Scattering Approach to Counting Statistics in Quantum Pumps

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We consider the Fermi gas in a non-equilibrium state obtained by applying an arbitrary time-dependent potential to the Fermi gas in the ground state. We present a general method that gives the quantum statistics of any single-particle quantity, such as the charge, total energy or momentum, in this non-equilibrium state. We show that the quantum statistics may be found from the solution of a matrix Riemann-Hilbert problem. We use the method to study how the finite measuring time modifies the distribution of the charge transferred through a biased quantum point contact.

I. INTRODUCTION

From a theoretical point of view, a quantum pump at zero temperature simply excites the Fermi gas from the ground state into a non-equilibrium state. The exact many particle state obtained after a few pumping cycles depends on the applied time-dependent potential. When the external potential changes slowly (this is commonly referred to as “adiabaticity” in the context of quantum pumps) the final non-equilibrium state is completely determined by the evolution of the scattering matrix at the Fermi energy $S(t)$. Therefore, it should be possible to obtain the distribution functions of various operators (like total energy, total momentum, total charge in one of the leads, etc) as functionals of $S(t)$. This turns out to be a non-trivial problem, which we address in the present paper.

Historically, the first quantity studied in the above setting was the total energy absorbed by the Fermi gas as a result of a sudden introduction of a single $s$-scatterer. The distribution function for absorbed energy shows a power-like divergence at low energies – the so called Fermi edge singularity (see e.g. Ref. 2 for details). The distribution of the absorbed energy for a more general situation, when the strength of the $s$-wave scatterer changes arbitrarily in time, was considered in Ref. 3. In all these cases the scattering matrix $S(t)$ commutes with itself at different times $S(t)S(t') = S(t')S(t)$. The non-commutative case, surprisingly, is much more difficult. The first non-perturbative treatment of the energy distribution for this case mapped the problem into a one-dimensional chiral determinant, and seemed not to be easily adapted for the computation of other quantities. In this paper we present an alternative approach which is much simpler and also readily applicable to distribution functions of any single particle operator.

We illustrate our method using the distribution of the charge transmitted through a contact, which has attracted a lot of attention recently. The second moment of the charge distribution (i.e. the shot noise power) for a biased quantum point contact was considered theoretically by Khlus and Lesovik, and the full counting statistics was obtained by Lesovik and Levitov (LL). The charge counting statistics for quantum pumps and quantum point contact under some specially chosen time-dependent bias were also considered.

The experimental measurements of the shot noise are now well established. The measurement of the higher cumulants, on the other hand, is still an unresolved problem actively discussed in the literature.

The question of what can really be measured in an experiment, and the closely related question of the back influence of a detector onto the measured system are still widely discussed in the literature. We use conceptually the simplest measurement scheme when the total charge $Q$ in the left lead is measured twice. Firstly before the pumping is started at time $t = 0$, and secondly after the pumping is finished at $t = t_f$. At both times the leads are decoupled, so $Q$ is a conserved quantity. The first measurement projects onto the state with a fixed number of electrons in the left lead. After that the pumping excites this state into a non-equilibrium state that is no longer an eigenstate of the charge operator. The statistics of the transmitted charge is given by the probability to find a particular number of electrons in the left lead during the second measurement.

Alternative approaches discussed in the literature include coupling the system to a spin and measuring the spin precession angle or measuring the statistics of the photon emission or measuring total dissipated energy. See also Ref. 24 for a general framework for measuring the full counting statistics of a quantum mechanical variable.

Our main result is the closed expression for the full counting statistics in terms of the scattering matrix $S(t)$ and the solution of an auxiliary matrix Riemann-Hilbert (RH) problem. We assume that $S(t)$ changes adiabatically, but otherwise do not restrict its time dependence. To demonstrate the power of the method we analyze in some detail the case of a biased quantum contact. In the limit of infinitely long pumping time the charge distribution is binomial. We compute the corrections to the binomial distribution when the pumping time is long but finite and discuss the physical meaning of these corrections.
The rest of the paper is organized as follows. In section II we relate the counting statistics to a determinant of a certain operator; in section III we compute this determinant using a solution of a matrix RH problem; in section IV we apply the general method to a biased quantum point contact and rederive LL’s answer. In section V we asymptotically solve the relevant RH problem and obtain corrections to LL’s answer due to finite measuring time. Some technical results are relegated to the appendix.

