Quantum entropies, Schur concavity and dynamical semigroups

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Abstract. Entropy plays a fundamental role in several branches of physics. In the quantum setting, one usually considers the von Neumann entropy, but other useful quantities have been proposed in the literature; e.g., the Rényi and the Tsallis entropies. The evolution of an open quantum system, described by a semigroup of dynamical maps (in short, a dynamical semigroup), may decrease a quantum entropy, for some initial condition. We will discuss various characterizations of those dynamical semigroups that, for every initial condition, do not decrease a general class of quantum entropies, which is defined using the notion of Schur concavity of a function. We will not assume that such a dynamical semigroup be completely positive, the physical justification of this condition being controversial. Therefore, we will consider semigroups of trace-preserving, positive — but not necessarily completely positive — linear maps. We will next focus on a special class of (completely positive) dynamical semigroups, the twirling semigroups, having applications in quantum information science. We will argue that the whole class of dynamical semigroups that do not decrease a quantum entropy can be obtained as a suitable generalization of the twirling semigroups.

1. Introduction and main ideas

Entropy is a fundamental concept in several branches of physics; in particular, in the theory of open quantum systems [1, 2] and in quantum information science [3]. In this setting, various quantities have been considered; e.g., the von Neumann, the Tsallis and the Rényi entropies [4].

A natural problem is to study the possible temporal evolutions of an open quantum system that do not decrease a certain entropy, for every initial condition of the system [5]. Under suitable hypotheses, the evolution of an open quantum system is well described by a dynamical semigroup [2, 6]; i.e., by a semigroup of operators [7] consisting of dynamical maps. Given a complex Hilbert space $\mathcal{H}$ (for the sake of simplicity, we will deal with a finite-dimensional — say, with a $N$-dimensional, $N \geq 2$ — Hilbert space), by a dynamical map we mean a linear map

$$\mathcal{Q}: \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}),$$

with $\mathcal{B}(\mathcal{H})$ denoting the space of linear operators in $\mathcal{H}$ (or $N \times N$ complex matrices), characterized by the following defining properties:

(i) $\mathcal{Q}$ is positive, i.e., $\mathcal{B}(\mathcal{H}) \ni \hat{A} \geq 0 \implies \mathcal{Q} \hat{A} \geq 0$;

(ii) $\mathcal{Q}$ is trace-preserving, i.e., $\text{tr}(\mathcal{Q} \hat{A}) = \text{tr}(\hat{A})$, for all $\hat{A} \in \mathcal{B}(\mathcal{H})$.
Note that, throughout the paper, we work in the Schrödinger picture.

One often assumes that $\mathcal{Q}$ is, more specifically, completely positive; i.e., that

(iii) for every $n = 1, 2, \ldots$, the map

$$\mathcal{Q} \otimes \text{Id}_n : \mathcal{B}(\mathcal{H}) \otimes \mathbb{C}^{n \times n} \to \mathcal{B}(\mathcal{H}) \otimes \mathbb{C}^{n \times n}$$

— where $\mathbb{C}^{n \times n}$ is regarded as the space $\mathcal{B}(\mathbb{C}^n)$ of $n \times n$ matrices and $\text{Id}_n$ is the identity map in this space — is positive.

This further assumption is usually justified with the possible — extremely weak, thus negligible — coupling of the relevant physical system with another system, a non-evolving $n$-dimensional ‘ancilla’ [6], $n \geq 2$. However, the validity of this and other similar arguments is controversial [8]. Moreover, the intriguing idea of describing the evolution of the universe itself, and the associated change of the entropy, by a dissipative master equation — see [9] and references therein — does not seem to entail, in any reasonable way, the notion of complete positivity.

In this paper, we will therefore consider the problem of characterizing the one-parameter semigroups of dynamical maps that do not decrease a suitable class of quantum entropies. To define this class of functions, we recall that the convex body of density operators in $\mathcal{H}$ — the physical states — can be endowed with a natural transitive, reflexive binary relation (a preorder) $\prec$, usually called majorization; see sect. 2.

**Definition 1.** Let $\mathcal{S}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$ be the convex body of density operators in $\mathcal{H}$ (the unit trace, positive operators), and let $\hat{\rho}_* := \mathbb{N}^{-1} \hat{I} \in \mathcal{S}(\mathcal{H})$, with $\hat{I}$ denoting the identity operator in $\mathcal{H}$, be the maximally mixed state. By a quantum entropy we mean a map $\mathcal{E} : \mathcal{S}(\mathcal{H}) \to \mathbb{R}$ such that

(E1) $\mathcal{E}$ is Schur concave w.r.t. the natural majorization relation in $\mathcal{S}(\mathcal{H})$, namely,

$$\hat{\omega} \prec \hat{\rho} \implies \mathcal{E}(\hat{\omega}) \geq \mathcal{E}(\hat{\rho});$$

(E2) $\mathcal{E}$ has a strict global maximum at $\hat{\rho}_*$.

The axioms (E1) and (E2) are coherent, because $\hat{\rho}_*$ is majorized by any other state. Examples of functions satisfying them — including the von Neumann entropy — will be discussed in sect. 2 (these entropies actually satisfy slightly more stringent properties). We now assign a precise meaning to the statement that a dynamical semigroup does not decrease a quantum entropy.

