Reactive Turing Machines with Infinite Alphabets

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Abstract. The notion of Reactive Turing machines (RTM) was proposed as an orthogonal extension of Turing machines with interaction. RTMs are used to define the notion of executable transition system in the same way as Turing machines are used to define the notion of computable function on natural numbers. RTMs inherited finiteness of all sets involved from Turing machines, and as a consequence, in a single step, an RTM can only communicate elements from a finite set of data. Some process calculi such as, e.g., the \( \pi \)-calculus, essentially use a form of infinite data in their definition and hence it immediately follows that transition systems specified in these calculi are not executable. On closer inspection, however, the \( \pi \)-calculus does not appear to use the infinite data in a non-computable manner.

In this paper, we investigate several ways to relax the finiteness requirement. We start by considering a variant of RTMs in which all sets are allowed to be infinite, and then refine by adding extra restrictions. The most restricted variant of RTMs in which the sets of action and data symbols are still allowed to be infinite is the notion of RTM with atoms. We propose a notion of transition systems with atoms, and show that every effective legal transition system with atoms is executable by RTM with atoms. In such a way, we show that processes specified in the \( \pi \)-calculus are executable by RTM with atoms.

1 Introductions

The Turing machine [19] is a machine model that formalizes which functions from natural numbers to natural numbers are effectively computable. In the recent decades, it is extensively discussed that it is not suitable to model the way of computing systems operate nowadays as functions on natural numbers. Modern computing systems continuously interact with their environment, and their operations are not supposed to terminate. However, Turing machines lack facilities to adequately deal with the above two important ingredients of modern computing: interaction and non-termination. In recent decades, quite a number of extended models of computation have been proposed to study the combination of computation and interaction (see, e.g., the collection in [11]).

The notion of Reactive Turing machines was proposed in [11] as an orthogonal extension of classical Turing machines with a facility to model interaction in the style of concurrency theory. It subsumes some other models based on computation and concurrency [14][15].

Reactive Turing machines serve to define which behaviours (labelled transition systems) can be executed by a computing system. We say that a transition system is executable if it is behaviourally equivalent to the transition system of a reactive Turing
machine. Note that the notion of executability is parameterised by the choice of a behavioural equivalence: if a behaviour specified in a transition system is not executable up to some fine notion of behavioural equivalence (e.g., divergence-preserving branching bisimilarity), it may still be executable up to some coarser notion of behavioural equivalence (e.g., the divergence-insensitive variant of branching bisimilarity). The entire spectrum of behavioural equivalences (see [9]) is at our disposal to draw precise conclusions.

RTMs can be used to characterise the absolute expressiveness of process calculi. In the theory of executability, we ask two interesting questions about a process calculus: (1) Is it possible to specify every executable behaviour in the process calculus?; and (2) Is every behaviour specified in the calculus executable? In [1], a one-to-one correspondence was established between the executable behaviours and the behaviours finitely definable (with a guarded recursive specification) in a process calculus with deadlock, a constant denoting successful termination, action prefix, non-deterministic choice, and parallel composition with handshaking communication; the result is up to divergence-preserving branching bisimilarity. In [14], it was established that every executable behaviour can be specified in the $\pi$-calculus [16] up to divergence-preserving branching bisimilarity.

But it was also observed that $\pi$-calculus processes are generally not executable: The $\pi$-calculus presupposes an infinite set of names, which gives rise to an infinite set of action labels. It is straightforward to define $\pi$-calculus processes that, in fact, execute an unbounded number of distinct actions. The infinity of the set of names is essential in the $\pi$-calculus both for the mechanism by which input of data is modelled, and for the mechanism by which the notion of private link between processes is modelled. But, one may also argue that these reasons are more syntactic than semantic; the mechanisms themselves are not essentially infinitary. In [14] it was already argued that if one abstracts, to some extent, from the two aspects for which the infiniteness is needed, then behaviour defined in the $\pi$-calculus are executable, at least up to branching bisimilarity.

Moreover, a notable number of process calculi leading to transition systems with infinite sets of labels were proposed in recent decades for various purposes, for instance, the psi-calculus [3], the value-passing calculus [6] and mCRL2 [12], etc. It is essential to extend the formalism of the reactive Turing machines to adapt to the behaviour with infinite sets of labels such as the transition systems of the models mentioned above.

In this paper, we explore a generalised notion of executability that allows an infinite alphabet of actions. First, we shall observe that allowing an infinite alphabet only makes sense if we also allow the set of data symbols (or, equivalently, the set of states) to be infinite. Putting no restrictions at all yields a notion of executability that is not discriminating at all: every countable transition system is executable by an infinitary RTM. The result has two immediate corollaries: Every effective transition system is executable up to divergence-preserving branching bisimilarity by an infinitary RTM with an effective transition relation, and every computable transition system is executable up to divergence-preserving branching bisimilarity by an infinitary RTM with a computable transition relation.

Then, we shall consider a more restricted notion of infinitary executability. Following the research in [7] about nominal sets for variable binding with infinite alphabets,
the notion of Turing machines with atoms is introduced in [5]. We define RTMs with atoms as an extension of Turing machines with atoms [5]. RTMs with atoms allow the sets involved in the definition to be infinite, but in a limited way; intuitively, the infinity can only be exploited to generate fresh names in an execution. By using the notion of legal and orbit-finite sets, the Turing machines with atoms are allowed to have infinite alphabets, and while keeping the transition relation finitely definable and finite up to atom automorphism.

To characterise the associated notion of executability, we propose a notion of transition system with atoms as a restricted version of transition systems. We show that every effective legal transition system with atoms is executable by an RTM with atoms modulo branching bisimilarity.

Finally, we apply the results to draw conclusions about the executability of process calculi. We shall prove that all $\pi$-calculus processes are executable by RTMs with atoms up to branching bisimilarity. On the other hand, in mCRL2 it is possible to define behaviours that are not executable by RTMs with atoms. We also show that behaviours definable in mCRL2 are executable by infinitary RTMs, but the RTMs do not necessarily have an effective transition relation.

The paper is organized as follows. In Section 2, the basic definitions of reactive Turing machines and divergence-preserving branching bisimilarity are recapitulated, and we also recall some theorems about executability in [1,14]. In Section 3, we investigate reactive Turing machines with infinite sets of labels, data symbols and transitions. In Section 4, we review the notion of sets with atoms from [4,5], propose the reactive Turing machines with atoms, and characterise the class of the transition systems that are executable by reactive Turing machines with atoms. In Section 5, the executability of some models leading to transition systems with infinite sets of labels are discussed. The paper concludes in Section 6 in which a hierarchy of executable transition systems with infinite sets is proposed.

2 Preliminaries

In this section, we briefly recap the theory of executability [1], which is based on RTMs in which all sets involved are finite. We shall generalise the finiteness condition for the sets in later sections.

The behaviour of discrete-event systems We use the notion of transition system to represent the behaviour of discrete-event systems. It is parameterised by a set $\mathcal{A}$ of action symbols, denoting the observable events of a system. We shall later impose extra restrictions on $\mathcal{A}$, e.g., requiring that it be finite or have a particular structure, but for now we let $\mathcal{A}$ be just an arbitrary abstract set. We extend $\mathcal{A}$ with a special symbol $\tau$, which intuitively denotes unobservable internal activity of the system. We shall abbreviate $\mathcal{A} \cup \{\tau\}$ by $\mathcal{A}_\tau$.

