Equality of orthogonal transvection group and elementary orthogonal transvection group

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Abstract: H. Bass defined orthogonal transvection group of an orthogonal module and elementary orthogonal transvection group of an orthogonal module with a hyperbolic direct summand. We also have the notion of relative orthogonal transvection group and relative elementary orthogonal transvection group with respect to an ideal of the ring. According to the definition of Bass relative elementary orthogonal transvection group is a subgroup of relative orthogonal transvection group of an orthogonal module with hyperbolic direct summand. Here we show that these two groups are the same in the case when the orthogonal module splits locally.

1 Introduction

In Section 5 of [10] L.N. Vaserstein proved that first row of an elementary linear matrix of even size (bigger than or equal to 4) is the same as the first row of a symplectic matrix of the same size w.r.t. an alternating form. This result motivated us to prove that the orbit of a unimodular row of even size under the action of elementary linear group is same as the orbit of a unimodular row of same size under the action of elementary symplectic group (see Theorem 4.1, [4]); we also proved a relative (to an ideal of the ring) version of this result (see Theorem 5.5, [4]). Generalising this result in the setting of finitely generated projective modules involving the transvection groups as defined by H. Bass, we proved that in the case of a symplectic module with a hyperbolic direct summand the orbits of any unimodular element from the symplectic module, under the actions of elementary linear transvection group and the elementary symplectic transvection group coincide (see Theorem 6.1, [5]). While proving the above result on equality of orbits of unimodular elements of symplectic modules, we observed that in the relative case to an ideal of the ring, the equality holds between the linear transvection group and the elementary linear transvection group (see Proposition 4.10, [5]). We also noticed that in the relative case to an ideal of the ring, the symplectic transvection group and the elementary symplectic transvection group coincide (see Theorem 5.23, [5]). In the absolute case the equalities for linear
transvection group, symplectic transvection group, and orthogonal transvection group with the corresponding elementary transvection groups were proved in [2]. In view of the above results it is natural to ask whether the equality of orthogonal transvection group and elementary orthogonal transvection group holds in the relative case to an ideal of the ring. In this article we prove the equality of these two groups in the case when the orthogonal module splits locally.

2 Preliminaries

In this article we will always assume that \( R \) is a commutative ring with unit. A row \( v = (v_1, \ldots, v_n) \in R^n \) is said to be unimodular if there are elements \( w_1, \ldots, w_n \) in \( R \) such that \( v_1 w_1 + \cdots + v_n w_n = 1 \). \( \text{Um}_n(R) \) will denote the set of all unimodular rows \( v \in R^n \). Let \( I \) be an ideal in \( R \). We denote by \( \text{Um}_n(R, I) \) the set of all unimodular rows of length \( n \) which are congruent to \( e_1 = (1, 0, \ldots, 0) \) modulo \( I \). (If \( I = R \), then \( \text{Um}_n(R, I) = \text{Um}_n(R) \)).

**Definition 2.1.** Let \( P \) be a finitely generated projective \( R \)-module. An element \( p \in P \) is said to be unimodular if there exists a \( R \)-linear map \( \phi : P \to R \) such that \( \phi(p) = 1 \). The collection of unimodular elements of \( P \) is denoted by \( \text{Um}(P) \).

Let \( P \) be of the form \( R \oplus Q \) and have an element of the form \( (1, 0) \) which correspond to the unimodular element. An element \( (a, q) \in P \) is said to be relative unimodular w.r.t. an ideal \( I \) of \( R \) if \( (a, q) \) is unimodular and \( (a, q) \) is congruent to \( (1, 0) \) modulo \( IP \). The collection of all relative unimodular elements w.r.t. an ideal \( I \) is denoted by \( \text{Um}(P, IP) \).

Let us recall that if \( M \) is a finitely presented \( R \)-module and \( S \) is a multiplicative set of \( R \), then \( S^{-1}\text{Hom}_R(M, R) \cong \text{Hom}_{R_S}(M_S, R_S) \) (Theorem 2.13", Chapter I, [3]). Also recall that if \( f = (f_1, \ldots, f_n) \in R^n := M \), then \( \Theta_M(f) = \{ \phi(f) : \phi \in \text{Hom}(M, R) \} = \sum_{i=1}^n Rf_i \). Therefore, if \( P \) is a finitely generated projective \( R \)-module of rank \( n \), \( m \) is a maximal ideal of \( R \) and \( v \in \text{Um}(P) \), then \( v_m \in \text{Um}_n(R_m) \). Similarly if \( v \in \text{Um}(P, IP) \) then \( v_m \in \text{Um}_n(R_m, I_m) \).

**Definition 2.2.** Elementary Linear Group: Elementary linear group \( E_n(R) \) denote the subgroup of \( \text{SL}_n(R) \) consisting of all elementary matrices, i.e. those matrices which are a finite product of the elementary generators \( E_{ij}(\lambda) = I_n + e_{ij}(\lambda), \ 1 \leq i \neq j \leq n, \ \lambda \in R \), where \( e_{ij}(\lambda) \in M_n(R) \) has an entry \( \lambda \) in its \( (i, j) \)-th position and zeros elsewhere.

In the sequel, if \( \alpha \) denotes an \( m \times n \) matrix, then we let \( \alpha^t \) denote its transpose matrix. This is of course an \( n \times m \) matrix. However, we will mostly be working with square matrices, or rows and columns.

**Definition 2.3.** The Relative Groups \( E_n(I) \), \( E_n(R, I) \): Let \( I \) be an ideal of \( R \). The relative elementary linear group \( E_n(I) \) is the subgroup of \( E_n(R) \) generated as a group by the elements \( E_{ij}(x), \ x \in I, \ 1 \leq i \neq j \leq n \).

The relative elementary linear group \( E_n(R, I) \) is the normal closure of \( E_n(I) \) in \( E_n(R) \).
(Equivalently, \( E_n(R, I) \) is generated as a group by \( E_{ij}(a)E_{ji}(x)E_{ij}(-a) \), with \( a \in R, x \in I, i \neq j \), provided \( n \geq 3 \) (see [12], Lemma 8)).

**Definition 2.4.** \( E_n^1(R, I) \) is the subgroup of \( E_n(R) \) generated by the elements of the form \( E_{11}(a) \) and \( E_{j1}(x) \), where \( a \in R, x \in I, \) and \( 2 \leq i, j \leq n \).

**Remark 2.5.** It is easy to check that if \( E \subseteq R \) and \( I \) be an ideal of \( R \) and \( I \) be an ideal of \( R \), then \( v = e_1\beta \), for some \( \beta \in E_n(R, I) \).

**Definition 2.6. Orthogonal Group:** The orthogonal group \( O_{2n}(R) \) with respect to the standard symmetric matrix \( \tilde{\psi}_{n} = \sum_{i=1}^{n}e_{2i-1,2i} + \sum_{i=1}^{n}e_{2i,2i-1} \) is the collection \( \{ \alpha \in \text{GL}_{2n}(R) \mid \alpha^t\tilde{\psi}_n\alpha = \tilde{\psi}_n \} \). For an ideal \( I \) of \( R \), \( O_{2n}(R, I) \) represents the kernel of the natural map \( O_{2n}(R) \rightarrow O_{2n}(R/I) \).

Let \( \sigma \) denote the permutation of the natural numbers \( \{1, 2, \ldots, 2n\} \) given by \( \sigma(2i) = 2i - 1 \) and \( \sigma(2i - 1) = 2i \).

