CLASSIFICATION PROBLEMS
AND MIRROR DUALITY

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In this paper we make precise, in the case of rank 1 Fano 3-folds, the following program: Given a classification problem in algebraic geometry, use mirror duality to translate it into a problem in differential equations; solve this problem and translate the result back into geometry.

The paper is based on the notes of the lecture series the author gave at the University of Cambridge in 2003. It expands the announcement [Go], providing the background for and discussion of the modularity conjecture.

We start with basic material on mirror symmetry for Fano varieties. The quantum D-module and the regularized quantum D-module are introduced in section 1. We state the mirror symmetry conjecture for Fano varieties. We give more conjectures implying, or implied by, the mirror symmetry conjecture. We review the algebraic Mellin transform of Loeser and Sabbah and define hypergeometric D-modules on tori.

In section 2 we consider Fano 3-folds of Picard rank 1 and review Iskovskikh’s classification into 17 algebraic deformation families. We apply the basic setup to Fano 3-folds to obtain the so called counting differential equations of type D3. We introduce DN equations as generalization of these, discuss their properties and take a brief look at their singularities.

In section 3, motivated by the Dolgachev-Nikulin-Pinkham picture of mirror symmetry for K3 surfaces, we introduce \((N,d)\) -modular families; these are pencils of K3 surfaces whose Picard-Fuchs equations are the counting D3 equations of rank 1 Fano 3-folds. The \((N,d)\) -modular family is the pullback of the twisted symmetric square of the universal elliptic curve over \(X_0(N)^W\) to a cyclic covering of degree \(d\).

Our mirror dual problem is stated in section 4: For which pairs \((N,d)\) is it possible for the Picard-Fuchs equation of the corresponding \((N,d)\) -modular family to be of type D3? Through a detailed analysis of singularities, we get a necessary condition on \((N,d)\), bringing the list down to 17 possibilities.

Identifying certain odd Atkin-Lehner, weight 2, level \(N\) Eisenstein series (that appear in section 5) with the sections of the bundle of relative differential 2-forms in our modular family, we compute the corresponding Picard-Fuchs equations and show them to be of type D3, recovering the matrix coefficients.

It turns out that the pairs \((N,d)\) that we get are exactly those for which there exists a rank 1 Fano 3-fold of index \(d\) and anticanonical degree \(2d^2N\). The Iskovskikh classification is revisited in section 6. We sketch a proof that the matrices we have recovered in section 5 via modular computations are, up to a scalar shift, the counting matrices of the corresponding Fanos.

In section 7 we briefly discuss further classification problems to which our approach may be applied.

1. Conjectures on Mirror Symmetry for Fano varieties.

1.1. The big picture and the small picture. There exist two different approaches to differential systems built from Gromov-Witten invariants of a variety. The full Frobenius manifold underlies vector bundles with connections whose construction requires knowledge of the big quantum cohomology and therefore of the whole system of multiple-pointed correlators (see chapter 2 in [Ma]). On the other hand, if we are content to restrict our study to the divisorial subdirection of the Frobenius manifold, only the small quantum cohomology is needed. It is still a strong invariant of a variety but it only requires knowledge of the three-pointed correlators. For this reason, it is easier to compute. The small quantum
differential system has the additional advantage of being representable as an algebraic D-module on the “Neron-Severi-dual” torus.

1.2. D-modules. Given a smooth scheme \( X/\mathbb{C} \) we denote by \( D_{b,\text{holo}} \) the full subcategory of cohomologically bounded cohomologically holonomic complexes of sheaves of left \( D_X \) modules. For morphisms \( f: X \to Y \) between smooth varieties, the “six operations” exist and provide a convenient language for the constructions that we are going to need. If \( f: X \to Y \) is smooth of relative dimension \( d \), then \( f_* K = Rf_*(K \otimes \omega_X \Omega_{X/Y}[d]) \), so that \( H^{-d}(f_* K) = H^D_{DR}(X/Y, K) \) with its Gauss-Manin connection.

We will need the following notion of pullback: if \( M \) is a flat \( D_Y \) module, then \( f^! M = f^+ M[\dim X - \dim Y] \), where \( f^+ M \) is the naive pullback of \( M \) as module with integrable connection. If \( G \) is a separated smooth group scheme over \( \mathbb{C} \) with group law \( \mu: G \times G \to G \), then the convolution of objects of \( D_{b,\text{holo}}(G) \) is defined by \( (K, L) \to K \ast L = \mu_* (K \boxtimes L) \) where \( K \boxtimes L \) is the external tensor product.

1.3. Three-point correlators. Let \( X \) be a Fano variety. Let \( T_{\text{NS}^\vee} \) be the torus dual to the lattice \( \text{NS}^\vee(X) \). Define a trilinear functional \( \langle \cdot, \cdot, \cdot \rangle \) on \( H(X) \) setting

\[
\langle \alpha, \beta, \gamma \rangle = \sum_{\chi \in \text{NS}^\vee(X)} \langle \alpha, \beta, \gamma \rangle_X \chi = \langle \alpha, \beta, \gamma \rangle_X \chi
\]

where \( \langle \alpha, \beta, \gamma \rangle_X \chi \) is “the expected number of maps” \(^1\) from \( \mathbb{P}^1 \) to \( X \) in the cohomology class \( \chi \) such that 0 maps into a general enough representative of \( \alpha \), 1 maps into a representative of \( \beta \), \( \infty \) maps into a representative of \( \gamma \). The functional \( \langle \cdot, \cdot, \cdot \rangle \) takes values in \( \mathbb{C}[\chi] \).

1.4. Algebra of quantum cohomology. Consider the trivial vector bundle \( \mathcal{H}(X) \) over \( T_{\text{NS}^\vee} \) with fiber \( H(X) \). Extend the Poincare pairing \( \langle \cdot, \cdot \rangle \) to the vector space of its sections \( H(X) \otimes \mathbb{C}[\chi] \).

Raising an index, we turn the trilinear form into a multiplication law on \( H(X) \otimes \mathbb{C}[\chi] : \)

\[
[\alpha \cdot \beta, \gamma] = \langle \alpha, \beta, \gamma \rangle
\]

1.5. Connection (the \( H^{2-} \) direction). Identify elements \( f \) in the lattice \( \text{NS}(X) \) with invariant derivations \( \partial_f \) on \( T_{\text{NS}^\vee} \) by the rule

\[
\partial_f(\chi) = f(\chi) \chi
\]

(In the left-hand side of the formula \( \chi \) is a function on a torus; on the right, it is an element of the lattice \( \text{NS}(X) \) and as such is paired with \( f \)).) Define a connection

\[
\nabla_{T_{\text{NS}^\vee}} : \Omega^0(\mathcal{H}(X)) \to \Omega^1(\mathcal{H}(X))
\]

in the vector bundle \( \mathcal{H}(X) \) by setting for any constant section \( \bar{\alpha} = \alpha \otimes 1 \in H(X) \otimes \mathbb{C}[\chi] \)

\[
\langle \partial_f, \nabla_{T_{\text{NS}^\vee}} \bar{\alpha} \rangle = (f \otimes 1) \cdot \bar{\alpha}
\]

(the derivation \( \partial_f \) is coupled with the vector-valued 1-form \( \nabla_{T_{\text{NS}^\vee}} \bar{\alpha} \) in the LHS).

1.6. Theorem. The connection \( \nabla \) is flat.

This turns the space \( H(X) \otimes \mathbb{C}[\chi] \) into a \( \mathcal{D} = D_{T_{\text{NS}^\vee}} \) module. We will denote it by \( Q \) and call it the quantum D-module.

1.7. The mirror symmetry conjecture states that the solution to the quantum D-module, convoluted with the canonical exponent, can be represented as a period in some family of varieties, called a (parametric) Landau-Ginzburg model. Let us make this more precise.

\(^1\)Technically, a Gromov-Witten invariant. [Mir VI-2.1]. Let \( \overline{M}_n(X, \chi) \) denote the compactified moduli space of maps of rational curves of class \( \chi \in \text{NS}^\vee \) with \( n \) marked points, and let \( \overline{M}_n(X, \chi)^{\text{virt}} \) be its virtual fundamental class of virtual dimension \( \text{vdim} \overline{M}_n(X, \chi) = \dim X - \deg_{K_X} \chi + n - 3 \). Let \( ev_i: \overline{M}_n(X, \chi) \to X \) denote the evaluation map at the \( i \)-th marked point. Then \( \langle \alpha, \beta, \gamma \rangle_\chi = ev_1^*(\alpha) \cdot ev_2^*(\beta) \cdot ev_3^*(\gamma) \cdot [\overline{M}_n(X, \chi)]^{\text{virt}} \) if \( \text{codim} \alpha + \text{codim} \beta + \text{codim} \gamma = \text{vdim} \overline{M}_3(X, \chi) \) and \( \langle \alpha, \beta, \gamma \rangle_\chi = 0 \) otherwise.
1.8. The exponent object on a one-dimensional torus. Let \( A^1 = \text{Spec } \mathbb{C}[t] \), \( G_m = \text{Spec } \mathbb{C}[t, t^{-1}] \), and let \( j : G_m \to A^1 \) denote the corresponding open immersion. Let \( \partial = \frac{\partial}{\partial t} \) and \( D = D_h = \frac{\partial}{\partial t} \) be the (invariant) derivations on \( A^1 \) and \( G_m \), respectively.

The D-modules \( E = D_{h^i}/D_{h^i}(\partial - 1) \), and its restriction to \( G_m \) \( j^*E = D_{G_m}/D_{G_m}(D - t) \), will be called the \( \text{exponent objects} \).

