Finding Singularities in Gravitational Lensing

Ashish Kumar Meena,1⋆ J. S. Bagla,1†

1Indian Institute of Science Education and Research Mohali, Knowledge City, Sector 81, Sahibzada Ajit Singh Nagar, Punjab 140306, India

ABSTRACT
The number of strong lens systems is expected to increase significantly in ongoing and upcoming surveys. With an increase in the total number of such systems we expect to discover many configurations that correspond to unstable caustics. In such cases, the instability can be used to our advantage for constraining the lens model. We have implemented algorithms for detection of different types of singularities in gravitational lensing. We validate our approach on a variety of lens models and then go on to apply it to the inferred mass distribution for Abell 697 as an example application. We propose to represent lenses using A3-lines and singular points (A4 and D4) in the image plane. We propose this as a compact representation of complex lens systems that can capture all the details in a single snapshot.

Key words: gravitational lensing: strong

1 INTRODUCTION

Strong gravitational lenses are unique probes of the Universe. By producing multiple images, they provide constraints on the lens mass distribution (Kneib & Natarajan 2011). The high magnification due to lensing gives us the opportunity to look further into the history of the Universe by observing magnified sources which otherwise would have remained unobserved, see e.g., Atek et al. (2018). In a given lensing system, the observed configuration and magnification of multiple images depends on properties of the lens and the location of the source with respect to the lens. The set of all points in the plane of the lens is called the image plane: here we are working in the small angle approximation. Each point on the image plane can be mapped to a plane at the source redshift, the so called source plane. For a given lens, and the distance to the source, there is a set of directions where the magnification is formally infinite. The set of points on the image plane representing these directions form the critical curves. As all sources have a finite size, magnification is always finite. The critical curves, mapped to the source plane form the caustics. High magnification images are formed if the source lies on or close to a caustic (Blandford et al. 1989; Schneider et al. 1992).

We have mentioned above that the critical curves and caustics correspond to infinite magnification. This happens because the lens mapping at these points is singular: a finite solid angle element in the image plane gets mapped to a line or a point in the source plane. The structure of the caustic depends on the form of the singularity: singularities of the lensing map can be classified using catastrophe theory (Berry & Upstill 1980; Poston & Stewart 1978; Gilmore 1981). The use of catastrophe theory in gravitational lensing was first discussed by Blandford & Narayan (1986) in case of elliptical lenses. Later this was discussed by Nityananda (1990), Kassiola et al. (1992), Schneider et al. (1992) (hereafter SEF) and Petters et al. (2001). Independently, classification of singularities in the same map in the context of Zel’Dovich approximation was done by Arnold et al. (1982).

One can divide singularities of the lensing map into two types: stable (fold and cusp) and unstable singularities (swallowtail, umbilics). Stable caustics are called so because a small perturbation in the lensing potential leads to a correspondingly small shift in the location of fold and cusp. On the other hand, the so called unstable caustics may disappear entirely on introduction of a small perturbation. In view of this, the focus of most of the studies has been on stable caustics with only a few efforts to improve our understanding of image formation and characteristics of unstable singularities in realistic lens maps (Bagla 2001; Xivry et al. 2009) though these caustics have been known and studied theoretically (Schneider et al. 1992).

In this work we propose that unstable caustics can be potentially useful to constrain lens models much more strongly than the stable caustics. The unstable caustics have a stronger variation of magnification around the singular points as compared to stable caustics. Further, if we can predict the location of unstable singularities in the image plane then these regions may be targeted for deep surveys to look for highly magnified sources (Yuan et al. 2012; Zheng et al. 2012; Coe et al. 2013; McLeod et al. 2015; Ebeling et al.)
The high magnification comes with a characteristic image formation, and due to the unstable nature of the singularity the characteristic image formation is visible only for a small range of source redshift. With upcoming facilities like EUCLID (Laurejs 2009), LSST (Ivezić, et al. 2008), Dark Energy Survey (DES) (Dark Energy Survey Collaboration, et al. 2016), JWST (Gardner, et al. 2006), the number of strong lenses will increase by more than an order of magnitude in the next decade. Thus the possibility of observing lensing near unstable singularities is higher and therefore it is timely that we carry out a detailed study. Preliminary results of this study were reported in Bagla (2001). We use algorithms described briefly in that work. We have developed and refined these algorithms further, and validated results of this study were reported in Bagla (2001). We use the definitions of singularities, e.g., see (Arnold et. al. 1982) and are similar to those reported in Hidding et. al. (2014) for the case of Zel’dovich approximation in two dimensions. These algorithms allow us to locate all singularities of the lensing map in the image plane starting from the lensing potential. We then proceed to analyse lens models with one or two major components and study the singularities. We also study variation in singularities in presence of perturbing shear. We illustrate characteristic image formations for each type of unstable singularity. This effort is complementary to an atlas of observed images in exotic lenses (Xivry et al. 2009) and makes the task of predicting possibility of such image formations much easier.

