N-dimensional static and evolving Lorentzian wormholes with cosmological constant

Mauricio Cataldo

Departamento de Física, Facultad de Ciencias, Universidad del Bío-Bío, Avenida Collao 1202, Casilla 5-C, Concepción, Chile.

Paola Meza† and Paul Minning‡

Departamento de Física, Facultad de Ciencias Físicas y Matemáticas, Universidad de Concepción, Casilla 160-C, Concepción, Chile.

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Abstract: We present a family of static and evolving spherically symmetric Lorentzian wormhole solutions in N+1 dimensional Einstein gravity. In general, for static wormholes, we require that at least the radial pressure has a barotropic equation of state of the form \( p_r = \omega_r \rho \), where the state parameter \( \omega_r \) is constant. On the other hand, it is shown that in any dimension \( N \geq 3 \), with \( \phi(r) = \Lambda = 0 \) and anisotropic barotropic pressure with constant state parameters, static wormhole configurations are always asymptotically flat spacetimes, while in 2+1 gravity there are not only asymptotically flat static wormholes and also more general ones. In this case, the matter sustaining the three-dimensional wormhole may be only a pressureless fluid. In the case of evolving wormholes with \( N \geq 3 \), the presence of a cosmological constant leads to an expansion or contraction of the wormhole configurations: for positive cosmological constant we have wormholes which expand forever and, for negative cosmological constant we have wormholes which expand to a maximum value and then recollapse. In the absence of a cosmological constant the wormhole expands with constant velocity, i.e without acceleration or deceleration. In 2+1 dimensions the expanding wormholes always have an isotropic and homogeneous pressure, depending only on the time coordinate.

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I. INTRODUCTION

It is well known the interest of studying gravitational fields in spacetimes with arbitrary dimensions, with or without cosmological constant. The theoretical properties of these multidimensional gravitational fields could be quite different from one dimension to another and so it is of much interest to get an insight into how the spacetime dimension may influence the gravitational dynamics. For example, recently, there was a lot of interest in low-dimensional gravity. The discovery of the existence of three-dimensional black hole solutions represents one of the main advances for low-dimensional gravity theories. While in (3+1)-dimensional gravity black hole solutions exist with or without cosmological constant, in (2+1) dimensions the cosmological constant plays a crucial role in their existence. From the point of view of the equilibrium configurations of stars the number of dimensions of spacetime also can influence this equilibrium. In Ref. [1] the authors show that dimensionality does increase the effect of mass but not the contribution of the pressure, which is the same in any dimension.

The efforts also have been directed to the extension of the analysis of (3+1)-solutions to higher dimensional spacetimes, which also have attracted the attention of the community. The interest is related for example to black hole physics [2], wormhole physics, Kaluza-Klein gravity, multidimensional and string/brane cosmology, among others. It is interesting to note for example, that there is a lack of uniqueness for black holes in higher dimensions, unlike the four dimensional counterparts. Specifically, the higher dimensional rotating black hole metric [3] is not unique, unlike the Kerr geometry in (3+1) dimensions [4].

The showed interest in higher dimensional spacetimes can be extended also to the study of wormhole physics, which has rapidly grown into an active area of research. Euclidean wormholes have been studied by Gonzalez-Diaz and by Jianjun and Sicong [5] for example. The Lorentzian ones have been studied in the context of the N-dimensional Einstein gravity [6] and Einstein-Gauss-Bonnet theory of gravitation [7]. Wormholes in the context of brane worlds are discussed in [8], while the construction of thin-shell electrically charged wormholes in d-dimensional general relativity with a cosmological constant is discussed in Ref. [9].

Evolving higher dimensional wormholes have been studied in Refs. [10] and [11]. The authors of Ref. [10] study wormhole solutions to Einstein gravity with an arbitrary number of time dependent compact dimensions and a matter-vacuum boundary. On the other hand, in Ref. [11] the authors consider non-static wormholes, mainly in 2+1 and 3+1 dimensions, with the required matter satisfying the weak energy conditions. The au-

*mcataldo@ubiobio.cl
†paolameza@udec.cl
‡pminning@udec.cl
thors explore several different scale factors and derive
the corresponding consequences.
In this paper, we shall obtain a family of static and
evolving spherically symmetric wormhole solutions, in
N+1 dimensional gravity, in the presence of a cosmolo-
gical constant and by imposing at least a barotropic
equation of state, with constant state parameter, on the
radial pressure.
The organization of the paper is as follows: In Sec. II
we give some characterization of Lorentzian wormholes.
In Sec. III the field equations for evolving wormholes
in N+1 gravity are formulated. In Sec. IV and Sec. V the
static and non-static N+1 dimensional wormholes are
discussed respectively for N ≥ 3. In Sec. VI the 2+1-
dimensional static and non-static wormholes are treated.
Finally, in Sec. VII we conclude with some remarks.

II. CHARACTERIZATION OF LORENTZIAN
WORMHOLES

On the purely gravitational side, the interest on worm-
hole geometries has been mainly focused on Lorentzian
wormholes, and was especially stimulated by the pioneer-
ing work of Morris, Thorne and Yurtsever [12], where
static, spherically symmetric Lorentzian wormholes were
defined and considered to be an exciting possibility for
constructing time machine models with these exotic ob-
jects. The metric ansatz of Morris and Thorne [13] for
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jects. Finally, in Sec. VII we conclude with some remarks.

The ± signs refer to the two asymptotically flat regions
which are connected by the wormhole. The equality sign
in (2) holds only at the throat.
Constraint 4: Asymptotic flatness condition, i.e. as
l → ±∞ (or equivalently, r → ∞) then b(r)/r → 0 and
φ(r) → 0.
Notice that these constraints provide a minimum set
of conditions which lead, through an analysis of the em-
bedding of the spacelike slice of (1) in a Euclidean space,
to a geometry featuring two asymptotically flat regions
connected by a bridge [13].
In this paper we are not including considerations about
the traversability constraints discussed by Morris and
Thorne [13].

