ALMOST PARACONTACT MANIFOLDS

GALIA NAKOVA AND SIMEON ZAMKOVOY

Abstract. In this paper eleven basic classes of almost paracontact manifolds are introduced and some examples are constructed.
MSC: 53C15, 5350, 53C25, 53C26, 53B30

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Introduction

As is well-known, in [GH] almost Hermitian manifolds are classified with respect to the decomposition in subspaces invariant under the action of the structural group $U(n)$. Thus we have an adequate framework for several types of almost Hermitian manifolds, previously defined by a number of authors in terms of geometric properties which retain some portion of Kähler geometry. The previous method was used in [N] for Riemannian almost product manifolds, and in [GB] for almost complex manifolds with Norden metric.

The geometry of almost contact manifolds is a natural extension in the odd dimensional case of almost Hermitian geometry. Similarly, the geometry of almost contact manifolds with $B$-metric can be considered as a natural extension in the odd dimensional case of geometry of almost complex Riemannian. A classification of almost contact manifolds with $B$-metric with respect to the covariant derivative of the fundamental tensor of type $(1,1)$ is made in [GMG]. The authors obtain eleven basic classes of almost contact manifolds with $B$-metric and construct some examples.

A classification of almost paraHermitian manifolds is made in [B].

The authors give examples of the primitive classes, which are based on general almost paraHermitian structure on the tangent bundles given in [C].
A classification of the almost paracontact Riemannian manifolds of type \((n, n)\) with respect to the covariant derivative of type \((1, 1)\)-tensor of the almost paracontact structure is made in [MS]. The authors consider almost paracontact Riemannian manifolds of type \((n, n)\) with positive definite Riemannian metric \(g\), which is compatible with almost paracontact structure and it satisfies the condition \(g(\varphi \cdot, \varphi \cdot) = g(\cdot, \cdot) - \eta(\cdot)\eta(\cdot)\).

The method used in the present paper is analogous of the one used in [GMG]. We give a classification of the almost paracontact manifolds with respect to the covariant derivative of the \((1, 1)\)-tensor of the almost paracontact structure. We consider almost paracontact pseudo-Riemannian manifolds with indefinite metric \(g\), which it compatible with almost paracontact structure and it satisfies the condition (1.2). We obtain eleven basic classes and construct some examples.

1. Preliminaries

A \((2n+1)\)-dimensional smooth manifold \(M^{(2n+1)}\) has an almost paracontact structure \((\varphi, \xi, \eta)\) if it admits a tensor field \(\varphi\) of type \((1, 1)\), a vector field \(\xi\) and a 1-form \(\eta\) satisfying the following compatibility conditions [KW, Zam]:

\[
\begin{align*}
(i) \quad & \varphi(\xi) = 0, \quad \eta \circ \varphi = 0, \\
(ii) \quad & \eta(\xi) = 1, \quad \varphi^2 = \text{id} - \eta \otimes \xi, \\
(iii) \quad & \text{let } D = \text{Ker } \eta \text{ be the horizontal distribution generated by } \eta, \text{ then the tensor field } \varphi \text{ induces an almost paracomplex structure on each fibre on } D.
\end{align*}
\]

Recall that an almost paracomplex structure on a \(2n\)-dimensional manifold is a \((1,1)\)-tensor \(J\) such that \(J^2 = 1\) and the eigensubbundles \(T^+, T^-\) corresponding to the eigenvalues \(1, -1\) of \(J\), respectively have equal dimension \(n\). The Nijenhuis tensor \(N\) of \(J\), given by \(N_J(X,Y) = [JX, JY] - J[MX, Y] - J[X, JY] + [X, Y]\), is the obstruction for the integrability of the eigensubbundles \(T^+, T^-\). If \(N = 0\) then the almost paracomplex structure is called paracomplex or integrable.

An immediate consequence of the definition of the almost paracontact structure is that the endomorphism \(\varphi\) has rank \(2n\), \(\varphi \xi = 0\) and \(\eta \circ \varphi = 0\) [Zam].

If a manifold \(M^{(2n+1)}\) with \((\varphi, \xi, \eta)\)-structure admits a pseudo-Riemannian metric \(g\) such that

\[
g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y),
\]

then we say that \(M^{(2n+1)}\) has an almost paracontact metric structure and \(g\) is called compatible metric. Any compatible metric \(g\) with a given almost paracontact structure is necessarily of signature \((n + 1, n)\) [Zam].

Setting \(Y = \xi\), we have \(\eta(X) = g(X, \xi)\).

The fundamental 2-form

\[
(1.3) \quad F(X, Y) = g(\varphi X, Y)
\]

is non-degenerate on the horizontal distribution \(D\) and \(\eta \wedge F^n \neq 0\).

We have the following [Zam].
Definition 1.1. If \( g(X, \varphi Y) = d\eta(X, Y) \) (where \( d\eta(X, Y) = \frac{1}{2}(X\eta(Y) - Y\eta(X) - \eta([X, Y])) \)) then \( \eta \) is a paracontact form and the almost paracontact metric manifold \((M, \varphi, \eta, g)\) is said to be paracontact metric manifold.

For a manifold \( M^{2n+1} \) with an almost paracontact metric structure \((\varphi, \xi, \eta, g)\) we can also construct a useful local orthonormal basis. Let \( U \) be a coordinate neighborhood on \( M \) and \( X_1 \) any unit vector field on \( U \) orthogonal to \( \xi \). Then \( \varphi X_1 \) is a vector field orthogonal to both \( X \) and \( \xi \), and \(|\varphi X_1|^2 = -1\). Now choose a unit vector field \( X_2 \) orthogonal to \( \xi, X_1 \) and \( \varphi X_1 \). Then \( \varphi X_2 \) is also vector field orthogonal to \( \xi, X_1, \varphi X_1 \) and \( X_2 \), and \(|\varphi X_2|^2 = -1\). Proceeding in this way we obtain a local orthonormal basis \((X_i, \varphi X_i, \xi), i = 1 \ldots n\) called a \( \varphi \)-basis.

