Concentration of Geodesics in Directed Bernoulli Percolation

Christian Houdré*  Chen Xu †

May 14, 2018

Abstract

For directed Bernoulli last passage percolation with i.i.d. weights on vertices over a $n \times n$ grid and for $n$ large enough, the geodesics are shown to be concentrated in a cylinder, centered on the main diagonal and of width of order $n^{(2\kappa+2)/(2\kappa+3)}\sqrt{\ln n}$, where $1 \leq \kappa < \infty$ is the curvature power of the shape function at $(1,1)$. The methodology of proof is robust enough to also apply to directed Bernoulli first passage site percolation, and further to longest common subsequences in random words.

1 Introduction

It has been initially conjectured in [14] that many percolation systems including undirected/directed, first/last passage percolation falls into the KPZ universality class. They are expected to satisfy the scaling relation: $\chi = 2\xi - 1$, where $\chi$ and $\xi$ are respectively the shape and the transversal fluctuations exponents. Moreover, $\chi$

*School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332-0160. Email: houdre@math.gatech.edu. Research supported in part by the grants # 246283 and # 524678 from the Simons Foundation.

†School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332-0160. Email: cxu60@math.gatech.edu.

MSC2010: Primary 60K35, 82B43.

Keywords: Last-passage percolation, geodesics, KPZ universality, first-passage percolation, longest common subsequence, random words, transversal exponent, concentration phenomenon.
can also be viewed as the asymptotic order of the standard deviation of the first/last passage time, while geodesics are expected to be confined to a cylinder around the diagonal of width of asymptotic order $n^\xi$. Specifically, on a two dimensional $n \times n$ grid, it is conjectured that $\chi = 1/3$ and $\xi = 2/3$. However, to date, it has only been shown that $\chi \leq 1/2$ and that $\xi \leq 3/4$, under various types of assumptions. The upper bound $3/4$ is obtained in [18] by showing that $2\xi \leq 1 + \chi'$, where $\chi'$ is an exponent closely related to $\chi$ and is itself upper-bounded by $1/2$. The relation $2\xi = 1 + \chi$ has also been recently proved under different definitions of $\xi$ and $\chi$ in [5] (see also [2]). As for the bounds for the shape fluctuations exponent $\chi$, fewer results are available. To date, a sublinear order $O(\sqrt{n/\ln n})$, in the context of first passage percolation (FPP) with various types of weight distributions has been shown in [4, 6]. For a list of other definitions and results on these topics, we refer the interested readers to the recent comprehensive survey [3].

Transversal fluctuations have also been studied in a related problem, i.e., the analysis of the longest common subsequences (LCSs) in two random words of length $n$. As well known, LCSs can be viewed as directed last passages in a two-dimensional percolation grid with dependent Bernoulli weights. It is proved in [11] that the optimal alignments corresponding to the LCSs also stay, with high probability, in a sector close to the diagonal. Moreover, when it comes to the shape fluctuation, i.e., the standard deviation of $LC_n$, the length of the LCSs, the results are more complete: First, by the Efron-Stein inequality, the shape fluctuations are upper bounded by $\sqrt{n}$, for arbitrary distributions on any finite dictionary. Second, a lower bound of order $\sqrt{n}$ has been obtained under various asymmetry assumptions (\cite{15 \ldots}). More noticeably, a central limit theorem has been proved for $LC_n$ in \cite{9}.

In the present paper, we mainly study the transversal fluctuations in directed last passage percolation (DLPP) and briefly extend it to other settings. Our methodology shows that, with high probability, geodesics in DLPP are confined to a cylinder, around the main diagonal, of width of order $n^{(2\kappa+2)/(2\kappa+3)}\sqrt{\ln n}$, where $1 \leq \kappa < +\infty$ is the curvature power of the shape function at $(1, 1)$.

The model under study is the classical one: DLPP on a $n \times n$ grid with $(n+1)^2$ vertices, each of which is associated with a Bernoulli random weight $w$, where $\mathbb{P}(w = 1) = s = 1 - \mathbb{P}(w = 0)$, $0 < s < 1$, and all the weights are independent. The last passage time $T(n, n)$ is the maximum of the sums of all the weights along all unit-step up-right paths on the grid, from $(0, 0)$ to $(n_1, n_2)$. For convenience, the path is
considered left – open – right – closed, i.e., the weight on (0, 0) is excluded:

\[ T(n, n) = \max_{\pi \in \Pi} \sum_{v \in \pi \setminus (0, 0)} w(v), \]

where \( \Pi \) is the set of all unit-step up-right paths from (0, 0) to \((n, n)\), and where each unit-step up-right path \( \pi \in \Pi \) is viewed as an ordered set of vertices, i.e., \( \pi = \{v_0 = (0, 0), v_1, \ldots, v_{2n} = (n, n)\} \) such that \( v_{i+1} = v_i \) (\( i \in [2n - 1] \)) is either \( e_1 := (1, 0) \) or \( e_2 := (0, 1) \), and \( w : v \to w(v) \in \{0, 1\} \) is the random weight associated with the vertex \( v \in [n] \times [n] \), where \( [n] := \{0, 1, 2, \ldots, n\} \). Hereafter directed path is short for unit-step up-right path and any directed path realizing the last passage time is called a geodesic. We also use the notation \( T(V_1, V_2) \) to denote the directed last passage time for a rectangular grid from the lower-left vertex \( V_1 \) to the upper-right vertex \( V_2 \) (\( w(V_1) \) is also excluded) and sometimes use coordinates to express \( V_1 \) and \( V_2 \), e.g., when \( V_1 = (i, j) \) and \( V_2 = (k, l) \), \( T(V_1, V_2) := T((i, j), (k, l)) \).

Let us now briefly describe the content of the paper: in the next section, we present properties of the shape function of DLPP and state our main result (Theorem 2.5). Section 3 first introduces a way of decomposing the entire grid into blocks in such a way that, with high probability, most of the blocks in any optimal decomposition are close-to-square shaped. Next, an intermediate rate of convergence result used in the proof of the main theorem is further obtained. Finally, we exhibit two lines \( \ell_1 \) and \( \ell_2 \) respectively above and below the main diagonal, bounding a sector within which, with high probability, geodesics are confined. Then, by finely tuning the slopes of these two bounding lines, we produce a concentration inequality for the fluctuations of the geodesics away from the main diagonal. In the concluding Section 4, extensions are briefly stated for the geodesics in directed first passage percolation (DFPP). Then, the case of LCSs is presented and some potential refinements are also discussed.

