The Irreducibility of the Spaces of Rational Curves on del Pezzo Manifolds

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We prove the irreducibility of the spaces of rational curves on del Pezzo manifolds of Picard rank 1 and dimension $n \geq 4$ by analyzing the fibers of evaluation maps. As a corollary, we prove Geometric Manin’s Conjecture in these cases.

1 Introduction

Throughout this paper, we work over an algebraically closed field $k$ of characteristic 0. A smooth projective variety $X$ is called Fano if the anticanonical divisor $-K_X$ is ample. Mori devised the bend and break method and proved the following theorem about rational curves on Fano manifolds:

**Theorem 1.1** ([31]). Let $X$ be a Fano manifold of dimension $n$. Then for any point $p \in X$, there is a rational curve $C$ on $X$ such that $p \in C$ and $-K_X \cdot C \leq n + 1$.

Hence, it is natural to study the space of rational curves on Fano manifolds. In particular, we are interested in the irreducibility of the spaces $\text{Hom}(\mathbb{P}^1, X, d)$ of rational curves of degree $d \geq 1$ and their dimensions.

Fano manifolds are classified by their indexes. For a Fano manifold $X$, the index of $X$ is the largest integer $r$ such that $-K_X = rH$ for some ample divisor $H$ on $X$. Such a divisor $H$ is called the fundamental divisor on $X$. The index of a Fano manifold of dimension $n$ is at most $n + 1$ (e.g., [19, Corollary 2.1.13]). Moreover, if the index of $X$ is $n + 1$, then $X \cong \mathbb{P}^n$, and if the index of $X$ is $n$, then $X$ is isomorphic to a smooth quadric $Q^n$ in $\mathbb{P}^{n+1}$. When $X = \mathbb{P}^n$ (resp. $Q^n$), the space $\text{Hom}(\mathbb{P}^1, X, d)$ is irreducible of dimension $(n + 1)d + n$ (resp. $nd + n$) for each $d \in \mathbb{Z}_{>0}$ (e.g., since $X$ is homogeneous and toric, it holds by the papers cited in the second paragraph from the bottom of Introduction). So we will consider del Pezzo manifolds, which are Fano manifolds such that $-K_X = (n - 1)H$ for some ample divisor $H$. Del Pezzo manifolds are completely classified (e.g., [19, Theorem 3.3.1]). In particular, $X$ is a del Pezzo manifold of Picard rank 1 if and only if $1 \leq H^n \leq 5$, and in this case $X$ is one of the following:

1. when $H^n = 1$, $X$ is a smooth sextic in $\mathbb{P}(1^n, 2, 3)$;
2. when $H^n = 2$, $X$ is a smooth quartic in $\mathbb{P}(1^{n+1}, 2)$;
3. when $H^n = 3$, $X$ is a smooth cubic in $\mathbb{P}^{n+1}$.

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(4) when $H^0 = 4$, $X$ is a smooth complete intersection of two quadrics in $\mathbb{P}^{n+2}$; and

(5) when $H^0 = 5$, $X$ is a smooth linear section of the Grassmannian $Gr(2, 5) \subset \mathbb{P}^9$.

In this paper, we study the spaces of rational curves on del Pezzo manifolds of dimension $n \geq 4$. Our aim is to prove the following theorem:

**Theorem 1.2.** Let $X$ be a del Pezzo manifold of Picard rank 1 and dimension $n \geq 4$ with an ample generator $H$. For any integer $d \geq 1$, the Kontsevich space $\overline{M}_{0,0}(X, d)$ parametrizing rational curves of $H$-degree $d$ is irreducible of the expected dimension $(n - 1)d + n - 3$.

For the definition of the Kontsevich spaces, see for example [16]. For the proof of this theorem, we mainly follow the methods in [14, 25]. In fact, [14] proved Theorem 1.2 for any smooth cubic hypersurface of dimension $\geq 4$, and by a similar argument, it is easily proved the theorem when $H^0 \geq 4$. Our main ingredient is to analyze the fibers of the evaluation map $ev_1 : \overline{M}_{0,1}(X, 1) \to X$ for the case when $H^0 \leq 2$.

One of the main difficulties for these cases is that the fundamental divisor is not very ample. Our idea is to study the subvarieties that are covered by lines through a fixed point instead of the fibers of $ev_1$.

For the case of degree 2, we conduct a little more precise study: for a point contained in the ramification locus of the double cover $f : X \to \mathbb{P}^n$, we give a necessary and sufficient condition for the fiber of $ev_1$ to have the expected dimension. Using the above result, we can run the induction on the degree $d$ as in the above references. The key lemma for the induction step is the movable bend and break, which asserts that any free curve on $X$ deforms to a chain of free lines. Note that we can focus on the study of lines thanks to the inequality in Theorem 1.1 since $(n - 1)H \cdot C \leq n + 1$ and $n \geq 4$ yield that $H \cdot C = 1$.

The study of the irreducibility of $\text{Hom}(\mathbb{P}^1, X, d)$ and their dimensions is motivated by Geometric Manin’s Conjecture. This conjecture predicts the growth rate of the number of components and their dimension as the degree $d$ grows. The precise formulation of the asymptotic formula, we need to remove the exceptional subsets. Reference [25] conjectured that this exceptional set is defined by two birational invariants $a$ and $b$. These invariants are defined in Section 2. Subvarieties with large $a$ and $b$ invariants would contain more rational curves than we expect. However, we prove that any del Pezzo manifold has no subvariety with large a invariant. Hence, the situation is rather simple. For details about Geometric Manin’s Conjecture, see [38].

The results about the number of components and their dimensions are known in various cases: smooth Fano hypersurfaces of certain degree ranges in [4, 10, 14, 17, and 35]; homogeneous varieties in [20] and [41]; Toric varieties in [8] and [9]; del Pezzo surfaces in [6, 39], and [40]; smooth Fano threefolds in [5, 11, 25, 27, and 36], del Pezzo fibrations in [28, 29], and so on.

The paper is organized as follows. In Section 2, we give the definitions of two invariants $a$ and $b$, and prove several properties about $a$-invariants on del Pezzo manifolds. Section 3 is the main part of this paper, we study the spaces of lines on del Pezzo manifolds passing through a fixed point. We mainly focus on del Pezzo manifolds of degree $\leq 2$. In Section 4, we run the induction step and prove the main theorem. The arguments are automatic by [14] and [25]. In Section 5, as a corollary of Theorem 1.2, we prove Geometric Manin’s Conjecture for our case.

## 2 $a$-, $b$-invariants

In this section, we will introduce the $a$-invariant and the $b$-invariant. Both invariants play an essential role to count the number of components of the scheme $\text{Hom}(\mathbb{P}^1, X)$ parametrizing morphisms from $\mathbb{P}^1$ to a smooth projective variety $X$. For details about $\text{Hom}(\mathbb{P}^1, X)$, see [21, I.1].

We let $N^0(X)$ be the set of $\mathbb{R}$-Cartier divisors modulo numerical equivalence. Let $\mathbb{R}^{-1}(X)$ be the set of pseudo-effective $\mathbb{R}$-divisors on $X$, which are elements in the closure of the set of classes of effective $\mathbb{R}$-divisors, and let $\text{Nef}^{-1}(X)$ be the set of classes of nef $\mathbb{R}$-divisors on $X$. Similarly, let $N_1(X)$ be the set of $\mathbb{R}$-1-cycles modulo numerical equivalence. Let $\mathbb{R}^{1}(X)$ be the set of pseudo-effective $\mathbb{R}$-1-cycles on $X$, and let $\text{Nef}^{1}(X)$ be the set of classes of nef $\mathbb{R}$-1-cycles on $X$.

**Definition 2.1** ([25] Definition 3.1). Let $X$ be a projective manifold and $L$ be a nef and big $\mathbb{Q}$-divisor on $X$. Then the $a$-invariant, or the Fujita invariant, $a(X, L)$ is defined by

$$a(X, L) = \inf\{t \in \mathbb{R} \mid K_X + tl \in \mathbb{R}^{-1}(X)\}.$$
This is a birational invariant by [18, Proposition 2.7]. Hence, we define \( a(X, L) \) for a singular variety \( X \) by

\[
a(X, L) = a(\tilde{X}, \varphi^* L),
\]

where \( \varphi : \tilde{X} \to X \) is a smooth resolution.

By [7], \( a(X, L) > 0 \) if and only if \( X \) is uniruled.

**Proposition 2.2** ([25] Proposition 4.2). Let \( X \) be a projective uniruled manifold and let \( L \) be a nef and big divisor on \( X \). For \( \alpha \in \text{Eff}_1(X) \) vanishing against \( K_X + a(X, L) L \), we take a component \( M \subset \text{Hom}(P^1, X) \) parametrizing morphisms \( f \) such that \( f_* P^1 = \alpha \). Consider the evaluation map

\[
ev : P^1 \times M \to X.
\]

If \( ev \) is not dominant, then the closure \( Y \) of the image of \( ev \) has an \( a \)-value \( a(Y, L) > a(X, L) \).

