Further results regarding the degree resistance distance of cacti

Jia-Bao Liu, Wen-Rui Wang, Yong-Ming Zhang, Xiang-Feng Pan

\textsuperscript{a} School of Mathematical Sciences, Anhui University, Hefei 230601, P. R. China
\textsuperscript{b} Department of Public Courses, Anhui Xinhua University, Hefei 230088, P. R. China

Abstract

A graph $G$ is called a cactus if each block of $G$ is either an edge or a cycle. Denote by $Cact(n; t)$ the set of connected cacti possessing $n$ vertices and $t$ cycles. In this paper, we show that there are some errors in [J. Du, G. Su, J. Tu, I. Gutman, The degree resistance distance of cacti, Discrete Appl. Math. 188 (2015) 16-24.], and we present some results which correct their mistakes. We also give the second-minimum and third-minimum degree resistance distances among graphs in $Cact(n; t)$, and characterize the corresponding extremal graphs as well.

AMS subject classifications: 05C12, 05C90

Keywords: Cactus; Resistance distance; Degree resistance distance; Kirchhoff index

1 Introduction

The graphs considered in this paper are finite, loopless, and contain no multiple edges. Given a graph $G$, let $V(G)$ and $E(G)$ be, respectively, its vertex and edge sets. The ordinary distance $d(u, v) = d_G(u, v)$ between the vertices $u$ and $v$ of the graph $G$ is the length of the shortest path between $u$ and $v$ [1]. For other undefined notations and terminology from graph theory, the readers are referred to [1].

The Wiener index $W(G)$ is the sum of ordinary distances between all pairs of vertices, that is, $W(G) = \sum_{\{u, v\} \subseteq V(G)} d(u, v)$. It is the oldest and one of the most thoroughly studied distance-based graph invariant. A modified version of the Wiener index is the degree distance defined as $D(G) = \sum_{\{u, v\} \subseteq V(G)} [d(u) + d(v)]d(u, v)$, where $d(u) = d_G(u)$ is the degree of the vertex $u$ of the graph $G$.

In 1993, Klein and Randić [2] introduced a new distance function named resistance distance, based on the theory of electrical networks. They viewed $G$ as an electric network $N$ by replacing each edge of $G$ with a unit resistor. The resistance distance between the vertices $u$ and $v$ of the graph $G$, denoted by $R(u, v)$, is then defined to be the effective resistance between the nodes $u$ and $v$.
v in N. If the ordinary distance is replaced by resistance distance in the expression for the Wiener index, one arrives at the Kirchhoff index [2, 3]

$$K_f(G) = \sum_{\{u,v\} \subseteq V(G)} R(u, v)$$

which has been widely studied [4, 5, 6, 7, 8, 9, 10]. In 1996, Gutman and Mohar [11] obtained the famous result by which a relationship is established between the Kirchhoff index and the Laplacian spectrum: $$K_f(G) = n \sum_{i=1}^{n-1} \frac{1}{\mu_i}$$, where $$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$$ are the eigenvalues of the Laplacian matrix of a connected graph G with n vertices. For more details on the Laplacian matrix, the readers are referred to [12, 13]. Bapat et al. has provided a simple method for computing the resistance distance in [14]. Palacios [15, 16, 17, 18, 19, 20] studied the resistance distance and the Kirchhoff indices of connected undirected graphs with probability methods. E. Bendito et al. [21] formulated the Kirchhoff index based on discrete potential theory. M. Bianchi et al. obtained the upper and lower bounds for the Kirchhoff index of an arbitrary simple connected graph G by using a majorization technique [31]. Besides, the Kirchhoff indices of some lattices are investigated in [23, 24, 25, 26, 27]. Similarly, if the ordinary distance is replaced by resistance distance in the expression for the degree distance, then one arrives at the degree resistance distance

$$D_R(G) = \sum_{\{u,v\} \subseteq V(G)} [d(u) + d(v)]R(u, v).$$

Palacios [28] named the same graph invariant additive degree-Kirchhoff index.

