STACKED CENTRAL CONFIGURATIONS WITH A HOMOGENEOUS POTENTIAL IN $\mathbb{R}^3$

YANGSHANSHAN LIU\textsuperscript{1} AND SHIQING ZHANG\textsuperscript{1∗}

\textsuperscript{1}Department of Mathematics, Sichuan University, Chengdu 610065, P.R. China

∗Corresponding author; E-mail: zhangshiqing@scu.edu.cn

Abstract. In this paper we generalize some results in [40] concerning stacked central configurations. We can deal with the general homogeneous potential $U_\alpha$ (containing the vortex case) in $\mathbb{R}^3$. We give the admissible set of $\alpha$ for a convex central configuration (with respect to the Newtonian potential i.e. $\alpha = 3$). We discuss some properties of the regular $n$-gon co-circular central configurations. We also find that the stacked property is particular for central configurations by studying the $S$-balanced configuration case.

stacked central configurations, celestial mechanics, $(n + 1)$-body problems, homogeneous potentials

Mathematics Subject Classification: 70F10, 70G10

1. Introduction

The classical $n$-body problems concern the motion of $n$ point particles with masses $m_i \in \mathbb{R}^+$ and positions $x_i = (x_{i,1}, \ldots, x_{i,d})^T \in \mathbb{R}^d (i = 1, \ldots, n; d = 2, 3)$ interacting under a central potential $U_\alpha$ given by

$$m_i \ddot{x}_{i,j} = \frac{\partial U_\alpha}{\partial x_{i,j}}, \quad i = 1, \ldots, n, \quad j = 1, \ldots, d,$$

where

$$U_\alpha = \sum_{i<j} \frac{m_i m_j}{r_{i,j}^{\alpha-2}}, \quad \alpha > 2,$$

and $r_{i,j} = |x_i - x_j|$ is the mutual distance between $x_i$ and $x_j$. The Newtonian potential corresponds to $\alpha = 3$. One can extend this $U_\alpha$ to the case $\alpha = 2$ via the logarithmic potential

$$U_2 = -\sum_{i<j} m_i m_j \ln(r_{i,j}),$$

which derives from the fluid vortices, where the parameters $m_i$, often denoted by $\Gamma_i$, i.e. the circulation (or vorticity), represent the strength of a vortex rotation and can be any real value. One can refer to [11] for some related terminologies and to [33] for more background and details. Here for convenient discussion, we only concentrate on the Newtonian case, i.e. $m_i > 0$. We call $x = (x_1, \ldots, x_n) \in \mathbb{R}^{dn} \setminus \Delta$ a collision free configuration in the configuration space $\mathbb{R}^{dn}$, with the singular set $\Delta = \{ x \in \mathbb{R}^{dn} | x_i = x_j, \forall i \neq j \}$ ruled out. Thus, a central configuration can be defined for $\alpha \geq 2$ as a special arrangement of the $n$ distinct positions $x_i$ satisfying the
nonlinear algebra equations

\[
\begin{align*}
\alpha > 2, & \quad \sum_{j \neq i, j=1}^{n} \frac{m_i m_j (x_j - x_i)}{r_{i,j}^\alpha} + \frac{\lambda}{\alpha - 2} m_i (x_i - c) = 0, \quad i = 1, \ldots, n, \\
\alpha = 2, & \quad \sum_{j \neq i, j=1}^{n} \frac{m_i m_j (x_j - x_i)}{r_{i,j}^2} + \lambda m_i (x_i - c) = 0, \quad i = 1, \ldots, n,
\end{align*}
\]

where

\[c = \frac{1}{m} \sum_{i=1}^{n} m_i x_i\]

is the center of mass, with \(m = \sum_{i=1}^{n} m_i\) the total mass. A more compact form of (1) is

\[
\nabla U_\alpha(x) + \lambda \nabla I(x) = 0,
\]

where

\[I(x) = \frac{1}{2} \sum_{i=1}^{n} m_i |x_i - c|^2 = \frac{1}{2m} \sum_{1 \leq i < j \leq n} m_i m_j r_{i,j}^2\]

is the moment of inertia of the system for \(\alpha > 2\) (resp. one-half the angular impulse for \(\alpha = 2\)), and by denoting \(L = \sum_{1 \leq i < j \leq n} m_i m_j\), i.e. the total angular vortex momentum for \(\alpha = 2\), the Lagrangian multiplier (or configuration constant) satisfies

\[2\lambda = \begin{cases} 
(\alpha - 2)U_\alpha(x)/I(x), & \alpha > 2, \\
L/I(x), & \alpha = 2.
\end{cases}\]

Central configurations play a key role in the Newtonian \(n\)-body problems, since they are used to construct self-similar (or homographic) solutions which may be the only explicit solutions for this problem and are also deeply relevant to the topology of the integral manifolds and to collision cases. When \(d = 2\) one may get a planar version usually called the relative equilibria (or the rigid motions). Since the system (1) (or (2)) is invariant under dilations, translations and rotations (or reflections), we usually deal with the equivalence classes of central configurations after defining an equivalence relation on the space \(\mathbb{R}^{dn}\setminus\Delta\), eliminating these symmetries.

There are several open problems around central configurations, and one of the most challenging problems is the conjecture about the finiteness of the number of equivalence classes of the central configurations (or the relative equilibria in \(\mathbb{R}^2\)) for any choice of \(n\) positive masses (especially for \(\alpha = 3\), the Newtonian case), which was collected in [36] by Smale as the sixth problem for the next century and in [2] by Albouy et al. the ninth open problem in celestial mechanics, respectively. The answer is obvious for \(n = 2\) and 3, and for \(n = 3\) it is well known that there are at most five equivalence classes of central configurations, three Eulerian collinear central configurations and two Lagrangian equilateral triangle central configurations (up to rotations, and only one up to reflections). For \(n = 4\), an upper bound 8472 was given by Hampton and Moeckel [23] until 2005. For \(n = 5\), only the generic finiteness has been solved both in the spacial case by Hampton and Jensen [22] and the planar case by Albouy and Kaloshin [4]. For \(n \geq 6\), less is known. Although the basic problem is not easy to attack, there have been quite a few nice results relevant to the finiteness problems with kinds of constraints (or symmetries) added, namely, either the fixed shapes or the mass conditions and so on. For example, for the collinear \(n\)-body problem, the answer is positive, i.e. the number is explicitly \(n! / 2\) (Moulton [32]). One can refer to [20,24,25,29] for more finiteness results.

Except for the Newtonian case, there are results concentrating on the homogeneous potential, and one can refer to [5,11–13,17,21,38,39].
It is natural to think about the relationships between the \( n \)-body and the \((n + 1)\)-body central configurations with some explicit facts, which may bring some inspirations for the study of central configurations. For example, the base of an \((n + 1)\)-body pyramidal central configuration is also a planar \( n \)-body central configuration \([34]\) (generalized from the \( n = 5 \) case \([14]\)), and the 4-body central configuration with three equal masses at the vertices of an equilateral triangle and an arbitrary mass at the geometry center, obviously, the three particles on the vertices form the famous Lagrangian central configuration. Furthermore, Zhang and Zhou \([41]\) discussed the \((n + 2)\)-body double pyramidal (or bipyramidal) central configuration by adding two more masses symmetrically orthogonal to the regular \( n \)-gon co-circular central configuration system.

In 2005, Hampton \([19]\) raised an interesting family of planar central configurations in 5-body problem which was called stacked central configurations such that a subset of the five points of a 5-body central configuration also form a 4-body central configuration. He mainly discussed a kind of stacked 5-body problem with two masses inside an equilateral triangle Lagrangian central configuration by giving some characterizations, and proposed two questions for \( n = 4 \) and 5, i.e.

1. In addition to symmetric collinear configurations, the square or a regular tetrahedron with a mass at its centre and the square pyramidal configuration, are there any 5-body central configurations with a subset forming a 4-body central configuration?

2. Are there any non-collinear 5-body central configurations such that all of whose 4-body subsets form a central configuration?

