OPERATOR-VALUED JACOBI PARAMETERS AND EXAMPLES OF OPERATOR-VALUED DISTRIBUTIONS

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Abstract. We consider \(B\)-valued distributions arising from sequences of Jacobi parameters. In particular, we construct \(B\)-valued free Meixner distributions. These include numerous known examples and one new family, of \(B\)-valued free binomial distributions, for which we are able to compute free convolution powers. Moreover, we develop a convenient combinatorial method for calculating the joint distributions of \(B\)-free random variables with Jacobi parameters, utilizing two-color non-crossing pairings. This leads to several new explicit examples of free convolution computations in the operator-valued setting. Additionally, we obtain a counting algorithm for certain distinguished subsets of the family of two-color non-crossing pairings using only free probabilistic techniques. Finally, we show that the class of distributions with Jacobi parameters is not closed under free convolution.

1. Introduction

Let \(\mu\) denote a probability measure on \(\mathbb{R}\) with finite moments. Then \(\mu\) is associated to two sequences of parameters \(\{\lambda_i, \alpha_i\}_{i \in \mathbb{N}}\) where \(\lambda_i, \alpha_i \in \mathbb{R}\) and \(\alpha_i \geq 0\) for all \(i \in \mathbb{N}\), the so-called Jacobi parameters (see [Ch78] for an overview). The moments of the measure \(\mu\) are calculated from these parameters using sums over Motzkin paths or non-crossing partitions (see [Fl80], [Vi84] and [ABo98]) and \(\mu\) has a moment generating function with continued fraction expansion

\[
M_\mu(z) = \frac{1}{1 - \lambda_0 z - \frac{\alpha_1 z^2}{1 - \lambda_1 - \frac{\alpha_2 z^2}{\cdots}}}.
\]

The study of \(B\)-valued probability was initiated by Voiculescu in [Vo95]. Indeed, let \(B\) denote a unital C\(^*\)-algebra and \(X\) a self-adjoint symbol. We define the non-commutative polynomials to be the algebraic free product of \(B\) and \(X\). Probability measures are replaced by completely positive, \(B\)-bimodular maps

\[\mu : B\langle X \rangle \mapsto B.\]

When provided with appropriate notions of boundedness, \(B\)-valued distributions may be realized in \(B\)-valued probability spaces, which are triples \((A, E, B)\) where \(B \subset A\) is a unital containment of C\(^*\) algebras and \(E : A \mapsto B\) is a conditional expectation. We say that random variables \(X_1, X_2 \in A\) are \(B\)-free if

\[E(P_1(X_{i_1})P_2(X_{i_2})\cdots P_n(X_{i_n})) = 0\]
whenever the $P_i(X) \in \mathcal{B}\langle X \rangle$ satisfy $E(P_j(X_i)) = 0$ for all $j = 1, \ldots, n$ and $i_1 \neq i_2, i_2 \neq i_3, \ldots, i_{n-1} \neq i_n$. If $X_1$ has distribution $\mu$ (that is $\mu(P(X)) = E(P(X_1))$ for all $P(X) \in \mathcal{B}\langle X \rangle$) and $X_2$ has distribution $\nu$ then we call the free convolution of $\mu$ and $\nu$ the distribution of the random variable $X_1 + X_2$. In symbols, this distribution is denoted $\mu \boxplus \nu$.

In this $\mathcal{B}$-valued setting, our notion of Jacobi parameters will be sequences $(\lambda_i, \alpha_i)_{i \in \mathbb{N}}$ where $\lambda_i \in \mathcal{B}$ are symmetric elements and $\alpha_i \in \mathcal{CP}(\mathcal{B})$, the set of completely positive self maps of $\mathcal{B}$.

In Proposition 1 we will show that these sequences of Jacobi parameters generate a $\mathcal{B}$-valued distribution. In Lemma 4 we recover the continued-fraction expansion together with a shift operation. The remainder of section 2 is dedicated to constructing Meixner distributions from Jacobi parameters. This class contains many of the ‘nicer’ distributions such as the semicircular, Bernoulli and free Poison distributions. Moreover, several convolution operations are calculated, culminating in Proposition 21, where the convolution powers of certain $\mathcal{B}$-valued Bernoulli distributions are computed explicitly. This is notable as there are at present very few explicit computations of this convolution operation in the literature.

In section 3 we show that the joint distribution of freely independent Jacobi distributed random variables has a remarkably simple combinatorial structure based the set of 2-color non-crossing pairings $\mathcal{TCNC}(n)$ (see Theorem 23). At least in this form, this formula is also new for the scalar valued case (but see [Mo09a] for a direct relation, in the scalar valued case, between free cumulants and Jacobi parameters). Moreover, in section 4 we consider what is, in some sense, atomic $\mathcal{B}$-valued distributions. In this case, the convolution operation is understood by considering certain subsets $\mathcal{TCNC}_{k,\ell}(n) \subset \mathcal{TCNC}(n)$ where $k$ and $\ell$ refer to a constraint on the types of nesting that can occur in these pairings. Remarkably, we obtain a recursive formula for the size of the sets $\mathcal{TCNC}_{k,\ell}(n)$ using free probabilistic methodology. It would be interesting to obtain a direct combinatorial argument for counting these sets.

Section 5 is concerned with the consequences of these theorems. We produce Example 31 where the simplest possible strictly $\mathcal{B}$-valued convolution operation is performed explicitly. Lastly, Example 32 shows that the convolution of two Bernoulli distributions produces a distribution that does not in general have Jacobi parameters, providing a negative answer to a question posed by Speicher.

2. JACOBI PARAMETERS AND CONTINUED FRACTION EXPANSIONS

We first briefly summarize the notation to be used throughout the section. See, for example, [ABFN13] for more details. Let $\mathcal{B}$ be a unital $C^*$-algebra. Denote $\mathcal{CP}(\mathcal{B})$ the completely positive maps on $\mathcal{B}$, and $\mathcal{B}\langle \mathcal{X} \rangle$ the non-commutative polynomials with coefficients in $\mathcal{B}$.

**Proposition 1.** Let $\lambda_i \in \mathcal{B}$, $i = 0, 1, \ldots$, be symmetric, and $\alpha_i \in \mathcal{CP}(\mathcal{B})$, $i = 1, 2, \ldots$. On the vector space $\mathcal{B}\langle \mathcal{X} \rangle$, define the $\mathcal{B}$-valued inner product

$$\langle b_0\mathcal{X}b_1\mathcal{X} \ldots \mathcal{X}b_n, c_0\mathcal{X}c_1\mathcal{X} \ldots \mathcal{X}c_k \rangle = \delta_{nk}b_n^*\alpha_1[b_{n-1}\alpha_2[\ldots \alpha_n[b_0^*c_0]c_1] \ldots c_{n-1}]c_n,$$

in particular $\langle b, c \rangle = b^*c$. This inner product may be degenerate, but we will only use it to compute moments. On the induced Hilbert bimodule, define operators

$$a^*(b_0\mathcal{X}b_1\mathcal{X} \ldots \mathcal{X}b_n) = \mathcal{X}b_0\mathcal{X}b_1\mathcal{X} \ldots \mathcal{X}b_n,$$
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\[ p(b_0 \mathcal{X} b_1 \mathcal{X} \ldots \mathcal{X} b_n) = \lambda_n b_0 \mathcal{X} b_1 \mathcal{X} \ldots \mathcal{X} b_n, \]
\[ a(b_0 \mathcal{X} b_1 \mathcal{X} \ldots \mathcal{X} b_n) = \alpha_n [b_0] b_1 \mathcal{X} \ldots \mathcal{X} b_n, \]
\[ a(b) = 0, \text{ and} \]
\[ X = a^* + p + a. \]
Then \( p \) and \( a^* + a \), and so also \( X \), are symmetric. Therefore \( \mu : \mathcal{B}(\mathcal{X}) \rightarrow \mathcal{B} \) defined by
\[ \mu[b_0 \mathcal{X} b_1 \mathcal{X} \ldots \mathcal{X} b_n] = \langle b_0 \mathcal{X} b_1 \mathcal{X} \ldots \mathcal{X} b_n, 1 \rangle \]
is a non-commutative distribution, that is, a completely positive \( \mathcal{B} \)-bimodule map \( \mathcal{B}(\mathcal{X}) \rightarrow \mathcal{B} \).
We denote
\[ \mu = J \left( \lambda_0, \lambda_1, \lambda_2, \lambda_3, \ldots \right). \] (1)

**Proof.** Clearly \( p \) is symmetric, and
\[ \langle a^*(b_0 \mathcal{X} b_1 \mathcal{X} \ldots \mathcal{X} b_n), c_0 \mathcal{X} c_1 \mathcal{X} \ldots c_n \mathcal{X} c_{n+1} \rangle = b_n^* a_1 \left[ b_{n-1}^* a_2 \ldots b_0 \alpha_{n+1} [1 c_0] c_1 \ldots c_n \right] c_{n+1} \]
\[ = \langle b_0 \mathcal{X} b_1 \mathcal{X} \ldots \mathcal{X} b_n, \alpha_{n+1} [c_0] c_1 \mathcal{X} \ldots c_n \mathcal{X} c_{n+1} \rangle \]
\[ = \langle b_0 \mathcal{X} b_1 \mathcal{X} \ldots \mathcal{X} b_n, a(c_0 \mathcal{X} c_1 \mathcal{X} \ldots c_n \mathcal{X} c_{n+1}) \rangle, \]
so \( a^* + a \) is also symmetric. Thus, \( X \) is symmetric, and \( \mu \) is its distribution with respect to a vector state.

**Example 2.**
\[ \mu[b_0] = b_0, \]
\[ \mu[b_0 \mathcal{X} b_1] = b_0 \lambda_0 b_1, \]
\[ \mu[b_0 \mathcal{X} b_1 \mathcal{X} b_2] = b_0 \lambda_0 b_1 \lambda_0 b_2 + b_0 \alpha_1 [b_1] b_2. \]

**Notation 3.** Define the moment generating function of \( \mu \),
\[ M_\mu(b) = \mu \left[ \sum_{n=0}^{\infty} (\mathcal{X} b)^n \right], \]
its Cauchy transform
\[ G_\mu(b) = b^{-1} M_\mu(b^{-1}), \]
and its free cumulant generating function implicitly via
\[ M_\mu(b) = 1 + R_\mu(b M_\mu(b)) \]
(compare with Corollary 5.4 in [PV13]).