III. N+1–DIMENSIONAL LORENTZIAN
WORMHOLES

A. The metric and the matter source

The evolving spherically symmetric wormhole in higher
dimensions may be obtained by a simple generalization
of the original Morris and Thorne metric [13] to a time-
dependent metric given by

\[ ds^2 = -e^{2\phi(t,r)} dt^2 + a^2(t) \left( \frac{dr^2}{1 - \frac{b(r)}{r}} + r^2 d\Omega_2^2 \right) + \sum_{i=1}^{N-1} \theta^{(\theta_i)} d\theta^{(\theta_i)} . \]  

(4)

where a(t) is the scale factor of the universe, and \( \theta^{(\mu)} \) are
the proper orthonormal basis whose one-forms are given by,

\[ \theta^{(t)} = e^{\phi(t,r)} dt, \]

\[ \theta^{(r)} = a(t) \frac{dr}{\sqrt{1 - \frac{b(r)}{r}}}, \]

\[ \theta^{(\theta_1)} = a(t) r d\theta_1, \]

\[ \theta^{(\theta_2)} = a(t) r \sin \theta_1 d\theta_2, \ldots, \]

\[ \theta^{(\theta_{N-1})} = a(t) r \prod_{i=1}^{N-2} \sin \theta_i d\theta_{N-1}. \]

It must be noticed that the metric [4] includes static
wormholes determined by the condition \( a(t) = a_0 = \text{const.} \). The constant \( a_0 \) may be absorbed by redefining
the shape function in the following form: \( b(r) \rightarrow r - a_0^2 (r - b(r)) \).
In general, for the metric (4), one might introduce a matter source described by an imperfect fluid. Of course in this case the energy-momentum tensor has also non-diagonal entries. However, we shall use the notion of phantom energy in a slightly more extended sense: We shall consider this traditionally homogeneous and isotropic exotic source to be generalized to an inhomogeneous and anisotropic fluid, but still with a diagonal energy-momentum tensor. This means that the only nonzero components of the energy-momentum tensor in this basis are

\[ T_{(t)(t)} = \rho(t, r), \quad T_{(r)(r)} = p_r(t, r) = -\tau(t, r), \]

and

\[ T_{(\theta)(\theta)} = \ldots = T_{(\theta_{N-1})(\theta_{N-1})} = p_l(t, r). \]

where the quantities \( \rho(t, r), \ p_r(t, r), \ \tau(t, r) (= -p_r(t, r)) \), and \( p_l(t, r) (= p_0(t, r)) \) are respectively the energy density, the radial pressure, the radial tension per unit area, and the lateral pressure as measured by observers who always remain at rest at constant \( r, \ \theta_l \).

B. The Einstein field and the conservation equations

For the evolving spherically symmetric wormhole metric (4), the Einstein field equations with cosmological constant \( \Lambda \) are given by

\[
\frac{N(N-1)}{2} e^{-2\phi(t, r)} H^2 + \frac{(N-1)}{2a^2 r^3} (rb' + (N-3)b) = \kappa \rho(t, r) + \Lambda, \tag{7}
\]

\[
\frac{(N-1)e^{-\phi(t, r)}H}{a} \sqrt{\frac{r-b(r)}{r}} \frac{\partial \phi}{\partial r} = 0, \tag{8}
\]

\[- (N-1) e^{-2\phi(t, r)} \left( \frac{N-2}{2} H^2 + \frac{\dot{a}}{a} - H \frac{\partial \phi}{\partial t} \right) - \frac{(N-1)(N-2)b}{2r^3 a^2} - \frac{(N-1)(r-b(r))}{r^2 a^2} \frac{\partial \phi}{\partial r} = \kappa p_r(t, r) - \Lambda, \tag{9}
\]

\[- (N-1) e^{-2\phi(t, r)} \left( \frac{N-2}{2} H^2 + \frac{\dot{a}}{a} - H \frac{\partial \phi}{\partial t} \right) - \frac{rb' + (2N-5)b - (2N-4)r \partial \phi}{2a^2 r^2} \frac{\partial \phi}{\partial r} - \frac{N-2}{2r^2 a^2} (rb' + (N-4)b) + \frac{(r-b)}{a^2 r} \left( \frac{\partial^2 \phi}{\partial r^2} + \frac{(\partial \phi)}{\partial r} \right)^2 = \kappa p_l(t, r) - \Lambda, \tag{10}
\]

where \( \kappa = 8\pi G \), \( H = \dot{a}/a \) and an overdot and a prime denote differentiation \( d/dt \) and \( d/dr \) respectively. Using the conservation equation \( \nabla_{\mu} T^\mu_{\nu} = 0 \) we have that,

\[
\frac{\partial \rho}{\partial t} + H (p_r + (N-1)p_l + N \rho) = 0, \tag{11}
\]

\[
\frac{\partial p_r}{\partial r} + \frac{\partial \phi}{\partial r} (\rho + p_r) - \frac{N-1}{r} (p_l - \omega_r) = 0. \tag{12}
\]

We shall study matter sources described with at least a barotropic equation of state for \( p_r(t, r) \). Thus we can write for the radial pressure

\[
p_r(t, r) = \omega_r \rho(t, r), \tag{13}
\]

where \( \omega_r \) is a constant state parameter. One can require the same for \( p_l(t, r) \). Thus in some cases we shall consider solutions with a lateral pressure given by

\[
p_l(t, r) = \omega_l \rho(t, r), \tag{14}
\]

where \( \omega_l \) is a constant state parameter.

In order to find solutions to the field equations we can see that the equation (8) plays a fundamental role. For the diagonal energy-momentum tensor (3) and (4), Eq. (8) implies that the solutions are separated into two branches: one static branch given by the condition \( H = \dot{a}/a = 0 \) and another non-static branch for \( \partial \phi(t, r)/\partial r = 0 \).

