A Solution of the P versus NP Problem based on specific property of clique function

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Abstract

Circuit lower bounds are important since it is believed that a super-polynomial circuit lower bound for a problem in NP implies that \( P \neq NP \) \[^1\]. Razborov has proved superpolynomial lower bounds for monotone circuits by using “method of approximation” \[^2\]. By extending this approach, researchers have proved exponential lower bounds for the monotone network complexity of several different functions \[^3\]-\[^6\]. But until now, no one could prove a non-linear lower bound for the non-monotone complexity of any Boolean function in NP. While we show that in this paper by replacement of each “Not” gates into constant “1” equivalently in standard circuit for clique problem, it can be proved that non-monotone network has the same or higher lower bound compared to the monotone one for computing the clique function. This indicates that the non-monotone network complexity of the clique function is super-polynomial which implies that \( P \neq NP \).
Introduction

An attempt to solve P versus NP Problem is to demonstrate whether a super-polynomial lower bound on the size of Boolean circuits solving NP-complete problem, like 3-SAT or Clique problem exists. In 1949, Shannon \[7\] proved that for almost all Boolean functions \( f: \{0, 1\}^n \rightarrow \{0, 1\} \), for computing \( f \) it requires at least \( \frac{2n}{n} \) gates. Unfortunately, his counting argument do not help to prove lower bounds for problems. In 1985, Razborov \[2\] and Andreev \[3\] successively proved an \( n^{\Omega(\log n)} \) lower bound on the monotone-size of the Clique function by using “method of approximation”. This was the first super-polynomial bound on the monotone-size of any explicit function and was improved to \( \omega(\frac{n^k}{\log^k n}) \) by Alon and Boppana \[4\] later. Some other works also proved super-polynomial lower bound on the monotone-size of clique-like functions with similar approach but more beautiful presentations \[5,6\]. But until now, no one could prove a non-linear lower bound for the non-monotone complexity of any Boolean function in NP.

In 2017, Norbert Blum \[8\] tried to extend the approximation approach to the non-monotone complexity of a Boolean function but failed at last. He tried to prove that any monotone Boolean function keep the characteristics of having same lower bound for non-monotone and monotone network which cannot be true due the
existence of functions like Tardos function and etc.\[^9\].

In this paper, we demonstrated that non-monotone complexity of the clique function is equal to or even larger than the monotone complexity based on a new approach. We showed that non-monotone networks with “NOT” gate can be transformed to a monotone networks equivalently without increment on the size of the circuit which is not only due to the monotone characteristics but also rely on the unique properties of the clique function.

**Preliminaries**

A Boolean circuit is a directed acyclic graph with gate nodes (or, simply gates) and input nodes. Operation AND or OR is associated with each gate whose indegree is 2 which is represented by “\(\wedge\)” and “\(\vee\)” in this paper for short, respectively. Not gate whose in-degree is 1 which is represented by “\(\neg\)” for short, like “\(\neg A\)” represents for NOT(A). A Boolean variable or a constant, namely, 0 or 1, is associated with each input node whose in-degree is 0. In particular, a circuit with no NOT gates is called monotone. A Boolean function of \(n\) variables is called monotone if \(f(w)\leq f(w')\) holds for any \(w, w' \in \{0,1\}^n\) such that \(w\leq w'\). Let \(M^n\) denote the set of all monotone functions of \(n\) variables. The size of a circuit \(C\), denoted \(\text{size}(C)\), is the number of gates in the circuit \(C\). The circuit complexity of a function \(f\), denoted by \(\text{size}(f)\), is the size of the smallest circuit
computing $f$.

For $1 \leq s \leq m$, let $\text{CLIQUE}(m,s)$ be the Boolean function of $n:={m \choose 2}$ variables representing the edges of an undirected graph $G = (V, E)$ on $m$ nodes (when the value of variable is 1 means the corresponding edge is connected and vice versa). $\text{CLIQUE}(m,s)(x)=1$ iff the corresponding graph $G$ contains a clique of size $s$. Let $C_f$ denote the circuit which compute the Boolean function $\text{CLIQUE}(m,s)$.

For any circuit network $\beta$, we can convert $\beta$ to an equivalent network $\beta'$ where all negations occur only at the input nodes and the size of $\beta$ is at most doubled[]. The equivalent network $\beta'$ is a so-called standard network where only input variables are negated. We consider a negated variable $\neg x_i$ as an input node $g$ with $\text{op}(g) = \neg x_i$. The standard circuit complexity $c_{st}(f)$ of a function $f \in \text{M}_n$ is the size of a smallest standard network which computes $f$. Note that the standard and the non-monotone complexity of a function $f$ differs at most by the factor two. Hence, for proving a super-linear lower bound for the non-monotone complexity of a Boolean function, we can restrict us to the consideration of standard networks.