**Lemma 4.** Denote
\[ \mu_n = J \left( \lambda_{n-1}, \lambda_n, \lambda_{n+1}, \ldots \right). \]
Then in the notation of Lemma 7.2 of [ABFN13],
\[ \mu_n = \mu(\lambda_n, \beta_n), \]
where \( \beta_n = \alpha_n \circ \mu_{n+1} \). Also, the moment generating function of \( \mu \) has a continued fraction expansion
\[ M_\mu(b) = \left( 1 - \lambda_0 b - \alpha_1 [b (1 - \lambda_1 b - \alpha_2 [b \ldots ]) b^{-1}] b \right)^{-1}. \]
More precisely, in the expansions of $M_\mu(b)$ and

$$\left(1 - \lambda_0 b - \alpha_1 \left[ b \left(1 - \lambda_1 b - \alpha_2 \left[ \ldots b \left(1 - \lambda_k b - \alpha_{k+1} b \right)^{-1}\right] b \right)^{-1}\right] b\right)^{-1}.$$ 

the first $n$ terms coincide, for each $k \geq n$.

Proof. For $\mu$ as in equation (1)

$$\mu[b_0 X \ldots X b_n] = \sum_{k=1}^{n} \sum_{1 \leq i_1 < i_2 < \ldots < i_k = n} b_0 \alpha_1 \left[ \mu_2[b_1 X \ldots X i_{i_1-1}] b_1 \alpha_1 \left[ \mu_2[b_{i_1+1} X \ldots X i_{i_2-1}] b_{i_2} \ldots \right] b_{i_{k-1}} \alpha_1 \left[ \mu_2[b_{i_{k-1}+1} X \ldots X b_{n-1}] b_n \right] b \right]$$

and so

$$M_\mu(b) = (1 - \lambda_0 b - \alpha_1 [bM_\mu_2(b)] b)^{-1},$$

The continued fraction expansion follows by induction. \square

Remark 5. Expanding relation (2), we can also write

$$\mu[b_0 X \ldots X b_n] = \sum_{\pi \in NC_{1,2}(n)} T_\pi(b_0, b_1, \ldots, b_n).$$

Here $NC_{1,2}(n)$ are non-crossing partitions of the set of $n$ $X$’s whose blocks are pairs or singletons, and $T_\pi$ is computed as follows. If a single $X$ is a block, it is replaced by a $\lambda$. If a pair of $X$’s form a block, they are replaced by an application of an $\alpha$ to the terms between these $X$’s. In each case, the index of $\lambda$ is the depth of the block in $\pi$, while the index of $\alpha$ is one greater than the depth of the block in $\pi$, with the outer blocks having depth zero. See Remark 3.2 in \cite{ABFN13} for a detailed description of a similar construction.

Relation (3) could also be derived directly from the definition of $\mu$ in Proposition 1.

Corollary 6. Let $\Phi$ be the transformation defined in Definition 6.8 of \cite{ABFN13}. If

$$\mu = J \left( \lambda_0, \lambda_1, \lambda_2, \lambda_3, \ldots \right).$$

then

$$\Phi[\mu] = J \left( 0, \lambda_0, \lambda_1, \lambda_2, \lambda_3, \ldots \right).$$

Proof. By Corollary 7.11 of \cite{ABFN13},

$$\Phi[\nu] = \mu_{(0, \nu)}.$$ 

So the result follows from Lemma 4. \square
Remark 7. Fix $d \in \mathbb{N}$. Define $\tilde{\alpha}_i = I_d \otimes \alpha_i$ to be the map on $M_d(\mathbb{C}) \otimes \mathcal{B} \cong M_d(\mathcal{B})$ and $\tilde{\lambda}_i$ to be a symmetric element $1_d \otimes \lambda_i \in M_d(\mathbb{C}) \otimes \mathcal{B}$. Also define $\tilde{\mu}$ to be the $M_d(\mathbb{C}) \otimes \mathcal{B}$-bimodule map
\[ I_d \otimes \mu : M_d(\mathcal{B}) \langle \mathcal{X} \rangle \to M_d(\mathcal{B}). \]

Then
\[ \tilde{\mu} = J \left( \tilde{\lambda}_0, \tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3, \ldots \right). \]

The proof of this fact (using the relation (3) and matrix units), is identical to that of Proposition 6.3 of Popa-Vinnikov. In the formulas below, we will thus prove the results for $d = 1$ and conclude that they hold for general $d$.

**Proposition 8.** If
\[ \mu = J \left( \lambda_0, \lambda_1, \lambda_2, \lambda_3, \ldots \right), \]
then
\[ \mu \circ \eta = J \left( \eta \lambda_0, \lambda_1, \lambda_2, \lambda_3, \ldots \right). \]

**Proof.** According to Theorem 7.5 of [ABFN13],
\[ \mu_{(\lambda_0, \beta_0)} = \mu(\eta \lambda_0, \eta \beta_0) = \mu(\eta \lambda_0, (\eta \alpha_1) \circ \mu_1). \]

The result follows from Lemma 4. \qed

The remainder of the section treats examples of $\mathcal{B}$-valued distributions which arise from Jacobi parameters.

**Proposition 9.** For $\lambda \in \mathcal{B}$ symmetric, the atomic distribution $\delta_\lambda$ has Jacobi parameter
\[ \mu = J \left( \lambda, 0, 0, 0, \ldots \right). \]

**Proof.** If $\mu$ is the distribution with these Jacobi parameters, then
\[ M_{\tilde{\mu}}(b) = (1 - \tilde{\lambda}b)^{-1}, \]
so that
\[ \mu[\mathcal{X}b_1 \mathcal{X} \ldots \mathcal{X}b_n] = \lambda b_1 \lambda \ldots \lambda b_n \]
and $\mu[P] = P(\lambda)$ for any $P \in \mathcal{B}\langle \mathcal{X} \rangle$. \qed

**Proposition 10.** For $\alpha \in \mathcal{CP}(\mathcal{B})$, the semicircular distribution with covariance $\alpha$ has Jacobi parameters
\[ \mu = J \left( 0, \alpha, 0, \alpha, \alpha, \ldots \right). \]
Proof. If $\mu$ is the distribution with these Jacobi parameters, then

$$M_{\tilde{\mu}}(b) = (1 - \tilde{\alpha}[bM_{\tilde{\mu}}(b)]b)^{-1},$$

or equivalently

$$M_{\tilde{\mu}}(b) = 1 + \tilde{\alpha}[bM_{\tilde{\mu}}(b)]bM_{\tilde{\mu}}(b).$$

In terms of the Cauchy transform, this says

$$bG_{\tilde{\mu}}(b) = 1 + \tilde{\alpha}[G_{\tilde{\mu}}(b)]G_{\tilde{\mu}}(b),$$

which is equation (1.2) from [HRFS07] (with $\eta$ from that paper being our $\alpha$). So $\mu$ is the centered $B$-valued semicircular distribution with covariance $\alpha$. Note also that its free cumulant generating function is $R_\mu(b) = \alpha[b]b$, as it should be. \qed

Example 11. For $\lambda_0, \lambda_1 \in B$ symmetric, and $\alpha \in CP(B)$, a general $B$-valued Bernoulli distribution is defined to have Jacobi parameters

$$\mu = J \left( \begin{array}{cc} \lambda_0, & \lambda_1, & 0, & 0, & 0, & \ldots \\ \alpha, & 0, & 0, & 0, & 0, & \ldots \end{array} \right).$$

More explicitly,

$$M_\mu(b) = (1 - \lambda_0 b - \alpha [b (1 - \lambda_1 b)^{-1} b]^{-1}.$$

The name is justified by two particular cases. First, if all $\lambda_i \equiv 0$, then

$$M_{\tilde{\mu}}(b) = (1 - \tilde{\alpha}[b]b)^{-1}.$$

Comparing this with Corollary 2.2 from [BPV10] (with slightly different notation), we see that $\mu$ is the centered $B$-valued Bernoulli law with covariance $\alpha$. The second particular case is given in the following lemma.

Lemma 12. For $0 < t < 1$ and $a, c \in B^{sa}$, the distribution

$$t\delta_a + (1 - t)\delta_c$$

is of the form in the preceding example, with

$$\lambda_0 = ta + (1 - t)c,$$
$$\lambda_1 = (1 - t)a + tc,$$
$$\alpha[b] = t(1 - t)(a - c)b(a - c).$$
Comparing with Definition 9.3 in [ABFN13] (which extends Definition 4.4.1 in [Spe98]), we thus have

$$(1 - t)\delta_0 + t\delta_\alpha \begin{array}{c} b_0 X \cdots X b_n \end{array} = tb_0 a \cdots a b_n = b_0(t + (1 - t))a \cdots (t + (1 - t))a b_{n-1} tab_n$$

$$= \sum_{k=1}^{n} \sum_{1 \leq i_1 < i_2 < \cdots < i_k = n} b_0 \left( (1 - t)ab_1 (1 - t)ab_2 \cdots ta \right) b_{i_1}$$

$$(1 - t)ab_{i_1+1}(1 - t)ab_{i_1+2} \cdots ta b_{i_2} \cdots (1 - t)ab_{i_{k-1}+1}(1 - t)ab_{i_{k-1}+2} \cdots ta b_n$$

$$= \sum_{k=1}^{n} \sum_{1 \leq i_1 < i_2 < \cdots < i_k = n} b_0 \left( ab_1 (1 - t)ab_2 \cdots t(1 - t)a \right) b_{i_1}$$

$$(ab_{i_1+1}(1 - t)ab_{i_1+2} \cdots t(1 - t)a) b_{i_2} \cdots (ab_{i_{k-1}+1}(1 - t)ab_{i_{k-1}+2} \cdots t(1 - t)a) b_n$$

$$= \sum_{k=1}^{n} \sum_{1 \leq i_1 < i_2 < \cdots < i_k = n} b_0 \alpha[b_1 \lambda_1 b_2 \cdots b_{i_1-1}] b_{i_1}$$

$$\alpha[b_{i_1+1} \lambda_1 b_{i_1+2} \cdots b_{i_2-1}] b_{i_2} \cdots \alpha[b_{i_{k-1}+1} \lambda_1 b_{i_{k-1}+2} \cdots b_{n-1}] b_n.$$ where $\alpha[0] = ta = \lambda_0$, which is precisely formula (2) for $\lambda_0, \lambda_1, \alpha$ as above and $\mu_2 = \delta_\lambda$.

