Three Dimensional SCFT from M2 Branes at Conifold Singularities

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Abstract

Recently it was conjectured that parallel branes at conical singularities are related to string/M theory on $AdS \times X$ where $X$ is an Einstein manifold. In this paper we consider coincident M2 branes near a conifold singularity when M theory is compactified on $AdS_4 \times Q^{1,1,1}$ for $Q^{1,1,1} = (SU(2) \times SU(2) \times SU(2))/\left(U(1) \times U(1)\right)$ as a seven dimensional Sasaki-Einstein manifold. We argue that M theory on $AdS_4 \times Q^{1,1,1}$ can be described in terms of a three dimensional superconformal field theory. We use the fact that the three dimensional self-mirror duality is preserved by exact marginal operators, as observed by Strassler.
1 Introduction

Recently, Maldacena argued that type IIB string theory on $AdS_5 \times S^5$ is equivalent to $\mathcal{N} = 4$ supersymmetric SU(N) gauge theory in four dimensions \[1\] in the ’t Hooft large N limit. The correspondence was made more precise in \[2, 3\] as stating a one to one correspondence between the Green’s functions in type IIB string theory on $AdS_5 \times S^5$ and the correlation functions of gauge invariant operators of the $\mathcal{N} = 4$ supersymmetric field theory.

Two other conjectures were made in \[1\] by relating M theory compactified on $AdS_4 \times S^7$ or $AdS_7 \times S^4$ and $\mathcal{N} = 8$ three dimensional field theory or $(0, 2)$ six dimensional field theory. The correspondence between the Green’s function in M theory and the correlation function of the field theory has been studied in \[4, 5, 6, 7\]. Maldacena’s conjectures have been generalized to theories with less supersymmetries by orbifolding \[8, 9, 10, 11, 12, 13, 14, 15, 16\] and to F theory \[18, 19, 20\].

One interesting generalization is to consider five-dimensional manifolds other than $S^5$ \[21, 22\]. (See also \[23, 29, 30\] for important discussions over these manifolds.) The fact that $S^5$ preserves the maximum amount of supersymmetry while Einstein manifolds does not in general implies that the field theory obtained in this way have fewer supersymmetry. In \[21\] Klebanov and Witten identified the field theory coming from string theory compactified on $AdS_5 \times X_5$ where $X_5$ was taken to be the homogeneous space $T^{1,1} = (SU(2) \times SU(2))/U(1)$. The theory on the worldvolume of D3 branes is $\mathcal{N} = 1$ superconformal field theory in four dimensions. It was also suggested the possibility to extend the result to M theory. In \[22, 23\], Morrison and Plesser investigated M or IIB theory compactified on $AdS_{p+2} \times H^{D-p-2}$ where $H$ are Einstein manifolds obtained as the horizons of the Gorenstein canonical singularities.

In a similar fashion to \[21\], we are going to consider a specific geometry $AdS_4 \times X^7$ where $X^7$ is a seven-dimensional compact Einstein manifold given by $Q^{1,1,1} = (SU(2) \times SU(2) \times SU(2))/(U(1) \times U(1))$. Historically, the nomenclature $Q^{p,q,r}$ manifolds is due to the classification of all Kaluza – Klein
coset $G/H$ seven-manifolds performed by Castellani, Romans and Warner in [24]. This classification both systematized already existing construction and introduced new manifolds. In particular the $Q^{pqr}$ manifolds had already been constructed and their supersymmetry derived by D’Auria, Fre’ and Van Nieuwenhuizen in [27] (see also [25, 26]). The use of $G/H$ Einstein manifolds to generate M2–brane solutions in connection with Conformal Field Theory interpretation has been discussed recently by Cersole et al. in [28].

We will present a Gorenstein canonical singularity $Y$ whose horizon manifold is $Q^{1,1,1}$. Hence $Y$ can be regarded as a limiting space of Calabi-Yau manifolds. We will also find interesting relations with theories with M2 branes placed at an orbifold singularity $S^5/Z_2 \times Z_2$. The homogeneous space $Q^{1,1,1} = (SU(2) \times SU(2) \times SU(2))/(U(1) \times U(1))$ may be obtained by blowing-up the orbifold $S^5/Z_2 \times Z_2$ which can be seen in type IIA as D2 branes in the presence of 2 D6 branes at $R^4/Z_2 \times R^4/Z_2$ singularity or as an elliptic model with D3 branes on a circle together with 2 NS5 branes and 2 D5 branes. This model is self-mirror and we use its self-duality to argue the existence of an exact marginal operator which gives a superconformal field theory.

In section 2 we begin with a brief introduction to Sasaki-Einstein manifolds and provide a Calabi-Yau Gorenstein canonical singularity $Y$ whose horizon is the Sasaki-Einstein manifold $Q^{111}$. In section 3 we show that topologically the horizon $Q^{1,1,1}$ is the same as the complex blow-up of the orbifold $S^5/Z_2 \times Z_2$ as two $S^3$ fibrations over $S^2 \times S^2$, which leads us to construct a field theory inspired by a theory on the complex blow-up of the orbifold $S^5/Z_2 \times Z_2$. In section 4 we discuss a field theory obtained on the brane world-volume placed on the Calabi-Yau Gorenstein canonical singularity $Y$. The field theory discussion was motivated by the recent work of Strassler [33].

