Entropy of random symbolic high-order bilinear Markov chains

S. S. Melnik and O. V. Usatenko

O. Ya. Usikov Institute for Radiophysics and Electronics
Academy of Science, 12 Proskura Street, 61805 Kharkiv, Ukraine

The main goal of this paper is to develop an estimate for the entropy of random stationary ergodic symbolic sequences with elements belonging to a finite alphabet. We present here the detailed analytical study of the entropy for the high-order Markov chain in the bilinear approximation. The appendix contains a short comprehensive introduction into the subject of study.

PACS numbers: 05.40.-a, 87.10+e, 07.05.Mh

In the paper [1], we have presented results of our study for the entropy of random long-range correlated symbolic sequences with elements belonging to a finite alphabet. As a plausible model, we have used the high-order additive Markov chain. Supposing that the correlations between random elements of the chain are weak we have expressed the conditional entropy of the sequence by means of symbolic pair correlation functions. Here we present detailed analytical calculations for the entropy of the high-order Markov chain in the bilinear approximation. The appendices contain a comprehensive introduction in the matter of high-order Markov chains.

I. HIGH-ORDER MARKOV CHAINS

Consider a semi-infinite random stationary ergodic sequence

\[ A = a_0, a_1, a_2, \ldots \]  

of symbols (letters) \( a_i \) taken from the finite alphabet

\[ \mathcal{A} = \{\alpha^1, \alpha^2, \ldots, \alpha^m\}, \quad a_i \in \mathcal{A}, \quad i \in \mathbb{N}_+ = \{0, 1, 2, \ldots\}. \]  

We use the notation \( a_i \) to indicate a position of the symbol \( a \) in the chain and the notation \( a^k \) to stress the value of the symbol \( a \in \mathcal{A} \).

We suppose that the symbolic sequence \( A \) is a high-order Markov chain [2–4]. Such sequences are also referred to as multi- or \( N \)-step [5, 6], or categorical [7] Markov’s chains. One of the most important and interesting application of the symbolic sequences is the probabilistic language model specializing in predicting the next item in the sequence by means of \( N \) previous known items. There the Markov chain is known as the \( N \)-gram model.

The sequence \( A \) is the \( N \)-step Markov’s chain if it possesses the following property: the probability of symbol \( a_i \) to have a certain value \( \alpha^k \in \mathcal{A} \) under condition that all previous symbols are given depends only on \( N \) previous symbols,

\[ P(a_i = \alpha| a_{i-N}, \ldots, a_{i-2}, a_{i-1}) = P(a_i = \alpha| a_{i-N}, \ldots, a_{i-2}, a_{i-1}). \]  

Sometimes the number \( N \) is also referred to as the order or the memory length of the Markov chain.

II. CONDITIONAL ENTROPY

To estimate the conditional entropy of stationary sequence \( A \) of symbols \( a_i \) one could use the Shannon definition [2] for entropy per block of length \( L \),

\[ H_L = - \sum_{a_1, \ldots, a_L \in \mathcal{A}} P(A_1) \log_2 P(A_1). \]  

Here \( P(A_1) = P(a_1, \ldots, a_L) \) is the probability to find \( L \)-word \( a_1^L \) in the sequence; hereafter we use the concise notation \( a_{i-1}^{-1} \) for \( N \)-word \( a_{i-N}, \ldots, a_{i-1} \). Instead the term word one often uses the words: subsequence, string or tuple. The conditional entropy, or the entropy per symbol, is given by

\[ h_L = H_{L+1} - H_L. \]  

This quantity specifies the degree of uncertainty of \( (L + 1) \)th symbol occurring and measures the average information per symbol if the correlations of \( (L + 1) \)th symbol with preceding \( L \) symbols are taken into account. The conditional entropy \( h_L \) can be represented in terms of the conditional probability function \( P(a_{L+1}|a_1^L) \),

\[ h_L = \sum_{a_1, \ldots, a_L \in \mathcal{A}} P(a_1^L) h(a_1^L) = h(a_1^L), \]  

where \( h(a_1^L) \) is the amount of information contained in the \( (L + 1) \)th symbol of the sequence conditioned on \( L \) previous symbols \( a_1^L \),

\[ h(a_1^L) = - \sum_{a_{L+1} \in \mathcal{A}} P(a_{L+1}|a_1^L) \log_2 P(a_{L+1}|a_1^L). \]  

The source entropy (or Shannon entropy) is the conditional entropy at the asymptotic limit, \( h = \lim_{L \to \infty} h_L \). This quantity measures the average information per symbol if all correlations, in the statistical sense, are taken into account, cf. with [10], Eq. (3).