II. DISTRIBUTION FUNCTION OF TRANSMITTED CHARGE

A. Notations

The free Hamiltonian $H_0$ describes non-interacting Fermi gas in two uncoupled leads. Let the single particle states $\psi_{ne}$ be the eigenvalues of the Hamiltonian $H_0$ with energy $\epsilon$, where $n = 1 \ldots N$ labels energy degenerate states in both the left and the right lead. In the simplest case when the leads are one dimensional $n = 1$ for the left lead and $n = 2$ for the right lead. The annihilation operator in the state $\psi_{ne}$ is denoted by $a_{ne}$. We assemble $N$ such operators in the $N$-dimensional column $a_{\epsilon}$

$$
a_{\epsilon} = \begin{pmatrix} a_{1\epsilon} \\ \vdots \\ a_{N\epsilon} \end{pmatrix}
$$

and use the notation $a_{\epsilon}^+ \equiv (a_{1\epsilon}^+, \ldots, a_{N\epsilon}^+)$ for the $N$-dimensional row of the creation operators. This gives for the free Hamiltonian

$$H_0 = \sum_{\epsilon} \epsilon a_{\epsilon}^+ a_{\epsilon}.$$ 

The time-dependent potential that couples the leads is described by an $N \times N$ matrix $M(t, \epsilon, \epsilon')$, so the full Hamiltonian has the form

$$H(t) = H_0 + \sum_{\epsilon, \epsilon'} a_{\epsilon}^+ M(t, \epsilon, \epsilon') a_{\epsilon'}.$$ 

Analogously, the total charge operator $Q$ in the left lead is given by

$$Q = \sum_{\epsilon} a_{\epsilon}^+ L a_{\epsilon},$$

where $L$ is the projector on the states in the left lead, and charge is measured in units of electron’s charge. For two one-dimensional leads $L = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$

B. The Characteristic Function

We want to compute the probability $P(q)$ that the charge in the left lead changes by $q$ during the time $t_f$.

The characteristic function $\chi(\lambda)$ is defined as

$$\chi(\lambda) = \sum_q P(q) e^{-i\lambda q}.$$ (3)

At zero bias the many particle ground state $|0\rangle$ corresponds to filling the states below the Fermi energy in both leads

$$|0\rangle = \prod_{\epsilon < 0} \prod_n (a_{ne}^+ |\rangle),$$ (4)

where $|\rangle$ is the true vacuum state with no electrons. We assume that $\epsilon$ takes some discrete values, so the total number of electrons is finite, and also set the Fermi energy to zero. At zero temperature we have

$$\chi(\lambda) = \langle 0 | U^+ e^{-i\lambda Q} U e^{i\lambda Q} |0\rangle.$$ 

where $U = U(t_f)$ is the many particle evolution operator which is the solution of the Schrödinger equation

$$i \frac{dU}{dt} = H(t) U(t), \quad U(0) = 1$$

Since the ground state $|0\rangle$ is the eigenstate of the operator $Q$, the characteristic function $\chi$ can be written as

$$\chi = \langle 0 | U^+ e^{-i\lambda Q} U |0\rangle \langle 0 | e^{i\lambda Q} |0\rangle.$$ (5)

C. One Particle Scattering Problem

To proceed, we separate the single-particle scattering problem from the computation of the expectation value over the ground state of the Fermi-gas. Since the Hamiltonian $H(t)$ is quadratic, we have

$$U a_{\epsilon}^+ = \sum_{\epsilon'} \sigma(\epsilon, \epsilon') a_{\epsilon'}^+ |\rangle,$$ (6)

where $\sigma(\epsilon, \epsilon')$ is some $N \times N$ matrix that fully describes the single particle time-dependent scattering problem. We introduce the operator $\tilde{\sigma}$ with matrix elements $\tilde{\sigma}(\epsilon, \epsilon')$ that acts on the full space of all possible states (i.e. in both channel space and energy-space). This operator is unitary $\tilde{\sigma}^{-1} = \tilde{\sigma}^\dagger$. We show that $\sigma$ can be related to the scattering matrix $S(t, \epsilon)$ on the instant value of the potential $M(t)$ for electron with energy $\epsilon$.

Consider the scattering states with energy $\epsilon$ propagating towards the contact $\psi_{\text{in}}$ and out of it $\psi_{\text{out}}$. Let us label them and fix their phases in such a way that far from the contact $\psi_{\text{ne}} = \psi_{\text{in}}^{\text{in}} + \psi_{\text{out}}^{\text{out}}$, where $\psi_{\text{ne}}$ is the eigenfunction for decoupled electrodes introduced previously. In the presence of the potential $M(t)$ the eigen space corresponding to the energy $\epsilon$ is still $N$-times degenerate. Far from the contact an eigen function has asymptotic form $\sum_a a_n \psi_{\text{in}} + b_n \psi_{\text{out}}$. The scattering matrix $S(t, \epsilon)$ relates the amplitudes of incoming and outgoing states.
Using (8) one can show (see e.g. [6]) that 
\[ \nu \]
leads to
\[ S = \begin{pmatrix} B & A \\ A & -B^* + \frac{i}{\pi} \end{pmatrix} \]  
(7)
where \( A \) is the transmission amplitude and \( B \) is the reflection amplitude for the electrons incident from the left. Eq (7) implies that the scattering potential is real (i.e. there is no magnetic field) and uses this fact to relate the scattering amplitudes for the electrons coming from the right and from the left. For example, consider the two half-infinite one-dimensional conductors occupying left and right half-line and separated by an infinite potential wall of width \( l \). Our choice of scattering states dictates \( A = 0 \) and \( B = 1 \). When the conductors are connected and potential wall is completely removed we get \( B = 0 \) and \( A = e^{ilp/\hbar} \), where \( p \) is the momentum of electrons with energy \( \epsilon \).

