On the representability of totally unimodular matrices on bidirected graphs

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Abstract

Seymour’s famous decomposition theorem for regular matroids states that any totally unimodular (TU) matrix can be constructed through a series of composition operations called $k$-sums starting from network matrices and their transposes and two compact representation matrices $B_1, B_2$ of a certain ten element matroid. Given that $B_1, B_2$ are binet matrices we examine the $k$-sums of network and binet matrices. It is shown that the $k$-sum of a network and a binet matrix is a binet matrix, but binet matrices are not closed under this operation for $k = 2, 3$. A new class of matrices is introduced the so called tour matrices, which generalises network, binet and totally unimodular matrices. For any such matrix there exists a bidirected graph such that the columns represent a collection of closed tours in the graph. It is shown that tour matrices are closed under $k$-sums, as well as under pivoting and other elementary operations on its rows and columns. Given the constructive proofs of the above results regarding the $k$-sum operation and existing recognition algorithms for network and binet matrices, an algorithm is presented which constructs a bidirected graph for any TU matrix.

Keywords: network matrices, binet matrices, matroid decomposition, signed graphic matroids.

1 Introduction

Totally unimodular matrices are a class of $\{0, \pm 1\}$ matrices which is of great importance to combinatorial optimisation since they describe a special class of polynomial time solvable integer programs. Specifically, every integer program which is defined by a totally unimodular constraint matrix can be solved as a linear program by relaxing the integrality constraint since the associated polyhedron is integral. Although there exist various equivalent characterisations for this class of matrices, it was Seymour’s decomposition theory [15] developed for the associated regular matroids, that yielded a polynomial time algorithm for recognising them. Seymour’s decomposition theorem states that all totally unimodular matrices can be constructed recursively by applying $k$-sum operations ($k = 1, 2, 3$) on network matrices, their transposes and two totally unimodular matrices $B_1$ and $B_2$. These sum operations, are essentially matrix operations which preserve certain structural properties. Combined with the fact that the matrices $B_1$ and $B_2$ are easily recognisable, and Tutte’s theory for recognising network matrices, Seymour’s theorem implies...
an algorithm to check whether a given matrix is totally unimodular or not. Moreover, and maybe even more importantly, it also provides a framework for graphical representation of totally unimodular matrices. Bidirected graphs are a generalisation of directed graphs, and can be represented algebraically by the so-called binet matrices in the same way network matrices represent directed graphs. Appa and Kotnyek [1] have shown that $B_1$ and $B_2$ can be represented on bidirected graphs since they have been proved to be binet. Since bidirected graphs generalise directed graphs, all the building blocks of totally unimodular matrices or their transposes are representable on bidirected graphs. In this work we show that every totally unimodular matrix has an associated bidirected graph representation, which provides a partial interpretation of the nice integrality property of the associated polyhedron and may provide the means of devising a combinatorial algorithm for solving the related integer programming problem.

Initially we show constructively that the $k$-sum of two network matrices is a network matrix and that of a network and a binet matrix is a binet matrix. However, for $k = 2, 3$ we show that the $k$-sum of two binet matrices is not necessarily a binet matrix. Based on this we can state that not all totally unimodular matrices are binet. To pursue graphical representability further a new class of $\{0, \pm 1\}$ matrices is introduced, the so-called tour matrices, which represent closed tours on bidirected graphs. We show that network matrices as well as $B_1$ and $B_2$ are tour matrices, and in contrast to binet matrices, it is also shown that tour matrices are closed under $k$-sums. This means that totally unimodular matrices not previously associated with bidirected graphs can now be represented on bidirected graphs.

The paper is organised as follows. Section 2 presents all the preliminary theory regarding network matrices, bidirected graphs and binet matrices, totally unimodular matrices as well as the definition of the $k$-sum operations. In section 3 we examine the operation of $k$-sums of network and binet matrices, where the most general case for $k = 3$ is treated and a graphical construction of the operation is presented. The negative result in this section is that binet matrices are not closed under $k$-sums. Tour matrices are defined in section 4.1 where various properties are proved. In section 4.2.1 we show that tour matrices are closed under $k$-sums, while in section 4.2.2 we gather all the results presented on the paper on an algorithm for constructing a bidirected graph of any TU matrix.

2 Preliminaries

2.1 Graphs and Network Matrices

A directed graph $G(V, E)$ consists of a finite set of nodes $V$ and a family $E$ of ordered pairs of $V$. For an edge $e = (u, v)$, $u$ and $v$ are called the end-nodes of $e$; $u$ is called the tail of $e$ and $v$ the head of $e$. We also say that $e = (u, v)$ leaves $u$ and enters $v$. The node-edge incidence matrix of a directed graph $G(V, E)$ is the $V \times E$ matrix $D_G$ with

$$D_G(v, e) = \begin{cases} 
-1 & \text{if } v \text{ is a tail of } e \\
+1 & \text{if } v \text{ is a head of } e \\
0 & \text{otherwise},
\end{cases}$$

for any $v \in V$ and any non-loop $e \in E$. If $e$ is a loop, we set $D_G(v, e) := 0$ for each vertex $v$. The definition for the network matrices goes as follows:

**Definition 1.** Let $D_G = [R|S]$ be the incidence matrix of a directed graph $G(V, E)$ minus an arbitrary row, where $R$ is a basis of the column space of $D_G$. The matrix $N_G = R^{-1}S$ is called a network matrix.

For material related to graphs and network matrices the reader is referred to [14].
2.2 Bidirected Graphs and Binet Matrices

A bidirected graph $\Sigma(V, E)$ is defined over a finite node set $V$ and an edge set $E \subseteq V \times V$. There are four types of edges: a link has two different end-nodes, a loop has two end-nodes that coincide, a half-edge has one end-node, and a loose edge which has no end-nodes.

Every edge is assigned a sign, so that half-edges are always negative; loose edges are always positive; links and loops can be positive or negative. The edges are oriented, i.e., we label the end-nodes of the edges by $+1$ or $-1$. The labels of a positive edge are different, those of a negative edge are the same. If an end-node of an edge is labeled with $+1$, then it is an in-node of the edge, otherwise an out-node. These names come from the graphical representation of bidirected graphs, where incoming and outgoing arrows on an edge represent positive and negative labels. For example in the bidirected graph shown in Figure 1, edge $r_1$ is a positive link; $r_3$ is a negative; $s_6$ is a negative loop; and $r_8$ is a half-edge. Loose edges and positive loops are not depicted in this illustration. A graph shown in Figure 1, edge $r_1$ is a positive link; $r_3$ is a negative; $s_6$ is a negative loop; and $r_8$ is a half-edge. Loose edges and positive loops are not depicted in this illustration. A walk in a bidirected graph is a sequence $(v_1, e_1, v_2, e_2, \ldots, e_{t-2}, v_{t-1}, e_{t-1}, v_t)$ where $v_i$ and $v_{i+1}$ are end-nodes of edge $e_i$ ($i = 1, \ldots, t - 1$), including the case where $v_i = v_{i+1}$ and $e_i$ is a half-edge. If $v_1 = v_t$, then the walk is closed. A walk which consists of only links and does not cross itself, that is $v_i \neq v_j$ for $i \neq j$, is a path. A closed walk which does not cross itself (except at $v_1 = v_t$) is called a cycle. That is, a cycle can be a loop, a half-edge or a closed path. The sign of a cycle is the product of the signs of its edges, so we have a positive cycle if the number of negative edges in the cycle is even, otherwise the cycle is a negative cycle. Obviously, a negative loop or a half-edge always makes a negative cycle. A bidirected graph is connected, if there is a path between any two nodes.

The node-edge incidence matrix of a bidirected graph $\Sigma(V, E)$ is the $V \times E$ matrix $D_\Sigma$ with

$$D_\Sigma(v, e) = \begin{cases} -1 & \text{if } v \text{ is an out-node of } e, \\ +1 & \text{if } v \text{ is an in-node of } e, \\ -2 & \text{if } e \text{ is a negative loop and } v \text{ is its out-node}, \\ +2 & \text{if } e \text{ is a negative loop and } v \text{ is its in-node}, \\ 0 & \text{otherwise}, \end{cases}$$

for any vertex $v \in V$ and any edge $e \in E$. The following operations are defined on bidirected graphs. Deletion of an edge is simply the removal of the edge; deletion of a node means that the node and all the edge-ends incident to it are removed. Thus incident half-edges or loops become loose edges, incident links become half-edges. Deletion of an edge or a node is equivalent to the deletion of the corresponding column or row from the node-edge incidence matrix. Switching at a node is the operation when all the labels at the incident edge-ends are changed to the opposite. It corresponds to the multiplication by $-1$ of a row in the incidence matrix. Finally, contracting an edge $e$ is the operation in which the end-nodes of $e$ are modified and $e$ is shrunk to zero length. For different types of edges contraction manifests itself differently. Specifically, if $e$ is a negative loop or a half-edge then the node incident to it is deleted together with all the edge-ends incident to it. If $e$ is a positive link, then its two end-nodes are identified and $e$ is deleted. If $e$ is a negative link, then first we switch at one of its end-nodes to make it a positive link, then contract it as defined for positive links. If $e$ is a positive loop then $e$ is simply deleted. Binet matrices are defined similarly as network matrices as follows:

Definition 2. Let $D_\Sigma$ be a full row rank node-edge incidence matrix of a bidirected graph $\Sigma$, $R$ be a basis of it and $D_\Sigma = [R|S]$. The matrix $B = R^{-1}S$ is called a binet matrix.

When in a bidirected graph $\Sigma$ the subgraph $\Sigma(R)$ is indicated for a basis $R$, we call it a binet representation or a binet graph. In Figure 1 the binet graph for basic edges $\{r_1, \ldots, r_8\}$ and non-basic
edges \{s_1, \ldots, s_6\} is shown, with the associated binet matrix. It is noted that in computing the entries of a binet matrix for a given basis, instead of using Definition 2 which involves the inverse of a matrix, there also exists a combinatorial algorithm described in [1, 4].

