Multipolar radiation and the gauge invariant reduction of multipole tensors

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Abstract. Compact general formulae for the energy, momentum and angular momentum radiated by confined systems of charges and currents are presented in terms of their multipole Cartesian tensors. Besides the usual electric and magnetic multipoles, a family of toroid (anapole) electromagnetic tensors, as well as some mean squared radii contributions are standout through a method given in previous works of the authors.

1. Introduction

The purpose of the present paper is to obtain general formulae for the energy, momentum and angular momentum loss in the case of a bounded charge and current distribution, in the form of multipolar expansions, using only fully symmetric and trace-free (STF) electric, magnetic and anapole (toroidal) multipole tensors, together with some mean squared radii. The gauge invariant technique of reducing the usual Cartesian electric and magnetic multipole moments to STF ones, presented in [2], is used for this end. In the section 2 the multipolar expansions of the scalar and vector potentials, $\Phi$ and $A$, are presented in terms of electric and magnetic Cartesian multipole moments. In section 3 the corresponding expansions of the electric and magnetic fields, $E$ and $B$, are given. The relevant terms for the purposes of this paper are only those that fall off as fast as $1/r$ and $1/r^2$ as $r \to \infty$ given in equations (11) and (12). In section 4 the total radiation intensity (energy loss) is calculated and given by equation (19) in the form of an expansion in terms of STF multipole moments. The same calculation is performed in section 5 and 6 respectively for the total momentum loss and angular momentum loss. In section 7, the approximation’s criteria are discussed and, finally, one presents explicit expressions for the radiated energy, momentum and angular momentum including all the electric and magnetic multipoles up to order three (octopoles). These results are compared with some incomplete expressions existing in literature.

In appendix A one can find the definition formulae for the reduced electric and magnetic multipole moments.
2. General Formulae

We shall accept in the following a compromise regarding the unit systems used for writing Maxwell’s equations and the different definitions and equations for electromagnetic theory. Let the Maxwell’s equations be written as

\[
\nabla \times B = \frac{\mu_0}{\alpha} j + \frac{\varepsilon_0 \mu_0}{\alpha} \frac{\partial E}{\partial t}, \quad \nabla \times E = -\frac{1}{\alpha} \frac{\partial B}{\partial t}
\]

\[
\nabla B = 0, \quad \nabla \cdot E = \frac{1}{\varepsilon_0 \mu_0} \rho \tag{1}
\]

where the positive parameters \(\varepsilon_0, \mu_0, \alpha\) are constants depending on the system of units. Considering the consequences of Maxwell equations, from physical reasons, only two independent constants will be needed because of the constraint

\[
\frac{\alpha^2}{\varepsilon_0 \mu_0} = c^2 \tag{2}
\]

c standing for the speed of light in vacuum.

If the Maxwell equations are written in the form above, than the fields \(E, B\) and the potentials \(\Phi, A\) are related by the equations

\[
B = \nabla \times A, \quad E = -\nabla \Phi - \frac{1}{\alpha} \frac{\partial A}{\partial t} \tag{3}
\]

while the retarded potentials are given by

\[
A(r, t) = \frac{\mu_0}{4\pi \alpha} \int_D \frac{j(\xi, t - R/c) \, d\xi^3}{R}, \quad \Phi(r, t) = \frac{1}{4\pi \varepsilon_0} \int_D \frac{\rho(\xi, t - R/c) \, d\xi^3}{R}, \quad R = |r - \xi|.
\]

One can write the equations and formulae above in the International unit system by taking \(\alpha = 1\) and the IS values for \(\varepsilon_0, \mu_0\). The same relations can be written in Heaviside-Lorentz unit system by taking \(\varepsilon_0 = 1, \mu_0 = 1, \alpha = c\) in cgs units while in Gauss unit system \(\varepsilon_0 = 1/4\pi, \mu_0 = 4\pi, \alpha = c\).

The goal of this compromise is to compare easily the results from the present paper with similar results in different issues written either in IS or in Gauss or Heaviside-Lorentz units.

Let us consider a charge \(\rho(r, t)\) and a current \(j(r, t)\) distributions having supports included in a finite domain \(D\). Choosing the origin \(O\) of the Cartesian axes in \(D\), the retarded vector and scalar potentials at a point outside \(D, r = x_i e_i\), will be given by the multipolar expansions

\[
\frac{4\pi}{\mu_0} A(r, t) = \nabla \times \sum_{n \geq 1} \frac{(-1)^{n-1}}{n!} \nabla^{n-1} \cdot \left[ \frac{1}{r} M^{(n)}(t_0) \right] + \frac{1}{\alpha} \sum_{n \geq 1} \frac{(-1)^{n-1}}{n!} \nabla^{n-1} \cdot \left[ \frac{1}{r} \dot{P}^{(n)}(t_0) \right]
\]

\[
= e_i \varepsilon_{ijk} \partial_j \sum_{n \geq 1} \frac{(-1)^{n-1}}{n!} \partial_{i_1} \ldots \partial_{i_{n-1}} \left[ M_{i_1 \ldots i_{n-1} k}(t_0) \right] r + \frac{1}{\alpha} e_i \sum_{n \geq 1} \frac{(-1)^{n-1}}{n!} \partial_{i_1} \ldots \partial_{i_{n-1}} \left[ \dot{P}_{i_1 \ldots i_{n-1}}(t_0) \right] r, \quad t_0 = t - \frac{r}{c}, \tag{4}
\]
The following formula can be easily obtained:

\[ 4\pi\varepsilon_0\Phi(r, t) = \sum_{n \geq 0} \frac{(-1)^n}{n!} \nabla^n \cdot \left[ \frac{P^{(n)}(t_0)}{r} \right] = \sum_{n \geq 0} \frac{(-1)^n}{n!} \partial_{i_1} \ldots \partial_{i_n} \left[ \frac{P_{i_1 \ldots i_n}(t_0)}{r} \right]. \]

The electric and magnetic moments are defined as

\[ \mathbf{P}^{(n)}(t) = \int_{\mathcal{D}} \xi^n \rho(\xi, t) d^3\xi : \mathbf{P}_{i_1 \ldots i_n} = \int_{\mathcal{D}} \xi_{i_1} \ldots \xi_{i_n} \rho(\xi, t) d^3\xi \]

\[ \mathbf{M}^{(n)}(t) = \frac{n}{(n+1)\alpha} \int_{\mathcal{D}} \xi^n \times \mathbf{j}(\xi, t) d^3\xi : \mathbf{M}_{i_1 \ldots i_n} = \frac{n}{(n+1)\alpha} \int_{\mathcal{D}} \xi_{i_1} \ldots \xi_{i_{n-1}} (\xi \times \mathbf{j})_{i_n} d^3\xi \]

It was shown in [2] that we can perform such transformations of the multipole tensors

\[ \mathbf{P}^{(n)} \rightarrow \mathbf{\tilde{P}}^{(n)}, \quad \mathbf{M}^{(n)} \rightarrow \mathbf{\tilde{M}}^{(n)}, \]

(where \( \mathbf{\tilde{P}}^{(n)}, \mathbf{\tilde{M}}^{(n)} \) are STF tensors) so that, if \( \mathbf{\tilde{A}} \) and \( \mathbf{\tilde{\Phi}} \) are obtained from equations (4) after the substitutions (6), the correspondence \( (\mathbf{A}, \Phi) \rightarrow (\mathbf{\tilde{A}}, \mathbf{\tilde{\Phi}}) \) is a gauge transformation.

In Appendix A are presented the principal results of the reducing procedure. In the following we will suppose that all the multipole moments are represented by reduced STF tensors.