This paper is organized as follows. In §2 we review the basics of the gravitational lensing and introduce the quantities that are useful for the following discussion. In §3 we review the classification of singularities and their properties. §4 contains a description of the algorithm used. Results are given in §5 for a variety of lenses. Summary and conclusions are presented in §6. We discuss possibilities for future work in this section.

2 THEORY

In this section we review the basics of gravitational lensing that are relevant for the following discussion. We use the formalism given in SEF. This is followed by an introduction to singularities in gravitational lensing.

The lens equation is a map between the image plane and the source plane. It can be written in a dimensionless form as:

$$y = x - \alpha(x),$$

where $x$ (in the image plane) and $y$ (in the source plane) are two-dimensional vectors with respect to the optic axis. The choice of optic axis is arbitrary. And $\alpha(x)$ is the scaled deflection angle for a light ray in lens plane at x. The scaled deflection angle $\alpha(x)$ is related to the projected lensing potential $\psi(x)$: $\alpha(x) = (D_{ds}/D_s)\nabla\psi(x)$. The projected lensing potential is given by,

$$\psi(x) = \frac{1}{\pi} \int d^2x' \kappa(x') \ln |x - x'|,$$

where

$$\kappa = \frac{\Sigma(x)}{\Sigma_{cr}}, \quad \Sigma_{cr} = \frac{c^2}{4\pi GD_d}.$$
Figure 1. Evolution of caustics and critical lines around a swallowtail singularity. The left column (A1, B1, C1) shows the caustics in source plane for three different redshift including redshift at which swallowtail singularity becomes critical (panel B1). The middle column (A2, B2, C2) shows the corresponding critical lines and the singularity map including A3-lines (red and dark green lines), swallowtail (violet point), hyperbolic umbilics (blue points). And the right column (A3, B3, C3) shows the image formation.

Plane is no longer one to one, and an infinitesimal solid angle in the image plane maps to zero solid angle in the source plane. There are two stable singularities: fold and cusp, and these are present in all situations when we have formation of multiple images. Thus, in general, a caustic in source plane represents cusps connected by folds. The corresponding curve in the image plane is called the critical curve and is expected to be smooth. In the following discussion, we will encounter other singularities, e.g., beak-to-beak, swallowtail, elliptic and hyperbolic umbilics, but these are not stable. These occur only for specific source redshifts with specific lens parameters Schneider et. al. (1992); Bagla (2001). At these unstable singularities, cusps are either created or destroyed or there is an exchange of cusp between radial and tangential caustics in such a way that the total number of the cusps in source plane always remains even. All these unstable singularities are point singularities and have characteristic image formations.

The classification of all these singularities is based on catastrophe theory. In the context of lensing, catastrophe theory describes the singularities in terms of derivatives of the Fermat potential, $\phi(x, y)$ (e.g. SEF, Petters et. al.)
Figure 2. Evolution of caustics and critical lines around a hyperbolic umbilic (purse). The left column (A1, B1, C1) shows the caustics in source plane for three different redshift including redshift at which purse singularity becomes critical (panel B1). The middle column (A2, B2, C2) shows the corresponding critical lines and the singularity map including $A_3$-lines (red and dark green lines) and purse (blue point). And the right column (A3, B3, C3) shows the image formation. One can notice the exchange of the cusp between radial and tangential caustics (panel B1) and the ring shaped image formation (panel B3) at hyperbolic umbilic. Kindly note that as the umbilics are symmetric, the image formation about either one will be the same apart from a reflection. Here we show images corresponding to one of the umbilics, as marked by the source position in the left column.

\begin{equation}
\phi(x, y) = \text{Const.} \left( \frac{1}{2} (x - y)^2 - \psi(x) \right).
\end{equation}

(2001)), which is related to the lensing potential $\psi(x)$ as,

\begin{equation}
\phi(x, y) = \text{Const.} \left( \frac{1}{2} (x - y)^2 - \psi(x) \right).
\end{equation}

