OPERATOR SPACES AND ARAKI-WOODS FACTORS
– A QUANTUM PROBABILISTIC APPROACH –

M. JUNGE

Abstract. We show that the operator Hilbert space OH introduced by Pisier embeds into the predual of the hyperfinite III$_1$ factor. The main new tool is a Khintchine type inequality for the generators of the CAR algebra with respect to a quasi-free state. Our approach yields a Khintchine type inequality for the $q$-gaussian variables for all values $-1 \leq q \leq 1$. These results are closely related to recent results of Pisier and Shlyakhtenko in the free case.

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0. Introduction and Notation

Probabilistic methods play an important role in the theory of operator algebras and Banach spaces. It is not surprising that a quantized theory of Banach spaces will require tools from quantum probability. This connection between noncommutative probability and the recent theory of operator spaces (sometimes called quantized Banach spaces) is well-established through the work of Haagerup, Pisier \cite{HP} and the general theory of Khintchine type inequalities by Lust-Piquard \cite{LP}, Lust-Piquard and Pisier \cite{LPP}. The importance of type III von Neumann algebras in this line of research was discovered only recently through the work of Pisier/Shlyakhtenko \cite{PS} on Grothendieck’s theorem for operator spaces and in \cite{J3}. Both papers make essential use of the theory of free probability. It is well-known that in free probability theory some probabilistic estimates, classically only valid for $p < \infty$, hold even for $p = \infty$. For example this holds for Biane/Speicher’s \cite{BS} work on stochastic process and Voiculescu’s inequality for sums of free random variables \cite{Voi}. In this paper we prove norm estimates for the sum of independent copies in noncommutative $L_1$ spaces in a quite general setting. This includes free random variables as in \cite{J3} and also classical commuting or anti-commuting random variables. Using a central limit procedure, similar as in \cite{J3}, we derive Khintchine type inequalities for the classical Araki-Wood factors. Although our results are motivated by the theory of operator spaces, the techniques used in the proof are (quantum) probabilistic in nature.

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Let us fix the notation required to state the main inequality of this paper. Let us assume that we have an inclusion of von Neumann algebras \( \mathcal{M} \subset A, B \subset \mathcal{N} \) and that \( E : \mathcal{M} \rightarrow \mathcal{M} \) is a faithful normal conditional expectation. Then \( A, B \) are called independent over \( \mathcal{M} \) (better over \( E \)) if 

\[
E(ab) = E(a)E(b)
\]

holds for all \( a \in A \) and \( b \in B \). Let us now assume that \( \mathcal{M} \subset M_1, ..., M_n \subset \mathcal{N} \) are von Neumann subalgebras. Then \( (\mathcal{M}_i) \) is called (increasingly) independent if \( M_i \) is independent of the von Neumann algebra \( \mathcal{M}_{i-1} \) generated by \( M_1, ..., M_{i-1} \) over \( \mathcal{M} \). This definition is due to Schürmann.

Our main inequality is an inequality for independent copies, allowing matrix valued coefficients. Let us therefore assume that \( \mathcal{M} \subset M \) and \( \alpha_i : M \rightarrow \mathcal{N} \) are faithful homomorphisms such that \( \alpha_i|\mathcal{M} = id \). We say that \( (\mathcal{M}, \alpha_1, ..., \alpha_n, \mathcal{N}, E) \) is a system of independent top-subsymmetric copies if the \( \alpha_i \)'s are independent and

\[
E(\alpha_i(a_1) \cdots \alpha_i(a_m)) = E(\alpha_j(a_1) \cdots \alpha_j(a_m))
\]

holds for all \( a_1, ..., a_m \in M \), and all functions \( i : \{1, ..., m\} \rightarrow \{1, ..., n\} \), \( j : \{1, ..., m\} \rightarrow \{1, ..., n\} \) such that \( i|\{1, ..., m\}\backslash A = j|\{1, ..., m\}\backslash A \), \( A \) has at most 2 elements, such that \( k \in A \) implies

\[
i_k = \max\{i_1, ..., i_m\} \quad , \quad j_k = \max\{j_1, ..., j_m\}.
\]

This means we are allowed to exchange at most two top values. In our applications we have often have much stronger assumptions, for example for free or independent copies. We say that such a system is conditioned if there is a faithful normal state \( \phi \) on \( \mathcal{N} \) such that \( \phi \circ E = \phi \) and

\[
\sigma_i^\phi(\alpha_i(M)) \subset \alpha_i(M)
\]

holds for all \( i = 1, ..., n \). This allows us to extends the maps \( \alpha_i \) to all \( L_p \) spaces. Our main inequality is an estimate for sums of independent copies in \( L_1 \):

**Theorem 0.1.** Let \( (\mathcal{M}, M, \alpha_1, ..., \alpha_n, \mathcal{N}, E) \) be a system of independent, conditioned top-subsymmetric copies. Then

\[
E\|\sum_{k=1}^n \varepsilon_k \alpha_k(x)\|_1 \sim \inf_{x=x_1+x_2+x_3} n\|x_1\|_1 + \sqrt{n}\|E(x_2^*x_2)^{1/2}\|_1 + \sqrt{n}\|E(x_3^*x_3)^{1/2}\|_1.
\]

Here \( (\varepsilon_k)_{k=1}^n \) is a sequence of independent Bernoulli variables with \( \text{Prob}(\varepsilon_k = \pm 1) = \frac{1}{2} \). We will use the symbol \( a \sim b \) if there exists an absolute constant \( c > 0 \) such that \( c^{-1}a \leq b \leq ca \) (of course independent of \( x \) in the theorem above). For non-tracial von Neumann algebras which occur in the context of free probability, we work with Haagerup’s \( L_1 \) spaces and the ‘natural’ extension of \( \alpha_k \) and \( E \) to these spaces (see e.g. \[\mathbb{M}, \mathbb{JX}\] ). We may replace \( \varepsilon_k \) by \( \varepsilon_k \otimes v_k \) where \( v_k \) are unitaries. This is important in the context of Speicher’s \[\text{Spec}\] interpolation technique for \( q \)-commutation relations.

An essential ingredient in our proof of Theorem 0.1 is the noncommutative Khintchine inequality due to Lust-Piquard and Pisier \[\text{LPP}\] :

\[
E\|\sum_{k=1}^n \varepsilon_k x_k\|_1 \sim \inf_{x_k=c_k+d_k} tr((\sum_{k=1}^n c_k^*c_k)^{1/2}) + tr((\sum_{k=1}^n d_k^*d_k)^{1/2})^2.
\]

Our passage from three terms above to two terms uses the central limit theorem. Given a sequence \( (x_k) \) of classical independent copies, the central limit theorem tells us that \( n^{-1/2} \sum_{k=1}^n x_k \) converges to a gaussian variable. Central limit theorems in quantum probability have a long
and impressive tradition starting with the work of Cushen/Hudson and [CH], see also Hudson [Hud], von Waldenfels [vW], Giri/von Waldenfels [GvW], Cockroft, Gudder and Hudson [CGH], Quaegebeur [Qua] and Hegerfeldt [Heg] among many others. Our interest in the central limit theorem is two-fold. First, we consider limits of the form

\[ u_n(x) = \sqrt{\frac{T}{n}} \sum_{k=1}^{n} u_k \otimes \pi_k(x) \]

where \( \pi_k : N \to N^{\otimes n} \) is the homomorphism which sends \( N \) in the \( k \)-th component and the \( u_k \in M_{2^n} \) are unitaries satisfying the CAR-relations \( u_k u_j = -u_j u_k \). Let \( \psi \) be a state on \( N \), and \( \tau_n \) be the normalized trace on \( M_{2^n} \). Using the classical combinatorial approach, we see that

\[
(0.1) \quad \lim_{n} (\tau_n \otimes \psi^{\otimes n})(u_n(x_1) \cdots u_n(x_m)) = \sum_{\sigma=\{(i_1,j_1),\ldots,(i_m,j_m)\}\in P_2(m)} (-1)^{I(\sigma)} \prod_{l=1}^{m} (T\psi)(x_{i_l}x_{j_l}).
\]

Here \( P_2(m) \) stands for the set of pair partitions of \( \{1,\ldots,m\} \) and \( I(\sigma) \) for the number of inversions. Using Speicher’s trick it is not difficult to replace \((-1)\) by any \( q \), \(-1 \leq q \leq 1 \). For \( q = 1 \) we see that algebraically the formal limit object \( u_\infty(x) \) satisfies

\[
[u_\infty(x), u_\infty(y)] = T\psi([x,y]).
\]

A suitable change of variables yields the classical commutation relations. One problem in our paper is to associate with the limit object \( u_\infty(x) \) a selfadjoint operator affiliated to a suitable von Neumann algebra. This is achieved using ultraproducts of von Neumann algebras and the fundamental work of Raynaud [Ray] (see section 4). The reader might object that generators for the CAR algebra are already bounded and hence there is no need for this ultraproduct procedure. This is where the second interest in the central limit procedure becomes apparent. We also want to guarantee that the norm estimates from Theorem 0.1 hold for the limit object in \( L_1 \). This requires extra knowledge on the action of the modular group and adds technical difficulties to this paper. Moreover, our approach works uniformly for all \(-1 \leq q \leq 1 \). In fact we find a simultaneous realization of all the \( q \)-commutation relations (which seems to be new). Note that for \( q = 1 \) the limit objects are classical gaussian variables which are indeed unbounded. In section 2 we establish the combinatorial aspect of the central limit theorem. The connection to the classical CAR and CCR relations and the Araki-Woods factors is established in section 3. In section 5 we prove a noncommutative version of the Hamburger moment problem which allows us to identify von Neumann algebras using combinatorial information as in (0.1). The proof of Theorem 0.1 is contained in section 6. Combining these results in section 7 we obtain a Khintchine type inequality for quasi-free states. More precisely, we consider the CAR-algebra \( \mathcal{A} \) generated by a sequence \( (a_k) \) sequence satisfying the canonical anti-commutation relations

\[ a_k a_j + a_j a_k = 0, \quad a_k a_j^* + a_j^* a_k = \delta_{kj}. \]

A quasi free state \( \phi_\mu \) is characterized by

\[
(0.2) \quad \phi_\mu(a_{i_1}^* \cdots a_{i_r}^* \otimes a_{j_1} \otimes \cdots \otimes a_{j_s}) = \delta_{r,s} \prod_{l=1}^{r} \delta_{i_l,j_l} \mu_i
\]

for all increasing sequences \( i_1 < i_2 < \cdots < i_r \) and \( j_1 < j_2 < \cdots < j_s \). Let us also recall the usual notation \( x.\phi(y) = \phi(xy) \). The Khintchine inequality for quasi-free states reads as follows:
Theorem 0.2. Let \((x_k) \subset \mathbb{M}_m\) be matrices. Then

\[
\| \sum_k x_k.tr \otimes a_k.\phi_\mu \| \leq \inf x_k = e_k + d_k \left( \sum_k \mu_k \bar{c}_k \frac{1}{2} \right) + \text{tr} \left( \sum_k \frac{\mu_k^2}{1 - \mu_k} d_k \bar{d}_k \frac{1}{2} \right).
\]

Analogous results hold for all \(-1 \leq q \leq 1\). The case \(q = 0\) in Theorem 0.2 in this form follows from \([PS2]\) but main ideas are already contained in \([PS]\) and somehow independently in \([J3]\). Using techniques from \([JX2]\) for Khintchine inequalities for the CAR generators in \(\ell_2\). Retrospectively, it seems that the Araki-Wood factors constructed from CAR-relations are a perfect fit for quotients of \(R \oplus C\) an interesting connection to operator space theory. We refer to \([JX2]\) for Khintchine inequalities for the CAR generators in \(L_p\). At the moment the result holds for \(p = 1\) and \(p > 1\), but the known constants \(c_p\) are not bounded as \(p \to 1\). It is an open problem how to close this gap. At the end of section 8 we construct a completely isomorphic embeddings.
of $O_{H_n}$ in the Brown algebra $B_{m}^{\text{nc}}$, the universal algebra of coefficients of an $m \times m$ unitary. Brown algebras have been introduced in the context of K-theory by \cite{Bro} and can be understood as noncommutative analogues of the full $C^*$-algebra of the free group. For an embedding of the infinite dimensional $OH$ we have to use a suitable direct limit of the Brown algebras. These result could be considered as a first step towards constructing a ‘concrete’ embedding of the operator space $OH$, a problem which remains open.

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1. Preliminaries

We use standard notation in operator algebras \cite{Tak1, Tak2, Tak3, KR1, KR2} and \cite{Str, SZ}. We refer to \cite{ER} and \cite{Ps1} for standard references on operator spaces. The relevant information on the projective tensor product for noncommutative $L_1$ spaces can also be found in \cite{13}. Let us recall very briefly some facts about Haagerup’s $L_p$ spaces associated to a von Neumann algebra $N$. Let $\phi$ be a normal faithful weight with modular group $\sigma^\phi_t$. We obtain a representation $\pi$ of $N$ on the Hilbert space $L_2(N, \phi)$, the completion of $N$ with respect to the scalar product $(x, y) = \phi(x^* y)$. On the crossed product $N \rtimes \sigma^\phi_t \mathbb{R}$ there exists a unique trace $\tau$ such that the dual action $\theta_s$ satisfies $\tau(\theta_s(x)) = e^{-s} \tau(x)$ for all $x$ in the positive cone of $N \rtimes \sigma^\phi_t \mathbb{R}$. Haagerup’s space $L_p(N) = L_p(N, \phi)$ is defined as the space of $\tau$ measurable operators $x$ affiliated to $N \rtimes \sigma^\phi_t \mathbb{R}$ such that $\theta_s(x) = e^{-s/p} x$. $L_p(N)$ is a left and right $N$ module. On the space $L_1(N)$ Haagerup defines a unique linear functional $tr : L_1(N) \to \mathbb{C}$ such that
\[ tr(Dy) = tr(yD) \]
holds for all $y \in N$ and $D \in L_1(N)$. This functional defines a one-to-one (linear) correspondence between the predual $N_*$ and $L_1(N)$ given by $\phi_D(y) = tr(xy)$. We call the operator $D$ the density of $\phi$. The norm in $L_p(N)$ is defined by
\[ \|x\|_p = \left[ \text{tr}(|x|^p) \right]^{\frac{1}{p}}. \]
It is important to note for $x \in L_p(N)$ the polar decomposition $x = u|x|$ satisfies $u \in N$ and $|x| \in L_p(N)$. $L_p(N)$ is a Banach space for $1 \leq p < \infty$ and $p$-normed for $0 < p \leq 1$. We refer to \cite{Ler} for proofs and more information. Let us consider a further semifinite von Neumann algebra $M$ with trace $\tau$. Then $\tau \otimes \phi$ is a normal faithful weight on the tensor product $M \bar{\otimes} N$. Using $\sigma^\tau \otimes \phi = 1 \otimes \sigma^\phi_t$ we may identify $(M \bar{\otimes} N) \rtimes \sigma^\tau \otimes \sigma^\phi_t \mathbb{R}$ and $M \bar{\otimes} (N \rtimes \sigma^\phi_t \mathbb{R})$. This implies that for $a \in L_p(M)$ and $x \in L_p(N)$ we have $x \otimes a \in L_p(M \bar{\otimes} N)$. Moreover, the tracial functional satisfies
\[ tr_{M \bar{\otimes} N} = \tau \otimes tr_N. \]
An easy application of Kaplansky’s density theorem shows that $L_p(M) \otimes L_p(N)$ is a dense subspace of $L_p(M \bar{\otimes} N)$. For more information on tensor products of noncommutative $L_p$ spaces in the non-tracial case we refer to \cite{12}.

The natural operator space structure on $L_1(N)$ is inherited from $N_*^{op}$. Let us recall that $N^{op}$ is the von Neumann algebra $N$ equipped with the reverse multiplication $x \circ y = yx$. Since $N$ and $N^{op}$ coincide as Banach spaces, the same holds true for $N_*$ and $N_*^{op}$. In particular, the
mapping \( \iota : L_1(N) \to N_{op}^{\pi} \), \( \iota(x)(y) = tr(xy) \) is well-defined. As explained in [JR], and [EM], the advantage of this embedding is the equality

\[(1.1) \quad L_1(M) \hat{\otimes} L_1(N) = L_1(M \otimes N)\]

which holds for all von Neumann algebras \( M \). We will also frequently use the following results of Effros and Ruan (see [ER])

\[(1.2) \quad (M \otimes N_\pi)^* = M \hat{\otimes} N \, .\]

Let us consider the special case \( M = B(\ell_2) \). Then, we see by complementation that

\[
\left\| \sum_k e_{1k} \otimes x_k \right\|_{L_1(B(\ell_2) \hat{\otimes} N)} = \left\| \sum_k e_{1k} \otimes x_k \right\|_{L_1(B(\ell_2) \otimes N)} = \left\| \left( \sum_k x_k \otimes e_{1k} \right)^{\frac{1}{2}} \right\|_{L_1(N)} .
\]

On the other hand, we deduce from (1.2) that

\[
\left\| \sum_k e_{1k} \otimes x_k \right\|_{R \hat{\otimes} L_1(N)} = \sup \left\{ \left\| \sum_k e_{1k} \otimes x_k, u \right\| : \| u : R \to N_{op} \|_{cb} \leq 1 \right\}
\]

\[
= \sup \{ \left\| \sum_k tr(x_k y_k) \right\| : \left\| \sum_k y_k^* \circ y_k \right\| \leq 1 \} = \sup \{ \left\| \sum_k tr(y_k x_k) \right\| : \left\| \sum_k y_k y_k^* \right\| \leq 1 \}
\]

\[
= \left\| \left( \sum_k x_k \otimes e_{1k} \right)^{\frac{1}{2}} \right\|_{L_1(N)} .
\]

This means that the column space in \( S_1 = L_1(B(\ell_2, tr)) \) carries the operator space structure of the row space in \( B(\ell_2) \).

At the end of these preliminaries we recall the definition of the operator space \( \text{OH} \). This space is obtained from \( C \) and \( R \) by complex interpolation. On the matrix level we have

\[
M_m(\text{OH}) = [M_m(C), M_m(R)]_{\frac{1}{2}} .
\]

Let \( (e_k) \) be the canonical basis of \( \text{OH} \). Then

\[
\left\| \sum_k x_k \otimes e_k \right\|_{M_m(\text{OH})} = \left\| \sum_k x_k \otimes \bar{x}_k \right\|^{\frac{1}{2}}_{M_m(\text{OH})} .
\]

Here \( \overline{B(H)} = B(\bar{H}) \). In many aspects the operator space \( \text{OH} \) is the appropriate analogue of \( \ell_2 \) in the category of operator spaces. We refer to [Ps1] for more details and information. For general Hilbert spaces we use the notations \( H^c = B(C, H) \), \( H^r = B(H, C) \) and \( H^{oh} = [H^c, H^r]_{\frac{1}{2}} \).

2. The algebraic central limit theorem

Due to the work of Hudson/Parthasarathy [HP], von Waldenfels [VW] and Hegerfeldt [Heg] it is well-known that the central limit theorem applies for CCR and CAR relations. More recently Speicher developed an approach which applies to the \( q \)-commutation relations as well. We will apply Speicher’s approach using the data given by a von Neumann algebra \( N \) and a weight. We will first start with a normal faithful state \( \psi \) on \( N \). For fixed \( n \in \mathbb{N} \), we consider the \( n \)-fold tensor product \( N_n = N^\otimes n \) and the faithful normal state \( \phi_n = \psi^\otimes n \) on \( N_n \). Let us denote by \( \pi_k : N \to N^\otimes n \) the homomorphism which sends \( N \) into the \( k \)-th copy

\[
\pi_k(x) = 1 \otimes \cdots \otimes x \otimes 1 \otimes \cdots \otimes 1 .
\]
As an additional reservoir we consider a finite von Neumann algebra $\mathcal{M}_{2^n}$ with normalized trace $\tau_n$ and selfadjoint contractions $v_1(n), ..., v_n(n)$. Let $T$ be an additional scaling factor. Then we consider the new random variable

$$u_n(x) = u_{n,T}(x) = \sqrt{\frac{T}{n}} \sum_{k=1}^{n} v_k(n) \otimes \tau_k(x).$$

We need some further notation to formulate the central limit procedure. Let $A \subset \{1, ..., m\}$ and $x_1, ..., x_m$ be elements in $N$. Then we use

$$\psi_A[x_1, ..., x_m] = \psi(\prod_{i \in A} x_i)$$

for the evaluation of $\psi$ of the ordered product $\prod_{i \in A} x_i$ arising from those indices which are in $A$. Let $\sigma = \{A_1, ..., A_r\}$ be a partition of $\{1, ..., m\}$. Then the multiplicative extension of $\psi$ is defined as

$$\psi_\sigma[x_1, ..., x_m] = \prod_{j=1}^{r} \psi_{A_j}[x_1, ..., x_m].$$

We denote by $P(m)$ the collection of all partitions of $\{1, ..., m\}$ and by $P_2(m)$ the set of pair partitions. For an element $(k_1, ..., k_m) \in \{1, ..., n\}^m$, we use the notation $(k_1, ..., k_m) \leq \sigma$ if

$$k_i = k_l \leftrightarrow \exists 1 \leq j \leq r \{i, l\} \subset A_j.$$

A rather general form of the central limit theorem may be formulated as follows. (The special cases discussed below are well-known). Let us recall that for an ultrafilter $U$ on an index set $I$ the limit $\lim_{U} a_i = a$ holds if for every $\varepsilon > 0$ the set $\{i : |a_i - a| < \varepsilon\} \in U$.

**Lemma 2.1.** Let $\psi$ be a state. Let $(v_k(n))_{k=1, ..., n}$ be contractions such that the singleton condition is satisfied:

$$\tau_n(v_k_1(n) \cdots v_k_m(n)) = 0$$

holds whenever one of the variables $k_1, ..., k_m$ occurs only once. Let $U$ be a free ultrafilter on $\mathbb{N}$ and

$$\beta(\sigma) = \lim_{n, U} n^{-\frac{m}{2}} \sum_{(k_1, ..., k_m) \leq \sigma} \tau_n(v_{k_1}(n) \cdots v_{k_m}(n)).$$

Let $x_1, ..., x_m \in N$. Then

$$\lim_{n, U} \tau_n \otimes \psi^{\otimes n}(u_{n,T}(x_1) \cdots u_{n,T}(x_m)) = \sum_{\sigma \in P_2(m)} \beta(\sigma)(T\psi)_\sigma[x_1, ..., x_m].$$

**Proof.** Let $\sigma = \{A_1, ..., A_r\}$ be a partition of $\{1, ..., m\}$ and $n \in \mathbb{N}$. Now, we may develop the terms

$$\tau_n \otimes \psi^{\otimes n}(u_{n,T}(x_1) \cdots u_{n,T}(x_m))$$

$$= \left(\frac{T}{n}\right)^{\frac{m}{2}} \sum_{k_1, ..., k_m = 1}^{n} \tau_n(v_{k_1}(n) \cdots v_{k_m}(n))\psi^{\otimes n}(\pi_{k_1}(x_1) \cdots \pi_{k_m}(x_m))$$

$$= \left(\frac{T}{n}\right)^{\frac{m}{2}} \sum_{\sigma \in P(m)} \sum_{(k_1, ..., k_m) \leq \sigma} \tau_n(v_{k_1}(n) \cdots v_{k_m}(n))\psi_\sigma[x_1, ..., x_m].$$
Let us fix a partition $\sigma$ and perform the limit for $n$ along the ultrafilter $U$. The singleton condition (2.1) implies that only partitions where all the subsets have cardinality bigger than 2 provide a non-trivial contribution. In particular, it suffices to consider partitions with cardinality $|\sigma| \leq m/2$. Since the $v_i(n)$'s are assumed to be contractions, we deduce from the Cauchy-Schwarz inequality that

$$\tau_n(v_{k_1}(n) \cdots v_{k_m}(n)) \leq \tau_n(v_{k_1}(n)v_{k_1}(n)^*)^{\frac{1}{2}} \prod_{j=2}^{m-1} \|v_{k_j}(n)\|_\infty \tau_n(v_{k_m}(n)^*v_{k_m}(n)^*)^{\frac{1}{2}} \leq 1.$$ 

Since there are at most $n(n-1) \cdots (n-r+1)$ many tuples $(k_1, ..., k_m)$ such that $(k_1, ..., k_m) \leq \sigma$, we deduce for $r < \frac{m}{2}$ that

$$\lim_{n,\mu} \frac{m}{2} \sum_{(k_1, ..., k_m) \leq \sigma} \tau_n(v_{k_1} \cdots v_{k_m}) = 0.$$ 

Therefore only the pair partitions with $|\sigma| = \frac{m}{2}$ provide a non-trivial contribution $\beta(\sigma)$. 

Remark 2.2. The additional parameter $T$ will be used later for strictly semifinite weights. We refer to [AB] for representation of arbitrary selfadjoint random variables represented with pair partitions.

In the next section it will be important to analyze a growth condition for central limits.

Corollary 2.3. For $x \in N$ we use the length function

$$|x| = \max \{(T\psi(x^*x))^{\frac{1}{2}}, (T\psi(xx^*))^{\frac{1}{2}}, \|x\|_\infty\}.$$ 

Let $x_1, ..., x_m \in N$. Then

$$|\tau_n \otimes \psi^\otimes n(u_{n,T}(x_1) \cdots u_{n,T}(x_m))| \leq m \frac{m}{2} \prod_{i=1}^{m} |x_i|.$$ 

holds for all $n \in \mathbb{N}$.

Proof. According to the proof of Lemma 2.1, we have

$$|\tau_n \otimes \psi^\otimes n(u_{n}(x_1) \cdots u_{n}(x_m))| \leq \sum_{\sigma \in P_{ns}(m)} \sum_{(k_1, ..., k_m) \leq \sigma, |\sigma| \leq \frac{m}{2}} |\tau_n(v_{k_1}(n) \cdots v_{k_m}(n))|(T\psi)_{\sigma}[x_1, ..., x_m].$$

Here $P_{ns}(m)$ stand for partitions not containing singletons. For fixed $\sigma$, we deduce from (2.3) and the fact that there are $n \cdots (n - |\sigma| + 1)$ many tuples satisfying $(k_1, ..., k_m) \leq \sigma$ that

$$n \frac{m}{2} \sum_{(k_1, ..., k_m) \leq \sigma} |\tau_n(v_{k_1}(n) \cdots v_{k_m}(n))| \leq \frac{n \cdots (n - |\sigma| + 1)}{n \frac{m}{2}} \leq 1.$$ 

The Cauchy-Schwarz inequality implies

$$|(T\psi)(y_1 \cdots y_r)| \leq ((T\psi)(y_1y_1^*)^{\frac{1}{2}}((T\psi)(y_2y_2^*) \cdots y_r y_r^*)^{\frac{1}{2}} \leq \prod_{i=1}^{r} |y_i|.$$ 

This implies $|(T\psi)_{A_j}[x_1, ..., x_m]| \leq \prod_{i \in A_j} |x_i|$ for $|A_j| \geq 2$. Thus we have

$$|(T\psi)_{\sigma}[x_1, ..., x_m]| \leq \prod_{i=1}^{m} |x_i|.$$
for partitions without singletons. Combining these estimates we get
\[ |τ_n \otimes ψ^⊗n (u_n(x_1) \cdots u_n(x_m))| ≤ \sum_{σ ∈ P(\mathcal{N}) | |σ| ≤ \frac{m}{2}} \prod_{i=1}^{m} |x_i|. \]

Finally, we note that there are not more than \( m^\frac{m}{2} \) partitions \( σ \) with \( |σ| \leq \frac{m}{2} \).