The first question has been solved by Yu and Zhu completely, see \([40]\) for more details. In their paper, Yu and Zhu gave the specific approaches to obtain a non-collinear \((n + 1)\)-body central configuration from a non-collinear \( n \)-body central configuration in \( \mathbb{R}^3 \), and raised some other questions concerning the co-circular and the co-spherical central configurations. The second question has still no answer in \( \mathbb{R}^3 \), although it is obvious in \( \mathbb{R}^d (d \geq 4) \), see Theorem 5 in this paper. Other results about the stacked central configurations are referred in \([7, 9, 10, 15, 16, 26, 28, 37]\).

In this paper, we mainly focus on how to generalize the \( n \) to \( n + 1 \) stacked central configurations, and generalize some results obtained by Yu and Zhu in \([40]\) to the homogeneous potential (also containing the vortex case). At the end of this paper we also discuss the stacked S-balanced configurations, which implies that the ”stack" property is particular for central configurations.

We arrange the contents of this paper as follows

1. In Section 2, we give the general results and the admissible set of \( \alpha \) for a convex central configuration with respect to the Newtonian potential;
2. In Section 3, we characterize the co-circular central configurations;
3. In Section 4, we discuss Hampton’s questions;
4. In Section 5, we focus on the regular \( n \)-gon co-circular central configurations and the pyramidal central configurations;
5. In Section 6, we study the stacked S-balanced configurations.

2. GENERAL RESULTS

We first introduce some notations following from \([40]\). Consider an \( n \)-body configuration \( x \) with the notation mentioned in Section 1 which can be extended to an \((n + 1)\)-body configuration by adding one mass, namely, \( m_0 \), with position \( x_0 \). We denote by \( \vec{x} = (x_0, x_1, \cdots, x_n) \) the
(n + 1)-body configuration. Then we call \( x \) the subconfiguration of \( \bar{x} \). Also, the corresponding total mass \( \bar{m} \), the center of mass \( \bar{c} \), the homogeneous potential \( \bar{U}_\alpha \), the moment of inertia \( \bar{I} \) and the total angular vortex momentum of the \((n + 1)\)-body configuration are listed as follows

\[
\bar{m} = m_0 + m, \\
\bar{c} = \frac{1}{\bar{m}} \sum_{i=0}^{n} m_i x_i = \frac{m_0 x_0 + mc}{m_0 + m}, \\
\bar{U}_\alpha = \begin{cases} 
\sum_{0 \leq i < j \leq n} \frac{m_i m_j}{r_{ij}^{\alpha-2}} = U_\alpha + \sum_{i=1}^{n} \frac{m_0 m_i}{r_{i,0}^{\alpha-2}}, & \alpha > 2, \\
\sum_{0 \leq i < j \leq n} m_i m_j \ln(r_{ij}) = U_2 + \sum_{i=1}^{n} m_0 m_i \ln(r_{i,0}), & \alpha = 2,
\end{cases}
\]

\[
\bar{I} = \frac{1}{2\bar{m}} \sum_{0 \leq i < j \leq n} m_i m_j |x_i - x_j|^2 = \frac{1}{2} \sum_{i=0}^{n} m_i |x_i - \bar{c}|^2, \\
\bar{L} = \sum_{0 \leq i < j \leq n} m_i m_j = L + m_0 m.
\]

Thus, for \( \alpha \geq 2 \), both the \( n \)-body and the \((n + 1)\)-body configuration are central configurations if and only if

\[
(3) \quad \alpha > 2, \quad \begin{cases} 
\sum_{j \neq i, j=1}^{n} m_i m_j (x_j - x_i) = -\frac{\lambda}{\alpha - 2} m_i (x_i - \bar{c}), & i = 1, \cdots, n, \\
\sum_{i=1}^{n} m_i m_0 (x_i - x_0) = -\frac{\lambda}{\alpha - 2} m_0 (x_0 - \bar{c}), \\
\sum_{j \neq i, j=1}^{n} m_i m_j \frac{x_j - x_i}{r_{ij}^{\alpha}} + m_i m_0 \frac{x_0 - x_i}{r_{i,0}^{\alpha}} = -\frac{\lambda}{\alpha - 2} m_i (x_i - \bar{c}), & i = 1, \cdots, n,
\end{cases}
\]

or

\[
(4) \quad \alpha = 2, \quad \begin{cases} 
\sum_{j \neq i, j=1}^{n} m_i m_j (x_j - x_i) = -\lambda m_i (x_i - \bar{c}), & i = 1, \cdots, n, \\
\sum_{i=1}^{n} m_i m_0 (x_i - x_0) = -\lambda m_0 (x_0 - \bar{c}), \\
\sum_{j \neq i, j=1}^{n} m_i m_j \frac{x_j - x_i}{r_{ij}^{2}} + m_i m_0 \frac{x_0 - x_i}{r_{i,0}^{2}} = -\lambda m_i (x_i - \bar{c}), & i = 1, \cdots, n,
\end{cases}
\]

where \( 2\lambda = \begin{cases} 
(\alpha - 2)U_\alpha(x)/\bar{I}(x), & \alpha > 2, \\
L/\bar{I}(x), & \alpha = 2,
\end{cases} \quad 2\bar{\lambda} = \begin{cases} 
(\alpha - 2)\bar{U}_\alpha(\bar{x})/\bar{I}(\bar{x}), & \alpha > 2, \\
\bar{L}/\bar{I}(\bar{x}), & \alpha = 2.
\end{cases} \)
Using the identical equation of the center of mass, i.e. $\bar{c} = \frac{m_0x_0 + m_0}{m_0 + m}$, we can reduce (3) and (4) with $\bar{c}$ eliminated to

$$\sum_{j \neq i,j=1}^{n} \frac{m_j(x_j - x_i)}{r_{i,j}^\alpha} = -\frac{\lambda}{\alpha - 2}(x_i - c), \quad i = 1, \ldots, n,$$

\hspace{1cm} for $\alpha > 2$, 

$$\sum_{i=1}^{n} \frac{m_i(x_i - x_0)}{r_{i,0}^\alpha} = -\frac{\bar{\lambda}}{\alpha - 2}(x_0 - c),$$

or

$$\left(\frac{m_0}{r_{i,0}^\alpha} - \frac{\lambda}{\alpha - 2} \frac{m_0}{m}\right)(x_0 - c) = \left(\frac{\lambda}{\alpha - 2} - \frac{\bar{\lambda}}{\alpha - 2} + \frac{m_0}{r_{i,0}^\alpha}\right)(x_i - c), \quad i = 1, \ldots, n,$$

\hspace{1cm} for $\alpha = 2$, 

where $2\lambda = \begin{cases} (\alpha - 2)\bar{U}_\alpha(x)/I(x), & \alpha > 2, \\ L/I(x), & \alpha = 2. \end{cases}$

We divide the results into two cases.

**Case 1:** $x_0 = c$.

**Theorem 1.** Suppose $x = (x_1, \ldots, x_n)$ to be a non-collinear central configuration with a homogeneous potential $U_\alpha$. Then for $\alpha \geq 2$, the corresponding $(n + 1)$-body configuration $\bar{x} = (x_0, x_1, \ldots, x_n)$ with $m_0$ located at the center of mass of $x$ is a central configuration if and only if $r_{i,0} = r_{j,0}, \forall i \neq j, i, j = 1, \ldots, n$.

