On Liouville type theorems for the stationary MHD and Hall-MHD systems

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Abstract

In this paper we prove a Liouville type theorem for the stationary magnetohydrodynamics (MHD) system in \( \mathbb{R}^3 \). Let \((v, B, p)\) be a smooth solution to the stationary MHD equations in \( \mathbb{R}^3 \). We show that if there exist smooth matrix valued potential functions \( \Phi, \Psi \) such that \( \nabla \cdot \Phi = v \) and \( \nabla \cdot \Psi = B \), whose \( L^6 \) mean oscillations have certain growth condition near infinity, namely

\[
\int_{B(r)} |\Phi - \Phi_{B(r)}|^6 \, dx + \int_{B(r)} |\Psi - \Psi_{B(r)}|^6 \, dx \leq Cr \quad \forall 1 < r < +\infty,
\]

then \( v = B = 0 \) and \( p = \text{constant} \). With additional assumption of

\[
r^{-8} \int_{B(r)} |B - B_{B(r)}|^6 \, dx \to 0 \quad \text{as} \quad r \to +\infty,
\]

similar result holds also for the Hall-MHD system.

AMS Subject Classification Number: 35Q30, 76D05, 76D03
Keywords: stationary magnetohydrodynamics equations, Liouville type theorem

1 Introduction

We consider the stationary magnetohydrodynamics equations in \( \mathbb{R}^3 \).

\[
(MHD) \begin{cases}
-\Delta v + (v \cdot \nabla) v = -\nabla p + (B \cdot \nabla) B & \text{in } \mathbb{R}^3, \\
-\Delta B + (v \cdot \nabla) B - (B \cdot \nabla) v = 0, \\
\nabla \cdot v = \nabla \cdot B = 0.
\end{cases}
\]

The system \((MHD)\) describe the steady state of the physical system of fluid flows of plasma. Here \( v = v(x) = (v_1, v_2, v_3) \) is the velocity field of the fluid flows, \( B = B(x) = (B_1, B_2, B_3) \)
is the magnetic field, and \( p = p(x) \) is the pressure of the flows. Note that if \( B = 0 \), then the system (MHD) reduces to the usual stationary Navier-Stokes system. In this paper we study the Liouville type problem for the system (MHD). The study is motivated by the similar Liouville problem for the stationary Navier-Stokes equations, which is an active research area in the community of mathematical fluid mechanics (see e.g. [7, 9, 10, 14, 15, 16, 12, 11, 6, 1, 3, 4] and references therein).

We say \( \Phi \in L^1_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^{n \times n}) \) is a potential function for vector field \( u \in L^1_{\text{loc}}(\mathbb{R}^n) \) if \( \nabla \cdot \Phi = u \), where the derivative is in the sense of distribution. In [14] Seregin proved Liouville type theorems for the Navier-Stokes equations under assumption on the potential functions of the velocity field, namely he showed the solution becomes trivial if the potential function of the velocity field belongs to \( BMO(\mathbb{R}^3) \) and the velocity itself belongs to \( L^6(\mathbb{R}^3) \). This was improved later in [15], removing the assumption for velocity belonging to \( L^6(\mathbb{R}^3) \). In recent paper [13] Schulz generalized Seregin’s earlier result of [14] to the system (MHD), where he proved that if the potential functions of the velocity and the magnetic field belongs to \( BMO(\mathbb{R}^3) \), while \( (v, B) \in L^6(\mathbb{R}^3) \), then \( v = B = 0 \). In this paper we prove a Liouville theorem for the system (MHD) under more relaxed conditions than [13]. We allow some order of growth at infinity for the mean oscillation for potentials of \( v \) and \( B \) without any integrability condition for them to obtain triviality of solutions. This result, on the other hand, could be regarded as a generalization of the authors’ previous result of [5] for the Navier-Stokes equations.

For a measurable set \( E \subset \mathbb{R}^n \) we denote by \( |E| \) the \( n \)-dimensional Lebesgue measure of \( E \), and for \( f \in L^1(E) \) we use the notation

\[
 f_E := \int_E f \, dx := \frac{1}{|E|} \int_E f \, dx.
\]

Our aim in this paper is to prove the following:

**Theorem 1.1.** Let \( (v, B, p) \in C^\infty(\mathbb{R}^3) \times C^\infty(\mathbb{R}^3) \times C^\infty(\mathbb{R}^3) \) be a solution of (MHD). Suppose there exist \( \Phi, \Psi \in C^\infty(\mathbb{R}^3; \mathbb{R}^{3 \times 3}) \) such that \( \nabla \cdot \Phi = v, \nabla \cdot \Psi = B \), and

\[
 (1.1) \quad \int_{B(r)} |\Phi - \Phi_{B(r)}|^6 \, dx + \int_{B(r)} |\Psi - \Psi_{B(r)}|^6 \, dx \leq Cr \quad \forall 1 < r < +\infty.
\]

Then, \( u = B = 0 \).

If \( v, B \in BMO^{-1}(\mathbb{R}^3) \), then there there exist \( \Phi, \Psi \) such that \( \nabla \cdot \Phi = v, \nabla \cdot \Psi = B \) and

\[
 \sup_{r > 1} \left( \int_{B(r)} |\Phi - \Phi_{B(r)}|^6 \, dx + \int_{B(r)} |\Psi - \Psi_{B(r)}|^6 \, dx \right) < +\infty,
\]

and the condition (1.1) holds obviously, and therefore we have the following immediate corollary of the above theorem.

**Corollary 1.2.** Let \( (v, B, p) \) be a smooth solution of (MHD) such that \( v, B \in BMO^{-1}(\mathbb{R}^3) \), then \( u = B = 0 \).
Next we consider the Hall-MHD system.

\[
\begin{align*}
(HMHD) \quad & - \Delta u + (u \cdot \nabla) u = - \nabla p + (B \cdot \nabla) B \quad \text{in} \quad \mathbb{R}^3, \\
& - \Delta B + (u \cdot \nabla) B - (B \cdot \nabla) u = \nabla \times ((\nabla \times B) \times B), \\
& \nabla \cdot u = \nabla \cdot B = 0.
\end{align*}
\]

This equations govern the the dynamics plasma flows of strong shear magnetic fields as in the solar flares, and have many important applications in the astrophysics. We refer [2] and references therein for a recent mathematical approaches and physical backgrounds for the system. The following is our Liouville type theorem for (HMHD).