Before we discuss concrete examples let us indicate how these estimates remain valid in the context of weights. We recall that a function \( ψ : N_+ \to [0, \infty] \) is called an n.s.f. weight if

\( n. \) \( ψ(\sup_i x_i) = \sup_i ψ(x_i) \) holds for every increasing net \( (x_i) \) of positive elements;

\( s. \) For every \( 0 \leq x \) one has \( ψ(x) = \sup_{0 \leq y \leq x} ψ(y) \).

\( f. \) \( ψ(x) = 0 \) implies \( x = 0 \).

For a weight \( ψ \) we may define \( n_ψ = \{ x ∈ N : ψ(xx^*) < ∞ \} \). We use the notation \( n_ψ^* = \{ x ∈ N : ψ(xx^*) < ∞ \} \) and \( (x, y)_ψ = ψ(x^*y) \). The completion of \( n_ψ \) with respect to the inner product norm is denoted by \( L_2(N, ψ) \). It is well-known that \( ψ \) extends to a linear functional on \( n_ψ^*n_ψ \), see \([Str]\) for details.

**Example 2.4.** Let \( (f_j) \) be a net of mutually orthogonal projections which sum up to the identity, i.e. \( \sum j f_j = 1 \). Let \( (ψ_j) \) be a family of normal faithful states on \( f_jNf_j \). Then \( ψ(x) = \sum j ψ(f_jxf_j) \) defines an n.s.f. weight.

The example in [24] is the prototype of a strictly semifinite normal faithful weight. A weight \( ψ \) is called strictly semifinite if there exists an increasing net \( (e_i) \) of projections in the centralizer \( N_ψ \) converging strongly to 1 and such that \( ψ(e_i) < ∞ \) holds for all \( i \). We recall that an element \( x \) belongs to the centralizer \( N_ψ \) if the modular group \( σ_i^ψ \) with respect to \( ψ \) satisfies \( σ_i^ψ(x) = x \) for all \( t ∈ \mathbb{R} \). In the example above the index set is the collection \( P_{<N_0}(I) \) of all finite subsets of \( I \) and the increasing family of projections is given by \( e_\sum j \in f_j \). We will use the following well-known lemma for a strictly semifinite weight with invariant projections \( (e_i) \). The proof follows immediately from the fact that \( xe_i \) and \( e_ix \) converges in \( L_2(N, ψ) \) to \( x \). To show the convergence of \( \lim_i xe_i = x \), we use standard modular theory, \( S(x) = x^* \), \( S = JΔ^\frac{1}{2} \) and observe that
\[
(2.4) \quad xe_i = (e_i x)^* = JΔ^\frac{1}{2} (e_i x) = Je_iΔ^\frac{1}{2} x = Je_iJx^*
\]
holds for all analytic elements \( x ∈ n_ψ \cap n_ψ^* \) and all \( i \).

**Lemma 2.5.** Let \( ψ \) be a strictly semifinite faithful normal weight and \( (e_j) \) as above. Then
\[
ψ(x_1 \cdots x_m) = \lim_i ψ(e_i x_1 e_i \cdots e_i x_m e_i)
\]
holds for all \( x_1, ..., x_m \in n_ψ \cap n_ψ^* \).

In this purely algebraic setting it is very convenient to use the algebraic notion of a tensor algebra
\[
A(V) = \sum_{n=0}^{∞} V^⊗n
\]
of a vector space \( V \). The formal multiplication is obtained by linear extension of
\[
(v_1 \otimes \cdots \otimes v_r) \times (v_{r+1} \otimes \cdots \otimes v_m) = v_1 \otimes \cdots \otimes v_r \otimes v_{r+1} \otimes \cdots \otimes v_m .
\]
The words of length 0, \( V^0 = \mathbb{C} \) (or \( V^0 = \mathbb{R} \)), correspond to the multiples of the identity. In our context we apply this to \( V = n_ψ \cap n_ψ^* \).
Corollary 2.6. Let $\psi$ be a strictly semifinite weight and $(e_i)_{i \in I}$ as above. Let $U'$ be a free ultrafilter on $I$ and $x_1, \ldots, x_m \in n_\psi \cap n_\psi^*$. Then the function
\[ \phi(x_1 \otimes \cdots \otimes x_m) = \lim_{j \to U'} \lim_{n \to U} \tau_n \otimes (\psi(e_j))^{\otimes n}(u_n,\psi(e_j)(e_j x_1 e_j) \cdots u_n,\psi(e_j)(e_j x_m e_j)) \]
extends to a linear functional on the tensor algebra $A(n_\psi \cap n_\psi^*)$ and satisfies
\[ \phi(x_1 \otimes \cdots \otimes x_m) = \sum_{\sigma \in P_2(m)} \beta(\sigma)\psi_\sigma[x_1, \ldots, x_m] \]
and for $|x| = \max\{|x|, \psi(x^* x)^{1/2}, \psi(xx^*)^{1/2}\}$
\[ |\phi(x_1 \otimes \cdots \otimes x_m)| \leq m^2 \prod_{i=1}^m |x_i|. \]

Proof. For fixed $j$ we use $T_j = \psi(e_j)$. Then $\psi_j(x) = \frac{\psi(x)}{T_j}$ is a normal faithful state on $e_j Ne_j$. Note that $T_j \psi_j$ is the restriction of $\psi$ to $e_j Ne_j$ and hence independent of $j$. According to Lemma 2.1 and Corollary 2.3 we know that (2.5) and (2.6) hold for all elements in $\bigcup_j e_j Ne_j$. We apply Lemma 2.5 and obtain that
\[ \psi_\sigma[x_1, \ldots, x_m] = \lim_j \psi_\sigma[e_j x_1 e_j, \ldots, e_j x_m e_j] \]
for all $x_j \in n_\psi \cap n_\psi^*$. Similarly, we deduce from $|x| = \max\{|x|, \psi(x^* x)^{1/2}, \psi(xx^*)^{1/2}\}$ that
\[ |x| = \lim_j \max\{|e_j x e_j|, T_j \psi_j((e_j x e_j)^*(e_j x e_j))^{1/2}, T_j \psi_j((e_j x e_j)(e_j x e_j)^*)^{1/2}\}. \]
Thus the assertion holds for arbitrary elements $x_1, \ldots, x_m \in n_\psi \cap n_\psi^*$. \hfill \blacksquare

We will now follow Speicher’s approach for the $q$-commutation relations. Let $(\Omega, \mu)$ be a probability space and $s_{ij} : \Omega \to \{1, -1\}$ be random variables such that $s_{ij} = s_{ji}$ and such that the family $(s_{ij})_{i < j}$ is independent. For fixed $\omega \in \Omega$ we define the Pauli-matrices
\[ v_{i,i}(\omega) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and for } i < j \quad v_{i,j}(\omega) = \begin{pmatrix} 1 & 0 \\ 0 & s_{i,j}(\omega) \end{pmatrix}. \]
For fixed $n \in \mathbb{N}$, we use
\[ v_j(\omega) = v_{1,j}(\omega) \otimes \cdots \otimes v_{j,j}(\omega) \otimes 1 \otimes \cdots \otimes 1 \in M_{2^n}. \]
This yields selfadjoint unitaries such that
\[ v_i(\omega)v_j(\omega) = s_{i,j}(\omega)v_j(\omega)v_i(\omega). \]
In the special case of a constant random variable $s_{i,j} = -1$ we obtain Clifford matrices. For a pair partition $\sigma = \{A_1, \ldots, A_{m/2}\}$ we write $(i, j) \in I(\sigma)$ if $i$ and $j$ are part of an inversion of $\sigma$. This means $\{i, l\} \subset A_{ji}$ and $\{r, j\} \subset A_{j_2}$ with $r < i < j < l$.

Lemma 2.7. The random variables $v_i \in L_\infty(\Omega, M_{2^n})$ satisfy the singleton condition (2.1) with respect to $\tau_n(x) = \int 2^{-n} \text{tr}(x(\omega)) d\mu(\omega)$. Moreover, for every pair partition $\sigma \in P_2(m)$:
\[ \beta(\sigma) = \lim_{n \to U} n \frac{1}{n} \sum_{(k_1, \ldots, k_m) \leq \sigma} \prod_{(i, j) \in I(\sigma)} E(s_{k_i, k_j}). \]
Proof. Let \((k_1, \ldots, k_m)\) be such that one index occurs only once. We fix \(\omega \in \Omega\) and consider
\[
2^{-n} \text{tr}(v_{k_1}(\omega) \cdots v_{k_m}(\omega)).
\]
Using the commutation relation \(v_{k_i}^2 = 1\) we may assume that all the variables occur only once and \(k_j\) is the largest index with this property. Then we find some \(v\) such that \(v_{k_1}(\omega) \cdots v_{k_m}(\omega) = (-1)^{\pm 1} v \otimes v_{k_j}(\omega) \otimes 1\). This implies that
\[
2^{-n} \text{tr}(v \otimes v_{k_j}(\omega) \otimes 1) = 2^{-n} \text{tr}(v) \text{tr}(v_{k_j}(\omega)) = 0.
\]
For the second assertion, we consider a pair partition \(\sigma = \{A_1, \ldots, A_{m/2}\}\). Let us consider \((k_1, \ldots, k_m) \leq \sigma\) and \(\omega \in \Omega\). We may assume that \(A_1 = \{1, j\}\). Now, we commute \(v_{k_j}(\omega)\) with all the unitaries \(v_{k_{j-1}}(\omega), \ldots, v_{k_2}(\omega)\). This yields a factor \(\prod_{i=2}^{j-1} s_{k_i, k_j}(\omega)\). If there is no inversion between \(i < j\) the index \(k_i\) occurs twice. Therefore, we get
\[
\prod_{i=2}^{j-1} s_{k_i, k_j}(\omega) = \prod_{i, (i, j) \in I(\sigma)} s_{k_i, k_j}(\omega).
\]
After this procedure we use \(v_{k_1}(\omega) v_{k_j}(\omega) = v_{k_j}(\omega)^2 = 1\). Thus we have eliminated all the inversions with \(j\) at the cost of the factor \(\prod_{i, (i, j) \in I(\sigma)} s_{k_i, k_j}(\omega)\). Continuing by induction, we find
\[
2^{-n} \text{tr}(v_{k_1}(\omega) \cdots v_{k_m}(\omega)) = \prod_{(i, j) \in I(\sigma)} s_{k_i, k_j}(\omega).
\]
Taking expectations yields the assertion by independence. \(\square\)

The following result is due to Speicher:

Corollary 2.8. \(\text{(Speicher)}\) Let \(\text{Prob}(s_{i,j} = 1) = p\) and \(\text{Prob}(s_{i,j} = -1) = 1 - p\). Then
\[
\beta(\sigma) = (2p - 1)^{I(\sigma)},
\]
holds for all \(\sigma \in P_2(m)\). Here \(I(\sigma)\) is the number of inversions of \(\sigma\).

Proof. This follows immediately from Lemma 2.7 \(\mathbb{E}s(k_i, k_j) = \alpha - (1 - \alpha) = 2\alpha - 1\) and
\[
\lim_{n} n^{-\frac{m}{2}} \sum_{(k_1, \ldots, k_m) \leq \sigma} = \lim_{n} n^{-\frac{m}{2}} n(n - 1) \cdots (n - \frac{m}{2}) = 1.
\]

Let us investigate the combinatorics including all values of \(-1 \leq q \leq 1\) simultaneously. We use the random variable
\[
s^q(t) = 1_{[-1, q]} - 1_{[q, 1]}
\]
on \([-1, 1]\) with respect to the Haar measure \(\frac{dt}{\pi}\). Note that \(\mathbb{E}s^q = \frac{1}{2}(2 - 2(1 - q)) = q\). Then \(v_{j,q}\) is constructed above using independent copies \(s^q_{i,j}\).

Corollary 2.9. Let \(u_{n,q}(x) = \sqrt{\frac{T}{n}} \sum_{j=1}^{n} v_{j,q} \otimes \pi_j(x)\). Let \(q_1, \ldots, q_m \in [-1, 1]\) and \(\rho\) be the partition given by sets where the \(q_i\)’s coincide. Then
\[
\lim_{n} \mathbb{E} \otimes \left( \frac{\psi_{\otimes n}}{T} \right) (u_{n,q_1}(x_1) \cdots u_{n,q_m}(x_m)) = \sum_{\sigma \in P_2(m), \sigma \leq \rho} t(\sigma, q_1, \ldots, q_m) \psi_{\sigma}[x_1, \ldots, x_m],
\]
where \(t(\sigma) = ((m/2))^{-1} \sum_{\gamma \in \text{Perm}(m/2)} \prod_{(i,j) \in I(\sigma)} [l_{\gamma_j q_i} + 1_{j \leq \gamma_i q_i}]\). Here we use the lexicographic order \((A_1, \ldots, A_{|\sigma|})\) for the sets in \(\sigma\), and for an inversion \((i, j) \in I(\sigma)\) with \(A_s = \{l, j\}\), \(A_t = \{i, k\}\) with \(l < i < j < k\) we write \(i \leq j \gamma\) if \(\gamma(t) < \gamma(s)\).
Proof. It suffices to analyze
\[ \lim_{n \to \infty} n^{-\frac{m}{2}} \sum_{(k_1, \ldots, k_m) \leq \sigma} \tau_n(v_{k_1,q_1} \cdots v_{k_m,q_m}). \]
As in Lemma 2.7, we find
\[ \tau_n(v_{k_1,q_1} \cdots v_{k_m,q_m}) = \prod_{(i,j) \in I(\sigma)} \epsilon(k_i,q_i, k_j,q_j) \tau_n(v_{k_1,q_{n(1)}} v_{k_1,q_{n(2)}} \cdots v_{k_m/2,q_{n(m-1)}} v_{k_m/2,q_{n(m)}}) \]
for a suitable permutation \( \pi \). Here \( I(\sigma) \) is the collection of inversions of \( \sigma \). The sign \( \epsilon(k_i,q_i, k_j,q_j) \) is calculated as follows. If \( k_i < k_j \), then we know that
\[
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & s_{k_i,k_j}^q \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & s_{k_i,k_j}^q \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]
Thus we find
\[ \epsilon(k_i,q_i, k_j,q_j) = 1_{k_i < k_j} q_j + 1_{k_j < k_i} q_i. \]
Thus the sign produces \( q_i \) or \( q_j \) depending on which color is considered the bigger one. This leads to the description of \( t(\sigma, q_1, \ldots, q_m) \) for \( \sigma \leq \rho \). If \( \sigma \not\leq \rho \), we will find terms of the form
\[ \frac{1}{n} | \sum_{j=k_{m/2-1}}^{n} \left[ \frac{1}{2} + \frac{1}{2} \mathbb{E} q_{m-1} q_m j - k_{m/2-1} \right] \leq \frac{1}{n|1-\alpha|} \]
where \( \alpha = \frac{1}{2} + \frac{1}{2} \mathbb{E} q_{m-1} q_m \) is in \((-1,1)\). These terms disappear in the limit. \( \square \)

3. CAR, CCR and \( q \)-COMMUTATION RELATIONS

The CCR and CAR relations and their quasi-free states are motivated by quantum mechanics. In this section we review algebraic properties of the family of \( q \)-commutation relation and the type of the associated von Neumann algebra. We will follow the central limit approach from the previous section and use the data provided by a weight on \( N = L_\infty(\Omega, \mu; \mathbb{M}_2) \). Let us start with the discrete case. We consider a sequence of positive numbers \( (\mu_j) \) such that \( 0 < \mu_j < 1 \) and the weight
\[ \psi(x) = \sum_j [(1 - \mu_j)x_{11}(j) + \mu_j x_{22}(j)] \]
on \( \ell_\infty(\mathbb{N}, \mathbb{M}_2) \). Obviously \( \psi \) is a strictly semifinite weight because the projections \( e_j = \delta_j \otimes 1_{\mathbb{M}_2} \) are invariant under the modular group. Here \( (\delta_j) \) denotes the sequence of unit vectors in \( \ell_\infty \) (where \( \delta_j(k) = \delta_{jk} \) is given by the Kronecker symbol).

For \(-1 \leq q \leq 1\), we may define a linear functional \( \phi_q \) on \( A(n_\psi \cap n_\psi^* \cap \mathbb{M}_2) \) by the formula
\[ \phi_q(x_1 \otimes \cdots \otimes x_m) = \sum_{\sigma \in F_2(m)} q^{I(\sigma)} \psi_\sigma [x_1, \ldots, x_m]. \]
We denote by \( I_q \) the ideal generated by
\[ \{ z \in A(n_\psi \cap n_\psi^*) : \forall a, b \in A(n_\psi \cap n_\psi^*) : \phi_q (a \otimes z \otimes b) = 0 \} \]
Then, we may define \( B_q = A(n_\psi \cap n_\psi^*) / I_q \). We denote by \( \pi_q \) the quotient homomorphism and observe that \( \phi_q \) induces a functional \( \hat{\phi}_q \) on \( B_q \) such that \( \hat{\phi}_q \pi_q = \phi_q \).
Lemma 3.1. Let $q = -1$ and $\alpha = 0$. Let $b_j = \delta_j \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then
\[ \pi_1(b_j \otimes b_k + b_k \otimes b_j) = 0, \quad \pi_1(b_j \otimes b^*_k + b^*_k \otimes b_j) = \delta_{kj} \]
and
\[ \phi_1(b^*_r \otimes \cdots \otimes b^*_1 \otimes b_{j_1} \otimes \cdots \otimes b_{j_s}) = \delta_{r,s} \prod_{l=1}^r \delta_{i_l,j_l} \mu_{i_l} \]
for all increasing sequences $i_1 < i_2 < \cdots < i_r$ and $j_1 < i_2 < \cdots < j_s$.

Proof. We first consider $y_1, \ldots, y_m \in n_\psi \cap n_\psi^*$ and deduce from Corollary 2.8 that
\[ \phi_1(y_1 \otimes \cdots \otimes y_m) = \sum_{\sigma \in P_\sigma} (-1)^{I(\sigma)} \psi_\sigma[y_1, \ldots, y_m]. \]

Now, we want to calculate the commutator between elements $y$ and $z$:
\[
\begin{align*}
\phi_1(y_1 \otimes \cdots \otimes y_{r-1} \otimes y \otimes z \otimes y_{r+1} \otimes \cdots \otimes y_m) \\
= \sum_{\sigma \in P_\sigma, \{r, r+1\} \in \sigma} (-1)^{I(\sigma)} \psi(yz) \psi_\sigma[y_1, \ldots, y_{r-1}, y_{r+1}, \ldots, y_m] \\
+ \sum_{\sigma \in P_\sigma, \{r, r+1\} \notin \sigma} (-1)^{I(\sigma)} \psi_\sigma[y_1, \ldots, y_{r-1}, y, z, y_{r+1}, \ldots, y_m].
\end{align*}
\]
For the second term we define a bijection $\rho$ on the partitions $\sigma$ by exchanging $r$ and $r + 1$. Since $\{r, r + 1\}$ is not contained in $\sigma$, we see that $\psi_\sigma[y_1, \ldots, y, z, \ldots y_m] = \psi_\rho(\sigma)[y_1, \ldots, y, z, \ldots y_m]$. However, $\rho$ changes the number of inversion by one. This implies
\[
\sum_{\sigma \in P_\sigma, \{r, r+1\} \notin \sigma} (-1)^{I(\sigma)} \psi_\sigma[y_1, \ldots, y, z, \ldots y_m] = - \sum_{\sigma \in P_\sigma, \{r, r+1\} \notin \sigma} (-1)^{I(\sigma)} \psi_\sigma[y_1, \ldots, z, y, \ldots y_m].
\]
By cancellation we obtain
\[
\phi_1(y_1 \otimes \cdots \otimes y_{r-1} \otimes y \otimes z \otimes y_{r+1} \otimes \cdots \otimes y_m) + \phi_1(y_1 \otimes \cdots \otimes y_{r-1} \otimes z \otimes y \otimes y_{r+1} \otimes \cdots \otimes y_m) \\
= \psi(yz + zy) \phi_1(y_1 \otimes \cdots \otimes y_{r-1} \otimes y_{r+1} \otimes \cdots \otimes y_m).
\]
This shows
\[ y \otimes z + z \otimes y - \psi(yz + zy) 1 \in I_{-1}. \]
However, it is easily checked that $\psi(b_j b_k + b_k b_j) = 0$ for all $j, k$ and $\psi(b_j b_k^* + b_k^* b_j) = 0$ for $j \neq k$. For $j = k$, we note that
\[ \psi(b_j^* b_k + b_k b_j^*) = (1 - \mu_k) + \mu_k = 1. \]
This completes the proof for the anticommutation relations. Equation (3.2) is a direct consequence of (3.3). Indeed, given a pair partition $\sigma = \{A_1, \ldots, A_m\}, m = r + s$, we see that only for $\{i_1, \ldots, i_r\} = \{j_1, \ldots, j_s\}$, we have a chance to match up all the coefficients. Since the subsequences are assumed to be increasing, we deduce that only the inversion-free partition $\{\{1, m\}, \{2, m-1\}, \ldots, \{m/2, m/2 + 1\}\}$ leads to a nontrivial contribution.
Remark 3.2. In the continuous case we consider weights of the form
\[ \psi(x) = \int [f_1(\omega)x_{11}(\omega) + f_2(\omega)x_{22}(\omega)] d\mu(\omega). \]

We assume in addition that \( f_1(\omega) + f_2(\omega) = 1 \). We use the elements \( b(\omega) = \begin{pmatrix} 0 & g(\omega) \\ 0 & 0 \end{pmatrix} \) and deduce from (3.4) that
\[ \pi_1(b(\omega) \otimes b(h) + b(h) \otimes b(\omega)) = 0, \quad \pi_1(b(\omega) \otimes b(h)^* + b(h)^* \otimes b(\omega)) = (\int \overline{h} g d\mu) \]
and
\[ \phi_1(b_{\rho}^r \otimes \cdots \otimes b_{\rho}^1 \otimes b_{\rho}^1 \cdots b_{\rho}^1) = \delta_{\rho,s} \det(\int \overline{g_i h_j f_2 d\mu}). \]

Remark 3.3. The type of the associated factor satisfying these commutation relations can be determined from the work of Araki-Woods [AW69]. Indeed, let us first consider a selfadjoint operator \( \rho \) on a real Hilbert space \( K \) such that \( 0 \leq \rho \leq 1 \). Let \( H = K + iK \) be the complexification and \( \{ u(f) : f \in H \} \) a field satisfying the CAR relations. We repeat the same construction for \( K \oplus K \) and use the well-known map
\[ u_\rho(f) = u((1 - \rho)^{\frac{1}{2}} f, 0) + u(0, \rho^{\frac{1}{2}} f)^*. \]
This is equivalent to [AW69 (12.22)] (see also [R83]). Indeed, one can check that
\[ u_\rho(f)u_\rho(g) + u_\rho(g)u_\rho(f) = 0 \quad \text{and} \quad u_\rho(f)u_\rho(g)^* + u_\rho(g)^*u_\rho(f) = (f, g), \]
Moreover, the moments with respect to the vacuum state \( \xi \) of the full system is given by
\[ (\xi, u_\rho(f_r)^* \cdot u_\rho(f_1)^* \cdots u_\rho(g_1) \cdots u_\rho(g_2)\xi) = \delta_{\rho,s} \det(e_j, \rho(g_i)). \]

The corresponding von Neumann algebra \( R(\rho) \) is defined by
\[ R(\rho) = \{ u_\rho(f) : f \in K \}'' \]
where the commutant is taken in the GNS-representation with respect to the vacuum state \( \xi \). For discrete \( \rho \) we see that \( R(\rho) \) is the ITPFI factor with \( R(\rho) \cong \otimes_{n \in \mathbb{N}} (M_2, \phi_n) \) where \( \phi_n(x) = (1 - \lambda_n)x_{11} + \lambda_n x_{22} \) is determined by the spectral values \( \{ \lambda_n : n \in \mathbb{N} \} \) of \( \rho \). (This can also be checked using the standard matrices satisfying the CAR relation and by considering the GNS-construction with respect to the tensor product state \( \phi = \otimes_{n \in \mathbb{N}} \phi_n \).) It follows from [AW69 section 12] that for \( \rho \) with continuous spectrum, \( R(\rho) \) is the hyperfinite III_1 factor. If there exists a \( 0 < \lambda < 1 \) such that for every \( n \in \mathbb{N} \) there exists \( k_n, l_n \) such that \( \lambda_n = \lambda^{k_n}/(\lambda^{k_n} + \lambda^{l_n}) \), then \( R(\rho) \) is a type III_\lambda factor (see [AW69 section 8]).

The CCR relations can be obtained in a similar manner.

