The influence of friction on behavior of a hyperelastic body in a conic channel

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Abstract. The model problem about the axisymmetric quasistatic motion of an elastic sphere considering the interaction with inner surface of a rigid cone die is investigated. The constitutive model is represented by the physically nonlinear generalization of the linear model of Hencky’s elastic material. The tensor of logarithmic strains is used by way of strain measure due to described below special properties. The influence of friction on macro characteristics of a process is studied using proposed numerical model of accounting of mixed boundary conditions on time-changing contact surface.

1. Introduction
The quantity of investigations on the modeling of nonlinear elastic behavior of elastomers has been greatly increased during last decade. First of all it deals with the increasing demand of various types calculations of mechanisms containing elastic elements in finite strain zone. The key questions are the following: the choice of material constitutive model, which is able to describe specific nonlinear effects; the identification of the most influential factors in mixed boundary conditions, which affect the qualitative results.

In terms of classical mechanics of solids elastomers are represented as hyperelastic incompressible solid \[1, 2\]. Mooney-Rivlin, Ogden, Yeoh models of materials are the most commonly used in hyperelasticity \[3\]. The new variants of constitutive equations of elastomers, which are well-agreed with known experimental data including biological tissues, were proposed in recent papers \[4–7\]. Indentation test is modern approach to verify new material models and to determine elastic constants of new types of materials \[8–11\]. However, the complexity of experimental data processing is the flip side of simplicity of experimental setup. The initial boundary value problem with boundary conditions of mixed type is to be solved. In this case, it is important to correctly consider factors that are of significant influence on qualitative result. The other factors can be discarded as inconsequential to simplify formulation of problem. The influence of a contact friction, a finite strain kinematics and geometrical parameters on the accuracy and robustness of elastic constants estimation from indentation test was investigated in papers \[8, 9\].

The motion of hyperelastic body inside variable cross-section channel was considered in the paper \[12\]. The results of numerical solving for isotropic material were obtained using original modification of finite element method with b-spline approximation. The analysis for various values of transverse strain coefficient of deformable body was done. The axisymmetric quasistatic
non-isothermal motion of an elastic sphere inside rigid tapering channel was considered in [13] in terms of coupled thermoelastic problem in initial configuration. The influence of Amonton-Coulomb law of friction on stress strain state of a body during motion was recommended to find out.

2. Kinematics of solids

The strain gradient $\Phi(\vec{X}, t) = \mathbf{\nabla} \varphi(\vec{X}, t)$ is a second rank tensor, which associate the material radius vector $\vec{X}$ of initial state and the spatial vectors $\vec{x}$ of current state within the law of motion $\vec{x} = \varphi(\vec{X}, t)$, $i = 1..3$:

$$\Delta \vec{x} = \Delta \vec{X} \cdot \Phi(M, t).$$  \hspace{1cm} (1)

The expression $\mathbf{\nabla} = \mathbf{e}_i \partial / \partial X^i$ introduces Hamilton operator of initial state. The components of strain gradient $[\Phi_{ij}] = \partial x_j / \partial X^i$ coincide with the components of Jacobi matrix of transformation $\varphi(\vec{X}, t)$. Formula $\mathbf{\nabla} = \Phi^{-1} \cdot \Phi$ associates Hamilton operators of current and initial states. Gradient $\Phi$ can be represented in the form of dot products of symmetric and orthogonal tensors as follows:

$$\Phi = \mathbf{U} \cdot \mathbf{R} = \mathbf{R} \cdot \mathbf{V},$$ \hspace{1cm} (2)

where $\mathbf{U}$ and $\mathbf{V}$ are left and right measures of distortion and $\mathbf{R}$ is orthogonal tensor of rotation.