2 Preliminaries and Main Results

In this section, we introduce the shape function \( g \) and a modification \( g_{\perp} \) (\( g\text{-perp} \)) along with some of their properties. It is well known that, by superadditivity and Fekete’s Lemma, the non-negative limit

\[ \lim_{n \to \infty} \frac{\mathbb{E}T(nx, ny)}{n} = \limsup_{n \to \infty} \frac{\mathbb{E}T(nx, ny)}{n} := g(x, y) \]
exist for any $x, y \in \mathbb{R}^+$. The function $g$ is typically called the shape function, and by a further application of superadditivity, it can be shown to be concave (see [17]). Instead of studying $g$ directly, we are more interested in its orthogonal modification, i.e., in the function $g_\perp$, given by $g_\perp(q) = g(1 - q, 1 + q)$, where $q \in (-1, +1)$. Since the transformation $(1 - q, 1 + q)$ is linear, it is trivial to transfer results from $g$ to $g_\perp$. Therefore, from [17]:

**Proposition 2.1.** $g_\perp$ is non-negative and concave.

By the invariance of $g$ under any permutation of its coordinates, i.e., since $g(x, y) = g(y, x)$, $g_\perp$ is symmetric about $q = 0$. Also $g_\perp((-1)^+) = g_\perp(1^-) = g(0, 2) = 2s$. Still, by concavity, $g_\perp$ is non-decreasing on $(-1, 0]$ and non-increasing on $[0, 1)$ and so $g_\perp$ attains its maximum at $q = 0$. The uniqueness of this maximum is not guaranteed but would follow from the strict concavity of the shape function $g$ at $(1, 1)$ which has been conjectured, in particular, for i.i.d. Bernoulli weights. To date, strict concavity has not been proved for any weight distribution. However, in the setting of undirected FPP, a class of weight distributions has been shown (see [7]) to produce a shape function having a flat edge around the direction $(1, 1)$. This class of weights is further studied and more properties of the associated shape function are obtained in [16, 22, 23, 1]. Our first result Theorem 3.4 stating that, with probability exponentially close to one, geodesics are bounded away from the upper-left and lower-right corners of the grid, does not require a strict-concavity assumption. Instead, it merely requires the existence of a threshold $t > 0$ such that if $q \in (-1, -t) \cup (t, 1)$, then $g_\perp(q) \preceq g_\perp(0) = g(1, 1)$, i.e., that $g_\perp$ is not identically constant on $(-1, 1)$. Before tackling this threshold problem, let us better estimate $g_\perp(0) = g(1, 1)$.

First, it is clear that any directed path from $(0, 0)$ to $(n, n)$ in a $n \times n$ grid covers exactly $2n$ vertices and the expected passage time associated with up-right path is $2ns$. But, clearly, the passage time associated with any such up-right path is at most the last passage time. Thus $\mathbb{E}T(n, n) \geq 2ns$. Therefore, $g_\perp(0) \geq 2s$, however this lower bound is strict.

**Lemma 2.2.** $g_\perp(0) - 2s \geq s(1 - s)$.

**Proof.** Consider the diagonal blocks in the $n \times n$ table, i.e., the $n$ blocks of size $1 \times 1$ on the diagonal as in Figure 2.1 Any up-right path on this block goes either up-right or right-up. Denote by $T^i_u$ the weight associated with the vertex at the upper-left corner of the $i$th $1 \times 1$ diagonal block, while $T^i_r$ is the weight associated with the
Figure 2.1:

The corresponding lower-right corner for \( i \in [n - 1] \) and \( T_d^j \) is the weight associated with the vertex on the diagonal for \( j \in [n] \). Then, all these \( 3n \) random weights are i.i.d. Bernoulli random variables with parameter \( s \). Moreover, the maximal passage time of all the paths going inside these blocks is a lower bound for the last passage time, i.e.,

\[
T(n, n) \geq \sum_{j=1}^{n} T_d^j + \sum_{i=0}^{n-1} \left( T_u^i \lor T_r^i \right).
\]

Hence,

\[
g(1, 1) \geq \lim_{n \to \infty} \frac{1}{n} \mathbb{E} \left( \sum_{j=1}^{n} T_d^j + \sum_{i=0}^{n-1} \left( T_u^i \lor T_r^i \right) \right)
\geq \lim_{n \to \infty} \frac{1}{n} \left( n s + n (1 - (1 - s)^2) \right)
= 3s - s^2.
\]

An explicit expression for \( g \) is known for geometric or exponential weights but not for Bernoulli weights (e.g., see \([21, 12, 20]\)). So to obtain a specific threshold \( t \),
as described above, we combine the lower bound on \( g(0) \) obtained in Lemma 2.2 with an upper bound on \( g \) obtained in [17].

**Proposition 2.3.** Let \( t = 1 - (g(1,1) - 2s)^2/8s(1-s) < 1 - s(1-s)/8 \). Then, for any \( q \in (-1,-t) \cup (t,1) \), \( g(q) \preceq g(0) = g(1,1) \).

**Proof.** First by Lemma 4.1 in [17],
\[
g(1,y) \leq (1+y)s + 2\sqrt{y(1+y)}\sqrt{s(1-s)}.
\]

Without loss of generality, assume \( q > 0 \), thus
\[
g(q) = (1+q)g\left(\frac{1-q}{1+q}, 1\right)
\]
\[
= (1+q)g(1, \frac{1-q}{1+q})
\]
\[
\leq (1+q)\left(\frac{2}{1+q}s + 2\sqrt{\frac{2(1-q)}{(1+q)^2} s(1-s)}\right)
\]
\[
= 2s + 2\sqrt{2 - 2q}\sqrt{s(1-s)}.
\]

(2.1)

When \( t = 1 - (g(1,1) - 2s)^2/8s(1-s)s \), the upper bound on \( g(q) \) given in (2.1) is equal to \( g(0) = g(1,1) \). Moreover, by Lemma 2.2
\[
g(1,1) - 2s = g(0) - 2s \geq s(1-s),
\]

which implies that \( t \leq 1 - s(1-s)/8 \).

It is commonly believed that, for Bernoulli weights, with sufficiently small parameter, \( g \) is strictly concave. In our setting instead, and in order to obtain our main result, the finiteness of the curvature power in the \((1,1)\) direction is imposed. This assumption is stronger than strict concavity since the curvature power of the shape function is defined as:

**Definition 2.4.** The shape function \( g \) is said to have curvature power \( \kappa(e) \) at \( e \in \mathbb{R}^+ \times \mathbb{R}^+ \), if \( g \) is differentiable at \( e \) and there exists \( \delta > 0 \), such that for any \( z \in \mathbb{R} \times \mathbb{R} \) such that \( |z| < \delta \) and \( z + e/g(e) \in L(e) \), where \( L(e) \) is a supporting line for \( g \) at \( e \),
\[
c|z|^\kappa(e) \leq |g(e + z) - g(e)| \leq C|z|^\kappa(e),
\]

6
for some positive constants $c$ and $C$ depending only on $\delta$. Otherwise, if $g$ is not differentiable at $e$, set $\kappa(e) = 1$. Hereafter, $\kappa$ is short for $\kappa((1,1))$.