Supposed stationarity of the random sequence under study together with decay of correlations, \( C_{\alpha,\beta}(r \to \infty) \to 0 \), see below definition [13], lead, according to the Slutsky sufficient conditions [11], to the mean-ergodicity. Due to the ergodicity, the ensemble average of any function \( f(a_{r_1}, a_{r_1+r_2}, \ldots, a_{r_1+\ldots+r_s}) \) of \( s \) arguments defined...
on the set $A$ of symbols can be replaced by the statistical (arithmetic, Cesàro’s) average over the chain. This latter property is very useful in numerical calculations since the averaging can be done over the sequence and the ensemble averaging can be avoided. Therefore, in our numerical as well as analytical calculations, we always apply averaging over the length of the sequence as it is implied in Eq. \(8\).

### III. CORRELATION FUNCTIONS AND WORDS

If the sequence, statistical properties of which we would like to analyze, is given, the conditional probability distribution function (CPDF) of $N$th order can be found by a standard method (written below for subscript $i = N + 1$)

\[
P(a_{N+1} = \alpha^k | a_1^N) = \frac{P(a_N^N, \alpha^k)}{P(a_1^N)}, \tag{8}
\]

where $P(a_N^N, \alpha^k)$ and $P(a_1^N)$ are the probabilities of the $(N + 1)$-subsequence $a_1^N$, $\alpha^k$ and $N$-subsequence $a_1^N$ occurring, respectively.

The conditional probability function completely determines all statistical properties of the random chain and the method of its generation. Equation \(8\) says that the CPDF is determined if we know the probability of $(N + 1)$-words occurring – the words containing $(N + 1)$ symbols without omissions among their indexes. Obviously, the average number of some word $a_1^N$ occurring in whole sequence exponentially decreases with the word length $L$. Let us evaluate the length $L_{\max}$ of word, that occurs on average one time. For given length $M$ of weakly correlated sequence with fixed dimension $m$ of the alphabet this length, evidently, is equal to $L_{\max} \approx M/ \ln m$.

To make this evaluation more precise we should take into account that the correlations decrease the number of typical words that one can encounter in the sequence and this phenomenon increases the length $L_{\max}$. From the famous result of the theory of information, known under the name of the Shannon-McMillan-Breiman theorem \([12]\), it follows

\[
L_{\max} \sim \frac{\log_2 M}{h}, \tag{9}
\]

where $h$ is the conditional entropy per letter of sequence under condition that all correlations are taken into account. This is a crucial point, because the correlation lengths of natural sequences of interest are usually of the same order as the sequence length, whereas the last inequality can only be fulfilled for the maximal lengths of the words $L_{\max} \lesssim 10$.

The words of the length $L \ll L_{\max}$ are well represented in the sequence, so that one can use the statistical approach to these objects and calculate directly the probabilities of their occurrence in the chain. By contrast, the statistics of longer words, $L \gtrsim L_{\max}$, are insufficient and the whole sequence for such words is not anymore probabilistic object. Some papers devoted to this question even put under doubt the correctness of the notion of “finite random sequence” \([13, 14]\).

So, if the correlation length $R_c$ of sequence is less than $L_{\max}$, then the random sequence should be considered as quasi-ergodic because the words of the length $L \leq R_c < L_{\max}$ provide statistically meaningful information for reconstructing the conditional probability function of the sequence.

We meet a completely different situation when $L_{\max} < R_c$. In this case the statistical properties of the studied sequence can be reconstructed only up to the length of order $L \ll L_{\max}$. Statistically important information on the properties of the sequence in the interval $L_{\max} < L < R_c$ is inaccessible in the frame of discussed likelihood estimation method.