**Proposition 13.** The centered free Poisson distribution with parameters $(\lambda, \alpha)$ has Jacobi parameters

$$\mu = J \left( \begin{array}{c} \lambda_0, \alpha, \lambda_0 + \lambda, \alpha, \lambda_0 + \lambda, \alpha, \ldots \end{array} \right).$$

**Proof.** If $\mu$ is the distribution with these Jacobi parameters, then

$$M_{\tilde{\mu}}(b) = (1 - \alpha \left[b(1 - \lambda b - \alpha[b \cdots b]^{-1}) b \right]^{-1} = (1 - \alpha [b M_{\tilde{\mu}}(b)] b)^{-1},$$

where $\nu$ is a semicircular distribution with mean $\lambda$ and covariance $\alpha$. So

$$M_{\nu}(b) = (1 - \lambda b - \alpha [b M_{\tilde{\mu}}(b)] b)^{-1}.$$ (4)

Combining equations (4) and (5), we get

$$M_{\nu}(b) = M_{\mu}(b)(1 - \lambda b M_{\mu}(b))^{-1},$$

so

$$M_{\mu}(b) = (1 - \alpha [b M_{\mu}(b)(1 - \lambda b M_{\mu}(b))^{-1}] b)^{-1}.$$

Thus

$$M_{\mu}(b) = 1 + \alpha [b M_{\mu}(b)(1 - \lambda b M_{\mu}(b))^{-1}] b M_{\mu}(b).$$

So $R_{\mu}(b) = \alpha[b(1 - \lambda b)^{-1}]b$ and by applying the arguments above to $\tilde{\mu}$ as in Remark 7,

$$R_{\tilde{\mu}}[b_1 X b_2 X \cdots X b_{n-1}] = \alpha[b_1 b_2 \lambda \cdots b_{n-1} \lambda].$$

Comparing with Definition 9.3 in [ABFN13] (which extends Definition 4.4.1 in [Spe98]), we see that $\mu$ is the $\mathcal{B}$-valued free Poisson distribution with parameters $(\lambda, \alpha)$. □

The following result is an immediate consequence of Proposition 17.
Corollary 14. Let $\mu_N$ be a Bernoulli distribution

$$
\mu_N = J\left(\frac{\lambda_0}{N} + o\left(\frac{1}{N}\right), \quad \frac{\lambda_0 + \lambda_1}{N} + o(1), \quad 0, \quad 0, \quad \ldots\right).
$$

Then $\mu_N^{\otimes N} \to \nu$, where $\nu$ is a free Poisson distribution

$$
\nu = J\left(\lambda_0, \quad \lambda_0 + \lambda_1, \quad \lambda_0 + \lambda_1, \quad \lambda_0 + \lambda_1, \quad \ldots\right).
$$

Note that Theorem 4.4.3 in [Spe98] proves the usual compound Poisson limit theorem, which implies a particular case of the above with $\lambda_0 = ta$, $\lambda_1 = a$ and $\alpha[b] = taba$.

Remark 15. For general (not necessarily symmetric) $\{\lambda_i\}$ and general (not necessarily positive) $\{\alpha_i\}$, we may still define

$$
\mu = J\left(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \ldots\right).
$$

via the combinatorial formula in Remark 3. This $\mu$ is now only an algebraic non-commutative distribution. Then numerous results above, notably Corollary 6, still hold. We may also define $B$-valued semicircular, free Poisson etc. distributions with such more general Jacobi parameters.

Example 16. Let $\lambda \in B$ be symmetric, and $\alpha$ a map on $B$ such that $I + \alpha \in \mathcal{CP}(B)$. The normalized free Meixner distribution with parameters $(\lambda, \alpha)$ is

$$
\mu = \Phi[\gamma_{\lambda, I + \alpha}],
$$

where $\gamma_{\lambda, \alpha}$ is the semicircular distribution with mean $\lambda$ and variance $\alpha$. In other words,

$$
\mu = J\left(0, \lambda, \lambda, \lambda, \ldots\right).
$$

Note that $\mu$ is positive even if $\alpha$ is not. For $\lambda \in B$ symmetric, $\eta \in \mathcal{CP}(B)$, and $\alpha$ such that $\eta + \alpha \in \mathcal{CP}(B)$, general (centered) free Meixner distributions are distributions with Jacobi parameters

$$
fM(\lambda, \alpha; \eta) = J\left(0, \lambda, \lambda, \lambda, \ldots\right).
$$

Note that $fM(0, 0; \eta)$ are the semicircular distributions; $fM(\lambda, 0; \eta)$ the free Poisson distributions; and $fM(\lambda, -\eta; \eta)$ the Bernoulli distributions.

Proposition 17. For fixed $\lambda, \alpha$, free Meixner distributions form a free convolution semigroup with respect to parameter $\eta$: whenever $\alpha + \eta, \alpha + \eta \in \mathcal{CP}(B)$,

$$
fM(\lambda, \alpha; \eta_1) \boxplus fM(\lambda, \alpha; \eta_2) = fM(\lambda, \alpha; \eta_1 + \eta_2)
$$

and if $I + \alpha \in \mathcal{CP}(B)$, then $fM(\lambda, \alpha; \eta) = fM(\lambda, \alpha; I) \boxplus \eta$. It also follows that for such $\alpha$ and the transformation $B_\eta$ (see below),

$$
B_\eta[fM(\lambda, \alpha, I)] = fM(\lambda, \eta + \alpha; I)
$$
Proof. Recall the following notation and results from Section 6 of [ABFN13]. For any linear map \( \alpha : \mathcal{B} \to \mathcal{B} \), one defined a transformation \( \mathbb{B}_\alpha \) on distributions, which satisfies
\[
(\mathbb{B}_\alpha[\mu])^{\oplus(I+\alpha)} = \mu^{\oplus(I+\alpha)}.
\]
For such \( \alpha \) and a symmetric \( \lambda \in \mathcal{B} \), we can define an (algebraic, not necessarily positive) semicircular distribution \( \gamma_{\lambda,\alpha} \) with mean \( \lambda \) and variance \( \alpha \). Finally, recall that for algebraic distributions,
\[
\mathbb{B}_\eta[\Phi[\mu]] = \Phi[\mu \boxplus \gamma_{0,\eta}].
\]
In particular, for the free Meixner distributions,
\[
\Phi[\gamma_{\lambda,\alpha}]^{\oplus\eta} = (\mathbb{B}_{\eta-1}[\Phi[\gamma_{\lambda,1+\alpha}]])^{\oplus\eta} = (\Phi[\gamma_{\lambda,1+\alpha} \boxplus \gamma_{0,\eta-1}])^{\oplus\eta} = (\Phi[\gamma_{\lambda,\alpha+\eta}])^{\oplus\eta}.
\]
In other words, if
\[
\mu = J \begin{pmatrix} 0, & \lambda, & \lambda, & \lambda, & \ldots \\ I, & I + \alpha, & I + \alpha, & I + \alpha, & \ldots \end{pmatrix},
\]
then
\[
\mu^{\oplus\eta} = J \begin{pmatrix} 0, & \lambda, & \lambda, & \lambda, & \ldots \\ \eta, & \eta + \alpha, & \eta + \alpha, & \eta + \alpha, & \ldots \end{pmatrix}
\]
whenever \( \eta, \eta + \alpha \in \mathcal{CP}(\mathcal{B}) \). In particular, these distributions form a semigroup with respect to \( \eta \). \( \square \)

**Proposition 18.** If \( \mu \) is a free normalized Meixner distribution \( fM(\lambda, \alpha; I) \), then
\[
b^{-1}R_\mu(b)b^{-1} = 1 + \lambda R_\mu(b)b^{-1} + \alpha[R_\mu(b)b^{-1}]R_\mu(b)b^{-1}.
\]

**Proof.** If
\[
\mu = J \begin{pmatrix} 0, & \lambda, & \lambda, & \lambda, & \ldots \\ I, & I + \alpha, & I + \alpha, & I + \alpha, & \ldots \end{pmatrix},
\]
then
\[
M_\mu(b) = (1 - bM_\nu(b)b^{-1})^{-1},
\]
where \( \nu \) is a semicircular distribution with mean \( \lambda \) and covariance \( I + \alpha \). So
\[
M_\nu(b) = 1 + \lambda bM_\nu(b) + (I + \alpha)[bM_\nu(b)]bM_\nu(b)
\]
and
\[
M_\mu(b)^{-1}M_\nu(b) = M_\nu(b) - bM_\nu(b)bM_\nu(b),
\]
thus
\[
M_\mu(b)^{-1}M_\nu(b) = 1 + \lambda bM_\nu(b) + \alpha[bM_\nu(b)]bM_\nu(b).
\]
Now using
\[
bM_\nu(b) = (1 - M_\mu(b)^{-1})b^{-1} = (M_\mu(b) - 1)(bM_\mu(b))^{-1},
\]
we get
\[
M_\mu(b)^{-1}b^{-1}(1 - M_\mu(b)^{-1})b^{-1} = 1 + \lambda(1 - M_\mu(b)^{-1})b^{-1} + \alpha[(1 - M_\mu(b)^{-1})b^{-1}](1 - M_\mu(b)^{-1})b^{-1},
\]
or equivalently
\[
(bM_\mu(b))^{-1}(M_\mu(b) - 1)(bM_\mu(b))^{-1} = 1 + \lambda(M_\mu(b) - 1)(bM_\mu(b))^{-1} + \alpha[(M_\mu(b) - 1)(bM_\mu(b))^{-1}](M_\mu(b) - 1)(bM_\mu(b))^{-1}.
\]
Thus
\[ b^{-1}R_\mu(b)b^{-1} = 1 + \lambda R_\mu(b)b^{-1} + \alpha[R_\mu(b)b^{-1}]R_\mu(b)b^{-1}. \]
\[ \square \]

**Example 19.** If
\[ \mu = J\left(\begin{array}{cccc}0, & 0, & 0, & \ldots \\2\alpha, & \alpha, & \alpha, & \ldots \end{array}\right), \]
in other words \( \mu = fM(0, -\alpha; 2\alpha) \), then it is natural to call \( \mu \) the \( B \)-valued arcsine distribution. Indeed, the Boolean cumulant transform
\[ B_\mu(b) = 1 - (M_\mu(b))^{-1} \]
is in this case
\[ B_\mu(b) = 2\alpha [M_\nu(b)] \]
where \( \nu \) is the centered semicircular distribution with variance \( \alpha \). Since for the semicircular distribution, \( B_\nu(b) = \alpha [M_\nu(b)] \), it follows that
\[ \mu = \nu \oplus 2\alpha. \]

On the other hand, if \( \alpha \) is invertible,
\[ \frac{1}{2} \alpha^{-1}[R_\mu(b)b^{-1}] = b - \frac{1}{4} bR_\mu(b)b^{-1}\alpha^{-1}[R_\mu(b)b^{-1}], \]
in other words
\[ \alpha^{-1}\left[\frac{1}{2} R_\mu(b)b^{-1}\right] = b - \frac{1}{2} R_\mu(b)b^{-1}\alpha^{-1}\left[\frac{1}{2} R_\mu(b)b^{-1}\right]. \]

Thus \( \mu^{\square(1/2)} \) is the centered Bernoulli distribution \( \rho \) with covariance \( \alpha \), and
\[ \mu = \rho \oplus 2\alpha. \]

It follows from Theorem 3.2 in [BPV10] that in the case when \( \alpha[b] = aba \) for some \( a \), this arcsine law is the same as in that paper, and in particular appears as the limit law in the monotone central limit theorem.

