ON COHOMOLOGICAL AND K-THEORETICAL HALL ALGEBRAS OF SYMMETRIC QUIVERS

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Abstract. We give a brief review of the cohomological Hall algebra CoHA $\mathcal{H}$ and the K-theoretical Hall algebra $\mathcal{K}A$ associated to quivers. In the case of symmetric quivers, we show that there exists a homomorphism of algebras (obtained from a Chern character map) $\mathcal{R} \rightarrow \hat{\mathcal{H}}^\sigma$ where $\hat{\mathcal{H}}^\sigma$ is a Zhang twist of the completion of $\mathcal{H}$. Moreover, we establish the equivalence of categories of “locally finite” graded modules $\hat{\mathcal{H}}^\sigma\text{-Mod}_{lf} \cong \mathcal{RQ}\text{-Mod}_{lf}$. Examples of locally finite $\hat{\mathcal{H}}^\sigma\text{-}$, resp. $\mathcal{RQ}\text{-}$ modules appear naturally as the cohomology, resp. K-theory, of framed moduli spaces of quivers.

1. Introduction

Let $Q$ be a finite oriented quiver with the set of vertices $I$ and with $a_{ij}$ arrows from $i \in I$ to $j \in I$, so that $a_{ij} \in \mathbb{Z}_{\geq 0}$. Fix a dimension vector $\gamma = (\gamma_i)_{i \in I} \in \mathbb{Z}^I_{\geq 0}$. Let $M_{\gamma}$ be the affine space of representations of $Q$ of dimension vector $\gamma$. The variety $M_{\gamma}$ carries a natural action of the group $G_{\gamma} := \prod_{i \in I} \text{GL}(\gamma_i, \mathbb{C})$.

Let $\mathcal{H} = \oplus_{\gamma} H_{G_{\gamma}}(M_{\gamma}, \mathbb{Q})$. There is a natural product on $\mathcal{H}$ coming from extension of representations. The corresponding $\mathbb{Z}^I_{\geq 0}$-graded algebra is called the cohomological Hall algebra (CoHA), it was introduced by Kontsevich and Soibelman [6].

Similarly, the $\mathbb{Z}^I_{\geq 0}$-graded algebra $\mathcal{R} = \oplus_{\gamma} K^0_{G_{\gamma}}(M_{\gamma})$ with multiplication defined in an analogous way is called the K-theoretical Hall algebra (KHA), it was introduced by Pădurariu [7, 8].

One can construct a natural (Chern character) homomorphism $\mathcal{R}$ to the completion $\hat{\mathcal{H}}$ of $\mathcal{H}$ which is an inclusion of $\mathbb{Z}^I_{\geq 0}$-graded spaces. However, in order to obtain a multiplicative map (in the case of a symmetric quiver) we need to twist the product on $\hat{\mathcal{H}}$ with an appropriate Zhang twist.

Theorem 1. (Theorem [21]) Assume that $Q$ is symmetric. Then there exists a group homomorphism $\sigma : \mathbb{Z}^I \rightarrow \text{Aut}(\hat{\mathcal{H}})$ such that (a twist of) the Chern character map $\mathcal{R} \rightarrow \hat{\mathcal{H}}^\sigma$ where $\hat{\mathcal{H}}^\sigma$ is the Zhang twist of $\hat{\mathcal{H}}$ is an injective homomorphism of $\mathbb{Z}^I_{\geq 0}$-graded algebras.

Remark 2. The twist of the Chern character map descends to associated graded algebras where it coincides with the usual Chern character map. In this case the homomorphism property was already established in [7].
Moreover, we show that certain module categories of $\hat{H}$ and $R$ are equivalent. By $\mathcal{H}$-$\text{Mod}_{lf}$, resp. $\mathcal{R}_Q$-$\text{Mod}_{lf}$, we denote the abelian category of “locally finite” graded $\mathcal{H}$-modules, resp. $\mathcal{R}_Q$-modules (cf. Definition 24).

**Proposition 3.** (Corollary 27) For a symmetric quiver the following categories of locally finite graded modules are equivalent

\[(1.1) \quad \mathcal{H}$-$\text{Mod}_{lf} \simeq \hat{\mathcal{H}}$-$\text{Mod}_{lf} \simeq \hat{\mathcal{H}}^\sigma$-$\text{Mod}_{lf} \simeq \mathcal{R}_Q$-$\text{Mod}_{lf}.\]

Locally finite graded $\hat{H}$, resp. $\mathcal{R}_Q$-modules appear naturally as the cohomology, resp. K-theory of the framed moduli space of $Q$.

**Proposition 4.** (Propositions 29, 31) Let $M_\gamma$ be the moduli space of representations of the augmented quiver $\tilde{Q}$ of dimension vector $\gamma = (1, \gamma)$. Then $\oplus_\gamma H(M_\gamma, Q) \in \mathcal{H}$-$\text{Mod}_{lf}$, $\oplus_\gamma K^0(M_\gamma)_Q \in \mathcal{R}_Q$-$\text{Mod}_{lf}$. If $Q$ is symmetric then these two modules correspond to each other under the equivalence \[(1.1),\] via the (twisted) Chern character (from Theorem 7).

2. **Review of cohomological Hall algebra CoHA**

We review the cohomological Hall algebra CoHA following [6]. Let $Q$ be a finite oriented quiver with the set of vertices $I$ and with $a_{ij}$ arrows from $i \in I$ to $j \in I$, so that $a_{ij} \in \mathbb{Z}_{\geq 0}$. Fix a dimension vector $\gamma = (\gamma_i)_{i \in I} \in \mathbb{Z}^I_{\geq 0}$. We have the affine space of representations of $Q$ in the vector space $\oplus_{i \in I} \mathbb{C}^{\gamma_i}$:

\[M_\gamma = \prod_{i,j \in I} \mathbb{C}^{a_{ij} \gamma_i \gamma_j}.\]

The variety $M_\gamma$ is acted upon via the conjugation by the group

\[G_\gamma := \prod_{i \in I} \text{GL}(\gamma_i, \mathbb{C}).\]

One is interested in the $G_\gamma$-equivariant cohomology of the space $M_\gamma$. For this one can use for example the following model for the classifying space of $G_\gamma$: recall that the infinite dimensional Grassmannian

\[Gr(d, \infty) := \lim_{\longrightarrow} Gr(d, \mathbb{C}^n)\]

is a classifying space for $\text{GL}(d, \mathbb{C})$. Then

\[BG_\gamma = \prod_{i \in I} B\text{GL}(\gamma_i, \mathbb{C}) = \prod_{i \in I} Gr(\gamma_i, \infty)\]

We have the standard universal $G_\gamma$-bundle $EG_\gamma \to BG_\gamma$, and hence also the universal fibration over $BG_\gamma$:

\[M_\gamma^{\text{univ}} := (EG_\gamma \times M_\gamma)/G_\gamma \to EG_\gamma/G_\gamma = BG_\gamma.\]

Define the $\mathbb{Z}^I_{\geq 0}$-graded $\mathbb{Q}$-vector space

\[\mathcal{H} = \mathcal{H}_Q := \bigoplus_{\gamma \in \mathbb{Z}^I_{\geq 0}} \mathcal{H}_\gamma,\]
where
\[ H_\gamma := H^\bullet_{G,\gamma}(M, \mathbb{Q}) = \bigoplus_{n \geq 0} H^n(M^\text{univ}, \mathbb{Q}). \]

**Remark 5.** Notice that the space \( M_\gamma \) is \( G_\gamma \)-equivariantly contractible to a point, hence \( H_\gamma \simeq H_{G,\gamma}(pt) \). However, we are going to make \( H \) into an algebra and the multiplication will depend on the spaces \( M_\gamma \).

### 2.1. Product on \( H \)

Fix two dimension vectors \( \gamma_1, \gamma_2 \in \mathbb{Z}_{\geq 0}^I \) and put \( \gamma = \gamma_1 + \gamma_2 \). Consider the affine subspace \( M_{\gamma_1, \gamma_2} \subset M_\gamma \), which consists of representations for which the standard subspaces \( \mathbb{C}^{\gamma_i} \subset \mathbb{C}^{\gamma_1} \) form a subrepresentation. The subspace \( M_{\gamma_1, \gamma_2} \) is preserved by the action of the parabolic subgroup \( G_{\gamma_1, \gamma_2} \subset G_\gamma \) which consists of transformations preserving subspaces \( \mathbb{C}^{\gamma_i} \subset \mathbb{C}^{\gamma_1} \), \( i \in I \). We have the natural morphisms of stacks
\[ M_{\gamma_1}/G_{\gamma_1} \times M_{\gamma_2}/G_{\gamma_2} \to M_{\gamma_1, \gamma_2} \to M_{\gamma_1}/G_{\gamma_1} \]
The maps \( i \) and \( \pi \) are proper and we define the multiplication
\[ m_{\gamma_1, \gamma_2} : H_{\gamma_1} \otimes H_{\gamma_2} \to H_\gamma \]
as the composition of the isomorphism
\[ p^* : H^\bullet_{G,\gamma_1}(M, \mathbb{Q}) \otimes H^\bullet_{G,\gamma_2}(M, \mathbb{Q}) \sim H^\bullet_{G,\gamma_1}(M, \mathbb{Q}) \]
with the push forward maps \( i_* \) and \( \pi_* \). Explicitly, the map \( i \) is the closed embedding
\[ (E_{G,\gamma} \times M_{\gamma_1, \gamma_2})/G_{\gamma_1, \gamma_2} \to (E_{G,\gamma} \times M_{\gamma_1, \gamma_2})/G_{\gamma_1, \gamma_2} \]
and \( \pi \) is the fiber bundle
\[ (E_{G,\gamma} \times M_{\gamma_1, \gamma_2})/G_{\gamma_1, \gamma_2} \to (E_{G,\gamma} \times M_{\gamma_1, \gamma_2})/G_{\gamma_1, \gamma_2} \]
with the fiber \( G_{\gamma}/G_{\gamma_1, \gamma_2} \) which is isomorphic to the product of Grassmannians \( \prod_{i \in I} Gr(\gamma^i_1, \mathbb{C}^{\gamma^i}). \)

Let
\[ c_1 = \dim_{\mathbb{C}} M_\gamma - \dim_{\mathbb{C}} M_{\gamma_1, \gamma_2}, \quad c_2 = \dim_{\mathbb{C}} G_{\gamma_1, \gamma_2} - \dim_{\mathbb{C}} G_{\gamma}. \]

Then
\[ i_* : H^\bullet_{G,\gamma_1, \gamma_2}(M_{\gamma_1, \gamma_2}, \mathbb{Q}) \to H^\bullet_{G,\gamma_1}(M_{\gamma_1, \gamma_2}, \mathbb{Q}), \quad \pi_* : H^\bullet_{G,\gamma_1, \gamma_2}(M_\gamma, \mathbb{Q}) \to H^\bullet_{G,\gamma_2}(M_\gamma, \mathbb{Q}). \]

So that
\[ m_{\gamma_1, \gamma_2} : H^\bullet_{G,\gamma_1}(M_{\gamma_1, \gamma_2}, \mathbb{Q}) \times H^\bullet_{G,\gamma_2}(M_{\gamma_1, \gamma_2}, \mathbb{Q}) \to H^\bullet_{G,\gamma_1}(M_\gamma, \mathbb{Q}). \]

### Proposition 6.

The product \( m = (m_{\gamma_1, \gamma_2})_{\gamma_1, \gamma_2} \) makes \( H \) into an associative \( \mathbb{Z}_{\geq 0} \)-graded algebra.

