Quantum Planar Hall Effect in Two-Dimensional Trigonal Crystals

Raffaele Battilomo,¹ Niccolò Scopigno,¹ and Carmine Ortix¹,²

¹Institute for Theoretical Physics, Center for Extreme Matter and Emergent Phenomena, Utrecht University, Princetonplein 5, 3584 CC Utrecht, Netherlands
²Dipartimento di Fisica “E. R. Caianiello”, Università di Salerno, IT-84084 Fisciano, Italy

It is well known that the planar Hall effect (PHE) is deeply intertwined with the anisotropic magnetoresistance (AMR) characterizing strongly spin-orbit coupled materials. The amplitude of the PHE is indeed precisely set by the AMR magnitude, and vanishes when the driving electric field is aligned with the external magnetic field. Here we demonstrate that two-dimensional trigonal crystals with strong spin-orbit coupling can display a PHE of a completely different nature. This effect has a quantum origin arising from the Berry curvature of the Bloch states, and survives even when the applied current is aligned with the planar magnetic field. Moreover when the electric and magnetic fields are aligned perpendicular to a mirror line of the crystal, the PHE can occur as a second-order response at both zero and twice the frequency of the applied electric field. We demonstrate that this non-linear PHE possesses a quantum part that originates from a Zeeman-induced Berry curvature dipole.

Introduction – The planar Hall effect (PHE) – the appearance of an in-plane transverse voltage in the presence of coplanar electric and magnetic fields – is a magnetotransport phenomenon occurring in strongly spin-orbit coupled materials that display a sizable anisotropy in the longitudinal magnetoconductance. It has been shown to arise in thin films of ferromagnetic semiconductors [1,3] and two-dimensional electron gases formed at perovskite oxide interfaces [4]. The recent discovery of Weyl semimetals [5–14] and three-dimensional electron gases formed at perovskite oxide interfaces [4]. The recent discovery of Weyl semimetals [5–14] and two-dimensional electron gases formed at perovskite oxide interfaces [4]. The recent discovery of Weyl semimetals [5–14] and two-dimensional electron gases formed at perovskite oxide interfaces [4]. The recent discovery of Weyl semimetals [5–14] and two-dimensional electron gases formed at perovskite oxide interfaces [4]. The recent discovery of Weyl semimetals [5–14] and two-dimensional electron gases formed at perovskite oxide interfaces [4]. The recent discovery of Weyl semimetals [5–14] and two-dimensional electron gases formed at perovskite oxide interfaces [4]. The recent discovery of Weyl semimetals [5–14] and two-dimensional electron gases formed at perovskite oxide interfaces [4]. The recent discovery of Weyl semimetals [5–14] and two-dimensional electron gases formed at perovskite oxide interfaces [4]. The recent discovery of Weyl semimetals [5–14] and two-dimensional electron gases formed at perovskite oxide interfaces [4]. The recent discovery of Weyl semimetals [5–14] and two-dimensional electron gases formed at perovskite oxide interfaces [4]. The recent discovery of Weyl semimetals [5–14] and two-dimensional electron gases formed at perovskite oxide interfaces [4]. The recent discovery of Weyl semimetals [5–14] and two-dimensional electron gases formed at perovskite oxide interfaces [4]. The recent discovery of Weyl semimetals [5–14] and two-dimensional electron gases formed at perovskite oxide interfaces [4]. The recent discovery of Weyl semimetals [5–14] and two-dimensional electron gases formed at perovskite oxide interfaces [4]. The recent discovery of Weyl semimetals [5–14] and two-dimensional electron gases formed at perovskite oxide interfaces [4]. The recent discovery of Weyl semimetals [5–14] and two-dimensional electron gases formed at perovskite oxide interfaces [4].

