QUALITATIVE BEHAVIOR OF A CLASS OF STOCHASTIC PARABOLIC PDES WITH DYNAMICAL BOUNDARY CONDITIONS

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Abstract. We consider non-linear parabolic stochastic partial differential equations with dynamical boundary conditions and with a noise which acts in the domain but also on the boundary and is presented by the temporal generalized derivative of an infinite dimensional Wiener process. We prove that solutions to this stochastic partial differential equation generate a random dynamical system. Under additional conditions we show that this system is monotone. Our main result states the existence of a compact global (pullback) attractor.

1. Introduction. Let \( \Omega \subset \mathbb{R}^d \) be a bounded \( C^\infty \)-smooth domain with the boundary \( \Gamma \). We assume that \( \Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \), where \( \Gamma_1 \), \( \Gamma_2 \) and \( \Gamma_3 \) are disjoint open and closed subsets of \( \Gamma \). We consider a system of quasi-linear parabolic stochastic partial differential equations (SPDE's) of the form

\[
\begin{align*}
\frac{du^i}{dt} &= - (A^i(x, \partial)u^i + f^i(u_1, \ldots, u_m)) \, dt + dW^i_0 \quad \text{on} \quad \Omega \times \mathbb{R}_+, \\
\frac{du^i}{dt} &= - (B^i(x, \partial)u^i + h^i(u_1, \ldots, u_m)) \, dt + dW^i_1 \quad \text{on} \quad \Gamma_1 \times \mathbb{R}_+, \\
B^i(x, \partial)u^i &= 0 \quad \text{on} \quad \Gamma_2 \times \mathbb{R}_+, \\
u^i |_{t=0} &= u^i_0,
\end{align*}
\]

(1)

where \( i = 1, \ldots, m \), \( W = (W^{(0)}; W^{(1)}) \) is a Wiener process with values in an appropriate function space, \( f \) and \( h \) are real functions (their properties are described below). The linear differential operations \( A^i(x, \partial) \) and \( B^i(x, \partial) \) have the form

\[
A^i(x, \partial) = - \sum_{k,j=1}^d \partial_{x_k} [a^i_{kj}(x) \partial_{x_j}] + a^i_0(x)
\]

(2)

and

\[
B^i(x, \partial) = \partial_{x_i} + c^i(x) = \sum_{k,j=1}^d \nu_{kj} a^i_{kj}(x) \partial_{x_j} + c^i(x),
\]

(3)

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where $\nu = (\nu_1, \ldots, \nu_d)$ is the outer normal to $\partial \Omega$. The pair $(A^i, B^i)$ generates a normally elliptic boundary value problem of second order for each $i = 1, \ldots, m$ (see, e.g., the survey [1] and the references therein). The simplest example of system (1) is the following problem for a single unknown function $u$ with dynamics on the boundary $\Gamma_1$:

\[
\begin{cases}
    d_u = (\Delta u - f(u)) \, dt + dW^{(0)} & \text{on } \Omega \times \mathbb{R}_+,
    \\
    d_u = -\left(\frac{\partial u}{\partial \nu} + h(u)\right) \, dt + dW^{(1)} & \text{on } \Gamma_1 \times \mathbb{R}_+,
    \\
    \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_2 \times \mathbb{R}_+, 
    \quad u = 0 & \text{on } \Gamma_3 \times \mathbb{R}_+, 
    \quad u|_{t=0} = u_0.
\end{cases}
\] (4)

Deterministic parabolic systems with dynamical boundary conditions arise in hydrodynamics and the heat transfer theory and were studied by many authors (see, e.g., [21] and also more recent publications [20, 17, 18] and the references therein). As an example of a SPDE with stochastic dynamical boundary conditions we point out the problem on stochastic forcing of surface gravity waves by random fluctuations of the atmospheric pressure field (see [24] and the references therein). Another application is studied in [16]. The existence and properties of solutions to a stochastic problem like (1) in the case $\Gamma_3 = \emptyset$ were studied in [13] in a more general situation.

Our main goal in this paper is to show that problem (1) generates a random dynamical system and to study its long-time dynamics by means of random attractors.

The paper is organized as follows.

In Section 2 we collect basic facts which we need for our further considerations. Following Arnold [3] we first introduce a concept of random dynamical systems (RDS) and describe its general properties. Then we introduce appropriate function spaces and operators and quote a result of Amann-Escher [2] concerning a deterministic linear version of problem (1). We also discuss here properties of Ornstein-Uhlenbeck processes. Section 3 deals with the well-posedness of problem (1). We rewrite (1) as a random PDE. Using ideas of the theory of monotone operators (see, e.g., [21]) we prove the existence and uniqueness theorem. This theorem makes it possible to prove that (1) generates an RDS in an appropriate function space. We also study regularity properties of solutions related to the parabolic nature of the problem. In Section 4 we prove that under some conditions the RDS generated by (1) is order preserving (monotone). Our main result on the existence of random (pullback) attractor is proved in Section 5. In the concluding Section 6 we consider an application of the results obtained to an equation of the form (4).

2. Preliminaries. This section contains some background material which we use in further considerations.

2.1. Random dynamical systems. Our intention is to describe (1) as a random dynamical system (RDS). So in this subsection, we are going to give some fundamental properties of these systems. We start to describe a general model of a noise:

**Definition 2.1.** A quadruple $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ is called a metric dynamical system if $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and the measurable mapping

$$\theta : (\mathbb{R} \times \Omega, \mathcal{B}(\mathbb{R}) \otimes \mathcal{F}) \to (\Omega, \mathcal{F})$$
satisfies the flow property
\[ \theta_0 = \text{id}_\Omega, \quad \theta_t \circ \theta_{\tau} = \theta_{t+\tau} \quad \text{for } t, \tau \in \mathbb{R}. \]
The measure \( \mathbb{P} \) is supposed to be invariant with respect to the flow \( \theta \).

We also have to describe random variables with a particular growth behavior.

**Definition 2.2.** A random variable \( X \geq 0 \) on \( (\Omega, \mathcal{F}, \mathbb{P}, \theta) \) is called tempered if there is a set of full measure such that
\[ \lim_{t \to \pm \infty} \frac{\log^+ X(\theta_t \omega)}{|t|} = 0. \]

We note that the set of \( \omega \)'s where this property holds is \( \theta \)-invariant. So we can modify \( X \) outside of this \( \theta \)-invariant set in an appropriate way and thus we can suppose that the temperedness property above holds for all \( \omega \in \Omega \).

We consider the metric dynamical system given by the Brownian motion. Let \( W \) be a continuous two-sided Wiener process with values in some separable Hilbert space \( U \). For the definition of a two-sided Wiener process see Arnold [3]. The distribution \( \mathbb{P} \) of this process is defined on \( \mathcal{F} = \mathcal{B}(C_0(\mathbb{R}, U)) \), where \( C_0(\mathbb{R}, U) \) is the Fréchet space of continuous functions on \( \mathbb{R} \) with values in \( U \) which are zero at zero.

We introduce the flow \( \{ \theta_t \}_{t \in \mathbb{R}} \) given by the Wiener shift
\[ \theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad t \in \mathbb{R}. \] (5)

If we interpret the above Wiener process in the canonical sense \( W(\cdot, \omega) = \omega(\cdot) \), then (5) is the well known helix property of a Wiener process:
\[ W(t+s, \omega) - W(s, \omega) = W(t, \theta_s \omega), \quad t, s \in \mathbb{R}. \]

We now introduce the concept of RDS.

**Definition 2.3.** Let \( H \) be some separable Banach space with norm \( \| \cdot \| \). A measurable mapping
\[ \phi : \mathbb{R}^+ \times \Omega \times H \to H \]

having the cocycle property
\[ \phi(0, \omega, x) = x, \quad \phi(t, \theta_{\tau} \omega, \cdot) \circ \phi(\tau, \omega, x) = \phi(t + \tau, \omega, x), \]

for \( t, \tau \in \mathbb{R}^+, \ x \in H \) and \( \omega \in \Omega \) is called RDS with respect to the metric dynamical system \( \theta \). We call this RDS continuous if \( H \ni x \to \phi(t, \omega, x) \in H \) is continuous for \( t \geq 0 \) and \( \omega \in \Omega \).

The cocycle property of an RDS is the generalization of the semi-group property of a deterministic dynamical system what we obtain if we delete in the above formulas the parameter \( \omega \). RDS (with continuous time \( t \)) are usually generated by the solution operator of a random or stochastic differential equation, see Arnold [3].

Let \( \phi_1 \) be a continuous RDS and let
\[ T : \Omega \times H \to H \]

be a mapping with the following properties: (i) for fixed \( \omega \) this mapping is a homeomorphism, and (ii) the mappings \( \omega \to T(\omega, x) \) and \( \omega \to T^{-1}(\omega, x) \) are measurable for \( x \in H \). Then the mapping
\[ (t, \omega, x) \to \phi_2(t, \omega, x) := T(\theta_{t} \omega, \phi_1(t, \omega, T^{-1}(\omega, x))) \] (7)
defines a continuous RDS which is called conjugate to \( \phi_1 \). Indeed, the cocycle property follows straightforwardly. Note that \( T \) and \( T^{-1} \) are measurable with respect to the \( \sigma \)-algebra of \( \Omega \times H \). This follows by Castaing and Valadier [7, Chap. III].

We intend to study the long time behavior of an RDS generated by (1) by means of random attractors. To introduce these attractors we need particular set systems that will be attracted by the random attractor.

A family \( D = \{ D(\omega) \}_{\omega \in \Omega} \) is called a random set if \( D(\omega) \) is closed and nonempty and, in addition,
\[
\omega \rightarrow \inf_{x \in D(\omega)} \text{dist}(x - y)
\]
is a random variable for every \( y \in H \). A random set \( D = \{ D(\omega) \}_{\omega \in \Omega} \) is said to be tempered if
\[
\omega \rightarrow \sup_{x \in D(\omega)} \|x\| \text{ is tempered.}
\]

The following definition of an attractor is motivated by the corresponding deterministic concept (see, e.g., [5, 9, 28] and the references therein).

**Definition 2.4.** Let \( D \) be the collection of all tempered random sets in \( H \). A random set \( A \in D \) with \( A(\omega) \) compact is called a random (pullback) attractor in \( D \) if
\[
\phi(t, \omega, A(\omega)) = A(\theta_t \omega) \quad \text{for every } t \geq 0, \omega \in \Omega \quad (8)
\]
and
\[
\lim_{t \to \infty} \text{dist}(\phi(t, \theta_{-t} \omega, D(\theta_{-t} \omega)), A(\omega)) = 0 \quad \text{for every } D \in D, \omega \in \Omega. \quad (9)
\]

By the invariance of \( \mathbb{P} \) we also have that \( \phi(t, \omega, D(\omega)) \) tends to \( A(\theta_t \omega) \) with respect to the convergence in probability when \( t \to \infty \). Relations (8) and (9) mean that the random attractor is a stationary set-valued random process which describes the long-time states of the RDS \( \phi \). We also note that instead of the sub-exponentially growing (tempered) sets one can also choose other collections of sets that will be attracted by a random attractor. For instance, one could define local random attractors introducing appropriate set collections.