IV. STATIC \( N + 1 \) WORMHOLE SOLUTIONS

In general, for the static case, we shall suppose that the shift and the shape functions, energy density and pressures are functions of the radial coordinate \( r \), and that only the radial pressure has a barotropic equation of state given by \( p_r(t, r) = \omega_r \rho(t, r) \). Then, the required condition for the static branch \( H = \dot{a}/a = 0 \) implies that the field equations take the following form:

\[
\frac{(N-1)}{2r^3} (rb' + (N-3)b) = \kappa \rho(t, r) + \Lambda, \tag{15}
\]

\[- \frac{(N-1)(N-2)b}{2r^3} + \frac{(N-1)(r-b(r))}{r^2 a^2} \frac{d\phi}{dr} = \kappa \omega_r \rho - \Lambda, \tag{16}
\]

\[- \frac{rb' + (2N-5)b - (2N-4)r \frac{d\phi}{dr}}{2a^2 r^2} - \frac{N-2}{2r^2 a^2} (rb' + (N-4)b) + \frac{(r-b)}{a^2 r} \left( \frac{\partial^2 \phi}{\partial r^2} + \frac{(\partial \phi)}{\partial r} \right)^2 = \kappa p_l(t, r) - \Lambda. \tag{17}
\]

In this case the conservation equation takes the form

\[
\omega_l \frac{d\rho}{dr} + \frac{d\phi}{dr} (1 + \omega_r) \rho - \frac{N-1}{r} (p_l - \omega_r \rho) = 0. \tag{18}
\]
In order to solve the field equations we can consider only Eqs. \([15, 16]\) and \([18]\). Thus we have four unknown functions of \(r\), i.e. \(\rho(r), p_t(r), b(r)\) and \(\phi(r)\), for three field equations. In order to construct solutions one can consider restricted choices for \(b(r)\) or \(\phi(r)\). So, we shall construct solutions by finding the lateral pressure \(p_t(r)\) from the conservation equation \([18]\), the energy density \(\rho(r)\) from Eq. \([15]\), and the shape function \(b(r)\) (giving a restricted form of \(\phi(r)\)) or the redshift function \(\phi(r)\) (giving a restricted form of \(b(r)\)) from a differential equation obtained from Eqs. \([15, 16]\). Thus, from Eqs. \([15]\) and \([16]\) we find that the shape function may be written in the form

\[
b(r) = \frac{C r^{2-N} (N-3)}{e^{2\phi(r)/\omega_r}} + \frac{2r^{2-N} (N-3)}{\omega_r (N-1) e^{2\phi(r)/\omega_r}} \int \left( (N-1) \phi' + \Lambda (1 + \omega_r) r \right) r^{-1} e^{2\phi(r)/\omega_r} \, dr,
\]

or equivalently the redshift function as

\[
\phi(r) = \int \left( \omega_r (N-3) + N - 2 \right) (N-1) b(r) r - 2 (1 + \omega_r) \Lambda r^3 + (N-1) \omega_r r b r.
\]

Now we shall consider specific static wormhole solutions.

**A. \(b(r) \sim r^\alpha\) solution**

First, let us consider the case \(b(r) = r_0 (r/r_0)\alpha\) with \(\Lambda = 0\). This choice permits us to consider the possibility of having an asymptotically flat space-time. By putting these expressions into Eq. \([20]\) we have that the shift function takes the form

\[
e^{\phi(r)} = \left( 1 - \left( \frac{r}{r_0} \right)^{1-\alpha} \right)^{\frac{N(1+\omega_r)-2-3\omega_r+\omega_r\alpha}{4(1-\alpha)}}, \quad (21)
\]

then the metric and the energy density are given by

\[
ds^2 = - \left( 1 - \left( \frac{r_0}{r} \right)^{1-\alpha} \right)^{\frac{N(1+\omega_r)-2-3\omega_r+\omega_r\alpha}{4(1-\alpha)}} \, dt^2 + \frac{dr^2}{1 - \left( \frac{r_0}{r} \right)^{1-\alpha}} + r^2 d\Omega^2_{N-1}, \quad (22)
\]

\[
\kappa \rho(r) = \frac{(N-1)(1+N-2)}{2\omega_r r_0^2} \left( \frac{r_0}{r} \right)^{3-\alpha},
\]

respectively. Note that by making \(N = 3\) we obtain the 3+1-wormhole solution discussed by Lobo in Ref. \([16]\).

From the metric \([22]\) we conclude that we have an asymptotically flat space-time if \(1 - \alpha > 0\) and \(N(1 + \omega_r) - 2 - 3\omega_r + \omega_r\alpha > 0\), since in this case we have that \(r^{\phi(r)} \to 1\) and \(b(r)/r \to 0\) for \(r \to \infty\). However this \(N\)-dimensional space-time is a non-traversable wormhole since an event horizon is located at \(r = r_0\). It is interesting to note that for \(N \geq 3\) this non-traversable wormhole may have a positive energy density by requiring that \(-\alpha < \alpha < 1\). In order to make this space-time a traversable wormhole we must require that \(e^{\phi(r)} = 1\) which implies that \(N(1 + \omega_r) - 2 - 3\omega_r + \omega_r\alpha = 0\), or equivalently \(\alpha = \frac{2+3\omega_r-N(1+\omega_r)}{N}\). Finally, the metric and the energy density take the following form:

\[
ds^2 = dt^2 - \frac{dr^2}{1 - \left( \frac{r_0}{r} \right)^{1+N-2}} - r^2 d\Omega^2_{N-1}, \quad (24)
\]

\[
\kappa \rho(r) = \frac{(N-1)(2-N)}{2\omega_r r_0^2} \left( \frac{r_0}{r} \right)^{\frac{N(1+\omega_r)-2}{\omega_r}} \cdot (25)
\]

