Nonperturbative deformation of D-brane states by the world sheet noncommutativity

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Abstract

By introducing the noncommutativity in the world sheet, we discuss a modification of the D-brane states in the closed string theory. In particular we show how the world sheet noncommutativity induces a nonperturbative effect to the D-brane states.

1 Introduction

The noncommutative spacetime provided by the string theory [1] has been discussed as a possible basis of the theory of quantum gravity [2]. In recent years, it has been shown that the noncommutative spaces can be constructed by the help of the concept of Hopf algebra [3] from certain commutative spaces which have Lie algebraic symmetries. We can consider the string theory, the D-brane dynamics and the M-theory on the noncommutative space, in which the noncommutativity plays an important role in the description of their nonperturbative aspects.

From this point of view, it will be interesting to investigate the effect of the noncommutativity of the string world sheet. In fact the two dimensional noncommutative conformal field theories have been discussed in various aspects, such as the perturbative approach [4], the harmonic oscillator formalism [5] and the derivation of the star product by virtue of the Hopf algebra [6]. In spite of many efforts, however, we are still far from obtaining some interesting results.

One of the main difficulties of these approaches owes to the fact that the noncommutativity of the world sheet introduces a dimensional constant which breaks explicitly the conformal symmetry of the theory. To recover the theory from this problem, one must commit oneself to many complicated formulae expanded into perturbative series.

On the other hand the world sheet noncommutativity possesses a notable feature, i.e., it does not affect the product of the left-movers, which are holomorphic, nor the product of the right-movers, which are anti-holomorphic, of the strings. Therefore the influence of the world sheet noncommutativity appears only in the product between the left-movers and the right-movers in the closed string theory. This means, for example, that the world sheet noncommutativity does not change the theory of open strings at all. Hence the world sheet noncommutativity is nothing to do with the spacetime quantization. In other words we must quantize the spacetime independent from the world sheet noncommutativity.
Considering the above background into account, we would like to study, in this note, some nonperturbative effects of the world sheet noncommutativity to the D-brane states. To this end we assume that all physical quantities can be represented by the string coordinates $X^\mu(\sigma, \tau)$ and that the effect of the world sheet noncommutativity appears only through their products. Moreover we assume that, although the left-moving string coordinate $X^\mu(\sigma + i\tau)$ and the right-moving string coordinate $\tilde{X}^\mu(\sigma - i\tau)$ do not commute any more, their components $\alpha^\mu_n$ and $\tilde{\alpha}^\mu_n$ commute, so that the world sheet noncommutativity does not change the conventional quantization rule of the spacetime. We will see that these assumptions enable us to put forward our investigation nonperturbatively. At the same time we must emphasize that we leave aside the problem of the breaking of the conformal symmetry.

In order to see the effect of the world sheet noncommutativity, the D-brane is a convenient object to study because the D-brane boundary state consists of a product of the left and right moving string coordinates. The star product associated with the world sheet noncommutativity is introduced locally. Nevertheless we will see that it produces a nonlocal deformation of the D-brane boundary state as a nonperturbative effect. In the D-brane correlators, for example, the noncommutativity parameter $\lambda$ appears through the elliptic functions as their modular parameter. All such results suggest that the world sheet noncommutativity changes the topology of the world sheet itself.

Since the D-brane is a much-discussed object and many results are accumulated, we should be able to compare our results with them. Although we do not even consider the fermionic partners and the ghosts in this note, our results are enough to see the effect of the world sheet noncommutativity.

The organization of this note is as follows. In section 2, we define the star product and consider the noncommutativity in the world sheet. In section 3, we evaluate correlators of the D-brane states as deformed by the world sheet noncommutativity. Finally, we summarize the results in section 4.

