THE RIEMANN HYPOTHESIS

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Abstract

The Riemann Hypothesis is a conjecture made in 1859 by the great mathematician
Riemann that all the complex zeros of the zeta function \( \zeta(s) \) lie on the ‘critical line’
\( \Re s = 1/2 \). Our analysis shows that the assumption of the truth of the Riemann
Hypothesis leads to a contradiction. We are therefore led to the conclusion that the
Riemann Hypothesis is not true.

1. The Zeta function of Riemann\(^1\) is the analytic function obtained by the analytic contin-
uation of the sum-function \( \zeta(s) \) of the infinite series

\[
\sum_{n=1}^{\infty} \frac{1}{n^s} \quad (s = \sigma + it)
\]

which is uniformly convergent in any finite region of the \( s \)-plane, defined by \( \sigma \geq 1 + \delta, \delta > 0 \).
The Riemann Zeta function \( \zeta(s) \) is analytic in the entire \( s \)-plane, except for a simple pole at
\( s = 1 \) with residue 1.

The function \( \zeta(s) \) defined as the sum-function of (1) is also known from the researches
of Euler to be defined as

\[
\zeta(s) = \prod_p \left( 1 - \frac{1}{p^s} \right)^{-1}, \quad \sigma > 1,
\]

the infinite product being taken for all prime numbers \( p \).

The Riemann Zeta function is given by

\[
\zeta(s) = \frac{e^{-is\pi} \Gamma(1-s)}{2\pi i} \int_C \frac{z^{s-1}}{e^z - 1} dz,
\]

where the contour \( C \) consists of the real axis from \( \infty \) to \( \rho \) \((0 < \rho < 2\pi)\), the circle \(|z| = \rho\) in
the positive sense round the origin and the real axis from \( \rho \) to \( \infty \), and \( z^{s-1} = \exp((s-1)\log z) \),
\( \log z \) being assumed real at the starting point of \( C \).

\(^1\)For the theory of the Riemann Zeta Function, see Titchmarsh [1], Edwards [1], Ivic [1], [2], Heath-Brown
[1], Bombieri [1], [2], Montgomery [1] and Conrey [1], [2]. Also see Chandrasekharan [1], Ramachandra [1]
and Selberg [1], [2]
It follows from (2) that $\zeta(s) \neq 0$ when $\sigma > 1$. It has been proved independently by Hadamard [1] and de la Vallée-Poussin [1] that $\zeta(s) \neq 0$ on the line defined by $\sigma = 1$. It is known that the only zeros of $\zeta(s)$ on the $\sigma$ axis are at $s = -2, -4, -6, \ldots$, and that all the complex zeros of $\zeta(s)$ lie within the ‘critical strip’ defined by $0 < \sigma < 1$ (See Titchmarsh [1]).

In his famous memoir “Über die Anzahl der Primzahlen unter einer gegebenen Grosse” of 1859 (for English translation, see Edwards [1]) the celebrated German mathematician Riemann has remarked that it is “very likely” that all the complex zeros of $\zeta(s)$ have real part equal to $1/2$. This is called the ‘Riemann Hypothesis’. This is equivalent to the statement that all the complex zeros of $\zeta(s)$ lie on the ‘critical line’ $R\sigma s = 1/2$.

In 1914 the distinguished British mathematician G. H. Hardy [1] proved that an infinity of complex zeros of $\zeta(s)$ lie on the critical line. Subsequently Hardy and Littlewood [1], Levinson [1], Selberg [1] and Conrey [1] have estimated the proportion of complex zeros of $\zeta(s)$ on the critical line to the number of complex zeros in the critical strip. Bohr and Landau [1] have also shown that in the arbitrarily thin strip defined by $\frac{1}{2} - \epsilon < \sigma < \frac{1}{2} + \epsilon$, $\epsilon > 0$, ‘almost all’ the complex zeros of $\zeta(s)$ lie. With the help of high-speed computers more and more zeros of $\zeta(s)$ have been found to conform to Riemann’s conjecture. However, the Riemann Hypothesis has so far been neither proved nor disproved.

For an account of recent developments concerning the Riemann Hypothesis, we may also refer to Marcus du Sautoy [1], Dan Rockmore [1] and John Friedlander [1].

2. Let $\mu(n)$ be the Möbius function defined as:

$$\mu(1) = 1, \quad \mu(n) = (-1)^k,$$

where $n$ is the product of $k$ different primes and $\mu(n) = 0$ if $n$ contains as factor any prime raised to power 2 of greater than 2. Let us write

$$M(x) = \sum_{n \leq x} \mu(n).$$

It was conjectured in 1897 by F. Mertens from numerical evidence that

$$M(n) < \sqrt{n} \quad (n > 1).$$

This conjecture, called the ‘Mertens Hypothesis’ was known to imply the Riemann Hypothesis. It was proved to be false in October, 1983 by Andrew Odlyzko and Hermann J. J. te Riele [1] after eight years of historic collaboration, with the help of both mathematical techniques and high-powered computation.

3. In 1912, Littlewood [1] proved

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Theorem A. The Riemann Hypothesis is equivalent to the statement that, for \( \epsilon > 0 \), \( M(x) = o \left( x^{\frac{1}{2} + \epsilon} \right) \), as \( x \to \infty \).

In 1948, Turan [1] proved that a sufficient condition for the truth of the Riemann Hypothesis is that the functions

\[
S_n(s) = \sum_{\nu=1}^{n} \frac{1}{\nu^s} \quad (s = \sigma + it)
\]

\( n = 1, 2, 3, \ldots \), should have no zeros in the half-plane \( \sigma > 1 \). He had shown that this condition (4) implies that, for all \( x > 0 \),

\[
\sum_{n \leq x} \frac{\lambda(n)}{n} \geq 0
\]

where \( \lambda(n) \) is the Liouville function, defined by: \( \lambda(n) = (-1)^r \), where \( n \) has \( r \) prime factors, a factor of multiplicity \( k \) being counted \( k \) times, and that (5) implies the truth of the Riemann Hypothesis. However, it was proved in 1958 by Haselgrove [1] that (5) is false in general. In 1968, Spira [1] found that \( S_{19}(s) \) has a zero in the region \( \sigma > 1 \).

