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Multiple Periodic Solutions for Odd Perturbations of the Discrete Relativistic Operator

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Abstract: We obtain the existence of multiple pairs of periodic solutions for difference equations of type

$$-\Delta \left( \frac{\Delta u(n-1)}{\sqrt{1-|\Delta u(n-1)|^2}} \right) = \lambda g(u(n)) \quad (n \in \mathbb{Z}),$$

where $$\Delta u(n) = u(n+1) - u(n)$$ is the usual forward difference operator, $$\lambda > 0$$ is a real parameter, $$g : \mathbb{R} \to \mathbb{R}$$ is a continuous odd function, and

$$\phi(y) = \frac{y}{\sqrt{1-y^2}} \quad (y \in (-1,1)).$$

In recent years, special attention has been paid to the existence and multiplicity of $$T$$-periodic solutions for problems with a discrete relativistic operator. Thus, for instance, in [1,2], variational arguments were employed to prove the solvability of systems of difference equations having the form

$$\Delta[\phi_N(\Delta u(n-1))] = \nabla_{\mathbf{a}} V(n,u(n)) + h(n) \quad (n \in \mathbb{Z}),$$

under various hypotheses upon $$V$$ and $$h$$ (coerciveness, growth restriction, convexity or periodicity conditions); here, $$\phi_N$$ is the $$N$$-dimensional variant of $$\phi$$, i.e.,

$$\phi_N(y) = \frac{y}{\sqrt{1-|y|^2}} \quad (y \in \mathbb{R}^N, |y| < 1).$$

The existence of at least $$N+1$$ geometrically distinct $$T$$-periodic solutions of (2) was proved in [3], under the assumptions that $$h$$ is $$T$$-periodic, $$\sum_{j=1}^{T} h(j) = 0$$, and the mapping $$V(n,x)$$ is $$T$$-periodic in $$n$$ and $$\omega_i$$-periodic ($$\omega_i > 0$$) with respect to each $$x_i$$ ($$i = 1, \ldots, N$$). For the proof, using an idea from the differential case [4], the singular problem (2) was reduced to an equivalent non-singular one to which classical Ljusternik–Schnirelmann category methods can be applied. In addition, under some similar assumptions on $$V$$ and $$h$$,
were obtained in [5] using Morse theory, conditions under which system (2) has at least $2^N$ geometrically distinct $T$-periodic solutions.

The motivation of the present study mainly comes from paper [6], where for problems involving Fisher-Kolmogorov nonlinearities of type

$$-\Delta[\phi(\Delta u(n-1))] = \lambda u(n)(1 - |u(n)|^q), \quad u(n) = u(n + T) \quad (n \in \mathbb{Z}),$$

with $q > 0$ fixed and $\lambda > 0$ a real parameter, it was proved that if $\lambda > 8mT$ for some $m \in \mathbb{N}$ with $2 \leq m \leq T$, then problem (3) has at least $m$ distinct pairs of nontrivial solutions. We also refer the interested reader to [6] for a discussion concerning the origin and steps in the study of this type of nonlinearity. In this respect, we shall see in Example 1 below that a sharper result holds true, namely,

(i) If $\lambda > 8\sin^2 \frac{m\pi}{T}$ with $0 \leq m \leq \left\{ \begin{array}{ll} (T - 1)/2 & \text{if } T \text{ is odd} \\ (T - 2)/2 & \text{if } T \text{ is even} \end{array} \right.$, then problem (3) has at least $2m + 1$ distinct pairs of nontrivial solutions.

(ii) If $T$ is even and $\lambda > 8$, then (3) has at least $T$ distinct pairs of nontrivial solutions.

Moreover, we prove in Theorem 2 that the above statements (i) and (ii) still remain valid for a larger class of periodic problems.

