Phase Structure of Non-Compact QED3 and the Abelian Higgs Model *

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Abstract

We review the phase structure of a three-dimensional, non-compact Abelian gauge theory (QED3) as a function of the number $N$ of 4-component massless fermions. There is a critical $N_c$ up to which there is dynamical fermion mass generation and an associated global symmetry breaking. We discuss various approaches to the determination of $N_c$, which lead to estimates ranging from $N_c = 1$ to $N_c = 4$. This theory with $N = 2$ has been employed as an effective continuum theory for the 2D quantum antiferromagnet where the observed Neel ordering corresponds to dynamical fermion mass generation. Thus the value of $N_c$ is of some physical interest. We also consider the phase structure of the model with a finite gauge boson mass (the Abelian Higgs model).

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1 Introduction

We review the dynamical generation of fermion mass in a three-dimensional, non-compact Abelian gauge theory (QED3) and its generalization to the case of a finite gauge boson mass. Attention was first drawn to this problem by Pisarski [1]. Some years ago, a study concluded that with the gauge symmetry unbroken, mass generation will occur, corresponding to the breaking of a certain global ”chiral” symmetry, if and only if the number $N$ of four-component fermions is no larger than a critical value $N_c$, estimated to be 4 [2, 3]. Since then, most studies have agreed that there is indeed such an $N_c$, but there has been little agreement as to its magnitude. Estimates have ranged from that of Refs. 2 and 3 to as low as $N_c = 1$ [4]. This uncertainty is typical of our rudimentary understanding of most strongly coupled quantum field theories.

The value of $N_c$ is a potentially important real-world question. It has been suggested that QED3 with $N = 2$ massless fermions can provide a continuum description of a 2D quantum antiferromagnet [5, 6]. Marston [7], Laughlin and Zou [8], and others have noted that the observed Neel ordering corresponds to dynamical fermion mass generation and its associated chiral symmetry breaking in QED3 with $N = 2$. More recent discussions of these ideas are provided by Kim and Lee [9], and Kleinert, Nogueira, and Sudbo [10]. For another application of non-compact QED3 to condensed matter systems see Ref. [11]. The suggestion that (non-compact) QED3 provides a continuum description of the 2D quantum antiferromagnet has, however, been challenged in a recent series of papers by Herbut et.al [12].

Several approaches have been brought to bear on the determination of $N_c$ in QED3. The original study [2, 3] was based on a questionable use of the $1/N$ expansion along with a continuum gap equation. Continuing to non-integral values of $N$, it led to the estimate $N_c \approx 128/3\pi^2 \approx 4.3$. This approach also led to some speculation on the nature of the phase transition as a function of $N$ [13], namely that because of the long-range force it is of infinite order. A lower value of $N_c$ is suggested by a conjectured inequality based on the counting of thermodynamic degrees of freedom [4]. It indicates that $N_c$ can be no greater than $3/2$. It is likely that the determination of $N_c$ will be settled only by numerical lattice studies. These studies have so far suggested that $N_c$ could be as low as 2 [14, 15], but a conclusive determination will need more refined simulations.

We first review the features of QED3 and discuss its symmetry properties and behavior for large $N$ where the $1/N$ expansion may be reliably employed. We discuss fermion mass generation, which takes place at low values of $N$. The conjectured constraint of Ref. 4 is then described and applied to QED3. We review recent lattice computations, discuss their sensitivity to finite size effects, and to provide some perspective on these simulations include a rough numerical estimate using the continuum gap equation. We also discuss dynamical fermion mass generation in a
related theory, the Abelian Higgs model in three-dimensions with the same fermion content as QED3, but where the gauge field is massive. We then summarize, revisit the relevance of these models to condensed matter systems, and describe some open questions.

2 QED3 and its Symmetry Breaking

The Lagrangian of the model is

$$L = \sum_{j=1}^{N} \bar{\psi}_j(i D)\psi_j - \frac{1}{4} F_{\mu\nu}F^{\mu\nu},$$

(1)

where $D_\mu = i\partial_\mu + eA_\mu$ is the covariant derivative, $e$ is the gauge coupling, and $\psi_j$ is a set of $N$ 4-component fermion fields. To explore the phase structure as a function of $N$, it is convenient to keep fixed the dimensionful quantity $\alpha \equiv e^2 N/8$. Being super-renormalizable, this theory is UV-complete, rapidly damped at momentum scales beyond $\alpha$. As an effective theory at lower momenta, one could nevertheless take the view that higher momentum physics is unknown, being integrated out into a tower of higher-dimension operators associated with the scale $\alpha$. We do not include such terms, which could modify the critical value of $N$ for chiral symmetry breaking. We return to this question in the summary when we discuss possible condensed matter applications of QED3.

