Commutative Fuzzy Geometry and Quantum Particle Dynamics

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Abstract

Commutative fuzzy geometry considered as the possible mathematical framework for reformulation of quantum-mechanical formalism in geometric terms. In this approach the states of massive particle \( m \) correspond to elements of fuzzy manifold called fuzzy points. In 1-dimensional case, due to manifold ultraweak (fuzzy) topology, \( m \) space coordinate \( x \) acquires principal uncertainty \( \sigma_x \) and described by positive, normalized density \( w(x, t) \). Analogous uncertainties appear for fuzzy point on 3-dimensional manifold. It’s shown that \( m \) states on such manifold are equivalent to vectors (rays) on complex Hilbert space, their evolution correspond to Shroedinger dynamics of nonrelativistic quantum particle.

1 Introduction

It’s well known that quantum mechanics (QM) can be consistently described by several alternative formalisms such as Shroedinger (standard) one, algebraic QM, functional integral, etc. [1]. In last years, possible reformulation of QM formalism in geometric terms attracted significant attention [2, 3]. Really, in some fields of physics, like optics and general relativity, geometric ideas have been very useful, so one can hope that they also can help to study some important problems of quantum physics, first of all, quantum theory of gravity and gauge fields. Up to now, several alternative formalisms were proposed for realization of QM geometrization: symplectic geometry, Hilbert manifolds, Kahler bundles, etc., see [3] and refs. therein. In this paper, we’ll study the approach based on fuzzy calculus, in particular, it exploits fuzzy geometry formalism [4, 5]. During last 50 years, the fuzzy set theory and other branches of fuzzy mathematics were applied in a wide range of scientific areas such as biology, economics and computer science. From the early days of its development, the significant similarity of that theory and QM was noticed [6, 7, 8]. It was argued that the parameter and proposition uncertainty (fuzziness), which is generic for fuzzy
mathematics, can be equivalent to QM uncertainty of particle coordinate, momentum and other observables [6, 7].

It’s notable that in modern QM the fuzzy (unsharp) observables became the important part of QM theory of measurements [9], hence QM formalism based on fuzzy methods can be interesting both from fundamental and applied angles. However, until now such studies were performed only in the context of fuzzy (multivalued) logics ([10] and refs. therein). In distinction, our approach deals with standard logics and based on the results of fuzzy topology and geometry [7, 8, 11]. In particular, this formalism exploits the system phase space equipped with fuzzy topology; it will be shown that the corresponding structure of system state space is equivalent to QM Hilbert space [12]. As the result, it permits to derive the quantum system evolution from geometric arguments. It will be considered here for nonrelativistic particle, such system traditionally exploited as testing ground for QM formalism studies [13]. In mathematics, the formalism exploited here called fuzzy geometry [8, 14], but in modern physics such term ascribed to some noncommutative field theories [15], so to avoid the confusion, it will be called here commutative fuzzy geometry (CFG).

Our paper organized as follows. In section 2, basic features of fuzzy topology and geometry are reviewed and their relations with set theory and topology are discussed. In section 3, the model of 1-dimensional particle evolution on CFG manifold considered and the resulting space of particle states is derived. In section 4, particle evolution equation in this model derived. In section 5, general formalism for 3-dimensional case constructed. Section 6 presents our concluding remarks.

2 Geometric Fuzzy Structures

Here we’ll consider the fuzzy structures important for formalism construction, for the detailed review on CFG and related topics see [8, 11, 14]. The properties of fuzzy objects can be introduced ad hoc, but its worth to start from consideration of their analogs in set theory which provides the useful link to the realm of fuzzy structures and illustrates their physical meaning.

Remind that in set theory the sets can be classified according their ordering structure; the simplest case presents the totally ordered set, for all its element pairs \( a_k, a_l \) the ordering relations \( a_k \leq a_l \) (or vice versa) fulfilled. In distinction, for partially ordered set (Poset), some its element pairs can obey to incomparability relations (IR) between them: \( a_j \sim a_k \). In this case, both \( a_j \leq a_k \) and \( a_k \leq a_j \) propositions are false [16]; as will be argued, in some aspects IR is the discrete analogue of fuzzy relations.

To illustrate this analogy, consider poset \( S^p = A^p \cup B \), which includes the subset of incomparable elements \( A^p = \{a_j\} \), and ordered subset \( B = \{b_i\} \). For the simplicity suppose that in \( B \) the element indexes grow correspondingly to their ordering, so that \( \forall i, b_i \leq b_{i+1} \); relations between an arbitrary pairs \( a_j, a_i \) or \( a_i, b_l \) can be ordered, as well as incomparable. Let’s consider open \( B \) interval \( \{b_l, b_n\} \) with \( l + 2 \leq n \), and suppose that \( A^p \) element \( a_j \) is confined in \( \{b_l, b_n\} \), i.e. \( b_l \leq a_j ; a_j \leq b_n \), and simultaneously \( a_j \) is incomparable with all other (internal) \( \{b_l, b_n\} \) elements: \( b_l \sim a_j \);
∀ i; l + 1 ≤ i ≤ n − 1. In this case, \( a_j \) is, in a sense, smeared over \( \{b_l, b_n\} \) interval, so this is analogue of \( a_j \) coordinate uncertainty, if to regard the sequence of \( B \) elements \( \{b_l\} \) as the discrete coordinate axe. Plainly, \( A^p, B \) element relations can be also described by the binary matrix \( M_{ij} \), such that \( M_{ij} = 0 \) if \( a_i, b_j \) are ordered, \( M_{ij} = 1 \) otherwise.