In 1885, T. J. Stieltjes [1] wrote to C. Hermite that he had succeeded in proving that, as \( x \to \infty \), \( M(x) = O \left( x^{\frac{1}{2}} \right) \), and that this implies the truth of the Riemann Hypothesis. As observed by Edwards [1], in later years Stieltjes was unable to reconstruct his proof. All he says about it is that it was very difficult, that it was based on arithmetic arguments concerning \( \mu(n) \), and that he put it aside “hoping to find a simpler proof of the Riemann Hypothesis, based on the theory of the Zeta function rather than on arithmetic.”

4. The object of this paper is to prove the following.

Theorem 1. The Riemann Hypothesis is not true. In other words, not all the complex zeros of \( \zeta(s) \) lie on the ‘critical line’: \( \sigma = 1/2 \) (\( \sigma = \text{Re} s \)).

Theorem 2. The ‘weakened Mertens Hypothesis’: For every \( \epsilon > 0 \),

\[
M(x) = o \left( x^{\frac{1}{2} + \epsilon} \right), \quad \text{as} \quad x \to \infty,
\]

is not true.

Remarks. In view of Theorem A, Theorem 2 is an immediate consequence of Theorem 1, which also shows that Turan’s condition stated as (4) in §3 is not satisfied. Some other immediate consequences of Theorem 1 have been stated in the last section entitled ‘Further Remarks’.
5.1 Lemmas. For the proof of Theorem 1 we shall need a number of lemmas. We give these below.

N.B. We shall sometimes write ‘RH’ for ‘Riemann Hypothesis’. It is also proper to state here that the assumption of RH is not necessarily essential, but makes the proof of certain lemmas easier, and since we assume the truth of the RH at the very outset of the proof of Theorem 1, we assume it, wherever convenient, in the lemmas. However, some lemmas are independent of the RH.

Lemma 1. Assume the RH. Then each interval \((\frac{1}{2} + in, \frac{1}{2} + i(n + 1))\), \(n\) large, of the ‘critical line’ contains a point \(\frac{1}{2} + iT\), say, such that \(|T - \gamma| > \frac{A}{\log n}\), where \(\gamma\) is the ordinate of any zero of \(\zeta(s)\) and \(A\) is a positive absolute constant. Thus, the interval of the ‘critical line’:

\[
\left(\frac{1}{2} + i(T - A/\log n), \frac{1}{2} + i(T + A/\log n)\right)
\]

is free from zeros of \(\zeta(s)\) (Titchmarsh [1], p.340.)

Proof: It is known that the number \(N\) of zeros of \(\zeta(s)\) in the interval \((\frac{1}{2} + in, \frac{1}{2} + i(n + 1))\) of the critical line is given by the formula

\[
N \leq A \log n, \quad \text{for} \quad n \geq n_0,
\]

where \(n_0\) is sufficiently large, assuming as we may that \(A_1\) is an absolute constant \(\geq 1\).

Now if the \(N\) zeros of \(\zeta(s)\) are denoted by \(z_1, z_2, ..., z_N\), then, assuming, if possible, that the Lemma 1 is false, there is no point \(T\) of the type described. Then certainly the midpoints of the segments: \((\frac{1}{2} + in, z_1), (z_1, z_2), ..., (z_N, \frac{1}{2} + i(n + 1))\) are such that their distances from the endpoints of the segments are each \(\leq A/\log n\), whatever positive constant \(A\) may be, so that the total length of the segment, viz. 1, satisfies

\[
1 \leq \frac{2A}{\log n} \times \text{no. of segments}
\]

\[
= \frac{2A}{\log n} \times (N + 1) \leq \frac{2A}{\log n}(A_1 \log n + 1)
\]

\[
< \frac{2A}{\log n}.2A_1 \log n
\]

\[
= 4AA_1.
\]

Hence \(4AA_1 > 1\) or \(A > \frac{1}{4A_1}\). Taking \(A = \frac{1}{4A_1}\), we get a contradiction. Hence the Lemma is true.

Lemma 2. Assume the RH. Let \(s = \sigma + it; \quad so = \frac{1}{2} + it_0\) is a zero of \(\zeta(s)\) of order \(m\) such that \(|t - t_0| < 1/\log \log t\), where \(t, t_0 > \tau\), a sufficiently large positive number. Then, uniformly for \(1/2 \leq \sigma \leq 2\),

\[
Rf[\log \zeta(s) - m \log(s - so)] < A \log t \frac{\log \log \log t}{\log \log t},
\]
where $A$ is an absolute positive constant.

**Proof.** The following theorem is known.

**Theorem B.** Under the Riemann Hypothesis, uniformly for $1/2 \leq \sigma \leq 2$, with $s = \sigma + it$, as $t \to \infty$,

$$\log \zeta(s) - \sum_{|t-\gamma|<1/\log_2 t} \log(s-\rho) = O\left(\frac{\log t \log_3 t}{\log_2 t}\right),$$

where $\rho = 1/2 + i\gamma$ runs through the zeros of $\zeta(s)$ and $\log_2 t = \log \log t$, $\log_3 t = \log \log_2 t$.

By Theorem B, uniformly for $1/2 \leq \sigma \leq 2$,

$$\log \frac{\zeta(s)}{\prod_{|t-\gamma|<1/\log_2 t} (s-\rho)} = O\left(\hat{\Omega}(t) \log_3 t\right),$$

where $\hat{\Omega}(t) \equiv \log t/\log_2 t$.