**Example 20.** If
\[ \mu = J\left(\begin{array}{cccc}0, & 0, & 0, & \ldots \\\eta, & \eta - \alpha, & \eta - \alpha, & \ldots \end{array}\right), \]
for \( \eta, \eta - \alpha \in \mathcal{CP}(\mathcal{B}) \), the distributions are the \( B \)-valued free binomial distributions (including Bernoulli for \( \eta = \alpha \)). They form a free convolution semigroup with respect to the parameter \( \eta \geq \alpha \). Using Remark 5, their moments are given explicitly as sums over non-crossing pairings, where the inner blocks have only one color \( \eta - \alpha \) and outer blocks have two colors \( \eta - \alpha \) and \( \alpha \) (note we are not assuming that \( \alpha \in \mathcal{CP}(\mathcal{B}) \)).

The following proposition computes explicitly the moments of free binomial distributions, arising as free convolution powers of (the distribution of) a special operator \( a \). See section 5 for a concrete example of such \( a \).

**Proposition 21.** Let \((\mathcal{A}, \mathcal{B}, E)\) be a n.c. probability space. Let \( a \in \mathcal{A} \) be such that \( E[a] = 0 \) and \( a\mathcal{B}a \subset \mathcal{B} \), and denote \( \alpha(b) = aba \). Then
\begin{enumerate}
  \item \( a \) has a Bernoulli distribution with parameter \( \alpha \).
  \item the map \( \alpha \) satisfies the special property that \( T_\pi \) (in the sense of equation (3)) does not depend on \( \pi \).
\end{enumerate}
(c) For \( t \geq 1 \), the odd moments of \( \mu_a^{\text{er}} \) are zero, and the even ones are

\[
\mu_a^{\text{er}}[b_0X \ldots Xb_n] = m_n(t)ab_0ab_1 \ldots ab_na.
\]

Here \( m_n(2) = \binom{2n}{n} \), and in general for \( n > 0 \),

\[
m_n(t) = t^{2n} - \frac{t}{2} \sum_{k=1}^{n} \frac{1}{2k-1} \binom{2k}{k} (t-1)^k t^{2(n-k)}.
\]

**Proof.** For part (a), we verify that for \( n \) even,

\[
E[b_0ab_1 \ldots ab_n] = b_0\alpha(b_1)b_2\alpha(b_3) \ldots \alpha(b_{n-1})b_n
\]

and for \( n \) odd,

\[
E[b_0ab_1 \ldots ab_n] = b_0\alpha(b_1)b_2\alpha(b_3) \ldots \alpha(b_{n-1})b_n E[a]b_n = 0.
\]

For part (b), it suffices to note that

\[
\alpha(b_0\alpha(b_1)b_2\alpha(b_3) \ldots \alpha(b_{n-1})b_n) = ab_0ab_1 \ldots ab_na = \alpha(b_0)\alpha(b_1)b_3 \ldots b_{n-1}\alpha(b_n).
\]

For part (c), we note that for even \( n \),

\[
\mu_a^{\text{er}} = J\begin{pmatrix} 0, & 0, & 0, & 0, & \ldots \\ t\alpha, & (t-1)\alpha, & (t-1)\alpha, & (t-1)\alpha, & \ldots \end{pmatrix},
\]

and so for even \( n \)

\[
\mu_a^{\text{er}}[b_0X \ldots Xb_n] = \sum_{\pi \in NC_{1,2}(n)} T_{\pi}(b_0, b_1, \ldots, b_n)
\]

\[
= \sum_{\pi \in NC_{1,2}(n)} t^{\text{Out}(\pi)}(t-1)^{\text{Inn}(\pi)} ab_0ab_1 \ldots ab_na = m_n(t)ab_0ab_1 \ldots ab_na,
\]

where \( m_n(t) \) is the \( n \)'th moment of the scalar-valued free binomial distribution with parameter \( t \). If \( t = 2 \), it is the arcsine distribution, and

\[
m_n(2) = \binom{2n}{n}.
\]

In general,

\[
\sum_{n=0}^{\infty} m_n(t)z^n = \frac{t-2-t\sqrt{1-4(t-1)z^2}}{2(t^2z^2 - 1)}.
\]

\[ \square \]

**3. Joint Distributions of B-free Jacobi-Distributed Random Variables.**

In this section, we provide a combinatorial description of the joint moments of freely independent distributions arising from symmetric Jacobi parameters.

Define the **two-color non-crossing pairings of size** \( 2n \) (in symbols, \( TCNC(n) \)) as the set of pairings of the set \( \{1, 2, \ldots, 2n\} \) where each of the pairs are also assigned one of two colors (red and blue, respectively).
Setting notation, let $\pi \in NC(n)$, $P(X) \in B(X)$ and let $\mu$ denote a symmetric Jacobi distribution with parameters $\{\alpha_i\}_{i=1}^{\infty}$. We define $E_\pi(P(X))$ to be the moment associated to the partition $\pi$. That is, for $\pi = \{(1, 4), (2, 3)\}$ and $\pi' = \{(1, 2), (3, 4)\}$ then
\[
E_\pi(Xb_1Xb_2Xb_3X) = \alpha_1(b_1\alpha_2(b_2)b_3), \quad E_{\pi'}(Xb_1Xb_2Xb_3X) = \alpha_1(b_1)b_2\alpha_1(b_3).
\]

Let $\pi \in TCNC(2n)$. Consider two symmetric Jacobi distributions with parameters $\{\alpha_i\}_{i=1}^{\infty}$ and $\{\beta_j\}_{j=1}^{\infty}$. Then $E_{\pi}$ is the moment calculated according to this partition where blue pairs are associated to the first Jacobi parameter and red the second. Thus, for $\pi = \{(1, 6)(2, 5)\}_b \cup \{(3, 4)\}_r$ and $\pi' = \{(2, 5)\}_b \cup \{(1, 6), (3, 4)\}_r$ (the $b$ and the $r$ assign the color), then
\[
E_{\pi}(Xb_1Xb_2Xb_3Xb_4Xb_5X) = \alpha_1(b_1\alpha_2(b_2\beta_1(b_3)b_4)b_5)
\]
\[
E_{\pi'}(Xb_1Xb_2Xb_3Xb_4Xb_5X) = \beta_1(b_1\alpha_2(b_2\beta_1(b_3)b_4)b_5).
\]

Crucially, nesting inside a pair of the opposite color implies that the algorithm for applying the automorphisms resets itself (that is, with partition $\pi'$, $\beta_1$ is applied to $b_3$ as opposed to $\beta_2$).

**Lemma 22.** Consider a sequence of $2n$ pairs $(1, i_1), (2, i_2), \ldots, (2n, i_{2n})$ where $i_j \in \{1, 2\}$ for $j = 1, 2, \ldots, 2n$. Assume that $i_k_{p} = i_{k_{p+1}} = \ldots = i_{k_{p+2\ell_p-1}}$ for some $k_p, \ell_p > 0$, $p = 1, \ldots, m$ and let $\ell = \ell_1 + \ldots + \ell_m$. Consider the set of partitions $\pi \in TCNC(n)$ whose coloring assigns blue (resp. red) to those $i_j = 1$ (resp. 2) with the additional property that $\pi$ fixes each of the intervals $[k_p, k_p + 2\ell_p - 1]$. There is a bijection between these pairings $\pi$ and
\[
TCNC(n - \ell) \times NC(\ell_1) \times NC(\ell_2) \times \cdots \times NC(\ell_m)
\]
where this bijection is induced by the following observations:
\[
\pi|[k_p, k_p + 2\ell_p - 1]
\]
is an element of $NC(\ell_p)$ (the non-crossing pairings of $2\ell_p$ elements) and
\[
\pi|[1, 2, \ldots, 2n] \setminus (\cup_{p=1}^{m}[k_p, k_p + 2\ell_p - 1])
\]
is an element of $TCNC(n - \ell)$ whose coloring respects the indices.

**Proof.** By assumption, $\pi|[k_p, k_p + 2\ell_p - 1]$ is a pairing. If there are two pairs $(a_1, b_1), (a_2, b_2)$ in the interval $[k_p, k_p + 2\ell_p - 1]$ such that $a_1 < a_2 < b_1 < b_2$, then this inequality will carry over to the full set and produce a crossing for $\pi$. Thus, $\pi|[k_p, k_p + 2\ell_p - 1]$ is an element of $NC(\ell_p)$. The restriction $\pi|[1, 2, \ldots, 2n] \setminus (\cup_{p=1}^{m}[k_p, k_p + 2\ell_p - 1])$ obviously respects coloring and, by the same reasoning as above, is a non-crossing pairing.

For the converse, pick elements $\pi' \in TCNC(n - \ell)$ and $\pi_p \in NC(\ell_p)$ for $p = 1, \ldots, m$. It is clear from a geometric standpoint that a pairing $\pi$ such that $\pi|[1, 2, \ldots, 2n] \setminus (\cup_{p=1}^{m}[k_p, k_p + 2\ell_p - 1]) = \pi'$ and $\pi|[k_p, k_p + 2\ell_p - 1] = \pi_p$ is non-crossing. By matching the colors, the result follows.