As in [6], our approach to problem (1) is variational and combines a Clark-type abstract result for convex, lower semicontinuous perturbations of $C^1$-functionals, based on Krasnoselskii’s genus. However, our technique here brings the novelty that it exploits the interference of the geometry of the energy functional with fine spectral properties of the operator $-\Delta^2$; recall that

$$\Delta^2u(n-1) := \Delta(\Delta u(n-1)) = u(n+1) - 2u(n) + u(n-1).$$

It is worth noting that in paper [7] analogous multiplicity results are obtained in the differential case for potential systems involving parametric odd perturbations of the relativistic operator. In addition, we mention the recent paper [8], where the authors obtain the existence and multiplicity of sign-changing solutions for a slightly modified parametric problem of type (1) using bifurcation techniques.

We conclude this introductory part by briefly recalling some topics in the frame of Szulkin’s critical point theory [9], which is needed in the sequel. Let $(Y, \| \cdot \|)$ be a real Banach space and $I : Y \to (-\infty, +\infty]$ be a functional having the following structure:

$$I = F + \psi,$$

where $F \in C^1(Y, \mathbb{R})$ and $\psi : Y \to (-\infty, +\infty]$ is proper, convex and lower semicontinuous. A point $u \in D(\psi)$ is said to be a critical point of $I$ if it satisfies the inequality

$$\langle F'(u), v - u \rangle + \psi(v) - \psi(u) \geq 0 \quad \forall \, v \in D(\psi).$$

A sequence $\{u_n\} \subset D(\psi)$ is called a (PS)-sequence if $I(u_n) \to c \in \mathbb{R}$ and

$$\langle F'(u_n), v - u_n \rangle + \psi(v) - \psi(u_n) \geq -\varepsilon_n \|v - u_n\| \quad \forall \, v \in D(\psi),$$

where $\varepsilon_n \to 0$. The functional $I$ is said to satisfy the (PS) condition if any (PS)-sequence has a convergent subsequence in $Y$.

Let $\Sigma$ be the collection of all symmetric subsets of $Y \setminus \{0\}$ which are closed in $Y$. The genus of a nonempty set $A \subset \Sigma$ is defined as being the smallest integer $k$ with the property that there exists an odd continuous mapping $h : A \to \mathbb{R}^k \setminus \{0\}$; in this case, we write $\gamma(A) = k$. If such an integer does not exist, then $\gamma(A) := +\infty$. Notice that if $A \subset \Sigma$ is homeomorphic to $S^{k-1}$ ($k - 1$ dimension unit sphere in the Euclidean space $\mathbb{R}^k$) by an odd homeomorphism, then $\gamma(A) = k$ ([10], Corollary 5.5). For other properties and more details on the notion of genus, we refer the reader to [10,11]. The following theorem is an immediate consequence of ([9], Theorem 4.3).
Theorem 1. Let \( I \) be of type (4) with \( \mathcal{F} \) and \( \psi \) even. In addition, suppose that \( I \) is bounded from below, satisfies the (PS) condition and \( I(0) = 0 \). If there exists a nonempty compact symmetric subset \( A \subset Y \setminus \{0\} \) with \( \gamma(A) \geq k \), such that
\[
\sup_{v \in A} I(v) < 0,
\]
then the functional \( I \) has at least \( k \) distinct pairs of nontrivial critical points.