Hence, an almost paracontact metric manifold \((M^{2n+1}, \varphi, \eta, \xi, g)\) is an odd dimensional manifold with a structure group \( U(n, \mathbb{R}) \times Id \), where \( U(n, \mathbb{R}) \) is the para-\( \mathbb{R} \)-unitary group isomorphic to \( \mathbb{GL}(n, \mathbb{R}) \).

A paracontact structure for which \( \xi \) is Killing vector field is called a \( K \)-paracontact structure.

Let \( \nabla \) be the Levi-Civita connection of the compatible metric \( g \). For all vectors \( X, Y, Z \in T_p M, p \in M \), we denote
\[
(1.4) \quad F(X, Y, Z) = g((\nabla_X \varphi) Y, Z).
\]
From (1.1) and (1.2) the tensor \( F \) has the following properties:
\[
(1.5) \quad F(X, Y, Z) = -F(X, Z, Y), \quad F(X, \varphi Y, \varphi Z) = F(X, Y, Z) + \eta(Y)F(X, Z, \xi) - \eta(Z)F(X, Y, \xi),
\]
for all vectors \( X, Y, Z \in T_p M \).

The following 1-forms are associated with \( F \):
\[
(1.6) \quad \theta(X) = g^{ij} F(e_i, e_j, X), \quad \theta^*(X) = g^{ij} F(e_i, \varphi e_j, X), \quad \omega(X) = F(\xi, \xi, X),
\]
where \( X \in T_p M, \{e_i, \xi\}, (i = 1, \ldots, 2n) \) is a basis of \( T_p M \), and \((g^{ij})\) is the inverse matrix of \((g_{ij})\).

2. The space of covariant derivatives of the structure \( \varphi \)

Let \( V \) be a \((2n+1)\)-dimensional vector space with almost paracontact structure \((\varphi, \xi, \eta)\) and metric \( g \) satisfying (1.2). For an arbitrary \( X \in V \) we have \( \varphi^2 X = X - \eta(X)\xi \iff X = \varphi^2 X + \eta(X)\xi \). Hence \( V \) admits a decomposition into a direct sum of vector subspaces
\[
V = \mathbb{D} \oplus \{\xi\},
\]
where \( \mathbb{D} = \text{Kern}\eta, \{\xi\} = (\text{Im}\eta)\xi \). Then for an arbitrary \( X \in V \) it follows \( X = hX + \eta(X)\xi \), where \( X \in \mathbb{D}, \eta(X)\xi \in \{\xi\} \). Denoting the restrictions of \( g \) and \( \varphi \) on \( \mathbb{D} \) with the same letters we obtain an \( 2n \)-dimension almost paracomplex manifold \((\mathbb{D}, \varphi, g)\).

Let \( \{e_1, \ldots, e_{2n}\} \) be any basis of \( \mathbb{D} \). Then \( \{e_1, \ldots, e_{2n}, \xi\} \) is a basis of \( V \) and for an arbitrary \( X \in V \) we have \( X = X^i e_i + \eta(X)\xi, i = 1, \ldots, 2n \).
We define the linear operators
\[ A_{e_i} : V \rightarrow V : \quad X \mapsto A_{e_i}X, \quad i = 1, \ldots, 2n; \]
\[ A_{\xi} : V \rightarrow \mathbb{D} : \quad X \mapsto A_{\xi}X, \]
having the following properties
\[ g(A_{e_i}X, e_j) = -g(A_{e_j}X, e_i), \quad i, j = 1, \ldots, 2n; \]
\[ A_{\varphi X} = -\varphi(A_{e_i}X) - g(A_{e_i}X, e_i)\xi; \]
\[ \eta(A_{e_i}X) = -g(A_{\xi}X, \varphi e_i); \]
\[ \eta(A_{\xi}X) = 0. \]

We consider the vector space \( \mathcal{F} \) of all tensors \( F \) of type (0,3) over \( V \), defined by
\[ F(X, Y, Z) = Y^i g(A_{e_i}X, Z) + \eta(Y)g(A_{\xi}X, \varphi Z), \]
where \( A_{e_i} \) \( (i = 1, \ldots, 2n) \) and \( A_{\xi} \) have the properties (2.2) - (2.5). It is easy to verify that the tensors \( F \) do not depend on the basis of \( V \). Using (2.6) and (2.2) - (2.5) we establish that the tensors \( F \in \mathcal{F} \) have the properties (1.5).

The compatible metric \( g \) induces on \( \mathcal{F} \) an inner product \( <,> \), defined by
\[ <F_1, F_2> = g^{i\alpha}g^{jr}g^{ks}F_1(f_i, f_j, f_k)F_2(f_\alpha, f_r, f_s) \]
for \( F_1, F_2 \in \mathcal{F} \) and \( \{f_1, \ldots, f_{2n+1}\} \) is a basis of \( V \).

The standard representation of the structure group \( U(n, \mathbb{R}) \times \operatorname{Id} \) in \( V \) induces a natural representation \( \lambda \) of \( U(n, \mathbb{R}) \times \operatorname{Id} \) in \( \mathcal{F} \):
\[ (\lambda(a)F)(X, Y, Z) = F(a^{-1}X, a^{-1}Y, a^{-1}Z), \quad a \in U(n, \mathbb{R}) \times \operatorname{Id}, \quad F \in \mathcal{F}, \quad X, Y, Z \in V, \]
so that
\[ <\lambda(a)F_1, \lambda(a)F_2> = <F_1, F_2>, \quad a \in U(n, \mathbb{R}) \times \operatorname{Id}, \quad F_1, F_2 \in \mathcal{F}. \]

