SHIFTED SCHUR FUNCTIONS II.
BINOMIAL FORMULA FOR
CHARACTERS OF CLASSICAL GROUPS
AND APPLICATIONS

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To A. A. Kirillov on his 60th birthday

Abstract. Let $G$ be any of the complex classical groups $GL(n)$, $SO(2n+1)$, $Sp(2n)$, $O(2n)$, let $\mathfrak{g}$ denote the Lie algebra of $G$, and let $Z(\mathfrak{g})$ denote the subalgebra of $G$-invariants in the universal enveloping algebra $U(\mathfrak{g})$. We derive a Taylor-type expansion for finite-dimensional characters of $G$ (the binomial formula) and use it to specify a distinguished linear basis in $Z(\mathfrak{g})$. The eigenvalues of the basis elements in highest weight $\mathfrak{g}$-modules are certain shifted (or factorial) analogs of Schur functions. We also study an associated homogeneous basis in $I(\mathfrak{g})$, the subalgebra of $G$-invariants in the symmetric algebra $S(\mathfrak{g})$. Finally, we show that the both bases are related by a $G$-equivariant linear isomorphism $\sigma: I(\mathfrak{g}) \to Z(\mathfrak{g})$, called the special symmetrization.

Introduction

The present paper is a continuation of our previous work [OO1] but can be read independently. Here we aim to transfer to the orthogonal and symplectic groups a part of results of [OO1] connected with the general linear group.

Let $G$ be one of the groups $SO(2n+1, \mathbb{C})$, $Sp(2n, \mathbb{C})$, let $\mathfrak{g}$ be the Lie algebra of $G$, $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$, and $Z(\mathfrak{g})$ be the center of $U(\mathfrak{g})$. The commutative algebra $Z(\mathfrak{g})$ is also known as the algebra of biinvariant differential operators on the group $G$, or the algebra of Laplace operators.

The irreducible finite-dimensional representations of the group $G$ are indexed by partitions $\lambda$ of length $\leq n$; we denote them by $V_\lambda$ and we view each $V_\lambda$ also as a $U(\mathfrak{g})$-module.

Our main object is a distinguished linear basis $\{T_\mu\} \subset Z(\mathfrak{g})$. Here the index $\mu$ ranges over partitions of length $\leq n$ and each basis element $T_\mu$ can be characterized, within a scalar multiple, as the unique element in $Z(\mathfrak{g})$ of degree $2|\mu|$ satisfying the following vanishing condition: the eigenvalue of $T_\mu$ in $V_\lambda$ equals 0 if $|\lambda| \leq |\mu|$ and $\lambda \neq \mu$ (moreover, it turns out that the eigenvalue is 0 if $\lambda_i < \mu_i$ for at least one $i$).

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1In the present Introduction, to simplify the discussion, we exclude the even orthogonal group $SO(2n, \mathbb{C})$. However, after minor modifications, all our constructions hold for this group as well.
For general \( \lambda \), we denote the eigenvalue of \( T_\mu \) in \( V_\lambda \) by \( t_\mu^*(\lambda) = t_\mu^*(\lambda_1, \ldots, \lambda_n) \). The functions \( t_\mu^* \) are inhomogeneous polynomials with a certain kind of symmetry; they can be identified with certain factorial analogs of the Schur polynomials. Note that the top homogeneous component \( t_\mu \) of the inhomogeneous polynomial \( t_\mu^* \) coincides with a Schur polynomial in the squares of the arguments:

\[
t_\mu(x_1, \ldots, x_n) = s_\mu(x_1^2, \ldots, x_n^2).
\]

The polynomials \( t_\mu^* \) arise in the binomial formula for the characters of \( G \). Denote by \( \chi_\lambda = \chi_\lambda(z_1, \ldots, z_n) \) the character of \( V_\lambda \); here \( z_1, \ldots, z_n \) are natural coordinates of a maximal torus of \( G \). Assume ‘additive’ variables \( x_1, \ldots, x_n \) are connected with the ‘multiplicative’ variables \( z_1, \ldots, z_n \) by the relation

\[
x_i = z_1^{1/2} - z_i^{-1/2}, \quad \text{i.e.,} \quad x_i^2 = z + z_i^{-1} - 2.
\]

The binomial formula is written as

\[
\frac{\chi_\lambda(z_1, \ldots, z_n)}{\chi_\lambda(1, \ldots, 1)} = \sum_\mu \frac{t_\mu^*(\lambda_1, \ldots, \lambda_n) t_\mu(x_1, \ldots, x_n)}{c_\pm(n, \mu)},
\]

where \( c_\pm(n, \mu) \) are simple normalization factors. This is a Taylor-type expansion for the character near the unit element of the group.\(^2\)

One more important property of the basis elements \( T_\mu \) is existence of a relation between the bases corresponding to different values of the parameter \( n \). We call this the coherence property of the canonical basis.

Then we study a counterpart of \( \{ T_\mu \} \) for the algebra \( I(\mathfrak{g}) \), the subalgebra of \( G \)-invariants in the symmetric algebra \( S(\mathfrak{g}) \). Note that \( I(\mathfrak{g}) \) is a graded algebra which is canonically isomorphic to \( \text{gr} \, Z(\mathfrak{g}) \), the graded algebra associated to the natural filtration in \( Z(\mathfrak{g}) \).

To the basis \( \{ T_\mu \} \) corresponds a homogeneous basis \( \{ T_\mu \} \) of \( I(\mathfrak{g}) \): each \( T_\mu \) coincides with the leading term of \( T_\mu \) (with respect to the identification \( I(\mathfrak{g}) = \text{gr} \, Z(\mathfrak{g}) \)). On the other hand, the basis \( \{ T_\mu \} \) can be described without reference to \( Z(\mathfrak{g}) \), in intrinsic terms of the \( G \)-module \( S(\mathfrak{g}) \). Note that the elements \( T_\mu \) admit an explicit expression as polynomials in natural generators of the Lie algebra \( \mathfrak{g} \).

Finally, we show that there exists a \( G \)-equivariant linear isomorphism

\[
\sigma: S(\mathfrak{g}) \to U(\mathfrak{g})
\]

which preserves leading terms and takes \( T_\mu \) to \( T_\mu \) for any \( \mu \). We call \( \sigma \) the special symmetrization for the orthogonal/symplectic Lie algebras.

The special symmetrization \( \sigma \) enters a family of ‘generalized symmetrizations’ studied in the paper [O2]. The results of [O2] provide explicit combinatorial formulas both for \( \sigma \) and its inverse \( \sigma^{-1} \). Since we dispose with an explicit expression for \( T_\mu \), this yields a certain formula for the elements \( T_\mu = \sigma(T_\mu) \).

We conclude this Introduction with a brief discussion of some related works.

For the general linear group \( GL(n, \mathbb{C}) \), the binomial theorem and distinguished bases \( \{ S_\mu \} \subset Z(\mathfrak{gl}(n, \mathbb{C})) \) and \( \{ S_\mu \} \subset I(\mathfrak{gl}(n, \mathbb{C})) \) with similar properties were earlier considered in [OO1]. The elements \( S_\mu \), which are called quantum inmanants,

\(^2\)Note that the binomial formula for characters of classical groups of type \( B \), \( C \), \( D \) is a particular case of an expansion of type \( BC \). Jacobi polynomials are [Lass].
appear in a higher version of the classical Capelli identity and admit a remarkable explicit expression, see the papers [Ok1, N, Ok2]. The eigenvalues of the elements $S_\mu$ in highest weight modules are described by certain polynomials $s^*_\mu(\lambda_1, \ldots, \lambda_n)$, called in [OO1] the shifted Schur polynomials. The both families of polynomials, $\{s^*_\mu\}$ and $\{t^*_\mu\}$, have similar properties.\footnote{However, in contrast to the polynomials $s^*_\mu$, the polynomials $t^*_\mu$ are not stable as $n \to \infty$.}

Let $\mu$ be of the form $(1^m)$ for the orthogonal group or of the form $(m)$ for the symplectic group ($m = 1, 2, \ldots$). Then the corresponding elements $T_\mu$ coincide with Laplace operators considered by A. Molev and M. Nazarov in [MN]. As shown in [MN], these elements also occur in a Capelli-type identity and admit an explicit expression in terms of the generators of the Lie algebra. An open problem is to generalize the results of [MN] to arbitrary $\mu$. Note that the explicit formulas found in [MN, Ok1, N, Ok2] differ from the formulas obtained via the special symmetrization.

Many of the results of [OO1] and the present paper should have counterparts for classical Lie superalgebras $\mathfrak{gl}(p|q)$, $\mathfrak{g}(n)$, and $\mathfrak{osp}(n|2m)$. Some work in this direction was made by A. Borodin and N. Rozhkovskaya [BR], A. Molev [Mo], V. Ivanov and A. Okounkov (see [I]).

In the present work (as in [OO1]), we used very elementary tools such as explicit formulas for the (ordinary and factorial) Schur polynomials or Weyl’s character formula. We believe our approach can be further developed by making use of a more fine technique. In particular, there exist more involved versions of the binomial formula, see [OO2], [Ok3], and [Ok4].

\section{Main notation}

Unless otherwise stated, the symbol $G(n)$ or $G$ will denote any of the complex classical groups of rank $n$

$$GL(n, \mathbb{C}), \quad Sp(2n, \mathbb{C}), \quad SO(2n + 1, \mathbb{C}), \quad SO(2n, \mathbb{C}),$$

which constitute the series A, C, B, D, respectively. By $\mathfrak{g}(n)$ or $\mathfrak{g}$ we denote the corresponding complex Lie algebras. In the case of the series D we also consider the nonconnected groups $G' = G'(n) = O(2n, \mathbb{C})$.

By $U(\mathfrak{g})$ we denote the universal enveloping algebra of $\mathfrak{g}$. For the series A, C, B, we denote by $Z(\mathfrak{g})$ the center of $U(\mathfrak{g})$ (which coincides with the subalgebra of $G$-invariants in $U(\mathfrak{g})$); for the series D the same symbol will denote the subalgebra of $G'$-invariants in $U(\mathfrak{g})$ (which is a proper subalgebra of the center).

Similarly, we define $I(\mathfrak{g}) \subset S(\mathfrak{g})$ as the subalgebra of $G$-invariants (or $G'$-invariants, for the series D) in the symmetric algebra of $\mathfrak{g}$.

$V_\lambda$ or $V_\lambda|_n$ is the irreducible finite-dimensional complex-analytic representation of $G$ with highest weight $\lambda$; it is also viewed as a $U(\mathfrak{g})$-module.

$\mathbb{C}[G]$ is the space of regular functions on $G$, where $G$ is viewed as an algebraic group over $\mathbb{C}$; in other words, $\mathbb{C}[G]$ is the linear span of matrix elements of all representations $V_\lambda$.

By $\mu$ we always denote a partition of length $l(\mu) \leq n$; it is also viewed as a Young diagram; $|\mu|$ denotes the number of boxes of the diagram $\mu$.\footnote{However, in contrast to the polynomials $s^*_\mu$, the polynomials $t^*_\mu$ are not stable as $n \to \infty$.}
§1. Binomial formula

Here we derive a multidimensional analog of the binomial formula

\[(1 + x)^k = \sum_{m=0}^{k} \frac{1}{m!} k(k-1) \cdots (k-m+1)x^m,\]

where the powers of a variable will be replaced by characters of \(G\).

