BUNDLES OF WEYL STRUCTURES AND INVARIANT
CALCULUS FOR PARABOLIC GEOMETRIES

ANDREAS ČAP AND JAN SLOVÁK

Abstract. For more than hundred years, various concepts were developed to understand the fields of geometric objects and invariant differential operators between them for conformal Riemannian and projective geometries. More recently, several general tools were presented for the entire class of parabolic geometries, i.e., the Cartan geometries modelled on homogeneous spaces $G/P$ with $P$ a parabolic subgroup in a semi-simple Lie group $G$. Similarly to conformal Riemannian and projective structures, all these geometries determine a class of distinguished affine connections, which carry an affine structure modelled on differential 1-forms $\Upsilon$. They correspond to reductions of $P$ to its reductive Levi factor, and they are called the Weyl structures similarly to the conformal case. The standard definition of differential invariants in this setting is as affine invariants of these connections, which do not depend on the choice within the class. In this article, we describe a universal calculus which provides an important first step to determine such invariants. We present a natural procedure how to construct all affine invariants of Weyl connections, which depend only tensorially on the deformations $\Upsilon$.

Differential invariants of various geometric structures are the core ingredients for numerous applications both in geometric analysis and mathematical physics. In particular, the invariants of conformal Riemannian manifolds attracted a lot of attention in the course of the last 100 years.

For smooth manifolds with an affine connection, the so called ‘first invariant theorem’ says that all the invariants are expressions built of the covariant derivatives of sections of natural bundles, the curvature and the torsion of the connection by means of algebraic tensorial invariants, cf. [20] for a modern treatment. Let us call them the affine differential invariants on smooth manifolds. The analogous ‘first invariant theorem’ for Riemannian geometries says that all differential invariants are built from affine invariants of the canonical Levi-Civita connection via (algebraic) invariants of the orthogonal group.

A conformal Riemannian geometry is defined as a class of conformally equivalent Riemannian metrics and so the above Riemannian first invariant theorem can be used to define conformal invariants. Thus, a conformal invariant is usually understood as a Riemannian invariant in terms of any metric from the conformal class, such that the change of the metric does not change the invariant. As proved by the extraordinary effort to understand such invariants for many decades, already the first invariant theorem is not easy in this case.

An equivalent definition of conformal structures treats them in terms of classical G-structures as reductions of the linear frame bundle to the structure group $G_0 = CO(n)$, the group of all conformally Euclidean linear transformations in the given dimension. Such a structure admits compatible torsion-free connections.

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which are classically called Weyl connections. This broader class of conformal connections was exploited by H. Weyl which motivates their name. It turns out that the Weyl connections form an affine space modelled on one-forms. Of course, the Levi-Civita connections of metrics in the conformal class are Weyl connections, they form an affine subspace modelled on exact one-forms. The study of conformal Riemannian invariants goes back to É. Cartan, T. Thomas, J.A. Schouten, and others (e.g., [13, 25, 26]). A lot of spectacular tricks to build invariant expressions have been developed, and some of them were turned into a quite effective calculus for conformal invariants by V. Wünsch, see [27].

Motivated by the rich geometry of conformal Riemannian manifolds, the Weyl structures and the preferred connections were introduced in the general framework of parabolic geometries in [9] (generalizing the approach from [17]). In particular, the notions of scales, closed and exact Weyl structures, and (Schouten’s) Rho-tensors were extended, and natural generalizations of classical normal coordinates in affine geometries were discussed.

Moreover, filtered analogues of classical G-structures which are equivalent to parabolic geometries and the general Cartan-Tanaka theory for all parabolic geometries are explained in great detail in [12], see also [5]. In this setting, the Weyl connections on a parabolic geometry of type $G/P$ correspond to reductions of the parabolic structure group $P$ of the canonical Cartan connection $\omega$ on the principal bundle $G \to M$ to its reductive Levi factor $G_0 \subset P$.

More recently, the geometry of the bundle of Weyl structures $\pi : A = G/G_0 \to M$ was studied carefully in the joint work [7] of the first author and T. Mettler. Here the canonical Cartan connection $\omega$ induces a canonical affine connection on the manifold $A$ as well as a canonical splitting $TA = L^- \oplus L^+$. There also is a nice relation between natural bundles over $M$ associated to $P$-representations and natural bundles over $A$. Using this, we will identify a natural class of differential invariants of the canonical connection on $A$ which induce affine differential invariants of the Weyl connections. The invariants obtained in this way transform tensorially in the one-forms $\Upsilon$ that parameterize the Weyl connections. We shall also prove a converse to this statement, which is much more subtle: If an affine invariant of Weyl connections, its curvature and its torsion transforms tensorially, then it comes from a natural affine differential invariant on $A$ as described above.

The paper is organized as follows: After a brief review of the main tools and concepts following [12], [7], we show in Section 2 how affine invariants of the canonical connection $D$ on $A$ can be used to construct affine invariants of the Weyl connections. This is based on an invariant concept of jets and by construction, the resulting invariants always change tensorially under a change of Weyl structure. We introduce the terminology “nearly invariant operators” for this behavior. In the last section, we prove that all nearly invariant operators are obtained in this way. This can be viewed as a week version of the ‘first invariant theorem’ (providing only an ansatz of possible expressions for invariant operators) and also a universal procedure realizing a generalization of the so-called Wünsch calculus in conformal geometry, which provides universal formulae for large classes of invariant operators, cf. [10].

1. Parabolic geometries and the bundle of Weyl structures

We shall follow the terminology and notation of [12] and [7]. The complete procedure deriving the canonical (i.e., properly normalized) Cartan connection from more elementary data on filtered manifolds corresponding to parabolic geometries was first worked out long ago by Tanaka in [24], more straightforward, simpler and more general versions appeared in [8], [12], [5]. We do not go into details on the
normalization here, but just use its implications for the curvature and torsion of Weyl connections. The calculus we develop actually does also work for non-normal Cartan connections. However in the non-normal case, in some of the results the notion of invariants have to be adapted.

1.1. The Cartan connections. A Cartan geometry of type $G/P$ is an absolute parallelism on a principal fiber bundle $G \to M$ with structure group $P$ encoded as a one-form $\omega \in \Omega^1(G, g)$. One requires suitable equivariancy properties with respect to the principal action of $P$, similar to those of the Maurer-Cartan form $\omega_G$ on $G \to G/P$. Thus, we may view Cartan geometries as curved deformations of the homogeneous spaces. More explicitly, the Cartan connection $\omega \in \Omega^1(G, g)$ is required to obey the following properties

1. $\omega(\xi_X)(u) = X$ for all $X \in \mathfrak{p}, u \in G$ (the connection reproduces the generators of fundamental vertical fields)
2. $(r^g)^*\omega = \text{Ad}(g^{-1}) \circ \omega$ for any $g \in P$ (the connection form is equivariant with respect to the principal action)
3. $\omega|_{\mathfrak{T}_uG} : T_uG \to \mathfrak{g}$ is a linear isomorphism for all $u \in G$ (the absolute parallelism condition).

A morphisms of Cartan geometries is a principal fiber bundle morphism $\phi : G \to G'$ with the property $\phi^*(\omega') = \omega$. Parabolic geometries are Cartan geometries of type $(G, P)$, where $P$ is a parabolic subgroup in a semisimple real Lie group $G$.

In the sequel, we shall consider a fixed parabolic subgroup $P \subset G$. It is well known, that $P$ uniquely determines a grading

\[ g = g_- \oplus g_0 \oplus g_+ = g_{-k} \oplus \cdots \oplus g_0 \oplus g_1 \oplus \cdots \oplus g_k \]

on the Lie algebra $\mathfrak{g}$ of the group $G$, such that $\mathfrak{p} = g_0 \oplus \mathfrak{g}_+ = g_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ is the Lie algebra of $P$. This comes with a so called grading element $E \in \mathfrak{p}$ with the property $\text{ad} E|_{\mathfrak{g}_i} = i \cdot \text{id}_{\mathfrak{g}_i}$ for all $i = -k, \ldots, k$.

