I. INTRODUCTION

Entropy can be spontaneously generated in a physical process. In addition, when a system is connected to an environment, there may also be a flux of entropy. Hence, the entropy of an open system, classical or quantum, evolves according to,

$$\frac{dS}{dt} = \Pi - \Phi,$$

where $\Pi \geq 0$ is the irreversible entropy production rate and $\Phi$ is the entropy flux from the system to the reservoir. Steady states are characterized by $dS/dt = 0$. These may either be equilibrium steady states (ESSs), characterized by $\Pi_{\text{ESS}} = \Phi_{\text{ESS}} = 0$, or non-equilibrium steady states (NESSs), which occur in systems connected to multiple baths, or which are externally driven. In this case $\Pi_{\text{NESS}} = \Phi_{\text{NESS}} > 0$, which means that entropy is being continuously produced within the system, all of which flows to the environment. Entropy production is hence the core concept in quantifying how far from equilibrium a process takes place, or in other words, how irreversible it is.

A particularly interesting class of non-equilibrium processes are those presenting driven-dissipative phase transitions (DDPTs) [1–3], which occur in quantum optical systems subject to a competition between dissipation and a coherent drive. When non-linear media is involved, these system may undergo a phase transition, where the NESS abruptly changes. This therefore falls in the class of dissipative transitions (also called non-equilibrium phase transitions in the classical context). Driven-dissipative transitions can be continuous or discontinuous [4–6] and are associated with the closing of the Liouvillian super-operator gap [3, 7]. They therefore offer the possibility of exploring, in a controlled way, many-body quantum phases without any classical analog. And for this reason, they have been the focus of significant theoretical studies [3, 6–35], as well as experimental investigations [36–39], in the last decade.

Since the entropy production rate $\Pi$ is the fundamental quantity that characterizes a non-equilibrium process, one may ask how does it behave dynamically, when the system is forced to adjust from one steady-state to another. A particularly interesting scenario is when the system is subject to a quantum quench; i.e., a sudden change in one of the system’s parameters [40]. In a thermodynamic language, this is a highly non-adiabatic process, analogous to the free expansion of a gas after a sudden change in volume. Hence, on top of the entropy production due to the intrinsic non-equilibrium nature of the NESS, there will also be a contribution due to this non-adiabatic transient dynamics. Quench dynamics have been the subject of intense research in the last two decades, although most of the work have been on closed systems undergoing unitary dynamics. Examples include the evolution of the expectation values of observables in quantum spin chains [40], the connection with work statistics [41–44], and the change in diagonal entropy [45–48]. Experimental studies have also been carried out in ultra-cold atoms in optical lattices [49–54], where the long coherent times allow one to assess the dynamics over extended time scales.

DDPTs, however, are usually associated with photon loss dissipation, and are thus inherently open. As a consequence, both the NESS, as well as the transient dynamics after a quench, should be associated with a finite entropy production. Very little is known about the behavior of the entropy production across DDPTs, however. Specially concerning the quench dynamics. Unfortunately, photon losses are not a standard thermal processes, since they effectively occur at zero-temperature. As a consequence, the standard formulation of entropy production does not apply [55]. Recently, we proposed a theory of entropy production suited for photon loss dissipation using concepts from quantum phase space [56]. This allowed us to incorporate not only thermal, but also vacuum fluctuations. This theory is suited for experiments in quantum optical setups, as illustrated in [57, 58]. Originally, it was formulated for Gaussian states and processes. But recently this has been extended for the non-Gaussian case [59]. This makes it particularly suited for dealing with DDPTs, specially those involving quenches, where Gaussianity is seldom preserved.

In Ref. [59] this formalism was applied to quantify the entropy production in a NESS. The theory, however, is also suited for describing entropy production dynamically. This
article aims to demonstrate the applicability, and usefulness, of the framework of [59] to study the dynamics of the entropy production in a quench scenario for DDPTs. We focus on the paradigmatic Kerr bistability model [3, 20, 21, 60] which describes an optical cavity filled with a non-linear medium, coherently pumped by a laser, and subjected to an incoherent single photon loss mechanism (see Fig. 1). This model presents a discontinuous DDPT as a function of the external laser drive, which at the mean field level can be associated with a bistable behavior of the cavity field. This analysis, as we hope to show, will provide an insightful, and timely, analysis of a problem that lies at the boundaries between statistical physics and quantum optics.

The paper is organized as follows: in Sec. II we introduce the Kerr bistability model, its thermodynamic limit and briefly discuss the spectral properties of the Liouvillian; in Sec. III we briefly review Ref. [59] and give the expressions for the entropy production/flux scenario. The formalism is then applied to the sudden quench dynamics scenario in Sec. IV. We first present a Gaussianity test, to show the intrinsic non-Gaussian physics and quantum optics.