The next step in transition to fuzzy structures is to change the set-theoretical relations between \( S^p \) elements to the fuzzy relations. To perform it, in place of \( M_{ij} \) one should put in correspondence to each \( a_j, b_i \) pair of \( S^p \) set the nonnegative, normalized weight function \( w_i^j \geq 0 \) with norm \( \sum_i w_i^j = 1 \). In fuzzy set theory, \( w_i^j \) characterizes the rate of closeness (membership) between \( a_j, b_i \) [4, 5]. In particular, analogously to \( M_{ij} \), \( w_i^j = 0 \) means that \( a_j, b_i \) are ordered relative to each other, i.e. their closeness is null, and if \( w_i^j = 1 \) they are equal. For the example considered above, one can ascribe arbitrarily: \( w_i^j = (n - l - 2)^{-1} \) to all \( b_i \) inside \( \{b_l, b_n\} \) interval, \( w_i^j = 0 \) for other \( b_i \), the interval width called tolerance scale [7]. In principle, the fuzzy relations can be introduced \textit{ad hoc} without any referring to the set partial ordering, but it’s worth to start from the analogy between them.

The similar structure can be introduced for the set of continuum power. As the example, consider the set \( S^f = A^p \cup X \) where \( A^p \) is the same discrete subset, \( X \) is the continuous ordered subset. In this case, \( A^p \) element \( a_i \) can be incomparable to some \( X \) elements \( \{x_u\} \), in particular, such \( x_u \) can constitute the interval on \( X \). If the flat metrics \( M(x, x') \) is defined on \( X \), then it’s equivalent to \( R^1 \) real number axe, and \( S^f \) is the fuzzy manifold denoted \( \tilde{R}^1 \). Then, the fuzzy relations between elements \( a_j, x \) are described by real, nonnegative functions \( w^j(x) \geq 0 \) with the norm \( \int w^j dx = 1 \) [5]. \( \{a_j\} \) called the fuzzy numbers \( \tilde{x}_j \) or in geometric framework, 1-dimensional fuzzy points (FPs) [4, 5, 14]. Plainly, ordered point \( x_c \in X \) is characterized by \( w^c(x) = \delta(x - x_c) \), hence the ordered points and FPs can be regarded formally on the same ground. Note that in fuzzy mathematics alternative FP definitions also are exploited, we use here one given in [5, 14].

In 3-dimensional case, one can consider the fundamental set: \( C^f = A^p \cup X' \) where \( A^p \) defined above, \( X' \) is the continuous set. Suppose that on \( X' \) 3—dimensional linear space \( R^3 \) with flat metrics \( M_{ij} \) is defined. Then FP \( a_j \) is described by the nonnegative function \( w^j(\vec{r}) \) with norm \( \int w^j d^3r = 1 \), such structure of \( C^f \) elements is fuzzy manifold, it denoted as \( \tilde{R}^3 \) [5, 14].

### 3 Particle States on Fuzzy Manifold

Here we’ll consider the model of particle evolution on fuzzy manifold called fuzzy mechanics (FM); it will be constructed as the minimal theory, i.e. at every step we’ll prefer the ansatz with minimal number of theory parameters. Such approach seems appropriate for QM reconstruction, since QM formalism contains only one theory parameter - Plank constant \( h \). We’ll suppose also that FM possesses space and time shift invariance and rotational invariance. In classical mechanics, the particle described as material point \( \vec{r}_a(t) \in R^3 \), whereas in FM formalism the particle \( m \)
corresponds to FP \( a(t) \) on fuzzy manifold \( \mathbb{R}^3 \) and characterized by normalized positive density \( w(\vec{r}, t) \) on \( \mathbb{R}^3 \). Beside \( w(x) \), \( m \) state \( \{ \zeta(t) \} \), called the fuzzy state, can depend, in principle, on other \( m \) degrees of freedom (DFs).

We shall start FM construction for 1-dimensional manifold, because in this case the theory premises are most simple and transparent. Let’s suppose that \( m \) state is prepared at some \( t_0 \) and consider \( m \) average velocity on \( \mathbb{R}^1 \)

\[
\vec{v}(t) = \frac{\partial}{\partial t} \int_{-\infty}^{\infty} x w(x, t) dx = \int_{-\infty}^{\infty} x \frac{\partial w}{\partial t} dx
\]

It’s reasonable to assume that in general \( \vec{v}(t) \) can be independent of \( w(x, t) \); we shall look for additional \( m \) DFs in form of real functions \( q_j(x, t) \); \( j = 1, ..., n \). Let’s suppose that in FM \( m \) state evolution is local, in particular,

\[
\frac{\partial w}{\partial t}(x, t) = \Phi[w(x, t), q_1(x, t), ..., q_n(x, t)]
\]

where \( \Phi \) is an arbitrary function. From \( w \) norm conservation it follows that

\[
\int_{-\infty}^{\infty} \Phi(x, t) dx = \int_{-\infty}^{\infty} \frac{\partial w}{\partial t}(x, t) dx = \frac{\partial}{\partial t} \int_{-\infty}^{\infty} w(x, t) dx = 0
\]

One can substitute: \( \Phi = -\partial_x J \), here \( J(x) \) is continuous, differentiable function, which obeys to the condition

\[
J(\infty, t) - J(-\infty, t) = 0
\]

i.e. there is no influence on \( w \) norm from infinity. Since \( w(x, t) \to 0 \) for \( x \to \pm\infty \), it follows from \( \mathbb{R}^1 \) reflection invariance that \( J(\pm\infty) = 0 \). Analogously to fluid dynamics, \( J \) can be decomposed formally as: \( J(x) = w(x)v(x) \), i.e. \( v(x) \) corresponds to 1-dimensional \( w \) flow velocity. In these terms eq. (2) can be rewritten in form of flow continuity equation [17]

\[
\partial_t w = -v \partial_x w - w \partial_x v
\]

We’ll assume that \( v(x, t) \) can be considered as independent \( m \) DF, and \( w, v \) functions possess the following continuity properties (CP): \( w(x, t), v(x, t) \in C^3(\mathbb{R}^1 \times [t_0, T]) \); here \( [t_0, T] \) is closed interval.