**Definition 2.3** ([25] Definition 3.4). Let \( X \) be a projective uniruled manifold and let \( L \) be a nef and big \( \mathbb{Q} \)-divisor. Then a generically finite dominant morphism \( f : Y \to X \) of degree \( d \geq 2 \) from a projective variety \( Y \) is an \( a \)-cover if \( a(Y, f^* L) = a(X, L) \).

**Remark 2.4.** Let \( X \) be a del Pezzo manifold of dimension \( n \) and let \( H \) be the fundamental divisor. Then, by definition, we have \( a(X, H) = n - 1 \).

On del Pezzo manifolds, \( a \)-invariants behave well:

**Lemma 2.5.** Let \( X \) be a del Pezzo manifold of Picard rank 1 and dimension \( n \geq 4 \) with an ample generator \( H \). Then \( X \) has no subvarieties \( Y \) such that \( a(Y, H) > a(X, H) \).

**Proof.** If the codimension of a subvariety \( Y \) is greater than 1, then \( a(Y, H) \leq n - 1 = a(X, H) \) by [26, Lemma 3.16]. Hence, we may assume that \( Y \) has codimension 1. By [26, Theorem 3.15], for general \( n - 3 \) hyperplanes \( H_1, \ldots, H_{n-3} \), the section \( X' := X \cap H_1 \cap \cdots \cap H_{n-3} \) is a smooth del Pezzo threefold. If there exists a subvariety \( Y' := Y \cap H_1 \cap \cdots \cap H_{n-3} \subset X' \) such that \( a(Y', H) > a(X', H) \). However, del Pezzo threefolds have no such subvarieties by [30, §6.3].

**Corollary 2.6.** Let \( X \) be a del Pezzo manifold of Picard rank 1 and dimension \( n \geq 4 \). For any \( d \geq 1 \) and any component \( M \subset \text{Hom}(P^1, X, d) \), the evaluation map \( P^1 \times M \to X \) is dominant.

The second invariant \( b \) is not so important for del Pezzo manifolds. However, we need to define it in order to formulate Geometric Manin’s Conjecture.

**Definition 2.7** ([25] Definition 3.2). Let \( X \) be a projective uniruled manifold. Let \( L \) be a nef and big \( \mathbb{Q} \)-divisor on \( X \). We define the \( b \)-invariant by

\[
b(X, L) = \dim F(X, L),
\]

where \( F(X, L) = \{ \alpha \in \text{Nef}_1(X) \mid (K_X + a(X, L) L) \cdot \alpha = 0 \} \). This is also a birational invariant by [18, Proposition 2.10]. Thus, we define \( b(X, L) \) for a singular variety \( X \) by

\[
b(X, L) = b(\tilde{X}, \varphi^* L),
\]

where \( \varphi : \tilde{X} \to X \) is a smooth resolution.

**Remark 2.8.** The two invariants \( a, b \) appeared in Manin’s Conjecture. For the notions of adelic line bundles and the height functions, see for example [12]. Let \( X \) be a smooth Fano variety over a number field \( F \) and \( \mathcal{L} = (L, || \cdot ||) \) be a nef and big line bundle with an adelic metrization.
For each integer $T \geq 1$ and a subset $Q \subset X(F)$, define the counting function by

$$N(Q, \mathcal{L}, T) = \# \{ x \in Q \mid H(x) \leq T \},$$

where $H$ is the height function associated to $\mathcal{L}$. Then Manin’s Conjecture asserts that there is a thin set $Z \subset X(F)$ such that

$$N(X(F) \setminus Z, \mathcal{L}, T) \sim cT^{2[X,L](\log T)^{k(X,L)-1}}, \quad \text{as} \quad T \to \infty,$$

where $c = c(F, X(F) \setminus Z, \mathcal{L})$ is Peyre’s constant introduced in [3, 33]. The references [2], [3], [15], and [33] contributed significantly to the formulation. The thin set $Z \subset X(F)$ was described conjecturally by using the invariants $a, b$ in [23, 24]. The reference [25] was inspired by them and stated a version of Manin’s conjecture for rational curves.

**Definition 2.9** ([25] Definition 3.5). Let $X$ be a projective uniruled manifold and let $L$ be a nef and big $\mathbb{Q}$-divisor. Then an $a$-cover $f : Y \to X$ is a face contracting morphism if the induced map $f_* : F(Y, f^*L) \to F(X, L)$ is not injective.

### 3 Lines Through a Fixed Point

We recall the notion of free rational curves, which will be frequently used in the remaining sections. See [21, II.3] for several results on free curves.

**Definition 3.1.** Let $X$ be a smooth projective variety. We say that a nonconstant morphism $f : \mathbb{P}^1 \to X$ is free if $H^1(\mathbb{P}^1, f^*T_X(-1)) = 0$.

In this section, we prove the base case of Theorem 1.2. For proving the induction step, we need to study fibers of the evaluation map of the space of lines:

**Theorem 3.2.** Let $X$ be a del Pezzo manifold of Picard rank $1$ and dimension $n \geq 4$ with an ample generator $H$. Let $ev : \mathcal{M}_{0,1}(X, 1) \to X$ be the evaluation map. Then

1. A general fiber of $ev$ is irreducible.
2. There is a finite subset $S$ of $X$ such that
   - If $p \notin S$, then $\dim(ev^{-1}(p)) = n - 3$.
   - If $p \in S$, then $\dim(ev^{-1}(p)) \leq n - 2$.

In particular, $\mathcal{M}_{0,0}(X, 1)$ is irreducible of dimension $2n - 4$.

**Remark 3.3.** For some dimension range, one can show (1) easily by an analysis of $a$-covers. Indeed, the generic fiber of $ev$ is smooth by [21, II.3.11 Theorem]. By [25, Proposition 5.15], the finite part of the Stein factorization of $ev$ is an $a$-cover. However, by [26, Theorem 11.1], there is no generically finite morphism with $a(Y, -f^*K_X) \geq a(X, -K_X)$ if $X$ satisfies one of the following:

- $H^a = 2, 3$, $\dim X \geq 4$, and $X$ is general in its moduli;
- $H^a = 4$ and $\dim X \geq 5$; or
- $H^a = 5$ and $\dim X = 6$, that is, $X \cong \text{Gr}(2, 5)$.

Hence, in these cases, $ev$ has connected fibers.

Before proving Theorem 3.2, we introduce several lemmas.

**Lemma 3.4.** Let $X$ be a del Pezzo manifold of Picard rank $1$ and dimension $n \geq 4$. Let $W \subset \mathcal{M}_{0,1}(X, 1)$ be an $m$-dimensional subvariety parametrizing lines passing through a fixed point. Then the members of $W$ cover a subvariety of $X$ of dimension $m + 1$.

**Proof.** Let $Z \subset X$ be the subvariety which is covered by lines parametrized by $W$. Suppose that $Z$ has dimension less than $m + 1$. Let $W' \subset \mathcal{M}_{0,1}(X, 1)$ be the $(m + 1)$-dimensional subvariety corresponding to $W$. Consider the evaluation map $ev : \mathcal{M}_{0,1}(X, 1) \to X$. Then any fiber of the restriction morphism
Lemma 3.5. Let $X$ be a del Pezzo manifold of Picard rank 1 and dimension $n \geq 4$ and let $\ev: \overline{M}_{0,1}(X, 1) \to X$ be the evaluation map. Then $\dim \ev^{-1}(p) \leq n - 2$ for any point $p \in X$.

Proof. Suppose that there is a point $p \in X$ such that $\dim \ev^{-1}(p) \geq n - 1$. Let $W$ be the image of an $(n - 1)$-dimensional component of $\ev^{-1}(p)$ under the morphism $\overline{M}_{0,1}(X, 1) \to \overline{M}_{0,0}(X, 1)$. Then $W$ has also dimension $n - 1$. If $W$ contains a point corresponding to a non-free line, then $W$ must have the expected dimension $n - 3$ by [21, II.3.5 Corollary]. Thus, $W$ parametrizes non-free lines. On the other hand, by Lemma 3.4, $X$ is covered by lines parametrized by $W$. However, it is a contradiction since non-free lines are contained in a closed subset of $X$ by [21, II.3.11 Theorem].

Lemma 3.6. Let $n \geq 4$ be an integer. If $X \subset \mathbb{P}^n$ is a smooth hypersurface, then any hyperplane section of $X$ is reduced and irreducible.

Proof. Assume that the section of $X$ cut by the hyperplane $(x_n = 0)$ is not irreducible. Then there is a polynomials $F, G, H$ such that $X$ is given by the equation $x_n F + GH = 0$.

Since $n \geq 4$, the set $S = V(x_n, F, G, H)$ is nonempty. Then, however, $X$ is singular at any point in $S$, which is a contradiction.

Firstly, we prove Theorem 3.2 for the case $H^n \geq 3$. In particular, when $H^n = 5$, $X$ is a smooth linear section of the Grassmannian $Gr(2, 5) \subset \mathbb{P}^9$. The papers such as [1], [13], [17], [34], and [37] will be of great help to study Grassmannians.

Proof of Theorem 3.2: $H^n \geq 3$ case. When $H^n = 3$, $X \subset \mathbb{P}^{n+1}$ is a smooth cubic hypersurface. In this case, the theorem is proved in [14].