While the grading (1) is not $P$-invariant, there is an associated filtration

\[ g = g_{-k} \supset g_{-k+1} \supset \cdots \supset g_0 = \mathfrak{p} \supset g_1 \supset g_2 \supset \cdots \supset g_k = \mathfrak{g}_k \]

defined by $g^l := \oplus_{i \geq l} \mathfrak{g}_i$, which is invariant under the adjoint action of $P$. The subgroup $G_0 \subset P$ of all elements whose adjoint action does preserve the grading has Lie algebra $g_0$. This is the reductive Levi factor of $P$. We also write $P_+ = \exp \mathfrak{p}_+$ for the nilradical of $P$, and similarly $G_- = \exp g_-$. In particular, for each $g \in P$, there are unique elements $g_0 \in G_0$, $\Upsilon \in \mathfrak{p}_+$, and $\Upsilon_i \in \mathfrak{g}_i$, $i = 1, \ldots, k$, such that

\[ g = g_0 \exp \Upsilon = g_0 \exp \Upsilon_1 \cdots \exp \Upsilon_k. \]

This decomposition reflects the fixed splitting of the filtration of $\mathfrak{p}_+$ by $P$-submodules, i.e. our fixed isomorphism $\text{gr}\mathfrak{p}_+ \to \mathfrak{p}_+$.

The Cartan connection $\omega$ provides the constant vector fields $\omega^{-1}(X) \in \mathfrak{X}(G)$ defined for all $u \in G$ and $X \in \mathfrak{g}$ by

\[ \omega(\omega^{-1}(X))(u) = X. \]

These generalize the left invariant vector fields on the homogeneous model $G \to G/P$ and enjoy the same equivariancy property, i.e.

\[ Tr^g \omega^{-1}(X)(u) = \omega^{-1} (\text{Ad}_{g^{-1}} \cdot X)(u \cdot g), \]

where $r^g$ is the principal right action by the element $g \in P$. In particular, due to our fixed splitting of $\mathfrak{g}$, there are the horizontal vector fields $\omega^{-1}(X)$ with $X \in \mathfrak{g}_-$. On the homogeneous model, the Maurer-Cartan equation reads as $d\omega + \frac{1}{2}[\omega, \omega] = 0$, while on a general geometry the same expression provides the two-form

\[ K = dw + \frac{1}{2}[\omega, \omega] \]
called the curvature. The equivariance properties of the Cartan connection imply that $K$ is always a horizontal two-form and so the curvature is completely determined by the curvature function $\kappa \in \mathcal{C}^\infty(\mathcal{G}, \Lambda^2 g^-_+ \otimes \mathfrak{g})$.

$$\kappa(u)(X, Y) = K(\omega^{-1}(X)(u), \omega^{-1}(Y)(u)) = [X, Y] - \omega(u)([\omega^{-1}(X), \omega^{-1}(Y)]).$$

Of course, the values of both the connection and the curvature function split according to the corresponding splitting of $\mathfrak{g}$ into

$$\omega = \omega_- + \omega_0 + \omega_+,$$

but the individual components are not $P$-equivariant. For instance, $\kappa_-$ is a well defined object only if its values are considered as elements of the quotient $\mathfrak{g}/\mathfrak{p}$. The latter component of the curvature is called the torsion of the Cartan connection.

Clearly, the curvature is the obstruction to the integrability of the horizontal distribution $\omega^{-1}(\mathfrak{g}_-)$ in $T\mathcal{G}$ and the Cartan connection is locally isomorphic to its homogeneous model if and only if the curvature vanishes, see e.g., [12, Section 1.5].

### 1.2. Natural bundles.

Every $P$-representation on a vector space $\mathbb{V}$ provides the homogeneous vector bundle $G \times_p \mathbb{V} \to G/P$ and, more generally, the associated vector bundles

$$\mathbb{V}M = \mathcal{G} \times_p \mathbb{V} \to M$$

with standard fiber $\mathbb{V}$ over all manifolds with a parabolic geometry of the type $G/P$. Shortly, we shall talk about $P$-modules $\mathbb{V}$ and the induced natural bundles. In the sequel, we shall restrict ourselves to $P$-modules with a diagonalizable action of the center of $G_0$.

For instance, the Cartan connection $\omega$ on $p : \mathcal{G} \to M$ identifies the tangent bundle $TM = \mathcal{G} \times_p \mathfrak{g}/\mathfrak{p} = \mathcal{G} \times_p \mathfrak{g}_-$, $(u, X) \mapsto Tp(\omega^{-1}(X)(u))$. Similarly, $T^*M = \mathcal{G} \times_p \mathfrak{p}_+$ and the duality is expressed by the Cartan-Killing form on $\mathfrak{g}$. Observe that for $i < 0$, we have the $P$-invariant subspace $\mathfrak{g}_i^+ \subseteq \mathfrak{g}/\mathfrak{p}$ induced by the filtration of $\mathfrak{g}$ from (2). This determines a smooth subbundle $T^iM \subseteq TM$, so we get a filtration of the cotangent bundle $T^*M$ by smooth subbundles.

A special class of natural bundles is induced by the $G$-modules viewed as $P$-modules. They are called the tractor bundles, see [1], [12] for detailed description in many specific geometries and historical links. A very special role is reserved for the adjoint tractor bundles $\mathcal{A}$ coming from the adjoint representation of $G$ on $\mathfrak{g}$. Of course, the Lie bracket itself is $\text{Ad}$-equivariant and thus there is the algebraic bracket $\{ , \}$ on the adjoint tractors.

It is well known that sections $\sigma$ of the induced bundles $\mathbb{V}M$ are in bijective correspondence with smooth functions $\tilde{\sigma} : \mathcal{G} \to \mathbb{V}$ which are $P$-equivariant in the sense that

$$s(u \cdot g) = g^{-1} \cdot s(u).$$

Here $u \cdot g$ denotes the principal right action of $g$ on $u$, while the bullet denotes the $P$-action on $\mathbb{V}$. We have already seen the curvature function $\kappa : \mathcal{G} \to \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g} = \Lambda^2 \mathfrak{p}_+ \otimes \mathfrak{g}$, representing a section of the adjoint tractor valued form $\kappa \in \Omega^2(M; \mathcal{A})$, cf. (4), which is the curvature of $\omega$ viewed as a two-form on $M$. Of course, the adjoint tractor bundles inherit all the $P$-invariant objects from $\mathfrak{g}$, including the metric defined by the Cartan-Killing form. Further, the 1-forms on $M$ live in the invariant subbundle $A^1$ corresponding to $\mathfrak{p}_+$, while the vector fields on $M$ can be viewed (with the help of the Cartan connection on $\mathcal{G}$ as noticed above) as sections of the quotient $\mathcal{A}/A^0$, where $A^0 = \mathcal{G} \times_p \mathfrak{p}$. In particular, the torsion $\kappa_-$ of the Cartan connection is a vector valued two-form on $M$, exactly as we are used to see the torsions of affine connections on manifolds.
1.3. The bundle of Weyl structures. We briefly remind the impacts of reductions of the parabolic structure groups to their reductive parts. The reader can find more details in [12] and [7].

Given the Cartan geometry \((G \to M, \omega)\), we can form the quotient \(G_0 := G/P_+\) which is a principal bundle over \(M\) with structure group \(P/P_+ \cong G_0\). Each reduction of \(G\) to the structure group \(G_0 \subset P\) can then be seen as a \(G_0\) equivariant smooth section \(\bar{s} : G_0 \to G\) of the quotient projection. It is well known that such reductions are in bijective correspondence with the smooth sections \(s\) of the bundle \(\pi : G/G_0 \to M\), which in our case is equal to \(\pi : A = G \times_P P/G_0 \to M\). We call \(\pi : A \to M\) the bundle of Weyl structures, see [18] and [7].

In particular, we see that Weyl structures always exist globally, and for two such sections \(\bar{s}\) and \(s\), there always exist unique equivariant functions \(\Upsilon : G_0 \to P_+\) and \(\Upsilon_i : G_0 \to G_i\) for \(i = 1, \ldots, k\), such that for all \(v \in G_0\),

\[\tilde{s}(v) = \bar{s}(v) \cdot \exp \Upsilon(v) = \bar{s}(v) \cdot \exp \Upsilon_1(v) \cdots \exp \Upsilon_k(v).\]

Now we come to several crucial observations.

First, \(G \to A\) is a principal fibre bundle with the structure group \(G_0\) and the tangent bundle is the associated bundle \(TA = G \times_{G_0} (g_- \oplus p_+)\) via the adjoint action of \(G_0\) on \(g\). In particular, \(TA\) naturally splits into two components \(TA = L^+ \oplus L^-\) corresponding to the \(G_0\)-invariant components \(g_-\) and \(p_+\) in \(g\). Moreover, \(L^+\) is the vertical bundle of \(\pi : A \to M\), while \(L^- = \pi^*TM\).

