STABILITY FOR THE INFINITY-LAPLACE EQUATION WITH VARIABLE EXPONENT

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Abstract: We study the stability for the viscosity solutions of the differential equation

\[ \sum_{i,j=1}^{n} u_{x_i} u_{x_j} u_{x_i x_j} + |\nabla u|^2 \ln(|\nabla u|) \langle \nabla u, \nabla \ln p \rangle = 0 \]

under perturbations of the function \( p(x) \). The differential operator is the so-called \( \infty(x) \)-Laplacian.

1 Introduction

The object of our study is the curious differential equation

\[ \sum_{i,j=1}^{n} u_{x_i} u_{x_j} u_{x_i x_j} + |\nabla u|^2 \ln(|\nabla u|) \langle \nabla u, \nabla \ln p \rangle = 0 \]  

(1)

in a bounded domain \( \Omega \) in \( \mathbb{R}^n \). Here \( p(x) \) is a positive function, the so-called variable exponent, and it is of class \( C^1(\Omega) \). The equation comes from the mini-max problem of determining

\[ \min_u \max_x \left\{ |\nabla u(x)|^{p(x)} \right\} . \]

The case of a constant \( p(x) = p \) reduces to the celebrated \( \infty \)-Laplace equation

\[ \Delta_\infty u \equiv \sum_{i,j=1}^{n} u_{x_i} u_{x_j} u_{x_i x_j} = 0 \]  

(2)
found by G. Aronsson. In [LL] the equation was derived as the limit of the
Euler-Lagrange equations for the variational integrals
\[
\left\{ \int_{\Omega} |\nabla u(x)|^{kp(x)} \, dx \right\}^{\frac{1}{k}}
\]
as \( k \to \infty \). Such integrals were first considered by Zhikov, cf. [Z]. See also
[RMU] for similar equations. For sufficiently smooth solutions the meaning
of the equation is that
\[
|\nabla u(x)|^{p(x)} = C
\]
along any fixed stream line (different stream lines may have different con-
stants attached). —In general, solutions have to be interpreted in the viscosity
sense, and we assume that the reader is acquainted with the basic theory
of viscosity solutions, see [CIL, K, C].

The viscosity solution with prescribed Lipschitz continuous boundary val-
ues is unique, cf. [LL]. Taking into account that, in contrast, uniqueness does
not always hold for the infinity-Poisson equation
\[
\Delta_{\infty} u = \varepsilon(x),
\]
as an example with a uniformly continuous sign-changing function \( \varepsilon(x) \) in
[LW] shows, the uniqueness for the curious equation (1) is pretty remark-
able. Therefore we have found it worth our while to study the stability under
variations of \( p(x) \).

Our first result is about a perturbation of the infinity-Laplace equation (2).

**Theorem 1** Let \( p \in C^1(\Omega) \) be a positive function and suppose that \( u \in C(\Omega) \)
is the viscosity solution of
\[
\Delta_{\infty} u + |\nabla u|^2 \ln(|\nabla u|) \langle \nabla u, \nabla \ln(p) \rangle = 0
\]
and that \( v \in C(\Omega) \) is the viscosity solution of
\[
\Delta_{\infty} v = 0,
\]
both having the same Lipschitz continuous boundary values \( f \). Then the es-
timate
\[
\|v - u\|_{L^\infty(\Omega)} \leq C_1 \|\nabla \ln p\|_{L^\infty(\Omega)} + C_2 \|\nabla \ln p\|_{L^\infty(\Omega)}^{\frac{1}{2}} \|\nabla \ln p\|_{L^\infty(\Omega)} \ln(c \|\nabla \ln p\|_{L^\infty(\Omega)}) \quad (3)
\]
is valid with constants depending only on \( \|f\|_{W^{1,\infty}(\Omega)} \) and \( \text{diam}(\Omega) \).
An interpretation is that when $p(x)$ deviates only little from a constant value, then $u$ is close to $v$. But, as we have pointed out, the perturbation term $|\nabla u|^2 \ln(|\nabla u|) \langle \nabla u, \nabla \ln(p) \rangle$ cannot be replaced by an arbitrary small perturbation $\varepsilon(x)$, despite the possibility to select $p(x)$ in any manner. — The exponent $\frac{1}{5}$ seems to be an artifact of the arrangements in our proof in section 4.