The \( \mathbb{Z}_{\geq 0} \)-graded algebra \((H, m)\) is called cohomological Hall algebra (CoHA).

### 2.2. Explicit description of CoHA

Let \( T \subset GL(d) \) be a maximal torus with the Lie algebra \( t \simeq \mathfrak{a}_d^d \). Let \( W = S_d = N(T)/T \) be the corresponding Weyl group. Recall that there exists a canonical isomorphism of graded algebras
\[ H^\bullet(BGL(d), \mathbb{Q}) = (\text{Sym}_\mathbb{Q} t^*)^W \simeq \mathbb{Q}[x_1, ..., x_d]^{S_d} \]
where \( \deg(x_i) = 2 \).
For a dimension vector $\gamma \in \mathbb{Z}_{\geq 0}^I$ introduce the variables $x_{i,\alpha}$, where $i \in I$ and $\alpha \in \{1, ..., \gamma^i\}$. Then we get a natural isomorphism
\begin{equation}
H_\gamma = \mathbb{Q}[[x_{i,\alpha}]_{i \in I, \alpha \in \{1, ..., \gamma^i\}}]\Pi_{i \in I} S_i^i,
\end{equation}
with the multiplication (2.1)
\[m_{\gamma_1, \gamma_2} : H_{\gamma_1} \otimes H_{\gamma_2} \to H_{\gamma_1 + \gamma_2}.
\]
We will denote the product $m_{\gamma_1, \gamma_2}(f_{\gamma_1}, f_{\gamma_2})$ by $f_{\gamma_1} \cdot f_{\gamma_2}$. The following theorem gives an explicit description of the multiplication in $H$:

**Theorem 7.** Given two polynomials $f_1(x') \in H_{\gamma_1}$, $f_2(x'') \in H_{\gamma_2}$, their product $f_1 \cdot f_2 \in H_\gamma$, $\gamma = \gamma_1 + \gamma_2$ equals to the following rational function in variables $(x'_{i,\alpha})_{i \in I, \alpha \in \{1, ..., \gamma^i\}}, (x''_{i,\alpha})_{i \in I, \alpha \in \{1, ..., \gamma^i\}}$:
\begin{equation}
\sum \sum_{i \in I, (\gamma^i, \gamma^i) - \text{shuffles}} f_1((x'_{i,\alpha})) f_2((x''_{i,\alpha}))(\Pi_{i,j \in I} \gamma_1^i \gamma_2^j)(\Pi_{i \in I} \gamma_1^i)(\Pi_{i \in I} \gamma_2^i).
\end{equation}

2.3. **Case of a symmetric quiver.** Let us first recall the definition of the Euler form for the quiver $Q$:

\[\chi_Q(\gamma_1, \gamma_2) = \sum_{i \in I} \gamma_1^i \gamma_2^j - \sum_{i,j \in I} a_{ij} \gamma_1^i \gamma_2^j.\]

For any representations $R_1, R_2$ of $Q$ with dimension vectors $\gamma_1, \gamma_2$ respectively, one has (see [1, Corollary 1.4.3])
\[\sum_i (-1)^i \dim \operatorname{Ext}_i^1(R_1, R_2) = \dim \operatorname{Hom}(R_1, R_2) - \dim \operatorname{Ext}_1^1(R_1, R_2) = \chi_Q(\gamma_1, \gamma_2).
\]

Note that
\begin{equation}
\chi_Q(\gamma_1, \gamma_2) = -c_1 - c_2.
\end{equation}

Assume that the quiver $Q$ is symmetric, i.e. $a_{ij} = a_{ji}, i, j \in I$. Then the Euler form $\chi_Q(\gamma_1, \gamma_2)$ is also symmetric. In this case we can make $H$ a $(\mathbb{Z}_{\geq 0}^I \times \mathbb{Z})$-graded algebra as follows: for a polynomial $f \in H_\gamma$ of degree $k$ we define its bidegree to be $(\gamma, 2k + \chi_Q(\gamma, \gamma))$. So one has

\[H = \bigoplus_{\gamma \in \mathbb{Z}_{\geq 0}^I} H_{\gamma, l}, \text{ where } \gamma \in \mathbb{Z}_{\geq 0}^I, l \in \mathbb{Z}.
\]

It follows from (2.1) and (2.5) that the product in $H$ is compatible with this bigrading, i.e. we have a bigraded algebra
\begin{equation}
(H, \cdot), \quad m : H_{\gamma, l} \otimes H_{\gamma', l'} \to H_{\gamma + \gamma', l + l'}.
\end{equation}

The formula (2.4) implies that for elements $a_\gamma \in H_{\gamma}$, $a_{\gamma'} \in H_{\gamma'}$ one has
\begin{equation}
a_\gamma \cdot a_{\gamma'} = (-1)^{\chi_Q(\gamma, \gamma')} a_{\gamma'} \cdot a_\gamma.
\end{equation}

One can twist the multiplication by a sign so that $H$ becomes super-commutative with respect to the $\mathbb{Z}$-grading. Namely define the homomorphism of abelian groups $\epsilon : \mathbb{Z}^I \to \mathbb{Z}/2\mathbb{Z}$ by the formula
\[\epsilon(\gamma) = \chi_Q(\gamma, \gamma) \mod 2.
\]
Thus $\epsilon(\gamma)$ is the parity of an $\gamma$ when specializing the $\mathbb{Z}_{\geq 0}^I \times \mathbb{Z}$-grading introduced above to a $\mathbb{Z}$-grading, given by the last factor. We have the bilinear form
\begin{equation}
\mathbb{Z}^I \times \mathbb{Z}^I \to \mathbb{Z}/2\mathbb{Z}, \quad (\gamma_1, \gamma_2) \mapsto (\chi_Q(\gamma_1, \gamma_2) + \epsilon(\gamma_1)\epsilon(\gamma_2)) \mod 2
\end{equation}
which induces a symmetric bilinear form $\beta$ on the space $(\mathbb{Z}/2\mathbb{Z})^I$, such that $\beta(\gamma, \gamma) = 0$ for all $\gamma \in (\mathbb{Z}/2\mathbb{Z})^I$. Hence there exists a bilinear form $\psi$ on $(\mathbb{Z}/2\mathbb{Z})^I$ such that
\begin{equation}
\psi(\gamma_1, \gamma_2) + \psi(\gamma_2, \gamma_1) = \beta(\gamma_1, \gamma_2).
\end{equation}

**Proposition 8.** Let $\psi$ be as in (2.9). Define the twisted product on $\mathcal{H}$ by the formula
\[ a_\gamma \star a_{\gamma'} = (-1)^{\psi(\gamma, \gamma')} a_\gamma \cdot a_{\gamma'} \quad \text{for} \quad a_\gamma \in \mathcal{H}_\gamma, \ a_{\gamma'} \in \mathcal{H}_{\gamma'}.
\]
Then the algebra $(\mathcal{H}, \star)$ is associative and supercommutative, i.e.
\[ a_\gamma \star a_{\gamma'} = (-1)^{\psi(\gamma, \gamma')} a_{\gamma'} \star a_\gamma.
\]
Moreover, different choices of the form $\psi$ as in (2.9) lead to canonically isomorphic graded supercommutative algebras.

**Proof.** The first claim follows easily from the definitions. To show the last claim we first construct $\psi$ that satisfies (2.9). For example, choose an order on the set $I$ and define $\psi$ on elements $(e_i)_{i \in I}$ of the standard basis of $(\mathbb{Z}/2\mathbb{Z})^I$ by
\[ \psi(e_i, e_j) = \beta(e_i, e_j) \quad \text{if} \quad i > j, \quad \psi(e_i, e_j) = 0 \quad \text{if} \quad i \leq j.
\]
Let $\psi'$ be another bilinear form on $(\mathbb{Z}/2\mathbb{Z})^I$ such that
\[ \psi'(\gamma_1, \gamma_2) + \psi'(\gamma_2, \gamma_1) = \beta(\gamma_1, \gamma_2)
\]
and let
\[ a_\gamma \star' a_{\gamma'} = (-1)^{\psi'(\gamma, \gamma')} a_\gamma \cdot a_{\gamma'} \quad \text{for} \quad a_\gamma \in \mathcal{H}_\gamma, \ a_{\gamma'} \in \mathcal{H}_{\gamma'}
\]
be the corresponding supercommutative multiplication on $\mathcal{H}$.

Notice that the bilinear form $\alpha = \psi - \psi' = \psi + \psi'$ is symmetric. Consider the standard basis $(e_i)_{i \in I}$ of the semigroup $\mathbb{Z}_{\geq 0}^I$ and define the function $\delta : \mathbb{Z}_{\geq 0}^I \to \mathbb{Z}/2\mathbb{Z}$ by
\[ \delta(0) = 0, \quad \delta(e_i) = 0, \quad \delta(e_{i_1} + \ldots + e_{i_n}) = \sum_{1 \leq s, t \leq n} \alpha(e_{i_s}, e_{i_t}) \quad \text{if} \quad n \geq 2.
\]
Then for any $\gamma, \gamma' \in (\mathbb{Z}/2\mathbb{Z})^I$ we have
\[ \alpha(\gamma, \gamma') = \delta(\gamma + \gamma') + \delta(\gamma) + \delta(\gamma')
\]
hence the map $\mathcal{H} \to \mathcal{H}, \ a_\gamma \mapsto (-1)^{\delta(\gamma)} a_\gamma$ defines an isomorphism of graded supercommutative algebras
\[ (\mathcal{H}, \star) \to (\mathcal{H}, \star'). \quad \square
\]

2.4. Freeness of CoHHA for a symmetric quiver. The following result was conjectured by Kontsevich-Soibelman and proved by Efimov in [3].