The following lemma shows the relation between attractors of conjugate RDS.

**Lemma 2.5.** Let \( \phi_1 \) be an RDS with a random attractor \( A_1 \) in the collection \( D \) introduced above. Suppose that the mapping \( T \) introduced in (6) has all the properties given above. In addition, we assume
\[
T(\omega, D(\omega)) \in D, \quad T^{-1}(\omega, D(\omega)) \in D \quad \text{for } D \in D.
\]
Then \( A_2(\omega) \equiv T(\omega, A_1(\omega)) \) is a random attractor of the RDS \( \phi_2 \) introduced in (7).

The following theorem contains sufficient conditions for the existence of random attractors, see Flandoli and Schmalfuß [19], Schmalfuß [25, 26].

**Theorem 2.6.** Let \( \phi \) be a continuous RDS and let \( B \in D \) be such that \( B(\omega) \) is compact, and for every \( D \in D \) and \( \omega \in \Omega \) there exists a moment \( t(\omega, D) \) such that for \( t \geq t(\omega, D) \)
\[
\phi(t, \theta_{-t} \omega, D(\theta_{-t} \omega)) \subset B(\omega). \quad (10)
\]
Then there exists a random attractor \( A = \{ A(\omega) \}_{\omega \in \Omega} \) which is unique in \( D \).

The set \( B \) in (10) is called (pullback) absorbing.
2.2. Ornstein–Uhlenbeck process. We now consider in a Hilbert space \( H \) the following stochastic linear differential equation
\[
d\eta + A\eta dt = dW, \tag{11}
\]
where \( W \) is a Wiener process in \( H \) with a trace class covariance operator \( K \) and \( A \) is a positive self-adjoint operator on \( H \). In this case \(-A\) is the generator of an analytic semigroup \( \{e^{-tA}\}_{t \geq 0} \) on \( H \). Below we denote by \( D(A^*) \) the domain of the operator \( A^*, s \geq 0 \). A mild solution to problem (11) has the form
\[
\eta(t) = \eta(0) + \int_0^t e^{-(t-\tau)A}dW, \quad t > 0,
\]
and is called an Ornstein–Uhlenbeck process which was studied by many authors (see, e.g., [11, Proposition 3.1] and the references therein). Below we use the following lemma.

**Lemma 2.7.** Let \( K \) be the covariance operator of the Wiener process \( W \) such that
\[
\text{tr}_H(KA^{2s-1+\varepsilon}) = \text{tr}_H(A^{s-\frac{1}{2}}K A^{s-\frac{1}{2}}) < \infty
\]
for some \( s \geq 0 \) and some (arbitrarily small) \( \varepsilon > 0 \). Then there exists a random Gaussian variable \( \eta \) with values in \( D(A^*) \) and defined for all \( \omega \in \Omega \) such that
\[
\Omega \times \mathbb{R} \ni (\omega, t) \mapsto \eta(\theta_t \omega) \in D(A^*)
\]
is a continuous stationary solution to stochastic equation (11). In addition, the random variable \( \|\eta\|_{D(A^*)} \) is tempered and
\[
\mathbb{E}\|\eta\|^2_{D(A^*)} = \frac{1}{2} \text{tr}_H\left(A^{s-\frac{1}{2}}KA^{s-\frac{1}{2}}\right) < \infty.
\]
Moreover, \( \eta(\theta_t \omega) \) can be represented by the stochastic integral
\[
\eta(\theta_t \omega) = \int_{-\infty}^t e^{-(t-\tau)A}dW(\omega) \quad \text{almost surely for every} \quad t \in \mathbb{R}.
\]

The proof follows from [11, Chap.5] and involves the perfection procedure suggested in [11, Proposition 3.1].

**Corollary 2.8.** Let \( \{\mathcal{F}_t\}_{t \in \mathbb{R}} \) be a filtration of \( \sigma \)-algebras
\[
\mathcal{F}_t = \{W(u) - W(v) : u, v \leq t\} \subset \mathcal{F}.
\]
Then \( \eta \) is \( \mathcal{F}_0 \)-measurable. Moreover, by particular properties of the Wiener shift (see, e.g., [3, Chap.2]) the mapping \( \omega \mapsto \eta(\theta_t \omega) \) is \( \mathcal{F}_t \)-measurable.

2.3. **Function spaces, linear differential operators and semigroups.** Let \( \Omega \subset \mathbb{R}^d \) be a bounded \( C^\infty \)-smooth domain with the boundary \( \Gamma \). We assume that \( \Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \), where \( \Gamma_1, \Gamma_2 \) and \( \Gamma_3 \) are disjoint open and closed subsets of \( \Gamma \). Let \( W^s_p(\Omega; \mathbb{R}^m) \) and \( W^s_p(\Gamma_1; \mathbb{R}^m) \) be the notations for Sobolev-Slobodetski spaces with values in \( \mathbb{R}^m \), see [29]. Below we also denote \( H^s(\Omega; \mathbb{R}^m) \equiv W^s_2(\Omega; \mathbb{R}^m) \) and \( H^s(\Gamma_1; \mathbb{R}^m) \equiv W^s_2(\Gamma_1; \mathbb{R}^m) \). We introduce the spaces
\[
\mathbb{L}_{p,q}(\Omega) = L_p(\Omega; \mathbb{R}^m) \times L_q(\Gamma_1; \mathbb{R}^m), \quad \mathbb{L}_p(\Omega) = L_{p,p}(\Omega), \quad p, q \geq 1,
\]
and
\[
\mathbb{W}^1_{p,1}(\Omega) = \left\{(u; \xi) \in W^1_p(\Omega; \mathbb{R}^m) \times W^{1-1/p}_p(\Gamma_1; \mathbb{R}^m) : \begin{array}{ll} u = \xi & \text{on} \ \Gamma_1, \\ u = 0 & \text{on} \ \Gamma_3 \end{array} \right\},
\]
where $1 < p < \infty$. We understand the boundary conditions for $u$ on $\Gamma_1$ and $\Gamma_3$ in the sense of the standard trace theory (see, e.g., [22]). For the norm of $L_{p,q}(\mathcal{O})$ we take

$$\|u\|_{L_{p,q}} \equiv \|u_1\|_{L_{p}(\mathcal{O})} + \|u_2\|_{L_{q}(\Gamma_1)}, \quad u = (u_1, u_2) \in L_{p,q}(\mathcal{O}),$$

and similarly for the other spaces. We also denote $\mathcal{H} \equiv L^2(\mathcal{O}), \mathcal{V} \equiv H^1_0(\mathcal{O}) \equiv W^{2,1}_{0}(\mathcal{O}).$ Below we use the notation $\gamma_i$ for the trace operator $u \mapsto u|_{\Gamma_i}$ and denote by $\| \cdot \|$ and $(\cdot, \cdot)$ the norm and the inner product in $\mathcal{H}$. We also note that $\mathcal{V}$ is densely and compactly embedded in $\mathcal{H}$ (see \cite{2} Lemma 3.1).

We consider the differential operations \cite{2} and \cite{3} and assume that $a^i_{kj}(x)$ and $a^0_i(x)$ belong to the space of continuous functions on $\mathcal{O}$ denoted by $C(\mathcal{O})$ and $a^0_0(x) > 0$ for almost all $x \in \mathcal{O}$. In addition, suppose that the matrix $(a^i_{kj}(x))_{k,j=1}^{d}$ is symmetric and uniformly positive definite for each $i$. The functions $c^i(x)$ are positive and belong to $C(\Gamma_1 \cup \Gamma_2)$.

On the space $\mathcal{V}$ we define a continuous symmetric bilinear form

$$a(U, V) = \sum_{i=1}^{m} \left\{ \sum_{k,j=1}^{d} \int_{\mathcal{O}} a^i_{kj}(x) \partial_{x_k} u^i(x) \partial_{x_j} v^i(x) dx + \int_{\mathcal{O}} a^i_0(x) u^i(x) v^i(x) dx \right\}$$

$$+ \sum_{i=1}^{m} \left\{ \int_{\Gamma_2} c^i(x) u^i(x) v^i(x) d\Gamma + \int_{\Gamma_1} c^i(x) \xi^i(x) \zeta^i(x) d\Gamma \right\}, \quad (12)$$

where $U = (u, \xi)$ and $V = (v, \zeta)$ are elements from $\mathcal{V}$. This form generates a positive self-adjoint operator $A$ in $\mathcal{H}$ with a compact resolvent, i.e. there exists an orthonormal basis $\{E_k\}$ in $\mathcal{H}$ such that

$$AE_k = \lambda_k E_k, \quad k = 1, 2, \ldots, \quad 0 < \lambda_1 \leq \lambda_2 \leq \ldots, \quad \lim_{k \to \infty} \lambda_k = \infty.$$ 

The domain of the operator $A$ can be described as follows. Let $\Phi = (f, g) \in \mathcal{H}$. Assume that $\Phi \in \mathcal{V}$ solves the equation $A\Phi = \Psi$. Then (see \cite{2}) $\Phi = (u, \gamma_1[u])$, where $u \in H^1(\mathcal{O}; \mathbb{R}^m)$ is a weak (variational) solution to the elliptic problem

$$A^i(x, \partial) u^i = f^i \quad \text{on} \quad \mathcal{O} \times \mathbb{R}_+, $$

$$B^i(x, \partial) u^i = g^i \quad \text{on} \quad \Gamma_1 \times \mathbb{R}_+, $$

$$B^i(x, \partial) u^i = 0 \quad \text{on} \quad \Gamma_2 \times \mathbb{R}_+, $$

$$u^i = 0 \quad \text{on} \quad \Gamma_3 \times \mathbb{R}_+, \quad (13)$$

for $i = 1, \ldots, m$. If coefficients $a^i_{kj}(x)$, $a^0_i(x)$ and $c^i(x)$ are $C^\infty$-smooth, then the elliptic regularity theory (see, e.g., \cite{22} Chap.2, Subsect.7.3 or \cite{29} Chap.5) gives us that

$$\Phi \in D(A) \Leftrightarrow \begin{cases}
\Phi = (u, \gamma_1[u]), \quad u = u_* + u_{**}, \\
u_* \in H^2(\mathcal{O}; \mathbb{R}^m), \quad B^i(x, \partial) u^i_* = 0 \text{ on } \Gamma_1 \cup \Gamma_2, \quad u^i_* = 0 \text{ on } \Gamma_3, \\
u_{**} \in H^{3/2}(\mathcal{O}; \mathbb{R}^m), \quad B^i(x, \partial) u^i_{**} = 0 \text{ on } \Gamma_2, \quad u^i_{**} = 0 \text{ on } \Gamma_3, \\
A^i(x, \partial) u^i_{**} = 0, \quad i = 1, \ldots, m.
\end{cases}$$