### 3. Expansions of the fields \( \mathbf{E} \) and \( \mathbf{B} \)

Let us write the multipole expansion of \( \mathbf{B} = \nabla \times \mathbf{A} \). Use of equation (4) and formulae

\[ \nabla \times (\nabla \times \mathbf{a}) = \nabla \cdot (\nabla \cdot \mathbf{a}) - \Delta \mathbf{a} \]

\[ [\Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}] \mathbf{f}(t - r/c) / r = 0, \text{ for } r \neq 0, \]

leads to the expansion

\[ \mathbf{B}(r, t) = \frac{\mu_0}{4\pi} \sum_{n \geq 1} \frac{(-1)^{n-1}}{n!} \nabla \cdot \left[ \nabla^n \cdot \left[ \frac{1}{r} \mathbf{\tilde{M}}^{(n)}(t_0) \right] \right] \]

\[ - \frac{\mu_0}{4\pi c^2} \sum_{n \geq 1} \frac{(-1)^{n-1}}{n!} \nabla^n \cdot \left[ \frac{1}{r} \mathbf{\tilde{M}}^{(n)}(t_0) \right] + \frac{\mu_0}{4\pi \alpha} \nabla \times \sum_{n \geq 1} \frac{(-1)^{n-1}}{n!} \nabla^n \cdot \left[ \frac{1}{r} \mathbf{\tilde{P}}^{(n)}(t_0) \right], \]

\[ t_0 = t - \frac{r}{c}. \]

For \( \mathbf{E} \) we get

\[ \mathbf{E}(r, t) = -\frac{\alpha}{4\pi\varepsilon_0 c^2} \nabla \times \sum_{n \geq 1} \frac{(-1)^{n-1}}{n!} \nabla^n \cdot \left[ \frac{1}{r} \mathbf{\tilde{M}}^{(n)}(t_0) \right] \]

\[ - \frac{1}{4\pi\varepsilon_0 c^2} \sum_{n \geq 1} \frac{(-1)^{n-1}}{n!} \nabla^n \cdot \left[ \frac{1}{r} \mathbf{\tilde{P}}^{(n)}(t_0) \right] - \frac{1}{4\pi\varepsilon_0} \nabla \sum_{n \geq 1} \frac{(-1)^{n-1}}{n!} \nabla^n \cdot \left[ \frac{1}{r} \mathbf{\tilde{M}}^{(n)}(t_0) \right] \]

In order to obtain the behavior of the fields \( \mathbf{E} \) and \( \mathbf{B} \) at large distances, we need the formula for the partial derivatives of a retarded arbitrary solution of the wave equation. The following formula can be easily obtained:

\[ \partial_{i_1} \ldots \partial_{i_n} \frac{1}{r} f(t_0) = \frac{1}{r} \frac{(-1)^n}{e^n} \nu_{i_1} \ldots \nu_{i_n} \frac{d^n f(t_0)}{dt^n} \]

\[ + \frac{1}{r^2} \frac{(-1)^n}{e^{n-1}} \left[ D_n \nu_{i_1} \ldots \nu_{i_n} \frac{d^{n-1} f(t_0)}{dt^{n-1}} - \nu_{\{i_1} \ldots \nu_{i_{n-2}} \delta_{i_{n-1}i_n} \frac{d^{n-1} f(t_0)}{dt^{n-1}} \right] + O\left(\frac{1}{r^3}\right) \]
where
\[ D_n = D_{n-1} + n, \quad D_0 = 0, \quad \nu = \frac{r}{r}. \quad (10) \]

In equation (10) and in the following we understand by \( A_{(i_1, \ldots, i_n)} \) the sum over all the permutations of the symbols \( i_q \) that give distinct terms. The use of the formula (9) in equations (11) and (13), and also of the well known properties of STF tensors, allows the ordering of the contributions in the expansions above as a 1/r power series.

\[
B(r, t) = \frac{\mu_0}{4\pi r} \sum_{n \geq 1} \frac{1}{n!} \left\{ \nu^n \cdot \tilde{M}_{(n)}^{(r, n+1)} \nu - [\nu^{n-1}, \tilde{M}_{(n)}^{(r, n+1)}] - \frac{c}{\alpha} \nu \times [\nu^{n-1}, \tilde{P}_{(n)}^{(r, n+1)}] \right\} + \frac{\mu_0}{4\pi r^2} \sum_{n \geq 2} \frac{1}{n! (n-1)!} \left\{ D_{n+1}[\nu^n \cdot \tilde{M}_{(n)}^{(r, n)}] - D_n[\nu^{n-1}, \tilde{M}_{(n)}^{(r, n)}] \right\} + \ldots (11)
\]

with the notation \( f_{(n)} = \frac{d^n f(t_0)}{dt^n} \) and

\[
E(r, t) = \frac{1}{4\pi \varepsilon_0} \sum_{n \geq 1} \frac{1}{n!} \left\{ \nu^n \cdot \tilde{P}_{(n)}^{(r, n+1)} \nu - [\nu^{n-1}, \tilde{P}_{(n)}^{(r, n+1)}] + \frac{\alpha}{c} \nu \times [\nu^{n-1}, \tilde{M}_{(n)}^{(r, n+1)}] \right\} + \frac{1}{4\pi \varepsilon_0 r^2} \sum_{n \geq 2} \frac{1}{n! (n-1)!} \left\{ D_{n+1}[\nu^n \cdot \tilde{P}_{(n)}^{(r, n)}] - D_n[\nu^{n-1}, \tilde{P}_{(n)}^{(r, n)}] + \frac{\alpha}{c} D_n \nu \times [\nu^{n-1}, \tilde{M}_{(n)}^{(r, n)}] \right\} + \ldots (12)
\]

Let us write \( E = E^{(1)} + E^{(2)} + O(1/r^3) \) where \( E^{(1)} \) and \( E^{(2)} \) are respectively, proportional with 1/r and 1/r^2 and, similarly, \( B = B^{(1)} + B^{(2)} + O(1/r^3) \). The parts \( E^{(1)} \) and \( B^{(1)} \) of the field are purely transverse fields, satisfying the properties

\[
\nu \cdot E^{(1)} = 0, \quad \nu \cdot B^{(1)} = 0; \quad E^{(1)} = \frac{c}{\alpha} B^{(1)} \times \nu, \quad \varepsilon_0 |E^{(1)}|^2 = \frac{1}{\mu_0} |B^{(1)}|^2. \quad (13)
\]

4. Radiation Intensity

The Poynting vector will be written in terms of \( E^{(1)} \) and \( B^{(1)} \) by considering the equations (13):

\[
S = \frac{\alpha}{\mu_0} (E \times B) = \varepsilon_0 |E^{(1)}|^2 c \nu + O(\frac{1}{r^3}) = \frac{1}{\mu_0} |B^{(1)}|^2 c \nu + O(\frac{1}{r^3})
\]

The total radiated power may be written as the limit of the integral on the sphere of radius \( r \) as \( r \to \infty \):

\[
\mathcal{I} = \lim_{r \to \infty} \int r^2 \nu \cdot S \, d\Omega(\nu) = \int r^2 \nu \cdot S \, d\Omega(\nu) = \frac{cr^2}{\mu_0} \int |B^{(1)}|^2 \, d\Omega(\nu)
\]

where

\[
< f(\nu) > = \frac{1}{4\pi} \int f(\nu) \, d\Omega(\nu).
\]

We obtain from equation (11) that

\[
r^2 |B^{(1)}|^2 = \left( \frac{\mu_0}{4\pi} \right)^2 \sum_{n, m \geq 1} \frac{1}{n! m!} \left\{ -\frac{1}{c^{n+m+2}} (\nu^n \cdot \tilde{M}_{(n)}^{(r, n+1)})(\nu^m \cdot \tilde{M}_{(m)}^{(r, m+1)}) \right\}
\]
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\[ + \frac{1}{\alpha^{n+m+2}} (\nu^{n} \cdot \tilde{M}_{n+1}^{(n)} + \nu^{m} \cdot \tilde{M}_{m+1}^{(m)}) + \frac{1}{\alpha^{n+m+1}} (\nu^{n} \cdot \tilde{M}_{n+1}^{(n)} + \nu \times (\nu^{m} \cdot \tilde{P}_{m+1}^{(m)})) \]
\[ + \frac{1}{\alpha^{n+m+1}} (\nu \times (\nu^{m} \cdot \tilde{P}_{m+1}^{(m)})) \cdot (\nu^{m} \cdot \tilde{M}_{m+1}^{(m)}) \]
\[ + \frac{1}{\alpha^{2}(n+m)} (\nu^{m} \cdot \tilde{P}_{m+1}^{(m)}), (\nu^{m} \cdot \tilde{P}_{m+1}^{(m)}) = \frac{1}{\alpha^{2}(n+m)} (\nu^{m} \cdot \tilde{P}_{m+1}^{(m)}) \]

The following formula is used to perform the average operations, [1]

\[ \langle \nu_1 \ldots \nu_n \rangle = \begin{cases} 0, & n = 2k + 1, \\
\frac{1}{(2k+1)!!} \delta_{i_1i_2 \ldots i_{n-1}i_n}, & n = 2k, \quad k = 0, 1, 2, \ldots \end{cases} \]

