DISTRIBUTION OF 3-REGULAR AND 5-REGULAR PARTITIONS

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ABSTRACT. In this paper we study the function $b_3(n)$ and $b_5(n)$, which denote the number of 3-regular partitions and 5-regular partitions of $n$ respectively. Using the theory of modular forms, we prove several arithmetic properties of $b_3(n)$ and $b_5(n)$ modulo primes greater than 3.

1. Introduction

The number of partitions of $n$ in which no parts are multiples of $k$ is denoted by $b_k(n)$, the $k$-regular partitions. $b_k(n)$ is also the number of partitions of $n$ into at most $k-1$ copies of each part.

We agree that $b_3(0) = b_5(0) = 1$ for convenience. Moreover, let $b_3(n) = b_5(n) = 0$ if $n \notin \mathbb{Z}_{\geq 0}$. The $k$-regular partitions has generating function as follows:

$$
\sum_{n=0}^{\infty} b_k(n)q^n = \prod_{n=1}^{\infty} \frac{1 - q^{kn}}{1 - q^n}.
$$

In 1919, Ramanujan found three remarkable congruences of $p(n)$ as follows

\begin{align*}
    p(5n + 4) &\equiv 0 \pmod{5}, \\
p(7n + 5) &\equiv 0 \pmod{7}, \\
p(11n + 6) &\equiv 0 \pmod{11}.
\end{align*}

In 2000, Ono [9] proved that for each prime number $m \geq 5$, there exists infinitely many arithmetic sequences $An + B$ such that

$$
p(An + B) \equiv 0 \pmod{m}.
$$

We call such congruences Ramanujan-type congruences. Subsequently Lovejoy [6] gave similar results for the function $Q(n)$, the number of partitions of $n$ into distinct parts. Following strategies of Ono and Lovejoy, we prove the following theorem.

**Theorem 1.1.** For each prime $m \geq 5$, there are infinitely many Ramanujan-type congruences of $b_3(n)$ and $b_5(n)$ modulo $m$.

Lovejoy and Penniston [7] study the distribution of $b_3(n)$ modulo 3. Recently, Keith and Zanello [5] study the parity of $b_3(n)$. As for the 5-regular partitions, Calkin
et al. [1], Hirschhorn and Sellers [4] study the parity of $b_5(n)$. Gordon and Ono [3] study the distribution of $b_5(n)$ modulo 5. Moreover, they prove that

\begin{equation}
(1.1) \quad b_5(5n + 4) \equiv 0 \pmod{5}.
\end{equation}

Up to now, we only know the distribution of $b_3(n)$ and $b_5(n)$ modulo primes mentioned above. In this paper, we study the distribution of $b_3(n)$ and $b_5(n)$ modulo primes $m \geq 5$. It is noteworthy that we still do not know anything about $b_5(n)$ modulo 3.

As a natural corollary of Theorem 1.1, we have

**Corollary 1.2.** If $m \geq 5$ is a prime and $k = 3, 5$, then there are infinitely many positive integers $n$ for which

\[ b_k(n) \equiv 0 \pmod{m}. \]

More precisely, we have

\[ \# \{0 \leq n \leq X : b_k(n) \equiv 0 \pmod{m} \} \gg X. \]

For other residue classes $i \not\equiv 0 \pmod{m}$, we provide a useful criterion to verify whether there are infinitely many $n$ such that $b_k(n) \equiv i \pmod{m}$.

**Proposition 1.3.** If $m \geq 5$ is a prime and there is one $k \in \mathbb{Z}$ such that

\[ b_3\left(mk + \frac{m^2 - 1}{12}\right) \equiv e \not\equiv 0 \pmod{m}, \]

then for each $i = 1, 2, \ldots, m - 1$, we have

\[ \# \{0 \leq n \leq X : b_3(n) \equiv i \pmod{m} \} \gg \frac{X}{\log X}. \]

Moreover, if such $k$ exists, then $k < 18(m - 1)$.