**Theorem 1.3.** Let \((v, B, p)\) be a smooth solution of (HMHD). Let us assume

\[
(1.2) \quad r^{-8} \int_{B(r)} |B - B_{B(r)}|^6 dx \to 0 \quad \text{as} \quad r \to +\infty,
\]

and there exist \(\Phi, \Psi \in C^\infty(\mathbb{R}^3, \mathbb{R}^{3 \times 3})\) such that \((1.1)\) holds. Then, \(u = B = 0\).

We believe that the additional condition \((1.2)\) in the case of (HMHD) system is sharp, and it requires extra efforts to deduce the Liouville property in Theorem 1.3. In fact, the iteration argument used below for the proof is not standard, and would be of interest in itself.

## 2 Proof of Theorem 1.1 and Theorem 1.3

In this section we prove Theorem 1.1 and Theorem 1.3 in a unified fashion. Let us consider the following unified system with a constant \(\alpha \geq 0\).

\[
\begin{align*}
(2.1) \quad & - \Delta u + (u \cdot \nabla) u = - \nabla p + (B \cdot \nabla) B \quad \text{in} \quad \mathbb{R}^3, \\
(2.2) \quad & - \Delta B + (u \cdot \nabla) B - (B \cdot \nabla) u = \alpha \nabla \times ((\nabla \times B) \times B), \\
(2.3) \quad & \nabla \cdot u = \nabla \cdot B = 0.
\end{align*}
\]

When \(\alpha = 0\) it reduces to (MHD), while for \(\alpha = 1\) it reduces to (HMHD).

We will use the assumption \((1.2)\) only for the estimate of the term involving \(\alpha\).

First let us show that \((1.1)\) implies the following estimate for the mean value. Let \(1 < R < +\infty\). Let \(\varphi \in C^\infty_c(B(R))\) denote a cut off function such that \(0 \leq \varphi \leq 1\) in \(B(R)\), \(|\nabla \varphi| \leq cR^{-1}\) and

\[
(2.4) \quad \int_{B(R)} \varphi dx \geq cR^3.
\]

For a function \(f \in L^1(B(R))\) we define the corresponding mean value

\[
f_{B(R), \varphi} = \frac{1}{\int_{B(R)} \varphi dx} \int_{B(R)} f \varphi dx.
\]
Using Hölder’s inequality and observing (2.4), we get for all \( f \in L^p(B(R)) \), \( 1 \leq p \leq +\infty \) the estimate

\[
|f_{B(R),\varphi}| \leq c\|f\|_{L^p(E)} \quad \text{for every measurable } B(R) \supset E \supset \text{supp}(\varphi).
\]

On the other hand, for every constant function \( a \in \mathbb{R} \) it holds \( a_{B(R),\varphi} = a \). This provides us with

\[
f - f_{B(R),\varphi} = f - a - (f - a)_{B(R),\varphi}.
\]

Thus, applying (2.5), we get for all \( a \in \mathbb{R} \) and for every measurable \( B(R) \supset E \supset \text{supp}(\varphi) \),

\[
\|f - f_{B(R),\varphi}\|_{L^p(E)} \leq c\|f - a\|_{L^p(E)}.
\]

In particular, (2.6) with \( a = f_E \) gives

\[
\|f - f_{B(R),\varphi}\|_{L^p(E)} \leq c\|f - f_E\|_{L^p(E)}.
\]

This, inequality will be used below for the application of both, the Poincaré inequality and the Sobolev-Poincaré inequality.

Using the generalized mean we have introduced above, we now get from (1.1) the following estimate for \( u \) and \( B \) respectively,

\[
|u_{B(R),\varphi}| \leq cR^{-\frac{\alpha}{8}}, \quad |B_{B(R),\varphi}| \leq cR^{-\frac{\alpha}{8}}.
\]

Indeed, applying integration by parts, Jensen’s inequality, and observing \((1.1)\) along with (2.4), we find

\[
|u_{B(R),\varphi}| \leq cR^{-\frac{\alpha}{8}} \left\| \nabla \cdot \left( \Phi - (\Phi)_{B(R)} \right) \varphi \right\|
\]

\[
= c \left\| \int_{B(R)} \left( \Phi - \Phi_{B(R)} \right) \cdot \nabla \varphi dx \right\| \leq R^{-1} \left( \int_{B(R)} |\Phi - \Phi_{B(R)}|^6 dx \right)^{\frac{1}{6}}
\]

\[
\leq cR^{-\frac{\alpha}{8}}.
\]

Whence, (2.8) for \( u \). The estimate for \( B \) follows by the same reasoning from (1.1).

For the sake of notational simplicity in our discussion below we use the following abbreviations

\[
\Theta(r) := \alpha \left( r^{-8} \int_{B(r)} |B - B_{B(r)}|^6 dx \right)^{\frac{1}{6}},
\]

\[
G(r) := \int_{B(r)} |\nabla u|^2 dx + \int_{B(r)} |\nabla B|^2 dx, \quad 0 < r < +\infty.
\]

Note that in case of \( \alpha = 1 \), in view of condition (1.2) for \( B \) it holds

\[
\lim_{r \to \infty} \Theta(r) = 0.
\]
Otherwise in case $\alpha = 0$ it holds $\Theta \equiv 0$.

Furthermore, during the proof below we make frequently use of the following elementary estimate, for all $\alpha, \beta, \gamma \in \mathbb{R}$ with $0 \leq \alpha \leq \beta \leq \gamma$ and for all $1 \leq \rho < R < +\infty$

\begin{equation}
R^\alpha (R - \rho)^{-\beta} \leq R^\gamma (R - \rho)^{-\gamma}.
\end{equation}

Indeed, since $\alpha \leq \beta$ we see that $R^\alpha \leq R^\beta$ for all $R \geq 1$. On the other hand, by means of $\beta \leq \gamma$ and $R(R - \rho)^{-1} > 1$ we infer that $R^\beta (R - \rho)^{-\beta} \leq R^\gamma (R - \rho)^{-\gamma}$ Accordingly, for all $1 \leq \rho < R < +\infty$ we have

$$R^\alpha (R - \rho)^{-\beta} \leq R^\beta (R - \rho)^{-\beta} \leq R^\gamma (R - \rho)^{-\gamma}.$$ 

Let $1 < r < +\infty$ be arbitrarily chosen, but fixed. Let $r \leq \rho < R \leq 2r$. We set $\bar{R} = \frac{R + \rho}{2}$.