In the description in the right hand side of (14) the function $u_*$ (resp. $u_{**}$) corresponds to solutions to (13) with $g^i \equiv 0$ (resp. $f^i \equiv 0$). It follows from (14) that $D(A) \subset \{(u, \gamma_1[u]) : u \in H^{3/2}(\mathcal{O}; \mathbb{R}^m)\}$ and

$$\|u\|_{H^{3/2}(\mathcal{O}; \mathbb{R}^m)} \leq C \|AU\|_\mathcal{H}, \quad U = (u, \gamma_1[u]) \in D(A).$$
Moreover, the bootstrap argument combined with an interpolation argument makes it possible to derive from (15) and (14) that
\[ D(A^s) \subset \left\{ (u, \gamma_1[u]) : u \in H^{\frac{1}{2}+s}(\partial) \right\}, \quad s \geq 1. \]
and
\[ D(A^s) \subset \begin{cases} H^{\frac{1}{2}+s}(\partial) \times H^s(\Gamma_1), & \text{for } 0 \leq s < 1. \\ H^{\frac{1}{2}+s}(\partial) \times H^s(\Gamma_1), & \text{for } s \geq 1. \end{cases} \]
Then it follows from (15) that in the case of \( C^\infty \)-smooth coefficients the eigenfunctions \( E_k \) of the operator \( A \) has the form
\[ E_k = (\epsilon_k; \gamma_1[\epsilon_k]), \quad k = 1, 2, \ldots, \]
where \( \epsilon_k \in C^\infty_{C_b}(\partial; \mathbb{R}^m) \equiv \{ u \in C^\infty(\partial; \mathbb{R}^m) : u|_{\Gamma_3} = 0 \}. \)

We also have the following assertion (see [2, Theorem 3.2]).

**Theorem 2.9 (Amann-Escher [2]).** The operator \( A \) generates a strongly continuous positive contraction semigroup \( e^{-tA} \) in the space
\[ C_0(\partial) \equiv \{ (u; \xi) \in C(\partial; \mathbb{R}^m) \times C(\Gamma_1; \mathbb{R}^m) : u = \xi \text{ on } \Gamma_1, u = 0 \text{ on } \Gamma_3 \}. \]
It has a unique extension to a strongly continuous positive contraction semigroup on \( L^p(\partial), 1 \leq p < \infty \), which is compact and analytic if \( 1 < p < \infty \).

We recall that the positivity of \( e^{-tA} \) means that for any nonnegative element \( V \) from \( L^p(\partial) \) we have that \( |e^{-tA}V|\geq0 \) for almost all \( x \in \partial \).

In what follows we consider this semigroup \( e^{-tA} \) generated by the form (14) in the space \( \mathcal{H} \). Below we also use the norm
\[ \|V\|_{\mathcal{H}} = (AV, V) = \|A^{1/2}V\|^2. \]

3. Well-posedness and RDS generation. In this section we rewrite (11) as a random PDE (i.e. a PDE with random coefficients but without noise). Then we prove the existence and uniqueness theorem and use it to show that (11) generates an RDS in both \( \mathcal{H} \) and \( C_0(\partial) \). However, the latter case requires the equality \( f(u) \equiv h(u) \).

3.1. Hypotheses concerning the non-linearities. We impose the following conditions on the non-linear terms \( f \) and \( h \) appearing in the equation.

(F1) the mapping \( f = (f^1, \ldots, f^m) : \mathbb{R}^m \to \mathbb{R}^m \) is continuous and, if \( \Gamma_3 \neq \emptyset \) then \( f^i(0) = 0 \);

(F2) for each \( i = 1, \ldots, m \) the function \( f^i(u) \) admits the representation
\[ f^i(u^1, \ldots, u^m) = f_0^i(u^i) + f_1^i(u^1, \ldots, u^m), \]
where \( f_1^i : \mathbb{R}^m \to \mathbb{R} \) is globally Lipschitz with Lipschitz constant \( L \) and \( f_0^i : \mathbb{R} \to \mathbb{R} \) is a locally Lipschitz function with the properties
\[ \alpha_1|s|^p - \beta_1 \leq s f_0^i(s) \leq \alpha_2|s|^p + \beta_2, \quad s \in \mathbb{R}, \]
and
\[ (s_1 - s_2) (f_0^i(s_1) - f_0^i(s_2)) \geq -\alpha_3|s_1 - s_2|^2, \quad s_1, s_2 \in \mathbb{R}, \]
where \( \alpha_i \) and \( \beta_i \) are positive constants and \( p \geq 2 \).

Concerning the boundary non-linearities we assume similar properties:

(H1) the mapping \( h = (h^1, \ldots, h^m) : \mathbb{R}^m \to \mathbb{R}^m \) is continuous;
for each \( i = 1, \ldots, m \) the function \( h^i(u) \) admits the representation
\[
h^i(u^1, \ldots, u^m) = h^i_0(u^i) + h^i_1(u^1, \ldots, u^m),
\]
where \( h^i_1 : \mathbb{R}^m \to \mathbb{R} \) is globally Lipschitz with Lipschitz constant \( L \) and \( h^i_0 : \mathbb{R} \to \mathbb{R} \) is a locally Lipschitz function possessing the properties
\[
\alpha_1 |s|^q - \beta_1 \leq sh^i_0(s) \leq \alpha_2 |s|^q + \beta_2 \quad s \in \mathbb{R},
\]
and
\[
(s_1 - s_2)\left(h^i_1(s_1) - h^i_0(s_2)\right) \geq -\alpha_3 |s_1 - s_2|^2 \quad s_1, s_2 \in \mathbb{R},
\]
where \( \alpha_i \) and \( \beta_i \) are positive constants and \( q \geq 2 \).

**Remark 3.1.** The assumptions concerning \( f^i_0 \) and \( h^i_0 \) are satisfied if, for instance, \( f^i_0(s) \) and \( h^i_0(s) \) are polynomials of odd degree with positive leading coefficients. In the case of smooth functions the relations (19) and (21) are equivalent to the assumption
\[
\frac{df^i_0(s)}{ds} \geq -\alpha_3, \quad \frac{dh^i_0(s)}{ds} \geq -\alpha_3, \quad i = 1, \ldots, m.
\]
We also note that for the sake of simplicity (and without loss of generality) we assume that (F2) and (H2) hold with the same coefficients \( \alpha_i, \beta_i \) and \( L \). However, we would like to emphasize that the exponents \( p \) and \( q \) can be different.

Below we will denote generic constants by \( c, c_1, \ldots \).

### 3.2. \( L_{p,q} \)-solutions

The assumptions on \( f \) and \( h \) make it possible to introduce the Nemytski operator
\[
B(U) = (f(u), h(\xi)), \quad U = (u, \xi) \in L_{p,q}(\mathcal{O})
\]
for \( p, q \geq 2 \).

**Lemma 3.2.** Relation (22) defines an operator
\[
B : L_{p,q}(\mathcal{O}) \to L_{p*,q*}(\mathcal{O})
\]
which is bounded on bounded sets, where
\[
p_* = \frac{p}{p-1}, \quad q_* = \frac{q}{q-1}.
\]
Moreover, for \( U = (u, \xi) \in L_{p,q}(\mathcal{O}) \) we have that
\[
\|B(U)\|_{L_{p*,q*}(\mathcal{O})} \leq c \left( 1 + \|u\|_{L_p(\mathcal{O})}^{p-1} + \|\xi\|_{L_q(\mathcal{O})}^{q-1} \right). \tag{24}
\]
In addition, for \( U_1 = (u_1, \xi_1), U_2 = (u_2, \xi_2) \in L_{p,q}(\mathcal{O}) \) we obtain
\[
\|(B(U_1), U_2)\| \leq c + \varepsilon \sum_{i=1}^{m} \left( \|u_1^i\|_{L_{p}(\mathcal{O})}^p + \|\xi_1^i\|_{L_{q}(\mathcal{O})}^q \right) + c\varepsilon \sum_{i=1}^{m} \left( \|u_2^i\|_{L_{p}(\mathcal{O})}^p + \|\xi_2^i\|_{L_{q}(\mathcal{O})}^q \right) \tag{25}
\]
for every \( \varepsilon > 0 \), and for any \( U = (u, \xi) \in L_{p,q}(\mathcal{O}) \) we have the following coercivity estimate
\[
(B(U), U) \geq \alpha_1 \sum_{i=1}^{m} \left( \|u^i\|_{L_{p}(\mathcal{O})}^p + \|\xi^i\|_{L_{q}(\mathcal{O})}^p \right) - L\|U\|^2 - c. \tag{26}
\]
Proof. It is easy to see from (18) and (20) that
\[ |f_i(s)| \leq c_1(1 + |s|^{p-1}), \quad |h_i(s)| \leq c_1(1 + |s|^{q-1}) \]  
for all \( i = 1, \ldots, m \). Therefore, since \( f_1 \) and \( h_1 \) are globally Lipschitz, we obtain (21).

We also have (24).

Let the coefficients \( \Psi \) be the solution of (18) and (20). Then
\[ L \Psi + B \tilde{U} = \tilde{v}, \quad L \Psi = \Psi \]  
for the covariance operator of \( \tilde{W} \).

Proof. It is easy to see from (18) and (20) that
\[ |f_i(s)| \leq c_1(1 + |s|^{p-1}), \quad |h_i(s)| \leq c_1(1 + |s|^{q-1}) \]  
for all \( i = 1, \ldots, m \). Therefore, since \( f_1 \) and \( h_1 \) are globally Lipschitz, we obtain (21).

We also have (24).

Let the coefficients \( \Psi \) be the solution of (18) and (20). Then
\[ L \Psi + B \tilde{U} = \tilde{v}, \quad L \Psi = \Psi \]  
for the covariance operator of \( \tilde{W} \).

Similarly, we obtain (24) applying the left inequalities in (18) and (20). \( \square \)

Using the notations \( \mathcal{V}_{p,q} = \mathcal{V} \cap L_{p,q} \) and \( \mathcal{V}_{p,q}^* = (\mathcal{V}_{p,q})^* = \mathcal{V}^* + L_{p,q} \), we are in a position to rewrite equation (11) as a stochastic evolution equation in the triple \( \mathcal{V}_{p,q} \subset H \subset \mathcal{V}_{p,q}^* \) of the form

\[ dU + (AU + B(U)) dt = dW, \quad U(0) = U_0 \in H, \]  
where \( W = (W^{(0)}, W^{(1)}) \) is a Wiener process in \( H \) with a trace class covariance operator \( K \). We understand solutions to (28) as mild solutions (see (2) in Remark 3.6 below).

