Sufficient Conditions for Conservativity of Minimal Quantum Dynamical Semigroups

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Abstract. The conservativity of a minimal quantum dynamical semigroup is proved whenever there exists a “reference” subharmonic operator bounded from below by the dissipative part of the infinitesimal generator. We discuss applications of this criteria in mathematical physics and quantum probability.

1. Introduction

A quantum dynamical semigroup (q.d.s.) \( \mathcal{T} = (\mathcal{T}_t)_{t \geq 0} \) on \( \mathcal{B}(h) \), the Banach space of bounded operators on a Hilbert space \( h \), is a \( w^* \)-continuous semigroup of completely positive linear maps on \( \mathcal{B}(h) \). Here \( I \) denotes the identity operator. A q.d.s. is conservative (or identity preserving, or Markovian) if \( \mathcal{T}_t(I) = I \).

Q.d.s. arise in the study of irreversible evolutions in quantum mechanics (see [2], [3], [4] and the references therein) and as a quantum analogue of classical Markovian semigroups in quantum probability (see [20], [21]).

In rather general cases the infinitesimal generator \( \mathcal{L} \) can be written (formally) as

\[
\mathcal{L}(X) = i [H, X] - \frac{1}{2} XM + \sum_{\ell=1}^{\infty} L_{\ell}^* X L_{\ell} - \frac{1}{2} MX,
\]

(1.1)

where \( M = \sum_{\ell=1}^{\infty} L_{\ell}^* L_{\ell} \) and \( H \) is a symmetric operator satisfying some conditions that will be made precise later. However, even if \( \mathcal{L}(I) = 0 \) for unbounded generators (1.1), the q.d.s. with the formal generator (1.1) may not be unique and conservative (see examples in [6], [7], [12]).

The study of conservativity conditions is important in quantum probability because they play a key role in the proof of uniqueness and unitarity of solutions of an Hudson–Parthasarathy quantum stochastic differential equation (see e.g. [7], [8], [14], [15], [20], [21]). Moreover they allow to deduce regularity conditions for trajectories of classical Markov processes (see, for example, [10], [17]) from an operator-theoretic approach.

On the other hand, when a q.d.s. is conservative, the predual semigroup (see [4], [12]) is trace preserving. A conservative irreversible Markov evolution on a von Neumann operator algebra can be considered as an analog of an isometric evolution in Hilbert space, and the infinitesimal generators of conservative q.d.s. play the same role in our approach as essentially self-adjoint operators generating unitary groups in Hilbert spaces.

Necessary and sufficient conditions for conservativity of a q.d.s. were obtained in [7],[8]. Some of these conditions, however, are excessive and difficult to check in practically interesting examples. A simplified form of sufficient conditions
was described in [9] and improved essentially in our previous paper [11] in view of applications to quantum stochastic calculus (see, for example, [15]). These conditions can be written formally as follows

$$i[H, M] \leq bM, \quad L(M) \leq bM$$

(1.2)

where $b$ is a constant.

These results are improved here in several aspects. In fact:

1. The second inequality (1.2) is assumed only for some self-adjoint operator $C$ bounded from below by $M$,

2. The operators $L_\ell$ need not to be closed or even closable, as well as the form represented by the operator $-M - 2iH$ (see Example 5.3),

Moreover the previous proof based on technical inequalities for contractive completely positive maps is replaced by a simple one based on a priori bounds for the resolvent of the minimal q.d.s. (see Theorem 3.1). The operator $C$ in the new sufficient condition can be considered as a “generalized” subharmonic operator for the q.d.s. $T$.

In Section 5 we apply our conditions to three q.d.s. The first one arises in a phenomenological model for a physical evolution (see [3], [4]); conservativity of this q.d.s. could not be proved by direct applications of the techniques developed in [9], [11]. The second one is the problem of constructing a quantum extension of the Brownian motion on $[0, +\infty]$ with partial reflection at 0 where a non-closable operator $L_\ell$ appears. In this case the appropriate operator $C$ turns out to be a singular perturbation of the second derivative on $[0, +\infty]$ by a delta function of the form studied in [1]. The third is the simplest example in which the quadratic form represented by the operator $-M - 2iH$ is not closed (see [18]).

### 2. The minimal quantum dynamical semigroup

Let $h$ be a complex separable Hilbert space endowed with the scalar product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ and let $B(h)$ be the Banach space of bounded operators in $h$. The uniform norm in $B(h)$ will be denoted by $\| \cdot \|_\infty$ and the identity operator in $h$ will be denoted by $I$. We shall denote by $D(G)$ the domain of an operator $G$ in $h$.

**Definition 2.1.** A quantum dynamical semigroup (q.d.s. in the sequel) in $B(h)$ is a family $\mathcal{T} = (\mathcal{T}_t)_{t \geq 0}$ of operators in $B(h)$ with the following properties:

i) $\mathcal{T}_0(X) = X$, for all $X \in B(h)$,

ii) $\mathcal{T}_{t+s}(X) = \mathcal{T}_t(\mathcal{T}_s(X))$, for all $s, t \geq 0$ and all $X \in B(h)$,

iii) $\mathcal{T}_t(I) \leq I$, for all $t \geq 0$,
iv) (complete positivity) for all \( t \geq 0 \), for all integer \( n \) and all finite sequences \((X_j)_{j=1}^n\), \((Y_l)_{l=1}^n\) of elements of \( \mathcal{B}(h) \) we have

\[
\sum_{j,l=1}^n Y_l^* \mathcal{T}_t(X_j^* X_j) Y_j \geq 0,
\]

v) (normality) for every sequence \((X_n)_{n \geq 0}\) of elements of \( \mathcal{B}(h) \) converging weakly to an element \( X \) of \( \mathcal{B}(h) \) the sequence \((\mathcal{T}_t(X_n))_{n \geq 0}\) converges weakly to \( \mathcal{T}_t(X) \) for all \( t \geq 0 \),

vi) (ultraweak continuity) for all trace class operator \( \rho \) in \( h \) and all \( X \in \mathcal{B}(h) \) we have

\[
\lim_{t \to 0^+} \text{Tr}(\rho \mathcal{T}_t(X)) = \text{Tr}(\rho X).
\]

We recall that:

a) as a consequence of properties iii, iv, (see e.g. [13]), for all \( X \in \mathcal{B}(h) \) and all \( t \geq 0 \), we have the inequality

\[
\| \mathcal{T}_t(X) \|_\infty \leq \| X \|_\infty. \tag{2.1}
\]

Thus, for all \( t \geq 0 \), \( \mathcal{T}_t \) is continuous also for the norm \( \| \cdot \|_\infty \).

b) as a consequence of properties iv, vi, for all \( X \in \mathcal{B}(h) \), the map \( t \to \mathcal{T}_t(X) \) is strongly continuous.

**Definition 2.2.** A q.d.s. \( \mathcal{T} \) is said to be conservative if \( \mathcal{T}_t(I) = I \) for all \( t \geq 0 \).

The bounded infinitesimal generator of a norm continuous q.d.s. was characterized by Gorini, Lindblad, Kossakowski and Sudarshan (see e.g. [21] Th. 30.12 p. 267). The characterization problem for arbitrary q.d.s. is still open (see e.g. [16]).