2 Near-Horizon Geometries of Cone Branes

Let $(X, g_X)$ be a Riemannian manifold of real dimension $2n - 1$ and $R_+$ be the open half-line $0 < r < \infty$. Then the metric cone $C(X)$ over $X$ is a
Riemannian manifold $\mathbb{R}_+ \times X$ with a metric

$$g_{C(X)} = dr^2 + r^2 g_X$$

(2.1)

We often add one point (which is called the vertex) to $C(X)$ corresponding to the location $r = 0$ and use the same notation $C(X)$ when there is no danger of confusion.

From a relation between Ricci curvature of $X$ and its metric cone $C(X)$, we can show that $X$ is Einstein with positive curvature if and only if then $C(X)$ is Ricci-flat [36, 21]. However $C(X)$ will be metrically singular at the vertex of the cone except the sphere, in which case $C(X)$ is actually flat.

We say that $(X, g_X)$ is Sasakian if the holonomy group of the metric cone $(C(X), g_{C(X)})$ reduces to a subgroup of $U(n)$. In particular, $(C(X), g_{C(X)})$ is Kähler. Therefore, $(X, g_X)$ is Sasaki-Einstein if and only if its metric cone $C(X)$ is Calabi-Yau (Kähler Ricci-flat).

Let $I$ be a parallel complex structure on $C(X)$, i.e. $I$ commutes with the Levi-Civita connection on $C(X)$. Then $\chi := I(\partial r)$ will be a unit Killing vector field on $X$ where $X$ is identified with $X \times \{1\}$. The Killing vector field $\chi$ on $X$ defines a foliation $\mathcal{F}$ whose leaves are the integral curves of $\chi$. $X$ is called a regular Sasakian manifold when these leaves are closed and have the same length. In this case, $\chi$ defines a $U(1)$ action on $X$ and $X$ can be understood as a circle bundle over the orbit space $(2d - 2)$ real dimensional manifold $M$, which is the space of the leaves. The CR structure on $X$ pushes down to give a complex structure on $M$ and the Sasakian condition on $X$ will guarantee that the complex structure will be Kähler. By a version of the Kodaira Embedding Theorem [37], $M$ will be a projective variety. Thus $C(X)$ is a $\mathbb{C}^*$-bundle over a projective variety $M$ and this can be realized as an affine cone over $M$ (which will be defined for a Sasaki-Einstein 7-manifold $Q^{1,1,1}$ later). It also can be shown that a $\mathbb{Q}$-factorial Fano variety. For details, we refer the reader to [34].

The above process can be reversed in the following sense. Consider a complex variety $Y \subset \mathbb{C}^n$ of complex dimension $d$ such that $0 \in Y$. Let
\( \rho : \mathbb{C}^n \to \mathbb{R} \) be the square of the usual distance function, namely,

\[
\rho(y) = |y_1|^2 + \cdots + |y_n|^2
\]

for \( y = (y_1, \ldots, y_n) \in \mathbb{C}^n \). It is easy to see that there exists an \( \epsilon_0 > 0 \) such that all the \((2d-1)\) real dimensional spheres \( S_\epsilon = \rho^{-1}(\epsilon) \) for \( \epsilon \geq \epsilon_0 > 0 \) are transversal to \( Y \) \[35\]. In particular, \( S_\epsilon \cap Y \) is smooth and has the equivalent Riemannian structure for all \( \epsilon \in (0, \epsilon_0) \). We denote it by \( L(Y, 0) \) and call it the link (or horizon) of \( Y \) at 0. In fact, integration along the Euler vector fields \( \rho \partial/\partial \rho \) produces an isometry

\[
\phi : L(Y, 0) \times (0, \epsilon_0) \to (B_{\epsilon_0} \setminus \{0\}) \cap Y
\]

such that \( \rho(\phi(x, t)) = t \) where \( B_{\epsilon_0} = \{x \in \mathbb{C}^n : \rho(x) < \epsilon_0\} \). Hence \( (B_{\epsilon_0} \setminus \{0\}) \cap Y \) is the metric cone over the link \( L(Y, 0) \). The link (or horizon) \( L(Y, 0) \) is Sasaki-Einstein if and only if the singularity \( (Y, 0) \) is Calabi-Yau. Let \( I \) be the complex structure on \( Y \). If \( L(Y, 0) \) is regular, then the Killing vector field \( \chi := I(\rho \partial/\partial \rho) \) defines a foliation whose leaves are circles of the same length. This implies that the \( \mathbb{C}^* \)-action on \( \mathbb{C}^n \) induces an action on \( Y \) and \( M \) is the quotient. Thus \( M \) is a projective variety and \( Y \) is an affine cone over \( M \).

