SIMULATING MAGNETOHYDRODYNAMICAL FLOW WITH CONSTRAINED TRANSPORT AND ADAPTIVE MESH REFINEMENT: ALGORITHMS AND TESTS OF THE AstroBEAR CODE

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ABSTRACT

A description is given of the algorithms implemented in the AstroBEAR adaptive mesh-refinement code for ideal magnetohydrodynamics. The code provides several high-resolution shock-capturing schemes which are constructed to maintain conserved quantities of the flow in a finite-volume sense. Divergence-free magnetic field topologies are maintained by machine precision by collating the components of the magnetic field on a cell-interface staggered grid and utilizing the constrained transport approach for integrating the induction equations. The maintenance of magnetic field topologies on adaptive grids is achieved using prolongation and restriction operators which preserve the divergence and curl of the magnetic field across collocated grids of different resolutions. The robustness and correctness of the code is demonstrated by comparing the numerical solution of various tests with analytical solutions or previously published numerical solutions obtained by other codes.

Key words: methods: numerical – MHD

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1. INTRODUCTION

The development of efficient and accurate numerical algorithms for astrophysical flow has become of great interest to the astrophysical community. Variable resolution approaches have provided an avenue for efficient simulation of hydrodynamical flow including multiphysical effects which involve substantial variation in length scale. Adaptive mesh refinement (AMR) has been recognized as one of the most versatile and efficient approaches to enable the simulation of multiscale phenomena for which fixed-resolution simulation is either impractical or impossible. AMR discretizations employ a hierarchy of grids at different levels. High resolution is applied only to those regions of the flow which would otherwise be subject to unacceptably large truncation error. The utility of the AMR approach is underscored by the extensive list of codes that are targeted toward astrophysical research which utilize AMR. The list includes AstroBEAR, Enzo (O’Shea et al. 2004), Flash (Fryxell et al. 2000), Orion (Truelove et al. 1998; Klein 1999; Crockett et al. 2005), Nirvana (Ziegler 2005a), Ramses (Fromang et al. 2006), RIEMANN (Balsara 2001), and AMRVAC (Keppens et al. 2003), and the list of codes for which the development AMR capability is in progress including Athena (Gardiner & Stone 2005) and Pluto (Mignone et al. 2007).

The simulation of magnetized flow is of particular interest to astrophysical researchers owing to the utility of numerical magnetohydrodynamics (MHD) in modeling a wide range of astrophysical phenomena. The leading line of recent research in this area has focused on the application of higher order Godunov methods to numerical MHD (Ryu & Jones 1995; Balsara 1998a). The conservative formulation and proper upwinding employed by these methods enable accurate simulation of strongly supersonic flow. Because of this unique capability, such methods are often referred to high-resolution shock-capturing (HRSC) methods.

While HRSC methods have long been recognized as the de facto standard for the simulation of supersonic hydrodynamical phenomena, their popularity among researchers interested in magnetized flow has been slowed because standard HRSC approaches to MHD fail to maintain the solenoidality constraint on the magnetic field (\[ \nabla \cdot B = 0 \]). If not corrected, local divergences in the magnetic field arising from this shortcoming usually grow rapidly, causing anomalous magnetic forces and unphysical plasma transport which eventually destroys the correct dynamics of the flow (Brackbill & Barnes 1980). Early practitioners of numerical MHD therefore relied heavily on finite difference methods as employed by codes such as Zeus (Stone & Norman 1992) which maintain the solenoidality constraint exactly despite their inferior shock-capturing ability (Falle 2002). Later works focused on improved HRSC which either eliminate the development of divergences in the magnetic field, or mitigate the effect of local divergence errors on the dynamics of the flow. In one such approach, a projection operator is devised, usually by solving a Poisson equation, which removes numerical divergences from the grid after each time step (Balsara 1998b; Jiang & Wu 1999; Kim et al. 1999; Zachary et al. 1994; Ryu et al. 1995). The primary limitation of this approach is that nontrivial boundary value problems become indeterminate (Ryu et al. 1998). In the second so-called “8-wave” approach first explored by Powell et al. (1999), alternative formulations of the MHD equations are constructed to prevent the local buildup of magnetic divergence by advecting monopoles to other regions of the grid where they are of less consequence to the dynamics of the flow. The work of Dedner et al. (2002) augments this approach by adding source terms to the system which act to counter the effect of local divergence in the magnetic field on the dynamics of the flow. A third approach known as constrained transport (CT) utilizes a multidimensional, divergence-preserving update procedure for the magnetic field components which are collated on a staggered grid centered on the computational volume interfaces (Evans & Hawley 1988; Ryu et al. 1998; Balsara & Spicer 2007).
The combination of AMR spatial discretizations with HRSC would seem a natural choice in order to satisfy the desire for a highly accurate, computationally efficient, and versatile strategy for the simulation of magnetized plasma flow. The implementation of the aforementioned adaptations to HRSC methods for MHD in an AMR framework, however, poses several challenges. Divergence-cleaning schemes utilizing a Poisson projection operator are ill-suited for AMR applications which combines an approximate projection operator based on a piecewise quadratic interpolation procedure by the application of additional refinement. Crockett et al. (2005), however, have constructed an approach suitable for AMR applications which combines an approximate projection operator with the divergence advection and damping the effects of local divergence of Powell et al. (1999) and Dedner et al. (2002).

Retaining the divergence-free property of the solution on hierarchical grids requires application of a divergence-preserving prolongation operator which interpolates the magnetic field from a coarse mesh to a finer mesh, a divergence-preserving restriction operator which maps the fine mesh magnetic field onto a coarser mesh, and that the evolution of the magnetic field be consistent between collocated meshes of different resolution. Two approaches to these challenges have emerged. Balsara (2001) has generalized the divergence-free reconstruction procedure of Balsara & Spicer (1999) to devise a prolongation operator based on a piecewise quadratic interpolation procedure that is divergence preserving in the Riemann MHD code. Li & Li (2004) present an adaptation of Balsara’s procedure that simplifies its implementation for problems involving arbitrary refinement ratios. Tóth & Roe (2002) have devised a prolongation operator by solving an algebraic system which enforces the maintenance of the volume-averaged curl and divergence between grids of different resolutions in the AMR hierarchy.

In this paper, we provide a concise description of the algorithms and tests of the AstroBEAR HRSC AMR MHD code. AstroBEAR is comprised of several numerical solvers, integration schemes, and radiative cooling modules for astrophysical fluids. The code’s AMR capability is derived from the AMR engine of the Bearclaw boundary embedded AMR package for conservation laws. This code utilizes the constrained transport approach to adapting HRSC methods to the MHD system of equations. To our knowledge, AstroBEAR is the first AMR code to utilize the prolongation operator of Tóth & Roe (2002) to maintain the $\nabla \cdot \mathbf{B} = 0$ constraint. By combining multiphysics capabilities relevant to simulation astrophysical plasma flow, AMR, and a wide selection of HRSC integration procedures, AstroBEAR will serve as a valuable research tool.

The authors intend that this paper will serve as a reference for future works that apply the code and provide a concise recipe for robust, reliable, and accurate HRSC solution strategies for MHD on AMR grid hierarchies. In Section 2, we describe the several HRSC schemes and divergence-preservation strategies implemented in the code. In Section 3, we provide an overview of the AMR algorithm, highlighting the stages which require special treatment of the magnetic field. In Section 4, we provide a concise description of the prolongation, restriction, and coarse to fine refluxing procedures required to preserve the divergence and consistency of the magnetic field across an AMR hierarchy of grids. In Section 5, we comment on the results of several test and example problems with particular emphasis on the relative strengths and weaknesses of the various HRSC schemes implemented in the code. In Section 6, we provide a synopsis and discussion of the main results of the paper.

2. NUMERICAL METHOD

AstroBEAR provides an AMR framework for the integration of conservation laws of the form

$$\frac{\partial}{\partial t} \mathbf{Q} + \frac{\partial}{\partial x} \mathbf{F}_x(Q) + \frac{\partial}{\partial y} \mathbf{F}_y(Q) + \frac{\partial}{\partial z} \mathbf{F}_z(Q) = \mathbf{S}(Q).$$

In this work, we focus on the equations of ideal MHD which are written in the conservative form as

$$\frac{\partial}{\partial t} \begin{bmatrix} \rho \\ \rho v_x \\ \rho v_y \\ \rho v_z \\ \rho E_x \\ \rho E_y \\ \rho E_z \\ \mathcal{E} \\ B_x \\ B_y \\ B_z \end{bmatrix} + \frac{\partial}{\partial x} \begin{bmatrix} \rho v_x + P + B_x^2/2 - B_y^2 \\ \rho v_x v_y - B_x B_y \\ \rho v_x v_z - B_x B_z \\ (E + P + B_x^2/2)v_x - B_x(B \cdot v) \\ 0 \\ -E_z \\ E_y \\ \mathcal{E} + P + B_x^2/2 \\ B_x \\ B_y \end{bmatrix} = \mathbf{S},$$

with gas density $\rho$, velocity $\mathbf{v}$, total volumetric energy density $\mathcal{E}$, thermal pressure $P$, magnetic field $\mathbf{B}$, and electric field $\mathbf{E}$. In the proceeding system of equations, we have chosen units for the electric and magnetic field in the plasma so that factors of $4\pi$ do not appear in the equations. The last three equations in the system follow from Faraday’s law,

$$\frac{\partial}{\partial t} \mathbf{B} + \nabla \times \mathbf{E} = 0,$$

implicit to which is the constraint that initially solenoidal magnetic field topologies remain solenoidal,

$$\nabla \cdot \mathbf{B} = 0.$$

The equations are brought to a closed form via Ohm’s law for a perfectly conducting medium,

$$\mathbf{E} = -\mathbf{v} \times \mathbf{B}.$$
and the polytropic equation of state for an ideal gas,

\[ P = (\gamma - 1)(\mathcal{E} - \rho v^2/2 - B^2/2). \]  

(5)

In the remainder of this section, we describe the details of the shock-capturing numerical schemes available in our code to integrate the solution to equations of the form of Equation (1) and modifications thereof that ensure the solenoidal constraint on the magnetic field. The purpose of this description is to provide a concise and complete illustration of the steps necessary to build the code. The details of any of the particular solution strategies may be obtained by consulting the original work credited for the particular strategy. For a more pedagogically oriented review of HRSC schemes, we refer the reader to the excellent books of Leveque (2002) and Toro (1999). In the remainder of this section, we consider only the solution method for the homogeneous part of the conservation law with \( S = 0 \). The effect of nonzero source terms which may be used to include additional physics is handled using the operator splitting technique described in Section 2.1 of Cunningham et al. (2005). The microphysical source terms included in the code to model the effects of radiative cooling, and time-dependent, nonequilibrium ionization, and \( H_2 \) chemistry are cataloged in the appendix to Cunningham et al. (2005). In Appendix A, we catalog the MHD source terms and modifications to the numerical scheme that are employed in applications involving cylindrical axisymmetric flow. We adopt the notation that superscripts denote time, the first subscript denotes the direction of vector components, and the last three subscripts denote the spatial location on the computational grid. The superscript is omitted from temporally varying quantities which are to be evaluated at time \( t \). Equations which demonstrate operations that take the same form in each of the \( x \), \( y \), and \( z \) directions are written only for the \( x \)-direction sweep. In these cases, the \( y \) sweep can be recovered by replacing \( y \rightarrow x \) and \( j \rightarrow i \), and the \( z \) sweep can be retrieved by replacing \( z \rightarrow x \) and \( k \rightarrow i \).