A very large class of q.d.s. with unbounded generators was constructed by Davies [12] (see also [23]) by considering operators \( G, L_\ell \) \((\ell = 1, 2, \ldots)\) satisfying the following technical assumption:

**A -** The operator \( G \) is the infinitesimal generator of a strongly continuous contraction semigroup \( P = (P(t))_{t \geq 0} \) in \( h \). The domain of the operators \((L_\ell)_{\ell=1}^\infty\) contains the domain \( D(G) \) of the operator \( G \). For all \( u, v \in D(G) \), we have

\[
\langle v, Gu \rangle + \langle Gv, u \rangle + \sum_{\ell=1}^\infty \langle L_\ell v, L_\ell u \rangle = 0. \tag{2.2}
\]
As shown in [11] Prop. 2.5 we could assume only that the domain of the operators $L_\ell$ contains a vector space $D$ which is a core for $G$ and that (2.2) holds for all $u, v \in D$.

For all $X \in \mathcal{B}(h)$ consider the sesquilinear form $\mathcal{L}(X)$ in $h$ with domain $D(G) \times D(G)$ given by

$$\langle v, \mathcal{L}(X)u \rangle = \langle v, XGu \rangle + \langle Gv, Xu \rangle + \sum_{\ell=1}^{\infty} \langle L_\ell v, XL_\ell u \rangle$$

(2.3)

Under the assumption $A$ it is possible to construct a q.d.s. $\mathcal{T}$ satisfying the equation

$$\langle v, \mathcal{T}_t(X)u \rangle = \langle v, Xu \rangle + \int_0^t \langle v, \mathcal{L}(\mathcal{T}_s(X))u \rangle \, ds$$

(2.4)

for all $u, v \in D(G)$ and all $X \in \mathcal{B}(h)$. As a first step one proves the following

**Proposition 2.3.** Suppose that condition $A$ holds and, for all $X \in \mathcal{B}(h)$, let $(\mathcal{T}_t(X))_{t \geq 0}$ be a strongly continuous family of elements of $\mathcal{B}(h)$ satisfying (2.1). The following conditions are equivalent:

(i) equation (2.4) holds for all $v, u \in D(G)$,

(ii) for all $v, u \in D(G)$ we have

$$\langle v, \mathcal{T}_t(X)u \rangle = \langle P(t)v, XP(t)u \rangle + \sum_{\ell=1}^{\infty} \int_0^t \langle L_\ell P(t-s)v, \mathcal{T}_s(X)L_\ell P(t-s)u \rangle \, ds.$$  

(2.5)

**Proof.** In order to show that condition i implies condition ii we fix $t$ and compute the derivative

$$\frac{d}{ds} \langle P(t-s)v, \mathcal{T}_s(X)P(t-s)u \rangle = \sum_{\ell=1}^{\infty} \langle L_\ell P(t-s)v, \mathcal{T}_s(X)L_\ell P(t-s)u \rangle$$

using equation (2.4). Clearly (2.5) follows integrating this identity on the interval $[0, t]$. We prove now that condition ii implies condition i. We recall first that $D(G^2)$ is a core for $G$ and, for all $v, u \in D(G^2)$, we compute the derivative

$$\frac{d}{dt} \langle v, \mathcal{T}_t(X)u \rangle = \langle P(t)v, XP(t)Gu \rangle + \langle P(t)Gu, XP(t)u \rangle$$

$$+ \sum_{\ell=1}^{\infty} \langle L_\ell v, \mathcal{T}_t(X)L_\ell u \rangle$$

$$+ \int_0^t \sum_{\ell=1}^{\infty} \langle L_\ell P(t-s)Gv, \mathcal{T}_s(X)L_\ell P(t-s)u \rangle \, ds$$

$$+ \int_0^t \sum_{\ell=1}^{\infty} \langle L_\ell P(t-s)v, \mathcal{T}_s(X)L_\ell P(t-s)Gu \rangle \, ds.$$
The right-hand side of the above equation coincides with $\langle v, \mathcal{L}(T_t(X))u \rangle$ by (2.4). Therefore (2.4), for $v, u \in D(G^2)$, follows by integration on $[0, t]$. Since $D(G^2)$ is a core for $G$ the proof is complete. ■

A solution of the equation (2.5) is obtained by the iterations

$$
\begin{align*}
\langle u, T_t^{(0)}(X)u \rangle & = \langle P(t)u, XP(t)u \rangle \\
\langle u, T_t^{(n+1)}(X)u \rangle & = \langle P(t)u, XP(t)u \rangle \\
& + \sum_{\ell=1}^{\infty} \int_0^t \langle L_{\ell}P(t-s)u, T_s^{(n)}(X)L_{\ell}P(t-s)u \rangle \, ds
\end{align*}
$$

(2.6)

(with $u \in D$). Indeed, for all positive elements $X$ of $\mathcal{B}(h)$ and all $t \geq 0$, the sequence of operators $\left( T_t^{(n)}(X) \right)_{n \geq 0}$ is non-decreasing. Therefore it is strongly convergent and its limits for $X \in \mathcal{B}(h)$ and $t \geq 0$ define the minimal solution $T^{(\text{min})}$ of (2.5). We refer to [7], [12], [13] for more details. The name “minimal” is justified by the fact that, given another solution $(T_t)_{t \geq 0}$ of (2.4), one can easily prove that

$$
T_t^{(\text{min})}(X) \leq T_t(X) \leq \|X\|I
$$

(2.7)

for all positive elements $X$ of $\mathcal{B}(h)$ and all $t \geq 0$ (see [7], [13]). The minimal solution however, in spite of (2.2), is possibly non-conservative (see e.g. [12] Ex. 3.3 p. 174, [6] Ex. 3.6, 3.7 p. 97).

The infinitesimal generator $\mathcal{L}^{(\text{min})}$ will be given by

$$
\mathcal{L}^{(\text{min})}(X) = w^* - \lim_{t \to 0^+} t^{-1} \left( T_t^{(\text{min})}(X) - X \right)
$$

for all $X \in \mathcal{B}(h)$ such that the limit exists in the ultraweak topology on $\mathcal{B}(h)$. Hence $\mathcal{L}$ is an extension of $\mathcal{L}^{(\text{min})}$. Moreover it can be shown that the semigroup $T^{(\text{min})}$ is conservative if and only if the identity operator $I$ belongs to the domain of $\mathcal{L}^{(\text{min})}$ and $\mathcal{L}^{(\text{min})}(I) = 0$ (see e.g. [9]).

3. A representation of the resolvent of the minimal q.d.s.

Let us consider the linear monotone maps $\mathcal{P}_\lambda : \mathcal{B}(h) \to \mathcal{B}(h)$ and $\mathcal{Q}_\lambda : \mathcal{B}(h) \to \mathcal{B}(h)$ defined by

$$
\langle v, \mathcal{P}_\lambda(X)u \rangle = \int_0^\infty \exp(-\lambda s) \langle P(s)v, XP(s)u \rangle \, ds
$$

(3.1)

$$
\langle v, \mathcal{Q}_\lambda(X)u \rangle = \sum_{\ell=1}^{\infty} \int_0^\infty \exp(-\lambda s) \langle L_{\ell}P(s)v, XL_{\ell}P(s)u \rangle \, ds
$$

(3.2)
for all $\lambda > 0$ and $X \in \mathcal{B}(h)$, $v, u \in D(G)$. It is easy to check that both $P_\lambda$ and $Q_\lambda$ are completely positive and normal contractions in $\mathcal{B}(h)$ (see e.g. [11] sect.2).

The resolvent of the minimal q.d.s. $(\mathcal{R}_\lambda^{(\text{min})})_{\lambda > 0}$ defined by

$$\langle v, \mathcal{R}_\lambda^{(\text{min})}(X)u \rangle = \int_0^\infty \exp(-\lambda s) \langle v, T_s^{(\text{min})}(X)u \rangle ds$$

(with $X \in \mathcal{B}(h)$ and $v, u \in h$) can be represented as follows:

**Theorem 3.1.** For every $\lambda > 0$ and $X \in \mathcal{B}(h)$ we have

$$\mathcal{R}_\lambda^{(\text{min})}(X) = \sum_{k=0}^\infty Q_k^\lambda (P_\lambda(X))$$

(3.3)

the series being convergent for the strong operator topology.

