THE INTEGRAL CLOSURE OF A PRIMARY IDEAL IS NOT ALWAYS PRIMARY

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Abstract. In 1936, Krull asked if the integral closure of a primary ideal is still primary. Fifty years later, Huneke partially answered this question by giving a primary polynomial ideal whose integral closure is not primary in a regular local ring of characteristic $p = 2$. We provide counterexamples to Krull's question regarding polynomial rings with any characteristics. We also find that the Jacobian ideal $J$ of the polynomial $f = x^6 + y^6 + x^4zt + z^3$ given by Briançon and Speder in 1975 is a counterexample to Krull’s question. Let $V_1$ be the hypersurface defined by $f = 0$ and $V_2$ be its singular locus. Briançon and Speder proved that Whitney equisingularity does not imply Zariski equisingularity by showing that the pair $(V_1 \setminus V_2, V_2)$ satisfies Whitney’s conditions around the origin but fails Zariski’s equisingular conditions. We discover that the pair $(V_1 \setminus V_2, V_2)$ fails Whitney’s conditions at the variety of the embedded prime of the integral closure $\bar{J}$, which means that $V_1$ is not Whitney regular along $V_2$. Moreover, we also show that Whitney stratification of this hypersurface is different from the stratification of isosingular sets given by Hauenstein and Wampler, which is related to Thom-Boardman singularity.

1. Introduction

Krull [14, p. 577] asked: Ist etwa bei einem Primärideal $q$ immer auch $\bar{q}$ Primärideal? For monomial ideals, the answer to Krull’s question is yes. The integral closure of a primary monomial ideal is always primary [13]. However, for non-monomial ideals, Huneke partially answered this question by giving a counterexample in the regular local ring $k[[x, y, z]]$ with char$(k) = 2$ [10, Example 3.7]. According to [11], there are no known counterexamples for rings of characteristic zero. The integral closure of ideals is related to Whitney equisingularity. For instance, Teissier [3, 18] gave an algebraic description for Whitney’s condition (b) using the integral closure of the sheaf of ideals, which started the modern equisingularity theory. Gaffney [4, 5, 6] generalized the theory of integral closure of ideals to modules, and made many applications in Whitney equisingularity.

Our main contributions are summarized below:

- We answer Krull’s question negatively by giving a sequence of primary ideals
  \[ I = \langle x^3, y^3, x^2y, x^2z^n - xy^2 \rangle, n \in \mathbb{Z}_+ \]
whose integral closures
  \[ \bar{I} = \langle x^3, y^3, x^2y, x^2z^n, xy^2 \rangle \]
are not primary over a field of characteristic zero or positive characteristics. Hence, taking integral closure of a polynomial primary ideal may create embedded primes. On the other hand, we also show that there are examples where the given polynomial ideal is not primary but its integral closure is primary. It implies that taking integral closure may also remove embedded primes. Therefore, the relation between a primary ideal and its integral closure is not clear yet.

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We show that the hypersurface defined by \( f(x, y, z, t) = x^6 + y^6 + x^4z^2 + z^3 = 0 \) is not Whitney regular along its singular locus. This hypersurface was given by Briançon and Speder [2] to show that Whitney equisingularity does not imply Zariski equisingularity for the set germs at the origin. Let \( V_1 \) be the hypersurface defined by \( f, V_2 = \text{Sing}(V_1) = \{(0, 0, 0, t) \mid t \in \mathbb{C}\} \) be the singular locus of \( V_1 \). It is known that the pair \((V_1 \setminus V_2, V_2)\) satisfies Whitney’s conditions (a) and (b) but fails Zariski equisingularity at the origin [2, 15, 20, 22], which makes the hypersurface \( V_1 \) as a counterexample for Zariski equisingularity problem [19, 21]. We consider Whitney equisingularity for all points of \( V_2 \) and show that the pair \((V_1 \setminus V_2, V_2)\) fails Whitney’s conditions at \( V_3 = \{(0, 0, 0, t) \mid 4t^3 + 27 = 0\} \subset V_2 \). The Jacobian ideal generated by the partial derivatives of \( f \) is primary, but its integral closure is not, which gives another counterexample to Krull’s question. The integral closure of the Jacobian ideal of \( f \) has an embedded prime which happens to be the vanishing ideal of \( V_3 \). Furthermore, we also show that Whitney stratification of \( V_1 \) is different from the stratification given by isosingular sets in [7].

