We review the construction of particle physics models in the framework of non-commutative geometry. We first give simple examples, and then progress to outline the Connes-Lott construction of the standard Weinberg-Salam model and our construction of the SO(10) model. We then discuss the analogue of the Einstein-Hilbert action and gravitational matter couplings. Finally we speculate on some experimental signatures of predictions specific to the non-commutative approach.

1. Introduction

The Weinberg-Salam model [1] of electroweak interactions is a milestone in the search for unity of all fundamental interactions. But although this model has passed all experimental tests at present energies, many challenges remain. To name just a few, we have to understand:

a-The role of the Higgs field necessary in the spontaneous breakdown of the $SU(2) \times U(1)$ gauge symmetry.

b-The fermionic mass matrices and family mixing, the gauge coupling constants, the mass and vacuum expectation value (vev) of the Higgs field.

c-Unifying gravity with the strong and electroweak interactions in a renormalizable theory.

There are many attempts to solve these problems using schemes such as grand unification, Kaluza-Klein compactification and string theory, all with and without supersymmetry. The virtues and shortcomings of these lines of research are now well known.

During the past few years, Connes has proposed a construction of particle physics models based on his formulation of non-commutative geometry [2]. This method addresses point a- raised above, in that it predicts the existence of the Higgs field and gives it a geometrical meaning [3]. This article is a short review of Connes’ non-commutative construction and intended for particle physicists. The mathematics used here will be the minimum needed. For the more mathematically oriented reader we refer to some of the available reviews [4]. Our plan is as follows. In section 2 we introduce the non-commutative construction and give simple examples. In section 3 we review the derivation of the standard model and in section 4 the grand unified
In section 5 we describe an analogue of the Einstein-Hilbert action and the gravitational matter couplings, and, under a natural geometrical assumption, obtain some predictions for the top quark mass and the Higgs mass.

2. The non-commutative construction

Connes’ non-commutative geometry is very general [2]. A non-commutative geometry is specified by the triple $(\mathcal{A}, \hbar, \mathcal{D})$, where $\hbar$ is a Hilbert space, $\mathcal{A}$ is an involutive algebra of operators on $\hbar$, and $\mathcal{D}$ is an unbounded self-adjoint operator on $\hbar$. Let $\Omega$ be the $\mathbb{Z}$ graded differential algebra of universal forms over $\mathbb{R}$ or $\mathbb{C}$: $\Omega = \bigoplus_n \Omega^n$, where $A = \Omega^0$ and $\Omega^n$ is the space of $n$-forms with operations i) $d : \Omega^n \rightarrow \Omega^{n+1}$, ii) $m : \Omega^n \otimes \Omega^m \rightarrow \Omega^{n+m}$. The algebra of universal forms over $\mathcal{A}$, $\Omega(\mathcal{A})$, is generated by $f$ and $df$, where $f \in \mathcal{A}$. The operator $d$ obeys Leibnitz rule, $d(fg) = (df)g + f(dt)$, where $f, g \in \mathcal{A}$, and $d^2 = 0$. An $n$-form in $\Omega^n(\mathcal{A})$ is given by $\sum_i a^i_0 da^i_1 \cdots da^n$, $a^i_0, \cdots a^i_n \in \mathcal{A}$.

An involutive representation of $\Omega(\mathcal{A})$ on $\hbar$ is provided by the map $\pi : \Omega(\mathcal{A}) \rightarrow B(\hbar)$ defined by $\pi(a_0 da_1 \cdots da_n) = a_0 [D, a_1] \cdots [D, a_n]$, \hspace{1cm} (2.1)

where $B(\hbar)$ is the algebra of bounded operators on $\hbar$. The non-commutativity resides in the fact that $ab$ is not necessarily equal, up to a sign, to $ba$. Let $E$ be a vector bundle determined by the vector space $\mathcal{E}$ of its sections. We will be mainly interested in the case $\mathcal{E} = \mathcal{A}$. Let $\rho$ be a self-adjoint element in the space $\Omega^1(\mathcal{A})$. It determines a connection with curvature $\theta = d\rho + \rho^2 \in \Omega^2(\mathcal{A})$. The Yang-Mills action functional is obtained using the Dixmier trace which permits the definition of integration and volume elements in non-commutative geometry. We set (see [2,3])

$$I_{YM} = Tr_w (\theta^2 D^{-4}), \hspace{1cm} (2.2)$$

The same quantity can be defined using the heat kernel expansion (see [5]); i.e.,

$$\lim_{\epsilon \rightarrow 0} \frac{Tr_H (\theta^2 e^{-\epsilon D^2})}{Tr_H (e^{-\epsilon D^2})}. \hspace{1cm} (2.3)$$

We illustrate these notions with two simple examples.

1- Let $\mathcal{A}_1 = C^\infty(M)$, the algebra of functions on a four-dimensional Riemannian manifold $M$, $\hbar$ the Hilbert space of spinors $L^2(M, \sqrt{g}d^4x)$ and $D_1 = \partial$, the Dirac operator on $\hbar$. The one-form $\rho = \sum_i a^i db^i$ has the image under $\pi$

$$\pi(\rho) = \sum_i a^i [D, b^i] = \sum_i a^i \partial b^i \equiv \gamma^\mu A_\mu. \hspace{1cm} (2.4)$$

Similarly for the two-form $d\rho$ we have

$$\pi(d\rho) = \sum_i [D, a^i] [D, b^i] = \sum \partial a^i \partial b^i. \hspace{1cm} (2.5)$$

The curvature $\pi(\theta) = \pi(d\rho) + \pi(\rho)^2$ is then given by

$$\pi(\theta) = \frac{1}{2} \gamma^\mu^\nu F_{\mu\nu} + X, \hspace{1cm} (2.6)$$
where $\gamma^{\mu\nu} = \frac{1}{2}(\gamma^{\mu}\gamma^{\nu} - \gamma^{\nu}\gamma^{\mu})$, $X = g^{\mu\nu}(A_\mu A_\nu + \sum_i \partial_\mu a_i^\dagger \partial_\nu b^i)$ is an "auxiliary field" and $F_{\mu\nu}$ is the field strength of $A_\mu$. Notice that $\pi(d\rho) = g^{\mu\nu}\sum_\delta \partial_\mu a_i^\dagger \partial_\nu b^i \neq 0$, is a scalar function. This is the reason behind the presence of the auxiliary field in $\pi(\theta)$. It is possible to work instead with the space $\Omega^2(\mathcal{A})$: Ker$^+_\pi + d$Ker$^-\pi$, but we will not do this now. The Yang-Mills action becomes

$$I_{YM} = \int d^4x\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + X^2\right).$$

(2.7)

After eliminating the auxiliary field $X$ by its equation of motion, it decouples from the action.