We obtain similar results for $b_5(n)$.

**Proposition 1.4.** Let $m \geq 5$ be a prime. If there exists one $k \in \mathbb{Z}$ such that

\[ b_5\left(mk + \frac{m^2 - 1}{6}\right) \equiv e \not\equiv 0 \pmod{m}, \]

then for each $i = 1, 2, \ldots, m - 1$, we have

\[ \# \{0 \leq n \leq X : b_5(n) \equiv i \pmod{m} \} \gg \frac{X}{\log X}. \]

Moreover, if such $k$ exists, then $k < 10(m - 1)$.

**Remark.** The congruence (1.1) show that our criterion is inapplicable for the case $m = 5$. However the case $m = 5$ is studied in [3].
2. Preliminaries on modular forms

First we introduce the $U$ operator. If $j$ is a positive integer, then

$$
\left( \sum_{n=0}^{\infty} a(n)q^n \right) | U(j) := \sum_{n=0}^{\infty} a(jn)q^n.
$$

Recalling that Dedekind's eta function is defined by

$$
\eta(z) = q^{1/2} \prod_{n=1}^{\infty} (1 - q^n),
$$

where $q = e^{2\pi iz}$.

If $m$ is a prime, then let $M_k(\Gamma_0(N), \chi)_m$ (resp. $S_k(\Gamma_0(N), \chi)_m$) denote the $\mathbb{F}_m$-vector space of the reductions mod $m$ of the $q$-expansions of modular forms (resp. cusp forms) in $M_k(\Gamma_0(N), \chi)$ (resp. $S_k(\Gamma_0(N), \chi)$) with integer coefficients.

Sometimes, we will use the notation $a \equiv_m b$ in the place of $a \equiv b \pmod{m}$ for convenience.

We need the following theorem to construct modular forms [2, Theorem 3]:

**Theorem 2.1** (B. Gordon, K. Hughes). Let

$$
f(z) = \prod_{\delta|N} q^{\delta r_\delta}(\delta z)
$$

be a $\eta$-quotient provided

(i) $\sum_{\delta|N} \delta r_\delta \equiv 0 \pmod{24};$

(ii) $\sum_{\delta|N} \frac{Nr_\delta}{\delta} \equiv 0 \pmod{24};$

(iii) $k := \frac{1}{2} \sum_{\delta|N} r_\delta \in \mathbb{Z},$

then

$$
f \left( \frac{az+b}{cz+d} \right) = \chi(d)(cz+d)^k f(z),
$$

for each $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ and $\chi$ is a Dirichlet character $\pmod{N}$ defined by

$$
\chi(n) := \left( \frac{(-1)^k \prod_{\delta|N} \delta^{r_\delta}}{n} \right), \text{ if } n > 0 \text{ and } (n, 6) = 1.
$$
If \( f(z) \) is holomorphic (resp. vanishes) at all cusps of \( \Gamma_0(N) \), then \( f(z) \in M_k(\Gamma_0(N), \chi) \) (resp. \( S_k(\Gamma_0(N), \chi) \)), since \( \eta(z) \) is never vanishes on \( \mathcal{H} \). The following theorem (c.f. \cite{8}) provide a useful criterion for compute the orders of an \( \eta \)-quotient at all cusps of \( \Gamma_0(N) \).