**Proof.** Consider the sequence $(\mathcal{R}_\lambda^{(n)})_{n \geq 0}$ of linear monotone maps $\mathcal{R}_\lambda^{(n)} : \mathcal{B}(h) \to \mathcal{B}(h)$ given by

$$\langle v, \mathcal{R}_\lambda^{(n)}(X)u \rangle = \int_0^\infty \exp(-\lambda s) \langle v, T_s^{(n)}(X)u \rangle ds$$

where the operators $T_s^{(n)}$ are defined by (2.6). Clearly (2.1) guarantees that $\mathcal{R}_\lambda^{(n)}$ is well defined. Moreover, for all positive elements $X$ of $\mathcal{B}(h)$, the sequence $(\mathcal{R}_\lambda^{(n)}(X))_{n \geq 0}$ is non-decreasing. Therefore, by the definition of minimal q.d.s., for all $u \in h$ we have

$$\langle u, \mathcal{R}_\lambda^{(\text{min})}(X)u \rangle = \sup_{n \geq 0} \langle u, \mathcal{R}_\lambda^{(n)}(X)u \rangle = \int_0^\infty \exp(-\lambda s) \langle u, T_s^{(\text{min})}(X)u \rangle ds.$$ 

The second equation (2.6) yields

$$\langle u, \mathcal{R}_\lambda^{(n+1)}(X)u \rangle = \int_0^\infty e^{-\lambda t} \langle P(t)v, XP(t)u \rangle dt$$

$$+ \sum_{\ell=1}^\infty \int_0^\infty e^{-\lambda t} dt \int_0^t \langle L_\ell P(t-s)u, T_s^{(n)}(X)L_\ell P(t-s)u \rangle ds$$

for all $u, v \in D(G)$. By the change of variables $(r, s) = (t - s, s)$ in the above double integral we have

$$\langle u, \mathcal{R}_\lambda^{(n+1)}(X)u \rangle = \langle u, \mathcal{P}_\lambda(X)u \rangle$$

$$+ \sum_{\ell=1}^\infty \int_0^\infty e^{-\lambda r} dr \int_0^\infty e^{-\lambda s} \langle L_\ell P(r)u, T_s^{(n)}(X)L_\ell P(r)u \rangle ds.$$
Thus we obtain the recursion formula
\[ R^{(n+1)}_\lambda(X) = \mathcal{P}_\lambda(X) + \mathcal{Q}_\lambda(R^{(n)}_\lambda(X)). \]

Iterating \( n \) times this equation we have
\[ R^{(n+1)}_\lambda(X) = \sum_{k=0}^{n+1} Q^k_\lambda(\mathcal{P}_\lambda(X)) \]
and (3.3) follows letting \( n \) tend to \(+\infty\). Clearly (3.3) also holds for an arbitrary element of \( \mathcal{B}(h) \) since each bounded operator can be written as a linear combination of positive self-adjoint operators.

The following proposition gives another useful relation between \( \mathcal{P}_\lambda, \mathcal{Q}_\lambda \) and \( R^{(\text{min})}_\lambda \).

**Proposition 3.2.** Suppose that the condition \( A \) holds and fix \( \lambda > 0 \). For all \( n \geq 1 \) we have
\[ \sum_{k=0}^{n} Q^k_\lambda(\mathcal{P}_\lambda(I)) + \lambda^{-1} Q^{n+1}_\lambda(I) = \lambda^{-1} I. \]  
(3.4)

**Proof.** For all \( u \in D(G) \) a straightforward computation yields
\[ \sum_{\ell=1}^{\infty} \int_{0}^{\infty} e^{-\lambda t} \| L_\ell P(t)u \|^2 dt = -2\Re \int_{0}^{\infty} e^{-\lambda t} \langle P(t)u, G P(t)u \rangle dt \]
\[ = - \int_{0}^{\infty} e^{-\lambda t} \frac{d}{dt} \| P(t)u \|^2 dt \]
\[ = \| u \|^2 - \lambda \int_{0}^{\infty} e^{-\lambda t} \| P(t)u \|^2 dt \]
(3.5)

Therefore we have
\[ \mathcal{P}_\lambda(I) + \lambda^{-1} \mathcal{Q}_\lambda(I) = \lambda^{-1} I. \]

This proves (3.4) for \( n = 0 \). Suppose it has been established for an integer \( n \). Applying the map \( \mathcal{Q}_\lambda \) to both sides of (3.4) yields
\[ \sum_{k=1}^{n+1} Q^k_\lambda(\mathcal{P}_\lambda(I)) + \lambda^{-1} Q^{n+2}_\lambda(I) = \lambda^{-1} \mathcal{Q}_\lambda(I) = \lambda^{-1} I - \mathcal{P}_\lambda(I). \]

This proves (3.4) for the integer \( n + 1 \) and completes the proof.

The representation formula for the resolvent of the minimal q.d.s. of Theorem 3.1 allows us to give a quick proof of the following necessary and sufficient condition for conservativity obtained by the first author in [7] (see also [11] Prop. 2.7).
Proposition 3.3. Suppose that the condition A holds and fix $\lambda > 0$. Then the sequence of positive operators $(Q^n(\lambda)_{n \geq 0}$ is non-increasing. Moreover the following conditions are equivalent:
(i) the q.d.s. $\mathcal{T}^{(\min)}$ is conservative,
(ii) $s-$ lim$_{n \to \infty} Q^n_{\lambda}(I) = 0$.

Proof. The sequence of positive operators $(Q^n_{\lambda}(I))_{n \geq 0}$ is non-increasing because (3.4) yields
\[ Q^n_{\lambda}(I) - Q^{n+1}_{\lambda}(I) = \lambda Q^n_{\lambda}(P_{\lambda}(I)). \]
Therefore it is strongly convergent to a positive operator $Y$. Letting $n$ tend to $+\infty$ in (3.4), we have
\[ R^{(\min)}_{\lambda}(I) + \lambda^{-1}Y = \lambda^{-1}I. \]
Now condition i can be clearly stated as: $R^{(\min)}_{\lambda}(I)$ coincides with $\lambda^{-1}I$. Therefore the desired equivalence follows. 

4. Sufficient conditions for conservativity
The minimal q.d.s. is conservative whenever, for a fixed $\lambda > 0$, the series
\[ \sum_{k=1}^{\infty} \langle u, Q^k_{\lambda}(I)u \rangle \tag{4.1} \]
is convergent for all $u$ in a dense subspace of $h$. In fact, in this case, condition (ii) of Proposition 3.3 holds because the sequence of positive operators $(Q^k_{\lambda}(I))_{k \geq 1}$ is non-increasing.

In this section we shall give easily verifiable conditions on the operators $G$, $L_\ell$ that guarantee convergence of (4.1). As a first step we shall prove an easy estimate. Let $R(n; G)$ be the resolvent operator $(nI - G)^{-1}$. The operator in $h$ with domain $D(G)$
\[ \sum_{\ell=1}^{\infty} (nL_\ell R(n; G))^* (nL_\ell R(n; G)) \]
has a unique bounded extension by virtue of identity (2.2) and well-known properties of resolvent operators. We shall denote this bounded extension by $F_n$. Notice that $F_n$ is a positive self-adjoint operator.