If \( A^{(n)}, B^{(m)} \) are STF tensors, one can show that

\[ \langle \nu^{k} \cdot A^{(n)} \rangle \cdot \langle \nu^{k'} \cdot B^{(m)} \rangle = \frac{k! \delta_{kk'}}{(2k+1)!!} \langle A^{(n)} \cdot B^{(m)} \rangle \]

For the calculation of the averaged square of the vector \( B^{(1)} \) one can apply the following formulae obtained from equation (14):

\[ \langle (\nu^{n} \cdot \tilde{M}^{(n)}) (\nu^{m} \cdot \tilde{M}^{(m)}) \rangle = \frac{n! \delta_{nm}}{(2n + 1)!!} \langle \tilde{M}^{(n)} \cdot \tilde{M}^{(n)} \rangle, \]
\[ \langle (\nu^{n-1} \cdot \tilde{M}^{(n)}) (\nu^{m-1} \cdot \tilde{M}^{(m)}) \rangle = \frac{(n-1)! \delta_{nm}}{(2n-1)!!} \langle \tilde{M}^{(n)} \cdot \tilde{M}^{(n)} \rangle \]

\[ \langle (\nu^{n-1} \cdot \tilde{M}^{(n)}) \cdot [\nu \times (\nu^{m-1} \cdot \tilde{P}^{(m)})] \rangle = \langle (\nu^{n-1} \cdot \tilde{M}^{(n)}) \rangle \cdot \nu^{m-1} (\nu^{m-1} \cdot \tilde{P}^{(m)}) \]
\[ = \varepsilon_{ijk} \nu_{i} \ldots \nu_{i-n-1} \nu_{i+1} \ldots \nu_{j-m-1} \tilde{M}_{i_{1}i_{2} \ldots i_{n-1}i_{n+1}}^{(n)} \tilde{P}_{j_{1}j_{2} \ldots j_{m+1}}^{(m)} = 0. \]

By introducing the expansion (15) of \( B \) in equation (14), and using the relations (15), one obtains

\[ I = \frac{\alpha^{2}}{4\pi \varepsilon_{0} c^{n}} \sum_{n \geq 1} \frac{n + 1}{n n! (2n + 1)!! c^{2n}} \left( \tilde{M}_{n+1}^{(n)} \tilde{B}_{n}^{(n)} (n) + \frac{\alpha^{2}}{c^{2}} \tilde{P}_{n+1}^{(n)} \tilde{P}_{n+1}^{(n)} \right) \]

5. Recoil Force

By considering the momentum current density tensor as

\[ T_{ij} = \frac{1}{2} (\varepsilon_{0} E_{i} + \frac{1}{\mu_{0}} B_{i}) \delta_{ij} - (\varepsilon_{0} E_{i} E_{j} + \frac{1}{\mu_{0}} B_{i} B_{j}), \]

then the recoil force is given by

\[ F_{R} = - \lim_{r \to \infty} \int_{\Sigma_{r}} (\nu \cdot T) dS = - \lim_{r \to \infty} r^{2} \int \nu_{i} T_{ij} d\Omega(\nu) e_{j} \]
\[ = - \lim_{r \to \infty} r^{2} \int \frac{1}{2} (\varepsilon_{0} E_{i} + \frac{1}{\mu_{0}} B_{i}) \nu d\Omega(\nu) = - \frac{4\pi}{\mu_{0}} (r^{2}|B^{(1)}|^{2} \nu). \]

Let us introduce the expansion (15) of \(|B^{(1)}|^{2}\) in equation (21) and consider the relations

\[ \langle (\nu^{n} \cdot \tilde{M}^{(n)}_{n+1} \cdot \nu^{m} \cdot \tilde{M}^{(m)}_{m+1}) \rangle \]
\[ = \frac{n! \delta_{m,n-1}}{(2n + 1)!!} (\tilde{M}^{(n)}_{n+1} \tilde{M}^{(n-1)}_{n} + \frac{(n + 1)! \delta_{m,n+1}}{(2n + 3)!!} (\tilde{M}^{(n)}_{n+1} \tilde{M}^{(n+1)}_{n+2})), \]
angular momentum current tensor

\[ \mu \]
density pseudovector

The angular momentum density current is associated with the 3rd-order tensor.

\[ \text{angular momentum density is defined by its Cartesian components} \]

\[ a \]
with the obvious notation,

\[ \text{radius} \]

\[ r \]
flux of the radiated electromagnetic angular momentum through the spherical surface of

\[ g \]
where the electromagnetic momentum density vector is

\[ \text{following formula can be obtained after performing some changes of summation indices:} \]

\[ \left\langle \nu \cdot \mathbf{M}^{(n)} \right\rangle \]
\[ \left\langle \nu \times \mathbf{P}^{(n)} \right\rangle \]
\[ \times e_\ell \varepsilon_{\ell i j k} \mathbf{M}^{(n)} \cdot n - 1 \cdot \mathbf{P}^{(n)} \]
\[ \delta_{n m} \]
with the obvious notation,

\[ \left\langle \mathbf{M}^{(n)} \cdot n - 1 \cdot \mathbf{P}^{(n)} \right\rangle_{j k} = \mathbf{M}_{i 1 \ldots i n - 1 j} \mathbf{P}_{i 1 \ldots i n - 1 k} \]

After observing that the terms that give nonzero contributions are only those containing a \( \delta \) factor, one gets

\[ F_R = -\frac{\mu_0}{4\pi} \sum_{n \geq 1} \frac{1}{c^{2 n} (2 n + 1)!!} \]
\[ \times \left\{ \frac{n + 1}{\alpha c n} \left( \mathbf{M}^{(n)}_{m 1 \ldots m n} \cdot \mathbf{M}^{(n)}_{m n} \right) + \frac{n + 2}{c^3 (2 n + 3)!! (n + 1)!} \left( \mathbf{M}^{(n)}_{m 1 \ldots m n + 1} \cdot \mathbf{M}^{(n)}_{m n + 1} \right) \right\} \]
\[ - \frac{2}{\alpha c n!} e_{\ell i j k} \left( \mathbf{M}^{(n)}_{m 1 \ldots m n} \cdot n - 1 \cdot \mathbf{P}^{(n)}_{m 1 \ldots m n + 1} \right)_{j k} + \frac{c (n + 1)}{\alpha^2 n!} \left( \mathbf{P}^{(n)}_{m 1 \ldots m n} \cdot \mathbf{P}^{(n)}_{m n} \right) \]
\[ + \frac{n + 2}{\alpha^2 c (2 n + 3)!! (n + 1)!} \left( \mathbf{P}^{(n)}_{m 1 \ldots m n} \cdot \mathbf{P}^{(n)}_{m n + 1} \right) \]

(23)

Since \( M^{(0)} = 0 \), and the total electric charge in \( D \) is constant \( (dP^{(0)}/dt = 0) \), then the following formula can be obtained after performing some changes of summation indices:

\[ F_R = -\frac{\mu_0}{2\pi c^3} \sum_{n \geq 1} \frac{1}{c^{2 n}} \left\{ \frac{n + 2}{(n + 1)! (2 n + 3)!!} \left[ \mathbf{M}^{(n)}_{m 1 \ldots m n + 1} \cdot \mathbf{M}^{(n)}_{m n + 1} + \frac{c^2}{\alpha^2} \left( \mathbf{P}^{(n)}_{m 1 \ldots m n + 1} \cdot \mathbf{P}^{(n)}_{m n + 1} \right) \right] \right\} \]
\[ - \frac{c^2}{\alpha n! (2 n + 1)!!} e_{\ell i j k} \left( \mathbf{M}^{(n)}_{m 1 \ldots m n} \cdot n - 1 \cdot \mathbf{P}^{(n)}_{m 1 \ldots m n + 1} \right)_{j k} \]

(24)