## 2 World sheet noncommutativity

In two dimensional space, the $\ast$-product can be represented by the differential operators of the coordinate as follows:

$$f(\sigma, \tau) \ast g(\sigma, \tau) = \lim_{(\sigma', \tau') \to (\sigma, \tau)} e^{-\lambda(\partial_{\sigma'}\partial_{\tau'} - \partial_{\tau'}\partial_{\sigma'})} f(\sigma, \tau) g(\sigma', \tau'),$$

with only one noncommutativity parameter $\lambda$. The commutator of the coordinates with respect to this product gives a constant value which means noncommutative space:

$$[\tau, \sigma]_{\ast} = 2\lambda.$$

In order to investigate the physical aspects, however, we have to discuss the theory of fields at different points and so we give a versatility to the $\ast$-product which acts on multiple different points. In the case of only two points, we extend it by the copy of (1) without taking the limitation $(\sigma', \tau') \to (\sigma, \tau)$:

$$f(\sigma, \tau) \ast g(\sigma', \tau') = e^{-\lambda(\partial_{\sigma}\partial_{\tau'} - \partial_{\tau}\partial_{\sigma'})} f(\sigma, \tau) g(\sigma', \tau').$$

Since this $\ast$-product is compatible with the trigonometric functions

$$e^{im\sigma + n\tau} \ast e^{im'\sigma' + n'\tau'} = e^{-i\lambda(mn' - nm')} e^{im\sigma + n\tau} e^{im'\sigma' + n'\tau'} \quad \forall m, n, m', n' \in \mathbb{Z},$$
it is possible to define any product of fields at different points through their Fourier expansion. In fact we can convince ourselves that the associativity

\[ e^{im_1\sigma_1+n_1\tau_1} \ast (e^{im_2\sigma_2+n_2\tau_2} \ast e^{im_3\sigma_3+n_3\tau_3}) \]

\[ = e^{-i\lambda(m_2n_3-n_2m_3)} e^{im_1\sigma_1+n_1\tau_1} \ast (e^{im_2\sigma_2+n_2\tau_2} e^{im_3\sigma_3+n_3\tau_3}) \]

\[ = e^{-i\lambda(m_1n_2-n_1m_2)} e^{im_1\sigma_1+n_1\tau_1} e^{-i\lambda(m_1n_3-n_1m_3)} e^{im_2\sigma_2+n_2\tau_2} e^{im_3\sigma_3+n_3\tau_3} \]

\[ = (e^{im_1\sigma_1+n_1\tau_1} \ast e^{im_2\sigma_2+n_2\tau_2}) \ast e^{im_3\sigma_3+n_3\tau_3} \]

is preserved, which is the sufficient property for our purpose.

Now we focus our attention to the string theory. It is described by the string coordinate, which is either holomorphic or anti-holomorphic function of the world sheet coordinate \((\sigma, \tau)\). Due to this fact an evaluation of the contribution of the world sheet noncommutativity is not difficult.

Let \(f_a(\sigma + i\tau)\) and \(g_a(\sigma - i\tau)\) be a holomorphic and an anti-holomorphic functions which can be expanded into the series of the complex variable \(z = e^{i(\sigma + i\tau)}\) and its conjugate, respectively, as follows

\[ f_a(\sigma + i\tau) = \sum_{n \in \mathbb{Z}} f_{a,n} e^{in(\sigma + i\tau)}, \quad g_a(\sigma - i\tau) = \sum_{n \in \mathbb{Z}} g_{a,n} e^{in(\sigma - i\tau)}. \]

Then the \(*\)-product (2) gives the following formulae,

\[ f_a(\sigma + i\tau) \ast f_b(\sigma' + i\tau') = f_a(\sigma + i\tau) f_b(\sigma' + i\tau'), \]

\[ g_a(\sigma - i\tau) \ast g_b(\sigma' - i\tau') = g_a(\sigma - i\tau) g_b(\sigma' - i\tau'), \]

\[ f_a(\sigma + i\tau) \ast g_b(\sigma' - i\tau') = \int_0^{2\pi} \int_0^{2\pi} \frac{d\sigma_1 d\sigma_2}{4\pi} e^{i(\sigma_1 - \sigma)(\sigma_2 - \sigma')/2\lambda} f_a(\sigma_1 + i\tau_1) g_b(\sigma_2 - i\tau_2). \]

In the same point limit this formalism has been already mentioned in [7]. We learn, from these results, that any product of two holomorphic functions, as well as two anti-holomorphic functions, is not affected by the world sheet noncommutativity. In other words the world sheet noncommutativity does not violate the analytic properties of each field. We have to attend our concern to the product only between a holomorphic and an anti-holomorphic components.