Hence, for $t > \tau$, where $\tau$ is a sufficiently large positive number,

$$\left| \log \frac{\zeta(s)}{\prod_{|t-\gamma|<1/\log_2 t} (s-\rho)} \right| = \left| \text{Rl} \log \frac{\zeta(s)}{\prod_{|t-\gamma|<1/\log_2 t} (s-\rho)} \right|$$

$$\leq \log \left| \frac{\zeta(s)}{\prod_{|t-\gamma|<1/\log_2 t} (s-\rho)} \right| < A\hat{\Omega}(t) \log_3 t.$$  \hfill (6)

Hence

$$\left| \frac{\zeta(s)}{\prod_{|t-\gamma|<1/\log_2 t} (s-\rho)} \right| < \exp(A\hat{\Omega}(t) \log_3 t),$$  \hfill (6)

and

$$\left| \frac{\zeta(s)}{(s-s_0)^m} \right| = \left| \frac{\zeta(s)}{\prod_{|t-\gamma|<1/\log_2 t} (s-\rho)} \right| \prod_{|t-\gamma|<1/\log_2 t, \rho \neq s_0} |s-\rho|. \hfill (7)$$

The number of factors in $\prod_{|t-\gamma|<1/\log_2 t} |s-\rho|$, taking into account the multiplicity of the zeros, does not exceed

$$N(t + \tilde{\delta}) - N(t - \tilde{\delta}) \quad (\tilde{\delta} = 1/\log_2 t)$$

which by Theorem 14.13 of Titchmarsh [1],

$$= O(\tilde{\delta} \log t) + O(\hat{\Omega}(t)) = O(\hat{\Omega}(t)).$$

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$^2$Titchmarsh [1], Th. 14.15
Thus the number of factors in the product \( \prod_{|t-\gamma|<\hat{\delta}, \rho \neq s_0} |s-\rho| \) is smaller than \( \hat{\delta}, \rho \neq s_0 \) does not exceed \( \hat{\Delta}\hat{\Omega}(t) \). Also, since

\[ s-\rho = \sigma - \frac{1}{2} + it - \gamma \]

and \( |t-\gamma| < \hat{\delta} \), we have \( |s-\rho| < A \) (assuming, as we may, that \( A \) is a positive absolute constant such that \( \log A > 0 \)). Hence from (6) and (7) we get

\[
\left| \frac{\zeta(s)}{(s-s_0)^m} \right| < \exp(A\hat{\Omega}(t) \log_3 t) A^{\hat{\Delta}\hat{\Omega}(t)}
\]

Hence, uniformly for \( 1/2 \leq \sigma \leq 2 \) and for \( t, t_0 > \tau \), where \( \tau \) is sufficiently large and positive,

\[ \log \left| \frac{\zeta(s)}{(s-s_0)^m} \right| < A\hat{\Omega}(t) \log_3 t, \]

i.e.

\[ \text{Re}[\log \zeta(s) - \log(s-s_0)^m] < A \frac{\log t}{\log_2 t} \log_3 t, \]

whence the lemma follows.

**Lemma 3.** Assume the RH. If \( s' = \sigma' + i\delta \) \( (\sigma' > 1, \delta > 0) \) is sufficiently close to the point \( s' = 1 \), i.e. \( 0 < \sigma' - 1 < \delta_1, 0 < \delta < \delta_1 \), where \( \delta_1 \) is sufficiently small, then

\[ \text{Re}(\log(s'-1) + \text{Re}(\log \zeta(s')) < A, \]

where \( A \) is an absolute positive constant.

**Proof:** When \( s' \) is close to the point \( s' = 1 \), then\(^3\)

\[ \zeta(s') = \frac{1}{s'-1} + \gamma + O(|s'-1|). \]

Hence

\[ (s'-1)\zeta(s') = 1 + \gamma(s'-1) + O(|s'-1|^2). \]

Thus

\[ \lim_{s'\to1} [(s'-1)\zeta(s')] = 1. \]

Hence

\[ \lim_{s'\to1} |(s'-1)\zeta(s')] = 1 \]

\(^3\)Titchmarsh [1], (2.1.6); Edwards [1], p. 70

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and, therefore, 
\[ |(s' - 1)\zeta(s')| < 1 + \epsilon \]
for any \( \epsilon > 0 \), however small, if \( 0 < |\sigma' - 1| < \delta_1 \), \( 0 < \delta < \delta_1 \), where \( \delta_1 \) is sufficiently small.

Hence
\[ \log |s' - 1| + \log |\zeta(s')| < \log(1 + \epsilon) \quad (\epsilon > 0), \]
which gives the result.

**Lemma 4.** For \( s' = \sigma' + it \), \( \sigma' > 1 \),
\[ \log \zeta(s') = \sum_p \frac{1}{p^{\sigma'}} + f(s') \]
where
\[ f(s') = \sum_p \sum_{m=2}^{\infty} \frac{1}{mp^{ms'}} \]
is regular for \( \sigma' > \frac{1}{2} \). Hence, for \( \sigma' \geq 1 \),
\[ |f(s')| \leq \sum_p \sum_{m=2}^{\infty} \frac{1}{p^m} = \sum_p \frac{1}{p(p-1)} = B, \]
say, a positive absolute constant. Thus, since \( |Rf(s')| \leq |f(s')| \),
\[ -B \leq Rf(s') \leq B. \]

**Lemma 5.** If \( \sigma' > 1 \), then, as \( \sigma' \to 1 \),
\[ \sum_p \frac{1}{p^{\sigma'}} \sim \log \frac{1}{\sigma' - 1}. \]

**Lemma 6.** For \( \delta > 0 \) and any prime number \( p \), \[ |\sin \left( \frac{\delta}{2} \log p \right) | \leq \frac{\delta}{2} \log p. \]

**Proof.** If \( 0 < \frac{\delta}{2} \log p \leq \frac{\pi}{2} \), \[ |\sin \left( \frac{\delta}{2} \log p \right) | = \sin \left( \frac{\delta}{2} \log p \right) \leq \frac{\delta}{2} \log p \] (see Copson [1], p. 136). If \( \frac{\pi}{2} < \frac{\delta}{2} \log p \), \[ |\sin \left( \frac{\delta}{2} \log p \right) | \leq 1 < \frac{\pi}{2} < \frac{\delta}{2} \log p. \]

**Lemma 7.** Assume the RH. Then for \( \sigma' = Rbs' > 1 \),
\[ \sum_p \frac{\log p}{p^{\sigma'}} = \frac{1}{\sigma' - 1} + O(1) = O\left( \frac{1}{\sigma' - 1} \right), \]

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4Titchmarsh [1], p. 46, (3.2.1) and p. 189; Edwards [1], p. 70
5Titchmarsh [1], p. 46, (3.2.2)
if \( \sigma' - 1 \) is bounded.

