Higher gradients estimates in Morrey spaces for weak solutions to linear ultraparabolic equations

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Abstract. The aim of this paper is to consider the linear ultraparabolic equation with bounded and VMO coefficients $a_{ij}(z)$. Assume that the operator $L_0$ obtained by freezing the coefficients $a_{ij}(z)$ at any point $z_0 \in \mathbb{R}^{N+1}$ is hypoelliptic. We first establish a Caccioppoli type inequality by choosing a cutoff function, a Sobolev type inequality by properties of the fundamental solution to $L_0$, and a Poincaré type inequality with a new cutoff function. Then $L^p$ estimate for weak solutions is derived by using the reverse Hölder inequality on homogeneous spaces. Finally, higher Morrey estimates for weak solutions to the above equation are shown by investigating a homogeneous ultraparabolic equation of variable coefficients with a nonhomogeneous boundary value condition, and a nonhomogeneous ultraparabolic equation of variable coefficients with homogeneous boundary value condition.

Key words: ultraparabolic equations, weak solutions, Caccioppoli inequality, Poincaré inequality, Sobolev inequality, $L^p$ estimates, Morrey estimates

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1 Introduction

In the paper, we consider the ultraparabolic equation of the kind

$$Lu = \sum_{i,j=1}^{m_0} \partial_{x_i} (a_{ij}(z) \partial_{x_j} u(z)) + \sum_{i,j=1}^{N} b_{ij} x_i \partial_{x_j} u(z) - \partial_t u(z) = g(z) + \sum_{j=1}^{m_0} \partial_{x_j} f_j(z), \quad (1.1)$$

where $z = (x, t) \in \mathbb{R}^{N+1}$, $1 \leq m_0 \leq N$, $b_{ij} \in \mathbb{R}$ ($i, j = 1, \ldots, N$), $g, f_j \in L^p(\Omega)$ or $L^{p,\lambda}(\Omega)$, $L^{p,\lambda}(\Omega)$ is a Morrey space, here $\Omega$ is a bounded domain in $\mathbb{R}^{N+1}$, $p \geq 2$, $0 \leq \lambda < Q + 2$, see Section 2 for the meaning of $Q$.

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The assumptions to (1.1) are

(H1) (ellipticity condition on \( \mathbb{R}^{m_0} \)) Let coefficients \( a_{ij}(z) \in VMO \cap L^\infty(\Omega) \) (see Section 2 for the definition of VMO), \( a_{ij}(z) = a_{ji}(z) \), satisfying that there exists a constant \( \Lambda > 1 \) such that for any \( z \in \mathbb{R}^{N+1}, \xi \in \mathbb{R}^{m_0}, \)

\[
\Lambda^{-1}|\xi|^2 \leq \sum_{i,j=1}^{m_0} a_{ij}(z)\xi_i\xi_j \leq \Lambda|\xi|^2.
\]

(H2) The constant matrix \( B = (b_{ij})_{i,j=1,...,N} \) in (1.1) has the form

\[
B = \begin{pmatrix}
0 & B_1 & 0 & \cdots & 0 \\
0 & 0 & B_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & B_r \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix},
\]

where \( B_k(k = 1, 2, \ldots, r) \) is a \( m_{k-1} \times m_k \) matrix with rank \( m_k \) and

\[
m_0 \geq m_1 \geq \cdots \geq m_r \geq 1, \quad \sum_{k=0}^{r} m_k = N.
\]

The equation (1.1) can be written as

\[
Lu = \text{div} (A(z)D_0 u) + Yu = g + \text{div} f,
\]

where \( D_0 = (\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_{m_0}}, 0, \ldots, 0) \), \( Y u = \langle x, BDu \rangle - \partial_t u, \) \( D = (\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_N}) \), \( f = (f_1, f_2, \ldots, f_{m_0}, 0, \ldots, 0) \), \( A(z) \) is a \( N \times N \) matrix, namely,

\[
A(z) = \begin{pmatrix}
A_0(z) & 0 \\
0 & 0
\end{pmatrix}, \quad A_0(z) = (a_{ij}(z))_{i,j=1,...,m_0}.
\]

Regularity for weak solutions to parabolic equations were provided by many authors including DiBenedetto \[5\], Friendman \[9\], Krylov \[15\], Ladyzhenskaya, Solonnikov, Ural’tseva \[16\], Lieberman \[18\] and references therein.

In recent decades, many scholars have concerned with regularity of weak solutions to ultraparabolic equations. These equations are closely related to finance, Brown motion, partial physics and human vision, etc. The classic linear parabolic equation is usually of the form

\[
\sum_{i=1}^{N} \partial_{x_i x_j} u(x,t) - \partial_t u(x,t) = f(x,t).
\]
But we see that (1.1) is strongly degenerate if $1 \leq m_0 < N$ and there is a drift $Yu$. These make research on regularity to (1.1) different from parabolic equation.

For the homogeneous ultraparabolic equation

$$Lu = \sum_{i,j=1}^{m_0} \partial_{x_i} (a_{ij}(z) \partial_{x_j} u(z)) + \sum_{i,j=1}^{N} b_{ij} x_i \partial_{x_j} u(z) - \partial_t u(z) = 0,$$

(1.2)

Polidoro in [23] got global lower bound of the fundamental solution to (1.2). The boundedness of weak solutions to (1.2) with measurable coefficients was investigated by Pascucci and Polidoro in [22] with Moser’s iteration method based on a combination of a Caccioppoli type estimate and the classical embedding Sobolev inequality. Wang and Zhang in [25] obtained Hölder estimates for weak solutions to (1.2) with measurable coefficients by deriving local a priori estimate to (1.2) and a Poincaré inequality of nonnegative weak lower solution.

To the following ultraparabolic equation

$$Lu = \sum_{i,j=1}^{m_0} \partial_{x_i} (a_{ij}(z) \partial_{x_j} u(z)) + \sum_{i,j=1}^{N} b_{ij} x_i \partial_{x_j} u(z) - \partial_t u(z) = \sum_{j=1}^{m_0} \partial_{x_j} F_j(x, t),$$

(1.3)

where $F_j \in L^p_{loc}(\mathbb{R}^{N+1}) (1 < p < \infty)$, coefficients $a_{ij}(z)$ belong to VMO spaces, Manfredini and Polidoro in [19] established $L^p$ estimates and Hölder continuity for weak solutions $u \in L^p_{loc}(\mathbb{R}^{N+1})$. If $F_j \in L^{p,\lambda}_{loc}(\mathbb{R}^{N+1}) (1 < p < \infty, 0 \leq \lambda < Q + 2)$ and coefficients $a_{ij}(z)$ belong to some VMO spaces, Polidoro and Ragusa in [24] derived Hölder regularity for weak solution $u \in L^p_{loc}(\mathbb{R}^{N+1})$ to (1.3). Bramanti, Cerutti and Manfredini [1] proved local $L^p$ estimates for second order derivatives $\partial_{x_i x_j} u (i, j = 1, \ldots, m_0)$ of strong solutions to the nondivergence ultraparabolic equation

$$\sum_{i,j=1}^{m_0} a_{ij}(z) \partial_{x_i x_j} u + \langle x, BD \rangle u - \partial_t u = f$$

with $a_{ij}(z)$ being in VMO and $f \in L^p$. The methods in [1, 19, 24] are based on the representation formulae for solutions and estimates of singular integral operators. More related results also see Cinti, Pascucci and Polidoro [4], Xin and Zhang [26], Zhang [27] and references therein.

The aim of this paper is to establish higher integrability for weak solution $u \in W^{1,1}_2(\Omega)$ to (1.1) with the method of a priori estimates. Results on higher integrability of parabolic equations see Byun and Wang [2], Fugazzda [10], Palagachev and Softova [21] and references therein. The first result here is the higher $L^p(p > 2)$ estimate. For this purpose, an
appropriate frame is homogeneous spaces. Bramanti, Cerutti and Manfredini [1] pointed out that the ball related to a quasidistance (see Section 2 below) is a homogeneous space and Gianazza [11] proved a reverse Hölder inequality on homogeneous spaces. These facts will play important roles. In spite of this, some new preliminary conclusions are needed. Inspired by the way in [22], we deduce a Caccioppoli type inequality and a Sobolev type inequality for weak solution to (1.1). Following to [7] and constructing suitable cutoff functions, a Poincaré type inequality for weak solution to (1.1) is obtained. And then we prove higher $L^p$ estimates for gradients of weak solutions to (1.1) by using these new inequalities and the reverse Hölder inequality on the homogeneous spaces in [11].

The second result is on higher integrability in Morrey spaces for gradients of weak solution $u \in W^{1,1}_2(\Omega)$ to (1.1). With the aid of the approach in the studying of parabolic equations (e.g., see [12]), we consider a homogeneous ultraparabolic equation of variable coefficients with a nonhomogeneous boundary value condition, i.e., (6.1) below, and a nonhomogeneous ultraparabolic equation of variable coefficients with homogeneous boundary value condition, i.e., (6.2) below. The $L^p$ estimate for gradients of weak solutions to (6.1) is obtained by proving a local $L^\infty$ estimate and a local $L^2$ estimate of homogeneous ultraparabolic equation of constant coefficient, (5.1) below. Then we establish a local $L^p$ estimate for gradients of weak solutions to (6.2). These results are of independent interest. Finally, higher integrability in Morrey spaces for gradients of weak solutions to (1.1) is deduced by using a known iteration lemma.