On the basis of definitions above, we want to study the free closed string theory on the noncommutative world sheet. The spacetime coordinates of a closed string are defined by

\[ X^\mu(\sigma + i\tau) = X_+^\mu(\sigma + i\tau) + X_-^\mu(\sigma + i\tau), \quad \bar{X}^\mu(\sigma - i\tau) = \bar{X}_+^\mu(\sigma - i\tau) + \bar{X}_-^\mu(\sigma - i\tau), \]

\[
\begin{align*}
X_+^\mu(\sigma + i\tau) &:= \frac{1}{2} \sigma_0^\mu - i \sum_{n=1}^{\infty} \frac{1}{n} \alpha_{-n}^\mu e^{-in(\sigma + i\tau)}, \\
X_-^\mu(\sigma + i\tau) &:= -2p_0^\mu(\sigma + i\tau) + i \sum_{n=1}^{\infty} \frac{1}{n} \alpha_n^\mu e^{in(\sigma + i\tau)}, \\
\bar{X}_+^\mu(\sigma - i\tau) &:= \frac{1}{2} \sigma_0^\mu - i \sum_{n=1}^{\infty} \frac{1}{n} \bar{\alpha}_{-n}^\mu e^{in(\sigma - i\tau)}, \\
\bar{X}_-^\mu(\sigma - i\tau) &:= 2p_0^\mu(\sigma - i\tau) + i \sum_{n=1}^{\infty} \frac{1}{n} \bar{\alpha}_n^\mu e^{-in(\sigma - i\tau)},
\end{align*}
\]
where we divide the left-moving and right-moving modes. Applying the general formulae (3), (4) and (5) we see immediately

\[
\partial_\sigma X^\mu (\sigma + i\tau) * \partial_{\sigma'} X^{\nu} (\sigma' + i\tau') = \partial_\sigma X^\mu (\sigma + i\tau) \partial_{\sigma'} X^{\nu} (\sigma' + i\tau'), \\
\partial_\sigma \tilde{X}^\mu (\sigma - i\tau) * \partial_{\sigma'} \tilde{X}^{\nu} (\sigma' - i\tau') = \partial_\sigma \tilde{X}^\mu (\sigma - i\tau) \partial_{\sigma'} \tilde{X}^{\nu} (\sigma' - i\tau'), \\
\partial_\sigma X^\mu (\sigma + i\tau) \partial_{\sigma'} \tilde{X}^{\nu} (\sigma' - i\tau') = \int_0^{2\pi} \int_0^{2\pi} \frac{d\sigma_1 d\sigma_2}{4\pi \lambda} e^{i(\sigma_1 - \sigma)(\sigma_2 - \sigma')/2\lambda} \partial_{\sigma_1} X^\mu (\sigma_1 + i\tau) \partial_{\sigma_2} \tilde{X}^{\nu} (\sigma_2 - i\tau').
\]

Despite of one’s expectation the noncommutativity does not introduce any quantum effect to the system of only left-moving or right-moving modes, while it deforms the product of two different components. Hence the quantization of the string coordinates must be achieved independent from the world sheet noncommutativity. We will adopt the standard quantization rule in which the components satisfy the commutation relations,

\[
[x_0^\mu, p_0^\nu] = i\eta^{\mu\nu}, \quad [\alpha^\mu_m, \alpha^\nu_n] = [\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n] = m\delta_{m+n,0} \eta^{\mu\nu}, \quad (6)
\]

\[
[\alpha^\mu_n, \tilde{\alpha}^\nu_m] = 0 \quad \forall n, m \neq 0, \quad (7)
\]

and annihilate the vacuum;

\[
p_0^\mu |0\rangle = 0, \quad \alpha^\mu_n |0\rangle = \tilde{\alpha}^\nu_n |0\rangle = 0 \quad \forall n > 0. \quad (8)
\]

Since the \(*\)-product gives rise to the mixing of the fields \(\partial_\sigma X^\mu (\sigma + i\tau)\) and \(\partial_\sigma \tilde{X}^{\nu} (\sigma - i\tau)\), they do not commute anymore. In order to recover the commutativity of these fields in the nature of the \(*\)-product, the authors in [6] required the conditions

\[
\alpha^\mu_n \tilde{\alpha}^\nu_m = e^{2\lambda nm} \tilde{\alpha}^\nu_m \alpha^\mu_n \quad \forall n, m \neq 0
\]

for all non-zero modes.