In this section we review the Kerr bistability model (KBM), which will be the main focus of this work. The model is described, in a frame rotating with the pump frequency $\omega_p$, by the Hamiltonian

$$H = \Delta a^\dagger a + i\mathcal{E}(a^3 - a) + \frac{U}{2} a^\dagger a^3 a a,$$

(2)

where $a^\dagger$ ($a$) is the single mode creation (annihilation) operator, obeying the bosonic algebra $[a, a^\dagger] = 1$. This model describes an optical cavity, filled with a non-linear medium responsible for an effective photon-photon interaction (described by the parameter $U$). The parameter $\Delta = \omega_c - \omega_p$ is the detuning between the cavity and pump frequencies. Finally, $\mathcal{E}$ describes the coherent drive amplitude.

The system is also subjected to single photon losses at rate $\kappa$ (see Fig. 1). These are described within the Born-Markov approximation, so that the state of the system $\rho$ evolves according to the Gorini–Kossakowski–Sudarshan–Lindblad (GKLS) master equation ($\hbar = 1$) [61],

$$\partial_t \rho = -i[H, \rho] + 2\kappa \left( \alpha^\dagger \alpha \rho - \frac{1}{2} [\alpha^\dagger, \alpha, \rho] \right) = \mathcal{L}(\rho),$$

(3)

where $\mathcal{L}$ is the Liouvillian super-operator.

**B. Thermodynamic limit**

The system described by Eq. (3) can undergo a phase transition for sufficiently large pump intensities $\mathcal{E}$. In order to clarify the nature of this transition, and also better connect it to the standard statistical mechanics literature, it is convenient to define a fictitious thermodynamic limit, which is where the transition actually takes place. That is, we introduce a dimensionless parameter $N$ and define the thermodynamic limit to correspond to $N \to \infty$. This parameter is introduced in a reparametrization of the pump intensity, as $\mathcal{E} = \sqrt{N} \epsilon$, where $\epsilon$ is finite. The first moment is known to behave as $\langle a \rangle \propto \mathcal{E}$. We therefore define $\langle a \rangle := \mu = \sqrt{N} \alpha$, where $\alpha$ is finite and represents the order parameter of the system. These parametrizations show that the pump term is extensive, i.e. $\langle H \rangle \propto O(N)$. In order to properly define the thermodynamic limit, we take as a physical assumption, that this should also hold for the average energy in general. That is, $\langle H \rangle \sim O(N)$. Hence, we must reparametrize all other parameters accordingly. In the case of (2), this amounts solely to a rescaling of $U$. Since $\langle a^\dagger a^3 a a \rangle \propto O(N^2)$, we therefore reparametrize $U = u/N [59]$.

To summarize, we introduce a parameter $N$ by rescaling $\mathcal{E} = \sqrt{N} \epsilon$ and $U = u/N$. Both $\epsilon$ and $u$ are now taken to be finite and the thermodynamic limit is defined as $N \to \infty$.

**C. Mean-field behavior and exact solution for the steady-state**

Within the mean-field approximation, one sets $\langle a^\dagger a a \rangle \approx N^{3/2}\alpha^2 \alpha$. As a consequence, the equation of motion for the scaled amplitude $\alpha$, becomes

$$\partial_t \alpha = -(\kappa + i\Delta + iu|\alpha|^2)\alpha + \epsilon,$$

(4)

When $\Delta < -\sqrt{3}\kappa$, the steady-state of this equation is known to present a bistable behavior [61]. This is illustrated in Fig. (2). The bistability region occurs for $\epsilon$ within the interval

$$\epsilon_\pm = \sqrt{n_\pm [\Delta + n_\pm u^2]}, \quad n_\pm = -\frac{2\Delta \pm \sqrt{\Delta^2 - 3k^2}}{3u}. \quad (5)$$

For $\epsilon_- < \epsilon < \epsilon_+$, Eq. (4) presents 3 steady-state solutions, two of which are stable.

At the level of the full master equation (3), however, the nature of this bistability is somewhat subtle. For, as shown in [61] (see also [20, 60, 62, 63]), the steady-state is always
unique and can, in fact, be solved analytically. Indeed, the moments of the system are found to be given by [61]

\[ \langle a^n \rangle = \sqrt{\frac{\xi^n \Gamma(\xi + n)}{\Gamma(n+1)}}, \]

where \( \xi = 2\Delta + iU \) and \( x = 2(i\Delta + \kappa)/iU \). Here \( \Gamma(a,b;c) \) and \( \Gamma(x) \) denote the hyper-geometric and gamma functions, respectively. The predictions of this exact solution are shown as the black-solid lines in Fig. (2). As can be seen, at a certain critical value \( \epsilon_c \) the system jumps abruptly from a state with low photon occupation, to another with high occupation. Henceforth, we shall refer to the state for \( \epsilon < \epsilon_c \) as the dark phase and that for \( \epsilon > \epsilon_c \) as the bright phase (there is no closed-form expression for \( \epsilon_c \), which has to be computed numerically).