Tomescu [29] determined the unicyclic and bicyclic graphs with minimum degree distance. In [30], the author investigated the properties of connected graphs having minimum degree distance. Bianchi et al. [31] gave some upper and lower bounds for $$D_R$$ whose expressions do not depend on the resistance distances. Yang and Klein gave formulae for the degree resistance distance of the subdivisions and triangulations of graphs [32]. For more work on $$K_f(G)$$, the readers are referred to [33, 34, 35, 36, 37, 38].

A graph G is called a cactus if each block of G is either an edge or a cycle. Denote by Cact(n; t) the set of cacti possessing n vertices and t cycles [39, 40]. In this paper, we determine the minimum degree resistance distance among graphs in Cact(n; t) and characterize the corresponding extremal graphs.

2 Preliminaries

Let $$R_G(u, v)$$ denote the resistance distance between u and v in the graph G. Recall that $$R_G(u, v) = R_G(v, u)$$ and $$R_G(u, v) \geq 0$$ with equality if and only if $$u = v$$.

For a vertex u in G, we define

$$K_f(v) = \sum_{u \in G} R_G(u, v) \quad \text{and} \quad D_v(G) = \sum_{u \in G} d_G(u)R_G(u, v).$$
In what follows, for the sake of conciseness, instead of $u \in V(G)$ we write $u \in G$. By the definition of $D_v(G)$, we also have

$$D_R(G) = \sum_{v \in G} d_G(v) \sum_{u \in G} R_G(u, v).$$

**Lemma 2.1 ([6]).** Let $G$ be a connected graph with a cut-vertex $v$ such that $G_1$ and $G_2$ are two connected subgraphs of $G$ having $v$ as the only common vertex and $V(G_1) \cup V(G_2) = V(G)$. Let $n_1 = |V(G_1)|$, $n_2 = |V(G_2)|$, $m_1 = |E(G_1)|$, $m_2 = |E(G_2)|$. Then

$$D_R(G) = D_R(G_1) + D_R(G_2) + 2m_2 K_f(G_1) + 2m_1 K_f(G_2) + (n_2 - 1)D_v(G_1) + (n_1 - 1)D_v(G_2).$$

**Definition 2.1 ([3]).** Let $v$ be a vertex of degree $p + 1$ in a graph $G$, such that $vv_1, vv_2, \ldots, vv_p$ are pendant edges incident with $v$, and $u$ is the neighbor of $v$ distinct from $v_1, v_2, \ldots, v_p$. We form a graph $G' = \sigma(G, v)$ by deleting the edges $vv_1, vv_2, \ldots, vv_p$ and adding new edges $uv_1, uv_2, \ldots, uv_p$. We say that $G'$ is a $\sigma$-transform of the graph $G$.

The $\sigma$-transform at $v$

**Lemma 2.2 ([6]).** Let $G' = \sigma(G, v)$ be a $\sigma$-transform of the graph $G$, $d_G(u) \geq 1$. Then $D_R(G) \geq D_R(G')$. Equality holds if and only if $G$ is a star with $v$ as its center.

**Lemma 2.3 ([6]).** Let $C_k$ be the cycle of size $k$, and $v \in C_k$. Then, $Kf(C_k) = \frac{k^3 - k}{12}, D_v(C_k) = \frac{k^3 - k}{3}, Kf_v(C_k) = \frac{k^2 - 1}{6}$ and $D_v(C_k) = \frac{k^2 - 1}{3}$.

**Definition 2.2 ([3]).** Let $G \in Cact(n;t)$, $t \geq 2$. A cycle $C$ of $G$ is said to be an end cycle if there is a unique vertex $v$ in $C$ which is adjacent to a vertex in $V(G) \setminus V(C)$. This unique vertex $v$ in $C$ is called the anchor of $C$.