On the other hand we have already shown that the quantization of the string coordinates must be done independent from the world sheet noncommutativity. Moreover our purpose of this note is to see the effect of the world sheet noncommutativity to the physical objects, rather than to discuss the conformal symmetry. Therefore, instead of following the argument of [6], we simply assume (7) and see how the deformation changes the physical quantities.

The contribution of the world sheet noncommutativity appears in the vertex operator. Since the fields \(X^\mu\) and \(\tilde{X}^{\nu}\) do not commute under the \(*\)-product, we have to fix their ordering. We will define the ordering rule of these fields such that the field \(X^\mu\) is always in the left of the field \(\tilde{X}^{\nu}\). According to this rule the tachyon vertex operator, for instance, is given by

\[
V^*_k (\sigma, \tau) := e^{ikX^+(\sigma+\tau)} e^{ik\tilde{X}^-(\sigma+\tau)} * e^{ikX^+(\sigma-\tau)} e^{ik\tilde{X}^-(\sigma-\tau)}
\]

\[
= \int_0^{2\pi} \int_0^{2\pi} \frac{d\sigma_1 d\sigma_2}{4\pi \lambda} e^{i(\sigma_1 - \sigma)(\sigma_2 - \sigma)/2\lambda} e^{ikX^+(\sigma_1 + \tau)} e^{ikX^-(\sigma_1 + \tau)} e^{ik\tilde{X}^+(\sigma_2 - \tau)} e^{ik\tilde{X}^-(\sigma_2 - \tau)}, \quad (9)
\]
where \( k \cdot X := k_\mu X^\mu = \eta_{\mu\nu} k^\mu X^\nu \) stands for the inner product. As an application of the vertex operator, we evaluate the tachyon propagator of a closed string:

\[
\langle 0 | V_k^*(\sigma, \tau) * V_{k'}^*(\sigma', \tau') | 0 \rangle = \delta(k + k') \int_0^{2\pi} \frac{d\sigma_1 d\sigma_2}{4\pi \lambda} \int_0^{2\pi} \frac{d\sigma'_1 d\sigma'_2}{4\pi \lambda} \Delta_k^\lambda(\sigma, \tau; \sigma_1, \sigma'_1; \tau, \tau'_1; 1), \tag{10}
\]

where

\[
\Delta_k^\lambda(\sigma, \tau; \sigma_1, \sigma'_1; \tau, \tau'_1; 1) := e^{i(\sigma - \sigma_1)(\sigma_2 - \tau)/4\lambda} e^{i(\sigma - \sigma_1)(\sigma'_2 - \tau')/4\lambda} e^{i(\sigma'_1 - \sigma)(\sigma'_2 - \tau')/4\lambda} e^{-i(\sigma'_1 - \sigma)(\sigma_2 - \tau)/4\lambda} \times \left( e^{-i(\sigma + i\tau)} - e^{-i(\sigma'_1 + i\tau')} \right)^{-k} \left( e^{i(\sigma_2 - i\tau)} - e^{i(\sigma'_2 - i\tau')} \right)^{-k}. \tag{11}
\]

### 3 D-brane correlators

It is reasonable to assume that the deformation of the physical objects by the world sheet noncommutativity is caused only through the spacetime coordinate. From this point of view we must express all physical objects in terms of the spacetime coordinates.

