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} e^{lt} = \sum_{n=0}^{\infty} S(n,k) \frac{t^n}{n!},
\]  
(12)

where \( k \in \mathbb{N} \) (see [7, 8, 9, 10, 17]).

Let \( \Delta \) be the shift difference operator with \( \Delta f(x) = f(x + 1) - f(x) \). By iterative process, we see that
\[
\Delta^n f(0) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(k), \text{ for } n \in \mathbb{N}.
\]  
(13)

From (12) and (13), we have
\[
\sum_{n=0}^{\infty} S(n,k) \frac{t^n}{n!} = \frac{1}{k!} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} e^{lt} = \sum_{n=0}^{\infty} \Delta^k \frac{t^n}{k!} \frac{n}{n!}, \text{ (see [7, 8, 9]).}
\]  
(14)

By comparing the coefficients on the both sides of (14), we get
\[
S(n,k) = \Delta^k \frac{n^m}{k!}, \text{ for } n, k \in \mathbb{Z}_+.
\]  
(15)
Now, we consider the \( q \)-extension of (13). Let \((Eh)(x) = h(x + 1)\) be the shift operator. Then the \( q \)-difference operator is defined by

\[
\Delta_n^q := (E - I)_q^n = \prod_{i=1}^{n}(E - Iq^{-i}) \quad \text{(see [7])},
\]

where \( I \) is an identity operator.

For \( f \in C[0, 1] \) and \( n \in \mathbb{N} \), we have

\[
\Delta_n^q f(0) = \sum_{k=0}^{n} \binom{n}{k}_q q^{k^2/2} \binom{k}{j}_q \sum_{j=0}^{\infty} S(n,k : q) \frac{t^n}{n!} = \sum_{n=0}^{\infty} S(n,k : q) \frac{t^n}{n!} \quad \text{(see [7, 8])}. \quad \text{(16)}
\]

By (16), we obtain the following theorem.

**Theorem 2.** For \( f \in C[0, 1] \) and \( n \in \mathbb{Z}_+ \), we have

\[
[|n|_q]^m \mathbb{B}_{n,q}(f \mid x) = \sum_{k=0}^{n} \binom{n}{k}_q x^k \Delta_k^q f \left( \frac{0}{|n|_q} \right).
\]

In the special case, \( f(x) = x^m \) \((m \in \mathbb{Z}_+)\), we obtain the following corollary.

**Corollary 3.** For \( x \in [0, 1] \) and \( m,n \in \mathbb{Z}_+ \), we have

\[
[n]_q^m \mathbb{B}_{n,q}(x^m \mid x) = \sum_{k=0}^{n} \binom{n}{k}_q x^k \Delta_k^q 0^m.
\]

By (17), we easily get

\[
S(n,k : q) = \frac{q^{-k^2}}{|k|_q!} \sum_{j=0}^{k} (-1)^j q^\binom{j}{2} \binom{k}{j}_q \binom{k-j}{n}_q = \frac{q^{-k^2}}{|k|_q!} \sum_{j=0}^{k} (-1)^{k-j} q^\binom{k-j}{2} \binom{k}{j}_q \binom{j}{n}_q = \frac{q^{-k^2}}{|k|_q!} \Delta_k^q 0^m. \quad \text{(18)}
\]

The equation (18) seems to be the \( q \)-extension of the equation (15). That is, \( \lim_{q \to 1} S(n,k : q) = S(n,k) \).

By Corollary 3 and (18), we obtain the following corollary.
Corollary 4. For $x \in [0, 1]$ and $m, n \in \mathbb{Z}_+$, we have

$$[n]_q^m B_{n,q}(x^m | x) = \sum_{k=0}^{n} \binom{n}{k} x^k [k]_q! q^{(k)} S(m, k : q).$$

From (1) and (5), for $0 \leq k \leq n$, we have

$$q^k (1 - q^{n-k-1} x) B_{k,n-1}(x, q) + x B_{k-1,n-1}(x, q)$$

$$= q^k (1 - q^{n-k-1} x) \binom{n-1}{k} x^k (1-x)^{n-1-k} + x \binom{n-1}{k-1} x^k (1-x)^{n-k}$$

$$= \binom{n}{k} x^k (1-x)^{n-k}.$$ \hspace{1cm} (19)