Let us agree about the choice of a Borel subgroup \(B \subset G\) and a maximal torus \(H \subset B\) (thus positive roots and dominant weights will be specified):

- For the series A, we take as \(B\) the subgroup of upper triangular matrices and as \(H\) the subgroup of diagonal matrices,

\[H = \{\text{diag}(z_1, \ldots, z_n)\}, \quad z_1, \ldots, z_n \in \mathbb{C}^*.\]

- For the series C–B–D we identity \(G\) with a subgroup in \(GL(N, \mathbb{C})\), where \(N = 2n\) or \(N = 2n + 1\),

\[G = \{g \in GL(N, \mathbb{C}) \mid g'Mg = M\}, \quad (1.1)\]

where \(M\) stands for the following symmetric (case B–D) or antisymmetric (case C) matrix of order \(N\):

\[
M = \begin{bmatrix}
0 & 1 & & & \\
\pm1 & 1 & \ddots & & \\
& \ddots & \ddots & \ddots & \\
& & \pm1 & 0 & \\
& & & & 1
\end{bmatrix}. \quad (1.2)
\]

Then as \(B\) and \(H\) we take the subgroups of upper triangular or diagonal matrices in \(G\). Thus

\[H = \{\text{diag}(z_1, \ldots, z_n, z_1^{-1}, \ldots, z_n^{-1})\}, \quad \text{case C–D}\]

\[H = \{\text{diag}(z_1, \ldots, z_n, 1, z_1^{-1}, \ldots, z_n^{-1})\}, \quad \text{case B}.\]

For all the series, the weights of \(H\) are of the form

\[z^\lambda = z_1^{\lambda_1} \cdots z_n^{\lambda_n}, \quad \text{where } \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n,\]

while the dominant weights are distinguished by the supplementary conditions

\[\lambda_1 \geq \cdots \geq \lambda_n, \quad \text{case A},\]

\[\lambda_1 \geq \cdots \geq \lambda_n \geq 0, \quad \text{case C–B},\]

\[\lambda_1 \geq \cdots \geq \lambda_{n-1} \geq |\lambda_n|, \quad \text{case D}.\]

The dominant weights for the series A are called the signatures and those for the series C–B are called the positive signatures (= partitions of length \(\leq n\)).
The irreducible character of $G$, indexed by a dominant weight $\lambda$ (i.e., the character of $V_\lambda = V_{\lambda[n]}$) will be denoted by the symbol

$$\chi_\lambda = \chi^{gl(n)}_\lambda, \quad \chi^{sp(2n)}_\lambda, \quad \chi^{so(2n+1)}_\lambda \quad \text{or} \quad \chi^{so(2n)}_\lambda.$$

For the series D, it is more convenient to deal with the reducible character

$$\chi^{o(2n)}_\lambda = \chi^{so(2n)}_\lambda + \chi^{so(2n)}_\lambda \quad \text{where} \quad \lambda_1 \geq \cdots \geq \lambda_n \geq 0.$$

When $\lambda_n = 0$, this is simply $2\chi^{so(2n)}_\lambda$, and when $\lambda_n > 0$, this coincides with the restriction to the group $G = SO(2n, \mathbb{C})$ of an irreducible character of the group $G' = O(2n, \mathbb{C})$.

Thus, we are working with the characters

$$\chi^{gl(n)}_\lambda, \quad \chi^{sp(2n)}_\lambda, \quad \chi^{so(2n+1)}_\lambda, \quad \chi^{o(2n)}_\lambda,$$

where $\lambda$ is either a signature (case A) or a positive signature (case C–B–D).

To each $\lambda$ we assign another weight $l = (l_1, \ldots, l_n)$ defined as

$$l = (\lambda_1 + n - 1, \lambda_2 + n - 2, \ldots, \lambda_n), \quad \text{case A}, \quad (1.3)$$

$$l = (\lambda_1 + n - 1 + \varepsilon, \lambda_2 + n - 2 + \varepsilon, \ldots, \lambda_n + \varepsilon), \quad \text{case C–B–D}, \quad (1.4)$$

where

$$\varepsilon = \begin{cases} 
1, & \text{case C}, \\
1/2, & \text{case B}, \\
0, & \text{case D}. 
\end{cases} \quad (1.5)$$

For the series C–B–D, $l$ just equals $\lambda$ plus the half-sum of positive roots, and for the series A, the same holds after restriction to $SL(n, \mathbb{C}) \subset GL(n, \mathbb{C})$.

Unless otherwise stated, we shall consider characters of $G$ as functions on the torus $H$, i.e., as functions of the coordinates $z_1, \ldots, z_n$. The following formulas follow from general Weyl’s character formula (the determinants below are of order $n$):

$$\chi^{gl(n)}_\lambda(z_1, \ldots, z_n) = \frac{\det[z_i^{l_i}]}{\Pi_{i<j}(z_i - z_j)},$$

$$\chi^{sp(2n)}_\lambda(z_1, \ldots, z_n) = \frac{\det[(z_i^{l_i} - z_j^{-l_i})/(z_j^{-1} - z_i^{-1})]}{\Pi_{i<j}(z_i + z_i^{-1} - z_j - z_j^{-1})},$$

$$\chi^{so(2n+1)}_\lambda(z_1, \ldots, z_n) = \frac{\det[(z_i^{l_i} - z_j^{-l_i})/(z_j^{1/2} - z_j^{-1/2})]}{\Pi_{i<j}(z_i + z_i^{-1} - z_j - z_j^{-1})},$$

$$\chi^{o(2n)}_\lambda(z_1, \ldots, z_n) = \frac{\det[z_i^{l_i} + z_j^{-l_i}]}{\Pi_{i<j}(z_i + z_i^{-1} - z_j - z_j^{-1})}.$$

Note that $\chi^{gl(n)}_\lambda(z_p, \ldots, z_n) = s_\lambda(z_1, \ldots, z_n)$, the Schur polynomial in $n$ variable, if $\lambda$ is a partition. The characters $\chi^{gl(n)}_\lambda$ are symmetric Laurent polynomials in $z_1, \ldots, z_n$, while the other characters are symmetric polynomial functions of the
variables $z_i + z_i^{-1}, i = 1, \ldots, n$. Note that the last claim fails when $\chi^{o(2n)}_\lambda$ is replaced by $\chi^{s(2n)}_\lambda$; this is the reason why we deal with the characters $\chi^{o(2n)}_\lambda$.

Let $a = (a_1, a_2, \ldots)$ be an arbitrary number sequence. The generalized factorial powers of a variable $x$ are defined as

$$(x|a)^k = \begin{cases} (x - a_1) \cdots (x - a_k), & k \geq 1, \\ 1, & k = 0. \end{cases}$$

When $a \equiv 0$ these are the ordinary powers, when $a = (0, 1, 2, \ldots)$ these are the falling factorial powers, and when $a = (0, -1, -2, \ldots)$ these are the raising factorial powers (or the Pochhammer symbol).

Further, the generalized factorial Schur polynomial in $n$ variables, indexed by a partition $\mu$ with $l(\mu) \leq n$, is defined as

$$s_\mu(z_1, \ldots, z_n|a) = \frac{\det[(x_j|a)^{\mu_i+n-i}]}{\det[(x_j|a)^{n-i}]} = \frac{\det[(x_j|a)^{\mu_i+n-i}]}{\prod_{i<j}(x_i - x_j)}.$$

When $a \equiv 0$ this turns into the standard formula for the ordinary Schur polynomial $s_\mu(x_1, \ldots, x_n)$, and in the general case $s_\mu(x_1, \ldots, x_n|a)$ is an inhomogeneous symmetric polynomial of degree $|\mu| = \mu_1 + \cdots + \mu_n$ and with the top homogeneous component equal to $s_\mu(x_1, \ldots, x_n)$:

$$s_\mu(x_1, \ldots, x_n|a) = s_\mu(x_1, \ldots, x_n) + \text{lower terms}.$$

It follows that the polynomials $s_\mu(x_1, \ldots, x_n|a)$ form a basis in the algebra of symmetric polynomials in $n$ variables.

The polynomials $s_\mu(x_1, \ldots, x_n|0, 1, \ldots)$ were introduced by Biedenharn and Louck and called the factorial Schur polynomials, see [BL1, BL2]. The general definition is due to Macdonald, see [M1] and [M2, I, §3, ex. 20] (note that Macdonald uses the notation $(x|a)^k = (x + a_1) \cdots (x + a_k)$).

**Theorem 1.1** (Binomial formula for $GL(n)$). Assume

$$z_1 = 1 + x_1, \ldots, z_n = 1 + x_n.$$

Then

$$\frac{\chi^{gl(n)}_\lambda(z_1, \ldots, z_n)}{\chi^{gl(n)}_\lambda(1, \ldots, 1)} = \sum_\mu \frac{s_\mu(l_1, \ldots, l_n|0, 1, 2, \ldots) s_\mu(x_1, \ldots, x_n)}{c(n, \mu)}. \quad (1.6)$$

Here $\lambda = (\lambda_1, \ldots, \lambda_n)$ is an arbitrary signature, $l_i = \lambda_i + n - i$ $(i = 1, \ldots, n)$, $\mu$ ranges over partitions of length $\leq n$, and

$$c(n, \mu) = \prod_{(i,j) \in \mu} (n + j - i) = \prod_{i=1}^n \frac{(\mu_i + n - i)!}{(n - i)!}, \quad (1.7)$$

where the first product is taken over all boxes $(i, j)$ of the Young diagram representing the partition $\mu$.

**Comment.** This version of the binomial formula was proposed by the authors, see [OO1, Theorem 5.1]. It is equivalent to the expansion

$$s_\lambda(1 + x_1, \ldots, 1 + x_n) = \sum d_{\lambda \mu} s_\mu(x_1, \ldots, x_n),$$
where the coefficients $d_{\lambda \mu}$ are given by

$$d_{\lambda \mu} = \det \left[ \begin{array}{cccc} \lambda_i + n - i & \mu_j + n - j \\ \mu_j + n - j \\ \end{array} \right]_{1 \leq i, j \leq n},$$

see Lascoux [Lasc], Macdonald [M2, I, §3, ex. 10].

Although the latter formula looks simpler than (1.6), formula (1.6) has a number of advantages, as it is explained in our paper [OO1]. E.g., an important idea is to consider $l_1, \ldots, l_n$ as variables rather than parameters, and then formula (1.6) makes in evidence the fact that the roles of $l_1, \ldots, l_n$ and of $x_1, \ldots, x_n$ are almost symmetric.

**Theorem 1.2** (Binomial formula for the series C–B–D). Assume the variables $z_i$ and $x_i$ are subject to the relations

$$x_i^2 = z_i + z_i^{-1} - 2, \quad i.e., \quad x_i = \pm(z_i^{1/2} - z_i^{-1/2}), \quad i = 1, \ldots, n.$$

Let $G$ be any of the groups of the series C, B, D and let us abbreviate $\chi_\lambda$ for $\chi_\lambda^{\text{sp}(2n)}$, $\chi_\lambda^{\text{so}(2n+1)}$ or $\chi_\lambda^{\text{o}(2n)}$, where $\lambda$ is a partition of length $\leq n$. Then we have

$$\frac{\chi_\lambda(z_1, \ldots, z_n)}{\chi_\lambda(1, \ldots, 1)} = \sum_{\mu} s_{\mu}(l_1^2, \ldots, l_n^2) s_{\mu}(z_1^2, \ldots, z_n^2) c_{\pm}(n, \mu). \quad (1.8)$$

Here $l_i = \lambda_i + n - i + \varepsilon$, $\varepsilon$ is defined by (1.5), $\mu$ ranges over partitions of length $\leq n$, and

$$c_{\pm}(n, \mu) = \prod_{(i, j) \in \mu} 4(n + j - i)(n \pm 1/2 + j - i), \quad (1.9)$$

where the sign “+” is taken for the series C, B, while the sign “−” is taken for the series D.

Formula (1.8) is a particular case of the expansion of a type $BC_n$ Jacobi polynomial with “Jack parameter” $\alpha = 1$ into a series of Schur polynomials. This expansion probably is well-known to experts (see, e.g., Lassalle [Lass, Théorème 9]). For sake of completeness we present a detailed proof.