For a column \(s\) of \(S\), let \(r_1, r_2, \ldots, r_t\) be the columns of \(R\) for which the corresponding component of vector \(R^{-1}s\) is non-zero. The vectors \(r_1, r_2, \ldots, r_t\) and \(s\) form a minimal linearly dependent set in \(\mathbb{R}\).

The subgraphs of \(\Sigma\) induced by sets of edges which correspond to minimally dependent sets of columns in \(A_\Sigma\) have to be one of the following three types as shown in [8, 22]:

(i) a positive cycle,

(ii) a graph consisting of two negative cycles which have exactly one common node,

(iii) a graph consisting of two node-disjoint negative cycles connected with a path which has no common node with the cycles except its end-nodes.

Graphs in categories (ii) and (iii) are called handcuffs of type I and II respectively. For example in the bidirected graph illustrated in Figure 1 the subgraph induced by the edges \(\{r_1, r_3, r_4, s_1\}\) is a positive cycle, while the sets of edges \(\{r_1, r_2, r_3, s_3\}\) and \(\{r_1, r_2, r_3, r_4, r_5, r_6, s_2\}\) induce handcuffs of type I and II respectively.

![Figure 1: An example of a binet graph, and its binet matrix.](image)

Some results concerning binet matrices which will be useful are the following. Proofs can be found in [1, 8].

**Theorem 3.** Binet matrices are closed under the following operations:

(a) Switching at a node of a binet graph.

(b) Multiplying a row or column with \(-1\).

(c) Deleting a row or a column.

(d) Pivoting (in \(\mathbb{R}\)) on a nonzero element.

Switching at a node does not change the matrix, the new binet graph represents the same matrix. Multiplying a row or column with \(-1\) is equivalent to reversing the orientation of the corresponding basic or non-basic edge. Deleting a column is simply deleting the corresponding non-basic edge, while deleting a row amounts to contracting the corresponding basic edge. Pivoting on an element in row \(r\) and column \(s\) means that these edges are exchanged in the basis.
2.3 Decomposition of Totally Unimodular Matrices

A matrix $A$ is totally unimodular if each square submatrix of $A$ has determinant 0, +1, or −1. There are numerous other characterisations of the class of TU matrices (see [11, 13]). The following decomposition theorem for TU matrices proved by Seymour [15] plays a central role in this work, and also yields a polynomial-time test for total unimodularity.

**Theorem 4.** Any totally unimodular matrix is up to row and column permutations and scaling by ±1 factors a network matrix, or the transpose of such a matrix, or the matrix $B_1$ or $B_2$ of (1) and (2), or may be constructed recursively from these matrices using matrix 1-, 2- and 3-sums (see Definition 5).

Matrices $B_1$ and $B_2$ are binet matrices, as it is indicated by the corresponding binet graphs shown in (1) and (2). The above theorem is essentially a direct consequence of a decomposition theory for matroids associated with TU matrices, the so-called regular matroids. Specifically Seymour characterised the class of regular matroids by defining certain operations called $k$-sums, such that every regular matroid can be decomposed into a set of elementary building blocks via these operations, if and only if these blocks satisfy certain properties.

\[
B_1 = \begin{pmatrix}
1 & 0 & 0 & 1 & -1 \\
-1 & 1 & 0 & 1 & 1 \\
1 & -1 & 1 & 0 & 1 \\
0 & 1 & -1 & 1 & 1 \\
0 & 0 & 1 & -1 & 1
\end{pmatrix}
\]

\[
B_2 = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 1
\end{pmatrix}
\]

In general, $k$-sum operations ($k = 1, 2, 3$) are defined in the more general theoretical framework of matroids, and here we basically treat the specialised version of this operation as applied to the compact representation matrices of regular matroids. Moreover, it can be shown that applying these operations on totally unimodular matrices preserves their total unimodularity.

**Definition 5.** If $A, B$ are matrices and $a, d$ and $b, c$ are column and row vectors of appropriate size in $\mathbb{R}$ then

1-sum: $A \oplus_1 B := \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$

2-sum: $\begin{pmatrix} A & a \\ c & 1 \end{pmatrix} \oplus_2 \begin{pmatrix} b \\ 1 \end{pmatrix} := \begin{bmatrix} A & ab \\ 0 & B \end{bmatrix}$

3-sum: $\begin{pmatrix} A & a & a \\ c & 0 & 1 \end{pmatrix} \oplus_3 \begin{pmatrix} 1 & 0 & b \\ d & d & B \end{pmatrix} := \begin{bmatrix} A & ab \\ dc & B \end{bmatrix}$ or

$\begin{pmatrix} A & 0 \\ b & 1 \end{pmatrix} \oplus_3 \begin{pmatrix} 1 & 1 & 0 \\ a & d & B \end{pmatrix} := \begin{bmatrix} A & 0 \\ D & B \end{bmatrix}$
where in the $\oplus^3$-sum row vectors $b$ and $c$ and column vectors $a$ and $d$ are submatrices of $D$ and the $2 \times 2$ matrix $\bar{D}$ is the intersection of rows $b$ and $c$ with columns $a$ and $d$. Further the rank of $D = [a|d]D^{-1}[\frac{1}{2}]$ is two. Note that there are two alternative definitions for 3-sum, distinguished by $\oplus_3$ and $\oplus^3$. The indices of the isolated columns and rows in the 2-sum and 3-sum operations, will be called connecting elements.

The definition of the $k$-sums may seem complicated at first glance, but they essentially provide a way to decompose a TU matrix into smaller TU matrices given that the matrix admits such a decomposition. Specifically suppose that we have a TU matrix $N$ which under row and column permutations can take the form

$$N = \begin{bmatrix} A & D_1 \\ D_2 & B \end{bmatrix}$$

(3)

and the following two conditions are satisfied:

(i) number of rows and columns of both $A$ and $B > k$,

(ii) rank($D_1$) + rank($D_2$) = $k$ − 1 where $D_1, D_2$ are viewed over GF(2).

Then the matrix $N$ of (3) can be decomposed under a $k$-sum operation into two matrices of smaller size which are submatrices of $N$, preserving total unimodularity. In the case of 3-sum we note from the definition that there are two alternative operations, reflecting the fact that condition (ii) above can be satisfied in two different ways (i.e. rank($D_1$) = rank($D_2$) = 1 or rank($D_1$) = 0, rank($D_2$) = 2). However it can be shown that when the matrices are TU both definitions of 3-sum are equivalent under pivoting in either GF(2) or $\mathbb{R}$. (The regular matroid decomposition theorem of Seymour, $k$-sums of matrices and their corresponding matroids, and decomposition theory for matroids in general is treated extensively in \[12, 17\].)

### 3 $k$-sum of Network and Binet Matrices

In this section we will examine the operation of $k$-sums of matrices, both network and binet. We will show whether the resulting matrix is a network or binet matrix, or does not belong to either class. Algebraic proofs as well as graphical representations of the associated operations on these matrices are presented.

#### 3.1 $k$-sums of Network Matrices

Here it is proved that network matrices are closed under the $k$-sum operations. Since network matrices are the compact representation matrices of graphic matroids, a direct consequence of these results is the well known fact (see \[12\]) that graphic matroids are closed under $k$-sums. However the analytical methodology in the proofs that will be given here, will be used in the sections that will follow where the binet, and the more general tour matrix case is treated. Moreover since the proof is constructive, it is used in the algorithm for composing the bidirected graph of a TU matrix which will be presented in section \[12\].

##### 3.1.1 Network $\oplus_3$ Network

The most general case of 3-sum will be examined since the other sum operations follow.
Lemma 6. If $N_1, N_2$ are network matrices such that

$$N_1 = e_3 \begin{bmatrix} e_1 & e_2 \\ A & a & a \\ c & 0 & 1 \end{bmatrix}, \quad N_2 = f_3 \begin{bmatrix} f_1 & f_2 \\ 1 & 0 & b \\ d & d & B \end{bmatrix},$$

then $N = N_1 \oplus_3 N_2$ is a network matrix.

Proof: Because of the definition of the 3-sum operation we have that in a possible graphical representation of $N_1$ the fundamental cycle of $e_1$ consists of the edges that correspond to non-zero elements in $a$. The fundamental cycle of $e_2$ has all these edges and $e_3$. This means that $e_1, e_2$ and $e_3$ should form a triangle. Similarly, $f_1, f_2$ and $f_3$ form a triangle in any network representation of $N_2$. Let now $[R_1|S_1]$ and $[R_2|S_2]$ be the incidence matrices associated with $N_1$ and $N_2$, respectively, where after permutations and/or multiplications of rows with $\pm 1$ we can write:

$$[R_1|S_1] = \begin{bmatrix} r_1 & -1 & s_1 & 0 & -1 \\ r'_1 & 1 & s'_1 & -1 & 0 \\ r''_1 & 0 & s''_1 & 1 & 1 \\ R'_1 & 0 & S'_1 & 0 & 0 \end{bmatrix}, \quad [R_2|S_2] = \begin{bmatrix} 0 & r_2 & -1 & -1 & s_2 \\ -1 & r'_2 & 0 & 1 & s'_2 \\ 1 & r''_2 & 1 & 0 & s''_2 \\ 0 & R'_2 & 0 & 0 & S'_2 \end{bmatrix} \quad (4)$$

where $0$ is a vector or matrix of zeros of appropriate size, $r_i, r'_i, r''_i, s_i, s'_i$ and $s''_i$ are row vectors and $R'_i, S'_i$ are matrices of appropriate size ($i = 1, 2$). By the definition of network matrices the following two equations hold:

$$R_1 N_1 = S_1, \quad R_2 N_2 = S_2 \quad (5)$$

For $N_1$ using (4) and (5) we have that:

$$\begin{bmatrix} r_1 & -1 \\ r'_1 & 1 \\ r''_1 & 0 \\ R'_1 & 0 \end{bmatrix} \begin{bmatrix} A & a \\ c & 0 & 1 \end{bmatrix} = \begin{bmatrix} s_1 & 0 & -1 \\ s'_1 & -1 & 0 \\ s''_1 & 1 & 1 \\ S'_1 & 0 & 0 \end{bmatrix}$$

where upon decomposing the block matrix multiplications we derive the following equations.

