AN ALTERNATING ANALOGUE OF $U(\mathfrak{gl}_n)$ AND ITS REPRESENTATIONS

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ABSTRACT. In 2010, V. Futorny and S. Ovsienko gave a realization of $U(\mathfrak{gl}_n)$ as a subalgebra of the ring of invariants of a certain noncommutative ring with respect to the action of $S_1 \times S_2 \times \cdots \times S_n$ where $S_j$ is the symmetric group on $j$ variables. An interesting question is what a similar algebra would be in the invariant ring with respect to a product of alternating groups. In this paper we define such an algebra, denoted $\mathcal{A}(\mathfrak{gl}_n)$, and show that it is a Galois ring. For $n = 2, 3$ we find generators and relations with some similarities to Kac-Moody algebras. We also discuss some techniques to construct Galois orders from Galois rings. Lastly, we study categories of finite-dimensional modules and generic Gelfand-Tsetlin modules over $\mathcal{A}(\mathfrak{gl}_n)$.

KEYWORDS: alternating group, enveloping algebra, Gelfand-Kirillov Conjecture, Gelfand-Tsetlin modules, weight modules.

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1. INTRODUCTION

The study of algebra-subalgebra pairs is an important technique used in the representation theory of Lie algebras [17, 2]. In 2010, Futorny and Ovsienko focused on so called semicommutative pairs $\Gamma \subset \mathcal{W}$, where $\mathcal{W}$ is an associative (noncommutative) $\mathbb{C}$-algebra and $\Gamma$ is an integral domain [17]. This situation generalizes the pair $(\Gamma, U(\mathfrak{gl}_n))$ where $\Gamma$ is the Gelfand-Tsetlin subalgebra $\Gamma = \mathbb{C}[\cup_{k=1}^n Z(U(\mathfrak{gl}_k))]$ [13, 2]. Galois rings and Galois orders were originally defined and studied by Futorny and Ovsienko in [13] and [10]. They form a collection of algebras that contains many important examples including: generalized Weyl algebras defined by independently by Bavula [1] and Rosenberg [20] in the early nineties, the universal enveloping algebra of $\mathfrak{gl}_n$, shifted Yangians and finite $W$-algebras [8], Coulomb branches [22], and $U_q(\mathfrak{gl}_n)$ [7]. Their structures and representations have been studied in [8, 5, 17], and [19].

In [9], Futorny and Ovsienko described $U(\mathfrak{gl}_n)$ as the subalgebra of the ring of invariants of a certain noncommutative ring with respect to the action of $S_1 \times S_2 \times \cdots \times S_n$ where $S_j$ is the symmetric group on $j$ variables such that $U(\mathfrak{gl}_n)$ was a Galois order with respect to its Gelfand-Tsetlin subalgebra $\Gamma$.

We recall in Galois theory, given a Galois extension $L/K$ with $\text{Gal}(L/K) = G$ the subgroups $\tilde{G}$ of $G$ correspond to intermediate fields $K$ with $\text{Gal}(L/K) = \tilde{G}$ with normal subgroups of particular interest. Since $S_n$ has only one normal subgroup for $n \geq 5$, one might wonder what the object similar to $U(\mathfrak{gl}_n)$ would be if we considered the invariants with respect to the normal subgroup $A_1 \times A_2 \times \cdots \times A_n$ where $A_j$ is the alternating group on $j$ variables. This paper describes such an algebra, denoted by $\mathcal{A}(\mathfrak{gl}_n)$ (see Definition 2.1). This provides the first natural example of a Galois ring whose ring $\Gamma$ is not a semi-Laurent polynomial ring, that is, a tensor product of polynomial rings and Laurent polynomial rings. Additionally, our symmetry group $A_1 \times A_2 \times \cdots \times A_n$ is not a complex reflection group. Our algebra $\mathcal{A}(\mathfrak{gl}_n)$ is an extension of $U(\mathfrak{gl}_n)$ by $n-1$ elements $V_2, \ldots, V_n$. In Proposition 2.2 we prove some properties of $\mathcal{A}(\mathfrak{gl}_n)$ that are quite similar to $U(\mathfrak{gl}_n)$. For example, it is shown that the "Weyl Group" of $\mathcal{A}(\mathfrak{gl}_n)$ is the alternating group $A_n$ in the sense that there is a natural extension $\tilde{\varphi}_{HC}$ of the Harish-Chandra homomorphism $\varphi_{HC}: Z(U(\mathfrak{gl}_n)) \to S(\mathfrak{h}) \cong \mathbb{C}[x_1, \ldots, x_n]$, such that

$$\tilde{\varphi}_{HC}: Z(\mathcal{A}(\mathfrak{gl}_n)) \cong \mathbb{C}[x_1, \ldots, x_n]^{A_n}.$$

Moreover, there is a chain of subalgebras $\mathcal{A}(\mathfrak{gl}_1) \subset \mathcal{A}(\mathfrak{gl}_2) \subset \cdots \subset \mathcal{A}(\mathfrak{gl}_n)$. In Section 3 we give multiple descriptions of $\mathcal{A}(\mathfrak{gl}_2)$ and it is proved as a Galois order. Example 4.2 shows that $\mathcal{A}(\mathfrak{gl}_n)$ is not a Galois order for $n \geq 3$. The rest of Section 4 provides a concise set of generators and relations for $\mathcal{A}(\mathfrak{gl}_3)$. In Section 5 we show that the category of finite-dimensional modules in not semi-simple and classify simple finite-dimensional weight modules. In Section 6 we provide a technique to turn a general Galois ring into a Galois order that is related to localization (see Theorem 6.2). We use this to prove that a family of simple examples are Galois orders (see Corollary 6.3) and that a localization of $\mathcal{A}(\mathfrak{gl}_n)$ is a (co-)principal Galois
order over the localized $\tilde{\Gamma}$ (see Definition \[1.12]\ and Corollary \[6.11]\). We use this localization to construct canonical Gelfand-Tsetlin modules over $\mathcal{A}(gl_n)$ in Section \[7]. Finally, in Section \[8] we compute the division ring of fractions and prove, for $n \leq 5$, $\mathcal{A}(gl_n)$ satisfies the Gelfand-Kirillov conjecture (see \[13]\). For the latter, we use Maeda’s positive solution to Noether’s problem for the alternating group $A_5$ \[18\], and Futorny-Schwarz’s Theorem 1.1 in \[12\].

1.1. Galois orders. Galois orders were introduced in \[9\]. We will be following the set up from \[15\]. Let $\Lambda$ be an integrally closed domain, $G$ a finite subgroup of $\text{Aut}(\Lambda)$, and $\mathcal{M}$ a submonoid of $\text{Aut}(\Lambda)$. We will adhere to the following assumptions for the entire paper:

\[ (A1) \quad (\mathcal{M}, \mathcal{M}^{-1}) \cap G = 1_{\text{Aut} \Lambda} \quad \text{(separation)} \]

\[ (A2) \quad \forall g \in G, \forall \mu \in \mathcal{M} : g \mu = g \circ \mu \circ g^{-1} \in \mathcal{M} \quad \text{(invariance)} \]

\[ (A3) \quad \Lambda \text{ is noetherian as a module over } \Gamma^G \quad \text{(finiteness)} \]

Let $L = \text{Frac}(\Lambda)$ and $\mathcal{L} = L \# \mathcal{M}$ the skew monoid ring, which is defined as the free left $L$-module on $\mathcal{M}$ with multiplication given by $a_1 \mu_1 \cdot a_2 \mu_2 = (a_1 \mu_1)(a_2 \mu_2)$ for $a_i \in L$ and $\mu_i \in \mathcal{M}$. As $G$ acts on $\Lambda$ by automorphisms, we can easily extend this action to $L$, and by (A2), $G$ acts on $\mathcal{L}$. So we consider the following $G$-invariant subrings $\Gamma = \Lambda^G$, $K = L^G$, and $\mathcal{K} = \mathcal{L}^G$.

A benefit of these assumptions is the following lemma.

**Lemma 1.1** (\[15\], Lemma 2.1 (ii), (iv) \& (v)).

(i) $K = \text{Frac}(\Gamma)$.

(ii) $\Lambda$ is the integral closure of $\Gamma$ in $L$.

(iii) $\Lambda$ is a finitely generated $\Gamma$-module and noetherian ring.

What follows are some definitions and propositions from \[9\].

**Definition 1.2** (\[9\]). A finitely generated $\Gamma$-subring $\mathcal{U} \subseteq \mathcal{K}$ is called a Galois $\Gamma$-ring (or Galois ring with respect to $\Gamma$) if $K \mathcal{U} = \mathcal{U} K = \mathcal{K}$.

**Proposition 1.3** (\[9\], Proposition 4.1). Assume a $\Gamma$-ring $\mathcal{U} \subseteq \mathcal{K}$ is generated by $u_1, \ldots, u_k \in \mathcal{U}$.

1. If $\bigcup_{i=1}^{k} \text{supp } u_i$ generate $\mathcal{M}$ as a monoid, then $\mathcal{U}$ is a Galois ring.

2. If $L \mathcal{U} = L \# \mathcal{M}$, then $\mathcal{U}$ is a Galois ring.

**Theorem 1.4** (\[9\], Theorem 4.1 (4)). Let $\mathcal{U}$ be a Galois $\Gamma$-ring, then the center $Z(\mathcal{U})$ of the algebra $\mathcal{U}$ equals $\mathcal{U} \cap K^\mathcal{M}$, where $K^\mathcal{M} = \{ k \in K \mid \mu(k) = k \forall \mu \in \mathcal{M} \}$.

**Definition 1.5** (\[9\]). A Galois $\Gamma$-ring $\mathcal{U}$ in $\mathcal{K}$ is a left (respectively right) Galois $\Gamma$-order in $\mathcal{K}$ if for any finite-dimensional left (respectively right) $K$-subspace $W \subseteq \mathcal{K}$, $W \cap \mathcal{U}$ is a finitely generated left (respectively right) $\Gamma$-module. A Galois $\Gamma$-ring $\mathcal{U}$ in $\mathcal{K}$ is a Galois $\Gamma$-order in $\mathcal{K}$ if $\mathcal{U}$ is a left and right Galois $\Gamma$-order in $\mathcal{K}$.

**Definition 1.6** (\[2\]). Let $\Gamma \subseteq \mathcal{U}$ be a commutative subalgebra. $\Gamma$ is called a Harish-Chandra subalgebra in $\mathcal{U}$ if for any $u \in U$, $\Gamma u$ is finitely generated as both a left and as a right $\Gamma$-module.

**Theorem 1.7** (\[9\], Theorem 5.2). Assume that $\mathcal{U}$ is a Galois ring, $\Gamma$ is finitely generated and $\mathcal{M}$ is a group.

