CMC foliations of closed manifolds

William H. Meeks III∗ Joaquín Pérez †

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Abstract

We prove that every closed, smooth n-manifold X admits a Riemannian metric together with a smooth, transversely oriented CMC foliation if and only if its Euler characteristic is zero, where by CMC foliation we mean a codimension-one, transversely oriented foliation with leaves of constant mean curvature and where the value of the constant mean curvature can vary from leaf to leaf. Furthermore, we prove that this CMC foliation of X can be chosen so that the constant values of the mean curvatures of its leaves change sign. We also prove a general structure theorem for any such non-minimal CMC foliation of X that describes relationships between the geometry and topology of the leaves, including the property that there exist compact leaves for every attained value of the mean curvature.

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1 Introduction.

This manuscript studies the existence, geometry and topology of smooth, transversely oriented foliations F of a smooth closed Riemannian n-manifold X, such that all of the leaves of F are two-sided hypersurfaces of constant mean curvature and where the value of the constant mean curvature can vary from leaf to leaf; in this setting, that the foliation is transversely oriented just means that there exists a smooth, unit vector field on X that is normal to the leaves of the foliation. Such a foliation is called a CMC foliation of X; this is a particular case of a tense foliation, defined as a smooth foliation of X by n − k submanifolds with parallel mean curvature vector, see e.g., Definition 1.36 in Rovenskii [14]. All manifolds and foliations appearing here will be assumed to be smooth unless otherwise stated.

By the next theorem, the vanishing of the Euler characteristic of a closed n-manifold is equivalent to the existence of a CMC foliation of the manifold with respect to some Riemannian metric.

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Theorem 1.1 (Existence Theorem for CMC Foliations) A closed $n$-manifold admits a non-minimal CMC foliation for some Riemannian metric if and only if its Euler characteristic is zero.

Since closed (topological) three-manifolds admit smooth structures and the Euler characteristic of any closed manifold of odd dimension is zero, the previous theorem has the following corollary.

Corollary 1.2 Every closed topological three-manifold admits a smooth structure together with a smooth Riemannian metric and a smooth non-minimal CMC foliation.

Theorem 1.1 is motivated by two seminal works. The first one, due to Thurston (Theorem 1(a) in [16]), shows that a necessary and sufficient condition for a smooth closed $n$-manifold $X$ to admit a smooth, codimension-one foliation $\mathcal{F}$ is for its Euler characteristic to vanish; for our applications, we will need to check that $\mathcal{F}$ can be chosen to be transversely oriented. The second one is the result by Sullivan (Corollary 3 in [15]) that given such a pair $(X, \mathcal{F})$ where $\mathcal{F}$ is orientable (see Definition 2.3), then $X$ admits a smooth Riemannian metric $g_X$ for which $\mathcal{F}$ is a minimal foliation (i.e., $\mathcal{F}$ is geometrically taut) if and only if for every compact leaf $L$ of $\mathcal{F}$ there exists a closed transversal that intersects $L$ (i.e., $\mathcal{F}$ is homologically taut); for our applications, we will need to prove that Sullivan’s hypothesis that the foliation $\mathcal{F}$ be orientable can be removed. Theorem 1.1 is also related to the question of prescribing a mean curvature function for a given transversely oriented, codimension-one foliation $\mathcal{F}$ of a closed $n$-manifold $X$: Walczak [17] asked the question of which smooth functions $f$ on $X$ can be written as mean curvature functions of the leaves of $\mathcal{F}$ with respect to some Riemannian metric on $X$. In this line, Oshikiri [11, 12] characterized the solutions to this problem when $X$ is orientable; in particular, under the hypothesis that $X$ is orientable, he described for which functions $f \in C^\infty(X)$ that are constant along the leaves of $\mathcal{F}$, there exists a Riemannian metric on $X$ that makes $\mathcal{F}$ a CMC foliation (see Theorem 2.5 below for a precise statement of Oshikiri’s result in the general setting).

An application of the divergence theorem shows that a geometrically taut foliation of a closed three-manifold $X$ cannot have a Reeb component\(^1\), since such a foliation can clearly not have a compact leaf that separates the manifold. Therefore, Novikov’s theorem [10], which implies that any foliation of the three-sphere admits a Reeb component, also implies that the three-sphere does not admit any geometrically taut foliations. By work of Novikov [10] and Rosenberg [13] (also see Corollary 9.1.9 in Candel and Conlon [2]), every closed orientable three-manifold $X$ admitting a codimension-one, transversely oriented foliation without Reeb components is either $\mathbb{S}^2 \times \mathbb{S}^1$ or irreducible\(^2\); in particular, such an $X$ is a prime three-manifold. As homologically taut foliations on a closed three-manifold are free of Reeb components, we deduce that every closed, non-prime three-manifold does not admit any homologically taut foliations, and so by Sullivan’s theorem such manifolds also do not admit any geometrically taut foliations.

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\(^1\)Classical Reeb components of foliations of closed three-manifolds are defined at the beginning of Section 3. In Definition 3.4 we will describe the notion of \textit{enlarged Reeb component} of a codimension-one foliation, which makes sense in all dimensions and differs for the classical definition in the case $n = 3$ as it does not contain a single compact leaf.

\(^2\)A three-manifold $X$ is \textit{irreducible} if every embedded two-sphere in $X$ bounds a three-ball in $X$. 

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(i.e., they do not admit any foliations that are minimal for some Riemannian ambient metric); however, every closed three-manifold admits a Riemannian metric together with a CMC foliation by Corollary 1.2.

We now explain the organization of the paper. In Section 2 we cover some of the basic definitions related to CMC foliations. In Section 3 we study the existence of codimension-one, \( SO(n-1) \times \mathbb{R} \)-invariant CMC foliations \( R_{n-1} \) of the Riemannian product of the real number line \( \mathbb{R} \) with the closed unit \((n-1)\)-disk \( \mathbb{D}(1) \subset \mathbb{R}^{n-1} \) with respect to a certain \( SO(n-1) \)-invariant metric, whose leaves are of one of two types: the leaves that intersect \( \mathbb{D}(r_1) \times \mathbb{R} \) (here \( \mathbb{D}(r_1) = \{ x \in \mathbb{R}^{n-1} \mid ||x|| < r_1 \} \) and \( 0 < r_1 < 1 \)) are rotationally symmetric hypersurfaces which are graphical over \( \mathbb{D}(r_1) \times \{0\} \) and asymptotic to the vertical \((n-1)\)-cylinder \( S^{n-2}(r_1) \times \mathbb{R} \); the remaining leaves of \( R_{n-1} \) are the vertical cylinders \( S^{n-2}(r) \times \mathbb{R}, \ r \in [r_1, 1] \). All leaves of \( R_{n-1} \) in \( \mathbb{D}^{n-1}(r_1) \times \mathbb{R} \) are vertical translates of a single such leaf (in particular, they all have the same constant mean curvature, equal to the constant value of the mean curvature of \( S^{n-2}(r_1) \times \mathbb{R} \)), while the (constant) mean curvature values of the cylinders \( S^{n-2}(r) \times \mathbb{R}, \ r \in [r_1, 1] \), vary from leaf to leaf. These foliations \( R_{n-1} \) give rise under the quotient action of \( Z \subset \mathbb{R} \) to what we call enlarged foliated Reeb components \( R_{n-1}/Z \), that are diffeomorphic to \( \mathbb{D}(1) \times S^1 \).

Section 4 will be devoted to proving Theorem 1.1 along the following lines. The sufficient implication follows directly from the divergence theorem. As for the necessary implication, the main result in [16] implies that a smooth, closed \( n \)-manifold \( X \) with Euler characteristic zero admits a smooth, transversely oriented foliation \( F' \) of codimension one. After a simple modification of \( F' \) by the classical technique of turbularization (explained in Subsection 4.1), \( F' \) can be assumed to have at least one non-compact leaf. In Section 4.2 we will prove the existence of a finite collection \( \Delta = \{ \gamma_1, \ldots, \gamma_k \} \) of pairwise disjoint, compact embedded arcs in \( X \) that are transverse to the leaves of \( F' \) and such that every compact leaf of the foliation intersects at least one of these arcs; this existence result will follow from work of Haefliger [5] on the compactness of the set of compact leaves of any codimension-one foliation of \( X \). We will then proceed to modify \( F' \) using again turbularization by introducing pairs of what we called in the previous paragraph "enlarged Reeb components", one pair of these enlarged Reeb components for each \( \gamma_i \in \Delta \). These modifications give rise to a new transversely oriented foliation \( F \) and a related function \( f \in C^\infty(X) \) that is constant along the leaves of \( F \) and that when \( X \) is orientable, satisfies Oshikiri's condition [11, 12] (see also Theorem 2.5 below) for there to exist a Riemannian metric on \( X \) that makes \( F \) a CMC foliation. Since the function \( f \) also changes sign on \( X \), Theorem 2.5 will produce the ambient metric on \( X \) such that the foliation \( F \) satisfies the properties stated in Theorem 1.1 when \( X \) is orientable. This analysis of the orientable case for \( X \) will be done in Section 4.3. In Section 4.4 we give a direct proof of the necessary implication of Theorem 1.1 that avoids the results of Oshikiri and also works when the manifold \( X \) is non-orientable. This direct proof depends on the rotationally invariant foliations constructed in Section 3, Theorem 2 in Moser [9], as well as on a generalization of Sullivan's theorem to the case of non-orientable codimension-one foliations, given in Theorem 4.3.

Finally, in Section 5 we will prove the Structure Theorem 1.3 given below on the geometry and topology of non-minimal CMC foliations of a closed \( n \)-manifold. Before stating this theorem, we fix some notation. For a CMC foliation \( F \) of a (connected)
closed Riemannian $n$-manifold $X$:

- $N_F$ denotes the unit normal vector field to $F$ whose direction coincides with the given transverse orientation.
- $H_F : X \to \mathbb{R}$ stands for the mean curvature function of $F$ with respect to $N_F$.
- $H_F(X) = [\min H_F, \max H_F]$ is the image of $H_F$.
- $C_F$ denotes the union of the compact leaves in $F$, which is a compact subset of $X$ by the aforementioned result of Haefliger [5].

**Theorem 1.3 (Structure Theorem for CMC Foliations)** Let $(X, g)$ be a closed connected Riemannian $n$-manifold which admits a non-minimal CMC foliation $F$. Then:

1. $\int_X H_F dV = 0$ and so, $H_F$ changes sign (here $dV$ denotes the Riemannian volume form with respect to $g$).

2. For $H$ a regular value of $H_F$, $H_F^{-1}(H)$ consists of a finite number of compact leaves of $F$ contained in $\text{Int}(C_F)$.

3. $X - C_F$ consists of a countable number of open components and the leaves in each such component $\Delta$ have the same mean curvature as the finite positive number of compact leaves in $\partial \Delta$; furthermore, every leaf in the closure of $X - C_F$ is stable. In particular, except for a countable subset of $H_F(X)$, every leaf of $F$ with mean curvature $H$ is compact, and for every $H \in H_F(X)$, there exists at least one compact leaf of $F$ with mean curvature $H$.