By symmetry, the supporting line $L((1,1))$ is in the direction of $(1, -1)$. Hence, the definition of $\kappa$ is equivalent to the fact that there exists $\delta > 0$, such that for any $z \in \mathbb{R} \times \mathbb{R}$ satisfying $z \cdot (1, 1) = 0$ and $|z| < \delta$,

$$c|z|^\kappa \leq |g((1,1) + z) - g(1,1)| \leq C|z|^\kappa,$$

for some positive constants $c$ and $C$ depending only on $\delta$. Then, requiring that $1 \leq \kappa < +\infty$ is in turn equivalent to: there exists $\delta > 0$ such that for any $q \in (-1,1)$ with $|q| < \delta$,

$$c|q|^\kappa \leq |g_\perp(q) - g_\perp(0)| \leq C|q|^\kappa,$$

(2.2)

for some positive constants $c$ and $C$, depending only on $\delta$.

As already indicated, it is believed (e.g., see [13]) that when $s < s_c$, where $s_c$ is the critical probability for the directed last passage percolation with i.i.d. Bernoulli weights, the shape function $g$ has curvature power $\kappa = 2$. But as mentioned before Lemma 2.2 in FPP, a class of weights has been shown to be such that $g$ has flat edges, i.e., there exist infinitely many $e$ such that $\kappa(e) = \infty$ (see [7]). It was first proved in [18] that there exists $q \in (-1,1)$ such that the lower inequality in (2.2) holds when $\kappa(1 - q, 1 + q) = 2$, while [5] shows that there is a (possibly different) $q$ at which the upper inequality in (2.2) holds when $\kappa(1 - q, 1 + q) = 2$. To finish this section, we state the main result of this paper.

**Theorem 2.5.** Let the curvature power $\kappa$ of the shape function $g$ at $(1,1)$ be such that $1 \leq \kappa < +\infty$. Then, in a $n \times n$ grid, with probability exponentially close to $1$, all the geodesics are within the cylinder, centered on the main diagonal and of width $O(n^{2\kappa+2}/\ln n)$.

As far as notations are concerned, and as usual, $a_n = O(b_n)$ is short for there exists a positive constant $C$ such that $|a_n| \leq C|b_n|$, for $n$ large enough; $a_n = \Theta(b_n)$ is short for there exist $0 < c < C < +\infty$ such that $cb_n \leq a_n \leq Cb_n$, for $n$ large enough; $a_n = \Omega(b_n)$ is short for there exists a constant $K > 0$ such that $a_n \geq Kb_n$, for $n$ large enough and finally, $a_n = o(b_n)$ is short for $\lim_{n \to +\infty} |a_n|/|b_n| = 0$. 

7
3 Proof of Main Results

In this section we start by introducing our main tools, i.e., decompositions and blocks, and then prove some concentration results which are further related to the concentration of geodesics needed to obtain our main result.

3.1 Blocks, Decompositions and Concentration

Throughout the rest of this manuscript, let $n = mk$ so that the $x$-axis of the grid is divided into $m$ segments each of equal length $k$. Meanwhile, the $y$-axis of the grid is also divided into $m$ segments. The $(m+1)$-tuples $\vec{r} = (r_0, r_1, ..., r_m)$ made of the end points of these consecutive segments on the $y$-edges, i.e., $r_0 = 0 \leq r_1 \leq r_2 \leq ... \leq r_{m-1} \leq r_m = n$, is called a decomposition of the $y$-axis. This decomposition leads to a decomposition of the grid into $m$ rectangular blocks, of which the $i$th ($i = 1, 2, ..., m$) block has lower-left corner $((i-1)k, r_{i-1})$ and upper-right corner $(ik, r_i)$, and is of size $k \times (r_i - r_{i-1})$. Moreover, the last passage time associated with the decomposition $\vec{r}$ is defined as the summation over all the $m$ last passage times in these $m$ blocks, i.e.,

$$T_n(\vec{r}) = \sum_{i=1}^{m} T(((i-1)k, r_{i-1}), (ik, r_i)).$$

By superadditivity, it is clear that $T(n, n) \geq T_n(\vec{r})$. Moreover, as explained next, there always exists a decomposition $\vec{r}^*_\eta$ such that $T(n, n) = T_n(\vec{r}^*_\eta)$, and such a decomposition is called optimal. Indeed, one can construct an optimal decomposition $\vec{r}^*_\eta$ by taking vertices $(ik, r_i)$, $i = 0, 1, ..., m$, on a geodesic for the entire $n \times n$ grid. Heuristically, any optimal decomposition $\vec{r}^*_\eta$ should, roughly, be evenly distributed over $n$, i.e., all the $m$ blocks in any optimal decomposition should be mostly square shaped at least with high probability. To be more precise, let us fix $0 < \eta < 1$ and $p_i > 0$ ($i = 1, 2$) such that $0 < p_1 < 1 < p_2$. Let $R_{\eta,p_1,p_2}$ be the deterministic set of decompositions $\vec{r}$ such that

$$\#\{i \in [m] : kp_1 \leq r_i - r_{i-1} \leq kp_2\} \geq (1 - \eta)m,$$

in words $R_{\eta,p_1,p_2}$ represents the decompositions having a proportion of at least $(1 - \eta)$ of those $m$ blocks close-to-square shaped, i.e., the decompositions for which the slope
of the block diagonal is close to 1, i.e., the non-skewed decompositions. Finally, let $A_{\eta,p_1,p_2}^n$ be the event that all the optimal decompositions are in $R_{\eta,p_1,p_2}$, i.e., if $\vec{r}$ is optimal, then
\[ \vec{r} \in R_{\eta,p_1,p_2}. \] (3.2)

Next, we show a lemma asserting that for any decomposition $\vec{r} = (r_0, r_1, ..., r_m) \in R_{\eta,p_1,p_2}^c$, the difference between the expected overall last passage time and the expected last passage times associated with $\vec{r}$ is at least linear in $n$. Before proving it, it is shown that even when the vertex weights, belonging to a particular set to be specified, are independently resampled, the absolute change in the last passage time can be upper-bounded by 1. To specify such a set, declare the vertices $\{V_i = (X_i, Y_i)\}_{i=1}^k$ to be strictly decreasing, if there exists a permutation $\pi$ of $\{1, 2, ..., k\}$ such that
\[ V_{\pi(1)} \prec V_{\pi(2)} \prec ... \prec V_{\pi(k)}, \]
where $V_i \prec V_j$ indicates that both $X_i < X_j$ and $Y_i < Y_j$. For example, on a $n \times n$ grid, the set of all the vertices on the reversed diagonal, i.e., $\{(n-i, i)\}_{i=0}^n$ is a strictly decreasing set and its cardinality is $n + 1$.

**Lemma 3.1.** Let a rectangular grid have lower-left vertex $V_1$ and upper-right vertex $V_2$ and let $S$ be a strictly decreasing set of vertices on the grid. Then, the absolute difference between the last passage times in the original weights setting and in the modified weights setting, where the weights on $S$ are independently resampled, is upper bounded by 1, i.e.,
\[ |T(V_1, V_2) - T^S(V_1, V_2)| \leq 1, \]
where $T(V_1, V_2)$ and $T^S(V_1, V_2)$ are respectively the last passage times before and after resampling.