D. Spectral properties of the phase transition

This apparent contradiction between the mean-field and exact solutions is resolved by analyzing the full spectrum of the Liouvillian (3), defined by [3, 7]

\[ \mathcal{L}(\rho_k) = \zeta_k \rho_k, \]

where \( \rho_k \) are the eigenmatrices associated with the eigenvalue \( \zeta_k \) (which may be complex, since \( \mathcal{L} \) is not Hermitian). The NESS, in particular, is associated with the zero eigenvalue, \( \zeta_0 \),

\[ \mathcal{L}(\rho_{NESS}) = 0 \]

It can be shown that, in general, \( \text{Re}\{\zeta_k\} \leq 0 \) [64], which ensures stability of the dynamics under GKLS evolution. We can thus order the eigenvalues as \( 0 = |\text{Re}\{\zeta_0\}| < |\text{Re}\{\zeta_1\}| < \ldots \). The Liouvillian gap is then defined as \( \lambda = |\text{Re}\{\zeta_1\}| \). Physically, it determines the slowest relaxation rate in the long-time limit. Or, what is equivalent, the tunneling time \( \tau_{\text{tun}} \) between the two branches of optical bistability [8].

As studied in detail in Refs. [3, 7, 21], in the Kerr model the gap closes asymptotically within the entire bistability region \( \epsilon \in [\epsilon_-, \epsilon_+] \). This is illustrated in Fig. 3. As a consequence, for any finite \( N \), there will always be a unique steady-state, but the gap between this and the first excited state becomes vanishingly small as \( N \) increases, thus giving rise to a bistable solution (see also [65, 66]). This hence reconciles the mean-field and exact solutions.

III. WEHRL ENTROPY PRODUCTION RATE FOR THE KBM

To study the thermodynamics of the KBM model, we use a phase space method based on the Husimi Q-function, introduced in Ref. [59]. A quantum-phase space formulation of the entropy production was first introduced in Ref. [56], and also experimentally assessed in Ref. [57]. The problem was formulated in terms of the Wigner function, which generates physical probability densities for Gaussian systems subjected to Gaussian dynamics. This procedure make it possible to circumvent the so called ultra-cold catastrophe [67]; that is, the divergence of the usual entropy production rate and entropy flux rate in the zero temperature limit. The Wigner function, however, can become negative, which would lead to complex entropies. To amend this, one can alternatively use the Husimi...
$Q$-function, defined as

$$Q(\mu, \bar{\mu}) = \frac{1}{\pi} (\mu | \rho | \mu), \quad (9)$$

where $| \mu \rangle$ is a coherent state and $\bar{\mu}$ denotes complex conjugation. The $Q$-function is such that $Q(\mu, \bar{\mu}) \geq 0$ [68] and can be operationally interpreted as the probability distribution of the outcomes of a heterodyne measurement [69] (see also Appendix A). As the basic entropic quantifier, we use the Wehrl entropy [70], defined as the Shannon entropy of the $Q$-function:

$$S(Q) = -\int d^2 \mu \ Q \ln Q. \quad (10)$$

Physically, $S(Q)$ quantifies the entropy of the system, convoluted with additional noise introduced by measuring in the coherent state basis. As a consequence, $S(Q)$ upper bounds the von Neumann entropy, $S(\rho) = -\text{Tr}[\rho \ln \rho]$, i.e. $S(\rho) \leq S(Q)$. The Wehrl entropy may thus be viewed as a semi-classical approximation to the von Neumann entropy. But despite these limitations, this is currently the only available framework for dealing with the thermodynamics of DDPTs.

We proceed by mapping the master equation Eq. (3) into a quantum Fokker-Planck equation

$$\partial_t Q = \mathcal{U}(Q) + \partial_\mu J(Q) + \partial_{\bar{\mu}} \bar{J}(Q) \quad (11)$$

where $\mathcal{U}(Q)$ and $J(Q)$ are differential operators associated with the unitary and dissipative parts of (3). The exact form of these quantities can be found using standard correspondence tables [68]. The latter, in particular, reads

$$J(Q) = -\kappa (\mu Q + \bar{\mu} \bar{Q}), \quad (12)$$

and can be interpreted as the irreversible quasi-probability current associated with the photon loss dissipator. One may verify that $J(Q)$ vanishes when $Q$ is the vacuum state, which is the fixed point of the dissipator (although not of the whole master equation, due to the presence of the unitary term). This corroborates its interpretation as a quasiprobability current [56].

The unitary term, on the other hand, is somewhat lengthier and reads

$$\mathcal{U}(Q) = \mathcal{U}_\Delta + \mathcal{U}_E + \mathcal{U}_U, \quad (13)$$

where

$$\mathcal{U}_\Delta = i \Delta (\mu \partial_\mu Q - \bar{\mu} \partial_{\bar{\mu}} \bar{Q}), \quad (14)$$

$$\mathcal{U}_E = -\mathcal{E} (\partial_\mu Q + \partial_{\bar{\mu}} \bar{Q}), \quad (15)$$

$$\mathcal{U}_U = \frac{i}{2} [2\mu^2 (\mu \partial_{\mu} Q - \bar{\mu} \partial_{\bar{\mu}} \bar{Q}) + \mu^2 \partial_\mu^2 Q - \bar{\mu}^2 \partial_{\bar{\mu}}^2 \bar{Q}]. \quad (16)$$

One notices, in particular, that the photon-photon interaction term, being quartic, leads to terms in $\mathcal{U}_U$ containing second-order derivatives. This is a special feature of quantum non-Gaussian models. In classical Fokker-Planck equations, second-order derivatives stem only from dissipation. In quantum models, on the other hand, one may in general obtain derivatives of arbitrary orders.