**Theorem 2.2** (Y. Martin). Let \( c, d \) and \( N \) be positive integers with \( d \mid N \) and \( (c, d) = 1 \). If \( f(z) \) is an \( \eta \)-quotient satisfying the conditions of Theorem 2.1, then the order of vanishing of \( f(z) \) at the cusp \( c/d \) is

\[
\frac{N}{24} \sum_{\delta \mid N} \frac{r_\delta(d^2, \delta^2)}{\delta(d^2, N)}.
\]

### 3. Ramanujan-type congruences

In this section, we will prove Theorem 1.1 via theory of modular forms. However, the generating function of regular partition function is not a modular form. But for primes \( m \geq 5 \), it turns out that for a properly chosen function \( h_m(n) \), then

\[
\sum_{n=0}^{\infty} b_k(h_m(n))q^n
\]

is the Fourier expansion of a cusp form modulo \( m \). In fact, we have

**Theorem 3.1.** Let \( m \geq 5 \) be a prime, then

\[
\sum_{n=0}^{\infty} b_3\left(\frac{mn-1}{12}\right)q^n \in S_{3m-3}(\Gamma_0(432), \chi_{12})_m,
\]

where \( \chi_{12}(n) = \left(\frac{n}{3}\right)\left(\frac{-4}{n}\right) \).

**Theorem 3.2.** Let \( m \geq 5 \) be a prime, then

\[
\sum_{n=0}^{\infty} b_5\left(\frac{mn-1}{6}\right)q^n \in S_{2m-2}(\Gamma_0(180), \chi_5)_m,
\]

where \( \chi_5(n) = \left(\frac{n}{5}\right) \).

**Proof of Theorem 3.1.** We begin with an \( \eta \)-quotient

\[
f(m; z) := \frac{\eta(3z)}{\eta(z)}\eta^a(3mz)\eta^b(mz),
\]

where \( m' := (m \mod 12) \), \( a := 9 - m' \) and \( b := m' - 3 \).

It is easy to verify that \( f(m; z) \equiv_m \eta^{am+1}(3z)\eta^{bm-1}(z) \) satisfies the conditions of Theorem 2.1. Moreover, one can compute via Theorem 2.2 that \( \eta^{am+1}(3z)\eta^{bm-1}(z) \) has
the minimal order of vanishing of \((m(3a+b)+2)/24\) at the cusp \(\infty\) and \((m(a+3b)-2)/24\) at the cusp 0.

Since \((m(3a+b)+2)/24 = (m(12 - m') + 1)/12 > 0\) and \((m(a+3b)-2)/24 = (mm' - 1)/12 > 0\), \(\eta^{am+1}(3z)\eta^{bm-1}(z) \in S_{3m}(\Gamma_0(3), \chi_3)\), where \(\chi_3(n) = \left(\frac{n}{3}\right)\). On the other hand,

\[
f(m; z) = \sum_{n=0}^{\infty} b_3(n) q^{n + \frac{m(3a+b)+2}{24}} \prod_{n=1}^{\infty} (1 - q^{3mn})^a (1 - q^{mn})^b.
\]

Thus,

\[
\eta^{am+1}(3z)\eta^{bm-1}(z) \mid U(m)
\]

\[(3.1) \equiv_m \left( \sum_{n=0}^{\infty} b_3(n) q^{n + \frac{m(3a+b)+2}{24}} \mid U(m) \right) \cdot \prod_{n=1}^{\infty} (1 - q^{3n})^a (1 - q^n)^b.
\]

As for LHS of (3.1),

\[
\sum_{n=0}^{\infty} b_3(n) q^{n + \frac{m(3a+b)+2}{24}} \mid U(m) = \sum_{n \geq 0}^* b_3(n) q^{\frac{24n + (3a+b)+2}{24m}},
\]

where \(\sum^*\) means take integral power coefficients of \(q\), i.e.

\[
24n + m(3a+b) + 2 \equiv 0 \pmod{24m}.
\]

It is easy to check that \(24 \mid 24n + m(3a+b) + 2\). Thus the condition becomes \(m \mid 12n + 1\).

As for RHS of (3.1), we have

\[
\eta^{am+1}(3z)\eta^{bm-1}(z) \mid T(m) = \eta^6(z)\eta^6(3z)g(m; z),
\]

where \(T(m)\) denotes usual Hecke operator acting on \(S_{3m}(\Gamma_0(3), \chi_3)\).