We are interested in the deformation of D-brane states by the world sheet noncommutativity. Thus we define the boundary state, which expresses a D-brane situated at the position \( y^j (j = 1, \cdots, d_\perp) \) on the \( d \)-dimensional Minkowski spacetime, by

\[
| \rho \rangle := B_\rho(0)|0\rangle \tag{12}
\]

The \( \rho_{\mu\nu} \) is a nonsingular matrix which specifies the boundary condition for each direction of the brane with the value \(-1\) or \(1\) corresponding to the Neumann or Dirichlet, respectively. The \( d_\perp \) is the number of directions which are the Dirichlet boundary. In this formula we used the notation \( f \overset{\perp}{\partial} g = \frac{1}{4} [f(\partial g) - (\partial f) g] \) and

\[
X^\mu(\sigma + i\tau, y) := \begin{cases} X^j(\sigma + i\tau) - y^j/2 & j = 1, \cdots, d_\perp \text{ Dirichlet,} \\ X^a(\sigma + i\tau) & a = d_\perp + 1, \cdots, d \text{ Neumann,} \end{cases}
\]

the same holds for \( \tilde{X}^\mu(\sigma - i\tau, y) \). We also assume that the operator \( p^j_0 \) is replaced by a \( c \)-number \( p^j \), which is integrated out in the Dirichlet directions, or is put zero in the Neumann directions. If we further impose the condition \( [X^j(\sigma) + \tilde{X}^j(\sigma)]|_{\sigma = 2\pi} = 0 \), we can indeed confirm that (12) coincides with the well-known form of the D-brane states (see, e.g., [8]):

\[
| \rho \rangle = \delta^{d_\perp}(x^j_0 - y^j) \exp \left( \rho_{\mu\nu} \sum_{n=1}^{\infty} \frac{1}{n!} \alpha_n^{\mu} \tilde{\alpha}_n^{\nu} \right) | 0 \rangle. \tag{13}
\]

Our expression of the boundary state (12) enables us to see the contribution of the world sheet noncommutativity, simply by replacing the product of the fields to the \(*\)-product with the ordering as defined before,

\[
B^*_\rho(\tau) := \int \frac{d^{d_\perp} p}{(2\pi)^{d_\perp}} \exp \left( i\rho_{\mu\nu} \int_0^{2\pi} \frac{d\sigma}{2\pi} X^\mu(\sigma + i\tau, y) * \overset{\perp}{\partial}_\sigma \tilde{X}^\nu(\sigma - i\tau, y) \right) :. \tag{13}
\]
Since this corresponds to the case \(13\), we obtain a compact expression of the deformed boundary state by using the elliptic theta function \(\vartheta_3\),

\[
B^*_\rho(\tau) = \int \frac{d^2\rho}{(2\pi)^2} \exp \left( i\rho_{\mu\nu} \int_0^{2\pi} \frac{d\sigma d\sigma'}{(2\pi)^2} X^\mu(\sigma + i\tau, y) \bar{\partial}_{\sigma + \sigma'} \bar{X}^\nu(\sigma' - i\tau, y) \vartheta_3 \left( \frac{\sigma - \sigma'}{2\pi}, \frac{2\lambda}{\pi} \right) \right);
\]

where \(\vartheta_3\) is defined as

\[
\vartheta_3 \left( \frac{\sigma_1 - \sigma_2}{2\pi}, \frac{2\lambda}{\pi} \right) = \frac{1}{2\lambda} \int_0^{2\pi} d\sigma \ e^{i(\sigma_1 - \sigma)(\sigma_2 - \sigma)/2\lambda}.
\]

Although the deformed boundary state is no longer an eigenstate of the spacetime coordinate, there exist \(X^\mu_{(\lambda)}\) and \(\bar{X}^\mu_{(\lambda)}\) which satisfy

\[
\left( X^j_{(\lambda)} + \bar{X}^j_{(\lambda)} \right)_{\tau=0} |\rho\rangle_* = y^j |\rho\rangle_* , \quad \partial_\tau \left( X^a_{(\lambda)} + \bar{X}^a_{(\lambda)} \right)_{\tau=0} |\rho\rangle_* = 0 .
\]

