Boundedness and Compactness of products of Toeplitz operators on the Bergman Space

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Abstract

In a celebrated conjecture, D. Sarason stated a necessary and sufficient condition on the symbols \( f, g \) in the Bergman space, \( L^2_a(\Delta) \) of the unit disk, \( \Delta \), for the product \( T_f T_g \) of associated Toeplitz operators to be bounded (respectively compact) on \( L^2_a(\Delta) \). K. Stroethoff and D. Zheng proved that these conditions are necessary. We prove the sufficiency of these conditions, thus solving Sarason’s conjecture.

Keywords Berezin Transforms, Bergman space, Toeplitz operator.

1 Introduction and Statement of Result

Let \( d\lambda \) denote the Lebesgue area measure on the unit disk \( \Delta \), normalized so that the measure of \( \Delta \) equals 1. The Bergman space \( L^2_a(\Delta, d\lambda) \) is the closed subspace of \( L^2(\Delta, d\lambda) \) consisting of functions analytic on the unit disk \( \Delta \). Let \( P \) be the Bergman projection from \( L^2_a(\Delta, d\lambda) \) onto \( L^2(\Delta, d\lambda) \) defined by

\[
(Pg)(z) = \int_{\Delta} g(w) K_w(z) \, d\lambda(w)
\]

where \( K_z(w) = \frac{1}{(1-z\overline{w})^2} \) is the Bergman reproducing kernel and the normalized Bergman kernel is \( k_z(w) = \frac{1-|z|^2}{(1-z\overline{w})^2} \). We will write \( \langle \cdot, \cdot \rangle \) for the usual inner product on \( L^2(\Delta, d\lambda) \). For a function \( f \) in \( L^2(\Delta) \), the Toeplitz operator \( T_f \) on \( L^2_a(\Delta) \) is defined by \( T_f g = P(fg) \) since \( P \) extends to an operator defined on \( L^1(\Delta, d\lambda) \). If \( g \in L^2_a(\Delta, d\lambda) \) then

\[
(T_f h)(w) = \int_{\Delta} \frac{g(z) h(z)}{(1-z\overline{w})^2} \, d\lambda(z).
\]

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For \( f, g \in L^2_a(\Delta, d\lambda) \) the product \( T_f T_g \) is defined on the dense subspace \( H^\infty(\Delta) \) of \( L^2_a(\Delta, d\lambda) \) by \( T_f T_g h = f T_g h \). Indeed, for \( h \in H^\infty(\Delta) \), \( T_f T_g h \) belongs to \( L^2_a(\Delta, d\lambda) \) and then
\[
T_f T_g h = P(f T_g h) = f T_g h
\]
since \( P \) reproduces functions belonging to \( L^1_a(\Delta, d\lambda) \).

We shall consider the problem posed by Sarason[10]: for which analytic functions \( f \) and \( g \) in \( L^2_a(\Delta, d\lambda) \) is the densely defined product \( T_f T_g \) bounded on \( L^2_a(\Delta, d\lambda) \). Several authors [4], [12], [9], [13], [14] and [15] have been working on this problem. Theorem 5.1 of Stroethoff and Zheng[13] shows that if \( T_f T_g \) is bounded then
\[
\sup_{w \in \Delta} \tilde{|f|^2}(w) \tilde{|g|^2}(w) < \infty
\]
where \( \tilde{|f|^2}(w) \) is the Berezin transform of \( |f|^2 \) to be defined later. There have been several attempts to show that this condition is sufficient see [9], [12], [13]. We are going to prove that this condition is sufficient. We have the following result:

**THEOREM 1** Let \( f, g \in L^2_a(\Delta, d\lambda) \), suppose
\[
\sup_{w \in \Delta} \tilde{|f|^2}(w) \tilde{|g|^2}(w) < \infty
\]
then \( T_f T_g \) extends to a bounded operator on \( L^2_a(\Delta, d\lambda) \).

This theorem together with the result of Stroethoff and Zheng[13] characterises boundedness of \( T_f T_g \).