Instead of using Fermat potential one can also use deformation tensor (which is completely determined by its eigenvalues and eigenvectors) to discuss different singularities that occur in gravitational lensing. In this way, one does not have to worry about the source parameters, which affects the Fermat potential. The benefit of using deformation tensor instead of Fermat potential is that one does not have to draw critical lines and caustics for all possible source redshift in order to find highly magnified regions in the image plane. And the study of deformation tensor gives a singularity map (in the lens plane) of all possible singularities that can occur
Figure 3. Evolution of caustics and critical lines near an elliptic umbilic (denoted by blue point in singularity map). At elliptic umbilic triangular shaped caustic corresponding to the tangential caustic (panel A1) goes to a point caustic (panel B1) and emerge as a triangular shaped caustic corresponding to radial caustic (panel C1). The corresponding image formation shows a Y-shaped seven image configuration.

for a given lens model for all source redshifts. In our analysis we do not focus on the source redshift but list all the singularities of the map. This approach enables us to focus on these singularities and attempt statistical analysis in typical lens models. This also brings in its own limitations: we are unable to discuss folds as these have an explicit source redshift dependence. This however can be recovered without much work after the singularities have been mapped.

3.1 $A_3$-lines

$A_3$-lines are the essential elements of the singularity map for a given lens model. In the image plane, these are the lines on which cusps form. As all point singularities are associated with creation, destruction or exchange of cusps, our first goal is to identify the $A_3$-lines for a lensing potential.

In the image plane $A_3$-lines pass through the points where the gradient of the eigenvalue of the deformation tensor is orthogonal to the corresponding eigenvector $n_1$,

$$n_1, \nabla_x \lambda = 0.$$  

(10)
Which implies that at $A_3$-lines the eigenvector $n_1$ is tangent to the corresponding eigenvalue contour (Arnold et al. 1982; Hidding et al. 2014). The reader may note that this is also true at points where the eigenvalues have extrema, however such points are isolated. At generic points along these lines an infinitesimal portion of the critical curve, which essentially is a contour level for the eigenvalue, is mapped onto itself as we go from the image plane to the source plane.

In case of a spherically symmetric lens, every point in the lens plane satisfies equation (10). As a result, a spherical symmetric lens gives a formation of point caustic in source plane (SEF) at any point.

In general, we observe two different sets of $A_3$-lines in lens plane, one for each eigenvalue of the deformation tensor. The points in the lens plane where $A_3$-lines and the corresponding eigenvalue contour (with $\alpha$ or $\beta = 1/\alpha$) cross each other correspond to the cusp singularities in source plane at that redshift.

These lines do not intersect each other though as we shall see, lines corresponding to the two eigenvalues can meet at degenerate points ($\alpha = \beta$). The presence of $A_3$-lines itself proves the stability of cusp singularities in lens mapping: changing the redshift of the source plane merely shifts the cusp to a neighbouring point.
3.2 Swallowtail Singularities

The characteristic image formation for a swallowtail singularity is an elongated arc. This arc is made up of four images. As we move away from swallowtail singularity the arc changes into multiple images. At a swallowtail singularity, the number of cusps in source plane change by two. In lens plane, swallowtail singularities mark the points where eigenvector $n_3$ of deformation tensor is tangent to the corresponding $A_3$-line. Which implies that at a swallowtail singularity the corresponding eigenvalue $\lambda$ reaches a local maxima along $A_3$-lines, but this is not a true local maxima Arnold et. al. (1982); Hidding et. al. (2014). We use this method to identify swallowtail singularity in lens maps.