These are defined through the redefinition of the non-zero modes of the fields such as

\[
\alpha^\mu_{\lambda, \pm n} := e^{\mp i\lambda n^2} \tilde{\alpha}^\mu_{\pm n} , \quad \tilde{\alpha}^\mu_{\lambda, \pm n} := e^{\mp i\lambda n^2} \alpha^\mu_{\pm n} \quad \forall n > 0 ,
\]

\[
X^\mu_{(\lambda)}(\sigma + i\tau) := \frac{1}{2} \phi_0^\mu - 2 \phi_0^\mu(\sigma + i\tau) + i \sum_{n=1}^\infty \frac{1}{n} \left( \alpha^\mu_{\lambda, n} e^{i(n(\sigma + i\tau)} - \alpha^\mu_{\lambda, -n} e^{-i(n(\sigma + i\tau)} \right) ,
\]

\[
\bar{X}^\mu_{(\lambda)}(\sigma - i\tau) := \frac{1}{2} \phi_0^\mu + 2 \phi_0^\mu(\sigma - i\tau) + i \sum_{n=1}^\infty \frac{1}{n} \left( \tilde{\alpha}^\mu_{\lambda, n} e^{-i(n(\sigma - i\tau)} - \tilde{\alpha}^\mu_{\lambda, -n} e^{i(n(\sigma - i\tau)} \right) .
\]

The modified coefficients still satisfy the same commutation relations \(6\) as well as \(7\):

\[
[\alpha^\mu_{\lambda, m}, \alpha^\nu_{\lambda, n}] = [\bar{\alpha}^\mu_{\lambda, m}, \bar{\alpha}^\nu_{\lambda, n}] = m \delta_{m+n,0} \eta^{\mu\nu} , \quad [\alpha^\mu_{\lambda, m}, \bar{\alpha}^\nu_{\lambda, n}] = 0 .
\]

Now we want to know the effect of the world sheet noncommutativity in physical objects. For example let us study a correlation function with respect to the deformed D-branes \(13\). The tachyon scattering off a D-brane is evaluated by the following expectation value

\[
\langle \rho, \tau | V^*_k(\sigma, \tau) * V^*_k(\sigma', 0) |\rho', 0\rangle_* = \langle 0 | B^*_\rho(\tau) V^*_k(\sigma, \tau) * V^*_k(\sigma', 0) B^*_\rho(0) | 0 \rangle .
\]

The vacuum expectation value on the right hand side can be evaluated easily if we rewrite the boundary state \(13\) by using auxiliary fields as

\[
B^*_\rho(\tau) = B_0(y^i) \int \mathcal{D}\xi \mathcal{D}\bar{\xi} \exp \left[ -i \int_0^{2\pi} \frac{d\sigma d\sigma'}{(2\pi)^2} \rho^{\mu\nu} \bar{\xi}_\mu(\sigma) \partial_{\sigma'} \xi_\nu(\sigma') \vartheta_3 \left( \frac{\sigma - \sigma'}{2\pi}, \frac{2\lambda}{\pi} \right) \right]
\]

\times \exp \left[ -i \int_0^{2\pi} \frac{d\sigma}{2\pi} \xi_\mu(\sigma) \partial_\sigma X^\mu(\sigma + i\tau) + i \int_0^{2\pi} \frac{d\sigma}{2\pi} \bar{\xi}_\mu(\sigma) \partial_\sigma \bar{X}^\mu(\sigma - i\tau) \right] ;
\]

where \(\rho^{\mu\nu}\) is the inverse matrix of \(\rho_{\mu\nu}\) and \(B_0(y^i)\) is an operator which specifies the position of the D-brane which is not affected by the world sheet noncommutativity. Note that \(\int_0^{2\pi} d\sigma \xi_\mu(\sigma) = \int_0^{2\pi} d\sigma \bar{\xi}_\mu(\sigma) = 0\) holds because the zero modes of the string coordinate vanish on the vacuum.
To proceed further we introduce a modified theta function