In general, the quantum \( \mathcal{D} \)-module \( Q \) is irregular. As such, it cannot possibly be of geometric origin, that is, arise from a Gauss-Manin connection of an algebraic family: Gauss-Manin connections are known to be regular. In order to make a suitable geometricity assertion one should pass to a regular object first.

1.9. Regularization. Consider the inclusion \( \mathbb{Z}K_X \to \text{NS}_X \). Dualizing twice, we have a morphism of tori \( \iota : G_m \to T_{\text{NS}^v} \) (the canonical torus map). Consider the exponent object \( j^*E \) on \( G_m \), and the pushforward \( \iota_*(j^*E) \). Define the \( \text{regularized quantum object} \) as follows

\[ Q^{\text{reg}} = Q * \iota_*(j^*E). \]

1.10. The mirror symmetry conjecture. The object \( Q^{\text{reg}} \) is of geometric origin.

1.11. This assertion in its strong interpretation means that for any irreducible constituent of \( Q^{\text{reg}} \) there exists a family of varieties \( \pi : E \to T_{\text{NS}^v} \) such that the restriction of that constituent to some open subset \( U \) is isomorphic to a constituent of \( \mathcal{R}^j\pi_!(O) \). In practise (e.g. 2.8–2.9) we will forget about the trivial constituents that may arise as a by-product of the convolution construction, and deal only with the essential subquotient of a single cohomology D-module of the regularized quantum object. We will call it the \( \text{regularized quantum D-module} \).

Let \( \iota_x : G_m \to \iota(G_m)x \) be an orbit of \( G_m \) in \( T_{\text{NS}^v} \). The Gauss-Manin connection in the Landau-Ginzburg model with parameter \( x \) is then essentially the pullback to \( G_m \) of the regularized quantum D-module with respect to \( \iota_x \).

1.12. Let us lay out broadly our classification strategy. It is logical to start with the Picard rank 1 case, as in this case \( T_{\text{NS}^v} \) is one-dimensional and the regularized quantum D-module is essentially a linear ordinary differential equation with polynomial coefficients.

Assume we are interested in finding all families of Fano varieties in a given class. (From our point of view, a class comprises varieties with similar cohomology structure. For instance, an interesting, if too narrow, class is that of \emph{minimal Fanos} of a given dimension, i.e. those whose non-trivial cohomology groups are just \( \mathbb{Z} \) in every even dimension. In the class of \emph{almost minimal Fanos} we allow nontrivial primitive cohomology in the middle dimension.) Assume that a variety \( X \) in the class is known, together with the values \( A_X = \{a_{ij}(X)\} \) of the three-point correlators between two arbitrary cohomology classes and the divisor class. Compute the regularized quantum D-module and represent it as \( \mathcal{D}_{G_m}/\mathcal{D}_{G_m}L_A \) for some \( L_A \in \mathcal{D}_{G_m} \). We will say that \( L_A \) is the \( \text{counting differential operator for } X \). Doint the same construction starting with a matrix variable \( A = \{a_{ij}\} \), we obtain a differential operator \( L_A \) depending on the set of parameters \( \{a_{ij}\} \). (We will do this in detail for almost minimal Fanos in getting what we call a differential operator of type DN.) Thus, we can restate the original classification problem as follows: determine which \( L_A \) can be counting differential operators \( L_A \) of some Fano variety \( X \).

1.13. Identifying counting operators. What are the properties that distinguish \( L_A \)'s as points in the affine space of all \( L_A \)? As we have seen, the mirror symmetry conjecture asserts that the \( L_A \)'s are of Picard-Fuchs type: we expect that there exist a pencil \( \pi : E \to G_m \) defined over \( \mathbb{Q} \) and \( \omega \) a meromorphic section of a sheaf of relative differential forms, such that a period \( \Phi \) of \( \omega \) satisfies \( L_A \Phi = 0 \). A believer in the mirror symmetry conjecture would therefore approach the problem of identifying the possible \( L_A \)'s by first telling which among all \( L_A \)'s are Picard-Fuchs. This will significantly narrow one’s search, as being Picard-Fuchs is a very strong condition.

1.14. Identifying Picard-Fuchs operators among all \( L_A \)'s apparently is not an algorithmic problem. The very first idea is to translate (and that can be done algorithmically) the basic properties that an (irreducible) variation of Hodge structures must have — regularity, polarizability, quasiunipotence of local monodromies — into algebraic conditions on the coefficients of the operator that represents
it. One might hope that these conditions cut out a variety of positive codimension from the affine space of all $L_A$’s, thereby facilitating further search. However, the hope is vain: a theorem proved recently by J. Stienstra and myself asserts that a generic DN equation is regular, polarizable and has quasiumitpotent local monodromies everywhere (see [21]).

Algebraic requirements being met by virtue of the construction, we have to shift the emphasis toward non-algebraizable conditions of analytic or of arithmetic nature imposed by the PF property.

It is known that if a differential equation $L_A \Phi = 0$ with coefficients in $\mathbb{Q}$ is of Picard-Fuchs type, then it is also

(H) Hodge (that is, describes an abstract variation of $\mathbb{Q}$-Hodge structures);

(GN) globally nilpotent \(^2\) in the sense of Dwork-Katz, see [19], [Ka-NC].

It is expected that, at least for small order $r$ and degree $d$, both (H) and (GN) are also sufficient conditions. Unfortunately, there is no algorithmic way, given $\alpha_{ij}$, to verify that (H) or (GN) holds: in the former case, because of the fact that (H) is, in particular, a condition on the global monodromy which depends transcendentally on the coefficients of the equation; in the latter case, because one does not know, given $\alpha_{ij}$, how to estimate the number of places $(p)$ of $\mathbb{Q}$ where the nilpotence of the $p$-curvature operator must be verified in order to conclude that global nilpotence holds.

1.15. In order to state the hypergeometric pullback conjecture, we will need some basic facts about hypergeometric D-modules. Roughly, a D-module is hypergeometric if the coefficients of the series expansion of its solution are products/quotients of the gamma-function applied to values of the form, introduced by Loeser and Sabbah ([LS-EDF]).

Let $\mathbb{C}[s] = \mathbb{C}[s_1, \ldots, s_p]$ be the ring of polynomials in $p$ variables and let $\mathbb{C}(s)$ be the corresponding fraction field.

1.16. Definition. A rational system of finite difference equations (FDE) is a finite dimensional $\mathbb{C}(s)$-vector space together with $\mathbb{C}$-linear automorphisms $\tau_1, \ldots, \tau_p$ that commute with each other and satisfy the relations

$$\tau_is_j = s_j \tau_i \text{ if } i \neq j,$$

$$\tau_is_i = (s_i + 1)\tau_i \forall i = 1, \ldots, p.$$  

1.17. If $\mathcal{M}(s)$ and $\mathcal{M}'(s)$ are rational systems of FDE, then so are $\mathcal{M}(s) \otimes \mathbb{C}(s)\mathcal{M}'(s), \hom_{\mathbb{C}(s)}(\mathcal{M}(s), \mathcal{M}'(s))$. Therefore, the set of isomorphism classes of 1-dimensional systems forms a group, which Sabbah and Loeser call the hypergeometric group and denote $\mathcal{H}(p)$.

Denote by $L$ a subset of non-zero linear forms on $\mathbb{C}[s]$ together with $\mathbb{C}$-linear automorphisms $\tau_1, \ldots, \tau_p$ that commute with each other and satisfy the relations

$$\tau_is_j = s_j \tau_i \text{ if } i \neq j,$$

$$\tau_is_i = (s_i + 1)\tau_i \forall i = 1, \ldots, p.$$  

1.18. Proposition. [LS-EDF 1.1.4]. Let $\sigma$ be a section of the projection $\mathbb{C} \rightarrow \mathbb{C}/\mathbb{Z}$. Then, the map

$$(\mathbb{C}^*)^p \times \mathbb{Z}^{[L \times \mathbb{C}/\mathbb{Z}]} \rightarrow \mathcal{H}(p)$$

that attaches to $[(c_1, \ldots, c_p), \gamma]$ the isomorphism class of the system satisfied by

$$(c_1)^{s_1} \ldots (c_p)^{s_p} \prod_{L \in L} \prod_{\alpha \in \mathbb{C}/\mathbb{Z}} \Gamma(L(s) - \sigma(\alpha))^{\gamma L, \alpha}$$

is a group isomorphism which does not depend on the choice of $\sigma$.

1.19. Let $T^p \simeq \mathbb{G}_m^p$ be a complex torus of dimension $p$. Put $D_i = t_i \frac{\partial}{\partial t_i}$. Let $\mathbb{C}[t, t^{-1}](D)$ denote the algebra of algebraic differential operators on the torus (here $t$ stands for $(t_1, \ldots, t_p)$ and $D$ for $(D_1, \ldots, D_p)$.

\(^2\)We briefly recall what global nilpotence is. Let $\partial \xi = \xi \partial M$ be an algebraic differential equation over $\mathbb{F}_p$. Consider $\partial \partial \xi = \partial \xi \partial M = \xi (M^2 + M \partial \xi)$, $\partial \partial \partial \xi$, etc. Then $\partial \partial \partial \partial \partial \partial \xi = C_p \xi$, for some matrix $C_p = C_p(M)$ which is called the $p$-curvature matrix. A differential equation $\partial \xi = \xi M$ over $\mathbb{Q}$ with $M$ having $p$-integral entries is said to be $p$-nilpotent if $C_p(M \mod p)$ is nilpotent. It is globally nilpotent if it is $p$-nilpotent for almost all $p$. 