In this paper, we are interested in a regular Sasaki-Einstein 7-manifold \( Q^{1,1,1} \) which is known to be a \( U(1) \) bundle over a threefold \( \mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1 \) with winding number \((1, 1, 1)\). Recall \[27\] that \( Q^{p,q,r} \) is a homogeneous space \((SU(2) \times SU(2) \times SU(2))/(U(1) \times U(1))\) where \( U(1) \times U(1) \) is embedded as follows: Consider an embedding of \( U(1) \times U(1) \times U(1) \) as the standard maximal torus of \((SU(2) \times SU(2) \times SU(2))\). Thus each \( U(1) \) is embedded into \( SU(2) \) via the third Pauli matrix on the Lie algebra level. Let \( h = u(1) \oplus u(1) \) be the Lie subalgebra orthogonal to the Lie subalgebra generated by \( p\sigma_{z,1} + q\sigma_{z,2} + r\sigma_{z,3} \) in the Lie algebra \( u(1) \oplus u(1) \oplus u(1) \) where \( \sigma_{z,i} \) are the generators of \( u(1) \). Then we embed \( h \) into \( su(2) \oplus su(2) \oplus su(2) \) by the embedding described above. Its quotient in the Lie group level is denoted by \( Q^{p,q,r} \). By taking further quotient of \( Q^{p,q,r} \) by the Lie subalgebra generated by \( p\sigma_{z,1} + q\sigma_{z,2} + r\sigma_{z,3} \), one can see that \( Q^{p,q,r} \) is a \( U(1) \)-bundle over \((SU(2)/U(1)) \times (SU(2)/U(1)) \times (SU(2)/U(1))\) which is \( \mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1 \). The complexification of this \( U(1) \)-bundle over \( \mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1 \) will be \( \mathcal{O}_{\mathbb{C}P^1}(-p) \otimes \mathcal{O}_{\mathbb{C}P^1}(-q) \otimes \mathcal{O}_{\mathbb{C}P^1}(-r) \). If we embed \( \mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1 \) by \( \mathcal{O}_{\mathbb{C}P^1}(p) \otimes \mathcal{O}_{\mathbb{C}P^1}(q) \otimes \mathcal{O}_{\mathbb{C}P^1}(r) \) into \( \mathbb{P}(\mathcal{H}^0(\mathcal{O}_{\mathbb{C}P^1}(p) \otimes \mathcal{O}_{\mathbb{C}P^1}(q) \otimes \mathcal{O}_{\mathbb{C}P^1}(r))) \)
and take the affine cone over the image, the vertex will be singular and the exceptional divisor over the vertex in the blowing-up will be $\mathcal{O}_{\mathbb{C}P^1}(-p) \otimes \mathcal{O}_{\mathbb{C}P^1}(-q) \otimes \mathcal{O}_{\mathbb{C}P^1}(-r)$.

We will provide more details for $Q^1.1.1$. Consider the following embedding

$$\sigma : \mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1 \hookrightarrow \mathbb{C}P^7 \quad (2.4)$$

by an very ample line bundle $\mathcal{O}_{\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1}(1, 1, 1)$, which can be expressed as a map sending

$$(s : t) \times (u : v) \times (w : x) \rightarrow (suw : sux : svw : svx : tuw : tvx : tvw : tvx) \quad (2.5)$$

in terms of the tri-homogeneous coordinates of $\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$ and the homogeneous coordinates of $\mathbb{C}P^7$. We denote the image by

$$Z := \sigma(\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1) \quad (2.6)$$

Let $q : \mathbb{C}^8 \setminus \{0\} \rightarrow \mathbb{C}P^7$ be the natural quotient map of $\mathbb{C}^8 \setminus \{0\}$ by the $\mathbb{C}^*$-action. Then the affine cone over $Z$

$$Y := q^{-1}(Z) \cup \{0\} \quad (2.7)$$

is the desired singularity of a Calabi-Yau fourfold. Thus $Y$ in $\mathbb{C}^8$ is a zero locus of the ideal $I$ generated by the kernel of the map

$$\sigma^* : \mathbb{C}[z_0, z_1, \ldots, z_7] \rightarrow \mathbb{C}[s, t, u, v, w, x], \quad (2.8)$$

$$\sigma^*(z_0) = suw, \sigma^*(z_1) = sux, \sigma^*(z_2) = svw, \sigma^*(z_3) = svx,$$

$$\sigma^*(z_4) = tuw, \sigma^*(z_5) = tvx, \sigma^*(z_6) = tvw, \sigma^*(z_7) = tvx.$$