The twople \( \varrho^0 = \{ w, v \} \) called the observational \( |\zeta\rangle \) representation, i.e. the list of independent \( m \) DFs, their algebra is undefined at the moment. Plainly, to make the formalism consistent, beside eq. (5), which describes \( w \) evolution, it’s necessary to find also equation for \( v \) evolution. However, \( \varrho^0 \) ansatz isn’t optimal for that purpose, so it’s instructive to look for alternative dynamical \( |\zeta\rangle \) representation \( \eta \), for which it will become more simple and straightforward. We’ll suppose that \( \eta \) corresponds to the set of real functions \( \{ \eta_i(x) \} \), \( i = 1, n_\eta \), plus some \( \eta_i \) algebra. If this is the case, the most general \( \eta \) ansatz is \( \eta_j(x) = \Upsilon^j_x(w, v) \) where \( \Upsilon^j_x \) are some \( w, v \) functionals and \( x \) is their parameter. For \( |\zeta\rangle \) characterized by two DFs \( w, v \), its natural to start
η ansatz search from complex η(x), not assuming yet that η is L²-normalized. Hence n_a = 2 and η(x) can be expressed as

$$\eta(x) = \Upsilon_x^1 + i\Upsilon_x^2 = \Omega_x(w, v)e^{i\lambda_x(w, v)}$$

(6)

where Ω_x, λ_x are real functionals. It’s natural to assume also that if w(x_a) = 0 for some x_a, then η(x_a) = 0 and vice versa. It can be easily shown that the corresponding minimal ansatz is Ω_x = f[w(x)] where f → 0 for w → 0.

In addition, we’ll suppose that in FM the particle m possesses holistic properties, namely, its evolution can be characterized also by the particle velocity ‘as the whole’ u(t) described by the corresponding normalized distribution w_u(u, t). w_u characterizes the instant w(x, t) variations, in particular, the shift of its centre of gravity \(\bar{x}(t)\) and variation of w half-width (r.m.s.) \(\sigma_x(t)\). It follows then

$$\bar{u} = \int_{-\infty}^{\infty} u w_u(u) du = \bar{v} = \int_{-\infty}^{\infty} v(x) w(x) dx$$

(7)

Since \(\bar{u} = \bar{v}\), we shall not assume beforehand that u is independent m DF. In place of u, below it will be convenient to use the variable \(p = \mu u\) where \(\mu\) is the theory parameter; its distribution denoted \(w_p(p)\). m state \(|\zeta\rangle\) presumably contains the information on the expectation value of any m observable \(Q\) in form of some η functional. In particular, \(w_p(p) = F_p(\eta)\) and it follows that \(F_p\) functional is related to η(x) Fourier transform. To prove it and calculate η, \(w_p\), let’s introduce the auxiliary function \(\varphi(p) = w_p^\dagger \exp(i\beta)\), here \(\beta(p)\) is the auxiliary real function, on which final \(w_p\) ansatz wouldn’t depend. We shall look for \(w_p, \beta\) such that \(\varphi\) fourier decomposition on X is equal to:

$$\varphi(p) = \int_{-\infty}^{\infty} \eta(x)e^{-ipx}dx = \int_{-\infty}^{\infty} f(w)e^{i\lambda_x-ipx}dx$$

(8)

\(w_p\) is normalized, so the application of Plancherele identity to that norm gives

$$\int_{-\infty}^{\infty} w_p(p)dp = \int_{-\infty}^{\infty} \varphi(p)\varphi^*(p)dp = \int_{-\infty}^{\infty} f^2(w)dx = 1$$

(9)

To calculate \(f(w)\), let’s define the function \(\Theta = f^2\) and consider its variation \(\delta\Theta\). As follows from eq. (9):

$$\int_{-\infty}^{\infty} \delta\Theta dx = \int_{-\infty}^{\infty} \frac{\partial\Theta}{\partial w} \delta w dx = 0$$

(10)

with additional \(\delta w\) constraint: \(\int \delta w dx = 0\). Let’s substitute \(\delta w = \partial_x \varpi\); under these conditions it’s possible to choose \(\varpi(\pm\infty) = 0\), then, such \(\varpi\) satisfies to the conditions of Du Bois-Reymond lemma treated in appendix, see also [18]. Its application gives:
Θ = w, so that \( f = \pm \frac{1}{2} w \). Now \( \bar{p} \) can be calculated anew from derivative Fourier transform [19]

\[
\bar{p} = \int_{-\infty}^{\infty} p \varphi(p) \varphi^*(p) dp = \int_{-\infty}^{\infty} \frac{\partial \lambda_x}{\partial x} f^2 dx = \int_{-\infty}^{\infty} \frac{\partial \lambda_x}{\partial x} w(x) dx
\]  

(11)

From its comparison with eq. (7) and equality \( \bar{p} = \mu \bar{u} \), it follows

\[
v(x) = \frac{1}{\mu} \frac{\partial \lambda_x}{\partial x} (w, v)
\]  