6. Angular Momentum Loss

The total angular momentum lost per unit time by a radiating system is given by the flux of the radiated electromagnetic angular momentum through the spherical surface of radius \( r \to \infty \). The 2nd-order antisymmetric tensor associated with the electromagnetic angular momentum density is defined by its Cartesian components \( m_{ij} = x_i q_j - x_j q_i \)

where the electromagnetic momentum density vector is \( q = (\varepsilon_0 / \alpha) (E \times B) \). The angular momentum density current is associated with the 3rd-order tensor \( \mu^{(3)} \), \( \mu_{ij} = x_i T_{jk} - x_j T_{ik} \) with \( T_{jk} \) given by equation (20). By introducing the angular momentum density pseudovector \( L \) with the components \( L_i = (1/2) \varepsilon_{ijk} m_{jk} \), and the 2nd-order angular momentum current tensor \( \mu^{(2)}_{ij} = (1/2) \varepsilon_{ijk} \mu^{(3)}_{kij} \), we obtain that

\[ \nu_k \mu^{(2)}_{ki} = -r \left[ \varepsilon_0 \left( \nu \cdot E \right) \right] (\nu \times E), \]

\[ + \frac{1}{\mu_0} \left( \nu \cdot B \right) (\nu \times B) \]


and further that
\[
\left(\frac{dL}{dt}\right)_{\text{rad}} = -\lim_{r \to \infty} \oint_{\Sigma_r} \nu_k \mu^{(2)}_{ki} \, dS_n
\]
\[
= 4\pi r^3 \left\{ \frac{1}{\mu_0} (\nu \cdot B^{(2)}) (\nu \times B^{(1)}) + \varepsilon_0 (\nu \cdot E^{(2)}) (\nu \times E^{(1)}) \right\}.
\] (25)

By using equations (10)-(12), one obtains
\[
r^2 \nu \cdot B^{(2)} = \frac{\mu_0}{4\pi \alpha} 4\pi \sum_{n \geq 1} \frac{n+1}{n! c^n} (\nu^n \cdot \tilde{M}^{(n)}_{\nu}),
\] (26)
\[
r^2 \nu \cdot E^{(2)} = \frac{1}{4\pi \varepsilon_0} \sum_{n \geq 1} \frac{n+1}{n! c^n} (\nu^n \cdot \tilde{P}^{(n)}_{\nu}),
\] (27)

and
\[
r \nu \times B^{(1)} = \frac{\mu_0}{4\pi c} \sum_{n \geq 1} \frac{1}{c^n n!} \left\{ -\frac{\alpha}{c} \nu \times (\nu^{n-1} \cdot \tilde{M}^{(n)}_{\nu}) - (\nu^n \cdot \tilde{P}^{(n)}_{\nu}) + (\nu^{n-1} \cdot \tilde{P}^{(n)}_{\nu}) \right\},
\] (28)
\[
r \nu \times E^{(1)} = \frac{\mu_0}{4\pi c} \sum_{n \geq 1} \frac{1}{c^n n!} \left\{ -\frac{c}{\alpha} \nu \times (\nu^{n-1} \cdot \tilde{P}^{(n)}_{\nu}) + (\nu^n \cdot \tilde{M}^{(n)}_{\nu}) - (\nu^{n-1} \cdot \tilde{M}^{(n)}_{\nu}) \right\},
\] (29)

We then have the following substitution rules:
\[
\left( \tilde{M}^{(n)}_{\nu} \rightarrow \frac{\varepsilon}{\alpha c} \tilde{P}^{(n)}_{\nu}, \tilde{P}^{(n)}_{\nu} \rightarrow -\tilde{M}^{(n)}_{\nu} \right) \Rightarrow \left( \nu \cdot B^{(2)} \rightarrow \nu \cdot E^{(2)}, \nu \times B^{(1)} \rightarrow \nu \times E^{(2)} \right).
\] (30)

After using equations (22) and (33), we obtain
\[
4\pi r^3 \langle (\nu^n \cdot \tilde{M}^{(n)}_{\nu}) (\nu^{m-1} \cdot \tilde{P}^{(m)}_{\nu}) \rangle = \frac{n! \delta_{m,n+1}}{(2n + 1)!!} (\tilde{M}^{(n)}_{\nu} \cdot \tilde{P}^{(n+2)}_{\nu})
\] (31)

and the substitution rules (30), we obtain
\[
4\pi r^3 \langle (\nu \cdot B^{(2)}) (\nu \times B^{(1)}) \rangle = \frac{\mu_0^2}{4\pi} \sum_{n \geq 1} \left[ -\frac{n+1}{n!(2n+1)!! \alpha^2 c^{2n-1}} e_i \varepsilon_{ijk} (\tilde{M}^{(n)}_{\nu} \cdot \tilde{P}^{(n-1)}_{\nu}) + \frac{n+2}{n!(2n+3)!! \alpha^2 c^{2n+1}} \tilde{M}^{(n+1)}_{\nu} \cdot \tilde{P}^{(n+2)}_{\nu} \right],
\] (32)
\[
4\pi r^3 \langle (\nu \cdot E^{(2)}) (\nu \times E^{(1)}) \rangle = \frac{\mu_0^2}{4\pi} \sum_{n \geq 1} \left[ -\frac{n+1}{n!(2n+1)!! \alpha^3 c^{2n-3}} e_i \varepsilon_{ijk} (\tilde{P}^{(n)}_{\nu} \cdot \tilde{M}^{(n-1)}_{\nu}) - \frac{n+2}{n!(2n+3)!! \alpha^3 c^{2n+1}} (\tilde{P}^{(n)}_{\nu} \cdot \tilde{M}^{(n+1)}_{\nu}) \right].
\] (33)

By introducing equations (32) and (33) in equation (25), we obtain
\[
\left(\frac{dL}{dt}\right)_{\text{rad}} = \frac{\mu_0}{4\pi} \sum_{n \geq 1} \left[ -\frac{n+1}{\alpha^2 c^{2n-1}(n-1)!(2n+1)!!} e_i \varepsilon_{ijk} (\tilde{M}^{(n)}_{\nu} \cdot \tilde{P}^{(n-1)}_{\nu}) + \frac{n+2}{\alpha^2 c^{2n+1} n!(2n+3)!!} (\tilde{M}^{(n)}_{\nu} \cdot \tilde{P}^{(n+1)}_{\nu}) \right]
\]
\[
- \frac{n+1}{\alpha^2 c^{2n-1}(n-1)!(2n+1)!!} e_i \varepsilon_{ijk} (\tilde{P}^{(n)}_{\nu} \cdot \tilde{M}^{(n-1)}_{\nu}) - \frac{n+2}{\alpha^2 c^{2n+1} n!(2n+3)!!} (\tilde{P}^{(n)}_{\nu} \cdot \tilde{M}^{(n+1)}_{\nu}) \right]
\] (34)
and, after some algebraic operations, we obtain
\[
\frac{dL}{dt} = \frac{\mu_0}{4\pi c} \sum_{n \geq 1} \frac{1}{c^{2n}} \left\{ -\frac{n + 1}{n!(2n + 1)!!} \epsilon_{ij} \right\} \times
\]
\[
\times \left[ \frac{c^2}{\alpha^2} \left( P^{(n)}_{n,n} \cdot n - 1, P^{(n)}_{n+1,n+1} \right)_{jk} + \left( M^{(n)}_{n,n} \cdot n - 1, M^{(n)}_{n,n+1} \right)_{jk} \right] +
\]
\[
+ \frac{n + 2}{\alpha n!(2n + 3)!!} \left[ P^{(n+1)}_{n+1,n+1} \cdot M^{(n)}_{n+1,n+1} - M^{(n+1)}_{n+1,n+1} \cdot P^{(n)}_{n+1,n+1} + M^{(n)}_{n,n} \cdot P^{(n+2)}_{n,n+2} - P^{(n+1)}_{n,n} \cdot M^{(n+1)}_{n,n+2} \right].
\]

7. Discussion and conclusions

The procedure, given in the present paper, for calculating the energy, momentum and angular momentum radiated by arbitrary sources is characterized by the mathematical simplicity. In our calculations only simple algebraic manipulation is necessary. The results expressed in different coordinate systems, for example spherical, may be obtained by adequate transformations. Complete and correct multipole analysis is given in some fundamental works as [6, 7]. In these works, the multipole expansion formulae are introduced using the canonical basis of the solutions of the scalar wave equation of Helmholtz. Although the results obtained by this formalism represent an exact description of the multipolar expansion applied to the electromagnetic field of arbitrary sources, the formulae are rather cumbersome. Due to the complexity of these formulae, the possibility to obtain erroneous results exists even in the case one calculates the contributions of the low order multipoles to the radiated field. This is the reason why we believe it is necessary to give corrections for the radiation formulae obtained in [6, 7] in addition to the corrections given in these papers to the results from [8, 9].