**Lemma 2.4 ([3]).** Let $G \in Cact(n;t)$, $t \geq 2$, be a cactus without cut edges. Let $C$ be an end cycle of $G$ and $v$ be its anchor. Let $u$ be a vertex of $C$ different from $v$. The graphs $G_1$ and $G_2$ are constructed by adding $r$ pendant edges to the vertices $u$ and $v$, respectively. Then $G_R(G_1) > G_R(G_2)$.
3 Some errors in [3] and corrections

In [3], J. Du, G. Su, J. Tu, I. Gutman proved that $G^0(n; t)$ is the unique element of $Cact(n; t)$, $t \geq 1$, having minimum degree resistance distance. Unfortunately, there are some computational errors in the process of the proof. We shall list the errors in [3] as Errors 3.1, 3.2 below.

Error 3.1 (Lemma 7 in [3])

$$D_R(C_h) - D_R(S) = \frac{h^2 - 8h + 3}{3}$$ and $|V(H)| - 1 = n - h - 1$.

Counterexample 1

If $h = 4$, according to the Lemma 7 in [3], the result is $-\frac{13}{3}$ and $n - 5$. In fact, the correct result is $-\frac{10}{3}$ and $n - 4$, which arrives at a contradiction.

Correction of Lemma 7 in [3]

Let $G = (V, E)$ be a graph belonging to $Cact(n; t)$, $t \geq 3$. Let $C_h$ be a cycle with $h \geq 4$ vertices, contained in $G$. Let there be a unique vertex $u$ in $C_h$ which is adjacent to a vertex in $V(G) \setminus V(C)$. Assuming that $uv, vw \in E(C)$, construct a new graph $G^* = G - vw + uw$ as shown in the following figure. Then, $D_R(G) > D_R(G^*)$. 
Let $S$ be the graph obtained by attaching to the vertex $u$ of $C_{h-1}$ the pendent vertex $v$. $D_R(C_h) - D_R(S) = \frac{h^2 - 8h + 6}{3}$ and $|V(H)| - 1 = n - h$.

Using Lemma 1, we have

$$D_R(G) = D_R(C_h) + D_R(H) + 2|E(H)|Kf_u(C_h) + 2hKf_u(H) + (|V(H)| - 1)D_u(C_h) + (h - 1)D_u(H),$$

$$D_R(G^*) = D_R(S) + D_R(H) + 2|E(H)|Kf_u(S) + 2hKf_u(H) + (|V(H)| - 1)D_u(S) + (h - 1)D_u(H).$$

Then

$$D_R(G) - D_R(G^*)$$

$$= D_R(C_h) - D_R(S) + 2(n + t - 1 - h)[Kf_u(C_h) - Kf_u(S)] + (n - h)[D_u(C_h) - D_u(S)]$$

$$= \frac{h^2 - 8h + 6}{3} + 2(n + t - 1 - h)\frac{2h - 7}{6} + (n - h)\frac{2h - 4}{3}$$

$$= \frac{h^2 - 8h + 6}{3} + (n - 1 - h)\frac{4h - 11}{3} + t\frac{2h - 7}{3} + \frac{2h - 4}{3}$$

$$\geq \frac{h^2 - 19}{3} + (n - 1 - h)\frac{4h - 11}{3} \quad \text{(by } t \geq 3).$$

If $h = 4$, then $D_R(G) - D_R(G^*) \geq \frac{5}{3}n - \frac{28}{3} > 0$.

If $h \geq 5$, then $D_R(G) - D_R(G^*) > (n - 1 - h)\frac{4h - 11}{3} > 0$.

This completes the proof.

**Error 3.2 (Theorem 1 in [3])**

$$D_R(G^0(n, t)) = -\frac{4}{3}t^2 - (\frac{8}{3}n - 6)t + 3n^2 - 7n + 4.$$

**Counterexample 2**

If $n = 5, t = 1$, according to the Theorem 1 in [3], the result is 50. In fact, the correct result is $44\frac{2}{3}$, which also arrives at a contradiction.