1. Assume $m^{-1}(\Gamma) \subseteq \Gamma$ (respectively $m(\Gamma) \subseteq \Gamma$). Then $\mathcal{U}$ is right (respectively left) Galois order if and only if $\mathcal{U} \cap \mathcal{M}$ is an integral extension of $\Gamma$.

2. Assume that $\Gamma$ is a Harish-Chandra subalgebra in $\mathcal{U}$. Then $\mathcal{U}$ is a Galois order if and only if $\mathcal{U} \cap \mathcal{M}$ is an integral extension of $\Gamma$.

The following are some useful results from \[15\].

**Proposition 1.8** (\[15\], Proposition 2.14). $\Gamma$ is maximal commutative in any left or right Galois $\Gamma$-order $\mathcal{U}$ in $\mathcal{K}$.

**Lemma 1.9** (\[15\], Lemma 2.16). Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be two Galois $\Gamma$-rings in $\mathcal{K}$ such that $\mathcal{U}_1 \subseteq \mathcal{U}_2$. If $\mathcal{U}_2$ is a Galois $\Gamma$-order, then so too is $\mathcal{U}_1$.

It is common to write elements of $L$ on the right side of elements of $\mathcal{M}$. 
Definition 1.10. For $X = \sum_{\mu \in \mathcal{M}} \mu \alpha_{\mu} \in \mathcal{L}$ and $a \in L$ defines the evaluation of $X$ at $a$ to be
\[ X(a) = \sum_{\mu \in \mathcal{M}} \mu(\alpha_{\mu} \cdot a) \in L. \]
Similarly defined is co-evaluation by
\[ X^\dagger(a) = \sum_{\mu \in \mathcal{M}} \alpha_{\mu} \cdot (\mu^{-1}(a)) \in L. \]

The following was independently defined by [21] called the universal ring.

Definition 1.11. The standard Galois $\Gamma$-order is as follows:
\[ \mathcal{K}_\Gamma := \{ X \in \mathcal{K} \mid X(\gamma) \in \Gamma \ \forall \gamma \in \Gamma \}. \]
Similarly we define the co-standard Galois $\Gamma$-order by
\[ \Gamma \mathcal{K} := \{ X \in \mathcal{K} \mid X^\dagger(\gamma) \in \Gamma \ \forall \gamma \in \Gamma \}. \]

Definition 1.12. Let $\mathcal{U}$ be a Galois $\Gamma$-ring in $\mathcal{K}$. If $\mathcal{U} \subseteq \mathcal{K}_\Gamma$ (resp. $\mathcal{U} \subseteq \Gamma \mathcal{K}$), then $\mathcal{U}$ is called a principal (resp. co-principal) Galois $\Gamma$-order.

In [15] it was shown that any (co-)principal Galois $\Gamma$-order is a Galois order in the sense of Definition 1.5.

2. Definition of the Alternating Analogue of $U(gl_n)$

2.1. Galois order realization of $U(gl_n)$. We first recall the realization of $U(gl_n)$ as a Galois $\Gamma$-order from [9]. Let $\Lambda = \mathbb{C}[x_{ki} \mid 1 \leq i \leq k \leq n]$, $S_n = S_1 \times S_2 \times \cdots \times S_n$, and $\Gamma = \Lambda^{S_n} = \mathbb{C}[e_{ki} \mid 1 \leq i \leq k \leq n]$. Here
\[ e_{ki} = e_{ki}(x_{k1}, \ldots, x_{kk}) = \sum_{1 \leq j_1 < \cdots < j_i \leq k} x_{kj_1} \cdots x_{kj_i}, \tag{1} \]
are the elementary symmetric polynomials. Also, let $L = \text{Frac}(\Lambda)$ and $K = \text{Frac}(\Gamma)$. Now, we construct a skew monoid ring. Let $\mathcal{M}$ be the subgroup of $\text{Aut}(\Lambda)$ generated by $\{ \delta^{ki} \}_{1 \leq i \leq k \leq n-1}$, where $\delta^{ki}$ is defined by
\[ \delta^{ki}(x_{ij}) = x_{ij} - \delta_{ik} \delta_{ij}. \tag{2} \]

We observe that $\mathcal{M} \cong \mathbb{Z}^{n(n-1)/2}$. Let $\mathcal{L} = L \# \mathcal{M}$ and $\mathcal{K} = (L \# \mathcal{M})^{S_n}$. In [9] the authors describe an embedding $\varphi : U(gl_n) \to \mathcal{K}$ defined by
\[ \varphi(E_k^\pm) = \sum_{i=1}^{k} (\delta^{ki})^{\pm1} a_{ki}^\pm, \quad \varphi(E_{kk}) = \sum_{j=1}^{k} (x_{kj} + j - 1) - \sum_{i=1}^{k-1} (x_{k-1,i} + i - 1), \tag{3} \]
where
\[ a_{ki}^\pm = \frac{1}{\prod_{j=1}^{k} (x_{ki} - x_{kj})}, \tag{4} \]
and $E_k^+ = E_{k+1,k}$, $E_k^- = E_{k+1,k}$ where $E_{ij}$ denotes the matrix units. Let $U_n = \varphi(U(gl_n))$. The algebra $U_n$ is a Galois $\Gamma$-order.

2.2. Defining $\mathcal{A}(gl_n)$. Let $\mathcal{A}_n = A_1 \times A_2 \times \cdots \times A_n$ and
\[ \tilde{\Gamma} = \Lambda_{\mathcal{A}_n} = \mathbb{C}[e_{ki}, V_{\ell} \mid 1 \leq i \leq k \leq n, 2 \leq \ell \leq n]. \tag{5} \]
Here
\[ V_{\ell} = V_{\ell}(x_{\ell1}, \ldots, x_{\ell\ell}) = \prod_{i<j} (x_{\ell i} - x_{\ell j}) \tag{6} \]
denotes the Vandermonde polynomial in the $\ell$ variables $x_{\ell1}, \ldots, x_{\ell\ell}$. Abstractly, $\tilde{\Gamma}$ is isomorphic to
\[ \mathbb{C}[T_{ki}, V_{\ell} \mid 1 \leq i \leq k \leq n, 2 \leq \ell \leq n]/(V_{\ell}^2 - D_{\ell}(T_{\ell1}, \ldots, T_{\ell\ell}) \mid 2 \leq \ell \leq n), \]
where $D_{\ell}(T_{\ell1}, \ldots, T_{\ell\ell})$ is the Vandermonde discriminant. Also, let $\tilde{K} = \text{Frac}(\tilde{\Gamma})$ and $\tilde{\mathcal{K}} = (L \# \mathcal{M})^{\mathcal{A}_n}$.
Definition 2.1. The alternating analogue of $U(gl_n)$, denoted $\mathcal{A}(gl_n)$, is defined as the subalgebra of $\mathcal{K}$ generated by $U_n \cup \{V_2, V_3, \ldots, V_n\}$. Explicitly, $\mathcal{A}(gl_n)$ is the subalgebra of $\mathcal{L}$ generated by

$$X_k^± = \sum_{i=1}^k (\delta_{ki})^{±1} a_{ki}^± \quad \text{for } k = 1, \ldots, n - 1,$$

$$X_{kk} = \sum_{i=1}^k (x_{kj} + j - 1) - \sum_{i=1}^{k-1} (x_{k-1,i} + i - 1) \quad \text{for } k = 1, \ldots, n,$$

$$V_k = V_k(x_{k1}, \ldots, x_{kk}) = \prod_{i<j}(x_{ki} - x_{kj}) \quad \text{for } k = 1, \ldots, n - 1,$$

where $a_{ki}^±$ are defined in \[4\].

The following proposition lists some basic properties of $\mathcal{A}(gl_n)$.

Proposition 2.2.

(i) $U(gl_1) \cong U_n \subset \mathcal{A}(gl_n)$,
(ii) $\mathcal{A}(gl_n)$ is a Galois $\Gamma$-ring,
(iii) $V_n$ is central in $\mathcal{A}(gl_n)$,
(iv) $Z(\mathcal{A}(gl_n)) \cong \mathbb{C}[x_1, \ldots, x_n]^{A_n}$,
(v) there is a chain of subalgebras $\mathcal{A}(gl_1) \subset \mathcal{A}(gl_2) \subset \cdots \subset \mathcal{A}(gl_n)$,
(vi) $\mathcal{A}(gl_n)$ is the minimal extension of $U(gl_n)$ with properties \[(iv)\] and \[(v)\].

Proof. \[1\] Clear because $\varphi$ is injective and $\mathcal{A}(gl_n)$ contains $\varphi(E_k^±), \varphi(E_k k)$.

(ii) Define $\mathcal{X}$ as follows:

$$\mathcal{X} = \{X_i^±, X_{ij}, V_j \mid 1 \leq i \leq n, 2 \leq j \leq n\}.$$

Since $X_i^± \in \mathcal{X}$, it is clear that $\bigcup_{u \in \mathcal{X}} \text{supp } u$ generates $\mathcal{M}$. Thus, $\mathcal{A}(gl_n)$ is a Galois $\Gamma$-ring for every $n \geq 1$ by Proposition \[3\].

(iii) As $\delta_{ki}$ fixes $x_{ij}$ if $\ell \neq k$ and $k \neq n$, it follows that $V_n$ is central in $\mathcal{A}(gl_n)$.

(iv) We first show that $Z(\mathcal{A}(gl_n)) = \mathbb{C}(U_n, V_n)$. $\mathbb{C}(Z(U_n), V_n)) \subseteq Z(\mathcal{A}(gl_n))$ is clear. Next, we observe that $\mathcal{A}(gl_n) \subseteq (L \# \mathcal{M})^{\mathbb{A}_n}$, where $L = \mathbb{C}(x_{ki} \mid 1 \leq i \leq k \leq n - 1|x_{n1}, \ldots, x_{nn})$. By Theorem \[4\] we have

$$Z(\mathcal{A}(gl_n)) = \mathcal{A}(gl_n) \cap (L \# \mathcal{M})^{\mathbb{A}_n} \subseteq \mathcal{M}^{\mathbb{A}_n} \subseteq \mathbb{C}(Z(U_n), V_n).$$

Consider the Harish-Chandra homomorphism $\varphi_{HC}: Z(U(gl_n)) \rightarrow \mathbb{C}[x_1, \ldots, x_n]^{S_n}$. We can define an extension of this map $\bar{\varphi}_{HC}: \mathcal{A}(gl_n) \rightarrow \mathbb{C}[x_1, \ldots, x_n]$ as follows:

$$\bar{\varphi}_{HC}(X) = \begin{cases} \varphi_{HC}(\varphi^{-1}(X)), & X \in Z(U_n), \\ \prod_{1<i<j} (x_i - x_j), & X = V_n. \end{cases} \quad (7)$$

In conjunction with Chevalley’s Theorem (see \[16\]), $\varphi_{HC}$ provides an isomorphism with $\mathbb{C}[x_1, \ldots, x_n]^{S_n}$. The claim follows by recalling that $\mathbb{C}[x_1, \ldots, x_n]^{A_n}$ is generated by the symmetric polynomials and the Vandermonde polynomial.