The following is the notion of weak solution to (1.1).

**Definition 1.1** If $u \in W^{1,1}_2(\Omega)$ and for any $\psi \in C_0^\infty(\Omega)$,

$$-\int_\Omega AD_0uD_0\psi dz + \int_\Omega \psi Y u dz = \int_\Omega (g\psi - f D_0\psi) dz,$$

then we say that $u$ is a weak solution to (1.1).

The main results of this paper are stated as follows.

**Theorem 1.1** Suppose that assumptions (H1) and (H2) hold. If $u \in W^{1,1}_2(\Omega)$ is a weak solution to (1.1), $g, f_j \in L^p(\Omega)$, then there exists a constant $\varepsilon_0 > 0$ such that for any $p \in \left[2, 2 + \frac{2Q}{Q+2}\varepsilon_0\right]$, we have $D_0 u \in L^p_{\text{loc}}(\Omega)$ and for any $\Omega' \subset \subset \Omega'' \subset \subset \Omega$,

$$\|D_0 u\|_{L^p(\Omega')} \leq c \left(\|D_0 u\|_{L^2(\Omega')} + \|g\|_{L^p(\Omega')} + \|f\|_{L^p(\Omega')}\right).$$

(1.4)

**Theorem 1.2** Under (H1) and (H2), let $u \in W^{1,1}_2(\Omega)$ be a weak solution to (1.1), $g, f_j \in L^{p,\lambda}(\Omega)$, then for any $p \in \left[2, 2 + \frac{2Q}{Q+2}\varepsilon_0\right]$, $\varepsilon_0$ as in Theorem 1.1, we have $D_0 u \in L^{p,\lambda}(\Omega)$. The following is the notion of weak solution to (1.1).
This paper is organized as follows. In Section 2, we describe some basic knowledge and some known material on the frozen operator $L_0$ of $L$ and the fundamental solution of $L_0$, and collect several useful lemmas which will be used later on. Section 3 is devoted to proofs of a Caccioppoli type inequality, a Sobolev type inequality and a Poincaré type inequality for weak solutions. In Section 4, the proof of Theorem 1.1 is given by using the inequalities in Section 3 and the reverse Hölder inequality in [11]. In Section 5, we derive a higher $L^p$ estimate for gradient of weak solutions to (5.1). In Section 6, the proof of Theorem 1.2 is ended by local $L^p$ estimate for gradient of weak solutions to (6.1) and (6.2).

2 Preliminaries

For any $z_0 \in \Omega \subset \mathbb{R}^{N+1}$, we denote the frozen operator of $L$ by

$$L_0 = \sum_{i,j=1}^{m_0} \partial_{x_i} (a_{ij}(z_0) \partial_{x_j}) + \sum_{i,j=1}^{N} b_{ij} x_i \partial_{x_j} - \partial_t. \quad (2.1)$$

Now one can introduce the following.

**Definition 2.1** For any $(x, t), (\xi, \tau) \in \mathbb{R}^{N+1}$, set a multiplication law

$$(x, t) \circ (\xi, \tau) = (\xi + E(\tau)x, t + \tau), \quad E(\tau) = \exp(-\tau B^T).$$

We say that $(\mathbb{R}^{N+1}, \circ)$ is a noncommutative Lie group with neutral element $(0, 0)$, the inverse of an element $(x, t) \in \mathbb{R}^{N+1}$ is

$$(x, t)^{-1} = (-E(-t)x, -t).$$

Authors in [17] claimed that the frozen operator $L_0$ is hypoelliptic and left invariant about the groups of translations and dilations. In this case, the dilations associated to $L_0$ are given by

$$\delta_\lambda = \text{diag} \left( \lambda I_{m_0}, \lambda^3 I_{m_1}, \ldots, \lambda^{2r+1} I_{m_r}, \lambda^2 \right), \quad \lambda > 0,$$

here $I_{m_k}$ denotes the $m_k \times m_k$ identity matrix, and

$$\det (\delta_\lambda) = \lambda^{Q+2},$$
with \( Q + 2 = m_0 + 3m_1 + \cdots + (2r + 1)m_r + 2 \). The number \( Q + 2 \) is called the homogeneous dimension of \( \mathbb{R}^{N+1} \), and \( Q \) the homogeneous dimension of \( \mathbb{R}^N \). Note that \( L_0 \) is \( \delta_\lambda \) homogeneous of degree 2, namely, for any \( \lambda > 0 \),

\[
L_0 \circ \delta_\lambda = \lambda^2 (\delta_\lambda \circ L_0).
\]

Due to \([14]\), the fundamental solution \( \Gamma_0(\cdot, \zeta) \) of \( L_0 \) has an explicit expression in the pole \( \zeta \in \mathbb{R}^{N+1} \), which is, for any \( z, \zeta \in \mathbb{R}^{N+1} \), \( z \neq \zeta \),

\[
\Gamma_0(z, \zeta) = \Gamma_0(\zeta^{-1} \circ z, 0), \tag{2.2}
\]

where

\[
\Gamma_0((x, t), (0, 0)) = \begin{cases} \\
\frac{1}{((4\pi)^N \det C(t))^{\frac{1}{2}}} \exp \left( -\frac{1}{4} \langle C^{-1}(t)x, x \rangle \right), t > 0, \\
0, & t \leq 0,
\end{cases}
\]

\[
C(t) = \int_0^t E(s)A_0E^T(s)ds.
\]

It is known that \( C(t) \) is strictly positive for every positive \( t \). In view of the invariance properties of \( L_0 \), we have that for any \( z \in \mathbb{R}^{N+1} \setminus \{0\} \) and \( \lambda > 0 \),

\[
\Gamma_0(\delta_\lambda(z), 0) = \lambda^{-Q} \Gamma_0(z, 0).
\]

We also observe that \( \Gamma_0 \) is \( \delta_\lambda \) homogeneous of degree \(-Q\). For \( i, j = 1, \ldots, m_0 \), \( D_{x_i} \Gamma_0 \) and \( D_{x_i x_j} \Gamma_0 \) are \( \delta_\lambda \) homogeneous of degree \(-(Q + 1)\) and \(-(Q + 2)\), respectively.

For any \((x, t) \in \mathbb{R}^{N+1}\), the homogeneous norm of \((x, t)\) with respect to \( \delta_\lambda \) is defined by

\[
\|(x, t)\| = \sum_{j=1}^N |x_j|^{\frac{\alpha_j}{m_j}} + |t|^{\frac{1}{2}},
\]

where \( \alpha_j = 1 \), if \( 1 \leq j \leq m_0 \); \( \alpha_j = 2k + 3 \), if \( m_k < j \leq m_{k+1} \) (\( 0 \leq k \leq r - 1 \)). For any \( z, \zeta \in \mathbb{R}^{N+1} \), we denote the quasidistance by

\[
d(z, \zeta) = \|\zeta^{-1} \circ z\|.
\]

**Lemma 2.2** ([6, Lemma 2.1]) For any bounded domain \( \Omega \subset \mathbb{R}^{N+1} \), \( d(z, \zeta) \) is a quasisymmetric quasidistance in \( \Omega \), if for any \( z, z', \zeta \in \mathbb{R}^{N+1} \),

\[
d(z, \zeta) \leq cd(\zeta, z), \quad d(z, \zeta) \leq c(d(\zeta, z') + d(z', \zeta)).
\]

The ball with respect to \( d \) centered at \( z_0 \) is denoted by

\[
B_R(z_0) = B(z_0, R) = \{ \zeta \in \mathbb{R}^{N+1} : d(z_0, \zeta) < R \}.
\]
Note clearly that $B(0, R) = \delta_R B(0, 1)$.

**Remark 2.3** Recalling [1, Remark 1.5], it holds that for any $z_0 \in \mathbb{R}^{N+1}$, $R > 0$,

$$|B(z_0, R)| = |B(0, R)| = |B(0, 1)| R^{Q+2},$$

$$|B(z_0, 2R)| = 2^{Q+2} |B(z_0, R)|,$$

and therefore the space $(\mathbb{R}^{N+1}, dz, d)$ is a homogeneous space. The fact allows us to employ known conclusions in homogeneous spaces.

If one does not need to concern the center of the ball, $B(z_0, R)$ can simply be written as $B_R$. For convenience, we usually consider the estimates on cubes instead of balls. Let us describe the notion of cubes. For any $x = (x', \bar{x})$, $x' = (x_1, \ldots, x_{m_0})$, $\bar{x} = (x_{m_0+1}, \ldots, x_N)$, the cube is denoted by

$$Q_R = \left\{ (x, t) \mid t_0 - R^2/2 \leq t \leq t_0 + R^2/2, |x'| \leq R, |x_{m_0+1}| \leq (\Lambda N^2 R)^3, \ldots, |x_N| \leq (\Lambda N^2 R)^{2r+1} \right\}.$$

Also, we write

$$I_R = \left[ t_0 - \frac{R^2}{2}, t_0 + \frac{R^2}{2} \right],$$

$$K_R = \left\{ x' \mid |x'| \leq R \right\},$$

$$S_R = \left\{ \bar{x} \mid |x_{m_0+1}| \leq (\Lambda N^2 R)^3, \ldots, |x_N| \leq (\Lambda N^2 R)^{2r+1} \right\},$$

then $Q_R = K_R \times S_R \times I_R$.