For large $N$, the theory remains weakly coupled at all momentum scales. This can be seen by computing the gauge boson propagator in the large-$N$ limit and extracting from it an effective, dimensionless running coupling $\bar{\alpha}(k)$. The large-$N$ form of the Euclidean propagator is $(k^2 + \alpha k)^{-1}$, where $k$ is the magnitude of the Euclidean three-momentum and the second term arises from the $N$ fermion loops. Multiplying the propagator by the coupling $e^2$ and by one power of $k$ to make a dimensionless quantity, we have

$$\bar{\alpha}(k) \equiv \frac{e^2 k}{k^2 + (e^2 N/8)k} = \frac{8}{N} \frac{\alpha}{\alpha + k}.$$  

(2)

This expression exhibits asymptotic freedom at large momentum, and conformal symmetry with an IR fixed point of strength $8/N$ as $k \to 0$. For large $N$, the coupling is always weak, and no dynamical fermion mass generation is expected.

For finite $N$, on the other hand, the infrared coupling becomes strong, governed by a strong IR fixed point, and fermion mass generation becomes possible. In the absence of fermion mass, the global symmetry associated with the $N$ four-component fermions is $U(2N)$. The dynamical generation of an equal mass for all the fermion flavors would break the global symmetry to $U(N) \otimes U(N)$ giving rise
to $2N^2$ Goldstone bosons [1]. More generally, mass terms for the Dirac spinors can violate parity (P) and time reversal symmetry (T), and the gauge field admits a corresponding P- and T-violating Chern-Simons mass term. These are not generated spontaneously [16]. We concentrate here on the parity conserving case.

Several studies over the years have led to the conclusion that a parity conserving mass is indeed generated dynamically for this theory providing that $N$ is no larger than a critical value $N_c$ [2]. An upper limit is expected, since, as we have noted, the theory is weakly coupled at all momentum scales in the large-$N$ limit. Since the Lagrangian of Eq. 1 is UV-complete, damped rapidly at momentum scales above $\alpha$, the value of $N_c$ is completely determined by the conformal, IR behavior.

### 3 A Conjectured Constraint

A conjectured constraint on the infrared structure of asymptotically free gauge theories [4] can be utilized to analyze the phase structure of QED3. This constraint, which takes the form of an inequality, states that for a wide class of such theories, the number of IR "degrees of freedom", defined using the thermal free energy per unit volume $F(T)$ of the theory, is less than or equal to the number of corresponding UV degrees of freedom.

These degree-of-freedom counts are defined by the quantities $f_{IR}$ and $f_{UV}$, given in terms of $F(T)$ by taking the the zero- and infinite-T limits respectively of the quantity $\frac{F(T)}{T^d} f(d)$. Here $f(d)$ is a function of the number of space time dimensions $d$, defined such that the contribution from a free bosonic field is 1. The inequality is then

$$f_{IR} \leq f_{UV}. \tag{3}$$

In general, $f_{IR}$ and $f_{UV}$ will exist providing that the theory is governed by IR and UV fixed points. The conjectured inequality was restricted, however, to asymptotically free theories, in which $f_{UV}$ can be computed from free field theory. It has been applied to a variety of four-dimensional theories, and found to be satisfied whenever $f_{IR}$ can be reliably computed [4]. This includes both IR-free theories and theories with weak IR fixed points.

For QED3, the asymptotic freedom leads to $f_{UV} = \frac{3}{4}(4N)+1$ [4]. The second term counts the single bosonic degree of freedom associated with a massless gauge boson in three space-time dimensions. In the first term, the $4N$ counts the number of fermionic degrees of freedom associated with the $N$ four-component fermions, and the $3/4$ represents the Boltzman weighting of fermions in three space-time dimensions.