(vi) We prove this result by induction on $n$. Since $\mathcal{A}(gl_1) = U(gl_1)$, the base step is clear. Assuming the claim holds for $\mathcal{A}(gl_{n-1})$, now consider an extension of $\mathcal{A}$ of $U(gl_n)$ satisfying \[(iv)\] and \[(v)\]. By \[(v)\] $\mathcal{A}$ contains $V_\ell$ for $\ell = 1, \ldots, n - 1$, and it contains $U(gl_n)$ by definition. From \[(iv)\] we get an element $\mathcal{X}$ that is central in $\mathcal{A}$ that maps to $\prod_{1<i<j} (x_i - x_j) \in \mathbb{C}[x_1, \ldots, x_n]^{A_n}$. This allows us to define an isomorphism $\tau: \mathcal{A} \rightarrow \mathcal{A}(gl_n)$ by sending $\{U(gl_n), V_\ell \mid \ell = 1, \ldots, n - 1\}$ to themselves and $V \mapsto V_n$. \[\square\]

Remark 1. In \[11\] another Galois algebra is described in the invariants of a Weyl algebra with respect to a single alternating group in Corollary 24 in \[11\].

3. The Structure of $\mathcal{A}(gl_2)$

In this section, we find a presentation for $\mathcal{A}(gl_2)$ as an extension of $U(gl_2)$ and as a generalized Weyl algebra as well as prove that it is a Galois $\Gamma$-order.

Lemma 3.1.

(i) $V_2$ commutes with every element of $U_2$.
(ii) $\mathcal{A}(gl_2) = U_2 \oplus (U_2 \cdot V_2)$
Proof. (i) Follows by Proposition 2.2 (iii)
(ii) Since $V_2$ commutes with everything in $U_2$,
\[
\mathcal{A}(\mathfrak{gl}_2) = \left\{ \sum_{j=0}^{\infty} u_j V_2^j \mid u_j \in U_2, \text{ at most finitely many } u_j \neq 0 \right\}.
\]
Since $V_2^2 \in U_2$, $\mathcal{A}(\mathfrak{gl}_2) = U_2 + U_2 \cdot V_2$. Now consider $(12)_2 := ((1), (12)) \in S_2$ acting on $\mathcal{A}$ by automorphisms. We have,
\[
(12)_2|u_2 = \text{Id}|u_2, \quad (12)_2|u_3v_2 = (-1) \cdot \text{Id}|u_2v_2.
\]
This implies that $\mathcal{A}(\mathfrak{gl}_2) = U_2 \oplus (U_2 \cdot V_2)$. □

We now give a presentation for $\mathcal{A}(\mathfrak{gl}_2)$ in terms of $U(\mathfrak{gl}_2)$.

**Proposition 3.2.** There is an isomorphism
\[
\widetilde{\varphi} : \frac{U(\mathfrak{gl}_2)[T_2]}{(T_2^2 - (-c_{21}^2 + 2c_{22} + 1))} \rightarrow \mathcal{A}(\mathfrak{gl}_2),
\]
where $T_2$ is an indeterminate and $c_{21}$ are the Gelfand invariants for $\mathfrak{gl}_2$. Explicitly
\[
\widetilde{\varphi}|U(\mathfrak{gl}_2) = \varphi, \quad \widetilde{\varphi}(T_2) = V_2,
\]
where $\varphi$ is the embedding from (3).

**Proof.** Let $p(T_2) = T_2^2 - (-c_{21}^2 + 2c_{22} + 1) \in U(\mathfrak{gl}_2)[T_2]$. Since $p(T_2)$ is degree two, $U(\mathfrak{gl}_2)[T_2]/(p(T_2))$ is free of rank 2 as a left $U(\mathfrak{gl}_2)$-module with basis $\{1, T_2\}$ where $T_2 = T_2 + (p(T_2))$. It follows Lemma 3.1 (ii) that $\mathcal{A}(\mathfrak{gl}_2)$ is also free of rank 2 with basis $\{1, V_2\}$ via the isomorphism $\varphi$ in (3). Therefore there is an isomorphism $\tilde{\varphi} : U(\mathfrak{gl}_2)[T_2]/(p(T_2)) \rightarrow \mathcal{A}(\mathfrak{gl}_2)$ as $U(\mathfrak{gl}_2)$-modules sending 1 to 1 and $T_2$ to $V_2$. Thus, it suffices to show that $\tilde{\varphi}(T_2^2) = V_2^2$.

To show this, we calculate the images of $c_{21}$ under $\varphi$:
\[
\varphi(c_{21}) = \varphi(E_{11} + E_{22}) = (x_{11}) + (x_{21} + x_{22} - x_{11} + 1) = x_{21} + x_{22} + 1,
\]
\[
\varphi(c_{22}) = \varphi(E_{11}^2 + E_1^+ E_1^- + E_1^- E_1^+ + E_{22}^2) = x_{21}^2 + x_{22}^2 + x_{21} + x_{22}.
\]
As such,
\[
\tilde{\varphi}(T_2^2) = \tilde{\varphi}(-c_{21}^2 + 2c_{22} + 1) = -\varphi(c_{21})^2 + 2\varphi(c_{22}) + 1
\]
\[
= -(x_{21} + x_{22} + 1)^2 + 2(x_{21}^2 + x_{22}^2 + x_{21} + x_{22}) + 1
\]
\[
= (x_{21} - x_{22})^2 = V_2^2.
\]
Therefore, $\tilde{\varphi}$ is an algebra isomorphism. □

**Theorem 3.3.** $\mathcal{A}(\mathfrak{gl}_2)$ is a Galois $\Gamma$-order.

**Proof.** We first observe that $\mathcal{A}(\mathfrak{gl}_2)$ is a Galois $\Gamma$-ring by Proposition 2.2 (ii). To prove that $\mathcal{A}(\mathfrak{gl}_2)$ is a Galois $\Gamma$-order, we will use Theorem 1.7. Since $\Gamma$ is a Harish-Chandra subalgebra of $U(\mathfrak{gl}_2)$, $\Gamma$ is a Harish-Chandra subalgebra of $\mathcal{A}(\mathfrak{gl}_2)$. Since $\mathfrak{h}_2$ is a group, all we need to show is that $\Gamma$ is maximal commutative in $\mathcal{A}(\mathfrak{gl}_2)$. This is clear because $\Gamma$ is maximal commutative in $U_2$, and $\Gamma$ is just an extension by a central element by Proposition 3.2. $\Gamma$ is maximal commutative in $\mathcal{A}(\mathfrak{gl}_2)$; therefore, $\mathcal{A}(\mathfrak{gl}_2)$ is a Galois $\Gamma$-order. □

The following shows that $\mathcal{A}(\mathfrak{gl}_2)$ is a generalized Weyl algebra [1], which gives another way to show it is a Galois order [9].

**Proposition 3.4.**
\[
\mathcal{A}(\mathfrak{gl}_2) \cong (C[h, c_{21}, c_{22}, T_2]/(p(T_2))) \langle \sigma, t \rangle
\]
where $\sigma(h) = h - 2$, $\sigma(a) = a$ for all $a \in \{c_{21}, c_{22}, T_2\}$, $t = \frac{1}{2}(c_{22} - h - \frac{1}{2}(h + c_{21})^2 - \frac{1}{2}(h - c_{21})^2)$

**Proof.** Straightforward, using [1]. □

We observe the following interesting property of $\mathcal{A}(\mathfrak{gl}_2)$ that we prove does not hold for general $n$ (see Proposition 4.10).
Proposition 3.5. \(\mathcal{A}(\mathfrak{gl}_2)\) has the property that \((\mathcal{A}(\mathfrak{gl}_2))^{\mathbb{Z}_2} = U_2\).

Proof. This becomes clear when we consider the direct sum decomposition shown in Lemma 3.1. Consider \(a + bV_2 \in \mathcal{A}(\mathfrak{gl}_2)\):

\[
a + bV_2 \in (\mathcal{A}(\mathfrak{gl}_2))^{\mathbb{Z}_2} \iff (12)_2(a + bV_2) = a + bV_2
\]

\[
\iff a - bV_2 = a + bV_2 \\
\iff b = 0 \\
\iff a + bV_2 = a \in U_2.
\]

Therefore, \((\mathcal{A}(\mathfrak{gl}_2))^{\mathbb{Z}_2} = U_2\). \(\square\)

4. The Structure of \(\mathcal{A}(\mathfrak{gl}_3)\)

Based on the result of the previous section, the next logical step is to see if similar results hold for \(\mathfrak{gl}_n\) with \(n \geq 3\). We will continue using the notation of the images of the generators of the \(U(\mathfrak{gl}_n)\) as before. As such:

\[
X_i^\pm := \varphi(E_i^\pm) \quad \text{and} \quad X_{ii} := \varphi(E_{ii}).
\]

4.1. Non-polynomial rational functions in \(\mathcal{A}(\mathfrak{gl}_3)\). Unlike in \(U(\mathfrak{gl}_3)\) and \(\mathcal{A}(\mathfrak{gl}_2)\), we can construct non-polynomial rational functions in \(\mathcal{A}(\mathfrak{gl}_3)\). It follows that for \(n \geq 3\), \(\mathcal{A}(\mathfrak{gl}_n)\) is not a Galois \(\Gamma\)-order, and the invariant property of \(\mathcal{A}(\mathfrak{gl}_2)\) does not hold.

