Optimized Synthesis of Snapping Fixtures
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Abstract
This paper deals with the following separability problem in 3D space: Given a rigid polyhedron \( P \) with \( n \) vertices, does a semi-rigid polyhedron \( G \) exist, such that both polyhedra can be transformed into an inseparable assembled state, where the fixture snaps on to \( P \), by applying a linear force and exploiting the mild flexibility of \( G \)? If such a flexible snapping polyhedron exists, devise an efficient and robust algorithm that constructs it. In simple words, we are looking for a semi-rigid polyhedron \( G \) such that when \( P \) and \( G \) are separate, we can push \( G \) towards \( P \), slightly bending \( G \) on the way, and obtain a configuration, where \( G \) is back in its original shape, and both \( P \) and \( G \) are inseparable as rigid bodies. We define certain properties such a pair of polyhedron and its snapping fixture may possess, and prove two theorems related to the pair. We introduce an algorithm that produces a snapping fixture with such properties in \( O(n^5) \) time, if a snapping fixture exists, and an efficient and robust implementation of this algorithm.

1 Introduction
A fixture is a device that holds a part in place. There are many types and forms of fixtures; they range from modular fixtures synthesized on a lattice to fixtures generated to suit a specific part. A fixture possesses some grasp characteristics. For example, a grasp with complete restraint prevents loss of contact, prevents any motion, and thus may be considered secure. Two primary kinematic restraint properties are form closure and force closure [10]. Both properties guarantee maintenance of contact under some conditions. However, the latter relies on contact friction; therefore, achieving force closure typically requires fewer contacts than achieving form closure. Fixtures with complete restraint are mainly used in manufacturing processes where preventing any motion is critical. Other types of fixtures can be found anywhere, for example, in the kitchen where a hook holds a cooking pan, or in the office where a pin and a bulletin board hold a paper still. This paper deals with a specific problem in this area; here, we are given a rigid object, referred to as the workpiece, and we seek for an automated process that designs a semi-rigid object, referred to as the holding fixture, such that, starting at a configuration where the workpiece and the holding fixture are separated, they can be pushed towards each other, applying a linear force and exploiting the mild flexibility of the fixture, into a configuration where both the workpiece and the fixture are inseparable as rigid bodies. Without additional computational effort, a hook, a nut, or a bolt can be added to a base part of the fixture (referred to as the palm and defined in Section 2.1). This results in a generic fixture that can be utilized in a larger system. Another advantage of the single-component flexible fixture is that it can easily be 3D-printed. We have 3D-printed several workpieces and suitable fixtures that our system has automatically generated. The objective of the algorithm is obtaining snapping fixtures, the skeleton of which has low complexity; for a precise definition see Section 2.4. With additional care that also accounts for properties of the material used to produce the fixtures, the smallest or lightest possible fixture can be synthesized, for a given workpiece. This can (i) expedite the production of the fixture using, e.g., additive manufacturing, (ii) minimize the weight of the produced fixture, and (iii) maximize the exposed area of the boundary of the workpiece when held by the fixture.

1.1 Background
Form closure has been studied since the 19th century. Early results showed that at least four frictionless contacts are necessary for grasping an object in the plane, and seven in 3D space. Specifically, it has been shown that four and seven contacts are necessary and sufficient for the form-closure grasp of any polyhedron in the 2D and 3D case, respectively [8 9].

Automatic generation of various types of fixtures, and in particular, the synthesis of form-closure grasps, are the subjects of a diverse body of research. Brost and Goldberg [3] proposed a complete algorithm for synthesizing modular fixtures of polygonal workpieces by locating three pegs (locators), and one clamp on a lattice. Their algorithm is complete in the sense that it examines all possible fixtures for an input polygon. Their results were obtained by generating all configurations of three.locators coincident to three edges, for each triplet of edges in the input polygon. For each such configuration, the algorithm checks whether form closure can be obtained by adding a single clamp.

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Our work uses a similar iteration strategy to obtain all possible configurations. In subsequent work Zhuang, Goldberg, and Wong [16] showed that there exists a non-trivial class of polygonal workpieces that cannot be held in form closure by any fixture of this type (namely, a fixture that uses three locators and a clamp). They also considered fixtures that use four clamps, and introduced two classes of polygonal workpieces that are guaranteed to be held in form closure by some fixture of this type. Some of our results are reminiscent of their results. Wallack and Canny [15] proposed another type of fixture called the vise fixture and an algorithm for automatically designing such fixtures. The vise fixture includes two lattice plates mounted on the jaws of a vise and pegs mounted on the plates. Then, the workpiece is placed on the plates, and form closure is achieved by activating the vise and closing the pins from both sides on the workpiece. The main advantage in this type of fixture is its simplicity of usage. Brost and Peters [4] extended the approach exploited in [3] to three dimensions. They provided an algorithm that generates suitable fixtures for three-dimensional workpieces. Wagner, Zhuang, and Goldberg [14] proposed a three-dimensional seven-contact fixture device and an algorithm for planning form-closure fixtures of a polyhedral workpiece with pre-specified pose. A summary of the studies in the field of flexible fixture design and automation conducted in the last century can be found in [1, 2]. Subsequent works studied other types of fixtures and provided algorithms for computing them, for example, unilateral fixtures [5], which are used to fix sheet-metal workpieces with holes. Other studies focused on grasping synthesis algorithms with autonomous robotic fingers, where a single robotic hand gets manipulated by motors and being used to grasp different workpieces; an overview of such algorithms can be found in [12]. A common dilemma for all the grasping and fixture design algorithms is defining and finding the optimal grasp. Several works, e.g., [7] and [11], discuss such quality functions and their optimization.