Lemma 3.4. Let \( N = \ell_\infty(M_2) \) and \( (\mu_k) \) as in Lemma 3.1. Let \( X_k = \delta_k \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) and \( Y_k = \delta_k \otimes \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \). Then
\[ \pi_1(X_k \otimes X_j - X_j \otimes X_k) = \delta_{kj} \quad \text{and} \quad \pi_1(Y_k \otimes Y_j - Y_j \otimes Y_k) = \delta_{kj} \]
and
\[ \pi_1(X_k \otimes Y_j - Y_j \otimes X_k) = \delta_{kj} 2i(2\mu_k - 1). \]
Moreover,
\[ \phi_1(e^{zX_k}e^{wY_j}) = e^{izw(2\mu_k - 1)\delta_{kj}}e^{\frac{z^2+w^2}{2}}. \]

**Proof.** Let us first consider arbitrary elements \(y_1, \ldots, y_{r-1}, y_{r+1}, \ldots, y_m\) and \(y, z \in N\). Now, we proceed as in Lemma 3.1. The only difference is that if we replace the partition \(\sigma\) containing \(\{i, r\}\) and \(\{r + 1, j\}\) by the one containing \(\{i, r + 1\}\) and \(\{j, r\}\) this does not produce a sign change. Therefore
\[
\phi_1(y_1 \otimes \cdots \otimes y_{r-1} \otimes y \otimes z \otimes y_{r+1} \otimes \cdots y_m) - \phi_1(y_1 \otimes \cdots \otimes y_{r-1} \otimes z \otimes y \otimes y_{r+1} \otimes \cdots y_m)
= \psi(yz - y)\phi_1(y_1 \otimes \cdots \otimes y_{r-1} \otimes y_{r+1} \otimes \cdots y_{m+2}).
\]

This implies
\[ (3.5) \quad [\pi_1(y), \pi_1(z)] = \psi([y, z]). \]

From
\[ \psi([X_k, X_j]) = \delta_{kj}, \quad \psi([Y_k, Y_j]) = \delta_{kj} \quad \text{and} \quad \psi([X_k, Y_j]) = \delta_{kj}2i(2\mu_k - 1) \]
we deduce the first three equations. For the last assertion we use the power series expansion. For a fixed even \(k\) we have
\[ \phi_1(y^\otimes k) = \sum_{\sigma \in P_2(k)} \psi(y^2)^k. \]

Let \(g\) be a \(N(0,1)\) distributed normal random variable and \(w \in \mathbb{C}\) such that \(w^2 = \psi(y^2)\). Then
\[ \mathbb{E}(wg)^k = w^k \sum_{\sigma \in P_2(k)} 1 = \phi_1(y^\otimes k). \]

This implies that
\[ (3.6) \quad \phi_1(e^y) = \sum_{k=0}^{\infty} \frac{\phi_1(y^\otimes k)}{k!} = \sum_{k=0}^{\infty} \frac{(wg)^k}{k!} = \mathbb{E}e^{wg} = e^{\frac{w^2}{2}} = e^{\psi(y^2)/2}. \]

If \(k \neq j\) then \(\pi_1(X_k)\) and \(\pi_1(Y_j)\) commute. This implies with (3.6)
\[ \phi_1(e^{zX_k}e^{wY_j}) = \phi_1(e^{zX_k+wY_j}) = e^{\psi((zX_k+wY_j)^2)/2} = e^{\frac{z^2+w^2}{2}}. \]

For \(k = j\), we use the Baker-Cambell-Hausdorff formula (see e.g. [OL, Section 3.3]) and obtain from (3.6) that
\[ (3.7) \quad e^{t(zX_k+wY_k)} = e^{-\frac{t^2zw\psi([X_k,Y_k])}{2}}e^{t^2X_ke^{twY_k}}. \]
holds as a formal power series in \(t\) with values in \(B_1\). On the other hand, we deduce from Corollary 2.3 that
\[ \phi_1(e^{t^{\otimes n}X_k}e^{twY_k}) = \sum_{n,m} \frac{t^{n+m}}{n!m!} \phi_1(X_k^{\otimes n}Y_k^{\otimes m}) \]
is absolutely converging because \(\phi_1(X_k^{\otimes n}Y_k^{\otimes m}) \leq |X_k|^n|Y_k|^m\). Similarly, we deduce that the map \(t \mapsto \phi_1(e^{t^{\otimes n}X_k}e^{twY_k})\) is an absolutely converging power series. This implies
\[ (3.8) \quad \phi_1(e^{t(zX_k+wY_k)}) = e^{-\frac{t^2zw\psi([X_k,Y_k])}{2}}\phi_1(e^{t^{\otimes n}X_k}e^{twY_k}) \]
for all \(t \in \mathbb{R}\). For \(t = 1\) we deduce the assertion from \(\psi([X_k,Y_k]) = 2i(2\mu_k - 1)\).
Remark 3.5. In the continuous case we consider the weight \( \psi(x) = \int [f_1 x_{11} + f_2 x_{22}] d\mu \) on \( N = L_\infty(\Omega, \mu; \mathcal{M}_2) \). The real Hilbert space \( H_R \) is given by functions \( f : \Omega \to \mathbb{C} \). Then, we define the real linear embedding

\[
j(f) = \begin{pmatrix} 0 & f \\ \bar{f} & 0 \end{pmatrix}.
\]

We obtain that

\[
[\pi_1(j(f)), \pi_1(j(g))] = \psi([j(f), j(g)]) = \int (fg - gf)(f_1 - f_2) d\mu.
\]

By the Baker-Cambell-Hausdorff formula (valid as a for formal power series in the algebra \( B_1 \)) this implies that

\[
\phi_1(a \otimes e^{-zj(f)}e^{wj(g)} \otimes b) = \phi_1(a \otimes e^{-zw} \int (fg - gf)(f_1 - f_2) d\mu e^{zj(f) +wj(g)} \otimes b)
\]

Here \( a, b \) is either 1 or a finite tensor \( a = x_1 \otimes \cdots \otimes x_m, b = y_1 \otimes \cdots \otimes y_m \). Note that in both cases the absolute convergence of the corresponding power series follows from the growth condition. Hence we may evaluate them at \( t = 1 \). Moreover, we deduce from \( \psi(j(f)^2) = \int |f|^2 d\mu \) that

\[
\phi_1(e^{zj(f)}) = e^{\frac{z^2}{2}} \int |f|^2 d\mu.
\]

The reader familiar with the classical CCR relations will have observed that the formulas are not the usual ones. Let us first recall these usual CCR relations and state some results on the corresponding type of the underlying von Neumann algebra. We will follow the representation in \[AW63\]. Let \( K \) be a real Hilbert space. The Weyl representation is given by a collection of unitaries \( \{ U(f) | f \in K \}, \{ V(f) | f \in K \} \) such that

\[
U(f + g) = U(f)U(g), \quad V(f + g) = V(f)V(g)
\]

and

\[
U(f)V(g) = e^{-i(f,g)}V(g)U(f).
\]

The corresponding quasi free state \( \phi \) is determined by the relation

\[
\phi(U(f)V(g)) = e^{-\frac{i(f,g)}{2} - \frac{||f||^2 + ||g||^2}{4}}.
\]

Given real subspaces \( K_1 \) and \( K_2 \subset K \) Araki introduced

\[
R(K_1, K_2/K) = \{ U(f)V(g) : f \in K_1, g \in K_2 \}''.
\]

Here the bicommutant is taken in the GNS representation of \( \{ U(f), V(g) : f, g \in K \} \) with respect to \( \phi \). In \[AW69\] section 12 we can find very precise information about the type of \( R(K_1, K_2/K) \). We shall assume in addition that \( K = K_1 \oplus K_1^\perp \) and that \( K_2 \) is the graph of an unbounded operator \( B \). If \( B*B \) has discrete spectrum, then \( R(K_1, K_2/K) \) may be constructed as an ITPFI factor. Let \( \rho \) be the operator such that \( B*B = 4\rho(1 + \rho) \). We refer to \[AW69\] section 12, section 8) for the following result.

**Theorem 3.6 (Araki-Woods).** If \( \rho \) has a continuous spectrum then \( R(K_1, K_2/K) \) is the hyperfinite III_1 factor. If the spectrum \( \{ \lambda_n ; n \in \mathbb{N} \} \) of \( \rho \) is discrete then \( R(K_1, K_2/K) \) is isomorphic to \( \otimes_{n \in \mathbb{N}}(B(t_2), \phi_n) \) where \( \phi_n(x) = tr(D_n x) \) and \( D_n \) has the eigenvalues \( \{ \mu_k^n(1 - \mu_n) : k = 0, 1, \ldots \} \) and \( \mu_n = \frac{\lambda}{1 + \lambda} \). Moreover, if \( 0 < \lambda < 1 \) such that \( \mu_n = \lambda^k n \) for some \( k_n \in \mathbb{N} \) Then \( R(K_1, K_2/K) \) is the type III_\lambda factor.
We will translate our algebraic relations into the usual CCR relations. For a measurable function \( f : \Omega \to \mathbb{R} \) we define
\[
    U(tf) = e^{it\pi_1(j(2^{-\frac{1}{2}}f))} \quad \text{and} \quad V(tg) = e^{it\pi_1(j(2^{-\frac{1}{2}}g))}
\]
as formal power series in \( B_1 \). The relations \( U(f + g) = U(f)U(g), \ V(f + g) = V(f)V(g) \) are obvious. We will show in Theorem 3.6 that this may be interpreted as unitary operators on a Hilbert space. According to Remark 3.5 we have
\[
    \phi_1(a \otimes U(tf)V(tg) \otimes b) = e^{-it^2 \int \rho(f_2-f_1)d\mu} \phi_1(a \otimes V(tg)U(tf) \otimes b).
\]
Similarly, we have
\[
    \phi_1(U(f)V(g)) = e^{-\frac{it}{2} \int f(g(f_2-f_1)d\mu} e^{-\frac{i}{2}(f^2 + g^2d\mu)}.
\]
We will now match this with the usual CCR-relations. We introduce \( K = L_2(\mu; \mathbb{R}) \oplus L_2(\mu; \mathbb{R}) \) and \( K_1 = \{(f, 0) : f \in L_2(\mu; \mathbb{R})\} \). We recall the assumption \( f_1 + f_2 = 1 \). Thus the operator
\[
    A(f) = (f_2 - f_1)f
\]
is a contraction on \( L_2(\mu; \mathbb{R}) \) and we may define
\[
    K_2 = \{(Ag, \sqrt{1-A^2}(g)) : g \in L_2(\mu, \mathbb{R})\}.
\]
Then \( K_2 \) is the graph of the operator \( B = A^{-1}\sqrt{1-A^2} \). The map \( g \mapsto (Ag, \sqrt{1-A^2}(g)) \) is an isometry and
\[
    ((f, 0), (Ag, \sqrt{1-A^2}(g))) = (f, Ag) = \int fg(f_2 - f_1)d\mu.
\]
In view of Theorem 3.6 we now solve the equation \( B^*B = 4(\rho(1+\rho)) \) for \( B = A^{-1}\sqrt{1-A^2} \) and find \( 2\rho = 1 + |A|^{-1} \). As an application, let us state the following result.

**Corollary 3.7.** Let \( K_1 \) and \( K_2 \) as above. If the operator \( (1 + |A|^{-1})(f) = (1 + |f_2 - f_1|^{-1})f \) has continuous spectrum, then \( R(K_1, K_2/K) \) is the hyperfinite \( III_1 \) factor.

At the end of this section, we will discuss the unifying approach introduced by Shlyakhtenko which allows us to describe the \( q \)-commutation relations for all \( -1 \leq q \leq 1 \). Here we shall assume that \( U_t \) is a one parameter unitary group on a real Hilbert space \( K \). On the complexification \( H = K + iK \) one may write \( U_t = A^{it} \) for a suitable positive nonsingular generator \( A \). The crucial information is contained in the new scalar product
\[
    (x, y)_U = (2A(1 + A)^{-1}x, y)
\]
on \( K + iK \). We assume the scalar product to be antilinear in the first component. We denote by \( H_U \) the completion of \( H \) with respect to this scalar product. Indeed, the following properties characterize \( (K, H_U, A) \).

i) \( H \supset K \) is a complex Hilbert space and \( (U_t) \) a one parameter group such that \( U_t = A^{it} \)
and \( U_t(K) \subset K \).

ii) \( K + iK \) is dense in \( H \) and \( K \cap iK = \{0\} \).

iii) The restriction of the real part of the scalar product on \( H \) induces the scalar product on \( K \).

iv) \( \text{Im}(x, y)_H = (i\frac{1-A^{-1}}{1+A}x, y)_K \) for all \( x, y \in K \).

Moreover, on a dense domain one has
\[
    (x, y)_U = (y, A^{-1}x)_U.
\]
We refer to [Shl] for more information on these conditions and the fact that these properties characterize the inclusion \( K \subset H \). The algebraic information of the factor \( \Gamma_q(K, U_t) \) is contained
in a generating family of bounded operators \( \{ s_q(h) : h \in K \} \) and the vacuum vector \( \xi \) satisfying

\[
(\xi, s_q(h_1) \cdots s_q(h_m) \xi) = \sum_{\sigma = \{i_1, j_1, \ldots, i_m, j_m\} \in \mathcal{P}_2(m)} q^{I(\sigma)} \prod_{l=1}^{m} (h_{i_l}, h_{j_l})_U
\]

for even \( m \). For odd \( m \) we have 0.

**Example 3.8.** Given a strictly semifinite weight \( \psi \) on a von Neumann algebra \( N \), we may consider the algebra \( B_q(n_\psi \cap n_\psi^*) \) and the scalar product given by

\[
(x, y)_\psi = \psi(x^*y).
\]

Then \( \phi_q \) defines a functional on \( A(n_\psi \cap n_\psi^*) \) satisfying

\[
\phi_q(x_1 \otimes \cdots \otimes x_m) = \sum_{\sigma \in \mathcal{P}_2(m)} q^{I(\sigma)} \psi[x_1, \ldots, x_m]
\]

which coincides with (3.15) for selfadjoint elements. Indeed, we consider the real Hilbert space \( K \) of selfadjoint elements in \( n_\psi \cap n_\psi^* \) and the complex Hilbert space \( L_2(N, \psi) \). On \( K \) we shall use the real scalar product

\[
(x, y)_K = \frac{\psi(xy) + \psi(yx)}{2}.
\]

The modular group \( \sigma^\psi_t \) is a one parameter group of unitaries on \( K \). From (3.14) we see that \( A = \Delta^{-1} \) is a generator such that \( \psi(xy) = \psi(yA^{-1}x) \) holds for analytic elements. Thus we consider \( U_t = \sigma^{-t} \) which leaves \( K \) invariant. For analytic selfadjoint elements \( x, y \) we have \( (x, y)_K = 1/2\psi((1 + \Delta^{-1})(xy)) \). This implies

\[
(i \frac{1 - A^{-1}}{1 + A^{-1}}(x, y)_K = (i \frac{1 - \Delta}{1 + \Delta}(x, y)_K = (i \frac{\Delta^{-1} - 1}{1 + \Delta^{-1}}(x, y)_K
\]

\[
= \frac{1}{2} \psi((1 + \Delta^{-1})(i \frac{\Delta^{-1} - 1}{1 + \Delta^{-1}}(xy))y = \frac{i}{2} [\psi(\Delta^{-1}(xy) - \psi(xy)]
\]

\[
= \frac{i}{2} [\psi(yx) - \psi(xy)] = \text{Im}(\psi(xy)).
\]

This yields

\[
(x, y)_U = \psi(xy)
\]

for selfadjoint elements. Thus the formula (3.15) extends the combinatorial formula (3.15) for arbitrary elements in \( n_\psi \cap n_\psi^* \). In the literature these extensions are obtained by considering the complex linear extension \( \tilde{s}_q \) (see [Shi] and [Hia]) of \( s_q \).

**Example 3.9.** Let us consider the special case \( N = L_\infty(\mu; \mathbb{M}_2) \) and

\[
\psi(x) = \int [f_1(\omega)x_{11}(\omega) + f_2(\omega)x_{22}(\omega)]d\mu(\omega).
\]

For \( f \in L_2(\mu; \mathbb{C}) \) we define \( j(f) = \begin{pmatrix} 0 & f \\ \bar{f} & 0 \end{pmatrix} \). It is easily checked that \( j \) is an isometric embedding and \( \Delta^{-it}(j(f)) = (\frac{\Delta}{\rho})^{it}f \) leaves \( K = L_2(\mu; \mathbb{C}) \) invariant. Since \( j \) is only real linear,
we shall consider $K$ as a real Hilbert space. The subspace $H = j(K) + ij(K)$ is spanned by elements of the form $j_2(f, g) = \begin{pmatrix} 0 & f \\ g & 0 \end{pmatrix}$. By restriction we find

$$\text{Re}(j(f)j(g)) = \text{Re} \int [\bar{f} \bar{g} f_1 + \bar{f} \bar{g} f_2] d\mu = \int \frac{\bar{f} \bar{g} + \bar{f} \bar{g}}{2} d\mu.$$ 

For the imaginary part we have

$$\text{Im}(j(f)j(g)) = \text{Im} \left( \int [\bar{f} \bar{g} f_1 + \bar{f} \bar{g} f_2] d\mu \right) = \int \frac{\bar{f} \bar{g} - \bar{f} \bar{g}}{2i} f_1 + \frac{\bar{f} \bar{g} - \bar{f} \bar{g}}{2i} f_2 d\mu$$

$$= \int \frac{i(f_2 - f_1)\bar{f} \bar{g} - i(f_2 - f_1)\bar{f} \bar{g}}{2} d\mu = \int \frac{1}{2} i(f_2 - f_1)\bar{f} \bar{g} + i(f_2 - f_1)\bar{f} \bar{g} d\mu$$

$$= \text{Re}(j(i(f_2 - f_1)\bar{f} \bar{g})) .$$

This shows that on $H$ the operator $A$ is given by $j_2(f, g) = j_2(f_2/f_1, f_1/f_2g)$ and $C = i\frac{A - A^*}{1 + A^*}$ is given by $C(j(f)) = j(i(f_2 - f_1)f)$.

We may consider Example 3.9 as the prototype for arbitrary inclusions using a well-known argument of Araki [Ara].

**Lemma 3.10.** Every triple $(K, H_U, A)$ is unitarily equivalent to

$$(K_0^R, K_0^C, id) \oplus (L_2(\Omega, \mu; \mathbb{C}), L_2(\Omega, \mu; \sqrt{f_2}\mathbb{C}) \oplus L_2(\Omega, \mu; \sqrt{f_1}\mathbb{C}), (M_{f_2/f_1}, M_{f_1/f_2})) .$$

Here $U_1$ acts as the identity on the real Hilbert space $K_0^R$ and $K_0^C = K_0^R + iK_0^R$ is the canonical complexification. On the orthogonal complement $K_1 = L_2(\Omega, \mu; \mathbb{C})$ the embedding is given by $j(f) = (f, f)$ and $U_1(f) = (f/f_1)^i f$ holds for suitable positive functions $f_1, f_2$ with $f_1 + f_2 = 1$.

**Proof.** We follow [Sh] Section 2 and [Ara]. On $K \oplus iK$ there exists a selfadjoint operator $S$ such that $U_t = e^{itS}$. Being a generator of a one parameter unitary group, we know that $B = iS$ is an unbounded operator on $K$. We may assume that $S$ is injective. Indeed, there is a projection $e$ corresponding to the kernel of $S$ and we might write $K = K_0 + K_1$ such that $eU_t = U_t e = eU_t e = e$ and $U_t$ acts non-trivially on $K_1$. On $K_0$ the inclusion $K_0 \subset H_U$ is given by the canonical inclusion $K_0 \subset K_0 + iK_0$. Thus for the rest of the proof we may assume $K = K_1$.

Note that $B^* B = S^* S$ is a selfadjoint real operator and thus we have a polar decomposition $B = V|S|$ (coming from $S = U|S|$ and $iU = V$) such that $V$ is a unitary on $K$. Since $S$ is selfadjoint we have

$$1 = (S|S|^{-1})^2 = U^2 = (-iV)^2.$$ 

This implies $V^2 = -1$. On the other hand we have $V^t V = 1$ and hence $V^t = -V$. Moreover, $S^* = S$ implies $B^t = -B$. Therefore we get

$$-V|S| = -B = B^t = |S|V^t = -|S|V .$$

Thus $V$ commutes with $S$. Using a maximal system of cyclic vectors, one can construct $K_+$ such that $K_+ + V(K_+) = K$ and $|S|(K_+) \subset K_+$. Using a maximal system of cyclic vectors again and functional calculus, we see that $|S|_{K_+}$ is unitarily equivalent to a multiplication operator $M_f$ on $L_2(\Omega, \mu; \mathbb{R})$ (see [KR1]). By orthogonality of $K_+$ and $K_- = V(K_+)$, we deduce that $(K, B)$ is unitarily equivalent to $(L_2(\mu; \mathbb{C}), iM_f)$ such that $K = K_+ \oplus K_-$ is real isomorphic to
$L_2(\mu; \mathbb{C})$ and $U_t = e^{itf}$. Now, we may use our standard model and set $f_1 = e^{-f}(1 + e^{-f})^{-1}$, $f_2 = (1 + e^{-f})^{-1}$, and

$$
\psi(x) = \int [f_1 x_{11} + f_2 x_{22}] d\mu
$$

According to Example 3.9, the corresponding action is given by $(f_2/f_1)^{it} = e^{itf} = U_t$. Moreover, $M_t(f_2 - f_1)$ also provides the correct imaginary part for $i\frac{1}{t+\lambda}$. 

As a direct application of [Hia, Theorem 3.3] we obtain the following information on types.

**Corollary 3.11.** Let $K = L_2(\mu; \mathbb{C})$ and $U_t(f) = (f_2/f_1)^{it}f$. Let $-1 < q < 1$ and $\Gamma_q(K, U_t)$ the $q$-deformed Araki-Woods factor. Let $G$ be the closed subgroup generated by the spectrum of $f_1/f_2$. Then $\Gamma_q(K, U_t)$ is of type $\text{II}_1$ if $G = \{1\}$, of type $\text{III}_\lambda$ if $G = \{\lambda^n : n \in \mathbb{Z}\}$, and of type $\text{III}_1$ if $G = \mathbb{R}_+$. 

### 4. Limit distributions

In the previous section, we have seen an algebraic construction of the CAR and CCR relations from the central limit theorem. In this part we will discuss some analytic properties which will be used to provide matrix models. We will first discuss ultraproducts of noncommutative $L_p$ spaces.

Given an index set $I$ and a family $(X_n)_{n \in I}$ of Banach spaces, we will use the notation $\prod b X_n$ for the subspace of bounded sequences in $\prod X_n$. The ultraproduct $\prod_{n \in \mathcal{U}} X_n$ is defined as the Banach space $\prod b X_n / J_{\mathcal{U}}$, where $J_{\mathcal{U}}$ is the closed subspace of $\prod b X_n$ given by

$$
J_{\mathcal{U}} = \{(x_n) | \lim_{n \in \mathcal{U}} \|x_n\| = 0\}.
$$

We will follow [Rav] and use the notation

$$(x_n)^* = (x_n) + J_{\mathcal{U}}.
$$

Let $(N_n)$ be a sequence of von Neumann algebras. Then, we may consider the ultraproduct $\prod_{n \in \mathcal{U}} L_1(N_n)$. Following the natural duality for operator spaces (see [JNRX]), we find an isometric embedding

$$
\iota : \prod_{n \in \mathcal{U}} L_1(N_n) \to (\prod b N_n^{\text{op}})^* \text{ given by } \iota(d_n)(x_n) = \lim_{n \in \mathcal{U}} tr_{N_n}(d_n x_n).
$$

Note $\iota$ is a completely isometric embedding and the range $V = \iota(\prod_{n \in \mathcal{U}} L_1(N_n))$ is left and right invariant by multiplication with elements $x \in \prod b N_n$. Therefore (see [Tak1]), we find a central projection $\tilde{z}_{\mathcal{U}} \in (\prod n N_n^{\text{op}})^* \text{ such that } V = \tilde{z}_{\mathcal{U}}(\prod n N_n^{\text{op}})^*$. We obtain a von Neumann algebra

$$
\tilde{N}_{\mathcal{U}} = \tilde{z}_{\mathcal{U}}(\prod n N_n^{\text{op}})^*.
$$

The von Neumann algebra $\tilde{N}_{\mathcal{U}}$ is the dual of the ultraproduct of preduals. The map $\iota$ provides a completely isometric isomorphism between $\prod_{n \in \mathcal{U}} L_1(N_n)$ and $L_1(\tilde{N}_{\mathcal{U}}) \cong (\tilde{N}_{\mathcal{U}})^*$. This argument (except for the additional complication with $\text{op}$ needed for the operator space level) is due to Groh [Gro]. For our applications $\tilde{N}_{\mathcal{U}}$ is still too big because, in general, it is not $\sigma$-finite. Therefore, we will in addition assume that $(\phi_n)$ is a sequence of faithful normal states such that $\phi_n$ is defined on $N_n$. Let $(D_n)$ be the corresponding sequence of densities associated to $(\phi_n)$. This notation will be fixed in the sequel. Clearly, $(D_n)^*$ is an element in $\prod_{n \in \mathcal{U}} L_1(N_n)$. We denote...
Lemma 4.1. Let $0 < p < \infty$. There is an (completely) isometric embedding $I_p : L_p(\mathcal{N}_\mathcal{U}) \to \prod_{n,\mathcal{U}} L_p(\mathcal{N}_n)$ satisfying

$$I_p(D_{\mathcal{U}}^{1-\theta} e_\mathcal{U}(x_n)^* e_\mathcal{U} D_{\mathcal{U}}^{\theta}) = (D_n^{1-\theta} x_n D_n^{\theta})^*.$$

Moreover, the subspace $W = \{ e_\mathcal{U}(x_n)^* e_\mathcal{U} \mid (x_n) \in \prod_b \mathcal{N}_n \}$ is strongly dense in $\mathcal{N}_\mathcal{U}$. If $\sigma_t^{\phi_n}$ and $\sigma_t^{\phi_\mathcal{U}}$ denote the modular group of $\phi_n$ and $\phi_\mathcal{U}$, respectively, then

$$(4.1) \quad \sigma_t^{\phi_\mathcal{U}}(e_\mathcal{U}(x_n)^* e_\mathcal{U}) = e_\mathcal{U}(\sigma_t^{\phi_n}(x_n))^* e_\mathcal{U}.$$ 

The classical central limit theorem provides unbounded random variables. If we work with unbounded operators in a non-tracial setting, we have to multiply unbounded operators without the help of the algebra of $\tau$-measurable operators. The context of $L_p$ spaces yields a very convenient substitute in our setting. Let $D$ be the density of a normal faithful state $\phi$ on a von Neumann algebra $\mathcal{N}$ and $x$ a selfadjoint operator affiliated to $\mathcal{N}$. We will say that $xD^{1-p} \in L_p(\mathcal{N})$ if there exists $y \in L_p(\mathcal{N})$ such that for any increasing sequence of intervals $(I_k)$ with $\bigcup_k I_k = \mathbb{R}$ we have

$$y = \lim_k 1_{I_k}(x) xD^{\frac{1}{p}}.$$

The next Lemma is an easy application of Kosaki’s interpolation result.

Lemma 4.2. Let $x$ be a selfadjoint operator affiliated to $\mathcal{N}$, $\phi$ a normal faithful state and $\mu(A) = \phi(1_A(x))$ the induced measure on $\mathbb{R}$. If $2 \leq p \leq \infty$ and the identity function $f(t) = t$ belongs to $L_p(\mu)$, then

$$xD^{\frac{1}{p}} \in L_p(\mathcal{N}).$$

In particular, let $p = 2m$. If $\int_{\mathbb{R}} t^{2m} d\mu(t)$ is finite, then $xD^{\frac{1}{p}}$ for all $p \leq m$.