4. a. Suppose that $L$ is a leaf of $F$ that contains a regular point of $H_F$. Then $L$ is compact, it consists entirely of regular points of $H_F$ and lies in the interior of $C_F$. Furthermore, $L$ and has index\textsuperscript{3} zero if and only if the function $g(\nabla H_F, N_F) = N_F(H_F)$ is negative along $L$, and if the index of $L$ is zero, then it also has nullity zero.

b. Suppose that $L$ is a leaf of $F$ that is disjoint from the regular points of $H_F$. Then the index of $L$ is zero, and if $L$ is a limit leaf\textsuperscript{4} of the CMC lamination of $X$ consisting of the compact leaves of $F$, then $L$ is compact with nullity one.

5. Any leaf of $F$ with mean curvature equal to $\min H_F$ or $\max H_F$ is stable and such a leaf can be chosen to be compact with nullity one.

**Remark 1.4** Let $(X, g)$ and $F$ be as in Theorem 1.3.

(i) Item 5 in Theorem 1.3 implies restrictions on the Ricci curvature of $(X, g)$; namely, $\text{Ric}(N_F) \leq -(n - 1) \max H_F^2$ at some point in $X$, which can be obtained from evaluating the index form of the Jacobi operator of a compact (stable) leaf with mean curvature equal to either $-\min H_F$, or $\max H_F$, at the constant function one on the leaf.

\textsuperscript{3}This index (resp. nullity) is the number of negative eigenvalues (resp. multiplicity of zero as an eigenvalue) of the Jacobi operator of $L$ viewed as a compact, two-sided hypersurface with constant mean curvature in $X$.

\textsuperscript{4}See Definition 2.2 for the definition of a limit leaf of a CMC lamination.
(ii) In the case that \( n = 3 \), \( X \) is orientable and \( \mathcal{F} \) is not topologically a product foliation of \( X = S^2 \times S^1 \) by spheres, then item 1 in Theorem 2.13 of [7] and item 5 in Theorem 1.3 imply that the scalar curvature of \((X,g)\) cannot be everywhere greater than \(-\frac{2}{3} \max H^2_F\).

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2 Preliminaries.

Definition 2.1 A smooth codimension-one lamination of a Riemannian \( n \)-manifold \( X \) is the union of a collection of pairwise disjoint, connected, injectively immersed hypersurfaces, with a certain local product structure. More precisely, it is a pair \((\mathcal{L}, \mathcal{A})\) satisfying:

1. \( \mathcal{L} \) is a closed subset of \( X \);
2. \( \mathcal{A} = \{ \varphi_\beta : D \times (0, 1) \to U_\beta \}_\beta \) is an atlas of (smooth) coordinate charts of \( X \) (here \( D \) is the open unit disk in \( \mathbb{R}^{n-1} \), \( (0, 1) \) is the open unit interval in \( \mathbb{R} \) and \( U_\beta \) is an open subset of \( X \)).
3. For each \( \beta \), there exists a closed subset \( C_\beta \) of \( (0, 1) \) such that \( \varphi_\beta^{-1}(U_\beta \cap \mathcal{L}) = D \times C_\beta \).

We will simply denote laminations by \( \mathcal{L} \), omitting the charts \( \varphi_\beta \) in \( \mathcal{A} \) unless explicitly necessary. A smooth lamination \( \mathcal{L} \) is said to be a foliation of \( X \) if \( \mathcal{L} = X \). Every lamination \( \mathcal{L} \) decomposes into a collection of disjoint, connected smooth hypersurfaces (locally given by \( \varphi_\beta(D \times \{t\}) \), \( t \in C_\beta \), with the notation above), called the leaves of \( \mathcal{L} \). Note that if \( \Delta \subset \mathcal{L} \) is any collection of leaves of \( \mathcal{L} \), then the closure of the union of these leaves has the structure of a lamination within \( \mathcal{L} \), which we will call a sublamination.

A smooth codimension-one lamination \( \mathcal{L} \) of \( X \) is said to be a CMC lamination if each of its leaves has constant mean curvature (possibly varying from leaf to leaf). Given \( H \in \mathbb{R} \), an \( H \)-lamination of \( X \) is a CMC lamination all whose leaves have the same mean curvature \( H \). If \( H = 0 \), the \( H \)-lamination is called a minimal lamination.

Definition 2.2 Given a smooth codimension-one lamination \( \mathcal{L} \) of a Riemannian \( n \)-manifold \( X \), a point \( p \in \mathcal{L} \) is a limit point if there exists a coordinate chart \( \varphi_\beta : D \times (0, 1) \to U_\beta \) as in Definition 2.1 such that \( p \in U_\beta \) and \( \varphi_\beta^{-1}(p) = (x, t) \) with \( t \) belonging to the accumulation set of \( C_\beta \). It is easy to show that if \( p \) is a limit point of a codimension-one lamination \( \mathcal{L} \) (resp. of a \( H \)-lamination), then the leaf \( L \) of \( \mathcal{L} \) passing through \( p \) consists entirely of limit points of \( \mathcal{L} \); in this case, \( L \) is called a limit leaf of \( \mathcal{L} \).

Definition 2.3 Let \( \mathcal{F} \) be a codimension-one foliation of a manifold \( X \). \( \mathcal{F} \) is called transversely orientable if there exists a continuous, nowhere zero vector field whose integral curves intersect transversely to the leaves of \( \mathcal{F} \). Once such a vector field has been chosen, we call \( \mathcal{F} \) a transversely oriented codimension-one foliation. \( \mathcal{F} \) is called orientable if there exists a smooth \((n-1)\)-form on \( X \) whose restriction to the tangent spaces of leaves of \( \mathcal{F} \) is never zero.
**Definition 2.4** Let $\mathcal{F}$ be a transversely oriented, codimension-one foliation of an $n$-manifold $X$ and let $\gamma$ be a smooth simple closed curve contained in a leaf $L$ of $\mathcal{F}$. After picking a Riemannian metric $g$ on $X$, we can define a normal fence above $\gamma$ as follows. Consider exponential coordinates for $(X, g)$ along $\gamma$. The right normal fence is the set
\[ A = \{ \exp_\gamma(t)(sN_F(\gamma(t))) \mid t \in S^1, s \in [0, \varepsilon] \}, \]
where $N_F$ is the positive unit normal vector to $F$ and $\varepsilon > 0$ is sufficiently small so that $A$ defines an embedded annulus in $X$. If the parameter $s$ moves in $[-\varepsilon, 0]$, then we call the annulus a left normal fence.

We finish this section by stating Oshikiri’s theorem [12] about which smooth functions on a compact oriented $n$-manifold $X$ can be viewed as mean curvature functions of a given transversely oriented, codimension-one foliation $F$ on $X$. To state this result, we first need some notation. A domain $D \subset X$ is called saturated if it is a union of leaves of $\mathcal{F}$. A compact, smooth saturated domain $D \subset X$ is called a $(+)$-foliated compact domain ($(+)$-fcd, for short) if the transverse orientation of $\mathcal{F}$ points outward everywhere on $\partial D$, and we call $D$ a $(-)$-foliated compact domain (or $(-)$-fcd) if the transverse orientation of $\mathcal{F}$ is inward pointing everywhere on $\partial D$. A smooth function $f: X \to \mathbb{R}$ is called admissible for $\mathcal{F}$ if there exists a Riemannian metric on $X$ such that $f$ is the oppositely signed mean curvature function of the leaves of $\mathcal{F}$ with respect to the given transversal orientation. In other words for each $x \in X$, $-f(x)$ is the mean curvature of the leaf $L_x$ of $\mathcal{F}$ passing through $x$ with respect to the unit normal vector field to $L_x$ whose direction coincides with the given transverse orientation.

**Theorem 2.5 ([12])** Let $\mathcal{F}$ be a codimension-one, transversely oriented foliation in a compact oriented $n$-manifold $X$, such that $\mathcal{F}$ contains at least one $(+)$-fcd. Then, a smooth function $f: X \to \mathbb{R}$ is admissible for $\mathcal{F}$ if and only if every minimal $(+)$-fcd contains a point where $f$ is positive, and every $(-)$-fcd contains a point where $f$ is negative.

Note that if a foliation $\mathcal{F}$ as in Theorem 2.5 does not contain any $(+)$-fcd, then it also does not contain any $(-)$-fcd and thus Corollary 6.3.4 in Candel and Conlon [1] implies that $\mathcal{F}$ is homologically taut.

### 3 Rotationally symmetric $H$-foliations of Reeb type.

**Definition 3.1** Let $D(R)$ the open disk of radius $R > 0$ in $\mathbb{R}^{n-1}$. For $z \in \mathbb{R}$, let $\Sigma_z \subset \mathbb{R}^{n-1} \times \mathbb{R}$ be the graph of the function
\[ f + z: D(1) \to \mathbb{R}, \quad f(x) = -\sec\left(\frac{\pi}{2}\|x\|^2\right), \quad x \in D(1), \]
and let $\Sigma_\infty$ be the cylinder $\partial D(1) \times \mathbb{R} \subset \mathbb{R}^n$.

1. Consider the foliated closed cylinder $D(1) \times \mathbb{R}$ with leaves $\Sigma_z$, $z \in (-\infty, \infty]$. Up to diffeomorphism, any such foliation is called a Reeb-type foliation of the cylinder.

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5Here, minimal refers to the partial order given by inclusion.
2. After passing to the quotient by the natural action of \( \mathbb{Z} \) acting on \( \mathbb{D}(1) \times \mathbb{R} \) by integer translations in the \( n \)-th variable, we obtain a compact \( n \)-manifold with boundary \( (\mathbb{D}(1) \times \mathbb{R})/\mathbb{Z} \); we call this foliated manifold a \textit{Reeb-type foliation} of \( \mathbb{D} \times S^1 \).

3. Given a foliation \( \mathcal{F} \) of an \( n \)-manifold \( X \), a compact saturated domain \( R \subset X \) is called a \textit{Reeb component} of \( \mathcal{F} \) if it is diffeomorphically equivalent to a Reeb-type foliation of \( \mathbb{D} \times S^1 \).

Assume for the moment that \( n = 2 \). We consider on \( \mathbb{D} = \mathbb{D}(1) \) polar coordinates \( r \in [0,1], \theta \in [0,2\pi) \) so that \( x = r \cos \theta, y = r \sin \theta \). Given \( H > 0 \), we will next construct a rotationally symmetric metric \( ds^2 \) over \( \mathbb{D} \) (depending on \( H \)) such that the Riemannian product \( (\mathbb{D} \times \mathbb{R}, ds^2 + dz^2) \) admits a rotationally symmetric CMC foliation \( \mathcal{F} \) which is topologically a Reeb-type foliation enlarged by a product foliation. By this we mean the following:

(E1) There exists \( r_1 \in (0,1) \) such that the leaves of \( \mathcal{F} \) in \( \mathbb{D}(r_1) \times \mathbb{R} \) are of the form \( \Sigma_z \), \( z \in \mathbb{R} \), where \( \Sigma_z \) is the vertical translate by \( z \in \mathbb{R} \) of a rotationally symmetric \( H \)-graph \( \Sigma \) over \( \mathbb{D}(r_1) \times \{0\} \), such that the generating curve \( \Gamma \subset \{ (x,0,z) \mid x \in [0,1], z \in \mathbb{R} \} \) of \( \Sigma \) can be globally parameterized as \( \Gamma(r) = (r,0,z(r)) \), with \( z: [0,r_1) \to \mathbb{R} \) being a smooth function that satisfies \( z'(0) = 0, z'(r) > 0 \) for all \( r \in (0,1) \) and \( z(r) \to +\infty \) as \( r \to r_1^{-} \). Note that the restriction of \( \mathcal{F} \) to \( \mathbb{D}(r_1) \times \mathbb{R} \) is an \( H \)-foliation with respect to the ambient metric \( ds^2 + dz^2 \).