Lemma 4.1. The following identity holds in \(\mathcal{A}(\mathfrak{gl}_3)\):

\[
\pm [X_1^\pm, V_2] = (\delta^{21})^{\pm 1} a_{21}^{\pm 1} - (\delta^{22})^{\pm 1} a_{22}^{\pm 1}.
\]

Proof. To show this, consider \(V_2 X_2^\pm:\)

\[
V_2 X_2^\pm = (x_{21} - x_{22})((\delta^{21})^{\pm 1} a_{21}^{\pm 1} + (\delta^{22})^{\pm 1} a_{22}^{\pm 1})
\]

\[
= (\delta^{21})^{\pm 1} a_{21}^{\pm 1}(x_{21} \pm 1 - x_{22}) + (\delta^{22})^{\pm 1} a_{22}^{\pm 1}(x_{21} - x_{22} + 1)
\]

\[
= X_2^\pm V_2 \pm ((\delta^{21})^{\pm 1} a_{21}^{\pm 1} - (\delta^{22})^{\pm 1} a_{22}^{\pm 1}).
\]

Therefore, \(\pm [X_1^\pm, V_2] = (\delta^{21})^{\pm 1} a_{21}^{\pm 1} - (\delta^{22})^{\pm 1} a_{22}^{\pm 1} \). \(\square\)

Let us denote the element described in (10) by \(\tilde{X}_2^\pm\). We define the following:

\[
A_{21}^+ := \frac{1}{2}(X_2^+ + \tilde{X}_2^+) = a_{21}^+ + \tilde{a}_{21}^+ \\
A_{21}^- := \frac{1}{2}(X_2^- + \tilde{X}_2^-) = (\delta^{21})^{-1} a_{21}^- \\
A_{22}^+ := \frac{1}{2}(X_2^+ - \tilde{X}_2^+) = (\delta^{22}) a_{22}^+ \\
A_{22}^- := \frac{1}{2}(X_2^- - \tilde{X}_2^-) = (\delta^{22})^{-1} a_{22}^-.
\]

By their definition, it is clear that they are in \(\mathcal{A}(\mathfrak{gl}_3)\).

The following example shows that if \(n \geq 3\), then \(\Gamma\) is not maximal commutative; hence, \(\mathcal{A}(\mathfrak{gl}_n)\) is not a Galois \(\Gamma\)-order by Proposition 1.3.

Example 4.2. The following element belongs to \(\mathcal{A}(\mathfrak{gl}_n)\) for \(n \geq 3\):

\[
A_{21}^+ A_{21}^- = \prod_{i=1}^3 (x_{3i} - x_{21} + 1) \cdot \frac{x_{11} - x_{21}}{x_{22} - x_{21}}
\]

This is a rational function; hence, it lies in \(\text{Cent}_{\mathcal{A}(\mathfrak{gl}_3)}(\tilde{\Gamma})\).

The following rather surprising fact shows that the property in Proposition 3.5 does not hold for larger \(n\).

Proposition 4.3. For \(n \geq 3\), \((\mathcal{A}(\mathfrak{gl}_n))^{\mathbb{Z}_n} \supseteq U_n\).

Proof. The fact that \(U_n \subset \mathcal{A}(\mathfrak{gl}_n)^{\mathbb{Z}_n}\) is obvious by definition. To show the containment is strict, we recall that because \(U_n\) is a Galois \(\Gamma\)-order, it is known that \(U_n \cap K = \Gamma\). Therefore, we consider \((\mathcal{A}(\mathfrak{gl}_n))^{\mathbb{Z}_n} \cap K\). Since \(U_n \subseteq U_n\) for every \(n \geq 3\), it suffices to show that \((\mathcal{A}(\mathfrak{gl}_3))^{\mathbb{Z}_n} \cap K = \Gamma\).

The object to prove this claim is constructed in the same way as in Example 4.2. It is quickly observed that

\[
A_{21}^+ A_{21}^- A_{22}^+ A_{22}^- = \prod_{i=1}^3 (x_{3i} - x_{21} + 1) \cdot \frac{x_{11} - x_{21}}{x_{22} - x_{21}} \cdot \frac{1}{x_{22} - x_{21}} \cdot \frac{1}{x_{22} - x_{21}} \cdot \frac{x_{11} - x_{22}}{x_{21} - x_{22}}.
\]
is invariant under the action of $S_3$. This element is clearly not in $\Gamma$, so this element is in $\mathcal{A}(\mathfrak{gl}_3)^{S_3} \cap K \setminus \Gamma$, thereby proving the claim.

4.2. Generators and relations for $n = 3$. Based on the previous subsection, we determine a set of generators and relations for $\mathcal{A}(\mathfrak{gl}_3)$, although we do not know if this constitutes a presentation.

**Proposition 4.4.** The algebra $\mathcal{A}(\mathfrak{gl}_3)$ is generated by \{X$_{11}, X_{22}, X_{33}, A^+_{11}, A^+_{22}, V_2, V_3\}$, where $A^\pm_{ij} := (\delta^j)^{\pm 1} a^\pm_{ij}$, $V_2 = x_{21} - x_{22}$, and $V_3 = \prod_{i<j}(x_{3i} - x_{3j})$. What follows is a list of known relations:

i) $[V_3, X] = 0$ for all $X \in \mathcal{A}(\mathfrak{gl}_3)$ (i.e. $V_3$ is central in $\mathcal{A}(\mathfrak{gl}_3)$),

ii) $[X, Y] = 0$ for all $X, Y \in \mathfrak{h} = \text{Span}_\mathbb{C} \{X_{11}, X_{22}, X_{33}, V_2, V_3\}$,

(iii) $[h, A^\pm_{ij}] = \pm \alpha_{ij}(h) A^\pm_{ij}$ for all $h \in \mathfrak{h}$ and $1 \leq j \leq i \leq 2$, where $\alpha_{ij}(h)$ are given by the following matrix:

\[
\begin{bmatrix}
    X_{11} & X_{22} & X_{33} & V_2 & V_3 \\
    1 & -1 & 0 & 0 & 0 \\
    \alpha_{21} & 0 & 1 & -1 & 1 & 0 \\
    \alpha_{22} & 0 & 1 & -1 & -1 & 0
\end{bmatrix},
\]

(iv) $[A^+_{21}, A^+_{22}] = 0$,

(v) $[A^+_{ii}, A^+_{ii}] = 0$ for $i = 1, 2$,

(vi) $[A^+_{11}, A^+_{11}] = X_{11} - X_{22}$,

(vii) $[A^+_{21}, A^+_{22}] + [A^+_{22}, A^+_{22}] = X_{22} - X_{33}$,

(viii) $[A^+_{11}, [A^+_{11}, A^+_{11}]] = 0$ for $i = 1, 2$,

(ix) $A^+_{22} V_2 A^+_{21} = A^+_{11} V_2 A^+_{21}$.

Proof. Any of the relations involving only elements from $U(\mathfrak{gl}_3)$ (such as (vi)) follow from $U(\mathfrak{gl}_3)$ relations by recalling that \{X$_{11}, X_{22}, X_{33}, A^+_{11}, A^+_{11}\}$ \in $\mathcal{A}(\mathfrak{gl}_3)$ correspond to \{E$_{11}, E_{22}, E_{33}, E_{12}, E_{21}\} \in U(\mathfrak{gl}_3)$. All that remains is to prove the relations involving new elements.

(i) This follows from Proposition 2.2 [ii].

(ii) This follows by observing that each is an element of $\Gamma$ which is a commutative ring.

(iii) By the statement at the beginning of this proof and (i), we only need to check the second two rows and the second to last column. Each is proved in an identical manner, we provide one below:

\[
V_2 \cdot A^+_{21} = (x_{21} - x_{22}) \cdot -\delta^{21} \prod_{i=1}^3 x_{3i} - x_{21}^2 \frac{x_{22} - x_{21}}{x_{22} - x_{21}}
= -\delta^{21} \prod_{i=1}^3 x_{3i} - x_{21} \frac{x_{22} - x_{21}}{x_{22} - x_{21}} \cdot (x_{21} - x_{22} + 1)
= A^+_{21} V_2 + A^+_{21}.
\]

Thus, $[V_2, A^+_{21}] = A^+_{21} = \alpha_{21}(V_2) A^+_{21}$.

(iv) Consider the following calculation:

\[
A^+_{21} A^+_{22} = -\delta^{21} \prod_{i=1}^3 x_{3i} - x_{21} \frac{x_{22} - x_{21}}{x_{22} - x_{21}} \cdot (\delta^{22})^{-1} \prod_{i=1}^3 x_{3i} - x_{21} \frac{x_{11} - x_{22}}{x_{21} - x_{22}}
= -\delta^{21} (\delta^{22})^{-1} \prod_{i=1}^3 x_{3i} - x_{21} \frac{x_{11} - x_{22}}{x_{22} - x_{21}} \cdot \frac{x_{22} - x_{21}}{x_{21} - x_{22}}
= -\delta^{21} (\delta^{22})^{-1} \prod_{i=1}^3 x_{3i} - x_{21} \frac{x_{11} - x_{22}}{x_{22} - x_{21}} \cdot \frac{x_{11} - x_{22}}{x_{21} - x_{22} + 1}
= (\delta^{22})^{-1} x_{11} - x_{22} \frac{x_{22} - x_{21}}{x_{22} - x_{21}} - \delta^{21} \prod_{i=1}^3 x_{3i} - x_{21} \frac{x_{11} - x_{22}}{x_{22} - x_{21}}
= A^+_{22} A^+_{21}.
\]

The other relation is proved similarly.

(v) Consider the following calculation:

\[
A^+_{11} A^+_{22} = -\delta^{11} (x_{21} - x_{11})(x_{22} - x_{11}) \cdot (\delta^{22})^{-1} \frac{x_{11} - x_{22}}{x_{21} - x_{22}}
\]

This demonstrates that (vii) holds.

We consider the relation \([E_{23}, E_{32}] = E_{22} - E_{33}\) mapped under \(\varphi\) from \(\mathcal{A}\):

\[
X_{22} - X_{33} = [X_2^+, X_3^-] = [A_{21}^+ + A_{22}^{-}, A_{21}^- + A_{22}^+]
\]

which has no \(x_{11}\)'s and as such commutes with \(A_{11}^-\). Thus, \([A_{11}^-, [A_{11}^-, A_{22}^-]] = 0\). The others are proved identically.

We observe that

\[
A_{11}^- A_{22} = (\delta_{11}^{-1}) \cdot (\delta_{22}^{-1})\frac{x_{11} - x_{22}}{x_{21} - x_{22}}
\]

\[
= (\delta_{11}^{-1} \delta_{22}^{-1})\frac{x_{11} - x_{22}}{x_{21} - x_{22}}
\]

\[
= (\delta_{22}^{-1})\frac{x_{11} - x_{22} + 1}{x_{21} - x_{22}} \cdot (\delta_{11}^{-1})^{-1}
\]

\[
A_{22}^- A_{11}^- = (\delta_{11}^{-1} \delta_{22}^{-1})^{-1} \frac{1}{x_{21} - x_{22}}
\]

This demonstrates that (vii) holds.

We observe that

\[
A_{11}^- A_{22} = (\delta_{11}^{-1}) \cdot (\delta_{22}^{-1})\frac{x_{11} - x_{22}}{x_{21} - x_{22}}
\]

\[
= (\delta_{11}^{-1} \delta_{22}^{-1})\frac{x_{11} - x_{22}}{x_{21} - x_{22}}
\]

\[
= (\delta_{22}^{-1})\frac{x_{11} - x_{22} + 1}{x_{21} - x_{22}} \cdot (\delta_{11}^{-1})^{-1}
\]

\[
A_{22}^- A_{11}^- = (\delta_{11}^{-1} \delta_{22}^{-1})^{-1} \frac{1}{x_{21} - x_{22}}
\]

which has no \(x_{11}\)'s and as such commutes with \(A_{11}^-\). Thus, \([A_{11}^-, [A_{11}^-, A_{22}^-]] = 0\). The others are proved identically.