Based on the definition of strain gradient (1) and its polar decomposition (2), the sets of finite strains measures can be constructed. According to the law of their change during rigid rotation one can attribute every strain measure to one of two classes. The invariant spatial measures transform as tensor $\mathbf{U}$ and indifferent material measures transform like $\mathbf{V}$. The set of spatial holonomic isotropic tensor measures is defined in issue [14]:

$$\varepsilon_A = (\mathbf{U} - \mathbf{U}^{-1}) \cdot \left[ (1 + c) \mathbf{U} + (1 - c) \mathbf{U}^{-1} \right]^{-1}, -1 \leq c \leq 1.$$ \hspace{1cm} (3)

The Seth-Hill set of strain measures is one of the most general formulation to obtain commonly known definitions in a uniform manner [15, 16]:

$$E_r(\mathbf{U}) = \begin{cases} \frac{1}{2r^2} (\mathbf{U}^{2r} - \mathbf{E}), & r = R\backslash 0; \\ \lim_{r \to 0} \left( \frac{1}{2r} (\mathbf{U}^{2r} - \mathbf{E}) \right) = \ln \mathbf{U}, r = 0. \end{cases}$$

$$\hat{E}_r(\mathbf{V}) = \begin{cases} \frac{1}{2r^2} (\mathbf{V}^{2r} - \mathbf{E}), & r = R\backslash 0; \\ \lim_{r \to 0} \left( \frac{1}{2r} (\mathbf{V}^{2r} - \mathbf{E}) \right) = \ln \mathbf{V}, r = 0. \end{cases}$$ \hspace{1cm} (4)

Table 1 contains strain measures associated with the various values of parameter $r = \{0, \pm 0.5, \pm 1\}$ in (4).

Let us consider the behavior of some strain measures from table 1 during uniaxial elongation when $\mathbf{U} = \lambda \mathbf{e}_1 \mathbf{e}_1 + \mathbf{e}_2 \mathbf{e}_2 + \mathbf{e}_3 \mathbf{e}_3$. In this case equation (4) can be expressed as $E_r(\mathbf{U}) = \hat{E}_r(\mathbf{V}) = \left\{ \begin{array}{ll} \frac{1}{2r} (\lambda^{2r} - 1), & r = R\backslash 0. \end{array} \right.$ The values of strain measures in the cases of extreme compression $\lambda \to 0$ and extreme elongation $\lambda \to \infty$ are:

$$\lim_{\lambda \to 0} E_r(\lambda) = \begin{cases} -\frac{1}{2r}, & r > 0; \\ -\infty, & r \leq 0. \end{cases} \quad \lim_{\lambda \to \infty} \hat{E}_r(\lambda) = \begin{cases} \infty, & r \geq 0; \\ \frac{1}{2r}, & r < 0. \end{cases}$$ \hspace{1cm} (5)
| $r$ | $E_r(U)$ | $E_r(V)$ | left tensor of logarithmic strains | right tensor of logarithmic strains |
|-----|-----------|-----------|-----------------------------------|-----------------------------------|
| 0   | $\Gamma = \ln U$ | $\Gamma_V = \ln V$ | $\frac{1}{2}(U^2 - E)$ | $\frac{1}{2}(V^2 - E)$ |
| 1   | $\varepsilon = \frac{1}{2}(U^2 - E)$ | $Cauchy - Green$ | Tensor | $\frac{1}{2}(V^2 - E)$ |
| -1  | $\alpha = \frac{1}{2}(E - U^{-2})$ | $Almansi$ tensor | $\mathbf{e} = \frac{1}{2}(E - V^{-2})$ | $Euler - Almansi$ tensor |
| 0.5 | $U - E$ | $Biot$ tensor | $V - E$ | $Almansi - Hamel$ tensor |
| -0.5 | $E - U^{-1}$ | $-$ | $E - V^{-1}$ | $-$ |

It follows from equation (5) that the strain measures with $r > 0$ have finite values under compression in the limit $\lambda \to 0$. The strain measures with $r < 0$ have horizontal asymptote $\frac{1}{r}$ in the limit $\lambda \to \infty$. It can be concluded that the behavior of true logarithmic strain tensor is the closest to the physical behavior of real material under uniaxial elongation among all measures of finite strains of the Seth-Hill set [17].

