Short Distance Freedom of Quantum Gravity

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Abstract

Fourth order derivative gravity in 3+1-dimensions is perturbatively renormalizable and is shown to describe a unitary theory of gravitons in a limited coupling parameter space. The running gravitational constant which includes graviton contribution is computed. Generically, gravitational Newton’s constant vanishes at short distances in this perturbatively renormalizable and unitary theory.

Keywords: Renormalization Group, Higher Derivative Gravity, Unitarity

1. Introduction and Theory

Newton’s gravitational constant $G$ in classical Einstein-Hilbert (EH) action is one of the smallest fundamental coupling constant with immense success in Cosmology [1, 2]. Yet it is less understood quantitatively when we quantize the theory. The primary reason being that in 3 + 1 space-time dimensions this coupling constant behaves as $(\text{Mass})^{-2}$ and the theory has ultra-violet (UV) divergences. It was known [3] that even quantum matter fields in the presence of background gravity demands that up to fourth order metric derivative terms need to be considered for UV regularization. Renormalization group (RG) studies of EH action [4] also suggest that we augment the gravitational action with fourth order derivative terms and witness UV renormalizability. Most general action functional up to fourth order derivative of metric $g_{\mu\nu}$ is given by $S$,

$$S = \int \frac{d^4x}{16\pi G} \left[ 2\Lambda - R + \frac{\omega R^2}{6M^2} - \frac{R_{\mu\nu}R^{\mu\nu}}{M^2} - \frac{R^2}{4M^2} \right],$$

where $R$ is the Ricci scalar and $R_{\mu\nu}$ is the Ricci tensor. The last term is also proportional to square of conformal Weyl tensor in 3 + 1 space-time dimensions, $\omega$ is dimensionless and $M$ has dimension of mass. It was first shown in [4] that the action $S$ is UV renormalizable using the $4 - \epsilon$ dimensional regularization scheme. The primary observation being the action is non-linear in metric and hence has many number of basic interactions between gravitons. However, in the UV regime the fluctuating metric field (in 3 + 1 space-time dimension) has zero mass dimension and all these terms due to general co-ordinate invariance can be tamed by renormalizing the coupling parameters in Eq. (1).

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In the action $S$, classically if $M^2 \to \infty$ and $\Lambda \to 0$, only the EH term dominates, thereby accounting for all the known success of EH action. A deeper look into quantizing the action using ADM formalism \[3\] showed that indeed $S$ corresponds to the well-known constraints, space-time translation and a host of other constraints. The Hamiltonian itself reduces to a surface integral just as in pure EH gravity. Furthermore, the Hamiltonian is shown to be positive if the last term in $S$ is absent and the coefficient of $R^2$ term is positive \[3, 8\]. It is also shown that in Minkowskian background metric, the theory describes massless spin-2 gravitons, a massive scalar and a massive spin-2 field.

Perturbation theory about a flat background $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ ($\eta_{\mu\nu} = \{1, -1, -1, -1\}$) was studied in \[3, 10\]. Here we investigate the action $S$ for $\Lambda = 0$ in the Landau gauge $\partial^\mu h_{\mu\nu} = 0$. The Feynman propagator in momentum space is,

\[
D_{\mu\nu,\alpha\beta} = \frac{i 16\pi G}{(2\pi)^4} \left[ \frac{(2P_2 - P_s)_{\mu\nu,\alpha\beta}}{q^2 + i\epsilon} + \frac{(P_s)_{\mu\nu,\alpha\beta}}{q^2 - M^2/\omega + i\epsilon} - \frac{2 (P_2)_{\mu\nu,\alpha\beta}}{q^2 - M^2 + i\epsilon} \right]
\]  

where $q$ is the momentum of fluctuating field $h_{\mu\nu}$. Various spin projectors are $(P_2)_{\mu\nu,\alpha\beta} = \frac{1}{2} [T_{\mu\alpha} T_{\nu\beta} + T_{\mu\beta} T_{\nu\alpha}] - \frac{1}{3} T_{\mu\nu} T_{\alpha\beta}$, $(P_s)_{\mu\nu,\alpha\beta} = \frac{1}{3} T_{\mu\nu} T_{\alpha\beta}$, where $T_{\mu\nu} = \eta_{\mu\nu} - q_\mu q_\nu/q^2$. The first term of Eq. (2) is the massless spin-2 graviton with two degrees of freedom, the second term is the scalar (Riccion) of mass $M/\sqrt{\omega}$ and the last term is that of massive spin-2 with mass $M$ ($M$-mode), which is in agreement with the general ADM analysis. Furthermore, the residues at the pole for graviton and Riccion are positive, while for the $M$-mode it is negative. The $M$-mode renders the theory non-unitary.

