Algebras of Quantum Variables for Loop Quantum Gravity

IV. A new formulation of the holonomy-flux $\ast$-algebra

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August 19, 2011

Abstract

In this article the holonomy-flux $\ast$-algebra, which has been introduced by Lewandowski, Okolow, Sahlmann and Thiemann [16], is modified. The new $\ast$-algebra is called the holonomy-flux cross-product $\ast$-algebra. This algebra is an abstract cross-product $\ast$-algebra. It is given by the universal algebra of the algebra of continuous and differentiable functions on the configuration space of generalised connections and the universal enveloping flux algebra associated to a surface set, and some canonical commutator relations. There is a uniqueness result for a certain path- and graph-diffeomorphism invariant state of the holonomy-flux cross-product $\ast$-algebra. This new $\ast$-algebra is not the only $\ast$-algebra, which is generated by the algebra of certain continuous and differentiable functions on the configuration space of generalised connections and the universal enveloping flux algebra associated to a surface set. The theory of abstract cross-product algebras allows to define different new $\ast$-algebras. Some of these algebras are presented in this article.

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1 Introduction

In [1] Ashtekar, Corichi and Zapata have introduced the concept of an algebra generated by holonomies along paths, which is Lie group-valued, and quantum fluxes associated to surfaces and paths, which take values in the Lie algebra of the Lie group. A further analysed *-algebra and representations of this *-algebra have been presented by Sahlmann [20, 21], or by Okolów and Lewandowski [17, 18]. Finally, the holonomy-flux *-algebra has been presented by the project group Lewandowski, Okolów, Sahlmann and Thiemann in [10]. In this article the holonomy-flux *-algebra is reformulated in a slightly different way such that the resulting *-algebra is different from the *-algebra presented by Lewandowski, Okolów, Sahlmann and Thiemann.

The ideas for the definition of the quantum configuration and momentum variables in the project AQV have been presented in [10] and are shortly introduced in section 2. In this article the Lie algebra-valued or enveloping algebra-valued quantum flux operators are used intensively. These operators replace and generalise the flux-like variables of Lewandowski, Okolów, Sahlmann and Thiemann. Moreover, new *-algebras are derived from certain functions depending on holonomies along paths in a graph and Lie algebra-valued or enveloping algebra-valued quantum flux operators and some canonical commutator relation among them. The concept of the construction is introduced in section 3.

The new *-algebras are in particular abstract cross-product algebras, which have been presented by Schmüdgen and Klimyk [15] in the context of Hopf algebras. The holonomy-flux *-algebra is generated by all multiplication operators defined by functions in $C(\mathcal{A})$ and left (or right) vector fields $e^L$ (or $e^R$) on $C^\infty(\mathcal{A})$. Similarly to the *-algebras in Quantum Mechanics the new holonomy-flux cross-product *-algebra is generated by the identity $1$, the holonomies along paths and the Lie algebra-valued quantum flux operators satisfying certain canonical commutator relations. If the surfaces are restricted to a certain set of surfaces, then this algebra is called the holonomy-flux cross-product *-algebra associated to a surface set. In contrast to the holonomy-flux *-algebra the construction of the holonomy-flux cross-product *-algebra is independent of the Hilbert space and the representation of the operators on the Hilbert space. For the definition of the holonomy-flux cross-product *-algebra associated to a graph $\Gamma$ and a surface set $\hat{S}$ the enveloping flux algebra $\mathcal{E}_{\hat{S},\Gamma}$ associated to a surface set $\hat{S}$ and a graph $\Gamma$ is necessary. Then this abstract cross-product *-algebra is given by the tensor vector space of the analytic holonomy $C^*$-algebra restricted to a graph and the enveloping flux algebra associated to a surface set equipped with a multiplication operation, which is derived from a certain action of enveloping flux algebra associated to a surface set on the analytic holonomy $C^*$-algebra restricted to a graph. In particular, it is used that the analytic holonomy $C^*$-algebra restricted to a graph is a right (or left) $\mathcal{E}_{\hat{S},\Gamma}$-module algebra.

The *-representation of the enveloping flux algebra associated to a surface set and a graph is given by infinitesimal representation of the flux group associated to the surface set $\hat{S}$ and the graph $\Gamma$ on the Hilbert space $\mathcal{H}_{\Gamma}$. The *-representation $\pi$ of the holonomy-flux cross-product *-algebra associated to the graph $\Gamma$ and the surface set $\hat{S}$ is given by this representation $dU^\pi_{\Gamma}$ of the enveloping flux algebra associated to the surface set and the graph and the representation $\Phi_M$ of the analytic holonomy $C^*$-algebra restricted to the graph. Consequently, an element $f_{\Gamma} \otimes E_{\mathcal{S}}(\gamma)$ is represented on the Hilbert space $\mathcal{H}_{\Gamma}$ by

$$\pi(f_{\Gamma} \otimes E_{\mathcal{S}}(\gamma)) := \frac{1}{2} \Phi_M(e^L(f_{\Gamma})) + \frac{1}{2} \Phi_M(f_{\Gamma}) dU^\pi_{\Gamma}(E_{\mathcal{S}}(\gamma))$$

where $e^L$ denotes the right-invariant vector field and $E_{\mathcal{S}}(\gamma)$ is an element of the enveloping flux algebra associated to a surface set $\hat{S}$ and a graph $\Gamma$. The representation extends to a representation of the holonomy-flux cross-product *-algebra associated to a surface set $\hat{S}$. In theorem 4.8 it is shown that, the corresponding state is the unique surface-orientation-preserving graph-diffeomorphism invariant state of the holonomy-flux cross-product *-algebra associated to the surface set $\hat{S}$. Moreover, for a restricted notion of graph-diffeomorphism invariance other *-representations and states are studied in section 4. Simple tensor products of the holonomy-flux cross-product *-algebra are shortly presented in section 6.
On the other hand, the Heisenberg double has been introduced by Schmüdgen and Klimyk [15]. The Heisenberg double $\mathcal{H}(\mathcal{C}^\infty(G), \mathcal{E})$ depends on a Lie algebra $G$ and the enveloping algebra $\mathcal{E}$ of the Lie algebra $g$ associated to $G$. The universal enveloping algebra $\mathcal{E}$ of the complexified Lie algebra $g^\mathbb{C}$ is equipped with an antilinear and antitmultiplicative involution $Y \mapsto Y^+$ such that $X^+ = -X$ for all $X \in g$. Therefore, the universal enveloping flux algebra itself is a unital $\ast$-algebra, which is isomorphic to a particular $O^\ast$-algebra in the sense of Inoue [8] and Schmüdgen [23].

The holonomy-flux cross-product $\ast$-algebra associated to surfaces presented in section 3 is regarded as an abstract cross-product algebra, which is constructed from holonomies ($G$-valued) and quantum fluxes ($g$- or $\mathcal{E}$-valued). But it is discovered in section 3 that, the holonomy-flux cross-product $\ast$-algebra is not equivalent to the Heisenberg algebra $\mathcal{H}(\mathcal{C}^\infty(\overline{G}_S, \mathcal{E}_S), \mathcal{E}_S)$ associated to a surface set and a graph, which is presented in section 3.2. Indeed there are three different $\mathcal{E}_S$-module algebras that define three different holonomy-flux cross-product $\ast$-algebras. The definitions of these algebras are related to different bilinear maps that define left (or right) module algebras and different multiplication operations on vector spaces that define cross-products.

In section 7, the construction of algebras of quantum configuration and momentum variables are compared for Quantum Mechanics, the Weyl $C^\ast$-algebras associated to a graph and surface sets and the holonomy-flux cross-product $\ast$-algebra associated to a graph and surface sets.

To summarise the holonomy-flux $\ast$-algebra [16] is reformulated and generalised to the holonomy-flux cross-product $\ast$-algebra by using the theory of abstract cross-product algebras invented by Schmüdgen and Klimyk [15]. In particular this concept is based on Hopf algebras and has been used in the context of quantum groups. A further project is the reformulation of the algebra defined by Okolów and Lewandowski [19] in this new context.

2 The basic quantum operators

2.1 Finite path groupoids and graph systems

Let $c : [0, 1] \to \Sigma$ be continuous curve in the domain $[0, 1]$, which is (piecewise) $C^k$-differentiable ($1 \leq k \leq \infty$), analytic ($k = \omega$) or semi-analytic ($k = s\omega$) in $[0, 1]$ and oriented such that the source vertex is $c(0) = s(c)$ and the target vertex is $c(1) = t(c)$. Moreover assume that, the range of each subinterval of the curve $c$ is a submanifold, which can be embedded in $\Sigma$. An edge is given by a reparametrisation invariant curve of class (piecewise) $C^k$. The maps $s_\Sigma, t_\Sigma : P_\Sigma \to \Sigma$ where $P_\Sigma$ is the path space are surjective maps and are called the source or target map.

A set of edges $\{e_i\}_{i=1,...,N}$ is called independent iff the only intersections points of the edges are source $s_\Sigma(e_i)$ or $t_\Sigma(e_i)$ target points. Composed edges are called paths. An initial segment of a path $\gamma$ is a path $\gamma_1$ such that there exists another path $\gamma_2$ and $\gamma = \gamma_1 \circ \gamma_2$. The second element $\gamma_2$ is also called a final segment of the path $\gamma$.

Definition 2.1. A graph $\Gamma$ is a union of finitely many independent edges $\{e_i\}_{i=1,...,N}$ for $N \in \mathbb{N}$. The set $\{e_1,...,e_N\}$ is called the generating set for $\Gamma$. The number of edges of a graph is denoted by $|\Gamma|$. The elements of the set $V_\Gamma := \{s_\Sigma(e_i), t_\Sigma(e_i)\}_{k=1,...,N}$ of source and target points are called vertices.

A graph generates a finite path groupoid in the sense that, the set $\mathcal{P}_\Gamma \Sigma$ contains all independent edges, their inverses and all possible compositions of edges. All the elements of $\mathcal{P}_\Gamma \Sigma$ are called paths associated to a graph. Furthermore the surjective source and target maps $s_\Sigma$ and $t_\Sigma$ are restricted to the maps $s,t : \mathcal{P}_\Gamma \Sigma \to V_\Gamma$, which are required to be surjective.

Definition 2.2. Let $\Gamma$ be a graph. Then a finite path groupoid $\mathcal{P}_\Gamma \Sigma$ over $V_\Gamma$ is a pair $(\mathcal{P}_\Gamma \Sigma, V_\Gamma)$ of finite sets equipped with the following structures:

(i) two surjective maps $s,t : \mathcal{P}_\Gamma \Sigma \to V_\Gamma$, which are called the source and target map,
(ii) the set $\mathcal{P}_\Gamma \Sigma^2 := \{(\gamma_i, \gamma_j) \in \mathcal{P}_\Gamma \Sigma \times \mathcal{P}_\Gamma \Sigma : t(\gamma_i) = s(\gamma_j)\}$ of finitely many composable pairs of paths,
(iii) the composition $\circ : \mathcal{P}_\Gamma \Sigma^2 \to \mathcal{P}_\Gamma \Sigma$, where $(\gamma_i, \gamma_j) \mapsto \gamma_i \circ \gamma_j$,
(iv) the inversion map $\gamma_i \mapsto \gamma_i^{-1}$ of a path,
(v) the object inclusion map $\iota : V_\Gamma \to \mathcal{P}_\Gamma \Sigma$ and
(vi) $\mathcal{P}_ΓΣ$ is defined by the set $\mathcal{P}_ΓΣ$ modulo the algebraic equivalence relations generated by
\begin{equation}
\gamma_i^{-1} \circ \gamma_i \simeq 1_{s(\gamma_i)} \text{ and } \gamma_i \circ \gamma_i^{-1} \simeq 1_{t(\gamma_i)}
\end{equation}

Shortly write $\mathcal{P}_ΓΣ \xrightarrow{\cong} V_Γ$.

Clearly, a graph $Γ$ generates freely the paths in $\mathcal{P}_ΓΣ$. Moreover the map $s \times t : \mathcal{P}_ΓΣ \to V_Γ \times V_Γ$ defined by $(s \times t)(γ) = (s(γ), t(γ))$ for all $γ \in \mathcal{P}_ΓΣ$ is assumed to be surjective ($\mathcal{P}_ΓΣ$ over $V_Γ$ is a transitive groupoid), too.

A general groupoid $G$ over $G^0$ defines a small category where the set of morphisms is denoted in general by $G$ and the set of objects is denoted by $G^0$. Hence in particular the path groupoid can be viewed as a category, since,

- the set of morphisms is identified with $\mathcal{P}_ΓΣ$,
- the set of objects is given by $V_Γ$ (the units)

From the condition (1) it follows that, the path groupoid satisfies additionally

(i) $s(γ_i \circ γ_j) = s(γ_i)$ and $t(γ_i \circ γ_j) = t(γ_j)$ for every $(γ_i, γ_j) ∈ \mathcal{P}_ΓΣ^2$

(ii) $s(v) = v = t(v)$ for every $v \in V_Γ$

(iii) $γ \circ 1_{s(γ)} = γ = 1_{t(γ)} \circ γ$ for every $γ ∈ \mathcal{P}_ΓΣ$ and

(iv) $γ \circ (γ_i, γ_j) = (γ \circ γ_i) \circ γ_j$

(v) $γ \circ (γ^{-1} \circ γ_i) = γ_i = (γ \circ γ) \circ γ^{-1}$

The condition (iii) implies that the vertices are units of the groupoid.

**Definition 2.3.** Denote the set of all finitely generated paths by
\[ \mathcal{P}_ΓΣ^{(n)} := \{(γ_1, ..., γ_n) ∈ \mathcal{P}_Γ × ... × \mathcal{P}_Γ : (γ_i, γ_{i+1}) ∈ \mathcal{P}_Γ^2, 1 ≤ i ≤ n - 1\} \]

The set of paths with source point $v ∈ V_Γ$ is given by
\[ \mathcal{P}_ΓΣ^v := s^{-1}\{(v)\} \]

The set of paths with target point $v ∈ V_Γ$ is defined by
\[ \mathcal{P}_ΓΣ^t := t^{-1}\{(v)\} \]

The set of paths with source point $v ∈ V_Γ$ and target point $u ∈ V_Γ$ is
\[ \mathcal{P}_ΓΣ^v_u := \mathcal{P}_ΓΣ^v \cap \mathcal{P}_ΓΣ^u \]

A graph $Γ$ is said to be disconnected if it contains only mutually pairs $(γ_i, γ_j)$ of non-composable independent paths $γ_i$ and $γ_j$ for $i \neq j$ and $i, j = 1, ..., N$. In other words for all $1 ≤ i, l ≤ N$ it is true that $s(γ_i) \neq t(γ_l)$ and $t(γ_i) \neq s(γ_l)$ where $i \neq l$ and $γ_i, γ_l ∈ Γ$.

**Definition 2.4.** Let $Γ$ be a graph. A subgraph $Γ'$ of $Γ$ is given by a finite set of independent paths in $\mathcal{P}_ΓΣ$.

For example let $Γ := \{γ_1, ..., γ_N\}$ then $Γ' := \{γ_1 \circ γ_2, γ_3^{-1}, γ_4\}$ where $γ_1 \circ γ_2, γ_3^{-1}, γ_4 ∈ \mathcal{P}_ΓΣ$ is a subgraph of $Γ$, whereas the set $\{γ_1, γ_1 \circ γ_2\}$ is not a subgraph of $Γ$. Notice if additionally $(γ_2, γ_4) ∈ \mathcal{P}_Γ^2$ holds, then $\{γ_1, γ_3^{-1}, γ_2 \circ γ_4\}$ is a subgraph of $Γ$, too. Moreover for $Γ := \{γ\}$ the graph $Γ^{-1} := \{γ^{-1}\}$ is a subgraph of $Γ$. As well the graph $Γ$ is a subgraph of $Γ^{-1}$. A subgraph of $Γ$ that is generated by compositions of some paths, which are not reversed in their orientation, of the set $\{γ_1, ..., γ_N\}$ is called an orientation preserved subgraph of a graph. For example for $Γ := \{γ_1, ..., γ_N\}$ orientation preserved subgraphs are given by $\{γ_1 \circ γ_2\}, \{γ_1, γ_2, γ_N\}$ or $\{γ_N^{-2} \circ γ_N^{-1}\}$ if $(γ_1, γ_2) ∈ \mathcal{P}_ΓΣ^2$ and $(γ_{N-2}, γ_{N-1}) ∈ \mathcal{P}_ΓΣ^2$.
Definition 2.5. A finite graph system \( \mathcal{P}_\Gamma \) for \( \Gamma \) is a finite set of subgraphs of a graph \( \Gamma \). A finite graph system \( \mathcal{P}_\Gamma \) for \( \Gamma' \) is a finite graph subsystem of \( \mathcal{P}_\Gamma \) for \( \Gamma \) iff the set \( \mathcal{P}_\Gamma \) is a subset of \( \mathcal{P}_\Gamma \) and \( \Gamma' \) is a subgraph of \( \Gamma \). Shortly write \( \mathcal{P}_\Gamma \leq \mathcal{P}_\Gamma \).

A finite orientation preserved graph system \( \mathcal{P}_\Gamma^* \) for \( \Gamma \) is a finite set of orientation preserved subgraphs of a graph \( \Gamma \).

Recall that, a finite path groupoid is constructed from a graph \( \Gamma \), but a set of elements of the path groupoid need not be a graph again. For example let \( \Gamma := \{\gamma_1 \circ \gamma_2\} \) and \( \Gamma' = \{\gamma_1 \circ \gamma_3\} \), then \( \Gamma' = \Gamma \cup \Gamma'' \) is not a graph, since this set is not independent. Hence only appropriate unions of paths, which are elements of a fixed finite path groupoid, define graphs. The idea is to define a suitable action on elements of the path groupoid, which corresponds to an action of diffeomorphisms on the manifold \( \Sigma \). The action has to be transferred to graph systems. But the action of bisection, which is defined by the use of the groupoid multiplication, cannot easily generalised for graph systems.

Problem 2.1: Let \( \hat{\Gamma} := \{\Gamma_i\}_{i=1,...,N} \) be a finite set such that each \( \Gamma_i \) is a set of not necessarily independent paths such that

(i) the set contains no loops and

(ii) each pair of paths satisfies one of the following conditions

\( \cdot \) the paths intersect each other only in one vertex,  
\( \cdot \) the paths do not intersect each other or  
\( \cdot \) one path of the pair is a segment of the other path.

Then there is a map \( \circ : \hat{\Gamma} \times \hat{\Gamma} \to \hat{\Gamma} \) of two elements \( \Gamma_1 \) and \( \Gamma_2 \) defined by

\[
\{\gamma_1, ..., \gamma_M\} \circ \{\tilde{\gamma}_1, ..., \tilde{\gamma}_M\} := \left\{ \gamma_i \circ \tilde{\gamma}_j : t(\gamma_i) = s(\tilde{\gamma}_j) \right\}_{1 \leq i, j \leq M}
\]

for \( \Gamma_1 := \{\gamma_1, ..., \gamma_M\}, \Gamma_2 := \{\tilde{\gamma}_1, ..., \tilde{\gamma}_M\} \). Moreover define a map \( \cdot^{-1} : \hat{\Gamma} \to \hat{\Gamma} \) by

\[
\{\gamma_1, ..., \gamma_M\}^{-1} := \{\gamma_1^{-1}, ..., \gamma_M^{-1}\}
\]

Then the following is derived

\[
\{\gamma_1, ..., \gamma_M\} \circ \{\gamma_1^{-1}, ..., \gamma_M^{-1}\} = \left\{ \gamma_i \circ \gamma_j^{-1} : t(\gamma_i) = t(\gamma_j) \right\}_{1 \leq i, j \leq M}
\]

\[
= \left\{ \gamma_i \circ \gamma_j^{-1} : t(\gamma_i) = t(\gamma_j) \text{ and } i \neq j \right\}_{1 \leq i, j \leq M}
\]

\[
\cup \{\overset{\circ}{\Pi}_{s_{\gamma_j}}\}_{1 \leq j \leq M}
\]

\[
\neq \cup \{\overset{\circ}{\Pi}_{s_{\gamma_j}}\}_{1 \leq j \leq M}
\]

The equality is true, if the set \( \hat{\Gamma} \) contains only graphs such that all paths are mutually non-composable. Consequently this does not define a well-defined multiplication map. Notice that, the same is discovered if a similar map and inversion operation are defined for a finite graph system \( \mathcal{P}_\Gamma \).

Consequently the property of paths being independent need not be dropped for the definition of a suitable multiplication and inversion operation. In fact the independence property is a necessary condition for the construction of the holonomy algebra for analytic paths. Only under this circumstance each analytic path is decomposed into a finite product of independent piecewise analytic paths again.

Definition 2.6. A finite path groupoid \( \mathcal{P}_\Gamma \Sigma \) over \( V_\Gamma \) is a finite path subgroupoid of \( \mathcal{P}_\Gamma \Sigma \) over \( V_\Gamma \) iff the set \( V_\Gamma \) is contained in \( V_\Gamma \) and the set \( \mathcal{P}_\Gamma \Sigma \) is a subset of \( \mathcal{P}_\Gamma \Sigma \). Shortly write \( \mathcal{P}_\Gamma \Sigma \leq \mathcal{P}_\Gamma \Sigma \).

Clearly for a subgraph \( \Gamma_1 \) of a graph \( \Gamma_2 \), the associated path groupoid \( \mathcal{P}_\Gamma_1 \Sigma \) over \( V_{\Gamma_1} \) is a subgroupoid of \( \mathcal{P}_{\Gamma_2} \Sigma \) over \( V_{\Gamma_2} \). This is a consequence of the fact that, each path in \( \mathcal{P}_{\Gamma_2} \Sigma \) is a composition of paths or their inverses in \( \mathcal{P}_{\Gamma_2} \Sigma \).
Definition 2.7. A family of finite path groupoids \( \{ \mathcal{P}_{\Gamma, \Sigma} \}_{i=1, \ldots, \infty} \), which is a set of finite path groupoids \( \mathcal{P}_{\Gamma, \Sigma} \) over \( V_{\Gamma, i} \), is said to be inductive if for any \( \mathcal{P}_{\Gamma, \Sigma}, \mathcal{P}_{\Gamma, \Sigma} \) exists a \( \mathcal{P}_{\Gamma, \Sigma} \) such that \( \mathcal{P}_{\Gamma, \Sigma}, \mathcal{P}_{\Gamma, \Sigma} \leq \mathcal{P}_{\Gamma, \Sigma} \).

A family of graph systems \( \{ \mathcal{P}_{\Gamma, \Sigma} \}_{i=1, \ldots, \infty} \), which is a set of finite path systems \( \mathcal{P}_{\Gamma, i} \) for \( \Gamma, \Sigma \), is said to be inductive if for any \( \mathcal{P}_{\Gamma, i}, \mathcal{P}_{\Gamma, i} \) exists a \( \mathcal{P}_{\Gamma, i} \) such that \( \mathcal{P}_{\Gamma, i}, \mathcal{P}_{\Gamma, i} \leq \mathcal{P}_{\Gamma, i} \).

Definition 2.8. Let \( \{ \mathcal{P}_{\Gamma, \Sigma} \}_{i=1, \ldots, \infty} \) be an inductive family of path groupoids and \( \{ \mathcal{P}_{\Gamma, \Sigma} \}_{i=1, \ldots, \infty} \) be an inductive family of graph systems.

The inductive limit path groupoid \( \mathcal{P} \) over \( \Sigma \) of an inductive family of finite path groupoids such that \( \mathcal{P} := \lim_{i \to \infty} \mathcal{P}_{\Gamma, \Sigma} \) is called the (algebraic) path groupoid \( \mathcal{P} \simeq \Sigma \).

Moreover there exists an inductive limit graph \( \Gamma_{\infty} \) of an inductive family of graphs such that \( \Gamma_{\infty} := \lim_{i \to \infty} \Gamma_{i} \).

The inductive limit graph system \( \mathcal{P}_{\Gamma_{\infty}} \) of an inductive family of graph systems such that \( \mathcal{P}_{\Gamma_{\infty}} := \lim_{i \to \infty} \mathcal{P}_{\Gamma_{i}} \).

Assume that, the inductive limit \( \Gamma_{\infty} \) of a inductive family of graphs is a graph, which consists of an infinite countable number of independent paths. The inductive limit \( \mathcal{P}_{\Gamma_{\infty}} \) of an inductive family \( \{ \mathcal{P}_{\Gamma_{i}} \} \) of finite graph systems contains an infinite countable number of subgraphs of \( \Gamma_{\infty} \) and each subgraph is a finite set of arbitrary independent paths in \( \Sigma \).

### 2.2 Holonomy maps for finite path groupoids, graph systems and transformations

In section [2.1], the concept of finite path groupoids for analytic paths has been given. Now the holonomy maps are introduced for finite path groupoids and finite graph systems. The ideas are familiar with those presented by Thiemann [24]. But for example the finite graph systems have not been studied before. Ashtekar and Lewandowski [2] have defined the analytic holonomy \( C^* \)-algebra, which they have based on a finite set of independent hoops. The hoops are generalised for path groupoids and the independence requirement is implemented by the concept of finite graph systems.

#### 2.2.1 Holonomy maps for finite path groupoids

Groupoid morphisms for finite path groupoids

Let \( G_{1} = \xrightarrow{\gamma} G_{1}^{0}, G_{2} = \xrightarrow{\gamma} G_{2}^{0} \) be two arbitrary groupoids.

**Definition 2.9.** A groupoid morphism between two groupoids \( G_{1} \) and \( G_{2} \) consists of two maps \( h : G_{1} \to G_{2} \) and \( h : G_{1}^{0} \to G_{2}^{0} \) such that

\[
\begin{align*}
(G1) & \quad h(\gamma \circ \gamma') = h(\gamma)h(\gamma') \quad \text{for all } (\gamma, \gamma') \in G_{1}^{(2)} \\
(G2) & \quad s_{2}(h(\gamma)) = h(s_{1}(\gamma)), \quad t_{2}(h(\gamma)) = h(t_{1}(\gamma))
\end{align*}
\]

A strong groupoid morphism between two groupoids \( G_{1} \) and \( G_{2} \) additionally satisfies

\[
\begin{align*}
(SG) & \quad \text{for every pair } (h(\gamma), h(\gamma')) \in G_{2}^{(2)} \text{ it follows that } (\gamma, \gamma') \in G_{1}^{(2)}
\end{align*}
\]

Let \( G \) be a Lie group. Then \( G \) over \( e_{G} \) is a groupoid, where the group multiplication \( \cdot : G^{2} \to G \) is defined for all elements \( g_{1}, g_{2}, g \in G \) such that \( g_{1} \cdot g_{2} = g \). A groupoid morphism between a finite path groupoid \( \mathcal{P}_{\Gamma, \Sigma} \) to \( G \) is given by the maps

\[
\begin{align*}
\mathfrak{h}_{\Gamma} : \mathcal{P}_{\Gamma, \Sigma} \to G, \quad h_{\Gamma} : V_{\Gamma} \to e_{G}
\end{align*}
\]

Clearly

\[
\begin{align*}
\mathfrak{h}_{\Gamma}(\gamma \circ \gamma') = \mathfrak{h}_{\Gamma}(\gamma)\mathfrak{h}_{\Gamma}(\gamma') \quad \text{for all } (\gamma, \gamma') \in \mathcal{P}_{\Gamma, \Sigma}^{(2)} \\
s_{\Gamma}(\mathfrak{h}_{\Gamma}(\gamma)) = h_{\Gamma}(s_{\Gamma, \Sigma}(\gamma)), \quad t_{\Gamma}(\mathfrak{h}_{\Gamma}(\gamma)) = h_{\Gamma}(t_{\Gamma, \Sigma}(\gamma))
\end{align*}
\]

(2)

But for an arbitrary pair \( (\mathfrak{h}_{\Gamma}(\gamma_{1}), \mathfrak{h}_{\Gamma}(\gamma_{2})) : (g_{1}, g_{2}) \in G^{(2)} \) it does not follows that, \( (\gamma_{1}, \gamma_{2}) \in \mathcal{P}_{\Gamma, \Sigma}^{(2)} \) is true. Hence \( \mathfrak{h}_{\Gamma} \) is not a strong groupoid morphism.
Definition 2.10. Let $\mathcal{P}_T\Sigma \rightrightarrows V_T$ be a finite path groupoid.

Two paths $\gamma$ and $\gamma'$ in $\mathcal{P}_T\Sigma$ have the same-holonomy for all connections iff

$$h_T(\gamma) = h_T(\gamma')$$

for all $(h_T, h_T)$ groupoid morphisms

$$h_T : \mathcal{P}_T\Sigma \to G, h : V_T \to \{e_G\}$$

Denote the relation by $\sim_{s.hol}$. 

Lemma 2.11. The same-holonomy for all connections relation is an equivalence relation.

Notice that, the quotient of the finite path groupoid and the same-holonomy relation for all connections replace the hoop group, which has been used in [2].

Definition 2.12. Let $\mathcal{P}_T\Sigma \rightrightarrows V_T$ be a finite path groupoid modulo same-holonomy for all connections equivalence.