**Definition 2.7. Elementary Orthogonal Group:** As in \( \S \)2 of [11] we define for \( z \in R, 1 \leq i \neq j \leq 2n, \)

\[
oe_{ij}(z) = 1_{2n} + ze_{ij} - ze_{\sigma(j)\sigma(i)} \quad \text{if } i \neq \sigma(j).
\]

It is easy to check that all these elements belong to \( O_{2n}(R) \). We call them *elementary orthogonal matrices* with respect to the standard symmetric matrix \( \tilde{\psi}_n \) over \( R \) and the subgroup of \( O_{2n}(R) \) generated by them is called the elementary orthogonal group \( EO_{2n}(R) \) with respect to the standard symmetric matrix \( \tilde{\psi}_n \).

**Definition 2.8. The Relative Group \( EO_{2n}(I), EO_{2n}(R, I) \):** Let \( I \) be an ideal of \( R \). The relative elementary group \( EO_{2n}(I) \) is the subgroup \( EO_{2n}(R) \) generated as a group by the elements \( oe_{ij}(x), x \in I \) and \( 1 \leq i \neq j \leq 2n \).

The relative elementary group \( EO_{2n}(R, I) \) is the normal closure of \( EO_{2n}(I) \) in \( EO_{2n}(R) \).

**Lemma 2.9.** \( EO_{2n}(R, I) \) is generated as a group by the elements of the form \( g oe_{ij}(x)g^{-1} \), where \( g \in EO_{2n}(R), x \in I, \) and either \( i = 1 \) or \( j = 1 \).

Proof: An element of the form \( g oe_{ij}(x)g^{-1} \in EO_{2n}(R, I), \) where \( g \in EO_{2n}(R), x \in I, \) and either \( i = 1 \) or \( j = 1 \). Consider an elementary generator \( oe_{kl}(a)oe_{ij}(x)oe_{kl}(-a) \) of \( EO_{2n}(R, I), \) where \( a \in R, x \in I, \) and \( i, j \neq 1 \). Then

\[
\begin{align*}
oe_{kl}(a)oe_{ij}(x)oe_{kl}(-a) &= oe_{kl}(a)[oe_{i1}(x), oe_{1j}(1)] \\
&= oe_{kl}(a)\{oe_{i1}(x) oe_{1j}(1) oe_{i1}(-x) oe_{1j}(-1)\} \\
&= oe_{kl}(a)oe_{i1}(x)oe_{kl}(a)\{oe_{1j}(1) oe_{i1}(-x) oe_{1j}(-1)\}
\end{align*}
\]

and hence the lemma follows. \( \square \)
**Definition 2.10.** The group $\text{EO}_2^n(R, I)$ is the subgroup of $\text{EO}_2^n(R)$ generated by the elements of the form $oe_{ij}(a)$ and $oe_{ij}(-a)$, where $a \in R$, $x \in I$ and $3 \leq i, j \leq 2n$.

In the following two lemmas we obtain some useful facts regarding elementary orthogonal groups. An analogous result in the linear case was proved in [7] and in the symplectic case was proved in the Appendix of [5].

**Lemma 2.11.** Let $R$ be a commutative ring and $I$ be an ideal of $R$. Then $\text{EO}_2^n(R, I) \subseteq \text{EO}_2^n(R, I)$, for $n \geq 3$.

Proof: Let us define $S_{ij} = \{oe_{ij}(a)oe_{ij}(x)oe_{ij}(-a) : a \in R, x \in I\}$. It suffices to show that $\text{EO}_2^n(R, I)$ contains the set $S_{ij}$ for all $1 \leq i \neq j \leq 2n$. Note that $S_{ij} = S_{\sigma(j)\sigma(i)}$ and $S_{ij} \subseteq \text{EO}_2^n(R, I)$, for $3 \leq j \leq 2n$. First we state the following identities

$$\begin{align*}
[g, hk] &= [g, h][g, k], \quad (1) \\
[g, hk] &= [g, h][h, k], \quad (2) \\
[g, h, k] &= [g, h, k], \quad (3)
\end{align*}$$

where $gh$ denotes $ghg^{-1}$ and $[g, h] = ghg^{-1}$. Using these identities and the commutator law $[oe_{ik}(a), oe_{kj}(b)] = oe_{ij}(ab)$ we establish the inclusion.

Note that $oe_{ij}(x), oe_{ij}(x) \in \text{EO}_2^n(R, I)$, for $3 \leq i, j \leq 2n$ and $x \in I$. For $3 \leq i, j \leq 2n$ and $x \in I$, we have $oe_{ij}(x) = [oe_{ij}(x), oe_{ij}(1)] \in \text{EO}_2^n(R, I)$. In the following computation we will express the generators of $S_{ij}$ in terms of $oe_{ij}(x)$ or $oe_{ij}(a)$, where $x \in I, a \in R$. Also, note that we will use $\oplus$ to represent elements of $\text{EO}_2^n(R, I)$.

$$\begin{align*}
\text{oe}_{ij}(a)oe_{ij}(x) &= \text{oe}_{ij}(a)[\text{oe}_{j2}(1), \text{oe}_{j2}(x)] \\
&= \text{oe}_{j2}(a)\text{oe}_{j2}(1), \text{oe}_{j2}(x)\text{oe}_{j2}(ax)] \\
&= \text{oe}_{j2}(a)[\text{oe}_{j2}(1), \text{oe}_{j2}(x)\text{oe}_{j2}(ax)]\text{oe}_{j2}(ax)] \\
&= \text{oe}_{j2}(a)\text{oe}_{j2}(1), \text{oe}_{j2}(x)\text{oe}_{j2}(ax)]\text{oe}_{j2}(ax)] \\
&= \text{oe}_{j2}(a)\text{oe}_{j2}(1), \text{oe}_{j2}(x)\text{oe}_{j2}(ax)]\text{oe}_{j2}(ax)] \\
&= \text{oe}_{j2}(a)\text{oe}_{j2}(1), \text{oe}_{j2}(x)\text{oe}_{j2}(ax)]\text{oe}_{j2}(ax)] \\
&= \text{oe}_{j2}(a)\text{oe}_{j2}(1), \text{oe}_{j2}(x)\text{oe}_{j2}(ax)]\text{oe}_{j2}(ax)] \\
&= \text{oe}_{j2}(a)\text{oe}_{j2}(1), \text{oe}_{j2}(x)\text{oe}_{j2}(ax)]\text{oe}_{j2}(ax)] \\
&= \text{oe}_{j2}(a)\text{oe}_{j2}(1), \text{oe}_{j2}(x)\text{oe}_{j2}(ax)]\text{oe}_{j2}(ax)] \\
&= \text{oe}_{j2}(a)\text{oe}_{j2}(1), \text{oe}_{j2}(x)\text{oe}_{j2}(ax)]\text{oe}_{j2}(ax)]
\end{align*}$$

Since $S_{12}, S_{23} \subseteq \text{EO}_2^n(R, I)$, therefore $S_{ij} \subseteq \text{EO}_2^n(R, I)$, for $3 \leq i, j \leq 2n$. Similarly $S_{ik}, S_{1k} \subseteq \text{EO}_2^n(R, I)$ will imply that $S_{ij} \subseteq \text{EO}_2^n(R, I)$, for $3 \leq i, j \leq 2n$. □
Proposition 2.12. Let $R$ be a commutative ring and let $I$ be an ideal of $R$. Then for $n \geq 3$ the following sequence is exact

$$1 \to EO_{2n}(R, I) \to EO^1_{2n}(R, I) \to EO^2_{2n}(R/I, 0) \to 1.$$ 

Thus $EO_{2n}(R, I)$ equals $EO^1_{2n}(R, I) \cap O_{2n}(R, I)$.

