A GENUS FORMULA FOR THE POSITIVE ÉTALE WILD KERNEL

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Abstract. Let $F$ be a number field and let $i \geq 2$ be an integer. In this paper, we study the positive étale wild kernel $WK^{\text{ét},+}_{2i-2} F$, which is the twisted analogue of the 2-primary part of the narrow class group. If $E/F$ is a Galois extension of number fields with Galois group $G$, we prove a genus formula relating the order of the groups $(WK^{\text{ét},+}_{2i-2} E)_G$ and $WK^{\text{ét},+}_{2i-2} F$.

1. Introduction

Let $F$ be a number field and let $p$ be a prime number. For a finite set $S$ of primes of $F$ containing the $p$-adic primes and the infinite primes, let $G_{F,S}$ be the Galois group of the maximal algebraic extension $F_S$ of $F$ which is unramified outside $S$. It is well known that for all integer $i \geq 2$, the kernel of the localization map $H^2(G_{F,S}, \mathbb{Z}_p(i)) \rightarrow \bigoplus_{v \in S} H^2(F_v, \mathbb{Z}_p(i))$ is independent of the choice of the set $S$ [Sc 79 §6, Lemma 1] [Ko 03 page 336]. This kernel is called the étale wild kernel [Ng 92, Ko 93], and denoted by $WK^{\text{ét},+}_{2i-2} F$.

Let $E/F$ be a Galois extension with Galois group $G$. For a fixed odd prime $p$, several authors have studied Galois co-descent and proved genus formulas [Ko-Mo 00, Gri 05, As-As, As], for the étale wild kernel, which is analogue to the Chevalley genus formula for the class groups. In this paper we settle the case $p = 2$. For this purpose, we use a slight variant of cohomology, the so-called totally positive Galois cohomology [Ka 93 §5]. More precisely, for all integer $i$, we are interested in the kernel of the map

$$H^2(G_{F,S}, \mathbb{Z}_2(i)) \rightarrow \bigoplus_{v \in S_f} H^2(F_v, \mathbb{Z}_2(i)).$$

Here $S_f$ denotes the set of finite primes in $S$ and $H^j(S, \ldots)$ denotes the $j$-th totally positive Galois cohomology groups (Section 2.1). When $i = 1$, this kernel is isomorphic to the 2-primary part of the narrow $S$-class group of $F$. For $i \geq 2$, we show that this kernel is independent of the set $S$; it is referred to as the positive étale wild kernel, and is denoted by $WK^{\text{ét},+}_{2i-2} F$. It is analogue to the narrow $S$-class group of $F$, and fits into an exact sequence (Proposition 2.6)

$$0 \rightarrow D^+_F(F^{\text{ét}},F^{\text{ét}}) \rightarrow \bigoplus_{v \mid \infty} H^1(F_v, \mathbb{Z}_2(i)) \rightarrow WK^{\text{ét},+}_{2i-2} F \rightarrow WK^{\text{ét}}_{2i-2} F \rightarrow 0$$

where $D^+_F$ (see [Ko 03 Definition 2.3]) is the étale Tate kernel and $D^+_F/F^{\text{ét}}$ is the kernel of the signature map

$$\text{sgn}_F : H^1(F, \mathbb{Z}_2(i))/2 \cong D^+_F/F^{\text{ét}} \rightarrow (\mathbb{Z}/2)^{r_1}.$$
• for \( i \) even, we have
\[ WK_{2i-2}^{\text{ét}} F \cong WK_{2i-2}^{\text{ét}} F. \]

• for \( i \) odd, we have an exact sequence
\[ 0 \to (\mathbb{Z}/2)^{\delta_i(F)} \to WK_{2i-2}^{\text{ét}} F \to WK_{2i-2}^{\text{ét}} F \to 0, \]
where \( \delta_i(F) \) is the 2-rank of the cokernel of the signature map \( sgn_F \).

For a place \( v \) of \( F \), let \( G_v \) denote the decomposition group of \( v \) in \( E/F \). Define the plus normic subgroup \( H_+^1(F, \mathbb{Z}_2(i)) \) to be the kernel of the map
\[ H_+^1(F, \mathbb{Z}_2(i)) \to \bigoplus_{v \in S} H^1_{\text{et}}(E_v, \mathbb{Z}_2(i)) \]
where \( N_{G_v} = \sum_{\sigma \in G_v} \sigma \) is the norm map, and if \( v \) is a prime of \( F \), we denote by \( w \) a prime of \( E \) above \( v \).

We prove the following genus formula for the positive étale wild kernel.

**Theorem.** Let \( E/F \) be a Galois extension of number fields with Galois group \( G \). Then for every \( i \geq 2 \), we have
\[
\frac{|(WK_{2i-2}^{\text{ét}} E)_G|}{|WK_{2i-2}^{\text{ét}} F|} = \frac{|X_{E/F}^{(i)}| \cdot \prod_{v \in S} |H_1(G_v, H^2(E_v, \mathbb{Z}_2(i)))|}{|H_1(G, H^0(E, \mathbb{Q}_2/\mathbb{Z}_2(1-i)))^\vee| \cdot |H_+^1(F, \mathbb{Z}_2(i)) : H^1_{\text{et}}(F, \mathbb{Z}_2(i))|}
\]

The group \( X_{E/F}^{(i)} \) (Definition 3.2) has order at most \( |H_2(G, H^0(E, \mathbb{Q}_2/\mathbb{Z}_2(1-i)))^\vee| \), and is trivial if the canonical morphism
\[ \kappa : H_2(G, \bigoplus_{w \in S} H^2(E_w, \mathbb{Z}_2(i))) \to H_2(G, H^0(E, \mathbb{Q}_2/\mathbb{Z}_2(1-i)))^\vee \]
is surjective.

In particular, if \( E/F \) is a relative quadratic extension of number fields, the order of the group \( X_{E/F}^{(i)} \) is at most 2. In this case we give, in the last section, a genus formula involving the positive Tate kernel \( D_F^{(i)} \). Roughly speaking, let \( F_\infty \) (resp. \( F_{v,\infty} \)) denote the cyclotomic \( \mathbb{Z}_2 \)-extension of \( F \) (resp. \( F_v \)) and let \( R_{E/F} \) be the set of both finite primes tamely ramified in \( E/F \) and 2-adic primes such that \( E_w \cap F_{v,\infty} \neq E_w \). Then for any odd integer \( i \geq 2 \), we have
(i) if \( E \subseteq F_\infty \), the positive étale wild kernel satisfies Galois descent;
(ii) if \( E \not\subseteq F_\infty \),
\[ \frac{|(WK_{2i-2}^{\text{ét}} E)_G|}{|WK_{2i-2}^{\text{ét}} F|} = \frac{2^{r(E/F)-1+t}}{|D_F^{(i)} : D_F^{(i)} \cap N_G E^*|} \]
where \( r(E/F) = |R_{E/F}| \) and \( t \in \{0,1\} \). Moreover, \( t = 0 \) if \( R_{E/F} \neq \emptyset \).

2. Positive étale wild kernel

2.1. **Totally positive Galois cohomology.** Let \( F \) be a number field and let \( S \) be a finite set of primes of \( F \) containing the set \( S_2 \) of dyadic primes and the set \( S_\infty \) of archimedean primes. For a place \( v \) of \( F \), we denote by \( F_v \) the completion of \( F \) at \( v \), and by \( G_{F_v} \) the absolute Galois group of \( F_v \).