Figure (1) illustrates the caustics and critical curves in source and lens plane around a redshift at which a swallowtail singularity becomes critical. The lens model used here is a two-component softened elliptical isothermal lens. The first column shows the formation of tangential (radial) caustics, denoted by thick (thin) lines, in the source plane for three different redshifts including redshift $z_s$, at which the swallowtail singularity becomes critical (panel B1). The second column shows the corresponding critical curves and the singularity map consisting of $A_3$-lines (red for $\alpha$ and dark green for $\beta$ eigenvalue) and other singularities in the lens plane. Position of the swallowtail singularity is denoted by a violet point on the $A_3$-line: this is the point where the $A_3$
line is tangential to the critical curve. The blue points denote
the position of hyperbolic umbilics, discussed in the follow-
ing subsection. The third column shows the corresponding
image formation in lens plane for a given source position in
source plane. To see the multiple image formation, we take
a circular source: a different color in each quadrant. Such a
multi-color source is helpful to recognize positive and nega-
tive parity images. The source is shown in the source plane
in panels in the left column. A circle is plotted around the
source for easy localisation, this circle is not used in the
lensing map. The top-left panel (A1) shows the caustics for
a redshift smaller than the \( z_{\text{cr}} \) with a circular source lying
outside to both caustics. In the lens plane (top-right panel
(A3)), we observe a single distorted image. As the source
redshift is set to \( z_{\text{cr}} \) (panel (B1)), we can see a kink (origin
of two extra cusps) in the tangential caustic near the source
position. In panel (B1), the centre of the source in source
plane lies on this kink. In the corresponding lens plane (B2)
swallowtail singularity three vectors: tangent to the A1-
tangent, tangent to the eigenvalue contour and the local eigen-
vector are parallel to each other. The corresponding image
formation (B3) shows formation of a tangential arc made
of four images. The magnification factor \((\mu(r))\) around
a swallowtail singularity is proportional to \( r^{-3/2} \), where \( r \)
is the distance from the singular point. Whereas in the case
of fold (cusp), the magnification factor proportional to the
\( r^{-1/2} \) (\( r^{-2/3} \)). Hence, the slope of the magnification factor
around the swallowtail singularities is steeper than fold and
cusp (Arnold et al. 1982).

As we further increase the source redshift (C1) newly
formed cusps in source plane move away from each other and
the corresponding arc in lens plane (C3) become more
stretched. One can see that the arc in the lens plane is made
of four images, two of them have positive parity and two
of them have negative parity. Due to the finite size of the
source, the images shown here are merging into one another.
And the image on the upper left corner has positive par-
ity. Eventually, the gradual increment in the source redshift
changes the arc into four individual images. Formation of
such giant arcs around swallowtail singularities has been al-
ready encountered in investigations of strong lens systems,
see, e.g., Saha et al. (1998); Suyu & Halkola (2010).

### 3.3 Umbilics

For a given lens model, the presence of umbilics in the corre-
sponding singularity map denote the points with zero shear
(\( \gamma \)) in lens plane. At these points, both of the eigenvalues
of the deformation tensor are equal to each other (\( \alpha = \beta \)).
The dependence on both eigenvalues simultaneously sepa-
rates these singularities from the A1-lines and swallowtail
singularities, which have dependency on one eigenvalue in
their definitions. The equality of both eigenvalues implies
that at umbilics, eigenvectors of the deformation tensor are
degenerate. As a result, any vector at these points can be-
have as an eigenvector. We can always choose the eigenvector
in such a way that A3-line condition is always satisfied (for
a quantitative analysis see Hidding et. al. (2014)). At these
points A3-lines corresponding to different eigenvalues meet
with each other. There are two types of umbilics present in
gravitational lens mapping: elliptic and hyperbolic umbil-
ics. This division of the umbilics depends on the sign of the
quantity \( s_D \).

\[
\begin{align*}
s_D &= \left( t_{11}^2 t_{22}^2 - 3 t_{12}^2 t_{22} - 6 t_{11} t_{12} t_{22} t_{12} + 4 t_{11} t_{22}^2 + 4 t_{22} t_{12} t_{12} \right),
\end{align*}
\]  

where \( t_{ij} \) is positive, the singularity is called
hyperbolic umbilic and if it is negative then the singularity
is elliptic umbilic. At umbilics, the number of cusps in the
source plane remains unchanged but there is an exchange
of one or three cusps between tangential and radial caustics
depending on the type of the umbilic. In case of a hyper-

bolic umbilic, one cusp is exchanged between the tangential
and the radial caustic: in the image plane an A3-line corre-
sponding to each of the two eigenvalues meet at this point.
Whereas three A3-lines of each of the two eigenvalues meet
at the elliptic umbilic in the image plane, and three cusps
are exchanged between the tangential and the radial caustic
in the source plane.