Independent of the dimensionality and the specific material at hand, the PHE does not satisfy the antisymmetry property of the conventional Hall conductivity, i.e. \( \sigma_{xy} \rho_{yx} = -1 \). Specifically, the transverse current follows a \( \cos \theta \sin \theta \) angular dependence, with \( \theta \) the relative angle between the coplanar electric and magnetic fields. Onsager’s reciprocity relations [26] then require the conductivity to be a symmetric tensor. The aim of this work is to demonstrate that two-dimensional (2D) trigonal crystals with strong spin-orbit coupling often possess an additional, but very different in nature, contribution to the PHE. It gives rise to an antisymmetric conductance, which, albeit not quantized, has a purely quantum origin since it stems from the anomalous velocity of Bloch electrons generated by Berry curvature [27]. This quantum PHE is independent of the relative direction between the driving electric field and the in-plane magnetic field. When the two fields are parallel the Hall current is entirely determined by the Berry curvature.

We show that when symmetries constrain the quantum PHE to vanish, transverse Hall currents are still present: they arise in the non-linear response regime and manifest as a second harmonic response to an oscillating electric field. In strict analogy with the non-linear Hall effect of time-reversal invariant materials [28–37], we find that this non-linear PHE has a geometric contribution that is directly related to the first moment of the Berry curvature, the so-called Berry curvature dipole [29]. This clearly distinguishes the non-linear PHE we discuss here with the one recently shown to exist on the surface of 3DTIs [25]. There, the Hall resistance originates from a surface non-linear spin-to-charge conversion of transverse currents that is only possible on a 3DTI surface. The non-linear PHE induced by the Berry curvature dipole instead represents a bulk property allowed also in conventional low-symmetric two-dimensional systems. To prove the existence of our novel quantum PHE in two-dimensional trigonal crystals, we study a microscopic model on the honeycomb lattice where an inversion-symmetry breaking mass allows for non-zero Berry curvature whereas spin-orbit coupling effects are considered in the form of a Rashba coupling [38]. The planar magnetic field generates a Zeeman coupling responsible of the Berry curvature-induced linear PHE, which vanishes only when the magnetic field is perpendicular to a mirror line of the crystal. In this situation, however, the Berry curvature dipole being still finite provides a non-linear quantum PHE.

Berry curvature contributions to “Hall” Effects – It is well known that in the presence of a non-vanishing Berry curvature the velocity of Bloch electrons acquires
an extra “anomalous” contribution [39]. In a generic 2D crystal this anomalous velocity is directly responsible for transverse Hall-like currents, with \( \sigma_H \propto \int f_k \Omega_z(k) \), even when external magnetic fields are not present [40] [41].

Here the integral \( \int f_k = \int d^2 k/(2\pi)^2 \) is performed over the Fermi surface of the occupied states, while \( \Omega_z(k) \) is the Berry curvature defined as the curl of the Berry connection \( A_j = -i \langle u_k | \partial_j | u_k \rangle \) with \( j = k_x, k_y \). Since time-reversal symmetry constrains the Berry curvature to be an odd function of momentum, \( \Omega_z(k) = -\Omega_z(-k) \), the Hall conductivity of a non-magnetic system is forced to vanish. The constraint, however, is no longer valid beyond linear response theory. Indeed, it has been recently shown that non-magnetic metals without inversion symmetry can host non-linear Hall like currents due to the finiteness of the Berry curvature dipole [29]. A non centro-symmetric metal subject to an AC electric driving field \( E_c = Re(E_c e^{i\omega t}) \) can in fact develop a non-linear current \( j_0 = Re(j_0^x + j_0^y e^{i2\omega t}) \) characterized by two Fourier components at zero and twice the frequency of the applied external field: \( j_0^x = \chi_{abc} E_b E_c^* \) and \( j_0^{2\omega} = \chi_{abc} E_b E_c^* \).