When $H^n = 4$, $X \subset \mathbb{P}^{n+2}$ is a smooth complete intersection of two quadrics. We will show in a similar way to [14] that $\ev$ is flat with irreducible general fibers. Let $p = (1 : 0 : \cdots : 0) \in X$ and we may assume that $T_pX = V(x_1, x_2)$. Then $X$ is given by two polynomials

$$
\begin{align*}
x_1 x + F(x_1, \ldots, x_{n+2}) &= 0, \\
x_0 x_2 + G(x_1, \ldots, x_{n+2}) &= 0,
\end{align*}
$$

where $F, G$ are homogeneous polynomials of degree 2. Any line $\ell$ on $\mathbb{P}^{n+2}$ passing through $p$ is written as

$$
\ell = \{(s : a_1 t : \cdots : a_{n+2} t) \mid (s : t) \in \mathbb{P}^1\}
$$

for $(a_1 : \cdots : a_{n+2}) \in \mathbb{P}^{n+1}$. Hence, $\ell$ is contained in $X$ if and only if

$$
\begin{align*}
a_1 st + F(a_1, \ldots, a_{n+2}) t^2 &= 0, \\
a_2 st + G(a_1, \ldots, a_{n+2}) t^2 &= 0,
\end{align*}
$$

for any $(s : t) \in \mathbb{P}^1$. Thus, we obtain that $\ev^{-1}(p) = V(x_1, x_2, F, G) \subset \mathbb{P}^{n+1}$, which is in particular, connected. Moreover, if $\dim \ev^{-1}(p) = n - 2$, then either of $F(0, 0, x_3, \ldots, x_{n+2})$ or $G(0, 0, x_3, \ldots, x_{n+2})$ is identically zero, or $F(0, 0, x_3, \ldots, x_{n+2})$ and $G(0, 0, x_3, \ldots, x_{n+2})$ have a common component. In any case, $X \cap V(x_1, x_2)$ has dimension $n - 1$ and degree at most 2. However, it contradicts the Lefschetz theorem (e.g., [22, Example...
Thus, $ev$ is flat with connected fibers. Finally, by [21, II.3.11 Theorem], the general fiber of $ev$ is smooth. Therefore, the claim holds when $H^n = 4$.

When $H^n = 5$, $X$ is a smooth linear section of the Grassmannian $Gr(2, 5)$. In particular, $\dim X \leq 6$. In this case as well, we will show that $ev$ is flat with irreducible general fibers. The smoothness of general fibers of $ev$ follows by [21, II.3.11 Theorem]. Hence it is enough to show that $ev$ is flat with connected fibers. Let $V$ be a 5-dimensional vector space and let $Gr(2, 5)$ be the Grassmannian of 2-dimensional subspaces of $V$. There is an embedding, called the Plücker embedding $Gr(2, 5) \to \mathbb{P}(\bigwedge^2 V) \cong \mathbb{P}^9$, which maps $\{v, w\} \to [v \wedge w]$. By [1], the space of lines on $Gr(2, 5)$ is isomorphic to the flag variety

$$F(1, 3, 5) = \{(V_1, V_3) \mid V_1 \subset V_3 \subset V, \dim V_i = i\},$$

that is, the line corresponding to $(V_1, V_3) \in F(1, 3, 5)$ is the Schubert variety

$$\sigma = \{[W] \in Gr(2, 5) \mid V_i \subset W \subset V_3\}.$$ 

Hence, the space of lines passing through $[W] \in Gr(2, 5)$ is isomorphic to

$$(V_1, V_3) \in F(1, 3, 5) \mid V_1 \subset W \subset V_3,$$

which is, moreover, isomorphic to $\mathbb{P}^1 \times \mathbb{P}^2$. Therefore, the claim holds when $\dim X = 6$.

Fix a basis $\{e_1, \ldots , e_5\}$ of $V$ and let $z_{ij} (1 \leq i < j \leq 5)$ be the corresponding coordinates of $\mathbb{P}^9$. Take a point $[W] = [\{e_1, e_2\}] \in Gr(2, 5)$. There is a bijection $\mathbb{P}^9(\bigwedge^2 V) \to \{Q \in \text{Mat}(V) \mid Q = -Q/k^*\}$, which maps a hyperplane $H = V(\sum_{1 \leq i < j \leq 5} a_{ij}z_{ij})$ to the skew symmetric matrix $Q_H := (a_{ij})_{1 \leq i < j \leq 5}$ (i.e., $a_{ij} = -a_{ji}$) modulo scalar multiplications. Then $Gr(2, 5) \cap H$ is singular at $[W]$ if and only if $T_{[W]}(Gr(2, 5) \cap H)$ is spanned by 7 points $[e_1 \wedge e_2], \ldots , [e_1 \wedge e_5], [e_2 \wedge e_3], \ldots , [e_2 \wedge e_5]$ by [1]. Hence, the condition $T_{[W]}(Gr(2, 5) \cap H)$ is equivalent that $W \subset \text{Ker} Q_H$ (e.g., [34, Corollary 1.6]). Since $Q_H$ is skew symmetric of size 5, it cannot be regular. Thus, $Gr(2, 5) \cap H$ is smooth if and only if $\dim \text{Ker} Q_H = 1$.

Suppose that $Gr(2, 5) \cap H$ is smooth and contains $[W]$. For $(\{x_1: x_2\}, \{x_3: x_4: x_5\}) \in \mathbb{P}^1 \times \mathbb{P}^2$, the corresponding line on $Gr(2, 5)$ connecting 2 points $[e_1 \wedge e_2]$ and $[\{x_1: x_2\} \wedge \{x_3: x_4: x_5\}]$ is contained in $Gr(2, 5) \cap H$ if and only if

$$(x_1e_1 + x_2e_2)Q_H(x_3e_3 + x_4e_4 + x_5e_5) = 0.$$ 

Hence, the space of lines on $Gr(2, 5) \cap H$ is isomorphic to an ample divisor $A_H := V(\sum_{i=1}^{2} \sum_{j=3}^{5} a_{ij}x_ix_j) \subset \mathbb{P}^1 \times \mathbb{P}^2$ of bidegree $(1, 1)$, where $Q_H = (a_{ij})_{i,j}$. Therefore, the claim holds when $\dim X = 5$. Furthermore, $A_H$ fails to be irreducible if and only if $\text{Ker} Q_H \subset W$.

Finally, we consider the intersection $X = Gr(2, 5) \cap H_1 \cap H_2$ of $Gr(2, 5)$ and two hyperplanes $H_1, H_2$. Another interpretation of $X$ is the base locus of the pencil $P \subset \mathcal{O}_{Gr(2, 5)}(1)$ spanned by $H_1$ and $H_2$. The smoothness of $X$ is equivalent to the smoothness of $Gr(2, 5) \cap H_a$ for any $H_a \in P$ (e.g., by the Jacobian criterion). We may assume that $[W] = [\{e_1, e_2\}] \in X$. Then the space of lines on $X$ passing through $[W]$ is isomorphic to the intersection of two ample divisors $A_{H_1} \cap A_{H_2} \subset \mathbb{P}^1 \times \mathbb{P}^2$, or one can write as $\bigcap_{H_a \in P} A_{H_a}$. If $A_{H_1} = A_{H_2}$, then for some $H_a$, we have $\dim \text{Ker} Q_{H_a} > 1$, a contradiction. Hence, the dimension of $A_{H_1} \cap A_{H_2}$ has to be 1 unless $\text{Ker} Q_{H_a} \subset W$ for any $H_a \in P$. Consider the morphism $c: P \to \mathbb{P}(V)$ given by $c(H_a) = \text{Ker} Q_{H_a}$. Then by [34, Proposition 6.3], $c$ is an embedding to a smooth conic. This implies that there exists $H_a \in P$ such that $\text{Ker} Q_{H_a} \not\subset W$. Thus, $A_{H_1} \cap A_{H_2}$ has dimension 1, which concludes the claim when $\dim X = 4$.