In order to discuss the evolution of the caustics and crit-
ical curves near a hyperbolic umbilic (because of the simplic-
ity of its singularity map) we use a one-component elliptical
lens. The evolution of caustics and critical curves near a hy-
perbolic umbilic is shown in figure 2. The A3-lines in the
singularity map (middle column) are denoted by red and
dark green lines for two different eigenvalues. The positions
of hyperbolic umbilic in lens plane is denoted by blue points,
at which two A3-lines (one for \( \alpha \) and one for \( \beta \) eigenvalue)
meet with each other. For a redshift smaller than the red-
shift at which hyperbolic umbilic becomes critical, \( z_{\text{cr}} \)
both (radial and tangential) caustics in source plane each
have two cusps (A1). As we increase the source redshift to \( z_{\text{cr}} \),
there is an exchange of cusp from radial caustic to tangential
dagnostic (panel B1) (For the single component elliptical lens
model, because of the symmetry of the lens model, both of
the hyperbolic umbilics become critical at the same redshift.
The symmetry is broken in presence of a second component
or shear.). The corresponding image formation (panel B3)
shows a single demagnified image with positive parity and an
loop formed by four images, two of them with positive par-
ity and two of them with negative parity. As we increase the
source redshift further, source plane has a diamond shaped
tangential caustic and a smooth radial caustic (panel C1)
and in lens plane the highly magnified ring shaped image
changes into four individual less magnified images (panel
C3). The ring and the cross (for higher redshifts) is not cen-
tered at the lens centre but is off centre. We have studied
the location of the ring by varying the mass profile of the
lens and we find that the ring is located where the projected
surface density begins to drop sharply. The magnification
factor \( |\mu| \) falls as \( r^{-3} \) around both umbilics as one moves
away from the singular point. Thus magnification factor falls
much more rapidly around umbilics than other singularities.
So far only one lens system (Abell 1703) with image forma-
tion near a hyperbolic umbilic has been seen (Xivry et. al.
2009).

Unlike the hyperbolic umbilic, at elliptic umbilic, there
are six A3-lines (three each for each of the two eigenval-
ues of the deformation tensor) meet with each other. For
an illustration, formation of an elliptic umbilic in case of
a two-component elliptical lens model is shown in the fig-
ure (3). We find that often, two of three A3-lines of one or
both eigenvalues form a small closed loop. This can be seen
in examples shown in figure 4. In panel (A1), we only see tangential caustics, and the source lies inside the triangular shaped caustic. Panel (A3) shows the characteristic image formation (seven images in a shape of Y) near an elliptic umbilic. The central image has positive parity. The next three images from the central image have negative parity. And the three outer images again have positive parity. As we increase the source redshift, the size of the triangular shaped caustic decrease and at the same time, it moves away from the source position. At a redshift \( z_p \), where elliptic umbilic become critical it become a points caustic (panel B1) and the source lies close to this point caustic. The corresponding images still form a Y-shaped structure in lens plane but with only five images. As we further increase the source redshift, the point caustic turns into a triangular shaped radial caustic (panel C3). Which implies that at elliptic umbilic there is an exchange of three cusps between tangential and radial caustic. In panel (C1), we moved the source inside the triangular caustic, to see whether it still gives a Y-shaped image formation. We get a different kind of image formation with central image rotated by \( \pi/2 \).

Figure 3, shows the singularity map close to the elliptic umbilic (shown by blue point). Swallowtail singularities are shown as violet points. The complete singularity map for figure 3 is given in figure 4 (panel A5).

4 ALGORITHM

We briefly discuss the algorithm used to find out the singularities for a given lens model, we focus on singularities that are discussed in above section. We set up a uniform grid in the lens plane for calculations of physical quantities in order to locate the singularities. The grid-size depends on the resolution required for the lens model, in general we require adequate resolution as we are dealing with non-linear combinations of second derivatives of the lensing potential, even the smallest features should be well resolved on the grid. We use finite difference methods to compute derivatives on the grid. To calculate the position of the umbilics in the lens plane, and intersections of these curves give us umbilics. We can classify the type of umbilics by counting the number of \( A_3 \)-lines that converge at this point.

5 RESULTS

We validate our algorithm by applying it to a single component lens where the potential can be expressed in a closed form and the image structure has been studied in detail. Then we apply it to multi-component lenses and study a variety of configurations. Lastly we apply it to one real lens model to validate the method. As mentioned above in subsection 3.1, this technique does not work in case of isolated spherically symmetric lenses because of the absence of cusp formation. We study elliptical lenses with one and two components. In case of one-component elliptical lens one get two \( A_3 \)-lines and two hyperbolic umbilics. Whereas for two-component lens the location of \( A_3 \)-lines and other singularities in lens plane depend on the lens configuration.

We also discuss the behaviour of swallowtail and umbilics in a lens model under external perturbations in case of one and two component elliptical lenses. This gives us the estimate about the amount of external shear under which a singularity does not vanish from the lens plane and hence gives us an idea about how stable/unstable these point singularities really are. In the following subsections, we will discuss the singularity maps for elliptical lenses and Abell 697 in detail. After that we study the stability of these different singularities in lens mapping.