Let \( X \in V \). Then for \( A_{e_i}X \in V, \ (i = 1, \ldots, 2n) \) we have
\[ A_{e_i}X = h(A_{e_i}hX) + \eta(X)h(A_{e_i}\xi) + \eta(A_{e_i}hX)\xi + \eta(X)\eta(A_{e_i}\xi)\xi. \]
Taking into account (2.6) and (2.8) we obtain
\[ F(X, Y, Z) = Y^i g(h(A_{e_i}hX), Z) - Z^i \eta(Y)\eta(A_{e_i}hX) + Y^i \eta(Z)\eta(A_{e_i}hX) \]
\[ + \eta(X)Y^i g(h(A_{e_i}\xi), Z) + \eta(X)\eta(Z)Y^i \eta(A_{e_i}\xi) - \eta(X)\eta(Y)Z^i \eta(A_{e_i}\xi). \]
Analogously as \textbf{GMG} we define the operators

\[ p_i : \mathcal{F} \rightarrow \mathcal{F}; \quad i = 1, 2, 3, 4, \]

\[ p_1(F)(X, Y, Z) = F(hX, hY, hZ); \]

\[ p_2(F)(X, Y, Z) = -\eta(Y)F(hX, hZ, \xi) + \eta(Z)F(hX, hY, \xi); \]

\[ p_3(F)(X, Y, Z) = \eta(X)F(\xi, hY, hZ); \]

\[ p_4(F)(X, Y, Z) = \eta(X)\eta(Y)F(\xi, \xi, hZ) - \eta(X)\eta(Z)F(\xi, \xi, hY). \]

**Lemma 2.1.** The operators \( p_i \) \((i = 1, 2, 3, 4)\) have the following properties

\( (i) \quad p_i \circ p_i = p_i, \quad i = 1, 2, 3, 4; \)

\( (ii) \quad \sum_{i=1}^{4} p_i = \text{id}; \)

\( (iii) \quad p_i \circ p_j = 0, \quad i \neq j, \quad i, j = 1, 2, 3, 4; \)

and \( p_i \) \((i = 1, 2, 3, 4)\) commute with \( \mathbb{U}(n, \mathbb{R}) \times \text{Id}. \)

We denote \( W_i = \text{Im}p_i \quad (i = 1, 2, 3, 4). \)

**Proposition 2.1.** (Partial decomposition) The decomposition

\[ \mathcal{F} = W_1 \oplus W_2 \oplus W_3 \oplus W_4 \]

is orthogonal and invariant under the action of the group \( \mathbb{U}(n, \mathbb{R}) \times \text{Id}. \)

**Proof.** From well known algebraic result and Lemma 2.1 we obtain the decomposition \( \mathcal{F} = W_1 \oplus W_2 \oplus W_3 \oplus W_4 \) that is \( \mathbb{U}(n, \mathbb{R}) \times \text{Id} \) - invariant. By direct computations using (2.7) we check that \( W_i \perp W_j, \quad i \neq j, \quad i, j = 1, 2, 3, 4. \)

From (2.9), (2.4) by explicit calculations we have

\[ p_1(F)(X, Y, Z) = F(hX, hY, hZ) = Y^i g(h(A_{e_i} hX), Z); \]

\[ p_2(F)(X, Y, Z) = -\eta(Y)F(hX, hZ, \xi) + \eta(Z)F(hX, hY, \xi) = -Z^i \eta(Y)\eta(A_{e_i} hX) + Y^i \eta(Z)\eta(A_{e_i} hX) = \]

\[ \eta(Y)g(A_{e_i} hX, \varphi Z) - \eta(Z)g(A_{e_i} hX, \varphi Y); \]

\[ (2.10) \]

\[ p_3(F)(X, Y, Z) = \eta(X)F(\xi, hY, hZ) = \eta(X)Y^i g(h(A_{e_i} \xi), Z); \]

\[ p_4(F)(X, Y, Z) = \eta(X)\eta(Y)F(\xi, \xi, hZ) - \eta(X)\eta(Z)F(\xi, \xi, hY) = \]

\[ -\eta(X)\eta(Y)Z^i \eta(A_{e_i} \xi) + \eta(X)\eta(Z)Y^i \eta(A_{e_i} \xi), \]

where \( X, Y, Z \in V \) and \( (i = 1, \ldots, 2n). \) Using (2.8), (2.9), (2.10) we obtain
**Proposition 2.2.** Let $A_{e_i} (i, \ldots, 2n)$ be the linear operators, defined by (2.1) and having the properties (2.2), (2.3), (2.4), (2.5). Then for an arbitrary $F \in \mathcal{F}$ and $X, Y, Z \in V$ we have

(i) $A_{e_i}X = h(A_{e_i}hX), (i = 1, \ldots, 2n) \iff F = p_1F$;

(ii) $A_{e_i}X = \eta(A_{e_i}hX)\xi = -g(A_{e_i}hX, \varphi e_i)\xi, (i = 1, \ldots, 2n) \iff F = p_2F$;

(iii) $A_{e_i}X = \eta(X)h(A_{e_i}\xi), (i = 1, \ldots, 2n) \iff F = p_3F$;

(iii) $A_{e_i}X = \eta(X)\eta(A_{e_i}\xi)\xi = \eta(X)g(A_{e_i}\xi, \varphi e_i)\xi, (i = 1, \ldots, 2n) \iff F = p_4F$.

3. The subspace $W_1$

From Proposition 2.2 we have

$$W_1 = \{ F \in \mathcal{F} : F = p_1F \iff A_{e_i}X = h(A_{e_i}hX) (i = 1, \ldots, 2n) \}.$$  

The condition $A_{e_i}X = h(A_{e_i}hX) (i = 1, \ldots, 2n)$ is equivalent to

(3.1) $A_{e_i}\xi = 0; \eta(A_{e_i}X) = 0 (i = 1, \ldots, 2n)$.

From equalities (2.4) and (3.1) we obtain

(3.2) $A_{e_i}X = 0$.