Let \( t \to \eta(\theta_t \omega) \) be a stationary solution in \( H \) to problem (11) with the operator \( A \) generated by the form \( a(U, V) \) given in (12). Introducing a new variable \( V(t) = U(t) - \eta(\theta_t \omega) \) we can rewrite (28) as the following random evolution equation

\[ \frac{dV}{dt} + AV + B(V + \eta(\theta_t \omega)) = 0, \quad t > 0, \quad V(0) = V_0 = U_0 - \eta(\omega) \in H. \]  

We consider this equation as a \( \omega \)-path-wise problem and use the standard definition of a solution \( V(t) \) in the sense of the theory of monotone operators (see (2) and also (1)). In Remark 3.6 we show that \( V(t) + \eta(\theta_t \omega) \) is a mild solution to (28).

For a regularity condition of \( \eta \) we notice the following

Lemma 3.3. Let the coefficients \( a^i_{kj}(x), a^0_i(x) \) and \( c^i(x) \) be \( C^\infty \)-smooth. Suppose that for the covariance operator of \( \tilde{W} \) we have that

\[ \text{tr}_H(K A^{2s_1-1+\varepsilon}) < \infty \]  
for some \( \varepsilon > 0 \). Then there exist numbers \( s_{p,q} \geq 0 \) and \( s_p \geq \max\{1, s_{p,p}\} \) such that (i) \( s_{2,2} = 0 \) and \( s_2 = 1 \), and (ii) the random variable \( \eta(\omega) = (\eta_1(\omega), \eta_2(\omega)) \) with values in appropriate function spaces on \( \mathcal{O} \times \Gamma_1 \) possesses the properties

- if \( s \geq s_{p,q} \), then
  \[ \rho_{p,q}(\omega) := \|\eta_1(\omega)\|^p_{L_p(\mathcal{O})} + \|\eta_2(\omega)\|^p_{L_p(\Gamma_1)} \]  
is a tempered random variable

and the function \( t \mapsto \rho_{p,q}(\theta_t \omega) \) is continuous for each \( \omega \in \Omega \);
• if \( s \geq s_p \), then 
\[
\rho_p(\omega) := \|A\eta(\theta_t \omega)\|_{L_p(O)}^p \text{ is a tempered random variable}
\]
and the function \( t \mapsto \rho_p(\theta_t \omega) \) is continuous for each \( \omega \in \Omega \).

**Proof.** By Lemma 2.27 we have that \( \| A^s \eta(\omega) \| \) is tempered. Therefore by (10) and by Sobolev's embeddings (see, e.g., [29, Chap.4]) we can find \( s_{p,q} \) such that \( D(A^s) \subset L_{p,q}(O) \) for every \( s \geq s_{p,q} \). This implies (31).

A similar argument can be applied in the second part.

We note that the constants \( s_{p,q} \) and \( s_p \) can be easily calculate from the embedding exponents (see [29, Chap.4]). \( \square \)

With respect to existence and uniqueness of (29) we can state

**Theorem 3.4.** Suppose we have (91). Then under the assumptions (F1), (F2), (H1), (H2) for any \( V_0 \in H \) there exists a unique solution \( V(t) \equiv V(t; \omega, V_0) \) to problem (29) such that for all \( \omega \in \Omega \) and \( T > 0 \)
\[
V(t) \in C([0,T]; H) \cap L_2(0,T; V)
\]
and
\[
V(t) \in L_p(0,T; L_p(O; \mathbb{R}^m)) \times L_q(0,T; L_q(\Gamma_1; \mathbb{R}^m)).
\] (32)

Moreover, there exists a deterministic constant \( b_0 > 0 \) such that
\[
\|V(t; \omega, V_0) - V(t; \omega, V_0')\| \leq \|V_0 - V_0'\| e^{b_0 t}, \quad t \geq 0, \omega \in \Omega.
\] (33)

**Proof.** Our argument is standard and follows the line described in [21, Chap.2] for parabolic problems with stationary homogeneous boundary conditions. However, dynamical boundary conditions require some non-trivial modifications.

Let \( \{\epsilon_k\} \) be a family of smooth functions from \( \overline{O} \) into \( \mathbb{R}^m \) such that (i) any finite family of elements from \( \{\epsilon_k\} \) is linearly independent, (ii) \( \{\epsilon_k; \gamma_1[\epsilon_k]\} \subset V \), and (iii) the set \( \mathcal{L} = \cup_{N \geq 1} L_N \) is dense in \( V \) and \( H \), where
\[
L_N \equiv \left\{ U \in H : U = \sum_{j=1}^N U_j \cdot (\epsilon_j; \gamma_1[\epsilon_j]), U_j \in \mathbb{R} \right\}.
\]

Such a set \( \{\epsilon_k\} \) exists because the space
\[
C_{\Gamma_3}(\overline{O}; \mathbb{R}^m) \equiv \{ u \in C^{\infty}(\overline{O}; \mathbb{R}^m) : u|_{\Gamma_3} = 0 \}
\]
is dense in \( H^1_{\Gamma_3}(O; \mathbb{R}^m) \) for any \( s \geq 1 \) (see [29, Chap.4]) and hence the space
\[
C_{\Gamma_3}(\overline{O}) \equiv \{ (u; \gamma_1[u]) : u \in C_{\Gamma_3}(\overline{O}; \mathbb{R}^m) \}
\]
is dense in \( V \). It is also clear that \( C^\infty_{\Gamma_3}(\overline{O}) \) is dense in \( L_{p,q}(O) \) (see the argument given in the proof of Lemma 3.1 in [2]). In the case of \( C^\infty \)-smooth coefficients of the differential operators \( A^s \) and \( B^s \) one can take \( E_k = (\epsilon_k; \gamma_1[\epsilon_k]) \) which are the eigenfunctions of \( A \).

We consider the Galerkin approximations to problem (29):
\[
V^N(t) \equiv V^N(t, x) = (u^N(t); \gamma_1[u^N(t)]), \quad u^N(t) = \sum_{j=1}^N u_j^N(t)\epsilon_j(x),
\]
where \( V^N(t) \) satisfies the equation
\[
(\partial_t V^N(t) + AV^N(t) + B(V^N(t) + \eta(\theta_t \omega)), \Phi_N) = 0, \quad t > 0,
\] (34)
for every $\Phi_N$ from the subspace $L_N$. We choose initial data $V^N(0) \in L_N$ such that $\|V^N(0) - V_0\| \to 0$ as $N \to \infty$. Relation (34) can be reduced to an ordinary differential equation for scalar functions $\{u^N_j(t)\}$ which has a local solution.

In the following calculations we omit the superscript $N$ for the sake of shortness.

By the chain rule we have that

$$\frac{1}{2} \frac{d}{dt} \|V\|_2^2 + \|B(V + \eta), V + \eta\| = (B(V + \eta), \eta).$$

Let $V = (v; \xi)$ and $\eta = (\eta_1; \eta_2)$. Then for $U_1 = V + \eta$ and $U_2 = \eta$ we obtain from (26) and the Young inequality

$$|(B(V + \eta), \eta)| \leq \frac{\alpha_1}{4} \sum_{i=1}^{m} \left[ \|v^i + \eta_1^i\|_{L^p(\Omega)}^p + \|\xi^i + \eta_2^i\|_{L^q(\Gamma_1)}^q \right] + c_1 \Xi(\theta_t \omega),$$

where

$$\Xi(\omega) = \sum_{i=1}^{m} \left[ 1 + \|\eta_1^i(\omega)\|_{L^p(\Omega)}^p + \|\eta_2^i(\omega)\|_{L^q(\Gamma_1)}^q \right].$$

On the other hand, by (26)

$$(B(V + \eta), V + \eta) \geq \alpha_1 \sum_{i=1}^{m} \left( \|v^i + \eta_1^i\|_{L^p(\Omega)}^p + \|\xi^i + \eta_2^i\|_{L^q(\Gamma_1)}^q \right) - (L + 1)\|V\|^2 - c_2\|\eta\|^2 - c_2.$$

Hence

$$\frac{d}{dt}\|V\|^2 + 2\|V\|^2 + \alpha_1 \sum_{i=1}^{m} \left( \|v^i\|_{L^p(\Omega)}^p + \|\xi^i\|_{L^q(\Gamma_1)}^q \right)$$

$$\leq c_3\|V\|^2 + c_3 \Xi(\theta_t \omega),$$

where $\Xi(\omega)$ is given by (35). In particular, we have that

$$\frac{d}{dt}\|V\|^2 \leq c_3\|V\|^2 + c_3 \Xi(\theta_t \omega).$$

Gronwall’s lemma yields that

$$\|V(t)\|^2 \leq \|V(0)\|^2 e^{c_3 t} + c_3 \int_0^t e^{c_3 (t - \tau)} \Xi(\theta_{\tau} \omega) d\tau.$$

Therefore from (36) after reintroducing the superscript $N$ we obtain the following a priori estimate

$$\|V^N(t)\|^2 + \int_0^t \left[ \|V^N(\tau)\|^2_2 + \sum_{i=1}^{m} \left( \|v^N_i(\tau)\|_{L^p(\Omega)}^p + \|\xi^N_i(\tau)\|_{L^q(\Gamma_1)}^q \right) \right] d\tau$$

$$\leq c_4\|V(0)\|^2 + c_5 \int_0^T \Xi(\theta_{\tau} \omega) d\tau$$

for any $t \in [0, T]$, where $c_4, c_5$ are non-random constants depending on $T$.

Our next step is to show that the limit points of the sequence $\{V^N\}$ of approximate solutions give us the solutions to problem (29). As usual (see the discussion in [21 Chap.2]) the main difficulty is to make a limit transition in the non-linear terms. We do this applying the same monotonicity ideas as in [21 Chap.2]. To do this we need the following Lemma.
Lemma 3.5. Let $p, q \geq 2$ and let $p_*, q_*$ be introduced in (23). There exists a $\mu > 0$ such that the mapping
\[
M_\mu(t)[V] = AV + B(V + \eta(t, \omega)) + \mu V,
\]
\[
M_\mu(t) : \mathcal{V} \cap L_{p,q}(\mathcal{O}) \rightarrow \mathcal{V}^* + L_{p,q}(\mathcal{O}), \quad t \geq 0
\]
is a strictly monotone hemi-continuous operator in the following sense:
(i) there exists a $\nu > 0$ such that
\[
(M_\mu(t)[V_1] - M_\mu(t)[V_2], V_1 - V_2) \geq \nu \|V_1 - V_2\|^2, \quad V_1, V_2 \in \mathcal{V} \cap L_{p,q}(\mathcal{O}), \quad (38)
\]
and
(ii) the scalar function $\lambda \mapsto (M_\mu(t)[V_1 + \lambda V_2], V_3)$ is continuous for all $V_i \in \mathcal{V} \cap L_{p,q}(\mathcal{O}), i = 1, 2, 3$.