(E2) The restriction of \( \mathcal{F} \) to \( (\mathbb{D} - \mathbb{D}(r_1)) \times \mathbb{R} \) is the product foliation by vertical cylinders \( (\partial \mathbb{D}(R)) \times \mathbb{R}, R \in [r_1,1] \). These cylinders also have constant mean curvature, which it might vary from leaf to leaf. Note that the restriction of \( \mathcal{F} \) to \( \mathbb{D}(r_1) \times \mathbb{R} \) is a Reeb-type foliation in the sense of item 1 of Definition 3.1.

\textbf{Definition 3.2} The analysis to produce the \( (SO(1) \times \mathbb{R}) \)-invariant Reeb-type foliation \( \mathcal{F} \) of \( \mathbb{D} \times \mathbb{R} \) will be entirely performed in the orbit space \( \{ (x,0,z) \mid x \in [0,1], z \in \mathbb{R} \} \) of \( \mathbb{D} \times \mathbb{R} \); in fact, the argument that follows can be easily generalized to the \( n \)-dimensional case, thereby producing an \( (SO(n-1) \times \mathbb{R}) \)-invariant foliation of \( \mathbb{D}(r_1) \times \mathbb{R} \) by vertical graphs \( \Sigma_z = \Sigma + (0,z), z \in \mathbb{R} \), so that each \( \Sigma_z \) is an \( SO(n-1) \)-invariant \( H \)-hypersurface which is a graph over its vertical projection over \( \mathbb{D}(r_1) \), together with the \( (n-1) \)-dimensional cylinders \( (\partial \mathbb{D}(R)) \times \mathbb{R}, R \in [r_1,1] \); this \( (SO(n-1) \times \mathbb{R}) \)-invariant foliation is what we will call an \textit{enlarged Reeb-type foliation} of the cylinder \( \mathbb{D} \times \mathbb{R} \). After quotienting by the natural \( \mathbb{Z} \)-action, the quotient foliation of \( \mathbb{D} \times S^1 \) will be called an \textit{enlarged Reeb-type foliation} of \( \mathbb{D} \times S^1 \), see Figure 3.

We return to the case \( n = 2 \). The following analysis of the ODE system that corresponds to a rotationally symmetric \( H \)-surface in \( \mathbb{D} \times \mathbb{R} \) is inspired by the paper [4] by Figueroa, Mercuri and Pedrosa. Every rotationally symmetric Riemannian metric on \( \mathbb{D} \) can be written in the form

\begin{equation}
 ds^2 = dr^2 + f(r)^2 d\theta^2, \quad (1)
\end{equation}

where \( f: [0,1] \to [0,\infty) \) is a smooth function that satisfies \( f'(0) = 1 \) and all whose derivatives of even order (including order zero) vanish at \( r = 0 \). Note that with respect
to $ds^2$, $\partial_r = \frac{\partial}{\partial r}$ is unitary and orthogonal to $\partial_\theta = \frac{\partial}{\partial \theta}$, and that $|\partial_\theta| = f$. It is well-known that the circle $\gamma_r$ of radius $r \in (0, 1]$ in $(\mathbb{D}, ds^2)$ has constant geodesic curvature given by

$$\kappa_r = \frac{f'(r)}{f(r)} \quad (2)$$

with respect to the inner pointing unit normal vector.

We consider on $\mathbb{D} \times \mathbb{R}$ the product metric $g = ds^2 + dz^2$, where $z \in \mathbb{R}$ represents height in the $\mathbb{R}$-factor. The circle group $SO(1)$ acts on $(\mathbb{D} \times \mathbb{R}, g)$ by isometries (rotations about the $z$-axis $\{r = 0\}$), with orbit space $\mathcal{B} = \{(r, z) \in \mathbb{R}^2 \mid 0 \leq r \leq 1\}$, and the induced metric by $g$ on $\mathcal{B}^* = \{(r, z) \mid 0 < r \leq 1\}$ is flat. Suppose that $s \mapsto \Gamma(s) = (r(s), z(s)) \in \mathcal{B}^*$ is a smooth curve parameterized by arc length. Let us denote by $\sigma(s)$ the angle that the velocity vector $\dot{\Gamma}(s)$ makes with the $\partial_r$ direction. The planar curvature of $\Gamma$ is $\kappa_\Gamma(s) = \dot{\sigma}(s)$, and the Frenet dihedron of $\Gamma$ can be written as

$$t = t(s) = (\dot{r}, \dot{z}) = (\cos \sigma, \sin \sigma), \quad n = n(s) = (-\sin \sigma, \cos \sigma).$$

Let $\Sigma \subset \mathbb{D}^* \times \mathbb{R}$ be the surface generated by $\Gamma$ after rotation about the $z$-axis, where $\mathbb{D}^* = \mathbb{D} - \{0\}$. We want to relate the mean curvature function $\mathcal{H}$ of $\Sigma$ with the geodesic curvatures of $\Gamma$ and of the circle of radius $r(s)$ in $(\mathbb{D}, ds^2)$. As $\Sigma$ is rotationally symmetric, we can do this by computing the mean curvature of $\Sigma$ at a point $p \in \Sigma \cap \{(x, 0, z) \mid 0 < x \leq 1\}$. After a slight abuse of notation, we will identify $\Sigma \cap \{(x, 0, z) \mid 0 < x \leq 1\}$ with $\Gamma$. The tangent plane $T_p \Sigma$ at such a point $p \in \Sigma$ admits the orthonormal basis $\{X = \frac{1}{f} \partial_\theta, \dot{\Gamma}\}$ and thus,

$$2\mathcal{H} = g \left( \nabla_X X, N \right) + g \left( \nabla_{\dot{\Gamma}} \dot{\Gamma}, N \right), \quad (3)$$

where $\nabla$ is the Riemannian connection of $g$ and $N$ stands for the unit normal vector to $\Sigma$ that coincides with $n$ at $p$ (the mean curvature function $\mathcal{H}$ is computed with respect
to \( N \). As \( X \) is a horizontal vector field on the Riemannian product \((\bar{\mathbb{D}}^s \times \mathbb{R}, g)\), then \( \nabla_X X \) is also horizontal and is given by \( \nabla_X X = (\nabla^D_X X, 0) \), where \( \nabla^D \) is the Riemannian connection in \((\mathbb{D}, ds^2)\). Since \( N = n = -\sin \sigma \partial_r + \cos \sigma \partial_z \), then the first term in the right-hand-side of (3) is

\[
g(\nabla_X X, N) = -\sin \sigma ds^2(\nabla^D_X X, \partial_r) = \sin \sigma \kappa_r = \sin \sigma \frac{f'(r)}{f(r)},
\]

where \( r = r(p) \). As for the second term of (3), since the vertical plane \( \{y = 0\} \) is totally geodesic in \((\bar{\mathbb{D}} \times \mathbb{R}, g)\), then \( g\left(\nabla_Y \tilde{\Gamma}, N\right) = \kappa = \dot{\sigma} \). In summary, given \( H \in \mathbb{R} \), the curve \( \Gamma = \Gamma(s) \) generates after rotation an \( H \)-surface if and only \( r(s), z(s), \sigma(s) \) satisfy the ODE system

\[
\begin{align*}
\dot{r} &= \cos \sigma \\
\dot{z} &= \sin \sigma \\
\dot{\sigma} &= 2H - \sin \sigma \frac{f'(r)}{f(r)}
\end{align*}
\]

The above system only makes sense in \( B^s \). Nevertheless, it is well-known that solutions of (4) that go to the inner boundary \( \{r = 0\} \) of \( B^s \) must enter perpendicularly (see e.g., Eells and Ratto [3] or Hsiang and Hsiang [6]). Three other basic properties of the system (4) are the following ones.

**P1.** Any vertical translation of a solution of (4) is also a solution of (4). In particular, (4) admits a first integral.

**P2.** If a solution \( \Gamma(s) \) of (4) defined in \((s_0 - \varepsilon, s_0)\) has vertical tangent line at \( s_0 \), i.e., \( \sigma(s_0) = \pm \pi/2 \), then \( \Gamma(s) \) can be extended by reflecting across the horizontal line \( \{z = z(s_0)\} \) to a solution of (4) defined in the interval \((s_0 - \varepsilon, s_0 + \varepsilon)\).

**P3.** The vertical line \( \Gamma(s) = (r, s) \) (with \( r \in (0, 1) \) fixed) is a solution of (4), and the value \( H \) of the mean curvature of the corresponding vertical cylinder \( C(r) = (\partial \mathbb{D}(r)) \times \mathbb{R} \) is \( H = \frac{f'(r)}{2f(r)} \).

Consider the smooth function \( h: [0, 1] \to \mathbb{R} \) given by \( h(r) = 2H \int_0^r f(u)du \). It is straightforward to show that

\[
J(s) = f(r(s)) \sin \sigma(s) - h(r(s))
\]

is a first integral of (4), i.e., \( J(s) \) is constant along any solution of (4). Rather than classifying all solutions of (4) by analyzing every possible value of \( J \), we will only study the case \( J = 0 \) as we are interested in producing an example of an \( H \)-surface satisfying certain conditions. To justify this choice of \( J \), note that in the particular cases \( f(r) = r \) (which produces the flat standard metric on \( \mathbb{D} \)), \( f(r) = \sin(r) \) (for \( S^2(1) \)) and \( f(r) = \sinh(r) \) (for \( \mathbb{H}^2(-1) \)), the choice \( J = 0 \) for the first integral gives rise for \( H > 0 \) to the unique \( H \)-sphere (up to congruences) in \( \mathbb{D} \times \mathbb{R}, S^2(1) \times \mathbb{R} \) and \( \mathbb{H}^2(-1) \times \mathbb{R} \), respectively. In our case, we will choose \( f \) so that the behavior of the profile curve \( \Gamma \) solution of (4) is similar to the one of a lower half-sphere, but with infinite length and asymptotic to a vertical line instead of achieving the vertical direction for its tangent line in a compact portion of curve.
Suppose that $\Gamma(s)$ is a solution of (4) with $J = 0$. Thus, $\sin \sigma = h/f$ and $\cos \sigma = \sqrt{1 - (h/f)^2}$. These equations only make sense if $h/f$ takes values in $[1, 1]$; in this line, note that $h(0) = 0$, $h'(0) = 2H f(0) = 0$, $h''(0) = 2H f'(0) = 2H$. Hence, L'Hôpital's rule insures that $(h/f)(0) = 0$, $(h/f)'(0) = H$. This implies that the number

$$r_1 = \sup \{ r \in (0, 1) \mid (h/f)(r) \in (0, 1) \text{ for all } r \in (0, r_1) \} \in (0, 1)$$

exists. Note that if $r_1 < 1$, then $(h/f)(r_1) = 1$; otherwise $r_1 = 1$ and we only can ensure that $\lim_{r \to 1^-} (h/f)(r) \leq 1$. Therefore, in $(0, r_1)$ we have

$$\frac{dz}{dr} = \frac{dz}{ds} \frac{ds}{dr} = \frac{h/f}{\sqrt{1 - (h/f)^2}}.$$  \hspace{1cm} (6)