**Definition 2.** We say that a dynamical semigroup $\{\mathcal{Q}_t : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})\}_{t \in \mathbb{R}^+}$ does not decrease a quantum entropy $\mathcal{E} : \mathcal{S}(\mathcal{H}) \to \mathbb{R}$ if

$$\mathcal{E}(\mathcal{Q}_t \hat{\rho}) \geq \mathcal{E}(\hat{\rho}),$$

for all $\hat{\rho} \in \mathcal{S}(\mathcal{H})$ and all $t \geq 0$. We say that $\{\mathcal{Q}_t\}_{t \in \mathbb{R}^+}$ is entropy-nondecreasing if it does not decrease every quantum entropy (satisfying axioms (E1) and (E2)).

Note that a dynamical semigroup $\{\mathcal{Q}_t\}_{t \in \mathbb{R}^+}$, by the semigroup property ($\mathcal{Q}_{t+s} = \mathcal{Q}_t \mathcal{Q}_s$), does not decrease the entropy $\mathcal{E}$ if and only if $\mathcal{E}(\mathcal{Q}_{t+s} \hat{\rho}) \geq \mathcal{E}(\mathcal{Q}_t \hat{\rho})$, for all $\hat{\rho} \in \mathcal{S}(\mathcal{H})$ and all $t, s \geq 0$. It is also worth anticipating one of the results outlined in sect. 3: a dynamical semigroup does not decrease a certain quantum entropy if and only if it is entropy-nondecreasing. Otherwise stated, the solution of our problem does not depend on the choice of a particular type of entropy.

The class of entropy-nondecreasing dynamical semigroups turns out to contain as a remarkable subclass — see sect. 4 — the so-called twirling semigroups [10–16]. This type of semigroups of operators, first considered by Kossakowski [17] in the early 1970s, are characterized by an integral expression involving a representation of a locally compact group and a convolution semigroup of probability measures [18] on that group.

A related interesting fact, discussed in sect. 4, is that the whole class of entropy-nondecreasing dynamical semigroups can be obtained as a suitable generalization of the twirling semigroups. Indeed, they admit an integral expression where the semigroups of probability measures are replaced with certain (more general) families of signed measures.
2. Quantum entropies and concavity

Given a density operator \( \hat{\rho} \in S(\mathcal{H}) \), the von Neumann entropy of \( \hat{\rho} \) is defined by

\[
\mathcal{S}(\hat{\rho}) := -\text{tr}(\hat{\rho} \ln \hat{\rho}) = -\sum_{k=1}^{N} p_k \ln p_k ,
\]

where \( \{p_1, \ldots, p_N\} \) is the whole set of the eigenvalues of \( \hat{\rho} \) (repeated taking into account degeneracy; here, \( 0 \ln 0 \equiv 0 \)). Notice that the definition of \( \mathcal{S}(\hat{\rho}) \) does not depend on the way the eigenvalues \( \{p_k\} \) of \( \hat{\rho} \) are ordered, and accordingly in the following \( \{p_k\} \) will be regarded as an unordered set. As it will be clear soon, \( \mathcal{S} \) is a quantum entropy in the sense of Definition 1.

Other examples of entropies considered in the literature are [4, 5, 19]:

- The Tsallis entropy \( \mathcal{T}_q \), labeled by the parameter \( q \), \( 0 < q \neq 1 \); namely,

\[
\mathcal{T}_q(\hat{\rho}) := \frac{1}{1-q} (\text{tr}(\hat{\rho}^q) - 1) = \tau_q(\{p_k\}) ,
\]

where the set \( \{p_k\} \) is defined as above and

\[
\tau_q(\{p_k\}) := \frac{1}{1-q} \left( \sum_{k=1}^{N} p_k^q - 1 \right) .
\]

Clearly, as previously noted for the von Neumann entropy, the quantity \( \tau_q(\{p_k\}) \) does not depend on the ordering of the elements of the probability distribution \( \{p_k\} \); i.e., \( \tau_q \) may be regarded as a symmetric function of the probability vector \( (p_1, \ldots, p_N) \). Observe that

\[
\mathcal{S}(\hat{\rho}) = \lim_{q \to 1} \mathcal{T}_q(\hat{\rho}) = \lim_{q \to 1} \tau_q(\{p_k\}) = -\sum_{k=1}^{N} p_k \ln p_k =: \tau_1(\{p_k\}) .
\]

It is then natural to set

\[
\mathcal{T}_1(\hat{\rho}) \equiv \mathcal{S}(\hat{\rho}) .
\]

For \( q = 2 \), the Tsallis entropy is directly related to the purity \( \mathcal{P}(\hat{\rho}) := \text{tr}(\hat{\rho}^2) \) [20], since

\[
\mathcal{T}_2(\hat{\rho}) = 1 - \mathcal{S}(\hat{\rho}) .
\]

- The Rényi entropy \( \mathcal{R}_q \), \( 0 < q \neq 1 \); namely,

\[
\mathcal{R}_q(\hat{\rho}) := \frac{1}{1-q} \ln \text{tr}(\hat{\rho}^q) = \frac{1}{1-q} \ln \left( \sum_{k=1}^{N} p_k^q \right) =: \varrho_q(\{p_k\}) .
\]

Once again we have:

\[
\mathcal{R}_1(\hat{\rho}) \equiv \mathcal{S}(\hat{\rho}) = \lim_{q \to 1} \mathcal{R}_q(\hat{\rho}) = \lim_{q \to 1} \varrho_q(\{p_k\}) .
\]

For \( q = 2 \), the Rényi entropy is related to the purity too: \( \mathcal{R}_2(\hat{\rho}) = -\ln \mathcal{P}(\hat{\rho}) \).

