LOCAL NULL CONTROLLABILITY OF COUPLED DEGENERATE SYSTEMS WITH NONLOCAL TERMS AND ONE CONTROL FORCE

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Abstract. In this paper, we are concerned with the internal control of a class of one-dimensional nonlinear parabolic systems with nonlocal and weakly degenerate diffusion coefficients. Our main theorem establishes a local null controllability result with only one internal control for a system of two equations. The proof, based on the idea developed by Fursikov and Imanuvilov, is obtained from the global null controllability of the linearized system provided by Lyusternik’s Inverse Mapping Theorem. This work extends the results previously treated by the authors for just one equation. For the system, the main issue is to obtain similar results with just one internal control, which requires a new Carleman estimate with the local term just depending on one of the state function.

1. Introduction. In this paper we will establish a local null controllability result for the degenerate system, with nonlocal terms, given by

\[
\begin{align*}
    u_t - \left( \mu_1 \left(x, \int_0^1 u \right) u_x \right)_x &+ f_1(t, x, u, v) = h \chi_\omega, & (t, x) \in (0, T) \times (0, 1), \\
v_t - \left( \mu_2 \left(x, \int_0^1 v \right) v_x \right)_x &+ f_2(t, x, u, v) = 0, & (t, x) \in (0, T) \times (0, 1), \\
    u(t, 0) & = u(t, 1) = v(t, 0) = v(t, 1) = 0, & t \in (0, T), \\
    u(0, x) & = u_0(x) \quad \text{and} \quad v(0, x) = v_0(x), & x \in (0, 1),
\end{align*}
\]

(1)

where \( T > 0 \) is given and \((u_0, v_0)\) is the initial data. Moreover, \( h \) is the control function, \((u, v)\) is the associated state and \( \chi_\omega \) represent the characteristic function.

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of $\omega = (\alpha, \beta) \subset (0, 1)$. Regarding the functions $\mu_1, \mu_2, f_1$ and $f_2$, we make the following assumptions:

**A.1** Let $\ell_1, \ell_2 : \mathbb{R} \to \mathbb{R}$ be $C^1$ functions with bounded derivative and suppose that $\ell_i(0) > 0$, for each $i = 1, 2$. We also consider $a \in C([0, 1]) \cap C^1((0, 1])$ satisfying $a(0) = 0$, $a > 0$ in $(0, 1]$, $a' \geq 0$ and

$$xd'(x) \leq ka(x), \ \forall x \in (0, 1] \text{ and some } K \in (0, 1).$$

(2)

In other words, the function $a$ behaves $x^\alpha$, with $\alpha \in (0, 1)$.

Throughout this article, we will consider the functions $\mu_1, \mu_2 : [0, 1] \times \mathbb{R} \to \mathbb{R}$, given by

$$\mu_1(x, r) = \ell_1(r)a(x) \quad \text{and} \quad \mu_2(x, r) = \ell_2(r)a(x).$$

And, for the sake of simplicity, we will also consider $\ell_i(0) = 1$, for each $i = 1, 2$.

**A.2** For $i \in \{1, 2\}$, we suppose that $f_i : [0, T] \times [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ is a $C^1$ function, with bounded derivatives, such that $f_i(t, x, 0, 0) = 0$. We also consider $b_{ij}(t, x) := \partial_{x}f_i(t, x, 0, 0) \in L^\infty((0, T) \times (0, 1))$, for any $i, j \in \{1, 2\}$. And, let us assume that there exists $\omega_1 \subset \subset \omega$ such that

$$\inf\{b_{21}(t, x); \ (t, x) \in [0, T] \times \omega_1\} > 0.$$

The main purpose of this work is to prove the local null controllability of (1) by means of one control. Precisely, we will obtain $h \in L^2((0, T) \times (0, 1))$ such that the associated state $(u, v) = (u(t, x), v(t, x))$ of (1) satisfies

$$u(T, x) \equiv v(T, x) \equiv 0 \text{ for any } x \in [0, 1],$$

at least if $\|(u_0, v_0)\|_{H^1_\alpha \times H^1_\alpha}$ is sufficiently small, where $H^1_\alpha$ is an appropriate weighted Hilbert space which will be defined later in Section 2.

The main difficulty comes from the fact that the diffusion coefficients degenerate at $x = 0$ and have nonlocal terms, namely

$$\left(\mu_1 \left(\cdot, \int_0^1 u \right) u_x\right)_x \quad \text{and} \quad \left(\mu_2 \left(\cdot, \int_0^1 u \right) u_x\right)_x$$

satisfy assumption (A.1).

It is important to remark that semilinear nondegenerate problems have been extensively studied over the last decades, see [16, 21, 22, 24, 27] for example.

However, it seems to us that there is also a large interest in degenerate operators when the degeneracy occurs at the boundary of the space domain. For instance, in [29], it was developed a study about the Prandtl system for stationary flows, in which the related boundary layer system was reduced to a quasilinear degenerate parabolic equation. Degenerate operators also appear in probabilistic models, see [17, 18], and in climate science, see [23].

In the context of degenerated systems, controllability was studied in the case of two coupled equations in [1, 7, 2]. Recently, Ait Benhassi et al., in [3], generalize the Kalman rank condition for the null controllability to $n$-coupled linear degenerate parabolic systems with $m$-controls.

On the other hand, as it was pointed out in [21], nonlocal terms type can be found in several natural phenomena, such as in the reaction-diffusion systems, see [9], and in nonlinear vibration theory, see [28] for example.

In [4], it was obtained the null controllability for the semilinear equation

$$u_t - (a(x)u_x)_x + f(t, x, u) = h(t, x)\chi_\omega, \text{ where } (t, x) \in (0, T) \times (0, 1).$$

(3)
Based on this work, in [13], we have considered (3), replacing the second-order term \((au_x)_x\) by a specific degenerate nonlocal operator. In that new context, we have achieved a local null controllability result. For systems of parabolic equations, the main issue is often to reduce the number of control functions acting on the system (see [3, 8, 11, 15], for example), besides that, as it was pointed out in [6], the problem of controlling coupled parabolic equations has a very different behavior with respect to the scalar case, for instance, boundary controllability is not equivalent to distributed controllability, approximate controllability is not equivalent to null controllability, and “the list of open problems is long and there is a lot of work to be done in order to fully understand this challenging subject” [6]. In this direction, the current work may be seen as a natural continuation of [13] and a first step in order to understand parabolic system with nonlocal and degenerate diffusion coefficients of the type \(\left(\mu \left(\cdot, \int_0^1 u \right) u_x \right)_x\).

Our main result is the following:

**Theorem 1.1.** Under the assumptions on \(\mu_1, \mu_2, f_1\) and \(f_2\), the nonlinear system (1) is locally null-controllable at any time \(T > 0\), i.e., there exists \(r > 0\) such that, whenever \(u_0, v_0 \in H^1_a\) and \(\| (u_0, v_0) \|_{H^1_a} \leq r\), there exists a control \(h \in L^2((0, T) \times \omega)\) associated to a state \((u, v)\) satisfying

\[u(T, x) = v(x, T) = 0, \text{ for every } x \in [0, 1].\]

The proof of Theorem 1.1 will follow standard arguments (see for instance [10, 19, 20, 21, 25]), based on Lyusternik’s Inverse Mapping Theorem, which can be found in [24] and [26]. To be more specific, we will see that the desired result is equivalent to find a solution to the equation

\[H(u, v, h) = (0, 0, u_0, v_0),\]  

where \(H : E \to F\) is a \(C^1\) mapping between two appropriate Hilbert spaces, defined by

\[H(u, v, h) = (H_1(u, v, h), H_2(u, v, h), u(\cdot, 0), v(\cdot, 0)),\]

where

\[H_1(u, v, h) = u_t - \left( \mu_1 \left( x, \int_0^1 u \right) u_x \right)_x + f_1(t, x, u, v) - h \chi_{\omega}\]

and

\[H_2(u, v, h) = v_t - \left( \mu_2 \left( x, \int_0^1 v \right) v_x \right)_x + f_2(t, x, u, v).\]

In order to use Lysternik’s Theorem, we need to prove that \(H'(0)\) is onto. It is equivalent to prove a global null controllability result to the linearization of (1) (see the system (11) below). This approach relies on a suitable Carleman estimate for the solutions of the adjoint problem associated to (11) (see Proposition 4).

This paper is organized as follows: Section 2 contains notations we use and a preliminary result. In Section 3, we prove a Carleman type inequality to solutions of (12), which also allows us to obtain an Observality inequality. Section 4 is concerned with the global null controllability of (11) as well as two crucial additional estimates. Section 5 is devoted to the proof of Theorem 1.1. In Section 6, we present some comments and remarks. At the end, we provide three appendices where we sketch the proof of wellposedness of problem (1) and two minors results we use throughout the paper.
2. Preliminaries. In this section we will state some notations and results which
are necessary to prove Theorem 1.1. At first, we need to introduce some weighted
spaces in Section 5. That result can be seen in [15]. Let us consider the
properties of these spaces.

As we pointed out in the Introduction, Lyusternik’s Theorem requires the proof
of a global null controllability result to a linear problem. Hence, we will present a
wellposedness result to this kind of problem that will be used later on the definition
of our spaces. We will also prove a null controllability result for the linear
system
\begin{align}
\begin{aligned}
&\begin{cases}
  \frac{\partial u}{\partial t} - (a(x)u)_x + b_{11}(t,x)u + b_{12}(t,x)v = G_1, & (t,x) \in (0,T) \times (0,1), \\
  \frac{\partial v}{\partial t} - (a(x)v)_x + b_{21}(t,x)u + b_{22}(t,x)v = G_2, & (t,x) \in (0,T) \times (0,1), \\
  u(t,1) = u(t,0) = v(t,0) = v(t,1) = 0, & t \in (0,T), \\
  u(0,x) = u_0(x) & x \in (0,1), \\
  v(0,x) = v_0(x) & x \in (0,1).
\end{cases}
\end{aligned}
\end{align}

Proposition 1. For all $G_1, G_2 \in L^2((0,T) \times (0,1))$ and $u_0, v_0 \in L^2(0,1)$, there
exists a unique weak solution $u, v \in C^0([0,T]; L^2(0,1)) \cap L^2(0,T; H^1_a)$ of (5) and
there exists a positive constant $C_T$ such that
\begin{align}
\sup_{t \in [0,T]} \left( \|u(t)\|_{L^2(0,1)}^2 + \|v(t)\|_{L^2(0,1)}^2 \right) + \int_0^T \left( \|\sqrt{a}u_x\|_{L^2(0,1)}^2 + \|\sqrt{a}v_x\|_{L^2(0,1)}^2 \right) = C_T \left( \|u_0\|_{L^2(0,1)}^2 + \|v_0\|_{L^2(0,1)}^2 + \|G_1\|_{L^2(0,T) \times (0,1)} + \|G_2\|_{L^2(0,T) \times (0,1)} \right).
\end{align}

Moreover, if $u_0, v_0 \in H^1_a$, then
\begin{align}
u, v \in H^1([0,T]; L^2(0,1)) \cap L^2(0,T; H^1_a) \cap C^0([0,T]; H^1_a),
\end{align}
and there exists a positive constant $C_T$ such that
\begin{align}
\sup_{t \in [0,T]} \left( \|u(t)\|_{H^1_a}^2 + \|v(t)\|_{H^1_a}^2 \right)
\leq C_T \left( \|u_0\|_{H^1_a}^2 + \|v_0\|_{H^1_a}^2 + \|G_1\|_{L^2(0,T) \times (0,1)} + \|G_2\|_{L^2(0,T) \times (0,1)} \right).
\end{align}

Now, let us present a Carleman estimate that will play an important role in
the next section. First, consider $\omega' = (\alpha', \beta') \subset \subset \omega$ and let $\psi : [0,1] \to \mathbb{R}$ be a $C^2$
function such that
\begin{align}
\psi(x) := \begin{cases}
\int_0^x \frac{\alpha'(y)}{\alpha(y)} dy, & x \in [0, \alpha') \\
- \int_{\beta'}^{\alpha'} \frac{\alpha'(y)}{\alpha(y)} dy, & x \in [\beta', 1].
\end{cases}
\end{align}
For $\lambda$ sufficiently large, define
\[ \theta(t) := \frac{1}{|t(T-t)|^{\frac{1}{t}}}, \quad \eta(x) := e^{\lambda(|\psi|_\infty + \psi)}, \quad \sigma(x, t) := \theta(t)\eta(x) \quad \text{and} \quad \varphi(x, t) := \theta(t)(e^{\lambda(|\psi|_\infty + \psi)} - e^{3\lambda|\psi|_\infty}). \quad (8) \]

Let us also consider the linear system
\[
\begin{align*}
-\xi_t - (a(x)\xi_x)_x + c(t,x)\xi &= F(t,x), & (t, x) &\in (0,T) \times (0,1), \\
\xi(t,1) &= \xi(t,0) = 0, & t &\in (0,T), \\
\xi(T, x) &= \xi_T(x), & x &\in (0,1),
\end{align*} \quad (9)
\]
where \( a \) satisfies assumption \( A.1 \), \( c \in L^\infty((0,T) \times (0,1)) \), \( F \in L^2((0,T) \times (0,1)) \)
and \( \xi_T \in L^2(0,1) \).

The following Carleman estimate, proved in [13], holds for the solution to (9):

**Proposition 2.** There exist \( C > 0 \) and \( \lambda_0, s_0 > 0 \) such that every solution \( \xi \) of (9) satisfies, for all \( s \geq s_0 \) and \( \lambda \geq \lambda_0 \),
\[
\int_0^T \int_0^1 e^{2s\varphi} \left( (s\lambda)\sigma a\xi_x^2 + (s\lambda)^2 a^2\xi^2 \right) dt \leq C \left( \int_0^T \int_0^1 e^{2s\varphi}|F|^2 + (\lambda s)^3 \int_0^T \int_0^1 e^{2s\varphi}a^3\xi^2 \right), \quad (10)
\]
where the constants \( C, \lambda_0, s_0 \) only depends on \( \omega, a, \|e\|_{L^\infty((0,T) \times (0,1))} \) and \( T \).