The correspondence $\tau_i = t_i$ and $s_i = -D_i$ identifies this algebra with the algebra $\mathbb{C}[s] \langle \tau, \tau^{-1} \rangle$ of finite difference operators (which is the quotient of the algebra freely generated by $\mathbb{C}[s]$ and $\mathbb{C}[\tau, \tau^{-1}]$ by the relations in Definition 1.16).

If $\mathcal{M}$ is a holonomic D-module on $T^p$, then its global sections form a $\mathbb{C}[t, t^{-1}]\langle D \rangle$-module. The algebraic Mellin transform $\mathcal{M}(\mathcal{M})$ of the D-module $\mathcal{M}$ is this module of global sections considered as $\mathbb{C}[s] \langle \tau, \tau^{-1} \rangle$-module. We say that $\mathcal{M}(\mathcal{M})$ is a holonomic algebraic system of DFE if $\mathcal{M}$ is holonomic.

1.20. **The algebraic Mellin transform theorem.** [LS-EDP 1.2.1] Let $\mathcal{M}$ be a holonomic algebraic system of FDE. Then $\mathcal{M}(s) = \mathbb{C}(s) \otimes_{\mathbb{C}[s]} \mathcal{M}$ is a rational holonomic system of FDE. Conversely, if $\mathcal{M}(s)$ is a rational holonomic system of FDE, then for any $\mathbb{C}[s] \langle \tau, \tau^{-1} \rangle$-submodule $\mathcal{M} \subset \mathcal{M}(s)$ such that $\mathcal{M}(s) = \mathbb{C}(s) \otimes_{\mathbb{C}[s]} \mathcal{M}$ there exists a holonomic algebraic system $\mathcal{M}' \subset \mathcal{M}$ such that $\mathcal{M}(s) = \mathbb{C}(s) \otimes_{\mathbb{C}[s]} \mathcal{M}'$.

1.21. **Proposition.** [LS-Ca] One has $\chi((G_m)^{\text{reg}}, \mathcal{M}) = \dim_{\mathbb{C}(s)} \mathcal{M}(\mathcal{M})(s)$.

Now we are ready to give a

1.22. **Definition.** A D-module $\mathcal{M}$ on $T^p$ is said to be hypergeometric if $\mathcal{M}(\mathcal{M})(s)$ has rank 1.

Every 1-dimensional $\mathbb{C}(s)$-vector space with invertible $\tau$-action contains a unique irreducible holonomic $\mathbb{C}[s] \langle \tau, \tau^{-1} \rangle$-module and every such module of generic rank one is obtained in this way.

Passing back to the subject of quantum D-modules, we are finally set to state

1.23. **The hypergeometric pullback conjecture.** Let $X$ be a Fano variety. We conjecture that for any constituent $C$ of the quantum D-module $Q$ there exists a torus $T_C$, a morphism of tori $h_C : T_{NS^+} \to T_C$ and a hypergeometric D-module $\mathcal{H}_C$ on $T_C$ such that $C$ is isomorphic to a constituent of the pullback $h^*\mathcal{H}_C$ on some open subset $U$ of $T_{NS^+}$.

1.24. **Remark.** One can show that the D-module $Q$ is essentially the restriction of the “extended first structural connection” onto the divisorial direction the Frobenius manifold associated to $X$ while $Q^{\text{reg}}$ corresponds to the “second structural connection”, see chapter 2 of [Ma].

2. The Iskovskikh classification and D3 equations.

Let $X$ be a Fano 3-fold with one-dimensional Picard lattice, and let $H = -K_X$ be the anticanonical divisor. V. A. Iskovskikh classified all deformation families of these varieties (see [IP]). Recall that if $X$ is a smooth rank 1 Fano variety and $G \in H^2(X, \mathbb{Z})$ is the positive generator then the index of $X$ is defined by $H = (\text{ind } X)G$.

2.1. **Theorem.** The possible pairs of invariants $(\frac{H^3}{2 \text{ind}^2 X}, \text{ind } X)$ are

(1, 1), (2, 1), (3, 1), (4, 1), (5, 1), (6, 1), (7, 1), (8, 1), (9, 1), (11, 1),
(1, 2), (2, 2), (3, 2), (4, 2), (5, 2), (3, 3), (2, 4).

To realize our strategy [1.12] for rank one Fano 3-folds, one must first compute the quantum D-module $Q$.

2.2. **Proposition.** The subspace of algebraic classes in the total cohomology $H^*(X)$ is stable under quantum multiplication by $H$. Therefore, the connection $\nabla$ restricts to the rank 4 subbundle of $\mathcal{H}(X)$ generated by the algebraic classes.

**Proof.** This follows easily from the “dimension axiom” (see the formula in the footnote on page 2).

2.3. We compute this divisorial submodule explicitly, according to the definition. Let us normalize $u_{ij}$ so that
\[ a_{ij} = \frac{1}{\log x_j} \cdot (j - i + 1) \cdot \text{the expected number of maps } \mathbb{P}^1 \to X \]
of degree \( j - i + 1 \) that send 0 to the class of \( H^{3-i} \), and \( \infty \) to the class of \( H^3 \).

The degrees of the variety and of curves on it are considered with respect to \( H \). Assume now for simplicity that \( X \) has index 1.

As always, \( \mathbb{G}_m = \text{Spec } \mathbb{C}[t, t^{-1}] \) and \( D = t \frac{\partial}{\partial t} \). Let \( h^i \) be the constant sections of \( \mathcal{H}(X) \) that correspond to the classes \( H^i \).

### 2.4. Proposition

The connection \( \nabla \) is given by

\[ D(h^0, h^1, h^2, h^3) = (h^0, h^1, h^2, h^3) \begin{pmatrix} a_{00} t & a_{01} t^2 & a_{02} t^3 & a_{03} t^4 \\ 1 & a_{11} t & a_{12} t^2 & a_{13} t^3 \\ 0 & 1 & a_{22} t & a_{23} t^2 \\ 0 & 0 & 1 & a_{33} t \end{pmatrix} \]

**Proof.** This follows from the definition.

### 2.5. Corollary

Put

\[ \hat{L}_A = \text{det}_{\text{right}} \left( D + \begin{pmatrix} a_{00} t & a_{01} t^2 & a_{02} t^3 & a_{03} t^4 \\ 1 & a_{11} t & a_{12} t^2 & a_{13} t^3 \\ 0 & 1 & a_{22} t & a_{23} t^2 \\ 0 & 0 & 1 & a_{33} t \end{pmatrix} \right) \]

where \( \text{det}_{\text{right}} \) means the “right determinant”, i.e. the one that expands as \( \sum \text{element \cdot its algebraic complement} \), the summation being over the rightmost column, and the algebraic complements being themselves right determinants. Then \( h^0 \) is annihilated by \( \hat{L} \).

**Proof.** This is a non-commutative version of Cayley-Hamilton.

### 2.6. Corollary

The quantum D-module \( Q \) is isomorphic to (a subquotient of) \( \mathcal{D}/\mathcal{D}\hat{L} \).

Having thus computed \( Q \), we proceed with regularization. We must convolute \( Q \) with the push-forward under the morphism \( \text{inv}: x \mapsto 1/x \) of the exponent object \( \mathcal{D}/(z \partial - z) \mathcal{D} \). Convolution with the exponent of the inverse argument on a torus is essentially \(^3\) the Fourier (-Laplace) transform, as the following formula suggests:

\[ (F(x) \ast \left( \frac{1}{x} e^{1/x} \right))(t) = \int F(y) \frac{y}{t} e^{y/t} \frac{dy}{y} = \frac{1}{t} (\text{FT}(F))(\frac{1}{t}) \]

More precisely, one has:

### 2.7. The Fourier transform on \( A^1 \).

[Ka-ESDE 2.10.0] The Fourier transform of a differential operator \( L = \sum f_i(t) \partial^i \in \mathcal{D}_{A^1} \) is defined by \( \text{FT}(L) = \sum f_i(\partial)(-t)^i \). The Fourier transform of the left D-module \( M = \mathcal{D}_{A^1}/\mathcal{D}_{A^1} L \) is \( \text{FT}(M) = \mathcal{D}_{A^1}/\mathcal{D}_{A^1} \text{FT}(L) \).

### 2.8. Proposition.

[Ka-ESDE 5.2.3, 5.2.3.1] Retain the notation of [L8]. Then for any holonomic D-module \( M \) on \( \mathbb{G}_m \) we have

\[ j^* \text{FT}(j_* \text{inv}_* (M)) \approx M * j^* E \quad \text{and} \quad \text{inv}_j^* \text{FT}(j_* M) \approx M * (\text{inv}_j^* E). \]

### 2.9. The second formula shows that the convolution is in fact a single D-module, though not in general an irreducible one. We need to isolate the essential subquotient, combing out the parasitic ones.