Now we analyze the \(\eta\)-product \(\eta^6(z)\eta^6(3z)\). By Theorem 2.1 and Theorem 2.2, \(\eta^6(z)\eta^6(3z)\) is a cusp form of weight 6 and level 3 and has the minimal order of vanishing of 1 at the two cusps of \(\Gamma_0(3)\). Since \(\eta(z)\) never vanishes on \(\mathcal{H}\), we can write

\[
\eta^{am+1}(3z)\eta^{bm-1}(z) \mid T(m) = \eta^6(z)\eta^6(3z)g(m; z),
\]

where \(g(m; z) \in M_{3m-6}(\Gamma_0(3), \chi_3)\).

In summary, we have

\[
\sum_{m \mid 12n+1} b_3(n) q^{\frac{24n + m(3a+b)+2}{24m}} \equiv_m \frac{\eta^6(z)\eta^6(3z)g(m; z)}{\prod_{n=1}^{\infty} (1-q^{3n})^a (1-q^n)^b}.
\]

Replacing \(q\) by \(q^{12}\) and then multiplying by \(q^{-(3a+b)/2}\) on both sides of (3.2), obtaining

\[
\sum_{m \mid 12n+1} b_3(n) q^{\frac{12n+1}{m}} \equiv_m \frac{\eta^{6-a}(36z)\eta^{6-b}(12z)g(m; 12z)}{\prod_{n=1}^{\infty} (1-q^{3n})^a (1-q^n)^b},
\]

namely,

\[
\sum_{n=0}^{\infty} b_3 \left( \frac{mn - 1}{12} \right) q^n \equiv_m \frac{\eta^{6-a}(36z)\eta^{6-b}(12z)g(m; 12z)}{\prod_{n=1}^{\infty} (1-q^{3n})^a (1-q^n)^b}.
\]
Using Theorem 2.1 and Theorem 2.2 again, one can verify that \( \eta^{6-a}(36z)\eta^{6-b}(12z) \) is a cusp form of weight 3 and level 432 and has the minimal order of vanishing of \( m' \) at the cusps \( c/d \) if \( d \in \{1, 2, 3, 4, 6, 8, 12, 16, 24, 48 \} \) and \( 12 - m' \) if \( d \) is other divisor of 432.

Therefore we obtain
\[
\eta^{6-a}(36z)\eta^{6-b}(12z) \in S_3(\Gamma_0(432), \chi_4),
\]
where \( \chi_4(n) = \left( \frac{4}{n} \right) \). Together with \( g(m; 12z) \in M_{3m-6}(\Gamma_0(36), \chi_3) \), we have
\[
\sum_{n=0}^\infty b_3 \left( \frac{mn-1}{12} \right) q^n \in S_{3m-3}(\Gamma_0(432), \chi_{12}).
\]

\[\square\]