Before tackling the main theorem below, we give an intuitive explanation for the result. Consider symmetric Jacobi random variables $X_1$ and $X_2$ with parameters $\{\alpha_i\}_{i=1}^{\infty}$ and $\{\beta_i\}_{i=1}^{\infty}$. Consider the expectation of $X_1b_1X_1b_2X_2b_3X_2b_4X_1b_5X_1b_6X_2b_7X_2$. According to Theorem 23 below, this moment should be equal to a sum of elements
\[
E_{\pi}(X_1b_1X_1b_2X_2b_3X_2b_4X_1b_5X_1b_6X_2b_7X_2)
\]
where $\pi \in \text{TCNC}(4)$ is such that the blue colors are assigned to the $X_1$'s and the red to the $X_2$'s. These correspond to the following partitions, labeled A, B and C from left to right:

\[
\begin{array}{c|c|c}
\text{A} & \text{B} & \text{C} \\
\hline
\begin{array}{cc}
\begin{array}{c}
\ \ \ \ \ \ \ \ \ \ \ \ \ \ \\
\ \ \ \ \ \ \ \ \ \ \ \ \ \\
\end{array}
&
\begin{array}{c}
\ \ \ \ \ \ \ \ \ \ \ \ \\
\ \ \ \ \ \ \ \ \ \ \\
\end{array}
&
\begin{array}{c}
\ \ \ \ \ \ \ \ \ \ \ \ \\
\ \ \ \ \ \ \ \ \ \ \\
\end{array}
\end{array}
\end{array}
\]

which produces the expectation

\[
\alpha_1(b_1)b_2\beta_1(b_3)b_4\alpha_1(b_5)b_6\beta_1(b_7) + \alpha_1(b_1)b_2\beta_1(b_3)b_4\alpha_1(b_5)b_6\beta_1(b_7) + \alpha_1(b_1)b_2\beta_1(b_3)b_4\alpha_1(b_5)b_6\beta_1(b_7).
\]

Approaching this same calculation through polarization, observe that if we define

\[
X_1b_jX_1 - \alpha_1(b_j) = (X_1b_jX_1)^{(0)}, \quad X_2b_jX_2 - \beta_1(b_j) = (X_2b_jX_2)^{(0)}
\]

\[
\alpha_1(b_j) = (X_1b_jX_1)^{(1)}, \quad \beta_1(b_j) = (X_2b_jX_2)^{(1)}
\]

we have the following equality, corresponding to equation (15) in our main proof,

\[
E(X_1b_1X_1b_2X_2b_3X_2b_4X_1b_5X_1b_6X_2b_7X_2)
= \sum_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4=0} E((X_1b_1X_1)^{(\epsilon_1)}b_2(X_2b_3X_2)^{(\epsilon_2)}b_4(X_1b_5X_1)^{(\epsilon_3)}b_6(X_2b_7X_2)^{(\epsilon_4)}).
\]

All but three of these terms disappear by freeness, namely

\[
E((X_1b_1X_1)^{(1)}b_2(X_2b_3X_2)^{(1)}b_4(X_1b_5X_1)^{(1)}b_6(X_2b_7X_2)^{(1)})
\]

\[
E((X_1b_1X_1)^{(0)}b_2(X_2b_3X_2)^{(1)}b_4(X_1b_5X_1)^{(0)}b_6(X_2b_7X_2)^{(1)})
\]

\[
E((X_1b_1X_1)^{(1)}b_2(X_2b_3X_2)^{(0)}b_4(X_1b_5X_1)^{(1)}b_6(X_2b_7X_2)^{(0)})
\]

and these produce the terms corresponding to the partitions where (11) is coupled with partition A, (12) with partition B and (13) with partition C in figure 8.

The bijection is somewhat more subtle than this simple example may suggest. Generally, the bijection matches the polarized terms with those $\pi$ that fix exactly those blocks with $\epsilon_i = 1$ and only those blocks (this will be the main claim in our proof, equation (17)).

To see that this produces the necessary expectation, we distribute (13),

\[
E(X_1b_1X_1b_2\beta_1(b_3)b_4X_1b_5X_1b_6\beta_1(b_7)) - E(\alpha_1(b_1)b_2\beta_1(b_3)b_4X_1b_5X_1b_6\beta_1(b_7))
\]

\[
- E(X_1b_1X_1b_2\beta_1(b_3)b_4\alpha_1(b_5)b_6\beta_1(b_7)) + E(\alpha_1(b_1)b_2\beta_1(b_3)b_4\alpha_1(b_5)b_6\beta_1(b_7))
\]

(this step is akin to equation (20) in the main proof). Now, by Remark 5 the first term of (14) is a sum

\[
\alpha_1(b_1)b_2\beta_1(b_3)b_4\alpha_1(b_5)b_6\beta_1(b_7) + \alpha_1(b_1)b_2\beta_1(b_3)b_4\alpha_1(b_5)b_6\beta_1(b_7)
\]

and the next three terms are, up to sign, all copies of

\[
\alpha_1(b_1)b_2\beta_1(b_3)b_4\alpha_1(b_5)b_6\beta_1(b_7).
\]

They cancel so that (14) is equal to

\[
\alpha_1(b_1)b_2\beta_1(b_3)b_4\alpha_1(b_5)b_6\beta_1(b_7).
\]
This cancellation step occurs in general and corresponds to the paragraph following equation (25) in the main proof.

**Theorem 23.** Let $X_1$ and $X_2$ denote $\mathcal{B}$-free random variables with symmetric Jacobi distribution. Then

$$E(X_{\epsilon_1} b_1 X_{\epsilon_2} \cdots b_{2n-1} X_{\epsilon_{2n}})$$

is equal to the sum of the terms

$$E_\pi(X_{\epsilon_1} b_1 X_{\epsilon_2} \cdots b_{2n-1} X_{\epsilon_{2n}})$$

where $\pi \in \mathcal{TCNC}(n)$, $\epsilon_i \in \{1, 2\}$ for $i = 1, 2, \ldots, 2n$ and the partition $\pi$ is such that all blue pairs are associated to pairs of $X_1$’s and all red pairs are associated to pairs of $X_2$’s.

**Proof.** Setting notation, we consider a family of monomials $P_i(X) \in \mathcal{B}(X)$ for $i = 1, \ldots, n$. We prove our theorem for

$$E(P_1(X) P_2(X) \cdots P_n(X))$$

where $i_j \in \{1, 2\}$ and $i_1 \neq i_2, i_2 \neq i_3, \ldots, i_{n-1} \neq i_n$. We refer to the monomial $P_j(X_{i_j})$ as the $j$th block. Let $d_j = \deg(P_j)$ and $d = \sum_{j=1}^n d_j$. We say that $\pi \in \mathcal{TCNC}(d/2)$ fixes the $j$th block if the elements in the interval

$$[d_1 + d_2 + \cdots + d_{j-1} + 1, d_1 + d_2 + \cdots + d_{j-1} + d_j]$$

are all paired with one another.

Proceeding by induction on the number of blocks, the case $n = 1$ is simply the computational algorithm for Jacobi distributed random variables in Remark 5 since only one color will be permitted and it will therefore collapse to $\mathcal{NC}(\deg(P_1)/2)$. Thus, we assume that the theorem holds for all $k < n$.

Polarizing our monomial, we have that

$$E(P_1(X_{i_1}) P_2(X_{i_2}) \cdots P_n(X_{i_n})) = \sum_{\epsilon_1, \ldots, \epsilon_n = 0} E(P_1(X_{i_1})^{(\epsilon_1)} P_2(X_{i_2})^{(\epsilon_2)} \cdots P_n(X_{i_n})^{(\epsilon_n)})$$  \hspace{1cm} (15)

where

$$P_j(X_{i_j})^{(0)} = P_j(X_{i_j}) - E(P_j(X_{i_j})) \quad P_j(X_{i_j})^{(1)} = E(P_j(X_{i_j}))$$

We isolate a single term

$$E(P_1(X_{i_1})^{(\epsilon_1)} P_2(X_{i_2})^{(\epsilon_2)} \cdots P_n(X_{i_n})^{(\epsilon_n)})$$  \hspace{1cm} (16)

The main idea for the proof is the claim that (16) is equal to the sum of the terms

$$E_\pi(P_1(X_{i_1}) P_2(X_{i_2}) \cdots P_n(X_{i_n}))$$  \hspace{1cm} (17)

where we sum over those partitions $\pi \in \mathcal{NC}(d)$ that fix exactly those blocks $P_j(X_{i_j})$ with $\epsilon_j = 1$ and no others. As a base case, if all of the $\epsilon_j \equiv 0$ then this term disappears by freeness. This squares with our main claim since any two-color non-crossing pairing must necessarily fix at least one block so that, according to our claim, this term should have no contribution.

Let $r_1, r_2, \ldots, r_k$ denote the indices with $\epsilon_{r_j} = 0$ in increasing order. Note that the first $r_1$ blocks of the polynomial

$$P_1(X_{i_1})^{(\epsilon_1)} P_2(X_{i_2})^{(\epsilon_2)} \cdots P_n(X_{i_n})^{(\epsilon_n)}$$
where \( \pi \in \text{TCNC} \) and we rewrite this as
\[
\pi \left[ P_{r_1} (X_{i_{r_1}}) - E(P_{r_1} (X_{i_{r_1}})) \right]
\]
where \( b_1 \in \mathcal{B} \). Continuing, we rewrite (16) as
\[
E(b_1 P_{r_1} (X_{i_{r_1}})^{(0)} b_2 \cdots P_{r_{k-1}} (X_{i_{r_{k-1}}})^{(0)} b_k P_{r_k} (X_{i_{r_k}})^{(0)} b_{k+1})
\]
Distributing each of the terms \( P_{r_j} (X_{i_{r_j}})^{(0)} \) for \( i = j, \ldots, k \), we define
\[
P_{r_j} (X_{i_{r_j}}) = P_{r_j} (X_{i_{r_j}}) ; \quad P_{r_j} (X_{i_{r_j}})^\dagger = -E(P_{r_j} (X_{i_{r_j}}))
\]
so that
\[
E(P_1 (X_{i_1})^t \cdots P_n (X_{i_n})^t)
\]
\[
\sum_{\delta_1, \ldots, \delta_k \in \{*, \dagger\}} E(b_1 P_{r_1} (X_{i_{r_1}})^{\delta_1} b_2 \cdots P_{r_{k-1}} (X_{i_{r_{k-1}}})^{\delta_{k-1}} b_k P_{r_k} (X_{i_{r_k}})^{\delta_k} b_{k+1})
\]
We focus on a single term,
\[
E(b_1 P_{r_1} (X_{i_{r_1}})^{\delta_1} b_2 \cdots P_{r_{k-1}} (X_{i_{r_{k-1}}})^{\delta_{k-1}} b_k P_{r_k} (X_{i_{r_k}})^{\delta_k} b_{k+1})
\]
The key observation is as follows. Assume that \( r_{j_1}, \ldots, r_{j_m} \) are such that \( \delta_{r_{j_i}} = \dagger \). Let
\[
p = \sum_{i=1}^{k} \deg (P_{r_i}) ; \quad d = \sum_{\{\delta_i = \dagger\}} \deg (P_{r_i}).
\]
By Remark 5 we have that
\[
P_{r_j} (X_{i_{r_j}})^\dagger = -E(P_{r_j} (X_{i_{r_j}})) = - \sum_{\pi \in \mathcal{N}\mathcal{C} (\deg(P_{r_j})/2)} E_\pi (P_{r_j} (X_{i_{r_j}}))
\]
(and if \( \deg(P_{r_j}) \) is odd, (22) is simply 0). Moreover, the inductive hypothesis tells us that (21) is equal to the sum
\[
E_\pi (b_1 P_{r_1} (X_{i_{r_1}})^{\delta_1} b_2 \cdots P_{r_{k-1}} (X_{i_{r_{k-1}}})^{\delta_{k-1}} b_k P_{r_k} (X_{i_{r_k}})^{\delta_k} b_{k+1})
\]
over all \( \pi \in \mathcal{N}\mathcal{CNC}(d/2) \) who are only pairing the elements with \( \delta_i = \ast \) (since those terms with \( \delta_i = \dagger \) are elements of \( \mathcal{B} \)). If we combine observations (22) and (23), we have precisely the statement of Lemma 22. Thus, we conclude that (21) is equal to the sum of the terms
\[
E_\pi (b_1 P_{r_1} (X_{i_{r_1}}) b_2 \cdots P_{r_{k-1}} (X_{i_{r_{k-1}}}) b_k P_{r_k} (X_{i_{r_k}}) b_{k+1})
\]
where we sum over all those \( \pi \in \mathcal{N}\mathcal{CTC}(p/2) \) where \( \pi \) fixes those blocks \( r_{j_1}, \ldots, r_{j_m} \) such that \( \delta_{r_{j_i}} = \dagger \). As we saw in (22), this set is in bijection with
\[
\pi \in \mathcal{N}\mathcal{CTC} \left( \frac{p-d}{2} \right) \times \mathcal{N}\mathcal{C}(d_1) \times \cdots \times \mathcal{N}\mathcal{C}(d_m)
\]
where \( \pi \) respects the coloring in the obvious sense.