(12)

For our model CP it follows \( \lambda_x(w, v) = \gamma(x) \) where \( \gamma \) is the functional:

\[
\gamma(x) = \mu \int_{-\infty}^{x} v(\xi) d\xi + c_\gamma
\]  

(13)

here \( c_\gamma \) is an arbitrary real number. The resulting \( m \) state \( |\zeta\rangle \) in \( x \)-representation is equal to:

\[
\eta(x) = w^{\frac{1}{2}}(x)e^{i\gamma}
\]  

(14)

so \( \eta(x) \) is \( L^2 \)-normalized vector (ray) of complex Hilbert space \( \mathcal{H} \), and it describes the complete set of \( m \) pure states. \( w_p(p) \) and \( \beta(p) \) can be calculated from eq. (8) as functions of \( w, \gamma \). In particular,

\[
w_p(p) = \left| \int_{-\infty}^{\infty} w^{\frac{1}{2}} e^{i\gamma - ipx} dx \right|^2
\]  

(15)

is independent of \( \beta(p) \), so \( w_p \) is just \( \eta \), i.e. \( w, v \) functional, the same is true for \( \beta_p \).

## 4 Linear Model of Fuzzy Dynamics

Resulting ansatz for \( m \) state dynamical representation can be written as \( \eta = g(x, t) \), where \( g(x, t) \) corresponds to standard QM wave function and \( \gamma(x, t) \) of (13) to the quantum phase. For \( g(x, t) \) state ansatz, which obeys to model CP given in sect. 3, \( v(x, t) \) can be treated here as the derivative of \( \gamma(x, t) \) of (13), so that \( \gamma(x, t) \) will be used for \( g(x, t) \) description on equal terms with \( v(x, t) \). The evolution equation for \( g \) presumably is of the first order in time, i.e.

\[
\frac{i\partial g}{\partial t} = \hat{H} g.
\]  

(16)

In general, \( \hat{H} \) can be nonlinear operator, for the simplicity we shall start with the linear \( \hat{H} \) and turn to nonlinear case later. Free \( m \) evolution is invariant relative to \( x \) space shift on arbitrary \( x_0 \) performed by the operator \( \hat{W}(x_0) = \exp(x_0 \frac{\partial}{\partial x}) \). Because
of it, corresponding operator \( \hat{H}_0 \) should commute with \( \hat{W}(x_0) \) for the arbitrary \( x_0 \), i.e. \( [\hat{H}_0, \frac{\hat{W}}{\partial x}] = 0 \). It holds only if \( \hat{H}_0 \) is differential polinom of the form

\[
\hat{H}_0 = -\sum_{l=1}^{n} b_{2l} \frac{\partial^{2l}}{\partial x^{2l}}
\]  

(17)

where \( b_{2l} \) are arbitrary real values, \( n \) is arbitrary number. From correspondence to classical mechanics, it supposed that the influence of potential field \( U \) on \( m \) evolution can be accounted in \( \hat{H} \) additively:

\[
\hat{H} = \hat{H}_0 + U(x, t)
\]  

(18)

where \( U \) is real, nonsingular function. Now eq. (16) can be rewritten as

\[
i \frac{\partial g}{\partial t} = (i \frac{\partial w^{\frac{1}{2}}}{\partial t} - w^{\frac{1}{2}} \frac{\partial \gamma}{\partial t}) e^{i\gamma} = e^{i\gamma} \hat{Z}g
\]  

(19)

where \( \hat{Z} = \exp(-i\gamma)\hat{H} \). Hence

\[
\frac{\partial w^{\frac{1}{2}}}{\partial t} = \text{im}(\hat{Z}g)
\]  

(20)

If to substitute \( v(x) \) by \( \gamma(x) \) in eq. (5) and transform it to \( w^{\frac{1}{2}} \) time derivative, it gives

\[
\frac{\partial w^{\frac{1}{2}}}{\partial t} = -\frac{1}{\mu} \frac{\partial w^{\frac{1}{2}}}{\partial x} \frac{\partial \gamma}{\partial x} - \frac{1}{2\mu} w^{\frac{3}{2}} \frac{\partial^2 \gamma}{\partial x^2}
\]  

(21)

Plainly, the right parts of equations (20) and (21) should coincide for arbitrary \( w, \gamma \), otherwise \( \hat{H} \) ansatz would be incompatible with \( w \) flow continuity described by eqs. (5, 21). Really, if they differ, it will impose the additional constraint on \( w, \gamma \) of the form \( \hat{L}_x(w, \gamma) = 0 \), where \( \hat{L}_x \) is the operator in \( x \)-derivatives. However, \( w, \gamma \) presumably are independent DFs, hence such constraint results in the open contradiction. Therefore, \( \hat{H} \) ansatz can be obtained from the term by term comparison of eqs. (20, 21). In particular, the equality of highest \( \gamma \) derivative for \( \text{im}(\hat{Z}g) \) and of left part of eq. (21) gives:

\[
-b_{2l} w^{\frac{1}{2}} \frac{\partial^2 \gamma}{\partial x^{2l}} = -\frac{1}{2\mu} w^{\frac{3}{2}} \frac{\partial^2 \gamma}{\partial x^2}
\]

It follows that \( b_2 = 1/2\mu \) and \( b_{2l} = 0 \) for \( l \geq 2 \), only in this case both expressions for \( \partial_t w^{\frac{1}{2}} \) would coincide. Therefore, \( g \) free evolution is described by single \( H_0 \) term: \( -\frac{1}{2\mu} \frac{\partial^2}{\partial x^2} \), and \( \hat{H} \) of eq. (18) is Schroedinger Hamiltonian for particle with mass \( \mu \). \( \gamma \) evolution equation can be extracted from eq. (19), and \( v \) evolution equation follows from it. Thus, the system of equations is obtained for \( w(x, t), v(x, t) \) functions, which obey to our model CP, under these conditions it’s equivalent to Schroedinger equation for \( g(x, t) \).

