Diameter estimates in Kähler geometry

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Abstract

Diameter estimates for Kähler metrics are established which require only an entropy bound and no lower bound on the Ricci curvature. The proof builds on recent PDE techniques for $L^\infty$ estimates for the Monge–Ampère equation, with a key improvement allowing degeneracies of the volume form of codimension strictly greater than one. As a consequence, we solve the long-standing problem of uniform diameter bounds and Gromov–Hausdorff convergence of the Kähler–Ricci flow, for both finite-time and long-time solutions.

1 | INTRODUCTION

The diameter is one of the most important geometric invariants defined by a metric. Bounds for the diameter are for example essential in the study of convergence of manifolds, which is of particular interest in moduli problems and geometric flows, where one hopes to arrive at a canonical model by taking limits. Unfortunately, in Riemannian geometry, there are very few tools for estimating the diameter, besides comparison theorems and the Bonnet–Myer theorem, which require that the Ricci curvature be strictly positive. The situation did not seem markedly different in Kähler geometry, although we can exploit there the fact that the potential can be viewed as the solution of a complex Monge–Ampère equation with right-hand side given by its volume form (see, e.g., Refs. [10, 16, 22, 25, 31]). However, there has been considerable progress recently in PDE methods for $L^\infty$ estimates for fully nonlinear equations [11, 12, 14]. These new methods turn out to be particularly amenable to geometric estimates, and have been shown to imply some promising estimates for noncollapse [17] and for the Green’s function [13].

The main goal of the present paper is to develop a general theory of diameter estimates in Kähler geometry. We shall be particularly interested in estimates, which require only an upper bound for the entropy of the volume form, but not a lower bound for the Ricci curvature. Our methods build on the PDE methods mentioned above [13, 14, 17], but with a crucial new ingredient required for
the most important geometric applications. Indeed, in such applications, it is important that the
diameter estimates be uniform with respect to suitable subsets, which can approach the boundary
of the Kähler cone. Now the methods of Refs. [13, 14, 17] typically apply to fully nonlinear
equations, which satisfy a specific structural condition, corresponding to a condition of nowhere
vanishing of the volume form in the case of Monge–Ampère. It has been recently shown by Harvey
and Lawson [18] that this condition is natural and applies to very broad classes of equations.
Nevertheless, for many applications which we shall consider in this paper, notably the Kähler–Ricci flow
and the analytic Minimal Model Program, it is necessary to allow the volume form to be arbitrarily
close to vanishing along subsets which are suitably small, such as a proper complex subvariety. It
is one of the main contributions of the present paper to obtain diameter estimates, which remain
uniform under such degenerations. This requires an argument by contradiction (Section 5), which
is quite different from those in Refs. [11, 13, 14], and which may be of independent interest.

We now state our general diameter estimates more precisely. Let \((X, \omega_X)\) be an \(n\)-dimensional
compact Kähler manifold equipped with a Kähler metric \(\omega_X\). Let \(\mathcal{K}(X)\) be the space of Kähler
metrics on \(X\). We define the \(p\)-Nash entropy of a Kähler metric \(\omega \in \mathcal{K}(X)\) associated to \((X, \omega_X)\) by

\[
\mathcal{N}_{X, \omega_X, p}(\omega) = \frac{1}{V_\omega} \int_X \left| \log \left( \frac{(V_\omega)^{-1} \omega^n}{\omega_X^n} \right) \right|^p \omega^n, \quad V_\omega = \int_X \omega^n = [\omega]^n,
\]

for \(p > 0\). If we write \(e^F = \frac{1}{V_\omega} \omega^n\), then

\[
\mathcal{N}_{X, \omega_X, p}(\omega) = \int_X e^F |F|^p \omega_X^n = \|e^F\|_{L^1 \log L^p(X, \omega_X)}
\]

and

\[
\mathcal{N}_{X, \omega_X, p}(\omega) \leq \int_{F \geq 0} e^F F^p \omega_X^n + C
\]

for some \(C = C(X, \omega_X, p) > 0\).

We introduce the following set of admissible functions for given parameters \(A, B, K > 0, p > n,\)

\[
\mathcal{V}(X, \omega_X, n, A, p, K) = \{ \omega \in \mathcal{K}(X) : [\omega] \cdot [\omega_X]^{n-1} \leq A, ~ \mathcal{N}_{X, \omega_X, p}(\omega) \leq K \}.
\]

Let \(\gamma\) be a non-negative continuous function. We further define a subset of \(\mathcal{V}(X, \omega_X, n, A, p, K)\) by

\[
\mathcal{W}(X, \omega_X, n, A, p, K; \gamma) = \left\{ \omega \in \mathcal{V}(X, \omega_X, n, A, p, K) : (V_\omega)^{-1} \omega_X^n \geq \gamma \right\}.
\]

We also define the Green’s function \(G(x, y)\) associated to a Riemannian manifold \((X, g)\) by (see,
e.g., Ref. [23])

\[
\Delta_g G(x, \cdot) = -\delta_x(\cdot) + (\text{Vol}_g(X))^{-1},
\]

where \(\Delta_g\) is the Laplace operator associated to \(g\). The following is the main theorem of our paper:
Theorem 1.1. Let $X$ be an $n$-dimensional connected Kähler manifold equipped with a Kähler metric $\omega_X$ and let $\gamma$ be a non-negative continuous function on $X$ satisfying

$$\dim_H \{ \gamma = 0 \} < 2n - 1, \quad \gamma \geq 0,$$

(1.4)

where $\dim_H$ is the Hausdorff dimension. Then for any $A, K > 0$ and $p > n$, there exist $C = C(X, \omega_X, n, A, p, K, \gamma) > 0$, $c = c(X, \omega_X, n, A, p, K, \gamma) > 0$ and $\alpha = \alpha(n, p) > 0$ such that for any $\omega \in \mathcal{W}(X, \omega_X, n, A, p, K; \gamma)$, we have the following bounds for

(a) The Green’s function:

$$\int_X |G(x, \cdot)| \omega^n + \int_X |\nabla G(x, \cdot)| \omega^n + \left( -\inf_{y \in X} G(x, y) \right) \Vol_\omega(X) \leq C$$

for any $x \in X$;

(b) The diameter

$$\diam(X, \omega) \leq C;$$

(c) The volume element: for any $x \in X$ and any $R \in (0, 1]$,

$$\frac{\Vol_\omega(B_\omega(x, R))}{\Vol_\omega(X)} \geq c R^\alpha.$$

This is the first general result on uniform diameter bounds and volume noncollapsing estimates for Kähler manifolds without any curvature assumption. We note that the assumption on the Hausdorff dimension of the set $\{ \gamma = 0 \}$ can be replaced by the weaker assumption that this set have small measure together with the connectedness of $\{ \gamma > 0 \}$ (c.f., Propositions 5.1 and 6.1), and that this is, in fact, how the theorem is proved. In practice, the theorem is often applied with $\{ \gamma = 0 \}$ supported on a proper analytic subvariety of $X$. It may be instructive to compare it with the situation in Riemannian geometry. There the sharp result is the theorem of Cheng and Li [6], where a lower bound for the Green’s function requires a lower bound on the Ricci curvature. Since $\text{Ric}(\omega) = -i \partial \overline{\partial} \log \omega^n$ in Kähler geometry, and we allow the lower bound $\gamma$ for the volume form $\omega^n$ to vanish, we see that Theorem 1.1 can give lower bounds for the Green’s function even when no lower bound for the Ricci curvature is available. As we shall see, this flexibility is particularly important in the study of the Kähler–Ricci flow and of fibrations of Calabi–Yau manifolds. As a consequence of Theorem 1.1, we obtain the following theorem, which can be viewed as a Kähler analog of Gromov’s precompactness theorem for metric spaces:

Theorem 1.2. Let $X$ be an $n$-dimensional connected Kähler manifold equipped with a Kähler metric $\omega_X$ and let $\gamma$ be a nonnegative continuous function on $X$ with

$$\dim_H \{ \gamma = 0 \} < 2n - 1.$$

Then for any $A, K > 0$, $p > n$ and any sequence $\{ \omega_j \}_{j=1}^\infty \subset \mathcal{W}(X, \omega_X, n, A, p, K; \gamma)$, after passing to a subsequence, $(X, \omega_j)$ converges in Gromov–Hausdorff topology to a compact metric space $(X_\infty, d_\infty)$. 
Geometric compactness is fundamental for understanding degeneration and moduli problems for complex Riemannian manifolds. Curvature bounds are usually necessary such as in the general theory of Cheeger-Colding [4]. Theorem 1.2 bypasses the curvature requirement to provide boundedness for families of Kähler manifolds. It might also be combined with techniques from Ref. [5] to explore the formation of singularities and achieve stronger geometric regularity for the limiting metric spaces.

2 APPLICATIONS TO THE KÄHLER–RICCI FLOW

We describe now the applications of Theorem 1.1 to the Kähler–Ricci flow and constant scalar curvature Kähler metrics. We shall be particularly interested in the analytic Minimal Model Program introduced in Refs. [24, 28] in relation to the formation of both finite time and long-time singularities in the Kähler–Ricci flow.

We first consider the following unnormalized Kähler–Ricci flow on a Kähler manifold $X$ with an initial Kähler metric $g_0$

$$\begin{align*}
\frac{\partial g}{\partial t} &= -\text{Ric}(g), \\
g|_{t=0} &= g_0.
\end{align*}$$

Let

$$T = \sup\{ t > 0 \mid [g_0] + t[K_X] > 0 \} \in \mathbb{R} \cup \{\infty\}.$$  

It is shown in Refs. [34, 36] that the Kähler–Ricci flow has a maximal solution $g(t)$ for $t \in [0,T)$.

2.1 The case of finite-time singularities

If $T < \infty$, the flow (2.1) must develop singularities at $t = T$. In this case, $\alpha_T = [g_0] + T[K_X]$ is a nef class on $X$. If $[g_0] \in H^2(X, \mathbb{Q})$, $\alpha_T$ is semi-ample by Kawamata’s base point free theorem and the numerical dimension of $\alpha_T$ coincides with its Kodaira dimension. In general, it is unclear if there exists a smooth semi-positive closed $(1,1)$-form in $\alpha_T$. The most interesting case is when $\alpha_T$ is big, that is, there exists a Kähler current in $\alpha_T$ or equivalently, $(\alpha_T)^n > 0$. Such a bigness condition can be also interpreted by the total volume along the Kähler–Ricci flow as

$$(\alpha_T)^n = \lim_{t \to T} \text{Vol}_{g(t)}(X) > 0.$$  

This bigness assumption for the limiting class is in fact generic for finite-time singularities.

It is conjectured in Ref. [28] as part of the analytic minimal model program that when $\alpha_T$ is big, $(X, g(t))$ should converge to a compact Kähler variety and the flow will extend uniquely through the singular time through a canonical metric surgery. Such a surgery corresponds to either a divisorial contraction or a flip in birational geometry. After suitable blow-ups, the singularity model is expected to be a transition from a shrinking soliton to an expanding soliton (c.f., Ref. [24]). This is confirmed in the case of Kähler surfaces by Refs. [31, 32], where it is shown that the Kähler–Ricci
flow contracts finitely many holomorphic $S^2$ of $(-1)$ self-intersection in Gromov–Hausdorff (GH) topology. The following diameter bound is the first step to understand the formation of finite time singularities for the Kähler–Ricci flow in general dimension. It has been a central problem in the whole theory for a long time:

**Theorem 2.1.** Let $(X, g_0)$ be a Kähler manifold equipped with a Kähler metric $g_0$. If $g(t)$ is the maximal solution of the Kähler–Ricci flow (2.1) for $t \in [0, T)$ for some $T \in \mathbb{R}^+$ and if the limiting class $[g_0] + T[K_X]$ is big, then there exist $C = C(X, g_0) > 0$, $c = c(X, g_0) > 0$ and $\alpha = \alpha(X, g_0) > 0$ such that for any $t \in [0, T)$,

$$\text{diam}(X, g(t)) \leq C,$$

$$\int_X |G_t(x, \cdot)|dV_{g(t)} + \int_X |\nabla G_t(x, \cdot)|dV_{g(t)} + \left( - \inf_{y \in X} G_t(x, y) \right) \text{Vol}_{g(t)}(X) \leq C,$$

$$\frac{\text{Vol}_{g(t)}(B_{g(t)}(x, R))}{\text{Vol}_{g(t)}(X)} \geq cR^{-\alpha},$$

for any $x \in X$ and $R \in (0, 1]$, where $G_t$ is the Green’s function for $(X, g(t))$.

We stress that Theorem 2.1 holds for general Kähler manifolds, and no projectiveness assumption is needed. We prove it in Section 8 by establishing a uniform upper bound for the $p$-Nash entropy for any $p > 0$ and a lower bound for the volume form along the flow. In addition to the assumptions of Theorem 2.1, if $[g_0] \in \mathbb{H}^2(X, \mathbb{Q})$ is a rational class, then the limiting big class $\alpha_T$ is semi-ample by Kawamata’s base point free theorem. The linear system $|m\alpha_T|$ for sufficiently large $m \in \mathbb{Z}^+$ induces a unique surjective birational morphism

$$\Phi : X \to Y \subset \mathbb{C}P^N,$$

where $Y$ is a projective variety birational to $X$ embedded in some $\mathbb{C}P^N$ via $\Phi$. We choose $g_Y$ to be the restriction of the Fubini–Study metric of $\mathbb{C}P^N$ to $Y$. Let $X^\circ$ be the regular set of $\Phi$ and $Y^\circ = \Phi(X^\circ)$. Then $\Phi$ is biholomorphic between $X^\circ$ and $Y^\circ$, both of which are Zariski open subsets of $X$ and $Y$, respectively.