For a discrete or a compact \( \mathbb{Z}_2[[G_{F,v}]] \)-module \( M \), we write \( M_+ \) for the cokernel of the map
\[ M \to \bigoplus_{v|\infty} \text{Ind}^{G_{F_v}} G F, M, \]
where $\text{Ind}_{G_{F_v}} G^F M$ denotes the induced module. Hence we have the exact sequence

$$0 \longrightarrow M \longrightarrow \oplus_{v \mid \infty} \text{Ind}_{G_{F_v}} G^F M \longrightarrow M_+ \longrightarrow 0.$$ 

Following [Ka 93, §5], we define the $n$-th totally positive Galois cohomology group $H^n_+(G_{F,S}, M)$ of $M$ by

$$H^n_+(G_{F,S}, M) := H^{n-1}(G_{F,S}, M_+).$$

Recall from [Ka 93, §5] some facts about the totally positive Galois cohomology.

**Proposition 2.1.** We have the following properties:

(i) There is a long exact sequence

$$\cdots \longrightarrow H^n_+(G_{F,S}, M) \longrightarrow H^n(G_{F,S}, M) \longrightarrow \oplus_{v \mid \infty} H^n(F_v, M) \longrightarrow H^{n+1}_+(G_{F,S}, M) \longrightarrow \cdots$$

(ii) $H^n_+(G_{F,S}, M) = 0$ for all $n \neq 1, 2$.

(iii) If $E/F$ is an extension unramified outside $S$ with Galois group $G$ then there is a cohomological spectral sequence

$$H^p(G, H^q_+(G_{E,S}, M)) \Rightarrow H^{p+q}_+(G_{F,S}, M).$$

□

We also have a Tate spectral sequence

$$H_p(G, H^q_+(G_{E,S}, M)) \Rightarrow H^{p+q}_+(G_{F,S}, M).$$

Hence, for a finite 2-primary Galois module $M$, we have an isomorphism

$$H^2_+(G_{E,S}, M)_G \cong H^2_+(G_{F,S}, M)_S$$

([We 06, Lemma 6.4]). In particular, by passing to the inverse limit, the corestriction map induces an isomorphism

$$H^2_+(G_{E,S}, \mathbb{Z}_2(i)) \cong H^2_+(G_{F,S}, \mathbb{Z}_2(i)).$$

(1)

Recall the local duality Theorem (e.g. [Mi 86, Corollary I.2.3]): For $n = 0, 1, 2$ and for every place $v$ of $F$, the cup product

$$H^n(F_v, M) \times H^{2-n}(F_v, M^*) \longrightarrow H^2(F_v, \mu_{2\infty}) \cong \mathbb{Q}_2/\mathbb{Z}_2, \quad \text{if } v \text{ is finite}$$

(2)

is a perfect pairing, where $\widehat{H}^n(F_v, \cdot)$ is the Tate cohomology group, $\mu_{2\infty}$ is the group of all roots of unity of 2-power order, and $(\cdot)^*$ means the Kummer dual: $M^* = \text{Hom}(M, \mu_{2\infty})$.

We have an analogue of the Poitou-Tate long exact sequence

**Proposition 2.2.** Let $S_f$ denote the set of finite places in $S$. Then there is a long exact sequence

$$\oplus_{v \in S_f} H^0(F_v, M) \longrightarrow H^2(G_{F,S}, M^*)^\vee \longrightarrow H^1_+(G_{F,S}, M) \longrightarrow \oplus_{v \in S_f} H^1(F_v, M) \downarrow$$

$$H^0(G_{F,S}, M^*)^\vee \longrightarrow \oplus_{v \in S_f} H^2(F_v, M) \longrightarrow H^2_+(G_{F,S}, M) \longrightarrow H^1(G_{F,S}, M^*)^\vee$$

where the subscript $(\cdot)^\vee$ refers to the Pontryagin dual: $M^\vee = \text{Hom}(M, \mathbb{Q}_2/\mathbb{Z}_2)$.

Proof. See [Ma 18, Proposition 2.6]. □
For a $\mathbb{Z}_2[[G_{F,S}]]$-module $M$ and $n = 1, 2$ we define the groups $\III_S^n(M)$ and $\III_S^{n+}(M)$ to be the kernels of the localization maps

$$\III_S^n(M) := \ker(H^n(G_{F,S}, M) \to \bigoplus_{v \in S_f} H^n(F_v, M))$$

and

$$\III_S^{n+}(M) := \ker(H^n(G_{F,S}, M) \to \bigoplus_{v \in S_f} H^n(F_v, M)).$$

We state a Poitou-Tate duality in the case $p = 2$ as a consequence of Proposition 2.2 and local duality (2).

**Corollary 2.3.** Let $n = 1, 2$. Then there is a perfect pairing

$$\III_S^{n+}(M) \times \III_S^{3-n}(M^*) \to \mathbb{Q}_2/\mathbb{Z}_2.$$

**Proof.** By Proposition 2.2 we have the exact sequences

$$0 \to \bigoplus_{v \in S_f} H^0(F_v, M) \to H^2(G_{F,S}, M^*)^\vee \to \III_S^{1+}(M) \to 0$$

and

$$\bigoplus_{v \in S_f} H^1(F_v, M) \to H^1(G_{F,S}, M^*)^\vee \to \III_S^{2+}(M) \to 0.$$ Dualizing these exact sequences and using the local duality (2), we get

$$\III_S^{1+}(M)^\vee \cong \III_S^2(M^*)$$ and \( \III_S^{2+}(M)^\vee \cong \III_S^1(M^*). \)

\[
\tag{2}
\]

\[
\tag{3}
\]

### 2.2. Signature.

In this subsection we recall some properties of the signature map (see e.g. [Ko 03], [As-Mo 18, §1.2]). For any real place $v$ of the number field $F$, let $i_v : F \to \mathbb{R}$ denote the corresponding real embedding. The natural signature maps $\text{sgn}_v : F^\bullet \to \mathbb{Z}/2\mathbb{Z}$ (where $\text{sgn}_v(x) = 0$ or 1 according to $i_v(x) > 0$ or not) give rise to the following surjective map

$$F^\bullet/F^\bullet^2 \to \bigoplus_{v \text{ real}} \mathbb{Z}/2\mathbb{Z}$$

with $x \mapsto (\text{sgn}_v(x))_{v \text{ real}}$.

The exact sequence of $G_F$-modules

$$0 \to \mathbb{Z}_2(i)^2 \to \mathbb{Z}_2(i) \to \mathbb{Z}/2(i) \to 0$$

gives rise to an exact sequence

$$0 \to H^1(F, \mathbb{Z}_2(i))/2 \to H^1(F, \mathbb{Z}/2(i)) \to H^2(F, \mathbb{Z}_2(i)),$$

where for an abelian group $A$, $A/2$ denotes the cokernel of the multiplication by 2 on $A$.

