The random walks of a Schwarzschild black hole

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Abstract

The purpose of this paper is to show that as a matter of principle, it is not appropriate to consider Schwarzschild black holes in thermal equilibrium with radiation because, even spinless and neutral holes undergo fluctuations of charge and angular momentum. Therefore, there will be a spread of these quantities around their zero-mean values. We calculate these spreads for a black hole in thermal contact with charged scalar particles and show that angular momentum fluctuations are governed by the size of the cavity, larger cavities yielding larger angular momentum fluctuations. This behaviour is expected if black hole angular momentum fluctuations stem from the random absorption and emission of quanta with random angular momenta from the thermal bath. Furthermore, in the limit $R/2M \gg 1$ charge fluctuations $\Delta Q^2/(\hbar c) \sim 1/(4\pi)$, that is to say, they become scale independent. This is expected if the underlying physics of these fluctuations is the random absorption and emission of charged quanta from the thermal bath. The independence of these fluctuations upon the elementary charge of the field is puzzling because it tells us that either the scale of the elementary charge is fixed by black hole physics to be $\alpha \sim 1/4\pi$ (this gives an elementary charge which is only three times the charge of the electron), or the underlying physical process responsible these fluctuations is not known and remains to be cleared up.

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1 Introduction

The fact that black holes are subjected to the same thermodynamical laws as are ordinary systems is one of the most intriguing findings and is likely to be our best window toward an elusive theory of quantum gravity. Therefore, exploring and exploiting this paradigm could yield important clues of the form and content of this still unveiled theory.

Thermodynamic equilibrium between black holes and radiation is of considerable interest and was matter of investigation in the past [1]-[4]. In particular for Schwarzschild holes, if was concluded that the vessel confining the radiation and black hole cannot be larger than some critical value because larger volumes render the (total) entropy a concave function rather than convex. Although these studies are very important for the understanding of the conditions of equilibrium of Schwarzschild black holes and radiation, they are incomplete in the sense that leave out an important ingredient, namely that it is not possible to prevent a neutral and non-rotating black hole from undergoing statistical fluctuations of charge and angular momentum around their zero-mean values. Put in another way, these fluctuations, although preserving the zero mean values of charge and angular momentum, produce mean square deviations of these quantities.

In this paper it is shown that these thermodynamical fluctuations do not result in any new instability, regardless what the external parameters are. Furthermore, we estimate both angular momentum and charge standard deviations.

This is done by first assuming a non-vanishing mean charge and mean angular momentum and by taking at the very end the limit were they do vanish. The paper is organized in the following way: in the next section we estimate angular momentum fluctuations and in the sequel charge fluctuations. The last section is devoted to the discussion of possible implications.

2 angular momentum fluctuations

Consider a black hole of angular momentum $J$ and mass $M$, in equilibrium with thermal radiation of massless scalar particles, inside a spherical vessel of radius $R$. The field operator associated to these quanta satisfies a Klein-Gordon equation, which is to be supplemented by a Dirichlet type boundary condition. For this reason

$$\Phi(\vec{x}, t) \approx j_l(\epsilon r)Y_{lm}(\theta, \phi)e^{-i\omega t}. \quad (1)$$

were $j_l(x)$ and $Y_{lm}(\theta, \phi)$ are the spherical Bessel and harmonic functions, respectively. The field spectrum is provided by the roots $x_{l,n} = \epsilon_{l,n}R$ of the spherical Bessel functions $j_l(x_{l,n}) = 0$.