Then the decomposition of $W_1$ over $V$ coincides with the decomposition of $W$ over $D$, where the vector space $W$ is defined by

$$W = \{ F \in \mathcal{F} : F(X, Y, Z) = -F(X, Z, Y) = F(X, \varphi Y, \varphi Z), X, Y, Z \in D \}.$$  

Taking into account (2.2) and (2.5), (2.9), (3.1), (3.2) we have

(3.3) $W = \{ F \in \mathcal{F} : F(X, Y, Z) = Y^i g(A_{e_i}X, Z) : g(A_{e_i}X, e_j) = -g(A_{e_j}X, e_i),

A_{\varphi e_i}X = -\varphi(A_{e_i}X), (i = 1, \ldots, 2n); X, Y, Z \in D \}.$

Using (3.3) and (1.6) we find

$$\theta(Z) = -Z^i \text{tr}A_{e_i}, \quad \theta^*(Z) = \theta(\varphi Z) = Z^i \text{tr}(A_{e_i} \circ \varphi), \quad (i = 1, \ldots, 2n).$$

We define the operators

$$m_i : W \longrightarrow W; \quad i = 1, 2;$$

$$m_1(F)(X, Y, Z) = \frac{1}{2} Y^i \{ g(A_{e_i}X, Z) - g(A_{e_i}\varphi X, \varphi Z) \} (i = 1, \ldots, 2n);$$

$$m_2(F)(X, Y, Z) = \frac{1}{2} Y^i \{ g(A_{e_i}X, Z) + g(A_{e_i}\varphi X, \varphi Z) \} (i = 1, \ldots, 2n).$$
Lemma 3.1. The operators $m_i \ (i = 1, 2)$ have the following properties

(i) $m_i \circ m_i = m_i, \ i = 1, 2$;

(ii) $\sum_{i=1}^{2} m_i = id$;

(iii) $m_1 \circ m_2 = m_2 \circ m_1 = 0$;

and $m_i \ (i = 1, 2)$ commute with $U(n, \mathbb{R})$.

We denote $W_{11} = Im m_1, \ F_3 = Im m_2$. Lemma 3.1 implies the decomposition $W = W_{11} \oplus F_3$ that is $U(n, \mathbb{R})$ - invariant. Using (2.7) we check that $W_{11} \perp F_3$.

Proposition 3.1. The decomposition

$$W = W_{11} \oplus F_3$$

is orthogonal and invariant under the action of the group $U(n, \mathbb{R})$.

After direct computations using definitions of $m_1, m_2, (3.3)$ we obtain

Proposition 3.2. For an arbitrary $F \in W$ we have

(i) $F = m_1 F \iff A_{e_i} \circ \varphi = \varphi \circ A_{e_i}, \ (i = 1, \ldots, 2n)$;

(ii) $F = m_1 F \iff F(X, Y, Z) = -F(\varphi X, \varphi Y, Z)$;

(iii) $F = m_2 F \iff A_{e_i} \circ \varphi = -\varphi \circ A_{e_i}, \ (i = 1, \ldots, 2n)$;

(iii) $F = m_2 F \iff F(X, Y, Z) = F(\varphi X, \varphi Y, Z)$.

From Proposition 3.2 the characteristic conditions of $W_{11}, F_3$ are

$$W_{11} = \{ F \in W : A_{e_i} \circ \varphi = \varphi \circ A_{e_i}, \ (i = 1, \ldots, 2n) \} \iff \{ F \in W : F(X, Y, Z) = -F(\varphi X, \varphi Y, Z) \};$$

$$F_3 = \{ F \in W : A_{e_i} \circ \varphi = -\varphi \circ A_{e_i}, \ (i = 1, \ldots, 2n) \} \iff \{ F \in W : F(X, Y, Z) = F(\varphi X, \varphi Y, Z) \}.$$  

(3.4)

(3.5)

We define the operator

$$m_3 : W_{11} \rightarrow W_{11};$$

$$m_3(F)(X, Y, Z) = F(X, Y, Z) - \frac{1}{2(n-1)} \left\{ g(X, \varphi Y)\theta_F(\varphi Z) - g(X, \varphi Z)\theta_F(\varphi Y) - g(\varphi X, \varphi Y)\theta_F(Z) + g(\varphi X, \varphi Z)\theta_F(Y) \right\}.$$  

(3.3)

Lemma 3.2. The operator $m_3$ has the following properties

(i) $m_3 \circ m_3 = m_3$;

(ii) $< m_3 F_1, F_2 > = < F_1, m_3 F_2 >, \ F_1, F_2 \in W_{11};$

and $m_3$ commutes with $U(n, \mathbb{R})$. 
If we denote $\mathcal{F}_1 = \text{Ker} m_3$ and $\mathcal{F}_2 = \text{Im} m_3$, then Lemma 3.2 implies Proposition 3.3.

**Proposition 3.3.** The decomposition

$$W_{11} = \mathcal{F}_1 \oplus \mathcal{F}_2$$

is orthogonal and invariant under the action of the group $U(n, \mathbb{R})$, where

$$\mathcal{F}_1 = \{ F \in W : F(X, Y, Z) = \frac{1}{2(n-1)} (g(X, \varphi Y)\theta(\varphi Z) - g(X, \varphi Z)\theta(\varphi Y)$$

$$-g(\varphi X, \varphi Y)\theta(Z) + g(\varphi X, \varphi Z)\theta(Y)) \},$$

$$\mathcal{F}_2 = \left\{ F \in W : A_{\xi_i} = \frac{\text{tr} A_{\xi_i}}{2n} \right\},$$

$$\mathcal{F}_2 = \{ F \in W : F(X, Y, Z) = -F(\varphi X, \varphi Y, Z), \; \theta = 0 \} \Leftrightarrow$$

$$\mathcal{F}_2 = \{ F \in W : A_{\xi_i} = 0, \; \theta = 0 \} \Leftrightarrow$$

Taking into account Proposition 3.1 and Proposition 3.3 we obtain Proposition 3.4.

**Proposition 3.4.** The decomposition

$$W = \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \mathcal{F}_3$$

is orthogonal and invariant under the action of the group $U(n, \mathbb{R}) \times \text{Id}$.