Second, the Cartan connection \(\omega = \omega_+ + \omega_0 + \omega_-\) can be viewed as an affine connection \(D\) on \(A\), with soldering form \(\omega_+ + \omega_-\) and principal connection form \(\omega_0\). The natural splitting \(TA = L^- \oplus L^+\) leads to two partial affine connections \(D^-\) and \(D^+\) by restriction. We also observe that, up to universal algebraic terms, the curvature of \(\omega\) on \(A\) encodes the torsion and curvature of \(D\) by means of the \(G_0\)-equivariant functions \(\bar{\kappa}_- + \bar{\kappa}_+\) and \(\bar{\kappa}_0\) from (3), respectively. Moreover, since \(\kappa\) is horizontal with respect to \(G\) \(\to M\), the torsion and curvature of \(D\) are given by universal algebraic terms if evaluated on one argument from \(L^+\). Evaluated on two arguments from \(L^+\), they provide, after algebraic corrections, the components \(T \in \Omega^2(A, L_-), Y \in \Omega^2(A, L^+),\) and \(W \in \Omega^2(A, G \times_P g_0)\). We call these components the universal torsion \(T\), universal Cotton York tensor \(Y\), and universal Weyl curvature \(W\), respectively, see [7] for details.

For each Weyl structure \(s\), we can also consider the pullback of the Cartan connection \(\check{s}^*\omega \in \Omega^1(G_0, g)\) which now splits into the three components naturally. Let us write

\[\theta^s = \check{s}^*\omega_- = \theta^s_k + \cdots + \theta^s_1 : TG_0 \to g_{-k} \oplus \cdots \oplus g_{-1}\]

\[\gamma^s = \check{s}^*\omega_0 : TG_0 \to g_0\]

\[P^s = \check{s}^*\omega_+ = P^s_1 + \cdots + P^s_k : TG_0 \to p_+ = g_1 \oplus \cdots \oplus g_k.\]

The filtration on \(TM\) introduced in Section 1.2 gives rise to the associated graded bundle \(\text{gr}(TM) = \bigoplus_{m}^{-1} T^m M/T^{m+1} M\). Now the form \(\theta^s\) provides an identification of \(G_0\) with a frame bundle for \(\text{gr}(TM)\) with structure group \(G_0\) and should
be understood as an analog of the soldering form. In particular, this defines the isomorphism $TM \simeq \text{gr}(TM)$, i.e. a splitting of the filtration
\[ TM = T^{-k}M \supset \cdots \supset T^{-1}M \]
determined by the choice of $s$.

The next component $\gamma^s$ is the connection form of a principal connection on $G_0$. Thus, the first two components provide a reductive Cartan connection $(\theta^0 + \gamma^s)$ of type $(G_0/G_0)/G_0$ on $M$. We call it the Weyl connection corresponding to the choice of the Weyl structure $s$. We shall write $\nabla^s$ for the covariant derivatives with respect to this connection on all vector bundles associated to $G_0$.

The last component $\mathcal{P}$ measures the difference between the Cartan connection $(\theta^0 + \gamma^s)$ on $G_0$ and the original Cartan connection $\omega$ on $G$, along the image of $s$. We view it as a one-form on $M$ valued in $\text{gr}T^*M$ and call it the Rho-tensor.

Third, let us take a representation $\mathcal{V}$ of $P$ and consider the induced bundle $G \times_P \mathcal{V} = \mathcal{V}M \to M$. Sections of this bundle correspond to $P$-equivariant maps $\tilde{\sigma} : P \to \mathcal{V}$, so they obviously also represent sections of the natural bundle $\mathcal{V}A = G \times_{G_0} \mathcal{V} \to A$. By the very construction, the natural bundles $\mathcal{V}A \to A$ are naturally identified with $\pi^*\mathcal{V}M$.

Next, the action of the grading element $E \in g_0$ decomposes $\mathcal{V}$ into $G_0$-invariant components
\[ \mathcal{V} = \mathcal{V}_0 \oplus \cdots \oplus \mathcal{V}_\ell \]
with the property that $X \cdot \mathcal{V}_j \subseteq \mathcal{V}_{i+j}$, for all $X \in g_i$ and $i = 1, \ldots, k$. For instance, the Lie algebra $g_\gamma$, viewed as $\text{gr}(g/p)$, or the entire $g$ are special cases illustrating the difference between the $P$-invariant filtrations and $G_0$-invariant induced gradings.

Thus while the natural bundles $\mathcal{V}M \to M$ come equipped with the natural filtrations only, the bundles $\mathcal{V}A \to A$ come with natural gradings. Once we choose a Weyl structure $s$, the bundles $\mathcal{V}M$ get graded, too (via the structure group reduction). This grading coincides with the pullback $s^*(\mathcal{V}A) \to M$, [7, Theorem 2.4].

Of course, the canonical affine connection $D$ induces linear connections on all natural bundles $\mathcal{V}A$. Now, the required $P$-equivariance of the maps $\tilde{\sigma}$ representing sections of $\mathcal{V}M$ directly implies that the sections of $\mathcal{V}M$ are exactly those sections $\tau$ of $\mathcal{V}A$, whose covariant derivatives satisfy $D\tau = -\xi \cdot \tau$ for all sections $\xi \in \Gamma(L^+)$, [7, Theorem 2.4].

For a section $\sigma \in \Gamma(\mathcal{V}M)$, the covariant derivative $D\sigma$ with respect to the canonical connection $D$ defines a map $D\sigma : \mathcal{V} \to (g_- \oplus p_+)^* \otimes \mathcal{V}$. Restricting to entries from $g_-$, one obtains an operation $\nabla^\omega$ which is called the invariant differential and which played a crucial role in [10] and [11]. However, this was viewed as an operation mapping sections of $\mathcal{V}M$ to equivariant functions on the Cartan bundle, the relation to the bundle of Weyl structures was not known at that time.

1.4. The curvatures. As mentioned in Section 1.1, the fundamental invariant of a Cartan geometry $(G, \omega)$ is the Cartan curvature $\kappa$. Via the curvature function discussed there, this admits direct interpretations both on $M$ and on $A$. $P$-equivariancy of the curvature functions implies that $\kappa$ can be viewed as a two-form on $M$ with values in the adjoint tractor bundle $\mathcal{A} = G \times_P g$. A choice of Weyl structure determines an isomorphism between the bundle $\Lambda^2 T^*M \otimes \mathcal{A}$ with its associated graded. The latter bundle decomposes into components according to the decomposition of $\Lambda^2 (g_-)^* \otimes g$ into $G_0$-irreducible representations.

On the level of $\mathcal{A}$, the decomposition into $G_0$-components is available canonically. In particular, the $G_0$-invariant decomposition $g = g_- \oplus g_+ \oplus g_0$ decomposes $G \times_{G_0} g$ with the first two summands corresponding to $L^- \oplus L^+ = TA$. Via the restriction of the adjoint action, $g_0$ canonically injects into endomorphisms of $g_- \oplus g_+$, so we can view $G \times_{G_0} g_0$ naturally as a subbundle $\text{End}_0(TA) \subset T^*A \otimes TA$. 
There is also a nice way to interpret the Rho-tensor via Weyl connections associated to a Weyl structure. Consider a Weyl structure \( D \) in order to obtain the torsion and the curvature of \( D \). More precisely, one has to add universal bundle maps induced by \((G_0\text{-equivariant})\) components of the Lie bracket on \( g \) to \( T + Y \) respectively to \( W \) in order to obtain the torsion and the curvature of \( D \).

Next, we want to relate the Cartan curvature to the torsion and curvature of the Weyl connections associated to a Weyl structure. Consider a Weyl structure \( s \), and indeed by \([7, \text{Theorem 2.12}] \) these equivalently encode the torsion and curvature of the Cartan connection \( \theta^s \) and \( \gamma^s \) on \( G_0 \). The full Cartan curvature (i.e., torsion and curvature) of this connection is also given by the structure equations

\[
T^s(\xi, \eta) = d\theta^s(\xi, \eta) + [\gamma^s(\xi), \theta^s(\eta)] + [\theta^s(\xi), \gamma^s(\eta)] + [\theta^s(\xi), \theta^s(\eta)]
\]

\[
R^s(\xi, \eta) = d\gamma^s(\xi, \eta) + [\gamma^s(\xi), \gamma^s(\eta)],
\]

where \( \xi, \eta \) are vector fields on \( G_0 \). Clearly the torsion \( T^s \) and the curvature \( R^s \) are horizontal two-forms which descend to well defined forms on the underlying manifold. Let us point out, that \( T^s \) is not the usual torsion of a linear connection but there is a correction involving natural bundle maps induced by \((G_0\text{-equivariant})\) Lie bracket on \( g \). Since the forms \( \theta^s \) and \( \gamma^s \) are pullbacks of \( \omega_- \) and \( \omega_0 \), the above structure equations are easily compared with those of \( \omega \).

