Lower Bounds of Algebraic Branching Programs and Layerization

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Abstract

In this paper we improve the lower bound of Chatterjee et al. (ECCC 2019) to an \(\Omega(n^2)\) lower bound for unlayered Algebraic Branching Programs. We also study the impact layerization has on Algebraic Branching Programs. We exhibit a polynomial that has an unlayered ABP of size \(O(n)\) but any layered ABP has size at least \(\Omega(n\sqrt{n})\). We exhibit a similar dichotomy in the non-commutative setting where the unlayered ABP has size \(O(n)\) and any layered ABP has size at least \(\Omega(n \log n - \log^2 n)\).

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1 Introduction

Arithmetic Circuit Complexity studies the complexity of arithmetic circuits that compute families of polynomials. The model was introduced by Valiant [10] to study the permanent and determinant. Both polynomials give rise to the standard classes of VP and VNP and distinguishing them is a major open problem. The exact question is, does there exist an arithmetic circuit, i.e., a circuit that computes with addition and multiplication gates on fields and variables, that has polynomial size and computes the permanent.

In this paper we will study small size classes in arithmetic circuit complexity. Next to VP and VNP there is the class of VF, namely formulas of polynomial size. A formula describes an arithmetic circuit where the underlying graph is a tree. This class can also be described via a different computation model which will be the focus of this paper.

Definition 1.1. Let \(G\) be a directed acyclic graph with one source node \(s\) and one sink node \(t\). Let the edges be labeled by affine linear forms over some field \(\mathbb{F}\) with variables in \(X\). We define the weight of an edge \(e\), \(w(e)\), to be the label. Then there is a polynomial computed by

\[
\sum_{p \text{ path from } s \text{ to } t} w(p)
\]

where \(w(p) = \prod_{e \in p} w(e)\). We call \(G\) an Algebraic Branching Program (ABP) with size \(m\) where \(m\) is the number of vertices.

We call the class of all polynomials computed by an ABP of polynomial size (in the number of variables) VBP. For more interesting connections, the reader is referred to excellent surveys by Mahajan [6] and Shpilka and Yehudayoff [9].

Recently, new results on ABPs lower bounds were given such as in [2, 4, 5]. And the community also discovered new general lower bounds (see [3]). However, there are two slightly different versions of ABPs. The first one is the unlayered model, as described in Definition 1.1. The second definition, has the graph \(G\) layered. Here layered means that the
vertices of the graph can be subdivided into sets $V_1, \ldots, V_d$ where there are no edges between vertices in $V_i$ and there are only edges between $V_i$ and $V_{i+1}$ for $i = 1, \ldots, d - 1$.

It is easy to see that any unlayered ABP can be made layered with a blowup of at most $s^2$. Just extend the graph with edges labelled by one, such that every path has length $s$. This increases the size to $O(s^2)$.

With this distinction in mind, we can look at recent lower bound results by Chatterjee et al. [3].

\begin{boldmath}
\textbf{Theorem 1.2} (Theorem 1.2 [3]). Let $F$ be a field of characteristic $\text{char}(F) \nmid n$. Then any layered ABP computing the polynomial $\sum_{i=1}^{n} x_i^n$ is of size at least $\Omega(n^2)$.
\end{boldmath}

\begin{boldmath}
\textbf{Theorem 1.3} (Theorem 1.3 [3]). Let $F$ be a field of characteristic $\text{char}(F) \nmid n$. Then any unlayered ABP computing the polynomial $\sum_{i=1}^{n} x_i^n$ is of size at least $\Omega(n \log n \log \log n + 1)$.
\end{boldmath}

Now the obvious question arises: Is the unlayered bound optimal because layerization has at least an $n^2 \log \log n$ lower bound? We give a partial answer to this question in the next theorem by giving a lower bound for layerization of ABPs.

\begin{boldmath}
\textbf{Theorem 1.4.} There exists a polynomial $f$ such that any unlayered ABP computing $f$ has size $O(n)$ and any layered ABP computing $f$ has size at least $\Omega(n \sqrt{n})$ and $\Omega(n^2)$ edges.
\end{boldmath}