**Proof.** It is known that \((s'-1)\zeta(s')\) is regular in all finite regions of the \(s'\)-plane. Also, under the RH, \((s'-1)\zeta(s')\) has no zero in the strip \(1/2 < \sigma' < 1\), while, without RH, \((s'-1)\zeta(s')\) has no zero in \(\sigma' \geq 1\), it being known that (see Lemma 3) \(\lim_{s' \to 1} [(s'-1)\zeta(s')] = 1\), and hence \(|(s'-1)\zeta(s')| > 1 - \epsilon\) for arbitrarily small \(\epsilon > 0\) (\(\epsilon < 1/2\)), for \(|s'-1| < \delta', \delta'\) sufficiently small, and also that \((s'-1)\zeta(s') \neq 0\) in the half-plane \(\sigma' > 1\) and, by virtue of the result of Hadamard [1] and de la Vallée-Poussin [1], \(\zeta(s')\) has no zero on \(\sigma' = 1\). Hence, in the entire half-plane \(Rt = \sigma' > 1/2\), the function

\[
\log [(s'-1)\zeta(s')]
\]

is regular, and hence for \(\sigma' > 1\),

\[
\frac{1}{\sigma' - 1} + \frac{\zeta'(\sigma')}{\zeta(\sigma')} = O(1) \quad (8)
\]

Also, for \(\sigma' > 1\),

\[
\log \zeta(s') = \sum_p \frac{1}{p^\sigma} + f(s')
\]

where \(f(s')\) is regular for \(\sigma' > 1/2\) (see Lemma 4). Hence for \(\sigma' > 1\),

\[
\frac{\zeta'(\sigma')}{\zeta(\sigma')} = \frac{d}{d\sigma'} \left( \sum_p \frac{1}{p^{\sigma'}} \right) + f'(\sigma')
\]

\[
= - \sum_p \frac{\log p}{p^{\sigma'}} + f'(\sigma')
\]

\[
= - \sum_p \frac{\log p}{p^{\sigma'}} + O(1). \quad (9)
\]

From (8) and (9), we have

\[
\sum_p \frac{\log p}{p^{\sigma'}} = - \frac{\zeta'(\sigma')}{\zeta(\sigma')} + O(1)
\]

\[
= \frac{1}{\sigma' - 1} - O(1) + O(1) = \frac{1}{\sigma' - 1} + O(1)
\]

\[
= O \left( \frac{1}{\sigma' - 1} \right),
\]

if \(\sigma' - 1\) is bounded.

**Lemma 8.** Assume the RH. Then, with \(s = \sigma + it\), \(\zeta(\frac{1}{2} + it) \neq 0\), uniformly for \(1/2 < \sigma \leq 2\),

\[
E \equiv \log \left| \frac{1}{\zeta(\sigma + it)} \right| - \log \left| \frac{1}{\zeta(1/2 + it)} \right| = O \left( \log t \left( \frac{\sigma - 1/2}{\delta} \right) \right),
\]

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where $0 < \delta = \min |t - \gamma|$, where $\gamma$ runs through the zeros of $\zeta(s)$ such that $|t - \gamma| < 1$.

**Proof.** Uniformly in $-1 \leq \sigma \leq 2$, as $t \to \infty$, \footnote{Titchmarsh [1], p. 340; see also Titchmarsh [1], Theorem 9.6(A)}

\[
\frac{\zeta'(s)}{\zeta(s)} = \mathcal{O}\left(\frac{\log t}{\delta}\right) + \mathcal{O}(\log t)
\]

where $\delta$ is as defined above, so that $0 < \delta < 1$.

Hence, uniformly for $1/2 < \sigma \leq 2$,

\[
|\mathcal{E}| = |\log |\zeta(\sigma + it)| - \log |\zeta(1/2 + it)||
\leq |\text{RI} \log \zeta(\sigma + it) - \text{RI} \log \zeta(1/2 + it)|
\leq |\log \zeta(\sigma + it) - \log \zeta(1/2 + it)|
\leq \int_{1/2}^{\sigma} \frac{\zeta'(x + it)}{\zeta(x + it)} dx
\leq \int_{1/2}^{\sigma} \frac{\zeta'(x + it)}{\zeta(x + it)} dx
= \mathcal{O}\left(\frac{\log t}{\delta} \int_{1/2}^{\sigma} dx\right) = \mathcal{O}\left(\frac{\log t}{\delta} \left(\sigma - \frac{1}{2}\right)\right).
\]

**Lemma 9.** \footnote{Titchmarsh [1], Th. 14.14 (A)} Assume the RH. Then, for all $t > \tau$, a sufficiently large positive number,

\[
|\zeta(1/2 + it)| \leq \hat{A} e^{\frac{\pi}{\text{RI}} A_3 t},
\]

where $\hat{A}$ and $A_3$ are absolute positive constants, such that $\hat{A} > 1$.

**Lemma 10.** If $a(t)$ and $k(t)$ are positive for $t > \tau_0$ (a positive constant) and $\theta$ is a positive constant, $a(t) \to \infty$ and $k(t) \to \infty$, as $t \to \infty$, such that

\[
\frac{a(t)}{k(t)} \to \infty \quad \text{as} \quad t \to \infty,
\]

and, for $t > \tau_0$, $b(t) = 1 + \frac{\theta}{k(t)}$, then, for all $t \geq \tau_1$ (a sufficiently large positive constant $> \tau_0$)

\[
a(t) + b(t) < a(t)b(t).
\]
Proof. It is enough to show that, for $a(t) > 1$,

$$b(t) > \frac{a(t)}{a(t) - 1}$$

and hence enough to show that

$$1 + \frac{\theta}{k(t)} > \frac{1}{1 - \frac{1}{a(t)}}$$

or

$$1 - \frac{1}{a(t)} + \frac{\theta}{k(t)} \left(1 - \frac{1}{a(t)}\right) > 1$$

or

$$\frac{\theta}{k(t)} \left(1 - \frac{1}{a(t)}\right) > \frac{1}{a(t)}$$

or

$$\theta > \frac{k(t)}{a(t) - 1} = \frac{1}{\frac{1}{k(t)} - \frac{1}{a(t)}}.$$  

Since $\frac{a(t)}{k(t)} \to \infty$ and $k(t) \to \infty$, the result follows from the assumption that $\theta$ is a constant $> 0$.