By (2), (7) and (8), we get

$$d B_{k,n}(x, q) \frac{d}{d_q x} = \binom{n}{k} x^k [n-k]_q (1-qx)^{n-k-1} + \binom{n}{k} [k]_q x^k (1-qx)^{n-k}.$$ \hspace{1cm} (20)

From the definition of Gaussian binomial coefficient (= $q$-binomials coefficient) and (2), we note that

$$\binom{n}{k} \frac{d}{d_q x} = [n]_q q^{-k} B_{k,n-1}(qx, q),$$ \hspace{1cm} (21)

and

$$\binom{n}{k} x^k [n-k]_q (1-qx)^{n-k-1} = [n]_q q^{-k} B_{k,n-1}(qx, q).$$

By (20) and (21), we see that

$$d B_{k,n}(x, q) \frac{d}{d_q x} = [n]_q q^{-k} (qB_{k-1,n-1}(qx, q) - B_{k,n-1}(qx, q)).$$ \hspace{1cm} (22)

Thus, we note that the $q$-derivative of the $q$- Bernstein polynomials of degree $n$ are also polynomial of degree $n - 1$. Therefore, by (19) and (22), we obtain the following recurrence formulae:

Theorem 5 (Recurrence formulae for $B_{k,n}(x, q)$). For $k, n \in \mathbb{Z}_+$ and $x \in [0, 1]$, we have

$$q^k (1 - q^{n-k-1} x) B_{k,n-1}(x, q) + x B_{k-1,n-1}(x, q) = B_{k,n}(x, q),$$

and

$$d B_{k,n}(x, q) \frac{d}{d_q x} = [n]_q q^{-k} (qB_{k-1,n-1}(qx, q) - B_{k,n-1}(qx, q)).$$

We also get from (5) and (6) that
\[
\frac{[n-k]_q}{[n]_q} B_{k,n}(x, q) + \frac{[k+1]_q}{[n]_q} B_{k+1,n}(x, q) \\
= (1 - x q^{n-k-1}) \binom{n-1}{k}_q x^k (1-x)^{n-k} + x \binom{n-1}{k}_q x^k (1-x)^{n-k-1} \\
= (1 - x q^{n-k-1}) B_{k,n-1}(x, q) + x B_{k,n-1}(x, q) \\
= B_{k,n-1}(x, q) + x [n-k-1]_q (1-q) B_{k,n-1}(x, q). 
\] (23)

By (23), we obtain the following theorem.

**Theorem 6.** For \( k, n \in \mathbb{Z}_+ \) and \( x \in [0,1] \), we have

\[
\frac{[n-k]_q}{[n]_q} B_{k,n}(x, q) + \frac{[k+1]_q}{[n]_q} B_{k+1,n}(x, q) = B_{k,n-1}(x, q) + x [n-k-1]_q (1-q) B_{k,n-1}(x, q). 
\]

From Theorem 6 we note that \( q \)-Bernstein polynomials can be written as a linear combination of polynomials of higher order.

For \( k, n \in \mathbb{N} \), we easily get from (5) that \( q \)-Bernstein polynomials can be expressed in the form

\[
\frac{[n-k+1]_q}{[k]_q} \left( \frac{x}{1-x q^{n-k}} \right) x^{k-1} (1-x)^{n-k+1} \binom{n}{k-1}_q \\
= \frac{[n]_q!}{[k]_q! [n-k]_q!} x^k (1-x)^{n-k} \\
= \binom{n}{k}_q x^k (1-x)^{n-k} \\
= B_{k,n}(x, q). 
\] (24)

By (24), we obtain the following proposition.

