ON GENERALIZATION AND ACCELERATION OF RANDOMIZED PROJECTION METHODS FOR LINEAR FEASIBILITY PROBLEMS

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ABSTRACT

Randomized Kaczmarz (RK), Motzkin Method (MM) and Sampling Kaczmarz Motzkin (SKM) algorithms are commonly used iterative techniques for solving linear system of inequalities (i.e., $Ax \leq b$). As linear systems of equations represents a modeling paradigm for solving many optimization problems, these randomized and iterative techniques are gaining popularity among researchers in different domains. In this work, we propose a Generalized Sampling Kaczmarz Motzkin (GSKM) method that unifies the iterative methods into a single framework. In addition to the general framework, we propose a Nesterov type acceleration scheme in the SKM method called as Probably Accelerated Sampling Kaczmarz Motzkin (PASKM). We prove the convergence theorems for both GSKM and PASKM algorithms in the $L_2$ norm perspective with respect to the proposed sampling distribution. Furthermore, from the convergence theorem of GSKM algorithm, we find the convergence results of several well known algorithms like Kaczmarz method, Motzkin method and SKM algorithm. We perform thorough numerical experiments using both randomly generated and real life (classification with support vector machine and Netlib LP) test instances to demonstrate the efficiency of the proposed methods. We compare the proposed algorithms with SKM, Interior Point Method (IPM) and Active Set Method (ASM) in terms of computation time and solution quality. In majority of the problem instances, the proposed generalized and accelerated algorithms significantly outperform the state-of-the-art methods.

Keywords: Kaczmarz Method · Randomized Projection · Sampling Kaczmarz Motzkin · Linear Feasibility · Nesterov’s Acceleration · Iterative Methods

1 Introduction

We consider the following Linear Feasibility (LF) problem:

$$Ax \leq b, \ b \in \mathbb{R}^m, \ A \in \mathbb{R}^{m \times n} \ (1)$$

We confine the scope of our work in the regime of thin/tall coefficient matrix $A (m \gg n)$, as iterative methods are more competitive for such problems. Note that, while almost all of the classical methods are deterministic in nature, recent advances [1][2] suggest that randomized iterative methods can outperform existing deterministic methods for solving many computational problems including linear feasibility, linear systems and convex optimization problems. From
a algorithmic point of view, our work connects three projection based algorithms—Kaczmarz, Motzkin, Sampling Kaczmarz Motzkin (SKM)—in one framework and explores the possibility of an accelerated algorithm. Before we delve into the contributions of this work, we give a brief descriptions of some of the classical and modern techniques related to solving LF problems with iterative methods.

**Randomized Kaczmarz (RK)** Kaczmarz method is one of the popular methods for solving linear systems due to its algorithmic simplicity [13]. Originally proposed in 1937 by Kaczmarz [13], Kaczmarz method remained hidden to the research community until the early 1980s, when Gordon et. al proposed Algebraic Reconstruction Techniques (ART) in the area of image reconstruction [14]. Later, it has found applications in several areas like computer tomography [15][16], digital signal processing [17], distributed computing [18][19] and many other engineering and physics problems. It has been rediscovered several times as a family of methods including component solution, successive projection, row-action and cyclic projection methods (see [20]). Given a current point \(x_k\), the Kaczmarz method generates new update \(x_{k+1}\) based on the orthogonal projection of \(x_k\) onto the hyperplane \(a_i^T x_k \leq b_i\),

\[
x_{k+1} = x_k - \frac{\delta (a_i^T x_k - b_i)^+}{\|a_i\|^2_2} a_i
\]

The difference between old and modern Kaczmarz scheme are the choice of projection hyperplanes in the update formula of equation (2) at each iteration and the choice of projection parameter \(\delta\). The original Kaczmarz method chooses hyperplanes by \(i \equiv k \mod m, i = 1, 2, 3, \ldots, m\) with parameter \(\delta = 1\). Strohmer et. al [1] showed that instead of using cyclic rules, convergence can be improved by choosing \(i\) from \(\{1, 2, \ldots, m\}\) at random with probability proportional to \(\|a_i\|^2_2\). This randomization scheme is very efficient for the linear system as well [2]. The projection parameter \(\delta\) can be chosen any value in the range of \((0, 2]\) [11].

**Motzkin Method (MM)** Another classical method for solving LF problems is the Motzkin method (MM) discovered by Motzkin et. al in the early 1950s [21][22]. The work of Motzkin was rediscovered several times by other researchers in the field of Machine Learning (ML). For instance, the so called perceptron algorithm in ML [23][24] can be classified as a member of Motzkin type methods. Furthermore, MM can be sought as Kaczmarz method with “maximal-residual control” or with “most violated constraint control” [20][26][27]. The MM starts with a initial point \(x_0\) and find the next update \(x_{k+1}\) as the projection of \(x_k\) onto the most violated hyperplane defined in the equation (1). Given the current point \(x_k\), find the next projection hyperplane \(a_i^*\) as the maximum violated constraint (i.e., select \(i^* = \text{arg max}_{i \in \{1, 2, \ldots, m\}} \|a_i^T x_k - b_i\|\)) and then update \(x_{k+1}\) as follows:

\[
x_{k+1} = (1 - \delta)x_k + \delta H(a_i^*, x_k)
\]

with the choice \(0 < \delta < 2\), where \(H(a_i^*, x_k)\) denotes orthogonal projection of \(x_k\) onto the hyperplane \(a_i^T x_k \leq b_i\). The analysis of the MM depends on the so called Hoffman constant (see Lemma 3.1 and Table 1). The main drawback of the standard MM is that it fails to terminate when the LF problem of (1) is infeasible. In the late 1980s, MM resurfaced for its connection to the ellipsoid method [28]. For the case of rational data its proven that the system can detect infeasibility and for totally unimodular matrices, the scheme gives strong polynomial time algorithms [29]. Recently, Chubanov [30][31] developed a modified method compared to the traditional relaxation type methods [22], where instead of projecting on the original hyperplane, one projects the new point to a induced hyperplane.

In recent times, Kacmzarz type methods gained immense popularity in the research community. The work of Strohmer et. al [1] encouraged numerous extensions and variants of the RK method (see [2][3][5][8][32]). For instance, in [5][33] authors analyzed variants of Kaczmarz method for a least square setup. The main breakthrough came from the work of Gower et. al, when they developed a generalized framework namely Gower-Richtarik (GR) sketch. The authors showed that several well known algorithms like Randomized Kaczmarz (RK), Randomized Newton (RN) and Randomized Coordinate Descent methods can be sought as special cases of the GR algorithm. For different choice of sampling distribution and a positive definite matrix, one can recover all of the above algorithms as special cases (see [8][10][34][35] for detail discussion).

Another area of research spurred when Gower et. al [36][37] provided the extension of the GR sketching method for the quasi newton type methods [36] and finding pseudo-inverse of a matrix [37]. They showed that almost all of the available quasi newton algorithms like Bad Broyden (BB), Powell-symmetric-Broyden (PSB), Good Broyden (GB), DavidonFletcherPowell (DFP) and BroydenFletcherGoldfarbShanno (BFGS) can be derived as special cases from the GR sketch. Several variants of acceleration have been explored for the GR algorithm [38][39]. Special block variants of RK methods have been analysed by Needell et. al [40][42]. From a linear programming perspective, Chubanov developed a polynomial time algorithm for solving the \(0-1\) linear system [30][33][34] and \(0-1\) LF problem [31]. In recent times, other variants of both RK and SKM algorithms have been developed that deals with acceleration related to important sampling [10][43][48].
Another type of acceleration namely Nesterov Accelerated Gradient (NAG), proposed by Nesterov in his seminal work [49], exhibits the worst-case rate of $O\left(\frac{1}{k^2}\right)$ for smooth convex function minimization compared to the original rate of $O\left(\frac{1}{k}\right)$. Since the introduction of Nesterov’s work, numerous work has been done on algorithmic development of the first-order accelerated methods (for a detailed discussion see [50–53]). Until then, Nesterov’s work has been integrated in several well known projection based algorithms like Coordinate Descent [53], Randomized Kaczmarz [52], Affine Scaling [54], Randomized Gossip [55], Sampling Kaczmarz Motzkin [56] and the references therein. Particularly, Morshed et. al [56] investigated the acceleration scheme of Nesterov in the SKM algorithm for $\delta = 1$.

In this work, we develop a generalized framework that unifies the iterative methods such as RK, MM, and SKM to solve LF problems. This general framework will provide an ideal platform for the researchers to experiment with wide range of iterative projection methods and design efficient algorithms to solve optimization problems in areas like artificial intelligence, machine learning, data mining and engineering. In addition to the general framework, we propose a Nesterov type acceleration scheme in the SKM method (0 < $\delta$ ≤ 2) that outperforms state-of-the-art methods in terms of computation time and solution quality. With the convergence analysis of GSKM algorithm, we synthesize the convergence analysis of RK, MM, and SKM methods into one convergence theorem. Although the proposed methods deal with the case of linear feasibility problem with systems of inequalities, it can be noted that with some modification, like the one stated in the work of Lewis et. al [2], one can apply this method to linear systems with both equality and inequality constraints.

The remaining of the paper is organized as follows. The proposed algorithms are discussed in section 2 and the convergence analysis of the proposed algorithms are given in section 3. In section 4 we preform extensive numerical experiments on simulated and real instances for a better understanding of the behaviour of the proposed generalized and accelerated schemes. In addition, we compared the effectiveness of the proposed acceleration schemes with the state-of-the-art techniques (i.e., SKM, IPM and ASM). And finally, the paper is concluded in section 5 with concluding remarks and future research directions.

2 Contributions & Preliminaries

2.1 Sampling Kaczmarz Motzkin

The SKM method (Algorithm 1) for solving LF problems, proposed by De Loera et. al [11], combines the ideas of both Kaczmarz and Motzkin method. The authors provided a generalized convergence Theorem and a certificate of feasibility which synthesizes the convergence analysis of Kaczmarz method and Motzkin method for solving LF problems. The proposed method requires only $O(n)$ memory storage and is much more efficient than the state-of-the-art techniques such as Kaczmarz type methods, IPMs and ASMs. The main advantage of SKM can be ascribed to its innovative way of projection plane selection. The hyperplane selection goes as follows: at iteration $k$ the SKM algorithm selects a collection of $\beta$ rows namely $\tau_k$ uniformly at random out of $m$ rows of the constraint matrix $A$, then out of these $\beta$ rows the row with maximum positive residual is selected (i.e., choose row $i^*$ as $i^* = \arg \max_{i \in \tau_k} \{ a_i^T x_k - b_i \}$) and finally the next point $x_{k+1}$ is updated as follows:

$$x_{k+1} = x_k - \delta \left(\frac{a_i^T x_k - b_i}{\|a_i\|^2}\right) a_i$$  \hspace{1cm} (4)

For simplicit analysis we denote the above sampling as $S_k$ at iteration $k$, i.e., at each iteration $k$ choose $\tau_k \sim S_k$ and denote $i^*$ as

$$i^* = \arg \max_{i \in \tau_k} \left( a_i^T x_k - b_i \right)$$

The SKM method generalizes RK and MM, and combines their strength in choosing a sampled constraints at each iteration. It has cheaper per iteration cost compared to Motzkin’s method and converges faster compared to the Kaczmarz method. Several extensions of SKM method in terms of acceleration [56], improved rate [57] has been proposed recently.

