Effects of linear Central potential induced by Lorentz symmetry breaking on a generalized Klein-Gordon Oscillator

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Abstract
We investigate the generalized Klein-Gordon oscillator under the Lorentz symmetry breaking effects where, a linear electric and constant magnetic field is considered and analyze its effects on the relativistic quantum oscillator. Furthermore, the behaviour of the quantum oscillator in the presence of a Cornell-type scalar potential is analyzed and the solution of the bound state is obtained. We see that the analytical solution to the generalized Klein-Gordon oscillator can be achieved and the angular frequency of the oscillator depends on the quantum numbers of the system.

Keywords: Lorentz symmetry violation, Relativistic wave-equations, scalar potential, electric & magnetic field, biconfluent Heun Equation.

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1 Introduction
Inspired by the Dirac oscillator model for spin $\frac{1}{2}$-fermionic field [11, 12], a relativistic oscillator model for spin-0 scalar field was proposed in Ref. [13, 14] which is known as the Klein–Gordon oscillator. For the Klein-Gordon oscillator, one can replace the momentum operator by $\vec{p} \rightarrow \vec{p} - i M \omega r \vec{r}$ in the Klein-Gordon equation. Here $\omega$ is the angular frequency of the oscillator,

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and $r$ being the axial/radial distance. The Klein–Gordon oscillator has been studied in the quantum mechanics, such as, in non-commutative phase space with a magnetic field [15], in a space-time with cosmic string [16], in a Gödel-type space-time [17], effects of linear and Coulomb-type central potentials [18, 19, 20], in Kaluza-Klein theory [21], in a space-time with screw dislocation under a scalar potential [22], in a space-time with torsion [23], with non-inertial effects [24], with position-dependent mass in rotating cosmic string space-time [25], in cosmic string space-time under a Cornell-type potential [26], in (1 + 2)-dimensional space-time [27], in a cosmic string space-time with spacelike dislocation under Cornell-type potential [28], effects of global monopole space-time [29].

In this paper, we study the generalized Klein-Gordon oscillator [30, 31, 32, 33, 34, 35, 36] under Lorentz symmetry breaking defined by a tensor $(K_F)_{\mu\nu\alpha\beta}$ that governs the Lorentz symmetry violation out of the Standard Model Extension [1, 2, 3, 4, 5, 6, 7, 8, 9]. The violation of the Lorentz symmetry is determined by a non-null component of tensor, a linear electric and constant magnetic field. Then, we search for the solution of the bound state of the generalized Klein-Gordon oscillator in the presence of a scalar potential by modifying the mass term. We see that the angular frequency of the oscillator depends on the quantum numbers of the system.

The Standard Model Extension (SME) [1, 2, 3, 4, 5, 6, 7, 8, 9] is an extension of the Standard Model of the fundamental interactions where, the effective Lagrangian corresponds to the usual Standard Model plus Lorentz-violating (LV) tensorial background coefficients. The gauge sector of SME consists of two terms called CPT-odd sector [1, 2] and CPT-even sector [37, 10]. The relativistic quantum systems under Lorentz symmetry violation has been studied in Refs. [10, 37, 38, 39, 40, 41, 42, 43, 44]. The Lorentz symmetry violation has investigated at a low energy scenario in Refs. [45, 46, 47].

The Klein-Gordon equation under Lorentz symmetry violation background
is given by \[ \mu_\mu \to \mu_\mu + \frac{\alpha}{4} (K_F)_{\mu\nu\alpha\beta} F^{\mu\nu}(x) F^{\alpha\beta}(x), \]

where \( \alpha \) is a constant, \( F_{\mu\nu}(x) \) is the electromagnetic field tensor, \((K_F)_{\mu\nu\alpha\beta}\) is a dimensionless tensor. It has the symmetries of the Riemann tensor \( R_{\mu\nu\alpha\beta} \) and zero on double trace, so it contains 19 independent real components.

Based on the coupling of the generalized Dirac-oscillator as done in Ref. [48], a generalized oscillator model to the Klein-Gordon field was given in Refs. [30, 31, 32, 33, 34, 35, 36] where the momentum operator is changed as

\[ p_\mu \to p_\mu - i M \omega X_\mu \]

where \( \omega \) is the oscillator frequency, \( X_\mu = (0, f(r), 0, 0) \) is the four-vector and \( f(r) \) is an arbitrary function.