We prove this by direct computation as follows:

\[
A_{22}^\pm V_2 A_{21}^\pm = (\delta_{21}^\pm)\delta_{22}^\pm \prod_{i=1}^{2\pm 1} \frac{(x_{2\pm 1,i} - x_{21})(x_{2\pm 1,i} - x_{22})}{x_{21} - x_{22}}
\]

\[
= -(\delta_{21}^\pm) \prod_{i=1}^{2\pm 1} x_{2\pm 1,i} - x_{21} \cdot (\delta_{22}^\pm) \prod_{i=1}^{2\pm 1} x_{2\pm 1,i} - x_{22}
\]

\[
= (\delta_{21}^\pm) \prod_{i=1}^{2\pm 1} x_{2\pm 1,i} - x_{21} \cdot \frac{V_2}{-V_2} (\delta_{22}^\pm) \prod_{i=1}^{2\pm 1} x_{2\pm 1,i} - x_{22}
\]

\[
= A_{21}^\pm V_2 A_{22}^\pm.
\]

This verifies that relation (ix) holds.

**Open Problem 1.** Determine whether the relations in Proposition 4.4 constitute a presentation for the algebra \(\mathcal{A}(\mathfrak{gl}_n)\).

5. Finite-Dimensional Modules over \(\mathcal{A}(\mathfrak{gl}_n)\)

Since, as was shown in Section 4, \(\mathcal{A}(\mathfrak{gl}_n)\) is not a Galois \(\widetilde{\Gamma}\)-order, techniques different from \(\mathcal{A}(\mathfrak{gl}_2)\) are required to study representations of \(\mathcal{A}(\mathfrak{gl}_n)\).

If we consider the case of \(n = 2\), we recall that \(\mathcal{A}(\mathfrak{gl}_2) \cong U(\mathfrak{gl}_2)[T_2]/(T_2^2 - (-c_{21}^2 + 2c_{22} + 1))\). As such, it makes sense to consider the induction and restriction functors between the categories of \(\mathcal{A}(\mathfrak{gl}_2)\)-modules and \(U(\mathfrak{gl}_2)\)-modules.
By applying the restriction functor to a given finite-dimensional simple module, we see that it decomposes to a direct sum of finite-dimensional simple $U(\mathfrak{gl}_2)$-modules, so the induction functor should help us to construct all of the possible finite-dimensional simple $\mathcal{A}(\mathfrak{gl}_2)$-modules.

**Proposition 5.1.** The finite-dimensional simple $\mathcal{A}(\mathfrak{gl}_2)$-modules are characterized by ordered pairs $(\lambda_2, \varepsilon_2)$, where $\lambda_2 := (\lambda_{21}, \lambda_{22}) \in \mathbb{C}^2$ is a weight for $U(\mathfrak{gl}_2)$ and $\varepsilon_2 \in \{1, -1\}$.

**Proof.** Recall that every finite-dimensional simple $U(\mathfrak{gl}_2)$-module is characterized a weight denoted by a pair of complex numbers $\lambda_2 = (\lambda_{21}, \lambda_{22})$; we will denote this module by $V(\lambda_2)$. We can induce such a module $V(\lambda_2)$ to a $\mathcal{A}(\mathfrak{gl}_2)$-module as follows,

$$\mathcal{A}(\mathfrak{gl}_2) \otimes_{U(\mathfrak{gl}_2)} V(\lambda_2).$$

So, it is important to describe $\mathcal{A}(\mathfrak{gl}_2)$ as a right $U(\mathfrak{gl}_2)$-module. By Proposition 5.2,

$$\mathcal{A}(\mathfrak{gl}_2) \cong \frac{U(\mathfrak{gl}_2)[T_2]}{\langle T_2^2 - (-c_{21}^2 + 2c_{22} + 1) \rangle} \cong U(\mathfrak{gl}_2) \oplus T_2 U(\mathfrak{gl}_2)$$

as right $U(\mathfrak{gl}_2)$-modules. Thus:

$$\mathcal{A}(\mathfrak{gl}_2) \otimes_{U(\mathfrak{gl}_2)} V(\lambda_2) \cong (U(\mathfrak{gl}_2) \oplus T_2 U(\mathfrak{gl}_2)) \otimes_{U(\mathfrak{gl}_2)} V(\lambda_2)$$

$$\cong (U(\mathfrak{gl}_2) \otimes_{U(\mathfrak{gl}_2)} V(\lambda_2)) \oplus (T_2 U(\mathfrak{gl}_2) \otimes_{U(\mathfrak{gl}_2)} V(\lambda_2))$$

$$\cong (1 \otimes_{U(\mathfrak{gl}_2)} V(\lambda_2)) \oplus (T_2 \otimes_{U(\mathfrak{gl}_2)} V(\lambda_2)).$$

As such, we can determine the action of $T_2$ on these modules now. For $v \in V(\lambda_2)$, we have that $T_2 (1 \otimes v) = T_2 \otimes v$, and $T_2 (T_2 \otimes v) = T_2 \otimes v = 1 \otimes T_2^2 v = (\lambda_{21} - \lambda_{22})^2 (1 \otimes v)$. Thus, $T_2$ can be characterized by the following matrix:

$$0 \quad (\lambda_{21} - \lambda_{22})^2 I$$

$$0 \quad 0 \quad 0 \quad - (\lambda_{21} - \lambda_{22}) I$$

so we can see that $\mathcal{A}(\mathfrak{gl}_2) \otimes_{U(\mathfrak{gl}_2)} V(\lambda_2)$ decomposes into the two eigenspaces of the action of $T_2$: $V(\lambda_2, +1) := \langle (\lambda_{21} - \lambda_{22})(1 \otimes v) + T_2 \otimes v \mid v \in V(\lambda_2) \rangle$ and $V(\lambda_2, -1) := \langle -(\lambda_{21} - \lambda_{22})(1 \otimes v) + T_2 \otimes v \mid v \in V(\lambda_2) \rangle$ both of which are clearly simple. It is also clear that as vector spaces $V(\lambda_2, \pm 1) \cong V(\lambda_2)$.

Conversely, if we have a finite-dimensional simple $\mathcal{A}(\mathfrak{gl}_2)$-module $V$ restricted to a $U(\mathfrak{gl}_2)$-module, it must remain simple, as $T_2$ is a central element. As such, $V \cong V(\lambda_2)$ for some weight $\lambda_2$. Thus, $V \cong V(\lambda_2, \varepsilon_2)$ for some $\varepsilon_2 \in \{ \pm 1 \}$. \qed

Next, we classify finite-dimensional simple weight modules over $\mathcal{A}(\mathfrak{gl}_n)$.

**Theorem 5.2.** Every finite-dimensional simple module over $\mathcal{A}(\mathfrak{gl}_n)$, on which $V_2, \ldots, V_{n-1}$ act diagonally, is of the form $V(\lambda_n, \varepsilon_n; \varepsilon_{n-1}, \ldots, \varepsilon_2)$ where $\lambda_n = (\lambda_{n1}, \lambda_{n2}, \ldots, \lambda_{nn})$ is a weight of $U(\mathfrak{gl}_n)$, $\varepsilon_j \in \{\pm 1\}^{r_{\lambda_j}}$, with $r_{\lambda_j}$ denoting the number of ways to fill the $j$-th row of Gelfand-Tsetlin pattern with fixed top row $\lambda_n$, and $j = 2, 3, \ldots, n$.

**Proof.** We define $V(\lambda_n, \varepsilon_n; \varepsilon_{n-1}, \ldots, \varepsilon_2)$ to be the $\mathcal{A}(\mathfrak{gl}_n)$-module which is $V(\lambda_n)$ as a $U(\mathfrak{gl}_n)$-module with the action of $V_j$ on a Gelfand-Tsetlin vector determined by $\varepsilon_j$ and $(V_j)^2 \in U(\mathfrak{gl}_n)$. It remains to show that every simple $\mathcal{A}(\mathfrak{gl}_n)$-module on which $V_j$ acts diagonally is isomorphic to one of these.

We prove this by induction on $n$. For the base case, $n = 3$, we have the following commutative diagram:

$$\mathcal{A}(\mathfrak{gl}_3) \to \mathcal{A}(\mathfrak{gl}_2) \quad \mathcal{A}(\mathfrak{gl}_2) \to \mathcal{A}(\mathfrak{gl}_3)$$

$$U(\mathfrak{gl}_3) \to U(\mathfrak{gl}_2) \quad U(\mathfrak{gl}_2) \to U(\mathfrak{gl}_3)$$

where each arrow is the restriction functor. If we consider a simple $V \in \mathcal{A}(\mathfrak{gl}_3)$ and its image in the bottom right corner, we see that $V \cong \bigoplus \lambda_3 \oplus \lambda_2 V(\lambda_2, \varepsilon_3) \in U(\mathfrak{gl}_2)$, where $\lambda_3$ and $\lambda_2$ are weights for $U(\mathfrak{gl}_3)$ and $U(\mathfrak{gl}_2)$, respectively, by the semi-simplicity of $U(\mathfrak{gl}_3)$ and $U(\mathfrak{gl}_2)$. Moreover, $V(\lambda_3)$ is the components of the restriction of $V(\lambda_3)$ to $U(\mathfrak{gl}_2)$. We know that $V_j$ must have a diagonal action by assumption. As such, we have $V \cong \bigoplus \lambda_3 \oplus \lambda_2 V(\lambda_2, \varepsilon_3) \lambda_3$ in the upper right corner by Proposition 5.1 where

...
$\varepsilon_2 = \varepsilon_2(\lambda_2)$ depends on $\lambda_2$. This is because otherwise the dimensions of the $\lambda_2$ weight spaces would not match. Since $V_2$ acts diagonally, $V_4$ is central, and the diagram commutes, it follows that $V \cong V(\lambda_3, \varepsilon_3, \varepsilon_2) \in \mathcal{A}(\mathfrak{g}l_2)_{mod}^f.d$. where $\varepsilon_3$ is determined as in Proposition 5.1, and $\varepsilon_2 = \{\varepsilon_2(\lambda_2)\}_{\lambda_2}$ is indexed by the number $r_{\lambda_3, 2}$.