We can now establish the following geometric convergence theorem for noncollapsing finite time solutions as one of the most challenging and important problems in the study of the Kähler–Ricci flow,

**Theorem 2.2.** Let $(X, g_0)$ be a Kähler manifold equipped with a Kähler metric $g_0 \in \mathbb{H}^2(X, \mathbb{Q})$. If $g(t)$ is the maximal solution of the Kähler–Ricci flow (2.1) for $t \in [0, T)$ for some $T \in \mathbb{R}^+$ and if the limiting class $[g_0] + T[K_X]$ is big, then for any sequence $t_j \to T^-$, after passing to a subsequence, $(X, g(t_j))$ converges in Gromov–Hausdorff topology to a compact metric space $(X_T, d_T)$. Furthermore, the convergence is smooth on $X^\circ$ and $\Phi|_{X^\circ}$ extends to a surjective Lipschitz map

$$\Phi_T : (X_T, d_T) \to (Y, g_Y).$$

In fact, $(X_T, d_T)$ is conjectured [28] to be homeomorphic to the projective variety $Y$ as well as the metric completion of $(X^\circ, g_T)$, where $g_T = \lim_{t \to T^-} g(t)$ is the limiting smooth Kähler
metric on $X^\circ$. This is indeed confirmed in the case of $\dim X = 2$ [31, 32] and special cases of small contractions [24].

2.2 The case of long-time solutions

It is well-known that the Kähler–Ricci flow has a long-time solution if and only if the canonical bundle $K_X$ is nef. The underlying manifold $X$ is then called a minimal model. The Kodaira dimension of $X$ is defined by

$$\text{Kod}(X) = \lim_{m \to \infty} \frac{\log h^0(X, mK_X)}{\log m}$$

if $h^0(X, mK_X) \neq 0$ for some $m \in \mathbb{Z}^+$. The Kodaira dimension of $X$ is always no greater than $n$ and is nonnegative as long as there exists one holomorphic pluricanonical section. The abundance conjecture predicts that if $X$ is minimal, $K_X$ must be semi-ample and hence the Kodaira dimension is always nonnegative. We would like now to obtain a uniform diameter bound for long-time solutions of the Kähler–Ricci flow.

We consider the following normalized Kähler–Ricci flow with initial metric $g_0$ if the Kodaira dimension of $X$ is non-negative.

$$\begin{aligned}
\frac{\partial g}{\partial t} &= -\text{Ric}(g) - g, \\
g \big|_{t=0} &= g_0.
\end{aligned}
$$

Obviously, the flow (2.4) exists for $t \in [0, \infty)$ since $K_X$ is nef.

**Theorem 2.3.** Let $(X, g_0)$ be an $n$-dimensional Kähler manifold with nef $K_X$ and non-negative Kodaira dimension. Let $g(t)$ be the solution of the normalized Kähler–Ricci flow (2.4). Then there exist $C = C(X, g_0) > 0$, $c = c(X, g_0) > 0$, and $\alpha = \alpha(X, g_0) > 0$ such that for any $t \geq 0$,

$$\text{diam}(X, g(t)) \leq C,$$

$$\int_X |G_t(x, \cdot)|dV_{g(t)} + \int_X |\nabla G_t(x, \cdot)|dV_{g(t)} + \left(\inf_{y \in X} G_t(x, y)\right)\text{Vol}_{g(t)}(X) \leq C,$$

$$\frac{\text{Vol}_{g(t)}(B_{g(t)}(x, R))}{\text{Vol}_{g(t)}(X)} \geq cR^\alpha,$$

for any $x \in X$ and $R \in (0, 1]$, where $G_t$ is the Green’s function for $(X, g(t))$.

The diameter is optimal for $X$ with positive Kodaira dimension. When $c_1(X) = 0$ and hence $\text{Kod}(X) = 0$, the diameter of $(X, g(t))$ decays at the exact rate $e^{-t/2}$. The uniform diameter bound in Theorem 2.3 is proved in Ref. [20] in the case when $K_X$ is semi-ample, and the proof is built on works of Refs. [1, 30], relying on the uniform scalar curvature bound obtained in Refs. [29, 40] (see also Ref. [37] in the case of general type). Our proof in the present case is based rather on Theorem 1.1. This has many advantages, since we do not need then any assumption on the scalar
curvature (for which bounds are not available in the nef case), nor on the projectiveness of $X$, nor on the abundance conjecture.

When the canonical bundle $K_X$ is semi-ample, the pluricanonical system $|mK_X|$ for sufficiently large $m \in \mathbb{Z}^+$ induces a unique surjective holomorphic map

$$\Phi : X \to Y \in \mathbb{C} \mathbb{P}^N,$$

where $Y$ is a projective variety embedded in $\mathbb{C} \mathbb{P}^N$ via $\Phi$. The projective normal variety $Y$ is uniquely determined as the canonical model of $X$ with $\dim Y = \text{Kod}(X)$. We let $Y^o$ be the set of regular points of $\Phi$, then $Y^o$ is an open dense Zariski subset of $Y$. We also let $X^o = \Phi^{-1}(Y^o)$. When $\dim Y = \dim X$, $X$ and $Y$ are birationally equivalent. When $\dim Y < \dim X$, $\Phi : X \to Y$ is a holomorphic fibration and its general fibre is a smooth Calabi–Yau manifold. There exists a unique twisted Kähler-Einstein metric $g_{\text{can}}$ on $Y^o$ [26, 27] satisfying

$$\text{Ric}(g_{\text{can}}) = -g_{\text{can}} + g_{WP},$$

where $g_{WP}$ is the Weil-Petersson metric for the Calabi–Yau fibration of $X^o$ over $Y^o$.

Applying Theorem 2.3, we obtain a new proof of the following convergence theorem [20] for the long-time solutions of the Kähler–Ricci flow (see also [35, 39]).

**Corollary 2.1.** Let $X$ be an $n$-dimensional Kähler manifold with semi-ample canonical line bundle $K_X$ and $\text{Kod}(X) > 0$. If $g(t)$ is the long-time solution of the normalized Kähler–Ricci flow (2.4) with any initial Kähler metric for $t \in [0, \infty)$, then $g(t)$ converges to the twisted Kähler–Einstein $g_{\text{can}}$ in $C^0(X^o)$ as $t \to \infty$, and for any $t_j \to \infty$, after possibly passing to a subsequence, $(X, g(t_j))$ converges in Gromov–Hausdorff topology to a compact metric space $(X_\infty, d_\infty)$.

Furthermore, $Y^o$ can be identified as an open dense subset of $(X_\infty, d_\infty)$ and the identity map from $Y^o \subset X_\infty$ to $Y^o \subset Y$ extends to a Lipschitz map $\Psi : (X_\infty, d_\infty) \to (Y, g_Y)$.

The Lipschitz map $\Psi$ is conjectured to be homeomorphic, which is indeed confirmed in the cases of $\text{Kod}(X) = 1, 2, n$ [20, 30, 37].

### 3 | BOUNDED SETS IN THE KÄHLER CONE

**Notational convention:** if $\omega = (g_{ij})$ is a Kähler metric and $\vartheta = (\vartheta_{ij})$ is a (1,1)-form, we denote $\text{tr}_\omega(\vartheta) = g^{ij}\vartheta_{ij}$, where $(g^{ij})$ is the inverse of $(g_{ij})$. For a number $p \in (1, \infty)$, we denote by $p^*$ the conjugate exponent of $p$, that is, $\frac{1}{p} + \frac{1}{p^*} = 1$.

**Proposition 3.1.** Let $(X, \omega_X)$ be an $n$-dimensional compact Kähler manifold equipped with a Kähler metric $\omega_X$ in a Kähler class $\alpha$. For any $k \geq 0$, and a cohomology class $\beta \in H^{1,1}(X, \mathbb{R})$, there exists a smooth representative $\vartheta \in \beta$ such that

$$\|\vartheta\|_{C^k(X, \omega_X)} \leq C,$$

for some constant $C = C(X, \omega_X, k, |\beta \cdot \alpha^{n-1}|, |\beta^2 \cdot \alpha^{n-2}|) > 0$. 


Proof. We will take $\theta$ to be the unique harmonic (1,1)-form in the class $\beta$, relative to the Kähler metric $\omega_X$, that is,

$$\Delta \bar{\partial} \theta = 0.$$ 

By the standard Bochner–Kodaira–Lichnerowicz formula, we have

$$0 = -\Delta \bar{\partial} \theta = \frac{1}{2}(\theta_{ik,jj} + \theta_{ik,\bar{j}\bar{j}}) + \theta_{mj}R_{i\bar{m}jk} - \frac{1}{2}\theta_{m\bar{k}}R_{i\bar{m}} - \frac{1}{2}\theta_{im}R_{m\bar{k}}, \quad (3.2)$$

where $R_{i\bar{m}jk}$, $R_{i\bar{m}}$ denote the Riemann and Ricci curvatures of the fixed metric $\omega_X$, and $\theta_{ik,jj}$, $\theta_{ik,\bar{j}\bar{j}}$ denote the covariant derivatives of $\theta_{ik}$ with respect to the connection induced by $\omega_X$. It is well-known that Equation (3.2) is a linear elliptic equation of the (1,1)-form $\theta_{ik}$.

Taking traces on both sides of Equation (3.2), we obtain

$$\Delta_{\omega_X} (\text{tr}_{\omega_X} \theta) = 0,$$

hence $\text{tr}_{\omega_X} \theta$ must be a constant. Then we have

$$c_1 = \int_X \theta \wedge \omega_X^{n-1} = \frac{V}{n} \text{tr}_{\omega_X} \theta,$$

where $V = \int_X \omega_X^n = [\omega_X]^n$. On the other hand, we also have

$$c_2 = \int_X \theta^2 \wedge (\omega_X)^{n-2} = C(n) \int_X ((\text{tr}_{\omega_X} \theta)^2 - |\theta|_{\omega_X}^2) \omega_X^n. \quad (3.3)$$

From this, we see that

$$\int_X |\theta|_{\omega_X}^2 \omega_X^n \quad (3.4)$$

is uniformly bounded depending only on $c_1$, $c_2$, and $\omega_X$. Applying Moser iteration to Equation (3.2), we get an $L^\infty$ bound for $\theta$. The uniform $C^{k,\alpha}$ estimates of $\theta$ then follow from the standard elliptic estimates applied to the linear elliptic equation (3.2).

Note that when $\beta$ is Kähler, $c_2$ in Equation (3.3) is positive and so Equation (3.4) is uniformly bounded depending only on $c_1$ and $\omega_X$. The following corollary is then an immediate consequence of Proposition 3.1 and its proof.

**Corollary 3.1.** Let $(X, \omega_X)$ be an $n$-dimensional Kähler manifold equipped with a Kähler metric $\omega_X$. Then the following hold.

(1) For any bounded set $U$ in the Kähler cone of $X$, there exists $C = C(U) > 0$ such that for any Kähler class $\beta \in U$,

$$\beta \cdot [\omega_X]^{n-1} < C.$$

(2) For any $A > 0$ and a Kähler class $\beta$ with

$$\beta \cdot [\omega_X]^{n-1} < A,$$
there exists $C = C(\omega_X, A) > 0$ such that
\[ C[\omega_X] - \beta \]
is a Kähler class.

**Corollary 3.2.** Let $(X, \omega_X)$ be an $n$-dimensional Kähler manifold equipped with a Kähler metric $\omega_X$. For any bounded set $U$ in the Kähler cone of $X$, there exists a smooth closed $(1,1)$-form $\vartheta \in \beta$ for any $\beta \in U$ with the following uniform properties.

1. There exists $C = C(U) > 0$ such that $\|\vartheta\|_{C^3(X,\omega_X)} \leq C$.
2. There exist $\alpha = \alpha(U) > 0$ and $C = C(U, \alpha) > 0$ such that for any $\varphi \in \text{PSH}(X, \vartheta)$,
\[
\int_X e^{-\alpha(\varphi - \text{sup}_X \varphi)}(\omega_X)^n \leq C,
\]

**Proof.** It suffices to prove Equation (2). By Proposition 3.1, there exists $B = B(U) > 0$ such that $\vartheta < B\omega_X$. Then for any $\varphi \in \text{PSH}(X, \vartheta)$, we have $\varphi \in \text{PSH}(X, B\omega_X)$. The corollary then follows by applying the $\alpha$-invariant of $B\omega_X$ (c.f. [19, 33]).

### 4 | $L^\infty$-ESTIMATES FOR COMPLEX MONGE-AMPÈRE EQUATIONS

**Proposition 4.1.** Let $(X, \omega_X)$ be an $n$-dimensional compact Kähler manifold equipped with a Kähler metric $\omega_X$. For any $K > 0$, $p > n$, there exist $C = C(X, \omega_X, n, p, K) > 0$ such that if $\vartheta$ a smooth closed $(1,1)$-form with $\vartheta \leq \omega_X$ and if $\omega = \vartheta + \sqrt{-1} \partial \bar{\partial} \varphi$ is a Kähler form satisfying
\[
\mathcal{N}_{X,\omega_X,p}(\omega) \leq K,
\]
then
\[
\|\varphi - \text{sup}_X \varphi - \mathcal{V}_\vartheta\|_{L^\infty(X)} \leq C,
\]
where $\mathcal{V}_\vartheta = \text{sup}\{u : u \in \text{PSH}(X, \vartheta), u \leq 0\}$ is the envelope of nonpositive $\vartheta$-psh functions.

**Proof.** By the assumption on $\vartheta$, we have $\text{PSH}(X, \vartheta) \subseteq \text{PSH}(X, \omega_X)$. Therefore, there exists $\alpha > 0$ and $C_\alpha > 0$ such that for any $\varphi \in \text{PSH}(X, \vartheta)$,
\[
\int_X e^{-\alpha(\varphi - \text{sup}_X \varphi)}\omega_X^n \leq C_\alpha.
\]
Then the proposition follows from the uniform $L^\infty$-estimates from Refs. [2, 7, 8, 10, 15], which generalize Kolodziej’s $L^\infty$ estimates in the case of a fixed background metric [21].