One can compute the ideal $I$ (for example, using Gröbner basis) and hence one can show that $Y$ is defined by the equations

$$z_0 z_3 - z_1 z_2 = z_0 z_5 - z_1 z_4 = z_0 z_6 - z_2 z_4 = 0, \quad (2.9)$$

$$z_0 z_7 - z_1 z_6 = z_0 z_7 - z_2 z_5 = z_0 z_7 - z_3 z_4 = 0,$$

$$z_1 z_7 - z_3 z_5 = z_2 z_7 - z_3 z_6 = z_4 z_7 - z_5 z_6 = 0.$$

One can see that the ideal $I$ can be generated by the quadrics from the following minimal free resolution of $\mathcal{O}_Z$:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^7}(-6) \rightarrow \bigoplus_{9} \mathcal{O}_{\mathbb{P}^7}(-4) \rightarrow \bigoplus_{16} \mathcal{O}_{\mathbb{P}^7}(-3) \rightarrow \bigoplus_{9} \mathcal{O}_{\mathbb{P}^7}(-2) \rightarrow \mathcal{O}_{\mathbb{P}^7} \rightarrow \mathcal{O}_Z \rightarrow 0. \quad (2.10)$$
It is easy to check all possible quadric generators for the ideal $I$. Now we want to show that $Y$ has a Gorenstein canonical singularity. This is needed to satisfy a requirement that $Y$ be at finite distance with respect to the natural Weil-Peterson metric on the moduli of Calabi-Yau manifolds as the limiting space in it [32, 39, 40]. Since $Z$ is projectively normal, $Y$ is normal. Thus $Y$ is toric because there is a $U(1)^4$-action on $Y$. (You may also explicitly compute the toric fan $\Delta$ which realizes this toric embedding. The fan $\Delta$ is nothing but a dual cone $\sigma$ in $\mathbb{Z}^6$ of the convex rational polyhedral cone $\tilde{\sigma}$ generated by $(1,0,1,0,1,0), (1,0,1,0,0,1), (1,0,0,1,1,0), (1,0,0,1,0,1), (0,1,1,0,1,0), (0,1,1,0,0,1), (0,1,0,1,1,0), (0,1,0,1,0,1)$ corresponding to the monomials which appears in (2.8). We refer to [41, 42] for a review of toric geometry.) Since the vertex of $Y$ has a rational singularity (again because it is toric), $Y$ has a canonical singularity when it is Gorenstein [38]. One can prove that $Y$ is Gorenstein by computing the dualizing sheaf $\omega_Y$ of $Y$ (which is the same as the bundle of holomorphic 4-forms on the smooth part of $Y$) using the fact that $Z$ is subcanonical. Note that since $Y$ is an affine toric variety, the dualizing sheaf $\omega_Y$ is in fact trivial which means that there is a non-vanishing holomorphic 4-form on $Y$. (You can also show this fact by a direct computation of $\omega_Y$ using the resolution (2.10).) As it is stated in [32], this means that there exists a non-vanishing holomorphic 4-form on $Y$ which extends to a holomorphic 4-form on any smooth resolution of $Y$.

In this setting, $Q^{1,1,1} = L(Y,0)$. Thus $Q^{1,1,1}$ can be taken to be the intersection of $Y$ with the unit sphere in $\mathbb{C}^8$:

$$|z_0|^2 + |z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 + |z_5|^2 + |z_6|^2 + |z_7|^2 = 1. \quad (2.11)$$

From the way the equations for $Y$ are derived, it is clear that the equations for $Y$ can be ‘solved’ by setting

$$z_0 = SC_1D_1, z_1 = SC_1D_2, z_2 = SC_2D_1, z_3 = SC_2D_2, \quad (2.12)$$
$$z_4 = TC_1D_1, z_5 = TC_1D_2, z_6 = TC_2D_1, z_7 = TC_2D_2. \quad (2.13)$$

where $(S,T), (C_1, C_2), (D_1, D_2)$ are the pairs of homogeneous coordinates for the three $\mathbb{C}P^1$ spaces. It is easy to check that this choice for $z_i$ satisfies equations (2.9).
If we write now $S = A_2B_1$ and $T = A_1B_2$, they are invariant under

$$(A_1, A_2) \rightarrow \lambda(A_1, A_2) \quad (B_1, B_2) \rightarrow \lambda^{-1}(B_1, B_2) \quad (2.13)$$

The $z_i$ are invariant under

$$(C_1, C_2) \rightarrow \gamma(C_1, C_2) \quad (D_1, D_2) \rightarrow \gamma^{-1}(D_1, D_2) \quad (2.14)$$

$\lambda$ and $\gamma$ are two complex numbers.

We can now choose the real parts of $\lambda$ and $\gamma$ to set:

$$|B_1|^2 + |C_1|^2 + |C_2|^2 = |A_1|^2 + |D_1|^2 + |D_2|^2 \quad (2.15)$$

and

$$|B_2|^2 + |C_1|^2 + |C_2|^2 = |A_2|^2 + |D_1|^2 + |D_2|^2 \quad (2.16)$$

To identify the manifold $Q^{1,1,1}$, we set $|C_1|^2 + |C_2|^2 = |D_1|^2 + |D_2|^2 = |S|^2 + |T|^2 = 1$ so the real isometry group is $SU(2) \times SU(2) \times SU(2)$ and this is to be divided by the angular parts of (2.13) and (2.14) which give 2 $U(1)$ groups so finally we obtain $Q^{1,1,1} = (SU(2) \times SU(2) \times SU(2))/(U(1) \times U(1))$.