Proof: Let $f : EO^1_{2n}(R, I) \to EO^2_{2n}(R/I, 0)$. Note that $\ker(f) = EO^1_{2n}(R, I) \cap O_{2n}(R, I)$. We shall prove that $\ker(f) = EO_{2n}(R, I)$. Let $E = \prod_{k=1}^{n} oe_{j_{k, 1}}(x_{k})oe_{1_{i_{k}}}(a_{k})$ be an element in the $\ker(f)$ and $E$ can be written as $oe_{j_{1, 1}}(x_{1}) \prod_{k=2}^{n} \gamma_{k}oe_{j_{k, 1}}(x_{k})\gamma_{k}^{-1}$, where $\gamma_{i}$ is equal to $\prod_{k=1}^{i-1} oe_{1_{i_{k}}}(a_{k}) \in EO_{2n}(R)$, and hence $\ker(f) \subseteq EO_{2n}(R, I)$. The reverse inclusion follows from the fact that $EO_{2n}(R, I) \subseteq EO^1_{2n}(R, I)$ (see Lemma 2.11).

\[\square\]

3 Local Global Principle for Relative Elementary Group

In this section we prove Lemma 3.1 and Lemma 3.4 which will be used in proving the main result in the final section. In Lemma 3.4 we obtain the Local-Global Principle for an extended ideal for slightly larger group $EO^1_{2n}(R, I)$ than the relative group $EO_{2n}(R, I)$. This group was introduced in the linear case by W. van der Kallen in [1]. The Local-Global Principle for an extended ideal in the linear, orthogonal and symplectic groups was proved in [1].

The line of arguments given in the proofs below closely follow that in the Local-Global principle for an extended ideal in the symplectic case in [5]. However, as far as the computational details are concerned, there are substantial deviations from [5] in many steps.

Lemma 3.1. Let $R$ be a commutative ring and $I$ be an ideal of $R$. Let $n \geq 3$. Let $\varepsilon = \varepsilon_{1} \ldots \varepsilon_{r}$ be an element of $EO^1_{2n}(R, I)$, where each $\varepsilon_{k}$ is an elementary generator. If $oe_{i_{j}}(Xf(X))$ is an elementary generator of $EO^1_{2n}(R[X], I[X])$, then

$$\varepsilon oe_{i_{j}}(Y^{4^{r}}Xf(Y^{4^{r}}X)) \varepsilon^{-1} = \prod_{t=1}^{s} oe_{i_{j_{t}}}(Yh_{t}(X, Y)),$$

where either $i_{t} = 1$ or $j_{t} = 1$ and $h_{t}(X, Y) \in R[X, Y]$, when $i_{t} = 1$; $h_{t}(X, Y) \in I[X, Y]$ when $j_{t} = 1$.

Proof: We prove the result using induction on $r$, where $\varepsilon$ is product of $r$ many elementary generators. Let $r = 1$ and $\varepsilon = oe_{pq}(a)$. Note that $a \in R$ when $p = 1$, and $a \in I$ when $q = 1$. Given that $oe_{ij}(Xf(X))$ is an elementary generator of $EO^1_{2n}(R[X], I[X])$. First we assume $i = 1$, hence $f(X) \in R[X]$. 

Case (1): Let $(p, q) = (1, j)$. In this case

$$oe_{1j}(a) oe_{ij}(Y^{4}Xf(Y^{4}X)) oe_{1j}(-a) = oe_{1j}(Y^{4}Xf(Y^{4}X)).$$

Case (2): Let $(p, q) = (1, \sigma(j))$. In this case

$$oe_{1\sigma(j)}(a) oe_{ij}(Y^{4}Xf(Y^{4}X)) oe_{1\sigma(j)}(-a) = oe_{1j}(Y^{4}Xf(Y^{4}X)).$$

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Case (3): Let \((p, q) = (1, k), k \neq j, \sigma(j)\). In this case
\[\text{oe}_{1k}(a) \text{oe}_{1j}(Y^4 X f(Y^4 X)) \text{oe}_{1k}(-a) = \text{oe}_{1j}(Y^4 X f(Y^4 X)).\]

Case (4): Let \((p, q) = (j, 1)\). In this case
\[\text{oe}_{j1}(a) \text{oe}_{j1}(Y^4 X f(Y^4 X)) \text{oe}_{j1}(-a) = \text{oe}_{j1}(a[Y^2 X f(Y^4 X)), \text{oe}_{kj}(Y^2)]
= \begin{bmatrix} \text{oe}_{jk}(aY^2 X f(Y^4 X)) \text{oe}_{1k}(Y^2 X f(Y^4 X)), \text{oe}_{k1}(aY^2), \text{oe}_{kj}(Y^2) \\
\text{oe}_{1k}(-Y^2 X f(Y^4 X)) \text{oe}_{jk}(-Y^2 X f(Y^4 X)), \text{oe}_{k1}(-Y^2), \text{oe}_{kj}(-aY^2) \\
\text{oe}_{1k}(Y^2 X f(Y^4 X)) \text{oe}_{jk}(Y^2) \text{oe}_{1k}(-Y^2 X f(Y^4 X)) \text{oe}_{kj}(-Y^2) \text{oe}_{kj}(Y^2) \\
\text{oe}_{1k}(-aY), \text{oe}_{1k}(Y X f(Y^4 X)) \text{oe}_{kj}(-Y^2) \text{oe}_{kj}(-aY^2) \\
\text{oe}_{1k}(Y^2 X f(Y^4 X)) \text{oe}_{1j}(Y^2 X f(Y^4 X)) \text{oe}_{1j}(-aY), \text{oe}_{1k}(Y^2 X f(Y^4 X)) \text{oe}_{1j}(-aY^2)
\end{bmatrix}.

Case (5): Let \((p, q) = (\sigma(j), 1)\). In this case
\[\text{oe}_{\sigma(j)}(a) \text{oe}_{1j}(Y^4 X f(Y^4 X)) \text{oe}_{\sigma(j)}(a) = \text{oe}_{\sigma(j)}(a[Y^2, \text{oe}_{kj}(Y^2 X f(Y^4 X)])
= \begin{bmatrix} \text{oe}_{\sigma(j)}(aY^2) \text{oe}_{1k}(Y^2) \text{oe}_{kj}(Y^2 X f(Y^4 X)) \\
\text{oe}_{\sigma(j)}(-Y^2 X f(Y^4 X)) \text{oe}_{1k}(-Y^2) \text{oe}_{kj}(-aY^2) \\
\text{oe}_{\sigma(j)}(aY^2) \text{oe}_{1j}(Y^4 X f(Y^4 X)) \text{oe}_{1j}(-aY) \text{oe}_{1j}(Y^2 X f(Y^4 X)) \text{oe}_{1j}(-aY^2)
\end{bmatrix}.

Case (6): Let \((p, q) = (k, 1), k \neq j, \sigma(j)\). In this case
\[\text{oe}_{k1}(a) \text{oe}_{1j}(Y^4 X f(Y^4 X)) \text{oe}_{k1}(-a) = \text{oe}_{k1}(aY^2), \text{oe}_{1j}(Y^4 X f(Y^4 X)) \text{oe}_{1j}(Y^4 X f(Y^4 X))
= \begin{bmatrix} \text{oe}_{k1}(aY^2), \text{oe}_{1j}(Y^2 X f(Y^4 X)) \text{oe}_{1j}(Y^4 X f(Y^4 X)) \\
\text{oe}_{k1}(aY^2) \text{oe}_{1j}(Y^4 X f(Y^4 X)) \text{oe}_{1j}(Y^4 X f(Y^4 X)) \text{oe}_{1j}(Y^4 X f(Y^4 X))
\end{bmatrix}.