The proof of this is given in Section 3. We also note that the polynomial they study, $\sum_{i=1}^{n} x_i^n$, is unlikely to have such a size blowup between layered and unlayered ABPs at it is homogeneous (Lemma 3.4). It is unclear how to transform a general ABP that computes a homogeneous polynomial into a homogeneous ABP with only a linear increase in size. This implies a possible improvement in [3, Theorem 1.3]. We show this improvement in Section 2 and prove an $\Omega(n^2)$ lower bound even in the case where the ABP is unlayered.

We also study variants of the layerization question, namely for the non-commutative model in Section 3.2. In this model, the variables do not commute multiplicatively and hence $x_1 x_2 - x_2 x_1$ is not a zero polynomial. All our layerization proofs also work in the monotone setting (Corollary 3.5).

## 2 ABP Lower Bound

In general, the complexity of ABP is given by families of polynomials $(f_n)$ that are computed by families of Algebraic Branching Programs $(P_n)$. Here, for all $n$, $f_n$ can have $p(n)$ variables where $p(n)$ is any polynomial. The parameterization of the branching programs and polynomial families will always be clear from the context. We sometimes skip talking about polynomial families when the polynomials and branching programs are parameterized by the number of variables.

Given a possible unlayered ABP with affine linear forms computing $\sum_{i=1}^{n} x_i^n$ we can transform it into split form.

\begin{boldmath}
\textbf{Definition 2.1.} We call an ABP in split-form if the edge set $E = E_{\text{const}} \uplus E_{\text{var}} \uplus E_{\text{split}}$ and $V = V_{\text{const}} \uplus V_{\text{var}} \uplus \{s, t\}$ where $s, t$ are the start and end points of the ABP and the following holds:
\begin{itemize}
  \item Edges in $E_{\text{const}}$ are labelled by an element in $F$.
  \item $V_{\text{const}} = \{u, v\mid (u, v) \in E_{\text{const}}\}$.
\end{itemize}
\end{boldmath}

\footnote{A homogeneous polynomial has all monomials of the same degree.}
Algorithm 1 Compacting Algorithm

1. From $t$ go backwards and find the border of degree $n$, i.e., the edge where the degree increases to $n + 1$ in $E_{\text{var}}$. Delete these edges.
2. Compact the vertices in $E_{\text{const}}$. Meaning, compute all paths with starting vertex in $E_{\text{var}}$ and ending vertex in $E_{\text{var}}$ going through $E_{\text{const}}$ and contract these to single edges with the computed constant.
3. Fold two following edges into one if one is in $E_{\text{const}}$ and the other in $E_{\text{var}}$. Also fold edges labelled with the constant one to the previous edge. Both these are only applicable if there are no other edges going out from the vertex that would be removed.

- Edges in $E_{\text{var}}$ are labelled with an element in $X$, the variable set.
- $V_{\text{var}} = \{ u, v \mid (u, v) \in E_{\text{var}} \}$.
- Edges in $E_{\text{split}}$ are labelled with the constant one.
- Edges in $E_{\text{split}} = \{ (u, v) \mid u \in V_{\text{const}}, v \in V_{\text{var}} \text{ or } u \in V_{\text{var}}, v \in V_{\text{const}} \}$.

It is trivial to transform any affine ABP into a split-form ABP with a constant size blowup. In essence, we double the ABP and compute the constant part and the variable part separately.

Theorem 2.2 (Theorem 3.4 [3]). Any branching program of formal degree $n$ computing $\sum_{i=1}^{n} x_{i}^{n} + R$ where $\deg(R) < n$ has at least $n^{2}$ vertices.

Theorem 2.3. Any split-form ABP computing $\sum_{i=1}^{n} x_{i}^{n}$ over $F$ with $\text{char}(F) \nmid n$ has size at least $\Omega(n^{2})$.