Differentiating (10) with respect to time, we find

$$\frac{dS}{dt} = -\int d^2 \mu \ (\partial_t Q) \ln Q. \quad (17)$$

Inserting (11) in (17), we can then split $dS/\,dt$ as [c.f. Eq. (1)]

$$\frac{dS}{dt} = \Pi_{\mu} + \Pi_{\bar{\mu}} - \Phi, \quad (18)$$

where

$$\Pi_{\mu} := -\int d^2 \mu \ \mathcal{U}(Q) \ln Q, \quad (19)$$

$$\Pi_{\bar{\mu}} - \Phi := -\int d^2 \mu \left[ \partial_\mu J(Q) + \partial_{\bar{\mu}} \bar{J}(Q) \right] \ln Q. \quad (20)$$

The term $\Pi_{\mu}$ represents how the unitary term contributes to $dS/\,dt$. This is a feature which is unique of phase-space entropies, such as (10), and is associated with their coarse-grained nature. It will be discussed further below. The terms $\Pi_{\bar{\mu}}$ and $\Phi$, on the other hand, refer to the splitting of the dissipative contribution into a term associated with an entropy production rate, $\Pi_{\bar{\mu}}$, and another identified as an entropy flux rate $\Phi$.

### A. Dissipative contribution

As shown in [56, 59], the correct splitting of $\Pi_{\bar{\mu}}$ and $\Phi$ in Eq. (20) should have the form

$$\Pi_{\bar{\mu}} = -\frac{2}{\kappa} \int d^2 \mu \frac{|J(Q)|^2}{Q} \geq 0, \quad (21)$$

$$\Phi = \int d^2 \mu \left( \bar{\mu} J(Q) + \mu \bar{J}(Q) \right) = 2\kappa \langle a^\dagger a \rangle. \quad (22)$$

The quantity $J(Q)/Q$ can be interpreted as a phase-space velocity [71], so that $\Pi_{\bar{\mu}}$ is seen to be proportional to the mean squared phase space velocity, $\langle |J(Q)|^2/Q \rangle$.

The entropy flux, Eq. (22), is associated with the photon loss rate. To see that, we start from Eq. (3) and compute the time-evolution of the cavity occupation $\langle a^\dagger a \rangle$, which reads

$$\frac{d\langle a^\dagger a \rangle}{dt} = E \langle a^\dagger a \rangle - 2\kappa \langle a^\dagger a \rangle. \quad (23)$$

The first term describes the effect of the external coherent pump in populating the cavity, while the second describes the loss rate to the environment. The entropy flux is thus found to be directly the photon loss rate, corroborating its interpretation as a “flux”.

Within the context of DDPTs, it is interesting to analyze the behavior of $\Pi_{\bar{\mu}}$ and $\Phi$ in the thermodynamic limit (Sec. II B). In this case it is convenient to define the fluctuation operator $\delta a \delta a = a^\dagger a - N|a|^2$. Substituting in Eq. (22) leads to

$$\Phi = \Phi_{\text{ext}} + \Phi_{\delta} = 2\kappa |a|^2 N + 2\kappa \langle \delta a^\dagger \delta a \rangle, \quad (23)$$

where $\Phi_{\text{ext}}$ and $\Phi_{\delta}$ represent the external and fluctuation contributions to the entropy flux, respectively.
where $\Phi_{\text{ext}} \propto O(N)$ is a clearly extensive contribution, while $\Phi_{\text{q}}$ is the contribution associated with quantum fluctuations.

A similar splitting can be done for the entropy production rate \( (21) \). We define the displaced phase space variable \( \nu = \mu - \alpha \sqrt{N} \), which leads to a splitting of the current as \( J = -\sqrt{N} \kappa \alpha Q + J_{\text{ext}}(Q) \), where \( J_{\text{ext}}(Q) = -\kappa \nu Q + \partial_t Q \). Substituting this in \( (21) \), we then find

\[
\Pi_f = \Pi_{\text{ext}} + \Pi_d = 2\kappa |\alpha|^2 N + \frac{2}{k} \int d^2 \nu \frac{|J_{\text{ext}}(Q)|^2}{Q}. \tag{24}
\]

As can be seen, the first contribution to the entropy production rate is \( \Pi_{\text{ext}} = \Phi_{\text{ext}} \). This means that the part of the entropy flow associated with the first moments \( \alpha \) produces an equal amount of entropy (and hence does affect the system entropy, according to Eq. \( (1) \)). The other contribution, \( \Pi_d \), concerns only the quantum fluctuations (we do not call it \( \Pi_q \) since there will also be a quantum contribution to \( \Pi \) from the unitary part, as we now discuss).