**Theorem 9.** For any symmetric quiver $Q$, the $(\mathbb{Z}_{\geq 0}^I \times \mathbb{Z})$-graded algebra $(\mathcal{H}, \star)$ is free supercommutative, generated by a $(\mathbb{Z}_{\geq 0}^I \times \mathbb{Z})$-graded vector space $V$ of the form $V = V^{\text{prim}} \otimes \mathbb{Q}[x]$,
where $x$ is a variable of bidegree $(0,2) \in \mathbb{Z}_{\geq 0}^I \times \mathbb{Z}$, and for any $\gamma \in \mathbb{Z}_{\geq 0}^I$ the space $V^\text{prim}_{\gamma,l}$ is finite dimensional and nonzero only for finitely many $l \in \mathbb{Z}$.

2.5. Example of the quiver with one vertex. Consider now the quiver with one vertex and $m \geq 0$ loops. The dimension vector $\gamma$ is simply a nonnegative integer.

We have $H_\gamma \simeq \mathbb{Q}[x_1,\ldots,x_\gamma]^S_\gamma$, and the multiplication in Theorem 7 is given as follows: if $f_1 \in H_{\gamma_1}$, $f_2 \in H_{\gamma_2}$, then the product $f_1 \cdot f_2$ is

$$\tag{2.10} \sum_{(\gamma_1,\gamma_2)\text{-shuffles}} f_1(x_1',\ldots,x_{\gamma_1}') f_2(x_1'',\ldots,x_{\gamma_2}'') \prod_{\alpha_1=1}^{\gamma_1} \prod_{\alpha_2=1}^{\gamma_2} (x_{\alpha_2}'' - x_{\alpha_1}')^{m-1}. $$

The formula (2.10) implies that the bigraded algebra $(\mathcal{H},\cdot)$ is commutative for odd $m$ and is supercommutative for even $m$. Moreover,

$$\chi_Q(\gamma_1,\gamma_2) = (1-m)\gamma_1\gamma_2$$

so the bilinear form $\beta$, induced from (2.8), is

$$\beta(\gamma_1,\gamma_2) = (1-m)\gamma_1\gamma_2(1+(1-m)\gamma_1\gamma_2) = 0$$

and therefore we may assume that the multiplications $(-) \cdot (-)$ and $(-) \ast (-)$ coincide.

In case $m = 0$, the bigraded algebra $\mathcal{H}$ is the exterior algebra on the graded vector space $\mathcal{H}_1 = \oplus_{k \geq 0} \mathcal{H}_{1,2k+1}$, where each $\mathcal{H}_{1,2k+1}$ is 1-dimensional.

2.6. Completed version of CoHA. Let $Q$ be a quiver. We define the completed cohomological Hall algebra $\hat{\mathcal{H}} = \hat{\mathcal{H}}_Q$ to be $\hat{\mathcal{H}} = \bigoplus_\gamma \hat{\mathcal{H}}_\gamma$, where

$$\hat{\mathcal{H}}_\gamma := \bigotimes_{i \in I} \prod_n H^n_{\text{GL}(\gamma_i)}(M_\gamma, \mathbb{Q}) \simeq \bigotimes_{i \in I} \prod_n H^n(\text{BGL}(\gamma_i), \mathbb{Q})$$

with the multiplication

$$\hat{\mathcal{H}}_\gamma \otimes \hat{\mathcal{H}}_{\gamma'} \to \hat{\mathcal{H}}_{\gamma + \gamma'}$$

which is induced from $(\mathcal{H},\cdot)$. This makes $\hat{\mathcal{H}}$ an associative $\mathbb{Z}_{\geq 0}$-graded algebra. Clearly $\mathcal{H}$ is a graded subalgebra of $\hat{\mathcal{H}}$.

Remark 10. Similarly to the natural isomorphism (2.3) we may identify the space $\hat{\mathcal{H}}_\gamma$ with the tensor product of rings of symmetric power series

$$\hat{\mathcal{H}}_\gamma = \bigotimes_{i \in I} \mathbb{Q}[\{x_{i,1},\ldots,x_{i,\gamma_i}\}]^{S_{\gamma_i}}.$$  

Then the formula for the product of two such functions $f_1 \in \hat{\mathcal{H}}_{\gamma_1}$, $f_2 \in \hat{\mathcal{H}}_{\gamma_2}$ is identical to (2.4).

Remark 11. If the quiver $Q$ is symmetric the formula (2.7) implies that a one-sided graded ideal in $\hat{\mathcal{H}}$ (or in $\mathcal{H}$) is a two-sided one.

3. Review of K-theoretical Hall algebra KHA

The K-theoretical Hall algebra was studied in [7] (in a more general setting). Let us recall the basic facts.
Let $Q$ be a quiver. Instead of considering the equivariant cohomology $H^*_G(M, \mathbb{Q})$ we now consider the equivariant algebraic $K$-group $\mathcal{R}_\gamma := K^G_0(M_\gamma)$, i.e. the Grothendieck group of the stack $M_\gamma/G_\gamma$, which is canonically isomorphic to the representation ring $\text{Rep}(G_\gamma)$. Define
\[ \mathcal{R} = \mathcal{R}_Q := \bigoplus_{\gamma \in \mathbb{Z}_{\geq 0}} \mathcal{R}_\gamma. \]
As in the case of the cohomological Hall algebra $\mathcal{H}$ we turn $\mathcal{R}$ into a $\mathbb{Z}_{\geq 0}$-graded ring with the multiplication
\[ \mu_{\gamma_1, \gamma_2} : \mathcal{R}_{\gamma_1} \times \mathcal{R}_{\gamma_2} \to \mathcal{R}_{\gamma_1 + \gamma_2} \]
defined similarly to the multiplication in $\mathcal{H}$. Namely, given dimension vectors $\gamma_1, \gamma_2$ and $\gamma = \gamma_1 + \gamma_2$ consider the diagram of stacks
\[ M_{\gamma_1}/G_{\gamma_1} \times M_{\gamma_2}/G_{\gamma_2} \xrightarrow{\mu} M_{\gamma_1, \gamma_2}/G_{\gamma_1, \gamma_2} \xrightarrow{\pi} M_{\gamma}/G_{\gamma}, \]
and define the multiplication $\mu_{\gamma_1, \gamma_2} : \mathcal{R}_{\gamma_1} \times \mathcal{R}_{\gamma_2} \to \mathcal{R}_{\gamma_1 + \gamma_2}$ as the composition of the induced maps
\[ \mu_{\gamma_1, \gamma_2} = \pi_* \cdot i_* \cdot p^* : K^G_0(M_{\gamma_1}) \times K^G_0(M_{\gamma_2}) \to K^G_0(M_{\gamma}). \]

**Proposition 12.** ([7]) *The multiplication $\mu : \mathcal{R} \otimes \mathcal{R} \to \mathcal{R}$ makes $\mathcal{R}$ into an associative algebra over $\mathbb{Z}$.***

The $\mathbb{Z}_{\geq 0}$-graded algebra $(\mathcal{R}, \mu)$ is called *K-theoretical Hall algebra* (KHA).

### 3.1. Explicit description of KHA.

Let $d$ be a positive integer and let $T \simeq (\mathbb{C}^*)^d \subset \text{GL}(d)$ be a maximal torus, say the subgroup of diagonal matrices. Let $z_1, \ldots, z_d$ be the standard coordinates on $T$. We have the canonical ring isomorphism $\text{Rep}(T) = \mathbb{Z}[z_1^{\pm 1}, \ldots, z_d^{\pm 1}]$. The action of the Weyl group $N(T)/T = S_d$ permutes the coordinates $z_i$ and the inclusion of groups $T \subset \text{GL}(d)$ induces the isomorphism
\[ \text{Rep}(\text{GL}(d)) = \text{Rep}(T)^{S_d} = \mathbb{Z}[z_1^{\pm 1}, \ldots, z_d^{\pm 1}]^{S_d}. \]

For a quiver $Q$ and a dimension vector $\gamma = (\gamma_i) \in \mathbb{Z}^I$ let $z_{i_1}, \ldots, z_{i_{\gamma_i}}$ be the standard coordinates on the diagonal torus in $\text{GL}(\gamma_i)$. Then
\[ \text{Rep}(G_\gamma) = \otimes_{i \in I} \text{Rep}(G_{\gamma_i}) = \mathbb{Z}[\{z_{i_{\alpha}}^{\pm 1}\}_{i \in I, \alpha \in \{1, \ldots, \gamma_i\}}]^{\prod_{i \in I} S_{\gamma_i}}. \]

Therefore we have the canonical isomorphism of groups
\[ \mathcal{R}_\gamma := K^G_0(M_\gamma) = \text{Rep}(G_\gamma) = \mathbb{Z}[\{z_{i_{\alpha}}^{\pm 1}\}_{i \in I, \alpha \in \{1, \ldots, \gamma_i\}}]^{\prod_{i \in I} S_{\gamma_i}}. \]

In these terms the product map
\[ \mu_{\gamma_1, \gamma_2} : \mathcal{R}_{\gamma_1} \times \mathcal{R}_{\gamma_2} \to \mathcal{R}_{\gamma_1 + \gamma_2} \]
is as follows.

**Theorem 13.** ([7]) *In the above notation let $\gamma_1 = (\gamma_i^1)$, $\gamma_2 = (\gamma_i^2)$ be dimension vectors and $f_1 \in \mathcal{R}_{\gamma_1}$, $f_2 \in \mathcal{R}_{\gamma_2}$. Then their product $\mu(f_1, f_2) \in \mathcal{R}_{\gamma_1 + \gamma_2}$ is the following function in the
variables \((z^{\underline{1}+1}_{i\alpha})_{i\in I,\alpha\in\{1,...,\gamma_1\}}\), \((z^{\underline{2}+1}_{i\alpha})_{i\in I,\alpha\in\{1,...,\gamma_2\}}\)

\[
\sum_{i\in I} \sum_{(\gamma_1',\gamma_2')} f_1(z^{\underline{1}+1}_{i\alpha}) f_2(z^{\underline{2}+1}_{i\alpha}) \prod_{i\in I} \frac{\prod_{\alpha_1=1}^{\gamma_1'} \prod_{\alpha_2=1}^{\gamma_2'} (1 - z^{\underline{1}+1}_{i\alpha_1} z^{\underline{2}+1}_{i\alpha_2})^{a_{ij}}}{\prod_{i\in I} \prod_{\alpha_1=1}^{\gamma_1'} \prod_{\alpha_2=1}^{\gamma_2'} (1 - z^{\underline{1}+1}_{i\alpha_1} z^{\underline{2}+1}_{i\alpha_2})}.
\]

Remark 14. Note that the substitution \(z_{i\alpha} \mapsto e^{x_{i\alpha}}\) (Chern character) defines an injective map \(\mathcal{R} \mapsto \hat{\mathcal{H}}\) of \(\mathbb{Z}^{\leq 0}\)-graded spaces. This map is additive, but is not a homomorphism of algebras. Using this change of variables we may and will consider

\[\Gamma\]

Remark map \(R\) corresponding equivalences of categories following [9]. Let \(\Gamma\) be a semi-group (not necessarily commutative) and let \(A = \oplus_{g \in \Gamma} A_g\) be a \(\Gamma\)-graded algebra.