Note that the operator \( \hat{L}_A \) is divisible in \( \mathbb{C}[t, \partial] \) by \( t \) on the left (because the rightmost column of the matrix is divisible by \( t \) on the left). Extend the D-module \( \mathcal{D}_{\mathbb{G}_m}/\mathcal{D}_{\mathbb{G}_m} t^{-1} \hat{L}_A \approx \mathcal{D}_{\mathbb{G}_m}/\mathcal{D}_{\mathbb{G}_m} \hat{L}_A \)

\(^3\)Since everything is considered on \( \mathbb{G}_m \), the functions \( \Phi \) and \( t \Phi \) are solutions to isomorphic D-modules.
naively to \( \mathbb{A}^1 \) as \( \mathcal{D}_{\mathbb{A}^1}/\mathcal{D}_{\mathbb{A}^1}t^{-1}\mathcal{L}_A \). Do the Fourier transform. We get a D-module that corresponds to the differential operator

\[
\partial^{-1} \det_{\text{right}} \begin{pmatrix}
-a_0 \partial & a_{01} \partial^2 & a_{02} \partial^3 & a_{03} \partial^4 \\
1 & a_{11} \partial & a_{12} \partial^2 & a_{13} \partial^3 \\
0 & 1 & a_{22} \partial^2 & a_{23} \partial^3 \\
0 & 0 & 1 & a_{33} \partial^3
\end{pmatrix}.
\]

FT

Pass to the inverse: under \( \text{inv} \), \( D \) is sent to \(-D\) and \( \partial \) to \(-t^2 \partial \). For further convenience we do two more things: shift the differential operator by \(-1\) on the torus \( (D \to 0 \) and \( \partial \to -\partial \) \) and multiply it by \( t \) on the right. The result is then what we call a counting differential operator of type \(D3\). Abstracting our situation to any dimension and arbitrary \( \{a_{ij}\} \), we introduce a

2.10. Definition. Let \( N \) be a positive integer. Let \( a_{ij} \in \mathbb{Q} \), \( 0 \leq i \leq j \leq N \). Let \( M \) be an \((N+1) \times (N+1)\) matrix such that for \( 0 \leq k, l \leq N \):

\[
M_{kl} = \begin{cases}
0, & \text{if } k > l + 1, \\
1, & \text{if } k = l + 1, \\
a_{kl} \cdot (Dt)^{l-k+1}, & \text{if } k < l + 1.
\end{cases}
\]

We will also assume that the set \( a_{ij} \) is symmetric with respect to the SW-NE diagonal: \( a_{ij} = a_{N-j,N-i} \).

Put

\[
\tilde{L} = \det_{\text{right}}(D - M).
\]

Since the rightmost column is divisible by \( D \) on the left, the resulting operator \( \tilde{L} \) is divisible by \( D \) on the left. Put

\[
\tilde{L} = DL.
\]

The differential equation \( L \Phi(t) = 0 \) will be called a determinantal equation of order \( N \), or just a \( DN \) equation. Sometimes we write \( DN_{0,0} \) to signify that \( 0 \) is a point of maximally unipotent monodromy, and that the local expansion \( \Phi = c_0 + c_1 t + \ldots \) of an analytic solution \( \Phi \) at \( 0 \) starts with a nonzero constant term. (One may have made other choices; for instance, the differential operator marked \( FT \) above is of type \( D3_{00,1} \) in this language.)

2.11. Example. A D3 equation expands as

\[
\begin{align*}
& D^3 - t(D + 1)(a_{00}D^2 + a_{11}D + a_{01}D + a_{00}) + \\
& + t^2 (D + 1)(a_{11}D^2 + a_{00}D^2 - a_{12}D^2 - 2a_{01}D^2 + \\
& + 8a_{11}a_{00}D - 2a_{12}D + 2a_{00}D^2 - 4a_{01}D + 2a_{11}^2D + 6a_{11}a_{00} + a_{00}^2 - 4a_{01} - \\
& - t^3 (D + 2)(D + 1)(a_{00}^2a_{11}^2 + a_{11}^2a_{00} + a_{00}^2) + a_{02} + a_{12}a_{00} - a_{11}a_{00} - a_{01}^2 + a_{00}^2 - 2a_{01}a_{11}a_{00}) \Phi(t) = 0
\end{align*}
\]

2.12. Definition. We say that two DN equations defined by \( a_{ij} \) and \( a_{ij}' \) are in the same class if there exists an \( a \) such that \( a_{ii} = a_{ii}' + a \) for \( i = 0, \ldots, N \) and \( a_{ij} = a_{ij}' \) for \( i \neq j \), i.e. if the matrices defined by \( a_{ij} \) and \( a_{ij}' \) differ by a scalar matrix.

Shifting the Fourier transformed differential operator \( FT \) on \( \mathbb{A}^1 \) corresponds exactly to shifting the \( \mathbb{D} \) matrix in its class.

2.13. Definition. We say that:

(i) a holonomic \( \mathcal{D} \)-module \( M \) is a variation of type \( DN \) if there exists a set of parameters \( A = \{a_{ij}\} \) such that \( \mathcal{D}/DL \approx M \). Here \( \approx \) denotes equivalence in the category of \( \mathcal{D} \)-modules up to modules with punctual support;

(ii) a constructible sheaf \( S \) is a variation of type \( DN \) if there exists a \( \mathcal{D} \)-module \( M \) of type \( DN \), such that \( H^{-1}(DR(M)) \approx S \). Here \( \approx \) denotes equivalence in the category of constructible sheaves up to sheaves with punctual support; \( DR \) is the Riemann-Hilbert correspondence functor.

2.14. Theorem. (i) A \( D \)-module \( \mathcal{D}/DL \) of type \( DN \) is holonomic with regular singularities;

(ii) it is self-adjoint;

(iii) the local monodromy around zero is maximally unipotent (i.e. is conjugate to a Jordan block of size \( N \)).
(iv) for a generic set \( A = \{a_{ij}\} \), the D-module \( \mathcal{D}/\mathcal{D}L_A \) has \( N \) non-zero singularities. The local monodromies at those singularities are symplectic (for \( N \) even) or orthogonal (for \( N \) odd) reflections, and the global monodromy is irreducible.

(v) the set \( A = \{a_{ij}\} \) can be recovered from the respective \( L_A \): if \( A \neq A' \), then \( L_A \neq L_{A'} \).

A proof can be found in a forthcoming paper by Jan Stienstra and myself.

2.15. **Definition.** We say that a DN variation \( M \) (resp. local system \( S \)) is of geometric origin if there exists a flat morphism \( \pi : \mathcal{E} \to \mathbb{G}_m \) of relative dimension \( d \) such that \( M \) (resp. \( S \)) is isomorphic to a subquotient of the variation arising in its middle relative cohomology (\( R^d\pi_*(\mathcal{O}) \), resp. \( R^d\pi_*(\mathcal{C}) \)) up to a D-module (resp. a sheaf) with punctual support.

2.16. **Remark.** Recall that we had assumed in 2.3 that the variety in question had index 1 before proceeding with the construction of the counting differential operator. What happens in the higher index cases? It turns out that the definition 2.10 with the values of \( a_{ij} \) as defined in 2.3 is still valid, in the sense that it yields a counting operator that corresponds to the pullback of the regularized quantum D-module with respect to the anticanonical isogeny \( \mathbb{G}_m \xrightarrow{\text{ind}} \mathbb{G}_m \) (see 2.10). We leave the proof to the reader. Use, for instance, the following property:

\[ \varphi^!(\varphi_*K \ast L) \approx K \ast (\varphi^!L). \]

2.17. **Proposition.** [Ka-ESDE 5.1.9 1b] Let \( G \) be a smooth separated group scheme of finite type, \( \varphi : G \to G \) a homomorphism. Then for any two objects \( K, L \) of \( D^b, \text{holo}(G) \) one has

\[ \varphi^!((\varphi_*K) \ast L) \approx K \ast (\varphi^!L). \]

2.18. In this language, the mirror symmetry conjecture for Fanos states: the counting DN equations of almost minimal Fano \( N \)-folds are of geometric origin. In order to recover all counting DN equations one should pose and then solve a mirror dual problem: find all geometric DN equations that possess some special property. In general, we do not know what that property is. However, in the D3 case we have an additional insight: a counting D3 should come from an \((N,d)\)-modular family.

2.19. **Definition.** A non-zero singularity of a D-module of type D3 is said to be:

(i) simple, if the local monodromy around that singularity is a reflection (i.e. conjugate to the operator \( \text{diag}(-1,1,1) \));

(ii) complex, if it is not simple and is of determinant 1;

(iii) very complex, if it is not simple and is of determinant -1;

3. \((N,d)\)-modular variation.

**Warning.** In this section \( N \) stands for level. This is not the \( N \) of the previous section, which denoted the order of a differential operator.

3.1. The quantum weak Lefschetz principle implies that the fibers of the Landau-Ginzburg model of a Fano variety are mirror dual to the sections of the anticanonical line bundle on it. For rank 1 Fano 3-folds, these sections are rank 1 K3 surfaces.

The first picture of mirror symmetry for families of K3 surfaces arose as an attempt to explain Arnold’s strange duality. Let \( L = 3U \oplus -2E_8 \) be the K3 lattice. For a wide class of primitive sublattices \( M \) of \( L \) there is a unique decomposition

\[ M^\perp = U \oplus M^D, \]

so that there is a duality between \( M \) and \( M^D \):

\[ (M^D)^\perp = U \oplus M. \]

The Picard lattices of mirror dual families of K3 surfaces are dual in this sense. Therefore, it is natural to expect that the dual Landau-Ginzburg model of a Fano 3-fold is a family of K3 surfaces of Picard rank 19. We recall that a Kummer K3 is the minimal resolution of the quotient of an Abelian surfaces by the canonical involution which sends \( x \) to \( -x \) in the group law.

\[ ^4 \text{One expects that the analog of Proposition 2.2 holds in even dimensions as well, so that all almost minimal } N \text{-folds are controlled by DN’s.} \]
The following construction was described in [PS], [Go-GP].