**Correction of Error 3.2**

It is obvious that the $D^0(n, t)$ consists of $n C_3$ and an $S_{n-2t}$, in which $n C_3$ and an $S_{n-2t}$ have
a common vertex \( v_1 \). Using Lemma 1, we have
\[
D_R(G^0(n,t)) = tD_R(C_3) + D_R(S_{n-2t}) + 2t(n + t - 4)Kf_{v_1}(C_3) + 6tKf_{v_1}(S_{n-2t}) + t(n - 3)Dv_1(C_3) + 2tDv_1(S_{n-2t})
\]
\[= 8t + (n - 2t)(n - 2t - 1) + 2(n - 2t - 1)(n - 2t - 2) + \frac{8}{3}t(n + t - 4) + 6t(n - 2t - 1) + \frac{8}{3}t(n - 3) + 2t(n - 2t - 1)
\]
\[= \frac{4}{3}t^2 + (\frac{4}{3}n - \frac{14}{3})t + 3n^2 - 7n + 4.
\]

In the following we shall consider the cacti with the second and the third-minimum degree resistance distances.

4 The second-minimum degree resistance distance

By Lemmas 2.2, 2.4 and Theorem 7 in [3], one can conclude that \( G \) which has the second-minimum degree resistance distance in \( Cact(n;t) \) must be one of the graphs \( G_3, G_4, \) and \( G_5 \) as shown in the Figure 1.

**Theorem 4.1** Among all graphs in \( Cact(n,t) \) with \( n \geq 7 \) and \( t \geq 1 \), the cactus with the second-minimum degree resistance distance is \( G_5 \).

**Proof.** (i): Let \( H_1 \) denote the common subgraph of \( G_3 \) and \( G^0(n,t) \). Thus, we can view graphs \( G_3 \) and \( G^0(n,t) \) as the graphs depicted in Figure 2.
Using Lemma 1, we have
\[ D_R(G^0(n, t)) = D_R(H_1) + D_R(S_3) + 4Kf_{v_1}(H_1) + 2(n + t - 3)Kf_{v_1}(S_3) + 2D_{v_1}(H_1) + (n - 3)D_{v_1}(S_3), \]
\[ D_R(G_3) = D_R(H_1) + D_R(P_3) + 4Kf_{v_1}(H_1) + 2(n + t - 3)Kf_{v_1}(P_3) + 2D_{v_1}(H_1) + (n - 3)D_{v_1}(P_3). \]
Here \( Kf_{v_1}(S_3) = 2, Kf_{v_1}(P_3) = 3, D_{v_1}(S_3) = 2, D_{v_1}(P_3) = 4. \)
Therefore,
\[ D_R(G_3) - D_R(G^0(n, t)) = 2(n + t - 3)(Kf_{v_1}(P_3) - Kf_{v_1}(S_3)) + (n - 3)(D_{v_1}(P_3) - D_{v_1}(S_3)) \]
\[ = 2(n + t - 2) + 2(n - 3) \]
\[ = 4n + 2t - 12. \]

(ii): Let \( H_2 \) denote the common subgraph of \( G_3 \) and \( G^0(n, t) \). Thus, we can view graphs \( G_3 \) and \( G^0(n, t) \) as the graphs depicted in Figure 3.

![Figure 2](image1)

