Rare meshes FEM scheme for quasi-stationary electromagnetic fields determination 3D problems

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Abstract. The initial-boundary value problem for the quasi-stationary magnetic approximation of the Maxwell equations in inhomogeneous media is studied. The considered problem is reduced to the variational problem of determining the vector magnetic potential. The special gauge for vector magnetic and scalar electrical potentials is used. The well-posedness of the problems is established under general conditions on the coefficients and the applicability of the projection methods for these problems is validated. For the numerical solution of this problem provides to use the effective rare mesh FEM scheme for 3D problems. This scheme is well-proven in 3D elasticity and plasticity problems solving.

1. Introduction
The solution of many actual technological problems leads to theoretical and numerical study of problems for quasi-stationary electromagnetic fields in physically heterogeneous media [1]-[3].

In the construction of numerical algorithms and in theoretical studies of the quasi-stationary problems vector magnetic and scalar electric potentials are traditionally used to describe the electromagnetic fields [4]-[10]. At the considering problems in terms of the potentials different gauges are employed, which ensure the uniqueness of the solution. In particular, in stationary and quasistationary problems the classical Coulomb gauge \( \text{div} \mathbf{A} = 0 \) and Lorenz gauge \( \text{div} \mathbf{A} + \mu \sigma \mathbf{\nabla} \phi = 0 \) are widely used [4], [6]-[8].

Problem statements are of practical interest, which allow to determine a magnetic potential independently of the electrical potential. In heterogeneous media the application of the classical Coulomb and Lorenz gauges leads to the study of the coupled system of equations for the potentials [7], [8]. In [8], [9] the modified Lorenz gauge is discussed for time-harmonic Maxwell equations, that decomposes the problem of finding the magnetic and electric potentials.

In the present paper the initial-boundary value problem for the time-dependent quasi-stationary magnetic approximation of the Maxwell equations is investigated. This problem is formulated in terms of potentials with special gauge, which generalizes the Lorenz gauge for the case of inhomogeneous media [11]. We prove the well-posedness of the problem and investigate the possibility of using the projection methods for its solution. The corresponding topics for the stationary problems are considered in [12].

2. Initial - boundary value problem
The quasi-stationary magnetic approximation of the Maxwell equations in the can be presented in the form [13]
\[ \text{curl } \mathbf{H}(x,t) = \mathbf{J}(x,t), \quad (1) \]
\[ \text{div } \mathbf{B}(x,t) = 0, \quad (2) \]
\[ \text{curl } \mathbf{E}(x,t) = -\frac{\partial}{\partial t} \mathbf{B}(x,t), \quad (3) \]
\[ \text{div } \mathbf{D}(x,t) = \rho(x,t), \quad (4) \]

where \((x,t) \in \mathcal{Q} = \Omega \times (0, T), \quad \Omega \subset \mathbb{R}^3, \quad T > 0; \quad \mathbf{H}, \ \mathbf{B}, \ \mathbf{E}, \ \mathbf{D}, \ \mathbf{J}: \mathcal{Q} \to \mathbb{R}^3 \quad \text{and} \quad \rho: \mathcal{Q} \to \mathbb{R}^1\) – unknown functions.

In linear media the following constitutive relations are valid:
\[ \mathbf{B} = \mu \mathbf{H}, \ \mathbf{D} = \varepsilon \mathbf{E}, \ \mathbf{J} = \sigma \mathbf{E} + \mathbf{J}^{\text{ext}}, \quad (5) \]

where \(\mu\) is a magnetic permeability, \(\varepsilon\) is a permittivity, \(\sigma\) is an electrical conductivity, \(\mathbf{J}^{\text{ext}}\) is an exterior current density.

It is assumed in this paper, that \(\Omega\) is an open bounded domain, homeomorphic to a ball, with a Lipschitz boundary \(\partial \Omega\). Let \(\nu(x)\) is unit normal vector in \(x \in \partial \Omega\). For function \(\mathbf{u}: \overline{\Omega} \to \mathbb{R}^3\) we denote by \(\mathbf{u}_\nu, \mathbf{u}_\tau\) the normal and tangent components of \(\mathbf{u}\) on \(\partial \Omega\).