We also address the problem with two positive exponents $p_1, p_2 \in C^1(\Omega)$, but now the result is weaker. Suppose that $u_\nu \in C(\Omega)$ is a viscosity solution of
\begin{equation}
\Delta_\infty u_\nu + |\nabla u_\nu|^2 \ln(|\nabla u_\nu|) \langle \nabla u_\nu, \nabla \ln(p_\nu) \rangle = 0
\end{equation}
in $\Omega$, $\nu = 1, 2$. If $u_1$ and $u_2$ have the same Lipschitz continuous boundary values, then
\begin{equation}
\|u_1 - u_2\|_{L^\infty(\Omega)} \leq \text{Const.} \frac{\|\ln(\|\nabla \ln p_2 - \nabla \ln p_1\|_{L^\infty(\Omega)})\|}{\kappa},
\end{equation}
where $\kappa > 0$ depends on $\max(p_\nu), \min(p_\nu)$. The constant depends on the boundary values and on the norms $\|\nabla \ln p_\nu\|_{\infty}$. Needless to say, the obtained modulus of stability appears to be far from sharp. Therefore we have only sketched out the proof in section 5. In the one-dimensional case a sharp bound is easily reached via the “first integral” $|u_\nu'(x)|^{p_\nu(x)} = C_\nu$.

## 2 Preliminaries

We briefly recall some basic concepts. Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ and suppose that $f : \partial \Omega \to \mathbb{R}$ is a Lipschitz continuous function satisfying
\[ |f(x) - f(y)| \leq L|x - y|. \]
By extension, we may as well assume that the inequality holds in the whole space, if needed. The abbreviation
\[ \Delta_\infty(x)u \equiv \Delta u + |\nabla u|^2 \ln(|\nabla u|) \langle \nabla u, \nabla \ln(p) \rangle \]
is convenient.\footnote{The suggestive subscript $\infty(x)$ symbolizes the “variable exponent infinity”.
}

To be on the safe side, we assume that $p \in C^1(\overline{\Omega})$, $p(x) > 0$. Then viscosity solutions to the equation (1) can be defined in the standard way.
Definition 2 We say that a lower semicontinuous function $v : \Omega \to (-\infty, \infty]$ is a viscosity supersolution if, whenever $x_0 \in \Omega$ and $\varphi \in C^2(\Omega)$ are such that

1. $\varphi(x_0) = u(x_0)$, and
2. $\varphi(x) < v(x)$, when $x \neq x_0$,

then we have

$$\Delta_{\infty(x_0)} \varphi(x_0) \leq 0.$$ 

The viscosity subsolutions have a similar definition; they are upper semicontinuous, the test functions touch from above and the differential inequality is reversed. Finally, a viscosity solution is both a viscosity supersolution and viscosity subsolution.

There is an alternative way of expressing the definition in terms of "semi-jets".

3 Auxiliary Equations

Following R. Jensen in [J] we introduce two auxiliary equations. For a constant exponent $p$ the situation is

$$\max\{\varepsilon - |\nabla u^+|, \Delta_{\infty} u^+\} = 0 \quad \text{Upper equation}$$
$$\Delta_{\infty} u = 0 \quad \text{Equation}$$
$$\min\{|\nabla u^-| - \varepsilon, \Delta_{\infty} u^-\} = 0 \quad \text{Lower equation}$$

where $\varepsilon > 0$. Given $\varepsilon > 0$, three viscosity solutions $u^-, u, u^+$ are constructed with the same boundary values $f$ so that

$$u^- \leq u \leq u^+$$
$$\|u^+ - u^-\|_{L^\infty(\Omega)} \leq \varepsilon \text{ diam}(\Omega)$$
$$\|\nabla u^\pm\|_{L^\infty(\Omega)} \leq K + \varepsilon = K_\varepsilon$$