We observe that with Theorem 1 we can easily characterise the boundedness of \( T_f \), \( f \in L^2_a(\Delta, d\lambda) \). If we set \( g = 1 \) in the theorem then we will have a well known result on the boundedness of \( T_f \) and \( T_g \), which is:

**Corollary 1** Let \( f \in L^2_a(\Delta, d\lambda) \). Then \( T_f \) is bounded if and only if \( T_g \) is bounded if and only if \( \sup_{v \in \Delta} \| f \circ \varphi_v \|_2 < \infty \) if and only if \( f \) is bounded.

We also solve the analogous problem of compactness of \( T_f T_g \) also in Stroethoff and Zheng[13]. To this end we prove the following result:

**THEOREM 2** Let \( f, g \in L^2_a(\Delta, d\lambda) \), and suppose that \( T_f T_g \) extends to a bounded on \( L^2_a(\Delta, d\lambda) \). Then \( T_f T_g \) is compact on \( L^2_a(\Delta, d\lambda) \) if and only if
\[
\lim_{|w| \to 1^-} \tilde{|g|^2}(w) \tilde{|f|^2}(w) = 0
\]

Theorem 6.2 of Stroethoff and Zheng[13] shows that if \( T_f T_g \) is compact for \( f, g \in L^2_a(\Delta) \), then \( \lim_{|w| \to 1^-} \tilde{|f|^2}(w) \tilde{|g|^2}(w) = 0 \) and Theorem 2 shows that this condition is sufficient thus solving the conjecture.

We also make a useful observation, by setting \( g = 1 \) in Theorem 2 we easily prove the following:
Corollary 2 Let $f \in L^2_a$. Then $Tf$ is compact if and only if $Tf$ is compact if and only if $\lim_{|v|\to 1^-} |f(v)| = 0$ if and only if

$$\lim_{|v|\to 1^-} \|f \circ \varphi_v\|_2 = 0.$$ 

2 Useful concepts and Local Estimates

For $w \in \Delta$, the fractional linear transformation $\varphi_w$ defined by

$$\varphi_w(z) = \frac{w - z}{1 - wz}$$

is an automorphism of the unit disk; in fact, the mappings are involutions: $\varphi_w^{-1} = \varphi_w$. The real Jacobian for the change of variable $v = \varphi_w(z)$ is equal to $|\varphi'_w(z)|^2 = |k_w(z)|^2$. The Berezin transform of a function $f \in L^2(\Delta)$ is the function $\tilde{f}$ defined on $\Delta$ by

$$\tilde{f}(w) = \int_{\Delta} f(z)|k_w(z)|^2 dA(z).$$

Thus it follows by change of variable $v = \varphi_w(z)$ we have

$$|\tilde{f}|^2(w) = \|f \circ \varphi_w\|_2^2$$

for every $f \in L^2(\Delta)$ and $w \in \Delta$.

Let $z, w \in \Delta$; then the Bergman metric $B(z, w)$ is given by

$$B(z, w) = \frac{1}{2} \log \left\{ \frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|} \right\}.$$ 

For $z \in \Delta$ and $\delta > 0$ we define

$$D(z, \delta) = \{ w \in \Delta : B(z, w) < \delta \}.$$ 

Then $D(z, \delta)$ is a Euclidean disk with centre $C$, and radius $R$, given by

$$C = \frac{1 - s^2}{1 - s^2|z|^2} z, \quad R = \frac{1 - |z|^2}{1 - s^2|z|^2} s \quad \text{where} \quad s = \frac{e^\delta - e^{-\delta}}{e^\delta + e^{-\delta}} = \tanh \delta.$$ 

We will denote the Lebesgue area measure of $D(z, \delta)$ by $\lambda(D(z, \delta))$.