In this framework, the observable \( p \) corresponds to the operator \( \hat{p} = -i\partial_x \) acting on \( g(x) \). Thus, \( x \) and \( p \) observables are described by the linear self-adjoint operators,
which obey to the commutation relation \([\hat{x}, \hat{p}] = i\). We’ll admit that in general \(m\) projective observables \(\{Q\}\) are the linear, self-adjoint operators on \(\mathcal{H}\) with standard extension to POVM [9]. As the result, \(\{Q\}\) expectation values don’t depend on \(c_\gamma\) of eq. (13).

5 General Fuzzy Dynamics

In the previous section, 1-dimensional FM formalism was derived from CFG premises assuming that \(|\zeta\rangle\) evolution is linear. Now this assumption will be dropped and the general situation studied, next, FM formalism will be extended on 3-dimensional case. The conditions of QM linearity initially were formulated by Wigner [20], however, they are extensively discussed up to now [21]. Recently, Jordan have shown that they are essentially weaker than Wigner theorem asserts [22]. Namely, if the following two conditions are fulfilled:

i) the evolution operator maps the set of all pure states one to one onto itself

ii) for the arbitrary mixture of orthogonal states \(\rho(t) = \sum_i P_i(t) \rho_i(t)\) it conserves all \(P_i\) constant in time.

It follows then that such evolution is linear. Both these conditions are in good correspondence with geometric framework proposed by FM formalism. Really, it was shown that the pure states \(|\zeta\rangle\) describe the evolution of geometric object FP \(m\) from its initial to final state. Plainly, such evolution shouldn’t result in probabilistic mixture of pure states. It’s also natural to assume that such evolution is unambiguous and reversible. Another arguments against QM nonlinearity involve the locality violations, in particular, superluminal signaling, which can occur in such theory for multiparticle systems [23].

Summing up the previous results, it follows that for \(m\) states, which obeys to model CP, the state evolution in FM formalism coincides, in fact, with standard QM evolution; now we shall prove their equivalence for more loose \(m\) state CP. In particular, it should be extended on \(m\) states, localized temporarily or permanently in the restricted space region, or in several disjoint regions. It was assumed previously that \(v(x)\) is independent \(m\) DF, but proposed formalism is compatible also with alternative hypothesis, namely, that \(v\) is unambiguous function of some other \(m\) state parameter, which in its turn is independent of \(w\). It was shown that for our model CP \(v(x)\) is proportional to \(\gamma(x)\) derivative, hence it’s reasonable to consider \(\gamma(x)\) as such fundamental \(m\) DF. In this case, \(m\) state observational representation becomes: \(g^s = \{w(x,t), \gamma(x,t)\}\), in the same time, \(g(x,t)\) can be retained as \(m\) state dynamical representation. As the result, such DF modification permits to apply for \(g(x,t)\) more loose CP than used previously in our model; in particular, CP of standard QM formalism can be imposed [1, 25]. In particular, such \(m\) states can be disjoint, the example of such disjointness gives the class of functions \(\{g^\prime\}\), such that \(g'(x,t) \in C^2(\Gamma \times [t_0, t_1])\); \(g'(x,t) \in C^2(R^1 \times [t_1, T])\); \(t_0 \leq t_1\), here \(\Gamma = \cup \Delta_j\), where \(\{\Delta_j\}\) is the finite set of nonintersecting open intervals on \(R^1\).

3-dimensional FM, in fact, doesn’t demand the serious modification of described
formalism, however, some new features appear. Besides topological distinctions between $R^1$ and $R^3$ spaces, they can be related both to the state disjointness and presence of nodes and noding regions. Hence the monodromy of evolution equations should be accounted properly. Below the formalism derivation is described briefly making the impact on that new moments.

On $\tilde{R}^3$ manifold the particle $m$ corresponds to FP $a(t)$ characterized by its state $|\zeta(t)\rangle$. As was shown in sect. 3, it depends on density $w(\vec{r}, t)$ and some other $m$ DFs $\{q_i(\vec{r}, t)\}$ supposed to be the real functions. Assuming that $w$ evolution depends on local parameters only, it can be expressed as:

$$\frac{\partial w}{\partial t}(\vec{r}, t) = -\Phi[w(\vec{r}, t), q_1(\vec{r}, t), ...q_k(\vec{r}, t)] \quad (22)$$

where $\Phi$ is an arbitrary local function. Then from $w$ norm conservation

$$\int \Phi(\vec{r}, t)d^3r = \int \frac{\partial w}{\partial t}(\vec{r}, t)d^3r = \frac{\partial}{\partial t} \int w(\vec{r}, t)d^3r = 0 \quad (23)$$

where integration performed over $R^3$. Substituting $\Phi = -\text{div} \vec{J}$ where $\vec{J}(\vec{r})$ is differentiable function, it follows that $w$ evolution described by the flow continuity equation

$$\frac{\partial w}{\partial t} = -\text{div} \vec{J} \quad (24)$$

assuming that $\vec{J}$ flow through surrounding infinite surface $S$ is zero:

$$\int_{S} \vec{J}d\vec{s} = 0$$

It means that $\vec{J}(\vec{r}) \to 0$ for any $|\vec{r}| \to \infty$. Really, $w(\vec{r}) \to 0$ for $|\vec{r}| \to \infty$, hence for any $\vec{r}_a$ such that $|\vec{r}_a| \to \infty$, it should be $\vec{J}(\vec{r}_a) \to 0$, otherwise it would violate global rotational invariance which should be fulfilled for localized systems. One can decompose formally $\vec{J} = w\vec{v}$ and suppose that $w$ flow velocity $\vec{v}(\vec{r})$ is independent $m$ DF. $m$ state $|\zeta\rangle$ supposedly depends on $w, \vec{v}$ DFs only, the twople $\mathfrak{g}^o = \{w, \vec{v}\}$ describes observational $|\zeta\rangle$ representation on $R^3$, and $m$ state CP assumed to be: $w(\vec{r}, t), \vec{v}(\vec{r}, t) \in C^3(R^3 \times [t_0, T])$.

Analogously to 1-dimensional case, one should find for $R^3$ the dynamical $|\zeta\rangle$ representation $\eta$ which supposedly described by the set of real functions $\{\eta_i(\vec{r})\}$, their algebra is undefined at this stage; it can correspond, for example, to quaternion. However, we’ll start its search from $\eta$ ansatz in form of complex $w, \vec{v}$ functional $\eta(\vec{r}) = \Upsilon_r(w, \vec{v})$. It follows that its minimal ansatz is $\eta = f(w) \exp[i\Lambda_r(w, \vec{v})]$ where $f(w) \to 0$ for $w \to 0$, $\Lambda_r(w, \vec{v})$ is real functional. As for $m$ evolution in $R^1$, $m$ state characterized also by velocity $\vec{u}(t)$ with corresponding distribution $w_u(\vec{u})$, so that

$$\langle \vec{u} \rangle = \int \vec{u}w_u(\vec{u})d^3u = \langle \vec{v} \rangle = \int \vec{v}(\vec{r})w(\vec{r})d^3r \quad (25)$$
m kinematical fuzzy momentum defined as: \( \vec{p} = \mu \vec{u} \). From that, analogously to calculations described in sect. 3, standard QM ansatz for m state on \( \mathbb{R}^3 \) follows: \( g(\vec{r}) = w^* \exp(i\gamma) \), for which \( \gamma(\vec{r}) \) obeys to the equality \( \mu \vec{u} = \text{grad}(\gamma) \). Thus, \( g(\vec{r}) \) is vector (ray) in complex Hilbert space \( \mathcal{H} \) and describes the complete set of m pure states.

Considering \( g \) linear evolution, for free m evolution its operator \( \hat{H}_0 \) should be the even polinom of the form

\[
\hat{H}_0 = -\sum_{l=1}^{n} b_{2l} \frac{\partial^{2l}}{\partial \vec{r}^{2l}}
\]

If potential field influence can be described by the addition of real function \( U(\vec{r}, t) \) to \( \hat{H}_0 \), so that

\[
i \frac{\partial g}{\partial t} = \hat{H} g = (\hat{H}_0 + U) g.
\]

Then, analogously to 1-dimensional ansatz, the term \( \partial_t w^\frac{d}{2} \) can be extracted from this equation and expressed via corresponding \( w, \gamma \vec{r} \)-derivatives. From their comparison with corresponding \( \hat{H} g \vec{r} \)-derivatives, the Schroedinger equation follows for m evolution. Applicability of Jordan theorem to 3-dimensional case is obvious, because the derivation of \( \hat{H} \) linearity doesn’t depend on manifold dimension.

Now one should extend these results for more loose CP, to perform it, \( \vec{v}(\vec{r}) \) DF can be replaced by another fundamental DF \( \gamma(\vec{r}) \), for which \( \vec{v} \) is its derivative. Thereon, m state observational representation becomes: \( \rho^0 = \{w(\vec{r}), \gamma(\vec{r})\} \), while \( g(\vec{r}) \) can be retained as m state dynamical representation. As the result, this DF modification permits to apply for m states CP of standard QM formalism, which are essentially more loose [1, 25]. In particular, it permits to incorporate into our formalism the disjoint m states.

To make such m state \( g(\vec{r}) \) unambigous, the additional constraint should be imposed on \( \gamma(\vec{r}) \), namely, for any closed loop \( l \)

\[
\oint_l \frac{\partial \gamma}{\partial \vec{r}} d\vec{r} = \frac{2\pi}{\mu} n_l
\]

Normally, \( n_l = 0 \), but if the node or nodal region located inside loop \( l \), then \( n_l = 0, \pm 1, \pm 2, ..., \) etc. [26]. We consider here only the stationary noding regions in form of infinite lines. The example is \( 2p_1 \), \( 2p_{-1} \) states of hydrogen atom where nodal region is the line of \( z \)-axis, \( n_l = \pm 1 \).