3.2. Consider the modular curve $X_0(N)$, and the “universal elliptic curve” over it. Strictly speaking, the universal elliptic curve is a fibration not over $X_0(N)$ but over a Galois cover with group $\Gamma$, e.g. $X(3N) - \{\text{cusps}\}$, such that one can choose a $\Gamma$-form of the universal elliptic curve; call it “the” universal elliptic curve and denote it by $E_t$. Denote by $W$ the Atkin-Lehner involution of $X_0(N)$. Consider the fibered product of $E_t$ with the $N$-isogenous universal elliptic curve $E_t^W$ over $X_0(N)$. We quotient out this relative abelian surface $V_t$ by the canonical involution $x \mapsto -x$ and then resolove to get a family of Kummer K3 surfaces. Let $X_0(N)^0$ stand for $X_0(N) - \{\text{cusps}\} - \{\text{elliptic points}\}$. Denote by $H(V_t^W)$ the cohomology of the generic fiber of $V_t$, that is, of the pullback of the family $V_t$ to the universal cover of the base.

The monodromy representation

$$\psi : \pi_1(X_0(N)^0) \to H^2(V_t^W)$$

is well defined. We are going to compute $\psi$ in terms of the tautological projective representation

$$\varphi : \pi_1(X_0(N)^0) \to PGL(H^1(E_t^W)) = PSL_2(\mathbb{Z}).$$

The monodromy that acts on $H^1$ of the fiber of the universal elliptic curve is given by a lift of $\varphi$ to a linear representation

$$\tilde{\varphi} : \gamma \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad c = 0 \mod N.$$  

Then, the monodromy that acts on $H^1$ of the fiber of the isogenous curve is:

$$\tilde{\varphi}_N : \gamma \mapsto \begin{pmatrix} d & -c \\ -bN & a \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -N & 0 \end{pmatrix},$$

where we have chosen symplectic bases $(e_1, e_2), (f_1, f_2)$ of $H^1(E_t^0), H^1(E_t^W)$ such that the matrix of the isogeny $W$ in these bases is

$$\begin{pmatrix} 0 & 1 \\ -N & 0 \end{pmatrix}.$$  

The cohomology ring of the generic fiber $V_t^W$ of our relative abelian surface is $H(E_t^0) \otimes H(E_t^W)$. The vector subspace of algebraic classes in $H^2(V_t^W)$ is generated by the pullbacks from the factors and the graph of the isogeny:

$$e_1 \wedge e_2 \otimes 1, 1 \otimes f_1 \wedge f_2, -e_1 \otimes f_1 - Ne_2 \otimes f_2.$$  

These classes are invariant under monodromy. The orthogonal lattice of transcendental classes is generated by

$$e_2 \otimes f_1, e_1 \otimes f_1 - Ne_2 \otimes f_2, e_1 \otimes f_2.$$  

Identifying the $e$’s and $f$’s with their pullbacks to the product, we write, abusing notation:

$$e_2 \wedge f_1, e_1 \wedge f_1 - Ne_2 \wedge f_2, e_1 \wedge f_2.$$  

In this basis the monodromy representation is

$$\psi : \gamma \mapsto Sym^2_N \varphi(\gamma) = \begin{pmatrix} d^2 & 2cd & -c^2/N \\ bd & bc + ad & -ac/N \\ -Nb^2 & -2Nb & a^2 \end{pmatrix}.$$  

(cf. [PS], [Go]).

Let $\bar{\omega}$ be a meromorphic section of the sheaf of relative holomorphic differential forms on the universal elliptic curve. Identify $e_1, e_2$ (resp. $f_1, f_2$) with cohomology classes in the pullback of the universal elliptic curve to the universal cover of the base. Denote by $\omega$ the pullback of $\bar{\omega}$. Introduce a coordinate $\tau$ on the universal cover by writing:

$$[\omega] = \tau e_1 + F e_2$$  

(where $F$ is a function on the universal cover) identifying it with the upper halfplane. The class $\omega^W$ is then:

$$[\omega^W] = F f_1 - N \tau F f_2.$$
Let $\omega$ and $\omega^W$ also denote, abusing notation, the pullbacks of the respective forms to $V_t$. Clearly,

$$[\omega \wedge \omega^W] = F^2 e_2 \wedge f_1 + \tau F^2 (e_1 \wedge f_1 - Ne_2 \wedge f_2) - \tau^2 NF^2 e_1 \wedge f_2.$$ 

Now, as $\omega$ is $\Gamma_0(N)$-equivariant,

$$\psi(\gamma) \begin{pmatrix} F^2(\tau) \\
\tau F^2(\tau) \\
- N\tau^2(F^2(\tau)) \end{pmatrix} = \begin{pmatrix} F^2(\gamma(\tau)) \\
\tau F^2(\gamma(\tau)) \\
- N\gamma(\tau)^2(F^2(\gamma(\tau))) \end{pmatrix},$$

where $\gamma(\tau) = \frac{a\tau + b}{c\tau + d}$. This is equivalent to the identity

$$F^2(\frac{a\tau + b}{c\tau + d}) = (c\tau + d)^2F^2(\tau).$$

Therefore, the period $F^2$ in our family of abelian surfaces, as a function of $\tau$, is a $\Gamma_0(N)$-automorphic function of weight $2$ on the upper halfplane. Now, for any $\Gamma_0(N)$-automorphic function of weight $2$ $G$, the quotient $\frac{G}{F^2}$ is $\Gamma_0(N)$-invariant on the upper halfplane, hence a rational function on $X_0(N)$. This identifies $G$ with a (meromorphic) section of the sheaf of relative holomorphic $2$-forms in our family.

Finally, delete the $W$-invariant points from $X_0(N)^o$ and let $X_0(N)^{W^0}$ be the quotient of the resulting curve by $W$. The involution $W$ extends to the fibration $V_t$ in an obvious way, and yields a family $V^W_t$ over $X_0(N)^{W^0}$. The fundamental group $X_0(N)^{W^0}$ is generated by $\pi_1(X_0(N)^o)$ and a loop $\iota$ around the point that is the image of a point $s$ on the upper halfplane stabilized by

$$\begin{pmatrix} 0 & 1 \\
-N & 0 \end{pmatrix}.$$ 

Extend $\psi$ to $\iota$, by setting $\psi(\iota) = \begin{pmatrix} 0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0 \end{pmatrix}$. The resulting representation is the monodromy representation of the family $V^W_t$ over $X_0(N)^{W^0}$.

If a relative holomorphic form in the family $V_t$ is a pullback from $V^W_t$, then, denoting its first period by $G$, one has

$$\psi(\iota) \begin{pmatrix} G(\tau) \\
\tau G(\tau) \\
- N\tau^2G(\tau) \end{pmatrix} = \begin{pmatrix} G(\frac{1}{\tau}) \\
\tau G(\frac{1}{\tau}) \\
- N\tau^2G(\frac{1}{\tau}) \end{pmatrix},$$

and $G$ is odd Atkin-Lehner, as, by definition,

$$G^W(\tau) = G(\frac{1}{\tau})N^{-1}\tau^{-2}.$$ 

Let now $N$ be a level such that the curve $X_0(N)^W$ is rational. We choose a coordinate $T$ on it such that $T = 0$ at the image of the cusp $(\infty)$; (the inverse of a Conway-Norton uniformizer, see [4.2] below) this defines an immersion of the torus $\iota: X_0(N)^W \hookrightarrow G_m^t = \text{Spec } \mathbb{C}[T, T^{-1}]$. Let $\text{Spec } \mathbb{C}[t, T^{-1}] = G_m \rightarrow G_m^t$ be the Kummer covering of degree $d$, given by the homomorphism $T \mapsto t^d$.

The pullback of the variation described above (that is, the pullback of the family itself, or the monodromy representation, or the $D$-module, depending on the context) to $G_m$ will be called the $(N,d)$-modular variation. Let us emphasize: $(N,d)$-modular variations are variations on tori, even if we speak of them as of variations on $\mathbb{P}^1$, as in the proof of Theorem [4.2] below.

In the case $N = 1$ the construction is modified, since the “Atkin-Lehner involution” $\begin{pmatrix} 0 & 1 \\
-N & 0 \end{pmatrix}$ acts trivially on $X_0(N)$. In this case we work with the fibered product of the “universal elliptic curve” over $X_0(1)$ with its quadratic twist with respect to the degree two branched covering ramified at the two elliptic points. In this case the relative 2-form can no longer be identified with a weight 2 level 1
modules function because of the sign multiplier. However, squaring the corresponding period, we get a bona fide modular function of weight 4 and level 1.

4. \((N,d)\)-modular D3 equations: the necessary condition.

4.1. Problem. Find all pairs \(N,d\) such that the \((N,d)\)-modular variation described in the previous section is of type \(D3\).

4.2. Theorem. (Necessary condition). If an \((N,d)\)-modular variation is of type D3, then the pair \((N,d)\) belongs to the set

\[
\mathcal{M} = \{(1,1), (2,1), (3,1), (4,1), (5,1), (6,1), (7,1), (8,1), (9,1), (11,1), (1,2), (2,2), (3,2), (4,2), (5,2), (3,3), (2,4)\}.
\]

Proof. We begin by noticing that no case with \(d > 5\) is possible as there would have to be at least 6 singularities.

Case \(d = 1\).

Assume \(N \neq 1\). We make the following remarks:

1. All ramification points of the quotient map

\[
\sigma : X_0(N) \longrightarrow X_0(N)^W
\]

map to singularities of the \((N,1)\)-modular variation. The corresponding local monodromy is projectively (dual to) the symmetric square of the element in \(\Gamma_0(N) + N\) that stabilizes this ramification point. This element is elliptic or cuspidal, therefore its symmetric square cannot be a scalar.