Returning to (20), we partition the right hand side the equation into \( k + 1 \) summands, \( \Omega_0, \Omega_1, \ldots, \Omega_k \) where \( \Omega_i \) corresponds to those terms with exactly \( i \) of the \( \delta_j = \dagger \). Let \( \pi \in \mathcal{TCNC}(p/2) \) fix \( m > 0 \) blocks in the following sense: the partition \( \pi \) fixes \( m \) of the
polynomials $P_{r_j}$ in the equation \([24]\) \((\text{the subtlety in this point is that, since we have collapsed the intermediate terms to the } b_i, \text{ the indices are no longer alternating})\). As we just saw in \([24]\), up to sign, there is a copy of

$$E_\pi(b_1 P_{r_1}(X_{i_{r_1}}) b_2 \cdots P_{r_{k-1}}(X_{i_{r_{k-1}}}) b_k P_{r_k}(X_{i_{r_k}}) b_{k+1})$$ \(\text{(25)}\)

inside of $\Omega_m$ arising from a term such as \([21]\) where the $\delta_j = \dagger$ correspond to the fixed blocks of $\pi$. Moreover, there is a canonical containment arising in Lemma \([22]\)

$$\mathcal{NCTC}\left(\frac{p-d}{2}\right) \times \bigcap_{i=1}^m \mathcal{NC}(d_i) \subset \mathcal{NCTC}\left(\frac{p-d + d_j}{2}\right) \times \bigcap_{i \neq j} \mathcal{NC}(d_i)$$

for each $j = 1, \ldots, m$. Thus, there are \(\binom{m}{1}\) copies of \([25]\) in $\Omega_{m-1}$. In this manner, $\Omega_j$ contributes \(\binom{m}{m-j}\) copies of \([25]\) to the summand. However, the contribution from the $\Omega_j$ alternate in sign so that, if $m > 0$, the total contribution of this partition with $m$ fixed polynomials is equal to

$$\sum_{j=0}^m (-1)^j \binom{m}{m-j} E_\pi(b_1 P_{r_1}(X_{i_{r_1}}) b_2 \cdots P_{r_{k-1}}(X_{i_{r_{k-1}}}) b_k P_{r_k}(X_{i_{r_k}}) b_{k+1}) = 0$$

since $(1 - 1)^m = 0$. Thus, if there are any additional fixed polynomials (beyond those where $\epsilon_j = 1$), the partition $\pi$ makes no contribution. By our induction hypothesis, the claim in \([17]\) holds.

Our theorem follows immediately. Indeed, the right hand side of \([15]\) turns into a sum of the terms

$$E_\pi(P_1(X_{i_1}) P_2(X_{i_2}) \cdots P_n(X_{i_n}))$$

with $\pi \in \mathcal{TNC}(d)$ where the assignment of values $\epsilon_i$ on the right hand side of \([15]\) corresponds exactly to the fixed blocks of $\pi$. The set $\mathcal{TNC}(d)$ is partitioned by the fixed blocks so that each $\pi$ is represented exactly one time. This completes our proof. \(\square\)

4. Analytic Computations

In this section we will consider sums of truncated Jacobi distributions. Let

$$\mu_k = J\left(0, 0, 0, 0, \ldots 0, 0, 0, \ldots\right)$$

$$\mu_\ell = J\left(0, 0, 0, 0, \ldots 0, 0, 0, \ldots\right)$$

We will describe the non-zero moments of $\mu_k \boxplus \mu_\ell$.

We begin by considering subsets $\mathcal{NC}_k(n) \subset \mathcal{NC}(n)$ whose partitions have depth less than $k$. That is, consider $\pi \in \mathcal{NC}(n)$ partitioned into pairs $\pi = \{(a_i, b_i)\}_{i=1}^n$. If there exist indices $i_1 < i_2 < \cdots < i_k$ such that

$$a_{i_1} < a_{i_2} < \cdots < a_{i_k} < b_{i_k} < \cdots < b_{i_2} < b_{i_1}$$

then we have that $\pi \in \mathcal{NC}(n) \setminus \mathcal{NC}_k(n)$.

We define subsets of $\mathcal{TNC}_k,\ell(n) \subset \mathcal{TNC}(n)$ as those elements where the blue elements have depth at most $k$ and the red elements have depth at most $\ell$, in the following precise
sense: Let $\pi \in \mathcal{TCNC}_{k,\ell}(n)$ with $\pi = \{(a_i, b_i)\}_{i=1}^p \cup \{(c_j, d_j)\}_{j=1}^{n-p}$ where the pairs $(a_i, b_i)$ are blue and the pairs $(c_j, d_j)$ are red. If there exist indices $i_1 < i_2 < \cdots < i_k$ such that

$$a_{i_1} < a_{i_2} < \cdots < a_{i_k} < b_{i_k} < \cdots < b_{i_2} < b_{i_1}$$

then, there exists a pair $(c_j, d_j)$ such that

$$a_{i_1} < c_j < a_{i_k} < b_{i_k} < \cdots < b_{i_2} < b_{i_1}.$$  

Moreover, if the red and blue are swapped than the same property must be true with $\ell$ replacing $k$. For the readers convenience, the figure above consists of the 20 elements in $\mathcal{TCNC}_{2,2}(3)$.

We provide a recursive definition of these sets. Indeed, $\mathcal{TCNC}_{k,\ell}(n)$ is the set of all two-color non-crossing partitions $\pi$ of $\{1, 2, \ldots, 2n\}$ whose coloring respects the pairing with the property that there exists an interval $I \subset \{1, 2, \ldots, 2n\}$ such that

(a) The elements of $I$ are blue (resp. red) and is bordered by red (resp. blue) elements.

(b) $\pi|_I \in \mathcal{NC}_k(|I|)$ (resp. $\pi|_I \in \mathcal{NC}_\ell(|I|)$).

(c) $\pi|_{\{1,2,\ldots,2n\}\setminus I} \in \mathcal{TCNC}_{k,\ell}(n - |I|/2)$.

**Lemma 24.** Let $X$ be a random variable with distribution $\mu_k$. Then

$$E(Xb_1X \cdots b_{2n-1}X) = \sum_{\pi \in \mathcal{NC}_k(n)} E_\pi(Xb_1X \cdots b_{2n-1}X).$$

**Proof.** We have that

$$E(Xb_1X \cdots b_{2n-1}X) = \sum_{\pi \in \mathcal{NC}(2n)} E_\pi(Xb_1X \cdots b_{2n-1}X).$$

If $\pi \in \mathcal{NC}(2n) \setminus \mathcal{NC}_k(2n)$ then the partition has depth of at least $k + 1$. This implies that the $k + 1$ automorphism will be applied. However, this is the 0 automorphism so this term vanishes. Our result follows. \hfill \Box

We have the following corollary to Theorem 23

**Corollary 25.** Assume that the random variables in the statement of Theorem 23 $X_1$ and $X_2$ have depth $k$ and $\ell$ respectively. Then the selection of $\pi$ may be restricted to the subset $\mathcal{TCNC}_{k,\ell}(n) \subset \mathcal{TCNC}(n)$.

We now produce a specific example of a convolution of these distributions with $B = \mathbb{C}$. This will also provide a convenient method for counting the size of the sets $\mathcal{TCNC}_{k,k}(n)$ through free probabilistic methods.
Consider a non-commutative probability space \((M_k(\mathbb{C}), \phi_k)\) where \(\phi_k(X) = e_{1,1}Xe_{1,1}\). Consider the self adjoint random variable

\[
X_k = \sum_{i=1}^{k-1} e_{i,i+1} + e_{i+1,i}.
\]

**Proposition 26.** We have that \(\phi_k(X_k^{2n}) = |\mathcal{NC}_k(n)|\).

**Proof.** We establish a bijection between the non-zero moment contributions and the elements in \(\mathcal{NC}_k(n)\). Consider an element \(\{(a_i, b_i)\}_{i=1}^n = \pi \in \mathcal{NC}_k(n)\). To each pair \((a_i, b_i)\) we define the depth of this pair to be the largest \(p\) such that

\[
a_{i_1} < a_{i_2} < \cdots < a_{i_p} < a_i < b_i < b_{i_p} < \cdots < b_{i_2} < b_{i_1}
\]

for some indices \(i_1, \ldots, i_p\). Thus, \(\mathcal{NC}_k(n)\) is simply the set of non-crossing partitions with no pairing of depth \(k\).

Now, observe that, given any \(i \in \{1, 2, \ldots, 2n\}\), if \(i\) and \(i + 1\) are both assigned to elements \(a_{j_i}\) and \(a_{j_{i+1}}\) (resp. \(b_{j_i}\) and \(b_{j_{i+1}}\)) [that is, they are both the left (resp. right) entry of a pair], then the depth of \((a_{j_i}, b_{j_i})\) is one smaller (resp. larger) than the depth of \((a_{j_{i+1}}, b_{j_{i+1}})\) since this implies that

\[
a_{j_i} < a_{j_{i+1}} < b_{j_{i+1}} < b_{j_i}.
\]

If \(i\) is assigned to \(b_{j_i}\) and \(i + 1\) is assigned to \(a_{j_{i+1}}\) then they have the same depth. The remaining scenario is impossible without introducing a crossing.

It follows from the observation that, given a pair \((a_i, b_i)\) of depth \(p\) in our partition \(\pi\), the map sending \(a_i\) to the matrix unit \(e_{p+1,p+2}\) and \(b_i\) to \(e_{p+2,p+1}\) produces product of \(2n\) matrix units equal to \(e_{1,1}\).

Moreover, given any such product of \(2n\) matrix units \(\{e_{i,i+1}, e_{i+1,i}\}_{i=1}^{k-1}\), it must arise as from an element of \(\mathcal{NC}_k(n)\). Indeed, given such a non-zero product

\[
e_{1,1} = \prod_{j=1}^{2n-1} e_{i_j,i_{j+1}}
\]

it follows that there must exist sequential elements with \(i_j = i_{j+2}\). Thus, removing this pair we are left with a product of \(2n - 2\) matrix units equal to \(e_{1,1}\). By induction, this produces an element of \(\tilde{\pi} \in \mathcal{NC}_k(n)\).