**Proof.** Consider the classical Jacobi polynomials $P_k^{(\alpha, \beta)}$ and put

$$\tilde{P}_k^{(\alpha, \beta)} = P_k^{(\alpha, \beta)}/P_k^{(\alpha, \beta)}(1).$$

Let $T_k$ and $U_k$ be the Chebyshev polynomials of the first and second kind, respectively. Then for any $k = 0, 1, \ldots$

$$\frac{z^{k+1} - z^{-k-1}}{z - z^{-1}} = U_k \left( \frac{z + z^{-1}}{2} \right) \sim \tilde{P}_k^{(1/2, 1/2)} \left( \frac{z + z^{-1}}{2} \right),$$

$$\frac{z^{k+1/2} - z^{-k-1/2}}{z^{1/2} - z^{-1/2}} \sim \tilde{P}_k^{(1/2, -1/2)} \left( \frac{z + z^{-1}}{2} \right),$$

$$z^k + z^{-k} = 2T_k \left( \frac{z + z^{-1}}{2} \right) \sim \tilde{P}_k^{(-1/2, -1/2)} \left( \frac{z + z^{-1}}{2} \right),$$

$$\frac{z^{k+2} - z^{-k-2}}{z^2 - z^{-2}} \sim \tilde{P}_k^{(-1, -1)} \left( \frac{z + z^{-1}}{2} \right).$$
where the symbol “∼” means equality within a number factor depending on \( k \) but not depending on \( z \) (see Szegő [S, (4.1.7) and (4.1.8)]).

Substituting these expressions into the formulas for the characters \( \chi_\lambda \) given above we obtain

\[
\chi_\lambda(z_1, \ldots, z_n) \sim \frac{\det [\tilde{P}^{(\alpha, \beta)}_{\lambda_i+n-i}(1 + t_j)]_{1 \leq i, j \leq n}}{\prod_{i<j}(t_i - t_j)}, \tag{1.10}
\]

where

\[
t_i = \frac{z_i + z_i^{-1}}{2} - 1, \quad i = 1, \ldots, n, \quad (\alpha, \beta) = \begin{cases} (1/2, 1/2), & \text{case C,} \\ (1/2, -1/2), & \text{case B,} \\ (-1/2, -1/2), & \text{case D,} \end{cases}
\]

and the symbol “∼” means equality within a factor not depending on \( t_1, \ldots, t_n \).

Recall a well-known identity: for any power series

\[
f_i(t) = \sum_{m=0}^{\infty} a_m^{(i)} t^m, \quad i = 1, \ldots, n,
\]

we have

\[
\det [f_i(t_j)]_{1 \leq i, j \leq n} \prod_{i<j}(t_i - t_j) = \sum_\mu \det [a_{\mu_i+n-i}^{(i)}] s_\mu(t_1, \ldots, t_n), \tag{1.11}
\]

where \( \mu \) ranges over partitions of length \( \leq n \) (see, e.g., Hua [Hua, Theorem 1.2.1]).

We shall apply this identity to

\[
f_i(t) = \tilde{P}^{(\alpha, \beta)}_{\lambda_i+n-i}(1 + t), \quad i = 1, \ldots, n,
\]

and we shall use the well-known expansion

\[
\tilde{P}^{(\alpha, \beta)}_k(1 + t) = \sum_{m \geq 0} \frac{k(k-1) \cdots (k-m+1)(k+\alpha+\beta+1) \cdots (k+\alpha+\beta+m)}{2^m m! (\alpha+1) \cdots (\alpha+m)} \cdot t^m
\]

(see, e.g., Szegő [S, (4.21.2)])).

Remark that in our situation \( \alpha + \beta + 1 = 2 \varepsilon \), so that

\[
(k - i)(k + \alpha + \beta + 1 + i) = (k + \varepsilon)^2 - (\varepsilon + i)^2, \quad i = 0, 1, \ldots.
\]

Further, remark that \( \alpha = \pm 1/2 \) according to the assumption on the sign made in the statement of the theorem. It follows that the above expansion can be rewritten as

\[
\tilde{P}^{(\alpha, \beta)}_k(1 + t) = \sum_{m \geq 0} \frac{((k + \varepsilon)^2 | \varepsilon^2, (\varepsilon + 1)^2, \ldots)^m t^m}{2^m m! (\pm 1/2 + 1) \cdots (\pm 1/2 + m)}, \tag{1.12}
\]

Formulas (1.10), (1.11), and (1.12) imply that

\[
\chi_\lambda(z_1, \ldots, z_n)
\sim \sum \det [(l_j^2 | \varepsilon^2, (\varepsilon + 1)^2, \ldots)^{\mu_j+n-j}] s_\mu(t_1, \ldots, t_n)
\prod_{i<j}(\mu_i+n-i)^!(\pm 1/2 + 1) \cdots (\pm 1/2 + \mu_i + n - i)),
\]

and the symbol “∼” means equality within a factor not depending on \( t_1, \ldots, t_n \).
Our construction is suggested by the binomial formula of §□ we obtain the desired formula.

The normalizing factor in this formula is determined by the condition that the constant term (i.e., the coefficient of $s_\varnothing(t_1, \ldots, t_n) \equiv 1$) should be equal to 1. After simple transformations, using the relations

$$s_\mu(t_1, \ldots, t_n) = 2^{-|\mu|} s_\mu(x_1^2, \ldots, x_n^2),$$

$$\frac{\det([l_i^2|\varepsilon^2, (\varepsilon + 1)^2, \ldots]^{\mu_j+j+n-j})}{\det([l_i^2|\varepsilon^2, (\varepsilon + 1)^2, \ldots]^{n-j})} = s_\mu(l_1^2, \ldots, l_n^2|\varepsilon^2, (\varepsilon + 1)^2, \ldots),$$

we obtain the desired formula. □

§2. A distinguished basis in $Z(\mathfrak{g})$

The aim of this section is to construct and characterize a linear basis in $Z(\mathfrak{g})$. Our construction is suggested by the binomial formula of §1.

Let $O_e(G)$ be the algebra of germs of holomorphic functions at the unity $e$ of the group $G$ and let $M_e(G)$ be the maximal ideal of $O_e(G)$ formed by germs vanishing at $e$. We may identity $U(\mathfrak{g})$ with the space of those linear functionals on $O_e(G)$ that vanish on $M_e(G)^m$, where $m$ is large enough. This defines a nondegenerate pairing $\langle \cdot, \cdot \rangle$ between $U(\mathfrak{g})$ and $\mathbb{C}[G]$ (see §0 for the definition of $\mathbb{C}[G]$). If $V$ is a finite-dimensional $G$-module (also viewed as a $U(\mathfrak{g})$-module), $V^*$ is the dual module, $\xi \in V$ and $\eta \in V^*$ are arbitrary vectors, and $f_{\xi\eta}(g) = \eta(g\xi)$ is the corresponding matrix coefficient, then we have

$$\langle X, f_{\xi\eta} \rangle = \eta(X\xi) \quad \text{for any } X \in U(\mathfrak{g}).$$

Let $I(G) \subset \mathbb{C}[G]$ be the subalgebra of invariants of the group $G$ (for series $A, C, B$) or the group $G'$ (for series $D$) with respect to its action on $G$ by conjugations; note that this definition is parallel to that of $Z(\mathfrak{g}) \subset U(\mathfrak{g})$, see §0.

Remark that both $U(\mathfrak{g})$ and $\mathbb{C}[G]$ are semisimple modules over $G$ or $G'$. It follows that there are canonical projections

$$\#: U(\mathfrak{g}) \to Z(\mathfrak{g}),$$

$$\#: \mathbb{C}[G] \to I(G);$$

these projections also can be defined as averaging over a compact form of the group $G$ or $G'$. The pairing $\langle \cdot, \cdot \rangle$ over $U(\mathfrak{g})$ and $\mathbb{C}[G]$ is invariant over $G$ or $G'$ and so defines a nondegenerate pairing between $Z(\mathfrak{g})$ and $I(G)$.

Lemma-Definition 2.1. Let $\mu$ range over the set of partitions of length $\leq n$. There exist central elements

$$S_\mu \in Z(\mathfrak{g}), \quad \deg S_\mu \leq |\mu| \quad (\text{series } A),$$

$$T_\mu \in Z(\mathfrak{g}), \quad \deg T_\mu \leq 2|\mu| \quad (\text{series } C, B, D),$$

uniquely specified by the condition

$$(f^\# |_H)(z_1, \ldots, z_n) = \sum_{\mu} \frac{\langle S_\mu, f \rangle}{c(n, \mu)} s_\mu(x_1, \ldots, x_n),$$

$$(f^\# |_H)(z_1, \ldots, z_n) = \sum_{\mu} \frac{\langle T_\mu, f \rangle}{c(n, \mu)} s_\mu(x_1^2, \ldots, x_n^2),$$
where $H \subset G$ is the diagonal torus defined in §1, $f \in C[G]$ stands for an arbitrary test function; $f^\# \in I(G)$ is the image of $f$ under the projection (2.2), and the variables $x_1, \ldots, x_n$ and the normalizing factors $c(n,\mu), c_\pm(n,\mu)$ are given by (1.6), (1.9).

**Proof.** The function $f^\# |_H$ is a symmetric Laurent polynomial in $z_1, \ldots, z_n$; in the case of the series $C, B, D$ it is also invariant under transformations $z_i \mapsto z_i^{-1}$. It follows that $f^\# |_H$ can be expanded into a series of polynomials $s_\mu(x_1, \ldots, x_n)$ or $s_\mu(x_1^2, \ldots, x_n^2)$, where $x_i = z_i - 1$ (for series $A$) or $x_i^2 = z_i + z_i^{-1} - 2$ (for series $C, B, D$). Given $\mu$, the map assigning to $f$ the $\mu$th coefficient in that expansion is a linear functional on $C[G]$ which only depends on a finite jet of $f$ at the point $e \in G$ (the order of the jet is equal to $|\mu|$ or $2|\mu|$). Hence this functional is an element of the algebra $U(\mathfrak{g})$ of degree $\leq |\mu|$ or $\leq 2|\mu|$. This proves existence of the elements $S_\mu$ or $T_\mu$. As will be clear in what follows, we have exact equalities

$$\deg S_\mu = |\mu|, \quad \deg T_\mu = 2|\mu|.$$

Finally, the elements $S_\mu, T_\mu$ belong to $Z(\mathfrak{g})$, because their values at any test function $f \in C[G]$ depend on $f^\#$ only. $\square$

Recall that $V_\lambda$ denotes the irreducible $G$-module with highest weight $\lambda = (\lambda_1, \ldots, \lambda_n)$, which is equally viewed as a $U(\mathfrak{g})$-module. Any central element $X \in U(\mathfrak{g})$ acts in $V_\lambda$ as a scalar operator const·1; the corresponding constant is called the eigenvalue of the central element $X$ in $V_\lambda$.

**Theorem 2.2.** Let $G = GL(n, \mathbb{C})$. The elements $S_\mu$ defined in Lemma 2.1 form a basis in $Z(\mathfrak{g})$ and are characterized (within a scalar multiple) by the following properties:

(i) $\deg S_\mu \leq |\mu|$

(ii) the eigenvalue of $S_\mu$ in a module $V_\lambda$ equals 0 for all partitions $\lambda$ such that $l(\lambda) \leq n, |\lambda| \leq |\mu|, \lambda \neq \mu$

(iii) the eigenvalue of $S_\mu$ in $V_\mu$ is nonzero.

**Theorem 2.3.** Let $G$ be a classical group of type $C, B, D$. The elements $T_\mu$ defined in Theorem 2.1 form a basis in $Z(\mathfrak{g})$ and are characterized (within a scalar multiple) by the following properties:

(i) $\deg T_\mu \leq 2|\mu|$

(ii) the eigenvalue of $T_\mu$ in a module $V_\lambda$ equals 0 for all partitions $\lambda$ such that $l(\lambda) \leq n, |\lambda| \leq |\mu|, \lambda \neq \mu$

(iii) the eigenvalue of $T_\mu$ in $V_\mu$ is nonzero.

**Comment.** Note a difference between these two claims: in Theorem 2.3 we used all irreducible $G$-modules to characterize our basis, while in Theorem 2.2 we used only a part of the modules (namely, the polynomial ones). This implies that for the series $A$, our basis $\{S_\mu\}$ is not invariant under the outer automorphism of $\mathfrak{gl}(n, \mathbb{C})$.