- Consider also the function

\[
\mathcal{A}_p : S(\mathcal{H}) \ni \hat{\rho} \mapsto (1 - \|\hat{\rho}\|_p) \in \mathbb{R} , \quad p > 1 ,
\]

where \( \| \cdot \|_p \) is the Schatten \( p \)-norm in \( \mathcal{B}(\mathcal{H}) \):

\[
\| \hat{A} \|_p := \text{tr}(|\hat{A}|^p)^{1/p} .
\]
With the same notation as above, we have:

$$\mathcal{A}_p(\hat{\rho}) = 1 - \left( \sum_{k=1}^{N} p_k^p \right)^{1/p} =: \alpha_p(\{p_k\}), \quad p > 1. \quad (14)$$

Note that the following relations hold:

$$\frac{1}{N} \sum_{k=1}^{N} p_k^{1-p} \leq \left( \sum_{k=1}^{N} p_k^p \right)^{1/p} \leq 1. \quad (15)$$

These inequalities imply that

$$0 \leq \alpha_p(\{p_k\}) \leq 1 - \frac{1}{N} \sum_{k=1}^{N} p_k^{1-p}, \quad (16)$$

where the upper inequality is saturated by the probability distribution \(\{1/N, \ldots, 1/N\}\) only. Thus, \(\mathcal{A}_p\) has a strict global maximum at the maximally mixed state \(\rho_*\). To describe the points where \(\mathcal{A}_p\) attains its minimum value it is convenient to label the eigenvalues \(\{p_k\}\) of \(\hat{\rho}\) in decreasing order — \(p_1 \geq \cdots \geq p_N \geq 0\) — in such a way to obtain an ordered probability vector \(\vec{p} = \vec{p}(\hat{\rho}) = (p_1, \ldots, p_N)\). Then, the lower inequality in (16) is saturated by every probability distribution \(\{p_k\}\) with corresponding ordered vector of the form \(\vec{p} = (1, 0, \ldots, 0)\); i.e., \(\mathcal{A}_p\) attains its minimum at every pure state. The same properties hold for the von Neumann, the Tsallis and the Rényi entropies too.

The notion of ordered vector induces a natural majorization relation [21, 22], a preorder. Indeed, as previously noted, with every vector \(x = (x_1, \ldots, x_n)\) in \(\mathbb{R}^n\) one can associate the vector \(\vec{x} = (\vec{x}_1, \ldots, \vec{x}_n)\) obtained by rearranging the coordinates in decreasing order: \(\vec{x}_1 \geq \cdots \geq \vec{x}_n\). Then, for \(x, y \in \mathbb{R}^n\), we say that \(x\) is majorized by \(y\) — \(x \prec y\) — whenever the following relations hold:

$$\sum_{k=1}^{j} \vec{x}_k \leq \sum_{k=1}^{j} \vec{y}_k, \quad 1 \leq j \leq n, \quad \text{and} \quad \sum_{k=1}^{n} \vec{x}_k = \sum_{k=1}^{n} \vec{y}_k. \quad (17)$$

If \(x = (x_1, \ldots, x_n)\) is a probability vector — \(x_1 \geq 0, \ldots, x_n \geq 0\) and \(\sum_{k=1}^{n} x_k = 1\) — we have:

$$p_* := (1/n, \ldots, 1/n) \prec (x_1, \ldots, x_n) \prec (1, 0, \ldots, 0). \quad (18)$$

If both \(x \prec y\) and \(y \prec x\) — in symbols, \(x \simeq y\) — then \(x = Py\), for some permutation matrix \(P\). Clearly, \(p_*\) is strictly majorized by any other vector belonging to the probability simplex \(\Delta_{n-1}\) in \(\mathbb{R}^n\), \(n \geq 2\), because \(\Delta_{n-1} \ni x \prec p_* \implies x = p_*\).

Note that majorization between (eigenvalue) vectors induces a corresponding majorization relation between density operators [1, 4]:

$$\hat{\omega} \prec \hat{\rho} \iff \vec{p}(\hat{\omega}) \prec \vec{p}(\hat{\rho}). \quad (19)$$

The basic idea is that, for \(\hat{\omega} \prec \hat{\rho}\) and \(\hat{\rho} \neq \hat{\omega}\) (\(\hat{\omega}\) strictly majorized by \(\hat{\rho}\)), the state \(\hat{\omega}\) will be ‘more mixed’ or ‘more chaotic’ than \(\hat{\rho}\) [1]. In particular, the maximally mixed state \(\hat{\rho}_* = \frac{1}{N} I \in S(\mathcal{H})\) is strictly majorized by any other state and every state is majorized by any pure state.