2. Every elliptic point or a cusp point \(p\) on \(X_0(N)\) maps to a complex or very complex point \(\sigma(p)\) on \(X_0(N)^W\). If \(\sigma(p)\) were an apparent singularity or a simple singularity, then the local monodromy around \(p\) would vanish, which is precluded by the reason given above in (1).

3. The point \(s\) on the upper halfplane is neither elliptic nor a cusp. It goes to a simple point on \(X_0(N)^W\). We defined the monodromy \(\nu\) (image of \(s\)) in the previous section to be a reflection.

4. If a D3 equation is \((N,1)\)-modular, then its set of non-zero singularities consists of either 4 simple points, or of 1 complex and 2 simple points, or of 1 very complex and 1 simple point. The non-zero singularities of a D3 equation are inverse to roots of a polynomial of degree 4, as can be seen from the expansion in Example 1.3. It has one simple singularity, according to (3). Any singularity of multiplicity 1 is simple. A singularity of multiplicity 3 is very complex because the determinant must be \(-1\) and it cannot be simple (otherwise the global monodromy would be generated by two reflections and therefore would be reducible).

5. The genus \(g\) of \(X_0(N)\) is related to the numbers of elliptic points \(\nu_2\) and \(\nu_3\) of order 2 and 3 on \(X_0(N)\) by the formula

\[
g = 1 + \frac{N}{12} \prod_{p|N} (1 + p^{-1}) - \frac{\nu_2}{4} - \frac{\nu_3}{3} - \frac{\nu_\infty}{2}.
\]

This is Proposition 1.40 from \([Sh]\).

These remarks show that \(g \leq 1\), (otherwise the variation would have at least 6 singularities according to (1)); that if \(g = 1\), then all of the singularities are simple (this is from (1) and (4)) and \(\nu_2 = 0, \nu_3 = 0, \nu_\infty = 2\) so \(N = 11\); and that if \(g = 0\), then \(N < 12\) (otherwise there would be too many singularities, which would contradict (2) and (4)). The last argument also shows that \(N \neq 10\), as in this case \(\nu_2 = 2\) and \(\nu_\infty = 4\).

Case \(d = 2\).

Again, assume \(N \neq 1\).

1. The genus \(g\) of \(X_0(N)\) is zero. If it were greater than zero, there would be at least four singularities besides the one at \(0\). Therefore, the \((N,2)\)-modular variation would have at least 7 singularities.

2. There may be no more than 3 cusps on \(X_0(N)\). Assume there are at least 4 cusps on \(X_0(N)\). Consider the ramification points of the Atkin-Lehner involution. One of them being \(s\), the other is
either a cusp or not a cusp. In the former case we get at least three cusps on \( X_0(N)^W \). Pulling them back we get at least 4 singularities of a D3 variation that are not simple, a contradiction. In the latter case, we get at least two cusps and at least two other singularities of the \((N,1)\)-modular variation. Pulling them back to the \((N,2)\)-modular variation we get either:

- at least four simple points and two non-simple points, or:
- at least two simple points and three non-simple points,

and in neither case can the resulting variation be of type D3.

(3) There may be no more than 1 order 3 elliptic point on \( X_0(N) \). Proof: same as above.

(4) There may be no more than 7 order 2 elliptic points on \( X_0(N) \). These would give at least 5 singularities on \( X_0(N)^W \) and therefore at least 7 singularities on the pullback.

(5) The level \( N \) is smaller than 48. This bad but easy estimate follows from the genus formula and the above remarks.

Having made these remarks, one proceeds (for instance) by inspecting the values \( g, \nu_2, \nu_3, \nu_\infty \) for all levels \( N < 48 \). One uses the formulas of [Sh, Proposition 1.43]:

\[
\nu_2 = \begin{cases} 0, & \text{if } N = 4k, \\ \frac{1}{2} \prod_{p|N} \left(1 + \left(\frac{-1}{p}\right)\right), & \text{if } N = 4k + 2, \\ \prod_{p|N} \left(1 + \left(\frac{-1}{p}\right)\right), & \text{if } 2 \nmid N; \end{cases}
\]

\[
\nu_3 = \begin{cases} 0, & \text{if } N = 9k, \\ \prod_{p|N} \left(1 + \left(\frac{-3}{p}\right)\right), & \text{if } 9 \nmid N; \end{cases}
\]

\[
\nu_\infty = \sum_{d|N} \varphi(\gcd(d,N/d)).
\]

where \( \varphi(n) \) is as usual the number of positive integers not exceeding \( n \) and relatively prime to \( n \).

One thus finds that the only levels that satisfy the requirements above are \( N = 2, 3, 4, 5 \).

**Case** \( d = 3 \).

Pulling back under the degree 3 map dramatically multiplies singularities; the analysis, which goes along the same rails, is this time much easier and leaves one with the only possibility of a curve of genus 0 that has just 2 cusps, 1 order 3 elliptic point and no order 2 points, which corresponds to level 3.

**Case** \( d = 4 \). Yet easier. The curve must be of genus 0 and have 2 cusps, 1 order 2 elliptic point and no order 3 points. The level is 2.

\[ \blacksquare \]

### 5. \((N,d)\)-modular D3 equations: a sufficient condition.

#### 5.1. **Theorem.**

For all pairs \((N,d)\) in \( M \) the corresponding \((N,d)\)-modular variation is of type D3.

**Proof.** Assume for simplicity that \( d = 1 \). Reshape the assertion this way: for any pair \((N,1)\) in \( M \) there is a period \( \Phi \) of a section of the line bundle \( \pi_* \Omega^2_{1/H0(N)^W} \) in our \((N,1)\)-modular variation that satisfies, as a multivalued function on \( X_0(N)^W \), a D3 equation with respect to a coordinate \( t \) on \( X_0(N)^W \). Given an expansion of \( \Phi(t) \) as a series in \( t \), it is easy to find the differential equation that it satisfies.

To be more specific, recall that we chose a coordinate \( T \) on \( X_0(N)^W \) such that \( T = 0 \) at the image of the cusp \( \{\infty\} \). The local monodromy at \( T = 0 \) of the cycles against which our fibrewise 2-form is integrated is conjugate to a unipotent Jordan block of size \( 3 \). Therefore, the analytic period \( \Phi = \Phi_0 \) is well defined as the integral against the monodromy-invariant cycle. In the same way, the logarithmic period \( \Phi_1 \), being the integral against a cycle in the second step of the monodromy filtration,

\[ \text{[32]} \]

\[ \text{[Go-GP]} \]
is well defined up to an integral multiple of the analytic period. This defines $\tau$ locally as $\frac{2\pi i}{\chi_0}$, and $q$ as $\exp(2\pi i \tau)$. 

Now $q$ being a local coordinate around 0, one can expand both $\Phi$ and $T$ as $q$-series. Note that the expansion of $T^{-1}$ is a $q$-series that is uniquely defined up to a constant term. The $q$-expansions of coordinates on $X_0(N)^W$ appeared in a paper by Conway and Norton [CN] and are called Conway-Norton uniformizers. The table 5.2 of the uniformizers for the levels that we need is taken from [CN].

Recall also that we have identified periods $\Phi$ with odd Atkin-Lehner weight 2 level $N$ modular functions in 3.2. Therefore, to prove our theorem explicitly one may: (1) produce a $q$-expansion of such a modular function $\Phi$; (2) fix the constant term in the uniformizer $T^{-1}$; (3) express $q$ in $T$; (4) expand $\Phi$ in $T$; (5) recover the differential equation that $\Phi$ satisfies with respect to $T$. If it is a D3 equation, we are done.

The same essentially goes for the cases $d = 2, 3, 4$, except that the coordinate on the Kummer covering is $t = T^{1/d}$ and the local parameter is $Q = q^{1/d}$. The tables 5.2 contain the $Q$-expansions of $\Phi$, the recovered D3 matrices and the eta-expansions of the $I$-function that we introduce below. The uniformizers, as we said, are next in the table 5.2. For level $N$ and index $d$, one should set the constant term of the uniformizer to $a_{11}$ in the $(N, 1)$ matrix in tables 5.3 (e.g. take $c = 744$ for level 1 and index 2).

### 5.2. The Conway-Norton uniformizers

The constant term is denoted indiscriminately by $c$ below. We put $i = q'^{24} \prod (1 - q^n)$ in this table.

| $N = 1$ | $N = 2$ | $N = 3$ | $N = 4$ | $N = 5$ |
|---------|---------|---------|---------|---------|
| $j + c$ | $\frac{124}{27} + 4096 \cdot \frac{24}{47} + c$ | $\frac{12}{37} + 729 \cdot \frac{72}{17} + c$ | $\frac{16}{17} + 256 \cdot \frac{34}{47} + c$ | $\frac{6}{5} + 125 \cdot \frac{56}{17} + c$ |
| $N = 6$ | $N = 7$ | $N = 8$ | $N = 9$ | $N = 11$ |
| $\frac{1}{2} \cdot \frac{3}{6} + 72 \cdot \frac{21}{6} + c$ | $\frac{1}{4} + 49 \cdot \frac{74}{17} + c$ | $\frac{14}{9} + 32 \cdot \frac{21}{17} + c$ | $\frac{13}{17} + 27 \cdot \frac{9}{17} + c$ | $\frac{13}{2} \cdot \frac{11}{17} + 16 \cdot \frac{224}{11} + 16 \cdot \frac{224}{11} + c$ |

### 5.3. Solution $\Phi$. Weighted ‘Eisenstein series’ $E_2$

In most of the cases, the form $\Phi$ will be expressed as a finite linear combination of “elementary Eisenstein series”

$$E_{2,i}(Q) \overset{\text{def}}{=} -\frac{1}{24} i (1 - 24 \sum_{n=1}^{\infty} \sigma(n)Q^n).$$

A sequence $e_1, e_2, e_3, \ldots$ determines the Eisenstein series

$$\sum e_j E_{2,j}(Q).$$

We use notation $\Phi = e_1 \cdot [1] + e_2 \cdot [2] + \ldots$ in the third column of the tables 5.6.