Numerical integration of the system of conservation laws is achieved using the finite-volume method. The basis of the finite-volume quadrature can be realized by discretizing the integral form of Equation (1), yielding the following unsplit procedure for advancing the conserved field forward in time by an increment \( \Delta t \):

\[
Q_{i,j,k}^{n+1} = Q_{i,j,k}^n + \frac{\Delta t}{\Delta x} \left( \bar{F}_{x,i,j,k}^n - \bar{F}_{x,i+1/2,j,k}^n \right) \\
+ \frac{\Delta t}{\Delta y} \left( \bar{F}_{y,i,j,k}^n - \bar{F}_{y,i,j+1/2,k}^n \right) \\
+ \frac{\Delta t}{\Delta z} \left( \bar{F}_{z,i,j,k}^n - \bar{F}_{z,i,j+1/2,k}^n \right),
\]

(6)

where \( \bar{F}_{x}^n \), \( \bar{F}_{y}^n \), and \( \bar{F}_{z}^n \) are suitably accurate numerical approximations to the intercell flux, spatially averaged over the intercell area and temporally averaged from \( t \) to \( t + \Delta t \). Construction of higher order schemes in more than one dimension can, in some cases, be simplified by utilizing a direction-split approach

\[
Q_{i,j,k}^{(0)} = Q_{i,j,k}, \quad Q_{i,j,k}^{(1)} = Q_{i,j,k} + \frac{\Delta t}{\Delta x} \left( \bar{F}_{x,i,j,k}^n - \bar{F}_{x,i+1/2,j,k}^n \right), \\
Q_{i,j,k}^{(2)} = Q_{i,j,k} + \frac{\Delta t}{\Delta y} \left( \bar{F}_{y,i,j,k}^n - \bar{F}_{y,i,j+1/2,k}^n \right), \\
Q_{i,j,k}^{(3)} = Q_{i,j,k} + \frac{\Delta t}{\Delta z} \left( \bar{F}_{z,i,j,k}^n - \bar{F}_{z,i,j+1/2,k}^n \right),
\]

(7)

In Equation (7), we have used the superscript in parenthesis to denote intermediate states between direction sweeps. The ordering of the component directions \((x_1, x_2, x_3)\) cycles over different permutations of the three coordinate directions on successive time steps. In two dimensions the ordering is \([(x, y), (y, x)]\), and in three dimensions we have found the most consistent results by cycling over both the cyclic and anticyclic permutations:

\[
[(x, y, z), (z, x, y), (y, z, x), (z, y, x), (x, z, y), (y, x, z)].
\]

The subject of the following subsections is the procedure to compute the numerical flux which is comprised of three steps: (1) reconstruction (interpolation) to zone edges (Section 2.1), (2) upwinding the solution of the Riemann problem at each zone edge (Section 2.2), and (3) temporal evolution of the field of conserved quantities (Section 2.3). Our code implements several methods for carrying out each of these steps which may be utilized in the combination that best tailors the integration strategy to the requirements of the application at hand.

### 2.1. Spatial Reconstruction

We construct a spatially second-order accurate integration procedure via suitable reconstruction of the state in each computational cell from the volume-averaged state within that cell and its neighbors. We define a “primitive variable” operator, \( L_p(Q_i) = P_i = [\rho, v_x, v_y, v_z, E - \rho v^2/2 - B^2/2, B_x, B_y, B_z]^T \), which converts the conserved state variables into a form more suitable for interpolation. Interpolation of the primitive variables, rather than the conserved, has the advantage that the reconstructed state at grid edges are guaranteed to have non-negative pressure. We write the reconstruction from the volume-averaged state variables collected at grid centers to the grid interface as

\[
P_{L,i+1/2} = P_i + \frac{1}{2} \phi_{+,i}, \quad P_{R,i-1/2} = P_i - \frac{1}{2} \phi_{-,i}.
\]

Figure 1 shows a cartoon schematic of the volume-averaged field of primitive variables on a one-dimensional grid (solid lines), the cell reconstruction (dotted lines), and the location of the left and right restricted states.

The grid interface conservative fields are then computed as

\[
Q_{L,i-1/2} = L_p^{-1}(P_{L,i-1/2}), \quad Q_{R,i+1/2} = L_p^{-1}(P_{R,i+1/2}.
\]

(9)

The code implements a user-selectable choice of three different second-order spatial reconstruction methods. The first is
the MUSCL reconstruction method of Van Leer (1979) using a slope limiter to maintain monotonicity,
\[ \phi_{\pm,i} = \text{LIMITER}(P_i - P_{i-1}, P_{i+1} - P_i). \] (10)

Many limiter functions may be constructed. AstroBEAR implements a choice of three limiters, which operate on the state vector parameters in Equation (10) in a componentwise fashion. In order of decreasing levels of diffusion introduced into the scheme, these limiters are the “min-mod” limiter,
\[ \text{MINMOD}(x, y) = \begin{cases} 0 & \text{if } xy < 0 \\ \text{MIN}(|x|, |y|) & \text{otherwise}, \end{cases} \] (11)
the limiter of Van Leer
\[ VL(x, y) = \begin{cases} 0 & \text{if } xy < 0 \\ \frac{xy}{x+y} & \text{otherwise}, \end{cases} \] (12)
and the “monotonized-centered” limiter
\[ MC(x, y) = \begin{cases} 0 & \text{if } xy < 0 \\ \text{MIN}(2x, \frac{1}{2}(x+y), 2y) & \text{otherwise}. \end{cases} \] (13)

The second reconstruction method is the local hyperbolic harmonic variation of the piecewise hyperbolic method (PHM) of Marquina (1994). The PHM reconstruction prescribes the components of \( \phi_{\pm,i} \) as
\[ \phi_{\pm,i} = \begin{cases} 0 & \text{if } |\delta_L| < 10^{-14} \text{ and } |\delta_R| < 10^{-14} \\ d\Delta x \eta_{\pm} & \text{otherwise}, \end{cases} \] (14)
where
\[ \eta_{\pm} = \begin{cases} 1 & \text{if } |\kappa| < 10^{-5} \\ 2\kappa^{-2} \left( \log \left( \frac{2\Delta x}{\kappa} \right) \pm \frac{2\Delta x}{\kappa} \right) & \text{otherwise} \end{cases} \]
\[ \kappa = 2 - \sqrt{1 + \psi} \text{ if } |\delta_L| < 10^{-14} \text{ or } \delta_L \leq 0 \text{ and } \delta_R \leq \delta_L, \]
\[ 1 - \sqrt{1 + \psi} \text{ if } |\delta_R| < 10^{-14} \text{ or } \delta_R \geq 0 \text{ and } \delta_L \leq \delta_R, \]
\[ \sqrt{\frac{\psi}{d}} - 1 \text{ if } \delta_L \leq \delta_R \]
\[ 1 - \sqrt{\frac{\psi}{d}} \text{ otherwise}, \]
\[ \delta_L = \frac{P_i - P_{i-1}}{\Delta x} \]
and
\[ \delta_R = \frac{P_{i+1} - P_i}{\Delta x}. \]

Note that \( \eta \) has a removable singularity about \( \kappa = 0 \) with \( \lim_{\kappa \to 0^+} \eta(\kappa) = \lim_{\kappa \to 0^-} \eta(\kappa) = 1 \). When evaluated with an 8-byte precision, the expression for \( \eta \) begins to diverge from its true solution for \( |\kappa| < 10^{-5} \) and we set \( \eta \to 1 \) in this region using the piecewise expression given above.

The third method is the reconstruction procedure of the piecewise parabolic method (PPM) of Colella & Woodward (1984). A detailed overview of the PPM method is available from Miller & Colella (2002) and Mignone et al. (2005). Appendix B of Mignone et al. (2005) already provides a concise description of the PPM reconstruction procedure which we do not repeat here. We note that in the numerical examples using the PPM reconstruction presented later in this paper, we have taken a different approach to numerical oscillations than Mignone et al. (2005). In particular, we maintain monotonicity via the MINMOD limiter (Equation (11)), rather than the more compressive Van Leer limiter, and we do not include the dissipation mechanisms of Section B.1 of Mignone et al. (2005).

2.2. Upwinded Numerical Flux

The numerical flux is computed by upwinding the waves associated with the Riemann problem defined by \( Q_{L,i-1/2} \) on the left and \( Q_{R,i-1/2} \) on the right of each computational cell interface. AstroBEAR implements three different methods for computing the upwinded flux, the Harten–Lax–van Leer discontinuity (HLLD) flux as described in Miyoshi & Kusano (2005), the Roe flux, and the Marquina flux.

The Roe flux method (Roe 1986) calls for the decomposition of the cell edge states into eigenmodes of a linearized system matrix. In the case of MHD, we utilize the approximate linearized Riemann solver of Ryu & Jones (1995). We write this decomposition in terms of the eigenvalues \( a_m(Q) \), right eigenvectors \( \mathbf{R}_m(Q) \), and left eigenvectors \( \mathbf{L}_m(Q) \) given in Section 2.2 of Ryu & Jones (1995) where the subscript \( m \) denotes the \( m \)-th eigenmode. Singularities which arise, in certain limiting cases, in the normalization of the eigensystem are avoided in the manner prescribed by Roe & Balsara (1996). The intercell flux across cell boundaries at constant \( x \) is computed as
\[ \tilde{F}_{x,i-1/2} = \frac{1}{2} \left( F_{x,1}(Q_{L,i-1/2}) + F_{x,1}(Q_{R,i-1/2}) \right) \]
\[ - \frac{1}{2} \sum_{m=1}^{s} \mathbf{L}_{m,i-1/2} \left( a_{m,i} \left[ Q_{R,i-1/2} - Q_{L,i-1/2} \right] \right) \mathbf{R}_{m,i-1/2}, \] (15)
where the eigendecomposition is carried out about a suitable average of the states to the left and right of the cell interface which we denote as \( (Q_{L,i-1/2}, Q_{R,i-1/2}) \):
\[ a_{m,i-1/2} = a_{m,i}(Q_{L,i-1/2}, Q_{R,i-1/2}), \]
\[ \mathbf{L}_{m,i-1/2} = \mathbf{L}_m(Q_{L,i-1/2}, Q_{R,i-1/2}), \]
\[ \mathbf{R}_{m,i-1/2} = \mathbf{R}_m(Q_{L,i-1/2}, Q_{R,i-1/2}). \]

The flux functions of Donat & Marquina (1996), first proposed by Shu & Osher (1989) for scalar equations, take the form
\[ \tilde{F}_{x,i-1/2} = \sum_{m=1}^{s} a^L_{i-1/2} \mathbf{R}_{m,i-1/2} + a^R_{i-1/2} \mathbf{R}_{m,i-1/2} \] (17)
and
\[ a^L_{i-1/2} = \begin{cases} \sum_{m=1}^{s} \mathbf{L}_{m,i-1/2} \left( F(Q_{L,i-1/2}) \right) a^L_{m,i-1/2} a^R_{m,i-1/2} > 0 & \text{if } a^L_{m,i-1/2} a^R_{m,i-1/2} > 0 \\ 0 & \text{otherwise,} \end{cases} \]
\[ a^R_{i-1/2} = \begin{cases} \sum_{m=1}^{s} \mathbf{L}_{m,i-1/2} \left( F(Q_{L,i-1/2}) \right) a^L_{m,i-1/2} a^R_{m,i-1/2} > 0 & \text{if } a^L_{m,i-1/2} a^R_{m,i-1/2} > 0 \\ 0 & \text{otherwise,} \end{cases} \]
\[ \frac{1}{2} \sum_{m=1}^{s} \text{MAX} \left( |a^L_{i-1/2}|, |a^R_{i-1/2}| \right) \times \mathbf{L}_{m,i-1/2} \left( Q_{L,i-1/2} \right) \] (18)
which is not true, in general, for the arithmetic average linearization. We therefore employ the arithmetic average linearization of Ryu & Jones (1995) for MHD applications and revert to the Roe average linearization only for purely hydrodynamic applications.