Proof. Since $A = A^*$ is positive, by the definition (22) of $B$ from the hypotheses in (F1), (F2), (H1), (H2) we have that
\[
(M_\mu(t)[V_1] - M_\mu(t)[V_2], V_1 - V_2) = (A(V_1 - V_2), V_1 - V_2) + (B(V_1 + \eta(t, \omega)) - B(V_2 + \eta(t, \omega)), V_1 - V_2) + \mu \|V_1 - V_2\|^2
\]
\[
\geq (-\alpha_3 - L + \mu) \|V_1 - V_2\|^2.
\]
Now, we can choose a $\mu$ such that (38) holds. Property (ii) follows easily from the Lebesgue Dominated Convergence Theorem. \hfill \Box

We consider the problem (29) as a problem with a monotone hemi-continuous operator of the form
\[
\frac{dV}{dt} + M_\mu(t)[V] = \mu V, \quad t > 0, \quad V(0) = V_0 \in \mathcal{H}.
\]
In addition, we can use the same argument as in Lions [21, Chap.2] to make a limit transition of the sequence $\{V^N(\cdot) : N \in \mathbb{N}\}$ of the approximate solutions defined in (34). Thus we can prove that for any $\omega \in \Omega$ there exists a function
\[
V(t) \in L_\infty(0, T; \mathcal{H}) \cap L_2(0, T; \mathcal{V})
\]
\[
V(t) \in L_p(0, T; L_p(\mathcal{O}; \mathbb{R}^m)) \times L_q(0, T; L_q(\Gamma_1; \mathbb{R}^m)), \quad (39)
\]
which satisfies (29) in the sense of distributions. Moreover, in the same way as in [21, Chap.2] we also have that
\[
\frac{dV}{dt} \in L_2(0, T; \mathcal{V}^*) + [L_{p_*}(0, T; L_{p_*}(\mathcal{O}; \mathbb{R}^m)) \times L_{q_*}(0, T; L_{q_*}(\Gamma_1; \mathbb{R}^m))] \quad (40)
\]
for any interval $[0, T]$. Consequently from (39) and (40), we can conclude that $V(t)$ belongs to $C([0, T]; \mathcal{H})$. The uniqueness easily follows from the coercivity estimate (38), which also implies relation (33). \hfill \Box

Solutions of (29) satisfying (39) and (40) are called $L_{p,q}$-solutions and if $p = q$ $L_p$-solutions.

Remark 3.6. It follows from (32) and (21) that $B(V + \eta) \in L_r(0, T; L_r(\mathcal{O}; \mathbb{R}^m))$, where $r = \min\{p_*, q_*\} \in (1, 2]$. Therefore one can see from Theorem 2.9 that $V(t)$ given by Theorem 3.3 is also a mild solution, i.e. it satisfies the relation
\[
V(t) = e^{-tA}V_0 - \int_0^t e^{-(t-\tau)A}B(V(\tau) + \eta(\theta, \omega))d\tau. \quad (41)
\]
This implies that the function \( U(t) = V(t) + \eta(\theta_t\omega) \) is a mild solution to problem (28), i.e. it satisfies the relation
\[
U(t) = e^{-tA}U_0 - \int_0^t e^{-(t-\tau)A}B(U(\tau))d\tau + \int_0^t e^{-(t-\tau)A}dW(\tau).
\] (42)

A direct consequence of Theorem 3.4 we have the following assertion.

**Corollary 3.7.** Under the hypotheses of Theorem 3.4 the problems (28) and (29) generate a continuous RDS \( \phi_U \) and \( \phi_V \) on \( \mathcal{H} \) which are conjugate, i.e.
\[
\phi_U(t,\omega,U_0) \equiv T(\theta_t\omega,\phi_V(t,\omega,T^{-1}(\omega,U_0))),
\] (43)
where \( T,T^{-1} : \Omega \times \mathcal{H} \to \mathcal{H} \) are random homeomorphisms given by
\[
T(\omega,V) \equiv V + \eta(\omega), \quad T^{-1}(\omega,V) = V - \eta(\omega).
\] (44)

### 3.3. Additional regularity

Under some additional hypotheses concerning problem (1) we can prove additional a priori estimates and thus regularity of \( \mathbb{L}_p \)-solutions. We use this regularity in the proof of the existence of a random attractor.

We first establish the following property of the Nemytskii operator \( B \) given by (22).

**Lemma 3.8.** Let \( p = q \geq 2 \). Suppose that in addition to \((F1), (F2), (H1), (H2)\) we have that \( f_0, f_1, h_0, h_1 \in C^1 \). Moreover, assume that there exists a \( c_1 > 0 \) such that
\[
|h(u) - f(u)| \leq c_1(1 + |u|), \quad u \in \mathbb{R}^m,
\] (45)

(which is only possible if \( p = q \)). In addition, \( U \in D(A) \) then for some \( c > 0 \)
\[
(B(U),AU) \geq -c - c\|U\|^2 - \frac{1}{4}\|AU\|^2.
\] (46)

**Proof.** We first note that under the conditions imposed we have that
\[
\frac{\partial f^i(u)}{\partial v^i} \geq -\alpha_3, \quad \frac{\partial h_0^i(u)}{\partial v^i} \geq -\alpha_3, \quad \left| \frac{\partial f^i(u)}{\partial v^j} \right| \leq L, \quad \left| \frac{\partial h_1^i(u)}{\partial v^j} \right| \leq L
\] (47)
for all \( i, j = 1, \ldots, m \) and \( v = (v^1, \ldots, v^m) \in \mathbb{R}^m \). On account of the definition of the operator \( A \) we have that
\[
(B(U),AU) = \sum_{i=1}^m \int_\Omega f^i(u)A^i(x,\partial)u^idx + \sum_{i=1}^m \int_{\Gamma_1} h^i(u)B^i(x,\partial)u^i d\Gamma
\]
\[
= \sum_{i=1}^m \sum_{k,j=1}^d \int_\Omega a_{kj}^i(x)\partial x_k[f^i(u)]\partial x_j u^i dx
\]
\[
+ \sum_{i=1}^m \sum_{k,j=1}^d \int_{\Gamma_1} [h^i(u) - f^i(u)]B^i(x,\partial)u^i d\Gamma
\]
\[
+ \sum_{i=1}^m \left( \int_\Omega a_{0}^i(x)f^i(u)u^i dx + \int_{\Gamma_1 \cup \Gamma_2} c^i(x)f^i(u)u^i d\Gamma \right).
\] (48)
Hence, applying the same argument as in (50) we find that

\[\sum_{k,j=1}^d \int_O a^k_j(x) \partial_{x_k} [f^i(u)] \partial_{x_j} u^i dx\]

\[= \int_O \frac{\partial f_0^i(u')}{\partial u^j} \sum_{k,j=1}^d a^k_j(x) \partial_{x_k} u^i \partial_{x_j} u^i dx + \sum_{k,j=1}^d \int_O a^k_j(x) \partial_{x_k} [f^i(u)] \partial_{x_j} u^i dx\]

\[\geq \sum_{k,j=1}^d \int_O a^k_j(x) \left[ -\alpha_3 \partial_{x_k} u^i + \sum_{l=1}^m \frac{\partial f_l(u)}{\partial u^i} \partial_{x_k} u^i \right] \partial_{x_j} u^i dx \geq -c_1 \|u\|_{H^1(\Omega, \mathbb{R}^m)}^2\]

\[\geq -c_2 \|U\|_V^2 \geq -c_3 \|U\|_V^2 - \frac{1}{12m} \|AU\|^2.\]

Here above we also use the obvious interpolation inequality

\[\|U\|_V^2 \leq \|U\|\|AU\|, \quad U \in D(A).\] (49)

By (45) we have that

\[
\int_{\Gamma_1} |h^i(u) - f^i(u)| |\mathcal{B}^i(x, \partial) u^i| d\Gamma \leq c_4 \left( 1 + \|u\|_{L^2(\Gamma_1)} \right) \|\mathcal{B}^i(x, \partial) u^i\|_{L^2(\Gamma_1)}.
\]

Therefore using the trace theorem, the structure of the operator $A$ described in (43) and also relation (49) we obtain that

\[
\int_{\Gamma_1} |h^i(u) - f^i(u)| |\mathcal{B}^i(x, \partial) u^i| d\Gamma \leq c_4 \left( 1 + \|u\|_{H^1(\Omega)} \right) \|AU\|_{H^1} \leq c_5 + c_6 \|U\|_V^2 + \frac{1}{12m} \|AU\|^2.
\] (50)

To estimate the last term in (48) we split $f$ into $f_0$ and $f_1$ according the hypothesis in (F2). For the integral with $f_1$ we use the fact that $f_1$ is globally Lipschitz and for the integral with us to $f_0$ we apply (43) and the positivity of of $a_0(x)$ and $c(x)$. Hence, applying the same argument as in (45) we find that

\[
\int_{\Gamma_{1,2}} a^i_0(x) f^i(u) u^i dx + \int_{\Gamma_{1,2}} c^i(x) f^i(u) u^i d\Gamma \geq -c_6 - c_0 \|U\|_V^2 - \frac{1}{12m} \|AU\|^2.
\]

Collecting all the estimates above in (48) we arrive to the conclusion. \(\Box\)

Lemma 5.8 makes it possible to prove the following conclusion.