For simplicity of further qualitative consideration let us fix our attention on the pair correlation function only. If we know statistics of $(N + 1)$-words, we also know the correlation function for $r \leq N$. Nevertheless, for the given sequence of length $M$, we can calculate $C_{\alpha \beta}(r)$ at $r$, which is of order of $M$. Really, for a weakly correlated sequence, the probability $P(a_i = \alpha, a_{i+r} = \beta)$ to have the pair of letters $\alpha$ and $\beta$ at the distance $r$ is equal to $p_{\alpha \beta}$. This quantity determines the number of pairs in the hole sequence. The number of pairs is a slowly decreasing function of $r$. As above we can evaluate the distance $r_{\max}$ between the pair of letters that occurs on average one time. Thus, from definition of the probability $P(a_i = \alpha, a_{i+r} = \beta) = N_{\alpha \beta}/(M - r)$, where $N_{\alpha \beta}$ is the number of pairs $ab$ in the interval $M - r$, we have $r_{\max} \sim M/1/(p_{\alpha \beta})$. It is clear that $r_{\max}$ can be much greater than $L_{\max}$.

For $k$-order correlation functions or $k$-words, the estimation is $r_{\max}^{(k)} \sim M - 1/(p_{\alpha_1 \ldots \alpha_k})$.

Let us note that in the frames of both methods we cannot take into account the correlation functions of order exceeding $L_{\max}$. This quantity determines both the maximal length of words, without or with omission of symbols among them in the sequence (in mathematics such sets are known under the name of cylinder ones), and the maximal order of correlation functions, which can be used to describe statistical properties of the sequence. In the general case the differences among the arguments of the correlation functions are limited by $r_{\max}^{(k)}$. The information about the region $L_{\max} \lesssim L \ll \min(R_c, r_{\max})$ is introduced in consideration by means of the memory functions, which are expressed through the correlation functions.

A method that allows us to use the information on the symbols spaced by a distance $r \ll \min(R_c, r_{\max})$, not only in narrower region with $r \ll L_{\max}$, is connected with the high-order additive and bilinear Markov chains, a construction proposed in Ref. \([5, 15, 16]\).
The bilinear Markov chain is determined by the conditional probability distribution function of the form

\[
P(a_i = \alpha|a_{i-N}^{i-1}) = p_\alpha + \sum_{r=1}^{N} \sum_{\beta \in A} F_{\alpha\beta}(r) [\delta(a_{i-r}, \beta) - p_\beta]
\]

\[
+ \sum_{1=r_1 < r_2} \sum_{\beta, \gamma \in A} F_{\alpha\beta\gamma}(r_1, r_2) [[\delta(a_{i-r_1}, \beta) - p_\beta][\delta(a_{i-r_2}, \gamma) - p_\gamma] - C_{\gamma\beta}(r_2 - r_1)],
\]

where \( p_\alpha \) is the relative number of symbols \( \alpha \) in the chain, or their probabilities of occurring,

\[
p_\alpha = \delta(a_i, \alpha).
\]

Here \( \delta(., .) \) is the Kronecker delta-symbol. The quantities \( F_{\alpha\beta}(r) \) and \( F_{\alpha\beta\gamma}(r_1, r_2) \) are the so called memory functions. In Appendix A some suggestions on the form of Eq. (10) and its properties are presented.

As a rule, the statistical properties of random sequences are determined by correlation functions. The symbolic correlation functions of the \( k \)th order are given by the following expression,

\[
C_{\beta_1...\beta_k}(r_1, r_2, \ldots, r_{k-1}) = [\delta(a_0, \beta_1) - p_{\beta_1}] \ldots [\delta(a_{r_1+...+r_{k-1}}, \beta_k) - p_{\beta_k}].
\]

The overline means a statistical average over an ensemble of sequences. Note that in some sense symbolic correlation functions-matrices are more general construction than numeric correlation functions. They can describe in more detail even numeric sequences.

There were suggested two methods for finding the memory functions of a sequence with a known conditional function. The first one \( 15 \) is based on the minimization of the “distance” between the conditional probability function, containing the sought-after memory function, and the given sequence \( A \) of symbols with a known correlation function,

\[
\text{Dist} = \sum_{\alpha} |\delta(a_i, \alpha) - P(a_i = \alpha|a_{i-N}^{i-1})|^2.
\]

In this equation a given sequence \( A \) is presented by the Kronecker delta-function, the unknown parameters are the memory functions of the CPDF.