**Proof.** Let $\Pi$ be the set of all up-right paths from $V_1$ to $V_2$. Since $S$ is a set of strictly decreasing vertices, for any path $\pi \in \Pi$ viewed as a set of vertices, the intersection between $\pi$ and $S$ is either empty or contains exactly one element, i.e., $\#(\pi \cap S) \leq 1$. Thus,
\[ T^S_{\pi}(V_1, V_2) - T_{\pi}(V_1, V_2) \leq 1, \]
where the upper bound is 1 if and only if there is a vertex $v \in \pi \cap S$ such that $w(v) = 0$ and $w^S(v) = 1$. Let $\pi^S_{\pi}$ be a geodesic after resampling $S$, i.e., $T^S_{\pi^S_{\pi}}(V_1, V_2) =
\( T^S(V_1, V_2) \). It follows that
\[
T^S(V_1, V_2) = T_\pi^S(V_1, V_2) \\
\leq T_\pi^S(V_1, V_2) + 1 \\
\leq \max_{\pi \in \Pi} T_\pi(V_1, V_2) + 1 \\
\leq T(V_1, V_2) + 1.
\]

Symmetrically, \( T(V_1, V_2) - T^S(V_1, V_2) \leq 1 \) and thus
\[
|T(V_1, V_2) - T^S(V_1, V_2)| \leq 1.
\]

\[ \square \]

For further convenience, we introduce a transformed shape function \( g_\lambda \) which depends on the slope of the main diagonal of the grid. Specifically, for \( p > 0 \), set
\[
g_\lambda(p) := \lim_{n \to \infty} \frac{\mathbb{E}T(n, np)}{n(1 + p)/2}.
\]

Now, recalling that \( g_\perp : q \in (-1, 1) \to g_\perp(q) \in (0, \infty) \) is defined via
\[
g_\perp(q) = g(1 - q, 1 + q) = \lim_{n \to \infty} \frac{\mathbb{E}T(n - nq, n + nq)}{n},
\]
it is clear that
\[
g_\lambda(p) = g_\perp \left( \frac{p - 1}{p + 1} \right),
\]
for \( p \in (0, +\infty) \). To prove a result showing that the difference of the expectations is at least linear in \( n \), in addition to Lemma 3.1, a rate of convergence result for \( \mathbb{E}T_n/n \) is also needed. This is stated and proved next, with a proof adapted from [19].

**Proposition 3.2.** \( 0 \leq g_\lambda(1) - \mathbb{E}T_n/n \leq c\sqrt{\ln n/n} \), where \( c > 0 \) is an absolute constant.

**Proof.** Consider the last passage time \( T_{kn} \) of site percolation on a \( kn \times kn \) grid. A sequence of vertices \( \mathbf{V} = (V_1 = (X_1 = 0, Y_1 = 0), V_2, ..., V_k = (X_k = kn, Y_k = kn)) \) is called a partition of the grid, if
\[
0 = X_1 \leq X_2 \leq ... \leq X_k = kn, \quad (3.3)
\]
\[0 = Y_1 \leq Y_2 \leq \ldots \leq Y_k = kn,\]  
\[||V_i - V_{i+1}||_1 = 2n,\]  
(3.4)  

where \(|| \cdot ||_1\) denotes the \(\ell_1\)-distance. Further, let the last passage time associated with some partition \(\vec{V}\) be

\[T(\vec{V}) = \sum_{i=0}^{k-1} T(V_i, V_{i+1}).\]

Then, as proved next,

\[T_{kn} = \max_{\text{partitions } \vec{V}} T(\vec{V}).\]  
(3.6)

First, it is clear that the identity is true if only (3.3) and (3.4) are imposed on partitions. To show it is fine to include (3.5), it suffices to show that any geodesic can be divided into \(k\) segments such that the \(\ell_1\)-distance between two ends of any segment is exactly \(2n\). Assume some geodesic is an ordered set of \(2kn + 1\) vertices \((W_0 = (0, 0), W_2, W_3, \ldots, W_{2kn} = (kn, kn))\). Notice that \(||W_i - W_{i+j}||_1 = j\), for any \(i, i+j \in [2kn]\). Therefore, this geodesic can be divided on \((V_0 = W_0, V_1 = W_{2n}, \ldots, V_k = W_{2kn})\) into \(k\) segments with \(||V_i - V_{i+1}||_1 = 2n\).

Next, consider a particular set of directed paths going from \((0, 0)\), through \((k, 2n - k)\), to \((2n, 2n)\) on a \(2n \times 2n\) grid. Then, by superadditivity, \(T(k, 2n - k) + T(2n - k, k) \leq T(2n, 2n)\). Further, thanks to symmetry, \(\mathbb{E}T(k, 2n - k) = \mathbb{E}T(2n - k, k)\). Hence, \(\mathbb{E}T(k, 2n - k) \leq \frac{1}{2} \mathbb{E}T(2n, 2n)\). So, \(\mathbb{E}(T(\vec{V})) \leq k \mathbb{E}(T(2n, 2n))/2\).

On the other hand, let us view \(T(\vec{V})\) as a function

\[T(\vec{V}) : (D_1, \ldots, D_{2kn}) \rightarrow T(\vec{V})(D_1, \ldots, D_{2kn}) \in \mathbb{N},\]

where \(\{D_j\}_{j=1}^{2kn}\) is the set of batches of the weights \(w(v)\) on the same reversed diagonal, i.e., \(D_j = \{w(v) \mid v \in \{x + y = j\} \cap [kn] \times [kn]\}\). Clearly, the independence of the weights yields the independence of the random vectors \(D_j, j = 1, \ldots, 2kn\). Further, any batch \(D_j\) is a strictly decreasing set of vertices and so by Lemma 3.1 independently resampling any one of these random vectors, say, as \(D'_{j_0}\) gives

\[|T(\vec{V})(D_1, \ldots, D_{j_0}, \ldots D_{2kn}) - T(\vec{V})(D_1, \ldots, D_{j_0}, \ldots, D_{2kn})| \leq 1.\]

Further, applying Hoeffding’s martingale inequality gives

\[\mathbb{P}\left(T(\vec{V}) - \mathbb{E}T(\vec{V}) \geq tkn\right) \leq \exp\left(-t^2kn\right),\]
and so,
\[ \mathbb{P} \left( T(\overrightarrow{V}) - \frac{k}{2} \mathbb{E} T_{2n} \geq tkn \right) \leq \exp \left( -t^2 kn \right). \]

In addition, by (3.6),
\[
\mathbb{P} \left( \frac{T_{kn}}{kn} - \frac{\mathbb{E} T_{2n}}{2n} \geq t \right) = \mathbb{P} \left( T_{kn} - \frac{k}{2} \mathbb{E} T_{2n} \geq tkn \right) \\
\leq \sum_{\text{partitions} \overrightarrow{V}} \mathbb{P} \left( T(\overrightarrow{V}) - \frac{k}{2} \mathbb{E} T_{2n} \geq tkn \right) \\
\leq \# \text{partitions} \exp \left( -t^2 kn \right). \tag{3.7}
\]