The choice of the scattering states is not unique. Another (more common) approach is to choose the scattering states in such a way that the scattering matrix for the fully opened channel is one. Denoting the scattering matrix in this representation by \( \hat{S} \) we see that \( \hat{S} = S S_{\text{open}}^{-1} \) where \( S_{\text{open}} \) is the scattering matrix for the fully open channel in the representation adopted in this paper.

The adiabaticity condition (common in describing “quantum pumps”) means that \( S(\tau, E) \) changes slowly both as a function of time and energy
\[ \hbar \frac{\partial S^{-1}}{\partial \tau} \frac{\partial S}{\partial E} \ll 1. \]  
(8)
Using (8) one can show (see e.g. [6]) that
\[ \hat{\sigma} = \hat{S} \]  
(9)
when \( \sigma \) acts on states near the Fermi energy. Here the operator \( \hat{S} \) is the Fourier transform of the scattering matrix, i.e. it has the matrix elements
\[ \hat{S}_{\epsilon \epsilon'} = \frac{1}{2\pi \nu} \int dt S(t)e^{i(\epsilon - \epsilon')t}, \]  
(10)
where \( \nu \) is the density of states near the Fermi energy per single channel. In the rest of the paper we often consider operators acting on both energy and channel indexes. For example, in addition to \( \hat{\sigma} \) and \( \hat{S} \) introduced above we use the operator \( \hat{L} \) that has matrix elements \( L_{\epsilon \epsilon'} \). We use hats to distinguish these operators from the corresponding matrix-valued functions. The notation “Tr” is used for trace in both channel and energy space, while the symbol “tr” is reserved to denote the trace in the \( N \times N \) channel space only.

D. Averaging Over the Fermi Distribution

The remaining task is to compute the average with respect to the Fermi ground state \( \psi \). We combine (6) with (2) and (3) and introduce the operator
\[ \hat{R} = \hat{\sigma} e^{-i\lambda \hat{L} \hat{\sigma}} = e^{-i\lambda \hat{\sigma}^* \hat{L}} \]  
(11)
to obtain
\[ U^+ e^{-i\lambda \hat{Q} U} |\psi\rangle = \sum_{\epsilon'} R(\epsilon, \epsilon') a^+_{\epsilon'} |\psi\rangle. \]

The Fermi distribution at zero temperature can be viewed as the diagonal operator \( \hat{f} \) with matrix elements \( f(\epsilon, \epsilon') = \delta_{\epsilon\epsilon'} \theta(-\epsilon) \). In the block notation that separates the states with positive and negative energies we have
\[ \hat{f} = \begin{pmatrix} \hat{1} & 0 \\ 0 & \hat{0} \end{pmatrix}. \]

In the same block notation the operator \( \hat{R} \) has the form
\[ \hat{R} = \begin{pmatrix} \hat{R}_{11} & \hat{R}_{12} \\ \hat{R}_{21} & \hat{R}_{22} \end{pmatrix}. \]  
(12)
Since the ground state \( \psi \) is simply a Slater determinant we have \( \langle 0 | U^+ e^{-i\lambda \hat{Q} U} |0\rangle = \det \hat{R}_{11} \) for the first term in (9). This expression is further simplified by noting that the operator \( \hat{1} - \hat{f} + \hat{f} \hat{R} \) has the same determinant as \( \hat{R}_{11} \) because in the block notation it takes the form
\[ \begin{pmatrix} \hat{R}_{11} & \hat{R}_{12} \\ 0 & \hat{1} \end{pmatrix}. \]  
(13)
The second term in (9) is simplified because \( \langle 0 | \rangle = \det (\hat{1} - \hat{f} + \hat{f} \hat{R}) e^{i\lambda \text{Tr} \hat{L} \hat{f}}. \)