A cube of centered at $(0, 0)$ is simply denoted by

$$Q_R(0, 0) = \left\{ (x, t) \mid |t| \leq R^2, |x_1| \leq R^\alpha, \ldots, |x_N| \leq R^\alpha N \right\}.$$

It is easy to find that there exists a constant $c_0 = c_0 (B, N) > 0$, such that

$$Q_{R/c_0}(0, 0) \subset B_R(0, 0) \subset Q_{c_0 R}(0, 0).$$

We state a result on $\delta_\lambda$ homogeneous functions in [8, 22].

**Lemma 2.4** Let $\alpha \in [0, Q + 2]$ and $G \in C(\mathbb{R}^{N+1} \setminus \{0\})$ be a $\delta_\lambda$ homogeneous function of degree $\alpha - Q - 2$. If $f \in L^p(\mathbb{R}^{N+1})$, $p \in [1, +\infty)$, then the function

$$G_f(z) \equiv \int_{\mathbb{R}^{N+1}} G(\zeta^{-1} \circ z) f(\zeta) d\zeta$$

is defined almost everywhere and there exists a constant $c = c(Q, P) > 0$ such that

$$\|G_f\|_{L^q(\mathbb{R}^{N+1})} \leq c \max_{\|z\|=1} |G(z)| \|f\|_{L^p(\mathbb{R}^{N+1})},$$

(2.3)
where \( \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q+2} \).

This lemma can be used to yield the following.

**Lemma 2.5** Let \( f \in L^{\frac{2(Q+2)}{Q+4}}(\mathbb{R}^{N+1}) \). There exists a positive constant \( c = c(Q) \) such that

\[
\| \Gamma_0(f) \|_{L^{\frac{2(Q+2)}{Q+4}}(\mathbb{R}^{N+1})} \leq c \| f \|_{L^{\frac{2(Q+2)}{Q+4}}(\mathbb{R}^{N+1})},
\]

\[
\| \Gamma_0(D_0f) \|_{L^{\frac{2(Q+2)}{Q+4}}(\mathbb{R}^{N+1})} \leq c \| f \|_{L^{\frac{2(Q+2)}{Q+4}}(\mathbb{R}^{N+1})},
\]

where \( \Gamma_0(f)(z) = \int_{\mathbb{R}^{N+1}} \Gamma_0(z, \zeta) f(\zeta) d\zeta \), \( \Gamma_0(D_0f)(z) = \int_{\mathbb{R}^{N+1}} \Gamma_0(z, \zeta) D_0f(\zeta) d\zeta \).

**Proof:** Since \( \Gamma_0 \) is homogeneous of degree \(-Q\) with respect to \( \delta_\lambda \), we immediately have (2.4) from Lemma 2.4 by taking \( \alpha = 2 \), \( q = \frac{2(Q+2)}{Q} \) and \( p = \frac{2(Q+2)}{Q+4} \). Noting that \( \partial_\lambda \Gamma_0 \) is homogeneous of degree \(-(Q+1)\) with respect to \( \delta_\lambda \), (2.5) holds by Lemma 2.4 with \( \alpha = 1 \), \( q = 2 \) and \( p = \frac{2(Q+2)}{Q+4} \).

**Definition 2.6** (Morrey space \( L^{p,\lambda} \)) Let \( \Omega \) be an open subset in \( \mathbb{R}^{N+1} \), \( 1 \leq p < +\infty \), \( \lambda > 0 \). We say that \( f \in L^p(\Omega) \) belongs to the Morrey space \( L^{p,\lambda}(\Omega) \), if

\[
\| f \|_{L^{p,\lambda}(\Omega)} = \sup_{z_0 \in \Omega, \rho > 0} \left( \frac{\rho^\lambda}{|\Omega \cap B_\rho(z_0)|} \int_{\Omega \cap B_\rho(z_0)} |f|^p dz \right)^{\frac{1}{p}} < \infty.
\]

**Definition 2.7** (Sobolev space \( W^{1,1}_p \)) Let \( \Omega \) be an open subset in \( \mathbb{R}^{N+1} \). The Sobolev space with respect to (1.1) is defined by

\[
W^{1,1}_p(\Omega) = \{ u \in L^p(\Omega) : \partial_\lambda u, Yu \in L^p(\Omega), i, j = 1, \ldots, m_0 \}
\]

with the norm

\[
\| u \|_{W^{1,1}_p(\Omega)} = \| u \|_{L^p(\Omega)} + \sum_{i=1}^{m_0} \| \partial_\lambda u \|_{L^p(\Omega)} + \| Yu \|_{L^p(\Omega)}.
\]

The space \( W^{1,1}_{p,0}(\Omega) \) is the closure of \( C_0^\infty(\Omega) \) in \( W^{1,1}_p(\Omega) \).

**Definition 2.8** (BMO and VMO spaces) For any \( a \in L^1_{loc}(\Omega) \), we set

\[
\eta_R(a) = \sup_{z_0 \in \Omega, 0 \leq \rho \leq R} \left( \frac{1}{|\Omega \cap B_\rho(z_0)|} \int_{\Omega \cap B_\rho(z_0)} |a(z) - a_{\Omega \cap B_\rho(z_0)}(z)| dz \right),
\]

where \( a_{\Omega \cap B_\rho(z_0)} = \frac{1}{|\Omega \cap B_\rho(z_0)|} \int_{\Omega \cap B_\rho(z_0)} a(z) dz \). If \( \sup_{R>0} \eta_R(a) < \infty \), we say \( a \in BMO(\Omega) \) (Bounded Mean Oscillation). Moreover, if \( \eta_R(a) \to 0 \) as \( R \to 0 \), we say \( a \in VMO(\Omega) \) (Vanishing Mean Oscillation).

It is stated two iteration lemmas.
Lemma 2.9 ([3]) Let \( \varphi(t) \) be a bounded nonnegative function on \([T_0, T_1]\), where \( T_1 > T_0 \geq 0 \). Suppose that for any \( s, t : T_0 \leq t < s \leq T_1 \), \( \varphi \) satisfies

\[
\varphi(t) \leq \theta_1 \varphi(s) + \frac{a_2}{(s - t)^\alpha} + b_2,
\]

where \( \theta_1, a_2, b_2 \) and \( \alpha \) are nonnegative constants, and \( \theta_1 < 1 \). Then for any \( T_0 \leq \rho < R \leq T_1 \),

\[
\varphi(\rho) \leq c \left( \frac{a_2}{(R - \rho)^\alpha} + b_2 \right),
\] (2.6)

where \( c \) depends only on \( \alpha \) and \( \theta_1 \).

Lemma 2.10 (see [13, 20]) Let \( H \) be a nonnegative increasing function. Suppose that for any \( \rho < R \leq R_0 = \text{dist}(z_0, \partial\Omega) \),

\[
H(\rho) \leq A_1 \left[ \left( \frac{\rho}{R} \right)^{a_1} + \varepsilon \right] H(R) + B_1 R^{b_1},
\]

where \( A_1, a_1 \) and \( b_1 \) are positive constants with \( a_1 > b_1 \). Then there exist positive constants \( \varepsilon_1 = \varepsilon_1(A_1, a_1, b_1) \) and \( c = c(A_1, a_1, b_1) \), such that if \( \varepsilon < \varepsilon_1 \), then

\[
H(\rho) \leq c \left[ \left( \frac{\rho}{R} \right)^{b_1} H(R) + B_1 \rho^{b_1} \right].
\] (2.7)

3 Preliminary inequalities

Theorem 3.1 (Caccioppoli type inequality) Let \( u \in W^{1,1}_2(\Omega) \) be a weak solution to (1.1). Then for any \( B_R \subset \Omega, \rho < R \), we have

\[
\int_{B_\rho} |D_0 u|^2 dz \leq \frac{c}{(R - \rho)^2} \int_{B_R} |u|^2 dz + c \int_{B_R} (|g|^2 + |f|^2) dz.
\] (3.1)

Proof: Let \( \xi(z) \in C_0^\infty(B_R) \) be a cutoff function satisfying

\[
\xi(z) = 1(|z| < \rho), \xi(z) = 0(|z| \geq R), 0 \leq \xi \leq 1, |\partial_x \xi|, |\partial_t \xi| \leq \frac{c}{R - \rho}(j = 1, \ldots, N).
\] (3.2)

Hence

\[
|Y \xi| = |xBD\xi - \partial_t \xi| \leq c |D\xi| + c |\partial_t \xi| \leq \frac{c}{R - \rho},
\]

and by the divergence theorem,

\[
\int_{B_R} Y (u_2\xi^2) dz = 0.
\]