### 4. The subspace $W_2$

From $W_2 = \{ F \in \mathcal{F} : F = p_2 F \}, \; \text{(2.10)}, \; \text{Proposition 2.2}$ and \text{(2.4)} it follows

$$W_2 = \{ F \in \mathcal{F} : F(X, Y, Z) = -\eta(Y)g(\varphi(A_{\xi}X), Y) +$$

$$\eta(Z)g(\varphi(A_{\xi}X), Y), \; A_{\xi} = 0 \}.$$

Using \text{(4.1)} and \text{(1.6)} we find

$$\theta(\xi) = \text{tr}(A_{\xi} \circ \varphi), \; \theta^*(\xi) = -\text{tr} A_{\xi}.$$
Lemma 4.1. The operators \( q_i (i = 1, 2) \) have the following properties

(i) \( q_i \circ q_i = q_i, \quad i = 1, 2; \)

(ii) \( \sum_{i=1}^{2} q_i = \text{id}; \)

(iii) \( q_1 \circ q_2 = q_2 \circ q_1 = 0; \)

and \( q_i (i = 1, 2) \) commute with \( U(n, \mathbb{R}) \times \text{Id}. \)

We denote \( W' = \text{Im} q_1, \ W'' = \text{Im} q_2. \) Lemma 4.1 implies the decomposition \( W_2 = W' \oplus W'' \) that is \( U(n, \mathbb{R}) \times \text{Id} \) invariant. Using (2.7) we check that \( W' \perp W''. \)

Proposition 4.1. The decomposition
\[
W_2 = W' \oplus W''
\]
is orthogonal and invariant under the action of the group \( U(n, \mathbb{R}) \times \text{Id}. \)

After direct computations using definitions of \( q_1, q_2, \) (4.1), (2.5) we obtain

Lemma 4.2. For an arbitrary \( F \in W_2 \) we have

(i) \( F = q_1 F \iff A_\xi \circ \varphi = \varphi \circ A_\xi; \)

(ii) \( F = q_1 F \iff F(X, Y, Z) = -F(\varphi X, \varphi Y, \varphi Z) - F(\varphi Y, X, Z); \)

(iii) \( F = q_2 F \iff A_\xi \circ \varphi = -\varphi \circ A_\xi; \)

Proposition 4.2. For an arbitrary \( F \in W_2 \) we have

\[
W' = \{ F \in W_2 : A_\xi \circ \varphi = \varphi \circ A_\xi \} \iff \{ F \in W_2 : F(X, Y, Z) = -F(\varphi X, \varphi Y, Z) - F(\varphi Y, X, Z) \};
\]

\[
W'' = \{ F \in W_2 : A_\xi \circ \varphi = -\varphi \circ A_\xi \} \iff \{ F \in W_2 : F(X, Y, Z) = -F(\varphi Y, X, Z) + F(\varphi X, \varphi Y, Z) \}.
\]

We define the operators
\[
r_i : W' \to W', \quad i = 1, 2;
\]
\[
r_1(F)(X, Y, Z) = -\frac{1}{2} \eta(Y) \{ g(\varphi(A_\xi X), Z) + g(\varphi X, A_\xi Z) \} + \frac{1}{2} \eta(Z) \{ g(\varphi(A_\xi X), Y) + g(\varphi X, A_\xi Y) \};
\]
\[
r_2(F)(X, Y, Z) = -\frac{1}{2} \eta(Y) \{ g(\varphi(A_\xi X), Z) - g(\varphi X, A_\xi Z) \} + \frac{1}{2} \eta(Z) \{ g(\varphi(A_\xi X), Y) - g(\varphi X, A_\xi Y) \}.
\]
Lemma 4.2. The operators $r_i (i = 1, 2)$ have the following properties

(i) $r_i \circ r_i = r_i$, $i = 1, 2$;

(ii) $\sum_{i=1}^{2} r_i = \text{id}$;

(iii) $r_1 \circ r_2 = r_2 \circ r_1 = 0$;

and $r_i (i = 1, 2)$ commute with $U(n, \mathbb{R}) \times \text{Id}$.

We denote $W'_1 = \text{Im} r_1$, $W'_2 = \text{Im} r_2$. Lemma 4.2 implies the decomposition $W' = W'_1 \oplus W'_2$ that is $U(n, \mathbb{R}) \times \text{Id}$-invariant. Using (2.7) we check that $W'_1 \perp W'_2$.

Proposition 4.3. The decomposition $W' = W'_1 \oplus W'_2$ is orthogonal and invariant under the action of the group $U(n, \mathbb{R}) \times \text{Id}$.

Having in mind definitions of $r_1, r_2$, (5.1), (4.2), (2.5) we obtain

Proposition 4.4. For an arbitrary $F \in W'$ we have

(i) $F = r_1 F \iff g(A_{\xi}, .) = g(., A_{\xi})$;

(ii) $F = r_1 F \iff F(X, Y, Z) = -F(Y, Z, X) + F(Z, X, Y) - 2F(\varphi X, \varphi Y, Z)$;

(iii) $F = r_2 F \iff g(A_{\xi}, .) = -g(., A_{\xi})$;

(iii) $F = r_2 F \iff F(X, Y, Z) = -F(Y, Z, X) - F(Z, X, Y)$.

From Proposition 4.4 the characteristic conditions of $W'_1, W'_2$ are

$$W'_1 = \left\{ F \in W' : g(A_{\xi}, .) = g(., A_{\xi}) \right\} \iff$$

$$\left\{ F \in W' : F(X, Y, Z) = -F(Y, Z, X) + F(Z, X, Y) - 2F(\varphi X, \varphi Y, Z) \right\};$$

$$W'_2 = \left\{ F \in W' : g(A_{\xi}, .) = -g(., A_{\xi}) \right\} \iff$$

$$\left\{ F \in W' : F(X, Y, Z) = -F(Y, Z, X) - F(Z, X, Y) \right\}.$$

We define the operator

$s : W'_1 \to W'_1$,

$s(F)(X, Y, Z) = F(X, Y, Z) + \frac{\theta_F^r(\xi)}{2n} \left\{ \eta(Y)g(X, \varphi Z) - \eta(Z)g(X, \varphi Y) \right\}$.