1.2 Our Results

We define certain properties that a pair of a workpiece and its snapping fixture should possess, and prove a theorem related to such pairs. Formally, we are given a simple and closed polyhedron $P$ of complexity $n$ that represents a workpiece. We introduce an algorithm that determines whether a simple and closed polyhedron $G$ that represents a fixture exists, and if so, it constructs it in $O(n^5)$ time. We also provide an efficient and robust implementation of this algorithm. In addition, we present two practical cases that utilize our algorithm and its implementation: One is the generation of a snapping fixture that mounts a device to an unmanned aerial vehicle (UAV), such as a drone. The other is the generation of a snapping fixture that mounts a precious stone to a jewel, such as a ring. The common objective in both cases is, naturally, the firm holding of the workpiece. In the first case, we are interested in a fixture with minimal weight. In the second case we are interested in a fixture with minimal volume, such that it minimally obscures the precious stone. Note that the obtained fixture prevents any linear motion, but does not necessarily prevent angular motion; hence, the held workpiece does not possess the form closure property per se. Handling angular motion is left for the future; see Section 6.1.

1.3 Outline

The rest of this paper is organized as follows. Terms and definitions for our snapping fixtures are provided in Section 2. Theoretical bounds and properties are provided also in Section 2. The synthesis algorithm is described in Section 4 along with the analysis of its complexity. Two applications are presented in Section 5. Finally, we point out some limitations of our planer and suggest future research in Section 6.

2 Terms and Definitions

In this section we describe the structure of our snapping fixtures and their properties.

2.1 Fixture Structure

**Definition 1** ($\alpha$-extrusion of a polygon and base polygon of an $\alpha$-extrusion). Let $L$ denote a polygon in 3D space, let $v$ denote the normal to the plane containing $L$, and let $v_\alpha$ denote the normal scaled to length $\alpha$. The $\alpha$-extrusion of $L$ is a polyhedron $P$ in 3D space, which is the extrusion of $L$ along $v_\alpha$. The polygon $L$ is referred to as the base polygon of $P$; see Figure 1.

For simplicity, we occasionally use the terms vertices, edges, and facets to abbreviate the references to their respective geometric embeddings (which are points, segments, and polygons, respectively). In particular, we use the abbreviation $\alpha$-extrusion of a facet $f$ of some polyhedron to refer to the $\alpha$-extrusion $P$ of the geometric embedding of the facet $f$, and we refer to the facet of $P$ that overlaps with $f$ as the base facet of the $\alpha$-extrusion $P$. 
Consider an input polyhedron $P$ that represents a workpiece. The structure of a fixture of $P$ resembles the structure of a hand; it is the union of a single polyhedral part referred to as the palm, several polyhedral parts, referred to as fingers, which are extensions of the palm, and semi-rigid joints that connect the palm and the fingers. Each finger consists of two polyhedral parts, namely, body and gripper, and the semi-rigid joint between the body and the gripper. The various parts, i.e., palm, bodies, and grippers are disjoint in their interiors. (In configurations where the joints bend, some parts may overlap with the joints.) In the following we describe these parts in detail.

Let $G$ denote a snapping fixture made of a palm, $k$ fingers, $F_1, F_2, \ldots, F_k$, and corresponding joints. The palm is an $\alpha_p$-extrusion of a facet $f_p$ of $P$. (The value of $\alpha_p$ is discussed below.) Consider a specific finger $F = F_i$ of $G$. The body of $F$ is defined by one of the neighboring facets of $f_p$, denoted $f_b$. The gripper of $F$ is defined by one of the neighboring facets of $f_b$, denoted $f_g$, $f_g \neq f_p$. Let $e_{pb}$ denote the common edge of $f_p$ and $f_b$, and let $e_{bg}$ denote the common edge of $f_b$ and $f_g$. Let $q_b$ denote the quadrilateral defined by the vertices incident to $e_{pb}$ and $e_{bg}$. Let $q_g$ denote the quadrilateral defined by the two vertices incident to $e_{bg}$ and their translations by $v$. The gripper is an $\alpha_g$-extrusion of $q_g$. The axis of the joint that connects the palm and the body of $F$ coincides with $e_{pb}$ and the axis of the joint that connects the body of $F$ with its gripper coincides with $e_{bg}$. The value $\alpha_p$ and the values $\alpha_b$ and $\alpha_g$ for each finger determine the trade-off between the strength and flexibility of the joints.\footnote{They depend on the material and shape of the fixture. In our implementation they can be determined by the user.}