**Proof.** With $x_0 = c$, first we notice that $\bar{c} = c$ and $I(x) = \bar{I}(\bar{x})$, and for $\alpha > 2$,

$$2\lambda = \bar{U}_\alpha(x)/\bar{I}(\bar{x}) = \left(U_\alpha(x) + \sum_{i=1}^{n} \frac{m_0m_i}{r_{i,0}^{\alpha-2}}\right)/I(x) = \frac{2\lambda}{\alpha - 2} + \left(\sum_{i=1}^{n} \frac{m_0m_i}{r_{i,0}^{\alpha-2}}\right)/I(x),$$

while for $\alpha = 2$,

$$2\bar{\lambda} = \bar{L}/\bar{I}(\bar{x}) = (L + m_0m)/I(x) = 2\lambda + m_0m/I(x).$$

Now assume that the corresponding $(n + 1)$-body configuration is a central configuration. Then for $\alpha > 2$ (resp. $\alpha = 2$), from the third equations of (5) (resp. (6)) we have for all $i = 1, \ldots, n$,

$$\left(\frac{\lambda}{\alpha - 2} - \frac{\bar{\lambda}}{\alpha - 2} + \frac{m_0}{r_{i,0}^\alpha}\right)(x_i - c) = 0 \quad \text{ (resp. } \left(\lambda - \bar{\lambda} + \frac{m_0}{r_{i,0}^2}\right)(x_i - c) = 0\text{),}$$

which imply

$$\frac{\lambda}{\alpha - 2} - \frac{\bar{\lambda}}{\alpha - 2} + \frac{m_0}{r_{i,0}^\alpha} = 0 \quad \text{ (resp. } \lambda - \bar{\lambda} + \frac{m_0}{r_{i,0}^2} = 0\text{),}$$

since $x_i - c = x_i - x_0 \neq 0$ holds for all $i = 1, \ldots, n$. Hence we easily get

$$r_{i,0} = r_{j,0}, \quad \forall i \neq j, i, j = 1, \ldots, n.$$
Conversely, we assume \( r_{i,0} = r_{j,0}, \forall i \neq j(i, j = 1, \ldots, n) \). Then for \( \alpha > 2 \) (resp. \( \alpha = 2 \)), the second equation in (5) (resp. in (6)) holds, since the corresponding left hand side can be simplified to

\[
\sum_{i=1}^{n} \frac{m_{i}m_{0}(x_{i} - x_{0})}{r_{i,0}^{\alpha}} = \frac{m_{0}}{r_{j,0}^{\alpha}} \sum_{i=1}^{n} m_{i}(x_{i} - c) = 0.
\]

We only need to check that the third equation in (5) (resp. in (6)) holds. By using (7) (resp. (8)) we have for \( \alpha > 2 \) (resp. \( \alpha = 2 \)) and \( \forall i = 1, \ldots, n \)

\[
\frac{2(\bar{\lambda} - \lambda)}{\alpha - 2} = \left( \sum_{i=1}^{n} \frac{m_{0}m_{i}}{r_{i,0}^{\alpha-2}} \right) / I(x) = \frac{2m_{0}}{r_{i,0}^{\alpha}},
\]

(resp. \( 2(\bar{\lambda} - \lambda) = m_{0}m/I(x) = \frac{2m_{0}}{r_{i,0}^{2}} \)).

Hence the right hand side of the third equation in (5) (resp. (6)) vanishes, which implies that the \( (n+1) \)-body configuration is a central configuration.

\[\Box\]

Case 2: \( x_{0} \neq c \).

**Theorem 2.** Suppose \( x = (x_{1}, \ldots, x_{n}) \) to be a non-collinear central configuration with a homogeneous potential \( U_{\alpha} \) and its center of mass locates at \( c \). Consider an \( (n+1) \)-body configuration \( \bar{x} \) obtained by adding mass \( m_{0} \) to the \( n \)-body system avoiding \( c \). Then for \( \alpha \geq 2 \), the configuration \( \bar{x} = (x_{0}, x_{1}, \ldots, x_{n}) \) forms a central configuration if and only if

\[
r_{i,0} = r_{j,0} = \begin{cases} 
\left( \frac{m(\alpha - 2)}{\lambda} \right)^{1/\alpha}, & \forall i \neq j(i, j = 1, \ldots, n), \quad \alpha > 2, \\
\left( \frac{m}{\lambda} \right)^{1/2}, & \alpha = 2.
\end{cases}
\]

**Proof.** First we consider \( \alpha > 2 \) and assume that the \( (n+1) \)-body configuration is a central configuration.

For the points that do not locate on the line determined by \( x_{0} \) and \( c \), denoted by \( x_{l} \), i.e. \( l \in L \subseteq \{1, 2, \ldots, n\} \), where \( L \neq \emptyset \), otherwise the configuration \( x \) turns to the collinear case, we conclude from the third equation in (5) that

\[
\frac{1}{r_{i,0}^{\alpha}} = \frac{1}{m} \frac{\bar{\lambda}}{\alpha - 2} = \frac{1}{m_{0}} \left( \frac{\bar{\lambda}}{\alpha - 2} - \frac{\lambda}{\alpha - 2} \right),
\]

which implies from the first equality that \( r_{i,0} = r_{h,0}, \forall l \neq h, l, h \in L \), and from the second that \( \lambda/m = \bar{\lambda}/m \). Hence the third equation in (5) can be simplified to

\[
\left( \frac{m_{0}}{r_{i,0}^{\alpha}} - \frac{\bar{\lambda}}{m} \frac{m_{0}}{m} \right) (x_{i} - x_{0}) = 0, \quad i = 1, \ldots, n.
\]

Since \( x_{i} - x_{0} \neq 0 \) for all \( i = 1, \ldots, n \), it implies that

\[
r_{i,0} = r_{j,0} = \left( \frac{m(\alpha - 2)}{\lambda} \right)^{1/\alpha} = \left( \frac{\bar{m}(\alpha - 2)}{\bar{\lambda}} \right)^{1/\alpha}, \quad \forall i \neq j(i, j = 1, \ldots, n).
\]

Conversely, suppose that \( r_{i,0} = r_{j,0} = \left( \frac{m(\alpha - 2)}{\lambda} \right)^{1/\alpha} \), \( \forall i \neq j(i, j = 1, \ldots, n) \) holds, and we have

\[
\frac{\bar{\lambda}}{\bar{m}} \frac{1}{\alpha - 2} = \frac{\bar{U}_{\alpha}(\bar{x})}{2\bar{m}I(\bar{x})} = \frac{U_{\alpha}(x) + \sum_{i=1}^{n} \frac{m_{0}m_{i}r_{i,0}^{2}}{r_{i,0}^{\alpha}}}{2mI(x) + \sum_{i=1}^{n} m_{i}m_{0}r_{i,0}^{2}} = \frac{2\lambda}{\alpha - 2} I(x) + \lambda \frac{1}{\alpha - 2} \sum_{i=1}^{n} m_{i}m_{0}r_{i,0}^{2} = 2mI(x) + \sum_{i=1}^{n} m_{i}m_{0}r_{i,0}^{2} = \frac{\lambda}{m} \frac{1}{\alpha - 2}.
\]
Then we only need to check that the second and the third equations in (5) hold. The former holds obviously, and for the third one, noticing that
\[
\frac{m_0}{r_{i,0}^\alpha} - \frac{\tilde{\lambda}}{\alpha - 2} \frac{m_0}{\bar{m}} = \frac{\lambda}{\alpha - 2} - \frac{\tilde{\lambda}}{\alpha - 2} + \frac{m_0}{r_{i,0}^\alpha} = 0,
\]
hence it holds for all \(i(i = 1, \cdots, n)\).

The proof is similar for \(\alpha = 2\). In fact, assume that (6) holds. First we still claim \(r_{i,0} = r_{j,0}, i \neq j(i, j = 1, \cdots, n)\) and \(\lambda/m = \bar{\lambda}/\bar{m}\), which forces
\[
r_{i,0} = \left(\frac{m}{\bar{\lambda}}\right)^{1/2}.
\]
Then the converse is obvious.

\[\square\]

Remark 1. In Theorem 2 we notice that \(r_{i,0}(i = 1, \cdots, n)\) is determined by the ratio \(\lambda/m = \bar{\lambda}/\bar{m}\), which is independent of the choice of \(m_0\).