Among \(p_t(r) = \omega \rho(r)\), we have for the lateral pressure

\[
\kappa p_t(r) = \frac{(N-2)(\omega_r + N - 2)}{2\omega_r r_0^2} \left( \frac{r_0}{r} \right)^{\frac{N(1+\omega_r)-2}{\omega_r}} . (26)
\]

Note that for \(\omega_r < -1\) and \(\omega_r > 0\) we have asymptotically flat wormholes, while for \(-1 < \omega_r < 0\) this does not occur. On the other hand, for dimensions \(N \geq 3\) we have a positive energy density if \(\omega_r < 0\), and a negative energy density for \(\omega_r > 0\). Thus we conclude that for \(\omega_r > 0\) we have asymptotically flat wormholes with negative energy density, while for \(\omega_r < -1\) (or \(-1 < \omega_r < 0\)) asymptotically flat wormholes supported by an energy of phantom (or quintessence) type, since we have a negative radial pressure \(p_r\). In this case, for \(N \geq 3\), the energy density \(\rho \to 0\) for \(r \to \infty\) since for \(\omega_r < -1\) (\(\omega_r > 0\)) always the inequality \(N(1+\omega_r)-2 < 0\) \((N(1+\omega_r)-2 > 0)\) takes place.

Non asymptotically flat solutions may be obtained by requiring \(\Lambda \neq 0\). These spacetimes are expressed through hypergeometric functions.
B. \( \phi(r) = 0 \) solution

Now we shall consider the general static solution for a barotropic radial pressure of the form (13) with \( \phi(r) = 0 \). Clearly, this is the more natural choice for the shift function, in order to have a finite \( e^\phi(r) \) throughout all the space–time. Thus by putting \( \phi(r) = 0 \) into Eq. (19) we find that

\[
b(r) = r_0 \left( \frac{r}{r_0} \right)^{\frac{2-N}{2} \omega_r - (N-3)} + \frac{2 \Lambda (\omega_r + 1) r^3}{(\omega_r N + N - 2)(N - 1)},
\]

and from Eqs. (16) and (18) we obtain for the energy density and the dimensional constraint the following expressions:

\[
\kappa \rho(r) = \left( \frac{N-1}{2} \right) \left( \frac{r_0}{r} \right) \left( \frac{N (\omega_r + 1) - 2}{2 \omega_r r_0^2} \right) + \frac{2 \Lambda}{(N (\omega_r + 1) - 2)},
\]

\[
\kappa \rho_{0}(r) = \left( \frac{N-2}{2} \right) \left( \frac{r_0}{r} \right) \left( \frac{N (\omega_r + 1) - 2}{2 \omega_r r_0^2} \right) + \frac{2 \Lambda \omega_r}{(N (\omega_r + 1) - 2)}.
\]

Clearly for \( \Lambda = 0 \) we obtain the previous asymptotically flat wormhole solution given by Eqs. (24)-(26). For \( \Lambda \neq 0 \) we do not have two asymptotically flat regions, and may have wormholes with two asymptotically de Sitter regions or two asymptotically anti-de Sitter regions, since as \( r \to \infty \) the cosmological term dominates. In other words, for very large values of the radial coordinate \( r \) the large-scale curvature of the spacetime must be taken into account [17]. On the other hand, it is remarkable that in this case we can have positive energy density not only for \( \omega_r < 0 \), but also for positive values of the state parameter \( \omega_r \) by requiring that \( \Lambda > \frac{4 \omega_r}{(N-1)(2-N)} (N (\omega_r + 1) - 2) \).

It is interesting to note that if we require that the lateral pressure has also a barotropic equation of state given by Eq. (14), i.e. \( p_l(r) = \omega_l \rho(r) \), then we obtain that the cosmological constant must vanish, i.e. \( \Lambda = 0 \), and the dimensional constraint

\[
N (\omega_r + 1) - \omega_l + \omega_r = 2
\]

must be fulfilled. In this case the lateral pressure may be written as

\[
p_l(r) = \left( \frac{2 - N - \omega_r}{N - 1}\right) \rho(r) = \frac{(2 - N - \omega_r)(2 - N)}{2 \omega_r r_0^2} \left( \frac{r_0}{r} \right)^{N(\omega_r + 1) - 2}.
\]

Thus, static wormhole configurations with \( e^\phi(r) = 1 \) and anisotropic barotropic pressure with constant state parameters are always, in any dimension \( N \neq 3 \), asymptotically flat spacetimes.

C. Pressure with constant state parameters

Now we shall consider the case \( \phi(r) \neq 0 \) and barotropic pressure with constant state parameters (13) and (14). This means that now we have three field equations for three unknown functions \( b(r) \), \( \phi(r) \), and \( \rho(r) \). By putting \( p_l(r) = \omega_l \rho(r) \) into Eq. (18) we obtain that the energy density is given by

\[
\rho(r) = C r^{(N-1)(\omega_l - \omega_r) - 2} e^{-\frac{(1+\omega_r)(\rho(r))}{\omega_r}},
\]

where \( C \) is an integration constant. From Eq. (15), and by taking into account the Eq. (32), we find that

\[
b(r) = 2r^{N+3} \int \left( \kappa C + \frac{N (\omega_l - \omega_r) - 2}{2 \omega_r (1+\omega_r)} + \Lambda \right) r^{N-1} dr
\]

\[
+ C_1 r^{-N-3},
\]

where \( C_1 \) is a new integration constant. In order to have the general solution for this case we can find the function \( \phi(r) \) by solving the differential equation \((10)\), by putting into it the expressions \((32)\) and \((33)\). Unfortunately, the obtained differential equation for \( \phi(r) \) is too complicated, so we shall give a particular solution by considering a restricted choice of the shift function \( \phi(r) \).