Proof. Let $M$ be the commutative von Neumann algebra generated by 1 and the spectral projections $e_A = 1_A(x)$ where $A$ ranges through the borel-measurable sets. Functional calculus provides a natural $*$-homomorphism $\pi : L_\infty(\mathbb{R}, \mu) \to \mathcal{N}$. For a bounded function $f \in L_\infty(\mu)$ we define

$$\pi_p(f) = \pi(f)D^{\frac{1}{p}}.$$

Let us recall Kosaki’s interpolation result [Kos]

$$L_p(\mathcal{N})D^{\frac{1}{p}-\frac{1}{p}} = [ND^{\frac{1}{2}}, L_2(\mathcal{N})]_{\frac{1}{p}}.$$

Here the inclusion map $\iota$ of $\mathcal{N}$ in $L_2(\mathcal{N})$ is given by the map $\iota(x) = xD^{\frac{1}{2}}$. We observe that $\pi_p(f)D^{\frac{1}{p}-\frac{1}{p}} = \iota(\pi(f))$. Thus, taking the inclusions into account, we see that the family...
(π_p)_{2 \leq p \leq \infty} is indeed induced by the 'same' operator π_2(f) = π(f)^{1/2}. Note also that π_p is an isometry for p = \infty and p = 2 (recall \|f\|_{L^2(\mu)} = [\phi((|f|^2))]^{1/2}). By interpolation, we deduce that π_p extends to a contraction on L_p(\mu). However, by the dominated convergence theorem, we see that for every increasing family of bounded intervals (I_k) and f \in L_p(\mu) we have
\[ L_p(\mu) - \lim_k 1_{I_k} f = f. \]

By continuity, we deduce that \lim_k π(1_{I_k}(x))D^{1/2}_{\mu} converges to π_p(f). This proves the first assertion. For p = 2m, we simply note that \int t^{2m}dμ(t) means that f(t) = t belongs to L_{2m}(\mu).

Since the inclusion map \( I_{2m,p} : L_{2m}(N) \to L_p(N) \) defined by \( I_{2m,p}(x) = xD^{1/2 - 1/2m}_{\mu} \) is continuous, we obtain XD^{1/2}_{\mu} \in L_p(N) for all p \leq m. □

As usual we denote by \( \sigma_\phi^t \) the modular group of a state (or weight) \( \phi \). An element \( x \in N \)
is called analytic if the function \( t \mapsto \sigma_\phi^t(x) \) extends to an analytic function with values in \( N \).

In this case we use the notation \( \sigma_z(x) \in N \). We denote by \( N_\alpha \subset N \) the subalgebra of analytic elements. The map \( \sigma_x : N_\alpha \to N \) is a homomorphism satisfying \( \sigma_x(x)^* = \sigma_z(x^*) \).

We denote by \( N_{\alpha_{x,a}} \) the real subalgebra of selfadjoint analytic elements.

**Proposition 4.3.** Let \( X \) be an index set and \( |x| : X \to [0, \infty) \) be a function. Let \( (N_n, \phi_n) \)
be a family of von Neumann algebras \( N_n \) with normal faithful states \( \phi_n \). For every \( n \in \mathbb{N} \) let \( (u_n(x))_{x \in X} \) be a family of analytic selfadjoint elements in \( N_n \) such that

i) There exists a constant \( C \) such that
\[ |\phi_n(u_n(x_1) \cdots u_n(x_m))| \leq (Cm)^m|x_1| \cdots |x_m| \]
holds for all \( n \in \mathbb{N} \) and \( x_1, \ldots, x_m \in X \).

ii) There exists a \( 2 < p < \infty \) such that for all \( x \in X \) there exists a constant \( c(x, k) \) such that
\[ \sup_n \left\| D^{1/2}_{\mu} \sigma_\phi^t(u_n(x)^k) \right\|_p \leq c(x, k). \]

Let \( \alpha : \bigcup_{m \geq 0} X^m \to \prod_{n, \mathcal{U}} L_2(N_n) \) be defined by \( \alpha(\emptyset) = (D^{1/2}_{\mu})^* \) and
\[ \alpha(x_1, \ldots, x_m) = (u_n(x_1) \cdots u_n(x_m))D^{1/2}_{\mu}. \]

Then there exists a family of selfadjoint operators \( (u(x))_{x \in X} \) affiliated to \( N_{\mathcal{U}} \) such that

a) If \( |z| < \delta(x, x_1, \ldots, x_m) \), then \( \sum_{k=0}^\infty \frac{z^k}{k!} u(x)^k \alpha(x_1, \ldots, x_m) \) converges absolutely in the ultraproduct \( \prod_{n, \mathcal{U}} L_2(N_n) \), and

b) \( \lim_{t \to \infty} e^{isu(x)} \alpha(x_1, \ldots, x_m) |_{t=0} = \alpha(x, x_1, \ldots, x_m) \), and

c) \( ((D^{1/2}_{\mu})^*, \alpha(x_1, \ldots, x_m)) = \lim_{n, \mathcal{U}} \phi_n(u_n(x_1) \cdots u_n(x_m)). \)

**Proof.** For fixed \( x \in X \), we see that \( U^x_\phi = (e^{isu_n(x)})^* \) is a unitary group in \( \tilde{N}_{\mathcal{U}} \).

Let us show that \( \alpha(x_1, \ldots, x_m) \) is in the domain of the generator. Let us fix \( n \in \mathbb{N} \) and define \( \alpha_n(x_1, \ldots, x_m) = u_n(x_1) \cdots u_n(x_m)D^{1/2}_{\mu} \).

According to our assumption
\[ \|\alpha_n(x_1, \ldots, x_m)\|_2^2 = \phi_n(u_n(x_m) \cdots u_n(x_1)u_n(x_1) \cdots u_n(x_m)) \leq (2Cm)^{2m} \prod_{i=1}^m |x_i|^2 \]
is uniformly bounded and hence $\alpha(x_1, ..., x_m)$ is an element in $L_2(\tilde{N}_\ell) \cong \prod_{n, \ell} L_2(N_n)$. Similarly, we have
\[
\left\| u_n(x_k^k \alpha_n(x_1, ..., x_m) \right\|_2^2 = \phi_n(u_n(x_m)) \cdots u_n(x_1)^2 u_n(x_1) \cdots u_n(x_m))
\]
\[
\leq (2C(m + k))^{2(m+k)} \prod_{j=1}^m |x_j|^2 |x|^{2k} \leq (2Cm)^{2m} \prod_{j=1}^m |x_j|^2 (2Cm)^k |x|^{2k}.
\]
Let $\delta = (4e C m (1 + |x|))^{-1}$, $c_m = (2C m) \prod_{j=1}^m |x_j|$, and $|z| \leq \delta$. Then we see that
\[
\sum_{k} \frac{z^k}{k!} u_n(x_k^k \alpha_n(x_1, ..., x_m)) \leq c_m \sum_{k=0}^{\infty} \frac{|z|^k}{k!} \leq \sum_{k=0}^{2k} 2^{-k}.
\]
Since these estimates are uniformly in $n$, we deduce assertion a) and that
\[
\frac{d}{dt} U_t(\alpha(x_1, ..., x_m))|_{t=0} = \alpha(x, x_1, ..., x_m)
\]
is well-defined. Therefore $\alpha(x_1, ..., x_m)$ is in the domain of the closed generator $u(x)$ of the unitary group $U^x_t$ (see [Kad, Theorem 5.6.36]). Let us note in passing that also $(\frac{1}{2} D_n^2)^* \tilde{N}_ell$ is in the domain of $u(x)$. We recall that $c_1$ is the support of $\phi_1 = \iota((D_n)^*)$. Now, we will use condition ii) to show that for $|t|$ small enough, we have $U^x_t e_{\ell} = e_{\ell} U^x_t = U^x_t e_{\ell}$. We will first show that for $|t| \leq \delta(x)$ we have
\[
(1 - e_{\ell}) U^x_t \left( \frac{1}{2} D_n^2 y_n \right)^* = 0
\]
for every bounded sequence $(y_n) \subset \prod_b N_n$. Again, we fix $n \in \mathbb{N}$ and note
\[
u_n(x)^k D^\frac{1}{2} = D^\frac{1}{2} \sigma_n^x (u_n(x)^k) = D^\frac{1}{2} D^\frac{1}{2} - \frac{1}{2} D^\frac{1}{2} \sigma_n^x (u_n(x)^k).
\]
By definition of the multiplication in $L_2(\tilde{N}_\ell) \cong \prod_{n, \ell} L_2(N_n)$ (see [Ray]), we deduce
\[
\left( u_n(x)^k D^\frac{1}{2} \right)^* = \left( D^\frac{1}{2} \sigma_n^x (u_n(x)^k) \right)^*.
\]
According to assumption ii) $(\frac{1}{2} D_n^2 \sigma_n^x (u_n(x)^k))^* \in \prod_{n, \ell} L_2(N_n)$ is well-defined and henceforth
\[
(1 - e_{\ell}) \left( u_n(x)^k D^\frac{1}{2} y_n \right)^* = (1 - e_{\ell}) \left( D^\frac{1}{2} \sigma_n^x (u_n(x)^k) \right)^* (y_n)^* = 0.
\]
However, for $|t| \leq \delta(x)$, we deduce from absolute convergence that
\[
(1 - e_{\ell}) U^x_t \left( \frac{1}{2} D_n^2 y_n \right)^* = \sum_{k=0}^{\infty} \frac{t^k}{k!} (1 - e_{\ell}) \left( u_n(x)^k D^\frac{1}{2} y_n \right)^* = 0.
\]
This shows ii). By density we find $(1 - e_{\ell}) U^x_t (D^\frac{1}{2})^* \tilde{N}_ell = 0$, i.e. $(1 - e_{\ell}) U^x_t e_{\ell} = 0$ for $|t| \leq \delta$. Using adjoints, we deduce $e_{\ell} U^x_t = e_{\ell} U^x_t e_{\ell} = U^x_t e_{\ell}$. For arbitrary $t$ we choose $m$ such that $|t| \leq m \delta$ and observe
\[
e_{\ell} U^x_t = e_{\ell} (U^x_t)^m = (U^x_t)^m e_{\ell} = U^x_t e_{\ell}.
\]
Therefore $e_{\ell} U^x_t e_{\ell}$ is a strongly continuous unitary group in $N_\ell$ and the generator $u(x)$ satisfies the assertion.
Remark 4.4. 1) The proof of Proposition 4.3 allows us to replace the bounded operators $u_n(x)$ by unbounded selfadjoint operators. In this case, we should require that $\alpha_n(x_1,\ldots,x_m)$ is in the domain of $u_n(x)$ and replace the moment condition by

$$|(D_n^{{1\over 2}}, u_n(x_1)\cdots u_n(x_m)D_n^{{1\over 2}})| \leq (Cm)^m \prod_{i=1}^m |x_i|.$$ 

In order to formulate the modular condition, we recall that $\sigma_1^\phi_n(x) = D_n^{{it}} x D_n^{{-it}}$ also defines an automorphism of $L_p(N_n)$. Therefore, we shall require that $u_n(x)D_n^{{1/p}}$ is analytic and that for $0 = {1\over p} + {1\over q}$

$$\sup_n \left\| \sigma_1^\phi_n (u_n(x)^k D_n^{{1\over p}}) \right\|_p \leq c(x,k).$$

Then we have

$$u_n(x)^k D_n^{{1\over 2}} = D_n^{{1\over p} - {1\over q}} \sigma_1^\phi_n (u_n(x)^k D_n^{{1\over q}}).$$

This allows us to deduce (4.5) and complete the proof as above.

2) For applications, it is often more convenient to replace condition ii) in Proposition 4.3 by a moment condition. Let us assume that $p = 2m$ is an even integer and $0 = {1\over p} + {1\over q}$. We consider the real and imaginary part

$$y_{n,k} = \sigma_1^\phi_n (u_n(x)^k) + \sigma_1^\phi_n (u_n(x)^k)^*,$$

$$z_{n,k} = \sigma_1^\phi_n (u_n(x)^k) - \sigma_1^\phi_n (u_n(x)^k)^*.$$

Then, we deduce from Lemma 4.2 that

$$\left\| y_{n,k} D_n^{{1\over p}} \right\|_p \leq \phi_n (y_{n,k})^p \leq \sum_{s_1,\ldots,s_p \in \{0,\ast\}} |\phi_n (\sigma_1^\phi_n (u_n(x)^k)^{s_1} \cdots \sigma_1^\phi_n (u_n(x)^k)^{s_p})|.$$

A similar estimate holds for $z_{n,k}$. This shows that moment estimate

$$\sup_{n,s_1,\ldots,s_p \in \{0,\ast\}} |\phi_n (\sigma_1^\phi_n (u_n(x)^k)^{s_1} \cdots \sigma_1^\phi_n (u_n(x)^k)^{s_p})| \leq c(x,k,p)$$

implies condition ii). Again this holds still in the context of unbounded operators as long as we can justify the domain issues for the operators $\sigma_1^\phi_n (u_n(x)^k)$ and justify the equation

$$\sigma_1^\phi_n (u_n(x)^k) D_n^{{1\over p}} = \sigma_1^\phi_n (u_n(x)^k D_n^{{1\over q}}).$$

At the end of this section we show that the limit objects in the algebraic central limit theorem may be realized as unbounded operators.

**Theorem 4.5.** Let $N$ be a von Neumann algebra and $\psi$ a strictly semifinite weight with associated projections $(e_j)$. Let $(v_k(n))_{k=1,\ldots,n \in \mathbb{N}}$ be a family of selfadjoint contractions such that $(v_k(n))_{k=1,\ldots,n} \subset (\mathcal{M}_{2^m}^\psi, \tau_n)$ are contained in a finite von Neumann algebra satisfying the singleton condition (2.1). Let $U$ be an ultrafilter and for a pair partition $\sigma \in P_2(m)$ the weight is given by

$$\beta(\sigma) = \lim_{n, U} n^{-m \over 2} \sum_{(k_1,\ldots,k_m) \leq \sigma} \tau_n(v_{k_1}(n) \cdots v_{k_m}(n)).$$

Consider $X = N_a \cap n_\psi \cap n_\psi$ and $|x| = \max \{ ||x||_\infty, \psi(x^*x)^{1\over 2}, \psi(xx^*)^{1\over 2} \}$. Then there exists a von Neumann algebra $M$, a normal faithful state $\phi$ with density $D$, a linear map $\alpha : A(X) \to L_2(M)$ and a family $\{u(x)\}_{x \in N_{a,a}}$ of selfadjoint operators affiliated to $M$ such that
i) \( \alpha(x_1 \otimes \cdots \otimes x_m) \) is in the domain of \( u(x) \) and
\[
u(x) \alpha(x_1 \otimes \cdots \otimes x_m) = \alpha(x \otimes x_1 \otimes \cdots \otimes x_m),
\]
i) \( D^\frac{1}{2} \) is in the domain of \( u(x) \) and
\[
u(x) D^\frac{1}{2} = \alpha(x),
\]
iii) \( (D^\frac{1}{2}, \alpha(x_1 \otimes \cdots \otimes x_m)) = \sum_{\sigma \in P_2(m)} \beta(\sigma) \psi_\sigma[x_1, \ldots, x_m], \)
iv) \( \sigma_t^0(u(x)) = u(\sigma_t^0(x)) \) holds for all \( t \in \mathbb{R}, \)
holds for all \( x_1, \ldots, x_m \in N_{\alpha(a,a)} \) and \( x \in N_{\alpha(a,a)}. \)

**Proof.** For fixed \( j \) and \( n \in \mathbb{N} \) we define \( T_j = \psi(e_j) \) and \( \phi_{n,j} = \tau_n \otimes \left( \frac{\psi}{T_j} \right)^n \) on \( M_{n,j} = M_{2^n} \otimes (e_j N e_j)^{\otimes n} \). For \( x \in N_{\alpha(a,a)} \) we define
\[
u_{n,j}(x) = \sqrt{\frac{T_j}{n}} \sum_{k=1}^n v_k(n) \otimes \pi_k(e_j x e_j).
\]
Since \( \tau_n \) is assumed to be a trace, we have

\[
\sigma_t^{\phi_n}(u_{n,j}(x)) = u_{n,j}(\sigma_t^0(x)).
\]

In order to check condition i) we apply Corollary 2.3 for \( e_j x e_j \). Let us show that \( |e_j x e_j| \leq |x| \).

Indeed, \( \|e_j x e_j\|_\infty \leq \|x\|_\infty \). Moreover, we deduce from 2.3 that
\[
\psi(e_j x^*e_j x) = \|e_j x^2\|_{L_2(N,\psi)} = \|J e_j J x\|_{L_2(N,\psi)} \leq \|x\|^2_{L_2(N,\psi)} = \psi(x^*x).
\]

Let us denote by \( D_{n,j} \) the density of \( \phi_{n,j} \). We will now apply Remark 4.4(2) in order to verify the condition ii) in Proposition 4.3. Indeed, we take \( p = 4 = q \) and \( x_j = e_j x e_j \). Then we consider
\[
y_{k,j} = \sigma_q(x_j^k) + \sigma_q(x_j^k)^* \quad \text{and} \quad z_{k,j} = \sigma_q(x_j^k) - \sigma_q(x_j^k)^*.
\]
We note that
\[
|y_{k,j}| \leq 2 |\sigma_q(x_j^k)| \leq 2 \|\sigma_q(x_j^k)\|_{\infty}^{k-2} \|\sigma_q(x_j^k)\|_\infty.
\]
However \( |\sigma_q(x_j)| = |e_j \sigma_q(x) e_j| \leq |\sigma_q(x)| \) implies
\[
|y_{k,j}| \leq 2 \|\sigma_q(x)\|_{\infty}^{k-2} |\sigma_q(x)|.
\]
For \( k = 1 \) we simply have \( |y_{k,j}| \leq 2 |\sigma_q(x)| \). Similarly, we find the estimate
\[
|z_{k,j}| \leq 2 \|\sigma_q(x)\|_{\infty}^{k-2} |\sigma_q(x)|.
\]
This implies with Lemma 4.2 and Lemma 2.3 that
\[
\left\| \sigma_q(x_j^k) D^\frac{1}{2} \right\|_4 \leq \left\| \frac{y_{k,j}}{2} D^\frac{1}{2} \right\|_4 + \left\| \frac{z_{k,j}}{2} D^\frac{1}{2} \right\|_4 \leq \frac{\phi_n(y_{k,j}^2) + \phi_n(z_{k,j}^2)}{2} \leq 4^2 \|y_{k,j} + z_{k,j}\|_4 \leq 32 \|\sigma_q(x)\|_{\infty}^{k-2} |\sigma_q(x)|.
\]

Therefore the condition ii) in Proposition 4.3 is satisfied uniformly in \( n \) and \( j \). Let \( \mathcal{U}' \) be an ultrafilter on \( I \) and \( \mathcal{U} = \mathcal{U}' \times \mathcal{U} \) the ultrafilter on \( I \times \mathbb{N} \) such that \( A \in \mathcal{U} \) iff \( \{j : n \in \mathbb{N} : (j, n) \in A\} \in \mathcal{U}' \). In terms of limits, this means \( \lim_{(j,n) \mathcal{U}} a_{j,n} = \lim_{j \mathcal{U}'} \lim_{n \mathcal{U}} a_{j,n} \). We apply
Proposition 4.3 to the double indexed family $u_{n,j}(x)$. By linearity it suffices to define the linear map $\alpha$ on $A(X)$ for tensors with the help of Raynaud’s isomorphism:

$$\alpha(x_1 \otimes \cdots \otimes x_m) = I_2^{-1} \left( (u_{n,j}(x_1) \cdots u_{n,j}(x_m) D_{n,j}^x)^{\cdot} \right).$$

The state $\phi_C$ is the ultraproduct state $(\phi_{n,j})^*$ with density $D_{n,j}$ in $L_1(N_{\overline{t}})$. Note that by definition of $N_{\overline{t}}$, the state $\phi_C$ is faithful. We deduce from Proposition 4.3 that $u(x)$ is affiliated to $N_{\overline{t}}$ and satisfies the domain properties i) and ii). The modular condition iv) follows from (4.7) and (4.1).

The moment condition iii) follows from Raynaud’s isomorphism, Lemma 4.1 and Corollary 2.6

$$(D_{ij}, \alpha(x_1 \otimes \cdots \otimes x_m)) = \lim_{(j,n) \in \mathcal{L}} (D_{n,j}^{1/2}, u_{n,j}(x_1) \cdots u_{n,j}(x_m) D_{n,j}^{1/2})$$

$$= \lim_{j \in \mathcal{L}, n \in \mathcal{L}} \phi_{n,j}(u_{n,j}(x_1) \cdots u_{n,j}(x_m)) = \lim_{j \in \mathcal{L}, \sigma \in \mathcal{P}(m)} \beta(\sigma) \psi_\sigma[e_j x_1 e_j, \ldots, e_j x_m e_j]$$

$$= \sum_{\sigma \in \mathcal{P}(m)} \beta(\sigma) \psi_\sigma[x_1, \ldots, x_m].$$

**Corollary 4.6.** Let $N$ be a von Neumann algebra and $\psi : N \to \mathbb{C}$ be a strictly semifinite normal faithful weight. Then there exists a von Neumann algebra $M[-1,1]$ with a normal faithful state $\phi$ and a family $\{U_t(x,q) : x \in N_{sa,a}, -1 \leq q \leq 1\}$ of unitary groups which generate $M$ with the following properties.

i) The generators $u_q(x) = \frac{d}{dt} U_t(x,q)$ satisfy

$$\phi(u_{q_1}(x_1) \cdots u_{q_m}(x_m)) = \sum_{\sigma \in \mathcal{P}(m), \sigma \leq \rho} t(\sigma, q_1, \ldots, q_m) \psi_\sigma[x_1, \ldots, x_m].$$

ii) For every subset $I \subset [-1,1]$ there is $\phi$-preserving conditional expectation $E_I : M \to M_I$, $M_I$ generated by $\{U_t(x,q) : x \in N_{sa,a}, q \in I\}$. For disjoint sets $I$ and $J$ the algebras $M_I$ and $M_J$ are independent over $\phi$ (in the sense of (6.1)).

**Proof.** The first assertion is a direct consequence of Corollary 2.9 and Theorem 4.5 but now applied for the family $v_{j,q}(n)$ indexed by the additional parameter $-1 \leq q \leq 1$. For a subset $I \subset [-1,1]$, we deduce from $\sigma^R I (e^{it v_{j,q}(x)}) = e^{it v_{j,q}(\sigma^R I (x)))}$ that $M_I$ is invariant under $\sigma^R I$ and hence we can apply Takesaki’s theorem (see e.g. [StR, Theorem 10.1]) and find a $\phi$-preserving conditional expectation. Now, we assume $I \cap J = \emptyset$, $q_1, \ldots, q_k \in I$, $q_{k+1}, \ldots, q_m \in J$. Then

$$\phi(u_{q_1}(x_1) \cdots u_{q_k}(x_k) u_{q_{k+1}}(x_{k+1}) \cdots u_{q_m}(x_m)) = \phi(u_{q_1}(x_1) \cdots u_{q_k}(x_k)) \phi(u_{q_{k+1}}(x_{k+1}) \cdots u_{q_m}(x_m))$$

follows from i). Using absolute convergence (uniform in $q$), we may extend this relation to polynomials in $U_t(x,q_k)$. Since $\phi$ is normal this implies $\phi(ab) = \phi(a) \phi(b)$ for $a \in M_J$, $b \in M_I$. 

We refer to [Bla] for a short introduction to $C(X)$-algebras. In our context we obtain the following continuity result (without constructing an embedding of $C[-1,1]$ in the center.)

**Corollary 4.7.** Let $T = \{(x,t) : x \in N_{sa,a}, t \in \mathbb{R}\}$ and $A(T)$ be the free algebra in $T$ noncommuting selfadjoint variables. Let $\pi_q : A(T) \to M$, $\pi_q((x,t)) = e^{it u_q(x)}$ the induced representation. The function

$$f_x(q) = \|x + \ker \pi_q\|$$

is lower semi-continuous.
from Corollary 2.1 and Corollary 2.8, we deduce for

\( p \in \mathbb{C}[x_1, \ldots, x_n] \), we can find noncommutative polynomials \( p_1, p_2 \) in the variables \( y_1, \ldots, y_k \in \mathbb{N}_{sa,a} \) such that

\[
(1 - \varepsilon) \| \pi_{q_0}(p) \| \leq \| (p_1(y_{q_0}(y_1)), \ldots, u_{q_0}(y_k), \pi_{q_0}(p)p_2(u_{q_0}(y_1), \ldots, u_{q_0}(y_k)) \|_2
\]

and \( \| p_1(u_{q_0}(y_1), \ldots, u_{q_0}(y_k)) \|_2 = 1 = \| p_1(u_{q_0}(y_1), \ldots, u_{q_0}(y_k)) \|_2 \). Using the combinatorial formula from Corollary 2.1 and Corollary 2.8, we deduce

\[
\lim_{q \to q_0} \| p_j(u_{q}(y_1), \ldots, u_{q}(y_k)) \|_2 = \| p_j(u_{q_0}(y_1), \ldots, u_{q_0}(y_k)) \|_2
\]

for \( j \in \{1, 2\} \). By linearity it suffices to show

\[
\lim_{q \to q_0} \phi(u_{q_0}(y_1), \ldots, u_{q_0}(y_k))e^{it_1 u_{q_0}(x_1)} \cdots e^{it_m u_{q_0}(x_m)} u_{q_0}(y_1') \cdots u_{q_0}(y_{k'})
\]

\[
= \phi(u_{q_0}(y_1), \ldots, u_{q_0}(y_k))e^{it_1 u_{q_0}(x_1)} \cdots e^{it_m u_{q_0}(x_m)} u_{q_0}(y_1') \cdots u_{q_0}(y_{k'})
\]

According to Corollary 2.8, we know that

\[
\| \phi(u_{q_0}(y_1), \ldots, u_{q_0}(y_k))u_{q_0}(x_1)^{l_1} \cdots u_{q_0}(x_m)^{l_m} u_{q_0}(y_1') \cdots u_{q_0}(y_{k'}) \|_2 \leq C(y_1, \ldots, y_k, y_1', \ldots, y_{k'}) (k + k')^{\frac{d}{2}} d^\frac{1}{2} \prod |x_j|^{l_j}
\]

where \( d = \sum_i l_i \). Using the absolute convergence of \( \sum_i \frac{a_i}{n} l_i^{\frac{1}{2}} \), we may now approximate the exponential functions by polynomials (uniformly in \( q \)). Hence the convergence follows from continuity in \( q \) of the combinatorial formula for the joint moments (see Corollary 2.1 and Corollary 2.8). For an arbitrary element of \( A(T) \) the assertion follows by approximation. \( \Box \)

5. A uniqueness result

The CAR and CCR relations are usually defined using Fock-space representations. We will prove a noncommutative version of the Hamburger moment problem which allows us to show that the ultraproduct construction and the Fock space models describe the same von Neumann algebra.