### 3.1 Proof of Theorem 3.2: $H^n = 2$ case

Let $X$ be a del Pezzo manifold of degree 2. Then $X$ is a smooth hypersurface of degree 4 in the weighted projective space $\mathbb{P}(1^{n+1}, 2)$. Hence, $X$ has the form

$$X = V(y^2 + F),$$
where \( F \in k[x_0, \ldots, x_n] \) is a homogeneous polynomial of degree 4 and \( \text{deg} y = 2 \). Then there is a double cover \( f : X \to \mathbb{P}^n \) branched over a smooth quartic \( B = V(F) \subset \mathbb{P}^n \). Then by the projection formula, a curve \( \ell \subset X \) is a line if and only if \( f_*\ell \subset \mathbb{P}^n \) is a line. This correspondence induces a natural transformation

\[
\overline{M}_{0,0}(X, 1) \to \text{Grass}(2, n + 1),
\]

of functors (notation as in [16, §.1] and [21, I.1]). Hence, it defines the morphism into a Grassmannian of lines in \( \mathbb{P}^n \)

\[
f_* : M := \overline{M}_{0,0}(X, 1) \to G(1, n) \cong \overline{M}_{0,0}(\mathbb{P}^n, 1),
\]

which is finite onto its image, say \( N \). Let \( \ell \subset \mathbb{P}^n \) be a line and \( \bar{\ell} \) be a component of \( f^{-1}(\ell) \). Applying the Hurwitz formula to the restriction morphism \( f|_{\bar{\ell}} \), we obtain that

\[
2g - 2 = -2 \text{deg}(f|_{\bar{\ell}}) + \text{deg}(R),
\]

where \( g \) is the genus of \( \ell \) and \( R \) is the ramification divisor. If \( \text{deg}(f|_{\bar{\ell}}) = 2 \), then \( H \cdot \bar{\ell} = O_{\mathbb{P}^n}(1) \cdot f_*\ell = 2 \). Thus, \( \ell \) is a line on \( X \) if and only if \( g = 0 \), \( \text{deg}(f|_{\bar{\ell}}) = 1 \), and \( \text{deg}(R) = 0 \). Thus, the image \( N \) of \( f_* \) is the set of lines bitangent to \( B \), or contained in \( B \). Note that there is a commutative diagram

\[
\begin{array}{ccc}
M' & \xrightarrow{ev} & X \\
\downarrow f_* & & \downarrow f \\
N' & \xleftarrow{ev_N} & \mathbb{P}^n,
\end{array}
\]

where \( M' = \overline{M}_{0,1}(X, 1) \) and \( N' \) be the image of \( f_* : M' \to \overline{M}_{0,1}(\mathbb{P}^n, 1) \).

**Proof of Theorem 3.2.1:** \( H^n = 2 \) case. By [21, II.3.11 Theorem], there is an open subset \( U \subset X \) such that any line on \( X \) intersecting \( U \) is free. Thus, the fiber \( ev^{-1}(p) \) is smooth for any point \( p \in U \).

Since a smooth and connected scheme is irreducible, it suffices to show that the general fiber is connected. Let \( p \in X \) be a point such that \( f(p) \not\in B \). Then we have \( ev^{-1}(p) \cong ev_{N_1}^{-1}(f(p)) \). We may assume that \( f(p) = (1 : 0 : \cdots : 0) \in \mathbb{P}^n \). We will calculate the fiber \( ev_{N_1}^{-1}(f(p)) \). The polynomial \( F \) defining the quartic \( B \) has the form

\[
F(x_0, \ldots, x_n) = x_0^4 + x_0^2x_1^2 + x_0^2x_2 + x_0x_3 + F_4,
\]

where \( F_i \in k[x_1, \ldots, x_n] \) are homogeneous of degree \( i \). Furthermore, by taking the linear transform \( x_0 \mapsto x_0 - \frac{1}{4}F_1 \), we may assume that \( F_1 = 0 \). We take a line \( \ell = \{(s : a_1t : \cdots : a_4t) \mid (s : t) \in \mathbb{P}^1 \} \) for \( a = (a_1 : \cdots : a_4) \in \mathbb{P}^{n-1} \). Then \( \ell \) is an element of \( N \) if and only if every solution of the equation

\[
s^4 + F_2(a)s^2t^2 + F_3(a)st^3 + F_4(a)t^4 = 0
\]

has an even multiplicity, that is, there exist \( \alpha, \beta, \gamma \in k \) such that

\[
s^4 + F_2(\alpha)s^2t^2 + F_3(\alpha)st^3 + F_4(\alpha)t^4 = (\alpha s^2 + \beta st + \gamma t^2)^2.
\]

Comparing the coefficients, we obtain the equations

\[
1 = \alpha^2,
\]

\[
0 = 2\alpha\beta,
\]

\[
F_2 = \beta^2 + 2\alpha\gamma,
\]

\[
F_3 = 2\beta\gamma,
\]

\[
F_4 = \gamma^2.
\]
Thus, we have
\[ F_3(a) = F_2^3(a) - 4F_4(a) = 0. \]

Thus, we obtain that \( e^{-1}(p) \cong e^{-1}_N(f(p)) \cong V(F_3, F_2^2 - 4F_4) \subset P^{n-1} \), which is connected. \( \square \)

**Proof of Theorem 3.2.2:** \( H^0 = 2 \) case. First, we give a characterization of a point \( p \in f^{-1}(B) \) such that the fiber dimension is \( n - 2 \). In fact, we will show that for a point \( p \in f^{-1}(B) \), \( \dim e^{-1}(p) = n - 2 \) if and only if \( B \cap T_{f(p)} \) is a cone with the vertex \( f(p) \). We may assume that \( f(p) = (1 : 0 : \cdots : 0) \in B \) and the tangent hyperplane to \( B \) at \( f(p) \) is given by \((x_n = 0)\). Then \( B \) is defined by a homogeneous polynomial

\[ F(x_0, \ldots, x_n) = x_0^3 + x_0^2x_2 + x_0x_3 + x_4, \]

where \( F_i \in k[x_1, \ldots, x_n] \) is a homogeneous polynomial of degree \( i \). We take a line \( \ell = \{(s : a_1t : \cdots : a_nt) \mid (s : t) \in P^1\} \) for \( a = (a_1 : \cdots : a_n) \in P^{n-1} \). Then \( \ell \) is an element of \( N \) if and only if every solution of the equation

\[ a_n s^4 t + F_2(a)s^2 t^2 + F_3(a)st^3 + F_4(a)t^4 = 0 \]

has an even multiplicity, and this is equivalent to

\[ a_n = F_3(a)^2 - 4F_4(a)F_2(a) = 0. \]

Therefore, the space of lines passing through \( f(p) \) is isomorphic to the closed subset \( V(x_n, F_2^2 - 4F_2F_4) \subset P^{n-1} \) of dimension at most \( n - 2 \). Set \( R_i = F_i(x_1, \ldots, x_{n-1}, 0) \) and \( R = F(x_0, \ldots, x_{n-1}, 0) \). Hence, \( R_i \) is a homogeneous polynomial of degree \( i \) with variables \( x_1, \ldots, x_{n-1} \) and

\[ R = x_0^3 R_2 + x_0 R_3 + R_4. \]

We consider the relation

\[ R_2^2 = 4R_2R_4 \]

We first assume that \( R_4 \neq 0 \). If \( R_2 \) divides \( R_3 \), then there is a homogeneous polynomial \( P \) of degree 1 such that \( R_3 = 2PR_2 \) and \( 4P^2 R_2^2 = 4R_2R_4 \), hence we have \( R_4 = P^2 R_2 \). Substituting them, we obtain that

\[ R = x_0^3 R_2 + 2x_0 PR_2 + P^2 R_2 = (x_0 + P)^2 R_2. \]

On the other hand, if \( R_2 \) does not divide \( R_3 \), then there is a homogeneous polynomials \( P, Q \) with \( \deg P = 1 \), \( \deg Q = 2 \) such that \( R_2 = P^2, R_3 = 2PQ \), and \( R_4 = Q^2 \). Indeed, \( R_2 \) is not an irreducible polynomial since \( R_2 \) does not divide \( R_3 \). Write \( R_3 = PP \), where \( P \) and \( P \) are homogeneous polynomials of degree 1. We have \( R_2^2 = 4PPR_4 \). If \( P \neq P \), then \( R_2 \) divides \( R_4 \), hence also \( R_3 \), a contradiction. Thus, \( R_2 = P^2 \) and one can also see the existence of \( Q \). Thus, we obtain that

\[ R = x_0^3 P^2 + 2x_0 PQ + Q^2 = (x_0P + Q)^2. \]

Therefore, in both cases, \( R \) is not an irreducible polynomial, which is a contradiction by Lemma 3.6, hence \( R_3 = 0 \). Hence, we obtain that \( R_2 = 0 \) or \( R_4 = 0 \). If \( R_4 = 0 \), then we have

\[ R = x_0^3 R_3, \]

which is a contradiction by Lemma 3.6 again. Therefore, \( R_2 = R_3 = 0 \) and \( R = R_4 \). This implies that the tangent hyperplane section of \( B \) at \( f(p) \) is a cone with vertex at \( f(p) \). Thus, there are only finitely many such points by a similar argument of the proof of [14, Corollary 2.2].

It remains to show that there are only finitely many points outside \( f^{-1}(B) \) where the fiber dimension may jump. Assume that there is a proper irreducible curve \( C \) on \( X \) where the fiber dimension is \( n - 2 \). Then \( f(C) \notin B \) by the above argument. Let \( Z \) be the reduced subscheme of \( X \) covered by lines intersecting with \( C \). The subscheme \( Z \) is the union of all subschemes \( Z_p \) which are spanned by lines through \( p \in C \). Since each \( Z_p \) has dimension \( n - 1 \) by Lemma 3.4, hence \( \dim Z \geq n - 1 \). If \( \dim Z = n \), then the family of
lines intersecting $C$ covers $X$. However, those lines are not free, which is a contradiction. Hence, we have $\dim Z = n - 1$. Now, for a point $p \in C$, we let $D_p$ be the reduced divisor on $X$ which is covered by lines through $p$. Then we claim that:

**Claim 3.7.** There is a nonempty divisor $D$ on $X$ such that $D \subset D_p$ for any point $p \in C$. We set $D^C$ to be the maximal divisor satisfying this property.