For a complete view of a workpiece and its snapping fixture consider Figure 2. Observe that for each finger $g$, coincides with $e_{pb}$ and coincides with $e_{bg}$. Avoiding overlaps is achieved by simultaneously shrinking the base facets $f_{g_1}$ and $f_{g_2}$. Now, the gripper grips only the tip of $f$ and the body is stretching only on a small portion of the workpiece facet. As another example, consider the body of a finger depicted in Figure 2(c) which is the $\alpha_b$-extrusion of the quadrilateral $q_b$ in the figure. It is defined by two points that lie in the interior of $e_{bg}$, as opposed to the formal definition above, where the endpoints of $e_{pb}$ and $e_{bg}$ define the corresponding formal quadrilateral. Also, in reality, parts are not fabricated separately, and the entire fixture is made of the same flexible material. Instead of rotating about the joint axes, the entire fingers bend. The differences, though, have no effect on the correctness of the proofs and algorithms (which adhere to the formal definitions) presented in the sequel. These structural changes and the extrusion values, merely determine the degree of flexibility and strength of the fixture; see Section 6.3 and Section 6.4 below.

### 2.2 The Configuration Space

The workpiece and its snapping fixture form an assembly. Each joint in the fixture connects two parts; it enables the rotation of one part with respect to the other about an axis. Each joint adds one degree of freedom (DOF) to the configuration space of the assembly. Thus, the configuration space of the assembly (which is the configuration space of the fixture, assuming the workpiece is stationary) has $6 + 2k$ DOFs, where $k$ indicates the number of fingers.

\footnote{Typically, these values are identical.}

\footnote{For example, In several of the fixtures that we produced, they were set to 5mm.}
In our context, the workpiece and its snapping fixture are considered assembled, if they are infinitesimally inseparable. When two polyhedra are infinitesimally inseparable, any linear motion applied to one of the polyhedra causes a collision between the polyhedra interiors.

**Definition 2** (Serving configuration). The workpiece and the fixture are in the serving configuration if (i) they are separated (that is, they are arbitrarily far away from each other), and (ii) there exists a vector $v$, such that when the fixture is translated by $v$, as a result of some force applied in the direction of $v$, exploiting the flexibility of the joints of the fixture, the workpiece and the fixture become assembled.

When the workpiece and its snapping fixture are separated, the fixture can be transformed without colliding with the workpiece to reach the serving configuration.
2.3 Spreading Degree

The spreading degree is the number of facets involved in the definition of a finger. In this paper we restrict ourselves to snapping fixtures that have fingers with spreading degree two, which means that the body of every finger is based on a single facet of \( P \). Every finger (the body and the gripper) stretches over two facets of \( P \). Naturally, fingers with a higher spreading-degree reach further. The figure to the right depicts an icosahedron, which does not admit a valid fixture with spreading degree two.

2.4 Fixture Planning

The basic objective of our fixture algorithm is obtaining fixtures with the minimal number of fingers. Our planner is of the exhaustive type. As explained in Section 3 we examine many different possible candidates of fingers, before it reaches a conclusion. As aforementioned, the planner generates fixtures of spreading degree two. Extending the planner to enable the generation of fixtures with an increased spreading degree (without further modifications), will directly increase the search space exponentially.

3 Bounds and Properties

In this section we provide some theoretical bounds and properties, including the introduction of our main theorem. We use these bounds and properties to analyze our algorithms and prove their correctness. We also use one known theorem and several known lemmas, which we nevertheless explicitly spell out for clarity.

3.1 Covering Set

**Definition 3** (Covering set). Let \( S = \{s_1, ..., s_{|S|}\} \) be a finite set of subsets of \( \mathbb{R}^d \) and \( C \) be a set of points in \( \mathbb{R}^d \). If \( \bigcup_{i=1}^{s_{|S|}} s_i \supseteq C \) then \( S \) is a covering set of \( C \).

**Theorem** (Helly’s theorem [6]). Let \( S = \{X_1, ..., X_n\} \) be a finite collection of convex subsets of \( \mathbb{R}^d \), with \( n > d \). If the intersection of every \( d+1 \) of these sets is nonempty, then the whole collection has a nonempty intersection; that is, \( \cap_{j=1}^{d+1} X_j \neq \emptyset \).

The contrapositive formulation of the theorem follows. If \( \cap_{j=1}^{d} X_j = \emptyset \) then there exists a subset \( R = \{X_{i_1}, ..., X_{i_{d+1}}\} \subseteq S \) such that \( |R| = d + 1 \) and \( \cap_{j=1}^{d+1} X_{i_j} = \emptyset \). In the succeeding proofs we use the following corollary:

**Corollary 1.** Let \( S = \{X_1, ..., X_n\} \) be a finite set of convex subsets of \( \mathbb{R}^d \). If \( \bigcup_{j=1}^{n} X_j = \mathbb{R}^d \) then there exists a subset \( R = \{X_{i_1}, ..., X_{i_{d+1}}\} \subseteq S \) such that \( |R| = d + 1 \) and \( \bigcup_{j=1}^{d+1} X_{i_j} = \mathbb{R}^d \).

The corollary holds because the intersection of a set of subgroups of \( \mathbb{R}^d \) is empty if and only if the union of their complement in \( \mathbb{R}^d \) is \( \mathbb{R}^d \).