For the sake of simplicity, we will introduce the operator
\[
I(s, \lambda, \xi) := \int_0^T \int_0^1 e^{2s\varphi} \left( (s\lambda)\sigma a\xi_x^2 + (s\lambda)^2 a^2\xi^2 \right) dt.
\]

Finally, let us present the following version of *Lyusternik’s Inverse Mapping Theorem* that can be found for instance in [5, 24].

**Theorem 2.1** (Lyusternik). Let \( E \) and \( F \) be two Banach spaces, \( H : E \to F \) a \( C^1 \) mapping and put \( \eta_0 = H(0) \). If \( H'(0) : E \to F \) is onto, then there exist \( r > 0 \) and \( \hat{H} : B_r(\eta_0) \subset F \to E \) such that
\[
H(\hat{H}(\xi)) = \xi, \quad \forall \xi \in B_r(\eta_0),
\]
that is, \( \hat{H} \) is a right inverse of \( H \). In addition, there exists \( K > 0 \) such that
\[
\|\hat{H}(\xi)\|_E \leq K\|\xi - \eta_0\|, \quad \forall \xi \in B_r(\eta_0).
\]

3. Carleman and observability inequalities. In order to prove that map \( H \) is onto, we have to prove a global null controllability result to the linearization of (1), given by
\[
\begin{align*}
\begin{cases}
\frac{u_t}{u_x} + b_{11}u + b_{12}v = h\chi_0 + g_1, & (t, x) \in (0,T) \times (0,1), \\
\frac{v_t}{v_x} + b_{21}u + b_{22}v = g_2, & (t, x) \in (0,T) \times (0,1), \\
u(t,1) = u(t,0) = v(t,0) = v(t,1), & t \in (0,T), \\
u(0,x) = u_0(x) \quad \text{and} \quad v(0,x) = v_0(x), & x \in (0,1),
\end{cases}
\end{align*} \quad (11)
\]
where \( g_1, g_2 \) and \( h \) belong to appropriate \( L^2 \)-weighted spaces which we will specify later on. To this purpose it is crucial to obtain an appropriate Carleman estimate for solutions to
There exist positive constants $C, \lambda_0$ and $s_0$ such that, for any $s \geq s_0$, $\lambda \geq \lambda_0$ and any $y_T, z_T \in L^2((0, T) \times (0, 1))$, the corresponding solution $(y, z)$ to (12) satisfies

$$
\int_0^T \int_0^1 e^{2s\varphi} ((s\lambda)\sigma a(y_x^2 + z_x^2) + (s\lambda)^2 \sigma^2 (y^2 + z^2)) \, dt \, dx
\leq C \left( \int_0^T \int_0^1 e^{2s\varphi} s^4 \lambda^4 (|F_1|^2 + |F_2|^2) + \int_0^T \int_\omega e^{2s\varphi} s^8 \lambda^8 \sigma^8 y^2 \, dt \, dx \right). \tag{13}
$$

**Proof.** Firstly, we rewrite the first equation in (12) as

$$
- y_t - (a(x)y_x)_x + b_{11}(t, x)y + b_{21}(t, x)z = F_1, \quad (t, x) \in (0, T) \times (0, 1),
$$

and by Theorem 2, we obtain

$$
y(t, 0) = y(t, 1) = z(t, 0) = z(t, 1) = 0, \quad t \in (0, T),
$$

$$
y(T, x) = y_T(x), \quad z(T, x) = z_T(x), \quad x \in (0, 1),
$$

which is the adjoint problem of (11).

**Proposition 3.** There exist positive constants $C, \lambda_0$ and $s_0$ such that, for any $s \geq s_0$, $\lambda \geq \lambda_0$ and any $y_T, z_T \in L^2((0, T) \times (0, 1))$, the corresponding solution $(y, z)$ to (12) satisfies

$$
\int_0^T \int_0^1 e^{2s\varphi} ((s\lambda)\sigma a(y_x^2 + z_x^2) + (s\lambda)^2 \sigma^2 (y^2 + z^2)) \, dt \, dx
\leq C \left( \int_0^T \int_0^1 e^{2s\varphi} s^4 \lambda^4 (|F_1|^2 + |F_2|^2) + \int_0^T \int_\omega e^{2s\varphi} s^8 \lambda^8 \sigma^8 y^2 \, dt \, dx \right). \tag{13}
$$

Proceeding in the same way for the second equation, we get an analogous inequality

$$
I(s, \lambda, z)
\leq C \left( \int_0^T \int_0^1 e^{2s\varphi} s^4 \lambda^4 (|F_1|^2 + |F_2|^2) + \int_0^T \int_\omega e^{2s\varphi} \lambda^3 s^3 \sigma^3 |y|^2 \, dt \, dx \right).
$$

Now, we add these two inequalities and take $s$ and $\lambda$ sufficiently large such that $I(s, \lambda, y)$ will absorb the integral depend on $|y|^2$, and $I(s, \lambda, z)$ the integral depend on $|z|^2$. This will give us the following inequality

$$
I(s, \lambda, y) + I(s, \lambda, z)
\leq C \left( \int_0^T \int_0^1 e^{2s\varphi} (|F_1|^2 + |F_2|^2) + \int_0^1 \int_\omega e^{2s\varphi} \lambda^3 s^3 \sigma^3 (|y|^2 + |z|^2) \right).
$$

Thus, in order to obtain (13), it is sufficient to show that there exists a small $\varepsilon > 0$ such that

$$
\int_0^1 \int_\omega e^{2s\varphi} \lambda^3 s^3 \sigma^3 |z|^2 \leq \varepsilon I(s, \lambda, z)
+ C \left( \int_0^T \int_0^1 e^{2s\varphi} s^4 \lambda^4 (|F_1|^2 + |F_2|^2) + \int_0^1 \int_\omega e^{2s\varphi} \lambda^8 s^8 \sigma^8 |y|^2 \right).
$$
Let us take $\chi \in C^\infty_0(\omega)$ satisfying $0 \leq \chi \leq 1$ and $\chi \equiv 1$ in $\omega_1$. Since $\inf b_{21} > 0$, we can easily see that
\[
\int_0^1 \int_0^{\omega_1} e^{2s\varphi} \lambda^3 \sigma^3 |z|^2 \leq C \int_0^T \int_\omega b_{21} e^{2s\varphi} \lambda^3 \sigma^3 |z|^2.
\]

Now, multiplying the first equation in (12) by $e^{2s\varphi} \lambda^3 \sigma^3 \chi z$ and integrating over $(0,T) \times (0,1)$, we get
\[
\int_0^T \int_0^1 \chi b_{21} e^{2s\varphi} \lambda^3 \sigma^3 |z|^2 = \int_0^T \int_0^1 \chi e^{2s\varphi} \lambda^3 \sigma^3 z F_1 + \int_0^T \int_0^1 \chi e^{2s\varphi} \lambda^3 \sigma^3 z y_t + \int_0^T \int_0^1 \chi e^{2s\varphi} \lambda^3 \sigma^3 (a y_z)_x z - \int_0^T \int_0^1 \chi b_{11} e^{2s\varphi} \lambda^3 \sigma^3 y z =: I_1 + I_2 + I_3 + I_4.
\]

Next, we need to estimate $I_1, I_2, I_3$ and $I_4$. Firstly, from Young’s inequality, we have
\[
I_1 \leq C \int_0^T \int_0^1 \chi e^{2s\varphi} \lambda^3 \sigma^3 |z||F_1|
\]
\[
\leq \varepsilon \int_0^T \int_0^1 e^{2s\varphi} \lambda^2 \sigma^2 |z|^2 + C \varepsilon \int_0^T \int_0^1 \chi e^{2s\varphi} \lambda^4 \sigma^4 |F_1|^2
\]
\[
\leq \varepsilon I(s, \lambda, z) + C \varepsilon \int_0^T \int_0^1 e^{2s\varphi} \lambda^4 \sigma^4 |F_1|^2.
\]

In the same way, since $b_{11}$ is bounded, it is immediate that
\[
I_4 \leq C \int_0^T \int_0^1 \chi e^{2s\varphi} \lambda^3 \sigma^3 |y||z|
\]
\[
\leq \varepsilon \int_0^T \int_0^1 e^{2s\varphi} \lambda^2 \sigma^2 |z|^2 + C \varepsilon \int_0^T \int_0^1 \chi e^{2s\varphi} \lambda^4 \sigma^4 |y|^2
\]
\[
\leq \varepsilon I(s, \lambda, z) + C \varepsilon \int_0^T \int_0^\omega e^{2s\varphi} \lambda^8 \sigma^8 |y|^2.
\]

Using integration by parts, we will split up $I_2$ and $I_3$ in several integrals. In fact,
\[
I_3 = - \int_0^T \int_0^1 \chi s^3 \lambda^3 \sigma^3 e^{2s\varphi} a \chi x y_z + \int_0^T \int_0^1 \left( a \chi e^{2s\varphi} \right)_x \lambda^3 \sigma^3 (x^3 \chi^3 y z + \int_0^T \int_0^1 s^3 \chi^3 \chi e^{2s\varphi} a \chi x y.
\]

and, recalling that $e^{2s\varphi}$ vanishes at 0 and $T$ and using the second equation of (12), we have
\[
I_2 = - \int_0^T \int_0^1 \left[ \chi s^3 \lambda^3 (e^{2s\varphi} \sigma^3)_z + \chi s^3 \lambda^3 \sigma^3 e^{2s\varphi} b_{22} \right] y z
\]
\[
+ \int_0^T \int_0^1 \chi s^3 \lambda^3 \sigma^3 e^{2s\varphi} a \chi x y - \int_0^T \int_0^1 \chi s^3 \sigma^3 \sigma_2 e^{2s\varphi} b_{12} y_2
\]
\[
+ \int_0^T \int_0^1 \chi s^3 \lambda^3 \sigma^3 e^{2s\varphi} y F_2.
\[
= - \int_0^T \int_0^1 \left[ \chi s^3 \lambda^3 (e^{2s\varphi} \sigma^3)_x + \chi s^3 \lambda^3 \sigma^3 e^{2s\varphi} b_{22} \right] yz \\
- \int_0^T \int_0^1 s^3 \lambda^3 (\chi e^{2s\varphi} \sigma^3)_x az_x y - \int_0^T \int_0^1 \chi s^3 \lambda^3 \sigma^3 e^{2s\varphi} az_x y_x \\
- \int_0^T \int_0^1 \chi s^3 \lambda^3 \sigma^3 e^{2s\varphi} b_{12} y^2 + \int_0^T \int_0^1 \chi s^3 \lambda^3 \sigma^3 e^{2s\varphi} y F_2.
\]  
(16)

Thus,
\[
J_2 + J_3 = \int_0^T \int_0^1 \left[ -\chi s^3 \lambda^3 (e^{2s\varphi} \sigma^3)_x - \chi s^3 \lambda^3 \sigma^3 e^{2s\varphi} b_{22} + s^3 \lambda^3 \left( a (\chi e^{2s\varphi} \sigma^3)_x \right)_x \right] yz \\
- 2 \int_0^T \int_0^1 \chi s^3 \lambda^3 \sigma^3 e^{2s\varphi} az_x y_x - \int_0^T \int_0^1 \chi s^3 \lambda^3 \sigma^3 e^{2s\varphi} b_{12} y^2 \\
+ \int_0^T \int_0^1 \chi s^3 \lambda^3 \sigma^3 e^{2s\varphi} y F_2
= J_1 + J_2 + J_3 + J_4.
\]

Now, it remains estimates these four integrals. It is immediate that
\[
J_3 \leq C \int_0^T \int_\omega e^{2s\varphi} s^8 \lambda^8 \sigma^8 y^2.
\]  
(18)

and, from Young’s inequality, that
\[
J_4 \leq \int_0^T \int_0^1 e^{2s\varphi} s^4 \lambda^4 \sigma^4 |F_2|^2 + \int_0^T \int_\omega e^{2s\varphi} s^8 \lambda^8 \sigma^8 y^2.
\]

In order to estimate $J_1$, we will analyze each term between brackets. Firstly, we observe that all the terms are multiplied by $\chi$, which vanishes outside of $\omega$. Clearly,
\[
|\chi s^3 \lambda^3 \sigma^3 e^{2s\varphi} b_{22}| \leq C \chi s^5 \lambda^5 \sigma^5 e^{2s\varphi}.
\]

Since $|\sigma_x| \leq C \lambda \sigma$, $|\sigma_{xx}| \leq C \lambda^2 \sigma$ and $a \in C^1(\omega)$, after distributing the derivatives with respect to $x$, we can see that
\[
|s^3 \lambda^3 (a (\chi e^{2s\varphi} \sigma^3)_x)_x| \leq C \chi s^5 \lambda^5 \sigma^5 e^{2s\varphi}.
\]

Likewise, the relations $|\varphi_1| \leq C \sigma^2$ and $|\sigma_1| \leq C \sigma^2$ yield
\[
-\chi s^3 \lambda^3 (e^{2s\varphi} \sigma^3)_x \leq C \chi s^5 \lambda^5 \sigma^5 e^{2s\varphi}.
\]

As a conclusion,
\[
J_1 \leq C \int_0^T \int_0^1 \chi s^5 \lambda^5 \sigma^5 e^{2s\varphi} |yz| \leq \varepsilon I(s, \lambda, z) + C \varepsilon \int_0^T \int_\omega s^8 \lambda^8 \sigma^8 e^{2s\varphi} y^2.
\]

The last step is to deal with $J_2$. To do this, we notice that
\[
J_2 \leq \varepsilon \int_0^T \int_0^1 e^{2s\varphi} s \lambda \sigma a z^2 + C \varepsilon \int_0^T \int_0^1 \chi s^5 \lambda^5 \sigma^5 \sigma_1 y^2 \\
\leq \varepsilon I(s, \lambda, z) + C \varepsilon \int_0^T \int_0^1 \chi s^5 \lambda^5 \sigma^5 \sigma_y y^2.
\]

Hence, we only need to estimate the last integral. Multiplying the first equation in (12) by $\chi^2 e^{2s\varphi} s^3 \lambda^5 \sigma^5 y$, integrate over $(0, T) \times (0, 1)$ and integrating by parts we get that
\[ \int_0^T \int_0^1 \chi^2 e^{2s\varphi} s^5 \lambda^5 \sigma^5 y^2 \leq \int_0^T \int_0^1 \left[ -\frac{1}{2} \chi^2 s^5 \lambda^5 (e^{2s\varphi} \sigma^5) + \frac{1}{2} s^5 \lambda^5 \left[a(e^{2s\varphi} \sigma^5)\right]_x - \chi^2 b_{11} e^{2s\varphi} s^5 \lambda^5 \sigma^5 \right] y^2 \]

We can see that all the integrals here are of the same type of those in (17). Following the same arguments developed there, we have the result. \(\square\)

Now we need to prove a Carleman inequality for solutions of problem (12) with weights which do not vanish at \(t = 0\). It is necessary in order to guarantee the null controllability results in Theorem 4.1 and Theorem 1.1. We will give more details in Remark 1.