2.2 Contributions

Generalized Sampling Kaczmarz Method (GSKM). For obtaining a generalized version of the SKM method, we suggest to use history information in updating the current update. In particular, we take two random iterates $x_{k-1}$ and $x_k$ generated by successive SKM iteration and then update the next iterate $x_{k+1}$ as affine combination of the previous two updates. Starting with $x_0 = x_1 \in \mathbb{R}^n$, for $k \geq 1$, we update

$$x_{k+1} = (1 - \xi) z_k + \xi z_{k-1}$$

Randomized Projection Methods for Linear Feasibility - February 21, 2020
Convergence Rate

E

RPK [1]

SKM [11]

Our convergence analysis suggests that for any

or GSKM method. GSKM method is formally provided in Algorithm 2 and the convergence analysis is provided in

recover the original SKM algorithm. For simpler representation we denote this method as generalized SKM method

with different parameter choice. To simplify the notation, we denote,

In the Table 1, we list the algorithms and their respective convergence Theorems recovered from the GSKM algorithm

Theorem [2] and the convergence analysis is provided in Section 3. Our convergence analysis suggests that for any 0 < \delta < 2, one should choose \xi from 0.5 - 0.5/\sqrt{h(\delta)} to 0.5 + 0.5/\sqrt{h(\delta)}, with h(\delta) denotes the rate of the SKM algorithm.

**Algorithm 2 GSKM Algorithm:**

\[ x_{k+1} = \text{GSKM}(A, b, x_0, K, \delta, \beta, \xi) \]

Choose 0 < \delta \leq 2, \xi \in Q;

Initialize \( x_1 = x_0, k = 0; \)

Choose a sample of \( \beta \) constraints, \( \tau_0, \) uniformly at random from the rows of matrix \( A. \)

From these \( \beta \) constraints, choose \( i_0 = \arg \max_{i \in \tau_0} \{ a_i^T x_0 - b_i, 0 \} \) and update:

\[ z_0 = x_0 - \delta \left( \frac{a_{i_0}^T x_0 - b_{i_0}}{\|a_{i_0}\|^2} \right) a_{i_0} \]

while \( k \leq K \) do

Choose a sample of \( \beta \) constraints, \( \tau_k, \) uniformly at random from the rows of matrix \( A. \)

From these \( \beta \) constraints, choose \( i^* = \arg \max_{i \in \tau_k} \{ a_i^T x_k - b_i, 0 \} \) and update:

\[ z_k = x_k - \delta \left( \frac{a_{i^*}^T x_k - b_{i^*}}{\|a_{i^*}\|^2} \right) a_{i^*} \]

\[ x_{k+1} = (1 - \xi) z_k + \xi z_{k-1}; \]

\[ k \leftarrow k + 1; \]

end while

return \( x \)

with \( z_k = x_k - \delta \left( \frac{a_{i_k^*}^T x_k - b_{i_k^*}}{\|a_{i_k^*}\|^2} \right) a_{i_k^*} \) is the \( k^{th} \) update of the SKM algorithm. Note that, by taking \( \xi = 0, \) one can recover the original SKM algorithm. For simpler representation we denote this method as generalized SKM method or GSKM method.

In the Table 1, we list the algorithms and their respective convergence Theorems recovered from the GSKM algorithm with different parameter choice. To simplify the notation, we denote,

\[ r_k = \|x_k - P\|, \eta = r_0 + \delta^2, \lambda_{\min} = \lambda_{\min}^+(A^T A), e_j(x) = a_j^T x - b_j \]

### Table 1: Algorithms obtained from GSKM.

| Parameters | Row selection | Convergence Rate | Algorithm |
|------------|---------------|------------------|-----------|
| \( \beta = 1, \delta = 1, \xi = 0 \) | \( P(i^*) = \frac{\|a_{i^*}\|^2}{\|A\|^2} \) | \( \mathbb{E} [r_k^2] \leq \left( 1 - \frac{\lambda_{\min}}{\|A\|^2} \right)^k r_0^2 \) | RK [1] |
| \( \beta = m, \delta = 1, \xi = 0 \) | \( i^* = \arg \max_j e_j(x_k) \) | \( r_k^2 \leq \left( 1 - \frac{\lambda_{\min}}{m} \right)^k r_0^2 \) | MM [22] |
| \( 0 < \delta \leq 2, \xi = 0 \) | \( \tau_k \sim \mathcal{S}_k \) | \( \mathbb{E} [r_k^2] \leq \left( 1 - \frac{\eta}{mL^2} \right)^k r_0^2 \) | SKM [11] |
**Probably Accelerated Sampling Kaczmarz Method (PASKM).** We propose an accelerated randomized projection method based on the SKM method and Nesterov accelerated gradient (NAG). Note that, NAG generates sequences \( \{y_k\} \) and \( \{v_k\} \) using the following update formulas:

\[
\begin{align*}
y_k &= \alpha_k v_k + (1 - \alpha_k) x_k \\
x_{k+1} &= y_k - \theta_k \nabla f(y_k) \\
v_{k+1} &= \omega_k v_k + (1 - \omega_k) y_k - \gamma k \nabla f(y_k)
\end{align*}
\]  

In equation (7), \( \nabla f \) is the gradient of the given function and \( \alpha_k, \omega_k, \theta_k \) are real step sequences of appropriate choice. Nesterov used updated values for the sequences \( \alpha_k, \omega_k, \theta_k \) and obtained better convergence rate in the context of standard descent. There are two available works directly involves applying Nesterov’s acceleration in Kaczmarz type methods, first one is by Wright et. al \([32]\) where accelerated RK method is proposed for linear systems, second one deals with applying acceleration in SKM for \( \delta = 1 \) \([56]\). In this work, we consider the general case \( 0 < \delta < 2 \) and develop a provably accelerated scheme for the SKM algorithm. The main difference between the proposed PASKM algorithm and the above mentioned methods is the choice of parameters update. We propose to use precomputed values for the parameters \( \omega, \gamma, \alpha \) for every iterate compared to the iterative parameter selection process in \([32][53][56]\). Instead of using the update formula described in equation (7), we use the following,

\[
\begin{align*}
y_k &= \alpha v_k + (1 - \alpha) x_k \\
x_{k+1} &= y_k - \delta \frac{(a_i^T y_k - b_i^*)^+}{\|a_i\|^2} a_{i^*} \\
v_{k+1} &= \omega v_k + (1 - \omega) y_k - \gamma \frac{(a_i^T y_k - b_i^*)^+}{\|a_{i^*}\|^2} a_{i^*}
\end{align*}
\]

with \( i^* \) chosen using \( i^* = \arg \max_{j \in \tau_k} e_j(x_{k-1}) \), where \( \tau_k \sim S_k \). And the formulas for the parameters are given by the equation (9)

\[
\omega = 1 - q \sqrt{\frac{\lambda \eta}{\tau}}, \quad \gamma = \sqrt{\eta \over \lambda \tau}, \quad \alpha = \frac{q \eta}{q \eta + \gamma \tau}
\]  

The PASKM method is formalised as Algorithm 3 and the detailed convergence analysis of the method is provided in Section 3. This method generally outperforms both the SKM and GSKM algorithms for almost all of the test instances considered in this work (see Section 4).

**Algorithm 3 PASKM Algorithm:** \( x_{k+1} = \text{PASKM}(A, b, x_0, K, \delta, \beta, \lambda, \tau, q) \)

**Initialize** \( v_0 \leftarrow x_0, k \leftarrow 0; \)

**while** \( k \leq K \) **do**

Choose \( \omega, \gamma, \alpha \) as follows:

\[
\begin{align*}
\eta &= 2\delta - \delta^2, \quad \omega = 1 - q \sqrt{\frac{\lambda \eta}{\tau}}, \quad \gamma = \sqrt{\frac{\eta}{\lambda \tau}}, \quad \alpha = \frac{q \eta}{q \eta + \gamma \tau}
\end{align*}
\]

Update \( y_k = \alpha v_k + (1 - \alpha) x_k \);

Choose a sample of \( \beta \) constraints, \( \tau_k \), uniformly at random from the rows of matrix \( A \). From these \( \beta \) constraints, choose \( i^* = \arg \max_{i \in \tau_k} \{a_i^T y_k - b_i, 0\}; \) Update

\[
\begin{align*}
x_{k+1} &= y_k - \delta \frac{(a_{i^*}^T y_k - b_{i^*})^+}{\|a_{i^*}\|^2} a_{i^*}; \\
v_{k+1} &= \omega v_k + (1 - \omega) y_k - \gamma \frac{(a_{i^*}^T y_k - b_{i^*})^+}{\|a_{i^*}\|^2} a_{i^*}
\end{align*}
\]

**end while**

**return** \( x \)
2.3 Notation

We follow the standard linear algebra notation in this work. \( \mathbb{R}^n \) denotes the \( n \) dimensional real space, \( \mathbb{R}^{m \times n} \) denotes the set of \( m \times n \) real valued matrices. For any matrix \( A \in \mathbb{R}^{m \times n} \), \( A^T \) denotes the transpose matrix \( A \), with \( tr(A) \), \( det(A) \), and \( diag(A) \) denotes the trace, determinant, and diagonal of matrix \( A \) respectively. We denote the rows of matrix \( A \) by \( a_i \) for \( i = 1, 2, ... m \). Furthermore, \( P = \{ x \in \mathbb{R}^n | Ax \leq b \} \) denotes the feasible region of the feasibility problem. The notation \( P(x) \) denotes the projection of \( x \) onto the feasible region \( P \). \( \| A \| \) is the spectral norm of the matrix \( A \) and \( \| A \|_F \) denote the Frobenius norm. For any function \( f : X \rightarrow Y \), we use \( \nabla f \) and \( \nabla^2 f \) to represent the gradient and Hessian of \( f \) respectively. Finally, \( (x, y)^T = x^T y \) denotes the standard inner product and \( \| x \| = \sqrt{\langle x, x \rangle} \) as the euclidean \( (L_2) \) norm. The notation \( x^+ \) denotes the positive part of any real number (i.e., \( x^+ = \max\{x, 0\} \)). For any two arbitrary matrices \( M \), \( N \), the notation \( M \succ N \) means that the matrix \( M - N \) is positive definite. For any \( x \in \mathbb{R}^n \), \( B \in \mathbb{R}^{n \times n} \) and \( M \succ 0 \), we denote,

\[
\| x \|_M^2 = x^T M x, \quad \| B \|_M = \max_{\| x \|_M = 1} \| B x \|_M
\]

2.4 Expectation

For the convergence analysis of Algorithm\([1]\) and its variations (any algorithm that uses that specific type of sampling distribution), we need to discuss a specific expectation calculation. First of all let us sort residual vector \((Ax_k - b)^+\) from smallest to largest for any random iterate \(x_k\) and denote \((Ax_k - b)^+_{i_1}\) as the \((\beta + j)^{th}\) entry on the list as the following.

\[
(Ax_k - b)^+_{i_0} \leq ... \leq (Ax_k - b)^+_{i_j} \leq ... \leq (Ax_k - b)^+_{i_m} = \beta
\]

Now, consider the list with all of the entries of the residual vector \((Ax_k - b)^+\), then we need to calculate the probability that a particular entry of the residual vector is selected at any given iteration. Note that, the probability that any sample is selected is \( \frac{1}{m} \) and each sample has equal probability of selection. Another intersecting fact can be noted that the size of the residual list controls the order and frequency that each entry of the residual vector will be expected to be selected. For instance, in the case of \( \beta = 1 \), the \( \beta^{th} = 1^{th} \) smallest entry will be selected (i.e., smallest entry of the whole list, RK method). And when \( \beta = m, \ m^{th} \) smallest entry or the largest entry will be selected (MM algorithm). Using the above discussion with the list of equation \([13]\), we have the following:

\[
\mathbb{E}_{\mathbb{S}_k} \left[ \| (a_i^T x_k - b_i)^+ \| \right] = \frac{1}{m} \sum_{j=0}^{m-\beta} \left( \frac{\beta - 1 + j}{\beta - 1} \right) \left| (a_i^T x_k - b_i)^+_{i_j} \right|^2
\]

Here, \( \mathbb{E}_{\mathbb{S}_k} \) denotes the required expectation in accordance with the sampling distribution \( \mathbb{S}_k \).