**Proof.** First, it follows that there is a divisor $D$ on $X$ such that $D \subset D_p$ for any point $p$ in an open subset $U \subset C$ since $D \subset Z$ and $\dim Z = n - 1$. We will show that $U = C$. We define a subset $\Gamma \subset C \times D$ by

$$\Gamma = \{(p, q) \in C \times D \mid \text{there exists a line } \ell \text{ through } p, q\}.$$  

Consider the following commutative diagram:

$$\begin{array}{ccc}
\overline{M}_{0,2}(X, 1) & \to & X, \\
\downarrow s_1 & & \downarrow s_2 \\
X & \to & X \times X \\
\downarrow p_1 & & \downarrow p_2 \\
& & X,
\end{array}$$

where $s$ is the evaluation map, $p_i$ is the $i$-th projection. Then we have $\Gamma = s(s^{-1}_1(C) \cap s^{-1}_2(D))$. Hence, it is closed in $C \times D$. On the other hand, $\Gamma$ contains $U \times D$. Thus, $\Gamma = C \times D$, that is, $D \subset D_p$ for any $p \in C$.  

Since $C \nsubseteq f^{-1}(B)$, we take two points $p, q \in C$ such that $f(p) \in B$ and $f(q) \notin B$. Any line on $\mathbb{P}^n$ tangent to $B$ at $f(p)$ is contained in $T_{f(p)}B \cong \mathbb{P}^{n-1}$. Since $\dim f^{-1}(p) = n - 2$, we have $f(D_p) = T_{f(p)}B$, which is irreducible. Since $f(D^C) \subset f(D_p)$ are $(n - 1)$-dimensional, we have $f(D^C) = T_{f(p)}B$. Now consider the projection morphism $\pi : T_{f(p)}B \setminus f(q) \to \mathbb{P}^{n-2}$ from $f(q)$. The restriction morphism to $B \cap T_{f(p)}B$ is finite since $f(q) \notin B$. Let $\ell$ be a line such that $f(q) \in \ell \subset T_{f(p)}B$. Then $\ell$ is bitangent to $B$ if and only if the fiber of $\pi|_{B \cap T_{f(p)}B}$ over the point $\pi(\ell)$ is non-reduced. Hence, in order to obtain $f(D_p) = T_{f(p)}B$, any fiber of this restriction has to be non-reduced. Thus, $B \cap T_{f(p)}B$ also has to be non-reduced. However, by Lemma 3.6, $B \cap T_{f(p)}B$ is reduced, a contradiction. Hence, we have $f(D_p) \neq f(D_p) = f(D^C)$ and thus there are only finitely many points where the fiber does not have the expected dimension.

### 3.2 Proof of Theorem 3.2: $H^n = 1$ case

Let $X$ be a del Pezzo manifold of degree $1$. Then $X$ is a smooth hypersurface of degree $6$ in the weighted projective space $\mathbb{P}(1^n, 2, 3)$. Using a homogeneous polynomial $F = y^3 + F_4y + F_6$ of degree $6$, where $F_i \in k[x_1, \ldots, x_{n-1}]$ is a homogeneous polynomial of degree $i$, and $\deg y = 2$, $X$ is written as

$$X = V(z^2 + F),$$

where $\deg z = 3$. Then there is a double cover $f : X \to \mathbb{P} := \mathbb{P}(1^n, 2)$ branched over a smooth sextic $Y = V(F)$ and the vertex $\nu := (0 : \cdots : 0 : 1) \in \mathbb{P}$. Let $Z := V(z)$ be the ramification divisor of $f$. Set $w := (0 : \cdots : 0 : 1 : 1) \in X$, which is the unique point mapping to the vertex $\nu \in \mathbb{P}$. For later use, we introduce some notations. Consider the projection $g : \mathbb{P} \setminus \{\nu\} \to \mathbb{P}^{n-1}$ from the vertex $\nu$. The restriction $g|_Y$ is finite of degree $3$. Let $R \subset Y$ be the ramification locus of $g|_Y$ and $B \subset \mathbb{P}^{n-1}$ be the branch locus. More explicitly, these are written as

$$R = V(3y^2 + F_4, 2y^3 - F_6) \subset Y,$$

$$B = V(4F_4^3 + 27F_6^2) \subset \mathbb{P}^{n-1}.$$

In addition, we set $T := V(F_4, F_6)$, over which $g|_Y$ is totally ramified. Therefore, there is a following diagram:

$$\begin{array}{ccc}
X & \to & Z \\
\downarrow f & & \downarrow g|_Y \\
\mathbb{P} & \to & \mathbb{P}^{n-1} \\
\downarrow g & & \downarrow \\
Y & \to & R \\
\downarrow \nu & & \downarrow \nu \\
B & \subset & T.
\end{array}$$
Let \( \ell \) be a line on \( X \). Then \( \mathcal{O}_\ell(1) \cdot f_* \ell = \mathcal{O}_X(1) \cdot \ell = 1 \) by the projection formula. Consider the restriction morphism \( f|_{\ell} \). We will call a connected 1-cycle \( \ell \subset \mathbb{P} \) to be a line if \( \mathcal{O}_\ell(1) \cdot \ell = 1 \).

1. If \( \deg f|_{\ell} = 1 \), then \( \alpha := f|_{\ell} \ell \) is a smooth line not passing through the vertex \( v \). By the Hurwitz formula, it is shown that \( \alpha \) is tritangent to \( Y \) or \( \alpha \subset Y \). Moreover, if \( \alpha \nsubseteq Y \), then the pull back \( f^*(\alpha) \) is a union of two distinct lines on \( X \). Since \( v \neq \alpha \), we have \( \mathcal{O}_{p^{-1}}(1) \cdot g_* \alpha = \mathcal{O}_p(1) \cdot \alpha = 1 \) by the projection formula. Hence, \( \deg g|_{\alpha} = 1 \) and \( g_* \alpha \) is a line on \( \mathbb{P}^{n-1} \).

2. If \( \deg f|_{\ell} = 2 \), then \( \alpha = 2 \beta \) is a double line, where \( \beta \) is the curve class satisfying \( \mathcal{O}_p(1) \cdot \beta = \frac{1}{2} \) and \( v \in \beta \). Moreover, \( \beta \) is tangent to \( Y \).

Let \( N_1 \) (resp. \( N_2 \)) be the locus of smooth lines of type (1) (resp. double lines of type (2)), and set \( N := N_1 \cup N_2 \). Since the inverse image of a line in \( N_2 \) is a line on \( X \) passing through \( w \), the dimension of \( N_2 \) is at most \( n - 2 \) by Lemma 3.5. Hence, \( N_2 \) cannot form a component of \( N \) since \( \dim N = \dim M_{0,0}(X,1) \geq 2n - 4. \)

Moreover, a general element of \( N \) has type (1) and not contained in \( Y \). Hence, the pushforward of \( f \) induces the double cover \( f_* : M := M_{0,1}(X,1) \to N \) and the commutative diagram

\[
\begin{array}{ccc}
M' & \xrightarrow{ev} & X \\
\downarrow f & \quad & \downarrow f \\
N' & \xrightarrow{ev_N} & \mathbb{P},
\end{array}
\]

where \( M' = \overline{M}_{0,1}(X,1) \) and \( N' \) be the image of \( f_* : M' \to M_{0,1}(\mathbb{P},1) \). We introduce two lemmas for the proof of Theorem 3.2(2):

**Lemma 3.8.** Notation as above. Then any fundamental divisor \( D \in |H| \) on \( X \) has only isolated singularities.

**Proof.** One can prove this in a similar way to [30, Lemma 6.6]. We give a proof for completeness.

Since \( \mathcal{O}_X(H) \cong \mathcal{O}(P^{n-2,1}(1)) \), we may assume that \( D \) is cut out by a hyperplane \( V(x_0) \). For each integer \( 0 \leq i \leq n - 1 \), we set \( U_i := (x_i \neq 0) \subset P^{1}(n, 2, 3) \), which is an open subset isomorphic to \( \mathbb{A}^{n+1} \cong \text{Spec}(k[x_0, \ldots, x_{n-1}, y, z]) \).

Set \( \bar{x}_j := x_j/x_i \) for \( j \neq i \), \( \bar{y} := y/x^2 \), and \( \bar{z} := z/x^3 \). Then we have \( D \subset \bigcup_{i=1}^{n-1} U_i \cup \{w\} \). For simplicity, we assume we are in \( U_{n-1} \). Then the Jacobian matrix for \( D \cap U_{n-1} \) is

\[
\begin{pmatrix}
1 & 0 & \cdots & 0 & 0 & 0 \\
\frac{\partial f}{\partial x_0} & \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_{n-2}} & \frac{\partial f}{\partial y} & 2z
\end{pmatrix},
\]

where \( \bar{f} = f/x^3_{n-1} \in k[x_0, \ldots, x_{n-2}, \bar{y}, \bar{z}] \). Then the singular locus \( \text{Sing}(D \cap U_{n-1}) \) is contained in \( V(\bar{z}) \). On the other hand, \( D \cap U_{n-1} \) is smooth along \( V(\bar{f}, \bar{z}) \) since \( Z \) is smooth. Running the same argument for any \( i \in \{1, \ldots, n-2\} \), we see that \( \text{Sing}(D) \subset Z \cup \{w\} \) and \( D \) is smooth along \( V(\frac{\partial f}{\partial x_i}) \cap Z \). However, \( \text{Sing}(D) \) meets \( V(\frac{\partial f}{\partial x_i}) \cap Z \) unless the locus is 0-dimensional. Thus, the claim holds.