2. Variational Approach and Preliminaries

To introduce the variational formulation for problem (1), let \( H_T \) be the space of all \( T \)-periodic \( Z \)-sequences in \( \mathbb{R} \), i.e., of mappings \( u : \mathbb{Z} \to \mathbb{R} \), such that \( u(n) = u(n + T) \) for all \( n \in \mathbb{Z} \). On \( H_T \), we consider the following inner product and corresponding norm:
\[
(u|v) := \sum_{j=1}^{T} u(j)v(j), \quad \|u\| = \left( \sum_{j=1}^{T} |u(j)|^2 \right)^{1/2},
\]
which makes it a Hilbert space. In addition, for each \( u \in H_T \), we set
\[
\bar{u} := \frac{1}{T} \sum_{j=1}^{T} u(j), \quad \tilde{u} := u - \bar{u}.
\]
It is not difficult to check that
\[
|\tilde{u}(i)| \leq T \frac{1}{2} \left( \sum_{j=1}^{T} |\Delta u(j)|^2 \right)^{1/2} \quad (i \in \{1, \ldots, T\}). \tag{5}
\]
Now, let the closed convex subset \( K \) of \( H_T \) be defined by
\[
K := \{ u \in H_T : |\Delta u|_{\infty} \leq 1 \},
\]
where \( |\Delta u|_{\infty} := \max_{i=1,\ldots,T} |\Delta u(i)| \). Then, from (5), one has
\[
|\bar{u}| - T \leq |u(i)| \leq |\bar{u}| + T \quad (i \in \{1, \ldots, T\}), \tag{6}
\]
for all \( u \in K \). We introduce the even functions
\[
\Psi(u) = \begin{cases} 
\sum_{j=1}^{T} \Phi[\Delta u(j)], & \text{if } u \in K, \\
+\infty, & \text{otherwise},
\end{cases}
\]
where \( \Phi(y) = 1 - \sqrt{1 - y^2} \) (\( y \in [-1, 1] \)) and
\[
\mathcal{G}_\lambda(u) = -\lambda \sum_{j=1}^{T} G(u(j)) \quad (u \in H_T),
\]
with \( G \) the primitive
\[
G(x) = \int_{0}^{x} g(\tau) d\tau \quad (x \in \mathbb{R}).
\]
It is not difficult to see that \( \Psi \) is convex and lower semicontinuous, while \( \mathcal{G}_\lambda \) is of class \( C^1 \), its derivative being given by
\[
\langle \mathcal{G}_\lambda'(u), v \rangle = -\lambda \sum_{j=1}^{T} g(u(j))v(j) \quad (u, v \in H_T).
Then, the functional $I_\lambda : H_T \to (-\infty, +\infty]$ associated to (1) is

$$I_\lambda = \Psi + G_\lambda$$

and it is clear that it has the structure required by Szulkin’s critical point theory. A solution of problem (1) is an element $u \in H_T$ such that $|\Delta u(n)| < 1$, for all $n \in \mathbb{Z}$, which satisfies the equation in (1). The following result reduces the search of solutions of problem (1) to finding critical points of $I_\lambda$.

**Proposition 1.** Any critical point of $I_\lambda$ is a solution of problem (1).

**Proof.** Let $e \in H_T$. By virtue of Lemmas 5 and 6 in [1], the problem

$$\Delta[\phi(\Delta u(n - 1))] = \pi + e(n), \quad u(n) = u(n + T) \quad (n \in \mathbb{Z})$$

has a unique solution $u_e$, which is also the unique solution of the variational inequality

$$\sum_{j=1}^{T} \left[ \Phi[\Delta v(j)] - \Phi[\Delta u(j)] + \pi(\pi - \pi) + e(j)(v(j) - u(j)) \right] \geq 0, \quad \forall v \in K \quad (7)$$

([6], Proposition 3.1). Next, let $w \in K$ be a critical point of $I_\lambda$. Then, for any $v \in K$, one has

$$\sum_{j=1}^{T} \left[ \Phi[\Delta v(j)] - \Phi[\Delta w(j)] - \lambda g(w(j))(v(j) - w(j)) \right] \geq 0,$$

which can be written as

$$\sum_{j=1}^{T} \left[ \Phi[\Delta v(j)] - \Phi[\Delta w(j)] + \pi(\pi - \pi) + e_w(j)(v(j) - w(j)) \right] \geq 0, \quad \forall v \in K, \quad (8)$$

with $e_w \in H_T$ being given by $e_w(n) = -\lambda g(w(n)) - \pi (n \in \mathbb{Z})$.

