Physical Exceptional Points without Degeneracy of Energy Levels

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Exceptional Point (EP) is an interesting physical phenomenon in non-Hermitian physics, at which for a singular matrix the eigenvalues and the eigenvectors coalesce. In this paper, we generalize the concept of physical EPs to that of physical EPs, an issue in certain protected sub-systems, for example, the defect edge states of non-Hermitian topological insulator, or the defect edge degenerate ground states in non-Hermitian systems with spontaneously symmetry breaking, et.al. For these physical EPs, the coalescence of eigenvectors may occur without requiring the eigenvalues degeneracy. In addition, for these physical EPs, more subtle structures are explored, including basis defectiveness, non-Hermitian weight vectors, and hidden quantum phase transitions. By taking the topologically protected edge states in non-Hermitian Su-Schrieffer-Heeger model as an example, we show the physical properties of different types of physical EPs.

The term ‘Exceptional Points’ (EPs) has been introduced by T. Kato over half a century ago. In mathematics, EPs are branch point singularities of the spectrum and eigenfunctions for non-Hermitian (NH) matrices. In physics, EPs are the points for NH Hamiltonian at which two or more eigenvalues and their corresponding eigenvectors coalesce together. Since the publication of paper by Bender and Boettcher, EPs become one of the most interesting features of NH physics. As a special example, for NH Hamiltonians with parity-time(PT)-symmetry, the spontaneous PT-symmetry breaking corresponds to a typical EP, at which energies levels become degenerate and eigenvectors coalesce. In experiments, the phenomena of EPs had been realized and simulated by different approaches.

It has been believed that in mathematics, for different NH matrices k-th order EPs have the same properties and can be represented by a matrix of Jordan normal form. In NH physics, the issue of EPs is related to a NH model that is described by a given Hamiltonian in corresponding Hilbert space. To emphasize the difference with EPs in mathematics (the issue about NH matrix), we call EPs in NH physical systems to be physical EPs. For physical EPs of protected sub-systems in NH models, we can generalize the definition of EPs by not requiring the eigenvalues degeneracy (see below discussion). Now, the identification of a physical EP comes from the coalescence of different eigenvectors. In this paper, from this starting point, we provide a systematic classification on physical EPs of protected sub-systems in NH models. In particular, we found that there may exist a new type of EPs (Basis-type of EPs), for which the eigenvectors become coalesce without energy degeneracy.

Protected NH sub-systems and their physical EPs: Before giving the discussion of physical EPs, we define protected NH sub-systems.

In some quantum many-body models, due to special conditions of symmetry/topology, there may exist protected sub-systems. For example, the topologically protected edge states of topological insulator, or the symmetry-protected degenerate ground states in systems with spontaneously symmetry breaking, et.al. For these physical EPs, the coalescence of eigenvectors may occur without requiring the eigenvalues degeneracy. In addition, for these physical EPs, more subtle structures are explored, including basis defectiveness, non-Hermitian weight vectors, and hidden quantum phase transitions. By taking the topologically protected edge states in non-Hermitian Su-Schrieffer-Heeger model as an example, we show the physical properties of different types of physical EPs.

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the quantum systems with coalescence of eigenstates. For example, for the case of $K = 2$, $|\psi^R\rangle$ and $|\psi^B\rangle$ are the eigenstates of the effective Hamiltonian $\hat{H}_{NH-S}$ of a protected sub-system and the corresponding energy levels are $E_+$ and $E_-$, respectively. At physical EPs, we have $|\psi^R\rangle = |\psi^B\rangle$. We need to emphasize that the two energy levels may not be necessary degenerate, i.e., $E_+ = E_-$ or $E_+ \neq E_-$ are all allowed.

**M-type of physical EPs:** Firstly, we show the properties of usual physical EPs in a general two-level NH sub-system, $\{H_{NH-S}, B_S\}$. Here, $H_{NH-S} = h_0 + \hat{h} \cdot \hat{\sigma}$ denotes the NH Hamiltonian, $H_{NH-S} \neq H_{NH-S}'$. $h_0$ is a complex number. $\hat{h} = (h^x, h^y, h^z)$ is a complex vector and $\hat{\sigma} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$ is the vector of Pauli matrices. $B_S = \{|\psi_0\rangle\} = \{|1\rangle, |0\rangle\}$ denotes a normal basis obeying orthogonal and normalization condition, i.e., $\langle 1 | 0 \rangle = 0$, $(1 \rangle = 0 | 0 \rangle = 1$.