To prove Theorems 2.2–2.3 we shall restate them as purely combinatorial claims. We shall need the Harish-Chandra homomorphism. Recall (see, e.g., Dixmier [D,

\[\text{The fact that for the series D there exist dominant highest weights } \lambda = (\lambda_1, \ldots, \lambda_n) \text{ with } \lambda_n < 0 \text{ is not relevant here, because for any element of } Z(o(2n)), \text{ its eigenvalue does not change under the transformations } \lambda \mapsto \alpha \lambda, \alpha > 0.\]
that it is defined for any reductive Lie algebra \( \mathfrak{g} \) over \( \mathbb{C} \). Fix a triangular decomposition \( \mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ \) where \( \mathfrak{h} \) is a Cartan subalgebra and \( \mathfrak{n}_+ \) and \( \mathfrak{n}_- \) are spanned by positive and negative root vectors. The Harish–Chandra homomorphism maps \( U(\mathfrak{g})^\mathfrak{h} \), the centralizer of \( \mathfrak{h} \) in \( U(\mathfrak{g}) \), onto \( U(\mathfrak{h}) = S(\mathfrak{h}) \). Let us identify \( S(\mathfrak{h}) \) with \( \mathbb{C}[\mathfrak{h}^*] \), the algebra of polynomial functions on the dual space \( \mathfrak{h}^* \). It is known that the restriction of the Harish–Chandra homomorphism to the center of \( U(\mathfrak{g}) \) is injective and its image in \( \mathbb{C}[\mathfrak{h}^*] \) consists of those polynomials \( f(\lambda) \) on \( \mathfrak{h}^* \) that are invariant under transformations

\[
\lambda \mapsto w(\lambda + \rho) - \rho, \quad \lambda \in \mathfrak{h}^*,
\]

where \( w \) ranges over the Weyl group \( W = W(\mathfrak{g}, \mathfrak{h}) \) and \( \rho \) denotes the half-sum of positive roots. Moreover, if \( X \) is a central element of \( U(\mathfrak{g}) \) and \( f_X \) is the corresponding invariant polynomial then the value of \( f_X \) at a point \( \lambda \in \mathfrak{h}^* \) coincides with the eigenvalue of \( X \) in the irreducible highest weight module over \( \mathfrak{g} \) with highest weight \( \lambda \). Although this is true for all \( \lambda \in \mathfrak{h}^* \) we shall deal with dominant weights \( \lambda \) only; note that any polynomial \( f(\lambda) \) is uniquely determined by its values on the set of dominant weights. Finally, note that \( \deg X = \deg f_X \).

After this digression let us return to the classical Lie algebras. Examine first the series A. Then the group \( W \) coincides with the symmetric group \( S(n) \) and the weight \( \rho \) satisfies the property \( \rho_i - \rho_{i+1} = 1, \quad 1 \leq i \leq n - 1 \). It follows that a polynomial \( f(\lambda_1, \ldots, \lambda_n) \) is invariant under the action (2.1) if and only if it satisfies the following symmetry property:

\[
f(\lambda_1, \ldots, \lambda_n) = f(\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1} - 1, \lambda_i + 1, \lambda_{i+2}, \ldots, \lambda_n)
\]

for \( i = 1, \ldots, n - 1 \). In our paper \([\text{OO}1]\) such polynomials were called shifted symmetric ones and the algebra of shifted symmetric polynomials in \( n \) variables was denoted by \( \Lambda^*(n) \). Clearly, \( f(\lambda_1, \ldots, \lambda_n) \) is shifted symmetric if and only if it is symmetric in the variables \( l_1, \ldots, l_n \) that are related to \( \lambda_1, \ldots, \lambda_n \) by (1.3). Note that the top homogeneous component of a shifted symmetric polynomial is a symmetric polynomial. The Harish–Chandra homomorphism induces an algebra isomorphism \( Z(\mathfrak{gl}(n)) \to \Lambda^*(n) \).

Next assume \( G \) is one of the classical groups of type C, B, D. Then the weight \( \rho \) has the form

\[
\rho = \rho_\varepsilon = (n - 1 + \varepsilon, n - 2 + \varepsilon, \ldots, \varepsilon), \quad \varepsilon = 1, 1/2, 0,
\]

and the group \( W \) is either the hyperoctahedral group \( H(n) = S(n) \times \{ \pm 1 \}^n \) (C–B case) or its subgroup \( H'(n) \subset H(n) \) of index 2 (D case); recall that the action of the generators of the subgroup \( \{ \pm 1 \}^n \) is of the form \( \lambda_i \mapsto -\lambda_i \). However, for the series D, we have replaced the center of \( U(\mathfrak{o}(2n, \mathbb{C})) \) by its proper subalgebra \( Z(\mathfrak{o}(2n, \mathbb{C})) \), which is just equivalent to replacement of \( H'(n) \) by the whole group \( H(n) \). This allows us to describe the algebra \( Z(\mathfrak{g}) \) in a uniform way for the three series C, B, D.

Denote by \( M^*(n) = M^*_\varepsilon(n) \) the algebra of polynomials in \( \lambda_1, \ldots, \lambda_n \) that can be written in the form

\[
f(\lambda_1, \ldots, \lambda_n) = g(l_1^2, \ldots, l_n^2), \quad l = \lambda + \rho_\varepsilon,
\]

where \( g \) is an arbitrary symmetric polynomial. Then we obtain an algebra isomorphism \( Z(\mathfrak{g}) \to M^*(n) \) induced by the Harish–Chandra homomorphism.
Lemma-Definition 2.4. For the series $A$, put
\[ s^*_\mu(\lambda_1, \ldots, \lambda_n) = s_\mu(l_1, \ldots, l_n \mid 0, 1, 2, \ldots) \]
and for the series $C$, $B$, $D$, put
\[ t^*_\mu(\lambda_1, \ldots, \lambda_n) = s_\mu(l_1^2, \ldots, l_n^2 \mid \varepsilon^2, (\varepsilon + 1)^2, \ldots), \]
where $\varepsilon = 1, 1/2, 0$, respectively. The Harish-Chandra homomorphism sends $S_\mu$ to $s^*_\mu$ and $T_\mu$ to $t^*_\mu$.

Proof. This follows at once from the above discussion if one compares the definition of the elements $S_\mu$, $T_\mu$ (Lemma-Definition 2.1), the claim of Theorems 1.1–1.2, and the above definition of $s^*_\mu$ and $t^*_\mu$. □

Since the map $Z(\mathfrak{gl}(n)) \to \Lambda^*(n)$ preserves degree and since $\deg s^*_\mu = |\mu|$, we have $\deg S_\mu = |\mu|$. Similarly, since the top homogeneous component of $t^*_\mu(\lambda_1, \ldots, \lambda_n)$ coincides with $s_\mu(\lambda_1^2, \ldots, \lambda_n^2)$, we have $\deg t^*_\mu = 2|\mu|$, so that $\deg T_\mu = 2|\mu|$.

Now it is clear that Theorems 2.2–2.3 are equivalent to the following claims about the polynomials $s^*_\mu$ and $t^*_\mu$.

Theorem 2.5. The polynomials $s^*_\mu(\lambda_1, \ldots, \lambda_n)$ and $t^*_\mu(\lambda_1, \ldots, \lambda_n)$ form a basis of the algebras $\Lambda^*(n)$ and $M^*(n)$, respectively. They can be characterized (within a scalar multiple) by the following properties:

(i) $s^*_\mu \in \Lambda^*(n)$ and $\deg s^*_\mu = |\mu|$, $t^*_\mu \in M^*(n)$ and $\deg t^*_\mu = 2|\mu|$;
(ii) $s^*_\mu(\lambda) = 0$ and $t^*_\mu(\lambda) = 0$ for all partitions $\lambda$ such that $l(\lambda) \leq n$, $|\lambda| \leq |\mu|$, $\lambda \neq \mu$;
(iii) $s^*_\mu(\mu) \neq 0$ and $t^*_\mu(\mu) \neq 0$.

Proof. The part of these claims concerning the polynomials $s^*_\mu$ was established in [Ok1]; see also [OO1, Theorem 3.3]. So we shall only consider the case of $t^*_\mu$ (which, however, is quite similar to that of $s^*_\mu$).

Let $M(n)$ denote the algebra of polynomials in $n$ variables $\lambda_1, \ldots, \lambda_n$, invariant under the natural action of the hyperoctahedral group $H(n)$ (the generators of $\{\pm 1\}^n \subset H(n)$ change the sign of the variables). Clearly, the polynomials
\[ t_\mu(\lambda_1, \ldots, \lambda_n) := s_\mu(\lambda_1^2, \ldots, \lambda_n^2) \]
form a homogeneous basis in $M(n)$.

On the other hand, $M(n)$ is canonically isomorphic to the graded algebra associated with the filtered algebra $M^*(n)$. Since $t_\mu$ coincides with the top component of $t^*_\mu$, we conclude that the polynomials $t^*_\mu$ form a basis in $M^*(n)$.

Let us check that the polynomials $t^*_\mu$ satisfy the conditions (i)–(iii). The condition (i) is evident and the conditions (ii), (iii) will be deduced from the explicit formula
\[ t^*_\mu(\lambda_1, \ldots, \lambda_n) = s_\mu(l_1^2, \ldots, l_n^2 \mid \varepsilon^2, (\varepsilon + 1)^2, \ldots) = \frac{\det [(l_j^2 \mid \varepsilon^2, (\varepsilon + 1)^2, \ldots)^{m_j+n-j}]}{\prod (l_j^2)}. \]
Note that the denominator is nonzero, because $l_1^2, \ldots, l_n^2$ strictly decrease. Thus, we have to analyze the vanishing properties of the numerator.

Assume $|\lambda| \leq |\mu|$ and $\lambda \neq \mu$. Then there exists $k \in \{1, \ldots, n\}$ such that $\lambda_k < \mu_k$, whence

$$\lambda_i < \mu_j \quad \text{for} \quad j \leq k \leq i.$$ 

For such couples $i, j$ the $(i, j)$th entry of the determinant in the numerator is

$$(l_i^2 \varepsilon^2, (\varepsilon + 1)^2, \ldots)^{\mu_j+n-j} \prod_{r=1}^{\mu_j+n-j} ((\lambda_i + n - i + \varepsilon)^2 - (\varepsilon + r - 1)^2) = 0,$$

so that the determinant vanishes.

If $\lambda = \mu$ then a similar argument shows that the matrix in the numerator is strictly upper triangular with nonzero diagonal entries, so that the determinant is nonzero.

Thus, we have verified the properties (i)–(iii) and it remains to prove that polynomials with such properties are unique. For $d = 2, 4, 6, \ldots$ consider the subspace $M_d^*(n) \subset M^*(n)$ formed by elements of degree $\leq d$. The existence property verified above implies that the linear functionals on $M_d^*(n)$ of the form

$$f \mapsto f(\lambda), \quad |\lambda| \leq d,$$

are linearly independent. On the other hand, their number just equals $\dim M_d^*(n)$, which implies the uniqueness claim. □

Note that Theorem 2.5 also can be directly deduced from the binomial formula, cf. second proof of Theorem 5.1 in [OO1].

§3. Coherence property

Until now we fixed a classical Lie algebra $\mathfrak{g}$. In this section we deal with the whole series $\{\mathfrak{g}(n)\}$ of classical Lie algebras (of type $A, C, B$ or $D$). We define natural embeddings $Z(\mathfrak{g}(n)) \to Z(\mathfrak{g}(n+1))$ and we show that the elements of the canonical bases are stable (within scalar multiples) with respect to these embeddings.

Since the Lie algebra will vary (inside a classical series), we shall use more detailed notation for the polynomials in $n$ variables introduced above:

$$s_{\mu|n}, \ s_{\mu|n}^*, \ t_{\mu|n}, \ t_{\mu|n}^*,$$

and the elements of the distinguished basis of $Z(\mathfrak{g}(n))$ will be denoted by $S_{\mu|n}$ or $T_{\mu|n}$.

For any $n$, we shall identify $\mathbb{C}^n$ with the subspace of $\mathbb{C}^{n+1}$ spanned by first $n$ basis vectors. Since $\mathfrak{g}(n)$ is realized in $\mathbb{C}^n$ (for series $A$) or in $\mathbb{C}^n \oplus \mathbb{C}^n$ (for series $C, B, D$), we obtain a natural Lie algebra embedding $\mathfrak{g}(n) \to \mathfrak{g}(n+1)$, which induces a natural algebra embedding $U(\mathfrak{g}(n)) \to U(\mathfrak{g}(n+1))$.