The paper is organized as follows. Section 2 is for basic definitions and properties of integral closures of ideals. In Section 3, we show a sequence of primary ideals whose integral closures are not primary over a field of characteristic zero or positive characteristics. Moreover, we also present an example to show that taking integral closure may remove embedded primes. Finally, we compute the integral closure \( \bar{J} \) of the Jacobian ideal of \( f \) and verify that the pair \((V_1 \setminus V_2, V_2)\) fails Whitney’s conditions at the variety of the embedded prime of \( \bar{J} \).

2. Basic properties

Let us first recall some basic definitions from [11].

**Definition 2.1.** Let \( I \) be an ideal in a ring \( R \). An element \( r \in R \) is said to be integral over \( I \) if there exists an integer \( n \) and elements \( a_i \in I^n \), \( i = 1, \ldots, n \), such that

\[
 r^n + a_1 r^{n-1} + a_2 r^{n-2} + \cdots + a_{n-1} r + a_n = 0.
\]

The set of all integral elements over \( I \) is called the **integral closure** of \( I \) and is denoted by \( \bar{I} \). If \( I = \bar{I} \), then \( I \) is called integrally closed. Let \( J \) be an ideal satisfying \( I \subset J \), we say that \( J \) is integral over \( I \) if \( \bar{I} \subset J \).

**Definition 2.2.** Let \( R \) be the polynomial ring \( k[x_1, \ldots, x_d] \). For any monomial \( m = x_1^{n_1} x_2^{n_2} \cdots x_d^{n_d} \), its exponent vector is \((n_1, \ldots, n_d) \in \mathbb{N}^d \). For any monomial ideal \( I \), the set of all exponent vectors of all the monomials in \( I \) is called the **exponent set** of \( I \).

The integral closure of a monomial ideal in a polynomial ring is still a monomial ideal [11, Proposition 1.4.2]. The following proposition is useful for computing the integral closure of a monomial ideal.

**Proposition 2.3.** [11, Proposition 1.4.6] The exponent set of the integral closure of a monomial ideal \( I \) equals all the integer lattice points in the convex hull of the exponent set of \( I \).

We give a simple example to show how to compute the integral closure of a monomial ideal using Proposition 2.3.

**Example 2.4.** Let \( R = k[x, y] \) be a polynomial ring over a field \( k \). Let \( I = \langle x^2, y^2 \rangle \) be a monomial ideal in \( R \). The exponent set of \( I \) is \( \text{exp} = \{(n_1, n_2) \in \mathbb{N}^2 \mid n_1 \geq 2, n_2 \geq 2\} \), see Figure 1(a). The convex hull of \( \text{exp} \) is \( \text{exp} \cup \{(1, 1)\} \), see Figure 1(b). Therefore, by Proposition 2.3, the integral closure of \( I \) is \( \bar{I} = \langle x^2, y^2, xy \rangle \).
In this section, we answer Krull’s question negatively by giving a sequence of primary ideals whose integral closures are not primary. The most amazing thing for us to notice is that the Jacobian ideal of the ideal defined by $f(x, y, z, t) = x^6 + y^6 + x^4z + z^3$ has already given another counterexample to Krull’s question.

3.1. A set of counterexamples to Krull’s question.

**Example 3.1.** Let $R = k[x, y, z]$ be a polynomial ring over an arbitrary field $k$. Let

$$I = \langle x^3, y^3, x^2y, x^2z - xy^2 \rangle,$$

where $I$ is a primary ideal and its radical ideal is $P = \sqrt{I} = \langle x, y \rangle$.

Let $\bar{I}$ be the integral closure of $I$. Using Proposition 2.3, we obtain

$$\bar{I} = \langle x^3, y^3, x^2y, x^2z, xy^2 \rangle.$$

Its primary decomposition is

$$\bar{I} = \langle x^2, y^3, xy^2 \rangle \cap \langle x^3, y, z \rangle,$$

which implies that $\bar{I}$ is not primary. One can also check it via Macaulay2, see Appendix A.1.

