Regularity and amenability of weighted Banach algebras and their second dual on locally compact groups

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Abstract. Let \( \omega \) be a weight function on a locally compact group \( G \) and let \( M_\ast(G, \omega) \) be the subspace of \( M(G, \omega)^\ast \) consisting of all functionals that vanish at infinity. In this paper, we first investigate the Arens regularity of \( M_\ast(G, \omega)^\ast \) and show that \( M_\ast(G, \omega)^\ast \) is Arens regular if and only if \( G \) is finite or \( \Omega \) is zero cluster. This result is an answer to the question posed and it improves some well-known results. We also give necessary and sufficient criteria for the weight function spaces \( Wap(G, 1/\omega) \) and \( Ap(G, 1/\omega) \) to be equal to \( C_b(G, 1/\omega) \). We prove that for non-compact group \( G \), the Banach algebra \( M_\ast(G, \omega)^\ast \) is Arens regular if and only if \( Wap(G, 1/\omega) = C_b(G, 1/\omega) \). We then investigate amenability of \( M_\ast(G, \omega)^\ast \) and prove that \( M_\ast(G, \omega)^\ast \) is amenable and Arens regular if and only if \( G \) is finite.

1 Introduction

Throughout this paper, \( G \) denotes a Hausdorff locally compact group with the group algebra \( L^1(G) \) and the measure algebra \( M(G) \). A weight on \( G \) is a continuous function \( \omega : G \to [1, \infty) \) such that \( \omega(e) = 1 \) and

\[
\omega(xy) \leq \omega(x) \omega(y)
\]

for all \( x, y \in G \), where \( e \) is the identity element of \( G \). Let the function \( \Omega : G \times G \to (0, 1] \) be defined as follows:

\[
\Omega(x, y) = \frac{\omega(xy)}{\omega(x)\omega(y)}.
\]

Let us recall that a complex-valued function \( F \) on \( G \times G \) is called cluster (respectively, zero cluster, positive cluster) if for every pair of sequences \( (x_n)_n \) and \( (y_m)_m \) of distinct elements in \( G \), we have

\[
\lim_{n} \lim_{m} F(x_n, y_m) = \lim_{m} \lim_{n} F(x_n, y_m), \tag{1}
\]

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It is well-known that $L^1(G, \omega)$ and $M(G, \omega)$ are Banach algebra and $M(G, \omega)$ is the dual space of $C_0(G, 1/\omega)$, the Banach space of all complex-valued continuous functions $f$ on $G$ such that $f/\omega$ vanishes at infinity, see for example [5, 23].

We say that $\lambda \in M(G, \omega)^*$ vanishes at infinity if for every $\varepsilon > 0$, there exists a compact subset $K$ of $G$, for which $|\langle \lambda, \mu \rangle| < \varepsilon$, where $\mu \in M(G, \omega)$ with $|\mu|(K) = 0$ and $||\mu||_\omega = 1$. We denote by $M_\ast(G, \omega)$ the subspace of $M(G, \omega)^*$ consisting of all functionals that vanish at infinity. In the case where, $\omega(x) = 1$ for all $x \in G$, we write the spaces

$$M_\ast(G, \omega) := M_\ast(G).$$

The space $M_\ast(G, \omega)$ is a norm closed subspace of $M(G, \omega)^*$ and so it is a $C^*$-algebra. Every element $f \in C_0(G, 1/\omega)$ may be regarded as an element in $M_\ast(G, \omega)$ by the pairing

$$\langle f, \mu \rangle = \int_G f d\mu \quad (M(G, \omega)).$$

Then $C_0(G, 1/\omega)$ is a closed subspace of $M_\ast(G, \omega)$. Also, the space $M_\ast(G, \omega)$ is left introverted in $M(G, \omega)^*$. This let us to endow $M_\ast(G, \omega)^*$ with the first Arens product. Then $M_\ast(G, \omega)^*$ with this product becomes to a Banach algebra [22]. For each $\phi \in L^1(G, \omega)$, let $\phi$ denote the functional in $M_\ast(G, \omega)^*$ defined by

$$\langle \phi, \lambda \rangle := \langle \lambda, \phi \rangle.$$

for all $\lambda \in M(G, \omega)^*$. This duality defines a linear isometric embedding from $L^1(G, \omega)$ into $M_\ast(G, \omega)^*$. One can prove that $L^1(G, \omega)$ is a closed ideal in $M_\ast(G, \omega)^*$ and $M_\ast(G, \omega)^* = L^1(G, \omega)$ if and only if $G$ is discrete [22]; see [21] for the case $\omega = 1$. Since $M(G, \omega)$ is a closed subspace of $M_\ast(G, \omega)^*$, an easy application of the Goldstine’s theorem shows that if $\Phi \in M_\ast(G, \omega)^*$, then there exists a net $(\mu_\alpha)_\alpha$ in $M(G, \omega)$ such that $\mu_\alpha \rightarrow \Phi$ in the weak*-topology of $M_\ast(G, \omega)^*$.

Let us recall that the first Arens product “$\diamond$” on the second dual of a Banach algebra $\mathfrak{A}$ is defined by

$$\langle \Phi \diamond \Psi, f \rangle = \langle \Phi, \Psi f \rangle,$$

in which

$$\langle \Psi f, a \rangle = \langle \Psi, fa \rangle \quad \text{and} \quad \langle fa, b \rangle = \langle f, ab \rangle$$

for all $\Phi, \Psi \in \mathfrak{A}^{**}$, $f \in \mathfrak{A}^*$ and $a, b \in \mathfrak{A}$. The Banach algebra $\mathfrak{A}$ is called Arens regular if for every $\Phi \in \mathfrak{A}^{**}$ the mapping $\Psi \mapsto \Phi \diamond \Psi$ is weak*-weak* continuous on $\mathfrak{A}^{**}$.
Several authors have studied the Arens regularity of weighted group algebras. For example, Crow and Young [3] showed that there exists a weighted function $\omega$ on $G$ such that $L^1(G, \omega)$ is Arens regular if and only if $G$ is discrete and countable. The second author and Vishki [25] proved that $L^1(G, \omega)$ is Arens regular if and only if $G$ is finite or $G$ is discrete and $\Omega$ is zero cluster. They showed that $L^1(G, \omega)$ is amenable and Arens regular if and only if $G$ is finite; see also [11]. These studies have continued for the other Banach algebras. See for example, [1, 26] for the Arens regularity of weighted semigroup algebras and [12, 13, 14, 15, 18] for the Arens regularity of Fourier algebras. See also [6, 29, 27, 28].