Proposition 4.1. For every $u \in h$ we have
\[ \sum_{k=1}^{\infty} \langle u, Q^k_{\lambda}(I)u \rangle \leq \lim inf_{n \to \infty} \langle u, R^{(\min)}_{\lambda}(F_n)u \rangle. \]
**Proof.** For $u \in D(G)$, $n \geq 1$, we have

$$
\langle u, \mathcal{P}_\lambda(F_n)u \rangle = \sum_{\ell=1}^{\infty} \int_0^\infty e^{-\lambda t} \| nL_\ell R(n; G)P(t)u \|^2 dt
= \sum_{\ell=1}^{\infty} \int_0^\infty e^{-\lambda t} \| L_\ell P(t)(nR(n; G)u) \|^2 dt
= \langle nR(n; G)u, n\mathcal{Q}_\lambda(I)R(n; G)u \rangle.
$$

Therefore the bounded operators $\mathcal{P}_\lambda(F_n)$ and $n^2R(n; G^*)\mathcal{Q}_\lambda(I)R(n; G)$ coincide. Moreover the sequence of operators $(\mathcal{P}_\lambda(F_n))_{n \geq 1}$ is uniformly bounded and converges strongly to $\mathcal{Q}_\lambda(I)$ by well-known properties of resolvent operators.

The maps $\mathcal{Q}_\lambda^k$ are normal (cf. Definition 2.1 v). We have then, for $u \in h$,

$$
\sum_{k=1}^{\infty} \langle u, \mathcal{Q}_\lambda^k(I)u \rangle \leq \liminf_{n \to \infty} \sum_{k=0}^{\infty} \langle u, \mathcal{Q}_\lambda^k(\mathcal{P}_\lambda(F_n))u \rangle = \liminf_{n \to \infty} \langle u, \mathcal{R}_\lambda^{(\min)}(F_n)u \rangle,
$$

by Fatou’s lemma and Theorem 3.1.

In order to estimate $\mathcal{R}_\lambda^{(\min)}(C)$ for self-adjoint operators $C$ we introduce now our key assumption

**C** - A positive self-adjoint operator $C$ satisfies Condition **C** if:

- the domain of its positive square root $C^{1/2}$ contains the domain $D(G)$ of $G$ and $D(G)$ is a core for $C^{1/2}$,
- the linear manifolds $L_\ell(D(G^2))$, $\ell \geq 1$, are contained in the domain of $C^{1/2}$,
- there exists a positive constant $b$ such that

$$
2\Re \left\langle C^{1/2}u, C^{1/2}Gu \right\rangle + \sum_{\ell=1}^{\infty} \left\langle C^{1/2}L_\ell u, C^{1/2}L_\ell u \right\rangle \leq b \left\| C^{1/2}u \right\|^2 \quad (4.2)
$$

for all $u \in D(G^2)$.

**Remark.** Condition **C** implies that, for each $u \in D(G^2)$, the function $t \to \| C^{1/2}P(t)u \|^2$ is differentiable and

$$
\frac{d}{dt} \left\| C^{1/2}P(t)u \right\|^2 = 2\Re \left\langle C^{1/2}P(t)u, C^{1/2}GP(t)u \right\rangle.
$$

Indeed, for each $u \in D(G)$ and each $\lambda > 0$, let $v = \lambda^{-1}R(\lambda; G)u$. The inequality (4.2) yields

$$
\left\| C^{1/2}u \right\|^2 = \left\| C^{1/2}v \right\|^2 - 2\lambda^{-1} \Re \left\langle C^{1/2}v, C^{1/2}Gu \right\rangle + \lambda^{-2} \left\| C^{1/2}Gv \right\|^2
\geq (1 - \lambda^{-1}b) \left\| C^{1/2}v \right\|^2 = (1 - \lambda^{-1}b) \left\| C^{1/2}\lambda R(\lambda; G)u \right\|^2.
$$
The above inequality also holds for \( u \in D(C^{1/2}) \) since \( D(G) \) is a core for \( C^{1/2} \). It follows that \( G \) is the infinitesimal generator of a strongly continuous semigroup on the Hilbert space \( D(C^{1/2}) \) (endowed with the graph norm). This is obtained by restricting the operators \( P(t) \) to \( D(C^{1/2}) \). Therefore the claimed differentiation formula follows.

Under assumption \( C \) we can prove a useful estimate of \( R^{(\min)}(\lambda)(C_\epsilon) \) where \((C_\epsilon)_{\epsilon > 0}\) is the family of bounded regularization \( C_\epsilon = C(I + \epsilon C)^{-1} \).

**Proposition 4.2.** Suppose that conditions \( A \) and \( C \) hold. Then, for all \( \lambda > b \) and all \( u \in D(G^2) \), we have

\[
(\lambda - b) \sup_{\epsilon > 0} \left\langle u, R^{(\min)}(\lambda)(C_\epsilon)u \right\rangle \leq \left\| C^{1/2}u \right\|^2 .
\]  

(4.3)

**Proof.** Let \( (R^{(n)}_\lambda)_{n \geq 0} \) be the sequence of monotone linear maps considered in the proof of Theorem 3.1. Clearly it suffices to show that, for all \( n \geq 0 \), \( \lambda > b \) and \( u \in D(G^2) \), the operator \( R^{(n)}_\lambda(C_\epsilon) \) satisfies

\[
(\lambda - b) \sup_{\epsilon > 0} \left\langle u, R^{(n)}_\lambda(C_\epsilon)u \right\rangle \leq \left\| C^{1/2}u \right\|^2 .
\]  

(4.4)

The above inequality holds for \( n = 0 \). In fact, integrating by parts, for all \( u \in D(G^2) \), we have

\[
\lambda \left\langle u, R^{(0)}_\lambda(C_\epsilon)u \right\rangle = \lambda \int_0^\infty e^{-\lambda t} \langle P(t)u, C_\epsilon P(t)u \rangle dt \\
\leq \lambda \int_0^\infty e^{-\lambda t} \left\| C^{1/2}P(t)u \right\|^2 dt \\
= \left\| C^{1/2}u \right\|^2 + 2Re \int_0^\infty e^{-\lambda t} \left\langle C^{1/2}P(t)u, C^{1/2}GP(t)u \right\rangle dt.
\]

The inequality (4.2) yields

\[
\lambda \left\langle u, R^{(0)}_\lambda(C_\epsilon)u \right\rangle \leq \left\| C^{1/2}u \right\|^2 + b \int_0^\infty e^{-\lambda t} \left\| C^{1/2}P(t)u \right\|^2 dt \\
= \left\| C^{1/2}u \right\|^2 + b \sup_{\epsilon > 0} \left\langle u, R^{(0)}_\lambda(C_\epsilon)u \right\rangle.
\]

This clearly implies (4.4) for \( n = 0 \). Suppose that (4.4) has been established for an integer \( n \); then, from the second equation (2.6) and the definition of \( R^{(n)}_\lambda \), we
have
\[ \langle u, \mathcal{R}_{\lambda}^{(n+1)}(C_\varepsilon)u \rangle = \langle u, \mathcal{P}_\lambda(C_\varepsilon)u \rangle \]
\[ + \sum_{l=1}^{\infty} \int_{0}^{\infty} e^{-\lambda t} \langle L_\ell P(t)u, \mathcal{R}_{\lambda}^{(n)}(C_\varepsilon)L_\ell P(t)u \rangle dt \]
\[ \leq \langle u, \mathcal{P}_\lambda(C_\varepsilon)u \rangle + \frac{1}{\lambda - b} \sum_{l=1}^{\infty} \int_{0}^{\infty} e^{-\lambda t} \left\| C^{1/2}L_\ell P(t)u \right\|^2 dt \]

Inequality (4.2) and integration by parts yield
\[ \sum_{l=1}^{\infty} \int_{0}^{\infty} e^{-\lambda t} \left\| C^{1/2}L_\ell P(t)u \right\|^2 dt \leq \int_{0}^{\infty} e^{-\lambda t} \left( -\frac{d}{dt} \left\| C^{1/2}P(t)u \right\|^2 \right) dt \]
\[ + b \int_{0}^{\infty} e^{-\lambda t} \left\| C^{1/2}P(t)u \right\|^2 dt \]
\[ = \left\| C^{1/2}u \right\|^2 - (\lambda - b) \int_{0}^{\infty} e^{-\lambda t} \left\| C^{1/2}P(t)u \right\|^2 dt \]
\[ \leq \left\| C^{1/2}u \right\|^2 - (\lambda - b) \langle u, \mathcal{P}_\lambda(C_\varepsilon)u \rangle . \]

Therefore (4.4) for \( n + 1 \) follows. The proof is complete. ■

We can now prove the main results of this paper.