Let $\zeta \in C^\infty(\mathbb{R}^n)$ be a cut off function, which is radially non-increasing with $\zeta = 1$ on $B(\rho)$ and $\zeta = 0$ on $\mathbb{R}^3 \setminus B(\bar{R})$ satisfying $|\nabla \zeta| \leq c(R - \rho)^{-1}$ and $|D^2 \zeta| \leq c(R - \rho)^{-2}$. We multiply (2.1) by $u\zeta^2$, integrate it over $B(R)$, and then we multiply (2.2) by $B\zeta^2$, integrating it over $B(R)$, and add them together. Then, after integration by parts we obtain

\begin{align*}
G(\rho) &= \int_{B(\rho)} |\nabla u|^2 dx + \int_{B(\rho)} |\nabla B|^2 dx \leq \frac{1}{2} \int_{B(\bar{R})} |u|^2 \Delta \zeta^2 dx \\
&\quad + \frac{1}{2} \int_{B(\bar{R})} |u|^2 u \cdot \nabla \zeta^2 dx + \int_{B(\bar{R})} (p - p_{B(\bar{R})}) u \cdot \nabla \zeta^2 dx - \int_{B(\bar{R})} B \cdot u (B \cdot \nabla) \zeta^2 dx \\
&\quad + \frac{1}{2} \int_{B(\bar{R})} |B|^2 \Delta \zeta^2 dx + \frac{1}{2} \int_{B(\bar{R})} |B|^2 u \cdot \nabla \zeta^2 dx + \int_{B(\bar{R})} u \cdot B (B \cdot \nabla) \zeta^2 dx \\
&\quad - \alpha \int_{B(\bar{R})} ((\nabla \times B) \times B) \cdot B \times \nabla \zeta^2 dx \\
\leq c(R - \rho)^{-2} \int_{B(\bar{R})} |u|^2 dx + c(R - \rho)^{-1} \int_{B(\bar{R})} |u|^3 dx \\
&\quad + c(R - \rho)^{-1} \int_{B(\bar{R})} |p - p_{B(\bar{R})}| |u| dx + c(R - \rho)^{-1} \int_{B(\bar{R})} |B|^2 |u| dx \\
&\quad + c(R - \rho)^{-2} \int_{B(\bar{R})} |B|^2 dx + \alpha c(R - \rho)^{-1} \int_{B(\bar{R})} |B|^2 |\nabla B| dx \\
&= I + II + III + IV + V + VI.
\end{align*}

In order to estimate $I$ we choose another cut off function $\psi \in C^\infty(\mathbb{R}^3)$, which is radially non-increasing with $\psi = 1$ on $B(\bar{R})$ and $\psi = 0$ on $\mathbb{R}^3 \setminus B(R)$ satisfying $|\nabla \psi| \leq c(R - \rho)^{-1}$.

Recalling that $u = \nabla \cdot \Phi$, applying integration by parts and Cauchy-Schwarz’ inequality, we
\[
\int_{B(R)} |u|^2 \psi^2 dx = \int_{B(R)} \partial_i (\Phi_{ij} - (\Phi_{ij})_{B(R)}) u_j \psi^2 dx
\]

\[
= - \int_{B(R)} (\Phi_{ij} - (\Phi_{ij})_{B(R)}) \partial_j u_j \psi^2 dx - \int_{B(R)} (\Phi_{ij} - (\Phi_{ij})_{B(R)}) u_j \partial_i \psi^2 dx
\]

\[
\leq c \left( \int_{B(R)} |\Phi - \Phi_{B(R)}|^2 dx \right)^{\frac{1}{2}} \left( \int_{B(R)} |\nabla u|^2 dx \right)^{\frac{1}{2}}
\]

\[
+ c(R - \rho)^{-1} \left( \int_{B(R)} |\Phi - \Phi_{B(R)}|^2 dx \right)^{\frac{1}{2}} \left( \int_{B(R)} |u|^2 \psi^2 dx \right)^{\frac{1}{2}}.
\]

Using Jensen’s inequality and Young’s inequality, we obtain

\[
\int_{B(R)} |u|^2 \psi^2 dx \leq cR \left( \int_{B(R)} |\Phi - \Phi_{B(R)}|^6 dx \right)^{\frac{1}{6}} \left( \int_{B(R)} |\nabla u|^2 dx \right)^{\frac{1}{2}}
\]

\[
+ cR^2 (R - \rho)^{-2} \left( \int_{B(R)} |\Phi - \Phi_{B(R)}|^6 dx \right)^{\frac{1}{2}}.
\]

Observing (1.1), and using Young’s inequality together with (2.10) with \( \gamma = 11 \), we infer

\[
I \leq cR^{5/3} (R - \rho)^{-2} \left( \int_{B(R)} |\nabla u|^2 dx \right)^{\frac{1}{3}} + cR^{10/3} (R - \rho)^{-4}
\]

\[
\leq \frac{1}{100} G(R) + cR^{11} (R - \rho)^{-11}.
\]

Similarly,

(2.12) \[ V \leq \frac{1}{100} G(R) + cR^{11} (R - \rho)^{-11}. \]

In order to estimate \( II \) we first estimate the \( L^3 \) norms of \( u \) and \( w \) as follows.