5.2 Proof of Theorem 1. We assume that the Riemann Hypothesis is true. We use the notations:

$$\log_2 t = \log \log t; \quad \log_3 t = \log(\log_2 t), \quad \Omega(t) = \frac{\log t \log_3 t}{\log_2 t}$$

Throughout $A$ will denote an absolute positive constant. For the sake of clarity we may sometimes use notations like $A^*, A_1, A_2, A'$ etc. for specific absolute constants. It is understood that the notation $A$ does not necessarily indicate the same positive absolute constant.

We consider the interval $(n, n + 1)$, where the positive integer $n$ is sufficiently large as per the requirements of our analysis. By Lemma 1, there will exist a point $\frac{1}{2} + iT$ on the critical line such that $T$ lies in the interval $(n, n + 1)$ and the region of the critical strip defined by the horizontal lines with ordinates: $(T - \frac{A^*}{\log n})$, $(T + \frac{A^*}{\log n})$ is free from zeros of the Zeta function $\zeta(s)$, $A^*$ being an absolute positive constant. Let $s_0 = \frac{1}{2} + it_0$ be the first zero of $\zeta(s)$ to be encountered just below $\frac{1}{2} + i \left(T - \frac{A^*}{\log n}\right)$. Let us write

$$t = t_0 + \delta \quad (10)$$

where $\delta$ is defined as

$$\delta = \frac{A^*}{\log(n + 1)} \quad (11)$$
where $A^*$ is a suitably small positive number to be further specified in the sequel such that

$$A^* < A^*.$$ \hfill (12)

By (11) and (12),

$$\delta < \frac{A^*}{\log n}.$$ \hfill (13)

Now by (10), (11), (12) and (13) and the definition of $T$, we have

$$t = t_0 + \delta < t_0 + \frac{A^*}{\log n} < T < n + 1;$$

$$\delta = \frac{A^*}{\log(n+1)} < \frac{A^*}{\log t} < \frac{1}{\log_2 t}.$$ \hfill (14)

for sufficiently large $t$, taking $n$ sufficiently large.

Let $\frac{1}{2} + it_1$ be the first zero of $\zeta(s)$ to be encountered just above $\frac{1}{2} + \left(T + \frac{A^*}{\log n}\right)$. Then

$$|t - t_1| = t_1 - t = t_1 - T + T - t,$$

so that

$$|t - t_1| > t_1 - T > \frac{A^*}{\log n},$$

while, by (13),

$$|t - t_0| = t - t_0 = \delta < \frac{A^*}{\log n}.$$

Hence

$$\delta = \min |t - \gamma|,$$ \hfill (15)

where $\gamma$ runs through the zeros of $\zeta(s)$ such that $|t - \gamma| < 1$.

By virtue of (14) and Lemma 2, uniformly for $1/2 \leq \sigma \leq 2$, writing $\Omega(t) = \frac{\log t \log_3 t}{\log_2 t}$, we have

$$\text{Re} \log \zeta(\sigma + it) - m \text{Re} \log \left(\sigma - \frac{1}{2} + i(t - t_0)\right) < A\Omega(t).$$ \hfill (16)

Hence uniformly in $1/2 \leq \sigma \leq 2$

$$\log |\zeta(\sigma + it)| - m \text{Re} \log \left(\sigma - \frac{1}{2} + i\delta\right) < A\Omega(t)$$
or
\[ \log | \zeta(\sigma + it) | - m \text{Re}(\sigma' - 1 + i\delta) < A\Omega(t) \]

where \( \sigma' - 1 = \sigma - \frac{1}{2} \).

Using Lemma 3, since \( \sigma - \frac{1}{2} = \sigma' - 1 \), we have, uniformly in \( \sigma : 0 < \sigma - \frac{1}{2} < \delta_1 \), with \( \delta < \delta_1 \),
\[ m \text{Re} \log \zeta(\sigma' + i\delta) < A\Omega(t) + mA + \log \left| \frac{1}{\zeta(\sigma + it)} \right| \]

or
\[ \text{Re} \log \zeta(\sigma' + i\delta) < \frac{A}{m} \Omega(t) + A + \frac{1}{m} \log \left| \frac{1}{\zeta(\sigma + it)} \right| \]

\[ < A\Omega(t) + \frac{1}{m} \log \left| \frac{1}{\zeta(\sigma + it)} \right| \]

Hence by Lemma 4, and using its notation,
\[ \text{Re} \sum_p \frac{1}{p^{\sigma' + i\delta}} + \text{Re} f(\sigma' + i\delta) < A\Omega(t) + \frac{1}{m} \log \left| \frac{1}{\zeta(\sigma + it)} \right| . \]

Appealing again to Lemma 4, with \( B \) having the same meaning as in Lemma 4,
\[ \text{Re} \sum_p \frac{1}{p^{\sigma' + i\delta}} < A\Omega(t) + B + \frac{1}{m} \log \left| \frac{1}{\zeta(\sigma + it)} \right| \]

or
\[ \sum_p \frac{\cos(\delta \log p)}{p^{\sigma'}} < A\Omega(t) + \frac{1}{m} \log \left| \frac{1}{\zeta(\sigma + it)} \right| \]

or
\[ \sum_p \frac{1}{p^{\sigma'}} < A\Omega(t) + \frac{1}{m} \log \left| \frac{1}{\zeta(\sigma + it)} \right| + 2 \sum_p \frac{\sin(\delta \log p)}{p^{\sigma'}} . \]

By Lemma 5, for \( \eta > 0 \), arbitrarily small \( (< \frac{1}{2}) \), \( \exists \sigma'_0(\eta) > 1 \) such that, uniformly for \( \sigma' : 1 < \sigma' < \sigma'_0(\eta) \), i.e. uniformly for \( \sigma : \frac{1}{2} < \sigma < \sigma_0(\eta) \), \( \sigma_0(\eta) = \sigma'_0(\eta) - \frac{1}{2} \),
\[ \sum_p \frac{1}{p^{\sigma'}} > (1 - \eta) \log \frac{1}{\sigma' - 1} . \]