Using Lemma 1, we have
\[ D_R(G^0(n, t)) = D_R(H_2) + D_R(P_2) + 2Kf_{v_1}(H_2) + 2(n + t - 2)Kf_{v_1}(P_2) + D_{v_1}(H_2) + (n - 2)D_{v_1}(P_2), \]
\[ D_R(G_4) = D_R(H_2) + D_R(P_2) + 2Kf_{v_2}(H_2) + 2(n + t - 2)Kf_{v_2}(P_2) + D_{v_2}(H_2) + (n - 2)D_{v_2}(P_2). \]
Here
\[ Kf_{v_1}(H_2) = n - \frac{2}{3}t - 2, Kf_{v_2}(H_2) = \frac{5}{3}n - \frac{2}{3}t - \frac{14}{3}, \]
\[ D_{v_1}(H_2) = n + \frac{2}{3}t - 2, D_{v_2}(H_2) = \frac{7}{3}n + 2t - \frac{26}{3}. \]
Therefore,
\[ D_R(G_4) - D_R(G^0(n, t)) = 2(Kf_{v_2}(H_2) - Kf_{v_1}(H_2)) + D_{v_2}(H_2) - D_{v_1}(H_2) \]
\[ = 2\left(\frac{2}{3}n - \frac{8}{3}\right) + \left(\frac{4}{3}n + \frac{4}{3}t - \frac{26}{3}\right) \]
\[ = \frac{8}{3}n + \frac{4}{3}t - 12. \]

(iii): Let \( H_2 \) denote the common subgraph of \( G_5 \) and \( G^0(n, t) \). Thus, we can represent these graphs as follows in Figure 4.
Using Lemma 1, we have

\[ D_R(G_0(n, t)) = D_R(H_3) + 8K_{v_1}(H_3) + 2(n + t - 5)K_{v_1}(S^3_4) + 3D_{v_1}(H_3) + (n - 4)D_{v_1}(S^3_4), \]

\[ D_R(G_5) = D_R(H_3) + D_R(C_4) + 8K_{v_1}(H_3) + 2(n + t - 5)K_{v_1}(C_4) + 3D_{v_1}(H_3) + (n - 4)D_{v_1}(C_4). \]

Here \( D_R(C_4) = \frac{70}{3}, \) \( D_R(S^3_4) = 20, \) \( K_{v_1}(C_4) = \frac{7}{3}, \) \( K_{v_1}(S^3_4) = \frac{5}{2}, \) \( D_{v_1}(C_4) = \frac{11}{7}, \) \( D_{v_1}(S^3_4) = 5. \)

Therefore,

\[
D_R(G_5) - D_R(G_0(n, t)) = D_R(C_4) - D_R(S^3_4) + 2(n + t - 5)(K_{v_1}(C_4) - K_{v_1}(S^3_4)) \\
+ (n - 4)(D_{v_1}(C_4) - D_{v_1}(S^3_4)) \\
= -\frac{10}{3} + \frac{1}{3}(n + t - 5) + \frac{4}{3}(n - 4) \\
= \frac{5}{3}n + \frac{t}{3} - \frac{31}{3}.
\]

By the above expressions for the degree resistance distances of \( G_3, G_4 \) and \( G_5, \) we immediately have the desired result.

From Theorem 4.1 we immediately have the following result.

**Corollary 4.2** For a graph \( G, \) not isomorphic to \( G_0(n, t), \) in \( \text{Cact}(n, t) \) with \( n \geq 7 \) and \( t \geq 1, \) it holds that \( D_R(G) \geq -\frac{10}{3} + \frac{1}{3}(n + t - 5) + \frac{4}{3}(n - 4) \), with equality if and only if \( G \cong G_5. \)

## 5 The third-minimum degree resistance distance

By the same reasonings as was used in Theorem 4.1, we conclude that the possible candidates having the third-minimum degree resistance distance must come from one of \( G_4, G_6 - G_{10}. \)

**Theorem 5.1** Among all graphs in \( \text{Cact}(n, t) \) with \( n \geq 25 \) and \( t \geq 1, \) the cactus with the third-minimum degree resistance distance is \( G_4. \)

**Proof.** By above discussions, we need only to determine the minimum cardinality among \( D_R(G_4), D_R(G_6), D_R(G_7), D_R(G_8), D_R(G_9) \) and \( D_R(G_{10}). \)
Let $H_4$ denote the common subgraph of $G_4$, $G_6$ and $G_7$. Thus, we can view graphs $G_4$, $G_6$ and $G_7$ as the graphs depicted in Figure 6.