Visiting the proof given in [3] with focus on Theorem 2.2 we see that the main technique is to transform an ABP of length greater than $n$ to an ABP of length $n$ with only controlled change of the computed polynomial such that Theorem 2.2 is still applicable. In the next proof, we will compact any split-form ABP and apply Theorem 2.2.

Proof. We apply Algorithm 1 to the split-form ABP $P$ to produce $P'$. We need to prove a couple of things for the algorithm.

1. Algorithm 1 Line 1 can find a border.
2. $P'$ has size $O(s)$.
3. $P'$ has depth $n$.
4. If $P$ computed $\sum_{i=1}^{n} x_{i}^{n}$ then $P'$ computes $\sum_{i=1}^{n} x_{i}^{n} + R$ where $\deg(R) < n$.

We start with Item 3. Let us first look at the ABP giving us the variables which is modified in Algorithm 1 Line 3. Here, as we cut of the computation at degree $n$, this gives us a depth of $n + n$, as there might be edges going through $E_{\text{const}}$. This gets then reduced to depth $n$ in step Algorithm 1 Line 2 and Algorithm 1 Line 3 as all these edges get folded into affine linear forms.

We can now look at Item 2. As every step decreases the size, we only need to look at the exception of Algorithm 1 Line 2. While this step increases the size of the ABP, Algorithm 1 Line 2 compacts every added edge to edges in the original ABP.

Let us continue with Item 1. As we have a split-form ABP there obviously exists such a border in $E_{\text{var}}$. as every edge increases the degree by one.

Finally, we come to the hardest part, proving Item 1. By Item 3 we do not produce monomials that have degree higher than $n$. As the lower degree monomials do not matter, the only thing left to prove is the degree $n$ homogeneous components. Algorithm 1 Line 2
removes all vertices in $E_{\text{const}}$, hence, we only need to look at all paths of length $n$ through $P'$. As we do not touch them in Algorithm 1 Line 1, we keep all paths of degree $n$ intact and by Algorithm 1 Line 3 we do not change the constants. ◀

Now by combining Theorem 2.3, the fact that any ABP can be transformed with size blowup of two and the previously proven lower bound in [3] for layered ABP gives us now the general lower bound for layered and unlayered ABP.

Corollary 2.4. Any ABP computing $\sum_{i=1}^{n} x_i$ over $F$ with $\text{char}(F) \nmid n$ has size at least $\Omega(n^2)$.

3 ABP Layerization is Optimal

In the following, we need the concept of a $p$-projection which is a well known reduction in arithmetic circuit complexity. Let $p,q$ be polynomials in $F[x_1,\ldots,x_n]$. We call $p$ being a projection of $q$, written $p \leq p q$, if $p = q(\alpha_1,\ldots,\alpha_n)$ where $\alpha_i \in F \cup \{x_1,\ldots,x_n\}$. It is easy to see that any polynomial $p$, that is a projection of $q$, and $q$ is computed by an ABP of size $s$ then $p$ is computed by an ABP of size $s$ without changing if the ABP is layered. This is easily achieved by changing the weights on the edges.

3.1 The General Case

We note that there is no constraint on the field for the results in this section. We introduce the Wi-Fi polynomial on variables $x_1,\ldots,x_n,y_1,\ldots,y_n,z_1,\ldots,z_n$. The polynomial of degree 9 is the evaluation of the ABP given in Figure 1. We define it as

$$f_{2n+1} = \sum_{i=1}^{n} x_i \prod_{j=1}^{i} y_j z_j.$$ 

It is clear that the polynomial has degree $2n + 1$, $3n$ edges and $2n$ vertices. Before we proof our bound, we need a couple of simple propositions. The following proof is a simple application of Baur-Strassen [1].

Proposition 3.1. Let $f$ be a polynomial. If all possible circuits $\phi$ computing simultaneously all partial derivatives have at least $m$ multiplication gates, then every circuit computing $f$ has at least $\Omega(m)$ multiplication gates.