Then we define
\[
R_\alpha = \left\{ \begin{array}{ll}
\left( \frac{m(\alpha - 2)}{\lambda} \right)^{1/\alpha}, & \alpha > 2, \\
\left( \frac{m}{\bar{\lambda}} \right)^{1/2}, & \alpha = 2,
\end{array} \right.
\]
whose property is similar to the case \(\lambda/m = 1\), see Figure 1 for a general look. In fact, if we denote by \(r_{\min}\) and \(r_{\max}\) the shortest and the longest distance of all \(r_{i,j}\)'s respectively, then for \(\alpha > 2\) (resp. \(\alpha = 2\)) we have
\[
\frac{m}{\lambda} = \frac{mI}{U_\alpha} = \frac{\sum_{i<j} m_im_j r_{i,j}^2}{\sum_{i<j} \frac{m_im_j}{r_{i,j}}} \in \left[ r_{\min}^{\alpha}, r_{\max}^{\alpha} \right],
\]
resp.
\[
\frac{m}{\lambda} = \frac{mI}{L} = \frac{\sum_{i<j} m_im_j r_{i,j}^2}{\sum_{i<j} m_im_j} \in \left[ r_{\min}^2, r_{\max}^2 \right],
\]
which implies for all \(\alpha > 2\), \(R_\alpha \in \left[ r_{\min}(\alpha - 2)^{1/\alpha}, r_{\max}(\alpha - 2)^{1/\alpha} \right]\), and with \(\alpha \to 2^+\), \(R_\alpha \to 0\). Correspondingly for \(\alpha = 2\), obviously we have \(R_\alpha \in \left[ r_{\min}, r_{\max} \right]\). With this observation, we give a necessary condition for a convex \(n\)-body central configuration with homogeneous potential \(U_\alpha\), generalizing the original Newtonian potential case.

Lemma 1. For every convex \(n\)-body central configuration with a homogeneous potential \(U_\alpha\), \(\alpha\) must satisfy
\[
\alpha \in \Lambda = \left\{ \alpha \left| (\alpha - 2)^{1/\alpha} \geq \frac{r_{\min}}{r_{\max}} \right. \right\} \cap [2, \infty).
\]
In other words, if \((\alpha - 2)^{1/\alpha} < \frac{r_{\text{min}}}{r_{\text{max}}},\) there are no convex central configurations with this homogeneous potential \(U_{\alpha}\).

We call \(\Lambda\) the admissible set of the convex central configurations.

**Proof.** Since the case \(\alpha = 2\) is obvious, we assume \(\alpha > 2\). We use reduction. If for some \(\alpha, (\alpha - 2)^{1/\alpha} < \frac{r_{\text{min}}}{r_{\text{max}}},\) then we have \(r_{\text{min}} > r_{\text{max}}(\alpha - 2)^{1/\alpha} \geq R_{\alpha}\). On the other hand, the equations \([3]\) of the central configurations can be written as

\[
\sum_{j \neq i} m_j \left( \frac{1}{r_{i,j}^\alpha} - \frac{1}{R_{\alpha}^\alpha} \right) (x_j - x_i) = 0, \quad i = 1, \ldots, n,
\]

which implies \(\frac{1}{r_{i,j}^\alpha} - \frac{1}{R_{\alpha}^\alpha} \leq 0\) for all \(r_{i,j}\). Since the planar (resp. spacial) central configuration is convex, then for each vertex \(i\), there does exist a line \(l_i\) (resp. a plane) passing through some vertices and "supporting" the whole configuration, i.e. all the other \(n - 1\) vertices lie on the same side of this line (resp. plane \(\Pi_i\)). Then by considering the projection \(P_i\) of all \(x_j - x_i\) to the orthogonal line \(l_i\) of \(l_i\) (resp. \(\Pi_i\)) passing through the vertex \(i\) we find

\[
P_i \left( \sum_{j \neq i} m_j \left( \frac{1}{r_{i,j}^\alpha} - \frac{1}{R_{\alpha}^\alpha} \right) (x_j - x_i) \right) = \sum_{j \neq i} m_j \left( \frac{1}{r_{i,j}^\alpha} - \frac{1}{R_{\alpha}^\alpha} \right) (P_i(x_j - x_i)) \neq 0,\]

which contradicts \([10]\).

---

**Remark 2.** For example, the square of the convex 4-body central configuration with equal masses, by direct computation we have that \(\alpha\) must satisfy \((\alpha - 2)^{1/\alpha} \geq 1/\sqrt{2}\), hence \(\alpha \in [\alpha^*, \infty) \cup \{2\}\), where \(\alpha^* \approx 2.431\). For the regular \(n\)-simplex central configurations we have \((\alpha - 2)^{1/\alpha} \geq 1\), hence \(\alpha \in [3, \infty) \cup \{2\}\).

The above lemma may not hold for non-convex cases in general.

### 3. Characterizing the co-circular central configurations via \(R_{\alpha}\)

As a kind of special case of the convex central configurations, we also obtain two interesting versions of Theorem \([1] and [2]\) for the co-circular and the co-spherical central configurations with \(R\) denoting the radius of the related circle or sphere.

We denote by \(r_{\text{min}}\) and \(r_{\text{max}}\) the shortest and the longest distance of the \(n\)-body central configuration, and \(r'_{\text{min}}\) and \(r'_{\text{max}}\) their counterparts of the \((n + 1)\)-body central configuration. Then we have

\[
r'_{\text{min}} \leq \frac{r'_{\text{min}}}{r'_{\text{max}}},
\]

since the shortest distance does not increase, i.e. \(r'_{\text{min}} \leq r_{\text{min}}\) while the longest does not decrease, i.e. \(r'_{\text{max}} \geq r_{\text{max}}\) with one mass added. In addition, the above inequality also holds for more than one masses added.

**Proposition 1.** For \(\alpha \in \Lambda\), where \(r_{\text{min}}\) and \(r_{\text{max}}\) are the shortest and the longest distance of the \(n\)-body central configuration respectively, there are two ways to get a coplanar \((n + 1)\)-body central configuration from an \(n\)-body co-circular central configuration with the radius \(R\), i.e.

1. If \(c\) coincides with the circle center, let \(x_0 = c\);
2. If \(c\) differs from the circle center, let \(x_0\) locate at the circle center and let \(R_{\alpha} = R\).

**Proposition 2.** For \(\alpha \in \Lambda\), where \(r_{\text{min}}\) and \(r_{\text{max}}\) are the shortest and the longest distance of the \(n\)-body central configuration respectively, there are three ways to get a spacial \((n + 1)\)-body central configuration \(\bar{x}\) from an \(n\)-body co-circular or co-spherical central configuration \(x\) with the related radius \(R\), i.e.
If $x$ is co-circular satisfying $R < R_\alpha$, let $x_0$ locate at the orthogonal axis passing through the circle center with $r_{i,0} = R_\alpha$;

(2) If $x$ is co-spherical satisfying that $c$ coincides with the sphere center, let $x_0 = c$;

(3) If $x$ is co-spherical satisfying that $c$ differs from the sphere center, let $x_0$ locate at the sphere center and $R = R_\alpha$.

Theorem 3. For $\alpha \in \Lambda$, where $r_{\text{min}}$ and $r_{\text{max}}$ are the shortest and the longest distance of the $n$-body central configuration $x$ respectively. If $x$ is co-circular with a sequential order (i.e. $x_1, \cdots, x_n$ are arranged clockwise or counterclockwise around a circle), then for each $i$ we have $r_{i,j} < R_\alpha(|i - j| = 1)$, and at least one $r_{i,j} > R_\alpha(|i - j| \geq 2)$.

Proof. The proof is similar to the one of Theorem 11 in [40], so we only list the outline of it.

First suppose that $r_{1,2}, r_{1,n} \geq R_\alpha$.

Second suppose that $r_{1,2} \geq R_\alpha$ and $r_{1,n} < R_\alpha$.

Since the above two cases can not hold, we conclude that $r_{1,2}, r_{1,n} < R_\alpha$. Then by [10], there is at least one $r_{1,k} > R_\alpha$, where $|k - 1| \geq 2$. This is true for all $i$. □

Corollary 1. For $\alpha \in \Lambda$, where $r_{\text{min}}$ and $r_{\text{max}}$ are the shortest and the longest distance of the $n$-body central configuration respectively, the co-circular $n$-body central configurations cannot locate entirely within a semi-circle.

Corollary 2. For $\alpha \in \Lambda$, where $r_{\text{min}}$ and $r_{\text{max}}$ are the shortest and the longest distance of the $n$-body central configuration respectively, for the $n$-body ($n = 4, 5, 6$) co-circular central configurations, the radius of the related circle $R$ satisfies $R < R_\alpha$.

4. Answers to Hampton’s Questions with the Homogeneous Potential

Hampton’s questions are mentioned in Section 1. In [40], Yu and Zhu give an exact answer to the first question for the Newtonian case $\alpha = 3$. We now give the similar answer for the general homogeneous potential cases, which can be viewed as direct results of Proposition 1 and Proposition 2.