The latter curvature forms are clearly related to the pullback \( s^* \kappa \) of the curvature of the Cartan connection \( \omega \) on \( G \) (i.e. also the torsion and curvature of \( D \)). The missing components of curvature are related to the P-tensor and they can be understood easily, too:

\[
s^*\kappa = T^s + R^s + Y^s + \partial P^s
\]

where we define the Cotton-York tensor of the Weyl structure on \( M \) as

\[
Y^s(\xi, \eta) = d^{\mathcal{V}} P^s(\xi, \eta) + P^s(\{\xi, \eta\}) + \{P^s(\xi), P^s(\eta)\},
\]

with \( d^{\mathcal{V}} \) denoting the covariant exterior differential, \( \{ , \} \) is the natural bracket on the adjoint tractors, and

\[
\partial P^s(\xi, \eta) = \{\xi, P^s(\eta)\} - \{\eta, P^s(\xi)\} - P^s(\{\xi, \eta\})
\]

is the \((G_0\text{-equivariant})\) Lie algebra cohomology differential. One then defines the Weyl curvature of the Weyl connection determined by \( s \) as \( W^s := R^s + (\partial P^s)_{|p} \).

As suggested by the notation, the quantities \( T^s, W^s \), and \( Y^s \) are directly related to the quantities on \( A \) discussed above. Indeed by \([7, \text{Proposition 2.10}] \) and \([12, \text{Section 5.2.9}] \) one has \( s^*T = T^s + (\partial P^s)_{|p} \), \( s^*Y = Y^s + (\partial P)_{|p} \), and \( s^*W = W^s \). There is also a nice way to interpret the Rho-tensor via \( A \). Viewing the projection onto the second factor in \( TA = L^- \oplus L^+ \) as an element of \( P \in \Omega^1(A, L^+) \), one can form \( s^*P \in \Omega^1(TM, \text{gr}(TM)) \) and by \([7, \text{Proposition 2.8}] \) this coincides with \( P^s \).

It will be very important for our results that the decomposition of the curvature and torsion of a Weyl connection into the pullback of the Cartan curvature and a part obtained from the Rho-tensor can be made explicit without reference to the Cartan bundle. This only works for normal parabolic geometries which anyway is the class usually studied. They are characterized by a normalization condition on their curvature \( \kappa \) which is usually phrased as \( \partial^s(\kappa) = 0 \). Here \( \partial^s \) is induced by a \( P\text{-equivariant} \) linear map \( \Lambda^2(g/p)^* \otimes g \to (g/p)^* \otimes g \), the so-called Kostant codifferential. This normalization condition then implies that the parabolic geometry is determined by some underlying geometric structure known as an infinitesimal flag structure, for example a conformal structure.

The detailed form of this is not important for our purposes. What it tells us, however, is that the right hand side of \((6)\) lies in the kernel of \( \partial^s \). This allows us to
express, $\partial^* \partial(P^*)$ in terms of the curvature and torsion of $\nabla^*$ and of $Y^*$ as defined above. Now on the one hand it turns out that $\partial^* \partial$ is invertible. On the other hand, the definition of $Y^*$ is chosen in such a way our equation can be solved iteratively homogeneity by homogeneity (with respect to the grading element), see Section 5.1 of [12] for details. Hence $P^*$ can be computed from curvature quantities, which in turn allows us compute the pullback of the Cartan curvature. In particular, in the case of conformal geometry this leads to the description of the Rho tensor as a trace-modification of the Ricci curvature as introduced by J.A. Schouten.

1.5. Normal Weyl structures. As we have seen, the Rho-tensor measures the deviation of the Cartan connection $\theta^a + \gamma^a$ on $G_0$ determined by a Weyl structure $s$, from the given Cartan connection $\omega$. Thus minimizing the values of $P^*$ and its derivatives in a point looks like a good idea. We can easily follow the way how the flow lines exist at least up to the time $t = 1$. In this way we obtain a horizontal embedding $X \mapsto \varphi_u(X) = Fl^{-1}_t\omega^{-1}(X)(u)$ and also a local section $p(\varphi_u(X)) \mapsto \varphi_u(X)$ of the Cartan bundle $p : \mathcal{G} \to M$ through $u$. Consequently there is a unique local Weyl structure $s_u$ through the frame $u$ defined by $s_u(p_0(\varphi_u(X))) = \varphi_u(X)$. (Here $p_0 : \mathcal{G} \to G_0$ is the natural projection and the $G_0$-orbit of this section defines the reduction.) We call them the normal Weyl structures. The images $\tilde{c}_u^X$ of the defining flow lines in $M$, i.e. $\tilde{c}_u^X(t) = p(Fl_t^{-1}\omega^{-1}(X))$, are called the generalized geodesics of the Cartan connection $\omega$. Thus, each choice of a frame $u \in \mathcal{G}$ over the point $p(u) = x \in M$ determines a uniquely defined “geodetical parametrization” from a neighborhood of $0$ in $T_x M$ to a neighborhood of $x$ in $M$.

This construction can be also nicely seen on $A$. We are restricting the standard normal coordinates of the affine connection $D$ to geodesics emanating in $L^*$-directions, which directly creates a (local) section $s : M \to A$, similarly as above. These sections evidently have the property, that the universal Rho tensor vanishes along those geodesics. Indeed, this leads to a complete characterization: The Weyl structures $s$ that are normal at $x$ are characterized by the fact that $P^*(c(t))(c'(t)) = 0$ for all generalized geodesics through $x = p(u) \in M$. In particular, this implies that for every integer $\ell \geq 0$ of covariant derivatives, and any tangent vectors $\xi_0, \ldots, \xi_\ell$, the full symmetrization of the expression

$$\nabla_{\xi_0}^* \cdots \nabla_{\xi_\ell}^* P(x)(\xi_0) \in \Omega^1(M)$$

over the $\xi$’s vanishes, see [9] for details.

2. The invariant calculus

Whatever concept of invariance we adopt, all objects built naturally in the category of Cartan connections of given type should be invariant. Thus, we can consider any (possibly non-linear) differential operator $\Phi : \Gamma(VA) \to \Gamma(WA)$ between two natural vector bundles over $A$ coming from $P$-modules $V$ and $W$, expressed in covariant derivatives with respect to $D$ and derivatives of the torsion and curvature of $D$. As noted in Section 1.3, we can view $\Gamma(VM)$ naturally as a subspace of $\Gamma(VA)$ (corresponding to $P$-equivariant functions among $G_0$-equivariant functions). Then it may happen that the restriction of $\Phi$ to this subspace has values
in \(\Gamma(WM) \subset \Gamma(WA)\). If this is the case, the restriction has to define an invariant operator.

However, choosing a Weyl structure \(s\), there is a possibility to “descend” any operator \(\Phi : \Gamma(VA) \to \Gamma(WA)\) as above to an operator \(\Phi^s : \Gamma(VM) \to \Gamma(WM)\). Namely, for a section \(\sigma \in \Gamma(VM)\) we can view \(\sigma\) as an element of \(\Gamma(VA)\) and form \(\Phi(\sigma) \in \Gamma(WA)\). Then we can take the pullback \(s^*\Phi(\sigma)\) and observe that there is a unique section \(s^*\Phi(\sigma) \in \Gamma(WM) \subset \Gamma(WA)\) such that \(s^*\Phi(\sigma) = s^*\Phi(\sigma)\). Equivalently, this can be phrased as \(\Phi(\sigma)|_{s(M)} = \Phi^s(\sigma)|_{s(M)}\). In the language of equivariant functions, this simply means that we take the \(G_0\)-equivariant function \(\Phi(\sigma) : G \to W\), restrict it to \(\tilde{s}(G_0) \subset G\) and then define \(\Phi^s(\sigma) : G \to W\) to be the \(P\)-equivariant extension of this \(G_0\)-equivariant function.

Observe that this construction immediately implies a property that will be of crucial importance in what follows, namely that for any \(\sigma \in \Gamma(VM)\) and any point \(x \in M\), the value \(\Phi^s(\sigma)(x)\) depends only on \(s(x)\), so the dependence of \(\Phi^s\) on \(s\) is of tensorial character.

An obvious first step in trying to understand this construction is to analyze the operator \(\Phi : \Gamma(VA) \to \Gamma((L^-)^* \otimes VA)\) defined by \(\Phi(\sigma) := D^-\sigma\) on some natural bundle. Choosing a Weyl structure \(s\), this descends to an operator \((D^-)^* : \Gamma(VM) \to \Gamma(T^*M \otimes VM)\) which has the chance to define a covariant derivative on \(VM\). This case is sorted out by a result which basically is proved as a part of Theorem 2.6 in [7]:

**2.1. Proposition.** Let \(V\) be any representation of \(P\) and consider the corresponding natural bundles \(VM \to M\) and \(VA \to A\), and the operator \(\Phi : \Gamma(VA) \to \Gamma((L^-)^* \otimes VA)\) defined by \(\Phi(\sigma) := D^-\sigma\). Then for any Weyl structure \(s : M \to A\), the operator \(\Phi^s : \Gamma(VM) \to \Gamma(T^*M \otimes VM)\) coincides with the Rho-corrected derivative \(\nabla^P\sigma\) corresponding to \(s\) as introduced in Section 5.1.9 of [12].