By definition the relative change in volume $\frac{dV}{dV_0}$ coincides with the third algebraic invariant of measure $U$ and associated with the first invariant of Hencky’s tensor $I_1(\Gamma)$ in the following manner:

$$\frac{dV}{dV_0} = I_3(U) = \lambda_1\lambda_2\lambda_3 = e^\theta,$$

where $\theta = I_1(\Gamma) = \ln \left(\frac{dV}{dV_0}\right)$.

The logarithmic strain tensor $\Gamma$ can be represented by the sum of the following form:

$$\Gamma = \frac{1}{3} \theta \mathbf{E} + \mathbf{\bar{G}}.$$  

The possibility to separate the volumetric deformations which coincide with the volumetric part of the tensor $\Gamma$ and deformations of form changes which coincide with the deviatoric part $\mathbf{\bar{G}}$ follows from equations (6) and (7).

Let us consider the natural invariants of the logarithmic strain tensor $\Gamma$ [18]:

$$\Gamma_0 = \frac{1}{3} \theta, \quad e = \sqrt{\Gamma \cdot \Gamma}, \quad \cos 3\alpha = \frac{3\sqrt{6} \det(\Gamma)}{e^3},$$

where $e$ is the intensity of form change, $\alpha$ is the angle of view of deformed state. Unlike natural invariant of other finite strain measures, invariants (8) have a clear physical meaning. Under pure volumetric deformations $e$ is equal to 0 and under process of pure form change $\theta$ is equal to 0. The angle $\alpha$ defines the orientation of form change vector $\mathbf{\varepsilon}$ on a deviatoric plane.

3. Equilibrium equations of solids

When considering only quasistatic processes it is convenient to use the condition of equilibrium flow of the deformation process [19]. It requires the simultaneous equality to zero of the main load vector and its variation rate with respect to time:

\[ \begin{align*} 
\frac{d}{dt} \mathbf{f} &= \mathbf{0}, \\
\frac{d}{dt} \mathbf{\dot{f}} &= \mathbf{0}.
\end{align*} \]
\[
\begin{align*}
\int_V \left( \nabla \cdot S + \rho \vec{F} \right) \cdot \delta \vec{v} dV &= 0, \\
\int_V \frac{d}{dt} \left( \nabla \cdot S + \rho \vec{F} \right) \cdot \delta \vec{v} + \dot{\theta} \left( \nabla \cdot S + \rho \vec{F} \right) \cdot \delta \vec{v} dV &= 0,
\end{align*}
\]

(9)

where \( S \) is the Cauchy true stress tensor; \( \vec{F} \) is the external field of bulk forces; \( \rho \) is the material density; \( \vec{v} \) is the velocity and \( V \) is the current body volume.

The system (9) can be transformed to the form:

\[
\int_V \left( \dot{\vec{S}} + \vec{S} \dot{\theta} - \vec{v} \nabla \cdot \vec{S} \right) \cdot \delta \left( \vec{v} \nabla \right) dV = \int_{\Sigma} \left( \dot{\vec{P}}^{(n)} + \vec{P}^{(n)} \left( \dot{\theta} - \vec{n} \cdot \vec{W} \cdot \vec{n} \right) \right) \cdot \delta \vec{v} d\Sigma.
\]

(10)

This equation must be supplemented by the relations that connect the rate of stress variations with the rate of strains as well as by the initial and boundary conditions respectively:

\[
\vec{S}(\vec{x}, t_0) = \vec{S}_0(\vec{x}).
\]

(11)

\[
\vec{P} = \vec{P}_0(\vec{x}, t) \quad \vec{x} \in \Sigma_P \quad \forall t > t_0,
\]

\[
\vec{u} = \vec{u}_0(\vec{x}, t) \quad \vec{x} \in \Sigma_u \quad \forall t > t_0.
\]

(12)

One can obtain the expression for the differential of the internal stresses work in the following form:

\[
d'(A^{(i)}) = -\frac{1}{\rho} \vec{S} \cdot \cdot \vec{v} \nabla dt.
\]

(13)