5.1 One-Component Elliptical Lens

We first consider a one component elliptical lens. This is a good model for an isolated lens that is dynamically relaxed, e.g., an isolated galaxy or a cluster of galaxies. The elliptical isothermal lens with a finite core has a potential of the form:

\[
\psi(x_1, x_2) = \psi_0 \sqrt{\frac{1}{r_0^2} + (1 - \epsilon) (x_1 - x_{01})^2 + (1 + \epsilon) (x_2 - x_{02})^2},
\]

(12)

where \( r_0 \) is the core radius of the lens, \( \epsilon \) is the ellipticity, \((x_{01}, x_{02})\) are the coordinates of the centre of the lens with respect to the optical axis and \( \psi_0 \) describes the strength of the lens.

The characteristics of this lens are described in detail.
Figure 6. Image formation and singularity map for Abell 697: The left panel shows caustics, the middle column shows critical curves in the image plane and the right column shows the lensed images. The top panel represent image formation near a swallowtail singularity at source redshift $z_s = 0.67$. The two middle panels represent image formation for a source at $z_s = 0.82$ and the bottom panel represent image formation for a source at redshift $z_s = 1.1$. 
in Blandford & Narayan (1986) (See Figure 10). The singularity map in lens plane for this lens model is shown in figure (2) (middle column (A2,B2,C2)). It has two $A_3$-lines (represented in red and dark green) for two eigenvalues of the deformation tensor, running along the major and the minor axis of the lens potential. This lens model also has two hyperbolic umbilics along the minor axis and their position depends on the lens parameters, primarily on the core radius $\eta_0$. Because of the elliptical symmetry in the lens model, both umbilics lie at the same distance from the centre of the lens. As a result, both umbilics become critical at the same source redshift. If we change the core radius, this distance from the centre and the redshift at which these hyperbolic umbilics becomes critical, also change.

5.2 Two-Component Elliptical Lens

Most realistic lenses have several components, though one of the components may dominate over others. In this section we consider two component lenses. We consider one primary (dominant) and one secondary component. The presence of secondary lens significantly affects the lensing due to primary lens. In order to include the effects of the secondary lens, one has to modify the lensing potential in equation (8).

If the secondary lens also an elliptical lens (for simplicity), then the lens potential becomes:

$$\psi(x_1, x_2) = \psi_p + \psi_s,$$

(13)

with $\psi_p$ and $\psi_s$ are one component elliptical lenses centered at different points in the image plane with different core radii and ellipticities and the major axes of the two components can be at an angle. $(\eta_0, \theta_0)$ are the corresponding core radii for the two potentials, $\epsilon_1$ ($\epsilon_2$) is the ellipticity and $(x_{11}, x_{12})$ ($\psi_{11}, \psi_{12}$) are the coordinates of the centre of the primary (secondary) lens with respect to the optical axis.

Different sets of lens parameters give different kinds of singularity map and image formation. For example, figure (1) and figure (3) represent singularity map for two-component lens models with two different set of lens parameters. Figure (4) shows some other possible singularity maps for two-component elliptical lens with a fixed primary lens and different (randomly picked) position and orientation of the secondary lens. As before, red and dark green lines represent the $A_3$-lines with swallowtail and umbilic points denoted by violet and blue points, respectively. One can see the dependency of unstable singularities on lens parameters: as we change the secondary lens the position and critical redshift for the unstable singularities also changes. From figure (4), we also gain some knowledge about the sensitivity of the unstable singularities to the lens parameters. All panels in figure (4) have hyperbolic umbilics and swallowtail (except first panel), whereas only three panels show elliptic umbilic. We infer that elliptic umbilics are more sensitive to the lens parameters than the swallowtail and the hyperbolic umbilic.

5.3 Stability

In general, finding an isolated gravitational lens with one or two components is highly unlikely. Real gravitational lenses reside in an environment made of several structures. These external local structure perturb the lensing potential, by introducing (constant) external convergence ($\kappa_{ext}$) and shear ($\gamma_{ext}$). As a result, the perturbed lensing potential is given by,

$$\psi(x_1, x_2) = \psi_p + \kappa_{ext} \left( x_1^2 + x_2^2 \right) + \frac{\gamma_{1}'}{2} \left( x_1^2 - x_2^2 \right) + \gamma_{2} x_1 x_2. \quad (14)$$

where $\psi_p$ is the potential of primary lens, given by equation (12) or (13) in case of elliptical lenses or given by some other profile and $\left( \gamma_{1}', \gamma_{2}' \right)$ denotes the component of external shear ($\gamma_{ext}$).