In this way we obtain
\[ \chi = \det (\hat{1} - \hat{f} + \hat{f} \hat{R}) e^{i\lambda \text{Tr} \hat{L} \hat{f}} \]  
(14)
Eq. (14) combined with (11) and (9) gives
\[ \chi = \det (1 - \hat{f} + \hat{f} \hat{S}^{-1} e^{-i\lambda \hat{L} \hat{S}}) e^{i\lambda \text{Tr} \hat{L} \hat{f}}. \]  
(15)
In the time representation \( \hat{S} \) and \( \hat{L} \) are multiplications by the matrix-valued functions \( S(t) \) and \( L \) respectively and \( \hat{f} \) has matrix elements
\[ f(t, t') = \frac{i}{2\pi} \frac{1}{t - t' + i0}. \]  
(16)
The representation of counting statistics for a biased quantum point contact through the determinant of type (15) was obtained in Ref. 12 and later adopted for parametric pumping13 and re-derived in different ways25. Our derivation is close in spirit to Ref. 12 and is probably the simplest. In addition it leads to a simple regularisation procedure which we discuss below.
E. Biased Quantum Pump

Finally, we briefly mention that in the case when the left lead is biased with respect to the right lead by a time-dependent potential \( V(t) \), a gauge transformation can be performed on all states in the left electrode
\[
a_k \to e^{iL \int_0^\tau V(\tau) d\tau} a_k.
\]
In the new gauge the ground state is again given by (4). The distribution function for this case can therefore be recovered from (15) by substituting the gauge transformed scattering matrix
\[
e^{iL \int_0^\tau V(\tau) d\tau} S e^{-iL \int_0^\tau V(\tau) d\tau}
\]
instead of \( S \).

F. Regularisation

Eq. (19) is only valid for the states near the Fermi surface. Since the determinant in (14) contains a non-vanishing contribution from the states deep below the Fermi surface, the transition from (14) to (15) is not quite correct. In particular, the determinant in (15) is formally infinite and so requires regularisation while Eq. (14) gives a finite answer. To find out the correct regulator we use the decomposition
\[
\ln \chi = \text{Tr} \left( \ln (\hat{1} - \hat{f} + \hat{f} \hat{R}) - \hat{f} \ln \hat{R} \right) + \text{Tr}(\hat{f} \ln \hat{R}) + i \lambda \text{Tr} \hat{L} \hat{f}.
\]
In the first term in (18) both the contributions from the states deep below the Fermi surface (when \( \hat{f} = 1 \)) and the states high above the Fermi surface (where \( \hat{f} = 0 \)) are zero, so Eq. (19) can be used. The two last terms in (18) can be re-arranged to give
\[
- i \lambda \text{Tr} \left( [\hat{\sigma}, \hat{f} \hat{L}] \right) \equiv - i \lambda Q_{av}
\]
where the square brackets denote the commutator. Again the contribution from the states away from the Fermi energy vanishes and we obtain
\[
Q_{av} = \text{Tr} \left( [\hat{S}, \hat{f}] \hat{S}^{-1} \hat{L} \right) = \frac{i}{2 \pi} \int \text{tr} \left( \frac{dS}{dt} S^{-1} L \right) dt
\]
To obtain the last equality in (19) we compute the trace in the time representation, where all operators, except \( \hat{f} \) are diagonal, and matrix elements of \( \hat{f} \) are given by (16). Comparing (19) with Brouwer’s formula\(^{25}\) for the average charge transmitted through a quantum pump, we identify \( Q_{av} \) with the average value of the total transmitted charge.

Combining these results together we get
\[
\ln \chi = \text{Tr} \left( \ln (\hat{1} - \hat{f} + \hat{f} \hat{R}) - \hat{f} \ln \hat{R} \right) - i \lambda Q_{av}.
\]
where in the time representation \( \hat{R} \) is multiplication by the matrix valued function
\[
R(t) = S^{-1}(t)e^{-i\lambda L} S(t).
\]
The first term in Eq. (20) contains the information about the second and higher moments of the charge distribution.

III. RELATION TO A RIEMANN-HILBERT PROBLEM

In this section we relate the operator trace
\[
Tr \equiv \text{Tr} \left( \ln(\hat{1} - \hat{f} + \hat{f} \hat{R}) - \hat{f} \ln \hat{R} \right)
\]
from Eq. (20) to a solution of an auxiliary \( N \times N \) matrix Riemann-Hilbert problem.