**Corollary 4.1.** Let $(X, \omega_X)$ be an $n$-dimensional compact Kähler manifold equipped with a Kähler metric $\omega_X$. For any $A, K > 0$ and $p > n$, if two Kähler forms $\omega_1$ and $\omega_2$ belong to the same Kähler class and
\[
\omega_1, \omega_2 \in \mathcal{V}(X, \omega_X, n, A, p, K),
\]

then there exist \( C = C(X, \omega_X, n, A, p, K) > 0 \) and \( \varphi \in C^\infty(X) \) such that

\[
\omega_2 = \omega_1 + \sqrt{-1} \partial \bar{\partial} \varphi, \| \varphi - \sup_X \varphi \|_{L^\infty(X)} \leq C.
\] (4.1)

**Proof.** Let \( \gamma \) be the Kähler class of \( \omega_1 \) and \( \omega_2 \). Then \( \gamma \cdot [\omega_X]^{n-1} < A \) by our assumption. By Proposition 3.1, we can choose a smooth closed (1,1)-form \( \theta \in \gamma \) (not necessarily positive) such that

\[
\| \theta \|_{C^3(X, \omega_X)}
\]
is uniformly bounded by a constant that only depends on \( A \). Furthermore, We define \( \varphi_i \in C^\infty(X) \) by

\[
\omega_i = \theta + \sqrt{-1} \partial \bar{\partial} \varphi_i, \sup_X \varphi_i = 0, \ i = 1, 2.
\]
Then \( \varphi_i \) satisfies the following complex Monge–Ampère equation

\[
\frac{(\theta + \sqrt{-1} \partial \bar{\partial} \varphi_i)^n}{[\theta]^n} = e^{F_i}(\omega_X)^n, \sup \varphi_i = 0.
\] (4.2)

Since \( \omega_i \in \mathcal{V}(X, \omega_X, n, A, p, K) \),

\[
\| e^{F_i} \|_{L^1 \log L^p(X, \omega_X)} \leq K, \ i = 1, 2.
\]

Let \( \mathcal{V}_\theta = \sup \{ u : u \in PSH(X, \theta), u \leq 0 \} \). Proposition 4.1 implies that there exists \( C = C(X, \omega_X, n, A, p, K) > 0 \) such that

\[
\| \varphi_i - \mathcal{V}_\theta \|_{L^\infty(X)} \leq C,
\]

and so

\[
\| \varphi_1 - \varphi_2 \|_{L^\infty(X)} \leq 2C.
\]
The corollary is proved, with \( \varphi = \varphi_1 - \varphi_2 \). \( \square \)

### 5 ESTIMATES FOR THE GREEN’S FUNCTIONS

Throughout this section, we fix the \( n \)-dimensional Kähler manifold \((X, \omega_X)\) and constants \( A, K > 0 \) and \( p > n \).

We will follow the approach on estimating the Green’s functions in Ref. [13], where similar estimates were derived for Kähler metrics \( \omega \) with total volume uniformly bounded away from zero and the relative volume form \( e^{F_\omega} \) bounded below by a positive constant. While these conditions restricted the applicability to certain geometric problems such as the Kähler–Ricci flow, our paper extends this in the sense that it allows for the degeneracy of Kähler classes \([\omega]\) and the possibility of a “small” vanishing locus for \( e^{F_\omega} \).

The following lemma is a natural extension of Lemma 2 in Ref. [13].
Lemma 5.1. Suppose $\omega \in \mathcal{V}(X, \omega_X, n, A, p, K)$. Let $v \in L^1(X, \omega^n)$ be a function that satisfies $\int_X v \omega^n = 0$ and
\begin{equation}
v \in C^2(\overline{\Omega_0}), \quad \Delta \omega v \geq -a \text{ in } \Omega_0\end{equation}
for some $a > 0$ and $\Omega_s = \{ v > s \}$ is the super-level set of $v$. Then there is a uniform constant $C = C(X, \omega_X, A, p, K) > 0$ such that
\begin{equation}
\sup_X v \leq C \left( a + \frac{1}{[\omega]_n} \int_X |v| \omega^n \right) .
\end{equation}

For the convenience of the readers, we sketch the proof of Lemma 5.1.

Proof. We follow closely the arguments in Ref. [13].

First, we observe that it suffices to prove the lemma in the case $a = 1$. This is because both the equation (5.1) and the desired inequality (5.2) are homogenous under a simultaneous rescaling of $a$ and $v$, $a \to 1, v \to \frac{v}{a}$.

Next, we may assume $\|v\|_{L^1(X, \omega^n)} \leq [\omega]^n$, otherwise, replace $v$ by $\hat{v} := v \cdot [\omega]^n / \|v\|_{L^1(X, \omega^n)}$, which still satisfies Equation (5.1) with the same $a = 1$ and $\|\hat{v}\|_{L^1(X, \omega^n)} = [\omega]^n$. It thus suffices to show $\sup_X v \leq C$ for some $C > 0$ with the dependence as stated in the lemma. By Proposition 3.1, we can choose a smooth closed $(1,1)$-form $\theta \in \mathcal{A}(\omega)$ with $\|\theta\|_{C^3(X, \omega_X)}$ uniformly bounded. We let $\omega = \theta + \sqrt{-1} \partial \bar{\partial} \varphi$ for $\varphi \in \text{PSH}(X, \theta)$ and $\sup_X \varphi = 0$.

(1) We fix a sequence of positive smooth functions $\eta_k : \mathbb{R} \to \mathbb{R}_+$ such that $\eta_k(x)$ converges uniformly and monotonically decreasingly to the function $x \cdot \chi_{\mathbb{R}_+}(x)$, as $k \to \infty$. We may choose $\eta_k(x) \equiv 1/k$ for any $x \leq -1/2$. As in Ref. [14], we make use of auxiliary Monge–Ampère equations. More precisely, for each $s \geq 0$ and large $k$, we consider the following specific complex Monge–Ampère equations [13]:
\begin{equation}
(\theta + \sqrt{-1} \partial \bar{\partial} \psi_{s,k})^n = [\omega]^n \frac{\eta_k(v - s)}{A_{s,k}} e^F \omega^n_X, \quad \sup_X \psi_{s,k} = 0,
\end{equation}
where
\begin{equation}
A_{s,k} = \int_X \eta_k(v - s) e^F \omega^n_X \to \int_{\Omega_s} (v - s) e^F \omega^n_X =: A_s \text{ as } k \to \infty.
\end{equation}

We have assumed that the open set $\Omega_s \neq \emptyset$ so $A_s > 0$. The assumption that $\|v\|_{L^1(X, \omega^n)}/[\omega]^n \leq 1$ implies that $A_s \leq 1$, hence $A_{s,k} \leq 2$ for large $k$.

(2) Recall that we have assumed that $a = 1$. We denote $\Lambda = C + 1$ where $C > 0$ is the constant in Corollary 4.1. Consider the test function
\begin{equation}
\Phi := -\varepsilon (-\psi_{s,k} + \varphi + \Lambda)^{n+1} + (v - s),
\end{equation}
where $\varepsilon > 0$ is chosen such that
\begin{equation}
\varepsilon^{n+1} = \left( \frac{n+1}{n^2} \right)^n (1 + \varepsilon n)^n A_{s,k}.
\end{equation}
It follows easily from $A_{s,k} \leq 2$ and Equation (5.5) that
\[ \varepsilon \leq C(n)A_{s,k}^{1/(n+1)}, \] (5.6)
for some $C(n) > 0$ depending only on $n$. The function $\Phi$ is a $C^2$ function on $\Omega_0$ and $-\psi_{s,k} + \varphi + \Lambda \geq 1$. As shown in Ref. [13], it follows from the maximum principle, the choice of $\varepsilon$ in Equation (5.5) and the equations of $\psi_{s,k}$ and $\varphi$ that $\Phi \leq 0$ on $X$.

(3) From $\Phi \leq 0$ and Equation (5.6), we have $(v - s)A_{s,k}^{-1/(n+1)} \leq C_1(-\psi_{s,k} + \varphi + \Lambda)^{n/(n+1)}$ on $X$, for some $C_1 > 0$ depending only on $n$. This together with the $\alpha$-invariant and Hölder–Young inequality (see [13] for more details) implies that for some uniform constant $C_2 > 0$
\[ r\Phi(s + r) \leq C_2\Phi(s)^{1+\delta_0}, \quad \text{for all } s \geq 0 \text{ and } r > 0, \] (5.7)
where we denote $\delta_0 = \frac{p-n}{np} > 0$ and $\Phi(s) = \int_{\Omega_s} e^F \omega^n_X$.

The assumption that $\|v\|_{L^1(X,\omega^n)} \leq [\omega]^n$ implies that Hence for any $s > 0$ we have
\[ \phi(s) = \int_{\Omega_s} e^F \omega^n_X \leq \frac{1}{s} \int_{\Omega_0} v e^F \omega^n_X \leq \frac{1}{s}. \] (5.8)

We can pick $s_0 = (2C_2)^{1/\delta_0}$ to ensure that $\phi(s_0)^{\delta_0} \leq 1/(2C_2)$. Given Equation (5.7), we can apply the De Giorgi type iteration argument of Kolodziej [21] to conclude that $\Phi(s) = 0$ for any $s > S_\infty$ with
\[ S_\infty = s_0 + \frac{1}{1 - 2^{-\delta_0}} = (2C_2)^{1/\delta_0} + \frac{1}{1 - 2^{-\delta_0}}. \]
This means that $\sup_X v \leq S_\infty$ and the lemma is proved.

The following corollary is an immediate consequence of Lemma 5.1.

**Corollary 5.1.** Suppose $\omega \in \mathcal{V}(X,\omega_X, n, A, p, K)$. If $v \in C^2(X)$ satisfies
\[ |\Delta \omega v| \leq 1 \text{ and } \int_X v \omega^n = 0, \]
then there is a uniform constant $C = C(X,\omega_X, n, A, p, K) > 0$ such that
\[ \sup_X |v| \leq C \left( 1 + \frac{1}{[\omega]^n} \int_X |v| \omega^n \right). \]

In order to bound $\frac{1}{[\omega]^n} \int_X |v| \omega^n$, we will have to impose a uniform lower bound for the normalized volume form
\[ ([\omega]^n)^{-1} \omega^n_{\omega_X} \omega^n. \]
In particular, we will consider $\omega \in \mathcal{W}(X,\omega_X, n, A, p, K, \gamma)$ for some non-negative continuous function $\gamma$. 
Lemma 5.2. Suppose $\gamma \geq 0$ is a continuous function on $X$ with $\{\gamma > 0\}$ being connected. Then for any open subset $V \subset \{\gamma > 0\}$, there exists a connected open subset $U$ of $X$ with

$$V \subset U \subset \{\gamma > 0\}.$$ 

Proof. Obviously, $\{\gamma > 0\}$ is path connected since it is open and connected. We choose a fixed base point $p \in V$. For any $q \in \overline{V}$, there exists a continuous path $C$ joining $p$ and $q$ in $\{\gamma > 0\}$. We can find an open tubular neighborhood $U_q$ of $C$ such that $U_q \subset \{\gamma > 0\}$. Then

$$\overline{V} \subset \cup_{q \in V} U_q$$

and we can find finitely many $q_1, q_2, ..., q_N \subset \overline{V}$ such that

$$\overline{V} \subset U = \cup_{j=1}^N U_{q_j}.$$ Then $U \subset \{\gamma > 0\}$ is open and connected since every $U_{q_j}$ is path connected with a common point $p$. The lemma is then proved. \qed

Lemma 5.3. There exists $\varepsilon = \varepsilon(X, \omega_X, n, A, p, K) > 0$ such that if

(1) $\gamma \geq 0$ is a continuous function on $X$ such that

$$||\gamma = 0||_{\omega_X} \leq \varepsilon, \{\gamma > 0\} \text{ is connected},$$

(2) $\omega \in \mathcal{W}(X, \omega_X, n, A, p, K; \gamma)$,

(3) $v \in C^2(X)$ satisfies

$$|\Delta_\omega v| \leq 1 \text{ and } \int_X v \omega^n = 0,$$

then there exists $C = C(X, \omega_X, n, A, p, K, \varepsilon, \gamma) > 0$ such that

$$\frac{1}{[\omega]^n} \int_X |v| \omega^n \leq C.$$

Proof. The proof is by contradiction. Suppose Lemma 5.3 fails. Then there exist a sequence of $\omega_j \in \mathcal{W}(X, \omega_X, n, A, p, K; \gamma)$ and $v_j \in C^2(X)$ satisfying

$$\Delta_\omega v_j = h_j, \quad \int_X v_j \omega_j^n = 0,$$

for some $|h_j| \leq 1$, and as $j \to \infty$

$$\frac{1}{[\omega_j]^n} \int_X |v_j| \omega_j^n = : N_j \to \infty. \quad (5.9)$$

We define $F_j$ by

$$e^{F_j} = ([\omega_j]^n)^{-1} \frac{\omega_j^n}{\omega_X^n}$$

and immediately we have

$$e^{F_j} \geq \gamma.$$
We now consider $\tilde{v}_j$ defined by $\tilde{v}_j = v_j/N_j$. Clearly, we have

$$|\Delta_{\omega_j} \tilde{v}_j| = |h_j|/N_j \to 0, \quad \frac{1}{|\omega_j|^n} \int_X |\tilde{v}_j| \omega^n_j = 1.$$  

Applying Lemma 5.1 to $\tilde{v}_j$, there exists a uniform $C > 0$ such that for all $j \geq 1$,

$$\sup_X |\tilde{v}_j| \leq C.$$  

(5.10)

From the equation of $\tilde{v}_j$ and integration by parts, we see that

$$\int_X |\nabla \tilde{v}_j|_{\omega_j}^2 e^{F_j} \omega^n_X = \frac{1}{|\omega_j|^n} \int_X |\tilde{v}_j|_{\omega_j}^2 \omega^n_j \to 0.$$  

(5.11)

Let $U_0 = \{ \gamma > 0 \}$. Suppose $|\{ \gamma = 0 \}|_{\omega_X} < \varepsilon$ for some sufficiently small $\varepsilon > 0$ to be determined. Then we can pick open connected subsets $U_{3\varepsilon} \subset \subset U_{2\varepsilon} \subset \subset \{ \gamma > 0 \}$ as in Lemma 5.2 such that

$$|X \setminus U_{2\varepsilon}|_{\omega_X} < 2\varepsilon, \quad |X \setminus U_{3\varepsilon}|_{\omega_X} < 3\varepsilon.$$  

Without loss of generality, we can assume that both $U_{2\varepsilon}$ and $U_{3\varepsilon}$ have smooth boundaries. Let

$$\delta_\varepsilon = \inf_{U_{2\varepsilon}} \gamma.$$  

(5.12)

Then we have $\inf_{U_{2\varepsilon}} e^{F_j} \geq \delta_\varepsilon^{-1}$ and

$$\int_{U_{2\varepsilon}} |\nabla \tilde{v}_j|_{\omega_X} \omega^n_X \leq \left( \int_{U_{2\varepsilon}} |\nabla \tilde{v}_j|_{\omega_j}^2 e^{F_j} \omega^n_X \right)^{1/2} \left( \int_{U_{2\varepsilon}} e^{-F_j} \text{tr}_{\omega_X} (\omega_j) \omega^n_X \right)^{1/2} \leq \delta_\varepsilon^{-1/2} \left( \int_X |\nabla \tilde{v}_j|_{\omega_j}^2 e^{F_j} \omega^n_X \right)^{1/2} \left( \int_X \omega^{-1}_X \wedge \omega_j \right)^{1/2} \leq (A\delta_\varepsilon)^{-1/2} \left( \int_X |\nabla \tilde{v}_j|_{\omega_j}^2 e^{F_j} \omega^n_X \right)^{1/2} \to 0$$

by Equation (5.11) as $j \to \infty$. Therefore, $\tilde{v}_j$ is uniformly bounded in $W^{1,1}(U_{2\varepsilon}, \omega_X)$ by the above estimate and Equation (5.10). By the Sobolev embedding theorem, after passing to a subsequence, we can assume that $\tilde{v}_j$ converges to $\tilde{v}_\infty$ in $L^1(U_{3\varepsilon}, \omega_X)$. Furthermore, since $\tilde{v}_j$ is uniformly bounded in $L^\infty(U_{3\varepsilon})$ and converges almost everywhere to $\tilde{v}_\infty$ in $U_{3\varepsilon}$, $\tilde{v}_\infty$ is also bounded in $L^\infty(U_{3\varepsilon}, \omega_X)$.