Since \( \# \text{partitions} = \binom{kn + k - 1}{k - 1} \) and \( \binom{p}{q} \leq \frac{p^p}{q^q (p - q)^{p - q}} \),

\[ \# \text{partitions} \leq \binom{kn + k}{k} \leq (kn + k)^{kn + k} / k^k (kn)^k \leq \exp(ck \ln n), \]

for some absolute constant \( c > 0 \). Combining this with (3.7) and taking \( t = \sqrt{2c \ln n / n} \) leads to
\[
\mathbb{P} \left( \frac{T_{kn}}{kn} - \frac{\mathbb{E} T_{2n}}{2n} \geq \sqrt{\frac{2c \ln n}{n}} \right) \leq \exp(-ck \ln n). \]

Hence,
\[
\frac{\mathbb{E} T_{kn}}{kn} - \frac{\mathbb{E} T_{2n}}{2n} \leq \sqrt{\frac{2c \ln n}{n}} + 2 \exp(-ck \ln n),
\]

and letting \( k \to \infty \) gives,
\[ \frac{\mathbb{E} T_{2n}}{2n} \geq \gamma^* - \sqrt{\frac{2c \ln n}{n}}. \]

In addition, for odd integers,
\[ \frac{\mathbb{E} T_{2n+1}}{2n + 1} = \frac{\mathbb{E} T_{2n}}{2n} - \frac{\mathbb{E} T_{2n+1}}{2n(2n + 1)} \geq \gamma^* - \sqrt{\frac{2c \ln n}{n}} - \frac{1}{n}. \]

\[ \square \]

To state our next lemma, recall that \( R_{n,p_1,p_2} = \{ r : \# \{ i \in [m] : kp_1 \leq r_i - r_{i-1} \leq kp_2 \} \geq (1 - \eta)m \}. \)
Lemma 3.3. Let $0 < \eta < 1$ and let $p_i$ $(i = 1, 2)$ be such that $0 < p_1 < 1 < p_2$, $g_\wedge(p_i) < g_\vee(1)$. Let $\delta^* = \min(g_\vee(1) - g_\wedge(p_1), g_\vee(1) - g_\wedge(p_2))$ and let $\delta^*\eta = \Omega(\sqrt{\log n/n})$. Then, for any $\vec{r} = (r_0, r_1, ..., r_\eta) \in R_n^{\vec{r}_{\eta,p_1,p_2}}$ and any $\delta \in (0, \delta^*)$,

$$
\mathbb{E}(T_n(\vec{r}) - T_n) \leq -\frac{\delta\eta n}{2},
$$

for all $n = n(\eta, \delta)$ large enough.

Proof. Let $p > 0$. By superadditivity, $g_\wedge(p)$ is well defined and finite. Moreover, for any $k \geq 1$,

$$
\frac{2\mathbb{E}T(k, kp)}{k(1 + p)} \leq g_\wedge(p). \tag{3.9}
$$

Since $g_\wedge$ is symmetric around $q = 0$ and concave and since $(1 - p)/(1 + p)$ is a monotone transformation in $p$, $g_\wedge$ is non-decreasing up to $p = 1$ and non-increasing thereafter. Proposition 2.3 shows that there exist $0 < p_1 < 1 < p_2$ such that for any $p \notin [p_1, p_2]$,

$$
g_\wedge(p) \leq \max(g_\wedge(p_1), g_\wedge(p_2)). \tag{3.10}
$$

Therefore, for any $p \notin [p_1, p_2]$, (3.9) and (3.10) lead to:

$$
\frac{2\mathbb{E}T(k, kp)}{k(1 + p)} \leq \max(g_\wedge(p_1), g_\wedge(p_2)) = g_\wedge(1) - \delta^*, \tag{3.11}
$$

where $0 < \delta^* := \min(g_\wedge(1) - g_\wedge(p_1), g_\wedge(1) - g_\wedge(p_2))$.

From here on, the proof proceeds as the proof, with its notation, of Lemma 2.1, in [11]. Since the weights are identically distributed, in the $i$th block $[(i - 1)k + 1, ik] \times [r_{i-1} + 1, r_i]$, letting $r_i - r_{i-1} := kp$ and assuming $(r_i - r_{i-1})/k = p \notin [p_1, p_2]$, then (3.11) gives

$$
g_\wedge(1) - \frac{2\mathbb{E}T(((i - 1)k + 1, r_{i-1} + 1), (ik, r_i))}{k + r_i - r_{i-1}} \geq \delta^*. \tag{3.12}
$$

Hence,

$$
\frac{1}{2}g_\wedge(1)(k + r_i - r_{i-1}) - \mathbb{E}T(((i - 1)k + 1, r_{i-1} + 1), (ik, r_i)) \geq \frac{1}{2}\delta^*k.
$$

Letting $M := \{i : r_i - r_{i-1} \notin [kp_1, kp_2]\}$, we then have

$$
\sum_{i \in M} \frac{1}{2}g_\wedge(1)(k + r_i - r_{i-1}) - \mathbb{E}T(((i - 1)k + 1, r_{i-1} + 1), (ik, r_i)) \geq \frac{1}{2}\delta^*k\eta m = \frac{1}{2}n\delta^*\eta, \tag{3.13}
$$
while, for any $i \in \mathcal{M}^c = \{i : r_i - r_{i-1} \in [kp_1, kp_2]\}$, (3.9) gives
\[
\frac{1}{2}g_{\kappa}(1)(k + r_i - r_{i-1}) - \mathbb{E}T(((i - 1)k + 1, r_{i-1} + 1), (ik, r_i)) \geq 0.
\]
Therefore,
\[
\sum_{i \in \mathcal{M}} \frac{1}{2}g_{\kappa}(1)(k + r_i - r_{i-1}) - \mathbb{E}T(((i - 1)k + 1, r_{i-1} + 1), (ik, r_i)) \leq g_{\kappa}(1)n - \mathbb{E}T_n(\overrightarrow{r}).
\]
Combining (3.13) and (3.14) leads to,
\[
g_{\kappa}(1)n - \mathbb{E}T_n(\overrightarrow{r}) \geq \frac{n\delta^* \eta}{2},
\]
when $\overrightarrow{r} = (r_0, r_1, ..., r_m) \in R_{\eta,p_1,p_2}^c$. Next, by Proposition 3.2 and since $\delta^* \eta = \Omega(\sqrt{\ln n/n})$, it follows that for any positive $\delta < \delta^*$ and for all $n$ large enough,
\[
0 \leq g_{\kappa}(1) - \frac{\mathbb{E}T_n(\overrightarrow{r})}{n} \leq c\sqrt{\frac{\ln n}{n}} \leq \frac{(\delta^* - \delta) \eta}{2},
\]
where $c > 0$ is an absolute constant. So combining (3.15) and (3.16), and for $\overrightarrow{r} = (r_0, r_1, ..., r_m) \in R_{\eta,p_1,p_2}^c$,
\[
\mathbb{E}(T_n(\overrightarrow{r}) - T_n) \leq -\frac{\delta \eta n}{2}.
\]

Before presenting the main result of this section, recall (see (3.2)) that $A_{\eta,p_1,p_2}^n$ is the event that all the optimal decompositions belong to $R_{\eta,p_1,p_2}$.