Therefore, by (8) and the uniqueness of the solution of (7), we obtain that, in fact, $w$ solves problem (1). \Box

**Proposition 2.** If $G$ is anticoercive, i.e.,

$$\lim_{|x| \to +\infty} G(x) = -\infty,$$

then $I_\lambda$ is bounded from below and satisfies the (PS) condition.

**Proof.** From (9) we have that $-G$, hence $G_\lambda$, are bounded from below on $\mathbb{R}$, respectively on $H_T$. This, together with the fact that $\Psi$ is bounded from below, ensure that the same is true for $I_\lambda$.

To see that $I_\lambda$ satisfies the (PS) condition, let $\{u_n\} \subset K$ be a (PS)-sequence. Assuming by contradiction that $\{\|u_n\|\}$ is not bounded, we may suppose, going, if necessary, to a subsequence, that $\|u_n\| \to +\infty$. Then, by virtue of (6) and (9), we deduce that $I_\lambda(u_n) \to -\infty$, contradicting the fact that $\{I_\lambda(u_n)\}$ is convergent. Consequently, $\{\|u_n\|\}$ is bounded. This, together with $|u_n| \leq T$ shows that $\{u_n\}$ is bounded in the finite-dimensional space $H_T$; hence, it contains a convergent subsequence. \Box
Remark 1. Notice that until here in this section, no parity assumptions on the continuous function \( g : \mathbb{R} \to \mathbb{R} \) must be required.

We end this section by reviewing some spectral properties of the operator \(-\Delta^2\), which is needed in the sequel. A real number \( \lambda \in \mathbb{R} \) is said to be an eigenvalue of \(-\Delta^2\) on \( H_T \), if there is some \( u \in H_T \setminus \{0_H\} \) such that

\[
-\Delta^2 u(n-1) = \lambda u(n), \quad (n \in \mathbb{Z}) \tag{10}
\]

and in this case, \( u \) is called eigensequence corresponding to the eigenvalue \( \lambda \). On account of the periodicity of \( u \), relation (10) is equivalent to the system

\[
\begin{align*}
-u(2) + 2u(1) - u(T) &= \lambda u(1) \\
-u(3) + 2u(2) - u(1) &= \lambda u(2) \\
&\vdots \\
-u(T) + 2u(T-1) - u(T-2) &= \lambda u(T-1) \\
-u(1) + 2u(T) - u(T-1) &= \lambda u(T).
\end{align*} \tag{11}
\]

If we consider the particular circulant matrix

\[
M_T := \begin{pmatrix}
2 & -1 & 0 & \cdots & 0 & 0 & -1 \\
-1 & 2 & -1 & \cdots & 0 & 0 & 0 \\
& \vdots & & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \cdots & -1 & 2 & -1 \\
-1 & 0 & 0 & \cdots & 0 & -1 & 2
\end{pmatrix}
\]

then, having in view (11), the eigenvalues of \(-\Delta^2\) are precisely the characteristic roots of \( M_T \). In addition, if \( y = (y_1, \ldots, y_T) \in \mathbb{R}^T \setminus \{0_{\mathbb{R}^T}\} \) is an eigenvector corresponding to a characteristic root \( \lambda \), then its extension \( u^y \in H_T \), defined by \( u^y(i) = y_i \) for \( i = \mathbb{T}, T \), is an eigensequence corresponding to the eigenvalue \( \lambda \). This means that an orthonormal basis of eigensequences \( u^1, \ldots, u^T \) can be constructed from an orthonormal basis of eigenvectors \( x^1, \ldots, x^T \) of \( M_T \) by extending \( x^i \) in \( H_T (i = \mathbb{T}, T) \) as above.