In general, we have $H_{NH-S} = h_0 + (\Re \hat{h}) \hat{\sigma}_{Re} + i(\Im \hat{h}) \hat{\sigma}_{Im}$ where $\hat{\sigma}_{Re}$ and $\hat{\sigma}_{Im}$ ($\hat{\sigma}_{Re} \cdot \hat{\sigma}_{Re} = 1$ and $\hat{\sigma}_{Im} \cdot \hat{\sigma}_{Im} = 1$) are the Pauli matrices corresponding to real part and imaginary part of $\hat{h}$, respectively. We divide $i(\Im \hat{h}) \hat{\sigma}_{Im}$ into two parts, $i(\Im \hat{h})^A \hat{\sigma}_{Im}^A$ and $i(\Im \hat{h})^C \hat{\sigma}_{Im}^C$ with $\hat{\sigma}_{Im}^A \hat{\sigma}_{Re} = 0$, and $\hat{\sigma}_{Im}^C \hat{\sigma}_{Re} = 0$. Then, the NH Hamiltonian becomes

$$H_{NH-S} = h_0 + \hat{h}_R \cdot \hat{\sigma}_{Re} + i\hat{h}_I^I \cdot \hat{\sigma}_{Im}^A, \quad (2)$$

where $\hat{h}_R = (\Re \hat{h}) + i(\Im \hat{h})^C$ and $\hat{h}_I^I = (\Im \hat{h})^A$. Under an inverse NH similarity transformation (NH ST) $\hat{S}_M(\beta^M)$, we transform the original NH Hamiltonian $H_{NH-S}$ into

$$H_0 = \hat{S}_M^{-1}(\beta^M) H_{NH-S} \hat{S}_M(\beta^M) \quad (3)$$

$$= h_0 + \sqrt{\hat{h}_R} (\hat{h}_R^I \cdot \hat{\sigma}_{Re}).$$

$\hat{S}_M(\beta^M) = e^{-\beta^M \cdot \hat{\sigma}^M}$ is the NH ST where $\hat{\sigma}^M = \frac{1}{2i}[\hat{\sigma}_{Im}^A, \hat{\sigma}_{Re}]$ and $\beta^M = \frac{1}{4} \ln \frac{\hat{h}_R^I + |\hat{h}_I^I|}{|\hat{h}_R^I|} \{\hat{\sigma}_{Im}^A |\hat{\sigma}_{Im}^C\}$ is the eigenvalue of $\hat{H}$ that is same to that of $H_{NH-S}$, i.e., $E_S = h_0 \pm \sqrt{\hat{h}_R^I}$. Under inverse NH ST, the basis of $H_{NH-S}$ is correspondingly changed, i.e., $\{|\psi_0\rangle\} \rightarrow \{|\psi^R\rangle\} = \{S_M(\beta^M) |\psi_0\rangle\}$.

At $\frac{\hat{h}_R^I}{|\hat{h}_R^I|} = 1$, the two energy levels are degenerate, i.e., $E_+ = h_0 = E_-$. It is obvious that this is a physical EP. Approaching it, the NH ST becomes singular, i.e., $|\psi^R\rangle = \hat{S}_M(\beta^M) |\psi_0\rangle = e^{-\beta^M \cdot \hat{\sigma}^M} |\psi_0\rangle$ with $\beta^M = \frac{1}{4} \ln \frac{\hat{h}_R^I + |\hat{h}_I^I|}{|\hat{h}_R^I|} \rightarrow \infty$. It is the singular NH ST $\hat{S}_M(\beta \rightarrow \infty)$ that leads to energy-level coalescing, i.e., $|\psi^R\rangle = |\psi^B\rangle$. Here, $|\psi^B\rangle$ and $|\psi^R\rangle$ are the eigenstates of $H_{NH-S}$ at $\frac{\hat{h}_R^I}{|\hat{h}_R^I|} = 1$. One can see that the key point of EPs is the existence of a singular NH ST that comes from a NH Hamiltonian $\hat{H}_{NH-S}$ obeying singular condition $|\hat{h}_R^I|_{\hat{h}_R^I} = 1$. For this reason, we call it Matrix-type of physical EP (or M-EP) and denote it by $\{\frac{\hat{h}_R^I}{|\hat{h}_R^I|} = 1, S_M(\beta^M \rightarrow \infty)\}$.