The value of $f_{IR}$ depends on whether the global symmetry is spontaneously broken. For large $N$, breaking is not expected, and only the massless fermions along
with the one gauge degree of freedom remain in the infrared spectrum. From Eq. 2, one sees that the infrared theory is described by a weak \((O(1/N))\) fixed point. The free-field result, \(f_{IR} = \frac{3}{4}(4N) + 1 (= f_{UV})\), can then be corrected perturbatively in \(1/N\), the next-to-leading term being \(O(1)\). The computation, similar in some ways to the corresponding perturbative computation in a 4D gauge theory, has not yet been done. The 4D result is negative, and if the same is true of the 3D \(1/N\) computation, the inequality \(f_{IR} \leq f_{UV}\) will be satisfied.

Now consider the more interesting possibility that for some finite \(N\) the global symmetry is broken. The fermions become massive and \(2N^2\) Goldstone bosons are formed. The infrared (massless) theory consists of only the Goldstone bosons and the gauge boson. Thus \(f_{IR} = 2N^2 + 1\) and the inequality will be satisfied only if \(N_c \leq 3/2\). If this upper limit on \(N_c\) is correct, then QED3 cannot be in the broken phase if \(N\) is larger then \(\frac{3}{2}\), meaning that the continuum gap equation [2] overestimates \(N_c\).

A natural question about this application of the inequality \(f_{IR} \leq f_{UV}\) has to do with the Mermin-Wagner-Coleman theorem [17]. We have determined \(f_{IR}\) in the broken phase by computing the free energy at a temperature \(0 < T << \alpha\), counting the Nambu-Goldstone bosons plus the gauge field as the relevant, non-interacting degrees of freedom, multiplying by \(1/T^3\) and an appropriate constant, and taking the limit \(T \to 0\). Now at finite \(T\), the IR behavior of this theory is that of the corresponding 2D theory. But the Mermin-Wagner-Coleman theorem states that there can be no spontaneous symmetry breaking, with its Nambu-Goldstone bosons, in 2D. This question was addressed by Rosenstein, Warr, and Park [18], who concluded that the symmetry is indeed unbroken at any non-zero \(T\), and that for small \(T\) the zero-temperature Nambu-Goldstone bosons develop small masses. In our notation, one finds \(m_{NG}^2 \sim T^2 \exp(-\alpha/T)\), arising from the derivative interactions among the (pseudo) Nambu-Goldstone bosons. Since this mass vanishes more rapidly than \(T\) as \(T \to 0\), it has no effect on the computation of \(f_{IR}\).

## 4 Lattice Studies

Lattice studies may be the most reliable method to analyze dynamical mass generation and directly determine \(N_c\) in QED3. Numerical simulations began over a decade ago, with a study in the quenched approximation giving preliminary evidence that the global chiral symmetry is spontaneously broken [19]. Recent advances in computing power have led to improved studies of mass generation in QED3. Simulations of 2-flavor QED3 by Hands, Kogut and Strouthos [14] on lattices of up to \(50^3\) sites report that the condensate is two orders of magnitude smaller than the quenched condensate value. They find that the value of the dimensionless condensate, defined
where $e$ is the dimensionful gauge coupling constant, is bounded above by $5 \times 10^{-5}$. They also analyze finite-size effects and find that this bound is stable for lattice sizes ranging from $10^3$ to $50^3$. They conclude that there is no decisive signal for chiral symmetry breaking for $N \geq 2$.

This conclusion is consistent with the conjectured inequality constraint described in the previous section, which led to $N_c \leq 3/2$ for QED3. On the other hand, this interpretation of the lattice results is likely premature. It has been noted by Gusynin and Reenders [20] that even a lattice of size $50^3$ provides a sufficient IR cutoff to affect substantially the value of $N_c$. They use the continuum gap equation first employed to note the existence of an $N_c$, modeling the effect of a finite lattice size by imposing an IR cutoff $\mu$ on the integral in the gap equation. Because of the scale-invariant form of the gap equation following from the dominance by the IR fixed point, the equation is logarithmic in character. A measure of the importance of the cutoff $\mu$ relative to the effective UV cutoff $\alpha$ is therefore $\ln \alpha/\mu$. As this quantity decreases from infinity, a growing portion of momentum space is truncated, and therefore a stronger gauge coupling is required to break the symmetry. Thus, $N_c$ should drop. Gusynin and Reenders find, for example, that with $\ln \alpha/\mu$ still as large as 6 ($\alpha/\mu \approx 400$), $N_c$ drops by more than 30%. A lattice of size $50^3$ corresponds to a much smaller $\alpha/\mu$, indicating that chiral symmetry breaking should not take place for $N = 2$.