Multiplying both sides of (1.1) by \( u\xi^2 \) and integrating on \( B_R \), we have

\[
\int_{B_R} \left[ -AD_0 u D_0 (u\xi^2) + u\xi^2 Y u \right] dz = \int_{B_R} [gu\xi^2 - f D_0 (u\xi^2)] dz
\]
Then for any \( c \in (0, 1) \) and \( \xi \in C_c^\infty(\mathbb{R}^n) \),
\[
\int_{B_R} A \xi^2 D_0 u D_0 u dz
= -2 \int_{B_R} A u \xi D_0 u D_0 \xi dz - \int_{B_R} u^2 \xi Y \xi dz - \int_{B_R} g u \xi^2 dz + \int_{B_R} f \xi^2 D_0 u dz
+ 2 \int_{B_R} f u \xi D_0 \xi dz.
\]
(3.3)

By using (H1) and Young’s inequality, it follows
\[
\Lambda^{-1} \int_{B_R} |D_0 u|^2 \xi^2 dz
\leq c \varepsilon \int_{B_R} |u|^2 |D_0 \xi|^2 dz + \varepsilon \int_{B_R} |D_0 u|^2 \xi^2 dz
+ c \varepsilon \int_{B_R} |g|^2 \xi^2 dz + \varepsilon \int_{B_R} |f|^2 \xi^2 dz
+ \varepsilon \int_{B_R} |D_0 \xi|^2 dz
\leq \int_{B_R} |u|^2 (c \varepsilon |D_0 \xi|^2 + |Y \xi| \xi + \varepsilon \xi^2 + \varepsilon |D_0 \xi|^2) dz
+ 2 \varepsilon \int_{B_R} |D_0 u|^2 \xi^2 dz + c \varepsilon \int_{B_R} (|g|^2 + |f|^2) \xi^2 dz.
\]
(3.4)

Choosing \( \varepsilon \) small enough such that \( \Lambda^{-1} - 2 \varepsilon > 0 \) and using the property of \( \xi \), one has
\[
\int_{B_R} |D_0 u|^2 \xi^2 dz
\leq \int_{B_R} |u|^2 \left( c \varepsilon |D_0 \xi|^2 + |Y \xi| \xi + \varepsilon \xi^2 + \varepsilon |D_0 \xi|^2 \right) dz
+ c \varepsilon \int_{B_R} (|g|^2 + |f|^2) \xi^2 dz
\leq \int_{B_R} |u|^2 \left( \frac{c \varepsilon}{(R - \rho)^2} + \frac{c \varepsilon}{R - \rho} + \varepsilon \xi^2 + \frac{\varepsilon}{(R - \rho)^2} \right) dz
+ c \varepsilon \int_{B_R} (|g|^2 + |f|^2) \xi^2 dz.
\]
Consequently (3.1) is proved.

**Theorem 3.2** (Sobolev type inequality) Let \( u \in W^{1,1}_2(\Omega) \) be a weak solution to (1.1).

Then for any \( B_R \subset \Omega, \rho < R \), it follows
\[
\|u\|_{L^2(B_R)}^2
\leq \frac{c}{R - \rho} \left( \|u\|_{L^2(B_R)}^{2Q+2} + \|D_0 u\|_{L^2(B_R)}^{2Q+2} + \|g\|_{L^2(B_R)}^{2Q+2} + \|f\|_{L^2(B_R)}^{2Q+2} \right).
\]
(3.5)

**Proof:** We represent \( u \) in terms of the fundamental solution \( \Gamma_0 \) of \( L_0 \) and apply the cutoff function \( \xi \) in (3.2). For any \( z \in B_R \),
\[
(\xi u)(z) = \int_{B_R} \langle A_0 D_0 (\xi u), D_0 \Gamma_0 \rangle dz \overset{\Delta}{=} I_1(z) + I_2(z) + I_3(z),
\]
(3.6)
where
\[ I_1(z) = \int_{B_R} [A_0 u D_0 \xi D_0 \Gamma_0 - \Gamma_0 u Y \xi] d\zeta, \]
\[ I_2(z) = \int_{B_R} [(A_0 - A) \xi D_0 u D_0 \Gamma_0 - \Gamma_0 A D_0 u D_0 \xi] d\zeta \]
and
\[ I_3(z) = \int_{B_R} [A D_0 u D_0 (\xi \Gamma_0) - \xi \Gamma_0 Y u] d\zeta. \]

It yields by using (2.4) and (2.5) that
\[
\|I_1\|_{L^2(B_R)} \leq 2 \left\| \int_{B_R} A_0 u D_0 \xi D_0 \Gamma_0 d\zeta \right\|_{L^2(B_R)} + 2 \left\| \int_{B_R} \Gamma_0 u Y \xi d\zeta \right\|_{L^2(B_R)}
\leq c \| \Gamma_0 (D_0 (u D_0 \xi)) \|_{L^2(B_R)} + c \| \Gamma_0 (u Y \xi) \|_{L^2(B_R)}
\leq c \|u D_0 \xi\|_{L^2(B_R)}^{2(Q+2)} + c \| \Gamma_0 (u Y \xi) \|_{L^2(B_R)^{2(Q+2)}} R
\leq \frac{c}{R - \rho} \|u\|_{L^2(B_R)}^{2(Q+2)} + \frac{cR}{R - \rho} \|u\|_{L^2(B_R)^{2(Q+2)}} R
\leq \frac{c}{R - \rho} \|u\|_{L^2(B_R)^{2(Q+2)}} R
\tag{3.7}
\]
and
\[
\|I_2\|_{L^2(B_R)} \leq 2 \left\| \int_{B_R} (A_0 - A) \xi D_0 u D_0 \Gamma_0 d\zeta \right\|_{L^2(B_R)} + 2 \left\| \int_{B_R} \Gamma_0 A D_0 u D_0 \xi d\zeta \right\|_{L^2(B_R)}
\leq c \| \Gamma_0 (D_0 (\xi D_0 u)) \|_{L^2(B_R)} + c \| \Gamma_0 (D_0 u D_0 \xi) \|_{L^2(B_R)}
\leq c \| \xi D_0 u\|_{L^2(B_R)}^{2(Q+2)} + c \| \Gamma_0 (D_0 u D_0 \xi) \|_{L^2(B_R)^{2(Q+2)}} R
\leq \frac{c}{R - \rho} \|D_0 u\|_{L^2(B_R)^{2(Q+2)}} R
\tag{3.8}
\]

Since \(u\) is a weak solution to (1.1), we infer that
\[ I_3(z) = \int_{B_R} [f D_0 (\xi \Gamma_0) - g \xi \Gamma_0] d\zeta = \int_{B_R} [f \xi D_0 \Gamma_0 + f \Gamma_0 D_0 \xi - g \xi \Gamma_0] d\zeta \]
and

\[
\|I_3\|_{L^2(B_R)} \leq c \left( \int_{B_R} f\xi D_0 \Gamma_0 d\xi \right)^2 + c \left( \int_{B_R} f\Gamma_0 D_0 \xi d\zeta \right)^2 + c \left( \int_{B_R} g\xi \Gamma_0 d\zeta \right)^2 \leq c\|\Gamma_0 (D_0 (f\xi))\|_{L^2(B_R)} + c\|\Gamma_0 (f D_0 \xi)\|_{L^2(B_R)} + c\|\Gamma_0 (g \xi)\|_{L^2(B_R)}
\]

\[
\leq c\|f\xi\|_{L^2(B_R)} + cR \left( \|\Gamma_0 (f D_0 \xi)\|_{L^2(B_R)} + c\|\Gamma_0 (g \xi)\|_{L^2(B_R)} \right)
\]

\[
\leq c\|f\xi\|_{L^2(B_R)} + cR \left( \|f D_0 \xi\|_{L^2(B_R)} + \|g \xi\|_{L^2(B_R)} \right)
\]

\[
\leq c\|f\|_{L^2(B_R)} + cR \left( \frac{c}{R - \rho} \|f\|_{L^2(B_R)} + \|g\|_{L^2(B_R)} \right).
\]

Inserting (3.7), (3.8) and (3.9) into (3.6), it obtains (3.5).