**Theorem 4.3.** Suppose that condition A holds and there exists an operator \( C \) satisfying condition C such that
\[ \langle u, F_n u \rangle \leq \langle u, Cu \rangle \]
for all \( u \in D(C), n \geq 1 \). Then the minimal q.d.s. is conservative.

**Proof.** Let \( \lambda > b \) fixed. Under the present hypotheses, for \( \varepsilon > 0 \), the bounded operators \( (F_n)_\varepsilon \) and \( C_\varepsilon \) satisfy the inequality \( (F_n)_\varepsilon \leq C_\varepsilon \) (see, e.g. [22] Chap. 8, Ex. 51, p.317). Applying Proposition 4.2, we obtain the estimate
\[ \sum_{k=1}^{\infty} \langle u, Q_k^{(I)}u \rangle \leq \liminf_{n \to \infty} \sup_{\varepsilon > 0} \langle u, \mathcal{R}_{\lambda}^{(\min)}((F_n)_\varepsilon)u \rangle \]
\[ \leq \sup_{\varepsilon > 0} \langle u, \mathcal{R}_{\lambda}^{(\min)}(C_\varepsilon)u \rangle \leq (\lambda - b)^{-1} \left\| C^{1/2}u \right\|^2 < +\infty \]

Therefore the minimal q.d.s. is conservative since condition (b) of Proposition 3.3 is fulfilled. ■

Notice that, in the above theorem, we did not assume that the quadratic form
\[ u \mapsto -2 \Re \langle u, Gu \rangle \]
with domain \( D(G) \) is closable (see Example 5.3).
Theorem 4.4. Suppose that assumptions A, C hold for some positive self-adjoint operator C and there exists a positive self-adjoint operator Φ in h such that:

(a) the domain of the positive square root $\Phi^{1/2}$ contains the domain of G and, for every $u \in D(G)$, we have

$$-2\Re \langle u, Gu \rangle = \sum_{\ell=1}^{\infty} \langle L_\ell u, L_\ell u \rangle = \langle \Phi^{1/2} u, \Phi^{1/2} u \rangle,$$

(b) the domain of C is contained in the domain of $\Phi$ and, for every $u \in D(C)$, we have

$$\langle \Phi^{1/2} u, \Phi^{1/2} u \rangle \leq \langle C^{1/2} u, C^{1/2} u \rangle.$$

Then the minimal q.d.s. is conservative.

Proof. Let $\lambda > b$ and $u \in D(G^2)$ fixed. For $\varepsilon > 0$, the bounded operators $\Phi_\varepsilon$ and $C_\varepsilon$ satisfy the inequality $\Phi_\varepsilon \leq C_\varepsilon$ (see, e.g. [22] Chap. 8, Ex. 51, p.317). Moreover, for $u \in D(G)$, we have

$$\sup_{\varepsilon > 0} \langle u, P_\lambda(\Phi_\varepsilon) u \rangle = \int_0^\infty e^{-\lambda t} \left\| \Phi^{1/2} P(t) u \right\|^2 dt = \sum_{\ell=1}^{\infty} \int_0^\infty e^{-\lambda t} \left\| L_\ell P(t) u \right\|^2 dt = \langle u, Q_\lambda(I) u \rangle.$$

This implies that the non-decreasing family of operators $(P_\lambda(\Phi_\varepsilon))_{\varepsilon > 0}$ is uniformly bounded and, since $D(G)$ is dense in h, it follows that it converges strongly to $Q_\lambda(I)$ as $\varepsilon$ goes to 0. The maps $Q_\lambda^k$ being normal we have

$$\sum_{k=0}^{\infty} \langle u, Q_\lambda^{k+1}(I) u \rangle = \sup_{\varepsilon > 0} \sum_{k=0}^{\infty} \langle u, Q_k(\lambda(\Phi_\varepsilon)) u \rangle = \sup_{\varepsilon > 0} \langle u, R_\lambda^{(\min)}(\Phi_\varepsilon) u \rangle$$

by Theorem 3.1. Applying Proposition 4.2 we obtain the estimate

$$\sum_{k=1}^{\infty} \langle u, Q_\lambda^k(I) u \rangle = \sup_{\varepsilon > 0} \langle u, R_\lambda^{(\min)}(\Phi_\varepsilon) u \rangle \leq \sup_{\varepsilon > 0} \langle u, R_\lambda^{(\min)}(C_\varepsilon) u \rangle \leq (\lambda - b)^{-1} \left\| C^{1/2} u \right\|^2$$

Therefore the minimal q.d.s. is conservative because condition (b) of Proposition 3.3 holds.

The following corollary gives a simpler and easily verifiable condition under stronger assumptions on the domain of the operator C.
Corollary 4.5. Suppose that assumption A holds and there exist a self-adjoint operator $C$ and a core $D$ for $G$ with the following properties:

(a) the domain of $G$ coincides with the domain of $C$ and for all $u \in D(G)$ there exists a sequence $(u_n)_{n \geq 0}$ of elements of $D$ such that both $(Gu_n)_{n \geq 0}$ and $(Cu_n)_{n \geq 0}$ converge strongly,

(b) there exists a positive self-adjoint operator $\Phi$ such that the domain of $\Phi$ contains the domain of $D$ and for all $u \in D$ and $n \geq 1$ we have the inequality

$$-2\Re \langle u, Gu \rangle = \langle u, \Phi u \rangle \leq \langle u, Cu \rangle ,$$

(c) for all $\ell \geq 1$, $L_\ell(D) \subseteq D(C)$,

(d) there exists a constant $b$ such that, for all $u \in D$, the following inequality holds

$$2\Re \langle Cu, Gu \rangle + \sum_{\ell=1}^{\infty} \langle L_\ell u, C L_\ell u \rangle \leq b \langle u, Cu \rangle . \quad (4.5)$$

Then the minimal q.d.s. is conservative.

Proof. The inequality of condition (b) obviously holds also for $u \in D(C)$ because of condition (a) and self-adjointness of $\Phi$. Therefore, in order to prove the corollary, it suffices to show that, under the above hypotheses, the operator $C$ satisfies assumption C and apply Theorem 4.4.

Let $(u_n)_{n \geq 0}$ be a sequence of elements of $D$ such that

$$\lim_{n \to \infty} Cu_n = Cu, \quad \lim_{n \to \infty} Gu_n = Gu.$$  

Condition (d) implies that $(C^{1/2}L_\ell u_n)_{n \geq 1}$ is a Cauchy sequence for $\ell \geq 1$. Therefore it is convergent and it is easy to deduce that (4.2) holds for $u \in D(G)$. 