Definition 1 (Labelled Transition System). An $\mathcal{A}_\tau$-labelled transition system is a triple $(S, \rightarrow, \dagger)$, where,
Definition 3 (Reactive Turing Machine).

A reactive Turing machine (RTM) is a triple $(S, \rightarrow, \uparrow)$, where

1. $S$ is a set of states,
2. $\rightarrow \subseteq S \times A \times S$ is an $A$-labelled transition relation. If $(s, a, t) \in \rightarrow$, we write $s \xrightarrow{a} t$.
3. $\uparrow \in S$ is the initial state.

In this paper, we shall use the notion of (divergence-preserving) branching bisimilarity \cite{10}, which is the finest behavioural equivalence in van Glabbeek’s linear time branching time spectrum \cite{9}.

In the definition of (divergence-preserving) branching bisimilarity we need the following notation: let $\rightarrow$ be an $A$-labelled transition relation on a set $\mathcal{S}$, and let $a \in A$; we write $s \xrightarrow{(a)} t$ for “$s \xrightarrow{a} t$ or $a = \tau$ and $s = t$”. Furthermore, we denote the transitive closure of $\xrightarrow{\tau}$ by $\xrightarrow{\ast}$ and the reflexive-transitive closure of $\xrightarrow{\tau}$ by $\xrightarrow{+}$.

Definition 2 (Branching Bisimilarity). Let $T_1 = (\mathcal{S}_1, \rightarrow_1, \uparrow_1)$ and $T_2 = (\mathcal{S}_2, \rightarrow_2, \uparrow_2)$ be transition systems. A branching bisimulation from $T_1$ to $T_2$ is a binary relation $\mathcal{R} \subseteq \mathcal{S}_1 \times \mathcal{S}_2$ such that for all states $s_1$ and $s_2$, $s_1 \mathcal{R} s_2$ implies

1. if $s_1 \xrightarrow{a} s_1'$, then there exist $s_2', s_2'' \in \mathcal{S}_2$ such that $s_2 \xrightarrow{(a)} s_2'$, $s_2' \mathcal{R} s_2''$ and $s_1 \mathcal{R} s_2'$;
2. if $s_2 \xrightarrow{a} s_2'$, then there exist $s_1', s_1'' \in \mathcal{S}_1$ such that $s_1 \xrightarrow{(a)} s_1'$, $s_1' \mathcal{R} s_2'$ and $s_1' \mathcal{R} s_2''$.

The transition systems $T_1$ and $T_2$ are branching bisimilar (notation: $T_1 \equiv^b T_2$) if there exists a branching bisimulation $\mathcal{R}$ from $T_1$ to $T_2$ s.t. $\uparrow_1 \mathcal{R} \uparrow_2$.

A branching bisimulation $\mathcal{R}$ from $T_1$ to $T_2$ is divergence-preserving if, for all states $s_1$ and $s_2$, $s_1 \mathcal{R} s_2$ implies

3. if there exists an infinite sequence $(s_{1,i})_{i \in \mathbb{N}}$ such that $s_1 = s_{1,0}$, $s_{1,i} \xrightarrow{\tau} s_{1,i+1}$ and $s_{1,i} \mathcal{R} s_{2,i}$ for all $i \in \mathbb{N}$, then there exists a state $s_2'$ such that $s_2 \xrightarrow{+} s_2'$ and $s_{1,i} \mathcal{R} s_2'$ for some $i \in \mathbb{N}$;
4. if there exists an infinite sequence $(s_{2,i})_{i \in \mathbb{N}}$ such that $s_2 = s_{2,0}$, $s_{2,i} \xrightarrow{\tau} s_{2,i+1}$ and $s_{1} \mathcal{R} s_{2,i}$ for all $i \in \mathbb{N}$, then there exists a state $s_1'$ such that $s_1 \xrightarrow{+} s_1'$ and $s_1' \mathcal{R} s_{2,i}$ for some $i \in \mathbb{N}$.

The transition systems $T_1$ and $T_2$ are divergence-preserving branching bisimilar (notation: $T_1 \equiv_{b}^d T_2$) if there exists a divergence-preserving branching bisimulation $\mathcal{R}$ from $T_1$ to $T_2$ such that $\uparrow_1 \mathcal{R} \uparrow_2$.

A theory of executability The notion of reactive Turing machine (RTM) was put forward in \cite{11} to mathematically characterise which behaviours are executable by a conventional computing system. In this section, we recall the definition of RTMs and the ensuing notion of executable transition system. The definition of RTMs is parameterised with the set $\mathcal{A}$, which we now assume to be a finite set. Furthermore, the definition is parameterised with another finite set $\mathcal{D}$ of data symbols. We extend $\mathcal{D}$ with a special symbol $\square \notin \mathcal{D}$ to denote a blank tape cell, and denote the set $\mathcal{D} \cup \{\square\}$ of tape symbols by $\mathcal{D}_0$.

Definition 3 (Reactive Turing Machine). A reactive Turing machine (RTM) is a triple $(\mathcal{S}, \rightarrow, \uparrow)$, where
1. $S$ is a finite set of states,
2. $\rightarrow \subseteq S \times \mathcal{D}_c \times \mathcal{A}_r \times \mathcal{D}_c \times \{L, R\} \times S$ is a finite collection of $(\mathcal{D}_c \times \mathcal{A}_r \times \mathcal{D}_c \times \{L, R\})$-labelled transition rules (we write $s \xrightarrow{a(d/e)^M} t$ for $(s, d, a, e, M, t) \in \rightarrow$),
3. $\uparrow \in S$ is a distinguished initial state.

Intuitively, the meaning of a transition $s \xrightarrow{a(d/e)^M} t$ is that whenever the RTM is in state $s$, and $d$ is the symbol currently read by the tape head, then it may execute the action $a$, write symbol $e$ on the tape (replacing $d$), move the read/write head one position to the left or to the right on the tape (depending on whether $M = L$ or $M = R$), and then end up in state $t$.

To formalise the intuitive understanding of the operational behaviour of RTMs, we associate with every RTM $M$ an $\mathcal{A}_r$-labelled transition system $\mathcal{T}(M)$. The states of $\mathcal{T}(M)$ are the configurations of $M$, which consist of a state from $S$, its tape contents, and the position of the read/write head. We denote by $\mathcal{D}_c = \{d \mid d \in \mathcal{D}_c\}$ the set of marked symbols; a tape instance is a sequence $\delta \in (\mathcal{D}_c \cup \hat{\mathcal{D}}_c)^*$ such that $\delta$ contains exactly one element of the set of marked symbols $\mathcal{D}_c$, indicating the position of the read/write head. We adopt a convention to concisely denote an update of the placement of the tape head marker. Let $\delta$ be an element of $\mathcal{D}_c$. Then by $\hat{\delta}$ we denote the element of $(\mathcal{D}_c \cup \hat{\mathcal{D}}_c)^*$ obtained by placing the tape head marker on the right-most symbol of $\delta$ (if it exists), and $\widehat{\delta}$ otherwise. Similarly $\hat{\delta}$ is obtained by placing the tape head marker on the left-most symbol of $\delta$ (if it exists), and $\widehat{\delta}$ otherwise.

