GEOMETRIC ALGEBRA AND QUADRILATERAL LATTICES

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Abstract. Motivated by the fundamental results of the geometric algebra we study quadrilateral lattices in projective spaces over division rings. After giving the noncommutative discrete Darboux equations we discuss differences and similarities with the commutative case. Then we consider the fundamental transformation of such lattices in the vectorial setting and we show the corresponding permutability theorems. We discuss also the possibility of obtaining in a similar spirit a noncommutative version of the B-(Moutard) quadrilateral lattices.

1. Introduction

1.1. Integrable discrete geometry. In the course of last ten years many results of the classical geometric approach to integrable partial differential equations [47, 41] has been transferred to the discrete setting (see [25] and references therein). The key role in the theory has been attributed to the multidimensional quadrilateral lattice [23], which is the discrete analog [43] of a conjugate net [13, 29]. It turns out that integrability of the quadrilateral lattice is encoded in a very simple geometric statement, visualized on Figure 1.

**Figure 1. The Geometric Integrability Scheme**

Despite of (or, thanks to) extremely simple formulation of the Geometric Integrability Scheme, the corresponding nonlinear discrete system (the discrete Darboux equations) turns out to be the generic discrete system integrable by the nonlocal $\partial$-dressing method [9]. Also the finite-gap integration scheme, a standard tool in the integrable systems theory [8], can be applied to that system in its pure form [1]. We mention that the differential Darboux equations, which have appeared first in projective differential geometry of multidimensional conjugate nets [12], play an important role [10, 28] in the...
multicomponent Kadomtsev–Petviashvili (KP) hierarchy, which is commonly considered \cite{14,32} as the fundamental system of equations in integrability theory.

Integrable reductions of the quadrilateral lattice (and thus of the discrete Darboux equations) arise from additional constraints which are compatible with the geometric integrability scheme. In \cite{20,21,22} we isolated the incidence geometry theorems which are responsible for the basic reductions of the quadrilateral lattice: B- and C-reductions providing geometric interpretation for BKP and CKP hierarchies \cite{15}, and the so called quadratic reduction.

On the geometric level there is no essential difference between the quadrilateral lattice construction and between its Darboux-type transformations \cite{36,33,27,35}. In particular, all classical transformations of conjugate nets \cite{29,28} have found their quadrilateral lattice analogs and have been shown to be reductions of the discrete analog of the fundamental transformation of Jonas.

Although the geometric integrability scheme was initially considered for real projective spaces, it is valid to projective spaces over other fields. In particular, finite field version together with the algebra-geometric method of construction of solutions to the corresponding discrete equations has been given in \cite{26,6}. The main idea of the present paper is that the geometric integrability scheme remains valid in projective spaces over division rings (called also skew fields, for details see \cite{11}), whose simplest example are quaternions. We would like to mention that division rings appear naturally in a generalization of the notion of determinant to matrices with noncommutative entries \cite{30}. The so called quasideterminants have been effective in many areas including noncommutative symmetric functions, noncommutative integrable systems, quantum algebras and Yangians, noncommutative algebraic geometry. Last but not least, the division ring of formal pseudodifferential operators lies in the heart of the Sato approach \cite{42} to integrable systems, see also \cite{40}. We should warn the Reader that a ring of square matrices usually is not a division ring (the sum of two invertible matrices does not have to be invertible or the zero matrix). Also, by the Wedderburn theorem, finite division rings are commutative.

The subject of noncommutative versions of integrable systems was studied in the literature in many papers, see, for example \cite{18,34} and references therein; we would like to stress that in the present paper noncommutativity is considered only on the level of dependent variables, i.e., the independent variables are still commutative ones. In relation to our work we would like to mention the paper \cite{38} where the noncommutative discrete KP equation was considered, and the papers \cite{7,46}. Moreover, in \cite{44} a quantization of the discrete Darboux equations was investigated. It should be also mentioned that already in the paper \cite{9} the discrete Darboux equations, together with some of their transformations, were considered in the matrix version within the non-local \(\partial\)-dressing method, thus in the noncommutative setting (for the differential matrix Darboux-Manakov–Zakharov equations see \cite{49}).

The main results of the paper were presented in my talk Geometric algebra and quadrilateral lattices during the ISLAND 3 (Integrable Systems: Linear and Nonlinear Dynamics) conference Algebraic Aspects of Integrable Systems, Port Ellen, Isle of Islay, Scotland (July, 2007).

1.2. Some basic facts from geometric algebra. Because the intended target of the paper consists of specialists from integrable systems theory we start from presenting some basic facts on the interplay between incidence geometry axioms and the corresponding algebraic structures (for details see \cite{34,2,3,5,4}).