Since $\lim_{j \to \infty} \int_{U_{2\varepsilon}} |\nabla \tilde{v}_j|_{\omega_X} \omega^n_X = 0$, we have for any test function $f \in C^\infty_0(U_{2\varepsilon})$

$$\left| \int_X \tilde{v}_j (\Delta f) \omega^n_X \right| \leq \left( \sup_X |\nabla f|_{\omega_X} \right) \int_{U_{2\varepsilon}} |\nabla \tilde{v}_j|_{\omega_X} \omega^n_X \to 0$$

as $j \to \infty$, where $\Delta$ is the Laplace operator with respect to $\omega_X$. Therefore by Weyl’s lemma for the Laplace equation, $\tilde{v}_\infty$ solves the Laplace equation $\Delta \tilde{v}_\infty = 0$ on $U_{3\varepsilon}$ and $\tilde{v}_\infty \in C^\infty(U_{3\varepsilon})$. For any
smooth vector field $Y \in C_0^\infty(U_{3\varepsilon})$, we have
\[
\int_X \langle \nabla \tilde{v}_\infty, Y \rangle_{\omega_X^n} \omega_X^n = -\int_X \tilde{v}_\infty \cdot \text{div}_{\omega_X^n} Y_{\omega_X^n} = -\lim_{j \to \infty} \int_X \tilde{v}_j \cdot \text{div}_{\omega_X^n} Y_{\omega_X^n} = \lim_{j \to \infty} \int_X \langle \nabla \tilde{v}_j, Y \rangle_{\omega_X^n} \omega_X^n = 0.
\]
Here the first and third lines follow from the divergence theorem, and the second equality holds since $\tilde{v}_j$ converge in $L^1(U_{3\varepsilon})$ to $\tilde{v}_\infty$ and the function $\text{div}_{\omega_X^n} Y$ is compactly supported in $U_{3\varepsilon}$. By taking $Y = \eta \nabla \tilde{v}_\infty$ for any nonnegative function $\eta \in C_0^\infty(U_{2\delta})$, we see immediately that $\nabla \tilde{v}_\infty \equiv 0$ on $U_{3\varepsilon}$, and hence $\tilde{v}_\infty$ is constant on $U_{3\varepsilon}$ since $U_{3\varepsilon}$ is connected.

We first derive a uniform positive lower bound for $\int_{U_{3\varepsilon}} |\tilde{v}_j| \omega_X^n$. In fact, by the Hölder–Young inequality (see, e.g., Equation (2.24) in Ref. [11]), there exists $C > 0$ such that for any $\delta > 0$ and smooth functions $u$ and $F$, we have
\[
|u|e^F = |\delta^{-1} u|e^{F+\log\delta} \leq \delta e^F(1 + |F| + |\log\delta|) + C\delta^{-1}|u|e^{\delta^{-1}|u|}.
\]
Applying Equation (5.13), we have
\[
1 = \frac{1}{[\omega_X^n]^n} \int_X |\tilde{v}_j| \omega_X^n = \int_X |\tilde{v}_j|e^{F_j} \omega_X^n
\leq \delta \int_X e^{F_j}(1 + |F_j| + |\log\delta|)\omega_X^n + C\delta^{-1} \int_X |\tilde{v}_j|e^{\delta^{-1}|u_j|} \omega_X^n
\leq \delta \int_X e^{F_j}(1 + |F_j| + |\log\delta|)\omega_X^n + C\delta^{-1} \int_{U_{3\varepsilon}} |\tilde{v}_j|e^{\delta^{-1}|u_j|} \omega_X^n + Ce^{C\delta^{-1}} \int_{X\setminus U_{3\varepsilon}} \omega_X^n
\leq \delta \int_X e^{F_j}(1 + |F_j| + |\log\delta|)\omega_X^n + C\delta^{-1} \int_{U_{3\varepsilon}} |\tilde{v}_j|e^{\delta^{-1}|u_j|} \omega_X^n + 2\varepsilon Ce^{C\delta^{-1}}
\leq \frac{1}{2} + C\delta([\omega_X^n])^{-1} \int_{U_{3\varepsilon}} |\tilde{v}_j| \omega_X^n,
\]
for all $j$ and some uniform constants $C = C(A, p, K)$ and $C_\delta = C_\delta(A, p, K) > 0$, if we choose $\delta = \delta(A, p, K) > 0$ sufficiently small such that
\[
\delta \int_X e^{F_j}(1 + |F_j| + |\log\delta|)\omega_X^n < \frac{1}{4},
\]
and then choose
\[
\varepsilon < \varepsilon_1 = \frac{e^{-C\delta^{-1}}}{8C}.
\]
Immediately we have
\[
([\omega_X^n])^{-1} \int_{U_{3\varepsilon}} |\tilde{v}_j| \omega_X^n \geq (2C_\delta)^{-1}
\]
for sufficiently large $j$. 
On the other hand, we can extend $\tilde{v}_\infty$ to a constant function on $X$. By applying the Hölder–Young inequality again and by the fact that on $U_{2\epsilon}$ is connected, for any $\epsilon' > 0$, there exists $C_1 > 0$ and $C_2 = C_2(A, p, K) > 0$ such that for sufficiently large $j$, we have

$$|\tilde{v}_\infty| \leq \frac{1}{[\omega_j]^n} \int_X (\tilde{v}_\infty - \tilde{v}_j) \omega_j^n$$

$$\leq \frac{1}{[\omega_j]^n} \int_{U_{3\epsilon}} |\tilde{v}_\infty - \tilde{v}_j| \omega_j^n + \frac{1}{[\omega_j]^n} \int_{X \setminus U_{3\epsilon}} |v_\infty - \tilde{v}_j| \omega_j^n$$

$$= \int_{U_{3\epsilon}} |\tilde{v}_\infty - \tilde{v}_j| e^{F_j} \omega_X^n + \int_{X \setminus U_{3\epsilon}} |v_\infty - \tilde{v}_j| e^{F_j} \omega_X^n$$

$$\leq \epsilon' \int_X e^{F_j} (1 + |F_j| + |\log \epsilon'|) \omega_X^n + C_1 (e')^{-1} e^{(e')^{-1} \sup_X |v_j - v_\infty|} \int_{U_{3\epsilon}} |\tilde{v}_j - \tilde{v}_\infty| \omega_X^n$$

$$+ C_1 (e')^{-1} e^{(e')^{-1} \sup_X (|v_\infty - v_j|)} \int_{X \setminus U_{3\epsilon}} |v_\infty - \tilde{v}_j| \omega_X^n$$

$$\leq C_2 (e')^{1/2} + e^{C_2 (e')^{-1}} \epsilon$$

$$< (4C_\delta)^{-1},$$

if we choose $\epsilon'$ and $\epsilon$ with

$$\epsilon' < (8C_2 C_\delta)^{-2}, \text{ and } \epsilon < \epsilon_2 = \left( 8e^{C_2 (e')^{-1}} C_\delta \right)^{-1}, \quad (5.16)$$

Since $v_j$ converges to $v_\infty$ in $L^1(U_{3\epsilon})$, for sufficiently large $j$, we have

$$([\omega_X]^n)^{-1} \int_{U_{3\epsilon}} |v_j| \omega_X^n < ([\omega_X]^n)^{-1} \int_{U_{3\epsilon}} |v_\infty| \omega_X^n + (4C_\delta)^{-1} < (2C_\delta)^{-1}.$$

This contradicts the lower bound $(5.15)$. From now on, we will fix the choice for

$$\epsilon = \frac{\min(\epsilon_1, \epsilon_2)}{2}$$

from Equations $(5.14)$ and $(5.16)$ for the parameter $\epsilon$ in the assumption of the lemma.

We have now completed the proof of the lemma. $\square$

Let $\omega$ be a Kähler metric on $X$. We let $G(x, \cdot)$ be the Green’s function of $(X, \omega)$ with base point $x$, for any $x \in X$.

**Lemma 5.4.** There exists $\epsilon = \epsilon(X, \omega_X, n, A, p, K) > 0$ such that if

1. $\gamma \geq 0$ is a continuous function on $X$ such that
   $$\{\gamma = 0\}_{\omega_X} \leq \epsilon, \{\gamma > 0\} \text{ is connected},$$

2. $\omega \in \mathcal{W}(X, \omega_X, n, A, p, K; \gamma),$
then there exists \( C = C(X, \omega_X, n, A, p, K, \varepsilon, \gamma) > 0 \) such that for any \( x \in X \)

\[
\int_X |G(x, \cdot)| \omega^n \leq C , \text{ and } \inf_{y \in X} G(x, y) \geq -\frac{C}{[\omega]^n},
\]

where \( G(x, \cdot) \) is the Green’s function of \((X, \omega)\).

**Proof.** We now fix \( \omega \in \mathcal{W}(X, \omega_X, n, A, p, K; \gamma) \) satisfying the assumption and \( x \in X \). Take a sequence of smooth functions \( h_k \), which are uniformly bounded and converge in \( L^q(X, \omega) \) for some fixed sufficiently large \( q > 0 \), to \(-\chi_{\{G(x, \cdot) \leq 0\}} + \frac{1}{[\omega]^n} \int G(x, \cdot) \leq 0 \) \( \omega^n \), where we denote \( \chi_E \) to be the characteristic function of a Borel set \( E \). We can also choose \( h_k \) to satisfy

\[
\sup_X |h_k| \leq 2, \quad \text{and} \quad \frac{1}{[\omega]^n} \int_X h_k \omega^n = 0.
\]

Immediately, there exists a unique smooth solution solving the linear equation

\[
\Delta \omega v_k = h_k, \quad \frac{1}{[\omega]^n} \int_X v_k \omega^n = 0.
\]

By Lemma 5.3, there exists \( C > 0 \) independent of \( k \) such that

\[
\sup_X |v_k| \leq C.
\]

Applying the Green’s formula, we have by the dominated convergence theorem

\[
v_k(x) = \int_X G(x, y)(-h_k(y))\omega^n(y) \to \int_{\{G(x, \cdot) \leq 0\}} G(x, \cdot)\omega^n
\]
as \( k \to \infty \). Combining this with the fact that

\[
\int_{\{G(x, \cdot) \geq 0\}} G(x, \cdot)\omega^n = -\int_{\{G(x, \cdot) \leq 0\}} G(x, \cdot)\omega^n.
\]

we easily find that \( \int_X |G(x, \cdot)| \omega^n \leq C \).

For the lower bound of the Green’s function, we apply Lemma 5.1 to the function \( v := -[\omega]^n \cdot G(x, \cdot) \) and \( a = 1 \). It then follows that

\[
-[\omega]^n \cdot \inf_X G(x, \cdot) \leq C([\omega]^n + \int_X |G(x, \cdot)| \omega^n) \leq C.
\]

This completes the proof of the lemma. \( \square \)

We observe that Lemma 5.4 implies a lower bound of the first nonzero eigenvalue of the Laplacian operator \( \Delta_\omega \). To see this, suppose \( \lambda_1 > 0 \) is such an eigenvalue and \( f \in C^\infty(X) \) is an associated eigenfunction normalized by \( \int_X f^2 \omega^n = [\omega]^n \). Then \( \Delta_\omega f = -\lambda_1 f \). If we let \( x_0 \in X \) be a maximum point of \( |f| \), by the Green’s formula we have

\[
0 \neq f(x_0) = \frac{1}{[\omega]^n} \int_X f \omega^n - \int_X G(x_0, \cdot) \Delta \omega f \omega^n = \lambda_1 \int_X G(x_0, \cdot) f \omega^n.
\]
Hence

\[ |f(x_0)| \leq \lambda_1 |f(x_0)| \int_X |G(x_0, \cdot)| \omega^n \leq C |f(x_0)| \lambda_1, \]

by Lemma 5.4. This immediately gives the uniform positive lower bound of \( \lambda_1 \).

For convenience of notation, we write

\[ G(x, \cdot) = G(x, \cdot) - \inf_{x,y\in X} G(x, y) + ([\omega]_n)^{-1} > 0. \]  

(5.17)

It is clear that \( \int_X G(x, \cdot) \omega^n \leq C \).

**Lemma 5.5.** There exist \( \varepsilon = \varepsilon(X, \omega_X, n, A, p, K) > 0 \) and \( \varepsilon' = \varepsilon'(n, p) > 0 \) such that if

(1) \( \gamma \geq 0 \) is a continuous function on \( X \) such that

\[ \{\gamma = 0\} \omega_X \leq \varepsilon, \{\gamma > 0\} \text{ is connected}, \]

(2) \( \omega \in \mathcal{W}(X, \omega_X, n, A, p, K; \gamma) \),

then there exists \( C = C(A, p, K, \varepsilon, \gamma, \varepsilon') > 0 \) such that for any \( x \in X \), we have

\[ \int_X G(x, \cdot)^{1+\varepsilon'} \omega^n \leq C([\omega]_n)^{-\varepsilon'}. \]

**Proof.** We fix \( x \in X \) and a small constant \( \varepsilon' > 0 \) to be determined. Fix a large \( k \gg 1 \) and consider a smooth positive function \( H_k \), which is a smoothing of \( \min\{G(x, \cdot), k\} \). Without loss of generality, we can assume that \( H_k \) converges increasingly to \( G(x, \cdot) \) as \( k \to \infty \). In particular, there exists \( C = C(A, p, K, \varepsilon, \gamma) > 0 \) such that for any \( k \)

\[ 0 < \int_X H_k e^F \omega_X^n \leq \frac{1}{[\omega]_n} \int_X G(x, \cdot) \omega_X^n \leq C [\omega]_n^n, \]

where \( F = ([\omega]_n)^{-1} \omega_X^n \).