Hence, uniformly for \( 0 < \sigma - \frac{1}{2} < \sigma_0 - \frac{1}{2} = \min \{ \sigma_0(\eta) - \frac{1}{2}, \delta_1 \} \), with \( 0 < \delta < \delta_1 \),
\[ (1 - \eta) \log \frac{1}{\sigma - \frac{1}{2}} < A\Omega(t) + \frac{1}{m} \log \left| \frac{1}{\zeta(\sigma + it)} \right| + 2 \sum_p \frac{|\sin(\frac{\delta}{2} \log p)|}{p^{\sigma'}} . \]
since \( \sin^2(\frac{\delta}{2} \log p) = | \sin(\frac{\delta}{2} \log p) |^2 \leq | \sin(\frac{\delta}{2} \log p) | \).

Hence, by Lemma 6, uniformly for \( 0 < \sigma - \frac{1}{2} < \sigma_0^* - \frac{1}{2}, \) with \( 0 < \delta < \delta_1, \)

\[
\log \frac{1}{\sigma - \frac{1}{2}} < \frac{A}{1 - \eta} \Omega(t) + \frac{A'}{1 - \eta} \log \frac{1}{\zeta(\sigma + it)} \bigg| + \frac{\delta}{1 - \eta} \sum_p \frac{\log p}{p^{\sigma'}}
\]

so that, by Lemma 7, uniformly for \( 0 < \sigma - \frac{1}{2} < \sigma_0^* - \frac{1}{2}, \) with \( 0 < \delta < \delta_1, \)

\[
\log \frac{1}{\sigma - \frac{1}{2}} < 2 \bar{A} \log \frac{1}{| \zeta(\sigma + it) |} + A' \frac{\delta}{\sigma - \frac{1}{2}}
\]

\[
\left( \bar{A} = \frac{1}{(1 - \eta)m} \leq \frac{1}{1 - \eta} < 2 \right. \left. \text{ and } A' \text{ is an absolute positive constant} \right)
\]

\[
= A_1 \Omega(t) + \bar{A} \log \frac{1}{| \zeta(\sigma + it) |} + \bar{A} \left[ \log \frac{1}{| \zeta(\sigma + it) |} - \log \frac{1}{| \zeta(\sigma + it) |} \right] + A' \frac{\delta}{\sigma - \frac{1}{2}}
\]

\[
\leq A_1 \Omega(t) + \bar{A} \log \frac{1}{| \zeta(\sigma + it) |} + 2 \log \frac{1}{| \zeta(\sigma + it) |} - \log \frac{1}{| \zeta(\sigma + it) |} + A' \frac{\delta}{\sigma - \frac{1}{2}}.
\]

Hence, by virtue of (15) and Lemma 8, for sufficiently large \( n, \) and hence \( t, \) uniformly for \( \sigma: 0 < \sigma - \frac{1}{2} < \sigma_0^* - \frac{1}{2}, \) with \( 0 < \delta < \delta_1, \)

\[
\log \left( \frac{| \zeta(\sigma + it) |}{\sigma - \frac{1}{2}} \right) < A_1 \Omega(t) + A_2 \log t \frac{\sigma - \frac{1}{2}}{\delta} + A' \frac{\delta}{\sigma - \frac{1}{2}}.
\]  

We write

\[
\hat{\sigma} - \frac{1}{2} = \frac{| \zeta(\frac{1}{2} + it) |}{\exp(A_1 \Omega + A_2 \log t + \theta)}
\]  

(18)

where \( \theta \) is a positive constant; \( \Omega = \Omega(t). \) We observe that, for all sufficiently large \( t, \)

\[
\left( \frac{A_1 \Omega + \theta}{A_1 \Omega} \right)^2 < \frac{A_2}{A' \log t}.
\]

Hence, for sufficiently large \( n, \) and hence \( t, \)

\[
\log(n + 1) \frac{A_1 \Omega + \theta}{A_1 \Omega} \left( \hat{\sigma} - \frac{1}{2} \right) < \log(n + 1) \frac{A_2}{A' A_1 \Omega + \theta} A_1 \Omega \log t \left( \hat{\sigma} - \frac{1}{2} \right).
\]

Therefore, there exists a positive number \( A^* \) such that

\[
\log(n + 1) \frac{A_1 \Omega + \theta}{A_1 \Omega} \left( \hat{\sigma} - \frac{1}{2} \right) < A^* < \log(n + 1) \frac{A_2}{A' A_1 \Omega + \theta} A_1 \Omega \log t \left( \hat{\sigma} - \frac{1}{2} \right).
\]  

(20)
For $n > 3$, $(n + 1) < 2(n - 1)$, so that

\[ R = \log(n + 1) \frac{A_2}{A'} \frac{A_1 \Omega}{A_1 \Omega + \theta} \log t \left( \frac{\sigma - 1}{2} \right) \]

\[
< \log(2(n - 1)) \frac{A_2}{A'} \frac{A_1 \Omega}{A_1 \Omega + \theta} \log t \left( \frac{\sigma - 1}{2} \right)\]

\[
< \log(2t) \frac{A_2}{A'} \log t \left( \frac{\sigma - 1}{2} \right)\quad [n - 1 < t]
\]

\[
< 2 \log t \frac{A_2}{A'} \log t \left( \frac{\sigma - 1}{2} \right)\quad [2 < n - 1 < t]
\]

\[
= \frac{2A_2}{A'} (\log t)^2 \left( \frac{\sigma - 1}{2} \right),
\]

since, for $t > n - 1 > 2$, $\log(2t) = \log 2 + \log t < 2 \log t$. Thus from (18) and Lemma 9,

\[ R < \frac{2A_2 \hat{A}^2}{A'} \exp \left( 2 \log_2 t + A_3 \hat{A} \frac{\log t}{\log_2 t} \right) \]

\[ = \frac{2A_2 \hat{A}^2}{A'} \exp \left( A_2 \log t + \frac{\log t}{\log_2 t} \left( A_1 \log_3 t - A_3 \hat{A} \right) - 2 \log_2 t + \theta \right). \]  

(22)

Now, since $\hat{A} = \frac{1}{(1-\eta)n} \leq \frac{1}{1-\eta} < 2$,

\[ A_1 \log_3 t - A_3 \hat{A} > A_1 \log_3 t - 2A_3, \]

which is positive for all sufficiently large $t$.