**Definition 4.** Let $M = (S, \rightarrow, \uparrow)$ be an RTM. The transition system $\mathcal{T}(M)$ associated with $M$ is defined as follows:

1. its set of states is the set $C_M = \{(s, \delta) \mid s \in S, \delta \text{ a tape instance}\}$ of all configurations of $M$;
2. its transition relation $\rightarrow \subseteq C_M \times \mathcal{A}_r \times C_M$ is the least relation satisfying, for all $a \in \mathcal{A}_r$, $d, e \in \mathcal{D}_c$ and $\delta_L, \delta_R \in \hat{\mathcal{D}}_c$:
   - $(s, \delta_L \delta_R) \xrightarrow{a} (t, \delta_L e \delta_R)$ iff $s \xrightarrow{a(d/e)^L} t$, and
   - $(s, \delta_L \delta_R) \xrightarrow{a} (t, \delta_L e \delta_R)$ iff $s \xrightarrow{a(d/e)^R} t$, and
3. its initial state is the configuration $(\uparrow, \widehat{\delta})$.

Turing introduced his machines to define the notion of effectively computable function in [19]. By analogy, the notion of RTM can be used to define a notion of effectively executable behaviour. Usually, we shall be interested in the executability up to some behavioural equivalence.

**Definition 5 (Executability).** A transition system is executable if it is behaviourally equivalent to a transition system associated with some RTM.

In [1], a characterisation of executability up to (divergence-preserving) branching bisimilarity is given that is independent of the notion of RTM.

In order to be able to recapitulate some results from [1] and [13] below, we need the following definitions, pertaining to the recursive complexity and branching degree of transition systems. Let $T = (S, \rightarrow, \uparrow)$ be a transition system. We say that $T$ is effective
if $\rightarrow$ is a recursively enumerable set up to some suitable encoding (Gödel numbering). The mapping $\text{out} : S \rightarrow 2^{A \times S}$ associates with every state its set of outgoing transitions, i.e., for all $s \in S$, $\text{out}(s) = \{(a, t) \mid s \xrightarrow{a} t\}$. We say that $T$ is computable if $\text{out}$ is a recursive function, again up to some suitable encoding. We call a transition system finitely branching if $\text{out}(s)$ is finite for every state $s$, and boundedly branching if there exists $B \in \mathbb{N}$ such that $|\text{out}(s)| \leq B$ for all $s \in S$.

The following results were established in [1].

**Theorem 1.** 1. For every finite set $\mathcal{A}_\tau$ and every boundedly branching computable $\mathcal{A}_\tau$-labelled transition system $T$, there exists an RTM $M$ such that $T \leftrightarrow^d b T(M)$. 2. For every finite set $\mathcal{A}_\tau$ and every effective $\mathcal{A}_\tau$-labelled transition system $T$ there exists an RTM $M$ such that $T \leftrightarrow^e b T(M)$.

Moreover, if a transition system is executable modulo $\leftrightarrow^d b$, then it is necessarily boundedly branching and computable. To illustrate the necessity, a negative result was established in [14].

**Theorem 2.** If a transition system $T$ has no divergence up to $\leftrightarrow^d b$ and is unboundedly branching up to $\leftrightarrow^d b$, then it is not executable modulo $\leftrightarrow^d b$.

### 3 Infinitary Reactive Turing Machines

In this section, we shall investigate the effect of lifting one or more of the finiteness conditions imposed on RTMs on the ensued notion of executability. We start with lifting the finiteness condition on the alphabet of actions and the transition relation only. We shall argue by means of an example that this extension is hardly useful, because it is still not possible to associate a different effect with each action. The next step is, therefore, to also allow an infinite set of data symbols. This, in turn, yields a notion of executability that is too expressive. Finally, we provide two intermediate notions of executability by restricting the transition relations associated with infinitary reactive Turing machines to be effective or computable. In this section, we allow $\mathcal{A}$ to be a countably infinite set of action labels.

**Infinitely many states or data symbols** Recall from Definition 3 that an RTM has a finite set of states $S$ and a finite transition relation. If we allow RTMs to have infinitely many actions, then, inevitably, we should at least also allow them to have an infinite transition relation. The following example illustrates that we then also either need infinitely many states or infinitely many data symbols.

**Example 1.** Consider an $\mathcal{A}_\tau$-labelled transition system $T = (S_T, \rightarrow_T, \uparrow_T)$, where

1. $S_T = \{\uparrow_T, \downarrow_T\} \cup \{s_x \mid x \in \mathcal{A}\}$,
2. $\rightarrow_T = \{((\uparrow_T, x, s_x) \mid x \in \mathcal{A}) \cup \{(s_x, x, \downarrow_T) \mid x \in \mathcal{A}\}$.

There does not exist an RTM with finitely many states and data symbols that simulates $T$ modulo branching bisimilarity.
Suppose \( M = (S, \rightarrow, \uparrow) \) is an RTM such that \( \mathcal{T}(M) \models_T T \), and we let \( \mathcal{A} = \{x_1, x_2, \ldots\} \).

The transitions \( \uparrow \rightarrow x_i \rightarrow s_{x_i}, \uparrow \rightarrow x_j \rightarrow s_{x_j}, \ldots \) lead to infinitely many states \( s_{x_1}, s_{x_2}, \ldots \), which are all mutually distinct modulo branching bisimilarity.

Let \( C = (\uparrow, \uparrow) \) be the initial configuration of \( M \). Assume that we have \( C \models_b \uparrow \rightarrow \), so \( C \) admits the following transition sequences: \( C \rightarrow^* x_i \rightarrow^* C_1 \rightarrow^* x_i \rightarrow^* C_2 \rightarrow^* x_i \rightarrow^* \ldots \), where \( C_1 \models_b s_{x_i}, C_2 \models_b s_{x_i}, \ldots \).

The transition rules of an RTM are of the form \( (s, a, d, e, M, t) \), where \( s, t \in S \), and \( d, e \in D_s \); we call the pair \( (s, d) \) the trigger of the rule. A configuration \( (s', \delta_L \hat{d} \delta_R) \) satisfies the trigger \( (s, d) \) if \( s = s' \) and \( d = d' \). Now we observe that a transition \( (s, a, d, e, M, t) \) gives rise to an \( a \)-transition from every configuration satisfying its trigger \( (s, d) \). Since \( S \) and \( D_s \) are finite sets, there are finitely many triggers.

So, in the infinite list of configurations \( C_1, C_2, \ldots \), there are at least two configurations \( C_i \) and \( C_j \), satisfying the same trigger \( (s, d) \); these configurations must have the same outgoing transitions.

Now we argue that we cannot have \( C_i \models_b s_{x_i} \) and \( C_j \models_b s_{x_j} \). Since \( C_i \rightarrow x_i \), a transition labelled by \( x_i \) is triggered by \( (s, d) \). As \( C_i \) also satisfies the trigger \( (s, d) \), we have the transition \( C_i \rightarrow x_i \). Hence \( C_i \models_b s_{x_i} \), and we get a contradiction to \( \mathcal{T}(M) \models_b T \).

**Infinitary reactive Turing machines** If we allow the set of control states or the set of data symbols to be infinite too, the expressiveness of RTMs is greatly enhanced. We introduce a notion of reactive Turing machine with an infinite alphabet as follows.