**Lemma 3.9.** Notation as above. Let \( D \in |3H| \) be a member of the form \( D = V(z + P) \cap X \), where \( P \in k[x_0, \ldots, x_{n-1}, y] \) is a homogeneous polynomial of degree 3. Then \( D \) has only isolated singularities.

**Proof.** As in the proof of Lemma 3.8, we use the notation \( U_i, \bar{x}_i, \bar{y}, \bar{z} \), and \( \bar{f} \). Since the degree of \( P \) is odd, we see that \( w \notin D \). Hence, we have \( D \subset \bigcup_{i=1}^{n-1} U_i \). For simplicity, we assume we are in \( U_{n-1} \). Then the Jacobian matrix for \( D \cap U_{n-1} \) is

\[
\begin{pmatrix}
\frac{\partial P}{\partial x_0} & \cdots & \frac{\partial P}{\partial x_{n-2}} & \frac{\partial P}{\partial y} & 1 \\
\frac{\partial P}{\partial x_0} & \cdots & \frac{\partial P}{\partial x_{n-2}} & \frac{\partial P}{\partial y} & 2z
\end{pmatrix},
\]

where \( \bar{P} = P/x^3_{n-1} \in k[x_0, \ldots, x_{n-2}, y, z] \). Then \( D \cap U_{n-1} \) is smooth along \( V(\bar{z}) \) since \( Z \) is smooth. Running the same argument for any \( i \in \{0, \ldots, n-2\} \), we see that \( D \) is smooth along \( Z \). However, the singular locus of \( D \) meets with \( Z \) unless the locus is 0-dimensional. Thus, the claim holds.
**Proof of Theorem 3.2.(1):** $H^n = 1$ case. As in the case of $H^n = 2$, the general fiber of $ev$ is smooth by [21, II.3.11 Theorem]. Hence, it suffices to show the connectedness of the general fiber. Let $p \in X$ map to a point $f(p) \not\in Y \cup g^{-1}(B)$. Then we have $ev^{-1}(p) \cong ev_\nu^{-1}(f(p))$ and any member of $N_2$ cannot pass through $f(p)$ since $f(p) \notin g^{-1}(B)$. We may assume that $f(p) = (1 : 0 : \cdots : 0 : b_0) \in \mathbb{P}$. Let $\ell$ be a line on $\mathbb{P}$ through $f(p)$ but not through $u$. We recall that a line on $\mathbb{P}$ is a connected 1-cycle $a$ such that $O_{\mathbb{P}}(1) \cdot a = 1$. Hence, by the projection formula, we see that $\text{deg} g|_{\ell} = 1$ and $g(\ell)$ is a line on $\mathbb{P}^{n-1}$. Since $gf(p) = (1 : 0 : \cdots : 0) \in g(\ell)$, one can write

$$g(\ell) = \{(s : a_1t : \cdots : a_{n-1}t) \mid (s : t) \in \mathbb{P}^1\}$$

for $(a_1 : \cdots : a_{n-1}) \in \mathbb{P}^{n-2}$. Thus, $\ell$ has the form

$$\ell = \{(s : a_1t : \cdots : a_{n-1}t : b_0s^2 + b_1st + b_2t^2) \mid (s : t) \in \mathbb{P}^1\}$$

for $(a_1 : \cdots : a_{n-1} : b_1 : b_2) \in \mathbb{P}(1^n, 2)$. We write the polynomial $F(\ell)$ in the variables $s, t$, substituting the parameter of $\ell$ in $F$, as

$$F(\ell) = \sum_{i=0}^{6} G_i s^{6-i} t^i.$$

Since $\ell$ contains $f(p)$, $\ell$ cannot be an element of $N_2$. Then $\ell$ is an element of $N$ if and only if every solution of the equation

$$F(\ell) = 0$$

has an even multiplicity, that is, there exist $\alpha, \beta, \gamma, \delta \in k$ such that

$$F(\ell) = (\alpha s^3 + \beta s^2 t + \gamma st^2 + \delta t^3)^2.$$

Comparing the coefficients, we obtain the equations

- $G_0 = \alpha^2$,
- $G_1 = 2\alpha \beta$,
- $G_2 = \beta^2 + 2\alpha \gamma$,
- $G_3 = 2\alpha \delta + 2\beta \gamma$,
- $G_4 = \gamma^2 + 2\beta \delta$,
- $G_5 = 2\gamma \delta$,
- $G_6 = \delta^2$.

By assumption, $G_0 = F(f(p)) \neq 0$, hence we may assume that $\alpha = 1$. Then we obtain

$$\beta = \frac{1}{2} G_1,$$

$$\gamma = \frac{1}{2} G_2 - \frac{1}{8} G_1^2,$$

$$\delta = \frac{1}{2} G_3 - \frac{1}{4} G_1 G_2 + \frac{1}{16} G_1^3.$$

Substituting them into $G_4, G_5, G_6$, we obtain three equations with variables $a_0, \ldots, a_{n-1}, b_1, b_2$. Thus, $ev_\nu^{-1}(f(p))$ is defined by three equations in $\mathbb{P}(1^n, 2)$. Since $\dim ev^{-1}(p) = n - 3$ except for finitely many points by Theorem 3.2.(2) proved later, the general fiber is a complete intersection, hence connected.

**Proof of Theorem 3.2.(2):** $H^n = 1$ case. For each point $p \in X$ such that $\dim ev^{-1}(p) = n - 2$, we let $D_p$ be the reduced divisor on $X$ which is covered by lines through $p$. Then we claim that:

**Claim 3.10.** For any irreducible component $D \subset D_p$, $D$ is singular at $p$. 


Theorem. Suppose that $p$ is a smooth point of $D$. Let $\phi: \tilde{D} \to D$ be a smooth resolution. Let $s: \mathbb{M}_{0,1}(\tilde{D}, \alpha) \to \tilde{D}$ be the evaluation map, where $\alpha$ is the class of the strict transforms of lines. Since $D$ is smooth at $p$, the fiber $\phi^{-1}(p)$ consists of a single point $\tilde{p}$, and $\dim s^{-1}(\tilde{p}) = \dim \ev^{-1}(p) = n - 2$. Moreover, since curves parametrized by $s^{-1}(\tilde{p})$ dominate $\tilde{D}$, there exists a free curve $[\tilde{\ell}] \in s^{-1}(\tilde{p})$ by [21, II.3.11 Theorem]. Thus,

$$\dim_{[\tilde{\ell}]} s^{-1}(\tilde{p}) = \dim_{[\tilde{\ell}]} \mathbb{M}_{0,1}(\tilde{D}, \alpha) - \dim D$$

by [21, II.3.5.4 Corollary]. On the other hand, [21, II.1.2 Theorem] yields that

$$\dim_{[\tilde{\ell}]} \mathbb{M}_{0,0}(\tilde{D}, \alpha) = -K_D \cdot \alpha + \dim \tilde{D} - 3.$$ 

Hence, we obtain that $-K_D \cdot \alpha = n$. Then $(K_D + n\phi^*H) \cdot \alpha = 0$. Thus, $K_D + n\phi^*H \in \mathfrak{a}\mathfrak{h}\mathfrak{f}^1(X)$ and hence $\alpha(D, H) = n > \alpha(X, H)$. However, this is impossible by Lemma 2.5.

Suppose that there is a proper irreducible curve $C$ on $X$ such that for any $p \in C$, $\dim \ev^{-1}(p) = n - 2$. Let $D^C$ be the divisor as defined in Claim 3.7. By Claim 3.10, we see that any component $D \subset D^C$ is singular along the curve $C$. In the remaining part of the proof, we show that the divisor $D^C$ cannot exist dividing into two cases: (i) $f(C) \subset g^{-1}(B) \cup Y$, or (ii) $f(C) \not\subset g^{-1}(B) \cup Y$.

(i) Suppose that $f(C) \subset g^{-1}(B) \cup Y$. If $f(C) \subset Y$, then $f(C)$ intersects with $R$. Also when $f(C) \subset g^{-1}(B)$, one can show that $f(C)$ intersects with $R$. Indeed, if $f(C)$ is contracted by $g$, then the assertion is clear since $f(C) \subset g^{-1}(B)$. Suppose $gf(C) \subset B$ is a curve. Then it suffices to show that $gf(C)$ intersects with $T$. Since $V(F_B)$ is an ample divisor on $\mathbb{P}^{n-1}$, one can take a point $a \in gf(C) \cap V(F_B)$. Then we have $\gamma(a) = 0$ since $gf(C) \subset B$. Hence, $a \in gf(C) \cap T$. For a point $p \in C$ such that $f(p) \in R$, we will analyze the divisor $D_p$. Consider the morphism $\gamma: R \to (\mathbb{P}^{n-1})^*$ defined by

$$\gamma(r) = \left(\frac{\partial F}{\partial x_0}(r), \ldots, \frac{\partial F}{\partial x_{n-1}}(r)\right),$$

which is well-defined since $Y$ is smooth and $\frac{\partial F}{\partial x_i}(r) = 0$ for any $r \in R$. We view $\gamma(r)$ as also the corresponding hyperplane in $\mathbb{P}^{n-1}$.