In summary, for a NH system at M-EPs, the merging of two energy levels and the merging of two eigenstates occur simultaneously. In these NH SSs $\{H_{NH-S}, B_S\}$, the coalescence of eigenstates comes from a singular Hamiltonian $H_{NH-S}$ rather than $B_S$.

**Basis-type of Physical EPs:** However, for protected sub-systems there exists a new type of physics EPs from basis definiteness. We call it basis-type of physical EPs (or B-EP).

We also take a two-level protected sub-system as an example that is described by $\{H_{NH-S}, B_S\}$. The effective (Hermitian or NH) Hamiltonian $H_{NH-S} = h_0 + \hat{h} \cdot \hat{\sigma}$ is assumed to obey non-singular condition $\frac{\hat{h}_R^I}{|\hat{h}_R^I|} \neq 1$. The abnormal basis $B_S = \{|\psi^R\rangle\}$ is defined by operating a singular NH ST on a normal basis, i.e., $B_S = \{|\psi^R\rangle\} = \{S_B |\psi_0\rangle\}$ where $|\psi_0\rangle$ is normal basis, and $S_B = e^{-\beta^B \cdot \hat{\sigma}^B}$ with real $\beta^B$. In the limit of $\beta^B \rightarrow \infty$, due to singular NH ST $S_B$, the basis becomes defective. As a result, we have a new type of EP – B-EP, of which the eigenvectors become coalesce into one eigenstate of $\beta^B$ as $|\psi^\rightarrow\rangle \rightarrow |\psi^-\rangle$ without energy degeneracy, i.e., $E_+ = h_0 + \sqrt{\hat{h}_R^I} \neq E_-$ = $h_0 - \sqrt{\hat{h}_R^I}$. We denote the B-EP by $\{\frac{\hat{h}_R^I}{|\hat{h}_R^I|} \neq 1, S_B(\beta^B \rightarrow \infty)\}$.

For the B-EPs, there exist unexpected structures – additional classification, NH weight vector and hidden quantum phase transitions. Now, the (Hermitian or NH) Hamiltonian $H_{NH-S}$ is assumed to be that with finite energy splitting (or $E_+ \neq E_-)$ and the abnormal basis $B_S = \{|\psi^R\rangle\}$ is assumed to be defective under a singular NH ST $S_B = e^{-\beta^B \cdot \hat{\sigma}^B}$ with $\beta^B \rightarrow \infty$.

We then do an inverse NH ST $S_S$ on the abnormal basis $B_S$ and get a description of protected sub-system with normal basis, i.e., $B_S' = \{|\psi^{R'}\rangle\} \rightarrow \{|\psi_0\rangle\} = \{S_B^{-1} |\psi^R\rangle\}$. Correspondingly, the Hamiltonian $H_{NH-S}$ is transformed into $H_{NH-S}^\alpha = S_B^{-1} H_{NH-S} S_B$. The original projected sub-system $\{H_{NH-S}, \{|\psi^R\rangle\}\}$ turns into $\{H_{NH-S}^\alpha, \{|\psi_0\rangle\}\}$. In particular, there are two possibilities of $H_{NH-S}^\alpha$: one possibility is all elements of $H_{NH-S}^\alpha$ are finite. We call it first class of B-EPs (or IB-EPs); the other is one or more element diverges, i.e., $(\hat{H}_{NH-S}^\alpha)_{ij} \rightarrow \infty$. We call it second class of B-EPs (or II-IB-EPs).

It was known that for B-EPs $\{E_+ \neq E_-, S_B(\beta^B \rightarrow \infty)\}$, due to defective basis, there is only one state $|\psi^R\rangle$ (or $|\psi^{R'}\rangle$) that distributes on two energy levels $E_+, E_-$ ($E_+ \neq E_-)$). To describe the distribution of the single defective state on two different energy levels, we introduce the concept of NH weight vectors. The weight vectors for the two energy levels are defined as $W = (W_1, W_2)$.
where $W_j = \sum_{i}^{2} \frac{(\psi_R^i | \psi_R^j)}{(\psi_R^i | \psi_R^i)}$ is state weight for $j$-th base $|\psi_R^j\rangle$ ($j = 1, 2$). From the fact of single state, we have $\sum_{j=1}^{2} W_j = 1$.