The second method for deriving the equations connecting the memory and correlation functions is a completely probabilistic straightforward calculation analogous to that used in \( 5, 16 \). These equations, despite its simplicity, can be analytically solved only in some particular cases: for one- or two-step chains, the Markov chain with a step-wise memory function and so on. To avoid the various difficulties in its solving we suppose that correlations in the sequence are weak (in amplitude, but not in length). An approximate solution for the memory function allows one to obtain the following simple formulas

\[
F_{\alpha\beta}(r) = \frac{C_{\alpha\beta}(r)}{p_\beta}, F_{\alpha\beta\gamma}(r_1, r_2) = \frac{C_{\alpha\beta\gamma}(r_2 - r_1, r_1)}{p_\beta p_\gamma}.
\]

Equation (10) together with Eq. (14) provide a tool for constructing weak correlated sequences with given pair and third-order correlation functions \( 15, 11 \). Note that \( i \)-independence of the function \( P(a_i = \alpha|a_{i-N}^{i-1}) \) provides homogeneity and stationarity of the sequences under consideration; and finiteness of \( N \) together with the strict inequalities

\[
0 < P(a_{i+N} = \alpha|a_i^{i+N-1}) < 1, i \in \mathbb{N}_+, \{0, 1, 2, \ldots\}
\]

provides, according to the Markov theorem (see, e.g., Ref. \( 17 \)), ergodicity of the sequences.

V. ENTROPY OF BILINEAR CHAIN

The conditional probability distribution function \( P(a_i = \alpha|a_{i-N}^{i-1}) \) determined by Eq. (10) gives the probability to have a symbol \( a_i = \alpha \) after \( N \)-word \( a_{i-N}^{i-1} \). Nevertheless, we would like to know the conditional entropy not only after \( N \)-word, but in all range of \( L \)-words. The conditional probability \( P(a_i = \alpha|a_{i-L}^{i-1}) \) for a word of length \( L < N \) can be obtained in the first approximation in the weak correlation parameter \( \Delta_\alpha(L) \) from general definition Eqs. (3) by means of a routine probabilistic reasoning presented in Appendix B. The rule to obtain the CPDF consists in replacement in Eq. (11) \( N \to L \)

\[
P(a_i = \alpha|a_{i-L}^{i-1}) = \begin{cases} P(a_i = \alpha|a_{i-N}^{i-1}), & L \geq N \\ P(a_i = \alpha|a_{i-N}^{i-1})|_{N-L}, & L < N. \end{cases}
\]

The first line follows from the markovian property of the CPDF, Eq. (3).

We suppose that the random sequence is weakly correlated and present CPDF, Eq. (11), as

\[
P(a_i = \alpha|a_{i-N}^{i-1}) = p_\alpha + \Delta_\alpha(a_{i-N}^{i-1}),
\]

where \( \Delta_\alpha(L) \) is the relative number of symbols \( \alpha \) in the chain.
admitting that the strong inequalities

$$|\Delta_\alpha| \ll p_\alpha,$$

are fulfilled.

Expanding the conditional entropy $h(a^L_1)$ in series with respect to the small $\Delta$ we have

$$h \approx h_0 + \sum_\alpha \frac{\partial h}{\partial P(\alpha)} \Delta_\alpha + \frac{1}{2} \sum_{\alpha\beta} \frac{\partial^2 h}{\partial P(\alpha)\partial P(\beta)} \Delta_\alpha \Delta_\beta,$$

where

$$h = h(a^L_1), h_0 = -\sum_{\alpha \in A} p_\alpha \log_2 p_\alpha, \quad P(\alpha) = P(a_{L+1} = \alpha|a^L_1).$$

It is important to note that the number of independent variables of the function $h$ is equal to $m-1$, because the sum,

$$\sum_{\alpha \in A} P(a_{L+1} = \alpha|a^L_1) = 1,$$

expressing the normalization condition of the CPDF, is fixed. By this reason, it is convenient to present Eq. (19) in the form,

$$h = -\frac{1}{\ln 2} \sum_{i=1}^{m-1} P(\alpha^i) \ln P(\alpha^i) - \frac{1}{\ln 2} (1 - \sum_{i=1}^{m-1} P(\alpha^i)) \ln(1 - \sum_{i=1}^{m-1} P(\alpha^i)),$$

containing independent variables only. After differentiation of Eq. (21) and substitution of obtained derivatives (taken at the point $P(a_i = \alpha|a_{i-L}^L = p_\alpha)$) in Eq. (19), we have

$$h = h_0 - \frac{1}{\ln 2} \sum_{\alpha \in A} \Delta_\alpha \ln p_\alpha - \frac{1}{2 \ln 2} \sum_{\alpha \in A} \frac{\Delta^2_\alpha}{p_\alpha},$$