2-For a two point space, we take $\mathcal{A}_2 = C \oplus C$, and $h = C^N \oplus C^N$ and the Dirac operator is $D_2 = \begin{pmatrix} 0 & K \\ K^* & 0 \end{pmatrix}$, where $K$ is an $N \times N$ matrix. The elements $a \in \mathcal{A}_2$ have the representation $a \rightarrow \text{diag}(a_1, a_2)$, $a_1, a_2 \in C$. Then

$$\pi(\rho) = \sum_i a_i^\dagger[D, b^i] = \begin{pmatrix} 0 & K\phi \\ K^*\phi^* & 0 \end{pmatrix},$$

(2.8)

where $\phi = \sum_\delta a_1^\dagger(b_2^\dagger - b_1^\dagger)$ and $\phi^* = \sum_\delta a_2^\dagger(b_1^\dagger - b_2^\dagger)$. Then $\pi(d\rho) = \sum_i[D, a_i^\dagger][D, b^i]$ is equal to

$$\pi(d\rho) = -\begin{pmatrix} KK^*(\phi + \phi^*) & 0 \\ 0 & K^*K(\phi + \phi^*) \end{pmatrix}.$$  

(2.9)

The Yang-Mills action is easily calculated to be

$$\text{tr}(\theta^2) = 2\text{Tr}(KK^*)^2(|\phi - 1|^2 - 1)^2,$$

(2.10)

It is seen to be of the same form as the Higgs potential for a scalar field $\phi$ and is positive definite. Notice that $[D, a] = \begin{pmatrix} 0 & K(b_2 - b_1) \\ K^*(b_1 - b_2) & 0 \end{pmatrix}$ is a difference operator in the discrete space.

3. The standard Weinberg-Salam model

With the simple tools introduced in the last section, we now show that it is possible to construct realistic action functionals. Not all models are possible, but for those ones which are, the Higgs structure is fixed. For lack of space we shall only describe the standard Weinberg-Salam model in this section and the grand unification SO(10) model in the next section. Our method is a modified variant of Connes’ construction (simplifying some computations [5]).

Combining examples 1 and 2, let the algebra be $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$ acting on the Hilbert space $h = h_1 \otimes h_2$, where $\mathcal{A}_1 = C^\infty(M)$, considered before, and $\mathcal{A}_2 = M_2(C) \oplus M_1(C)$ the algebras of $2 \times 2$ and $1 \times 1$ matrices. The Hilbert space is that of spinors of the form $L = \begin{pmatrix} l \\ e \end{pmatrix}$ where $l$ is a doublet and $e$ is a singlet. The spinor $L$ satisfies the chirality condition $\gamma_5 \otimes \Gamma_1 L = L$, where $\Gamma_1 = \text{diag}(1_2, -1)$ is the grading
operator. This implies that $l = l_L$ is left-handed and $e = e_R$ is right-handed, and so we can write $l_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$. The Dirac operator is $D = D_1 \otimes 1 + \Gamma_1 \otimes D_2$, so that

$$D_l = \begin{pmatrix} \partial \otimes 1_2 & \gamma_5 M_{12} \otimes k \\ \gamma_5 M_{21} \otimes k^* & \partial \end{pmatrix},$$  

(3.1)

where $M_{21} = M_{12}^*$ and $k$ is a family mixing matrix. The geometry is that of a four-dimensional manifold $M$ times a discrete space of two points. The column $M_{12}$ in $D$, the vev of the Higgs field, is taken to be $M_{12} = \mu \begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv H_0$. The elements $a \in \mathcal{A}$ have the representation $a \rightarrow \text{diag}(a_1, a_2)$ where $a_1$ and $a_2$ are $2 \times 2$ and $1 \times 1$ unitary matrix-valued functions, respectively. The self-adjoint one-form $\rho$ has the representation

$$\pi_l(\rho) = \begin{pmatrix} A_1 \otimes 1_3 & \gamma_5 H \otimes k \\ \gamma_5 H^* \otimes k^* & A_2 \otimes 1_3 \end{pmatrix},$$

(3.2)

where $A_1 = \sum_i a_i^j \partial b_i^j$, $A_2 = \sum_i a_i^j \partial b_i^j$ and $H = H_0 + \sum_i a_i^j H_0 b_i^j$. In a world without quarks, the generalized tracelessness condition $\text{Tr}(\Gamma_1 \pi(\rho)) = 0$ allows the gauge fields to be written in the form $A_1 = -\frac{i}{2} g_2 \sigma^a A_a + ig_1 B$, $A_2 = 2ig_1 B$ where $g_1, g_2$ are the U(1) and SU(2) gauge couplings. The leptonic action $< L, (D + \rho)L >$ gives the correct lepton couplings to the gauge and Higgs fields. However, to be realistic, the quarks and the SU(3) gauge group must be introduced. This can be achieved by taking a bimodule structure relating two algebras $\mathcal{A}$ and $\mathcal{B}$, where the algebra $\mathcal{B}$ is taken to be $M_1(C) \oplus M_3(C)$, commuting with the action of $\mathcal{A}$, and the mass matrices in the Dirac operator are taken to be zero when acting on elements of $\mathcal{B}$. Then the one-form $\eta$ in $\Omega^1(\mathcal{B})$ has the simple form $\pi_l(\eta) = B_1 \text{diag}(1, 1, 1)$, where $B_1$ is a U(1) gauge field associated with $M_1(C)$. The quark Hilbert space is that of the spinor

$$Q = \begin{pmatrix} u_L \\ d_L \\ d_R \\ u_R \end{pmatrix}.$$  

The representation of $a \in \mathcal{A}$ is: $a \rightarrow \text{diag}(a_1, a_2, \bar{a}_2)$ where $a_1$ is a $2 \times 2$ matrix-valued function and $a_2$ is a complex-valued function. The Dirac operator acting on the quark Hilbert space is

$$D_q = \begin{pmatrix} \gamma^\mu (\partial \mu + \ldots) \otimes 1_2 \otimes 1_3 & \gamma_5 \otimes M_{12} \otimes k' & \gamma_5 \otimes \bar{M}_{12} \otimes k'' \\ \gamma_5 \otimes M_{21}^* \otimes k^* & \gamma_5 \otimes \bar{M}_{12}^* \otimes k'^* & \gamma_5 \otimes \bar{M}_{12} \otimes k''^* \\ \gamma_5 \otimes \bar{M}_{12}^* \otimes k''^* & \gamma^\mu (\partial \mu + \ldots) \otimes 1_3 & 0 \\ \gamma_5 \otimes \bar{M}_{12} \otimes k'' & 0 & \gamma^\mu (\partial \mu + \ldots) \otimes 1_3 \end{pmatrix} \otimes 1_3,$$  

(3.3)

where $k'$ and $k''$ are $3 \times 3$ family mixing matrices, and $\bar{M}_{12} = \mu \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Then the one-form in $\Omega^1(\mathcal{A})$ has the representation

$$\pi_q(\rho) = \begin{pmatrix} A_1 \otimes 1_3 & \gamma_5 H \otimes k' & \gamma_5 \bar{H} \otimes k'' \\ \gamma_5 H^* \otimes k'^* & A_2 \otimes 1_3 & 0 \\ \gamma_5 \bar{H}^* \otimes k''^* & 0 & A_2 \otimes 1_3 \end{pmatrix},$$

(3.4)

where $\bar{H}_a = \epsilon_{ab} H^b$. On the algebra $\mathcal{B}$ the Dirac operator has zero mass matrices, and the one form $\eta$ in $\Omega^1(\mathcal{B})$ has the representation $\pi_q(\eta) = B_2 \text{diag}(1, 1, 1)$ where
$B_2$ is the gauge field associated with $M_3(C)$. Imposing the unimodularity condition on the algebras $A$ and $B$ relates the U(1) factors in both algebras [3]: $\text{tr}(A_1) = 0$, $A_2 = B_1 = -\text{tr}B_2 = \frac{i}{2}g_1B$. We can then write

$$A_1 = -\frac{i}{2}g_2A^\sigma\sigma_a$$

$$B_2 = -\frac{i}{6}g_1B - \frac{i}{2}g_3V^i\lambda_i$$

where $g_3$ is the SU(3) gauge coupling constant, and $\sigma^a$ and $\lambda^i$ are the Pauli and Gell-Mann matrices, respectively. The fermionic action for the leptons is

$$<L,(D + \rho + \eta)L> = \int d^4x \sqrt{g}\left(L(D_l + \pi_l(\rho) + \pi_l(\eta))L\right), \quad (3.5)$$

and, for the quarks it is

$$<Q,(D + \rho + \eta)Q> = \int d^4x \sqrt{g}\left(Q(D_q + \pi_q(\rho) + \pi_q(\eta))Q\right), \quad (3.6)$$

and these can be easily checked to reproduce the standard model lepton and quark interactions with the correct hypercharge assignments.