A projective plane is a set, whose elements are called points and a set of subsets, called lines, satisfying the following four axioms:
P1 Two distinct points lie on one and exactly one line;
P2 Two distinct lines meet in precisely one point;
P3 There exist four points with no three collinear.

It is known that axioms P1-P3 make possible to introduce on the plane coordinates from an algebraic structure called the ternary ring. If, in addition to P1-P3, the Desargues axiom holds: (see Figure 2):

P4 If two triangles are perspective from a point then they are perspective from a line;

then axioms P1-P4 imply possibility of coordinatization of the plane in terms of a division ring. If, instead, one adds to the axioms P1-P3 the so called Pappus’ axiom:

P4’ If the six vertices of a hexagon lie alternately on two lines, then the three points of intersection of pairs of opposite sides are collinear;

then one has coordinates in commutative division ring, i.e. in a field.

For more dimensional projective spaces the basic incidence axioms, analogous to P1–P3, are enough to show that the spaces are actually coordinatized by division rings, i.e., there is no need

Figure 2. The Desargues configuration: the triangles $\triangle ABC$ and $\triangle A'B'C'$ are perspective from the point $O$, and are perspective from the line $l$.

Figure 3. The Pappus configuration: the vertices of the hexagon $ABCDEF$ lie alternately on two (coplanar) lines $k$ and $l$, and the three points $K, L, M$ of intersection of pairs of opposite sides are collinear.
for the Desargues axiom (which becomes a theorem). In order to have a projective geometry over a field one has to add the Pappus axiom (or its equivalent formulations).

2. Quadrilateral lattice in spaces over division rings (affine description)

Because the Geometric Integrability Scheme is valid in projective spaces over division rings, this motivates us to consider quadrilateral lattices in such spaces.

2.1. The Laplace and Darboux equations. Consider a multidimensional quadrilateral lattice, i.e., a mapping \( x : \mathbb{Z}^N \to \mathbb{P}^M(\mathbb{D}) \) with all the elementary quadrilaterals planar [23]; here \( \mathbb{Z}^N \) is \( N \geq 3 \) dimensional integer lattice, and \( \mathbb{P}^M(\mathbb{D}) \) is \( M \geq N \) dimensional right projective space over division ring \( \mathbb{D} \) (we multiply vectors by scalars from right). It turns out that the theory of quadrilateral lattices in spaces over division rings does not differ considerably from the standard case where \( \mathbb{D} \) was assumed to be commutative. One should be only careful with the order of coefficients.

Below we will use the affine description of the quadrilateral lattice. Recall that the affine space \( \mathbb{A}^M = \mathbb{P}^M \setminus H_\infty \) is the projective space with removed a fixed hyperplane \( H_\infty \subset \mathbb{P}^M \) (called the hyperplane at infinity; see, for example [12]). Two lines of \( \mathbb{A}^M \) called parallel if they intersect in a point of \( H_\infty \).

In the affine gauge the lattice is represented by a mapping \( x : \mathbb{Z}^N \to \mathbb{D}^M \), the planarity condition can be formulated in terms of the Laplace equations

\[
T_i T_j x - x = (T_i x - x) A_{ij} + (T_j x - x) A_{ji}, \quad i \neq j, \quad i, j = 1, \ldots, N,
\]

where \( T_i \) is the translation operator in the \( i \)-th direction. Then the coefficients \( A_{ij} : \mathbb{Z}^N \to \mathbb{D} \) satisfy, by compatibility of the system (2.1),

\[
A_{jk} T_k A_{ji} = 1 + (A_{ji}-1) T_j A_{ik} + (A_{jk}-1) T_j A_{ki}, \quad i, j, k \text{ distinct.}
\]

The \( i \leftrightarrow k \) symmetry of RHS of (2.2) implies existence of the potentials \( H_i, i = 1, \ldots, N \), (called the Lamé coefficients) such that

\[
A_{ij} = T_i (H_i^{-1} T_j H_i), \quad i \neq j.
\]

If we introduce the suitably scaled tangent vectors \( X_i : \mathbb{Z}^N \to \mathbb{D}^M, i = 1, \ldots, N \), by equations

\[
\Delta_i x = X_i T_i H_i,
\]

(here \( D_i = T_i - \text{id} \)) and the rotation coefficients \( Q_{ij} : \mathbb{Z}^N \to \mathbb{D}, i \neq j \), by

\[
\Delta_i H_j = Q_{ij} T_i H_i, \quad i \neq j,
\]

then equations (2.1) can be rewritten as a first order system

\[
\Delta_i X_i = X_i T_i Q_{ij}, \quad i \neq j.
\]

The compatibility condition for the system (2.6) (or its adjoint (2.5)) gives the following form of the MQL (or discrete Darboux) equations

\[
\Delta_k Q_{ij} = Q_{ik} T_k Q_{jk}, \quad i, j, k \text{ distinct.}
\]

Remark. The above equations (up to small modification which, in our language, results from considering left-vector spaces) appeared first in the matrix setting in [3].