In general, the Roe flux provides the least diffusive formulation. This is because the flux formulations based on the method of Donat & Marquina (1996) revert to the more diffusive local Lax–Frederichs upwinding for transonic eigenmodes. This additional component of numerical diffusion is advantageous for simulating certain astrophysical phenomena. It is sufficient to prevent the development of rarefaction shocks without the need to introduce an “entropy fix” (Harten et al. 1976). It is also sufficient to dampen the development of carbuncles, even–odd decoupling, and related numerical pathologies (Sutherland et al. 2003) that are particularly problematic in grid-aligned flows which exhibit strong radiative cooling.

### 2.3. Temporal Reconstruction

The temporal update operation for the grid-centered values is given by Equation (6). Replacing \( n \to t \) and utilizing any of the methods for computing the numerical flux in the previous section achieves an integration that is first order in time. We implement four different methods to obtain second-order temporal accuracy by performing this update using time-centered estimates of the numerical flux.

The MUSCL–Hancock predictor–corrector temporal discretization achieves second-order accuracy by advancing the grid-face interpolated states by a half-time-step using a one-dimensional predictor step. The predictor step is carried forward according to

\[
Q_{i+\Delta t/2}^{(t)} = Q_{i, i+1/2}^{(t)} + \frac{\Delta t}{2\Delta x} \left[ F_{i+1/2} - F_{i} \right] + \frac{\Delta t}{2 \Delta x} S(Q_{i+1/2})
\]

Note that the predictor step uses the cell-centered, volume-averaged fluxes. The Riemann problem at the cell interfaces is not solved and the upwinded flux at the cell faces are not needed for this step.

The MUSCL–Hancock corrector step calls for the construction of a second-order, time-centered numerical flux which is computed by applying any of the upwinding procedures of Section 2.2 using \( Q_{i, i-1/2}^{(t)} \) and \( Q_{i, i+1/2}^{(t)} \) as the left and right states for the Riemann problem at the \( i - 1/2 \) cell interface. We note the update to the time-centered state described here, the use of this procedure in more than one dimension requires the application of the operator split integrator to retain the second-order accuracy of the scheme for multidimensional flow. The fully second-order accurate update is therefore carried out via application of Equation (7) with the time-centered flux with \( n = \Delta t/2 \).

The application of the predictor–corrector schemes such as the MUSCL–Hancock approach described above will in some cases necessitate the application of a protection procedure to ensure pressure and density positivity of the predictor interface.
This flux may be used to carry forward a fully second-order, time-step problems per grid cell per time step as the MUSCL–Hancock cost due to the solution of twice the number of Riemann problems. However, the method entails somewhat greater computational costs due to the solution of twice the number of Riemann problems. In general, the MUSCL–Hancock method can be expected to produce more accurate results than the van Leer scheme, especially in regions of strong gradients. For this reason, the MUSCL–Hancock scheme is often used in conjunction with other schemes, such as the van Leer scheme. The method is described in detail in Colella (1990) and enhancements include the multidimensional corner transport upwind (CTU) scheme (Shu 1988). In the first step, \( \tilde{F}_{x,i-1/2} \) and \( Q_{x,i-1/2} \) are computed via the application of a first-order update step given by Equation (6) with \( n = \Delta t \) and any operator split microphysical effects (e.g., source terms). In the second step, \( \tilde{F}(Q^*_{x,i-1/2}) \) is computed by a second application of the spatial interpolation and upwinding procedure on the grid of \( Q^*_{x,i-1/2} \) data. The second-order, time-centered fluxes are then computed via the interpolation formula

\[
\tilde{F}^t_{x,i-1/2} = \frac{1}{2} \left( \tilde{F}(Q^*_{x,i-1/2}) + \tilde{F}(Q^*_{x,i+1/2}) \right).
\]

This flux may be used to carry forward a fully second-order, unsplit update via Equation (6). The unsplit nature of the Runge–Kutta time stepping is a significant advantage of the method. The Runge–Kutta scheme also has a comparative advantage in that it is pressure positivity-preserving and therefore more robust. However, the method entails somewhat greater computational cost due to the solution of twice the number of Riemann problems per grid cell per time step as the MUSCL–Hancock approach. In addition, the scheme suffers a somewhat restrictive time-step stability condition. The maximum numerically stable time-step \( \Delta t \) is computed in terms of the Courant condition as

\[
\Delta t < \text{MAX} \left[ \frac{a_{m,i-1/2,j,k}}{\Delta x} + \text{MAX} \left( \frac{a_{m,i,j-1/2,k}}{\Delta y}, \frac{a_{m,i,j,k-1/2}}{\Delta z} \right) \right].
\]

In practice, we estimate the next time-step increment from the maximum wave speed encountered during the preceding integration sweeps as

\[
dt_{\text{next}} = \text{CFLMAX} \left[ \frac{a_{m,i-1/2,j,k}}{\Delta x} + \frac{a_{m,i,j-1/2,k}}{\Delta y} + \frac{a_{m,i,j,k-1/2}}{\Delta z} \right],
\]

where CFL is a user-tunable parameter. We typically choose CFL \( \sim 0.8 \) for one-dimensional calculations and CFL \( \sim 0.4 \) for multidimensional problems. Future revisions of the code will include the multidimensional corner transport upwind (CTU) reconstruction method of Colella (1990) and enhancements to this method for the integration of the MHD equations by Gardiner & Stone (2005). By explicitly including the effect of transverse-propagating waves at each grid interface, the CTU scheme retains numerical stability for larger time steps, CFL < 1, while capturing greater accuracy.

### 2.4. Constrained Transport Scheme

While Faraday’s law (Equation (3)) guarantees solenoidality of \( \partial B/\partial t \), and therefore the maintenance of solenoidal magnetic field topologies, the Godunov-based conservative field update procedures described in the previous section do not provide any such guarantee. This is because no provision has been made in the construction of the integration procedure that would enforce the divergence constraint on the magnetic field. Each of the fluxes used in the conservative update procedure are second-order approximations to the exact area-averaged fluxes where

\[
\begin{align*}
F_x &= \tilde{F}_x + O(\Delta x^3) \\
F_y &= \tilde{F}_y + O(\Delta y^3) \\
F_z &= \tilde{F}_z + O(\Delta z^3).
\end{align*}
\]

Therefore, the divergence of the magnetic field after a conservative update will also contain high-order truncation errors, with \( \nabla \cdot B = O(\Delta x^3) + O(\Delta y^3) + O(\Delta z^3) \). Local departures from \( \nabla \cdot B = 0 \) usually grow rapidly, causing anomalous magnetic forces and unphysical plasma transport which eventually destroys the correct dynamics of the flow (Brackbill & Barnes 1980). Two strategies have emerged for adapting Godunov-based MHD schemes so that the divergence-free constraint is exactly maintained. In the first approach a projection operator is devised, usually by solving a Poisson equation which removes numerical divergences from the grid (Balsara 1998b; Jiang & Wu 1999; Kim et al. 1999; Zachary et al. 1994; Ryu et al. 1995). Solving the Poisson equation is somewhat computationally expensive and particularly algorithmically complex on AMR grid hierarchies. The second approach utilizes a more multidimensional approach toward the numerical quadrature of Faraday’s law by utilizing a conservative formulation of Stoke’s theorem to represent magnetic field components at staggered collocation points (Balsara & Spicer 1999; Dai & Woodward 1998; Ryu et al. 1998). Following the nomenclature first commissioned by Evans & Hawley (1988), this approach is commonly referred to as CT in the literature. The AstroBEAR code utilizes the CT approach to maintain a divergence-free field, primarily due to the limitations of the first approach in AMR applications. Furthermore, Balsara & Kim (2004) have demonstrated the superiority of the staggered grid approach in the context of a stringent astrophysically motivated test problem involving the interplay of strong shocks with radiative cooling.

The basis of the CT approach is realized by applying of Stoke’s theorem to Faraday’s law and integrating over each face of a control volume. This yields an expression for the face-average normal component of \( \partial B/\partial t \) at each control volume interface. Spatial discretization of the line integral around the control volume interfaces calls for the average electric field parallel to each edge of the control volume. Temporal discretization reveals an explicit update procedure for the normal component of the magnetic field at each computational volume interface. The resulting discretized equations,

\[
B^t_{x,i-1/2,j,k} = B^t_{x,i-1/2,j,k} + \frac{\Delta t}{\Delta y \Delta z} \\
\times \left( \Delta y E_{y,i-1/2,j,k+1/2} - \Delta y E_{y,i-1/2,j,k-1/2} - \Delta z E_{z,i-1/2,j,k+1/2} + \Delta z E_{z,i-1/2,j,k-1/2} \right)
\]
At the end of each CT update step, the volume-centered cell solution for the magnetic field, computed during the Gudonov update, is discarded in favor of the solution provided by the CT update. At this stage in the algorithm, the cell-centered magnetic field is recomputed according to the procedure requires that the components of the magnetic field be collated at the center of the zone faces to which they are orthogonal and that the component of the electric field parallel to each computational cell interface be known. The spatial locations of the desired electric and magnetic field components are illustrated in Figure 2.