Assuming that the system has no net angular momentum, conservation of energy and angular momentum requires that

$$0 = j(\Omega, T) + J$$

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\[ E = e(\Omega, T) + M \] (2)

where \( j(\Omega, T) \) and \( e(\Omega, T) \) are the field’s mean angular momentum and mean energy. More precisely,

\[ j(\Omega, T) = \sum_{l,m,n} \frac{m}{e^{(\epsilon - m\Omega)/T} - 1}, \] (3)

and

\[ e(\Omega, T) = \sum_{l,m,n} \frac{\epsilon}{e^{(\epsilon - m\Omega)/T} - 1}. \] (4)

In the above equations \( \Omega \) is a Lagrange multiplier whose purpose is the implementation of angular momentum conservation [eq. (2)]. It has a clear physical meaning, as the angular velocity the system must counter-rotate in order to compensate black hole angular momentum fluctuations.

The total entropy \( S \) receives contributions, one from the field entropy

\[ s(\Omega, T) = \sum_{l,m,n} \left[ (\bar{n} + 1) \log(\bar{n} + 1) - \bar{n} \log \bar{n} \right], \] (5)

where \( \bar{n} \) is the mean number of quanta sitting in the mode indexed by \((l, m, n)\):

\[ \bar{n}(\Omega, T) = \frac{1}{e^{(\epsilon - m\Omega)/T} - 1}, \] (6)

and the second from black hole entropy,

\[ S_{bh} = \frac{1}{4} 4\pi (r_+^2 + a^2), \] (7)

where \( a = J/M \) and \( r_+ = M + \sqrt{M^2 - a^2} \) is the horizon radius.

We shall be dealing with an ensemble parametrized by \( \Omega \) and \( T \). After evaluating the relevant thermodynamical calculations we want to reproduce the limit \( J \to 0 \) which is implemented by taking the limit \( \Omega \to 0 \). Under these circumstances, the conditions for stable thermal equilibrium are

\[ \left( \frac{\partial S}{\partial T} \right)_{\Omega = 0} = 0; \left( \frac{\partial^2 S}{\partial T^2} \right)_{\Omega = 0} < 0 \] (8)

\[ \left( \frac{\partial S}{\partial \Omega} \right)_{T, \Omega = 0} = 0; \left( \frac{\partial^2 S}{\partial \Omega^2} \right)_{T, \Omega = 0} < 0 \] (9)

\(^2\)The condition on the mixed derivative is omitted because \( \left( \frac{\partial^2 S}{\partial T \partial \Omega} \right)_{\Omega = 0} = 0 \) automatically.
Work out these conditions is much simplified because \( e(\Omega, T), s(\Omega, T) \) are even functions of \( \Omega \) while \( j(\Omega, T) \) is odd. Therefore, \( j(0, T) = 0 \), as well as the first derivatives \( \left( \frac{\partial e}{\partial T} \right)_{\Omega=0} = \left( \frac{\partial s}{\partial T} \right)_{\Omega=0} = 0 \). The computation of these derivatives while committed to the above constraints [eq’s (2)] translate the above inequalities into:

\[
\left( \frac{\partial S}{\partial T} \right)_{\Omega=0} = \left( \frac{\partial s}{\partial T} \right)_{\Omega=0} - 8\pi M \left( \frac{\partial e}{\partial T} \right)_{\Omega=0} = 0
\]

(10)

Inspecting this equation having an eye on the first law of thermodynamics, we can immediately infer the common temperature of the system, \( T = (8\pi M)^{-1} \). Similarly, for the second derivative:

\[
\left( \frac{\partial^2 S}{\partial^2 T} \right)_{\Omega=0} = \left( \frac{\partial^2 s}{\partial^2 T} \right)_{\Omega=0} - T^{-1} \left( \frac{\partial^2 e}{\partial^2 T} \right)_{\Omega=0} + 8\pi \left( \frac{\partial e}{\partial T} \right)^2_{\Omega=0}.
\]

(11)

The first law allows a simplification of this expression, because

\[
\left( \frac{\partial^2 s}{\partial^2 T} \right)_{\Omega=0} = \frac{\partial}{\partial T} \left( T^{-1} \frac{\partial e}{\partial T} \right)_{\Omega=0} = -T^{-2} \left( \frac{\partial e}{\partial T} \right)^2_{\Omega=0} + T \left( \frac{\partial^2 e}{\partial^2 T} \right)_{\Omega=0}.
\]