Furthermore, the response function \( \chi_{abc} \), which can be expressed as \( \chi_{abc} = -\epsilon_{abc} e^3 \tau D_{bd}/2(1 + i\omega \tau) \) with \( \epsilon_{abc} \) being the Levi-Civita tensor and \( \tau \) the scattering time, explicitly contains the Berry curvature dipole defined as:

\[
D_{bd} = \int_k f_0 (\partial_k \Omega_d),
\]

where \( f_0 \) is the Fermi-Dirac distribution. A finite Berry curvature dipole can only exist in low symmetric crystals. In two-dimensional systems, in particular, since the Berry curvature is a pseudoscalar the corresponding dipole behaves as a pseudovector contained in the two-dimensional crystalline plane. Hence, a system with a non-vanishing Berry curvature dipole can possess at most one mirror symmetry. If such a symmetry is present, the dipole will be then directed perpendicular to the mirror line. Time-reversal symmetry instead does not pose any constraint on the Berry curvature dipole. The crux of the story is that, as we show below, in the presence of a single mirror symmetry, the integral of the Berry curvature weighed by the equilibrium Fermi distribution is forced to vanish even if time-reversal symmetry is broken. This implies that the Berry-phase dependent contribution to the transverse currents are exclusively non-linear.

Quantum (non)linear planar Hall effect – Non-linear Hall currents of quantum origin can occur in systems where time-reversal symmetry is broken by an externally applied planar magnetic field, therefore leading to the notion of a quantum non-linear PHE. Consider, for instance, a non-magnetic two-dimensional system subject to a planar magnetic field perpendicular to a mirror line of the crystal, which, without loss of generality, we assume to map a point with coordinates \( \{x, y\} \) to \( \{-x, y\} \). Since the external planar magnetic field preserves the mirror symmetry \( \mathcal{M}_x \), the Berry curvature will obey the symmetry constraint \( \Omega_z(k_x, k_y) = -\Omega_z(-k_x, k_y) \). Furthermore, the Fermi surface must be symmetric with respect to the mirror line, and therefore the integral of the Berry curvature is forced to vanish. In addition, considering the driving electric field to be parallel to the applied magnetic field, the conventional PHE, if present, does not provide any contribution guaranteeing the complete absence of Hall currents in the linear response regime. The Berry curvature dipole of \( \Omega_z(k_x, k_y) \), instead may still be finite and can thus generate non-linear transverse currents. Even more importantly, producing a non-vanishing dipole does not require a crystalline symmetry content as low as the one required in time-reversal symmetric conditions. This is because the externally applied planar magnetic field breaks all rotational and additional mirror symmetries thus partially relaxing the necessary conditions for a finite dipole. As a result, the non-linear Hall currents generated by the Berry curvature dipole are entirely controlled by the external magnetic field, i.e. when the field is set to zero also the transverse currents vanish. We wish to remark that due to the absence of time-reversal symmetry there exists an additional second-order but Berry-phase independent contribution to the transverse Hall conductivity. This semiclassical contribution is distinguished from the quantum non-linear PHE since the corresponding response function is expected to scale with a different power of the scattering time \( \tau \) [see the Supplemental Material].

We now show that the quantum non-linear PHE naturally arises in strongly spin-orbit coupled 2D crystals with \( C_3 \) symmetry. First, we notice that a planar magnetic field is invariant under the combined \( C_2 T \) symmetry, where \( C_2 \) indicates the twofold rotation around the axis perpendicular to the crystalline plane and \( T \) is the internal time-reversal symmetry. The presence of \( C_2 T \) symmetry then forces the Berry curvature to be identically zero: \( \Omega_z(k) \equiv 0 \). As a result, only trigonal crystals, which do not contain a twofold rotation symmetry, can display a planar magnetic-field induced Berry curvature dipole. Another necessary condition for the appearance of a finite dipole is the presence of a sizable spin-orbit coupling, which ensures that the crystal Hamiltonian \( \hat{H}_0 \) and the Zeeman coupling term \( \hat{H}_Z = B \cdot \hat{\sigma} \) do not commute. This prevents the possibility of separating the Bloch eigenfunctions of the full Hamiltonian \( \hat{H} = \hat{H}_0 + \hat{H}_Z \) into a spinorial part \( \chi_s \), regulated only by the Zeeman term, and an orbital wavefunction \( \psi_{orb}(k_x, k_y) \), where all the momentum dependence is stored: for eigenstates of that form the Berry curvature is indeed independent from the Zeeman coupling and retains the trigonal symmetry of the pristine crystal also in presence of the externally applied magnetic field. This forces the corresponding Berry curvature dipole to vanish. Finally, we notice that the non-linear PHE can occur only if the \( SU(2) \) spin symmetry in \( \hat{H}_0 \) is completely broken. A residual \( U(1) \) spin symmetry – as ensured by a mirror plane symme-
try $\mathcal{M}_z$ – would in fact imply that $\mathcal{H}_0$ commutes with the spin rotation $U_\alpha = e^{i\alpha \sigma_z/2}$. This operator rotates the planar magnetic field by an angle $\alpha$ according to $\tilde{B} = R_\alpha(B)$, but since $U_\alpha$ does not explicitly contain a momentum dependence, the two Hamiltonians $\mathcal{H}(\tilde{B})$ and $\mathcal{H}' = U_\alpha^\dagger \mathcal{H}(B) U_\alpha = \mathcal{H}(B')$ have the same Berry curvature dipole. On the other hand, the dipole is forced to be parallel to the external magnetic field when the latter is orthogonal to a mirror line [42]. If we choose $\tilde{B}$ and $B'$ to be perpendicular to different mirror lines (any two among the three of the $C_{3v}$ crystal), the only allowed vector compatible with such constraint is the null one. Hence, the Berry curvature dipole must vanish thereby proving that the non-linear planar Hall effect necessitates a complete breaking of the spin-rotation symmetry.