We will say a \( \ast \)-probability space \((A, \ast, \phi)\) is given by unital \( \ast \)-algebra \( A \) over the complex numbers and a positive linear functional \( \phi : A \to \mathbb{C} \) with \( \phi(1) = 1 \). Here \( \ast \)-algebra means that \( \ast \) is an antilinear involution and that

\[
\mathcal{A} = \mathcal{A}_{sa} + i \mathcal{A}_{sa} \quad \text{where} \quad \mathcal{A}_{sa} = \{ a \in A | a^* = a \}
\]

is called the selfadjoint part of \( \mathcal{A} \). The functional \( \phi \) is called positive, if \( \phi(a^*a) \geq 0 \) and \( \phi(a^*) = \overline{\phi(a)} \). A representation of \( \mathcal{A} \) is given by a generating set \( S \subset \mathcal{A}_{sa} \),

i) a Hilbert space \( H \), a unit vector \( \xi \), and a linear map

\[
\alpha : A \to H \quad \text{with} \quad \alpha(1) = \xi;
\]

ii) a map \( \pi : S \cup \{1\} \to \mathcal{A}(H) \), \( \mathcal{A}(H) \) the set of selfadjoint densely defined operators on \( H \), such that \( \pi(1) = 1 \), and \( \alpha(A) \) is a subset of the domain of \( \pi(a)^k \) for all \( a \in S \), \( k \in \mathbb{N} \), and

\[
\pi(a)\xi = \alpha(a) , \quad \int t^k(\alpha(b), dE_t^{\pi(a)} \alpha(b)) = (\alpha(b), \alpha(a b b))
\]

holds for all \( a \in S \) with spectral resolution \( E_t^{\pi(a)} \) and for all \( b \in \mathcal{A} \).
iii) Moreover, we require

\[ \phi(a^* b) = (\alpha(a), \alpha(b))_H \]

for all \( a, b \in \mathcal{A} \).

Let us say that \( \mathcal{A} \) satisfies the growth condition if there exists a generating subset \( S \subset A_{sa} \) such that \( S \cup \{1\} \) generates \( \mathcal{A} \) as an algebra and there exists a length function \( |\cdot| : S \to (0, \infty) \) such that

\[ |\phi(a_1 \cdots a_n)| \leq n^n \prod_{i=1}^n |a_i| \]

holds for all \( a_1, \ldots, a_n \in S \). We will show that under these assumptions the von Neumann algebra generated by the spectral projections of elements \( \pi(a), a \in S \) is uniquely determined. We will need the following formulation of the Hamburger moment problem (see e.g. [Kad]).

**Theorem 5.1** (Hamburger moment problem). Let \( \mu_1 \) and \( \mu_2 \) be regular Borel measures on \( \mathbb{R} \) such that the moments

\[ \int t^k d\mu_1(t) = m_k = \int t^k d\mu_2(t) \]

coincide. If there exists a constant \( c \) such that \( m_k \leq c^{k+1} k^k \), then \( \mu_1 = \mu_2 \). Moreover, under these assumptions on \( m_k \) the polynomials are dense in \( L_p(\mu_1) \) for every \( p < \infty \).

**Sketch of Proof.** It is well-known that the Hamburger moment problem has a positive solution under these conditions. We will only indicate the proof of the density assertion needed in this paper. We consider the function \( h(x) = \frac{\sin(x)}{x} \) and the translates \( h_{\lambda,s}(x) = h(s(x - \lambda)) \). One first shows that for \( 2 \leq p < \infty \) and \( s(p) = (6cep)^{-1} \) the functions \( (h_{\lambda,s})_{s < s(p)} \) belong to the closure of the polynomials in \( L_p(\mathbb{R}, \mu_1) \). Indeed, let \( q_m(x) = \sum_{k=0}^m (-1)^k \frac{s^{2k}(x - \lambda)^{2k}}{(2k+1)!} \) and \( p \in 2\mathbb{N} \). Assume \( m > |\lambda| + 1 \). Then

\[ \|h_{\lambda,s} - q_m\|_p \leq \sum_{k>m} \frac{s^{2k}}{(2k+1)!} \left\| (x - \lambda)^{2k}\right\|_p \leq \sum_{k>m} \frac{s^{2k}}{(2k+1)!} c^{2k+1} (2kp + |\lambda|)^{2k} \]

\[ \leq c^{\frac{1}{p}} \sum_{k>m} \frac{(cs)^{2k}}{(2k+1)!} (2kp + k)^{2k} \leq c^{\frac{1}{p}} \sum_{k>m} \frac{(3cps)^{2k}}{2k} \frac{1}{2k(2k+1)}. \]

Here, we used the estimate \( \| (x - \lambda)^{2k}\|_p \leq c^{2k+1} (2kp + |\lambda|)^{2k} \) for \( k > |\lambda| + 1 \). Let \( s < s(2p) \). Now we show by induction on \( m \) that for every polynomial \( q \) the functions \( h_{\lambda_1,s} \cdots h_{\lambda_m,s} q \) belong to the closure of the polynomials in the \( L_p(\mathbb{R}, \mu_1) \). In particular, the algebra \( \mathcal{A} \) generated by \( (h_{\lambda,s})_{s < s(2p), \lambda \in \mathbb{R}} \) is in the closure of the polynomials in \( L_p(\mu_1, \mathbb{R}) \). Finally, we deduce from the Stone-Weierstrass theorem that \( \mathcal{A} \) is dense in \( C(\mathbb{R} \cup \{\infty\}) \) and hence in \( L_p(\mu_1) \) because \( \mu_1 \) is finite. We deduce that \( 1_{(a,b)} \) is in the \( L_p \) closure of polynomials for all \(-\infty < a < b \leq \infty \) and \( p < \infty \). By regularity of the Borel measure \( \mu \) the step functions are dense in \( L_p \) and this completes the proof.

Let us now verify that the growth condition allows us to apply the Hamburger moment problem. We consider a representation \( (\pi, \alpha, \xi) \) of \( \mathcal{A} \), and \( a \in S \) and \( b \in \mathcal{A} \). Then we may define the regular Borel measure (see e.g. [Kad], I Theorem 5.2.6) \( \mu \) on the real line given by

\[ \mu(B) = \int_B (\alpha(b), d\pi_t(a) \alpha(b)). \]
Clearly, \( \mu \) is a finite measure satisfying
\[
\mu(\mathbb{R}) = \phi(b^*b).
\]
The following estimate follows similarly as in (1.2) and we leave it to the reader.

**Lemma 5.2.** Let \( b \in A \) and \( a \in S. \) Then there exists a constant \( C \) such that for all \( k \in \mathbb{N}_0, \lambda \in \mathbb{R} \)
\[
| \int_{\mathbb{R}} (x + \lambda)^k d\mu(x) | \leq C^{k+1}(k + |\lambda|)^k.
\]

**Theorem 5.3.** Let \((A, \phi)\) be a *-probability space satisfying the growth condition with respect to \( S. \) Let \((\pi_1, \alpha_1, \xi_1)\) and \((\pi_2, \alpha_2, \xi_2)\) be representations. Let \( \mathcal{N}_1, \mathcal{N}_2 \) be the von Neumann algebras generated by the spectral projections of \( \{\pi_1(x)\}_{x \in \mathcal{S}}, \{\pi_2(x)\}_{x \in \mathcal{S}} \), respectively. Assume in addition that the restrictions \( \phi_1(x) = (\xi_1, x\xi_1), \phi_2 = (\xi_2, x\xi_2), \) are faithful on \( \mathcal{N}_1, \mathcal{N}_2, \) respectively. Then there is a normal homomorphism \( \pi : \mathcal{N}_1 \to \mathcal{N}_2 \) such that \( \phi_2 \circ \pi = \phi_1. \)

**Proof.** Let \( A_\phi \) be the pre-Hilbert space \( A \) equipped with the scalar product \( (b, c) = \phi(b^*c) \) and \( H_\phi \) its completion. Let us first consider a single representation \( \alpha_1 : A_\phi \to H_1. \) Then \( \alpha_1 \) extends to an isometric isomorphism between \( H_\phi \) and the closure of \( \alpha_1(A). \) Given \( b \in A, a \in S, \) we may consider the measure
\[
\mu_1(B) = \int_B (\alpha_1(b), dE_{\pi_1}^{\pi_1(a)}(\alpha_1(b))) \ .
\]
The measure is a regular Borel measure (see e.g. [Kad, I Theorem 5.2.6]) and satisfies the growth condition. Moreover, given a bounded function \( f, \) there exists a polynomial \( q_n \) such that \( \|f - q_n\|_{L^2(\mu_1)} \leq \frac{1}{n}. \) Therefore, we deduce
\[
\| f(\pi_1(a))\alpha_1(b) - q_n(\pi_1(a))\alpha_1(b) \|_2 = \|f - q_n\|_{L^2(\mu_1)} \leq n^{-1}.
\]
However,
\[
q_n(\pi_1(a))(\alpha_1(b)) = \alpha_1(q_n(a)b)
\]
and thus \( q_n(a)(b) \) forms a Cauchy sequence in \( H_\phi \) converging to some element \( h \in H_\phi. \) Thus we get
\[
f(\pi_1(a))(\alpha_1(b)) = \lim_n q_n(\pi_1(a))(\alpha_1(b)) = \lim_n \alpha_1(q_n(a)b) = \alpha_1(h).\]
We observe that \( K_1 = \text{cl}(\alpha_1(A)) \) is an invariant subspace for \( \mathcal{N}_1 \) and hence there exists a projection \( e_1 \in \mathcal{N}_1' \) such that \( K_1 = e_1H_1. \) Then \( \alpha_1 \) provides a unitary between \( H_\phi \) and \( K_1. \) We apply the same argument to \( \alpha_2 \) and obtain \( e_2 \in \mathcal{N}_2', \) \( K_2 = e_2H_2 \) and an isometry \( \alpha_2 : H_\phi \to K_2. \)
We define the unitary \( u = \alpha_2\alpha_1^{-1} : e_1H_1 \to e_2H_2 \) and want to show
\[
u^{-1}f(\pi_2(a))u = f(\pi_1(a)) \]
holds for all bounded measurable functions \( f \) and \( a \in S. \) Let \( a \in S \) and \( b \in A. \) The argument above shows that
\[
\alpha_1(h) = f(\pi_1(a))\alpha_1(b) = \lim_n \alpha_1(q_n(a)b) \text{ and } \alpha_2(h) = f(\pi_2(a))\alpha_2(b) = \lim_n \alpha_2(q_n(a)b).\]
Therefore, it suffices to observe that the sequence \((q_n(a))\) from above can be chosen to work for \(\pi_1(a)\) and \(\pi_2(a)\) simultaneously. This follows immediately from the fact that the measure \(\mu_2\) given by

\[
\mu_2(B) = \int_B \left( \alpha_2(b), dE^{\pi_2(a)}(\alpha_2(b)) \right)
\]

has the same moments as \(\mu_1\) and thus \(\mu_1 = \mu_2\) by Theorem 5.1. This completes the proof of (5.4). We can now define the state preserving normal \(*\)-homomorphism \(\tilde{\pi} : e_1N_1 \rightarrow e_2N_2\) given by \(\tilde{\pi}(x) = u^{-1}xu\). The conclusion follows from the fact \(N_1\) and \(e_1N_1\) are isomorphic and \(N_2\) and \(e_2N_2\) are isomorphic. Indeed, the induction \(\rho_1 : N_1 \rightarrow e_1N_1\) defined by \(\rho_1(x) = e_1x\) is a normal \(*\)-homomorphism. For a positive element \(\rho_1(x) = 0\) implies

\[
\phi_1(x) = (\xi_1, \rho_1(x)\xi_1) = 0.
\]

Since \(\phi_1\) is faithful, \(\rho_1\) is injective and therefore a normal isomorphism (see [Dix, I.4.3 Corollary 1]). The same applies for \(\rho_2\). Hence \(\pi = \rho_2^{-1}\tilde{\pi}\rho_1\) yields the state preserving homomorphism.

**Remark 5.4.** If we assume only that \(\phi_1\) is faithful we still have an isomorphism \(\pi : N_1 \rightarrow e_2N_2\).

In the following we will often have some control on the modular group:

**Proposition 5.5.** Let \((A, \phi)\) be a \(*\)-probability space with generating system \(S\) satisfying the growth condition. Let \((\sigma_t)_{t \in \mathbb{R}}\) be a family of maps such that \(\sigma_t(S) \subset S\). Let \((\pi, \alpha, \xi)\) be a representation of \((A, \phi)\) and \(M\) be a von Neumann algebra such that \(\pi(x)\) is affiliated to \(M\) for all \(x \in S\) and \(\phi_\xi(y) = (\xi, y\xi)\) is faithful on \(M\). Assume that

\[
\pi(\sigma_t(x)) = \sigma_t^{\phi_\xi}(\pi(x))
\]

holds for all \(t \in \mathbb{R}\) and \(x \in S\). Let \(N\) be the subalgebra of \(M\) generated by the spectral projections of \(\pi(S)\). Then there is a conditional expectation \(E : M \rightarrow N\) onto \(N\) such that \(\phi \circ E = \phi\).

**Proof.** Let \(x \in S\) and \(B \subset \mathbb{R}\) be measurable. Since \(\sigma_t^{\phi_\xi}\) is an automorphism on \(M\), we deduce

\[
\sigma_t^{\phi_\xi}(1_B(\pi(x))) = 1_B(\sigma_t^{\phi_\xi}(\pi(x))) = 1_B(\pi(\sigma_t(x)))
\]

Therefore \(\sigma_t^{\phi_\xi}\) leaves \(N\) invariant. The result follows by an application of Takesaki’s theorem (see [Str, Theorem 10.1]).

As an application we will show that the von Neumann algebras \(\Gamma_q(K, U_t)\) can be obtained from the central limit procedure. This is an important link for our norm estimates.

**Corollary 5.6.** Let \((K, H, U_t)\) be as in section 2. Then \(\Gamma_q(K, U_t)\) is isomorphic to a complemented subalgebra \(N \subset M\) obtained in Theorem 5.3. More precisely, there exists a homomorphism \(\pi : \Gamma_q(K, U_t) \rightarrow M\) such that \(\phi \circ \pi\) is the vacuum vector state \(\phi_{\text{vac}}, \sigma_t^{\phi} \circ \pi = \pi \circ \sigma_t^{\phi_{\text{vac}}}\) and

\[
(5.5) \quad \pi(s_q(h)D_{\text{vac}}) = u_q(h)D_{\frac{1}{\beta}}^h
\]

holds for all \(h \in K\). Here \(u_q\) is the map \(u\) constructed for \(\beta(\sigma) = q^{\beta(\sigma)}\).
Proof. According to Lemma 3.10 we may assume $K = K_{0}^{\mathbb{R}} \oplus L_{2}(\Omega_{1}, \mu; \mathbb{R})$. Using a conditional expectation at the end of our proof, we may assume that the dimension of $K_{0}$ is even and hence given by $K_{0} = L_{2}(\Omega_{0}, \mu_{0}; \mathbb{C})$. Then we use the disjoint union $\Omega = \Omega_{0} \cup \Omega_{1}$ with the measure $\mu = \mu_{0} + \mu_{1}$. We extend the functions $f_{1}$ and $f_{2}$ defined on $\Omega_{1}$ via Lemma 3.10 by $f_{1}(\omega) = f_{2}(\omega) = \frac{1}{2}$ on $\Omega_{0}$. We work with our standard model $N = L_{\infty}(\Omega, \mu; \mathbb{M}_{2})$ from section 3. We apply the central limit procedure Theorem 1.5 for the partition function $\beta(\sigma) = q^{\sigma}$ using Speicher’s random model (see Corollary 2.8). Our generating set is given by $S = \{ j(f) : f \in L_{\infty}(\Omega, \mu; \mathbb{C}) \}$.

Let $N(S)$ be the von Neumann algebra generated by the spectral projections of elements $u_{q}(s)$, $s \in S$. Since $U_{t}$ leaves $S$ invariant, we can apply Proposition 3.3 and find a normal faithful state preserving conditional expectation $E : M \to N(S)$. Using the combinatorial formulæ 3.15 and 3.11, we see $u_{q}(S)$ and $s_{q}(S)$ have the same moment formulæ and satisfy the growth condition. By Corollary 2.8 and Theorem 5.3 we conclude that $N(S)$ and the subalgebra $\Gamma_{q}(S) \subset \Gamma_{q}(K, U_{t})$ generated by $s_{q}(S)$ in the Fock space construction are isomorphic via an isomorphism respecting the state and the modular group. Moreover $L_{\infty}(\Omega, \mu) \cap L_{2}(\Omega, \mu)$ is dense in $L_{2}(\Omega, \mu)$ and the map $u_{q,2} : L_{2}(\Omega, \mu) \to L_{2}(\Gamma_{q}(K, U_{t}))$, $u_{q,2}(f) = s_{q}(f)D_{0}^{1/2}$ is an isometry. Here $D_{0}^{1/2}$ is the density of the state given by the vacuum. For $-1 \leq q < 1$ we know that $s_{q}(f)$ is bounded and hence $s_{q}(f) = \text{SOT} - \lim s_{q}(f_{k})$ also belongs to the von Neumann algebra $\Gamma_{q}(S)$ (see [Sh] and [Ha]). Thus $\Gamma_{q}(S) = \Gamma_{q}(K, U_{t})$ and (5.5) holds for arbitrary elements by approximation (see e.g. [Ha] Lemma 2.3).

For $q = 1$ the argument is slightly different because we can no longer define $\Gamma_{1}(K, U_{t})$ as generated by the unbounded operators $\{ s_{1}(h) : h \in K \}$. In that case we recall Segal’s formulation of the commutation relations given by a complex Hilbert space $H_{U}$ and a family of unitaries $W(h)$:

\begin{equation}
W(h_{1} + h_{2}) = e^{-\frac{i}{2} \text{Im}(h_{1}, h_{2})} \quad \text{and} \quad (\Omega, W(h)\Omega) = e^{-\frac{1}{2} \| h \|^{2}}.
\end{equation}

Then we have

$\Gamma_{1}(K, U_{t}) = R_{\text{Segal}}(K/H_{U}) = \{ W(h) : h \in K \}$. In Example 3.9 we have calculated the form $B(f, g) = \text{Im} \psi(j(f), j(g)) = (A(f), g)$ which determines $\Gamma_{1}(K, U_{t})$. Here we have $A(f) = (f_{2} - f_{1})f$. For $f \in L_{\infty}(\mu; \mathbb{C}) \cap L_{2}(\mu; \mathbb{C})S$ we know by Theorem 1.5 that the unitary $W(f) = e^{iu_{1}(j^{(2-1/2)f})} \in N(S)$. Using Remark 3.3 and the fact that $\alpha(A(S))$ is dense in $L_{2}(N(S))$ (see also the proof of Theorem 5.3), we find the same commutation relations as in (5.6). Since the commutation relations uniquely determine $R_{\text{Segal}}(K/H_{U})$, we see that the $\Gamma_{1}(K, U_{t})$ and the algebra $\tilde{N}(S) \subset N(S)$ generated by the family $\{ \tilde{W}(f) : f \in S \} \subset N(S)$ are isomorphic with respect to a state preserving homomorphism. (Alternatively, we may apply Theorem 5.3 to the real and imaginary parts of these unitaries). Note that $\tilde{N}(S)$ is invariant under the modular group and thus Takesaki’s theorem (see [Str], Theorem 10.1) yields a normal state preserving conditional expectation onto $\tilde{N}(S)$ (in fact we have $\tilde{N}(S) = N(S)$). Formula (5.5) is obtained by differentiation for elements $s \in S$. In order to show that (5.5) holds for arbitrary elements in $K$, we recall that the distribution of $u_{1}(j(f))$ with respect to $\phi$ is the normal distribution $N(0, \| f \|^{2})$. Thus we have $\| u_{1}(j(f)) \|_{p} \leq c\sqrt{p + 1}\| f \|_{2}$. According to Lemma 4.2 we deduce $\| u_{1}(j(f))D_{0}^{\frac{1}{2}} \| \leq c\sqrt{p + 1}\| f \|_{2}$. By density we obtain (5.5).
Remark 5.7. In order to determine the type of $\Gamma_1(K, U_t)$, we may transform Segal’s representation into Weyl’s representation $W(h) = e^{i/2(Re(h), Im(h))}U(Re(h))V(Im(h))$. According to Kaimanovich and Rivas we get

$$\Gamma_1(K, U_t) = \{W(h) : h \in K\}'' = \{U(Re(h))V(Im(h)) : h \in K\}'' = R(K_1, \rho(K_1)/K).$$

Here $K_1, K_2$ and $\rho$ are chosen according to the definitions before Corollary 3.17 such that $(f, \rho(g)) = (f, f_2 - f_1)g$. Thus the type depends only on the spectrum of $1 + |f_2 - f_1|^{-1}$.

Remark 5.8. 1) As an application we obtain a generalization of a very recent result of Nou [Nou]: The von Neumann algebras $\Gamma_\beta(K, U_t)$ are QWEP or equivalently allow ‘matrix models’. Although it is open whether every von Neumann algebra is QWEP, it is sometimes difficult to verify this property explicitly. Let us recall that a $C^*$-algebra has WEP if $A \subseteq A^{**} \subset B(H)$ and there is a contraction $P : B(H) \to A^{**}$ such that $P|_A = id|_A$. A $C^*$-algebra $B$ is QWEP if there exists a $C^*$-algebra $A$ with WEP and a two sided ideal $I \subset A$ such that $B = A/I$. Since the QWEP property is stable under ultraproducts and conditional expectations (see e.g. [13]), we deduce that the algebra $\mathcal{N}(S)$ is QWEP because in this case the von Neumann algebra $M$ in Theorem 4.5 is constructed from an ultraproduct of type I von Neumann algebras. The same argument applies for the algebra $M[-1, 1]$ from Corollary 4.6.

2) More generally we may define a von Neumann algebra $\Gamma_\beta(K, U_t)$ given by a partition function $\beta$ constructed as in 2.2 and an inclusion $K \subset H_U$ as above. We define a positive functional on $A(K)$ by

$$(5.7) \quad \phi_\beta(x_1 \otimes \cdots \otimes x_n) = \sum_{\sigma = \{i_1, j_1\}, \ldots, \{i_m, j_m\} \in P_2(m)} \beta(\sigma) \prod_{l=1}^n (x_{i_l}, x_{j_l})U.$$ 

This induces a Hilbert space $(A(K), \langle , \rangle_\beta)$ obtained by completion of $(a, b)_\beta = \phi_\beta(a^*b)$. According to Lemma 4.10 it suffices to consider our standard model $N = L_\infty(\Omega, \mu; \mathcal{M}_2)$ with functions $f_1 + f_2 = 1$. The growth condition is satisfied for $|f| = \max\{\|f\|_2, \|f\|_\infty\}$ and $S$ the set of self-adjoint elements of the form $x = e_{12} \otimes f + e_{21} \otimes \tilde{f}$, $f \in L_\infty$. Hence the ultraproduct construction Theorem 4.5 provides unbounded operators $s_\beta(x)$ affiliated to $N_U$. Using Proposition 5.5, we see that the von Neumann algebra $\mathcal{N}_\beta(S)$ generated by the spectral projections admits a state preserving conditional expectation. In order to extend this representation to all elements in $K$, we introduce the new length function $|x|_U = \|x\|_U$ and note that

$$\lim_j \phi_\beta(a_j \otimes e^{2\pi j} \otimes b_j) = \phi_\beta(a \otimes e^{2\pi x} \otimes b)$$ 

holds whenever we have convergence $\lim_j x_j = x$, $\lim_j a_j = a$, $\lim_j b_j = b$ in $H_U$. By density we find an isomorphism $u$ between the completion of $A(K)$ with respect to the scalar product induced by $\phi_\beta$ and $L_2(\mathcal{N}(S), \phi_{U_t})$. For a selfadjoint element $x = e_{12} \otimes f + e_{21} \otimes \tilde{f}$ we choose an approximating sequence with $f_j \in L_\infty$ and define $U_t^x = w^* - \lim_j e^{it\beta(x_j)}$ using an ultra-limit in the weak operator topology. For $b \in A(K)$, we may also define the measure $\mu = w^* - \lim_j \mu^j_b$ of the measures $\mu^j_b(A) = \int_A (u(b), dE^{\pi(x_j)}(u))$. Using the length function $|x|_U$ we see that the growth condition is uniformly satisfied for all $j$, hence also for $\mu$. From this we deduce strong convergence $U_t^x = \lim_j e^{itx_j}$ and multiplicativity $U_{s+t}^x = U_s^x U_t^x$. We find a von Neumann algebra $\Gamma_\beta(K, U_t)$ generated by $(e^{it\beta(x)})_{x \in K}$, complemented in $N_\beta$. Moreover, by construction $\Gamma_\beta(K, U_t)$ is QWEP and 5.7 holds for the generators. It would be interesting to know what kind of functions $\beta$ defined on the pair partitions have similar properties.
6. An inequality for sums of independent copies

In this section we will prove the main inequality of the paper based on an appropriate concept of independent copies. Let us fix von Neumann algebras $M \subset \mathcal{N}$, and a normal, faithful, conditional expectation $E : \mathcal{N} \to \mathcal{M}$. Let $\mathcal{M} \subset A, B \subset \mathcal{N}$ be subalgebras. $A$ and $B$ are said to be independent over $E$ if

$$E(ab) = E(a)E(b)$$

holds for all $a \in A$ and $b \in B$. We are indebted to C. Köstler (see [Kös]) for pointing out the condition ii) in the following Lemma:

**Lemma 6.1.** Let $A$ and $B$ independent over $E$. Let $\phi_M$ be a normal faithful state on $M$ such that $\sigma_i^\phi \circ E(A) \subset A$.

i) There exists a conditional expectation $E_A : \mathcal{N} \to A$ such that

$$E_A(b) = E(b)$$

for all $b \in B$.

ii) Let $a_1, a_2 \in A$ and $b \in B$. Then

$$E(a_1ba_2) = E(a_1E(b)a_2).$$

iii) Let $a, b \in M_m(A)$. Then

$$\| (id \otimes E)(a^*b^*ba) \| \leq \| (id \otimes E)(a^*a) \| \| (id \otimes E)(b^*b) \|.$$

**Proof.** If $\phi_M$ is a normal faithful state on $M$, then $\phi = \phi_M \circ E$ is a normal faithful state on $\mathcal{N}$ such that $\sigma_i^\phi \circ E = E \circ \sigma_i^\phi$ (see [Con]). Then Takesaki’s theorem implies that $E$ is the unique conditional expectation such that

$$\phi(ab) = \phi(aE(b))$$

holds for all $a \in M$ and $b \in \mathcal{N}$. By assumption on the modular group, we may apply Takesaki’s theorem (see [Str, Theorem 10.1]) and find a normal conditional expectation $E_A : \mathcal{N} \to A$ which is characterized by

$$\phi(ab) = \phi(aE_A(b))$$

for all $a \in A, b \in \mathcal{N}$. However, by independence and the module property of $E$ we deduce

$$\phi(aE_A(b)) = \phi(ab) = \phi_M(E(ab)) = \phi_M(E(aE(b))) = \phi_M(E(aE(b))) = \phi(aE(b))$$

for all $a \in A$ and $b \in B$. Now, we prove ii). Let us consider $a_1 \in A$, a $\phi$-analytic element $a_2 \in A$ and $m \in M$. Then, we deduce from i)

$$\phi(mE(a_1ba_2)) = \phi(ma_1ba_2) = \phi(\sigma_i^\phi(a_2)ma_1b) = \phi(\sigma_i^\phi(a_2)ma_1E_A(b))$$

$$= \phi(\sigma_i^\phi(a_2)ma_1E(b)) = \phi(ma_1E(b)a_2) = \phi(mE(a_1E(b)a_2)).$$

Since this is true for all $m \in M$, we deduce ii) for analytic elements $a_2$. By approximation with bounded nets of analytic elements in the strong topology, the assertion follows for general $a_2$. For the proof of iii) we first note that $M_m(A)$ and $M_m(B)$ are independent over $id_{M_m} \otimes E$. Indeed, we deduce from linearity that

$$(id_{M_m} \otimes E)(ab) = \left[ E\left( \sum_k a_{ik}b_{kj} \right) \right]_{ij} = \left[ \sum_k E(a_{ik})E(b_{kj}) \right]_{ij} = (id_{M_m} \otimes E)(a)(id_{M_m} \otimes E)(b).$$
If $\phi$ is a normal faithful state, then $\phi_m = \frac{1}{m} \otimes \phi$ is a normal faithful state on $M_m(\mathcal{N})$. It is easily checked that the assumption on the modular group on $A$ implies the same assumption on the modular group for $M_m(A)$. Hence condition ii) holds on the matrix level. By positivity we deduce

$$(id_{M_m} \otimes E)(a^* b^* ba) = (id_{M_m} \otimes E)(a^* (id_{M_m} \otimes E)(b^* b) a) \\ \leq \|(id_{M_m} \otimes E)(b^* b)\| (id_{M_m} \otimes E)(a^* a).$$

Taking norms implies the assertion.