It is interesting to notice that a sequence of counterexamples exist according to the following theorem.

**Theorem 3.2.** Let $n$ be a positive integer, and

$$I = \langle x^3, y^3, x^2y, x^2z^n - xy^2 \rangle, \ n \in \mathbb{Z}_+.$$

be a polynomial ideal in $k[x, y, z]$. Then the ideal $I$ is primary and its integral closure

$$\bar{I} = \langle x^3, y^3, x^2y, x^2z^n, xy^2 \rangle$$

is not primary.

Theorem 3.2 follows from three claims below. Some preliminary results from [1, 16] are summarized in Appendix B.

- **Claim 1.** The ideal $I = \langle x^3, y^3, x^2y, x^2z^n - xy^2 \rangle \subset k[x, y, z]$ is primary, where $n \in \mathbb{Z}_+$.

**Proof of Claim 1.** Let $P = \sqrt{I} = \langle x, y \rangle$ and $Q$ be the primary component of $I$ with $\sqrt{Q} = P$. By Lemma B.4, $Q = I : z^\infty$. We prove that $I = Q$. 

Since $I \subseteq Q$, we only need to prove that $Q \subseteq I$. Suppose that $I \subseteq Q$, and let $g \in Q$ and $g \notin I$. Then $z^m g \in I$ for some $m \geq 1$. The set of generators $G = \{x^3, y^3, x^2 y, x^2 z n - xy^2\}$ is a Gröbner basis of $I$ with respect to the lexicographic order $x > y > z$. Without loss of generality, we assume that $g$ is reduced with respect to $G$. We may also assume that $m \geq n$, because $z^m g \in I$ implies that $z^{m+\ell} g \in I$ for any $\ell \geq 0$. Therefore, we can choose an exponent not less than $n$. Let $\text{LM}(G) = \{x^3, y^3, x^2 y, x^2 z n\}$ be the set of leading monomials of polynomials in $G$. Since $G$ is a Gröbner basis of $I$, $z^m \text{LM}(g) \in \{\text{LM}(G)\} = \{x^3, y^3, x^2 y, x^2 z n\}$, which implies that either $x^2$ or $y^3$ divides $\text{LM}(g)$. On the other hand, $\text{LM}(g)$ is not divisible by any element in $\text{LM}(G)$ because $g$ is reduced with respect to $G$. Therefore, $\text{LM}(g) = x^2 z^{n-\ell}$ for some $1 \leq \ell \leq n$, and so
\[
g = c x^2 z^{n-\ell} + xg_1(y, z) + g_0(y, z)
\]
for some $c \in k \setminus \{0\}$ and $g_1(y, z), g_0(y, z) \in \mathbb{Q}[y, z]$. Let $r$ be the remainder on division of $z^m g$ by the polynomial $x^2 z^n - xy^2$,
\[
r = z^m g - (x^2 z^n - xy^2) c z^{m-\ell}
\]
\[
= z^m (c x^2 z^{n-\ell} + xg_1(y, z) + g_0(x, y)) - (x^2 z^n - xy^2) c z^{m-\ell}
\]
\[
= z^m (xg_1(y, z) + g_0(y, z)) + cxy^2 z^{m-\ell}.
\]

As $\deg x(r) < 2$ and $\deg y(r) < 3$, no term of $r$ is divisible by any element in $\text{LM}(G)$, which means $r$ is the normal form of $z^m g$ with respect to $G$. Recall that $z^m g \in I$, we have $r = 0$, which leads to $c = 0$ and $xg_1 + g_0 = 0$, and so $g = 0$. This is a contradiction. Claim 1 is proved. 

- **Claim 2.** For $n \in \mathbb{Z}_+$, the integral closure of the ideal $I = \langle x^3, y^3, x^2 y, x^2 z n - xy^2 \rangle \subseteq k[x, y, z]$ is $\bar{I} = \langle x^3, y^3, x^2 y, x^2 z n, xy^2 \rangle$.

**Proof of Claim 2.** Let $I_1 = \langle x^3, y^3 \rangle \subseteq I$, then by Proposition 2.3, the monomial $xy^2$ is in the integral closure of $I_1$, and so $xy^2$ is in the integral closure of $I$. Therefore $I + \langle xy^2 \rangle \subseteq \bar{I}$. On the other hand, $I + \langle xy^2 \rangle = \langle x^3, y^3, x^2 y, x^2 z n, xy^2 \rangle$ and it is integrally closed according to Proposition 2.3, which leads to $I + \langle xy^2 \rangle = \bar{I}$. Claim 2 is proved. 