From ([12], p. 38), we know that the characteristic roots of \( M_T \), hence the eigenvalues of \(-\Delta^2\), are \( 4 \sin^2 \frac{i\pi}{T} \) (\( i = 0, T-1 \)). We can label them according to the parity of \( T \) as follows:

\[
\text{Todd : } \quad \lambda_0 = 0, \quad \lambda_{2k-1} = \lambda_{2k} = 4 \sin^2 \frac{k\pi}{T}, \quad k = 1, \ldots, \frac{T-1}{2};
\]

\[
\text{Even : } \quad \lambda_0 = 0, \quad \lambda_{2k-1} = \lambda_{2k} = 4 \sin^2 \frac{k\pi}{T}, \quad k = 1, \ldots, \frac{T-2}{2}, \quad \lambda_{T-1} = 4.
\]

In both cases, we consider an orthonormal basis \( e^0, \ldots, e^{T-1} \) in \( H_T \), such that \( e^i \) is an eigensequence corresponding to \( \lambda_i \) (\( i = 0, T-1 \)). Observe that, by multiplying equality (10) by arbitrary \( v \in H_T \) and using summation by parts formula, one obtains that if \( u \in H_T \) and \( \lambda \in \mathbb{R} \) satisfy (10), then

\[
\sum_{j=1}^{T} \Delta u(j) \Delta v(j) = \lambda (u|v).
\]

This yields

\[
\sum_{j=1}^{T} \Delta e^i(j) \Delta e^k(j) = \lambda_k \delta_{ik} \quad (i, k \in \{0, \ldots, T-1\}), \tag{12}
\]
where $\delta_{ik}$ stands for the Kronecker delta function.

3. Main Result

Our main result is given in the following.

**Theorem 2.** Assume that $g : \mathbb{R} \to \mathbb{R}$ is a continuous odd function and that $G$ satisfies (9) together with
\[
\liminf_{x \to 0} \frac{2G(x)}{x^2} \geq 1.
\] (13)

Then, the following hold true:

(i) If $\lambda > 8 \sin^2 \frac{m \pi}{T}$ (which $= 2\lambda_{2m}$) with $0 \leq m \leq \begin{cases} \frac{T-1}{2} & \text{if } T \text{ is odd} \\ \frac{T-2}{2} & \text{if } T \text{ is even} \end{cases},$ (14)

then problem (1) has at least $2m + 1$ distinct pairs of nontrivial solutions.

(ii) If $T$ is even and $\lambda > 8$ (which $= 2\lambda_{T-1}$), (15)

then (1) has at least $T$ distinct pairs of nontrivial solutions.

**Proof.** We show (i) in the odd case because the even case follows by exactly the same arguments, and under assumption (15), a quite similar strategy works by simply replacing “2m” with “$T - 1$.”

Thus, let $0 \leq m \leq \frac{T-1}{2}. On account of Theorem 1 and Propositions 1 and 2, we have to prove that there exists a nonempty compact symmetric subset $A_m \subset H_T \setminus \{0\}$ with $
\gamma(A_m) \geq 2m + 1,$ such that
\[
\sup_{v \in A_m} I_\lambda(v) < 0.
\] (16)

Since $\lambda > 2\lambda_{2m},$ we can choose $\varepsilon \in (0,1),$ so that $\lambda > 2\lambda_{2m}/(1 - \varepsilon).$ Then, by virtue of (13), there exists $\delta > 0$ such that
\[
2G(x) \geq (1 - \varepsilon)x^2 \quad \text{as } |x| \leq \delta.
\] (17)

Next, we introduce the set
\[
A_m := \left\{ \sum_{k=0}^{2m} \alpha_k e^k : \alpha_0^2 + \cdots + \alpha_{2m}^2 = \rho^2 \right\},
\]
where $\rho$ is a positive number, which is chosen $\leq \min\left\{ \frac{1}{2\sqrt{2m+1}}, \delta \right\}.$