We extend \(\tilde{\pi}\) to an element \(\pi \in \mathcal{NC}_k(n)\) by inserting the removed pairing which we call \((a_n, b_n)\) with \(i_j = a_n\) and \(i_j + 1 = b_n\) so that \(\pi = \tilde{\pi} \cup (a_n, b_n)\). It is clear that this is still a non-crossing partition, but it does require argument that \(\pi\) will still have depth less than \(k\).

Observe that if \(i_j = i_{j+2} = k - 1\) then \(i_{j+1} = k\). This implies that, \(i_{j-1}\) and \(i_{j+3}\) are either \(k - 2\) or \(k\) (if this does not occur, then the depth is trivially smaller than \(k\)). If \(i_{j-1} = k\), then our bijection is such that \(j - 1\) is assigned to the right element of a pair, \(b_{\ell}\) for some \(\ell\). Thus, it follows from our definition of depth that the pairs \((a_n, b_n)\) and \((a_{\ell}, b_{\ell})\) are of the same depth, namely \(k - 1\). If \(i_{j-1} = k - 2\) then the matrix unit it is assigned to is \(e_{k-2,k-1}\) so, by construction of our bijection, it is assigned to an element \(a_{\ell}\) in a pairing \((a_{\ell}, b_{\ell})\) of depth \(k - 2\). This implies that \((a_n, b_n)\) is of depth \(k - 1\) so that the depth of the partition is (possibly) increased, but still strictly smaller than \(k\). This completes our proof.
Corollary 27. We have that \(|\mathcal{TNC}_{k,\ell}(n)| = \nu_k \boxplus \nu_\ell(t^{2n})\) where \(\nu_k\) and \(\nu_\ell\) are the probability measures arising in [20].

Proof. This is simply a combination of Corollary 25 and Proposition 26.

We establish the convention that \(\nu_1 = \delta_0\), the Dirac mass at 0.

Lemma 28. For \(n > 1\), we have that

\[
G_{\nu_n}(z) = \frac{1}{z - G_{\nu_{n-1}}(z)}. \tag{26}
\]

Proof. Recall that

\[
G_{\nu_n}(z) = \phi_{1,1}((z - X_n)^{-1}) \tag{27}
\]

Observe that

\[
z1_k - X_k = \begin{bmatrix}
z & -1 & 0 & 0 & \cdots & 0 & 0 \\
-1 & z & -1 & 0 & \cdots & 0 & 0 \\
0 & -1 & z & -1 & \cdots & 0 & 0 \\
\vdots & & & & \ddots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & z & -1 \\
0 & 0 & 0 & 0 & \cdots & -1 & z
\end{bmatrix} \tag{28}
\]

Letting \((z1_k - X_k)_{i,j}\) denote the subminor obtained by removing the \(i\)th row and the \(j\)th column, we have that

\[
\phi_{1,1} \circ (z1_k - X_k)^{-1} = \frac{\det((z1_k - X_k)_{1,1})}{\det((z1_k - X_k))} = \frac{\det((z1_k - X_k)_{1,1})}{z \det((z1_k - X_k)_{1,1}) + \det((z1_k - X_k)_{1,2})} \tag{29}
\]

Observe that

\[
(z1_k - X_k)_{1,2} = \begin{bmatrix}
-1 & -1 & 0 & 0 & \cdots & 0 & 0 \\
0 & z & -1 & 0 & \cdots & 0 & 0 \\
0 & -1 & z & -1 & \cdots & 0 & 0 \\
\vdots & & & & \ddots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & z & -1 \\
0 & 0 & 0 & 0 & \cdots & -1 & z
\end{bmatrix} \tag{30}
\]

So that

\[
\det((z1_k - X_k)_{1,2}) = -\det((z1_{k-1} - X_{k-1})_{1,1}) \tag{31}
\]

Moreover, we have that

\[
\det((z1_k - X_k)_{1,1}) = \det(z1_{k-1} - X_{k-1}) \tag{32}
\]

Plugging [31] and [32] into [29] and factoring out the numerator, we have that

\[
\phi_{1,1} \circ (z1_k - X_k)^{-1} = \frac{1}{z - \frac{\det((z1_{k-1} - X_{k-1})_{1,1})}{\det((z1_{k-1} - X_{k-1}})}} = \frac{1}{z - G_{\nu_{k-1}}(z)} \tag{33}
\]

\(\Box\)

Let \(\mu_{n,n} = \nu_n \boxplus \nu_n\). The following Corollary will prove useful in computing the convolved distribution.
Corollary 29. For all $z \in \mathbb{C}^+$, we have
\[ F_{\mu_{n,n}}(z + G_{\nu_{n-1}}(z)) = z - G_{\nu_{n-1}}(z). \]

Proof. On an appropriate domain, we have that
\[ F_{\mu_{n,n}}^{(-1)}(z) = 2F_{\nu_n}^{(-1)} - z \]
\[ \Rightarrow F_{\mu_{n,n}}^{(-1)}(F_{\nu_n}(z)) = 2z - F_{\nu_n}(z) \]
\[ \Rightarrow F_{\mu_{n,n}}^{(-1)}(z - G_{\nu_{n-1}}(z)) = z + G_{\nu_{n-1}}(z) \]
where the last implication follows from Lemma 28. Our claim follows on an appropriate domain and on all of $\mathbb{C}^+$ through continuation. \qed

We set notation before proving the main result of this section. For $q < p$, we define $P_O(p, q)$ to denote the set of interval partitions of $\{1, 2, \ldots, p\}$ into $q$ distinct blocks of odd length. Given an element $\pi \in P_O(p, q)$ we define $\pi_i$ to be the $i$th block, in ascending order, for $i = 1, 2, \ldots, q$. We let $|\pi_i|$ denote the length of this interval.

Theorem 30. Let $M_n^{(k)}$ and $m_n^{(\ell)}$ denote the $n$th moments for the measures $\mu_{k,k}$ and $\nu_{\ell}$, respectively. The measure $\mu_{k,k}$ is symmetric and we have the following recursive formula for the even moments:
\[ M_{2n}^{(k)} = S_{n,k} - T_{n,k} \] (33)
where
\[ S_{n,k} = 2 \sum_{i=n}^{2n-1} \binom{2n-1}{i} \sum_{\pi \in P_O(i, 2n-i)} m_{|\pi_1|-1}^{(k-1)} m_{|\pi_2|-1}^{(k-1)} \cdots m_{|\pi_{2n-i}|-1}^{(k-1)} \] (34)
\[ T_{n,k} = \sum_{j=1}^{n-2} M_{2j}^{(k)} \sum_{p=(n-j)-1}^{2(n-j)-1} \left( \frac{2(n-j) - 1}{p} \right) [R_{p+1,j,n,k} - R_{p,j,n,k}] \] (35)
\[ R_{p,j,n,k} = \left( \sum_{\pi \in P_O(p, 2(n-j)-p)} (m_{|\pi_1|-1}^{(k-1)} m_{|\pi_2|-1}^{(k-1)} \cdots m_{|\pi_{2n-i}|-1}^{(k-1)}) \right) \] (36)

Proof. Consider the Cauchy transform
\[ G_{\mu_{k,k}}(z) = \sum_{p=0}^{\infty} \frac{M_{2p}^{(k)}}{z^{2p+1}}. \] (37)
Rewriting and taking limits, we have

\[ M_{2n}^{(k)} = \lim_{|z| \to \infty} z^{2n+1} \left[ G_{\nu k}(z) - \frac{M_0^{(k)}}{z} + \frac{M_2^{(k)}}{z^3} + \cdots + \frac{M_{2(n-1)}^{(k)}}{z^{2n-1}} \right] \]

\[ = \lim_{|z| \to \infty} (z + G_{\nu k-1}(z))^{2n+1} \left[ G_{\nu k}(z + G_{\nu k-1}(z)) - \sum_{j=0}^{n-1} \frac{M_{2j}^{(k)}}{z^{2j}} (z + G_{\nu k-1}(z))^{2j+1} \right] \]

\[ = \lim_{|z| \to \infty} \left[ z - G_{\nu k-1}(z) \right] \left[ (z + G_{\nu k-1}(z))^{2n+1} \left[ \frac{1}{z - G_{\nu k-1}(z)} - \sum_{j=0}^{n-1} \frac{M_{2j}^{(k)}}{z^{2j}} (z + G_{\nu k-1}(z))^{2j+1} \right] \right] \]

where equality is justified since \(|z + G_{\nu k-1}(z)| \to \infty\) as \(|z| \to \infty\). Equality is justified since this is a product of convergent limits and

\[ \lim_{|z| \to \infty} \frac{z - G_{\nu k-1}(z)}{z + G_{\nu k-1}(z)} = 1 \]

as well as Corollary. Since convergence of is established, we need only identify the constant terms to identify the limit. We break this into two pieces, letting

\[ S(z) = (z + G_{\nu k-1}(z))^{2n} - (z - G_{\nu k-1}(z))^{2n-1}\]

and setting the remaining terms in equal to \(T(z)\). We will establish our theorem by showing that the constant term for \(S(z)\) is equal to \(S_{n,k}\) and the constant term for \(T(z)\) is equal to \(T_{n,k}\).