Proof of Theorem 3.2. For a fixed prime \( m \), let
\[
f(m; z) := \frac{\eta(5z)}{\eta(z)} \eta^a(5mz)\eta^b(mz),
\]
where \( m' := (m \mod 6) \) and \( a := 5 - m', b := m' - 1 \). It is easy to show that
\[
f(m; z) \equiv \eta^{am+1}(5z)\eta^{bm-1}(z) \pmod{m} \quad \text{and} \quad \eta^{am+1}(5z)\eta^{bm-1}(z) \in S_{2m}(\Gamma_0(5), \chi_5),
\]
where \( \chi_5(n) = \left( \frac{n}{5} \right) \). On the other hand,
\[
f(m; z) = \sum_{n=0}^\infty b_5(n) q^{\frac{24n+6m(5a+b)}{24} + 4} \prod_{n=1}^\infty (1 - q^{5mn})a(1 - q^{5mn})^b.
\]
Acting the \( U(m) \) operator on \( f(z) \) and since \( U(m) \equiv T(m) \pmod{m} \), obtaining
\[
\sum_{n=0}^\infty b_5(n) q^{\frac{24n+6m(5a+b)}{24} + 4} | U(m) = \frac{\eta^{am+1}(5z)\eta^{bm-1}(z)}{\prod_{n=1}^\infty (1 - q^{5mn})a(1 - q^{5mn})^b} \pmod{m},
\]
where \( T(m) \) denotes usual Hecke operator acting on \( S_{2m}(\Gamma_0(5), \chi_5) \). As for the LHS of (3.3), we have
\[
\sum_{n=0}^\infty b_5(n) q^{\frac{24n+6m(5a+b)}{24} + 4} | U(m) = \sum_{n=0}^\infty b_5(n) q^{\frac{24n+6m(5a+b)}{24m}}.
\]
Using Theorem 2.1 and 2.2, one can verify that \( \eta^4(5z)\eta^4(z) \in S_4(\Gamma_0(5)) \) and have the order of 1 at all cusps. Thus we can write \( \eta^{am+1}(5z)\eta^{bm-1}(z) | T(m) = \eta^4(5z)\eta^4(z)g(m; z) \), where \( g(m; z) \in M_{2m-4}(\Gamma_0(5), \chi_5) \). Hence
\[
\sum_{n=0}^\infty b_5(n) q^{\frac{6n+1}{6m}} \equiv \eta^{4-a}(5z)\eta^{4-b}(z)g(m; z) \pmod{m}.
\]
Replacing \( q \) by \( q^6 \) shows that
\[
\sum_{n=0}^\infty b_5(n) q^{\frac{6n+1}{6m}} \equiv \eta^{4-a}(30z)\eta^{1-b}(6z)g(m; 6z) \pmod{m}.
\]
Since $b_5(n)$ vanishes for non-integer $n$, so
\[
\sum_{n=0}^{\infty} b_5 \left( \frac{mn-1}{6} \right) q^n \equiv \eta^{4-a}(30z)\eta^{4-b}(6z)g(m; 6z) \pmod{m}.
\]
Moreover, one can verify that $\eta^{4-a}(30z)\eta^{4-b}(6z) \in S_2(\Gamma_0(180))$. Together with $g(m; 6z) \in M_{2m-4}(\Gamma_0(30), \chi_5)$, we have
\[
\sum_{n=0}^{\infty} b_5 \left( \frac{mn-1}{6} \right) q^n \in S_{2m-2}(\Gamma_0(180), \chi_5)_m.
\]

We need some important results due to Serre (c.f. [10, (6.4)]), which are the critical factors of the existence of Ramanujan-type congruences.

**Theorem 3.3** (J.-P. Serre). The set of primes $l \equiv -1 \pmod{Nm}$ such that
\[
f \mid T(l) \equiv 0 \pmod{m}
\]
for each $f(z) \in S_k(\Gamma_0(N), \psi)_m$ has positive density, where $T(l)$ denotes the usual Hecke operator acting on $S_k(\Gamma_0(N), \psi)$.

Now Theorem 1.1 is an immediately corollary of the next two theorems.

**Theorem 3.4.** Let $m \geq 5$ be a prime. A positive density of the primes $l$ have the property that
\[
b_3 \left( \frac{mln-1}{12} \right) \equiv 0 \pmod{m}
\]
for each nonnegative integer $n$ coprime to $l$.

**Theorem 3.5.** Let $m \geq 5$ be a prime. Then a positive density of primes $l$ have the property that
\[
b_5 \left( \frac{mln-1}{6} \right) \equiv 0 \pmod{m}
\]
satisfied for each integer $n$ coprime to $l$.

**Proof of Theorem 3.4.** Let
\[
F(m; z) = \sum_{n=0}^{\infty} b_3 \left( \frac{mn-1}{12} \right) q^n,
\]
then $F(m; z) \in S_{3m-3}(\Gamma_0(432), \chi_{12})_m$.