Specification of the CT update procedure is completed via a suitable construction of the required electric field components at each grid edge. The fluxes of the MHD equation (Equation (2)) can be expressed in terms of the electric field using Ohm’s law (Equation (4)). The numerical intercell fluxes computed during the conservative update step described in the preceding subsections provide a second-order, shock-capturing approximation to the components of the electric field at the center of each computational cell interface. The CT update scheme, however, calls for the electric field at the grid edges. A reconstruction of the cell-face electric fields to the cell edges that retains second-order accuracy is given by

\[ E_{x,i,j-1/2,k-1/2}^{+\Delta t/2} = k \left( j_{x,i,j-1/2,k}^{\Delta t/2} + j_{x,i,j-1/2,k-1/2} \right) \]

where the cell-face electric fields have been written in terms of cell interface fluxes. Many procedures exist for averaging the cell interface flux components to construct the cell-edge electric field components. Setting \( \tilde{f}_s = \tilde{F}_s \) and \( k = 1/4 \) recovers the CT scheme of Balsara & Spicer (1999). The CT scheme of Ryu et al. (1998) can be expressed in the form of Equation (34) by retaining only the advective part of the intercell flux,

\[ f_{x6,7,8} = v_y [B_y, B_z]^T \]

and setting \( k = 1/2 \). The upwinded, time-centered, cell-face fluxes \( \tilde{f}_{x6,7,8}^{+\Delta t/2} \) are constructed from the cell-centered flux components \( f_x \) according to the same procedure given in Section 2.2. This update procedure retains the upwinding abilities of the conservative update scheme while maintaining a divergence-free solution to the magnetic field via the correspondence of the components of the magnetic flux with the components of the electric field called for by the CT update procedure. Construction of the electric field from the components of the intercell numerical flux therefore has the advantage of introducing very little numerical dissipation into the solution. Of the schemes tested by Tóth (2000), those that utilize this approach to reconstructing the electric field at grid edges produce the most accurate results. One implementation detail that should be noted is that the update of the components of the magnetic field collated along the boundary of the computational domain requires the flux components that are parallel to that boundary extending into the first row of ghost cells along the boundary. The conservative update procedure must therefore extend one row of computational cells into the ghost cell region during integration sweeps transverse to the boundary even though no conservative update is applied to the boundary cells. Extension into the ghost region ensures that the components of the numerical flux required in the CT update step are computed.

At the end of each CT update step, the volume-centered cell solution for the magnetic field, computed during the Gudonov update, is discarded in favor of the solution provided by the CT update. At this stage in the algorithm, the cell-centered magnetic field is recomputed according to the
procedure
\[
\tilde{B}_{x,i,j,k} = \frac{1}{2} (B_{x,i+1/2,j,k} + B_{x,i-1/2,j,k}) \\
\tilde{B}_{y,i,j,k} = \frac{1}{2} (B_{y,i+1/2,j,k} + B_{y,i-1/2,j,k}) \\
\tilde{B}_{z,i,j,k} = \frac{1}{2} (B_{z,i,j+1/2,k} + B_{z,i,j-1/2,k})
\] (36)

Even though the solution of the magnetic field is advanced in time at cell interfaces, we retain the grid of volume-averaged magnetic field components in order to compute the magnetic pressure at the cell centers. Also, the volume-averaged magnetic field and total energy are synchronized with the cell-face magnetic field after every CT update step in order to preserve the volume-averaged thermal energy:

\[
\tilde{E}_{i,j,k} = \tilde{E}_{i,j,k} - \frac{B_{x,i,j,k}^2}{2} + \frac{B_{y,i,j,k}^2}{2}.
\] (37)

The total energy is thereby adjusted such that the CT update preserves thermal energy. This optional step avoids numerical difficulty with negative thermal pressure that arise in strongly magnetized problems at the expense of energy conservation. We view this as an acceptable tradeoff, particularly for astrophysical applications that are not energy conserving due to radiative energy losses.

Because the normal component of the magnetic field is known at cell faces, it is not necessary to interpolate these values during the spatial reconstruction step of Section 2.1. Recalling that we have defined the sixth, seventh, and eighth components of the primitive field vector as the \(x\), \(y\), and \(z\) components of the magnetic field (Equation (8)), we set

\[
P_{L,6,i-1/2} = P_{R,6,i-1/2} = b_{x,i-1/2}
\]

\[
P_{L,7,j-1/2} = P_{R,7,j-1/2} = b_{y,j-1/2}
\]

\[
P_{L,8,k-1/2} = P_{R,8,k-1/2} = b_{z,k-1/2}
\] (38)

in lieu of the spatial reconstruction procedure when using an unsplit scheme (Equation (6)). With a direction split scheme (Equation (7)), the partial update of the cell-centered field from the split reconstruction procedure is small and because we apply a divergence-free CT update for the magnetic field, the effect of this small error does not grow rapidly. The effect of this error appears as inexact evolution of the \(z\)-component of the magnetic field in the magnetic field loop advection test of Gardiner & Stone (2005) as shown in Section 5.2. Gardiner & Stone (2005) introduced a two-dimensional unsplit scheme and later a three-dimensional extension thereof (Gardiner & Stone 2008) where they devise an unsplit reconstruction that is not subject to this error at the expense of considerably increased algorithmic and computational complexity. Balsara (2004) has shown that unsplit MHD schemes are less diffusive in some tests than the dimensionally split counterpart. Because of this, we recommend employing unsplit Runge–Kutta schemes for exact divergence-free reconstructed states for AstroBEAR MHD applications. Nevertheless, the direction split MUSCL–Hancock update has proven to be reliable in earlier purely hydrodynamic works (Cunningham et al. 2006a, 2006b; Dennis et al. 2008; Yirak et al. 2008), and for this reason, we leave this option available for MHD applications. We note that the MUSCL–Hancock scheme is efficient in terms of computational cost, requiring only one Riemann solver per grid cell per time step while retaining stability for CFL \(< 1\) and that exhaustive testing of this and other (Ryu et al. 1998) direction split MHD schemes have shown good results.

2.5. Summary or Numerical Methods

The menu of options available in our code includes three spatial reconstruction methods (Section 2.1): linear (MUSCL), piezewise hyperbolic (PHM), and piecewise parabolic (PPM), two temporal reconstruction methods (Section 2.3): MUSCL–Hancock and Runge–Kutta, four different Riemann solvers/upwinding procedures (Section 2.2): the HLLD flux, the Roe flux, the Marquina flux, and an adaptation of the Marquina flux that is better suited for magnetized flow involving compound wave structures and two different CT schemes for preserving the solenoidality constraint on magnetized flows (Section 2.4); the method of Balsara & Spicer (1999) and the method of Ryu et al. (1995). The user of the code may choose the method for each of these operations which are summarized in Table 1. The advantage of this approach is that a given simulation may be performed using several different methods. The solution strategy which is optimal for the physical regime of a given simulation may be readily applied.

3. ADAPTIVE MESH REFINEMENT

The central feature of the BEARCLAW framework on which AstroBEAR is based is that it provides a framework for AMR. Under AMR, regions of the flow that are susceptible to large discretization errors are carried forward on a computational grid of higher resolution, while flow features not requiring high resolution for adequate numerical convergence are carried
The first step of the update procedure, “Set Ghost,” calls for the initialization of the ghost cells that are exterior to all grids on the given level. Figure 4 shows an example of an AMR hierarchy containing one grid on the root level (level=0), and one refined level (level=1). The interior of each grid is delineated by solid lines and the extended grid which include the interior and ghost region of each grid are delineated by dashed lines.

The “Grid Adapt” procedure determines the arrangement of and initialization of a new AMR hierarchy on a given level that tracks the evolution of flow features in the solution. A user-specified truncation error-estimation procedure is applied to each grid that is one level coarser. AstroBEAR uses either the maximum absolute value of the primitive vector gradient or Richardson extrapolation (Berger & Oliger 1984; Berger & Colella 1989) as the options for error estimation. Zones on the parent grids with estimated error greater than a user-specified tolerance are flagged for refinement. We employ the patchwise clustering algorithm of Berger & Rigoutsos (1991) to determine the arrangement of refined grid patches that optimally overlays all of the zones flagged for refinement. For easier parallel implementation, we require that each grid has only one parent. In Figure 5, the new arrangement of grid patches on the first refinement level with interior boundaries and ghost boundaries delineated by solid and dashed lines, respectively. The regions coinciding with the previous arrangement of AMR patches are shaded in gray. Interior regions of the new grid arrangement that coincide with interior regions of the previous patchwork of grids on the same level are initialized by copying the field values from the previous grids. The previous patchwork of grids on this level are then released from memory. The ghost zones, and interior zones that do not coincide with the interior of the previous grid patches are initialized from the

**Table 1**

| Spatial Reconstruction | Temporal Reconstruction | Flux Function | CT Scheme |
|------------------------|-------------------------|--------------|-----------|
| MUSCL                  | MUSCL–Hancock           | Roe          | Ryu et al. (1995) |
| PFM                    | Runge–Kutta             | Marquina     | Balsara & Spicer (1999) |
| PPH                    |                         | Adapted Marquina | HLLD |

**Figure 3.** Schematic of the update procedure for an AMR hierarchy of three levels of refinement where each level has a refinement ratio of 2. Curved horizontal arrows represent integration of all grids on a given refinement level. Gray vertical arrows represent restriction of refined grids to their parent level. Black vertical arrows represent prolongation of the solution from the coarse level to its parent level.

**Figure 4.** Example of an AMR hierarchy containing one grid on the root level (level=0), and one refined level (level=1). The interior of each grid is delineated by solid lines and the extended grid which include the interior and ghost region of each grid are delineated by dashed lines.

The ghost zones, and interior zones that do not coincide with the interior of the previous grid patches are initialized from the

forward on a computational grid of lower resolution. Two approaches to AMR have emerged: (1) the block-based method of Berger & Oliger (1984) and Berger & Colella (1989) which constructs a patchwork of refined grids that optimally covers all of the cells on the next-coarsest level that are heuristically identified for refinement, and (2) an alternative approach where individual computational cells are refined or derefined separately (Khokhlov 1998). Our code employs the former approach. In this section, we give an overview of the AMR algorithm and stages of the AMR algorithm that require special attention in handling grid face magnetic field components. For discussion, we use the term “parent grid” to designate an underlying block on the next coarser level, “parent level” to designate all of the grid blocks that are one level coarser, “child grids” to designate those grid blocks that are one level finer and “child level” to refer to grids that are of higher resolution by one refinement ratio than the current grid. Advancement of an AMR hierarchy of grids is carried out according to the pseudo-code algorithm given in Appendix B. A schematic of the update procedure is shown in Figure 3 for an AMR hierarchy of three levels of refinement where each level has a refinement ratio of 2. Curved horizontal arrows represent integration of all grids on a given refinement level. The algorithm is adaptive in time, with each level advanced in time increments 

\[
\Delta t_{\text{level}} = \Delta t_{\text{level-1}} / r, \ 
\]

where \( r \) is the refinement ratio of the level. Gray vertical arrows represent restriction and refluxing of the solution on refined grids to their parent level. Black vertical arrows represent prolongation of the solution from the coarse level to its parent level.

The first step of the update procedure, “Set Ghost,” calls for the initialization of the ghost cells that are exterior to all grids on the given level. Figure 4 shows an example of an AMR hierarchy containing one grid on the root level (level=0), and one refined level (level=1). The interior of each grid is delineated by solid lines and the extended grid which include the interior and ghost region of each grid are delineated by dashed lines. The face-centered magnetic field components that coincide with boundaries delineating the interior of each grid are treated as interior cells. We classify ghost cells into three categories: (1) same-level ghost cells that coincide with the interior of another grid on the same level appear white in the figure, (2) physical ghost cells that lie outside of the computational domain appear dark gray in the figure, and (3) child-level ghost cells that coincide with the interior of grids on the parent level appear light gray in the figure. Same-level ghost zones are initialized to the state of the interior of the coincident grid. Physical ghost cells are initialized according to user-specified boundary conditions, the code provides three physical boundary options: constant extrapolation, reflecting, or periodic. The initialization of parent-level ghost cells is carried out as a part of the “Grid Adapt” procedure which will be discussed below.