$$\begin{bmatrix} r_1 \\ r'_1 \\ r''_1 \end{bmatrix} A + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} c = \begin{bmatrix} s_1 \\ s'_1 \\ s''_1 \end{bmatrix}, \quad \begin{bmatrix} r_1 \\ r'_1 \\ r''_1 \end{bmatrix} a = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \quad (6)$$

Similarly, for $N_2$ using (4) and (5) we have

$$\begin{bmatrix} 0 & r_2 \\ -1 & r'_2 \\ 1 & r''_2 \\ 0 & R'_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & b \\ d & d & B \end{bmatrix} = \begin{bmatrix} -1 & -1 & s_2 \\ 0 & 1 & s'_2 \\ 1 & 0 & s''_2 \\ 0 & 0 & S'_2 \end{bmatrix}$$
so that
\[
\begin{bmatrix}
0 \\
-1 \\
1
\end{bmatrix} + \begin{bmatrix}
r_2' \\
r_2'' \\
r_2'''
\end{bmatrix} d = \begin{bmatrix}
-1 \\
0 \\
1
\end{bmatrix}, \quad \begin{bmatrix}
r_2' \\
r_2'' \\
r_2'''
\end{bmatrix} d = \begin{bmatrix}
-1 \\
0 \\
1
\end{bmatrix}
\]  
(7)

Also, we have that:
\[
\begin{bmatrix}
r_1' \\
r_1'' \\
r_1'''
\end{bmatrix} B = \begin{bmatrix}
s_1' \\
S_1' \\
S_2'
\end{bmatrix}, \quad R_2' d = 0, \quad R_2' B = S_2'
\]

Using block matrix multiplication and equations in (6) and (7), it is easy to show that the following equality holds:
\[
\begin{bmatrix}
r_1 \\
r_1' \\
r_1'' \\
r_1'''
\end{bmatrix} = \begin{bmatrix}
r_2 \\
r_2' \\
r_2'' \\
r_2'''
\end{bmatrix} \begin{bmatrix}
A \\
\begin{bmatrix} ab \end{bmatrix}
\end{bmatrix} = \begin{bmatrix}
s_1' \\
S_1' \\
S_2'
\end{bmatrix} \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}
\]  
(8)

The matrix \([R'|S']\) is the incidence matrix of a directed graph since each column contains a +1 and a −1. It remains to be shown that the matrix \(\hat{R}\) obtained by deleting one row of \(R'\) is non-singular. If we delete the first row of \(R'\) we have that:
\[
\hat{R} = \begin{bmatrix}
r_1' & r_2' \\
r_1'' & r_2'' \\
R_1' & 0 \\
0 & R_2'
\end{bmatrix}
\]

If we delete the first row from \(R_1\) then we obtain the matrix
\[
\begin{bmatrix}
r_1' & 1 \\
r_1'' & 0 \\
R_1' & 0
\end{bmatrix}
\]
which is a non-singular one. Expanding now the determinant of that matrix along the last column we can see that the matrix
\[
\begin{bmatrix}
r_1'' \\
R_1'
\end{bmatrix}
\]
is also non-singular. Therefore, within the submatrix
\[
\begin{bmatrix}
r_1' \\
r_1'' \\
R_1'
\end{bmatrix}
\]
of \(\hat{R}\), \(r_1'\) can be written as a linear combination of the other rows:
\[
r_1' + u r_1'' + q R_1' = 0
\]  
(9)

where \(u\) is a scalar, and \(q\) is a column vector of appropriate size with elements in \(\mathbb{R}\). Also, we have that \(u \neq 0\) since if we delete \(e_3\) in \(R_1\) then the matrix obtained corresponds to a forest in which the nodes which correspond to rows \(r_1'\) and \(r_1''\) belong to the same tree of that forest. We denote the determinant of \(\hat{R}\) by \(\det[\hat{R}]\). Using (4) we get:
\[
\det[\hat{R}] = \det \begin{bmatrix}
r_1' \\
r_1'' \\
r_1'''
\end{bmatrix} R_1' \quad \det \begin{bmatrix}
r_2' \\
r_2'' \\
r_2'''
\end{bmatrix} R_2'
\]  
\[
= \det \begin{bmatrix}
r_2' \\
r_2'' \\
r_2'''
\end{bmatrix} R_2' = \det \begin{bmatrix}
r_2' \\
r_2'' \\
r_2'''
\end{bmatrix} R_2'
\]  
(10)
So, matrix $\hat{R}$ is block diagonal and its blocks are square. Thus:

$$\det[\hat{R}] = \det\begin{bmatrix} r_1' & r_1'' \\ R_1' \end{bmatrix} \det\begin{bmatrix} r_2' + u r_2'' \\ R_2' \end{bmatrix} = \det\begin{bmatrix} r_1'' \\ R_1' \end{bmatrix} \left( \det\begin{bmatrix} r_2' \\ R_2' \end{bmatrix} + u \det\begin{bmatrix} r_2'' \\ R_2' \end{bmatrix} \right)$$

(11)

If we delete from $R_2$ its first row then the matrix so obtained is non-singular and, since it is a submatrix of a TU matrix, it has to be TU as well, i.e. its determinant should be equal to $\pm 1$. Expanding the determinant of that matrix along its first column we take:

$$\det\begin{bmatrix} r_2' \\ R_2' \end{bmatrix} + u \det\begin{bmatrix} r_2'' \\ R_2' \end{bmatrix} = \pm 1$$

(12)

Furthermore $\det\begin{bmatrix} r_2' \\ R_2' \end{bmatrix}, \det\begin{bmatrix} r_2'' \\ R_2' \end{bmatrix} \in \{0, \pm 1\}$ since the corresponding matrices are TU. From (12) we see that exactly one of these matrices has a nonzero determinant. Combining this with (11) and the fact that $u \neq 0$ we have that $\hat{R}$ is nonsingular.

Finally, it is obvious that the matrix $[R'|S']$ contains a $-1$ and a $+1$ in each column since its columns are columns of $[R_1|S_1]$ and $[R_2|S_2]$. We can conclude that the 3-sum of two network matrices is a network matrix with incidence matrix $[R'|S']$.

**Theorem 7.** Network matrices are closed under $k$-sums ($k = 1, 2, 3$).

**Proof:** For $k = 1$ it is straightforward. For $k = 2$ it is enough to observe that if $N_1 = \begin{bmatrix} A & a \\ \end{bmatrix}$, $N_2 = \begin{bmatrix} b \\ \end{bmatrix}$ are network matrices, then the matrices $\bar{N}_1 = \begin{bmatrix} A & a & a \\ 0 & 0 & 1 \end{bmatrix}$ and $\bar{N}_2 = \begin{bmatrix} 1 & 0 & b \\ 0 & 0 & B \end{bmatrix}$ are network matrices too, since we have only duplicated columns and added unitary rows and columns. But then $N_1 \oplus_2 N_2 = \bar{N}_1 \oplus_3 \bar{N}_2$ which we know from Lemma 6 to be network. For the alternative 3-sum operation, since network matrices are closed under pivoting the result follows.

### 3.2 $k$-sums of Network and Binet Matrices

In this section we examine the $k$-sums between network and binet matrices. We prove that the result is always a binet matrix and we provide the associated bidirected graph representations.

#### 3.2.1 Network $\oplus_3$ Binet

Let’s assume that $N_2$ of Lemma 6 is a binet matrix instead of a network matrix; then in a possible representation of it, its edges could be not only links but also loops and half-edges. Most importantly, because of the structure of matrix $N_2$ we have that the edges $f_1$, $f_2$ and $f_3$ should be of a specific type (loop, link, or half-edge) in order to form a binet representation of $N_2$. We examine below all the possible cases.

If $f_3$ is a link in the cycle (and then we can assume that it is a positive link), then $f_1$ and $f_2$ cannot be half-edges, because the fundamental circuit of a half-edge uses all the cycle edges, and the values on the cycle edges determined by the fundamental circuit are halves, so there can be neither 0 nor 1 in the row $f_3$ and columns $f_1$ and $f_2$ of $N_2$. Furthermore, $f_2$ cannot be a loop, because the fundamental circuit of any loop uses all cycle edges, despite the 0 in the corresponding position of the matrix. So either both $f_1$ and $f_2$ are links, or $f_1$ is a loop and $f_2$ is a link. If they are both links, then they are both positive
or both negative. Otherwise the fundamental circuit of one of them would use the negative edge in the cycle, the other would not, and they use the same edges except for the positive $f_3$. Moreover, $f_1$, $f_2$ and $f_3$ must form a triangle, so by a switching at a node we can make both $f_1$ and $f_2$ positive.

If $f_3$ is a loop, then $f_1$ cannot be a half-edge, because then the entry in row $f_3$ and column $f_1$ of $N_2$ would be a half. If $f_1$ is a loop, then vector $d$ of $N_2$ contains $\pm 2$ entries, but this is impossible because then $f_2$ would be an edge whose fundamental circuit uses non-cycle edges twice but does not use the basic cycle (which is $f_3$). So $f_1$ must be a link, which implies that $f_2$ is also a link, and $f_1$ is negative and $f_2$ is positive, because the fundamental circuit of $f_1$ uses the basic cycle, that of $f_2$ does not.

If $f_3$ is a half-edge, then $f_2$ must be a positive link, as its fundamental circuit does not use the basic cycle formed by $f_3$. This also implies that $f_1$ is a half-edge.