Then, it is not difficult to see that the odd mapping $H : A_m \to S^{2m}$ defined by
\[
H\left( \sum_{k=0}^{2m} \alpha_k e^k \right) = \left( \frac{\alpha_0}{\rho}, \frac{\alpha_1}{\rho}, \ldots, \frac{\alpha_{2m}}{\rho} \right)
\]
is a homeomorphism between $A_m$ and $S^{2m};$ therefore, $\gamma(A_m) = 2m + 1.$

We have that $A_m \subset K.$ Indeed, let $v = \sum_{k=0}^{2m} \alpha_k e^k \in A_m.$ Then, for all $j \in \{1, \ldots, T\},$ we obtain
\[
|\Delta v(j)| \leq \sum_{k=0}^{2m} |\alpha_k e^k (j + 1)| + \sum_{k=0}^{2m} |\alpha_k e^k (j)| \leq 2 \sum_{k=0}^{2m} |\alpha_k|
\]
\[
\leq 2\sqrt{2m + 1} \left( \sum_{k=0}^{2m} \alpha_k^2 \right)^{1/2} = 2\rho \sqrt{2m + 1}
\] (18)
and since \( \rho \leq 1/(2\sqrt{2m+1}) \), one has \( |\Delta v|_\infty \leq 1 \), which shows that \( v \in K \). On the other hand, using (12), we obtain

\[
\sum_{j=1}^{T} |\Delta v(j)|^2 = \sum_{j=1}^{T} \Delta \left( \sum_{k=0}^{2m} a_k \xi^k(j) \right)^2 = \sum_{j=1}^{T} \left( \sum_{k=0}^{2m} a_k \Delta \xi^k(j) \right)^2
\]

\[
= \sum_{j=1}^{T} \left( \sum_{k=0}^{2m} \alpha_k \Delta \xi^k(j)^2 + \sum_{i,k=0}^{2m} a_i a_k \Delta \xi^k(j) \Delta \xi^i(j) \right)
\]

\[
= \sum_{k=0}^{2m} \lambda_k a_k^2 \leq \lambda_{2m} \sum_{k=0}^{2m} a_k^2 = \lambda_{2m} \rho^2. \tag{19}
\]

In addition, it is clear that

\[
\sum_{j=1}^{T} |v(j)|^2 = |v|^2 = (v|v) = \sum_{k=0}^{2m} \alpha_k^2 = \rho^2. \tag{20}
\]

Then, from (17), (19), (20) and \( |v(j)| \leq \rho \leq \delta (j \in \{1, \ldots, T\}) \), it follows that

\[
I_\lambda(v) = \Psi(v) + G_\lambda(v) \leq \sum_{j=1}^{T} |\Delta v(j)|^2 - \frac{\lambda}{2} (1 - \epsilon) \sum_{j=1}^{T} |v(j)|^2
\]

\[
\leq \rho^2 \lambda_{2m} - \frac{\lambda}{2} (1 - \epsilon) \rho^2 = \rho^2 \frac{2 \lambda_{2m} - \lambda (1 - \epsilon)}{2} < 0.
\]

Therefore, (16) holds true and the proof of (i) is complete. \( \square \)

**Example 1.** If (14) holds true, then problem (3) has at least \( 2m + 1 \) distinct pairs of nontrivial solutions. In addition, if \( T \) is even, under assumption (15), problem (3) has at least \( T \) distinct pairs of nontrivial solutions. Notice that besides the trivial solution, problem (3) always has the pair of constant solutions \( u \equiv \pm 1 \), and these are the only constant nontrivial solutions of (3). Therefore, problem (3) has at least \( 2m \) (resp. \( T - 1 \)) distinct pairs of nonconstant solutions if hypothesis (14) is satisfied (resp. (15) holds true).

Consider the eigenvalue type problem

\[
- \Delta [\phi(\Delta u(n-1))] = \lambda u(n) + h(u(n)), \quad u(n) = u(n + T) \quad (n \in \mathbb{Z}) \tag{21}
\]

and set \( H(x) = \int_{0}^{x} h(\tau) d\tau \) (\( x \in \mathbb{R} \)).