Consider a function \(m \in C^\infty([0, T])\) satisfying
\[
\left\{ \begin{array}{l}
m(t) \geq t^4(T - t)^4, \quad t \in (0, T/2]; \\
m(t) = t^4(T - t)^4, \quad t \in [T/2, T]; \\
m(0) > 0,
\end{array} \right.
\]
and define
\[
\tau(t) := \frac{1}{m(t)}, \quad \zeta(x,t) := \tau(t)(x) \quad \text{and} \quad A(t,x) := \tau(t)(x)(e^{\lambda(|\psi| + \psi)} - e^{\lambda|\psi|}).
\]
where \((t,x) \in [0, T] \times [0, 1]\). As usual, we introduce the operators
\[
\Gamma(s, \xi, \vartheta) := \int_0^T \int_0^1 e^{2sA} \left[ s\lambda \zeta a \left( |\xi_x|^2 + |\vartheta_x|^2 \right) + (s\lambda)^2 \zeta^2 \left( |\xi|^2 + |\vartheta|^2 \right) \right],
\]
\[
\Gamma_1(s, \xi, \vartheta) := \int_0^{T/2} \int_0^1 e^{2sA} \left[ s\lambda \zeta a \left( |\xi_x|^2 + |\vartheta_x|^2 \right) + (s\lambda)^2 \zeta^2 \left( |\xi|^2 + |\vartheta|^2 \right) \right]
\]
and
\[
\Gamma_2(s, \xi, \vartheta) := \int_0^T \int_0^1 e^{2sA} \left[ s\lambda \zeta a \left( |\xi_x|^2 + |\vartheta_x|^2 \right) + (s\lambda)^2 \zeta^2 \left( |\xi|^2 + |\vartheta|^2 \right) \right].
\]

**Proposition 4.** There exist positive constants \(C, \lambda_0\) and \(s_0\) such that, for any \(s \geq s_0, \lambda \geq \lambda_0\) and any \(y, z \in L^2((0, T) \times (0, 1))\) the corresponding solution \((y, z)\) to (12) satisfies
\[
\int_0^T \int_0^1 e^{2sA} \left[ s\lambda \zeta a \left( |y_x|^2 + |z_x|^2 \right) + (s\lambda)^2 \zeta^2 \left( |y|^2 + |z|^2 \right) \right]
\leq C \left( \int_0^T \int_0^1 e^{2sA} s^4 \lambda^4 \zeta^4 (|F_1|^2 + |F_2|^2) + \int_0^T \int_\omega e^{2sA} s^8 \lambda^8 \zeta^8 |y|^2 \right).
\]

**Proof.** In order to estimate \(\Gamma_2(s, y, z)\), let us observe that \(e^{2s\varphi} \sigma^n \leq C e^{2sA} \zeta^n\) for all \((t, x) \in [0, T] \times [0, 1]\) and \(n \geq 0\). Since \(\tau = \theta\) and \(A = \varphi\) in \([T/2, T]\), Carleman inequality (13) implies
\[
\Gamma_2(s, y, z) \leq C \left( \int_0^T \int_0^1 e^{2sA} s^4 \lambda^4 \zeta^4 (|F_1|^2 + |F_2|^2) + \int_0^T \int_\omega e^{2sA} s^8 \lambda^8 \zeta^8 |y|^2 \right).
\]
Thus, integrating inequality (22) from $-y$ to $z$, respectively, and integrating over $[0, 1]$, we obtain
\[
-\frac{d}{dt} \left( \|y\|^2_{L^2([0,1])} + \|z\|^2_{L^2([0,1])} \right) - C \left( \|y\|^2_{L^2([0,1])} + \|z\|^2_{L^2([0,1])} \right) + 2 \left( \|\sqrt{a}y_x\|^2_{L^2([0,1])} + \|\sqrt{a}z_x\|^2_{L^2([0,1])} \right) \leq \|F_1\|^2_{L^2([0,1])} + \|F_2\|^2_{L^2([0,1])},
\]
which implies
\[
-\frac{d}{dt} \left[ e^{CT} \left( \|y\|^2_{L^2([0,1])} + \|z\|^2_{L^2([0,1])} \right) \right] \leq e^{CT} \left( \|F_1\|^2_{L^2([0,1])} + \|F_2\|^2_{L^2([0,1])} \right).
\]
Integrating from $t \in [0, T/2]$ to $t + T/4$, we get
\[
\|y\|^2_{L^2([0,1])} + \|z\|^2_{L^2([0,1])} \leq e^{CT} \int_0^{T/4} \left( \|F_1\|^2_{L^2([0,1])} + \|F_2\|^2_{L^2([0,1])} \right)
+ e^{3CT/4} \left( \|y(t + T/4)\|^2_{L^2([0,1])} + \|z(t + T/4)\|^2_{L^2([0,1])} \right).
\]
Hence, we conclude that
\[
\int_0^{T/2} \left( \|y\|^2_{L^2([0,1])} + \|z\|^2_{L^2([0,1])} \right) \leq e^{CT} \int_0^{3T/4} \left( \|F_1\|^2_{L^2([0,1])} + \|F_2\|^2_{L^2([0,1])} \right)
+ e^{3CT} \int_{T/4}^{3T/4} \left( \|y\|^2_{L^2([0,1])} + \|z\|^2_{L^2([0,1])} \right). \tag{21}
\]
Now, integrating inequality (20) over $[0, t]$, where $t \in [0, T]$, we take
\[
\int_0^t \left( \|\sqrt{a}y_x\|^2_{L^2([0,1])} + \|\sqrt{a}z_x\|^2_{L^2([0,1])} \right) \leq \frac{1}{2} \left( \|y\|^2_{L^2([0,1])} + \|z\|^2_{L^2([0,1])} \right)
+ C \left[ \int_0^t \left( \|y\|^2_{L^2([0,1])} + \|z\|^2_{L^2([0,1])} \right) + \int_0^t \left( \|F_1\|^2_{L^2([0,1])} + \|F_2\|^2_{L^2([0,1])} \right) \right]. \tag{22}
\]
In order to establish our next inequality, first we recall that if a function $f$ is non-negative, then the function $t \mapsto \int_0^t f(t)$ is non-decreasing. As a consequence, for all $t \in [T/2, 3T/4]$, we have that
\[
\int_0^{T/2} \left( \|\sqrt{a}y_x\|^2_{L^2([0,1])} + \|\sqrt{a}z_x\|^2_{L^2([0,1])} \right) \leq \int_0^t \left( \|\sqrt{a}y_x\|^2_{L^2([0,1])} + \|\sqrt{a}z_x\|^2_{L^2([0,1])} \right)
\leq \int_0^{3T/4} \left( \|\sqrt{a}y_x\|^2_{L^2([0,1])} + \|\sqrt{a}z_x\|^2_{L^2([0,1])} \right).
\]
Thus, integrating inequality (22) from $T/2$ to $3T/4$ and using (21) we have
\[
\int_0^{3T/4} \left( \|\sqrt{a}y_x\|^2_{L^2([0,1])} + \|\sqrt{a}z_x\|^2_{L^2([0,1])} \right)
\leq C \left[ \int_{T/2}^{3T/4} \left( \|y\|^2_{L^2([0,1])} + \|z\|^2_{L^2([0,1])} \right) + \int_0^{3T/4} \left( \|F_1\|^2_{L^2([0,1])} + \|F_2\|^2_{L^2([0,1])} \right) \right]. \tag{23}
\]
Finally, we observe that $e^{2sA} (s\lambda\zeta)^n$ and $e^{2s\varphi} (s\lambda\sigma)^n$ are bounded in $[0, T/2]$ and $[T/4, 3T/4]$. 

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respectively, for all \( n \in \mathbb{Z} \). Hence, \((21), (23)\) and Carleman Inequality \((13)\) imply
\[
\Gamma_1(s, y, z) = \int_0^{T/2} \int_0^1 e^{2sA} [s\lambda z a (|y_x|^2 + |z_x|^2) + (s\lambda)^2 \zeta^2 (|y|^2 + |z|^2)] \\
\leq C \left( \int_{T/4}^{3T/4} \int_0^1 (|y|^2 + |z|^2) + \int_0^{3T/4} \int_0^1 (|F_1|^2 + |F_2|^2) \right) \\
\leq C \left( \int_{T/4}^{3T/4} \int_0^1 e^{2sA}(s\lambda)^2 \sigma^2 (|y|^2 + |z|^2) \\
+ \int_0^{3T/4} \int_0^1 e^{2sA}(s\lambda)^4 \sigma^4 (|F_1|^2 + |F_2|^2) \right) \\
\leq C \left( \int_0^T \int_0^1 e^{2sA(s\lambda)^4 \zeta^2} (|F_1|^2 + |F_2|^2) + \int_0^T \int_{\omega} e^{2sA(s\lambda)^8 \zeta^8}|y|^2 \right),
\]
which concludes the proof. \( \square \)

**Corollary 1.** There exist positive constants \( C, \lambda_0 \) and \( s_0 \) such that, for any \( s \geq s_0 \), \( \lambda \geq \lambda_0 \) and any \( y, z \in L^2(0, 1) \) the corresponding solution \((y, z)\) to \((12)\), with \( F_1 \equiv F_2 \equiv 0 \), satisfies
\[
\|y(0)\|_{L^2(0, 1)}^2 + \|z(0)\|_{L^2(0, 1)}^2 \leq C \int_0^T \int_{\omega} e^{2sA(s\lambda)^8 \zeta^8}|y|^2. \tag{24}
\]

**Proof.** Using standard energy inequalities for each equations in \((12)\), we obtain
\[
-\frac{d}{2dt} \left( \|y\|_{L^2(0, 1)}^2 + \|z\|_{L^2(0, 1)}^2 \right) \leq 2C \left( \|y\|_{L^2(0, 1)}^2 + \|z\|_{L^2(0, 1)}^2 \right).
\]
Hence, we get that
\[
\|y(0)\|_{L^2(0, 1)}^2 + \|z(0)\|_{L^2(0, 1)}^2 \leq e^{4CT} \left( \|y(0)\|_{L^2(0, 1)}^2 + \|z(0)\|_{L^2(0, 1)}^2 \right). \tag{25}
\]
Finally, integrating the last inequality in \([0, 3T/4]\), recalling that \( e^{2sA(s\lambda)^2 \zeta^2} \) is bounded from below in \([0, 3T/4]\) and using \((19)\) with \( F_1 \equiv F_2 \equiv 0 \), we obtain
\[
\|y(0)\|_{L^2(0, 1)}^2 + \|z(0)\|_{L^2(0, 1)}^2 \leq C \int_0^{3T/4} \int_0^1 e^{2sA(s\lambda)^2 \zeta^2} (|y|^2 + |z|^2) \leq C \int_0^T \int_{\omega} e^{2sA(s\lambda)^8 \zeta^8}|y|^2.
\]
\( \square \)

4. **Global null controllability for the linear system.** The goal of this section is to prove a null controllability result for the linear system \((11)\) and establish some important additional estimates. In order to state this result, we need to define the weights functions:
\[
\rho = e^{-sA}, \quad \rho_0 = e^{-sA}, \quad \hat{\rho} = e^{-sA}, \quad \rho_* = e^{-sA},
\]
which satisfy \( \rho_* \leq C\hat{\rho} \leq C\rho_0 \leq C\rho \) and \( \hat{\rho}^2 = \rho_* \rho_0 \).

**Theorem 4.1.** If \( u_0, v_0 \in H^1_a(0, 1) \) and the functions \( g_1 \) and \( g_2 \) fulfill
\[
\int_0^T \int_0^1 \rho_0^2 (|g_1|^2 + |g_2|^2) < +\infty,
\]
then the system (11) is null-controllable. More precisely, there exists a control $h \in L^2((0, T) \times \omega)$ with associated state $(u, v)$ satisfying
\[ \int_0^T \int_\omega \rho_n^2 |h|^2 < +\infty \text{ and } \int_0^T \int_0^1 \rho_n^2(|u|^2 + |v|^2) < +\infty. \] (26)
In particular, $u(T, x) = v(T, x) = 0$, for all $x \in [0, 1]$.

**Remark 1.** (a) Recalling that $\rho_0(t) \to +\infty$, as $t \to T^-$, and $\rho_0(0) > 0$ (since $m(0) > 0$), the second relation in (26) guarantees $u(T, x) = v(T, x) = 0$.

(b) If we had chosen $m \in C^\infty([0, T])$ satisfying $m(0) = 0$, we would verify $\rho_0(t) \to +\infty$, as $t \to 0^+$. As a consequence, the second relation in (26) would imply $u(0, x) = v(0, x) \equiv 0$. However, in general, this fact is not true, because $u_0 \in H^1_0(0, 1)$ and $v_0 \in H^1_0(0, 1)$ must be taken arbitrarily.