By applying the formulae for the radiated quantities, some care is necessary for obtaining a correct grouping of the different multipolar terms in a given approximation. For understanding this problem it is sufficient to refer to the harmonic time variation of the charges and currents. Let us consider
\[
\rho(r,t) = \rho_0(r) e^{-i\omega t}, \quad j(r,t) = j_0(r) e^{-i\omega t}.
\]

By considering the magnetic multipole tensor \( \mathbf{M}(t) \), the \( n \)-order time derivative may be written as
\[
\frac{d^n}{dt^n} \mathbf{M}^{(n)} = \left( \frac{-i\omega}{\alpha} \right)^n n \int_D \xi^n \times \mathbf{j}_0(\xi) d^3\xi \sim \omega^n d^n
\]
where \( d \) is the linear dimension of the domain \( D \). By introducing the wave length of the emitted radiation, \( \lambda = 2\pi c/\omega \), we may write
\[
\frac{1}{c^n} \frac{d^n}{dt^n} \mathbf{M}^{(n)} \sim \left( \frac{d}{\lambda} \right)^n.
\]

The parameters \( \omega \) and \( d \) are related, as order of magnitude, to the charges velocities, \( \omega d \sim v \), such that \( \omega d/c \sim v/c \). In what concerns the electric multipole’s contributions, we notice that the contribution of the time derivative of the \( n \)-order electric multipole
is of the same order of magnitude as the contribution of the \((n - 1)th\) order magnetic multipole because

\[
\frac{d}{dt} P_{i_1...i_n}(t) = \int_D j \cdot \nabla (\xi_{i_1} \cdots \xi_{i_n}) d^3 \xi = \int_D \xi_{i_1} \cdots \xi_{i_{n-1}} j_{i_n} d^3 \xi.
\]

Consequently, as is pointed out also in [5], in the case of the radiation intensity, in equations (19), (24) and (35), a consistent expansion should take into account, alongside with the magnetic multipoles up to a given order \(\mu\), the electric multipoles up to the order \(\varepsilon = \mu + 1\).

For a comparison with some results from the literature, for example [6, 7], we give below the results of the expansions of the total radiated power, recoil force and time variation, we stand out only the parameters \(d, c, \omega\). In the case \((\mu, \varepsilon) = (4, 5)\) pointing out the criteria for grouping the different multipolar terms for a given approximation.

In order to understand this problem, we give for the case of the radiated power a detailed description of the procedure used. We denote by \(\mathcal{I}(\mu, \varepsilon)\) the radiated power obtained by the recursive procedure beginning from the order \(\mu\) for the magnetic multipoles and from the order \(\varepsilon\) for the electric ones. In the case of the harmonic time variation, we stand out only the parameters \(d, c, \omega\). In the case \((\mu, \varepsilon) = (4, 5)\), considering the \(c\) dependence of the coefficients given by equations (A.20), and using equations (A.22)-(A.26), we can write

\[
\begin{align*}
\mathcal{P}^{(1)} & \sim a_{10} + a_{11} \frac{\omega^2 d^2}{c^2} + a_{12} \frac{\omega^4 d^4}{c^4} + \ldots, \\
\mathcal{P}^{(2)} & \sim a_{20} d + a_{21} \frac{\omega^2 d^3}{c^2} + \ldots, \\
\mathcal{P}^{(3)} & \sim a_{30} d^2 + \frac{\omega^2 d^4}{c^2} + \ldots, \\
\mathcal{P}^{(4)} & \sim a_{40} d^3 + \ldots, \\
\mathcal{P}^{(5)} & \sim a_{50} d^4 + \ldots; \\
\mathcal{M}^{(1)} & \sim a'_{10} d + a'_{11} \frac{\omega^2 d^3}{c^2} + \ldots, \\
\mathcal{M}^{(2)} & \sim a'_{20} d^2 + a'_{21} \frac{\omega^2 d^4}{c^2} + \ldots, \\
\mathcal{M}^{(3)} & \sim a'_{30} d^3 + \ldots, \\
\mathcal{M}^{(4)} & \sim a'_{40} d^4 + \ldots
\end{align*}
\]

where \(a_{ij}\) and \(a'_{ij}\) are functions of time proportional with \(\sin \omega t\) or \(\cos \omega t\), and the magnitude order of a common factor is not considered. We point out that in equations (38) a contribution to the absolute order of magnitude of the final results gives the current density \(j \sim v \sim \omega d\). This factor present in all the expressions from equations (38), associated with \(1/c\) factors from equations (19), (24) and (35), gives a factor \((d/\lambda)^2 \sim (v/c)^2\) in the expressions of the corresponding expansions. This common factor is irrelevant four our considerations. If we consider equations (38) in the multipolar expansion (19), we can write in the case \((\mu, \varepsilon) = (4, 5)\) the following result for the relative orders of magnitude of the different terms

\[
\mathcal{I}^{(4,5)} \sim A_0 + A_2 \zeta^2 + A_4 \zeta^4 + A_6 \zeta^6 + A_8 \zeta^8, \quad \zeta = d/\lambda.
\]

From equations (A.15) and (A.16), one can see that \(\mathcal{I}^{(5,6)}\) can be written in the form of a similar expansion in which \(A_0, A_2,\) and \(A_4\) are unchanged, and only \(A_6, A_8,\ldots\) will be different. Consequently, for a consistent approximation of \(\mathcal{I}\) using the multipole
expansions up to \((\mu, \varepsilon) = (4, 5)\) one should keep only the terms of the order \((d/\lambda)^n\) for \(n < 4\).

Similar considerations may be applied for the evaluation of the recoil force and radiated angular momentum. From the results given in [6] and [7], it seems that the authors claim to give the first 3 terms of the \(d/\lambda\) expansion of the radiated power, the first two terms of the expansion of the recoil force and again the first 3 terms of the expansion of the angular momentum loss. Our evaluations give for these cases, very different results. It seems that, in the above mentioned papers, the approximation criteria are not applied consistently. The comprising of the factor \(1/c\) in the definitions of different parameters as, for example, for the toroid dipole, seems to be one of the sources of errors. We point out the benefit of using a system-free expression of Maxwell’s equations, as in the present paper. So, the factors \(c\) included in different definitions in Gauss system of units is represented here by \(\alpha\) which is no counting to the evaluation of approximation order. This fact was observed also in [4] in the case of the energy loss.

We give bellow our results for the above mentioned approximations.

In the case of the radiated power, it is necessary to calculate \(\mathcal{I}^{(4,5)}\) and retain only the powers of \(\zeta\) up to four. One obtains

\[
\mathcal{I} = \frac{1}{4\pi c_0^3} \left[ \frac{2}{3} \ddot{\mathbf{p}}^2 - \frac{4}{3c^2} \dddot{\mathbf{p}} \cdot \dddot{\mathbf{t}} + \frac{1}{20c^2} \dddot{\mathbf{p}}^{(2)} \cdot \dddot{\mathbf{p}} + \frac{2\alpha^2}{3c^2} \dddot{\mathbf{m}}^2 + \frac{2}{3c^4} \dddot{\mathbf{t}}^2 + \frac{4}{3c^4} \dddot{\mathbf{p}} \cdot \dddot{\mathbf{T}}^{(2)} \right.
\]

\[
- \frac{1}{10c^4} \dddot{\mathbf{p}} \cdot \dddot{\mathbf{T}}^{(1)} + \frac{2}{945c^4} \dddot{\mathbf{p}}^{(3)} \cdot \dddot{\mathbf{p}} + \frac{4\alpha^2}{3c^4} \dddot{\mathbf{m}} \cdot \dddot{\mathbf{\mu}} + \frac{\alpha^2}{20c^4} \dddot{\mathbf{M}}^{(2)} \cdot \dddot{\mathbf{M}}^{(2)} \bigg] + O(\zeta^6). \tag{40}
\]