An important geometric fact, which lies in the heart of integrability of the quadrilateral lattice, is the multidimensional consistency of the geometric integrability scheme. Its four dimensional version reads as follows (see Fig. 4).

Four Dimensional Consistency of the Geometric Integrability Scheme. Given points \( x_0, x_1, x_2, x_3 \) and \( x_4 \) in general position in \( \mathbb{P}^M \), \( M \geq 4 \), choose generic points \( x_{ij} \in \{x_0, x_i, x_j\}, 1 \leq i < j \leq 4 \), on the corresponding planes. Using the planarity condition construct the points \( x_{ijk} \in \{x_0, x_i, x_j, x_k\}, 1 \leq i < j < k \leq 4 \) — the remaining vertices of the four (combinatorial) cubes. Then there are four different ways to construct the point \( x_{1234} \), which is the last vertex of the (combinatorial) hypercube. However all of them give the same result due to the fact that the point \( x_{1234} \) is the
unique intersection point of the four three dimensional subspaces \( \langle x_1, x_{12}, x_{13}, x_{14} \rangle, \langle x_2, x_{12}, x_{23}, x_{24} \rangle, \langle x_3, x_{13}, x_{23}, x_{34} \rangle, \) and \( \langle x_4, x_{14}, x_{24}, x_{34} \rangle \) of the four dimensional subspace \( \langle x_0, x_1, x_2, x_3, x_4 \rangle \).

2.2. The backward data and the connection factors. The backward tangent vectors \( \tilde{X}_i \), the backward Lamé coefficients \( \tilde{H}_i, \ i = 1, \ldots, N \) and the backward rotation coefficients \( \tilde{Q}_{ij} \) are defined with the help of the backward shifts \( T^{-1}_i \). They are again chosen in such a way that the \( T^{-1}_i \) variation of \( \tilde{X}_j \) is proportional to \( \tilde{X}_i \) only:

\[
\Delta_i \tilde{X}_j = (T_i \tilde{X}_i) \tilde{Q}_{ij}, \quad i \neq j .
\]

Then

\[
\Delta_i x = (T_i \tilde{X}_i) \tilde{H}_i,
\]

and

\[
\Delta_j \tilde{H}_i = (T_j \tilde{Q}_{ij}) \tilde{H}_j, \quad i \neq j .
\]

The new functions \( \tilde{Q}_{ij} \) satisfy the backward Darboux (MQL) equations

\[
\Delta_i \tilde{Q}_{ij} = (T_k \tilde{Q}_{ik}) \tilde{Q}_{kj}, \quad i, j, k \ \text{distinct}.
\]

Remark. Notice that, opposite to the commutative case \[33, 24\], the backward Darboux equations are not the same like the forward Darboux equations \[2.7\].

The connection factors \( \rho_i : \mathbb{Z}^N \to \mathbb{D} \) are the proportionality coefficients between \( X_i \) and \( T_i \tilde{X}_i \) (both vectors are proportional to \( \Delta_i x \)):

\[
X_i = -(T_i \tilde{X}_i) \rho_i, \quad T_i H_i = -\rho_i^{-1} \tilde{H}_i, \quad i = 1, \ldots, N.
\]

Going around an elementary quadrilateral it is not difficult to show that

\[
\rho_j T_j Q_{ij} = (T_i \tilde{Q}_{ij}) \rho_i,
\]

and

\[
T_j \rho_i = \rho_i (1 - (T_i Q_{ji})(T_j \tilde{Q}_{ij})) = (1 - (T_j \tilde{Q}_{ij})(T_i \tilde{Q}_{ji})) \rho_i, \quad i \neq j .
\]
Remark. In the commutative case there exists yet another potential (the $\tau$-function of the quadrilateral lattice) such that

$$\rho_{i} = \frac{T_{i} \tau}{\tau},$$

which is an immediate consequence of

$$\frac{T_{i} \rho_{j}}{\rho_{j}} = \frac{T_{j} \rho_{i}}{\rho_{i}}.$$  \hspace{1cm} (2.15)

The last equation does not hold in the noncommutative case because, in general $(T_{i}Q_{ji})(T_{j}Q_{ij}) \neq (T_{j}Q_{ij})(T_{i}Q_{ji})$.