Our goal is to find the $\mathcal{N} = 2$ superconformal theory which is dual to the M theory compactified on $AdS_4 \times Q^{1,1,1}$, seen as the infrared limit of the theory of $N$ coincident M2 branes placed at a conifold singularity of $M_3 \times Y$.

3 Comparison to an $\mathbb{R}^4/\mathbb{Z}_2 \times \mathbb{R}^4/\mathbb{Z}_2$ orbifold.

In order to perform an important check over our theory, we compare the conifold to a $\mathbb{R}^4/\mathbb{Z}_2 \times \mathbb{R}^4/\mathbb{Z}_2$ orbifold background. Consider an action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ on $\mathbb{R}^8$ as follows:

$$(1, 0) \cdot (x_1, \ldots, x_8) = (-x_1, -x_2, -x_3, -x_4, x_5, x_6, x_7, x_8) \quad (3.17)$$

$$(0, 1) \cdot (x_1, \ldots, x_8) = (x_1, x_2, x_3, x_4, -x_5, -x_6, -x_7, -x_8) \quad (3.18)$$
where \((1, 0)\) (resp. \((0, 1)\)) is the generator of the first (resp. second) factor of the group \(\mathbb{Z}_2 \times \mathbb{Z}_2\) and \(x_1, \ldots, x_n\) are the coordinates of \(\mathbb{R}^8\). This will induce an action on \(S^7\) in \(\mathbb{R}^8\) given by an equation \(x_1^2 + x_2^2 + \cdots + x_7^2 = 1\). The twisted sector mode of string theory on \(AdS_3 \times S^7/\mathbb{Z}_2 \times \mathbb{Z}_2\) is the blowup of the orbifold singularity of \(S^7/\mathbb{Z}_2 \times \mathbb{Z}_2\).

In order to understand the geometry of this blowup, we study the blowup of an orbifold singularity \(\mathbb{R}^8/\mathbb{Z}_2 \times \mathbb{Z}_2\) in the complex sense via identification of \(\mathbb{C}^4\) with \(\mathbb{R}^8\). First note that

\[
\mathbb{R}^8/\mathbb{Z}_2 \times \mathbb{Z}_2 \cong \mathbb{R}^4/\mathbb{Z}_2 \times \mathbb{R}^4/\mathbb{Z}_2 \quad (3.19)
\]

where \(\mathbb{Z}_2\) acts on \(\mathbb{R}^4\) by \(-1\). Let

\[
\pi : Bl(\mathbb{R}^8/\mathbb{Z}_2 \times \mathbb{Z}_2) \rightarrow \mathbb{R}^8/\mathbb{Z}_2 \times \mathbb{Z}_2 \quad (3.20)
\]

\[
\pi' : Bl(\mathbb{R}^4/\mathbb{Z}_2) \rightarrow \mathbb{R}^4/\mathbb{Z}_2 \quad (3.21)
\]

be the complex blowups of the orbifold singularities \(\mathbb{R}^8/\mathbb{Z}_2 \times \mathbb{Z}_2\) and \(\mathbb{R}^4/\mathbb{Z}_2\) respectively. By (3.19), we have

\[
Bl(\mathbb{R}^8/\mathbb{Z}_2 \times \mathbb{Z}_2) \cong Bl(\mathbb{R}^4/\mathbb{Z}_2) \times Bl(\mathbb{R}^4/\mathbb{Z}_2). \quad (3.22)
\]

The space \(Bl(\mathbb{R}^4/\mathbb{Z}_2)\) can be regarded as the total space of a line bundle \(\mathcal{O}_{\mathbb{C}P^1}(-1)\). Hence it is a complex line bundle over \(S^2\). Therefore there is a vector bundle map

\[
q : Bl(\mathbb{R}^8/\mathbb{Z}_2 \times \mathbb{Z}_2) \rightarrow S^2 \times S^2 \quad (3.23)
\]

with fibers \(\mathbb{R}^4\). We define the complex blowup of the orbifold singularity \(S^7/\mathbb{Z}_2 \times \mathbb{Z}_2\) via the map \(\pi\) in (3.20):

\[
Bl(S^7/\mathbb{Z}_2 \times \mathbb{Z}_2) = \pi^{-1}(S^7/\mathbb{Z}_2 \times \mathbb{Z}_2). \quad (3.24)
\]

Thus we have the following diagram:
Now we study the complex blowup $Bl(S^7/\mathbb{Z}_2 \times \mathbb{Z}_2)$. There is a map

$$q_S : Bl(S^7/\mathbb{Z}_2 \times \mathbb{Z}_2) \to S^2 \times S^2$$

(3.25)

which is a composition of the natural inclusion $i_{Bl}$ and the map $q$ in (3.23). Since $i_{Bl}$ and $q$ are transversal, the map $q_S$ in (3.25) is a smooth fibration and it is easy to see the fiber is $S^3$. In a summary, we have the following fibrations:

$$\begin{array}{ccc}
Bl(S^7/\mathbb{Z}_2 \times \mathbb{Z}_2) & \mathcal{C} & i_{Bl} & Bl(R^8/\mathbb{Z}_2 \times \mathbb{Z}_2) \\
S^3 & \mathcal{C} & & R^4 \\
& \longrightarrow & & S^2 \times S^2
\end{array}$$