In this paper, we investigate the Arens regularity of $M^*(G, \omega)$ and the relation between it, the weighted function spaces and amenability. In Section 2, we give an answer to the question presented in [22] and prove that $M^*(G, \omega)$ is Arens regular if and only if $G$ is finite or $\Omega$ is zero cluster. This result is an improvement of Theorem 2 of [25]. We also show that $M^*(G)$ is Arens regular if and only if there exists a weight function $\omega$ on $G$ such that $M^*(G, \omega)$ is $C^*$-algebra; or equivalently, $G$ is finite. In Section 3, we prove that $G$ is weight regular if and only if $G$ is a countable discrete group. For a normal subgroup $N$ of $G$, we show that if $G$ is weight regular, then $G/N$ is weight regular and $N$ is countable and open. Section 4 is devote to weighted function spaces $\text{Wap}(G, 1/\omega)$ and $\text{Ap}(G, 1/\omega)$. We give necessary and sufficient condition for these weighted function spaces to be equal to $C_b(G, 1/\omega)$. For instance, we show that $\text{Wap}(G, 1/\omega) = C_b(G, 1/\omega)$ if and only if $G$ is compact or $\Omega$ is zero cluster. As a consequence of this result, we prove that $M^*(G, \omega)$ is Arens regular if and only if $\text{Wap}(G, 1/\omega) = C_b(G, 1/\omega)$, when $G$ is non-compact. In Section 5, we investigate amenability of $M^*(G, \omega)$ and prove that $M^*(G, \omega)$ is amenable if and only if $G$ is a discrete amenable group and $\omega^*$ is bounded. We also show that $M^*(G, \omega)$ is Arens regular and amenable if and only if $G$ is finite.

2 Arens regularity of $M^*(G, \omega)$

The following lemma is needed to prove our results.

**Lemma 2.1** Let $\omega$ be a weight function on a locally compact group $G$. If $\Omega$ is zero cluster, then $G$ is discrete.

Proof. Suppose that $G$ is a non-discrete group. Let $\mathcal{U}$ be the family of all neighborhood of $e$ directed by upward inclusion, i.e.,

$$U_1 \geq U_2 \iff U_1 \subseteq U_2 \quad (U_1, U_2 \in \mathcal{U}).$$

Assume that $U \in \mathcal{U}$. Since $G$ is non-discrete, $U$ is infinite. So we can choose $x_U \in U$ such that $x_U \neq e$. Then the net $(x_U)_{U \in \mathcal{U}}$ of distinct points of $G$ converges to the
identity element $e$. Indeed, if $W$ is a neighborhood of $e$, then for every $U \geq W$, we have
\[ x_U \in U \subseteq W. \]
Now, using continuity of $\omega$ together with $\omega(e) = 1$, both iterated limits $\Omega(x_U, x_V)$ converge to 1. By Proposition 2.1 in [5], there exist subsequences $(x_{U_n})_{n \in \mathbb{N}}$ and $(x_{V_m})_{m \in \mathbb{N}}$ of $(x_U)_{U \in \mathcal{U}}$ such that
\[ \lim_n \lim_m \Omega(x_{U_n}, x_{V_m}) = 1 = \lim_m \lim_n \Omega(x_{U_n}, x_{V_m}). \]
Hence $\Omega$ can not be zero cluster, a contradiction. So zero clusters may exist only on discrete groups. $\square$

Let $L^\infty(G, 1/\omega)$ be the space of all measurable functions $f$ on $G$ with
\[ \|f\|_{\infty, \omega} = \|f/\omega\|_{\infty} < \infty, \]
where $\|\cdot\|_{\infty}$ is the essential supremum norm. We denote by $L^\infty_0(G, 1/\omega)$ the subspace of $L^\infty(G, 1/\omega)$ consisting of all functions $f \in L^\infty(G, 1/\omega)$ that vanish at infinity. It is well-known from [20] that the dual space of $L^\infty_0(G, 1/\omega)^*$ is a Banach algebra with the first Arens product; see also [17, 19]. One can show that $L^\infty_0(G, 1/\omega)^*$ is isomorphic with the set of all $F \in M_*(G, \omega)^*$ with
\[ \langle F, \lambda \rangle = \langle F, \lambda_0 \rangle \]
for all $\lambda \in M_*(G, \omega)^*$, where $\lambda_0 = \lambda|_{L^1(G, \omega)}$; see [22].

The first author and Moghimi [22] proved that if $M_*(G, \omega)^*$ is Arens regular, then $G$ is discrete. We are now in a position to prove the main result of this paper which is an improvement of Theorem 2 of [25] and is an answer to the open question presented in [22].

**Theorem 2.2** Let $\omega$ be a weight function on a locally compact group $G$. Then the following assertions are equivalent.

(a) $M_*(G, \omega)^*$ is Arens regular.  
(b) $L^1(G, \omega)$ is Arens regular.  
(c) $M(G, \omega)$ is Arens regular.  
(d) $L^1(G, \omega)^{**}$ is Arens regular.  
(e) $M(G, \omega)^{**}$ is Arens regular.  
(f) $L^\infty_0(G, \omega)^*$ is Arens regular.  
(g) $G$ is finite or $\Omega$ is zero cluster.  

In this case, $G$ is discrete and countable.
Proof. Assume that $M^*_s(G, \omega)$ is Arens regular. Since $L^1(G, \omega)$ is a closed ideal in $M^*_s(G, \omega)$, it follows from Corollary 2.6.18 in [6] that $L^1(G, \omega)$ is Arens regular. So (a) implies (b). It is well-known from [25] that $L^1(G, \omega)$ is Arens regular if and only if $G$ is finite or $G$ is discrete and $\Omega$ is zero cluster. From this and Lemma 2.1 follows that the statements (b) and (g) are equivalent. From Theorem 4.7 and Corollary 4.11 in [20] and Lemma 2.1 we see that the other statements are equivalent.