**Theorem 3.3** (Poincaré type inequality) Let \( u \in W^{1,1}_2(\Omega) \) be a weak solution to (1.1). Then for any \( B_R \subset \Omega, \rho < R, \) one has

\[
\int_{B_R} |u|^2 dz \leq \frac{cR^4}{(R - \rho)^2} \int_{B_R} |D_0 u|^2 dz + cR^2 \int_{B_R} (|g|^2 + |f|^2) dz.
\]

**Proof:** Introduce two cutoff functions \( \varsigma(x), \eta(t) \in C^\infty_0(Q_R) \) satisfying

\[
\varsigma(x) = 1(|x| < \rho), \quad \varsigma(x) = 0(|x| \geq R),
\]

\[
0 \leq \varsigma \leq 1, \quad |\partial_x \varsigma| \leq \frac{c}{R - \rho} (j = 1, \ldots, N);
\]

\[
\eta(t) = \begin{cases} \frac{2t - 2(t_0 - R^2/2)}{R^2 - \rho^2}, t \in \left[ t_0 - \frac{R^2}{2}, t_0 - \frac{\rho^2}{2} \right), \\ 1, t \in \left[ t_0 - \frac{\rho^2}{2}, t_0 + \frac{R^2}{2} \right]. \end{cases}
\]

Multiplying both sides of (1.1) by \( u\varsigma^2(x)\eta(t) \) and integrating on \( Q'_R = K_R \times S_R \times I'_R \) \( (I'_R = \left[ t_0 - \frac{R^2}{2}, s \right], s \leq t_0 + \frac{R^2}{2}) \), we have

\[
\int_{Q'_R} [-A D_0 u D_0 (u\varsigma^2\eta) + x B u \varsigma^2 \eta D_0 u - u^2 \eta \partial_t u] dz = \int_{Q'_R} [gu \varsigma^2 \eta - f D_0 (u \varsigma^2 \eta)] dz.
\]

Noting

\[
\int_{Q'_R} u\varsigma^2\eta \partial_t u dz = \frac{1}{2} \int_{Q'_R} \varsigma^2 (u^2 \eta)_t dz - \frac{1}{2} \int_{Q'_R} u^2 \varsigma^2 \eta_t dz,
\]

\[
\int_{Q'_R} x B u \varsigma^2 \eta D_0 u dz = \frac{1}{2} \int_{Q'_R} x BD (u^2 \varsigma^2 \eta) dz - \int_{Q'_R} x Bu^2 \varsigma \eta D_0 \varsigma dz,
\]

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it implies by inserting (3.12) and (3.13) into (3.11) that

\[
\frac{1}{2} \int_{Q_{R'}} u^2 \varsigma^2 \eta_t dz \\
= \int_{Q_{R'}} A \varsigma^2 \eta D_0 u D_0 \varsigma d z + 2 \int_{Q_{R'}} A u \varsigma \eta D_0 u D_0 \varsigma d z - \frac{1}{2} \int_{Q_{R'}} x B D (u^2 \varsigma^2 \eta) d z \\
+ \int_{Q_{R'}} x B u^2 \varsigma \eta D_\varsigma d z + \frac{1}{2} \int_{Q_{R'}} \varsigma^2 (u^2 \eta) d z + \int_{Q_{R'}} g u \varsigma^2 \eta d z \\
- \int_{Q_{R'}} f \varsigma^2 \eta D_0 u d z - 2 \int_{Q_{R'}} f u \varsigma \eta D_0 \varsigma d z \\
= \int_{Q_{R'}} A \varsigma^2 \eta D_0 u D_0 \varsigma d z + 2 \int_{Q_{R'}} A u \varsigma \eta D_0 u D_0 \varsigma d z - \int_{Q_{R'}} Y \left( \frac{1}{2} u^2 \varsigma^2 \eta \right) d z \\
+ \int_{Q_{R'}} x B u^2 \varsigma \eta D_\varsigma d z + \int_{Q_{R'}} g u \varsigma^2 \eta d z - \int_{Q_{R'}} f \varsigma^2 \eta D_0 u d z - 2 \int_{Q_{R'}} f u \varsigma \eta D_0 \varsigma d z. 
\]

(3.14)

By the divergence theorem and the property of \( \varsigma \), it follows

\[
\int_{Q_{R'}} Y \left( \frac{1}{2} u^2 \varsigma^2 \eta \right) d z = 0.
\]

Hence we have by Young’s inequality that

\[
\frac{1}{2} \int_{Q_{R'}} u^2 \varsigma^2 \eta_t d z \\
\leq \Lambda \int_{Q_{R'}} |D_0 u|^2 \varsigma^2 \eta d z + \varepsilon \int_{Q_{R'}} |u|^2 |D_0 \varsigma|^2 \eta d z + c_\varepsilon \int_{Q_{R'}} |D_0 u|^2 \varsigma^2 \eta d z \\
+ c \int_{Q_{R'}} |u|^2 |D_\varsigma| \varsigma \eta d z + c_\varepsilon \int_{Q_{R'}} |g|^2 \varsigma^2 \eta d z + \varepsilon \int_{Q_{R'}} |u|^2 \varsigma^2 \eta d z + c_\varepsilon \int_{Q_{R'}} |f|^2 \varsigma^2 \eta d z \\
+ \varepsilon \int_{Q_{R'}} |D_0 u|^2 \varsigma^2 \eta d z + c_\varepsilon \int_{Q_{R'}} |f|^2 \varsigma^2 \eta d z + \varepsilon \int_{Q_{R'}} |u|^2 |D_0 \varsigma|^2 \eta d z \\
\leq \int_{Q_{R'}} |u|^2 (2 \varepsilon |D_0 \varsigma|^2 \eta + c |D_\varsigma|^2 \varsigma \eta + \varepsilon \varsigma^2 \eta) d z + c \int_{Q_{R'}} |D_0 u|^2 \varsigma^2 \eta d z \\
+ c_\varepsilon \int_{Q_{R'}} (|g|^2 + |f|^2) \varsigma^2 \eta d z. 
\]

(3.15)
In the light of properties of \( \varsigma, \eta \) and (3.15), it yields
\[
\int_{Q_r} |u|^2 dz \leq \int_{Q_{r'}} |u|^2 \varsigma^2 dz \leq c \left( R^2 - \rho^2 \right) \int_{Q_{r'}} |u|^2 \varsigma^2 \eta dz \\
\leq \left( R^2 - \rho^2 \right) \int_{Q_{r'}} |u|^2 (2\varepsilon |D_0\varsigma|^2 \eta + c|D\varsigma|^2 \varsigma \eta + \varepsilon \varsigma^2 \eta) dz \\
+ c \left( R^2 - \rho^2 \right) \int_{Q_{r'}} |D_0u|^2 \varsigma^2 \eta dz + c \varepsilon \left( R^2 - \rho^2 \right) \int_{Q_{r'}} (|g|^2 + |f|^2) \varsigma^2 \eta dz
\]
\[
\leq \int_{Q_r} |u|^2 \left( \frac{2\varepsilon (R^2 - \rho^2) \eta}{(R - \rho)^2} + \frac{c(R^2 - \rho^2) \varsigma \eta}{(R - \rho)^2} + \varepsilon (R^2 - \rho^2) \varsigma^2 \eta \right) dz \\
+ \frac{cR^2(R - \rho)^2}{(R - \rho)^2} \int_{Q_r} |D_0u|^2 dz + c \varepsilon R^2 \int_{Q_r} (|g|^2 + |f|^2) dz
\]
\[
\leq \theta_1 \int_{Q_r} |u|^2 dz + \frac{cR^4}{(R - \rho)^2} \int_{Q_r} |D_0u|^2 dz + c \varepsilon R^2 \int_{Q_r} (|g|^2 + |f|^2) dz,
\]
(3.16)
where \( \theta_1 = \frac{2\varepsilon (R^2 - \rho^2) \eta}{(R - \rho)^2} + \frac{c(R^2 - \rho^2) \varsigma \eta}{(R - \rho)^2} + \varepsilon (R^2 - \rho^2) \varsigma^2 \eta \). Choosing \( \varepsilon \) small enough, it ensures \( 0 < \theta_1 < 1 \) and we have from Lemma 2.9 that
\[
\int_{Q_r} |u|^2 dz \leq \frac{cR^4}{(R - \rho)^2} \int_{Q_r} |D_0u|^2 dz + c \varepsilon R^2 \int_{Q_r} (|g|^2 + |f|^2) dz.
\]
(3.17)
Now (3.17) and \( B_{\rho/c_0} \subset Q_\rho \subset Q_R \subset B_{c_0 R} \) imply (3.10).

## 4 Proof of Theorem 1.1

Let us first describe a known result.

**Lemma 4.1** (reverse Hölder inequality, [11]) Let \( \hat{g} \) and \( \hat{f} \) be nonnegative functions on \( \Omega \) and satisfy
\[
\hat{g} \in L^{\hat{q}}(\Omega), \quad \hat{f} \in L'^{\hat{r}}(\Omega), \quad 1 < \hat{q} < \hat{r}.
\]
If there exist constants \( b_2 \) and \( \theta_2 \) with \( b_2 > 1 \) such that for any \( B_{2R} \subset \Omega \), the inequality holds
\[
\frac{1}{|B_R|} \int_{B_R} \hat{g}^{\hat{q}} dz \\
\leq b_2 \left[ \left( \frac{1}{|B_{4R/3}|} \int_{B_{4R/3}} \hat{g} dz \right)^{\hat{q}} + \frac{1}{|B_{4R/3}|} \int_{B_{4R/3}} \hat{f}^{\hat{q}} dz \right] + \theta_2 \frac{1}{|B_{4R/3}|} \int_{B_{4R/3}} \hat{g}^{\hat{q}} dz,
\]
then there exist positive constants \( \theta_0 = \theta_0(\hat{q}, \Omega) \) and \( \varepsilon_0 \) such that if \( \theta_2 < \theta_0 \), then for any \( \hat{p} \in [\hat{q}, \hat{q} + \varepsilon_0) \), it follows \( \hat{g} \in L_{loc}^{\hat{p}}(\Omega) \) and
\[
\left( \frac{1}{|B_R|} \int_{B_R} \hat{g}^{\hat{p}} dz \right)^{\frac{1}{\hat{p}}} \leq c \left[ \left( \frac{1}{|B_{2R}|} \int_{B_{2R}} \hat{g}^{\hat{q}} dz \right)^{\frac{1}{\hat{q}}} + \left( \frac{1}{|B_{2R}|} \int_{B_{2R}} \hat{f}^{\hat{p}} dz \right)^{\frac{1}{\hat{p}}} \right],
\]
(4.1)
where \( c \) and \( \varepsilon_0 \) depend on \( b_2, \dot{q}, \theta_2 \) and \( Q \).