We now consider the following linear equation on \( X \)

\[
\begin{align*}
\Delta_\omega u_k &= -(H_k)\varepsilon' + \frac{1}{[\omega]_n} \int_X (H_k)\varepsilon' \omega^n, \\
\frac{1}{[\omega]_n} \int_X u_k \omega^n &= 0.
\end{align*}
\]

(5.18)

Equation (5.18) admits a unique smooth solution since the smooth function on the right-hand side of the first equation has integral 0. We cannot apply the maximum principle to \( u_k \) directly since the term \( -(H_k)\varepsilon' \) on the right-hand side of Equation (5.18) is unbounded.

By Proposition 3.1, we can pick a smooth closed (1,1)-form \( \chi \in [\omega] \) such that \( \|\chi\|_{C^3(X, \omega_X)} \) is uniformly bounded by some constant that only depends on \( A \) and \( \omega_X \). We let

\[ \omega = \chi + \sqrt{-1} \partial \bar{\partial} \varphi, \sup_X \varphi = 0, \]
and let
\[ \hat{H}_k := [\omega]^n \cdot H_k. \]

Then we consider the following auxiliary complex Monge–Ampère equation, which admits a smooth solution by [38]
\[
\frac{1}{[\omega]^n}(\chi + \sqrt{-1} \partial \bar{\partial} \psi_k)^n = \frac{(\hat{H}_k)^{nz'} + 1}{\int_X ((\hat{H}_k)^{nz'} + 1) \omega^n} \omega^n = \frac{(\hat{H}_k)^{nz'} + 1}{B_k} e^F \omega^n_X, \tag{5.19}
\]
with
\[ \sup_X \psi_k = 0, B_k = \int_X ((\hat{H}_k)^{nz'} + 1) e^F \omega^n_X. \]

By the Hölder inequality, there exists \( C = C(A, p, K, \epsilon', \gamma) > 0 \) for sufficiently small \( 0 < \epsilon' < n^{-1} \) such that for all \( k \), we have
\[
C^{-1} \leq \int_X \gamma \omega^n_X \leq B_k \leq \int_X e^F \omega^n_X + \left( \int_X e^F \omega^n_X \right)^{1-n\epsilon'} \left( \int_X \hat{H}_k e^F \omega^n_X \right)^{nz'} \leq C. \tag{5.20}
\]

For fixed \( p' \in (n, p) \), the \( p' \)th entropy of the function \( ((\hat{H}_k)^{nz'} + 1)e^F / B_k \) on the right-hand side of Equation (5.19) satisfies
\[
\frac{1}{B_k} \int_X ((\hat{H}_k)^{nz'} + 1) \left| - \log B_k + F + \log(1 + (\hat{H}_k)^{nz'}) \right|^{p'/p} e^F \omega^n_X \leq \int_X \hat{H}_k e^F \omega^n_X \tag{5.21}
\]
\[
\leq \left| \log B_k \right|^{p'/p} B_k \int_X ((\hat{H}_k)^{nz'} + 1) e^F \omega^n_X + \frac{1}{B_k} \int_X ((\hat{H}_k)^{nz'} + 1) \left( \log((\hat{H}_k)^{nz'}) + 1 \right)^{p'/p} e^F \omega^n_X \]
\[ + \frac{1}{B_k} \int_X ((\hat{H}_k)^{nz'} + 1) |F|^{p'} e^F \omega^n_X. \]

The first integral on the right hand side in Equation (5.21) is bounded due to the estimate of the constant \( B_k \) in Equation (5.20), the Hölder inequality and the uniform \( L^1 \)-bound of
\[
\int_X \hat{H}_k e^F \omega^n_X \leq \int_X G(x, \cdot) \omega^n
\]
by Lemma 5.4.

The second integral on the right-hand side in Equation (5.21) is also uniformly bounded by a similar argument since
\[
\frac{1}{B_k} \int_X ((\hat{H}_k)^{nz'} + 1) \left| \log((\hat{H}_k)^{nz'}) + 1 \right|^{p'/p} e^F \omega^n_X \leq C \int_X ((\hat{H}_k)^{nz'} + 1) e^F \omega^n_X \leq C,
\]
by the Hölder inequality and the calculus inequality \( (\log(1 + x))^{p'} \leq Cx^{n'/n} \) for any \( x > 0 \). We have also chosen \( \epsilon' > 0 \) small so that \( (n + 1)\epsilon' < 1 \).

To deal with the last integral in Equation (5.21), we observe that by Young’s inequality
\[
|((\hat{H}_k)^{nz'} + 1)|F|^{p'} \leq \frac{|F|^p}{p/p'} + \frac{(\hat{H}_k)^{nz'} + 1)^{(p/p')^*}}{(p/p')^*},
\]
where \((p/p')^* > 1\) is the conjugate exponent of \(p/p' > 1\). Hence the last term in Equation (5.21) satisfies

\[
\frac{1}{B_k} \int_X ((\hat{H}_k)^{nc'} + 1)|F|^{p'e^F}\omega^n_X
\]

\[
\leq C \int_X |F|^{p'e^F}\omega^n_X + C \int_X ((\hat{H}_k)^{nc'(p/p')^*} + 1)e^F\omega^n_X
\]

\[
\leq C,
\]

if we choose \(\varepsilon'\) small so that \(nc'(p/p')^* < 1\).

From now on, we fix a small \(\varepsilon' > 0\) that meets the requirements above and so the \(p'\)th entropy of the function on the right-hand side of Equation (5.19) is uniformly bounded. We apply Corollary 4.1 to conclude that

\[
\sup_X |\psi_k - \varphi| \leq C,
\]

for some uniform constant \(C = C(A, p, K, \varepsilon, \gamma, \varepsilon') > 0\).

We now consider the function

\[
v_k := (\psi_k - \varphi) - \frac{1}{[\omega]^n} \int_X (\psi_k - \varphi)\omega^n + \varepsilon'' u_k,
\]

where \(\varepsilon'' > 0\) is a suitable constant to be chosen later. It follows from the definition that \(\frac{1}{[\omega]^n} \int_X v_k\omega^n = 0\) and \(v_k\) is a smooth function.

Let \(\omega\psi_k = \chi + \sqrt{-1}\partial\bar{\partial}\psi_k\). We then calculate the Laplacian of \(v\) in Equation (5.23) and there exists \(C > 0\) such that

\[
\Delta_\omega v_k = \text{tr}_\omega (\omega\psi_k) - n + \varepsilon''\Delta_\omega u_k
\]

\[
\geq n \left(\frac{\omega^n_{\psi_k}}{\omega^n}\right)^{1/n} - n - \varepsilon''(H_k)^{\varepsilon'} + \frac{\varepsilon''}{[\omega]^n} \int_X (H_k)^{\varepsilon'}\omega^n
\]

\[
= nB_k^{-1/n}((\hat{H}_k)^{nc'} + 1)^{1/n} - n - \varepsilon''(H_k)^{\varepsilon'} + \frac{\varepsilon''}{([\omega]^n)} \int_X (H_k)^{\varepsilon'}\omega^n
\]

\[
\geq nC^{-1}([\omega]^n)^{\varepsilon'}(H_k)^{\varepsilon'} - n - \varepsilon''(H_k)^{\varepsilon'} \geq -n,
\]

if we choose \(\varepsilon'' = nC^{-1}([\omega]^n)^{\varepsilon'}\). We apply the Green’s formula to the function \(v_k\) at \(x\)

\[
v_k(x) = \frac{1}{[\omega]^n} \int_X v_k\omega^n + \int_X G(x, \cdot)(-\Delta_\omega v_k)\omega^n = \int_X G(x, \cdot)(-\Delta_\omega v_k)\omega^n
\]

\[
\leq n \int_X G(x, \cdot)\omega^n \leq C,
\]

where the last inequality follows from the uniform \(L^1(X, \omega^n)\)-bound of \(G(x, \cdot)\). It then follows from Equation (5.22) that

\[
u_k(x) \leq C([\omega]^n)^{-\varepsilon'}
\]
for a uniform constant $C > 0$. We now apply the Green’s formula to $u_k$ at $x \in X$

$$u_k(x) = \frac{1}{[\omega]^n} \int_X u_k \omega^n + \int_X G(x, \cdot)(-\Delta \omega_k) \omega^n$$

$$= \int_X G(x, \cdot) \left( (H_k)^{\epsilon'} - \frac{1}{[\omega]^n} \int_X (H_k)^{\epsilon'} \omega^n \right) \omega^n.$$

It then follows that

$$\int_X G(x, \cdot)(H_k)^{\epsilon'} \omega^n \leq u_k(x) + C \frac{1}{[\omega]^n} \int_X (H_k)^{\epsilon'} \omega^n \leq C([\omega]^n)^{-\epsilon'} + C \left( \frac{1}{[\omega]^n} \int_X H_k \omega^n \right)^{\epsilon'} \leq 2C([\omega]^n)^{-\epsilon'},$$

for some uniform constant $C > 0$. Letting $k \to \infty$ and applying the monotone convergence theorem, we can conclude that

$$\int_X G(x, \cdot)^{1+\epsilon'} \omega^n \leq C([\omega]^n)^{-\epsilon'},$$

for some uniform constant $C > 0$. The proof of the lemma is now completed.

We observe the following elementary estimate, which follows easily from the Green’s formula.

**Lemma 5.6.** Under the same assumptions of Lemma 5.5, for any $\beta > 0$, we have

$$\sup_{x \in X} \int_X \frac{\left| \nabla_y G(x, y) \right|^2 \omega(y)}{G(x, y)^{1+\beta}} \omega^n(y) \leq \frac{[\omega]^n \beta}{\beta}. \quad (5.24)$$

*Proof.* Fix $\beta > 0$ and a point $x \in X$. The function $u(y) := G(x, y)^{-\beta}$ is a continuous function with $u(x) = 0$ and $u \in C^\infty(X \setminus \{x\})$. By the definition of $G$ in Equation (5.17), for any $y \in X$, we have

$$0 \leq u(y) \leq ([\omega]^n)^{\beta}. \quad (5.25)$$

Applying the Green’s formula, we have

$$0 = u(x) = \frac{1}{[\omega]^n} \int_X u \omega^n + \int_X G(x, \cdot)(-\Delta \omega_u) \omega^n$$

$$= \frac{1}{[\omega]^n} \int_X u \omega^n - \beta \int_X \frac{\left| \nabla G(x, \cdot) \right|^2 \omega}{G(x, \cdot)^{1+\beta}} \omega^n.$$

In the last inequality, we apply the integration by parts using the asymptotic behavior of $G(x, y)$ near $y = x$. The lemma then follows easily from Equation (5.25).

Finally, we are ready to derive the uniform $L^1(X, \omega^n)$ bound for the gradient of $G(x, \cdot)$.
**Lemma 5.7.** Under the same assumptions of Lemma 5.5, for any $s \in [1, \frac{2 + 2\epsilon'}{2 + \epsilon'})$, there is a uniform constant $C = C(s) > 0$ such that for any $x \in X$, we have

$$
\int_X |\nabla G(x, \cdot)|_\omega^s \omega^n \leq \frac{C}{([\omega]^n)^{s-1}}.
$$

(5.26)

**Proof.** It suffices to prove the same estimate for $G(x, \cdot)$. For fixed $x \in X$, we regard $G(y) := G(x, y)$ as a function of $y$. By fixing $s \in [1, \frac{2 + 2\epsilon'}{2 + \epsilon'})$ and applying Hölder inequality, we have

$$
\int_X |\nabla G(x, \cdot)|_\omega^s \omega^n \leq \left( \int_X |\nabla G|_\omega^2 \omega^n \right)^{s/2} \left( \int_X G^{1+\epsilon'} \omega^n \right)^{(2-s)/2}
$$

$$
\leq C([\omega]^n)^{\beta s/2} ([\omega]^n)^{-\epsilon'(2-s)/2}
$$

$$
= C([\omega]^n)^{1-s},
$$

where $\beta > 0$ is chosen by $1 + \beta = (1 + \epsilon') \frac{2-s}{s}$. We apply the estimates in Lemmas 5.5 and 5.6 for the second inequality in Equation (5.27).

Combining the estimates above, we have established the following main result of this section.

**Proposition 5.1.** For any $A, K > 0$ and $p > n$, there exist $\epsilon = \epsilon(X, \omega_X, n, A, p, K, \gamma) > 0$ and $\epsilon' = \epsilon'(n, p) > 0$ such that if

1. $\gamma \geq 0$ is a continuous function on $X$ such that

   $$
   |\{\gamma = 0\}|_{\omega_X} \leq \epsilon, \\{\gamma > 0\}\text{ is connected},
   $$

2. $\omega \in \mathcal{W}(X, \omega_X, n, A, p, K; \gamma)$,
3. $s \in [1, \frac{2 + 2\epsilon'}{2 + \epsilon'}]$,

there exist $C_1 = C(X, \omega_X, n, A, p, K, \gamma, \epsilon) > 0$, $C_2 = C_2(X, \omega_X, n, A, p, K, \gamma, \epsilon, \epsilon') > 0$ and $C_3 = C_3(X, \omega_X, n, A, p, K, \gamma, \epsilon, \epsilon', s) > 0$ such that

$$
\inf_{y \in X} G(x, y) \geq -C_1([\omega]^n)^{-1},
$$

$$
\int_X |G(x, \cdot)|^{1+\epsilon'} \omega^n \leq C_2([\omega]^n)^{-\epsilon'},
$$

$$
\int_X |\nabla G(x, \cdot)|^{1+s} \omega^n \leq C_3([\omega]^n)^{1-s},
$$

for any $x \in X$. 
6 | DIAMETER AND VOLUME ESTIMATES

In this section, we will establish the following diameter and volume estimates by applying Proposition 5.1.