**Proposition 3.9.** Assume (F1)–(F2), (H1)–(H2) with $p = q \geq 2$. Let the coefficients of the elliptic pair $(A', \mathcal{B}')$ be $C^\infty$-smooth and suppose that (15) holds. Moreover, suppose that the Wiener process $W$ has the covariance $K$ such that

\[\text{tr}_H KA^{2s-1+\varepsilon} < \infty, \quad \text{for some} \quad \varepsilon > 0, \quad s \geq s_p,\]

where $s_p \geq 1$ is the parameter from Lemma 5.3. Then the solution $V(t)$ to problem (29) given by Theorem 3.4 possesses the properties

\[V(t) \in C_w(\mathbb{R}^+ \setminus \{0\}; \mathcal{V}) \cap L^2_{\text{loc}}(\mathbb{R}^+ \setminus \{0\}; D(A)) \quad \text{for every} \quad \omega \in \Omega, \]

(51)

where $C_w(\mathbb{R}^+ \setminus \{0\}; \mathcal{V})$ is the space of weakly continuous functions with values in $\mathcal{V}$. Moreover, for every $T > 0$ and $\omega \in \Omega$ we have the inequality

\[t \cdot \|V(t)\|_V^2 + \int_0^t \tau \|AV(\tau)\|^2 d\tau \leq c_1 \|V(0)\|^2 + c_2 \int_0^T \Xi_*(\theta_\omega) d\tau, \]

(52)
for any \( t \in (0, T) \), where \( c_i \) are non-random constants depending on \( T \) and

\[
\Xi_\ast(\omega) = 1 + \|\eta(\omega)\|_{L^p(\mathcal{O})}^p + \|A\eta(\omega)\|_{L^p(\mathcal{O})}^p.
\]

Proof. By Lemma 3.3 it follows from the conditions on \( K \) that

\[
\eta \in D(A), \quad \text{and} \quad A\eta \in L^p(\mathcal{O}).
\]

Let \( V \equiv V^N \) be a solution of (52) constructed by the eigenfunctions of \( A \) denoted by \( E_k = (\gamma_k, \gamma_1, \ldots, \gamma_k) \) (this type of approximation is possible because \( (A', B') \) are \( C^\infty \)-smooth elliptic pairs). Taking \( \Phi_N = AV \) in (51) and integrating this equation we have for \( 0 \leq s \leq t \)

\[
\|V(t)\|_{L^2}^2 + 2 \int_s^t \|AV\|_p^2 d\tau + 2 \int_s^t (B(V + \eta), A(V + \eta))d\tau \\
\leq \|V(s)\|_{L^2}^2 + 2 \int_s^t \|AV\|_p^2 d\tau + \|A\eta\|_{L^p(\mathcal{O})}^2 + \frac{1}{2} \|AV\|_{L^p(\mathcal{O})}^2 - \frac{1}{2} \|A\eta\|_{L^p(\mathcal{O})}^2.
\]

By (54) and the Young inequality

\[
\|B(V + \eta), A(V + \eta)\| \leq \|V\|_{L^p(\mathcal{O})}^p + c \left( 1 + \|\eta\|_{L^p(\mathcal{O})}^p + \|A\eta\|_{L^p(\mathcal{O})}^p \right).
\]

Therefore from (54) we have that

\[
\|V(t)\|_{L^2}^2 + \int_s^t \|AV\|_p^2 d\tau \leq \|V(s)\|_{L^2}^2 + c \int_s^t \left[ \|V(\tau)\|_{L^2}^2 + \|AV(\tau)\|_{L^p(\mathcal{O})}^p \right] d\tau \\
+ c \int_s^t \Xi_\ast(\theta, \omega) d\tau,
\]

where \( \Xi_\ast(\omega) \) is given by (53). Integrating with respect to \( s \) over the interval \([0, t]\) gives us

\[
t\|V(t)\|_{L^2}^2 + \int_0^t \tau \|AV(\tau)\|_p^2 d\tau \leq \int_0^t \left[ \|V(\tau)\|_{L^2}^2 + \|AV(\tau)\|_{L^p(\mathcal{O})}^p \right] d\tau \\
+ c_1 \int_0^t \Xi_\ast(\theta, \omega) d\tau,
\]

for all \( t \in [0, T] \), where \( c_1 \) and \( c_2 \) depend on \( T \). Using (54) we obtain (55) for the approximate solutions \( V^N \) which, after limit transition, implies that (55) holds for the solutions given by Theorem 3.4 for almost all \( t \in (0, T) \). In particular, we have that \( V \in L^\infty([0, T]; \mathcal{V}) \). Therefore, due to [10] by Lions and Magenes [22] Chap. 3, Lemma 8.1 we can conclude that \( V(t) \in C_w((0, T]; \mathcal{V}) \) and establish (55) for all \( t \in (0, T) \). \( \square \)

3.4. \( \mathcal{C}_0 \)-solutions. Now we consider problems (29) and (30) in the space \( \mathcal{C}_0(\mathcal{O}) \) given by (17). We assume that the boundary non-linearities \( h_i \) are traces of the corresponding non-linear terms \( f_i \), i.e., we assume that

\[
h_i(u_1, \ldots, u_m) = f_i(u_1, \ldots, u_m) \quad \text{for all} \quad u_1, \ldots, u_m \in \mathbb{R}, \quad i = 1, \ldots, m.
\]

We start with the following lemma.
Lemma 3.10. If (55) holds, then under the conditions (F1) and (F2) the Nemytski operator B given by (22) is a continuous mapping from $C_0(\bar{O})$ into itself which is bounded on bounded sets. Moreover, there exists a $\mu > 0$ such that the mapping $B_\mu(U) := B(U) + \mu U$ is accretive in $C_0(\bar{O})$, i.e.,

$$\|U - V\|_{C_0(\bar{O})} \leq \|U - V + \alpha [B_\mu(U) - B_\mu(V)]\|_{C_0(\bar{O})}$$

for any $U, V \in C_0(\bar{O})$ and for all $\alpha \geq 0$.

Proof. The argument is the same as in [14, Chap.7] and [8, Chap.6] because $B_\mu$ is accretive if and only if $-B_\mu$ is dissipative.

We also assume in this subsection that the random variable $\eta(\omega)$ takes values in $C_0(\bar{O})$ and for any $\omega \in \Omega$ we have that

$$\eta(\theta_t \omega) \in C(\mathbb{R}; C_0(\bar{O})).$$

By (15) in the case of the smooth elliptic pairs $\left(A_i; B_i\right)$ this property is satisfied if the correlation operator $K$ possesses the property (30) with $s \geq \max\{1, (d-1)/2\}$.

Theorem 3.11. Let (55) and (56) be valid. Assume also that the conditions (F1) and (F2) hold. Then for any $V_0 \in C_0(\bar{O})$ problem (29) has a unique mild solution in $C(\mathbb{R}^+; C_0(\bar{O}))$, i.e. (41) holds for all $t \geq 0$ and $\omega \in \Omega$ as an equality in $C_0(\bar{O})$.

Proof. We can apply exactly the same argument as in the proof of Theorem 7.13 in [14] which deals with another random version of an abstract stochastic equation of the form (28). We do not repeat this argument.

Thus in the same way as in the case of $L_{p,q}$-solutions we obtain the following conclusion.

**Corollary 3.12.** Under the conditions of Theorem 3.11 Problems (28) and (29) generate a continuous RDS $\phi_U$ and $\phi_V$ on $C_0(\bar{O})$ which are conjugate, i.e. relation (30) holds for any $U_0 \in C_0(\bar{O})$ with the same random homeomorphism $T$ as in Corollary 3.7.

Remark 3.13. Considering the mild solution $V(t)$ given by Theorem 3.11 as a generalized solution to linear problem

$$\frac{dV}{dt} + AV = f(t), \quad \text{with} \quad f(t) := -B(V(t) + \eta(\theta_t \omega)) \in C(\mathbb{R}^+; C_0(\bar{O})), \quad \text{for all} \ t > 0 \ 	ext{and} \ \omega \in \Omega$$

one can easily see from the standard methods for mild solutions that $V(t) \in \mathcal{V}$ for

$$\int_0^T \|V(\tau)\|^2_d \, d\tau \leq C_T(\omega) \quad \text{for every} \ T > 0.$$

Therefore $V$ is also a $L_{p}(O)$-solution. By the uniqueness theorem for $L_p(O)$-solutions we have that under the hypotheses of Theorem 3.11 the space $C_0(\bar{O})$ is an invariant set for the RDS $\phi_V$ and $\phi_U$ in $\mathcal{H}$. Moreover, in this case, since $C_0(\bar{O})$ is dense in $L_p(O)$, by (30) we can approximate any $L_p(O)$-solution by $C_0(\bar{O})$-solutions. We will use this observation later.
4. Monotonicity. In this section we show that under special conditions the RDS generated by \( \Phi \) (and by \( \Phi^* \)) is order preserving.

Let \( X \) be a real separable Banach space with a cone \( X^+ \subset X \). By definition, \( X^+ \) is a closed convex set in \( X \) such that \( \lambda v \in X^+ \) for all \( \lambda \geq 0 \), \( v \in X^+ \) and \( X^+ \cap (-X^+) = \{0\} \). The cone \( X^+ \) defines a partial order relation on \( X \) via \( x \leq y \) if \( y - x \in X^+ \) which is compatible with the vector space structure of \( X \). We write \( x < y \) when \( x \leq y \) and \( x \neq y \).

Below we deal with the spaces \( L^2(\Omega) \) and \( C_0(\Omega) \) and the cones \( L^+_2(\Omega) \) and \( C^+_0(\Omega) \) of nonnegative functions.

We recall the following notion (see [4], [10] and [12]).

**Definition 4.1.** An RDS \( \phi \) on some Banach space \( X \) with a cone \( X^+ \) is said to be order preserving (or monotone) if the relation \( U_2 - U_1 \in X^+ \), \( U_1, U_2 \in X \) implies that

\[
\phi(t, \omega, U_2) - \phi(t, \omega, U_1) \in X^+, \quad \omega \in \Omega, \quad t \geq 0.
\]

We note that monotone deterministic systems were studied by many authors (see, e.g., [27] and the references therein). We also refer to [10] for a general consideration of order preserving RDS.

Our main result in this section is the following cooperativity property for the functions \(-f \) and \(-h\).

(M) the mappings \( u \rightarrow -f(u) \) and \( u \rightarrow -h(u) \) are cooperative, i.e. for every \( i = 1, \ldots, m \) we have that

\[
-f^i(u^1, \ldots, u^m) \geq -f^i(v^1, \ldots, v^m)
\]

and

\[
-h^i(u^1, \ldots, u^m) \geq -h^i(v^1, \ldots, v^m)
\]

for any \( u = (u^1, \ldots, u^m) \) and \( v = (v^1, \ldots, v^m) \) from \( \mathbb{R}^m \) possessing the property

\[
u^j \geq v^j \quad \text{if} \quad j \neq i \quad \text{and} \quad u^i = v^i.
\]

Our main result in this section is the following conclusion.

**Theorem 4.2.** Let the assumptions (F1), (F2), (H1), (H2), (M) be in force. Assume that \( \Phi \) holds. We also suppose that

(i) either \( f \) and \( h \) are globally Lipschitz with Lipschitz constant \( L_f, L_h \);

(ii) or else \( \Phi \) and \( \Phi^* \) are valid and \( f_i^0 \in C^1(\mathbb{R}) \) for each \( i = 1, \ldots, m \).

Then the RDS \( \phi_{\Phi} \) and \( \phi_{\Phi^*} \) generated by \( \Phi \) and \( \Phi^* \) are order preserving in the space \( \mathcal{H} = L^2(\Omega) \) (with respect to the cone \( \mathcal{H}^+ \) of nonnegative functions).

**Proof.** Since the random homeomorphisms \( T, T^{-1} : \Omega \times \mathcal{H} \rightarrow \mathcal{H} \) given by \( \Phi \) preserve the order relation generated by the cone \( \mathcal{H}^+ \), by relation (56) we need to prove Theorem 4.2 for \( \phi_{\Phi} \) only.