Hence from (22), for sufficiently large $n$, and hence $t$,

\[ R < \frac{2A_2 \hat{A}^2}{A'} \exp \left( A_2 \log t + \frac{\log t}{\log_2 t} \left( A_1 \log_3 t - 2A_3 \right) - 2 \log_2 t + \theta \right) \]

\[ < A^\circ \]

(23)

however small $A^\circ$ may be, since $\hat{A} > 1$, as assumed in Lemma 9.

From (20) and (23) it follows that

\[ A^* < A^\circ, \]

which had been stated in (12), as an assumption.

From (18) and (20), it follows that for sufficiently large $n$ and hence $t$,
\[ \delta = \frac{A^*}{\log(n+1)} < \frac{A_2}{A'} \log t \left( \frac{\sigma - 1}{2} \right) \]

< \frac{A_2 \hat{A}^2}{A' \exp \left( A_2 \log t + \frac{\log t}{\log_2 t} (A_1 \log_3 t - 2A_3) - \log_2 t + \theta \right)} \]

< \frac{A_2 \hat{A}^2}{A' \exp \left( A_2 \log t + \frac{\log t}{\log_2 t} (A_1 \log_3 t - 2A_3) - \log_2 t + \theta \right)} \]

< \sigma_0^* - \frac{1}{2} \] (25)

however small \( \sigma_0^* - \frac{1}{2} \) may be.

Now, from (17) it follows that, for sufficiently large \( n \), and hence \( t \), uniformly in \( \sigma : 0 < \sigma - \frac{1}{2} < \delta < \sigma_0^* - \frac{1}{2} \),

\[ \frac{1}{A'} \log \left( \frac{|\zeta(\frac{1}{2} + it)| \hat{A}}{\sigma - 1/2} \right) < \frac{1}{A'} \left( A_1 \Omega + A_2 \log t \frac{\sigma - 1/2}{\delta} \right) + \frac{\delta}{\sigma - 1/2} \] (26)

Let us now take

\[ \sigma - \frac{1}{2} = \frac{A_1 \Omega}{A_1 \Omega + \theta} \delta. \] (27)

Hence from (25)

\[ \sigma - \frac{1}{2} < \delta < \sigma_0^* - \frac{1}{2}. \] (28)

We use the notations:

\[ a(t) \equiv \frac{1}{A'} \left( A_1 \Omega + A_2 \log t \frac{\sigma - 1/2}{\delta} \right), \] (29)

\[ b(t) \equiv \frac{\delta}{\sigma - 1/2} = \frac{A_1 \Omega + \theta}{A_1 \Omega} = 1 + \frac{\theta}{A_1 \Omega} = 1 + \frac{\theta}{k(t)}, \] (30)

where
\[ k(t) \equiv A_1 \Omega(t). \] (31)

Evidently, by (27) and (29), \( a(t) \rightarrow \infty \), as \( t \rightarrow \infty \); also

\[ k(t) \rightarrow \infty, \text{ as } t \rightarrow \infty. \] (32)

Also, as \( t \rightarrow \infty \),

\[
\frac{a(t)}{k(t)} = \frac{1}{A} \left( 1 + \frac{A_2 \log t}{A_1 \Omega \cdot A_1 \Omega + \theta} \right) \\
= \frac{1}{A} \left( 1 + \frac{A_2 \log t}{A_1 \log t \cdot A_1 \Omega + \theta} \right) \rightarrow \infty
\] (33)

Therefore, by Lemma 10,

\[ a(t) + b(t) < a(t) \cdot b(t). \]

Thus from (26), we get, for \( \sigma - \frac{1}{2} \) as per (27), using (25),

\[
\log \left( | \zeta \left( \frac{1}{2} + it \right) | ^\sigma \right) < \left( A_1 \Omega + A_2 \log t \cdot \frac{\sigma - \frac{1}{2}}{\delta} \right) \cdot \frac{\delta}{\sigma - \frac{1}{2}} \\
= A_1 \Omega \cdot \frac{\delta}{\sigma - \frac{1}{2}} + A_2 \log t \\
= A_1 \Omega \cdot \frac{A_1 \Omega + \theta}{A_1 \Omega} + A_2 \log t \\
= A_1 \Omega + A_2 \log t + \theta \] (34)

or

\[
\log \left( | \zeta \left( \frac{1}{2} + it \right) | ^\sigma \frac{(A_1 \Omega + \theta)}{A_1 \Omega \cdot \delta} \right) < A_1 \Omega + A_2 \log t + \theta, \] (35)

whence we have

\[
\delta > A_1 \Omega + \theta \cdot \frac{| \zeta \left( \frac{1}{2} + it \right) | ^\sigma}{A_1 \Omega \cdot \exp(A_1 \Omega + A_2 \log t + \theta)} = \frac{A_1 \Omega + \theta}{A_1 \Omega} \left( \hat{\sigma} - \frac{1}{2} \right) \] (36)

(which is equivalent to the first inequality in (20) by the definition of \( \delta \) (see (11)) and (18), the definition of \( \hat{\sigma} - \frac{1}{2} \)).