Our next lemma is an application of a result by D. Luecking\cite{5} which gives a necessary and sufficient condition for a positive Borel measure $\mu$ on $\Delta$ to satisfy the following property: the Bergman space $L^p_\mu(\Delta)$, which is the space of analytic function in $L^p(\Delta, d\lambda)$, is embedded in $L^q(\Delta, d\mu)$ for $p > q > 0$. Precisely, Luecking’s result is the following:
THEOREM 3 Suppose $\delta < \frac{1}{2}$ and $0 < q < p$. Define the function $k(w) := \frac{\mu(D(w)\delta)}{\lambda(D(w)\delta)}$, the estimate

$$\left( \int |f|^q d\mu \right)^{\frac{1}{q}} \leq C \left( \int |f|^p d\lambda \right)^{\frac{1}{p}} \text{ for all } f \in L^p_\alpha(\Delta, \lambda)$$

holds if and only if $k$ is in $L^s(\Delta, d\lambda)$, where $\frac{1}{s} + \frac{q}{p} = 1$. Moreover, $C = c\|k\|_s^{\frac{1}{q}}$.

From now on, we shall keep $\delta$ fixed and we shall write $D(z)$ instead of $D(z, \delta)$, for $z \in \Delta$. Furthermore, for $\epsilon > 0$, define an operator $S$ by

$$(Sf)(z) = \int \frac{1}{1 - |z|^2} \frac{1}{1 - |v|^2} \frac{1}{(1 - |\varphi_z(v)|^2)^{2\epsilon - 1}} |f(v)| d\lambda(v). \quad (1)$$

Lemma 1 Let $p > 1$ and $0 < \epsilon < \frac{1}{2p'}$, where $p'$ is the conjugate exponent of $p$. Then there exist a constant $C = C(p, \epsilon)$ such that for all $f \in L^p_\alpha(\Delta)$, the following estimate holds:

$$|Sf(z)| \leq C(K_z(z))^{\epsilon} \|f\|_p.$$  

proof. Let

$$d\mu(v) = \frac{1}{1 - |z|^2} \frac{1}{1 - |v|^2} \frac{1}{(1 - |\varphi_z(v)|^2)^{2\epsilon - 1}} dA(v).$$

Then from a similar calculation in [1] we obtain

$$\mu(D(w)) = \int_{D(w)} \frac{1}{1 - |z|^2} \frac{1}{1 - |v|^2} \frac{1}{(1 - |\varphi_z(v)|^2)^{2\epsilon - 1}} dA(v) \leq \frac{C}{(1 - |z|^2)(1 - |w|^2)(1 - |\varphi_z(w)|^2)^{2\epsilon - 1}} \lambda(D(w)).$$

Next, if $k(w) = \frac{\mu(D(w))}{\lambda(D(w))}$ then

$$k(w) \leq \frac{C}{(1 - |z|^2)(1 - |w|^2)(1 - |\varphi_z(w)|^2)^{2\epsilon - 1}(1 - |z|^2)^{2\epsilon - 1}} \leq \frac{C'}{(1 - |w|^2)^{2\epsilon}(1 - |z|^2)^{2\epsilon}} = \frac{C'}{(1 - |w|^2)^{2\epsilon}} K_z(z)^{\epsilon}.$$ 

Furthermore, we see that $k$ belongs to $L^{p'}(\Delta, d\lambda)$ if $0 < \epsilon < \frac{1}{2p'}$. Thus by Luecking [3], there exists a constant $C = C(p, \epsilon) > 0$ such that

$$\int |f| d\mu \leq C \|f\|_p(K_z(z))^{\epsilon}$$

which proves our lemma.
Remark 1 Let \( f \in L^2_a \) and \( u \in \Delta \).

\[
I(u) := \int_{\Delta} |f(v)||K_v(u)|(K_v(v))' d\lambda(v) = |S(f \circ \varphi_u)(u)|.
\]