(12)

and, consequently

\[
\left( \frac{\partial^2 S}{\partial^2 T} \right)_{\Omega=0} = -T^{-2} \left( \frac{\partial e}{\partial T} \right)^2_{\Omega=0} + 8\pi \left( \frac{\partial e}{\partial T} \right)^2_{\Omega=0}.
\]

(13)

In the thermodynamical limit \( e = aT^4V \) and

\[
\left( \frac{\partial^2 S}{\partial^2 T} \right)_{\Omega=0} = -4aTV(1 - 32\pi aT^5V),
\]

(14)

which is the well known result setting a critical volume \( V < V_c = (32\pi aT^5)^{-1} \) for stable equilibrium [2].

Regarding the next two conditions, as expected \( \left( \frac{\partial S}{\partial \Omega} \right)_{\Omega=0} = 0 \) is satisfied automatically. The only missing condition is:

\[
\left( \frac{\partial^2 S}{\partial^2 \Omega} \right)_{\Omega=0} = \left( \frac{\partial^2 s}{\partial^2 \Omega} \right)_{\Omega=0} - T^{-1} \left( \frac{\partial^2 e}{\partial^2 \Omega} \right)_{\Omega=0} - \frac{2\pi}{M^2} \left( \frac{\partial j}{\partial \Omega} \right)^2_{\Omega=0} \leq 0.
\]

(15)
Introducing two new functions $\Sigma_{1,2}(T)$

\begin{align*}
\Sigma_1(T) &= \sum_{l,m,n} m^2 \frac{e^{x/T}}{(e^{x/T} - 1)^2} \geq 0, \quad (16) \\
\Sigma_2(T) &= \sum_{l,m,n} \epsilon^m m^2 \frac{e^{x/T} + e^{2x/T}}{(e^{x/T} - 1)^3} \geq 0, \quad (17)
\end{align*}

the quantities displayed in the r.h.s of eq. (15) can be cast as:

\begin{align*}
\left( \frac{\partial j}{\partial \Omega} \right)_{\Omega=0} &= T^{-1} \Sigma_1(T) \quad (18) \\
\left( \frac{\partial^2 e}{\partial \Omega^2} \right)_{\Omega=0} &= T^{-2} \Sigma_2(T), \quad (19)
\end{align*}

and finally

\begin{align*}
\left( \frac{\partial^2 s}{\partial \Omega^2} \right)_{\Omega=0} &= T^{-3} \Sigma_2(T) - T^{-2} \Sigma_1(T). \quad (20)
\end{align*}

With these results, eq. (15) boils down to

\begin{align*}
\left( \frac{\partial^2 S}{\partial \Omega^2} \right)_{\Omega=0} &= -T^{-2} \Sigma_1(T) - 2^7 \pi^3 \Sigma_1^2(T), \quad (21)
\end{align*}

telling us that the system is stable under angular velocity fluctuations.

We turn next to the computation of angular momentum fluctuations. Because

\[ \left( \frac{\partial^2 S}{\partial \Omega^2} \right)_{\Omega=0} = 0, \]

if follows that the frequency variance is

\[ (\Delta \Omega)^2 = \left( \frac{\partial^2 S}{\partial \Omega^2} \right)^{-1}_{\Omega=0} \quad (22) \]

which yields for the angular momentum fluctuations:

\[ (\Delta J)^2 = \left( \frac{\partial j}{\partial \Omega} \right)^2_{\Omega=0} (\Delta \Omega)^2 = \frac{\Sigma_1(T)}{1 + 2^7 \pi^3 T^2 \Sigma_1(T)}. \quad (23) \]