Since we have

$$H^1(F, \mathbb{Z}/2(i)) \cong H^1(F, \mathbb{Z}/2(1))(i - 1) \cong F^\bullet/F^\bullet^2(i - 1)$$

there exists a subgroup $D_F^{(i)}$ (the étale Tate kernel) of $F^\bullet$ containing $F^\bullet^2$ such that

$$D_F^{(i)}/F^\bullet^2 \cong H^1(F, \mathbb{Z}_2(i))/2.$$

We will consider the restriction of the above signature map to the quotient $D_F^{(i)}/F^\bullet^2$:

$$\text{sgn}_F : D_F^{(i)}/F^\bullet^2 \to (\mathbb{Z}/2)^{r_1},$$

where $r_1$ is the number of real places of $F$.

\[\tag{3} \]
Let $D_F^{+i}/F^\bullet^2$ be the kernel and $(\mathbb{Z}/2)^{\delta(F)}$ be the cokernel of $\text{sgn}_F$, respectively. So we have an exact sequence

$$0 \longrightarrow D_F^{+i}/F^\bullet^2 \longrightarrow D_F^{(i)} / F^\bullet^2 \xrightarrow{\text{sgn}_F} (\mathbb{Z}/2)r_1 \longrightarrow (\mathbb{Z}/2)^{\delta(F)} \longrightarrow 0.$$  

If $i$ is an even integer, the signature map

$$\text{sgn}_F : D_F^{(i)} / F^\bullet^2 \longrightarrow (\mathbb{Z}/2)r_1 \quad (4)$$

is trivial [As-Mo 18, Proposition 1.2], and then $D_F^{+i} = D_F^{(i)}$.

2.3. Positive étale wild kernel. Following [Ng 92, Ko 93], the étale wild kernel $WK_{2i-2}^\text{ét}F$ is the group

$$WK_{2i-2}^\text{ét}F := \ker( H^2(G_{F,S}, Z_p(i)) \longrightarrow \bigoplus_{v \in S} H^2(F_v, \mathbb{Z}_p(i)) ).$$

For $i \geq 2$, it is well known that the étale wild kernel $WK_{2i-2}^\text{ét}F$ is independent of the set $S$ containing the $p$-adic primes and the infinite primes ([Sc 79, §6, Lemma 1], [Ko 03, page 336]).

There have been much work on the Galois co-descent for the étale wild kernel at odd primes [Ko-Mo 00, Gri 05, As-As, As]. The case $p = 2$ has been studied essentially in the classical case $i = 2$ [Ko-Mo 00, Ko-Mo 03, Gri 05]. The situation for $p = 2$ is more complicated, since the cohomology groups $H^k(R, \mathbb{Z}_2(i))$ and $H^k(R, \mathbb{Q}_2/\mathbb{Z}_2(i))$ do not necessarily vanish, and the group $H^2(G_{E,S}, \mathbb{Z}_2(i))$ does not satisfy Galois co-descent. This motivates the following definition of the positive étale wild kernel.

Let $S_f$ denote the set of finite primes in $S$ and let $\mathcal{O}_{F,S}$ be the ring of $S$-integers of $F$. For all $i \in \mathbb{Z}$, recall the last three terms of the Poitou-Tate exact sequence (Proposition 2.2):

$$H^2_+(G_{F,S}, \mathbb{Z}_2(i)) \longrightarrow \bigoplus_{v \in S_f} H^2(F_v, \mathbb{Z}_2(i)) \longrightarrow H^0(G_{F,S}, \mathbb{Q}_2/\mathbb{Z}_2(1-i)^\vee) \longrightarrow 0$$

**Definition 2.4.** Let $i \in \mathbb{Z}$. We define the positive étale wild kernel $WK_{2i-2}^\text{ét}+\mathcal{O}_{F,S}$ to be the kernel of the localization map

$$H^2_+(G_{F,S}, \mathbb{Z}_2(i)) \longrightarrow \bigoplus_{v \in S_f} H^2(F_v, \mathbb{Z}_2(i)).$$

**Remark 2.5.** For $i = 1$, the group $WK_0^\text{ét}+\mathcal{O}_{F,S}$ is isomorphic to the 2-part of the narrow $S$-class group $A_{F,S}^+$ of $F$. In particular it depends on the set $S$. Indeed, on the one hand Corollary 2.3 shows that

$$WK_0^\text{ét}+\mathcal{O}_{F,S} \cong \Pi_S^1(\mathbb{Q}_2/\mathbb{Z}_2).$$

On the other hand

$$\Pi_S^1(\mathbb{Q}_2/\mathbb{Z}_2) = \ker( H^1(G_{F,S}, \mathbb{Q}_2/\mathbb{Z}_2) \longrightarrow \bigoplus_{v \in S_f} H^1(F_v, \mathbb{Q}_2/\mathbb{Z}_2) )$$

$$= \ker( \text{Hom}(G_{F,S}, \mathbb{Q}_2/\mathbb{Z}_2) \longrightarrow \bigoplus_{v \in S_f} \text{Hom}(G_{F_v}, \mathbb{Q}_2/\mathbb{Z}_2) )$$

$$\cong (A_{F,S}^+)^\vee.$$  

It follows that $WK_0^\text{ét}+\mathcal{O}_{F,S} \cong A_{F,S}^+$.  

Hence the positive étale wild kernel plays a similar role as the 2-primary part of the narrow $S$-class group. We restrict our study to the case $i \geq 2$, and we will show that $WK_{2i-2}^\text{ét}+\mathcal{O}_{F,S}$ is independent of the set $S$ containing the 2-adic primes and the infinite primes. However, all the results remain true for $i \neq 1$ if we assume the finiteness of the Galois cohomology group $H^2(G_{F,S}, \mathbb{Z}_2(i))$. Note that for $i = 0$, this finiteness is equivalent to the Leopoldt
conjecture, and for \( i \geq 2 \) this is true as a consequence of the finiteness of the \( K \)-theory groups \( K_{2i-2}O_{F,S} \) and the connection between \( K \)-theory and étale cohomology via Chern characters [So 79, Dw-Fr 85].

The following proposition gives the link between the kernels \( WK_{2i-2}^{\text{ét}} O_{F,S} \) and \( WK_{2i-2}^{\text{ét}} F \).

**Proposition 2.6.** For all integer \( i \geq 2 \), there exists an exact sequence

\[
0 \longrightarrow D_F^{(i)} / F^* \longrightarrow D_F^{(i)} / F^* \longrightarrow \oplus_{v \mid \infty} H^1(F_v, \mathbb{Z}_2(i)) \longrightarrow WK_{2i-2}^{\text{ét}} O_{F,S} \longrightarrow WK_{2i-2}^{\text{ét}} F \longrightarrow 0.
\]

In particular,
- if \( i \) is even, there is an isomorphism
  \[
  WK_{2i-2}^{\text{ét}} O_{F,S} \cong WK_{2i-2}^{\text{ét}} F;
  \]
- if \( i \) is odd, we have an exact sequence:
  \[
  0 \longrightarrow (\mathbb{Z}/2)^{\delta_i(F)} \longrightarrow WK_{2i-2}^{\text{ét}} O_{F,S} \longrightarrow WK_{2i-2}^{\text{ét}} F \longrightarrow 0
  \]

where \( \delta_i(F) \) is the 2-rank of the cokernel of the signature map \( \text{sgn}_F \).