**Step 1. Lipschitz case.** We first consider the case (i). Following Martin [28] Sect.8.6, p.367 ff we need to check the condition

\[
\lim_{\tau \to +0} d_{\mathcal{H}} \left( \frac{U - V - \tau [B(U + \eta(\omega)) - B(V + \eta(\omega))]}{\tau}; \mathcal{H}^+ \right) = 0 \quad (57)
\]

for any \( U, V \in \mathcal{H} \) such that \( U \geq V \), where \( \mathcal{H}^+ = \{ U \in \mathcal{H} : U \geq 0 \} \) and \( d_{\mathcal{H}}(z; S) \) is the distance between an element \( z \) and a set \( S \) with respect to the metric in \( \mathcal{H} \).

For \( U = (u; \xi) \) and \( V = (v; \zeta) \) from \( \mathcal{H} \) we have that

\[
Z_\tau \equiv \frac{U - V - \tau [B(U + \eta(\omega)) - B(V + \eta(\omega))]}{\tau} \equiv (z_\tau; \chi_\tau), \quad (58)
\]
where \( z_\tau = (z^{1}_{\tau}, \ldots, z^{m}_{\tau}) \) with
\[
z^{i}_{\tau}(x) = \frac{u^{i}(x) - v^{i}(x) - \tau [f^{i}(u(x) + \eta_{1}(\omega, x)) - f^{i}(v(x) + \eta_{1}(\omega, x))]}{\tau}, \quad x \in \mathcal{O},
\]
and \( \chi_{\tau} = (\chi^{1}_{\tau}, \ldots, \chi^{m}_{\tau}) \) with
\[
\chi^{i}_{\tau}(x) = \frac{\xi^{i}(x) - \zeta^{i}(x) - \tau [h^{i}(\xi(x) + \eta_{2}(\omega, x)) - h^{i}(\zeta(x) + \eta_{2}(\omega, x))]}{\tau}, \quad x \in \Gamma_{1}.
\]
Below we denote for shortness \( f^{i}(t, u) \equiv f^{i}(u(x) + \eta_{1}(\omega, x)) \) and suppress the dependence on \( x \). Using the obvious relation
\[
z^{i}_{\tau} = \tau^{-1} \left( u^{i} - v^{i} - \tau [f^{i}(t, u) - f^{i}(t, v)] \right) = \tau^{-1} \left( u^{i} - v^{i} - \tau [f^{i}(t, u^{1}, u^{2}, \ldots, u^{m}) - f^{i}(t, v^{1}, u^{2}, \ldots, u^{m})] \right) - \left( f^{i}(t, u^{1}, \ldots, v^{i}, \ldots, u^{m}) - f^{i}(t, v^{1}, \ldots, v^{i}, \ldots, v^{m}) \right),
\]
from cooperativity condition (M) we have that
\[
z^{i}_{\tau} \geq \tau^{-1} \left( u^{i} - v^{i} - \tau [f^{i}(t, u^{1}, \ldots, u^{i}, \ldots, u^{m}) - f^{i}(t, u^{1}, \ldots, v^{i}, \ldots, u^{m})] \right).
\]
Therefore from the Lipschitz condition (i) we obtain
\[
z^{i}_{\tau} \geq \tau^{-1} \left[ u^{i} - v^{i} - \tau \cdot L_{f} \cdot (u^{i} - v^{i}) \right] \geq \tau^{-1} (u^{i} - v^{i}) \left[ 1 - \tau \cdot L_{f} \right],
\]
where \( L_{f} \) is the corresponding Lipschitz constant for \( f \). Thus \( z^{i}_{\tau}(x) \geq 0 \) for \( \tau \) small enough. Similarly, we obtain that \( \chi^{i}_{\tau} \geq 0 \) for \( \tau \) small enough and for almost all \( x \in \Gamma_{1} \). Consequently, we have that \( Z_{\tau} \in \mathcal{H}^{+} \) for \( \tau \) small enough. This implies (57).

**Step 2. The Case \( f \equiv h \).** In the case (ii) we apply the following assertion which is also of independent interest.

**Proposition 4.3.** Let (55) and (56) be valid. Assume also that the conditions (F1) and (F2) hold and \( f_{i}^{0} \in C^{1}(\mathbb{R}) \) for each \( i = 1, \ldots, m \). Then the RDS \( \phi_{\tau} \) and \( \phi_{U} \) generated by (29) and (28) are order preserving in \( C_{0}(\overline{\mathcal{O}}) \) (with respect to the cone \( C_{0}^{+}(\mathcal{O}) \) of non-negative continuous functions).

**Proof.** We assume \( U \geq V \). In this case we need to check relation (57) in the space \( C_{0}(\overline{\mathcal{O}}) \) (instead of \( \mathcal{H} \)). With the same notation as in the previous step, from (59) we have that
\[
z^{i}_{\tau}(x) \geq \tau^{-1} \left( [1 - \tau \cdot L](u^{i}(x) - v^{i}(x)) - \tau [f_{0}^{i}(t, u^{i}(x)) - f_{0}^{i}(t, v^{i}(x))] \right),
\]
where \( L \) is a Lipschitz constant for \( f_{1} \). Since
\[
f_{0}^{i}(t, u^{i}(x)) - f_{0}^{i}(t, v^{i}(x)) = (u^{i}(x) - v^{i}(x)) \int_{0}^{1} \partial_{\lambda}[f_{0}^{i}](t, \lambda u^{i}(x) + (1 - \lambda)v^{i}(x)) d\lambda,
\]
we have that
\[
|f_{0}^{i}(t, u^{i}(x)) - f_{0}^{i}(t, v^{i}(x))| \leq c(u^{i}(x) - v^{i}(x)),
\]
where the constant \( c \) depends on \( t, u \) and \( v \), but does not depend on \( x \in \mathcal{O} \). Therefore (60) implies that
\[
z^{i}_{\tau}(x) \geq \tau^{-1} [1 - \tau \cdot L - \tau c] (u^{i}(x) - v^{i}(x)) \geq 0, \quad x \in \mathcal{O},
\]
for \( \tau \) small enough. The same argument gives that \( \chi^{i}_{\tau}(x) \geq 0 \) for \( \tau \) small enough and \( x \in \Gamma_{1} \).

To conclude the proof of Theorem 3.2 we use Proposition 4.3 and the fact that \( C_{0}(\overline{\mathcal{O}}) \) is dense in \( \mathcal{H} \).
Under the hypotheses of Theorem 4.2 we are also in a position to establish a comparison principle for RDS generated by problems (29) and (28) with different nonlinearities $f$ and $h$. More precisely, the following assertion holds.

**Proposition 4.4.** Assume that \{${\tilde f}^i, {\tilde h}^i$\} and \{${\tilde f}^i, {\tilde h}^i$\} are two collections of (nonlinear) functions satisfying the hypotheses of Theorem 4.2. Let $\phi_U$ (resp. $\tilde\phi_U$) be the RDS generated by (28) with $B$ given by (22) (resp. with $B = \tilde B$, where $\tilde B$ is defined by (22) with \{${\tilde f}^i, {\tilde h}^i$\} instead of \{${f}^i, {h}^i$\}). Assume that for every $i = 1, \ldots, m$ we have that

\[
{f}^i(u^1, \ldots, u^m) \leq {\tilde f}^i(u^1, \ldots, u^m) \quad \text{and} \quad {h}^i(u^1, \ldots, u^m) \leq {\tilde h}^i(u^1, \ldots, u^m)
\]

for any $u = (u^1, \ldots, u^m) \in \mathbb{R}^m$. Then we have

\[
\phi_U(t, \omega, U_0) \geq \tilde\phi_U(t, \omega, U_0) \quad \text{for all} \quad U_0 \in \mathcal{H}, \quad \omega \in \Omega, \quad t \geq 0.
\]

**Proof.** We use the same method as in the proof of Theorem 4.2 and, instead of $\phi_U$ and $\tilde\phi_U$, consider the cocycles $\phi_V$ and $\tilde\phi_V$ generated by the corresponding (random) problems of the form (29).

According to Sect.8.6, instead of (57), we need to check the condition

\[
\lim_{\tau \to +0} d_{\mathcal{H}} \left( \tilde Z_\tau(U, V); \mathcal{H}^+ \right) = 0
\]

for any $U, V \in \mathcal{H}$ such that $U \geq V$, where $\mathcal{H}^+ = \{U \in \mathcal{H} : U \geq 0\}$ and

\[
\tilde Z_\tau(U, V) \equiv \frac{U - V - \tau \left[ B(U + \eta(\omega)) - \tilde B(V + \eta(\omega)) \right]}{\tau}.
\]

As above $d_{\mathcal{H}}(z; S)$ is the distance between an element $z$ and the set $S$ in $\mathcal{H}$. Since

\[
B(U + \eta) - \tilde B(V + \eta) = \left[ B(U + \eta) - B(V + \eta) \right] + \left[ B(V + \eta) - \tilde B(V + \eta) \right],
\]

it follows from (61) that

\[
B(U + \eta) - \tilde B(V + \eta) \geq B(U + \eta) - B(V + \eta)
\]

for any $U, V \in \mathcal{H}$. Therefore $\tilde Z_\tau(U, V) \leq Z_\tau$, where $Z_\tau$ is given by (58). Consequently, after a simple calculation we can see that

\[
d_{\mathcal{H}} \left( \tilde Z_\tau(U, V); \mathcal{H}^+ \right) \leq d_{\mathcal{H}} \left( Z_\tau; \mathcal{H}^+ \right).
\]

Thus relation (62) follows from (57). This completes the proof of Proposition 4.4.

We note that comparison principles are important tools in the study of long-time dynamics of monotone systems (see discussions in [27] and [10]).

5. **Random attractors.** In this section we are going to show that (1) has a random attractor. We start with the dissipativity of the system. We add the following hypotheses:

(F3) Either $p > 2$ or there exists a constant $c > 0$ such that

\[
\sum_{i=1}^m f_i^j(u^1, \ldots, u^m)u^i \geq -c, \quad u^j \in \mathbb{R},
\]

where $f_i^j$ is a nonlinear function.
(H3) Either $q > 2$ or there exists a constant $c > 0$ such that
\[
\sum_{i=1}^{m} h_i^1(u^1, \ldots, u^m) u^i \geq -c, \quad u^j \in \mathbb{R}.
\]  
(64)

We emphasize that relation (63) is only necessary in the case $p = 2$. The same is true concerning (64). Moreover, we need conditions (F3) and (H3) to prove the dissipativity of the RDS $\phi_V$ only in the following assertion.