If this analysis is qualitatively correct, that is, if the continuum IR cutoff correctly models the finite size of a lattice, then only a larger lattice would be able to determine accurately the value of $N_c$ for a theory such as QED3, governed by a (conformal) IR fixed point. This same remark would apply to studies of the conformal phase transition as a function of the number of fermion species in four-dimensional gauge theories.

## 5 Gap Equation Estimates

Suppose that it becomes possible to carry out simulations with such large lattices that finite size effects are no longer important, and suppose that the results of the simulations for $\sigma$ (Eq. 4) with $N = 2$ continue to be bounded as in Ref. [14]. What would one conclude? Here we present a rough argument using the continuum gap equation indicating that even then it could be unclear whether $N = 2$ is in the symmetric or broken phase.

The spirit of the gap equation approach is to use the large-N form of the kernel of this equation (the large-N gauge-boson propagator), and then to continue
to finite $N$. The reliability of this approach, which first suggested the existence of a finite $N_c (\approx 128/3\pi^2)$, is not clear because the higher terms in the kernel are not parametrically small. There is, however, some evidence that the corrections are small numerically [3]. We use this approach to estimate the condensate for $N = 2$, expecting only that the estimate is order-of-magnitude.

The theory is UV-complete, rapidly damped at momentum scales beyond $\alpha$. Thus, to a good approximation, the gap equation can be written down employing the $\alpha \to \infty$ form of the large-$N$ kernel, with $\alpha$ then used as a UV cutoff. The resulting equation is

$$
\Sigma(p) = \frac{16}{3N\pi^2p} \int^\alpha \frac{kdk\Sigma(k)}{k^2 + \Sigma^2(k)} [p + k - |p - k|],
$$

(5)

where we make use of the non-local Kondo-Nakatani gauge for which the leading large-$N$ form of the wave function renormalization is unity. The critical $N$ determined from this equation is $N_c = 128/3\pi^2$.

Numerical and analytical solutions of this equation [21] show that for a range of $N$ such that $N_c/N > 1$, the overall scale of the solution $\Sigma(p)$ of Eq.(5), set by $\Sigma(0)$, is much less than $\alpha$, and for $N = 2$,

$$
\frac{\Sigma(0)}{\alpha} \approx 1.7 \times 10^{-2}.
$$

(6)

In the range of momentum $\Sigma(p) < p < \alpha$ the solution has the following approximate form

$$
\Sigma(p) = \frac{\Sigma(0)^{\frac{3}{2}}}{p^{\frac{1}{2}}} \sin\left[\frac{1}{2} \sqrt{\left(N_c/N - 1\right)}(\ln\left(\frac{p}{\Sigma(0)}\right) + \delta)\right],
$$

(7)

where $\Sigma(0)$ is used to scale the log and $\delta$ is a phase.

We use these formulae to estimate the dimensionless fermion condensate of Eq. 4. The fermion condensate $< \bar{\psi}\psi >$ is given by the integral

$$
< \bar{\psi}\psi > = \int^\alpha \frac{d^3k}{(2\pi)^3} \frac{4\Sigma(k)}{k^2} = \frac{2}{\pi^2} \int^\alpha dk\Sigma(k).
$$

(8)

Noting that the sine function is bounded above by 1 in the region of integration up to $\alpha$, we see that the dimensionless condensate $\sigma$ of Eq. 4 ($= <N^2\bar{\psi}\psi>/64\alpha^2$) is bounded above by

$$
\frac{N^2}{16\pi^2} \left(\frac{\Sigma(0)}{\alpha}\right)^{\frac{3}{2}}.
$$

With $N = 2$, $N_c = 128/3\pi^2$ and $\frac{\Sigma(0)}{\alpha} \approx 1.7 \times 10^{-2}$, this expression becomes $6 \times 10^{-5}$ which is slightly above the upper bound of $5 \times 10^{-5}$ found in lattice simulations [14]. We have numerically estimated the actual value of the condensate to be $4.4 \times 10^{-5}$ which falls below the lattice bound [22].
This suggests that a dimensionless condensate (4) of order $10^{-5}$ may naturally arise in QED3 for $N$ of order 2 even when $N_c$ is as large as $128/3\pi^2 \approx 4$. It indicates that even if the numerical simulations are performed on a very large lattice, so that finite size effects are unimportant, a precise computation will be required to decide whether $N = 2$ is in the broken or symmetric phase.