2 Generalized KG-oscillator Under the Effects of Lorentz Symmetry Violation

The generalized Klein-Gordon oscillator under the effects of Lorentz symmetry breaking (1) becomes

\[ p_\mu \to (p_\mu + i M \omega X^\mu \mu_\mu - i M \omega X_\mu) + \frac{\alpha}{4} (K_F)_{\mu\nu\alpha\beta} F^{\mu\nu}(x) F^{\alpha\beta}(x). \]

Therefore, the generalized Klein-Gordon oscillator equation is given by (\( c = \hbar = 1 \)):

\[ [(p_\mu + i M \omega X^\mu \mu_\mu - i M \omega X_\mu) + \frac{\alpha}{4} (K_F)_{\mu\nu\alpha\beta} F^{\mu\nu}(x) F^{\alpha\beta}(x) - M^2] \Psi = 0. \]

We consider the Minkowski flat space-time in the cylindrical coordinates \((t, r, \phi, z)\), where coordinates have their usual ranges given by

\[ ds^2 = -dt^2 + dr^2 + r^2 d\phi^2 + dz^2. \]
Therefore, the generalized Klein-Gordon oscillator under the effects of the Lorentz symmetry violation equation (4) becomes

\[
- \frac{\partial^2}{\partial t^2} + \left( \frac{1}{r} \frac{\partial}{\partial r} + M \omega f(r) \right) \left( r \frac{\partial}{\partial r} - M \omega r f(r) \right) + \frac{\partial^2}{\partial z^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} \right] \Psi \\
+ \frac{\alpha}{4} (K_F)_{\mu\nu\alpha\beta} F^{\mu\nu}(x) F^{\alpha\beta}(x) \Psi = M^2 \Psi.
\] (6)

Using the properties of the tensor \((K_F)_{\mu\nu\alpha\beta}\) given in Refs. [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 37, 38, 39, 40, 41, 42, 43, 44], we can rewrite equation (6) as

\[
- \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} - M^2 \omega^2 f^2(r) - M \omega \left( f'(r) + \frac{f(r)}{r} \right) \] \Psi \\
+ \left[ -\frac{\alpha}{2} (\kappa_{DE})_{ij} E^i E^j + \frac{\alpha}{2} (\kappa_{HB})_{jk} B^i B^j - \alpha (\kappa_{DB})_{jk} E^i B^j \right] \Psi \\
= M^2 \Psi.
\] (7)

Let us consider a possible scenario of the Lorentz symmetry violation determined by only one non-null component of the tensor \((\kappa_{DB})_{jk}\) as being \((\kappa_{DB})_{13} = \kappa = \text{const}\) and by a field configuration given by \([41, 42]\):

\[
\vec{B} = B_0 \hat{z}, \quad \vec{E} = \frac{\lambda}{2} r \hat{r}
\] (8)

where \(B_0\) is a constant, \(\hat{z}\) is a unit vector in the z-direction and \(\lambda\) is a constant associated with linear charge density of electric charge along the axial direction.

Hence Equation (7) becomes

\[
\left[ -\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} - M^2 \omega^2 f^2(r) - M \omega \left( f'(r) + \frac{f(r)}{r} \right) \right] \Psi \\
- \frac{\alpha \lambda B_0 \kappa}{2} r \Psi = M^2 \Psi.
\] (9)

Let the solution to the Eq. (9) is

\[
\Psi(t, r, \phi, z) = e^{i(-E t + l \phi + k z)} \psi(r),
\] (10)

where \(E, l, k\) have their usual meaning.
Substituting the solution (10) into the Eq. (9), we obtain the following radial wave-equation for $\psi(r)$:

$$\psi''(r) + \frac{1}{r} \psi'(r) + \left[ E^2 - k^2 - \frac{l^2}{r^2} - M^2 \omega^2 f^2 - M \omega \left( f' + \frac{f}{r} \right) \right] \psi(r)$$

$$- \frac{\alpha \lambda B_0 \kappa}{2} r \psi(r) = M^2 \psi(r).$$

(11)

To study the generalized KG-oscillator under the effects of Lorentz symmetry violation, we have chosen the following function [48]

$$f(r) = b_1 r + \frac{b_2}{r},$$

(12)

where $b_1 > 0, b_2 > 0$ are arbitrary constants. This type of function has been studied in the relativistic quantum systems Refs. [49, 50, 51, 30, 31, 32, 33, 34, 35].