where $K$ depends only on the Lipschitz constant $L$ of $f$. The virtue of the auxiliary equations is that

$$\varepsilon - |\nabla u^+| \leq 0, \quad |\nabla u^-| - \varepsilon \geq 0$$

in the viscosity sense. We refer to [J] and [LL] about the construction via variational integrals.
We need a strict supersolution. We will construct a function \( g(u^+) \approx u^+ \) such that \( \Delta_\infty g(u^+) < 0 \). To this end we use the following approximation of the identity

\[
g(t) = \frac{1}{\alpha} \ln \left( 1 + A(e^{\alpha t} - 1) \right), \quad A > 1, \alpha > 0 \tag{9}
\]

taken from [JLM] and [LL]. For \( t > 0, A > 1, \alpha > 0 \) we have

\[
0 < g(t) - 1 < \frac{A - 1}{\alpha}
\]

\[
(A - 1)e^{-\alpha t} < g'(t) < A - 1
\]

\[
g''(t) = -\alpha (g'(t) - 1),
\]

which are easy to verify.

**Lemma 3** Let \( v > 0 \) and consider \( w = g(v) \). If

\[
\varepsilon - |\nabla v| \leq 0 \quad\text{and}\quad \Delta_\infty v \leq 0
\]

in the viscosity sense, then the inequality

\[
\Delta_\infty w \leq -\alpha(A - 1)A^{-1}e^{-\alpha\|v\|\infty}\varepsilon^4 \equiv -\mu \tag{10}
\]

holds in the viscosity sense.

**Proof:** Formally, the equation for \( w = g(v) \) is

\[
\Delta_\infty w = g'(v)^3 \Delta_\infty v + g''(v)g'(v)^2|\nabla v|^4 \\
\leq 0 + g''(v)g'(v)^2|\nabla v|^4 \\
= -\alpha(A - 1)A^{-1}e^{-\alpha\varepsilon}g'(v)^4|\nabla v|^4 \\
\leq -\alpha(A - 1)A^{-1}e^{-\alpha\varepsilon}1^4\varepsilon^4.
\]

To conclude the proof, one has to pass the calculation over to test functions. □

We will apply the lemma on \( w = g(u^+) \) and we assume that \( f > 0 \) so that the encountered functions are non-negative. It holds that

\[
\min_{\partial \Omega} (f) = \min_{\partial \Omega} (u) \leq u \leq u^+ \leq \max_{\partial \Omega} (f) + \varepsilon \text{diam}(\Omega) \tag{11}
\]
by the maximum principle and (8). Fix
\[ \alpha = \frac{1}{\|u^+\|_\infty}. \]
Estimate (10) in the lemma above becomes
\[ \Delta_\infty g(u^+) \leq -\mu = -\frac{(A - 1)\varepsilon^4}{Ae\|u^+\|_\infty}. \]

4 Proof of the Stability

Suppose that \( u_1 \) is a viscosity (sub)solution of
\[ \Delta_\infty u_1 + |\nabla u_1|^2 \ln(|\nabla u_1|) \langle \nabla u_1, \nabla \ln p_1 \rangle = 0 \]
and that \( u_2 \) is a viscosity (super)solution of
\[ \Delta_\infty u_2 = 0, \quad (p_2 = \text{constant}) \]
both with boundary values \( f \). Adding the same constant to \( f, u_1, \) and \( u_2 \), we may assume that \( f \geq 0 \) and \( u_2 \geq 0 \). Given \( \varepsilon > 0 \), write
\[ v_2 = u_2^+, \quad w_2 = g(v_2) = g(u_2^+). \]
We obtain the estimate
\[ u_1 - u_2 = (u_1 - w_2) + (w_2 - v_2) + (v_2 - u_2) \]
\[ < (u_1 - w_2) + \frac{A - 1}{\alpha} + \varepsilon \text{diam}(\Omega). \]
(13)
The last two terms could be made as small as we please, but the term \( u_1 - w_2 \) requires our attention, since there \( w_2 \) depends also on \( A \) and \( \varepsilon \).