To complete the proof, note that $G = \bigcup_{n=1}^{\infty} A_n$, where $A_n = \{ x \in G : \omega(x) \leq n \}$. If $G$ is uncountable, then $A_m$ is infinite for some $m \in \mathbb{N}$. For every $x, y \in A_m$, we have $\Omega(x, y) \geq 1/m^2$. This implies that $\Omega$ can not be zero cluster. \qed

Example 2.3 (i) Let $\alpha \geq 0$ and for every $n \in \mathbb{Z}$

$$\omega_\alpha(n) = (1 + |n|)^\alpha.$$ One can prove that $\Omega_\alpha$ is zero cluster if and only if $\alpha > 0$. It follows from Theorem 2.2 that $M^*_s(\mathbb{Z}, \omega_\alpha)$ is Arens regular if and only if $\alpha > 0$. In the case where $\alpha = 0$, the Banach algebra $M^*_s(\mathbb{Z})$ is not Arens regular.

(ii) Let $\alpha, \beta > 0$. For every $m, n \in \mathbb{Z}$ we define

$$\omega(m, n) = (1 + |m|)^\alpha(1 + |n|)^\beta.$$ Set $x_m = (m, 0)$ and $y_n = (0, n)$. Then $\Omega(x_m, x_n) = 1$. Hence $M^*_s(\mathbb{Z}^2, \omega)$ is not Arens regular.

Remark 2.4 Let $\omega$ be a weight function on a locally compact group $G$. If $\Omega$ is either positive-cluster or $\Omega > \alpha$ for some $\alpha > 0$, or $\omega$ is multiplicative, then $\Omega$ can not be zero cluster. So by Theorem 2.2, the Banach algebra $M^*_s(G, \omega)$ is Arens regular if and only if $G$ is finite.

Baker and the second author [1] gave the following result for the discrete convolution semigroup algebra $\ell^1(S, \omega)$. In the following, we prove this result for locally compact groups.

Corollary 2.5 Let $\omega$ be a weight function on infinite locally compact group $G$. Then the following assertions are equivalent.

(a) $M^*_s(G, \omega)$ is Arens regular.

(b) $L^1(G, \omega)$ is Arens regular.
(c) $G$ is discrete and for every $A \subseteq G$ and each pair of sequences $(x_n)$ and $(y_n)$ in $G$,
\[ \{ \chi_A(x_ny_m)\Omega(x_n, y_m) : n < m \} \cap \{ \chi_A(x_ny_m)\Omega(x_n, y_m) : n > m \} \neq \emptyset. \]

(d) $G$ is discrete and for each pair of sequences $(x_n)$ and $(y_n)$ in $G$ there exist subsequences $(a_n)$ and $(b_n)$ of $(x_n)$ and $(y_n)$, respectively, such that at least one of the following statements hold.

1. $\lim_n \lim_m \Omega(a_n, b_m) = 0 = \lim_m \lim_n \Omega(a_n, b_m)$.
2. Either the rows or the columns of the matrix $(a_n b_m)$ are constant and distinct.
3. The matrix $(a_n b_m)$ is constant.

As an immediate consequence of Corollary 3.5 in [1] and Theorem 2.2, we give the next result.

**Corollary 2.6** Let $\omega_1$ and $\omega_2$ be weight functions on locally compact infinite group $G$. Then the following statements hold.

(i) If $\Omega_1 \geq \alpha \Omega_2$ for some $\alpha > 0$ and $M_*(G, \omega_1)^\ast$ is Arens regular, then $M_*(G, \omega_2)^\ast$ is Arens regular.

(ii) If there exist positive numbers $\alpha$ and $\beta$ such that $\alpha \omega_1 \leq \omega_2 \leq \beta \omega_1$, then $M_*(G, \omega_1)^\ast$ is Arens regular if and only if $M_*(G, \omega_2)^\ast$ is Arens regular.

Let us recall that a Banach algebra $\mathfrak{A}$ is called a **dual Banach algebra** if there exists a closed submodule $E$ of the dual module $\mathfrak{A}^\ast$ such that $E^\ast = \mathfrak{A}$. It is well-known that $(\mathfrak{A}^\ast\ast, \phi)$ is a dual Banach algebra if and only if $\mathfrak{A}$ is Arens regular; see for example Corollary 2.16 in [5]. From this together with Theorem 2.2, we have the following result.

**Corollary 2.7** Let $\omega$ be a weight function on a locally compact group $G$. Then $((M_*(G, \omega)^\ast)^\ast, \phi)$ is a dual Banach algebra if and only if $G$ is finite or $\Omega$ is zero cluster.

For a weight function $\omega$ on $G$, we define $\omega^\ast(x) = \omega(x)\omega(x^{-1})$ for all $x \in G$. It is easy to see that $\omega^\ast$ is a weight function on $G$.

**Proposition 2.8** Let $G$ be a locally compact group. Then the following assertions are equivalent.

(a) $M_*(G)^\ast$ is Arens regular.

(b) For every weight function $\omega$ on $G$, the Banach algebra $M_*(G, \omega)^\ast$ is Arens regular.

(c) There exists a weight function $\omega$ on $G$ such that $M_*(G, \omega)^\ast$ is Arens regular and $\omega^\ast$ is bounded.

(d) There exists a weight function $\omega$ on $G$ such that $M_*(G, \omega)^\ast$ is reflexive and $\omega^\ast$ is bounded.

(e) There exists a weight function $\omega$ on $G$ such that $M_*(G, \omega)^\ast$ is a $C^\ast$-algebra.