3 Main Results

In this section, we present the convergence analysis of the proposed algorithms.

**Lemma 3.1.** (Hoffman [58], Theorem 4.4 in [2]) Let \( x \in \mathbb{R}^n \) and \( P \) be the feasible region, then there exists a constant \( L > 0 \) such that the following identity holds:

\[
\| x - P \| \leq L \| (Ax - b)^+ \|
\]

The constant \( L \) is the so-called Hoffman constant. Note that, for a consistent system (i.e., \( Ax = b \), unique solution \( x^* \)), \( L \) can be expressed in terms of the smallest singular value of matrix \( A \), i.e.,

\[
L^2 = \frac{1}{\| A^{-1} \|^2} = \frac{1}{\lambda_{\min}^2 (A^T A)}
\]

**Lemma 3.2.** (Lemma 2.1 in [14]) Let \( \{ x_k \}, \{ y_k \} \) be real non-negative sequences such that \( x_{k+1} \geq x_k > 0 \) and \( y_{k+1} \geq y_k \geq 0 \), then

\[
\sum_{k=1}^{n} x_k y_k \geq \sum_{k=1}^{n} \overline{x} y_k, \quad \text{where} \quad \overline{x} = \frac{1}{n} \sum_{k=1}^{n} x_k
\]

6
For the next four Lemmas (Lemma 3.3 to Lemma 3.6), for any \( x \in \mathbb{R}^n \), let us define, \( s \) as the number of zero entries in the residual \((Ax - b)^+\), which also corresponds to the number of satisfied constraints. We also denote the set of sampled \( \beta \) constraints as \( \tau \sim S \) and \( V = \max\{m - s, m - \beta + 1\} \). Let’s take

\[
i^* = \arg\max_{i \in \tau \sim S} \{a_i^T x - b_i, 0\} = \arg\max_{i \in \tau \sim S} (A\tau x - b_i)^+
\]

(15)

**Lemma 3.3.** Let \( \lambda_j \) be the \( j \) th eigenvalue of the matrix \( W = \mathbb{E}_S[a_i a_i^T] \), then for all \( j \), \( 0 \leq \lambda_j \leq 1 \)

**Proof.** Since, \( W \) is positive semidefinite, we can write \( \lambda_j \geq 0 \) for all \( j \). Also as the mapping \( F : X \rightarrow \lambda_{\max}(X) \) is convex, using Jensen’s inequality we have,

\[
F(W) = \lambda_{\max}(W) = \lambda_{\max}\left[\mathbb{E}_S[(a_i a_i^T)_{\tau}]\right] \leq \mathbb{E}_S\left[\lambda_{\max}\left((a_i a_i^T)_{\tau}\right)\right] \leq 1
\]

This proves the Lemma.

**Lemma 3.4.** For any \( 1 \leq \beta \leq m \), we have the following

\[
\mathbb{E}_S[a_i a_i^T] \leq \frac{\beta}{m} A^T A
\]

**Proof.** Using the expectation expression given in first section we have,

\[
\mathbb{E}_S[a_i a_i^T] = \frac{1}{(m)_{\beta}} \sum_{k=0}^{m-\beta} \left( \frac{\beta - 1 + k}{\beta - 1} \right) a_{ik} a_{ik}^T
\]

\[
\leq \frac{(m-1)}{(m-1)} \sum_{k=0}^{m-\beta} a_{ik} a_{ik}^T
\]

\[
\leq \frac{\beta}{m} \sum_{i=1}^{m} a_i a_i^T = \frac{\beta}{m} A^T A
\]

This proves the Lemma.

**Lemma 3.5.** For any \( x \in \mathbb{R}^n \) with \( i^* \) defined in (15), we have the following

\[
\frac{1}{mL^2} \|x - P\|^2 \leq \mathbb{E}_S\left[\|(a_i^T x - b_i)^+\|^2\right] \leq \min\left\{1, \frac{\beta}{m} \lambda_{\max}\right\} \|x - P\|^2
\]

**Proof.** Using the definition of the expectation from (14), we have,

\[
\mathbb{E}_S\left[\|(a_i^T x - b_i)^+\|^2\right] = \mathbb{E}_S\left[\|(a_i^T x - b_i)^+\|^2_{\infty}\right]
\]

\[
= \frac{1}{(m)_{\beta}} \sum_{k=0}^{m-\beta} \left( \frac{\beta - 1 + k}{\beta - 1} \right) \|a_{ik}^T x - b_{ik}\|^2
\]

\[
\geq \frac{1}{(m)_{\beta}} \sum_{k=0}^{m-\beta} \left( \frac{\beta - 1 + k}{\beta - 1} \right) \|a_{ik}^T x - b_{ik}\|^2
\]

\[
\geq \frac{1}{m - \beta + 1} \sum_{k=0}^{m-\beta} \|(Ax - b)_{ik}^+\|^2
\]

\[
\geq \frac{1}{m - \beta + 1} \min\left\{\frac{m - \beta + 1}{m - s}, 1\right\} \|(Ax - b)^+\|^2
\]

\[
= \frac{1}{V} \|(Ax - b)^+\|^2 \geq \frac{1}{mL^2} \|x - P\|^2
\]

Here, we used Lemma 3.2. Now, since \( AP \leq b \) we have the following,

\[
\mathbb{E}_S\left[\|(a_i^T x - b_i)^+\|^2\right] = \frac{1}{(m)_{\beta}} \sum_{k=0}^{m-\beta} \left( \frac{\beta - 1 + k}{\beta - 1} \right) \|a_{ik}^T x - b_{ik}\|^2
\]
Theorem 3.7. Let 

\[ \frac{1}{m} \sum_{k=0}^{m-\beta} \left( \beta - 1 + k \right) |a_k^T x - a_k^T P|^2 \]

\[ = \frac{1}{m} (x - P)^T \sum_{k=0}^{m-\beta} \left( \beta - 1 + k \right) a_k a_k^T (x - P) \]

\[ = (x - P)^T E_\phi[Z](x - P) \leq \min \left\{ 1, \frac{\beta}{m} \lambda_{max} \right\} \|x - P\|^2 \]

Combining the above identities we get,

\[ \frac{1}{mL^2} \|x - P\|^2 \leq E_\phi \left[ \|a_i^T x - b_i\|_2^2 \right] \leq \min \left\{ 1, \frac{\beta}{m} \lambda_{max} \right\} \|x - P\|^2 \]

which proves the Lemma.

\[ \square \]

Lemma 3.6. For any \( x \in \mathbb{R}^n \) and \( 0 < \delta \leq 2 \), we have the following,

\[ E_\delta \left[ \|z - P\|^2 \right] = E_\delta \left[ \|x - P - \delta (a_i^T x - b_i) + a_i\|_2^2 \right] \leq h(\delta) \|x - P\|^2 \]

where \( h(\delta) = 1 - \frac{2\delta - \delta^2}{mL^2} < 1 \)

\[ \text{Proof. Using the identity } a_i^T (x - P) \geq a_i^T x - b_i \text{, we get the following,} \]

\[ E_\delta \left[ \|x - P - \delta (a_i^T x - b_i) + a_i\|_2^2 \right] = E_\delta \left[ \|x - P - \delta (A_i x - b_i) + a_i\|_2^2 \right] \]

\[ = \|x - P\|^2 + \delta^2 E_\delta \left[ \|A_i x - b_i\|_2^2 \right] - 2\delta E_\delta \left[ (A_i x - b_i) + a_i (x - P) \right] \]

\[ \leq \|x - P\|^2 + \delta^2 E_\delta \left[ \|A_i x - b_i\|_2^2 \right] - 2\delta E_\delta \left[ (A_i x - b_i) + (a_i^T x - b_i) \right] \]

\[ = \|x - P\|^2 - (2\delta - \delta^2) E_\delta \left[ \|A_i x - b_i\|_2^2 \right] \]

\[ \leq \|x - P\|^2 - \frac{2\delta - \delta^2}{mL^2} \|x - P\|^2 = h(\delta) \|x - P\|^2 \]

Here, we used the lower bound of the expected value expression from Lemma 3.5

\[ \square \]

Before we delved into the main Theorems let us define the following parameters for any \( 0 \leq \phi_1, \phi_2 < 1 \):

\[ \phi = -\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}, \quad \rho = \phi + \phi_1 \]

\[ C_1 = \frac{1 + \rho}{\phi + \rho}, \quad C_2 = \frac{1 - \rho}{\phi + \rho}, \quad D_1 = \frac{\rho + \phi_2}{\phi + \rho}, \quad D_2 = \frac{\phi - \phi_2}{\phi + \rho} \]

(16)

**Theorem 3.7.** Let \( \{G_k\} \) be a non-negative real sequence satisfying the following

\[ G_{k+1} \leq \phi_1 G_k + \phi_2 G_{k-1}, \forall k \geq 1 \]

\( G_0 = G_1 \geq 0 \)

if \( \phi_1, \phi_2 \geq 0 \) then the following bounds hold:

1. (Lemma 9 in [59]) Let, \( \phi \) is the largest root of \( \phi^2 + \phi \phi_1 - \phi_2 = 0 \), then

\[ G_{k+1} \leq (1 + \phi)(\phi + \phi_1) G_0, \forall k \geq 1 \]

2. Define, \( \rho = \phi + \phi_1 \), then we have the following

\[ \begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix} \leq \begin{cases} 
C_1 \rho^{k+1} + C_2 \phi^{k+1} & k \text{ even} \\
C_1 \rho^k - C_2 \phi^k & k \text{ odd}
\end{cases} \]

\[ D_1 \rho^{k-1} + D_2 \phi^{k-1} \]

where, \( 0 \leq \phi, \phi_2 < 1 \) and \( 0 < \rho = \phi + \phi_1 < 1 \).
Proof. Since, \( \phi_1, \phi_2 \geq 0 \), we can write the largest root \( \phi \) of the equation \( \phi^2 + \phi_1 \phi - \phi_2 = 0 \) easily.

\[
\phi = \frac{-\phi_1 + \sqrt{\phi_1^2 + 4 \phi_2}}{2}
\]

And also \( \phi \geq \frac{-\phi_1 + \phi_2}{2} = 0 \). And we also have \( \phi_2 = \phi (\phi + \phi_1) \). Then using the given recurrence we have,

\[
G_{k+1} + \phi G_k \leq (\phi + \phi_1)G_k + \phi_2 G_{k-1} = (\phi + \phi_1)(G_k + \phi G_{k-1})
\]

\[
\vdots
\]

\[
\leq (\phi + \phi_1)^k (G_1 + \phi G_0)
\]

\[
= (\phi + \phi_1)^k (1 + \phi) G_0
\]