Using Lemma 1, we have

\[
D_R(G_4) = D_R(H_4) + D_R(G_0) + 10Kf_{v_3}(H_4) + 2(n + t - 6)Kf_{v_3}(G_0) + 4D_{v_3}(H_4) + (n - 5)D_{v_3}(G_0),
\]

\[
D_R(G_6) = D_R(H_4) + D_R(S_5^4) + 10Kf_{v_3}(H_4) + 2(n + t - 6)Kf_{v_3}(S_5^4) + 4D_{v_3}(H_4) + (n - 5)D_{v_3}(S_5^4).
\]

Here $D_R(G_0) = \frac{142}{3}$, $D_R(S_5^4) = 43$, $Kf_{v_3}(G_0) = 4$, $Kf_{v_3}(S_5^4) = \frac{17}{4}$, $D_{v_3}(G_0) = 6$, $D_{v_3}(S_5^4) = \frac{15}{2}$.

Therefore,

\[
D_R(G_6) - D_R(G_4) = D_R(S_5^4) - D_R(G_0) + 2(n + t - 6)(Kf_{v_3}(S_5^4) - Kf_{v_3}(G_0))
\]
\[
+ (n - 5)(D_{v_3}(S_5^4) - D_{v_3}(G_0))
\]
\[
= -\frac{13}{3} + \frac{1}{2}(n + t - 6) + \frac{3}{2}(n - 5)
\]
\[
= 2n + \frac{t}{2} - \frac{89}{6} > 0.
\]
Using Lemma 1, we have

\[ \text{DR}(G_7) = \text{DR}(H_4) + \text{DR}(S_6^4) + 10Kf_{v_3}(H_4) + 2(n + t - 6)Kf_{v_3}(S_5^4) + 4D_{v_3}(H_4) + (n - 5)D_{v_3}(S_5^4). \]

Here \( Kf_{v_3}(S_6^4) = \frac{9}{2}, D_{v_3}(S_6^4) = 8. \)

Therefore,

\[ \text{DR}(G_7) - \text{DR}(G_6) = \frac{1}{2}(n + t - 6) + \frac{1}{2}(n - 5) = n + \frac{1}{2}t - \frac{11}{2} > 0. \]

Then \( \text{DR}(G_7) > \text{DR}(G_6) > \text{DR}(G_4). \)

Similar to the relationship between \( \text{DR}(G_5) \) and \( \text{DR}(G_0(n,t)) \), we have

\[ \text{DR}(G_8) = \text{DR}(G_5) - \text{DR}(G_0(n,t)) = \frac{5}{3}n + \frac{t}{3} - \frac{31}{3}. \]

Therefore,

\[ \text{DR}(G_8) - \text{DR}(G_4) = \frac{2}{3}n - \frac{2}{3}t - \frac{26}{3}. \]

Because of \( t \leq \frac{n-1}{2} \), when \( n \geq 25 \), \( \text{DR}(G_8) - \text{DR}(G_4) > 0. \)

Similar to the relationship between \( \text{DR}(G_5) \) and \( \text{DR}(G_0(n,t)) \), we have

\[ \text{DR}(G_9) - \text{DR}(G_4) = \frac{5}{3}n + \frac{t}{3} - \frac{31}{3}. \]

Therefore,

\[ \text{DR}(G_9) - \text{DR}(G_8) = n + t - \frac{5}{3} > 0. \]

Then \( \text{DR}(G_9) > \text{DR}(G_8) > \text{DR}(G_4). \)

Similar to the method of \( \text{DR}(G_5) - \text{DR}(G_0(n,t)) \), we have

\[ \text{DR}(G_{10}) = \text{DR}(C_5) - \text{DR}(S_6^4) + 2(n + t - 6)(Kf_{v_1}(C_5) - Kf_{v_1}(S_5^4)) + (n - 5)(D_{v_1}(C_5) - D_{v_1}(S_5^4)) = -3 + (n + t - 6) + 2(n - 5) = 3n + t - 19 > 0. \]

Then \( \text{DR}(G_{10}) > \text{DR}(G_5) > \text{DR}(G_4). \)

By the above several inequalities, we immediately have the desired result.
From Theorem 5.1 we immediately have the following result.