**Theorem 4.2** Let \( u \in W^{1,1}_2(\Omega) \) be a weak solution to (1.1) in \( \Omega \). Then for any \( p \in \left[ 2, 2 + \frac{2Q}{q+2} \varepsilon_0 \right] \), we have \( D_0 u \in L^p_{\text{loc}}(\Omega) \) and for any \( B_R \subseteq B_{2R} \subseteq \Omega \),

\[
\left( \frac{1}{|B_R|} \int_{B_R} |D_0 u|^p \, dz \right)^{1/p} \leq c \left[ \left( \frac{1}{|B_{2R}|} \int_{B_{2R}} |D_0 u|^2 \, dz \right)^{1/2} + \left( \frac{1}{|B_{2R}|} \int_{B_{2R}} (|g|^2 + |f|^2)^{\frac{p}{2}} \, dz \right)^{\frac{1}{p}} \right]. \tag{4.2}
\]

**Proof:** By using Hölder’s inequality, it implies

\[
\int_{B_{1R/9}} |D_0 u|^{2(\frac{Q+2}{Q+4})} \, dz \leq \left( \int_{B_{1R/9}} |D_0 u|^2 \, dz \right)^{1/2} \left( \int_{B_{1R/9}} |D_0 u|^{\frac{2Q}{Q+4}} \, dz \right)^{\frac{1}{2}} \leq \left( \int_{B_{1R/9}} |D_0 u|^2 \, dz \right)^{1/2} |B_{1R/9}|^{\frac{Q+4}{Q+2}} \left( \int_{B_{1R/9}} |D_0 u|^{\frac{2Q}{Q+4}} \, dz \right)^{\frac{Q+4}{Q+2}}. \tag{4.3}
\]

Combining (3.5) and (4.3), we get

\[
\int_{B_{10R/9}} |u|^2 \, dz \leq \frac{c}{R^2} \left[ \left( \int_{B_{1R/9}} |u|^{\frac{2Q}{Q+4}} \, dz \right)^{\frac{Q+4}{Q+2}} + \left( \int_{B_{1R/9}} |D_0 u|^{\frac{2Q}{Q+4}} \, dz \right)^{\frac{Q+4}{Q+2}} \right] \left( \int_{B_{1R/9}} |u|^2 \, dz \right)^{\frac{1}{2}} \left( \int_{B_{1R/9}} |D_0 u|^{\frac{2Q}{Q+4}} \, dz \right)^{\frac{1}{2}} \right)^2
\]

\[
+ \left[ \left( \int_{B_{1R/9}} |D_0 u|^2 \, dz \right)^{\frac{1}{2}} \left( \int_{B_{1R/9}} |u|^2 \, dz \right)^{\frac{1}{2}} \left( \int_{B_{1R/9}} |D_0 u|^{\frac{2Q}{Q+4}} \, dz \right)^{\frac{1}{2}} \right]^2
\]

\[
+ \left[ \left( \int_{B_{1R/9}} |D_0 u|^2 \, dz \right)^{\frac{1}{2}} \left( \int_{B_{1R/9}} |u|^2 \, dz \right)^{\frac{1}{2}} \left( \int_{B_{1R/9}} |D_0 u|^{\frac{2Q}{Q+4}} \, dz \right)^{\frac{1}{2}} \right]^2
\]

\[
\leq c \int_{B_{11R/9}} |u|^2 \, dz + \frac{c}{R} \left( \int_{B_{11R/9}} |D_0 u|^2 \, dz \right)^{\frac{1}{2}} \left( \int_{B_{11R/9}} |D_0 u|^{\frac{2Q}{Q+4}} \, dz \right)^{\frac{1}{2}} + c \int_{B_{11R/9}} (|f|^2 + |g|^2) \, dz. \tag{4.4}
\]
Noting (3.1), (3.10) and (4.4), it follows
\[ \int_{B_R} |D_0 u|^2 \, dz \]
\[ \leq \frac{c}{R^2} \int_{B_{11R/9}} |u|^2 \, dz + \frac{c}{R^2} \left( \int_{B_{11R/9}} |D_0 u|^2 \, dz \right)^{\frac{Q+4}{2(4Q+27)}} \left( \int_{B_{11R/9}} |D_0 u|^{\frac{2Q}{Q+27}} \, dz \right)^{\frac{1}{2}} + c \int_{B_{10R/9}} (|f|^2 + |g|^2) \, dz + c \int_{B_{10R/9}} (|g|^2 + |f|^2) \, dz \]
\[ \leq c \int_{B_{AR/3}} |D_0 u|^2 \, dz \]
\[ + \frac{c}{R^2} \left( \int_{B_{4R/3}} |D_0 u|^2 \, dz \right)^{\frac{Q+4}{2(4Q+27)}} \left( \int_{B_{4R/3}} |D_0 u|^{\frac{2Q}{Q+27}} \, dz \right)^{\frac{1}{2}} \]
\[ + \frac{c}{R^2} \int_{B_{4R/3}} (|g|^2 + |f|^2) \, dz \]
\[ \leq c \left( \int_{B_{AR/3}} |D_0 u|^2 \, dz \right) + c R^{-\frac{4Q}{Q+2}} \left( \int_{B_{AR/3}} |D_0 u|^{\frac{2Q}{Q+27}} \, dz \right)^{\frac{Q+4}{2}} \]
\[ + \frac{c}{R^2} \left( \int_{B_{4R/3}} (|g|^2 + |f|^2) \, dz \right). \] (4.5)

Let \( \hat{g} = |D_0 u|^{\hat{q}}, \hat{q} = \frac{2Q}{Q+2}, \hat{q} = \frac{2}{\hat{q}} = \frac{Q+2}{Q} > 1, \hat{f} = (|g|^2 + |f|^2)^{\frac{Q}{Q+2}}, \) then we rewrite (4.5) in the form
\[ \frac{1}{|B_R|} \int_{B_R} \hat{g} \, dz \]
\[ \leq c \left[ \left( \int_{B_{AR/3}} \hat{g} \, dz \right)^{\frac{1}{\hat{q}}} + \frac{1}{|B_{AR/3}|} \int_{B_{AR/3}} \hat{f} \, dz \right] + \frac{c}{|B_{4R/3}|} \int_{B_{4R/3}} \hat{g} \, dz. (4.6) \]

It shows from Lemma 4.1 that for any \( \hat{p} \in [\hat{q}, \hat{q} + \varepsilon_0), \)
\[ \left( \frac{1}{|B_R|} \int_{B_R} \hat{g} \, dz \right)^{1/\hat{p}} \leq c \left[ \left( \frac{1}{|B_{2R}|} \int_{B_{2R}} \hat{g} \, dz \right)^{1/\hat{q}} + \left( \frac{1}{|B_{2R}|} \int_{B_{2R}} \hat{f} \, dz \right)^{1/\hat{p}} \right], \]
which means
\[
\left( \frac{1}{|B_R|} \int_{B_R} |D_0u|^p \, dz \right)^{\frac{1}{p}} \leq c \left[ \left( \frac{1}{|B_{2R}|} \int_{B_{2R}} |D_0u|^2 \, dz \right)^{\frac{Q}{Q+2}} + \left( \frac{1}{|B_{2R}|} \int_{B_{2R}} (|g|^2 + |f|^2)^{\frac{p}{q}} \, dz \right)^{\frac{1}{p}} \right].
\]

(4.7)

Setting \( p = \tilde{p} q \in \left[ 2, 2 + \frac{2Q}{Q+2} \varepsilon_0 \right) \), we finish the proof.

**Proof Theorem 1.1:** The conclusion follows from Theorem 4.2 and the cutoff function technique.

### 5 Homogeneous ultraparabolic equation

In this section, we consider the following homogeneous ultraparabolic equation

\[
div (AD_0u) + Yu = 0.
\]

(5.1)

To obtain \( L^p \) estimates for gradients of weak solutions to (5.1), we divide (5.1) into two parts. Let \( v \) be a weak solution to the following Dirichlet boundary value condition to the homogeneous ultraparabolic equation with constant principal part:

\[
\begin{cases}
  \text{div} (A_R D_0 v) + Y v = 0, \text{in} \ B_R, \\
  v = u, \quad \text{on} \ \partial B_R.
\end{cases}
\]

(5.2)

Then \( w = u - v \) satisfies the Dirichlet boundary value condition to the nonhomogeneous ultraparabolic equation with constant principal part:

\[
\begin{cases}
  \text{div} (A_R D_0 w) + Y w = \text{div} ((A_R - A) D_0 u), \text{ in } B_R, \\
  w = 0, \quad \text{on} \ \partial B_R,
\end{cases}
\]

(5.3)

where \( A_R = \frac{1}{|B_R|} \int_{B_R} A \, dz \).