The “Grid Adapt” procedure determines the arrangement of and initialization of a new AMR hierarchy on a given level that tracks the evolution of flow features in the solution. A user-specified truncation error-estimation procedure is applied to each grid that is one level coarser. AstroBEAR uses either the maximum absolute value of the primitive vector gradient or Richardson extrapolation (Berger & Oliger 1984; Berger & Colella 1989) as the options for error estimation. Zones on the parent grids with estimated error greater than a user-specified tolerance are flagged for refinement. We employ the patchwise clustering algorithm of Berger & Rigoutsos (1991) to determine the arrangement of refined grid patches that optimally overlays all of the zones flagged for refinement. For easier parallel implementation, we require that each grid has only one parent. In Figure 5, the new arrangement of grid patches on the first refinement level with interior boundaries and ghost boundaries delineated by solid and dashed lines, respectively. The regions coinciding with the previous arrangement of AMR patches are shaded in gray. Interior regions of the new grid arrangement that coincide with interior regions of the previous patchwork of grids on the same level are initialized by copying the field values from the previous grids. The previous patchwork of grids on this level are then released from memory. The ghost zones, and interior zones that do not coincide with the interior of the previous grid patches are initialized from the

The regions coinciding with the previous arrangement of AMR patches are shaded in gray. Interior regions of the new grid arrangement that coincide with interior regions of the previous patchwork of grids on the same level are initialized by copying the field values from the previous grids. The previous patchwork of grids on this level are then released from memory. The ghost zones, and interior zones that do not coincide with the interior of the previous grid patches are initialized from the
parent grid via a prolongation operator. The code prolongs the cell-centered conserved fields using interpolation from the parent grid. The cell-face grid of magnetic field components are initialized using a divergence-preserving prolongation operator in order to preserve the integrity of the CT update procedure. We will discuss this operator in detail in Section 4.2.

The “integrate” step advances the solution on each grid on the given level using one of the integration procedures discussed in Section 2 as specified by the user. The time-adaptive nature of the AMR engine imposes the difficulty that boundary information from parent grids is not available during each step of child grid integration cycle. As shown in Figure 3 for a refinement ratio of 2, child ghost zones are temporally synchronized with their parent grids only every other time cycle. To accommodate this, we incorporate an “extended” arrangement of ghost cells. Each refined grid carries a strip of ghost zones that extends the width \( r \times m_{bc} \) cells beyond the interior of the grid, where \( r \) is the refinement ratio and \( m_{bc} \) is the number of ghost cells required by the integration stencil. The MUSCL–Hancock reconstruction operator (Section 2.1) without direction splitting requires two ghost cells, yielding \( m_{bc} = 2 \). An extra ghost zone is required in multidimensional problems when using direction splitting and CT in order to compute the flux components necessary to compute the electromagnetic force (EMF) at the grid corners, yielding \( m_{bc} = 3 \). The Runge–Kutta temporal integration method (Section 2.3) has been implemented via additional rows of ghost cells to fully update the interior region with \( m_{bc} = 4 \).

The update of the extended region of ghost cells is illustrated in Figure 6 for a refinement ratio \( r = 2 \). The initial representation of the field is at time \( t \). The first integration cycle carries all cells interior to the first ring of \( m_{bc} \) ghost zones shaded in dark gray forward in time from \( t \) to \( t + dt/2 \). On the second integration cycle, the interior cells shaded in dark gray are carried forward in time from \( t \) to \( t + dt/2 \). The second integration step can be carried out because the outermost ring of \( m_{bc} \) ghost zones are outside of the domain of influence on the interior cells.

The “Synchronize to Parent” call carries forward the synchronization of the solution on a given level to its parent grid. The synchronization procedure is composed of two steps: application of coarse to fine level refluxing and restriction of the solution. In the refluxing step, a spatiotemporal average of the flux from child grid interfaces along coarse/fine grid boundaries is compared to the flux as computed during the integration of the coarse grid. A correction is applied to the parent grid so that the effective flux across this boundary is equal to that as computed on the child grid. The restriction step calls for the coarsening and injection of field data from the interior of the child grids into their respective parent grids. We employ volume-weighted average restriction of cell-centered conserved fields. The restriction of face-centered magnetic field components requires special attention to maintain the divergence-free property of the magnetic field which will be discussed in the following section.

4. DIVERGENCE-PRESERVING RESTRICTION AND PROLONGATION OPERATORS

In Section 2.4, we discussed the importance of satisfying the \( \nabla \cdot \mathbf{B} = 0 \) in order to maintain the integrity of the numerical solution to the MHD equations and demonstrated an adaptation to Godunov-based schemes that preserves the divergence of the magnetic field throughout the calculation. The use of AMR imposes an additional challenge that must be overcome to maintain the divergence of the magnetic field throughout the calculation. Specifically, the restriction step, which maintains the consistency on coarse levels with the finer levels of refinement in the grid, and the prolongation step, which initializes coarse representations of the solution to finer grids in regions that have been flagged for refinement must not introduce divergence errors into the solution. The volume average and bilinear interpolation procedures utilized for the restriction and prolongation of the cell-centered conserved fields cannot be adapted to operate on the grid-edge magnetic fields in a divergence-preserving manner. Two approaches to the divergence-preserving prolongation have emerged. Balsara (2001) has generalized the divergence-free reconstruction procedure of Balsara & Spicer (1999) to devise a piecewise quadratic interpolant that is divergence preserving. Li & Li (2004) present an adaptation of Balsara’s procedure that simplifies its implementation for problems involving arbitrary refinement ratios. Tóth & Roe (2002) devised a prolongation procedure by solving an algebraic system which enforces the maintenance of the volume-averaged curl and divergence over a coarse grid cell.

In the following subsections, we present the restriction and prolongation formulae employed in our code. In particular, we present the prolongation procedure of Tóth & Roe (2002) in a less compact form than that of the original authors which are more readily transcribed into computer code.
Figure 7. Illustration of the locations on a two-dimensional grid of the staggered electric and magnetic field components with a refined grid, delineated with dotted lines, that has been embedded within one coarse grid cell, delineated by solid lines.

To simplify implementation, our code only allows refinement ratios of two between successive levels of refinement.

4.1. Restriction

Figure 7 illustrates the locations of the magnetic field components utilized by the CT update procedure where a refined grid (denoted with dotted lines) has been embedded within one coarse grid cell (denoted with solid lines) for a two-dimensional calculation. We denote the magnetic field components on coarse grid faces as \(\beta\), on refined cell faces that are coincident with coarse grid faces (the exterior faces) as \(b\), and on refined cell faces that do not coincide with coarse grid faces (the interior faces) as \(B\). We have adapted a short hand notation where the first, second, and third character in the superscript to the refined grid magnetic field components either represent the location above (+), below (−), or aligned with (0) the center of the coarse cell in each of the \(x\), \(y\), and \(z\) directions.

We require that the fine-to-coarse grid synchronization step maintains the cell interface magnetic field in an area-averaged sense

\[
\begin{align*}
\beta_{x,i,j+1/2,k} &= \frac{1}{2} (b_{x,i,j+1/2}^{++} + b_{x,i,j+1/2}^{+-}) \\
\beta_{y,i,j+1/2,k} &= \frac{1}{2} (b_{y,i,j+1/2}^{+\pm} + b_{y,i,j+1/2}^{-\pm}) \\
\beta_{z,i,j+1/2,k} &= \frac{1}{2} (b_{z,i,j+1/2}^{+\pm} + b_{z,i,j+1/2}^{-\pm})
\end{align*}
\]

in two dimensions, and

\[
\begin{align*}
\beta_{x,i,j+1/2,k} &= \frac{1}{4} (b_{x,i,j+1/2}^{++} + b_{x,i,j+1/2}^{+-} + b_{x,i,j+1/2}^{-+} + b_{x,i,j+1/2}^{--}) \\
\beta_{y,i,j+1/2,k} &= \frac{1}{4} (b_{y,i,j+1/2}^{+\pm} + b_{y,i,j+1/2}^{-\pm} + b_{y,i,j+1/2}^{+\pm} + b_{y,i,j+1/2}^{-\pm}) \\
\beta_{z,i,j+1/2,k} &= \frac{1}{4} (b_{z,i,j+1/2}^{+\pm} + b_{z,i,j+1/2}^{-\pm} + b_{z,i,j+1/2}^{+\pm} + b_{z,i,j+1/2}^{-\pm})
\end{align*}
\]

in three dimensions. Equations (41) and (42) must be satisfied while preserving the divergence of the magnetic field along coarse/fine grid boundaries. Simultaneous satisfaction of these properties along coarse/fine grid interfaces and this restriction condition are accomplished by applying a suitable restriction of the electric field from fine to coarse grids before performing the CT update (Equations (31)) on the coarse level. Manipulation of Equations (34) subject to the constraint provided by Equations (41) and (42) at all times yield the desired electric field restriction operator as

\[
E_{t}^{i+1/2} = \frac{1}{2} \left( E_{t}^{i+1/2} + E_{t}^{i+3/4} \right)
\]

in two dimensions, and

\[
\begin{align*}
E_{x,i,j+1/2,k-1/2} &= \frac{1}{4} \left( e_{x,i,j+1/2,k-1/2}^{+} + e_{x,i,j+1/2,k-1/2}^{-} + e_{x,i,j+1/2,k-1/2}^{+\pm} + e_{x,i,j+1/2,k-1/2}^{-\pm} \right) \\
E_{y,i,j+1/2,k-1/2} &= \frac{1}{4} \left( e_{y,i,j+1/2,k-1/2}^{+} + e_{y,i,j+1/2,k-1/2}^{-} + e_{y,i,j+1/2,k-1/2}^{+\pm} + e_{y,i,j+1/2,k-1/2}^{-\pm} \right) \\
E_{z,i,j+1/2,k-1/2} &= \frac{1}{4} \left( e_{z,i,j+1/2,k-1/2}^{+} + e_{z,i,j+1/2,k-1/2}^{-} + e_{z,i,j+1/2,k-1/2}^{+\pm} + e_{z,i,j+1/2,k-1/2}^{-\pm} \right)
\end{align*}
\]

in three dimensions, where \(E\) is the electric field on the coarse grid and \(e\) is electric field of the refined grid. In Figure 3, note that level 0 is advanced by an increment from \(t\) to \(t + dt\) using the electric field computed at time \(t + dt/2\) in one step, while the next child level, level 1, is integrated by the same increment in two steps using the electric field at time \(t + dt/4\) to integrate from \(t\) to \(t + dt/2\) and electric field at time \(t + 3dt/4\) to integrate from \(t + dt/2\) to \(t + dt\). The temporal averaging of the electric field is necessary due to the temporal refinement capability of the code. The time averaging in Equations (43) and (44) ensures that the evolution of the magnetic field across level boundaries remains divergence-free and consistent in the sense of Equations (41) and (42).