Lemma 4 We have
\[ u_1 - w_2 \leq C_\varepsilon^3 \|u_2^+\|_\infty^2 \varepsilon^{-4} \ln\left(\frac{C_\varepsilon}{\varepsilon}\right) \|\nabla \ln p_1\|_\infty, \]
(14)
where \( C_\varepsilon = C(1 + \varepsilon) \).
Proof: Let $\sigma = \max(u_1 - w_2)$. If $\sigma \leq 0$ there is nothing to prove. Assume thus that $\sigma > 0$. In order to use the Theorem on Sums for viscosity solutions, we double the variables writing

$$M_j = \sup_{x \in \Omega, y \in \Omega} \left( u_1(x) - w_2(y) - \frac{j}{2} |x - y|^2 \right)$$

as usual. Then $M_j \geq \sigma$ (take $x = y$ to see this). The supremum is attained at some points $x_j, y_j$. Now $|x_j - y_j| \to 0$ as $j \to \infty$ and

$$x_j \to \hat{x}, \quad y_j \to \hat{y} = \hat{x},$$

at least for a subsequence. We claim that $\hat{x}$ is an interior point of $\Omega$. Indeed, if $\hat{x} \in \partial \Omega$ then

$$u_1(\hat{x}) - w_2(\hat{x}) = (u_1(\hat{x}) - v_2(\hat{x})) + (v_2(\hat{x}) - w_2(\hat{x})) = 0 + (v_2(\hat{x}) - g(v_2(\hat{x}))) \leq 0$$

and hence

$$u_1(\hat{x}) - w_2(y) = (u_1(\hat{x}) - w_2(y)) + (w_2(\hat{x}) - w_2(y)) \leq w_2(\hat{x}) - w_2(y),$$

which, by continuity, is less than $\sigma/2 < \sigma \leq M_j$ provided that $|\hat{x} - y|$ is small. Hence $\hat{x} \in \Omega$.

We conclude that also $x_j$ and $y_j$ are interior points for large indices $j$. We need the bounds

$$\varepsilon \leq j|x_j - y_j| \leq C \varepsilon.$$  \hspace{1cm} (16)

The upper bound follows from

$$u_1(x_j) - w_2(y_j) - \frac{j}{2} |x_j - y_j|^2 \geq u_1(x_j) - w_2(x_j), \quad \frac{j}{2} |x_j - y_j|^2 \leq w_2(x_j) - w_2(y_j) \leq \|g'(v_2) \nabla v_2\|_\infty |x_j - y_j| \leq AK \varepsilon |x_j - y_j|,$$

where we used that $g'(v_2) < A$. We had $K = K + \varepsilon$ and we will later see that $A \leq 2$. Then $C \varepsilon = 2K \varepsilon$ will do. The lower bound is deduced from the fact that $\varepsilon - |\nabla w_2| \leq 0$ in the viscosity sense $\nabla w_2 = g'(v_2) \nabla v_2$, $1 \leq g'(v_2)$, $\varepsilon \leq |\nabla v_2|$. To wit,

$$u_1(x_j) - w_2(y_j) - \frac{j}{2} |x_j - y_j|^2 \geq u_1(x_j) - w_2(y) - \frac{j}{2} |x_j - y|^2,$$
from which it follows that the function
\[ \varphi(y) = w_2(y_j) + \frac{j}{2} |x_j - y_j|^2 - \frac{j}{2} |x_j - y|^2 \]
touches \( w_2(y) \) from below at the point \( y_j \). Thus \( \varepsilon \leq |\nabla \varphi(y_j)| \), and this is the desired inequality, indeed.