3. Transformations of the quadrilateral lattice

Due to its vectorial character, the theory of transformations of quadrilateral lattices transfers to the noncommutative case almost without changes. Therefore mostly we just state the relevant formulas (the proofs are by direct verification along lines given in [36, 27, 35]).

Given the solution $Y_{i} : \mathbb{Z}^{N} \rightarrow \mathbb{D}^{K}$ of the linear system (2.6), and given the solution $Y_{i}^{*} : \mathbb{Z}^{N} \rightarrow (\mathbb{D}^{K})^{*}$, of the linear system (2.5); we recall that elements of $\mathbb{D}^{K}$ we represent by column vectors, and elements of its dual $(\mathbb{D}^{K})^{*}$ as row vectors. These allow to construct the linear operator valued potential $\Omega(Y, Y^{*}) : \mathbb{Z}^{N} \rightarrow M_{K}^{N}(\mathbb{D})$, defined by

$$\Delta_{i} \Omega(Y, Y^{*}) = Y_{i} \otimes T_{i} Y_{i}^{*}, \hspace{1cm} i = 1, \ldots, N;$$

similarly, one defines $\Omega(X, Y^{*}) : \mathbb{Z}^{N} \rightarrow M_{N}^{K}(\mathbb{D})$ and $\Omega(Y, H) : \mathbb{Z}^{N} \rightarrow \mathbb{D}^{K}$ by

$$\Delta_{i} \Omega(X, Y^{*}) = X_{i} \otimes T_{i} Y_{i}^{*}, \hspace{1cm} (3.2)$$

$$\Delta_{i} \Omega(Y, H) = Y_{i} \otimes T_{i} H_{i}. \hspace{1cm} (3.3)$$

We remark that because we multiply vectors from the right then covectors are multiplied from the left. This makes the tensor products above well defined.

Proposition 1. If $\Omega(Y, Y^{*})$ is invertible then the vector function $x' : \mathbb{Z}^{N} \rightarrow \mathbb{D}^{N}$ given by

$$x' = x - \Omega(X, Y^{*})\Omega(Y, Y^{*})^{-1}\Omega(Y, H), \hspace{1cm} (3.4)$$

represent a quadrilateral lattice (the vectorial fundamental transform of $x$), whose Lamé coefficients $H'_{i}$, normalized tangent vectors $X'_{i}$ and rotation coefficients $Q'_{ij}$ are given by

$$H'_{i} = H_{i} - Y_{i}^{*} \Omega(Y, Y^{*})^{-1}\Omega(Y, H), \hspace{1cm} (3.5)$$

$$X'_{i} = X_{i} - \Omega(X, Y^{*})\Omega(Y, Y^{*})^{-1}Y_{i}, \hspace{1cm} (3.6)$$

$$Q'_{ij} = Q_{ij} - Y_{j}^{*} \Omega(Y, Y^{*})^{-1}Y_{i}. \hspace{1cm} (3.7)$$

Moreover, the backward data and the connection coefficients transform according to

$$\hat{H}'_{i} = \hat{H}_{i} + \rho_{i} Y_{i}^{*} \Omega(Y, Y^{*})^{-1}\Omega(Y, H), \hspace{1cm} (3.8)$$

$$\hat{X}'_{i} = \hat{X}_{i} + \Omega(X, Y^{*})\Omega(Y, Y^{*})^{-1}Y_{i} \rho_{i}^{-1}, \hspace{1cm} (3.9)$$

$$\hat{Q}'_{ij} = \hat{Q}_{ij} - \rho_{i} Y_{j}^{*} \Omega(Y, Y^{*})^{-1}Y_{j} \rho_{j}^{-1}, \hspace{1cm} (3.10)$$

$$\rho'_{i} = \rho_{i}(1 + T_{i} Y'_{i} \Omega(Y, Y^{*})^{-1}Y_{i}). \hspace{1cm} (3.11)$$

Remark. We would like to mention that the above formulas can be put into a form using the so called quasideterminants [30], like it was done, for example, in [31] for a non-Abelian Toda lattice.