We now investigate the issue of the unitarity of the higher derivative gravitational action Eq. \(1\) in the context of background Minkowskian perturbation theory. The main observation being that in quantum theory, all the couplings cease to be constant and will start running with momentum $\mu$. Thus effectively at energy scale $\mu$ we have running couplings $G(\mu)$, $M^2(\mu)$ and $\omega(\mu)$. This is an efficient way to incorporate the quantum effects in the renormalizable theory. In such a context if $M^2(\mu)/\mu^2 > 1$ for all $\mu^2$, then in any scattering process involving gravitons and Riccions we will never encounter the $M$-mode particle as an intermediate state to any order in Feynman loop expansion. Consequently the negative norm does not violate unitarity of the scattering matrix in graviton and Riccion subspace (physical GR). To reiterate consider a very high energy graviton scattering off with another graviton or Riccion. In principal there is enough energy to produce an $M$-mode, however the couplings and the masses are evolving with energy. In the renormalized theory where $M^2(\mu^2)/\mu^2 > 1$, the mass of $M$-mode is always greater than the available energy, hence it is never realized on shell. In a scattering process of high energy graviton and Riccion going to other gravitons and Riccions with different momenta, $M$-mode can give off mass shell contributions, which affects the real part of the transition amplitude and never to the imaginary part (Cutkosky cut) \[11\]. Since $M^2(\mu^2)/\mu^2 > 1$ implies it cannot go on-shell. Indeed we can reinterpret $M$-mode as an effective ghost normalizing the functional integral as conjectured in \[9\]. We will show that this can be explicitly realized in a suitably defined parameter space of the action $S$.

2. Renormalization Group Analysis

We first make the observation that our choice of parameterization in Eq. \(1\) naturally shows that the Feynman loop expansion is the same as perturbation theory in small $G$, namely $G$
effectively plays the same role as $\hbar$. To one-loop the beta function of couplings have been computed using Schwinger-Dewitt technique \[12\]. Using dimensional regularization in $4 - \epsilon$ dimensions and minimal subtraction \[13, 14\] in the background Landau gauge, the RG flows are the following \[15\]:

\[
\begin{align*}
\frac{d}{dt} \left( \frac{1}{M^2 G} \right) & = -\frac{133}{10\pi}, \\
\frac{d}{dt} \left( \frac{\omega}{M^2 G} \right) & = \frac{5}{3\pi} \left( \omega^2 + 3\omega + \frac{1}{2} \right), \\
\frac{d}{dt} \left( \frac{1}{G} \right) & = \frac{5M^2}{3\pi} \left( \omega - \frac{7}{40\omega} \right), \\
\frac{d}{dt} \left( \frac{2\Lambda}{G} \right) & = \frac{M^4}{2\pi} \left( 5 + \frac{1}{\omega^2} \right) - \frac{4M^2\Lambda}{3\pi} \left( 14 + \frac{1}{\omega} \right)
\end{align*}
\]

where $t = \ln(\mu/\mu_0)$ and all rhs contain the leading contribution in $G$ ($M^2 G$ is also taken to be small), with higher powers coming from higher loops being neglected. Using Eqs. (3 and 4) we solve for the running of $\omega$,

\[
\frac{d\omega}{dt} = \frac{5M^2 G}{3\pi} \left( \omega^2 + \frac{549}{50}\omega + \frac{1}{2} \right) = \frac{5M^2 G}{3\pi} (\omega + \omega_1)(\omega + \omega_2)
\]

where $\omega_1 = 0.0457$ and $\omega_2 = 10.9343$. Eq. (7) shows that $-\omega_1$ and $-\omega_2$ are two fixed points, with the former being repulsive and latter attractive under UV evolution or increasing $t$. As we see from our propagator Eq. (2) only $\omega$ positive can be in the physical domain. That is both the fixed points are in the unphysical domain. The rhs of Eq. (7) is always positive for $\omega > 0$. This means that $\omega$ is a monotonic increasing function of $t$ and vice versa. Eq. (3) readily allows us to express $M^2 G$ in terms of $t$ with which we can integrate Eq. (7) to obtain,

\[
t = T \left[ 1 - \left( \frac{\omega + \omega_2}{\omega + \omega_1}, \frac{\omega_0 + \omega_1}{\omega_0 + \omega_2} \right) \right]^{\alpha},
\]

where $T = \frac{10\pi}{133M^2\mu_0^2}$ and $\alpha = 399/50(\omega_2 - \omega_1)$, with subscript 0 meaning that the coupling parameters are evaluated at $t = 0$ or $\mu = \mu_0$.