For a fix prime $m \geq 5$, let $S(m)$ denote the of primes $l$ such that
\[
f \mid T(l) \equiv 0 \pmod{m}
\]
for each $f \in S_{3m-3}(\Gamma_0(432), \chi_{12})$. By Theorem 3.3, $S(m)$ contains a positive density of primes. So if $l \in S(m)$, we have
\[
F(m; z) \mid T(l) \equiv 0 \pmod{m}.
\]
Then by the theory of Hecke operator we have

\[ F(m; z) \mid T(l) = \sum_{n=0}^{\infty} \left( b_3 \left( \frac{mln - 1}{12} \right) + \left( \frac{3}{7} \right) l^{3m-4} b_3 \left( \frac{mn - l}{12l} \right) \right) q^n \equiv 0 \pmod{m}. \]

Since \( b_3(n) \) vanishes when \( n \) is not an integer, \( b_3((mn - l)/12l) = 0 \) for each \( n \) coprime to \( l \) and \( l \neq m \). Thus

\[ b_3 \left( \frac{mln - 1}{12} \right) \equiv 0 \pmod{m} \]

satisfied for each integer \( n \) coprime to \( l \) with \( l \neq m \). Moreover, the set of such primes \( l \) has a positive density of primes.

\[ \square \]

**Proof of Theorem 3.5.** Let

\[ F(m; z) = \sum_{n=0}^{\infty} b_5 \left( \frac{mn - 1}{6} \right) q^n \in S_{2m-2}(\Gamma_0(180), \chi_5)_m. \]

By Theorem 3.3, the set of primes \( l \) such that

\[ F(m; z) \mid T(l) \equiv 0 \pmod{m} \]

has positive density, where \( T(l) \) denotes Hecke operator acting on \( S_{2m-2}(\Gamma_0(180), \chi_5) \). Moreover, by the theory of Hecke operator, we have

\[ \sum_{n=0}^{\infty} F(m; z) \mid T(l) = \sum_{n=0}^{\infty} \left( b_5 \left( \frac{mln - 1}{6} \right) + \left( \frac{l}{5} \right) l^{2m-3} b_5 \left( \frac{mn - l}{6l} \right) \right) q^n. \]

Since \( b_5(n) \) vanishes for non-integer \( n \), \( b_5((mn - l)/6l) = 0 \) when \( (n, l) = 1 \) and \( l \neq m \). Thus we obtain

\[ b_5 \left( \frac{mln - 1}{6} \right) \equiv 0 \pmod{m} \]

satisfied for each integer \( n \) with \( (n, l) = 1 \) and \( l \neq m \). Moreover, the set of such primes \( l \) has a positive density of primes.

\[ \square \]

Since the number of selections of \( l \) is infinite, choose \( l > 3 \). Replacing \( n \) by \( 12nl + ml + 12 \), then we have \( b_5(ml^2n + ml + (m^2l^2 - 1)/12) \equiv 0 \pmod{m} \) satisfied for each nonnegative integer \( n \). Similar way can be applied to \( b_5(n) \). Hence we obtain Theorem 1.1. Moreover, since the choices of \( l \) is infinite, together with the Chinese Remainder Theorem and previous results, we obtain

**Corollary 3.6.** If \( m \) is a squarefree integer, then there are infinitely many Ramanujan-type congruences of \( b_3(n) \) modulo \( m \); if \( k \) is a squarefree integer coprime to 3, then there are infinitely many Ramanujan-type congruences of \( b_5(n) \) modulo \( k \).
4. Distribution on nonzero residues

Following Lovejoy [6], we need the following theorem due to Serre [10].

**Theorem 4.1** (J.-P. Serre). The set of primes \( l \equiv 1 \pmod{Nm} \) such that
\[
a(nl^r) \equiv (r+1)a(n) \pmod{m}
\]
for each \( f(z) = \sum_{n=0}^{\infty} a(n)q^n \in S_k(\Gamma_0(N),\psi)_m \) has positive density, where \( r \) is a positive integer and \( n \) is coprime to \( l \).