Thus $Bl(S^7/\mathbb{Z}_2 \times \mathbb{Z}_2)$ is an $S^3$ bundle over $S^2 \times S^2$. Now we claim that both $Q^{1,1,1}$ and $Bl(S^7/\mathbb{Z}_2 \times \mathbb{Z}_2)$ are topologically trivial $S^3$ bundles over $S^2 \times S^2$. In order to show that $Bl(S^7/\mathbb{Z}_2 \times \mathbb{Z}_2)$ is topologically trivial $S^3$ bundle, note that the blow-up $Bl(S^7/\mathbb{Z}_2 \times \mathbb{Z}_2)$ can be achieved in two steps. First we consider a blow-up $Bl(S^7/\mathbb{Z}_2)$, where $\mathbb{Z}_2$ acts only on the first four coordinates by $-1$. Then the situation is the same as in [21] except we have two more extra coordinates. Thus it will be a trivial $S^5$ bundle over $S^2$. Now we further blow-up the space $Bl(S^7/\mathbb{Z}_2)/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts on the last four coordinates. This will act on the fiber $S^5$ and it will be the trivial bundle over $S^2$. Since the space $Bl(S^7/\mathbb{Z}_2 \times \mathbb{Z}_2)$ is the same as the space we obtain by the success of two blow-ups we described, we can conclude that $Bl(S^7/\mathbb{Z}_2 \times \mathbb{Z}_2)$ is topologically $S^3 \times S^2 \times S^2$. In the case of $Q^{1,1,1} = (SU(2) \times SU(2) \times SU(2))/(U(1) \times U(1))$, the triviality can be proved by observing that the ‘forgetting’ map of the last $SU(2)$ will give a fibration of $S^3$ over $S^2 \times S^2$, which is trivial.
4 $\mathcal{N} = 2$ field theory in 3 dimensions

In this section we are going to build the field theory corresponding to the M theory compactified on the conifold $Y$. In order to do this we are going to use the above identification of both $Q^{1,1,1}$ and the blow-up of the orbifold $Bl(S^7/Z_2 \times Z_2)$ as $S^3$ bundles over $S^2 \times S^2$.

 Orbifold theory

We begin by briefly reviewing the discussion of [21]. Klebanov and Witten considered $AdS_5 \times S^5/Z_2$ background where the $Z_2$ acts on four of the six coordinates of $S^5$. In terms of branes, this corresponds to D3 branes moving on a $C^2/Z_2 \times C$ space which after taking a T-duality on one of the directions of $C^2/Z_2$ gives an elliptic model with D4 branes between NS5 branes. The gauge group is $SU(N) \times SU(N)$, at each NS5 branes we have an $(N, \bar{N})$ field. The rotation of NS5 branes correspond to adding a mass for the adjoint field which gives a $N = 1$ theory in which the $(N, \bar{N})$ and $(\bar{N}, N)$ fields correspond to two $(N, \bar{N})$ fields and two $(\bar{N}, N)$ fields. There is one SU(2) group acting on the two $(N, \bar{N})$ fields and one SU(2) group acting on the two $(\bar{N}, N)$ fields.

We use the same argument in our case where we start with M2 branes on $C^2/Z_2 \times C^2/Z_2$ space. We want to go to type IIB string theory in 10 dimension. We consider this two $A_1$ singularity to be given by a $Z_2$ orbifold combined with two D6 branes. In 11 dimensions we start with 2 types of KK monopoles, say in the $(x^3, x^4, x^5, x^6)$ directions and $(x^7, x^8, x^9, x^{10})$ directions respectively, and with M2 branes in the $(x^1, x^2)$ directions. Reduction to 10 dimensions gives KK monopoles in the $(x^3, x^4, x^5, x^6)$ directions, D6 branes in $(x^1, x^2, x^3, x^4, x^5, x^6)$ directions and D2 branes in the $(x^1, x^2)$ directions. If the $x^3$ direction is compact, we make a T - duality with respect to it and we obtain

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*The field theory discussion was suggested to us by Matthew Strassler and we refer to [6] for a detailed and interesting discussion about exactly marginal operators in three dimensions.

†This choice for orbifold was suggested to us by Matthew Strassler who pointed out that by making this choice we would obtain self mirror-symmetrical models where we could have a nice identification of marginal operators.
a configuration with D3 branes on \((x^1, x^2, x^3)\) directions, D5 branes in the \((x^1, x^2, x^4, x^5, x^6)\) directions and NS5 branes in \((x^1, x^2, x^7, x^8, x^9)\) directions.