Let us consider the case

\[
e^\phi(r) = \left( \frac{r}{r_0} \right)^{\alpha}.
\]

This choice clearly may ensure the absence of horizons and singularities for \( 0 < r_0 < r < \infty \), since the shift function is finite throughout the space–time. In this case the field equations require that \( \Lambda = 0 \), so in order to have a shift function of the form \((34)\) the cosmological constant must vanish. Thus the metric takes the form

\[
ds^2 = -\left( \frac{T_0}{r} \right)^{N-2} dt^2 + \frac{dr^2}{1 + \frac{1}{\omega_r} - \frac{c}{(r_0)} r^2} + r^2 d\Omega_{N-1}^2,
\]

where \( C \) is a constant of integration, the energy density is given by

\[
\kappa \rho(r) = -\frac{(N - 1)(N - 2)}{2 r_0^2 \omega_r} \left( \frac{r_0}{r} \right)^{2},
\]

and the constraint

\[
\omega_l = \frac{-N + N \omega_r + 4 \omega_r}{2(N - 1)}.
\]

was used. Clearly for dimensions \( N \geq 3 \) we have a positive energy density if \( \omega_r < 0 \), and a negative energy density for \( \omega_r > 0 \).

One can rewrite the wormhole metric \((35)\) in a more appropriate form. From the condition \( g_{rr}(r = r_0) = 0 \),
we obtain for the integration constant $C = 1 + 1/\omega_r$. Then, we have for the metric component $g_{rr}^{-1}(r) = (1 + 1/\omega_r)(1 - 1/(r/r_0)(N - 2))$, and the wormhole throat is located at $r_0$.

Note that this wormhole is not asymptotically flat, so in order to this wormhole connects two different asymptotically flat regions we need to match this solution to an exterior N-dimensional vacuum spacetime, i.e. to the N+1-dimensional Schwarzschild solution [3].

This wormhole spacetime has an interesting feature to be remarked: if we rescale the coordinate time $t$ and the radial coordinate $r$ we can rewrite the metric to

$$ds^2 = -\left(\frac{r_0}{r}\right)^{N-2}dt^2 + \frac{dr^2}{\left(\frac{r_0}{r}\right)^N} + \left[1 + \frac{1}{\omega_r}\right]r^2d\Omega^2_{N-1},$$

This metric has a solid angle deficit, which depends on the value of the state parameter.

It is interesting to note that if we want an isotropic solution with the shift function of the form (34), we obtain from Eq. (37), by putting $\omega_r = \omega_l$, that

$$\omega = \frac{2 - N}{2 + N}.$$  \hspace{1cm} (39)

Thus the metric is given by

$$ds^2 = -\left(\frac{r_0}{r}\right)^{N-2}dt^2 + \frac{dr^2}{4(2-N)}\left(\frac{r_0}{r}\right)^N + r^2d\Omega^2_{N-1},$$

where the energy density and the isotropic pressure are given

$$\rho(r) = \frac{(N-1)(N+2)}{2\kappa r_0^2\left(\frac{r_0}{r}\right)^2},$$

$$p(r) = \frac{(N-1)(2-N)}{2\kappa r_0^2\left(\frac{r_0}{r}\right)^2}.$$  \hspace{1cm} (40)

This energy density is always positive. In order to have a wormhole we must require $C = \frac{1}{2+N}$, ensuring that $g_{rr}^{-1}(r = r_0) = 0$. However, it can be shown that in this case the metric component $g_{rr}$ is positive for $0 < r_0 < r_0$ and negative for $r > r_0$, so the metric (40) does not represent a wormhole spacetime for $N \geq 3$.

Now we shall consider the non-static branch of the solutions. As we stated above in order to have non-static wormholes we must require $\partial \phi(t, r)/\partial r = 0$. This condition implies that the redshift function can only be a function of $t$, i.e. $\phi(t, r) = f(t)$, and then we can rescale the time coordinate $t$, so without any loss of generality we shall require $\phi(t, r) = f(t) = 0$. In this case we are interested in solutions having a barotropic anisotropic pressure with constant state parameters given by Eqs. (13) and (14).

Now, from the conservation equation (12) we obtain

$$\rho(t, r) = \rho_0a^{-(\omega_r + (N-1)\omega_l + N)}r^{\frac{N-1}{2}(\omega_l - \omega_r)},$$

where $\rho_0$ is an integration constant, and by subtracting equations (9) and (10) we have that

$$N - 2\sqrt{2\kappa^2 \rho_0} (rb' - 3b) = \frac{\kappa \rho_0 \omega_r}{a(\omega_r + (N-1)\omega_l + N)}.$$  \hspace{1cm} (45)

It is straightforward to see that in order to have a solution for the shape function $b = b(r)$ the following constraint must be imposed:

$$\omega_r + (N-1)\omega_l + N = 2,$$  \hspace{1cm} (46)

on the state parameters $\omega_r$ and $\omega_l$, thus obtaining for the shape function and the energy density

$$b(r) = C_1r^3 - \frac{2\kappa \rho_0 \omega_r}{(N-2)(N-1)}r^{\frac{N-1}{2}(\omega_l - \omega_r)},$$

$$\rho(t, r) = \rho_0 a^{-\frac{N-1}{2}(\omega_r - \omega_l)}.$$  \hspace{1cm} (47)

where $C_1$ is an integration constant. Now, from equation (17) we obtain the following differential equation for the scale factor

$$\dot{a} = \pm \sqrt{\frac{2\Lambda a^2}{N(N-1)} - C_1}.$$  \hspace{1cm} (49)

The solution for this equation depends on the signs of the cosmological constant $\Lambda$ and the integration constant $C_1$, as we display in Table II.