Replacing the current state variables with the initial state variables in (13) allows us to put each strain measure from (4) in accordance with some stress measure. Obtained in such a manner pairs of strain and stress measures are called energy conjugated. It is commonly known that the logarithmic strain tensor \( \Gamma \) and the generalized corotational stress tensor

\[
\Sigma_R = \frac{dV}{dV_0} R \cdot S \cdot R^{-1}
\]

are energy conjugated in isotropic material [19]:

\[
d'(A^{(i)}) = -\frac{1}{\rho} \Sigma_R \cdot \cdot d\Gamma.
\]

(14)

Using tensor \( \Sigma_R \) as stress measure we obtain the resolving equation in the following form:

\[
\int_V \left( \frac{d\vec{v}}{dt} R^{-1} \cdot \Sigma_R \cdot R - \Omega \cdot S + S \cdot \Omega - \vec{v} \nabla \cdot S \right) \cdot \delta \left( \vec{v} \nabla \right) dV =
\]

\[
\int_{\Sigma} \left( \dot{\vec{P}}^{(n)} + \vec{P}^{(n)} \left( \dot{\theta} - \vec{n} \cdot \vec{W} \cdot \vec{n} \right) \right) \cdot \delta \vec{v} d\Sigma,
\]

(15)

where \( \Omega \) is antisymmetric angular velocity tensor.

4. The model of hyperelastic material

The variant of constitutive equation for physically nonlinear isotropic material was described in paper [20]:

\[
\Sigma_R = \sigma_0 \vec{E} + \tau_e \vec{\Gamma} + \tau_q \vec{Q}.
\]

(16)

where \( \sigma_0 = K \theta + \frac{C_1}{6V_0} \theta^2 + \frac{C_2}{6V_0} \varepsilon^2, \tau_e = 2G + \frac{C_2}{3V_0} \theta, \tau_q = C_3 \) are the functions of invariants of tensor \( \Gamma \) (8); \( K, G, C_1, C_2, C_3 \) are the material constants.

If we put the values \( C_1 = C_2 = C_3 = 0 \) into the (16), the constitutive equation reduces to commonly known Hencky’s material model:
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\[ \Sigma_R = K \theta E + 2G \dot{\Gamma}. \] (17)

Note that the equations (16) will coincide with the Murnaghan model if we replace the
invariants of Cauchy strain tensor \( \mathbf{e} \) with the invariants of logarithmic strain tensor \( \tilde{\mathbf{\Gamma}} \) [19].

To have an opportunity to use the equation (16) within formulations in terms of velocities,
let’s find the derivative of (16) with respect to time parameter:

\[ \Sigma_R = \left( (K - \frac{2}{3}G) \dot{\theta} + \left( \frac{C_1}{3\sqrt{3}} - \frac{2C_2}{9\sqrt{3}} + \frac{1}{3}C_3 \right) \theta \dot{\theta} + \left( \frac{C_1}{3\sqrt{3}} - \frac{2}{3}C_3 \right) \mathbf{e} \mathbf{e} \right) \mathbf{E} + \\
+ \left( \frac{C_1}{3\sqrt{3}} - \frac{2}{3}C_3 \right) \theta + \left( 2G + \left( \frac{C_1}{3\sqrt{3}} - \frac{2}{3}C_3 \right) \theta \right) \mathbf{\Gamma} + C_3 \left( \mathbf{\tilde{\Gamma}} \cdot \mathbf{\Gamma} + \mathbf{\Gamma} \cdot \mathbf{\tilde{\Gamma}} \right). \] (18)

With a high degree of certainty most elastomers are incompressible bodies with \( \theta = 0 \). Let us compare the uniaxial tension diagram for model (16) with the diagrams of other known
models [3] taking into account the equality of the initial shear modulus \( G \). Table 2 contains
axial stresses with respect to elongation ratio \( \lambda \) for various models. The equations from table 2
are illustrated in the figure 1.