Here we introduce a theorem of Sturm (Theorem 1 of [11]), which provide a useful criterion for deciding when modular forms with integer coefficients are congruent to zero modulo a prime via finite computation.

**Theorem 4.2** (J. Sturm). Suppose \( f(z) = \sum_{n=0}^{\infty} a(n)q^n \in M_k(\Gamma_0(N),\chi)_m \) such that
\[
a(n) \equiv 0 \pmod{m}
\]
for all \( n \leq \frac{kN}{12} \prod_{p|N} \left(1 + \frac{1}{p} \right) \). Then \( a(n) \equiv 0 \pmod{m} \) for all \( n \in \mathbb{Z} \).

**Proof of Theorem 1.3.** If there is one \( k \in \mathbb{Z} \) such that
\[
b_3 \left( mk + \frac{m^2-1}{12} \right) \equiv e \not\equiv 0 \pmod{m},
\]
let \( s = 12k + m \). Since \( b_3(n) \) vanishes for negative \( n \), we have \( mk + \frac{m^2-1}{12} \geq 0 \). Hence \( s = 12k + m > 0 \) and
\[
b_3 \left( \frac{ms-1}{12} \right) = b_3 \left( mk + \frac{m^2-1}{12} \right) \equiv e \pmod{m}.
\]
For a fix prime \( m \geq 5 \), let \( R(m) \) denote the set of primes \( l \) such that
\[
a(nl^r) \equiv (r+1)a(n) \pmod{m}
\]
for each \( f(z) = \sum_{n=0}^{\infty} a(n)q^n \in S_{3m-3}(\Gamma_0(432),\chi_{12})_m \), where \( r \) is a positive integer and \( n \) is coprime to \( l \). By the proof of Theorem 3.4 we have \( \sum_{n=0}^{\infty} b_3 \left( \frac{mn-1}{12} \right) q^n \in S_{3m-3}(\Gamma_0(432),\chi_{12})_m \). Since \( R(m) \) is infinite by Theorem 4.1, choose \( l \in R(m) \) such that \( l > s \), then
\[
b_3 \left( \frac{mlr-1}{12} \right) \equiv (r+1)b_3 \left( \frac{ms-1}{12} \right) \equiv (r+1)e \pmod{m}.
\]
Now we fix \( l \), choose \( \rho \in R(m) \) such that \( \rho > l \), then
\[
(4.1) \quad b_3 \left( \frac{m\rho n-1}{12} \right) \equiv 2b_3 \left( \frac{mn-1}{12} \right) \pmod{m}
\]
satisfied for each \( n \) coprime to \( \rho \). For each \( i = 1, 2, \ldots, m - 1 \), let \( r_i \equiv i(2e)^{-1} - 1 \pmod{m} \) and \( r_i > 0 \). Let \( n = l^s \) in (4.1), we obtain

\[
 b_3 \left( \frac{mnl^s - 1}{12} \right) \equiv 2b_3 \left( \frac{ml^s - 1}{12} \right) \equiv 2(r_i + 1)e \equiv i \pmod{m}.
\]

Since the variables except \( \rho \) are fixed, it suffice to prove that the estimate of the choices of \( \rho \gg X/\log X \) and which is easily derived from Theorem 4.1 and the Prime Number Theorem.

Moreover, by Sturm’s Theorem, if \( b_3 \left( \frac{mn-1}{12} \right) \equiv 0 \pmod{m} \) for each \( n \leq 216(m - 1) \), then \( b_3 \left( \frac{mn-1}{12} \right) \equiv 0 \pmod{m} \) for all \( n \in \mathbb{Z} \). Since \( b_3(n) \) vanishes if \( n \) is not an integer, it suffice to compute those \( n \) of the form \( 12j + m \) for \( 12j + m \leq 216(m - 1) \). This implies \( j < 18(m - 1) \). In addition,

\[
 b_3 \left( \frac{m(12j + m) - 1}{12} \right) = b_3 \left( m\dot{j} + \frac{m^2 - 1}{12} \right).
\]

Thus if such \( k \) exist, then \( k < 18(m - 1) \).