To finish the induction we look at a similar diagram:

$\mathcal{A}(\mathfrak{g}l_n)_{mod}^f.d. \rightarrow \mathcal{A}(\mathfrak{g}l_{n-1})_{mod}^f.d. \rightarrow \cdots \rightarrow \mathcal{A}(\mathfrak{g}l_2)_{mod}^f.d. \rightarrow \mathcal{A}(\mathfrak{g}l_1)_{mod}^f.d.$

Following the image of a simple $V \in \mathcal{A}(\mathfrak{g}l_n)_{mod}^f.d.$ and using identical arguments, we observe that:

$V \cong \bigoplus_{\lambda_n} \bigoplus_{\lambda_{n-1}} V(\lambda_{n-1}, \varepsilon_{n-1}, \varepsilon_{n-2}, \ldots, \varepsilon_2)_{\lambda_n} \in \mathcal{A}(\mathfrak{g}l_{n-1})_{mod}^f.d.$

By the induction hypothesis,$$
V \cong \bigoplus_{\lambda_n} \bigoplus_{\lambda_{n-1}} V(\lambda_{n-1}, \varepsilon_{n-1}, \varepsilon_{n-2}, \ldots, \varepsilon_2)_{\lambda_n} \in \mathcal{A}(\mathfrak{g}l_{n-1})_{mod}^f.d. \tag{6.10}
$$

Finally by $V_n$ central, $V_j$ acting diagonally for $j = 2, \ldots, n - 1$, and the diagram commuting, it follows that $V \cong V(\lambda_n, \varepsilon_n, \varepsilon_{n-1}, \ldots, \varepsilon_2).$ \hfill \square

The following example demonstrates that $\mathcal{A}(\mathfrak{g}l_n)_{mod}^f.d.$ is not semi-simple for every $n \geq 2$.

**Example 5.3.** We recall that $V_2^2$ must act diagonally on any $\mathcal{A}(\mathfrak{g}l_2)$-modules $V$ because $\text{Res}^{gr}_{U(\mathfrak{g}l_2)} V$ can be viewed as a direct sum of irreducible $U(\mathfrak{g}l_2)$-modules and $V_2^2$ is a quadratic polynomial of Gelfand invariants in $U(\mathfrak{g}l_2)$. Let $V = V(0) \oplus V(0)$ where $U(\mathfrak{g}l_2)$ acts trivially. This means that $V_2^2$ must act as $\text{Id}_V$. We define the following action of $V_2$

$\nu_2 \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 & \alpha \\ 0 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

with $0 \neq \alpha \in \mathbb{C}$. It is clear then that $V_2^2$ acts as the identity on $V$, but the subrepresentation $W = \{(v_1, 0) \mid v_1 \in V(0)\}$ is not a direct summand of $V$ as a $\mathcal{A}(\mathfrak{g}l_2)$-module.

**6. A Technique for Creating Galois Orders from Galois Rings Via Localization**

In this section, we describe a technique that allows us to turn a Galois ring into a Galois order involving localization. We use this technique on a toy example and a localized version of $\mathcal{A}(\mathfrak{g}l_n)$ denoted $\mathcal{A}(\mathfrak{g}l_n)$ (see Definition 6.10).

**6.1. The General Result.** We recall that Proposition 5.8 states that $\Gamma$ is maximal commutative in a Galois $\mathcal{L}$-order. We observe that for a general Galois $\mathcal{L}$-ring $\mathcal{U}$, while $\Gamma$ might not be maximal commutative, its centralizer $C_{\mathcal{U}}(\Gamma)$ in $\mathcal{U}$ will be $\mathcal{L}$. This can be seen from the following remark:

**Remark 2.** For Galois $\mathcal{L}$-ring $\mathcal{U}$, the centralizer of $\Gamma$ in $\mathcal{U}$ denoted $C_{\mathcal{U}}(\Gamma)$ is equal to $\mathcal{U} \cap K$.

First we define a subring of $L$ that is needed in our result.

**Definition 6.1.** Let $\mathcal{U}$ be a subalgebra of $\mathcal{L}$. We define the *ring of coefficients* of $\mathcal{U}$:

$D_{\mathcal{U}} := \{\alpha \in L \mid \exists X \in \mathcal{U} \text{ such that } \alpha \text{ is a left coefficient of some } \mu \in \text{supp}_{\mathcal{U}} X\}_{\text{ring}}$

Similarly, we define the *opposite ring of coefficients* of $\mathcal{U}$ denoted $D_{\mathcal{U}}^{op}$ using right coefficients.

Now for the result.

**Theorem 6.2.** Let $G$ be arbitrary and $\mathcal{U}$ be a Galois $\mathcal{L}$-ring in $(L \# \mathcal{M})^G$. If $C = C_{\mathcal{U}}(\Gamma)$ is the $G$ invariants of localization of $\Lambda$ with respect to a set that is $\mathcal{M}$-invariant, that is $C = (S^{-1}\Lambda)^G$ where $S$ is $\mathcal{M}$-invariant, and $D_{\mathcal{U}}$ is a finitely generated module over $C$, then $\mathcal{U}$ is a Galois $C$-order in $(L \# \mathcal{M})^G$. Moreover, if $D_{\mathcal{U}} \subseteq S^{-1}\Lambda$ (resp. $D_{\mathcal{U}}^{op} \subseteq S^{-1}\Lambda$), then $\mathcal{U}$ is a (co-)principal Galois $C$-order.
Proof. First, we find a $\Lambda'$ such that $(\Lambda', G, \mathcal{M})$ satisfies the assumptions in Section 1.1. We define $\Lambda' = \overline{C}$ the integral closure of $C$ in $L$. We observe that $C = (S^G)^{-1}\Gamma$. As such, $C$ is a localization, and it follows that:

$$\overline{C} = (S^G)^{-1}\Gamma = S^{-1}\Lambda.$$  \hspace{1cm} (11)

Since $S$ is $\mathcal{M}$-invariant and $\overline{C}$ is integral over $C$, it follows that $\overline{\mathcal{M}}$ and $G$ are subgroups of $\text{Aut}(\Lambda')$. The first two assumptions clearly hold, and the third follows by $\Lambda' = S^{-1}\Lambda$.

We have that $\mathcal{U}$ is a Galois $C$-ring since it is a Galois $\Gamma$-ring and $\text{Frac}(C) = \text{Frac}(\Gamma) = K$. All that remains is to show that $\mathcal{U}$ is a Galois $C$-order. We consider $W \subset \mathcal{L}$ a finite-dimensional left $L$-subspace and aim to show that $W \cap \mathcal{U}$ is finitely generated as a left $C$-module. $W$ has a finite basis $w_1, \ldots, w_n$ such that:

$$W = \{ \sum \alpha_i w_i \mid \alpha_i \in L \}.$$  

Note that for each $i$, $w_i = \sum_{\mu \in \mathcal{M}} \beta_i, \mu \mu$; as such, since $C$ is a localization of a Noetherian ring and therefore Noetherian, WLOG we can assume $w_i = \mu_i$ for some $\mu_i \in \mathcal{M}$. Hence:

$$W = \sum_i L\mu_i.$$  

So, $W \cap \mathcal{U} \subset \sum_i D_{\mathcal{U}} \mu_i$, and is therefore finitely generated. A similar argument justifies the claim if $W$ is instead a right $L$-module. Therefore, $U$ is a Galois $C$-order.

If additionally we assume $D_{\mathcal{U}} \subset S^{-1}\Lambda$, we need to show that $X(c) \subset C$ for all $X \in \mathcal{U}$ and $c \in C$. So, we consider an arbitrary $c \in C$ and $X \in \mathcal{U}$. By Lemma 2.19 in [15], it follows that $X(c) \subset K$. Since $C = (S^G)^{-1}\Gamma$, it follows that $X(c) \subset S^{-1}\Lambda$. As such:

$$X(c) \subset S^{-1}\Lambda \cap K = (S^{-1}\Lambda)^G = C.$$  \hspace{1cm} (12)

Thus $X(c) \subset C$. If instead $D_{\mathcal{U}}^\text{op} \subset S^{-1}\Lambda$, a similar argument shows that $X^\dagger(c) \subset C$, thereby proving the claim. \hfill $\square$

The above theorem also gives an alternate proof to one direction of Corollary 2.15 in [15].

6.2. A toy example. In this subsection, we provide a family of simple examples of Galois rings to which Theorem 6.2 can be applied.

Let $\Lambda = C[x], \delta \in \text{Aut} \Lambda$ such that $\delta(x) = x - 1, \mathcal{M} = \langle \delta \rangle^\text{grp}$, and $G$ the trivial group. Then, let $\mathcal{L} = L\# \mathcal{M}$ be the skew-monoid ring and $f(x) \in C[x]$ such that $f(0) \neq 0$. We define $X, Y \in \mathcal{L}$ such that:

$$X := \frac{\delta f(x)}{x} \quad \text{and} \quad Y := \delta^{-1}.$$  \hspace{1cm} (13)

Let $U_f = \mathcal{C}(\Lambda, X, Y)_{\text{alg}}$ and $C_{U_f}(\Lambda) = C_{U_f}$ the centralizer of $\Lambda$ in $U_f$. We note, as $G$ is trivial, that $\Lambda = \Gamma$. First, we will show that $U_f$ is Galois $\Gamma$-ring.

Proposition 6.3. The algebra $U_f$ is a Galois $\Gamma$-ring in $L\# \mathcal{M}$.

Proof. This immediately follows from Proposition 1.3 letting $\mathcal{X} = \{ X, Y \}$. \hfill $\square$

In order to apply Theorem 6.2, we need to describe $C_{U_f}$. The next three lemmas are used to do just that.

Lemma 6.4. For any $f(x)$ such that $f(0) \neq 0$, we have $\frac{1}{x}, \frac{1}{x - 1} \in C_{U_f}$.

Proof. First, we show that $\frac{1}{x} \in C_{U_f}$. Now, $f(x) = a_nx^n + \cdots + a_1x + a_0$ with $a_0 \neq 0$ by assumption. As such:

$$\frac{f(x)}{x} = a_nx^{n-1} + a_{n-1}x^{n-2} + \cdots + a_1 + a_0 \frac{1}{x}$$

$$\Rightarrow \frac{1}{x} = a_0^{-1} \left( \frac{f(x)}{x} - (a_nx^{n-1} + a_{n-1}x^{n-2} + \cdots a_1) \right).$$

This shows that $\frac{1}{x} \in C_{U_f}$. To see that $\frac{1}{x - 1} \in C_{U_f}$, we follow a similar division algorithm argument with $\frac{f(x - 1)}{x - 1}$. \hfill $\square$
Lemma 6.5. For any \( f(x) \) such that \( f(0) \neq 0 \) and \( k \geq 1 \), we have \( \frac{1}{x+k} \in \mathcal{I}_f \).