### 5.4. Non-uniqueness

We have proved Theorem 5.1 by producing some modular function $\Phi$ and some Conway-Norton uniformizer $T^{-1}$ of level $N$ such that $\Phi$ expanded in $T$ satisfies a D3 equation. Is the pair $\Phi, T^{-1}$ that we have produced determined by this condition uniquely? The answer is in general no, even if $\Phi$ is known to be an Eisenstein series: at certain composite levels the space spanned by Eisenstein series has dimension higher than 1, and it is possible to find two different Eisenstein series and two uniformizers (that differ by a constant term) such that the respective expansions give rise to different D3 matrices.

The extra piece that we use to characterize the matrices and the solutions $\Phi$ in the tables 5.6 uniquely is:
5.5. The miraculous eta-product formula. Define \( I = \Phi \cdot t^{\frac{N+1}{2}} \). Let \( H_j(Q) = Q^{j/24} \prod (1 - Q^{j^n}) \). It turns out that \( I \) expands as a finite product of series of the form \( \prod H_j^h(Q) \) in a remarkably uniform way:

\[
I = H_d(Q)^2 H_{Nd}(Q)^2.
\]

We reflect this phenomenon in the fourth column of the table. The notation used is \( I = 1^h \cdot 2^h \cdots \). No intrinsic explanation of the eta-product formula is known to the author.

5.6. Level, matrix, solution, \( I \)-function.

\[
d = 1
\]

| \( N \) | \( a_{ij} \) | \( \Phi (|j| = -\frac{1}{24} E_{2,j}(Q)) \) | \( I (j = H_j(Q)) \) |
|---|---|---|---|
| 1 | 120 137520 119681280 21690374400 | \( \sqrt{E_d(q)} \) | \( 1^21^2 \) |
| 0 | 744 650016 119681280 | | |
| 0 | 0 744 137520 | | |
| 0 | 0 0 120 | | |
| 2 | 24 3888 504576 18323712 | +24 \cdot [1] - 24 \cdot [2] | \( 1^22^2 \) |
| 0 | 104 13600 504576 | | |
| 0 | 0 104 3888 | | |
| 0 | 0 0 24 | | |
| 3 | 12 792 43632 793152 | +12 \cdot [1] - 12 \cdot [3] | \( 1^23^2 \) |
| 0 | 42 2340 43632 | | |
| 0 | 0 42 792 | | |
| 0 | 0 0 12 | | |
| 4 | 8 304 9984 121088 | +8 \cdot [1] - 8 \cdot [4] | \( 1^24^2 \) |
| 0 | 24 800 9984 | | |
| 0 | 0 24 304 | | |
| 0 | 0 0 8 | | |
| 5 | 6 156 3600 33120 | +6 \cdot [1] - 6 \cdot [5] | \( 1^25^2 \) |
| 0 | 16 380 3600 | | |
| 0 | 0 16 156 | | |
| 0 | 0 0 6 | | |
| 6 | 5 96 1692 12816 | +5 \cdot [1] - 1 \cdot [2] + 1 \cdot [3] - 5 \cdot [6] | \( 1^26^2 \) |
| 0 | 12 216 1692 | | |
| 0 | 0 12 96 | | |
| 0 | 0 0 5 | | |
| 7 | 4 64 924 5936 | +4 \cdot [1] - 4 \cdot [7] | \( 1^27^2 \) |
| 0 | 9 140 924 | | |
| 0 | 0 9 64 | | |
| 0 | 0 0 4 | | |
| 8 | 4 48 576 3328 | +4 \cdot [1] - 2 \cdot [2] + 2 \cdot [4] - 4 \cdot [8] | \( 1^28^2 \) |
| 0 | 8 96 576 | | |
| 0 | 0 8 48 | | |
| 0 | 0 0 4 | | |
| 9 | 3 36 378 1944 | +3 \cdot [1] - 3 \cdot [9] | \( 1^29^2 \) |
| 0 | 6 72 378 | | |
| 0 | 0 6 36 | | |
| 0 | 0 0 3 | | |
| 10/11 | 12/5 24 198 880 | +12/5 \cdot [1] - 12/5 \cdot [11] | \( 1^211^2 \) |
| 0 | 22/5 44 198 | | |
| 0 | 0 22/5 24 | | |
| 0 | 0 0 12/5 | | |
5.7. Remark. The functions $\Phi$ and $I$ that describe in these tables the cases with the same level $N$, but with different $d$, are equal.
5.8. Respective differential operators.

\[ d = 1 \]

\begin{tabular}{|l|l|}
\hline
1 & \[ D^3 - 24 t (1 + 2 D) (6 D + 5) (6 D + 1) \] \\
2 & \[ D^4 - 8 t (1 + 2 D) (4 D + 3) (4 D + 1) \] \\
3 & \[ D^4 - 6 t (1 + 2 D) (3 D + 2) (3 D + 1) \] \\
4 & \[ D^3 - 8 t (1 + 2 D)^3 \] \\
5 & \[ D^4 - 2 t (1 + 2 D) (11 D^2 + 11 D + 3) - 4 t^2 (D + 1) (2 D + 3) (1 + 2 D) \] \\
6 & \[ D^3 - t (1 + 2 D) (17 D^2 + 17 D + 5) + t^2 (D + 1)^3 \] \\
7 & \[ D^3 - 4 t (1 + 2 D) (3 D^2 + 3 D + 1) + 16 t^2 (D + 1)^3 \] \\
8 & \[ D^3 - 3 t (1 + 2 D) (3 D^2 + 3 D + 1) - 27 t^2 (D + 1)^3 \] \\
9 & \[ D^3 - 2/5 t (2 D + 1) (17 D^2 + 17 D + 6) - \frac{126}{125} t^2 (D + 1) (11 D^2 + 22 D + 12) = \right) - \frac{126}{125} t^2 (D + 1) (11 D^2 + 22 D + 12) - \]
\hline
\end{tabular}

\[ d = 2 \]

\begin{tabular}{|l|l|}
\hline
1 & \[ D^3 - 192 t^2 (3 D + 5) (3 D + 1) (D + 1) \] \\
2 & \[ D^4 - 64 t^2 (2 D + 3) (2 D + 1) (D + 1) \] \\
3 & \[ D^4 - 12 t^2 (3 D + 2) (3 D + 4) (D + 1) \] \\
4 & \[ D^4 - 64 t^2 (D + 1)^3 \] \\
5 & \[ D^4 - 4 t^2 (D + 1) (11 D^2 + 22 D + 12) - 16 t^4 (D + 3) (D + 2) (D + 1) \] \\
\hline
\end{tabular}

\[ d = 3 \]

\begin{tabular}{|l|l|}
\hline
3 & \[ D^3 - 54 t^3 (2 D + 3) (D + 2) (D + 1) \] \\
\hline
\end{tabular}

\[ d = 4 \]

\begin{tabular}{|l|l|}
\hline
2 & \[ D^3 - 256 t^4 (D + 3) (D + 2) (D + 1) \] \\
\hline
\end{tabular}

6. A conjecture on counting matrices. The Iskovskikh classification revisited.

6.1. Corollary of theorems 2.1 and 4.2. The \( d \)-Kummer pullback of the Picard-Fuchs equation of the twisted symmetric square of the universal elliptic curve over \( X_0(N)^W \) is of type D3 if and only if there exists a family of rank 1 Fano 3-folds of index \( d \) and anticanonical degree \( 2d^2 N \).

6.2. Modularity conjecture. The counting matrix of a generic Fano 3-fold in the Iskovskikh family with parameters \( (N,d) \) is in the same class as the corresponding matrix in tables 5.8.

More concretely, the conjecture states that the matrix \( a_{ij} \) of normalized Gromov-Witten invariants of a Fano 3-fold with invariants \( (N,d) \) can be obtained in the following uniform way. Let \( T = T(q) \) be the inverse of the suitable Conway-Norton uniformizer on \( X_0(N) \) (that is, the one with the “right” constant term). Consider

\[ \Phi = (q^{1/24} \prod (1 - q^n)q^{N/24} \prod (1 - q^{N^n}))^2 T^{- \frac{N+1}{12}}. \]
Then \( \Phi \) satisfies a D3 equation with respect to \( t = T^1 \). Recover the matrix of \( a_{ij} \) that corresponds to this equation (e.g. by looking at the expansion in Example 2.11), and normalize it by subtracting \( a_{00} I \).

This uniform description is somewhat unexpected, since it does not have an obvious translation in terms of the geometry of Fano 3-folds. Let us now take a more detailed view at the Iskovskikh classification, according to the index and the degree.