**Proposition 7.** For \( n, k \in \mathbb{N} \) and \( x \in [0,1] \), we have

\[
B_{k,n}(x, q) = \frac{[n-k+1]_q}{[k]_q} \left( \frac{x}{1-x q^{n-k}} \right) B_{k-1,n}(x, q). 
\]

The \( q \)-Bernstein polynomials of degree \( n \) can be written in terms of power basis \( \{1, x, x^2, \cdots, x^n\} \). By using the definition of \( q \)-Bernstein polynomial and \( q \)-binomial theorem, we get

\[
B_{k,n}(x, q) = \binom{n}{k}_q x^k (1-x)^{n-k} = \binom{n}{k}_q x^k \sum_{i=0}^{n-k} \binom{n-k}{i}_q (-1)^i q^{i(i+1)/2} x^i \\
= \sum_{i=0}^{n-k} \binom{n-k}{i}_q \binom{n}{k}_q (-1)^i q^{i(i+1)/2} x^i \\
= \sum_{i=k}^{n} \binom{n-k}{i-k}_q \binom{n}{k}_q (-1)^{i-k} q^{(i-k)(i-k+1)/2} x^i. 
\] (25)
By simple calculation, we easily see that
\[
\binom{n}{k}_q \binom{n-k}{i-k}_q = \binom{n}{i}_q \binom{i}{k}_q.
\] (26)

Therefore, by (25) and (26), we obtain the following theorem.

**Theorem 8.** For \(k, n \in \mathbb{Z}_+\) and \(x \in [0,1]\), we have
\[
B_{k,n}(x, q) = \sum_{i=k}^{n} \binom{n}{i}_q \binom{i}{k}_q (-1)^{i-k} q^{(i-k)} x^i.
\]

We get from the properties of \(q\)-Bernstein polynomials that
\[
\sum_{k=1}^{n} \frac{k}{(1)_q} B_{k,n}(x, q) = \sum_{k=1}^{n} \frac{k}{[n]_q} \binom{n}{k}_q x^k (1-x)^{n-k}
\]
\[
= \sum_{k=1}^{n} \binom{n-1}{k-1}_q x^k (1-x)^{n-k}
\]
\[
= x \sum_{k=0}^{n-1} \binom{n-1}{k}_q x^k (1-x)^{n-k-1} = x,
\]
and that
\[
\sum_{k=2}^{n} \frac{k}{(2)_q} B_{k,n}(x, q) = \sum_{k=2}^{n} \frac{n-2}{(k-2)_q} x^k (1-x)^{n-k}
\]
\[
= x^2 \sum_{k=0}^{n-2} \binom{n-2}{k}_q x^k (1-x)^{n-k-2} = x^2.
\]

Continuing this process, we obtain
\[
\sum_{k=1}^{n} \frac{k}{(1)_q} B_{k,n}(x, q) = x^i.
\]

Therefore, we obtain the following theorem.

**Theorem 9.** For \(k, i \in \mathbb{Z}_+\) and \(x \in [0,1]\), we have
\[
\sum_{k=1}^{n} \frac{k}{(1)_q} B_{k,n}(x, q) = x^i.
\]

Now we define \(q\)-Bernoulli polynomials of order \(k\) as follows:
\[
\left( \frac{z}{e^z-1} \right)^k e_q(zx) = \sum_{n=0}^{\infty} \beta^{(k)}_n(x, q) z^n [n]_q, \quad k \in \mathbb{N}.
\] (27)

From the generating function (27) of \(q\)-Bernoulli polynomials and (3), we derive
Therefore, by (6) and (30), we obtain the following theorem.

From (27) and (28), we easily get

\[ \beta^{(k)}_n(x, q) = \sum_{m=0}^{n} \left( \frac{m}{m} \right) q^{m} x^{n-m} B^{(k)}_m, \]  

where \( B^{(k)}_m \) are the \( m \)-th ordinary Bernoulli numbers of order \( k \).

From (26) and (27), we note that

\[ \beta^{(k)}_n(x, q) = \sum_{m=0}^{n} \left( \frac{m}{m} \right) q^{m} x^{n-m} B^{(k)}_m, \]  

where \( B^{(k)}_m \) are the \( m \)-th Bernoulli numbers of order \( k \).