- **Claim 3.** The ideal $\bar{I} = \langle x^3, y^3, x^2 y, x^2 z n, xy^2 \rangle \subseteq k[x, y, z]$ is not primary, where $n \in \mathbb{Z}_+$.

**Proof of Claim 3.** Because $x^2 z n \in I$ while $x^2 \notin \bar{I}$ and $(z^n)^m \notin \bar{I}$ for any $m \in \mathbb{Z}_+$, the ideal $\bar{I}$ is not primary. Claim 3 is proved. 

In contrast, there exist non-primary ideals whose integral closures are primary. For instance,

\[
I = \langle x^2, y^2, xy \rangle = \langle x^2, xy, y^2 \rangle \cap \langle x^2, y^2, z \rangle
\]
is not primary but its integral closure

\[
\bar{I} = \langle x^2, y^2, xy \rangle
\]
is primary using Proposition 2.3.

### 3.2. Whitney stratification of the hypersurface defined by $f = x^6 + y^6 + x^4 z t + z^3$.

The following example was given in [2] by Briançon and Speder to show that Whitney equisingularity does not imply Zariski equisingularity. The Jacobian ideal $J$ of $f$ is primary and its integral closure $\bar{J}$ of $J$ is not, which gives another counterexample to Krull’s question. Moreover, the embedded prime of $\bar{J}$ happens to be the vanishing ideal of the points where Whitney equisingularity fails.
Example 3.3. Let \( f = x^6 + y^6 + x^4zt + z^3 \in \mathbb{Q}[x, y, z, t] \), and its Jacobian ideal
\[
J = \langle x^4t + 3z^2, x^4z, y^5, 3x^3 + 2x^3zt \rangle.
\]
We first show that \( J \) is a primary ideal, while the integral closure
\[
\tilde{J} = \langle 3x^2yz + 2yz^2t, 3x^3z + 2xz^2t, x^4t + 3z^2, y^1z, x^4z, y^5, 3x^3y^2 + 2y^3zt, 3x^3y^2 + 2xy^2zt, 9x^4y - 4yz^2t^2, 3x^5 + 2x^3zt, x^3yzt, x^4y^3, 4y^3zt^3 + 27y^3z, xy^3zt^2, 2x^3y^2t^2 - 9xy^2z, 4xy^4t^3 + 27xy^4 \rangle
\]
is not primary. Its associated primes are
\[
\langle z, y, x \rangle \text{ and } \langle z, y, x, 4t^3 + 27 \rangle.
\]
One can verify the result by Macaulay2, see Appendix A.2.

Before we show that the hypersurface defined by \( f = 0 \) is not Whitney regular along its singular locus. Let us recall the definitions of Whitney’s conditions (a) and (b).

Let \( X \) and \( Y \) be two smooth manifolds in \( k^n, k = \mathbb{R} \text{ or } \mathbb{C} \). Suppose \( X \cap Y = \emptyset \) and \( Y \) is contained in the closure of \( X \). Let \( p \) be a point in \( Y \).

- The pair \((X, Y)\) is said to satisfy Whitney’s condition (a) at \( p \) if for any sequence \( p_\epsilon \in X \), \( p_\epsilon \to p \) and \( T_{p_\epsilon}X \to T \), then \( T_{p_\epsilon}Y \subset T \).

- The pair \((X, Y)\) is said to satisfy Whitney’s condition (b) at \( p \) if for any sequences \( p_\epsilon \in X \) and \( q_\epsilon \in Y \) such that \( p_\epsilon \to p \), \( q_\epsilon \to q \), \( T_{p_\epsilon}X \to T \), and the lines \( \overrightarrow{p_\epsilon q_\epsilon} \) converges to a line \( \ell \) in the projective space \( \mathbb{P}^{n-1} \), then \( \ell \subset T \). Here \( \overrightarrow{p_\epsilon q_\epsilon} \) denotes the unique line through the two points \( p_\epsilon \) and \( q_\epsilon \).

