TOPOLOGICAL SEMIGROUPS AND UNIVERSAL SPACES RELATED TO EXTENSION DIMENSION

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Abstract. It is proved that there is no structure of left (right) cancelative semigroup on $[L]$-dimensional universal space for the class of separable compact spaces of extensional dimension $\leq [L]$. Besides, we note that the homeomorphism group of $[L]$-dimensional space whose nonempty open sets are universal for the class of separable compact spaces of extensional dimension $\leq [L]$ is totally disconnected.

1. Preliminaries

Let $L$ be a CW-complex and $X$ a Tychonov space. The Kuratowski notation $X\tau L$ means that, for any continuous map $f: A \rightarrow L$ defined on a closed subset $A$ of $X$, there exists an extension $\bar{f}: X \rightarrow L$ onto $X$. This notation allows us to define the preorder relation $\preceq$ onto the class of CW-complexes: $L \preceq L'$ iff, for every Tychonov space $X$, $X\tau L$ implies $X\tau L'$.

The preorder relation $\preceq$ naturally generates the equivalence relation $\sim$: $L \sim L'$ iff $L \preceq L'$ and $L' \preceq L$. We denote by $[L]$ the equivalence class of $L$.

The following notion is introduced by A. Dranishnikov (see, [3] and [4]). The extension dimension of a Tychonov space $X$ is less than or equal to $[L]$ (briefly, $\text{ext} - \text{dim}(X) \leq [L]$) if $X\tau L$.

We say that a Tychonov space $Y$ is said to be a universal space for the class of compact metric spaces $X$ with $\text{ext} - \text{dim}(X) \leq [L]$ if $Y$ contains a topological copy of every compact metric space $X$ with $\text{ext} - \text{dim}(X) \leq [L]$. See [1] and [2] for existence of universal spaces.

In what follows we will need the following statement which appears in [3] as Lemma 3.2.

**Proposition 1.1.** Let $i_0 = \min\{i : \pi_i(L) \neq 0\}$. Then $\text{ext} - \text{dim}(S^{i_0}) \leq [L]$.

2. Main theorem

Recall that a semigroup $S$ (whose operation is denoted as multiplication) is called a left cancelation semigroup if $xy = xz$ implies $y = z$ for every $x, y, z \in S$.

**Theorem 2.1.** Let $L$ be a connected CW-complex and let $Y$ be a universal space for the class of compact metric spaces $X$ with $\text{ext} - \text{dim}(X) \leq [L]$. If $\text{ext} - \text{dim}(Y) = [L]$, then there is no structure of left (right) cancelation semigroup on $Y$ compatible with its topology.

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Proof. Suppose the contrary and let $Y$ be a left cancelation semigroup. Let $\alpha(\coprod_{j=1}^{\infty} S_i^0)$ be the Alexandrov compactification of the countable topological sum of copies $S_i^0$ of the sphere $S^0$, where $i_0 = \min\{i : \pi_i(L) \neq 0\}$. By the countable sum theorem for extension dimension and Proposition 4, $\dim(\text{Homeo}(\coprod_{j=1}^{\infty} S_i^0)) \leq |L|$ and, since $Y$ is universal, $Y$ contains a copy of $\alpha(\coprod_{j=1}^{\infty} S_i^0)$. We will assume that $\alpha(\coprod_{j=1}^{\infty} S_i^0) \subset Y$. Besides, since $\dim(Y) \geq [S^1]$, we see that $Y$ contains an arc $J$. Let $a, b$ be endpoints of $J$. There exists $j_0$ such that $a S_i^{j_0} \cap b S_i^{j_0} = \emptyset$. By Proposition 3, there exists a map $f: a S_i^{j_0} \cup b S_i^{j_0} \to L$ such that $f|a S_i^{j_0}$ is a constant map and $f|b S_i^{j_0}$ is not null-homotopic. Extend map $f$ to a map $\tilde{f}: Y \to L$. Let $g: [0, 1] \to J$ be a homeomorphism, then the map $F: S_i^{j_0} \times [0, 1] \to L$, $\tilde{f}(x, t) = g(t)x$, is a homotopy that contradicts to the fact that $f|b S_i^{j_0}$ is not null-homotopic.

The homeomorphism group $\text{Homeo}(X)$ of a space $X$ is endowed with the compact-open topology.

**Theorem 2.2.** Suppose $\dim(X) = |L|$ and every nonempty open subset of $X$ is universal for the class of separable metric spaces $X$ with $\dim(X) \leq |L|$. Then the homeomorphism group $\text{Homeo}(X)$ is totally disconnected.

**Proof.** Suppose the contrary. Let $h \in \text{Homeo}(X)$, $h \neq \text{id}_X$. There exists $x \in X$ such that $h(x) \neq x$ and, therefore, there exists a neighborhood $U$ of $x$ such that $h(U) \cap U = \emptyset$. Since $U$ is universal for the class of separable metric spaces $X$ with $\dim(X) \leq |L|$, there exists an embedding of $S^{i_0}$ into $U$, where $i_0$ is as in Proposition 4. We may suppose that $S^{i_0} \subset U$. There exists a map $f: S^{i_0} \cup h(S^{i_0}) \to L$ such that the restriction $f|S^{i_0}$ is not null-homotopic while the restriction $f|h(S^{i_0})$ is null-homotopic. Since $\dim(X) \leq |L|$, there exists an extension $\tilde{f}: X \to L$ of the map $f$. The set

$$W = \{g \in \text{Homeo}(X) : \tilde{f}|g(S^{i_0}) \text{ is not null-homotopic} \}$$

is an open and closed subset of $\text{Homeo}(X)$. We see that $W$ is a neighborhood of $h$ that does not contain $h$.

### 3. Open problems

Note that the case $L = S^n$ corresponds to the case of covering dimension. In this case, the topology of homeomorphism groups of some universal spaces has been investigated by many authors (see the survey [3]).

In particular, it is known (see [3] and [4]) that the homeomorphism group of the $n$-dimensional Menger compactum $M^n$ (note that $M^n$ satisfies the conditions of Theorem 2.2 with $L = S^n$) is one-dimensional.

Let $|L| \geq [S^1]$ and $X$ be as in Theorem 2.2. Is $\dim(\text{Homeo}(X)) \geq 1$?

Another version: Is there $X$ that satisfies the conditions of Theorem 2.2 and such that $\dim(\text{Homeo}(X)) \geq 1$?

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