\[
\int_{B(R)} |u|^3 \psi^3 dx = \int_{B(R)} \partial_i (\Phi_{ij} - (\Phi_{ij})_{B(R)}) u_j |u|^2 \psi dx
\]

\[
= - \int_{B(R)} (\Phi_{ij} - (\Phi_{ij})_{B(R)}) \partial_j (u_j |u|^2) \psi dx - \int_{B(R)} (\Phi_{ij} - (\Phi_{ij})_{B(R)}) u_j \partial_i |u|^2 \psi dx
\]

\[
\leq c \left( \int_{B(R)} |\Phi - \Phi_{B(R)}|^6 dx \right)^{\frac{1}{6}} \left( \int_{B(R)} |u|^3 \psi^3 dx \right)^{\frac{1}{3}} \left( \int_{B(R)} |\nabla u|^2 dx \right)^{\frac{1}{2}}
\]

\[
+ c(R - \rho)^{-1} \left( \int_{B(R)} |\Phi - \Phi_{B(R)}|^3 dx \right)^{\frac{1}{3}} \left( \int_{B(R)} |u|^3 \psi^3 dx \right)^{\frac{2}{3}}.
\]
Using Young’s inequality, we get
\[ \int_{B(R)} |u|^3 \psi^3 \, dx \leq c \left( \int_{B(R)} |\Phi - \Phi_{B(R)}|^6 \, dx \right)^{\frac{1}{6}} \left( \int_{B(R)} |\nabla u|^2 \, dx \right)^{\frac{3}{4}} + cR^\frac{3}{2}(R - \rho)^{-\frac{3}{2}} \left( \int_{B(R)} |\Phi - \Phi_{B(R)}|^6 \, dx \right)^{\frac{1}{6}}. \] (2.13)

Multiplying (2.13) by \((R - \rho)^{-1}\) combined with the hypothesis (1.1), and using (2.10) \(\gamma = 4\), we infer
\[ (R - \rho)^{-1} \int_{B(R)} |u|^3 \psi^3 \, dx \leq R(R - \rho)^{-1} G(R)^\frac{3}{4} + cR^4(R - \rho)^{-4}. \] (2.14)

Applying Young’s inequality, and (2.10) with \(\gamma = 11\) one has
\[ II \leq \frac{1}{100} G(R) + cR^{11}(R - \rho)^{-11}. \]

Similarly to (2.13), we also obtain
\[ (R - \rho)^{-1} \int_{B(R)} |B|^3 \psi^3 \, dx \leq R(R - \rho)^{-1} G(R)^\frac{3}{4} + cR^4(R - \rho)^{-4}. \] (2.15)

Next, using (2.13), (2.14), we shall estimate \(\int_{B(R)} |u||B|^2 \psi^3 \, dx\). Applying Young’s inequalities, and (2.13), (2.14) we have
\[ (R - \rho)^{-1} \int_{B(R)} |u||B|^2 \psi^3 \, dx \leq (R - \rho)^{-1} \int_{B(R)} (|u|^3 + |B|^3) \psi^3 \, dx \leq cR(R - \rho)^{-1} G(R)^\frac{3}{4} + cR^4(R - \rho)^{-4}. \]

Applying Young’s inequality, and again using (2.10) with \(\gamma = 11\), we arrive at
\[ IV \leq \frac{1}{100} G(R) + cR^{11}(R - \rho)^{-11}. \]

We now estimate III. Using Hölder’s inequality and Young’s inequality, we infer
\[ III \leq c(R - \rho)^{-1} \int_{B(R)} |p - p_{B(R)}|^\frac{3}{2} \, dx + c(R - \rho)^{-1} \int_{B(R)} |u|^3 \, dx. \] (2.16)

We are now going to estimate the pressure term. For this purpose let us define the functional \(F \in W^{-1,\frac{3}{2}}(B(\overline{R}))\), by means of
\[ \langle F, \varphi \rangle = \int_{B(\overline{R})} (\nabla u - u \otimes u + B \otimes B) : \nabla \varphi \, dx, \quad \varphi \in W^{1,3}_0(B(\overline{R})). \]
Since \((u, w, p)\) is a solution to (2.1), it follows that
\[
\langle F, \varphi \rangle = \int_{B(\overline{R})} (p - p_{B(\overline{R})}) \nabla \cdot \varphi \, dx \quad \forall \varphi \in W^{1,3}_0(B(\overline{R})).
\]
Thus, we have \(F = -\nabla (p - p_{B(\overline{R})})\) with \(p - p_{B(\overline{R})} \in L^{3/2}(B(\overline{R}))\) and \(\int_{B(\overline{R})} (p - p_{B(\overline{R})}) \, dx = 0\).

Consulting Sohr [17, Lemm 2.1.1], we get the estimate
\[
(2.17) \quad \int_{B(\overline{R})} |p - p_{B(\overline{R})}|^{3/2} \, dx \leq c \|F\|^{3/2}_{W^{-1,3/2}(B(\overline{R}))},
\]
with a constant \(c > 0\) independent of \(\overline{R}\). On the other hand, we estimate by the aid of Hölder’s inequality
\[
(2.18) \quad \|F\|^{3/2}_{W^{-1,3/2}(B(\overline{R}))} \leq \|\nabla u - u \otimes u + B \otimes B\|^{3/2}_{L^{3/2}(B(\overline{R}))} \leq cR^{\frac{3}{2}} \left( \int_{B(\overline{R})} |\nabla u|^2 \, dx \right)^{\frac{1}{2}} + c \int_{B(\overline{R})} |u|^3 \, dx + c \int_{B(\overline{R})} |B|^3 \, dx.
\]

Combining (2.16), (2.17), and (2.18), and using Young’s inequality along with (2.10) for \(\gamma = 11\), we obtain
\[
(2.19) \quad III \leq \frac{1}{100} G(R) + cR^{11}(R - \rho)^{-11} + c(R - \rho)^{-1} \int_{B(\overline{R})} |u|^3 \, dx + c(R - \rho)^{-1} \int_{B(\overline{R})} |B|^3 \, dx.
\]

Note that (2.13), (2.14) provide us with the estimate
\[
(2.20) \quad (R - \rho)^{-1} \int_{B(\overline{R})} |u|^3 \psi^3 \, dx + (R - \rho)^{-1} \int_{B(\overline{R})} |B|^3 \psi^3 \, dx \leq cR(R - \rho)^{-1} G(R)^{\frac{3}{2}} + cR^4(R - \rho)^{-4}.
\]