Next, we define the averaging operator

$$\text{Av}_{n,n+1}: Z(\mathfrak{g}(n)) \to Z(\mathfrak{g}(n+1))$$

as the composition

$$Z(\mathfrak{g}(n)) \circ U(\mathfrak{g}(n)) \circ U(\mathfrak{g}(n+1)) \to Z(\mathfrak{g}(n+1)).$$
where the latter arrow is the projection \# defined in (2.1). Similarly, for any couple
\( n < m \), we define the averaging operator

\[
\text{Av}_{nm} : Z(g(n)) \to Z(g(m)),
\]

which also coincides with composition of averagings

\[
Z(g(n)) \to Z(g(n+1)) \to \cdots \to Z(g(m)).
\]

**Theorem 3.1.** Let \( c(n,\mu) \) and \( c_{\pm}(n,\mu) \) be the normalizing factors occurring in
the binomial formula (see (1.7), (1.9)). For any \( n < m \) we have

\[
\text{Av}_{nm} \left( \frac{1}{c(n,\mu)} S_{\mu|n} \right) = \frac{1}{c(m,\mu)} S_{\mu|m},
\]

\[
\text{Av}_{nm} \left( \frac{1}{c_{\pm}(n,\mu)} T_{\mu|n} \right) = \frac{1}{c_{\pm}(m,\mu)} T_{\mu|m}.
\]

We call this the coherence property of the basis \( \{S_{\mu}\} \) or \( \{T_{\mu}\} \). For the series A
the coherence property was established in our paper [OO1, \S 10], by three different
methods which can also be transferred to the series C, B, D. We shall give now one
of the proofs, which is based on the binomial formula.

**Proof.** We assume \( \{g(n)\} \) is one the series C, B, D. First of all we shall restate
Theorem 1.2. Let \( g \in G \) and \( X \in g \) be related by

\[
g^{1/2} - g^{-1/2} = X,
\]

where \( g \) is close to \( e \in G \) and \( X \) is close to \( 0 \in g \). Let \( f \in I(G) \) be a test function
and let \( f' = f'(X) \) be the corresponding function in a neighborhood of \( 0 \in g \). Write \( \pm x_1, \pm x_2, \ldots \) for the eigenvalues of the matrix \( X \) and define

\[
T_{\mu}(X) = s_{\mu}\left(x_1^2, x_2^2, \ldots \right), \quad X \in g, \ l(\mu) \leq \text{rank} \ g.
\]

One can expand \( f'(X) \) into a series of the invariant polynomials \( T_{\mu}(X) \), which
converges in a neighborhood of the origin. We claim that this expansion can be
written as

\[
f'(x) = \sum_{l(\mu) \leq \text{rank} \ g} \langle \tilde{T}_{\mu}, f \rangle T_{\mu}(X), \quad (3.1)
\]

where

\[
\tilde{T}_{\mu} := \frac{1}{c_{\pm}(\text{rank} \ g, \mu)} T_{\mu}.
\]

Indeed, since \( I(G) \) is spanned by the characters \( \chi_\lambda \) (here we use the same
abbreviation as in Theorem 1.2), we may assume \( f = \chi_\lambda / \chi_\lambda(e) \). But then our
claim is just a restatement of the binomial formula. To see this we have to recall
the definition of the elements \( T_{\mu} \) and use the fact that for any element \( A \in Z(g) \)
(in particular, for \( A = T_{\mu} \)), \( \langle A, \chi_\lambda / \chi_\lambda(e) \rangle \) equals the eigenvalue of \( A \) in the \( U(g) \)-
module corresponding to \( \chi_\lambda \).
Now let \( n < m \) and let \( G(n) \hookrightarrow G(m) \) be the natural embedding. Let \( f \in I(G(m)) \) be a test function and \( f|_{G(n)} \) be its restriction to \( G(n) \), which is clearly an element of \( I(G(n)) \). Let us write the expansion (3.1) for \( f \) and \( f|_{G(n)} \):

\[
 f'(X) = \sum_{l(\mu) \leq m} \langle \bar{T}_\mu |m|, f \rangle T_{\mu|m}(X), \quad X \in g(m),
\]

\[
 (f|_{G(n)})'(Y) = \sum_{l(\mu) \leq n} \langle \bar{T}_\mu |n|, f|_{G(n)} \rangle T_{\mu|n}(Y), \quad Y \in g(n).
\]

Assume \( X = Y \in g(n) \). Then the left-hand sides of both expansions coincide. On the other hand, by the definition of the polynomial functions \( T_\mu \) and the stability property of the Schur functions, we have

\[
 T_{\mu|m}(Y) = \begin{cases} 
 T_{\mu|n}(Y), & \text{if } l(\mu) \leq n, \\
 0, & \text{if } l(\mu) > n.
\end{cases}
\]

It follows

\[
 \langle \bar{T}_\mu |n|, f|_{G(n)} \rangle = \langle \bar{T}_\mu |m|, f \rangle, \quad f \in I(G(m)), \ l(\mu) \leq n.
\]

More generally, for any \( f \in \mathbb{C}[G] \) we have

\[
 \langle \bar{T}_\mu |m|, f \rangle = \langle \bar{T}_\mu |m|, f^\# \rangle = \langle \bar{T}_\mu |n|, f^\# |_{G(n)} \rangle,
\]

which is equivalent to the claim of the theorem. \( \square \)

**Remark 3.2.** It was shown in [OO1], that the coherence property of the basis \( \{S_\mu\} \) is equivalent to the following relation satisfied by the polynomials \( s_\mu^* \): for any partition \( \mu \) of length \( \leq n \) and any signature \( \Lambda \) of length \( n+1 \),

\[
 \frac{\chi^{gl(n+1)}(e)}{c(n+1, \mu)} s_{\mu|n+1}^*(\Lambda_1, \ldots, \Lambda_{n+1}) = \sum_{\lambda < \Lambda} \frac{\chi^{gl(n)}(e)}{c(n, \mu)} s_{\mu|n}^*(\lambda_1, \ldots, \lambda_n),
\]

where \( \lambda \) denote a signature of length \( n \) and \( \lambda < \Lambda \) means the Gelfand–Tsetlin interlacing condition

\[
 \Lambda_1 \geq \lambda_1 \geq \Lambda_2 \geq \cdots \geq \Lambda_{n+1} \geq \lambda_n \geq \Lambda_{n+1}.
\]

A similar relation, which is equivalent to the coherence property of \( \{T_\mu\} \), holds for the polynomials \( t_{\mu|n+1}^* \):

\[
 \frac{\chi^{sl(2n+2)}(e)}{c_\pm(n+1, \mu)} t_{\mu|n+1}^*(\Lambda_1, \ldots, \Lambda_{n+1}) = \sum_{\lambda} [\chi_\Lambda : \chi_\lambda] \frac{\chi^{so(2n+3)}(e)}{c_\pm(n, \mu)} t_{\mu|n}^*(\lambda_1, \ldots, \lambda_n).
\]

Here \( \Lambda \) is a partition of length \( \leq n+1 \), \( \mu \) is a partition of length \( \leq n \), \( \chi_\Lambda \) is one of the characters of \( G(n+1) \) (i.e., \( \chi^{sp(2n+2)}_\Lambda \), \( \chi^{so(2n+3)}_\Lambda \) or \( \chi^{o(2n+2)}_\Lambda \)), \( \chi_\lambda \) has the same meaning (but for the group \( G(n) \) of rank \( n \)), and \( [\chi_\Lambda : \chi_\lambda] \) denotes the multiplicity of \( \chi_\lambda \) in the decomposition of \( \chi_\Lambda \) as restricted to \( G(n) \subset G(n+1) \).

In [OO1, §10] we gave a direct derivation of the relation (3.2). The relation (3.3) can be directly verified by a similar (but more complicated) argument. This gives another approach to the coherence property.

\[5\]This multiplicity is given by the well-known ‘branching rules’ for symplectic and orthogonal groups, see Želobenko [Z, §§129, 130].
§4. A Distinguished Basis in \( I(\mathfrak{g}) \)

In this section, we study a distinguished basis in \( I(\mathfrak{g}) \subset S(\mathfrak{g}) \) (see section 0 for the definition of \( I(\mathfrak{g}) \)).

We equip \( \mathfrak{g} \) with an invariant inner product: for any \( X, Y \in \mathfrak{g} \),

\[
\langle X, Y \rangle = \begin{cases} 
\text{tr } XY, & \text{for the series A,} \\
\frac{1}{2} \text{tr } XY, & \text{for the series C, B, D.}
\end{cases}
\]

We shall identify \( \mathfrak{g} \) with its dual space \( \mathfrak{g}^* \) by making use of the product \( \langle \cdot, \cdot \rangle \). This will allow us to interpret \( S(\mathfrak{g}) \) as the algebra of polynomial functions on \( \mathfrak{g} \) and then elements of \( I(\mathfrak{g}) \) will become invariant polynomial functions on \( \mathfrak{g} \).

Assume \( X \) ranges over \( \mathfrak{g} \). In the case of the series A we shall denote by \( x_1, \ldots, x_n \) the eigenvalues of \( X \). In the case of the series C, D the eigenvalues of \( X \) may be written as \( \pm x_1, \ldots, \pm x_n \), and for the series B one has to add one 0.

In this notation we set

\[
S_\mu(X) = S_\mu|_n(X) = s_\mu(x_1, \ldots, x_n),
\]

\[
T_\mu(X) = T_\mu|_n(X) = t_\mu(x_1, \ldots, x_n) = s_\mu(x_1^2, \ldots, x_n^2).
\]

Clearly, \( \{S_\mu\} \) or \( \{T_\mu\} \) is a homogeneous basis in \( I(\mathfrak{g}) \).

Note that \( I(\mathfrak{g}) \) may be identified with \( \text{gr } Z(\mathfrak{g}) \), the graded algebra associated with the fibered algebra \( Z(\mathfrak{g}) \). The next claim is immediate from definitions.

**Proposition 4.1.** Under the identification \( I(\mathfrak{g}) = \text{gr } Z(\mathfrak{g}) \), each basis element \( S_\mu \in I(\mathfrak{g}) \) or \( T_\mu \in I(\mathfrak{g}) \) coincides with the leading term of the basis element \( S_\mu \in Z(\mathfrak{g}) \) or \( T_\mu \in Z(\mathfrak{g}) \), respectively. \( \square \)

Now write \( \mathfrak{g}(n) \) instead of \( \mathfrak{g} \) and consider the chain \( \cdots \hookrightarrow \mathfrak{g}(n) \hookrightarrow \mathfrak{g}(n+1) \hookrightarrow \cdots \) of natural embeddings that we already discussed in §3. Note that these embedding are isometric with respect to the inner product \( \langle \cdot, \cdot \rangle \). Therefore, for each couple \( n < m \) there is a natural projection \( \mathfrak{g}(m) \rightarrow \mathfrak{g}(n) \) and so an algebra morphism \( S(\mathfrak{g}(m)) \rightarrow S(\mathfrak{g}(n)) \), which in turn induces an algebra morphism

\[
\text{Proj}_{nm}: I(\mathfrak{g}(m)) \rightarrow I(\mathfrak{g}(n)), \quad n < m.
\]

**Proposition 4.2** (Stability of the bases \( \{S_\mu\}, \{T_\mu\} \)). Assume \( n < m \). If \( l(\mu) \leq n \) then

\[
\text{Proj}_{nm}(S_\mu|_n) = S_\mu|_n, \quad \text{Proj}_{nm}(T_\mu|_m) = T_\mu|_n.
\]

If \( n < l(\mu) \leq m \) then the result is zero.

**Proof.** This follows at once from the stability property of the Schur polynomials. \( \square \)

On the other hand, for any \( n < m \) we can define linear maps

\[
\text{Avr}_{nm}: I(\mathfrak{g}(n)) \rightarrow I(\mathfrak{g}(m))
\]

(the **averaging operators**) just in the same way as for the invariants in the enveloping algebras, see §3.
Proposition 4.3 (Coherence property). Assume $n < m$ and $l(\mu) \leq n$. Then
\[ \text{Avr}_{nm}(S_{\mu|n}) = \frac{c(n, \mu)}{c(m, \mu)} S_{\mu|m}, \quad \text{Avr}_{nm}(T_{\mu|n}) = \frac{c_{\pm}(n, \mu)}{c_{\pm}(m, \mu)} T_{\mu|m}. \]

In particular, the averaging operators are injective.