Remark. Another simple sufficient condition for conservativity can be easily obtained by substituting (b) in the above corollary with the following hypothesis:

(b’) for all $u \in D$ and $n \geq 1$ we have the inequality

$$-2\Re \langle nR(n; G)u, nGR(n; G)u \rangle = \langle u, F_n u \rangle \leq \langle u, Cu \rangle .$$

The proof can be easily done by applying Theorem 4.3.

5. Applications and examples

In this section we apply our results to study conservativity of three minimal q.d.s.: one arising from a physical model and another from extension problems
of classical Markovian semigroups to non-abelian algebras. We consider semigroups of diffusion type since the minimal q.d.s. of jumps and drift type leaves the abelian algebra of multiplication operators invariant. Therefore the conservativity problem for the minimal q.d.s. can be reduced to a problem in classical probability.

5.1 Q.d.s. in a model for heavy ion collision.

As a first example we apply the conservativity condition of Corollary 4.5 to the minimal q.d.s. proposed by Alicki (see, for example, [3], [4]) to describe phenomenologically a quantum system with dissipative heavy ion collisions. This problem can not be solved by applying the tools developed in [9], [11].

Let $h = L^2(\mathbb{R}^3; \mathcal{F})$ and let $m \in ]0, +\infty[, \alpha \in \mathbb{R}$. We denote by $\partial_\ell (\ell = 1, 2, 3)$ differentiation with respect to the $\ell$-th coordinate. Let $V : \mathbb{R}^3 \to \mathbb{R}, W : \mathbb{R}^3 \to \mathbb{R}$ be two functions with the following properties:

1. $V$ can be written as the sum of a bounded function and a square integrable function, $V$ is differentiable and the partial derivatives $\partial_\ell V$ are bounded,
2. $W$ is bounded and $\sup_{x \in \mathbb{R}^3} |W(x)|^2 < (m\alpha^2)^{-1},$

$W$ is twice differentiable and the following functions are bounded continuous

$$x \to x_\ell W(x), \quad x \to x_\ell \partial_\ell W(x), \quad x \to \partial^{2}_\ell W(x) \quad \ell = 1, 2, 3. \quad (5.1)$$

Consider the operators $H_0, V, L_\ell, G$ with domain $H^2(\mathbb{R}; \mathcal{F})$

$$H_0 u = -\frac{1}{2m} \Delta u, \quad (V u)(x) = V(x) u(x),$$

$$L_\ell u = W(x) (x_\ell + \alpha \partial_\ell) u, \quad G u = -i(H_0 + V) u - \frac{1}{2} \sum_{\ell=1}^{3} L^*_\ell L_\ell u$$

and let $L_\ell = 0$ for $\ell \geq 4$. The arguments of [19] Ch. V, Sect. 3 show that the operator $G$ is a relatively bounded perturbation of $H_0$ with relative bound smaller than 1 and the linear manifold $D$ of infinitely differentiable functions with compact support is a core for $G$. The operator $G$ is the infinitesimal generator of a strongly continuous contraction semigroup in $h$ by [19] Th. 2.7 p. 499 and following remarks. Thus the basic assumption $A$ holds because it suffices to check identity (2.2) for $v, u \in D$ and, in this case, (2.2) is trivial.

We show that the minimal q.d.s. is conservative applying Corollary 4.5. The most natural choice of the operator $C$ is the following

$$D(C) = H^2(\mathbb{R}^3; \mathcal{F}), \quad Cu = c (-\Delta + 1) u$$
where $c$ is a suitable constant to be determined. In fact hypothesis (a) obviously holds because $G$ and $C$ are relatively bounded one respect to the other and $D$ is a core for both. By virtue of von Neumann’s theorem (see [19] Th. 3.24 p. 275) (b) is satisfied. Hypothesis (c) is trivially fulfilled. In order to check (d) notice first that it suffices to check (4.5) for all $u \in D$, i.e. to estimate the quadratic form associated with the formal operator

$$CG + G^* C + \sum_{\ell=1}^{3} L_\ell^* C L_\ell = i[V, C] + \frac{1}{2} \sum_{\ell=1}^{3} (L_\ell^*[C, L_\ell] + [L_\ell^*, C]L_\ell).$$

This turns out to be a second order differential operator with bounded coefficients. Hence, for $u \in D$, we have

$$2\text{Re} \langle Cu, Gu \rangle + \sum_{\ell=1}^{\infty} \langle L_\ell u, CL_\ell u \rangle \leq b \langle u, Cu \rangle$$

where $b$ depends only on the supremum of the partial derivatives $\partial_\ell V$, $\ell = 1, 2, 3$, and of the functions (5.1). Therefore hypothesis (d) of Corollary (4.5) also holds and the minimal q.d.s. is conservative.

> ¿From the above discussion it is clear that our result can be applied to a large class of Lindblad type perturbations of pure hamiltonian evolutions arising in physical models (see, for instance, [4]).

### 5.2 Extension of classical Brownian motions with partial reflection.

Let $h = L^2((0, +\infty); \mathcal{D})$, let $\alpha \in \mathbb{R}$ and let $g$ be a function in $h$. Define the parameter $\theta = \|g\|^2/(2\alpha)$. Consider the operators $G$ and $L_\ell$

$$D(G) = \{ u \in H^2((0, +\infty); \mathcal{D}) \mid u'(0) = \theta u(0) \} \quad Gu = \frac{1}{2} u'',$$

$$D(L_1) = H^1((0, +\infty); \mathcal{D}) \quad L_1 u = u',$$

$$D(L_2) = H^1((0, +\infty); \mathcal{D}) \quad L_2 u = \frac{u(0)}{\sqrt{2\alpha}} g,$$

and let $L_\ell = 0$ for all $\ell \geq 3$. In [5] (Prop. 4.3 and Th. 2.4) it has been shown that:

1. Our basic assumption A is satisfied.
2. The operator $G$ is negative and self-adjoint.
3. The restriction of the map $\mathcal{L}$ defined by (2.3) to multiplication operators by a regular bounded real function $f$ on $[0, +\infty[$, coincides with the infinitesimal generator $A$ of a brownian motion on $[0, +\infty[$ with partial reflection at the
boundary point \( \{0\} \) and partial reentrance in \([0, +\infty[\) with reentrance density \( x \rightarrow |g(x)|^2 \).

\[
D(A) = \left\{ f \in C^2_b([0, +\infty[; \mathbb{R}) \mid \alpha f'(0) + \int_0^\infty (f(x) - f(0)) |g(x)|^2 \, dx = 0 \right\}
\]

\[
(Af)(x) = \frac{1}{2} f''(x)
\]

where \( C^2_b([0, +\infty[; \mathbb{R}) \) denotes the vector space of real functions on \([0, +\infty[\) bounded with bounded continuous derivatives up to the second order.

4. The minimal q.d.s. is an extension to \( \mathcal{B}(h) \) of the Markovian semigroup of the classical stochastic process if and only if it is conservative.