Note that as $(h/f)(0) = 0$ and $h/f \in [0, 1]$ in $[0, r_1)$, then (6) implies that $z = z(r)$ is an increasing function of $r \in [0, r_1)$ (strictly increasing if $r > 0$), whose graph intersects the vertical line $\{ r = 0 \}$ orthogonally. We next compute the length of $\Gamma$:

$$\text{length}(\Gamma)_R^0 = \int_0^R \sqrt{1 + \left( \frac{dz}{dr} \right)^2} dr = \int_0^R \frac{dr}{\sqrt{1 - (h/f)^2}} dr,$$  \hspace{1cm} (7)

so we want to choose $f$ so that $\lim_{r \to r_1^-} \text{length}(\Gamma)_R^0 = +\infty$. To do this, consider the smooth function

$$f : [0, r_1) \to \mathbb{R}, \quad f(r) = 2H \frac{e^{2H \arcsin(r-r_1)}}{\sqrt{1 - (r_1 - r)^2}}.$$  \hspace{1cm} (8)

Note that with definition (8), $f$ fails to satisfy the conditions to define a smooth metric at $r = 0$ (for instance, $f(0) > 0$). Thus we will need to truncate $f$ in some interval $[0, r_0)$ with $0 < r_0 < r_1$ and glue the above expression of $f$ with a suitably defined function in $[0, r_0)$. For the moment we will only work in an interval of the form $[r_0, r_1)$ with $0 < r_0 < r_1$. A direct computation gives that

$$h(r) = 2H \int_0^r f(u) du = 2He^{2H \arcsin(r-r_1)} = f(r)\sqrt{1 - (r_1 - r)^2},$$

and thus,

$$\text{length}(\Gamma)_R^{r_0} = \int_{r_0}^R \frac{dr}{r_1 - r} = \log \left( \frac{r_1 - r_0}{r_1 - R} \right),$$

which limits to $+\infty$ if $R \to r_1^-$, as desired.

We conclude that with the choice of $f$ given by (8), every solution $\Gamma$ of (4) can be globally parameterized as $\Gamma(r) = (r, z(r))$ with $z : [0, r_1) \to \mathbb{R}$ a smooth function that satisfies $z'(0) = 0$, $z'(r) > 0$ for all $r \in (0, 1)$ and $z(r) \to +\infty$ as $r \to r_1^-$. We now fix $r_0 \in (0, r_1)$ and substitute the definition of $f$ in (8) in $[0, r_0]$ by a smooth positive function so that the resulting function, also called $f$, is of class $C^\infty$ at $r = r_0$ and $f'(0) = 1$, $f''(0) = 0$ for all $k \in \mathbb{N} \cup \{0\}$, and $h/f$ keeps taking values in $[0, 1)$. Hence, plugging this new function $f : [0, r_1) \to (0, \infty)$ in (1) we define a smooth, $SO(1)$-invariant metric in $\mathbb{D}(r_1)$. By uniqueness of solutions of the ODE system (4) with given initial values, we also conclude that the restriction of the above graphical solution $\Gamma$ to
admits a transversely oriented smooth foliation on \( X \).

Nowhere zero smooth vector field, then item (b) of Theorem 1 in [16] implies that there exists a solution \( \Gamma(r) = (r, z(r)) \) over \([0, r_1]\) with \( z(r) \) strictly increasing in \((0, r_1)\) and \( z'(0) = 0 \).

We next extend \( f \) to \([r_1, 1]\). Note that (8) defines a smooth function at \( r = r_1 \), so simply extend \( f \) smoothly to \([0, 1]\) by a positive function in \([r_1, 1]\), also denoted by \( f \), all whose derivatives coincide with the corresponding ones of (8) at \( r_1 \). Plugging this function \( f \) in (1) we obtain a smooth, rotationally symmetric metric \( ds^2 \) on \( D \).

Recall that for any choice of \( f \), the vertical cylinder \( C(r) \) of radius \( r \) has constant mean curvature \( \frac{f'(r)}{f(r)} \) with respect to \( ds^2 + dz^2 \), by property P3 above. As the graphical solution \( \Gamma(r) = (r, z(r)), r \in [0, r_1) \) constructed in the last paragraph generates an \( H \)-surface \( \Sigma \) which is smoothly asymptotic to the upper end of the vertical cylinder \( C(r_1) \), then the mean curvature of \( C(r_1) \) is also \( H \), so this must be the value of \( \frac{f'(r_1)}{\frac{f(r_1)}{2}} \) (this can be checked by direct computation in (8)). Finally, we define the desired foliation \( F \) as the vertical translates of \( \Sigma \) in \( \mathbb{D}(r_1) \times \mathbb{R} \) together the collection of vertical cylinders \( C(r), r \in [r_1, 1] \).

**Remark 3.3** (A) Let \( r_2 \in (r_1, 1) \). If we choose the extension of \( f \) to \([r_1, 1]\) so that it additionally becomes constant in \([r_2, 1]\), then the corresponding mean curvatures of the cylinders \( C(r), r \in [r_2, 1] \) are zero and the ambient metric on \((\mathbb{D}(1) - \mathbb{D}(r_2)) \times \mathbb{R}\) is flat. Note that in the above construction, \( H > 0 \), \( 0 < r_1 < r_2 \) are arbitrary.

(B) The construction of the \( H \)-foliation \( F \) of \( \mathbb{D}(1) \times \mathbb{R} \) induces an \( H \)-foliation of \( \mathbb{D}(1) \times (\mathbb{R}/\lambda \mathbb{Z}) \) for any value of \( \lambda > 0 \). Note that the \((n - 1)\)-dimensional volume of the compact hypersurface \((\partial \mathbb{D}(\delta)) \times (\mathbb{R}/\lambda \mathbb{Z})\) is independent of \( \delta \in [r_2, 1] \) and of \( H > 0 \), and it can be prescribed arbitrarily by picking the appropriate value of \( \lambda \).

**Definition 3.4** We say that a codimension-one foliation \( F \) of a smooth \( n \)-manifold \( X \) contains an enlarged Reeb component \( \Omega \subset X \) if \( \Omega \) is a smooth saturated domain and there exists a diffeomorphism \( \phi: \mathbb{D} \times S^1 \to \overline{\Omega} \) such that the pullback foliation \( \phi^* (F|_{\overline{\Omega}}) \) is an enlarged Reeb-type foliation of \( \mathbb{D} \times S^1 \), according to Definition 3.2. We refer to the image foliated region \( \phi(\mathbb{D} \times S^1) \subset X \) as an enlarged Reeb component of \( F \). Note that every enlarged Reeb component of \( F \) contains a classical Reeb component, according to Definition 3.1.

### 4 The proof of Theorem 1.1.

Let \( X \) be a closed smooth \( n \)-manifold. The existence of a smooth, transversely oriented codimension-one foliation of \( X \) implies that the Euler characteristic of \( X \) vanishes (apply the Poincaré-Hopf index theorem to the unit normal vector field to the transversely oriented foliation with respect to an arbitrarily chosen metric on \( X \)). Therefore the necessary implication in Theorem 1.1 is clear. In fact, item (a) of Theorem 1 in Thurston [16] shows that \( X \) admits a smooth foliation if and only if it has vanishing Euler characteristic. Since such an \( X \) with Euler characteristic zero always admits a nowhere zero smooth vector field, then item (b) of Theorem 1 in [16] implies that there exists a transversely oriented foliation on \( X \).

Henceforth, to prove the sufficient implication of Theorem 1.1 we can assume that \( X \) admits a transversely oriented smooth foliation \( F \). We will also fix an auxiliary...
4.1 Turbularization along a closed transversal.

Consider $S^1$ to be the quotient $\mathbb{R}/\mathbb{Z}$ with the orientation induced by the usual orientation on $\mathbb{R}$ and let $\Gamma: S^1 \to X$ be a smooth curve transverse to $\mathcal{F}$, which exists by the following elementary argument. Consider a maximal integral curve $\gamma$ of the unit normal field $N_F$ to $\mathcal{F}$. If $\gamma$ is closed, then we are done. If $\gamma$ never closes, then the compactness of $X$ implies that $\gamma$ enters twice (actually, infinitely many times) inside some product coordinate chart $U = \mathbb{D} \times (0, 1)$ of the foliation. Thus, $(X - U) \cap \gamma$ contains a subarc $\hat{\gamma}$, one of whose end points $\hat{\gamma}(0)$ lies in $\mathbb{D} \times \{0\}$ and the other one $\hat{\gamma}(1)$ lies in $\mathbb{D} \times \{1\}$. By basically joining $\hat{\gamma}(0)$ with $\hat{\gamma}(1)$ by a ‘straight line segment’ in $U$ and then smoothing the resulting embedded closed curve, we construct the desired closed transversal $\Gamma: S^1 \to X$ to $\mathcal{F}$. Finally, after replacing $\Gamma$ by a small perturbation of its two-sheeted cover will assume that a small regular neighborhood of $\Gamma$ is orientable. We can assume without loss of generality that the inner product with respect to $g$ of the velocity vector field to $\Gamma$ with $N_F$ is positive.

Given $\sigma \in \{+, -\}$, $\Gamma$ and $\mathcal{F}$, we next describe a method for modifying $\mathcal{F}$ in a small regular neighborhood of $\Gamma$ giving rise to a new foliation $\mathcal{F}(\Gamma, \sigma)$ of $X$, by means of a standard technique called turbularization. Later we will introduce some metric aspects in the turbularization process that will be useful in our goal to prove the necessary implication of Theorem 1.1.

First choose $\varepsilon > 0$ sufficiently small so that the closed $\varepsilon$-neighborhood $V(\Gamma, 8\varepsilon)$ of $\Gamma$ is parameterized by

$$\Phi: \mathbb{D}(8\varepsilon) \times S^1 \to V(\Gamma, 8\varepsilon),$$

where $\mathbb{D}(r) = \{x \in \mathbb{R}^{n-1} \mid \|x\| \leq r\}$ for each $r > 0$. We can also take $\Phi$ so that the restricted foliation $\mathcal{F}|_{V(\Gamma, 8\varepsilon)}$ of $V(\Gamma, 8\varepsilon)$ consists of $(n-1)$-dimensional disks leaves of the form $\Phi(\mathbb{D}(8\varepsilon) \times \{\theta\})$, $\theta \in S^1$; note that each orbit of the action of $S^1$ on $V(\Gamma, 8\varepsilon)$, induced by pushing forward via $\Phi$ the product action of $S^1$ on $\mathbb{D}(8\varepsilon) \times S^1$, intersects each leaf of $\mathcal{F}|_{V(\Gamma, 8\varepsilon)}$ transversely in a single point. In particular, given $r \in (0, 8\varepsilon]$, the compact hypersurface

$$\mathbb{T}(r) = \Phi(\partial \mathbb{D}(r) \times S^1)$$

obtained as the orbit of the action of $S^1$ on $\partial \mathbb{D}(r)$, intersects each of the disks leaves of $\mathcal{F}|_{V(\Gamma, 8\varepsilon)}$ transversely in an embedded $(n-2)$-sphere. Note that $SO(n-1) \times S^1$ acts naturally on $\mathbb{D}(8\varepsilon) \times S^1$ (hence on $V(\Gamma, 8\varepsilon)$ via $\Phi$) so that each $(A, \theta) \in SO(n-1) \times S^1$ acts by rotation by $A$ in $\mathbb{D}(8\varepsilon)$ and by translation by $\theta$ in $S^1$.