Therefore, a very natural property to be considered for defining a quantum entropy is the so-called Schur convexity, which has been indeed used in Definition 1. Recall that a function \(\mathcal{F}: S(\mathcal{H}) \rightarrow \mathbb{R}\) is said to be Schur concave if

$$\hat{\omega}, \hat{\rho} \in S(\mathcal{H}), \quad \hat{\omega} \prec \hat{\rho} \implies \mathcal{F}(\hat{\omega}) \geq \mathcal{F}(\hat{\rho}); \quad (20)$$

in particular, we say that \(\mathcal{F}\) is strictly Schur concave if \(\hat{\omega} \prec \hat{\rho}\), with \(\hat{\omega} \neq \hat{\rho}\) — i.e., \(\vec{p}(\hat{\omega}) \neq \vec{p}(\hat{\rho})\) — implies that \(\mathcal{F}(\hat{\rho}) > \mathcal{F}(\hat{\omega})\).
Remark 1. Since the maximally mixed state $\hat{\rho}_m$ is strictly majorized by any other state, a strictly Schur concave function $\mathcal{E} : \mathcal{S}(\mathcal{H}) \to \mathbb{R}$ satisfies both the axioms (E1) and (E2) in Definition 1. It is quite natural, moreover, to assume that $\mathcal{E}(\hat{\rho}) = \mathcal{E}(\hat{\omega})$, for $\hat{\rho} \approx \hat{\omega}$ (i.e., for $\hat{\rho} = \hat{\omega}$). This condition is equivalent to the unitary invariance of $\mathcal{E}$: $\mathcal{E}(\hat{\rho}) = \mathcal{E}(\hat{U}\hat{\rho}\hat{U}^*)$, for every $\hat{\rho} \in \mathcal{S}(\mathcal{H})$ and every unitary operator $\hat{U}$ in $\mathcal{H}$. We will call a unitarily invariant, strictly Schur concave real function on $\mathcal{S}(\mathcal{H})$ a proper quantum entropy.

The following result — see [5] — provides a convenient recipe for generating proper quantum entropies ad libitum.

**Proposition 1.** If $h$ is a real function on $\mathbb{R}^n$, strictly decreasing — alternatively, strictly increasing — w.r.t. each of its arguments, and $g_1, \ldots, g_n$ are strictly convex — respectively, strictly concave — continuous real functions on the interval $[0, 1]$, then

$$\mathcal{S}(\mathcal{H}) \ni \hat{\rho} \mapsto (h(tr(g_1(\hat{\rho})), \ldots, tr(g_n(\hat{\rho})))) =: \mathcal{E}(\hat{\rho})$$

is a proper quantum entropy.

**Proposition 2.** The families of functions $\{\mathcal{T}_q\}_{q>0}$, $\{\mathcal{R}_q\}_{q>0}$ — that include the von Neumann entropy ($q = 1$) — and $\{\mathcal{S}_p\}_{p>1}$ are proper quantum entropies.

**Proof.** Apply Proposition 1. E.g., in the case of the von Neumann entropy, we have: $n = 1$, $g : [0, 1] \ni \xi \mapsto -\xi \ln \xi$ (concave) and $h(x) = x$ (increasing). \quad \Box

Remark 2. By Klein’s inequality [4], one can show that the von Neumann entropy is also concave, in the ordinary sense: for every pair $\hat{\rho}, \hat{\omega} \in \mathcal{S}(\mathcal{H})$, $\mathcal{S}(\hat{\rho} + (1 - \epsilon)\hat{\omega}) \geq \epsilon \mathcal{S}(\hat{\rho}) + (1 - \epsilon) \mathcal{S}(\hat{\omega})$, $\epsilon \in [0, 1]$. The same property holds for the entropy $\mathcal{S}_p$ too, for all $p > 1$. For the Tsallis entropy $\mathcal{T}_q$, $q > 0$, the standard concavity holds, in general, in a slightly weaker sense: the associated function $\tau_q$ (see (7)), regarded as a symmetric function on the the probability simplex $\Delta_{N-1} \subset \mathbb{R}^N$, is concave. For the Rényi entropy $\mathcal{R}_q$, even this weaker property is no longer true, in general (i.e., for a sufficiently large $q > 1$): see Remark 9 of [5].

In connection with the previous remark, it is worth noting the following fact:

**Proposition 3.** A function $\mathcal{F} : \mathcal{S}(\mathcal{H}) \to \mathbb{R}$ is unitarily invariant — $\mathcal{F}(\hat{\rho}) = \mathcal{F}(\hat{U}\hat{\rho}\hat{U}^*)$, for every $\hat{\rho} \in \mathcal{S}(\mathcal{H})$ and every unitary operator $\hat{U}$ in $\mathcal{H}$ — if and only if it is of the form

$$\mathcal{F}(\hat{\rho}) = \varphi(\{p_k\}),$$

(22)

where $\{p_k\}$ is the whole set of the eigenvalues of $\hat{\rho}$, repeated taking into account degeneracy, and $\varphi$ can be regarded as a symmetric function on the probability simplex $\Delta_{N-1} \subset \mathbb{R}^N$. Moreover, if $\mathcal{F}$ is of the form (22) and the function $\varphi$ is concave on $\Delta_{N-1}$ — in particular, if $\mathcal{F}$ is concave on $\mathcal{S}(\mathcal{H})$ and of the form (22) — then $\mathcal{F}$ is Schur concave.