We will now discuss the notion of subsymmetric copies needed for our key inequality. This notion is closely related to the order invariance discussed in [11]. We refer to [Küm] for more information on the notion of white noise in the continuous setting which seems to be closely related. We will consider an inclusion of three von Neumann algebras

$$M \subset \mathcal{M} \subset \mathcal{N}.$$  

We will always assume that we have normal faithful conditional expectations $E : \mathcal{N} \to M$ and $E : \mathcal{N} \to \mathcal{M}$ such that $E|_{\mathcal{M}} \circ E = E$. A system of subsymmetric copies $(M, \mathcal{M}, \mathcal{N}, \alpha_1, \ldots, \alpha_n, E)$ over $M$ (strictly speaking over $E$) is given by faithful normal isomorphisms $\alpha_i : \mathcal{M} \to \mathcal{M}_i \subset \mathcal{N}$ such that $\alpha_i \circ E = E \circ \alpha_i$ holds for all $i = 1, \ldots, n$ and

$$(6.2) \quad E(\alpha_{i_1}(a_1) \cdots \alpha_{i_m}(a_m)) = E(\alpha_{j_1}(a_1) \cdots \alpha_{j_m}(a_m))$$

holds for all $a_1, \ldots, a_m \in \mathcal{M}$ and order-equivalent functions $1, j : \{1, \ldots, m\} \to \{1, \ldots, n\}$. We recall that two functions $1, j$ are order equivalent if

$$i_k \leq i_l \iff j_k \leq j_l$$

holds for all $1 \leq k, l \leq m$. Subsymmetric, not necessarily symmetric, copies appear naturally in the context of iterated crossed products (see [13]). For our proof we need a slightly weaker assumption. We say that $(M, \alpha_1(\mathcal{M}), \ldots, \alpha_n(\mathcal{M}), \mathcal{N}, E)$ are top-subsymmetric copies if

$$(6.2) \quad E(\alpha_{i_1}(a_1) \cdots \alpha_{i_m}(a_m)) = E(\alpha_{\sigma(i_1)}(a_1) \cdots \alpha_{\sigma(j_m)}(a_m))$$

holds for every function $i : \{1, \ldots, m\} \to \{1, \ldots, n\}$ and every permutation $\sigma$ of $\{1, \ldots, n\}$. It is clear that this condition implies top-subsymmetry by considering an inversion $(i_k, j_k)$ of the largest elements. We observe that (6.2) induces an automorphisms $\alpha_{(k, l)}$ of the von Neumann
Example 6.2. Let $\mathcal{N} = \mathbb{M} \otimes N_{\otimes_n}$ and $\mathcal{M} = \mathbb{M} \otimes \pi_n(N)$. We define an automorphism $\alpha_i$ by

$$
\alpha_i(m \otimes x_1 \otimes \cdots \otimes x_{i-1} \otimes x_i \otimes x_{i+1} \otimes \cdots \otimes x_n) = m \otimes x_1 \otimes \cdots \otimes x_{i-1} \otimes x_n \otimes x_{i+1} \otimes \cdots \otimes x_i.
$$

If $\phi_\mathbb{M}$, $\phi_N$ are faithful normal states then $\phi = \phi_\mathbb{M} \otimes \phi_N^{\otimes n}$ is a faithful normal state on $\mathcal{N}$. The conditional expectation $E$ is given by $id \otimes \phi_N^{\otimes n}$. Then it is easily checked that $(\mathbb{M} \otimes 1, \mathcal{M}, \mathcal{N}, (\alpha_i)_{i=1,\ldots,n}, \phi)$ is a system of independent copies. The same argument applies in the free product situation $\mathcal{N} = \mathbb{M} \otimes \ast_{i=1,\ldots,n}(N, \phi)$ where $\mathcal{M} = \mathbb{M} \otimes \pi_n(N)$ is given by the $n$-th copy. In the next example we extend this to $q$-independent copies.

Example 6.3. Let $N = l_\infty^n (\ell_\infty(M_2))$ and

$$
\psi(x) = \sum_{i=1}^n \sum_{j \in N} [(1 - \mu_j)x_{11}(i, j) + \mu_j x_{22}(i, j)].
$$

We assume that $(v_k(n))$ satisfies the assumptions of Theorem 4.4 and denote by $\beta$ the resulting partition function. We consider the algebra $N_\beta \subset N_\mathbb{M}$ generated by the spectral projections of the selfadjoint operators $\{u(x)\}_{x \in N_{sa, a}}$ from Proposition 4.3. Recall that by Corollary 2.3 the growth condition is satisfied and hence $N_\beta$ is uniquely determined by the partition function $\beta$. Now, we consider the $\psi$-preserving automorphisms

$$
\gamma_i(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) = (x_1, \ldots, x_{i-1}, x_n, x_{i+1}, \ldots, x_i).
$$

According to Theorem 5.3 we find $\phi_{\mathcal{M}}$-preserving automorphisms $\alpha_i : N \rightarrow N$ such that

$$
\alpha_i(u(x)) = u(\gamma_i(x)).
$$

Let $f_n$ be the projection onto the last coordinate in $l_\infty^n$. Let $\mathcal{M} \subset N_\beta$ be the subalgebra generated by $\{u(x)\}_{f_n x f_n \in (f_n N_{sa, a})}$. Using $\sigma^{\phi_\mathcal{M}}(u(x)) = u(\sigma_\psi(x))$ we may apply Proposition 5.5 and find a conditional expectation $E : N_\beta \rightarrow \mathcal{M}$. Then $(\mathbb{C}, N_\beta, \alpha_1(\mathcal{M}), \ldots, \alpha_n(\mathcal{M}), \phi_\beta)$ is a system of independent copies over $\phi_\mathcal{M}$. Tensoring with matrices $\mathbb{M}_m$, we find examples of independent copies over $id_{\mathbb{M}_m} \otimes \phi_\mathcal{M}$.

For our proof we need some facts from modular theory. Unless stated otherwise we will assume that $\mathbb{M}$ is $\sigma$-finite. Let $\phi_\mathbb{M}$ be a normal faithful state. Then we find normal faithful states $\phi_\mathcal{M} = \phi_\mathbb{M} \circ E|_M$ and $\phi = \phi_\mathbb{M} \circ E$ on $\mathcal{M}$ and $\mathcal{N}$ respectively. Due to the work of Connes [Con] it is well-known that then

$$
\sigma_i^{\phi_\mathcal{M}}(\mathbb{M}) \subset \mathbb{M}, \quad \sigma_i^\phi(\mathbb{M}) \subset \mathbb{M}
$$

holds for the modular group of the corresponding states. Moreover, from $E = E^2$, we deduce $\phi = \phi_\mathcal{M} \circ E$ and hence we also have

$$
\sigma_i^\phi(\mathcal{M}) \subset \mathcal{M}.
$$

Lemma 6.4. Let $(\mathbb{M}, \alpha_1(\mathcal{M}), \ldots, \alpha_n(\mathcal{M}), \mathcal{N}, E)$ be a system of symmetric independent copies and $\mathcal{N} = \mathcal{M}_n$ the von Neumann algebra generated by $\alpha_1(\mathcal{M}), \ldots, \alpha_n(\mathcal{M})$. Then there exist conditional expectations $\mathcal{E}_i : \mathcal{N} \rightarrow \mathcal{M}_i$ such that

$$
\mathcal{E}_i(x) = E(x)
$$
holds for all \( x \in \alpha_{i+1}(M) \). Moreover, we have
\[
\alpha_i \circ \sigma_i^{\phi,M} = \sigma_i^{\phi} \circ \alpha_i.
\]
and
\[
\phi \circ \alpha_i = \phi \quad \text{and} \quad \mathcal{E}_i = \alpha_i \circ \mathcal{E} \circ \alpha_{i+1}^{-1}
\]
for all \( i = 1, \ldots, n \).

Proof. We observe that \( \alpha_{1,i} : M_n \to M_n \) is an automorphism of \( M_n \) such that \( \alpha_{1,i}|_{M} = \alpha_i \). Moreover, our assumption implies that we have \( E \circ \alpha_{1,i} = E \). This allows us to define \( E_i = \alpha_i \mathcal{E} \alpha_{1,i} \). We observe that \( \phi|_{\alpha_i(M)} \circ \mathcal{E}_i = \phi \). By Connes’s result \([\text{Con}]\) we find (6.7). The equations (6.8) follow from \( \alpha_i|_{M} = \text{id} \) and the uniqueness of the \( \sigma_i^{\phi} \) invariant conditional expectation. Since \( \mathcal{M}_i \) is generated by \( \alpha_1(M), \ldots, \alpha_i(M) \) we deduce that \( \mathcal{M}_i \) is also \( \sigma_i^{\phi} \) invariant. This allows us to apply Lemma to apply Lemma (6.11) and deduce (6.6).

We will say that \((\mathbb{M}, \alpha_1(M), \ldots, \alpha_n(M), \mathcal{N}, E)\) is a conditioned system of top-subsymmetric copies of in addition
\[
\sigma_i^{\phi \circ E}(\alpha_i(M)) \subset \alpha_i(M)
\]
holds for a normal faithful state \( \phi_{\mathbb{M}} \circ E \) on \( \mathbb{M} \). We will keep this (minimal) assumption for the rest of this section and the notation \( \phi = \phi_{\mathbb{M}} \circ E \). Let us note that in particular, the von Neumann algebra \( \mathcal{M}_i \) generated by \( \alpha_1(M), \ldots, \alpha_i(M) \) satisfies
\[
\sigma_i^{\phi}(\mathcal{M}_i) \subset \mathcal{M}_i
\]
In particular, we find \( \sigma_i^{\phi} \)-invariant conditional expectations \( E_i : \mathcal{N} \to \alpha_i(M) \) and \( \mathcal{E}_i : \mathcal{N} \to \mathcal{M}_i \). Let us fix some further notations which will be used in the proof. We denote by \( D \) the density of the faithful normal state \( \phi \). For an element \( x \in \mathcal{N} \), we may define
\[
\alpha_i(xD) = \alpha_i(x)D.
\]
Then
\[
\|\alpha_i(xD)\|_1 = \sup_{\|y\| \leq 1} |\phi(y \alpha_i(x))| = \sup_{\|y\| \leq 1} |\phi(E_i(y) \alpha_i(x))| = \sup_{\|y\| \leq 1} |\phi(\mathcal{E}(y) x)| = \|xD\|_{L_1(M)}.
\]
We refer to \([\text{JX1}]\) for the natural extension of \( E \) and \( \mathcal{E}_i, i = 1, \ldots, n \) on \( L_p(\mathcal{N}) \). We refer to \([\text{J1}]\) for the space \( L_1^*(M, E) \) to be the completion of \( \mathcal{M}D \) under then norm
\[
\|xD\|_{L_1^*(M, E)} = \|DE(x^*x)D\|_{L_2^*(M)}^{1/2} \equiv \|(DE(x^*x)D)^{1/2}\|_{L_1(M)}.
\]
It is easily checked that \( L_1^*(M, E) = L_2(M)L_2(\mathbb{M}) \). Moreover, we have a natural duality bracket between \( L_1^*(M, E) \) and the Hilbert \( C^* \)-module \( L_\infty(M, E) \) defined as the completion of \( M \) with respect to the norm \( \|x\|_{L_\infty(M, E)} = \|E(x^*x)|^{1/2} \). Indeed, we have
\[
|\text{tr}(x^*y)| \leq \|E(x^*x)\|_{L_2(\mathbb{M})}^{1/2} \|E(y^*y)\|_{L_2(\mathbb{M})}^{1/2}.
\]
For more information on this Cauchy-Schwarz type inequality and the fact that \( L_\infty(M, E) \) embeds isometrically into the antilinear dual of \( L_1^*(M, E) \), we refer to \([\text{J1}]\). We will also use the notation \( L_1^*(M, E) = L_2(\mathbb{M})L_2(M) \) for the completion of \( D_M \) with respect to the norm
The following 3-term quotient norm is the central object in this section
\[
\|x\|_{L_1^r(M,E)} = \|x^*\|_{L_1^r(M,E)}.
\]
Here, we allow \(x_2 \in L_1^r(M,E)\) and \(x_3 \in L_1^r(M,E)\). The parameter \(\varepsilon > 0\) will be chosen in a convenient way later. We will also use the symbol \(I_{n,\varepsilon}(M,E,\phi)\) for the space \(L_1(M)\) equipped with this norm. In our context it is convenient to use the following antilinear duality
\[
\langle \langle y, x \rangle \rangle_n = ntr(y^*x).
\]

The following Lemma is a minor modification of [L3, Lemma 6.9].

**Lemma 6.5.** The dual space of \(I_{n,\varepsilon}(M,E,\phi)\) with respect to the duality bracket \(\langle \langle \cdot , \cdot \rangle \rangle_n\) is \(\mathcal{M}\) and the norm is given by
\[
\|y\|_{I_{n,\varepsilon}^*} = \max \|y\|_{\mathcal{M}}, \varepsilon^{-\frac{1}{2}}\sqrt{n} \|E(y^*y)\|_{\mathcal{M}}^\frac{1}{2}, \varepsilon^{-\frac{1}{2}}\sqrt{n} \|E(yy^*)\|_{\mathcal{M}}^\frac{1}{2}\}.
\]

The main inequality of this section is to show that for \(\varepsilon = 0.01\) we have
\[
\|x\|_{I_{n,\varepsilon}} \leq c E \left\| \sum_{i=1}^n \varepsilon_i \alpha_i(x) \right\|_{L_1(\mathcal{N})}.
\]

Here \((\varepsilon_i)_{i=1}^n\) are independent Bernoulli variables, i.e. \(\text{Prob}(\varepsilon_i = \pm 1) = \frac{1}{2}\). Of course, we will first apply the noncommutative Khintchine inequality (see [LPP]). In [LPP] the Khintchine inequality was formulated for the characters \((e^{2\pi \sqrt{-1} t^2})_i\). Together with the well-known contraction principle (see [LTI, LTII]) we may replace the characters \((e^{2\pi \sqrt{-1} t^2})_i\) by Rademacher variables \((\varepsilon_i)_i\). This implies
\[
E \left\| \sum_{i=1}^n \varepsilon_i \alpha_i(x) \right\|_1 \leq \inf_{\alpha_i(x) = \alpha_i + d_i} \left\| \sum_{i=1}^n (c_i^* c_i)^\frac{1}{2} \right\|_1 + \left\| \sum_{i=1}^n (d_i d_i^*)^\frac{1}{2} \right\|_1
\leq 2(1 + \sqrt{2}) \sum_{i=1}^n \varepsilon_i,\alpha_i(x) \right\|_1.
\]

Therefore a lower estimate for \(\sum_{i=1}^n \varepsilon_i \alpha_i(x)\) may be obtained by finding elements in \(y_1, \ldots, y_n \in \mathcal{N}\) whose row and column norms are controlled simultaneously. The first impulse is to use the elements \(y_1 = \alpha_1(y), \ldots, y_n = \alpha_n(y)\). However, there we have no good norm estimate for the square function of these elements. The key idea in our proof is to define the following elements starting from a contraction \(y \in \mathcal{M}\):
\[
a = \sqrt{1 - yy^*}, \quad b = \sqrt{1 - y^*y},
\]
\[
Y_i = \alpha_i(y),
\]
\[
A_i = \alpha_1(a) \alpha_2(a) \cdots \alpha_{i-1}(a) \quad \text{and} \quad B_i = \alpha_{i-1}(b) \cdots \alpha_2(b) \alpha_1(b).
\]
Note \(a, b\) are well-defined and \(A_i\) and \(B_i\) are contractions.

**Lemma 6.6.** With the definitions above, we have
\[
\left\| \sum_{i=1}^n (A_i Y_i B_i)(A_i Y_i B_i)^* \right\|_{\mathcal{N}} \leq 1, \quad \left\| \sum_{i=1}^n (A_i Y_i B_i)^*(A_i Y_i B_i) \right\|_{\mathcal{N}} \leq 1.
\]
Proof. By symmetry it suffices to consider the first term. Since the \( B_i \)'s are contractions, we have

\[
\sum_{i=1}^{n} (A_i Y_i B_i) (A_i Y_i B_i)^* = \sum_{i=1}^{n} A_i Y_i B_i B_i^* Y_i^* A_i^* \leq \sum_{i=1}^{n} A_i Y_i Y_i^* A_i^* .
\]

We will inductively rewrite this sum as follows:

\[
1 - \sum_{i=1}^{n} A_i Y_i^* A_i^* = 1 - \alpha_1(yy^*) - \sum_{i=2}^{n} A_i Y_i^* A_i^* = \alpha_1(a) \alpha_2(a) \cdots \alpha_{i-1}(a) \alpha_i(yy^*) \alpha_{i-1}(a) \cdots \alpha_2(a) \alpha_1(a) \\
= \alpha_1(a) \alpha_2(a) \cdots \alpha_{i-1}(a) \alpha_i(yy^*) \alpha_{i-1}(a) \cdots \alpha_2(a) \alpha_1(a) \\
= \alpha_1(a) \left( 1 - \sum_{i=2}^{n} \alpha_2(a) \cdots \alpha_{i-1}(a) \alpha_i(yy^*) \alpha_{i-1}(a) \cdots \alpha_2(a) \alpha_1(a) \right) \\
= \cdots \\
= \alpha_1(a) \alpha_2(a) \cdots \alpha_n(a) \alpha_n(a) \cdots \alpha_1(a) \geq 0 .
\]

This implies \( \sum_{i=1}^{n} A_i Y_i Y_i^* A_i^* \leq 1 \) and the assertion is proved.

The next (slightly technical) lemma explains why we need independent copies.

Lemma 6.7. Let \( \varepsilon < e^{-1} \) and \( \|y\|_{K_n^{a,\varepsilon}} \leq 1 \). Let \( i \in \{1, \ldots, n\} \). Then we have

\( i \) \( E(\alpha_1(a) \alpha_2(a) \cdots \alpha_i(a)) = E(a)^i, \) \( E(\alpha_1(a) \alpha_{i-1}(a) \cdots \alpha_1(a)) = E(a)^i, \)

\( E(\alpha_i(b) \cdots \alpha_2(b) \alpha_1(b)) = E(b)^i, \) \( E(\alpha_1(b) \alpha_2(b) \cdots \alpha_i(b)) = E(b)^i. \)

\( ii \) \( \|1 - E(a)\| \leq \frac{\varepsilon}{n}, \) \( \|1 - E(b)\| \leq \frac{\varepsilon}{n}. \)

\( iii \) \( \| \sum_{i=1}^{n} 1 - E(a)^{i-1} \| \leq e\varepsilon n, \) \( \| \sum_{i=1}^{n} 1 - E(b)^{i-1} \| \leq e\varepsilon n. \)

\( iv \) \( \left\| \sum_{i=1}^{n} E \left( (1 - A_i) (1 - A_i)^* \right) \right\| \leq 2e\varepsilon n, \) \( \left\| \sum_{i=1}^{n} E \left( (1 - B_i)^* (1 - B_i) \right) \right\| \leq 2e\varepsilon n .
\)

Proof. It suffices to prove the inequalities involving \( a \)'s. In order to prove \( i \), we note that by \( \sigma_i^\phi \)-invariance, the conditional expectation \( E \) is unique. Therefore \( E = E \circ \mathcal{E}_i \). From (6.6) and independence we deduce that

\[
E(\alpha_1(a) \cdots \alpha_i(a)) = \left( E \left( \mathcal{E}_{i-1}(\alpha_1(a) \cdots \alpha_{i-1}(a)) \alpha_i(a) \right) \right) = \left( E \left( \alpha_1(a) \cdots \alpha_{i-1}(a) \right) \alpha_i(a) \right) \\
= \alpha_1(a) \cdots \alpha_{i-1}(a) E(a). 
\]

By induction, we get

\[
E(\alpha_1(a) \cdots \alpha_i(a)) = E(a)^i.
\]

For the proof of \( ii \), we note that for \( 0 \leq t \leq 1 \), we have \( 1 - \sqrt{1-t} \leq t \). By functional calculus, we obtain from the assumption on \( y \)

\[
1 - E(a) = 1 - E(\sqrt{1-yy^*}) = \left( 1 - \sqrt{1-yy^*} \right) \leq E(yy^*) \leq \frac{\varepsilon}{n}.
\]
In order to prove iii), we first recall that for \( t \leq e^{-1} \), we have \( 1 - t \geq \exp(-et) \). By functional calculus, this implies

\[
1 \geq E(a) \geq 1 - \frac{\varepsilon}{n} \geq \exp\left(-\frac{e\varepsilon}{n}\right).
\]

Hence, for all \( i = 1, \ldots, n \), we have

\[
E(a)^i \geq \exp(-e\varepsilon).
\]

Using \( 1 - \exp(-t) \leq t \), this yields

\[
\sum_{i=1}^{n}[1 - E(a)^{i-1}] \leq n(1 - \exp(-e\varepsilon)) \leq ne\varepsilon.
\]

For the proof of the last assertion iv), we deduce with \( \|a\| \leq 1 \) and i)

\[
E((1 - \alpha_1(a) \cdots \alpha_{i-1}(a))(1 - \alpha_1(a) \cdots \alpha_{i-1}(a))^*)
= 1 - E(\alpha_1(a) \cdots \alpha_{i-1}(a)) - E(\alpha_{i-1}(a) \cdots \alpha_1(a)) + E(\alpha_1(a) \cdots \alpha_{i-1}(a)\alpha_{i-1}(a) \cdots \alpha_1(a))
\leq 2 - E(\alpha_1(a) \cdots \alpha_{i-1}(a)) - E(\alpha_{i-1}(a) \cdots \alpha_1(a))
= 2(1 - E(a)^{i-1}).
\]

Therefore, iii) implies

\[
\left\| \sum_{i=1}^{n} E\left((1 - \alpha_1(a) \cdots \alpha_{i-1}(a))(1 - \alpha_1(a) \cdots \alpha_{i-1}(a))^*\right) \right\| \leq 2ne\varepsilon.
\]

Lemma 6.8. Let \( E_n : \mathcal{N} \to \alpha_n(M) \) be the \( \sigma_i^\phi \) invariant conditional expectation and \( A_i, B_i \) as above. Let \( e\varepsilon \leq 2 \) and \( z \in \mathcal{M} \). Then

\[
\left\| \sum_{i=1}^{n} \alpha_n^{-1}E_n\left((1 - A_i)\alpha_n(z)(1 - B_i)\right) \right\|_{\mathcal{K}_n^{\varepsilon}} \leq 2ne\varepsilon \|z\|_{\mathcal{K}_n^{\varepsilon}}.
\]

**Proof.** We first consider the norm estimate in \( \mathcal{M} \). By the Cauchy-Schwarz inequality [11] and Lemma 6.1 i), we deduce

\[
\left\| \sum_{i=1}^{n} \alpha_n^{-1}E_n((1 - A_i)\alpha_n(z)(1 - B_i)) \right\|_{\mathcal{N}}
\leq \left\| \sum_{i=1}^{n} E_n((1 - A_i)(1 - A_i)^*) \right\|_{\mathcal{N}}^{\frac{1}{2}} \left\| \sum_{i=1}^{n} E_n((1 - B_i)^*\alpha_n(z^*z)(1 - B_i)) \right\|_{\mathcal{N}}^{\frac{1}{2}}
\leq \|z^*z\|^{\frac{1}{2}} \left\| \sum_{i=1}^{n} E_n((1 - A_i)(1 - A_i)^*) \right\|_{\mathcal{N}}^{\frac{1}{2}} \left\| \sum_{i=1}^{n} E_n((1 - B_i)^*(1 - B_i)) \right\|_{\mathcal{N}}^{\frac{1}{2}}
\leq \|z^*z\|^{\frac{1}{2}} \left\| \sum_{i=1}^{n} E((1 - A_i)(1 - A_i)^*) \right\|_{\mathcal{N}}^{\frac{1}{2}} \left\| \sum_{i=1}^{n} E((1 - B_i)^*(1 - B_i)) \right\|_{\mathcal{N}}^{\frac{1}{2}}.
\]

Here we used \( 1 - A_i, 1 - B_i \in \mathcal{M}_{n-1} \) for all \( i = 1, \ldots, n \). Then Lemma 6.7 implies that

\[
(6.12) \left\| \sum_{i=1}^{n} E((1 - A_i)(1 - A_i)^*) \right\| \leq 2en\varepsilon \quad \text{and} \quad \left\| \sum_{i=1}^{n} E((1 - B_i)^*(1 - B_i)) \right\| \leq 2en\varepsilon.
\]
Thus we get
\[ \left\| \sum_{i=1}^{n} \alpha_i^{-1} E_n((1 - A_i)\alpha_n(z)(1 - B_i)) \right\|_\mathcal{M} \leq 2e\varepsilon n \|z\|_\mathcal{M}. \]

Now, we prove the estimates for the conditioned norm. We observe that
\[ B = \sum_{i=1}^{n} e_{i,1} \otimes (1 - B_i) \in \mathcal{M}_{n-1} \quad \text{and} \quad A = \sum_{i=1}^{n} e_{1,i} \otimes (1 - A_i) \in \mathcal{M}_{n-1}. \]

Since the $A_i$'s are contractions, we deduce
\[ \|A\| \leq \sqrt{n} \sup_{1 \leq i \leq n} \|1 - A_i\| \leq 2\sqrt{n}. \]

Similarly, $\|B\| \leq 2\sqrt{n}$. Now we apply Kadison’s inequality $E_n(x^*E_n(x) \leq E_n(x^*x)$ (see [Pau], Lemma 6.1 (iii) and (6.12):
\[
\left\| E\left(\left(\sum_{i=1}^{n} \alpha_i^{-1} E_n((1 - A_i)\alpha_n(z)(1 - B_i))\right)^* \left(\sum_{i=1}^{n} \alpha_i^{-1} E_n((1 - A_i)\alpha_n(z)(1 - B_i))\right)\right)\right\|
\leq \left\| E\left(\left(B^* (1 \otimes \alpha_n(z)^*)A^* \right)E_n(A(1 \otimes \alpha_n(z))B)\right)\right\|
\leq \left\| A^* A \right\| \left\| E(B^* (1 \otimes \alpha_n(z)^* \alpha_n(z))B)\right\|
\leq 4n \left\| \sum_{i=1}^{n} E((1 - B_i)^*(1 - B_i)) \right\| \left\| E(\alpha_n(z^* z)) \right\| \leq 8n^2 e\varepsilon \left\| E(z^* z) \right\|. \]

The argument in the row case is identical. Since $\sqrt{2e} \leq \varepsilon$ and $\varepsilon \leq \sqrt{\varepsilon}$, we get
\[ \left\| \sum_{i=1}^{n} \alpha_i^{-1} E_n((1 - A_i)\alpha_n(z)(1 - B_i)) \right\|_{\mathcal{K}_{n,\delta}^*} \leq 2ne\sqrt{\varepsilon} \|z\|_{\mathcal{K}_{n,\delta}} \]
for all $\delta > 0$. Using $\delta = \varepsilon$ yields the assertion. 