The quantities $\Delta_\alpha$ depend on a concrete $L$-word $a_1^L$ preceding the generated symbol $a_{L+1} = \alpha$. To obtain the conditional entropy $h_L$ we should substitute (22) into (10) to average it. As a result we have

$$h_L = h_0 - \frac{1}{2 \ln 2} \sum_{\alpha \in A} \frac{\Delta^2_\alpha}{p_\alpha}. \quad (23)$$

In the case of weak correlations, we can calculate the dispersion of $\Delta_\alpha$ after expressing the memory functions by means of correlation functions,

$$\Delta_\alpha = \sum_{r=1}^L \sum_{\beta} C^{\beta\alpha}_r \left[\delta(a_{i-r}, \beta) - p_\beta\right]$$

$$+ \sum_{r_1 < r_2} \sum_{\beta,\gamma} C^{\beta\gamma\alpha}_{\beta\gamma} \left[\delta(a_{i-r_1}, \beta) - p_\beta\right] \left[\delta(a_{i-r_2}, \gamma) - p_\gamma\right] - C^{\beta\gamma}_{\beta\gamma} (r_2 - r_1).$$

Under calculation of $\Delta_{\alpha}^2$, there appear three terms. The first one is

$$D^{(11)}_\alpha = \sum_{\substack{r,r'=1 \beta,\beta'}} \sum_{\substack{\beta,\gamma,\beta'}} \frac{C^{\beta\alpha}_r}{p_\beta} \frac{C^{\beta\gamma\alpha}_{\beta\gamma}}{p_\beta} \left[\delta(a_{i-r}, \beta) - p_\beta\right] \left[\delta(a_{i-r'}, \beta') - p_\beta\right] \frac{C^{\beta\gamma}_{\beta\gamma}}{p_\beta} (r - r').$$

Here the second term is equal zero, because $\Delta_\alpha$ is taken at the “equilibrium point” $P(a_i = \alpha|a_{i-L}^L = p_\alpha)$. As a result, we have

$$h_L = h_0 - \frac{1}{2 \ln 2} \sum_{\alpha \in A} \frac{\Delta^2_\alpha}{p_\alpha}. \quad (23)$$
In the weak-correlation approximation, the main contribution in $D_\alpha^{(11)}$ gives the term with the function $C_{\beta\beta'}(r - r')$ taken at $r' = r$, 

$$
D_\alpha^{(11)} = \frac{1}{p_\beta} \sum_{r=1}^{L} \sum_\beta C_{\beta\alpha}(r) C_{\beta'\gamma\alpha}(r, r_1, r_2) C_{\beta'\gamma\beta}(r - r, r_1 - r).
$$

(26)

where we have taken into account $C_{\alpha\beta}(0) = p_\alpha \delta(\alpha, \beta) - p_\alpha p_\beta, C_{\beta,\beta}(0) \simeq p_\beta$. In the same way, we obtain a contribution to $D_\alpha^{(12)}$:

$$
D_\alpha^{(12)} = \sum_{r=1}^{L} \sum_\beta C_{\beta\alpha}(r) C_{\beta'\gamma\alpha}(r, r_1, r_2) C_{\beta'\gamma\beta}(r - r, r_1 - r).
$$

(27)

Since none of the correlators has zero arguments (in the third term one of arguments may take a zero value, but others are not equal zero) the hole expression is small with respect to the term $D_\alpha^{(11)}$ in the limiting case of small correlations.

For the last contribution $D_\alpha^{(22)}$, one gets:

$$
D_\alpha^{(22)} = \sum_{1=r_1 < r_2}^{L} \sum_\beta C_{\beta\alpha}(r, r_1) C_{\beta'\gamma\alpha}(r_2, r_1) C_{\beta'\gamma\beta}(r_2 - r_1, r_1 - r_2).
$$

(28)

where all arguments of four-order correlation functions, for convenience of further analysis, are expressed by means of the distances between the current and generated $a_i$ symbols.