**Lemma 5.1** Let \( v \in W^{1,1}_2 (\Omega) \) be a weak solution to (5.2). Then for any \( B_R \subset \Omega \), one has

\[
\sup_{B_{R/2}} |v|^2 \leq \frac{c}{R^{Q+2}} \int_{B_R} |v|^2 \, dz.
\]

(5.4)

**Proof:** It is true from Corollary 1.4 of [22].

Furthermore, we have

**Lemma 5.2** Let \( v \in W^{1,1}_2 (\Omega) \) be a weak solution to (5.2). Then for any \( B_R \subset \Omega \), \( \rho < R \), it follows

\[
\int_{B_{\rho}} |v|^2 \, dz \leq c \left( \frac{\rho}{R} \right)^{Q+2} \int_{B_R} |v|^2 \, dz.
\]

(5.5)
Proof: When \( \frac{R}{2} \leq \rho < R \), the result is evident. Now it is enough to treat the case \( \rho < \frac{R}{2} \). But by Lemma 5.1, it yields

\[
\int_{B_{\rho}} |v|^2 dz \leq |B_{\rho}| \sup_{B_{\rho}} |v|^2 \leq |B_{\rho}| \sup_{B_{R/2}} |v|^2
\]

\[
\leq |B_{\rho}| \frac{\epsilon}{R^{q+2}} \int_{B_{R}} |v|^2 dz \leq c \left( \frac{\rho}{R} \right)^{Q+2} \int_{B_{R}} |v|^2 dz.
\]

On the gradient of \( v \), we have

**Lemma 5.3** Let \( v \in W^{1,1}_2(\Omega) \) be a weak solution to (5.2). Then for any \( B_R \subset \Omega \), \( \rho < R \), it follows

\[
\int_{B_{\rho}} |D_0 v|^2 dz \leq c \left( \frac{\rho}{R} \right)^Q \int_{B_{R}} |D_0 v|^2 dz.
\]

**Proof:** Combining Theorem 3.1, Theorem 3.3 \((g=f=0)\) and (5.5), we arrive at

\[
\int_{B_{\rho/2}} |D_0 v|^2 dz \leq \frac{\rho}{R} \int_{B_{\rho}} |v|^2 dz \leq \frac{\rho}{R^2} (Q) \int_{B_R} |v|^2 dz
\]

\[
\leq \frac{\rho}{R^3} (Q+2) R^2 \int_{B_{2R}} |D_0 v|^2 dz \leq c \left( \frac{\rho}{R} \right)^Q \int_{B_{2R}} |D_0 v|^2 dz.
\]

**Lemma 5.4** Let \( v \in W^{1,1}_2(\Omega) \) be a weak solution to (5.2). Then for any \( p \in \left[2, 2 + \frac{2Q}{Q+2} \varepsilon_0\right), B_R \subset \Omega, \rho < R \), we have

\[
\int_{B_{\rho}} |D_0 v|^p dz \leq c \left( \frac{\rho}{R} \right)^{Q+2-p} \int_{B_{R}} |D_0 v|^p dz.
\]

**Proof:** By Theorem 4.2 \((g=f=0)\) and (5.6),

\[
\left( \frac{1}{|B_{\rho/2}|} \int_{B_{\rho/2}} |D_0 v|^p dz \right)^\frac{1}{p} \leq c \left( \frac{1}{|B_{\rho}|} \int_{B_{\rho}} |D_0 v|^2 dz \right)^\frac{1}{2} \leq c \left( \frac{1}{|B_{\rho}|} \left( \frac{\rho}{R} \right)^Q \int_{B_{R}} |D_0 v|^2 dz \right)^\frac{1}{2}.
\]

From Hölder’s inequality, it implies

\[
\int_{B_{\rho/2}} |D_0 v|^p dz \leq c |B_{\rho/2}| \left( \frac{1}{|B_{\rho}|} \left( \frac{\rho}{R} \right)^Q \int_{B_{R}} |D_0 v|^2 dz \right)^\frac{p}{2} \leq c \left( \frac{B_{\rho}}{|B_{\rho/2}|} \left( \frac{\rho}{R} \right)^{2Q} |B_{R}| \int_{B_{R}} |D_0 v|^2 dz \right)^\frac{p}{2} \leq c \left( \frac{\rho}{R} \right)^Q \int_{B_{R}} |D_0 v|^p dz
\]

and the proof is ended.

The main result of this section is

**Theorem 5.5** Let \( u \in W^{1,1}_2(\Omega) \) be a weak solution to (5.1). Then for any \( p \in \left[2, 2 + \frac{2Q}{Q+2} \varepsilon_0\right), \varepsilon_0 \) is the constant in Theorem 1.1, \( \frac{p-2}{p}(Q+2) < \mu < Q, B_R \subset \Omega, \rho < R \), one has

\[
\int_{B_{\rho}} |D_0 u|^p dz \leq c \left( \frac{\rho}{R} \right)^{2(Q+2)-p(Q+2-\mu)} \int_{B_{R}} |D_0 u|^p dz.
\]
**Proof:** When \( \frac{1}{2}R \leq \rho < R \), (5.8) is clearly true. The remainder is to treat the case \( \rho < \frac{1}{2}R \).

Multiplying both sides of (5.3) by \( w \) and integrating on \( B_R \), it observes

\[
- \int_{B_R} \mathcal{A} R_0 w D_0 w dz + \int_{B_R} w \mathcal{Y} w dz = - \int_{B_R} (A_R - A) D_0 u D_0 w dz, \quad (5.9)
\]

and from the divergence theorem,

\[
\int_{B_R} w \mathcal{Y} w dz = \frac{1}{2} \int_{B_R} \mathcal{Y} (w^2) dz = 0.
\]

By (H1) and Young’s inequality, we have from (5.9) that

\[
\Lambda^{-1} \int_{B_R} |D_0 w|^2 dz \leq c \epsilon \int_{B_R} |A_R - A|^2 |D_0 u|^2 dz + \epsilon \int_{B_R} |D_0 w|^2 dz. \quad (5.10)
\]

Choosing \( \epsilon \) small enough such that \( \Lambda^{-1} - \epsilon > 0 \), then (5.10) implies

\[
\int_{B_{2\rho}} |D_0 u|^2 dz \leq 2 \int_{B_{2\rho}} |D_0 v|^2 dz + 2 \int_{B_{2\rho}} |D_0 w|^2 dz \\
\leq c \left( \frac{\rho}{R} \right)^Q \int_{B_R} |D_0 v|^2 dz + c \int_{B_R} |D_0 w|^2 dz \\
\leq c \left( \frac{\rho}{R} \right)^Q \int_{B_R} |D_0 u|^2 dz + c \int_{B_R} |D_0 w|^2 dz \\
\leq c \left( \frac{\rho}{R} \right)^Q |B_R| \int_{B_R} |D_0 u|^p dz \int_{B_R} |D_0 w|^p dz + c \left( \frac{\rho}{R} \right)^Q \eta_R (a_{ij}) \int_{B_R} |D_0 u|^p dz \int_{B_R} |D_0 w|^p dz \int_{B_R} |D_0 u|^p dz \\
\leq c \left[ \left( \frac{\rho}{R} \right)^Q + \eta_R (a_{ij}) \right] \left( |B_R| \int_{B_R} |D_0 u|^p dz \right)^{\frac{2}{p}}. \quad (5.12)
\]

It shows owing to Theorem 4.2 (\( g = f = 0 \)) that

\[
\left( |B_R| \int_{B_R} |D_0 u|^p dz \right)^{\frac{2}{p}} \leq c \int_{B_{2\rho}} |D_0 u|^2 dz \\
\leq c \left[ \left( \frac{\rho}{R} \right)^Q + \eta_R (a_{ij}) \right] \left( |B_R| \int_{B_R} |D_0 u|^p dz \right)^{\frac{2}{p}}. \quad (5.13)
\]
Denoting \( H(\rho) = \left( |B_\rho| \frac{p^2}{2} \int_{B_\rho} |D_0 u|^p dz \right)^{\frac{2}{p}}, \ H(R) = \left( |B_R| \frac{p^2}{2} \int_{B_R} |D_0 u|^p dz \right)^{\frac{2}{p}}, \ a_1 = Q, \) 
\( B_1 = 0 \) in Lemma 2.10, we know that there exists \( b_1 = \mu \left( \frac{p^2}{p} (Q + 2) < \mu < Q \right) \) such that
\[
\left( |B_\rho| \frac{p^2}{2} \int_{B_\rho} |D_0 u|^p dz \right)^{\frac{2}{p}} \leq c \left( \frac{\rho}{R} \right)^{\mu} \left( |B_R| \frac{p^2}{2} \int_{B_R} |D_0 u|^p dz \right)^{\frac{2}{p}}.
\]
(5.14)
Inserting \( \frac{|B_R|}{|B_\rho|} \leq c \left( \frac{R}{\rho} \right)^{-Q-2} \) into (5.14), it attains (5.8).