**Proposition 6.1.** For any $A, K > 0$ and $p > n$, there exist $\varepsilon = \varepsilon(X, \omega_X, n, A, p, K) > 0$ and $\varepsilon' = \varepsilon'(n, p) > 0$ such that if

1. $\gamma \geq 0$ is a continuous function on $X$ such that
   $\{|\gamma = 0| \omega_X \leq \varepsilon, \{|\gamma > 0\} \text{ is connected},$
2. $\omega \in \mathcal{W}(X, \omega_X, n, A, p, K; \gamma),$

there exist $\alpha = \alpha(n, p)$, $C = C(X, \omega_X, n, A, p, K, \gamma, \varepsilon) > 0$ and $c = c(X, \omega_X, n, A, p, K, \gamma, \varepsilon, \alpha) > 0$ such that

\[
\text{diam}(X, \omega) \leq C, \quad (6.1)
\]
\[
\frac{\text{Vol}_{\omega}(B_{\omega}(x, R))}{[\omega]^n} \geq cR^2, \quad (6.2)
\]

for any $x \in X$ and $R \in (0, 1]$.

**Proof.** We first prove the diameter bound. Since $(X, \omega)$ is compact and complete, there exist a pair of points $x_0, y_0 \in X$ such that $d_{\omega}(x_0, y_0) = \text{diam}(X, \omega)$. We define the 1-Lipschitz function $d(\cdot)$ with respect to $\omega$ on $X$ by

\[
d(y) = d_{\omega}(x_0, y).
\]

Apply the Green’s formula to $d$ at a point $x \in X$. We obtain

\[
d(x) = \frac{1}{[\omega]^n} \int_X d(y)\omega^n(y) + \int_X \langle \nabla_y G(x, y), \nabla d(y) \rangle_{\omega(y)\omega^n(y)}. \quad (6.3)
\]

By letting $x = x_0$, we have $d(x_0) = 0$ and

\[
\frac{1}{[\omega]^n} \int_X d(y)\omega^n(y) = - \int_X \langle \nabla_y G(x_0, y), \nabla d(y) \rangle_{\omega(y)\omega^n(y)} \leq \int_X |\nabla_y G(x_0, y)|_{\omega(y)\omega^n(y)}.
\]

Finally, we establish the uniform diameter by applying Equation (6.3) to $x = y_0$ with

\[
\text{diam}(X, \omega) = d(y_0)
\]

\[
= \frac{1}{[\omega]^n} \int_X d(y)\omega^n(y) + \int_X \langle \nabla_y G(y_0, y), \nabla d(y) \rangle_{\omega(y)\omega^n(y)}
\]

\[
\leq \int_X |\nabla_y G(x_0, y)|_{\omega(y)\omega^n(y)} + \int_X |\nabla_y G(y_0, y)|_{\omega(y)\omega^n(y)}
\]

\[
\leq C,
\]

for some uniform constant $C > 0$ by Proposition 5.1.
We now turn to the noncollapsing estimate for metric balls in \((X, \omega)\). Fix a point \(x \in X\) and a number \(R \in (0, 1]\). Let \(B(x, R) \subset X\) be the geodesic ball in \((X, \omega)\) with center \(x\) and radius \(R > 0\). We choose a smooth cutoff function \(\eta\) with support in \(B(x, R)\) satisfying
\[
\eta \equiv 1, \quad \text{on } B \left( x, \frac{R}{2} \right), \quad \sup_{X} |\nabla \eta|_{\omega} \leq \frac{4}{R}.
\]
We let \(d(y) = d_\omega(x, y)\) be the geodesic distance from \(x\) to \(y \in X\). Applying the Green’s formula to the Lipschitz function \(d \cdot \eta\), we have for any \(z \in X\)
\[
d(z)\eta(z) = \frac{1}{[\omega]^n} \int_X d(y)\eta(y)\omega^n(y) + \int_X \langle \nabla_y G(z, y), \eta(y)\nabla d(y) + d(y)\nabla \eta(y) \rangle_{\omega(y)} \omega^n(y). \tag{6.4}
\]
Take \(s = \frac{2+1.5\varepsilon'}{2+\varepsilon'} > 1\) for \(\varepsilon'\) from the assumption in Proposition 5.1. We apply Equation (6.4) to a point \(\hat{z} \in X \setminus B(x, R)\). Then \(d(\hat{z})\eta(\hat{z}) = 0\) and by Lemma 5.7, we have
\[
\frac{1}{[\omega]^n} \int_X d(y)\eta(y)\omega^n(y) \leq \int_X |\nabla_y G(\hat{z}, y)|_{\omega(y)} \left( \eta(y) + d(y)|\nabla \eta(y)|_{\omega(y)} \right) \omega^n(y)
\leq 5 \left( \int_X |\nabla_y G(\hat{z}, y)|^{s\varepsilon'}_{\omega(y)} \omega^n(y) \right)^{1/s^*} \cdot (\text{Vol}_{\omega}(B(x, R)))^{1/s^*}
\leq C([\omega]^{-s^{-1}}/[\omega]^n) \cdot (\text{Vol}_{\omega}(B(x, R)))^{1/s^*}
\leq C \left( \frac{\text{Vol}_{\omega}(B(x, R))}{[\omega]^n} \right)^{1/s^*},
\]
where \(s^* = \frac{s}{s-1}\) is the conjugate exponent of \(s\). Next we apply Equation (6.4) to a point \(\hat{z} \in \partial B(x, R/2)\) where \(d(\hat{z})\eta(\hat{z}) = R/2\). Applying the above estimate along with the same argument, we have
\[
\frac{R}{2} \leq \frac{1}{[\omega]^n} \int_X d(y)\eta(y)\omega^n(y) + \int_X |\nabla_y G(\hat{z}, y)|_{\omega(y)} \left( \eta(y) + d(y)|\nabla \eta(y)|_{\omega(y)} \right) \omega^n(y)
\leq C \left( \frac{\text{Vol}_{\omega}(B(x, R))}{[\omega]^n} \right)^{1/s^*},
\]
for some uniform constant \(C > 0\). This immediately gives a lower bound of the volume of \(B(x, R)\),
\[
\frac{\text{Vol}_{\omega}(B(x, R))}{[\omega]^n} \geq cR^\alpha,
\]
for some uniform constants \(\alpha = s^*(n, p) > 0\) and \(c = c(A, p, K, \gamma, \varepsilon', \alpha) > 0\). \qed

Remark. We briefly explain an application of the noncollapsing estimate (c) of Theorem 1.1 to the precompactness in Gromov–Hausdorff (GH) topology. Let \((X, \omega_j)\) be a sequence of Kähler metrics
satisfying the assumptions in Theorem 1.1. By Gromov's precompactness theorem, it suffices to verify the following:

for any $\varepsilon > 0$, there exists an $N(\varepsilon) > 0$ which is independent of $j$ such that there exists an $\varepsilon$-dense set $\{x_{j}^{a}\}_{a=1}^{M_{j}}$ in the metric space $(X, \omega_{j})$ with $M_{j} \leq N(\varepsilon)$.

In fact, suppose $\{x_{j}^{a}\}_{a=1}^{M_{j}}$ is an $\varepsilon$-dense set in the metric space $(X, \omega_{j})$, by which we mean a maximal collection of points where any two of them have distance at least $\varepsilon$. By definition, the geodesic balls $\{B_{j}(x_{j}^{a}, \varepsilon/2)\}_{a}$ are pairwise disjoint, hence by (c) of Theorem 1.1,

$$c(\varepsilon/2)^{\alpha}(\lfloor \omega \rfloor^{n})M_{j} \leq \sum_{a=1}^{M_{j}} \text{Vol}_{\omega_{j}}(B_{\omega_{j}}(x_{j}^{a}, \varepsilon/2)) \leq (\lfloor \omega \rfloor^{n}) = \text{Vol}(X, \omega_{n}^{n}),$$

so $M_{j} \leq c^{-1}(\varepsilon/2)^{-\alpha} =: N(\varepsilon)$.

This shows that up to a subsequence, the metric spaces $(X, \omega_{j})$ converge in GH topology to a compact metric space $(Z, d_{Z})$.

Now we can complete the proof of Theorem 1.1.

**Proof of Theorem 1.1.** It suffices to show that if $S$ is a closed subset of $X$ with $\dim_{\mathcal{M}} S < 2n - 1$, then $X \setminus S$ is connected. Since the Čech cohomological dimension is always no greater than the topological dimension, which is no greater than Minkowski dimension, we have

$$\tilde{H}^{2n-1}(S) = \tilde{H}^{2n}(S) = 0$$

and so by Poincaré–Alexander–Lefschetz duality $H_{1}(X, X \setminus S) = \tilde{H}^{2n-1}(S) = 0$ and $H_{0}(X, X \setminus S) = \tilde{H}^{2n}(S) = 0$ (c.f. Theorem 8.3, Chapter VI, in Ref. [3]). The exact sequence for reduced cohomology gives

$$0 = H_{1}(X, X \setminus S) \to \tilde{H}_{0}(X \setminus S) \to \tilde{H}_{0}(X) \to H_{0}(X, X\setminus S) = 0.$$

Therefore, $\tilde{H}_{0}(X \setminus S) = \tilde{H}_{0}(X) = \mathbb{Z}$ and so $X \setminus S$ is connected. Then Theorem 1.1 is a direct consequence of Propositions 5.1 and 6.1. \hfill \Box

### 7 A UNIFORM SOBOLEV INEQUALITY

In this section, we will prove a special Sobolev-type inequality for Kähler metrics satisfying the assumption in Proposition 5.1. The main feature of this inequality is the uniformity of the constants.

We first improve Lemma 5.3 in the following lemma.

**Lemma 7.1.** For any $A, K > 0$ and $p > n$, there exist $\varepsilon = \varepsilon(X, \omega_{X}, n, A, p, K) > 0$ and $\varepsilon' = \varepsilon'(n, p) > 0$ such that if
(1) $\gamma \geq 0$ is a continuous function on $X$ such that
\[ |\{\gamma = 0\}|_{\omega_X} \leq \varepsilon, \{\gamma > 0\} \text{ is connected}, \]

(2) $\omega \in \mathcal{W}(X, \omega_X, n, A, p, K; \gamma)$,

then there exists $C = C(A, p, K, \varepsilon, \gamma, \varepsilon') > 0$ such that for any $v \in C^\infty(X)$ satisfying
\[ \frac{1}{[\omega]^n} \int_X |\Delta v|^{(1+\varepsilon')^s} \omega^n \leq 1, \quad \int_X v \omega^n = 0, \]

we have
\[ \sup_X |v| \leq C. \]

**Proof.** Let $p = 1 + \varepsilon'$ and $q = p^s$. Then applying Proposition 5.1, there exists $C = C(A, p, K, \varepsilon, \gamma, \varepsilon') > 0$ such that
\[
 v(x) = -\int_X G(x, y) \Delta v(y) \omega^n(y) \\
 \leq \left( \int_X |G(x, \cdot)|^p \omega^n \right)^{1/p} \left( \int_X |\Delta v|^q \omega^n \right)^{1/q} \\
 \leq C([\omega]^n)^{-\varepsilon'/p} \left( \int_X |\Delta v|^q \omega^n \right)^{1/q} \\
 \leq C([\omega]^n)^{-\varepsilon'/p + 1/q} \\
 \leq C,
\]

since $\frac{\varepsilon'}{p} + \frac{1}{q} = 1 - \frac{\varepsilon' + 1}{p} = 0$. \hfill \square

We can now apply Lemma 7.1 to derive a Sobolev-type inequality with large exponents.

**Lemma 7.2.** For any $A, K > 0$ and $p > n$, there exist $\varepsilon = \varepsilon(X, \omega_X, n, A, p, K) > 0$ and $\varepsilon' = \varepsilon'(n, p) > 0$ such that if

(1) $\gamma \geq 0$ is a continuous function on $X$ such that
\[ |\{\gamma = 0\}|_{\omega_X} \leq \varepsilon, \{\gamma > 0\} \text{ is connected}, \]

(2) $\omega \in \mathcal{W}(X, \omega_X, n, A, p, K; \gamma)$,

(3) $s \in \left( 1, \frac{2+2\varepsilon'}{2+\varepsilon'} \right)$,

then there exists $C = C(A, p, K, \varepsilon, s) > 0$ such that for any $u \in C^\infty(X)$ satisfying $\int_X u \omega^n = 0$,
\[ \|u\|_{L^\infty(X)} \leq C \left( \frac{1}{[\omega]^n} \int_X |\nabla u|^s \omega^n \right)^{\frac{1}{s}}. \]
Proof. By the Green’s formula and integration by parts, we have

\[
|u(x)| = \left| \int_{X \setminus \{x\}} \langle \nabla G(x, \cdot), \nabla u(\cdot) \rangle \omega^n \right|
\]

\[
\leq \left( \int_X |\nabla G(x, \cdot)|^s \omega^n \right)^{\frac{1}{s}} \left( \int_X |\nabla u|^{\frac{s}{s-1}} \omega^n \right)^{\frac{s-1}{s}}
\]

\[
\leq C(\omega)^{\frac{s-1}{s}} \left( \int_X |\nabla u|^{\frac{s}{(s-1)} \omega^n} \right)^{\frac{s-1}{s}},
\]

for some uniform constant \( C = C(A, p, K, \varepsilon, \gamma, s) > 0 \), after applying Proposition 5.1.

We remark that Proposition 6.1 can also be proved by directly applying Lemma 7.2.

8 | FINITE TIME SOLUTIONS OF THE KÄHLER–RICCI FLOW

We will prove Theorem 2.1 in this section by applying Theorem 1.1. The key is to bound the \( p \)-Nash entropy from above and the volume form from below along the Kähler–Ricci flow.