(ii) Suppose that $f(C) \not\subset g^{-1}(B) \cup Y$. Note that for any $p \in X$ such that $f(p) \not\equiv g^{-1}(B)$, $D_p$ cannot contain the point $u$ since a line $\ell \subset C$ contains $u$ if and only if $\ell \not\subset N_2$. Therefore, $D^C$ cannot contain any divisor that is the pullback of a divisor on $\mathbb{P}^{n-1}$. Let $D \subset D^C$ be an irreducible component. Then $gf(D) = \mathbb{P}^{n-1}$. Let $\iota: X \to X$ be the involution defined by

$$(x_0 : \cdots : x_{n-1} : y : z) \mapsto (x_0 : \cdots : x_{n-1} : y : -z).$$

Note that $f = f \circ \iota$ and $\iota|_Z = \id_Z$. Since $u \not\in D$, we see that $\deg f|_{\ell} = 1$ for any line $\ell \subset D$. Hence, if $q \in C \cap Z$, then $\iota(q) \neq D$. Therefore, $D \neq \iota(D)$. Now fix a point $p \in C \cap Z$ and let $M_0$ be the irreducible component of $\ev^{-1}(p)$, which parametrizes lines dominating $D$. Then we obtain that

$$M_0 \cap M_{\iota(D)} \subset \{\text{lines contained in } Z\}$$

since $D$ and $\iota(D)$ are different components of $D_p$. Furthermore, we claim that:

Claim 3.11. $D \cap \iota(D) \subset Z$.

Proof. Pick a point $q \in D \cap Z$. It suffices to show that $q \not\in \iota(D)$. Since $p \in C$ and $q \in D$, one can take a line $\ell \subset D$ through both $p$ and $q$. The image $\iota \cdot f(\ell)$ is a member of $N$ such that $\{f(p), f(q)\} \subset \ell \subset f(D)$. Since $u \not\in f(D)$, $\ell$ is a member of $N_2$. We now assume that there exists a line $\ell' \subset N_1 \setminus \{\ell\}$ such that $f(p) \in \ell' \subset f(D)$. If $g(\ell') \neq g(\ell')$, then $\ell'$ cannot contain $f(q)$ since $g(\ell) \cap g(\ell') = [gf(p)] \neq [gf(q)]$. Suppose that $g(\ell) = g(\ell')$. As we view $\ell$ and $\ell'$ as the curves in $\mathbb{P}(1^2, 2) \cong g^{-1}(\mathbb{P}(1^2, 2))$, then $\ell, \ell' \in |\mathcal{O}_{\mathbb{P}(1^2, 2)}(2)|$ and hence we have the intersection number $\ell \cdot \ell' = \tfrac{1}{2} \cdot 2^2 = 2$ on the surface $\mathbb{P}(1^2, 2)$. On the other hand, both $\ell$ and $\ell'$ are tangent to $Y$ at $f(p)$ since $\ell, \ell' \subset N_1$. Hence, $\ell$ and $\ell'$ intersect at $f(p)$ with multiplicity 2. Thus, $\ell'$ cannot contain $f(q)$. This implies that $\ell$ is the unique line such that $\{f(p), f(q)\} \subset \ell \subset f(D)$. The inverse
image of $\ell$ is the union of two lines $\tilde{\ell}, \iota(\tilde{\ell})$. Since $q \in \tilde{\ell} \setminus Z$, we see that $q \notin \iota(\tilde{\ell})$. We also see that $\iota(\tilde{\ell}) \notin M_D$ since

$$\iota(\tilde{\ell}) \in M_{(D)} \setminus \{\text{lines contained in } Z\}.$$ 

Therefore, $\iota(D)$ cannot contain $q$, hence the claim holds. \hfill \blacksquare

We retake a point $p \in C \setminus Z$. Let $\tilde{\ell}$ be a line on $X$ such that $p \in \tilde{\ell} \subset D$. Then we claim that:

**Claim 3.12.** $D \cdot \iota(\tilde{\ell}) = 3$.

**Proof.** By Claim 3.11, we have

$$\tilde{\ell} \cap \iota(\tilde{\ell}) \subset D \cap \iota(\tilde{\ell}) \subset D \cap \iota(D) \subset Z.$$

Set $\ell := f(\tilde{\ell}) = f(\iota(\tilde{\ell}))$. Since $f(D \cap \iota(\tilde{\ell})) \subset \ell$, we have

$$D \cap \iota(\tilde{\ell}) \subset f^{-1}(\ell) \cap Z = \tilde{\ell} \cap \iota(\tilde{\ell}).$$

Thus, we obtain a set-theoretic equality $\tilde{\ell} \cap \iota(\tilde{\ell}) = D \cap \iota(\tilde{\ell})$. Via an isomorphism $\mathbb{P}^1 \cong \ell$, we obtain $\mathbb{P}(1^2, 3) \cong \tilde{f}^{-1}(\ell)$, where $\tilde{f} : \mathbb{P}(1^2, 3) \dashrightarrow \mathbb{P}(1^2, 2)$ is the projection map. If we view $\tilde{\ell}$ and $\iota(\tilde{\ell})$ as the curves in $\mathbb{P}(1^2, 3)$, then $\tilde{\ell}, \iota(\tilde{\ell}) \in |O_{\mathbb{P}(1^2, 3)}(3)|$ and hence we have the intersection number $\tilde{\ell} \cdot \iota(\tilde{\ell}) = \frac{1}{2} \cdot 3^2 = 3$ on the surface $\mathbb{P}(1^2, 3)$. Thus, in order to prove that $D \cdot \iota(\tilde{\ell}) = 3$, it is enough to show that the intersection multiplicity at each point $q_0 \in D \cap \iota(\tilde{\ell})$ is equal to that of $q_0 \in \tilde{\ell} \cap \iota(\tilde{\ell})$.

Suppose that there is a line $\ell' \in N_1 \setminus \{\ell\}$ such that $f(q_0) \in \ell' \subset f(D)$. We assume that $g(\tilde{\ell}) = g(\ell')$. As in the proof of Claim 3.11, we see that $f(p) \neq \ell'$. We recall that $f(D)$ is covered by lines in $N_1$ passing through $f(p)$. We also recall that any line $\alpha \in N_1$ satisfies $\deg g|_{\alpha} = 1$. Hence, we see that $g^{-1}(gf(p)) \cap f(D) = (f(p))$. This implies that the unique point in $g^{-1}(gf(p)) \cap \ell'$ is not contained in $f(D)$, which contradicts the assumption $\ell' \subset f(D)$. Therefore, $g(\tilde{\ell}) \neq g(\ell')$. Then $\ell$ and $\ell'$ intersect at $f(q_0)$ with different tangent directions. Summarizing the argument, for any line $\ell'$ on $X$ distinct from $\tilde{\ell}$ such that $q_0 \in \ell' \subset D$, the intersection multiplicity at $q_0 \in \ell \cap \iota(\tilde{\ell})$ is 1. Therefore, the intersection multiplicity at $q_0 \in D \cap \iota(\tilde{\ell})$ is equal to that of $q_0 \in \tilde{\ell} \cap \iota(\tilde{\ell})$, as required. \hfill \blacksquare

By Claim 3.12, we have $D \in |3H|$. Write

$$D = V(cz + P) \cap X,$$

where $c \in k$ and $P \in k[x_0, \ldots, x_{n-1}, y]$ is a homogeneous polynomial of degree 3. Since $D \neq \iota(D)$, we may assume that $c = 1$. Therefore, by Lemma 3.9, $D$ has only isolated singularities, which contradicts Claim 3.10.

**Remark 3.13.** Applying the above argument, one can give a very short proof for Theorem 3.2.2 when $H^n = 2$: if $C$ is a proper irreducible curve where the fiber has dimension $n - 2$, then for any point $p \in C \cap f^{-1}(B)$, the divisor $D_p$ is a fundamental divisor. Hence, the rest is to prove that any fundamental divisor has only isolated singularities.

The reason why we gave the first proof is because one can see that for a point $p \in f^{-1}(B)$, $\dim ev^{-1}(p) = n - 2$ if and only if $f(p)$ is a cone point of $B$. This result is similar to the characterization for smooth cubics $X = X_3 \subset \mathbb{P}^{n+1}$: for a point $p \in X$, $\dim ev^{-1}(p) = n - 2$ if and only if $p$ is an Eckardt point ([14, Lemma 2.1 and Definition 2.3]). Unfortunately, the author was not able to give a similar characterization for $H^n = 1$.