**Proof.** On the one hand, the exact sequence

\[
0 \longrightarrow \mathbb{Z}_2(i) \longrightarrow \mathbb{Z}_2(i) \longrightarrow \mathbb{Z}/2(i) \longrightarrow 0
\]
gives rise to an exact commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & H^1(G_{F,S}, \mathbb{Z}_2(i))/2 & \longrightarrow & H^1(G_{F,S}, \mathbb{Z}/2(i)) & \longrightarrow & H^2(G_{F,S}, \mathbb{Z}_2(i)) \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \oplus_{v \mid \infty} H^1(F_v, \mathbb{Z}_2(i))/2 & \longrightarrow & \oplus_{v \mid \infty} H^1(F_v, \mathbb{Z}/2(i)) & \longrightarrow & \oplus_{v \mid \infty} H^1(F_v, \mathbb{Z}_2(i))
\end{array}
\]

where the vertical maps are the localization maps. Since

\[
2H^1(F_v, \mathbb{Z}_2(i)) = 0
\]

for all infinite place \( v \) of \( F \), we get

\[
H^1(G_{F,S}, \mathbb{Z}_2(i))/2 \longrightarrow H^1(G_{F,S}, \mathbb{Z}/2(i)) \longrightarrow \oplus_{v \mid \infty} H^1(F_v, \mathbb{Z}_2(i)) \longrightarrow \oplus_{v \mid \infty} H^1(F_v, \mathbb{Z}/2(i))
\]

Observe that the composite

\[
D_F^{(i)} / F^* \cong H^1(G_{F,S}, \mathbb{Z}_2(i))/2 \longrightarrow \oplus_{v \mid \infty} H^1(F_v, \mathbb{Z}_2(i)) \longrightarrow \oplus_{v \mid \infty} H^1(F_v, \mathbb{Z}/2(i)) \cong (\mathbb{Z}/2)^{\delta_i(F)}
\]

is the signature map

\[
\text{sgn}_F : H^1(F, \mathbb{Z}_2(i))/2 \longrightarrow (\mathbb{Z}/2)^{\delta_i(F)}
\]

and then

\[
D_F^{(i)} / F^* \cong \ker( H^1(G_{F,S}, \mathbb{Z}_2(i))/2 \longrightarrow \oplus_{v \mid \infty} H^1(F_v, \mathbb{Z}_2(i)) ).
\]

(5)

On the other hand, by the definition of totally positive Galois cohomology, we have the exact sequence

\[
H^1(G_{F,S}, \mathbb{Z}_2(i)) \longrightarrow \oplus_{v \mid \infty} H^1_v \longrightarrow H^2_+ (G_{F,S}, \mathbb{Z}_2(i)) \longrightarrow H^2(G_{F,S}, \mathbb{Z}_2(i)) \longrightarrow \oplus_{v \mid \infty} H^2_v \longrightarrow 0,
\]

where for \( n = 1 \) or \( 2 \), \( H^2_v \) denotes the cohomology group \( H^n(F_v, \mathbb{Z}_2(i)) \). Since

\[
2H^1(F_v, \mathbb{Z}_2(i)) = 0
\]
for all infinite place $v$ of $F$, we get

$$H^1(G_{F,S}, \mathbb{Z}_2(i))/2 \longrightarrow \bigoplus_{v \mid \infty} H^1_v \longrightarrow H^2_+(G_{F,S}, \mathbb{Z}_2(i)) \longrightarrow H^2(G_{F,S}, \mathbb{Z}_2(i)) \longrightarrow \bigoplus_{v \mid \infty} H^2_v.$$ 

Therefore, we have the following exact commutative diagram

$$
\begin{array}{cccccc}
H^1(G_{F,S}, \mathbb{Z}_2(i))/2 & \longrightarrow & \bigoplus_{v \mid \infty} H^1_v & \longrightarrow & H^2_+(G_{F,S}, \mathbb{Z}_2(i)) & \longrightarrow & H^2(G_{F,S}, \mathbb{Z}_2(i)) & \longrightarrow & \bigoplus_{v \mid \infty} H^2_v \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \bigoplus_{v \in S, f} H^2_v & \longrightarrow & \bigoplus_{v \in S} H^2_v & \longrightarrow & \bigoplus_{v \mid \infty} H^2_v 
\end{array}
$$

By the snake lemma and (5), we obtain the exact sequence

$$0 \longrightarrow D_F^{\pm(i)}/F^* \longrightarrow D_F^{i}/F^* \longrightarrow \bigoplus_{v \mid \infty} H^1(F_v, \mathbb{Z}_2(i)) \longrightarrow WK^{\text{ét}, +}_{2i-2} \mathcal{O}_{F,S} \longrightarrow WK^{\text{ét}}_{2i-2}F \longrightarrow 0.
$$

Since $\bigoplus_{v \mid \infty} H^1(F_v, \mathbb{Z}_2(i))$ is isomorphic to $(\mathbb{Z}/2)^{r_1}$ if $i$ is odd, and is trivial if $i$ is even, we obtain

- for $i$ even, $WK^{\text{ét}, +}_{2i-2} \mathcal{O}_{F,S} \cong WK^{\text{ét}}_{2i-2}F$, and
- for $i$ odd, we have the exact sequence

$$0 \longrightarrow (\mathbb{Z}/2)^{\delta_i(F)} \longrightarrow WK^{\text{ét}, +}_{2i-2} \mathcal{O}_{F,S} \longrightarrow WK^{\text{ét}}_{2i-2}F \longrightarrow 0, \tag{6}$$

where $\delta_i(F)$ is the 2-rank of the cokernel of the signature map $\text{sgn}_F$.

The following corollary shows that $WK^{\text{ét}, +}_{2i-2} \mathcal{O}_{F,S}$ is in fact independent of the set $S$ containing $S_{2\infty} = S_2 \cup S_{\infty}$ for $i \geq 2$.

**Corollary 2.7.** For $i \geq 2$, the positive étale wild kernel $WK^{\text{ét}, +}_{2i-2} \mathcal{O}_{F,S}$ is independent of the set $S$ containing $S_{2\infty}$.

**Proof.** Since $\delta_i(F)$ and $WK^{\text{ét}}_{2i-2}F$ are independent of the set $S$ [Ko 03 page 336], the exact sequence (5) shows that the order of $WK^{\text{ét}, +}_{2i-2} \mathcal{O}_{F,S}$ is also independent of $S$. Therefore it suffices to prove that there exists an injective map from $WK^{\text{ét}, +}_{2i-2} \mathcal{O}_{F,S_{2\infty}}$ to $WK^{\text{ét}, +}_{2i-2} \mathcal{O}_{F,S}$. But this follows from the exact commutative diagram

$$
\begin{array}{cccccc}
0 & \longrightarrow & WK^{\text{ét}, +}_{2i-2} \mathcal{O}_{F,S_{2\infty}} & \longrightarrow & H^2_+(G_{F,S_{2\infty}}, \mathbb{Z}_2(i)) & \longrightarrow & \bigoplus_{v \mid 2} H^2(F_v, \mathbb{Z}_2(i)) \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & WK^{\text{ét}, +}_{2i-2} \mathcal{O}_{F,S} & \longrightarrow & H^2_+(G_{F,S}, \mathbb{Z}_2(i)) & \longrightarrow & \bigoplus_{v \in S, f} H^2(F_v, \mathbb{Z}_2(i))
\end{array}
$$

where the middle vertical map is the inflation map.

From now on, we make the notation

$$WK^{\text{ét}, +}_{2i-2} \mathcal{O}_{F,S} := WK^{\text{ét}}_{2i-2}F, \text{ for } i \geq 2.
$$

We finish this subsection by giving a description of $WK^{\text{ét}, +}_{2i-2}F$ as an Iwasawa module.

Let $X'_{2\infty}$ be the Galois group of the maximal unramified $p$-extension of the cyclotomic $\mathbb{Z}_p$-extension of $F$, which is completely decomposed at all primes above $p$. It is well known that ([Sc 79 Lemma 1,§6]) for any odd prime $p$

$$WK^{\text{ét}}_{2i-2}F \cong X'_{2\infty}(i - 1)\text{Gal}(F(\mu_{p^\infty})/F). \tag{7}$$

In the next proposition, we prove an analogue result in the case $p = 2$. 
Let $F_\infty$ be the cyclotomic $\mathbb{Z}_2$-extension of $F$ with Galois group $\Gamma = \text{Gal}(F_\infty/F)$, and let $X^+_{\infty}$ be the Galois group of the maximal 2-extension of $F_\infty$, which is unramified at finite places and completely decomposed at all primes above 2.