| Neo Hookean material | Axial stresses \( S \) | Initial shear modulus \( G \) |
|----------------------|----------------------|----------------------|
| \( S_{nh} = 2C_{10} \frac{\lambda^3 - 1}{\lambda} \) | \( 6C_{10} \) |
| \( S_{mr} = 2 \left( C_{10} \lambda^2 + C_{01} \lambda + 3C_{11} (\lambda^3 - \lambda^2 - \lambda + 1) \right) \left( 1 - \lambda^{-3} \right) \) | \( 6C_{10} + 6C_{01} \) |
| \( S_{yeo} = 2 \left( C_{10} \lambda^2 + 2C_{20} (\lambda^3 - 3\lambda + 2) \right) \left( 1 - \lambda^{-3} \right) + 6C_{30} (\lambda^6 - 6\lambda^4 + 4\lambda^3 + 9\lambda^2 - 12\lambda + 4) \left( 1 - \lambda^{-3} \right) \) | \( 6C_{10} \) |
| \( S_{yeo} = 2 \left( \lambda^3 - 3\lambda + 2 \right) \left( \lambda - \lambda^{-2} \right) \left( 6C_{10} + 6C_{01} \right) \) | \( 6C_{10} + 6C_{01} \) |
| \( S_{th} = 2 \left( C_{01} + C_{10} e^{\frac{-11(\lambda-1)(\lambda+2)}{\lambda^3}} \right) \left( \lambda^2 - \frac{1}{\lambda^2} \right) \left( 6C_{10} + 6C_{01} \right) \) | \( 6C_{10} + 6C_{01} \) |
| \( \text{Fung} \) | \( S_{yn} = G \ln(\lambda) \) | \( G \) |
| \( \text{Ishihara} \) | \( S_{ng} = G \ln(\lambda) + \frac{3C_{10}}{4} \ln(\lambda)^2 \) | \( G \) |

Table 2. Uniaxial stress state for some incompressible materials.

The significant differences between models, shown in figure 1, in large strain zone may indicate
the possibility of describing experimental data of any shape by choosing the most suitable model.
It is possible to obtain a minimum relative error in the case of best experimental data fitting
with an array of values of the model parameters. Moreover, the magnitude of error, comparable
with the measurement error during the experiments, can be obtained for several material models
simultaneously. In this case, it is necessary to estimate the number of significant model constants,
the program of experiments to determine model constants, and the computational complexity
of the model. In this sense, the models using logarithmic strains are distinctive among all
hyperelastic models. The Hencky tensor allows one to naturally separate the processes of
volumetric deformation and deformations of form changes under finite strain values. The elastic
constants will have an explicit physical meaning. The constant $C_1$ in (16) takes into account the second order changes in the volume and can be determined from the experiment on hydrostatic compression. The constant $C_2$ allows one to take into account the dilatation phenomena in the material: the appearance of hydrostatic stress under a pure deformation of form changes. If the angle of view of deformed state is equal to $\alpha = \frac{\pi}{6} + \frac{\pi n}{3}$ in some process of form changing, we can conclude that this process is one of two: pure simple shear or biaxial tension-compression. In this case the double dot product of tensors is $\tau_R \cdot Q = A_3 Q \cdot Q \neq 0$. It indicates the misalignment of tensors $dev\tau_R$ and $devH$ or the mutual deviation of the angles of view of stress and strain states.

5. The procedure to set up boundary conditions in contact zone

The finite element method [21] is used to discretize the equations (11)–(12), (15), (18) and allows to reduce the variational problem to the problem of solving of system of linear algebraic equations in terms of nodal velocities. In general case the contact zone between deformable body and rigid stamp is variable and must be determined. Considering axisymmetric formulation the form of stamp on the plane $Orz$ is defined with piecewise smooth curve in a parametric form: $\psi: r = r(\xi), \ z = z(\xi)$, where $\xi \in [\xi_0, \xi_1]$ is a monotonic parameter.

The physical contact assumes that two bodies along the surface $\Sigma_C$ do not interpenetrate:

$$\vec{\rho} \cdot \vec{n} < 0, \ \forall \vec{r} \in V \quad (19)$$

and the mutual normal contact forces are non negative if an adhesion is neglected:

$$\vec{\sigma} \cdot \vec{n} \geq 0, \ \forall \vec{r} \in \Sigma_c, \quad (20)$$

where $\vec{\rho}$ is a normal vector to $\Sigma_C$, $\vec{n}$ is an external normal vector to $\psi$, $\vec{\sigma} = S \cdot \vec{n}$ is a contact force vector.