A holonomy map for a finite path groupoid $\mathcal{P}_T\Sigma$ over $V_T$ is a groupoid morphism consisting of the maps $(h_T, h_T)$, where $h_T : \mathcal{P}_T\Sigma \to G, h_T : V_T \to \{e_G\}$. The set of all holonomy maps is abbreviated by $\text{Hom}(\mathcal{P}_T\Sigma, G)$.

For a short notation observe the following. In further sections it is always assumed that, the finite path groupoid $\mathcal{P}_T\Sigma \rightrightarrows V_T$ is considered modulo same-holonomy for all connections equivalence although it is not stated explicitly.

Admissible maps and equivalent groupoid morphisms

Now consider a finite path groupoid morphism $(h_T, h_T)$ from a finite path groupoid $\mathcal{P}_T\Sigma$ over $V_T$ to the groupoid $G$ over $\{e_G\}$, which is contained in $\text{Hom}(\mathcal{P}_T\Sigma, G)$.

Consider an arbitrary map $\Phi : \mathcal{P}_T\Sigma \to G$. Then there is a groupoid morphism defined by

$$\Theta_T(\gamma) := \Phi(\gamma)h_T(\gamma)\Phi(\gamma^{-1})^{-1}$$

for all $\gamma \in \mathcal{P}_T\Sigma$ (3)

if and only if

$$\Theta_T(\gamma_1 \circ \gamma_2) = \Theta_T(\gamma_1)\Theta_T(\gamma_2)$$

for all $(\gamma_1, \gamma_2) \in \mathcal{P}_T\Sigma^{(2)}$ holds. Then $\Theta_T \in \text{Hom}(\mathcal{P}_T\Sigma, G)$.

Hence for all $(\gamma_1, \gamma_2) \in \mathcal{P}_T\Sigma^{(2)}$ it is necessary that

$$\Theta_T(\gamma_1 \circ \gamma_2) = \Phi(\gamma_1 \circ \gamma_2)h_T(\gamma_1 \circ \gamma_2)\Phi(\gamma_2^{-1} \circ \gamma_1^{-1})^{-1}$$

$$= \Phi(\gamma_1 \circ \gamma_2)h_T(\gamma_1)h_T(\gamma_2)\Phi(\gamma_2^{-1} \circ \gamma_1^{-1})^{-1}$$

$$\Phi(\gamma_1 \circ \gamma_2) = \Phi(\gamma_1)h_T(\gamma_1)h_T(\gamma_2)\Phi(\gamma_2^{-1} \circ \gamma_1^{-1})^{-1}$$

is satisfied. Therefore the map is required to fulfill

$$\Phi(\gamma_1) = \Phi(\gamma_1 \circ \gamma_2), \Phi(\gamma_2^{-1}) = \Phi((\gamma_1 \circ \gamma_2)^{-1})$$

and

$$\Phi(\gamma_1^{-1})^{-1}\Phi(\gamma_2) = e_G \text{ for all } (\gamma_1, \gamma_2) \in \mathcal{P}_T\Sigma^{(2)}$$

in particular,

$$\Phi(\gamma_1^{-1})^{-1}\Phi(\gamma) = e_G \text{ for all } (\gamma^{-1}, \gamma) \in \mathcal{P}_T\Sigma^{(2)}$$

(4)

for every refinement $\gamma_1 \circ \gamma_2$ of each $\gamma$ in $\mathcal{P}_T\Sigma$ and $\gamma_1$ being an initial segment of $\gamma_1 \circ \gamma_2$ and $\gamma_2^{-1}$ an final segment of $(\gamma_1 \circ \gamma_2)^{-1}$. In comparison with Fleischhack’s definition in [7, Def. 3.7] such maps are called admissible.

Definition 2.13. The set of maps $\Phi_T : \mathcal{P}_T\Sigma \to G$ satisfying (4) for all pairs of decomposable paths in $\mathcal{P}_T\Sigma^{(2)}$ is called the set of admissible maps and is denoted by $\text{Map}_A(\mathcal{P}_T\Sigma, G)$. 7
Consider a graph $gr : V_G \to G$ such that 
$$(gr, h_T) \in \text{Map}(V_G, G) \times \text{Hom}(P_T \Sigma, G)$$
which is also called a local gauge map. Then the map $\tilde{\Theta}_T$ defined by
$$\tilde{\Theta}_T(\gamma) := gr(s(\gamma))h_T(\gamma)gr(t(\gamma))^{-1}$$
for all $\gamma \in P_T \Sigma$ is a groupoid morphism. This is a result of the computation:
$$\tilde{\Theta}_T(\gamma \gamma_2) = gr(s(\gamma_1))h_T(\gamma_1 \gamma_2)gr(t(\gamma_1))^{-1}$$
$$= gr(s(\gamma_1))h_T(\gamma_1)gr(t(\gamma_1))^{-1}gr(s(\gamma_2))h_T(\gamma_2)gr(t(\gamma_2))^{-1}$$
since $t(\gamma_1) = s(\gamma_2)$.

**Definition 2.14.** Two groupoid morphisms $(\eta_T, h_T)$ and $(\Theta_T, h_T)$, or respectively $(\tilde{\Theta}_T, h_T)$, between the groupoids $P_T$ over $V_T$ and the groupoid $G$ over $\{e_G\}$, which are defined for $(gr, h_T) \in \text{Map}(P_T \Sigma, G) \times \text{Hom}(P_T \Sigma, G)$ by [5], or respectively for $(gr, h_T) \in \text{Map}(V_T, G) \times \text{Hom}(P_T \Sigma, G)$ by [5], are said to be similar or equivalent groupoid morphisms.

### 2.2.2 Holonomy maps for finite graph systems

Ashtekar and Lewandowski [2] have presented the loop decomposition into a finite set of independent hoops (in the analytic category). This structure is replaced by a graph, since a graph is a set of independent edges. Notice that, the set of hoops that is generated by a finite set of independent hoops, is generalised to the set of finite graph systems. A finite path groupoid is generated by the set of edges, which defines a graph $\Gamma$, but a set of elements of the path groupoid need not be a graph again. The appropriate notion for graphs constructed from sets of paths is the finite graph system, which is defined in section 2.1. Now the concept of holonomy maps is generalised for finite graph systems. Since the set, which is generated by a finite number of independent edges, contains paths that are composable, there are two possibilities to identify the image of the holonomy map for a finite graph system on a fixed graph with a subgroup of $G^{\Sigma}$. One way is to use the generating set of independent edges of a graph, which has been also used in [2]. On the other hand, it is also possible to identify each graph with a disconnected subgraph of a fixed graph, which is generated by a set of independent edges. Notice that, the author implements two situations. One case is given by a set of paths that can be composed further and the other case is related to paths that are not composable. This is necessary for the definition of an action of the flux operators. Precisely the identification of the image of the holonomy maps along these paths is necessary to define a well-defined action of a flux element on the configuration space. This issue has been studied in [11, 9].

First of all consider a graph $\Gamma$ that is generated by the set $\{\gamma_1, ..., \gamma_N\}$ of edges. Then each subgraph of a graph $\Gamma$ contain paths that are composition of edges in $\{\gamma_1, ..., \gamma_N\}$ or inverse edges. For example the following set $\Gamma' := \{\gamma_1 \circ \gamma_2 \circ \gamma_3, \gamma_4\}$ defines a subgraph of $\Gamma := \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$. Hence there is a natural identification available.

**Definition 2.15.** A subgraph $\Gamma'$ of a graph $\Gamma$ is always generated by a subset $\{\gamma_1, ..., \gamma_M\}$ of the generating set $\{\gamma_1, ..., \gamma_N\}$ of independent edges that generates the graph $\Gamma$. Hence each subgraph is identified with a subset of $\{\gamma_1^\pm_1, ..., \gamma_N^\pm_1\}$. This is called the natural identification of subgraphs.

**Example 2.1:** For example consider a subgraph $\Gamma' := \{\gamma_1 \circ \gamma_2, \gamma_3 \circ \gamma_4, ..., \gamma_{M-1} \circ \gamma_M\}$, which is identified naturally with a set $\{\gamma_1, ..., \gamma_M\}$. The set $\{\gamma_1, ..., \gamma_M\}$ is a subset of $\{\gamma_1, ..., \gamma_N\}$ where $N = |\Gamma|$ and $M \leq N$.

Another example is given by the graph $\Gamma'' := \{\gamma_1, \gamma_2\}$ such that $\gamma_2 = \gamma_1' \circ \gamma_2'$, then $\Gamma''$ is identified naturally with $\{\gamma_1, \gamma_2, \gamma_3, ..., \gamma_{N-1}\}$.

**Definition 2.16.** Let $\Gamma$ be a graph, $P_T$ be the finite graph system. Let $\Gamma' := \{\gamma_1, ..., \gamma_M\}$ be a subgraph of $\Gamma$.

A holonomy map for a finite graph system $P_T$ is a given by a pair of maps $(h_T, h_T)$ such that there exists a holonomy map $\tilde{h}_T(\eta_T, h_T)$ for the finite path groupoid $P_T \Sigma \rightarrow V_T$ and

\[ h_T : P_T \rightarrow G^{\Sigma}, \quad h_T(\{\gamma_1, ..., \gamma_M\}) = (h_T(\gamma_1), ..., h_T(\gamma_M), e_G, ..., e_G) \]

\[ h_T : V_T \rightarrow \{e_G\} \]

---

1In the work the holonomy map for a finite graph system and the holonomy map for a finite path groupoid is denoted by the same pair $(h_T, h_T)$.
The set of all holonomy maps for the finite graph system is denoted by \( \text{Hom}(\mathcal{P}_\Gamma, G^{[\Gamma]}) \).

The image of a map \( h_\Gamma \) on each subgraph \( \Gamma' \) of the graph \( \Gamma \) is given by

\[
(h_\Gamma(\gamma_1), ..., h_\Gamma(\gamma_M), e_G, ..., e_G)
\]

is an element of \( G^{[\Gamma]} \). The set of all images of maps on subgraphs of \( \Gamma \) is denoted by \( \tilde{A}_\Gamma \).

The idea is now to study two different restrictions of the set \( \mathcal{P}_\Gamma \) of subgraphs. For a short notation of a "set of holonomy maps for a certain restricted set of subgraphs of a graph" in this article the following notions are introduced.

**Definition 2.17.** If the subset of all disconnected subgraphs of the finite graph system \( \mathcal{P}_\Gamma \) is considered, then the restriction of \( \tilde{A}_\Gamma \), which is identified with \( G^{[\Gamma]} \) appropriately, is called the **non-standard identification of the configuration space**. If the subset of all natural identified subgraphs of the finite graph system \( \mathcal{P}_\Gamma \) is considered, then the restriction of \( \tilde{A}_\Gamma \), which is identified with \( G^{[\Gamma]} \) appropriately, is called the **natural identification of the configuration space**.

A comment on the non-standard identification of \( \tilde{A}_\Gamma \) is the following. If \( \Gamma':=\{\gamma_1 \circ \gamma_2\} \) and \( \Gamma'':=\{\gamma_2\} \) are two subgraphs of \( \Gamma:=\{\gamma_1, \gamma_2, \gamma_3\} \). The graph \( \Gamma' \) is a subgraph of \( \Gamma \). Then evaluation of a map \( h_\Gamma \) on a subgraph \( \Gamma' \) is given by

\[
h_\Gamma(\Gamma') = (h_\Gamma(\gamma_1 \circ \gamma_2), h_\Gamma(s(\gamma_2)), h_\Gamma(s(\gamma_3))) = (h_\Gamma(\gamma_1)h_\Gamma(\gamma_2), e_G, e_G) \in G^3
\]

and the holonomy map of the subgraph \( \Gamma'' \) of \( \Gamma' \) is evaluated by

\[
h_\Gamma(\Gamma'') = (h_\Gamma(s(\gamma_1)), h_\Gamma(s(\gamma_2)), h_\Gamma(s(\gamma_3))) = (h_\Gamma(\gamma_2), e_G, e_G) \in G^3
\]

**Example 2.2:** Recall example 2.2. For example for a subgraph \( \Gamma':=\{\gamma_1 \circ \gamma_2, \gamma_3 \circ \gamma_4, ..., \gamma_{M-1} \circ \gamma_M\} \), which is naturally identified with a set \( \{\gamma_1, ..., \gamma_M\} \). Then the holonomy map is evaluated at \( \Gamma' \) such that

\[
h_\Gamma(\Gamma') = (h_\Gamma(\gamma_1), h_\Gamma(\gamma_2), ..., h_\Gamma(\gamma_M), e_G, ..., e_G) \in G^N
\]

where \( N = |\Gamma'| \). For example, let \( \Gamma':=\{\gamma_1, \gamma_2\} \) such that \( \gamma_2 = \gamma_1 \circ \gamma'_2 \) and which is naturally identified with \( \{\gamma_1, \gamma'_1\} \). Hence

\[
h_\Gamma(\Gamma') = (h_\Gamma(\gamma_1), h_\Gamma(\gamma'_1), h_\Gamma(\gamma'_2), e_G, ..., e_G) \in G^N
\]

is true.

Another example is given by the disconnected graph \( \Gamma':=\{\gamma_1 \circ \gamma_2 \circ \gamma_3, \gamma_4\} \), which is a subgraph of \( \Gamma:=\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\} \). Then the non-standard identification is given by

\[
h_\Gamma(\Gamma') = (h_\Gamma(\gamma_1 \circ \gamma_2 \circ \gamma_3), h_\Gamma(\gamma_4), e_G, e_G) \in G^4
\]

If the natural identification is used, then \( h_\Gamma(\Gamma') \) is identified with

\[
(h_\Gamma(\gamma_1), h_\Gamma(\gamma_2), h_\Gamma(\gamma_3), h_\Gamma(\gamma_4)) \in G^4
\]

Consider the following example. Let \( \Gamma':=\{\gamma_1, \alpha, \gamma_2, \gamma_3\} \) be a graph such that
Then notice the sets $\Gamma_1 := \{\gamma_1 \circ \alpha, \gamma_3\}$ and $\Gamma_2 := \{\gamma_1 \circ \alpha^{-1}, \gamma_3\}$. In the non-standard identification of the configuration space $\mathcal{A}_{\Gamma''}$ it is true that,

$$\begin{align*}
h_{\Gamma''}(\Gamma_1) &= (h_{\Gamma''}((\gamma_1 \circ \alpha)), h_{\Gamma''}(\gamma_3), e_G, e_G) \in G^4,
h_{\Gamma''}(\Gamma_2) &= (h_{\Gamma''}(\gamma_1 \circ \alpha^{-1}), h_{\Gamma''}(\gamma_3), e_G, e_G) \in G^4
\end{align*}$$

holds. Whereas in the natural identification of $\bar{\mathcal{A}}_{\Gamma''}$

$$\begin{align*}
h_{\Gamma''}(\Gamma_1) &= (h_{\Gamma''}(\gamma_1), h_{\Gamma''}(\alpha), h_{\Gamma''}(\gamma_3), e_G) \in G^4,
h_{\Gamma''}(\Gamma_2) &= (h_{\Gamma''}(\gamma_1), h_{\Gamma''}(\alpha^{-1}), h_{\Gamma''}(\gamma_3), e_G) \in G^4
\end{align*}$$

yields.

The equivalence class of similar or equivalent groupoid morphisms defined in definition 2.14 allows to define the following object. The set of images of all holonomy maps of a finite graph system modulo the similar or equivalent groupoid morphisms equivalence relation is denoted by $\bar{\mathcal{A}}_{\Gamma}/\bar{\mathcal{G}}_{\Gamma}$.

### 2.2.3 Transformations in finite path groupoids and finite graph systems

The aim of this section is to clarify the graph changing operators in LQG framework and the role of finite diffeomorphisms in $\Sigma$. Therefore operations, which add, delete or transform paths, are introduced. In particular translations in a finite path graph groupoid and in the groupoid $G$ over $\{e_G\}$ are studied.

**Transformations in finite path groupoid**

**Definition 2.18.** Let $\varphi$ be a $C^k$-diffeomorphism on $\Sigma$, which maps surfaces into surfaces. Then let $(\Phi_\Gamma, \varphi_\Gamma)$ be a pair of bijective maps, where $\varphi|_{V_\Gamma} = \varphi_\Gamma$ and

$$\Phi_\Gamma : \mathcal{P}_\Gamma \Sigma \to \mathcal{P}_\Gamma \Sigma \text{ and } \varphi_\Gamma : V_\Gamma \to V_\Gamma$$

such that

$$\begin{align*}
(s \circ \Phi_\Gamma)(\gamma) &= (\varphi_\Gamma \circ s)(\gamma),
(t \circ \Phi_\Gamma)(\gamma) &= (\varphi_\Gamma \circ t)(\gamma) \text{ for all } \gamma \in \mathcal{P}_\Gamma \Sigma
\end{align*}$$

holds that $(\Phi_\Gamma, \varphi_\Gamma)$ defines a groupoid morphism.

Call the pair $(\Phi_\Gamma, \varphi_\Gamma)$ a **path-diffeomorphism of a finite path groupoid** $\mathcal{P}_\Gamma \Sigma$ over $V_\Gamma$. Denote the set of finite path-diffeomorphisms by $\text{Diff}(\mathcal{P}_\Gamma \Sigma)$.

Notice that, for $(\gamma, \gamma') \in \mathcal{P}_\Gamma \Sigma^{(2)}$ it is true that

$$\Phi_\Gamma(\gamma \circ \gamma') = \Phi_\Gamma(\gamma) \circ \Phi_\Gamma(\gamma')$$

requires that

$$\begin{align*}
(t \circ \Phi_\Gamma)(\gamma) &= (s \circ \Phi_\Gamma)(\gamma')
\end{align*}$$

Hence from $[5]$ and $[9]$ it follows that, $\Phi_\Gamma(1_v) = 1_{\varphi_\Gamma(v)}$ is true.

A path-diffeomorphism $(\Phi_\Gamma, \varphi_\Gamma)$ is lifted to $\text{Hom}(\mathcal{P}_\Gamma \Sigma, G)$. The pair $(h_\Gamma \circ \Phi_\Gamma, h_\Gamma \circ \varphi_\Gamma)$ defined by

$$\begin{align*}
h_\Gamma \circ \Phi_\Gamma : \mathcal{P}_\Gamma \Sigma \to G, & \quad \gamma \mapsto (h_\Gamma \circ \Phi_\Gamma)(\gamma) \\
h_\Gamma \circ \varphi_\Gamma : V_\Gamma \to \{e_G\}, & \quad (h_\Gamma \circ \varphi_\Gamma)(v) = e_G
\end{align*}$$

such that

$$\begin{align*}
s_{\text{Hol}}((h_\Gamma \circ \Phi_\Gamma)(\gamma)) &= (h_\Gamma \circ \varphi_\Gamma)(s(\gamma)) = e_G, \\
t_{\text{Hol}}((h_\Gamma \circ \Phi_\Gamma)(\gamma)) &= (h_\Gamma \circ \varphi_\Gamma)(t(\gamma)) = e_G \text{ for all } \gamma \in \mathcal{P}_\Gamma \Sigma
\end{align*}$$

whenever $(h_\Gamma, h_\Gamma) \in \text{Hom}(\mathcal{P}_\Gamma \Sigma, G)$ and $(\Phi_\Gamma, \varphi_\Gamma)$ is a path-diffeomorphism, is a **holonomy map for a finite path groupoid** $\mathcal{P}_\Gamma \Sigma$ over $V_\Gamma$.  


Definition 2.19. A left-translation in the finite path groupoid $P_\Gamma\Sigma$ over $V_\Gamma$ at a vertex $v$ is a map defined by

$$L_\theta : P_\Gamma \Sigma^v \to P_\Gamma \Sigma^w, \quad \gamma \mapsto L_{\theta}(\gamma) := \theta \circ \gamma$$

for some $\theta \in P_\Gamma \Sigma^w$ and all $\gamma \in P_\Gamma \Sigma^v$.

In analogy a right-translation $R_\theta$ and an inner-translation $I_{\theta, \varphi}$ in the finite path groupoid $P_\Gamma \Sigma$ over $V_\Gamma$ at a vertex $v$ can be defined.

Remark 2.20. Let $(\Phi_\Gamma, \varphi_\Gamma)$ be a path-diffeomorphism on a finite path groupoid $P_\Gamma \Sigma$ over $V_\Gamma$. Then a left-translation in the finite path groupoid $P_\Gamma \Sigma$ over $V_\Gamma$ at a vertex $v$ is defined by a path-diffeomorphism $(\Phi_\Gamma, \varphi_\Gamma)$ and the following object

$$L_{\Phi_\Gamma} : P_\Gamma \Sigma^v \to P_\Gamma \Sigma^w, \quad \gamma \mapsto L_{\Phi_\Gamma}(\gamma) := \Phi_\Gamma(\gamma)$$

Furthermore a right-translation in the finite path groupoid $P_\Gamma \Sigma$ over $V_\Gamma$ at a vertex $v$ is defined by a path-diffeomorphism $(\Phi_\Gamma, \varphi_\Gamma)$ and the following object

$$R_{\Phi_\Gamma} : P_\Gamma \Sigma_v \to P_\Gamma \Sigma_{\varphi_\Gamma(v)}, \quad \gamma \mapsto R_{\Phi_\Gamma}(\gamma) := \Phi_\Gamma(\gamma)$$

Finally an inner-translation in the finite path groupoid $P_\Gamma \Sigma$ over $V_\Gamma$ at the vertices $v$ and $w$ is defined by

$$I_{\Phi_\Gamma} : P_\Gamma \Sigma^v_w \to P_\Gamma \Sigma_{\varphi_\Gamma(w)}, \quad \gamma \mapsto I_{\Phi_\Gamma}(\gamma) = \Phi_\Gamma(\gamma)$$

where $(s \circ \Phi_\Gamma)(\gamma) = \varphi_\Gamma(v)$ and $(t \circ \Phi_\Gamma)(\gamma) = \varphi_\Gamma(w)$.

In the following considerations the right-translation in a finite path groupoid is focused, but there is a generalisation to left-translations and inner-translations.

Definition 2.21. A bisection of a finite path groupoid $P_\Gamma \Sigma$ over $V_\Gamma$ is a map $\sigma : V_\Gamma \to P_\Gamma \Sigma$, which is right-inverse to the map $s : P_\Gamma \Sigma \to V_\Gamma$ (i.o.w. $s \circ \sigma = \text{id}_{V_\Gamma}$) and such that $t \circ \sigma : V_\Gamma \to V_\Gamma$ is a bijective map. The set of bisections on $P_\Gamma \Sigma$ over $V_\Gamma$ is denoted $B(P_\Gamma \Sigma)$.

Remark 2.22. Discover that, a bisection $\sigma \in B(P_\Gamma \Sigma)$ defines a path-diffeomorphism $(\varphi_\Gamma, \Phi_\Gamma) \in \text{Diff}(P_\Gamma \Sigma)$, where $\varphi_\Gamma = t \circ \sigma$ and $\Phi_\Gamma$ is given by the right-translation $R_{\sigma(v)} : P_\Gamma \Sigma_v \to P_\Gamma \Sigma_{\varphi_\Gamma(v)}$ in $P_\Gamma \Sigma \cong V_\Gamma$, where $R_{\sigma(v)}(\gamma) = \Phi_\Gamma(\gamma)$ for all $\gamma \in P_\Gamma \Sigma_v$ and for a fixed $v \in V_\Gamma$. The right-translation is defined by

$$R_{\sigma(v)}(\gamma) := \begin{cases} \gamma \circ \sigma(v) & v = t(\gamma) \\ \gamma \circ \text{Id}_{t(\gamma)} & v \neq t(\gamma) \end{cases}$$

whenever $t(\gamma)$ is the target vertex of a non-trivial path $\gamma$ in $\Gamma$. For a trivial path $\text{Id}_v$ the right-translation is defined by $R_{\sigma(v)}(\text{Id}_v) = \text{Id}_{(t\sigma)(v)}$ and $R_{\sigma(v)}(\text{Id}_w) = \text{Id}_w$ whenever $v \neq w$. The right-translation $R_{\sigma(v)}$ is required to be bijective. Before this result is proven in lemma 2.22 notice the following considerations.

Note that, $(R_{\sigma(v)}, t \circ \sigma)$ transfers to the holonomy map such that

$$(h_\Gamma \circ R_{\sigma(t(\gamma'))}(\gamma \circ \gamma') = h_\Gamma(\gamma \circ \gamma' \circ (t(\gamma'))))$$

is true. There is a bijective map between a right-translation $R_{\sigma(v)} : P_\Gamma \Sigma_v \to P_\Gamma \Sigma_{(t\sigma)(v)}$ and a path-diffeomorphism $(\varphi_\Gamma, \Phi_\Gamma)$. In particular observe that, $\sigma \in B(P_\Gamma \Sigma_v)$ and $(\varphi_\Gamma, \Phi_\Gamma) \in \text{Diff}(P_\Gamma \Sigma_v)$. Simply speaking the path-diffeomorphism does not change the source and target vertex at the same time. The path-diffeomorphism changes the target vertex by a (finite) diffeomorphism and, therefore, the path is transformed.

Bisections $\sigma$ in a finite path groupoid can be transfered, likewise path-diffeomorphisms, to holonomy maps. The pair $(h_\Gamma \circ \Phi_\Gamma, h_t \circ \varphi_\Gamma)$ of the maps defines a pair of maps $(h_\Gamma \circ \Phi_\Gamma, h_t \circ \varphi_\Gamma)$ by

$$h_\Gamma \circ \Phi_\Gamma : P_\Gamma \Sigma_v \to G$$

which is a holonomy map for a finite path groupoid $P_\Gamma \Sigma$ over $V_\Gamma$.\footnote{Note that in the infinite case of path groupoids an additional condition for the map $t \circ \sigma : \Sigma \to \Sigma$ has to be required. The map has to be a diffeomorphism. Observe that, the map $t \circ \sigma$ defines the finite diffeomorphism $\varphi_\Gamma : V_\Gamma \to V_\Gamma$.}
Lemma 2.23. *The set $\mathcal{B}(\mathcal{P}_T \Sigma)$ of bisections on the finite path groupoid $\mathcal{P}_T \Sigma$ over $V_T$ forms a group.*

**Proof:** The group multiplication is given by

$$(\sigma \ast \sigma')(v) = \sigma'(v) \circ \sigma(t(\sigma'(v))) \text{ for } v \in V_T$$

whenever $\sigma'(v) \in \mathcal{P}_T \Sigma^v_{(v)}$ and $\sigma(t(\sigma'(v))) \in \mathcal{P}_T \Sigma^{(t \sigma')(v)}_{(v)}$.

Clearly the group multiplication is associative. The unit id is equivalent to the object inclusion $v \mapsto 1_v$ of the groupoid $\mathcal{P}_T \Sigma \rightrightarrows V_T$, where $1_v$ is the constant loop at $v$, and the inversion is given by

$$\sigma^{-1}(v) = \sigma((t \circ \sigma)^{-1}(v))^{-1} \text{ for } v \in V_T.$$ 

\[
\square
\]

The group property of bisections $\mathcal{B}(\mathcal{P}_T \Sigma)$ carries over to holonomy maps. Using the group multiplication $\ast$ of $G$ conclude that

$$(h_T \circ R_{(\sigma \ast \sigma')(v)}(1_v)) = h_T \circ (R_{( \sigma',v)} \circ R_{(t(\sigma')(v))})(1_v) = h_T(\sigma'(v)) \cdot h_T(\sigma(t(\sigma'(v)))) \text{ for } v \in V_T$$

is true.

**Remark 2.24.** *Moreover right-translations define path-diffeomorphisms, i.e. $R_{(\sigma,v)} = \Phi_T$ and $\varphi_T = t \circ \sigma$ whenever $v \in V_T$. But for two bisections $\sigma_T, \sigma'_T \in \mathcal{B}(\mathcal{P}_T \Sigma)$ the object $\sigma_T(\cdot) \circ \sigma'_T(\cdot)$ is not comparable with $(\sigma_T \ast \sigma'_T)(\cdot)$. Then for the composition $\Phi_T(\gamma) \circ \Phi_T(\gamma')$, there exists no path-diffeomorphism $\Phi$ such that $\Phi_T(\gamma) \circ \Phi_T(\gamma') = \Phi(\gamma \circ \gamma')$ yields in general. Moreover generally the object $\Phi_T(\gamma) \circ \Phi_T(\gamma') = \Phi(\gamma \circ \gamma')$ is not well-defined.

But the following is defined

$$R_{(\sigma \ast \sigma')(v)}(\gamma) = \Phi_T(\gamma) \circ \Phi_T(\mathbb{1}_{\varphi'_{-v}}) = (\Phi_T \ast \Phi_T)(\gamma)$$

whenever $\gamma \in \mathcal{P}_T \Sigma_v$, $(\varphi_T, \Phi_T) \in \text{Diff}(\mathcal{P}_T \Sigma_v)$ and $(\varphi'_{-v}, \Phi'_{-v}) \in \text{Diff}(\mathcal{P}_T \Sigma^v_{\varphi_{-v}})$ are path-diffeomorphisms such that $\varphi_T = t \circ \sigma$, $\Phi_T = R_{\varphi'_{-v}}(\cdot)$ and $\Phi'_{-v} = t \circ \sigma'$, $\Phi'_{-v} = R_{\varphi'_{-v}}(\cdot)$.