**Corollary 5.2** For a graph $G$, not isomorphic to $G^0(n,t), G_5$, in $Cact(n,t)$ with $n \geq 25$ and $t \geq 1$, it holds that $D_R(G) \geq -\frac{4}{3}t^2 + \left(\frac{4}{3}n - \frac{10}{3}\right)t + 3n^2 - \frac{13}{3}n - 8$, with equality if and only if $G \cong G_4$.

**Acknowledgments**

The work of J. B. Liu was partly supported by the Natural Science Foundation of Anhui Province of China under Grant No. KJ2013B105 and the National Science Foundation of China under Grant Nos. 11471016 and 11401004; The work of X. F. Pan was partly supported by the National Science Foundation of China under Grant Nos. 10901001, 11171097, and 11371028.

**References**

[1] J.A. Bondy, U.S.R. Murty, Graph Theory with Applications Macmillan Press, New York, 1976.

[2] D.J. Klein, M. Randić, Resistance distance, J. Math. Chem. 12 (1993) 81-95.

[3] J. Du, G. Su, J. Tu, I. Gutman, The degree resistance distance of cacti, Discrete Appl. Math. 188 (2015) 16-24.

[4] X. Gao, Y. Luo, W. Liu, Resistance distances and the Kirchhoff index in Cayley graphs, Discrete Appl. Math. 159 (2011) 2050-2057.

[5] X. Gao, Y. Luo, W. Liu, Kirchhoff index in line, subdivision and total graphs of a regular graph, Discrete Appl. Math. 160 (2012) 560-565.

[6] I. Gutman, L. Feng, G. Yu, On the degree resistance distance of unicyclic graphs, Trans. Comb. 1 (2) (2012) 27-40.

[7] A. Ilić, D. Stevanović, L. Feng, G. Yu, P. Dankelmann, Degree distance of unicyclic and bicyclic graphs, Discrete Appl. Math. 159 (2011) 779-788.

[8] L.H. Feng, G. Yu, K. Xu, Z. Jiang, A note on the Kirchhoff index of bicyclic graphs, Ars Comb. 114 (2014) 33-40.

[9] L.H. Feng, G. Yu, W. Liu, Further results regarding the degree Kirchhoff index of a graph, Miskolc Mathematical Notes, 15 (2014), 97-108.

[10] C. Bu, B. Yan, X. Zhou, J. Zhou, Resistance distance in subdivision-vertex join and subdivision-edge join of graphs, Linear Algebra Appl. 458 (2014) 454-462.

[11] I. Gutman, B. Mohar, The quasi-Wiener and the Kirchhoff indices coincide, J. Chem. Inf. Computer Sci. 36 (1996) 982-985.

[12] J.B. Liu, X.F. Pan, Asymptotic incidence energy of lattices, Physica A 422 (2015) 193-202.

[13] J.B. Liu, X.F. Pan, F.T. Hu, F.F. Hu, Asymptotic Laplacian-energy-like invariant of lattices, Appl. Math. Comput. 253 (2015) 205-214.

[14] R.B. Bapat, I. Gutman, W. Xiao, A simple method for computing resistance distance, Z. Naturforsch. 58a, (2003) 494-498.

[15] J.L. Palacios, J.M. Renom, Bounds for the Kirchhoff index of regular graphs via the spectra of their random walks, Int. J. Quantum Chem. 110(9) (2010) 1637-1641.
[16] J.L. Palacios, Resistance distance in graphs and random walks, Int. J. Quantum Chem. 81 (2001) 29-33.

[17] J.L. Palacios, Foster’s formulas via probability and the Kirchhoff index, Methodol. Comput. Appl. 6 (4) (2004) 381-387.