Substituting the function (12) into the Eq. (11), we obtain the following radial wave-equation equation:

$$\left[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} + \Lambda - M^2 \omega^2 b_1^2 r^2 - \frac{j^2}{r^2} - br \right] \psi(r) = 0,$$

(13)

where

$$\Lambda = E^2 - M^2 - k^2 - 2 M \omega b_1 - 2 M^2 \omega^2 b_1 b_2,$$

$$j = \sqrt{l^2 + M^2 \omega^2 b_1^2},$$

$$b = \frac{\alpha \lambda B_0 \kappa}{2}.$$

(14)

Transforming $x = \sqrt{M \omega b_1} r$ into the Eq. (16), we obtain the following equation:

$$\left[ \frac{d^2}{dx^2} + \frac{1}{x} \frac{d}{dx} + \zeta - x^2 - \frac{j^2}{x^2} - \theta x \right] \psi(x) = 0,$$

(15)

where

$$\zeta = \frac{\Lambda}{M \omega b_1}, \quad \theta = \frac{b}{(M \omega b_1)^{\frac{3}{2}}}.$$

(16)
Now, we use the appropriate boundary conditions that the wave functions is regular both at \( x \to 0 \) and \( x \to \infty \). Suppose the possible solution to the Eq. (15) is
\[
\psi(x) = x^j e^{-\frac{1}{2} (x+\theta) x} H(x).
\] (17)
Substituting the solution (17) into the Eq. (15), we obtain the following equation
\[
H''(x) + \left[ \frac{1 + 2 j}{x} - 2 x - \theta \right] H'(x) + \left[ -\frac{\theta}{2} (1 + 2 j) + \Theta \right] H(x) = 0,
\] (18)
where \( \Theta = \zeta + \frac{\theta^2}{4} - 2 (1 + j) \).

Equation (18) is the biconfluent Heun’s differential equation \[52, 32, 33, 36, 34, 58, 59\] with \( H(x) \) is the Heun polynomials function.

The above equation (18) can be solved by the Frobenius method. Writing the solution as a power series expansion around the origin \[60\]:
\[
H(x) = \sum_{i=0}^{\infty} d_i x^i.
\] (19)
Substituting the power series solution into the Eq. (18), we obtain the following recurrence relation
\[
d_{n+2} = \frac{1}{(n+2)(n+2+2 j)} \left[ \theta (n+j+\frac{3}{2}) d_{n+1} - (\Theta - 2 n) d_n \right].
\] (20)
With few coefficients are
\[
d_1 = \frac{\theta}{2} d_0,
\]
\[
d_2 = \frac{1}{4 (1 + j)} \left[ \theta (j+\frac{3}{2}) d_1 - \Theta d_0 \right].
\] (21)

The power series expansion \( H(x) \) becomes a polynomial of degree \( n \) by imposing the following two conditions \[52, 32, 33, 36, 34\]
\[
\Theta = 2n, \quad (n = 1, 2, \ldots)
\]
\[
d_{n+1} = 0.
\] (22)
By analyzing the first condition, we obtain following equation for the energy eigenvalue $E_{n,l}$:

$$E_{n,l} = \pm \sqrt{M^2 + k^2 + 2M^2 \omega^2 b_1 b_2 + 2M \omega b_1 \left(n + 2 + \sqrt{l^2 + M^2 \omega^2 b_2^2}\right)} - \left(\frac{\alpha \lambda B_0 \kappa}{4M \omega b_1}\right)^2. \quad (23)$$

Note that Eq. (23) is not the general expression of the relativistic energy eigenvalues of the generalized KG-oscillator field.

$$E_{n,l} = \pm \sqrt{M^2 + k^2 + 2M \omega b_1 (n + 2 + |l|)} - \left(\frac{\alpha \lambda B_0 \kappa}{4M \omega b_1}\right)^2. \quad (24)$$

which is similar to the result obtained in Ref. [42] (see Eq. (19) in the Ref. [42]).