Remark. As it was shown in [27] for the commutative case, other Darboux-type transformations of the quadrilateral lattice, like the Laplace, Combescore, Lévy, adjoint Lévy or the radial transformations, can be obtained as reductions of the fundamental transformation. There are no obstructions which would prevent the geometric reasoning applied in [27] to transfer such a statement to the noncommutative case.
The vectorial fundamental transformation can be considered as superposition of \( \text{dim} \mathcal{V} \) (scalar) fundamental transformations; on intermediate stages the rest of the transformation data should be suitably transformed as well. Such a description contains already the principle of permutability of such transformations, which follows from the following observation [27].

**Proposition 2.** Assume the following splitting of the data of the vectorial fundamental transformation

\[
Y_i = \begin{pmatrix} Y^a_i \\ Y^b_i \end{pmatrix}, \quad Y^*_i = \begin{pmatrix} Y^a_{i\alpha} & Y^b_{i\beta} \end{pmatrix},
\]

associated with the partition \( \mathbb{D}^K = \mathbb{D}^{K_\alpha} \oplus \mathbb{D}^{K_\beta} \), which implies the following splitting of the potentials

\[
\Omega(Y, H) = \begin{pmatrix} \Omega(Y^a, H) \\ \Omega(Y^b, H) \end{pmatrix}, \quad \Omega(Y, Y^*) = \begin{pmatrix} \Omega(Y^a, Y^*_a) & \Omega(Y^b, Y^*_a) \\ \Omega(Y^b, Y^*_a) & \Omega(Y^b, Y^*_b) \end{pmatrix},
\]

\[
\Omega(X, Y^*) = \begin{pmatrix} \Omega(X, Y^*_a) \\ \Omega(X, Y^*_b) \end{pmatrix}.
\]

Then the vectorial fundamental transformation is equivalent to the following superposition of vectorial fundamental transformations:

1) Transformation \( x \rightarrow x^{(a)} \) with the data \( Y^a_i, Y^*_a \) and the corresponding potentials \( \Omega(Y^a, H), \Omega(Y^a, Y^*_a), \Omega(X, Y^*_a) \)

\[
x^{(a)} = x - \Omega(X, Y^*_a)\Omega(Y^a, Y^*_a)^{-1}\Omega(Y^a, H),
\]

\[
X^{(a)}_i = X_i - \Omega(X, Y^*_a)\Omega(Y^a, Y^*_a)^{-1}Y^*_i,
\]

\[
H^{(a)}_i = H_i - Y^*_a\Omega(Y^a, Y^*_a)^{-1}\Omega(Y^a, H).
\]

2) Application on the result the vectorial fundamental transformation with the transformed data

\[
y^{(b,a)}_i = Y^b_i - \Omega(Y^b, Y^*_a)\Omega(Y^a, Y^*_a)^{-1}Y^*_i,
\]

\[
y^{(a)}_{ab} = Y^b_{ab} - \Omega(Y^a, Y^*_a)\Omega(Y^a, Y^*_a)^{-1}\Omega(Y^a, Y^*_b),
\]

and potentials

\[
\Omega(Y^b, H)^{(a)} = \Omega(Y^b, H) - \Omega(Y^b, Y^*_a)\Omega(Y^a, Y^*_a)^{-1}\Omega(Y^a, H) = \Omega(Y^b(a), H^{(a)}),
\]

\[
\Omega(Y^b, Y^*_b)^{(a)} = \Omega(Y^b, Y^*_b) - \Omega(Y^b, Y^*_a)\Omega(Y^a, Y^*_a)^{-1}\Omega(Y^a, Y^*_b) = \Omega(Y^b(a), Y^*_b^{(a)}),
\]

\[
\Omega(X, Y^*_b)^{(a)} = \Omega(X, Y^*_b) - \Omega(X, Y^*_a)\Omega(Y^a, Y^*_a)^{-1}\Omega(Y^a, Y^*_b) = \Omega(X^{(a)}, Y^*_b^{(a)}),
\]

i.e.,

\[
x' = x^{(a,b)} = x^{(a)} - \Omega(X, Y^*_a)^{(a)}[\Omega(Y^b, Y^*_b)^{(a)}]^{-1}\Omega(Y^b, H)^{(a)}.
\]