Having established the occurrence of a quantum non-linear PHE when the system is characterized by a residual mirror symmetry, we now consider the situation in which the external planar magnetic field is not constrained to be orthogonal to one of the three mirror lines of the $C_{3v}$ crystal. Since the presence of the planar magnetic field reduces the point group to the trivial group $C_1$, the Berry curvature does not obey any constraint, and therefore the net anomalous velocity is not forced to vanish. This consequently leads to the possibility of a purely Zeeman-induced quantum PHE in the linear response regime, which represents an antisymmetric contribution to the resistivity tensor and therefore displays a $2\pi$ periodic angular dependence. Furthermore, it is important to notice that for the integral of the Berry curvature weighed by the equilibrium Fermi distribution function to be non zero the spin-rotation symmetry needs to be completely broken – in a crystal with a $\mathcal{M}_z$ mirror plane, the combined $\mathcal{M}_zT$ symmetry, which is still preserved with a planar magnetic field, forces the Berry curvature to be an odd function. Hence, as for its non-linear counterpart, also the quantum PHE in linear response can only occur in strongly spin-orbit coupled crystals. It is thus expected to coexist with the conventional Berry-phase independent contribution to the PHE, which, as stated above, represents instead a symmetric part of the resistivity tensor. These different symmetry properties of the quantum and semiclassical contributions to the linear PHE imply that the semiclassical linear contribution to the PHE can be isolated in experiments by taking measurements with both positive and negative $B$. Instead, since the quantum contribution is independent of the angle between the electric and magnetic field, in a configuration where they are parallel it is the only term that survives.

Model – Next, we present a concrete microscopic model on the honeycomb lattice where both the linear and the non-linear quantum PHE are realized. The tight-binding Hamiltonian reads,

$$\mathcal{H}_{\text{cry}} = -t \sum_{i=1}^{3} \left[ \cos(k \cdot \delta_i) \tau_x + \sin(k \cdot \delta_i) \tau_y \right] \otimes \sigma_0 + \frac{\Delta}{2} \tau_z \otimes \sigma_0 + \mathcal{H}_R,$$

where $\sigma$ and $\tau$ refer to the spin and sublattice degrees of freedom respectively, and $\{\delta_1, \delta_2, \delta_3\} = \{(0, a/\sqrt{3}), (a/2, -a/2\sqrt{3}), (-a/2, -a/2\sqrt{3})\}$ are the nearest neighbours with $a$ the honeycomb lattice constant. In the Hamiltonian of Eq. 2 the first term containing nearest neighbour spin-independent hopping respects the $C_6v$ point group symmetry of the honeycomb lattice, which is generated by the three-fold rotation symmetry $\mathcal{C}_{3v} = \tau_x \otimes e^{i\pi \sigma_z/6}$, the twofold rotation symmetry $\mathcal{C}_2 = \tau_x \otimes e^{i\pi \sigma_z/2}$, and the mirror symmetry $\mathcal{M}_z = \tau_0 \otimes e^{i\pi \sigma_z/2}$. In order to reduce the crystalline symmetry to be trigonal, we have introduced the $\mathcal{C}_3$ and inversion-symmetry breaking mass $\Delta$. In graphene, the latter term is naturally realized by placing the graphene flake on lattice-matched substrates, such as hexagonal boron nitride [43,44]. Finally, the last term in