A. General case

We use
\[
Tr = \int_0^\lambda d\lambda \text{Tr} \left( (1 - \hat{f} + \hat{f} \hat{R})^{-1} \hat{f} - \hat{f} \hat{R}^{-1} \right) \frac{dR}{d\lambda}.
\]
To invert \( 1 - \hat{f} + \hat{f} \hat{R} \) we need to solve the following auxiliary RH problem. Find a matrix-valued function \( Y(t) \) of the complex variable \( t \) such that \( Y \)
- is analytic on the complement of the cut \((0, t_f)\)
- obeys
\[
Y_-(t)Y_+^{-1}(t) = R(t)e^{i\lambda L} \text{ when } t \in (0, t_f),
\]
where \( Y_{\pm}(t) = Y(t \pm i0) \)
- tends to 1 at infinity
\[
Y \to 1 \text{ when } |t| \to \infty.
\]
Using (16) and analyticity of \( Y_{\pm} \) we observe that \( Y_{\pm} \) has the following properties
\[
\hat{f} \hat{Y}_- \hat{f} = \hat{f} \hat{Y}_-,
\hat{f} \hat{Y}_+ \hat{f} = \hat{Y}_+ \hat{f}.
\]
Using the above and substituting \( \hat{R} = \hat{Y}_- \hat{Y}_+^{-1} e^{-i\lambda L} \) one can verify
\[
(1 - \hat{f} + \hat{f} \hat{R})^{-1} = e^{i\lambda L} \hat{Y}_+ \left( (1 - \hat{f}) \hat{Y}_+^{-1} e^{-i\lambda L} + \hat{f} \hat{Y}_- \right)
\]
which after substitution in (24) gives
\[
Tr = \int_0^\lambda d\lambda \frac{d\lambda}{2\pi} \int dt \text{Tr} \left( e^{i\lambda L} \frac{dY_+}{dt} Y_+^{-1} e^{-i\lambda L} S^{-1} L S \right)
\]
Combining (25) with (20) and (18) we obtain our main result
\[
\ln \chi = \int_0^\lambda \frac{d\lambda}{2\pi} \int dt \text{Tr} \left\{ d \left( S e^{i\lambda L} Y_+ \right) (S e^{i\lambda L} Y_+)^{-1} L \right\}
\]
where \( Y \) is analytic in the complement of the cut \((0, t_f)\), obeys \( Y_0^{-1} = S^{-1} e^{-i\alpha t} S e^{i\beta t} \) along the cut and goes to 1 at infinity.

### B. Charge Transfer per Cycle for Periodic Pumping

Eq. \ref{eq:Y} can also be used when the scattering matrix is periodic \( S(t) = S(t + T) \) and we are interested in the limit of the large number of cycles (i.e. \( t_f/T \gg 1 \)). The RH problem in this case can be somewhat simplified by rolling the infinite real axis into a unit circle with the mapping \( z = e^{2\pi i t/T} \). After that the RH problem \ref{eq:RH} reduces to finding two functions \( Y_\pm(z) \) such that

- \( Y_+ \) is analytic for \(|z| < 1 \) and \( Y_- \) is analytic for \(|z| > 1 \)
- on the unit circle \(|z| = 1\)
  \[
  Y_- Y_+^{-1} = S^{-1}(z) e^{-i\lambda t} S(z) e^{i\lambda t} \tag{27}
  \]
- \( Y_- \to 1 \) when \(|z| \to \infty\)

The distribution of charge transmitted during time \( t_f \) in the limit \( t_f/T \gg 1 \) is now given by

\[
\ln \chi = \frac{t_f}{T} \int_0^\lambda \frac{d\lambda}{2\pi} \left\{ \frac{dz}{|z|=1} \sum_0^\alpha \frac{d}{dz} \left( \frac{d^2}{dz^2} \right) \left( \frac{d}{dz} \right) \right\} \left( S e^{i\lambda L} Y_+ \right)^{-1} L \tag{28}
\]

### IV. COUNTING STATISTICS FOR A BIASED QUANTUM POINT CONTACT

In this section we apply the technique introduced above to the quantum point contact described by the scattering matrix \ref{eq:S} and biased by the time-independent potential \( V \).

#### A. General formulas

After the gauge transformation \ref{eq:gauge}, the scattering matrix takes the form

\[
S(t) = e^{iLVT} S_0 e^{-iLVT}, \tag{29}
\]

where the time-independent matrix \ref{eq:S0} is denoted by \( S_0 \). The matrix \( S^{-1} e^{-i\lambda L} S e^{i\lambda L} \) can be represented as

\[
S^{-1} e^{-i\lambda L} S e^{i\lambda L} = e^{i\lambda L} R_0 e^{-i\lambda L} \tag{30}
\]

where \( R_0 = S_0^{-1} e^{-i\lambda L} S_0 e^{i\lambda L} \) is time-independent. It is convenient to decompose \( R_0 \) into a product of upper and lower-triangular matrices

\[
R_0 = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix} \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} , \tag{31}
\]

where

\[
\alpha = |A|^2 e^{i\lambda} + |B|^2 \tag{32}
\]

\[
\beta = \frac{A^* B (1 - e^{i\lambda})}{a} \tag{33}
\]