**Theorem 3.4.** Let $0 < \eta < 1$ and let $p_i$ ($i = 1, 2$) be such that $0 < p_1 < 1 < p_2$, $g_{\kappa}(p_i) < g_{\kappa}(1)$. Let $\delta^* = \min(g_{\kappa}(1) - g_{\kappa}(p_1), g_{\kappa}(1) - g_{\kappa}(p_2))$ and let $\delta^* \eta = \Omega(\sqrt{\log n/n})$. Let the integer $k$ be such that $(1 + \ln k)/k \leq \delta^* \eta^2/16$, where $\delta \in (0, \delta^*)$. Then,
\[
\mathbb{P}(A_{\eta,p_1,p_2}^n) \geq 1 - \exp\left(-n\left(-\frac{1 + \ln k}{k} + \frac{\delta^* \eta^2}{16}\right)\right),
\]
for all $n = n(\eta, \delta)$ large enough.
Proof. The beginning of this proof, which is similar to that of Theorem 2.2 in [11], is only sketched. By superadditivity, the decomposition $\vec{r}$ is optimal if and only if

$$T_n(\vec{r}) \geq T_n.$$  \hfill (3.17)

Assume now that the event $A_{n,p_1,p_2}^n$ does not hold. Then there exists an optimal decomposition $\vec{r}_*^n$ such that $\vec{r}_*^n \in R^{c}_{n,p_1,p_2}$, i.e.,

$$(A_{n,p_1,p_2}^n)^c = \bigcup_{\vec{r} \in R^{c}_{n,p_1,p_2}} \{ \vec{r} = \vec{r}_*^n \}$$

$$= \bigcup_{\vec{r} \in R^{c}_{n,p_1,p_2}} \{ T_n(\vec{r}) - T_n \geq 0 \},$$

hence,

$$\mathbb{P}((A_{n,p_1,p_2}^n)^c) \leq \sum_{\vec{r} \in R^{c}_{n,p_1,p_2}} \mathbb{P}(T_n(\vec{r}) - T_n \geq 0).$$ \hfill (3.18)

Then, by Lemma 3.3, for any decomposition $\vec{r} \in R^{c}_{n,p_1,p_2}$,

$$\mathbb{E}(T_n(\vec{r}) - T_n) \leq -\frac{\delta \eta n}{2},$$

and so

$$\mathbb{P}(T_n(\vec{r}) - T_n \geq 0) \leq \mathbb{P} \left( T_n(\vec{r}) - T_n - \mathbb{E}(T_n(\vec{r}) - T_n) \geq \frac{\delta \eta n}{2} \right).$$ \hfill (3.19)

for all $n$ large enough. Next, as in the proof of Proposition 3.2, we view the random variable $T_n(\vec{r}) - T_n := \Delta$ as a function

$$\Delta : \{D_1, \ldots, D_{2n}\} \rightarrow \Delta(D_1, \ldots, D_{2n}) \in \mathbb{Z} \cap [-2n, 2n],$$

where $\{D_j\}_{j=1}^{2n}$ is the set of batches of the weights $w(v)$ on the same reversed diagonal, i.e., $D_j = \{w(v) \mid v \in \{x+y = j\} \cap [n] \times [n]\}$. So by Lemma 3.1 again, independently resampling any one of these random vectors, say, as $D_{j_0}'$ gives

$$|\Delta(D_1, \ldots, D_{j_0}', \ldots D_{2n}) - \Delta(D_1, \ldots, D_{j_0}, \ldots, D_{2n})|$$

$$\leq |T_{n}^{j_0}(\vec{r}) - T_n(\vec{r})| + |T_{n}^{j_0} - T_n| \leq 2,$$ \hfill (3.20)
where $T^0_n(\vec{r})$ and $T^0_n$ are respectively the last passage time associated with $\vec{r}$ and the overall last passage time with the weights in $D^0_n$ resampled.

Finally, Hoeffding’s martingale inequality applied to $\Delta(D_1,\ldots,D_{2n})$ and $(3.19)$ yield
\[
\mathbb{P}(T_n(\vec{r}) - T_n \geq 0) \leq \exp\left(-\frac{\delta^2 \eta^2 n}{16}\right),
\]
geq for $\vec{r} \in R^c_{\eta,p_1,p_2}$. Further, by $(3.18)$,
\[
\mathbb{P}((A^n_{\eta,p_1,p_2})^c) \leq (ek)^m \exp\left(-\frac{\delta^2 \eta^2 n}{16}\right) \\
\leq \exp\left(-n\left(-\frac{1 + \ln k}{k} + \frac{\delta^2 \eta^2}{16}\right)\right),
\]
since
\[
\#R^c_{\eta,p_1,p_2} \leq \binom{n}{m} \leq \frac{n^m}{n!} \leq \left(\frac{en}{m}\right)^m,
\]
when $n$ is large enough.

**Remark 3.5.** Note that above, when applying Hoeffding’s martingale inequality and if only a single weight had been independently resampled then, the exponential concentration would have failed to hold, since this naively constructed martingale would have had a length of size $\Theta(n^2)$. This justifies and motivates resampling weights in batches.

### 3.2 Proof of the Main Result

Heuristically, if most blocks in an optimal decomposition are close-to-square shaped, then all the vertices on the diagonals of these blocks are close to the main diagonal of the grid and therefore all the corresponding geodesics going through these vertices do not deviate much from it. Further, the parameters such as $k$, $\delta$ and $\eta$ can be fixed in an optimal way so that the cylinder, in which geodesics are confined, is as small as possible.

**Proof of Theorem 2.5.** Let $D^n_{\eta,p_1,p_2}$ be the event that all the geodesics are above the line $\ell_1: y = p_1 x - p_1 n \eta - p_1 k$ and below the line $\ell_2: y = p_2 x + p_2 n \eta + p_2 k$. We first show that the probability of this event is exponentially close to 1. Again the proof is similar to the corresponding result in [III] and as such only sketched. We
start with a few definitions: denote by $D_n^a$ the event that all the geodesics are above the line $\ell_1: y = p_1x - p_1n\eta - p_1k$ and by $D_n^b$ the event that they are below the line $\ell_2: y = p_2x + p_2n\eta + p_2k$. Then $D_{\eta,p_1,p_2}^n = D_a^n \cap D_b^n$, hence

$$\mathbb{P}((D_{\eta,p_1,p_2}^n)^c) \leq \mathbb{P}((D_a^n)^c) + \mathbb{P}((D_b^n)^c).$$

Moreover, as shown next,

$$A_{\eta,p_1,p_2}^n \subset D_a^n, \quad A_{\eta,p_1,p_2}^n \subset D_b^n,$$

so that by Theorem 3.4

$$\mathbb{P}((D_{\eta,p_1,p_2}^n)^c) \leq 2 \exp \left( -n \left( -\frac{1 + \ln k}{k} + \frac{\delta^2 \eta^2}{16} \right) \right).$$

(3.22)

To prove (3.21), at first we prove that $A_{\eta,p_1,p_2}^n \subset D_a^n$. This last inclusion is obtained by considering three cases which depend on $x$: If $(x,y)$ is on one of the geodesics in the event $A_{\eta,p_1,p_2}^n$, namely, $x = uk$, where $u \in \mathbb{N} = \{0, 1, 2, \ldots\}$, and $uk \leq n\eta$; $x = uk$, where $u \in \mathbb{N}$ and $uk > n\eta$; and there exists $u \in \mathbb{N}$ such that $uk < x < (u+1)k$. Before we move on to verify the inclusion case by case, recall again that $A_{\eta,p_1,p_2}^n$ corresponds to geodesic decompositions belonging $R_{\eta,p_1,p_2}$, i.e., such that the number of $i \in [m]$ with $(r_{i+1} - r_i) \in [p_1 k, p_2 k]$ is at least $(1 - \eta)m$, where $m$ is the total number of blocks and $mk = n$ (see (3.1)).