Lemma 4.3. The operator $s$ has the following properties

(i) $s \circ s = s$;

(ii) $<sF_1, F_2> = <F_1, sF_2>$, $F_1, F_2 \in W'_1$;

and $s$ commutes with $U(n, \mathbb{R}) \times \text{Id}$.
If we denote \( F_5 = \text{Kers} \) and \( F_6 = \text{Ims} \), then Lemma \ref{lem:4.3} implies

**Proposition 4.5.** The decomposition
\[
W'_1 = F_5 \oplus F_6
\]
is orthogonal and invariant under the action of the group \( U(n, \mathbb{R}) \times \text{Id} \), where

\[
F_5 = \left\{ F \in W'_1 : F(X,Y,Z) = -\frac{\theta^*(\xi)}{2n} \{ \eta(Y)g(X,\varphi Z) - \eta(Z)g(X,\varphi Y) \} \right\} \iff \left\{ F \in W'_1 : \text{tr} A_\xi = 0 \right\},
\]
\[
F_6 = \left\{ F \in W'_1 : \theta^*(\xi) = 0 \right\} \iff \left\{ F \in W'_1 : \text{tr} A_\xi = 0 \right\}.
\]

We define the operator
\[
t : W'_2 \rightarrow W'_2;
\]
\[
t(F)(X,Y,Z) = F(X,Y,Z) - \frac{\theta_F(\xi)}{2n} \{ \eta(Y)g(\varphi X,\varphi Z) - \eta(Z)g(\varphi X,\varphi Y) \}.
\]

**Lemma 4.4.** The operator \( t \) has the following properties

1. \( t \circ t = t \);
2. \( <tF_1,F_2> = <F_1,tF_2> \), \( F_1,F_2 \in W'_2 \);

and \( t \) commutes with \( U(n, \mathbb{R}) \times \text{Id} \).

If we denote \( F_4 = \text{Kert} \) and \( F_7 = \text{Imt} \), then Lemma \ref{lem:4.4} implies

**Proposition 4.6.** The decomposition
\[
W'_2 = F_4 \oplus F_7
\]
is orthogonal and invariant under the action of the group \( U(n, \mathbb{R}) \times \text{Id} \), where

\[
F_4 = \left\{ F \in W'_2 : F(X,Y,Z) = \frac{\theta(\xi)}{2n} \{ \eta(Y)g(\varphi X,\varphi Z) - \eta(Z)g(\varphi X,\varphi Y) \} \right\} \iff \left\{ F \in W'_2 : \text{tr}(A_\xi \circ \varphi) = 0 \right\},
\]
\[
F_7 = \left\{ F \in W'_2 : \theta(\xi) = 0 \right\} \iff \left\{ F \in W'_2 : \text{tr}(A_\xi \circ \varphi) = 0 \right\}.
\]
Now we consider the subspace $W''$ of $W_2$. We define the operators

$$l_i : W'' \to W'', \quad i = 1, 2;$$

$$l_1(F)(X, Y, Z) = -\frac{1}{2} \eta(Y) \{g(\varphi(A_\xi X), Z) - g(\varphi X, A_\xi Z)\} + \frac{1}{2} \eta(Z) \{g(\varphi(A_\xi X), Y) - g(\varphi X, A_\xi Y)\};$$

$$l_2(F)(X, Y, Z) = -\frac{1}{2} \eta(Y) \{g(\varphi(A_\xi X), Z) + g(\varphi X, A_\xi Z)\} + \frac{1}{2} \eta(Z) \{g(\varphi(A_\xi X), Y) + g(\varphi X, A_\xi Y)\}.$$

**Lemma 4.5.** The operators $l_i$ ($i = 1, 2$) have the following properties

(i) $l_i \circ l_i = l_i, \quad i = 1, 2$;

(ii) $\sum_{i=1}^{2} l_i = \text{id}$;

(iii) $l_1 \circ l_2 = l_2 \circ l_1 = 0$;

and $l_i$ ($i = 1, 2$) commute with $U(n, \mathbb{R}) \times \text{Id}$.

We denote $\mathcal{F}_9 = \text{Im} l_1, \mathcal{F}_8 = \text{Im} l_2$. Lemma 4.5 implies the decomposition $W'' = \mathcal{F}_8 \oplus \mathcal{F}_9$ that is $U(n, \mathbb{R}) \times \text{Id}$ - invariant. Using (2.7) we check that $\mathcal{F}_8 \perp \mathcal{F}_9$.

**Proposition 4.7.** The decomposition

$$W'' = \mathcal{F}_8 \oplus \mathcal{F}_9$$

is orthogonal and invariant under the action of the group $U(n, \mathbb{R}) \times \text{Id}$.

Taking into account definitions of $l_1, l_2$, (5.1), (4.3), (2.5) we obtain

**Proposition 4.8.** For an arbitrary $F \in W''$ we have

(i) $F = l_1 F \iff g(A_\xi \ldots) = g(\ldots, A_\xi)$;

(ii) $F = l_1 F \iff F(X, Y, Z) = -F(Y, Z, X) - F(Z, X, Y)$;

(iii) $F = l_2 F \iff g(A_\xi \ldots) = -g(\ldots, A_\xi)$;

(iii) $F = l_2 F \iff F(X, Y, Z) = -F(Y, Z, X) + F(Z, X, Y) + 2F(\varphi X, \varphi Y, Z)$.

From Proposition 4.8 the characteristic conditions of $\mathcal{F}_8, \mathcal{F}_9$ are

$$\mathcal{F}_8 = \left\{ F \in W'' : g(A_\xi \ldots) = -g(\ldots, A_\xi) \right\} \iff$$

(4.6)$$\left\{ F \in W' : F(X, Y, Z) = -F(Y, Z, X) + F(Z, X, Y) + 2F(\varphi X, \varphi Y, Z) \right\}.$$
\[ \mathcal{F}_9 = \left\{ F \in W'': g(A_\xi, .) = g(., A_\xi) \right\} \Longleftrightarrow \left\{ F \in W'': F(X, Y, Z) = -F(Y, Z, X) - F(Z, X, Y) \right\}. \]

Finally, we denote \( \mathcal{F}_{10} = W_3 \) and \( \mathcal{F}_{11} = W_4 \). Taking into account Proposition 2.1, Proposition 3.3, Proposition 4.1, Proposition 4.3, Proposition 4.5, Proposition 4.6, Proposition 4.7 we obtain

**Theorem 4.1.** The decomposition

\[ \mathcal{F} = \mathcal{F}_1 \oplus \ldots \oplus \mathcal{F}_{11} \]

is orthogonal and invariant under the action of the group \( U(n, \mathbb{R}) \times Id \).