Theorem 4. The only three types of 5-body stacked central configurations from 4-body ones with homogeneous potential $U_\alpha$ are

(1) the square with equal masses at each vertex and an arbitrary mass $m_0$ at the geometric center with $\alpha \in [\alpha^*, \infty) \cup \{2\}$, where $\alpha^* \in \Lambda$ satisfies $(\alpha^* - 2)^{1/\alpha^*} = 1/\sqrt{2}$, i.e. $\alpha^* \approx 2.431$;

(2) the regular 4-simplex with equal masses and an arbitrary mass $m_0$ at the geometry center with $\alpha \in [3, \infty) \cup \{2\}$;

(3) the pyramidal central configurations with the base formed by 4-body co-circular central configurations and an arbitrary mass $m_0$ on the vertical line passing through the circle center, whose vertical distance to the circle equals $H = \sqrt{R_\alpha^2 - R^2}$ with $\alpha \in \Lambda$, where $r_{\text{min}}$ and $r_{\text{max}}$ are the shortest and the longest distance of the 4-body co-circular central configuration respectively.

Unfortunately, there is no answer to the second question in $\mathbb{R}^d (d \leq 3)$. But if we consider in $\mathbb{R}^d (d \geq 4)$, the answer is explicit.
Theorem 5. The only type of an \((n + 1)\)-body central configuration satisfying that any of its \(n\)-body subconfiguration forms a central configuration is the regular \((n + 1)\)-simplex in \(\mathbb{R}^d\), for all \(n \leq d\). Especially for \(n = 4\), the only type of the 5-body central configurations is the 5-simplex in \(\mathbb{R}^d\) \((d \geq 4)\) with the homogeneous potential \(U_\alpha\) satisfying \(\alpha \in [3, \infty) \cup \{2\}\).

5. FROM n-BODY CENTRAL CONFIGURATIONS TO PYRAMIDAL CENTRAL CONFIGURATIONS

The \((n + 1)\)-body pyramidal central configuration is co-spherical since its base is co-circular (see [1]).

5.1. From n-body central configurations to \((n + 1)\)-body pyramidal central configurations.

By choosing some \(m_0\) we can make \(\bar{c}\), the center of mass of this pyramidal central configuration coincide with the sphere center. To avoid confusion, we denote by \(r\) the circle containing the base of the pyramidal central configuration, then we have

Proposition 3. For \(\alpha \in \Lambda\), where \(r_{\text{min}}\) and \(r_{\text{max}}\) are the shortest and the longest distance of the \(n\)-body co-circular central configuration \(x\) respectively, an \((n + 1)\)-body pyramidal central configuration obtained from \(x\) with the homogeneous potential \(U_\alpha\) satisfies that its center of mass \(\bar{c}\) coincides with the sphere center if and only if \(R_\alpha > \sqrt{2}r\).

Proof. It holds if and only if the sphere center is inside the pyramidal, i.e. the height of the pyramidal \(H = \sqrt{R_\alpha^2 - r^2} > R\), where \(R\) satisfies \((H - R)^2 = R^2 - r^2\). □

5.2. From n-body central configurations to \((n+2)\)-body pyramidal central configurations.

Let us try to get an \((n+2)\)-body co-spherical central configuration from an \((n+1)\)-body pyramidal central configuration which has been extended from an \(n\)-body co-circular one, then we have the following result.

Proposition 4. For \(\alpha \in \Lambda\), where \(r_{\text{min}}\) and \(r_{\text{max}}\) are the shortest and the longest distance of the \(n\)-body co-circular central configuration \(x\) respectively. Suppose the \((n + 1)\)-body pyramidal central configuration \(\bar{x}\) is extended from \(x\), satisfying that \(\bar{c}\), its center of mass differs from the sphere center. Then we get an \((n+2)\)-body central configuration \(\hat{x}\) by putting one more arbitrary mass \(\hat{m} > 0\) to the sphere center if and only if \(R_\alpha = R = \frac{2}{\sqrt{3}}r\).

Proof. From (3) in Proposition 2 we know that \(R_\alpha = R\). Then the triangle connecting \(m_i, m_0, \hat{m}\) is equilateral. Hence we get the equality immediately. □

5.3. From regular n-gon central configurations to \((n + 1)\)-body pyramidal central configurations.

Chenciner’s conjecture [6]: Does there exist planar choreographies (with equal time spacing) whose masses are not all equal? In other words, is there any central configuration of the \(n\)-body problem, different from the \(n\)-gon with equal masses, with all the masses lying on a common circle with origin at its center of mass? Hampton [18] gave the uniqueness of the case \(n = 4\). Corbera and Valls [8] gave some characterization of the general case for \(n\)-body problems, and it is still open for \(n \geq 5\). For the regular \(n\)-gon central configuration itself, Cors et al. [11], Wang [38] (generalizing the work of [35] by Perko and Walter) and Hampton [21] studied the homogeneous potential case, and gave some characterization from different aspects.

As a special type of the co-circular central configurations, the regular \(n\)-gon central configurations possess some nice properties. Within this subsection, we mainly concern the \((n + 1)\)-body pyramidal central configurations extended from a regular \(n\)-gon one. Suppose the radius of the corresponding circle is \(r\).
Proposition 5. Assume $n \geq 4$. For $\alpha > 2$, there are no regular $n$-gon central configurations with equal masses if

$$(\alpha - 2)^{1/\alpha} < \frac{r_{\min}}{r_{\max}} = \begin{cases} \sin \frac{\pi}{n}, & n \text{ is even}, \\ 2 \tan \frac{\pi}{2n}, & n \text{ is odd}, \end{cases}$$

where $r_{\min} = 2r \sin \frac{\pi}{n}$ and $r_{\max} = \begin{cases} 2r, & n \text{ is even}, \\ r \sin \frac{\pi}{n} \tan \frac{\pi}{2n}, & n \text{ is odd}. \end{cases}$

Proof. For $\alpha > 2$ we can conclude directly from Lemma 1. □

We denote by

$$\Lambda_n = \left\{ \alpha \mid \alpha > 2, (\alpha - 2)^{1/\alpha} \geq \begin{cases} \sin \frac{\pi}{n}, & n \text{ is even}, \\ 2 \tan \frac{\pi}{2n}, & n \text{ is odd}, \end{cases} \right\} \cup \{2\}.$$  

the admissible set of the regular $n$-gon central configuration with homogeneous potential $U_\alpha$.

Since $\sin \frac{\pi}{n}$ and $2 \tan \frac{\pi}{2n}$ are both less than 1 when $n \geq 4$, and with $n$ increasing, the values quickly approach zero, which forces $(\alpha - 2)^{1/\alpha}$ to be close to zero, too. Hence the inequality in Proposition 5 holds for $\alpha$ near 2.

Next we discuss the relationship between $R_\alpha$ and $r$, which determines whether or not an $(n+1)$-body pyramidal central configuration obtained from a regular $n$-gon one does exist. By computer assistant we have

$$R_\alpha = \frac{n}{A_\alpha(n)},$$  

where

$$A_\alpha(n) = \left\{ \sum_{m=1}^{n-1} \left( \csc \frac{m \pi}{n} \right)^{\alpha-2}, \alpha \in (2, +\infty) \cap \Lambda_n, \right\}$$  

and $r$ is the radius of the circle containing the $n$ bodies. Then

$$R_\alpha = \left( \frac{n}{A_\alpha(n)} \right)^{1/\alpha} = 2r \left( \frac{n(\alpha - 2)}{2 \sum_{m=1}^{n-1} \left( \csc \frac{m \pi}{n} \right)^{\alpha-2}} \right)^{1/\alpha},$$

and only if $R_\alpha > r$, the pyramidal central configuration does exist, which implies

$$\frac{n(\alpha - 2)}{\sum_{m=1}^{n-1} \left( \csc \frac{m \pi}{n} \right)^{\alpha-2}} > \left( \frac{1}{2} \right)^{\alpha-1}.$$  

For example, when $\alpha = 3, r = 1$, it coincides with the Newtonian case discussed in [31] that when $R_3 = \frac{n}{A_3(n)} > 1 \Leftrightarrow n \leq 472$ by giving an asymptotic expansion of $\sum_{m=1}^{n-1} \csc \frac{m \pi}{n}$ as a main component in $A_3(n)$, then a regular $n$-gon central configuration can extend to a pyramidal central configuration since the function $\frac{A_3(n)}{n}$ is increasing.