6.3. The Iskovskikh classification revisited.

\[
\begin{array}{|c|}
\hline
d & \text{Description} \\
\hline
1 & \text{hypersurface of degree 6 in } \mathbb{P}(1,1,1,3) \\
2 & \text{quartic in } \mathbb{P}^4 \\
3 & \text{complete intersection of a quadric and a cubic in } \mathbb{P}^5 \\
4 & \text{complete intersection of 3 quadrics in } \mathbb{P}^6 \\
5 & \text{a section of the Grassmannian } G(2,5) \text{ by a quadric and a codimension 2 plane} \\
6 & \text{a section of the orthogonal Grassmannian } O(5,10) \text{ by a codimension 7 plane} \\
7 & \text{a section of the Grassmannian } G(2,6) \text{ by a codimension 5 plane} \\
8 & \text{a section of the lagrangian Grassmannian } L(3,6) \text{ by a codimension 3 plane} \\
9 & \text{a section of } G_2/\mathbb{P} \text{ by a codimension 2 plane} \\
10 & \text{a section of } G_2/\mathbb{P} \text{ by a codimension 2 plane} \\
11 & \text{a section of } G_2/\mathbb{P} \text{ by a codimension 2 plane} \\
\hline
\end{array}
\]

6.4. Remark. The description of families \((6,1), (8,1), (9,1)\) as hyperplane sections in Grassmannians is due to Sh. Mukai [Mu].

6.5. How can one prove the modularity conjecture?

The uniformity of the assertion calls for a uniform proof, but I do not know how such a proof might work.

The only way I know how to prove the conjecture is to explicitly calculate the quantum cohomology of Fano 3-folds on a case by case basis.

Kuznetsov calculated the quantum cohomology of \( V_{22} \). All other cases are complete intersections in weighted projective spaces or Grassmannians of simple Lie groups.

For complete intersections in usual projective space, Givental’s result allows to compute the D3 equations and the result agrees with the conjecture. Przyjalkowski [Pr] has recently extended Givental’s result to the cases of smooth complete intersections in weighted projective spaces and established the predictions in the cases \( (N,d) \in \{(1,1), (1,2), (2,2)\} \).
In the remaining cases we use the quantum Lefschetz principle to reduce the computation of the quantum D-module of a hyperplane section to that of the ambient variety.

6.6. Theorem. (Quantum Lefschetz hyperplane section theorem, Coates-Givental-Lee-Gathmann, see e.g. [Ga].) Let $Y$ be a section of a very ample line bundle $L$ on $X$. We assume that both varieties are of Picard rank 1. Let $ι : G_m \to T_{NS\nu}$ be the morphism of tori double dual to the map $\mathbb{Z}[L] \to NS_X$. For $λ \in \mathbb{C}^*$ let $[λ] : G_m \to G_m$ be the corresponding translation. For $α \in \mathbb{C}^*$ let $[α] : \mathbb{A}^1 \to \mathbb{A}^1$ be the corresponding multiplication morphism. Then the quantum D-modules are related as follows:

(i) if the index of $Y > 1$, there exists $λ$ in $\mathbb{C}^*$ such that

$$Q_Y \text{ is a subquotient of } [λ]^*(Q_X*_{ι*}(j^*E))$$

(ii) if the index of $Y = 1$, there exist $λ$ and $α$ in $\mathbb{C}^*$ such that

$$Q_Y \text{ is a subquotient of } [λ]^*(Q_X*_{ι*}(j^*E)) \otimes j^*([α], E).$$

6.7. The quantum cohomology of ordinary, orthogonal and lagrangian Grassmannians is known (Givental-Kim-Siebert-Tian-Peterson-Kresch-Tamvakis). Przyjalkowski calculated the quantum Lefschetz reduction for the cases $(5,1), (7,1)$, confirming the conjecture.

Note that we do not need the whole cohomology structure: we just need to know quantum multiplication by the divisor classes, and this can be computed using Peterson’s quantum Chevalley formula [FW]. I calculated the quantum Lefschetz reduction for the cases $(6,1), (8,1), (9,1)$ and the results again agreed with the ones predicted by the conjecture.

To our knowledge, quantum multiplication by the divisor class on $V_5$ (case $(5,2)$) was first computed by Beauville [Bea]. We refer the reader to [BM] which makes use of Beauville’s and Kuznetsov’s results.

To summarize, we have checked the conjecture in all 17 cases by a case by case analysis. This proof, however, does not explain why the conjecture is true. A more uniform approach, yet to be discovered, would presumably start from the embedded K3 rather than the ambient space.

6.8. Remark. If the conjecture is true, then there is a mysterious relation between varieties of different index, as implied by Remark [AY].

7. What next?

7.1. Classification of smooth rank 1 Fano 4-folds. This is an open question. For a variety of index $\geq 2$ one can pass to the hyperplane section (which has to be a Fano 3-fold) and thus reduce the problem to lower dimension. On the other hand, the classification of index one Fano 4-folds seems to be beyond reach of today’s geometric methods. Our program, if carried out in this case, would suggest a blueprint of a future classification.

As a first step one must show that rank 1 Fano 4-folds do give rise to equations of type D4. The dimension argument that we used in [22] to show that the subspace generated by $H^n, n = 0, \ldots, \dim X$, is stable under quantum multiplication by $H$ no longer works. Still, the assertion is true in dimension 4. The next step is to classify counting D4 equations. Unlike D2 and D3 variations, whose differential Galois group is $Sl_2 = Sp_2 = So_3$, a variation of type D4 is controlled by $Sp_4$, and has in general no chance of being modular. Thus, as we remarked in [213] in the D4 case we lack the consequences of modularity that enabled us first to state the correct mirror dual problem, and then effectively to handle it in the D3 case.

With no idea of what the mirror dual problem might be, one can still rely on the basic conjectures of chapter 1 to compose a list of candidate D4 equations. If the list is not too long and it contains all D4 equations, the problem is reduced to weeding out the extra non-counting D4 equations that have sneaked into the list.

Which D4 equations are of Picard-Fuchs type? Of the approaches that we discuss in [14] establishing the $\mathcal{Q}$-Hodge or even the $\mathcal{R}$-Hodge property of a differential equation, given its coefficients, seems hopeless. On the other hand, a necessary, though not sufficient, condition for global nilpotence is that the...
\( p \)-curvatures are nilpotent for sufficiently many prime \( p \). In principle \(^6\), one needs to guess the upper bound \( h_{\text{max}} \) of the height (\( h = \text{max}(p,q) \) for \( p/q \in \mathbb{Q} \) in lowest terms) of possible Gromov-Witten invariants \( a_{ij} \), and then run the above search over the corresponding box.

A non-systematic search for D4 equations whose analytic solution expands as a series in \( \mathbb{Z}[[t]] \) was pioneered by Almkvist, van Enckevort, van Straten and Zudilin, [AZ], [ES]. See [ES] for a systematic approach to recognizing a given globally nilpotent D4 equation as the mirror DE of a Calabi-Yau family by computing invariants of its global monodromy.

The hypergeometric pullback conjecture suggests a (presumably) more restrictive candidate list, but it is not clear how one can identify these among all D4 equations, without further assumptions.

### 7.2. Del Pezzo surfaces and D2’s.

The only rank 1 del Pezzo is \( \mathbb{P}^2 \), so it might seem that our program is just not applicable here. However, it turns out (Orlov and Golyshov, unpublished) that the three-dimensional subspace of the total cohomology generated by the classical powers of the anticanonical class is stable under quantum multiplication by the anticanonical class, and it gives rise to D2 equations for del Pezzo surfaces of degrees 9, 6, 5, 4, 3, 2, 1. In [Go-GP] the parametric D2 equation was identified with a particular case of the classical Heun equation that had been studied by Beukers [Beh] and Zagier [Za]. Zagier had run a search over a large box for D2 equations with analytic solution in \( \mathbb{Z}[[t]] \), see the list in [Za]. Our counting D2 equations are hypergeometric in degrees 4, 3, 2, 1 and are hypergeometric pullbacks in degrees 9, 6, 5.

The classification of del Pezzo surfaces is of course well known; however, it might be interesting to understand the significance of the non-D2 equations arising as canonical pullbacks in degrees 8 and 7.

### 7.3. Singular Fano 3-folds.

The classification of singular Fano 3-folds of Picard rank 1 is of interest in birational geometry, see [CPR]. Corti has suggested to extend the mirror approach to the classification of \( \mathbb{Q} \)-Fano 3-folds with prescribed (say terminal, or canonical) singularities. One expects that to a \( \mathbb{Q} \)-Fano 3-fold one can associate a differential equation that reflects its properties in much the same way as D3 equations do for smooth 3-folds. In order to construct it as a counting DE one would have to rely on a theory of Gromov-Witten invariants of singular varieties, which is not yet sufficiently developed. A provisional solution is to model the construction of such a DE on the known smooth examples, formally generalizing them in the simplest cases such as complete intersections in weighted projective spaces.

An instance of a pair of mirror dual problems in this setup is due to Corti and myself. Let \( \mathbb{P}(w_0, w_1, w_2, w_3) \) be a weighted projective space, \( d = \sum w_j \). The operator

\[
\prod_{j=0}^{3} (w_i^{w_j} (D - w_i - 1) (D - w_i - 2) \ldots D) - d!t(D + \frac{1}{d})(D + \frac{2}{d}) \ldots (D + \frac{d-1}{d})(D + 1)
\]

gives rise to a hypergeometric D-module, whose essential constituent we call the anticanonical Riemann-Roch D-module. It is easy to show that the monodromy of this D-module respects a real orthogonal form. The problem of classification of weighted \( \mathbb{P}^3 \) with canonical singularities happens to admit a mirror dual problem: to classify the anticanonical Riemann-Roch D-modules such that the form above is of signature \( (2, n - 2) \).

I am obliged to Helena Verrill who checked the modular formulas and made a number of suggestions.

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\(^6\)But not in practice. The generic D4 depends on 9 parameters, and the computation involved needs unrealistic resources.