From (26) and (27), we note that

\[ \frac{(t x)^k}{[k]_q!} e_q((1 - x)_q t) = \frac{x^k(e^t - 1)^k}{[k]_q!} \left( \frac{t}{e^t - 1} \right)^k e_q((1 - x)_q t) \]
\[ = \frac{k!}{[k]_q!} x^k \left( \sum_{m=0}^{\infty} S(m, k) \frac{t^m}{m!} \right) \left( \sum_{n=0}^{\infty} \beta^{(k)}_n((1 - x)_q, q) \frac{t^n}{[n]_q!} \right) \]
\[ = \frac{k!}{[k]_q!} x^k \sum_{m=0}^{\infty} \left( \sum_{l=m}^{\infty} \frac{[m]_q!}{m!} S(m, k) \left( \frac{l}{m} \right) \beta^{(k)}_{l-m}((1 - x)_q, q) \right) \frac{t^l}{[l]_q!}. \]  

Therefore, by (6) and (30), we obtain the following theorem.

**Theorem 10.** For \( k, l \in \mathbb{Z}_+ \) and \( x \in [0, 1] \), we have

\[ B_{k,l}(x, q) = \frac{k!}{[k]_q!} x^k \sum_{m=0}^{l} \frac{[m]_q!}{m!} S(m, k) \beta^{(k)}_{l-m}((1 - x)_q, q) \left( \frac{l}{m} \right)_q, \]

where \( \beta^{(k)}_{l}((1 - x)_q, q) \) are called the \( l \)-th \( q \)-Bernoulli polynomials.

From (15) and Theorem 10, we have the following corollary.

**Corollary 11.** For \( k, l \in \mathbb{Z}_+ \) and \( x \in [0, 1] \), we have

\[ B_{k,l}(x, q) = \frac{x^k}{[k]_q!} \sum_{m=0}^{l} \frac{[m]_q!}{m!} \left( \frac{l}{m} \right)_q \beta^{(k)}_{l-m}((1 - x)_q, q) \Delta^{k}_0. \]  

It is well known that

\[ x^n = \sum_{k=0}^{n} \binom{x}{k} k! S(n, k), \text{ (see [7])}. \]  

(31)
By (31) and Theorem 9, we easily see that

\[ \sum_{k=0}^{i} \binom{k}{i} \frac{x^k}{k!} S(i, k) = \sum_{k=i}^{n} \binom{n}{i} q^k B_{k,n}(x, q). \]

3. A matrix representation for $q$-Bernstein polynomials

Given a polynomial is written as a linear combination of $q$-Bernstein basis functions:

\[ B_q(x) = C_0^q B_{0,n}(x, q) + C_1^q B_{1,n}(x, q) + \cdots + C_n^q B_{n,n}(x, q). \]  

(32)

It is easy to write (32) as a dot product of two vectors:

\[ B_q(x) = \left( B_{0,n}(x, q), B_{1,n}(x, q), \ldots, B_{n,n}(x, q) \right) \cdot \left( \begin{array}{c} C_0^q \\ C_1^q \\ \vdots \\ C_n^q \end{array} \right). \]  

(33)

Now, we can convert (33) to

\[ B_q(x) = \left( 1, x, \ldots, x^n \right) \cdot \left( \begin{array}{cccc} b_{0,0}^q & 0 & \cdots & 0 \\ b_{1,0}^q & b_{1,1}^q & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_{n,0}^q & b_{n,1}^q & \cdots & b_{n,n}^q \end{array} \right) \cdot \left( \begin{array}{c} C_0^q \\ C_1^q \\ \vdots \\ C_n^q \end{array} \right), \]

where $b_{i,j}^q$ are the coefficients of the power basis that are used to determine the respective $q$-Bernstein polynomials.

From (5) and (6), we note that

\begin{align*}
B_{0,2}(x, q) &= (1 - x)_q^2 = \sum_{l=0}^{2} \binom{2}{l} (-1)^l q^{(l)} = 1 - [2]_q x + q x^2 \\
B_{1,2}(x, q) &= \binom{2}{1}_q x(1 - x)_q = [2]_q x(1 - x) = [2]_q x - [2]_q x^2 \\
B_{2,2}(x, q) &= x^2.
\end{align*}

In the quadratic case ($n = 2$), the matrix can be represented by

\[ B_q(x) = \left( 1, x, x^2 \right) \cdot \left( \begin{array}{ccc} 1 & 0 & 0 \\ -[2]_q & [2]_q & 0 \\ q & -[2]_q & 1 \end{array} \right) \cdot \left( \begin{array}{c} C_0^q \\ C_1^q \\ C_2^q \end{array} \right). \]

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