**Proposition 2.8.** Let $i \geq 2$ be an integer. If either $i$ is odd, or $i$ is even and $\sqrt{-1} \in F$, then

$$WK_{2i-2}^{\text{ét}} F \cong X^+_{\infty}(i-1)\Gamma.$$ 

In particular, in both cases we recover that the group $WK_{2i-2}^{\text{ét}} F$ is independent of the set $S$ containing $S_{2\infty}$.

**Proof.** First observe that, if $i$ is odd or $i$ is even and $\sqrt{-1} \in F$, then

$$H^1(\Gamma, Q_2/\mathbb{Z}_2(1-i)) = 0. \quad (8)$$

Indeed, in both cases, $Q_2/\mathbb{Z}_2(1-i)$ is a $\Gamma$-module and using [Se 68, §XIII.1, Proposition 1], we see that

$$H^1(\Gamma, Q_2/\mathbb{Z}_2(1-i)) \cong \left(Q_2/\mathbb{Z}_2(1-i)\right)/(\gamma - 1).\left(Q_2/\mathbb{Z}_2(1-i)\right),$$

where $\gamma$ is a topological generator of $\Gamma$. Hence

$$H^1(\Gamma, Q_2/\mathbb{Z}_2(1-i)) = 0.$$

Now we consider the following exact commutative diagram

$$\begin{array}{cccccc}
H^1(\Gamma, Q_2/\mathbb{Z}_2(j)) & \rightarrow & H^1(G, Q_2/\mathbb{Z}_2(j)) & \rightarrow & H^1(G_\infty, Q_2/\mathbb{Z}_2(j)) & \rightarrow 0 \\
\bigoplus_{v \in S_f} H^1(\pi, Q_2/\mathbb{Z}_2(j)) & \rightarrow & \bigoplus_{v \in S_f} H^1(F, Q_2/\mathbb{Z}_2(j)) & \rightarrow & \bigoplus_{v \in S_f} H^1(F_\infty, Q_2/\mathbb{Z}_2(j)) & \rightarrow 0 \\
\end{array}$$

where $j = 1 - i$, $H^1(F, Q_2/\mathbb{Z}_2(j)) = \bigoplus_{w | v} H^1(F_{w, \infty}, Q_2/\mathbb{Z}_2(j))$, and $\Gamma_v$ denotes the decomposition group of $v$ in $F_\infty/F$. By [8], we have

$$H^1(\Gamma, Q_2/\mathbb{Z}_2(j)) = 0 \text{ and } H^1(\Gamma_v, Q_2/\mathbb{Z}_2(j)) = 0 \text{ for all } v \in S_f,$$

and then

$$\text{III}_{S}(Q_2/\mathbb{Z}_2(j)) = \ker(H^1(G_\infty, Q_2/\mathbb{Z}_2(j))^\Gamma \rightarrow \bigoplus_{v \in S_f} H^1(F_\infty, Q_2/\mathbb{Z}_2(j))^\Gamma)$$

$$= \text{Hom}(X^+_{\infty}, Q_2/\mathbb{Z}_2)(-j)^\Gamma.$$

Hence, using the duality

$$WK_{2i-2}^{\text{ét}} F \cong \text{III}_{S}(Q_2/\mathbb{Z}_2(1-i))^\vee$$

(Corollary 2.3), we obtain the isomorphism

$$WK_{2i-2}^{\text{ét}} F \cong X^+_{\infty}(i-1)\Gamma.$$

\[\square\]

Let $F_\infty = \cup_n F_n$ be the cyclotomic $\mathbb{Z}_2$-extension of $F$ and for $n \geq 0$, $G_n = \text{Gal}(F_n/F)$. The above description of the positive étale wild kernel, leads immediately to the following corollary:

**Corollary 2.9.** If either $i$ is odd, or $i$ is even and $\sqrt{-1} \in F$, then the positive étale wild kernel satisfies Galois co-descent in the cyclotomic $\mathbb{Z}_2$-extension:

$$(WK_{2i-2}^{\text{ét}} F)_n \cong WK_{2i-2}^{\text{ét}} F.$$

Compare to [Ko-Mo 00] Theorem 2.18, which deals with the case $i = 2$ and $\sqrt{-1} \in F$. If $p$ is odd, the Galois co-descent holds in the cyclotomic tower as a consequence of Schneider’s description of the étale wild kernel.
3. Genus formula

Let \( E / F \) be a Galois extension of number fields with Galois group \( G \). Let \( S \) denote the set of infinite places, 2-adic places and those which ramify in \( E / F \). We denote also by \( S \) the set of places of \( E \) above places in \( S \). In the sequel we assume that \( i \geq 2 \).

By the definition of \( WK_{2i-2}^\text{ét} F \) and Proposition 2.2 we have the exact sequence

\[
0 \longrightarrow WK_{2i-2}^\text{ét} F \longrightarrow H^2(G_{E,S}, \mathbb{Z}_2(i)) \longrightarrow \bigoplus_{v \in S_f} H^2(F_v, \mathbb{Z}_2(i)) \longrightarrow 0,
\]

where \( \bigoplus_{v \in S_f} H^2(F_v, \mathbb{Z}_2(i)) \) denotes the kernel of the surjective map

\[
\bigoplus_{v \in S_f} H^2(F_v, \mathbb{Z}_2(i)) \longrightarrow H^0(G_{F,S}, \mathbb{Q}/\mathbb{Z}_2(1 - i))\check{\cdot}.
\]

Then the corestriction map induces the exact commutative diagram

\[
\begin{array}{c}
(WK_{2i-2}^\text{ét} E)_G \longrightarrow H^2(G_{E,S}, \mathbb{Z}_2(i))_G \longrightarrow (\bigoplus_{w \in S_f} H^2(E_w, \mathbb{Z}_2(i)))_G \longrightarrow 0 \\
0 \longrightarrow WK_{2i-2}^\text{ét} F \longrightarrow H^2(G_{E,S}, \mathbb{Z}_2(i)) \longrightarrow \bigoplus_{v \in S_f} H^2(F_v, \mathbb{Z}_2(i)) \longrightarrow 0
\end{array}
\]  \( (9) \)

where the middle vertical map is an isomorphism by (1). Using the snake lemma, we get

- \( \text{coker} N_i \cong \ker N'_i \), where \( N'_i \) denotes \( N_i \) with \( \alpha \) the homology map
  \[
  \bar{\alpha} : H_1(G, H^2(G_{E,S}, \mathbb{Z}_2(i))) \longrightarrow H_1(G, \bigoplus_{w \in S_f} H^2(E_w, \mathbb{Z}_2(i))).
  \]