Hence the result is true when \(i = 1\) and \(\varepsilon\) is an elementary generator. Carrying out similar calculation one can show the result is true when \(j = 1\) and \(\varepsilon\) is an elementary generator. Let us assume that the result is true when \(\varepsilon\) is product of \(r\) many elementary generators, i.e, \(\varepsilon_2 \ldots \varepsilon_r \text{oe}_{ij}(Y^{r-1} X f(Y^{r-1} X)) \varepsilon_{r-1} \ldots \varepsilon_2^{-1} = \prod_{t=1}^k \text{oe}_{p_t q_t}(Y g_t(X, Y))\), where either \(p_t = 1\) or \(q_t = 1\). Note that \(g_t(X, Y) \in R[X, Y]\) when \(p_t = 1\) and \(g_t(X, Y) \in I[X, Y]\) when \(q_t = 1\).
We now establish the result when \( \varepsilon \) is product of \( r \) many elementary generators. We have

\[
\varepsilon \circ e_{i_1j_1}(Y^{4^r} X f(Y^{4^r} X)) \varepsilon^{-1} = \varepsilon_1 \varepsilon_2 \ldots \varepsilon_r \circ e_{i_1j_1}(Y^{4^r} X f(Y^{4^r} X)) \varepsilon^{-1} = \varepsilon_1 \left( \prod_{t=1}^{k} e_{p_tq_t}(Y^4 g_t(X,Y)) \right) \varepsilon^{-1} = \prod_{t=1}^{k} e_{p_tq_t}(Y^4 g_t(X,Y)) = \prod_{t=1}^{s} e_{i_tj_t}(Y h_t(X,Y)).
\]

To get the last equality one needs to repeat the calculation which was done for a single elementary generator. Note that at the last line either \( i_t = 1 \) or \( j_t = 1 \). Also, note that \( h_t(X,Y) \in R[X,Y] \), when \( i_t = 1 \) and \( h_t(X,Y) \in I[X,Y] \), when \( j_t = 1 \). \( \square \)

**Notation 3.2.** Let \( M \) be a finitely presented \( R \)-module and \( a \) be a non-nilpotent element of \( R \). Let \( R_a \) denote the ring \( R \) localised at the multiplicative set \( \{ a^i : i \geq 0 \} \) and \( M_a \) denote the \( R_a \)-module \( M \) localised at \( \{ a^i : i \geq 0 \} \). Let \( \alpha(X) \) be an element of \( \text{End}(M[X]) \). The localization map \( i : M \rightarrow M_a \) induces a map \( i^*: \text{End}(M[X]) \rightarrow \text{End}(M[X]_a) = \text{End}(M_a[X]) \). We shall denote \( i^*(\alpha(X)) \) by \( \alpha(X)_a \) in the sequel.

We need the following two lemmas.

**Lemma 3.3.** Let \( M \) be a finitely presented \( R \)-module and \( I \) be an ideal of \( R \). Let \( \alpha(X), \beta(X) \in \text{End}(M[X], IM[X]) = \ker(\text{End}(M[X]) \rightarrow \text{End}(M[X]/IM[X])) \), with \( \alpha(0) = \beta(0) \). Let \( a \) be a non-nilpotent element in \( R \) such that \( \alpha(X)_a = \beta(X)_a \) in \( \text{End}(M_a[X], IM_a[X]) \). Then \( \alpha(a^N X) = \beta(a^N X) \) in \( \text{End}(M[X], IM[X]), \) for \( N \gg 0 \).

**Lemma 3.4.** Let \( R \) be a commutative ring and \( I \) be an ideal of \( R \). Let \( n \geq 3 \). Let \( a \) be a non-nilpotent element in \( R \) and \( \alpha(X) \) be in \( \text{EO}_{2n}(R_a[X], I_a[X]) \), with \( \alpha(0) = \text{Id.} \) Then there exists \( \alpha^*(X) \in \text{EO}_{2n}(R[X], I[X]) \), with \( \alpha^*(0) = \text{Id.} \), such that \( \alpha^*(X) \) localises to \( \alpha(bX) \), for \( b \in (a^d), \ d \gg 0 \).

The proofs as in Lemma 3.3 in [1] and in Lemma 3.4 in [1] work verbatim for Lemma 3.3 and Lemma 3.4 as above, respectively, and hence the proofs are omitted.

The following result was proved in [1]. We next apply Lemma 3.4 to record a different proof.

**Theorem 3.5.** Let \( R \) be a commutative ring and \( I \) be an ideal of \( R \). Let \( n \geq 3 \). Let \( a \) be a non-nilpotent element in \( R \) and \( \alpha(X) \) be in \( \text{EO}_{2n}(R_a[X], I_a[X]) \), with \( \alpha(0) = \text{Id.} \) Then there exists \( \alpha^*(X) \in \text{EO}_{2n}(R[X], I[X]) \), with \( \alpha^*(0) = \text{Id.} \), such that \( \alpha^*(X) \) localises to \( \alpha(bX) \), for \( b \in (a^d), \ d \gg 0 \).

Proof: Follows from Lemma [2.12] and Lemma 3.4. \( \square \)

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4 Orthogonal Modules and Orthogonal Transvections

In this section we prove Theorem 4.22 which is the main result of this paper. We begin with a sequence of definitions.

Definition 4.1. Let $M$ be an $R$-module. A bilinear form on $M$ is a function $\beta : M \times M \to R$ such that $\beta(x, y)$ is $R$-linear as a function of $x$ for fixed $y$, and $R$-linear as a function of $y$ for fixed $x$. The pair $(M, \beta)$ is called bilinear form module over $R$. $\beta$ is called an inner product if it satisfies non-degeneracy condition, i.e, the natural map induced by $\beta$ from $P \to P^*$ is an isomorphism. In this case the pair $(M, \beta)$ is called inner product module over $R$. A bilinear form or inner product $\beta$ is called symmetric if $\beta(x, y) = \beta(y, x)$, for all $x, y \in M$. An inner product module $(M, \beta)$ will be called inner product space if $M$ is finitely generated and projective over $R$.

Definition 4.2. An orthogonal $R$-module is a pair $(P, \langle \cdot \rangle)$, where $P$ is a finitely generated projective $R$-module of even rank and $\langle \cdot, \cdot \rangle : P \times P \to R$ is a non-degenerate symmetric bilinear form. This is also known as symmetric inner product space.

Definition 4.3. Let $(P_1, \langle \cdot \rangle_1)$ and $(P_2, \langle \cdot \rangle_2)$ be two orthogonal $R$-modules. Their orthogonal sum is the pair $(P, \langle \cdot \rangle)$, where $P = P_1 \oplus P_2$ and the inner product is defined by $\langle (p_1, p_2), (q_1, q_2) \rangle = \langle p_1, q_1 \rangle_1 + \langle p_2, q_2 \rangle_2$.

Definition 4.4. There is a non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ on the $R$-module $R \oplus R^*$, namely $\langle (a_1, f_1), (a_2, f_2) \rangle = f_2(a_1) + f_1(a_2)$. The orthogonal module $R \oplus R^*$ with this symmetric bilinear form is denoted by $\mathbb{H}(R)$ and called hyperbolic plane. Note that $\mathbb{H}_n(R)$ is the orthogonal sum of $n$-copies of $\mathbb{H}(R)$.

Definition 4.5. An isometry of an orthogonal module $(P, \langle \cdot, \cdot \rangle)$ is an automorphism of $P$ which fixes the bilinear form. The group of isometries of $(P, \langle \cdot, \cdot \rangle)$ is denoted by $O(P)$.