This proves the first part of the Theorem. Also note that \( \phi_1 + \phi_2 < 1 \) we have,

\[
\phi + \phi_1 = \frac{\phi_1 + \sqrt{\phi_1^2 + 4 \phi_2}}{2}
\]

\[
< \frac{\phi_1 + \sqrt{\phi_1^2 + 4(1 - \phi_1)}}{2}
\]

\[
= \frac{\phi_1 + 2 - \phi_1}{2} = 1
\]

For the second part, notice that from the recurrence inequality, we can deduce the following matrix inequality,

\[
\begin{bmatrix}
G_{k+1} & \phi_1 G_k \\
G_k & \phi_2
\end{bmatrix}
\leq
\begin{bmatrix}
\phi_1^2 + \phi_2 & \phi_1 \phi_2 \\
\phi_1 & \phi_2
\end{bmatrix}
\begin{bmatrix}
G_{k-1} \\
G_{k-2}
\end{bmatrix}
\]

(17)

The Jordan decomposition of the matrix in the above expression is given by,

\[
\begin{bmatrix}
\phi_1^2 + \phi_2 & \phi_1 \phi_2 \\
\phi_1 & \phi_2
\end{bmatrix}
= \begin{bmatrix}
-\phi & \phi + \phi_1 \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
\phi^2 & 0 \\
0 & \rho^2
\end{bmatrix}
\begin{bmatrix}
-\frac{1}{\phi_1 + 2 \phi} & 1 + \frac{1}{2(\phi_1 + 2 \phi)} \\
\frac{1}{\phi_1 + 2 \phi} & 1 - \frac{1}{2(\phi_1 + 2 \phi)}
\end{bmatrix}
\begin{bmatrix}
G_{k-1} \\
G_{k-2}
\end{bmatrix}
\]

(18)

Case 1: \( k \) even

\[
\begin{bmatrix}
G_{k+1} \\
G_k
\end{bmatrix}
\leq
\begin{bmatrix}
\phi_1^2 + \phi_2^2 + \phi_1 \phi_2 + \phi_1^2 \phi_1 + \phi_2^2 \\
\phi_1 & \phi_2
\end{bmatrix}
\begin{bmatrix}
G_{k-1} \\
G_{k-2}
\end{bmatrix}
\]

\[
\vdots
\]

\[
= \begin{bmatrix}
\phi_1^2 + \phi_2^2 + \phi_1 \phi_2 + \phi_1^2 \phi_1 + \phi_2^2 \\
\phi_1 & \phi_2
\end{bmatrix}
\begin{bmatrix}
G_1 \\
G_0
\end{bmatrix}
\]

\[
= \begin{bmatrix}
-\phi & \phi + \phi_1 \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
\phi^k & 0 \\
0 & \rho^k
\end{bmatrix}
\begin{bmatrix}
-\frac{1}{\phi_1 + 2 \phi} & 1 + \frac{1}{2(\phi_1 + 2 \phi)} \\
\frac{1}{\phi_1 + 2 \phi} & 1 - \frac{1}{2(\phi_1 + 2 \phi)}
\end{bmatrix}
\begin{bmatrix}
G_1 \\
G_0
\end{bmatrix}
\]

\[
= \begin{bmatrix}
(1 + \phi) \rho^{k+1} + (1 - \phi - \phi_1) \phi^{k+1} \\
(1 + \phi) \rho^{k} - (1 - \phi - \phi_1) \phi^{k}
\end{bmatrix}
\begin{bmatrix}
\frac{G_0}{\phi_1 + 2 \rho} \\
\frac{C_1 \rho^{k+1} + C_2 \phi^{k+1}}{C_1 \rho^{k} - C_2 \phi^{k}}
\end{bmatrix}
\]

(19)

Here, we used \( G_0 = G_1 \) and the definitions from equation (16).

Case 2: \( k \) odd

\[
\begin{bmatrix}
G_{k+1} \\
G_k
\end{bmatrix}
\leq
\begin{bmatrix}
\phi_1^2 + \phi_2^2 + \phi_1 \phi_2 + \phi_1^2 \phi_1 + \phi_2^2 \\
\phi_1 & \phi_2
\end{bmatrix}
\begin{bmatrix}
G_{k-1} \\
G_{k-2}
\end{bmatrix}
\]

\[
\vdots
\]
Let us define \( z \) as the number of zero entries in the residual \( \xi \). Here, we used the inequality \( G_2 \leq \phi_1 G_1 + \phi_2 G_0 \). Now combining the relations from equation (19) and (20), we can prove the second part of Theorem 3.7.

Now, for any \( \xi \in \mathbb{R} \) let us define the sets \( Q_1, Q_2, Q_3 \) as the following.

\[
Q_1 = \{ \xi \mid (1 - \xi) \geq 0 \}
\]

\[
Q_2 = \{ \xi \mid (1 - \xi) < 0, (2\xi - 1)^2 h(\delta) < 1 \}
\]

\[
Q = \{ \xi \mid (2\xi - 1)^2 h(\delta) < 1 \} = Q_1 \oplus Q_2
\]

**Theorem 3.8.** Let \( \{ x_k \} \) be the sequence of random iterates generated by algorithm [2] starting with \( x_0 = x_1 \). With the choice of parameters \( 0 < \delta \leq 2 \) and \( 0 \leq \xi \leq 1(\xi \in Q_1) \), the sequence of iterates \( \{ x_k \} \) converges and the following results hold:

1. Take \( \phi_1 = (1 - \xi) h(\delta) \), \( \phi_2 = \xi h(\delta) \) and \( \rho, \phi \) as in equation (16), then

\[
E \left[ \left\| x_{k+1} - P \right\|^2 \right] \leq \rho^k (1 + \phi) \left\| x_0 - P \right\|^2
\]

2. With \( \phi_1 = (1 - \xi) h(\delta) \), \( \phi_2 = \xi h(\delta) \) we have

\[
E \left[ \left\| x_{k+1} - P \right\|^2 \right] \leq \begin{cases} 
C_1 \rho^{k+1} + C_2 \phi^{k+1} & \text{if } k \text{ even} \\
C_1 \rho^k - C_2 \phi^k & \text{if } k \text{ odd}
\end{cases}
\]

where, the constants \( C_1, C_2, D_1, D_2 \) are defined in equation (16) and the solution set \( \delta = 0 \leq \phi_1, \phi_2 < 1 \) and \( 0 < \rho = \phi + \phi_1 < 1 \). Moreover, \( \rho \geq h(\delta) \), with equality \( \rho = h(\delta) \) at \( \xi = 0 \).

**Proof.** Let us define \( P \) as the projection operator onto the feasible region \( P = \{ x \in \mathbb{R}^n \mid Ax \leq b \} \), and denote \( s_k \) as the number of zero entries in the residual \( (Ax_k - b) \), which also corresponds to the number of satisfied constraints. We also define \( V_k = \max\{ m - s_k, m - \beta + 1 \} \). Now, from the update formula of Algorithm [2] we have \( z_k = x_k - (A \tau x_k - b \tau) \), where,

\[
i^* = \arg\max_{i \in \tau_k} \{ a_i^T x_k - b_i, 0 \} = \arg\max_{i \in \tau_k} (A \tau_k x_k - b \tau_i)^+
\]

Similarly, the previous update formula can be written as, \( z_{k-1} = x_{k-1} - (A \tau_{k-1} x_{k-1} - b \tau_{k-1})^+ a_j \), where,

\[
j^* = \arg\max_{j \in \tau_{k-1}} \{ a_j^T x_{k-1} - b_j, 0 \} = \arg\max_{j \in \tau_{k-1}} (A \tau_{k-1} x_{k-1} - b \tau_{k-1})^+
\]

Let us define \( r_{k+1} = \left\| x_{k+1} - P \right\| \), then we have,

\[
r_{k+1}^2 = \left\| x_{k+1} - P(x_{k+1}) \right\|^2 \leq \left\| x_{k+1} - P(x_k) \right\|^2
\]

\[
= \left\| (1 - \xi) z_k + \xi z_{k-1} - P \right\|^2
\]

\[
= \left\| (1 - \xi) \left\{ x_k - P - \delta (a_i^T x_k - b_i)^+ a_i \right\} \right\|^2
\]

\[2\text{Note that, } \xi = 0 \text{ gives us the SKM method} \]
\[ + \xi \left\{ x_{k-1} - P - \delta (a_j^T x_{k-1} - b_j) a_j \right\} \|^2 \]
\[ \leq (1 - \xi) \| x_k - P - \delta (a_j^T x_k - b_j) a_j \|^2 \]
\[ + \xi \| x_{k-1} - P - \delta (a_j^T x_{k-1} - b_j) a_j \|^2 \]
\[ \text{ (25)} \]

We used the fact that the function \( \| \cdot \|^2 \) is convex and \( 0 \leq \xi \leq 1 \). Now, taking expectation in both sides of the equation \[25\] and using Lemma \[3.6\] we get the following,
\[ \mathbb{E} [r_{k+1}^2] \leq (1 - \xi) \mathbb{E}_{\delta_k} \left[ \| x_k - P - \delta (a_j^T x_k - b_j) a_j \|^2 \right] \]
\[ + \xi \mathbb{E}_{\delta_{k-1}} \left[ \| x_{k-1} - P - \delta (a_j^T x_{k-1} - b_j) a_j \|^2 \right] \]
\[ \leq (1 - \xi) \| h(\delta) x_k - P \|^2 + \xi \| h(\delta) x_{k-1} - P \|^2 \]
\[ \text{ (26)} \]

where, \( h(\delta) = 1 - \eta / m \), \( \eta = 2\delta - \delta^2 \). Taking expectation again in equation \[26\] and letting \( G_{k+1} = \mathbb{E} [r_{k+1}^2] \), we get the following,
\[ G_{k+1} \leq \phi_1 G_k + \phi_2 G_{k-1} \]
\[ \text{ (27)} \]

Since, \( \phi_1, \phi_2 \geq 0, \phi_1 + \phi_2 < 1 \) and \( z_0 = z_1 \), using first part of Theorem \[3.7\] we have the following,
\[ \mathbb{E} [r_{k+1}^2] = G_{k+1} \leq (1 + \phi)(\phi + \phi_1)G_0 = (1 + \phi)\rho^k \mathbb{E} [r_0^2] \]
\[ \text{ (28)} \]

which proves the first part of Theorem \[3.8\]. Furthermore, using second part of Theorem \[3.7\] and equation \[27\], we get the second part of Theorem \[3.8\].

**Corollary 3.8.1.** (Theorem 1.3 in [11]) Let \( \{x_k\} \) be the sequence of random iterates generated by algorithm 2 starting with \( z_0 = x_0 \in \mathbb{R}^n \). With \( 0 < \delta \leq 2 \), the sequence of iterates \( \{x_k\} \) converges and the following result holds,
\[ \mathbb{E} [\|x_k - P\|^2] \leq [h(\delta)]^k \|x_0 - P\|^2 \]

**Proof.** Take \( \xi = 0 \), then we have
\[ \phi_1 = h(\delta), \phi_2 = 0, \phi = -\phi_1 + \sqrt{\phi_1^2 + 4\phi_2} / 2 = 0, \rho = \phi + \phi_1 = h(\delta) \]
\[ \text{ (29)} \]

And the constants of equation \[16\] can be simplified as follows,
\[ C_1 = \frac{1}{h(\delta)}, C_2 = \frac{1}{h(\delta)} - 1, D_1 = 1, D_2 = 0 \]
\[ \text{ (30)} \]

Now using the parameter values of equation \[29\] and \[30\], we get the following relation,
\[ D_1 \rho^k - D_2 \phi^k = C_1 \rho^{k+1} + C_2 \phi^{k+1} = \rho^k = (h(\delta))^k \]
\[ \text{ (31)} \]

And finally, using identity \[31\] in Theorem 3.8, we get the result of Corollary 3.8.1.