We first determine \( \text{coker} N_i \), and then we give a criterion of the surjectivity of the morphism \( N_i \). For this, the exact commutative diagram

\[
\begin{array}{c}
(\bigoplus_{w \in S_f} H^2(E_w, \mathbb{Z}_2(i)))_G \longrightarrow (\bigoplus_{w \in S_f} H^2(E_w, \mathbb{Z}_2(i)))_G \longrightarrow (H^0(E, \mathbb{Q}_2/\mathbb{Z}_2(1 - i))\check{\cdot})_G \\
\bigoplus_{v \in S_f} H^2(F_v, \mathbb{Z}_2(i)) \longrightarrow \bigoplus_{v \in S_f} H^2(F_v, \mathbb{Z}_2(i)) \longrightarrow H^0(F, \mathbb{Q}_2/\mathbb{Z}_2(1 - i))\check{\cdot}
\end{array}
\]  \( (10) \)

shows that

\[
\ker N'_i \cong \text{coker}( H_1(G, \bigoplus_{w \in S_f} H^2(E_w, \mathbb{Z}_2(i))) \longrightarrow H_1(G, H^0(E, \mathbb{Q}_2/\mathbb{Z}_2(1 - i)))\check{\cdot}).
\]

Now we give a description of \( H_1(G_v, H^2(E_w, \mathbb{Z}_2(i))) \) and \( H_1(G, H^0(E, \mathbb{Q}_2/\mathbb{Z}_2(1 - i)))\check{\cdot} \). We need some notation

- \( E_\infty \) : the cyclotomic \( \mathbb{Z}_2 \)-extension of \( E \).
- \( G_v \) : the decomposition group of \( v \) in \( E / F \).
- \( \Gamma_v \) : the decomposition group of \( v \) in \( F_\infty / F \).
- \( H \) : the 2-part of the abelianization of \( \text{Gal}(E_\infty / F_\infty) \).
- \( H_v \) : the 2-part of the abelianization of \( \text{Gal}(E_w, F_\infty) \).
- \( L^+_{\infty} \) : the maximal abelian 2-extension of \( F_\infty \), which is unramified at finite places and completely decomposed at all primes above 2.

**Proposition 3.1.** Let \( i \geq 2 \) be an integer and let \( v \) be a finite place of \( F \). If either \( i \) is odd, or \( i \) is even and \( \sqrt{-1} \in F \), then we have

1. \( H_1(G_v, H^2(E_w, \mathbb{Z}_2(i))) \cong H_v(i - 1)\Gamma_v \) and \( H_1(G, H^0(E, \mathbb{Q}_2/\mathbb{Z}_2(1 - i)))\check{\cdot} \cong H(i - 1)\Gamma \).
2. \( \text{coker} N_i \cong \text{Gal}(L^+_{\infty} \cap E_\infty / F_\infty)(i - 1)\Gamma \).

In particular, the map \( N_i \) is surjective if and only if \( L^+_{\infty} \cap E_\infty = F_\infty \).
Proof. Using the assumption and [8], we have
\[ H^1(\Gamma, Q_2/Z_2(1-i)) = 0 \text{ and } H^1(\Gamma_v, Q_2/Z_2(1-i)) = 0. \]
Then (1) can be proved with the same argument of Proposition 2.1 of [As-As]. The second assertion is a direct consequence of the first one. \qed

To prove a genus formula for the positive étale wild kernel, we give a description of \( \ker N_i \cong \coker \alpha \).

Consider the following exact commutative diagram
\[
\begin{array}{c}
H_2(G, \bigoplus_{w \in S_I} H^2(E_w, Z_2(i))) \\
\downarrow \kappa \\
H_2(G, H^0(E, Q_2/Z_2(1-i))^\vee) \\
\downarrow \\
H_1(G, H^2_\Sigma(G_{E,S}, Z_2(i))) \\
\downarrow \alpha \\
H_1(G, \bigoplus_{w \in S_I} H^2(E_w, Z_2(i))) \\
\downarrow \theta \\
H_1(G, H^0(E, Q_2/Z_2(1-i))^\vee).
\end{array}
\]

Then we have an exact sequence
\[
0 \longrightarrow \ker \tilde{\alpha} \longrightarrow \ker \alpha \longrightarrow \coker \kappa \longrightarrow \coker \tilde{\alpha} \longrightarrow \coker \alpha \longrightarrow \Im \theta \longrightarrow 0. \tag{12}
\]

**Definition 3.2.** We define the module \( X^{(i)}_{E/F} \) as
\[
X^{(i)}_{E/F} := \Im( \coker \kappa \longrightarrow \coker \tilde{\alpha} ),
\]
where \( \kappa \) is the homology map
\[
H_2(G, \bigoplus_{w \in S_I} H^2(E_w, Z_2(i))) \longrightarrow H_2(G, H^0(E, Q_2/Z_2(1-i))^\vee).
\]

So
\[
|X^{(i)}_{E/F}| \leq |H_2(G, H^0(E, Q_2/Z_2(1-i))^\vee)|.
\]

We have the following comparison between \( |(WK_{2i-2} E)_G| \) and \( |WK_{2i-2} F| \).

**Proposition 3.3.** For any integer \( i \geq 2 \), we have
\[
\frac{|(WK_{2i-2} E)_G|}{|WK_{2i-2} F|} = \frac{|X^{(i)}_{E/F}| \cdot |\coker \alpha|}{|H_1(G, H^0(E, Q_2/Z_2(1-i))^\vee)|}.
\]

**Proof.** On the one hand, by (12) and the definition of \( X^{(i)}_{E/F} \), we have an exact sequence
\[
0 \longrightarrow X^{(i)}_{E/F} \longrightarrow \coker \tilde{\alpha} \longrightarrow \coker \alpha \longrightarrow \Im \theta \longrightarrow 0.
\]

On the other hand the exact commutative diagram (10) shows that
\[
\ker N_i' = \coker( H_1(G, \bigoplus_{w \in S_I} H^2(E_w, Z_2(i))) \longrightarrow H_1(G, H^0(G_{E,S}, Q_2/Z_2(1-i))^\vee)) = \coker \theta.
\]
Since
\[
\ker N_i \cong \ker N_i' \quad \text{and} \quad |\Im \theta|, |\coker \theta| = |H_1(G, H^0(E, Q_2/Z_2(1-i))^\vee)|,
\]
we obtain
\[
\frac{|(WK^\text{ét}_{i-2} + E)_G|}{|WK^\text{ét}_{i-2} F|} = \frac{|X^{(i)}_{E/F}| \cdot |\text{coker}\alpha|}{|H_1(G, H^0(E, Q_2/Z_2(1 - i))^\vee)|}.
\]
\[\square\]

Now we are going to compute |coker|α. For every \(q \in \mathbb{Z}\), we have an isomorphism
\[
\hat{H}^q(G, H^2_+(G_{E,S}, \mathbb{Z}_2(i))) \cong \hat{H}^{q+2}(G, H^1_+(E, \mathbb{Z}_2(i))),
\]
given by cup-product ([CKPS, Proposition 3.1]), where \(\hat{H}^*(\ldots, \ldots)\) denotes the Tate cohomology. Then the commutative diagram
\[
\begin{array}{ccc}
H_1(G, H^2_+(G_{E,S}, \mathbb{Z}_2(i))) & \xrightarrow{\alpha} & H_1(G, \oplus_{w \in S_f} H^2(E_w, \mathbb{Z}_2(i))) \\
\downarrow i & & \downarrow i \\
\hat{H}^0(G, H^1_+(E, \mathbb{Z}_2(i))) & \xrightarrow{\beta} & \hat{H}^0(G, \oplus_{w \in S_f} H^1(E_w, \mathbb{Z}_2(i)))
\end{array}
\]
shows that
\[
\text{coker}\alpha \cong \text{coker}\beta.
\]

**Definition 3.4.** Let \(H^1_+^{1/N}(F, \mathbb{Z}_2(i))\) denote the kernel of the map
\[
H^1_+(F, \mathbb{Z}_2(i)) \rightarrow \oplus_{v \in S_f} \frac{H^1(F_v, \mathbb{Z}_2(i))}{N_{G_v} H^1(E_v, \mathbb{Z}_2(i))}
\]
where \(N_{G_v} = \sum_{\sigma \in G_v} \sigma\) is the norm map.