The radial wave-functions are given by

$$\psi_{n,l}(x) = x\sqrt{l^2 + M^2 \omega^2 b_2^2} \frac{1}{\sqrt{x + \frac{\alpha \lambda B_0 \kappa}{(M \omega b_1)^2}}} x H(x). \quad (25)$$

Now, we evaluate the individual energy level and eigenfunction one by one as in Refs. [52, 32, 33, 36, 34]. For example, $n = 1$, we have $\Theta = 2$ and $d_2 = 0$ which implies

$$\Rightarrow \frac{2}{\theta (j + \frac{3}{2})} d_0 = \frac{\theta}{2} d_0$$

$$\Rightarrow \omega_{1,l} = \left[\frac{(\alpha \lambda B_0 \kappa)^2 (j + \frac{3}{2})^{\frac{3}{2}}}{4M b_1}\right]^{\frac{1}{2}} \quad (26)$$

a constraint on the angular frequency $\omega_{1,l}$ of the oscillator. Note that its value changes for each quantum number $n$ and $l$ of the system, so we have labeled $\omega \rightarrow \omega_{n,l}$.

Therefore, the ground state energy level for the radial mode $n = 1$ is
given by
\[ E_{1,l} = \pm \sqrt{M^2 + k^2 + 2M\omega_{1,l}b_1 \left( M\omega_{1,l}b_2 + 3 + \sqrt{l^2 + M^2\omega_{1,l}^2b_2^2} \right) - \left( \frac{\alpha \lambda B_0 \kappa}{\sqrt{\frac{3}{2} + \sqrt{l^2 + M^2\omega_{1,l}^2b_2^2}}} \right)^2}. \]  

(27)

And the ground state eigenfunction is
\[ \psi_{1,l}(x) = x\sqrt{l^2 + M^2\omega_{1,l}b_2^2} e^{-\frac{1}{2}(x + 2d_1)x} (1 + d_1x). \]  

(28)

where for simplicity \( d_0 = 1 \) and
\[ d_1 = \frac{2}{\left(\sqrt{l^2 + M^2\omega_{1,l}b_2^2 + \frac{3}{2}}\right)^2}. \]  

(29)

The lowest energy state (27) plus the ground state wave-function (28)–(29) with the restriction (26) on the angular frequency of the oscillator is defined for the radial mode \( n = 1 \).

3 Generalized KG-oscillator Under the Effects of Lorentz Symmetry Violation subject to Cornell-type potential

We introduce a scalar potential \( S(r) \) in the generalized KG-oscillator by modifying the mass term \( M \rightarrow M + S(r) \) under the effects of the Lorentz symmetry violation. Therefore, the radial wave-equation for \( \psi(r) \) the Eq. (11) becomes
\[ \psi''(r) + \frac{1}{r}\psi'(r) + \left[ E^2 - k^2 - \frac{l^2}{r^2} - M^2\omega^2 f^2 - M\omega \left( f' + \frac{f}{r} \right) \right] \psi(r) - \frac{\alpha \lambda B_0 \kappa}{4} r \psi(r) = (M + S(r))^2 \psi(r). \]  

(30)

In this work, we have chosen a Cornell-type scalar potential given by
\[ S(r) = \eta_L r + \frac{\eta_c}{r}, \]  

(31)

8
where $\eta_L > 0, \eta_c > 0$ are arbitrary constants.

Substituting the above potential (31) and the function (12) into the radial wave-equation (30), we have

\[
\left[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} + \tilde{\Lambda} - \Omega^2 r^2 - \tilde{j}^2 \frac{\eta}{r^2} - \frac{a}{r} - \tilde{b} r \right] \psi(r) = 0, \tag{32}
\]

where we have defined

\[
\tilde{\Lambda} = E^2 - M^2 - k^2 - 2 M \omega b_1 - 2 M^2 \omega^2 b_1 b_2 - 2 \eta_L \eta_c, \\
\Omega = \sqrt{M^2 \omega^2 b_1^2 + \eta_L^2}, \\
\tilde{j} = \sqrt{j^2 + \eta_c^2}, \\
a = 2 M \eta_c, \\
\tilde{b} = b + 2 M \eta_L. \tag{33}
\]