With the help of weight vectors, we describe the distribution of the single state on two energy levels for B-EPs $(E_+ \neq E_- \triangleq \beta \to \infty)$ with $\mathcal{S}_B = e^{-\beta^M \sigma^B \eta_1}$. For the IB-EPs, the Hamiltonian must be $\hat{H}_{\text{NH-S}} = \hbar_0 + \lambda \sigma^B$ with $\lambda \neq 0$. Now, the weight vector is $\mathbf{W} = (1, 0)$ for $\beta^B > 0$ or $\mathbf{W} = (0, 1)$ for $\beta^B < 0$; On the other hand, for the IIB-EPs, the Hamiltonian must be $\hat{H}_{\text{NH-S}} = \hbar_0 + \lambda \sigma^B + \eta (\sigma^B)^\dagger$ with $\lambda^2 + \eta^2 \neq 0$ and $((\sigma^B)^\dagger, \sigma^B) = 0$. Now, one element of $\hat{H}_{\text{NH-S}}$ diverges. The weight vector is obtained as $\mathbf{W} = \frac{1}{2} (1, 1)$. The detailed discussion is provided in supplementary materials. Therefore, during tuning the Hamiltonian $\hat{H}_{\text{NH-S}}$ or $\hat{H}_{\text{NH-M}}^\beta$, if a system of IB-EPs is changed into to another with IIB-EPs, there must exist a hidden quantum phase transition (QPT) with a sudden changing of the weight vectors from $\mathbf{W} = (1, 0)$ or $\mathbf{W} = (0, 1)$ to $\mathbf{W} = \frac{1}{2} (1, 1)$.

In addition, to verify this type of hidden QPTs and show the defectiveness of the abnormal basis, we calculate the state similarity of two eigenstates, $\Lambda = \langle \psi_R^I | \psi_R^J \rangle$ where $|\psi_R^I\rangle$ and $|\psi_R^J\rangle$ are the eigenstates for the two edge states under self-normalization basis, i.e., $\langle \psi_R^I | \psi_R^I \rangle = \langle \psi_R^J | \psi_R^J \rangle = 1$. We found that in IB-EP, $\Lambda = 0$ and in IIB-EP, $\Lambda = 1$. A hidden phase transition occurs with sudden changing between of the state similarity from $\Lambda = 0$ to $\Lambda = 1$. See the detailed discussion in supplementary materials.

In summary, for a NH system at B-EPs (both IB-EPs and IIB-EPs), the merging of two eigenstates may occur without energy degeneracy. Between IB-EPs and IIB-EPs, there must exist a hidden QPT. In these PNHSSs $\{\hat{H}_{\text{NH-S}}, B_S\}$, the coalescence of eigenstates comes from defective basis $B_S$ rather than singular Hamiltonian $\hat{H}_{\text{NH-S}}$.

**Hybrid-type of physical EPs:** For a NH protected sub-system, there may exist both B-EPs and M-EPs. Now, we call it hybrid-type of physical EP (or H-EP).

We also take a two-level protected sub-system as an example $\{\hat{H}_{\text{NH-S}}, B_S\}$. On the one hand, the Hamiltonian must be NH that is described by $\hat{H}_{\text{NH-S}} = \hbar_0 + \hbar \cdot \sigma$ obeying singular condition $\frac{\tilde{h}_0}{\tilde{h}_0} = 1$; on the other hand, the basis $B_S$ is defective and don’t obey usual normalization condition, i.e., $\mathcal{B}_S = \{ |\psi_R^1\rangle \} = \{ \mathcal{S}_B | \psi_0 \rangle \}$ where $\mathcal{S}_B = e^{-\sigma^B \eta_1}$ with $\beta^B \to \infty$.