The following conditions can be specified on $\Sigma_C$:

– the condition of bonded surfaces:

$$\vec{v} = 0, \ \forall \vec{r} \in \Sigma_C, \quad (21)$$
the condition of free slipping:

\[ \vec{v} \cdot \vec{n} = 0, \quad \vec{\sigma} \cdot \vec{\tau} = 0, \quad \forall \vec{r} \in \Sigma_C. \]  

(22)

The restrictions imposed on the values of tangential loads obey the law of dry friction of Amonton-Coulomb:

\[
\begin{align*}
|\sigma_\tau| < \mu |\sigma_n| & \Rightarrow \vec{v}_\tau \cdot \vec{\tau} = 0, \\
|\sigma_\tau| = \mu |\sigma_n| & \Rightarrow \sigma_\tau |\vec{v}_\tau \cdot \vec{\tau}| + \vec{v}_\tau \mu |\sigma_n| = 0,
\end{align*}
\]

(23)

where \( \mu \) is a friction coefficient, \( \sigma_n = \vec{\sigma} \cdot \vec{n} \) is a normal component of contact force and \( \sigma_\tau = \vec{\sigma} \cdot \vec{\tau} \) is a tangential component of contact force.

The modification of the procedure of step-by-step loading method is proposed to take into account the time-changing in \( \Sigma_C \) as follows (see figure 2). A narrow strip is introduced on the outside of the stamp. The width of a strip \( \Delta \) is small enough in comparison with the characteristic size of the finite element. The bonding condition (21) is set up for all contact nodes falling into the strip zone \( \Delta \) at the beginning of the loading step. The solution of the boundary value problem is carried out. Next, the contact correctness conditions are checked for all boundary nodes of the grid: if at least one node does not satisfy one of the conditions (19), (20), the integration step is reduced and the integration procedure is repeated. The cycle continues until conditions (19), (20) are satisfied for all nodes. After that, all border nodes and nodes that fall into the strip zone \( \Delta \) are checked. These nodes are considered as contact and kinematic constraints (21) or (22) are set for them. For each element whose nodes are considered as contact, the current stamp reaction values \( \vec{\sigma} \) are calculated. If the condition (23) is not satisfied in any element on the boundary, then the tangent load \( \sigma_\tau = -\mu |\sigma_n| \text{sign}(\vec{v} \cdot \vec{\tau}) \) is specified on its contact edge, and the contacting nodes are assumed to be freely slipping according to (22).

![Figure 2. The scheme of contact zone changing.](image)

6. The results of numerical modeling

The loading scheme of a problem about the interaction of a hemisphere with a conical shape stamp, the initial and boundary conditions are shown in the figure 3.

The modeling of the hemisphere indentation into the cone was performed for various combinations of the coefficient of friction \( \mu \): 0.1, 0.2... and cone angle \( \beta \): 30°, 45°, 60°, as well as with the conditions of bonded surfaces (21) and free slipping (22). The inhomogeneous finite element mesh consisting of 7560 quadrangular axisymmetric elements with condensation in the
proposed contact region was generated inside the circle sector. The loading process was divided into 750 steps. Calculations were made for a model of weakly compressible material with the following values of elastic constants: \( K = 2.16 \text{MPa}, \ G = 1 \text{MPa}, \ C_1 = C_2 = 0, \ C_3 = -4 \text{MPa} \). Figure 4 contains the example of the deformed finite element mesh with highlighted nodal forces in contact zone. The results of numerical modeling are represented in the figure 5 in the form of dependences of the main load vector \( P \) applied to the hemisphere with respect to the relative displacement \( D/R_0 \).

To perform the analysis of the curves in figures 5 we will give the distributions of the shear components of the Cauchy true stress tensor for the values \( \beta = 30^\circ, 60^\circ \) and \( \mu = 0, \infty \) (see figures 6 and 7).