**Definition 6.** An infinitary reactive Turing machine \((RTM^\infty)\) is a triple \((S, \rightarrow, \uparrow)\), where

1. \( S \) is a countable set of states,
2. \( \rightarrow \subseteq S \times D_S \times \mathcal{A}_r \times D_S \times (L, R) \times S \) is a countable collection of \((D_S \times \mathcal{A}_r \times D_S \times (L, R))\)-labelled transition rules (we write \( s \rightarrow a[d/e]M \) for \( (s, a, d, e, M, t) \in \rightarrow \)),
3. \( \uparrow \in S \) is a distinguished initial state.

**Executability by an RTM^\infty** By analogy to Definition 5, we define the executability with respect to RTM^\infty's.

**Definition 7.** A transition system is executable by an RTM^\infty if it is behaviourally equivalent to a transition system associated with some RTM^\infty.

The following theorem illustrates the expressiveness of RTM^\infty's, we show that every countable transition system is executable by an RTM^\infty modulo \( \equiv_b^\infty \).

**Theorem 3.** For every infinite set \( \mathcal{A}_r \) and every countable \( \mathcal{A}_r \)-labelled transition system \( T \), there exists an RTM^\infty \( M \) such that \( T \equiv_b^\infty \mathcal{T}(M) \).

**Proof.** Let \( T = (S_T, \rightarrow_T, \uparrow_T) \) be an \( \mathcal{A}_r \)-labelled countable transition system, and let \( [\_] : S_T \rightarrow \mathbb{N} \) be an injective function encoding its states as natural numbers. Then, an RTM with infinite sets of action symbols and data symbols \( M(T) = (S, \rightarrow, \uparrow) \) is defined as follows.
1. $\mathcal{S} = \{s, t, \uparrow\}$ is the set of control states.

2. $\rightarrow$ is a finite $(\mathcal{D}_\tau \times \mathcal{A} \times \mathcal{D}_\tau \times \{L, R\})$-labelled transition relation, and it consists of the following transitions:
   
   (a) $(\uparrow, \tau, \Box, \Box, \uparrow T, R, s)$,
   
   (b) $(s, \tau, \Box, \Box, L, t)$, and
   
   (c) $(t, a, [s_1], [s_2], R, s)$ for every transition $s_1 \xrightarrow{a} T s_2$.

3. $\uparrow \in \mathcal{S}$ is the initial state.

Note that a transition step $s_1 \xrightarrow{a} s_2$ is simulated by a sequence $(t, [s_1] \Box) \xrightarrow{a} (s, [s_2] \Box) \xrightarrow{\tau} (s, [s_2] \Box)$.

Then one can verify that $T(\mathcal{M}(T)) \leftrightarrow_b T$.

As a consequence, we trivially get an extremely expressive model. One may argue that RTM$^\infty$s are hardly useful since the countable transition relation used to define RTM$^\infty$ need not even be computable or effective.

We provide two intermediate results here. We distinguish with two cases for the infinite set of transition relations to define the RTM$^\infty$.

We say that a transition relation is effective, if for every pair of a control state and a data symbol $(s, d)$, the set of subsequent transitions is recursively enumerable, i.e., the set $\{(a, e, M, t) \mid s \xrightarrow{[d][e] M} t\}$ is recursively enumerable with respect to some encoding. Note that by the proof of Theorem 3 if the transition system is effective, then the set of transitions $\{(t, a, [s_1], [s_2], R, s) \mid s_1 \xrightarrow{a} s_2\}$ becomes recursively enumerable. For the other transitions, they are also trivially recursively enumerable. Hence, we get an effective transition relation. We derive the following corollary for the executability of effective transition systems from Theorem 3.

**Corollary 1.** For every infinite set $\mathcal{A}_\tau$ and every effective $\mathcal{A}_\tau$-labelled transition system $T$, there exists an RTM$^\infty$ $\mathcal{M}$ with an effective transition relation such that $T \leftrightarrow_b T(\mathcal{M})$.

We say that a transition relation is computable, if for every pair of a control state and a data symbol $(s, d)$, the set of subsequent transitions is computable, i.e., the set $\{(a, e, M, t) \mid s \xrightarrow{[d][e] M} t\}$ is recursive up to some encoding. Note that by the proof of Theorem 3 if the transition system is computable, then the set of transitions $\{(t, a, [s_1], [s_2], R, s) \mid s_1 \xrightarrow{a} s_2\}$ becomes recursive. For the other transitions, they are also trivially recursive. Hence, we get a computable transition relation. Then from Theorem 3 we also derive a corollary for the executability of computable transition systems.

**Corollary 2.** For every infinite set $\mathcal{A}_\tau$ and every computable $\mathcal{A}_\tau$-labelled transition system $T$, there exists an RTM$^\infty$ $\mathcal{M}$ with a computable transition relation such that $T \leftrightarrow_b T(\mathcal{M})$. 
In [7], nominal sets were introduced for variable binding with infinite alphabets. The notion of sets with atoms was used to define Turing machines with atoms in [5]. We also use sets with atoms for an extension of reactive Turing machines; we introduce a notion of reactive Turing machine with atoms (RTM\(\alpha\)). RTM\(\alpha\) is a natural intermediate between RTMs and RTM\(\infty\)s. On the one hand, RTM\(\alpha\)s will be more expressive than RTMs, since they will admit infinite alphabets, whereas RTMs do not. On the other hand, RTM\(\alpha\)s will be less expressive than RTM\(\infty\)s, because there will be restrictions imposed that make them finitely representable, in fact finite up to \(\alpha\)-conversion.

Sets with atoms  Let \(\mathbb{A}\) be a countably infinite set; we call its elements atoms. An atom automorphism is a bijection (permutation) on \(\mathbb{A}\). A set with atoms is any set that contains atoms or other sets with atoms, in a well-founded way. Sets with atoms might contain infinitely many distinct elements. We only consider legal and orbit-finite sets with atoms, which will be introduced below.

For a set with atoms \(X\) and an atom automorphism \(\pi\), by \(\pi(X)\) we denote the set obtained by application of \(\pi\) to every atom contained in the elements of \(X\), recursively. For a set \(S \subseteq \mathbb{A}\), if an atom automorphism \(\pi\) is the identity on \(S\), then we call such \(\pi\) an \(S\)-automorphism. Moreover, we say that \(S\) supports a set with atoms \(X\) if \(X = \pi(X)\) for every \(S\)-automorphism \(\pi\). A set with atoms is called legal if it has a finite support, each of its elements has a finite support, and so on recursively. A set with atoms may be infinite, nevertheless, legality restricts the elements that determine the set. For instance, a finite or co-finite set is legal; however, the set of all odd numbers is not legal.

For the notion of orbit-finite set, we adopt the definition in [4,5]. Let \(x\) be an element in a set with atoms \(X\), the \(x\)-orbit is the set

\[\{y \in X \mid y = \pi(x) \text{ for some atom automorphism } \pi\} .\]

A set with atoms \(X\) is partitioned into disjoint orbits: elements \(x\) and \(y\) are in the same orbit iff \(\pi(x) = y\) for some atom automorphism \(\pi\). For example \(\mathbb{A}^2\) decomposes into two orbits, the diagonal and its complement; and \(\mathbb{A}^\ast\) has infinitely many orbits as the elements from \(\mathbb{A}, \mathbb{A}^2, \ldots\) all fall into disjoint orbits. A set with atoms that is partitioned into finitely many orbits is called an orbit-finite set. Orbit-finiteness restricts the number of partitions of a set with atoms with respect to atom automorphism.