\[ \text{FIG. 1. Berry curvature } \Omega \text{ (a,c) and dipole density } \partial_{k_x} \Omega \text{ (b,d) of the conduction bands corresponding to the Hamiltonian of Eq. 2 in the absence (a,b) and presence (c,d) of Rashba spin-orbit coupling (} \lambda_R/t = 10^{-2} \text{). The magnetic field (} B/t = 10^{-3} \text{) has been placed along the zig-zag direction, } \alpha = 0 \text{, preserving the mirror symmetry } \mathcal{M}_z \text{. The two valleys at } K \text{ and } K' \text{ are related by } \mathcal{M}_z \text{ and hence contribute identically to the Berry curvature dipole. The inversion breaking mass has been taken to be } \Delta/t = 5 \times 10^{-2} \text{. In plots (b) and (d) light colors correspond to positive values while darker colors correspond to negative ones.} \]
Eq. 2 is a Rashba-like spin-orbit coupling term that fully breaks the $SU(2)$ spin symmetry and therefore allows for a non-vanishing Berry curvature dipole when an external planar magnetic field is applied. The Rashba term can be written as $\mathcal{H}_R = \sqrt{3}\lambda_R \sum_{j=1}^{3}[\sin(k \cdot \delta_1)\tau_x \otimes (\sigma_y \delta_{i,1} - \sigma_x \delta_{i,2}) + \cos(k \cdot \delta_1)\tau_y \otimes (\sigma_y \delta_{i,1} - \sigma_x \delta_{i,2})]$, with the strength of the Rashba coupling $\lambda_R$ that in graphene is controlled by the strength of the perpendicular electric field, and the local curvature of the graphene sheet.

We finally account for the external planar magnetic field introducing the Zeeman coupling term $\mathcal{H}_Z = B \tau_0 \otimes (\sigma_x \cos \alpha + \sigma_y \sin \alpha)$ where $\alpha$ is the angle from the zig-zag direction of the honeycomb lattice. For $\alpha = 2n\pi/6$ with $n \in \mathbb{N}$ the magnetic field preserves one mirror symmetry thus allowing only for a Berry curvature dipole. In the absence of spin-orbit interaction, i.e. for $\lambda_R \equiv 0$, the Zeeman coupling leads to a closing of the half-filling gap at the critical strength $B_c \equiv \Delta/2$, above which the system becomes a nodal semimetal generated by the crossing of two bands belonging to different spin sectors. A finite value of the Rashba spin-orbit coupling changes the crossings into anticrossings, and thus the system has a finite half-filling gap as long as the strength of the applied magnetic field is of the same order of magnitude as the inversion-symmetry breaking mass $\Delta$. For larger values of the applied magnetic field $B \simeq 2\Delta$, the half-filling gap closes but we will neglect this regime in the remainder.

More importantly, a finite value of $\lambda_R$ changes the distribution of the Berry curvature allowing for a non-zero Berry curvature dipole. This is explicitly demonstrated in Fig. 3 where we show the local Berry curvature, computed using the method outlined in Ref. 47, both in the absence and in the presence of the Rashba spin-orbit interaction. We find that effect of the Rashba spin-orbit coupling is twofold. First, it boosts the Berry curvature by reducing the splitting between the two conduction and valence bands. Second, it shifts the dipole distribution away from being centred around the high symmetry points $K$ and $K'$, hence allowing for an overall finite dipole. Fig. 2 shows the behavior of the ensuing Berry curvature dipole as a function of the carrier density for various values of the external planar magnetic field. We generally find that increasing the external magnetic field strength boosts the amplitude of the dipole over a larger range of carrier density. The dipole also displays a characteristic non-monotonous behavior, similar to the one theoretically predicted and experimentally observed in the time-reversal non-linear Hall effect, with various sign reversals, which implies that the quantum contribution to the transverse current changes direction. We note that a similar non-monotonous behavior is also found in the semiclassical contribution to the non-linear Hall conductance as shown in the Supplemental Material.