**Proof.** Consider \(\tilde{s}(G_0) \subset G\) and restrict the component \(\omega_\rho\) to this subset. Then this can be uniquely extended to a principal connection form \(\gamma^s\) on \(G\) and by definition, the Rho-corrected derivative is the covariant derivative induced by this principal connection. Viewing \(\sigma \in \Gamma(VM)\) as a \(P\)-equivariant function \(G \to V\), the function \(G \to L(g_{\rho}, V)\) corresponding to \(\nabla^P\sigma\) thus is given by taking the derivative of \(\sigma\) with respect to horizontal vector fields along \(\tilde{s}(G_0)\) and then equivariantly extending to \(G\). But this exactly means that, along \(\tilde{s}(G_0)\), this coincides with the \(G_0\)-equivariant function associated to \(D^-\sigma\), see Section 1.3, which proves the claim. \(\Box\)

In view of our observation above, this gives an alternative argument for the fact that the transformation rule for Rho-corrected derivatives is tensorial in the one-form \(\Upsilon\) describing the change between two Weyl-structures, see part (2) of [12, Proposition 5.1.9].

The name Rho-corrected derivative is motivated by the relation of this operation (determined by some Weyl structure \(s\)) to the Weyl connection \(\nabla^s\). This can be written as
\[
\nabla^P_\xi \sigma = \nabla^{\xi}_\rho \sigma + \rho^!(\xi) \bullet \sigma
\]
for any vector field \(\xi \in \mathfrak{X}(M)\), since the difference of the horizontal lifts of \(\xi \in T_xM\) for \(\gamma^s\) and \(\gamma\) on the image \(\tilde{s}(G_0)\) is the fundamental vector field corresponding to \(-\omega_{P+}(\xi)\), and \(\sigma\) is \(P\)-equivariant. Here the bullet comes from the bundle map \(T^*M \times VM \to VM\) induced by the infinitesimal representation \(p_+ \times \mathbb{V} \to \mathbb{V}\). A bit of care is needed in interpreting this formula, however, since both the Weyl connection \(\nabla^s\) and the \(P^!(\xi)\) are actually carried over from the associated graded bundles \(\text{gr}(VM)\) and \(\text{gr}(T^*M)\) via the isomorphism induced by \(s\). This also explains how the transformation law under a change of Weyl structure can be tensorial in the one-form \(\Upsilon\) in spite of the explicit occurrence of a Rho-tensor (whose transformation is not tensorial, see Section 3.1 below). Hence under the isomorphism \(\Gamma(VM) \cong \Gamma(\text{gr}(VM))\) induced by a Weyl
structure $s$, the operator $(D^-)^s$ corresponds to a universal expression in terms of $\nabla^s$ and the Rho-tensor $P^s$ of $s$.

We next want to obtain similar descriptions for more general operators built up from $D^-$ and the torsion and curvature of $D$. The first step is to understand the iterated derivatives $(D^-)^k$ with respect to $D^-$. This is tricky, because the argument in the proof of Proposition 2.1 was based on the fact that we start with $\sigma \in \Gamma(VM)$, i.e. with a $P$-equivariant function. However, the function corresponding to $D^- \sigma$ is not $P$-equivariant any more, so we cannot simply iterate. A neat way around this is via an invariant notion of jets that was crucial for the developments in [10] and in [11].

2.2. First order jets. In geometry, a differential operator $\Gamma(VM) \to \Gamma(WM)$ of order at most $k$ can be equivalently viewed as a vector bundle homomorphism $J^kVM \to WM$, where $J^kVM$ is the $k$th jet prolongation of $VM$. Thus, in order to understand the operators on $VA$ and $VM$, let us look at jets.

Let us first consider Klein geometries, i.e., the flat model Cartan geometries $G \to G/P$ with the Maurer-Cartan form $\omega$. There, natural vector bundles are exactly the homogenous bundles $V = G \times P V$ determined by $P$-modules $V$. Any jet prolongation of a homogeneous bundle is homogeneous, too, so there is a $P$-module $J^kV$ such that $J^kV = G \times P J^kV$. The $P$-module $J^kV$ can be obtained as the fiber of $J^kV$ over the origin $o \in G/P$. Thus, the invariant operators are in bijective correspondence with the intertwining maps $J^kV \to W$ between the $P$-modules. In a dual picture, this provides an identification of linear invariant operators with the singular vectors in generalized Verma modules for all parabolic models, see e.g. [14].

Let us now consider a parabolic geometry $G \to M$ with Cartan connection $\omega$. In 1.3, we identified the covariant derivative of the canonical affine connection $\nabla$ restricted to $L^-$ with the so called invariant differential $\nabla^\omega$ acting on $\mathbb{V}$-valued functions on $G$ via the constant vector fields $\omega^{-1}(X)$, $X \in g_-$. This is just the usual definition of the covariant derivative of affine connections in its frame form.

Although $\nabla^\omega$ does not map $P$-equivariant functions $\sigma$ into $P$-equivariant results, its extension $D^\omega: C^\infty(G, V) \to C^\infty(G, g^* \otimes V)$, $D^\omega\sigma(u)(X) = \omega^{-1}(X)(u)\sigma \in V$

does. This operator is called the fundamental derivative. It is very well understood in terms of the adjoint tractor bundles and leads to the so called tractor calculus, see [12, Sections 1.5.7-8].

The operator $D^\omega: \Gamma(VM) \to \Gamma(WM)$ can be understood as the universal differential operator and the standard fiber $\mathbb{W}$ of the target is $g^* \otimes V$. We may also consider the analogue of the first jet prolongation by mapping $\sigma$ to $(\sigma, D^\omega \sigma)$. The natural action of $g \in P$ on this target module $\mathbb{V} \oplus (g^* \otimes \mathbb{V})$ is

$$g(v, \varphi) = (g \cdot v, B \mapsto g \cdot \varphi(Ad_{g^{-1}}B)).$$

The definition of $D^\omega$ together with equivariance of the function $\sigma$ that is differentiated readily implies that for $X \in p \subset g$, we get $D^\omega\sigma(u)(X) = -X \cdot \sigma(u)$, where in the right hand side, we use the infinitesimal action of $p$ on $\mathbb{V}$. Correspondingly, there is a $P$-submodule $J^1V \subset V \oplus g^* \otimes V$, and $\sigma \mapsto (\sigma, D^\omega \sigma)$ induces a functorial isomorphism $J^1VM \to G \times P J^1V$. This construction works for all Cartan geometries of type $G/P$. See [12, Section 1.5.10] for details.

Now as a $G_0$-module, $g_0 = g_- \oplus p$ which leads to an isomorphism $J^1V \cong_{G_0} V \oplus (g_-)^* \otimes V$ via restriction in the second component. Consequently, we can naturally identify $G \times G_0 J^1V \to A$ with $VA \oplus (L^-)^* \otimes VA$. Now for a section $\sigma \in \Gamma(VM)$ the value of the one-jet operator $j^1\sigma \in \Gamma(J^1VM)$ defines a section of $G \times G_0 J^1V \to A$, which under this identification corresponds to $(\sigma, D^- \sigma)$. 


2.3. Higher order jets. The iteration of the first jet prolongation leads to the non-holonomic jet prolongations and most of the redundancies can be removed by considering semi-holonomic jets. It turns out that the target space is a natural bundle with the fiber called semiholonomic jet module $J^kV$ induced by $V$. Hence there is a $k$th order invariant semiholonomic jet operator, $k = 1, 2, \ldots$ with values in $G \times P J^kV$, see [10] for details. Now similarly as in order 1, we get an isomorphism of $G_0$-modules

$$J^kV \cong_{G_0} V \oplus (g_-)^* \otimes V \oplus \cdots \oplus (g_-)^* \otimes V$$

which is induced by restriction of multilinear maps. In particular, since any bundle map induced by a $G_0$-homomorphism is parallel for $D$ and hence for $D^\perp$, for a section $\sigma \in \Gamma(VM)$, the semi-holonomic $k$-jet operator $J^k\sigma$ corresponds to $(\sigma, D^-\sigma, \ldots, (D^-)^k\sigma)$ under this identification. We will implement this description by viewing $J^kV$ as the $G_0$-module $\oplus (g_-)^* \otimes V \oplus \cdots \oplus (g_-)^* \otimes V$ endowed with a (very complicated) extension of the action to $P$. Still, the infinitesimal action of $p_+$ is given by a $G_0$-equivariant map $p_+ \otimes J^kV \to J^kV$ in this picture.