Now, it must be noted that the radial coordinate in this
solution may be rescaled in order to absorb the integration constant \( C_1 \). In this case the metric is given by
\[
\frac{dr^2}{1 - k r^2} + \frac{2 \kappa \rho}{\omega r} r^{N-2(1+\omega)} + r^2 d\Omega^2_{N-1},
\]
where \( k = 0 \) for \( C_1 = 0 \), \( k = -1 \) for \( C_1 < 0 \) and \( k = 1 \) for \( C_1 > 0 \). We summarize all possible scale factors for the found wormhole solutions in Table II.

| \( a(t) \) | \( k \) | \( A \) |
|-----------------|--------|--------|
| const           | 0      | 0      |
| \( t + const \) | -1     | 0      |
| \( a_0 e^{\pm \sqrt{N(N-1)/2}t} \) | 0 > 0  | 0 > 0  |
| \( \sqrt{N(N-1)/2} t + \phi_0 \) | -1 < 0 | -1 < 0 |
| \( \sqrt{N(N-1)/2} t + \phi_0 \) | 1 > 0  | 1 > 0  |

TABLE II: The table shows all the possible scale factors for the general solution (55) of an evolving Lorentzian wormhole in \( N+1 \) dimensions with the radial tension and the tangential pressure having barotropic equations of state with constant state parameters.

Now we shall rewrite the wormhole metric (55) in a more appropriate form. From the condition \( g_{rr}^{-1}(r = r_0) = 0 \), we obtain for the integration constant \( \rho_0 \)
\[
\rho_0 = \frac{(N - 1)(N - 2)(kr_0^2 - 1) N-2 + (N-3)\omega r}{2\kappa \omega r} t_0^{N-2 + (N-3)\omega r},
\]
yielding for the shape function and the metric component \( g_{rr} \)
\[
b(r) = r_0 \left( \frac{r}{r_0} \right)^{\frac{N - 2 + (N-3)\omega r}{\omega r}} + \\
k r^3 \left( 1 - \left( \frac{r}{r_0} \right)^{\frac{N(1+\omega r) - 2}{\omega r}} \right),
\]
and
\[
a^2(t) g_{rr}^{-1} = 1 - \left( \frac{r}{r_0} \right)^{\frac{(N-2)(1+\omega r)}{\omega r}} -
\]
respectively. This implies that the wormhole throat is located at \( r_0 \), and the energy density is given by
\[
\kappa \rho(t, r) = \frac{(N - 1)(N - 2)(kr_0^2 - 1) N-2 + (N-3)\omega r}{2\kappa \omega r} t_0^{N-2 + (N-3)\omega r}.
\]

It is clear that for \( N \geq 3 \) the cosmological constant directly controls the behavior of the scale factor \( a(t) \) and not the shape function \( b(r) \), which mainly is controlled by the state parameter \( \omega_r \). In order to have an evolving wormhole we must require \( \omega_r < -1 \) or \( \omega_r > 0 \) (in both of these cases, in the \( g_{rr} \) metric component (55), \( (N - 2)(1 + \omega_r)/\omega_r > 0 \) and \( (N(1 + \omega_r) - 2)/\omega_r > 0 \), implying that the phantom energy can support the existence of evolving wormholes in the presence of a cosmological constant, and the energy density vanishes at \( r \to \infty \). On the other hand, clearly for \( \omega_r < -1 \) or \( \omega_r > 0 \) the metric (55) at spatial infinity \( (r \to \infty) \) has slices \( t = const \) which are \( N \)-dimensional spaces of constant curvature: open for \( k = -1 \), flat for \( k = 0 \) and closed for \( k = 1 \). This implies that for \( r \to \infty \) the metric (55) is foliated with spaces of constant curvature.

Now some words about the energy conditions. It is well known that, in all cases, the violation of the weak energy condition (WEC)
\[
\rho \geq 0, \quad p_\rho + p_t \geq 0,
\]
is a necessary condition for a static wormhole to exist. In the case of our non-static solution we must require \( \omega_r < -1 \) or \( \omega_r > 0 \) in order to have evolving wormholes. Thus in general, for \( k = 0 \) or \( k = -1 \) and \( \rho > 0 \), we must require \( \omega_r < -1 \), so the WEC is always violated (this is independent of the value of the cosmological constant), while for \( k = 1 \) (in the case \( \Lambda > 0 \) and \( \rho > 0 \), we may require \( \omega_r < -1 \) for \( r^2 < 1 \) (and the WEC is always violated) or require \( 0 < \omega_r < 1 \) for \( r^2 > 1 \) (and the violation of WEC is avoided). Unfortunately this latter case must be ruled out for the consideration of evolving wormhole configurations.

For \( N = 3 \) we obtain the evolving wormhole solutions discussed in Refs. [18] and [19], where the traversability criteria for these four dimensional wormholes are also considered.

### VI. 2+1 EVOLVING LORENTZIAN WORMHOLES

As we can see from evolving \( N+1 \) wormhole solutions the shape function \( b(r) \) in Eq. (47) is not well defined for \( N = 2 \). On the other hand, by studying more accurately the field equations for evolving wormholes in any dimensions (11)–(14) we conclude that the nature of
such wormholes for \( N = 2 \) and \( N \geq 3 \), are quite different. Effectively, in this case the condition \( \phi(r) = 0 \) must be required. Thus, the pressure sustaining the three-dimensional evolving wormholes must be always isotropic and homogeneous, i.e. of the form \( p_r(t) = p_t(t) = p(t) \), while for \( N \geq 3 \) the pressure must be always inhomogeneous and anisotropic, i.e. given by \( p_r(t, r) \) and \( p_t(t, r) \). In other words, it can be seen from the field equations \( (11) \) that for \( N \geq 3 \) the requirements \( \phi(r) = 0 \), \( p_r = p_r(t) \) and \( p_t = p_t(t) \) immediately implies that the shape function \( b(r) \) must vanish, and then we must consider a pressure of the form \( p_r(t, r) \) and \( p_t(t, r) \). For \( N = 2 \) clearly this does not occur.