Turning now to the computation of $\Sigma_1$, we recall the asymptotic expansion of the spherical Bessel function

\[ j_l(x) \approx x^{-1} \cos(x - (l + \frac{1}{2})(\frac{\pi}{2} - \frac{\pi}{4})) \quad (24) \]
Therefore, the roots are approximately localized at
\[ x_{ln} \approx \left( \frac{l}{2} + n \right) \pi \]  
(25)

Indexing the modes by a new triple of quantum numbers \( m, l \) and \( s = l/2 + n \) \((-l \leq m \leq l, l \leq 2s\)) is a convenient expedient because \( \epsilon = \epsilon_s = s\pi/R \) and the modes become degenerate with respect to the the remaining quantum numbers \( m, l \). With the aid of the identity:
\[ \sum_{l=0}^{2s} \sum_{m=-l}^{l} m^2 = \frac{2}{3} s(s + 1)(2s + 1)^2, \]  
(26)
we can reexpress
\[ \Sigma_1(T) = \frac{2}{3} \sum_s s(s + 1)(2s + 1)^2 \frac{e^{s\pi/T}}{(e^{s\pi/T} - 1)^2}. \]  
(27)

Next, we go to the thermodynamical limit \( R \to \infty \). Introducing the auxiliary quantities \( x = \epsilon/T \) and \( \alpha = RT/\pi \),
\[ \Sigma_1 \approx \frac{2}{3} \alpha^2 \int_0^\infty x(\alpha x + 1)(2\alpha x + 1)^2 \frac{e^x}{(e^x - 1)^2} dx. \]  
(28)
Assuming that the vessels’ radius is always much larger than the black hole, which is also equivalent to a thermal wave length much larger than the size of the cavity \( (\alpha >> 1) \), then, the leading term in this expression is
\[ \Sigma_1(T) \approx \frac{8}{3} \frac{R^5T^5}{\pi^3} \left[ \frac{1}{\pi^2} \int_0^\infty x^4 \frac{e^x}{(e^x - 1)^2} dx \right]. \]  
(29)

Because the function
\[ a(\lambda) = \frac{1}{\pi^2} \int_0^\infty \frac{x^3}{e^{\lambda x} - 1} \]  
(30)
is such that \( -a'(1) \) reproduces the bracket in the previous equation and \( a(\lambda) = a(1)\lambda^{-4} \), where \( a(1) = \pi^2/60 \) is the Stephan-Boltzmann’s constant:
\[ \Sigma_1(T) \approx \frac{8}{45\pi} R^5T^5. \]  
(31)
Thus
\[ (\Delta J)^2 = \frac{8}{45\pi} \frac{R^5T^5}{1 + 2^{10}\pi^2/45 R^5T^7}. \]  
(32)
For a spherical cavity the bound on the critical volume translates into
\[ T^2 \leq \left( \frac{45}{32\pi^4} \right) (RT)^{-3}, \]
which in the limit \( R/2M \gg 1 \) yields a lower bound on the these fluctuations:
\[ (\Delta J)^2 \sim \frac{2\pi^2}{45} R^3 T^3. \quad (33) \]
This shows that the fluctuations are governed by the size of the cavity.

3 charge fluctuations

In this section we shall deal with charge fluctuations. For this end we consider a vessel of volume \( V \) containing scalar charged quanta in equilibrium with a black hole. Similarly to the previous procedure, we assume that the black hole has a nonvanishing charge, consider the equilibrium conditions and only at the end take the limit where the charge does vanish. Energy and charge bookkeeping requires
\[ E = \eta(\mu, T) V + M \]
\[ 0 = \rho(\mu, T) + Q, \quad (35) \]
were \( Q \) and \( M \) are the black hole’s mass and charge; \( \rho(\mu, T) \) and \( \eta(\mu, T) \) are the charge and energy densities of the field, respectively:
\[ \rho(\mu, T) = \frac{q}{(2\pi)^3} \left[ \int \frac{d^3k}{e^{(\epsilon-\mu)/T} - 1} - \int \frac{d^3k}{e^{(\epsilon+\mu)/T} - 1} \right], \quad (36) \]
\[ \eta(\mu, T) = \frac{1}{(2\pi)^3} \left[ \int \frac{e^{\epsilon d^3k}}{e^{(\epsilon-\mu)/T} - 1} + \int \frac{e^{\epsilon d^3k}}{e^{(\epsilon+\mu)/T} + 1} \right], \quad (37) \]
and \( q \) is the elementary charge of the field. We assumed that the total system is neutral. Here, the chemical potential \( \mu \) implements the charge conservation constraint and has the physical meaning of an electrostatic energy associated to the complete screening the black hole charge.