From the construction it follows easily that for $VM = G \times_P V$ the invariant semi-holonomic jet operator defines an injective bundle map $J^kV \to \bar{V}$ endowed with a (very complicated) extension of the action to $P$. Still, the infinitesimal action of $p_+$ is given by a $G_0$-equivariant map $p_+ \otimes J^kV \to J^kV$ in this picture.

This makes the construction of invariant differential operators via algebraic techniques subtle. On the one hand, any $P$-equivariant map $J^kV \to \bar{W}$ induces a $k$th order invariant differential operator on locally flat geometries, but not on general curved geometries. On the other hand, any $P$-equivariant map $J^kV \to \bar{W}$ gives rise to an invariant differential operator on all Cartan geometries via composition with the invariant semi-holonomic jet operator) but, even in the linear case, not all invariant differential operators are obtained in this way. One obtains many operators in this way, however, in particular all operators that arise in BGG-sequences are of this character, see e.g. [14] and [11]. We will use the semi-holonomic jet prolongations in a different way here.

2.4. Expansion of $(D^-)^k$. Now we are ready to prove our first main result, which is a higher order analog of Proposition 2.1, so we want to study the operators $((D^-)^k)^s$ for a Weyl structure $s$ and any $k \geq 1$. We will show that, under the isomorphism to the associated graded, this operator can be written via a “universal formula” in the following sense. Fix an initial representation $V$ of $P$ and an order $k$. Then we need finitely many $G_0$-equivariant linear maps $A_i$ with values in $\otimes^k g_- \otimes V$, such that for each Weyl structure $s$ and any section $\sigma \in \Gamma(VM)$, we can write the image of $s^*((D^-)^k(\sigma))$ in the associated graded (under the isomorphism determined by $s$) as $(\nabla^s)^k s^* \sigma + \sum_i A_i(T_i)$. Here $\nabla^s$ denotes the Weyl connection, $A_i$ is the bundle map induced by $A_i$ and $T_i$ is an iterated tensor product of factors of the form $(\nabla^s)^l s^* \sigma$ and $(\nabla^s)^l P^s$ with $0 \leq \ell < k$ that is independent of $s$. (The form of such a tensor product also determines the domain of the map $A_i$.)

2.5. Theorem. For any representation $V$ and any $k \geq 1$, there is a universal expression for the image of $s^*((D^-)^k(\sigma))$ in the associated graded under the isomorphism determined by $s$ as $(\nabla^s)^k s^* \sigma$ plus a sum of terms containing iterated Weyl derivatives of $\sigma$ and $P^s$ in the sense described above. Moreover, in any tensor product in this expansion, the total number of covariant derivatives is $< k$. 
Proof. We proceed by induction on $k$. For $k = 1$, Proposition 2.1 provides an expansion in the claimed form with $i = 1$, $A_1 : (g_{-})^* \otimes p_+ \otimes \nabla \to (g_{-})^* \otimes \nabla$, the tensor product of the identity with the infinitesimal representation, and $T_1 = P^* \otimes \sigma$.

Assuming that the result has been proved for all $\ell \leq k$, we consider the invariant semi-holonomic $k$-jet operator $j^k$, which via our description of $J^k \nabla$ is given by $j^k \sigma = (\sigma, D^- \sigma, \ldots, (D^-)^k \sigma)$. Of course, the components of the pullback $s^* j^k \sigma$ are just $s^* D^- j^k \sigma$ for $0 \leq \ell \leq k$. Hence by induction hypothesis, we can express the image of $s^* j^k \sigma$ in the associated graded under the isomorphism induced by $s$ as $(\nabla^k)^* s^* \sigma$ plus a sum $\sum_i A_i(T_i)$ of universal terms as described above which involve covariant derivatives of $s^* \sigma$ and of $P^*$. Moreover, the total number of covariant derivatives occurring in each $T_i$ is $< k$.

Since $j^k \sigma$ lies in the subspace $\Gamma(\mathcal{G} \times_P J^k \nabla)$, we can apply Proposition 2.1 to conclude that $s^* (D^- j^k \sigma) = \nabla^k (s^* j^k \sigma) + P^* \bullet s^* j^k \sigma$. But naturality of $D^-$ implies that viewing $(g_{-})^* \otimes J^k \nabla$ as $\otimes_{i=0}^{k+1} \otimes \sigma^r (g_{-})^* \otimes \nabla$, the components of $D^- j^k \sigma$ are just $(D^-)^r \sigma$ for $r = 1, \ldots, k + 1$. In particular, we can extract an expression for $s^* (D^-)^{k+1} \sigma$ from $s^* (D^- j^k \sigma)$ and the known expressions for the iterated derivatives of lower order.

Expanding $s^* j^k \sigma = (\nabla^k)^* s^* \sigma + \sum_i A_i(T_i)$ and applying $\nabla^\kappa$, we obtain $(\nabla^\kappa)^{k+1} s^* \sigma$ from the first summand. On the other hand $P^* \bullet (\nabla^k)^* s^* \sigma$ is of the form that we allow for the additional terms in the expansion of $s^* ((D^-)^{k+1} \sigma)$. Thus we can conclude the proof by showing that the expressions $\nabla^\kappa(A_i(T_i))$ and $P^* \bullet A_i(T_i)$ all lead to expressions of the form we allow. For the first case, we can use that the bundle map $A_i$ is induced by a $G_0$-equivariant linear map and hence it is parallel for $\nabla^\kappa$. Thus we obtain $(\id \otimes A_i)(\nabla^\kappa T_i)$ and of course $\nabla^\kappa T_i$ can be expanded as a tensor product of factors $(\nabla^\kappa)^r$ and $(\nabla^\kappa)^r P^*$ with a total of at most $k$ covariant derivatives by our assumptions on $T_i$. For the other term, we can write $P^* \bullet A_i(T_i)$ as $B_i(P^* \otimes T_i)$. Here $B_i = (\id \otimes \sigma \tau) \circ (\id \otimes \id \otimes A_i)$, where $\tau$ denotes the infinitesimal representation, so this is again of the required form.



2.6. Involving curvature quantities. As described in Section 1.4, the torsion and the curvature of the canonical connection $D$ on $A$ can be equivalently encoded via the curvature $\kappa$ of the canonical Cartan connection $\omega$. Now the Cartan curvature $\kappa$ is a section of $\mathcal{G} \times_P V \to M$, where $V := (\Lambda^2 (g/p)^* \otimes g)$, so Theorem 2.5 applies to $\kappa$. Hence for any $k \in \mathbb{N}$, we can describe the pullback $s^* (D^-)^k \kappa$ in terms of $(\nabla^k)^* s^* \kappa$ and universal terms depending on iterated Weyl derivatives of order less than $k$ of $s^* \kappa$ and of the Rho-tensor $P^*$. We have seen, that on $A$, $\kappa$ decomposes as $T + W + Y$ according to the values in $g = g_- \oplus g_0 \oplus p_+$. Of course, these and further decompositions coming from $G_0$-invariant operations are, on the level of $A$, preserved by the operator $D^-$. They correspond to decompositions of $s^* \kappa$ on $M$ (depending on $s$) which in turn are preserved by the Weyl connection $\nabla^\kappa$. We also discussed in Section 1.4 how to decompose $s^* \kappa$ into its components $s^* T = T^s + (\partial P^*)_{g_-}, s^* W = R^s + (\partial P^*)_{g_0}$, and $s^* Y = Y^s + (\partial P^*)_{p_+}$, which are related to the torsion and curvature of $\nabla^s$.

Otherwise put, if we consider a component $K$ of the curvature and torsion of $D$, and form $(D^-)^k K$, we can recover this from $(D^-)^k \kappa$ by $G_0$-equivariant operations. Applying the same operations to $s^* (D^-)^k \kappa$ we conclude that we can express $s^* (D^-)^k K$ as $(\nabla^k)^* s^* K$ plus a universal expression in the sense discussed in Section 2.4 involving iterated derivatives of order $< k$ of components of $s^* \kappa$ and of $P^*$.

But this is sufficient to pull back polynomial invariant operators (in a sense that will be made precise) constructed from $D^-$ along Weyl structures and obtain a special class of affine invariants of Weyl structures. So we start with arbitrary representations $\nabla$ and $\nabla$ of $P$ and we are looking for an operator $\Phi : \Gamma(\mathcal{V}A) \to \mathcal{V}A$. 