If $f_3$ is a non-basic link, then $f_1$ cannot be a loop, as then it would have $\pm 2$ on $f_3$ in the fundamental circuit. So either $f_1$ is a link and then $f_2$ is a link or a loop; or $f_1$ is a half-edge in which case $f_2$ is also a half-edge.

Therefore the cases that may appear are the following six:

(a) $f_3$ is a positive link in the cycle and $f_1$, $f_2$ are positive links;

(b) $f_3$ is a positive link in the cycle, $f_1$ is a negative loop and $f_2$ is a negative link;

(c) $f_3$ is a negative loop, $f_1$ is a negative link and $f_2$ is a positive link;

(d) $f_1$, $f_3$ are half-edges and $f_2$ is a positive link;

(e) $f_3$ is a non-cycle link, $f_1$ is a link and $f_2$ is a link or a negative loop; and

(f) $f_3$ is a non-cycle link and $f_1$, $f_2$ are half-edges.

**Lemma 8.** If $N_1$ is a network matrix and $N_2$ is a binet matrix such that

$$N_1 = \begin{bmatrix} e_1 & e_2 \\ e_3 & A & a & a \end{bmatrix}, \quad N_2 = \begin{bmatrix} f_1 & f_2 \\ 1 & 0 & b \\ d & d & B \end{bmatrix},$$

then $N = N_1 \oplus_3 N_2$ is a binet matrix.

**Proof:** Since $N_1$ is a network matrix we have that $e_1$, $e_2$ and $e_3$ should form a triangle. Therefore, w.l.o.g. we can assume for all the cases that the incidence matrix associated with the network matrix $N_1$ is the following one:

$$[R_1|S_1] = \begin{bmatrix} e_3 & e_1 & e_2 \\ r_1 & -1 & s_1 \ 0 & -1 \\ r'_1 & 1 & s'_1 \ -1 & 0 \\ r''_1 & 0 & s''_1 \ 1 & 1 \\ R'_1 & 0 & S'_1 \ 0 & 0 \end{bmatrix},$$

where $0$ is a zero matrix, $r_i$, $r'_i$, $r''_i$, $s_i$, $s'_i$ and $s''_i$ are vectors and $R'_i$ and $S'_i$ are matrices of appropriate size $(i = 1, 2)$. 

10
Case (a): For case (a) we have that the incidence matrix associated with the binet matrix $N_2$ can have the following form:

$$[R_2|S_2] = \begin{bmatrix} f_3 & f_1 & f_2 \\ 0 & r_2 & -1 & -1 & s_2 \\ -1 & r'_2 & 0 & 1 & s'_2 \\ 1 & r''_2 & 1 & 0 & s''_2 \\ 0 & R'_2 & 0 & 0 & S'_2 \end{bmatrix}$$

The proof for this case is very similar to the one regarding the 3-sum of two network matrices in Lemma 6. Because of the structure of matrix $N_2$, we have that $f_1$, $f_2$, and $f_3$ should form a triangle in any binet representation of $N_2$. Although we omit the full proof for this case because of its similarity to the one of Lemma 6, we provide the incidence matrix matrix $[R'|S']$ of the binet graph associated with the binet matrix $N$ produced by the 3-sum:

$$[R'|S'] = \begin{bmatrix} r_1 & r_2 & s_1 & s_2 \\ r'_1 & r'_2 & s'_1 & s'_2 \\ r''_1 & r''_2 & s''_1 & s''_2 \\ R'_1 & 0 & S'_1 & 0 \\ 0 & R'_2 & 0 & 0 & S'_2 \end{bmatrix}$$  \hspace{1cm} (14)

Case (b): For this case we have that the incidence matrix associated with the binet matrix $N_2$ can have the following form:

$$[R_2|S_2] = \begin{bmatrix} f_3 & f_1 & f_2 \\ -1 & r_2 & -2 & -1 & s_2 \\ 1 & r'_2 & 0 & -1 & s'_2 \\ 0 & R'_2 & 0 & 0 & S'_2 \end{bmatrix}$$  \hspace{1cm} (15)

Initially, we convert the network representation $[R_1|S_1]$ of $N_1$ to a binet representation in which $e_2$ is a loop. This can be done so by introducing an artificial link parallel to $e_2$ and then contracting it. Thus, $e_1$ becomes a negative link, as contraction involves switching at the node to which $e_1$ and $e_2$ are incident. Graphically this case is illustrated in Figure 3 which shows such an alternative binet representation of the matrix represented by the directed graph in Figure 2. Therefore, the incidence matrix $[R_1|S_1]$ of the binet graph associated with $N_1$ can have the following form:

$$[R_1|S_1] = \begin{bmatrix} e_3 & e_1 & e_2 \\ r_1 & -1 & s_1 & -1 & -2 \\ r'_1 & 1 & s'_1 & -1 & 0 \\ R'_1 & 0 & S'_1 & 0 & 0 \end{bmatrix}$$  \hspace{1cm} (16)

We have that the following equations hold:

$$R_1N_1 = S_1, \quad R_2N_2 = S_2$$  \hspace{1cm} (17)

From (16) and (17) we have that:

$$\begin{bmatrix} r_1 \\ r'_1 \\ R'_1 \end{bmatrix} = \begin{bmatrix} A & a & a \\ c & 0 & 1 \end{bmatrix} \begin{bmatrix} e_3 \\ e_1 \\ e_2 \\ s_1 \\ s'_1 \\ S'_1 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ s_1 & -1 & 0 \\ S'_1 & 0 & 0 \end{bmatrix}$$
where upon decomposing the block matrix multiplications we derive the following equations.

\[
\begin{bmatrix}
  r_1 \\
  r'_1
\end{bmatrix}
A +
\begin{bmatrix}
  -1 \\
  1
\end{bmatrix}
= \begin{bmatrix}
  s_1 \\
  s'_1
\end{bmatrix},
\begin{bmatrix}
  r_1 \\
  r'_1
\end{bmatrix}
A = \begin{bmatrix}
  -1 \\
  -1
\end{bmatrix},
\begin{bmatrix}
  r_1 \\
  r'_1
\end{bmatrix}
a +
\begin{bmatrix}
  -1 \\
  1
\end{bmatrix}
= \begin{bmatrix}
  -2 \\
  0
\end{bmatrix},
R_1'A = S_1',
R_1'a = 0
\]

(18)

From (15) and (17) we have that:

\[
\begin{bmatrix}
  -2 \\
  1
\end{bmatrix}_0 r_2
\begin{bmatrix}
  -1 \\
  0
\end{bmatrix}_1 d
= \begin{bmatrix}
  -2 \\
  0
\end{bmatrix}_0 s_2
\begin{bmatrix}
  -1 \\
  0
\end{bmatrix}_1 s'_2
\]

and

\[
\begin{bmatrix}
  -1 \\
  1
\end{bmatrix}_0 b
\begin{bmatrix}
  -2 \\
  0
\end{bmatrix}_1 d
= \begin{bmatrix}
  -1 \\
  0
\end{bmatrix}_0 s_2
\begin{bmatrix}
  -1 \\
  0
\end{bmatrix}_1 s'_2
\]

(19)

Using block matrix multiplication and the equations in (18) and (19), the following equality holds:

\[
\begin{bmatrix}
  r_1 \\
  r'_1
\end{bmatrix}
A +
\begin{bmatrix}
  s_1 \\
  s'_1
\end{bmatrix}
= \begin{bmatrix}
  r_2 \\
  r'_2
\end{bmatrix}
d
\begin{bmatrix}
  r_2 \\
  r'_2
\end{bmatrix}
d
= \begin{bmatrix}
  s_2 \\
  s'_2
\end{bmatrix}
\]

and \([R'|S']\) is the incidence matrix associated with \(N\).

**Case (c):** This case is very similar to case (b). Here we have again to find an alternative binet representation of \(N_1\). This can be obtained if we take the representation where \(e_1\) is a loop in a binet representation of \(N_1\). In this case the incidence matrix associated with a binet representation of \(N_1\) can be:

\[
[R_1|S_1] = \begin{bmatrix}
  r_1 & s_1 & e_1 & e_2 \\
  1 & 1 & 0 & 1 \\
  r'_1 & s'_1 & 2 & 1 \\
  0 & 0 & S_1' & 0
\end{bmatrix}
\]

and w.l.o.g. we can also assume that the incidence matrix associated with the binet matrix \(N_2\) is:

\[
[R_2|S_2] = \begin{bmatrix}
  f_3 & f_1 & f_2 \\
  0 & r_2 & 1 & 1 & s_2 \\
  2 & r'_2 & 1 & -1 & s'_2 \\
  0 & 0 & S_2'
\end{bmatrix}
\]

Using the same methodology as we did in cases (a) and (b) it can be shown that for case (c) a incidence matrix associated with matrix \(N\), i.e. such that \(R'N = S'\), is:

\[
[R'|S'] = \begin{bmatrix}
  r_1 & r_2 & s_1 & s_2 \\
  r'_1 & r'_2 & s'_1 & s'_2 \\
  R_1' & S_1' & 0 \\
  0 & R_2' & S_2'
\end{bmatrix}
\]
Case (d): Similarly, the incidence matrix associated with $N_2$ can be:

$$[R_2|S_2] = \begin{bmatrix} f_3 & f_1 & f_2 \\ -1 & r_2 & -1 & 0 & s_2 \\ 0 & r_2' & 1 & -1 & s_2' \\ 0 & R_2' & 0 & 0 & S_2' \end{bmatrix}$$

We can delete the third row from matrix $[R_1|S_1]$ of (13) in order to get a binet representation of matrix $N_1$. Therefore, we can assume that in this case the incidence matrix associated with $N_1$ can be:

$$[R_1|S_1] = \begin{bmatrix} e_3 & e_1 & e_2 \\ r_1 & -1 & s_1 & 0 & -1 \\ r_1' & 1 & s_1' & -1 & 0 \\ R_1' & 0 & S_1' & 0 & 0 \end{bmatrix}$$

Using the same methodology as we did in all the previous cases it is easy to show that a incidence matrix associated with $N$ is:

$$[R'|S'] = \begin{bmatrix} r_1 & r_2 & s_1 & s_2 \\ r_1' & r_2' & s_1' & s_2' \\ R_1' & 0 & S_1' & 0 \\ 0 & R_2' & 0 & S_2' \end{bmatrix}$$

(20)

Case (e) is directly analogous to the case (a) and (b) where $f_2$ is a link and $f_2$ is a loop, respectively. Case (f) is directly analogous to the case (d). For this reason we omit the proof for these cases.