Finally, we have computed the linear quantum contribution to the PHE for $\alpha \neq 2n\pi/6$. We find that the integral of the Berry curvature weighed by the equilibrium Fermi distribution contributes to the PHE with an angular dependence that only depends on the relative direction between the magnetic field and the principal crystallographic direction, and changes sign under a $\pi$ rotation of the planar magnetic field, in perfect agreement with our general analysis. This dependence is different than the semiclassical contribution $\sigma_{xy} = e^2\tau \int_k v_x v_y (-\partial f_0/\partial \varepsilon_k)$, which we find to depend exclusively on the angle between the coplanar electric and magnetic field [see the Supplemental Material] and follows the usual PHE $\cos \theta \sin \theta$ behavior, thus vanishing when the applied fields are aligned.

Conclusions — In short, we have shown that two-dimensional trigonal crystals with sizable spin-orbit coupling subject to planar magnetic fields display a quan-
tum contribution to the PHE that has been overlooked so far. This effect is rooted in the geometric properties of the Bloch states encoded in the Berry curvature and appears in the linear response regime whenever the planar magnetic field does not lie on a principal crystallographic direction. It can be effectively decoupled from the conventional PHE since it survives even when the driving electric fields and the planar magnetic field are aligned. In a configuration in which the coplanar fields are aligned and perpendicular to one of the mirror lines of the crystal, transverse Hall currents still exist and appear at second order in the driving electric field. The resulting non-linear planar Hall effect is comprised by a semiclassical first moment of the Berry curvature, the Berry curvature dipole. Finally, the quantum planar Hall effect uncovered in this work should be present in a large number of two-dimensional material structures with Rashba-type spin-orbit interaction, including bilayer graphene due to its enhanced Berry curvature dipole.

C.O. acknowledges support from a VIDI grant (Project 680-47-543) financed by the Netherlands Organization for Scientific Research (NWO).