Reactive Turing machine with atoms  For the definition of reactive Turing machines with atoms, we presuppose a countable set of atoms \(\mathbb{A}\), and we assume that the sets of action symbols \(\mathcal{A}_\tau\), the set of data symbols \(\mathcal{D}_\square\), and the set of control states \(\mathcal{S}\) are legal and orbit-finite sets with atoms from \(\mathbb{A}\).

Definition 8 (Reactive Turing Machine with atoms). A reactive Turing machine with atoms (RTM\(\alpha\)) \(M\) is a triple \((\mathcal{S}, \rightarrow, \uparrow)\), where

1. \(\mathcal{S}\) is a legal and orbit-finite set of states,
2. $\rightarrow \subseteq S \times D \times \mathcal{A} \times \mathcal{D} \times \{L, R\} \times S$ is a legal and orbit-finite $(D \times \mathcal{A} \times D \times \{L, R\})$-labelled transition relation (we write $s \xrightarrow{a[d/e]} t$ for $(s, a, d, e, M, t) \in \rightarrow$).

3. $\uparrow \in S$ is a distinguished initial state.

Now we explain some intuitions of putting legality and orbit-finiteness as restrictions to the sets used in RTM$^A$.

Legality ensures that the sets depend on finitely many atoms. Note that the RTM$^\infty$ we used in the proof of Theorem 3 is not necessarily an RTM$^A$, since the set of its transitions is not necessarily legal. We consider the set of the transitions

$$\{(t, a, [s_1], [s_2], R, s) \mid s_1 \xrightarrow{a} s_2\}.$$

These transitions are used to simulate countable transition systems. For an arbitrary infinite transition system, the set of its transitions may not have a finite support. For instance, we consider a transition system that only consisting (exclusively) of the transitions

$$s_1 \xrightarrow{1} s_3 \xrightarrow{3} \ldots s_{2n+1} \xrightarrow{2n+1} \ldots .$$

It produces a sequence of transitions labelled by odd numbers, however, the set of all odd numbers does not have a finite support. Hence the set of transitions above does not necessarily have a finite support, which means it is not legal.

Orbit-finiteness also plays an important role as it restricts the forms of the elements included in a set with atoms. For an RTM$^A$, intuitively, it restricts the type of interaction but still allows infinitely many distinct values as input. For instance, if we allow the action symbols $\mathcal{A}_\tau$ and the data symbols $\mathcal{D}_\Box$ to be a string of arbitrary lengths, namely, $\mathcal{A}_*$, then we are able to define an RTM$^\infty$ to communicate with an alphabet of any string and write any string as a data symbol as follows: the machine receives any string of arbitrary length as an input and write it on the tape by a set of transitions of the form $s \xrightarrow{a[a]} t$, where $s, t$ are control states and $a \in \mathcal{A}_*$. However, such a set is not orbit-finite, as $a$ is from a set with infinitely many orbits. In such a way, the orbit-finiteness restricts the length of symbols in a single interaction.

By analogy to the executability with respect to RTMs in Definition 5 we associate with every RTM$^A$ a labelled transition system, and define a notion of executability with respect to RTM$^A$.

**Definition 9.** A transition system is executable by an RTM$^A$ if it is behaviourally equivalent to a transition system associated with some RTM$^A$.

**Transition systems with atoms** Next we investigate the class of transition systems that are executable by an RTM$^A$. We consider transition systems with legal and orbit-finite sets of labels $\mathcal{A}_\tau$. We define the notion of legal transition system with atoms as follows:

**Definition 10.** Let $K \subset \mathcal{A}_\tau$. A labelled transition system $T = (S_T, \rightarrow_T, \uparrow_T)$ is $K$-supported, if it satisfies the following conditions:

1. $S_T, \rightarrow_T$ are sets with atoms with support $K$; and
We include the lemma from the definition of the support.

**Proof.**

We remark that if \( \rightarrow_T \) is a legal transition system with atoms, then we can number every equivalence class (orbit) in \( X \) by a natural number as \( X \) is countable. We encode \( \pi(x) \) by an injection from the equivalence class of \( x \) to the natural number that numbers \( [x] \). We assume that there is an encoding from \( A \) to \( \mathbb{N} \), and there is an encoding of lists. We encode \( \text{atom}(x) \) by the encoding of lists of encodings of atoms. We define the encoding of an element \( x \) by a pair consisting of the encoding of the function of \( \text{orb}_{[x]} \), and the encoding of the list of atoms \( \text{atom}(x) \) as follows:

\[
[x] = [([\text{orb}_{[x]}], \text{atom}(x))].
\]

We remark that if \( X \) is, e.g., the set of labels of a \( \pi \)-calculus process, then one may derive \( \text{atom} \) and \( \text{orb}_{[x]} \) as well as their encodings trivially.

**Valid operations of RTM\(^A\)s on sets with atoms**

To obtain the valid operations of RTM\(^A\)s on sets with atoms, we show the following property on the transitions of an RTM\(^A\).

**Lemma 1.** Let \( M = (S_M, \rightarrow_M, \uparrow_M) \) be an RTM\(^A\). Then there exists a finite set of atoms \( K \subset A \) such that, for every \((s,a,d,e,M,t) \in \rightarrow_M\) and for every \( K \)-automorphism \( \pi_K \), we have \( \pi_K(s,a,d,e,M,t) \in \rightarrow_M \).

**Proof.** As \( \rightarrow_M \) is a legal set. We take \( K \) to be its the minimal support. Then we conclude the lemma from the definition of the support.
Now we investigate to what extent, an RTM can manipulate elements from sets with atoms. We first have the following fact:

**Lemma 2.** Given an RTM $M$, and we suppose that currently the tape instance is a set of atoms $K \subseteq \mathcal{A}$. Then within a sequence of executions, $M$ can change its tape instance to $(K, K')$ with $K' \subseteq \mathcal{A}$. Then for every atom in $K'$, it is obtained by one of the following two ways:

1. duplicating an atom $a$ from $K$, and adding it to $K'$
2. creating a fresh atom $a$ which is not in $K$, and adding it to $K'$.

_Proof._ We assume the current tape instance is $(K, K'')$, and we show the two ways to add a new atom to $K''$.

For the first case, we suppose that $a$ is the atom to be duplicated, and $\square$ is the destination of the duplication. The machine could accomplish the task by the transitions $\tau^{\text{copy}a} \rightarrow \text{copy}a \rightarrow \tau^{\text{finish}}$, which is realized by the following set of transitions.

$$
\{(\text{copy}, \tau, a, a, R, \text{copy}a) \mid a \in \mathcal{A}\}
\{(\text{copy}_a, \tau, b, b, R, \text{copy}_a) \mid a, b \in \mathcal{A}\}
\{(\text{copy}_a, \tau, \square, a, R, \text{finish}) \mid a \in \mathcal{A}\}
$$

This is a legal and orbit-finite set of transitions. Moreover, it is triggered only if an atom $a$ is already on the tape.