On the one hand, according to the abnormal basis $B_S$, there exists singular NH ST from abnormal basis $\mathcal{S}_B = e^{-\sigma^B \eta_1}$; On the other hand, at $\frac{\tilde{h}_0}{\tilde{h}_0} = 1$, the two energy levels become degenerate, i.e., $E_+ = E_-$. According to above discussion, another NH ST $\mathcal{S}_M = e^{-\beta^M \sigma^M}$ becomes singular, $\beta^M = \frac{1}{2} \ln \frac{|\tilde{h}_0| + |\tilde{h}_m|}{|\tilde{h}_0| - |\tilde{h}_m|} \to \infty$. Now, in the limit of $\beta^M, \beta^B \to \infty$, we have composite singular NH ST as $\mathcal{S}_B \mathcal{S}_M$. If $\mathcal{S}_B \mathcal{S}_M \neq 1$, we have a H-EP. In the following parts, we use $\{ |\tilde{h}_0\rangle \langle \tilde{h}_0|, 1, \mathcal{S}_B (\beta^B \to \infty), \mathcal{S}_M (\beta^M \to \infty) \}$ to denote it.

In summary, for a NH system at H-EPs, the two energy levels become degenerate for single (defective) quantum state. In these PNHSSs $\{\hat{H}_{\text{NH-S}}, B_S\}$, the coalescence of eigenstates comes from both a singular Hamiltonian $\hat{H}_{\text{NH-S}}$ and a defective basis $B_S$.

**Example: Physics of EPs for topological edge states in 1D NH Su-Schrieffer-Heeger model:** We take 1D nonreciprocal Su-Schrieffer-Heeger (SSH) model as an example to illustrate the different types of physics EPs. The Bloch Hamiltonian for a nonreciprocal SSH model under periodic boundary condition (PBC) is given by

$$\hat{H}_{\text{PBC}}(k) = \sum_k c_k^\dagger \tau_x (t_1 + t_2 \cos k) c_k$$

$$+ \sum_k c_k^\dagger \tau_y (t_2 \sin k + i \gamma) c_k + i \varepsilon \sum_k c_k^\dagger \tau_z c_k$$

where $c_k^\dagger = (c_k^A, c_k^B)$. $\tau$’s are the Pauli matrices acting on the (A or B) sublattice subspace. $t_1$ and $t_2$ describe the intra-cell and inter-cell hopping strengths, respectively. $\gamma$ describes the unequal intra-cell hoppings. $\varepsilon$ denotes the imaginary staggered potential on the two sublattices. $t_1, t_2, \gamma, \varepsilon$ are all real. In this paper, we set $t_2 = 1$

Due to the NH skin effect, the physics properties for 1D nonreciprocal SSH model under open boundary condition (OBC) are characterized by $\hat{H}_{\text{OBC}}(k)$ rather than $\hat{H}_{\text{PBC}}(k)$ $[12]$. Consequently, the effective hopping parameters become $t_1 = \sqrt{(t_1 + \gamma) (t_1 - \gamma)}$, and $t_2 = t_2$. To characterize the topological properties of the NH topological system, the non-Bloch topological invariant $\psi$ of $\hat{H}_{\text{OBC}}(k)$. In the region of $|\tilde{t}_1| < |\tilde{t}_2|$, $\tilde{w} = 1$, the system is a topological insulator; In the region of $|\tilde{t}_1| > |\tilde{t}_2|$, $\tilde{w} = 0$, the system is a normal insulator. Quantum phase transition occurs at $|\tilde{t}_1| = |\tilde{t}_2|$, at which bulk energy gap under OBC is closed.

In topological phase with $\tilde{w} = 1$, there exist two edge states. The two edge states make up a topologically PNHSS denoted by $\{\hat{H}_{\text{NH-S}}, B_S\}$. Under the biorthogonal set, the basis $B_S$ is denoted as $\{ |\psi_R^1\rangle \} = \{ |\psi_1^1\rangle | \psi_1^2\rangle \}$. The effective Hamiltonian $\hat{H}_{\text{NH-S}}$ is written as $\{ h_{11} h_{12} h_{21} h_{22} \}$ where $h_{ij} = \langle \psi_1^i | \hat{H} | \psi_1^j \rangle$, $i, j = 1, 2$ $[13]$. Through straightforward calculations, we can analytically get the effective Hamiltonian as