For each of the aforementioned cases it is obvious that $[R'|S']$ is a incidence matrix of a bidirected graph, since the set of columns of this matrix is a combination of columns in $[R_1|S_1]$ and $[R_2|S_2]$. The rows/columns of $R'$ in each case are linearly independent, something that can be proved in much the same way as we did for the $R'$ in Lemma 8. Alternatively, the non-singularity of $R'$ stems also from the graphical explanation we give in the following section. Specifically, since there is one-to-one correspondence between the $R'$ and the associated bidirected graph, it can be shown that the graph induced by the edges corresponding to the columns of $R'$ form a negative 1-tree in the unique bidirected graph associated with $[R'|S']$ found in each case.

Graphical Representation of Network $\oplus_3$ Binet:

An illustration regarding case (a) is depicted in Figure 2 where the triangles $(f_1, f_2, f_3)$ and $(e_1, e_2, e_3)$ are glued together and their edges are deleted from the unified graph. In this way, we obtain a bidirected graph whose associated incidence matrix is the one given by (14). In case (b), we convert the network representation of $N_1$ to a binet representation in which $e_2$ is a loop. As described in the proof of Lemma 5, this can be done by introducing an artificial link parallel to $e_2$ and then contracting it. In this way $e_1$ becomes a negative link, since contraction involves switching at the node at which $e_1$ and $e_2$ are incident, but this does not affect the gluing of $e_1$ and $f_2$ since $f_2$ is also a negative link because its fundamental circuit uses the negative link of the basic cycle. This case is illustrated in Figure 3.

That figure shows the alternative binet representation of the matrix represented by the directed graph in Figure 2. For case (c) see Figure 4 for an illustration. To make a similar representation for $N_1$, we can convert $e_1$ to a loop with a contraction. The binet graph representing $N_1$ in Figure 4 is an alternative representation to the directed graph in Figure 2. In case (d) the three edges $f_1, f_2$ and $f_3$ are positioned as in Figure 5. We can have a similar position of edges $e_1, e_2, e_3$ if we delete a node that is incident to $e_1$ and $e_2$. The leftmost graph in Figure 5 shows such a binet representation of the network.
Figure 2: The binet representation of the 3-sum of a network and a binet matrix. The case when $f_1, f_2, f_3$ are links.

Figure 3: The binet representation of the 3-sum of a network and a binet matrix. The case when $f_1$ is a loop, $f_2$ is a negative link, $f_3$ is a positive link.

matrix represented by the directed graph in Figure 2. Finally, cases (e) and (f) can be handled with the techniques described previously. If an edge among $f_1, f_2, f_3$ is a loop, then contract an artificial edge in the directed graph representation of $N_1$ to make the corresponding edge a loop. If two edges among $f_1, f_2, f_3$ are half-edges, then delete an appropriate node from the directed graph.

3.2.2 Binet $\oplus_3$ Network

A very similar analysis of the cases can be done here. The role of $e_1, e_2$ and $e_3$ is analogous to $f_1, f_2$ and $f_3$ as in the previous section. All the cases can be handled in much the same way, by finding a suitable alternative representation of $N_2$ as we did for $N_1$ in the proof of Lemma 8.

Figure 4: The binet representation of the 3-sum of a network and a binet matrix. The case when $f_3$ is a loop.
Figure 5: The binet representation of the 3-sum of a network and a binet matrix. The case when $f_3$ is a half-edge.

**Lemma 9.** If $N_1$ is a binet matrix and $N_2$ is a network matrix such that

$$N_1 = \begin{bmatrix} e_1 & e_2 \\ e_3 & A & a & a \\ c & 0 & 1 \end{bmatrix}, \quad N_2 = \begin{bmatrix} e_3 & f_1 & f_2 \\ f_3 & 1 & 0 & b \\ d & d & d & B \end{bmatrix},$$

then $N = N_1 \oplus_3 N_2$ is a binet matrix.

**Theorem 10.** The $k$-sum of a network (binet) matrix and a binet (network) matrix is binet ($k = 1, 2, 3$).

**Proof:** The proof is similar to that of Theorem 7 since binet matrices are also closed under duplication of columns and rows, addition of unitary rows and pivoting.

### 3.3 $k$-sums of Binet Matrices

Here we prove that the $k$-sum ($k = 2, 3$) of two binet matrices is not necessarily a binet matrix. Furthermore, an analogous statement can be made for the associated matroids, the so-called signed-graphic matroids. Using a counterexample, we show that the 2-sum of two binet, non-network and totally unimodular matrices, namely $B_1$ and $B_2$ of (1) and (2), is not a binet matrix. The column of $B_1$ as well as the row of $B_2$ used in our 2-sum counterexample are indicated below:

$$B_1 = \begin{bmatrix} A \mid a \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & -1 & 1 \\ 1 & 0 & 0 & 1 & -1 \\ -1 & 1 & 0 & 0 & 1 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} b \\ B \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

Let $M$ be the 2-sum of $B_1$ and $B_2$ which according to the 2-sum definition is:

$$M = \begin{bmatrix} A \mid ab \\ 0 \mid B \end{bmatrix} = \begin{bmatrix} s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 & s_9 \\ r_1 & 0 & 0 & 1 & -1 & 1 & 1 & 1 & 1 \\ r_2 & 1 & 0 & 0 & 1 & -1 & -1 & -1 & -1 \\ r_3 & -1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ r_4 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ r_5 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ r_6 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ r_7 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ r_8 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ r_9 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}.$$
Assume that $M$ is a binet matrix and that $r_i$ and $s_i$ ($i = 1 \ldots 9$) label the basic and non-basic edges, respectively, in a binet representation of $M$. Matrix $M$ is integral and since it is also binet then any possible binet representation of $M$ up to switching should be one of the following two types [8] (Lemmas 5.10 and 5.12):

**Type I:** Every basic cycle is a half-edge, and all other basic edges are directed.

**Type II:** There are no half-edges in the binet graph, the basis is connected and there is only one bidirected edge in the basis.

We will show that $M$ has neither of the above two representations, thereby it can not be binet. We make use of the following Lemma in [8]:

**Lemma 11.** Let us suppose that a binet matrix $B$ is totally unimodular. Then it is a network matrix if and only if it has a binet representation in which each basic cycle is a half-edge.

**Lemma 12.** Matrix $M = B_1 \oplus_2 B_2$ does not have a binet representation of type I or type II.

**Proof:** Suppose that $M$ has a binet representation of type I. Combining the fact that $M$ is totally unimodular with Lemma [11] we have that $M$ is a network matrix. It is well-known that any submatrix of a network matrix is a network matrix itself (e.g. see [11]). $B_1$ is a submatrix of $M$ which is known to be non-network. Thus, $M$ can not have a binet representation of type I.

Assume that $M$ has a binet representation $\Sigma$ of type II. Let $\Sigma_R$ be the subgraph of $\Sigma$ induced by the edges in $R = \{r_1, \ldots, r_9\}$ ($\Sigma_R$ is also called the basis graph of $\Sigma$). Let also $C$ be the set of edges that constitute the unique cycle in $\Sigma_R$, i.e. $C$ is the edge set of the basic cycle of the binet graph $\Sigma$. Because of column $s_5$ of $M$ the subgraph of $\Sigma_R$ induced by the basic edges in $S = \{r_1, r_2, r_3, r_6, r_7, r_8, r_9\}$ is connected. Our first claim is that $C \subseteq S$. If we assume the contrary, i.e. that $C \not\subseteq S$, then the edges in $S$ should form a path in $\Sigma_R$. Moreover, observe that each non-basic edge of the set $\{s_6, s_7, s_8, s_9\}$ is using edges of $S$ in order to create the associated fundamental circuit in $\Sigma$. Combining this with the fact that the edges in $S$ induce a path of $\Sigma$, we have that \[
\begin{pmatrix}
ab & B
\end{pmatrix}
\] must be a network matrix. But this can not happen since this matrix contains $B_2$ as a submatrix which is not a network matrix and thus, our claim is true. Thus, $C \subseteq S$ and furthermore, since there is only one cycle in $\Sigma_R$, we have that $\{r_4, r_5\} \not\subseteq C$.

Let $D = \{r_1, r_2, r_3\}$ and $E = S - C = \{r_6, r_7, r_8, r_9\}$; our second claim is that $C \not\subseteq D$. If we assume the contrary, i.e. that $C \subseteq D$ then because of column $s_5$ of $M$ we have that the corresponding fundamental circuit in $\Sigma$ should be either a handcuff of type I or a handcuff of type II. However, it can not be a handcuff of type II since then a $\pm 2$ would appear in $M$ (see Algorithm 1 in [1]). Therefore, it is a handcuff of type I and thus the basic edges in $(D - C) \cup E$ induce a path in the basis graph. Thus, the edges in $E$ and one or more edges of $D$ are the parts of this path in the basis graph. Moreover, from the fundamental circuits of $\Sigma$ described by the columns of \[
\begin{pmatrix}
A & 0
\end{pmatrix}
\] part of $M$ we have that the subgraph $\Sigma_T$ of $\Sigma_R$ induced by the set of edges in $T = \{r_1, r_2, r_3, r_4, r_5\}$ is connected. Observe now that the edges in $D$ appear in all the fundamental circuits of $\Sigma$ corresponding to the columns of \[
\begin{pmatrix}
ab & B
\end{pmatrix}
\]. Therefore, because of the structure of these fundamental circuits and since $\Sigma_T$ is connected we have that in $\Sigma_R$ the following conditions must be satisfied: (i) $r_6$ and $r_9$ are adjacent, (ii) $r_6$ and $r_7$ are adjacent, (iii) $r_7$ and $r_8$ are adjacent, and (iv) $r_8$ and $r_9$ are adjacent. We show now that this can not happen. Assume, w.l.o.g., that $r_9$ is on the right side of $r_6$ then because of (ii) $r_7$ should be put on the left side of $r_6$. 