Let $r$ equals 1. Numerical results indicate that for $\alpha \geq 3$, the maximal integer $n$ which makes $A_\alpha(n)/n < 1$ seems to decrease with $\alpha$ increasing. For example, for $\alpha = 3.5$, this $n$ equals 97, for $\alpha = 4$, this $n$ fails to 48 and for $\alpha = 5$, it is 24. While for $2 < \alpha < 3$, the relationship between $\alpha$ and the maximal $n$ seems converse. For example, when $\alpha = 2.6$, the corresponding $n$ equals 413, and for $\alpha = 2.61$, $n$ rises to 520. But it is still not easy to give a similar estimation.
Hence we have $\alpha > U$ this type with the homogeneous potential $U$. From Proposition 5, we know that the corresponding equations of the central configuration of $(1)$ holds for any arbitrary positive mass $\tilde{m}$, and the $\alpha$ may also effect the error estimation. One can see Figure 2 and 3 for a numerical look.

5.4. From a regular $n$-gon central configurations to $(n + 2)$-body double pyramidal central configurations. Zhang and Zhou [41], Yu and Zhu [40] discussed the double pyramidal (or bipyramidal) central configurations, and in this subsection, we follow their notations. Suppose the masses of the regular $n$-gon central configuration are equal, i.e. $m_1 = m_2 = \cdots = m_n = 1$, locating sequentially along the equator, and the two masses added to the $n$-body system $x$ are $m_{n+1} = m_{n+2} = \tilde{m}$, locating at the two poles respectively. Let the origin coincide with the sphere center, then the position of these $n + 2$ points are $x_k = (\cos \frac{2k\pi}{n}, \sin \frac{2k\pi}{n}, 0)$ for $k = 1, \cdots, n$, $x_{n+1} = (0, 0, 1)$ and $x_{n+2} = (0, 0, -1)$. The radius of the sphere is $r$.

From Proposition 5 we know that the corresponding equations of the central configuration of this type with the homogeneous potential $U_\alpha$ are equivalent to

$$\alpha > 2,$$

$$-\frac{\tilde{\lambda}}{\alpha - 2} x_1 = \sum_{k\neq 1}^n \frac{x_{j} - x_{1}}{r_{j, 1}^{\alpha}} + \frac{\tilde{m}}{r_{n+1, 1}^{\alpha}} x_{n+1} - \frac{\tilde{m}}{r_{n+2, 1}^{\alpha}} x_{n+2} = -\left( A_\alpha(n) + \frac{2\tilde{m}}{(\sqrt{2}r)^{\alpha}} \right) x_1,$$

$$-\frac{\tilde{\lambda}}{\alpha - 2} x_{n+1} = \sum_{k=1}^{n} \frac{x_{k} - x_{n+1}}{r_{n+1, k}^{\alpha}} + \frac{\tilde{m}}{r_{n+2, n+1}^{\alpha}} x_{n+2} = -\left( \frac{n}{(\sqrt{2}r)^{\alpha}} + \frac{\tilde{m}}{2\alpha - 1 r^{\alpha}} \right) x_{n+1},$$

and

$$- \frac{\lambda}{\alpha - 2} x_{1} = - \frac{A_\alpha(n)}{n} n x_1 = \sum_{k\neq 1}^n \frac{x_{j} - x_{1}}{r_{k, 1}^{\alpha}}.$$\n
Hence we have

$$\frac{n}{r^{\alpha} A_\alpha(n)} = \left( 1 + \frac{\tilde{m}}{r^{\alpha} A_\alpha} \left( 1 - \frac{2}{(\sqrt{2})^{\alpha}} \right) \right) (\sqrt{2})^{\alpha} > (\sqrt{2})^{\alpha},$$

i.e.

$$\frac{R_\alpha(n)}{r} = \left( \frac{n}{A_\alpha(n)} \right)^{1/\alpha} / r > \sqrt{2}$$

holds for any arbitrary positive mass $\tilde{m}$, where $\tilde{\lambda}$ (resp. $\lambda$) is the multiplier of the $(n + 2)$-body (resp. regular $n$-gon) central configuration and $A_\alpha(n)$ is from [12].
(2) For $\alpha = 2$ it is similar that the corresponding equations of (14), (15) and (16) are
\[
-\tilde{\lambda}x_1 = \sum_{k \neq 1}^{n} \frac{x_j - x_1}{r_{j,1}^2} + \tilde{m} \frac{x_{n+1} - x_1}{r_{n+1,1}^2} + m \frac{x_{n+2} - x_1}{r_{n+2,1}^2} = -\left(A_2(n) + \frac{\tilde{m}}{r^2}\right)x_1,
\]
and
\[
-\tilde{\lambda}x_{n+1} = \sum_{k=1}^{n} \frac{x_k - x_{n+1}}{r_{n+1,k}^2} + \tilde{m} \frac{x_{n+2} - x_{n+1}}{r_{n+2,n+1}^2} = -\left(n + \frac{\tilde{m}}{2r^2}\right)x_{n+1},
\]
Hence we have
\[
-\lambda x_1 = -\frac{A_2(n)}{n} nx_1 = \sum_{k \neq 1}^{n} \frac{x_j - x_1}{r_{k,1}^2}.
\]
Hence we have
\[
\frac{n}{A_2(n)} = 2r^2 + \frac{\tilde{m}}{A_2(n)} > 2r^2,
\]
i.e.
\[
\frac{R_2(n)}{r} = \left(\frac{n}{A_2(n)}\right)^{1/2} > \sqrt{2}
\]
holds for any arbitrary positive mass $\tilde{m}$, and it is consistent with (17).

**Proposition 6.** The regular $n$-gon central configuration can extend to an $(n + 2)$-body double pyramidal central configuration with the homogeneous potential $U_\alpha$ if and only if $R_2(n)/r > \sqrt{2}$ holds for $\alpha \in \Lambda_n \cap \{2\}$. Especially for Newtonian case, i.e. $\alpha = 3$, $R_2(n)/r > \sqrt{2} \iff 3 \leq n \leq 8$.

Since the center of mass of the $(n + 2)$-body double pyramidal central configuration coincides with the sphere center, we can naturally obtain an $(n + 3)$-body central configuration by putting an arbitrary mass at the sphere center according to Proposition 2.

6. **Stacked S-balanced configurations in $\mathbb{R}^d$**

In this section we would like to illustrate the particular property of stacked central configurations by studying the generalized S-balanced configurations. For convenience, we only discuss the Newtonian potential case, i.e. $\alpha = 3$.

As it has been shown in [30] (earlier in [3]) that central configuration is not necessary for relative equilibria, i.e. as long as there exists a $d \times d$ skew-symmetric matrix $\beta$ such that
\[
\nabla_i U(x) - \beta^2(x_i - c) = 0, \ i = 1, \cdots, n,
\]
we can obtain a relative equilibria solution $x(t) = c(t) + e^{t\beta}(x_0 - c_0)$, and $x$ is called balanced in $\mathbb{R}^d$ or $d$-balanced.

Now let $S = -\beta^2$, then it is easy to see that $S$ is positive semi-definite, i.e. all the eigenvalues of $S$ are nonaggetive. Without loss of generality, we assume $S$ to be positive definite, further, diagonal, i.e. $S = \text{diag}(\sigma_1, \cdots, \sigma_d)$, $\sigma_i > 0, i = 1, \cdots, d$, since the motion of the system actually takes place in the "effective" dimensions determined by $C(x)$, i.e. the centered position space generated by the vectors $\{x_i - c\}_{i=1}^n$, and the spectral basis is equivalent to the canonical one in $\mathbb{R}^d$. Then we have

**Definition 1.** A configuration $x$ in $\mathbb{R}^d$ is an S-balanced configuration (abbreviated SBC) if it satisfies
\[
\nabla_i U(x) + \lambda S x_i = 0, \ i = 1, \cdots, n,
\]
where \( S = \text{diag}(\sigma_1, \ldots, \sigma_d), \sigma_i > 0, i = 1, \ldots, d \) is a diagonal, positive definite symmetric matrix, and \( \lambda > 0 \) a constant. Equivalently we have

\[
\nabla U(x) + \lambda \hat{S} M(x - c) = 0,
\]

where \( \hat{S} = \text{diag}(S, \ldots, S) \) is an \( dn \times dn \) block-diagonal matrix with \( n \) blocks \( S \).