Definition 15. [9] A set of degree preserving linear automorphisms of \(A\), say \(\sigma = \{\sigma_g | g \in \Gamma\}\), is called a twisting system of \(A\) if

\[
\sigma_l(\sigma_s(x)) = \sigma_l(x)\sigma_l(y)
\]

for all \(g, s, l \in \Gamma\) and \(x \in A_g, y \in A_s\).

Given a twisting system \(\sigma\) we can define the (left) twist \(\sigma A\) of \(A\) as a \(\Gamma\)-graded algebra which equals \(A\) as a graded vector space and has multiplication

\[
x \circ y = \sigma_s(x)y
\]

for all \(x \in A_g, y \in A_s\).

Similarly, in Definition 15 we say that \(\sigma\) is a right twisting system if instead of (4.1) the following holds

\[
\sigma_l(y \sigma_s(z)) = \sigma_l(y) \sigma_l(z)
\]

for all \(g, s, l \in \Gamma\) and all \(y \in A_g, z \in A_l\). In this case one can similarly define the right twist \(A^\sigma\) of \(A\) with multiplication

\[
y \bullet z = y\sigma_s(z).
\]

Both \(\sigma A\) and \(A^\sigma\) are associative algebras due to (4.1), (4.3).
For a graded algebra $B$ denote by $B$-Mod (resp. Mod-$B$) the abelian categories of graded left (resp. right) $B$-modules (with degree preserving morphisms).

The main point of the above twisting constructions is the following remark in [9].

**Proposition 16.** [9] Let $A$ be a $\Gamma$-graded algebra and let $\sigma$ be a twisting system (resp. right twisting system) of $A$. Then there is a natural equivalence of categories $A$-Mod $\simeq \sigma A$-Mod (resp. $\text{Mod} - A \simeq \text{Mod} - A^\sigma$).

**Proof.** For the convenience of the reader we recall the construction of equivalences here. The equivalences are defined as follows (see [9, Thm 3.1] for details): Given a graded left (resp. right) $A$-module $M = \bigoplus_{g \in \Gamma} M_g$ one defines the structure of a left $\sigma A$-module (resp. right $A^\sigma$-module) on $M$ by the formula

$$a \circ m = \sigma_s(a)m \quad (\text{resp.} \quad m \bullet a = m\sigma_s(a))$$

for $a \in A$ and $m \in M_g$. □

**Example 17.** Suppose that $\Gamma$ is a semigroup and $A$ is a $\Gamma$-graded algebra. Let $\text{Aut}(A)$ denote the group of degree preserving algebra automorphisms of $A$. Suppose that we are given a homomorphism of semigroups

$$\sigma : \Gamma \to \text{Aut}(A), \quad g \mapsto \sigma_g.$$

Then the collection $\{\sigma_g\}$ is a right twisting system of $A$. If the semigroup $\Gamma$ is abelian then $\{\sigma_g\}$ is also a twisting system of $A$; hence in this case we have both the left and right twists of $A$ defined and we have the equivalences of categories in Proposition 16.

5. Twists of $\hat{H}$

Let $Q$ be a quiver. In the notation of Section 4 let the semi-group $\Gamma$ be the group $\mathbb{Z}^I$. We consider the algebra $\hat{H} = \bigoplus_{\gamma \in \mathbb{Z}^I} \hat{H}_{\gamma}$ as a $\mathbb{Z}^I$-graded algebra (with $\hat{H}_\gamma = 0$ for $\gamma \notin \mathbb{Z}^I_{\geq 0}$) and $\text{\hat{H}}$-Mod and $\text{Mod-\hat{H}}$ denote the categories of left and right $\mathbb{Z}^I$-graded $\hat{H}$-modules respectively.

Recall that $\hat{H}_{\gamma}$ is canonically identified with the tensor product of rings of symmetric power series (cf. Remark 10)

$$\hat{H}_{\gamma} = \bigotimes_{i \in I} Q[[x_{i,\alpha} \alpha \in \{1,\ldots,\gamma_i\}]]^{S_{\gamma_i}}$$

Denote

$$x_{\gamma_i} := x_{i,1} + \ldots + x_{i,\gamma_i}.$$ 

Assume that for each $i \in I$ we are given an additive function $l_i : \mathbb{Z}^I \to \mathbb{Z}$. Then for $\tau \in \mathbb{Z}^I$ and $\gamma \in \mathbb{Z}^I_{\geq 0}$ define

$$a_{\gamma}^\tau := \prod_{i \in I} \exp(x_{\gamma_i})^{l_i(\tau)}.$$ 

Clearly $a_{\gamma}^\tau$ is $S_{\gamma}$-invariant and hence belongs to $\hat{H}_{\gamma}$. We have

$$a_{\gamma_1}^{\tau_1}(x')a_{\gamma_2}^{\tau_2}(x'') = a_{\gamma_1 + \gamma_2}^{\tau_1 + \tau_2}(x', x''),$$ 

$$a_{\gamma_1}^{\tau_1}a_{\gamma_2}^{\tau_2} = a_{\gamma_1 + \gamma_2}^{\tau_1 + \tau_2}.$$
For each $\tau \in \mathbb{Z}^I$ define a degree preserving map $\sigma_\tau : \hat{H} \to \hat{H}$ such that
\[
\sigma_\tau(f) := a_\gamma^\tau f = \left( \prod_{i \in I} \exp(x_i x_i) \right) \mu(\tau, \gamma) f \quad \text{for} \quad f \in \hat{H}_\gamma.
\]

**Lemma 18.** For each $\tau \in \mathbb{Z}^I$ the map $\sigma_\tau$ is an algebra automorphism of $\hat{H}$. The correspondence $\tau \mapsto \sigma_\tau$ is a group homomorphism $\sigma : \mathbb{Z}^I \to \text{Aut}(\hat{H})$.

**Proof.** By Remark 10 the product $f_1 \cdot f_2$ of $f_1 \in \hat{H}_{\gamma_1}$, $f_2 \in \hat{H}_{\gamma_2}$ is given by the formula (2.4). The first assertion now follows from the $S_\gamma$-invariance of $a_\gamma^\tau$ and the first formula in (5.1). The second assertion follows from the second formula in (5.1). □

**Corollary 19.** The homomorphism of groups $\sigma : \mathbb{Z}^I \to \text{Aut}(\hat{H})$ gives rise to the left and right twists, $\sigma^\hat{H}$ and $\hat{H}^\sigma$ of the $\mathbb{Z}^I$-graded algebra $\hat{H}$.

**Proof.** This follows from the fact that the group $\mathbb{Z}^I$ is commutative as explained in Example 17. □

We will also need a small tweak of the above construction. Namely, let
\[
\mu : \mathbb{Z}^I \times \mathbb{Z}^I \to \mathbb{Z}/2\mathbb{Z}
\]
be a bilinear form (where $\mathbb{Z}/2\mathbb{Z}$ is considered with the multiplicative operation). Put
\[
\tilde{a}_\gamma^\tau := a_\gamma^\tau \mu(\tau, \gamma).
\]
We have an analogue of (5.1)
\[
(5.2) \quad \tilde{a}_\gamma^\tau(x')\tilde{a}_\gamma^{\tau_2}(x'') = \tilde{a}_\gamma^\tau + \gamma_2(x', x''),
\]
\[
\tilde{a}_\gamma^\tau \tilde{a}_\gamma^{\tau_2} = \tilde{a}_\gamma^{\tau + \tau_2}.
\]
For each $\tau \in \mathbb{Z}^I$ define the degree preserving map $\tilde{\sigma}_\tau : \hat{H} \to \hat{H}$ as follows
\[
\tilde{\sigma}_\tau(f) := \tilde{a}_\gamma^\tau f = \left( \prod_{i \in I} \exp(x_i x_i) \right) \mu(\tau, \gamma) f \quad \text{for} \quad f \in \hat{H}_\gamma.
\]
Similarly as in Lemma 18 Corollary 19 we obtain the following lemma.

**Lemma 20.** (1) For each $\tau \in \mathbb{Z}^I$ the map $\tilde{\sigma}_\tau$ is an algebra automorphism of $\hat{H}$. The correspondence $\tau \mapsto \tilde{\sigma}_\tau$ is a group homomorphism $\tilde{\sigma} : \mathbb{Z}^I \to \text{Aut}(\hat{H})$.

(2) The homomorphism of groups $\tilde{\sigma} : \mathbb{Z}^I \to \text{Aut}(\hat{H})$ gives rise to the left and right twists, $\tilde{\sigma}^\hat{H}$ and $\hat{H}^{\tilde{\sigma}}$ of the $\mathbb{Z}^I$-algebra $\hat{H}$.

6. Relation between the graded algebra $R$ and the twisted algebras $\hat{H}^\sigma$ and $\hat{H}^{\tilde{\sigma}}$ in case of a symmetric quiver

Let $a_{ij} \in \mathbb{Z}_{\geq 0}$, $a_{ij} = a_{ji}$, $i, j \in I$. In the notation of §5 define the additive maps $l_i : \mathbb{Z}^I \to \mathbb{Z}$, $i \in I$ to be
\[
l_i(\tau) := \tau^i - \sum_{j \in I} a_{ij} \tau^j.
\]
As explained in §5 this gives elements

\[ a_{\gamma} = \prod_{\iota \in I} \exp(x_{\iota})^{\gamma_{\iota} - \sum_{j \in I} a_{i,j} \tau_j} \in \hat{H}_{\gamma} \]

and defines a homomorphism of groups \( \sigma : \mathbb{Z}^I \to \text{Aut}(\hat{H}) \), where

\[ \sigma_\tau(f) = a_{\gamma} f = \left( \prod_{\iota \in I} \exp(x_{\iota})^{\gamma_{\iota} - \sum_{j \in I} a_{i,j} \tau_j} \right) f \quad \text{for} \quad f \in \hat{H}_{\gamma}. \]

By Corollary 19 we obtain the corresponding (right) twist \((\hat{\sigma}, \bullet)\) of the \(\mathbb{Z}^I\)-graded algebra \(\hat{H}\), where

\[ f_1 \bullet f_2 = f_1 \cdot (a_{\gamma}^{\bullet}(x'') f_2) = f_1 \cdot \left( \prod_{\iota \in I} \exp(x_{\iota}')^{\gamma_{\iota}' - \sum_{j \in I} a_{i,j} \gamma_j} \right) f_2 \]

for \( f_1(x') \in \hat{H}_{\gamma_1}, f_2(x'') \in \hat{H}_{\gamma_2} \).