**Proof.** The transformation rules for the intermediate data and potentials are consequence of proposition [1]. Denote

\[
\Omega(Y, Y^*) = \begin{pmatrix} \Omega^a & \Omega^b \\ \Omega^a & \Omega^b \end{pmatrix}, \quad \Omega(Y, H) = \begin{pmatrix} \Omega^a \\ \Omega^b \end{pmatrix}, \quad \Omega(X, Y^*) = \begin{pmatrix} \Omega^a & \Omega^b \end{pmatrix},
\]

and notice that

\[
\Omega = \begin{pmatrix} \Omega^a \\ \Omega^b \end{pmatrix} \begin{pmatrix} 1_a & 0 \\ 0 & 1_b \end{pmatrix} \begin{pmatrix} \Omega^a \\ \Omega^b \end{pmatrix}^{(a)} = \begin{pmatrix} \Omega^a \\ \Omega^b \end{pmatrix} \begin{pmatrix} \Omega^a & \Omega^b \\ \Omega^a & \Omega^b \end{pmatrix}^{(a)},
\]

which gives

\[
\Omega^{-1} = \begin{pmatrix} \Omega^a & \Omega^b \\ \Omega^a & \Omega^b \end{pmatrix}^{-1} = \begin{pmatrix} \Omega^a & \Omega^b \\ \Omega^a & \Omega^b \end{pmatrix} \begin{pmatrix} \Omega^a(a) & \Omega^b(a) \\ \Omega^a(a) & \Omega^b(a) \end{pmatrix}^{-1} = \begin{pmatrix} 1_a & 0 \\ 0 & 1_b \end{pmatrix}.
\]

Inserting such \( \Omega^{-1} \) into formula (3.23) we obtain

\[
x' = x - \begin{pmatrix} \Omega^a & \Omega^b \\ \Omega^a & \Omega^b \end{pmatrix}^{(a)} \begin{pmatrix} \Omega^a(a) & \Omega^b(a) \\ \Omega^a(a) & \Omega^b(a) \end{pmatrix}^{-1} \begin{pmatrix} \Omega^a \\ \Omega^b \end{pmatrix}^{(a)},
\]
thus equation (3.23).

Remark. The same result \( x' = x^{(a,b)} = x^{(b,a)} \) is obtained exchanging the order of transformations, exchanging also the indices \( a \) and \( b \) in formulas (3.15)-(3.23).

Remark. The scalar, i.e. \( K = 1 \), fundamental transformation preserves in the noncommutative case its geometric meaning as a transformation between two quadrilateral lattices such that \( x, x' \ T_i, T_j \) are coplanar. Therefore also in the noncommutative case the fundamental transformation can be considered as a construction of a new level (in the new dimension direction) of the quadrilateral lattice. In particular, in the case \( K = 2, K_a = K_b = 1 \), any point \( x \) of the lattice and its transforms \( x^{(a)}, x^{(b)} \) and \( x^{(a,b)} \) are coplanar.

4. The B-(Moutard) quadrilateral lattice

We will concentrate below on the B-(Moutard) quadrilateral lattice which provides geometric interpretation of the discrete BKP equations. We will study implications of the corresponding additional (apart from the Geometric Integrability Scheme) incidence geometric structures, which assure integrability of the above mentioned reduction, on the possibility of deriving their noncommutative versions. The main result of this Section is that the multidimensional consistency of the reduction holds if and only if the division ring is commutative.

In the geometric considerations below we assume generality of configurations, i.e., only those explicitly stated (and their consequences) hold. In particular, the subspace

\[[x_0, x_1, x_2, x_3, x_4] = \langle x_{1234}, x_{123}, x_{124}, x_{134}, x_{234} \rangle\]

of the hypercube in the Four Dimensional Consistency of the Geometric Integrability Scheme has dimension four.

4.1. The B-quadrilateral lattice. The B-quadrilateral lattice was defined geometrically in [21] in the commutative case (we consider for a moment the projective space over a (commutative) field \( F \)) as follows.

**Definition 1.** A quadrilateral lattice \( x : \mathbb{Z}^N \to \mathbb{P}^M(F) \) is called the **B-quadrilateral lattice** if for any triple of different indices \( i, j, k \) the points \( x, T_i T_j x, T_j T_k x \) and \( T_i T_k x \) are coplanar.

In [21] it was also shown that the homogeneous coordinates \( x : \mathbb{Z}^N \to \mathbb{P}^{M+1} \) satisfy (in appropriate gauge) the system of discrete Moutard equations [17, 39]

\[
(4.1) \quad T_i T_j x - x = \frac{(T_i \tau T_j \tau)}{(T_i \tau T_j \tau)} (T_i x - T_j x), \quad 1 \leq i < j \leq N,
\]

where the \( \tau \)-function above is the square root of the \( \tau \)-function of the quadrilateral lattice mentioned in the last remark of section 2.2. The compatibility condition of the linear system (4.1) is Miwa’s discrete BKP system [37]

\[
(4.2) \quad \tau T_i T_j T_k \tau = (T_i T_j \tau T_k \tau) - (T_i T_k \tau T_j \tau) (T_j T_k \tau T_i \tau) + (T_j T_k \tau T_i \tau), \quad 1 \leq i < j < k \leq N.
\]

Because the B-reduction condition is imposed on the elementary hexahedra level, to show integrability of the B-quadrilateral lattice it is important to check its four dimensional compatibility with the Geometric Integrability Scheme. The four dimensional consistency of the BQL-constraint was proved algebraically in [21] in the commutative case. We will show geometrically that, in contrary to the quadrilateral lattice case, one cannot obtain directly the noncommutative integrable B-quadrilateral lattice.