Here we apply the main result of this paper to show that the minimal q.d.s. constructed from the above operators \( G \) and \( L_\ell \) is conservative whenever \( g \) belongs to \( H^1((0, +\infty); \mathcal{C}) \). We prove first the following

**Lemma 5.1.** For all \( u \in H^1((0, +\infty); \mathcal{C}) \) and \( \lambda > 0 \) we have

\[
\lim_{\lambda \to +\infty} (\lambda R(\lambda; G)u)(0) = u(0), \quad s - \lim_{\lambda \to +\infty} \lambda L_1 R(\lambda; G)u = L_1 u.
\]

**Proof.** An elementary computation yields

\[
(\lambda R(\lambda; G)u)(x) = \sqrt{\frac{\lambda}{2}} \int_0^\infty \exp \left(-\sqrt{2\lambda}|x-s|\right) u(s) \, ds
\]

\[
+ \sqrt{\frac{\lambda}{2}} \frac{\sqrt{2\lambda} - \theta}{\sqrt{2\lambda} + \theta} \int_0^\infty \exp \left(-\sqrt{2\lambda}(x+s)\right) u(s) \, ds,
\]

\[
(\lambda L_1 R(\lambda; G)u)(x) = -\lambda \int_0^\infty \text{sgn}(x-s) \exp \left(-\sqrt{2\lambda}|x-s|\right) u(s) \, ds
\]

\[
- \lambda \frac{\sqrt{2\lambda} - \theta}{\sqrt{2\lambda} + \theta} \int_0^\infty \exp \left(-\sqrt{2\lambda}(x+s)\right) u(s) \, ds
\]

where \( \text{sgn}(x-s) = 1 \) if \( x \geq s \) and \( \text{sgn}(x-s) = -1 \) if \( x < s \). Therefore the first limit is easily computed. Integrating by parts both the above integrals we have

\[
(\lambda L_1 R(\lambda; G)u)(x) = (\lambda R(\lambda; G)L_1 u)(x) + \frac{\theta \sqrt{2\lambda}}{\sqrt{2\lambda} + \theta} \exp \left(-\sqrt{2\lambda}x\right) u(0)
\]

\[
- \sqrt{2\lambda} \frac{\sqrt{2\lambda} - \theta}{\sqrt{2\lambda} + \theta} \int_0^\infty \exp \left(-\sqrt{2\lambda}(x+s)\right) u'(s) \, ds
\]

The first term converges to the desired limit, for the strong topology on \( h \), by a well-known property of the resolvent operators and the second clearly vanishes as \( \lambda \) goes to \(+\infty\).
Disregarding the factor \((\sqrt{2\lambda} - \theta) / (\sqrt{2\lambda} + \theta)\) goes to 1 as \(\lambda\) goes to \(+\infty\) and using the Schwarz inequality we can estimate the third term by

\[
2\lambda \int_0^\infty \exp \left( -2\sqrt{2\lambda} x \right) dx \cdot \left| \int_0^\infty \exp \left( -\sqrt{2\lambda s} \right) u'(s) ds \right|^2
\]

\[
= \frac{\sqrt{2\lambda}}{2} \left| \int_0^\infty \exp \left( -\sqrt{2\lambda s} / 2 \right) \cdot \left( \exp \left( -\sqrt{2\lambda s} / 2 \right) u'(s) \right) ds \right|^2
\]

\[
\leq \frac{\sqrt{2\lambda}}{2} \int_0^\infty \exp \left( -\sqrt{2\lambda s} \right) ds \cdot \int_0^\infty \exp \left( -\sqrt{2\lambda s} \right) |u'(s)|^2 ds
\]

\[
= \frac{1}{2} \int_0^\infty \exp \left( -\sqrt{2\lambda s} \right) |u'(s)|^2 ds
\]

The right-hand side vanishes as \(\lambda\) goes to \(+\infty\) by Lebesgue’s theorem.

This completes the proof. \(\blacksquare\)

**Lemma 5.2.** Let \(C\) be the positive self-adjoint operator \(-2G\). The vector space \(H^1((0, +\infty); \mathcal{E}')\) is contained in the domain of \(C^{1/2}\) and

\[
\left\|C^{1/2}u\right\|^2 = \|L_1 u\|^2 + \|L_2 u\|^2
\]

for all \(u \in H^1((0, +\infty); \mathcal{E}')\).

**Proof.** Let \(u \in H^1((0, +\infty); \mathcal{E}')\) and let \(u_\lambda = \lambda R(\lambda; G)u\). The vector \(u_\lambda\) belongs to the domains of \(C\) and \(G\). Moreover, for all \(\lambda, \mu > 0\), (2.2) yields

\[
\left\|C^{1/2}(u_\lambda - u_\mu)\right\|^2 = \|L_1(u_\lambda - u_\mu)\|^2 + \|L_2(u_\lambda - u_\mu)\|^2.
\]

Therefore the family of vectors \((C^{1/2}u_\lambda)_{\lambda > 0}\) is Cauchy by Lemma 5.1. Thus \(u\) belongs to the domain of \(C^{1/2}\). Moreover (5.2) holds for all vectors \(u_\lambda\) because it is equivalent to (2.2) for vectors belonging to \(D(G)\). Letting \(\lambda\) go to \(+\infty\) we see that (5.2) holds also for the vector \(u\). \(\blacksquare\)

The above Lemma shows that the operator \(C\) is a singular perturbation of \(-d^2/dx^2\) by a delta function at the point 0 studied also in [1]. We prove now the stated result

**Theorem 5.3.** The minimal q.d.s. constructed from the above operators \(G\) and \(L_\ell\) is conservative whenever \(g \in H^1((0, +\infty); \mathcal{E}')\).

**Proof.** We check assumption \(C\) for the operator \(C = -2G\) in order to apply Theorem 4.4. All vectors \(u \in D(G)\) belong to \(H^2((0, +\infty); \mathcal{E}')\); hence \(L_1 u\) belongs
to $H^1((0, +\infty); \mathcal{G})$. Moreover, by Lemma 5.2, we have

$$2\Re \langle Cu, Gu \rangle + \sum_{\ell=1}^{2} \left\| C^{1/2} L_\ell u \right\|^2 = -\|u''\|^2 + \sum_{\ell=1}^{2} \left( \|L_1(L_\ell u)\|^2 + \|L_2(L_\ell u)\|^2 \right)$$

$$= \frac{|u(0)|^2}{2\alpha} \left( \|g'\|^2 + \theta^2 \|g\|^2 + \frac{|g(0)|^2}{2\alpha} \right).$$

Thus, when $g = 0$, assumption C obviously holds. If $g \neq 0$ then Lemma 5.2 yields the inequality

$$(2\alpha)^{-1} |u(0)|^2 = \|g\|^{-2} \|L_2 u\|^2 \leq \|g\|^{-2} \left\| C^{1/2} u \right\|^2.$$ 

Hence we have

$$2\Re \langle Cu, Gu \rangle + \sum_{\ell=1}^{2} \left\| C^{1/2} L_\ell u \right\|^2 \leq \left( \frac{\|g'\|^2}{\|g\|^2} + \theta^2 + \frac{|g(0)|^2}{2\alpha \|g\|^2} \right) \left\| C^{1/2} u \right\|^2.$$ 

Therefore assumption C holds. The proof is complete letting $\Phi = C$ and applying Theorem 4.4. □

5.3 Non closable forms.

We study a minimal q.d.s. constructed from operators $G, L_\ell$ so singular that the quadratic form $u \to -2\Re \langle u, Gu \rangle$ with domain $D(G)$ is not closable. This problem also can not be solved applying the tools developed in [9], [11].