The next argument provides the lower estimate.

**Proposition 6.9.** Let $\varepsilon = \frac{1}{100}$ and $N$ be a von Neumann algebra with normal faithful tracial state $\tau$. Let $v_1, \ldots, v_n$ be unitaries in $N$. Then for every $x \in L_1(N)$
\[ \|x\|_{\mathcal{K}_{n,\varepsilon}} \leq 4 \inf_{v_1, \tau \otimes \alpha_i(x) = c_i + d_i} \left( \left\| \sum_{i=1}^{n} c_i^* c_i \right\|_{L_1(N \otimes N)} + \left\| \left( \sum_{i=1}^{n} d_i d_i^* \right)^{\frac{1}{2}} \right\|_{L_1(N \otimes N)} \right). \]

**Proof.** We may assume $\|x\|_{\mathcal{K}_{n,\varepsilon}} = 1$. We denote the right hand side without the factor 4 by INF. By the Hahn-Banach theorem, we can find $y \in \mathcal{M}$ such that
\[ n \operatorname{tr}(y^* x) = 1 \quad \text{and} \quad \|y\|_{\mathcal{K}_{n,\varepsilon}} = 1. \]

For any decomposition $v_i, \tau \otimes \alpha_i(x) = c_i + d_i$, we deduce from Lemma 6.6
\[ \left\| \sum_{i=1}^{n} \operatorname{tr}\left( (v_i \otimes A_i Y_i B_i)^* (v_i \tau \otimes \alpha_i(x)) \right) \right\| = \left\| \sum_{i=1}^{n} \operatorname{tr}\left( (v_i \otimes A_i Y_i B_i)^* (c_i + d_i) \right) \right\| \]
we obtain that

Similarly, we obtain

following algebraic identity.

We will now complete the argument using an error estimate. For general $y, a, b$ we will use the following algebraic identity.

$$y - aby = y - yb + (1 - a)yb = y(1 - b) + (1 - a)y - (1 - a)y(1 - b).$$

Let us introduce $\beta_i(x) = v_i \tau \otimes \alpha_i(x)$ and $y_i = v_i \otimes Y_i$. Then, we deduce from the above that

$$|ntr(y^* x)| = \left| \sum_{i=1}^{n} tr(y_i^* \alpha_i(x)) \right| = \left| \sum_{i=1}^{n} tr(y_i^* \beta_i(x)) \right|$$

$$\leq \sum_{i=1}^{n} tr((A_i y_i B_i)^* \beta_i(x)) + \sum_{i=1}^{n} tr((y_i^* \beta_i(x)) - tr((A_i y_i B_i)^* \beta_i(x))$$

$$\leq \text{INF} + \left| \sum_{i=1}^{n} tr(((A_i - 1)y_i(1 - B_i))^* \beta_i(x)) \right|$$

$$+ \left| \sum_{i=1}^{n} tr((1 - A_i) y_i^* \beta_i(x)) \right| + \left| \sum_{i=1}^{n} tr((y_i(1 - B_i))^* \beta_i(x)) \right|.$$

The three error terms will be treated separately. In order to estimate the first term on the last line, we remind the reader of (6.9) and that the trace is invariant under the conditional expectations. Using Lemma 6.7(i) together with the fact that $(M, \| \cdot \|_{K_{n,e}})$ is a left $M$ module, we obtain that

$$| \sum_{i=1}^{n} tr(y_i^*(1 - A_i)^* \beta_i(x)) | = \left| \sum_{i=1}^{n} tr(\alpha_i(y)^*(1 - A_i)^* \alpha_i(x)) \right| = \left| \sum_{i=1}^{n} tr(\alpha_i(xy)^* E_i(1 - A_i)) \right|$$

$$= \left| \sum_{i=1}^{n} tr(\alpha_i(xy)^* E(1 - A_i)) \right| = \left| \sum_{i=1}^{n} tr(y^* E(1 - A_i)^* x) \right| = |tr(y^*(\sum_{i=1}^{n}[1 - E(a)^{i-1}]) x)|$$

$$\leq \frac{1}{n} \left\| \left( \sum_{i=1}^{n} [1 - E(a)^{i-1}] \right) y \right\|_{K_{n,e}} \| x \|_{K_{n,e}} \leq \varepsilon \varepsilon.$$

Similarly, we obtain

$$\left| \sum_{i=1}^{n} tr((y_i(1 - B_i))^* \beta_i(x)) \right| \leq \varepsilon \varepsilon.$$

For the remaining term we first top-subsymmetry and then Lemma 6.8. This yields that

$$\left| \sum_{i=1}^{n} tr(((1 - A_i)y_i(1 - B_i))^* \beta_i(x)) \right| = \left| \sum_{i=1}^{n} tr(((1 - A_i)\alpha_i(y)(1 - B_i))^* \alpha_i(x)) \right|$$

$$= \left| \sum_{i=1}^{n} tr(((1 - A_i)\alpha_n(y)(1 - B_i))^* \alpha_n(x)) \right| = \left| \sum_{i=1}^{n} tr\left(E_n((1 - A_i)\alpha_n(y)(1 - B_i))^* \alpha_n(x) \right|.$$
Lemma 6.10. Let $\text{unitaries in } \mathbb{B}_{n,\varepsilon}^*$. 

Proof. We deduce from the triangle inequality for $c$ dependent copies. Let $\varepsilon > 0$. For $\varepsilon = 0.01$, we have $2\varepsilon (\varepsilon + \sqrt{\varepsilon}) \leq \frac{2}{\varepsilon}$ and thus $\frac{1}{\varepsilon} \leq \text{INF}$. This completes the proof. 

For the sake of completeness, we also prove the converse implication.

Lemma 6.10. Let $x \in L_1(\mathcal{M})$. Then 

$$\inf_{v_i, \tau \otimes \alpha_i(x) = c_i + d_i} \left( \sum_{i=1}^{n} c_i^* c_i \right)^{\frac{1}{2}} \leq \sum_{i=1}^{n} \|v_i \tau \otimes \alpha_i(x)\| = n \|x\|_{L_1(\mathcal{M})}.$$ 

Proof. We define $c_i = v_i, \tau \otimes \alpha_i(x)$. Then we have 

$$\left\| \sum_{i=1}^{n} c_i^* c_i \right\| \leq \sum_{i=1}^{n} \|v_i \tau \otimes \alpha_i(x)\| = n \|x\|_{L_1(\mathcal{M})}.$$ 

Moreover, let us recall that $\|c^* c\| \leq \|E(c^* c)\|^{\frac{1}{2}}$ (see [11]). Since the $v_i$'s are unitaries, we have 

$$\left\| \sum_{i=1}^{n} c_i^* c_i \right\| \leq \sum_{i=1}^{n} \|E(\alpha_i(x)^* \alpha_i(x))\| = n \|E(x^* x)\|^{\frac{1}{2}}.$$ 

Similarly, we deduce $\left\| \sum_{i=1}^{n} c_i^* c_i \right\| \leq n \|E(x x^*)\|^{\frac{1}{2}}$. Hence for any decomposition $x = x_1 + x_2 + x_3$ we deduce from the triangle inequality for $c_i = v_i, \tau \otimes \alpha_i(x_1 + x_2)$ and $d_i = v_i, \tau \otimes \alpha_i(x_3)$ that 

$$\left\| \sum_{i=1}^{n} c_i^* c_i \right\| \leq \sum_{i=1}^{n} \|x_1\| + \sqrt{n} \|E(x_2^* x_2)^{\frac{1}{2}}\| + \sqrt{n} \|E(x_3^* x_3)^{\frac{1}{2}}\|.$$ 

A combination of Proposition 6.9 and Lemma 6.10 yields the main results of this section.

Theorem 6.11. Let $(\mathcal{M}, \mathcal{M}, \mathcal{N}, (\alpha_i)_{i=1}^{n}, E, \phi)$ be a system of conditioned top-subsymmetric independent copies. Let $N$ be a von Neumann algebra with normalized faithful trace $\tau$ and $v_i$ be unitaries in $N$. Then $x \in L_1(\mathcal{M})$. Then 

$$\frac{1}{40} \inf_{x=x_1 + x_2 + x_3} \left( n \|x_1\|_{L_1(\mathcal{N})} + \sqrt{n} \|E(x_2^* x_2)^{\frac{1}{2}}\|_{L_1(\mathcal{M})} + \sqrt{n} \|E(x_3^* x_3)^{\frac{1}{2}}\|_{L_1(\mathcal{M})} \right)$$ 

$$\leq \inf_{v_i \tau \otimes \alpha_i(x) = c_i + d_i} \left( \sum_{i=1}^{n} c_i^* c_i \right)^{\frac{1}{2}} + \left( \sum_{i=1}^{n} d_i^* d_i \right)^{\frac{1}{2}} \leq \inf_{x=x_1 + x_2 + x_3} n \|x_1\|_{L_1(\mathcal{N})} + \sqrt{n} \|E(x_2^* x_2)^{\frac{1}{2}}\|_{L_1(\mathcal{M})} + \sqrt{n} \|E(x_3^* x_3)^{\frac{1}{2}}\|_{L_1(\mathcal{M})}.$$ 

Proof. For $\varepsilon = 0.01$, we have $\|x\|_{\mathbb{B}_{n,1}} \leq 10 \|x\|_{\mathbb{B}_{n,\varepsilon}}$. Hence Proposition 6.9 implies the first inequality and the converse inequality is Lemma 6.10.
Corollary 6.12. Let \((\mathcal{M}, \mathcal{M}, \mathcal{N}, (\alpha_i)_{i=1}^n, E, \phi)\) be a system of conditioned top-subsymmetric independent copies. Let \(N\) be a von Neumann algebra with normalized faithful trace \(\tau\) and \(\alpha_i\) be independent Bernoulli random variables. Then

\[
\inf_{x = x_1 + x_2 + x_3} \left( n \left\| x_1 \right\|_{L_1(\mathcal{N})} + \sqrt{n} \left\| E(x_2^* x_2)^{\frac{1}{2}} \right\|_{L_1(\mathcal{M})} + \sqrt{n} \left\| E(x_3 x_3^*)^{\frac{1}{2}} \right\|_{L_1(\mathcal{M})} \right)
\]

\[
\sim 200 \mathbb{E} \left\| \sum_{i=1}^n \varepsilon_i v_i \otimes \alpha_i(x) \right\|_{L_1(\mathcal{N})}.
\]

Proof. This follows immediately from the noncommutative Khintchine inequality, see \((6.11)\).

Our main application is obtained for tensor products.

Corollary 6.13. Let \(\mathcal{M}\) and \(M\) be von Neumann algebras with normal, faithful states \(\phi_\mathcal{M}\) and \(\psi\), respectively. Let \(N\) be a finite von Neumann algebra with faithful normal trace \(\tau\) and \(v_1, \ldots, v_n \in N\) be unitaries. Let \(D_n\) be the density of \(\phi_\mathcal{M} \otimes \psi^{\otimes n}\), \(D\) be the density of \(\phi_\mathcal{M} \otimes \psi\). Then

\[
\mathbb{E} \left\| \sum_{k=1}^n \varepsilon_k v_k \cdot \tau \otimes (1 \otimes \pi_k)(y) D_n \right\|_{L_1(\mathcal{N} \otimes M \otimes M^{\otimes n})} \sim 200 \inf_{y_D = y_1 D + y_2 D + y_3} \left( n \left\| y_1 D \right\|_1 + \sqrt{n} \left\| (D E(y_2 y_2^*) D)^{\frac{1}{2}} \right\|_1 + \sqrt{n} \left\| (D E(y_3 y_3^*) D)^{\frac{1}{2}} \right\|_1 \right)
\]

holds for every \(y \in (\mathcal{M} \otimes M)_{a}\). Here the infimum is taken over analytic elements.

Proof. Let us assume \(200 \mathbb{E} \left\| \sum_{k=1}^n \varepsilon_k v_k \cdot \tau \otimes (1 \otimes \pi_k)(y) D \right\|_{L_1(\mathcal{N} \otimes M \otimes M^{\otimes n})} \leq 1\). For \(x = y D \in L_1(\mathcal{M} \otimes M)\) we consider a decomposition

\[
y D = x_1 + x_2 + x_3
\]

where \(x_1, x_2\) and \(x_3\) satisfy the corresponding norm estimates given by Theorem \((6.10)\). Then we may approximate \(x_2\) and \(x_3\) by elements of the form

\[
x_2 = y_2 D + x_2' \quad \text{and} \quad x_3 = D y_3 + x_3'
\]

such that \(y_2\) and \(y_3\) are analytic and

\[
\left\| x_2' \right\|_{L_1(\mathcal{M} \otimes M, E)} \leq \frac{\varepsilon}{n} \quad \text{and} \quad \left\| x_3' \right\|_{L_1(\mathcal{M} \otimes M, E)} \leq \frac{\varepsilon}{n}.
\]

This yields

\[
y D = x_1 + x_2' + x_3' + y_2 D + D y_3 \quad \text{and} \quad x_1 + x_2' + x_3' = (y - y_2 - \sigma_{-i}(y_3)) D.
\]

Since the inclusions \(L_1(\mathcal{M} \otimes M, E) \subset L_1(\mathcal{M} \otimes M)\) and \(L_1(\mathcal{M} \otimes M, E) \subset L_1(\mathcal{M} \otimes M)\) are contractive, we deduce

\[
n \left\| x_1 + x_2' + x_3' \right\|_1 \leq n \left\| x_1 \right\|_1 + n \left\| x_2' \right\|_{L_1(\mathcal{M} \otimes M, E)} + n \left\| x_3' \right\|_{L_1(\mathcal{M} \otimes M, E)} \leq (1 + 2n).
\]

Thus the assertion follows with \(y_1 = y - y_2 - \sigma_{-i}(y_3)\).
7. Khintchine Type Inequalities

In this section we combine the central limit procedure with the norm estimates from the previous section. Let us recall that for a sequence \( \mu = (\mu_k) \subset (0,1) \) the von Neumann algebra \( \mathcal{N}(\mu) \) is the completion of \( \otimes_k \mathbb{M} \) with respect to the GNS construction of the tensor product state \( \phi_\mu = \otimes_k \phi_{\mu_k} \).

**Theorem 7.1.** Let \( (a_k) \) be the generators of the CAR algebra, \( \phi_\mu \) the quasi-free state satisfying (1) and \( D_\mu \in L_1(\mathcal{N}(\mu)) \) the density of \( \phi_\mu \). Let \( \mathbb{M} \) be a von Neumann algebra and \( x_k \in L_1(\mathbb{M}) \). Then

\[
\| \sum_k D_\mu^k a_k D_\mu^k \otimes x_k \|_{L_1(\mathcal{N}(\mu)) \otimes L_1(\mathbb{M})} \sim \inf \| \sum_k (1 - \mu_k) c_k^* c_k \|_{L_1(\mathbb{M})} + \| \sum_k \mu_k d_k d_k^* \|_{L_1(\mathbb{M})}^{\frac{3}{2}}.
\]

We will need some notations from \([13]\). For a Hilbert space \( H \) and a von Neumann algebra \( \mathcal{M} \) we denote by \( H^\wedge \otimes L_1(\mathcal{M}) \) the closure of \( H \otimes L_1(\mathcal{M}) \) with respect to norm

\[
\| x \|_{H^\wedge \otimes L_1(\mathcal{M})} = \| (x^*, x) \|^{\frac{1}{2}}.
\]

Here the \( L_2^\wedge(\mathcal{M}) \)-valued scalar product is defined by

\[
\langle \sum_k h_k \otimes x_k, \sum_j l_j \otimes y_j \rangle = \sum_{k,j} (h_k, l_j) x_k y_j
\]

and the \( * \) operation is given by \( (\sum_k h_k \otimes x_k)^* = \sum_k h_k^* \otimes x_k^* \). We refer the reader to [13, section 1] for more details on the operator space structure and to [JS] for more information on \( L_2^\wedge \)-valued scalar products. Our motivation for this abstract definition is the Hilbert space \( \ell_2(\mu) \) with scalar product \( (h,l) = \sum_k \mu_k h_k l_k \). Let us denote by \( (e_k) \) the standard unit vectors basis. Then

\[
(7.1) \quad \left\| \sum_k e_k \otimes x_k \right\|_{\ell_2^\wedge(\mu) \otimes L_1(\mathcal{M})} = \left\| \sum_k \mu_k x_k^* x_k \right\|_{L_1(\mathcal{M})}^{\frac{3}{2}}.
\]

Similarly, we recall that \( H^c \otimes L_1(\mathcal{M}) \) is the completion of \( H \otimes L_1(\mathcal{M}) \) with respect to the norm

\[
\| x \|_{H^c \otimes L_1(\mathcal{M})} = \left\| (x, x^*)^{\frac{1}{2}} \right\|.
\]

More precisely, for \( x = \sum_k h_k \otimes x_k \) we have

\[
(x, x^*) = \sum_{k,l} (h_k, h_l) x_k x_l^* = \sum_{k,l} (h_l, h_k) x_k x_l^*.
\]

For \( H = \ell_2 \) this is consistent with \( H^c \otimes L_1(\mathcal{M}) \cong \{ \sum_k e_k \otimes x_k : x_k \in L_1(\mathcal{M}) \} \subset L_1(B(\ell_2) \otimes \mathcal{M}) \). In particular, we have

\[
(7.2) \quad \left\| \sum_k e_k \otimes x_k \right\|_{\ell_2^\wedge(\mu) \otimes L_1(\mathcal{M})} = \left\| \sum_k \mu_k x_k^* x_k \right\|_{L_1(\mathcal{M})}^{\frac{3}{2}}.
\]

The proof of Theorem 7.1 will use the central limit procedure and also work for the \( q \)-commutation relations for \(-1 \leq q \leq 1\). By approximation, we shall assume that \( \mathbb{M} \) is \( \sigma \)-finite. Again by
approximation, it suffices to prove Theorem 1.1 for a finite number \( m \) of generators. This leads to the following data: \( N = \ell_{\infty}^m(\mathbb{M}_2) \) with weight \( \psi(x) = \sum_{k=1}^{m} [(1 - \mu_k)x_{11}(k) + \mu_k x_{22}(k)] \).

We define \( T = m \) and
\[
u_n, T(x) = \sqrt{\frac{T}{n}} \sum_{i=1}^{n} v_i \otimes \pi_i(x)
\]
where \( v_i \in \mathbb{M}_{2^n} \) are Clifford matrices satisfying \( v_i v_j = - v_j v_i \). We use the normal faithful state \( \phi_n = \tau_n \otimes (\psi)^{\otimes n} \) on \( N_n = \mathbb{M}_{2^n} \otimes N^{\otimes n} \). We recall that on the Haagerup \( L_2 \) space \( L_2(N, tr) \) we use \( (x, y) = tr(x^*y) \).

**Lemma 7.2.** Let \( \phi_M \) be a normal faithful state on \( \mathbb{M} \). Let \( D \) be the density of \( \phi_M \otimes \psi \), \( D_M \) be the density of \( \phi_M \), and \( D_n \) be the density of \( \phi_n \otimes \phi_M \). Let \( y \in N \otimes \mathbb{M} \). Then
\[
\left\| D_n^r(u_n, T \otimes id)(y)D_n^l \right\|_{L_1(N_n \otimes \mathbb{M})} \sim_{200} \inf_{y = y_1 + y_2 + y_3} \sqrt{Tn} \left\| D_n^r y_1 D_n^l \right\|_1
\]
\[
+ \left\| (D_n^r \otimes D_n^l) y_2 (1 \otimes D_n^l) \right\|_{L_1(N, tr) \otimes L_1(\mathbb{M})} + \left\| (1 \otimes D_n^l) y_3 (D_n^r \otimes D_n^l) \right\|_{L_1(N, tr) \otimes L_1(\mathbb{M})}.
\]

**Proof.** Let us first mention the well-known absorption principle
\[
\mathbb{E} \left\| \sum_k \varepsilon_k \otimes v_k \otimes y_k \right\|_{L_1(N_n \otimes \mathbb{M})} = \left\| \sum_k v_k \otimes y_k \right\|_{L_1(\mathbb{M} \otimes \mathbb{M})}.
\]
Indeed, the unitaries \( w_k = \varepsilon_k \otimes v_k \in L_\infty(\{-1, 1\}^n) \otimes \mathbb{M}_{2^n} \) also satisfy the CAR relations
\( w_k w_j = - w_j w_k, w_k = w_k^* \) and \( w_k^2 = 1 \). Therefore \( \pi(v_k) = w_k \) extends to a trace preserving homomorphism from \( \mathbb{M}_{2^n} \) onto the algebra \( B_n \) generated by \( w_k \). Using the trace preserving conditional expectation onto \( B_n \), we deduce
\[
\mathbb{E} \left\| \sum_k \varepsilon_k \otimes v_k \otimes y_k \right\|_{L_1(N_n \otimes \mathbb{M})} = \left\| \sum_k w_k \otimes y_k \right\|_{L_1(B_n \otimes \mathbb{M})} = \left\| \sum_k v_k \otimes y_k \right\|_{L_1(B_n \otimes \mathbb{M})}.
\]

By approximation it suffices to consider analytic elements \( y \in N \otimes \mathbb{M} \). We deduce from Corollary 6.13 applied to \( z = \sigma_{-t}(y) \) that
\[
\left\| D_n^r(id \otimes u_n, T)(y)D_n^l \right\|_{L_1(\mathbb{M} \otimes N_n)} = \sqrt{\frac{T}{n}} \left\| \sum_{i=1}^{n} v_i, \tau \otimes (1 \otimes \pi_i(z))D_n \right\|
\]
\[
\sim_{200} \sqrt{\frac{T}{n}} \left( \inf_{zD = z_1D + z_2D + z_3} n \left\| z_1D \right\|_1 + \sqrt{n} \left\| (DE(z_2^*z_2)^{1/2}D^{1/2} \right\|_1 + \sqrt{n} \left\| DE(z_3^*z_3^{1/2}D^{1/2} \right\|_1 \right).
\]
Here \( E(a \otimes z_2) = T^{-1}\psi(a)z_2 \) is the conditional expectation onto \( \mathbb{M} \). Given \( z = \sum_k b_k \otimes z_k \) we observe that
\[
\left\| z(D_n^{1/2} \otimes D_{\phi_M}) \right\|_{L_1(N, tr) \otimes L_1(\mathbb{M})} = \left\| (D_n^{1/2} \otimes D_{\phi_M})z^* \otimes z(D_n^{1/2} \otimes D_{\phi_M}) \right\|_1^{1/2}.
\]
where $v$ holds for all $u, y$.

According to [J2, Lemma 7.2 and Lemma 7.3] we deduce for subalgebra $N$:

$$\ell_y = \lim_{n \to \infty} \inf \left\| \sum_k (D_{\psi}^{1/2} b_k^* D_{\phi_{m2}} z_k^* z_l D_{\phi_{m2}}) \right\|^{1/2} = \sqrt{T} \left\| DE(z^* z) D \right\|^{\frac{1}{2}} = \sqrt{T} \left\| (DE(z^* z) D)^{\frac{1}{2}} \right\|_1.$$ 

A similar calculation shows that for $z = \sum_{k} b_k \otimes z_k$ we have

$$\sqrt{T} \left\| DE(z z^*) D \right\|^{\frac{1}{2}} = \left\| \sum_k (D_{\psi}^{1/2} b_k D_{\phi_{m2}} z_k z_l^* D_{\phi_{m2}}) \right\|^{\frac{1}{2}} = \left\| \sum_k (D_{\psi}^{1/2} b_k D_{\phi_{m2}} L_2(N, tr) D_{\phi_{m2}} z_k z_l^* D_{\phi_{m2}}) \right\|^{\frac{1}{2}} = \left\| (D_{\psi}^{1/2} \otimes D_{\phi_{m2}}) z \right\|_{L_2^2(N, tr) \otimes L_1(M)}.$$ 

Since the infimum is taken over analytic elements, we may define $y_1 = \sigma_{i/2}(z_1), y_2 = \sigma_{i/2}(z_2)$ and $y_3 = \sigma_{-i/2}(z_3)$ such that

$$D_{\psi}^{1/2} y D_{\psi}^{1/2} = z D = z_1 D + z_2 D + D z_3 = D_{\psi}^{1/2} y_1 D_{\psi}^{1/2} + D_{\psi}^{1/2} y_2 D_{\psi}^{1/2} + D_{\psi}^{1/2} y_3 D_{\psi}^{1/2}.$$ 

This implies $y = y_1 + y_2 + y_3$. For arbitrary elements the assertion follows by approximation.

The following Lemma is an easy adaptation of [J3, Proposition 6.2]. Indeed, in the proof we may use that $N$ is finite dimensional and that the modular operator $S(x) = x^*$ is bounded on $L_2(N, \psi)$. Then the proof given for $M = M_m$ in [J3] is easily adapted to this setting.