\[ \Theta(\sigma, \tau, \lambda) := \sum_{n=-\infty}^{\infty} e^{in\sigma} e^{-|n|\tau} e^{-2i\lambda n^2}, \quad \Theta(\sigma, 0, \lambda) = \psi_{3}\left(\frac{\sigma}{2\pi}, -\frac{2\lambda}{\pi}\right), \]

and define a function \( G \) with \( \varepsilon, \beta \in \mathbb{R} \) by

\[ G^{(\varepsilon)}_{\beta}(\sigma, \tau, \lambda) := -i \sum_{n=1}^{\infty} \frac{e^{in\sigma} + \varepsilon e^{-in\sigma}}{1 - \beta e^{-2i\lambda n^2} e^{in\tau}} + \frac{c(\varepsilon, \beta)}{1 - \beta} \sigma, \quad c(\varepsilon, \beta) = \begin{cases} 1 & \varepsilon = -1, \beta \neq 1 \\ 0 & \text{otherwise} \end{cases} \tag{17} \]

such that it obeys

\[ \int_{0}^{2\pi} \frac{d\sigma''}{2\pi} \left[ 2\pi \delta(\sigma - \sigma'') - \beta \Theta(\sigma - \sigma'', -\tau, \lambda) \right] \partial_{\sigma''} G^{(1)}_{\beta}(\sigma'', \tau, \lambda) = 2\pi \delta(\sigma - \sigma'), \tag{18} \]

in the case \( \varepsilon = -1 \). These are sufficient to calculate the Gaussian integration over the auxiliary fields.

For the sake of simplicity, let us specify the boundary conditions to the case \((\rho_{\mu\nu}) = (\rho'_{\mu\nu}) = \text{diag}(\delta_{jk}, -\eta_{ab})\). Then the expectation value results in

\[ \langle \rho(\tau) \rangle = \delta(k + k') \int_{0}^{2\pi} \frac{d\sigma_{1}}{4\pi \lambda} \int_{0}^{2\pi} \frac{d\sigma_{2}}{4\pi \lambda} \Delta^{k}_{(\lambda)}(\sigma; \sigma_{1}, \sigma_{2}; \sigma', \sigma', \sigma_{1}', \sigma_{2}') \]

\[ \times \left( \frac{\Phi_{(\lambda)}^{+}(\sigma_{1} - \sigma_{1}' + i\tau, \tau) \Phi_{(\lambda)}^{+}(\sigma_{2} - \sigma_{2}' - i\tau, \tau)}{\Phi_{(\lambda)}^{-}(\sigma_{1} - \sigma_{2} + i\tau, \tau) \Phi_{(\lambda)}^{-}(\sigma_{1}' - \sigma_{2}' - i\tau, \tau)} \right)^{-k' S - k} \tag{19} \]

where we used the definition \( [11] \) and denoted that

\[ \Phi_{(\lambda)}^{\pm}(\sigma, \tau) := \exp \left( i \left[ G_{1}^{(1)}(\sigma, \tau, \lambda) \pm G_{-1}^{(1)}(\sigma, \tau, \lambda) \right] \right), \quad S = \begin{pmatrix} 1_{d_{\perp}} & 0 \\ 0 & -1_{d_{\parallel}} \end{pmatrix}, \]

and \( d_{\parallel} = d - d_{\perp} \) is the number of dimensions dominated by the D-brane.

From the analysis we carried out in this section we can easily read off some of the effects of the world sheet noncommutativity to the D-brane correlators. For example, the change of the free energy part of the D-branes, which can be obtained by the calculation of \( \langle 0 | B_{\nu}^{*}(\tau) B_{\nu}'^{*}(0) | 0 \rangle \), is simply given by the following modification of the Dedekind eta function:

\[ \eta(q) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^{n}) \rightarrow q^{1/24} \prod_{n=1}^{\infty} (1 - e^{4i\lambda n^2 q^{n}}) \quad \text{where} \quad q = e^{-2\tau}. \tag{20} \]

The effect of the world sheet noncommutativity to the tachyon propagator between the D-branes are separated into two parts. One is the free tachyon propagator \( \Delta^{k}_{(\lambda)} \). The deformation of this part is caused by the appearance of the \( \ast \)-product in all vertex operators and their products, and is nothing to do with the boundary states.
The other part of the effects of the world sheet noncommutativity is in \( \Phi^\pm_{(\lambda)} \) of (19). In contrast to the free tachyon propagators we notice that their \( \lambda \)-dependence comes only from the deformation of the D-brane states \( B^*_\rho(\tau) \) of (14). In the derivation of \( \Phi^\pm_{(\lambda)} \) the \( * \)-products between tachyon vertices are irrelevant. Therefore we are able to deal with the effects of the world sheet noncommutativity on the vertex operators and the D-brane states, independently.