We can conclude from figures 6 and 7 that the effect of contact friction on the stress fields and, therefore, on the macrocharacteristics of the process increases with an increase in the angle \( \beta \). The fields of shear stresses \( S_{rz} \) at \( \beta = 30^\circ \) have a similar distribution pattern when considering different conditions at the contact boundary. When \( \beta = 60^\circ \) and considering the condition of bonded surfaces the observed maximum values of the tangential stresses are an order of magnitude higher than the values of stresses obtained under the condition of free slipping for similar values of radial and axial strains.
According to the results of numerical experiments with spherical weakly compressible bodies interacting with conical matrices of various geometries, it was found that taking into account contact friction has a significant effect on the characteristics of stress-strain state at large values of $\beta$. When $\beta = 60^\circ$ the magnitude of the main force $P$ required to move the deformable body differs by more than 3 times with a change in the coefficient of friction $\mu$. When $\beta = 30^\circ$ the solution with $\mu = 0.5$ already coincides with the solution with the condition of bonded surfaces. In this case, the magnitude of the main force $P$ required to ensure the same movement of the body differs by no more than 20% depending on the value $\mu$. In an extreme case, when
considering the problem of the interaction of a sphere with a rigid plane (β = 0°), the friction in the contact zone can be neglected up to large strains [22].

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References

[1] Wang J, Song Z and Dai H H 2016 Int. Journal of Solids and Structures 78-79 101–109
[2] Freidenberg A, Lee C W, Durant B, Nesterenko V, Stewart L and Hegemier G 2013 Int. Journal of Impact Engineering 60 58–66
[3] Mihai L and Goriely A 2017 Proc. R. Soc. A 473
[4] Cai R, Holweck F, Feng Z Q and Peyraut F 2016 Int. Journal of Solids and Structures 84 1–16
[5] Mansouri M, Darijani H and Baghani M 2017 Experimental Mechanic 57 195–206
[6] Latorre M and Montáns F J 2014 Int. Journal of Solids and Structures 51 (7–8) 1507–1515
[7] Korba A and Barkey M 2017 Proc. of the ASME 2017 12th Int. MSEC2017 (Los Angeles, CA, USA)
[8] Zhang Q and Yang Q S 2017 Mechanics Research Communications 84 55–59
[9] Wu C, Lin K and Juang J 2016 Tribology Int. 97 71–76
[10] Suzuki R, Ito K, Lee T and Oghara N 2017 Journal of the Mechanical Behavior of Biomedical Materials 65 753–760
[11] Zafiropoulou V, Zisis T and Giannakopoulos A 2016 European Journal of Mechanics - A/Solids 58 221–232
[12] Duddu R, Lavier L, Hughes T and Calo V 2012 Int. Journal for Numerical methods in engineering 89 (6) 762–785
[13] Astapov Y, Glagolev V, Khristich D, Markin A and Sokolova M 2016 Int. Journal of Applied Mechanics 8 (8) 1650099
[14] Brovko G 2013 Journal of engineering mathematics 78 37–53
[15] Seth B 1962 IUTAM Symposium on Second Order Effects in Elasticity, Plasticity and Fluid Mechanics (Haifa)
[16] Hill R 1968 J. Mech. Phys. Solids 16 (4) 222–242
[17] Neff P, Eidel B and Martin R 2016 Arch. Rational Mech. Anal. 222 507–572
[18] Ilyushin A 1963 Plasticity. Basics of General Mathematical Theory (Moscow: Academy of Sciences of the USSR Publishing)
[19] Markin A and Sokolova M 2015 Thermomechanics of Elastoplastic Deformation (Cambridge: Cambridge International Science Publishing)
[20] Astapov Y and Khristich D 2019 J. Phys.: Conf. Series 1203 10
[21] Zienkiewicz O and Taylor R 2000 The finite element method. Fifth edition (Butterworth-Heinemann a division of Reed Educational and Professional Publishing Ltd, Woburn, MA, USA).
[22] Dintwa E, Tijskens E, and Ramon H 2008 Granular Matter 10 209221