The effect of the external convergence ($\kappa_{ext}$) is equivalent to the addition of a constant mass sheet in the lens model, which simply changes the total strength of the primary lens. As a result, the critical redshift for unstable singularities changes, but neither the unstable singularities vanish nor the location of $A_3$-lines in lens plane changes due to the presence of external convergence. On the other hand, the presence of external shear ($\gamma_{ext}$) shifts the location of $A_3$-lines significantly and as a result it changes the singularity map for a given lens model. The presence of external shear can also introduce or remove point singularities. The effect of external shear with a fixed value of external convergence in case of a one-component elliptical lens model, (equation (12)) is shown in figure (5). One can see that, for non-zero external shear, two extra hyperbolic umbilics occur in the lens plane along the major or minor axis depending on the values of shear components. As we increase the amount of external shear, this extra pair of umbilics move towards already existing umbilics and merge with them. This implies that introducing a finite amount of external shear can also remove the already existing point singularities from the singularity map and it is possible (in highly symmetric case) to have a singularity map without any point singularities. The amount of external shear $\gamma_{ext} = \sqrt{\gamma_{1}'^2 + \gamma_{2}'^2}$, under which point singularities shift but remain in the lens plane depends on the type of the singularity. In case of hyperbolic umbilic, it is of the order of $10^{-3}$. Similarly, the amount of external shear for which a swallowtail (elliptic umbilic) shifts but survives in the lens plane is of the order of $10^{-4}$ ($10^{-5}$). But for some particular directions of external shear, the swallowtail and elliptic umbilics show extra stability, i.e., the magnitude of external shear under which these singularities remain in the lens plane attain a higher value than the other cases. This reinforces the impression from the qualitative study in the last subsection that elliptic umbilic is less stable as compared to the hyperbolic umbilic and swallowtail.

5.4 Abell 697

After testing our approach with simple model lenses, we apply the algorithm to a real lens to illustrate the utility and efficacy of our approach. We work with the cluster lens Abell 697 ($z = 0.282$). We use the data for the lens from RELICS (Cibirka, et al. 2018; Coe, et al. 2019). The reason for choosing the Abell 697 for the preliminary analysis is the relative simplicity of the critical lines in the lens plane. The study of more complicated lensed is under consideration, and the results for a large set of clusters will be presented in a forthcoming paper along with a statistical analysis of occurrence of point singularities. The cosmological parameters used in the calculation of different angular diameter
distances are: \( H_0 = 70 \text{km s}^{-1}\text{Mpc}^{-1}, \Omega_{\Lambda} = 0.7, \Omega_m = 0.3 \). Figure 6 shows the singularity map along with the critical lines and caustics in image and source plane for Abell 697. Here we only considered the central region of Abell 697 with size 440 x 440 pixels (1 pixel = 0.06") (Cibirka, et al. 2018).

We can see that the dominant component here is like an elliptical lens and there is a lot of small scale structure contributed by other components in the lens. The role of other components is to increase the length of \( A_3 \)-lines and also to introduce point singularities.

The top panel in figure 6 shows the image formation near a swallowtail singularity for a source at redshift \( z_s = 0.67 \). The second and third panel shows the image formation for a source at redshift \( z_s = 0.82 \) for two different source positions. The bottom panel shows the image formation for a source at redshift \( z_s = 1.1 \). As one can see from the bottom panel, one pair of hyperbolic umbilic is still outside the critical curves. This means that the critical redshift for this pair is higher than the 1.1. Locations of these singularities are optimal sites for searching for faint sources at high redshifts.

6 CONCLUSIONS

We have analyzed stable and unstable singularities that can occur in gravitational lensing. In order to locate these singularities, we have implemented algorithms which take lens potential as an input. We have applied our algorithm in case of simple lens models as well as a real lens. Singularity map, which comprises all these singularities provides a compact representation of the given lens model in the lens plane. The presence of these unstable singularities in the singularity map can be used to constrain the lens model if we can find a lensed source in the vicinity. Magnification is very large in the vicinity of these singularities and each of these singularities has a characteristic image formation that can be used to identify the singularities. Further, the regions with \( A_3 \)-lines and point singularities are obvious targets for deep surveys that use gravitational lenses to search for very faint sources at high redshifts.