(f) $G$ is finite.
Proof. First note that if \( \omega \) is a weight function on \( G \) such that \( \omega^* \) is bounded, then there exists \( \alpha > 0 \) such that

\[
\alpha \omega(x) \omega(y) \leq \omega(xy)
\]

for all \( x, y \in G \). This shows that \( \Omega \) can not be zero cluster. We also note that there exists a weight function \( \omega \) on \( G \) such that \( \Omega \) can not be zero cluster. From these facts and Theorem 2.2, we infer that the assertions (a)-(d) and (f) are equivalent. Now, let (e) hold. Then for every \( x \in G \), we have

\[
\|\delta_x \ast \delta_x^*\|_\omega = \|\delta_x\|_{\omega^*}^2.
\]

This implies that \( \omega = \Delta^{1/2} \), where \( \Delta \) is the modular function of \( G \). So

\[
\omega(xy) = \omega(x) \omega(y)
\]

for all \( x, y \in G \). Therefore, \( \Omega = 1 \). By Theorem 2.2, (e) holds. \( \square \)

Let \( \omega \) be a weight function on a locally compact group \( G \). By Proposition 2.8, if \( M_*(G)^* \) is Arens regular, then \( M_*(G, \omega)^* \) is Arens regular. The converse, however, is not true.

Example 2.9 Let \( \omega(n) = 1 + |n| \) for all \( n \in \mathbb{Z} \). Then \( \Omega \) is zero cluster and so \( M_*(\mathbb{Z}, \omega)^* \) is Arens regular. But \( M_*(\mathbb{Z})^* \) isn’t Arens regular.

As an immediate consequence of proposition 2.8 we have the following result.

Corollary 2.10 Let \( \omega \) be a weight function on a locally compact group \( G \). Then the following assertions are equivalent.

(a) \( M_*(G, \omega)^* \) is Arens regular and \( \omega^* \) is bounded.

(b) \( M_*(G, \omega)^* \) is reflexive and \( \omega^* \) is bounded.

(c) \( G \) is finite.

Example 2.3(i) shows that Corollary 2.10 is not true without the assumption that \( \omega^* \) is bounded.

3 Weight regularity of locally compact groups

A locally compact group \( G \) is called weight regular if there exists a weight function \( \omega : G \to [1, \infty) \) such that \( M_*(G, \omega)^* \) is Arens regular.

Theorem 3.1 Let \( G \) be a locally compact group. Then the following assertions are equivalent.

(a) \( G \) is weight regular.

(b) \( G \) is countable and discrete.

(c) \( G \) is finite or there exists a weight function \( \omega : G \to [1, \infty) \) such that \( \Omega \) is zero cluster.
Proof. The implications (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (a) follow from Theorem 2.2. The implication (b) $\Rightarrow$ (c) follows from Corollary 6.1.5 of [9]. \qed

As a consequence of Theorem 3.1 we have the following result.

**Corollary 3.2** Let $G$ be a locally compact infinite group. If $G$ is compact or there exists a convergent net of distinct points of $G$, then $G$ is not weight regular. Furthermore, there is no weight function $\omega$ on $G$ such that $M_{*}(G, \omega)^*$ is Arens regular.

**Proof.** Let $(x_{\alpha})_{\alpha \in A}$ be a convergent net of distinct points of $G$. If $G$ is weight regular, then $G$ is discrete. So $(x_{\alpha})$ is eventually constant, a contradiction. To complete, the proof note that if $G$ is an infinite compact group, then any net of distinct points of $G$, has a convergent subnet. \qed

**Example 3.3** By Theorem 3.1, the additive group $\mathbb{Z}$ is weight regular, however, $\mathbb{R}$ and the tours group $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ are not weight regular. So there is no weight function $\omega$ on $\mathbb{R}$ (respectively, $\mathbb{T}$) such that $M_{*} (\mathbb{R}, \omega)^*$ and $L^{1}(\mathbb{R}, \omega)$ (respectively, $M_{*} (\mathbb{T}, \omega)^*$ and $L^{1}(\mathbb{T}, \omega)$) are Arens regular.

From Theorems 2.2 and 3.1 we have the following result due to Craw and Young [3].

**Corollary 3.4** Let $G$ be a locally compact group. Then there exists a weight function $\omega$ on $G$ such that $L^{1}(G, \omega)$ is Arens regular if and only if $G$ is countable and discrete.

**Proposition 3.5** Let $G_{1}$ and $G_{2}$ be locally compact groups and $\psi : G_{1} \rightarrow G_{2}$ be a group homomorphism. Then the following statements are hold.

(i) If $G_{1}$ is weight regular, then $\text{Im} \psi$ is weight regular.

(ii) If $\psi$ is epimorphism and $G_{1}$ is weight regular, then $G_{2}$ is weight regular.

(iii) If $\psi$ is epimorphism and $M_{*}(G_{1})^*$ is Arens regular, then $G_{2}$ is weight regular.

(iv) If $\psi$ is monomorphism and $M_{*}(G_{2})^*$ is Arens regular, then $G_{1}$ is weight regular. In these cases, $\psi$ is continuous.

**Proof.** Let $G_{1}$ be weight regular. Then there exists a weight function $\omega$ on $G_{1}$ such that $M_{*}(G_{1}, \omega)^*$ is Arens regular. Define the weight function $\omega_{2}$ on $\text{Im} \psi$ by

$$\omega_{2}(\psi(t)) = \inf \omega_{1}(\psi^{-1}(\psi(t)))$$

for all $t \in G_{1}$. Note that there is $0 < \alpha < 1$ such that for every $t \in G_{1}$

$$\alpha \leq \omega_{1}(t) \leq \omega_{2}(\psi(t)) + \alpha^2.$$
So

\[(1 - \alpha)\omega_1(t) \leq \omega_2(\psi(t)) \leq \omega_1(t).\]

This implies that

\[\Omega_2(\psi(t), \psi(s)) \leq \frac{1}{1 - \alpha^2} \Omega_1(s, t)\]

Now, Corollary 2.6 proves (i). The statements (ii) and (iii) follow from (i).

Finally, let \(M_*(G_2)^*\) be Arens regular. Then \(G_2\) is finite. If \(\psi\) is monomorphism, then \(G_1\) is finite and so it is regular. \(\square\)

Let \(\{G_i\}_{i \in I}\) be a family of locally compact groups and \(\pi_j : \Pi_{i \in I}G_i \rightarrow G_j\) be the canonical projection, for \(j \in I\). It is clear that \(\pi_j\) is onto. Hence the following result holds.

**Corollary 3.6** Let \(\{G_i\}_{i \in I}\) be a family of locally compact groups. If \(\Pi_{i \in I}G_i\) is weight regular, then \(G_i\) is weight regular for all \(i \in I\).