Moreover for $(\gamma, \gamma') \in \mathcal{P}_T \Sigma_v$ and $\gamma' \in \mathcal{P}_T \Sigma_v$ it is true that

$$\Phi_T(\gamma \circ \gamma') = \Phi_T(\gamma) \circ \Phi_T(\mathbb{1}_{\varphi'_{-v}}) = \Phi_T(\gamma) \circ \Phi_T(\mathbb{1}_{\varphi'_{-v}}) = \Phi_T(\gamma) \circ (\Phi_T \ast \Phi_T)(\gamma')$$

holds.

Then the following lemma easily follows.

**Lemma 2.25.** *Let $\sigma$ be a bisection contained in $\mathcal{B}(\mathcal{P}_T \Sigma)$ and $v \in V_T$. The pair $(R_{(\sigma,v)}, t \circ \sigma)$ of maps such that

$$R_{(\sigma,v)} : \mathcal{P}_T \Sigma_v \rightarrow \mathcal{P}_T \Sigma_{(t \circ \sigma)(v)}, \quad s \circ R_{(\sigma,v)} = (t \circ \sigma) \circ s$$

$$t \circ \sigma : V_T \rightarrow V_T, \quad t \circ R_{(\sigma,v)} = (t \circ \sigma) \circ t$$

defined in remark 2.22 is a path-diffeomorphism in $\mathcal{P}_T \Sigma \rightrightarrows V_T$.*

**Proof:** This follows easily from the derivation

$$R_{(\sigma(\gamma'))}(\gamma \circ \gamma') = \gamma \circ \gamma' \circ \sigma(t(\gamma')) = R_{(\sigma(\gamma'))}(\gamma) \circ R_{(t(\gamma'))}(\gamma')$$

$$R_{(\sigma(\gamma'))}(1_{\gamma'}) = R_{(\sigma(\gamma'))}(1_{\gamma'}) \circ R_{(\sigma(\gamma'))}(1_{\gamma'}) = 1_{\gamma'} \circ \gamma \circ \sigma(t(\gamma))$$

$$R_{(\sigma(\gamma'))}(\gamma(\cdot) \circ 1_{\gamma'}) = R_{(\sigma(\gamma'))}(\gamma(\cdot)) \circ R_{(\sigma(\gamma'))}(1_{\gamma'}) = \gamma \circ \sigma(t(\gamma)) \circ 1_{(t \circ \sigma)(\gamma)})$$

The inverse map satisfies

$$R_{(\sigma,v)}^{-1}(\gamma \circ \sigma(v)) = R_{(\sigma^{-1},v)}(\gamma \circ (\sigma(v))) = \gamma \circ (\sigma(v) \circ \sigma^{-1}(v)) = \gamma$$
whenever \( v = t(\gamma) \),
\[
R_{\sigma(v)}^{-1}(\gamma) = \gamma
\]
whenever \( v \neq t(\gamma) \) and
\[
R_{\sigma(v)}^{-1} \mathbb{1}_{(t \circ \sigma)(v)} = \mathbb{1}_v
\]

Moreover derive
\[
(s \circ R_{\sigma(v)})(\gamma') = ((t \circ \sigma) \circ s)(\gamma')
\]
for all \( \gamma' \in P_{\Gamma} \Sigma_v \) and a fixed bisection \( \sigma \in \mathcal{B}(P_{\Gamma} \Sigma) \).

Notice that, \( L_{\sigma(v)} \) and \( I_{\sigma(v)} \) similarly to the pair \( (R_{\sigma(v)}, t \circ \sigma) \) can be defined. Summarising the pairs \( (R_{\sigma(v)}, t \circ \sigma), (L_{\sigma(v)}, t \circ \sigma) \) and \( (I_{\sigma(v)}, t \circ \sigma) \) for a bisection \( \sigma \in \mathcal{B}(P_{\Gamma} \Sigma) \) are path-diffeomorphisms of a finite path groupoid \( P_{\Gamma} \Sigma \to \Gamma \).

In general a right-translation \( (R_{\sigma}, t \circ \sigma) \) in the finite path groupoid \( P_{\Gamma} \Sigma \) over \( \Sigma \) for a bisection \( \sigma \in \mathcal{B}(P_{\Gamma} \Sigma) \) is defined by the bijective maps \( R_{\sigma} \) and \( t \circ \sigma \), which are given by
\[
R_{\sigma} : P_{\Gamma} \Sigma \to P_{\Gamma} \Sigma, \quad s \circ R_{\sigma} = s
\]
\[
t \circ \sigma : V_{\Gamma} \to V_{\Gamma}, \quad t \circ R_{\sigma} = (t \circ \sigma) \circ t
\]
\[
R_{\sigma}(\gamma) := \gamma \circ \sigma(t(\gamma)) \quad \forall \gamma \in P_{\Gamma} \Sigma; \quad R_{\sigma}^{-1} := R_{\sigma^{-1}}
\]

For example for a fixed suitable bisection \( \sigma \) the right-translation is \( R_{\sigma}(\mathbb{1}_v) = \gamma \), then \( R_{\sigma}^{-1}(\gamma) = \gamma \circ \gamma^{-1} = \mathbb{1}_v \) for \( v = s(\gamma) \). Clearly the right-translation \( (R_{\sigma}, t \circ \sigma) \) is not a groupoid morphism in general.

**Definition 2.26.** Define for a given bisection \( \sigma \in \mathcal{B}(P_{\Gamma} \Sigma) \), the **right-translation in the groupoid** \( G \) **over** \( \{ e_G \} \) through
\[
h_{\Gamma} \circ R_{\sigma} : P_{\Gamma} \Sigma \to G, \quad \gamma \mapsto (h_{\Gamma} \circ R_{\sigma})(\gamma) := h_{\Gamma}(\gamma \circ \sigma(t(\gamma)))
\]
\[
h_{\Gamma} \circ t \circ \sigma : V_{\Gamma} \to e_G
\]

Furthermore for a fixed \( \sigma \in \mathcal{B}(P_{\Gamma} \Sigma) \) define the **left-translation in the groupoid** \( G \) **over** \( \{ e_G \} \) by
\[
h_{\Gamma} \circ L_{\sigma} : P_{\Gamma} \Sigma \to G, \quad \gamma \mapsto h_{\Gamma}(\sigma((t \circ \sigma)^{-1}(s(\gamma))) \circ \gamma) = h_{\Gamma}(\sigma((t \circ \sigma)^{-1}(s(\gamma)))) \cdot h_{\Gamma}(\gamma)
\]
\[
h_{\Gamma} \circ t \circ \sigma : V_{\Gamma} \to e_G
\]

and the **inner-translation in the groupoid** \( G \) **over** \( \{ e_G \} \)
\[
h_{\Gamma} \circ I_{\sigma} : P_{\Gamma} \Sigma \to G, \quad \gamma \mapsto h_{\Gamma}(\sigma((t \circ \sigma)^{-1}(s(\gamma))) \circ \gamma \circ \sigma(t(\gamma))) = h_{\Gamma}(\sigma((t \circ \sigma)^{-1}(s(\gamma)))) \cdot h_{\Gamma}(\gamma) \cdot h_{\Gamma}(\sigma(t(\gamma)))
\]
\[
h_{\Gamma} \circ t \circ \sigma : V_{\Gamma} \to e_G
\]
such that \( I_{\sigma} = L_{\sigma^{-1}} \circ R_{\sigma} \).

The pairs \( (R_{\sigma}, t \circ \sigma) \) and \( (L_{\sigma}, t \circ \sigma) \) are not groupoid morphisms. Whereas the pair \( (I_{\sigma}, t \circ \sigma) \) is a groupoid morphism, since for all pairs \( (\gamma, \gamma') \in P_{\Gamma} \Sigma^{(2)} \) such that \( t(\gamma) = s(\gamma') \) it is true that \( \sigma(t(\gamma)) \circ \sigma((t \circ \sigma)^{-1}(t(\gamma))) = \mathbb{1}_{t(\gamma)} \). Notice that, in this situation \( \sigma(t(\gamma)) = \sigma(t(\gamma \circ \gamma')) \) is satisfied.

**Proposition 2.27.** The map \( \sigma \mapsto R_{\sigma} \) is a group isomorphism, i.e. \( R_{\sigma \circ \sigma'} = R_{\sigma} \circ R_{\sigma'} \) and where \( \sigma \mapsto t \circ \sigma \) is a group isomorphism from \( \mathcal{B}(P_{\Gamma} \Sigma) \) to the group of finite diffeomorphisms \( \text{Diff}(V_{\Gamma}) \) in a finite subset \( V_{\Gamma} \) of \( \Sigma \).

The maps \( \sigma \mapsto L_{\sigma} \) and \( \sigma \mapsto I_{\sigma} \) are group isomorphisms.
There is a generalisation of path-diffeomorphisms in the finite path groupoid, which coincide with the graphomorphism presented by Fleischhack in [7]. In this approach the diffeomorphism \( \varphi : \Sigma \to \Sigma \) changes the source and target vertex of a path \( \gamma \). Consequently the path-diffeomorphism \( (\Phi, \varphi) \), which implements the inner-translation \( I_\varphi \) in the path groupoid \( P \Sigma \rightrightarrows \Sigma \), is a graphomorphism in the context of Fleischhack. Some element of the set of graphomorphisms is directly related to a right-translation \( R_\sigma \) in the path groupoid. Precisely for every \( \sigma \in \Sigma \) and \( \sigma \in \mathcal{B}(P \Sigma) \) the pairs \( (R_\sigma(v), t \cdot \sigma) \), \( (L_\sigma(v), t \cdot \sigma) \) and \( (I_\sigma(v), t \cdot \sigma) \) define graphomorphism. Furthermore the right-translation \( R_\sigma(v) \), the left-translation \( L_\sigma(v) \) and the inner-translation \( I_\sigma(v) \) are required to be bijective maps, and hence the maps cannot map non-trivial paths to trivial paths. This property restricts the set of all graphomorphism, which is generated by these translations. In particular in this article graph changing operations, which change the number of edges of a graph, are studied. Hence the left- or right-translation in a finite path groupoid is used in the further development. Notice that in general, these objects do not define graphomorphism. Finally notice that, in particular for the graphomorphism \( (R_\sigma(v), t \cdot \sigma) \) and a holonomy map for the path groupoid \( P \Sigma \rightrightarrows \Sigma \) a similar relation \([13]\) holds. The last equation is fundamental for the construction of \( C^* \)-dynamical systems, which contain the analytic holonomy \( C^* \)-algebra restricted to a finite path groupoid \( P \Sigma \rightrightarrows \Sigma \) and a point norm continuous action of the finite path-diffeomorphism group \( \text{Diff}(V_\Gamma) \) on this algebra. Clearly the right-, left- and inner-translations \( R_\sigma \), \( L_\sigma \) and \( I_\sigma \) are constructed such that \([13]\) generalises. But note that, in the infinite case considered by Fleischhack the action of the bisections \( B(P \Sigma) \) on the analytic holonomy \( C^* \)-algebra.

### Transformations in finite graph systems

To proceed it is necessary to transfer the notion of bisections and translations to finite graph systems. A right-translation \( R_\sigma \) is a mapping that maps graphs to graphs. Each graph is a finite union of independent edges. This causes problems. Since the definition of right-translation in a finite graph system \( P \Gamma \) is often not well-defined for all bisections in the finite graph system and all graphs. For example if the graph \( \Gamma := \{ \gamma_1, \gamma_2 \} \) is disconnected and the bisection \( \tilde{\sigma} \) in the finite path groupoid \( P \Sigma \) over \( V_\Gamma \) is defined by \( \tilde{\sigma}(s(\gamma_1)) = \gamma_1 \), \( \tilde{\sigma}(s(\gamma_2)) = \gamma_2 \), \( \tilde{\sigma}(t(\gamma_1)) = \gamma_1^{-1} \) and \( \tilde{\sigma}(t(\gamma_2)) = \gamma_2^{-1} \) where \( V_\Gamma := \{ s(\gamma_1), t(\gamma_1), s(\gamma_2), t(\gamma_2) \} \). Let \( \Gamma \) be the set given by the elements \( \{ \gamma_1, \gamma_2, \gamma_1^{-1}, \gamma_2^{-1} \} \). Then notice that, a bisection \( \sigma \), which maps a set of vertices in \( V_\Gamma \) to a set of paths in \( P \Sigma \), is given for example by \( \sigma(V_\Gamma) = \{ 1, \gamma_2, \gamma_1^{-1}, \gamma_2^{-1} \} \). In this case the right-translation \( R_{\sigma(V_\Gamma)}(\tilde{\sigma}(\Gamma)) \) is equivalent to \( \{ 1, \gamma_2, \gamma_1^{-1}, \gamma_2^{-1} \} \), which is not a set of independent edges and hence not a graph. Loosely speaking the graph-diffeomorphism acts on all vertices in the set \( V_\Gamma \) and hence implements four new edges. But a bisection \( \sigma \), which maps a subset \( V := \{ s(\gamma_1), s(\gamma_2) \} \) of \( V_\Gamma \) to a set of paths, leads to a translation \( R_{\sigma(V)}(\{ s(\gamma_1), s(\gamma_2) \}) = \{ \gamma_1, \gamma_2 \} \), which is indeed a graph. Define \( \Gamma' := \{ 1 \} \) and \( V' = \{ s(\gamma_1) \} \). Then observe that, for a restricted bisection, which maps a set \( V' \) of vertices in \( V_\Gamma \) to a set of paths in \( P \Sigma \), the right-translation becomes \( \Gamma_{\sigma(V')} = \{ \gamma_1 \} \), which defines a graph, too. Notice that \( \Gamma_{\sigma(V')} \) is a subgraph of \( \Gamma' \). Hence in the simplest case new edges are emerging. The next definition of the right-translation shows that composed paths arise, too.

#### Definition 2.28.

Let \( \Gamma \) be a graph, \( P \Sigma \rightrightarrows \Sigma \) be a finite path groupoid and let \( P \Gamma \) be a finite graph system. Moreover the set \( V_\Gamma \) is given by \( \{ v_1, ..., v_{2N} \} \).

A bisection of a finite graph system \( P \Sigma \) is a map \( \sigma : V_\Gamma \to P \Sigma \) such that there exists a bisection \( \tilde{\sigma} \in \mathcal{B}(P \Sigma) \) such that \( \sigma(V_\Gamma) = \{ \tilde{\sigma}(v_i) : v_i \in V_\Gamma \} \) whenever \( V_\Gamma \) is a subset of \( V_\Sigma \).

Define a restriction \( \sigma_{\Gamma'} : V_{\Gamma'} \to P_{\Gamma'} \) of a bisection \( \sigma \) in \( P \Sigma \) by

\[
\sigma_{\Gamma'}(V) := \{ \tilde{\sigma}(w_k) : w_k \in V \}
\]

for each subgraph \( \Gamma' \) of \( \Gamma \) and \( V \subseteq V_{\Gamma'} \).

A right-translation in the finite graph system \( P \Sigma \) is a map \( R_{\sigma,V} : P \Sigma \to P \Sigma \), which is given by a bisection \( \sigma : V_{\Gamma'} \to P_{\Gamma'} \) such that

\[
R_{\sigma,V}(\Gamma') = R_{\sigma,v}(\{ \gamma_1^n, ..., \gamma_M^n \}) \upharpoonright w_i := \{ s(\gamma_1), ..., s(\gamma_M) \} \upharpoonright w_i \in \{ s(\gamma_1)' \} \to \Sigma \to \Sigma \}
\]

\[
\left\{ \begin{array}{l}
\gamma_1^n, ..., \gamma_1^n \circ \tilde{\sigma}(t(\gamma_1^n)), ..., \gamma_M^n \circ \tilde{\sigma}(t(\gamma_M^n)) \upharpoonright w_i := \tilde{\sigma}(w_i) : \\
\gamma_i^n \in \{ s(\gamma_1), ..., s(\gamma_M) \} \subseteq \Sigma, \quad \tilde{\sigma}(w_i) \end{array} \right.
\]

\[
= \Gamma_\sigma
\]

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where \( \bar{\sigma} \in \mathcal{B}(\mathcal{P}_F \Sigma) \), \( K := |\Gamma'| \) and \( M := |\Gamma''| \). \( V_F^\circ \) is the set of all source vertices of \( \Gamma' \) and such that \( \Gamma'' := \{ \gamma_1'', \ldots, \gamma_M'' \} \) is a subgraph of \( \Gamma' := \{ \gamma_1', \ldots, \gamma_K' \} \) and \( \Gamma''_0 \) is a subgraph of \( \Gamma''. \)

Derive that, for \( \bar{\sigma}(t(\gamma_i)) = \gamma_i^{-1} \) it is true that \( (t \circ \bar{\sigma})(s(\gamma_i^{-1})) = s(\gamma_i) = (t \circ \sigma)(t(\gamma_i)) \) holds.

**Example 2.3:** Let \( \Gamma \) be a disconnected graph. Then for a bisection \( \bar{\sigma} \in \mathcal{B}(\mathcal{P}_F \Sigma) \) such that \( \sigma(t(\gamma_i)) = \gamma_i^{-1} \) for all \( 1 \leq i \leq |\Gamma| \) it is true that

\[
R_{\sigma}(\Gamma) = \left\{ \gamma_1 \circ \bar{\sigma}(t(\gamma_1)), \ldots, \gamma_N \circ \bar{\sigma}(t(\gamma_N)), \mathbb{I}_{s(\gamma_1)} \circ \bar{\sigma}(s(\gamma_1)) \right\}
\]

yields. Set \( \Gamma' := \{ \gamma_1', \ldots, \gamma_M' \} \), then derive

\[
R_{\sigma}(\Gamma') = \left\{ \gamma_1' \circ \bar{\sigma}(t(\gamma_1')), \ldots, \gamma_M' \circ \bar{\sigma}(t(\gamma_M')), \mathbb{I}_{s(\gamma_1')} \circ \bar{\sigma}(s(\gamma_1')) \right\}
\]

if \( \Gamma = \Gamma' \cup \{ \gamma_1, \ldots, \gamma_{N-M} \} \).

To understand the definition of the right-translation notice the following problem.

**Problem 2.1:** Consider a subgraph \( \Gamma \) of \( \bar{\Gamma} := \{ \gamma_1, \gamma_2, \gamma_3, \gamma_4 \} \), a map \( \bar{\sigma} : V_F \rightarrow \mathcal{P}_F \Sigma \). Then the map

\[
R_{\bar{\sigma}}(\Gamma) = \{ \gamma_1 \circ \gamma_1^{-1}, \gamma_2 \circ \mathbb{I}_{t(\gamma_2)}, \gamma_3 \circ \mathbb{I}_{t(\gamma_3)}, \mathbb{I}_{s(\gamma_1)} \circ \gamma_4 \} := \Gamma_{\bar{\sigma}}
\]

is not a right-translation. This follows from the following fact. Notice that, the map \( \sigma \) maps \( t(\gamma_1) \mapsto s(\gamma_1) \), \( t(\gamma_2) \mapsto t(\gamma_2) \), \( t(\gamma_3) \mapsto t(\gamma_3) \) and \( s(\gamma_1) \mapsto t(\gamma_4) \). Then the map \( \bar{\sigma} \) is not a bisection in the finite path groupoid \( \mathcal{P}_F \Sigma \) over \( V_F \), and does not define a right-translation \( R_{\bar{\sigma}} \) in the finite graph system \( \mathcal{P}_F \).

This is a general problem. For every bisection \( \bar{\sigma} \) in a finite path groupoid such that a graph \( \Gamma := \{ \gamma \} \) is translated to \( \{ \gamma \circ \bar{\sigma}(t(\gamma), \bar{\sigma}(s(\gamma))) \} \). Hence either such translations in the graph system are excluded or the definition of the bisections has to be restricted to maps such that the map \( t \circ \bar{\sigma} \) is not bijective. Clearly, the restriction of the right-translation such that \( \Gamma \) is mapped to \( \{ \gamma \circ \bar{\sigma}(t(\gamma), \mathbb{I}_{s(\gamma)}) \} \) implies that a simple path orientation transformation is not implemented by a right-translation.

Furthermore there is an ambiguity for graph containing to paths \( \gamma_1 \) and \( \gamma_2 \) such that \( t(\gamma_1) = t(\gamma_2) \). Since in this case a bisection \( \sigma \), which maps \( t(\gamma_1) \) to \( t(\gamma_3) \), the right-translation is \( \{ \gamma_1 \circ \gamma_3, \gamma_2 \circ \gamma_3 \} \), is not a graph anymore.

**Example 2.4:** Otherwise there is for example a subgraph \( \Gamma' \) of \( \bar{\Gamma} := \{ \gamma_1, \gamma_2, \gamma_3, \gamma_4 \} \) and a bisection \( \bar{\sigma}_{\Gamma} \) such that

\[
\Gamma'_{\bar{\sigma}} := \{ \gamma_1 \circ \gamma_1^{-1}, \gamma_2 \circ \mathbb{I}_{s(\gamma_2)}, \gamma_3 \circ \mathbb{I}_{s(\gamma_3)} \}
\]
Notice that, \( t(\gamma_1) \mapsto s(\gamma_1), t(\gamma_2) \mapsto t(\gamma_2), t(\gamma_3) \mapsto t(\gamma_3) \) and \( t(\gamma_4) \mapsto t(\gamma_4) \). Hence the the map \( \tilde{\sigma} : V_\Gamma \to P_\Gamma \Sigma \) is bijective map and consequently a bisection. The bisection \( \sigma_\Gamma \) in the graph system \( P_\Gamma \) defines a right-translation \( R_{\sigma_\Gamma} \) in \( P_\Gamma \).

Moreover for a subgraph \( \Gamma'' := \{\gamma_2, \gamma_3\} \) of the graph \( \tilde{\Gamma} := \{\gamma_1, \gamma_2, \gamma_3\} \) there exists a map \( \sigma_\Gamma : V_\Gamma \to P_\Gamma \) such that

\[
R_{\sigma_\Gamma}(\Gamma'') = \{\gamma_2, \gamma_3, \tilde{\sigma}(s(\gamma_1))\} = \{\gamma_2, \gamma_3, \gamma_1\}
\]

where \( t(\gamma_2) \mapsto t(\gamma_2), t(\gamma_3) \mapsto t(\gamma_3) \) and \( s(\gamma_1) \mapsto t(\gamma_1) \). Consequently in this example the map \( \tilde{\sigma}_\Gamma \) is a bisection, which defines a right-translation in \( P_\Gamma \).

Note that, for a graph \( \Gamma \) such that \( \tilde{\Gamma} \) and \( \tilde{\Gamma} \) are subgraphs the bisection \( \sigma_\Gamma \) extends to a bisection \( \sigma \) in \( P_\Gamma \)

Moreover the bisections of a finite graph system are transferred, analogously, to bisections of a finite path groupoid \( \Pi \Sigma \rightleftharpoons V_\Gamma \) to the group \( G^{[\Gamma]} \). Let \( \sigma \in \mathcal{B}(P_\Gamma) \) and \((\mathfrak{h}_\Gamma, h_\Gamma) \in \text{Hom}(P_\Gamma, G^{[\Gamma]})) \). Thus there are two maps

\[
\mathfrak{h}_\Gamma \circ R_\sigma : P_\Gamma \to G^{[\Gamma]}
\quad \text{and}\quad
h_\Gamma \circ (t \circ \sigma) : V_\Gamma \to \{e_G\}
\]

which defines a holonomy map for a finite graph system if \( \sigma \) is suitable.

Now, a similar right-translation in a finite graph system in comparison to the right-translation \( R_{\sigma(v)} \) in a finite path groupoid \( \Pi \Sigma \rightleftharpoons V_\Gamma \) is studied. Let \( \sigma_\Gamma : V_\Gamma \to P_\Gamma \) be a restriction of \( \sigma_\Gamma \in \mathcal{B}(P_\Gamma) \). Moreover let \( V \) be a subset of \( V_\Gamma \), let \( \Gamma'' \) be a subgraph of \( \Gamma '' \) and \( \Gamma''' \) be a subgraph of \( \Gamma'' \). Then a right-translation is given by

\[
R_{\sigma_{\Gamma''}(V)}(\Gamma''')
\]

Loosely speaking, the action of a path-diffeomorphism is somehow localised on a fixed vertex set \( V \).

For example note that for a subgraph \( \Gamma' := \{\gamma \circ \gamma'\} \) of \( \Gamma := \{\gamma, \gamma'\} \) and a subset \( V := \{t(\gamma')\} \) of \( V_\Gamma \), it is true that

\[
(\mathfrak{h}_\Gamma \circ R_{\sigma_{\Gamma''}(V)})(\gamma \circ \gamma') = (\mathfrak{h}_\Gamma \circ R_{\sigma_{\Gamma''}(V)})(\gamma)' \cdot (h_\Gamma \circ R_{\sigma_{\Gamma''}(V)})(\gamma)' = h_\Gamma(\gamma) \cdot (h_\Gamma \circ R_{\sigma_{\Gamma''}(V)})(\gamma)' = h_\Gamma(\gamma \circ \gamma' \circ \sigma(t(\gamma'))
\]

yields whenever \( \sigma_\Gamma \in \mathcal{B}(P_\Gamma \Sigma) \). For a special bisection \( \tilde{\sigma}_\Gamma \) it is true that,

\[
(\mathfrak{h}_\Gamma \circ R_{\sigma_{\Gamma}})(\gamma) = \mathfrak{h}_\Gamma(\gamma \circ \gamma') = (\mathfrak{h}_\Gamma \circ R_{\sigma_{\Gamma}})(\gamma) \cdot (h_\Gamma \circ R_{\sigma_{\Gamma}})(\gamma)' \]

holds whenever \( \tilde{\sigma}_\Gamma \in \mathcal{B}(P_\Gamma \Sigma) \) that defines the bisection \( \tilde{\sigma} \) in \( P_\Gamma \). Then the last statement is true, since \( R_{\sigma_{\Gamma}}(\gamma) = \gamma \circ \gamma'^{-1} \) requires \( \tilde{\sigma} : t(\gamma') \mapsto s(\gamma') \) and \( R_{\sigma_{\Gamma}}(\gamma) = \gamma \circ \gamma' \) needs \( \tilde{\sigma} : t(\gamma) \mapsto t(\gamma') \), where \( s(\gamma') = t(\gamma) \). Then \( R_{\sigma_{\Gamma}}(\gamma) \) and \( R_{\sigma_{\Gamma}}(t(\gamma')) \) coincide if \( \tilde{\sigma}(t(\gamma)) = \sigma_\Gamma(t(\gamma)) \) and \( \tilde{\sigma}(t(\gamma')) = \mathbb{1}_{t(\gamma')} \) holds.

**Problem 2.2** Let \( \Gamma' \) be a subgraph of the graph \( \Gamma \), \( \sigma_{\Gamma'} \) be a bisection in \( P_\Gamma \), \( \sigma_{\Gamma'} : V_{\Gamma'} \to P_{\Gamma'} \) be a restriction of \( \sigma_\Gamma \in \mathcal{B}(P_\Gamma) \). Moreover let \( V \) be a subset of \( V_{\Gamma'} \), let \( \Gamma'' := \{\gamma \circ \gamma'\} \) be a subgraph of \( \Gamma' \). Let \( \langle \gamma, \gamma' \rangle \in \mathcal{P}_\Gamma \Sigma^{(2)} \).

Then even for a suitable bisection \( \sigma_{\Gamma'} \) in \( P_\Gamma \) it follows that,

\[
R_{\sigma_{\Gamma'}(V)}(\gamma \circ \gamma') \neq R_{\sigma_{\Gamma'}(V)}(\gamma) \cdot R_{\sigma_{\Gamma'}(V)}(\gamma')
\]
yields. This is a general problem. In comparison with problem 2.1.1, the multiplication map $\circ$ is not well-defined and hence

$$R_{\sigma_{\tau}(V)}(\gamma) \circ R_{\sigma_{\tau}(V)}(\gamma')$$

is not well-defined. Recognize that, $R_{\sigma_{\tau}(V)} : \mathcal{P}_T \rightarrow \mathcal{P}_T$.

Consequently in general it is not true that,

$$\left( h \circ R_{\sigma_{\tau}(V)}(\gamma) \circ R_{\sigma_{\tau}(V)}(\gamma') \right) = (h \circ R_{\sigma_{\tau}(V)}(\gamma)) \cdot (h \circ R_{\sigma_{\tau}(V)}(\gamma'))$$

yields.

With no doubt the left-translation $L_{\sigma_{\tau}}$, and the inner automorphisms $I_{\sigma_{\tau}}$ in a finite graph system $\mathcal{P}_T$ for every $\Gamma' \in \mathcal{P}_T$ are defined similarly.

**Definition 2.29.** Let $\sigma_{\tau} \in \mathcal{B}(\mathcal{P}_T)$ be a bisection in the finite graph system $\mathcal{P}_T$. Let $R_{\sigma_{\tau}}(V)$ be a right-translation, where $V$ is a subset of $V_{\Gamma}$.