Given $\sigma \in \{+, -\}$, let $S_{\sigma}$ be a connected, complete, non-compact smooth hypersurface with compact boundary in $V(\Gamma, 8\varepsilon)$ with the following properties:

S1. $S_{\sigma}$ is contained in $\Phi((\mathbb{D}(8\varepsilon) - \mathbb{D}(4\varepsilon)) \times S^1)$ and it is of revolution, i.e., if $\Phi(x, \theta) \in S_{\sigma}$, then $\Phi(Ax, \theta) \in S_{\sigma}$ for any $A \in SO(n-1)$.

S2. $S_{\sigma}$ is graphical (with respect to the orbits of the $S^1$-action on the second factor) over the annulus $\Phi((\mathbb{D}(6\varepsilon) - \mathbb{D}(4\varepsilon)) \times S^1)$. 

Riemannian metric $g$ on $X$. Our goal will be to make a possibly different choice of $\mathcal{F}$ and $g$ so that $\mathcal{F}$ is a non-minimal CMC foliation with respect to $g$. 

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S3. The intersection of $S_\sigma$ with $\Phi(\mathbb{D}(8\varepsilon) - \mathbb{D}(6\varepsilon) \times S^1)$ equals the annulus $\Phi(\mathbb{D}(8\varepsilon) - \mathbb{D}(6\varepsilon)) \times \{0\}$ (recall that this annulus is part of a leaf of $\mathcal{F}|_{V(\Gamma, 8\varepsilon)}$).

S4. $S_\sigma$ is smoothly asymptotic in $V(\Gamma, 8\varepsilon)$ to the hypersurface $T(4\varepsilon)$.

S5. The orientation of $T(4\varepsilon)$ induced by the annular graph $S_\sigma$ coincides with to the inward pointing unit normal to the solid region $\Phi(\mathbb{D}(4\varepsilon) \times S^1)$ when $\sigma = +$, and to the outward pointing unit normal when $\sigma = -$; see Figure 4.1 for the case of $S_\sigma = S_+$.

We are now in a position to describe $\mathcal{F}(\Gamma, \sigma)$. Note that Property S2 above implies that the family of translates $S_\sigma(\theta)$ of $S_\sigma$ by elements $\theta \in S^1$ in the second factor of the $(SO(n-1) \times S^1)$-action on $V(\Gamma, 8\varepsilon)$, are all disjoint. These translates $S_\sigma(\theta)$, $\theta \in S^1$, together with the hypersurfaces $\mathcal{T}(t)$, $t \in (2\varepsilon, 4\varepsilon]$ and with the leaves of $\mathcal{F} \cap [X - \Phi(\mathbb{D}(6\varepsilon) \times S^1)]$ produce a smooth, transversely oriented foliation of $X - \Phi(\mathbb{D}(2\varepsilon) \times S^1)$. In turn, this foliation can be extended to a transversely oriented foliation $\mathcal{F}(\Gamma, \sigma)$ of $X$ by attaching an enlarged Reeb component $\Omega = \Omega(\Gamma, \sigma)$ along the boundary $\mathcal{T}(2\varepsilon)$ of $X - \Phi(\mathbb{D}(2\varepsilon) \times S^1)$ and so that the $S^1$-action on $\Phi(\mathbb{D}(8\varepsilon) - \mathbb{D}(2\varepsilon)) \times S^1$ agrees with the natural translational action of $S^1$ on $\Omega$.

4.2 Constructing the desired foliation of $X$.

Using the above construction of $\mathcal{F}(\Gamma, \sigma)$ from $\mathcal{F}$, $\Gamma$ and $\sigma$, we will show how to obtain a transversely oriented, codimension-one foliation of $X$ such that with respect to some Riemannian metric on $X$ (to be defined in Sections 4.3 and 4.4), the foliation is CMC. We start with a smooth, transversely oriented, codimension-one foliation $\mathcal{F}$ on $X$. If every leaf of $\mathcal{F}$ is compact, then $\mathcal{F}$ is equivalent to a bundle over a circle (this is a
consequence of the Reeb Stability Theorem, see e.g., Theorem 2.4.1 in [1]); in this case, there exists a closed transversal $\Gamma \subset X$ which intersects every leaf of $\mathcal{F}$ in a single point. After doing turbularization along $\Gamma$, we will henceforth assume that $\mathcal{F}$ contains a non-compact leaf.

**Lemma 4.1** There exists a finite collection $\Delta = \{\gamma_1, \ldots, \gamma_k\}$ of pairwise disjoint, compact embedded arcs that are transverse to the leaves of $\mathcal{F}$ and positively oriented with respect to $N_\mathcal{F}$, such that:

1. Every compact leaf of $\mathcal{F}$ intersects at least one of the $\gamma_i$.

2. For each $i = 1, \ldots, k$, the end points of $\gamma_i$ lie on non-compact leaves of $\mathcal{F}$.

**Proof.** Let $\mathcal{C}_\mathcal{F}$ be the set of compact leaves of $\mathcal{F}$. After picking a metric on $X$, we can endow $\mathcal{C}_\mathcal{F}$ with the structure of a compact metric space with the induced distance between compact leaves (compactness of $\mathcal{C}_\mathcal{F}$ follows from Haefliger [5], also see Theorem 6.1.1 in [1]). Given a leaf $L \in \mathcal{C}_\mathcal{F}$, then either $L$ lies in a maximal, compact oriented 1-parameter family $I = \{L_t \mid t \in [0, 1]\} \subset \mathcal{C}_\mathcal{F}$ or it fails to have this property.

In the first case, the leaves $L_0, L_1$ are limits of non-compact leaves of $\mathcal{F}$. Pick points $p(0), p(1)$ in non-compact leaves of $\mathcal{F}$ sufficiently close to $L_0, L_1$ so that there exists a positively oriented transversal arc $\gamma_L$ joining $p(0)$ with $p(1)$ and which intersects exactly once each of the leaves in $I$. If on the contrary, $L$ does not lie in any compact oriented 1-parameter family $I$ as before, then a slight modification of the above arguments applies to $L$ in order to find an arbitrarily short, positively oriented transversal arc $\gamma_L$ intersecting $L$ exactly once and with end points $p(0), p(1)$ in non-compact leaves of $\mathcal{F}$.

Hence we have associated to each $L \in \mathcal{C}_\mathcal{F}$ an open arc $\gamma_L$ that intersects $L$ with end points in $X - \mathcal{C}_\mathcal{F}$. For each open transversal arc $\gamma_L$ as before, consider the set of leaves $A(\gamma_L)$ in $\mathcal{C}_\mathcal{F}$ that intersect $\gamma_L$. Clearly $A(\gamma_L)$ is an open set in $\mathcal{C}_\mathcal{F}$. As $\mathcal{C}_\mathcal{F}$ is compact, then we can extract a finite open subcover from the family $\{A(\gamma_L) \mid L \in \mathcal{C}_\mathcal{F}\}$, and the lemma follows by choosing the transversal arcs associated to the finite subcover. $\square$

Our next goal is to modify $\mathcal{F}$ by turbularization along pairs $\Gamma_1, \Gamma_2$ of disjoint closed transversals associated to each $\gamma_i \in \Delta$ with the notation of Lemma 4.1. We next explain how to associate such a pair $\Gamma_1, \Gamma_2$ to $\gamma_i$. Consider one of the oriented arcs $\gamma \in \Delta$ appearing in Lemma 4.1. Note that $\gamma$ intersects $\mathcal{C}_\mathcal{F}$ in a compact set. As we travel along $\gamma$ with its orientation, there exists a first point $p(\gamma)$ in $\gamma$ that lies in a compact leaf of $\mathcal{F}$, and a last point $q(\gamma)$ in $\gamma$ that lies in a compact leaf of $\mathcal{F}$. We label by $L(p(\gamma)), L(q(\gamma))$ the (compact) leaves of $\mathcal{F}$ passing through $p(\gamma)$ and $q(\gamma)$, respectively. Note that

- $L(p(\gamma))$ has non-trivial holonomy on its side containing the starting point of $\gamma$.
- $L(q(\gamma))$ has non-trivial holonomy on its side containing the end point of $\gamma$.

Thus, we may assume that there exists a closed embedded curve $c_1 \subset L(p(\gamma))$, a left normal fence above $c_1$ and non-compact leaves of $\mathcal{F}$ limiting to $L(p(\gamma))$ on the local side of $L(p(\gamma))$ that contains the initial point of $\gamma$. In fact, after a small perturbation of $\gamma$ we may assume that this normal fence is chosen to contain the subarc $\gamma_{p(\gamma)}^{0}$ of $\gamma$.
between the beginning point of $\gamma$ and $p(\gamma)$ (after possibly shortening $\gamma$). A standard argument allows us to find a positively oriented closed transversal $\Gamma_{p(\gamma)}$ to $\mathcal{F}$ that lies in the left normal fence, intersects $\gamma^{0}_{p(\gamma)}$ at a single point and is topologically parallel to $c_{1}$, see Figure 3.

Next we perform turbulization around $\Gamma_{p(\gamma)}$, constructing a foliation $\mathcal{F}(\Gamma_{p(\gamma)}, -)$ of $X$ as we explained in Section 4.1; in particular, $\mathcal{F}(\Gamma_{p(\gamma)}, -)$ contains an enlarged Reeb component $\Omega(\Gamma_{p(\gamma)}, -) \subset \Phi(\overline{B}(3\varepsilon) \times S^{1})$ (here we are using the notation in (9)) and $\mathcal{F}(\Gamma_{p(\gamma)}, -)$ coincides with the previous foliation $\mathcal{F}$ outside $\Phi(\overline{B}(6\varepsilon) \times S^{1})$.

We next do a similar construction on the “future” side of $q(\gamma)$, i.e., there exists a closed embedded curve $c_{2} \subset L(q(\gamma))$, a right normal fence above $c_{2}$ containing the subarc $\gamma|_{1}^{q(\gamma)}$ of $\gamma$ between $q(\gamma)$ and the ending point of $\gamma$, and a positively oriented closed transversal $\Gamma_{q(\gamma)}$ to $\mathcal{F}$ that lies in the normal fence, intersects $\gamma|_{1}^{q(\gamma)}$ at a single point and is topologically parallel to $c_{2}$, and then we do turbulization around $\Gamma_{q(\gamma)}$ by constructing a new foliation $\mathcal{F}(\Gamma_{q(\gamma)}, +)$ of $X$ that contains an enlarged Reeb component $\Omega(\Gamma_{q(\gamma)}, +)$ and such that $\mathcal{F}(\Gamma_{q(\gamma)}, +)$ coincides with the previous foliation $\mathcal{F}(\Gamma_{p(\gamma)}, -)$ outside an embedded neighborhood of $\Gamma_{q(\gamma)}$. Note that $\mathcal{F}(\Gamma_{q(\gamma)}, +)$ also contains the enlarged Reeb component $\Omega(\Gamma_{p(\gamma)}, -)$ and that both enlarged Reeb components $\Omega(\Gamma_{p(\gamma)}, -), \Omega(\Gamma_{q(\gamma)}, +)$ can be assumed to be disjoint.