Inserting the estimate of (2.20) into (2.19), applying Young’s inequality, and using (2.10), we find
\[
III \leq \frac{1}{100} G(R) + cR^{11}(R - \rho)^{-11}.
\]
It remains to estimate $VI$. Applying Hölder’s inequality, we estimate
\[
VI \leq c\alpha (R - \rho)^{-1} \int_{B(R)} |B|^2 |\nabla B| \psi dx
\]
\[
\leq c\alpha (R - \rho)^{-1} \left( \int_{B(R)} |B|^6 dx \right)^{\frac{1}{6}} \left( \int_{B(R)} |B|^3 \psi^3 dx \right)^{\frac{1}{3}} \left( \int_{B(R)} |\nabla B|^2 dx \right)^{\frac{1}{2}}
\]
\[
\leq c\alpha (R - \rho)^{-1} \left( \int_{B(R)} |B - B_{B(R),\varphi}|^6 dx \right)^{\frac{1}{6}} \left( \int_{B(R)} |B|^3 \psi^3 dx \right)^{\frac{1}{3}} \left( \int_{B(R)} |\nabla B|^2 dx \right)^{\frac{1}{2}}
\]
\[
+ c\alpha R^{\frac{1}{2}} (R - \rho)^{-1} |B_{B(R),\varphi}| \left( \int_{B(R)} |B|^3 \psi^3 dx \right)^{\frac{1}{3}} \left( \int_{B(R)} |\nabla B|^2 dx \right)^{\frac{1}{2}}.
\]
Applying (2.7) with $E = B(R)$, and (2.8) both with $\varphi = \psi$, we infer from the above estimate
\[
VI \leq c\alpha (R - \rho)^{-1} \left( \int_{B(R)} |B - B_{B(R)}|^6 dx \right)^{\frac{1}{6}} \left( \int_{B(R)} |B|^3 \psi^3 dx \right)^{\frac{1}{3}} \left( \int_{B(R)} |\nabla B|^2 dx \right)^{\frac{1}{2}}
\]
\[
+ c\alpha R^{-\frac{1}{3}} (R - \rho)^{-1} \left( \int_{B(R)} |B|^3 \psi^3 dx \right)^{\frac{1}{3}} \left( \int_{B(R)} |\nabla B|^2 dx \right)^{\frac{1}{2}}
\]
\[
\leq cR^{\frac{1}{2}} (R - \rho)^{-\frac{1}{6}} \Theta(R) \left( (R - \rho)^{-1} \int_{B(R)} |B|^3 \psi^3 dx \right)^{\frac{1}{3}} \left( \int_{B(R)} |\nabla B|^2 dx \right)^{\frac{1}{2}}
\]
\[
+ cR^{-\frac{1}{3}} (R - \rho)^{-\frac{1}{3}} \left( (R - \rho)^{-1} \int_{B(R)} |B|^3 \psi^3 dx \right)^{\frac{1}{3}} \left( \int_{B(R)} |\nabla B|^2 dx \right)^{\frac{1}{2}}.
\]
(2.21)
Combining (2.20) with (2.21), and applying Young’s inequality, and using (2.10) several times with $\gamma = 11$, we find
\[
VI \leq cR^{\frac{1}{2}} (R - \rho)^{-\frac{1}{6}} \Theta(R) \left( R^{\frac{1}{2}} (R - \rho)^{-\frac{1}{3}} G(R)^{\frac{1}{2}} + R^{\frac{2}{3}} (R - \rho)^{-\frac{1}{3}} G(R)^{\frac{1}{2}} \right) G(R)^{\frac{1}{2}}
\]
\[
+ cR^{-\frac{1}{3}} (R - \rho)^{-\frac{1}{3}} \left( R^{\frac{1}{3}} (R - \rho)^{-\frac{1}{3}} G(R)^{\frac{1}{2}} + R^{\frac{2}{3}} (R - \rho)^{-\frac{1}{3}} G(R)^{\frac{1}{2}} \right) G(R)^{\frac{1}{2}}
\]
\[
\leq cR^{\frac{20}{3}} (R - \rho)^{-4} \Theta^4(R) + cR^{\frac{16}{3}} (R - \rho)^{-4} \Theta^2(R) + c(R - \rho)^{-4} + cR^2 (R - \rho)^{-4}
\]
\[
+ \frac{1}{100} G(R)
\]
\[
\leq \frac{1}{100} G(R) + cR^{\frac{20}{3}} (R - \rho)^{-4} \Theta^4(R) + cR^{11} (R - \rho)^{-11}.
\]
Inserting the estimates of $I, \cdots, VI$ into the right hand side of (2.11), again using $R \geq 1$, and applying Young’s inequality, we are led to
\[
G(\rho) \leq \frac{1}{2} G(R) + R^{\frac{20}{3}} (R - \rho)^{-4} \Theta^4(R) + cR^{11} (R - \rho)^{-11}.
\]
(2.22)
Applying the iteration Lemma in [8, V. Lemma 3.1], from (2.22) we deduce that
\[ G(\rho) \leq cR^2(\rho - r)^{-4}\Theta^4(R) + cR^4(\rho - r)^{-11}, \quad \forall r \leq \rho < R \leq 2r. \]

In the case \( \alpha = 0 \) (MHD equations) we see that the first term on the right-hand side vanishes, and thus, \( G \) is bounded. In case \( \alpha = 1 \) the inequality (2.23) with \( \rho = r \) and \( R = 2r \) gives
\[ G(r) \leq cr^{\frac{5}{3}} \quad \forall 1 \leq r < +\infty. \]

Since our aim is to show that \( \nabla u \) and \( \nabla B \) are in \( L^2(\mathbb{R}^3) \) we still need to improve the above estimate. To do this we proceed as follows. Let \( 0 < \tau < 1 \) be sufficiently small, specified below. Observing (2.9), we may choose \( r_0 > 1 \) such that \( \Theta(r) \leq \tau^2 \) for all \( r \geq r_0 \). Let \( r \geq r_0 \) be arbitrarily chosen, but fixed. Taking into account (2.23), we repeat the estimation of \( VI \). Starting from the first inequality in (2.21) applying Sobolev’s-Poincaré inequality, we find
\[
VI \leq c(R - \rho)^{-1} \left( \int_{B(R)} |B|^3\psi^3 dx \right)^{\frac{1}{3}} \left( \int_{B(R)} |\nabla B|^2 dx \right)^{\frac{1}{3}} \left( \int_{B(R)} |\nabla B|^2 dx \right)^{\frac{1}{3}} \\
+ cR^{-\frac{4}{3}}(R - \rho)^{-1} \left( \int_{B(R)} |B|^3\psi^3 dx \right)^{\frac{1}{3}} \left( \int_{B(R)} |\nabla B|^2 dx \right)^{\frac{1}{3}} \\
\leq c(R - \rho)^{-\frac{2}{3}} \left( (R - \rho)^{-1} \int_{B(R)} |B|^3\psi^3 dx \right)^{\frac{1}{3}} G(R)^{\frac{1}{3}}G(R)^{\frac{1}{3}} \\
+ cR^{-\frac{4}{3}}(R - \rho)^{-\frac{2}{3}} \left( (R - \rho)^{-1} \int_{B(R)} |B|^3\psi^3 dx \right)^{\frac{1}{3}} G(R)^{\frac{1}{3}}.
\]