Let us recall that a sequence \(G_1 \xrightarrow{f} G_2 \xrightarrow{g} G_3\) of group homomorphisms is said to be exact if \(\text{Im } f = \ker g\). An exact sequence of the form \(0 \rightarrow G_1 \xrightarrow{f} G_2 \xrightarrow{g} G_3 \rightarrow 0\) is called short exact. If there exists a group homomorphism \(h : G_2 \rightarrow G_1\) such that \(hf = 1_{G_2}\), then the short exact sequence is called split.

**Proposition 3.7** Let \(G_1, G_2\) and \(G_3\) be locally compact groups, \(0 \rightarrow G_1 \xrightarrow{f} G_2 \xrightarrow{g} G_3 \rightarrow 0\) be a short exact sequence of group homomorphisms and \(G_2\) be weight regular. Then the following statements hold.

(i) \(G_1\) is countable and \(G_3\) is weight regular.

(ii) If the given sequence is split, then \(G_1\) and \(G_3\) are weight regular.

**Proof.** Note that if the given sequence is short exact, then \(g\) is onto. Also, if it is split, then \(h\) is injective. These facts together with Theorem 3.1 and Proposition 3.5 prove the result. \(\square\)

In the sequel, we present a consequence of Proposition 3.7.

**Corollary 3.8** Let \(N\) be a normal subgroup of locally compact group \(G\). Then the following statements hold.

(i) If \(G\) is weight regular, then \(G/N\) is weight regular and \(N\) is is countable and open.

(ii) If \(G\) is weight regular and the sequence \(0 \rightarrow N \xrightarrow{\iota} G \xrightarrow{\pi} G/N \rightarrow 0\) is split, then \(G/N\) and \(N\) are weight regular, where \(\iota\) is the inclusion map and \(\pi\) is the quotient map.
Proof. It is easy to see that the sequence \(0 \to N \xrightarrow{i} G \xrightarrow{\pi} G/N \to 0\) is short exact. So \(G/N\) is weight regular by Proposition 3.7. From the weight regularity of \(G\) and \(G/N\) we infer that \(G\) is countable and \(G/N\) is discrete. Hence \(N\) is countable and open. So (i) holds. The statement (ii) follows at once from Proposition 3.7. \(\square\)

We finish this section with the following result.

**Proposition 3.9** Let \(G_i\) and \(G'_i\), for \(i = 1, 2, 3\), be locally compact groups and the sequences \(0 \to G_1 \xrightarrow{f} G_2 \xrightarrow{g} G_3 \to 0\) and \(0 \to G'_1 \xrightarrow{f'} G'_2 \xrightarrow{g'} G'_3 \to 0\) be short exact. Let there exist group homomorphisms \(\alpha_i : G_i \to G'_i\), for \(i = 1, 2, 3\), such the obtained diagram is commutative, i.e., \(\alpha_2 f = f' \alpha_1\) and \(\alpha_3 g = g' \alpha_2\). Then the following statements hold.

(i) If \(\alpha_1\) and \(\alpha_3\) are group epimorphisms and \(G_2\) is weight regular, then \(G'_2, G_3\) and \(G'_3\) are weight regular. Furthermore, \(G_1, G_2\) and \(G'_1\) are countable.

(ii) If \(\alpha_1\) and \(\alpha_3\) are group epimorphisms and \(M_*(G_2)^*\) is Arens regular, then \(G_i\) and \(G'_i\) are weight regular for \(i = 1, 2, 3\).

(iii) If \(\alpha_1\) and \(\alpha_3\) are group monomorphisms and \(M_*(G'_2)^*\) is Arens regular, then \(G_i\) and \(G'_i\) are weight regular for \(i = 1, 2, 3\).

Proof. (i) Since \(g\) and \(\alpha_3\) are surjective and the diagram is commutative, \(g' \alpha_2\) is surjective and hence
\[
\text{Im } g' \alpha_2 = C' = \text{Im} g'.
\]

So, if \(b' \in B'\), then there exists \(b \in B\) such that
\[
\alpha_2(b) - b' \in \ker g' = \text{Im } f'.
\]

But \(\alpha_1\) is surjective and \(f' \alpha_1 = \alpha_2 f\). Thus
\[
\text{Im } f' = \text{Im } f' \alpha_1 = \text{Im } \alpha_2 f.
\]

Therefore, \(\alpha_2(b) - b' \in \text{Im } \alpha_2 f\). This shows that
\[
\alpha_2(b) - b' = \alpha_2 f(a)
\]

for some \(a \in A\). It follows that \(\alpha_2(b - f(a)) = b'\). Hence \(\alpha_2\) is surjective. Now, apply Proposition 3.7.

(ii) This is an immediate consequence of (i).

(iii) By the assumption, \(\alpha_3\) is injective and \(\alpha_3 g = g' \alpha_2\). This implies that
\[
\ker \alpha_2 \subseteq \ker g = \text{Im } f.
\]

By commutativity, \(f' \alpha_1 = \alpha_2 f\). Since \(f'\) and \(\alpha_1\) are injective, \(\alpha_2 f\) is injective. Hence \(\alpha_2\) is injective. By Proposition 3.7, the statement (iii) holds. \(\square\)
Weighted function spaces

Let \( C_b(G) \) (respectively, \( LUC(G) \)) be the space of all bounded continuous (respectively, uniformly continuous) functions on \( G \). Let \( C_b(G, 1/\omega) \) denote the space of all functions \( f \) on \( G \) such that \( f/\omega \in C_b(G) \). A function \( f \in C_b(G, 1/\omega) \) is called \( \omega \)-weakly almost periodic (respectively, \( \omega \)-almost periodic) if the set
\[
\left\{ \frac{xf}{\omega(x)\omega} : x \in G \right\}
\]
is relatively weakly (respectively, norm) compact in \( C_b(G) \), where \( xf(y) = f(yx) \) for all \( x,y \in G \). The set of all \( \omega \) (respectively, \( \omega \)-weakly) almost periodic on \( G \) is denoted by \( Ap(G, 1/\omega) \) (respectively, \( Wap(G, 1/\omega) \)). It is clear that
\[
Ap(G, 1/\omega) \subseteq Wap(G, 1/\omega) \subseteq C_b(G, 1/\omega).
\]
The equality may obtain for compact groups, however, it isn’t necessary. Note that if \( G \) is compact and \( f \in C_b(G, 1/\omega) \), then the mapping
\[
x \mapsto \frac{xf}{\omega(x)\omega}
\]
from \( G \) into \( C_b(G) \) is continuous. This implies that \( f \in Ap(G, 1/\omega) \). So the equality holds. In the sequel, we give necessary and sufficient condition under which the equality holds.