Let us consider the contraction semigroup in $h = L^2(0, +\infty)$

$$P(t)u(x) = u(x + t)$$

with infinitesimal generator $G$ given by

$$D(G) = H^1(0, +\infty), \quad Gu = u'$$

Let $L_\ell = 0$ for $\ell \geq 2$ and let $L_1$ be the operator in $h$

$$D(L) = H^1(0, +\infty), \quad Lu = u(0)g$$

where $g \in h$ and $\|g\| = 1$. Clearly condition A holds. Let $C$ be the self-adjoint operator in $h$

$$D(C) = \{ u \in H^2(0, +\infty) \mid u'(0) = u(0) \}, \quad Cu = -2u'',$$
Applying Lemma 5.2 we can prove that:

(a) the domain of $C^{1/2}$ contains $H^1(0, +\infty) = D(G)$,

(b) for all $u \in H^1(0, +\infty) = D(G)$ we have

$$-2\Re \langle u, Gu \rangle = |u(0)|^2 \leq \|C^{1/2}u\|^2 = 2 \left( \|u'\|^2 + |u(0)|^2 \right),$$

(c) for all $u \in D(G^2) = H^2(0, +\infty)$ and $g \in H^1((0, +\infty); \mathcal{F})$ we have

$$2\Re \langle Cu, Gu \rangle + \left\| C^{1/2}Lu \right\|^2 = -2 \langle u'', u' \rangle - 2 \langle u', u'' \rangle + 2|u(0)|^2 \left\| C^{1/2}g \right\|^2$$

$$= 2|u(0)|^2 \left( 1 + |g(0)|^2 + \|g'\|^2 \right) \leq 2 \left( 1 + |g(0)|^2 + \|g'\|^2 \right) \left\| C^{1/2}u \right\|^2.$$

Therefore conditions A and C hold whenever $g \in H^1((0, +\infty); \mathcal{F})$.

By Theorem 4.3, in order to show that the minimal q.d.s. constructed from the above operators $G$ and $L$ is conservative it suffices to check the inequality

$$\langle u, F_n u \rangle \leq 2 \left( \|u'\|^2 + |u(0)|^2 \right) = \langle u, Cu \rangle$$

for $u \in D(C)$, $n \geq 1$ where $F_n$ is the unique bounded extension of $|nLR(n; G)|^2$.

A straightforward computation yields

$$\|nLR(n; G)u\|^2 = \left| n \int_0^\infty e^{-nt}u(t)dt \right|^2$$

Integrating by parts for $u \in D(C)$ and using the Schwarz inequality we have

$$\left| n \int_0^\infty e^{-nt}u(t)dt \right|^2 = \left| u(0) + \int_0^\infty e^{-nt}u'(t)dt \right|^2 \leq 2|u(0)|^2 + \frac{1}{n} \|u'\|^2.$$

Therefore the hypotheses of Theorem 4.3 are satisfied and the minimal q.d.s. is conservative.

It is worth mentioning here that the restriction of this q.d.s. to the abelian algebra of multiplication operators by a bounded function coincides with the infinitesimal generator of a classical stochastic process that can be described as follows: a point moves on $]0, +\infty[$ towards 0 with constant speed, when it reaches 0 it jumps back on an interval $(a, b)$ with probability $\|g1_{(a,b)}\|^2$. Journé showed in [18] that the above minimal q.d.s. is conservative for every $g \in h$.

**Comment added in proof.** The use of the “reference” operator $C$ in (4.2) dominating $\Phi$ was done independently also by A.S. Holevo in [24]. We arrive to this observation
from the resolvent analyses and we find this assumption quite relevant from the physical point of view because it allows to deal with noncomparable operators $H$ and $\Phi$. However in [24] additional hypotheses were used: (1) the form $u \rightarrow -2\Re\langle u, Gu \rangle$ is closed, (2) the operators $L_\ell$ are closed, (3) $\|Hu\| \leq \|Cu\|$ for $u$ in a common core for $G, G^*$ and $C$. The last assumption means essentially that $H$ is dominated by $C$. On the other hand tangible interpretations of assumptions (1) and (2), which are not fulfilled in Example 5.2 and 5.3, is unclear.

References
[1] S. Albeverio, F. Gesztesy, R. Hoegh-Krohn, H. Holden: Solvable Models in Quantum Mechanics, Springer, Berlin-Heidelberg-New York, (1988).
[2] L. Accardi, R. Alicki, A. Frigerio, Y.G. Lu: An invitation to the weak coupling and low density limits. Quantum Probability and related topics VI, (1991), 3–61.
[3] R. Alicki, A. Frigerio: Scattering theory for quantum dynamical semigroups II. Ann. Inst. Henri Poincaré XXXVIII, n. 2, (1983), 187-197.
[4] R. Alicki, K. Lendi: Quantum dynamical semigroups and applications. Lect. Notes Phys. 286, Springer Verlag, Berlin Heidelberg, New York (1987).
[5] B.V.R. Bhat, F. Fagnola, K.B. Sinha: On quantum extensions of semigroups of brownian motions on an half-line. Russian J. Math. Phys. 4, (1996), 13–28.
[6] B.V.R. Bhat, K.B. Sinha: Examples of unbounded generators leading to non-conservative minimal semigroups. Quantum Probability and Related Topics IX (1994), 89–103.
[7] A.M. Chebotarev: Sufficient conditions for conservativity of dynamical semigroups. Theor. Math. Phys. 80, 2 (1989).
[8] A.M. Chebotarev: The theory of conservative dynamical semigroups and its applications. Preprint MIEM n.1. March 1990.
[9] A.M. Chebotarev: Sufficient conditions of the conservativity of a minimal dynamical semigroup. Math. Notes 52, (1993), 1067–1077.
[10] A.M. Chebotarev: Application of quantum probability to classical stochastics. Univ. degli Studi di Roma ”Tor Vergata”, Centro V. Volterra, March 1996, Preprint N 246.
[11] A.M. Chebotarev, F. Fagnola: Sufficient Conditions for Conservativity of Quantum Dynamical Semigroups. J. Funct. Anal. 118 (1993), 131–153.
[12] E.B. Davies: Quantum dynamical semigroups and the neutron diffusion equation. Rep. Math. Phys. 11 (1977), 169–188.
[13] F. Fagnola: Chebotarev’s sufficient conditions for conservativity of quantum dynamical semigroups. *Quantum Probability and Related Topics* **VIII** (1993) 123–142.

[14] F. Fagnola: Characterization of isometric and unitary weakly differentiable cocycles in Fock space. Preprint UTM n.358, Trento, October 1991. *Quantum Probability and Related Topics* **VIII**, (1993), 143–164.

[15] F. Fagnola: Diffusion processes in Fock space. *Quantum Probability and Related Topics* **IX** (1994), 189–214.

[16] A.S. Holevo: On the Structure of Covariant Dynamical Semigroups. *J. Funct. Anal.* **131** (1995), 255–278.

[17] K. Ichihara: Explosion problems for symmetric diffusion processes. in “Stochastic Processes and Their Applications” (K. Itô and T. Hida Eds.) *Lect. Notes Math.* **1203** (1986), 75–89.

[18] J.-L. Journé: Structure des cocycles markoviens sur l’espace de Fock. *Probab. Th. Rel. Fields* **75**, (1987), 291–316.

[19] T. Kato: *Perturbation Theory for Linear Operators*. Springer. 1966.

[20] P.-A. Meyer: *Quantum Probability for Probabilists*. *Lect. Notes Math.* **1538**. Springer Verlag, Berlin Heidelberg, New York 1993.

[21] K.R. Parthasarathy: *An Introduction to Quantum Stochastic Calculus*. Monographs in Mathematics, Birkhäuser, Basel 1992.

[22] M. Reed, B. Simon: *Methods of Modern Mathematical Physics*. Vol. I: Functional Analysis. Academic Press. New York and London 1972.

[23] Kalyan B. Sinha: Quantum Dynamical Semigroups. in “Operator Theory: Advances and Applications”, Vol. 70, Birkhäuser Verlag, Basel, 1994.

[24] A.S. Holevo: Stochastic differential equations in Hilbert space and quantum Markovian evolution. In: S. Watanabe, M. Fukujima, Yu. V. Prokhorov, A.N. Shiryaev (eds.) *Proceedings of the VII Japan-Russia Symposium*, p. 122–131, World Scientific 1996.