To define a left twist of \(\hat{H}\) we will use

\[ b_{\gamma}(x) := (a_{\gamma}(x))^{-1} = \prod_{\iota \in I} \exp(-x_{\iota})^{\gamma_{\iota} - \sum_{j \in I} a_{i,j} \tau_j}, \]

\[ \mu(\tau, \gamma) := (-1)^{\sum_{i,j \in I} \tau_i \gamma_j + \sum_{i,j \in I} a_{i,j} \gamma_j}, \]

\[ \tilde{b}_{\gamma} := b_{\gamma} \mu(\tau, \gamma) \in \hat{H}_{\gamma}. \]

By Lemma 20 we obtain the corresponding (left) twist \((\hat{\sigma}, \circ)\) of the \(\mathbb{Z}^I\)-graded algebra \(\hat{H}\), where

\[ f_1 \circ f_2 = (\tilde{b}_{\gamma}^{\circ}(x') f_1) \cdot f_2 \]

for \( f_1(x') \in \hat{H}_{\gamma_1}, f_2(x'') \in \hat{H}_{\gamma_2} \).

Our main observation is the following theorem.

**Theorem 21.** Assume that the quiver \(Q\) is symmetric, i.e. \(a_{ij} = a_{ji}\) for all \(i, j \in I\). Then there exist injective degree preserving homomorphisms of \(\mathbb{Z}^I\)-graded algebras

\[ h^\sigma : \mathcal{R} \to \hat{H}^\sigma, \quad \hat{\sigma} h : \mathcal{R} \to \hat{\sigma} \hat{H}. \]

**Proof.** By Remark 13 we may and will consider the algebra \(\mathcal{R}\) as a graded subspace of \(\hat{H}\) (via the Chern character map \(ch : \mathcal{R} \to \hat{H}\)) where the multiplication \(f_1 \cdot \mathcal{R} f_2\) of \(f_1(x'_{i,\alpha}) \in \mathcal{R}_{\gamma_1}\) and \(f_2(x''_{i,\alpha}) \in \mathcal{R}_{\gamma_2}\) is given by the formula

\[ \sum_{i \in I} \sum_{(\gamma_{\iota}\gamma_{\iota}') - \text{shuffles}} f_1(x'_{i,\alpha}) f_2(x''_{i,\alpha}) \frac{\prod_{i,j \in I} \gamma_{\iota}^{ij} \prod_{\alpha_1=1}^{\gamma_{\iota}^{ij}} \prod_{\alpha_2=1}^{\gamma_{\iota}^{ij}} (1 - e^{x'_{i,\alpha_1} - x''_{i,\alpha_2} a_{i,j}})}{\prod_{i \in I} \gamma_{\iota}^{ij} \prod_{\alpha_1=1}^{\gamma_{\iota}^{ij}} \prod_{\alpha_2=1}^{\gamma_{\iota}^{ij}} (1 - e^{x'_{i,\alpha_1} - x''_{i,\alpha_2}})} \]
Choose a linear order on the finite set $I$ and let $<$ be the induced lexicographic order on $I \times \mathbb{Z}_{\geq 0}$. For any $\gamma \in \mathbb{Z}^I_{\geq 0}$ we put

$$k_\gamma((i,a)_{i\in I, 1 \leq a \leq \gamma_i}) := \frac{\prod_{i,j \in I} \prod_{a_1=1}^{\gamma_i} \prod_{a_2=1}^{\gamma_j} (x_{j,a_2} - x_{i,a_1})^{a_{ij}}}{\prod_{i \in I} \prod_{a_1=1}^{\gamma_i} \prod_{a_2=1}^{\gamma_1} (x_{i,a_2} - x_{i,a_1})},$$

$$K_\gamma((i,a)_{i\in I, 1 \leq a \leq \gamma_i}) := k_\gamma((e^{x_{i,a}})_{i\in I, 1 \leq a \leq \gamma_i}),$$

$$\eta_\gamma((i,a)_{i\in I, 1 \leq a \leq \gamma_i}) := \frac{k_\gamma((i,a)_{i\in I, 1 \leq a \leq \gamma_i})}{K_\gamma((i,a)_{i\in I, 1 \leq a \leq \gamma_i})}.$$  

Then $\eta_\gamma$ is $S_I$-invariant (as $k_\gamma$, $K_\gamma$ are invariant up to the same sign). (Notice that $\eta_\gamma$ is a power series with the constant coefficient 1.)

When we use the notation $k_{\gamma_1+\gamma_2}((x_{i',a})_{i\in I, 1 \leq a \leq \gamma_1'}, (x_{i'',a})_{i\in I, 1 \leq a \leq \gamma_2'})$ it is understood that for a fixed index $i \in I$ the variables are ordered as $(x_{i,1}, x_{i,1}, x_{i,1}, \ldots, x_{i,1})$. Similarly for $K_{\gamma_1+\gamma_2}$ and $\eta_{\gamma_1+\gamma_2}$. Moreover, let us denote

$$k_{\gamma_1,\gamma_2}((x_{i',a})_{i\in I, 1 \leq a \leq \gamma_1'}, (x_{i'',a})_{i\in I, 1 \leq a \leq \gamma_2'}) := \frac{\prod_{i,j \in I} \prod_{a_1=1}^{\gamma_i} \prod_{a_2=1}^{\gamma_j} (x_{j,a_2} - x_{i,a_1})^{a_{ij}}}{\prod_{i \in I} \prod_{a_1=1}^{\gamma_i} \prod_{a_2=1}^{\gamma_2} (x_{i,a_2} - x_{i,a_1})^{a_{ij}}},$$

$$K_{\gamma_1,\gamma_2}((x_{i',a})_{i\in I, 1 \leq a \leq \gamma_1'}, (x_{i'',a})_{i\in I, 1 \leq a \leq \gamma_2'}) := k_{\gamma_1,\gamma_2}((e^{x_{i,a}})_{i\in I, 1 \leq a \leq \gamma_1'}, (e^{x_{i,a}})_{i\in I, 1 \leq a \leq \gamma_2'}).$$

We further denote

$$L_{\gamma_1,\gamma_2}((x_{i',a})_{i\in I, 1 \leq a \leq \gamma_1'}, (x_{i'',a})_{i\in I, 1 \leq a \leq \gamma_2'}) := \frac{\prod_{i,j \in I} \prod_{a_1=1}^{\gamma_i} \prod_{a_2=1}^{\gamma_j} (1 - e^{x_{i,a_1} - x_{i,a_2}})^{a_{ij}}}{\prod_{i \in I} \prod_{a_1=1}^{\gamma_i} \prod_{a_2=1}^{\gamma_2} (1 - e^{x_{i,a_1} - x_{i,a_2}})^{a_{ij}}}.$$  

When it is clear from the context we slightly abuse the notation and write $k_{\gamma_1,\gamma_2}$ (or $k_{\gamma_1,\gamma_2}(x',x'')$) for $k_{\gamma_1,\gamma_2}((x_{i',a})_{i\in I, 1 \leq a \leq \gamma_1'}, (x_{i'',a})_{i\in I, 1 \leq a \leq \gamma_2'})$. Similarly for $K_{\gamma_1,\gamma_2}$, $L_{\gamma_1,\gamma_2}$.

Define the additive map $h^\sigma : \mathcal{R} \to \hat{\mathcal{H}}^\sigma$ to be:

$$f \mapsto \eta_\gamma f, \quad \text{for} \quad f \in \mathcal{R}_\gamma \subset \hat{\mathcal{H}}_\gamma.$$  

We claim that the map $h^\sigma$ is a (injective) homomorphism of graded algebras

$$h^\sigma : \mathcal{R} \to \hat{\mathcal{H}}^\sigma.$$  

Indeed, let $f_1((x_{i',a})_{i\in I, 1 \leq a \leq \gamma_1'}) \in \mathcal{R}_{\gamma_1}$, $f_2((x_{i'',a})_{i\in I, 1 \leq a \leq \gamma_2'}) \in \mathcal{R}_{\gamma_2}$. By Remark 4 we need to check that

$$(\eta_{\gamma_1} f_1) \cdot (\eta_{\gamma_2} f_2) = \eta_{\gamma_1+\gamma_2}(f_1 \cdot_R f_2).$$

The LHS of (6.4) is by definition equal to

$$\sum_{I \in I} \sum_{(i',i'') \in \text{shuffles}} (\eta_{\gamma_1} f_1)(\eta_{\gamma_2} f_2) k_{\gamma_1,\gamma_2} a_{i'',i'}^{\gamma_2'}(x'').$$
which by Lemma 22 below (by (6.5), (6.6), respectively) and the $S_{\gamma_1+\gamma_2}$-invariance of $\eta_{\gamma_1+\gamma_2}$ equals

$$
\sum_{\ell \in I} \sum_{(\gamma_1^\ell, \gamma_2^\ell)\text{-shuffles}} \eta_{\gamma_1+\gamma_2} f_1 f_2 K_{\gamma_1, \gamma_2} a_{\gamma_1}^{\gamma_2}(x'') \\
= \eta_{\gamma_1+\gamma_2} \sum_{\ell \in I} \sum_{(\gamma_1^\ell, \gamma_2^\ell)\text{-shuffles}} f_1 f_2 L_{\gamma_1, \gamma_2} a_{\gamma_1}^{\gamma_2}(x'') (a_{\gamma_1}^{\gamma_2}(x''))^{-1} \\
= \eta_{\gamma_1+\gamma_2} \sum_{\ell \in I} \sum_{(\gamma_1^\ell, \gamma_2^\ell)\text{-shuffles}} f_1 f_2 L_{\gamma_1, \gamma_2} \\
= \eta_{\gamma_1+\gamma_2} (f_1 \cdot \mathcal{R} f_2)
$$

This proves (6.4).