Indeed by making the change of variable \( v = \varphi_u(v') \) in the integral \( I(u) \) we obtain

\[
I(u) = \int_{\Delta} |f \circ \varphi_u(v')||K_{v'}(u)| \frac{1}{(1 - |\varphi_u(v')|^2)^{2\alpha}} d\lambda(v')
\]

\[
= \int_{\Delta} |f \circ \varphi_u(v)| \frac{1}{1 - |v|^2} \frac{1}{1 - |u|^2} \frac{1}{(1 - |\varphi_u(v)|^2)^{2\alpha - 1}} d\lambda(v) = |S(f \circ \varphi_u)(u)|.
\]

Lemma 2 For \( g, f \in L^2_a(\Delta) \) then

1. \( g(z)K_z(u) = P(\overline{g}K_z)(u) = K_z(u)P(\overline{g} \circ \varphi_z)(\varphi_z(u)) \);
2. \( (T_fT_gK_z)(w) = f(w)g(z)K_z(w) \);
3. \( (T_fT_gK_z)(w) = (T_gT_fK_w)(z) \)

Proof.

1. \[
P(\overline{g}K_z)(u) = \int_{\Delta} \overline{g}(w)K_z(w)K_w(u) d\lambda(w)
\]

\[
= \langle K_z, gK_u \rangle = \overline{g}(z)K_z(u).
\]

For the second equality we use the fact that \( P \overline{g} = \overline{g}(0) \), for all \( g \in L^2_a \), to get

\[
P(\overline{g} \circ \varphi_z)(\varphi_z(u)) = \overline{g} \circ \varphi_z(0) = \overline{g}(z).
\]

2. By definition, \( (T_fT_gK_z)(w) = f(w)T_gK_z(w) \) and thus the result follows from 1. above.

3. By 2. and 1. we have

\[
(T_fT_gK_z)(w) = f(w)\overline{g}(z)K_z(w).
\]

On the other hand 2. shows

\[
(T_gT_fK_w)(z) = g(z)f(w)K_w(z) = f(w)\overline{g}(z)K_z(w)
\]

which gives the desired equality.

We will end this section by stating a theorem known as Schur’s Lemma which will be very useful to us.

Suppose \( (X, \mu) \) is a measurable space and \( K(x, y) \) is a non negative function on \( X \times X \). Let \( T \) be the integral operator induced by \( K(x, y) \), that is

\[
Tf(x) = \int_X K(x, y)f(y) d\mu(y).
\]
THEOREM 4 Suppose $K(x, y)$ is nonnegative (measurable) function on $X \times X$, $T$ is the integral operator induced by $K(x, y)$, and $1 < p < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. If there exist constants $C_1, C_2 > 0$ and a positive (measurable) function $\psi(x)$ on $X$ such that

$$\int_X K(x, y)\psi(y)^q \, d\mu(y) \leq C_1\psi(x)^q$$

$$\int_X K(x, y)\psi(x)^p \, d\mu(x) \leq C_2\psi(x)^p$$

for all $x$ and $y$ in $X$, then $T$ is bounded on $L^p(X, \mu)$ with norm less than or equal to $C = \max\{C_1, C_2\}$.

3 Proof of Theorem

Let $f, g \in L^2_\alpha$ then for all $h \in H^\infty$. Then Lemma 2 assertion 2 implies

$$T_f T_\psi h(v) = \int_\Delta h(u)(T_f T_\psi K_u)(v) \, dA(u).$$

This implies that $T_f T_\psi$ is an integral operator on $L^2_\alpha$ with kernel $(T_f T_\psi K_u)(v)$. Thus by Schur’s Lemma if there exist a positive measurable function $\psi$ on $\Delta$ and constants $C_1, C_2$ such that

$$\int_\Delta |T_f T_\psi K_u(v)|\psi(v)^2 \, d\lambda(v) \leq C_2\psi(u)^2$$

(2)

for all $u \in \Delta$ and

$$\int_\Delta |T_f T_\psi K_u(v)|\psi(u)^2 \, d\lambda(u) \leq C_2\psi(v)^2$$

(3)

for all $v \in \Delta$, then $T_f T_\psi$ is bounded on $L^2_\alpha$. For $\epsilon > 0$, take $\psi(v)^2 = (K_v(v))^\epsilon$ then to get (2) we use Lemma 2(2,3) and Remark 4 to get