**Proof.** Let $\lambda : \{p_k\} \mapsto \{p_k\}$ be the function which assigns to each density operator in $\mathcal{H}$ the (unordered) set of its eigenvalues, repeated according to degeneracy. The relation $\lambda(\hat{\rho}) = \lambda(\hat{\omega})$ is equivalent to the fact that $\hat{\omega} = \hat{U}\hat{\rho}\hat{U}^*$, for some unitary operator $\hat{U}$ in $\mathcal{H}$. Therefore, if $\mathcal{F}$ is unitarily invariant, then $\lambda(\hat{\rho}) = \lambda(\hat{\omega}) \Rightarrow \mathcal{F}(\hat{\rho}) = \mathcal{F}(\hat{\omega})$. It follows that in this case $\mathcal{F} = \varphi \circ \lambda$, for some map $\varphi$ which can be regarded as symmetric function on the probability simplex $\Delta_{N-1}$. The reverse implication is clear. At this point, to complete the proof it is sufficient to observe that a concave function on a convex symmetric domain in $\mathbb{R}^n$ — e.g., on the probability simplex $\Delta_{N-1}$ — is Schur concave if (and only if) it is symmetric; see Remark 10 of [5]. \quad \Box

We stress that Remark 2 and Proposition 3 provide further support to our previous claim that a very natural defining property for a quantum entropy is the Schur concavity (rather than the standard concavity).
3. Dynamical semigroups that do not decrease a quantum entropy

In this section, given a quantum entropy $\mathcal{E}$ on $\mathcal{S}(\mathcal{H})$ (satisfying the axioms (E1) and (E2) in Definition 1) — e.g., the von Neumann entropy, or some of the entropies $\mathcal{J}_q$, $\mathcal{R}_q$, $\mathcal{A}_q$ defined in the previous section — we will provide various characterizations of the dynamical semigroups that do not decrease this quantity.

Like every semigroup of operators [7], a dynamical semigroup $\{\Omega_t : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})\}_{t \in \mathbb{R}^+}$ — a continuous (w.r.t. any norm topology in $\mathcal{B}(\mathcal{H})$) one-parameter semigroup of trace-preserving, positive linear maps — is completely characterized by its infinitesimal generator $\mathcal{L}$:

$$\mathcal{L} = \lim_{t \downarrow 0} t^{-1}(\Omega_t - \mathrm{Id}), \quad \Omega_t = \exp(\mathcal{L} t).$$

We will denote by $\langle \cdot, \cdot \rangle_{\text{HS}}$ the Hilbert-Schmidt scalar product in $\mathcal{B}(\mathcal{H})$, and the adjoint of a linear map in $\mathcal{B}(\mathcal{H})$ will be understood as relative to this pairing.

Without assuming, in general, the complete positivity, the following result holds [5]:

**Theorem 1.** Let $\{\Omega_t : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})\}_{t \in \mathbb{R}^+}$ be a (continuous) semigroup of linear maps, with generator $\mathcal{L}$. Then, the following properties are equivalent:

- $\{\Omega_t\}_{t \in \mathbb{R}^+}$ is positive, trace-preserving and does not decrease the entropy $\mathcal{E}$;
- $\{\Omega_t\}_{t \in \mathbb{R}^+}$ is positive, trace-preserving and entropy-nondecreasing;
- $\{\Omega_t\}_{t \in \mathbb{R}^+}$ is positive, trace-preserving and purity-nonincreasing: for every $t \geq 0$ and every $\hat{\rho} \in \mathcal{S}(\mathcal{H})$, $\mathcal{P}(\Omega_t, \hat{\rho}) \leq \mathcal{P}(\hat{\rho})$;
- $\{\Omega_t\}_{t \in \mathbb{R}^+}$ is positive, trace-preserving and, for every $t \geq 0$ and every $\hat{\rho} \in \mathcal{S}(\mathcal{H})$, $\Omega_t \hat{\rho} < \hat{\rho}$;
- $\{\Omega_t\}_{t \in \mathbb{R}^+}$ is positive, trace-preserving and leaves the maximally mixed state invariant: $\Omega_t \hat{\rho}_\text{max} = \hat{\rho}_\text{max}$, $t \geq 0$;
- for every set $\{\hat{\psi}_j\} \subset \mathcal{B}(\mathcal{H})$ of mutually orthogonal rank-one (selfadjoint) projectors forming a resolution of the identity $- \sum_j \hat{\psi}_j \hat{\psi}_k = \hat{I}$ —

$$\mathrm{tr} \left( \hat{\psi}_j \left( \mathcal{L} \hat{\psi}_k \right) \right) \geq 0, \text{ for } j \neq k,$$

and

$$\sum_{j=1}^{N} \mathrm{tr} \left( \hat{\psi}_j \left( \mathcal{L} \hat{\psi}_k \right) \right) = 0 = \sum_{k=1}^{N} \mathrm{tr} \left( \hat{\psi}_j \left( \mathcal{L} \hat{\psi}_k \right) \right);$$

- for every pair of mutually orthogonal (selfadjoint) projectors $\hat{P}, \hat{Q} \in \mathcal{B}(\mathcal{H})$,

$$\langle \hat{P}, \mathcal{L} \hat{Q} \rangle_{\text{HS}} \geq 0 \text{ and } \langle \hat{I}, \mathcal{L} \hat{P} \rangle_{\text{HS}} = 0 = \langle \hat{P}, \mathcal{L} \hat{I} \rangle_{\text{HS}}.$$  

The reader will have noted in particular that, as anticipated in the introduction, the property of a dynamical semigroup of not decreasing a certain quantum entropy is equivalent to the property of being entropy-nondecreasing; i.e., of not decreasing every quantum entropy.