The four-order correlation function takes a maximal value under condition of coincidence of two pair of its arguments: $r_1 = r'_1$, $r_2 = r'_2$, when it takes the value $C_{\beta\beta'}(0) C_{\gamma\gamma'}(0)$. The second term in the square bracket is small with respect to the first one. From here one obtains for $D_\alpha^{(22)}$:

$$
D_\alpha^{(22)} = \sum_{1=r_1 < r_2}^{L} \sum_\beta \frac{C_{\beta\alpha}(r, r_1)}{p_\beta p_\gamma}.
$$

So, taking into account the obtained above expressions for $D_\alpha^{(11)}$ $D_\alpha^{(22)}$, the average deviation $\Delta_\alpha$ is presented by means of second and third-order correlation functions. Substituting their expressions in Eq. (26), we get the desired result for the conditional entropy in the limiting case of weak second and third order correlations, 

$$
h_L = h_0 - \frac{1}{2\ln 2} \left[ \sum_{r=1}^{L} \sum_\alpha \frac{C_{\alpha\beta}(r)}{p_\alpha p_\beta} + \sum_{r_1 < r_2} \sum_\alpha \frac{C_{\alpha\beta}(r_1, r_2)}{p_\alpha p_\beta p_\gamma} \right].
$$

(29)

If the length of block exceeds the memory length, $L > N$, the conditional probability $P(a_i = \alpha|a_{i-1}^{i-1})$ depends only on $N$ previous symbols, see Eqs. (6) and (10). Then, it is easy to show from (12) that the conditional entropy remains constant at $L \geq N$. This property can be used for the numeric definition of the sequence memory length.

### Appendix A

Accepting definition (3) of the high-order Markov chain as a starting point, we present in this section different models for the conditional probability distribution function (CPDF) of symbolic random sequences. It is helpful to present it as a finite polynomial series containing $N$ Kronecker delta-symbols,

$$
P(a_i = \alpha|a_{i-1}, \ldots, a_{i-N}) = \sum_{\beta_1, \ldots, \beta_N \in \mathcal{A}} \prod_{r=1}^{N} \delta(a_{i-r}, \beta_r).
$$

(A1)

This form of CPDF express some “independence” of the random variables $a$ and the spatial coordinates $i$. The function $F_{\alpha;\beta_1, \ldots, \beta_N}$ is referred to as the generalized memory function and the Kronecker delta-symbols play the role of the indicator function of random variable $a_{i-r}$ converting symbols to numbers 0 or 1.

Let us decouple the memory function $F_{\alpha;\beta_1, \ldots, \beta_N}$ and present it in the form of the sum of memory functions of $k$th order, $F^{(k)} = F_{\alpha;\beta_1, \ldots, \beta_k}(r_1, \ldots, r_k)$, 

$$
F_{\alpha;\beta_1, \ldots, \beta_N} = \sum_{k=0}^{N} \prod_{r=1}^{k} F_{\alpha;\beta_1, \ldots, \beta_k}(r_1, \ldots, r_k),
$$

(A2)

where all symbols $r_s$ at the right hand side of Eq. (A2) are different, ordered, 

$$
1 \leq r_1 < r_2 < \ldots < r_k \leq N,
$$

(A3)
and contain all different subsets \( \{r_1, \ldots, r_k\} \) picked out from the set \( \{1, \ldots, N\} \). The coordinates \( r_k \) of the memory function \( F_{\alpha; \beta, \ldots, \beta_k}(r_1, \ldots, r_k) \) indicate positions of elements \( a_{i-r_k} \) taking on the values \( \beta_k \).

Uncorrelated sequence known also as a discrete white noise or the Bernoulli scheme is defined by the past-independent function

\[
P(a_i = \alpha|a_{i-N}^{-1}) = P(a_i = \alpha) = p_\alpha.
\]

(A4)

It is the simplest and most well studied random sequence. This sequence can be obtained by taking into account of random variable \( F \) ordinary markovian

\[
\text{white noise}
\]

or the

\[
\text{ordinary function}
\]

from the set \( \{0, 1\} \). \( \text{Ordinary markovian} \)

\[
\text{random sequence}
\]

\[
\text{additive white noise}
\]

or the Bernoulli scheme is defined by the past-independent function

\[
P(a_i = \alpha|a_{i-N}^{-1}) = P(a_i = \alpha) = p_\alpha.
\]

(A5)

The CPDF of this sequence is obtained from Eq. (A2) by taking into account two terms \( F^{(0)} = F_\alpha \) and \( F^{(1)} = F_{\alpha\beta}(1) \).