**Theorem 4.1** Let \( \omega \) be a weight function on a locally compact infinite group \( G \). Then the following statements hold.

(i) \( Wap(G, 1/\omega) = C_b(G, 1/\omega) \) if and only if \( G \) is compact or \( \Omega \) is zero cluster.

(ii) \( Ap(G, 1/\omega) = C_b(G, 1/\omega) \) if and only if \( G \) is either compact or discrete and \( \Omega \in C_0(G \times G) \).

**Proof.** (i) Let \( G \) be a non-compact group and \( Wap(G, 1/\omega) = C_b(G, 1/\omega) \). Then
\[
LUC(G, 1/\omega) = C_b(G, 1/\omega),
\]
where \( LUC(G, 1/\omega) \) is the set of all \( f \in C_b(G, 1/\omega) \) such that the map \( x \mapsto xf/\omega \) from \( G \) into \( C_b(G, 1/\omega) \) is norm continuous. Note that \( Wap(G, 1/\omega) \) is a subspace of \( LUC(G, 1/\omega) \). It is well-known from [24] that \( LUC(G, 1/\omega) = C_b(G, 1/\omega) \) if and only if \( G \) is compact or discrete; see also [2]. These facts show that \( G \) is discrete. It follows from Corollary 3.8 (ii) in [1] that \( L^1(G, \omega) \) is Arens regular. By Theorem 2.2, \( \Omega \) is zero cluster.

Conversely, let \( \Omega \) be zero cluster. In view of Lemma 2.1 and Theorem 2.2, \( G \) is discrete and \( L^1(G, \omega) \) is Arens regular. Applying Corollary 3.8 (ii) in [1], again, we have \( Wap(G, 1/\omega) = C_b(G, 1/\omega) \).
(ii) Let $G$ be non-compact and $Ap(G, 1/\omega) = C_b(G, 1/\omega)$. Using (i) and Lemma 2.1, $G$ is discrete. Now, the result is proved if we only note that for discrete infinite group $G$, $Ap(G, 1/\omega) = C_b(G, 1/\omega)$ if and only if $\Omega \in C_0(G \times G)$; see Corollary 3.18 (iii) in [1].

Example 4.2 For every $n \in \mathbb{Z}$, we define $\omega(n) = 1 + |n|$. Then $\Omega \in C_0(\mathbb{Z} \times \mathbb{Z})$. So

$$Ap(\mathbb{Z}, 1/\omega) = C_b(\mathbb{Z}, 1/\omega) = Wap(\mathbb{Z}, 1/\omega).$$

Proposition 4.3 Let $G$ be a non-compact group and $\omega$ be a weight function on $G$. Then the following assertions are equivalent.

(a) $Wap(G, 1/\omega) = C_b(G, 1/\omega)$.

(b) $M_\omega(G, \omega)^*$ is Arens regular.

(c) $Wap(G, 1/\omega) = LUC(G, 1/\omega)$.

(d) $\Omega$ is zero cluster.

Proof. It follows from Theorems 2.2 and 4.1 that the statements (a), (b) and (d) are equivalent. By Theorem 2.2 (ii) in [24] and Lemma 2.1 the statements (c) and (d) are equivalent. Finally, if $Ap(G, 1/\omega) = C_b(G, 1/\omega)$, then $Wap(G, 1/\omega) = C_b(G, 1/\omega)$. Hence $\Omega$ is zero cluster. By Theorem 3.1, $G$ is weight regular.

Theorem 4.4 Let $G$ and $G'$ be non-compact groups, $N$ be a normal subgroup of $G$ and $\omega$ and $\omega_p$ be weight functions on $G$ and $G \times G'$, respectively. Then the following statements hold.

(i) If $\psi : G \to G'$ is a group epimorphism and $Wap(G, 1/\omega) = C_b(G, 1/\omega)$, then there exists a weight function $\omega'$ on $G'$ such that $Wap(G', 1/\omega') = C_b(G', 1/\omega')$.

(ii) If $Wap(G, 1/\omega) = C_b(G, 1/\omega)$, then there exists a weight function $\omega_q$ on $G/N$ such that $Wap(G/N, 1/\omega_q) = C_b(G/N, 1/\omega_q)$.

(iii) If $Wap(G \times G', 1/\omega_p) = C_b(G \times G', 1/\omega_p)$, then there exist weight functions $\omega_0$ and $\omega'_0$ on $G$ and $G'$, respectively, such that $Wap(G, 1/\omega_0) = C_b(G, 1/\omega_0)$ and $Wap(G', 1/\omega'_0) = C_b(G', 1/\omega'_0)$.

Proof. Let $\psi : G \to G'$ be a group epimorphism and $Wap(G, 1/\omega) = C_b(G, 1/\omega)$. It follows from Propositions 4.3 that $G'$ is weight regular. By Proposition 3.5, $G'$ is weight regular. Again, by Proposition 4.3, we obtain $Wap(G', 1/\omega') = C_b(G', 1/\omega')$ for some a weight function $\omega'$ on $G'$. So (i) holds. The statements (ii) and (iii) follow from Proposition 4.3 together with Corollary 3.8 and Corollary 3.6, respectively.

Let $\mathfrak{A}$ be a Banach algebra. Then $f \in \mathfrak{A}^*$ is called weakly almost periodic (respectively, almost periodic) if the map $a \mapsto af$ from $\mathfrak{A}$ into $\mathfrak{A}^*$ is weakly compact respectively compact, where $\langle af, b \rangle = \langle f, ba \rangle$ for all $b \in \mathfrak{A}$. The spaces of all weakly almost periodic (respectively, almost periodic) functionals on $\mathfrak{A}$ are denote by $WAP(\mathfrak{A})$ and $AP(\mathfrak{A})$, respectively.
Theorem 4.5 Let \( \omega \) be a weight function on a locally compact infinite group \( G \). Then the following assertions are equivalent.