\( \Gamma(WA) \) such that we can write \( \Phi(\sigma) = \sum_i A_i(T_i) \), where \( A_i \) is a natural bundle map induced by a \( G_0 \)-equivariant map with values in \( W \) and \( T_i \) is an iterated tensor product of factors of the form \( (D^-)^{\ell}(\sigma) \) and \( (D^-)^{\ell}K \) for a component \( K \) of \( \kappa \). The form of this tensor product is required to be independent of \( \sigma \) and it determines the representation on which the map inducing \( A_i \) is defined.

2.7. Theorem. Let \( V \) and \( W \) be representations of \( P \) and let \( \Phi : \Gamma(VA) \to \Gamma(WA) \) be a polynomial invariant operator constructed from \( D^- \) in the sense introduced above. Then for any Weyl structure \( s \), the operator \( \Phi^s : \Gamma(VM) \to \Gamma(WM) \) is induced, under the isomorphism to the associated graded bundles determined by \( s \), by a polynomial invariant operator constructed from \( \nabla^s \), its curvature and its torsion in the same sense. Moreover, this correspondence has the property that for two Weyl structures \( s \) and \( \bar{s} \) and a point \( x \in M \) such that \( s(x) = \bar{s}(x) \) one obtains \( \Phi^s(\sigma)(x) = \Phi^{\bar{s}}(\sigma)(x) \) for any \( \sigma \in \Gamma(VM) \).

Proof. Theorem 2.5 provides us with an expansion for \( (D^-)^{\ell}(\sigma) \) for any \( \ell \) involving \( G_0 \)-equivariant maps acting on tensors. The discussion in Section 2.6 gives us analogous expressions for \( (D^-)^{\ell}(K) \) for any component \( K \) of \( \kappa \). Since tensor products of \( G_0 \)-equivariant maps are \( G_0 \)-equivariant, we also get expressions for iterated tensor products of such terms. Applying \( G_0 \)-equivariant maps to such tensor products just gives rise to compositions of \( G_0 \)-equivariant maps, which are equivariant, too. The last statement has already been observed for any operator of the form \( \Phi^s \) in the beginning of Section 2.

3. The nearly invariant operators

3.1. The transformation rules. The natural approach to invariants of parabolic geometries is via Weyl-structures, i.e. to consider differential operators and differential invariants defined using the Weyl connections \( \nabla^s \), their torsion and their curvature, and request that they are independent of the choice of \( s \). Thus, we should understand how the gradings, the covariant derivatives and Rho tensors change under the change of the Weyl structures.

Considering two Weyl structures as reductions \( \bar{s}_1, \bar{s}_2 : G_0 \to G \), clearly there must be a function \( \Upsilon : G_0 \to \mathfrak{p}_+ \) such that for all \( u \in G_0, g_0 \in G_0 \),

\[
\bar{s}_1(u) = s_1(u) \cdot \exp \Upsilon(u),
\]

\[
\bar{s}_2(u g_0) = s_1(u) \cdot \exp \Upsilon(u) g_0 = s_1(u) g_0 \cdot \exp(A d_{s_1} \Upsilon(u)).
\]

Thus the function \( \Upsilon \) represents a one-form on \( M \) and, fixing \( s_1 \), this is a bijective correspondence between \( \Upsilon \in \Omega^1(M) \) and all Weyl structures. This reflects the affine bundle structure of \( A \). Of course, these one-forms are also represented as functions \( \Upsilon : G \to \mathfrak{p}_+ \) with the right equivariance property, and then the functions \( f = \exp \Upsilon : G \to P/G_0 \simeq P_+ \) with the equivariance property \( f(u \cdot (g_0 g_0')) = g_0^{-1} g_0'^{-1} f(u) g_0 \) can be directly seen as the corresponding sections \( s : M \to A = G \times_P P/G_0 \). See [7, Proposition 2.2] for a detailed exposition.

The formulae for the transformations in terms of the functions \( \Upsilon \) are explained in detail in [12, Sections 5.1.6-9]. They look pretty complicated and we shall not need them explicitly here. Just in the simplest case of trivial filtrations (i.e., \( |1| \)-graded \( \mathfrak{g} \)) we obtain for vector fields \( \xi \) on \( M \) and sections \( \sigma \) of irreducible natural bundles (the hat indicates the transformed objects)

\[
\hat{\nabla}_\xi \sigma = \nabla_\xi \sigma - \{ \Upsilon, \xi \} \bullet \sigma
\]

\[
\hat{P}(\xi) = P(\xi) + \nabla_\xi \Upsilon + \frac{1}{2} \text{ad}(\Upsilon)^2(\xi).
\]
Differentiating (9) again, derivatives of $\Upsilon$ appear, and the relatively nice transformation rule (10) indicates a chance to balance the formulae by adding ‘correction terms’ including $P$ in order to keep the transformation rules algebraic in the parameters $\Upsilon$. This is exactly what happens with the Rho-corrected derivatives appearing in Proposition 2.1.

About a hundred years back this was the motivation for considering Schouten’s Rho tensor in conformal Riemannian geometry.

In the sequel we shall consider polynomial invariant operators $\Psi_s : \Gamma(VM) \to \Gamma(WM)$, constructed by a fixed universal expression $\Psi$ from from $\nabla^s$, its curvature and its torsion in the way discussed in Section 2.7. We say that $\Psi$ is a nearly invariant operator if the transformation formula for $\Psi_s$ under the change of the Weyl structure $s$ is tensorial in $\Upsilon$. Otherwise put, the operator $\Psi$ is nearly invariant if and only if for Weyl structures $s_1, s_2$ and a point $x \in M$ such that $s_1(x) = s_2(x)$ we get $\Psi_{s_1} \sigma(x) = \Psi_{s_2} \sigma(x)$ for every $\sigma \in \Gamma(VM)$.

Theorem 2.7 shows that, via $\Phi \mapsto \Phi^s$, any polynomial affine invariant of the canonical connection $D^-$ on $A$ gives rise to a nearly invariant operator on $M$.

3.2. Derivatives of the Rho-tensor. The final key step towards understanding nearly invariant operators is analogous to the well known fact that in affine differential invariants one may use symmetrized iterated covariant derivatives rather then iterated covariant derivatives. We need an analogous symmetrization argument for iterated covariant derivatives of the Rho tensor associated to a Weyl structure. This will allow us to effectively use normal Weyl structures in the study of nearly invariant operators.

Recall from Section 1.5 that a characteristic of normal Weyl structures is the vanishing of certain symmetrizations of iterated covariant derivatives of the Rho-tensor. We will refer to these as form-symmetrized iterated derivatives. Fixing a Weyl-structure $s$ with Weyl-tensor $P$ and Weyl connections $\nabla$, we view $P$ as a one-form on $M$ with values in $\text{gr}(T^*M)$. The $k$th covariant derivative $(\nabla)^k P$ is then a section of $\otimes^{k+1} T^*M \otimes T^*M$ (or the associated graded of this bundle). Now we define the $k$th form symmetrized Weyl-derivative $\mathcal{F}^k(P) \in \Gamma(\text{gr}(S^{k+1} T^*M \otimes T^*M))$ as the symmetrization of $(\nabla^s)^k P$ over the first $k+1$ indices. For convenience, we put $\mathcal{F}^0(P) = P$.

3.3. Theorem. For any $k$, there is a universal expression for $(\nabla)^k P - \mathcal{F}^k(P)$ obtained via bundle maps induced by $\text{Gr}_0$-equivariant maps from tensor products whose factors are of the form $\mathcal{F}^l(P)$ and $(\nabla^s)^K$ for $l < k$, where $K$ is a component of the pullback of the Cartan curvature along $s$.

Proof. Throughout this proof, we write $\nabla$ for $\nabla^s$. We start by computing $(\nabla P - \mathcal{F}^1(P))(\xi, \eta)$ for vector fields $\xi, \eta \in \mathfrak{X}(M)$ via

$$\nabla_\xi P(\eta) - \nabla_\eta P(\xi) = \nabla_\xi(P(\eta)) - P(\nabla_\xi \eta) - \nabla_\eta(P(\xi)) + P(\nabla_\eta \xi).$$

The definition of the torsion of a Weyl connection reads as

$$T(\xi, \eta) = \nabla_\xi \eta - \nabla_\eta \xi - [\xi, \eta] + \{\xi, \eta\},$$

so we can insert this to rewrite the terms in which a derivative goes into Rho. Moreover, from section 1.4, we know that

$$Y^s(\xi, \eta) = \nabla_\xi(P(\eta)) - \nabla_\eta(P(\xi)) - P([\xi, \eta]) + P(\{\xi, \eta\}) + \{P(\xi), P(\eta)\}$$

can be recovered from the pullback of the Cartan curvature and a component of $P$. Together with the above, we conclude that

$$\nabla P = \mathcal{F}^1(P) + \frac{1}{2} Y - \frac{1}{2} [T, P] - \frac{1}{2} \{P, P\}.$$
This proves the theorem in the case \( k = 1 \). To complete the proof we show by induction on \( k \) that \( \nabla F^k(P) - F^{k+1}(P) \) admits a universal expansion as claimed in theorem with \( \ell \leq k \). Since, by induction, one Weyl derivative of any term allowed in step \( k \) leads to a term allowed in step \( k + 1 \), this recursively implies that claim of the theorem.