Definition 4.6. Let $(P, \langle \cdot, \cdot \rangle)$ be an orthogonal module. H. Bass defined orthogonal transvection of an orthogonal module $(P, \langle \cdot, \cdot \rangle)$ is an automorphism of the form

$$\tau(p) = p - \langle u, p \rangle v + \langle v, p \rangle u,$$

where $u, v \in P$ are isotropic, i.e, $\langle u, u \rangle = \langle v, v \rangle = 0$ with $\langle u, v \rangle = 0$, and either $u$ or $v$ is unimodular. It is easy to check that $(\tau(p), \tau(q)) = \langle p, q \rangle$, i.e, $\tau \in O(P)$ and $\tau$ has an inverse $\sigma(p) = p + \langle u, p \rangle v - \langle v, p \rangle u$.

The subgroup of $O(P)$ generated by the orthogonal transvections is called orthogonal transvection group and denoted by $\text{Trans}_O(P, \langle \cdot, \cdot \rangle)$ (see [3] or [4]).

Now onwards $Q$ will denote $(R^2 \oplus P)$ with induced form on $(\mathbb{H}(R) \oplus P)$, and $Q[X]$ will denote $(R[X]^2 \oplus P[X])$ with induced form on $(\mathbb{H}(R[X]) \oplus P[X])$.

Definition 4.7. The orthogonal transvections of $Q = (R^2 \oplus P)$ of the form

$$(a, b, p) \mapsto (a, b + \langle p, q \rangle, p - aq),$$
or of the form
\[ (a, b, p) \mapsto (a + \langle p, q \rangle, b, p - bq), \]
where \(a, b \in R\) and \(p, q \in P\), are called elementary orthogonal transvections. Let us denote the first isometry by \(\rho(q)\) and the second one by \(\mu(q)\). It can be verified that the elementary orthogonal transvections are orthogonal transvections on \(Q\). Indeed, consider \((u, v) = ((0, 1, 0), (0, 0, q))\) to get \(\rho(q)\) and consider \((u, v) = ((1, 0, 0), (0, 0, q))\) to get \(\mu(q, \beta)\).

The subgroup of \(\text{Trans}_O(Q, \langle \cdot \rangle)\) generated by elementary orthogonal transvections is denoted by \(\text{ETrans}_O(Q, \langle \cdot \rangle)\).

**Definition 4.8.** Let \(I\) be an ideal of \(R\). The group of relative orthogonal transvections to an ideal \(I\) is generated by the orthogonal transvections of the form \(\sigma(p) = p - \langle u, p \rangle v + \langle v, p \rangle u\), where either \(u \in IP\) or \(v \in IP\). The group generated by relative orthogonal transvections is denoted by \(\text{Trans}_O(P, IP, \langle \cdot \rangle)\).

**Definition 4.9.** Let \(I\) be an ideal of \(R\). The elementary orthogonal transvections of \(Q\) of the form \(\rho(q), \mu(q)\), where \(q \in IP\), are called relative elementary orthogonal transvections to an ideal \(I\).

The subgroup of \(\text{ETrans}_O(Q, \langle \cdot \rangle)\) generated by relative elementary orthogonal transvections is denoted by \(\text{ETrans}_O(Q, \langle \cdot \rangle)\). The normal closure of \(\text{ETrans}_O(Q, \langle \cdot \rangle)\) in \(\text{ETrans}_O(Q, \langle \cdot \rangle)\) is denoted by \(\text{ETrans}_O(Q, \langle \cdot \rangle)\).

**Remark 4.10.** Let \(P = \bigoplus_{i=1}^{2n} Re_i\) be a free \(R\)-module with \(R = 2R\). The non-degenerate symmetric bilinear form \(\langle \cdot, \cdot \rangle\) on \(P\) corresponds to a symmetric matrix \(\varphi\) with respect to the basis \(\{e_1, e_2, \ldots, e_{2n}\}\) of \(P\) and we write \(\langle p, q \rangle = p^t \varphi q\).

In this case the orthogonal transvection \(\tau(p) = p - \langle u, p \rangle v + \langle v, p \rangle u\), where either \(u \in IP\) or \(v \in IP\), corresponds to the matrix \((I_{2n} - vu^t \varphi + uv^t \varphi)\) and the group generated by them is denoted by \(\text{Trans}_O(P, IP, \langle \cdot \rangle)\).

Also in this case \(\text{ETrans}_O(Q, \langle \cdot \rangle)\) will be generated by the matrices of the form \(\rho_{\varphi}(q) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & q \varphi \\ -q \varphi & 0 & 1 \end{pmatrix}_{2n} \), and \(\mu_{\varphi}(q) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -q \varphi & 1 \end{pmatrix}_{2n} \).

Note that for standard symmetric matrix \(\tilde{\varphi}_n\) and for \(q = (q_1, \ldots, q_{2n}) \in R^{2n}\) with \(q^t \tilde{\varphi}_n q = 0\), we have

\[
\rho_{\tilde{\varphi}_n}(q) = \prod_{i=3}^{2n+2} o e_{i1}(-q_{i-2}), \tag{4}
\]

\[
\mu_{\tilde{\varphi}_n}(q) = \prod_{i=3}^{2n+2} o e_{i1}(-q_{\sigma_i(i-2)}). \tag{5}
\]

In the following four lemmas we shall use the assumptions and notations in the statement of Remark 4.10.

**Lemma 4.11.** Let \(R\) be a commutative ring with \(R = 2R\), and \(I\) be an ideal of \(R\). Let \((P, \langle \cdot, \cdot \rangle)\) be an orthogonal \(R\)-module with \(P\) free \(R\)-module of rank \(2n, n \geq 2\)
and $Q = R^2 \oplus P$ with the induced form on $\mathbb{H}(R) \oplus P$. If the symmetric bilinear form $(\cdot, \cdot)$ correspond (w.r.t. some basis) to $\tilde{\varphi}_n$, the standard symmetric matrix, then $\text{Trans}_O(Q, IQ, (\cdot, \tilde{\varphi}_{n+1}) = EO_{2n+2}(R, I)$.

Proof: For proof see §2 of [11].

**Lemma 4.12.** Let $R$ be a commutative ring with $R = 2R$, and let $I$ be an ideal of $R$. Let $(P, (\cdot, \cdot))$ be an orthogonal $R$-module with $P$ free $R$-module of rank $2n$, $n \geq 2$ and $Q = R^2 \oplus P$ with the induced form on $\mathbb{H}(R) \oplus P$. If the symmetric bilinear form $(\cdot, \cdot)$ correspond (w.r.t. some basis) to $\varphi_n$, the standard symmetric matrix, then $\mathcal{E}_{\text{Trans}}(Q, IQ, (\cdot, \varphi_{n+1}) = EO_{2n+2}(R, I)$.

Proof: We first show $\mathcal{E}_{\text{Trans}}(Q, IQ, (\cdot, \tilde{\varphi}_{n+1})$ is a subset of $EO_{2n+2}(R, I)$. An element of $\mathcal{E}_{\text{Trans}}(Q, IQ, (\cdot, \tilde{\varphi}_{n+1})$ is of the form $T_1(q)T_2(s)T_1(q)^{-1}$, where $q \in R^{2n}$, $s \in I^{2n} \subseteq R^{2n}$. Here $T_1$ and $T_2$ can be either of $\rho_{\tilde{\varphi}_n}$ or $\mu_{\tilde{\varphi}_n}$. Using equations (4) and (5) we show that either of the above elements belong to $EO_{2n+2}(R, I)$, and hence $\mathcal{E}_{\text{Trans}}(Q, IQ, (\cdot, \tilde{\varphi}_{n+1}) \subseteq EO_{2n+2}(R, I)$.