The configuration we obtain is an elliptic, self mirror model having \(N\) D3 branes on a circle intersecting two non-coincident D5 branes and two non-coincident NS branes, with the above orientation. The theory obtained is an \(\mathcal{N} = 4\) supersymmetric field theory with a gauge group \(U(N) \times U(N)\) and matter given by one field \(A_1\) in the \((\bar{N}, 1)\) representation, one field \(B_1\) in the \((N, 1)\) representation, 1 field \(A_2\) in the \((1, \bar{N})\) representation, 1 field \(B_2\) in the \((1, N)\) representation and two fields \(C_i, D_i\) in the \((\bar{N}, N)\) representation where \(i = 1, 2\) (we discuss the fields in the \(\mathcal{N} = 2\) language). The model also contains fields in the adjoint representation of the two \(U(N)\), denoted by \(\Phi\) and \(\bar{\Phi}\). The \(A_i, B_i\) fields are given by strings with one end on D3 and one end on either of the two D5 branes, the \(C_i, D_i\) fields are given by strings with both ends on the D3 branes stretched across either of the two NS branes. The superpotential is

\[
g(A_1\Phi B_1 + A_2\bar{\Phi} B_2) + g\text{Tr}[\Phi(C_1 D_1 + C_2 D_2)] + g\text{Tr}[\bar{\Phi}(D_1 C_1 + D_2 C_2)]
\] (4.1)

To arrive to a \(\mathcal{N} = 2\) theory, we add a mass term for the \(\Phi\) and \(\bar{\Phi}\). This corresponds to rotating the NS branes in the \((x^5, x^6, x^8, x^9)\) directions. But the rotation of the D5 brane induces a change in the coupling between the hyper-multiplets A and B and the adjoint multiplets. In order to preserve the self-duality, we need to rotate the NS5 branes and the D5 branes by the same angle. In other words, we need to add the following to the superpotential:

\[
\frac{m}{2}(\text{Tr}(\Phi^2 + \bar{\Phi}^2) + \text{Tr}(A_1 B_1 - A_2 B_2)^2 + \text{Tr}(C_1 D_1 - C_2 D_2)^2)
\] (4.2)

In equation (4.2), \(m\) is the mass of the adjoint field and the tangent of the rotation angle by which the D5 branes are rotated. If we add all the above discussed terms and integrate out the adjoint field \(\Phi\), the following superpotential is obtained:

\[
g^2 \frac{2m}{2m} [\epsilon^{ij} \epsilon^{kl} \text{Tr}(A_i B_k A_j B_l) + \epsilon^{ij} \epsilon^{mn} \text{Tr}(C_i D_m C_j D_n)]
\] (4.3)

This is the term obtained by rotating all the branes by the same angle, considering the terms which appear and then integrating out the massive
fields (only the adjoint field acquires mass). Before rotation, the theory was $\mathcal{N} = 4$ and we could exchange the positions of the NS and D5 branes. After rotation, in order to be able to be allowed to do that, all the branes need to be rotated by the same angle. This implies that all the terms in (4.2) are needed in order to recover a self-dual theory. As explained in [33] we actually require that the $\mathbb{Z}_2$ symmetry exchanging the two $\mathbb{Z}_2$ orbifolds is preserve which determines a marginal deformation from one self-dual model to another. The theory has a line of fixed points which contains the $\mathcal{N} = 4$ theory.

So, the field theory obtained by taking M theory on $S^7/\mathbb{Z}_2 \times \mathbb{Z}_2$ is in our case equivalent to the $\mathcal{N} = 2$ three dimensional theory obtained by adding the superpotential (4.3) to the $U(N) \times U(N)$ gauge theory with the matter content given by the fields A, B, C, D.

We want to make an important observation here. The brane configuration described is invariant under an S-duality which replaces the NS branes by D5 branes, the D5 branes by NS branes and leaves the D3 branes invariant. This S-duality replaces the A and B fields by C and D fields and vice-versa. In equation (4.1), $C_i$ and $D_i$ appear in the first term and $A_i$ and $B_i$ appear in the second term. Equation (4.2) remains invariant under exchanging of $A$, $B$ and $C$, $D$ fields. By remembering that the two $U(1)$ groups of the conifold act on pairs $A, B$ and $C, D$, the S-duality just inverts this two $U(1)$ groups. One of the $SU(2)$ groups will act upon $(A_1, A_2)$, one on $(B_1, B_2)$ and the third one on $S' = C_2D_1, T' = C_1D_2$.

Conifold Theory

We are discussing the field theory as obtained by starting from the conifold theory.

The parameterization of the conifold is given in terms of the fields $S, T, C_iD_i$ as done in section 2. As done at the end of section 2, we consider $S = A_2B_1, T = A_1B_2$. We consider a $U(1)$ gauge theory with $\mathcal{N} = 2$ in three dimensions and introduce $C_1, D_1$ as chiral multiplets with charge 1 and $C_2, D_2$
as chiral multiplets with charge -1. The fields $A_2, B_2$ are neutral under this gauge group. We have equation (2.15):

$$|B_1|^2 + |C_1|^2 + |C_2|^2 = |A_1|^2 + |D_1|^2 + |D_2|^2$$

(4.4)