Then the pair $(\Phi_{\tau}, \varphi_{\tau})$ defined by $\Phi_{\tau} = R_{\sigma_{\tau}}(V)$ (or, respectively, $\Phi_{\tau} = L_{\sigma_{\tau}}(V)$, or $\Phi_{\tau} = I_{\sigma_{\tau}}(V)$) for a subset $V \subseteq V_{\Gamma}$ and $\varphi_{\tau} = t \circ \sigma_{\tau}$ is called a graph-diffeomorphism of a finite graph system. Denote the set of finite graph-diffeomorphisms by $\text{Diff}(\mathcal{P}_T)$.

Let $\Gamma'$ be a subgraph of $\Gamma$ and $\sigma_{\tau}$ be a restriction of bisection $\sigma_{\tau}$ in $\mathcal{P}_T$. Then for example another graph-diffeomorphism $(\Phi_{\tau}', \varphi_{\tau}')$ in $\text{Diff}(\mathcal{P}_T)$ is defined by $\Phi_{\tau}' = R_{\sigma_{\tau}}(V)$ for a subset $V \subseteq V_{\Gamma'}$ and $\varphi_{\tau}' = t \circ \sigma_{\tau}$.

Remembering that the set of bisections of a finite path groupoid forms a group (refer to 2.23) one may ask if the bisections of a finite graph system form a group, too.

**Proposition 2.30.** The set of bisections $\mathcal{B}(\mathcal{P}_T)$ in a finite graph system $\mathcal{P}_T$ forms a group.

**Proof**: Let $\Gamma$ be a graph and let $V_{\Gamma}$ be equivalent to the set $\{v_1, ..., v_{2N}\}$.

First two different multiplication operations are studied. The studies are comparable with the results of the definition 2.28 of a right-translation in a finite graph system. The easiest multiplication operation is given by $*_1$, which is defined by

$$(\sigma *_{1} \sigma')(V_{\Gamma}) := \{(\tilde{\sigma} * \tilde{\sigma}')(v_1), ..., (\tilde{\sigma} * \tilde{\sigma}')(v_{2N}) : v_i \in V_{\Gamma} \}$$

where $*$ denotes the multiplication of bisections on the finite path groupoid $\mathcal{P}_T \Sigma \equiv V_{\Gamma}$. Notice that, this operation is not well-defined in general. In comparison with the definition of the right-translation in a finite graph system one has to take care. First the set of vertices doesn’t contain any vertices twice, the map $\sigma$ in the finite path system is bijective, the mapping $\sigma$ maps each set to a set of vertices containing no vertices twice and the situation in problem 2.11 has to be avoided.

Fix a bisection $\tilde{\sigma}$ in a finite path groupoid $\mathcal{P}_T \Sigma \equiv V_{\Gamma}$. Let $V_{\sigma'}$ be a subset of $V_{\Gamma}$ where $\Gamma := \{\gamma_1, ..., \gamma_N\}$ and for each $v_i$ in $V_{\sigma'}$ it is true that $v_i \neq v_j$ and $v_i \neq (t \circ \sigma')(v_j)$ for all $i \neq j$. Define the set $V_{\sigma,\sigma'}$ to be equal to a subset of all vertices $\{v_k \in V_{\sigma'} : 1 \leq k \leq 2N\}$ such that each pair $(v_i, v_j)$ of vertices in $V_{\sigma,\sigma'}$ satisfies $t \circ (\tilde{\sigma} * \tilde{\sigma}')(v_i) \neq t \circ \sigma'(v_j)$ and $t \circ \sigma'(v_i) \neq t \circ (\tilde{\sigma} * \tilde{\sigma}')(v_j)$ for all $i \neq j$. Define

$$W_{\sigma,\sigma'} := \left\{ v_i \in \left\{ V_{\sigma} \cap V_{\sigma'} \right\} : \forall i, j, \ 1 \leq i, j \leq l \right\}$$

The set $V_{\sigma,\sigma',\sigma}$ is a subset of all vertices $\{v_k \in V_{\sigma,\sigma'} : 1 \leq k \leq 2N\}$ such that each pair $(v_i, v_j)$ of vertices in $V_{\sigma,\sigma',\sigma}$ satisfies $t \circ (\tilde{\sigma} * \tilde{\sigma}')(v_i) \neq t \circ \sigma'(v_j)$ and $t \circ \sigma'(v_i) \neq t \circ (\tilde{\sigma} * \tilde{\sigma}')(v_j)$ for all $i \neq j$.

Consequently define a second multiplication on $\mathcal{B}(\mathcal{P}_T)$ similarly to the operation $*_1$. This is done by the following definition. Set

$$(\sigma *_{2} \sigma')(V_{\Gamma}) := \{(\tilde{\sigma} * \tilde{\sigma}')(v_1), ..., (\tilde{\sigma} * \tilde{\sigma}')(v_{2N}) : v_i, ..., v_k \in V_{\sigma,\sigma'}, 1 \leq k \leq 2N \}$$

$$\cup \{\tilde{\sigma}(w_1), \sigma'(w_1), ..., \tilde{\sigma}(w_l), \sigma'(w_l) : w_1, ..., w_l \in W_{\sigma,\sigma'}, 1 \leq l \leq 2N \}$$

$$\cup \{p_1, ..., p_n : p_1, ..., p_n \in V_{\sigma'} \cap \{W_{\sigma,\sigma'} \cup V_{\sigma,\sigma'}\}, 1 \leq n \leq 2N \}$$
Hence the inverse is supposed to be \( \sigma^{-1}(V_T) = \sigma((t \circ \sigma)^{-1}(V_T))^{-1} \) such that

\[
(\sigma \ast_2 \sigma^{-1})(V_{\sigma^{-1}}) = \{(\sigma \ast \sigma^{-1})(v_1), ..., (\sigma \ast \sigma^{-1})(v_{2N}) : v_i \in V_{\sigma \circ \sigma^{-1}} \} \\
\cup \{\sigma(w_1), \sigma'(w_1)^{-1}, ..., \sigma(w_l), \sigma'(w_l)^{-1} : w_1, ..., w_l \in W_{\sigma \circ \sigma^{-1}}, 1 \leq l \leq 2N \} \\
\cup \{ \mathbb{1}_{p_1}, ..., \mathbb{1}_{p_n} : p_1, ..., p_n \in V_{\sigma'} \setminus \{V_{\sigma \circ \sigma^{-1}} \cup W_{\sigma \circ \sigma^{-1}} \}, 1 \leq n \leq 2N \}
\]

Notice that, the problem 2.2 is solved by a multiplication operation \( \ast_2 \), which is defined similarly to \( \ast \). Hence the equality of (17) is available and consequently (18) is true. Furthermore a similar remark to (15) can be done.

**Example 2.5:** Now consider the following example. Set \( \Gamma' := \{\gamma_1, \gamma_3\} \), let \( \Gamma := \{\gamma_1, \gamma_2, \gamma_3\} \) and \( V_T := \{s(\gamma_1), t(\gamma_1), s(\gamma_2), t(\gamma_2), s(\gamma_3), t(\gamma_3) : s(\gamma_i) \neq s(\gamma_j), t(\gamma_i) \neq t(\gamma_j) \text{ for } \forall i \neq j \} \).

Set \( V \) be equal to \( \{s(\gamma_1), s(\gamma_2), s(\gamma_3)\} \). Take two maps \( \sigma \) and \( \sigma' \) such that \( \sigma'(V) = \{\gamma_1, \gamma_3\} \), \( \sigma(V) = \{\gamma_2\} \), where \( (t \circ \sigma)(s(\gamma_3)) = t(\gamma_3), \sigma'(s(\gamma_3)) = \gamma_3, \sigma'(s(\gamma_1)) = \gamma_1 \) and \( \tilde{\sigma}(t(\gamma_3)) = \gamma_2 \). Then \( s(\gamma_3) \in V_{\sigma \circ \sigma', \sigma'} \) and \( s(\gamma_1) \in W_{\sigma \circ \sigma', \sigma'} \). Derive

\[
(\sigma \ast_1 \sigma')(V) = \{\gamma_3 \circ \gamma_2, \gamma_1\}
\]

Then conclude that,

\[
(\sigma \ast_2 \sigma')(V_T) = \{\gamma_3 \circ \gamma_2, \gamma_1\}
\]

holds. Notice that

\[
(\sigma \ast_2 \sigma')(V) \neq (\sigma' \ast_2 \sigma)(V) = \{\gamma_2, \gamma_1, \gamma_3\}
\]

is true. Finally obtain

\[
(\sigma \ast_2 \sigma^{-1})(V_T) = \{\gamma_3 \circ \gamma_3^{-1}, \gamma_1 \circ \gamma_1^{-1}\} = \{\mathbb{1}_{s(\gamma_3)}, \mathbb{1}_{s(\gamma_1)}\}
\]

Let \( \sigma'(V_T) = \{\gamma_1, \gamma_3\} \) and \( \tilde{\sigma}(V_T) = \{\gamma_2, \gamma_4\} \). Then notice that,

\[
(\tilde{\sigma} \ast_1 \sigma')(V_T) = \{\gamma_3 \circ \gamma_2, \gamma_1\}
\]

and

\[
(\tilde{\sigma} \ast_2 \sigma')(V_T) = \{\gamma_3 \circ \gamma_2, \gamma_1\}
\]

yields.

Furthermore assume supplementary that \( t(\gamma_3) = t(\gamma_1) \) holds. Then calculate the product of the maps \( \sigma \) and \( \sigma' \):

\[
(\sigma \ast_1 \sigma')(V) = \{\gamma_3 \circ \gamma_2, \gamma_1 \circ \gamma_2\} \notin \mathcal{P}_T
\]

and

\[
(\sigma \ast_2 \sigma')(V_T) = \{\mathbb{1}_{t(\gamma_1)}, \mathbb{1}_{t(\gamma_3)}\} \in \mathcal{P}_T
\]

The group structure of \( \mathfrak{B}(\mathcal{P}_T) \) transfers to \( G \). Let \( \tilde{\sigma} \) be a bisecton in the finite path groupoid \( \mathcal{P}_T \xrightarrow{\sigma} V_T \), which defines a bisection \( \sigma \) in \( \mathcal{P}_T \) and let \( \tilde{\sigma}' \) be a bisecton in \( \mathcal{P}_T \xrightarrow{\sigma'} V_T \), which defines another bisecton \( \sigma' \) in \( \mathcal{P}_T \). Let \( V_{\sigma, \sigma'} \) be equal to \( V_T \), then derive

\[
\mathfrak{h}_T((\sigma \ast_2 \sigma')(V_T)) = \{\mathfrak{h}_T((\tilde{\sigma} \ast \sigma')(v_1)), ..., \mathfrak{h}_T((\tilde{\sigma} \ast \sigma')(v_{2N}))\}
\]

\[
= \mathfrak{h}_T(\sigma'(V_T) \circ \tilde{\sigma}(t(\sigma'(V_T)))) = \{\mathfrak{h}_T(\sigma'(v) \circ \tilde{\sigma}(t(\sigma'(v_1)))), ..., \mathfrak{h}_T(\sigma'(v_N) \circ \tilde{\sigma}(t(\sigma'(v_N))))\}
\]

\[
= \{\mathfrak{h}_T(\sigma'(v))\mathfrak{h}_T(\tilde{\sigma}(t(\sigma'(v_1)))), ..., \mathfrak{h}_T(\sigma'(v_N))\mathfrak{h}_T(\tilde{\sigma}(t(\sigma'(v_N))))\}
\]

(19)

Consequently the right-translation in the finite product \( G^{[\Gamma]} \) is definable.
Definition 2.31. Let $\sigma_{\gamma'}$ be in $\mathfrak{B}(\mathcal{P}_\Gamma)$, $\Gamma'$ a subgraph of $\Gamma$, $\Gamma''$ a subgraph of $\Gamma'$ and $R_{\sigma_{\gamma'}}$ a right-translation, $L_{\sigma_{\gamma'}}$ a left-translation and $I_{\sigma_{\gamma'}}$ an inner-translation in $\mathcal{P}_\Gamma$.

Then the right-translation in the finite product $G^{[\Gamma]}$ is given by

$$\mathfrak{h}_\Gamma \circ R_{\sigma_{\gamma'}} : \mathcal{P}_\Gamma \to G^{[\Gamma]}, \quad \Gamma'' \mapsto (\mathfrak{h}_\Gamma \circ R_{\sigma_{\gamma'}})(\Gamma'')$$

Furthermore define the left-translation in the finite product $G^{[\Gamma]}$ by

$$\mathfrak{h}_\Gamma \circ L_{\sigma_{\gamma'}} : \mathcal{P}_\Gamma \to G^{[\Gamma]}, \quad \Gamma'' \mapsto (\mathfrak{h}_\Gamma \circ L_{\sigma_{\gamma'}})(\Gamma'')$$

and the inner-translation in the finite product $G^{[\Gamma]}$

$$\mathfrak{h}_\Gamma \circ I_{\sigma_{\gamma'}} : \mathcal{P}_\Gamma \to G^{[\Gamma]}, \quad \Gamma'' \mapsto (\mathfrak{h}_\Gamma \circ I_{\sigma_{\gamma'}})(\Gamma'')$$

such that $I_{\sigma_{\gamma'}} = L_{\sigma_{\gamma'}}^{-1} \circ R_{\sigma_{\gamma'}}$.

Lemma 2.32. It is true that $R_{\sigma_{\gamma'} \circ \sigma_{\gamma''}} = R_{\sigma_{\gamma'}} \circ R_{\sigma_{\gamma''}}$, $L_{\sigma_{\gamma'} \circ \sigma_{\gamma''}} = L_{\sigma_{\gamma'}} \circ L_{\sigma_{\gamma''}}$ and $I_{\sigma_{\gamma'} \circ \sigma_{\gamma''}} = I_{\sigma_{\gamma'}} \circ I_{\sigma_{\gamma''}}$ for all bisections $\sigma_{\gamma'}$ and $\sigma_{\gamma''}$ in $\mathfrak{B}(\mathcal{P}_\Gamma)$.

There is an action of $\mathfrak{B}(\mathcal{P}_\Gamma)$ on $G^{[\Gamma]}$ by

$$(\zeta_{\sigma_{\gamma'}} \circ \mathfrak{h}_\Gamma)(\Gamma'') := (\mathfrak{h}_\Gamma \circ R_{\sigma_{\gamma'}})(\Gamma'')$$

whenever $\sigma_{\gamma'} \in \mathfrak{B}(\mathcal{P}_\Gamma)$, $\Gamma'' \in \mathcal{P}_\Gamma$, and $\Gamma' \in \mathcal{P}_\Gamma$. Then for another $\tilde{\sigma} \in \mathfrak{B}(\mathcal{P}_\Gamma)$ it is true that,

$$( (\zeta_{\sigma_{\gamma'}} \circ \zeta_{\sigma_{\gamma''}}) \circ \mathfrak{h}_\Gamma)(\Gamma'') = (\mathfrak{h}_\Gamma \circ R_{\sigma_{\gamma'} \circ \sigma_{\gamma''}})(\Gamma'') = (\zeta_{\sigma_{\gamma'} \circ \sigma_{\gamma''}} \circ \mathfrak{h}_\Gamma)(\Gamma'')$$

yields.

Finally the left or right-translations in a finite path groupoid can be studied in the context of natural or non-standard identification of the configuration space. This new concept leads to two different notions of diffeomorphism-invariant states. The actions of path- and graph-diffeomorphism and the concepts of natural or non-standard identification of the configuration space was not used in the context of LQG before.

2.3 The Lie algebra-valued quantum flux operators associated to surfaces and graphs

The quantum analogue of a classical connection $A_x(v)$ is given by the holonomy along a path $\gamma$ and is denoted by $\mathfrak{h}(\gamma)$. The quantum flux operator $E_S(\gamma)$, which replaces the classical flux variable $E(S, f^S)$, is given by a map $E_S$ from a graph to the Lie algebra $\mathfrak{g}$. Let $\mathbf{Exp}$ be the exponential map from the Lie algebra $\mathfrak{g}$ to $G$ and set $U_t(E_S(\gamma)) := \mathbf{Exp}(tE_S(\gamma))$. Then the quantum flux operator $E_S(\gamma)$ and the quantum holonomies $\mathfrak{h}(\gamma)$ satisfy the following canonical commutator relation:

$$E_S(\gamma)\mathfrak{h}(\gamma) = \frac{i}{\hbar} \left. \frac{d}{dt} \right|_{t=0} U_t(E_S(\gamma))\mathfrak{h}(\gamma)$$

where $\gamma$ is a path that intersects the surface $S$ in the target vertex of the path and lies below with respect to the surface orientation of $S$. 
In this section different definitions of the quantum flux operator, which is associated to a fixed surface \( S \), are presented. For example, the quantum flux operator \( E_S \) is defined to be a map from a graph \( \Gamma \) to a direct sum \( \mathfrak{g} \oplus \mathfrak{g} \) of the Lie algebra \( \mathfrak{g} \) associated to the Lie group \( G \). This is related to the fact that, one distinguishes between paths that are ingoing and paths that are outgoing with respect to the surface orientation of \( S \). If there are no intersection points of the surface \( S \) and the source or target vertex of a path \( \gamma \) of a graph \( \Gamma \), then the map maps the path \( \gamma \) to zero in both entries. For different surfaces or for a fixed surface different maps refer to different quantum flux operators. Furthermore, the quantum flux operators are also defined as maps form the graph \( \Gamma \) to direct sum \( \mathcal{E} \oplus \mathcal{E} \) maps refer to different quantum flux operators. Respectively, the quantum flux operators are also defined as maps form the graph \( \Gamma \) to direct sum \( \mathcal{E} \oplus \mathcal{E} \) of the universal enveloping algebra \( \mathcal{E} \) of \( \mathfrak{g} \).

**Definition 2.33.** Let \( \hat{S} \) be a finite set \( \{S_i\} \) of surfaces in \( \Sigma \), which is closed under a flip of orientation of the surfaces. Let \( \Gamma \) be a graph such that each path in \( \Gamma \) satisfies one of the following conditions

- the path intersects each surface in \( \hat{S} \) in the source vertex of the path and there are no other intersection points of the path and any surface contained in \( \hat{S} \),
- the path intersects each surface in \( \hat{S} \) in the target vertex of the path and there are no other intersection points of the path and any surface contained in \( \hat{S} \),
- the path intersects each surface in \( \hat{S} \) in the source and target vertex of the path and there are no other intersection points of the path and any surface contained in \( \hat{S} \),
- the path does not intersect any surface \( S \) contained in \( \hat{S} \).

Then define the intersection functions \( \iota_L : \hat{S} \times \Gamma \to \{\pm 1, 0\} \) such that

\[
\iota_L(S, \gamma) := \begin{cases} 
1 & \text{for a path } \gamma \text{ lying above and outgoing w.r.t. } S \\
-1 & \text{for a path } \gamma \text{ lying below and outgoing w.r.t. } S \\
0 & \text{the path } \gamma \text{ is not outgoing w.r.t. } S 
\end{cases}
\]

and the intersection functions \( \iota_R : \hat{S} \times \Gamma \to \{\pm 1, 0\} \) such that

\[
\iota_L(S, \gamma) := \begin{cases} 
-1 & \text{for a path } \gamma' \text{ lying above and ingoing w.r.t. } S \\
1 & \text{for a path } \gamma' \text{ lying below and ingoing w.r.t. } S \\
0 & \text{the path } \gamma' \text{ is not ingoing w.r.t. } S 
\end{cases}
\]

whenever \( S \in \hat{S} \) and \( \gamma \in \Gamma \).

Define a map \( \sigma_L : \hat{S} \to \mathfrak{g} \) such that

\[
\sigma_L(S) = \sigma_L(S^{-1})
\]

whenever \( S \in \hat{S} \) and \( S^{-1} \) is the surface \( S \) with reversed orientation. Denote the set of such maps by \( \tilde{\sigma}_L \).

Respectively, the map \( \sigma_R : \hat{S} \to \mathfrak{g} \) such that

\[
\sigma_R(S) = \sigma_R(S^{-1})
\]

whenever \( S \in \hat{S} \). Denote the set of such maps by \( \tilde{\sigma}_R \). Moreover, there is a map \( \sigma_L \times \sigma_R : \hat{S} \to \mathfrak{g} \oplus \mathfrak{g} \) such that

\[
(\sigma_L, \sigma_R)(S) = (\sigma_L, \sigma_R)(S^{-1})
\]

whenever \( S \in \hat{S} \). Denote the set of such maps by \( \bar{\sigma} \).

Finally, define the **Lie algebra-valued quantum flux set for paths**

\[
\mathcal{E}_S : \bigcup_{\sigma_L \times \sigma_R \in \mathcal{E}} \bigcup_{\gamma \in \hat{S}} \{ (E_L, E_R) \in \text{Map}(\Gamma, \mathfrak{g} \oplus \mathfrak{g}) : (E_L, E_R)(\gamma) := (\iota_L(S, \gamma) \sigma_L(S), \iota_R(S, \gamma) \sigma_R(S)) \}
\]

where \( \text{Map}(\Gamma, \mathfrak{g} \oplus \mathfrak{g}) \) is the set of all maps from the graph \( \Gamma \) to the direct sum \( \mathfrak{g} \oplus \mathfrak{g} \) of Lie algebras.

Observe that, \((\iota_L \times \iota_R)(S^{-1}, \gamma) = (-\iota_L \times -\iota_R)(S, \gamma)\) holds for every \( \gamma \in \Gamma \).

Remark that, the condition \( E_L(\gamma) = E_R(\gamma^{-1}) \) is not required.

**Example 2.6:** Analyse the following example. Consider a graph \( \Gamma \) and two disjoint surface sets \( \hat{S} \) and \( \hat{T} \).
Then the elements of \( g_{S,T} \) are for example given by the maps \( E^L_i \times E^R_i \) for \( i = 1, 2 \) such that

\[
\begin{align*}
E_1(\gamma) &:= (E^L_1, E^R_1)(\gamma) = (\iota_L(S_1, \gamma)\sigma_L(S_1), \iota_R(S_1, \gamma)\sigma_R(S_1)) = (X_1, 0) \\
E_1(\tilde{\gamma}) &:= (E^L_1, E^R_1)(\tilde{\gamma}) = (\iota_L(S_1, \tilde{\gamma})\sigma_L(S_1), \iota_R(S_1, \tilde{\gamma})\sigma_R(S_1)) = (X_1, 0) \\
E_2(\gamma) &:= (E^L_2, E^R_2)(\gamma) = (\iota_L(S_2, \gamma)\sigma_L(S_2), \iota_R(S_2, \gamma)\sigma_R(S_2)) = (X_2, 0) \\
E_2(\tilde{\gamma}) &:= (E^L_2, E^R_2)(\tilde{\gamma}) = (\iota_L(S_2, \tilde{\gamma})\sigma_L(S_2), \iota_R(S_2, \tilde{\gamma})\sigma_R(S_2)) = (X_2, 0) \\
E_3(\gamma) &:= (E^L_3, E^R_3)(\gamma) = (\iota_L(S_3, \gamma)\sigma_L(S_3), \iota_R(S_3, \gamma)\sigma_R(S_3)) = (0, -Y_3) \\
E_3(\tilde{\gamma}) &:= (E^L_3, E^R_3)(\tilde{\gamma}) = (\iota_L(S_3, \tilde{\gamma})\sigma_L(S_3), \iota_R(S_3, \tilde{\gamma})\sigma_R(S_3)) = (0, -Y_3) \\
E_4(\gamma) &:= (E^L_4, E^R_4)(\gamma) = (\iota_L(S_4, \gamma)\sigma_L(S_4), \iota_R(S_4, \gamma)\sigma_R(S_4)) = (0, Y_4) \\
E_4(\tilde{\gamma}) &:= (E^L_4, E^R_4)(\tilde{\gamma}) = (\iota_L(S_4, \tilde{\gamma})\sigma_L(S_4), \iota_R(S_4, \tilde{\gamma})\sigma_R(S_4)) = (0, Y_4)
\end{align*}
\]

This example shows that, the surfaces \( \{S_1, S_2\} \) are similar, whereas the surfaces \( \{T_1, T_2\} \) produce different signatures for different paths. Moreover, the set of surfaces are chosen such that one component of the direct sum is always zero.

For a particular surface set \( \tilde{S} \), the set

\[
\bigcup_{\sigma_L, \sigma_R \in \sigma} \bigcup_{S \in \tilde{S}} \{ (E^L, E^R) \in \text{Map}(\Gamma, g \oplus g) : (E^L, E^R)(\gamma) := (\iota_L(S, \gamma)\sigma_L(S), 0) \}
\]

can be identified with

\[
\bigcup_{\sigma_L, \sigma_R \in \sigma} \bigcup_{S \in \tilde{S}} \{ E \in \text{Map}(\Gamma, g) : E(\gamma) := \iota_L(S, \gamma)\sigma_L(S) \}
\]

The same is observed for another surface set \( \tilde{T} \) and the set \( g_{\tilde{T}, \Gamma} \) is identifiable with

\[
\bigcup_{\sigma_R \in \sigma} \bigcup_{T \in \tilde{T}} \{ E \in \text{Map}(\Gamma, g) : E(\gamma) := \iota_R(T, \gamma)\sigma_R(T) \}
\]

The intersection behavior of paths and surfaces plays a fundamental role in the definition of the flux operator. There are exceptional configurations of surfaces and paths in a graph. One of them is the following.

**Definition 2.34.** A surface \( S \) has the **surface intersection property for a graph** \( \Gamma \) iff the surface intersects each path of \( \Gamma \) once in the source or target vertex of the path and there are no other intersection points of \( S \) and the path.

This is for example the case for the surface \( S_1 \) or the surface \( S_3 \), which are presented in example 2. Notice that in general, for the surface \( S \) there are \( N \) intersection points with \( N \) paths of the graph. In the example the evaluated map \( E_1(\gamma) = (X_1, 0) = E_1(\tilde{\gamma}) \) for \( \gamma, \tilde{\gamma} \in \Gamma \) if the surface \( S_1 \) is considered.

The property of a path lying above or below is not important for the definition of the surface intersection property for a surface. This indicates that the surface \( S_4 \) in the example 2 has the surface intersection property, too.
Let a surface \( S \) does not have the surface intersection property for a graph \( \Gamma \), which contains only one path \( \gamma \). Then for example the path \( \gamma \) intersects the surface \( S \) in the source and target vertices such that the path lies above the surface \( S \). Then the map \( E^L \times E^R \) is evaluated for the path \( \gamma \) by

\[
(E^L \times E^R)(\gamma) = (X, -Y)
\]

Hence, simply speaking the surface intersection property reduces the components of the map \( E^L \times E^R \), but for different paths to different components.

Now, consider a bunch of sets of surfaces such that for each surface there is only one intersection point.

**Definition 2.35.** A set \( \bar{S} \) of \( N \) surfaces has the **surface intersection property for a graph** \( \Gamma \) with \( N \) independent edges iff it contain only surfaces, for which each path \( \gamma_i \) of a graph \( \Gamma \) intersects each surface \( S_i \) only once in the same source or target vertex of the path \( \gamma_i \), there are no other intersection points of each path \( \gamma_i \) and each surface in \( \bar{S} \), and there is no other path \( \gamma_j \) that intersects the surface \( S_i \) for \( i \neq j \) where \( 1 \leq i, j \leq N \).

Then for example consider the following configuration.

**Example 2.7:**

The sets \( \{S_6, S_7\} \) or \( \{S_5, S_8\} \) have the surface intersection property for the graph \( \Gamma \). The images of a map \( E \) is

\[
E_3(\bar{\gamma}) = (X_3, 0), \quad E_9(\gamma) = (0, Y_9)
\]

Note that simply speaking, the property indicates that each map reduces to a component of \( E^L \times E^R \) but for different surfaces the map reduces to \( E^L \) or \( E^R \).

A set of surfaces that has the surface intersection property for a graph can be further specialised by restricting the choice to paths lying ingoing and below with respect to the surface orientations.

**Definition 2.36.** A set \( \bar{S} \) of \( N \) surfaces has the **simple surface intersection property for a graph** \( \Gamma \) with \( N \) independent edges iff it contains only surfaces, for which each path \( \gamma_i \) of a graph \( \Gamma \) intersects only one surface \( S_i \) only once in the target vertex of the path \( \gamma_i \), the path \( \gamma_i \) lies above and there are no other intersection points of each path \( \gamma_i \) and each surface in \( \bar{S} \).

**Example 2.8:** Consider the following example.
The sets \{S_9, S_{11}\} or \{S_{10}, S_{12}\} have the simple surface intersection property for the graph \(\Gamma\). Calculate
\[
E_9(\gamma) = (0, -Y_9), \quad E_{11}(\gamma) = (0, -Y_{11})
\]

In this case the set \(g_{S,\Gamma}\) reduces to
\[
\bigcup_{\sigma_\Gamma \in \sigma_\Gamma} \bigcup_{S \in \hat{S}} \left\{ E \in \text{Map}(\Gamma, g) : E(\gamma) := -\sigma_L(S) \text{ for } \gamma \cap S = t(\gamma) \right\}
\]

Notice that, the set \(\Gamma \cap \hat{S} = \{t(\gamma_i)\}\) for a surface \(S_i \in \hat{S}\) and \(\gamma_i \cap S_j \cap S_k = \emptyset\) for a path \(\gamma_i\) in \(\Gamma\) and \(i \neq j\).