One-step Markov chain or the ordinary markovchain is given by the two-parameter transition probability matrix function \( p_{\alpha\beta} \),

\[
P(a_i = \alpha|a_{i-N}^{-1}) = P(a_i = \alpha|a_{i-1} = \beta) = p_{\alpha\beta}.
\]

(A5)

The CPDF of this sequence is obtained from Eq. (A2) by taking into account two terms \( F^{(0)} = F_\alpha \) and \( F^{(1)} = F_{\alpha\beta}(1) \).

Additive high-order Markov chain. For this random sequence the CPDF takes on the “linear form” with respect to the Kronecker delta-symbols,

\[
P_{\text{add}}(a_i = \alpha|a_{i-N}, \ldots, a_{i-2}, a_{i-1}) = p_\alpha + \sum_{r=1}^N \sum_{\beta \in A} F_{\alpha\beta}(r) [\delta(a_{i-r}, \beta) - p_\beta].
\]

(A6)

The additivity means that the previous symbols \( a_{i-N}^{-1} \) exert an independent effect on the probability of the symbol \( a_i = \alpha \) occurring. The conditional probability function in form (A5) can reproduce correctly the pair (two-point) correlations in the chain. The higher-order correlators and all correlation properties of higher orders cannot be reproduced correctly by means of the memory function \( F_{\alpha\beta}(r) \). To understand better how we can obtain Eq. (A5), let us consider its simpler forms.

Binary additive high-order Markov chain with step-wise memory function. For the binary state space \( a_i \in \{0, 1\} \) the conditional probability distribution function to have the symbol “1” after \( N \)-word containing \( k \) unities, is supposed to be of the form,

\[
P(a_{N+1} = 1 \mid \underbrace{1 1 \ldots 1}_{m} \underbrace{0 0 \ldots 0}_{N-m} ) = \frac{1}{2} + \mu \frac{2k}{N} - 1.
\]

(A7)

Here the correlation parameter \( \mu \) belongs to the region determined by inequality \(-1/2 < \mu < 1/2\). It is exactly solvable model [3, 13, 19].

Bilinear high-order Markov chain. The CPDF of this chain Eq. (A10) is the direct generalization of Eq. (A6). More detailed explanation is given in Ref. [10].

Appendix B

Here we prove Eq. (10) using Eq. (10) as a starting point. From the definition of the CPDF it follows

\[
P(a_i = \alpha|W) = \frac{P(W, \alpha)}{P(W)}, \quad W = a_{i-N}^{-1}.
\]

(B1)

Adding symbol \( a_{i-N} = \beta \) to the word \( W, \alpha \), we have

\[
P(a_i = \alpha|W) = \frac{\sum_{\beta \in A} P(\beta, W, \alpha)}{P(W)}.
\]

(B2)
Replacing here the probabilities \( P(\beta, W, \alpha) \) by the CPDF \( P(a_i = \alpha | \beta, W) \) from the equation similar to that of Eq. (B1),
\[
P(a_i = \alpha | \beta, W) = \frac{P(\beta, W, \alpha)}{P(\beta, W)}, \tag{B3}
\]
we obtain
\[
P(a_i = \alpha | W) = \frac{1}{P(W)} \sum_{\beta \in A} P(\beta, W) \{ P(a_i = \alpha | \beta, W) \}. \tag{B4}
\]

We obtain them after separation \( P(a_i = \alpha | a_{i-N+1}^{i-1}) \) from the term \( P(a_i = \alpha | a_{i-N}^{i-1}) \). In the paper \(^{[1]}\) it was shown that the term Eq. (B5) is small with respect to the others terms of the sum over \( r \in (1,...,N - 1) \) in Eq. (10). So, we should show that the term (B6) also is small.