Inserting the estimate (2.20) into the right-hand side of (2.23), we arrive at
\[
VI \leq c(R - \rho)^{-\frac{2}{3}} \left( R^4(R - \rho)^{-\frac{1}{4}}G(R)^{\frac{1}{4}} + R^4(R - \rho)^{-\frac{3}{4}} \right)G(R)^{\frac{1}{4}}G(R)^{\frac{1}{4}} \\
+ cR^{-\frac{4}{3}}(R - \rho)^{-\frac{2}{3}} \left( R^4(R - \rho)^{-\frac{1}{4}}G(R)^{\frac{1}{4}} + R^4(R - \rho)^{-\frac{3}{4}} \right)G(R)^{\frac{1}{4}} \\
= cR^4(R - \rho)^{-1}G(R)^{\frac{1}{4}}G(R)^{\frac{1}{4}}G(R)^{\frac{1}{4}} + cR^4(R - \rho)^{-2}G(R)^{\frac{1}{4}}G(R)^{\frac{1}{4}} + cR^4(R - \rho)^{-\frac{1}{2}}G(R)^{\frac{1}{4}}G(R)^{\frac{1}{4}} \\
+ c(R - \rho)^{-1}G(R)^{\frac{1}{4}} + cR(R - \rho)^{-2}G(R)^{\frac{1}{4}}.
\]

Estimating \( G(R)^{\frac{1}{4}} \) by means of (2.23) with \( \rho = R \), using (2.10) with \( \gamma = 11 \), and applying Young’s inequality, we are provided with
\[
VI \leq cR^4(R - \rho)^{-\frac{2}{3}}\Theta(R)G(R)^{\frac{1}{4}}G(R)^{\frac{1}{4}} + cR^4(R - \rho)^{-\frac{5}{3}}G(R)^{\frac{1}{4}}G(R)^{\frac{1}{4}}G(R)^{\frac{1}{4}} \\
+ cR^2(R - \rho)^{-\frac{2}{3}} + cR^3(R - \rho)^{-\frac{5}{3}} + cR^4(R - \rho)^{-4} + \frac{1}{100}G(R) \\
\leq \frac{1}{50}G(R) + cR^{11}(R - \rho)^{-11}\Theta(R)G(R) + cR^3(R - \rho)^{-3}G(R)^{\frac{1}{4}} \\
+ cR^4(R - \rho)^{-11} \\
\leq \frac{1}{25}G(R) + cR^{11}(R - \rho)^{-11}\Theta(R)G(R) + cR^4(R - \rho)^{-11}.
\]
Again inserting the estimates of $I, \cdots, VI$ into the right hand side of (2.11), and applying Young’s inequality, we are led to

\begin{equation}
G(\rho) \leq \frac{1}{2} G(R) + R^{11} (R - \rho)^{-11} \theta(R) G(R) + c R^{11} (R - \rho)^{-11}.
\end{equation}

Once more applying the iteration Lemma in [8, V. Lemma 3.1], from (2.26) we deduce that

\begin{equation}
G(\rho) \leq c R^{11} (R - \rho)^{-11} (\tau^2 G(2r) + 1), \quad \forall r \leq \rho \leq 2r.
\end{equation}

In particular, this inequality with $\rho = r$ and $R = 2r$ (2.27) reads

\begin{equation}
G(r) \leq c \tau^2 G(2r) + c.
\end{equation}

with an absolute constant $c > 0$. We take $0 < \tau < 1$ such that $c \tau \leq 1$, and iterate (2.30) $k$-times starting with some $R \geq r_0$. This gives

\begin{equation}
G(R) \leq \tau^k G(2^k R) + c \sum_{i=0}^{k-1} \tau^i.
\end{equation}

We may choose $0 < \tau \leq 2^{-3}$. With this choice along with (2.24) we get

\begin{equation}
G(R) \leq 2^{-3k} G(2^k R) + 2c \leq c 2^{-3k} R^{\frac{8}{3} 2^{k+1}} + 2c \leq R^{\frac{8}{3} 2^{k-3k} + 2c}.
\end{equation}

After letting $k \to +\infty$ and then passing $R \to +\infty$, we find

\begin{equation}
\int_{\mathbb{R}^3} |\nabla u|^2 dx + \int_{\mathbb{R}^3} |\nabla B|^2 dx < +\infty.
\end{equation}

Note that from (2.20) with $\rho = r$ and $R = 2\rho$, using (2.31), provides us with the estimate

\begin{equation}
\frac{1}{r^3} \left( \int_{B(r)} |u|^3 dx + \int_{B(r)} |B|^3 dx \right) \leq c \quad \forall 1 < r < +\infty.
\end{equation}

Next, we claim that

\begin{equation}
\frac{1}{r^3} \left( \int_{B(3r) \setminus B(2r)} |u|^3 dx + \int_{B(3r) \setminus B(2r)} |B|^3 dx \right) = o(1) \quad \text{as} \quad r \to +\infty.
\end{equation}