It is also worth observing, with regard to the last two equivalent properties in Theorem 1, that in dimension two a more explicit, but still relatively simple, characterization of the infinitesimal generator of an entropy-nondecreasing dynamical semigroup is provided by the following result [5]:

**Proposition 4.** For $\dim(\mathcal{H}) = 2$, the general form of the generator of an entropy-nondecreasing dynamical semigroup is given by

$$\mathcal{L} \hat{A} = -i \sum_{j=1}^{3} h_j [\hat{S}_j, \hat{A}] + \sum_{j,k=1}^{3} \varepsilon_{jk} \left( \hat{S}_j \hat{A} \hat{S}_k - \frac{1}{2} (\hat{S}_j \hat{S}_k \hat{A} + \hat{A} \hat{S}_j \hat{S}_k) \right),$$

$$j, k = 1, 2, 3; \quad h_j, \varepsilon_{jk} \in \mathbb{R}; \quad \varepsilon_{12} = \varepsilon_{23} = \varepsilon_{31} = 1.$$
Proposition 5. The infinitesimal generators $L$ is positive semidefinite, symmetric real matrix. Thus, such a generator $L$ is selfadjoint — if and only if $h_1 = h_2 = h_3 = 0$.

Assuming now the complete positivity, one can prove a slightly stronger result [5], w.r.t. Theorem 1. Indeed, the generators of completely positive dynamical semigroups admit a complete classification, the Gorini-Kossakowski-Lindblad-Sudarshan canonical form [23, 24]. Therefore, one can describe, within this general classification, the typical form of the generators that give rise to an entropy-nondecreasing temporal evolution.

Theorem 2. Let $\{\Omega_t: B(\mathcal{H}) \to B(\mathcal{H})\}_{t \in \mathbb{R}^+}$ be a completely positive dynamical semigroup with generator $L$. Then, the following properties are equivalent:

- $\{\Omega_t\}_{t \in \mathbb{R}^+}$ does not decrease the entropy $E$;
- $\{\Omega_t\}_{t \in \mathbb{R}^+}$ is entropy-nonincreasing;
- $\{\Omega_t\}_{t \in \mathbb{R}^+}$ is purity-nonincreasing;
- for every $t \geq 0$ and every $\tilde{\rho} \in S(\mathcal{H})$, $\Omega_t \tilde{\rho} < \tilde{\rho}$;
- $\{\Omega_t\}_{t \in \mathbb{R}^+}$ leaves the maximally mixed state invariant: $\Omega_t \hat{\rho}_s = \hat{\rho}_s$, $t \geq 0$;
- the adjoint semigroup of $\{\Omega_t\}_{t \in \mathbb{R}^+}$ is trace-preserving;
- $L$ is of the form

$$L \dot{A} = -i[H, A] + \bar{g} \dot{A} - \frac{1}{2} \left( (\bar{g} \dot{I} + \dot{A} (\bar{g} \dot{I})) \right) ,$$

where $H$ is a selfadjoint operator and $\bar{g}$ a completely positive map in $B(\mathcal{H})$ such that

$$\bar{g} \dot{I} = \bar{g}^* \dot{I} ;$$

- $L$ is of the form

$$L \dot{A} = -i[H, A] + \sum_{k=1}^{n^2-1} \gamma_k \left( \dot{F}_k A \dot{F}_k^* - \frac{1}{2} (\dot{F}_k^* \dot{F}_k A + A \dot{F}_k^* \dot{F}_k) \right) ,$$

where $H$ is a trace-less selfadjoint operator, $\gamma_1 \geq 0, \ldots, \gamma_{n^2-1} \geq 0$, and $\dot{F}_1, \ldots, \dot{F}_{n^2-1}$ are trace-less operators such that

$$\langle \dot{F}_j, \dot{F}_k \rangle_{HS} = \delta_{jk}, \quad j, k = 1, \ldots, n^2 - 1 ,$$

and

$$\sum_{k=1}^{n^2-1} \gamma_k \dot{F}_k \dot{F}_k^* = \sum_{k=1}^{n^2-1} \gamma_k \dot{F}_k^* \dot{F}_k .$$

Once again, in dimension two one has a somewhat simpler result for the characterization of the infinitesimal generators [5]:

Proposition 5. For $\dim(\mathcal{H}) = 2$, the general form of the infinitesimal generator of an entropy-nondecreasing, completely positive dynamical semigroup is given by (27), where $\hat{S}_1, \hat{S}_2, \hat{S}_3$ are the spin operators, $h_1, h_2, h_3$ are arbitrary real numbers and $\mathcal{K} := (\kappa_{jk})$ is a positive semidefinite, symmetric real matrix.
4. Twirling semigroups and beyond

We will now focus on a type of completely positive dynamical semigroups first considered by Kossakowski [17] in the early 1970s. Not only are these semigroups of operators interesting for applications (e.g., in quantum information [12, 14, 15]), but they also form a remarkable class of entropy-nondecreasing dynamical semigroups. Moreover, they are in some sense “prototypical”, because every entropy-nondecreasing dynamical semigroup can be realized as a generalization of this type of completely positive dynamical semigroups, in a sense to be clarified below.