The recoil force is given by

\[
4\pi c_0 F = -\frac{1}{c^3} \left[ \frac{1}{5} \dddot{\mathbf{p}} \cdot \dddot{\mathbf{p}}^{(2)} - \frac{2\alpha}{3} \dddot{\mathbf{m}} \times \dddot{\mathbf{p}} - \frac{2\alpha}{3c^2} (\dddot{\mathbf{\mu}} \times \dddot{\mathbf{p}} - \dddot{\mathbf{m}} \times \dddot{\mathbf{t}}) + \frac{\alpha^2}{5c^2} \dddot{\mathbf{m}} \cdot \dddot{\mathbf{M}}^{(2)} \right.
\]

\[
- \frac{1}{5c^2} (\dddot{\mathbf{p}} \cdot \dddot{\mathbf{T}}^{(2)} + \dddot{\mathbf{T}}^{(1)} \cdot \dddot{\mathbf{p}}^{(2)}) + \frac{4}{315c^2} \dddot{\mathbf{p}}^{(2)} \cdot \dddot{\mathbf{p}} - \frac{\alpha}{30c^2} \epsilon_{ijk} \dddot{\mathbf{M}}_{aj} \dddot{\mathbf{p}}_{qk} + O(\zeta^5). \tag{41}
\]

and the radiated angular momentum

\[
4\pi c_0 \frac{d\mathbf{L}}{dt} = \frac{1}{c^3} \left[ -\frac{2}{3} \dddot{\mathbf{p}} \times \dddot{\mathbf{p}} + \frac{2}{3c^2} \dddot{\mathbf{p}} \times \dddot{\mathbf{t}} + \frac{2}{3c^2} \dddot{\mathbf{t}} \times \dddot{\mathbf{p}} - \frac{2\alpha^2}{3c^2} \dddot{\mathbf{m}} \times \dddot{\mathbf{m}} + \frac{\alpha}{5c^2} \dddot{\mathbf{m}} \cdot \dddot{\mathbf{p}}^{(2)} \right.
\]

\[
+ \frac{\alpha}{5c^2} \dddot{\mathbf{m}} \cdot \dddot{\mathbf{M}}^{(2)} - \frac{1}{10c^2} \epsilon_{ijk} \dddot{\mathbf{p}}_{aj} \dddot{\mathbf{p}}_{qk} - \frac{2}{3c^2} \dddot{\mathbf{p}} \times \dddot{\mathbf{T}}^{(2)} - \frac{2}{3c^2} \dddot{\mathbf{t}} \times \dddot{\mathbf{t}}
\]

\[
- 2\alpha^2 \dddot{\mathbf{\mu}} \times \dddot{\mathbf{m}} - \frac{2}{3c^4} \dddot{\mathbf{T}}^{(2)} \times \dddot{\mathbf{p}} - \frac{2\alpha}{3c^2} \dddot{\mathbf{m}} \times \dddot{\mathbf{\mu}} + \frac{\alpha}{5c^2} \dddot{\mathbf{\mu}} \cdot \dddot{\mathbf{p}}^{(2)} - \frac{\alpha}{5c^2} \dddot{\mathbf{\mu}} \cdot \dddot{\mathbf{M}}^{(2)}
\]

\[
- \frac{2\alpha}{3c^2} \dddot{\mathbf{p}} \cdot \dddot{\mathbf{M}}^{(2)} + \frac{\alpha}{5c^2} \dddot{\mathbf{t}} \cdot \dddot{\mathbf{M}}^{(2)} - \frac{1}{10c^4} \epsilon_{ijk} \dddot{\mathbf{p}}_{aj} \dddot{\mathbf{T}}^{(2)} + \frac{1}{10c^4} \epsilon_{ijk} \dddot{\mathbf{T}}^{(2)} \dddot{\mathbf{q}}_j \dddot{\mathbf{p}}_{qk} - \frac{2\alpha}{10c^4} \epsilon_{ijk} \dddot{\mathbf{M}}_{aj} \dddot{\mathbf{M}}_{qk} \dddot{\mathbf{p}}^{(3)} - \frac{2\alpha}{10c^4} \dddot{\mathbf{p}} \cdot \dddot{\mathbf{M}}^{(3)} - \frac{2\alpha}{10c^4} \dddot{\mathbf{M}}^{(3)} \cdot \dddot{\mathbf{p}}^{(2)}
\]

\[
- \frac{2}{315c^6} \epsilon_{ijk} \dddot{\mathbf{p}}_{qj} \dddot{\mathbf{p}}_{qk} \bigg] + O(\zeta^6). \tag{42}
\]
The following notations was introduced in the last three equations:
\[ p, m, \mu, t \] for \( P^{(1)}, M^{(1)}, \mu^{(1)}, T^{(1)}_i \), and \( T_{(2)} = e_i T^{(1)}_{(2)i} \) (43)

For comparison, we give below the results from \[ [6, 7] \] with our notation but written in Gauss unit system. So, the result from \[ [6] \] is \[ \\]

\[
\mathcal{I}_{DT} = \frac{2}{3c^3} |\vec{p} - \frac{1}{c^2} \vec{t}|^2 + \frac{2}{3c^3} |\vec{m}|^2 + \frac{1}{20c^5} (\vec{P}_{ij} - \frac{1}{c^2} \vec{T}_{(1)ij})(\vec{P}_{ij} - \frac{1}{c^2} \vec{T}_{(1)ij})
+ \frac{2}{945c^7} (\cdots M_{ijk} \cdots + \cdots M_{ijk} \cdots).
\]

(44)

In this expression are present terms as \( M_{ijk}, \cdots M_{ijk} \) and \( T_{(1)ij}, \cdots T_{(1)ij} \) of order six in \( \xi \) and are missed terms of the order four as \( \vec{P}_{ijk}, \vec{P}_{ijk} \) for example. This expression may not to be considered as an expansion up to the order six in \( \xi \) because in this case many terms must be added.

The result for \( \mathcal{I} \) from \[ [7] \] is given by
\[
\mathcal{I}_{RV} = \frac{2}{3c^3} \dot{p}^2 + \frac{2}{3c^3} \dot{m}^2 - \frac{4}{3c^5} \vec{P} \cdot \vec{t} + \frac{2}{3c^5} \vec{t}^2 + \frac{4}{3c^3} \vec{m} \cdot \vec{\mu}
+ \frac{1}{20c^7} \vec{P}_{ij} \vec{P}_{ij} + \frac{1}{20c^7} M_{ij} \tilde{M}_{ij}.
\]

(45)

In equation (45), the terms \( \dot{t}^2 \) and \( \vec{m} \cdot \vec{\mu} \) represent corrections of order four but many terms of the same order are missing compared with equation (40). Similar conclusions may be formulated regarding the results for recoil force and angular momentum.

### Appendix A. Gauge invariant reduction of multipole Cartesian tensors

The transformations \[ [6] \] involve a sequence of operations meant to obtain the symmetric and traceless part of the tensors implied. Let an \( n \)-rank tensor \( L^{(n)} \) of magnetic type i.e. symmetric in the first \( n - 1 \) indices and verifying the property \( L_{i_1...i_k-1,j}i_{k+1}...i_{n-1},j = 0, k = 1, n - 1 \). The symmetric part of this tensor is given by

\[
L_{(\text{sym})i_1...i_n} = \frac{1}{n} [L_{i_1...i_n} + L_{i_2i_1...i_n} + \ldots + L_{i_1...i_n i_{n-1}}] = L_{i_1...i_n} - \frac{1}{n} \sum_{\lambda=1}^{n-1} \varepsilon_{i_1...i_n} \varepsilon_{i_1...i_n} \varepsilon_{i_{n-1}} \mathcal{N}^{(n)}_{i_1...i_{n-1} q} L^{(n)}_{q}
\]

where \( \mathcal{N}^{(n)} \) is the component with the index \( i_{\lambda} \) suppressed.

The operator \( \mathcal{N} \) defines a correspondence between \( L^{(n)} \) and a tensor of rank \( (n-1) \):

\[
L^{(n)} \rightarrow \mathcal{N}[L^{(n)}] : \left[ \mathcal{N}[L^{(n)}] \right]_{i_1...i_{n-1}} = \mathcal{N}_{i_1...i_{n-1}} L^{(n)} = \varepsilon_{i_{n-1} i_{n-2} i_n} \varepsilon_{i_1...i_{n-2} i_{n-1}} \mathcal{N}^{(n)}_{i_1...i_{n-1} q} L^{(n)}_{q}
\]

(A.1)

which is fully symmetric in the first \( n-2 \) indices and the contractions of the last index with all the previous indices give null results. So, the tensor \( \mathcal{N}[L^{(n)}] \) is of the type \( M^{(n-1)} \). Particularly,

\[
\mathcal{N}^{2k}[M(n)] = \frac{(-1)^k n}{(n+1)n} \int_{D} \xi^{2k} n^{-2k} \times j d^3 \xi,
\]

\[ \dagger \] For comparison with our formulae one takes \( 4\pi \varepsilon_0 \rightarrow 1, \alpha \rightarrow c \)
\[
\mathcal{N}^{2k+1}[\mathbf{M}^{(n)}] = \frac{(-1)^k n}{(n+1)\alpha} \int_D \xi^{2k}\xi^{n-2k-1} \times (\xi \times j) d^3\xi, \ k = 0, 1, 2 \ldots
\]

(A.2)

Let a fully symmetric tensor \( \mathbf{S}^{(n)} \) and the \textit{detracer operator} \( \mathcal{T} \) introduced in [4]. This operator acts on a totally symmetric tensor \( \mathbf{S}^{(n)} \) so that \( \mathcal{T}[\mathbf{S}^{(n)}] \) is a fully symmetric and traceless tensor of rank \( n \). The \textit{detracer theorem} states that \( \mathbf{S}^{(n)} \) is a solution of the homogeneous wave equation for \( \mathbf{S}^{(n)} \).