Finally, we repeat the above process for each transversal arc $\gamma_{i} \in \Delta$ appearing in Lemma 4.1, increasing the number of pairwise disjoint enlarged Reeb components (two for each $\gamma_{i}$) until producing a foliation $\mathcal{F}'$ that contains $2k$ enlarged Reeb components $\Omega(\Gamma_{p(\gamma_{i})}, -) \subset \Phi^{*}_{2}(\overline{B}(3\varepsilon) \times S^{1}), \Omega(\Gamma_{q(\gamma_{i})}, +) \subset \Phi_{1}(\overline{B}(3\varepsilon) \times S^{1}), i = 1, \ldots, k$, where
Figure 4: The enlarged Reeb components $\Omega(\Gamma_{p(\gamma)}, -)$, $\Omega(\Gamma_{q(\gamma)}, +)$ crossing each of the open transversal arcs $\gamma \in \Delta$.

$\Gamma^1 = \Gamma_{q(\gamma)}$, $\Gamma^2 = \Gamma_{p(\gamma)}$ for each open transversal arc $\gamma = \gamma_i \in \Delta$, and the $\Phi^i_j : \mathbb{D}(8\varepsilon) \times S^1 \to V(\Gamma^i_j, 8\varepsilon)$ parameterize embedded tubular neighborhoods of the closed transversals $\Gamma^i_j$ (clearly we can assume that $\varepsilon > 0$ is common to all these tubular neighborhoods), see Figure 4.

4.3 Producing the desired metric on $X$: the case $X$ is orientable.

The final step in our construction is to produce a smooth Riemannian metric on $X$ that makes $\mathcal{F}'$ a CMC foliation. To do so we will first analyze the case that $X$ is orientable, in which the argument is simpler and based on Oshikiri’s condition (Theorem 2.5). In next section we will give a different proof based on Sullivan’s arguments in [15] which does not require either orientability for $X$ or Oshikiri’s result.

Consider the smooth function $f$ on $X$ which equals the mean curvature function of the enlarged Reeb foliations $\Omega(\Gamma_{p(\gamma)}, -)$, $\Omega(\Gamma_{q(\gamma)}, +)$ of $\mathcal{F}'$ and is extended by zero to the complement of the union of these enlarged Reeb components. We now check that $f$ satisfies the conditions of Theorem 2.5 with respect to the foliation $\mathcal{F}'$. First observe that, with the notation introduced at the end of Section 4.2, the portion $R(\Gamma^i_j, \pm)$ of each of the enlarged Reeb components $\Omega(\Gamma^i_j, \pm)$ given by the closure of their simply-connected leaves (this is, after unwrapping $\Gamma^i_j$, $R(\Gamma^i_j, \pm)$ lifts to $\mathbb{D}(r_1) \times \mathbb{R}$ with the notation of item (E1) just before Definition 3.2), we have that:

1. $R(\Gamma^i_j, \pm)$ is a Reeb component of $\mathcal{F}'$ (see item 3 of Definition 3.1).
(R2) \( R(\Gamma^i_j, \pm) \) is a minimal (+) or (−)-fcd.

(R3) \( f \) has points inside \( R(\Gamma^i_j, \pm) \) with the same sign as the character of this Reeb component.

We now verify that there are no other minimal (+) or (−)-fcd in \( \mathcal{F}' \). Consider a smooth, foliated compact domain \( D \) of \( \mathcal{F}' \) which is minimal under inclusion, and which is different from the above Reeb components \( R(\Gamma^i_j, \pm) \) of \( \mathcal{F}' \). We claim that \( D \) contains a boundary component where the transversal orientation points inward and another boundary component where the transversal orientation points outward (this would imply that there are no minimal (+) or (−)-fcd in \( \mathcal{F}' \) different from the Reeb components \( R(\Gamma^i_j, \pm) \), which in turn implies that Oshikiri’s condition holds for \( f \) and \( \mathcal{F}' \)). By Lemma 4.1, there exists a positively oriented transversal arc \( \gamma_i \in \Delta \) that intersects transversely a boundary component \( \partial_1 \) of \( D \) at some point and by construction, \( \gamma_i \) contains a subarc \( s \) that enters the Reeb component \( R(\Gamma^i_1, +) \) near the ending point of \( \gamma_i \) and exits its related “opposite” Reeb component \( R(\Gamma^i_2, -) \) near the initial point of \( \gamma_i \). As \( s \) also intersects \( \partial_1 \) at some point \( s(t_0) \) and the end points of \( s \) lie outside of \( D \), then \( s \cap D \) must contain a further subarc \( s' \) with one of its boundary points being \( s(t_0) \) and its other end point is \( s(t_1) \) which lies in a second boundary component of \( D \) along which \( \mathcal{N} \) is oppositely pointing with respect to \( D \). Thus our claim in proved.

Finally, when \( X \) is orientable then Theorem 2.5 implies that there exists a smooth Riemannian metric \( g_1 \) on \( X \) whose mean curvature function is \( f \), thereby proving the sufficient implication of Theorem 1.1 in this special case for \( X \).

4.4 Producing the desired metric on \( X \): the general case.

We next prove existence of a metric \( g_1 \) on \( X \) that makes \( \mathcal{F}' \) a CMC foliation, and which coincides up to scalings with the previously constructed smooth metrics from Section 3 in the finitely many enlarged Reeb components of \( \mathcal{F} \) of the form \( \Omega(\Gamma^q(\gamma), -) \), \( \Omega(\Gamma^p(\gamma), +) \) with the notation at the end of Section 4.2. This proof does not use Oshikiri’s result and also works in the case \( X \) is non-orientable.

The key idea is, roughly speaking, to remove part of the enlarged Reeb components from \( \mathcal{F}' \) (so that a neighborhood of the resulting boundary is still a product foliation), glue appropriately the resulting boundary leaves in pairs to obtain a new compact \( n \)-manifold \( \tilde{X} \) without boundary where \( \mathcal{F}' \) induces a smooth codimension-one foliation \( \mathcal{F} \); then we will construct a smooth Riemannian metric \( \tilde{g} \) on \( \tilde{X} \) that makes \( \mathcal{F} \) minimal (by application of Sullivan’s Theorem possibly generalized to the non-orientable case, which will be proved in Section 4.5). Finally, after pulling back \( \tilde{g} \) to the complement of the Reeb components and gluing this pulled back metric with the ones constructed in Section 3 on the enlarged Reeb components (appropriately scaled by positive constants), we will obtain the desired smooth Riemannian metric. We now give details on this construction.

Recall that with the notation of Section 4.2, we have two closed, positively oriented transversals \( \Gamma^1_i = \Gamma^q(\gamma), \Gamma^2_i = \Gamma^p(\gamma) \) associated to each open transversal arc \( \gamma = \gamma_i \in \Delta \). With the notation in Section 4.1, we also have related coordinates \( \Phi^j_i(\overline{D}(8\varepsilon) \times S^1) \) for the replaced tubular neighborhoods of the \( \Gamma^j_i, j = 1, 2 \).
We consider the manifold $\widehat{X}$ obtained as the quotient manifold

$$\widehat{X} = \left[ X - \bigcup_{i=1}^{k} \left( \Phi_{i}^{1}(\mathbb{D}(3\varepsilon) \times S^{1}) \cup \Phi_{i}^{2}(\mathbb{D}(3\varepsilon) \times S^{1}) \right) \right] / \sim,$$

where $\sim$ is the equivalence relation induced by the map $\Phi_{i}^{1}(x, \theta) \in \Phi_{i}^{1}(\partial \mathbb{D}(3\varepsilon) \times S^{1}) \mapsto \Phi_{i}^{2}(x, -\theta) \in \partial \Phi_{i}^{2}(\mathbb{D}(3\varepsilon) \times S^{1})$. It is straightforward to check:

1. $\widehat{X}$ is orientable when $X$ is orientable.

2. The transversely oriented foliation $\mathcal{F}'$ on $X$ induces a transversely oriented foliation $\mathcal{F}$ of $\widehat{X}$.

3. Every compact leaf of $\mathcal{F}$ is induced by a compact leaf of $\mathcal{F}'$ and has a closed transversal passing through it (namely, the quotient by the equivalence relation $\sim$ of a suitable subarc of one of the $\gamma_{i} \in \Delta$; note that for this to make sense, we possibly need to adjust the $\theta$-variable in one of the two local ‘cylindrical’ coordinates $(x, \theta)$ defined by (9) so that the intersection of $\gamma_{i}$ with $\Phi_{i}^{1}[\mathbb{D}(3\varepsilon) \times S^{1}]$ and $\Phi_{i}^{2}[\mathbb{D}(3\varepsilon) \times S^{1}]$ has coordinates $(x_{0}, 0)$ in both local systems).

4. The foliation $\mathcal{F}$ is homologically taut, and hence Sullivan’s Theorem (or rather, its generalization Theorem 4.3 below) implies that $\widehat{X}$ admits a smooth Riemannian metric $g_{\widehat{X}}$ such that all of the leaves of the foliation $\mathcal{F}$ are minimal.

We next consider the $n$-manifold with boundary

$$\tilde{X} = X - \bigcup_{i=1}^{k} \left[ \Phi_{i}^{1}(\mathbb{D}(3\varepsilon) \times S^{1}) \cup \Phi_{i}^{2}(\mathbb{D}(3\varepsilon) \times S^{1}) \right].$$

The next assertion uses the rotationally symmetric metric $ds^{2}$ defined by equation (1) for the function $f = f(r)$ given by (8) and extended to $r \in [0, 1]$ as explained in Remark 3.3 and in the paragraph just before this remark. Recall that by Remark 3.3, given $\lambda > 0$, the metric $ds^{2} + dz^{2}$ on $[\mathbb{D}(1) - \mathbb{D}(r_{2})] \times (\mathbb{R}/\lambda \mathbb{Z})$ restricts to a product metric on $[\mathbb{D}(1) - \mathbb{D}(r_{2})] \times \mathbb{R} \times (\mathbb{R}/\lambda \mathbb{Z})$. In the sequel it will be useful to write this product Riemannian manifold in a different manner. Consider the diffeomorphism

$$\chi: \left[ \mathbb{D}(1) - \mathbb{D}(r_{2}) \right] \times (\mathbb{R}/\lambda \mathbb{Z}) \to \left( \mathbb{S}^{n-2}(r_{2}) \times (\mathbb{R}/\lambda \mathbb{Z}) \right) \times [r_{2}, 1], \quad \chi(ra, b) = (r_{2}a, b, r),$$

where $a \in \mathbb{S}^{n-2}(1)$ and $r \in [r_{2}, 1]$. Denote by $g_{\mu}$ the product metric on $\mathbb{S}^{n-2}(r_{2}) \times (\mathbb{R}/\lambda \mathbb{Z})$ that makes $\chi$ an isometry from $ds^{2} + dz^{2}$ to $g_{\mu} \times dr^{2}$. Note that the function $f$ in (8) depends on the choice of $H > 0$, but this dependence does not affect the metric $\tilde{g}$ appearing in Assertion 4.2.