**Corollary 1.** If the continuous function \( h : \mathbb{R} \to \mathbb{R} \) is odd and

\[
\lim_{x \to -\infty} \frac{H(x)}{x^2} \geq 0, \quad \lim_{x \to +\infty} \frac{H(x)}{x^2} = -\infty,
\]

then the conclusions (i) and (ii) of Theorem 2 remain valid with (21) instead of (1).

**Proof.** Theorem 2 applies to the problem

\[
- \Delta [\phi(\Delta u(n-1))] = \lambda \left( u(n) + \frac{h(u(n))}{\lambda} \right), \quad u(n) = u(n + T) \quad (n \in \mathbb{Z}).
\]
Theorem 2 can be employed to derive the multiplicity of nontrivial solutions of autonomous non-parametric problems having the form
\[-\Delta [\phi(\Delta u(n-1))] = f(u(n)), \quad u(n) = u(n+T) \quad (n \in \mathbb{Z}). \tag{22}\]

Setting $F(x) = \int_0^x f(\tau) d\tau$ ($x \in \mathbb{R}$), we have the following.

**Corollary 2.** Assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous odd function and that
\[
\lim_{x \to +\infty} F(x) = -\infty. \tag{23}\]

Then, the following hold true:

(i) If
\[
\liminf_{x \to 0} \frac{F(x)}{x^2} > 4 \sin^2 \frac{m\pi}{T} \quad \text{with} \quad 0 \leq m \leq \begin{cases} 
(T-1)/2 & \text{if } T \text{ is odd} \\
(T-2)/2 & \text{if } T \text{ is even}
\end{cases}, \tag{24}
\]
then problem (22) has at least $2m + 1$ distinct pairs of nontrivial solutions.

(ii) If $T$ is even and
\[
\liminf_{x \to 0} \frac{F(x)}{x^2} > 4, \tag{25}\]
then (22) has at least $T$ distinct pairs of nontrivial solutions.

**Proof.** From (24), there exists $\overline{\lambda} > 0$ such that
\[
\liminf_{x \to 0} \frac{2F(x)}{x^2} \geq \overline{\lambda} > 8 \sin^2 \frac{m\pi}{T}
\]
and the result follows from Theorem 2 with $g(x) = f(x)/\overline{\lambda}$; a similar argument works when (25) is fulfilled. \(\square\)

**Example 2.** Let $f_a : \mathbb{R} \to \mathbb{R}$ be given by
\[
f_a(x) = 2x \sin |x|^{-\frac{1}{2}} - \frac{x| |x|^{-\frac{1}{2}} \cos |x|^{-\frac{1}{2}}}{2} + 2ax - 4x^3 \quad (x \in \mathbb{R}).
\]

Then,
\[
F_a(x) = x^2 \left( \sin |x|^{-\frac{1}{2}} + a - x^2 \right) \quad (x \in \mathbb{R})
\]
and by Corollary 2, we obtain that, if
\[
a > 1 + 4 \sin^2 \frac{m\pi}{T} \quad \text{with} \quad 0 \leq m \leq \begin{cases} 
(T-1)/2 & \text{if } T \text{ is odd} \\
(T-2)/2 & \text{if } T \text{ is even}
\end{cases},
\]
then the equation
\[-\Delta [\phi(\Delta u(n-1))] = f_a(u(n)) \quad (n \in \mathbb{Z}) \tag{26}\]
has at least $2m + 1$ distinct pairs of nontrivial $T$-periodic solutions, while if $T$ is even and $a > 5$, then (26) has at least $T$ distinct pairs of nontrivial $T$-periodic solutions.

**Remark 2.** A multiplicity result for odd perturbations of the discrete p-Laplacian operator is obtained in [13] using a Clark-type result in the frame of the classical critical point theory.

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