**Proposition 5.1.** Assume (F1)–(F3) and (H1)–(H3). Then under condition (31) there exists a random set $C \in \mathcal{D}$ such that (10) holds (with $B(\omega) = C(\omega)$), i.e.
\[
\phi_V(t, \theta^{-t} \omega, D(\theta^{-t} \omega)) \subset C(\omega) \quad \text{for } t \geq t_0(\omega, D)
\]  
(65)

**Proof.** Under Hypotheses (F3) and (H3) instead of (26) we have the relation
\[
(B(U), U) \geq \alpha_1 \sum_{i=1}^{m} \left( \|u^i(t)\|_p^p + \|\xi^i(t)\|_{L_q(\mathcal{G}_1)}^q \right) - c, \quad U = (u, \xi) \in L_{p,q}(\mathcal{O}).
\]  
Therefore the same argument as in the proof of (36) gives us that
\[
\frac{d}{dt} ||V(t)||_2^2 + 2 ||V(t)||_2^2 + \alpha_1 \sum_{i=1}^{m} (\|u^i(t)\|_p^p + \|\xi^i(t)\|_{L_q(\mathcal{G}_1)}^q) \leq c_1 \Xi(\theta t \omega),
\]  
(66)

where $\alpha_1, c_1 > 0$ and $\Xi(\omega)$ is given by (35). Since $\|V\|_{L_2}^2 \geq \lambda_1 \|V\|_{\mathcal{H}}^2$ ($\lambda_1 > 0$ is the first eigenvalue of $A$) this implies that
\[
\|V(t)\|_2^2 \leq e^{-2\lambda_1 t} \|V(0)\|_2^2 + c_1 \int_0^t e^{-2\lambda_1 (t-\tau)} \Xi(\theta t \omega) d\tau.
\]  

Now we can apply the idea presented in [6]. First we note that the one dimensional random differential equation
\[
\psi' + 2\lambda_1 \psi = c_1 \Xi(\theta t \omega)
\]  
(67)

has a unique stationary solution $t \rightarrow \psi_*(\theta t \omega)$ defined by the (tempered) random variable
\[
\psi_*(\omega) = c_1 \int_{-\infty}^{0} e^{2\lambda_1 \tau} \Xi(\theta t \omega) d\tau.
\]

If $\psi(t, \omega, x)$ is the random dynamical system generated by the above differential equation (67) with initial condition $x \in \mathbb{R}$ then by a direct calculation (see [6] for details) we have that
\[
\lim_{t \to \infty} |\psi(t, \theta^{-t} \omega, x) - \psi_*(\omega)| = 0
\]
\[
\psi(t, \omega, \psi_*(\omega)) = \psi_*(\theta t \omega), \quad t \geq 0.
\]  
(68)

Comparing (66) and (67) we have
\[
\|V(t)\|_2^2 \leq \psi(t, \omega, \|V(0)\|_2^2) \quad \text{for } t \geq 0
\]  
which shows that the ball in $\mathcal{H}$
\[
C(\omega) = B_\mathcal{H}(0, \sqrt{2\psi_*(\omega)}) \in \mathcal{D}
\]  
is pullback $\mathcal{D}$-absorbing. The invariance property in (65) follows from the second line in (68).

With respect to the dynamics of (29) we have the following assertion.

**Theorem 5.2.** Assume (F1)–(F3), (H1)–(H3) with $p = q \geq 2$. Let the coefficients of the elliptic pair $(A^i, B^i)$ be $C^\infty$-smooth and suppose that (45) holds. In addition, assume that the Wiener process $W$ has the covariance $K$ such that

$$\text{tr} HKA^2 s^{-1+\varepsilon} < \infty,$$

for some $\varepsilon > 0$, $s \geq s_p$, where $s_p \geq 1$ is the parameter from Lemma 3.3. Let $D$ be the collection of all tempered random sets in $H$ (see Subsect. 2.1). Then the random dynamical system $\phi_V$ generated by (29) has a random pullback attractor $A$ with respect to $D$.

**Proof.** We have to check the assumptions of Theorem 2.6. So it is sufficient to show the existence of a compact absorbing set $B \in D$.

Under the conditions imposed we can apply Proposition 3.9 which with $t = 1$ gives us that

$$\|V(1)\|_V^2 \leq c_1 \|V(0)\|^2 + c_2 \int_0^1 \Xi_\omega(\theta\tau) d\tau$$

for a solution $V(t)$, where $\Xi_\omega$ is given by (53). Define

$$B(\omega) \equiv \phi_V(1, \theta_{-1}\omega, C_{\theta_{-1}\omega}) \subset C(\omega).$$

(70)

By (69) $B(\omega)$ is compact. Indeed, $V$ is compactly embedded in $H$, such that $B(\omega)$ is bounded in $V$. Since $C$ is pullback $D$-absorbing, so is $B$. The inclusion (70) shows that $B \in D$. Hence we can apply Theorem 2.6. 

We are now in a position to prove the existence of a random attractor for RDS generated by (28), or equivalently by (29).

**Corollary 5.3.** Under the conditions of Theorem 5.2 the random dynamical system $\phi_U$ in $H$ generated by (28) possesses a random attractor in the collection $D$ of all tempered sets from $H$.

**Proof.** By Corollary 3.7 the RDS $\phi_U$ and $\phi_V$ are conjugate:

$$\phi_U(t, \omega, x) \equiv T(\theta t \omega, \phi_V(t, \omega, T^{-1}(\theta t \omega, x))),$$

where $T, T^{-1} : \Omega \times H \to H$ are random homeomorphisms given by (44). It is clear that the both $T$ and $T^{-1}$ map tempered sets into tempered sets. Therefore we can apply Lemma 2.6. 

6. **Example.** In a smooth bounded domain $O \subset \mathbb{R}^d$ we consider the following problem for a single unknown function $u(t, x)$ with dynamics on the boundary $\Gamma = \partial O$:

$$
\begin{cases}
du = \left(\Delta u - u^{2r+1} + \lambda u\right) dt + dW^{(0)} & \text{on } O \times \mathbb{R}_+, \\
du = -\left(\frac{\partial u}{\partial \nu} + cu + u^{2r+1} - \lambda u\right) dt + dW^{(1)} & \text{on } \Gamma \times \mathbb{R}_+, \\
u(0, x) = u_0(x) & \text{on } O, \quad u(0, x) = \xi_0(x) & \text{on } \Gamma.
\end{cases}
$$

(71)

Here $r > 0$ is an integer and $\lambda$ and $c > 0$ are real numbers.
Comparing this problem with the equations formulated in \cite{ChueshovSchmalfuSS16} we have that $m = 1$, $\Gamma_1 = \Gamma$, $\Gamma_2 = \Gamma_3 = \emptyset$, and
\begin{align*}
\mathcal{A}(x, \partial)u &= -\Delta u, \quad \mathcal{B}(x, \partial)u = -\frac{\partial}{\partial y} u + cu, \\
B(U) &= B(u, \xi) = (u^{2r+1} - \lambda u, \xi^{2r+1} - \lambda \xi), \quad U = (u, \xi).
\end{align*}

It is clear that the hypotheses in (F1)-(F3) and (H1)-(H3) are satisfied with $p = q = 2r + 2 > 2$. We choose $s \geq s_{2r+2}$ in the assumption given by \cite{ChueshovSchmalfuSS20}. Then we can apply Corollary \ref{cor:existence} to see that (71) generates a random dynamical system $\phi_U$ on $\mathcal{H}$. By Theorem \ref{thm:existence}, this system is order preserving under the condition $s \geq \max\{1, (d-1)/2\}$.

Applying Theorem \ref{thm:existence} and Corollary \ref{cor:existence} we obtain the following assertion.

**Theorem 6.1.** The random dynamical system $\phi_U$ generated by (71) has a random pullback attractor $\mathcal{A}$ attracting every set from the collection $\mathcal{D}$ of all tempered random sets. In addition, if we assume that
\begin{equation*}
\lambda < \lambda_1 := \inf \left\{ \int_{\mathcal{O}} |\nabla u|^2 dx + c \int_{\Gamma} |u|^2 d\Gamma : u \in H^1(\mathcal{O}), \|u\|^2_{L_2(\mathcal{O})} + \|u\|^2_{L_2(\Gamma)} = 1 \right\},
\end{equation*}
then the random attractor $\mathcal{A}$ is a singleton, i.e., there exists a tempered random variable $U(\omega)$ in $\mathcal{H} = L_2(\mathcal{O})$ such that $\mathcal{A}(\omega) = \{U(\omega)\}$ for each $\omega \in \Omega$.

**Proof.** The existence of the attractor follows from Corollary \ref{cor:existence}.

For the second conclusion we first consider the random system (29) which corresponds the model given in (71). In the case considered we obviously have that
\[-(B(V_1 + \eta) - B(V_2 + \eta), V_1 - V_2) \leq \lambda \|V_1 - V_2\|^2.
\]

Thus the same argument as in the proof of relation (33) yields that
\begin{equation}
\|V_1(t) - V_2(t)\|^2 \leq \|V_{0,1} - V_{0,2}\|^2 e^{2(-\lambda_1 + \lambda)t} \tag{72}
\end{equation}
for any two solutions to (29) with the initial conditions where $V_{0,1}, V_{0,2} \in \mathcal{H}$. For the attractor $\mathcal{A}_V$ of the RDS $\phi_V$ generated by (29) we have
\begin{equation*}
\mathcal{A}_V(\omega) = \phi_V(t, \theta_{-t}\omega, \mathcal{A}_V(\theta_{-t}\omega))
\end{equation*}
for every $t \geq 0$ and $\mathcal{A}_V(\theta_{-t}\omega) \subset C(\theta_{-t}\omega) = B_{\mathcal{H}}(0, \sqrt{2}\psi_s(\theta_{-t}\omega))$. Hence, by (72) we have that
\[\max_{x, y \in \mathcal{A}_V(\omega)} \|x - y\|^2 \leq e^{-2(\lambda_1 - \lambda)t} \psi_s(\theta_{-t}\omega),\]
where the right hand side tends to zero for $t \to \infty$ by the temperedness of $\psi_s$. Thus $\mathcal{A}_V$ is singleton. By Lemma \ref{lem:existence}, the same is true for $\mathcal{A}$.

**Remark 6.2.** By the invariance property of the attractor $\mathcal{A}$ the random variable $U(\omega)$ in the statement of Theorem 6.1 is an equilibrium, i.e. it possesses the property $\phi(t, \theta_{-t}\omega, U(\omega)) = U(\theta_t\omega)$ for all $t \geq 0$ and $\omega \in \Omega$. Thus the second part of Theorem 6.1 states the existence of a globally asymptotically stable equilibrium $U$ under the condition $\lambda < \lambda_1$. Due to the uniqueness of the random attractor $\mathcal{A}$, this equilibrium is unique in the class of all tempered random variables. The situation when $\lambda \geq \lambda_1$ is more complicated. In this case using the comparison principle stated in Proposition \ref{prop:comparison} and the methods presented in \cite{ChueshovSchmalfuSS17} Sect.3.5) one can also prove the existence of an equilibrium for the RDS generated by (71). However the uniqueness of this equilibrium is still an open question.
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