**Assertion 4.2** Let $\tilde{g}$ be the smooth pulled back metric and let $\tilde{F}$ be the pulled back minimal foliation on $\tilde{X}$ obtained respectively from $g_{\widehat{X}}$, $\mathcal{F}$ on $\widehat{X}$. Then, there exist numbers $\lambda_{1}, \ldots, \lambda_{k} > 0$ such that $\tilde{g}$ can be chosen so that the each of the domains $\left( \Phi_{j}[\mathbb{D}(4\varepsilon) - \mathbb{D}(3\varepsilon)] \times S^{1}], \tilde{g} \right)$ is isometric to the Riemannian product

$$\left( \mathbb{S}^{n-2}(r_{2}) \times (\mathbb{R}/\lambda_{i} \mathbb{Z}) \right) \times [r_{2}, 1], g_{\mu} \times dr^{2}.$$
by a diffeomorphism ψ: Φ′ j(\([\mathbb{D}(4\varepsilon) - \mathbb{D}(3\varepsilon)] \times S^1\)) \to (S^{n-2}(r_2) \times (\mathbb{R}/λ_l\mathbb{Z})) \times [r_2, 1] such that the pullback by ψ of the product foliation \(\{[S^{n-2}(r_2) \times (\mathbb{R}/λ_l\mathbb{Z})] \times \{δ\} \mid δ \in [r_2, 1]\}\) is the restriction of \(\hat{F}\) to \(\Phi′ j(\([\mathbb{D}(4\varepsilon) - \mathbb{D}(3\varepsilon)] \times S^1\))

\[\text{Proof.}\] We first prove the assertion under the additional hypothesis that \(\mathcal{F}'\) is orientable, which, since it is transversely oriented, would hold if \(X\) were orientable. The assertion will follow from analyzing the proof of the main theorem in [15], more specifically, the paragraph in the proof of this result where Sullivan shows that homological tautness implies geometrical tautness. To do this, he first uses the homological tautness of \(\hat{F}\) to produce via the Hahn-Banach theorem a closed \((n-1)\)-form \(ω\) on \(\hat{X}\) whose restriction to the leaves of \(\hat{F}\) is positive. Then he considers the pointwise projections \(P_ω: T\hat{X} \to T\hat{F}\) given by the process of purification, i.e.,

\[P_ω(v)\l(ω|\hat{F}\r) = (v|ω)|_\hat{F}, \quad v \in T_x\hat{X},\]  

(10)

where \(T\hat{X}\) denotes that tangent bundle to \(\hat{X}\), \(T\hat{F}\) is the subbundle of \(T\hat{X}\) tangent to the leaves of \(\hat{F}\), \(\hat{F}\) is the hyperplane tangent to \(\hat{F}\) at a point \(\hat{x} \in \hat{X}\) and \(\l\) means contraction. Next he defines the purification of \(ω\) as \(P_ω^*(ω|T\hat{F})\) (in our codimension-one case, \(P_ω^*(ω|T\hat{F})\) coincides with \(ω\)). Now the ambient smooth metric on \(\hat{X}\) for which \(\hat{F}\) is minimal is constructed by simply taking the orthogonal direct sum

\[g_\hat{X} = g^1 \oplus g^2\]  

(11)

of any metric \(g^1\) on the 1-dimensional distribution \(\{\mathrm{kernel}(P_ω)\}\) with any metric \(g^2\) on the subbundle \(T\hat{F}\) whose \((n-1)\)-volume form coincides with \(ω|\hat{F}\). As \(\tilde{g}\) is the pulled back metric of \(g_\hat{X}\), the decomposition (11) can be written as well for \(\tilde{g}\). In our proof of Assertion 4.2 we will choose these metrics \(g^1, g^2\) appropriately.

Consider a nowhere zero smooth vector field \(V\) on \(\Phi′ j(\([\mathbb{D}(4\varepsilon) - \mathbb{D}(3\varepsilon)] \times S^1\])\) that generates the distribution \(\{\mathrm{kernel}(P_ω)\}\). Note that \(V\) is \(\tilde{g}\)-orthogonal to the leaves of \(\hat{F}\). After multiplying \(V\) by a positive function defined on \(\Phi′ j(\([\mathbb{D}(4\varepsilon) - \mathbb{D}(3\varepsilon)] \times S^1\])\), we can assume that

\[\varphi_t = T^i_j(3\varepsilon + t)\]  

(12)

where \(\{\varphi_t\}_t\) denotes the local 1-parameter group of diffeomorphisms generated by \(V\) and \(T^i_j(3\varepsilon + t) = \Phi′ j(\partial/\partial (3\varepsilon + t) + S^1)\), for each \(t \in [0, \varepsilon]\), \(i = 1, \ldots, k\), \(j = 1, 2\). Then we can identify \(V = \partial/\partial\varepsilon\). After changing the metric \(g^1\), we can also assume that \(\partial/\partial\varepsilon\) has \(g^1\)-length equal to \((1-r_2)/\varepsilon\) in \(\Phi′ j(\([\mathbb{D}(4\varepsilon) - \mathbb{D}(3\varepsilon)] \times S^1\])\). Two geometrical consequences of this change of metric in the orthogonal direction to \(T\hat{F}\) are that the distance between \(T^i_j(3\varepsilon)\) and \(T^i_j(3\varepsilon + t)\) depends linearly on \(t \in [0, \varepsilon]\) and that the integral curves of \(\partial/\partial\varepsilon\) are geodesics of \(g_{\hat{X}}\). Also note that (12) implies that we can define natural “coordinates” \((x, t)\) in \(\Phi′ j(\([\mathbb{D}(4\varepsilon) - \mathbb{D}(3\varepsilon)] \times S^1\])\) so that \(x \in T^i_j(3\varepsilon), t \in [0, \varepsilon]\), and we can consider the projection

\[\Pi: T^i_j(3\varepsilon) \times [0, \varepsilon] \to T^i_j(3\varepsilon), \quad \Pi(x, t) = x.\]

Once the above change of \(g^1\) is done, we will change the “tangential part” \(g^2\) of \(g_{\hat{X}}\) appropriately. Consider the induced metric \(g_{3\varepsilon}\) by \(g_{\hat{X}}\) on the compact hypersurface \(T^i_j(3\varepsilon)\). By item (B) of Remark 3.3, we can pick a positive number \(λ_i\) so that

\[\mathrm{Vol}(S^{n-2}(r_2) \times (\mathbb{R}/λ_i\mathbb{Z})) = \mathrm{Vol}(T^i_j(3\varepsilon), g_{3\varepsilon}).\]
By the first theorem in Moser [9], there exists a diffeomorphism \( \xi : T^1_j(3\varepsilon) \to \mathbb{S}^{n-1}(r_2) \times (\mathbb{R}/\lambda_i\mathbb{Z}) \) such that
\[
\xi^*dV_{n-1} = \omega|_{T^1_j(3\varepsilon)},
\]
where \( dV_{n-1} \) is the \((n-1)\)-volume form associated to the product metric \( g_\mu \).

Now consider the smooth 1-parameter family of \((n-1)\)-forms \( \{\alpha_t \mid t \in [0,\varepsilon]\} \) on \( \mathbb{S}^{n-1}(r_2) \times (\mathbb{R}/\lambda_i\mathbb{Z}) \) defined by
\[
\left[ (\xi \circ \Pi)|_{T^1_j(3\varepsilon) \times \{t\}} \right]^* \alpha_t = \omega|_{T^1_j(3\varepsilon+t)}, \quad t \in [0,\varepsilon].
\]

Note that (13) and (14) imply that \( \alpha_0 = dV_{n-1} \). Since \( \omega \) is a closed \((n-1)\)-form, then (13) and Stokes’ theorem imply that
\[
\int_{\mathbb{S}^{n-1}(r_2) \times (\mathbb{R}/\lambda_i\mathbb{Z})} \alpha_t = \int_{\mathbb{S}^{n-1}(r_2) \times (\mathbb{R}/\lambda_i\mathbb{Z})} dV_{n-1} \quad \text{for all } t \in [0,\varepsilon].
\]

By Theorem 2 in [9], there exists a smooth 1-parameter family of diffeomorphisms \( \phi_t : \mathbb{S}^{n-1}(r_2) \times (\mathbb{R}/\lambda_i\mathbb{Z}) \to \mathbb{S}^{n-1}(r_2) \times (\mathbb{R}/\lambda_i\mathbb{Z}) \), \( t \in [0,\varepsilon] \), such that \( \phi_0 \) is the identity and for each \( t \in [0,\varepsilon] \), \( \phi_t^*dV_{n-1} = \alpha_t \). A direct computation gives that the pullback metric \( \psi^*(g_\mu \times dr^2) \) by the map
\[
\psi : T^1_j(3\varepsilon) \times [0,\varepsilon] \to (\mathbb{S}^{n-1}(r_2) \times (\mathbb{R}/\lambda_i\mathbb{Z})) \times [r_2,1], \quad \psi(x,t) = (\phi_t(\xi(x)), \frac{1-r_2}{\varepsilon}t + r_2)
\]
satisfies the following properties:

1. The \((n-1)\)-volume form of the restriction of \( \psi^*(g_\mu \times dr^2) \) to the leaves of \( \overline{\mathcal{F}} \) agrees with \( \omega|_{\overline{\mathcal{F}}} \).

2. \( \frac{\partial}{\partial t} = \psi_*(\frac{\partial}{\partial t}) \) is orthogonal to the foliation \( \{([\mathbb{S}^{n-2}(r_2) \times (\mathbb{R}/\lambda_i\mathbb{Z})] \times \{\delta\} \mid \delta \in [r_2,1]\} \) with respect to the metric \( g_\mu \times dr^2 \).

So, after identification by \( \psi \), Sullivan’s construction can be performed with the metrics \( g^1 = dr^2 \) and \( g^2 = g_\mu \) in \( \Phi_j^i(\mathbb{D}(4\varepsilon) - \mathbb{D}(3\varepsilon)) \times S^1 \) and smoothly extended to the remainder of \( \overline{\mathcal{X}} \), which finishes the proof of the assertion when \( \mathcal{F}' \) is orientable.

If \( \mathcal{F}' \) is not orientable, then the above arguments can be generalized in a straightforward manner, using the covering space techniques applied in the proof of Theorem 4.3 below, to construct the desired minimal metric on \( \overline{X} \).

To finish the proof of the existence of the ambient metric \( g_1 \) that makes \( \mathcal{F}' \) a CMC foliation, simply define \( g_1 \) to be equal to \( \overline{g} \) on \( \overline{X} \) and equal to the \((SO(n-1) \times (\mathbb{R}/\lambda_i\mathbb{Z}))\)-invariant metric defined in Section 4.1 in each of the domains \( \Phi_j^i(\mathbb{D}(3\varepsilon) \times S^1) \), \( i = 1, \ldots, k \), \( j = 1, 2 \), where \( \lambda_i > 0 \) is defined in Assertion 4.2. We remark that the non-zero constant values \( H_j^i \) of the mean curvatures in the Reeb foliations of \( \Phi_j^i(\mathbb{D}(2\varepsilon) \times S^1) \) (\( H_j^1 < 0 \) for \( j = 1 \) and \( H_j^2 > 0 \) for \( j = 2 \)) can be chosen arbitrarily, see Remark 3.3(B).

4.5 Sullivan’s theorem for non-orientable foliations.

**Theorem 4.3** A smooth codimension-one foliation of a closed manifold is geometrically taut if and only if it is homologically taut.
Proof. Let \( \mathcal{F} \) be a codimension-one foliation of a closed \( n \)-manifold \( X \). If \( \mathcal{F} \) can be oriented, then Sullivan’s Theorem implies that it is geometrically taut. Suppose now that \( \mathcal{F} \) cannot be oriented and let \( \Pi: \tilde{X} \to X \) be the two-sheeted cover of \( X \) such that the pulled back foliation \( \tilde{\mathcal{F}} \) by the projection \( \Pi \) is orientable. Let \( \sigma: \tilde{X} \to \tilde{X} \) be the order-two covering transformation. Also fix an orientation for the leaves of \( \tilde{\mathcal{F}} \).