4.2. Prolongation

The prolongation step initializes a newly refined grid from its parent grid that is coarser by one level of refinement. The prolongation of the face-centered magnetic field is carried out in two steps. In the first stage, the exterior faces that coincide with the edges of an already refined grid block are set by copying the field values from the coincident face of the already refined grid. The exterior faces that do not coincide with the edges of an already refined region are computed via a bilinear interpolation of the coarse representation of the field given by

\[
\begin{align*}
b_{x}^{++} &= \beta_{x,i \pm 1/2,j} + \frac{1}{2} \delta_{y} \beta_{y} \\
b_{x}^{+-} &= \beta_{x,i \pm 1/2,j} - \frac{1}{2} \delta_{y} \beta_{y} \\
b_{x}^{-+} &= \beta_{x,i \pm 1/2,j} + \frac{1}{2} \delta_{y} \beta_{y} \\
b_{x}^{--} &= \beta_{x,i \pm 1/2,j} - \frac{1}{2} \delta_{y} \beta_{y}
\end{align*}
\]
in two dimensions, and

\[
\begin{align*}
b_{x}^{\pm} & = \beta_{x,i \pm 1/2,j,k} + \frac{1}{2}(\delta_{i} \beta_{x} + \delta_{j} \beta_{x}) \\
b_{y}^{-} & = \beta_{y,i,j + 1/2,k} + \frac{1}{2}(\delta_{j} \beta_{y} + \delta_{k} \beta_{y}) \\
b_{z}^{+} & = \beta_{z,i,j + 1/2,k} + \frac{1}{2}(\delta_{j} \beta_{z} + \delta_{k} \beta_{z}) \\
b_{x}^{\pm-} & = \beta_{x,i \pm 1/2,j,k-1} + \frac{1}{2}(\delta_{i} \beta_{x} - \delta_{k} \beta_{x}) \\
b_{y}^{+} & = \beta_{y,i,j + 1/2,k-1} + \frac{1}{2}(\delta_{j} \beta_{y} - \delta_{k} \beta_{y}) \\
b_{z}^{-} & = \beta_{z,i,j + 1/2,k-1} + \frac{1}{2}(\delta_{j} \beta_{z} - \delta_{k} \beta_{z}) \\
b_{x}^{\pm+} & = \beta_{x,i \pm 1/2,j,k+1} + \frac{1}{2}(\delta_{i} \beta_{x} - \delta_{k} \beta_{x}) \\
b_{y}^{-} & = \beta_{y,i,j + 1/2,k+1} + \frac{1}{2}(\delta_{j} \beta_{y} - \delta_{k} \beta_{y}) \\
b_{z}^{+} & = \beta_{z,i,j + 1/2,k+1} + \frac{1}{2}(\delta_{j} \beta_{z} - \delta_{k} \beta_{z})
\end{align*}
\]

in three dimensions, where we compute the spatial jumps using the same slope limiter (Equations (11)–(13)) as the base scheme:

\[
\begin{align*}
\delta_{i} \beta_{x} = & \text{LIMITER}(\beta_{x,i+1,j-1/2,k-1/2} - \beta_{x,i-1/2,j-1/2,k-1/2}) \\
\delta_{j} \beta_{x} = & \text{LIMITER}(\beta_{x,i-1/2,j+1/2,k-1/2} - \beta_{x,i+1/2,j-1/2,k-1/2}) \\
\delta_{k} \beta_{x} = & \text{LIMITER}(\beta_{x,i-1/2,j-1/2,k+1} - \beta_{x,i+1/2,j-1/2,k-1})
\end{align*}
\]

In the second stage of the prolongation procedure, we interpolate from the exterior faces of the refined grid cell to the interior faces. In three dimensions, the 12 refined intercell faces that are interior to the coarse cell are constructed via an interpolation from the magnetic field components collated at the 24 refined grid exterior face centers that coincide with a coarse cell interface. Following work of Tóth & Roe (2002), we derive the desired interpolation operator by imposing the constraint that the divergence and curl of the magnetic flux be conserved. For readability and conciseness, we illustrate the steps required to derive such an interpolation procedure in detail only for the two-dimensional case. We then present the analogous solution in three dimensions. The volume-averaged divergence, computed via the application of Gauss’s law around the perimeter of the exterior faces is

\[
d_{00} = \frac{1}{2}(\bar{b}_{x}^{++} + \bar{b}_{x}^{+-} - \bar{b}_{x}^{-+} + \bar{b}_{x}^{--} + \bar{b}_{y}^{++} + \bar{b}_{y}^{+-} - \bar{b}_{y}^{-+} - \bar{b}_{y}^{--})
\]

where we have used the bar accent to denote division by the discretization width in the normal direction on the coarse level (see Figure 7), i.e., $\bar{b}_{x} = b_{x}/Ax$ and $\bar{b}_{y} = b_{y}/Ay$. The curl of the field is constructed by bilinear interpolation of the requisite exterior field components to the origin of the refined cell as

\[
c_{z}^{00} = \frac{\Delta y}{2\Delta x} (\bar{b}_{y}^{++} - \bar{b}_{y}^{+-} + \bar{b}_{y}^{-+} - \bar{b}_{y}^{--} - \bar{b}_{y}^{++} - \bar{b}_{y}^{+-} + \bar{b}_{y}^{-+} - \bar{b}_{y}^{--}).
\]

The divergence centered at each of the four refined cell interiors is

\[
\begin{align*}
D_{x}^{00} & = 2(\bar{b}_{x}^{00} - \bar{b}_{x}^{0-} - \bar{b}_{x}^{0+} - \bar{b}_{x}^{-+} - \bar{b}_{x}^{+-} - \bar{b}_{x}^{--} - \bar{b}_{x}^{+-} - \bar{b}_{x}^{++}) \\
D_{y}^{00} & = 2(\bar{b}_{y}^{00} - \bar{b}_{y}^{0-} - \bar{b}_{y}^{0+} - \bar{b}_{y}^{-+} - \bar{b}_{y}^{+-} - \bar{b}_{y}^{--} - \bar{b}_{y}^{+-} - \bar{b}_{y}^{++}) \\
D_{z}^{00} & = 2(\bar{b}_{z}^{00} - \bar{b}_{z}^{0-} - \bar{b}_{z}^{0+} - \bar{b}_{z}^{-+} - \bar{b}_{z}^{+-} - \bar{b}_{z}^{--} - \bar{b}_{z}^{+-} - \bar{b}_{z}^{++})
\end{align*}
\]

and the curl at the origin of the refinement cells implied by the interior field values is

\[
c_{z}^{00} = \frac{\Delta y}{2\Delta x} (\bar{b}_{y}^{00} - \bar{b}_{y}^{0-} - \bar{b}_{y}^{0+} - \bar{b}_{y}^{-+} - \bar{b}_{y}^{+-} - \bar{b}_{y}^{--} - \bar{b}_{y}^{+-} - \bar{b}_{y}^{++}).
\]

The desired interpolation procedure is determined by imposing the condition that each of the refined grid cells contributes equally to the divergence of the refinement region, $d_{00} = D_{x}^{00} = D_{y}^{00} = D_{z}^{00}$, and that the curl implied by the interior faces equals the curl interpolated from the exterior faces, $c_{z}^{00} = C_{z}^{00}$. Because only three of the four divergence conditions are linearly independent, we have a system of four independent linear equations for the four desired interior field values. The solution to the system, in matrix form, is

\[
\begin{align*}
\mathbf{b}_{\text{ext}} & = \begin{bmatrix} \bar{b}_{x}^{--} & \bar{b}_{x}^{-+} & \bar{b}_{x}^{+-} & \bar{b}_{x}^{++} & \bar{b}_{y}^{--} & \bar{b}_{y}^{-+} & \bar{b}_{y}^{+-} & \bar{b}_{y}^{++} & \bar{b}_{z}^{--} & \bar{b}_{z}^{-+} & \bar{b}_{z}^{+-} & \bar{b}_{z}^{++} \end{bmatrix}^T \\
\mathbf{b}_{\text{int}} & = \begin{bmatrix} \bar{b}_{x}^{00} & \bar{b}_{y}^{00} & \bar{b}_{z}^{00} \end{bmatrix}^T
\end{align*}
\]

(52)

In three dimensions, we follow the same procedure to derive the solution for 12 interior refined cell face fields from 24 exterior refined cell face fields. In particular, we have eight equations (seven linearly independent) for the volume-integrated divergence over each of the eight refined cell interiors, $d_{00} = D_{x}^{0-} = D_{x}^{0+} = D_{y}^{0-} = D_{y}^{0+} = D_{z}^{0-} = D_{z}^{0+} = D_{z}^{++}$, and six equations (five linearly independent) by computing the three components of the curl, each at two positions relative to the center of the coarse cell, $c_{z}^{00} = C_{x}^{0} = C_{y}^{0} = C_{z}^{0}$. The solution of this system for the desired interior field values yields the desired prolongation procedure which can be written in the form of Equation (52) as

\[
\begin{align*}
\mathbf{b}_{\text{ext}} & = \begin{bmatrix} \bar{b}_{x}^{--} & \bar{b}_{x}^{-+} & \bar{b}_{x}^{+-} & \bar{b}_{x}^{++} & \bar{b}_{y}^{--} & \bar{b}_{y}^{-+} & \bar{b}_{y}^{+-} & \bar{b}_{y}^{++} & \bar{b}_{z}^{--} & \bar{b}_{z}^{-+} & \bar{b}_{z}^{+-} & \bar{b}_{z}^{++} \end{bmatrix}^T \\
\mathbf{b}_{\text{int}} & = \begin{bmatrix} \bar{b}_{x}^{00} & \bar{b}_{y}^{00} & \bar{b}_{z}^{00} \end{bmatrix}^T
\end{align*}
\]

(53)
5. NUMERICAL TESTS, EXAMPLES, AND DISCUSSION

The AstroBEAR numerical schemes have been tested against a suite of test problems, including the one-dimensional tests of Ryu & Jones (1995) and the two-dimensional tests of Ryu et al. (1995). Except for the failure of the original Marquina flux formulation to converge for problems involving compound MHD wave structures as discussed in Section 2.2, each of the methods recover results that are equivalent to those of earlier researchers barring minor differences due to different levels of numerical truncation error. In the remainder of this section, we present the results of some of these tests. The results of a few popular hydrodynamic shock tubes are presented which illustrate the differences between and limitations of each of the reconstruction and upwinding methods. In these cases, we provide commentary which is intended to provide guidance as to the optimal choice of methods given the expected physical regime of a particular simulation. We also reproduce the results of several of the two-dimensional tests of Ryu et al. (1995) using AMR. These tests demonstrate the success of the divergence-preserving restriction and prolongation operators presented in Section 4.