In the first of these cases, $p_1x - p_1n\eta \leq 0$ and therefore,

$$y \geq 0 \geq p_1x - p_1n\eta \geq p_1x - p_1n\eta - p_1k.$$

In the second case, by the very definition of $R_{\eta,p_1,p_2}$, there are at most $\eta m$ blocks having side length $(r_{i+1} - r_i)$ less than $p_1 k$. Since $x = uk$, in the worst case, all these $\eta m$ blocks appear among the first $u$ blocks. Hence, at least $u - \eta m$ blocks of the first $u$ blocks have side length at least equal to $p_1 k$. Therefore,

$$y \geq (u - \eta m)p_1 k = p_1(uk) - p_1\eta mk = p_1x - p_1n\eta \geq p_1x - p_1\eta n - p_1 k.$$

In the third and last case, since $x_1 := uk < x < (u+1)k$, then

$$x - x_1 < k.$$
From the first two cases, \( y_1 \geq p_1x_1 - p_1\eta n \). Moreover, a geodesic is a directed path and so \( y \geq y_1 \) since \( x > x_1 \). Hence, by (3.23)

\[
y \geq y_1 \geq p_1x_1 - p_1\eta n \geq p_1x - p_1k - p_1n\eta.
\]

Symmetrically, a reversed inequality can be proved for the upper bounding line \( y = p_2x + p_2n\eta + p_2k \), and then (3.22) follows.

Now, let \( k = n^\alpha, p_{1,2} = 1 \pm n^{-\beta} \), for \( 0 < \alpha, \beta < 1 \) and so, \( \delta^* = \min(g_\lambda(1) - g_\lambda(p_1), g_\lambda(1) - g_\lambda(p_2)) = cn^{-\kappa\beta} \), for some constant \( c > 0 \). Further, set \( \delta = \delta^*/2 = cn^{-\kappa\beta}/2 \) and let \( \eta = 4\sqrt{2n^{\kappa\beta-\alpha/2}}\sqrt{1+\alpha\ln n/c} \) in (3.22) be such that \( 2\kappa\beta < \alpha \). Then, the condition \( \delta^*\eta = \Theta(n^{-\alpha/2}\sqrt{\ln n}) = \Omega(\sqrt{\ln n/n}) \) is satisfied, since \( \alpha < 1 \). Hence,

\[
\mathbb{P}(D_{\alpha,\beta}^n) \geq 1 - 2\exp(-(1 + \alpha\ln n)n^{1-\alpha}),
\]

where \( D_{\alpha,\beta}^n \) is the event that all the geodesics are above the line \( y = (1 - n^{-\beta})(x - cn^{1+\kappa\beta-\alpha/2}\sqrt{1+\alpha\ln n} - n^\alpha) \) and below the line \( y = (1 + n^{-\beta})(x + cn^{1+\kappa\beta-\alpha/2}\sqrt{1+\alpha\ln n} + n^\alpha) \).

Lastly, we will fix the orders of \( \alpha \) and \( \beta \) so that the cylinder has minimal width and so that the condition \( 2\kappa\beta < \alpha \) is satisfied. Notice that the distances at which the lines \( \ell_{1,2} \) are from the main diagonal is of the same order as the Euclidean distance from their intercepts on the left and right edges of the grid to, respectively, the lower-left vertex \( V_1 \) and the upper-right vertex \( V_2 \) as pictured in Figure 3.1. For the lower bounding line \( \ell_1 \), denoting its intercept on the left edge by \( U_1^1 \),

\[
|U_1^1V_1| = (1 - n^{-\beta})(cn^{1+\kappa\beta-\alpha/2}\sqrt{1+\alpha\ln n} + n^\alpha)
= \Theta(n^{1+\kappa\beta-\alpha/2}\sqrt{\ln n} + n^\alpha).
\]

Then denoting its intercept on the right edge by \( U_2^1 \), whose \( y \)-coordinate is \( (1 - n^{-\beta})(n - cn^{1+\kappa\beta-\alpha/2}\sqrt{1+\alpha\ln n} - n^\alpha) \),

\[
|U_2^1V_2| = (1 - n^{-\beta})n - (1 - n^{-\beta})(n - cn^{1+\kappa\beta-\alpha/2}\sqrt{1+\alpha\ln n} - n^\alpha)
= n(n^{-\beta} - n^{-\beta'}) + (1 - n^{-\beta})(cn^{1+\kappa\beta-\alpha/2}\sqrt{1+\alpha\ln n} + n^\alpha)
= \Theta(n^{1-\beta} + n^{1+\kappa\beta-\alpha/2}\sqrt{\ln n} + n^\alpha),
\]

since \( \beta' > \beta \). Therefore the distance from \( \ell_1 \) to the diagonal is of order

\[
n^{(1-\beta)\vee(1+\kappa\beta-\alpha/2)\vee\alpha}\sqrt{\ln n}.
\]
Symmetrically, a similar result holds true for the upper line $\ell_2$. The minimizing order occurs for $1 - \beta = 1 + \kappa \beta - \alpha/2 = \alpha$, i.e.,

$$\alpha = (2\kappa + 2)\beta = \frac{2\kappa + 2}{2\kappa + 3} > 2\kappa \beta.$$  

Setting $\alpha = (2\kappa + 2)/(2\kappa + 3)$ and $\beta = 1/(2\kappa + 3)$ in the event $D_{\alpha,\beta}^n$ gives

$$\mathbb{P}
\left(D_{\frac{2\kappa + 2}{2\kappa + 3}, \frac{1}{2\kappa + 3}}^n \geq 1 - 2 \exp\left(- \left(1 + \frac{2\kappa + 2}{2\kappa + 3} \ln n\right) n^{1/(2\kappa + 3)}\right),
\right)$$

which completes the proof.

\[\square\]

4 Concluding Remarks

By symmetry, it is clear that our methodology for proving the concentration of the geodesics in DLPP is also applicable to the concentration of geodesics in Bernoulli directed first passage site percolation. In DFPP, one studies the minimum of the passage times instead of maximum. In that context, the shape functions $g, g_\perp$ and $g_{\parallel}$ are convex instead of concave. Then, a version of Lemma 3.3 with the inequality (3.8) reversed holds true. Further, a version of Lemma 3.1 replacing last passage time by first passage time is still true. Then, so are Theorem 3.4, Proposition 2.3, Proposition 3.2 and Theorem 2.5. Combining all these results finally leads to:

**Theorem 4.1.** In directed Bernoulli first passage site percolation, let the curvature power $\kappa$ of the shape function $g$ at $(1,1)$ be such that $1 \leq \kappa < +\infty$. Then, in a $n \times n$ grid, with probability exponentially close to $1$, all the geodesics are within the $O(n^{2/3} \sqrt{\ln n})$.