[1] H. X. Tang, R. K. Kawakami, D. D.Awschalom, and M. L. Roukes, Phys. Rev. Lett. 90, 107201 (2003)
[2] M. Bowen, K.-J. Friedland, J. Herfort, H.-P. Schönherr, and K. H. Ploog, Phys. Rev. B 71, 172401 (2005)
[3] Z. Ge, W. L. Lim, S. Shen, Y. Y. Zhou, X. Liu, J. K. Furdyna, and M. Dobrowolska, Phys. Rev. B 75, 014407 (2007)
[4] N. Wadehra, R. Tomar, R. M. Varma, R. K. Gopal, and K. H. Ploog, Phys. Rev. B 85, 195320 (2012)
[5] Z. K. Liu, B. Zhou, Y. Zhang, Z. J. Wang, H. M. Weng, D. Prabhakaran, S.-K. Mo, Z. X. Shen, Z. Fang, X. Dai, Z. Hussain, and Y. L. Chen, Science 343, 864 (2014)
[6] S.-M. Huang, S.-Y. Xu, I. Belopolski, C.-C. Lee, G. Chang, B. Wang, N. Alidoust, G. Bian, M. Neupane, C. Zhang, S. Jia, A. Bansil, H. Lin, and M. Z. Hasan, Nature Communications 6, 7373 (2015)
[7] H. Weng, C. Fang, C. Schirmer, and M. Z. Hasan, Phys. Rev. X 5, 011020 (2015)
[8] B. Q. Lv, H. M. Weng, B. B. Fu, X. P. Wang, H. Miao, J. Ma, P. Richard, X. C. Huang, L. X. Zhao, G. F. Chen, Z. Fang, X. Dai, T. Qian, and H. Ding, Phys. Rev. X 5, 031013 (2015)
[9] B. Q. Lv, N. Xu, H. M. Weng, J. Z. Ma, P. Richard, X. C. Huang, L. X. Zhao, G. F. Chen, C. E. Matt, F. Bist, V. N. Strocov, J. Moset, Z. Fang, X. Dai, T. Qian, M. Shi, and H. Ding, Nature Physics 11, 724 (2015)
[10] S.-Y. Xu, I. Belopolski, N. Alidoust, M. Neupane, G. Bian, C. Zhang, R. Sankar, G. Chang, Z. Yuan, C.-C. Lee, S.-M. Huang, H. Zheng, J. Ma, D. S. Sanchez, B. Wang, A. Bansil, F. Chou, P. P. Shibayev, H. Lin, S. Jia, and M. Z. Hasan, Science 349, 613 (2015)
[11] S.-Y. Xu, N. Alidoust, I. Belopolski, Z. Yuan, G. Bian, T.-R. Chang, H. Zheng, V. N. Strocov, D. S. Sanchez, G. Chang, C. Zhang, D. Mou, Y. Wu, L. Huang, C.-C. Lee, S.-M. Huang, B. Wang, A. Bansil, H.-T. Jeng, T. Neupert, A. Kaminski, H. Lin, S. Jia, and M. Zahid Hasan, Nature Physics 11, 748 (2015)
[12] N. P. Armitage, E. J. Mele, and A. Vishwanath, Rev. Mod. Phys. 90, 015001 (2018)
[13] A. Lau and C. Ortix, Phys. Rev. Lett. 122, 186801 (2019)
[14] A. A. Burkov, Phys. Rev. B 96, 041110 (2017)
[15] S. Nandy, G. Sharma, A. Tarafpider, and S. Tewari, Phys. Rev. Lett. 119, 176804 (2017)
[16] L. P. He, X. C. Hong, J. K. Dong, J. Pan, Z. Zhang, J. Zhang, and S. Y. Li, Phys. Rev. Lett. 113, 246402 (2014)
[17] T. Liang, Q. Gibson, M. N. Ali, M. Liu, R. J. Cava, and N. P. Ong, Nature Materials 14, 280 (2015)
[18] C.-L. Zhang, S.-Y. Xu, I. Belopolski, Z. Yuan, Z. Lin, B. Tong, G. Bian, N. Alidoust, C.-C. Lee, S.-M. Huang, T.-R. Chang, G. Chang, C.-H. Hsu, H.-T. Jeng, M. Neupane, D. S. Sanchez, H. Zheng, J. Wang, H. Lin, C. Zhang, H.-Z. Lu, S.-Q. Shen, T. Neupert, M. Zahid Hasan, and S. Jia, Nature Communications 7, 10735 (2016)
[19] Q. Li, D. E. Kharzeev, C. Zhang, Y. Huang, I. Pletikosic, A. V. Fedorov, R. D. Zhong, J. A. Schneeloch, G. D. Gu, and T. Valla, Nature Physics 12, 550 (2016)
[20] J. Xiong, S. K. Kuswa, T. Liang, J. W. Kriazan, M. Hirschberger, W. Wang, R. J. Cava, and N. P. Ong, Science 350, 413 (2015)
[21] M. Hirschberger, S. Kuswa, Z. Wang, Q. Gibson, S. Liang, C. A. Belvin, B. A. Bernveig, R. J. Cava, and N. P. Ong, Nature Materials 15, 1161 (2016)
[22] M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. 82, 3045 (2010)
[23] A. A. Taskin, H. F. Legg, F. Yang, S. Sasaki, Y. Kannai, K. Matsumoto, A. Rosch, and Y. Ando, Nature Communications 8, 1340 (2017)
[24] P. He, S. S.-L. Zhang, D. Zhu, S. Shi, O. G. Heinonen, G. Vignale, and H. Yang, Phys. Rev. Lett. 123, 016801 (2019)
[25] L. Onsager, Phys. Rev. 37, 405 (1931)
[26] D. Xiao, M.-C. Chang, and Q. Niu, Rev. Mod. Phys. 82, 1959 (2010)
[27] J. E. Moore and J. Orenstein, Phys. Rev. Lett. 105, 026805 (2010)
[28] I. Sodemann and L. Fu, Phys. Rev. Lett. 115, 216806 (2015)
[29] J.-S. You, S. Fang, S.-Y. Xu, E. Kaxiras, and T. Low, Phys. Rev. B 98, 121109 (2018)
[30] Y. Zhang, J. van den Brink, C. Felser, and B. Yan, 2D Materials 5, 044001 (2018)
[31] J. Son, H.-H. Kim, Y. H. Ahn, H.-W. Lee, and J. Lee, arXiv e-prints , arXiv:1907.00010 (2019)
[32] S.-Y. Xu, Q. Ma, H. Shen, V. Fatemi, S. Wu, T.-R. Chang, G. Chang, A. M. M. Valdivia, C.-K. Chan, Q. D. Gibson, J. Zhou, Z. Liu, K. Watanabe, T. Taniguchi, H. Lin, R. J. Cava, L. Fu, N. Gedik, and P. Jarillo-Herrero, Nature Physics 14, 900 (2018)
[33] Z. Z. Du, C. M. Wang, H.-Z. Lu, and X. C. Xie, Phys. Rev. Lett. 121, 266601 (2018)
[35] Q. Ma, S.-Y. Xu, H. Shen, D. MacNeill, V. Fatemi, T.-R. Chang, A. M. Mier Valdivia, S. Wu, Z. Du, C.-H. Hsu, S. Fang, Q. D. Gibson, K. Watanabe, T. Taniguchi, R. J. Cava, E. Kaxiras, H.-Z. Lu, H. Lin, L. Fu, N. Gedik, and P. Jarillo-Herrero, Nature 565, 337 (2019).