**Remark 3.** As we may learn clearly in [30], an important fact about SBCs should be noticed that every central configuration is balanced provided it is contained in an even-dimensional subspace in \( \mathbb{R}^d \). When \( d = 3 \), namely the physical case, an S-balanced but non-central configuration desitnes to be planar. For non-central planar SBCs, we may obtain relative equilibria in \( \mathbb{R}^3 \) at least, i.e. a double dimension space, but we obtain nothing in just \( \mathbb{R}^2 \).

With this definition, we may naturally define the \textit{S-weighted momentum of inertia} of an SBC

\[
I_S(x) = (x - c)^T \hat{S} M(x - c) = \sum_{i=1}^{n} m_i |x_i - c|^2_S,
\]

where the new inner product and norm in \( \mathbb{R}^d \) are respectively

\[
\langle \xi, \eta \rangle = \xi^T S \eta, \quad |\xi|^2_S = \xi^T S \xi.
\]

With the similar notations mentioned before, both \( x \) and \( \bar{x} \) are SBCs if and only if

\[
\sum_{j \neq i, j=1}^{n} m_i m_j \frac{x_j - x_i}{|x_j - x_i|^3} = -\lambda m_i S(x_i - c), i = 1, \ldots, n,
\]

\[
\sum_{j=1}^{n} m_j m_0 \frac{x_j - x_0}{|x_j - x_0|^3} = -\bar{\lambda} m_0 S(x_0 - \bar{c}),
\]

\[
\sum_{j \neq i, j=1}^{n} m_i m_j \frac{x_j - x_i}{|x_j - x_i|^3} - m_i m_0 \frac{x_i - x_0}{|x_i - x_0|^3} = -\bar{\lambda} m_i S(x_i - \bar{c}), i = 1, \ldots, n,
\]

where \( \lambda = U/I_S \) and \( \bar{\lambda} = \bar{U}/\bar{I}_S \). Notice that \( c, x_0, \bar{c} \) lie on the same line, i.e.

\[
m(\bar{c} - x_0) = m(c - x_0),
\]

then [26] can be simplified as

\[
\sum_{j \neq i, j=1}^{n} m_j \frac{x_j - x_i}{\delta_{ij}} = -\lambda S(x_i - c), i = 1, \ldots, n,
\]

\[
\sum_{j=1}^{n} m_j \frac{x_j - x_0}{\delta_{j0}} = -\bar{\lambda} S(x_0 - \bar{c}),
\]

\[
\left( (\bar{\lambda} - \lambda) S - \frac{m_0}{\bar{\delta}_{j0}} I \right) (x_i - c) = \left( \frac{\lambda m_0}{\bar{\delta}} S - \frac{m_0}{\delta_{j0}} I \right) (x_0 - c), i = 1, \ldots, n.
\]

We now consider the relationship between \( x_0 \) and \( c \). We denote by \( \delta \) and \( \bar{\delta} \) their configuration dimension respectively, i.e. \( \dim(x) = \dim C(x) = \delta \) and \( \dim(\bar{x}) = \dim C(\bar{x}) = \bar{\delta} \). Obviously, \( C(x) \subseteq C(\bar{x}) \).

**Proposition 7.** Assume \( x_0 = c \). Then \( x \) as well as \( \bar{x} \) are SBCs if and only if \( S|_{C(x)} = \gamma I_{\delta \times \delta}, \gamma = \frac{m_0}{(\lambda - \lambda) r_{i,0}^3} \), i.e. both of them are central configurations in the subspace \( C(x) \subseteq \mathbb{R}^d \), where \( S|_{C(x)} \) denotes the restriction of \( S \) on \( C(x) \). Hence \( r_{i,0} = r_{j,0} \) holds for \( i \neq j, i, j = 1, \ldots, n \).
Proposition 8. Assume $x_0 = c$, we have $x_0 = c = \bar{c}$, $x_i \neq c$ and $\delta = \bar{\delta}$. The third equation of (28) becomes

$$\left((\bar{\lambda} - \lambda)S - \frac{m_0}{r_i^3}I\right)(x_i - c) = 0, i = 1, \cdots, n,$$

where $x_1 - c, \cdots, x_n - c$ is the basis of $\mathcal{C}(x)$. Since $\dim \mathcal{C}(x) = \delta$, without loss of generality, we suppose $x_1 - c, \cdots, x_\delta - c$ to be maximal linearly independent, and obviously they are basis of $\mathcal{C}(x)$ as well. Let $(X - C)_{|\mathcal{C}(x)} = (x_1 - c) \cdots (x_\delta - c)$ be a matrix composed of $\delta$ column vectors $x_1 - c, \cdots, x_\delta - c$. Notice that $\delta \leq d$, then we can further delete the zero lines (if there is any) to get a $\delta \times \delta$ matrix $(X - C)_{|\delta \times \delta}$. It is easy to see that this matrix is non-degenerate. Then from the linear algebra, we have

$$\left((\bar{\lambda} - \lambda)S - \frac{m_0}{r_i^3}I\right)_{|\mathcal{C}(x)} (X - C)_{|\delta \times \delta} = 0,$$

where $\left((\bar{\lambda} - \lambda)S - \frac{m_0}{r_i^3}I\right)_{|\mathcal{C}(x)}$ denotes the matrix after deleting the lines operating trivially on $(X - C)_{|\delta \times \delta}$. For a fixed $i$, we have $\sigma_k = \sigma_l$ for $k \neq l$ in any two non-trivial dimensions, which implies $S_{|\mathcal{C}(x)} = I_{\delta \times \delta}$, hence the $n$-body configuration $x$ is a central configuration in $\mathbb{R}^3$, not to mention in a larger space $\mathbb{R}^d$. Then for $i \neq j$, $i, j = 1, \cdots, n$ we have $r_{i,0} = r_{j,0}$. By Theorem 1 (or Theorem 2 in [40]), it is clear that both $x$ and $\bar{x}$ are central configurations. \hfill \Box

Proposition 8. Assume $x_0 \neq c$.

1. If $\delta = \bar{\delta}$, then $x$ as well as $\bar{x}$ are SBCs if and only if $S_{|\mathcal{C}(x)} = I_{\delta \times \delta}$, i.e. both of them are central configurations in the subspace $\mathcal{C}(x) \subseteq \mathbb{R}^d$, where $S_{|\mathcal{C}(x)}$ denotes the restriction of $S$ on $\mathcal{C}(x)$. Hence $r_{i,0} = r_{j,0} = (\frac{m_i}{\lambda_i})^{1/3} = (\frac{m_j}{\lambda_j})^{1/3}$ holds for $i \neq j(i, j = 1, \cdots, n)$.

2. If $\delta < \bar{\delta}$, then $x$ as well as $\bar{x}$ are SBCs if and only if $S_{|\mathcal{C}(\bar{x})} = I_{\delta \times \delta}$, i.e. both of them are central configurations in the subspace $\mathcal{C}(\bar{x}) \subseteq \mathbb{R}^d$, where $S_{|\mathcal{C}(\bar{x})}$ denotes the restriction of $S$ on $\mathcal{C}(\bar{x})$. Hence $r_{i,0} = r_{j,0} = (\frac{m_i}{\lambda_i})^{1/3} = (\frac{m_j}{\lambda_j})^{1/3}$ holds for $i \neq j(i, j = 1, \cdots, n)$.

Proof. The proof is similar. \hfill \Box

From the above results we find that the the “stacked” property seems unique to the central configurations, which means we can’t extend a ”pure” SBC to another SBC by adding one mass to an appropriate position unless they are central configurations.