(a) \( \text{WAP}(M_\omega(G,\omega)^*) = (M_\omega(G,\omega))^\text{**} \).
(b) \( \text{Wap}(G,1/\omega) = C_b(G,1/\omega) \) and \( G \) is discrete.
(c) \( \Omega \) is zero cluster.
(d) \( M_\omega(G,\omega)^* \) is Arens regular.

Proof. The implications (a)\( \Leftrightarrow \) (d) and (c)\( \Rightarrow \) (d) follow from Theorem 2.14 in [5] and Theorem 2.2. Let (d) hold. Since \( G \) is infinite, by Theorem 2.2, \( \Omega \) is zero cluster. So (b) follows from Lemma 2.1 and Theorem 4.1(i). That is, (d)\( \Rightarrow \) (b). If (b) holds, then by Theorem 4.1(i), \( G \) is finite or \( \Omega \) is zero cluster. By assumption, \( \Omega \) is zero cluster. That is, (b)\( \Rightarrow \) (c). \( \square \)

Theorem 4.6 Let \( \omega \) be a weight function on a locally compact infinite group \( G \). Then the following assertions are equivalent.

(a) \( \text{AP}(M_\omega(G,\omega)^*) = (M_\omega(G,\omega))^\text{**} \).
(b) \( \text{Ap}(G,1/\omega) = C_b(G,1/\omega) \) and \( G \) is discrete.
(c) \( G \) is discrete and \( \Omega \in C_0(G \times G) \).

Proof. If \( \text{AP}(M_\omega(G,\omega)^*) = (M_\omega(G,\omega))^\text{**} \), then \( \text{WAP}(M_\omega(G,\omega)^*) = (M_\omega(G,\omega))^\text{**} \). By Theorem 4.5, \( G \) is discrete. Hence \( \text{AP}(\ell^1(G,\omega)) = \ell^\infty(G,1/\omega) \). Therefore, \( G \) is discrete and \( \Omega \in C_0(G \times G) \). That is, (a) implies (c). By Theorem 4.1, the statements (b) and (c) are equivalent. The implication (c)\( \Rightarrow \) (a) is clear. \( \square \)

Using a routine argument, the next result is established. So we omit it.

Proposition 4.7 Let \( \omega \) be a weight function on \( G \). Then the following statements hold.

(i) \( C_0(G,1/\omega) = C_b(G,1/\omega) \) if and only if \( G \) is compact.
(ii) \( L^\infty(G,1/\omega) = C_b(G,1/\omega) \) if and only if \( G \) is discrete.
(iii) \( L^\infty_0(G,\omega) = C_b(G,1/\omega) \) if and only if \( G \) is finite.

5 Amenability of \( M_\omega(G,\omega)^* \)

Let us recall that the Banach algebra \( M_\omega(G,\omega)^* \) is called amen able if every continuous derivation from \( M_\omega(G,\omega)^* \) into \( E^* \) is inner for all Banach \( M_\omega(G,\omega)^* \)–module \( E \).

Theorem 5.1 Let \( \omega \) be a weight function on locally compact group \( G \). Then the following assertions are equivalent.

(a) \( M_\omega(G,\omega)^* \) is amenable.
(b) \( M(G,\omega) \) is amenable.
(c) \( G \) is a discrete amenable group and \( \omega^* \) is bounded.
In this case, \( M_\omega(G,\omega)^* = L^1(G) \).
Proof. Since $C_0(G, 1/\omega)$ is a closed subspace of $M_*(G, \omega)$, we imply that
$$M_*(G, \omega)^* = M(G, \omega) \oplus M(G, \omega)_0,$$
where
$$M(G, \omega)_0 = \{ \Psi \in M_*(G, \omega)^*: \Psi|_{M(G, \omega)} = 0 \}.$$

Let $\Phi \in M_*(G, \omega)^*$. Then there exists a net $(\mu_\alpha)_\alpha$ in $M(G, \omega)$ such that $\mu_\alpha \to \Phi$ in the weak* topology of $M_*(G, \omega)$
$$\langle \Phi \circ \Psi, \nu \rangle = \lim_{\alpha} \langle \mu_\alpha \circ \Psi, \nu \rangle = \lim_{\alpha} \langle \Psi, \nu^* \mu_\alpha \rangle = 0.$$  

Consequently, $M(G, \omega)_0$ is a left ideal in $M_*(G, \omega)^*$. On the hand, $\Phi = \mu + \Phi_0$ for some $\mu \in M(G, \omega)$ and $\Phi_0 \in M(G, \omega)_0$. Then
$$\Psi \circ \Phi = \Psi \circ (\mu + \Phi_0) = \Psi \circ \mu + \Psi \circ \Phi_0.$$  

It is clear that $\Psi \circ \mu \in M(G, \omega)_0$. Since $M(G, \omega)_0$ is a left ideal in $M_*(G, \omega)^*$, we have $\Psi \circ \Phi_0 \in M(G, \omega)_0$. So $M(G, \omega)_0$ is a right ideal in $M_*(G, \omega)^*$. Therefore, $M(G, \omega)_0$ is an ideal in $M_*(G, \omega)^*$.

Now, if $M_*(G, \omega)^*$ is amenable, then
$$\frac{M_*(G, \omega)^*}{M(G, \omega)_0} \cong M(G, \omega)$$
is amenable; see for example [26]. So (a) implies (b).

The second author and Vishki [25] showed that $M(G, \omega)$ is amenable if and only if $G$ is a discrete amenable and $\omega^*$ is bounded. Hence (b) and (c) are equivalent.

Let us recall that Gronback [16] proved that $L^1(G, \omega)$ is amenable if and only if $G$ is amenable and $\omega^*$ is bounded. The first author and Moghimi [22] prove that $G$ is discrete if and only if $M_*(G, \omega)^* = L^1(G, \omega)$. These facts show that (c) implies (a).