[36] J. I. Facio, D. Efremov, K. Koepernik, J.-s. You, I. Sodemann, and J. V. D. Brink, Phys. Rev. Lett. 121, 246403 (2018).

[37] R. Battilomo, N. Scopigno, and C. Ortix, Phys. Rev. Lett. 123, 196403 (2019).

[38] Y. A. Bychkov and E. I. Rashba, Journal of Physics C: Solid State Physics 17, 6039 (1984).

[39] R. Karplus and J. M. Luttinger, Phys. Rev. 95, 1154 (1954).

[40] F. D. M. Haldane, Phys. Rev. Lett. 93, 206602 (2004).

[41] N. Nagaosa, J. Sinova, S. Onoda, A. H. MacDonald, and N. P. Ong, Rev. Mod. Phys. 82, 1539 (2010).

[42] Y. Gao, S. A. Yang, and Q. Niu, Phys. Rev. Lett. 112, 166601 (2014).

[43] G. Giovannetti, P. A. Khomyakov, G. Brocks, P. J. Kelly, and J. van den Brink, Phys. Rev. B 76, 073103 (2007).

[44] C. R. Woods, L. Britnell, A. Eckmann, R. S. Ma, J. C. Lu, H. M. Guo, X. Lin, G. L. Yu, Y. Cao, R. V. Gorbachev, A. V. Kretinin, J. Park, L. A. Ponomarenko, M. I. Katsnelson, Y. N. Gornostyrev, K. Watanabe, T. Taniguchi, C. Casiraghi, H.-J. Gao, A. K. Geim, and K. S. Novoselov, Nature Physics 10, 451 EP (2014).

[45] C. L. Kane and E. J. Mele, Phys. Rev. Lett. 95, 226801 (2005).

[46] D. Huertas-Hernando, F. Guinea, and A. Brataas, Phys. Rev. B 74, 155426 (2006).

[47] T. Fukui, Y. Hatsugai, and H. Suzuki, Journal of the Physical Society of Japan 74, 1674 (2005).

[48] G. D. Mahan, Many-Particle Physics, 2nd ed. (Plenum, New York, N.Y., 1993).