It’s difficult to include \( \gamma(\vec{r}) \) into consistent geometric picture, due to the presence of indefinite component \( c_\gamma \) in it. Meanwhile, in CFG formalism some FP properties can be described by bilocal functions, i.e. \( \vec{r}_1, \vec{r}_2 \) correlations [4, 5]. By the analogy, \( \gamma(\vec{r}) \) can be replaced in m state description by the dynamical correlation \( \kappa(\vec{r}_1, \vec{r}_2) \) defined as

\[
\kappa(\vec{r}_1, \vec{r}_2) = \gamma(\vec{r}_1) - \gamma(\vec{r}_2).
\]

so that \( \kappa \) doesn’t depend on \( c_\gamma \). Hence for arbitrary \( \vec{r}_1 \neq \vec{r}_2 \)

\[
v(\vec{r}_1, t) = \frac{1}{\mu} \frac{\partial \kappa(\vec{r}_1, \vec{r}_2, t)}{\partial \vec{r}_1}.
\]
It’s notable that in standard QM, $\kappa(\vec{r}_1, \vec{r}_2)$ correlation appears in the description of quantum states by density matrixes, i.e. the positive, trace one operators on $\mathcal{H}$ [1]. Namely, the density matrix of pure state is equal to

$$\rho(\vec{r}_1, \vec{r}_2) = [w(\vec{r}_1)w(\vec{r}_2)]^{1/2} e^{i \kappa(\vec{r}_1, \vec{r}_2)} \quad (28)$$

so that $\rho(\vec{r}, \vec{r}) = w(\vec{r})$. For arbitrary observable $Q$ it gives $\bar{Q} = \text{Tr} \hat{Q} \rho$; $\rho$ evolution obeys to Liouville equation $\dot{\rho} = [\hat{H}, \rho]$ which is formally bilocal. In standard QM, the density matrix is considered, in fact, as auxiliary object exploited mainly for the description of mixed states and statistical ensembles. Alternatively, the state space of algebraic QM formalism is the density matrix space, which considered as fundamental states [1]. Our analysis indicates that in geometric QM formulation the density matrixes of pure states can be treated as fundamental dynamical representation of state $|\zeta\rangle$. Hence now the observational $|\zeta\rangle$ representation becomes $g^\zeta_k = \{w(\vec{r}_1), \kappa(\vec{r}_1, \vec{r}_2)\}$. Yet, for pure states $\rho(\vec{r}_1, \vec{r}_2) = g(\vec{r}_1) * \bar{g}(\vec{r}_2)$, hence $g(\vec{r})$ can be used as $\rho(\vec{r}_1, \vec{r}_2)$ description whenever it doesn’t violate described conditions. The similar properties possess the particle states on projective Hilbert space, in particular, such state ansatz doesn’t depend on $\gamma(\vec{r})$ indefinite component $c_\gamma$; in its framwork, the state evolution is also described by bilocal equation [2].

6 Discussion

Planck constant $\hbar = 1$ in this formalism, but the same value ascribed to it in Lorentz-Heaviside (relativistic) unit system, in which velocity of light $c = 1$. In FM framework, $\hbar$ connects $\vec{r}$, $\vec{p}$ scales and doesn’t have any additional meaning. The superposition principle doesn’t need to be postulated separately in such formalism. Rather, as follows from our results, the sum of two physical $m$ states $g_{1,2}$ with proper complex coefficients $a_{1,2}$ can be considered as the physical $m$ state also. In our approach, the state space is defined by the underlying geometry and corresponding dynamics i.e. is derivable concept. For states of nonrelativistic particle $m$ it was found to be equivalent to Hilbert space $\mathcal{H}$; however, for other systems the resulting state space supposedly can differ from $\mathcal{H}$ analogously to algebraic QM where the state space is defined by the observable algebra [1]. The flow velocity $\vec{v}(\vec{r})$ isn’t observable, but its value consistently defined as the ratio of $\vec{J}(\vec{r}), w(\vec{r})$ expectation values [27]; here $w(\vec{r})$ observable is described by the projection operator $\hat{\Pi}(\vec{r})$; observable $\vec{J}(\vec{r})$ defined in [27, 28].

The particle evolution in QM in some aspects is similar to the motion of continuous media, this analogy is exploited in hydrodynamical QM model [26]. Namely, in its framework, the Schroedinger equation is postulated and from it the flow continuity equation and other continuous media properties of quantum systems derived. However, it was shown some years ago that the model results differ from standard QM for disjoint initial states, and so it can’t give the complete description of quantum systems [24]. In FM formalism, FP density evolution also described by flow
continuity equation, but despite some formal similarity, the basic theory premises are principally different.

In standard QM the evolution equations are or postulated *ad hoc* or derived assuming Galilean invariance of system states [1, 28]. In FM the Schroedinger equation for massive particle was derived assuming only space-time shift and rotational invariance which are essentially weaker assumptions. Meanwhile, it’s well known that Galilean invariance can be derived from Schroedinger equation if a reference frame is associated with free system $S$ with mass $M_s \to \infty$ and which initial state $\Psi(t_0)$ is the wave packet with the width $\sigma_s \to 0$ [28, 29].

In this paper it was shown that QFG formalism can be considered as the mathematical basis for the consistent description of quantum particle dynamics. Novel features of considered formalism can be revealed in most simple way from the comparison with Schroedinger QM formalism. From the formal side, standard QM exploits two fundamental structures of different nature: the space-time manifold $R^3 \times T$ and function space $H$ defined on $R^3$. In distinction, minimal FM formalism involves only one basic structure - the fuzzy manifold $\tilde{R}^3 \times T$, nonrelativistic particles are $\tilde{R}^3$ elements - fuzzy points. Their evolution induces the physical states which are equivalent to $H$ Dirac vectors. Resulting dynamics of massive particles described by Shroedinger equation, which is the basis of universal quantum dynamics. The quantum-classical transition in such theory is essentially more simple than in standard QM, it’s just the transition from $\tilde{R}^3$ fuzzy manifold to $R^3$ manifold, on which the classical particles correspond to material points $\vec{r}(t)$. Here only the evolution of nonrelativistic particle was considered in FM formalism, but all QM formalism is, in fact, based on its analysis, and there are no obvious obstacles, which forbid to construct universal quantum dynamics in this approach.