Back to Example 3.3, recall that \( f = x^6 + y^6 + x^4zt + z^3 \in \mathbb{Q}[x, y, z, t] \) and the Jacobian ideal of \( f \) is \( J = \langle x^4t + 3z^2, x^4z, y^5, 3x^3 + 2x^3zt \rangle \). The radical ideal of \( J \) is \( \sqrt{J} = \langle x, y, z \rangle \). Let \( V_1 \) be the hypersurface defined by \( f = 0 \) and \( V_2 \) be its singular locus. Then we have
\[
V_2 = \{(0, 0, 0, t) \mid t \in \mathbb{C} \} \subset \mathbb{C}^4.
\]

We compute the set of points where the pair \((V_1 \setminus V_2, V_2)\) does not satisfy Whitney’s conditions using the criterion in [12, Lemma 2.8]. We discover three such points defined by the variety
\[
V_3 = \{(0, 0, 0, t) \in \mathbb{C}^4 \mid 4t^3 + 27 = 0 \}.
\]
These points can also be found by running algorithms in [8, 9]. The following is a formal proof for the statement.

Theorem 3.4. The pair \((V_1 \setminus V_2, V_2)\) does not satisfy Whitney’s condition (a) and (b) at the points in \( V_3 \).

Proof. Since Whitney’s condition (b) implies Whitney’s condition (a). In the following, we only prove the pair \((V_1 \setminus V_2, V_2)\) does not satisfy Whitney’s condition (a).

Let \( p = (0, 0, 0, \xi) \) be a point in \( V_3 \), where \( \xi = (-\sqrt[3]{27/4})\omega \) and \( \omega \) is one of the cube roots of unity, i.e. \( \omega^3 = 1 \). Consider the sequence of points
\[
p_\epsilon = (\epsilon, 0, c\epsilon^2, \xi) \text{ where } \epsilon \neq 0 \text{ and } c = (\sqrt[3]{1/2})\omega^2.
\]
Note that \( \xi = -3c^2 \) and \( c^3 = 1/2 \). For any \( \epsilon \neq 0 \), we have \( p_\epsilon \in V_1 \setminus V_2 \) because
\[
\begin{align*}
\epsilon^6 + 0 + \epsilon^4(c\epsilon^2)\xi + (c\epsilon^2)^3 &= (1 + c^2 + c^3)\epsilon^6, \\
&= (1 - 3c^2 + c^3)\epsilon^6, \\
&= (1 - 2c^3)^6 = 0.
\end{align*}
\]
Also, $p_\epsilon \to p$ as $\epsilon \to 0$. The Jacobian matrix of $f$ is
\[
\text{Jac}(f) = \begin{pmatrix}
\frac{\partial f}{\partial x}, & \frac{\partial f}{\partial y}, & \frac{\partial f}{\partial z}, & \frac{\partial f}{\partial t}
\end{pmatrix}
\]
\[
= \begin{pmatrix}
6x^5 + 4x^3zt, & 6y^5, & 3z^2 + x^4t, & x^4z
\end{pmatrix}
\]
Evaluating the Jacobian matrix of $f$ at $p_\epsilon = (\epsilon, 0, cc^2, \xi)$, $\xi = -3c^2$, we get
\[
\text{Jac}(f)|_{p_\epsilon} = \begin{pmatrix}
6\epsilon^5 + 4\epsilon^3(cc^2)\xi, & 0, & 3(cc^2)^2 + \epsilon^4, & \epsilon^4(cc^2)
\end{pmatrix}
\]
\[
= \begin{pmatrix}
6\epsilon^5 - 12\epsilon^3\epsilon^5, & 0, & 3\epsilon^2\epsilon^4 + \epsilon^4(-3\epsilon^2), & \epsilon^6
\end{pmatrix}
\]
\[
= (0, \ 0, \ 0, \ \epsilon^6).
\]
The tangent space of $V_1$ at $p_\epsilon$, denoted as $T_{p_\epsilon}V_1$, is the kernel of $\text{Jac}(f)|_{p_\epsilon}$. Therefore, we have
\[
T_{p_\epsilon}V_1 = \{ (x, y, z, t) \in \mathbb{C}^4 \mid t = 0 \},
\]
which is not related to $\epsilon$. Let
\[
T = \{ (x, y, z, t) \in \mathbb{C}^4 \mid t = 0 \},
\]
then
\[
T_{p_\epsilon}V_1 \to T \text{ as } \epsilon \to 0.
\]
Recall that
\[
V_2 = \{ (0, 0, 0, t) \in \mathbb{C}^4 \}
\]
a linear space, so the tangent space of $V_2$ at $p$ is $V_2$ itself. Now we have
\[
T_pV_2 = V_2 \not\subset T,
\]
which violates Whitney’s condition (a).