The entropy densities of the species are
\[ s_\pm = \int \frac{d^3k}{(2\pi)^3} \left[ ((\bar{n}_\pm + 1) \log(\bar{n}_\pm + 1) - \bar{n}_\pm \log \bar{n}_\pm \right], \quad (38) \]
with
\[ \bar{n}_\pm = \frac{1}{e^{(\epsilon\pm\mu)/T} - 1}. \quad (39) \]
The system’s entropy is
\[ S = (s_+ + s_-) V + \frac{1}{4} 4\pi r_+^2, \quad (40) \]
were $r_+ = M + \sqrt{M^2 - Q^2}$.

The ensemble is now parametrized by $\mu$ and $T$. Because $Q$ is an odd function of $\mu$, the limit $Q \to 0$ which we are interested in is reproduced by $\mu \to 0$.

It is a trivial exercise to compute:

$$\left( \frac{\partial S}{\partial T} \right)_{\mu=0} = 8aT^2V(1 - 8\pi MT)$$

Whose vanishing reproduces the black hole temperature $T^{-1} = 8\pi M$. The second derivative provides the critical volume:

$$\left( \frac{\partial^2 S}{\partial T^2} \right)_{\mu=0} = -8aTV(1 - 64\pi VaT^5),$$

which is half of the previous one, because it is inversely proportional to the number of species. Likewise, the first derivative $\left( \frac{\partial S}{\partial \mu} \right)_{\mu=0} = 0$ vanishes automatically and does not provide any new condition. Again, the calculation of the second derivative is much simplified by the fact that the first derivatives of the field entropy and energy with respect to $\mu$ vanish for $\mu = 0$. Calling $s = s_+ + s_-$,

$$\left( \frac{\partial^2 S}{\partial \mu^2} \right)_{\mu=0} = V\left( \frac{\partial^2 s}{\partial \mu^2} \right)_{\mu=0} - \frac{V}{T}\left( \frac{\partial^2 \eta}{\partial \mu^2} \right)_{\mu=0} - 4\pi V^2\left( \frac{\partial \rho}{\partial \mu} \right)_{\mu=0}$$

Calling $x_\pm = (\epsilon \mp \mu)/T$, by virtue of eqs.(38) and (39)

$$\left( \frac{\partial s_\pm}{\partial x} \right) = \int \frac{d^3k}{(2\pi)^3} \frac{\partial n}{\partial x}$$

Thus,

$$\left( \frac{\partial^2 s}{\partial \mu^2} \right)_{\mu=0} = 2T \int \frac{d^3x}{(2\pi)^3} \left[ \frac{\partial n}{\partial x} + x \frac{\partial^2 n}{\partial x^2} \right]$$

$$\left( \frac{\partial^2 \eta}{\partial \mu^2} \right)_{\mu=0} = 2T^2 \int \frac{d^3x}{(2\pi)^3} x \frac{\partial^2 n}{\partial x^2}$$

$$\left( \frac{\partial \rho}{\partial \mu} \right)_{\mu=0} = -2qT^2 \int \frac{d^3x}{(2\pi)^3} \frac{\partial n}{\partial x}$$