Given a locally compact (second countable, Hausdorff) topological group $G$, a continuous unitary representation $\mathcal{V}: G \rightarrow U(\mathcal{H})$ of $G$ in $\mathcal{H} = U(\mathcal{H})$ denoting the unitary group of $\mathcal{H}$, endowed with the strong topology — and a convolution semigroup of probability measures $\{\mu_t\}_{t \in \mathbb{R}^+}$ on $G$ [10, 18], the family of maps $\{\Omega_t: \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})\}_{t \in \mathbb{R}^+}$, with

$$\Omega_t \hat{A} := \int_G \mathcal{V}(g) \hat{A} \mathcal{V}(g)^* \, d\mu_t(g),$$

(35)

turns out to be a completely positive dynamical semigroup; a twirling semigroup [10–12]. Typically, $G$ will be a Lie group, and in fact it can be shown that the pair $(\mathcal{V}, \{\mu_t\}_{t \in \mathbb{R}^+})$ generating $\{\Omega_t\}_{t \in \mathbb{R}^+}$ is not unique: to obtain a generic twirling semigroup, one can set $G = SU(\mathfrak{n})$ and choose $\mathcal{V}$ as the defining representation (with the identification of $U(\mathfrak{n})$ with $U(\mathcal{H})$), where $\{\mu_t\}_{t \in \mathbb{R}^+}$ spans the whole set of convolution semigroups of probability measures on $SU(\mathfrak{n})$ [10].

Clearly, the maximally mixed state $\hat{\rho}$ is left invariant by the twirling semigroup $\{\Omega_t\}_{t \in \mathbb{R}^+}$, which is then entropy-nondecreasing (by Theorem 2). The generator of $\{\Omega_t\}_{t \in \mathbb{R}^+}$ can be written as

$$\mathcal{L} \hat{A} = -i[H, \hat{A}] + \mathfrak{D} \hat{A} - \frac{1}{2} \left((\mathfrak{D} \mathcal{I}) \hat{A} + \hat{A} (\mathfrak{D} \mathcal{I})\right), \quad \hat{A} \in \mathcal{B}(\mathcal{H}),$$

(36)

with $\hat{H}$ denoting a trace-less selfadjoint operator (determining the Hamiltonian component of the generator) and $\mathfrak{D}$ a completely positive map (the dissipative component of $\mathcal{L}$) of the form

$$\mathfrak{D} = \sum_{k=1}^{N^2-1} \gamma_k \hat{L}_k(\cdot) \hat{L}_k + \gamma_0 \mathcal{I},$$

(37)

where $\gamma_0, \ldots, \gamma_{N^2-1}$ are non-negative numbers, $\hat{L}_1, \ldots, \hat{L}_{N^2-1}$ trace-less selfadjoint operators such that $\hat{L}_j \hat{L}_k^* = \delta_{jk}$ and $\mathcal{I}$ a random unitary map in $\mathcal{B}(\mathcal{H})$; namely, $\mathcal{I}$ admits a decomposition of the type

$$\mathcal{I} \hat{A} = \sum_{j=1}^{N^2} p_j \hat{U}_j \hat{A} \hat{U}_j^*,$$

(38)

for some set $\{\hat{U}_j\}_{j=1}^{N^2}$ of unitary operators in $\mathcal{H}$ and some probability distribution $\{p_j\}_{j=1}^{N^2}$. Note that the operators

$$\hat{U}_1, \ldots, \hat{U}_{N^2}, \hat{L}_1, \ldots, \hat{L}_{N^2-1}$$

(39)

are normal; hence, the condition $\mathfrak{D} \mathcal{I} = \mathfrak{D}^* \mathcal{I}$ is satisfied (coherently with Theorem 2). Conversely, every map $\mathcal{L}$ of the form (36)–(37) is the generator of a twirling semigroup [10].

The class of all twirling semigroups turns out to coincide with the class of random unitary semigroups acting in $\mathcal{B}(\mathcal{H})$ [10, 12], namely, of those completely positive dynamical semigroups whose elements are random unitary maps. It is known [25], on the other hand, that for $N = \text{dim}(\mathcal{H}) = 2$ every completely positive dynamical map in $\mathcal{B}(\mathcal{H})$ which leaves $\hat{\rho}$ invariant is actually a random unitary map (while for $N > 2$ this is no longer true). By this reasoning and by Theorem 2, we conclude that, for $N = 2$, the class of twirling semigroups simply coincides with the class of entropy-nondecreasing, completely positive dynamical semigroups. Therefore, we have the following result:
Proposition 6. In the case where \( \dim(\mathcal{H}) = 2 \), a completely positive dynamical semigroup \( \{\Omega_t : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})\}_{t \in \mathbb{R}^+} \) is entropy-nondecreasing if and only if it is a twirling semigroup. In the integral form of such a semigroup of operators, one can set \( G = SU(2) \) and choose \( \mathcal{V} : G \to U(\mathcal{H}) \) as the defining representation.