The construction of the homomorphism $\tilde{\delta} h$ is similar and we only sketch it. In fact we only need a small modification of the previous argument. Similarly to the functions $k_\gamma$, $K_\gamma$ ... we define the new functions:

$$
\tilde{k}_\gamma (x) := k_\gamma (x); \quad \tilde{K}_\gamma (x) := K_\gamma (-x); \quad \tilde{\eta}_\gamma (x) := \tilde{k}_\gamma (x)/\tilde{K}_\gamma (x) \\
\tilde{k}_{\gamma_1, \gamma_2} (x', x'') := k_{\gamma_1, \gamma_2} (x', x''); \quad \tilde{K}_{\gamma_1, \gamma_2} (x', x'') := K_{\gamma_1, \gamma_2} (-x', -x'') \\
\tilde{L}_{\gamma_1, \gamma_2} (x', x'') := L_{\gamma_1, \gamma_2} (x', x'').
$$

Define an additive map $\tilde{\delta} h : \mathcal{R} \to \tilde{\delta} \tilde{\mathcal{H}}$ as

$$
f \mapsto \tilde{\eta}_\gamma f, \quad f \in \mathcal{R}_\gamma.
$$

Exactly as above (using Lemma 23 below instead of Lemma 22) one proves that $\tilde{\delta} h$ is a homomorphism of graded algebras. This completes the proof of the theorem. \qed

**Lemma 22.** The above functions are connected as follows:

(6.5) \hspace{1cm} \eta_{\gamma_1+\gamma_2} ((x'_{i, \alpha})_{i \in I, 1 \leq \alpha \leq \gamma_1^i}, (x''_{i, \alpha})_{i \in I, 1 \leq \alpha \leq \gamma_2^i})

$$
= \eta_1 \left( ((x'_{i, \alpha})_{i \in I, 1 \leq \alpha \leq \gamma_1^i}, (x''_{i, \alpha})_{i \in I, 1 \leq \alpha \leq \gamma_2^i}) \right) \frac{k_{\gamma_1, \gamma_2}}{K_{\gamma_1, \gamma_2}}.
$$

(6.6) \hspace{1cm} K_{\gamma_1, \gamma_2} ((x'_{i, \alpha})_{i \in I, 1 \leq \alpha \leq \gamma_1^i}, (x''_{i, \alpha})_{i \in I, 1 \leq \alpha \leq \gamma_2^i}) = L_{\gamma_1, \gamma_2} (a_{\gamma_1}^{\gamma_2}(x''))^{-1}.

**Proof.** To see (6.5), consider first the ratio

(6.7) \hspace{1cm} \frac{k_{\gamma_1+\gamma_2} (x', x'')} {k_{\gamma_1} (x') k_{\gamma_2} (x'')} \\
which is equal to the triple product

(6.8) \hspace{1cm} \left[ \prod_{i \in I} \prod_{\alpha_1 = 1}^{\gamma_1^i} \prod_{\alpha_2 = 1}^{\gamma_2^i} (x''_{i, \alpha_2} - x'_{i, \alpha_1})^{a_{\gamma_1}^{-1}} \right] \left[ \prod_{j > i} \prod_{\alpha_1 = 1}^{\gamma_1^j} \prod_{\alpha_2 = 1}^{\gamma_2^j} (x''_{j, \alpha_2} - x'_{i, \alpha_1})^{a_{\gamma_1}} \right] \left[ \prod_{j > i} \prod_{\alpha_1 = 1}^{\gamma_1^j} \prod_{\alpha_2 = 1}^{\gamma_2^j} (x'_{j, \alpha_2} - x''_{i, \alpha_1})^{a_{\gamma_1}} \right].
Components \((x'' - x')\) of the first two factors appear also in \(k_{\gamma_1,\gamma_2}\), whereas instead of components \((x'_{i,\alpha_1} - x''_{i,\alpha_2})a_{ij}\) of the third factor we see \((x''_{i,\alpha_2} - x'_{i,\alpha_1})b_{ji}\) in \(k_{\gamma_1,\gamma_2}\). Because \(a_{ij} = a_{ji}\) we conclude that the ratio \((6.7)\) differs from \(k_{\gamma_1,\gamma_2}(x',x'')\) by the sign factor

\[
\prod_{j>i,\alpha_1=1}^{\gamma_1} \prod_{\alpha_2=1}^{\gamma_2} (-1)^{a_{ij}}
\]

The same sign factor appears in the comparison of the ratio

\[
\frac{K_{\gamma_1,\gamma_2}(x',x'')}{K_{\gamma_1}(x')K_{\gamma_2}(x'')}
\]

and \(K_{\gamma_1,\gamma_2}(x',x'')\). This proves the equality \((6.5)\).

For the equality \((6.6)\) notice that

\[
K_{\gamma_1,\gamma_2}/L_{\gamma_1,\gamma_2} = \frac{\prod_{i,j\in I} \prod_{\alpha_1=1}^{\gamma_1} \prod_{\alpha_2=1}^{\gamma_2} (e^{x''_{i,\alpha_2}}a_{ij}^{-1})}{\prod_{i\in I} \prod_{\alpha_1=1}^{\gamma_1} \prod_{\alpha_2=1}^{\gamma_2} (e^{x'_{i,\alpha_2}}a_{ij}^{-1})} = \prod_{i\in I} \prod_{\alpha_2=1}^{\gamma_2} (e^{x''_{i,\alpha_2}}a_{ij}^{-1})^{-\gamma_1} = \prod_{i\in I} \prod_{\alpha_2=1}^{\gamma_2} (e^{x''_{i,\alpha_2}}a_{ij}^{-1})^{-\gamma_1} + \sum_i a_{ij}^{-1}.
\]

Since we assumed that \(a_{ij} = a_{ji}\), the equality \((6.6)\) follows. \(\square\)

\[\textbf{Lemma 23.} \text{ The above functions are connected as follows:} \]

\[
(6.9) \quad \tilde{\eta}_{\gamma_1,\gamma_2}(x',x'') = \tilde{\eta}_{\gamma_1}(x') \tilde{\eta}_{\gamma_2}(x'') \frac{\tilde{K}_{\gamma_1,\gamma_2}(x',x'')}{K_{\gamma_1,\gamma_2}(x',x'')},
\]

\[
(6.10) \quad \tilde{K}_{\gamma_1,\gamma_2}(x',x'') = L_{\gamma_1,\gamma_2}(x',x'')(\tilde{b}_{\gamma_2})^{-1}.
\]

7. \textbf{Categories of locally finite modules and their equivalences}

Although some of the material in this section makes sense for general quivers, we assume for simplicity that the quiver \(Q\) is symmetric, so that the results of Section 6 can be applied. In particular we use the notation \(\hat{\mathcal{H}}\) and \(\hat{\mathcal{H}}^\sigma\) from Section 6.

For \(\gamma \in \mathbb{Z}_{\geq 0}^I\) and \(n \in \mathbb{Z}_{\geq 0}\) denote by \(\hat{\mathcal{H}}_{\gamma,n}^\geq \subset \hat{\mathcal{H}}_{\gamma,n}\) (resp. \(\hat{\mathcal{H}}_{\gamma,n}^\leq \subset \hat{\mathcal{H}}_{\gamma,n}\), resp. \(\hat{\mathcal{H}}^\sigma_{\gamma,n}^\geq \subset \hat{\mathcal{H}}^\sigma_{\gamma,n}\)) the space of polynomials (resp. of power series) without monomials of degree \(< n\).

Given a \(\mathbb{Z}^I\)-graded \(\mathcal{H}\) (resp. \(\hat{\mathcal{H}}\), resp. \(\hat{\mathcal{H}}^\sigma\) - ...) module \(M = \oplus_{\gamma} M_{\gamma}\) and \(\tau \in \mathbb{Z}^I\) we define the twist \(M(\tau)\) to be the graded module such that \(M(\tau)_{\gamma} = M_{\tau-\gamma}\).

\[\textbf{Definition 24.} \text{ A graded (left or right) ideal } I \subset \mathcal{H} \text{ is large if for every } \gamma \in \mathbb{Z}_{\geq 0}^I \text{ there exists } n_\gamma \in \mathbb{Z}_{\geq 0} \text{ such that } I_\gamma \supset \mathcal{H}_{\gamma,n_\gamma}^{\geq}.\]

A graded (left or right) \(\mathcal{H}\)-module \(M\) is \textit{locally finite} if there exists a surjection of graded modules

\[
\theta : \bigoplus \mathcal{H}(\tau) \rightarrow M
\]

such that for every summand \(\mathcal{H}(\tau)\), \(\ker \theta|_{\mathcal{H}(\tau)}\) is the twist of a large ideal in \(\mathcal{H}\). Denote by \(\mathcal{H}\)-Mod_{lf} \subset \mathcal{H}\)-Mod be the full subcategory of locally finite modules.
Similarly one defines large ideals and locally finite graded \( \hat{\mathcal{H}} \), \( \hat{\mathcal{H}}^\sigma \), \( \check{\mathcal{H}} \)-modules.

**Lemma 25.** (1) The inclusion of graded algebras \( \mathcal{H} \subset \hat{\mathcal{H}} \) induces by restriction of scalars the equivalence of categories

\[
\mathcal{H}\text{-Mod}_{lf} \simeq \hat{\mathcal{H}}\text{-Mod}_{lf}.
\]

(2) The equivalences of categories \( \hat{\mathcal{H}}\text{-Mod} \simeq \sigma \hat{\mathcal{H}}\text{-Mod}, \hat{\mathcal{H}}\text{-Mod} \simeq \check{\mathcal{H}}\text{-Mod} \) ... from Proposition 17 induce the equivalences of the corresponding categories of locally finite modules

\[
\hat{\mathcal{H}}\text{-Mod}_{lf} \simeq \sigma \hat{\mathcal{H}}\text{-Mod}_{lf}, \quad \hat{\mathcal{H}}\text{-Mod}_{lf} \simeq \check{\mathcal{H}}\text{-Mod}_{lf}.
\]

**Proof.** (1) Given a large graded ideal \( I \subset \mathcal{H} \) the ideal \( \hat{I} := \hat{\mathcal{H}} \cap \mathcal{H} \) is also large. Moreover, the natural map \( \mathcal{H}/I \to \hat{\mathcal{H}}/\hat{I} \) is an isomorphism. This implies that a locally finite \( \hat{\mathcal{H}} \)-module gives by restriction of scalars a locally finite \( \mathcal{H} \)-module. So we get the fully faithful functor (7.1)

\[
\hat{\mathcal{H}}\text{-Mod}_{lf} \to \mathcal{H}\text{-Mod}_{lf}.
\]

We need to prove that this functor is essentially surjective. Let \( I \subset \mathcal{H} \) be a large ideal. Let \( \hat{I}_\gamma \subset \hat{\mathcal{H}}_\gamma \) be the completion of \( I_\gamma \) in the adic topology of the power series ring \( \mathcal{H}_\gamma \). Then \( \hat{I} := \oplus_q \hat{I}_\gamma \) is a large graded ideal in \( \hat{\mathcal{H}} \) (because the product map \( \mathcal{H}_\gamma \times \hat{\mathcal{H}}_\tau \to \mathcal{H}_{\gamma+\tau} \) is continuous). Moreover (since \( I \) is large) we have \( I \cap \mathcal{H} = I \) and the natural map \( \mathcal{H}/I \to \hat{\mathcal{H}}/\hat{I} \) is an isomorphism. This implies that any locally finite \( \mathcal{H} \)-module is in fact a locally finite \( \hat{\mathcal{H}} \)-module and proves the essential surjectivity of the functor (7.1).