### B. Unitary contribution

Next we turn to the unitary term \( \Pi_{\text{q}} \) in \( (19) \). An interesting feature of the Husimi $Q$-function is that the only unitary terms which contribute to \( d\rho/dt \) are those involving second-derivatives. That is, the terms associated with \( \Delta a^\dagger a \) and \( \partial^2 a^\dagger - a \), Eqs. \( (14) \) and \( (15) \), vanish. This can be proven integrating by parts multiple times. The same is also true for the first two terms in Eq. \( (16) \). Thus, the only non-trivial contributions come from the last two terms. Integrating by parts multiple times, these can be written as

\[
\Pi_{\text{q}} = \frac{iU}{2} \int d^2 \mu \left( \mu^2 (\partial_\mu Q)^2 - \hat{\mu}^2 (\partial_{\hat{\mu}} Q)^2 \right). \tag{25}
\]

This contribution to the entropy production stems from the second-order derivatives in the quantum Fokker-Planck equation. Usually, second-order derivatives are associated with diffusion. Since, in this case, these derivatives refer to the unitary part, this cannot be standard diffusion, which would not conserve energy. Instead, the structure appearing in the last two terms of Eq. \( (16) \) represents a special type of diffusion, which is compatible with unitarity. This type of phenomena was studied in [72, 73], in the context of the Dicke model, where the authors showed that it corresponds to a type of decoherence, linked with the coarse-graining introduced by the phase-space representation.

Unlike \( \Pi_f \), which is always non-negative (as expected by the 2nd law), the sign of \( \Pi_{\text{q}} \) is not well defined. This, in a sense, is expected since this quantity measures the effect of the unitary dynamics on $S(Q)$. And one cannot expect that all unitary dynamics always lead to an increase in the Wehrl entropy. Notwithstanding, we have recently carried out a study of this type of term in the context of dynamical phase transitions in the Lipkin-Meshkov-Glick model [74]. There, we found that even though \( \Pi_{\text{q}} \) did not have a well defined sign, this became asymptotically the case in the thermodynamic limit. Thus, our expectation is that for critical models, this term should tend to be non-negative in the thermodynamic limit (although it is not clear whether this could somehow be proved in general).

We have also carried out a detailed analysis of $\Pi_{\text{q}}$ in the NESS of the KBM model [59]. We found that, quite surprisingly, it behaved similarly to what is expected from the classical entropy production in across non-equilibrium transitions [75–79].

### IV. QUANTUM ENTROPY PRODUCTION DYNAMICS

#### A. Quench protocol and computational details

We now arrive at the core of our paper, where we probe the dynamics of the different entropic quantities during a quench protocol of the KBM model. The quench makes the system evolve from the initial NESS, with pump $\epsilon_i$, to a final NESS, with pump $\epsilon_f$. Both initial and final states are already nonequilibrium in nature. In addition, however, there will also be an entropy production associated with the quench dynamics, a highly non-adiabatic process.

From this point on, all quantities will be given in units of $\kappa = 1/2$. We fix $\Delta = -2$ and $u = 1$. We also start all quenches from $\epsilon_i = 0.5$. The only free parameters are then $\epsilon_f$ and $N$. We will make frequent reference to Fig. 2, which shows the different values we chose for $\epsilon_f$. For these parameters, the mean-field bistability region occurs between $\epsilon_c = 0.701373$ and $\epsilon_5 = 1.16616$ [c.f. Eq. \( (5) \)]. Moreover, the critical point (determined numerically) is at $\epsilon_c = 0.933$. The system is thus always initialized in the dark phase, $\epsilon_i = 0.5 < \epsilon_c$. We recall the nomenclature we adopted: the dark phase refers to the first branch, where $\epsilon < \epsilon_c$, and the bright phase refers to the upper branch, where $\epsilon > \epsilon_c$. Moreover, the region between $[\epsilon_c, \epsilon_5]$ is referred to as the bistability region.

All simulations are done by expressing the relevant operators in the Fock basis, using a sufficiently large number of basis states $n_{\text{max}}$ to ensure convergence. Manipulations involving the Liouvillian \( \mathcal{L} \) are then carried out using standard vectorization methods [80]. The quench protocol we investigated can be summarized as follows:

1. The system is initialized in the steady-state of the Liouvillian $\mathcal{L}_i$, associated with $\epsilon = \epsilon_i$. That is, $\rho_i$ is the solution of $\mathcal{L}(\rho_i) = 0$.
2. At time $t = 0$ we abruptly change the pump amplitude to a certain value $\epsilon_f$, which defines a new Liouvillian $\mathcal{L}_f$.
3. For $t > 0$ the state evolution is governed by the final Liouvillian according to $\rho_t = e^{\mathcal{L}_f t} \rho_i$,

which is the formal solution of \( (3) \). $\rho_i$ is computed numerically by discretizing the evolution in small time steps, usually $\Delta t = 0.2$. 


4. At each time step we compute \( \langle a \rangle \) and \( \langle a\,^\dagger a \rangle \). From the former we find \( \alpha = \langle a \rangle / N \) and from the latter we obtain the net entropy flux rate (22), \( \Phi = 2k \langle a\,^\dagger a \rangle \).

5. The Husimi \( Q \)-function is computed numerically at each time step, by constructing approximate coherent states,

\[
|\mu\rangle = e^{-|\mu|^2/2} \sum_{n=0}^{n_{\text{max}}} \frac{\mu^n}{n!} |n\rangle .
\]  

We compute \( Q(\mu, \bar{\mu}) \) for a sufficiently fine grid of points \((\mu, \bar{\mu})\) in the complex plane.