According to the Theorem on Sums there exist symmetric \( n \times n \)-matrices \( X_j \) and \( Y_j \) such that
\[ (j(x_j - y_j), X_j) \in J^{2,+} u_1(x_j), \]
\[ (j(x_j - y_j), Y_j) \in J^{2,-} w_2(y_j) \]
where \( J^{2,+} u_1(x_j) \) and \( J^{2,-} w_2(y_j) \) are the closures of the super- and subjets. (Caution: supersolutions are tested with subjets.) For the jets and their closures we refer to [CIL], [C], [K]. The meaning of the notion is that we can rewrite the equations as
\[

g^2 \langle Y_j(x_j - y_j), x_j - y_j \rangle \leq -\mu,
\]
\[
\sum_{i,j} g \langle X_j(x_j - y_j), x_j - y_j \rangle + \sum_{j} g |x_j - y_j| \ln(j|x_j - y_j|) \langle x_j - y_j, \nabla \ln p_1(x_j) \rangle \geq 0,
\]
\[
|j| x_j - y_j | \geq \varepsilon,
\]
\[
|j|x_j - y_j| \leq C_\varepsilon.
\]

It follows that
\[
\sum_{i,j} g \langle (Y_j - X_j)(x_j - y_j), x_j - y_j \rangle \leq -\mu + C_\varepsilon^3 \ln\left(\frac{C_\varepsilon}{\varepsilon}\right) \|\nabla \ln p_1\|_\infty.
\]
The left-hand member is a positive semidefinite quadratic form, since \( Y_j - X_j \geq 0 \), in other words it is non-negative. Thus
\[
\mu \leq C_\varepsilon^3 \ln\left(\frac{C_\varepsilon}{\varepsilon}\right) \|\nabla \ln p_1\|_\infty.
\]
Recall the expression for \( \mu \) in (12). The above estimate can be written as
\[
\frac{(A - 1)\varepsilon^4}{Ae \|u_j^\perp\|_\infty} \leq C_\varepsilon^3 \ln\left(\frac{C_\varepsilon}{\varepsilon}\right) \|\nabla \ln p_1\|_\infty.
\]
We fix \( A > 1 \) so that
\[
\frac{A - 1}{\varepsilon} = \sigma.
\]
where we had $\alpha^{-1} = \|u_2^+\|_\infty$. Then

$$\sigma \leq A\epsilon e^{-4}C^3 \ln\left(\frac{C\epsilon}{\epsilon}\right) \|\nabla \ln p_1\|_\infty \|u_2^+\|^2_\infty.$$ 

Further, $A \leq 2$ so that $A\epsilon$ can be absorbed into the constant $C\epsilon$. (Indeed, $\sigma = \max(u_1 - w_2) \leq \max(u_1) = u_1(\xi)$ where $\xi$ is some boundary point. Now

$$A = 1 + \alpha \sigma \leq 1 + \frac{u_1(\xi)}{\|u_2^+\|_\infty} \leq 1 + \frac{u_1(\xi)}{u_2^+(\xi)} = 2,$$

because the functions have the same boundary values.) This concludes the proof of the estimate (14). □

We return to (13). Using (14) we obtain

$$u_1 - u_2 \leq \sigma + \sigma + \epsilon \text{diam}(\Omega)$$

$$\leq 2\epsilon^{-4}C^3 \ln\left(\frac{C\epsilon}{\epsilon}\right) \|u_2^+\|^2_\infty \|\nabla \ln p_1\|_\infty + \epsilon \text{diam}(\Omega).$$

It remains to determine $\epsilon$ nearly optimally. To simplify, we use (11):

$$u_2^+ \leq u_2 + \epsilon \text{diam}(\Omega) \leq \|f\|_\infty + \epsilon \text{diam}(\Omega).$$

We have to optimize

$$(C+\epsilon)^3 \ln\left(\frac{C+\epsilon}{\epsilon}\right) (\|f\|_\infty + \epsilon)^2 \|\nabla \ln p_1\|_\infty \epsilon^{-4} + \epsilon \text{diam}(\Omega),$$

which, renaming constants, is the same as an expression of the form

$$\left(\frac{C+\epsilon}{\epsilon}\right)^5 \ln\left(\frac{C+\epsilon}{\epsilon}\right) \|\nabla \ln p_1\|_\infty \epsilon + \epsilon a.$$ 