$$\int_\Delta |T_f T_\psi K_u(v)|\psi(v)^2 \, d\lambda(v) = \int_\Delta |T_g T_\psi K_v(u)|(K_v(v))^\epsilon \, d\lambda(v)$$

$$= |g(u)| \int_\Delta |f(v)||K_v(u)|(K_v(v))^\epsilon \, d\lambda(v)$$

$$= |g(u)||S(f \circ \varphi_u)(u)|.$$

Thus if $\epsilon < \frac{1}{2}$, Lemma 1 and the fact that $|g(u)| \leq \|g \circ \varphi_u\|_2$ for all $g \in L^2_\alpha$, implies

$$\int_\Delta |T_f T_\psi K_u(v)(u)|h(v)^2 \, d\lambda(v) \leq C|g(u)|(K_u(u))^\epsilon \|f \circ \varphi_u\|_2$$

$$\leq C\|f \circ \varphi_u\|_2\|g \circ \varphi_u\|_2(K_u(u))^\epsilon$$

$$\leq C\sup_{u \in \Delta}\|f \circ \varphi_u\|_2\|g \circ \varphi_u\|_2\psi(u)^2 = C_1\psi(u)^2.$$
where \( C_1 = C \sup_{u \in \Delta} \| f \circ \varphi_u \|_2 \| g \circ \varphi_u \|_2 \). To get (3) we use Lemma 2(2) to have

\[
\int_{\Delta} |T_f T_{\overline{f}} K_u(v)| h(u)^2 \, d\lambda(u) = |f(v)| \int_{\Delta} |T_{\overline{f}} K_u(v)| (K_u(u))^t \, dA(u) = |f(v)| \int_{\Delta} |g(u)||K_u(v)| (K_u(u))^t \, dA(u).
\]

Using the same argument as above we have that

\[
\int_{\Delta} |T_f T_{\overline{f}} K_u(v)| \psi(u)^2 \, d\lambda(u) \leq C \| f \circ \varphi_u \|_2 \| g \circ \varphi_u \|_2 \psi(v)^2 \leq C \sup_{v \in \Delta} \| f \circ \varphi_v \|_2 \| g \circ \varphi_v \|_2 \psi(v)^2 = C_2 h(v)^2
\]

Since \( \sup_{v \in \Delta} \| f \circ \varphi_v \|_2 \| g \circ \varphi_v \|_2 < \infty \) we conclude by Schur’s Lemma that \( T_f T_{\overline{f}} \) is bounded on \( L_a^2 \) and

\[
\| T_f T_{\overline{f}} \| \leq C \sup_{v \in \Delta} |f|^2(v) |g|^2(v).
\]

**Proof of Theorem 2**

Let \( f, g \in L_a^2(\Delta, d\lambda) \) then for all \( h \in L_a^2(\Delta, d\lambda) \) we have that

\[
T_f T_{\overline{f}} h(v) = \int_{\Delta} h(u)(T_f T_{\overline{f}} K_u)(v) \, d\lambda(u)
\]

Now for \( r \in (0, 1) \) we define the operator \( S_r \) on \( L_a^2(\Delta, d\lambda) \) by

\[
S_r h(v) = \int_{r \Delta} h(u) T_f T_{\overline{f}} K_u(v) \, d\lambda(u)
\]

then \( S_r \) is an integral operator on \( L_a^2(\Delta, d\lambda) \) with kernel \( \chi_{r \Delta}(u) T_f T_{\overline{f}} K_u(v) \) which is compact on \( L_a^2(\Delta, d\lambda) \). Indeed,

\[
\int_{\Delta} \int_{\Delta} |(T_f T_{\overline{f}} K_u)(v) \chi_{r \Delta}(u)|^2 \, d\lambda(u) \, d\lambda(v) \leq \int_{r \Delta} \int_{\Delta} |(T_f T_{\overline{f}} K_u)(v)|^2 \, d\lambda(v) \, d\lambda(u) \leq C \int_{r \Delta} \| K_u \|^2 d\lambda(u) < \infty
\]