Proof. This follows from Theorem 3.1 and Proposition 4.1.

Theorem 4.4 (Characterization of the basis). Let $n$ be fixed and $\mu$ range over partitions of length $\leq n$. The basis elements $S_{\mu|n}$ (for the series $A$) or $T_{\mu|n}$ (for the series $C, B, D$) are the unique, within a scalar multiple, elements of $I(\mathfrak{g}(n))$ that are eigenvectors of all linear mappings $\text{Proj}_{nm} \circ \text{Avr}_{nm}$.

Proof. By Propositions 4.2 and 4.3, we have for any $m > n$
\[ \text{Proj}_{nm}(\text{Avr}_{nm}(S_{\mu|n})) = \frac{c(n, \mu)}{c(m, \mu)} S_{\mu|n}, \]
\[ \text{Proj}_{nm}(\text{Avr}_{nm}(T_{\mu|n})) = \frac{c_{\pm}(n, \mu)}{c_{\pm}(m, \mu)} T_{\mu|n}. \]

Thus, each basis element is an eigenvector for all the maps $\text{Proj}_{nm} \circ \text{Avr}_{nm}$. It remains to show that the eigenvalues corresponding to two different basis vectors are distinct for certain $m$. But this follows from the explicit expressions for $c(n, \mu)$ and $c_{\pm}(n, \mu)$ given in §1. □

In [OO1, (2.10)], we obtained an explicit expression of the basis elements $S_{\mu|n} \in I(\mathfrak{gl}(n))$ through the matrix limits $E_{ij} \in \mathfrak{gl}(n)$, $1 \leq i, j \leq n$:

Proposition 4.5. Let $\mu$ be a partition of length $\leq n$, $k = |\mu|$, and $\chi^\mu$ be the irreducible character of the symmetric group $\mathfrak{S}(k)$ that is indexed by $\mu$. Then
\[ S_{\mu|n} = (k!)^{-1} \sum_{i_1, \ldots, i_k=1}^n \sum_{s \in \mathfrak{S}(k)} \chi^\mu(s) E_{i_1 i_{s(1)}} \cdots E_{i_k i_{s(k)}}. \]

We shall give now a similar formula for the series $C, B, D$. Assume $\mathfrak{g} = \mathfrak{g}(n)$ is of type $C, B, D$ and realize it as an involutive subalgebra in $\mathfrak{gl}(N, \mathbb{C})$, where $N = 2n$ or $N = 2n + 1$, as indicated in §1. We shall assume the canonical basis in $\mathbb{C}^N$ is labeled by the numbers $i = -n, -n + 1, \ldots, n - 1, n$, where 0 is included for the series $B$ only, and we put
\[ F_{ij} = E_{ij} - \theta_{ij} E_{-j, -i}, \]
where $\theta_{ij} \equiv 1$ for the series $B, D$ and $\theta_{ij} = \text{sgn}(i) \text{sgn}(j)$ for the series $C$. Note that for our embedding $\mathfrak{g}(n) \hookrightarrow \mathfrak{gl}(N, \mathbb{C})$ the elements $F_{ij}$ form a basis in $\mathfrak{g}(n)$.

Given a partition $\mu \vdash k$ we define a central function $\varphi^\mu$ on the symmetric group $\mathfrak{S}(2k)$ as follows. If a permutation $s \in \mathfrak{S}(2k)$ is such that all its cycles are even (so that the circle type of $s$ may be written as $2\rho$ for a certain $\rho \vdash k$) then we put
\[ \varphi^\mu(s) = \chi^\mu_{\rho}, \]
where $\chi^\mu_{\rho}$ denotes the value of the irreducible character $\chi^\mu$ at any permutation with cycle type $\rho$. Otherwise (i.e., if $s \in \mathfrak{S}(2k)$ has at least one cycle of odd length) $\varphi^\mu(s) = 0$.

The next claim is an exact analog of Proposition 4.5.
Proposition 4.6. Assume $g = g(n)$ is of type C, B, D. Let $\mu$ be an arbitrary partition of length $\leq n$ and $k = |\mu|$. Then

$$T_\mu|_n = \frac{1}{(2k)!} \sum_{i_1, \ldots, i_{2k} = -n}^{n} \sum_{s \in G(2k)} \varphi^\mu(s) F_{i_1 i_{s(1)}} \cdots F_{i_{2k} i_{s(2k)}}.$$  

Proof. Let us view $T_\mu|_n$ as an invariant polynomial function $T_\mu|_n(X)$, where $X$ ranges over $g(n)$ and denote by $\pm x_1, \ldots, \pm x_n$ the eigenvalues of $X$ (for the series B we exclude the zero eigenvalue). We shall use standard notation of Macdonald’s book [M2]: $p_\rho$ are the power sums and $z_\rho = 1 \cdot 2^{-l(\rho)} \chi_\rho^{p_\rho}(x_1, -x_1, \ldots, x_n, -x_n)$ for $\rho = (1^{k_1} 2^{k_2} \cdots)$.

By definition of the basis $\{T_\mu\}$ we have

$$T_\mu|_n(X) = s_\mu(x_1^2, \ldots, x_n^2) = \sum_{\rho \vdash k} z_\rho^{-1} \chi_\rho^{p_\rho}(x_1^2, \ldots, x_n^2)$$

$$= \sum_{\rho \vdash k} z_\rho^{-1} 2^{-l(\rho)} \chi_\rho^{p_\rho}(x_1, -x_1, \ldots, x_n, -x_n)$$

$$= \sum_{\rho \vdash k} z_\rho^{-1} \chi_\rho^{p_\rho}(x_1, -x_1, \ldots, x_n, -x_n).$$

Let $s$ range over elements of $G(2k)$ with even cycle type $2\rho$ (where $\rho \vdash k$ depends on $s$). Then the latter expression can be written as

$$T_\mu|_n(X) = \frac{1}{(2k)!} \sum_{s} \varphi^\mu(s) p_{2\rho}(x_1, -x_1, \ldots, x_n, -x_n).$$

On the other hand, for any (even) $m$, the invariant polynomial function

$$p_m(X) = \text{tr}(X^m) = p_m(x_1, -x_1, \ldots, x_n, -x_n)$$

corresponds to the element

$$\sum_{i_1, \ldots, i_m = -n}^{n} F_{i_1 i_2} F_{i_2 i_3} \cdots F_{i_m i_1} \in I(g(n)),$$

which also can be written as

$$\sum_{i_1, \ldots, i_m = -n}^{n} F_{i_1 i_{s(1)}} \cdots F_{i_{m} i_{s(m)}}$$

for any cyclic permutation $s$ of the indices $1, \ldots, m$. This concludes the proof. □

There exists another characterization of the bases $\{S_\mu\}$, $\{T_\mu\}$, which is based on the following observation, which is undoubtedly well known.
Proposition 4.7. Define a classical group $\tilde{G}$ containing $G$ as follows.
If $G = GL(n, \mathbb{C})$ then $\tilde{G} = GL(n, \mathbb{C}) \times GL(n, \mathbb{C})$ and the embedding $G \hookrightarrow \tilde{G}$ has the form $g \mapsto (g, (g')^{-1})$.

If $G$ is one of the groups $SO(2n + 1, \mathbb{C})$ or $Sp(2n, \mathbb{C})$, $SO(2n, \mathbb{C})$ then $\tilde{G} = GL(N, \mathbb{C})$, where $N = 2n + 1$ or $N = 2n$, respectively.

Then the adjoint action of $G$ in $\mathfrak{g}$ can be extended to a linear action of the group $\tilde{G} \supset G$ such that the induced representation of $\tilde{G}$ in the symmetric algebra $S(\mathfrak{g})$ is a multiplicity free polynomial representation.

Proof. Assume $G = GL(n, \mathbb{C})$. Using the bijection $E_{ij} \leftrightarrow e_j \otimes e_i$, where $\{e_i\}$ stands for the natural basis of $\mathbb{C}^n$, we identify $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ with $\mathbb{C}^n \otimes \mathbb{C}^n$. The action of the group $\tilde{G} = GL(n, \mathbb{C}) \times GL(n, \mathbb{C})$ in the vector space $\mathfrak{g} = \mathbb{C}^n \otimes \mathbb{C}^n$ is the natural one. Clearly, its restriction to the subgroup $G = \{(g, (g')^{-1})\}$ is equivalent to the adjoint action. It is well-known that action of the group $GL(n, \mathbb{C}) \times GL(n, \mathbb{C})$ in $S(\mathbb{C}^n \otimes \mathbb{C}^n)$ is a multiplicity free polynomial representation, see, e.g., Želobenko [Z, §56] or Howe [H, 2.1].

Assume $G = Sp(2n, \mathbb{C})$ and put $N = 2n$. One can identify $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$ with $S^2(\mathbb{C}^n)$ in such a way that the adjoint action of $G$ in $\mathfrak{g}$ will coincide with the natural action of $\tilde{G} = GL(n, \mathbb{C})$ in $S^2(\mathbb{C}^n)$, restricted to $G \subset \tilde{G}$. It is well known [H, 3.1] that the natural action of $GL(N, \mathbb{C})$ in $S(S^2(\mathbb{C}^n))$ is a multiplicity free polynomial representation.

Assume now $G = SO(N, \mathbb{C})$, where $N = 2n + 1$ or $N = 2n$. Then one can identify $\mathfrak{g}$ with $\Lambda^2(\mathbb{C}^N)$ in such a way that the adjoint action of $G$ in $\mathfrak{g}$ will coincide with the restriction to $G$ of the natural action of the group $\tilde{G} = GL(N, \mathbb{C})$ in $\Lambda^2(\mathbb{C}^N)$. On the other hand, it is well known [H, 3.8] that $S(\Lambda^2 \mathbb{C}^N)$ is a multiplicity free polynomial $GL(N, \mathbb{C})$-module.

Finally, note that in this argument, we could replace the special orthogonal group $SO(N, \mathbb{C})$ by the complete orthogonal group $O(N, \mathbb{C})$.

The next result provides us with an alternative characterization of the bases $\{S_\mu\}$, $\{T_\mu\}$.

Theorem 4.8. Let $G$ be a classical group and $\tilde{G} \supset G$ be as in Proposition 4.7. The basis elements $S_\mu \in I(\mathfrak{g})$ or $T_\mu \in I(\mathfrak{g})$ are the unique (within a scalar multiple) elements of $I(\mathfrak{g}) \subset S(\mathfrak{g})$ that generate, under the action of $\tilde{G}$, irreducible submodules of $S(\mathfrak{g})$.

Proof. Examine the case $G = GL(n, \mathbb{C})$. The representation of the group $\tilde{G} = GL(n, \mathbb{C}) \times GL(n, \mathbb{C})$ in the space $S(\mathbb{C}^n \otimes \mathbb{C}^n)$ is decomposed in the direct sum of the irreducibles $V_{\mu n} \otimes V_{\mu n}$, where $\mu$ ranges over partitions of length $\leq n$ (see [Z, §56] or [H, 2.1.2]). Each component $V_{\mu n} \otimes V_{\mu n}$ clearly contains a unique (within a scalar multiple) vector $S'_\mu$, invariant under the subgroup $G = \{(g, (g')^{-1})\}$. It is also evident that the elements $S'_\mu$ can be characterized as unique elements of $I(\mathfrak{g})$ generating irreducible $G$-submodules of $S(\mathfrak{g})$. Thus, we have to check that $S'_\mu = S_\mu$ (within a scalar multiple).

By Theorem 4.4, it suffices to show that each $S'_\mu$ is an eigenvector for the maps $\text{Proj}_{nm} \circ \text{Avr}_{nm}$, $m = n + 1, n + 2, \ldots$, but this follows from the fact that

$$\text{Avr}_{nm}(V_{\mu n}) \subset V_{\mu |m|}, \quad \text{Proj}_{nm}(V_{\mu |m|}) = V_{\mu |n|}$$

which in turn follows from the very definition of these maps.
For other groups $G$ the argument is similar.