6. The entropy production rates \( \Pi_I \) and \( \Pi_{II} \) in Eqs. (21) and (25) are computed numerically, by integrating the corresponding functions over the complex plane grid. Derivatives of the Husimi function can be computed efficiently using Bargmann state [68], as detailed in Appendix B.

These simulations are computationally costly and the cost increases significantly with \( N \). The reason is twofold. First, larger values of \( N \) require a larger number of basis elements \( n_{\text{max}} \) to ensure convergence. Second, larger \( N \) also requires larger grids in the complex plane, which significantly increases the cost of the numerical integrations involved in computing \( \Pi_I \) and \( \Pi_{II} \).

We begin in Sec. IV B by showing that, in general, the dynamics is not Gaussian. This is interesting, since in the thermodynamic limit this would be the case for the NESS. However, due to the highly non-adiabatic nature of the quench, the intermediate states will in general deviate significantly from Gaussianity. Next we turn to the properties of the entropy production rate and flux rate. In Sec. IV C we study the entropy flux rate \( \Phi \) in Eq. (23) and in Sec. IV C we study the component \( \Pi_I \) [of \( \Pi_I \), Eq. (24)], and the unitary contribution \( \Pi_{II} \) in (25).

**B. Gaussianity of the state during the dynamics**

We can quantify the degree of non-Gaussianity in \( \rho_t \) by comparing it with a corresponding Gaussian state \( \rho_t^G \) having the same first and second moments. Following [81], we do this using the quantum Kullback-Leibler divergence

\[
\mathcal{G}(\rho_t) = D(\rho_t||\rho_t^G) = \text{tr} \left\{ \rho_t \ln \rho_t - \rho_t^G \ln \rho_t^G \right\} .
\]  

The state is Gaussian if \( \mathcal{G} = 0 \) and non-Gaussian otherwise. The results for \( \mathcal{G}(\rho_t) \), for the quenches laid out in Fig. 2, are presented in Fig. 4.

The quench in figure 4(a) goes to \( \epsilon_f = 0.6 < \epsilon_- \) and thus remains in the dark phase, outside the bistability region. We observe that most evolutions remain nearly Gaussian, except \( N = 1 \). This is consistent with the fact that Gaussianity, even in the NESS, is only expected in the thermodynamic limit. In figure 4(b) the quench goes to \( \epsilon_f = 0.8 \). Again, it is still in the dark phase, but now it lies inside the bistability region and below the critical point; that is, \( \epsilon_- < \epsilon_f < \epsilon_+ \). Now we observe that \( N = 5 \) is not even close to being Gaussian, but one does roughly find Gaussianity for \( N = 10 \) onward.

The real action starts in Fig. 4(c), where \( \epsilon_f = \epsilon_- \). The corresponding final NESS will be roughly a mixture of the dark and bright phases. We observe that the dynamics is markedly non-Gaussian for any value of \( N \). But for larger values of \( N \) one still finds that there is a tendency toward becoming Gaussian.

The situation changes completely in figure 4(d), where we show results for \( \epsilon_f = 1.1 \), thus representing a quench from the dark to the bright phase, but within the bistability region, \( \epsilon_- < \epsilon_f < \epsilon_+ \). (see Fig. 2). In this case we find that \( \mathcal{G}(\rho_t) \) increases with \( N \). This indicates that, in this case, a Gaussian limit would never be reached, no matter the values of \( N \) used. It therefore represents the most dramatic example, over all quenches, of a markedly non-Gaussian dynamics. A similar behavior is observed in Fig 4(e), where \( \epsilon_f = \epsilon_+ \). Unlike (d), however, we now see that \( \mathcal{G}(\rho_t) \) tends to fall for long times. This is related to the time-scales of the problem, which are much slower in Fig. 4(d).

Finally, in figure 4(f) we show results for \( \epsilon_f = 1.3 > \epsilon_+ \). It is found that, for intermediate times, the state is non-Gaussian for any size \( N \), but quickly tends back toward zero (notice the different horizontal scale in comparison with (d) and (e)).

The above results therefore clearly show that, during the quench dynamics, the state will in general be highly non-Gaussian. This justifies the use of the full numerical algorithm described above, in contrast to, say, Gaussianization techniques. It also highlights one of the advantages of using
Figure 5. \( \Phi/N \), Eq. (22), as a function of time. The dashed black lines correspond to the NESS values at \( \epsilon_i \) and \( \epsilon_f \) (in the thermodynamic limit). Other details are as in Fig. 4.

the Husimi function, which is non-negative for any quantum state (in contrast, for instance, with the Wigner function which can become negative).

C. Entropy flux rate dynamics

Fig. 5 summarizes the results for the entropy flux rate (22). This quantity is dominated by the first term, which is extensive in \( N \). The dashed black lines represents the values of \( \Phi_{\text{NESS}} \) computed from the exact solution (6). In Figs. 5(c) and 5(d), the MFA solution, Eq. (5), is represented by red dash-dotted lines, for comparison.