We consider the unnormalized Kähler–Ricci flow (2.1) on a Kähler manifold \( X \) with an initial Kähler metric \( g_0 \). Suppose the flow develops finite time singularity. Without loss of generality by rescaling, we can assume that the singular time is given by

\[ T = \sup\{ t > 0 \mid [\omega_0] + t[K_X] > 0 \} = 1. \]

By choosing a smooth closed (1,1)-form \( \chi \in K_X \), the Kähler–Ricci flow (2.1) is equivalent to the following parabolic complex Monge–Ampère flow.

\[
\begin{cases}
\frac{\partial \varphi}{\partial t} = \log \left( \frac{\omega_0 + t \chi + \sqrt{-1} \partial \bar{\partial} \varphi}{\Omega} \right)^n, \\
\varphi|_{t=0} = 0,
\end{cases}
\]

(8.1)

where \( \Omega \) is a smooth volume form on \( X \) satisfying

\[ \sqrt{-1} \partial \bar{\partial} \log \Omega = \chi \in [K_X]. \]

We let \( \omega_t = \omega_0 + t \chi \) and \( \omega = \omega(t) = \omega_t + \sqrt{-1} \partial \bar{\partial} \varphi \).

Lemma 8.1. There exists \( C > 0 \) such that

\[ \varphi \leq C, \quad \frac{\partial \varphi}{\partial t} \leq C \]

on \( X \times [0, 1) \).
Proof. The upper bound for $\varphi$ follows directly from the maximum principle. Let

$$u = t \frac{\partial \varphi}{\partial t} - \varphi - nt.$$  

Then $u$ satisfies

$$\left( \frac{\partial}{\partial t} - \Delta \right) u = -tr_\omega(\omega_0) \leq 0.$$  

By the maximum principle,

$$\sup_{X \times [0, 1)} u \leq \sup_X u(\cdot, 0) = 0$$  

and so $\frac{\partial \varphi}{\partial t}$ is also uniformly bounded from above. The lemma immediately follows by considering $t \in [1/2, 1)$ since $\frac{\partial \varphi}{\partial t}$ is uniformly bounded for $t \in [0, 1/2]$.

We can now view the Monge–Ampère flow as a family of complex Monge–Ampère equations

$$(\omega_t + \sqrt{-1} \partial \bar{\partial} \varphi)^n = e^{\frac{\partial \varphi}{\partial t}} \Omega$$

for $t \in [0, 1)$. If $[\omega_0] + [K_X]$ is big,

$$\lim_{t \to 1} [\omega_t]^n = ([\omega_0] + [K_X])^n > 0.$$  

**Lemma 8.2.** There exists $\psi \in \text{PSH}(X, \chi)$ such that $\omega_0 + \chi + \sqrt{-1} \partial \bar{\partial} \psi$ is a Kähler current on $X$. Furthermore, $\psi$ has analytic singularities and is smooth outside the locus of its singularities.

**Proof.** The lemma is a consequence of Ref. [9] (The regularization theorem 3.2). □

We can assume that $\omega_0 + \chi + \sqrt{-1} \partial \bar{\partial} \psi > \varepsilon \omega_0$ for some $\varepsilon > 0$.

**Lemma 8.3.** There exists $C > 0$ such that on $X \times [0, 1)$, we have

$$\varphi \geq \psi - C.$$  

**Proof.** Let $u = \varphi - \psi$. Then $u$ is bounded below and tends to $\infty$ near the singular locus of $\psi$ for each $t \in [0, 1)$. $u$ satisfies the evolution equation

$$\frac{\partial u}{\partial t} = \log \frac{\left( \omega_0 + \chi + \sqrt{-1} \partial \bar{\partial} \psi - (1-t)\chi + \sqrt{-1} \partial \bar{\partial} u \right)^n}{\Omega}.$$  

Let

$$t' = \inf\{0 < t < 1 \mid \varepsilon \omega_0 > 2(1-s)\chi, \text{ for all } s \in (t, 1)\}.$$
Obviously, $t' < 1$. Suppose $\inf_{X(\max(1/2,t'),t_0)} u = u(z_0,t_0)$. Then $\psi$ is smooth at $z_0$. By applying the maximum principle, we have at $(z_0,t_0)$

$$
\frac{\partial u}{\partial t} \geq \log \left( \frac{\omega_0 + \chi + \sqrt{-1} \delta \bar{\psi} - (1 - t_0)\chi}{\Omega} \right)^n
\geq \log \left( \frac{\varepsilon \omega_0 - (1 - t_0)\chi}{\Omega} \right)^n
\geq \log \frac{\omega_0^n}{\Omega} - C,
$$

for some uniform $C > 0$. Therefore $u$ is uniformly bounded below for $t \in [0,1)$, principle. The lemma then immediately follows.

**Lemma 8.4.** There exist $A, C > 0$ such that

$$
\frac{\partial \psi}{\partial t} \geq A\psi - C.
$$

**Proof.** Let $u = \frac{\partial \varphi}{\partial t} + 2A(\varphi - \psi)$ for some fixed $A > 2\varepsilon^{-1} > 0$. Then the evolution for $u$ is given by

$$
\left( \frac{\partial}{\partial t} - \Delta \right) u = 2A \frac{\partial \varphi}{\partial t} + tr_{\omega}(2A\omega_1 + 2A\sqrt{-1} \delta \bar{\psi} + \chi) - 2nA
\geq 2A \frac{\partial \varphi}{\partial t} + tr_{\omega}(2A\varepsilon \omega_0 + (1 - 2A(1 - t))\chi) - 2nA
\geq 2A \frac{\partial \varphi}{\partial t} + 2A \varepsilon tr_{\omega}(\omega_0) - 2nA
\geq 2A \frac{\partial \varphi}{\partial t} + A \varepsilon \left( \frac{\omega_0^n}{\omega^n} \right)^{1/n} - 2nA
\geq 2A \frac{\partial \varphi}{\partial t} + e^{-n-1} \frac{\partial \varphi}{\partial t} - 2nA.
$$

by choosing $A >> \varepsilon^{-1}$ for $t > 1 - A^{-1}$. Let $p$ be the minimum point of $u$ at $t > 1 - A^{-1}$. Then $p$ does not lie in the singular locus of $\psi$ and by applying the maximum principle, we have

$$
\frac{\partial \varphi}{\partial t}(p) \geq -C
$$

for some uniform constant $C > 0$. Hence $u(p)$ is uniformly bounded below by applying Lemma 8.3. The lemma then immediately follows.

**Corollary 8.1.** For any $p > n$, there exists $C > 0$ such that for all $t \in [0,1)$

$$
N_{\mathcal{X},\omega_0,p}(\omega(t)) \leq C.
$$
Furthermore, for any $p > n$, there exist $A, B, K > 0$ such that for all $t \geq 0$,

$$\omega(t) \in \mathcal{W}(X, \omega_0, n, A, p, K; e^{B\psi-B}),$$

where $\psi$ is defined in Lemma 8.2.

Proof. By combining the previous lemmas, there exist $C_1, C_2 > 0$ such that on $X \times [0, 1)$, we have

$$C_2^{-1} e^{C_1\psi}(\omega_0)^n \leq \omega^n \leq C_2(\omega_0)^n.$$

The corollary immediately follows. □

Proof of Theorem 2.1. The assumption of Theorem 1.1 is satisfied due to Corollary 8.1. Theorem 2.1 immediately follows. □

Proof of Theorem 2.2. Since the initial Kähler class is rational, $X$ is projective and by Kawamata’s base point free theorem, the limiting class $\alpha_T$ is semi-ample. Let $\Phi : X \rightarrow Y$ be the unique birational morphism induced by the linear system $|m\alpha_T|$ for sufficiently large $m \in \mathbb{N}^+$. The smooth convergence of $g(t)$ on $X^\circ$ with uniform bounded potential is well-known [28]. We let $g_T$ be the limiting smooth Kähler metric on $X^\circ$. In addition, the parabolic Schwarz lemma from Refs. [26, 28] gives a uniform lower bound of $g(t)$, that is, there exists $C > 0$ such that for all $t \in [0, T)$,

$$g(t) \geq C^{-1} \Phi^* g_Y$$

on $X$. Equivalently, on $X \times [0, T)$, we have

$$|\nabla \Phi| \leq C$$

with respect to $g(t)$ and $g_Y$. By applying the convergence result in Theorem 1.1, for any $t_j \rightarrow T^-$, after passing to a subsequence, we can assume $(X, g(t_j))$ converges in GH sense to a compact metric space $(X_T, d_T)$. Obviously, the convergence is smooth on $X^\circ$ because of smooth convergence of $g(t)$ to $g_T$ on $X^\circ$. We will now extend the biholomorphism $\Phi : X^\circ \rightarrow Y^\circ$ to $\Phi_T : X_T \rightarrow Y$ by the following construction. For any $p_T \in X_T$ and any two sequence of points $p_j, p'_j \in (X, g(t_j)) \rightarrow p_T$ in GH distance,

$$d_Y(\Phi(p_j), \Phi(p'_j)) \leq C d_{g(t_j)}(p_j, p'_j) \rightarrow 0.$$

By compactness of $(Y, d_Y)$ and the above estimate, $\Phi(p_j)$ and $\Phi(p'_j)$ converge to the same point $q \in Y$. Therefore, we define

$$\Phi_T(p_T) = q.$$

$\Phi_T : X_T \rightarrow Y$ is then a well-defined surjective map as an extension of $\Phi$.

For any $p_T, p'_T \in X_T$, we can choose $p_j, p'_j \in (X, g(t_j))$ such that $p_j$ and $p'_j$ converge to $p_T$ and $q_T$ in GH distance.

$$d_Y(\Phi(p_T), \Phi(p'_T)) \leq \lim_{j \rightarrow \infty} d_Y(\Phi(p_j), \Phi(p'_j))$$
\[ \leq C \lim_{j \to \infty} d_{g(t_j)}(p_j, q_j) \]
\[ = C d_T(p_T, p'_T). \]

We have now completed the proof. \qed

9 | LONG-TIME SOLUTIONS OF THE KÄHLER-RICCI FLOW

We will prove Theorem 2.3 in this section as an application of Theorem 1.1. As in the previous section, we will bound the $p$-Nash entropy from above and the volume form from below along the Kähler–Ricci flow.

Let $X$ be a Kähler manifold with nef $K_X$ and nonnegative Kodaira dimension. For any smooth closed $(1,1)$-form $\chi \in K_X$, we can find a smooth volume form $\Omega$ such that

\[ \chi = \sqrt{-1} \partial \bar{\partial} \log \Omega, \quad \int_X \Omega = 1. \]

We let $\omega_t = (1 - e^{-t})\chi + e^{-t}\omega_0$. The numerical dimension of $K_X$ is defined by

\[ \kappa = \kappa(X) = \max\{k \geq 0 : [K_X]^k \neq 0 \text{ in } H^{k,k}(X, \mathbb{R})\}. \]

We will assume $\text{Kod}(X) \geq 0$, the numerical dimension $\kappa(X) \geq \text{Kod}(X) \geq 0$. The normalized Kähler–Ricci flow (2.4) is equivalent to the following parabolic complex Monge–Ampère equation

\[
\begin{cases}
\frac{\partial \varphi}{\partial t} = \log \frac{e^{(n-\kappa)t}(\omega_t + \sqrt{-1} \partial \bar{\partial} \varphi)^n}{\Omega} - \varphi, \\
\varphi|_{t=0} = 0.
\end{cases}
\] (9.1)

We let $\omega = \omega(\cdot, t) = \omega_t + \sqrt{-1} \partial \bar{\partial} \varphi$ solving the Monge–Ampère flow (9.1).

We first derive the volume growth for $(X, g(t))$.

Lemma 9.1. There exists $C > 0$ such that for all $t \geq 0$, we have

\[ C^{-1} e^{-(n-\kappa)t} \leq [\omega_t]^n \leq C e^{-(n-\kappa)t}. \]

Proof. Let $\alpha = [\omega_0] - K_X$. Then by the definition of $\kappa$, we have

\[ (K_X)^k \cdot \alpha^{n-k} > 0 \]

and

\[ e^{(n-\kappa)t}[\omega_t]^n = \sum_{l=0}^{n} C_l^n e^{(l-\kappa)t}(K_X)^l \cdot \alpha^{n-l} \]
\[ = \sum_{l=0}^{\kappa} C_l^n e^{(l-\kappa)t}(K_X)^l \cdot \alpha^{n-l} \]
\[ = C^n(K_X)^\kappa \cdot \alpha^{n-\kappa} + O(e^{-t}). \]

This proves the lemma. \qed
Lemma 9.2. There exists $C > 0$ such that for all $t \in [0, \infty)$,

$$-C \leq \sup_X \varphi(\cdot, t) \leq C, \quad \sup_X \left( \frac{\partial \varphi}{\partial t} + \varphi \right)(\cdot, t) \leq C.$$ 

Proof. By Jensen inequality and Lemma 9.1,

$$\frac{\partial}{\partial t} \left( \int_X \varphi(\cdot, t) \Omega \right) = \int_X \left( \log \frac{e^{(n-\kappa)t} \omega^n}{\Omega} \right) \Omega - \int_X \varphi(\cdot, t) \Omega$$

$$\leq \log \left( \int_X e^{(n-\kappa)t} \omega^n \right) - \int_X \varphi(\cdot, t) \Omega$$

$$\leq \log \left( e^{(n-\kappa)t} \Omega^n \right) - \int_X \varphi(\cdot, t) \Omega$$

$$\leq C - \int_X \varphi(\cdot, t) \Omega.$$ 

Hence $\int_X \varphi(\cdot, t) \Omega$ is uniformly bounded above. Since $\varphi \in \text{PSH}(X, \omega_t) \subset \text{PSH}(X, A\omega_0)$ for some fixed sufficiently large $A > 0$, by the mean value theorem for plurisubharmonic functions, there exists $C > 0$ such that for any $t \in [0, \infty)$ and $x \in X$,

$$\varphi(x, t) \leq \int_X \varphi(\cdot, t) \Omega + C.$$ 

This proves the uniform upper bound for $\varphi$.