The isomorphism (13) shows that
\[
\text{Im}(\alpha) \cong H^1_+(F, \mathbb{Z}_2(i))/H^1_+^{1/N}(F, \mathbb{Z}_2(i)).
\]
Hence
\[
|\text{coker}\alpha| = \frac{\prod_{v \in S_f} |H_1(G_v, H^2(E_w, \mathbb{Z}_2(i)))|}{|H^1_+(F, \mathbb{Z}_2(i)) : H^1_+^{1/N}(F, \mathbb{Z}_2(i))|}.
\]
This yields our main result:

**Theorem 3.5.** Let \(E/F\) be a Galois extension of number fields with Galois group \(G\). Then for every \(i \geq 2\), we have
\[
\frac{|(WK^\text{ét}_{i-2} + E)_G|}{|WK^\text{ét}_{i-2} F|} = \frac{|X^{(i)}_{E/F}| \cdot \prod_{v \in S_f} |H_1(G_v, H^2(E_w, \mathbb{Z}_2(i)))|}{|H_1(G, H^0(E, Q_2/Z_2(1 - i))^\vee)| \cdot |H^1_+(F, \mathbb{Z}_2(i)) : H^1_+^{1/N}(F, \mathbb{Z}_2(i))|}.
\]
\[\square\]

**Remark 3.6.** The above genus formula involves the order of the group \(X^{(i)}_{E/F}\), which seems to be difficult to compute. If we make the following hypothesis (H):

"the morphism
\[
\kappa : H_2(G, \oplus_{w \in S_f} H^2(E_w, \mathbb{Z}_2(i))) \rightarrow H_2(G, H^0(E, Q_2/Z_2(1 - i))^\vee)
\]
is surjective",

then \(X^{(i)}_{E/F}\) is trivial.
This hypothesis is satisfied if there is a prime \(v_0\) of \(F\) such that:

- \(H^0(E, Q_2/Z_2(1 - i)) \cong H^0(E_{w_0}, Q_2/Z_2(1 - i))\)
- \(v_0\) is an undecomposed 2-adic prime, or \(v_0\) is a totally and tamely ramified prime in \(E/F\).
Remark 3.7. The groups $H_1(G_v, H^2(E_w, \mathbb{Z}_2(i)))$ can be easily computed (at least if $i$ is odd, or $i$ is even and $\sqrt{-1} \in F$ by Proposition \[7.7\]). The difficult part here is the norm index $[H^1_+(F, \mathbb{Z}_2(i)) : H^1_+(N)(F, \mathbb{Z}_2(i))]$. When $E/F$ is a relative quadratic extension, we obtain a genus formula involving the norm index $[D_F^{(i)} : D_F^{(i)} \cap N_G E^*]$. Moreover, if $F$ has at most one 2-adic prime undecomposed in $E/F$, we use \[As-Mo 18, \S 4\] to give an explicit description of this norm index in terms of the ramification in $E/F$.

4. Relative quadratic extension case

In this section we focus on relative quadratic extensions of number fields $E/F$ with Galois group $G$. For such extensions, we give a genus formula for the positive étale wild kernel involving the norm index $[D_F^{(i)} : D_F^{(i)} \cap N_G E^*]$, where $D_F^{(i)}$ is the positive Tate kernel.

First recall that for every even integer $i \geq 2$, Proposition \[2.6\] says that the étale wild kernel and the positive étale wild kernel coincide. A genus formula has been obtained by Kolster-Movahhedi \[Ko-Mo 00, \text{Theorem 2.18}\] for $i = 2$, and by Griffiths \[Gri 05, \S 4.3\], as a generalization, for any even integer $i \geq 2$. This genus formula can be used to determine families of abelian 2-extensions with trivial 2-primary Hilbert kernel \[Ko-Mo 03, \text{Le 04, Gr-Le 09}\].

Throughout this section we keep the notations of the previous sections and we assume that the integer $i \geq 2$ is odd.

We need to calculate the order of $\text{coker} \alpha$ (see Proposition \[3.3\]). Since $G$ has order 2, we have the following exact commutative diagram

$$
\begin{array}{ccc}
\hat{H}^0(G, H^1_+(E, \mathbb{Z}_2(i))) & \xrightarrow{\beta} & \oplus_{v \in S_f} \hat{H}^0(G_v, H^1(E_w, \mathbb{Z}_2(i))) \\
\downarrow \beta' & & \downarrow \beta' \\
H^1_+(F, \mathbb{Z}_2(i))/2/N_G(H^1_+(E, \mathbb{Z}_2(i))/2) & \xrightarrow{\oplus_{v \in S_f} H^1(F_v, \mathbb{Z}_2(i))/2/N_{G_v}(H^1(E_w, \mathbb{Z}_2(i))/2) \\
\end{array}
$$

Likewise as in the global case, there exists a subgroup $D_v^{(i)}$ of $F_v^*$ containing $F_v^{*2}$ such that $H^1(F_v, \mathbb{Z}_2(i))/2 \cong D_v^{(i)}/F_v^{*2}$ for each $v \in S_f$. Then, we have a natural isomorphism

$$H^1(F_v, \mathbb{Z}_2(i))/2/N_{G_v}(H^1(E_w, \mathbb{Z}_2(i))/2) \cong D_v^{(i)}/F_v^{*2} N_{G_v}(D_w^{(i)})$$

where $w$ is a prime of $E$ above $v$. Hence,

$$\text{coker} \alpha \cong \text{coker} \beta \quad (\text{by } \text{(13)})$$

$$\cong \text{coker} \beta'$$

$$\cong \text{coker} (\delta : H^1_+(F, \mathbb{Z}_2(i))/2/N_G(H^1_+(E, \mathbb{Z}_2(i))/2) \xrightarrow{\oplus_{v \in S_f} D_v^{(i)}/F_v^{*2} N_{G_v}(D_w^{(i)}))}$$

On the one hand, there exists a surjective map

$$H^1_+(F, \mathbb{Z}_2(i))/2/N_G(H^1_+(E, \mathbb{Z}_2(i))/2) \xrightarrow{\oplus_{v \in \infty} H^1(F_v, \mathbb{Z}_2(i))} D_F^{(i)}/F^{*2} N_G D_E^{(i)}$$