6 Abelian Higgs Model

We conclude by discussing fermion mass generation when the Abelian gauge field has a non-zero mass. This theory has been used as a continuum description of the competition between long range antiferromagnetic order and superconducting order in planar cuprate systems [23], [24]. It can also be used to interpolate between QED3 and the 2+1 dimensional Thirring model, the latter having been studied in recent lattice simulations [14]. Related studies, using continuum gap equation methods and invoking the idea of hidden local gauge symmetry may be found in references [25] [26]. Finally, it can be used to study the effect of an IR cutoff on QED3 [24], similarly to the use of a simple IR cutoff as described in Section 3 [20].

The Lagrangian of the model is

$$\mathcal{L} = \mathcal{L}_{QED3} + \frac{1}{2} D_\mu \Phi^* D^\mu \Phi - \lambda (\Phi^* \Phi - Nv)^2$$

where $\mathcal{L}_{QED3}$ is given in equation (1), and $\Phi$ is a complex scalar field with vacuum expectation value $v$. The addition of the scalar field does not change the global symmetry structure of the theory. We remove the Higgs boson from the spectrum by taking the limit $\lambda \to \infty$ with $v$ fixed. This leaves the theory UV-complete. The gauge boson mass is $M = e\sqrt{N}v$. To explore the phase structure as a function of $N$, we keep fixed the quantities $\alpha \equiv e^2 N/8$ and $v^2$. The gauge boson mass $M$ is then also fixed.

As in QED3, we define a dimensionless running gauge coupling $\tilde{\alpha}(k)$. The coefficient of $g_{\mu\nu}$ in the Euclidean gauge-boson propagator takes the form $(k^2 + M^2 + \alpha k)^{-1}$ to leading order in $1/N$, where $k$ is the magnitude of the Euclidean three-momentum and the last term arises from the $N$ fermion loops. The $k_\mu k_\nu$ term of the propagator is dropped since it couples to a conserved current. Multiplying the propagator by the coupling $e^2$ and by $k$ to make it dimensionless, we have

$$\tilde{\alpha}(k) \equiv \frac{e^2 k}{k^2 + M^2 + (e^2 N/8)k} = \frac{8}{N} \frac{k}{(k^2/\alpha) + G^{-1} + k},$$

where $G \equiv \alpha/M^2$. This effective coupling vanishes in both the UV and IR limits. It reaches a maximum no larger than $8/N$, attainable only when $M << \alpha$. As $M/\alpha$ increases, the maximum value decreases monotonically. Thus, for any $M/\alpha$, the model is weakly coupled in the large-$N$ limit at all momentum scales.
It is reasonable to expect that, as in QED3, a parity conserving mass is generated for $N$ below some critical value $N_c$. Assuming an $N_c$ to exist, what can it depend on? Since the forces are strongly damped at scales beyond $\alpha$, no additional cutoff is necessary to define the theory. Thus $N_c$ can be a function of only the dimensionless ratio $M/\alpha$. This dependence describes the boundary of a phase diagram for the theory.

Consider first the limit $M/\alpha \to 0$. On the one hand, this may be thought of as taking $M \to 0$ with $\alpha$ fixed, giving QED3. As discussed earlier, estimates of $N_c$ in this limit have ranged from $3/2$ to approximately $4$. The limit $M/\alpha \to 0$ may also be taken by letting $\alpha \to \infty$ with $G \equiv \alpha/M^2$ fixed. This leads to the 2+1 dimensional Thirring model, meaning that $N_c$ for this model is the same as that for QED3. This assumes that the Thirring model is treated with any UV cutoff taken very large relative to $G^{-1}$. For finite $N$, this can be implemented using lattice techniques. (For large $N$, these two ways of taking the limit $M/\alpha \to 0$ may be considered by examining $\hat{\alpha}(k)$ (Eq. 11). In the case $M \to 0$ with $\alpha$ fixed (QED3), there appears an IR ($k << \alpha$) fixed point of strength $8/N$. In the case $\alpha \to \infty$ with $G^{-1}$ fixed, there is a UV ($k >> G^{-1}$) fixed point of the same strength).

As $M/\alpha$ is increased from zero, low momentum components are damped out. If the force driving the symmetry breaking is attractive at all scales (as in the large $N$ approximation), a stronger infrared coupling is required to trigger dynamical symmetry breaking, and therefore $N_c$ will decrease. At some value of $M/\alpha$, $N_c$ drops below unity meaning that symmetry breaking will not take place at all [23]. This critical curve should be similar to that determined by Gusynin and Reenders [20], with the gauge boson mass replacing the explicit IR cutoff. The character of the phase transition along this critical curve is also of interest. For $M/\alpha = 0$, as discussed earlier, because the force is of infinite range, the continuum gap equation suggests that the transition is of infinite order [13]. With $M > 0$, the force becomes of finite range, and the transition is of second order.