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If we take
\[ \delta' = A_1 \Omega + \theta \left( \hat{\sigma} - \frac{1}{2} \right), \]  
then, from (36) we have
\[ \delta' < \delta. \]  
(38)
Thus
\[ \hat{\sigma} - \frac{1}{2} = A_1 \Omega + \theta \delta' < A_1 \Omega + \theta \delta < \delta \]  
(39)
Since \( \hat{\sigma} - \frac{1}{2} < \delta \), it follows from (25) that
\[ \hat{\sigma} - \frac{1}{2} < \sigma^* - \frac{1}{2} \]  
(40)
so that by virtue of (17) and (25),
\[ \log \left( \left| \frac{\zeta \left( \frac{1}{2} + it \right)}{\sigma - 1/2} \right|^{A_1} \right) < A_1 \Omega + A_2 \log t \frac{\hat{\sigma} - \frac{1}{2}}{\sigma - \frac{1}{2}} + A' \frac{\delta}{\sigma - \frac{1}{2}}. \]  
(41)
We show below that
\[ A_2 \log t \frac{\hat{\sigma} - \frac{1}{2}}{\delta} + A' \frac{\delta}{\sigma - \frac{1}{2}} < A_2 \log t \frac{\hat{\sigma} - \frac{1}{2}}{\delta'} + A' \frac{\delta'}{\sigma - \frac{1}{2}}. \]  
(42)
The inequality (42) holds if
\[ A_2 \log t \left( \hat{\sigma} - \frac{1}{2} \right) \left( \frac{1}{\delta'} - \frac{1}{\delta} \right) + A' \frac{\delta' - \delta}{\sigma - \frac{1}{2}} > 0 \]
or
\[ A_2 \log t \left( \hat{\sigma} - \frac{1}{2} \right)^2 - A' \delta' \delta > 0 \]
or, using (37), if
\[ A_2 \log t \frac{A_1 \Omega}{A_1 \Omega + \theta} \left( \hat{\sigma} - \frac{1}{2} \right) - A' \delta > 0 \quad \text{[(10) : } \delta = A^*/\log(n + 1)\text{].} \]
or, by (10),
\[ A_2 \log t \frac{A_1 \Omega}{A_1 \Omega + \theta} \left( \hat{\sigma} - \frac{1}{2} \right) > A' \frac{A^*}{\log(n + 1)} \]
or
\[ A^* < \log(n + 1) \frac{A_2}{A'} \log t \frac{A_1 \Omega}{A_1 \Omega + \theta} \left( \hat{\sigma} - \frac{1}{2} \right), \]  
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which holds by virtue of (20).

Thus (42) is proved, and from (41) we have

$$\log \left( \frac{|\zeta(\frac{1}{2} + it)|^{A}}{\sigma - 1/2} \right) < A_1\Omega + A_2 \log t \frac{\hat{\sigma} - \frac{1}{2}}{\delta'} + A' \frac{\delta'}{\sigma - \frac{1}{2}}. \quad (43)$$

Hence

$$\frac{1}{A'} \log \left( \frac{|\zeta(\frac{1}{2} + it)|^{A}}{\sigma - 1/2} \right) < \frac{1}{A'} \left( A_1\Omega + A_2 \log t \frac{\hat{\sigma} - \frac{1}{2}}{\delta'} \right) + \frac{\delta'}{\sigma - \frac{1}{2}}. \quad (44)$$

We note that, by (37),

$$\hat{\sigma} - \frac{1}{2} = \frac{A_1\Omega}{A_1\Omega + \theta} \frac{1}{A'} \log \left( \frac{|\zeta(\frac{1}{2} + it)|^{A}}{\sigma - 1/2} \right) < A_1\Omega + A_2 \log t \frac{\hat{\sigma} - \frac{1}{2}}{\delta'} + A' \frac{\delta'}{\sigma - \frac{1}{2}}.$$

We take

$$\hat{a}(t) \equiv \frac{1}{A'} \left( A_1\Omega + A_2 \log t \frac{\hat{\sigma} - \frac{1}{2}}{\delta'} \right),$$

$$\hat{b}(t) \equiv \frac{\delta'}{\sigma - \frac{1}{2}},$$

$$k(t) = A_1\Omega(t).$$

We find that

$$\hat{b}(t) = 1 + \frac{\theta}{k(t)},$$

and

$$\hat{a}(t) \to \infty, \quad k(t) \to \infty, \quad \text{as} \quad t \to \infty$$

and, as in (33),

$$\frac{\hat{a}(t)}{k(t)} \to \infty, \quad \text{as} \quad t \to \infty.$$

Hence, applying Lemma 10, we have

$$\hat{a}(t) + \hat{b}(t) < \hat{a}(t)\hat{b}(t).$$

Hence, from (44) we get

$$\log \left( \frac{|\zeta(\frac{1}{2} + it)|^{A}}{\sigma - 1/2} \right) < A_1\Omega \frac{\delta'}{\sigma - \frac{1}{2}} + A_2 \log t = A_1\Omega(t) + A_2 \log t + \theta. \quad (45)$$

Using (18), we get

$$\log e^{A_1\Omega + A_2 \log t + \theta} < A_1\Omega + A_2 \log t + \theta,$$
which is obviously false.

This contradiction shows that the assumption that the Riemann Hypothesis is true is not tenable. Thus the proof of Theorem 1 is complete.

6. Further Remarks
(I) It is known that the Riemann Hypothesis is equivalent to the statement that, for any $\epsilon > 0$, and all sufficiently large $x > 0$, the relative error in the Prime Number Theorem:

$$\pi(x) \sim \text{li}(x)$$

where \(\text{li}(x) = \lim_{\epsilon \to 0} \left[ \int_{0}^{1-\epsilon} \frac{dt}{\log t} + \int_{1+\epsilon}^{x} \frac{dt}{\log t} \right] ,$$

is less that \(x^{-\frac{1}{2}+\epsilon}\).

In view of our Theorem 1, this statement is false.

(II) It is known that the following assertion is equivalent to the Riemann Hypothesis. The series

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$

is convergent for $\sigma = \Re s > \frac{1}{2}$. In view of Theorem 1, this assertion is false.

(III) The following necessary and sufficient conditions for the truth of the Riemann Hypothesis were given respectively by Hardy and Littlewood and M. Riesz. By Theorem 1, these are false.

$$(\alpha)\quad \sum_{k=1}^{\infty} \frac{(-x)^k}{k! \zeta(2k+1)} = O(x^{-\frac{1}{2}}), \quad \text{as} \quad x \to \infty;$$

$$(\beta)\quad \sum_{k=1}^{\infty} \frac{(-1)^{k+1}x^k}{(k-1)\zeta(2k)} = O(x^{\frac{1}{2}+\epsilon}), \quad \text{as} \quad x \to \infty,$$

for every $\epsilon > 0$.

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