The system \((1)-(5)\) is considered with the boundary condition
\[ \mathbf{H}_\nu(x,t) = 0, \quad x \in \partial \Omega, \quad t \in (0, T), \quad (6) \]

and the initial condition
\[ \mathbf{H}(x,0) = \mathbf{h}(x), \quad x \in \Omega. \quad (7) \]

We assume, that \(\sigma = \sigma(x), \ \mu = \mu(x)\) and \(\varepsilon = \varepsilon(x)\) are symmetric \(3 \times 3\) matrices of measurable functions on \(\Omega\), satisfying for almost all \(x \in \Omega\) and for all \(\xi \in \mathbb{R}^3\) the conditions
\[ \sigma_1 \xi^2 \leq (\sigma(x) \xi, \xi) \leq \sigma_2 \xi^2, \quad \mu_1 \xi^2 \leq (\mu(x) \xi, \xi) \leq \mu_2 \xi^2, \quad \epsilon_1 \xi^2 \leq (\varepsilon(x) \xi, \xi) \leq \epsilon_2 \xi^2, \]

where \(\epsilon_i, \mu_i, \sigma_i, \ (i = 1, 2)\) are given positive numbers, \(\mathbf{J}^{\text{ext}}: \mathcal{Q} \to \mathbb{R}^3, \ \mathbf{h}: \Omega \to \mathbb{R}^3\) are square integrable functions.

The generalized solutions of the problems are considered, that is all equalities have to be satisfied in the sense of the trace theory [15], and boundary conditions have to be satisfied in the sense of the trace theory [15].

The following Hilbert spaces with the respective scalar products are defined [15]:
\[ H(\text{div}; \Omega) = \left\{ \mathbf{u} \in [L_2(\Omega)]^3 : \text{div } \mathbf{u} \in L_2(\Omega) \right\}, \ K(\text{div}; \Omega) = \left\{ \mathbf{u} \in [L_2(\Omega)]^3 : \text{div } \mathbf{u} = 0 \right\}, \]
\[ (\mathbf{u}, \mathbf{v})_{\text{div}, \Omega} = \int_{\Omega} (\mathbf{u} \cdot \mathbf{v}) dx + \int_{\Omega} \text{div } \mathbf{u} \text{ div } \mathbf{v} dx, \]
\[ H(\text{curl}; \Omega) = \left\{ \mathbf{u} \in [L_2(\Omega)]^3 : \text{curl } \mathbf{u} \in [L_2(\Omega)]^3 \right\}, \ K(\text{curl}; \Omega) = \left\{ \mathbf{u} \in [L_2(\Omega)]^3 : \text{curl } \mathbf{u} = 0 \right\}, \]
\[ (\mathbf{u}, \mathbf{v})_{\text{curl}, \Omega} = \int_{\Omega} (\mathbf{u} \cdot \mathbf{v}) dx + \int_{\Omega} \text{curl } \mathbf{u} \cdot \text{curl } \mathbf{v} dx. \]

\(H_0(\text{div}; \Omega), \ H_0(\text{curl}; \Omega)\) denote the closures of the set of test vector-functions in \(H(\text{div}; \Omega)\) and \(H(\text{curl}; \Omega)\) respectively, \(K_0(\text{div}; \Omega) = K(\text{div}; \Omega) \cap H_0(\text{div}; \Omega)\).
The solution of the problem (1)-(7) is the set of functions $H \in L_2(0,T,H_0(\text{curl};\Omega))$, $B \in L_2\left(0,T,K(\text{div};\Omega)\right)$, $J \in L_2\left(0,T,K_0(\text{div};\Omega)\right)$, $E \in L_2\left(0,T,\{L_2(\Omega)\}^3\right)$, $D \in L_2\left(0,T,\{L_2(\Omega)\}^3\right)$, $\rho \in L_2\left(0,T,H^{-1}(\Omega)\right)$, satisfying (1), (3)–(5), (7).

The equality (3) for $B \in L_2\left(0,T,K(\text{div};\Omega)\right)$, $E \in L_2\left(0,T,\{L_2(\Omega)\}^3\right)$ implies that $H = \mu^{-1}B \in C\left(0,T,\{L_2(\Omega)\}^3\right)$ [14], that is the initial condition (7) makes sense.

3. Problem for vector potential

Relations (2), (3) allow to introduce the vector magnetic potential $A$ and the scalar electric potential $\phi$ as new unknown variables by formula [13]

$$B = \text{curl}A, \quad E = -\text{grad} \phi - \frac{\partial}{\partial t}A.$$  \hfill (9)

In this case, the system (1)-(5) is reduced to one equation

$$\partial A + \mu^{-1} \text{curl} A = -\sigma \text{grad} \phi + J^{\text{ext}}.$$  \hfill (10)

Equation (10) is provided by the boundary condition corresponding to (6)

$$\left(\mu^{-1} \text{curl} A\right)_t(x,t) = 0, \quad x \in \partial \Omega, \quad t \in (0,T),$$  \hfill (11)

and the initial condition

$$A(x,0) = a(x), \quad x \in \Omega,$$  \hfill (12)

where $a \in \{L_2(\Omega)\}^3$ – given function such that $\text{curl}a = \mu h$.