**Proof.** For each $n \in \mathbb{N}^*$, we define
\[ A_n(t, x) = \frac{A(T - t)^4}{(T - t)^4 + \frac{1}{n}}, \quad (t, x) \in [0, T] \times [0, 1]. \]
We also consider
\[ \rho_n = e^{-sA_n}, \quad \rho_{0, n} = \rho_n \zeta^{-2} \text{ and } \rho_{*, n} = \rho_n \zeta^{-4} m_n, \quad \text{where } m_n = \begin{cases} 1, & x \in \omega, \\ n, & x \notin \omega. \end{cases} \]

These weight functions are built in such a way that $\rho_{0, n}$ and $\rho_{*, n}$ are bounded from below by a constant depending only on $T$ and from above by another one depending on $n$ and $T$, see Lemma C.1. It will allows us to obtain a sequence $(u_n, v_n, h_n)_{n \in \mathbb{N}^*}$ of solutions to (11) which will converge to a solution of (11) satisfying (26).

To do that, for any functions $u, v, h \in L^2((0, T) \times (0, 1))$, let us define the functional
\[ J_n(u, v, h) = \frac{1}{2} \int_0^T \int_0^1 \rho_{0, n}^2 (|u|^2 + |v|^2) + \frac{1}{2} \int_0^T \int_0^1 \rho_{*, n}^2 |h|^2. \]
Since each $J_n$ is lower semi-continuous, strictly convex and coercive (see Appendix C), Proposition 1.2 in [14] yields a unique $(u_n, v_n, h_n)$ that minimizes $J_n(u, v, h)$ subject to the condition $C = \{(u, v, h) \in [L^2(0, T) \times (0, 1)]^3 : (u, v, h) \text{ solves (11)}\}$. In this case, $(u_n, v_n, h_n)$ satisfies (11) and, by virtue of Lagrange’s Principle, there exist functions $p_n, q_n$ solving the following system
\[ \begin{aligned}
-p_{nt} - (ap_{nx})_x + b_{11} p_n + b_{21} q_n &= -\rho_{0, n}^2 u_n, \quad (t, x) \in (0, T) \times (0, 1), \\
-q_{nt} - (aq_{nx})_x + b_{12} p_n + b_{22} q_n &= -\rho_{0, n}^2 v_n, \quad (t, x) \in (0, T) \times (0, 1), \\
p_n(T, x) &= q_n(T, x) = q_n(t, 1) = 0, \quad t \in (0, T), \\
q_n(T, x) &= q_n(T, x) = 0, \quad x \in (0, 1), \\
p_n &= \rho_{*, n}^2 h_n \chi, \quad \in (0, T) \times (0, 1). 
\end{aligned} \] (27)

We want to prove that $(J_n(u_n, v_n, h_n))_{n=1}^\infty$ is a numerical bounded sequence. To do that, we will use Carleman and Observability inequalities to prove that $J_n(u_n, v_n, h_n) \leq C \sqrt{J_n(u_n, v_n, h_n)}$.

In fact, multiplying the PDEs in (27) by $u_n$ and $v_n$, integrating over $(0, T) \times (0, 1)$ and using integration by parts, we have
\[ 0 = \int_0^T \int_0^1 [-p_{nt} - (ap_{nx})_x + b_{11} p_n + b_{21} q_n + \rho_{0, n}^2 u_n] u_n \]
Hence, since \( \rho^* J_u \int_0^T \int_0^1 u_{nt} - (au_{nx})_x + b_{11}u_n + b_{12}v_n + \rho_{0,n}^2 v_n \) \( v_n \)

\[= \int_0^T \int_0^1 [u_{nt} - (au_{nx})_x + b_{11}u_n + b_{12}v_n] p_n + \int_0^T \int_0^1 \rho_{0,n}^2 |u_n|^2 + \int_0^1 p_n(0, x) u_0 \]

\[+ \int_0^T \int_0^1 |v_{nt} - (av_{nx})_x + b_{21}u_n + b_{22}v_n| q_n \]

\[+ \int_0^T \int_0^1 \rho_{0,n}^2 |v_n|^2 + \int_0^1 q_n(0, x) v_0 \]

\[= \int_0^T \int_0^1 (h_n \chi_\omega + g_1)p_n + \int_0^T \int_0^1 \rho_{0,n}^2 |u_n|^2 + \int_0^1 p_n(0, x) u_0 \]

\[+ \int_0^T \int_0^1 g_2 q_n + \int_0^T \int_0^1 \rho_{0,n}^2 |v_n|^2 + \int_0^1 q_n(0, x) v_0 \]

Hence, since \( p_n = \rho^* h_n \chi_\omega \), we obtain

\[J_n(u_n, v_n, h_n) = \frac{1}{2} \int_0^T \int_0^1 \rho_{0,n}^2 (|u_n|^2 + |v_n|^2) + \frac{1}{2} \int_0^T \int_0^1 \rho_{0,n}^2 |h_n|^2 \chi_\omega \]

\[= -\frac{1}{2} \int_0^T \int_0^1 (h_n \chi_\omega + g_1)p_n - \frac{1}{2} \int_0^1 p_n(0, x) u_0 \]

\[= -\frac{1}{2} \int_0^T \int_0^1 g_2 q_n - \frac{1}{2} \int_0^1 q_n(0, x) v_0 + \frac{1}{2} \int_0^T \int_0^1 p_n h_n \chi_\omega \]

Using Hölder inequality, we have

\[J_n(u_n, v_n, h_n) \leq \frac{1}{2} \|g_1 p_0\|_{L^2} \|\rho_{0,n}^{-1} p_n\|_{L^2} + \frac{1}{2} \|g_2 p_0\|_{L^2} \|\rho_{0,n}^{-1} q_n\|_{L^2} \]

\[+ \frac{1}{2} \|p_n(0, \cdot)\|_{L^2(0, 1)} \|u_0\|_{L^2(0, 1)} + \frac{1}{2} \|q_n(0, \cdot)\|_{L^2(0, 1)} \|v_0\|_{L^2(0, 1)}. \]

Applying the classical Cauchy-Schwartz inequality \( |a \cdot b| \leq |a| |b| \), where \( a, b \in \mathbb{R}^4 \)

and then using hypothesis \( \int_0^T \int_0^1 \rho_{0,n}^2 (|g_1|^2 + |g_2|^2) < +\infty \), we get

\[J_n(u_n, v_n, h_n) \]

\[\leq \frac{1}{2} \left( \|g_1 p_0\|_{L^2}^2 + \|g_2 p_0\|_{L^2}^2 + \|u_0\|^2 + \|v_0\|^2 \right)^{1/2} \]

\[\cdot \left( \|\rho_{0,n}^{-1} p_n\|_{L^2}^2 + \|\rho_{0,n}^{-1} q_n\|_{L^2}^2 + \|p_n(0, \cdot)\|_{L^2(0, 1)}^2 + \|q_n(0, \cdot)\|_{L^2(0, 1)}^2 \right)^{1/2} \]

\[\leq C \left( \int_0^T \int_0^1 \rho_{0,n}^{-2} (p_n^2 + q_n^2) + \|p_n(0, \cdot)\|_{L^2(0, 1)}^2 + \|q_n(0, \cdot)\|_{L^2(0, 1)}^2 \right)^{1/2}. \]

Now, it is enough to prove that each term in the last inequality is bounded by \( J(u_n, v_n, h_n) \). In order to estimate the first term, we will apply Carleman inequality (19) to the solution \( (p_n, q_n) \) to (27) and then use that \( \rho_{0,n}^2 \rho_n^4 \leq \rho_n^2 \) and \( p_n = \rho_{0,n}^2 h_n \chi_\omega \) in \( (0, T) \times (0, 1) \).
Indeed,
\[
\int_0^T \int_0^1 \rho_0^{-2}(p_n^2 + q_n^2)
\]
\[
\leq \frac{1}{(s\lambda)^2} \int_0^T \int_0^1 e^{2sA(s\lambda)^2}\zeta^2(p_n^2 + q_n^2)
\]
\[
\leq C \left( \int_0^T \int_0^1 e^{2sA(s\lambda)^4}\rho_{0,n}^4(|u_n|^2 + |v_n|^2) + \int_0^T \int_\omega e^{2sA(s\lambda)^8}|p_n|^2 \right)
\]
\[
\leq C \left( \int_0^T \int_0^1 \rho^{-2}\rho_{0,n}^4(|u_n|^2 + |v_n|^2) + \int_0^T \int_\omega \rho^{-2}s\lambda^8\rho_{*,n}|h_n|^2 \right)
\]
\[
\leq C \left( \int_0^T \int_0^1 \rho_{0,n}^2(|u_n|^2 + |v_n|^2) + \int_0^T \int_\omega \rho_{*,n}^2|h_n|^2 \right)
\]
\[
= CJ_n(u_n, v_n, h_n). \]

The remain terms are readily estimated from the Observability Inequality (24). In fact,
\[
\|p_n(0, \cdot)\|_{\ell_2(0,1)}^2 + \|q_n(0, \cdot)\|_{\ell_2(0,1)}^2 \leq C \left( \int_0^T \int_\omega e^{2sA(s\lambda)^8}|p_n|^2 \right)
\]
\[
\leq CJ_n(u_n, v_n, h_n). \]

Hence, we have proven that \(J_n(u_n, v_n, h_n) \leq C\sqrt{J_n(u_n, v_n, h_n)}\). As a consequence, \((J_n(u_n, v_n, h_n))_{n\in\mathbb{N}}\) is a bounded sequence. Since \(\rho_{0,n}^2 \geq C_T\) and \(\rho_{*,n}^2 \geq C_T m_n\), we deduce that
\[
\|u_n\|_{\ell_2}^2 + \|v_n\|_{\ell_2}^2 + \int_0^T \int_\omega |h_n|^2 + n \int_0^T \int_{\{0,1\}\backslash\omega} |h_n|^2 \leq CJ_n(u_n, v_n, h_n) \leq C.
\]

It implies that there exist \(u, v \in L^2((0,T) \times (0,1))\) and \(h \in L^2((0,T) \times \omega)\) such that
\[
u_n \to u, \quad v_n \to v \quad \text{in} \quad L^2((0,T) \times (0,1)) \quad \text{and} \quad h_n \chi_\omega \to h \quad \text{in} \quad L^2((0,T) \times \omega),
\]
up to subsequences. From this, we take
\[
\rho_{0,n} u_n \to \rho_0 u \quad \rho_{0,n} v_n \to \rho_0 v \quad \text{and} \quad \rho_{*,n} h_n \chi_\omega \to \rho_* h \chi_\omega \quad \text{in} \quad L^2((0,T) \times (0,1)). \quad (28)
\]

Consequently, passing to limits as \(n \to +\infty\), we conclude that \((u, v, h)\) solves (11). Furthermore, (26) follows from (28) and this establishes the result. \qed

The next step is to prove two crucial estimates which will needed later.

**Proposition 5.** Assume the same hypothesis of Theorem 4.1. Then
\[
\int_0^T \int_0^1 \rho^2 a(|u_x|^2 + |v_x|^2) \leq C \left( \int_0^T \int_0^1 \rho_0^2(|u|^2 + |v|^2) + \int_0^T \int_\omega \rho_*^2|h|^2 \right)
\]
\[
+ C \left( \int_0^T \int_0^1 \rho_0^2(|g_1|^2 + |g_2|^2) + \|u_0\|_{H^4}^2 + \|v_0\|_{H^4}^2 \right). \quad (29)
\]
Proof. Let us multiply the first equation in (11) by $\hat{\rho}^2 u$ and the second one by $\hat{\rho}^2 v$, and let us integrate over $[0, 1]$. In this case, we obtain

$$
\int_0^1 \hat{\rho}^2 [u_t u + v_t v] - \int_0^1 \hat{\rho}^2 [(au_x)_x u + (av_x)_x v]
$$

$$
= - \int_0^1 \hat{\rho}^2 [(b_{11} u + b_{12} v) u + (b_{21} u + b_{22} v) v] + \int_0^1 \hat{\rho}^2 h \chi_\omega u + \int_0^1 \hat{\rho}^2 [g_1 u + g_2 v].
$$

Clearly, the terms in the right hand side of (30) can be estimated as follows:

$$
\left| \int_0^1 \hat{\rho}^2 [(b_{11} u + b_{12} v) u + (b_{21} u + b_{22} v) v] \right| \leq C \int_0^1 \hat{\rho}^2 (|u|^2 + |v|^2),
$$

$$
\int_0^1 \hat{\rho}^2 h \chi_\omega u = \int_\omega (\rho_0 u)(\rho, h) \leq \frac{1}{2} \int_\omega \rho_0^2 |u|^2 + \frac{1}{2} \int_\omega \rho_0^2 |h|^2
$$

and

$$
\int_0^1 \hat{\rho}^2 [g_1 u + g_2 v] \leq \frac{1}{2} \int_0^1 \hat{\rho}^2 (|g_1|^2 + |u|^2 + |g_2|^2 + |v|^2)
$$

$$
\leq C \left( \int_0^1 \rho_0^2 (|g_1|^2 + |g_2|^2) + \int_0^1 \rho_0^2 (|u|^2 + |v|^2) \right).
$$

Now, let us deal with the left hand side of (30). Notice that

$$
\int_0^1 \hat{\rho}^2 [u_t u + v_t v] = \frac{1}{2} \frac{d}{dt} \int_0^1 \hat{\rho}^2 (|u|^2 + |v|^2) - \int_0^1 \hat{\rho}^2 (|u|^2 + |v|^2)
$$

$$
:= \frac{1}{2} \frac{d}{dt} \int_0^1 \hat{\rho}^2 (|u|^2 + |v|^2) - I
$$

and

$$
- \int_0^1 \hat{\rho}^2 [(au_x)_x u + (av_x)_x v] = \int_0^1 [(\hat{\rho}^2 u_x)(au_x) + (\hat{\rho} v_x)(av_x)]
$$

$$
= 2 \int_0^1 \hat{\rho}^2 a (uu_x + v v_x) + \int_0^1 \hat{\rho}^2 a (u_x^2 + v_x^2) := J + \int_0^1 \hat{\rho}^2 a (u_x^2 + v_x^2).
$$