**Definition 4** (unit circle, semicircle, open semicircle). A unit circle is a circle with unit radius; a semicircle is half of a circle, and an open semicircle excluding its two endpoints. Similarly, a unit sphere is a sphere with unit radius; a hemisphere is half of a sphere, and an open hemisphere is a hemisphere excluding the great circle that comprises its boundary curve.

The following four lemmas, [4][5], based on the analysis in [7], are used below to prove Theorem 1.

**Lemma 1.** Let \( S \) be a finite set of open unit semicircles. If \( S \) is a covering set of a closed unit semicircle \( \tilde{A} \), then there exists \( R \subseteq S \) such that \( R \) is a covering set of \( \tilde{A} \) and \( |R| \in \{2, 3\} \).

**Proof.** It is obvious that one open unit semicircle cannot cover a closed unit semicircle. Let \( A \) denote the interior of \( \tilde{A} \). (\( \tilde{A} \) is an open unit semicircle.) There are two cases: (i) \( A \in S \) and \( A \in R \) for every covering set \( R \subseteq S \) of \( \tilde{A} \). It implies that every covering set \( R \subseteq S \) must contain two additional open semicircles that cover the two boundary points of \( \tilde{A} \), respectively. These two semicircles together with \( A \) constitute a covering set of \( \tilde{A} \) of size three. (ii) There exists a covering set \( S' \subseteq S \), where \( \tilde{A} \notin S' \). Let \( S'_A = \{s \cap \tilde{A} \mid s \in S'\} \) be the set of intersections of the elements of \( S' \) and \( \tilde{A} \). Let \( \Pi^1 \) denote the extended central projection that maps the closed semicircle \( \tilde{A} \) to the **affinely extended real number line** [8]. \( \Pi^1(p) = (x, w) : \tilde{A} \to \mathbb{R}^1 \), where the points in \( \mathbb{R}^1 \) are represented in homogeneous coordinates \( (x, w) \).

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3 The set \( \mathbb{R}^1 \cup \{+\infty, -\infty\} \) is referred to as the affinely extended real number line.
Notice that for every \( s \in S'_H \), \( s \) covers one of the boundary points of \( \tilde{A} \); therefore, \( \Pi^1(s) \) is an open ray covering either \((-1,0)\) or \((+1,0)\). \( S'_H \) covers \( \tilde{A} \); therefore, the set of its images \( S'' = \{ \Pi^1(s) \mid s \in S'_H \} \) covers \( \mathbb{P}^1 \). By Helly’s theorem, there exists a subset \( R'' \subseteq S'' \) of size two that covers \( \mathbb{R}^1 \). Thus, the set of preimages of \( R'' \) covers \( \tilde{A} \).

**Lemma 2.** Let \( S \) be a finite set of open unit semicircles. If \( S \) is a covering set of the unit circle \( \mathbb{S}^1 \), then there exists \( R \subseteq S \) such that \( R \) is a covering set of \( \mathbb{S}^1 \) and \( |R| \in \{3,4\} \).

**Proof.** Let \( s \in S \) be an arbitrary open unit semicircle in \( S \). The remaining elements \( S \setminus \{s\} \) of \( S \) must cover the complement \( \tilde{s} \) of \( s \), which is a closed unit semicircle. By lemma 1 there exists \( R' \subseteq S \setminus \{s\} \) that covers \( \tilde{s} \), and \( |R'| \in \{2,3\} \). Thus, \( R' \cup \{s\} \) covers \( \mathbb{S}^1 \), and \( |R' \cup \{s\}| \in \{3,4\} \).

**Lemma 3.** Let \( S \) be a finite set of open unit hemispheres. If \( S \) is a covering set of a closed unit hemisphere \( \tilde{A} \), then there exists \( R \subseteq S \), such that \( R \) is a covering set of \( \tilde{A} \) and \( |R| \in \{3,4,5\} \).