We begin with \(S(z)\). Observe that

\[ S(z) = 2G_{\nu k-1}(z)(z + G_{\nu k-1}(z))^{2n-1} = 2 \sum_{i=0}^{n-1} \binom{2n-1}{i} z^i G_{\nu k-1}(z)^{2n-i} \]

Now, isolating \(z^0 G_{\nu k-1}(z)^{2n-1}\), our task devolves to identifying the constant term of this Laurent series. As \(G_{\nu k-1}(z)^{2n-1} = O(z^{2n})\), we only receive contributions for \(i \geq n\) so that we focus on

\[ 2 \sum_{i=n}^{2n-1} \binom{2n-1}{i} z^i G_{\nu k-1}(z)^{2n-i} \]

Now observe that

\[ G_{\nu k-1}(z)^{2n-i} = \left( \sum_{p=0}^{\infty} \frac{m_{p}}{z^{p+1}} \right)^{2n-i} \]

and we must identify the coefficient of the \(z^{-i}\) term. But this is exactly

\[ \sum_{p \in \mathcal{P}_0(i,2n-i)} m_{\nu_k-1}^{(k-1)} m_{\nu_{k-1}-1}^{(k-1)} \cdots m_{\nu_{2n-i}-1}^{(k-1)} \]
Our second claim is that the constant term for 
where the last equality also follows from the definition of \( \pi \) since, given 
where we may restrict the range in 42 to \( S \), this will have no constant term unless 

We can immediately discard the \( j = n - 1 \) term since this is equal to 

Isolating a single term for fixed \( j \), we have that 

\[
M_{2j}^{(k)}(z - G_{\nu_{k-1}}(z)) (z + G_{\nu_{k-1}}(z))^{2(n-j)-1}.
\]

We can immediately discard the \( j = n - 1 \) term since this is equal to 

Putting the pieces together, we have proven our second claim and, therefore, the theorem.

\[\square\]

5. Concrete Examples

We now establish additional concrete results based on the Theorems proven in the previous sections. We begin by calculating the values for \( |\mathcal{T}C\mathcal{N}C_{k,k}(n)| \) based on the recursive algorithm in Theorem 30.
Going through one of the computations that drives Theorem 30, we consider $|\mathcal{TCNC}(2,2)(3)| = M^{(2)}_3$. Utilizing the same reasoning from equalities 39 through 41, we have that this moment is equal to

$$
\lim_{|z| \to \infty} (z - G_{\nu_1}(z))(z + G_{\nu_1}(z))^6 \left( G_{\mu_2,2}(z + G_{\nu_1}(z)) - \frac{1}{z + G_{\nu_1}(z)} - \frac{2}{(z + G_{\nu_1}(z))^3} - \frac{6}{(z + G_{\nu_1}(z))^5} \right)
$$

Recalling that $G_{\mu_2,2}(z + G_{\nu_1}(z)) = z - G_{\nu_1}(z)$, we distribute these terms,

$$M^{(2)}_3 = \lim_{|z| \to \infty} (z + G_{\nu_1}(z))^6 - (z - G_{\nu_1}(z))(z + G_{\nu_1}(z))^5 - 2(z - G_{\nu_1}(z))(z + G_{\nu_1}(z))^3 - 6(z - G_{\nu_1}(z))(z + G_{\nu_1}(z))$$

We need only isolate the constant terms. Once again, $(z - G_{\nu_1}(z))(z + G_{\nu_1}(z))$ contributes nothing. Consider

$$(z + G_{\nu_1}(z))^6 - (z - G_{\nu_1}(z))(z + G_{\nu_1}(z))^5 = 2G_{\nu_1}(z)(z + G_{\nu_1}(z))^5$$

$$= 2G_{\nu_1}(z)(z^5 + 10z^4G_{\nu_1}(z) + 10z^3G_{\nu_1}(z) + \cdots)$$

and note that the $\cdots$ terms make no contribution to the constant as their degree is too low. The constant term is equal to $2[m_4^{(1)} + 5(2m_0^{(1)}m_2^{(1)}) + 10(m_0^{(1)})^3]$. By a similar argument, the term

$$2(z - G_{\nu_2}(z))(z + G_{\nu_2}(z))^3$$

contributes $-2(2m_2^{(1)})$ to the constant. Now, since $\nu_1 = \delta_0$, we have $m_0^{(1)} = 1$ and $m_i^{(1)} = 0$ for all $i > 0$. Thus, the only contributing term is $20(m_0^{(1)})^3 = 20$, matching Figure 1 and our table above.

**Example 31.** We isolate a special case of Proposition 21 as it is simple concrete example of a non-commutative convolution that can be computed through the traditional Cauchy transform methodology.

Let $E : M_2(\mathbb{C}) \mapsto \mathcal{D}$ denote the non-commutative probability space generated by the conditional expectation of $M_2$ onto the diagonal subalgebra. Let $X = e_{1,2} + e_{2,1}$. Observe that for $b = \lambda e_{1,1} + \gamma e_{2,2}$ we have that

$$XbX = \alpha(b) = \gamma e_{1,1} + \lambda e_{2,2} \in \mathcal{D}$$

so that the hypotheses of Proposition 21 are satisfied. We let $\mu$ denote the distribution of $X$. Calculating the various transforms, we have

$$G_{\mu}(b) = \sum_{n=0}^{\infty} \begin{pmatrix} [\lambda(\gamma\lambda)^n]^{-1} & 0 \\ 0 & [\gamma(\lambda\gamma)^n]^{-1} \end{pmatrix} = \begin{pmatrix} \frac{1}{\lambda - \gamma - 1} & 0 \\ 0 & \frac{1}{\gamma - \lambda - 1} \end{pmatrix}$$
we conclude that

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\[ F_\mu(b) = \begin{pmatrix} \lambda - \frac{1}{\gamma} & 0 \\ 0 & \gamma - \frac{1}{\lambda} \end{pmatrix} \]  

(49)

\[ F_\mu^{(-1)}(b) = \left( \begin{array}{cc} \frac{1}{2} \left[ \lambda + \sqrt{\lambda^2 + 4\frac{\alpha}{\gamma}} \right] & \frac{1}{2} [\gamma + \sqrt{\gamma^2 + 4\frac{\gamma}{\lambda}}] \\ 0 & \frac{1}{2} \left[ \lambda - \sqrt{\lambda^2 + 4\frac{\alpha}{\gamma}} \right] \end{array} \right) \]  

(50)

Utilizing the linearizing property for Voiculescu transforms, that is,

\[ F_{\mu \oplus \mu}^{(-1)}(b) = \varphi_{\mu \oplus \mu}(b) + b = \varphi_\mu(b) + \varphi_\mu(b) + 2F_\mu^{(-1)}(b) - b \]

we conclude that

\[ F_{\mu \oplus \mu}^{(-1)}(b) = \left( \begin{array}{cc} \sqrt{\lambda^2 + 4\frac{\alpha}{\gamma}} & 0 \\ 0 & \sqrt{\gamma^2 + 4\frac{\gamma}{\lambda}} \end{array} \right). \]  

(51)

Taking the compositional inverse, we have

\[ F_{\mu \oplus \mu}(b) = \left( \begin{array}{cc} \sqrt{\lambda^2 - 4\frac{\alpha}{\gamma}} & 0 \\ 0 & \sqrt{\gamma^2 - 4\frac{\gamma}{\lambda}} \end{array} \right). \]  

(52)

Letting \( \lambda = \gamma = z \), the entries are precisely the \( F \)-transform of the arcsine distribution. This, coupled with observation \( \Box \) in Proposition 21 allows to reprove the main result in that theorem from more basic principles in this special case.

**Example 32.** We construct examples of Jacobi distributions \( \mu_1 \) and \( \mu_2 \) such that \( \mu_1 \oplus \mu_2 \) is not a Jacobi distribution.

Indeed, let \( \mu_1 \) and \( \mu_2 \) be symmetric Bernoulli distributions with respective morphisms \( \alpha_1 \) and \( \alpha_2 \). That is,

\[ \mu_i = J \begin{pmatrix} 0, & 0, & 0, & 0, & \ldots \end{pmatrix}. \]  

(53)

We assume that

\[ \mu = J \begin{pmatrix} 0, & 0, & 0, & 0, & \beta_1, & \beta_2, & \beta_3, & \beta_4, & \ldots \end{pmatrix}. \]  

(54)

satisfies \( \mu = \mu_1 \oplus \mu_2 \) and show that \( \alpha_1 \) and \( \alpha_2 \) may be chosen so that this precipitates a contradiction.

According to Theorem 23, we have the following equalities:

\[ \mu(Xb_0X) = \beta_1(b_0) \]  

(55)

\[ \mu_1 \oplus \mu_2(Xb_1X) = \alpha_1(b_0) + \alpha_2(b_0) \]  

(56)

\[ \mu(Xb_1Xb_2Xb_3X) = \beta_1(b_1\beta_2(b_2)b_3) + \beta_1(b_1)b_2\beta_1(b_3) \]  

(57)

\[ \mu_1 \oplus \mu_2(Xb_1Xb_2Xb_3X) = \alpha_1(b_1\alpha_2(b_2)b_3) + \alpha_2(b_1\alpha_1(b_2)b_3) + \alpha_1(b_1)b_2\alpha_1(b_3) + \alpha_1(b_1)b_2\alpha_2(b_3) \]

(58)

+ \alpha_1(b_1)b_2\alpha_2(b_3) + \alpha_2(b_1)b_2\alpha_1(b_3) + \alpha_2(b_1)b_2\alpha_2(b_3)

Now, 55 and 56 combine to imply that \( \beta_1 = \alpha_1 + \alpha_2 \). At this point, equality of equations 57 and 58 becomes completely untenable in most non-commutative settings. Indeed, letting \( \alpha_1 = \alpha \) from Example 31, \( \alpha_2 = Id \) and \( b_1 = b_2 = b_3 = c_{1,1} \) we obtain an easy contradiction.
References

[ABo98] Accardi, Luigi and Boeijko, Marek, *Interacting Fock Spaces and Gaussianization of Probability Measures*, Infinite Dimensional Analysis, Quantum Probability and Related Topics **01** (1998), no. 4, 663-670.

[ABFN13] Michael Anshelevich, Serban T. Belinschi, Maxime Fevrier, and Alexandru Nica, *Convolution powers in the operator-valued framework*, Trans. Amer. Math. Soc. **365** (2013), no. 4, 2063–2097. MR

[BPV10] Serban T. Belinschi, Mihai Popa, and Victor Vinnikov, *On the operator-valued analogues of the semicircle, arcsine and Bernoulli laws*, arXiv:1008.5205v2 [math.OA], 2010.

[Ch78] Chihara, T. S., *An introduction to orthogonal polynomials*, Mathematics and its Applications, Vol. **13**, Gordon and Breach Science Publishers, New York-London-Paris (1978). MR

[Fl80] Flajolet, P., *Combinatorial aspects of continued fractions*, Discrete Math. **32** (1980), no. 2, 125–161. MR

[HRFS07] J. William Helton, Reza Rashidi Far, and Roland Speicher, *Operator-valued semicircular elements: solving a quadratic matrix equation with positivity constraints*, Int. Math. Res. Not. IMRN (2007), no. 22, Art. ID rnm086, 15. MR

[Mlo09a] Wojciech Młotkowski, *Combinatorial relation between free cumulants and Jacobi parameters*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **12** (2009), no. 2, 291–306. MR2541398

[PV13] Mihai Popa and Victor Vinnikov, *Non-commutative functions and the non-commutative free Lévy-Hinčin formula*, Adv. Math. **236** (2013), 131–157. MR2541398

[Spe98] Roland Speicher, *Combinatorial theory of the free product with amalgamation and operator-valued free probability theory*, Mem. Amer. Math. Soc. **132** (1998), no. 627, x+88. MR2541398

[Vi84] Gérard Viennot, *Une théorie combinatoire des polynômes orthogonaux généraux*, (unpublished notes).

[Vo95] Voiculescu, Dan, *Operations on certain non-commutative operator-valued random variables*, Recent advances in operator algebras (Orléans, 1992), Astérisque **232** (1995), 243–275. MR2541398

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