On the other hand, the set of surfaces can be such that each path of a graph intersects all surfaces of the set in the same vertex. This contradicts the assumption that, each path of a graph intersects only one surface once.

**Definition 2.37.** Let \(\Gamma\) be a graph that contains no loops.

A set \(\hat{S}\) of surfaces has the same surface intersection property for a graph \(\Gamma\) iff each path \(\gamma_i\) in \(\Gamma\) intersects with all surfaces of \(\hat{S}\) in the same source vertex \(v_i \in V_\Gamma\) \((i = 1, \ldots, N)\), all paths are outgoing and lie below each surface \(S \in \hat{S}\) and there are no other intersection points of each path \(\gamma_i\) and each surface in \(\hat{S}\).

A surface set \(\hat{S}\) has the same right surface intersection property for a graph \(\Gamma\) iff each path \(\gamma_i\) in \(\Gamma\) intersects with all surfaces of \(\hat{S}\) in the same target vertex \(v_i \in V_\Gamma\) \((i = 1, \ldots, N)\), all paths are ingoing and lie above each surface \(S \in \hat{S}\) and there are no other intersection points of each path \(\gamma_i\) and each surface in \(\hat{S}\).

Recall the example 2.6. Then the set \(\{S_1, S_2\}\) has the same surface intersection property for the graph \(\Gamma\).

Then the set \(g_{S,\Gamma}\) reduces to
\[
\bigcup_{\sigma_\Gamma \in \sigma_\Gamma} \bigcup_{S \in \hat{S}} \left\{ E \in \text{Map}(\Gamma, g) : E(\gamma) := -\sigma_L(S) \text{ for } \gamma \cap S = s(\gamma) \right\}
\]

Notice that, \(\gamma \cap S_1 \cap \ldots \cap S_N = s(\gamma)\) for a path \(\gamma\) in \(\Gamma\), whereas \(\Gamma \cap \hat{S} = \{s(\gamma_i)\}_{1 \leq i \leq N}\). Clearly, \(\Gamma \cap S_i = s(\gamma_i)\) holds for a surface \(S_i\) in \(\hat{S}\).

Simply speaking the physical intuition is that, fluxes associated to different surfaces should act on the same path. Notice that both properties can be restated for other surface and path configurations. Hence, a surface set can have the simple or same surface intersection property for paths that are outgoing and lie above (or ingoing and below, or outgoing and below). The important fact is related to the question if the intersection vertices are the same for all surfaces or not.

In section 2.1 the concept of finite graph systems is introduced. The following remark shows that, the properties simply generalises to this new structure.

**Remark 2.38.** A set \(\hat{S}\) has the surface intersection property for a finite graph system \(P_\Gamma\) iff the set \(\hat{S}\) has the surface intersection property for each subgraph of \(\Gamma\) and \(\Gamma\).

A set \(\hat{S}\) has the same surface intersection property for a finite orientation preserved graph system \(P_\Gamma^o\) associated to a graph \(\Gamma\) (with no loops) iff the set \(\hat{S}\) has the same surface intersection property for the graph \(\Gamma\).

A set \(\hat{S}\) has the simple surface intersection property for a finite orientation preserved graph system \(P_\Gamma^o\) associated to a graph \(\Gamma\) iff the set \(\hat{S}\) has the simple surface intersection property for the graph \(\Gamma\).

**Definition 2.39.** Let \(\hat{S}\) be a surface set and \(\Gamma\) be a graph such that the only intersections of the graph and each surface in \(\hat{S}\) are contained in the vertex set \(V_\Gamma\).

Then the set of images \(\{E(\gamma) : E \in g_{S,\Gamma}\}\) of flux maps for a fixed path \(\gamma\) in \(\Gamma\) is denoted by \(g_{S,\gamma}\).

---

\footnote{Let \(\hat{S}\) be equal to \(S\). Then notice that the property of all graphs being orientation preserved subgraphs is necessary, since, for a subgraph \(\Gamma' := \{\gamma'\}\) of the graph \(\{\gamma'\}\) is a subgraph of \(\Gamma\), too. Consequently, if there is a surface \(S\) intersecting a path \(\gamma'\) such that \(\gamma'\) is ingoing and lies above, then \(S\) intersects the path \(\gamma'\) such that \(\gamma'\) is outgoing and lies above. This implies that, the surface \(S\) cannot have the same surface intersection property for each subgraph of \(\Gamma\).}
**Proposition 2.40.** Let $\tilde{S}$ be a set of surfaces and $\Gamma$ be a fixed graph (with no loops) such that the set $\tilde{S}$ has the same surface intersection property for a graph $\Gamma$. Moreover, let $\bar{T}$ be a set of surfaces and $\Gamma$ be a fixed graph such that the set $\bar{T}$ has the simple surface intersection property for a graph $\Gamma$.

Then the set $\mathfrak{g}_{\tilde{S},\gamma}$ is equipped with a structure, which is induced from the Lie algebra structure of $\mathfrak{g}$, such that it forms a Lie algebra. The the set $\mathfrak{g}_{\bar{T},\gamma}$ is equipped with a structure to form a Lie algebra, too.

**Proof :**

**Step 1: linear space over $\mathbb{C}$**

Consider a path $\gamma$ in $\Gamma$ that lies above and ingoing w.r.t. the surface orientation of each surface $S$ in $\tilde{S}$ and ingoing and above with respect to $T$. Then there is a map $E_S$ such that

$$E_S(\gamma) = -X$$

There exists an operation $\cdot$ given by the map $s : \mathfrak{g}_{\tilde{S},\gamma} \times \mathfrak{g}_{\tilde{S},\gamma} \rightarrow \mathfrak{g}_{\tilde{S},\gamma}$ such that

$$(E^1_1(\gamma), E^1_2(\gamma)) \mapsto s(E^1_1(\gamma), E^1_2(\gamma)) := E^1_1(\gamma) + E^1_2(\gamma) = -\sigma^1_L(S_1) - \sigma^2_L(S_2) = -\sigma^3_L([S])$$

since $\sigma^i_L \in \tilde{\sigma}_L$ and where $[S]$ denotes an arbitrary representative of the set $\tilde{S}$. Respectively it is defined

$$(E^1_1(\gamma), E^1_2(\gamma)) \mapsto s(E^1_1(\gamma), E^1_2(\gamma)) := E^1_1(\gamma) + E^1_2(\gamma) = -\sigma^1_R(T) - \sigma^2_R(T) - \sigma^3_R(T)$$

whenever $\sigma^i_R \in \sigma_R$ and $T \in \bar{T}$. There is an inverse

$$E(\gamma) - E(\gamma) = X - X = 0$$

and a null element

$$E(\gamma) + E_0(\gamma) = X$$

whenever $E_0(\gamma) = -\sigma_L(S) = 0$. Notice the following map

$$\mathfrak{g}_{\tilde{S},\gamma} \times \mathfrak{g}_{\tilde{S},\gamma} \ni (E_1(\gamma), E_2(\gamma)) \mapsto E_1(\gamma) + E_2(\gamma) \in \mathfrak{g}$$

is not considered, since, this map is not well-defined. One can show easily that $(\mathfrak{g}_{\tilde{S},\gamma}, +)$ is an additive group. The scalar multiplication is defined by

$$\lambda \cdot E(\gamma) = \lambda X$$

for all $\lambda \in \mathbb{C}$ and $X \in \mathfrak{g}$. Finally, prove that $(\mathfrak{g}_{\tilde{S},\gamma}, +)$ is a linear space over $\mathbb{C}$.

**Step 2: Lie bracket**

is defined by the Lie bracket of the Lie algebra $\mathfrak{g}$ and

$$[E_1(\gamma), E_2(\gamma)] := [X_1, X_2]$$

for $E_1(\gamma), E_2(\gamma) \in \mathfrak{g}_{\tilde{S},\gamma}$ and $\gamma \in \Gamma$.

\[\blacksquare\]

If a surface set $\tilde{S}$ does not have the same or simple surface intersection property for the graph $\Gamma$, then the surface set can be decomposed into several sets and the graph $\Gamma$ can be decomposed into a set of subgraphs. Then for each modified surface set there is a subgraph such that required condition is fulfilled.

**Definition 2.41.** Let $\tilde{S}$ a set of surfaces and $\Gamma$ be a fixed graph (with no loops) such that the set $\tilde{S}$ has the same (or simple) surface intersection property for a graph $\Gamma$.

The universal enveloping Lie algebra of the Lie algebra $\mathfrak{g}_{\tilde{S},\gamma}$ of electric fluxes for paths of a path $\gamma$ in $\Gamma$ and all surfaces in $\tilde{S}$ is called the universal enveloping flux algebra $B_{\tilde{S},\gamma}$ associated to a path and a finite set of surfaces.

For a detailed construction of the universal enveloping flux algebra associated a surface set and a path refer to the section 3 and definition 3.6.

Now, the definitions are rewritten for finite orientation preserved graph systems.
**Definition 2.42.** Let \( \tilde{S} \) be a surface set and \( \Gamma \) be a graph such that the only intersections of the graph and each surface in \( \tilde{S} \) are contained in the vertex set \( V_\Gamma \). \( \mathcal{P}_\Gamma \) denotes the finite graph system associated to \( \Gamma \). Let \( \mathcal{E} \) be the universal Lie enveloping algebra of \( g \).

Define the set of *Lie algebra-valued quantum fluxes for graphs*

\[
\mathfrak{g}_{\tilde{S},\Gamma} := \bigcup_{\sigma_L \times \sigma_R \in \sigma} \bigcup_{S \in \tilde{S}} \left\{ E_{S,\sigma} \in \text{Map}(\mathcal{P}_\Gamma, \bigoplus_{|E_T|} g) : E_{S,\sigma} := E_S \times \cdots \times E_S \right\}
\]

where \( E_S(\gamma) := (\iota_L(\gamma,S)\sigma_L(S),\iota_R(\gamma,S)\sigma_R(S)) \), \( E_S \in \mathfrak{g}_{\tilde{S},\sigma}, S \in \tilde{S}, \gamma \in \Gamma \)

Moreover, define

\[
\mathcal{E}_{\tilde{S},\Gamma} := \bigcup_{\sigma_L \times \sigma_R \in \sigma} \bigcup_{S \in \tilde{S}} \left\{ E_{S,\sigma} \in \text{Map}(\mathcal{P}_\Gamma, \bigoplus_{|E_T|} \mathcal{E}) : E_{S,\sigma} := E_S \times \cdots \times E_S \right\}
\]

where \( E_S(\gamma) := (\iota_L(\gamma,S)\sigma_L(S),\iota_R(\gamma,S)\sigma_R(S)) \), \( E_S \in \mathcal{E}_{\tilde{S},\sigma}, S \in \tilde{S}, \gamma \in \Gamma \)

The set of all images of the linear hull of all maps in \( \mathfrak{g}_{\tilde{S},\Gamma} \) for a fixed surface set \( \tilde{S} \) and a fixed graph \( \Gamma \) is denoted by \( \mathfrak{g}_{\tilde{S},\Gamma} \). The set of all images of the linear hull of all maps in \( \mathfrak{g}_{\tilde{S},\Gamma} \) for a fixed surface set \( \tilde{S} \) and a fixed subgraph \( \Gamma' \) of \( \Gamma \) is denoted by \( \mathfrak{g}_{\tilde{S},\Gamma'} \).

Note that, the set of Lie algebra-valued quantum fluxes for graphs can be generalised for the inductive limit graph system \( \mathcal{P}_{\Gamma_\infty} \). This follows from the fact that each element of the inductive limit graph system \( \mathcal{P}_{\Gamma_\infty} \) is a graph.

**Proposition 2.43.** Let \( \tilde{S} \) be a set of surfaces and \( \mathcal{P}_L^\infty \) be a finite orientation preserved graph system such that the set \( \tilde{S} \) has the same surface intersection property for a graph \( \Gamma \) (with no loops).

The set \( \mathfrak{g}_{\tilde{S},\Gamma} \) forms a Lie algebra and is called the *Lie flux algebra associated a graph and a finite surface set*. The *universal enveloping flux algebra* \( \mathcal{E}_{\tilde{S},\Gamma} \) associated a graph and a finite surface set is the enveloping Lie algebra of \( \mathfrak{g}_{\tilde{S},\Gamma} \).

**Proof.** This follows from the observation that, \( \mathfrak{g}_{\tilde{S},\Gamma} \) is identified with

\[
\bigcup_{\sigma_L \times \sigma_R \in \sigma} \bigcup_{S \in \tilde{S}} \left\{ E_{S,\sigma} \in \text{Map}(\mathcal{P}_L^\infty, \bigoplus_{|E_T|} g) : E_{S,\sigma} := E_S \times \cdots \times E_S \right\}
\]

where \( E_S(\gamma) := \sigma_L(S), E_S \in \mathfrak{g}_{\tilde{S},\sigma}, S \in \tilde{S}, \gamma \in \Gamma \)

and the addition operation

\[
E_{S_1,\Gamma}(\Gamma) + E_{S_2,\Gamma}(\Gamma) := (E_{S_1}(\gamma_1) + E_{S_2}(\gamma_1), \ldots, E_{S_1}(\gamma_N) + E_{S_2}(\gamma_N))
\]

\[
= (-\sigma_L^1(S_1) - \sigma_L^2(S_2), \ldots, -\sigma_L^1(S_1) - \sigma_L^2(S_2))
\]

holds whenever \( \Gamma := \gamma_1, \ldots, \gamma_N \).

Notice that indeed it is true that,

\[
\mathfrak{g}_{\tilde{S},\Gamma} = \mathfrak{g}_{S,\Gamma}
\]

for every \( S_i \in \tilde{S} \). The more general definition is due to physical arguments.

**Proposition 2.44.** Let \( \hat{T} \) be a set of surfaces and \( \mathcal{P}_L^\infty \) be a finite orientation preserved graph system such that the set \( \hat{T} \) has the simple surface intersection property for \( \Gamma \).

The set \( \mathfrak{g}_{\hat{T},\Gamma} \) forms a Lie algebra.
Notice this follows from the fact that, $\mathfrak{g}_{\mathcal{F},\Gamma}$ reduces to

$$\bigcup_{\sigma_L \in \sigma_L} \left\{ E_{T,\Gamma} \in \text{Map}(\mathcal{P}_{T,\Gamma}) : E_{T,\Gamma} := E_{T_1} \times \ldots \times E_{T_N} \right\},$$

where $E_{T_i}(\gamma_i) := -\sigma_L(T_i)$, $E_S \in \mathfrak{g}_{\mathcal{F},\Gamma}$, $T_i \in \mathcal{T}$,

$$\gamma_i \cap T_i = t(\gamma_i), \gamma \in \Gamma \right\}$$

since,

$$E_{S,\Gamma}(\Gamma) + \ldots + E_{S_N,\Gamma}(\Gamma) = \left(E_{T_1}(\gamma_1), 0, \ldots, 0\right) + \left(0, E_{T_2}(\gamma_2), 0, \ldots, 0\right) + \ldots + \left(0, \ldots, 0, E_{T_N}(\gamma_N)\right)$$

$$= \left(E_{T_1}(\gamma_1), \ldots, E_{T_N}(\gamma_N)\right) := E_{T,\Gamma}(\Gamma)$$

yields.

If it is additionally required that $E_{S}^{E}(\gamma_1) = -E_{S}^{R}(\gamma_2)$ holds, then actions of $\mathfrak{g}_{\mathcal{F},\Gamma}$ on a configuration space have to be very carefully implemented.

The Lie flux algebra and the universal enveloping flux algebra for the inductive limit graph system $\mathcal{P}_{T,\infty}$ and a fixed suitable surface set $\tilde{S}$ are denoted by $\hat{\mathfrak{g}}_S$ and $\hat{\mathcal{E}}_S$.

### 3 The holonomy-flux cross-product *-algebras

#### 3.1 The holonomy-flux cross-product *-algebra

Recall that, there is a big bunch of actions on the naturally identified configuration space $\mathcal{A}_\Gamma$, which is connected to the requirement of paths lying above or below the surface and are ingoing or outgoing w.r.t. the surface orientation of a surface $S$. Recall the Lie flux algebra $\hat{\mathfrak{g}}_{\mathcal{F},\Gamma}$, which is given by the evaluation of all maps for a fixed finite orientation preserved graph system associated to a graph $\Gamma$ and a suitable surface set $\tilde{S}$. Refer to 2.3 for a precise definition. Let $\hat{\mathfrak{g}}_{\mathcal{F},\Gamma}^{C}$ be the complexified Lie flux algebra, then $\hat{\mathcal{E}}_{\mathcal{F},\Gamma}^{C}$ denotes the universal enveloping flux algebra.

For simplicity the investigations start with a graph $\Gamma$, which contains only one path $\gamma$, and one surface $S$. Clearly the following definitions can be generalised to finite orientation preserved graph system associated to an arbitrary graph $\Gamma$ and a suitable surface set $\tilde{S}$.

**Definition 3.1.** Let the graph $\Gamma$ contains only a path $\gamma$ and $S$ be a surface such that the path lies below and outgoing w.r.t. the surface orientation of this surface. Set $E_{S}(\Gamma) := X_{S}$. Then the **right-invariant flux vector field** $e^{T}_\gamma$ is defined by

$$[E_{S}(\Gamma), f_{\Gamma}] := e^{T}_\gamma(f_{\Gamma})$$

where

$$e^{T}_\gamma(f_{\Gamma})(h_{\Gamma}(\gamma)) = \frac{d}{dt} \bigg|_{t=0} f_{\Gamma}(\exp(tX_{S})h_{\Gamma}(\gamma)) \text{ for } X_{S} \in \mathfrak{g}, \gamma(\gamma) \in G, t \in \mathbb{R}$$

whenever $f_{\Gamma} \in C^{\infty}(\mathcal{A}_\Gamma)$ and $E_{S}(\Gamma) \in \hat{\mathfrak{g}}_{\mathcal{F},\Gamma}$.

Respectively for a path $\gamma$ lying above and outgoing w.r.t. the surface orientation it is true that,

$$e^{T}_\gamma(f_{\Gamma})(h_{\Gamma}(\gamma)) = \frac{d}{dt} \bigg|_{t=0} f_{\Gamma}(\exp(-tX_{S})h_{\Gamma}(\gamma)) \text{ for } X_{S} \in \mathfrak{g}, \gamma(\gamma) \in G, t \in \mathbb{R}$$

holds if $-E_{S}(\Gamma) := X_{S}$. Since $E_{S} \in \mathfrak{g}_{\mathcal{F},\Gamma}$ there exists a skew-adjoint operator $E_{S}(\Gamma)^{\dagger}$ that satisfies

$$[E_{S}(\Gamma)^{\dagger}, f_{\Gamma}] = e^{T}_\gamma(f_{\Gamma})$$

where

$$[E_{S}(\Gamma)^{\dagger}, f_{\Gamma}] = [E_{S-1}(\Gamma), f_{\Gamma}]$$

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A quantum flux operator of a surface $\tilde{S}$ and a path $\gamma$ lying below and outgoing with respect to the surface orientation of $S$ can be changed by a flip of the path orientation such that the path $\gamma$ lies below and ingoing.

Recall the map $\gamma : C^\infty(\tilde{A}_f) \to C^\infty(\tilde{A}_f)$ s.t.

$$f_\gamma(h_r(\gamma_1), ..., h_r(\gamma_n)) = f_\gamma(h_r(\gamma_1), ..., h_r(\gamma_n)) = f_\gamma(h_r(\gamma)^{-1}, ..., h_r(\gamma)^{-1})$$

**Definition 3.2.** Define the **surface and graph orientation flip operator** as a map $\mathfrak{S} : C^\infty(\tilde{A}_f) \times g_{S,f} \to C^\infty(\tilde{A}_f) \times g_{S,f}$

$$\mathfrak{S}(f, E_S(\Gamma)) = \left(\hat{f}, E_{S^{-1}}(\Gamma)\right) = \left(\hat{f}, E_{S}(\Gamma)\right), \quad \mathfrak{S}(f, E_S(\Gamma)^+) = \left(\hat{f}, E_{S}(\Gamma)\right)$$

$$\mathfrak{S}(f, E_S(\Gamma)) = \left(\hat{f}, E_{S^{-1}}(\Gamma)\right)$$

$$(\mathfrak{S} \circ pr_\gamma)(f_r, E_S(\Gamma)) = \mathfrak{S}(f_r) = \hat{f}_\gamma,$$

$$(\mathfrak{S} \circ pr_\gamma)(f_r, E_S(\Gamma)) = \mathfrak{S}(E_S(\Gamma)) = E_{S^{-1}}(\Gamma)$$

Notice that, $\hat{f}_\gamma = \mathfrak{S}(f_r)$ holds whenever $f_r \in C^\infty(\tilde{A}_f)$.

**Definition 3.3.** Let the graph $\Gamma$ contains only a path $\gamma$ and $S$ be a surface such that the path lies below and outgoing w.r.t. the surface orientation of this surface. Set $s(\gamma) = v$ and $-E_S(\Gamma) := Y_S$.

The **left-invariant flux vector field** $e_{\tilde{R}}$ is realized as the following commutator

$$[E_S(\Gamma), f_r] := e_{\tilde{R}}(f_r)$$

where

$$e_{\tilde{R}}(f_r)(h_r(\gamma)) = \left.\frac{d}{dt}\right|_{t=0} f_r(h_r(\gamma) \exp(-tY_S)) \text{ for } Y_S \in g, t \in \mathbb{R} \quad (23)$$

whenever $f_r \in C^\infty(\tilde{A}_f)$ and $E_S \in g_{S,f}$. There exists a skew-adjoint operator $E_S(\Gamma)^+$ such that

$$[E_S(\Gamma)^+, f_r] := e_{\tilde{R}}(f_r)$$

where

$$e_{\tilde{R}}(f_r)(h_r(\gamma)) = \left.\frac{d}{dt}\right|_{t=0} f_r(h_r(\gamma) \exp(tY_S)) \text{ for } Y_S \in g, t \in \mathbb{R} \quad (24)$$

is satisfieds.

Summarising, the flux operators are implemented as differential operators $e_{\tilde{T}}$ (or $e_{\tilde{R}}$) on $\tilde{A}_f$ commuting with the right (or left) shifts.

**Definition 3.4.** Let $A$ be an (associative complex) algebra.

A **homomorphism of a Lie algebra** $\mathfrak{g}$ in $A$ is a map $\tilde{\tau} : \mathfrak{g} \to A$ such that

$$\tilde{\tau}(\alpha X + \beta Y) = \alpha \tilde{\tau}(X) + \beta \tilde{\tau}(Y),$$

$$\tilde{\tau}([X,Y]) = \tilde{\tau}(X)\tilde{\tau}(Y) - \tilde{\tau}(Y)\tilde{\tau}(X)$$

whenever $X, Y \in \mathfrak{g}$ and $\alpha, \beta \in \mathbb{R}$.

With no doubt, there is a map $\tilde{\tau}_1 : g_{S,f} \to C(\tilde{A}_f)$ defined by $\tilde{\tau}_1(E_S(\Gamma))f_r = [E_S(\Gamma), f_r]$ for $f_r \in C^\infty(\tilde{A}_f)$, which is indeed a homomorphism of a Lie flux algebra $g_{S,f}$ associated to a suitable surface set $S$ and a graph in $C^\infty(\tilde{A}_f)$.

**Lemma 3.5.** Fix an element $E_S(\Gamma) \in g_{S,f}$. Let $\tilde{\tau}_1 : C^\infty(\tilde{A}_f) \to C^\infty(\tilde{A}_f)$ be a map such that $\tilde{\tau}_1(E_S(\Gamma))(f_r) := [E_S(\Gamma), f_r]$ or, equivalently, $\tilde{\tau}_1(E_S(\Gamma))(f_r) := e_{\tilde{T}}(f_r)$ for each function $f_r \in C^\infty(\tilde{A}_f)$.

Then $\tilde{\tau}_1 \circ \mathfrak{S}$ defines an *-isomorphism $\mathcal{F}$ on $C^\infty(\tilde{A}_f)$ to $C^\infty(\tilde{A}_f)$ by

$$(\mathcal{F} \circ e_{\tilde{T}})(f_r) = [E_{S^{-1}}(\Gamma), f_r] = e_{\tilde{R}}(f_r)$$

which implements a flip of the path orientation.
Hence in general there is a bilinear map \( \tau \) such that
\[
e^T(f_\Gamma)(h_\Gamma(\gamma)) = \left. \frac{d}{dt} \right|_{t=0} f_\Gamma(\exp(tX_S)h_\Gamma(\gamma)) = \left. \frac{d}{dt} \right|_{t=0} \tilde{f}_\Gamma(h_\Gamma(\gamma))^{-1} \exp(-tX_S)
\]
and
\[
(\tau_1 \circ \tilde{\mathcal{S}})(E_{S_0}(\Gamma))(\{E_S(\Gamma), f_\Gamma\}) = e^T(f_\Gamma)(h_\Gamma(\gamma)) = e^R(\tilde{f}_\Gamma)(h_\Gamma(\gamma))^{-1}
\]
holds where \( S_0 \) is a surface, which does not intersect any path in \( \Gamma \) and \( \mathbb{1}_\Gamma \) is the constant function for any \( \Gamma \).

Proof: This is true, since,
\[
e^T(f_\Gamma)(h_\Gamma(\gamma)) = \left. \frac{d}{dt} \right|_{t=0} f_\Gamma(\exp(tX_S)h_\Gamma(\gamma)) = \left. \frac{d}{dt} \right|_{t=0} \tilde{f}_\Gamma(h_\Gamma(\gamma))^{-1} \exp(-tX_S)
\]

There is also an \(*\)-isomorphism \( \tilde{F} \) presented by
\[
(\tilde{F} \circ e^\tau)(f_\Gamma) = [E_{S^{-1}}(\Gamma), f_\Gamma] = e^R(f_\Gamma)
\]
connected to a flip of the path and surface orientation.

Now, the focus lies on quantum fluxes, which takes values in the enveloping algebra of \( \mathfrak{g} \).

**Definition 3.6.** Let \( \tilde{S} \) be a surface set such that \( \tilde{S} \) has the same surface intersection property for a graph \( \Gamma \).

Then the **tensor algebra of flux operators** is defined by
\[
\mathcal{T}(\tilde{S}) := \bigoplus_{k=0}^\infty \mathfrak{g}_{\tilde{S},T}^C \otimes \mathfrak{g}_{\tilde{S},T}
\]
There is a natural inclusion \( j : \mathfrak{g}_{\tilde{S},T} \rightarrow \mathcal{T}(\tilde{S}), E_S(\Gamma) \mapsto (E_S(\Gamma))^{\otimes 1} \). Denote by \( \mathcal{E}_{\tilde{S},T} \) the **universal enveloping \(*\)-algebra for flux operators** generated by the quotient of \( \mathcal{T}(\tilde{S}) \) and a two sided ideal \( I \) expressed by
\[
I = \left\{ j(E_{S_1}(\Gamma)) \otimes j(E_{S_2}(\Gamma)) - j(E_{S_2}(\Gamma)) \otimes j(E_{S_1}(\Gamma)) - j([E_{S_1}(\Gamma), E_{S_2}(\Gamma)]) : E_{S_1}, E_{S_2} \in \mathfrak{g}_{\tilde{S},T}^C, S_K \in \tilde{S}, K = 1, 2 \right\}
\]
The antilinear and antimultiplicative involution \( ^+ \) is given by
\[
(E_{S_1}(\Gamma) \times \ldots \times E_{S_k}(\Gamma))^+ = E_{S_1}^+(\Gamma) \times \ldots \times E_{S_k}^+(\Gamma),
\]
\[
E_{S_k}(\Gamma)^+ = -E_{S_k}(\Gamma) \text{ for } E_{S_k}(\Gamma) \in \mathfrak{g}_{\tilde{S},T} \text{ and } K = 1, \ldots, k
\]
Recall the structure of the enveloping algebra of \( \mathfrak{g} \). Moreover there is a bilinear map
\[
\tau_1 : C^\infty(\tilde{A}_\Gamma) \times \mathfrak{g}_{\tilde{S},T}^C \rightarrow C^\infty(\tilde{A}_\Gamma)
\]
such that for \( (f_\Gamma, E_{S}(\Gamma)) \in C^\infty(\tilde{A}_\Gamma) \times \mathfrak{g}_{\tilde{S},T}^C \), such that it is true that,
\[
\tau_1(f_\Gamma, E_{S}(\Gamma)) = [E_S(\Gamma), f_\Gamma]
\]
holds, which is further generalised to
\[
\tau_2 : C^\infty(\tilde{A}_\Gamma) \times \mathfrak{g}_{\tilde{S},T}^C \otimes \mathfrak{g}_{\tilde{S},T}^C \rightarrow C^\infty(\tilde{A}_\Gamma)
\]
such that
\[
\tau_2(f_\Gamma, E_{S_1}(\Gamma) \cdot E_{S_2}(\Gamma)) = -[E_{S_1}(\Gamma), [E_{S_2}(\Gamma), f_\Gamma]]
\]
Hence in general there is a bilinear map \( \tau : C^\infty(\tilde{A}_\Gamma) \times \mathcal{E}_{\tilde{S},T} \rightarrow C^\infty(\tilde{A}) \)
\[
\tau(f_\Gamma, E_{S_1}(\Gamma) \cdot \ldots \cdot E_{S_n}(\Gamma)) = [E_{S_1}(\Gamma), \ldots, [E_{S_n}(\Gamma), f_\Gamma], \ldots]
\]
such that \( \tilde{\tau} : \mathcal{E}_{\tilde{S},T} \rightarrow C^\infty(\tilde{A}) \) where \( \tilde{\tau}(E_S(\Gamma))f_\Gamma = [E_S(\Gamma), f_\Gamma] \) for \( f_\Gamma \in C^\infty(\tilde{A}_\Gamma) \) is a unit-preserving homomorphism.