5.1. One Dimension

Figure 8 shows the density resulting from the Sod (1978) shock tube problem denoted as “test problem 1” in the book by Toro (1999) using several spatial reconstruction methods. The initial left and right Riemann problem states, number of computational zones, and final time of the solution are presented in Table 2. In all three cases, the Runge–Kutta temporal integration and the flux formulation of Roe was used. We note that the result of this test is not very sensitive to the choice of temporal reconstruction method or flux formulations. The shock tube consists from left to right of a backward propagating rarefaction, a contact discontinuity, and a forward propagating shock wave. The numerical diffusion caused by the truncation error of each of the spatial reconstruction methods is readily apparent in the figure. We characterize the diffusion of the method by the width of the contact discontinuity. The three methods are presented in order of increasing diffusion. The least diffusive method, PPM, resolves the contact across three zones, and the most diffusive method, MUSCL, resolves the contact across eight zones. We note that the diffusion of the PPM and MUSCL methods may be improved by choosing a more compressive slope limiter than the MINMOD limiter used here at the expense of introducing oscillations near sharp discontinuities in the flow. Naturally, those methods which exhibit the least numerical diffusion are the most accurate. However, the PPM and PHM methods tend to “overshoot” the solution near sharp discontinuities, thereby introducing small oscillations into the solution. Such oscillations may drive numerical pathologies in simulations of flow with very low pressure near discontinuities such as persistently negative pressures. We note that some researchers have managed to reduce such oscillations by introducing additional sources of diffusion to the base scheme (see, for example Section B1 of Mignone et al. 2005). Simulation of astrophysical phenomena involving strong radiative cooling is particularly susceptible to this problem. In such cases, the optimal solution strategy is determined as a tradeoff between the desire for simultaneously low numerical diffusion and low oscillation.

Figure 9 shows the result of the “1-2-0-3” strong expansion shock tube of Einseifld et al. (1991), denoted as “test problem 2” in the book by Toro (1999). The initial left and right Riemann problem states, the number of computational zones, and the final time of the solution are presented in Table 3. This shock tube launches two rarefaction waves: one propagating to the left and one propagating to the right. The upper three panels show the test results utilizing the MUSCL–VL slope spatial and Runge–Kutta temporal reconstruction methods with different flux upwinding procedures. This test problem was designed to illustrate the failure mode common to linearized Riemann solvers evident by the anomalous oscillation in velocity and anomalous rise in specific thermal energy ($E_{th}/\rho$) about the center of the grid in panel 9(a). These anomalies are caused by the addition of small amounts of thermal energy to the solution in regions where the pressure becomes negative. We note that specific thermal energy anomalies such as these can result in a cascade of numerical pathologies in multiphysics simulations involving temperature-dependent microphysics. Einseifld et al. (1991) showed that the more diffusive HLLE Riemann solver, under certain conditions, guarantees pressure positivity and therefore reduces such anomalies. Gardiner & Stone (2005) have also demonstrated success applying the HLLE solver only in those regions where the linearized solver produces negative pressure or density. In panel 9(b) we show that application of the Marquina flux formulation also resolves the anomalous behavior. In Section 2.2, we presented an adaptation of the Marquina flux formulation which is better suited to magnetized flow involving compound wave structures. The result of this calculation utilizing the adapted flux formulation (panel 9(c)) shows that the adaptation retains the desirable features of the Marquina flux for this test problem.

In Figure 9, we reproduce the MHD shock tube test from Figure 2(a) of Ryu & Jones (1995). Following the notation of Ryu & Jones (1995), we denote the orientation angle of the magnetic field, $\Psi = \tan^{-1}(B_x/B_y)$. The initial left and right Riemann problem states, number of computational zones and final time of the solution is presented in Table 3. This Riemann problem demonstrates the ability of the code to correctly capture a number of MHD discontinuities. The discontinuities
in the flow from left to right are (1) fast shock, (2) rotational discontinuity, (3) slow shock, (4) contact discontinuity, (5) slow shock, (6) rotational discontinuity, and (7) fast shock. We note that the PHM spatial reconstruction employed for this test generally results in a level of truncation error slightly lower than that of Ryu & Jones (1995) with a slightly more diffusive result about of rotational discontinuities due to the lack of a rotational discontinuity steepening procedure like that of Equations (2.96)–(2.98) of Ryu & Jones (1995). We note that when applied to this particular problem, the MUSCL and PPM spatial reconstruction methods result in lower and higher truncation error, respectively, in a manner that is similar to Sod problem discussed earlier.

5.2. Two Dimensions

In this section, we present the results of several two-dimensional simulations to demonstrate the robustness of the divergence-preserving scheme in AMR applications. The simulations presented in this section which utilize the AMR functionality of the code apply an AMR hierarchy of 3 levels using a refinement ratio of 2 between levels.

We begin with the problem of the advection of a weak magnetic field loop following the prescription given in Gardiner & Stone (2005) and Gardiner & Stone (2008). The initial conditions given in Table 5. The initial face-centered magnetic field is generated from the analytic prescription of the vector potential \( \mathbf{B} = \nabla \times \mathbf{A} \) at cell corners using a centered difference formula on a grid extending from \( 0 \leq L_x \leq 2, 0 \leq L_y \leq 1 \) with resolution \( 2^N \times N \). In the left column of Figure 11, we show magnetic field lines (red) over a grayscale representation of the current density \( \mathbf{J} = \nabla \times \mathbf{B} \) for the initial condition (top) and after the advection of the loop through the periodic domain at \( t = 1 \) for both the Runge–Kutta (center) and MUSCL–

| Table 2 |
|-----------------|-----------------|
| **Sod Shock Tube Parameters** | Left, \( x < 0.5 \) | Right, \( x \geq 0.5 \) |
| \( \rho \) | 1 | 0.125 |
| \( v_x \) | 0 | 0 |
| \( P \) | 1 | 0.1 |
| \( \gamma \) | 1.4 | 1.4 |
| Grid zones | 128 |
| CFL | 0.9 (0.5 for PPM) |
| \( \delta_{final} \) | 0.25 |
Hancock temporal integration (bottom) with a resolution of $N = 128$. In both cases, monotonized-centered limited linear spatial reconstruction with the flux function of Roe were used. The problem contains an initially singular current sheet along the surface and a singular spike at the center which is very sensitive to error in the evolution of the magnetic field. The propagation of these singular features through the grid serves as a very stringent test of the MHD update algorithm. We find that both methods maintain the correct location of these singularities and maintain magnetic field contours that are smooth and nearly symmetric about the current spike.

The top and center portions of the right column of Figure 11 show the evolution of the spatially averaged magnetic energy normalized to the initial analytic value $B_0$ for the MUSCL–Hancock and Runge–Kutta integration approaches, respectively, for different grid resolutions. We find that both methods show comparable accuracy and the expected convergence properties for this quantity. However, the direction split MUSCL–Hancock scheme shows inexact evolution of the axial component of the magnetic field $B_z$ and that this error converges slowly as shown in the lower right panel of Figure 11 whereas the unsplit Runge–Kutta scheme maintains $B_z = 0$ exactly to machine precision. This error is due to the failure of the direction split scheme to produce the an exactly divergence-free representation of the magnetic field for the time-centered predictor state as discussed in the last paragraph of Section 2.4. In particular, as noted by Gardiner & Stone (2008), the evolution of the
Figure 11. Left: Field loop advection magnetic field lines (red) over a grayscale representation of the current density $\mathbf{J} = \nabla \times \mathbf{B}$ for the initial condition (top) and after the advection of the loop through the periodic domain at $t = 1$ for both the Runge–Kutta (center) and MUSCL–Hancock temporal integration (bottom). Right: Evolution of the spatially averaged magnetic energy normalized to the initial analytic value $B_0$ for the MUSCL–Hancock (top) and Runge–Kutta (center) integration. The grid resolutions shown correspond to $N = 32$, 64 and 128 for the dash-dot, dash and solid lines, respectively. Evolution of the spatial average of the normalized axial component of the magnetic field $|B_z|$ for the direction split MUSCL–Hancock integrator (bottom). The plot shows decreasing error for the three resolutions $N = 32$, 64 and 128.

(A color version of this figure is available in the online journal.)

The $z$-component of the magnetic induction equation reduces to $\frac{\partial B_z}{\partial t} = v_z(\partial B_x/\partial x + \partial B_y/\partial y)$ so that with finite $v_z$ the exact evolution of $B_z$ will only be maintained for unsplit schemes with exactly divergence-free reconstruction.

The next set of simulations show the propagation of a cylindrical, supersonic cloud through a magnetized medium where the magnetic field is oriented parallel to the propagation of the cloud (Figure 12, top), perpendicular to the propagation of the cloud (Figure 13, middle), and $45^\circ$ to the propagation of the cloud (Figure 13, bottom). These simulations have been carried out using PHM spatial reconstruction, MUSCL–Hancock temporal reconstruction, the adapted Marquina flux and the CT evolution of the magnetic field of Ryu et al. (1995). Each of these simulations employ constant extrapolation conditions along each boundary. The AMR level with the finest resolution achieves an effective resolution of 32 cells per cloud radius. The initial cloud density is smoothed linearly to that of the ambient environment and the velocity is smoothed over the outer four computational cells of the cloud. The figures show the result of each simulation at two evolutionary times, $t = 2t_c$ in the left panel and $t = 4t_c$ in the right panel where $t_c = 2r_c \sqrt{\chi}/v_c$ is the cloud crushing time where $r_c$ is the cloud radius, $v_c$ is the initial cloud speed and $\chi$ is the density contrast of the cloud against the ambient environment (see Jones et al. 1996.
Figure 12. Simulation results of a supersonic cylindrical cloud moving through a magnetized medium with initial magnetic field oriented parallel, perpendicular, and diagonal to the propagation of the cloud from top to bottom. The logarithm of the density distribution is presented in grayscale, red lines delineate the magnetic field lines, and blue lines delineate regions of AMR-enhanced resolution. (A color version of this figure is available in the online journal.)

The density distribution is presented in grayscale, red lines delineate the magnetic field lines and blue lines delineate regions of AMR-enhanced resolution. We note that the turbulent wake behind the 45° cloud and the associated early onset of tearing mode instability and magnetic reconnection pose some degree of numerical difficulty which requires the use of the more accurate and robust Runge–Kutta method over the faster MUSCL–Hancock method. The simulations use the same initial physical parameters as the simulations presented in Ryu et al. (1998) and Jones et al. (1996) which are presented in Table 5.

Because the shocked cloud simulations presented here do not utilize a moving mesh, the final time \( t = 4t_{bc} \) of the simulations is somewhat earlier than that of the earlier works \( t = 6t_{bc} \) of Ryu et al. (1995) and Jones et al. (1996). The density distribution and magnetic field lines at \( t = 2t_{bc} \) show agreement with results of the earlier calculations at the same evolutionary time. As pointed out in Ryu et al. (1995) and Jones et al. (1996), the nonlinear evolution of the cloud depends sensitively on the exact initial perturbations which develop out of the geometric mismatches between the cloud and the computational grid. The agreement of the AMR simulations presented here with the earlier works demonstrates the robustness and accuracy of the divergence-preserving restriction and prolongation procedures presented in Section 4.