Now we can suppose that the $c_{st}$ which compute the Boolean function $\text{CLIQUE}(m,s)$ can be written as:
\[ Cf_n (\text{CLIQUE}(m, s)) = f(x_1, x_2, x_3, \ldots, x_n, \neg x_1, \neg x_2, \neg x_3, \ldots, \neg x_n) \]

Where \( x_1, x_2, x_3, \ldots, x_n \) are variables and \( \neg x_1, \neg x_2, \neg x_3, \ldots, \neg x_n \) are negated variables. \( f(x_1, x_2, x_3, \ldots, x_n, \neg x_1, \neg x_2, \neg x_3, \ldots, \neg x_n) \) represents some combinary form of both variables and negated variables with “\( \land \)” and “\( \lor \)” gates.

For example, \( f(x_1, x_2, x_3, \ldots, x_n, \neg x_1, \neg x_2, \neg x_3, \ldots, \neg x_n) \) can be given as:

\[
f(x_1, x_2, x_3, \ldots, x_n, \neg x_1, \neg x_2, \neg x_3, \ldots, \neg x_n) = (x_1 \land x_2 \lor \neg x_3) \land x_4 \land \neg x_5 \lor \neg x_1 \lor x_2 \lor x_5 \land x_7 \land x_8 \lor \ldots
\]

(1)

It is just a rough example of \( f(x_1, x_2, x_3, \ldots, x_n, \neg x_1, \neg x_2, \neg x_3, \ldots, \neg x_n) \). Note that for same function, \( f(x_1, x_2, x_3, \ldots, x_n, \neg x_1, \neg x_2, \neg x_3, \ldots, \neg x_n) \) may have different forms. The minimized circuit size of different forms can also be very different.

**Proofs of the equivalence of the monotone and non-monotone complexity for the clique function**

It has been proved that the lower bounds for the monotone network complexity of the clique function is exponential. Now we will demonstrate that any non-monotone network with “NOT” gate for the clique function can be transformed to an equivalent monotone circuit without increment of the circuit size.

For any form of \( f(x_1, x_2, x_3, \ldots, x_n, \neg x_1, \neg x_2, \neg x_3, \ldots, \neg x_n) \), we can transform it into the similar form like shown below by exacting the
negated variable $-x_i$:

$$f(x_1, x_2, x_3, ... x_n, -x_i, -x_2, -x_3, ... -x_n)$$

$$= -x_i \land ((x_6 \lor -x_7 \lor x_8) \land x_9 \land -x_{10} \lor x_{11} \lor x_{12} \lor ... \lor (x_1 \land x_2 \land -x_3) \land x_4 \land x_5 \land -x_5)$$

$$\lor (x_1 \land (x_5 \lor -x_{15} \lor x_{18}) \lor -x_{16}) \land x_9 \land x_{10} \land x_{13} + ...$$

(2)

Note that the exacted negated variable can be an arbitrary one.

The operation following the $-x_i$ will always be “$\land$” gate when $f(x_1, x_2, x_3, ... x_n, -x_i, -x_2, -x_3, ... -x_n)$ represents a monotone function. Eq. (2) is just an example not a strict expression like Eq. (1) but it does have universal significance. To write it in a more formal manner, Eq. (2) can be expressed as:

$$f(x_1, x_2, x_3, ... x_n, -x_i, -x_2, -x_3, ... -x_n)$$

$$= -x_i \lor \text{Term}A \lor \text{Term}B \lor \text{Term}C + ...$$

(3)

where $\text{Term}A$, $\text{Term}B$, $\text{Term}C ...$ represent arbitrary combinations of variables and operations except the extracted negated variable which is $-x_i$ in this case. Any Boolean circuit network for monotone functions can be expressed in this form by extracting one of its negated variables.

Then the theorem below will demonstrate that the negated variable $-x_i$ can be replaced by constant 1 without influence the value of $f(x_1, x_2, x_3, ... x_n, -x_i, -x_2, -x_3, ... -x_n)$ for Clique function with all different inputs. This means $f(x_1, x_2, x_3, ... x_n, -x_i, -x_2, -x_3, ... -x_n)$ can still compute CLIQUE(m,s) function with negated variable $-x_i$ replaced by constant 1.
Theorem 1 Let \( f(x_1, x_2, \ldots, x_n, \neg x_1, \neg x_2, \ldots, \neg x_n) \) be a standard network which computes CLIQUE(m,s) Boolean function. Then the following hold:

By replacement of one of the negated variables \( \neg x_i \) (\( i = 1 \ldots n \)) in \( f(x_1, x_2, \ldots, x_n, \neg x_1, \neg x_2, \ldots, \neg x_n) \) into constant 1, the new network \( f(x_1, x_2, \ldots, x_n, 1, \neg x_2, \ldots, \neg x_n) \) still computes CLIQUE(m,s) function correctly.

Proof of Theorem 1:

Let’s focus on the first term in (2) (also \( \neg x_i \land \text{TermA} \) in Eq. 3) which is:

\[
\text{Term1} = \neg x_i \left( (x_6 \lor \neg x_7 \lor x_8) \land x_9 \land \neg x_{10} \lor x_{11} \lor x_{12} \lor \ldots \right) \quad (4)
\]

This is the only term containing \( \neg x_i \). By considering the characteristics of the truth table of the CLIQUE function, we can analysis the influence of the replacement of \( \neg x_i \) to constant 1. As can be seen directly, Term1 consists of two part. \( \neg x_i \) is the first part and the left formulas is the second part which is

\[
\text{Term1part2} = \text{TermA} = \left( (x_6 \lor \neg x_7 \lor x_8) \land x_9 \land \neg x_{10} \lor x_{11} \lor x_{12} \lor \ldots \right) .
\]

According to the value of the second part of Term1, we distinguish two cases.