In Figs. 5(a) and 5(b) the system relaxes to the exact NESS after a relatively short transient (when contrasted to (d)-(f)). This is related to the fact that both the initial and final NESSs belong to the same manifold of dark-phase states. The behaviour of the flux is in agreement with the non-Gaussianity in Fig. 4. In Fig. 5(c), where the quench goes to the critical point \( \epsilon_i \), the system tends relaxes to the NESS predicted by the MFA solution in Eq. (5) when \( N \rightarrow \infty \). Recall that this quench was the first to present a non-Gaussian dynamics.

In Fig. 5(d), where the quench is toward the bright phase, we begin to observe a non-trivial dependency on \( N \) (all plots refer to \( \Phi/N \)). Recall that, in this case, the degree of non-Gaussianity increases with \( N \). One also notices that in this case the system does not relax to the exact NESS, nor to the MFA solution, even for the significantly larger simulation times (compare the horizontal axis with the other curves). Given a sufficient amount of time, the system would likely relax. However, this requires significantly longer simulations, which are outside the numerical capabilities of our code.

Finally, in Figs. 5(e) and (f), one observes a much more orderly transition between the two NESSs, as depicted by the dashed black lines. The transient time scales also tend to be relatively \( N \)-independent, except for \( N = 1 \).

In Fig. 6 we present the plots for the quantum entropy flux \( \Phi_q \) in Eq. (23). This quantity behaves similarly to \( \Pi_d \), which will be discussed in the next section. Thus, we shall comment more thoroughly on it below. At this point, we just call attention to the overall vertical scale of \( \Phi_q \), in comparison with \( \Phi \) in Eq. (5). The latter is normalized by \( N \). Notwithstanding, for certain quenches one finds that \( \Phi_q \) can reach values which are comparable in magnitude to \( \Phi_{\text{ext}} \). This is particularly true for \( \epsilon_f > \epsilon_c \) (Figs. 6(d)-(f)).

D. Quantum entropy production dynamics

In this section we present and compare the dynamics of the quantum entropy production rate due to dissipation, \( \Pi_d \) (Fig. 7), and due to the unitary contribution, \( \Pi_{Ud} \) (in Fig. 8). We begin by comparing the two quantities in the cases of (a) \( \epsilon_f = 0.6 \), (b) \( \epsilon_f = 0.8 \) and (c) \( \epsilon_f = \epsilon_c \). In these three cases, we observe that \( \Pi_d \) and \( \Pi_{Ud} \) are of the same order of magnitude. Physically, this means that irreversibility due to dissipation is comparable to the one introduced by the coarse-graining. We call attention to a non-trivial \( N \)-dependency in \( \Pi_d \) in Fig. 7(c), where the curve for \( N = 5 \) is actually above \( N = 1 \), unlike all other cases. It is not clear to us, precisely why this happens.

Conversely, for the quenches to (d) \( \epsilon_f = 1.1 \), (e) \( \epsilon_f = \epsilon_c \)
and (f) $\epsilon_f = 1.3$, we find that $\Pi_d \gg \Pi_{Q}\epsilon$; that is, dissipation becomes the main source of irreversibility. In all these cases both $\Pi_d$ and $\Pi_{Q}\epsilon$ present an $N$-dependent transient, peak at some instant of time and then eventually relaxing back to the NESS value. Notice how the maximal values reached by $\Pi_d$ are significantly higher than those of the NESS. This allows one to conclude that the transition from a dark to a bright NESS is accompanied by a significant production of entropy. Please note also the horizontal scales, showing that the relaxation times in Figs. 7(d) and 8(d) are significantly larger. Conversely, in Figs. 7(f) and 8(f), the relaxation times reduce significantly. Notwithstanding, the maximum entropy production rate is still very high.

These results show that the thermodynamic response of the system depends sensibly on the final pump value $\epsilon_f$. When the quench is to the same phase we find that $\Pi_d \approx \Pi_{Q}\epsilon$, while when we abruptly change the phase $\Pi_d \gg \Pi_{Q}\epsilon$: in the former, the contributions of the dissipation and uncertainty introduced by the coarse-graining are comparable and in the later the dissipation dominates over the measurement procedure. When the system is quenched toward the bright phase, one finds a strong dependency on $N$, together with a peak of the entropy production rates before the relaxation to the NESS. Whether or not $\epsilon_f$ lies inside the bistability region, plays an essential role in the time-scales involved, which become significantly slower due to the metastable character of this region.

V. CONCLUSION AND OUTLOOK

Entropy production is the key concept characterizing systems out of equilibrium. And much is still unknown about how it behaves as a system undergoes a non-equilibrium transition. In this paper we have employed a phase-space formalism to characterize the entropy production of the Kerr bistability model, a prototypical model of driven-dissipative quantum optical systems, presenting a discontinuous transition. This paper complements [59], which studied the NESS of this model. Here our focus was on dynamical aspects of the entropy production in a sudden quench scenario.

We showed that, in general, the state during the dynamics is highly non-Gaussian. By analyzing the quenches to different representative configurations, we have found that the thermodynamic response of the system presents at least 3 markedly different behaviours. For quenches within the same phase, the unitary and dissipative contributions to the entropy production are of the same order of magnitude and the time-scales involved are all relatively short and $N$-independent. For quenches from the dark to the bright phase, but still inside the bistability region, we find that $\Pi_d \gg \Pi_{Q}\epsilon$ and the time-scales become significantly longer. Finally, for quenches from the dark to the bright phase, but outside the bistability region, one still has $\Pi_d \gg \Pi_{Q}\epsilon$, but now the time-scales become once again relatively short and $N$-independent.