The bosonic actions are the square of the curvature in the lepton and quark spaces, and are given, respectively, by

$$I_l = \text{Tr}(C_l(\theta_\rho + \theta_\eta)^2 D_l^{-4})$$

$$I_q = \text{Tr}(C_q(\theta_\rho + \theta_\eta)^2 D_q^{-4}). \quad (3.7)$$

To compute the bosonic action, we use a general formula, derived in [5], based on a Dirac operator where the discrete space has $N$ points:

$$D = \begin{pmatrix}
\emptyset \otimes 1 \otimes 1 & \gamma_5 \otimes M_{12} \otimes K_{12} & \ldots & \gamma_5 \otimes M_{1N} \otimes K_{1N} \\
\gamma_5 \otimes M_{21} \otimes K_{21} & \emptyset \otimes 1 \otimes 1 & \ldots & \gamma_5 \otimes M_{2N} \\
\vdots & \vdots & \ddots & \vdots \\
\gamma_5 \otimes M_{N1} \otimes K_{N1} & \gamma_5 \otimes M_{N2} \otimes K_{N2} & \ldots & \emptyset \otimes 1
\end{pmatrix}, \quad (3.8)$$

where the $K_{mn}$ are $3 \times 3$ matrices commuting with the $a_i$ and $b_i$. The Yang-Mills action associated with this operator is

$$I_B = \sum_{m=1}^{N} \text{Tr}\left(\frac{1}{2}F_{\mu\nu}^mF^{\mu\nu}m - \sum_{p \neq m} |K_{mp}|^2|\phi_{mp} + M_{mp}|^2 - (Y_m + X'_m)^2\right)$$

$$+ \sum_{p \neq m} |K_{mp}|^2|\partial_\mu(\phi_{mp} + M_{mp}) + A_{\mu m}(\phi_{mp} + M_{mp}) - (\phi_{mp} + M_{mp})A_{\mu p}|^2$$

$$- \sum_{n \neq m} \sum_{p \neq m,n} |K_{mp}K_{pn}(\phi_{mp} + M_{mp})(\phi_{pn} + M_{pn} - M_{mp}M_{pn} - X_{mn})|^2), \quad (3.9)$$
where the $A^m$ are the gauge fields in the $m - m$ entry of $\pi(\rho)$ and $\phi_{mn}$ are the scalar fields in the $m - n$ entry of $\pi(\rho)$. The $X_{mn}$, $X'_{mn}$ and $Y_m$ are fields whose unconstrained elements are auxiliary fields that can be eliminated from the action. Their expressions in terms of the $a^i$ and $b^i$ are

$$X_{mn} = \sum_i a^i_m \sum_{p \neq m,n} K_{mp}K_{pn}(M_{mp}M_{pn}b^i_n - b^i_mM_{mp}M_{pn}), \quad m \neq n, \quad (3.10)$$

$$X'_{mm} = \sum_i a^i_m \partial^2 b^i_m + (\partial^\mu A^m_\mu + A^{\mu m} A^m_\mu), \quad (3.11)$$

$$Y_m = \sum_{p \neq m} \sum_i a^i_m |K_{mp}|^2 |M_{mp}|^2 b^i_m. \quad (3.12)$$

Using Eqs (3.9)-(3.12) for the leptons and quarks separately, yields an action containing the kinetic terms for the $U(1)$, $SU(2)$ and $SU(3)$ gauge fields, as well as the kinetic energy and potential of the Higgs field. The most complicated step is the elimination of the auxiliary fields, but this only changes the coefficients of the Higgs potential, not its form. By writing $C_1 = \text{diag}(c_1, c_1, c_2)$ and $C_q = \text{diag}(c_3, c_3, c_4, c_4)$, the bosonic action depends on the constants $c_1, c_2, c_3, c_4, g_1, g_2, g_3$ as well as on the Yukawa couplings. Normalizing the kinetic energies of the $SU(3)$, $SU(2)$ and $U(1)$ gauge fields fixes three of the constants $c_1, \ldots, c_4$ in terms of $g_1, g_2, g_3$. In the special case when $c_1 = c_2 = c_3 = c_4$, one gets a constraint on the gauge coupling constants as well as fixed values for the Higgs mass and top quark mass [3]. These relations cannot be maintained after quantization, as can be seen from the renormalization group equations for the coupling constants and the masses [6]. We shall not assume any such relations among the $c'$s. The Higgs sector is then parametrized in terms of two parameters $\lambda$ and $m$ which are functions of of the $c'$s, $k, s$ and $H_0$. The bosonic part of the standard model becomes

$$L_b = -\frac{1}{4} (F_{\mu\nu}^3 F^{\mu\nu 3} + F_{\mu\nu}^2 F^{\mu\nu 2} + F_{\mu\nu}^1 F^{\mu\nu 1}) + D_\mu (H + M_{12})^\dagger D_\nu (H + M_{12})g^{\mu\nu} - \frac{\lambda}{24} ||H + M_{12}||^2 - |M_{12}|^2. \quad (3.13)$$

The cosmological constant comes out to be zero, naturally, at the classical level.

4. **SO(10) unification model.**

The way the strong interactions are introduced in the standard model suggests that a more unified picture would be preferable. The starting point is the Hilbert space of spinors and the Dirac operator acting on this space. The arrangement of fermions determines the structure of the discrete space. We place the fermions in the $16_s$ spinor representation of SO(10) [7]. This is a 32-component spinor subject to the space-time and SO(10) chirality

$$(\gamma_5)^{\beta}_{\alpha} \psi_{\beta\dot{\alpha}} = \psi_{\alpha\dot{\alpha}} \quad (4.1)$$

$$\Gamma_{11}^{\beta}_{\alpha} \psi_{\alpha\dot{\beta}} = \psi_{\beta\dot{\alpha}}.$$ 

where $\Gamma_{11} = -i\Gamma_0 \Gamma_1 \cdots \Gamma_9$. This reduces the independent spinor components to two for the space-time indices, and to sixteen for the SO(10) indices. The general
fermionic action is given by

\[ \overline{\psi}_p \hat{p} (\theta + A^{IJ} \Gamma_{IJ})_{a \alpha} \beta^p \psi_s + \psi^T_p C^{\alpha \beta} H^{pq} \psi_q \]  