**Theorem 3.** Multidimensional consistency of the B-quadrilateral lattice constraint holds if and only if the division ring \( D \) is commutative.

**Proof.** It is an immediate consequence of two Lemmas below. \( \square \)

**Lemma 4.** The B-constraint is multidimensionally consistent if and only if for any triple of different indices \( i, j, k \) the points \( T_i x, T_j x, T_k x \) and \( T_i T_j T_k x \) are coplanar as well.
Lemma 5. Under hypotheses of the Geometric Integrability Scheme, assume that \( x_0, x_{12}, x_{13} \) and \( x_{23} \) are coplanar. Then the following is true: \( \mathbb{D} \) is commutative (hence a field) if and only if the points \( x_1, x_2, x_3 \) and \( x_{123} \) are coplanar as well (see Figure 5).

Proof of Lemma 5. Consider a hypercube with planar faces as in Four Dimensional Consistency of the Geometric Integrability Scheme. It consists with four “initial hexahedra” shearing vertex \( x_0 \), and the four “final hexahedra” shearing vertex \( x_{1234} \).

To demonstrate the first implication consider three “final hexahedra” containing the vertex \( x_{123} \). Because by the B-reduction condition
\[
\{x_{123}, x_{134}, x_{124}\}, \quad \{x_{123}, x_{124}, x_{234}\}, \quad \{x_{123}, x_{134}, x_{234}\},
\]
then the three planes above (and therefore the points \( x_1, x_2 \) and \( x_3 \)) are contained in the three dimensional subspace \( \langle x_{123}, x_{124}, x_{134}, x_{234} \rangle \). Notice that this subspace contains also \( x_4 \) (as a point of the plane \( \langle x_{124}, x_{134}, x_{234} \rangle \)). Another three dimensional subspace \( \langle x_{123}, x_{124}, x_{134}, x_{234} \rangle \) is the subspace of the initial hexahedron containing \( x_{123} \). By construction (according to the Geometric Integrability Scheme) it contains also the points \( x_1, x_2 \) and \( x_3 \). Both subspaces are different (one contains \( x_4 \) and the other does not), and belong to the four dimensional subspace \( \langle x_0, x_1, x_2, x_3, x_4 \rangle \) of the hypercube. Therefore their intersect in a plane. Therefore the points \( x_1, x_2, x_3 \) and \( x_{123} \) are coplanar, i.e., the implication holds for one of the “initial hexahedra”; the statement for three others can be shown analogously.

To show the backward implication we apply similar arguments, but for a “final hexahedron” of the hypercube — this time let us concentrate on that containing \( x_1 \). Notice that, by the assumption, the three planes
\[
\langle x_1, x_2, x_3 \rangle, \quad \langle x_1, x_2, x_4 \rangle, \quad \langle x_1, x_3, x_4 \rangle.
\]
cannot contain, respectively, \( x_{123} \), \( x_{124} \) and \( x_{134} \). They belong therefore to the subspace \( \langle x_1, x_2, x_3, x_4 \rangle \) of dimension three, which contains also the point \( x_{234} \) (as a point of the plane \( \langle x_2, x_3, x_4 \rangle \)). Another three dimensional subspace \( \langle x_{12}, x_{13}, x_{14} \rangle \) of the final hexahedron we are considering, by construction (according to the Geometric Integrability Scheme) contains the points \( x_{123}, x_{124} \) and \( x_{134} \). Notice that this subspace cannot contain \( x_{234} \), because it would contain then all the vertices of the hypercube. Both subspaces are different, and belong to the four dimensional subspace
\[
\langle x_0, x_1, x_2, x_3, x_4 \rangle = \langle x_{1234}, x_{123}, x_{124}, x_{134}, x_{234} \rangle,
\]
then they both intersect in a plane. This plane contains the points \( x_1, x_{123}, x_{124} \) and \( x_{134} \), which shows that the hexahedron under investigation satisfies the B-reduction condition; the statement for three others can be shown analogously.