**Theorem 3.9.** Let \( \{x_k\} \) be the sequence of random iterates generated by algorithm 2 starting with \( x_0 = x_1 \). With the choice of parameters \( 0 < \delta \leq 2 \) and \( \xi \in Q_2 \), the sequence of iterates \( \{x_k\} \) converges and the following results hold,

1. Take \( \phi_1 = (1 - 2\xi)(1 - \xi)h(\delta), \phi_2 = -\xi(1 - 2\xi)h(\delta) \) in equation \[16\], then,
\[ \mathbb{E} [\|x_k - P\|^2] \leq \rho^k (1 + \phi) \|x_0 - P\|^2 \]

2. With the constants \( C_1, C_2, D_1, D_2 \) defined in equation \[16\], we have,
\[ \mathbb{E} \left[ \frac{\|x_{k+1} - P\|^2}{\|x_k - P\|^2} \right] \leq \begin{cases} 
C_1 \rho^{k+1} + C_2 \phi^{k+1} & \text{for } k \text{ even} \\
C_1 \rho^k - C_2 \phi^k & \text{for } k \text{ odd} \\
D_1 \rho^{k-1} + D_2 \phi^{k-1} & \text{for } k \text{ odd} 
\end{cases} \|x_0 - P\|^2 
\]

where, \( 0 \leq \phi, \phi_1, \phi_2 < 1 \) and \( 0 < \rho = \phi + \phi_1 < 1 \).
Proof. From equation (24), we have the following,
\[ r_{k+1}^2 = \| (1 - \xi) (z_k - P) + \xi (z_{k-1} - P) \|^2 \]
\[ = (1 - \xi)^2 \| z_k - P \|^2 + \xi^2 \| z_{k-1} - P \|^2 + 2 \xi (1 - \xi) (z_k - P, z_{k-1} - P) \]  
(32)

Now, the following identity holds,
\[ 2 (z_k - P, z_{k-1} - P) = \| z_k - P \|^2 + \| z_{k-1} - P \|^2 - \| z_k - z_{k-1} \|^2 \]
\[ \geq \| z_k - P \|^2 + \| z_{k-1} - P \|^2 - 2 \| z_k - P \|^2 - 2 \| z_{k-1} - P \|^2 \]
\[ = -\| z_k - P \|^2 - \| z_{k-1} - P \|^2 \]  
(33)

Using the identity of equation (33) in equation (32), we have,
\[ r_{k+1}^2 = (1 - \xi)^2 \| z_k - P \|^2 + \xi^2 \| z_{k-1} - P \|^2 + 2 \xi (1 - \xi) (z_k - P, z_{k-1} - P) \]
\[ \leq [(1 - \xi)^2 - \xi (1 - \xi)] \| z_k - P \|^2 + [\xi^2 - \xi (1 - \xi)] \| z_{k-1} - P \|^2 \]
\[ = (1 - \xi) (1 - 2 \xi) \| z_k - P \|^2 - \xi (1 - 2 \xi) \| z_{k-1} - P \|^2 \]  
(34)

Taking expectation in equation (34) and using Lemma 3.6 we have,
\[ \mathbb{E} \left[ r_{k+1}^2 \right] \leq (1 - \xi) (1 - 2 \xi) \mathbb{E} \| z_k - P \|^2 - \xi (1 - 2 \xi) \mathbb{E} \| z_{k-1} - P \|^2 \]
\[ \leq (1 - \xi) (1 - 2 \xi) h(\delta) \| x_k - P \|^2 - \xi (1 - 2 \xi) h(\delta) \| x_{k-1} - P \|^2 \]  
(35)

Now, taking expectation again in equation (35) and letting \( G_{k+1} = \mathbb{E} \left[ r_{k+1}^2 \right], \) \( \phi_1 = (1 - 2 \xi) (1 - \xi) h(\delta), \) \( \phi_2 = -\xi (1 - 2 \xi) h(\delta) \) we get the following,
\[ G_{k+1} \leq \phi_1 G_k + \phi_2 G_{k-1} \]  
(36)

Since, for any \( \xi \in Q_2 \) we have \( \phi_1, \phi_2 \geq 0 \) and \( (1 - 2 \xi)^2 h(\delta) < 1 \), we have,
\[ \phi_1 + \phi_2 = (1 - 2 \xi)^2 h(\delta) < 1, \quad \rho = \phi_1 + \phi < 1 \]

Also we know, \( G_0 = \mathbb{E} \left[ \| x_0 - P \|^2 \right] = \mathbb{E} \left[ \| x_1 - P \|^2 \right] = G_1 \). Then, using first part of Theorem 3.7, we have the following,
\[ \mathbb{E} \left[ r_{k+1}^2 \right] = G_{k+1} \leq (1 + \phi) (\phi + \phi_1)^k G_0 = (1 + \phi) \rho^k \mathbb{E} \left[ r_0^2 \right] \]

which proves the first part of Theorem 3.9. Furthermore, using second part of Theorem 3.7 and equation (36), we get the second part of Theorem 3.9. \( \square \)

The next Theorem is an extension of the result obtained in [11] and to a certain extend it can be taken as an extension of Telgen’s result [28]. Theorem gives one a certificate of feasibility after a finite number of GSKM iterations. Before delving into the Theorem, we will provide some known Lemmas for the SKM algorithm which holds for GSKM algorithm too. We refer interested readers to the work of De-Loira et al. [11] for a detailed proof of these Lemmas (Lemma 3.10 to Lemma 3.12).

Lemma 3.10. (Lemma 1 in [11]) Define, \( \theta(x) = \max_i (a_i^T x - b_i) \) as the maximum violation of point \( x \in \mathbb{R}^n \) and the length of the binary encoding of a linear feasibility problem with rational data-points as
\[ \sigma = \sum_i \sum_j \ln (|a_{ij}| + 1) + \sum_i \ln (|b_i| + 1) + \ln (mn) + 2 \]
Then if the rational system \( Ax \leq b \) is infeasible, for any \( x \in \mathbb{R}^n \), the maximum violation \( \theta(x) \) satisfies the following lower bound
\[ \theta(x) \geq \frac{2}{2\sigma} \]

Lemma 3.11. (Lemma 4 in [11]) If \( P \) is \( n \)-dimensional (full-dimensional) then the sequence of iterates \( \{ x_k \} \) generated by an GSKM method converge to a point \( x \in P \).

Proof. Since, by assumption, \( P \) is full dimensional, then the rest of the the proof follows the same argument as Lemma 4 in [11]. \( \square \)
Lemma 3.12. ([60]) If the rational system $Ax \leq b$ is feasible, then there is a feasible solution $x^*$ whose coordinates satisfy $|x^*_j| \leq \frac{2}{2^n}$ for $j = 1, \ldots, n$.

Theorem 3.13. Suppose $A, b$ are rational matrices with binary encoding length, $\sigma$, and that we run an GSKM method $(0 < \delta \leq 2, \xi \in \mathbb{Q})$ on the system $Ax \leq b$ ($\|a_i\| = 1, i = 1, 2, \ldots, m$) with $x_0 = 0$. Suppose the number of iterations $k$ satisfies the following lower bound:

$$\frac{4\sigma - 4 - \log n + \log(1 + \phi)}{\log \left(\frac{1}{p}\right)} < k$$

If the system $Ax \leq b$ is feasible, then,

$$p \leq H(\sigma, \phi, k, \rho) = \sqrt{\frac{1 + \phi}{n}} 2^{2\sigma - 2} \rho^{\frac{k}{2}}$$

where $p$ is the probability that the current update $x_k$ is not a certificate of feasibility. And $\rho$ defined in Theorem 3.8 and Theorem 3.9 for the choice $\xi \in S_1$ and $\xi \in S_2$ respectively. Also note that, the function $H(\sigma, \phi, k, \rho)$ is a decreasing function with respect to $k$.

Proof. Note that, since $Ax \leq b$ is feasible, then from Lemma 3.12 we know that there is a feasible solution $x^*$ with $|x^*_j| \leq \frac{2}{2^n}$ for $j = 1, \ldots, n$. Thus, we have,

$$\|x_0 - P\| \leq \|x^*\| \leq \frac{2^{\sigma - 1}}{\sqrt{n}}$$

(37) as $x_0 = 0$. then if the system $Ax \leq b$ is infeasible, by using Lemma 3.10 we have,

$$\theta(x) \geq 2^{1-\sigma}$$

This implies when GSKM runs on the system $Ax \leq b$, the system is feasible when $\theta(x) < 2^{1-\sigma}$. Furthermore, since every point of the feasible region $P$ is inside the half space defined by $H_i = \{x \mid a_i^Tx \leq b_i\}$ for all $i = 1, 2, \ldots, m$ we have the following,

$$\theta(x) = \left[\max_i \{a_i^T(x - b_i)\}\right]^+ \leq \|a_i^T(x - P)\| \leq \|x - P\|$$

(38)

Then, whenever the system $Ax \leq b$ is feasible, we have,

$$\mathbb{E}[\theta(x_k)] \leq \mathbb{E}[\|x_{k+1} - P\|] \leq \left[\mathbb{E}[\|x_{k+1} - P\|^2]\right]^{\frac{1}{2}} \leq \sqrt{1 + \phi \rho^{\frac{k}{2}}} \|x_0 - P\| \leq \sqrt{1 + \phi \rho^{\frac{k}{2}}} \frac{2^{\sigma - 1}}{\sqrt{n}}$$

(39)

Here, we used Theorems 3.8 & 3.9 and the identities from equations (37) & (38). Now, for detecting feasibility we need to have, $\mathbb{E}[\theta(x_k)] < 2^{1-\sigma}$. That gives us,

$$\sqrt{1 + \phi \rho^{\frac{k}{2}}} \frac{2^{\sigma - 1}}{\sqrt{n}} < 2^{1-\sigma}$$

Simplifying the above identity further we get the following lower bound on $k$,

$$k > \frac{4\sigma - 4 - \log n + \log(1 + \phi)}{\log \left(\frac{1}{p}\right)}$$

Moreover, if the system $Ax \leq b$ is feasible, then the probability of not having a certificate of feasibility is bounded as follows,

$$p = \mathbb{P}(\theta(x_k) \geq 2^{1-\sigma}) \leq \frac{\mathbb{E}[\theta(x_k)]}{2^{1-\sigma}} \leq \sqrt{\frac{1 + \phi}{n}} \frac{2^{2\sigma - 2} \rho^{\frac{k}{2}}}{\sqrt{n}}$$

Here, we used the Markov’s inequality $\mathbb{P}(x \geq t) \leq \frac{\mathbb{E}[x]}{t}$. This completes the proof of Theorem 3.13
Remark 3.14. Note that instead of a normalised system if we consider an non-normalised system $\overline{A}x \leq \overline{b}$, $\|\overline{a}_i\| \neq 1$ for some $i$, then suppose the number of iterations $k$ satisfies the following lower bound,

$$4\sigma - 4 - \log n + \log(1 + \phi) + 2 \log \psi < k$$

where $\sigma$ is the binary encoding length for $\overline{A}, \overline{b}$. If the system $\overline{A}x \leq \overline{b}$ is feasible, then,

$$p \leq \sqrt{\frac{1 + \phi}{n} \cdot 2^{2\sigma - 2} \psi p^2}$$

where $p$ = probability that the current update $x_k$ is not a certificate of feasibility and $\psi = \max_j \|\overline{a}_i\|$.