It is well-known from [30] that the mapping $\phi \mapsto \phi \omega$ from $L^1(G, \omega)$ onto $L^1(G)$ is an isometric isomorphism of Banach spaces. Since $G$ is discrete, we have
$$M_*(G, \omega) = L^1(G, \omega) = L^1(G).$$

So the assertions (a), (c) and (d) are equivalent. \qed

Theorem 5.2 Let $\omega$ be a weight function on locally compact group $G$. Then the following assertions are equivalent.

(a) $M_*(G, \omega)^*$ is amenable and Arens regular
(b) $M(G, \omega)^{***}$ is amenable.
(c) $M(G, \omega)^{**}$ is amenable.
(d) $L^1(G, \omega)^{**}$ is amenable.
(e) $G$ is finite.
Proof. From Theorems 2.2 and 5.1 we infer that (a) and (e) are equivalent. Since
$L^1(G,\omega)$ is an ideal in $M_*(G,\omega)^*$ and $M(G,\omega)$, respectively, it follows that
$L^1(G,\omega)^{**}$ is an ideal in $M_*(G,\omega)^{**}$ and $M(G,\omega)^{**}$, respectively. Hence (b) and (c) imply (d). It
follows from Theorem 4 in [25] that (d) implies (e). Trivially, (e) imply (b) and (c). □

References

[1] J. W. Baker and A. Rejali, On the Arens regularity of weighted convolution algebras, J. London
Math. Soc., (2) 40 (1989) 535–546.

[2] R. B. Burkel, Weakly Almost Periodic Functions on Semigroups, Gordon and Breach, New York,
1970.

[3] I. G. Craw and N. J. Young, Regularity of multiplication in weighted group and semigroup
algebras, Quart. J. Math. Oxford, 25 (1974) 351–358.

[4] H. G. Dales, F. Ghahramani and A. Y. A. Helemskii, The amenability of measure algebras, J.
London Math. Soc., (2) 66 (2002) 213–226.

[5] H. G. Dales and A. T. Lau, The second duals of Beurling algebras, Mem. Amer. Math. Soc., 177
(836) (2005).

[6] M. Daws, Arens regularity of the algebra of operators on a Banach space, Bull. London Math.
Soc., 36 (2004) 493–503.

[7] S. Degenfeld-Schonburg and R. Lasser, Multipliers on $L^p$-spaces for hypergroups, Rocky Moun-
tain J. Math., 43 (4) (2013) 1115-1139.

[8] J. Duncan and S. A. R. Hosseiniun, The second dual of a Banach algebra, Proc. Roy. Soc.
Edinburgh, A 84 (1979) 309-325.

[9] H. A. M. Dzinotyiweyi, The analogue of the group algebra for topological semigroups, Research
Notes in Mathematics, 98. Pitman (Advanced Publishing Program), Boston, MA, 1984.

[10] H. R. Ebrahimi Vishki, B. Khodsiani and A. Rejali, Arens Regularity of certain weighted semi-
group algebras and countability, Semigroup Froum, 92 (2016) 304–310.

[11] B. Forrest, Arens regularity and discrete groups, Pacific J. Math., 151 (2) (1991) 217–227.

[12] B. Forrest, Arens regularity and the $A_p(G)$ algebras, Proc. Amer. Math. Soc., 119 (2) (1993)
595–598.

[13] C. C. Graham, Arens regularity and weak sequential completeness for quotients of the Fourier
algebra, Illinois J. Math., 44 (4) (2000) 712–740.

[14] C. C. Graham, Arens regularity and the second dual of certain quotients of the Fourier algebra,
Q. J. Math., 52 (1) (2001) 13–24.

[15] C. C. Graham, Arens regularity for quotients $A_p(E)$ of the Herz algebra, Bull. London Math.
Soc., 34 (4) (2002) 457–468.

[16] N. Gronback, Amenability of weighted convolution algebras on locally compact groups, Trans.
Amer. Math. Soc., 319 (1990) 765–775.

[17] A. T. Lau and J. Pym, Concerning the second dual of the group algebra of a locally compact
group, J. London Math. Soc., 41 (1990) 445–460.
[18] A. T. Lau and A. Ulger, Some geometric properties on the Fourier and Fourier-Stieltjes algebras of locally compact groups, Arens regularity and related problems, Trans. Amer. Math. Soc., 337 (1) (1993) 321–359.

[19] S. Maghsoudi, M. J. Mehdipour and R. Nasr-Isfahani, Compact right multipliers on a Banach algebra related to locally compact semigroups, Semigroup Forum, 83 (2) (2011) 205–213.

[20] S. Maghsoudi, R. Nasr-Isfahani and A. Rejali, Strong Arens irregularity of Beurling algebras with a locally convex topology, Arch. Math., 86 (5) (2006) 437–448.

[21] D. Malekzadeh Varnosfaderani, Derivations, Multipliers and Topological Centers of Certain Banach Algebras Related to Locally Compact Groups, Ph.D. thesis, University of Manitoba, 2017.

[22] M. J. Mehdipour and GH. R. Moghimi, The existence of non-zero compact right multipliers and Arens regularity of weighted Banach algebras, preprint.

[23] H. Reiter and J. D. Stegeman, Classical Harmonic Analysis and Locally Compact Groups, London Math. Society Monographs, 22, Clarendon Press, Oxford, 2000.

[24] A. Rejali, Weighted function spaces on topological groups, Bull. Iranian Math. Soc., 22 (2) (1996) 43–63.

[25] A. Rejali and H. R. Vishki, Regularity and amenability of the second dual of weighted group algebras, Proyecciones, 26 (2007) 259–267.

[26] V. Runde, Lectures on amenability, Lecture Notes in Mathematics 1774, Springer Verlag, Berlin, 2002.

[27] A. Ulger, Arens regularity of the algebra $C(K, A)$, J. London Math. Soc., (2) 42 (1990) 354–364.

[28] A. Ulger, Some stability properties of Arens regular bilinear operators, Proc. Edinburgh Math. Soc., (2) 34 (1991) 443–454.

[29] A. Ulger, Arens regularity of weakly sequentially complete Banach algebras, Proc. Amer. Math. Soc., 127 (1999) 3221–3227.

[30] M. C. White, Characters on weighted amenable groups, Bull. London Math. Soc., 23 (1991) 375–380.

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