The following corollary implies that due to the universality structure of \( \mathcal{E}_{\tilde{S}} \), this map \( \tilde{\tau} \) is unique.
Corollary 3.7. Let $A$ be a unital algebra and $\tilde{\tau}$ be a homomorphism of a Lie algebra $\mathfrak{g}$ into $A$. Then there exists a unique unit-preserving homomorphism of the universal enveloping flux algebra $E$ of $\mathfrak{g}$ into $A$ which extends $\tilde{\tau}$.

Lemma 3.8. Let $\tilde{S}$ be a set of surfaces which has the same intersection surface property for a finite orientation preserved graph system associated to $\Gamma$.

Then $C^\infty(\tilde{\mathcal{A}}_\Gamma)$ is a left $\tilde{\mathcal{E}}_{\tilde{S},\Gamma}$-module algebra. The action of $\tilde{\mathcal{E}}_{\tilde{S},\Gamma}$ on $C^\infty(\tilde{\mathcal{A}}_\Gamma)$ is given by $E_S(\Gamma) \triangleright f_\Gamma := e^{\tilde{\tau}}(f_\Gamma)$.

Proof : This following from the fact that, $C^\infty(\tilde{\mathcal{A}}_\Gamma)$ is a left $\tilde{\mathcal{E}}_{\tilde{S},\Gamma}$-module, which is defined by the map

$$E_S(\Gamma) \triangleright f_\Gamma := e^{\tilde{\tau}}(f_\Gamma) = \tau(f_\Gamma, E_S(\Gamma)) \text{ for } E_S(\Gamma) \in \tilde{\mathcal{E}}_{\tilde{S},\Gamma}, f_\Gamma \in C^\infty(\tilde{\mathcal{A}}_\Gamma)$$

which is obviously bilinear and $1 \triangleright f_\Gamma = f_\Gamma$ is satisfied. Moreover

$$E_S(\Gamma) \triangleright (E_S(\Gamma) \triangleright f_\Gamma) = (E_S(\Gamma) \cdot E_S(\Gamma)) \triangleright f_\Gamma$$

holds. Furthermore it turns out to be left $\tilde{\mathcal{E}}_{\tilde{S},\Gamma}$-module algebra, since, additionally,

$$E_S(\Gamma) \triangleright (f_\Gamma k_\Gamma) = (e^{\tilde{\tau}}(f_\Gamma))k_\Gamma + f_\Gamma(e^{\tilde{\tau}}(k_\Gamma))$$

and $E_S(\Gamma) \triangleright 1 = 0$ for all $E_S(\Gamma) \in \mathfrak{g}_{\tilde{S},\Gamma}$ yields.

Lemma 3.9. Let $\tilde{S}$ be a set of surfaces which has the appropriate same intersection surface property for a finite orientation preserved graph system associated to $\Gamma$.

Then $C^\infty(\tilde{\mathcal{A}}_\Gamma)$ is a right $\tilde{\mathcal{E}}_{\tilde{S},\Gamma}$-module algebra. The action of $\tilde{\mathcal{E}}_{\tilde{S},\Gamma}$ on $C^\infty(\tilde{\mathcal{A}}_\Gamma)$ is given by $E_S(\Gamma) \triangleleft f_\Gamma := e^{\tilde{\tau}}(f_\Gamma)$.

Finally, the definition of the holonomy-flux *-algebra in LQG by the authors [16] Def.2.7 is rewritten for the case of a fixed graph $\Gamma$. The vector space $C^\infty(\tilde{\mathcal{A}}_\Gamma) \otimes \tilde{\mathcal{E}}_{\tilde{S},\Gamma}$ is equipped with the multiplication

$$(f_1^r \otimes E_S(\Gamma)) \cdot (f_2^r \otimes E_S(\Gamma)) = -\tau(f_1^r, E_S(\Gamma)) \otimes E_S(\Gamma) \cdot E_S(\Gamma)$$

such that a Lie algebra bracket is derived

$$[f_1^r \otimes E_S(\Gamma), f_2^r \otimes E_S(\Gamma)] =$$

$$- (\tau(f_2^r, E_S(\Gamma)) - \tau(f_1^r, E_S(\Gamma))) F \otimes [E_S(\Gamma), E_S(\Gamma)]$$

Notice if $S_1$ and $S_2$ are disjoint the commutator on $C^\infty(\tilde{\mathcal{A}}_\Gamma) \otimes \tilde{\mathcal{E}}_{\tilde{S},\Gamma}$ is zero. Calculate the commutator

$$[(f_\Gamma \otimes 1), (1 \otimes E_S(\Gamma))] = -\tau(f_\Gamma, E_S(\Gamma)) \otimes E_S(\Gamma)$$

Additionally, the algebra is equipped with an involution such that this algebra is a unital associative *-algebra. In this work the algebra is slightly modified.

Definition 3.10. Let $\tilde{S}$ be a set of surfaces which has the appropriate same intersection surface property for a finite orientation preserved graph system associated to $\Gamma$.

The holonomy-flux cross-product *-algebra associated a graph $\Gamma$ and a surface set $\tilde{S}$ is given by the left or right cross-product *-algebra

$$C^\infty(\tilde{\mathcal{A}}_\Gamma) \triangleright_L \tilde{\mathcal{E}}_{\tilde{S},\Gamma} \text{ or } C^\infty(\tilde{\mathcal{A}}_\Gamma) \triangleright_R \tilde{\mathcal{E}}_{\tilde{S},\Gamma}$$

which is defined by the vector space $C^\infty(\tilde{\mathcal{A}}_\Gamma) \otimes \tilde{\mathcal{E}}_{\tilde{S},\Gamma}$ with the multiplication given by

$$(f_1^r \otimes E_S(\Gamma)) \cdot_L (f_2^r \otimes E_S(\Gamma)) = f_1^r(E_S(\Gamma) \triangleright f_2^r) \otimes E_S(\Gamma) + f_1^r f_2^r \otimes E_S(\Gamma) \cdot E_S(\Gamma)$$
Clearly for different surface sets there are a lot of different holonomy-flux cross-product
∗
γ
lies above and there are no other intersection points of each other path is different from the commutator (30) of the holonomy-flux 

Moreover let ˘
holonomy-flux cross-product
∗
S
Γ
be a set of independent edges such that every surface

Summarising, the unital holonomy-flux cross-product ∗-algebra

The holonomy-flux cross-product ∗-algebra associated a surface set

which are the inductive limit of the families

or respectively

(31)

Derive for suitable surface

and a graph

the following commutator relation between elements of the holonomy-flux cross-product ∗-algebra

which can be compared to the definition usually used in LQG, which is illustrated in (29). The definitions do not coincide, since, in LQG the ACZ- holonomy-flux algebra

is defined by the multiplication

This shows that, the holonomy-flux cross-product ∗-algebra is a modified holonomy-flux ∗-algebra if it is compared with the ∗-algebra presented in [16].

Notice

holds. Observe that, the commutator

is different from the commutator of the holonomy-flux ∗-algebra.

Clearly for different surface sets there are a lot of different holonomy-flux cross-product ∗-algebras. For example let

be a set of N surfaces and let

be a graph with

independent edges such that every surface

intersects only one path

of a graph

only once in the target vertex of the path

the path

lies above and there are no other intersection points of each other path

and the surface

in

(i ≠ j).

Moreover let

be a set of

surfaces and let

be a graph with

independent edges such that every surface
Then the sets $\tilde{S}$ and $\tilde{T}$ have the simple surface intersection property for $\Gamma$. Then there exists two different holonomy-flux cross-product $^*\text{-algebras}$ $C^\infty(\hat{A}) \rtimes_L \hat{E}_S$ and $C^\infty(\hat{A}) \rtimes_R \hat{E}_T$. Consider $C^\infty(\hat{A}_T)$ as a $^*\text{-subalgebra}$ of the analytic holonomy $C^*$-algebra $\mathfrak{A}_T := C(\hat{A}_T)$. Moreover refer to Sakai \[22\] or Bratteli and Robinson \[3\] for the definition of $^*\text{-derivations}$. 

**Lemma 3.11.** For any graph $\Gamma$ and a surface $S$, which has the same intersection surface property for a finite orientation preserved graph system associated to $\Gamma$, the object

$$i[E_S(\Gamma)^* + E_S(\Gamma), f_\Gamma] =: \delta^2_{S,\Gamma}(f_\Gamma)$$

defines a unbounded symmetric $^*\text{-derivation}$ $\delta^2_{S,\Gamma}$ on $C(\hat{A}_T)$ with domain $C^\infty(\hat{A}_T)$, in other words $\delta^2_{S,\Gamma} \in \text{Der}(C^\infty(\hat{A}_T), C(\hat{A}_T))$.

Since multiplier algebra of a unital and commutative $^*\text{-algebra}$ is the algebra itself, the elements $E_S(\Gamma)$ are not contained in the multiplier algebra of $\mathfrak{A}_T = C(\hat{A}_T)$. Hence the derivation defined in the lemma \[3.11\] is not inner.

Following the notion of infinitesimal representations $d\ U$ in a $C^*$-algebra introduced by Woronowicz \[26\] p.8 the flux operators can be also understood in the following way. Recall the unbounded operators $e_{\tilde{E}_T} e_{\tilde{L}}^T$ defined in \[21\], \[24\]. These operators are infinitesimal representations or differentials of the Lie flux group $\hat{G}_S$ in $\mathcal{K}(L^2(\hat{A}_T, \mu_\Gamma))$. They correspond to the set $\text{Rep}(\hat{G}_S, \mathcal{K}(L^2(\hat{A}_T, \mu_\Gamma)))$ of unitary representations $U$ of $\hat{G}_S$, which are analysed in \[11\], \[9\]. Therefore rewrite

$$e_{\tilde{L}}^T (f_\Gamma) := d(U(E_s(\Gamma))f_\Gamma) \text{ for } f_\Gamma \in D(d\ U) \text{ and } E_S(\Gamma) \in \hat{E}_S,$$

The domain of the infinitesimal representations $d\ U$ is defined by

$$D(d\ U) := \{ f_\Gamma \in C(\hat{A}_T) : \text{the mapping } \hat{G}_S \ni \rho s_\Gamma(\Gamma) \mapsto \| U(\rho s_\Gamma(\Gamma))f_\Gamma \| \text{ is a } C^\infty(\hat{G}_S, \mathcal{T}) \text{- function} \}$$

which is a dense subset in $C(\hat{A}_T)$.

### 3.2 Heisenberg holonomy-flux cross-product $^*$-algebras

The structures of Hopf algebras have been presented by Schmüdgen and Klimyk \[15\]. The authors have rewritten the algebra of Quantum mechanics in terms of the Hopf $^*$-algebra $(\text{Pol}(\mathbb{R}^n), \Delta)$ of coordinate functions. Then $\text{Pol}(\mathbb{R}^n) \rtimes \mathbb{R}^n$ is the Heisenberg algebra of Quantum Mechanics, where the elements are differential operators with polynomial coefficients. In this section similar algebras for Loop Quantum Gravity are studied.

First of all in mathematics a further cross-product, which is called the Heisenberg double, using the properties of bialgebras has been constructed. A particular bialgebra is a Hopf algebra. Let $G$ be either a connected compact Lie group and $\mathcal{G}$ a simple matrix Lie group. Therefore consider either the Hopf $^*$-algebra $(C^\infty(G), \Delta)$, or the Hopf $^*$-algebra $(\text{Pol}(\mathcal{G}), \Delta)$ of coordinate functions on the group $\mathcal{G}$, or the Hopf $^*$-algebra $(\text{Rep}(G), \Delta)$ of representatitive functions on the group $G$. Then restrict the $^*$-algebra $\text{Pol}(\mathcal{G})$ or $\text{Rep}(G)$ to a $^*$-subalgebra of $C^\infty(G)$, which is denoted by $\text{Pol}^\infty(\mathcal{G})$ or $\text{Rep}^\infty(G)$. Suppose that $(\ldots) : \mathcal{E} \times \text{Pol}^\infty(\mathcal{G}) \to \mathbb{C}$ denotes the universal enveloping flux algebra of $G$. The dual pairing is defined by

$$\langle E, f \rangle := \frac{d}{dt} \bigg|_{t=0} f(e_G \exp(tE)) \text{ for } E \in \mathcal{E} \text{ and } f \in \text{Rep}^\infty(G)$$

Then the Heisenberg double $\text{Rep}^\infty(G) \rtimes_H \mathcal{E}$ is defined by the bilinear map

$$E \triangleright_H f := \langle E, 1 \rangle f + \langle E, f \rangle = \langle E, f \rangle$$

and the multiplication

$$(f_1, E_1) \cdot_H (f_2, E_2) := \langle E_1, 1 \rangle f_1 f_2 \odot E_2 + \langle 1, f_2 \rangle f_1 \otimes E_1 E_2$$

These objects are used to define similar objects for LQG.
Definition 3.12. Let $G$ be either a connected compact Lie group or a simple matrix Lie group. Moreover let $\bar{\mathcal{A}}_\Gamma$ for a graph $\Gamma$ be the set of generalised connections for $G$ such that $\mathcal{A}_\Gamma$ is identified with $G^N$ naturally, where $N = |E_\Gamma|$. Suppose that $\bar{S}$ has the simple intersection surface property for a finite orientation preserved graph system associated to $\Gamma$. Then $\bar{\mathfrak{g}}$, $\mathfrak{g}$ is identified with $\mathfrak{g}^N$.

The Heisenberg representation-holonomy-flux $^\ast$-algebra of the graph $\Gamma$ and the surface set $\bar{\mathcal{S}}$ is given by

$$\text{Rep}^\infty(\mathcal{A}_\Gamma) \rtimes_H \mathcal{E}_{\bar{\mathcal{S}},\Gamma}$$

The Heisenberg polynomial-holonomy-flux $^\ast$-algebra of the graph $\Gamma$ and the surface set $\bar{\mathcal{S}}$ is given by

$$\text{Pol}^\infty(\mathcal{A}_\Gamma) \rtimes_{H,\text{pol}} \mathcal{E}_{\bar{\mathcal{S}},\Gamma}$$

The Heisenberg holonomy-flux $^\ast$-algebra of the graph $\Gamma$ and the surface set $\bar{\mathcal{S}}$ is given by

$$C^\infty(\mathcal{A}_\Gamma) \rtimes_H \mathcal{E}_{\bar{\mathcal{S}},\Gamma}$$

Notice that, an element of $\text{Pol}^\infty(G^N)$ is a matrix element $(h_\Gamma)_{ij}$ of a $M \times M$ matrix. This elements are called coordinate functions $v_j^i(h_\Gamma) = (h_\Gamma)_{ij}$ on a simple matrix Lie group $G$. Then an element of $\text{Pol}^\infty(G^N) \rtimes \mathcal{E}^N$ is for example given by

$$(h_\Gamma)_{ij} \in \text{Pol}^\infty(G^N) \rtimes \mathcal{E}^N$$

where by natural identification $\mathfrak{h}_\Gamma := h_\Gamma(\Gamma)$ is an element of $\mathfrak{g}^N$ and $E_S(\Gamma)$ is an element of the universal enveloping flux algebra $\mathcal{E}^N$ of the Lie group $G^N$. In this case the map

$$(h_\Gamma)_{ij} \in \text{Pol}^\infty(G^N) \rtimes \mathcal{E}^N$$

is bilinear and defines the left $\mathcal{E}^N$-module algebra $\text{Pol}^\infty(G^N)$. Clearly these Heisenberg cross-product algebras defined above are not equivalent to a holonomy-flux cross-product $^\ast$-algebra and they are in particular Heisenberg doubles in the sense of Schm"udgen and Klimyk.

Similarly to the different automorphic actions on the $C^\ast$-algebra $C(\bar{\mathcal{A}}_\Gamma)$ there are a lot of different Heisenberg doubles depending on the number of intersections and the orientations of the surface and paths.

4 Representations and states of the holonomy-flux cross-product $^\ast$-algebra

Surface-orientation-preserving graph-diffeomorphism-invariant states of the holonomy-flux cross-product $^\ast$-algebra

A $^\ast$-representations of the universal enveloping flux algebra $\mathcal{E}_{\bar{\mathcal{S}},\Gamma}$ is given by the infinitesimal representation $dU$ of a unitary representation $U$ of $G_{\bar{\mathcal{S}},\Gamma}$ in $C(\bar{\mathcal{A}}_\Gamma)$. In general $^\ast$-representations can be defined on arbitrary $^\ast$-algebra, but there is no necessary condition that a unitary representation $U$ of the Lie group $G_{\bar{\mathcal{S}},\Gamma}$ on a Hilbert space exists such that the commutator is equivalent to the infinitesimal representation. Mathematically $^\ast$-representations of Lie algebras are required to recover the structure of the Lie algebra.

Definition 4.1. Let $\mathcal{D}$ be a dense subspace of a Hilbert space $\mathcal{H}$. A $^\ast$-representation of a Lie algebra $\mathfrak{g}$ on $\mathcal{D}$ is a mapping $\pi$ of $\mathfrak{g}$ into $L(\mathcal{D})$ such that

(i) $\pi(\alpha X + \beta Y) = \alpha \pi(X) + \beta \pi(Y)$

(ii) $\pi([X,Y]) = \pi(X)\pi(Y) - \pi(Y)\pi(X)$

(iii) $\pi(\phi) = \langle \phi, \pi(X^+) \phi \rangle$

whenever $X,Y \in \mathfrak{g}$, $\alpha, \beta \in \mathbb{R}$ and $\phi, \varphi \in \mathcal{D}$ where $L(\mathcal{D})$ vector space of linear mappings of $X$ into $X$.  

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Notice that, from $\pi(X) \in L(D)$ and property [iii] it follows that $\pi(X) \in \mathcal{L}^+(D)$ (refer to Appendix).

Let $\hat{S}$ be a surface set with same surface intersection property for a finite orientation preserved graph system associated to a graph $\Gamma$.

In LQG the flux operators $E_{S}(\Gamma)$ are represented as differential operators $dU$ on the Hilbert space $\mathcal{H}_{\Gamma} := L^2(\hat{A}_\Gamma, \mu_\Gamma)$ of square integrable functions on $\hat{A}_\Gamma$. The domain $\mathcal{D}(dU)$ of the infinitesimal representations $dU$ is the set of all functions $\psi_\Gamma$ in $L^2(\hat{A}_\Gamma, \mu_\Gamma)$ such that for each $E_{S}(\Gamma) \in \mathfrak{g}_{\hat{S},\Gamma}$ the limit

$$dU(E_{S}(\Gamma))\psi_\Gamma := \lim_{t \to 0} \frac{(U(\exp(tE_{S}(\Gamma))) - 1)\psi_\Gamma}{t}$$

exists weakly in $L^2(\hat{A}_\Gamma, \mu_\Gamma)$. Notice that, there is a domain $\mathcal{D}(dU)$ for all infinitesimals that corresponds to unitary representations $U \in \text{Rep}(\hat{G}_{\hat{S},\Gamma},\mathcal{K}(L^2(\hat{A}_\Gamma, \mu_\Gamma)))$. The requirement of the existence of the limit is equivalent to the condition that, the function $\rho_\Gamma(\Gamma) \mapsto \langle U(\rho_{S,\Gamma}(\Gamma)\psi_\Gamma, \phi_\Gamma) \rangle$ is in $C^\infty(\hat{G}_{\hat{S},\Gamma})$ for each $\phi_\Gamma \in L^2(\hat{A}_\Gamma, \mu_\Gamma)$. With no doubt $C^\infty(\hat{A}_\Gamma) \subset \mathcal{D}(dU)$ holds.

Now recognize a short remark. The domain of the unbounded operator $dU$ can be rewritten in the following way. Let $\{X_{S_1}, \ldots, X_{S_d}\}$ be a basis of $\mathfrak{g}_{\hat{S},\Gamma}$ where $S_1, \ldots, S_d \in \hat{S}$. Then by a corollary [23 Cor.10.1.10] the domain $\mathcal{D}(dU)$ is equivalent to the set of all elements $\psi_\Gamma \in L^2(\hat{A}_\Gamma, \mu_\Gamma)$ such that for all $X_{S_k}$ where $k = 1, \ldots, d$ and $\phi_\Gamma \in L^2(\hat{A}_\Gamma, \mu_\Gamma)$ the function $\mathbb{R} \ni t \mapsto \langle U(\exp(tX_{S_k})\psi_\Gamma, \phi_\Gamma) \rangle$ is in $C^\infty(\mathbb{R})$. Notice that, $t \mapsto U(\exp(tX_{S_k})\psi_\Gamma)$ is a unitary representation of the Lie group $\mathbb{R}$ for each element $X_{S_k}$ of the basis of $\mathfrak{g}_{\hat{S},\Gamma}$, too. Consequently it is obvious that, $U \in \text{Rep}(\mathbb{R},\mathcal{K}(\mathcal{H}_{\Gamma}))$ for each $X_{S_k}$. The operators $X_{S_k}$ corresponding infinitesimal representation $dU(X_{S_k})$ are called the infinitesimal generators of $U$. This reformulation can be used to understand the connection between the construction of the holonomy-flux $^*$-algebra of Lewandowski, Okolów, Sahlmann and Thiemann [16] and the Weyl $C^*$-algebra of Fleischhack [7].

Summarising the operators $e^\Gamma$ and $e^\bar{\Gamma}$ are defined on a dense linear subspace $\mathcal{D}(dU)$ of the Hilbert space $\mathcal{H}_{\Gamma} := L^2(\hat{A}_\Gamma, \mu_\Gamma)$, and their adjoint operators $e^{\bar{\Gamma}}, e^{\Gamma}$ defined on $\mathcal{D}(dU^*)$. In particular, $e^\Gamma$ and $e^{\bar{\Gamma}}$ are elements of the set

$$\mathcal{L}^+(\mathcal{D}(dU)) := \{dU \in \mathcal{L}(\mathcal{D}(dU)) : U \in \text{Rep}(\hat{G}_{\hat{S},\Gamma},\mathcal{K}(\mathcal{H}_{\Gamma})), \mathcal{D}(dU) \subset \mathcal{D}(dU^*), dU^*dU \subset \mathcal{D}(dU)\}$$

where $\mathcal{L}^+(\mathcal{D}(dU)) \subset \mathcal{L}(\mathcal{D}(dU))$ and $\mathcal{L}(\mathcal{D}(dU))$ denotes the set of all linear operators from $\mathcal{D}(dU)$ to $\mathcal{D}(dU)$ and $\mathcal{D}(dU^*)$ the domain of the adjoint of the linear operator $dU$.

In analogy to the result of Schm"udgen in [23 Prop.10.1.6] the following proposition holds.

**Proposition 4.2.** Let $\hat{S}$ be a surface set with same surface intersection property for a finite orientation preserved graph system associated to a graph $\Gamma$.

Let $\hat{E}_{\hat{S},\Gamma}$ be the universal enveloping Lie flux $^*$-algebra for a surface set $\hat{S}$ and $U$ a unitary representation of $\hat{G}_{\hat{S},\Gamma}$ on the Hilbert space $L^2(\hat{A}_\Gamma, \mu_\Gamma)$.

Then $(dU(E_{S}(\Gamma)))(f_{\Gamma}) := e^\Gamma(f_{\Gamma})$ defines a $^*$-representation $dU$ of $\hat{E}_{\hat{S},\Gamma}$ on a dense subdomain $\mathcal{D}(dU)$ of the Hilbert space $L^2(\hat{A}_\Gamma, \mu_\Gamma)$.

**Proof:** Let $\bar{\tau}_\Gamma$ be a homomorphism of the Lie algebra $\mathfrak{g}_{\hat{S},\Gamma}$ in $C^\infty(\hat{A}_\Gamma)$. Furthermore $d\bar{U}$ is a $^*$-homomorphism of $\mathfrak{g}_{\hat{S},\Gamma}$ into the $O^*$-algebra $\mathcal{L}^+(\mathcal{D}(dU))$ such that $d\bar{U}(1) = 1$. Then $d\bar{U}$ defines a $^*$-representation of $\mathfrak{g}_{\hat{S},\Gamma}$ on the domain $\mathcal{D}(dU)$. There exists a unique extension of $d\bar{U}$ to an homomorphism $dU$ of the $^*$-algebra $\hat{E}_{\hat{S},\Gamma}$ into the $^*$-algebra $\mathcal{L}^+(\mathcal{D}(dU))$, which defines a $^*$-representation of $\hat{E}_{\hat{S},\Gamma}$ by corollary 3.7.

Consequently one shows that, the map $\mathfrak{g}_{\hat{S},\Gamma} \ni X_S \mapsto dU(X_S)$ is a $^*$-representation of $\mathfrak{g}_{\hat{S},\Gamma}$ on $\mathcal{D}(dU)$. For a suitable surface $S$ and a graph $\Gamma$ set $E_{S}(\Gamma) = X_S$. First derive that,

$$\langle dU(X_S)\varphi_\Gamma, \phi_\Gamma \rangle = \left. \frac{d}{dt} \right|_{t=0} \langle U(\exp(tX_S))\psi_\Gamma, \phi_\Gamma \rangle = \left. \frac{d}{dt} \right|_{t=0} \langle \psi_\Gamma, U^*(\exp(tX_S))\varphi_\Gamma \rangle = \left. \frac{d}{dt} \right|_{t=0} \langle \psi_\Gamma, U(\exp(tX_S))\varphi_\Gamma \rangle = -\langle \varphi_\Gamma, dU(X_S)\phi_\Gamma \rangle$$

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yields for \( \psi_T, \varphi_T \in \mathcal{H}_\Gamma \). Remember that, \( X^+_S = -X_S \) for \( E_S(\Gamma) \in \mathfrak{g}_{S,\Gamma} \) to conclude \( dU(X_S)^* = -dU(X_S) = dU(X^+_S) \).

Hence the crucial property is \( (ii) \) Let \( X \mapsto U(\exp(tX)) \) be weakly continuous, then derive
\[
\langle (dU(X) dU(Y) - dU(Y) dU(X)) \psi_T, \varphi_T \rangle \\
= \left. \frac{d}{dt} \right|_{t=0} \langle U(\exp(-tX) \exp(-sY)) \psi_T, \varphi_T \rangle \\
= \left. \frac{d}{ds} \right|_{s=0} \langle U(\exp(-sY) \exp(-tX)) \psi_T, \varphi_T \rangle \\
= \langle (Y_S X_S - X_S Y_S) \psi_T, \varphi_T \rangle = \langle [Y_S, X_S] \psi_T, \varphi_T \rangle \\
= \left. \frac{d}{dt} \right|_{t=0} \langle U(t[Y_S, X_S]) \psi_T, \varphi_T \rangle \\
= \left. \frac{d}{dt} \right|_{t=0} \langle U(-t[X_S, Y_S]) \psi_T, \varphi_T \rangle
\]

Remark that, the unbounded operator \( dU \) of \( \hat{E}_{S,\Gamma} \) and the operator \( dU(E_S(\Gamma)) \) for a fixed element \( E_S(\Gamma) \in \hat{E}_{S,\Gamma} \) are not equivalent, since for example the domains are different. Observe that, for an infinitesimal generator \( dU(E_S(\Gamma)) \) of the strongly continuous one-parameter unitary group \( \mathbb{R} \ni t \mapsto U(\exp(tE_S(\Gamma))) \) on \( L^2(\mathcal{A}_\Gamma, \mu_\Gamma) \) define the self-adjoint \( i dU(E_S(\Gamma)) \) on the domain \( D(dU(E_S(\Gamma))) \). Clearly the subset \( D(dU) \) is contained in \( D(dU(E_S(\Gamma))) \). Therefore different special flux operators \( E_S(\Gamma) \) or all flux operators \( E_S(\Gamma) \) can be analysed.

Moreover the operators the \( dU \) and \( dU(E_S(\Gamma)) \) have different self-adjointness properties. Indeed the *-representation \( dU \) on \( D(dU) \) is self-adjoint \( [23, \text{Cor.10.2.3}] \), whereas \( dU(E_S(\Gamma)) \) for any hermitian elliptic element \( E_S(\Gamma) \) of \( \hat{E}_{S,\Gamma} \) on the domain \( D(dU) \) is essentially self-adjoint \( [23, \text{Cor.10.2.5}] \).