since \( T_f T_{\overline{f}} \) is bounded on \( L_a^2(\Delta, d\lambda) \), and the last integral is over the compact set \( r \Delta \) of \( \Delta \). Thus \( S_r \) is a Hilbert-Schmidt operator and hence compact on \( L^2(\Delta, d\lambda) \) and hence on \( L_a^2(\Delta, d\lambda) \). We will now show that

\[
\| T_f T_{\overline{f}} - S_r \|^2 \to 0 \text{ as } r \to 1^-. 
\]

For \( h \in L_a^2 \),

\[
(T_f T_{\overline{f}} - S_r) h(v) = \int_{\Delta/r \Delta} T_f T_{\overline{f}} K_u(v) h(u) \, d\lambda(u)
\]

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which implies \( (T_f T_f^* - S_r) \) is an integral operator on \( L^2_\alpha \) with kernel \( (T_f T_f^* K_u)(v) \).

By Schur’s theorem, if there exist a positive measurable function \( \psi \) on \( \Delta \) and constants \( c_1, c_2 \) such that

\[
\int_{\Delta} |T_f T_f^* K_u(v)\chi_{\Delta/r}(u)|\psi(v) d\lambda(v) \leq c_1 h(u)
\]

for all \( u \in \Delta \) and

\[
\int_{\Delta} |T_f T_f^* K_u(v)\chi_{\Delta/r}(u)|\psi(u) d\lambda(u) \leq c_2 h(v)
\]

for all \( v \in \Delta \), then

\[
\|T_f T_f^* - S_r\|_2^2 \leq c_1 c_2.
\]

For \( \epsilon > 0 \), take \( \psi(v) = (K_v(v))^\epsilon \) then to get \( \|T_f T_f^* - S_r\|_2 \leq c_1 c_2 \) we use Lemma 2(2,3) to have

\[
\int_{\Delta} |T_f T_f^* K_u(v)\chi_{\Delta/r}(u)|h(v) d\lambda(v) \leq \chi_{\Delta/r}\int_{\Delta} |T_f T_f^* K_u(v)\chi_{\Delta/r}(u)|h(v) d\lambda(v) = \chi_{\Delta/r}\int_{\Delta} |f(v)|K_v(u)|K_v(v)|h(u) d\lambda(v).
\]

A similar argument as in the proof of Theorem 1 shows that

\[
\int_{\Delta} |T_f T_f^* K_u(v)\chi_{\Delta/r}(u)|\psi(v) d\lambda(v) \leq C\chi_{\Delta/r}\|f \circ \varphi_u\|_2\|g \circ \varphi_u\|_2(K_u(u))^\epsilon
\]

\[
\leq C\sup_{|u| > \rho} \|f \circ \varphi_u\|_2\|g \circ \varphi_u\|_2\psi(u) = c_1 \psi(u).
\]

In a similar manner we show that

\[
\int_{\Delta} |T_f T_f^* K_u(v)\chi_{\Delta/r}(u)|h(u) d\lambda(u) \leq C\sup_{u \in \Delta} \|f \circ \varphi_u\|_2\|g \circ \varphi_u\|_2\psi(v) = c_2 \psi(v).
\]

Thus by Schur’s lemma

\[
\|T_f T_f^* - S_r\|_2 \leq c_1 c_2
\]

where \( c_1 = C\sup_{|u| > \rho} \|f \circ \varphi_u\|_2\|g \circ \varphi_u\|_2 \) and \( c_2 = C\sup_{u \in \Delta} \|f \circ \varphi_u\|_2\|g \circ \varphi_u\|_2 \). Also since \( c_1 \to 0 \) as \( r \to 1^- \) we have that

\[
\|T_f T_f^* - S_r\|_2 < \epsilon
\]

This shows that \( T_f T_f^* \) is compact.

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