$$\hat{H}_{\text{NH-S}} = \tilde{\Delta} \sigma^x + i \varepsilon \sigma^z$$

where $\tilde{\Delta} = \frac{t_2^2 - t_1^2}{t_2} (\frac{t_1}{t_2})^N$. The two energy levels for the two eigenstates $|\psi_R^1\rangle$ and $|\psi_R^2\rangle$ are $E = \pm \sqrt{\Delta^2 - \varepsilon^2}$. 
Then, based on \(\{\hat{H}_{\text{NH-S}}, \{\psi^R\}\}\), we study the physical EPs for topologically PNHHSS in 1D nonreciprocal SSH model. Fig.1(a) is an illustration of an effective two-level model for physical EPs: the system at four red dots belongs to M-EPs; that on the solid black lines belongs to H-EP; that in the yellow region belongs to IB-EP; that in the grey region belongs to IIB-EP. Let us separately discuss the different types of physics EPs:

1. **M-EPs**: Firstly, we consider the M-EPs by setting \(\varepsilon\) to be a pure real value and \(\gamma = 0\). Now, the basis becomes normal, i.e., \(B_\varepsilon = \{\psi^R\} \to \{\psi^0\}\). The effective two-level model is reduced into \(\hat{H}_{\text{NH-S}} = i\varepsilon \sigma^z + \Delta_0 \sigma^x\) where \(\Delta_0 = \frac{(\tilde{t}_1 - \tilde{t}_2)}{t_1} \left(\frac{t_2}{t_1}\right)^N\). A spontaneous \(PT\)-symmetry-breaking transition occurs at \(|\varepsilon| = \Delta_0\). The energy levels become degenerate, i.e., \(E = \pm \sqrt{\Delta^2 - \Delta_0^2} \to 0\), and the NH ST becomes singular, i.e., \(\hat{S}_M = e^{-\beta M} \sigma^z\) with \(\beta M = \frac{i}{\hbar} \ln \left(\frac{\Delta_{\text{out}} +\varepsilon}{\Delta_{\text{in}} - \varepsilon}\right) \to 0\). Because \(\hat{H}_{\text{NH-S}}\) obeys singular condition \(|\hat{E}\varepsilon| = 1\), we have an M-EP denoted by \(\{\hat{E}\varepsilon|\hat{E}\varepsilon\} = 1\), \(\hat{S}_M (\beta M \to \infty)\). In Fig.1(b), for the case of \(N = 20\), \(\gamma = 0.694\) and \(\varepsilon = 0.001\), the energy levels degeneracy at \(t_1 = 0\) comes from M-EP. Now, the state similarity \(\Lambda = |\langle \psi^R_+ | \psi^R_+ \rangle| = 1\) indicates the coalescence of the two edge states.

2. **B-EPs**: Secondly, we consider the B-EPs by setting \(|\Delta| \neq |\varepsilon|\) where \(\gamma \neq 0\), \(\Delta = \frac{t_1 - t_2}{t_1} \left(\frac{t_2}{t_1}\right)^N\). A pure real value. Now, the basis becomes abnormal, i.e., \(B_S = \{\psi^R\} = \{\psi^0\}\) with \(\hat{S}_B = e^{-\beta B} \sigma^z\) and \(\beta B = Nq_0\). Here, \(q_0 = \frac{1}{2} \ln \left(\frac{\Delta_{\text{out}}}{\Delta_{\text{in}} - \varepsilon}\right)\) is imaginary wave vector that characterizes the NH skin effect. However, \(\hat{H}_{\text{NH-S}}\) does not obey singular condition \(|\hat{E}\varepsilon| \neq 1\) and the energy levels aren’t degenerate, i.e., \(E = \pm \sqrt{\Delta^2 - \varepsilon^2} \neq 0\). In thermodynamic limit \(N \to \infty\), the protected sub-system may be regarded as a B-EP; On the other hand, for the case of \(N = 20\), \(\gamma = 0.694\) and \(\varepsilon = 0.001\), we have \(\Lambda = 1\) that indicates the coalescence of the two edge states.