Corollary 3.14.1. (Theorem 1.5 in [11]) Suppose $\overline{A}, \overline{b}$ are rational matrices with binary encoding length, $\sigma$, and that we run an GSKM method ($0 < \delta \leq 2$, $\xi = 0$) on the system $\overline{A}x \leq \overline{b}$ ($\|\overline{a}_i\| \neq 1$ for some $i$) and $x_0 = 0$. Suppose the number of iterations $k$ satisfies the following lower bound,

$$4\sigma - 4 - \log n + 2 \log \psi < k$$

where $\sigma$ is the binary encoding length for $\overline{A}, \overline{b}$. If the system $\overline{A}x \leq \overline{b}$ is feasible, then,

$$p \leq \sqrt{\frac{1}{n} \cdot 2^{2\sigma - 2} \psi |h(\delta)|^2}$$

where $p$ = the probability that the current update $x_k$ is not a certificate of feasibility and $\psi = \max_j \|\overline{a}_i\|$.

Proof. For, $\xi = 0$ we have,

$$\phi = 0, \rho = \phi + \phi_1 = h(\delta)$$

Now, considering Theorem 3.13 with the above parameter choice, we can get the bound of Corollary 3.14.1.

Lemma 3.15. With, $\eta = 2\delta - \delta^2$ and $x_{k+1}$, $y_k$ defined in equation (11), we have,

$$\mathbb{E}_\phi \left[ \left\| \overline{a}_T^T \overline{y} - \overline{b}_T \right\|^2 \right] \leq \frac{1}{\eta} \|y_k - P\|^2 - \mathbb{E} \left[ \|x_{k+1} - P\|^2 \right]$$

Proof. Using the update formula of (11) with $\overline{a}_T^T \overline{y} - \overline{a}_T^T P \geq \overline{a}_T^T \overline{y} - \overline{b}_T$, we have,

$$\|x_{k+1} - P(x_{k+1})\|^2 \leq \|x_{k+1} - P(y_k)\|^2$$

$$= \|y_k - P - \delta \left( \frac{\overline{a}_T^T \overline{y} - \overline{b}_T}{\|\overline{a}_T\|^2} \right)\|^2$$

$$= \|y_k - P\|^2 + \|\overline{a}_T^T \overline{y} - \overline{b}_T\|^2 - 2\delta \left( \frac{\overline{a}_T^T \overline{y} - \overline{b}_T}{\|\overline{a}_T\|^2} \right) \|y_k - P\|^2$$

$$\leq \|y_k - P\|^2 - (2\delta - \delta^2) \left( \frac{\overline{a}_T^T \overline{y} - \overline{b}_T}{\|\overline{a}_T\|^2} \right) \|y_k - P\|^2$$

(40)

Now, taking expectation with respect to $y_k$ in both sides of equation (40) and simplifying we get,

$$\mathbb{E}_\phi \left[ \left\| \overline{a}_T^T \overline{y} - \overline{b}_T \right\|^2 \right] \leq \frac{1}{\eta} \|y_k - P\|^2 - \mathbb{E} \left[ \|x_{k+1} - P\|^2 \right]$$

This proves the above lemma.

Theorem 3.16. Let, $v_{k+1}$ and $x_{k+1}$ are generated by Algorithm 3. Then with the choice of parameters given in equation (10), we have,

$$\mathbb{E} \left[ \|v_{k+1} - P\|^2 + \frac{1}{\lambda} \|x_{k+1} - P\|^2 \right] \leq \omega^k \mathbb{E} \left[ \|v_0 - P\|^2 + \frac{1}{\lambda} \|x_0 - P\|^2 \right]$$

with the step size choice $0 < \delta < 2$ and $\omega = \left( 1 - q \sqrt{\frac{\sigma}{\tau}} \right)^k$. This theorem implies that the PASKM algorithm converges linearly with a rate of $\omega$, which accumulates to a total of $\mathcal{O}(\frac{1}{\epsilon} \sqrt{\frac{\sigma}{\tau}} \log 1/\epsilon)$ iterations to bring the given error of Theorem below $\epsilon > 0$.  

14
**Definition:** Let us define $W = E_{\mathcal{S}} [a_i^r a_i^{r*}]$ and $\lambda, \tau$ as follows,

$$\lambda = \inf_{z \in \text{Range}(A^T)} \langle W z, z \rangle, \quad \tau = \sup_{z \in \text{Range}(A^T)} \frac{\langle W^\dagger z, z \rangle}{\|z\|^2}$$  \hspace{1cm} (41)

With simple simplification one can derive the following identities,

$$\lambda = \frac{1}{\|W^\dagger\|} = \lambda_{\min}(W), \quad 1 \leq \frac{1}{\|W\|} \leq \tau \leq \|W^\dagger\| = \frac{1}{\lambda}$$  \hspace{1cm} (42)

**Proof.** Let us define, $u_{k+1} = \|v_{k+1} - P\|_W^2$. Using the update formula of $v_{k+1}$ from equation (12), we have,

$$u_{k+1}^2 = \|v_{k+1} - P(v_{k+1})\|_W^2 \leq \|v_{k+1} - P(y_k)\|_W^2$$

$$= \|\omega y_k + (1 - \omega)y_k - P(\gamma a_i^r y_k - b_i^r)^{+} a_i^{r*}\|_W^2$$

$$= \frac{\|\omega y_k + (1 - \omega)y_k - P\|_W^2}{I_1} + \gamma^2 \frac{\|a_i^r (a_i^T y_k - b_i^r)^{+} a_i^{r*}\|_W^2}{I_2}$$

$$- 2\gamma \left\langle \omega y_k + (1 - \omega)y_k - P, a_i^r (a_i^T y_k - b_i^r)^{+} a_i^{r*} \right\rangle_{W^\dagger}$$

$$= I_1 + \gamma^2 I_2 - 2\gamma I_3$$  \hspace{1cm} (43)

Since $\|\cdot\|_W^2$ is a convex function and $0 < \omega < 1$, we can bound the expected first term as follows,

$$E_{\mathcal{S}_k}[I_1] = \|\omega y_k + (1 - \omega)y_k - P\|_W^2$$

$$= \|\omega(y_k - P) + (1 - \omega)(y_k - P)\|_W^2$$

$$\leq \omega \|y_k - P\|_W^2 + (1 - \omega)\|y_k - P\|_W^2$$

$$\leq \omega u_k^2 + \frac{1 - \omega}{\lambda} \|y_k - P\|^2$$  \hspace{1cm} (44)

Taking expectation with respect to the sampling distribution in the second term of the equation (43) and using the definition of $\tau$, we get,

$$\gamma^2 E_{\mathcal{S}_k} [\|(a_i^r y_k - b_i^{r*})^{+}\|_W^2] \leq \gamma^2 \tau E_{\mathcal{S}_k} [\|(a_i^r y_k - b_i^{r*})^{+}\|_W^2]$$

$$\leq \frac{\gamma^2 \tau}{\eta} \left[ \|y_k - P\|^2 - E \|x_{k+1} - P\|^2 \right]$$  \hspace{1cm} (45)

Here, we used Lemma 3.15 Let $q > 0$ such that,

$$q = \frac{\langle z, E[ai(a_i^T v + a_i^T P - b_i)^{+}] \rangle_{W^\dagger}}{\langle z, v \rangle}$$  \hspace{1cm} (46)

It can be easily shown that $q \leq 1$. Note that, for the case of consistent linear systems, the value of $q$ can be easily calculated. Take any $x^* \in \mathbb{R}^n$ such that $Ax^* = b$, then we have

$$W^\dagger E[a_i(a_i^T v + a_i^T P - b_i)^{+}] = W^\dagger E[a_i(a_i^T v + a_i^T x^* - b_i)]$$

$$= W^\dagger E[a_i a_i^{r*}] = W^\dagger W v = v$$

Here we used the identity, $W^\dagger W v = v$ since $v \in \text{Range}(W)$. Then we can calculate $q$ as follows,

$$q = \inf_{v \neq v} \frac{\langle z, E[a_i(a_i^T v + a_i^T P - b_i)^{+}] \rangle_{W^\dagger}}{\langle z, v \rangle} = \inf_{v \neq v} \frac{\langle z, v \rangle}{\langle z, v \rangle} = 1$$

Now, for our framework, with the choice $v = y - P$ and $z = y - x$ in equation (46), for any $\psi \geq 0$ we get the following inequality,

$$\langle y - P + \psi(y - x), E_{\mathcal{S}}[a_i(a_i^T y - b_i)^{+}] \rangle_{W^\dagger} \geq q \|y - P\|^2 + \psi q(y - x, y - P)$$  \hspace{1cm} (47)

Now, taking expectation in the third term of (43) and using (47) we get,

$$-2\gamma E_{\mathcal{S}_k}[I_3] = -2\gamma \left\langle \omega y_k + (1 - \omega)y_k - P, E_{\mathcal{S}_k} [a_i^r (a_i^T y_k - b_i^r)^{+}] \right\rangle_{W^\dagger}$$
\[
= -2\gamma \left( \frac{\omega}{\alpha} [y_k - (1 - \alpha)x_k] + (1 - \omega)y_k - P, E_{S_k} \left[ a_i^T (a_i^T y_k - b_i^T) \right] \right)_{W^1}
\]
\[
= -2\gamma \left( \frac{\omega(1 - \alpha)}{\alpha} (y_k - x_k) + y_k - P, E_{S_k} \left[ a_i^T (a_i^T y_k - b_i^T) \right] \right)_{W^1}
\]
\[
\leq -2q\gamma \|y_k - P\|^2 - \frac{2q\gamma \omega(1 - \alpha)}{\alpha} (y_k - x_k, y_k - P)
\] (48)

For further simplification, we will use the following parallelogram identity,
\[
2(y_k - x_k, y_k - P) = \|y_k - x_k\|^2 + \|y_k - P\|^2 - \|x_k - P\|^2
\] (49)

Now, substituting the values of equation (44), (45) & (48) in equation (43) and taking conditional expectation we get the following,
\[
E[u_{k+1}^2] = I_1 + \gamma^2 E_{S_k} [I_2] - 2\gamma E_{S_k} [I_3]
\]
\[
= \omega u_k^2 + (1 - \omega) \|y_k - P\|^2_{W^1} + \frac{\gamma^2 \tau}{\eta} \left[ \|y_k - P\|^2 - E \left[ \|x_k + 1 - P\|^2 \right] \right] - 2q\gamma \|y_k - P\|^2 + \frac{q\gamma \omega(1 - \alpha)}{\alpha} \left( \|x_k - P\|^2 - \|y_k - x_k\|^2 - \|y_k - P\|^2 \right)
\]
\[
\leq \omega u_k^2 + \frac{(1 - \omega)}{\lambda} \|y_k - P\|^2 + \frac{\gamma^2 \tau}{\eta} \left( \|y_k - P\|^2 - E \left[ \|x_k + 1 - P\|^2 \right] \right) - 2q\gamma \|y_k - P\|^2 + \frac{q\gamma \omega(1 - \alpha)}{\alpha} \left( \|x_k - P\|^2 - \|y_k - P\|^2 \right)
\] (50)

Now, let’s choose the following values for the parameters \(\omega, \gamma, \alpha\) as follows,
\[
\omega = 1 - q \sqrt{\frac{\lambda \eta}{\tau}}, \quad \gamma = \sqrt{\frac{\eta}{\lambda \tau}}, \quad \alpha = \frac{q \eta}{q \eta + \gamma \tau}
\] (51)