It seems that FM approach is based on essentially more natural axiomatic than standard QM. Its main postulate is the principal uncertainty of some system parameters, like coordinate or momentum, such uncertainty has geometric origin and isn’t connected *a priori* with wave-like system properties. Rather, in FM the wave-like system evolution is stipulated by these geometric features. It indicates that topology of physical space-time can differ from topology of Euclidian or Minkowski spaces and to be comparatively more weak.

Plainly, general relativity is essentially geometric theory, meanwhile, up to now the attempts to quantize gravity meet serious difficulties even at axiomatic level. Hence, if consistent geometric QM formalism will be constructed, it can help, in principle, to develop quantum theory of gravity. Currently, the main impact of QM geometrization studies is done on the exploit of Hilbert manifolds ([3] and refs. therein), however, the results obtained in this approach have quite abstract form, and their applicability to particular physical problems isn’t obvious. The considered FM formalism possesses simple and logical axiomatics which origin is basically geometrical, hence it can become the appropriate part of QM geometrization program, its implications can be important also for the analysis of QM foundations.
Appendix

Here the proof of Du Bois-Reimond lemma is considered according to [30], it extended also for functions defined on $\mathbb{R}^3$.

Lemma: if for continuous function $N(x)$ and arbitrary continuous, differentiable function $\tau(x)$ for which $\tau(a) = \tau(b) = 0$, $(a = -\infty, b = \infty$ permitted) and $\tau' = \frac{\partial \tau}{\partial x}$

$$I(\tau) = \int_a^b N(x)\tau' dx = 0$$

then $N(x)$ is constant on $[a, b]$ interval. Suppose that on the opposite, $N(x)$ varies on $[a, b]$, then it should be at least two points $c_1, c_2$ for which $N$ values differ, for example, $N(c_2) < N(c_1)$. Let’s $d_1, d_2$ to be the numbers for which the following inequality holds

$$N(c_2) < d_2 < d_1 < N(c_1)$$

For large enough $n$ it’s always possible to construct the nonintersecting intervals $[x_0, x_0 + \pi/n], [x_1, x_1 + \pi/n]$ confined in $[a, b]$ and such that inside $[x_0, x_0 + \pi/n]$ the inequality $N(x) > d_1$ holds, and inside another one: $N(x) < d_2$. Let’s chose $\tau'(x)$ as follows:

$$\tau' = \sin^3[n(x - x_0)] \text{ on } [x_0, x_0 + \pi/n];$$
$$\tau' = -\sin^3[n(x - x_1)] \text{ on } [x_1, x_1 + \pi/n];$$
$$\tau' = 0 \text{ on the rest of } [a, b].$$

In that case, function $\tau(x)$ and its derivative are continuous. In addition, lemma premises suppose that $I(\tau) = 0$, but for admitted $N$ variations in considered interval it follows that for the same integral

$$I(\tau) = \int_a^b N(x)\tau'(x) dx > (d_1 - d_2) \int_0^{\pi} \sin^3 n x \, dx > 0 \quad (29)$$

Thus $N(x)$ should be constant on $[a, b]$ interval.

For $\mathbb{R}^3$ the proof is given here only for functions defined on unbounded volume. For our problem it’s possible to substitute $\delta w(\vec{r}) = \text{div} \vec{E}(\vec{r})$. Under these conditions the relation between $\delta w$ and $\vec{E}$ described by the solution of Laplace equation well known in electrodynamics [31]. Then one should prove that if for continuous scalar $N(\vec{r})$ and any continuous vector $\vec{E}(\vec{r})$ for which $\text{div} \vec{E} \rightarrow 0$ for $|\vec{r}| \rightarrow \infty$

$$I_w(\vec{E}) = \int N \text{div} \vec{E} \, d^3 r = 0$$

then $N(\vec{r})$ is constant on $\mathbb{R}^3$.

Suppose that on the opposite, $N(\vec{r})$ varies on $\mathbb{R}^3$, then it should be at least two points $\vec{r}_1, \vec{r}_2$ for which $N$ values differ, for example, $N(\vec{r}_2) < N(\vec{r}_1)$. Let’s $d_1, d_2$ to be the numbers for which the following inequality holds

$$N(\vec{r}_2) < d_2 < d_1 < N(\vec{r}_1)$$
For large enough \( n \) it’s always possible to construct the nonintersecting spheres \( S_{1,2} \):
\[
|\vec{r} - \vec{r}_{1,2}| \leq \frac{\pi}{2n},
\]
such that inside \( S_1 \) the inequality \( N(\vec{r}) > d_1 \) holds, and inside \( S_2 \), \( N(\vec{r}) < d_2 \). Let’s choose \( \text{div}\vec{E} \) as follows:
\[
\text{div} \vec{E} = \cos^3(n|\vec{r} - \vec{r}_1|) \text{ in } S_1;
\]
\[
\text{div} \vec{E} = -\cos^3(n|\vec{r} - \vec{r}_2|) \text{ in } S_2;
\]
\[
\text{div} \vec{E} = 0 \text{ outside of } S_{1,2}.
\]
From lemma assumption \( I_w(\vec{E}) = 0 \), but for admitted \( N \) difference in considered \( S_{1,2} \) regions it follows
\[
I_w(\vec{E}) = \int N\text{div}\vec{E} \, d^3r > (d_1 - d_2) \int_0^{\frac{\pi}{2n}} \cos^3(n|\vec{r}|) \, d^3r > 0
\]
Thus, \( N(\vec{r}) \) should be constant.

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