Acknowledgements. The paper is partially supported by NSF of China (No. 12071316).

REFERENCES

[1] Alain Albouy, On a paper of Moeckel on central configurations, Regular and Chaotic Dynamics 8 (2003), no. 2, 133–142.
[2] Alain Albouy, Hildeberto E. Cabral, and Alan A. Santos, Some problems on the classical n-body problem, Celestial Mechanics and Dynamical Astronomy 113 (2012), no. 4, 369–375.
[3] Alain Albouy and Alain Chenciner, Le probl`eme des n corps et les distances mutuelles, Inventiones Mathematicae 131 (1997), no. 1, 151–184.
[4] Alain Albouy and Vadim Kaloshin, Finiteness of central configurations of five bodies in the plane, Annals of Mathematics 176 (2012), no. 1, 535–588.
[5] Martha Alvarez-Ramírez, Alan Almeida Santos, and Claudio Vidal, On co-circular central configurations in the four and five body-problems for homogeneous force law, Journal of Dynamics and Differential Equations 25 (2013), no. 2, 269–290.
[6] Alain Chenciner, Are there perverse choreographies?, New advances in celestial mechanics and hamiltonian systems, 2004, pp. 63–76.
The straight line solutions of the problem of n bodies
Forest Ray Moulton,
Finiteness of relative equilibria of the four-body problem
Marshall Hampton and Richard Moeckel,
The n-vortex problem: Analytical techniques
Paul K. Newton,
Pyramidal central configurations and perverse solutions
Tiancheng Ouyang, Zhifu Xie, and Shiqing Zhang,
16 STACKED CENTRAL CONFIGURATIONS WITH A HOMOGENEOUS POTENTIAL IN R^3

[7] Montserrat Corbera, Jaume Llibre, and Ernesto Pérez-Chavela, Spatial bi-stacked central configurations formed by two dual regular polyhedra, Journal of Mathematical Analysis and Applications 413 (2014), no. 2, 648–659.
[8] Montserrat Corbera and Claudia Valls, On centered co-circular central configurations of the n-body problem, Journal of Dynamics and Differential Equations 31 (2019dec), no. 4, 2053–2060.
[9] J. Lino Cornelio, M. Álvarez-Ramírez, and Josep M. Cors, A family of stacked central configurations in the planar five-body problem, Celestial Mechanics and Dynamical Astronomy 129 (2017), no. 3, 321–328.
[10] J. Lino Cornelio, Martha Álvarez-Ramírez, and Josep M Cors, A special family of stacked central configurations: Lagrange plus euler in one, Journal of Dynamics and Differential Equations 31 (2019), no. 2, 711–718.
[11] Josep M. Cors, Glen R. Hall, and Gareth E. Roberts, Uniqueness results for co-circular central configurations for power-law potentials, Physica D: Nonlinear Phenomena 280 (2014), 44–47.
[12] Yiyang Deng and Marshall Hampton, Spatial equilateral chain central configurations of the five-body problem with a homogeneous potential, Celestial Mechanics and Dynamical Astronomy 134 (2022), no. 2, 1–11.
[13] Thiago Dias, New equations for central configurations and generic finiteness, Proceedings of the American Mathematical Society 145 (2017), no. 7, 3069–3084.
[14] Nelly Fayçal, On the classification of pyramidal central configurations, Proceedings of the American Mathematical Society (1996), 249–258.
[15] A.C. Fernandes and Claudio Vidal, Stacked central configurations in the 5-vortex problem, Journal of Nonlinear Science 31 (2021), no. 5, 1–22.
[16] Antonio Carlos Fernandes and Luís Fernando Mello, On stacked central configurations with n bodies when one body is removed, Journal of Mathematical Analysis and Applications 405 (2013), no. 1, 320–325.
[17] Antonio Carlos Fernandes, Luís Fernando Mello, and Claudio Vidal, On the uniqueness of the isosceles trapezoidal central configuration in the 4-body problem for power-law potentials, Nonlinearity 33 (2019), no. 1, 388–407.
[18] Marshall Hampton, Co-circular central configurations in the four-body problem, Equadiff 2003, 2005, pp. 993–998.
[19] _____, Stacked central configurations: new examples in the planar five-body problem, Nonlinearity 18 (2005), no. 5, 2299–2304.
[20] _____, Finiteness of kite relative equilibria in the five-vortex and five-body problems, Qualitative Theory of Dynamical Systems 8 (2009), no. 2, 349–356.
[21] _____, Planar n-body central configurations with a homogeneous potential, Celestial Mechanics and Dynamical Astronomy 131 (2019), no. 5, 1–27.
[22] Marshall Hampton and Anders Jensen, Finiteness of spatial central configurations in the five-body problem, Celestial Mechanics and Dynamical Astronomy 109 (2011apr), no. 4, 321–332.
[23] Marshall Hampton and Richard Moeckel, Finiteness of relative equilibria of the four-body problem, Inventiones Mathematicae 163 (2006), no. 2, 289–312.
[24] _____, Finiteness of stationary configurations of the four-vortex problem, Transactions of the American Mathematical Society 361 (2009), no. 3, 1317–1332.
[25] Eduardo S.G. Leandro, Finiteness and bifurcations of some symmetrical classes of central configurations, Archive for Rational Mechanics and Analysis 167 (2003), no. 2, 147–177.
[26] Jaume Llibre, Luís Fernando Mello, and Ernesto Pérez-Chavela, New stacked central configurations for the planar 5-body problem, Celestial Mechanics and Dynamical Astronomy 110 (2011), no. 1, 43–52.
[27] Luís Fernando Mello, Felipe Emanoel Chaves, Antonio Carlos Fernandes, and Braulio Augusto Garcia, Stacked central configurations for the spatial six-body problem, Journal of Geometry and Physics 59 (2009), no. 9, 1216–1226.
[28] Luís Fernando Mello and Antonio Carlos Fernandes, Stacked central configurations for the spatial seven-body problem, Qualitative Theory of Dynamical Systems 12 (2013), no. 1, 101–114.
[29] Richard Moeckel, Generic finiteness for dziobek configurations, Transactions of the American Mathematical Society 353 (2001), no. 11, 4673–4686.
[30] _____, Central configurations, Central configurations, periodic orbits, and hamiltonian systems, 2015, pp. 105–167.
[31] Richard Moeckel and Carles Simó, Bifurcation of spatial central configurations from planar ones, SIAM Journal on Mathematical Analysis 26 (1995), no. 4, 978–998.
[32] Forest Ray Moulton, The straight line solutions of the problem of n bodies, The Annals of Mathematics 12 (1910), no. 1, 1–17.
[33] Paul K. Newton, The n-vortex problem: Analytical techniques, Springer New York, 2001.
[34] Tiancheng Ouyang, Zhifu Xie, and Shiqing Zhang, Pyramidal central configurations and perverse solutions, Electronic Journal of Differential Equations 2004 (2004), no. 106, 281–286.
[35] L. M. Perko and E. L. Walter, *Regular polygon solutions of the n-body problem*, Proceedings of the American Mathematical Society 94 (1985), no. 2, 301–309.

[36] Steve Smale, *Mathematical problems for the next century*, The Mathematical Intelligencer 20 (1998), no. 2, 7–15.

[37] Xia Su and Chunhua Deng, *Two classes of stacked central configurations for the spatial 2n+1-body problem: nested regular polyhedra plus one*, Journal of Geometry and Physics 76 (2014), 1–9.

[38] Zhiqiang Wang, *Regular polygon central configurations of the n-body problem with general homogeneous potential*, Nonlinearity 32 (2019), no. 7, 2426–2440.

[39] Mervin Woodlin and Zhifu Xie, *Collinear central configurations in the n-body problem with general homogeneous potential*, Journal of Mathematical Physics 50 (2009), no. 102901, 1–8.

[40] Xiang Yu and Shuqiang Zhu, *Classification of (n + 1, 1)-stacked central configurations in R^3*, Journal of Nonlinear Science 31 (2021), no. 11, 1–21.

[41] Shiqing Zhang and Qing Zhou, *Double pyramidal central configurations*, Physics Letters A 281 (2001), no. 4, 240–248.