Case 1: The value of Term1part2 is 0.

It is obvious that when the value of Term1part2 is 0, the replacement of \( \neg x_i \) to constant 1 will not have any influence to \( f(x_1, x_2, \ldots, x_n, 1, \neg x_2, \ldots, \neg x_n) \). Because no matter what the value of
\(-x_i\) is, the value of Term1 will always be 0.

Case 2: The value of Term1 part2 is 1.

This the key part of the proposed method. When the value of Term1 part2 is 1, it means the value of 
\(f(x_1, x_2, x_3...x_n, \neg x_1, \neg x_2, \neg x_3...\neg x_n)\)
is also 1. Because when \(-x_i\) also takes the value of 1, the value of Term1 is 1 which leads to 
\(f(x_1, x_2, x_3...x_n, \neg x_1, \neg x_2, \neg x_3...\neg x_n)=1\).

However, \(-x_i\) equals to 1 means \(x_i\) equals to 0. This means the corresponding edge is disconnected which implies that \(-x_i\) has no contribution to the size of clique. Thus the “1” value of 
\(f(x_1, x_2, x_3...x_n, \neg x_1, \neg x_2, \neg x_3...\neg x_n)\) comes totally from Term1 part2. Thus, as long as Term1 part2 equals to 1, the value of 
\(f(x_1, x_2, x_3...x_n, \neg x_1, \neg x_2, \neg x_3...\neg x_n)\) will become 1 no matter what value \(-x_i\) is. This can also be seen from the truth table of the Clique function.

When both Term1 part2 and \(-x_i\) take the value of 1, it corresponds to several inputs with outputs 1 in the truth table like purple rows shown in the Table. 1. It is not difficult to find that if one change the value of \(x_i\) to 1 (i.e. \(-x_i=0\)) meanwhile keep values of all variables contained in Term1 part2 unchanged, the outputs remain “Truth” like blue rows in the Table. 1. This indicates that when the value of Term1 part2 equals to 1, the 
\(f(x_1, x_2, x_3...x_n, \neg x_1, \neg x_2, \neg x_3...\neg x_n)=1\) no matter what the value of \(-x_i\) is.
This is due to the reason that $\text{Term}_{1 \cdot part 2} = 1$ already means clique with size larger than $s$ exists. So the output value can always be 1 as long as the inputs satisfy $\text{Term}_{1 \cdot part 2} = 1$. Thus $\neg x_i$ can also be replaced by constant 1 without influence on the output of circuit when the value of $\text{Term}_{1 \cdot part 2}$ is chosen to be 1.

| $x_1$ | $x_2$ | $x_3$ | ... | $x_{n-1}$ | $x_n$ | $Y$  |
|-------|-------|-------|------|-----------|-------|------|
| 1     | 1     | 1     | ...  | 1         | 1     | 1    |
| 1     | 1     | 1     | ...  | 1         | 0     | 1    |
| 1     | 1     | 1     | ...  | 0         | 1     | 1    |
| 1     | 1     | 1     | ...  | 0         | 0     | 1    |

Table 1 Truth table of Clique function (purple rows stand for the inputs which make the $\text{Term}_1 = 1$; blue rows stand for the inputs which satisfy $\text{Term}_{1 \cdot part 2} = 1$ while $\neg x_i = 0$).

Considering both situations, we have proved the Theorem 1.
Then, we extend the replacement to all of negated variables one by one. The following Theorem 2 can be naturally proved.

**Theorem 2** Let \( f(x_1, x_2, \ldots, x_n, \neg x_1, \neg x_2, \neg x_3, \ldots, \neg x_n) \) be a standard network which computes CLIQUE\((m,s)\) function. Then by replacement of all of the negated variables in \( f(x_1, x_2, \ldots, x_n, \neg x_1, \neg x_2, \neg x_3, \ldots, \neg x_n) \) into constant 1, the new network \( f(x_1, x_2, \ldots, x_n, 1, 1, \ldots, 1) \) still computes CLIQUE\((m,s)\) function correctly.

Theorem 2 indicates that any standard network which computes CLIQUE\((m,s)\) Boolean function can be transformed to an equivalent monotone circuit by replacement of all the negated variables to constant 1. It is obvious that this process will not increase the complexity of the circuit. This means that standard network do not have smaller circuit size than the monotone one for Clique function. For the reason that the circuit size of monotone network of Clique function has proven to be exponential, we can conclude that non-monotone network complexity of the clique function is also super-polynomial which implies that P\( \neq \)NP.
Conclusion

To sum up, the non-monotone complexity of clique function is proven to be larger than its monotone network complexity by using equivalent replacement of negated variables in non-monotone circuit to constant value of 1 according to specific properties of the clique function. Because the monotone complexity of clique function has proven to be exponential, the non-monotone circuit complexity must be super-polynomial which implies that P≠NP.

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