6 Proof of Theorem 1.2

Based on the discussion in the preceding section, let \( v \) be a weak solution to the following problem
\[
\begin{aligned}
\text{div} (AD_0 v) + Y v &= 0, \quad \text{in} \quad B_R, \\
v &= u, \quad \text{on} \quad \partial B_R,
\end{aligned}
\]
(6.1)
then \( w = u - v \) satisfies
\[
\begin{aligned}
\text{div} (AD_0 w) + Y w &= g + \text{div} f, \quad \text{in} \quad B_R, \\
w &= 0, \quad \text{on} \quad \partial B_R.
\end{aligned}
\]
(6.2)

Theorem 6.1 Let \( w \in W^{1,1}_{2,0} (\Omega) \) be a weak solution to (6.2). Then for any \( B_{2R} \subset \Omega \), one has
\[
\int_{B_R} |D_0 w|^2 dz \leq c \int_{B_{2R}} (|g|^2 + |f|^2) dz.
\]
(6.3)

Proof: Multiplying both sides of (6.2) by \( w \) and integrating on \( B_R \),
\[
- \int_{B_R} AD_0 w D_0 w dz + \int_{B_R} w Y w dz = \int_{B_R} g w dz - \int_{B_R} f D_0 w dz.
\]
(6.4)
By (H1), the divergence theorem and Young’s inequality, we have
\[
\Lambda^{-1} \int_{B_R} |D_0 w|^2 dz \leq c_\varepsilon \int_{B_R} |g|^2 dz + \varepsilon \int_{B_R} |w|^2 dz + c_\varepsilon \int_{B_R} |f|^2 dz + \varepsilon \int_{B_R} |D_0 w|^2 dz.
\]
(6.5)
Since by using (3.10),
\[
\int_{B_R} |w|^2 dz \leq c R^2 \int_{B_{2R}} |D_0 w|^2 dz + c R^2 \int_{B_{2R}} (|g|^2 + |f|^2) dz,
\]
(6.6)
it implies
\[
\int_{B_R} |D_0 w|^2 dz \\
\leq c_\varepsilon R^2 \int_{B_{2R}} |D_0 w|^2 dz + c \varepsilon R^2 \int_{B_{2R}} (|g|^2 + |f|^2) dz \\
+ c_\varepsilon \int_{B_R} (|g|^2 + |f|^2) dz + \varepsilon \int_{B_R} |D_0 w|^2 dz \\
\leq \varepsilon \int_{B_{2R}} |D_0 w|^2 dz + c_\varepsilon \int_{B_{2R}} (|g|^2 + |f|^2) dz.
\]
Then for any $\rho \leq R$,

$$\int_{B_\rho} |D_0 w|^2 dz \leq \int_{B_R} |D_0 w|^2 dz \leq \varepsilon \int_{B_{2R}} |D_0 w|^2 dz + c_e \int_{B_{2R}} |g|^2 dz + c_e \int_{B_{2R}} |f|^2 dz \leq \varepsilon \int_{B_{2R}} |D_0 w|^2 dz + \frac{c_e R^2}{(2R-\rho)^2} \int_{B_{2R}} |g|^2 dz + c_e \int_{B_{2R}} |f|^2 dz.$$ 

Now due to Lemma 2.9, we obtain

$$\int_{B_\rho} |D_0 w|^2 dz \leq \frac{c R^2}{(2R-\rho)^2} \int_{B_{2R}} |g|^2 dz + c \int_{B_{2R}} |f|^2 dz,$$

and the conclusion holds with $\rho = R$.

**Theorem 6.2** Let $w \in W^{1,1}_{2,0}(\Omega)$ be a weak solution to (6.2). Then for any $p \in \left[2, 2 + \frac{2\alpha}{Q+2}\varepsilon_0\right)$, we have $D_0 w \in L^p_{\text{loc}}(\Omega)$, and for any $B_R \subset B_{4R} \subset \Omega$,

$$\int_{B_R} |D_0 w|^p dz \leq c \int_{B_{4R}} (|g|^p + |f|^p) dz. \quad (6.7)$$

**Proof:** By (4.2) and (6.3), it follows

$$\int_{B_R} |D_0 w|^p dz \leq c |B_R| \left[ \left( \frac{1}{|B_{2R}|} \int_{B_{2R}} |D_0 w|^2 dz \right)^{\frac{p}{2}} + \left( \frac{1}{|B_{2R}|} \int_{B_{2R}} (|g|^2 + |f|^2) dz \right)^{\frac{p}{2}} \right]^p \leq c |B_R| \left[ \left( \frac{c}{|B_{2R}|} \int_{B_{4R}} (|g|^2 + |f|^2) dz \right)^{\frac{p}{2}} + \left( \frac{1}{|B_{2R}|} \int_{B_{2R}} (|g|^2 + |f|^2) dz \right)^{\frac{p}{2}} \right]^p \leq c |B_R| \left[ \left( \frac{1}{|B_{2R}|} \int_{B_{4R}} (|g|^p + |f|^p) dz \right)^{\frac{1}{p}} + \left( \frac{1}{|B_{2R}|} \int_{B_{2R}} (|g|^p + |f|^p) dz \right)^{\frac{1}{p}} \right]^p \leq c \int_{B_{4R}} (|g|^p + |f|^p) dz.$$ 

**Theorem 6.3** Let $u \in W^{1,1}_2(\Omega)$ be a weak solution to (1.1). Then for any $p \in \left[2, 2 + \frac{2\alpha}{Q+2}\varepsilon_0\right)$, we have $D_0 u \in L^p_{\text{loc}}(\Omega)$ and for any $B_R \subset B_{4R} \subset \Omega$,

$$\int_{B_R} |D_0 u|^p dz \leq c \left[ \left( \frac{\rho}{R} \right)^{Q+2-\lambda} \int_{B_{4R}} |D_0 u|^p dz + \rho^{Q+2-\lambda} \left( \|g\|^p_{L^{p,\lambda}} + \|f\|^p_{L^{p,\lambda}} \right) \right]. \quad (6.8)$$
Proof: Combining Theorem 5.5 and Theorem 6.2 indicates
\[
\int_{B_{p}} |D_{0}u|^{p} dz \leq 2 \int_{B_{p}} |D_{0}v|^{p} dz + 2 \int_{B_{p}} |D_{0}w|^{p} dz
\]
\[
\leq c \left( \frac{\rho}{R} \right)^{2(Q+2)-p(Q+2-\mu)} \int_{B_{R}} |D_{0}v|^{p} dz + 2 \int_{B_{p}} |D_{0}w|^{p} dz
\]
\[
\leq c \left( \frac{\rho}{R} \right)^{2(Q+2)-p(Q+2-\mu)} \int_{B_{R}} |D_{0}u|^{p} dz + c \int_{B_{R}} (|g|^{p} + |f|^{p}) dz
\]
\[
\leq c \left( \frac{\rho}{R} \right)^{2(Q+2)-p(Q+2-\mu)} \int_{B_{R}} |D_{0}u|^{p} dz + c \frac{|B_{4R}|}{R^{\lambda}} \left( \|g\|_{L^{p,\lambda}}^{p} + \|f\|_{L^{p,\lambda}}^{p} \right)
\]
\[
\leq c \left( \frac{\rho}{R} \right)^{2(Q+2)-p(Q+2-\mu)} \int_{B_{R}} |D_{0}u|^{p} dz + cR^{Q+2-\lambda} \left( \|g\|_{L^{p,\lambda}}^{p} + \|f\|_{L^{p,\lambda}}^{p} \right). \quad (6.9)
\]
Let \( H(\rho) = \int_{B_{\rho}} |D_{0}u|^{s} dz \), \( H(R) = \int_{B_{R}} |D_{0}u|^{s} dz \), \( a_{1} = \frac{2(Q+2)-s(Q+2-\mu)}{2} \), \( b_{1} = Q + 2 - \lambda \), \( B_{1} = c \left( \|g\|_{L^{p,\lambda}}^{p} + \|f\|_{L^{p,\lambda}}^{p} \right) \), \( 0 < \lambda < Q + 2 \). Note that there exists \( \mu, Q + 2 - \frac{2\lambda}{p} < \mu < Q \) such that \( a_{1} > b_{1} \). Hence we can conclude from Lemma 2.10 that
\[
\int_{B_{p}} |D_{0}u|^{p} dz \leq c \left( \frac{\rho}{R} \right)^{Q+2-\lambda} \int_{B_{R}} |D_{0}u|^{p} dz + \rho^{Q+2-\lambda} \left( \|g\|_{L^{p,\lambda}}^{p} + \|f\|_{L^{p,\lambda}}^{p} \right).
\]

Proof Theorem 1.2: The result of Theorem 1.2 follows in virtue of Theorem 6.3 and the cutoff function technique.

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