(4.2)

where \( C \) is the charge conjugation matrix, \( p, q = 1, 2, 3 \) are family indices, and \( H \) is some appropriate combination of Higgs fields breaking the subgroup \( SU(2) \times U(1) \) of \( SO(10) \) at low energies. An exception of a Higgs field that breaks the symmetry at high energies and yet couples to fermions is the one that gives a Majorana mass to the right handed neutrinos. The other Higgs fields needed to break the \( SO(10) \) symmetry at high energies should not couple to the fermions so as not to give the quarks and leptons super heavy masses. The simplest picture corresponds to the spinor \( \Psi = \begin{pmatrix} P_+ \psi \\ P_+ \psi \\ P_- \psi^c \\ P_- \psi^c \end{pmatrix} \) where \( \psi^c \equiv B C \psi^T \), \( B \) is the \( SO(10) \) conjugation matrix satisfying \( B^{-1} \Gamma_I B = -\Gamma^T_I \) and \( P_\pm = \frac{1}{2}(1 \pm \Gamma_{11}) \). However, it turns out that the model associated with this arrangement, although elegant, is not realistic, because the Cabbibo angle vanishes [8]. The correct model is the one with the spinor

\[
\Psi = \begin{pmatrix} P_+ \psi \\ P_+ \psi \\ P_- \psi^c \\ P_- \psi^c \\ \lambda \\ \lambda^c \end{pmatrix},
\]

(4.3)

where \( \lambda \) is a singlet fermion that will couple to the right-handed neutrino in the 16\(_s\). The algebra \( \mathcal{A} \) is equal to \( \mathcal{A}_1 \otimes \mathcal{A}_2 \) where \( \mathcal{A}_1 = C^\infty(\hat{M}) \), and

\[ \mathcal{A}_2 \equiv P_+ \text{Cliff}(SO(10))P_+ \oplus R. \]

(4.4)

The involutive map \( \pi \) is taken to be

\[
\pi(a) = \pi_0(a) + \pi_0(a) + \pi_0(a) + \pi_0(a) + \pi_1(a) + \pi_1(a),
\]

(4.5)

acting on the Hilbert space \( \hat{h} = h_1 \otimes (h_2^{(1)} \oplus \cdots \oplus h_2^{(6)}) \) where \( h_2^{(i)} \cong h_2, \quad i = 1, \ldots, 4, \) \( h_2 \) is the 32 dimensional Hilbert space on which \( \mathcal{A}_2 \) acts, and \( h_2^{(i)} \cong C, \quad i = 5, 6. \) Let \( h \) be the subspace of \( \hat{h} \) which is the image of the orthogonal projection onto elements of the form \( (4.3). \) On \( \hat{h} \) the self-adjoint Dirac operator has the form \( (3.8) \), for \( N=6. \) From Eq \( (4.5) \) we have the permutation symmetry \( 1 \leftrightarrow 2, 3 \leftrightarrow 4, 5 \leftrightarrow 6, \) and the conjugation symmetry \( 1 \leftrightarrow 3, \) and the one-form \( \pi(\rho) \) reads

\[
\pi(\rho) = \begin{pmatrix}
A & \gamma_5 M K_{12} & \gamma_5 N K_{13} & \gamma_5 N K_{14} & \gamma_5 H K_{15} & \gamma_5 H K_{16} \\
\gamma_5 M K_{12} & A & \gamma_5 N K_{23} & \gamma_5 N K_{24} & \gamma_5 H K_{25} & \gamma_5 H K_{26} \\
\gamma_5 N^* K_{31} & \gamma_5 N^* K_{32} & A & \gamma_5 M K_{34} & \gamma_5 H^* K_{35} & \gamma_5 H^* K_{36} \\
\gamma_5 N^* K_{41} & \gamma_5 N^* K_{42} & B A B^{-1} & A & \gamma_5 M K_{43} & \gamma_5 H^* K_{45} \\
\gamma_5 H^* K_{51} & \gamma_5 H^* K_{52} & \gamma_5 H^* K_{53} & B A B^{-1} & A & \gamma_5 M K_{43} \\
\gamma_5 H^* K_{61} & \gamma_5 H^* K_{62} & \gamma_5 H^* K_{63} & \gamma_5 H^* K_{64} & \gamma_5 H^* K_{65} & A
\end{pmatrix},
\]

(4.6)
where the new functions $A, \mathcal{M}, \mathcal{N}$ and $H$ are given in terms of the $a^i$ and $b^i$ by

\[
A = P_+\left(\sum_i a^i \partial b^i\right)P_+
\]

\[
\mathcal{M} + \mathcal{M}_0 = P_+\left(\sum_i a^i \mathcal{M}_0 b^i\right)P_+
\]

\[
\mathcal{N} + \mathcal{N}_0 = P_+\left(\sum_i a^i \mathcal{N}_0 b^i B^i B^{-1}\right)P_-
\]

\[
H + H_0 = P_+\left(\sum_i a^i H_0 b^i\right)
\]

(4.7)

and $\mathcal{M}' = B\mathcal{M}B^{-1}, \ H' = B\mathcal{H}$. We can expand these fields in terms of the $SO(10)$ Clifford algebra as follows:

\[
A = P_+\left(ia + a^{IJ}\Gamma_{IJ} + ia^{IJKL}\Gamma_{IJKL}\right)P_+
\]

\[
\mathcal{M} = P_+\left(m + im^{IJ}\Gamma_{IJ} + m^{IJKL}\Gamma_{IJKL}\right)P_+
\]

\[
\mathcal{N} = P_+\left(n^I\Gamma_I + n^{IJK}\Gamma_{IJK} + n^{IJKLM}\Gamma_{IJKLM}\right)P_-
\]

(4.8)

The self-adjointness condition on $\pi(\rho)$ implies, after using the hermiticity of the $\Gamma_I$ matrices, that all the fields $a$ and $m$ appearing in the expansion of $A, \mathcal{M}$ are real, because both are self-adjoint, while those in $\mathcal{N}$ are complex. Imposing the reality condition on the coefficients of the Clifford algebra expansion of the gauge field $A$ forces $a = 0 = a^{IJKL}$, reducing the gauge group from $U(8)$ to $SO(10)$. The symmetry breaking pattern that breaks the gauge group $SO(10)$ must be coded into the Dirac operator $D$. The Higgs fields at our disposal are $\mathcal{M}, \mathcal{N}$ and $H$. In terms of $SO(10)$ representations these are $1, 45, 210$ in $\mathcal{M}$, complex $10, 120$ and $126$ in $\mathcal{N}$ and $16, 8$, in $H$. To be explicit we shall work in a specific $\Gamma$ matrix representation. The $32 \times 32$ $\Gamma$ matrices are represented in terms of tensor products of five sets of Pauli matrices $\sigma_i, \tau_i, \eta_i, \rho_i, \kappa_i$ where $i = 1, 2, 3$. The $\Gamma$ matrices are given by

\[
\Gamma_i = \kappa_1 \rho_3 \eta_i, \quad \Gamma_{i+3} = \kappa_1 \rho_1 \sigma_i \\
\Gamma_{i+6} = \kappa_1 \rho_2 \tau_i, \quad \Gamma_0 = \kappa_2, \quad \Gamma_{11} = \kappa_3
\]

(4.9)

where $i = 1, 2, 3$, and where we have omitted the tensor product symbols. In this basis, an $SO(10)$ chiral spinor will take the form $\psi_+ = \begin{pmatrix} \chi_+ \\ 0 \end{pmatrix}$ where $\chi$ is a $16_\chi$. The $SO(10)$ conjugation matrix is defined by $B \equiv -\Gamma_1 \Gamma_3 \Gamma_4 \Gamma_6 \Gamma_8$ which, in the basis of equation (4.9), becomes

\[
B = \kappa_1 \rho_2 \eta_2 \tau_2 \sigma_2 \equiv \kappa_1 b
\]