### B. Long time limit \( t_f V \gg 1 \)

The scattering matrix \ref{eq:S0} is periodic with period \( T = 2\pi/V \) and in the limit \( t_f V \gg 1 \) we can apply the results of section IIIA. The RH problem \ref{eq:RH} takes the form

\[
Y_- Y_+^{-1} = \begin{pmatrix} \alpha & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{a} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \beta \theta \tag{34}
\]

where \( \theta = e^{i\lambda V} \) and we used \ref{eq:alpha} and \ref{eq:beta}.

The solution of RH problem \ref{eq:Y} is obvious now

\[
Y_- = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & a \end{pmatrix} .
\]

Substituting \( Y_+ \) into \ref{eq:chi} we get

\[
\ln \chi(\lambda) = \frac{t_f V}{2\pi} \ln a \tag{35}
\]

with \( a \) given by \ref{eq:a}. The above formula is LL’s answer\textsuperscript{[11]} for the counting statistics of a biased quantum point contact. Thus we have verified that measuring charge in one lead after the leads are disconnected yields the same counting statistics (in the limit \( t_f V \gg 1 \)) as the measurement scheme based on a coupling to a spin, as suggested by LL.

### V. CORRECTIONS TO THE COUNTING STATISTICS DUE TO FINITE MEASURING TIME

#### A. General considerations

Eq. \ref{eq:chi} gives the leading term in the expansion of \( \ln \chi \) in \( t_f V \). The correction is proportional to \( \ln t_f V \), and to compute it we need to solve RH problem \ref{eq:RH}.

\[
Y_- Y_+^{-1} = \begin{pmatrix} 1 & 0 \\ \alpha e^{-i\lambda t} & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix} \begin{pmatrix} 1 & \beta e^{i\lambda t} \tag{36}
\]

This RH problem is no longer periodic, since we need to take into account explicitly the finite length of the cut. We are not aware of an analytical solution for RH problem \ref{eq:RH}. Instead, we find an approximate solution in the limit \( t_f V \gg 1 \). To do this we use the saddle-point method introduced for RH problems in Ref. \ref{eq:SPM}.
The idea is to modify RH problem 35 in such a way that the cuts become vertical and the oscillating factors $e^{±iVt}$ turn into decaying ones. To implement the plan, we construct an auxiliary matrix-valued function $W$ and look for the solution in the form $Y = Wξ$. It turns out that $ξ$ obeys another RH problem, where all cuts are vertical, and the jump functions decay along the cuts as $e^{-1/t|V|}$. As a result $ξ$ is close to 1 away from the branch points 0 and $t_f$, and $W$ yields the desired approximate solution to RH problem 35.

B. approximate solution of RH problem 35

It is convenient to introduce the function

$$\psi(z) = \exp \left[ \frac{\ln a}{2\pi i} \left( \frac{0}{1} \frac{a}{0} \right) \ln \frac{z}{-t_f} \right].$$

which has the same region of analyticity as $Y$, tends to 1 at infinity and obeys

$$\psi_+^{-1} - \psi_-^{-1} = \left( \frac{a}{0} \frac{1}{a} \right)$$

along the cut $(0, t_f)$.

The approximate solution $W(t)$ is constructed as analytic in the complement of three cuts. The first two cuts are along the two vertical lines $\text{Re} \ t = 0$ and $\text{Re} \ t = t_f$ and the third cut is along the real axis between 0 and $t_f$ (see Fig. 1). The cuts divide the complex plane into four regions: the half-plane $\text{Re} \ t < 0$ (region 1), the upper part of the strip $0 < \text{Re} \ t < t_f$ (region 2); the lower part of the same strip (region 3) and the half-plane $\text{Re} \ t > t_f$ (region 4). Denoting the value of the function $W$ inside the region $i$ as $W_i(t)$ we define $W$ in the following way:

$$W_1 = \psi, \quad W_2 = \left( \begin{array}{cc} -1 & \beta e^{iVt} \beta e^{iVt} \\
0 & 1 \end{array} \right) -1 \psi \\
W_3 = \psi, \quad W_4 = \left( \begin{array}{cc} 1 & 0 \alpha e^{-iVt} \alpha e^{-iVt} \\
0 & 1 \end{array} \right) \psi.$$

Comparing 37 and 39 with 35 we note the important property

$$W_+ - W_+^{-1} = Y - Y_+^{-1}$$

along the cut $(0, t_f)$.