Eq. (7) transforms any evolution in $t$ to evolution in $\omega$. Using Eqs. (5 and 7) we get,

\[
\frac{d\ln G}{d\omega} = -\frac{\omega - \frac{7}{40\omega}}{(\omega + \omega_1)(\omega + \omega_2)}.
\]

$G$ gets maximized at $\omega = \sqrt{7/40}$, which lies in the physical domain. For convenience we choose this point to be our reference point $\mu_0$ or $t = 0$ and integrate Eq. (9) to obtain,

\[
\frac{G}{G_0} = \frac{\omega_0}{\omega} \left( \frac{1 + \omega_1/\omega}{1 + \omega_1/\omega_0} \right)^{A1} \left( \frac{1 + \omega_2/\omega}{1 + \omega_2/\omega_0} \right)^{A2},
\]

where $A1 = -0.3473$ and $A2 = -1.0027$. Eqs. (9 and 10) show that for large $\omega$ or $t$, $G \sim 1/\omega$, thereby vanishing for large $t$ \[23\], while for small $\omega$, $G \sim \omega^{7/20}$ reaching a peak at $\omega_0 = \sqrt{7/40}$. 


Similarly, using Eqs. (4 and 5) we obtain the running of $M^2/\omega$, which along with with Eq. (7) is integrated to give,

$$\frac{M^2}{\omega} = \frac{M_0^2}{\omega_0} \left( \frac{1 + \omega_1/\omega}{1 + \omega_1/\omega_0} \right)^{B_1} \left( \frac{1 + \omega_2/\omega}{1 + \omega_2/\omega_0} \right)^{B_2},$$  \hspace{1cm} (11)

where $B_1 = 1.0802$ and $B_2 = 0.2698$. We can now analyze $M^2/\mu^2$ from Eq. (11). Alternatively, it is instructive to note that from Eqs. (3, 5 and 7) we obtain the running of $\ln(\frac{M^2}{\mu^2})$,

$$\frac{d}{d\omega} \ln \left( \frac{M^2}{\mu^2} \right) = \left( \omega + \frac{399}{100} - \frac{7}{40\omega} \right) \left( \frac{1}{\omega + \omega_1} \right) \left( \frac{1}{\omega + \omega_2} \right),$$  \hspace{1cm} (12)

This shows that $M^2/\mu^2$ reaches a minima for $\omega = \omega^*$ given by

$$\left( \omega^* + \frac{399}{100} - \frac{7}{40\omega^*} \right) = \frac{6\pi}{5\mu^2G^*}. $$  \hspace{1cm} (13)

Hence by demanding

$$\frac{M^2}{\mu^2} = \frac{6\pi}{5\mu^2G^*} > 1$$  \hspace{1cm} (13)

we make the $M$-mode not realizable in the physical GR sector of the theory. The inequality is easily achieved by choosing $\mu^2G^*$ appropriately. Perturbative loop expansion requires that $M^2G$ is small. Therefore $M$ is a sub-Planckian mass, yet the running mass as dictated by interactions makes it physically not realizable even in post Planckian regime.

Analogous to Eq. (11) from Eqs. (4, 5 and 7) we obtain,

$$\frac{d}{d\omega} \ln \left( \frac{M^2}{\mu^2} \right) = \frac{3\pi}{5\mu^2G^*} + \frac{6\pi}{5\mu^2G^*} \left( \omega + \omega_1 \right) \left( \omega + \omega_2 \right),$$  \hspace{1cm} (14)

showing that the Riccian mass relative to $\mu$ decreases monotonically. By a suitable choice we can make the Riccian to be physically realizable or not. So we conclude that there exists unitary physical subspace only with the gravitons or along with Riccians.