The solution of the problem (10)–(12) is the functions $A \in L_2\left(0,T,H(\text{curl};\Omega)\right)$, $\phi \in L_2\left(0,T,H^1(\Omega)\right)$, satisfying (10), (12) in the sense of the distribution on $(0,T)$ with the values in the Hilbert spaces and (11) in the sense of the trace theory, that is $\mu^{-1} \text{curl} A(t) \in H_0(\text{curl};\Omega)$ for almost all $t \in (0,T)$.

Suppose that $A, \phi$ is the solution of (10)–(12) and $A \in C^1\left(0,T,H(\text{curl};\Omega)\right)$. Then from (10) we obtain for all $v \in H(\text{curl};\Omega)$

$$\frac{d}{dt} \int_\Omega (\sigma A \cdot v)dx + \int_\Omega (\mu^{-1} \text{curl} A \cdot \text{curl} v)dx = -\int_\Omega (\sigma \text{grad} \phi \cdot v)dx + \int_\Omega (J^{\text{ext}} \cdot v)dx.$$  \hfill (13)

The solution of the problem (10)–(12) is obviously not unique. The following gauge is discusses:

$$\phi(x,t) = -\kappa \text{div} \varphi A(x,t), \quad (x,t) \in Q, \quad (\varphi A)_t(x,t) = 0, \quad x \in \partial \Omega, \quad t \in (0,T),$$  \hfill (14)

where $\kappa > 0$ is arbitrary constant.

We introduce the scalar product for $\{L_2(\Omega)\}^3$ as $\langle u, v \rangle = \int_\Omega (u \cdot v)dx$. The norm, generated by this scalar product, is equivalent to the usually norm of $\{L_2(\Omega)\}^3$, the resulting Hilbert space is denoted by $\{L_2(\sigma;\Omega)\}^3$. The following Gilbert space is also defined:

$$W(\sigma,\Omega) = \{u \in H(\text{curl};\Omega); \sigma u \in H_0(\text{div};\Omega)\},$$
The space \( W(\sigma; \Omega) \) is dense and continuous injected in \( \{L_2(\sigma; \Omega)\}^3 \), the space \( \left(\{L_2(\sigma; \Omega)\}^3\right)^* \) can be identified with dense subspace of \( W^*(\sigma; \Omega) \). By identifying \( \{L_2(\sigma; \Omega)\}^3 \) and \( \left(\{L_2(\sigma; \Omega)\}^3\right)^* \), we arrive at the inclusions

\[
W(\sigma; \Omega) \subseteq \{L_2(\sigma; \Omega)\}^3 = \left(\{L_2(\sigma; \Omega)\}^3\right)^* \subseteq W^*(\sigma; \Omega).
\]

The scalar product in \( \{L_2(\sigma; \Omega)\}^3 \) of \( u \in \{L_2(\sigma; \Omega)\}^3 \) and \( v \in W(\sigma; \Omega) \) is the same as the value of functional \( u \) at the element \( v \) in the sense of the duality between \( W^*(\sigma; \Omega) \) and \( W(\sigma; \Omega) \).

Using (13) we obtain that the problem (10)–(12), (14) is reduced to the following problem: to find a function \( A \in L_2(0,T,W(\sigma; \Omega)) \), satisfying (12), such that for all \( v \in W(\sigma; \Omega) \)

\[
\frac{d}{dt} \left( \int_\Omega (\sigma A \cdot v) \, dx + \int_\Omega (\mu^{-1} \text{curl} A \cdot \text{curl} v) \, dx + \kappa \int_\Omega (\text{div} \sigma A \cdot \text{div} v) \, dx \right) = \int_\Omega (f^{ext} \cdot v) \, dx.
\]  

**Theorem 1.** For any \( a \in \{L_2(\Omega)\}^3 \), \( J^{ext} \in \{L_2(Q)\}^3 \) there exists a unique solution \( A \in L_2(0,T,W(\sigma; \Omega)) \) of the problem (12), (15). Moreover, \( A \in C(0,T,\{L_2(\sigma; \Omega)\}^3) \) and

\[
\sup_{t \in (0,T)} \|A(t)\|_\sigma + \|A\|_{L_2(0,T,W)} \leq C_T \left( \left\|J^{ext}\right\|_{L_2(Q)} + \|a\|_\sigma \right).
\]

If \( a \in W(\sigma; \Omega) \), then \( A \in L_2(0,T,\{L_2(\sigma; \Omega)\}^3) \),

\[
\|A\|_{L_2(0,T,\{L_2(\sigma; \Omega)\}^3)} \leq C_{T,1} \left( \left\|J^{ext}\right\|_{L_2(Q)} + \|a\|_\sigma \right).
\]

The constants \( C_T > 0 \), \( C_{T,1} > 0 \) not depend on \( a \), \( J^{ext} \).