Furthermore, it is interesting to know that Example 3.3 also provides an example to show that Whitney stratification is different from the stratification given by isosingular sets in [7]. Let $V_0 = \{ (0, 0, 0, 0) \}$ be the origin. The stratification of $V_1$ given by isosingular sets is
\[
\Sigma_S := (V_1 \setminus V_2, \ V_2 \setminus V_0, \ V_0),
\]
and the minimal Whitney Stratification of $V_1$ is
\[
\Sigma_W := (V_1 \setminus V_2, \ V_2 \setminus V_3, \ V_3).
\]
The origin is an isosingular point [7, Definition 5.1] while all three points in $V_3$ are not. Moreover, the hypersurface $V_1$ fails Zariski’s equisingular conditions at the origin [17]. Since Zariski equisingularity implies Whitney equisingularity [15, 17], we know that the hypersurface $V_1$ fails to be Zariski equisingular along $V_2$ at least at four points including the origin and three points in $V_3$ (1).

In Example 3.3, the hypersurface defined by $f = x^6 + y^6 + x^4zt + z^3$ is in 4-dimensional space. Let $H_f$ be the hypersurface defined by $f$ in $\mathbb{R}^4$. We cannot draw a figure of $H_f$, however, we can consider the following “slice” of the $H_f$:
\[
H_f \cap \{ (x, y, z, t) \in \mathbb{R}^4 \mid y = 0 \},
\]
which is homeomorphic to the surface
\[
H_g = \{ (x, z, t) \in \mathbb{R}^3 \mid g = x^6 + x^4zt + z^3 = 0 \}
\]
in 3-dimensional space. We can see how the surface $H_g$ looks like around the point $(0, 0, -\sqrt[3]{27/4})$ (see Figure 2), which gives a glimpse of how the hypersurface $H_f$ behaves around the real point $(0, 0, 0, -\sqrt[3]{27/4})$. We also find that the Jacobian ideal of the polynomial $g = x^6 + x^4zt + z^3$ is a counterexample to Krull’s question.
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4. Conclusion

We give counterexamples of primary ideals whose integral closures are not primary. Some of our examples are valid for any characteristics; this fills the gap left by Huneke’s example that works only in positive characteristics. Another counterexample is the Jacobian ideal $J$ of $f(x, y, z, t) = x^6 + y^6 + x^4zt + z^3$. The hypersurface $V_1$ defined by $f = 0$ is given by Briançon and Speder to show that Whitney equisingularity does not imply Zariski equisingularity for the set germs at the origin. Let $V_2 = \{(0, 0, 0, t)\}$ be the singular locus of $V_1$, $\bar{J}$ be the integral closure
of \( J \), and \( V_3 = \{(0,0,0,t) \mid 4t^3 + 27 = 0\} \) be the variety of the embedded prime of \( \tilde{J} \). It turns out that the variety \( V_3 \) contains exactly all the points where the pair \((V_1 \setminus V_2, V_2)\) fails Whitney’s conditions. Moreover, we show that the stratification defined by isosingular sets is different from Whitney stratification. In particular, the pair \((V_1 \setminus V_2, V_2)\) satisfies Whitney’s conditions at the origin \( V_0 = \{(0,0,0,0)\} \), while \( V_0 \) is an isosingular point. On the other hand, all three points in \( V_3 \) are not isosingular points. It is interesting to notice that the hypersurface is not Zariski equisingular along its singular locus at both \( V_0 \) and \( V_3 \).

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APPENDIX A.