Next we summarize the characterization conditions for the factors \( \mathcal{F}_i \) \((i = 1, \ldots, 11)\). Let \( X, Y, Z \in V \). Then

\[ \mathcal{F}_1 : F(X, Y, Z) = \frac{1}{2n} \{ g(X, \varphi Y) \theta(hZ) + g(X, \varphi Z) \theta(hY) \}, \]

\[ \mathcal{F}_2 : F(\varphi X, \varphi Y, Z) = -F(X, Y, Z); \quad \theta = 0, \]

\[ \mathcal{F}_3 : F(\varphi X, \varphi Y, Z) = F(X, Y, Z), \]

\[ \mathcal{F}_4 : F(X, Y, Z) = \frac{\theta(\xi)}{2n} \{ \eta(Y) g(X, \varphi Z) - \eta(Z) g(X, \varphi Y) \}, \]

\[ \mathcal{F}_5 : F(X, Y, Z) = -\frac{\theta^*(\xi)}{2n} \{ \eta(Y) g(X, \varphi Z) - \eta(Z) g(X, \varphi Y) \}, \]

\[ \mathcal{F}_6 : F(X, Y, Z) = -F(\varphi X, \varphi Y, Z) - F(\varphi X, Y, \varphi Z) = -F(Y, Z, X) + F(Z, X, Y) - 2F(\varphi X, \varphi Y, Z); \quad \theta^*(\xi) = 0, \]

\[ \mathcal{F}_7 : F(X, Y, Z) = -F(\varphi X, \varphi Y, Z) - F(\varphi X, Y, \varphi Z) = -F(Y, Z, X) - F(Z, X, Y); \quad \theta(\xi) = 0, \]
$F_8 : F(X, Y, Z) = F(\varphi X, \varphi Y, Z) + F(\varphi X, Y, \varphi Z) = -F(Y, Z, X) + F(Z, X, Y) + 2F(\varphi X, \varphi Y, Z)$.

$F_9 : F(X, Y, Z) = F(\varphi X, \varphi Y, Z) + F(\varphi X, Y, \varphi Z) = -F(Y, Z, X) - F(Z, X, Y)$.

$F_{10} : F(X, Y, Z) = \eta(X)F(\xi, \varphi Y, \varphi Z)$.

$F_{11} : F(X, Y, Z) = \eta(X)\{\eta(Y)\omega(Z) - \eta(Z)\omega(Y)\}.$

5. Basic classes of almost paracontact manifolds and some examples

Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost paracontact manifold. The tensor $F$, defined by (1.4) we can write in the form (2.6), where the linear operators $A_{e_i}$ ($i, \ldots, 2n$) and $A_\xi$ are defined by

$A_{e_i}X = (\nabla_X \varphi)e_i, \quad (i, \ldots, 2n); \quad A_\xi X = \nabla_X \xi.$

We verify immediately that so defined operators $A_{e_i}$ ($i, \ldots, 2n$) and $A_\xi$ have the properties (2.2) ÷ (2.5). Using the decomposition of the space $\mathcal{F}$ over $V = T_pM, p \in M$, we define the corresponding subclasses of the class of almost paracontact manifolds with respect to the covariant derivative of the structure tensor field $\varphi$.

An almost paracontact manifold is said to be in the class $\mathcal{F}_i$ ($i = 1, \ldots, 11$) if the tensor $F(X, Y, Z) = g(\nabla_X \varphi)Y, Z$ belongs to the class $\mathcal{F}_i$ over $V = T_pM$ for each $p \in M$.

In a similar way we define the classes $\mathcal{F}_i \oplus \mathcal{F}_j$. It is clear that $2^{11}$ classes of almost paracontact manifolds are possible.

The class $\mathcal{F}_0$ of almost paracontact manifolds is defined by the condition $F(X, Y, Z) = 0$. This special class belongs to everyone of the defined classes.

**Example 5.1.** Let $(M^5, \varphi, \xi, \eta, g)$ be an almost paracontact metric manifold. We consider a $\varphi$-basis $\{e_1, e_2, \varphi e_1, \varphi e_2, \xi\}$ of $T_pM, p \in M$ such that $g(e_i, e_i) = g(\varphi e_i, \varphi e_i) = 1, \quad i = 1, 2$.

We denote the matrices of the the operators $A_{e_i}$ and $A_{\varphi e_i}$ ($i = 1, 2$) with respect to the basis $\{e_1, e_2, \varphi e_1, \varphi e_2, \xi\}$ by $A_i$ ($i = 1, 2$) and $A_j$ ($j = 3, 4$) respectively. We define

$A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ a & b & c & d & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -c & -d & -a & -b & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -a & -b & -c & -d & 0 \\ 0 & 0 & 0 & 0 & 0 \\ c & d & a & b & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$
where \( a, b, c, d \) are functions over \( M \).

From the definitions of the matrixes \( A_j \) \((j = 1, 2, 3, 4)\) we have \( \eta(A_{e_i}, X) = \eta(A_{\varphi e_i}, X) = 0 \) \((i = 1, 2)\). From \((2.4)\) it follows \( A_{\xi}X = 0 \). Using \((2.6)\) we compute

\[
F(X, Y, Z) = (aX^1 + bX^2 + cX^3 + dX^4) (Y^1 Z^2 - Y^2 Z^1 + Y^3 Z^4 - Y^4 Z^3) +
(cX^1 + dX^2 + aX^3 + bX^4) (Y^1 Z^4 - Y^2 Z^3 + Y^3 Z^2 - Y^4 Z^1),
\]

where \( X = X^i e_i + X^{i+2} \varphi e_i + \eta(X) \xi, \ Y = Y^i e_i + Y^{i+2} \varphi e_i + \eta(Y) \xi, \Z = Z^i e_i + Z^{i+2} \varphi e_i + \eta(Z) \xi \) \((i = 1, 2)\). We verify that

\[
F(X, Y, Z) = F(\varphi X, \varphi Y, Z),
\]

which is the characterization condition of the class \( \mathcal{F}_3 \).