Figure 1 Wi-Fi graph of degree 9.
Proof. We will follow mostly the proof of Baur and Strassen \cite{baur1983} and the exposition by Saptharishi \cite{saptharishi2008} but take care to count the number of multiplication gates.

Let a circuit $\phi$ computing a polynomial $f$ be given. We assume that $x_1, x_2$ are the variables feeding into a gate $v$. Deleting these two edges gives us a circuit $\phi'$ computing $f'$ of size $s-2$ and multiplication gates $m-1$. By induction, we can now assume we have a circuit of size $5(s-2)$ with $5(m-1)$ multiplication gates that computes all first order derivatives of $f'$. Let $v = x_1 \oplus x_2$. Since

$$\frac{\partial f}{\partial x_i} = \left( \frac{\partial f'}{\partial x_{i,v=x_1 \oplus x_2}} \right) + \left( \frac{\partial f'}{\partial v} \right)_{v=x_1 \oplus x_2} \left( \frac{\partial (x_1 \oplus x_2)}{\partial x_i} \right).$$

It is easy to see that the case of $\oplus = +$ does not add any addition gates, as

$$\frac{\partial f}{\partial x_1} = \left( \frac{\partial f'}{\partial x_1} \right)_{v=x_1 + x_2} + \left( \frac{\partial f'}{\partial v} \right)_{v=x_1 + x_2} x_2,$$

$$\frac{\partial f}{\partial x_2} = \left( \frac{\partial f'}{\partial x_2} \right)_{v=x_1 + x_2} + \left( \frac{\partial f'}{\partial v} \right)_{v=x_1 + x_2} x_1,$$

$$\frac{\partial f}{\partial x_i} = \left( \frac{\partial f'}{\partial x_i} \right)_{v=x_1 + x_2}.$$

Let us now take a closer look at the multiplication gates.

$$\frac{\partial f}{\partial x_1} = \left( \frac{\partial f'}{\partial x_1} \right)_{v=x_1 x_2} + \left( \frac{\partial f'}{\partial v} \right)_{v=x_1 x_2} x_2,$$

$$\frac{\partial f}{\partial x_2} = \left( \frac{\partial f'}{\partial x_2} \right)_{v=x_1 x_2} + \left( \frac{\partial f'}{\partial v} \right)_{v=x_1 x_2} x_1,$$

$$\frac{\partial f}{\partial x_i} = \left( \frac{\partial f'}{\partial x_i} \right)_{v=x_1 x_2}.$$ 

This adds 3 additional multiplication gates. As in the original proof our size bound is now $5s$. Looking at the last few equations, we see that the multiplication bound is $5m - 5 + 3 \leq 5m$. This finishes the Baur-Strassen proof.

The rest is a simple proof. Assume $\phi$ has less than $m/5$ multiplication gates. Then computing the partial derivatives as above gives us less than $m$ multiplication gates. This violates the assumption of the proposition.

The next proposition was stated by Chatterjee et al. \cite{chatterjee2010}.

\begin{proposition} \cite{chatterjee2010} \end{proposition}

\begin{corollary} \end{corollary}

\begin{theorem} \end{theorem}

This is proven by a simple contraposition argument from Proposition \ref{hierarchy-theorem}.
Every single partial derivative has size $i$. Hence, the circuit computing $g$ has $\Omega(n^2)$ multiplication gates via Proposition 3.1. By Corollary 3.3 all layered ABPs computing $g$ have at least $\Omega(n^2)$ edges. From this we can easily now give a size lower bound.

Let $w_1, \ldots, w_n$ be the number of vertices in layer $i$ and $d$ the number of layers. The size of the ABP is $\sum_{i=1}^{d} w_i$. For this, the number of edges is given by $\sum_{i=1}^{d-1} w_i w_{i+1}$. We know that this is equal to $n^2$. Hence, with the following elementary equations we get our size lower bound of $\Omega(n^2)$. For clarity, we omit the constant in the $\Omega$ notation and assume large enough $n$.

$$\sum_{i=1}^{d-1} w_i w_{i+1} \geq n^2$$

$\Rightarrow nw_iw_{i+1} \geq n^2$ where $j$ is the index such that $w_jw_{j+1}$ is smallest.