Summing up, we have just checked that

$$
\frac{1}{2} \frac{d}{dt} \int_0^1 \hat{\rho}^2 (|u|^2 + |v|^2) + \int_0^1 \hat{\rho}^2 a (|u_x|^2 + |v_x|^2)
$$

$$
\leq C \left( \int_0^1 \rho_0^2 (|u|^2 + |v|^2) + \int_\omega \rho_0^2 |h|^2 + \int_0^1 \rho_0^2 (|g_1|^2 + |g_2|^2) \right) + |I| + |J|
$$

(31)

Next, we will estimate $I$. Firstly, we put

$$
A(t, x) = \tau(t) (e^{\lambda |\psi|_\infty^-\psi} - e^{3\lambda |\psi|_\infty^-} ) = \zeta (t, x),
$$

where $\eta(x) := \frac{e^{\lambda |\psi|_\infty^-\psi} - e^{3\lambda |\psi|_\infty^-}}{\eta(x)}$ is a bounded function on $[0, 1]$. Secondly, we observe that

$$
\hat{\rho}(\hat{\rho})_t = e^{-sA} \zeta^{-5/2} \left[ -se^{-sA} A_t \zeta^{-5/2} + e^{-sA} \left( \frac{-5}{2} \right) \zeta^{-7/2} \zeta_t \right]
$$
and, for any \( t \in [0, T) \), we have

\[
|\hat{\rho}(\hat{\rho})_t| \leq C \rho_0^2 \left| \frac{s \zeta^{-3} \hat{\eta}(x) - \frac{5}{2} \zeta^{-4}}{\eta^2} \right|
\]

From this, we obtain

\[
|\mathcal{I}| \leq C \int_0^1 \rho_0^2(|u|^2 + |v|^2).
\]

Now, in order to deal with \( \mathcal{J} \), we consider the estimate

\[
|\mathcal{J}| \leq \frac{1}{2} \int_0^1 \left| \rho^2 a(|ux|^2 + |v_x|^2) + \hat{\rho}^2_x a(|u|^2 + |v|^2) \right| := \frac{1}{2} \int_0^1 \rho^2 a(|ux|^2 + |v_x|^2) + \hat{\mathcal{J}}
\]

and recall that \( A_x = \zeta_x = \lambda \psi_x \) in \([0, T] \times [0, 1]\). Hence,

\[
\hat{\rho}^2_x = \left( -e^{-sA} s A_x \zeta^{-5/2} - \frac{5}{2} e^{sA} \zeta^{-7/2} \right)^2 \leq C \left( e^{-2sA} s^2 A_x^2 \zeta^{-5} + \frac{25}{4} e^{-2sA} \zeta^{-7} \right)
\]

\[
= Ce^{-2sA} \zeta^{-2} \psi_x^2 \left( s^2 \lambda^2 \zeta^{-3} + \frac{25}{4} \lambda^2 \zeta^{-3} \right) = C \rho_0^2 \psi_x^2 \left( s^2 \lambda^2 \zeta^{-1} + \frac{25}{4} \lambda^2 \zeta^{-3} \right)
\]

\[
\leq C \rho_0^2
\]

and we get \( \hat{\mathcal{J}} \leq C \int_0^1 \rho_0^2 a(|u|^2 + |v|^2) \).

Recalling inequality (31), we conclude that

\[
\frac{1}{2} \frac{d}{dt} \int_0^1 \rho^2(|u|^2 + |v|^2) + \frac{1}{2} \int_0^1 \rho^2 a(|ux|^2 + |v_x|^2)
\]

\[
\leq C \left( \int_0^1 \rho_0^2(|u|^2 + |v|^2) + \int_\omega \rho^2_x |h|^2 + \int_0^1 \rho_0^2(|g_1|^2 + |g_2|^2) \right)
\]

and, integrating in time, we obtain the desired result.

\[\square\]

**Proposition 6.** Assume the hypothesis of Theorem 4.1 and suppose that \( h \) and \((u, v)\) satisfy (26). Then

\[
\int_0^T \int_0^1 \rho^2_x(|u_t|^2 + |v_t|^2 + |(au_{xt})_x|^2 + |(av_{xt})_x|^2) \leq C \int_0^T \int_0^1 \rho_0^2(|u|^2 + |v|^2) dt
\]

\[
+ C \left( \int_0^T \int_\omega \rho^2_x |h|^2 + \int_0^T \int_0^1 \rho_0^2(|g_1|^2 + |g_2|^2) + \|u_0\|_{H^2}^2 + \|v_0\|_{H^2}^2 \right).
\]

**Proof.** Multiplying the first equation in (11) by \( \rho^2_x u_t \) and the second one by \( \rho^2_x v_t \), we take
\[
\int_0^1 \rho_*^2(\|u_t\|^2 + |v_t|^2) - \int_0^1 \rho_*^2[(au_x)_x u_t + (av_x)_x v_t]
\]
\[
= - \int_0^1 \rho_*^2[(b_{11}u + b_{12}v)u_t + (b_{21}u + b_{22}v)v_t] + \int_0^1 \rho_*^2 h \chi_\omega u_t + \int_0^1 \rho_*^2(g_1 u_t + g_2 v_t).
\]
Notice that,
\[
- \int_0^1 \rho_*^2[(au_x)_x u_t + (av_x)_x v_t]
\]
\[
= \frac{1}{2} \frac{d}{dt} \int_0^1 \rho_*^2 a(\|u_x\|^2 + |v_x|^2)
\]
\[
- \frac{1}{2} \int_0^1 (\rho_*^2)_t a(\|u_x\|^2 + |v_x|^2) + \int_0^1 (\rho_*^2)_x a(\|u_t\|^2 + |v_t|^2)
\]
\[
=: \frac{1}{2} \frac{d}{dt} \int_0^1 \rho_*^2 a(\|u_x\|^2 + |v_x|^2) - K.
\]
Proceeding as in the proof of Proposition 5,
\[
\int_0^1 \rho_*^2(\|u_t\|^2 + |v_t|^2) + \frac{1}{2} \frac{d}{dt} \int_0^1 \rho_*^2 a(\|u_x\|^2 + |v_x|^2)
\]
\[
\leq C \left( \int_0^1 \rho_*^2(\|u\|^2 + |v|^2) + \int_0^1 \rho_*^2|h|^2 + \int_0^1 \rho_*^2(\|u_1\|^2 + |g_2|^2) + \frac{3}{8} \int_0^1 \rho_*^2(\|u_t\|^2 + |v_t|^2) + |K| \right). \tag{33}
\]
Using Young’s inequality with \(\varepsilon\), we have
\[
K = \frac{1}{2} \int_0^1 (\rho_*^2)_t a(\|u_x\|^2 + |v_x|^2)
\]
\[
- \int_0^1 [((\rho_*^2)_t \rho_*^{-1} au_x)(\rho_* u_t) + ((\rho_*^2)_x \rho_*^{-1} av_x)(\rho_* v_t)] \tag{34}
\]
\[
\leq C \int_0^1 [\|\rho_*^2\|_t + (\|\rho_*^2\|_x \rho_*^{-2} a(\|u_x\|^2 + |v_x|^2)) + \frac{1}{8} \int_0^1 \rho_*^2(\|u_t\|^2 + |v_t|^2)].
\]
Since \(\|\zeta_t\| \leq C \zeta^2\) and \(\zeta_x = A_x = \lambda \zeta \psi_x\) in \([0, T] \times [0, 1]\), we have
\[
[(\rho_*^2)_t] = \rho^2|2s A\lambda \zeta^{-3} + 8\zeta^{-4} \zeta_t| \leq C \rho^2(2s \zeta^{-1} + 8 \zeta^{-2})
\]
and
\[
(\rho_*^2)_x \rho_*^{-2} \leq C(4s^2 e^{-4sA} A^2 \lambda^2 - 16 + 64e^{-4sA} \zeta^{-16} \zeta_2^2) \rho_*^2
\]
\[
= C \rho^2(4s^2 A^2 \lambda^2 \zeta^{-3} + 64 \zeta^{-5} \zeta_2^2)
\]
\[
= C \rho^2(4s^2 \lambda^2 \psi_x^2 \zeta^{-1} + 64 \zeta^{-3} \lambda^2 \psi_x^2),
\]
which imply
\[
K \leq C \int_0^1 \rho^2 a(\|u_x\|^2 + |v_x|^2) + \frac{1}{8} \int_0^1 \rho_*^2(\|u_t\|^2 + |v_t|^2).
\]
Thus, from (33), applying Proposition 5 and using \(\rho_* \leq C \rho_0\), we get
\[
\int_0^T \int_0^1 \rho_*^2(\|u_t\|^2 + |v_t|^2) \leq C \left( \int_0^T \int_0^1 \rho_0^2(\|u\|^2 + |v|^2) + \int_0^T \int_0^1 \rho_*^2|h|^2 \right)
\]
\[
+ C \left( \int_0^T \int_0^1 \rho_*^2(\|u_x\|^2 + |v_x|^2) + \frac{1}{8} \int_0^1 \rho_*^2(\|u_t\|^2 + |v_t|^2) \right).
In order to conclude the proof, it remains to estimate \( \int_0^T \int_0^1 \rho_s^2(|u_x|)^2 + |(av_x)_x|^2 \). In fact, it is enough to multiply the first equation in (11) by \(-\rho_s^2(av_x)_x\) and the second one by \(-\rho_s^2(av_x)_x\), and proceed as in the first part of this proof. \( \square \)

5. Main result. In this section, our goal is to prove Theorem 1.1. Let us define the functions spaces

\[
E := \left\{ (u, v, h) \in \left[ L^2((0, T) \times (0, 1)) \right]^2 \times L^2((0, T) \times \omega) : \right. \\
\left. u(t, \cdot), v(t, \cdot) \text{ are absolutely continuous in } [0, 1], \ a.e. \text{ in } [0, T], \right. \\
\left. u_t, u_x, (av_x)_x, \rho, h \in L^2((0, T) \times (0, 1)), \right. \\
\left. \rho_0 u, \rho_0 u_t - (av_x)_x - h\chi_\omega, \rho_0 v, \rho_0 v_t - (av_x)_x \in L^2((0, T) \times (0, 1)), \right. \\
\left. u(t, 1) = v(t, 1) = u(t, 0) = v(t, 0) = 0 \ a.e \text{ in } [0, T], \ \text{and } u(0, \cdot), v(0, \cdot) \in H^1_a \right\},
\]

and

\[ F := G \times H^1_a \times H^1_a, \]

where

\[ G := \{ g \in L^2((0, T) \times (0, 1)) : \rho_0 g \in L^2((0, T) \times (0, 1)) \} . \]

We also consider the Hilbertian norm

\[
\| (u, v, h) \|_E^2 := \int_0^T \int_0^1 \rho_0^2(|u|^2 + |v|^2) + \int_0^T \rho_s^2|h|^2 \left. \right. + \int_0^T \rho_0^2|u_t - (av_x)_x - h\chi_\omega|^2 + \int_0^T \rho_0^2|v_t - (av_x)_x|^2 \\
\left. \right. + \| u(0, \cdot) \|_{H^1_a}^2 + \| v(0, \cdot) \|_{H^1_a}^2 .
\]

The proof that \( E \) is a Hilbert space is given in Appendix B.

Now, we set the mapping \( H : E \to F \), given by

\[ H(u, v, h) = (H_1(u, v, h), H_2(u, v, h), u(\cdot, 0), v(\cdot, 0)), \]

where

\[ H_1(u, v, h) = u_t - \left( \mu_1 \left( x, \int_0^1 u \right) u_x \right)_x + f_1(t, x, u, v) - h\chi_\omega \]

and

\[ H_2(u, v, h) = v_t - \left( \mu_2 \left( x, \int_0^1 v \right) v_x \right)_x + f_2(t, x, u, v) . \]

Applying Lyusternik’s Inverse Mapping Theorem, see [5], we will prove that \( H \) has a right inverse mapping defined in a small ball contained in \( F \). Due to the choice of the spaces \( E \) and \( F \), the existence of that inverse mapping will imply the local null controllability of (1). Before doing it, we will establish some results which will guarantee that \( H \) satisfies the hypotheses of Lyusternik’s Theorem.

**Lemma 5.1.** Define \( \beta(x) = e^{\lambda(|v|_\infty + \psi)} - e^{3\lambda|v|_\infty} \) and \( \tilde{\beta} = \max_{x \in [0, 1]} \beta(x) \). There exists \( s > 0 \) such that, if \( s\tilde{\beta} < M < 0 \), then

\[
\sup_{t \in [0, T]} \left\{ e^{s M} \left[ \left( \int_0^1 u \right)^2 + \left( \int_0^1 v \right)^2 \right] \right\} \leq C \| (u, v, h) \|_E^2 .
\]
for all \((u, v, h) \in E\).

**Proof.** In fact, for each \((u, v, h) \in E\), consider \(q_1 : [0, T] \rightarrow \mathbb{R}\) and \(q_2 : [0, T] \rightarrow \mathbb{R}\), given by

\[
q_1(t) = e^{-\frac{m(t)}{2}} \int_0^1 u(t, x) \quad \text{and} \quad q_2(t) = e^{-\frac{m(t)}{2}} \int_0^1 v(t, x).
\]

Taking \(k > 0\), we quickly get \(e^{-\frac{m(t)}{2}} \leq 8! [m(t)]^8 / k^8\), for any \(t \in [0, T]\). Since \(A = \tau \beta\), taking \(s > 0\) such that \(2s(\beta - \beta) > k\), we have

\[
- \frac{2M}{m(t)} + 2sA = - \frac{2M}{m(t)} + \frac{2sB}{m(t)} \leq - \frac{2s(\beta - \beta)}{m(t)} < - \frac{k}{m(t)}
\]

and

\[
e^{-\frac{2M}{m(t)}} < e^{-2sA} e^{-\frac{M}{m(t)}} \leq C e^{-2sA} \tau^{-8} \leq C \rho^2,
\]

for any \(t \in [0, T]\). From this point, we may argue as in [13] (see Lemma 4.4, on page 533), in order to check that \(q_1, q_2 \in H^1(0, T)\) and

\[
\|q_1\|_{H^1(0, T)} + \|q_2\|_{H^1(0, T)} \leq C \|(u, v, h)\|_E^2.
\]

Therefore, the desired result is a consequence of the continuous embedding \(H^1(0, T) \hookrightarrow C(0, T)\).