**Example 5.2.** Let \((M^5, \varphi, \xi, \eta, g)\) be an almost paracontact metric manifold. We consider a \( \varphi \)-basis \( \{e_1, e_2, \varphi e_1, \varphi e_2, \xi\} \) of \( T_pM, p \in M \) such that

\[
g(e_i, e_i) = -g(\varphi e_i, \varphi e_i) = 1, \quad i = 1, 2.
\]

We denote the matrixes of the operators \( A_{e_i}, A_{\varphi e_i} \) \((i = 1, 2)\) and \( A_{\xi} \) with respect to the basis \( \{e_1, e_2, \varphi e_1, \varphi e_2, \xi\} \) by \( A_i \) \((i = 1, 2)\), \( A_j \) \((j = 3, 4)\) and \( A \) respectively. We define

\[
A_1 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
-a & -b & -c & -d & 0
\end{pmatrix}, \quad A_2 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
-e & -f & -b & -c & 0
\end{pmatrix},
\]

\[
A_3 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
-a & -b & -d & -e & 0
\end{pmatrix}, \quad A_4 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
-b & -c & -e & -f & 0
\end{pmatrix},
\]

\[
A = \begin{pmatrix}
a & b & d & e & 0 \\
b & c & e & f & 0 \\
d & -e & -a & -b & 0 \\
e & -f & -b & -c & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

where \( a, b, c, d, e, f \) are functions over \( M \). Using \((2.6)\) we compute

\[
F(X, Y, Z) = \eta(Y) \{a(X^1 Z^3 + X^3 Z^1) + b(X^1 Z^4 + X^2 Z^3 + X^3 Z^2 + X^4 Z^1) +
\}
\[
c(X^2 Z^4 + X^4 Z^2) + d(X^1 Z^1 + X^3 Z^3) + e(X^1 Z^2 + X^2 Z^4 + X^3 Z^4 + X^4 Z^3) +
\]
\[
f(X^2 Z^2 + X^4 Z^4)\} - \eta(Z) \{a(X^1 Y^3 + X^3 Y^1) +
\]
\[
...
where \( X = X^i e_i + X^{i+2} \varphi e_i + \eta(X) \xi, \ Y = Y^i e_i + Y^{i+2} \varphi e_i + \eta(Y) \xi, \ Z = Z^i e_i + Z^{i+2} \varphi e_i + \eta(Z) \xi \), \( i = 1, 2 \). We verify that
\[
F(X,Y,Z) = F(\varphi X, \varphi Y, Z) + F(\varphi X, Y, \varphi Z) = -F(Y, Z, X) - F(Z, X, Y),
\]
which is the characterization condition of the class \( \mathcal{F}_9 \).

**Example 5.3.** Let \( (M^5, \varphi, \xi, \eta, g) \) be an almost paracontact metric manifold. We consider a \( \varphi \)-basis \( \{e_1, e_2, \varphi e_1, \varphi e_2, \xi\} \) of \( T_p M, p \in M \) such that
\[
g(e_i, e_i) = -g(\varphi e_i, \varphi e_i) = 1, \quad i = 1, 2.
\]
We denote the matrixes of the the operators \( A_{e_i} \) and \( A_{\varphi e_i} \) \( (i = 1, 2) \) with respect to the basis \( \{e_1, e_2, \varphi e_1, \varphi e_2, \xi\} \) by \( A_i \) \( (i = 1, 2) \) and \( A_j \) \( (j = 3, 4) \) respectively. We define
\[
A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & -a \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},
\]
\[
A_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -b & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},
\]
where \( a, b \) are functions over \( M \).

From the definitions of the matrixes \( A_j \) \( (j = 1, 2, 3, 4) \) we have \( \eta(A_{e_i}, X) = \eta(A_{\varphi e_i}, X) = 0 \) \( (i = 1, 2) \). From (2.4) it follows \( X^{\xi} X = 0 \). Using (2.6) we compute
\[
F(X,Y,Z) = \eta(X) \{ Y^1(aZ^2 - bZ^4) + Y^2(bZ^3 - aZ^1) + Y^3(aZ^4 - bZ^2) + Y^4(bZ^1 - aZ^3) \},
\]
where \( X = X^i e_i + X^{i+2} \varphi e_i + \eta(X) \xi, \ Y = Y^i e_i + Y^{i+2} \varphi e_i + \eta(Y) \xi, \ Z = Z^i e_i + Z^{i+2} \varphi e_i + \eta(Z) \xi \), \( i = 1, 2 \). We verify that
\[
F(X,Y,Z) = \eta(X)F(\xi, \varphi Y, \varphi Z),
\]
which is the characterization condition of the class \( \mathcal{F}_{10} \).

**Remark 5.1.** Taking into account the characterization of the classes \( \mathcal{F}_i \) \( (i = 1, \ldots, 11) \) by the linear operators \( A_{e_i} \) \( (i = 1, \ldots, 2n) \) and \( A_{\xi} \), we can construct examples for the rest of the classes too. Using the matrixes of the operators with respect to a \( \varphi \)-basis we obtain also that an almost paracontact manifold of dimension 3 can not belong to the classes \( \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_6 \).
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(Nakova) University of Veliko Tarnovo "St. Cyril and St. Methodius", Faculty of Education, T. Tarnovski 2 str., 5003 Veliko Tarnovo, Bulgaria

E-mail address: gnakova@yahoo.com

(Zamkovoy) University of Sofia "St. Kl. Ohridski", Faculty of Mathematics and Informatics, Blvd. James Bourchier 5., 1164 Sofia, Bulgaria

E-mail address: zamkovoy@fmi.uni-sofia.bg