3. **H-EPs**: Thirdly, we consider the H-EPs by setting \(\varepsilon\) to be a pure real value, \(\gamma \neq 0\), and \(|\Delta| = |\varepsilon|\). On the one hand, the basis is abnormal, \(B_S = \{\psi^R\} = \{\psi^0\}\) with a singular NH ST \(\hat{S}_B = e^{-\beta B} \sigma^z\) and \(\beta B = Nq_0 \to \infty\). In thermodynamic limit \(N \to \infty\), the protected sub-system may be regarded as a B-EP; On the other hand, for the case of \(\Lambda = 1\), the same sub-system can be regarded as a M-EP with degenerate energy levels and another singular NH ST \(\hat{S}_B (\beta M \to \infty)\). Therefore, this protected sub-system belongs to a H-EP denoted by \(\{\hat{E}\varepsilon|\hat{E}\varepsilon\} = 1, \hat{S}_B (\beta B = Nq_0 \to \infty)\). In Fig.1(b), the energy levels degeneracy at \(t_1 \neq 0\) comes from H-EP and \(\Lambda = 1\) indicates the coalescence of the two edge states.

Finally, we discuss the hidden QPT between the two classes of B-type EPs (see the dash lines in Fig.1(a)). After doing a similarity transformation \(\hat{S}_B\), the effective two-level model \(\hat{H}_{\text{NH-S}}\) turns into another NH one, i.e.,

\[
\hat{H}_{\text{NH-S}} = (\hat{S}_B)^{-1} \hat{H}_{\text{NH-S}} (\hat{S}_B) = \hat{\Delta}^+ \sigma^+ + \hat{\Delta}^- \sigma^- + i\varepsilon \sigma^z
\]

where \(\hat{\Delta}^\pm = \hat{\Delta} \exp(-Nq_0)\), \(\hat{\Delta}^- = \hat{\Delta} \exp(Nq_0)\). In thermodynamic limit \(N \to \infty\), there exist three phases: phase with \(\hat{\Delta}^+ \to \infty\), \(\hat{\Delta}^- \to 0\), phase with \(\hat{\Delta}^+ \to 0\), \(\hat{\Delta}^- \to \infty\), phase with \(\hat{\Delta}^+ \to 0\), \(\hat{\Delta}^- \to 0\). At \(\hat{\Delta}^\pm = 1\) or \(t_1 \pm t_2 = 1\), the hidden QPT occurs. In the phases with \(\hat{\Delta}^+ \to \infty\), \(\hat{\Delta}^- \to 0\) or \(\hat{\Delta}^+ \to 0\), \(\hat{\Delta}^- \to \infty\), the weight vector is \(\vec{W} = (1, 1)\); In the phase with \(\hat{\Delta}^+ \to 0\), \(\hat{\Delta}^- \to 0\), the weight vector is \(\vec{W} = (0, 1)\).
The fidelity susceptibility is a general purpose probe of phase transitions. We can use fidelity susceptibilities of edge states $\chi_\pm = F_\pm (\gamma, \delta) = |\langle \psi^R_\pm \rangle | \langle \psi^R_\pm (\gamma + \delta) \rangle |$ to characterize the occurrence of the hidden QPTs. Fig.1(c) shows the results that indicate the hidden phase transitions for the case of $N = 20$, $t_1 = 0.6$, $\varepsilon = 0.001$.

In addition, to verify this type of hidden QPTs and show the defectiveness of the abnormal basis, we calculate the state similarity $\Lambda = |\langle \psi^R | \psi^R \rangle |$. A hidden phase transition occurs with sudden changing between IB-EP ($\Lambda = 0$) and IIB-EP ($\Lambda = 1$). The numerical results for state similarity $\Lambda$ for the case of $N = 20$, $\varepsilon = 0.001$ are given in Fig.1(d).

**Conclusion and discussion:** In this paper, we points out that the phenomenon of physical EPs for protected sub-systems is much more interesting than we once meet. For protected two-level sub-systems $\{ H_{NH-S}, B_5 \}$, there may exist three types of EPs - M-EPs (singular $H_{NH-S}$, normal $B_5$), B-EPs (normal $H_{NH-S}$, defective $B_5$), and H-EPs (singular $H_{NH-S}$, defective $B_5$). In particular, for B-EPs, the eigenvectors become coalesce without energy degeneracy. In addition, there are two classes of B-EPs - IB-EPs and IIB-EPs. Between IB-EPs and IIB-EPs, there exist a hidden QPT. By taking the topologically protected edge states in NH SSH model as an example, we explore the physical properties of physical EPs. In the future, we will study higher order physical EPs in the protected NH sub-systems and try to develop a complete theory for physical EPs.

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