(4.10)

where the matrix $b = \rho_2 \eta_2 \tau_2 \sigma_2$ is the conjugation matrix in the space of the sixteen component spinors. The action of $B$ on a chiral spinor is then $B\psi_+ = \begin{pmatrix} 0 \\ b\chi_+ \end{pmatrix}$. The advantage of this system of matrices is that $bC\chi_+/^T$, have the same form as $\chi_+$ but is right-handed not left-handed. To correctly associate the components of $\chi_+$ with quarks and leptons, we consider the action of the charge operator $[7]$ on $\chi_+$:

\[
Q = -\frac{1}{6}(\sigma_3 + \tau_3 + \rho_3 \tau_3 \sigma_3) + \frac{1}{2} \eta_3
\]

(4.11)

8
which gives
\[ Q\chi_+ = \text{diag}(0, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, -1, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 1, -\frac{2}{3}, -\frac{2}{3}, -\frac{2}{3}, 0)\chi_+ \]  
(4.12)

Thus the components of the left handed spinor \( \chi_+ \) are written as the column
\[ \chi_+ = (n_L, u^1_L, u^2_L, u^3_L, e_L, d^1_L, d^2_L, d^3_L, (d^3_R)^c, (d^2_R)^c, (d^1_R)^c, (e_R)^c, (u^3_R)^c, -(u^2_R)^c, -(u^1_R)^c, (n_R)^c) \]
(4.12)

where the \( c \) in this equation stands for the usual charge conjugation, e.g. \( d^c = Cd'^T \). The upper and lower components in \( \chi \) are mirrors, with the signs chosen so that the spinor \( bC\chi_+^T \) has exactly the same form as \( \chi_+ \), but with the left-handed and right-handed signs, \( L \) and \( R \), interchanged. We now specify the vevs \( M_0, N_0 \) and \( H_0 \). The group \( SO(10) \) is broken at high energies by \( M \) which contains the complex representations 45 and 210. By taking the vev of the 210 to be \( \mathcal{M}^{0123} = O(M_G) \), the \( SO(10) \) symmetry is broken to \( SO(4) \times SO(6) \) which is isomorphic to \( SU(4)_c \times SU(2)_L \times SU(2)_R \). The \( SU(4)_c \) is further broken to \( SU(3)_c \times U(1)_c \) by the vev of the 45. Therefore we write [8]
\[ P_+M_0P_+ = P_+(M_G\Gamma_{0123} - iM_1(\Gamma_{45} + \Gamma_{78} + \Gamma_{69}))P_+ = \frac{1}{2}(1 + \kappa_3)(-M_G\rho_3 + M_1(\sigma_3 + \tau_3 + \rho_3\tau_3\sigma_3)). \]
(4.13)

Therefore \( M_0 \) breaks \( SO(10) \) to \( SU(3)_c \times U(1)_c \times SU(2)_L \times SU(2)_R \) which is also of rank five. The rank is reduced by giving a vev to the components of 126 that couple to the right-handed neutrino. Therefore the vev of \( N_0 \) must contain the term
\[ M_2(\frac{1}{2\sqrt{3}})(\kappa_1 + i\kappa_2)(\rho_1 + i\rho_2)(\eta_1 + i\eta_2)(\tau_1 + i\tau_2)(\sigma_1 + i\sigma_2) \]
(4.14)

The vev of \( N_0 \) break \( U(1)_c \times SU(2)_R \) to \( U(1)_Y \), and the surviving group would be the familiar \( SU(3)_c \times SU(2)_L \times U(1)_Y \). This breaking is also obtained for an \( H_0 \) whose vev is \( H_0 = M_3 \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \). As we shall explain shortly, \( M_1, M_2 \) and \( M_3 \) must be related for the model to be consistent. The only generators that leave \( M_0 \), the part of \( N_0 \) given by 4.14 and \( H_0 \) invariant are those of the standard model. The eight \( SU(3) \) generators are given by \( (1 - \rho_3\tau_3)\sigma_i, (1 - \rho_3\sigma_3)\tau_i, \rho_3(\tau_1\sigma_1 + \tau_2\sigma_2) \) and \( \rho_3(\tau_2\sigma_1 - \tau_1\sigma_2) \). The \( SU(2)_L \) generators are \( \frac{1}{2}(1 + \kappa_3\rho_3)\eta' \). Finally the \( U(1)_Y \) generator is related to the charge operator \( Q \) by \( Q = \frac{1}{2}Y + T^3_L \), where the action of the \( SU(2)_L \) isospin \( T^3_L \) on \( \chi_+ \) is given by \( T^3_L = \frac{1}{2}(1 + \rho_3)\eta_3 \).

For the last stage of symmetry breaking of \( SU(2)_L \times U(1)_Y \) we can use the field \( \mathcal{N} \) which contains the complex representations 10, 120 and 126. The most general vev that preserves the group \( SU(3)_c \times U(1)_Y \) is
\[ P_+N_0P_{-\kappa_1} = \frac{1}{2}(1 + \kappa_3)(s + pp_3\eta_3 + ap\rho_3 + a'\eta_3)
+ (b' + bp_3\eta_3 + ep\eta_3 + f\rho_3)(\sigma_3 + \tau_3 + \rho_3\tau_3\sigma_3)
+ M_2(\frac{1}{2\sqrt{3}})(\rho_1 + i\rho_2)(\eta_1 + i\eta_2)(\tau_1 + i\tau_2)(\sigma_1 + i\sigma_2), \]
(4.15)
where all terms containing $\eta_3$ break $SU(2)_L \times U(1)_Y$ and $s, p, a, a', b, b', e, f$ are $O(M_W)$. The fermionic action is simply given by

$$I_{f-\text{mass}} = \langle \Psi, (D + \rho)\Psi \rangle = -\int d^4x \left( \left( (s + p + 3(e + f))K_{(pq)} + (a + a' + 3(b + b'))K_{[pq]} \right) N^q_R N^q_L \right.$$  
+ $\left( (s + p - (e + f))K_{(pq)} + (a + a' - (b + b'))K_{[pq]} \right) u^T_R u_L$  
+ $\left( (s - p - 3(e - f))K_{(pq)} + (a - a' - 3(b - b'))K_{[pq]} \right) e^T_R e_L$  
+ $\left( (s - p + e - f)K_{(pq)} + (a - a' + b - b')K_{[pq]} \right) d^T_R d_L$  
+ $\left( (\sqrt{2}M_3 K_{pq} N^q_R \lambda_L + M_2 K''_{(pq)} (N^q_R)^T C^{-1} N^q_c) + h.c. \right)$, 

(4.16)

where we have denoted the family mixing matrices $K_{13}, K_{15}$ and $K_{56}$ by $K, K', K''$, respectively. The symmetric and antisymmetric parts of $K_{pq}$ are denoted by $K_{(pq)}$ and $K_{[pq]}$, respectively. Since we have three neutral fields, $N_L, N^c_R$ and $\lambda_L$, and their mass eigenstates are mixed, the mass matrix must be diagonalised. Ignoring the mixing due to the generation matrices, the mass matrix of the neutral fields is of the form

$$\begin{pmatrix}
N_L & N^c_R & \lambda_L \\
N^c_R & 0 & m \\
\lambda_L & 0 & M_2 \\
0 & M_2 & M_3
\end{pmatrix},$$

(4.17)

and we shall assume a mass hierarchy $m \ll M_2, M_3$, and $M_2 \sim M_3$. Diagonalisation of the matrix (4.17) produces two massive fields whose masses are of order $M_2$, and the third will be a massless left-handed neutrino.