We have studied the dependency of unstable singularities on lens parameters as well as on the external shear. The magnitude of external shear under which these singularities remain in the singularity map is different for different singularities. This is of the order of, \( 10^{-3}, 10^{-4}, 10^{-5} \) in case of hyperbolic umbilic, swallowtail and elliptic umbilic, respectively. Thus the elliptic umbilic is most sensitive to perturbations in lensing potential and hence is the most unstable.

We have validated our approach with simple lens models and one real lens: Abell 697. We are studying other lens systems using our approach and an atlas of lens singularities and their statistical analysis will be presented in a forthcoming paper. Such an atlas can be of use for refinement of lens models with further observations and also for targeting specific regions in searches for very faint sources at high redshifts.

7 ACKNOWLEDGEMENTS

AKM would like to thank CSIR for financial support through research fellowship No.524007. This research has made use of NASA’s Astrophysics Data System Bibliographic Services. We acknowledge the HPC\textsuperscript{\textregistered}USERM, used for some of the computations presented here. Authors thank Professor D Narasimha for useful discussions and comments. JSB thanks Professor Varun Sahni and Professor Sergei Shandarin for insightful discussions on singularities and caustics, and also for useful comments on the manuscript. Authors also thank Ms Soniya Sharma who worked on some aspects of this problem for her MS thesis.

REFERENCES

Abdelsalam, H. M., Saha, P., Williams, L. L. R. 1998, MNRAS, 294, 744
AbdelSalam, H. M., Saha P., Williams L. L. R., 1998, AJ, 116, 1541
Arnold V.I., Shandarin S.F. & Zeldovich Y.B., 1982, Geophys. Astrophys. Fluid Dynamics 20, 111
Atek H., Richard J., Kneib J.-P., Scherer D., 2018, MNRAS, 479, 5184
Bagla J. S., 2001, in Brainerd T. G., Kochanek C. S., eds, ASP Conf. Ser. Vol. 237, Gravitational Lensing: Recent Progress and Future Goals. Astron. Soc. Pac., San Francisco, p. 77
Berry M. V., Updill C., 1980, PrOpt, 18, 257
Blandford R. & Narayan R., 1986, APJ, 310, 568.
Blandford R. D., Kochanek C. S., Kovner I., Narayan R., 1989, Sci, 245, 824
Cibirka N., et al., 2018, ApJ, 863, 145
Coe D., et al., 2013, ApJ, 762, 32
Coe D., et al., 2019, arXiv e-prints, arXiv:1903.02002
Dark Energy Survey Collaboration, et al., 2016, MNRAS, 460, 1270
Ebeling H., Stockmann M., Richard J., Zabl J., Brammer G., Toft S., Man A., 2018, ApJ, 852, L7
Gardner J. P., et al., 2016, SSRV, 123, 485
Gilmore L., 1981, Catastrophe theory for scientists and engineers, Wiley, New York
Hidding J., Shandarin S. F., van de Weygaert R., 2014, MNRAS, 437, 3442
Ivezić Ž., et al., 2008, arXiv e-prints, arXiv:0805.2366
Kassiola A., Kovner I. & Fort B., 1992, ApJ 400, 41
Kneib J.-P., Natarajan P., 2011, A&ARv, 19, 47
Laureijs R., 2009, arXiv e-prints, arXiv:0912.0914
McLeod D. J., McLure R. J., Dunlop J. S., Robertson B. E., Ellis R. S., Targett T. A., 2015, MNRAS, 450, 3032
Nityananda R., 1990, LNP, 1, LNP...360
Orban de Xivry, G., Marshall, P. 2009, MNRAS, 399, 2
Petters A. O., Levine H., Wambsganss J., 2001, in Petters A. O., Levine H., Wambsganss J., eds, Progress in Mathematical Physics. Vol. 21. Singularity Theory and Gravitational Lensing. Birkhäuser, Boston
Poston T., Stewart I., 1978, Catastrophe Theory and its application, Pitman, New York
Saha, P., Williams, L. L. R. 1997, MNRAS, 292, 148
Saha, P. 2000, AJ, 120, 1654
Schneider P., Ehlers J., Falco E. E., 1992, Gravitational Lenses. Birkh"auser, Boston
Walter E., Blandford R. D., 1999, ApJ, 510, 151
Wambsganss J., 2001, in Petters A. O., Levine H., Wambsganss J., eds, Progress in Mathematical Physics. Vol. 21. Singularity Theory and Gravitational Lensing. Birkhäuser, Boston
Zheng W., et al., 2012, Nature, 489, 406