Finally for general elliptic elements in \( \hat{E}_{S,\Gamma} \), the adjoint operator \( dU(E_S(\Gamma))^* \) is equivalent to the closure w.r.t. the graph topology of \( dU(E_S^+(\Gamma)) \), \( [23, \text{Cor.10.2.7}] \). For an abelian or compact Lie group \( G \) it turns out that, the adjoint \( dU(E_S(\Gamma))^* \) is equivalent to the closure w.r.t. the graph topology of \( dU(E_S^+(\Gamma)) \) for all \( E_S(\Gamma) \in \hat{E}_{S,\Gamma} \).

Summarising the issue of domains of the different differential operators have to be carefully analysed.

According to the observations of the Lie flux group \( C^*-\)algebra, the universal enveloping flux *-algebra \( \hat{E}_{S,\Gamma} \) can be considered. This algebra itself can be shown to be equivalent to the algebra of differential operators on \( C^\infty(G_{S,\Gamma}) \). Observe that, due to the different structure of \( G_{S,\Gamma} \) and \( \mathcal{A}_\Gamma \) the identification of both sets is valid only for suitable surface sets and graphs.

**Proposition 4.3.** Let \( G \) be a compact Lie group and the set \( \tilde{S} \) has the same intersection surface property for a finite orientation preserved graph system associated to a graph \( \Gamma \). Set \( N = |E_\Gamma| \) and identify \( \mathcal{A}_\Gamma \) with \( G^N \) naturally.

Then the universal enveloping Lie flux *-algebra \( \hat{E}_{S,\Gamma} \) is *-isomorphic to the \( O^*-\)algebra \( D_\Gamma(G_{S,\Gamma}) \) of differential operators on \( C^\infty(G^N) \) in the Hilbert space \( L^2(G^N, \mu_N) \), where \( D_\Gamma(G_{S,\Gamma}) \) is the algebra of all right-invariant differential operators \( dU_T^L(\mathfrak{g}_{S,\Gamma}) \big| _{C^\infty(G^N)} \) on \( G^N \).

Summarising there are different involutive algebras, like the analytic holonomy algebra associated a graph, the universal enveloping Lie flux *-algebra associated a graph and a surface set or the holonomy-flux cross-product *-algebra associated a graph represented on the Hilbert space \( \mathcal{H}_\Gamma \).

**Theorem 4.4.** Let \( \tilde{S} \) be a surface set having the same intersection surface property for a finite orientation preserved graph system associated to \( \Gamma \).

There exists the following *-representations of the analytic holonomy \( C^*-\)algebra \( C^\infty(\mathcal{A}_\Gamma) \), the universal enveloping Lie flux *-algebra \( \hat{E}_{S,\Gamma} \) and the holonomy-flux cross-product *-algebra \( C^\infty(\mathcal{A}_\Gamma) \rtimes_L \hat{E}_{S,\Gamma} \) for a
proof, whenever \( \psi_T \in C^\infty(\mathcal{A}_r) \). For two surfaces \( S_1 \cap S_2 = \emptyset \) the representation satisfies

\[
\pi((f_1 \otimes E_S(\Gamma))\psi_T) = \frac{1}{2} [E_S(\Gamma), f_1]\psi_T + \frac{1}{4} f_1^2[E_S(\Gamma), f_1^2] [E_S(\Gamma), \psi_T]
\]

whenever \( \psi_T \in C^\infty(\mathcal{A}_r) \).

The representation \( \pi \) of the holonomy-flux cross-product \( ** \)-algebra is called the Heisenberg representation of \( C^\infty(\mathcal{A}_r) \times L \mathcal{E}_{S,\Gamma} \) on \( \mathcal{H}_r \).

Proof : The following computations show that, \( \pi \) is a \( ** \)-representation of \( C^\infty(\mathcal{A}_r) \times L \mathcal{E}_{S,\Gamma} \) on the domain \( C^\infty(\mathcal{A}_r) \):

\[
\pi((f_1 \otimes E_S(\Gamma))\psi_T) = \frac{1}{2} [E_S(\Gamma), f_1]\psi_T + \frac{1}{2} f_1^2[E_S(\Gamma), \psi_T]
\]

for \( f_1, f_2 \in C^\infty(\mathcal{A}_r), E_S(\Gamma), E_S(\Gamma), E_S(\Gamma) \in \mathcal{E}_{S,\Gamma}, \lambda_1, \lambda_2 \in \mathbb{C} \).

For two surfaces \( S_1 \cap S_2 = \emptyset \) calculate

\[
\pi((f_1 \otimes E_S(\Gamma))\psi_T) = \frac{1}{2} [E_S(\Gamma), f_1]\psi_T + \frac{1}{4} f_1^2[E_S(\Gamma), f_1^2] [E_S(\Gamma), \psi_T]
\]

From another point of view the bracket \([E_S(\Gamma), \cdot] \) defines a \( ** \)-derivation of the analytic holonomy \( C^* \)-algebra \( C(\mathcal{A}_r) \) for a graph \( \Gamma \). Moreover in general such \( ** \)-derivations can be implemented by automorphisms on \( C(\mathcal{A}_r) \). This point of view is more general than the consideration of differential operators.

For a simplification restrict the following computations to a suitable surface \( S \) and a graph \( \Gamma := \{ \gamma \} \).

Lemma 4.5. Let \( \Phi_M \) be a representation of \( C(\mathcal{A}_r) \) on \( \mathcal{H}_r \) and \( \alpha \in \text{Act}(\mathcal{G}_{S,\Gamma}, C(\mathcal{A}_r)) \) defined in [12 Section 3.1] or [22 Section 6.1], where \( \rho_{S,\Gamma}(\Gamma) = \exp(t E_S(\Gamma)) \in \mathcal{G}_{S,\Gamma} \) for \( t \in \mathbb{R} \). Let \( \Gamma = \{ \gamma \} \) and \( S \) be suitable and set \( E_S(\Gamma) := X_S \).

Then it is true that,

\[
\omega_M^r(\alpha^r_{\exp(X_S)}(f_r)) = \int_{\mathcal{A}_r} f_r(\exp(t X_S)\mathfrak{h}_r(\gamma)) d \mu_r(\mathfrak{h}_r(\gamma)) = \int_{\mathcal{A}_r} f_r(\mathfrak{h}_r(\gamma)) d \mu_r(\mathfrak{h}_r(\gamma)) = \omega_M^r(f_r)
\]

yields for all \( t \in \mathbb{R} \) and \( f_r \in C(\mathcal{A}_r) \).
There is a general property of states on an arbitrary (untial) $C^*$-algebra $A$ represented on a Hilbert space and a group of $^*$-automorphisms $\alpha$.

**Corollary 4.6.** Let

$$\delta_{S,\Gamma}(f_\Gamma) := i[E_S(\Gamma_{t})^+ E_S(\Gamma), f_\Gamma]$$

be a $^*$-derivation such that $\delta_{S,\Gamma} \in \text{Der}(C^\infty(\mathcal{A}_\Gamma), C(\mathcal{A}_\Gamma))$. Let $\alpha \in \text{Act}(\mathcal{G}_{S,\Gamma}, C(\mathcal{A}_\Gamma))$.

Then for each element $E_S(\Gamma) \in \mathcal{E}_{S,\Gamma}$ the limit

$$\delta_{S,\Gamma}(f_\Gamma) := \lim_{t \to 0} \frac{\alpha_t^* \exp(E_S(\Gamma_{t})^+ E_S(\Gamma)) (f_\Gamma) - f_\Gamma}{t} \text{ for } f_\Gamma \in C^\infty(\mathcal{A}_\Gamma)$$

exists in norm topology and $\delta_{S,\Gamma} = \delta_{S,\Gamma}$.

Then the state $\omega_M^\Gamma$ on $C(\mathcal{A}_\Gamma)$ presented in [12, Corollary 3.28] or [9, Corollary 6.1.41] satisfies

$$\omega_M^\Gamma(\delta_{S,\Gamma}(f_\Gamma)) = 0$$

for all $f_\Gamma \in C^\infty(\mathcal{A}_\Gamma)$. Hence there is a cyclic vector $\Omega_M^\Gamma$ of the GNS-representation associated to this state such that

$$E_S(\Gamma_{t})^+ E_S(\Gamma) \Omega_M^\Gamma = 0 \text{ for all } E_S \in \mathcal{E}_{S,\Gamma}$$

where $E_S(\Gamma)$ is a skew-adjoint operator with domain $\mathcal{D}(E_S(\Gamma))$.

Notice that, the flux operator $iE_S^+ (\Gamma) E_S(\Gamma)$ is self-adjoint and positive in $\mathcal{H}_\Gamma$. The unbounded $^*$-derivation given by $\delta_{S,\Gamma}$ is symmetric.

**Proof:** First observe that, $\omega_M^\Gamma(\alpha_t^* \exp(E_S(\Gamma_{t})^+ E_S(\Gamma)) (f_\Gamma)) = \omega_M^\Gamma(f_\Gamma)$ holds for all $t \in \mathbb{R}$ and $f_\Gamma \in C(\mathcal{A}_\Gamma)$.

This follows from lemma 4.5. For $f_\Gamma \in C(\mathcal{A}_\Gamma)$ the $^*$-automorphisms $\alpha$ are implementable as a one-parameter group $\mathbb{R} \ni t \mapsto \alpha_t^* E_S(\Gamma_{t})^+ E_S(\Gamma)(f_\Gamma) \in C(\mathcal{A}_\Gamma)$ for each $E_S(\Gamma) \in \mathcal{E}_{S,\Gamma}$, which is weakly continuous.

Then the norm limit [39] exists for suitable $f_\Gamma$ in the domain $C^\infty(\mathcal{A}_\Gamma)$. The symmetric derivation is therefore given by

$$\delta_{S,\Gamma}(f_\Gamma) = \frac{d}{dt}igg|_{t=0} \alpha_t^* iE_S(\Gamma_{t})^+ E_S(\Gamma)(f_\Gamma) \text{ for } f_\Gamma \in C^\infty(\mathcal{A}_\Gamma)$$

The operator $iE_S(\Gamma_{t})^+ E_S(\Gamma)$ is the generator of the unbounded symmetric $^*$-derivation $\delta_{S,\Gamma}$ on $\mathcal{H}_\Gamma$ by definition.

Recall the state $\omega_M^\Gamma$ of $C(\mathcal{A}_\Gamma)$ presented in [12, Proposition 3.29] or [9, Proposition 6.1.40], then the derivation $\delta_{S,\Gamma}$ satisfies

$$\omega_M^\Gamma(\delta_{S,\Gamma}(f_\Gamma)) = \left\langle \Omega_M^\Gamma, \frac{d}{dt}igg|_{t=0} \phi_M(\alpha_t^* \exp(E_S(\Gamma_{t})^+ E_S(\Gamma)) (f_\Gamma)) \Omega_M^\Gamma \right\rangle = 0$$

for $f_\Gamma \in C^\infty(\mathcal{A}_\Gamma)$. There exists a covariant representation $(\phi_M, U)$ of $(\mathcal{G}_{S,\Gamma}, \mathfrak{H}_\Gamma, \alpha)$ in $\mathcal{L}(\mathcal{H}_\Gamma)$ such that

$$\phi_M(\alpha_t^* \exp(E_S(\Gamma_{t})^+ E_S(\Gamma)) (f_\Gamma)) = U(\exp(tE_S(\Gamma_{t})^+ E_S(\Gamma))) \phi_M(f_\Gamma) U(\exp(-tE_S(\Gamma_{t})^+ E_S(\Gamma)))$$

and, hence, $U(\exp(tE_S(\Gamma_{t})^+ E_S(\Gamma))) \Omega_M^\Gamma = \Omega_M^\Gamma$ for all $t \in \mathbb{R}$ and $iE_S^+ (\Gamma) E_S(\Gamma) \Omega_M^\Gamma = 0$.

The next derivation can be defined only for a suitable family of graphs and a suitable surface set.

Define the $^*$-derivation on the domain $C^\infty(\mathcal{A})$ of the $C^*$-algebra $C(\mathcal{A})$ by

$$\delta(f) := i[E_S(\Gamma_{t})^+ E_S(\Gamma_{\infty}), f] \text{ for } f \in C^\infty(\mathcal{A}), E_S \in \mathcal{E}_S \text{ and } \Gamma_{\infty} \in \mathcal{T}_{\Gamma_{\infty}}$$
Proposition 4.7. Let \( \Gamma_\infty \) be the inductive limit of a family of graphs \( \{ \Gamma_i \} \). Let \( \tilde{\mathcal{S}} \) be a finite set of surfaces in \( \Sigma \) such that

(i) such that the surface set \( \tilde{\mathcal{S}} \) has the same surface intersection property for each graph of the family,
(ii) the inductive limit structure preserves the same surface intersection property for \( \tilde{\mathcal{S}} \) and
(iii) each surface in \( \tilde{\mathcal{S}} \) intersects the inductive limit graph \( \Gamma_\infty \) only in a finite number of vertices.

Then \( \mathcal{P}^{\mathcal{E}}_{\Gamma_\infty} \) is the inductive limit of an inductive family \( \{ \mathcal{P}^{\mathcal{E}}_{\Gamma_i} \} \) of finite orientation preserved graph systems. Let \( \tilde{\mathcal{A}}_\Gamma \) be identified in the natural way with \( G^{\mathcal{N}}_\ast \).

Then the limit

\[
\alpha^t_{\exp(iE^+_S(\Gamma_\infty)E_S(\Gamma_\infty))}(f) := \lim_{j \to \infty} \alpha^t_{\exp(iE^+_S(\Gamma_j)E_S(\Gamma_j))}(f) \quad \text{for } f \in C^{\infty}(\tilde{\mathcal{A}}), \quad E_S \in \mathcal{E}_S \quad \text{and} \quad \Gamma_j \in \mathcal{P}_{\Gamma_\infty}
\]

exists for each \( t \in \mathbb{R} \) in norm topology. Consequently the limit

\[
\delta_S(f) := \lim_{t \to 0} \frac{\alpha^t_{\exp(iE^+_S(\Gamma_\infty)E_S(\Gamma_\infty))}(f) - f}{t} \quad \text{for } f \in C^{\infty}(\tilde{\mathcal{A}})
\]

exists in norm topology and \( \delta_S = \delta_S \) for \( S \in \tilde{\mathcal{S}} \).

Finally, for each \( * \)-derivation \( \delta_S \) the state satisfies

\[
\omega_M(\delta_S(f)) = 0
\]

for all \( f \in C^{\infty}(\tilde{\mathcal{A}}) \) and for \( S \in \tilde{\mathcal{S}} \).

Proof: On the inductive limit of the family of \( C^{\ast} \)-algebras \( \{(C(\tilde{\mathcal{A}}_\Gamma), \beta_{\Gamma, \Gamma'}) : \mathcal{P}^{\mathcal{E}}_{\Gamma_j} \subseteq \mathcal{P}^{\mathcal{E}}_{\Gamma_i} \} \) the action of \( \alpha(\rho_S, \Gamma, (\Gamma_i)) \) on \( C(\tilde{\mathcal{A}}_{\Gamma_j}) \) for \( \rho_S, \Gamma_j := \exp(iE_S(\Gamma_j)E_S(\Gamma_j)) \in \tilde{G}_{\Sigma, \Gamma_j} \) is non-trivial if each surface \( S \) of \( \tilde{\mathcal{S}} \) intersect the graph \( \Gamma_i \) in vertices of the graph \( \Gamma_j \). In particular, there is a graph \( \Gamma_j \) having the maximal number of intersection vertices with any surface \( S \) in \( \tilde{\mathcal{S}} \). Consequently for a graph \( \Gamma_j+1 \) that contains \( \Gamma_j \) the flux \( E_S(\Gamma_j+1 \setminus \Gamma_j) = 0 \). Furthermore derive

\[
\alpha^t_{\exp(iE^+_S(\Gamma_\infty)E_S(\Gamma_\infty))}(f) := \lim_{k \to \infty} \alpha^t_{\exp(iE^+_S(\Gamma_k)E_S(\Gamma_k))}(f)
\]

\[
= \lim_{k \to \infty} \alpha^t_{\exp(iE^+_S(\Gamma_k)E_S(\Gamma_k))}(\beta_{\Gamma^\prime}f_{\Gamma^\prime})
\]

\[
= \alpha^t_{\exp(iE^+_S(\Gamma_j)E_S(\Gamma_j))}(\beta_{\Gamma^\prime}f_{\Gamma^\prime}) \quad \text{for } f \in C^{\infty}(\tilde{\mathcal{A}}), \quad f = \beta_{\Gamma^\prime}f_{\Gamma^\prime} \quad \text{and} \quad E_S \in \mathcal{E}_S
\]

whenever \( \Gamma_k \leq \Gamma^\prime \) for \( 1 \leq k \leq j \). Furthermore conclude that,

\[
\delta_S(f) = i[E_S(\Gamma_j^\prime)E_S(\Gamma_j), f] + \lim_{i \to \infty} i[E_S(\Gamma_j^\prime)E_S(\Gamma_j), E_S(\Gamma_j+1 \setminus \Gamma_j), f]
\]

\[
= \beta_{\Gamma_j} \circ (i[E_S(\Gamma_j^\prime)E_S(\Gamma_j), f_{\Gamma_j^\prime}]) \quad \text{for } f \in C^{\infty}(\tilde{\mathcal{A}}), \quad f = \beta_{\Gamma_j}f_{\Gamma_j} \quad \text{and} \quad E_S \in \mathcal{E}_S
\]

holds. Hence there is a \( * \)-homomorphism \( \beta_{\Gamma, \Gamma^\prime} \) from \( C^{\infty}(\tilde{\mathcal{A}}_{\Gamma_j}) \) to \( C^{\infty}(\tilde{\mathcal{A}}_{\Gamma^\prime}) \) such that \( \beta_{\Gamma, \Gamma^\prime} \circ \delta_{S, \Gamma} \circ \beta_{\Gamma^\prime, \Gamma^{-1}} = \delta_{S, \Gamma^\prime} \) is a \( * \)-derivation from \( C^{\infty}(\tilde{\mathcal{A}}_{\Gamma_j}) \) into \( C(\tilde{\mathcal{A}}_{\Gamma^\prime}) \) and

\[
\omega_M(\delta_{S, \Gamma}(f_{\Gamma^\prime})) = \omega_M(\beta_{\Gamma, \Gamma^\prime} \circ \delta_{S, \Gamma})(f_{\Gamma^\prime}) = \omega_M(\delta_{S, \Gamma^\prime}(f_{\Gamma^\prime})) = 0
\]

(42)

Finally, derive

\[
\omega_M(\delta_S(f)) = \omega_M(\beta_{\Gamma_j} \circ \delta_{S, \Gamma_j})(f_{\Gamma_j}) = \beta_{\Gamma_j} \omega_M(\delta_{S, \Gamma_j}(f_{\Gamma_j})) = 0
\]

(43)

whenever \( f \in C^{\infty}(\tilde{\mathcal{A}}) \).

Recall that, \( \omega_M \) is graph-diffeomorphism invariant if the natural identification of \( \tilde{\mathcal{A}}_\Gamma \) with \( G^{\mathcal{N}}_\ast \) is used. Recall that a real-valued, linear and \( * \)-preserving functional \( \omega \) on a \( * \)-algebra associated to a \( * \)-representation \( \pi \) has to be positive.
Theorem 4.8. Let $\Gamma_\infty$ be the inductive limit of a family of graphs $\{\Gamma_j\}$. Let $\check{S}$ be a finite set of surfaces in $\Sigma$ such that

(i) the surface set $\check{S}$ has the same surface intersection property for each graph of the family,
(ii) the inductive limit structure preserves the same surface intersection property for $\check{S}$ and
(iii) each surface in $\check{S}$ intersects the inductive limit graph $\Gamma_\infty$ only in a finite number of vertices.

Then $\mathcal{P}_\Gamma^\infty$ is the inductive limit of a inductive family $\{\mathcal{P}_\Gamma^\infty\}$ of finite orientation preserved graph systems. Let $\mathcal{S}_\Gamma$ is identified in the natural way with $G^N$. Denote the center of $\mathcal{E}_{\check{S}}$ by $\mathcal{Z}(\mathcal{E}_{\check{S}})$.

The state $\bar{\omega}_M$ associated to the GNS-representation $(\mathcal{H}_\Gamma, \pi, \Omega)$ given in theorem 4.4 is a surface-orientation-preserving graph-diffeomorphism invariant state for a fixed set of surfaces $\check{S}$ on the holonomy-flux cross-product $^\ast$-algebra $C^\infty(\check{A}) \rtimes_L \mathcal{E}_{\check{S}}$ such that

$$\bar{\omega}_M(f \otimes E_S(\Gamma)) = \beta_{\Gamma_j}^\Gamma \omega_{\check{S}}^\Gamma(f_{\Gamma_j} \otimes E_S(\Gamma)) = 0$$

for all $E_S \in \mathcal{Z}(\mathcal{E}_{\check{S}})$

Moreover the state $\bar{\omega}_M$ on $C^\infty(\check{A}) \rtimes_L \mathcal{E}_{\check{S}}$ is the unique state, which is surface-orientation-preserving graph-diffeomorphism invariant.

Proof : First recall that,

$$\zeta_\sigma \circ \alpha(\rho_{S,\Gamma}(\Gamma)) \neq \alpha(\rho_{S,\Gamma}(\Gamma \circ \zeta_\sigma))$$

yields for every $\sigma \in \mathfrak{B}(\mathcal{P}_\Gamma^\infty)$ and $\rho_{S,\Gamma} := \exp(i E_S(\Gamma)^+ E_S(\Gamma)) \in G_{S,\Gamma}$. Hence the problem carry over to

$$\delta_{S,\Gamma} \circ \zeta_\sigma \neq \zeta_\sigma \circ \delta_{S,\Gamma}$$

Consequently in the following the center $\mathcal{Z}(\mathcal{E}_{S,\Gamma})$ and the center $\mathcal{Z}(\mathcal{E}_{\check{S}})$ are considered.

Moreover a surface-orientation-preserving graph-diffeomorphism invariant state for a fixed set of surfaces $\check{S}$ means that

$$\bar{\omega}_M(\zeta_\sigma(f \otimes E_S(\Gamma))) = \bar{\omega}_M(f \otimes E_S(\Gamma)) \text{ for all } \sigma \in \mathfrak{B}_{S,\infty}(\mathcal{P}_\Gamma^\infty) \text{ and } E_S(\Gamma) \in \mathcal{Z}(\mathcal{E}_{\check{S}})$$

holds. To show that $\bar{\omega}_M$ satisfies this property derive the following

$$\bar{\omega}_M(f \otimes E_S(\Gamma)) = \beta_{\Gamma_j}^\Gamma \omega_{\check{S}}^\Gamma(f_{\Gamma_j} \otimes E_S(\Gamma)) = \langle \Omega^\Gamma_{\check{S}}, \pi(f_{\Gamma_j} \otimes E_S(\Gamma)) \Omega^\Gamma_{\check{S}} \rangle = \frac{1}{2} \langle \Omega^\Gamma_{\check{S}}, [E_S(\Gamma), f_{\Gamma_j}] \Omega^\Gamma_{\check{S}} \rangle + \frac{1}{2} \langle \Omega^\Gamma_{\check{S}}, f_{\Gamma_j} dU(E_S(\Gamma)) \Omega^\Gamma_{\check{S}} \rangle

= \frac{1}{2} \langle \Omega^\Gamma_{\check{S}}, \delta_{S,\Gamma}(f_{\Gamma_j}) \Omega^\Gamma_{\check{S}} \rangle + \frac{1}{2} \langle \Omega^\Gamma_{\check{S}}, f_{\Gamma_j} |U(\exp(t E_S(\Gamma)))\rangle \Omega^\Gamma_{\check{S}} \rangle = 0$$

for all $f \in C(\check{A})$ and $E_S \in \mathcal{E}_{\check{S}}$.

Recognize that

$$\bar{\omega}_M(f \otimes 1) = \omega^\Gamma_M(f)$$

yields whenever $\omega_M$ is a state on $C(\check{A})$. With no doubt a $^\ast$-derivation is also defined for all $E_S(\Gamma) \in \mathcal{Z}(\mathcal{E}_{S,\Gamma})$ for $j = 1, \ldots, \infty$ such that corollary 4.6 and proposition 4.7 hold. Clearly it is true that

$$\bar{\omega}_M(f \otimes E_S(\Gamma))^\ast(f \otimes E_S(\Gamma)) = \langle \Omega_M, \pi_T((f^\ast \otimes E_S(\Gamma)^\ast)^\ast(f \otimes E_S(\Gamma))) \rangle \Omega^\Gamma_M$$

$$= \langle \Omega^\Gamma_M, \frac{1}{4} [E_S(\Gamma), f_{\Gamma_j}|E_S(\Gamma)^\ast, f_{\Gamma_j}] |\Omega^\Gamma_M \rangle + \langle \Omega^\Gamma_M, \frac{1}{4} |E_S^\Gamma(\Gamma), f_{\Gamma_j}|E_S^\Gamma(\Gamma), f_{\Gamma_j}] |\Omega^\Gamma_M \rangle

= \langle \Omega^\Gamma_M, \frac{1}{4} [E_S^\Gamma(\Gamma), f_{\Gamma_j}|E_S(\Gamma), f_{\Gamma_j}] |\Omega^\Gamma_M \rangle - \langle \Omega^\Gamma_M, \frac{1}{4} f_{\Gamma_j}|E_S(\Gamma), f_{\Gamma_j}|E_S^\Gamma(\Gamma), f_{\Gamma_j}] |\Omega^\Gamma_M \rangle

= 0$$

yields and therefore $\bar{\omega}_M$ is a $\mathfrak{B}_{S,\infty}(\mathcal{P}_\Gamma^\infty)$-invariant state on $C^\infty(\check{A}) \rtimes_L \mathcal{E}_{\check{S}}$.

Let $\omega'_{\check{S}}$ be another state of the holonomy-flux cross-product $^\ast$-algebra $C^\infty(\check{A}) \rtimes_L \mathcal{Z}(\mathcal{E}_{\check{S}})$ such that $\omega'_{\check{S}}(f \otimes 1) = \omega_M(f)$ for all $f \in C(\check{A})$. Recall [12, Corollary 3.60] or [9, Corollary 6.4.3] that states that, $\omega_M$ is the unique
state on \( C(\bar{A}) \) being invariant under the translation of \( \hat{G}_{S,T} \) and graph-diffeomorphisms \( \text{Diff}(\mathcal{P}_{{\bar{F}}}) \) for every \( 1 \leq i \leq \infty \). Then it is assumed that,

\[
\omega'_M(\alpha_{{\exp}(E_S(\Gamma_\infty)+E_S(\Gamma_\infty))}(f)) \neq \omega'_M(f) \quad \forall f \in C(\bar{A}) \tag{44}
\]
yields for some \( t_0 \in \mathbb{R} \). Consequently, derive \( \omega'_M(\delta_S(f)) \neq 0 \).

But from \( \text{(44)} \) it follows for a suitable graph-diffeomorphism \((\varphi, \Phi) \in \text{Diff}_{S,\text{surf}}(\mathcal{P}_{{\bar{F}}}) \) such that \( \varphi(S) = S' \) for \( S, S' \in \hat{S} \) that

\[
\omega'_M(\alpha_{(\varphi, \Phi)}(\alpha_{{\exp}(E_S(\Gamma_\infty))}(f))) = \beta^\varphi_t \omega'_M(\alpha_{(\varphi, \Phi)}(\alpha_{{\exp}(E_S(\Gamma_\infty))}(f)))
\]

\[
= \beta^\varphi_t \omega'_M(\alpha_{{\exp}(E_S(\Gamma_\infty))}(f)\gamma(\Gamma_\infty)) = \omega'_M(\alpha_{{\exp}(E_S(\Gamma_\infty))}(f))
\]

\[
\neq \beta^\varphi_t \omega'_M(\alpha_{{\exp}(E_S(\Gamma_\infty))}(f)\gamma(\Gamma_\infty)) = \omega'_M(\alpha_{{\exp}(E_S(\Gamma_\infty))}(f))
\]

yields whenever \( \Phi(\Gamma) = \Gamma' \) and \( f \in C(\bar{A}) \). In other words the state \( \omega' \) is only surface preserving graph-diffeomorphism invariant. The state \( \omega'_M \) is not invariant under surface-orientation-preserving graph-diffeomorphisms and, hence, general graph-diffeomorphisms. Consequently, the state \( \omega'_M \) is equal to \( \omega_M \) where \( \omega_M(\delta_S(f)) = 0 \) for all \( f \in C^\infty(\bar{A}) \) and for all \( \delta_S \in \text{Der}(C^\infty(\bar{A}), C(\bar{A})) \).