We consider two cases. The case of a large $\|\nabla \ln p_1\|_\infty$ is plain. Namely, if $a \leq 32 \|\nabla \ln p_1\|_\infty$ we just take $\epsilon = 1$ and obtain immediately a majorant of the form $C_1\|\nabla \ln p_1\|_\infty$. If not, we can determine $\epsilon$ from the equation

$$\left(\frac{C+\epsilon}{\epsilon}\right)^5 \|\nabla \ln p_1\|_\infty = a.$$ 

This yields a majorant like

$$C_2\|\nabla \ln p_1\|_\infty^{\frac{1}{5}} \ln(c \|\nabla \ln p_1\|_\infty).$$
Combining the two cases we arrive at the desired estimate (3), yet so far only for \(\max(u_1 - u_2)\). The corresponding estimate for \(\max(u_2 - u_1)\) is still missing; the situation is not symmetric. To complete the proof, observe that

\[
\begin{align*}
u_2(x) - u_1(x) = (k - u_1(x)) - (k - u_2(x))
\end{align*}
\]

where the constant is large enough to make the new viscosity solution \(k - u_2\) positive; \(k = \max(f)\) will do. Now \(\Delta_\infty(k - u_2(x)) = 0\) and the situation has been reduced to the previous case. —Instead, we could have repeated the proof, this time using the Lower Equation (7).

\[\square\]

5 Two Varying Exponents

In the case of two exponents none of which is constant, an extra complication arises: the parameter \(\alpha\) must be taken very large, say \(\alpha \approx \varepsilon^{-1}\), and then the exponential factor in the counterpart to (12) is extremely small. This weakens the final result.

In principle, the proof is a repetition of the previous one. Only an outline is provided below. First, the auxiliary equations in section 3 are modified so that \(\Delta_\infty\) is replaced by \(\Delta_\infty(x)\). As in [LL] one then obtains the estimate

\[
\|u^+ - u^-\|_\infty \leq B\varepsilon^\kappa
\]

where \(\kappa > 0\) (either \(\kappa = \min(p(x))\) or \(\kappa = \max(p(x))\)). Second, we need a strict supersolution to equation (1).

**Lemma 5** Consider \(w = g(v)\) for \(v > 0\). If

\[
\varepsilon - |\nabla v| \leq 0 \quad \text{and} \quad \Delta_\infty(x) v \leq 0
\]

in the viscosity sense, then

\[
\Delta_\infty(x) w \leq -\varepsilon^3 (A - 1) A^{-1} e^{-\|\nabla \ln p\|_\infty} \|v\|_\infty \varepsilon^{-1} \equiv -\mu
\]

in the viscosity sense.

**Proof:** A routine calculation yields

\[
\begin{align*}
\Delta_\infty(x) w &\leq g'(v)^3 (g'(v) - 1) |\nabla v|^3 \{ -\alpha |\nabla v| + |\nabla \ln p| \} \\
&\leq g'(v)^3 (g'(v) - 1) |\nabla v|^3 \{ -\alpha \varepsilon + \|\nabla \ln p\|_\infty \}.
\end{align*}
\]
Given $\varepsilon > 0$, we fix $\alpha = \alpha(\varepsilon)$ so that
$$-\alpha \varepsilon + \|\nabla \ln p\|_\infty = -1.$$  

The estimate (18) readily follows. $\square$

Suppose now that $u_\nu$ is a viscosity solution of the equation (4), $\nu = 1, 2$. We assume that $u_1 = u_2 = f$ on $\partial \Omega$. By adding a constant, we reach the situation that $u_2^+ \geq u_2 > 0$. Write
$$v_2 = u_2^+, \quad w_2 = g(v_2).$$

Now
$$u_1 - u_2 = (u_1 - w_2) + (w_2 - v_2) + (v_2 - u_2)$$
$$\leq (u_1 - w_2) + \frac{A - 1}{\alpha} + B \varepsilon^\kappa.$$

**Lemma 6** We have
$$u_1 - w_2 \leq 2 \varepsilon^{-2} A C^3 \varepsilon \ln \left( \frac{C \varepsilon}{\varepsilon} \right) e^{(1 + \|\nabla \ln p_2\|_\infty)} \|v_2\|_\infty \varepsilon^{-1} \|\nabla \ln p_2 - \nabla \ln p_1\|_\infty. \quad (19)$$