Transforming $x = \sqrt{\Omega} r$ in the above equation (32), we have

\[
\left[ \frac{d^2}{dx^2} + \frac{1}{x} \frac{d}{dx} + \tilde{\zeta} - x^2 - \frac{\tilde{j}^2}{x^2} - \frac{\eta}{x} - \tilde{\theta} x \right] \psi(x) = 0, \tag{34}
\]

where

\[
\tilde{\zeta} = \frac{\tilde{\Lambda}}{\Omega}, \quad \eta = \frac{a}{\sqrt{\Omega}}, \quad \tilde{\theta} = \frac{\tilde{b}}{\Omega^2}. \tag{35}
\]

Suppose the possible solution to the Eq. (34) is

\[
\psi(x) = x^\tilde{j} e^{-\frac{\tilde{\theta}}{2} (x+\tilde{\theta}) x} H(x). \tag{36}
\]

Substituting the solution (36) into the Eq. (34), we obtain the following equation

\[
H''(x) + \left[ \frac{1 + 2 \tilde{j}}{x} - 2 x - \tilde{\theta} \right] H'(x) + \left[ -\frac{\beta}{x} + \tilde{\Theta} \right] H(x) = 0, \tag{37}
\]

where

\[
\tilde{\Theta} = \tilde{\zeta} + \frac{\tilde{\theta}^2}{4} - 2 (1 + \tilde{j}) \quad , \quad \beta = \eta + \frac{\tilde{\theta}}{2} (1 + 2 \tilde{j}). \tag{38}
\]
Equation (37) is the biconfluent Heun’s differential equation \[52, 32, 33, 36, 34, 58, 59\] with \( H(x) \) is the Heun polynomials function.

Considering the power series method considered earlier, we obtain the following recurrence relations

\[
d_{n+2} = \frac{1}{(n+2)(n+2+2j)} \left[ \left\{ \beta + \tilde{\theta} (n+1) \right\} d_{n+1} - (\tilde{\Theta} - 2n) d_n \right]. \tag{39}
\]

With few coefficients are

\[
d_1 = \left( \frac{\eta}{1+2j} + \frac{\tilde{\theta}}{2} \right) d_0,
\]

\[
d_2 = \frac{1}{4(1+j)} \left[ (\beta + \tilde{\theta}) d_1 - \tilde{\Theta} d_0 \right]. \tag{40}
\]

The power series expansion \( H(x) \) becomes a polynomial of degree \( n \) by imposing the following two conditions \[52, 32, 33, 36, 34\]

\[
\tilde{\Theta} = 2n, \quad (n = 1, 2, ...)
\]

\[
d_{n+1} = 0. \tag{41}
\]

By analyzing the first condition, we obtain following equation of the energy eigenvalue \( E_{n,l} \):

\[
E_{n,l}^2 = M^2 + k^2 + 2M^2 \omega^2 b_1 b_2 + 2M \omega b_1 + 2\Omega \left( n + 1 + \sqrt{l^2 + M^2 \omega^2 b_2^2 + \eta_c^2} \right) \\
+ 2 \eta_L \eta_c - \frac{\tilde{b}_2^2}{4 \Omega^2}. \tag{42}
\]

Note that Eq. (42) is not the general expression of the relativistic energy eigenvalues of a generalized KG-oscillator.

The corresponding wave-functions are given by

\[
\psi_{n,l}(x) = x^{\frac{l}{2}} e^{-\frac{1}{2} \left[ x + \frac{\tilde{b}_2}{\eta_c} \right]^2} H(x). \tag{43}
\]

Note that for \( \eta_L \rightarrow 0 \) and \( b_2 \rightarrow 0 \), the energy eigenvalues expression (42) and the corresponding radial wave-function (43) reduces to the result found in Ref. 42.
Now, we evaluate the individual energy levels and eigenfunctions one by one as in [52, 32, 33, 36, 34]. For example, \( n = 1 \), we have \( \Theta = 2 \) and \( c_2 = 0 \) which implies

\[
\Rightarrow \frac{2}{\beta + \tilde{\theta}} d_0 = \left( \frac{\eta}{1 + 2 j} + \frac{\tilde{\theta}}{2} \right) d_0
\]

\[
\Rightarrow \Omega_{1,l}^3 - \left[ \frac{a_1^2}{2 \left( 1 + 2 j \right)} \right] \Omega_{1,l}^2 - a b \left( \frac{1 + \tilde{j}}{1 + 2 j} \right) \Omega_{1,l} - \frac{\tilde{b}_1^2}{8} (3 + 2 \tilde{j}) = 0
\]

a constraint on the potential parameter \( \Omega_{1,l} \).