16
Moreover, because of (iii) $r_8$ should be on the left side of $r_7$. But now condition (iv) can not be satisfied. Thus, our assumption that $C \subseteq D$ is not correct and this completes the proof of our second claim.

Since we have shown that $\{r_4, r_5\} \notin C$ and that $C \notin D$ we have that $\Sigma_T$ is a tree in $\Sigma_R$. We show now that any two edges in $D$ do not share a common end-node. Note that the following procedure can be used in much the same way for any pair of edges in $D$. Specifically, suppose that $r_1$ and $r_2$ share an end-node and without loss of generality suppose that $r_2$ stands on the right side of $r_1$. Consider the fundamental circuits of $\Sigma$ determined by the columns of the

\[
\begin{bmatrix}
A \\
0
\end{bmatrix}
\]

part of $M$. Because of the columns $s_3$ and $s_4$ we have that $r_5$ stands on the left side of $r_1$. Moreover, because of the columns $s_1$ and $s_3$ we have that $r_4$ has a common end-node with $r_1$ and $r_2$. But now, we can not satisfy the fundamental circuit defined by $s_2$ because edge $r_1$ is in the middle of $r_4$ and $r_5$. Thus, we can conclude that any two edges of $D$ do not share a common end-node. However, we have that $\Sigma_T$ (which contains $r_4$ and $r_5$) is a tree and that the edges in $S$ (which does not contain $r_4$ and $r_5$) induce a connected subgraph in $\Sigma_R$ containing a basic cycle. This can only happen if $\Sigma_R$ contains at least two cycles. In other words, in order to find a binet graph satisfying the circuits described by the columns of $M$ we have that $\Sigma_R$ should contain at least two cycles. This is in contradiction with the fact that connected binet graphs contain at most one basic cycle in the basis graph. Therefore, $M$ does not have a binet representation of type II.

In general we can state the following theorem.

**Theorem 13.** Totally unimodular binet matrices are not closed under $k$-sums for $k = 2, 3$.

**Proof:** For $k = 2$ the Lemma [12] provides a counterexample. For $k = 3$ it is enough to observe that for $c = 0$ in the Definition [3] the 3-sum of two matrices reduces to the 2-sum of some submatrices obtained by the deletion of columns and rows. Since binet and TU matrices are closed under row and column deletions, the result follows.

4 Tour Matrices

In this section a new class of matrices is introduced, that of tour matrices, in order to represent some important classes of matrices on bidirected graphs. In what follows, we also prove some elementary properties of tour matrices and show that they are closed under $k$-sums.

4.1 Definition and Elementary Properties

Let $[Q|S]$ be the incidence matrix of a bidirected graph $\Sigma$. We denote by $\Sigma(Q)$ and $\Sigma(S)$ the subgraphs of $\Sigma$ induced by the edges that correspond to the columns of $Q$ and $S$, respectively. A tour in a bidirected graph is a walk in which no edge is repeated. A closed tour is a tour in which the first and last node coincide or the first and last edge are half edges.

**Definition 14.** Let $\Sigma$ be a bidirected graph with $[Q|S]$ its incidence matrix. A $\{0, \pm 1\}$ matrix $B$ with rows indexed by the columns of $Q$ and columns indexed by the columns of $S$, such that

1. $QB = S$
2. $Q$ is full row rank

is called a tour matrix.
The edges in $\Sigma(Q)$ are called prime and the edges in $\Sigma(S)$ are called non-prime. When in a bidirected graph representing a tour matrix $B$ the prime and non-prime edges are labeled, we call it a tour representation or a tour graph of $B$.

**Lemma 15.** Let $B$ be an $m \times n$ tour matrix of a bidirected graph $\Sigma$ with incidence matrix $[Q|S]$ and $Q(b_i)$ be the set of edges in $Q$ indexed by the nonzero entries in the column $b_i$ of $B$ ($i = 1, \ldots, n$). Then the subgraph induced by $Q(b_i) \cup s_i$ is a collection of closed tours in $\Sigma$, where $s_i$ is the $i$-th column of $S$.

*Proof:* Since $Qb_i - s_i = 0$ for all $i \in (1, \ldots, n)$ and $q_j, s_i \in \{0, \pm 1, \pm 2\}^n, b_i \in \{0, \pm 1\}^n$ for all $q_j \in Q(b_i)$ we have that the degree of every vertex in the subgraph induced by $Q(b_i) \cup s_i$ is even, therefore its connected components are Eulerian. Thus, the subgraph induced by $Q(b_i) \cup s_i$ is a collection of closed tours. \qed

In the following lemmas we provide some elementary operations which if applied to a tour matrix then the matrix obtained is also tour.

**Lemma 16.** If $\Sigma$ is a tour representation of a tour matrix $B$ then switching at a node of $\Sigma$ keeps $B$ unchanged.

*Proof:* Switching at a node $v$ in a bidirected graph $\Sigma$ is interpreted as multiplying by $-1$ the row of the incidence matrix $D = [Q|S]$ which corresponds to node $v$. Let $Q'$ and $S'$ be the matrices obtained multiplying by $-1$ the aforementioned row of $D$. Since $QB = S$, from matrix multiplication we also have that $Q'B = S'$. \qed

**Lemma 17.** Tour matrices are closed under the following operations:

(a) Permuting rows or columns.
(b) Multiplying a row or column by $-1$.
(c) Duplicating a row or column.
(d) Deleting a row or column.

*Proof:* If $B$ is a tour matrix then by definition we have that $QB = S$, where $D = [Q|S]$ is the incidence matrix of a bidirected graph $\Sigma$ associated with $B$. Let $B'$ be the matrix obtained by applying one of the above operations on $B$. We show in each case that $B'$ is a tour matrix by providing the associated incidence matrix $D' = [Q'|S']$.

(a) When permutation of columns takes place let $Q' = Q$ and $S'$ be the matrix obtained from $S$ by permuting the columns of $S$ in the same way that columns of $B$ were permuted. When permutation of rows takes place let $S' = S$ and $Q'$ be the matrix obtained from $Q$ by permuting its columns in the same way that rows of $B$ were permuted. From matrix multiplication rules we have that $Q'B = S'$ and that $D' = [Q'|S']$ is the incidence matrix of a bidirected graph in both cases.

(b) If row $e$ of $B$ is multiplied by $-1$ then let $Q'$ be $Q$ with column $e$ multiplied by $-1$ and $S' = S$. If we multiply a column $f$ of $B$ by $-1$ then let $Q' = Q$ and $S'$ be $S$ with column $f$ multiplied by $-1$. Obviously in both cases $B'$ is a tour matrix since from matrix multiplication rules we have that $Q'B' = S'$.

(c) If we duplicate a column $f$ in $B$, let $Q' = Q$ and $S'$ be $S$ with column $f$ duplicated. It is easy to check then that $B'$ satisfies the conditions of a tour matrix.

Row duplication is a bit more involved. We have four cases corresponding to the different types of edges, and in each case we will alter the bidirected graph to correspond to the new tour matrix. If row $f$ to be duplicated is a positive loop, simply add a positive loop to any node of the signed graph. If the
prime edge $f$ is a negative loop (see (i) in Figure 6), then add a zero row $t$ in $[Q|S]$ and a zero column $f'$ in $Q$ to obtain $[Q'|S']$ and set

$$Q'_{s,f} = Q'_{t,f} = Q_{s,f}/2 \quad \text{and} \quad Q'_{s,f'} = -Q'_{t,f'} = Q_{s,f}/2.$$ 

If the prime edge $f$ is a link (see (ii) in Figure 6) then duplicate row $s$ in $[Q|S]$ to create a new row $t$, and make all the elements of row $s$ zero except the element in position $f$. In row $t$ make the element in position $f$ zero. Finally add a new column $f'$ in $Q'$ and set

$$Q'_{t,f'} = -Q'_{s,f'} = Q_{s,f}.$$ 

Finally, if the prime edge $f$ is a half-edge (see (iii) in Figure 6) then then add a zero row $t$ in $[Q|S]$ and a zero column $f'$ in $Q$ to obtain $[Q'|S']$ and set

$$Q'_{s,f} = -Q'_{s,f'} = Q'_{t,f'} = Q_{s,f}.$$ 

In all cases, the matrix $[Q'|S']$ is the incidence matrix of a bidirected graph by construction, and $Q'B' = S'$. 

(d) Deletion of a column in a tour matrix is simply the deletion of the corresponding non-prime edge in the corresponding bidirected graph. Deletion of a row $f$, differs according the type of the corresponding prime edge $f$. If $f$ is a positive loop, or a link, then contract $f$ in the bidirected graph. If $f$ is a negative loop then make all adjacent links to the end-node of $f$ half-edges adjacent to their other end-node, while all adjacent loops and half edges become positive loops at some other arbitrary node, and delete $f$ and its end-node. In all cases it is easy to verify that the new bidirected graph corresponds to the tour matrix with a column(row) deleted. 

We should note here that multiplying a row (column) by -1 in a tour matrix, graphically is equivalent to reversing the direction of the corresponding prime (respectively non-prime) edge in the associated bidirected graph. On the other hand, duplicating a column amounts to creating a parallel non-prime edge to the tour graph.

Given a matrix $\begin{bmatrix} 1 & c \\ b & D \end{bmatrix}$ in $\mathbb{R}$, a pivot is the matrix $\begin{bmatrix} -1 & c \\ b & D - bc \end{bmatrix}$ (see [13]).