**Proof of Theorem 1.4.** The proof is similar to the proof above so we omit it.

\( \square \)

### 5. Examples of Ramanujan-type congruences

By Theorem 4.2 we find that

\[
 \sum_{n=0}^{\infty} b_3 \left( \frac{mn-1}{12} \right) q^n \mid T(l) \equiv 0 \pmod{m}
\]

for the pairs \( (m, l) = (5, 61), (7, 71), (11, 12553) \). An elementary computation yields that

**Proposition 5.1.**

\[
 b_3(18605n + 127) \equiv 0 \pmod{5},
\]

\[
 b_3(35287n + 207) \equiv 0 \pmod{7},
\]

\[
 b_3(1733355899n + 126576) \equiv 0 \pmod{11}.
\]

Our method is not available to the case \( m = 3 \), but one can prove that there are infinitely many Ramanujan-type congruences modulo 3 via results of Lovejoy and Penniston [7, Corollary 4].
Proposition 5.2. If $m$ is a prime of the form $12k + 1$, then
\[ b_3 \left( m^3 n + \frac{m^2 - 1}{12} \right) \equiv 0 \pmod{3}. \]

For example, we obtain
\[ b_3(2197n + 14) \equiv 0 \pmod{3}. \]

As for $b_5(n)$, we compute that
\[ \sum_{n=0}^{\infty} b_5 \left( \frac{mn - 1}{6} \right) q^n \mid T(l) \equiv 0 \pmod{m} \]
satisfied for $(m, l) = (7, 17), (11, 41), (13, 16519)$. An elementary computation yields that

Proposition 5.3.
\[ b_5(2023n + 99) \equiv 0 \pmod{7}, \]
\[ b_5(18491n + 75) \equiv 0 \pmod{11}, \]
\[ b_5(3547405693n + 35791) \equiv 0 \pmod{13}. \]

Moreover, the congruence $b_5(5n + 4) \equiv 0 \pmod{5}$ implies that
\[ \sum_{n=0}^{\infty} b_5 \left( \frac{5n - 1}{6} \right) q^n \mid T(l) \equiv 0 \pmod{5} \]
satisfied for each prime $l$.

6. More on $k$-regular partitions

In this paper, we prove that for $b_k(n)(k = 3, 5)$ and each prime $m \geq 5$ that there are infinitely many Ramanujan-type congruences modulo $m$. In fact, we conjecture that

Conjecture 6.1. For $b_k(n)(k = 3, 5)$ and each positive integer $m$ that there are infinitely many Ramanujan-type congruences modulo $m$.

We also have the following conjecture analogous to Newman’s Conjecture.

Conjecture 6.2. If $m$ is an integer, $k = 3, 5$, then for each residue class $r \pmod{m}$ there are infinitely many integers $n$ for which $b_k(n) \equiv r \pmod{m}$.
Though Ramanujan-type congruences modulo primes $m \geq 5$ exist, one may need enormous computation to find some. We encourage interested readers to find examples of congruences modulo other primes.

One can modify our proof to get some partial results of $b_{11}(n)$, the number of 11-regular partitions of $n$. In fact, if $p$ is a prime for which $p > 5$ and $p \equiv 5, 7 \pmod{12}$, then $b_{11}(n)$ has infinitely many Ramanujan-type congruences modulo $p$. For example, we obtain $b_{11}(43687n + 230) \equiv 0 \pmod{7}$. However, one can do better since from [3] we have $b_{11}(11n + 6) \equiv 0 \pmod{11}$. We intent to take up these in a future paper.

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