The bosonic action can be read from Eq (3.9), for $N=6$. The only complicated step is the elimination of the auxiliary fields, and one finds that the vev’s used cannot be arbitrary but must be related for the potential to survive and the model to be consistent. These relations are $M_G = M_1$ and $M_1 M_2 = -\frac{K_{15} K_{15}'}{2K_{12} K_{13}} M_3^2$. The bosonic action is

$$-4g^2 F_{\mu\nu}^{IJ} F^{\mu\nu, IJ} + 2|K_{12}|^2 \text{Tr}\left( (D_\mu (\mathcal{M} + \mathcal{M}_0))^2 \right) + 8|K_{13}|^2 \text{Tr}\left( |D_\mu (\mathcal{N} + \mathcal{N}_0)|^2 \right) + 12|K_{15}|^2 \left| D_\mu (H + H_0) \right|^2 - V(\mathcal{M}, \mathcal{N}, \mathcal{H}),$$

(4.18)
where the potential $V(\mathcal{M}, \mathcal{N}, H)$ is

\[ (\text{Tr}|K_{12}|^4 - (\text{Tr}|K_{12}|^2)^2)\text{Tr}\left(|\mathcal{M} + \mathcal{M}_0|^2 - |\mathcal{M}_0|^2\right)^2 + 4\left|K_{13}K_{12}((\mathcal{M} + \mathcal{M}_0)(\mathcal{N} + \mathcal{N}_0) + (\mathcal{N} + \mathcal{N}_0)B(\overline{\mathcal{M}} + \overline{\mathcal{M}_0})B^{-1}) + 2K_{15}\overline{K}_{15}((H + H_0)B(H + H_0))\right|^2 + 8\left|K_{12}K_{15}(\mathcal{M} + \mathcal{M}_0)(H + H_0) + 2K_{13}\overline{K}_{15}(\mathcal{N} + \mathcal{N}_0)B(H + H_0) - u(H + H_0)\right|^2 + 16(\text{Tr}|K_{15}|^4 - (\text{Tr}|K_{15}|^2)^2)\left|H^* + H_0^*\right|^2 - M_3^2\right|^2 + 16\text{Tr}|K_{15}|^4\left|H^* + H_0^*\right|^2 - M_3^2\right|^2, \]

and $u = 2K_{13}K_{15}(s + p - 3(b + b') + 2(a + a') + M_2) - 2K_{12}K_{25}M_1$. We deduce that the SO(10) model is an attractive model. Its construction is completely dictated by the arrangement of the fermions, their representations, and the Dirac operator acting on them. The nature of the Higgs fields is completely fixed, and their vev’s constrained by the requirement that the potential is non-trivial for the consistency of the theory. The mass matrix of the fermions is realistic.

5. Gravity in non-commutative geometry

A natural question to ask is how to introduce gravity in the framework of non-commutative geometry. An answer to this question requires a generalisation of the basic notions of Riemannian geometry. Connes has proposed to define metric properties of a non-commutative space corresponding to an involutive unital algebra $\mathcal{A}$ in terms of K-cycles over $\mathcal{A}$ [2-3]. In [9] it was shown that every K-cycle over $\mathcal{A}$ yields a notion of "cotangent bundle" associated to $\mathcal{A}$ and a Riemannian metric on the cotangent bundle. One can also introduce analogues of the spin connection, torsion, Riemann curvature tensor, Ricci tensor, and scalar curvature. This allows one to write the generalized Einstein-Hilbert action. Here we shall only describe the gravity action for a two sheeted space, and refer the reader to [9] and [4] for details. We shall also derive, heuristically, an experimental signature of the effect of the geometry on the standard model, which turns out to be a constraint on the Higgs mass and top quark mass.

Consider a space-time $X$ consisting of two copies of a four-dimensional manifold $M$: $X = M \times Z_2$. The algebra $\mathcal{A}$ is given by $\mathcal{A} = C^\infty(M) \otimes \mathcal{A}_1 \oplus C^\infty(M) \otimes \mathcal{A}_2$, where $\mathcal{A}_1 = \mathcal{A}_2 = C$. The elements of $\mathcal{A}$ are operators of the form $\text{diag}(1 \otimes a_1, 1 \otimes a_2)$ where $a_i$, $i = 1, 2$ are smooth function on $M$, and 1 is the identity in the Clifford algebra, $\text{Cliff}(T^*M)$, of Dirac matrices over $M$. We consider even K-cycles $(\pi, H, D, \Gamma)$ for $\mathcal{A}$, with $\pi = \pi_1 \oplus \pi_2$, where $\pi_i$ is a representation of $C^\infty(M) \otimes \mathcal{A}_i$ on a Hilbert space $L^2(S_i, \tau_i dv)$, where $S_i$ is a bundle of spinors on $M$ with values in a finitely generated, projective hermitian left $\mathcal{A}_i$ module $E_i$, $\tau_i$ is a normalized trace on $\mathcal{A}_i$ and $dv$ is the volume element on $M$. Then $h$ is defined by $h = L^2(S_1, \tau_1, dv) \oplus L^2(S_2, \tau_2, dv)$. The Dirac operator is taken to be

$$D = \begin{pmatrix} \nabla_M \otimes 1 & \gamma_5 \otimes \phi \\ \gamma_5 \otimes \phi^* & \nabla_M \otimes 1 \end{pmatrix},$$

(5.1)
where $\nabla_M$ is the standard covariant Dirac operator on $M$. The $Z_2$ grading on $M$ is given by $\Gamma = \begin{pmatrix} \gamma_5 & 0 \\ 0 & -\gamma_5 \end{pmatrix}$. The "cotangent bundle" $\Omega^1_D(A) = \text{Omega}^1(A)/\ker \pi$ is a free left and right $A$ module, with a basis $\{e^N\}_{N=1}^5$ given by

$$e^a = \begin{pmatrix} \gamma^a & 0 \\ 0 & \gamma^a \end{pmatrix}, \quad e^5 = \begin{pmatrix} 0 & \gamma_5 \\ -\gamma_5 & 0 \end{pmatrix}, \quad a = 1, 2, 3, 4. \quad (5.2)$$

The hermitian structure on $\Omega^1_D(A)$ is given by the trace of $8 \times 8$ matrices, normalized such that $tr 1 = 1$. Hence

$$< e^N, e^M > = \text{tr}(e^N(e^M)^*) = \delta^{NM}. \quad (5.3)$$

For a one-form $\rho = \sum_i a_i db_i$ in $\Omega^1_D(A)$, $\pi(\rho)$ is parametrized by

$$\pi(\rho) = \begin{pmatrix} \gamma^\mu \rho^1_\mu & \gamma_5 \phi \rho_5 \\ -\gamma_5 \phi \tilde{\rho}_5 & \gamma^\mu \rho_{2\mu} \end{pmatrix}, \quad (5.4)$$

where $\rho^1_\mu = \sum_i a_i \partial_\mu b_i$, $\rho_5 = \sum_i a_i (b_{i2} - b_{i1})$, and similarly for $\rho_{2\mu}$ and $\tilde{\rho}_5$. Evaluating $\pi(dp) = \sum_i [D,a_i][D,b_i]$, we obtain

$$\pi(dp) = -\left( \begin{array}{cc} g^{\mu\nu} \partial_\mu a_i \partial_\nu b_i & 0 \\ 0 & g^{\mu\nu} \partial_\mu a_i b_i \end{array} \right). \quad (5.5)$$