Does the conjectured inequality $f_{IR} \leq f_{UV}$ provide any information about $N_c$ as a function of $M/\alpha$? With $\alpha = \frac{e^2N}{8}$ finite, the theory is asymptotically free. A free-field computation then gives $f_{UV} = \frac{3}{4}(4N) + 2 = 3N + 2$ [4]. The second term counts the 2 bosonic degrees of freedom associated with a massive gauge boson in three space-time dimensions. As in QED3, the value of $f_{IR}$ depends on whether the global symmetry is spontaneously broken. For large $N$, breaking is not expected, and, if $M$ is nonzero, the infrared theory consists of only the (non-interacting) massless fermions. Then $f_{IR} = \frac{3}{4}(4N)$, and the inequality is satisfied. Now suppose that for some finite $N$ the global symmetry is broken. The fermions become massive, and for finite gauge boson mass $M$ the IR theory consists of only the $2N^2$ non-interacting Goldstone degrees of freedom. Thus $f_{IR} = 2N^2$, and the inequality demands that $N_c$ be no greater than 2. But the inequality already demands that $N_c$ be no greater than $3/2$ when $M = 0$, and we expect that $N_c$ will decrease as $M/\alpha$ increases. So
it would seem that the conjectured inequality has nothing useful to say about the shape of this critical curve.

Lattice simulations of the Abelian Higgs model for an appropriate range of values of $M/\alpha$ could determine the shape of this critical curve. Simulations have been done only for QED3 and the Thirring model, which correspond to the limit $M/\alpha \rightarrow 0$. We discussed in Section 3 the sensitivity of the QED3 simulation to finite size effects. A numerical study of the Thirring model was reported by Hands and Lucini [27]. We noted above that providing this model is treated with a UV cutoff large compared to the inverse four-fermion coupling $G^{-1}$, the value of the critical coupling $N_c$ should approach that of QED3. The lattice simulations of Ref. [27] do take the ultraviolet cutoff larger than $G^{-1}$, and find chiral symmetry breaking for $N \approx 3$. Their results remain sensitive to the UV cutoff, but we anticipate that $N_c$ will only increase as the cutoff is made larger (as the continuum limit is approached). This suggests that $N_c > 3$ for the continuum Thirring model, and therefore also for QED3.

We end this section by digressing to use the large-N Thirring model to explain why the inequality [4] was restricted to asymptotically free theories. The model becomes non-interacting in the infrared ($k << G^{-1}$) so $f_{IR} = 3N$. Having taken $\alpha \rightarrow \infty$, the resultant weak UV fixed point means that $f_{UV}$ exists and may be computed in the $1/N$ expansion. To leading order, $f_{UV} = 3N$. The computation of the next-to-leading corrections should be the same as that of $f_{IR}$ in QED3, since the latter is governed by an IR fixed point identical in strength to the UV fixed point of the Thirring model. If the result is negative as anticipated, then $f_{IR} > f_{UV}$ for this model. Another 2+1 dimensional model with a UV fixed point, leading to $f_{IR} > f_{UV}$, was studied by Sachdev [28]. In light of these examples, the conjectured inequality was restricted to asymptotically free theories.

7 Summary and Discussion

A key question for a non-compact Abelian gauge theory in three space-time dimensions is the critical number $N_c$ of four-component fermions below which there is fermion mass generation with an associated chiral symmetry breaking, and above which the fermions remain massless. Over the years, estimates of $N_c$ with the gauge symmetry unbroken (QED3) have ranged from infinity to one. Recent studies agree that it is finite and about 4, but some estimates continue to be as low as 1. Since this theory is UV-complete, damping rapidly at momentum scales above $\alpha \equiv e^2N/8$, the value of $N_c$ depends only of the IR behavior of the theory, where conformal symmetry sets in.