Having in mind that

$$\int \frac{d^3x}{(2\pi)^3} \frac{\partial n}{\partial x} = - \int \frac{d^3x}{(2\pi)^3} \frac{e^x}{(e^x - 1)^2} = -\frac{1}{6}$$

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and putting all these pieces together:

\[
\frac{\partial^2 S}{\partial T^2} = -\frac{1}{9} TV \left(3 + 4\pi V q^2 T^3\right) \leq 0,
\]

which ensures stability under electrostatic energy fluctuations. Paralleling the calculation of angular momentum fluctuations we estimate charge fluctuations as follows:

\[
\Delta Q^2 = -\left(V \frac{\partial p}{\partial \mu}\right)^2 \left[\frac{\partial^2 S}{\partial \mu^2}\right]^{-1} \approx \frac{q^2 VT^3}{(3 + 4\pi V q^2 T^3)}.
\]

Because the reservoir is always much larger than the hole \(T^3V \gg 1\), it follows that

\[
\frac{\Delta Q^2}{\hbar c} \approx \frac{1}{4\pi},
\]

independently of the size of the cavity.

4 assessesments

We can sum up our results by telling that Schwarzschild black holes undergo a considerable amount of fluctuations both in angular momentum and charge which should not be neglected. Because the mean angular momenta squared of the particles in the thermal bath \(J^2 \sim p^2 l^2\) where \(b\) is the impact parameter, one would expect the angular momentum spread to scale roughly linearly with the size of the cavity. We obtained that it actually scales with power \(3/2\), not far from our intuitive feeling. Thus, we can regard black hole angular momentum fluctuations to be the outcome of the random absorption and emission of quanta with random angular momenta from the thermal bath.

Furthermore, if charge fluctuations were the outcome of random absorption and emission of charged quanta, then one would expect the black hole charge spread to be independent on the size of the cavity. As a matter of fact, in the limit \(R/2M \gg 1\) we obtained that \(\Delta Q^2/(\hbar c) = 1/(4\pi)\) lending support to this interpretation. Nevertheless, there is a hidden puzzle here. If we insist that this is the appropriate physical description of the spread then this spread had to be proportional to the elementary charge of the field, which our calculations showed not to be the case. This poses a serious dilemma, because either black hole physics sets atomatically the scale of the elementary charge to be such that \(\alpha \sim 1/4\pi\) (this yields an elementary charge which is only three times the charge of the electron), or the physical process behind these fluctuations is a mystery which remains to be cleared up.
A major advantage of dealing with black holes in thermal contact with radiation is that we can gloss over the issue of the hole's absorptivity $\Gamma(\omega, m, q)$, whose computation is in general very cumbersome. The reason is that after many and many reflections against a grey body (black holes are grey bodies [6]-[8]) inside the cavity, the outgoing radiation becomes truly black body, in which case $\Gamma_{\text{effective}} = 1$. This brings about a tremendous simplification of the problem but has its price, we cannot scrutinize black hole fluctuations at Planckian scales because the stability condition combined with the fact that $2M << R$ imposes that $T >> T_p$. If these fluctuations survive at these large energy scales, is currently under investigation by a very different strategy.

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References

[1] P. C. W. Davies (1977), Proc. R. Soc. London., A353, 499.
[2] P. C. W. Davies (1978), Rep. Progr. Phys., 41, 1313.
[3] S. W. Hawking (1976), Phys. Rev.D13, 199.
[4] P. Hur (1977) Mon. Not. R. Astron. Soc. 180, 379.
[5] L. D. Landau and E. M.Lifshitz Statistical Physics (Reading: Addison-Wesley 1970)
[6] J. D. Bekenstein and A. Meisels (1977), Phys. Rev.D15, 2775.
[7] J. D. Bekenstein and M. Schiffer(1994), Phys. Rev. Lett. 72, 2512
[8] M. Schiffer (1995) Gen. Rel. Grav. 27, 1.