In fact we can derive the commutative limit \( \lambda \to 0 \) of (19) as

\[
\lim_{\lambda \to 0} \langle \rho, \tau \mid V^*_k(\sigma, \tau) \ast V^*_k(\sigma', 0) \mid \rho', 0 \rangle_* = \delta(k + k') e^{-2k \cdot S \cdot \zeta(1)} e^{-2\pi(k^2 + k'^2)} \left| \frac{\vartheta'_1(0, i\tau)/2\pi}{\vartheta_1\left(\frac{\sigma + i\tau - \sigma'}{2\pi}, \frac{i\tau}{\pi}\right)} \right|^{-4k^2} \left| 1 - e^{i(\sigma + i\tau - \sigma')} \right|^{-4k'^2},
\]

where \( 2k^2 := k^2 - k \cdot S \cdot k \). This agrees with the result which we obtain by the calculation of the same object by putting \( \lambda = 0 \) in the \( \Delta^k_{(\lambda)} \) of (11) and in the \( \vartheta_3 \) function of (14).

4 Conclusion

By introducing the noncommutativity in the world sheet, we discussed a modification of the D-brane states in the closed string theory. In particular we have shown, in this note, how the world sheet noncommutativity induces a nonperturbative effect to the D-brane states. The results were based on our assumption that the conventional quantization rules (6) and (7) are not changed by the world sheet noncommutativity. The world sheet noncommutativity induces the noncommutativity of only the product of the fields \( \partial_\sigma X^\mu(\sigma + i\tau) \) and \( \partial_\sigma \tilde{X}^\mu(\sigma - i\tau) \).

This assumption owes to the fact that the \( * \)-product (2) does not give rise to the quantization of the spacetime. We must quantize the string coordinate independently from the world sheet noncommutativity. Once we adopt this assumption as our starting point of our discussion we can evaluate the effects of the world sheet noncommutativity nonperturbatively. For example we found that the D-brane boundary states are modified simply replacing the \( \delta \) function by the elliptic function \( \vartheta_3 \) whose modulus is the noncommutativity parameter \( \lambda \) itself. This phenomenon can be interpreted as a consequence of the topology change of the world sheet from a sphere to a torus. We emphasize that it will be difficult to see this result by the perturbative calculations.

Meanwhile we have found that the deformed D-brane states have a nice feature given by (15) in contrast to one’s misgivings. Namely the modification of the D-branes is equivalent to the deformation of the string coordinates \( X^\mu \to X^\mu_{(\lambda)} \). Moreover the simple formula of the modified D-brane states enables us to evaluate D-brane correlators nonperturbatively, as we have shown in section 3. The result (19) tells us how the world sheet noncommutativity appears in the D-brane physics. In particular the \( \lambda \)-dependence of the D-brane states appears in the correlators only through the function \( G^{(\varepsilon)}_{\beta}(\sigma, \tau, \lambda) \) of (17).

We have not discussed about the breaking of the conformal symmetry by the introduction of the noncommutativity parameter \( \lambda \). In order to clarify this problem within the framework of the quantum field theory, however, we must specify a model to be considered and study the relationship between the symmetry and the quantization procedure. Our results are expected to contribute to a better understanding of such problems.

Since the purpose of this note is to show the possibility of evaluating the nonperturbative effects of the world sheet noncommutativity, we have ignored not only the supersymmetric
partners but also the ghosts. In order to discuss physical insights of our results we must take them all into account, which we are going to discuss in our forthcoming paper.

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