We are now looking for the solution of the original RH problem 35 in the form $Y = Wξ$. Combining 35 with the definition of $W$ we conclude that $ξ$ is analytic in each of the three regions $\text{Re} \ t < 0$ (i.e. region 1), $0 < \text{Re} \ t < t_f$ (this is regions 2 and 3 combined together) and $\text{Re} \ t > t_f$ (i.e. region 4). Due to condition 38 the function $ξ$ does not have a cut between regions 2 and 3. The other conditions on the function $ξ$ come from the requirement that $Y$ is continuous on the vertical cuts of $W$. Using

the notation $ξ_i$ for the value of $ξ$ in the region $i$, this condition gives on the upper half of the cut $\text{Re} \ t = 0$

$$ξ_2ξ_2^{-1} = W_2^{-1}W_2 = \psi^{-1} \left( \begin{array}{cc} 1 & 0 \beta e^{iVt} \beta e^{iVt} \\
0 & 1 \end{array} \right) -1 \psi, \quad (39)$$

and on the lower half of this cut

$$ξ_3ξ_3^{-1} = W_3^{-1}W_3 = \psi^{-1} \left( \begin{array}{cc} 1 & 0 \alpha e^{-iVt} \alpha e^{-iVt} \\
0 & 1 \end{array} \right) \psi. \quad (40)$$

Importantly, the right hand sides of Eqs. 39, 40 are exponentially close to 1 when $t$ goes to infinity along the cuts. The difference is of the order $e^{-1/t|V|}$ and can be neglected when $|t| \gg 1/V$. The same happens along the other cut at $\text{Re} \ t = t_f$. Using this fact alone we prove (see appendix A) that the function $ξ(t)$ is close to 1 when $t$ is far away from 0 and $t_f$. More precisely, in the region

$$|t| \gg \frac{1}{V} \quad \text{and} \quad |t - t_f| \gg \frac{1}{V} \quad (41)$$

$ξ$ is close to 1, so $Y$ is close to $W$, as required. Explicitly, we obtain for $Y_+$

$$Y_+(t) = \begin{cases} \psi_+(t) & \text{when } t < 0 \\
\psi_+(t) & \text{when } 0 < t < t_f \\
\psi_+(t) & \text{when } t_f < t \end{cases} \quad (42)$$

where $ψ$ is given by 36. Eq. 42 is not valid in the vicinity of the end points 0 and $t_f$, where conditions 11 are not met. It turns out that we do not need the exact behavior of $Y_+$ in those regions to obtain the $\ln t_f V$ term in $\ln χ$.

C. Computation of the distribution function and interpretation of the result

The function $Y_+(t)$ enters $\ln χ$ through the combination $d\ln Y_+^{-1}$ (see Eq. 26). There are, therefore, two contribution to $\ln χ$. The first contribution comes from differentiating the factor $e^{iVt}$ in 42 and reproduces Eq. 64 which is linear in $t_f V$. The second contribution comes from differentiating $ψ$ in 42 and is proportional to $\ln t_f V$. Combining both contributions together
we obtain
\[ \ln \chi = \frac{t_f V}{2\pi} \ln a + \frac{\ln t_f V}{2\pi^2} \ln^2 a, \]  
(43)
where \( a \) is given by \[ (32) \] .

Finally, there are contributions from the regions \( t < 1/V \) and \(|t - t_f| < 1/V \) where the asymptotic solution constructed above is not valid. These contributions are of order \( \ln(\tau V) \), where \( \tau \) is the switching time. Assuming \( \tau \sim 1/V \) these extra logarithms can be neglected. Combining all the conditions specified above we find the validity region for Eq. 13
\[ \tau \sim 1/V \ll t_f, \]  
(44)

The characteristic function \[ (13) \] corresponds (with logarithmic accuracy) to the probability distribution
\[ P(k) = \sum_n p(n)B(n, k), \]  
(45)
which is a convolution of a Gaussian distribution for the number of attempts
\[ p(n) \propto e^{-\pi^2(n-t_f^2V)/(4\ln t_f V)} \]  
(46)
and the binomial distribution
\[ B(n, k) = \frac{n!}{k!(n-k)!}|A|^{2k}|B|^{2(n-k)}. \]  
(47)
for the number of successes. The mean number of attempts is given by \( t_f V/2\pi \) and the variance is logarithmic \( \sigma^2 \sim \ln t_f V \).

There are, therefore, two sources of shot noise in a quantum point contact. One source is the fluctuations in the number of incident electrons (number of attempts to go through the contact) and the other is the fluctuations in the number of reflections (number of failures). According to \[ (15) \] these two sources are statistically independent. The physical picture of the two statistically independent sources of quantum shot noise was suggested (but not proven) by LL. Note, that in order to confirm the statistical independence we needed to obtain the second term in Eq. 13 and analyze its dependence on the transmission amplitude \( A \).