To show the other inclusion we recall that $EO_{2n+2}(R, I)$ is generated by the elements $g \circ \epsilon_i(x)g^{-1}$, where $g \in EO_{2n+2}(R), x \in I$, and either $i = 1$ or $j = 1$ (see Lemma 2,9). Using commutator relation $[\epsilon_i(ab), \epsilon_k(b)] = \epsilon_i(ab)$ and the equations (4), (5) we can show that $EO_{2n+2}(R, I) \subseteq \mathcal{E}_{\text{Trans}}(Q, IQ, (\cdot, \tilde{\varphi}_{n+1})$, and hence the equality is established. □

**Lemma 4.13.** Let $P$ be a free $R$-module of rank $2n$. Let $(P, (\cdot, \varphi))$ and $(P, (\cdot, \varphi^*)$ be two orthogonal $R$-modules with $\varphi = \varepsilon^t \varphi^* \varepsilon$, for some $\varepsilon \in GL_{2n}(R)$. Then

\[ \text{Trans}_O(P, (\cdot, \varphi)) = \varepsilon^{-1} \text{Trans}_O(P, (\cdot, \varphi^*) \varepsilon, \]

\[ \mathcal{E}_{\text{Trans}}(Q, (\cdot, \tilde{\varphi}_{1, \perp \varphi}) = (I_2 \perp \varepsilon)^{-1} \mathcal{E}_{\text{Trans}}(Q, (\cdot, \tilde{\varphi}_{1, \perp \varphi}) (I_2 \perp \varepsilon). \]

Proof: In the free case for orthogonal transvections we have

\[ (I_{2n} - vu^t \varphi + uv^t) = \varepsilon^{-1} (I_{2n} - \tilde{v}u^t \varphi^* - \tilde{u}v^t \varphi^*) \varepsilon, \]

where $\tilde{u} = \varepsilon u$ and $\tilde{v} = \varepsilon v$. Hence the first equality follows.

For elementary orthogonal transvections we have

\[ (I_2 \perp \varepsilon)^{-1} \rho_{\varphi^*}(q)(I_2 \perp \varepsilon) = \rho_{\varphi}(\varepsilon^{-1}q), \]

\[ (I_2 \perp \varepsilon)^{-1} \mu_{\varphi^*}(q)(I_2 \perp \varepsilon) = \mu_{\varphi}(\varepsilon^{-1}q), \]

hence the second equality follows. □

**Lemma 4.14.** Let $I$ be an ideal of $R$ and $P$ be a free $R$-module of rank $2n$. Let $(P, (\cdot, \varphi))$ and $(P, (\cdot, \varphi^*)$ be two orthogonal $R$-modules with $\varphi = \varepsilon^t \varphi^* \varepsilon$, for some $\varepsilon \in GL_{2n}(R)$. Then

\[ \text{Trans}_O(P, IP, (\cdot, \varphi)) = \varepsilon^{-1} \text{Trans}_O(P, IP, (\cdot, \varphi^*) \varepsilon, \]

\[ \mathcal{E}_{\text{Trans}}(Q, IQ, (\cdot, \tilde{\varphi}_{1, \perp \varphi}) = (I_2 \perp \varepsilon)^{-1} \mathcal{E}_{\text{Trans}}(Q, IQ, (\cdot, \tilde{\varphi}_{1, \perp \varphi}) (I_2 \perp \varepsilon). \]
Remark 4.18. In view of Proposition 4.15 and above lemma, for any split orthogonal inner product space has matrix $ETrans$ module $(\alpha)$. Then, there exists $\alpha$ with the induced form on Lemma 4.19.

Proof: Using Lemma 4.11 Lemma 4.12 and Lemma 4.14 we get,

$$\text{Trans}_O(Q, IQ, (\cdot, \cdot)_{\tilde{v}_1 \perp \varphi}) = (I_2 \perp \epsilon)^{-1} \text{Trans}_O(Q, IQ, (\cdot, \cdot)_{\tilde{v}_{n+1}}) (I_2 \perp \epsilon)$$

and

$$\text{ETrans}_O(Q, IQ, (\cdot, \cdot)_{\tilde{v}_1 \perp \varphi}) = (I_2 \perp \epsilon)^{-1} \text{ETrans}_O(Q, IQ, (\cdot, \cdot)_{\tilde{v}_{n+1}}) (I_2 \perp \epsilon)$$

and hence the equality is established.

Definition 4.16. An orthogonal module $(P, (\cdot, \cdot))$ over the ring $R$ is called split if there exists a submodule $N \subseteq P$ such that $N$ is a direct summand of $P$ and such that $N$ is precisely equal to its orthogonal complement $N^\perp = \{p \in P : (p, n) = 0 \text{ for all } n \in N\}$.

Moreover, an orthogonal module $(P, (\cdot, \cdot))$ over the ring $R$ is called locally split if $(P_m, (\cdot, \cdot))$ is a split orthogonal $R_m$-module for every maximal ideal $m$ of $R$.

Lemma 4.17. (See Lemma 6.3, Chapter I in [5]) Let $R$ be a ring such that every finitely generated projective module over $R$ is free. Then an inner product space over $R$ is split if and only if it possesses a basis so that the associated inner product matrix has the form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. If we also assume that 2 is a unit in the ring, then every split inner product space has matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ with respect to a suitable basis.

Remark 4.18. In view of Proposition 4.15 and above lemma, for any split orthogonal module $(P, (\cdot, \cdot))$ over a local ring $(R, m)$ with $R = 2R$, we have $\text{Trans}_O(Q, IQ, (\cdot, \cdot)_{\tilde{v}_1 \perp \varphi}) = \text{ETrans}_O(Q, IQ, (\cdot, \cdot)_{\tilde{v}_1 \perp \varphi})$. Here $I$ is an ideal of the ring $R$.

Next we establish dilation principle for relative elementary orthogonal transvection group.

Lemma 4.19. Let $R$ be a commutative ring with $R = 2R$, and let $I$ be an ideal of $R$. Let $(P, (\cdot, \cdot))$ be an orthogonal $R$-module with rank of $P$ is $2n$, $n \geq 2$, and $Q = R^2 \oplus P$ with the induced form on $\mathbb{H}(R) \oplus P$. Suppose that $a$ is a non-nilpotent element of $R$ such that $P_a$ is a free $R_a$ module, $(P_a, (\cdot, \cdot))$ is split orthogonal $R_a$-module, and the bilinear form $(\cdot, \cdot)$ corresponds to the symmetric matrix $\varphi$ (w.r.t. some basis). Let $\alpha(X) \in \text{Aut}(Q[X])$, with $\alpha(0) = Id$, and $\alpha(X) \in \text{ETrans}_O(Q_a[X], IQ_a[X], (\cdot, \cdot)_{\tilde{v}_1 \perp \varphi})$. Then, there exists $\alpha^*(X) \in \text{ETrans}_O(Q[X], IQ[X], (\cdot, \cdot))$, with $\alpha^*(0) = Id$, such that $\alpha^*(X)$ localises to $\alpha(bX)$, for $b \in (a^d)$, $d \gg 0$. 