Lemma 18. Totally unimodular tour matrices are closed under pivoting.
Proof: Let \( T = \begin{bmatrix} 1 & c \\ b & D \end{bmatrix} \) be a totally unimodular tour matrix associated with a bidirected graph \( \Sigma \) with incidence matrix \([f \ Q | e \ S]\). By definition

\[
[f \ Q] T = [e \ S]
\]

and the columns \( f \) and \( e \) correspond to the prime and non-prime edges respectively. Consider the bidirected graph \( \Sigma' \) with incidence matrix \([e \ Q | -f \ S]\), that is \( \Sigma \) with edge \( f \) having its endpoints reversed in sign. We will show that matrix \( B = \begin{bmatrix} -1 & c \\ b & D - bc \end{bmatrix} \) is a tour matrix associated with \( \Sigma' \).

Initially let us show that

\[
[e \ Q] B = [-f \ S]
\]

We know from (21) that \( f + \sum_i b_i q_i = e \), where \( q_i \) is the \( i \)th column of \( Q \). Therefore

\[
-f = -e + \sum_i b_i q_i,
\]

which shows that the first column of \( B \) is a collection of tours in \( \Sigma' \). Take any other column \( j \) of \( B \). If \( c_j = 0 \) the relationship (22) follows. If \( c_j = +1 \) then we know from (21) that

\[
f + \sum_i d_{ij} q_i = s_j,
\]

and the corresponding product in (22) will be

\[
e + \sum_i (d_{ij} - b_i) q_i.
\]

Partition the indices of the differences in the above summation into three sets: \( I_1 \) which corresponds to indices where both \( d_{ij}, b_i \neq 0 \), \( I_2 \) where \( d_{ij} \neq 0 \) and \( b_i = 0 \) and \( I_3 \) where \( d_{ij} = 0 \) and \( b_i \neq 0 \). Replacing \( e \) by (23) we have

\[
e + \sum_i (d_{ij} - b_i) q_i = f + \sum_{i \in I_1} b_i q_i + \sum_{i \in I_2} (d_{ij} - b_i) q_i + \sum_{i \in I_3} d_{ij} q_i - \sum_{i \in I_3} b_i q_i
\]

\[
= f + \sum_i d_{ij} q_i = s_j
\]

Similarly for the case where \( c_j = -1 \) (or alternatively use (b) of Lemma 17).

Given that totally unimodular matrices are closed under pivoting, \( B \) will be a \( \{0, \pm 1\} \) matrix.

Lemma 19. Network matrices are tour matrices.

Proof: Consider a network matrix \( N \in \{0, \pm 1\} \) of a directed graph \( G \) with incidence matrix \([R|S]\). We will show that \( N \) can be viewed as a binet matrix by providing a binet representation of it.

Let \( e \) be any column of \( S \). In what follows we will show that there exists a binet representation in which edge \( e \) is a loop at any one of its endpoints. View the graph \( G \) as a bidirected graph \( \Sigma \) with only positive links. Add a negative link \( f \) parallel to \( e \) and as a result we have that the binet matrix associated with \( \Sigma \) is equal to the original network matrix \( N \) plus an all-zero row. Deleting this all-zero row we get the original matrix \( N \), while the equivalent graphical operation would be the contraction of edge \( f \). Contraction of \( f \) involves switching at one end-node of \( f \) (say at \( v \)), since \( f \) is a negative link. This way \( e \) becomes a negative loop (see Figure 7).
Figure 7: Inserting a negative edge $f$, and then contracting it by switching at $v$.

In matrix terms, we have that starting from $[R|S] = v\begin{bmatrix} e & -1 \\ R & 1 \\ S & 0 \end{bmatrix}$ by the aforementioned procedure we obtain $[R'|S'] = v\begin{bmatrix} e \\ R' & 2 \\ 0 & 0 \end{bmatrix}$, where $R'N = S'$ and $[R'|S']$ is a incidence matrix associated with a binet representation of $N$. Therefore we have found a bidirected graph $[R'|S']$ where $R'N = S'$, and $R'$ is full-row rank.

Furthermore, it is known that any binet matrix which is TU and non-network should have a binet representation $\Sigma$ that does not contain half-edges (see Lemma 22 and Theorem 24 in [1]). Therefore we can state the following corollary.

**Corollary 20.** TU binet matrices are tour matrices.

From Corollary 20, it is evident that $B_1$ and $B_2$ are tour matrices. Combining this with Lemma 19, Theorem 4 and the fact that zero columns are preserved, we have the following.

**Corollary 21.** All the building blocks of TU matrices are tour matrices and their transposes.

### 4.2 Bidirected Graph Representation of TU matrices

In this section we will show that all TU matrices have a bidirected graph representation since they are a subclass of tour matrices. This is illustrated in the following “pathological” case by the usage of positive loops, which in general allow a somewhat arbitrary insertion of prime edges and thereby rows in a given matrix.

**Theorem 22.** All TU matrices are tour matrices.

**Proof:** Let $B \in \{0, \pm 1\}^n \times m$ be a totally unimodular matrix. By Ghouila-Houri characterisation of TU matrices (see [3], we have that there exists a vector $x^T \in \{\pm 1\}^n$ such that $x^T B = y^T \in \{0, \pm 1\}^m$; that is multiplying the rows by $\pm 1$ the resulting matrix has columns which sum up to $\{0, \pm 1\}$. Therefore we can have $\begin{bmatrix} x^T \\ x^T \end{bmatrix} B = \begin{bmatrix} y^T \\ y^T \end{bmatrix}$ and $[Q|S] = \begin{bmatrix} x^T & y^T \\ x^T & y^T \end{bmatrix}$, is the incidence matrix of a bidirected graph since the sum of each column is less or equal to $|2|$. If the first column of $Q$ is $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$ replace it with
\[
\begin{bmatrix}
-2 \\
0
\end{bmatrix}, \text{ while if it is } 
\begin{bmatrix}
1 \\
1
\end{bmatrix} \text{ replace it with } 
\begin{bmatrix}
2 \\
0
\end{bmatrix}
\text{ to obtain a new matrix } Q', \text{ and set } S' = Q'B. \text{ Then } [Q'|S'] \text{ is also the incidence matrix of a bidirected graph with } B \text{ its tour matrix.}
\]

However a tour matrix may have multiple bidirected graph representations, and in the proof of Theorem 22 the bidirected graph so constructed does not have enough structural information with respect to the linear independence of the columns of the associated matrix. We know from Seymour’s decomposition Theorem 4 that a TU matrix is composed by \( k \)-sums from matrices which do have a bidirected graph representation, therefore in view of Corollary 21 there must exist a richer in structure bidirected graph representation. Moreover, the building blocks in the \( k \)-sum composition do have bidirected graphs which do not have positive loops. In order to obtain this representation, we have to examine the way the \( k \)-sum operations behave on tour matrices.

### 4.2.1 The \( k \)-sum Operations on Tour Matrices

In what follows we present results on the \( k \)-sums of tour matrices. The case of only 3-sum will be shown as we did in the previous sections, since the other sum operations could be reduced to it by the addition of unitary rows and duplication of columns.

**Lemma 23.** If \( K, L \) are tour matrices, then there exist tour matrices \( K', L' \) such that \( K \oplus_3 L \) is a row submatrix of \( K' \oplus_3 L' \) where the connecting elements are all positive links.

**Proof:** Let

\[
K = \begin{bmatrix}
e_1 & e_2 \\
e_3 & A & a & a \\
& c & 0 & 1
\end{bmatrix}, \quad L = \begin{bmatrix}
f_1 & f_2 \\
f_3 & 1 & 0 & b \\
& d & d & B
\end{bmatrix}.
\]

For all possible edge type configurations of \( f_1, f_2 \) and \( f_3 \) we will apply graphical operations on the tour graph of \( L \), so that the resulting graph will be the tour graph of a tour matrix \( L' \) that will contain \( L \) as a submatrix.

**Case (a):** Consider the case where \( f_1 \) is a negative loop, \( f_2 \) is a negative link and \( f_3 \) is a positive link. Because of the first two columns of \( L \) we have that these edges must be of the following form: \( f_1 = \{v, v\} \), \( f_2 = \{u, v\} \) and \( f_3 = \{u, v\} \) (see Figure 8). The graphical operation is the following: we split the end-node \( v \) of \( f_1 \) into two nodes \( v_1 \) and \( v_2 \) and add a new basic positive link \( f' = \{v_1, v_2\} \).

In the new bidirected graph \( f_1, f_2 \) are negative links, and \( f_3 \) is a positive link, while for all other edges having end-node \( v \) we replace \( v \) by \( v_1 \). Up to switchings, the tour matrix \( L' \) associated with this graph is:

\[
f' = \begin{bmatrix}
1 & 0 & b \\
d & d & B \\
1 & 1 & 0
\end{bmatrix}, \text{ where the connecting elements } f_1, f_2 \text{ and } f_3 \text{ are all positive links.}
\]

**Case (b):** Let now \( f_1 = \{u\}, f_3 = \{v\} \) be half-edges and \( f_2 = \{u, v\} \) a positive link (see Figure 9). The graphical operation in this case is the following: add a new vertex \( w \) in the bidirected graph, replace the half-edges \( f_1 \) and \( f_3 \) by positive links \( f_1 = \{u, w\} \) and \( f_3 = \{v, w\} \), and add a negative loop \( f' = \{w, w\} \).

The new tour matrix \( L' \) associated with this graph will be:

\[
f' = \begin{bmatrix}
1 & 0 & b \\
d & d & B \\
0 & 0 & b
\end{bmatrix}.
\]
Case (c): For the case where \( f_1, f_2 \) are negative loops and \( f_3 \) a positive loop the graphical operation is similar to the ones described previously, and is depicted in Figure 10. The new tour matrix \( L' \) associated

with the so constructed graph will be:

\[
\begin{pmatrix}
f_1 & f_2 \\
f_3 & \begin{pmatrix} 1 & 0 & b \\ d & d & B \\ 0 & 0 & b \\ 1 & 1 & 0 \end{pmatrix} \\
\end{pmatrix}
\]

Case (d): The case where \( f_1 \) is a negative link, \( f_2 \) a negative loop and \( f_3 \) a half-edge, can be easily verified that is not possible, due to the structure of \( L \).