VI. DISTRIBUTION FUNCTION OF THE DISSIPATED ENERGY

We mention without derivation that the characteristic function \( \chi_e(\lambda) = \int P(E)e^{-i\lambda E}dE \) for the distribution of energy dissipated during the pumping, is given by the expression
\[ \ln \chi_e(\lambda) = \frac{i}{2\pi} \int_0^\lambda d\lambda \int dt \text{tr} \left\{ \frac{dY_e}{dt} Y_e^{-1} R_e^{-1} \frac{dR_e}{d\lambda} \right\} - i\lambda E_{av}, \]  
(48)
which is very similar in form to Eq. 26. The analog of the Riemann-Hilbert problem \[ 24 \] is \( Y_e Y_e^{-1} = R_e \), where
\[ R_e(\tau) = S^{-1}(\tau)S(\tau + \lambda). \]  
(49)
The average dissipated energy 28
\[ E_{av} = \frac{1}{4\pi} \int_0^{t_f} tr \left( \frac{dS}{d\tau} S^{-1} \right)d\tau. \]  
(50)
appears analogously to \( Q_{av} \) during the regularisation of the determinant.

Similarly, the distribution function of the values of any single-particle operator is given by expression similar to Eq. 26, 13 with appropriate replacement of \( R \) and \( Q_{av} \).

VII. CONCLUSION

It is clear from the derivation, that the charge operator \( Q \) in equation 15 can be replaced with any single-particle operator. This will modify the counter-term 19, as well as the relation 21 between \( R \) and \( S \). Also Eq. 26 that relates the characteristic function with the solution of the RH problem is modified. On the other hand, the general form of the answer remains the same, i.e. the distribution function is still expressed as an integral over an expression containing the solution of a particular matrix RH problem.

The general method is used to compute the distribution of the charge transmitted through a biased quantum point contact in finite time. We confirm LL’s picture of the two sources of the charge fluctuation (see section V C) and prove that the transmission attempts are statistically independent from the reflections when the total observation time is much longer than inverse bias.

APPENDIX A: ASYMPTOTIC EXPANSION FOR \( \xi \)

To complete the discussion in section V B we need to find the function \( \xi(t) \) with the following properties :

- \( \xi \) is analytic in the complement of the two vertical cuts \( \text{Re} t = 0 \) and \( \text{Re} t = t_f \)
- \( \xi \) obeys
\[ \xi_1 \xi_i^{-1} = W_1^{-1} W_i \]  
(A1)
along the cut \( \text{Re} t = 0 \), where \( i = 2, 3 \) and notations are the same as in section V B
- \( \xi \) obeys \( \xi_4 \xi_i^{-1} = W_4^{-1} W_i \) along the cut \( \text{Re} t = t_f \)
- \( \xi \to 1 \) when \(|t| \to \infty \)
In this appendix we show that $\xi$ is close to 1 in the limit $V t_f \gg 1$ everywhere except in the two small regions near $t = 0$ and $t = t_f$

The general method can be illustrated by finding a function $\eta$ with just one vertical cut along the upper part of imaginary axis ($\text{Re} t = 0, \text{Im} t > 0$) obeying the condition $\eta \eta^{-1} = W_1^{-1} W_2$ along that cut. The case with two cuts is exactly the same.

Analyticity of $\eta$ together with the boundary condition at infinity leads to the representation

$$\eta(z) = 1 + \frac{1}{2\pi i} \int_{\text{Re} t' = 0; \text{Im} t' > 0} \frac{\eta(t' - 0) - \eta(t' + 0)}{t' - z} dt'.$$

Using the above we obtain the integral equation for the values of $\eta(t)$ on the line Re $t = 0$

$$\eta(t) = 1 + \frac{1}{2\pi i} \int_{\text{Re} t' = 0; \text{Im} t' > 0} \frac{(1 - W_2^{-1}(t') W_1(t')) \eta(t')}{t' - t + 0} dt'$$

Because of exponential decay of the kernel along the cut

$$1 - W_2^{-1} W_1 = -e^{V t_f} \psi^{-1} \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix} \psi,$$

the integral in Eq (A2) converges at $t' \sim 1/V \ll t$, so in the limit $V t \gg 1$ we can neglect $t'$ in the denominator and get the expansion

$$\eta_1 = 1 + \frac{C}{V t} + \cdots,$$

where

$$C = \frac{V}{2\pi i} \int_{\text{Re} t' = 0; \text{Im} t' > 0} (1 - W_2^{-1}(t') W_1(t')) \eta(t') dt' \sim 1$$

is some constant.