**Proof.** Let \( \mathbb{H} \) denote the interior of \( \tilde{A} \) (\( \mathbb{H} \) is an open unit hemisphere) and \( \partial \mathbb{H} \) denote the boundary of \( \mathbb{H} \) (\( \partial \mathbb{H} \) is a great circle). Similar to the proof of Lemma 1, there are two cases: (i) \( \mathbb{H} \in S \) and \( \mathbb{H} \in R \) for every covering set \( R \subseteq S \) of \( \mathbb{H} \). It implies that every covering set \( R \subseteq S \) must contain additional open hemispheres that cover \( \partial \mathbb{H} \). Let \( S_{\partial \mathbb{H}} = \{ s \cap \partial \mathbb{H} \mid s \in S \} \) be the set of intersections of the elements of \( S \) and \( \partial \mathbb{H} \). Note that an intersection of a unit open hemisphere and a great circle is either empty or an open unit semicircle. Therefore \( S_{\partial \mathbb{H}} \) is a set of open unit semicircles lying on the same plane. By Lemma 2 there exists a covering set \( R_{\partial \mathbb{H}} \subseteq S_{\partial \mathbb{H}} \) of \( \partial \mathbb{H} = \mathbb{S}^1 \), such that \( |R_{\partial \mathbb{H}}| \in \{3,4\} \). This implies that there exists a covering set \( R \subseteq S \) of \( \mathbb{H} \), such that \( |R| \in \{4,5\} \). (ii) There exists a covering set \( S' \subseteq S \), where \( \mathbb{H} \notin S' \). Let \( S'_H = \{ s \cap \mathbb{H} \mid s \in S' \} \) be the set of intersections of the elements of \( S' \) and \( \mathbb{H} \). Let \( \Pi^2 \) denote the extended central projection that maps the closed hemisphere \( \mathbb{H} \) to an extended plane obtained by adjoining all signed slopes to \( \mathbb{R}^2 \) (a generalization of the affinely extended real number line, to the plane), \( \Pi^2(p) = (x,y,w) : \mathbb{H} \rightarrow \mathbb{P}^2 \), where the points in \( \mathbb{P}^2 \) are represented in homogeneous coordinates \((x,y,w)\). Notice that every \( s \in S'_H \) is a semi-open spherical wedge; see the figure above and to the right. The wedge is bounded by two semicircles \( \tilde{A}_1 \) and \( \tilde{A}_2 \) (in the figure), where \( \tilde{A}_1 \) lies in \( \partial \mathbb{H} \). The intersection of \( \tilde{A}_2 \) and \( \tilde{A}_1 \) is empty, and the intersection of \( \hat{A}_1 \) and \( s \) is an open semicircle; therefore, \( \Pi^2(s) \) is an open halfplane. \( S'' \) covers \( \mathbb{H} \); therefore, the set of its images \( S'' = \{ \Pi^2(s) \mid s \in S'' \} \) covers \( \mathbb{P}^2 \). By Helly’s theorem, there exists a minimal subset \( R_{\Pi^2} \subseteq S'' \) of size at most three that covers \( \mathbb{R}^2 \). If \( |R_{\Pi^2}| = 2 \), that is, two open halfplanes, say \( h_1 \) and \( h_2 \) comprise \( R_{\Pi^2} \), then they must be parallel: \( h_1 : ax + by + c_1 > 0 \) and \( h_2 : ax + by + c_2 > 0 \). In this case they do not cover the points \((-b,a)\) and \((b,-a)\) in \( \mathbb{P}^2 \). Thus, the pair of preimages of \( R_{\Pi^2} \) covers \( \mathbb{H} \) except for two antipolar points. Covering these antipolar points requires two additional elements from \( S'' \), which yields a covering set of size four. If \( |R_{\Pi^2}| = 3 \), then none of the halfplanes in \( R_{\Pi^2} \) (which cover \( \mathbb{R}^2 \)) are parallel, and they also cover \( \mathbb{P}^2 \). Thus, the set of preimages of \( R_{\Pi^2} \) covers \( \mathbb{H} \), which yields a covering set of size three.

**Lemma 4.** Let \( S \) be a finite set of open unit hemispheres. If \( S \) is a covering set of the unit sphere \( \mathbb{S}^2 \), then there exists \( R \subseteq S \) such that \( R \) is a covering set of \( \mathbb{S}^2 \) and \( |R| \in \{4,5,6\} \).

**Proof.** Let \( s \in S \) be an arbitrary open unit hemisphere in \( S \). The remaining elements \( S \setminus \{s\} \) of \( S \) must cover the complement \( \tilde{s} \) of \( s \), which is a closed unit hemisphere. By lemma 3 there exists \( R' \subseteq S \setminus \{s\} \) that covers \( \tilde{s} \), and \( |R'| \in \{3,4,5\} \). Thus, \( R' \cup \{s\} \) covers \( \mathbb{S}^2 \), and \( |R' \cup \{s\}| \in \{4,5,6\} \).

**Corollary 2.** Let \( R \) be a set of six open unit hemispheres that cover the unit sphere \( \mathbb{S}^2 \). If \( R \) is minimal, then it must consist of three pairs of open unit hemispheres, such that the closure of every pair is the entire unit sphere.

**Proof.** Assume, by contradiction that \( R \) contains an open unit hemispheres \( h \), such that the interior of its complement is not in \( R \). Observe that the complement of \( h \) is equivalent to \( \mathbb{H}^2 \). This is exactly case (ii) in the proof of Lemma 3. Here, there exists a covering set \( R' \) of \( \mathbb{H}^2 \), such that \( |R'| \in \{3,4\} \). It implies that \( |R| \) is at most five, a contradiction.

When a facet \( f \) of the workpiece partially coincides with a facet of the fixture, the workpiece cannot translate in any direction that forms an acute angle with the (outer) normal to the plane containing \( f \) (without colliding with the fixture). This set of blocking directions comprises an open unit hemisphere denoted as \( h(f) \). Similarly, \( H(F) = \{ h(f) \mid f \in F \} \) denotes the mapping from a set of facets to the set of corresponding open unit hemispheres; see, e.g., [13]. Let \( F' \) denote the set of facets of the workpiece that are coincident with facets of the fixture in some fixed configuration. If the union of all blocking directions covers the unit sphere in that configuration, formally stated \( \mathbb{S}^2 = \bigcup H(F') \), then the workpiece cannot translate at all.