Proof. Let \( m \) be the order of \( (x+k) \) in \( \prod_{j=0}^{k-1} f(x+j) \). Then consider the following:

\[
Y^{k+1}(XY)^mX^{k+1} = \delta^{-k-1} \left( \frac{f(x-1)}{x-1} \right)^m \delta^{k+1} \prod_{j=0}^{k-1} \frac{f(x+j)}{x+j} \\
= \left( \frac{f(x+k)}{x+k} \right)^m \prod_{j=0}^{k-1} \frac{f(x+j)}{x+j} \\
= \left( \frac{f(x+j)}{x+j} \right)^{m+1} \prod_{j=0}^{k-1} \frac{f(x+j)}{x+j}
\]

As such, there are \( m \) factors of \( (x+k) \) in the numerator and \( m+1 \) factors in the denominator. Thus, multiplying by \( \prod_{j=0}^{k-1} (x+j) \) and using a division algorithm argument, it follows that \( \frac{1}{x+k} \in \mathcal{I}_f \). \( \square \)

Lemma 6.6. For any \( f(x) \) such that \( f(0) \neq 0 \) and \( k \geq 2 \), we have \( \frac{1}{x-k} \in \mathcal{I}_f \).

Proof. Let \( m \) be the order of \( (x-k) \) in \( \prod_{j=1}^{k-1} f(x-j) \). Then consider the following:

\[
X^k(YX)^mY^k = \delta^k \prod_{j=0}^{k-1} \frac{f(x+j)}{x+j} \left( \frac{f(x)}{x} \right)^m \delta^{-k} \\
= \prod_{j=0}^{k-1} \frac{f(x+j)-k}{x+j-k} \left( \frac{f(x-k)}{x-k} \right)^m \\
= \left( \frac{f(x-k)}{x-k} \right)^{m+1} \prod_{j=1}^{k-1} \frac{f(x-\ell)}{x-\ell}.
\]

As such, there are \( m \) factors of \( (x-k) \) in the numerator and \( m+1 \) factors in the denominator. Thus, multiplying by \( \prod_{j=1}^{k-1} (x-j) \) and using a division algorithm argument, it follows that \( \frac{1}{x-k} \in \mathcal{I}_f \). \( \square \)

Proposition 6.7. If \( f(x) \) is a polynomial such that \( f(0) \neq 0 \), then \( \mathcal{I}_f = \mathbb{C}[x] \left[ \frac{1}{x+k} \mid k \in \mathbb{Z} \right] \).

Proof. \( \mathcal{I}_f \supseteq \mathbb{C}[x] \left[ \frac{1}{x+k} \mid k \in \mathbb{Z} \right] \) by Lemmas 6.4, 6.5, and 6.6. To show the reverse inclusion, we observe that for \( Z \in \mathcal{I}_f \), \( Z \) must be of \( \delta \) degree \( 0 \) with regards to \( \delta \) that is:

\[
Z = \sum_{k=1}^{m} g_k(x) \prod_{n=0}^{\infty} (X^n Y^n)^{k_n} (Y^n X^n)^{k_n} \\
= \sum_{k=1}^{m} g_k(x) \prod_{\ell=-\infty}^{\infty} \frac{f(x+\ell)}{(x+\ell)}^{k_\ell} \\
= \sum_{k=1}^{m} G_k(x) \prod_{\ell=-\infty}^{\infty} \frac{1}{(x+\ell)^{k_\ell}} \in \mathbb{C}[x] \left[ \frac{1}{x+k} \mid k \in \mathbb{Z} \right],
\]

where \( k_\ell \neq 0 \) for at most finitely many terms. Thus \( \mathcal{I}_f \subseteq \mathbb{C}[x] \left[ \frac{1}{x+k} \mid k \in \mathbb{Z} \right] \). \( \square \)
We can now prove that \( U_f \) is a Galois \( C_{U_f} \)-order using Theorem 6.2.

**Corollary 6.8.** The algebra \( U_f \) is a principal and co-principal Galois \( C_{U_f} \)-order in \( L\# M \).

**Proof.** Proposition 6.7 gives us that the main supposition of Theorem 6.2. All that remains to show is \( D_{U_f}, D_{U_f}^{op} \subset S^{-1}\Lambda = C_{U_f} \) in this case. However, this is clear since \( U_f \) is generated by \( X, Y, \) and \( \Lambda \).

6.3. **Localizing \( \mathscr{A}(\mathfrak{g}_n) \).** In this subsection, we construct a localization of \( \mathscr{A}(\mathfrak{g}_n) \) denoted \( \widetilde{\mathscr{A}}(\mathfrak{g}_n) \), to which Theorem 6.2. can be applied.

In order to construct this localization, we describe shifted Vandermonde polynomials using the following notation:

**Notation.** Let \( V_k \) be the Vandermonde in the \( x_{ki} \) variables. Let \( l := (l_1, l_2, \ldots, l_{k-1}) \in \mathbb{Z}^{k-1} \). We denote the \((l-)\)-shifted \( V_k \) as follows:

\[
V_{k,l} := \prod_{i<j} (x_{ki} - x_{kj} + \sum_{n=i}^{j-1} l_n).
\]

This notation makes sense because for \( i < j \):

\[
x_{ki} - x_{kj} = (x_{ki} - x_{k,i+1}) + (x_{k,i+1} - x_{k,i+2}) + \cdots + (x_{kj-1} - x_{kj}).
\]

Therefore, any shift of \( V_k \) is uniquely determined by the shifts of \( x_{ki} - x_{k,i+1} \) for \( i = 1, 2, \ldots, k-1 \).

Now to construct our localization.

**Definition 6.9.** Let \( S := \{ V_{k,l} \mid l \in \mathbb{Z}^{k-1}; \ k = 2, \ldots, n-1 \} \). We observe that \( S \) is a multiplicatively closed set in \( \Lambda \), and \( \mathscr{A}(\mathfrak{g}_n) \subset (S^{-1}\Lambda)^{\mathscr{M}_{\mathfrak{g}_n}} \). We also note that \( S \) is the smallest \( \mathscr{M} \)-invariant multiplicatively closed set that contains \( V_2, \ldots, V_{n-1} \).

As Example 4.2. demonstrates, \( C_{\mathscr{A}(\mathfrak{g}_2)}(\widehat{\Gamma}) \subset (S^{-1}\Lambda)^{\mathfrak{g}_n} \). It is not known if this containment is strict, so this motivates the construction of the following localization of \( \mathscr{A}(\mathfrak{g}_n) \).

**Definition 6.10.** Our new algebra of interest in \( \widetilde{\mathscr{A}} \) is \( \widetilde{\mathscr{A}} := \mathbb{C}(U_n, (S^{-1}\Lambda)^{\mathfrak{g}_n})_{\text{alg}} \). Notice this coincides with the definitions of \( \mathscr{A}(\mathfrak{g}_2) \) for \( n = 2 \).

**Remark 3.** It follows from Lemma 2.10 in [15] that \( \widetilde{\mathscr{A}}(\mathfrak{g}_n) \) is a Galois \( \widehat{\Gamma} \)-ring since it contains \( \mathscr{A}(\mathfrak{g}_n) \). Moreover, \( C_{\widetilde{\mathscr{A}}(\mathfrak{g}_n)}(\widehat{\Gamma}) = (S^{-1}\Lambda)^{\mathfrak{g}_n} \) as well.

**Remark 4.** In \( \widetilde{\mathscr{A}}(\mathfrak{g}_n) \), relation \( (ix) \) from Section 4.2. can be rewritten either as

\[
(iix) \quad A_{21}^{\pm} A_{22}^{\pm} = \frac{\pm 2}{\lambda_{21}^{\pm} + 1} A_{21}^{\pm} A_{22}^{\pm}, \quad \text{or}
\]

\[
(iix)' \quad A_{22}^{\pm} A_{21}^{\pm} = \frac{\lambda_{22}^{\pm} + 1}{\lambda_{22}^{\pm} + 1} A_{22}^{\pm} A_{21}^{\pm}.
\]

**Corollary 6.11.** The subalgebra \( \widetilde{\mathscr{A}}(\mathfrak{g}_n) \subset \widehat{\mathfrak{g}} \) is both a principal and co-principal Galois \( (S^{-1}\Lambda)^{\mathfrak{g}_n} \)-order.

**Proof.** It is clear by construction that \( \widetilde{\mathscr{A}}(\mathfrak{g}_n) \) satisfies the main supposition of Theorem 6.2. Also, it follows from the definition of the \( a_{ki}^{\pm} \)’s in [1] that \( D_{\widetilde{\mathscr{A}}(\mathfrak{g}_n)}, D_{\widetilde{\mathscr{A}}(\mathfrak{g}_n)}^{op} \subset S^{-1}\Lambda \). We can therefore apply Theorem 6.2.

In [22], it was shown that every (co-)principal Galois order has a corresponding (co-)principal flag order. This leads us to the following:

**Open Problem 2.** What is the corresponding (co-)principal flag order of \( \widetilde{\mathscr{A}}(\mathfrak{g}_n) \)?

7. **(Generic) Gelfand-Tsetlin Modules over \( \mathscr{A}(\mathfrak{g}_n) \)**

7.1. **Some general results.** Following the techniques in [3] and [15], we construct canonical simple Gelfand-Tsetlin modules over \( \mathscr{A}(\mathfrak{g}_n) \). We need the following additional assumptions for these next two results:

\begin{itemize}
  \item[(A4)] \( \Lambda \) is finitely generated over an algebraically closed field \( k \) of characteristic 0,
  \item[(A5)] \( G \) and \( M \) act by \( k \)-algebra homomorphisms on \( \Lambda \).
\end{itemize}

Let \( \hat{\Gamma} \) be the set of all \( \Gamma \)-characters (i.e., \( k \)-algebra homomorphisms \( \xi : \Gamma \to k \)).
Definition 7.1. Let be a Galois -ring in . A left -module is said to be a Gelfand-Tsetlin module (with respect to ) if acts locally finitely on . Equivalently:

\[ V = \bigoplus_{\xi \in \hat{\Gamma}} V_\xi, \quad V_\xi = \{ v \in V : (\ker \xi)^N v = 0, N \gg 0 \}. \]

Similarly, one can define a right Gelfand-Tsetlin modules.

The details for the following lemma can be found in [2].

Lemma 7.2. Let be a Galois -ring in .

(i) Any submodule and any quotient of a Gelfand-Tsetlin module is a Gelfand-Tsetlin modules.

(ii) Any -module generated by generalized weight vectors is a Gelfand-Tsetlin module.

Theorem 7.3 ([15], Theorem 3.3 (ii)). Let be a Galois -order in , then the left cyclic -module has a unique simple quotient , Moreover, a Gelfand-Tsetlin over with is the canonical simple left Gelfand-Tsetlin -module associated to .