So now we shall discuss separately wormhole spacetimes in 2+1 dimensions.

In three dimensional gravity the metric for an evolving wormhole is given by

\[
ds^2 = -e^{2\phi(t,r)}dt^2 + a^2(t) \left( \frac{dr^2}{1 - 2\phi/r^2} + r^2 d\theta^2 \right). \quad (56)
\]

The field equations may be directly obtained by putting \( N = 2 \) into Eqs. \( (7)-(12) \). As in the \( N \)-dimensional case the solutions are separated into two branches: one static branch given by the condition \( a(t) = \text{const} \) and another non-static branch for \( \partial \phi(t,r)/\partial r = 0 \).

\section*{A. Static three-dimensional branch}

In general, for the static branch the solutions may be obtained by putting \( N = 2 \) into Eqs. \( (10) \) and \( (20) \), and then we can impose a restricted form of the shape function \( b(r) \) or the redshift function \( e^{\phi(r)} \). For \( \phi(r) = 0 \) we obtain that the pressure is isotropic and constant, and in the presence of the cosmological constant takes the value \( \kappa p_r = \kappa p_t = \Lambda \). In this case the energy density is given by

\[
\kappa \rho(r) = \frac{rb^\prime - b}{2r^3} - \Lambda. \quad (57)
\]

Note that a static wormhole sustained by a pressureless fluid is only possible in 2+1 (i.e. \( \Lambda = 0 \)). For dimensions \( N \geq 3 \), the requirements \( \phi(r) = 0 \) and \( p_r = p_t = 0 \) implies that the shape function must vanish. Let us now consider a specific asymptotically flat wormhole given by \( b(r) = r_0(r/r_0)^\alpha \). Thus the pressureless fluid has an energy density given by

\[
\kappa \rho(r) = \frac{(\alpha - 1)}{2r_0^2(r/r_0)^{3-\alpha}} - \Lambda. \quad (58)
\]

In this case, we must require \( \alpha < 1 \) in order to have a three-dimensional wormhole, thus for \( \Lambda = 0 \) the energy density is always negative. For \( \Lambda \neq 0 \) we can demand that \( \Lambda < \frac{\alpha - 1}{2r_0^2} < 0 \) in order to have a positive energy density for \( r \geq r_0 \).

On the other hand, for the shape function given by \( b(r) = r_0(r/r_0)^\alpha \) and \( \phi(r) \neq 0 \) the solution has the form \( (21)-(23) \) with \( N = 2 \), obtaining a 2+1-dimensional non-traversable wormhole with an event horizon located at \( r_0 \). In this case the energy density is always negative, to the contrary of the real possibility of having a positive energy density for \( N \geq 3 \).

It must be noted that three-dimensional static wormhole configurations are discussed by Perry and Mann in Ref. \( [20] \), where the constraints on the field equations to obtain wormholes are presented and further constraints on traversibility are discussed.

\section*{B. Non-static three-dimensional branch}

Let us now discuss the 2+1-non-static branch with \( \partial \phi(t,r)/\partial r = 0 \). In this case the Einstein field equations are given by

\[
H^2 + \frac{(rb^\prime - b)}{2r^3a^2} = \kappa \rho + \Lambda. \quad (59)
\]

\[
-\frac{\ddot{a}}{a} = \kappa p_r - \Lambda. \quad (60)
\]

\[
-\frac{\ddot{a}}{a} = \kappa p_t - \Lambda. \quad (61)
\]

\[
\frac{\partial \rho}{\partial t} + H (p_r + p_t + 2\rho) = 0. \quad (62)
\]

\[
\frac{\partial p_r}{\partial r} - \frac{(p_t - p_r)}{r} = 0. \quad (63)
\]

It’s clear from Eqs. \( (60) \) and \( (61) \) that only an isotropic pressure is permitted, then we shall write \( p_r = p_t = p \). Thus from Eq. \( (63) \) we obtain that \( \partial p/\partial r = 0 \), which implies that the pressure has the general form \( p = p(t) \). In the following we shall discuss the cases \( p = \text{const} \) and \( p = p(t) \).

1. Case \( p = \text{const} \) and \( \rho(t,r) \)

By putting \( p = \text{const} \) into Eqs. \( (60) \) and \( (62) \) we obtain for the scale factor and the energy density

\[
a(t) = C_1 \sin(\sqrt{\kappa p - \Lambda} t) + C_2 \cos(\sqrt{\kappa p - \Lambda} t), \quad (64)
\]

\[
\rho(t,r) = \frac{\kappa \rho_0(r)}{a^2} - p, \quad (65)
\]

respectively. Now by putting the scale factor and energy density from Eqs. \( (64) \) and \( (65) \) into Eq. \( (59) \) we obtain the following expression for the shape function:

\[
\frac{b(r)}{r} = r^2 \left( C_1^2 + C_2^2 \right) (\Lambda - \kappa p) + 2\kappa \int r \rho_0(r) dr + C, \quad (66)
\]
where $C$ is an integration constant. By giving a restricted form of $\rho_0(r)$ we can obtain the shape function. One can also impose a restricted form of $b(r)$ and obtain the form of the function $\rho_0(r)$ with the help of the expression

$$
\kappa \rho_0(r) = \frac{1}{2r} \left( \frac{b(r)}{r} \right)' + \left( C_1^2 + C_2^2 \right) (\kappa p - \Lambda),
$$

(67)

where $' = d/dr$. Let us consider a specific wormhole solution given by $b(r) = r_0(r/r_0)^{\alpha}$. Thus we obtain

$$
\kappa \rho_0(r) = \frac{(\alpha - 1)}{2r_0^2(r/r_0)^{1-\alpha}} + \left( C_1^2 + C_2^2 \right) (\kappa p - \Lambda). \tag{68}
$$

As we know $\alpha < 1$ for having $b(r)/r \leq 1$, and in order to have a positive energy density we must require that $\kappa p > -\frac{2\alpha (C_1^2 + C_2^2)}{\alpha - 1} + \Lambda$. Note that in this case even for $\Lambda = 0$ we can have $\rho(r) > 0$.