As a consequence of Lemma 5.1, we deduce the useful result below:

**Corollary 2.** There exists \(C > 0\) such that

\[
\int_0^T \int_0^1 \rho_0^2 \left( \int_0^1 \bar{u} \right)^2 \|(au_x)_x\|^2 + \int_0^T \int_0^1 \rho_0^2 \left( \int_0^1 \bar{v} \right)^2 \|(av_x)_x\|^2 \leq C \|(u, v, h)\|_E^2 \|(\bar{u}, \bar{v}, \bar{h})\|_E^2,
\]

for any \((u, v, h), (\bar{u}, \bar{v}, \bar{h}) \in E\).

**Proof.** Take \((u, v, h), (\bar{u}, \bar{v}, \bar{h}) \in E\) and let \(M < 0\) be the constant mentioned in Lemma 5.1. Since \(\rho_0^2 \rho_x^2 = \zeta^6\) and \(\zeta^6 \leq \frac{45}{23M} e^{-\frac{2m}{M}}\), applying Proposition 6 and Lemma 5.1, we obtain

\[
\int_0^T \int_0^1 \rho_0^2 \left( \int_0^1 \bar{u} \right)^2 \|(au_x)_x\|^2 \leq \sup_{t \in [0, T]} \left\{ e^{-\frac{2M}{m}} \left( \int_0^1 \bar{u} \right)^2 \right\} \int_0^T \int_0^1 \rho_0^2 \|(au_x)_x\|^2 \leq C \|(u, v, h)\|_E^2 \|(\bar{u}, \bar{v}, \bar{h})\|_E^2.
\]

Analogously, a similar estimate also holds to \(\int_0^T \int_0^1 \rho_0^2 \left( \int_0^1 \bar{v} \right)^2 \|(av_x)_x\|^2\).

**Proposition 7.** The mapping \(H : E \rightarrow F\) has the following properties:

(a) \(H\) is well defined;

(b) For each \((u, v, h) \in E\), let us define \(\bar{f}_i = D_i f_j(t, x, u, v)\), with \(i = 3, 4\) and \(j = 1, 2\). Then, the linear mapping \(T : E \rightarrow G\) and \(S : E \rightarrow G\), given by

\[
T(\bar{u}, \bar{v}, \bar{h}) = \bar{u}_t - \ell_1 \left( \int_0^1 u \right) \left( \int_0^1 \bar{u} \right) (au_x)_x - \ell_1 \left( \int_0^1 u \right) (a\bar{u}_x)_x + f_1^3 \bar{u} + f_1^4 \bar{v} - \bar{h} \chi_w,
\]

and

\[
S(\bar{u}, \bar{v}, \bar{h}) = \bar{v}_t - \ell_2 \left( \int_0^1 u \right) \left( \int_0^1 \bar{v} \right) (av_x)_x - \ell_2 \left( \int_0^1 u \right) (a\bar{v}_x)_x + f_2^3 \bar{u} + f_2^4 \bar{v} - \bar{h} \chi_w,
\]

where \(\chi_w\) is a function with \(\chi_w = 0\) on \(\partial \Omega\).
Proof. (a) For each \((u, v, h) \in E\), we must check that \(H(u, v, h) \in F\). Of course, \(H_3(u, v, h) = u(\cdot, 0) \in H^1_0(0,1)\) and \(H_4(u, v, h) = v(\cdot, 0) \in H^1_0(0,1)\). Besides, recalling the Assumptions (A.1) and (A.2), and using Corollary 2, we take
\[
\int_0^T \int_0^1 \rho_0^2 H_1(u, v, h)\,dt = \int_0^T \int_0^1 \rho_0^2 \left| u_t - \ell_1 \left( \int_0^1 u \right) (au)_x + f_1(t, x, u, v) - h\omega \right|^2 \,dt \\
\leq 3 \int_0^T \int_0^1 \rho_0^2 |u_t - (au)_x - h\omega|^2 + 3 \int_0^T \int_0^1 \rho_0^2 \left| \ell_1 \left( \int_0^1 u \right) (au)_x + f_1(t, x, u, v) - f_1(t, x, 0, 0) \right|^2 \\
+ C \int_0^T \int_0^1 \rho_0^2 (|u|^2 + |v|^2) \\
\leq C\left(\|u\|_{L^2} + \|[u, v, h]\|_{E}\right)
\]
Thus, for any \((u, v, h) \in E\), we take
\[
H_1(u, v, h) = \int_0^T \int_0^1 \rho_0^2 H_1(u, v, h)\,dt = H_2(u, v, h) = \int_0^T \int_0^1 \rho_0^2 H_1(u, v, h)\,dt.
\]
Therefore, \(H_1(u, v, h) \in G\) and, in a similar way, we also have \(H_2(u, v, h) \in G\).
(b) In this part, fix \((u, v, h) \in E\) and \(\lambda \neq 0\), we take
\[
1/\lambda \left[ H_1(u + \lambda \tilde{u}, v + \lambda \tilde{v}, h + \lambda \tilde{h}) - H_1(u, v, h) \right] - \lambda \left[ \int T_0 t \int_0^1 \rho_0 \|A\|^2 \|u\|_{L^2} + \lambda \|	ilde{u}, \tilde{v}, \tilde{h}\|_{E} \right]
\]
We will see that \(A, B, C\) converge to zero in \(G\), as \(\lambda \to 0\). Indeed, taking into account (A.1) and Mean Value Theorem, for each \(\lambda \neq 0\), there exists \(u_\lambda = u(\cdot, t)\) such that \(u_\lambda \to \int_0^1 u\) for any \(t \in [0, T]\),
\[
\int_0^T \int_0^1 \rho_0^2 |A_\lambda|^2 = \int_0^T \int_0^1 \rho_0^2 \left| \ell_1'(u_\lambda) - \ell_1 \left( \int_0^1 u \right) \right|^2 \left( \int_0^1 \tilde{u} \right)^2 \|\omega\|_{L^2}^2 \to 0
\]
and
\[
\int_0^T \int_0^1 \rho_0^2 |B_\lambda|^2 = \lambda^2 \int_0^T \int_0^1 \rho_0^2 \left| \ell_1'(u_\lambda) \right|^2 \left( \int_0^1 \tilde{u} \right)^2 \|\omega\|_{L^2}^2 \to 0
\]
and
\[
\int_0^T \int_0^1 \rho_0^2 |C_\lambda|^2 \leq C\lambda^2 \|u, v, h\|_{E}^2 \|\tilde{u}, \tilde{v}, \tilde{h}\|_{E}^2 \to 0.
\]
as \( \lambda \to 0 \). On the other hand, for each \( \lambda \neq 0 \), we can apply again Mean Value and Lebesgue’s Theorem, in order to obtain \( w_\lambda = w_\lambda(t, x) \) satisfying: 
\( w_\lambda \to (t, x, u, v) \), for any \((t, x) \in (0, T) \times (0, 1)\), and 
\[
\int_0^T \int_0^1 \rho_0^2 |C_\lambda|^2 = \int_0^T \int_0^1 \rho_0^2 [D_3 f_1(w_\lambda) \bar{u} + D_4 f_1(w_\lambda) \bar{v}] - [\bar{f}_1^1 u + \bar{f}_1^2 v]^2
\leq \int_0^T \int_0^1 \rho_0^2 [D_3 f_1(w_\lambda) - \bar{f}_1^1]^2 |\bar{u}|^2 + \int_0^T \int_0^1 \rho_0^2 [D_4 f_1(w_\lambda) - \bar{f}_1^2]^2 |\bar{v}|^2 \to 0,
\]
as \( \lambda \to 0 \).

As a consequence, \( T \) is the Gateaux derivative of \( H_1 \) at \((u, v, h) \in E \). Likewise, \( S \) is the Gateaux derivative of \( H_2 \) at \((u, v, h) \in E \).

\[ \square \]

**Proposition 8.** The mapping \( H : E \to F \) is continuously differentiable.

**Proof.** Clearly, \( H_3, H_4 \in C^1(E, H_1^1) \). Now, take \((u, v, h) \in E \) and let \(((u^n, v^n, h^n))_{n=1}^\infty \) be a sequence which converges to \((u, v, h) \in E \). For each \((\bar{u}, \bar{v}, \bar{h}) \in B_1(0) \subset E \), we have proved in Proposition 7 that 
\[ H'(u, v, h)(\bar{u}, \bar{v}, \bar{h}) = \bar{u}_t - \ell_1^1 \left( \int_0^1 u \right) \left( \int_0^1 \bar{u} \right) (au_x)_x - \ell_1^1 \left( \int_0^1 u \right) (a\bar{u}_x)_x + \bar{f}_1^{11} \bar{u} + \bar{f}_1^{12} \bar{v} - \bar{h} \chi \omega, \]
and 
\[ H'(u^n, v^n, h^n)(\bar{u}, \bar{v}, \bar{h}) = \bar{u}_t - \ell_1^1 \left( \int_0^1 u^n \right) \left( \int_0^1 \bar{u} \right) (a\bar{u}_x)_x - \ell_1^1 \left( \int_0^1 u^n \right) (a\bar{u}_x)_x + D_3 f_1(t, x, u^n, v^n) \bar{u} + D_4 f_1(t, x, u^n, v^n) \bar{v} - \bar{h} \chi \omega. \]

Thus, 
\[
(H'_1(u^n, v^n, h^n) - H'_1(u, v, h))(\bar{u}, \bar{v}, \bar{h}) = -\ell_1^1 \left( \int_0^1 u^n \right) \left( \int_0^1 \bar{u} \right) [a(u^n - u)]_x
- \left[ \ell_1^1 \left( \int_0^1 u^n \right) \ell_1^1 \left( \int_0^1 u \right) \right] (a\bar{u}_x)_x
- \left[ \ell_1^1 \left( \int_0^1 u^n \right) - \ell_1^1 \left( \int_0^1 u \right) \right] (a\bar{u}_x)_x
+ [D_3 f_1(t, x, u^n, v^n) - D_3 f_1(t, x, u, v)] \bar{u} + [D_4 f_1(t, x, u^n, v^n) - D_4 f_1(t, x, u, v)] \bar{v}
:= X_1^n + X_2^n + X_3^n + X_4^n + X_5^n.
\]

From assumption A.1 and Corollary 2, we get 
\[
\int_0^T \int_0^1 \rho_0^2 |X_1^n|^2 \leq C \int_0^T \int_0^1 \rho_0^2 \left( \int_0^1 \bar{u} \right)^2 |[a(u^n - u)]_x|^2
\leq C \|(u^n - u, v^n - v, h^n - h)\|^2_E \to 0
\]
and 
\[
\int_0^T \int_0^1 \rho_0^2 |X_2^n|^2 \leq C \int_0^T \int_0^1 \rho_0^2 \left( \int_0^1 (u^n - u) \right)^2 |(a\bar{u}_x)_x|^2
\leq C \|(u^n - u, v^n - v, h^n - h)\|^2_E \to 0.
\]
as \( n \to +\infty \).

On the other hand, due to Lemma 5.1 and assumptions A.1 and A.2, we obtain
\[
\int_0^T \int_0^1 \rho_0^n |X^n_2|^2 \\
\leq \sup_{t \in [0,T]} \left\{ e^{-\frac{2M}{a_1}} \left( \int_0^1 \bar{u} \right)^2 \right\} \int_0^T \int_0^1 \rho_0^n \left| \ell_1 ' \left( \int_0^1 u^n \right) - \ell_1 ' \left( \int_0^1 u \right) \right|^2 |(au_x)_x|^2 \\
\leq \int_0^T \int_0^1 \rho_0^n \left| \ell_1 ' \left( \int_0^1 u^n \right) - \ell_1 ' \left( \int_0^1 u \right) \right|^2 |(au_x)_x|^2 
\to 0,
\]

and
\[
\int_0^T \int_0^1 \rho_0^n |X^n_4|^2 \\
\leq C \left( \int_0^T \int_0^1 \rho_0^n |D_3f_1(t, x, u^n, v^n) - D_3f_1(t, x, u, v)|^2 |\bar{u}|^2 \right)^{\frac{1}{2}} \left( \int_0^T \int_0^1 \rho_0^n |\bar{u}|^2 \right)^{\frac{1}{2}} \\
\leq C \left( \int_0^T \int_0^1 \rho_0^n |D_3f_1(t, x, u^n, v^n) - D_3f_1(t, x, u, v)|^2 |\bar{u}|^2 \right)^{\frac{1}{2}} \to 0,
\]
as \( n \to +\infty \), where we have also applied Lebesgue’s Theorem. Clearly, \( \int_0^T \int_0^1 \rho_0^n |X^n_5|^2 \)
is similar to \( \int_0^T \int_0^1 \rho_0^n |X^n_3|^2 \) and we conclude that \( H_1' \) is a continuous mapping.
Analogously, this conclusion remains valid to \( H_2' \). In this case, the proof is complete.

**Proof of Theorem 1.1.** From Propositions 7 and 8, we already know that the \( H \in C^1(E, F) \). We state that \( H'(0, 0, 0) : E \to F \) is onto. In fact, consider \( b_{11}(t, x) = D_3f_1(t, x, 0, 0), b_{22}(t, x) = D_4f_2(t, x, 0, 0), b_{21}(t, x) = D_3f_2(t, x, 0, 0) \) and \( b_{22}(t, x) = D_4f_2(t, x, 0, 0) \) in (11). Thus, given \((g_1, g_2, u_0, v_0) \in F\), we apply Theorem 4.1 in order to obtain \((u, v, h)\) which solves (11) and satisfies the relations in (26). As a result, \((u, v, h) \in E\) and \(H'(0, 0, 0)(u, v, h) = (g_1, g_2, u_0, v_0)\), as we were supposed to check.