7.2. The case of . We note that for , is not finitely generated as a -algebra. This prevents us from using all of the results as is, but all is not lost. The main arguments of Theorem 7.3 rests on:

\[ \operatorname{Hom}_{\Gamma}(\Gamma/m, \Gamma) \cong \operatorname{Hom}_{\Gamma}(\Gamma/m \otimes \Gamma, k) \cong k. \]

If we want a similar result for the left cyclic module , where is a Gelfand-Tsetlin module over the canonical simple left Gelfand-Tsetlin -module associated to , then the left cyclic -module satisfies the Gelfand-Kirillov Conjecture for each .

Proof. The key difference in this proof compared to Theorem 7.3 is observing that

\[ S^{-1}\hat{\Gamma}/S^{-1}m \cong S^{-1}(\Gamma/m) \cong k. \]

Otherwise, the proof follows the same structure.

Since is created by localizing and , we can view any -module as a -module by precomposing with the embedding \( \iota : \mathcal{A}(gl_n) \hookrightarrow \mathcal{A}(\mathfrak{g}l_n) \).

8. Gelfand-Kirillov Conjecture for \( \mathcal{A}(\mathfrak{g}l_n) \)

In this section we will discuss for which the algebras and satisfy the Gelfand-Kirillov Conjecture. This is related to the Noncommutative Noether Problem for the alternating group , as discussed in [12].

The Gelfand-Kirillov Conjecture deals with the birational equivalence between universal enveloping algebras and Weyl algebras.

Lemma 8.1. \( \operatorname{Frac}(\mathcal{A}(\mathfrak{g}l_n)) = \operatorname{Frac}(\mathcal{A}(gl_n)) \).

Proof. This follows because is created by localizing and .

Hence, and either both will or will not satisfy the Gelfand-Kirillov Conjecture for each .

Proposition 8.2. For each , \( \operatorname{Frac}(\mathcal{A}(\mathfrak{g}l_n)) \cong \mathbb{C}(x_1, \ldots, x_n)^A_n \otimes \bigotimes_{k=1}^{n-1} \left( \operatorname{Frac}(\mathcal{W}_k(\mathbb{C})) \right)^A_k \) where is the -dimensional Weyl algebra over .

Proof. It is clear by construction that:

\[ \operatorname{Frac}(\mathcal{A}(\mathfrak{g}l_n)) = \operatorname{Frac}(L_{\mathcal{H}^n}) = \operatorname{Frac}((L \# \mathcal{M})_{\mathcal{H}^n}). \quad (14) \]

Since \( L = \operatorname{Frac}(\Lambda) \):

\[ \operatorname{Frac}((L \# \mathcal{M})_{\mathcal{H}^n}) \cong \operatorname{Frac}((\Lambda \# \mathcal{M})_{\mathcal{H}^n}). \quad (15) \]
We now recall that $\mathcal{M}$ is generated by $\delta^{ki}$’s and $\delta^{ki}$ fixes $x_{ij}$ if $\ell \neq k$. As such, we have:

$$\text{Frac}(\Lambda^\# \mathcal{M}) \cong \text{Frac}(\Lambda_n \otimes_{k=1}^{n-1} \Lambda_k^\# \mathcal{M}_k^\#)$$

where $\Lambda_k = \mathbb{C}[x_{k1}, \ldots, x_{kk}] \subset \Lambda$ and $\mathcal{M}_k = \langle \delta^{ki} | 1 \leq i \leq k \rangle_{\text{grp}} \leq \mathcal{M}$. Now, the $k$-th component of $\Lambda_n$ acts only on the $k$-th component of the tensor product. Therefore:

$$\text{Frac}(\Lambda_n \otimes_{k=1}^{n-1} \Lambda_k^\# \mathcal{M}_k^\#) \cong \text{Frac}(\Lambda_n^A \otimes_{k=1}^{n-1} (\Lambda_k^\# \mathcal{M}_k^\#)^A_k).$$

Finally, since $A_k$ is finite for each $k$ we have:

$$\text{Frac}(\Lambda_n^A \otimes_{k=1}^{n-1} (\Lambda_k^\# \mathcal{M}_k^\#)^A_k) \cong (\text{Frac}(\Lambda_n^A))^A \otimes_{k=1}^{n-1} (\text{Frac}(\Lambda_k^\# \mathcal{M}_k^\#))^A_k.$$  

Combining the equations (17)-(18), we have:

$$\text{Frac}(\mathcal{A}(\mathfrak{gl}_n)) \cong (\text{Frac}(\Lambda_n^A))^A \otimes_{k=1}^{n-1} (\text{Frac}(\Lambda_k^\# \mathcal{M}_k^\#))^A_k.$$  

We finish the proof by observing that $\text{Frac}(\Lambda_n^A) \cong \mathbb{C}[x_1, \ldots, x_n]$ and $\Lambda_k^\# \mathcal{M}_k^\# \cong \mathcal{W}_k(\mathbb{C})$ by sending $\delta^{ki}x_{ki} \mapsto X_i$ and $(\delta^{ki})^{-1} \mapsto Y_i$.  

We recall for readers both the classical Noether’s problem and the noncommutative Noether’s problem as stated in [12]. The classical problem asks, given a finite group $G$ and a rational function field $k(x_1, \ldots, x_n)$ over field $k$ such that $G$ acts linearly on $k(x_1, \ldots, x_n)$, is $k(x_1, \ldots, x_n)^G$ a purely transcendental extension of $k$. The noncommutative problem exchanges the rational function field with the skew field of fractions of a Weyl algebra and asks if the $\mathcal{M}$ is generated by $\delta^{ki}$’s and $\delta^{ki}$ fixes $x_{ij}$.

**Theorem 8.3 (Theorem 1.1 in [12]).** If $G$ satisfies the Commutative Noether’s problem, then $G$ satisfies the Noncommutative Noether’s Problem.

Noether’s problem for $A_n$ is still open for $n \geq 5$. However, we obtain the following result:

**Theorem 8.4.** If the alternating groups $A_1, A_2, \ldots, A_n$ provide a positive solution to Noether’s problem, then $\mathcal{A}(\mathfrak{gl}_n)$ satisfies the Gelfand-Kirillov conjecture.

**Proof.** This follows immediately by Theorem 8.3 applied to Proposition 8.2.  

Hence, as a corollary to Theorem 8.4 and Maeda’s results in [18], we have:

**Corollary 8.5.** For $n \leq 5$, $\mathcal{A}(\mathfrak{gl}_n)$ satisfies the Gelfand-Kirillov Conjecture.

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**References**

[1] V. V. Bavula. Generalized Weyl algebras and their representations. St. Petersbg. Math. J., 4(1):71–92, 1992.
[2] Y. A. Drozd, V. M. Futorny, and S. A. Ovsienko. Harish-chandra subalgebras and gelfand-zetlin modules. Finite Dimensional Algebras and Related Topics, 424, 1994.
[3] Nick Early, Volodymyr Mazorchuk, and Elizaveta Vishnyakova. Canonical Gelfand-Zeitlin modules over orthogonal Gelfand-Zeitlin algebras. arXiv e-prints, page arXiv:1709.01553, Sep 2017.
[4] Farkhod Eshmatov, Vyacheslav Futorny, Sergiy Ovsienko, and Joao Fernando Schwarz. Noncommutative noether’s problem for complex reflection groups. 2015.
[5] V. Futorny and J. Schwarz. Quantum Linear Galois Algebras. arXiv e-prints. arXiv:1804.08120 Apr 2018.
[6] Vyacheslav Futorny, Dimitar Grantcharov, Luis Enrique Ramirez, and Pablo Zadunaisky. Gelfand-Tsetlin Theory for Rational Galois Algebras. arXiv e-prints, page arXiv:1801.09516 Jan 2018.
[7] Vyacheslav Futorny and Jonas Hartwig. Solution of a q-difference noether problem and the quantum gelfand–kirillov conjecture for $\mathfrak{gl}_n$. Mathematische Zeitschrift, 276, 02 2014.
[8] Vyacheslav Futorny, Alexander Molev, and Serge Ovsienko. The gelfand–kirillov conjecture and gelfand–tsetlin modules for finite w-algebras. *Advances in Mathematics*, 223(3):773 – 796, 2010.

[9] Vyacheslav Futorny and Serge Ovsienko. Galois orders in skew monoid rings. *Journal of Algebra*, 324(4):598 – 630, 2010.

[10] Vyacheslav Futorny and Serge Ovsienko. Fibers of characters in gelfand-tsetlin categories. *Transactions of the American Mathematical Society*, 366, 08 2014.

[11] Vyacheslav Futorny and João Schwarz. Algebras of Invariant Differential Operators. *arXiv e-prints* page arXiv:1804.05029, May 2019.

[12] Vyacheslav Futorny and João Schwarz. Noncommutative Noether’s Problem vs Classical Noether’s Problem. *arXiv e-prints*, page arXiv:1805.01809, Nov 2018.

[13] Israel M. Gelfand and Michael L. Tsetlin. Finite-dimensional representations of the group of unimodular matrices. *Dokl. Akad. Nauk Ser. Fiz.*, 71:825–828, 1950.

[14] Izrail M. Gelfand and Alexander A. Kirillov. Sur les corps liés aux algèbres enveloppantes des algèbres de lie. *Publications Mathématiques de l'IHÉS*, 31:5–19, 1966.

[15] J. T. Hartwig. Principal Galois orders and Gelfand-Zeilin modules. *ArXiv e-prints*, October 2017.

[16] James E. Humphreys. *Introduction to Lie algebras and representation theory*, volume 9 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1978. Second printing, revised.

[17] J. Lepowsky and G. W. McCollum. On the determination of irreducible modules by restriction to a subalgebra. *Trans. Amer. Math. Soc.*, 176:45–57, 1973.

[18] Takashi Maeda. Noether’s problem for a5. *Journal of Algebra*, 125(2):418 – 430, 1989.

[19] Volodymyr Mazorchuk and Elizaveta Vishnyakova. Harish-Chandra modules over invariant subalgebras in a skew-group ring. *arXiv e-prints*, page arXiv:1811.00532, Nov 2018.

[20] Alexander L. Rosenberg. *Noncommutative algebraic geometry and representations of quantized algebras*, volume 330 of *Mathematics and its Applications*. Kluwer Academic Publishers Group, Dordrecht, 1995.

[21] Elizaveta Vishnyakova. A geometric approach to 1-singular Gelfand-Tsetlin $\mathfrak{gl}_n$-modules. *arXiv e-prints*, page arXiv:1704.00170, Apr 2017.

[22] Ben Webster. Gelfand-Tsetlin modules in the Coulomb context. *arXiv e-prints*, page arXiv:1904.05415, Apr 2019.

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