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Proof: We have $P_a \cong R^{2n}_a$. Let $e_1, \ldots, e_{2n+2}$ be the standard basis of $Q_a$ with respect to which the bilinear form on $Q_a$ will correspond to $\tilde{\psi}_1 \perp \varphi$. Since $(P_a, \langle \cdot, \cdot \rangle)$ is a split orthogonal $R$-module with $R_a = 2R_a$, we have $\varphi = \varepsilon^\ell \tilde{\psi}_n \varepsilon$, for some $\varepsilon \in \text{GL}_{2n}(R_a)$ by Lemma 4.17. Therefore, $\text{ETranso}(Q_a[X], IQ_a[X], \langle \cdot, \cdot \rangle \tilde{\psi}_1 \perp \varphi) = (I_2 \perp \varepsilon)^{-1} \text{EO}_{2n+2}(R_a[X], I_a[X]) (I_2 \perp \varepsilon)$ by Lemma 4.12 and Lemma 4.14. Hence, $\alpha(X)_a = (I_2 \perp \varepsilon)^{-1} \beta(X) (I_2 \perp \varepsilon)$, for some $\beta(X) \in \text{EO}_{2n+2}(R_a[X], I_a[X])$, with $\beta(0) = Id$. By Lemma 2.12 we have

$$\text{EO}_{2n+2}(R_a[X], I_a[X]) = \text{EO}_{2n+2}^1(R_a[X], I_a[X]) \cap \text{O}_{2n+2}(R_a[X], I_a[X]).$$

Hence we can write $\beta(X) = \prod \gamma_i \text{oc}_{i,j_i}(X f_i(X)) \gamma_i^{-1}$, where either $i_1 = 1$, or $j_1 = 1$, and $\gamma_i \in \text{EO}_{2n+2}^1(R_a, I_a)$. Note that $f_i(X) \in R_a[X]$, when $i_1 = 1$ and $f_i(X) \in I_a[X]$, when $j_1 = 1$. Using Lemma 3.1 we get $\beta(Y^d X) = \prod \text{oc}_{i,j} Y h_k(X, Y, Y')/a^m$, with either $i_k = 1$ or $j_k = 1$. Note that $h_k(X, Y) \in R[X, Y]$, when $i_k = 1$ and $h_k(X, Y) \in I[X, Y]$, when $j_k = 1$. We have

$$\text{oc}_{i,j_k}(Y h_k(X, Y)/a^m) = I_{2n+2} - (-Y h_k(X, Y)/a^m) e_1 e_{\sigma(j_k)} \tilde{\psi}_{n+1} + (-Y h_k(X, Y)/a^m) e_1 e_{\sigma(j_k)} \tilde{\psi}_{n+1}, \text{ for } j_k \geq 3,$$

$$\text{oc}_{i,j_k}(Y h_k(X, Y)/a^m) = I_{2n+2} - (-Y h_k(X, Y)/a^m) e_1 e_{\sigma(j_k)} \tilde{\psi}_{n+1} + (-Y h_k(X, Y)/a^m) e_2 e_{\sigma(j_k)} \tilde{\psi}_{n+1}, \text{ for } i_k \geq 3.$$

Let $\varepsilon_1, \ldots, \varepsilon_{2n}$ be the columns of the matrix $\varepsilon \in \text{GL}_{2n}(R_a)$. Let $\tilde{e}_i, \tilde{e}_i^t$ denote the column vector $(I_2 \perp \varepsilon) e_i$ of length $2n + 2$. Note that $\tilde{e}_1 = e_1, \tilde{e}_2 = e_2$, and $\tilde{e}_i = (0, 0, \tilde{e}_{i-2})$, for $i \geq 3$. Using Lemma 4.13 we can write $\alpha(Y^d X)_a$ as product of elements of the form

$$I_{2n+2} - (-Y h_k(X, Y)/a^m) \tilde{e}_i \tilde{e}_i^t = \mu_\varphi((Y h_k(X, Y)/a^m) \tilde{\psi}_{n+1} \tilde{e}_1 \tilde{e}_1^t \phi_0 \phi) + (-Y h_k(X, Y)/a^m) \tilde{e}_2 \tilde{e}_2^t \phi_0 \phi,$$

$$I_{2n+2} - (-Y h_k(X, Y)/a^m) \tilde{e}_i \tilde{e}_i^t = \rho_\varphi(-Y h_k(X, Y)/a^m) \tilde{\psi}_{n+1} \tilde{e}_1 \tilde{e}_1^t \phi_0 \phi + (-Y h_k(X, Y)/a^m) \tilde{e}_2 \tilde{e}_2^t \phi_0 \phi.$$

for $i_k, j_k \geq 3$. Note that $\alpha(Y^d X)_a \in \text{ETranso}(Q_a[X, Y], IQ_a[X, Y], \langle \cdot, \cdot \rangle \tilde{\psi}_1 \perp \varphi)$, hence $\alpha(Y^d X)_a = id \text{ mod } (IQ_a[X, Y])$. Since $\rho_\varphi$ and $\mu_\varphi$ satisfy the splitting property $\rho_\varphi(q_1 + q_2) = \rho_\varphi(q_1) \rho_\varphi(q_2)$ and $\mu_\varphi(q_1 + q_2) = \mu_\varphi(q_1) \mu_\varphi(q_2)$, we get $\alpha(Y^d X)_a$ is product of elements of the form $T_1(Y f_k(X, Y)/a^m) \tilde{e}_k T_2(Y g_k(X, Y)/a^m) \tilde{e}_p$, where $T_1, T_2$ are either $\rho_\varphi$ or $\mu_\varphi$, $f_k(X, Y) \in R[X, Y]$, $g_k(X, Y) \in I[X, Y]$, and $p, q, k \geq 3$.

Let $s \geq 0$ be an integer such that $\varepsilon_i = a^s \varepsilon_i \in P$ for all $i = 1, \ldots, 2n$. Let $d = s + m$. Therefore $\alpha((a^d Y)^d X)_a$ is product of elements of the form $T_1((a^d Y) f_k(X, a^d Y)/a^m) \tilde{e}_p)$ $T_2((a^d Y g_k(X, a^d Y)/a^m) \tilde{e}_q) T_1(-(a^d Y f_k(X, a^d Y)/a^m) \tilde{e}_p)$ Substituting $Y = 1$ we get $\alpha(a^d X)_a$ is product of elements of the forms

$$T_1(a^s f_k(X) \tilde{e}_p) T_2(a^s g_k(X) \tilde{e}_q) T_1(-(a^m f_k(X) \tilde{e}_p).$$
Let us set \( \alpha^*(X) \) to be the product of elements of the forms
\[
T_1(f'_k(X)\varepsilon_{p_k})T_2(g_k(X)\varepsilon_{q_k})T_1(-f'_k(X)\varepsilon_{p_k}).
\]
From the construction it is clear that \( \alpha^*(X) \) belongs to ETrans\(_O\)(\(Q[X], IQ[X], \langle \cdot, \cdot \rangle\)), \(\alpha^*(0) = Id\), and \(\alpha^*(X)\) localises to \(\alpha(bX)\), for some \(b \in (\alpha^d)\), \(d \gg 0\).

**Lemma 4.20.** Let \(R\) be a commutative ring with \(R = 2R\), and let \(I\) be an ideal of \(R\). Let \((P, \langle \cdot, \cdot \rangle)\) be a locally split orthogonal \(R\)-module with \(P\) is of rank \(2n\), \(n \geq 2\), and \(Q = R^2 \oplus P\) with the induced form on \(\mathbb{H}(R) \oplus P\). Let \(\alpha(X) \in O(Q[X])\), with \(\alpha(0) = Id\). If \(\alpha(X)_m \in \text{ETrans}_O(Q_m[X], IQ_m[X], \langle \cdot, \cdot \rangle)\) for each maximal ideal \(m\) of \(R\), then \(\alpha(X) \in \text{ETrans}_O(Q[X], IQ[X], \langle \cdot, \cdot \rangle)\).