(2) This is clear. \( \square \)

We now repeat the above definitions for the graded algebra \( \mathcal{R} \). For each \( \gamma \in \mathbb{Z}^I_{\geq 0} \) consider \( \mathcal{R}_\gamma \) as the space of symmetric polynomials of the variables \( z \) (Section 3.1). Let \( \mathcal{R}_\gamma^\geq \subset \mathcal{R}_\gamma \) be the subspace of polynomials divisible by some \( (z_i - 1) \ldots (z_i - 1) \). Note that under the embedding \( \mathcal{R} \subset \hat{\mathcal{H}} \) we have \( \mathcal{R}_\gamma^\geq = \mathcal{R} \cap \mathcal{H}_\gamma^\geq \) (and similarly for the ring homomorphisms \( \hat{\mathcal{H}} \to \hat{\mathcal{H}} \) and \( h^\sigma : \mathcal{R} \to \hat{\mathcal{H}}^\sigma \)). We say that a graded (left or right) ideal \( J \subset \mathcal{R} \) is large if for every \( \gamma \) there exists \( n_\gamma \in \mathbb{Z}_{\geq 0} \) such that \( J_\gamma \subset \mathcal{R}_\gamma^{\geq n_\gamma} \). It follows that the inverse image under \( \hat{\mathcal{H}} \) or \( h^\sigma \) of a large ideal is also large. Exactly as in Definition 24 we define the notion of a locally finite graded \( \mathcal{R} \)-module. Let

\[
\mathcal{R}\text{-Mod}_{lf} \subset \mathcal{R}\text{-Mod}
\]

be the full subcategory of locally finite \( \mathcal{R} \)-modules. For a better statement in the next lemma we consider the algebra \( \mathcal{R}_Q := \mathcal{R} \otimes_{\mathbb{Z}} Q \) with the obvious extension of the above definitions.

Similarly to Lemma 25 we have the following result.

**Lemma 26.** The homomorphisms of graded algebras \( \hat{\mathcal{H}} \to \hat{\mathcal{H}} \) and \( h^\sigma : \mathcal{R} \to \hat{\mathcal{H}}^\sigma \) (Theorem 21) induce by restriction of scalars the equivalences of categories

\[
\hat{\mathcal{H}}\text{-Mod}_{lf} \simeq \mathcal{R}_Q\text{-Mod}_{lf}, \quad \hat{\mathcal{H}}^\sigma\text{-Mod}_{lf} \simeq \mathcal{R}_Q\text{-Mod}_{lf}
\]

and similarly for right modules.

**Proof.** Exactly the same as that of Lemma 25. \( \square \)
Corollary 27. For a symmetric quiver we have the equivalences of categories
\[ \mathcal{H}\text{-Mod}_{lf} \simeq \mathcal{H}\text{-Mod}_{lf} \simeq \hat{\mathcal{H}}\text{-Mod}_{lf} \simeq \mathcal{R}_{Q}\text{-Mod}_{lf}. \]

Proof. This follows from Lemmas \[25, 26\] and Proposition \[16\]. \[\Box\]

Example 28. Let \( M \) be a \( \mathbb{Z}^I \)-graded cyclic locally finite \( \mathcal{H} \) (or \( \hat{\mathcal{H}} \)) module with a surjection of \( \mathcal{H} \)-modules \( \overline{-} : \mathcal{H} \to M \). We want to describe explicitly the corresponding graded \( \mathcal{R} \)-module structure on \( M \) given by Corollary \[27\].

Let \( f_1 \in \mathcal{H}_\gamma \), \( f_2 \in \mathcal{H}_\tau \) and \( m = \overline{f_2} \in M_{\tau} \). Then
\[ f_1 m = \overline{f_1 \cdot f_2} \]
where \( f_1 \cdot f_2 \) is the product in \( \mathcal{H} \). So the \( \hat{\mathcal{H}} \)-module structure on \( M \) is as follows:
\[ f_1 \circ m = \overline{f_1 \circ f_2} = (\overline{b_i f_1}) \cdot \overline{f_2}. \]

Finally, considering \( \mathcal{R} \) as a subspace of \( \mathcal{H} \) as before (via the Chern character map), for \( f \in \mathcal{R}_\gamma \) one has
\[ fm = \hat{\mathcal{H}}(f) \circ m = (\overline{\eta f}) \circ m = (\overline{b_i \eta f}) \cdot f_2. \]

8. Example: Cohomology and K-theory of framed moduli as cyclic locally finite modules over \( \mathcal{H} \) and \( \mathcal{R} \)

8.1. We recall the graded \( \mathcal{H} \)- and \( \mathcal{R} \)-modules corresponding to framed moduli of \( Q \). A good reference is \[5\].

Let \( Q \) be a finite directed quiver with the set of vertices \( I \) and \( a_{ij} \) arrows from vertex \( i \) to vertex \( j \). Consider the augmented quiver \( \hat{Q} \), which is obtained from \( Q \) by adding a vertex \( * \) and one arrow from \( * \) to any vertex in \( Q \), i.e. \( a_{is} = 1 \), \( a_{is} = 0 \) for any \( i \in I \).

One uses the augmented quiver to construct moduli spaces of "framed" representations of \( Q \) with a dimension vector \( \gamma \). More precisely, consider the augmented dimension vector \( \tilde{\gamma} := (1^*, \gamma) \) of \( \hat{Q} \). Denote by \( \hat{M}_\gamma \) the space of \( \hat{Q} \)-representations with dimension vector \( \tilde{\gamma} \). It consists of a \( Q \)-representation \( (V^i)_{i \in I} \) and a choice of a vector \( v \in \oplus V^i \). The action of \( G_\gamma \) on \( \hat{M}_\gamma \) extends naturally to an action on \( \hat{M}_\gamma \).

Choose the stability condition
\[ \theta : \{ * \cup I \} \to \mathbb{Z}, \quad \theta(*) = -\sum_i \gamma^i, \quad \text{and} \quad \theta(i) = 1 \quad \text{for all} \quad i \in I \]
so that \( \theta(\tilde{\gamma}) = 0 \). Recall that a representation \( \hat{V} \) of \( \hat{Q} \) with dimension vector \( \tilde{\gamma} \) is \( \theta \)-semi-stable (resp. stable) if \( \theta(\tilde{\gamma}) = 0 \) and for every subrepresentation \( \hat{V}' \subset \hat{V} \) with dimension vector \( \tilde{\gamma}' \) one has \( \theta(\tilde{\gamma}') \geq 0 \) (resp. > 0 for all proper subrepresentations \( N' \)). Actually in this case a representation is stable if it is semi-stable. We denote by \( \hat{M}_\gamma^{st} \subset \hat{M}_\gamma \) the open subset consisting of \( \theta \)-stable representations of \( \hat{Q} \). Obviously a representation \( \hat{V} = (v \in \oplus V^i) \in \hat{M}_\gamma^{st} \) is stable if and only if the vector \( v \) generates the \( Q \)-representation \( (V^i) \).

The \( G_\gamma \)-action on \( \hat{M}_\gamma \) restricts to an action on \( \hat{M}_\gamma^{st} \). Moreover, the \( G_\gamma \)-action on \( \hat{M}_\gamma^{st} \) is free and there exists a geometric quotient
\[ \hat{M}_\gamma^{st} \to \hat{M}_\gamma^{st} \sslash G_\gamma =: \mathcal{M}_\gamma. \]
which is a principal $G_\gamma$-bundle. The quotient $\mathcal{M}_\gamma$ is a smooth quasi-projective variety (when nonempty). It is the moduli space of representations of $Q$ with dimension vector $\gamma$ and a choice of a generator.

Forgetting the generator gives the map

$$\tau : \tilde{M}_\gamma \to M_\gamma$$

which is a $G_\gamma$-equivariant vector bundle. Like the graded space $\mathcal{H} = \oplus_\gamma H_{G_\gamma}(M_\gamma, \mathbb{Q})$ is naturally a $\mathbb{Z}_{\geq 0}$-graded $\mathcal{H}$-module, so is the the space

$$\bigoplus_\gamma H_{G_\gamma}(\tilde{M}_\gamma, \mathbb{Q}).$$

Indeed, let $\gamma_1, \gamma_2 \in \mathbb{Z}_{\geq 0}$ and put $\gamma = \gamma_1 + \gamma_2$. In the notation of Section 2 let

$$\tilde{M}_\gamma \supset \tilde{M}_{\gamma_1, \gamma_2} := \tau^{-1}(M_{\gamma_1, \gamma_2})$$

We have the obvious commutative diagram of stacks

$$\begin{array}{ccc}
M_{\gamma_1}/G_{\gamma_1} \times \tilde{M}_{\gamma_2}/G_{\gamma_2} & \xrightarrow{\mathcal{p}} & \tilde{M}_{\gamma_1, \gamma_2}/G_{\gamma_1, \gamma_2} \\
\downarrow \text{id} \times \tau & & \downarrow \tau \\
M_{\gamma_1}/G_{\gamma_1} \times M_{\gamma_2}/G_{\gamma_2} & \xrightarrow{\mathcal{p}} & M_{\gamma_1, \gamma_2}/G_{\gamma_1, \gamma_2} \\
\downarrow \tau & & \downarrow \tau \\
M_{\gamma}/G_{\gamma} & \xrightarrow{\mathcal{p}} & M_{\gamma}/G_{\gamma}
\end{array}$$

(8.1)

Taking the cohomology of stacks in the upper row of the diagram (8.1) gives the map

$$H_{G_{\gamma_1}}(M_{\gamma_1}) \times H_{G_{\gamma_2}}(\tilde{M}_{\gamma_2}) \xrightarrow{\mathcal{p}_*+\mathcal{p}^*} H_{G_{\gamma}}(\tilde{M}_\gamma)$$

which defines the graded $\mathcal{H}$-module structure on $\oplus_\gamma H_{G_\gamma}(\tilde{M}_\gamma, \mathbb{Q})$. Moreover, the equivariant vector bundle $\tau : \tilde{M}_\gamma \to M_\gamma$ induces an isomorphism

$$\tau^* : \mathcal{H}_\gamma = H_{G_\gamma}(M_\gamma) \xrightarrow{\sim} H_{G_\gamma}(\tilde{M}_\gamma)$$

which identifies the $\mathcal{H}$-modules $\mathcal{H} \simeq \oplus_\gamma H_{G_\gamma}(M_\gamma, \mathbb{Q})$.