### Conditions for a surface preserving graph-diffeomorphism-invariant state of the holonomy-flux cross-product \(*\)-algebra

In theorem \[4.8\] the uniqueness of the state is referred to the assumption that, \( \omega_M \) is surface-orientation graph-diffeomorphism-invariant state of the holonomy-flux cross-product \(*\)-algebra \( C^\infty(\bar{A}) \times_L \mathcal{Z}(\mathcal{E}_S) \). Now this requirement is relaxed to surface preserving graph-diffeomorphisms. In the case of the holonomy-flux cross-product \( C^\ast\)-algebra \( C^\infty(\bar{A}) \times_L \mathcal{Z}(\mathcal{E}_S) \), a surface preserving graph-diffeomorphism-invariant state \( \omega_{E(\mathcal{S})} \) is presented in \[12\] Proposition 4.6], \[16\] Proposition 7.2.6]. Hence the question arise if there exists another state on \( C^\infty(\bar{A}) \times_L \mathcal{Z}(\mathcal{E}_S) \) satisfying weaker conditions.

It is assumed that, the flux operators are implemented as \(*\)-derivations \( \delta_S \) on the domain \( \mathcal{D}(\delta_S) \) of the unital \( C^\ast\)-algebra \( C(\bar{A}) \), which are generators of a strongly continuous one-parameter group \( t \mapsto \alpha(t) \) of \(*\)-automorphisms of \( C(\bar{A}) \). In this case the derivation is of the form \( \delta_S(f) = i[E_S(\Gamma_\infty) + E_S(\Gamma_\infty), f] \) for \( E_S(\Gamma_\infty) \in \mathfrak{g}_S \) and where \( iE_S(\Gamma_\infty) + E_S(\Gamma_\infty) \) is some unbounded symmetric operator with domain \( \mathcal{D} \) on the Hilbert space \( \mathcal{H}_S \), such that \( \mathcal{D}(\delta_S) \mathcal{D} \subset \mathcal{D} \). Then a new state \( \tilde{\omega} \) on \( C^\infty(\bar{A}) \times_L \mathcal{Z}(\mathcal{E}_S) \), which is not of the form \( \tilde{\omega}(\delta_S(f)) = 0 \), is required to satisfy a set of three conditions:

**First condition:**

Require the state \( \tilde{\omega} \) to be \( \text{Diff}_{S,\text{surf}}(\mathcal{P}_{{\bar{F}}}) \)-invariant, i.o.w.

\[
\tilde{\omega}(\alpha_{(\varphi, \Phi)}(\delta_S(f))) = \beta^\varphi_t \tilde{\omega}(\delta_S(f)) = \beta^\varphi_t \tilde{\omega}(f) = \tilde{\omega}(f)
\]

\[
\tilde{\omega}(\alpha_{(\varphi, \Phi)}(\delta_S(\gamma(f)))) = \beta^\varphi_t \tilde{\omega}(\delta_S(\gamma(f))) = \beta^\varphi_t \tilde{\omega}(\delta_S(f)) = \tilde{\omega}(\delta_S(f))
\]

for all \( f \in C(\bar{A}) \), \( f = \beta^\varphi_t \circ \gamma, \delta_S \in \text{Der}(\mathcal{D}(\delta_S), C(\bar{A})) \) and \((\varphi, \Phi, \Gamma) \in \text{Diff}_{S,\text{surf}}(\mathcal{P}_{{\bar{F}}}) \).

**Second condition:**

Furthermore the state need to have the property

\[
\tilde{\omega}(\delta_S(f)) \neq 0 \quad \forall f \in C(\bar{A}) \tag{47}
\]

which is equivalent to the requirement that for some \( t \in \mathbb{R} \)

\[
\tilde{\omega}(\alpha_{{\exp}(E_S(\Gamma_\infty))}(t)(f)) \neq \omega(f) \quad \forall f \in C(\bar{A}) \text{ and } E_S(\Gamma_\infty) \in \mathcal{E}_S \gamma
\]
yields.

**Third condition:**

The state is assumed to fulfill

\[
|\tilde{\omega}(\delta_S(f))| \leq c (\tilde{\omega}(f^\ast f) + \tilde{\omega}(f f^\ast))^{1/2} \quad \forall f \in C(\bar{A}) \text{ and some } c > 0 \tag{48}
\]
Clearly the ansatz is to search for representations of $\mathcal{E}_{S,Γ}$ on $H_Γ$ that are not $G_{S,Γ}$-integrable, i.o.w. these representations are not equal to the infinitesimal representation $d\,U$ of some unitary representation $U$ of $G_{S,Γ}$. The author does not know such a representation. Note that, in comparison to Dziendzikowski and O kolów [6] this representation is called the non-standard representation. But for representations of the universal enveloping algebra, which are not $G_{S,Γ}$-integrable, the relation to the unitary Weyl elements, which define the Weyl algebra for surfaces, is not clear.

5 Summary of different holonomy-flux cross-product *-algebras

In the following section three different $\mathcal{E}_{S,Γ}$-module algebras are presented.

First of all remember that $C^∞(\mathcal{A}_Γ)$ is a left (or right) $\mathcal{E}_{S,Γ}$-module algebra with the map

$$E_{S,Γ} \triangleright f_Γ := e^L(f_Γ)$$

or resp. $E_{S,Γ} \triangleleft f_Γ := e^R(f_Γ)$ for $f_Γ \in C^∞(\mathcal{A}_Γ), E_{S,Γ} \in \mathcal{E}_{S,Γ}$

where $e^L$ (or resp. $e^R$) denotes the right- (or left-) invariant vector field. Then the vector space $C^∞(\mathcal{A}_Γ) \otimes \mathcal{E}_{S,Γ}$ with the multiplication operation

$$(f_1^1 \otimes E_{S,Γ}, f_1^2 \otimes E_{S,Γ}) = f_1^1(E_{S,Γ} \triangleright f_1^2) \otimes E_{S,Γ} + f_1^1 f_1^2 \otimes E_{S,Γ} \cdot E_{S,Γ}$$

for $f_1^1 \in C^∞(\mathcal{A}_Γ), E_{S,Γ} \in \mathcal{E}_{S,Γ}$ and $i = 1, 2$, or respectively

$$(f_1^1 \otimes E_{S,Γ}, f_1^2 \otimes E_{S,Γ}) = (E_{S,Γ} \triangleleft f_1^1) f_1^2 \otimes E_{S,Γ} + f_1^1 f_1^2 \otimes E_{S,Γ} \cdot E_{S,Γ}$$

for $f_1^1 \in C^∞(\mathcal{A}_Γ), E_{S,Γ} \in \mathcal{E}_{S,Γ}$ and $i = 1, 2$, defines the holonomy-flux cross-product *-algebra $C^∞(\mathcal{A}_Γ) \rtimes L \mathcal{E}_{S,Γ}$ or resp. $C^∞(\mathcal{A}_Γ) \rtimes R \mathcal{E}_{S,Γ}$. The elements satisfy the canonical commutator relation

$$[E_{S,Γ}, f_Γ] := E_{S,Γ} \triangleright f_Γ - f_Γ \triangleleft E_{S,Γ} \in \mathcal{E}_{S,Γ}$$

For a vector $ψ \in H_Γ := L^2(\mathcal{A}_Γ, μ_Γ)$ there representation $π_Ψ$, of the analytic holonomy algebra $C^∞(\mathcal{A}_Γ)$ is given by $π_Ψ(f_Γ)ψ := f_Γ \cdot ψ$ and the representation $π_Ψ$ of the enveloping flux algebra $\mathcal{E}_{S,Γ}$ is presented by $π_Ψ(E_{S,Γ})ψ := E_{S,Γ} \triangleright ψ$ for $f_Γ \in C^∞(\mathcal{A}_Γ), E_{S,Γ} \in \mathcal{E}_{S,Γ}$, $ψ \in H_Γ$

Then the Heisenberg representation of $C^∞(\mathcal{A}_Γ) \rtimes L \mathcal{E}_{S,Γ}$ on the Hilbert space $H_Γ$ is defined by

$$π([E_{S,Γ}, f_Γ])ψ := (E_{S,Γ} \triangleright f_Γ) \cdot ψ - f_Γ \cdot (E_{S,Γ} \triangleright ψ)$$

for $f_Γ \in C^∞(\mathcal{A}_Γ), E_{S,Γ} \in \mathcal{E}_{S,Γ}, ψ \in H_Γ$

Notice that $E_{S,Γ} \triangleright ψ := d\,U(E_{S,Γ})ψ$ for $E_{S,Γ} \in \mathcal{E}_{S,Γ}$, $ψ \in H_Γ$

defines a representation $d\,U$ of the enveloping algebra $\mathcal{E}_{S,Γ}$ on $H_Γ$. This is also called the infinitesimal representation of the Lie flux group $G_{S,Γ}$ on $H_Γ$. The element $d\,U(E_{S,Γ}^+, E_{S,Γ})$ itself is an essential self-adjoint operator on this Hilbert space.

Another left (or resp. right) $\mathcal{E}_{S,Γ}$-module algebra is given by $C^∞(\mathcal{A}_Γ)$ with the bilinear map

$$E_{S,Γ} \triangleright_H f_Γ := e^L(f_Γ|_{(e_0,...,e_0)}) := (f_Γ, E_{S,Γ})$$

for $f_Γ \in C^∞(\mathcal{A}_Γ), E_{S,Γ} \in \mathcal{E}_{S,Γ}$

and where $(\cdot, \cdot) : C^∞(\mathcal{A}_Γ) \otimes \mathcal{E}_{S,Γ} \to \mathbb{C}$. Then the Heisenberg holonomy-flux cross-product *-algebra $C^∞(\mathcal{A}_Γ) \rtimes_H \mathcal{E}_{S,Γ}$ is given by the vector space $C^∞(\mathcal{A}_Γ) \otimes \mathcal{E}_{S,Γ}$ with the multiplication operation

$$(f_Γ, E_{S,Γ}) \triangleright_H (f_Γ, E_{S,Γ}) := (E_{S,Γ}, 1) f_Γ f_Γ + (1, f_Γ) f_Γ \otimes E_{S,Γ}$$

for $f_Γ \in C^∞(\mathcal{A}_Γ), E_{S,Γ} \in \mathcal{E}_{S,Γ}$ (49)

The elements satisfy the canonical commutator relation

$$E_{S,Γ} f_Γ = e^L(f_Γ|_{(e_0,...,e_0)}) - f_Γ E_{S,Γ}$$

for $f_Γ \in C^∞(\mathcal{A}_Γ), E_{S,Γ} \in \mathcal{E}_{S,Γ}$

This algebra is indeed a Heisenberg double in the sense of Schmüdgen and Klimyk [15]. The Heisenberg representation of $C^∞(\mathcal{A}_Γ) \rtimes_H \mathcal{E}_{S,Γ}$ on $H_Γ$ is given by

$$π([E_{S,Γ}, f_Γ])ψ := e^L(f_Γ|_{(e_0,...,e_0)})ψ - f_Γ \cdot e^L(ψ)$$

for $f_Γ \in C^∞(\mathcal{A}_Γ), E_{S,Γ} \in \mathcal{E}_{S,Γ}, ψ \in H_Γ$
The third possibility is given by the left (or resp. right) $\tilde{E}_{S,\Gamma}$-module algebra $C^\infty(\tilde{A}_{\Gamma})$ with

$$E_{S,\Gamma} \triangleright f_{\Gamma} := \epsilon(f_{\Gamma}) E_{S,\Gamma} \text{ for } f_{\Gamma} \in C^\infty(\tilde{A}_{\Gamma}), E_{S,\Gamma} \in \tilde{E}_{S,\Gamma}$$

where the Hopf algebra $(C^\infty(\tilde{A}_{\Gamma}), S)$ is considered with antipode $S$, comultiplication $\Delta$ and counit $\epsilon : C^\infty(\tilde{A}_{\Gamma}) \rightarrow \mathbb{C}$. Then the **simple Holonomy-flux cross-product** *-algebra $C^\infty(\tilde{A}_{\Gamma}) \rtimes \tilde{E}_{S,\Gamma}$ is defined by the vector space $C^\infty(\tilde{A}_{\Gamma}) \otimes \tilde{E}_{S,\Gamma}$ with multiplication operation

$$(f_{\Gamma} \otimes E_{S,\Gamma}) \cdot (f_{\Gamma} \otimes E_{S,\Gamma}) = f_{\Gamma} f_{\Gamma} \otimes E_{S,\Gamma} E_{S,\Gamma} \text{ for } f_{\Gamma} \in C^\infty(\tilde{A}_{\Gamma}), E_{S,\Gamma} \in \tilde{E}_{S,\Gamma}$$

The elements satisfying the canonical commutator relation given by

$$E_{S,\Gamma} f_{\Gamma} = \epsilon(f_{\Gamma}) E_{S,\Gamma} - f_{\Gamma} E_{S,\Gamma} \text{ for } f_{\Gamma} \in C^\infty(\tilde{A}_{\Gamma}), E_{S,\Gamma} \in \tilde{E}_{S,\Gamma}$$

The Heisenberg representation of $C^\infty(\tilde{A}_{\Gamma}) \rtimes \tilde{E}_{S,\Gamma}$ on $H_{\Gamma}$ is presented by

$$\pi([E_{S,\Gamma}, f_{\Gamma}])\psi = \epsilon(f_{\Gamma}) e_{L}(\psi) = f_{\Gamma} e_{L}(\psi) \text{ for } f_{\Gamma} \in C^\infty(\tilde{A}_{\Gamma}), E_{S,\Gamma} \in \tilde{E}_{S,\Gamma}, \psi \in H_{\Gamma}$$

The algebra $C^\infty(\tilde{A}) \rtimes_{L} \tilde{D}_{S}(G_{S})$ is an $O^*$-algebra and is called the **holonomy-flux cross-product** $O^*$-algebra **associated a surface set** $S$ on $C^\infty(\tilde{A})$ in $H_{\infty}$.

### 6 Tensor products of the holonomy-flux cross-product *-algebra

The structure of the holonomy-flux cross-product *-algebra can be slightly modified in the following way.

**Definition 6.1.** The **modified holonomy-flux cross-product** *-algebra restricted to a graph $\Gamma$ and a surface set $\tilde{S}$ is given by

$$C(G_{S,\Gamma}) \otimes \left( C^\infty(\tilde{A}_{\Gamma}) \rtimes_{L} \tilde{E}_{S,\Gamma} \right)$$

where the tensor product $\otimes$ is the minimal tensor product of $C^\ast$-algebras.

The **modified holonomy-flux cross-product** *-algebra associated a surface set $\tilde{S}$ is equivalent to the inductive limit of the family

$$\left\{ \left( C(G_{S,\Gamma}) \otimes \left( C^\infty(\tilde{A}_{\Gamma}) \rtimes_{L} \tilde{E}_{S,\Gamma} \right), \beta_{\Gamma, \Gamma'}, \beta_{\Gamma, \Gamma'} \right) : \mathcal{P}_{\Gamma} \leq \mathcal{P}_{\Gamma'} \right\}$$

Then for the state $\tilde{\omega}_{M}^{\Gamma}$ on $C(G_{S,\Gamma}) \otimes \left( C^\infty(\tilde{A}_{\Gamma}) \rtimes_{L} \tilde{E}_{S,\Gamma} \right)$ it is true that

$$\tilde{\omega}_{M}^{\Gamma}(f(\rho_{S,\Gamma}(\Gamma))\delta_{S,\Gamma}(f_{\Gamma}))$$

$$= \int_{G_{S,\Gamma}} d\mu_{S,\Gamma}(\rho_{S,\Gamma}(\Gamma)) f(\rho_{S,\Gamma}(\Gamma)) \delta(\rho_{S,\Gamma}(\Gamma), \exp(E_{S}(\Gamma))) \langle \Omega_{M}, \delta_{S,\Gamma}(f_{\Gamma}) \Omega_{M} \rangle$$

$$= f(\exp(E_{S}(\Gamma))) \omega_{M}^{\Gamma}(\delta_{S,\Gamma}(f_{\Gamma}))$$

holds where $\delta(g_{1}, g_{2})$ is the delta function on $G_{S,\Gamma}$.

**Definition 6.2.** The **modified intersection-holonomy-flux cross-product** *-algebra restricted to a graph $\Gamma$ and a surface set $\tilde{S}$ is given by

$$C(V_{T}^{S}) \otimes \left( C^\infty(\tilde{A}_{\Gamma}) \rtimes_{L} \tilde{E}_{S,\Gamma} \right)$$

where $V_{T}^{S} = V_{T} \cap S$ and the tensor product $\otimes$ is the minimal tensor product of $C^\ast$-algebras.

Then for the state $\tilde{\omega}_{M}^{\Gamma}$ on $C(V_{T}^{S}) \otimes \left( C^\infty(\tilde{A}_{\Gamma}) \rtimes_{L} \tilde{E}_{S,\Gamma} \right)$ it is true that

$$\tilde{\omega}_{M}^{\Gamma}(f(v_{1}, ..., v_{M})\delta_{S,\Gamma}(f_{\Gamma})) = f(v_{1}, ..., v_{M}) \omega_{M}^{\Gamma}(\delta_{S,\Gamma}(f_{\Gamma}))$$

yields where $v_{1}, ..., v_{M} \in V_{T}^{S}$.

Clearly these states are not surface-orientation-preserving graph-diffeomorphism invariant, but the states are surface preserving graph-diffeomorphism invariant.
The author has argued in [10] that, for example for Quantum Mechanics different algebras are obtained by using different generating sets of abstract operators. The aim of the construction of the Weyl $\mathcal{C}^*$-algebra, which has been invented in [11], and the holonomy-flux cross-product $\ast$-algebras defined in this article, is to use a common setup. Both algebras are generated by functions depending on holonomies along paths, and group- or Lie algebra-valued quantum flux operators. These abstract operators satisfy some canonical commutator relations, which are called Heisenberg relations if the unbounded configuration and momentum operators are studied, or Weyl relations if the bounded configuration and momentum operators are used. Clearly by choosing different sets of operators other $\ast$-algebras or respectively $\mathcal{C}^*$-algebras can be constructed.

For example if the functions depending on the quantum flux group associated to surfaces, and the functions depending on holonomies along paths, are considered, then these operators generate the holonomy-flux cross-product $\mathcal{C}^*$-algebra, which has been presented in [12]. The abstract operators are represented on a common Hilbert spaces as self-adjoint Hilbert space operators. The exponentiated Lie algebra-valued quantum flux operator are implemented by an unitary weakly continuous representation of the group $\mathbb{R}$ on the Hilbert space. The Lie algebra-valued quantum flux operator is related to the infinitesimal representation of this unitary weakly continuous representation. This flux operator is unbounded and self-adjoint. In a Hilbert space independent framework automorphisms of and derivations for the algebra of quantum variables play a fundamental role. In particular strongly continuous one-parameter group of $\ast$-automorphisms defines a derivation, which is given by the commutator of two self-adjoint operators.
| Configuration space | Quantum Mechanics | Weyl alg. for surfaces and holonomy-flux cross-prod. *-alg. |
|---------------------|------------------|-------------------------------------------------|
| \( \mathbb{R}^n \) | \( \mathbb{R}^n \) | \( \tilde{A} := \lim_{\gamma \in P, \gamma \to \infty} A_\gamma \) |
| Momentum space      | \( x_i \)        | \( \tilde{E}_S \) or \( \tilde{G}_S \) (where \( G \) compact connected Lie group) |
| Configuration variable I | \( f(x) \) for \( f \in C_0(\mathbb{R}^n) \) | \( h(\gamma_i) \) for \( h \in \text{Hom}(P, G), \gamma_i \in P \) |
| Momentum variable I | \( p_i \)         | \( f(h(\gamma)) \) for \( f \in C(\tilde{A}) \) |
| Configuration variable II | \( \exp(tp_i) \) | \( E_{S_i}(\gamma_i) \) for \( E_{S_i} \in \mathcal{E}_S, \gamma_i \in P, S_i \in \tilde{S} \) |
| Momentum variable II | \( H = \sum_i \frac{p_i^2}{2m} + V(x_1, \ldots, x_n) \) | \( \rho_{S_j}(\gamma_i) \) for \( \rho_{S_j} \in G_S, \gamma_i \in P, S_j \in \tilde{S} \) |
| Dynamical Hamiltonian | \( \mathcal{H} := L^2(\mathbb{R}^n, \chi_1 \leq \chi \leq n \ d x_k) \) | \( H = \sum_i Tr((h(\alpha_i) - h(\alpha_i)^{-1}) h(\gamma_i)[h(\gamma_i)^{-1}, V]) \) |
| Hilbert space       | \( \pi(x_i) = x_i \) | \( V = \sum_{\gamma \in \mathcal{P}} E_{S_1}(\gamma_1)E_{S_2}(\gamma_2)E_{S_3}(\gamma_3) \) |
| self-adjoint Hilbert space operator | \( \pi(\exp(tp_i)) = U_{p_i}(t) \) | for \( \alpha_i, \gamma_i \in P, S_m \in \tilde{S} \) |
| unitary Hilbert space operator | \( (U_{p_i}(t)\psi)(x_i) := \psi(x_i - tp_i) \) for \( \psi \in \mathcal{H} \) | \( \mathcal{H}_\infty := L^2(\tilde{A}, d \mu_\infty) \) |
| unitary Hilbert space operator | Fourier transform \( \mathcal{F} \) | \( \pi(h(\gamma)) = h(\gamma) \) |
| *-automorphism       | \( \pi(p_i) = -i \frac{\partial}{\partial x_i} \) such that \( \pi(p_i) \psi = \mathcal{F}p_i\mathcal{F}^{-1}\psi = -i \frac{\partial}{\partial x_i} \psi \) for \( \psi \in D(p_i) \) | \( \pi(\exp(tE_{S_i}(\gamma_i))) = U_t(\rho_{S_i}(\gamma_i)) \) |
| unitary transformation | \( \pi(H), \pi(\exp(tH)) = U_H(t) \) | \( (U_t(\rho_{S_i}(\gamma_i))\psi)(h(\gamma)) := \psi(\exp(tE_{S_i}(\gamma_i))h(\gamma)) \) for \( \psi \in \mathcal{H} \) |
| self-adjoint Hilbert space operator | \( \mathbb{R} \ni t \mapsto U_H(t) \) such that \( \pi(\zeta_t(f)) = V_{\pi(f)}V_{\pi(f^*)}^*, \zeta_t \in \text{Aut}(C(\tilde{A})) \) | \( \pi(E_{S_i}(\gamma_i)^{+}E_{S_j}(\gamma_i)) = -i \frac{d}{dt} U_t(\rho_{S_i}(\gamma_i)^{+}E_{S_j}(\gamma_i)) \) such that \( \frac{d}{dt} \bigg|_{t=0} U_t(\rho_{S_i}(\gamma_i)^{+}E_{S_j}(\gamma_i))\psi = i \pi(E_{S_i}(\gamma_i)^{+}E_{S_j}(\gamma_i))\psi \) for \( \psi \in D(E_{S_i}(\gamma_i)^{+}E_{S_j}(\gamma_i)) \) |
| Unitary Hilbert space operator | \( \pi(\exp(sx_i)) = V_x(s) \) |
|-------------------------------|--------------------------------|
| Self-adjoint Hilbert space operator | \( \pi(f) = f \) for \( f \in C(\mathcal{A}) \) |
| Strongly continuous 1-parameter group of *-automorphisms | \( \mathbb{R} \ni t \mapsto \alpha_t(E_{\gamma_i}(\gamma)) \in \text{Aut}(C(\mathcal{A})) \) |
| Canonical Commutator Relations | \([E_{\gamma_i}(\gamma), h(\gamma_i)] = \frac{d}{dt} \big|_{t=0} \exp(tE_{\gamma_i}(\gamma))h(\gamma_i) - h(\gamma_i)E_{\gamma_i}(\gamma) \) |
| *-automorphism | \([E_{\gamma_i}(\gamma), f] = \frac{d}{dt} \big|_{t=0} \alpha_t(E_{\gamma_i}(\gamma))(f) \) |
| Strongly contin. 1-parameter group of *-automorphism | \( \pi(\alpha(\rho_{\gamma_i}(\gamma_i))(f)) = U(\rho_{\gamma_i}(\gamma_i))\pi(f)U^*(\rho_{\gamma_i}(\gamma_i)) \) |
| Uniqueness of the GNS-representation | \( \alpha(\rho_{\gamma_i}(\gamma_i)) \circ \zeta_\sigma = \zeta_\sigma \circ \alpha(\rho_{\gamma_i}(\gamma_i)) \) |
| \( \mathcal{H}, \pi, \Omega \) irreducible, cyclic and regular representation of \( C_0(\mathbb{R}^n) \) | For all \( \sigma \in \mathfrak{B}(P_T) \) and \( \rho_{\gamma_i} \in \mathcal{Z}_{S_{\gamma_i}} \) |
| such that \( \omega(f) = \langle \Omega, \pi(f)\Omega \rangle \) and \( U_H(t)\Omega = \Omega \) | \( \mathcal{H}_{\infty}, \pi, \Omega \) irreducible and regular GNS-representation of \( C(\mathcal{A}) \) |
| Symmetric *-derivation with domain \( D(\delta) \) | such that \( \omega_M(f) = \langle \Omega, \pi(f)\Omega \rangle \), |
| Uniqueness of the state | \( V_\sigma\Omega = \Omega \) for all \( \sigma \in \mathfrak{B}(P_T) \) |

\( \mathcal{H}, \Phi, \Omega_M \) irreducible and regular GNS-repr. of \( \text{Weyl}_Z(\mathcal{S}) \)

such that \( \omega_M(W) = \langle \Omega_M, \Phi(W)\Omega_M \rangle \)

\( V_\sigma\Omega_M = \Omega_M \) for all \( \sigma \in \mathfrak{B}(P_T) \)

and \( U(\rho_{\gamma_i}(\gamma_i))\Omega_M = \Omega_M \) for all \( \rho_{\gamma_i} \in \mathcal{Z}_{S_{\gamma_i}} \) and \( \gamma_i \in \mathcal{P} \)

\( \omega_M(\delta_S(f)) = 0 \)

\( \omega_M \) is the unique state on \( C^\infty(\mathcal{A}) \rtimes_L \mathcal{Z}(\mathcal{E}_S) \)

such that \( \omega_M \circ \alpha_\sigma = \omega_M \) for all \( \sigma \in \mathfrak{B}(P_T) \)

\( \omega_M \circ \alpha_t(E_{\gamma_i}(\gamma_i)+E_{\gamma_i}(\gamma_i)) = \omega_M \) for all \( E_{\gamma_i}(\gamma_i) \in \mathcal{Z}(\mathcal{E}_S) \)
8 Appendix

The theory of $O^*$-algebras has been developed by Schmüdgen [23] and Inoue [8]. In this appendix only the basic objects are collected.

Definition of $O^*$-algebras

Let $D$ be a dense subspace in a Hilbert space with inner product $\langle , \rangle$. By $\mathcal{L}(D)$ (respect. $\mathcal{L}_c(D)$) denote the set of all (closable) linear operators from $D$ to $D$ and

$$\mathcal{L}^+(D) = \{ A \in \mathcal{L}(D) : D \subset D(A^*), A^*D \subset D \}$$

Then with the operations $AB$, $A+B$ and $\lambda A$ the set $\mathcal{L}(D)$ forms an algebra. The set $\mathcal{L}^+(D)$ forms a $^*$-algebra with involution $A \mapsto A^+ = A^*|_D$.

**Proposition 8.1.** [23 Prop 2.1.10] Let $A \in \mathcal{L}^+(D)$ and let $A$ be closed. Then $\mathcal{L}^+(D)$ is equal to the algebra $\mathcal{L}(H)$ of bounded linear operators on a Hilbert space $H$.

Observe

$$\mathcal{L}^+(D) \subset \mathcal{L}_c(D) \subset \mathcal{L}(D)$$

**Definition 8.2.** A subalgebra of $\mathcal{L}(D)$ contained in $\mathcal{L}_c(D)$ is said to be an $O$-algebra on $D$ in $H$, and a $^*$-subalgebra of $\mathcal{L}^+(D)$ is said to be an $O^*$-algebra on $D$ in $H$.

Representations of $^*$-algebras

**Definition 8.3.** Let $\mathfrak{A}$ be a $^*$-algebra with unit $1$ and let $D$ be a dense subspace of a Hilbert space $H$.

The map $\pi : \mathfrak{A} \rightarrow \mathcal{L}(D)$ is a $^*$-representation of a $^*$-algebra $\mathfrak{A}$ on a Hilbert space $H$ if

(i) there exists a dense subset $\mathcal{D}$ of $H$ such that

$$\mathcal{D} \subset \bigcap_{A \in \mathfrak{A}} (D(\pi(A)) \cap D(\pi(A)^*))$$

(ii) for every $A, B \in \mathfrak{A}$ and $\lambda \in \mathbb{C}$

$$\pi(A + B) = \pi(A) + \pi(B), \quad \pi(\lambda A) = \lambda \pi(A)$$

$$\pi(AB) = \pi(A)\pi(B), \quad \pi(A^*) = \pi(A)^*$$

$$\pi(1) = I$$

Acknowledgements

The work has been supported by the Emmy-Noether-Programm (grant FL 622/1-1) of the Deutsche Forschungsgemeinschaft.

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