For the second case, the machine creates a fresh atom, by the following set of transitions,

$$
\{(\text{fresh}, \tau, \square, a, L, \text{check}a) \mid a \in \mathcal{A}\}
\{(\text{check}_a, \tau, b, b, L, \text{check}a) \mid a \neq b \land b \neq \square \land a, b \in \mathcal{A}\}
\{(\text{check}_a, \tau, a, a, L, \text{refresh}a) \mid a \in \mathcal{A}\}
\{(\text{refresh}_a, \tau, b, b, R, \text{refresh}_a) \mid a \neq b \land a, b \in \mathcal{A}\}
\{(\text{refresh}_a, \tau, a, b, L, \text{check}b) \mid a \neq b \land a, b \in \mathcal{A}\}
\{(\text{check}_a, \tau, \square, \square, R, \text{finish}) \mid a \in \mathcal{A}\}
$$

The machine first creates an arbitrary atom, and checks every atom on the tape if it is identical with the created one. We suppose that $\square$ indicates the end of the sequence of atoms on tape. If the check procedure succeeds, the creation is finished, otherwise, the machine creates another atom and check again. We also verify that the above transitions form a legal and orbit-finite set.

Next we show that an RTM can be able to produce the element $x$ by evaluating $\text{orb}[x](\text{atom}(x))$, if $x$ is from an orbit-finite set. Given a legal and orbit-finite set with atoms $X$, we have the following lemma:

**Lemma 3.** For every legal and orbit-finite set with atoms $X$, there exists an RTM such that for every $x \in X$, it produces an $x$ labelled transition, given that atom$(x)$ and $[\text{orb}(x)]$ are written on the tape.
Proof. We define an RTM^A M = (S_M, →_M, ↑_M), and we show that M suffices the requirement.

According to the assumption, we suppose that some state start, the tape instance is [orb_{1^3}] atom(x). We first show that within finitely many steps, the machine is able to write x on the tape.

Note that X is an orbit-finite set, which means that there are finitely many distinct orbits that construct the set X. Therefore, there are finitely many different possible values of [orb_{1^3}]. The machine is able to associate with each value a program that calculates orb_{1^3}(atom(x)), which produces the elements from that specific orbit according to the string of atoms. The machine enters the programme by entering the state orb_{1^3}.

\[
\{(\text{start}, \tau, [\text{orb}_{1^3}], [\text{orb}_{1^3}], R, \text{orb}_{1^3}) \mid x \in X\}
\]

As X is orbit-finite, there is an upper-bound for the length of the string atom(x) for every x ∈ X. Therefore, the machine is able to represent these strings using a set of tuples as data symbol. We suppose that every element in x-orbit uses n atoms in its structure, then atom(x) is a tuple of n atoms. Now we consider the following set of transitions:

\[
\{(\text{orb}_{1^3}, \text{orb}_{1^3}(a_1, \ldots, a_n), (a_1, \ldots, a_n), \text{orb}_{1^3}(a_1, \ldots, a_n), R, \text{finish} \mid a_1 \ldots a_n \in \text{Atom}\}
\]

These transitions will create the element \text{orb}_{1^3}(a_1, \ldots, a_n) from a tuple (a_1, \ldots, a_n). As there is an upper bound of n, this set of transitions is orbit-finite.

Moreover, we show that the machine is able to create such tuples from the strings of atoms atom(x), given that each atom is written on one tape cell, and ordered from left to right as the order of the atoms in the string.

The machine constructs the tuple by duplicating the elements from each tape cell to the tuple one by one, using the transitions as follows:

\[
\begin{align*}
&\{(\text{orb}_{1^3}, \tau, a, a, R, \text{orb}_{1^3,a}) \mid a \in \mathbb{A}\} \\
&\{(\text{orb}_{1^3,a}, \tau, b, b, R, \text{orb}_{1^3,a}) \mid a, b \in \mathbb{A}\} \\
&\{(\text{orb}_{1^3,a}, \tau, \text{orb}_{1^3,a}, (a), L, \text{orb}_{1^3,a}) \mid a \in \mathbb{A}\} \\
&\{(\text{orb}_{1^3,a}, \tau, (a_1, \ldots, a_{i-1}), (a_1, \ldots, a_{i-1}, a_i), L, \text{orb}_{1^3,a}) \mid a_1, \ldots, a_i \in \mathbb{A}, 2 \leq i \leq n\} \\
&\{(\text{orb}_{1^3,a}, \tau, b, b, L, \text{orb}_{1^3,a}) \mid a, b \in \mathbb{A}, a \neq b\} \\
&\{(\text{orb}_{1^3,a}, \tau, a, a, R, \text{orb}_{1^3,a}) \mid a \in \mathbb{A}\}
\end{align*}
\]

The machine first finds the atom to duplicate, and uses orb_{1^3,a} to register the atom. Then it moves the tape head to the tuple and adds the atom to that tuple. Finally, the tape goes back to the atom it duplicated and starts another duplication. This procedure ends by entering the state where the orb_{1^3}(a_1, \ldots, a_n) is produced. We verify that this set of transition is legal and orbit-finite. Moreover, there are finitely many orbits for the set X, which means that the machine needs finitely many programs. Hence, we have obtained an RTM^A M that meets the requirement.

By applying a π^k-automorphism on the element x, we derive the following corollary.
Corollary 3. For every legal and orbit-finite set with atoms $X$, there exists an $RTM^\#$, such that for every $x \in X$, and for every $K$-automorphism $\pi_K$, it produces an $\pi_K(x)$ labelled transition, given that atom($\pi_K(x)$) and $[\text{orb}_{[\tau]}]$ are written on the tape.

We will use this fact to show that an RTM$^\#$ is able to produce $\mathcal{A}_t$-labelled transitions if $\mathcal{A}_t$ is a legal and orbit-finite set.

**Executability of RTM$^\#$** We first recall the notion of effective transition systems. Given a transition system, the set of outgoing transitions from a state $s$ is denoted by $\text{out}(s) = \{(a, t) \mid s \xrightarrow{a} t\}$. We say that the transition system is effective if the function $\text{out}$ is recursively enumerable. For a legal transition system with atoms, we say that it is effective, if the encodings of the sets of states and labels exist, and given the encoding of $s$, the encoding of the set of pairs in $\text{out}(s)$ can be effectively enumerated.

We first show that the assumptions of effective legal labelled transition systems with atoms are sufficient to prove that a transition system is executable by an RTM$^\#$ modulo $\equiv_b$.

**Lemma 4.** For every legal and orbit-finite set $\mathcal{A}_t$ and every effective legal $\mathcal{A}_t$-labelled transition system with atoms $T$, there exists an RTM$^\#$ $M$ such that $T \equiv_b \mathcal{T}(M)$.

**Proof.** Let $T = (S_T, \rightarrow_T, \sharp_T)$ be an effective legal $\mathcal{A}_t$-labelled transition system with atoms, and $K \subset \mathcal{A}$ is a minimal support of $T$. We show that there exists an RTM$^\#$ $M = (S_M, \rightarrow_M, \sharp_M)$ such that $\mathcal{T}(M) \equiv_b T$.

As $T$ is effective, for every state $s \in S_T$, the set $\text{out}(s) = \{(a, t) \mid s \xrightarrow{a} t\}$ is recursively enumerable. We use this fact to simulate the transition system. We describe the simulation by 5 steps.

1. $\sharp_M \Rightarrow \text{enumerate}$:

Initially, the tape is empty. Hence the initial configuration is $(\sharp_M, \emptyset)$. For simplicity, we do not denote the position of the tape head in the tape instances if not necessary.