Let $\psi \in C^\infty(\mathbb{R}^3)$ be a cut off function for the annulus $B(3r) \setminus B(2r)$ in $B(4r) \setminus B(r)$, i.e. $0 \leq \psi \leq 1$ in $\mathbb{R}^3$, $\psi = 0$ in $\mathbb{R}^3 \setminus (B(4r) \setminus B(r))$, $\psi = 1$ on $B(3r) \setminus B(2r)$ and $|\nabla \psi| \leq cr^{-1}$ . Recalling that $u = \nabla \cdot \Phi$, and applying integration by parts, using Hölder’s inequality along
with \( \mathbf{1.1} \) we calculate

\[
\int_{B(4r) \setminus B(r)} |u|^3 \psi^3 \, dx
\]

\[
= \int_{B(4r) \setminus B(r)} \partial_j (\Phi_{ij} - (\Phi_{ij})_{B(4r)}) u_i |u| \psi^3 \, dx
\]

\[
= - \int_{B(4r) \setminus B(r)} (\Phi_{ij} - (\Phi_{ij})_{B(4r)}) \partial_j (u_i |u|) \psi^3 \, dx - \int_{B(4r) \setminus B(r)} (\Phi_{ij} - (\Phi_{ij})_{B(4r)}) (u_i |u|) \partial_j \psi^3 \, dx
\]

\[
\leq c \left( \int_{B(4r)} |\Phi - \Phi_{B(4r)}|^6 \, dx \right)^{\frac{1}{6}} \left( \int_{B(4r) \setminus B(r)} |u|^3 \psi^3 \, dx \right)^{\frac{1}{3}} \left( \int_{B(4r) \setminus B(r)} |\nabla u|^2 \, dx \right)^{\frac{1}{2}}
\]

\[
+ cr^{-1} \left( \int_{B(4r)} |\Phi - \Phi_{B(4r)}|^6 \, dx \right)^{\frac{1}{6}} \left( \int_{B(4r) \setminus B(r)} |u|^3 \psi^3 \, dx \right)^{\frac{1}{3}} \left( \int_{B(4r) \setminus B(r)} |u|^2 \, dx \right)^{\frac{1}{2}}
\]

\[
\leq cr^{\frac{2}{3}} \left( \int_{B(4r) \setminus B(r)} |u|^3 \psi^3 \, dx \right)^{\frac{1}{3}} \left( \int_{B(4r) \setminus B(r)} |\nabla u|^2 \, dx \right)^{\frac{1}{2}}
\]

(2.34)

\[
+ cr^{-\frac{1}{2}} \left( \int_{B(4r) \setminus B(r)} |u|^3 \psi^3 \, dx \right)^{\frac{1}{3}} \left( \int_{B(4r) \setminus B(r)} |u|^2 \, dx \right)^{\frac{1}{2}}.
\]

First, we note that (2.8) with \( R = 4r \) and \( \varphi = \psi \) implies

(2.35) \( |u|_{B(4r),\psi} \leq cr^{-\frac{3}{2}} \).

By the triangular inequality and (2.7) with \( E = B(4r) \setminus B(r) \) we have

\[
\left( \int_{B(4r) \setminus B(r)} |u|^2 \, dx \right)^{\frac{1}{2}} \leq \left( \int_{B(4r) \setminus B(r)} |u - u_{B(4r),\psi}|^2 \, dx \right)^{\frac{1}{2}} + \left( \int_{B(4r) \setminus B(r)} |u_{B(4r),\psi}|^2 \, dx \right)^{\frac{1}{2}}
\]

\[
\leq c \left( \int_{B(4r) \setminus B(r)} |u - u_{B(4r) \setminus B(r)}|^2 \, dx \right)^{\frac{1}{2}} + c r^2 |u_{B(4r),\psi}|.
\]

Using the Poincaré inequality and (2.35), we find

(2.36) \( \left( \int_{B(4r) \setminus B(r)} |u|^2 \, dx \right)^{\frac{1}{2}} \leq cr \left( \int_{B(4r) \setminus B(r)} |\nabla u|^2 \, dx \right)^{\frac{1}{2}} + cr^{\frac{3}{2}}. \)

Inserting (2.36) into the last term of (2.34), and dividing the result by \( \left( \int_{B(4r) \setminus B(r)} |u|^3 \psi^3 \, dx \right)^{\frac{1}{3}} \), we find

(2.37) \( \int_{B(4r) \setminus B(r)} |u|^3 \psi^3 \, dx \leq cr \left( \int_{B(4r) \setminus B(r)} |\nabla u|^2 \, dx \right)^{\frac{3}{4}} + cr^{\frac{3}{2}}. \)
Repeating the estimates from (2.34) to (2.37) for $B$, we obtain

$$\int_{B(4r) \setminus B(r)} |B|^3 \psi^3 \, dx \leq cr \left( \int_{B(4r) \setminus B(r)} |\nabla B|^2 \, dx \right)^{\frac{3}{8}} + cr^\frac{1}{2}.$$

Thus, observing (2.31), we obtain the claim (2.33).

Let $1 < r < +\infty$ be arbitrarily chosen. By $\zeta \in C^\infty_0(\mathbb{R}^n)$ we denote a cut off function, which is radially non-increasing with $\zeta = 1$ on $B(2r)$ and $\zeta = 0$ on $\mathbb{R}^3 \setminus B(3r)$ such that $|\nabla \zeta| \leq cr^{-1}$ and $|D^2 \zeta| \leq cr^{-2}$. We multiply (2.1) by $u \zeta$, and integrate it over $B(3r)$, and similarly multiply (2.2) by $B \zeta$ and integrate it over $B(3r)$, and adding them together, we find after integration by parts.