In Figure 13, we reproduce the light cylindrical MHD jet simulation of Ryu et al. (1995) and Lind et al. (1989). A jet
with a top-hat profile is injected into a uniformly magnetized environment by imposing the physical jet conditions presented in Table 7 inside of $r \leq r_j$ along the $z = 0$ boundary. The symmetry of the problem dictates the use of reflecting boundary conditions along the $r = 0$ boundary. All other boundaries utilize constant extrapolation. The simulation was carried forward on an AMR hierarchy with four levels of refinement utilizing the PHM spatial integration method with Runge–Kutta temporal integration, the Roe flux upwinding method, and the CT magnetic field evolution of Ryu et al. (1998). The finest AMR level has a resolution of 32 cells per jet radius. The figures show the result of the simulation at 5 evolutionary times, $t = 2.43, 6.57, 10.62, 14.76,$ and 18.00. Note these are the same evolutionary times shown in the results of Ryu et al. (1998).
constant extrapolation and the wind state is held constant in the environment. The upper and lower boundaries of the cylindrical clouds with 100 times the density of the ambient environment are given in Table 8. Figures 14 and 15 show the results of the simulations at several evolutionary stages \( (t = 0, 92, 184, \text{ and } 369 \text{ yr}) \). The simulations were carried forward using MUSCL spatial reconstruction, Runge–Kutta temporal integration, the adapted Marquina flux upwinding and the CT method of Ryu et al. (1995). An operator split energy sink source term is used to include the effects of radiative energy loss via atomic line cooling using the cooling function of Dalgarno & McCray (1972). The density distribution is presented in grayscale, red lines delineate the magnetic field lines and blue lines delineate regions of AMR-enhanced resolution. Figure 14 shows the case where the magnetic field oriented parallel to the direction of wind and Figure 15 shows the case where the ambient environment is threaded with a magnetic field that is oriented perpendicular to the direction of an unmagnetized wind. The turbulent nature of the flow pattern that emerges in simulations, compounded sock compression ratios as high as \( \sim 12 \) which are achieved via radiative losses results in the development of persistently converging flow. Such flows are particularly problematic for codes that do not maintain the solenoidal constraint on the magnetic field as local divergences tend to accumulate in highly compressed regions of the flow (Balsara & Kim 2004). These simulations, therefore, demonstrate the robustness of the methods described in this paper for the simulation of such flows.

### 5.3. Three Dimensions

Circulary polarized Alfvén waves are an exact nonlinear solution to the MHD equations. We follow an approach similar to that of Gardiner & Stone (2008) and Tóth (2000) by rotating the one-dimensional prescription of the problem (Table 9) onto a three-dimensional periodic grid of size \( \sqrt{6}/2 \times \sqrt{6}/2 \times \sqrt{6}/2 \) with resolution \( N \times 2N \times 2N \) via the rotation

\[
\begin{bmatrix}
 x
 y
 z
\end{bmatrix} = \begin{bmatrix}
 \cos(\alpha) \cos(\beta) & -\sin(\alpha) & -\cos(\alpha) \sin(\beta)
 \\
 \sin(\alpha) \cos(\beta) & \cos(\alpha) & -\sin(\alpha) \sin(\beta)
 \\
 \sin(\beta) & 0 & \cos(\beta)
\end{bmatrix} \begin{bmatrix}
 x_1
 y_1
 z_1
\end{bmatrix}
\]

(54)

where \( \alpha = \arcsin(1/\sqrt{3}) \) and \( \beta = \arcsin(1/\sqrt{6}) \) so that the initial state is periodic with the grid and the wavevector points along the diagonal of the computational domain. Solutions have been computed for one crossing time \( (t = 1) \) using monotonized-centered limited linear spatial reconstruction, the Roe flux function, and both the unsplit Runge–Kutta and direction split MUSCL–Hancock temporal integrators. The left panel of Figure 16 shows the convergence of the volume-averaged norm of the \( L_1 \) error vector as measured with respect
Figure 14. Shock propagation through multiple clouds with magnetic field oriented parallel to the direction of shock propagation at evolutionary time $t = 0, 92.184$, and 369 yr. The logarithm of the density distribution in cm$^{-3}$ is presented in grayscale, red lines delineate the magnetic field lines, blue lines delineate regions of AMR-enhanced resolution and the cloud diameter is used as the length unit.

(A color version of this figure is available in the online journal.)
Figure 15. Shock propagation through multiple clouds with magnetic field oriented perpendicular to the direction of shock propagation at evolutionary time $t = 0, 92, 184, \text{ and } 369 \text{ yr}$. The logarithm of the density distribution in cm$^{-3}$ is presented in grayscale, red lines delineate the magnetic field lines, blue lines delineate regions of AMR-enhanced resolution and the cloud diameter is used as the length unit.

(A color version of this figure is available in the online journal.)
Figure 16. Circularly polarized Alfvén wave in three dimensions. Left: convergence of the $L_1$ error for the direction split MUSCL–Hancock temporal reconstruction (dotted) and the unsplit Runge–Kutta temporal reconstruction (solid). Right: transverse component of the magnetic field (right) after one grid crossing ($t = 1$) for grid resolutions $N = 8$ (dot), 16 (dash-dot), 32 (dash), and 64 (solid).

Table 9

| Parameter | Value |
|-----------|-------|
| $\rho$    | 1     |
| $v_1$     | 0     |
| $v_2$     | $\alpha \sin(2\pi x_1)$ |
| $v_3$     | $\alpha \cos(2\pi x_1)$ |
| $P$       | 1     |
| $B_1$     | 1     |
| $B_2$     | $\alpha \sin(2\pi x_1)$ |
| $B_3$     | $\alpha \cos(2\pi x_1)$ |
| $a$       | 0.1   |
| $R$       | 0.3   |
| $\gamma$  | 5/3   |

$\Delta t = 1/(3N)$ (MUSCL–Hancock)
$\Delta t = 1/(6N)$ (Runge–Kutta)

for grid resolutions of $N = 8$, 16, 32, and 64. Consistent with the results of other authors on this test problem (Tóth 2000; Gardiner & Stone 2005, 2008), the solution error arises mainly from the magnetic field components that are transverse to the wave propagation direction. The right panel of Figure 16 shows the steady convergence of the amplitude of the transverse components of the magnetic $B_T = \sqrt{B_{x1}^2 + B_{z1}^2}$ field after rotating back into the unrotated system.

$$\epsilon = \frac{1}{4N^3} \left( \sum_{i,j,k} \left| q(i,j,k,nq) - q(i,j,k,nq) \right|^2 \right)^{1/2}$$

for the unsplit Runge–Kutta case. Both integration techniques converge in a manner consistent with the expected second-order accuracy with least-squared power law induces $\epsilon \propto N^{-1.92}$ for the MUSCL–Hancock scheme and $\epsilon \propto N^{-2.49}$ for the Runge–Kutta scheme.

6. CONCLUSION

The staggered grid CT schemes described in this paper enable the application of HRSC methods to magnetized flow.

In this paper, we have demonstrated that a wide cross section of HRSC schemes for general conservation laws may be adapted for magnetized flow while preserving the divergence-free constraint on the magnetic field topology exactly by conserving the surface integral of magnetic flux over each computational cell in an upwind fashion. The use of such schemes on multiresolution AMR grids is encumbered by the requirement that the prolongation and restriction steps preserve the divergence-free topology of the magnetic fields. In this paper, we have described the application of prolongation and restriction operators which maintain such topologies to machine precision.

The numerical schemes discussed here have been implemented and tested in the AstroBEAR AMR code. The code utilizes a modular design, enabling the user to choose from various methodologies to tailor the numerical integration strategy to the requirements of the application at hand. The robustness of this approach to HRSC MHD on AMR grid structures, and relative advantages of the various numerical schemes implemented in the code are demonstrated in the context of several numerical example problems. The description of the numerical schemes presented in this paper provides a concise recipe for their implementation which will enable the reproduction of these outcomes by other researchers and the interpretation of future works derived from the AstroBEAR code.

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APPENDIX A

CYLINDRICAL AXISYMMETRY

The conservative update procedure for integrating the ideal MHD equations (Section 2) may be readily extended to the case of cylindrical axisymmetric flows via a change in the co-
ordinate variables, all differential terms in $\theta$ vanish. We handle the geometric source term, $S$, separately from the conservative update of the homogeneous part of the system using an operator split approach. The source term step $\frac{\partial S}{\partial t} = S$ is integrated via a fourth-order Rosenbrock integration scheme for stiff systems of ordinary differential equations (Press et al. 1992) using an adaptive time step to maintain the accuracy of the solution to a user-specified level (usually 1 part in 10$^4$). Symmetry dictates that such simulations be carried out in a half-meridional plane ($r > 0$) and that reflecting boundary conditions be applied along the $r = 0$ plane.

The CT of Section 2.4 and prolongation/restriction of Section 4 of grid-interface magnetic field components, $B_r$ and $B_z$, may also be readily adapted for cylindrical axisymmetric flow. Readers interested in the extension to more complicated magnetic field takes the form

$$\nabla \cdot B = \frac{1}{r} \frac{\partial B_r}{\partial r} + \frac{\partial B_z}{\partial z} = 0,$$  \hspace{1cm} (A2)

and the component of the curl of the magnetic field orthogonal to the symmetry plane takes the form

$$[\nabla \times B]_\theta = \frac{\partial B_z}{\partial z} - \frac{\partial B_r}{\partial r}.$$  \hspace{1cm} (A3)

These operators take the same form as the Cartesian case under the change of variable $B_r \rightarrow r B_r$. Therefore, CT update formulae of Section 2.4 and the prolongation and restriction formulae of Section 4 are written for the case of axisymmetry by replacing $(f_x, f_y) \rightarrow (f_r, f_\theta)$, $E_z \rightarrow E_\theta$, $(B_x, B_y) \rightarrow (r B_r, B_z)$, and $(b_x, b_y) \rightarrow (r b_r, b_z)$. The CT integration procedure (Equations (31)) can be written for axisymmetric geometry by replacing $B_r, B_z, E_\theta, r, \theta$ vanishing. We handle the geometric source term, $S$, separately from the conservative update of the homogeneous part of the system using an operator split approach. The source term step $\frac{\partial S}{\partial t} = S$ is integrated via a fourth-order Rosenbrock integration scheme for stiff systems of ordinary differential equations (Press et al. 1992) using an adaptive time step to maintain the accuracy of the solution to a user-specified level (usually 1 part in 10$^4$). Symmetry dictates that such simulations be carried out in a half-meridional plane ($r > 0$) and that reflecting boundary conditions be applied along the $r = 0$ plane.

**APPENDIX B**

**PSEUDO-CODE LISTING FOR THE AMR ENGINE**

```fortran
SUBROUTINE AMR(level, dt)
  IF(level = 0)
    nsteps = 1
    Set Ghost(level)
  ELSE
    nsteps = r
  END IF
  DO n = 1, nsteps
    Distribute(level)
    IF(level < MaxLevel)
      Grid Adapt(level + 1)
      Set Ghost(level+1)
    ELSE
      END IF
    END DO
END SUBROUTINE AMR
```

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