In the following, for simplifying the notation, all arguments of the operator \( \mathcal{T} \) are dropped.

Here the equation \( \Delta \mathbf{S}^{(n)} = \mathbf{S}^{(n)} \) that used in [4] by a factor 1/(2n - 1)!!

(A.3)

where \( [n/2] \) denotes the integer part of \( [n/2] \), \( A_{(i_1 \ldots i_n)} \) is the sum over all permutations of the symbols \( i_q \) which give distinct terms, and \( \mathbf{S}^{(n;m)}_{(i_2m+1 \ldots i_n)} \) denotes the components of the \( (n - 2m) \)th-order tensor obtained from \( \mathbf{S}^{(n)} \) by contracting \( m \) pairs of symbols \( i \).

It is useful to introduce here another operator \( \Lambda \) by the relationship

(A.4)

where \( \Lambda[\mathbf{S}^{(n)}] \) defines a fully symmetric tensor of rank \( n - 2 \). From this definition together with the theorem (A.3), we obtain

(A.5)

In the following, for simplifying the notation, all arguments of the operator \( \Lambda \) should be considered as a symmetrized tensor i.e. \( \Lambda[\mathbf{T}^{(n)}] = \Lambda[\mathbf{T}_{sym}^{(n)}] \) for any tensor \( \mathbf{T}^{(n)} \). The same applies to the operator \( \mathcal{T} \) as well.

The following four transformation properties of the multipole tensors and potentials may be used for establishing the results from [2].

I. Let the transformation of the \( n \)th-order magnetic tensor:

(A.6)

Let us substitute in the expansion of the potential \( \mathbf{A} \) the tensor \( \mathbf{M}^{(n)} \) by \( \mathbf{M}_{(L)}^{(n)} \) obtaining

(A.7)

Here the equation \( \Delta \mathbf{A} = (1/c^2)\partial^2/\partial t^2 \mathbf{A} \) is considered. The function \( \Psi \) is a solution of the homogeneous wave equation for \( r \neq 0 \) and the corresponding

\( \Lambda \) in this equation, the definition of the symmetric and traceless part of the tensor \( \mathbf{S}^{(n)} \) differs from that used in [4] by a factor 1/(2n - 1)!!.
eliminated by the transformation $S$.

We obtain

$$\frac{\alpha n - 1}{c^2 n^2}.$$  \hspace{1cm} (A.8)

Introducing the transformed potentials produced by the substitution $P^{(n-1)} \rightarrow P^{(n-1)}$, we obtain

$$\begin{align*}
A[M^{(n)}_{(a)} \rightarrow M^{(n)}_{(a)}; P^{(n-1)} \rightarrow P^{(n-1)}] &= A + \nabla \Psi, \\
\Phi[P^{(n-1)} \rightarrow P^{(n-1)}] &= \Phi - \frac{\partial \Psi}{\partial t}.
\end{align*}$$

So, the transformation \[A\] produces changes in the potentials which, up to a gauge transformation, are compensated by the transformation \[A\].

II. Let the transformation of the magnetic vector of rank $n$:

$$P^{(n)} \rightarrow P^{(n)}_{(L)}, \quad (L)_{i_1 \ldots i_n} = P_{i_1 \ldots i_n} - \frac{1}{n} \sum_{\lambda=1}^{n-1} \epsilon_{i_1 \ldots i_n} \nabla^{(\lambda)}[L^{(n)}(t_0)].$$

We obtain

$$A[P^{(n)} \rightarrow P^{(n)}_{(L)}] = A + \frac{\mu_0}{4\pi} \frac{(-1)^{n-1}(n-1)}{n! n} \nabla \times \left[ \nabla^{n-2} \cdot \left( \frac{1}{r} \nabla [L^{(n)}] \right) \right], \quad \Phi[P^{(n)} \rightarrow P^{(n)}_{(L)}] = \Phi.$$  \hspace{1cm} (A.9)

The change of the vector potential is compensated by the transformation

$$M^{(n-1)} \rightarrow M^{(n-1)} + a_2(n) \nabla [L^{(n)}], \quad a_2(n) = \frac{n-1}{\alpha n^2} = -\frac{c^2}{\alpha^2} a_1(n).$$

(A.10)

III. Let the transformation of the magnetic vector of rank $n$:

$$M^{(n)} \rightarrow M^{(n)}_{(S)}, \quad M_{(S)i_1 \ldots i_n} = M_{i_1 \ldots i_n} - \sum_{D(i)} \delta_{i_1 i_2} \Lambda_{i_3 \ldots i_n} [S^{(n)}(t_0)].$$

(A.11)

where $S^{(n)}$ is a fully symmetric tensor. The alteration of the vectorial potential is eliminated by the transformation

$$M^{(n-2)} \rightarrow M^{(n-2)} + b(n) \tilde{\Lambda} [S^{(n)}], \quad b(n) = \frac{n-2}{2c^2 n}.$$  \hspace{1cm} (A.12)

IV. The transformation

$$P^{(n)} \rightarrow P^{(n)}_{(S)}, \quad P_{(S)i_1 \ldots i_n} = P_{i_1 \ldots i_n} - \sum_{D(i)} \delta_{i_1 i_2} \Lambda_{i_3 \ldots i_n} [S^{(n)}]$$

(A.13)

produces the following changes of the potentials:

$$A[P^{(n)} \rightarrow P^{(n)}_{(S)}] = A - \frac{\mu_0}{4\pi} \frac{(-1)^{n-1}}{n!} e_i \partial_{i_1} \ldots \partial_{i_{n-1}} \left[ \frac{1}{r} \sum_{D(i)} \delta_{i_1 i_2} \Lambda_{i_3 \ldots i_n} [S^{(n)}] \right]$$

$$= A + \nabla \Psi - \frac{\mu_0}{4\pi} \frac{(-1)^{n-1}(n-2)(n-1)}{2n! c^2} \nabla \cdot \left[ \frac{1}{r} \tilde{\Lambda} \cdot [S^{(n)}] \right]$$

with $\Psi'$, as $\Psi$, satisfying the homogeneous wave equation and

$$\Phi[P^{(n)} \rightarrow P^{(n)}_{(S)}] = \Phi + \frac{\mu_0}{4\pi} \frac{(-1)^{n-1}n(n-1)}{2c^2 n!} \nabla \cdot \left[ \frac{1}{r} \tilde{\Lambda} \cdot [S^{(n)}] \right].$$

Let the transformation

$$P^{(n-2)} \rightarrow P^{(n-2)} = P^{(n-2)} + b(n) \tilde{\Lambda} [S(t_0)]$$

(A.14)

with $b(n)$ given by equation \[A.12\]. The effect of the transformation \[A.14\] on the potential $A$ is the compensation of the extra-gauge term. So $A[P^{(n)} \rightarrow P^{(n)}_{(S)}; P^{(n-2)} \rightarrow P^{(n-2)}] = \Phi[P^{(n)} \rightarrow P^{(n)}_{(S)}; P^{(n-2)} \rightarrow P^{(n-2)}] = \Phi[P^{(n)} \rightarrow P^{(n)}_{(S)}; P^{(n-2)} \rightarrow P^{(n-2)}]$. 


\( A + \nabla \Psi' \) but it is easy to see that the modification of the scalar potential \( \Phi \) produced by the transformation \( (A.13) \) together with the modification due to the transformation \( (A.13) \) give \( \Phi[\mathcal{P}(n) \rightarrow \mathcal{P}'(n) \mathcal{S}^{(n)} ; \mathcal{P}'(n-2) \rightarrow \mathcal{P}'(n-2)] = \Phi - \partial \Psi'/\partial t \) the total effect of the transformations \( (A.13) \) and \( (A.14) \) being a gauge transformation of the potentials.