One sees that, for a suitable choice of $a_i, b_i$ subject to the constraint $\pi(\rho) = 0$, any expression of the form $\text{diag}(X_1, X_2)$ can be obtained, where $X_1, X_2$ are scalar functions. Therefore, we can express $\pi(dp)$ modulo auxiliary fields in terms of its components:

$$\pi(dp) = \begin{pmatrix} \gamma^\mu \partial_\mu \rho^1_\nu & \phi \gamma^\mu \gamma_5 \partial_\mu \rho_{5} + \rho^1_\mu - \rho_{2\mu} \\ -\phi \gamma^\mu \gamma_5 \partial_\mu \rho_{5} + \rho^1_\mu - \rho_{2\mu} & \gamma^\mu \partial_\mu \alpha_{2\nu} \end{pmatrix}. \quad (5.6)$$

This is a representative of $\pi(dp)$ in $\pi(\Omega^2(A))/\pi(\text{Ker}(\pi|_{\Omega^1(A)}))$ orthogonal to the auxiliary fields. Let $\nabla$ be a connection on $\Omega^1_D(A)$ and $\omega^N_M \in \Omega^1_D(A)$ defined by $\nabla e^N = -\omega^N_M \otimes_A e^M$. The components of $\pi(\nabla)$ in the basis $\{e^N\}_{N=1}^5$ are given by

$$\omega^N_M = \begin{pmatrix} \gamma^\mu \rho^N_{1\mu} & \gamma_5 \phi \rho^N_{5} \\ -\gamma_5 \phi \tilde{\rho}^N_M & \gamma^\mu \rho^N_{2\mu} \end{pmatrix}. \quad (5.7)$$

Hermiticity of $\nabla$ then implies that

$$\omega^N_{i\mu M} = -\omega^M_{i\mu N}, \quad i = 1, 2, \quad \tilde{\rho}^N_M = -\rho^M_N. \quad (5.8)$$

Let $T^N \in \Omega^2_D(A)$ be the components of the torsion $T(\nabla)$ defined by $T^N = T(\nabla)e^N$. Then

$$T^N = de^N + \omega^N_M e^M \quad (5.9)$$
Similarly define \( R^N_M \in \Omega^2_D(A) \) by \( R(\nabla)e^N = R^N_M \otimes A e^M \) where \( R(\nabla) \) is the Riemann curvature of \( \nabla \) defined by \( R(\nabla) := -\nabla^2 \). Then

\[
R^N_M = d\omega^N_M + \omega^N_P \omega^P_M.
\] (5.10)

Imposing the condition that the torsion \( T(\nabla) \) vanishes gives

\[
\begin{align*}
\omega_{1\mu b} &= \omega_{2\mu b} = \omega_{a b}, \\
\omega_{1\mu 5} &= -\omega_{2\mu 5} = \phi l^a_b e^b_\mu, \\
l^a_b &= l^b_a, \quad l^5_a &= -l^a_5, \quad l^5_a e^a_\mu = -\partial_\mu \phi^{-1},
\end{align*}
\] (5.11)

where \( \omega_{a b} \) is the classical Levi-Civita connection derived from the metric \( g_{\mu \nu} = e^a_\mu \delta_{a b} e^b_\nu \) on \( M \). The analogue of the Einstein-Hilbert action is

\[
I(\nabla) := \kappa^{-2} < R^N_M e^M, e_N > + \Lambda < 1, 1 >
\]

\[
= \kappa^{-2} \int_M \text{tr}(R^N_M e^M(e_N)^*) + \Lambda \int 1,
\] (5.12)

where \( \kappa^{-1} \) is the Planck scale. This action is then calculated to be

\[
I(\nabla) = \kappa^{-2} \int_M \left( 2r - 4\phi \nabla_\mu \partial^\mu \phi^{-1} + 4\phi^2 l^a_b l^5_b \right)
\]

\[
+ \phi^2 ((l^a_b)^2 - l^5_a l^a_5) \sqrt{g} d^4 x + 2\Lambda \int_M \sqrt{g} d^4 x,
\] (5.13)

where \( r \) is the scalar curvature of the classical Levi-Civita connection. The fields \( l^a_b \) and \( l^5_5 \) decouple, and by setting \( \phi = e^{-\kappa \sigma} \) one finds

\[
I(\nabla) = 2 \int_M (\kappa^{-2} r - 2\partial_\mu \sigma \partial^\mu \sigma + \Lambda) \sqrt{g} d^4 x.
\] (5.14)

Therefore a theory of gravity on \( M \times Z_2 \) is equivalent to general relativity on \( M \), with an additional massless scalar field \( \sigma \) that couples to the metric of \( M \). To better understand the role of the field \( \sigma \) we can study the coupling of gravity to the Yang-Mills sector [10]. In the case of the standard model the field \( \phi = e^{-\kappa \sigma} \) replaces the electroweak scale. In other words, the vev of the field \( \phi \) determines the electroweak scale. This simple result has some unexpected consequences. To determine the \( \sigma \) dependence in the Yang-Mills action of the standard model, we consider the \( \sigma \) dependence in the Dirac operator. For example, the leptonic Dirac operator is

\[
D_l = \left( \begin{array}{c}
\gamma^a e^\mu_a (\partial_\mu + \ldots) \otimes 1_2 \otimes 1_3 \\
\gamma_5 e^{-\kappa \sigma} \otimes M_{12} \otimes k \\
\gamma_5 e^{-\kappa \sigma} \otimes M_{12} \otimes k^* \\
\gamma^a e^\mu_a (\partial_\mu + \ldots) \otimes 1_3
\end{array} \right).
\] (5.15)

From this one can easily verify that the bosonic part of the standard model is

\[
L_b = -\frac{1}{4} \left( F_{\mu \nu}^3 F^{\mu \nu 3} + F_{\mu \nu}^2 F^{\mu \nu 2} + F_{\mu \nu}^1 F^{\mu \nu 1} \right)
\]

\[
+ D_\mu (H + M_{12})^* D_\nu (H + M_{12}) g^{\mu \nu} e^{-2\kappa \sigma}
\]

\[
- \frac{\lambda}{24} \left| H + M_{12} \right|^2 - \left| M_{12} \right|^2 e^{-4\kappa \sigma}.
\] (5.16)
The $\sigma$ dependence in Eq (5.16) is a consequence of the "Weyl invariance" of the action (3.7) under rescaling of the Dirac operator $D \to e^{-w}D$, as this implies $g_{\mu\nu} \to e^{2w}g_{\mu\nu}$ and $\kappa\sigma \to \kappa\sigma + w$. This can be easily seen from the scalings: $\pi(\rho) \to e^{-w}\pi(\rho)$ and $\pi(\theta) \to e^{-2w}\pi(\theta)$. By redefining $H + M_{12} \to e^{\kappa\sigma}H$, the $H$ dependent terms in (5.16) become

$$D_\mu H^* D^\mu H + \kappa \partial_\mu (H^* H) \partial^\mu \sigma + \kappa^2 H^* H \partial_\mu \partial^\mu \sigma - \frac{\lambda}{24} (\mathcal{H}^* H)^2 - \mu^2 e^{-2\kappa\sigma}.$$  (5.17)