Similarly, the space $\oplus_\gamma H_{G_\gamma}(\tilde{M}_{st}^\gamma, \mathbb{Q})$ is a graded $\mathcal{H}$-module. Indeed, in the previous notation define

$$\tilde{M}_{\gamma_1, \gamma_2}^{st} := \tilde{M}_{\gamma_1, \gamma_2} \cap \tilde{M}_{\gamma_2}^{st}.$$ 

Then the projection $\tilde{M}_{\gamma_1, \gamma_2} \to M_{\gamma_1} \times \tilde{M}_{\gamma_2}^{st}$ restricts to the projection

$$\tilde{M}_{\gamma_1, \gamma_2}^{st} \to M_{\gamma_1} \times \tilde{M}_{\gamma_2}^{st}$$

and we have the commutative diagram of stacks

$$\begin{array}{ccc}
M_{\gamma_1}/G_{\gamma_1} \times \tilde{M}_{\gamma_2}^{st}/G_{\gamma_2} & \xrightarrow{\mathcal{p}} & \tilde{M}_{\gamma_1, \gamma_2}^{st}/G_{\gamma_1, \gamma_2} \\
\downarrow \text{id} \times j & & \downarrow j \\
M_{\gamma_1}/G_{\gamma_1} \times M_{\gamma_2}/G_{\gamma_2} & \xrightarrow{\mathcal{p}} & M_{\gamma_1, \gamma_2}/G_{\gamma_1, \gamma_2} \\
\downarrow j & & \downarrow j \\
M_{\gamma}/G_{\gamma} & \xrightarrow{\mathcal{p}} & M_{\gamma}/G_{\gamma}
\end{array}$$

(8.2)

where $j : \tilde{M}_{st}^\gamma \to \tilde{M}_\gamma$ is the open embedding. As before we obtain the map

$$H_{G_{\gamma_1}}(M_{\gamma_1}) \times H_{G_{\gamma_2}}(\tilde{M}_{st}^{\gamma_2}) \xrightarrow{\mathcal{p}_*+\mathcal{p}^*} H_{G_{\gamma}}(\tilde{M}_{\gamma}^{st})$$
which endows the space
\[ \tilde{\mathcal{H}}^\text{st} := \bigoplus_{\gamma} H_{G_\gamma}(\tilde{M}^\text{st}_\gamma, \mathbb{Q}) \]
with a structure of a $\mathbb{Z}_{\geq 0}$-graded $\mathcal{H}$-module. It comes with a morphism of $\mathcal{H}$-modules
\[ j^* : \mathcal{H} \to \tilde{\mathcal{H}}^\text{st} \]
induced by the open embedding $j : \tilde{M}^\text{st}_\gamma \hookrightarrow \tilde{M}_\gamma$.

**Proposition 29.** The morphism of graded $\mathcal{H}$-modules $j^* : \mathcal{H} \to \tilde{\mathcal{H}}^\text{st}$ is surjective and its kernel is a large ideal in $\mathcal{H}$. In particular, $\tilde{\mathcal{H}}^\text{st}$ is a cyclic locally finite $\mathcal{H}$-module.

**Proof.** We use the ideas from [5]. Fix $\gamma \in \mathbb{Z}_{\geq 0}$. For a smooth complex algebraic variety $Y$ with a $G_\gamma$-action one has the equivariant Chow group $A_{G_\gamma}(Y)$ [2]. It comes together with the cycle map to equivariant cohomology $A_{G_\gamma}(Y) \to H_{G_\gamma}(Y, \mathbb{Q})$. The open embedding $j : \tilde{M}^\text{st}_\gamma \hookrightarrow \tilde{M}_\gamma$ and the quotient map $q : \tilde{M}^\text{st}_\gamma \to \mathcal{M}_\gamma$ give rise to a commutative diagram
\[ \begin{array}{ccc}
A(M_\gamma)_\mathbb{Q} & \overset{q^*}{\to} & A_{G_\gamma}(\tilde{M}^\text{st}_\gamma)_\mathbb{Q} \\
\downarrow & & \downarrow \\
H(M_\gamma, \mathbb{Q}) & \overset{q^*}{\to} & H_{G_\gamma}(\tilde{M}^\text{st}_\gamma, \mathbb{Q}) \\
\downarrow & & \downarrow \\
H(\mathcal{M}_\gamma, \mathbb{Q}) & \overset{q^*}{\to} & H_{G_\gamma}(\tilde{M}_\gamma, \mathbb{Q})
\end{array} \] (8.3)
where the maps $q^*$ are isomorphisms, since $q$ is a principal $G_\gamma$-bundle.

The map $A_{G_\gamma}(\tilde{M}^\text{st}_\gamma)_\mathbb{Q} \overset{j^*}{\to} A_{G_\gamma}(\tilde{M}_\gamma)_\mathbb{Q}$ is clearly surjective and the right vertical arrow is an isomorphism. So to prove that the map $H_{G_\gamma}(\tilde{M}^\text{st}_\gamma, \mathbb{Q}) \overset{j^*}{\to} H_{G_\gamma}(\tilde{M}_\gamma, \mathbb{Q})$ is surjective it suffices to show that the equivariant cycle map $A^*_{G_\gamma}(\tilde{M}^\text{st}_\gamma)_\mathbb{Q} \to H_{G_\gamma}(\tilde{M}^\text{st}_\gamma, \mathbb{Q})$ is an isomorphism. Recall that the variety $\mathcal{M}_\gamma$ has a cell decomposition [3]. This implies that the cycle map $A(M_\gamma)_\mathbb{Q} \to H(M_\gamma, \mathbb{Q})$ is an isomorphism, and completes the proof of surjectivity of the map $j^*$. To finish the proof of the proposition it suffices to notice that for each $\gamma \in \mathbb{Z}_{\geq 0}$ we have $\mathcal{H}_{\geq 0} \subset \ker j^*$ (because $\mathcal{M}_\gamma$ is a finite dimensional manifold). \qed

**Corollary 30.** The $\mathcal{H}$-module $\tilde{\mathcal{H}}^\text{st}$ can be considered as a cyclic locally finite $\mathcal{H}$-module via the equivalence $\mathcal{H}\text{-Mod}_{lf} \simeq \tilde{\mathcal{H}}\text{-Mod}_{lf}$ from Corollary [24]. In particular, the surjection of $\mathcal{H}$-modules $j^* : \mathcal{H} \to \tilde{\mathcal{H}}^\text{st}$ induces the surjection of $\mathcal{H}$-modules
\[ j^* : \mathcal{H} \to \tilde{\mathcal{H}}^\text{st}. \]

**8.2.** Similarly, the framed moduli spaces $\{\mathcal{M}_\gamma\}$ give rise to a graded $\mathcal{R}$-module. Namely, the equivariant vector bundle $\tau : \tilde{M}_\gamma \to \mathcal{M}_\gamma$ induces an isomorphism
\[ \mathcal{R}_\gamma = K_0^{G_\gamma}(\mathcal{M}_\gamma) \overset{\tau^*}{\to} K_0^{G_\gamma}(\tilde{M}_\gamma) \]
and diagrams of stacks (8.1), (8.2) give rise to a graded $\mathcal{R}$-module structure on the vector space
\[ \mathcal{R}^\text{st} := \bigoplus_{\gamma \in \mathcal{I}} K_0^{G_\gamma}(\tilde{M}^\text{st}_\gamma) \]

together with a morphism $j^* : \mathcal{R} \to \mathcal{R}^\text{st}$ of graded $\mathcal{R}$-modules, coming from the open embedding $j : \tilde{M}^\text{st}_\gamma \hookrightarrow \tilde{M}_\gamma$. 

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1. [5] Reference 5
2. [2] Reference 2
3. [3] Reference 3
4. [24] Reference 24
Proposition 31. (1) The homomorphism of graded $R_Q$-modules $j^* : R_Q \to R_Q^{st}$ is surjective and its kernel is a large ideal. In particular, $R_Q^{st}$ is a cyclic locally finite $R_Q$-module.

(2) Assume that the quiver $Q$ is symmetric. The locally finite $H$- and $R_Q$-modules $H^{st}$ and $R_Q^{st}$ correspond to each other under the equivalence of categories $H$-Mod$_{lf} \simeq R_Q$-Mod$_{lf}$ from Corollary 27.

Proof. (1) The (linear) Chern character map $ch : R \to \hat{H}$ descends to the map $ch : R^{st} \to H^{st}$ in the commutative diagram

$$
\begin{array}{ccc}
R & \xrightarrow{ch} & \hat{H} \\
\downarrow j^* & & \downarrow j^* \\
R^{st} & \xrightarrow{ch} & H^{st}
\end{array}
$$

As in the proof of Proposition 29 consider the isomorphisms $H^{st}_\gamma \simeq H(M_\gamma, Q)$ and $R_Q^{st} \simeq K^0(M_\gamma)$. The map $ch : K^0(M_\gamma)_Q \to H^{st}_\gamma$ is an isomorphism, because the space $M_\gamma$ has a cell decomposition. Therefore, the map $ch : R_Q^{st} \to H^{st}$ is also an isomorphism. It follows that $j^* : R_Q \to R_Q^{st}$ is a surjection of $R_Q$-modules. Moreover, for every $\gamma$ and $n$ we have $ch^{-1}(H^{st}_\gamma) = R_Q^{st}$. This implies that the kernel of the map $j^* : R_Q \to R_Q^{st}$ is a large ideal, i.e. $R_Q^{st}$ is a cyclic locally finite $R_Q$-module.

(2) Since the map $ch : R_Q \to \hat{H}$ is not a homomorphism of algebras, the induced map $ch : R_Q^{st} \to H^{st}$ is not a morphism of $R_Q$-modules. But we may consider the map $j^* : \hat{H} = \hat{H}^{st} \to \hat{H}^{st}$ as a surjection of $\hat{H}$-modules (Proposition 10). Recall that the map

$$\hat{H} : R \to \hat{H}, \quad f \mapsto \eta_\gamma ch(f), \quad \text{for } f \in R_\gamma
$$

is a homomorphism of algebras (see the end of the proof of Theorem 21). Now the key point is the following: for each $\gamma$ the inverse image map $j^*_\gamma : H^{st}_\gamma \to R_Q^{st}$ is a ring homomorphism with respect to the natural ring structures on $\hat{H}_\gamma$ and $H^{st}_\gamma$. Therefore, $\eta_\gamma(\ker j^*_\gamma) = \ker j^*_\gamma$. It follows that the algebra homomorphism $h : R \to \hat{H}$ descends to the isomorphism of $R_Q$-modules

$$R_Q^{st} \to H^{st}, \quad m \mapsto \eta_\gamma ch(m), \quad \text{for } m \in R_\gamma Q.$$

This completes the proof of the proposition. \qed

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