This question has a potential real-world interest, because of the possibility that QED3 with $N = 2$ can describe the physics of the planar anti-ferromagnet [5].
The observed antiferromagnetic (Neel) ordering corresponds to chiral symmetry breaking in QED3 \[7, 8\]. Numerical simulations \[29\] of the planar antiferromagnet also indicate Neel ordering. Herbut and collaborators \[12\] have challenged the idea that non-compact QED3 is relevant to the planar antiferromagnet, noting that the underlying theory is compact QED3 with its topological (instanton) structure. Some argue, however, that for large enough N, instantons and anti-instantons are bound at distance scales of order \(1/\alpha\) by a logarithmic term in the action \[10\]. Then for \(N = 2\) the non-compact theory could be employed at larger distances. But Herbut et al \[12\] claim that the finite density of instantons and anti-instantons screen the logarithmic term in the action and that instantons are therefore unsuppressed at all N. If this is the case, the non-compact theory cannot be used for this condensed matter system \[30\].

In general, the question of whether the value of \(N_c\) as determined by (UV-complete) QED3 is relevant to condensed matter systems depends on the role of new, short-distance physics. With the physical lattice spacing of order \(1/\alpha = N/8e^2\), physics at this scale can be represented in the effective theory at momentum scales below \(\alpha\) by the addition of a set of higher-dimension operators. The importance of these terms depends on their strength. We have analyzed this question in the framework of the continuum gap equation in 4-dimensions \[31\], which has much the same structure as QED3 treated in the \(1/N\) expansion. With a four-fermion additive term, there is a critical four-fermion coupling (attractive) of order unity above which this term by itself can trigger spontaneous chiral symmetry breaking. As long as the magnitude of this coupling, either attractive or repulsive, is less than one quarter of its critical value \[32, 33\], the additive term is unimportant, not affecting the value of \(N_c\). This limit may not be quantitatively accurate, but it seems reasonable that a limit of this order exists. Whether this limit is satisfied for condensed matter systems such as the planar antiferromagnet is not clear. Thus even if the non-compact theory does provide a continuum description of the antiferromagnet, the value of \(N_c\) for this system may or may not be determined accurately by QED3 itself.

Returning to our summary of QED3, we have briefly reviewed the various approaches to the determination of \(N_c\). A recently conjectured inequality constraining asymptotically free quantum field theories \[4\], when applied to QED3, suggests that \(N_c \leq 3/2\). This bound disagrees with the estimate \(N_c \approx 4\) emerging from the continuum gap equation approach which originally suggested the existence of a finite \(N_c\).

Recent lattice simulations of \(N = 2\) QED3 \[14\] indicate that the fermion condensate is very small, possibly signaling that QED3 is in the symmetric phase, and therefore that \(N_c < 2\). This conclusion depends, however, on the importance of finite-size effects. A recent analysis of Gusynin and Reenders \[20\] using the continuum gap equation indicates that a lattice size of \(50^3\), used in the simulations,
provides a strong enough suppression of IR physics to reduce \( N_c \) below 2, thus accounting for the observation of a small upper bound for the condensate. Using the solution of the gap equation with no IR cutoff, we have made a rough estimate of the fermion condensate, assuming that \( N_c \) is larger – approximately 4 as suggested by the continuum gap equation studies \([3]\). Interestingly, even though \( N = 2 \) is then well into the broken phase, the condensate is estimated to be very small, below the upper bound of Ref. \([14]\). This indicates that even if simulations are conducted on a large enough lattice to make finite size effects unimportant, a precise determination of the condensate is needed to decide whether QED3 with \( N = 2 \) is in the symmetric or broken phase.

The value of \( N_c \) in the Abelian Higgs model is also of interest. This model leads to either QED3 or the 2+1-dimensional Thirring model in the limit \( M/\alpha \to 0 \) depending on how the limit is taken. But since \( N_c \) depends on only the ratio \( M/\alpha \) the value of \( N_c \) must be the same in both models. Lattice simulations of the Thirring model on a \( 12^3 \) lattice lead to \( N_c \approx 3 \) \([27]\). This value remains sensitive to the UV cutoff (the lattice spacing), and should increase toward the \( N_c \) of QED3 as the continuum limit is approached. On the other hand the inequality (3) applied to the Abelian Higgs model yields \( N_c \leq 2 \).

To summarize, after more than fifteen years there is still no definitive answer to the question of the critical number of four-component fermions in QED3 (and its generalization to the Abelian Higgs model) marking the boundary between broken and unbroken chiral symmetry. This question is of interest because it could be relevant to the behavior of certain condensed matter systems and because the answer requires an understanding of a strongly coupled, UV-complete quantum field theory.

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