It is straightforward to show that all possible edge type configurations for the connecting edges of \( L \), fall into one of the above described cases where the new tour matrix \( L' \) will contain either a row \( f' \) or \( f'' \) or both. Applying the above graphical operations and switchings on both \( K \) and \( L \), we can therefore obtain \( K' \) and \( L' \) were the connecting elements of \( K' \oplus_3 L' \) are positive links \( e_1, e_2, e_3 \) and \( f_1, f_2, f_3 \), while the matrix \( K' \oplus_3 L' \) contains \( K \oplus_3 L \) as a row submatrix.

Lemma 24. If \( K, L \) are tour matrices such that

\[
K = \begin{pmatrix} e_1 & e_2 \\ e_3 & \begin{pmatrix} A & a \\ c & 0 \end{pmatrix} \end{pmatrix}, \quad L = \begin{pmatrix} f_1 & f_2 \\ f_3 & \begin{pmatrix} 1 & 0 & b \\ d & d & B \end{pmatrix} \end{pmatrix}
\]
then $M = K \oplus_3 L$ is a tour matrix.

Proof: Let $D_1 = [Q_1|S_1]$ and $D_2 = [Q_2|S_2]$ be incidence matrices associated with $K$ and $L$ and $\Sigma(D_1)$ and $\Sigma(D_2)$ be the associated tour graphs. By Lemma 23 and (d) of Lemma 17 we can assume that the connecting elements $e_1, e_2, e_3$ and $f_1, f_2, f_3$ are all positive links in the tour graphs.

By Lemma 17 the incidence matrices $D_1$ and $D_2$ associated with $K$ and $L$ can have the following form:

$$[Q_1|S_1] = \begin{bmatrix} q_1 & -1 & e_1 & e_2 & \vdots & q_3 & f_3 & f_1 & f_2 \\ q_1' & 1 & s_1' & -1 & 0 & v' & [Q_2|S_2] = v'' & -1 & q_2' & 0 & 1 & s_2' \\ q_2'' & 0 & s_1'' & 1 & 1 & y & 1 & q_2'' & 1 & 0 & s_2'' & 0 \end{bmatrix}$$

(24)

where $0$ is a vector or matrix of zeroes of appropriate size, $q_i, q_i', q_i'', s_i, s_i'$ and $s_i''$ are row vectors and $Q_i', S_i'$ are matrices of appropriate size ($i = 1, 2$). Also, $u, v$ and $y$ label the three first rows of $D_1$ and consequently the corresponding nodes of $\Sigma(D_1)$. Similarly, $u', v', y'$ label the first three rows of $D_2$ and the corresponding nodes of $\Sigma(D_2)$. We have that the following equations hold:

$$Q_1K = S_1, \quad Q_2L = S_2$$

(25)

For $K$ using (24) and (25) we have that:

$$\begin{bmatrix} q_1 & -1 \\ q_1' & 1 \\ q_1'' & 0 \end{bmatrix} A + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} c = \begin{bmatrix} s_1 \\ s_1' \\ s_1'' \end{bmatrix}, \quad \begin{bmatrix} q_1 \\ q_1' \\ q_1'' \end{bmatrix} a = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

(26)

From the above equation we take the following equations:

$$\begin{bmatrix} q_1 \\ q_1' \\ q_1'' \end{bmatrix} A + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} c = \begin{bmatrix} s_1 \\ s_1' \\ s_1'' \end{bmatrix}, \quad \begin{bmatrix} q_1 \\ q_1' \\ q_1'' \end{bmatrix} a = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Similarly, for $L$ using (24) and (25) we have that:

$$\begin{bmatrix} 0 & q_2 \\ -1 & q_2' \\ 1 & q_2'' \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ b \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

(27)

From the above equation we take the following equations:

$$\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} q_2 \\ q_2' \\ q_2'' \end{bmatrix} d = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} q_2 \\ q_2' \\ q_2'' \end{bmatrix} d = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} b + \begin{bmatrix} q_2 \\ q_2' \\ q_2'' \end{bmatrix} B = \begin{bmatrix} s_2 \\ s_2' \\ s_2'' \end{bmatrix}, \quad Q_2' d = 0, \quad Q_2' B = S_2'$$

(27)
Using block matrix multiplication and equations in (23) and (24), it is easy to show that the following equation holds:

\[
\begin{bmatrix}
q_1' & q_2' \\
q_1'' & q_2'' \\
Q_1' & 0 \\
0 & Q_2'
\end{bmatrix}
\begin{bmatrix}
A & ab \\
dc & B
\end{bmatrix} =
\begin{bmatrix}
s_1' & s_2' \\
s_1'' & s_2'' \\
S_1' & 0 \\
0 & S_2'
\end{bmatrix}
\]

Clearly \( D' = [Q'|S'] \) is incidence matrix of a bidirected graph. \( \square \)

Let us examine the structure of the bidirected graph \( \Sigma(D') \) so obtained, from the \( k \)-sum operation on tour matrices. From [28] we have that \( \Sigma(D') \) is obtained by gluing \( \Sigma(D_1) \) and \( \Sigma(D_2) \) such that \( u \) and \( u', v \) and \( v' \), \( y \) and \( y' \) become single nodes \( u, v \) and \( y \) respectively, and deleting edges \( e_1, e_2, e_3, f_1, f_2 \) and \( f_3 \) from the unified graph. In other words, this can also be seen as gluing together the \( \Sigma(D_1) \) and \( \Sigma(D_2) \) along the triangles \( (e_1, e_2, e_3) \) and \( (f_1, f_2, f_3) \) so that \( e_1 \) meets \( f_3 \), \( e_2 \) meets \( f_1 \) and \( e_3 \) meets \( f_2 \) and deleting the glued triangle from the unified graph. Obviously, we can say that in \( \Sigma(D') \) the edge \( e_3 \) which was deleted is substituted by the tour associated with \( f_2 \) in \( \Sigma(D_2) \) and that the \( f_3 \) which was deleted is substituted by the tour associated with \( e_1 \) in \( \Sigma(D_1) \). Therefore, now any tour that used \( e_3 \) will instead go through the tour associated with \( f_2 \) giving rise to the non-zero part of \( dc \) in \( K \oplus_3 L \). The tours that went through \( f_3 \) use the tour of \( e_1 \) in the unified graph, as determined by the \( ab \) part of \( K \oplus_3 L \). All other tours remain unchanged, as it is expressed by the fact that if \( c \) or \( b \) had a zero element then \( dc \) or \( ab \) has an all-zero column in the same position.

From Lemmata [24] and [18] and the fact that 1-, and 2-sum operations are special cases of the 3-sum operation we obtain the following theorem:

**Theorem 25.** Totally unimodular tour matrices are closed under \( k \)-sums for \( k = 1, 2, 3 \).

### 4.2.2 Graph Algorithm

We are now ready to present an algorithm which given a totally unimodular matrix \( N \) will construct a bidirected graph \( \Sigma \) or equivalently an incidence matrix, where the columns in \( N \) represent collection of closed tours.

1. Given a TU matrix \( N \), by Seymour’s Theorem [3] we can decompose it via \( k \)-sums into matrices \( N_1, \ldots, N_n \). A separation algorithm for this can be found in the book by Truemper [17].

2. For each matrix \( N_i \) one of the following cases will be true:

   2.1 Check whether \( N_i \) is a network matrix, and if so construct the associated incidence matrix \( D_{\Sigma_i} \). This can be done by the Tutte’s recognition algorithm which results from his decomposition theory for graphic matroids [19] [3].

   2.2 Check whether \( N_i \) is a binet matrix, and if so construct the associated incidence matrix \( D_{\Sigma_i} \). Similarly with step 2.1, a decomposition theory for binary signed graphic matroids given in [2] can be used in this step. Alternatively one can also use the algorithm given in [10].

   2.3 If neither of the above cases is true, then \( N_i \) is the transpose of a network matrix which is not binet. In this case construct the bidirected graph representation given in the proof of Theorem [22].
3 Starting from the incidence matrices $D_{\Sigma_i}$ resulted from step 2 and the $k$-sum decomposition indicated in step 1, compose the incidence matrix of $N$ using the matrix operations so defined in the constructive proofs of Lemmata 8 and 24.

All of the above steps can be performed in polynomial time with respect to the size of the matrix $N$.

The fact that case 2.3 in the above algorithm is possible, that is the existence of a transpose of a network matrix which is not binet, is verified by a recent work of Slilaty [16] where he identifies a set of 29 excluded minors for a cographic matroid to be signed graphic. Examination of the aforementioned excluded minors, reveals that all are a 2- or 3-sum of two binet matrices without positive loops, therefore by Lemma 24 our matrices with a bidirected graph representation without positive loops. However, we were unable to generalise this to an arbitrary non-binet transpose of a network matrix, therefore we use the trivial bidirected graph representation given in the proof of Theorem 22.

5 Concluding Remarks

Totally unimodular matrices characterise a class of well solved integer programming problems, due to the integrality property of the associated polyhedron. In this paper we exploit the decomposition theorem of Seymour for totally unimodular matrices, and provide a graphical representation for every such matrix in a bidirected graph, such that the structural information of the decomposition building blocks is mostly retained. In order to do this, we examine the effect of the $k$-sum operations on network matrices, their transposes and binet matrices, and show that the aforementioned classes of matrices are not closed under these composition operations. A new, more general, class of matrices is introduced called tour matrices, which is proved to be closed under $k$-sums, and it has an associated bidirected graph representation in the sense that the columns of a tour matrix represent a collection of closed tours.

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