The geometric proof of Lemma 5 can be obtained by application: (i) its equivalence with certain theorem concerning the so called quadrangular set of points \[21\], an (ii) equivalence of that theorem with validity of the Pappus’ configuration \[12\]. Below we give a direct algebraic proof.
Algebraic proof of Lemma 5. In what follows, by $x \in \mathbb{D}_n^{M+1}$ we denote the homogeneous coordinates of a point $x \in \mathbb{P}^M(\mathbb{D})$; recall that we deal with right vector spaces.

The coplanarity of the four points $x_0$, $x_1$, $x_2$ and $x_{12}$ can be algebraically expressed as the linear relation

$$x_0\alpha + x_1\beta + x_2\gamma + x_{12}\delta = 0,$$

where, by the generality assumption (no three of the points are collinear), all the coefficients do not vanish. Suitably rescaling the homogeneous coordinates of the points we can transfer above equation to the form

$$(4.3) \quad x_{12} = x_0 + x_1 + x_2.$$

In the equation expressing coplanarity of the points $x_0$, $x_1$, $x_3$ and $x_{13}$ we can again rescale the homogeneous coordinates of $x_3$ and $x_{13}$ to get

$$(4.4) \quad x_{13} = x_0 + (x_1 + x_3)a.$$

However, the coplanarity of $x_0$, $x_2$, $x_3$ and $x_{23}$ can be expressed, by plaing with the gauge of $x_{23}$, at most as

$$(4.5) \quad x_{23} = x_0 + x_2b + x_3c.$$

Then the additional condition of coplanarity of $x_0$, $x_{12}$, $x_{13}$ and $x_{23}$, which is equivalent to existence of $\lambda, \mu, \nu \in \mathbb{D}$ such that the expression

$$x_{12}\lambda + x_{13}\mu + x_{23}\nu$$

is proportional to $x_0$, gives $c = -b$.

The homogeneous coordinates of the point $x_{123}$ are given by

$$x_{123} = x_1A + x_{12}B + x_{13}C = x_2\tilde{A} + x_{23}\tilde{B} + x_{12}\tilde{C} = x_3A' + x_{13}B' + x_{23}C',$$

where the nine coefficients $A, \ldots, C'$ (to be determined) are given up to a common factor. Using equations (4.3), (4.4) and (4.5) with $c = -b$ we obtain decomposition of $x_{123}$ in terms of the basis vectors $x_0, x_1, x_2, x_3$

$$x_{123} = x_0(B + C) + x_1(A + B + aC) + x_2B + x_3aC,$$

$$= x_0(\tilde{B} + \tilde{C}) + x_1\tilde{C} + x_2(\tilde{A} + b\tilde{B} + \tilde{C}) - x_3b\tilde{B},$$

$$= x_0(B' + C') + x_1aB' + x_2bC' + x_3(A' + aB' - bC').$$

In consequence we obtain eight equations

$$B + C = \tilde{B} + \tilde{C} = B' + C', \quad A + B + aC = \tilde{C} = aB',$$

$$B = \tilde{A} + b\tilde{B} + \tilde{C} = bC', \quad aC = -b\tilde{B} = A' + aB' - bC',$$

which allow to find the coefficients $A, \ldots, C'$.

The additional requirement $x_{123} \in \langle x_1, x_2, x_3 \rangle$ algebraically means that the coefficients in front of $x_0$ in the above decompositions vanish. Neglecting three of the above equations which simply express $A$, $\tilde{A}$ and $A'$ in terms of six other coefficients $B, \ldots, C'$, we obtain the system

$$B + C = \tilde{B} + \tilde{C} = B' + C' = 0,$$

$$\tilde{C} = aB', \quad B = bC', \quad aC = -b\tilde{B},$$

which allows for nontrivial solution if and only if $ab = ba$. \qed
5. Conclusions and Discussion

Motivated by validity of the Geometric Integrability Scheme in projective spaces over division rings we investigated basic properties of the quadrilateral lattices in such spaces and the corresponding version of the discrete Darboux equations. In particular, we showed that basic ingredients of the vectorial fundamental transformation of quadrilateral lattices transfer to such a setting almost without changes (one has to take care of correct ordering only). We would like to mention that in the incidence geometry one considers also more general spaces over rings \[45\], which should provide geometric interpretation for the matrix Darboux equations.

We also investigated possibility of obtaining the noncommutative version of the B-(Moutard) quadrilateral lattices. It turns out that the additional incidence geometry assumptions which imply integrability of such lattices hold if and only if the division ring under consideration is commutative (hence a field). The question remains open for more general geometries over rings.

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