Note that since \( \mathcal{F} \) is homologically taut then \( \tilde{\mathcal{F}} \) is also homologically taut. By the proof of Sullivan’s Theorem, there exists a closed \((n-1)\)-form \( \omega \) on \( \tilde{X} \) that is positive on the oriented tangent spaces to the leaves of \( \tilde{\mathcal{F}} \). Note that the pulled back \((n-1)\)-form \( \sigma^*\omega \) is negative on the oriented tangent spaces to the leaves of \( \tilde{\mathcal{F}} \) and so, the form \( \tilde{\omega} = \omega - \sigma^*\omega \) is positive on the tangent spaces to the leaves of \( \tilde{\mathcal{F}} \). Clearly, \( \tilde{\omega} \) is also closed on \( \tilde{X} \). By the proof of Sullivan’s Theorem applied to \( \tilde{\omega} \), there exists a metric \( g' \) on \( \tilde{X} \) such that the \( \tilde{\omega} \) restricts to be the \((n-1)\)-volume form on the leaves of \( \tilde{\mathcal{F}} \). Let \( \tilde{g} = g' + \sigma^*g' \) be the related \( \sigma \)-invariant metric on \( \tilde{X} \) and let \( g \) be the corresponding quotient metric on \( X \). Since \( \sigma^*\tilde{\omega} = -\tilde{\omega} \), it follows that the \( \tilde{g} \)-isometry \( \sigma \) leaves invariant the kernel of the purification operator \( P_{\tilde{\omega}} \), defined as in (10). It follows by Rummler’s calculation described in [15] that \( \tilde{\mathcal{F}} \) is a minimal foliation with respect to \( \tilde{g} \); thus, under quotient, the foliation \( \mathcal{F} \) is minimal with respect to \( g \), and the theorem is proved.

Remark 4.4 With straightforward modifications, the proof of the Theorem 4.3 generalizes to show that Sullivan’s theorem holds for any codimension-\( k \) foliation of a closed \( n \)-manifold, not just in the case that \( k = 1 \). The only difference in the proof is that after obtaining the form \( \tilde{\omega} \), one replaces it by its purified form \( \tilde{\omega}' \) as described in [15], which continues to be closed and to satisfy \( \sigma^*\tilde{\omega}' = -\tilde{\omega}' \).

5 The proof of the Structure Theorem 1.3.

This section is devoted to the proof of Theorem 1.3. Let \( (X, g) \) be a closed, connected Riemannian \( n \)-manifold which admits a non-minimal CMC foliation \( \mathcal{F} \). During the proof, the reader should keep in mind that since \( \mathcal{F} \) is assumed to be transversely oriented, every compact leaf \( L \) of \( \mathcal{F} \) is two-sided and has a closed regular neighborhood diffeomorphic to \( L \times [0,1] \). The proof of the theorem uses the result by Haefliger [5] that the set of compact leaves of the transversely oriented foliation \( \mathcal{F} \) is sequentially compact in the sense that any sequence of compact leaves has a subsequence that converges smoothly with multiplicity one to a compact leaf of \( \mathcal{F} \). In particular, the union \( \mathcal{C}_\mathcal{F} \) of the compact leaves of \( \mathcal{F} \) is a compact subset of \( X \).

The divergence \( \text{Div}(N_\mathcal{F}) \) of the unit normal field to \( \mathcal{F} \) is equal to \( -(n-1)H_\mathcal{F} \), where we are using the notation just before the statement of Theorem 1.3. Therefore, by the divergence theorem, one obtains the well-known formula

\[
\int_X H_\mathcal{F} \, dV = 0,
\]

which completes the proof of the first item in Theorem 1.3.

We next prove item 2 of the theorem. Since the foliation \( \mathcal{F} \) is smooth, then \( H_\mathcal{F}: X \to \mathbb{R} \) is smooth as well. By Sard’s theorem and the compactness of \( X \), the subset of regular values of \( H_\mathcal{F} \) is an open subset of the interval \((\min H_\mathcal{F}, \max H_\mathcal{F}) \subset \mathbb{R} \) whose complement has measure zero in \([\min H_\mathcal{F}, \max H_\mathcal{F}] \). By the implicit function
theorem and the compactness of $X$, for each regular value $H$ of $H_F$, $H_F^{-1}(H)$ consists of a finite number of compact leaves of $F$ contained in $\text{Int}(C_F)$. This completes the proof of item 2.

We now proceed with the proof of item 3. Since manifolds are second countable and $C_F$ is a compact subset of $X$, then $X - C_F$ has a countable number of components, all of which are open. It follows that if $H_F$ restricted to a component $\Delta$ of $X - C_F$ were not constant, then by Sard’s Theorem there would be a regular value $H_0$ of $H_F$ different from any of the finite number of values of $H_F$ on the finite number of compact boundary components of $\Delta$. By item 2 of this theorem, $\Delta \cap (H_F)^{-1}(H_0)$ contains a compact leaf of $F$; this is a contradiction since $\Delta$ lies in $X - C_F$. Thus, $H_F$ is constant in every component of $X - C_F$, from which the first part of the first sentence in item 3 of the theorem follows by the continuity of $H_F$. In particular, except for the countable subset of $H_F(X)$ corresponding to the set of values of $H_F$ on the components of $X - C_F$, every leaf of $F$ with mean curvature different from one of these special values is compact. Furthermore, if $H_\Delta \in \mathbb{R}$ is the value of $H_F$ on a component $\Delta$ of $X - C_F$, then by the continuity of $H_F$, $H_\Delta$ is the value of the mean curvature of any compact leaf in the boundary of $\Delta$. Thus, for every $H \in H_F(X)$ there exists at least one compact leaf of $F$ of mean curvature $H$.

To complete the proof of item 3, it remains to demonstrate that every leaf in the closure of $X - C_F$ is stable. To do this, first consider a component $\Delta$ of $X - C_F$. We have already proved that the leaves in the closure of $\Delta$ have the same constant mean curvature $H$. Since the closure of $\Delta$ in $X$ has the structure of an $H$-lamination of $X$ where every leaf is a limit leaf, then the main theorem in [8] implies that every leaf in the closure of $\Delta$ is a stable $H$-hypersurface. Suppose that $L$ is a leaf in the closure of $X - C_F$ which is not a leaf in the closure of any component of $X - C_F$; in this case, every point $x \in L$ is the limit in $X$ of a sequence of points $x_n \in X$, each of which lies in the boundary $\partial \Delta_n$ of a component $\Delta_n$ of $X - C_F$. In this case, Haefliger’s compactness result for the set of closed leaves of $F$ implies that $L$ is compact and it is the smooth limit (with multiplicity one) of a sequence of compact $H_n$-stable leaves $L_n \subset \partial \Delta_n$. By the continuity of $H_F$, the $H_n$ converge to the mean curvature of $L$. Since for $n$ large any smooth unstable subdomain in $L$ can be lifted normally to a smooth unstable subdomain in $L_n$, the instability of $L$ would contradict the assumption that the $L_n$ are $H_n$-stable leaves. Therefore, $L$ must also be stable, which completes the proof of item 3 of the theorem.

To prove item 4, suppose that $L$ is a leaf of $F$ that passes through a regular point of $H_F$. By the local product structure of a smooth foliation, the gradient $\nabla H_F$ is seen to be non-zero everywhere along $L$; in particular, $L$ is contained in the set of regular points of $H_F$. By item 3, the closure of $X - C_F$ consists entirely of critical points of $H_F$. Since every non-compact leaf of $F$ is contained in $X - C_F$, then we conclude that $L$ must be compact. As the set of regular points of $H_F$ is open, then the same arguments show that $L$ lies in the interior of $C_F$ and that $\nabla H_F$ is non-zero in a neighborhood $U(L)$ of $L$ in $X$. By an application of the implicit function theorem to $H_F$, $U(L)$ can be taken so that $F$ restricts to $U(L)$ as a product foliation of compact leaves diffeomorphic to $L$. This product foliation can be considered to be a smooth normal variation $L_t$ of $L$ through compact leaves of $F$, whose variational field is $V = f\mathbb{N}_F$ for $f = g(\frac{d}{dt}|_{t=0} L_t, N_F)$ and
with
\[ Jf = (n - 1) \frac{d}{dt} \bigg|_{t=0} H(L_t) \neq 0 \]
everywhere on \( L \), where \( J = \Delta + \|\sigma\|^2 + \text{Ric}(N_F) \) is the Jacobi operator of \( L \) (here \( \|\sigma\|^2 \) denotes the square of the norm of the second fundamental form of \( L \) and \( \text{Ric} \) the ambient Ricci curvature). The foliation property of the variation \( t \mapsto L_t \) insures that \( f \) has constant sign on \( L \), say \( f > 0 \). If \( Jf < 0 \) on \( L \), then Lemma 2.1 in [7] insures that \( L \) is stable (in fact, with nullity zero). Conversely, if \( Jf > 0 \) on \( L \), then the index form \( Q(f, f) = -\int_L fJf \) is strictly negative, and so \( L \) is unstable. Thus, item 4a of Theorem 1.3 holds.

Suppose now that \( L \) is a leaf of \( F \) that is disjoint from the regular points of \( H_F \). One possibility for \( L \) is that it is contained in the closure of \( X - C_F \), in which case item 3 implies \( L \) is stable, and so, the index of \( L \) is zero. Otherwise \( L \) lies in the interior of \( C_F \), which means that \( L \) is a limit leaf of the CMC lamination of \( X \) consisting of compact leaves of \( F \). As \( L \) lies in a 1-parameter family \( \{L_t\} \) of compact leaves whose mean curvatures \( H_t \) at \( L \) agree up to first order with the mean curvature of \( L \) (since \( L \) is a subset of critical points of \( H_F \)), the normal variational vector field \( F \) to the variation at the leaf \( L \) is a nowhere zero Jacobi field on \( L \). It follows in this case that \( L \) is stable and has nullity one. This completes the proof of item 4b.

Finally we prove item 5. Suppose that \( L \) is a leaf of \( F \) with mean curvature in \( \{\min H_F, \max H_F\} \). Then \( L \) is stable by item 4b. Next consider a sequence of regular values \( r_n \) of \( H_F \) that are tending to, but smaller than, the value \( \max H_F \). By item 2, there exist compact leaves \( L_n \) of \( F \) with \( H_F(L_n) = r_n \), and by compactness of \( C_F \), after replacing by a subsequence, the \( L_n \) converge to a compact leaf \( L \) with mean curvature \( \max H_F \). Since \( L \) is disjoint from the regular values of \( H_F \) by item 4a, then the last part of item 4b implies that the nullity of \( L \) is one. A similar argument proves the existence of a compact stable leaf \( L' \) of nullity one and mean curvature \( \min H_F \), which completes the proof of Theorem 1.3.

William H. Meeks, III at profmeeks@gmail.com
Mathematics Department, University of Massachusetts, Amherst, MA 01003
Joaquín Pérez at jperez@ugr.es
Department of Geometry and Topology, University of Granada, Granada, Spain

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