Hence, by **Lyusternik’s Inverse Mapping Theorem** (Theorem 2.1), there exist \( r > 0 \) and a mapping \( \hat{H} : B_r(0) \subset F \to E \) such that
\[
H(\hat{H}(y)) = y \text{ for each } y \in B_r(0) \subset F.
\]

In particular, if \((\bar{u}_0, \bar{v}_0) \in H_1^1 \times H_2^1 \) and \( \| (\bar{u}_0, \bar{v}_0) \|_{H_1^1 \times H_2^1} \) < \( r \), we conclude that \((\bar{u}, \bar{v}, \bar{h}) = \hat{H}(0, 0, \bar{u}_0, \bar{v}_0) \in E \) solves \( H(\bar{u}, \bar{v}, \bar{h}) = (0, 0, \bar{u}_0, \bar{v}_0) \). Since \( \int_0^T \int_0^1 \rho_0^n |\bar{u}|^2 + |\bar{v}|^2 < +\infty \), we get \( \bar{u}(T, x) = \bar{v}(T, x) = 0 \) for any \( x \in [0, 1] \), following the result.

6. **Some additional comments.** As a first comment, we note that, in assumption A.1, we have taken a weak type of degeneracy and so that Dirichlet boundary conditions are required in (1). However, if we had chosen strong type degeneracy, see [4], (1) it would be treated with Neumann conditions. In this context, we believe that analogous results can be obtained.

Another interesting question is concerned with global null controllability to (1), which does not seem to be simple. Perhaps, this kind of result relies on a global inverse mapping theorem, see [12], but much more refined estimates are necessary.
Under some changes in the Lemma 5.1 and following the arguments presented here, Theorem 1.1 can also be obtained if we consider (1) with the diffusion coefficients
\[(\mu_1(x,u)u_x)_x\text{ and } (\mu_1(x,v)v_x)_x.\]

Other important topics arise from our current research:

- It would be very nice to obtain Theorem 1.1 without imposing \(\mu_1\) and \(\mu_2\) have separated variables. Nevertheless, it is still an open problem.
- In the system (1), we can replace each nonlinearity \(f_i(t,x,u,v)\) by \(f_i(t,x,u,v,u_x,v_x)\), with \(i \in \{1,2\}\), in order to analyse whether it is possible to prove results about null controllability.
- Previously, in [13], we have obtained a local null controllability result for degenerate parabolic equations with nonlocal terms, which implies, throughout standard arguments, a local null boundary controllability result. However, the same fact can not be directly deduced for systems with a reduced number of controls, see [6]. In other words, the boundary controllability of

\[
\begin{cases}
  u_t - \left(\mu_1 \left(x, \int_0^1 u \right) u_x \right)_x + f_1(t,x,u,v) = 0, & (t,x) \in (0,T) \times (0,1), \\
  v_t - \left(\mu_2 \left(x, \int_0^1 v \right) v_x \right)_x + f_2(t,x,u,v) = 0, & (t,x) \in (0,T) \times (0,1), \\
  u(t,1) = v(t,0) = v(t,1) = 0, & u(t,0) = h(t) \quad t \in (0,T), \\
  u(0,x) = u_0(x) \quad \text{and} \quad v(0,x) = v_0(x), & x \in (0,1). 
\end{cases}
\]

is a very interesting unknown issue.

Appendix A. Wellposedeness of (1). In this section, we will apply Galerkin’s method in order to obtain a unique solution to (1). Precisely, let us consider the functions \(a, \ell_1, \ell_2, f_1\) and \(f_2\) as in the assumptions A.1 and A.2. Additionally, let us suppose that

\[0 < \ell_0 \leq \ell_1, \ell_2 \leq L_1,\]

where \(\ell_0\) and \(L_0\) are two positive constants. We observe that there exist \(C_0 > 0\) and \(L_2 > 0\) such that

\[|f_i(t,x,s_1,s_2)| \leq C_0(|s_1| + |s_2|)\]

and

\[|\ell_i'(s)| \leq L_2,\]

for any \((t,x,s_1,s_2) \in [0,T] \times [0,1] \times \mathbb{R} \times \mathbb{R}\) and \(s \in \mathbb{R}\), with \(i \in \{1,2\}\). Under this conditions, we will prove the following result:

**Theorem A.1.** Take \(T > 0\). If \(F_1, F_2 \in L^2((0,T) \times (0,1))\) and \(u_0, v_0 \in H^1_0(0,1)\), then there exists a unique weak solution of

\[
\begin{cases}
  u_t - \ell_1 \left(\int_0^1 u \right) (au_x)_x + f_1(t,x,u,v) = F_1, & (t,x) \in (0,T) \times (0,1), \\
  v_t - \ell_2 \left(\int_0^1 v \right) (av_x)_x + f_2(t,x,u,v) = F_2, & (t,x) \in (0,T) \times (0,1), \\
  u(t,0) = u(t,1) = v(t,0) = v(t,1) = 0, & t \in (0,T), \\
  u(0,x) = u_0(x) \quad \text{and} \quad v(0,x) = v_0(x), & x \in (0,1). 
\end{cases}
\] (36)

**Proof.** Let \((w_i)_{i=1}^{\infty}\) be an orthonormal basis of \(H^1_0(0,1)\) such that

\[-(a(x)w_{ix})_x = \lambda_i w_i.\]
Fix $m \in \mathbb{N}^*$. Due to Carathéodory’s theorem, there exist absolutely continuous functions $g_{im} = g_{im}(t)$ and $h_{im} = h_{im}(t)$, with $i \in \{1, \cdots, m\}$ and $t \in [0, T]$, such that the functions

\[ t \in [0, T] \mapsto u_m(t) = \sum_{i=1}^{m} g_{im}(t) \psi_i \in H^1(0,1) \]

and

\[ t \in [0, T] \mapsto v_m(t) = \sum_{i=1}^{m} h_{im}(t) \psi_i \in H^1(0,1) \]

satisfy

\[
\begin{aligned}
(u_{mt}, w) - \ell_1 \left( \int_0^1 u_m \right) (a u_{mx}, w) \\
+ (f_1(t, x, u_m, v_m), w) = (F_1, w), \quad (t, x) \in (0, T) \times (0, 1),
\end{aligned}
\]

\[
\begin{aligned}
(v_{mt}, \tilde{w}) - \ell_2 \left( \int_0^1 v_m \right) (a u_{mx}, \tilde{w}) \\
+ (f_2(t, x, u_m, v_m), \tilde{w}) = (F_2, \tilde{w}), \quad (t, x) \in (0, T) \times (0, 1),
\end{aligned}
\]

for any $w, \tilde{w} \in [w_1, \cdots, w_m]$, where $(\cdot, \cdot)$ denotes the inner product in $L^2(0,1)$. Next, our goal is to prove three energy estimates to $u_m$ and $v_m$.

Firstly, taking $w = u_m$ and $\tilde{w} = v_m$ in (37), we obtain

\[
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \left( \| u_m \|^2_{L^2(0,1)} + \| v_m \|^2_{L^2(0,1)} \right) \\
+ \ell_1 \left( \int_0^1 u_m \right) \| \sqrt{a} u_{mx} \|^2_{L^2(0,1)} + \ell_2 \left( \int_0^1 v_m \right) \| \sqrt{a} v_{mx} \|^2_{L^2(0,1)} \\
\leq C_1(\| u_m \|^2_{L^2(0,1)} + \| v_m \|^2_{L^2(0,1)}) + C_2(\| F_1 \|^2_{L^2(0,1)} + \| F_2 \|^2_{L^2(0,1)}).
\end{aligned}
\]

Hence, Gronwall’s inequality yields

\[
\max_{t \in [0, T]} \| u_m(t) \|^2_{L^2(0,1)} + \max_{t \in [0, T]} \| v_m(t) \|^2_{L^2(0,1)}
\]

\[
\quad + 2L_0 \int_0^T (\| \sqrt{a} u_{mx} \|^2_{L^2(0,1)} + \| \sqrt{a} v_{mx} \|^2_{L^2(0,1)})
\]

\[
\leq C \left( \| u_0 \|^2_{L^2(0,1)} + \| v_0 \|^2_{L^2(0,1)} + \int_0^T (\| F_1 \|^2_{L^2(0,1)} + \| F_2 \|^2_{L^2(0,1)}) \right)
\]

\[
= \mathcal{K}.
\]

Now, let us consider $w = u_{mt}$ and $\tilde{w} = v_{mt}$. Thus,

\[
\| u_{mt}(t) \|^2_{L^2(0,1)} + \| v_{mt}(t) \|^2_{L^2(0,1)}
\]

\[
\quad + \frac{1}{2} \frac{d}{dt} \left( \ell_1 \left( \int_0^1 u_{mt} \right) \| \sqrt{a} u_{mx} \|^2_{L^2(0,1)} + \ell_2 \left( \int_0^1 v_{mt} \right) \| \sqrt{a} v_{mx} \|^2_{L^2(0,1)} \right)
\]

\[
\leq C_0(\| u_m \|_{L^2(0,1)} + \| v_m \|_{L^2(0,1)}) \| u_{mt} \|_{L^2(0,1)} + \| v_{mt} \|_{L^2(0,1)}
\]

\[
+ C(\| F_1 \|_{L^2(0,1)} + \| F_2 \|_{L^2(0,1)}) \| u_{mt} \|_{L^2(0,1)} + \| v_{mt} \|_{L^2(0,1)}
\]

\[
+ \frac{1}{2} \int_0^1 u_{mt} \ell_1 \left( \int_0^1 u_m \right) \| \sqrt{a} u_{mx} \|^2_{L^2(0,1)}
\]
As a conclusion, \( \tilde{\varphi} \leq C_0 \left( \| u_m \|_{L^2(0,1)} + \| \varphi_{\nu m} \|_{L^2(0,1)} \right) \left( \| u_{mt} \|_{L^2(0,1)} + \| v_{mt} \|_{L^2(0,1)} \right) \\
+ C \left( \| F_1 \|_{L^2(0,1)} + \| F_2 \|_{L^2(0,1)} \right) \left( \| u_{mt} \|_{L^2(0,1)} + \| v_{mt} \|_{L^2(0,1)} \right) \\
+ L_2 \left( \| u_{mt} \|_{L^2(0,1)} \| \sqrt{u_{\nu m}} \|_{L^2(0,1)}^2 + \| v_{mt} \|_{L^2(0,1)} \| \sqrt{v_{\nu m}} \|_{L^2(0,1)}^2 \right) \\
\leq C(\varepsilon) \left( \| u_m \|_{L^2(0,1)} + \| v_m \|_{L^2(0,1)} + \| F_1 \|_{L^2(0,1)} + \| F_2 \|_{L^2(0,1)} \right) \\
+ \varepsilon \left( \| u_{mt} \|_{L^2(0,1)} + \| v_{mt} \|_{L^2(0,1)} \right) \\
+ L_2 \left( \| \sqrt{u_{\nu m}} \|_{L^2(0,1)}^2 + \| \sqrt{v_{\nu m}} \|_{L^2(0,1)}^2 \right)^2. \\
Thus, integrating in \([0, T]\), we get \\
\int_0^T \left( \| u_m \|_{L^2(0,1)} + \| F_1 \|_{L^2(0,1)} + \| F_2 \|_{L^2(0,1)} \right) \\
+ \int_0^T \left( \| \sqrt{u_{\nu m}} \|_{L^2(0,1)} + \| \sqrt{v_{\nu m}} \|_{L^2(0,1)} \right) \\
\leq (\varepsilon T + \varepsilon \left( \| u_{\nu m} \|_{L^2(0,1)} + \| v_{\nu m} \|_{L^2(0,1)} \right) e^{\frac{\varepsilon T}{\rho_0}} e^{\frac{L_2}{\rho_0} S} \\
\leq (\varepsilon T + \varepsilon \left( \| u_{\nu m} \|_{L^2(0,1)} + \| v_{\nu m} \|_{L^2(0,1)} \right) e^{\frac{L_2}{\rho_0} S}.

(39)

for any \( t \in [0, T] \), where we have used estimate (38) and applied Gronwall’s inequality.

Finally, we will prove the last estimate which is necessary to build a weak solution of (36).

In fact, taking \( w = -(u_{\nu m})_x \) and \( w = -(v_{\nu m})_x \) in (37), we have

\[
\frac{1}{2} \frac{d}{dt} \left( \| \sqrt{u_{\nu m}} \|_{L^2(0,1)}^2 + \| \sqrt{v_{\nu m}} \|_{L^2(0,1)}^2 \right) + L_0 \left( \| (u_{\nu m})_x \|_{L^2(0,1)}^2 + \| (v_{\nu m})_x \|_{L^2(0,1)}^2 \right) \\
\leq C_0 \left( \| u_m \|_{L^2(0,1)} + \| v_m \|_{L^2(0,1)} \right) \left( \| (u_{\nu m})_x \|_{L^2(0,1)} + \| (v_{\nu m})_x \|_{L^2(0,1)} \right) \\
+ C \left( \| F_1 \|_{L^2(0,1)} + \| F_2 \|_{L^2(0,1)} \right) \left( \| (u_{\nu m})_x \|_{L^2(0,1)} + \| (v_{\nu m})_x \|_{L^2(0,1)} \right) \\
+ \varepsilon \left( \| (u_{\nu m})_x \|_{L^2(0,1)} + \| (v_{\nu m})_x \|_{L^2(0,1)} \right).
\]

As a conclusion,

\[
\| \sqrt{u_{\nu m}} \|_{L^2(0,1)}^2 + \| \sqrt{v_{\nu m}} \|_{L^2(0,1)}^2 \\
\leq 2C(\varepsilon) \left( \| F_1 \|_{L^2(0,1)} + \| F_2 \|_{L^2(0,1)} \right) + \left( \| \sqrt{u_{\nu m}} \|_{L^2(0,1)} + \| \sqrt{v_{\nu m}} \|_{L^2(0,1)} \right) \\
=: \mathcal{K}_2.
\]

(40)
Assumptions A.1 and A.2, where just ℓ

Remark 2. Above, we have proved the well-posedness of (1), by assuming that

Since \( K_1, K_2 \) and \( K_3 \) do not depend on \( m \in \mathbb{N}^* \), the three estimates (38), (39) and (40) imply that the sequences \( (u_m)_{m=1}^{\infty} \) and \( (v_m)_{m=1}^{\infty} \) are bounded in \( L^2(0, T; L^2(0, 1)) \cap H^1(0, T; H^2_0(0, 1)) \).