Let \( F \) denote the set of all facets of \( G \). Let \( F_P \) denote the singleton that consists of the base facet of the palm of \( G \), and let \( f_b \) and \( f_g \), \( 1 \leq i \leq k \), denote the base facet of the body and the base facet of the gripper, respectively, of the \( i \)-th
finger of \( G \), where \( k \) indicates the number of fingers. Let \( \mathcal{F}_B = \{f_b \mid 1 \leq i \leq k\} \) and \( \mathcal{F}_G = \{f_g \mid 1 \leq i \leq k\} \) denote the set of the base facets of the body parts of the fingers of \( G \) and the set of the base facets of the gripper parts of the fingers of \( G \), respectively. Let \( \mathcal{F}_{PBG} \) denote the set of all base facets of the parts of \( G \), that is \( \mathcal{F}_{PBG} = \mathcal{F}_P \cup \mathcal{F}_B \cup \mathcal{F}_G \). Let \( \mathcal{F}_{PB} \) denote the set of all base facets of the parts of \( G \) excluding the base facets of the grippers, that is, \( \mathcal{F}_{PB} = \mathcal{F}_P \cup \mathcal{F}_B \).

If the fixture resists any linear force applied on the workpiece while in the assembled state and there exists a collision free path (in the configuration space) between any separated configuration and the assembled configuration then our fixture is valid. We relax the second condition for practical reasons and instead of requiring a full path, we require a path of infinitesimal length. Formally we get:

1. \( \mathbb{S}^2 = \bigcup \mathcal{H}(\mathcal{F}_{PBG}) \).
2. \( \mathbb{S}^2 \neq \bigcup \mathcal{H}(\mathcal{F}_{PB}) \).

If the second condition holds, a serving state exists (assuming the flexibility of the joints cancels out the problem induced by the presence of the grippers).

A candidate finger of an input polyhedron \( P \) is a valid finger of at least one possible fixture \( G \) of \( P \).

**Observation 1.** The number of candidate fingers of an input polyhedron \( P \) is linear in the number of vertices of \( P \).

**Proof.** Let \( e \) be an edge of \( P \) and let \( f_e \) and \( f_e' \) be the two faces incident to \( e \). Two fingers can be built on \( e \). The base facet of the body and the base facet of the gripper of one finger coincides with \( f_e \) and \( f_e' \), respectively. In order to construct the other finger, the roles of these facets exchange; that is, the base facet of the body and the base facet of the gripper of one finger coincides with \( f_e' \) and \( f_e \), respectively. Every candidate finger is built on a single edge. Thus, the number of candidate fingers is at most \( 3n - 6 \), where \( n \) is the number of vertices of the polyhedron. Thus, the number of candidate fingers is at most \( 6n - 12 \).

**Theorem 1.** Four fingers are always sufficient and sometimes necessary for a valid snapping fixture.

**Proof.** Consider a polyhedron \( P \). Let \( G \) be a valid fixture of \( P \), and assume that \( G \) has more than four fingers. We show that it is possible to construct a valid snapping fixture of \( P \) that has (i) the same palm as \( G \), and (ii) four fingers that are a subset of the fingers of \( G \). Consider the closed hemisphere \( \mathbb{H} = \mathbb{S}^2 \setminus \mathcal{H}(\mathcal{F}_P) \). By Corollary \ref{corollary}, \( \mathbb{S}^2 = \bigcup \mathcal{H}(\mathcal{F}_{PBG}) \). We get that \( \mathbb{H} \subseteq \mathcal{F}_{BG} \). In other words, \( \mathcal{H}(\mathcal{F}_{BG}) \) is a covering set of \( \mathbb{H} \). By Lemma \ref{lemma} there exists a subset \( R \subseteq \mathcal{H}(\mathcal{F}_{BG}) \), such that (i) \( R \) is a covering set, and (ii) \( |R| \in \{3,4,5\} \). We prove separately for \( |R| \in \{3,4\} \) and \( |R| = 5 \).

If \( |R| = 3,4 \), there exist \( i \in \{3,4\} \) hemispheres that correspond to \( i \) base facets of \( i \) bodies and grippers, respectively, of at most four fingers, which we choose as the fingers of \( G' \).

If \( |R| = 5 \), then as seen in the proof of Lemma \ref{lemma} \( R \) contains an open hemisphere \( \mathbb{H}_g \), such that \( \mathbb{H}_g = h(f_g) \) and the base facet of the palm and \( f_g \) are parallel. In a polyhedron, two parallel facets cannot be neighbours; thus, \( f_g \) must be the base facet of a gripper of some finger \( F \). Let \( f_b \) denote the base facet of the body of the finger \( F \) and set \( \mathbb{H}_b = h(f_b) \). Observe, that \( R_1 = R \setminus \{\mathbb{H}_g\} \) must be a covering set of the unit circle \( \partial \mathbb{H}_g \), and \( |R_1| = 4 \). Observe that \( \partial \mathbb{H}_g \neq \partial \mathbb{H}_b \); thus, \( R_2 = R_1 \setminus \{\mathbb{H}_b\} \) is a covering set of a closed semicircle \( \bar{A} \) and \( |R_2| = 3 \). Following a deduction similar to the above, there exist three hemispheres that correspond to three base facets of three bodies or grippers, respectively, of at most three fingers, which we choose as the fingers of \( G' \) in addition to \( F \).