[18] J.L. Palacios, J. M. Renom, Another look at the degree-Kirchhoff index, Int. J. Quantum Chem. 111 (14) (2011) 3453-3455.

[19] J.L. Palacios, On the Kirchhoff index of regular graphs, Int. J. Quantum Chem. 110 (7) (2010) 1307-1309.

[20] J.L. Palacios, On the Kirchhoff index of graphs with diameter 2, Discrete Appl. Math. 184 (2015) 196-201.

[21] E. Bendito, A. Carmona, A.M. Encinas, J.M. Gesto, A formula for the Kirchhoff index, Int. J. Quantum Chem. 108 (6) (2008) 1200-1206.

[22] M. Bianchi, A. Cornaro, J.L. Palacios, A. Torriero, Bounds for the Kirchhoff index via majorization techniques, J. Math. Chem. 51 (2) (2013) 569-587.

[23] S. Li, W.G. Yan, T. Tian, Some physical and chemical indices of the Union Jack lattice, J. Stat. Mech. P10004 (2015) 1-14.

[24] X.Y. Liu, W.G. Yan, The triangular kagomé lattices revisited, Physica A 392 (2013) 5615-5621.

[25] L.Z. Ye, On the Kirchhoff index of some toroidal lattices, Linear Multilinear A. 59 (6) (2011) 645-650.

[26] Z. Zhang, Some physical and chemical indices of clique-inserted lattices, J. Stat. Mech. Theory Exp. 10 (2013) P10004.

[27] J.B. Liu, X.F. Pan, J. Cao, F.F. Hu, A note on some physical and chemical indices of clique-inserted lattices, J. Stat. Mech. Theory Exp. 6 (2014) P06006.

[28] J.L. Palacios, Upper and lower bounds for the additive degree-Kirchhoff index, MATCH Commun. Math. Comput. Chem. 70 (2013) 651-655.

[29] I. Tomescu, Unicyclic and bicyclic graphs having minimum degree distance, Discrete Appl. Math. 156 (2008) 125-130.

[30] I. Tomescu, Properties of connected graphs having minimum degree distance, Discrete Math. 309 (2009) 2745-2748.

[31] M. Bianchi, A. Cornaro, J.L. Palacios, A. Torriero, New upper and lower bounds for the additive degree-Kirchhoff index, Croat. Chem. Acta 86 (2013) 363-370.

[32] Y. Yang, D.J. Klein, Resistance distance-based graph invariants of subdivisions and triangulations of graphs, Discrete Appl. Math. 181 (2015) 260-274.

[33] K. Xu, M. Liu, K.C. Das, I. Gutman, B. Furtula, A survey on graphs extremal with respect to distance-based topological indices, MATCH Commun. Math. Comput. Chem. 71 (2014) 461-508.

[34] Y. Yang, The Kirchhoff index of subdivisions of graphs, Discrete Appl. Math. 171 (2014) 153-157.

[35] Y. Yang, X. Jiang, Unicyclic graphs with extremal Kirchhoff index, MATCH Commun. Math. Comput. Chem. 60 (2008) 107-120.

[36] Y. Yang, H. Zhang, Some rules on resistance distance with applications, J. Phys. A 41 (2008) 445203 (12 pp).
[37] Y. Yang, D.J. Klein, A recursion formula for resistance distances and its applications, Discrete Appl. Math. 161 (2013) 2702-2715.

[38] J. Huang, S. Li, On the normalised Laplacian spectrum, degree-Kirchhoff index and spanning trees of graphs, Bull. Aust. Math. Soc. 91 (2015) 353-367.

[39] H. Wang, H. Hua, D. Wang, Cacti with minimum, second-minimum, and third-minimum Kirchhoff indices, Math. Commun. 15 (2010) 347-358.

[40] H. Liu, M. Lu, A unified approach to extremal cacti for different indices, MATCH Commun. Math. Comput. Chem. 58 (2007), 193-204.