We find that the solution of $\Lambda$ in the RG flows is dictated by $M^2$ and $G$. From Eq. (10) because of $M^4$ term $\Lambda$ increase in the UV region. These terms can be explicitly eliminated by adding fermion fields to our action. For example adding two spin-$\frac{1}{2}$ Dirac fields with mass $(5/4)^{1/4}M$ and $M/\sqrt{2}$ these additional fermionic fields can be interpreted as ghosts normalizing the functional integral). Consequently the other coefficients in Eqs. (3, 5 and 7) do change in rhs making $\omega_0, \omega_1, \omega_2$ and $\omega^*$ shift slightly but all conclusions remains unaltered. Furthermore, in this modified theory the cosmological constant can be made to vanish for large $t$.

We now look at the generality of the results. Adding matter fields to our action $S$ modifies the RG flow Eqs. (3, 5 and 4). In particular in rhs we will have terms which are dependent on mass $m$ and couplings defined within the matter theory but independent of $M$ and $\omega$. For example to one-loop in matter fields these have been computed $[10, 17, 15]$. Assuming $m/M < 1$, where $m$ is a typical matter field mass, we notice that $\omega_0, \omega_1, \omega_2$, and $\omega^*$ all shift by a small amount and all our conclusion remain unaltered. We remark that in the absence of higher derivative terms i.e. the $R^2$ and conformal term, running of $G$ is dominated only by matter fields and $G$ increases in the UV regime $[17]$. In the renormalizable theory given by $S$, the Riccian and $M$-mode reverse this tendency and make it vanish.

All our calculations are done in the Landau gauge. These calculations can in principal be done in other gauges, which have additional gauge parameters $[14, 15]$. They do not affect Eqs.
and in Eq. (5) there are additional terms in rhs proportional to $M^2$ and $M^2/\omega$, but not $M^2\omega$. Consequently $\omega_1$ and $\omega_2$ are unaltered but $\omega_0$ shifts. It should also be noted that in all these gauge parameter choices, for large $\omega$, $G$ still vanishes as $1/\omega$, while for small $\omega$, $G$ does vanish with a different power (as opposed to $\omega^{7/20}$ in the Landau gauge).

The mass of the Riccion $M/\sqrt{\omega}$ is seen to be gauge parameter dependent, however in all these cases it has positive norm. We also looked at the unitarity issue in Prentki gauge ($\partial_i h_{\mu i} = 0$, where $i = 1, 2, 3$) [18]. In this gauge the Faddeev-Popov ghost is not propagating and we have exactly gravitons, Riccion with positive norm and mass $M/\sqrt{\omega}$ and spin-2 $M$-mode with negative norm and mass $M$, just as in Landau gauge. In addition there is a vector particle with positive norm and mass $M$ (same as $M$-mode, RG flow equations have not been computed in this gauge). We see that in all these gauges the physical GR sector scattering matrix is unitary.

Our RG analysis is at $G = \Lambda = 0$. However it is should be noted that higher derivative gravity action Eq. (1) was also considered at a non-trivial fixed point where both $G$ and $\Lambda$ are non-vanishing [19, 20]. At this fixed point unitarity issue was considered [20, 21] at one-loop in a generalized de-Donder gauge using non-minimal regularization scheme. By doing Wilson’s RG analysis on Euclidean functional integral, it is shown that the fixed point enjoys necessary spectral positivity properties over a finite range of parameter $\omega$.

We conclude that for large $\omega$, $G \sim 1/\omega$ is a gauge invariant result. In the UV regime $1/M^2 G$ is the coefficient of conformal term in the action and this is finite while $R$ and $R^2$ coefficients are becoming large as $\omega$. $R^2$ being always positive makes functional integral better behaved as well. In addition the renormalized action enforces $R$ not to fluctuate away from zero. In infrared the EH term $R$ naturally is dominant along with cosmological constant. In infrared if $\Lambda = 0$ (by tuning) the vanishing of $G$ again enforces that $R$ fluctuate around zero.