Therefore, there exist subsequences \( (u_{m_j})_{j=1}^{\infty} \) of \( (u_m)_{m=1}^{\infty} \) and \( (v_{m_j})_{j=1}^{\infty} \) of \( (v_m)_{m=1}^{\infty} \), such that

\[ u_{m_j} \to u \quad \text{and} \quad v_{m_j} \to v, \quad \text{as} \quad m \to \infty \]

weakly in \( L^2(0, T; L^2(0, 1)) \cap H^1(0, T; H^2_0(0, 1)) \). By standard arguments, we can conclude that \( (u, v) \) is a weak solution of (36).

\[ \square \]

Remark 2. Above, we have proved the well-posedness of (1), by assuming that \( \ell_1(r) \geq C > 0 \), for \( i = 1, 2 \). However, Theorem 1.1 has been obtained under the Assumptions A.1 and A.2, where just \( \ell_1(0) > 0 \) is required. This last hypotheses is sufficient to prove that \( H \subset C^1(E; F) \) is well defined and that \( H'(0, 0, 0) \subset L(E; F) \) is onto.

Appendix B. \( E \) is a Hilbert space. In this section we will prove that \( E \) is a Hilbert space. Let us recall that

\[ E := \left\{ (u, v, h) \in [L^2((0, T) \times (0, 1))]^2 \times L^2((0, T) \times \omega) : \right. \\
\quad u(t, \cdot), v(t, \cdot) \ \text{are absolutely continuous in} \ [0, 1], \ \text{a.e. in} \ [0, T], \\
\quad u_t, u_x, (au_x)_x, \rho_s h \in L^2((0, T) \times (0, 1)), v_t, v_x, (av_x)_x \in L^2((0, T) \times (0, 1)), \\
\quad \rho_0 u, \rho_0 [u_t - (au)_x - h \chi_\omega], \rho_0 v, \rho_0 [v_t - (av)_x] \in L^2((0, T) \times (0, 1)), \\
\quad u(t, 1) \equiv v(t, 1) \equiv u(t, 0) \equiv v(t, 0) \equiv 0 \ \text{a.e in} \ [0, T], \ \text{and} \ u(0, \cdot), v(0, \cdot) \in H^1_0 \} \\
\]

with the following norm induced by an inner product

\[ \| (u, v, h) \|_E^2 := \int_0^T \int_0^1 \rho_0^2 |u|^2 + |v|^2 + \int_0^T \int_\omega \rho_s^2 |h|^2 \\
\quad + \int_0^T \int_0^1 \rho_0^2 |u_t - (au)_x - h \chi_\omega|^2 + \int_0^T \int_0^1 \rho_0^2 |v_t - (av)_x|^2 \\
\quad + \| u(0, \cdot) \|^2_{H^1_0} + \| v(0, \cdot) \|^2_{H^1_0}. \]

Proof. Let \( (u_n, v_n, h_n)_{n=1}^{\infty} \) be a Cauchy sequence in \( E \). In particular,

\[ \begin{cases}
(\rho_0 u_n)_{n=1}^{\infty}, (\rho_0 v_n)_{n=1}^{\infty}, (\rho_s h_n \chi_\omega)_{n=1}^{\infty}, (\rho_0 |u_{n_t} - (au_{n_x})_x - h_n \chi_\omega|)_{n=1}^{\infty}, \\
(\rho_0 |v_{n_t} - (av_{n_x})_x|)_{n=1}^{\infty}, \ \text{are Cauchy in} \ L^2((0, T) \times (0, 1)).
\end{cases} \tag{41} \]

And, since \( \rho_0, \rho_s \geq C_T \), we also have that

\[ \begin{cases}
(u_n)_{n=1}^{\infty}, (v_n)_{n=1}^{\infty}, (h_n \chi_\omega)_{n=1}^{\infty}, (u_{n_t} - (au_{n_x})_x - h_n \chi_\omega)_{n=1}^{\infty}, \\
(v_{n_t} - (av_{n_x})_x)_{n=1}^{\infty}, \ \text{are Cauchy in} \ L^2((0, T) \times (0, 1)) \tag{42}
\end{cases} \]

In particular, there exists \( h \in L^2((0, T) \times \omega) \) such that

\[ h_n \to h \ \text{in} \ L^2((0, T) \times \omega). \tag{43} \]
Now, let us set
\[
\begin{aligned}
g_{1,n} &:= u_n - (a u_{n_x})_x + b_{11} u_n + b_{12} v_n - h_n \chi \omega, \\
g_{2,n} &:= u_n - (a v_{n_x})_x + b_{21} u_n + b_{22} v_n, \\
u_{0,n} &:= u_n(0,\cdot), \quad v_{0,n} := v_n(0,\cdot).
\end{aligned}
\]
In this case, we can see that \((u_n, v_n)\) is a weak solution to
\[
\begin{aligned}
&g_{1,n} := u_n - (a u_{n_x})_x + b_{11} u_n + b_{12} v_n = h_n \chi \omega + g_{1,n}, \quad (t, x) \in (0, T) \times (0, 1), \\
&g_{2,n} := u_n - (a v_{n_x})_x + b_{21} u_n + b_{22} v_n = g_{2,n}, \quad (t, x) \in (0, T) \times (0, 1), \\
u_n(t, 1) = u_n(t, 0) = v_n(t, 0) = v_n(t, 1) = 0, \quad t \in (0, T), \\
u_n(0, x) = u_{0,n}(x) \quad \text{and} \quad v_n(0, x) = v_{0,n}(x), \quad x \in (0, 1),
\end{aligned}
\]
where \(g_{1,n}, h_n \chi \omega, g_{2,n} \in L^2((0, T) \times (0, 1))\) and \(u_{0,n}, v_{0,n} \in H^1_0\). Hence, Proposition 1 gives us that
\[
u_n, v_n \in H^1(0, T; L^2(0, 1)) \cap L^2(0, T; H^2_0) \cap C^0([0, T]; H^1_0)
\]
as well as the inequality (6). Therefore, from (42) together with the third term of (41), we have that \((u_n)_{n=1}^{\infty}, (v_n)_{n=1}^{\infty}\) are Cauchy sequences in the Banach space \(H^1(0, T; L^2(0, 1)) \cap L^2(0, T; H^2_0) \cap C^0([0, T]; H^1_0)\). As a consequence, there exist \(u, v\) such that
\[
u_n \to u \quad \text{and} \quad v_n \to v \quad \text{in} \quad H^1(0, T; L^2(0, 1)) \cap L^2(0, T; H^2_0) \cap C^0([0, T]; H^1_0).
\]
This convergence together with (43) guarantees that
\[
u_n, v_n, h_n \to (u, v, h) \quad \text{in} \quad E.
\]

Appendix C. Some properties of \(J_n\). In this section, we will prove that the functional \(J_n\) defined in Theorem 4.1 is lower semi-continuous, strictly convex and coercive. For convenience, let us recall its definition:
\[
J_n(u, v, h) = \frac{1}{2} \int_0^T \int_0^1 \rho_{0,n}(u^2 + v^2) + \frac{1}{2} \int_0^T \int_0^1 \rho_{s,n} |h|^2 \, dx + \frac{1}{2} \left( \|\rho_{0,n} u\|^2_{L^2_x} + \|\rho_{0,n} v\|^2_{L^2_x} + \|\rho_{s,n} h\|^2_{L^2_x} \right)
\]
where \((u, v, h) \in [L^2((0, T) \times (0, 1))]^3\).

Since \(J_n\) is a sum of squared norms, it is strictly convex. In order to prove the remaining properties, we will need the following lemma.

Lemma C.1. There exist constants \(C_{n,T} > 0\), depending on \(n\) and \(T\), and \(C_T > 0\), depending only on \(T\), such that
\[
0 < C_T \leq \rho_{0,n} \leq C_{n,T} \quad \text{in} \quad [0, T] \times [0, 1],
\]
and
\[
0 < C_T \leq \rho_{s,n} \leq C_{n,T} \quad \text{in} \quad [0, T] \times [0, 1].
\]

Proof. We will prove the estimates to \(\rho_{0,n}\), those corresponding to \(\rho_{s,n}\) are analogous. Firstly, we note that \(\zeta\) is bounded from below by a positive constant depending only on \(T\), which we will denote by \(m_T\).

Secondly, note that we can rewrite \(A\) as
\[
A = -\left( \frac{e^{3\lambda |\psi|}}{\eta} - 1 \right) \zeta := -\beta(x) \zeta,
\]
where \( \beta(x) = e^{\frac{\beta |x|}{\eta \lambda_p}} - 1 \) is bounded and strictly positive for \( \lambda \) large enough, i.e., there exist constants \( \beta_0, \beta_1 > 0 \) such that
\[
0 < \beta_0 \leq \beta(x) \leq \beta_1, \; \forall x \in [0, 1].
\]

Thus,
\[
-\beta_1 \zeta \leq A \leq -\beta_0 \zeta < 0.
\]

If \( t \in [0, T/2] \), we can see that \( \zeta(t) \) is bounded from above and below by positive constants which depend only on \( T \), that is, there exist constants \( m_T, M_T > 0 \), depending only on \( T \), such that
\[
0 < m_T \leq \zeta \leq M_T.
\]

Now, notice that
\[
\frac{T^4}{16(T^4 + 1)} \leq \frac{(T - t)^4}{(T - t)^4 + \frac{1}{n}} \leq 1
\]
and, since \( A_n = A \frac{(T - t)^4}{(T - t)^4 + \frac{1}{n}} < 0 \), we have that
\[
0 < s \beta_0 m_T \frac{T^4}{16(T^4 + 1)} \leq -s A_n \leq s \beta_1 M_T \Rightarrow 1 \leq e^{-s A_n} \leq e^{s \beta_1 M_T}.
\]

Therefore, since \( \rho_{0,n} = e^{-s A_n} \zeta^{-2} \), we get that
\[
0 < \frac{1}{M_T^2} \leq \rho_{0,n} \leq e^{s \beta_1 M_T} \frac{1}{m_T^2}.
\]

If \( t \in [T/2, T] \), we have that
\[
A_n = -\beta \zeta \frac{(T - t)^4}{(T - t)^4 + \frac{1}{n}} = -\beta \frac{\eta}{t^2} \frac{1}{(T - t)^4 + \frac{1}{n}}.
\]

And, as a consequence,
\[
\frac{\beta_0}{T^4 \left( \frac{T}{T^4 + 1} \right)} \leq -A_n \leq \frac{\beta_1 |\eta|_{ \infty } 16 n}{T^4}.
\]

Hence,
\[
e^{-s A_n} \zeta^{-2} \leq e^{s \beta_1 |\eta|_{ \infty } 16 n} m_T^{-2} := C_{n,T}.
\]

Therefore, since \( e^x \geq \frac{x^2}{3!} \) for all \( x > 0 \), we finally conclude that
\[
e^{-s A_n} \zeta^{-2} \geq -\frac{s^3 A_n^3 \zeta^{-2}}{3!} = \frac{s^3 \beta_0^3 A_n^3}{3!} (-A_n) \geq \frac{s^3 \beta_0^3}{T^4 \left( \frac{T}{16} + 1 \right)} := C_T.
\]

\[\square\]

**Proposition 9.** \( J_n \) is lower semi-continuous and coercive.

**Proof.** Firstly, note that for each \( n \in \mathbb{N}^* \), the last lemma gives us that the norms in the definition of \( J_n \) are equivalents to the norms in \( L^2((0, T) \times (0, 1)) \).

Given a sequence \( (u_k, v_k, h_k)_{k=1}^\infty \) in \( \left[ L^2((0, T) \times (0, 1)) \right]^3 \) such that
\[
(u_k, v_k, h_k) \to (u, v, h) \quad \text{in} \quad \left[ L^2((0, T) \times (0, 1)) \right]^3,
\]
we have
\[
\rho_{0,n} u_k \to \rho_{0,n} u, \quad \rho_{0,n} v_k \to \rho_{0,n} v \quad \text{and} \quad \rho_{0,n} h_k \to \rho_{0,n} h \quad \text{in} \quad L^2((0, T) \times (0, 1)),
\]
as \( k \to +\infty \), where \( n \) is fixed. In particular,
\[
\| \rho_{0,n} u_k \|_{L^2} \to \| \rho_{0,n} u \|_{L^2}, \quad \| \rho_{0,n} v_k \|_{L^2} \to \| \rho_{0,n} v \|_{L^2} \quad \text{and} \quad \| \rho_{0,n} h_k \|_{L^2} \to \| \rho_{0,n} h \|_{L^2}.
\]
As a consequence,

\[ J_n(u, v, h) = \frac{1}{2} \left( \| \rho_{0,n} u \|_{L^2}^2 + \| \rho_{0,n} v \|_{L^2}^2 + \| \rho_{n} h \|_{L^2}^2 \right) \]

\[ = \lim_{k \to \infty} \frac{1}{2} \left( \| \rho_{0,n} u_k \|_{L^2}^2 + \| \rho_{0,n} v_k \|_{L^2}^2 + \| \rho_{n} h_k \|_{L^2}^2 \right) \]

\[ = \lim_{k \to \infty} J_n(u_k, v_k, h_k), \]

which proves that \( J_n \) is continuous and consequently lower semi-continuous.

Analogously, given a sequence \( (u_k, v_k, h_k)_{k=1}^{\infty} \) in \( L^2((0,T) \times (0,1)) \) such that

\[ \|(u_k, v_k, h_k)\|_{L^2((0,T) \times (0,1))} \to +\infty, \]

we have that

\[ J_n(u_k, v_k, h_k) = \frac{1}{2} \left( \| \rho_{0,n} u_k \|_{L^2}^2 + \| \rho_{0,n} v_k \|_{L^2}^2 + \| \rho_{n} h_k \|_{L^2}^2 \right) \]

\[ \geq C_T \left( \| u_k \|_{L^2}^2 + \| v_k \|_{L^2}^2 + \| h_k \|_{L^2}^2 \right) \to +\infty. \]

Therefore, \( J_n \) is coercive.

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