A polyhedron that admits the lower bound is depicted in Figure \ref{figure}. Proving that a snapping fixture for this polyhedron with less then four fingers does not exists is done using our planner. We exhaustively searched the configurations space and did not find a valid snapping fixture.

**Observation 2.** A fixture with only one finger does not exist.

**Proof.** Let \( G \) be a fixture with only one finger. Then, \( |\mathcal{H}(\mathcal{F}_{PBG})| = 3 \). However, by Lemma \ref{lemma} the minimum size of a covering set of \( \mathbb{S}^2 \) is four.

While the lower bound is tight, there exists a polyhedron that has a snapping fixture that has only two fingers; see Figure \ref{figure}.
Figure 4: Different views of a polyhedron that has one valid fixture with four fingers.

4 Algorithm

A snapping fixture \( G \) (of spreading degree two) is formally defined by a pair that consists of (i) an index \( i \) of a facet of \( P \), and (ii) a set of pairs of indices \((j_1, \ell_1), (j_2, \ell_2), \ldots, (j_\ell, \ell_\ell)\) of facets of \( P \). The palm of \( G \) is the \( \alpha \)-extrusion of the facet \( f_i \). Each member pair of indices \((j, \ell)\) define a finger of \( G \). The body and gripper of the finger are the \( \alpha \)-extrusions of the facets \( f_j \) and \( f_\ell \), respectively.

Procedure 1. snappingFixture(\( P \)) The procedure accepts a polyhedron \( P \) as input and returns a fixture \( G \) of \( P \) of the best quality according to given optimization criteria (see below); see Algorithm 1. The algorithm consists of two phases. In the first phase we compute a data structure \( M \) that associates palms and their candidate fingers. In the second phase we identify subsets of fingers for each palm stored in \( M \) that together form a potential valid fixture and extract the fixture with the best quality over all potential fixtures.

Algorithm 1 Snapping fixture generation

Input: A simple polyhedron \( P \) that consists of \( m \) facets \( \{f_1, f_2, \ldots, f_m\} \).
Output: A snapping fixture \( G \) of \( P \), if there exists one, with the best quality.

1: procedure snappingFixture(\( P \))
2: for \( i \leftarrow 1, m \) do
3: \( M[i] \leftarrow \emptyset \)
4: for all \( j, f_j \in \text{neighbours}(f_i) \) do
5: for all \( \ell, f_\ell \in \text{neighbours}(f_j) \& \ell \neq i \) do
6: \( M[i] \leftarrow M[i] \cup \{(j, \ell)\} \)
7: end for
8: end for
9: end for
10: \( B \leftarrow \text{null} \)
11: for \( i \leftarrow 1, m \) do
12: for all \( S, S \in \text{subsets}(M[i], 4) \) do
13: \( F \leftarrow (f_i, S) \)
14: if \( \text{validFixture}(F) \) then
15: \( B \leftarrow \text{findBest}(B, F) \)
16: end if
17: end for
18: end for
19: return \( F \)
20: end procedure
Procedure 2. **neighbours**\((f)\)** The procedure accepts a facet \(f\) of a polyhedron and returns all the neighbouring facets of \(f\).

Procedure 3. **subsets**\((S,c)\)** The procedure accepts a set \(S\) and a number \(c\) and returns all subsets of \(S\) of cardinality at most \(c\).

Procedure 4. **validFixture**\((F)\)** The procedure accepts a snapping fixture and determines whether it is a valid snapping fixture based on Conditions 2 and 3 defined in Section 3.

Procedure 5. **findBest**\((B,F)\)** The procedure accepts two valid snapping fixtures and returns the one with better quality. A natural way to rank fixtures is to compare their number of fingers. We have implemented several versions of this procedure that differ in the tiebreaking.

**Theorem 2.** Algorithm \(A\) runs in \(O(n^5)\) time, where \(n\) is the number of vertices of the input polyhedron.

**Proof.** During the first phase we perform three loops that iterate over facets. In the first loop we iterate over all facets, and in the second nested loop we iterate over all neighbours of facets visited in the first loop essentially visiting each edge twice; see the proof of Observation 1. The number of facets and the number of edges is linear in \(n\); thus, the overall time complexity of the first phase is \(O(n^2)\).

The second phase dominates the time complexity. Recall that a potential fixture passed to **validFixture**\((F)\) and to **findBest**\((B,F)\) encoded by \((f,S)\), where \(f\) denotes a facet and \(S\) denotes a set, the cardinality of which is at most 4, has at most 4 fingers. Therefore, both functions run in constant time. For every possible palm the function **validFixture** is invoked once per every subset of candidate fingers of size at most 4. As the number of candidate fingers is linear in \(n\) (see Observation 1), the total complexity of this phase is \(O(n((n)^2 + (n)^3 + (n)^4)) = O(n \cdot n^4) = O(n^5)\).

5 Two Applications

We present two applications that utilize our algorithm and its implementation.