By gauging how far the system is from equilibrium, together with the relative contribution from unitary and dissipative dynamics, this study has therefore helped shed light on the intricate interplay between criticality and dissipation in non-equilibrium transitions. It also serves to tighten the connection
between quantum optical models and statistical mechanics.

Finally, we believe that these results also show the flexibility of the Wehrl entropy production framework in dealing with driven-dissipative quantum optical systems far from equilibrium. In future research, we plan to consider infinitesimal quenches, as well as models presenting continuous transitions. From a computational point of view, these studies would benefit tremendously from more efficient methods of computing the Husimi \( Q \)-function and the associated phase space quantities. This would be an absolute necessity if one is to extend these studies, for instance, to multimode systems such as the Dickey model or the optical parametric oscillators.

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Appendix A: Heterodyne measurements

In order to clarify the physical content of the Wehrl entropy, Eq. (10), we briefly review here the interpretation of the Husimi \( Q \)-function as the probability distribution of the outcomes of a heterodyne measurement. We start by considering a generalized measurement, described by the continuous set of Kraus operators

\[ M_\mu = \frac{1}{\sqrt{\pi}} \mu |\mu\rangle \langle \mu |. \]  

(A1)

This set is properly normalized, which is a consequence of the (over)completeness of the coherent states basis,

\[ \int d^2 \mu \, M^\dagger_\mu M_\mu = \int d^2 \mu \, \frac{\mu |\mu\rangle \langle \mu |}{\pi} = 1. \]  

(A2)

Thus, the probability of obtaining outcome \( \mu \) will be given by,

\[ p_\mu = \text{tr} \left( M_\mu \rho M^\dagger_\mu \right) = \frac{1}{\pi} |\langle \mu | \rho | \mu \rangle |^2 = Q(\mu, \bar{\mu}). \]  

(A3)

which is precisely the Husimi \( Q \)-function. The Wehrl entropy (10) can therefore be interpreted in an operational sense, as the entropy associated with the outcome probability distribution of an heterodyne measurement. This highlights its coarse-grained character, as it also encompasses the additional noise introduced by the measurement.

Appendix B: Computation of \( \Pi_J \) and \( \Pi_{\mu\mu} \) from Bargmann state

To compute \( \Pi_J \) and \( \Pi_{\mu\mu} \) numerically, we make use of the Bargmann state, defined as,

\[ |\mu\rangle = \exp \left( \frac{|\mu|^2}{2} \right) |\mu\rangle. \]  

(B1)

where \( |\mu\rangle \) is the usual coherent state. This state has the useful property that the action of the derivatives with respect \( \mu \) and \( \bar{\mu} \) are mapped into the action of the creation/annihilation operators as,

\[ \partial_\mu |\mu\rangle = a^\dagger |\mu\rangle \]  

\[ \partial_\bar{\mu} |\mu\rangle = \langle \mu | a. \]  

(B2)

We can use this to write the phase-space currents (12) in a computationally more convenient way. First we express the \( Q \)-function as

\[ Q(\mu, \bar{\mu}) = \frac{\exp \left[ -|\mu|^2 \right]}{\pi} (\mu | \rho | \mu). \]  

(B3)

Using Eq. (B2) to compute \( \partial_\mu Q \), we then find

\[ \partial_\mu Q = -\mu Q + \frac{1}{\pi} (\mu | a \rho | \mu) \]  

(B4)

where, in the second line, we already recast the state in terms of usual coherent states. We then find that

\[ J(Q) = -\frac{\kappa}{\pi} (\mu | a \rho | \mu) \]  

(B5)

which allows us to write \( \Pi_J \) in Eq. (21) as,

\[ \Pi_J = \frac{2\kappa}{\pi} \int d^2 \mu \frac{(|\mu| a \rho | \mu)|^2}{\langle \mu | a \rho | \mu \rangle}. \]  

(B6)

Now, we turn to the computation of Eq. (25). First, we note it can be rewritten as,

\[ \Pi_{\mu\mu} = U \int d^2 \mu \, \text{Im} \left\{ \frac{\partial^2 (\partial_\mu Q)^2}{Q} \right\}. \]  

(B7)

Hence, using Eq. (B4) and the definition of \( Q(\mu, \bar{\mu}) \), we obtain

\[ \Pi_{\mu\mu} = U \int d^2 \mu \, \text{Im} \left\{ \frac{|\mu|^2}{\pi} \left( (\mu | a \rho | \mu)^2 + 2\bar{\mu} \langle \mu | a \rho | \mu \rangle \bar{\mu} + \frac{\mu | a \rho | \mu |^2}{(\bar{\mu} | a \rho | \mu)} \right) \right\} \]  

During the simulations, the quantities \( \langle \mu | a \rho | \mu \rangle \) and \( \langle \mu | a \rho | \mu \rangle \) are directly computed from \( \rho_t \).

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