The case \( k = 0 \) has been sorted out above, so we assume that \( k > 0 \) and our claim has been proved \( \ell < k \).

Since \( F^{k-1}(P) \in \Gamma(S^k T^* M \otimes T^* M) \) is symmetric in its first \( k \)-indices, we can compute \( F^kP \) as \( \frac{1}{k+1} \sum_{i=0}^{k} \nabla a_i F a_0 ... a_i a_k b \). Using symmetry of \( F \) to rewrite \( \nabla a_i F a_0 ... a_k b \), we conclude that we can rewrite \( \nabla F^k(P) - F^{k+1}(P) \) as

\[
\frac{1}{k+1} \sum_{i=1}^{k} \left( \nabla a_0 F a_1 ... a_i a_k b - \nabla a_i F a_0 a_1 ... a_k b \right).
\]

If \( k = 1 \), then the terms \( F \) already are of the form \( \nabla P \). For \( k > 1 \), the induction hypothesis implies that, up to terms of the form we allow in our expansions, we may replace the occurrences of \( F \) by \( \nabla F^{k-2}(P) \). But then each term in our sum becomes twice an alternation of a double covariant derivative of \( F^{k-2} \). (Recall that \( F^0(P) = P \).

But now for any tensor field \( t \), the alternation of \( \nabla^2 t \) maps \( \xi, \eta \in X(M) \) to

\[
\nabla \xi \nabla \eta t - \nabla \xi \eta t - \nabla \eta \xi t + \nabla \xi \eta t.
\]

Inserting from (11) and rearranging terms, we see that this can be rewritten as

\[
R(\xi, \eta) \bullet t - \nabla_{J(\xi, \eta)} t + \nabla (\xi, \eta) t,
\]

where in the first term the bullet denotes the tensorial action of the curvature. Now the last two terms in (13) are obtained from \( G_0 \)-equivariant operations acting on \( T \otimes \nabla t \) and of \( \nabla t \), respectively, and \( T \) can be obtained from the pullback of the Cartan curvature and from \( P \). In our situation \( t = F^{k-2}(P) \), so by induction we can rewrite \( \nabla t \) as a sum of terms of the allowed forms, so we see that these two summands altogether only produce terms of the allowed form.

For the first terms in (13), we can rewrite the curvature \( R \) of \( \nabla = \nabla^\bullet \) according to section 1.4 as \( W - (\partial P)_{g_0} \), where \( W \) is a component of the pullback of the Cartan curvature. But then the resulting terms can be obtained via \( G_0 \)-equivariant maps from \( W \otimes t \) respectively from \( P \otimes t \) and since \( t \) equals \( F^{k-2}(P) \), this leads to allowed terms only.

\( \square \)

3.4. Nearly invariant operators. Let us collect the information on affine invariants of the Weyl connections we have obtained so far. As discussed in Section 3.1, such an operator comes from a universal expression \( \Psi \) which involves bundle maps induced from \( G_0 \)-equivariant linear maps and certain tensor products. If the operator acts on \( \Gamma(V M) \) these tensor products contain factors that are iterated covariant derivatives of either a section \( \sigma \in \Gamma(V M) \) or a component of the curvature of the involved connection (for which we will insert all Weyl connections). Now as discussed in Sections 1.4 and 2.6, we can universally decompose the curvature and torsion of the Weyl connections into components of the pullback of the Cartan connection and into components of the Rho-tensor. Since these decompositions are
induced by $G_0$-equivariant maps, the same applies to iterated derivatives of (components of) the torsion and the curvature. Moreover, we can use Theorem 3.3 to rewrite iterated derivatives of the Rho-tensor via components of the pullback of the Cartan curvature and form-symmetrized iterated derivatives of the Rho-tensor and as before, this extends to components of the Rho-tensor. The upshot of this is that we may assume that all our factors in the tensor products are

- iterated covariant derivatives of $\sigma \in \Gamma(VM)$
- iterated covariant derivatives of components of the pullback of the Cartan curvature
- components of form symmetrized iterated covariant derivatives of the Rho tensor.

Armed with this observation, we are ready to formulate and proof the main result of the paper.

3.5. Theorem. The nearly invariant operators are exactly the universal expansions obtained via $\Phi \mapsto \Phi^s$ from polynomial affine differential invariants of the natural covariant derivative $D^-$ and its curvature and torsion on $A$.

Proof. Theorem 2.7 shows that affine differential invariants of $D^-$ and the curvature and torsion of $D$ induce nearly invariant operators.

To prove the converse inclusion, let us take a universal expression $\Psi$ which gives rise to a nearly invariant operator which is formed in the way discussed above. Now let $\Psi^1$ be the universal expression obtained by removing from $\Psi$ all terms in which the tensor product contains a component of a form symmetrized covariant derivative of the Rho tensor. For each summand in this expression count the total number of covariant derivatives in all factors showing up in the tensor product. Let $k$ be the maximal number that occurs and let $\Psi_1$ be the sum of all terms in which the total number of covariant derivatives equals $k$. Then there is an obvious affine invariant $\Phi_1$ of $D^-$ on $A$ that corresponds to $\Psi_1$ (in which we simply replace all covariant derivatives by $D^-$). Via $\Phi_1 \mapsto \Phi_1^s$, we obtain a nearly invariant operator defined on $\Gamma(VM)$ and we subtract this from $\Psi$.

On the one hand, the result by construction is a nearly invariant operator $\tilde{\Psi}$. On the other hand, Theorem 2.5 implies that in $\Phi_1^s$, we obtain exactly the same terms involving a total number of $k$ covariant derivatives as in $\Psi_1$ while for all other terms, the total number of covariant derivatives that occur is strictly less than $k$. Otherwise put, the operator $\tilde{\Psi}$ has the property that in any term that does not contain a component of a form symmetrized iterated covariant derivative of the Rho tensor, there are less than $k$ covariant derivatives in total.

This shows that we can iterate our procedure by applying the same construction to $\tilde{\Psi}$. After finitely many steps, we arrive at an expression for which the terms that do not involve any components of form symmetrized iterated covariant derivatives of Rho also do not contain any covariant derivatives and hence evidently are obtained from an affine invariant on $A$.

Hence we conclude that our original operator $\Psi$ can be written as the sum of an operator of the form $\Phi \mapsto \Phi^s$ and a nearly invariant operator $\tilde{\Psi}$ for which any term involves a component of a form symmetrized iterated covariant derivative of Rho. But now we can easily conclude that the nearly invariant operator $\tilde{\Psi}$ vanishes identically. Indeed, take any Weyl structure $s$ and consider $\Psi_s(\sigma)$ and a point $x \in M$. Then from Section 1.5 we know that we can find a Weyl structure $\tilde{s}$ which is normal at $x$ and satisfies $\tilde{s}(x) = s(x)$. But by normality, any term in $\Psi_{\tilde{s}}$ vanishes at $x$ and hence $0 = \tilde{\Psi}_{\tilde{s}}(\sigma)(x)$, and this equals $\Psi_{\tilde{s}}(\sigma)(x)$ since $\tilde{\Psi}$ is nearly invariant. This completes the proof. □
3.6. Remark. In our discussion, we have focused on the case of normal parabolic geometries, which equivalently encode some underlying structure and hence provides applications to invariants of that structure. The concept of Weyl structures as well as the basic calculus we develop here are available for general Cartan geometries and basically also the results we have proved above allow for extensions to this more general setting. The basic difference is that the relation between the pullback $s^*\kappa$ of the Cartan curvature, the curvature of the Weyl connection $\nabla^s$ determined by $s$ and the Rho-tensor $\Psi^s$ becomes more complicated. So to extend the results, one would have to modify the definitions of affine invariants that are used in Theorems 2.7 and 3.5 appropriately. We do not go into details on this here.

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A.C.: Faculty of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, 1090 Wien, Austria
J.S.: Department of Mathematics and Statistics, Masaryk University, Kotlářská 2a, 611 37 Brno, Czech Republic
Email address: Andreas.Cap@univie.ac.at, slovak@math.muni.cz