The machine first writes the encoding of the initial state $\sharp_T$ on the tape. By the assumption of the encoding of sets with atoms, $[\sharp_T] = [\text{orb}_{[\tau]}][\text{atom}(\sharp_T)]$. As $T$ is legal, $\sharp_T$ consists of finitely many atoms. The machine is also able to write $\text{atom}(\sharp_T)$ on the tape. This procedure is represented as follows,

$$(\sharp_M, \emptyset) \xrightarrow{i} (\text{enumerate}, [\text{orb}_{[\tau]}][\text{atom}(\sharp_T)]\text{atom}(\sharp_T))$$  

2. enumerate $\Rightarrow$ generate:

In the control state enumerate, we assume that the tape instance is $[\text{orb}_{[\tau]}][\text{atom}(s)]\text{atom}(\pi_K(s))$, and the machine starts to simulate the state $\pi_K(s)$ from this configuration, where $s \in S$ and $\pi_K$ is some $K$-automorphism. Note that, for the simulation of $\sharp_T$, the tape instance is $[\text{orb}_{[\tau]}][\text{atom}(\sharp_T)]\text{atom}(\sharp_T)$.

For state $s$, its subsequent transitions are $\{(s, a, t) \in \rightarrow_T\}$ and by the proposition of transition system with atoms, the set of transitions from $\pi_K(s)$ is $\pi_K(\{(s, a, t) \in \rightarrow_T\})$.

We use this fact to enumerate the transitions from $s$ and take a $\pi_K$ automorphism by the the atoms $\text{atom}(\pi_K(s))$ which is already on the tape, and hence we obtained the transitions from $\pi_K(s)$. 

Then the machine enumerates $out(s)$, and writes $([a], [t])$ (which are represented by $[orb_{[a]}], [atom(a)], [orb_{a}], [atom(t)]$ respectively) on the tape whenever a transition $(s, a, t)$ is enumerated.

The step of enumerate is represented as follows,

$$(\text{enumerate, } [orb_{[a]}][atom(s)][atom(\pi_K(s))]) \rightarrow^*$$

$$(\text{generate, } [orb_{[a]}][atom(s)][orb_{a}][atom(a)][orb_{t}][atom(t)][atom(\pi_K(s))]).$$

3. $\text{generate} \Rightarrow \text{action}$:

In the control state $\text{generate}$, the machine produce the new sets of atoms for the action label and the subsequent state of the transition.

We use $\pi_{K,\cup,s}$ to denote some arbitrary atom automorphism that preserves the support $K$ and the atoms from the state $\pi_K(s)$, and $\pi_{K,\cup,\cup,s}$ to denote some arbitrary atom automorphism that preserves $K$, $\pi_K(s)$ and $atom(\pi_K(s))$. Then the machine non-deterministically produces $atom(\pi_{K,\cup,s}(a))$ and $atom(\pi_{K,\cup,\cup,s}(t))$ according to $atom(s)$, $[atom(s)]$, $[atom(a)]$ and $[atom(t)]$, for some arbitrary $\pi_{K,\cup,s}$ and $\pi_{K,\cup,\cup,s}$.

The machine first generates $atom(\pi_{K,\cup,s}(a))$. For every element $[x] \in [atom(a)]$, we distinguish with two cases:

(a) if $[x]$ also in $[atom(s)]$, then the machine find the the corresponding element from $atom(\pi_K(s))$ and duplicate that one to fills the position of $[x]$ in $atom(\pi_{K,\cup,s}(a))$;

(b) if $[x]$ not in $[atom(s)]$, then the machine create a fresh atom and fills the position of $[x]$ in $atom(\pi_{K,\cup,s}(a))$.

Then the machine then generates $atom(\pi_{K,\cup,\cup,s}(t))$. For every element $[x] \in [atom(t)]$, we also distinguish with two cases:

(a) if $[x]$ is also in $[atom(s)]$ or $[atom(a)]$, then the machine find the the corresponding element from $atom(\pi_K(s))$ or $atom(\pi_{K,\cup,s}(a))$ and duplicate that one to fills the position of $[x]$ in $atom(\pi_{K,\cup,\cup,s}(t))$;

(b) if $[x]$ is not in $[atom(s)]$ nor in $[atom(a)]$, then the machine create a fresh atom and fills the position of $[x]$ in $atom(\pi_{K,\cup,\cup,s}(t))$.

By Lemma\footnote{2} duplication and creating of atoms are valid operations for RTM$^A$. We denote this step as follows,

$$(\text{generate, } [orb_{[a]}][orb_{a}][orb_{t}][atom(s)][atom(a)][atom(t)][atom(\pi_K(s))]) \rightarrow^*$$

$$(\text{action, } [orb_{[a]}][orb_{a}][orb_{t}][atom(s)][atom(a)][atom(t)][atom(\pi_K(s))][atom(\pi_{K,\cup,s}(a))[atom(\pi_{K,\cup,\cup,s}(t))]).$$

4. $\text{action} \Rightarrow \text{transition}$:

Hence, the machine will produce a label $\pi_{K,\cup,s}(a)$ from $atom(\pi_{K,\cup,s}(a))$ and $[orb_{[a]}]$. By Corollary\footnote{3} this step is computable by an RTM$^A$. According to the property of transition systems with atoms, the transition to be simulated $(\pi_K(s), \pi_{K,\cup,s}(a), \pi_{K,\cup,\cup,s}(t)) \in \rightarrow_T$. Moreover, from Lemma\footnote{4} and the procedure of generating new atoms, it is guaranteed that every transition $(\pi_K(s), a', t') \in \rightarrow_T$ satisfying that there exists some $\pi_{K,\cup,s}$ such that $\pi_{K,\cup,s}(\pi_K(s), a, t) = (\pi_K(s), a', t')$ are produced in the previous step.

This procedure of producing the action label is represented as follows,
5. transition ⇒ enumerate.

Finally the machine simulates the transition or continues the enumeration.

\[
\begin{align*}
(\text{transition}, \llbracket \text{orb}[a] \rrbracket \llbracket \text{orb}[a] \rrbracket \llbracket \text{atom)(s)\rrbracket \llbracket \text{atom(a)\rrbracket \llbracket \text{atom(t)\rrbracket,} \\
&\text{atom(π}_K\text{(s)\rrbracket \llbracket \text{atom(π}_K\text{(t)π}_K\text{(a)\rrbracket) \rightarrow} \ast
\end{align*}
\]

We can verify that in before the step that actually make the labelled transition, the transition system of the machine preserves its states modulo \( \equiv \_b \) by a sequence of \( \tau \)-transitions which leads back to the configuration \( \text{(enumerate, } [\text{orb}[a]] [\text{atom(t)}] \text{atom(π}_K\text{(s))}) \). Moreover, every transition with a label \( π_K(a) \) is indeed one of the transitions from the state \( π_K(s) \) by our assumption for transition systems with atoms. Note that the \( π_K(a) \) labelled transition is also included if \( π_K(a) \) happens to map \( π_K(a) \) to its identity. Therefore, we conclude that \( T \equiv \_b T(M) \).

We also show that the requirements of effective legal transition systems with atoms are necessary to prove that a transition system is executable an by RTM\(^k \) modulo \( \equiv \_b \).

**Lemma 5.** For every RTM\(^k \) M, the associated transition system \( T(M) \) is an effective legal transition system with atoms.

**Proof.** Let \( M = (S_M, \rightarrow_M, \uparrow_M) \), and \( \mathcal{A}_r \) is its set of labels with a minimal support \( K \). It is obvious that \( T(M) \) is effective.