In Eq. (B6), let us consider the factor
\[
\sum_{\beta \in A} P(\beta, W) [\delta(\beta, \rho) - p_\rho]
\]
and present it in the form
\[
P(\rho, W)(1 - p_\rho) - P(\overline{\rho}, W)p_\overline{\rho}, \tag{B7}
\]
where the symbol \( \overline{\rho} \) stands for an event NOT-\( \rho \). It is intuitively clear that in the zero approximation in \( \Delta \) (i.e., for uncorrelated sequence, when \( P(\rho, W) \approx P(\rho)P(W) \)) this term equals zero, \( P(\rho, W)(1 - p_\rho) - P(\overline{\rho}, W)p_\overline{\rho} \approx P(W)p_\rho(1 - p_\rho) - p_\overline{\rho}p_\overline{\rho} \). In the next approximation this term is of order of \( \Delta \). These two statements can be verified by using the condition of compatibility for the Chapman-Kolmogorov equation (see, for example, Ref. \(^{[20]}\)),
\[
P(a_{i-N+1}^i) = \sum_{a_{i-N}^i} P(a_{i-N}^i)P_N(a_i | a_{i-N}^{i-1}). \tag{B8}
\]

The term in Eq. (B6) containing \( C(\gamma, 1 - N - r_1) \) is of the same order as considered above Eq. (B7), because of the inequality \( N \neq r_1 \) in Eq. (10) is fulfilled.

Hence, we should neglect both terms Eq. (B5) and Eq. (B6); they are of the second order in \( \Delta \). So, Eq. (16) is proven for \( L = N - 1 \). By induction, the equation can be written for arbitrary \( L < N \).

---

1. S. S. Melnik, O. V. Usatenko, Phys. Rev. E 93, 062144 (2016).
2. A. Raftery, J. R. Stat. Soc. B 47, 528 (1985).
3. M. Seifert, A. Gohr, M. Strickert, I. Grosse, PLoS Computat. Biol, 8, e1002286 (2012).
4. P. C. Shields, The ergodic theory of discrete sample paths (Graduate studies in mathematics, 13, 1996).
5. S. S. Melnik, O. V. Usatenko, V. A. Yampol’skii, and V. A. Golick, Phys. Rev. E 72, 026140 (2005).
6. O. V. Usatenko, S. S. Apostolov, Z. A. Mayzelis, and S. S. Melnik, Random Finite-Valued Dynamical Systems: Additive Markov Chain Approach (Cambridge Scientific Publisher, Cambridge, 2010).
7. O. V. Usatenko, V. A. Yampol’skii, Phys. Rev. Lett. 90, 110601 (2003).
8. R. Hosseini, N. Leb, J. Zideka, Journal of Statistical Theory and Practice, 5, 261 (2011).
9. C. E. Shannon and W. Weaver, The Mathematical Theory of Communication (University of Illinois Press, Urbana, Illinois, 1949).
10. P. Grassberger, arXiv:physics/0207023 [physics.data-an].
11. See, e.g., A. M. Yaglom, Correlation theory of stationary and related random functions (Springer-Verlag, New York, 1987).
12. T. M. Cover, J. A. Thomas, Elements of Information Theory, second edition (New York, Wiley, 2006).
13. https://math.dartmouth.edu/ doyle/docs/random /random.pdf
14. V. A. Uspensky, A. Kh. Shen, Math. Systems Theory, 29, 271 (1996).
15. S. S. Melnyk, O. V. Usatenko, V. A. Yampol’skii, Physica A 361, 405 (2006).
16. S. S. Melnik and O. V. Usatenko, arXiv.org > physics > arXiv:1703.07764 to be published in Phys. Rev. E.
[17] A. N. Shiryaev, *Probability* (Springer, New York, 1996).
[18] O. V. Usatenko, V. A. Yampol’skii, K. E. Kechedzhy and S. S. Mel’nyk, Phys. Rev. E 68, 061107 (2003).
[19] S. S. Melnyk, O. V. Usatenko, V. A. Yampol’skii, S. S. Apostolov, Z. A. Mayselis, J. Phys. A: Math. Gen. 39, 14289 (2006).

[20] C. W. Gardiner: *Handbook of Stochastic Methods for Physics, Chemistry, and the Natural Sciences*, Springer Series in Synergetics, Vol. 13 (Springer-Verlag, Berlin, 1985).