$$\int_{B(3r)} |\nabla u|^2 \, dx + \int_{B(3r)} |\nabla B|^2 \, dx \leq \frac{1}{2} \int_{B(3r) \setminus B(2r)} |u|^2 \Delta \zeta^2 \, dx$$

$$+ \frac{1}{2} \int_{B(3r) \setminus B(2r)} |u|^2 \cdot \nabla \zeta^2 \, dx + \int_{B(3r) \setminus B(2r)} (p - p_{\overline{B(\mathbb{R})}}) \cdot u \cdot \nabla \zeta^2 \, dx$$

$$- \int_{B(3r) \setminus B(2r)} B \cdot u (B \cdot \nabla) \zeta^2 \, dx + \frac{1}{2} \int_{B(3r) \setminus B(2r)} |B|^2 \Delta \zeta^2 \, dx$$

$$+ \frac{1}{2} \int_{B(3r) \setminus B(2r)} |B|^2 \cdot \nabla \zeta^2 \, dx + \int_{B(3r) \setminus B(2r)} u \cdot B (B \cdot \nabla) \zeta^2 \, dx$$

$$- \alpha \int_{B(3r) \setminus B(2r)} ((\nabla \times B) \times B) \cdot B \times \nabla \zeta^2 \, dx$$

$$\leq cr^{-2} \int_{B(3r) \setminus B(2r)} |u|^2 \, dx + cr^{-1} \int_{B(3r) \setminus B(2r)} |u|^3 \, dx + cr^{-1} \int_{B(3r) \setminus B(2r)} |p - p_{\overline{B(\mathbb{R})}}| |u| \, dx$$

$$+ cr^{-1} \int_{B(3r) \setminus B(2r)} |B|^2 |u| \, dx + cr^{-2} \int_{B(3r) \setminus B(2r)} |B|^2 \, dx + \alpha cr^{-1} \int_{B(3r) \setminus B(2r)} |B|^2 |\nabla B| \, dx$$

(2.38)  

$$= J_1 + \cdots + J_6.$$  

Using (2.32), we immediately get  

$$J_1 + J_5 \leq cr^{-1} \left( \int_{B(3r)} |u^3| \, dx \right)^{\frac{2}{3}} + cr^{-1} \left( \int_{B(3r)} |B|^3 \, dx \right)^{\frac{2}{3}} \leq cr^{-\frac{1}{3}} = o(1) \text{ as } r \to +\infty.$$
From (2.33) it follows that $J_2 = o(1)$ as $r \to +\infty$. For $J_3$ we observe

$$J_3 \leq cr^{-1} \left( \int_{B(3r)} |p - p_{B(3r)}|^\frac{3}{2} dx \right)^\frac{2}{3} \left( \int_{B(3r) \setminus B(2r)} |u|^3 dx \right)^\frac{1}{3}$$

$$\leq c \left( r^{-1} \int_{B(3r)} |p - p_{B(3r)}|^\frac{3}{2} dx \right)^\frac{2}{3} \left( r^{-1} \int_{B(3r) \setminus B(2r)} |u|^3 dx \right)^\frac{1}{3}$$

$$\leq c \left( r^{-1} \int_{B(3r)} |p - p_{B(3r)}|^\frac{3}{2} dx \right)^\frac{2}{3} o(1).$$

Using the estimate (2.17) for $3r$ in place of $R$, we obtain

$$r^{-1} \int_{B(3r)} |p - p_{B(3r)}|^\frac{3}{2} dx \leq cr^{-1} \int_{B(3r)} |\nabla u|^2 dx + cr^{-1} \int_{B(3r)} |u|^3 dx + cr^{-1} \int_{B(3r)} |B|^3 dx.$$

$$\leq cr^{-\frac{1}{3}} \left( \int_{B(3r)} |\nabla u|^2 dx \right)^\frac{3}{2} + cr^{-1} \int_{B(3r)} |u|^3 dx + cr^{-1} \int_{B(3r)} |B|^3 dx.$$

By virtue of (2.31) and (2.32) the right-hand side of the above inequality is bounded for $r \geq 1$. This shows that $J_3 = o(1)$ as $r \to +\infty$.

For $J_4$ by the Hölder inequality and (2.33) we have

$$J_4 \leq cr^{-1} \left( \int_{B(3r) \setminus B(2r)} |B|^3 dx \right)^\frac{2}{3} \left( \int_{B(3r) \setminus B(2r)} |u|^3 dx \right)^\frac{1}{3}$$

$$\leq c \left( r^{-1} \int_{B(3r) \setminus B(2r)} |B|^3 dx \right)^\frac{2}{3} \left( r^{-1} \int_{B(3r) \setminus B(2r)} |u|^3 dx \right)^\frac{1}{3} = o(1)$$

as $r \to +\infty$. Finally for $J_6$, by the Hölder and the Sobolev-Poincaré inequalities together with (2.8), we have

$$J_6 \leq \alpha cr^{-1} \left( \int_{B(3r) \setminus B(2r)} |B|^4 dx \right)^\frac{1}{3} \left( \int_{B(3r) \setminus B(2r)} |\nabla B|^2 dx \right)^\frac{1}{3}$$

$$\leq \alpha cr^{-\frac{1}{2}} \left( \int_{B(3r) \setminus B(2r)} |B|^6 dx \right)^\frac{1}{3} \left( \int_{B(3r) \setminus B(2r)} |\nabla B|^2 dx \right)^\frac{1}{3}$$

$$\leq \alpha cr^{-\frac{1}{2}} \left( \int_{B(3r) \setminus B(2r)} |B - B_{B(3r) \setminus B(2r)}, \xi|^6 dx \right)^\frac{1}{3} \left( \int_{B(3r) \setminus B(2r)} |\nabla B|^2 dx \right)^\frac{1}{3}$$

$$+ cor^\frac{1}{2} |B_{B(3r) \setminus B(2r)}, \xi|^2 \left( \int_{B(3r) \setminus B(2r)} |\nabla B|^2 dx \right)^\frac{1}{3}.$$
\[
\leq \alpha cr^{-\frac{1}{2}} \left( \int_{B(3r) \setminus B(2r)} |\nabla B|^2 \, dx \right)^{\frac{3}{2}} + \alpha r^{-\frac{7}{6}} \left( \int_{B(3r) \setminus B(2r)} |\nabla B|^2 \, dx \right)^{\frac{1}{2}} = o(1)
\]
as \( r \to +\infty \). Inserting the above estimates of \( J_1, \cdots, J_6 \) into the right-hand side of (2.38), we deduce that

\[
\int_{B(r)} |\nabla u|^2 \, dx + \int_{B(r)} |\nabla B|^2 \, dx = o(1) \quad \text{as} \quad r \to +\infty.
\]

Accordingly, \( u \equiv \text{const} \) and \( B \equiv \text{const} \) and by means of (2.32) it follows that \( u = B = 0 \).}

**Acknowledgements**

Chae was partially supported by NRF grant 2016R1A2B3011647, while Wolf has been supported supported by NRF grant 2017R1E1A1A01074536. The authors declare that they have no conflict of interest.

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