4. Let the gauge invariant process of reducing the multipole tensors begin from the order \( n = \varepsilon \) in the case of the electric tensors and from \( n = \mu \) for the magnetic ones, and go downward up to lowest value, \( n = 1 \). Generally, we may choose \( \varepsilon > \mu \) as will be seen in the following. In \( \textbf{3} \) are given the formulae for the results \( \tilde{\mathcal{P}}_n \) and \( \tilde{\mathcal{M}}_n \):

\[
\tilde{\mathcal{P}}^{(n)} = \mathcal{P}^{(n)} + T \left\{ \sum_{k=1}^{[\epsilon-n]/2} A_k^{(n)} \frac{d^{2k}}{dt^{2k}} \Lambda^k [\mathcal{P}^{(n+2k)}] \right. \\
+ \left. \sum_{k=0}^{[\mu-n-1]/2} B_{k,l}^{(n)} \Lambda^l \mathcal{N}^{2k-2l+1} [\mathcal{M}^{(n+1+2k)}] \right\}, \quad (A.15)
\]

\[
\tilde{\mathcal{M}}^{(n)} = \mathcal{M}^{(n)} + T \left\{ \sum_{k=1}^{[\epsilon-n]/2} C_{k,l}^{(n)} \Lambda^l \mathcal{N}^{2k-2l} [\mathcal{M}^{(n+2k)}] \right\}, \quad (A.16)
\]

where

\[
A_k^{(n)} = \prod_{l=1}^{k} b(n + 2l),
\]

\[
B_{k,l}^{(n)} = \prod_{q=1}^{l} b(n + 2q) \prod_{h=0}^{k-l} a_1(n + 1 + 2k - 2h) \prod_{s=0}^{k-l-1} a_2(n + 2k - 2s), \quad (A.17)
\]

\[
C_{k,l}^{(n)} = \prod_{q=1}^{l} b(n + 2q) \prod_{h=0}^{k-l-1} a_1(n + 2k - 2h) \prod_{s=0}^{k-l-1} a_2(n - 1 + 2k - 2s) \quad (A.18)
\]

and

\[
a_1(n) = -\frac{\alpha}{c^2} \frac{n-1}{n^2}, \quad a_2(n) = \frac{1}{\alpha} \frac{n-1}{n^2} = -\frac{c^2}{\alpha^2} a_1(n), \quad b(n) = \frac{n - 2}{2c^2 n}. \quad (A.19)
\]

By introducing equations \( (A.19) \) in equations \( (A.17) \) and \( (A.18) \), we obtain

\[
A_k^{(n)} = \frac{1}{2k c^{2k}} \frac{n}{n + 2k},
\]

\[
B_{k,l}^{(n)} = \frac{(-1)^{k-l+1} \alpha}{2 l c^{2k+2}} \frac{n(n + 2l)!}{(n + 2k + 1)(n + 2k + 1)!},
\]

\[
C_{k,l}^{(n)} = \frac{(-1)^{k-l}}{2 l c^{2k}} \frac{n(n + 2l)!}{(n + 2k)(n + 2k)!}. \quad (A.20)
\]

\( \mathcal{P}^{(n)} \) and \( \mathcal{M}^{(n)} \) stand for the "static" expressions of the reduced multipole tensors:

\[
\mathcal{P}^{(n)}(t) = T[\mathcal{P}^{(n)}] = \frac{(-1)^n}{(2n - 1)!!} \int_D \rho(r, t) r^{2n+1} \nabla r^{1/3} x,
\]

\[
\mathcal{M}^{(n)}(t) = T[\mathcal{M}^{(n)}] = \frac{(-1)^n}{\alpha(n + 1)(2n - 1)!!} \sum_{\lambda=1}^{n} \int_D r^{2n+1} [j(r, t) \times \nabla_{i_{1}} \hat{e}_{i_{1}}] \frac{1}{r} d^3 x.
\]
In formulae above one should consider that $\prod_{k=1}^{L} F_k = 1$ if $L < l$. In the case $\varepsilon = \mu + 1$, we may write
\[
\hat{P}^{(n)} = P^{(n)} + \sum_{k=1}^{\frac{[\varepsilon-n]/2}{e^{2k}}} \frac{(-1)^k}{e^{2k}} \frac{d^{2k-1}}{dt^{2k-1}} T^{(n)}_{(k)},
\]
\[
T^{(n)}_{(k)} = (-1)^k e^{2k} T \left[ A^{(n)}_k \Lambda^{k} [\hat{P}^{(n+2k)}] + \sum_{l=0}^{k-1} B^{(n)}_{k-l} \Lambda^{2k-2l-1} [M^{(n+2k-1)}] \right],
\]
where besides the usual electric and magnetic multipole moments, a third multipole family, the toroid moments and, generally, mean-square radii of various orders, are involved [6, 7].

For applications, we give the results of the gauge invariant reduction of the electric and magnetic multipole tensors, beginning the reduction procedure from the rank $\varepsilon = 5$ for the electric tensors, and from $\mu = 4$ for the magnetic ones. From equation (A.21) we have for the electric multipole tensors,
\[
\hat{P}^{(1)} = P^{(1)} - \frac{1}{c^2} \hat{T}^{(1)} + \frac{1}{c^4} \hat{\hat{T}}^{(1)},
\]
\[
[T^{(1)}]_i = -c^2 T \left\{ A^{(1)}_1 \Lambda [\hat{P}^{(3)}] + B^{(1)}_{0,0} N[M^{(2)}] \right\}_i = \frac{1}{10} \int_{D} [ (\xi \cdot j) \xi - 2 \xi^2 j_i ] d^3 \xi,
\]
\[
[T^{(1)}]_i = c^4 T \left\{ A^{(1)}_2 \Lambda^2 [\hat{P}^{(5)}] + B^{(1)}_{1,0} N^3 [M^{(4)}] + B^{(1)}_{1,1} \Lambda N[M^{(4)}] \right\}_i
\]
\[
= -\frac{1}{280} \int_{D} [ 2 \xi^2 (\xi \cdot j) \xi_i - 3 \xi^4 j_i ] d^3 \xi,
\]
\[
\hat{P}^{(2)} = P^{(2)} - \frac{1}{c^2} \hat{T}^{(2)},
\]
\[
[T^{(2)}]_{ik} = -c^2 T \left\{ A^{(2)}_1 \Lambda [\hat{P}^{(4)}] + B^{(2)}_{0,0} N[M^{(3)}] \right\}_{ik}
\]
\[
= \frac{1}{42} \int_{D} \left[ 4 (\xi \cdot j) \xi_i \xi_k - 5 \xi^2 (\xi_i j_k + \xi_k j_i) + 2 \xi^2 (\xi \cdot j) \delta_{ik} \right] d^3 \xi,
\]
\[
\hat{P}^{(3)} = P^{(3)} - \frac{1}{c^2} \hat{T}^{(3)},
\]
\[
[T^{(3)}]_{ikl} = -c^2 T \left\{ A^{(3)}_1 \Lambda [\hat{P}^{(5)}] + B^{(3)}_{0,0} N[M^{(4)}] \right\}_{ikl}
\]
\[
= \frac{1}{60} \int_{D} \left[ 5 (\xi \cdot j) \xi_i \xi_k \xi_l - 5 \xi^2 (\xi_i j_k j_l + \xi_k j_i \xi_l + \xi_l j_i \xi_k) + \xi^4 \delta_{ikjl} \right] d^3 \xi,
\]
\[
\hat{P}^{(4)} = P^{(4)}, \quad \hat{P}^{(5)} = P^{(5)}.
\]

The magnetic multipole tensors are given by
\[
\hat{M}^{(1)}_i = \mathcal{M}^{(1)}_i + T \left\{ C^{(1)}_{1,0} N^2 [M^{(3)}] + C^{(1)}_{1,1} \Lambda [\hat{M}^{(3)}] \right\}_i = m_i + \frac{1}{c^2} \tilde{\mu}_i,
\]
\[
\hat{M}^{(2)}_{ik} = \mathcal{M}^{(2)}_{ik} + T \left\{ C^{(2)}_{1,0} N^2 [M^{(4)}] + C^{(2)}_{1,1} \Lambda [\hat{M}^{(4)}] \right\}_{ik} = \mathcal{M}^{(2)}_{ik} + \frac{1}{c^2} \tilde{\mu}_{ik},
\]
\[
\hat{M}^{(3)} = \mathcal{M}^{(3)}, \quad \hat{M}^{(4)} = \mathcal{M}^{(4)}.
\]
where
\[ m_i = \mathcal{M}_i^{(1)}, \quad \mu_i = \frac{1}{20\alpha} \int_{\mathcal{D}} \xi^2 (\xi \times j)_i d^3 \xi, \quad \mu_{ik} = \frac{1}{42\alpha} \int_{\mathcal{D}} \xi^2 \xi_{(i} (\xi \times j)_{k)} d^3 \xi \] (A.27)

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