Let $G = Sp(2n, \mathbb{C})$ and put $N = 2n$. The representation of $\tilde{G} = GL(N, \mathbb{C})$ in $S(S^2(\mathbb{C}^N))$ is the direct sum of the irreducibles $V_{M|N}$, where $M$ is a Young diagram with even rows (the number of rows does not exceed $N$); on the other hand, an irreducible polynomial $GL(N, \mathbb{C})$-module $V_{M|N}$ contains a vector invariant under $Sp(2n, \mathbb{C})$, where $n = N/2$, if and only if all the columns of $M$ are even, and then such a vector is unique within a scalar multiple (see [H, 3.1] or [M2, VII, (6.11)]).

Thus, an irreducible component $V_{M|N} \subset S(S^2(\mathbb{C}^{2n}))$ has a nonzero (and then one-dimensional) intersection with $I(\mathfrak{g})$ if and only if $M$ can be written as

$$M = 2\mu \cup 2\mu = (2\mu_1, 2\mu_2, 2\mu_2, \ldots, 2\mu_n, 2\mu_n),$$

where $\mu$ is a partition of length $\leq n$. Let $T'_\mu$ be any nonzero element of the one-dimensional space $V_{2\mu \cup 2\mu \cup 2n} \cap I(\mathfrak{g})$. The same argument as above shows that $T'_\mu = T_\mu$, within a scalar multiple.

(Note, however, a difference with the case $G = GL(n, \mathbb{C})$ examined above. For $G = GL(n, \mathbb{C})$, we saw that each irreducible $\tilde{G}$-submodule contained a $G$-invariant, while now only a part of components possess $G$-invariants.)

Now let $G = SO(N, \mathbb{C})$, where $N = 2n+1$ or $N = 2n$, and remark that in both cases $I(\mathfrak{g})$ coincides with the subspace of $S(\mathfrak{g})$ formed by the $G'$-invariants, where $G' = O(N, \mathbb{C})$. The representation of $\tilde{G} = GL(N, \mathbb{C})$ in the space $S(A^2(\mathbb{C}^N))$ decomposes into the direct sum of irreducible modules $V_{M|N}$, where $M$ is a diagram with even columns. On the other hand, a $G'$-invariant in $V_{M|N}$ exists (and then is unique, within a scalar multiple) if and only if $M$ has even rows (see [H, 3.8] or [M2, VII, (3.14)]). Thus we again obtain that $M$ has the form $2\mu \cup 2\mu$ and conclude the proof as above. □

§5. A RELATIONSHIP BETWEEN THE BASES IN $Z(\mathfrak{g})$ AND $I(\mathfrak{g})$

Here we exhibit a linear isomorphism $S(\mathfrak{g}) \to U(\mathfrak{g})$, called the special symmetrization, which maps $I(\mathfrak{g})$ onto $Z(\mathfrak{g})$ and takes the canonical basis in $I(\mathfrak{g})$ to that of $Z(\mathfrak{g})$. The main result is Theorem 5.2. We reduce it to Proposition 5.3 which in turn is reduced to Proposition 5.4. These two propositions are of independent interest.

We shall heed the concept of generalized symmetrization proposed by Olshanski [O2]. Let us identify the algebra $U(\mathfrak{g})$ with a subspace of the dual of $O_c(G)$ (see the beginning of §2). Similarly, we shall identify $S(\mathfrak{g})$ with a subspace of the dual space to $O_0(\mathfrak{g})$, the space of germs of holomorphic functions defined at a neighborhood of the origin $0 \in \mathfrak{g}$.

Assume we are given a map $F : \mathfrak{g} \to G$ with the following properties:

(i) $F$ is holomorphic and defined in a neighborhood of the origin, invariant under the adjoint representation;

(ii) $F$ takes $0 \in \mathfrak{g}$ to $e \in G$;

(iii) the differential of $F$ at the origin is the identical map $\mathfrak{g} \to \mathfrak{g}$;

(iv) $F$ is equivariant with respect to the group $G$ acting by the adjoint representation in $\mathfrak{g}$ and by conjugations on itself.

Then $F$ induces an isomorphism of local rings $O_0(\mathfrak{g}) \to O_c(G)$, which by duality determines a linear isomorphism

$$\sigma : S(\mathfrak{g}) \to U(\mathfrak{g}).$$
Note that this construction has a sense for any complex Lie group; in fact it also works for formal groups.

When $F$ is the exponential map, the corresponding map $\sigma$ coincides with the standard symmetrization map. Following [O2], in the general case we call $\sigma$ a \textit{generalized symmetrization}.

A generalized symmetrization $\sigma$ shares a number of properties of the standard symmetrization. In particular, $\sigma$ preserves leading terms and also induces a bijection between invariants of the group.

Thus, for a classical group $G$, $\sigma$ induces a linear isomorphism $I(g) \rightarrow Z(g)$ (for the series $D$ we shall assume $F$ is invariant under $G' \supset G$).

\textbf{Theorem 5.1 ([OO1, Theorem 14.1])}. Assume $g = \mathfrak{gl}(n, \mathbb{C})$ and $F(X) = 1 + X$, and let $\sigma$ be the corresponding generalized symmetrization. Then

$$\sigma(S_{\mu|n}) = S_{\mu|n} \quad \text{for any } \mu, \ell(\mu) \leq n. \quad \square$$

This generalized symmetrization was introduced in [O1]; then it was used in the note [KO] and called the \textit{special symmetrization}. We aim to find a similar generalized symmetrization for the series $C$, $B$, $D$.

\textbf{Theorem 5.2}. Let $g = g(n)$ be any classical Lie algebra of type $C, B, D$, realized as an involutive subalgebra of $\mathfrak{gl}(N, \mathbb{C})$, where $N = 2n$ or $N = 2n + 1$. Define $F: g \rightarrow G$ by the relation

$$F(X)^{1/2} - F(X)^{-1/2} = X \quad \text{for } X \in g,$$

i.e.,

$$F(X) = 1 + X^2/2 + ((1 + X^2/2)^2 - 1)^{1/2} = 1 + X^2/2 + X(1 + X^2/4)^{1/2} = 1 + X + \cdots,$$

where the eigenvalues of the matrix $X$ are supposed to be sufficiently small.

Let $\sigma$ be the corresponding generalized symmetrization. Then

$$\sigma(T_{\mu|n}) = T_{\mu|n} \quad \text{for any } \mu, \ell(\mu) \leq n.$$

Note that $F$ is indeed a well-defined (local) map from $g$ to $G$. We call $\sigma$ the \textit{special symmetrization} for the classical Lie algebras of type $C$, $B$, $D$. We refer to [O2] for explicit formulas for $\sigma: S(g) \rightarrow U(g)$ and its inverse $\sigma^{-1}: U(g) \rightarrow S(g)$.

Note that the proof of the theorem given below differs from the argument used in [OO1] for the series $A$. On the other hand, the argument given below also holds for the series $A$. We do not know if the approach of [OO1, Theorem 14.1], can be transferred to the series $C$, $B$, $D$.

\textbf{Proof}. Let $\varphi \in O_e(G)$ stand for a test element. By definition, we have to prove the equality

$$\langle T_{\mu|n}, \varphi \rangle = \langle T_{\mu|n}, \varphi \circ F \rangle,$$

where the brackets on the left-hand side denote the pairing between $U(g)$ and $O_e(G)$ while the brackets on the right-hand side denote the pairing between $S(g)$ and $O_e(G)$.
Let \( O_e(G) \) and \( O_0(\mathfrak{g}) \) denote the subspaces of invariants with respect to the action of the group \( G \) (or the group \( G' \), for the series D). Choose a compact form \( K \subset G \) (or \( K' \subset G' \)). Averaging over \( K \) (or \( K' \)) determines an invariant projection

\[
O_e(G) \to O_e(G)_{\text{inv}}.
\]

Using it we reduce our problem to the case \( \varphi \in O_e(G)_{\text{inv}} \).

We shall apply the binomial formula (Theorem 1.2) which can be conveniently written as

\[
(\varphi \circ F)(X) = \sum_{l(\mu) \leq n} \frac{\langle T_{\mu|n}, \varphi \rangle}{c_{\pm}(n, \mu)} T_{\mu|n}(X),
\]

where \( \varphi \in O_e(G)_{\text{inv}} \). Indeed, if \( \varphi \) has the form \( \varphi = \chi_{\lambda}/\chi_{\lambda}(e) \), where \( \chi_{\lambda} \) is one of the characters \( \chi_{\lambda}^{sp(2n)}, \chi_{\lambda}^{so(2n+1)}, \chi_{\lambda}^{o(2n)} \) then the above relation just coincides with the binomial formula, because of the definition of the map \( F \) and the relation

\[
\langle T_{\mu|n}, \chi_{\lambda}/\chi_{\lambda}(e) \rangle = t^*_{\mu|n}(\lambda_1, \ldots, \lambda_n).
\]

Further, since the linear span of the characters \( \chi_{\lambda} \) is dense in \( O_e(G)_{\text{inv}} \) with respect to the adic topology defined by the unique maximal ideal of \( O_e(G)_{\text{inv}} \), our expansion holds for any \( \varphi \in O_e(G)_{\text{inv}} \).

On the other hand, any element \( \psi \in O_0(\mathfrak{g})_{\text{inv}} \) can be uniquely written as a series

\[
\psi = \sum_{l(\mu) \leq n} c_\mu(\psi) T_{\mu|n}, \quad c_\mu(\psi) \in \mathbb{C},
\]

converging in the adic topology of \( O_0(\mathfrak{g})_{\text{inv}} \). By taking \( \psi = \varphi \circ F \) and comparing the both expansion we reduce the problem to the following relation

\[
c_\mu(\psi) = \frac{\langle T_{\mu|n}, \psi \rangle}{c_{\pm}(n, \mu)}, \quad \psi \in O_0(\mathfrak{g})_{\text{inv}}.
\]

Finally, without loss of generality we may assume \( \psi = T_{\nu|n} \) for a certain partition \( \nu \) of length \( \leq n \), because \( \{T_{\nu|n}\} \) is a topological basis in \( O_0(\mathfrak{g})_{\text{inv}} \).

Thus, the final reduction of the problem is the following claim.

**Proposition 5.3.** Let \( \mu, \nu \) be partitions of length \( \leq n \). Then

\[
\langle T_{\mu|n}, T_{\nu|n} \rangle = \delta_{\mu\nu} c_{\pm}(n, \mu).
\]

**Comment.** The brackets in the left-hand side can be understood in two different but equivalent ways. First, they represent the pairing between a distribution supported at the origin and a polynomial function, i.e.,

\[
\langle T_{\mu|n}, T_{\nu|n} \rangle = \langle \varphi(T_{\mu|n}), T_{\nu|n} \rangle(0).
\]
where $\partial(T_{\mu|n})$ stands for the differential operator on $\mathfrak{g}$ with constant coefficients that corresponds to $T_{\mu|n} \in S(\mathfrak{g})$. Second, the brackets may denote the canonical extension to $S(\mathfrak{g})$ of the inner product on $\mathfrak{g}$.

**Proof.** Probably, the proposition could be proved starting from the explicit expression for $T_{\mu|n}$ given in Proposition 4.6 but we prefer to use another argument.

Let us extend to $S(\mathfrak{g})$ the inner product $\langle \cdot, \cdot \rangle$ in $\mathfrak{g}$. Then one can associate with $(S(\mathfrak{g}), \langle \cdot, \cdot \rangle)$ a reproducing kernel $E(X,Y)$, where $X,Y \in \mathfrak{g}$. By definition, $E(X,Y) = \sum_\alpha \psi_\alpha(X) \psi^*_\alpha(Y)$, where $\{\psi_\alpha\}$ is an arbitrary homogeneous basis in $S(\mathfrak{g})$ and $\{\psi^*_\alpha\}$ is the dual basis. Note that $E(X,Y)$ does not depend on the choice of the basis.

Similarly, to the inner product space $(I(\mathfrak{g}), \langle \cdot, \cdot \rangle)$ also corresponds a reproducing kernel $F(X,Y)$.