By Lemma 5 it follows that \( T(M) \) is a legal transition system with atoms.

To conclude, we have the following theorem stating that the class of executable transition systems by RTM\(^k \)s modulo \( \equiv \_b \) are exactly the set of effective legal transition systems with atoms.

**Theorem 4.** A transition system \( T \) is executable by an RTM\(^k \) modulo \( \equiv \_b \) iff there exists a legal and orbit-finite set \( \mathcal{A}_r \) and an effective legal \( \mathcal{A}_r \)-labelled transition system with atoms \( T' \), and \( T \equiv \_b T' \).

### 5 Related Work

\textbf{π-calculus} The motivation of introducing the notion of RTM with infinite alphabets comes from the discussion of the executability of the π-calculus in [14]. The transition systems of the π-calculus processes use infinite sets of names for their labels, whereas RTMs use finite sets of labels. We have introduced infinite alphabets to RTM in order to break that difference in the formalism of the two models. As the transition systems associated with π-calculus processes are computable, by Theorem 3, we straightforwardly conclude that the transition system of a π-calculus process is executable by an RTM\(^{∞}\) modulo \(\leftrightarrow_{b}\).

\textbf{Corollary 4.} For every π-calculus process \(P\), the transition system \(T(P)\) is executable by an RTM\(^{∞}\) with a computable transition relation modulo \(\leftrightarrow_{b}\).

We show that we can improve this result to the executability defined by RTM\(^{α}\)'s modulo \(\leftrightarrow_{b}\).

Now we consider the set of names \(N\) of a π-calculus process as the set of atoms. Note that \(N\) itself is a legal and orbit-finite set with atoms. Moreover, a label from the transition system of a π-calculus process is a pair two names or \(τ\). Hence the set of labels \(A_\tau\) for every π-calculus process is a legal and orbit-finite set with atoms. Moreover, the transition relations of all π-calculus processes are preserved under \(α\)-conversion. As mentioned in [18], the set of free variables of a π-calculus process is invariant under execution. We consider the set of free names in a π-term as the support of the associated transition system. The set of free names is finite, thus we get a finite support of the transition system. We conclude that the transition systems associated with π-calculus processes are legal transition systems with atoms. By using the fact that the operational semantics of the π-calculus leads to an effective transition relation, we conclude from Theorem 4 with the following result.

\textbf{Corollary 5.} For every π-calculus process \(P\), the transition system \(T(P)\) is executable by an RTM\(^{α}\) modulo \(\leftrightarrow_{b}\).

\textbf{mCRL2} The formal specification language mCRL2 [12] is widely used to specify and analyze the behaviour of distributed systems. It uses abstract data types that leads to transition systems with infinitely many labels. A question is raised to what extent, the transition systems specified by mCRL2 are executable. The actions in an mCRL2 specification may contain vectors of data of any arbitrary lengths, which leads to a set of actions with infinitely many orbits. Despite the infiniteness of orbits of the labels, we can also specify transition systems that does not have a finite support in mCRL2 (e.g. the transition system used to prove Theorem 3). Therefore, we conclude that such transition systems are not executable by any RTM\(^{α}\).

\textbf{Corollary 6.} There exists an mCRL2 specification \(P\), such that the transition system \(T(P)\) is not executable by any RTM\(^{α}\) modulo \(\leftrightarrow_{b}\).

Moreover, the transition systems specified in mCRL2 are countable but not necessarily effective, therefore, we have the following corollary.

\textbf{Corollary 7.}
1. For every mCRL2 specification $P$, the transition system $T(P)$ is executable by an RTM$^\infty$ modulo $\leftrightarrow_b$.

2. There exists an mCRL2 specification $P$, such that the transition system $T(P)$ is not executable by any RTM$^\infty$ with an effective/computable transition relation modulo $\leftrightarrow_b$.

Psi-calculus A psi-calculi were introduced in [3] as an extension of the $\pi$-calculus with nominal data types for data structures and for logical assertions representing facts about data. The adoption of nominal data types provides a natural characterisation of the behaviour executed by RTM$^A$. Moreover, in [2], an encoding of the $\pi$-calculus is proposed using the psi-calculus, suggesting that the psi-calculus is at least as expressive as the $\pi$-calculus. It could be interesting to figure out the relationship between the transition systems associated with the psi-calculus and the transition systems associated with RTM$^A$. We conjecture that as long as the logical assertions used in the psi-calculus are semi-decidable, the transition systems of the psi-calculus processes are executable by RTM$^A$’s modulo $\leftrightarrow_b$.

Nominal Transition systems Followed by the work of the psi-calculus, a notion of nominal transition systems was introduced in [17]. The transition relations of nominal transition systems satisfy the requirements of transition systems with atoms naturally. We did not use the notion of nominal transition systems since the predicates for states in Hennessy-Milner logic is not necessary in proving the executability. By assuming the existence of a suitable encoding on nominal transition systems, we could also define the type of effective nominal transition systems. In a way, we can show that the effective nominal transition systems are executable modulo by RTM$^A$’s $\leftrightarrow_b$.

Value-passing calculus The value-passing calculus [6] is a process calculus in which the contents of communications are values chosen from some data domain. Hence, transition systems of a value-passing calculus process usually need infinitely many labels (such as natural numbers). Some structures are naturally imposed on the set of atoms, e.g. the set of natural numbers has a structure of total ordering. It could be an interesting future work to investigate an adequate notion of executability for the transition systems associated with the value-passing calculus.

6 Conclusions

We summarize the executable labelled transition systems associated with RTMs with infinite sets/ legal and orbit-finite sets with atoms/ finite sets (denoted by RTM$^\infty$/ RTM$^A$/ RTM, respectively) as a hierarchy of executable transition systems. We figure out the classes of transition systems with respect to various notions of executability defined in Figure 1. Note that all the inclusion relation over the above sets of transition systems are interpreted as the inclusion of the regions in Figure 1.

We summarize the corresponding transition system of each notion of executability as follows:
1. by Theorem 1, the class of executable transition systems by RTMs modulo $\leftrightarrow^d_b$ is the boundedly branching computable transition system with a finite set of labels;
2. the class of executable transition systems by RTMs modulo $\leftrightarrow_b$ is the effective transition system with a finite set of labels;tem by Theorem 4 the class of executable transition systems by RTM$^A$'s modulo $\leftrightarrow_b$ is the effective legal transition system with atoms;
3. by Corollary 2 the class of executable transition systems by RTM$^\infty$'s modulo $\leftrightarrow^d_b$ is the computable transition system;
4. by Corollary 1 the class of executable transition systems by RTM$^\infty$'s with an effective transition relation modulo $\leftrightarrow^d_b$ is the effective transition system; and
5. by Theorem 3 the class of executable transition systems by RTM$^\infty$'s modulo $\leftrightarrow^d_b$ is the countable transition system.

Finally, we propose some future work on this issue.

1. The precise characterisation of the transition systems executable by RTM$^A$'s modulo $\leftrightarrow^d_b$ is still open. Further restrictions should be imposed to make it possible to generate all possible transitions of an arbitrary state in the transition system from a single configuration of an RTM$^A$.
2. We could impose some structures on the sets of atoms, e.g., by imposing a total order on the a set of atoms. Moreover, we could investigate a notion of RTMs with natural numbers, and try to make a connection to the behaviour specified in the value-passing calculus.

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