A.1.

\begin{verbatim}
i1 : R = QQ[x, y, z];
i2 : I = ideal (x^3, y^3, x^2 * y, x^2 * z - x * y^2);
i3 : isPrimary (I)
o3 = true
i4 : primaryDecomposition (integralClosure (I))
o4 = {ideal(x^2, y^3, xy^2), ideal(z, y, x^3)}
\end{verbatim}

Here, “QQ” is the rational number field \( \mathbb{Q} \), and one can replace \( \mathbb{Q} \) by other computational fields of positive characteristics available in Macaulay2.

A.2.

\begin{verbatim}
i1 : R = QQ[x, y, z, t];
i2 : I = ideal (x^6 + y^6 + x^4 * z * t + z^3);
i3 : J = trim ideal ( jacobian (I) )
o3 = ideal(x^4t + 3z^2, x^4z, y^5, 3x^5 + 2x^3zt)
i4 : associatedPrimes (J)
o4 = {ideal (z, y, x)}
i5 : associatedPrimes ( integralClosure (J) )
o5 = {ideal (z, y, x), ideal (z, y, x, 4t^3 + 27)}
\end{verbatim}

APPENDIX B.

**Definition B.1.** [16, Definition 2.3] An ideal \( I \) of a polynomial ring over \( \mathbb{Q} \) is called a pseudo-primary ideal if \( \sqrt{I} \) is a prime ideal.

**Proposition B.2.** [16, Proposition 2.11] Let \( I \) be a pseudo-primary ideal in \( \mathbb{Q}[x_1, \ldots, x_n] \) with \( \sqrt{I} = P \) and let \( Q \) be the primary component of \( I \) with \( \sqrt{Q} = P \). Suppose that a subset \( U \) of \{\( x_1, \ldots, x_n \)\} is a maximally independent set modulo \( P \). Then \( Q = IQ(U)[\{x_1, \ldots, x_n\} \setminus U] \).
Proposition B.3. [1, Proposition 8.94] Let $k$ be a field, $I$ be an ideal of $k[x_1, \ldots, x_n]$, $U$ be any subset of $\{x_1, \ldots, x_n\}$, and $G$ be a Gröbner basis of $I$ with respect to an inverse block order $<$ in $\{x_1, \ldots, x_n\}$ such that $U \ll \{x_1, \ldots, x_n\} \setminus U$. Set
\[ f = \text{lcm}\{\text{LC}(g) \mid g \in G\}, \]
where $\text{LC}(g) \in k[U]$ is the leading coefficient of $g$ as an element in $k(U)[\{x_1, \ldots, x_n\} \setminus U]$ with respect to the restriction $<'$ of $<$ to $\{x_1, \ldots, x_n\} \setminus U$. Then $I_k(U)[\{x_1, \ldots, x_n\} \setminus U] = I : f^\infty$.

Lemma B.4. Let $R = k[x, y, z]$, $n \in \mathbb{Z}_+$, and $I = \langle x^3, y^3, x^2y, x^2z^n - xy^2 \rangle \subset R$ be an ideal, $P = \sqrt{I} = \langle x, y \rangle$. Let $Q$ be the primary component of $I$ with $\sqrt{Q} = P$. Then
\[ Q = I : z^\infty \]
where $I : z^\infty = \{g \in k[x, y, z] \mid z^m g \in I \text{ for some } m\}$.

Proof. Definition B.1 and the proof of Proposition B.2 are valid for polynomial rings over any field. $I$ is a pseudo-primary ideal since $P = \sqrt{I} = \langle x, y \rangle$ is a prime ideal. The set $U = \{z\}$ is a maximally independent set modulo $P$. By Proposition B.2, $Q = I_k(z)[x, y]$.

The set of generators $G = \{x^3, y^3, x^2y, x^2z^n - xy^2\}$ is a Gröbner basis of $I$ with respect to the lexicographic order $x > y > z$. The leading coefficients of the polynomials in $G$ as elements in $k(z)[x, y]$ are
\[ \text{LC}(x^3) = \text{LC}(y^3) = \text{LC}(x^2y) = 1, \quad \text{LC}(x^2z^n - xy^2) = z^n. \]
Therefore $z^n = \text{lcm}\{\text{LC}(g) \mid g \in G\}$. By Proposition B.3, $Q = I : (z^\infty)^\infty = I : z^\infty$. \qed