**Proof:** Denote $\sigma = \max(u_1 - u_2)$. We may assume that $\sigma > 0$. Double the variables as in (15). By the Theorem on Sums we again obtain symmetric matrices $X_j$ and $Y_j$ so that $X_j \leq Y_j$ and

\begin{align*}
&j^2 \langle Y_j(x_j - y_j), x_j - y_j \rangle \\
&\quad + j^3 |x_j - y_j|^2 \ln(j|x_j - y_j|) \langle x_j - y_j, \nabla \ln p_2(y_j) \rangle \leq -\mu, \\
&j^2 \langle X_j(x_j - y_j), x_j - y_j \rangle \\
&\quad + j^3 |x_j - y_j|^2 \ln(j|x_j - y_j|) \langle x_j - y_j, \nabla \ln p_1(x_j) \rangle \geq 0, \\
&j|x_j - y_j| \geq \varepsilon, \\
&j|x_j - y_j| \leq C \varepsilon.
\end{align*}

Write $\ln p_1(x_j) = \ln p_1(x_j) - \ln p_2(x_j) + \ln p_2(x_j)$ and arrange the equations. It follows that

\begin{align*}
0 &\leq j^2 \langle (Y_j - X_j)(x_j - y_j), (x_j - y_j) \rangle \leq -\mu \\
&\quad + j^3 |x_j - y_j|^2 \ln(j|x_j - y_j|) \langle x_j - y_j, \nabla \ln p_2(x_j) - \nabla \ln p_2(y_j) \rangle \\
&\quad + j^3 |x_j - y_j|^2 \ln(j|x_j - y_j|) \langle x_j - y_j, \nabla \ln p_1(x_j) - \nabla \ln p_2(x_j) \rangle \\
&\leq -\mu + C^3 \varepsilon \ln \left( \frac{C \varepsilon}{\varepsilon} \right) \|\nabla \ln p_2(x_j) - \nabla \ln p_2(y_j)\| \\
&\quad + C^3 \varepsilon \ln \left( \frac{C \varepsilon}{\varepsilon} \right) \|\nabla \ln p_1 - \nabla \ln p_2\|_\infty.
\end{align*}
As $j \to \infty$, $x_j - y_j \to 0$, so that by continuity

$$\mu \leq C_\varepsilon^3 \ln \left( \frac{C_\varepsilon}{\varepsilon} \right) \| \nabla \ln p_1 - \nabla \ln p_2 \|_\infty. \quad (20)$$

To conclude the proof, we fix $A > 1$ so that

$$\frac{A - 1}{\alpha} = \frac{\sigma}{2}$$

and insert the expression for $\mu$ given in (18). Hence

$$\varepsilon^3 \sigma \alpha \frac{A - 1}{\alpha} e^{-\left(1 + \| \nabla \ln p_2 \|_\infty \right) \| v_2 \|_\infty} \varepsilon^{-1}$$

$$\leq C_\varepsilon^3 \ln \left( \frac{C_\varepsilon}{\varepsilon} \right) \| \nabla \ln p_1 - \nabla \ln p_2 \|_\infty.$$  

The estimate follows, since $\alpha \approx 1/\varepsilon$. □

In order to finish the proof of (4) we choose $\varepsilon$ in the inequality

$$u_1 - u_2 \leq 4\varepsilon^2 C_\varepsilon^3 \ln \left( \frac{C_\varepsilon}{\varepsilon} \right) e^{\left(1 + \| \nabla \ln p_2 \|_\infty \right) \| v_2 \|_\infty} \varepsilon^{-1} \| \nabla \ln p_1 - \nabla \ln p_2 \|_\infty + B\varepsilon^\kappa$$

so that

$$e^{\left(1 + \| \nabla \ln p_2 \|_\infty \right) \| v_2 \|_\infty} \varepsilon^{-1} \| \nabla \ln p_1 - \nabla \ln p_2 \|_\infty \approx \varepsilon^{\kappa + 2}.$$  

We omit the calculation.

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