Between these two kernels there is an evident relation

$$F(X,Y) = \int E(X, \text{Ad} u \cdot Y) \, du = \int E(\text{Ad} u \cdot X, Y) \, du,$$

where the integral is taken over the compact form $K \subset G$ (or $K' \subset G'$) equipped with the normalized Haar measure.

Further, taking as $\{\psi_\alpha\}$ the basis of monomials formed from a basis in $\mathfrak{g}$, we obtain that $E(X,Y) = e^{\langle X,Y \rangle}$ whence $F(X,Y) = \int e^{\langle X, \text{Ad} u \cdot Y \rangle} \, du$.

On the other hand, the claim of the proposition means that

$$F(X,Y) = \sum_{l(\mu) \leq n} \frac{T_{\mu|n}(X) T_{\mu|n}(Y)}{c_{\pm}(n,\mu)},$$

because $\{T_{\mu|n}\}$ is a homogeneous basis in $I(\mathfrak{g})$.

Thus, we have reduced our problem to the following claim.

**Proposition 5.4.** Let $u$ range over a compact form $K \subset G$ (or $K' \subset G'$, for the series $D$) and let $du$ denote the normalized Haar measure. The following expansion holds

$$\int e^{\langle X, \text{Ad} u \cdot Y \rangle} \, du = \sum_{l(\mu) \leq n} \frac{T_{\mu|n}(X) T_{\mu|n}(Y)}{c_{\pm}(n,\mu)}, \quad X,Y \in \mathfrak{g} = \mathfrak{g}(n). \quad (5.1)$$

**Proof.** We shall show that this can be obtained from the binomial formula (Theorem 1.2) by a limit transition.

First of all, without loss of generality we may assume $X$ and $Y$ are diagonal matrices with diagonal entries $\pm x_i$ and $\pm y_i$, respectively (as usual, we omit the zero entry in the case of the series $D$). Then in the right-hand side one can replace...
\( T_{\mu|n}(X) \) by \( t_{\mu|n}(x_1, \ldots, x_n) \) and \( T_{\mu|n}(Y) \) by \( t_{\mu|n}(y_1, \ldots, y_n) \). After this the right-hand side becomes very similar to the right-hand side of the binomial formula: the only difference is that in the binomial formula, we have \( t^*_{\mu|n}(\lambda_1, \ldots, \lambda_n) \) instead of \( t_{\mu|n}(y_1, \ldots, y_n) \).

Let us compare now the left-hand sides of both formulas (in the discussion below one has to replace \( K \) by \( K' \) for the series \( D \)).

In the binomial formula (1.8) occurs the normalized irreducible character \( \chi_\lambda/\chi_\lambda(e) \), which can be viewed as a spherical function for the Gelfand pair \( (K \times K, K) \). The left-hand side of formula (5.1) also can be viewed as a spherical function for a Gelfand pair. Namely, this pair consists of the Cartan motion group \( K \rtimes \mathfrak{k} \) (the semidirect product of \( K \) and its Lie algebra \( \mathfrak{k} \) viewed as a \( K \)-module) and its subgroup \( K \). The vector \( x = (\pm x_i) \) can be considered as the argument of the spherical function and \( y = (\pm y_i) \) is the parameter.

Now we shall use the well-known relation between the spherical functions of a symmetric space (say, of compact type) and the spherical functions of the corresponding Cartan motion group (see, e.g., Dooley and Rice [DR]).

In our context, this relation looks as follows. Let \( \varepsilon \) be a small parameter which then will tend to 0. Assume \( g \in K \) has the form \( g = 1 + \varepsilon X + O(\varepsilon) \), where \( X \in \mathfrak{k} \), and the partition \( \lambda \) has the form \( \lambda = \varepsilon^{-1} y + O(\varepsilon^{-1}) \). Then in the limit \( \varepsilon \to 0 \) the normalized character indexed by \( \lambda \) turns into the spherical function for the Cartan motion group, indexed by \( y \).

Since the top homogeneous component of the polynomial \( t^*_{\mu|n} \) coincides with \( t_{\mu|n} \), the right-hand side of the binomial formula (1.8) will turn, after this limit transition, into the right-hand side of formula (5.1).

This concludes the proof of Proposition 5.4 and at the same time of Proposition 5.3 and of Theorem 5.2.

§6. Appendix: Bispherical functions on \( Sp(2n, \mathbb{C}) \backslash GL(2n, \mathbb{C})/O(2n, \mathbb{C}) \)

In this section we discuss a curious fact suggested by the results of §4.

Consider two Gelfand pairs \( (\tilde{G}, G) \):

\[
(GL(N, \mathbb{C}), O(N, \mathbb{C})), \quad N = 2n + 1 \text{ or } N = 2n,
\]

and

\[
(GL(N, \mathbb{C}), Sp(N, \mathbb{C})), \quad N = 2n.
\]

The spherical functions of these pairs are matrix elements \( g \mapsto (V(g)\xi, \xi) \), where \( g \) ranges over \( \tilde{G} \), \( V \) as an irreducible finite-dimensional \( \tilde{G} \)-module, and \( \xi \) is a \( G \)-invariant vectors. As is well-known these spherical functions can be identified (with an appropriate parametrization of the double \( G \)-cosets in \( \tilde{G} \)) as Jack polynomials \( P_{\mu(\alpha)}^{(\alpha)} \) in \( n \) variables, where the parameter \( \alpha \) takes the value \( \alpha = 2 \) for the former pair and the value \( \alpha = 1/2 \) for the latter one, and \( \mu \) ranges over partitions of length \( \leq n \).

\(^6\)Since the both sides of (5.1) are invariant with respect to the action of the hyperoctahedral group on \( y_1, \ldots, y_n \), we may assume \( y_1 \geq \cdots \geq y_n \geq 0 \), so that the approximation of the vectors \( \varepsilon^{-1} y \) by partitions does exist.
The aim of this Appendix is to examine what happens when these two pairs are “mixed” in the following sense. We set

\[ \tilde{G} = GL(2n, \mathbb{C}), \quad G_1 = O(2n, \mathbb{C}), \quad G_2 = Sp(2n, \mathbb{C}), \]

and assume \( V \) is an irreducible \( \tilde{G} \)-module admitting both a \( G_1 \)-invariant \( \xi \) and a \( G_2 \)-invariant \( \eta \). Then the dual module \( V^* \) also possesses a \( G_2 \)-invariant \( \eta^* \) and we form the matrix element

\[ \varphi(g) = \langle V(g) \xi, \eta^* \rangle, \quad g \in \tilde{G}, \]

which we call a bispherical function. We shall calculate \( \varphi \) in terms of a natural parametrization of the \( (G_2, G_1) \)-cosets in \( \tilde{G} \) and we shall see that \( \varphi \) is a Schur polynomial (note that the Schur polynomials are the Jack polynomials with \( \alpha = 1 \)).

Without loss generality we may assume \( V \) is a polynomial module. We shall specify the embeddings \( G_1 \hookrightarrow \tilde{G} \) as indicated in \$1.$

**Theorem 6.1.** Let \( V \) be an irreducible polynomial representation of \( \tilde{G} = GL(2n, \mathbb{C}) \) admitting both a \( G_1 \)-invariant \( \xi \) and a \( G_2 \)-invariant \( \eta \) (where \( G_1 = O(2n, \mathbb{C}) \) and \( G_2 = Sp(2n, \mathbb{C}) \)), i.e.,

\[ V = V_{2\mu \cup 2\mu|2n}, \]

where \( \mu \) is a partition of length \( \leq n \) and

\[ 2\mu \cup 2\mu = (2\mu_1, 2\mu_1, \ldots, 2\mu_n, 2\mu_n). \]

Then the spherical function

\[ \varphi_\mu(g) = \langle V_{2\mu \cup 2\mu|2n}(g) \xi, \eta^* \rangle, \]

where \( g \in \tilde{G} \) and \( \eta^* \) stands for a \( G_2 \)-invariant in \( V^* \), is proportional to the Schur polynomial \( s_\mu \) in \( n \) variables under a suitable parametrization of the double cosets \( G_2gG_1 \) in \( \tilde{G} \).

**First proof.** Thus proof is based on Theorem 4.8. Let \( g \mapsto g' \) denote transposition of \( 2n \times 2n \) matrices that corresponds to the symmetric form \( M \) preserved by the subgroup \( G_1 = O(2n, \mathbb{C}) \), see \$1 \ for the choice of \( M \). I.e., \( g \mapsto g' \) is transposition with respect to the secondary diagonal. The map \( g \mapsto gg' \) defines a bijection between the cosets \( gG_1 \subset \tilde{G} \) and (nondegenerate) \( 2n \times 2n \) matrices, symmetric relative the secondary diagonal so one can write

\[ \varphi_\mu(g) = \psi_\mu(gg'), \]

where \( \psi_\mu \) is a polynomial function on symmetric matrices.

The group \( \tilde{G} \) acts on symmetric matrices, and under this action \( \psi_\mu \) is specified by two properties: first, it transforms according to \( V_{2\mu \cup 2\mu} \) and, second, it is \( G_2 \)-invariant.

Next, we identify the space of symmetric matrices with the Lie algebra \( sp(2n, \mathbb{C}) \) or, which is better, with its dual space. Under this identification, the functions \( \psi_\mu \) become elements of the space \( I(sp(2n, \mathbb{C})) \), and application of Theorem 4.8 implies that these are just the basis elements \( T_\mu \) (within a scalar multiple, as usual).
Given a \( n \)-tuple \((x_1, \ldots, x_n)\) of nonzero complex numbers, form a diagonal matrix,
\[
g(x) = \text{diag}(x_1, \ldots, x_n, x_n, \ldots, x_1).
\]
Then
\[
g(x)(g(x))' = \text{diag}(x_1^2, \ldots, x_n^2, x_n^2, \ldots, x_1^2).
\]
Under the above identification between symmetric matrices and elements of the (dual space to) the Lie algebra \( sp(2n, \mathbb{C}) \) the latter matrix turns into
\[
\text{diag}(x_1^2, \ldots, x_n^2, -x_n^2, \ldots, -x_1^2)
\]
which is an element of the Cartan subalgebra of \( sp(2n, \mathbb{C}) \).

It follows
\[
\varphi_\mu(g(x)) = \psi_\mu(g(x)(g(x))') = \text{const } s_\mu(x_1^4, \ldots, x_n^4),
\]
by the definition of the elements \( T_\mu \) for the series \( C \). To conclude the proof we remark that each double \((G_2, G_1)\)-coset in \( \tilde{G} \) has the form \( G_2 g(x) G_1 \) with a certain \( x = (x_1, \ldots, x_n) \). □

**Second proof (sketch).** One can choose a compact form \( \tilde{G}^u \subset \tilde{G} \) (isomorphic to \( U(2n) \)) such that \( G_i^u = \tilde{G}^u \cap G_i \) is a compact form of \( G_i \) for \( i = 1, 2 \). We also can arrange so that the matrices \( g(x) \) be in \( \tilde{G}^u \) provided \( |x_1| = \cdots = |x_n| = 1 \). Then each \((G_2^u, G_1^u)\)-coset in \( \tilde{G}^u \) is of the form \( G_2^u g(x) G_1^u \), so that we can use \( x_1, \ldots, x_n \) as parameters of the \((G_2^u, G_1^u)\)-cosets in \( \tilde{G}^u \).

A direct computation shows that the “radial part” of the Haar measure of the compact group \( \tilde{G}^u \), expressed in these parameters, has the density
\[
w(x) = \text{const } \prod_{1 \leq i < j \leq n} |x_i^4 - x_j^4|^2.
\]
It follows that the bispherical functions are symmetric orthogonal polynomials in \( x_1^4, \ldots, x_n^4 \) with weight \( w(x) \).

Finally, analysis of the weights of \( V_{2\mu_2/2\mu_1}^{2n} \) shows that the leading term of the bispherical function (with respect to the lexicographic order on monomials in \( x_1^4, \ldots, x_n^4 \)) is equal to
\[
\text{const } x_1^{4\mu_1} \cdots x_n^{4\mu_n}.
\]
Hence our orthogonal polynomials coincide with the Schur polynomials \( s_\mu(x_1^4, \ldots, x_n^4) \). □

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