3. Discussion

The physical domain $0 \leq \omega < \infty$ translates to a finite range in $t$, namely

$$\frac{t_{\text{min}}}{T} = 1 - \left(\frac{\omega_2 \omega_0 + \omega_1}{\omega_1 \omega_0 + \omega_2}\right)^{\alpha} \leq \frac{t}{T} < 1 - \left(\frac{\omega_0 + \omega_1}{\omega_0 + \omega_2}\right)^{\alpha} \equiv \frac{t_{\text{max}}}{T}$$  \hspace{1cm} (15)
From one-loop analysis we see that there is no solution for $\omega$ from Eq. (8) if $t > t_{\text{max}}$. For $t < t_{\text{min}}$, $\omega$ becomes negative i.e. $M^2/\omega$ the Ricciion mass square becomes negative signaling the instability of the vacuum. This infrared issue needs to be studied, namely the effects of the cosmological constant. To the leading order in Feynman loop expansion, at both extremes $G$ vanishes as $1/\omega$ for $t \to t_{\text{max}}$ or $\omega^{7/20}$ for $t \to t_{\text{min}}$.

Including inflationary scenarios the entire cosmological phenomenon ranges from as early as $10^{-42}$ s to the current age of the universe $10^{18}$ s, a span of about 130 e-folds. In the higher derivative gravity theory we have essentially the following dimensionful parameters $G_0$, $M_0$ and $\mu_0$ ($\omega_0$ is set by the maximum of $G$). These can be traded for one dimensionful parameter $G_0$ and two dimensionless parameters $t_{\text{max}}$ and $M_*/\mu_*$. By choosing $t_{\text{max}}$ large enough for example about 100, we plotted $G/G_0$ in Fig. 1 and $M/\mu$ and $M/\sqrt{\omega \mu}$ in Fig. 2. For $t_{\text{max}} = 100$, if $M_*/\mu_* > 22.52$ then both $M$-mode and Ricciion are not in physical GR sector. In the Fig. 2 the dashed line plot represent such a case for $M_*/\mu_* = 25$. If $1 < M_*/\mu_* < 22.52$, then Ricciion is realized. In the Fig. 2 the solid lines represent an extreme case for $M_*/\mu_* = 1.05$. From Fig. 2 we notice that Ricciion is physically realizable in high energy scattering processes at most for about three e-folds before $t_{\text{max}} = 100$.

From Fig. 1 we note that the entire cosmology may span around the maximum of $G$ accounting for sub-Planckian regime. Eventually in the post-Planckian regime $> 10^{19}$ GeV ($10^{-43}$ s), the gravitation coupling goes to zero at some finite energy $\mu_0 e^{t_{\text{max}}}$ (from Eq. 5) for $\omega > \sqrt{7/40}$ and $M^2 G$ positive, $G$ exponentially reaches zero and remains at zero). Naturally the question of physics beyond this scale is raised.

To discuss the range of validity of this analysis, we rewrite the propagator Eq. (2)

$$D_{\mu\nu,\alpha\beta} = \frac{i 16\pi}{(2\pi)^4} \left[ \frac{M^2 G}{\omega} (P_2)_{\mu\nu,\alpha\beta} - \frac{M^2 G}{q^2 - M^2/\omega + i\epsilon} - \frac{2 (P_2)_{\mu\nu,\alpha\beta}}{q^2 - M^2 + i\epsilon} \right] \cdot \frac{1}{q^2 + i\epsilon} \quad (16)$$

From Eq. (1) and Eq. (16) we see that there are three perturbative parameters $G$, $M^2 G$ and $M^2 G$ and in the UV limit the important dimensionless perturbative parameters are latter two.
Under these circumstances we find $M^2G/\omega$ and $G$ both vanish at a finite energy scale $\mu_0 e^{t_{\text{max}}}$. These results are valid if $M^2G$ is small as shown in the figures. However from Eq. (11) we also know that eventually $M^2G$ does become large before it reaches Landau singularity. We envisage two possible alternatives. Higher loop corrections may introduce terms like $(M^2G)\ln\omega$ etc, such that the Landau singularity is avoided and $t_{\text{max}}$ approach infinity. Another alternative could be $M^2G$ becomes of order one, consequently we need to consider the conformal term in Eq. (11) non-perturbatively.

Finally we conclude that the action $S$, Eq. (11) describes a perturbative quantum gravity as self consistent, renormalizable and unitary theory of gravitons and the curvature cannot become singular, in particular it cannot fluctuate wildly at sub-Planckian or post Planckian regimes consistent with known cosmology. Its predictions asymptotically beyond Planck scale needs to be investigated further.

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[23] $G$ is indeed a nontrivial function of $\ln \mu/\mu_0 = t$ through $\omega(t)$, consequently the criticisms alluded in “M. M. Anber and J. F. Donoghue, arXiv:1111.2875 [hep-th],” do not pertain to our analysis.