Indeed, the exact sequence

$$\oplus_{v \in \infty} H^0(F_v, \mathbb{Z}_2(i)) \longrightarrow H^1_+(F, \mathbb{Z}_2(i)) \longrightarrow H^1(F, \mathbb{Z}_2(i)) \longrightarrow \oplus_{v \in \infty} H^1(F_v, \mathbb{Z}_2(i))$$

induces the exact sequence

$$H^1_+(F, \mathbb{Z}_2(i))/2 \longrightarrow H^1(F, \mathbb{Z}_2(i))/2 \longrightarrow \oplus_{v \in \infty} H^1(F_v, \mathbb{Z}_2(i)).$$
Then, by the definition of $D_{F}^{+}(i)$, we have a surjective map
\[ H_{+}^{1}(F, \mathbb{Z}_{2}(i))/2 \rightarrow D_{F}^{+}(i)/F^{*2} \rightarrow 0. \]
Hence, the surjectivity of the map (13) follows from the exact commutative diagram
\[
\begin{aligned}
H_{+}^{1}(E, \mathbb{Z}_{2}(i))/2 & \rightarrow D_{E}^{+}(i)/E^{*2} \rightarrow 0 \\
\downarrow N_{G} & \downarrow N_{G} \\
H_{+}^{1}(F, \mathbb{Z}_{2}(i))/2 & \rightarrow D_{F}^{+}(i)/F^{*2} \rightarrow 0 \\
\downarrow & \\
H_{+}^{1}(F, \mathbb{Z}_{2}(i))/2/N_{G}(H_{+}^{1}(E, \mathbb{Z}_{2}(i))/2) & \rightarrow D_{F}^{+}(i)/F^{*2}N_{G}D_{E}^{+}(i) \rightarrow 0
\end{aligned}
\]
On the other hand, since $i$ is odd, the canonical surjection map
\[ \psi^{(i)}_{v} : D_{v}^{+}/F_{v}^{*2}N_{G_{v}}(D_{w}^{(i)}) \rightarrow D_{v}^{(i)}/D_{v}^{(i)} \cap N_{G_{v}}(E_{w}^{*}) \]
is an isomorphism, as a consequence of \cite[Lemma 4.2.1]{Gri05}. Therefore,
\[ \text{Im}(\delta) \cong \text{Im}(D_{F}^{+}(i)/F^{*2}N_{G}D_{E}^{+}(i) \rightarrow \oplus_{v \in S_{J}}D_{v}^{(i)}/D_{v}^{(i)} \cap N_{G_{v}}(E_{w}^{*})) , \]
it follows that
\[ |\text{Im}(\delta)| = [D_{F}^{+}(i) : D_{F}^{+}(i) \cap N_{G}E^{*}] \]
where, in the last equality, $v$ runs through all places of $F$. Then, using the Hasse norm theorem ($G$ is cyclic), we get
\[ |\text{Im}(\delta)| = [D_{F}^{+}(i) : D_{F}^{+}(i) \cap N_{G}E^{*}]. \]
Therefore, we obtain
\[ |\text{coker}\alpha| = \frac{\prod_{v \in S_{J}}|H_{1}(G_{v}, H^{2}(E_{w}, \mathbb{Z}_{2}(i)))|}{[D_{F}^{+}(i) : D_{F}^{+}(i) \cap N_{G}E^{*}].} \]
Moreover, the order of the group $X_{E/F}^{(i)}$ (see Definition 3.2) is at most 2:
\[ |X_{E/F}^{(i)}| = 2^{t}, \ t \in \{0, 1\}. \]
Indeed, by the definition of $X_{E/F}^{(i)}$, we have
\[ |X_{E/F}^{(i)}| \leq |H_{2}(G, H^{0}(E, \mathbb{Q}_{2}/\mathbb{Z}_{2}(1 - i))^{\vee})| \leq 2, \]
since $G$ is cyclic of order 2. Hence, by Proposition 3.3 we get
\[ \frac{|(WK_{2i-2}^{\text{et},+} E)_{G}|}{|WK_{2i-2}^{\text{et},+} F|} = \frac{2^{t} \cdot \prod_{v \in S_{J}}|H_{1}(G_{v}, H^{2}(E_{w}, \mathbb{Z}_{2}(i)))|}{|H_{1}(G, H^{0}(E, \mathbb{Q}_{2}/\mathbb{Z}_{2}(1 - i))^{\vee})| \cdot [D_{F}^{+}(i) : D_{F}^{+}(i) \cap N_{G}E^{*}]} \]
where $t \in \{0, 1\}$.}

First, assume that $E \subseteq F_{\infty}$. Then for every finite prime $v$, we have
\[ H_{1}(G_{v}, H^{2}(E_{w}, \mathbb{Z}_{2}(i))) = 0, \]
by Proposition 3.1. Since $G$ is cyclic, we also have
\[ |H_{2}(G_{v}, H^{2}(E_{w}, \mathbb{Z}_{2}(i)))| = |H_{1}(G_{v}, H^{2}(E_{w}, \mathbb{Z}_{2}(i)))| = 1, \]
and then, using the commutative diagram (I), we see that
cokerα = 0.

Then the map

\[ N_i : (WK_{2i-2}^{\text{ét},+}E)_G \longrightarrow WK_{2i-2}^{\text{ét},+}F \]

is an isomorphism, as a consequence of Proposition 3.1 and the fact that ker \( N_i \cong \text{coker} \bar{\alpha} \).

Now, if \( E \not\subseteq F_\infty \), then

\[ |H_1(G, H^0(E, Q_2/Z_2(1-i)))| = 2 \]

and if \( v \mid 2 \) is a finite ramified prime in \( E/F \) or \( v \) is a 2-adic prime such that \( E_w \cap F_{v,\infty} \neq E_w \), then

\[ |H_1(G_v, H^2(E_w, Z_2(i)))| = 2. \]

Since \( G \) is cyclic, \( |(WK_{2i-2}^{\text{ét},+}E)_G| = |(WK_{2i-2}^{\text{ét},+}E)^G| \). We can now formulate the genus formula for a relative quadratic extension:

**Proposition 4.1.** Let \( E/F \) be a relative quadratic extension of number fields with Galois group \( G \) and let \( R_{E/F} \) be the set of finite primes tamely ramified in \( E/F \) or 2-adic primes such that \( E_w \cap F_{v,\infty} \neq E_w \). Then for any odd positive integer \( i \geq 2 \)

(i) if \( E \subseteq F_\infty \) then the positive étale wild kernel satisfies Galois codescent, and

(ii) if \( E \not\subseteq F_\infty \),

\[ \frac{|(WK_{2i-2}^{\text{ét},+}E)_G|}{|WK_{2i-2}^{\text{ét},+}F|} = \frac{2^r(E/F) - 1 + t}{[D_F^{+(i)} : D_F^{+(i)} \cap NG^*]}, \]

where \( r(E/F) = |R_{E/F}| \) and \( t \in \{0, 1\} \).

Recall that under the hypothesis (H) the group \( X_{E/F}^{(i)} \) is trivial. Moreover, if \( E/F \) is a relative quadratic extension of number fields, the hypothesis (H) is satisfied precisely when the set \( R_{E/F} \) is nonempty. We obtain

**Corollary 4.2.** Let \( E/F \) be a relative quadratic extension of number fields such that \( E \not\subseteq F_\infty \) with Galois group \( G \). If \( R_{E/F} \neq \emptyset \) then we have

\[ \frac{|(WK_{2i-2}^{\text{ét},+}E)_G|}{|WK_{2i-2}^{\text{ét},+}F|} = \frac{2^r(E/F) - 1}{[D_F^{+(i)} : D_F^{+(i)} \cap NG^*]}. \]

**Example 4.3.** If \( F \) has at most one 2-adic prime undecomposed and the set of ramified primes is nonempty, then using [As-Mo 12, Theorem 2.4] and [As-Mo 18, Proposition 4.11], we get

\[ [D_F^{+(i)} : D_F^{+(i)} \cap NG^*] = 2t_i^+, \]

where \( t_i^+ = \dim \text{Im}(D_F^{+(i)}/F^{s^2} \longrightarrow \oplus_{v \in T \setminus S_2} D_v^{(i)}/F_v^{s^2}) \) with \( T = \{v \text{ ramified in } E/F \} \cup S_2 \).

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