We can easily see that \(\frac{1}{\lambda} = \frac{\gamma^2 \tau}{\eta} = \frac{q \gamma(1 - \alpha)}{\alpha}\). Using the parameter choice of (51), we have,
\[
\frac{1 - \omega}{\lambda} + \frac{\gamma^2 \tau}{\eta} - 2q\gamma - \frac{q\gamma \omega(1 - \alpha)}{\alpha} \leq 0
\] (52)

Now, using all of the above relations (equation (51), (52)) in equation (50), we get the following,
\[
E \left[ u_{k+1}^2 + \frac{\gamma^2 \tau}{\eta} \|x_{k+1} - P\|^2 \right] \leq \omega \left( u_k^2 + \frac{q \gamma(1 - \alpha)}{\alpha} \|x_k - P\|^2 \right)
\]
\[
+ \left[ \frac{1 - \omega}{\lambda} + \frac{\gamma^2 \tau}{\eta} - 2q\gamma - \frac{q\gamma \omega(1 - \alpha)}{\alpha} \right] \|y_k - P\|^2 \leq 0
\]
\[
\leq \left( 1 - q \sqrt{\frac{\lambda \eta}{\tau}} \right) \left( u_k^2 + \frac{\gamma^2 \tau}{\eta} \|x_k - P\|^2 \right)
\] (53)

Finally, taking expectation again with tower rule and substituting \(\frac{\gamma^2 \tau}{\eta} = \frac{1}{\lambda}\) we have,
\[
E \left[ u_{k+1}^2 + \frac{1}{\lambda} \|x_k - P\|^2 \right] \leq \left( 1 - q \sqrt{\frac{\lambda \eta}{\tau}} \right)^k \left( u_0^2 + \frac{1}{\lambda} \|x_0 - P\|^2 \right)
\]

This proves the Theorem. Furthermore, for faster convergence we need to choose parameters such that, \(\omega\) becomes as small as possible.

**Remark 3.17.** Note that, one can optimize the parameter choice by considering the following optimization problem,
\[
\min \omega \quad s.t. \quad \frac{1 - \omega}{\lambda} + \frac{\gamma^2 \tau}{\eta} - 2q\gamma - \frac{q\gamma \omega(1 - \alpha)}{\alpha} \leq 0
\]
\[
1 \leq \tau \leq \frac{1}{\lambda}, \quad \alpha \geq \frac{q \eta}{q \eta + \gamma \tau}, \quad \gamma > 0
\]
\[
0 < \alpha, \omega < 1, \quad 0 < q, \eta \leq 1
\]
4 Numerical Experiments

In this section, we discuss the numerical experiments performed to show the computational efficiency of the proposed algorithms (Algorithm 2 and 3). As mentioned before, we limit our focus on the over-determined systems regime (i.e., \( m \gg n \)) where iterative methods are competitive in general. However, from our experiments we see similar computational behaviour for the under-determined systems as well.

4.1 Experiment Specifications

We implemented the proposed GS-KM and PASKM algorithms in MATLAB R2018b and performed the experiments in a Dell Precision 7510 workstation with 32GB RAM, Intel Core i7-6820HQ CPU, processor running at 2.70 GHz. To analyze computational performance, we perform the numerical experiments for a wide range of instances including both randomly generated and real-life test problems.

- **Randomly generated problems**: Gaussian and highly correlated systems
- **Real life test instances**: Standard ML data sets and Sparse Netlib LP instances

We compare SKM with two versions of the proposed GS-KM and PASKM algorithms for a better understanding of the algorithmic behavior. In Table 2 we provide the parameter choices for GS-KM and PASKM algorithms. Throughout the numerical experiments section, we compared SKM with GS-KM-1, GS-KM-2 and PASKM-1, PASKM-2. And finally, we investigate the performance behavior of the proposed GS-KM and PASKM methods with state-of-the-art methods such as, Interior point methods (IPMs) and Active set methods (ASMs) for several Netlib LP instances. The total CPU time consumption is calculated in seconds (s). For a fair comparison, we ran the algorithms 10 times and report the averaged performance throughout the experiments. Moreover, all the algorithms start from same initial point that is far away from the feasible region.

| Algorithms | GS-KM (\( \xi \in \mathbb{Q} \)) Algorithm 2 | PASKM (\( \lambda, \tau \)) Algorithm 3 |
|------------|---------------------------------|---------------------------------|
| Parameters | \( \beta \leq m \) \( 0 \leq \delta \leq 2 \) | \( \beta \leq m \) \( 0 \leq \delta \leq 2 \) |
| SKM       | \( \xi = 0 \) \( \xi = -0.1 \) \( \xi = -0.2 \) | \( \xi = 0.5 \) \( \lambda = \frac{\lambda(A^T A)}{m} \) \( 1 \leq \tau \leq \frac{1}{\lambda} \) |
| GS-KM-1   | \( \lambda = 0.5 \) \( 1 \leq \tau \leq \frac{1}{\lambda} \) | \( \lambda = 0.5 \) \( \tau = 1 \) |
| GS-KM-2   | \( \lambda = 0.5 \) \( 1 \leq \tau \leq \frac{1}{\lambda} \) | \( \lambda = 0.5 \) \( \tau = 1 \) |
| PASKM-1   | \( \lambda = 0.5 \) \( 1 \leq \tau \leq \frac{1}{\lambda} \) | \( \lambda = 0.5 \) \( \tau = 1 \) |
| PASKM-2   | \( \lambda = 0.5 \) \( 1 \leq \tau \leq \frac{1}{\lambda} \) | \( \lambda = 0.5 \) \( \tau = 1 \) |

4.2 Experiments on Randomly Generated Instances

We considered the linear feasibility \( Ax \leq b \), where the entries of matrices \( A \in \mathbb{R}^{m \times n} \) and \( b \in \mathbb{R}^m \) are chosen randomly from a certain distribution. To maintain the system consistency (i.e., \( b \in R(A) \)), we first generated vectors \( x_1, x_2 \in \mathbb{R}^n \) at random from the corresponding distributions, then set \( b \) as the convex combination of vectors \( A x_1 \) and \( A x_2 \) (i.e., \( b = \sigma A x_1 + (1-\sigma) A x_2, \ 0 \leq \sigma \leq 1 \)). Two types of random data sets are considered: highly correlated, and Gaussian. For the correlated systems, data matrices \( A \) and \( x_1, x_2 \) are chosen uniformly at random between \([0.9, 1.0]\) (i.e., \( a_{ij}, x_j \in [0.9, 1.0], \ 0 \leq \sigma \leq 1 \)). For the Gaussian system data matrices, \( A \) and \( x_1, x_2 \) are chosen uniformly at random from standard normal distribution (i.e., \( a_{ij}, x_j \in \mathcal{N}(0,1), \forall i, j \)). Moreover, the vector \( b \in \mathbb{R}^m \) are generated by following the above mentioned procedure.

**CPU time vs Sample size \( \beta \)** We first compared the total CPU time consumption of the proposed algorithms (GS-KM-1, GS-KM-2, PASKM-2, PASKM-2) with the original SKM algorithm. The comparison is carried out by varying the sample size \( \beta \) from 1 to the total row size \( m \). The positive residual error tolerance is chosen as \( 10^{-05} \) (i.e., \( \| Ax - b \|_2 \leq 10^{-05} \)). The comparison is carried out for \( \beta = 0.2, 0.5, 0.8 \) and \( 1.5 \). In Figure 1 we compared the above mentioned algorithms for two randomly generated highly correlated linear feasibility problems of size \( 20000 \times 1000 \) and \( 50000 \times 4000 \). From Figure 1 we see that the proposed GS-KM-1, PASKM-1, PASKM-2 algorithms outperform the SKM algorithm in terms of average CPU time when \( \beta = 0.2, 0.5, 0.8 \). For \( \beta = 1.5 \), the performance of SKM, PASKM-1 and PASKM-2 are fairly similar whereas, the performance of GS-KM-1 and GS-KM-2 are worse compared to all other algorithms. We present the time versus sample size plot for two randomly generated Gaussian system of size \( 2000 \times 500 \) and \( 5000 \times 1000 \) in Figure 2. All the algorithms shows similar performance pattern as shown in the correlated systems (Figure 1) for the choice \( 0 < \delta < 1 \). However, for the case of \( \delta = 1.5 \), GS-KM-1 performs...
Randomized Projection Methods for Linear Feasibility - February 21, 2020

Figure 1: Sample size $\beta$ VS average CPU time comparison among SKM, GSKM, PASKM variants for $\delta = 0.2, 0.5, 0.8, 1.5$ on correlated systems. Problem size: $20000 \times 1000$ (Top panel), $50000 \times 4000$ (Bottom panel).

Figure 2: Sample size $\beta$ VS average CPU time comparison among SKM, GSKM, PASKM variants for $\delta = 0.2, 0.5, 0.8, 1.5$ on Gaussian systems. Problem size: $2000 \times 500$ (Top panel), $5000 \times 1000$ (Bottom panel).

better than all other algorithms in both problems while PASKM-1 outperform SKM for the smaller sized problem. In a nutshell, we can conclude that for the choice of $0 < \delta < 1$, PASKM-1, PASKM-2 and GSKM-1 outperform the original SKM method. And in that region, PASKM-2 is the best performing algorithm. Moreover, for $1 \leq \delta \leq 2$, we believe with correct parameter choice one can find better performing variants of GSKM and PASKM compared to SKM algorithm. Note that, the best sample size choice for all of the considered methods occur at $1 < \beta \ll m$. This amplifies the importance of the special sampling distribution selection.

Positive residual error $\| (Ax - b)^+ \|_2$ VS No. of iterations and Time Now, we compare the respective convergence trend for the considered algorithms with respect to number of iterations and CPU time consumption. We choose positive residual error $\| (Ax - b)^+ \|_2$ as the convergence measure and considered $5000 \times 1000$ Gaussian system. We
Figure 3: Positive residual error $\| (Ax - b)^+ \|_2$ VS No. of iteration comparison among SKM, GSKM, PASKM variants for $\delta = 0.2, 0.5, 0.8, 1.5$ and $\beta = 1, 50, 100, 1000, 5000$ on $5000 \times 1000$ Gaussian system.

carried out the analysis for several choice of sample sizes, $\beta = 1, 100, 1000, m$ and the choice of $\delta$ values remains same as before. In Figure 3 we provide the respective convergence results for different sample sizes and different projection parameters. We plot positive residual error VS iteration and positive residual error VS time in Figure 3 and 4 respectively. From Figures 3 and 4 we see that irrespective of sample size, $\| (Ax - b)^+ \|_2$ converges to zero much faster for GSKM-1 and PASKM-2 compared to SKM. As expected, the choice $\beta = 1$ produces the slowest rate and the choice $\beta = 100$ produces the best convergence graph.
Fraction of satisfied constraints (FSC) VS No. of iterations and Time  To investigate the generated solution quality of the above mentioned algorithms of Table 2, we measure the number of satisfied constraints at each iteration, for that we define,

\[
\text{Fraction of Satisfied Constraints (FSC)} = \frac{\text{Number of satisfied constraints}}{\text{Total number of constraints (m)}}
\]