The allowed values of the angular frequency of the oscillator for the radial mode \( n = 1 \) is

\[
\omega_{1,l} = \frac{1}{M b_1} \sqrt{\Omega_{1,l}^2 - \eta_c^2}.
\] (45)

Therefore, the ground state energy level for the radial mode \( n = 1 \) is given by

\[
E_{1,l}^2 = M^2 + k^2 + 2 M \omega_{1,l} b_1 (M \omega_{1,l} b_2 + 1) + 2 \Omega_{1,l} \left( 2 + \sqrt{l^2 + M^2 \omega_{1,l}^2 b_2^2 + \eta_c^2} \right) + 2 \eta_L \eta_c - \left( \frac{a b_0 \lambda \kappa}{2} + 2 M \eta_L \right)^2.
\] (46)

And the ground state eigenfunction is

\[
\psi_{1,l}(x) = x^{1/2 + \Omega_{1,l}^2 b_2^2 + \eta_c^2} e^{-\frac{x + \left( \frac{a b_0 \lambda \kappa}{2} + 2 M \eta_L \right)}{\Omega_{1,l}^2}} (1 + d_1 x).
\] (47)

where we have chosen \( d_0 = 1 \) and

\[
d_1 = \frac{1}{\sqrt{\Omega_{1,l}}} \left[ \frac{M \eta_c}{\left( \frac{1}{2} + \sqrt{l^2 + M^2 \omega_{1,l}^2 b_2^2 + \eta_c^2} \right)} + \frac{(a b_0 \lambda \kappa/2 + M \eta_L)}{\Omega_{1,l}} \right].
\] (48)

The lowest energy state Eq. (43) plus the ground state wave-function Eqs. (44)–(46) with the restriction (45) that gives the possible values of the angular frequency of the oscillator is defined for the radial mode \( n = 1 \).
4 Conclusions

The Klein-Gordon oscillator under the effects of the Lorentz symmetry violation was studied in Ref. [42]. Inspired by this work, we have analysed the behaviour of the generalized Klein-Gordon oscillator by choosing a function \( f(r) = b_1 r + \frac{b_2}{r} \) in the equation under the effects of a linear central potential induced by Lorentz symmetry violation. Furthermore, we have introduced a Cornell-type scalar potential and analyzed the behaviour of the relativistic quantum oscillator. In section 2, we have chosen the function \( f(r) = b_1 r + \frac{b_2}{r} \) and derived the radial wave equation. For a suitable wave function, Heun’s biconfluent differential equation is derived from this radial wave equation. Substituting the power series in the biconfluent differential equation and finally truncating it, the non-compact expression of the energy eigenvalues Eq. (23) and the radial wave function Eq. (25) is obtained. By imposing the truncating condition \( d_{n+1} = 0 \), the possible values of the oscillator frequency Eq. (26), the energy level Eq. (27), and the radial wave function Eqs. (28)–(29) associated with the lowest state of the quantum system defined by \( n = 1 \) is obtained. We noted that for \( b_2 \to 0 \) in the function \( f(r) \), the energy eigenvalue expression (23) is very similar to the result obtained in Ref. [42] (see Eq. (19) in Ref. [42]).

In section 3, we have introduced a Cornell-type scalar potential \( S(r) \) in the equation by modifying the mass term under the effects of Lorentz symmetry violation. Following a similar procedure, we have obtained the ground state energy level (46) and the lowest state wave function Eqs. (47)–(48) with the restriction (45) on the angular frequency of the oscillator. Here also, for \( b_2 \to 0 \) and \( \eta_L \to 0 \), the energy eigenvalues expression (43) and the wave-function (44) is similar to the result obtained in Ref. [42]. Thus the linear potential term \( \eta_L r \) present in the scalar potential \( S(r) \) and the extra Coulomb-like term \( \frac{b_2}{r} \) in the function \( f(r) \) modified the energy spectrum in comparison to the previous result. In both cases, we have seen that for the function \( f(r) = b_1 r + \frac{b_2}{r} \), and the Lorentz symmetry violation parameters
(α, λ, B₀, κ) (in section 2), and the Cornell-type scalar potential considered in section 3 modified the energy levels and the wave-function associated with each radial mode.

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