Proof: One can suitably choose an element \(a_m\) from \(R\setminus m\) such that \(\alpha(X)_m \in \text{ETrans}_O(Q_m[X], IQ_m[X])\). Let us set \(\gamma(X,Y) = \alpha(X + Y)\alpha(Y)^{-1}\). Note that \(\gamma(X,Y)\) belongs to \(\text{ETrans}_O(Q_m[X], IQ_m[X], \langle \cdot, \cdot \rangle)\), and \(\gamma(0,Y) = Id\). From Lemma 4.19 it follows that \(\gamma(b_m X, Y) \in \text{ETrans}_O(Q[X,Y], IQ[X,Y])\), for \(b_m \in (a^d_m)\), where \(d \gg 0\). Note that the ideal generated by \(a^d_m\)'s is the whole ring \(R\). Therefore, \(c_1a^d_{m_1} + \cdots + c_k a^d_{m_k} = 1\), for some \(c_i \in R\). Let \(b_m = c_i a^d_{m_i} \in (a^d_{m_i})\). It is easy to see that \(\alpha(X) = \prod_{i=1}^{k-1} \gamma(b_{m_i}X, T_i)\gamma(b_{m_i}X, 0)\), where \(T_i = b_{m_i}X + \cdots + b_{m_k}X\). Each term in the right hand side of this expression belongs to \(\text{ETrans}_O(Q[X], IQ[X])\) and hence \(\alpha(X) \in \text{ETrans}_O(Q[X], IQ[X])\).

We now establish equality of the orthogonal transvection group and the elementary orthogonal transvection group (in the relative case to an ideal) of a locally split orthogonal \(R\)-module with \(R = 2R\). An absolute version of this result (i.e, when \(I = R\)) was proved in [2] (see Theorem 3.10). Before proving the main result we establish a lemma to show that orthogonal transvections are homotopic to identity.

**Lemma 4.21.** Let \((P, \langle \cdot, \cdot \rangle)\) be an orthogonal \(R\)-module and \(\alpha \in \text{Trans}_O(P, \langle \cdot, \cdot \rangle)\). Then there exists \(\beta(X) \in \text{Trans}_O(P[X], \langle \cdot, \cdot \rangle)\) such that \(\beta(1) = \alpha\) and \(\beta(0) = Id\).

Proof: As \(\alpha \in \text{Trans}_O(P, \langle \cdot, \cdot \rangle)\), it is product of orthogonal transvections of the form \(\tau\), where \(\tau\) takes \(p \in P\) to \(p - \langle u, p \rangle v + \langle v, p \rangle u\), where \(u, v \in P\) are isotropic with \(\langle u, v \rangle = 0\), and either \(u\) or \(v\) is unimodular. Define \(\tau X\) as the map which takes \(p \in P\) to either \(p - \langle u, p \rangle vX + \langle v, X \rangle p\) or \(p - \langle uX, p \rangle v + \langle u, p \rangle uX\). This choice depends on whether \(u\) is unimodular or \(v\) is unimodular. Note that \(uX\) represents \(u\) times \(X\) and \(vX\) represents \(v\) times \(X\). Also, note that \(uX, vX \in P[X]\). We set \(\beta(X)\) to be the product of elements of the form \(\tau X\), whenever \(\tau\) appears in the expression of \(\alpha\). Then \(\beta(1) = \alpha\) and \(\beta(0) = Id\).

**Theorem 4.22.** Let \(R\) be a commutative ring with \(R = 2R\), and let \(I\) be an ideal of \(R\). Let \((P, \langle \cdot, \cdot \rangle)\) be a locally split orthogonal \(R\)-module with \(P\) is of rank \(2n\), \(n \geq 2\), and \(Q = R^2 \oplus P\) with the induced form on \(\mathbb{H}(R) \oplus P\). Then \(\text{Trans}_O(Q, IQ, \langle \cdot, \cdot \rangle) = \text{ETrans}_O(Q, IQ, \langle \cdot, \cdot \rangle)\).

Proof: We have \(\text{ETrans}_O(Q, IQ, \langle \cdot, \cdot \rangle) \subseteq \text{Trans}_O(Q, IQ, \langle \cdot, \cdot \rangle)\). We need to show other inclusion. Let us choose \(\alpha\) from \(\text{Trans}_O(Q, IQ, \langle \cdot, \cdot \rangle)\). By Lemma 4.21 there
exists \( \alpha(X) \) in \( \text{Trans}_O(Q[X], IQ[X], \langle \cdot, \cdot \rangle) \) such that \( \alpha(1) = \alpha \) and \( \alpha(0) = \text{Id} \). Note that \( \text{Trans}_O(Q_m[X], IQ_m[X], \langle \cdot, \cdot \rangle) \) \( \tilde{\psi}_1 \perp \phi_m \) = \( E\text{Trans}_O(Q_m[X], IQ_m[X], \langle \cdot, \cdot \rangle) \), for each maximal ideal \( m \) of \( R \) (follows from Remark 4.18). Hence \( \alpha(X)_m \) belongs to \( E\text{Trans}_O(Q_m[X], IQ_m[X], \langle \cdot, \cdot \rangle) \), for each maximal ideal \( m \) of \( R \). Therefore, \( \alpha(X) \in \text{Trans}_O(Q[X], IQ[X], \langle \cdot, \cdot \rangle) \) (see Lemma 4.20). Substituting \( X = 1 \) we get the result. \( \square \)

In closing we make Remark 4.24 below for which we need the following elementary observation.

**Lemma 4.23.** Let \( (P, \langle \cdot, \cdot \rangle) \) be a split orthogonal \( R \)-module. Then \( (P_m, \langle \cdot, \cdot \rangle) \) is a split orthogonal \( R_m \)-module for every maximal ideal \( m \) of \( R \).

**Proof:** Let us consider an equivalent form of the definition of split orthogonal \( R \)-modules as it is stated in §6, Chapter I, [9]. The orthogonal module \( (P, \langle \cdot, \cdot \rangle) \) is split if it is direct sum of two submodules \( M \) and \( N \) which are dually paired to \( R \) by the inner product,

\[
M \xrightarrow{\cong} \text{Hom}_R(N, R), \quad \text{and} \quad N \xrightarrow{\cong} \text{Hom}_R(M, R)
\]

and such that \( N \) is self orthogonal, i.e., \( \langle N, N \rangle = 0 \). Tensoring with \( R_m \) we get \( P_m = M_m \oplus N_m \). Moreover, \( P \) projective will imply both \( M \) and \( N \) are projective and hence finitely presented (\( M \) finitely presented means there exists an exact sequence \( R^k \rightarrow R^l \rightarrow M \), for suitable natural numbers \( k, l \)). Therefore, by Proposition 2.13” in Chapter I, [8] we get

\[
M_m \xrightarrow{\cong} \text{Hom}_{R_m}(N_m, R_m), \quad \text{and} \quad N_m \xrightarrow{\cong} \text{Hom}_{R_m}(M_m, R_m)
\]

Also, \( N \) is self orthogonal will imply \( N_m \) is self-orthogonal and hence \( (P_m, \langle \cdot, \cdot \rangle) \) is a split orthogonal \( R_m \)-module for every maximal ideal \( m \) of \( R \). \( \square \)

**Remark 4.24.** In view of the above lemma the result as in Theorem 4.22 holds when \( (P, \langle \cdot, \cdot \rangle) \) is assumed to be a split orthogonal \( R \)-module.

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