It is interesting to note that this wormhole, for every slice $a(t) = a_0 = const$, and for $p = \Lambda = 0$ with $\alpha = 1/2$, reproduces the asymptotically flat and static wormhole solution discussed in Ref. [20].

2. Case $p = p(t)$ and $\rho(t,r)$

By direct integration of Eq. (62) we have that

$$
\kappa p(t,r) = \frac{F(r)}{a^2(t)} - \frac{2\kappa}{a^2(t)} \int \left( a(t) p(t) \frac{da(t)}{dt} \right) dt. \tag{69}
$$

Since $p(t)$ is an arbitrary function, this implies that the general form of the energy density is $\rho(t,r) = \frac{F(r)}{a^2(t)} + C(t)$.

By putting the expression $p(t) = -\dot{a}(t)/a(t) + \Lambda$ into Eq. (69) we obtain finally

$$
\kappa p(t,r) = \frac{F(r)}{a^2(t)} + \left( \frac{\dot{a}}{a} \right)^2 - \Lambda. \tag{70}
$$

By substituting $\rho(t,r)$ from the above equation in Eq. (62) we obtain that

$$
F(r) = \frac{1}{2r} \left( \frac{b(r)}{r} \right)'. \tag{71}
$$

Clearly, for having a solution we must give a restricted form of the pressure in order to find the form of the scale factor. For example, for $p(t) = p = const$ we obtain the discussed above solution (34), (42) and (47). If we consider the pressure given by $p(t) = t^\omega$, then the scale factor takes the form $a(t) = C_1 t^{1/2(1+\sqrt{1-4\Lambda})} + C_2 t^{1/2(1-\sqrt{1-4\Lambda})}$ for $\alpha = -2$, and

$$
a(t) = C_1 \sqrt{t} BesselJ \left( \frac{\alpha + 2}{2} \right) (t^\omega) + C_2 \sqrt{t} BesselY \left( \frac{\alpha + 2}{2} \right) (t^\omega), \tag{72}
$$

for $\alpha \neq -2$, where $BesselJ$ and $BesselY$ are the Bessel functions of the first and second kinds respectively. In order to have a traversable wormhole we must give a restricted form of the shape function satisfying all wormhole constraints, as for example, $b(r) = r_0(r/r_0)^{\alpha}$. Clearly in this case from Eq. (71) we have $F(r) < 0$ for $\alpha < 1$, thus the energy density (70) may be negative or positive during the evolution. This mainly depends on the relation between the terms $F(r)/a^2$ and $H^2$. As an example let us consider the case of a power law scale factor $a(t) = t^\omega$. In this case the energy density will be given by

$$
\kappa \rho(r) = \frac{(\alpha - 1)}{2r_0^2(r/r_0)^{3-\alpha} t^{2\beta}} + \frac{\beta^2}{t^2} - \Lambda. \tag{73}
$$

Clearly in this case for $\Lambda \neq 0$ there exist values of the parameters which ensure the positivity of the energy density during all evolution of the scale factor. For $\Lambda = 0$ we always can find a value of the cosmic time $t = t_0 > 0$, where the energy density vanishes, thus $\rho(t,r)$ may be negative for $0 < t < t_0$ or $t > t_0$.

It must be noticed that the case $p(t)$ and $\rho(t)$ must be excluded from consideration since in this case we must require that $b(r) = 0$, and then we can not have a wormhole configuration.

VII. DISCUSSION

In this paper we have obtained $N+1$-dimensional solutions for the Einstein field equations which describe static and evolving spherically symmetric Lorentzian wormholes. In general, for static wormholes, we require that at least the radial pressure has a barotropic equation of state of the form $p_r = \omega_r \rho$, where the state parameter $\omega_r$ is constant, and for evolving wormholes we also require a barotropic equation of state $p_1 = \omega_1 \rho$ with constant state parameter $\omega$ for the lateral pressure.

For static wormholes it is shown that, in any dimension $N \geq 3$, with $\phi(r) = \Lambda = 0$ and anisotropic barotropic pressure with constant state parameters, they are always asymptotically flat spacetimes, while in $2+1$ gravity the static wormholes may have more general asymptotic spaces. In this case, the matter sustaining the three-dimensional wormhole may be only a pressureless fluid.

The nature of evolving wormholes in $2+1$-dimensions and $N+1$-dimensions, with $N \geq 3$, are quite different. For evolving wormholes in any dimensions the condition $\phi(r) = 0$ must be required. However, this constraint implies that the pressures sustaining the tree-dimensional wormhole configurations must be always homogeneous, i.e. only must depend on the cosmological time, while for $N \geq 3$ these pressures must be always inhomogeneous, i.e. of the form $p(t,r)$, in order to have an evolving wormhole spacetime. This can be seen from the field equations (77)-(10) (for $N \geq 3$) by requiring $\phi(r) = 0$, $p_r = p_r(t)$ and $p_1 = p_1(t)$. This immediately implies that...
the shape function \( b(r) \) must vanish. In the case of evolving wormholes with \( N \geq 3 \), the presence of a cosmological constant leads to an expansion or contraction of the wormhole configurations: for positive cosmological constant we have wormholes which expand forever and, for negative cosmological constant we have wormholes which expand to a maximum value and then recollapse. In the absence of a cosmological constant the wormhole expands with constant velocity, i.e. without acceleration or deceleration. In 2+1 dimensions the expanding wormholes always have an isotropic and homogeneous pressure, depending only on the time coordinate.

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