To gain a better intuitive view of the concentration order, let, as commonly believed, $\kappa = 2$. Then, the order is $O(n^{6/7} \sqrt{\ln n})$. Again, it is conjectured that the correct order should be $O(n^{2/3})$ and a currently available bound for the exponent $\xi$ is $3/4$, which has been shown in [18], in the setting of first passage percolation on grids in arbitrary dimension.

It is further worth mentioning that our methodology can also be adapted to produce the order of the closeness to the diagonal for the optimal alignments corresponding to the LCSs of two random words of size $n$. In that setting, it is known that
Figure 3.1:
the curvature power of the shape function of the LCSs at (1, 0) and (0, 1) is equal to 1 (see the proof of Lemma 2.1 in [11]). However, the value of κ (the curvature power at (1, 1)) remains unknown but we conjecture it to be equal to 2, as in the percolation models. Adapting our methods leads to:

**Theorem 4.2.** In the longest common subsequences problem, let the curvature power κ of the shape function g at (1, 1) be such that $1 \leq \kappa < +\infty$. Then, with probability exponentially close to 1, all the alignments corresponding to the longest common subsequences of two random words of length n are within the cylinder, centered on the main diagonal and of width of order $O(n^{\frac{2\kappa+2}{2\kappa+3}}\sqrt{\ln n})$.

Let the exponent of transversal fluctuations ξ be:

$$\xi = \inf\{\gamma > 0 : \lim \inf_{n \to +\infty} \mathbb{P}(A_n^{\gamma}) = 1\},$$

where $A_n^{\gamma}$ is the event that all the optimal alignments are confined to a cylinder centered on the main diagonal and of width of order $n^\gamma$. Therefore, from Theorem 4.2 for LCSs, $\xi \leq \frac{(2\kappa + 2)}{(2\kappa + 3)}$. Moreover, as previously mentioned, the shape fluctuations exponent for LCSs has been shown to be $\chi = 1/2$, i.e., $\text{Var}(LC_n) = \Theta(n)$, for various asymmetric discrete distribution on any finite dictionary (see [10, 8, 15, ...]). But, by the conjectured KPZ universality relation with curvature power κ,

$$\chi = \kappa \xi - (\kappa - 1).$$

This leads, for $\chi = 1/2$, to $\xi = \frac{(2\kappa - 1)}{(2\kappa)}$ which we conjecture to be equal to 3/4.

**Acknowledgments**

Many thanks to M. Damron and R. Gong for their bibliographical help and their numerous comments which greatly helped to improve this manuscript.

**References**

[1] Antonio Auffinger and Michael Damron. Differentiability at the edge of the percolation cone and related results in first-passage percolation. *Probability Theory and Related Fields*, 156(1-2):193–227, 2013.
[2] Antonio Auffinger and Michael Damron. A simplified proof of the relation between scaling exponents in first-passage percolation. *The Annals of Probability*, 42(3):1197–1211, 2014.

[3] Antonio Auffinger, Michael Damron, and Jack Hanson. *50 Years of First-Passage Percolation*, volume 68 of *University Lecture Series*. American Mathematical Society, 2017.

[4] Itai Benjamini, Gil Kalai, and Oded Schramm. First passage percolation has sublinear distance variance. *The Annals of Probability*, 31(4):1970–1978, 2003.

[5] Sourav Chatterjee. The universal relation between scaling exponents in first-passage percolation. *The Annals of Mathematics*, 177(2):663–697, 2013.

[6] Michael Damron, Jack Hanson, and Philippe Sosoe. Sublinear variance in first-passage percolation for general distributions. *Probability Theory and Related Fields*, 163(1-2):223–258, 2015.

[7] Richard Durrett and Thomas M Liggett. The shape of the limit set in Richardson’s growth model. *The Annals of Probability*, 9(2):186–193, 1981.

[8] Ruoting Gong, Christian Houdré, and Jüri Lember. Lower bounds on the generalized central moments of the optimal alignments score of random sequences. *Journal of Theoretical Probability*, pages 1–41, Dec 2017.

[9] Christian Houdré and Ümit Işlak. A central limit theorem for the length of the longest common subsequences in random words. *arXiv preprint arXiv:1408.1559v4*, 2017.

[10] Christian Houdré and Jingyong Ma. On the order of the central moments of the length of the longest common subsequences in random words. *High Dimensinal Probability: The Cargèse Volume*, Progress in Probability 71, Birkhäuser:105–137, 2016.

[11] Christian Houdré and Heinrich Matzinger. Closeness to the diagonal for longest common subsequences. *Electronic Communications in Probability*, 21(2):1–19, 2016.
[12] W. Jockusch, J. Propp, and P. Shor. Random Domino Tilings and the Arctic Circle Theorem. *arXiv preprint arXiv: math/9801068*, January 1998.

[13] Harry Kesten. Aspects of first passage percolation. In *École d’Été de Probabilités de Saint Flour XIV-1984*, pages 125–264. Springer, 1986.

[14] Joachim Krug and Herbert Spohn. Kinetic roughening of growing surfaces. In C. Godreche, editor, *Solids far from equilibrium*, pages 479–582. Cambridge University Press, Cambridge, 1991.

[15] Jüri Lember and Heinrich Matzinger. Standard deviation of the longest common subsequence. *The Annals of Probability*, 37(3):1192–1235, 2009.

[16] R Marchand. Strict inequalities for the time constant in first passage percolation. *The Annals of Applied Probability*, 12(3):1001–1038, 2002.

[17] James B Martin. Limiting shape for directed percolation models. *The Annals of Probability*, 32(4):2908–2937, 2004.

[18] Charles M. Newman and Marcelo S.T. Piza. Divergence of shape fluctuations in two dimensions. *The Annals of Probability*, 23(3):977–1005, 1995.

[19] WanSoo T. Rhee. On rates of convergence for common subsequences and first passage time. *The Annals of Applied Probability*, 5(1):44–48, 1995.

[20] Hermann Rost. Non-equilibrium behaviour of a many particle process: density profile and local equilibria. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 58(1):41–53, 1981.

[21] Timo Seppäläinen. Hydrodynamic scaling, convex duality, and asymptotic shapes of growth models. *Markov Processes and Related Fields*, 4(1):1–26, 1998.

[22] Yu Zhang. Shape fluctuations are different in different directions. *The Annals of Probability*, 36(1):331–362, 2008.

[23] Yu Zhang. On the concentration and the convergence rate with a moment condition in first passage percolation. *Stochastic Processes and their Applications*, 120(7):1317–1341, 2010.