Note that, at any particular iteration we have, \(0 \leq \text{FSC} \leq 1\). In Figure 5 and 6 we plot the value of FSC with respect to No. of iterations and CPU time consumption of each algorithms respectively. From Figures 5 and 6 we can see
Figure 5: No. of iteration vs fraction of satisfied constraints (FSC) comparison among SKM, GSKM, PASKM variants for $\delta = 0.2, 0.5, 0.8, 1.5$ and $\beta = 1, 50, 100, 1000, 5000$ on $5000 \times 1000$ Gaussian system.

that the choice of $\beta = 1$ is the worst choice for all algorithms as the improvement of FSC is much slower compared to other choices of $\beta$. And for the choice $\beta = 100$, we get the best solution quality for each algorithm. Our proposed GSKM-1 and PASKM-2 algorithms outperform the other methods significantly for $0 < \delta < 1$ but, for $\delta = 1.5$ only PASKM-2 performs similar to SKM.
4.3 Experiments on Real Life Instances

In this subsection, we consider some nonrandom, real life test instances. For the sake of unbiased performance analysis, we consider the following two types of real life data-sets: standard Machine Learning (ML) data-sets for Support Vector Machine (SVM) classifier [11,61,62], and sparse linear feasibility problems extracted from benchmark Netlib LP problems [63].
SVM classifier instances We first consider two linear feasibility problems arising from binary classification with SVM. We compare the proposed algorithms with SKM to the linear classification problem using SVM model for the following two data sets: 1) Wisconsin (diagnostic) breast cancer data set and 2) Credit card default data set. The Wisconsin breast cancer data set consists of data points whose features are calculated from images. There are two types of data points: 1) malignant and 2) benign cancer cells. As shown by researchers [11, 64], the SVM classifier problem can be re-written as equivalent homogeneous system of linear inequalities, \( Ax \leq 0 \) which represents the separating hyperplane between malignant and benign data points. The constraint matrix \( A \) has 569 rows (data points) and 30 columns (features). Since the data set is not perfectly separable, we allow a tolerance for the positive residual \( \| (Ax)^+ \| \). For our experiments, we fixed the tolerance as \( 10^{-3} \) (i.e., we ran the algorithm until \( \| (Ax)^+ \| \leq 10^{-3} \) is satisfied).

Similarly, we consider the credit card default data set described in [11, 61]. This data set consists of features denoting the payment profile of an user and binary variables describing payment condition in a certain billing cycle: 1 for payment made on time and 0 for late payment. The SVM classification problem for the data set can be transformed into an equivalent homogeneous system of inequalities, \( Ax \leq 0 \) like before. The solution \( x^* \) denotes the coefficients of the separating hyperplane between on-time and default data points. The transformed data matrix \( A \) has 30000 rows (30000 user profiles) and 23 columns (22 profile features). As the data set is not separable, like the previous problem we allow a tolerance error. In this case, we ran the algorithms until the condition: \( \frac{\| (Ax)^+ \|}{\| (A0)^+ \|} \leq 10^{-3} \) is satisfied.

CPU time vs Sample size \( \beta \) We plot the CPU time consumption VS sample size \( \beta \) graphs for SVM problems in Figure 7. To be consistent with our previous experiments, we choose \( \delta = 0.2, 0.5, 0.8, 1.5 \). From Figure 7 we see that the proposed GSKM-1, PASKM-2 algorithms outperform the other algorithms including SKM for \( \delta = 0.2, 0.5, 0.8 \). However, for \( \delta = 1.5 \), GSKM variants performs best—GSKM-1 with smaller sample sizes and GSKM-2 with larger sample sizes. On the other hand, both SKM and PASKM-2 follows similar trend across different sample sizes with PASKM-2 marginally outperforming SKM. Another interesting point can be noted that the comparison graphs for the credit card data set is not as smooth as the breast cancer data set graphs, which we can attributed to the irregularity of the constraint matrix \( A \).

Netlib LP instances We also investigate the comparative performance of the proposed algorithms with SKM on real life sparse data sets. For this experiment, we consider some Netlib LP [63] test instances. Each of these problems are formulated as standard linear programming problem \( (Ax = b, l \leq x \leq u) \). We transform the above-mentioned problem as an equivalent feasibility problem given below:

\[
\bar{A}x \leq \bar{b}, \quad \bar{A} = \begin{bmatrix} A & I \\ -I & \end{bmatrix}, \quad \bar{b} = \begin{bmatrix} b \\ u \\ -b \end{bmatrix}
\]

CPU time vs Sample size \( \beta \) Now, we plot the CPU time consumption VS sample size \( \beta \) graphs for five Netlib LP instances in Figure 8. Later in subsection 4.4 we consider total ten Netlib LP instances including the five considered
here. In Figure 8, we provide comparison graphs for the following Netlib LP test instances: lp-brandy, lp-addlittle, lp-scorpion, lp-bandm, lp-recipe. Furthermore, we consider different error tolerances for these problems (see Table 3 for details).

Figure 8: Average CPU time VS Sample size $\beta$ comparison among SKM, GSKM, PASKM variants for $\delta = 0.2, 0.5, 0.8, 1.5$ on Netlib LP instances.
From Figure 8, we see that the proposed GSKM-1, PASKM-1, PASKM-2 algorithms outperform the SKM algorithm for $\delta = 0.2, 0.5, 0.8$. In the case of $\delta = 1.5$, the performance of SKM, GSKM-1 and PASKM-2 are fairly similar for the problems lp-scorpion, lp-banddm and lp-recipe. For lp-brandy and lp-adl1t all, of the proposed variants of GSKM and PASKM outperform the original SKM.

### 4.4 Comparison with IPM and ASM for Netlib LP instances

In this subsection, we compare the performance of GSKM and PASKM variants with SKM and benchmark commercial solvers for solving Netlib LP test instances. We follow the standard framework used by De Loera et al. [11] and Morshed et al. [56] in their work for linear feasibility problems. The problem instances are transformed from standard LP problems (i.e., $\min c^T x$ subject to $Ax = b$, $l \leq x \leq u$ with optimum value $p^*$) to an equivalent linear feasibility formulation (i.e., $Ax \leq b$, where $\bar{A} = [A^T - A^T I - I c]^T$ and $\bar{b} = [b^T - b^T u^T - l^T p^*]^T$). For all of the experiments we compared the proposed algorithms for $0 < \delta < 1$, since from our experiments in subsection 4.2 and 4.3 this is the domain where the proposed GSKM and PASKM variants significantly outperform SKM method.

In Table 3, we list the total CPU time in seconds for each of the above mentioned algorithms of Table 2. In addition to that, we provide the CPU time for Interior point method (IPM) and Active set method (ASM) algorithms for solving the selected Netlib LP problems. For a better and fair comparison, the pseudo-code of the proposed methods and SKM are written in MATLAB and Optimization Toolbox function `fmincon` is used to implement IPM and ASM methods. We first solve the linear feasibility problem ($Ax \leq b$) with SKM, GSKM and PASKM variants and record the CPU time consumption in Table 3. Note that, we can’t use `fmincon`’s IPM and ASM algorithms directly to solve the linear feasibility problem ($\min 0, s.t. Ax \leq b$) since both methods fail to solve the linear feasibility problems. The reason for that is, in IPM the Karush Kuhn Tucker (KKT) system at each iteration becomes singular, and ASM stops in the first step of finding a feasible solution. For a fair comparison, in Table 3 we list the total CPU consumption

| Instance | Dimensions | GSKM $\times 10^{-2}$ | PASKM $\times 10^{-2}$ | SKM $\times 10^{-2}$ | Interior Point | Active set | $\beta$ | $\epsilon \times 10^{-2}$ |
|----------|------------|-----------------------|------------------------|----------------------|----------------|-----------|-------|------------------|
| adlittle | $389 \times 138$ | 0.027 | **0.02** | 0.032 | 2.16 | 4.96 | 150 | 0.1 |
| agg      | $2207 \times 615$ | 0.22 | 0.21 | 0.23 | 66.54* | 315.91* | 50 | 1 |
| bandm    | $1555 \times 472$ | 9.82 | **3.21** | 9.2 | 14.57 | 529.43* | 50 | 1 |
| blend    | $337 \times 114$ | 1.48 | **0.74** | 1.28 | 2.28 | 4.62 | 50 | 0.1 |
| brandy   | $1047 \times 303$ | **0.53** | 0.57 | 14.06 | 16.97 | 63.11 | 1 | 1 |
| degen2   | $2403 \times 757$ | 26.26 | **12.62** | 20.73 | 7.13 | 21038 | 100 | 1 |
| finnis   | $3123 \times 1064$ | 0.53 | 0.54 | **0.527** | 66.16* | 237750* | 10 | 0.1 |
| recipe   | $591 \times 204$ | 0.60 | **0.23** | 0.52 | 0.89 | 63.24 | 50 | 0.1 |
| scorpion | $1709 \times 466$ | 156.9 | **55.9** | 125 | 17.68 | 8.02 | 50 | 1 |
| stocfor1 | $565 \times 165$ | 1.05 | **0.49** | 0.95 | 2.13 | 2.52 | 50 | 0.1 |

Table 3: CPU time comparisons among the state-of-the-art methods (using MATLAB’s `fmincon` function) solving LP, and SKM, GSKM and PASK solving LF. * implies that the solver was unable to solve the problem with predetermined accuracy within 100,000 function evaluations. CPU time of the best performing algorithm for a problem is represented in bold letters.

3. we note the best possible time from our previous experiments.
Instead of moving forward with the sequences \( x_{k+1}, y_{k+1} \) and \( v_{k+1} \), for any \( T \gg 1 \) we can skip \( T \) iterations and update \( x_{k+T}, y_{k+T} \) and \( v_{k+T} \) using a generalized recurrence relation that can enhance the computational efficiency.

5 Conclusion

In this work, we propose a general algorithmic framework (GSKM) for solving linear feasibility problem that unifies algorithms such as Randomized Kaczmarz, Motzkin Method and Sampling Kaczmarz Motzkin. We synthesize the convergence analysis of these three methods into one convergence theorem. In addition to the general framework, we propose a Nesterov type acceleration scheme in the SKM method called as PASKM. Our proposed PASKM method provides a bridge between Nesterov type acceleration of Machine Learning to sampling Kaczmarz methods for solving linear feasibility problem. To show the effectiveness of the proposed algorithms, we performed a wide range of numerical experiments on various types of random and standard benchmark data sets. For a better understanding of the behaviour of the proposed algorithms, we numerically analyze two variants for both GSKM and PASKM algorithms in comparison with the original SKM method. Furthermore, we compare our proposed methods to commercially available methods such as IPM and ASM. In majority of the test instances, the proposed algorithms significantly outperform the state-of-the-art methods. Furthermore, as shown in our numerical experiments, correct choice of parameters can lead to much faster accelerated methods for different types of test instances.

Future Research In future, the proposed algorithms and their theoretical analysis can be adopted effectively to various types of extensions such as, sparse variants, optimal parameter and sampling strategy selection based algorithms. First, we plan to extend our work to design efficient sparse variations of the proposed methods that can handle large-scale real-world problems with greater sparsity in the data matrix \( A \). Second, we intend to design a test instance dependent scheme for identifying optimal parameter selection (i.e., \( \beta, \delta, \xi, \lambda, \tau \)) for both GSKM and PASKM. For the GSKM algorithm, adaptive parameter selection (i.e., \( \beta_k, \delta_k, \xi_k \)) policy can be a great area of future research. One can also derive connecting ideas between the proposed GSKM and induced projection plane generation of Chubanov [30,31] which can produce faster algorithms. Finally, we aspire to develop adaptive sampling strategies and integrate greedy Kaczmarz [48] type method into the GSKM framework to further speed up the convergence.

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