5.1 Minimal Weight Fixtures

![Figure 5: Opposite views pf a micro switch and a low-weight snapping fixture.](image)

Generating lightweight fixtures that could be mounted on a UAV has been a major challenge ever since the first UAV was introduced. The desire for robust and efficient solutions to this problem rapidly scaled up during the last decade with the introduction of small drones, the weight of devices that can be mounted on which, is limited. Naturally, the device must be securely attached to the drone; however, at the same time, the holding mechanism should weigh as little as possible. Figure 5 shows a fixture generated for a micro-switch sensor, one of the most common sensors in the field of robotics and automation. Figure 6 shows the fabricated fixture (3D printed) permanently attached to a drone. It holds a micro-switch. While the micro-switch if firmly held, it can be easily replaced.

5.2 Minimal Obscuring Fixtures

One of the objectives of jewelry making is to expose the gems mounted on a jewel, such as a ring, and reveal their allure. As with the minimal-weight fixture, the mounted gem must be securely attached to the jewel. While the weight of the holding mechanism can be compromised, its volume cannot. In particular, the fixture must obscure the gem as little as possible, so that the gem surface is exposed as much as possible. Figure 7 shows a pendant with an integrated
5.3 3D Printing Considerations

We used various materials for generating snapping fixtures, such as, ABS, PLA, PETG, Nylon 12 and Sterling silver. All generated fixtures properly snapped and firmly held the workpieces. However, low quality prints (made of ABS, PLA, or PETG) occasionally broke after repeated or incautious uses. We noticed that increasing the infill density and orienting the prints such that the joint axes and the printing plate are not parallel increase the fixture durability. Also, we compensated for the limited precision of printers by scaling up the fixture to create a gap of up to 0.2mm between the fixture and the workpiece.

$^4$3D printed wax and lost-wax method where used to generate fixtures made of Sterling Silver.
6 Limitations and Future Research

6.1 Form Closure

Our planner synthesizes fixtures that do not necessarily prevent angular motion. Such fixtures are rarely obtained. Nevertheless, the figure to the right depicts a workpiece and a snapping fixture (synthesized by our planner), such that the workpiece can escape the assembled configuration using torque. However, other snapping fixture of this workpiece that guarantee form-closure of the workpiece do exist (and offered by our planner). Devising efficient synthesis algorithms for guaranteeing form closure is left for future research.

6.2 Spreading Degree

Increasing the spreading degree (see Section 2.3) will enable the synthesis of fixtures for a larger range of workpieces. Future research could result with (i) a classification of polyhedra according to the minimal spreading degree required for their snapping fixtures, and (ii) algorithms for synthesis of fixtures with a larger fixed spreading degree or even unlimited.

6.3 Joint Flexibility

The flexibility of the joints is an important consideration in the design. In order to construct a snapping fixture, the joint that connects the body of a finger to the palm, and the joint that connects the gripper of a finger to its body must allow the rotation of the respective parts about the respective axes when force is applied. Some of the subtleties of this flexibility are discussed below. For simplicity we move the discussion to the plane, where our workpiece and snapping fixture are polygons.

Let’s focus on one finger. Consider the configuration where the finger is about to snap. Assume, for further simplicity, that the joint that connects the body and the gripper of the finger is rigid, and consider only the joint that connects the finger with the palm, as depicted in the figure to the right. This configuration occurs a split of a second before the assembly reaches the assembled state when transformed, starting at the serving configuration. Let $\theta$ denote the angle between the finger and the workpiece. Note that in the assembled configuration $\theta$ equals 0 for all fingers. Let $\theta_c$ denote the joint threshold angle, that is, the maximum bending angle the finger can tolerate without breaking. The threshold angle of every joint depends on the material and thickness of the region around joint. $\theta$ is an angle of a triangle with one edge lying on the body base-facet and another edge lying on the gripper base-facet. Let $a$ and $b$ denote the lengths of these edges, respectively, and let $\eta$ be the angle between them. The finger will break when $\theta > \theta_c$. Applying the law of sines, we get $b = \frac{a \sin \theta}{\sin(\pi - \theta - \eta)} = \frac{a \sin \theta}{\sin(\theta + \eta)}$, which implies a maximal value $b \leq \frac{\min(a) \sin \theta}{\sin(\theta + \eta)}$. On the other hand, the characteristics of the material of the finger determine the minimal value of $b$ that guarantees a secured grasp of the workpiece by the gripper. The construction of a fixture $G$ is feasible, only if selecting a proper value $b$ for every finger of $G$ is possible. We remark that the full analysis in space is more involved, and for now our planner does not take into account material properties such as flexibility.

6.4 Gripping Strength

Another consideration in the fixture design is the gripping strength. The gripping strength of a finger is based on the angle between the palm and the body of the finger and on the angle between the body of the finger and the gripper of the finger. The gripping strength is in opposite relation with these angles; that is, the smaller each one of these angles is the stronger the gripping is. While our planner currently does not take in account strength considerations, it could be used as a criterion in ranking valid snapping fixtures of a given workpiece.

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Figure 9: Examples of snapping fixtures automatically generated by our planner for a tetrahedron, a cube, an octahedron, and a dodecahedron.