$\Rightarrow w_jw_{j+1} \geq n$ where $j''$ is the index such that $w_{j''}$ is minimal between $w_j$ and $w_{j+1}$.

Hence, as every $w_i$ has size at least $\sqrt{n}$ and $d \geq n$ we get a bound of $\Omega(n\sqrt{n})$.

\[\text{Lemma 3.4. Every Homogeneous ABP of size } s \text{ can be made into a layered ABP of size } 2s.\]

\textbf{Proof.} Assume that there are no consecutive edges labeled with a constant. This can be easily achieved.

Let us define the distance of a vertex $v$, $d(v)$, to be the minimum length of a path from $s$ to $v$. Assume we have a vertex of some degree $d$ with at least two inputs $v_1, v_2$ with $d(v_1) < d(v_2) \leq d(v)$ and $d(v)$ being minimal. Then, as $[v]$, the polynomial computed at $v$ is homogeneous, and $[v_1], [v_2]$ are homogeneous, they need to have the same degree, as otherwise $[v_1] + [v_2]$ would be non-homogeneous. We now modify the ABP such that $d(v_1) = d(v_2)$ in the obvious way.

Left to check is the increase in size. As every edge consist of either a constant, we added at most one edge in the above step. This finishes the proof.

\[\text{Corollary 3.5. All our layerization results also hold in the monotone setting.}\]

\subsection{Non-Commutative ABP}

One of the major problems in showing a similar theorem for non-commutative ABPs is that the current known general lower bounds only work for homogeneous ABPs. While we could modify this to only look at the highest degree homogeneous component, this will not be enough to show an interesting lower bound.

However, we could show a non-optimal lower bound for the homogeneous case, implying that Lemma 3.4 does not hold in the non-commutative case.

We will use the Nisan Matrix $M_i(f)$ (see [7] for a rigorous definition). The Nisan Matrix defines $M_i(f)$ as having the rows and columns indexed by monomials of size $i$ and $d - i$ respectively over just the variables, where $d$ is the degree. The coefficient at the coordinate $(m, m')$ is the coefficient of $m \cdot m'$ in $f$. Then for any homogeneous polynomial $f$, the non-commutative ABP has size exactly $\sum_{i=1}^{n} \text{rank}(M_i(f))$.

\[\text{Theorem 3.6. The polynomial } P \text{ needs non-commutative layered ABPs of size } \Omega(n \log n - \log^2 n) \text{ but has unlayered non-commutative ABP of size } O(n).\]
Proof. We will look at the projection of the polynomial $P$, namely, the polynomial
\[ P' = \sum_{i=\log n}^{n-\log n} x_i \prod_{j=1}^{\log n} y_j z_j. \]
This polynomial is an easy projection from the Wi-Fi polynomial and is homogeneous of degree $\log n$. We assume the variables be ordered in the following way:
\[ x_1, \ldots, x_n, y_1, \ldots, z_1, \ldots, y_{\log n}, z_{\log n}. \]
We take the layered ABP of size $s$ and homogenize it. This is a standard technique and increases the size by at most the degree, i.e., $\log n$. This results in an ABP $P''$.

Let us now look at one summand. For every $\log n \leq i \leq n - \log n$, the matrix $M_i(P'')$ has rank at least $\min\{\log n/2 - i, i\}$. This gives us a total size lower bound of $\sum_{i=\log n/4}^{\log 3n/4} i = \log^2 n/4$. As we have $n - 2\log n$ summands, all independent, the final bound is $(n - 2\log n)\log^2 n/4 = n\log^2 n - 2\log^3 n$. The independence is obvious as $x_i$ will partition the matrix into different rows. Now with the above increase of $\log n$, the size is now bounded from below by $n\log n - 2\log^2 n$. \hfill \lhd

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While the polynomial is homogeneous, our ABP computing is does not necessarily need to be homogeneous.