## 4 Rational Curves of Arbitrary Degree

In this section, we prove the movable bend and break and Theorem 1.2 for arbitrary $d \geq 1$. First, we can apply the argument in [14, Proposition 2.5], hence we obtain:

**Proposition 4.1.** Let $X$ be a del Pezzo manifold of Picard rank 1 and dimension $n \geq 4$ with an ample generator $H$. Let $ev_1 : \overline{M}_{0,1}(X, d) \to X$ be the evaluation map for each integer $d \geq 1$. Let $S \subset X$ be the finite set as in Theorem 3.2. Then for any $d \geq 1$,
• if $p \notin S$, then $\dim(\text{ev}_d^{-1}(p)) = (n - 1)d - 2$, and
• if $p \in S$, then $\dim(\text{ev}_d^{-1}(p)) \leq (n - 1)d - 1$.

Furthermore, any component $M$ of $\overline{M}_{0,0}(X, d)$ generically parametrizes free curves and has the expected dimension $(n - 1)d + n - 3$. We will prove the movable bend and break introduced in [25]. We give a proof for completeness.

**Theorem 4.2** (Movable bend and break). Let $X$ be a del Pezzo manifold of Picard rank $1$ and dimension $n \geq 4$ with an ample generator $H$. Then, any free curve deforms to a chain of free lines.

**Proof.** Fix an integer $d \geq 1$. Let $M \subset \overline{M}_{0,0}(X, d)$ be a component and we take a free curve $(C, f) \in M$. By [21, II.3.11 Theorem], there is a closed subset $V \subset X$ such that any non-free curve of degree at most $d$ is contained in $V$. By Mori’s bend and break lemma (e.g., [21, II.5.5 Corollary]), the locus of stable maps with reducible domains has codimension $1$. Thus, $f$ can be deformed to a stable map with a reducible domain $g : C_1 + C_2 \to X$. Let $A \ni g$ be a component of $\overline{M}_{0,1}(X, d_1) \times X \overline{M}_{0,1}(X, d_2)$, where $d_1 + d_2 = d$. We now assume that $A$ does not parametrize chains of two free curves. We calculate the dimension of $A$ dividing into two cases.

1. Suppose that any stable map in the image of $A \to \overline{M}_{0,1}(X, d_1)$ has a reducible domain. Then by Proposition 4.1, we have
   \[
   \dim A \leq (n - 1)d_1 - 3 + (n - 1)d_2 - 2 + n = (n - 1)d + n - 5.
   \]

2. Suppose that general stable map in the image of $A \to \overline{M}_{0,1}(X, d_1)$ is irreducible but non-free. Then the node $p \in C_1 \cap C_2$ maps to a point in $V$. Hence, by Proposition 4.1, we have
   \[
   \dim A \leq (n - 1)d_1 - 2 + (n - 1)d_2 - 2 + (n - 1) = (n - 1)d + n - 5.
   \]

Thus, the codimension of $A$ in $M$ is greater than $1$, which means that general stable map with a reducible domain is a chain of two free curves. Thus, the assertion follows by [25, Lemma 5.9] and induction on the degree $d$. 

**Proof of Theorem 1.2.** For the case $d = 1$, we have proved it in Theorem 3.2. Let $d > 1$ and let $M \subset \overline{M}_{0,0}(X, d)$ be a component. Take a free curve $(C, f) \in M$. Then by Theorem 4.2, $(C, f)$ deforms to a chain of free lines of length $d$. Let $U \subset \overline{M}_{0,0}(X, 1)$ be an open sublocus of free lines. Then the fiber product
   \[
   \Delta := U' \times_X U'' \times_X \cdots \times_X U'' \times_X U'\]

is the locus of chains of free lines, where $U' \subset \overline{M}_{0,1}(X, 1)$ and $U'' \subset \overline{M}_{0,2}(X, 1)$ are corresponding subloci. By [25, Lemma 5.7], the projections from $\Delta$ to each factor are dominant and flat. In addition, by Theorem 3.2, the general fiber of the evaluation map $\text{ev} : \overline{M}_{0,1}(X, 1) \to X$ is irreducible. Thus, the locus $\Delta$ of chains of free lines is irreducible. Then we see that $M$ is the unique component of $\overline{M}_{0,0}(X, d)$ containing $\Delta$, hence $\overline{M}_{0,0}(X, d) = M$. 

**5 Geometric Manin’s Conjecture**

As a corollary of Theorem 1.2, we will prove Geometric Manin’s Conjecture for our case. In [25], the authors defined Manin components, which they propose should be counted in the conjectural asymptotic formula. For details about this conjecture, see [38]. We recall that a smooth projective variety $X$ is weak Fano if $-K_X$ is a nef and big divisor.

**Definition 5.1** ([38] Definition 4.3). Let $X$ be a weak Fano manifold. A generically finite morphism $f : Y \to X$ from a projective manifold is a breaking morphism if either
(i) $a(Y, -f^*K_X) > a(X, -K_X)$, or
(ii) $f$ is an $a$-cover with Iitaka dimension $\kappa(Y, K_Y - f^*K_X) > 0$, or
(iii) $f$ is a face contracting morphism.

An irreducible component $M \subset \text{Hom}(\mathbb{P}^1, X)$ is an accumulating component if there is a breaking morphism $f : Y \to X$ and a component $N \subset \text{Hom}(\mathbb{P}^1, Y)$ such that $f$ induces a dominant generically finite map $N \dashrightarrow M$. A Manin component is a component that is not accumulating.

**Conjecture 5.2** (Geometric Manin’s Conjecture ([38])). Let $X$ be a weak Fano manifold. Then there is an integer $m \geq 1$ and a nef integral 1-cycle $\alpha \in \text{Nef}_1(X)_{\mathbb{Z}}$ such that for any nef integral 1-cycle $\beta \in \alpha + \text{Nef}_1(X)_{\mathbb{Z}}$, the space $\text{Hom}(\mathbb{P}^1, X, \beta)$ has exactly $m$ Manin components.

**Lemma 5.3.** Let $X$ be a del Pezzo manifold of Picard rank 1 and dimension $n \geq 4$. Then $X$ has no $a$-cover.

**Proof.** Let $L$ be an ample sheaf on $X$ such that $\omega_X \cong L^{n-1}$. Assume that there exists an $a$-cover $f : Y \to X$ with $\kappa := \kappa(Y, K_Y + a(Y, f^*L))$. By [26, Theorem 3.15], the general member $H \in |L|$ is a del Pezzo manifold of dimension $n - 1$. Since $f$ can be replaced by the composition with a birational morphism, we may assume that $K_Y := f^{-1}(H)$ is also smooth. In this situation, we obtain the isomorphism as in the proof of [26, Theorem 3.15]:

$$H^2(Y, m(K_Y + a(Y, f^*L))) \to H^0(H_Y, m(K_{H_Y} + (a(Y, f^*L) - 1)f^*L))$$

for sufficiently divisible integer $m \geq 1$. Hence, $a(H_Y, f^*L) = a(Y, f^*L) - 1$ and $\kappa(H_Y, K_{H_Y} + a(H_Y, f^*L)) = \kappa(Y, K_Y + a(Y, f^*L))$. Thus, the restriction $f|_{H_Y}$ is again an $a$-cover. Repeating this process, we eventually obtain an $a$-cover $f' : Y' \to X$ to a del Pezzo threefold such that $\kappa(Y', K_{Y'}) = \kappa$. Then by the proof of [25, Lemma 7.2], the Iitaka dimension $\kappa$ is equal to 2.

Let $\phi : Y \to W$ be the Iitaka fibration for $K_Y - f^*K_X$. By the above argument, a general fiber $Y_0$ of $\phi$ has dimension $n - 2$ and $a(Y_0, f^*L) = n - 1$. Then the image $f(Y_0)$ also has $a(f(Y_0), H) = n - 1$. Let $v : \hat{Z} \to f(Y_0)$ be the normalization. Then we see that $(\hat{Z}, v^*H)$ is isomorphic to $(\mathbb{P}^{n-2}, \mathcal{O}(1))$. Then, in fact, $v$ is an isomorphism since any strict sublinear system of $|\mathcal{O}(1)|$ cannot define a morphism. Hence, $f(Y_0)$ is isomorphic to $\mathbb{P}^{n-2}$. Thus, it suffices to prove that $X$ cannot be dominated by projective $(n - 2)$-spaces. Let $Z \subset X$ be a projective $(n - 2)$-space. For any point $p \in Z$, the space of lines in $Z$ passing through $p$ has dimension $n - 3$. Hence, it is a component of the fiber of the evaluation map $\text{ev}_1 : \mathbb{P}^1 \times (X, 1) \to X$. However, the general fiber of $\text{ev}_1$ is irreducible, and the lines in such a fiber do not cover $Z$ by the explicit description of the fibers as in the proof of Theorem 3.2 for $H^n \neq 3$, and [14, Lemma 2.1] for $H^n = 3$. Thus, the assertion holds.

**Theorem 5.4.** Let $X$ be a del Pezzo manifold of Picard rank 1 and dimension $n \geq 4$. Then for any $d \geq 1$, the unique component of $\text{Hom}(\mathbb{P}^1, X, d)$ is a Manin component.

**Proof.** By Lemma 2.5 and Lemma 5.3, there is no breaking morphism. Thus, any component $\text{Hom}(\mathbb{P}^1, X, d)$ is a Manin component.

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