Now we let $u = \frac{\partial \varphi}{\partial t} - e^{-t} \varphi$. Then the evolution for $u$ is given by

$$\frac{\partial u}{\partial t} = \Delta u - u - e^{-t} \frac{\partial \varphi}{\partial t} - e^{-t} tr_\omega (\omega_t + \omega_0 - \chi) + n - \kappa + ne^{-t}$$

$$= \Delta u - u + e^{-t} \log \frac{\Omega}{\omega^n} - e^{-t} tr_\omega (\omega_0 + e^{-t}(\omega_0 - \chi))$$

$$+ e^{-t} \frac{\partial \varphi}{\partial t} + n - \kappa + e^{-t}(n - (n - \kappa)t).$$

Then there exist $C_1, C_2 > 0$ such that for all $t \geq 0$,

$$\frac{\partial u}{\partial t} \leq \Delta u - u - \frac{1}{2} e^{-t} \left( tr_\omega (\omega_0) - \log \frac{\omega^n_0}{\omega^n} \right) + C_1$$

$$\leq \Delta u - u - C_2.$$ 

The uniform upper bound for $u$ immediately follows from the maximum principle. Therefore,

$$\frac{\partial \varphi}{\partial t} + \varphi = u + (1 + e^{-t})\varphi$$

is uniformly bounded above.
We now prove the lower bound for $\sup_X \varphi(\cdot, t)$. By the upper bound of $\frac{\partial \varphi}{\partial t} - e^{-t} \varphi$ and by Lemma 9.1, there exist $c_1, c_2 > 0$ such that for all $t \geq 0$, we have

$$
e^{(1+e^{-t})} \sup_X \varphi(\cdot, t) \geq c_1 e^{(1+e^{-t})} \sup_X \varphi + \sup_X \left( \frac{\partial \varphi}{\partial t} - e^{-t} \varphi \right) \geq \int_X e^{\frac{\partial \varphi}{\partial t}} \varphi \Omega = \int_X \varphi^{(n-k)} \omega^n \geq c_2.$$

This completes the proof of the theorem. □

Let

$$\mathcal{V}_t(x) = \sup \{ \varphi(x) | \omega_t + \sqrt{-1} \partial \bar{\partial} \varphi \geq 0, \varphi \leq 0 \} \quad (9.2)$$

be the extremal function associated to $\omega_t$ for any $t \in [0, \infty)$. We let $\mathcal{V}_\infty$ be the extremal function associated to $\chi$. Since we assume $\kappa > 0$, there exists a holomorphic section $\sigma \in |mK_X|$ for some sufficiently large $m$. Let $h$ be the hermitian metric on $mK_X$ with $\text{Ric}(h) = m\chi$ and $\sup_X |\sigma|^2_h = 1$. Then

$$\mathcal{V}_\infty \geq \frac{1}{m} \log |\sigma|^2_h \quad (9.3)$$

because

$$\chi + \frac{1}{m} \sqrt{-1} \partial \bar{\partial} \log |\sigma|^2_h = \frac{1}{m} [\sigma] \geq 0.$$ 

The following lemma is obvious by definition.

**Lemma 9.3.** For any $t_1 \leq t_2$,

$$\mathcal{V}_\infty \leq \mathcal{V}_{t_2} \leq \mathcal{V}_{t_1}.$$ 

**Lemma 9.4.** There exists $C > 0$ such that on $X \times [0, \infty)$, we have

$$\varphi(\cdot, t) \geq \mathcal{V}_t - C \geq \mathcal{V}_\infty - C. \quad (9.4)$$

**Proof.** By Lemmas 9.1 and 9.2, there exists $C > 0$ such that on $X \times [0, \infty)$, the normalized volume measure is uniformly bounded above by a smooth volume form

$$\frac{\omega^n}{|\omega|^n} \leq C \Omega.$$ 

The lemma then follows from Proposition 4.1 and the uniform lower bound of $\sup_X \varphi(\cdot, t)$. □
Lemma 9.5. There exists $C > 0$ such that on $X \times [0, \infty)$, we have

$$\frac{\partial^2 \varphi}{\partial t^2} + \frac{\partial \varphi}{\partial t} \leq C.$$ 

Proof. Let $u = \frac{\partial \varphi}{\partial t} + \varphi$. Let $R$ be the scalar curvature of $\omega$. Since the scalar curvature of the Kähler–Ricci flow is uniformly bounded below, there exists $C > 0$ such that on $X \times [0, \infty)$, we have

$$\Delta u = \Delta \left( \log \frac{\omega^n}{\Omega} \right) = -R - tr_\omega(\chi) \leq C - tr_\omega(\chi).$$

On the other hand,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 \varphi}{\partial t^2} + \frac{\partial \varphi}{\partial t} \leq \Delta u + tr_\omega(\chi) - \kappa \leq C - \kappa.$$

This proves the lemma.

Lemma 9.6. Suppose $f = f(x)$ is a smooth function for $x \geq 0$ and it satisfies the differential inequality

$$f'' + f' \leq 1, \quad |f| \leq A$$

for all $x \geq 0$ and for some fixed $A > 1$. Then for any $x \geq 0$, we have,

$$f' \geq -5A.$$

Proof. Suppose the lemma fails. Then there exists $x_0 \geq 0$, such that

$$f'(x_0) < -5A.$$ 

Since $|f| \leq A$, there must exist an $a \in (0, 1)$ such that

$$f'(x_0 + a) = -2A.$$ 

Otherwise, $f(x) < f(x_0) - 2A \leq -A$ for $x \in (x_0, x_0 + 1)$, contradicting the assumption. By our assumption, $(f' + f - x)' \leq 0$, and so

$$f'(x_0 + a) + f(x_0 + a) - (x_0 + a) \leq f'(x_0) + f(x_0) - x_0.$$ 

This implies that

$$-2A = f'(x_0 + a) \leq f'(x_0) + f(x_0) - f(x_0 + a) + a \leq -5A + 2A + 1 = -3A + 1.$$ 

This leads to contradiction as $A > 1$. 

Lemma 9.7. There exists $C > 0$ such that on $X \times [0, \infty)$, we have

$$\frac{\partial \varphi}{\partial t} \geq 5V_\infty - C.$$  

Proof. By Lemmas 9.6 and 9.4, there exists $C > 0$ such that for any $x \in X$ and $t \geq 0$, we have

$$\frac{\partial^2 \varphi}{\partial t^2} + \frac{\partial \varphi}{\partial t} \leq C, \quad -V_\infty - C \leq \varphi \leq C$$

The lemma is then proved by directly applying Lemma 9.6. \(\square\)

We then immediately have the following corollary by combining Lemmas 9.2 and 9.7 and Equation (9.3).

Corollary 9.1. There exists $C = C(X, g_0) > 0$ such that on $X \times [0, \infty)$, we have

$$C^{-1} \exp \left( \frac{6}{m} \log |\sigma|^2 \right) \leq \frac{1}{[\omega]^n} \frac{\omega}{\Omega} \leq C.$$  

Proof. By Lemma 9.1, there exists $C > 0$ such that

$$C^{-1} \exp \left( \frac{\partial \varphi}{\partial t} + \varphi \right) \leq \frac{1}{[\omega]^n} \frac{\omega}{\Omega} \leq C \exp \left( \frac{\partial \varphi}{\partial t} + \varphi \right).$$  

The corollary is then a direct consequence of Lemmas 9.7 and 9.4 and Equation (9.3). \(\square\)

Corollary 9.1 implies the bound for $p$-Nash entropy and a pointwise lower bound for $\omega(t)^n$.

Corollary 9.2. For any $p > n$, there exists $C > 0$ such that for all $t \geq 0$

$$N_{X, \omega_0, p}(\omega(t)) \leq C.$$  

Furthermore, for any $p > n$, there exist $A, B, K > 0$ such that for all $t \geq 0$,

$$\omega(t) \in W(X, \omega_0, n, A, p, K; B^{-1}|\sigma|^2),$$  

where $\psi$ is defined in Lemma 8.2.

Proof of Theorem 2.3. The assumption of Theorem 1.1 is satisfied due to Corollary 9.2. Theorem 2.3 immediately follows. \(\square\)

Proof of Corollary 2.1. With the diameter bound of Theorem 2.3 and the scalar curvature bound of Ref. [29], the corollary can be proved by applying the same argument of Ref. [20]. \(\square\)

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REFERENCES

1. R. Bamler, *Entropy and heat kernel bounds on a Ricci flow background*, arXiv:2008.07093.
2. S. Boucksom, P. Eyssidieux, V. Guedj, and A. Zeriahi, *Monge-Ampré equations in big cohomology classes*, Acta Math. 205 (2010), no. 2, 199–262.
3. G. E. Bredon, *Topology and geometry*, Graduate Texts in Mathematics, vol. 139, Springer-Verlag, New York, 1997, pp. xiv+557.
4. J. Cheeger and T. H. Colding, *On the structure of space with Ricci curvature bounded below I*, J. Differential. Geom. 46 (1997), 406–480.
5. J. Cheeger, T. H. Colding, and G. Tian, *On the singularities of spaces with bounded Ricci curvature*, Geom. Funct. Anal. 12 (2002), 873–914.
6. S. Y. Cheng and P. Li, *Heat kernel estimates and lower bound of eigenvalues*, Comment. Math. Helv. 56 (1981), 327–338.
7. J. P. Demailly and N. Pali, *Degenerate complex Monge-Ampère equations over compact Kähler manifolds*, Intern. J. Math. 21 (2010), no. 3, 357–405.
8. P. Eyssidieux, V. Guedj, and A. Zeriahi, *A priori $L^\infty$-estimates for degenerate complex Monge-Ampère equations*, Int. Math. Res. Not. IMRN 1 (2008), rnn 070, 8 pp.
9. J. P. Demailly and M. Paun, *Degenerate complex Monge-Ampère equations over compact Kähler manifolds*, Internat. J. Math. 21 (2010), no. 3, 357–405.
10. X. Fu, B. Guo, and J. Song, *Geometric estimates for complex Monge-Ampère equations*, J. Reine Angew. Math. 765 (2020), 69–99.
11. B. Guo and D. H. Phong, *On $L^\infty$ estimates for fully non-linear equations*, arXiv:2204.12549.
12. B. Guo and D. H. Phong, *Uniform entropy and energy bounds for fully non-linear equations*, to appear in Comm. Anal. Geom. arXiv:2207.08983.
13. B. Guo, D. H. Phong, and J. Sturm, *Green’s functions and complex Monge-Ampère equations*, to appear in J. Differential Geom. arXiv:2202.04715.
14. B. Guo, D. H. Phong, and F. Tong, *On $L^\infty$ estimates for complex Monge-Ampère equations*, Ann. of Math. 198 (2023), no. 1, 393–418.
15. B. Guo, D. H. Phong, F. Tong, and C. Wang, *On $L^\infty$ estimates for Monge-Ampère and Hessian equations on nef classes*, Anal. PDE, in press, arXiv:2111.14186.
16. B. Guo, D. H. Phong, F. Tong, and C. Wang, *On the modulus of continuity of solutions to complex Monge-Ampère equations*, arXiv:2112.02354.
17. B. Guo and J. Song, *Local noncollapsing for complex Monge-Ampère equations*, J. Reine Angew. Math. (Crelle’s J.) 793 (2022), 225–238.
18. R. Harvey and H. B. Lawson, *Determinant majorization and the work of Guo-Phong-Tong and Abja-Olive*, Calc. Var. 62, (2023), 153.
19. L. Hörmander, *An introduction to complex analysis in several variables*, Van Nostrand, Princeton, NJ, 1973.
20. W. Jian and J. Song, *Diameter and Ricci curvature estimates for long-time solutions of the Kähler-Ricci flow*, Geom. Funct. Anal. 32 (2022), 1335–1356.
21. S. Kolodziej, *The complex Monge-Ampère equation*, Acta Math. 180 (1998), 69–117.
22. Y. Li, *On collapsing Calabi-Yau fibrations*, J. Differ. Geom. 117 (2021), no. 3, 451–483.
23. R. Schoen and S. T. Yau, *Lectures on differential geometry*, vol. 2, International press, Cambridge, MA, 1994.
24. J. Song, *Ricci flow and birational surgery*, arXiv:1304.2607.
25. J. Song, *Riemannian geometry of Kähler-Einstein currents*, arXiv:1404.0445.
26. J. Song and G. Tian, *The Kähler-Ricci flow on surfaces of positive Kodaira dimension*, Invent. Math. 170 (2007), no. 3, 609–653.
27. J. Song and G. Tian, *Canonical measures and Kähler-Ricci flow*, J. Am. Math. Soc. 25 (2012), no. 2, 303–353.
28. J. Song and G. Tian, *The Kähler-Ricci flow through singularities*, Invent. Math. 207 (2017), no. 2, 519–595.
29. J. Song and G. Tian, *Bounding scalar curvature for global solutions of the Kähler-Ricci flow*, Am. J. Math. 138 (2016), no. 3, 683–695.
30. J. Song, G. Tian, and Z. Zhang, *Collapsing behavior of Ricci-flat Kähler metrics and long-time solutions of the Kähler-Ricci flow*, arXiv:1904.08345.
31. J. Song and B. Weinkove, *Contracting exceptional divisors by the Kähler-Ricci flow*, Duke Math. J. **162** (2013), no. 2, 367–415.

32. J. Song and B. Weinkove, *Contracting divisors by the Kähler-Ricci flow II*, Proc. Lond. Math. Soc. (3) **108** (2014), no. 6, 1529–1561.

33. G. Tian, *On Kähler-Einstein metrics on certain Kähler manifolds with $C_1(M) \geq 0$*, Invent. Math. **89** (1987), no. 2, 225–246.

34. G. Tian and Z. Zhang, *Convergence of Kähler-Ricci flow on lower-dimensional algebraic manifolds of general type*, Int. Math. Res. Not. IMRN **2016** (2016), no. 21, 6493–6511.

35. V. Tosatti, B. Weinkove, and X. Yang, *The Kähler-Ricci flow, Ricci-flat metrics and collapsing limits*, Am. J. Math. **140** (2018), no. 3, 653–698.

36. H. Tsuji, *Existence and degeneration of Kähler-Einstein metrics on Minimal Algebraic Varieties of General Type*, Math. Ann. **281** (1988), 123–133.

37. B. Wang, *The local entropy along Ricci flow, Part A: the no-local-collapsing theorems*, Camb. J. Math. **6** (2018), no. 3, 267–346.

38. S. T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I*, Commun. Pure Appl. Math. **31** (1978), 339–411.

39. Z. Zhang, *On degenerate Monge-Ampère equations over closed Kähler manifolds*, Int. Math. Res. Not. **2006** (2006), 63640.

40. Z. Zhang, *Scalar curvature bound for Kähler-Ricci flows over minimal manifolds of general type*, Int. Math. Res. Not. (2009), DOI 10.1093/imrn/rnp073.