But this is just the D auxiliary field of the U(1) vector multiplet, by considering terms which do not involve the adjoint scalar obtained after reduction from $\mathcal{N} = 1, D = 4$, so the moduli space of vacua is part of the conifold. If we consider now another one U(1) gauge theory with $\mathcal{N} = 2$ in three dimensions and introduce $B_2, C_1, D_1$ as chiral multiplets with charge 1 and $C_2, D_2, A_2$ with charge -1. The fields $A_1, B_1$ are neutral under this gauge group. We have equation (2.16):

$$|B_2|^2 + |D_1|^2 + |D_2|^2 = |A_2|^2 + |C_1|^2 + |C_2|^2$$

(4.5)

which can be again interpreted as a D auxiliary field of the U(1) vector multiplet. In the paper [21], one of the two gauge groups lives on the worldvolume of their D3 branes so the conifold is identified with the moduli space of vacua of one of the U(1) groups. Here we do not have any U(1) group on the two-branes worldvolume. So we need to identify the moduli spaces of both U(1) groups with the two branches of the conifold (2.15), (2.16). It is now clear why do we need two equations for the conifold as compared with [21] where there is only one.

The result is that we have a theory with the gauge group $U(1) \times U(1)$ with chiral multiplets $A_1, B_1, A_2, B_2$ and $C_1, C_2$ and $D_1, D_2$ with charges (-1, 0), (1, 0), (0, -1), (0, 1), (1, -1), (-1, 1) respectively. The chiral multiplets describe the M2 brane motion on the conifold. The model can be considered to describe the low energy behavior of a threebrane on $M_3 \times Y$.

If we have $N$ M2 branes the gauge theory is generalized to a $U(N) \times U(N)$ gauge theory with the same field content as for the orbifold discussion. Now the chiral fields $C_i, D_i$ are matrices, the chiral fields $A_i$ are row vectors and $B_i$ are column vectors. It is then natural to use the $S, T$ fields in order to have all the fields given by matrices. If all the matrices are diagonal, the diagonal entries of $C_i, D_i$ give us the positions of the $N$ M2 branes at distinct points on the conifold. The gauge group is broken to a product of U(1) factors. But there are the extra-diagonal entries which need to be given
masses in order to integrate out the unwanted massless chiral multiplets and we are going to do it by introducing a superpotential that does so. The superpotential should preserve the symmetry of the conifold \( Y \) i.e. \( SU(2) \times SU(2) \times SU(2) \times U(1)_R \) symmetry. The \( U(1)_R \) is the R-symmetry inherited by the \( \mathcal{N} = 2 \) three dimensional theory from its reduction from 4 dimensions. All the fields \( A_i, B_i, C_i, D_i \) have charge 1/2 and the fields \( S, T \) have charge 1. A superpotential \( SU(2) \times SU(2) \times SU(2) \) invariant and having \( U(1)_R \) charge 2 can be written as:

\[
W = \lambda (\text{Tr}(ST - TS) + \epsilon^{ij} \epsilon^{kl} \text{Tr}(C_i D_k C_j D_l))
\] (4.6)

where all the fields are now \( N \times N \) matrices. If we now use the definition of \( S \) and \( T \), we can rewrite the superpotential as:

\[
\frac{g^2}{2m} \left[ \epsilon^{ij} \epsilon^{kl} \text{Tr}(A_i B_k A_j B_l) + \epsilon^{ij} \epsilon^{mn} \text{Tr}(C_i D_m C_j D_n) \right]
\] (4.7)

where the products \( A_i B_k \) are to be understood as \( N \times N \) matrices. This has charge 2 and is \( SU(2) \times SU(2) \times SU(2) \) invariant. The off diagonal components receive mass from the superpotential (plus Higgs mechanism).

We observe that the above superpotential is the same with the one of (4.3). This is a marginal operator which takes us from a conformal theory to a new conformal field theory.

We now have the ingredients to state the result of this paper:

\textit{M theory on } \( \text{AdS}_4 \times Q^{1,1,1} \text{ is equivalent to the theory obtained by starting with } U(N) \times U(N) \text{ theory with two copies of } (N, \bar{N}) \oplus (\bar{N}, N) \text{ and four fields in } (N, 1), (\bar{N}, 1), (1, N), (1, \bar{N}) \text{ flowing to an infrared fixed point and then perturbed by the potential (4.3).} \)

We end this section with a discussion over the conifold description of the configuration which is the S-dual of the one considered before. By remembering that the two \( U(1) \) groups of the conifold act on pairs \( A, B \) and \( C, D \), the S-duality just inverts this two \( U(1) \) groups. One of the \( SU(2) \) groups will act upon \( (A_1, A_2) \), one on \( (B_1, B_2) \) and the third one on \( S' = C_2 D_1, T' = C_1 D_2 \).
As a conclusion, in this paper we extended the idea of comparing string/M
theory at conifold singularities and superconformal field theories to the case
of $AdS_4 \times Q^{1,1,1}$ which gives superconformal field theory on three dimensions.
One important development is the one also described in [21] i.e. the case of
$AdS_4 \times V_{5,2}$ where $V_{5,2}$ is the seven dimensional Einstein homogeneous space
$SO(5)/SO(3)$ obtained as a link (or horizon) of the singularity of a quadric
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