Let the operator \( G: W(\sigma; \Omega) \rightarrow W^*(\sigma; \Omega) \) associates with \( u \in W(\sigma; \Omega) \) the element of \( W^*(\sigma; \Omega) \) such that for all \( v \in W(\sigma; \Omega) \)

\[
\langle Gu, v \rangle = \int_\Omega (\mu^{-1} \text{curl} A \cdot \text{curl} v) \, dx + \kappa \int_\Omega (\text{div} \sigma A \cdot \text{div} v) \, dx.
\]

Then \( Gu \in L_2(0,T,W^*(\sigma; \Omega)) \) for \( u \in L_2(0,T,W(\sigma; \Omega)) \). If \( A \in L_2(0,T,W(\sigma; \Omega)) \) is a solution of the problem (12), (15), then (15) for all \( v \in L_2(0,T,W(\sigma; \Omega)) \) can be written as

\[
\frac{d}{dt} \langle A, v \rangle + \langle GA, v \rangle = \left\langle \sigma^{-1} J^{ext}, v \right\rangle.
\]

Hence \( A' \in L_2(0,T,W^*(\sigma; \Omega)) \) and, therefore, \( A \in C(0,T,\{L_2(\sigma; \Omega)\}^3) \) [14].

The proof of the existence of the solution of the problem is based on the Faedo - Galerkin method [16], the feasibility of which follows from the following inequalities which generalize the obtained in [17] estimates for scalar products of vector fields in a star-shape domain.

**Theorem 2.** Let \( \Omega \subset \mathbb{R}^3 \) is open bounded Lipschitz domain, homeomorphic to a ball. There exists a positive constant \( C(\Omega) \), which depends only on \( \Omega \), that the inequality
\[ \left\| (u, v) \right\|_{L^2(O)} \leq C(\Omega) \left( \left\| u \right\|_{L^2(O)} + \left\| \nabla v \right\|_{L^2(O)} + \left\| \nabla u \right\|_{L^2(O)} \right) \]

is true for any \( u \in H_0(\text{curl}; \Omega) \), \( v \in H(\text{div}; \Omega) \) and for any \( u \in H(\text{curl}; \Omega) \), \( v \in H_0(\text{div}; \Omega) \).

We will establish the equivalence the problem in terms of potentials and origin initial-boundary value problem. The following statements are valid.

**Theorem 3.** Let \( A \in L_2(\Omega; 0, T, W(\sigma; \Omega)) \) is the solution of the problem (12), (15), where \( \sigma \in W(\sigma; \Omega) \), and \( \varphi = -\kappa \text{div} A \). Then \( \mu^{-1} \text{curl} A \in L_2(\Omega; 0, T, H_0(\text{curl}; \Omega)) \), \( \varphi \in L_2(\Omega; 0, T, H^1(\Omega)) \) and (14) is fulfilled, that is \( A \), \( \varphi \) is the solution of the problem (10) –(12), (14).

From theorem 3 the next theorem follows.

**Theorem 6.** Let \( \sigma \in W(\sigma; \Omega) \), \( \mu^{-1} \text{curl} \sigma = h \). Then the problem (1)-(7) has unique solutions. Moreover the relations (9) are valid, where \( A \), \( \varphi \) is the solution of the problem (10 –(12), (14).

4. Numerical scheme

Rare mesh scheme [18] proved the high quality of the numerical solution for 3D problems of elasticity and plasticity theory [19-21]. Scheme constructing technology is exactly the same as described in [22]. The scheme is based on the traditional scheme of the linear 4-node finite element in the form of a tetrahedron with a linear approximation of functions in the element. Integration in the element uses one point of integration that corresponds to well known reduced integration technique by O. Zienkiewich [23]. Finite element mesh is composed of a hexahedron, and the elements are located in the center of each hexahedron so that the edges of the tetrahedron are the hexahedron faces diagonals. As a result, each base hexahedron contains only one calculation element. Ullage hexahedron and it’s contains extensive parameters are appended to the central tetrahedral element. To solve the problems of the quasi-stationary magnetic fields definition rare mesh scheme is applied to approximation of variational equality (15). For the time integration of the system used Crank-Nicholson numerical scheme. The explicit form of the numerical scheme for the stationary problem in the case of the uniform mesh is given in [12].

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