The potential in Eq (5.17) could be rewritten in the familiar form

$$V_0 = \frac{\lambda}{24} (H^* H)^2 - \frac{1}{2} m^2 (H^* H) + \frac{3}{2\lambda} m^4,$$  (5.18)

where we have set $m^2 = \frac{\lambda\mu^2}{6} e^{-2\kappa\sigma}$, so that $m$ is now a field and not just a parameter. The potential $V_0$ is of the same form as that of the standard model. We assume that, after renormalization, the bosonic action takes the same form as $I_I + I_q$. In the absence of some understanding of symmetries, it is not possible to prove this assumption at the quantum level. Let $\phi$ be the component of the Higgs field that develops a vev. We are then mainly interested in the potential

$$V_0 = \frac{\lambda}{24} \phi^4 - \frac{1}{2} m^2 \phi^2 + \frac{3}{2\lambda} m^4.$$  (5.19)

Minimizing with respect to $\phi$ and $m$ yields the same asymmetric phase $\phi^2 = \frac{2\lambda}{\lambda} m^2$, and the weak scale, $e^{-\kappa\sigma}$, is undetermined at the classical level. The quantum corrections to the potential are given, in the one-loop approximation, by the effective Coleman-Weinberg [11] potential of the standard model [12]:

$$V_1 = \frac{1}{16\pi^2} \left( \frac{1}{4} H^2 (\ln \frac{H}{M^2} - \frac{3}{2}) + \frac{3}{4} G^2 (\ln \frac{G}{M^2} - \frac{3}{2}) + \frac{3}{2} W^2 (\ln \frac{W}{M^2} - \frac{5}{6}) + \frac{3}{4} Z^2 (\ln \frac{Z}{M^2} - \frac{5}{6}) - 3 T^2 (\ln \frac{T}{M^2} - \frac{3}{2}) \right),$$  (5.20)

where

$$H = -m^2 + \frac{1}{2} \lambda \phi^2, \quad G = -m^2 + \frac{1}{6} \lambda \phi^2,$$

$$W = \frac{1}{4} g_2^2 \phi^2, \quad Z = \frac{1}{4} (g_2^2 + g_1^2) \phi^2, \quad T = \frac{1}{2} h^2 \phi^2,$$

and $M$ is the renormalization scale. Minimizing the total potential $V_0 + V_1$ with respect to the fields $\phi$ and $m$, after rescaling

$$G = \overline{G} M^2, \quad H = \overline{H} M^2, \quad T = \overline{T} M^2,$$  (5.20)

the asymmetric solution is given by the solution to the following two equations:

$$0 = \overline{G} + \frac{M^2}{32\pi^2 \phi^2} (\overline{H} - \overline{G}) (\overline{H} (\ln \overline{H} - 1) + 3\overline{G} (\ln \overline{G} - 1)), \quad (5.21)$$

$$0 = \overline{G} + \frac{3M^2}{32\pi^2 \phi^2} (\overline{H} - \overline{G}) (\overline{H} (\ln \overline{H} - 1) + \overline{G} (\ln \overline{G} - 1)) - \frac{\overline{g_2^2 + g_1^2}}{64\pi^2} + \frac{3g_2^4 \phi^2}{128\pi^2 M^2} (\ln \frac{g_2^2 \phi^2}{4M^2} - \frac{1}{3}) - \frac{3M^2}{4\pi^2 \phi^2} \overline{T}^2 (\ln \overline{T} - 1). \quad (5.22)$$
At the scale $M = m_Z$, the mass of the $Z$-particle, the coupling constants $g_1, g_2$ as well as the vev $\phi$ are known from experimental data, corrected with the help of the renormalization group equations [12]:

$$g_2 = 0.650, \quad g_1 = 0.358, \quad \phi = 246 \text{ Gev.} \quad (5.23)$$

The only unknowns in the minimization equations are $\lambda, m$ and the square of the top quark mass $T = m_t^2$. These equations, being complicated functions of $\overline{H}$ and $\overline{G}$, can only be solved numerically, for various values of $T$. The numerical solutions are easily obtained using Mathematica. The Higgs mass can be determined from the formula $m_H^2 = \frac{\partial^2 V}{\partial \phi^2}$ which gives

$$m_H^2 = M^2 \left( (\overline{H} - \overline{G}) + \frac{9M^2}{16\pi^2 \phi^2} (\overline{H} - \overline{G})^2 (\ln \overline{H} + \frac{1}{3} \ln \overline{G}) \right.$$

$$+ \frac{3g_2^4 \phi^2}{64\pi^2 M^2} \ln \frac{g_2^2 \phi^2}{4M^2} - \frac{3M^2}{2\pi^2 \phi^2} \overline{T}^2 \ln \overline{T} \left. \right). \quad (5.24)$$

We now quote the results: There are only two classes of solutions, for $\overline{G} \ll \overline{H}$ and for $\overline{H} \ll \overline{G}$. In the first case, we find that there are only two narrow bands for the top quark mass where solutions exist. The first band is $0.365 \leq T \leq 0.455, \ \overline{G} \ll \overline{H}$, corresponding to a top quark mass

$$54.90 \text{ Gev} \leq m_t \leq 61.35 \text{ Gev}, \quad (5.25)$$

which is already ruled out experimentally. The second band is very narrow: $2.57 \leq \overline{T} \leq 2.61, \ \overline{G} \ll \overline{H}$, corresponding to the top quark mass

$$146.23 \text{ Gev} \leq m_t \leq 147.37 \text{ Gev}, \quad (5.26)$$

and a Higgs mass $117.26 \text{ Gev} \leq m_H \leq 142.61 \text{ Gev}$. Clearly this band of values for the top quark mass lies within the present experimental average of [13]

$$m_t = 149 + \left( ^{+21}_{-47} \right) \text{ Gev.} \quad (5.27)$$

The second class of solutions occurs when $1.30 \leq \overline{T} \leq 2.61, \ \overline{H} \ll \overline{G}$, corresponding to the top quark mass

$$104.07 \text{ Gev} \leq m_t \leq 147.48 \text{ Gev}, \quad (5.28)$$

and a Higgs mass $1208 \text{ Gev} \geq m_H \geq 1197 \text{ Gev}$. However, since $\overline{H} \ll \overline{G}$, and since the coupling constant $\lambda = O(-100)$, the potential, in this domain, becomes unbounded from below, signaling the break down of the perturbative regime. Requiring stability of the electroweak potential excludes this solution. Therefore the only acceptable solution is (5.26) which is remarkably constrained, considering the wide range of possibilities that one might have, a priori. We note that the field $\sigma$ becomes massive with the square of the mass given by: $m_\sigma^2 = \frac{\partial^2 V}{\partial \sigma^2}$. This is equal to

$$m_\sigma^2 = \kappa^2 m^2 \left( 2\phi^2 \frac{\overline{H} - 4\overline{G}}{\overline{H} - \overline{G}} + \frac{M^2}{16\pi^2} \left( \overline{H}(1 - \ln \overline{H}) + 3\overline{G}(1 - \ln \overline{G}) \right) \right). \quad (5.29)$$
For the physically acceptable solutions we have $\overline{\mathcal{H}} = O(1)$, $\overline{\mathcal{G}} = O(10^{-4})$ and $m^2 = O(M^2)$. Then we find from Eq (5.29) that

$$m^2_\sigma = O(\kappa^2 M^4),$$

so that $m_\sigma = O(10^{-15})$ Gev, which is unobservable. These predictions have at best a heuristic value, since the problem of fixing the form of the cosmological constant at the one-loop level by imposing natural geometrical constraints is not understood. However, they do suggest that gravitational effects may play a role in understanding masses of fermions and Higgses.

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