Interestingly, replacing in (35) the convolution semigroup of probability measures \( \{\mu_t\}_{t \in \mathbb{R}^+} \) with a suitable family of signed measures, one gets the whole class of entropy-nondecreasing dynamical semigroups. Precisely, the following result holds [5]:

**Theorem 3.** A family of maps \( \{\Omega_t : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})\}_{t \in \mathbb{R}^+} \) is a one-parameter semigroup of entropy-nondecreasing, trace-preserving, positive linear maps if and only if it can be expressed in the form

\[
\Omega_t \hat{A} = \int_G \mathcal{V}(g) \hat{A} \mathcal{V}(g)^* \ d\varsigma_t(g),
\]

where \( \{\mathcal{V}, \{\varsigma_t\}_{t \in \mathbb{R}^+}\} \) is a pair formed by a continuous unitary representation \( \mathcal{V} : G \to U(\mathcal{H}) \) of a locally compact group \( G \) and by a family \( \{\varsigma_t\}_{t \in \mathbb{R}^+} \) of finite, signed Borel measures on \( G \) satisfying the conditions (always verified if \( \{\varsigma_t\}_{t \in \mathbb{R}^+} \) is a convolution semigroup of probability measures):

1. \( \varsigma_0 \) coincides with the Dirac measure at the identity of \( G \) and \( \varsigma_t(G) = 1 \), for all \( t \geq 0 \);  
2. for some — hence, for every — orthonormal basis \( \Psi \equiv \{\psi_k\}_{k=1}^N \) in \( \mathcal{H} \),

\[
\lim_{t \downarrow 0} \int_G v_{jklm}(\Psi; g) \ d\varsigma_t(g) = \delta_{jk} \delta_{lm}
\]

and

\[
\int_G v_{jklm}(\Psi; g) \ d(\varsigma_s \circ \varsigma_t)(g) = \int_G v_{jklm}(\Psi; g) \ d\varsigma_{s+t}(g), \quad \forall s, t > 0,
\]

where \( v_{jklm}(\Psi; g) := \langle \psi_j, \mathcal{V}(g)\psi_k \rangle \langle \mathcal{V}(g)\psi_l, \psi_m \rangle \) and the measure \( \varsigma_s \circ \varsigma_t \) is the convolution of \( \varsigma_s \) with \( \varsigma_t \);  
3. for every orthonormal basis \( \Psi \equiv \{\psi_k\}_{k=1}^N \) in \( \mathcal{H} \),

\[
\lim_{t \downarrow 0} t^{-1} \int_G v_{jkj}(\Psi; g) \ d\varsigma_t(g) \geq 0, \quad j \neq k.
\]

(It can be shown that the limit (43) exists, provided that (M2) is verified.)

Here, one can always assume that \( G = SU(N) \) and choose \( \mathcal{V} \) as the defining representation.

It seems natural to call a semigroup of operators of the form (40), where the family of signed measures \( \{\varsigma_t\}_{t \in \mathbb{R}^+} \) satisfies conditions (M1)–(M3), a *generalized twirling semigroup*. A dynamical semigroup is entropy-nondecreasing if and only if it is a generalized twirling semigroup.

5. Conclusions

An interesting problem is the characterization of the dynamical semigroups that do not decrease a quantum entropy, for every initial state. Studying this problem one is led in a natural way to consider a general class of quantum entropies satisfying two simple axioms, relying on the notion of Schur concavity [5]; see Definition 1. The von Neumann, the Tsallis and the Rényi entropies all satisfy these axioms. They actually satisfy slightly more stringent hypotheses: they are *proper* quantum entropies; namely, unitarily invariant, strictly Schur concave functions on \( S(\mathcal{H}) \). It turns out that, however, the solution of our problem does not depend neither on the fact of considering, in particular, a proper entropy, nor on any specific choice in the whole class of quantum entropies (Theorem 1). A dynamical semigroup that does not decrease a certain entropy is “entropy-nondecreasing *tout court*.” In hindsight, this is quite natural. Such
a dynamical semigroup simply maps any state into a not less mixed state; thus, the maximally mixed state $\rho_s$ is stationary. Conversely, if $\rho_s$ is left invariant by a dynamical semigroup, then this is entropy-nondecreasing. In the case of completely positive, entropy-nondecreasing dynamical semigroups, on can exhibit an explicit characterization of the generators, within the Gorini-Kossakowski-Lindblad-Sudarshan classification; see condition (34) in Theorem 2. Banks et al [9] found the selfadjointness of the operators $\hat{F}_1, \ldots, \hat{F}_{d^2-1}$ to be a condition sufficient to ensure that the associated dynamical semigroups do not decrease the von Neumann entropy. Actually this condition amounts to selecting a subclass out of the class of twirling semigroups; see (36)–(37). For $\dim(H) = 2$, the twirling semigroups saturate the class of entropy-nondecreasing, completely positive dynamical semigroups (Proposition 6). In general — $\dim(H) \geq 2$, no assumption of complete positivity — the class of entropy-nondecreasing dynamical semigroups coincides with the generalized twirling semigroups (Theorem 3), where the semigroups of probability measures of the standard twirling semigroups are replaced with suitable families of signed measures. This is somewhat reminiscent of the formalism of Wigner functions, where the classical probability distributions are replaced with (signed) quantum quasi-distributions [15,26].

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