**Lemma 7.3.** Keep the notations from the previous Lemma. Let $U$ be an ultrafilter on the integers. Then

$$\lim_{n \to U} \left\| D_{\psi}^{1/2} (u_{n,T} \otimes id)(y) D_{\psi}^{1/2} \right\|_{L_2^2(N(T) \otimes L_1(M))} \sim 200 \inf_{y = y_2 + y_3} \left\| (D_{\psi}^{1/2} \otimes D_{\phi_{m2}}) y_2 (1 \otimes D_{\phi_{m2}}) \right\|_{L_2^2(N(T) \otimes L_1(M))} + \left\| (1 \otimes D_{\phi_{m2}}) y_3 (D_{\psi}^{1/2} \otimes D_{\phi_{m2}}) \right\|_{L_2^2(N(T) \otimes L_1(M))}.$$ 

**Proof of Theorem 7.1** We fix $m \in \mathbb{N}$. We denote by $(\delta_k)$ the unit vector basis in $\ell_2^m$ and $N = \ell_2^m(M_2)$. We consider the selfadjoint subspace $S \subset N$ generated by the elements $x_k = \delta_k \otimes e_{12}$. We apply the central limit procedure Theorem 4.5 to $u_{n,T}(x) = T^{1/2} n^{-1/2} \sum_{j=1}^{n} v_j \otimes \pi_j(x)$ where $v_j v_l = -v_l v_j$ are anticommuting unitaries. We obtain an ultraproduct state $\phi_U$ and a subalgebra $N(S) \subset (\prod_{n \notin U} (M_{2^n} \otimes N_S))_*$ and a map $u_{-1} : S \to N_U$ such that

$$\sigma_{U}^{\phi_U}(u_{-1}(x)) = u_{-1}(\sigma_{U}^{\psi}(x))$$

holds for all $t \in \mathbb{R}$ and $x \in S$. In particular, we find a $\phi_U$-preserving conditional expectation $E : N_U \to N(S)$. This yields a completely isometric embedding $\iota : L_1(N(S)) \to L_1(N_U)$. According to [J2, Lemma 7.2 and Lemma 7.3], we deduce for $y = \sum_t f_t \otimes x_t$ that

$$\left\| D_{\psi}^{1/2} (u_{-1} \otimes id)(y) D_{\psi}^{1/2} \right\|_{L_2^2(N(S)) \otimes L_1(M)} = \left\| D_{\psi}^{1/2} (u_{-1} \otimes id)(y) D_{\psi}^{1/2} \right\|_{L_1(N(U)) \otimes L_1(M)}$$

$$= \lim_{n \to U} \left\| D_{\psi}^{1/2} (u_{n,T} \otimes id)(y) D_{\psi}^{1/2} \right\|_{L_2^2(N(T) \otimes L_1(M))}.$$ 

$$\sim 200 \inf_{y = y_2 + y_3} \left\| (D_{\psi}^{1/2} \otimes D_{\phi_{m2}}) y_2 (1 \otimes D_{\phi_{m2}}) \right\|_{L_2^2(N(T) \otimes L_1(M))} + \left\| (1 \otimes D_{\phi_{m2}}) y_3 (D_{\psi}^{1/2} \otimes D_{\phi_{m2}}) \right\|_{L_2^2(N(T) \otimes L_1(M))}.$$
Now, we consider the subspace $K_{12} = \text{span}\{\delta_k \otimes e_{12} : k = 1, \ldots, m\} \subset L_2(N, tr)$. It is very easily shown that the orthogonal projection

$$P_{12} \left( \begin{array}{cc} f_{11} & f_{12} \\ f_{21} & f_{22} \end{array} \right) = \left( \begin{array}{cc} 0 & f_{12} \\ 0 & 0 \end{array} \right)$$

satisfies

$$\left\| P_{12} \otimes id_{L_1(M)} : L_2^c(N, tr) \hat{\otimes} L_1(M) \rightarrow L_2^c(N, tr) \hat{\otimes} L_1(M) \right\| \leq 1$$

and

$$\left\| P_{12} \otimes id_{L_1(M)} : L_2^c(N, tr) \hat{\otimes} L_1(M) \rightarrow L_2^c(N, tr) \hat{\otimes} L_1(M) \right\| \leq 1 .$$

Thus given $y = \sum_k \delta_k \otimes e_{12} \otimes y_k$ and any decomposition $y = \tilde{y}_2 + \tilde{y}_3$, we may define

$$y_2 = (P \otimes id)(\tilde{y}_2) = \sum_k \delta_k \otimes e_{12} \otimes v_k \quad \text{and} \quad y_3 = (P \otimes id)(\tilde{y}_3) = \sum_k \delta_k \otimes e_{12} \otimes w_k .$$

Then we have

$$\left\| (D^\frac{1}{2}_\psi \otimes D^\frac{1}{2}_\delta)(1 \otimes D^\frac{1}{2}_\delta)y_2(1 \otimes D^\frac{1}{2}_\delta) \right\|_{L_2^c(N, tr) \hat{\otimes} L_1(M)} = \left\| (P_{12} \otimes id)(D^\frac{1}{2}_\psi \otimes D^\frac{1}{2}_\delta)\tilde{y}_2(1 \otimes D^\frac{1}{2}_\delta) \right\|_{L_2^c(N, tr) \hat{\otimes} L_1(M)}$$

$$\leq \left\| (D^\frac{1}{2}_\psi \otimes D^\frac{1}{2}_\delta)\tilde{y}_2(1 \otimes D^\frac{1}{2}_\delta) \right\|_{L_2^c(N, tr) \hat{\otimes} L_1(M)} .$$

The same argument works for $y_3$. Hence it suffices to consider decompositions of the form $y_2$ and $y_3$ above. We note that

$$(D^\frac{1}{2}_\psi(\delta_k \otimes e_{12}), D^\frac{1}{2}_\psi(\delta_k \otimes e_{12}))_{tr} = tr((\delta_l \otimes e_{21})D^\cdot(\delta_k \otimes e_{12})) = \delta_{kl}(1 - \mu_k) .$$

Thus

$$\left\| (D^\frac{1}{2}_\psi \otimes D^\frac{1}{2}_\delta)y_2(1 \otimes D^\frac{1}{2}_\delta) \right\|_{L_2^c(N, tr) \hat{\otimes} L_1(M)} = \left\| \sum_k (1 - \mu_k)(D^\frac{1}{2}_\psi v_k D^\frac{1}{2}_\delta)^*(D^\frac{1}{2}_\psi v_k D^\frac{1}{2}_\delta) \right\|^\frac{1}{2} .$$

For the $y_3$-term we find

$$((\delta_l \otimes e_{12})D^\cdot(\delta_k \otimes e_{12}))_{tr} = tr(D^\cdot(\delta_k \delta_l \otimes e_{21}e_{12})) = \delta_{kl}\mu_k .$$

Thus by approximation of $x_k$ with analytic elements of the form $D^{1/2}_{\phi_{12}} y_k D^{1/2}_{\phi_{12}}$, we obtain

$$\left\| \sum_k D^{\frac{1}{2}}_{\phi_{12}}(u_{-1}(\delta_k \otimes e_{12}))D^{\frac{1}{2}}_{\phi_{12}} \otimes x_k \right\|_{L_1(N(S)) \hat{\otimes} L_1(M)}$$

$$ \sim_200 \inf_{x_k = c_k + d_k} \left\| \left( \sum_k (1 - \mu_k)c_k^* c_k \right)^{\frac{1}{2}} + \left( \sum_k \mu_k d_k^* d_k \right)^{\frac{1}{2}} \right\| .$$

(7.3)

Now, we have to identify $u_{-1}(\delta_k \otimes e_{12})$ with the standard generators satisfying the CAR relations. Indeed, we recall from Theorem 5.33 that we may assume that the map $\alpha : A(S) \rightarrow L_2(N(S))$ has dense range. This enables us to apply Lemma 3.1 and to conclude that for $b_k = \delta_k \otimes e_{12}$ we have

$$u_{-1}(b_k)u_{-1}(b_j) + u_{-1}(b_j)u_{-1}(b_k) = 0 , \quad u_{-1}(b_k)u_{-1}(b_j)^* + u_{-1}(b_j)^*u_{-1}(b_k) = \delta_{kj}$$

and

$$\phi_{tr}(u_{-1}(b_k^*) \cdots u_{-1}(b_j^*) u_{-1}(b_{j_1}) \cdots u_{-1}(b_{j_r})) = \delta_{r,s} \prod_{l=1}^r \delta_{i_l,j_l}\mu_{i_l}.$$
We deduce from Corollary 5.6 and analyticity that

\[ \text{and } j : L_2(N, \psi) \rightarrow L_2(N, \psi). \]

We have seen in Corollary 5.6 that the von Neumann algebra \( \Gamma_q(K, U_t) \) and the von Neumann algebra \( N(S) \) generated by \( \{ u_q(j(f)) : f \in \ell^m_\infty \} \) are isomorphic and \( \pi(s_q(f)) = u_q(f) \). By construction, \( N(S) \subset (\prod_n (L\infty(\Omega_n; M_n) \otimes N^{\otimes n}*))^* \). Moreover, we have a \( \phi_{tU} \)-preserving conditional expectation and hence a natural inclusion

\[ L_1(N(S)) \subset \prod_{n,t} L_1(\Omega_n; M_n \otimes N^{\otimes n}). \]

We deduce from Corollary 5.6 and analyticity that

\[
\| \sum_l D^\frac{\delta}{\phi} u_q(f_l) D^\frac{\delta}{\phi} \otimes x_l \|_{L_1(N(S)) \otimes L_1(\mathbb{M})} = \| \sum_l D^\frac{\delta}{\phi} u_q(j(f_l)) D^\frac{\delta}{\phi} \otimes x_l \|_{L_1(N(S)) \otimes L_1(\mathbb{M})} = \lim_{n,t} \| \sum_l D^\frac{\delta}{\phi} u_n, T(f_l) D^\frac{\delta}{\phi} \otimes x_l \|_{L_1(\Omega_n; L_1(\Omega_{n+1} \otimes N^{\otimes n})) \otimes L_1(\mathbb{M})}
\]

holds for all \( e \in \ell^m_\infty \) and \( x_l \in L_1(\mathbb{M}) \). Here \( u_{n,T} = \sqrt{\frac{T}{n}} \sum_{j=1}^n v_j(n) \otimes \pi_j(x) \) and the \( v_j(n) \) are Speicher’s random matrices (see Corollary 2.8). We consider the special case \( f_2l = \delta_l \) and \( f_{2l+1} = -i\delta_l \). The \( q \)-gaussian analogue of the elements \( a_k \) are then given by

\[ a_k(q) = \frac{1}{2}[s_q(\delta_k) - is_q(i\delta_k)] \cong \frac{1}{2}(u_q(j(\delta_k)) - iu_q(j(i\delta_k))) = u_q(\delta_k \otimes e_{12}). \]

Therefore the argument in the proof of Theorem 7.1 shows that

\[
\left\| \sum_k D^\frac{i}{\phi} v a_k(q) D^\frac{i}{\phi} \otimes x_k \right\|_{L_1(N_\infty) \otimes L_1(\mathbb{M})} \sim \inf \left\| \sum_k (1 - \mu_k) c_k^* c_k \right\|_{L_1(\mathbb{M})} + \left\| \sum_k \mu_k d_k d_k^* \right\|_{L_1(\mathbb{M})}
\]

where the infimum is taken over \( x_k = c_k^* d_k \). For \( q = 1 \) we shall understand \( s_1(f) D^\frac{1}{\phi} \) as \( \frac{d}{dt} W(f)^{1/2} D^\frac{1}{\phi}_{t=0} \). Here \( \{ W(f) : f \in \ell^m_\infty(\mathbb{C}) \} \) is a family of unitaries in the Segal representation (see the proof of Corollary 5.6). We may then extend \( s_1 \) by linearity.

**Remark 7.5.** In the continuous case we may assume that \( K = L_2(\mu; \mathbb{C}) \), the inclusion map is given by \( j : K \rightarrow L_2(N, \psi), j(f) = f \otimes e_{12} + \tilde{f} \otimes e_{21} \) and the weight is given by \( \psi(x) = \int_\Omega [f_1 x_1 + f_2 x_2] d\mu \). As usual we assume \( f_1 + f_2 = 1 \). For \( f L_2(\mu; \mathbb{R}) \) we shall now define

\[ a_q(f) = \frac{1}{2}(s_q(j(f)) - is_q(j(i_f))). \]

Then we have

\[
\left\| \sum_l D^\frac{i}{\phi} a_q(f_l) D^\frac{i}{\phi} \otimes x_l \right\|_{L_1(\Gamma_q(K, U_t)) \otimes L_1(\mathbb{M})} \sim \left\| \int c(\omega)^* c(\omega) f_1(\omega) d\mu(\omega) \right\|_{L_1(\mathbb{M})} + \left\| (\int d(\omega) d(\omega)^* f_2(\omega) d\mu(\omega)) \right\|_{L_1(\mathbb{M})}.
\]

Let us first assume that \( f_2 = \sum_k f_2(k) 1_{A_k} \) is a finite simple function. Then (7.4) follows by approximation of the \( f_l \)'s by simple functions \( f_l L_2(\Omega_m, \Sigma, \mu) \) where \( \Sigma \) is a finite \( \sigma \)-algebra on \( \Omega_m = A_1 \cup \cdots \cup A_m \) generated by \( A_1, \ldots, A_m \). In that case the unitary group \( U_t = (f_2/f_1)^t \)
leaves the subspace $L_2(\Omega_m, \Sigma, \mu; \mathbb{C})$ invariant. This in turn implies that we have a conditional expectation $E : \Gamma_q(K, U_t) \to \Gamma_q(L_2(\Omega, \Sigma, \mu; \mathbb{C}), U_t)$ and therefore the norm estimates in $L_1$ are preserved. By density we then obtain (7.4) for infinite simple functions. For general $f_2$ we consider the sequence

$$f_2^q = \sum_{k+1 \leq r} 2^k 1_{\{2^k \leq f_2 < 2^{k+1}\}} + \sum_{k+1 \leq r} (1 - 2^k) 1_{\{2^k \leq f_2 < 2^{k+1}\}}$$

which converges to $f_2$ everywhere. Then we may consider the density $\psi_r$ given by $(1 - f_2, f_2)$ and the vacuum state $\omega_r$ on $\Gamma_q(K, U_t(r))$. We use the ultraproduct

$$M_{\mathcal{U}} \subset e_{\mathcal{U}}(\prod_{r \in \mathcal{U}} \Gamma_q(K, U_t(r))^*)$$

formed with respect to the support $e_{\mathcal{U}}$ of the ultraproduct state $(\omega_r)^\ast$. We define a generating system $S \subset L_2(\mu; \mathbb{C})$ of all bounded functions with support contained in one of the sets $\Omega_m = \{\omega : 2^{-m} \leq f_2(\omega) \leq 1 - 2^{-m}\}$. The advantage of this generating set is that for $f \in S$ the family $\sigma_t^{\psi_r} (f)$ is uniformly bounded and $\sigma_t^{\psi_r} (S) \subset S$ for all $r$. Moreover, the family $(s_{q,r}(f))_{r \in \mathbb{N}}$ satisfies the moment conditions in Remark 7.6 and therefore we find operators $s_{q,\mathcal{U}}(f)$ affiliated to $M_{\mathcal{U}}$ such that

$$(D_{\mathcal{U}}^{\frac{1}{2}}, s_{q,\mathcal{U}}(f_1) \cdots s_{q,\mathcal{U}}(f_m) D_{\mathcal{U}}^{\frac{1}{2}}) = \lim_{r \in \mathcal{U}} (D_{\omega_r}^{\frac{1}{2}}, s_{q,r}(f_1) \cdots s_{q,r}(f_m) D_{\omega_r}^{\frac{1}{2}}) = \lim_{r \in \mathcal{U}} \sum_{\sigma = \{i_1, j_1\}, \ldots, \{i_m, j_m\}} q^{I(\sigma)} \prod_{l=1}^{m} \psi_r(j(f_{i_l}) j(f_{j_l}))$$

According to Theorem 5.3 the subalgebra $M(S) \subset M_{\mathcal{U}}$ generated by the spectral projections of the elements $\{s_{q,\mathcal{U}}(f) : f \in S\}$ is isomorphic to $\Gamma_q(S, U_t)$. Our choice of $S$ also guarantees that $M(S)$ is invariant under the modular group of the ultraproduct state. Therefore, we have a completely isometric embedding of $L_1(\Gamma_q(S, U_t)) \subset L_1(M_{\mathcal{U}})$. However, for every $r$ the estimate (7.4) holds up to a factor $c(r)$ with $\lim_r c(r) = 1$. Thus the estimates also hold in the limit and hence for $\Gamma_q(S, U_t)$. By density, we may then extend it to $\Gamma_q(K, U_t)$.

**Remark 7.6.** Let us note that for fixed $0 < \lambda < 1$ we may find a function $f_2^\lambda$ with $1/c(\lambda)f_1 \leq f_1^\lambda \leq c(\lambda)f_1$, $1/c(\lambda)f_2 \leq f_2^\lambda \leq c(\lambda)f_2$ such that $f_1^\lambda / f_2^\lambda \in \{\lambda^n : n \in \mathbb{Z}\}$. Thus we obtain the same estimates in a factor of type III$_\lambda$, see Theorem 8.11 for $-1 < q < 1$. For the border case $q = \pm 1$ we may perform a similar construction using Remark 8.5 or Theorem 8.6.

8. Applications to operator spaces

In this section we assume the reader to be familiar with operator spaces theory. We are now well-prepared for the proof of Corollary 0.3.

**Theorem 8.1.** Let $Q$ be a quotient of $R \oplus \mathbb{C}$. Then $Q$ embeds into the predual of the hyperfinite III$_\lambda$, $0 < \lambda \leq 1$. 


Proof. We will start with a general characterization of quotients of $R \oplus C$. Such a quotient has a direct decomposition

$$Q \approx R_n \oplus C_m \oplus (R \oplus C/\text{gr}(D\lambda))$$

where $n, m \in \mathbb{N}_0 \cup \{\infty\}$ and $\text{gr}(D\lambda) = \{(e_{k,1}, \lambda_k e_{k,1}) : k \in \mathbb{N}\}$ is the graph of a diagonal operator. Here $\approx$ stand for a completely isomorphim $u$ with $\|u\|_{cb}\|u^{-1}\|_{cb} \leq 4$. Note that the sequence $(\lambda_k)$ might be finite as well. Let us briefly sketch the proof of this decomposition. Indeed, we consider the linear subspace $S = Q^\perp \subset C \oplus R$ and the projections $\pi_C$ and $\pi_R$ on the column and row component. By splitting off $R_m$ and $C_m$, we may assume that $\pi_C(S^\perp)$ and $\pi_R(S^\perp)$ are dense. Then, we mod out by $S \cap \{0\} \times R$ and $S \cap C \times \{0\}$. By homogeneity this does not change the operator space structure. We obtain $\tilde{S} \subset H^r \oplus K^c$ such that $\tilde{S}$ is the graph of an operator with domain in a separable Hilbert space $H$. Using homogeneity again we may assume that $\tilde{S} \subset H^r \oplus H^c$ is the graph of a positive operator $T$ on $H^r$. By the spectral theorem (see e.g. [Kad, Theorem 5.6.2]) $T$ is unitarily equivalent to a multiplication operator $M_f(f) = fg$ on some $L_2(\Omega, \mu)$. Using a small perturbation, we may assume that $f$ is an infinite sum of characteristic functions (changing the operator space structure of $L_2^c(\mu) \oplus L_2^c(\mu)/\text{graph}(M_f)$ only by a constant $(1 + \varepsilon)$). By assumption $L_2(\mu)$ is separable and hence a suitable choice of the basis yields a diagonal operator. For more details on this argument, known to Xu and the author for quite some time, see also [Ps2].

Thus in the following, we have to find an embedding of $Q(\lambda) = R \oplus C/\text{graph}(D\lambda)$ such that $(\lambda_k)$ are positive integers. We observe that $Q(\lambda)$ has a basis $(f_k)$ such that the quotient mapping $q : R \oplus C \to Q(\lambda)$ is given by $q((e_{1k}, 0)) = -\lambda_k f_k$ and $q((0, e_{k1})) = f_k$. This implies that

$$\left\| \sum_k x_k \otimes f_k \right\|_{Q(\lambda) \otimes L_1(\mathbb{M}_m)} = \inf_{x_k = -\lambda_k c_k + d_k} \left\| \sum_k c_k^* c_k \right\|_{L_1(\mathbb{M}_m)} + \left\| \sum_k d_k d_k^* \right\|_{L_1(\mathbb{M}_m)}.$$  

Let $\mu_k$ to be determined later. The change of variables $\hat{c}_k = -(1 - \mu_k)^{-1/2} c_k$ and $\hat{d}_k = \mu_k^{-1/2} d_k$ yields

$$\left\| \sum_k x_k \otimes f_k \right\|_{Q(\lambda) \otimes L_1(\mathbb{M}_m)} = \inf \left\| \sum_k (1 - \mu_k) \hat{c}_k \hat{c}_k^* \right\|_{L_1(\mathbb{M}_m)} + \left\| \sum_k \mu_k \hat{d}_k \hat{d}_k^* \right\|_{L_1(\mathbb{M}_m)}.$$  

Here the infimum is taken over $\mu_k^{-1/2} x_k = -\mu_k^{-1/2} \lambda_k c_k + \mu_k^{-1/2} c_k = \hat{c}_k + \hat{d}_k$. For the last equality to hold we choose $\mu_k = (1 + \lambda_k^{-2})^{-1}$. Then Theorem 7.1 implies that $w(f_k) = \mu_k^{-1/2} \lambda_k^{-1/2} a_k D_\mu^{-1/2}$ extends to a complete isomorphism $w : Q(\lambda) \to L_1(\mathbb{N}(\mu)) = N(\mu)^{op}$. However, $N(\mu)$ and $N(\mu)^{op}$ are hyperfinite von Neumann algebras. If we want to accommodate the additional pieces $R_m$ and $C_m$, we can use $N = B(\ell_2) \oplus N(\mu)^{op} \subset B(\ell_2) \otimes N(\mu)^{op}$. Using a conditional expectation, we can replace $\mu$ by a sequence $\mu'$ such that for every rational $0 < \lambda < 1$ there are infinitely many $\mu_k$‘s with $\mu_k = \lambda/1 + \lambda$. According to [AW69, section8], we deduce that $N(\mu')$ and $N(\mu')^{op}$ are of type III$_1$. Then $B(\ell_2) \otimes N(\mu')^{op} \cong N(\mu')^{op}$ and we find a complete embedding in the predual of a hyperfinite III$_1$ factor. Due to the results of Haagerup, Rosenthal and Sukovitch [HRS], the predual of the hyperfinite III$_1$ factor also embeds into the predual of the hyperfinite III$_\lambda$ factor for all $0 < \lambda < 1$.  

Remark 8.2. We learned from Haagerup that a von Neumann algebra admits a normal conditional expectation onto a copy of the hyperfinite III$_1$ factor if and only if the flow of weights, i.e. the restriction of the dual automorphism group on the center of the core, admits a normal
invariant measure. In particular, for those von Neumann algebras \( N \) we have an embedding of the predual of the hyperfinite factor in the predual of \( N \). This implies that \( R \oplus C/Q \) completely embeds in \( N_* \). It is open whether every quotient of \( R \oplus C \) completely embeds into the predual of every type III factor.

**Remark 8.3.** In our applications, we will often consider a concrete quotient \( Q = Q(\lambda, \nu) \) of \( R \oplus C \) with basis \((f_k)\) satisfying

\[
\left\| \sum_k f_k \otimes x_k \right\|_{Q(\lambda, \nu) \hat{} L_1(\mathbb{M}_m)} = \inf_{x_k = c_k + d_k} \left\| \sum_k \lambda_k c_k^* c_k \right\|_{1} + \left\| \sum_k \nu_k d_k^* d_k \right\|_{1}
\]

\[
= \inf_{\sqrt{\lambda_k + \nu_k} x_k = c_k + d_k} \left\| \sum_k \frac{\lambda_k}{\lambda_k + \nu_k} c_k^* c_k \right\|_{1} + \left\| \sum_k \frac{\nu_k}{\lambda_k + \nu_k} d_k^* d_k \right\|_{1}
\]

\[
\sim_{200} \sum_k \sqrt{\lambda_k + \nu_k} D^\frac{1}{2}_\mu a_k D^\frac{1}{2}_\mu \otimes x_k \bigg\|_{L_1(\mathcal{N}(\lambda, \nu) \hat{} L_1(\mathbb{M}_m))}
\]

for all \( x_k \in L_1(\mathbb{M}) \). Here we used \( \mu_k = \frac{\mu}{\lambda_k + \nu_k} \) and \( \phi_{\mu_k}(x) = (1 - \mu_k)x_{11} + \mu_k x_{22} \). Then \( D_\mu \) is the density of the quasi-free state \( \phi_\mu \) on \( \mathcal{N}(\mu) \). We obtain an isomorphism \( v^op : Q \to L_1(\mathcal{N}(\mu)) \) defined by

\[
v^op(f_k) = \sqrt{\lambda_k + \nu_k} D^\frac{1}{2}_\mu a_k D^\frac{1}{2}_\mu .
\]

Let us make this even more explicit. We may use the transposition map \( \pi^t(x_1 \otimes \cdots \otimes x_m \otimes 1 \cdots) = x_1^0 \otimes \cdots \otimes x_m^0 \otimes 1 \cdots \) and obtain an isomorphism \( \pi^t : \mathcal{N}_2 \to \mathcal{N}_2^op \) such that \( \phi_\mu \pi^t = \phi_\mu \). Therefore \( \pi^t \) extends to an isomorphism \( \pi^t : \mathcal{N}(\mu) \to \mathcal{N}(\mu)^{op} \). Note that \( a_k^t = a_k^\ast \) and hence

\[
v^op(f_k)(\pi^t(x)) = (\lambda_k + \nu_k)\frac{1}{2} tr(D^\frac{1}{2}_\mu a_k D^\frac{1}{2}_\mu \pi^t(x)) = [(\lambda_k + \nu_k)\mu_k(1 - \mu_k)^{-1}]^\frac{1}{2} tr(a_k D^\frac{1}{2}_\mu \pi^t(x))
\]

\[
= ([\lambda_k + \nu_k] \mu_k(1 - \mu_k)^{-1})^\frac{1}{2} \phi_\mu(\pi^t(x)) = ([\lambda_k + \nu_k] \mu_k(1 - \mu_k)^{-1})^\frac{1}{2} \phi_\mu(a_k^t x)
\]

\[
= ([\lambda_k + \nu_k] \mu_k(1 - \mu_k)^{-1})^\frac{1}{2} \phi_\mu(a_k^t x) = ([\lambda_k + \nu_k](1 - \mu_k)\mu_k)^{-1})^\frac{1}{2} \phi_\mu(a_k^t x).
\]

This implies that \( v^op \pi^t(f_k) = [(\lambda_k + \nu_k)\sqrt{\lambda_k}]^{1/2} a_k^\ast \phi_\mu = [(\lambda_k + \nu_k)\lambda_k/\nu_k]^{1/2} a_k^\ast \phi_\mu \) is a complete isomorphism in the predual of \( \mathcal{N}(\mu) \).

**Remark 8.4.** We refer to [P82] for a characterization of operator spaces in \( Q(R \oplus C) \) which embed in the predual of a semifinite hyperfinite factor.

Before studying further applications to operator spaces, we provide an application of operator space theory to norm inequalities for linear functionals.

**Corollary 8.5.** Let \( \phi_\mu \) and \( \phi_\nu \) be quasi free states on the CAR algebra. Let \((b_{ij})\) be a matrix with finitely many non-zero entries. Then

\[
\left\| \sum_{ij} b_{ij} (a_i \phi_\mu \otimes a_j \phi_\nu) \right\|_{(\mathcal{N}(\mu) \otimes \mathcal{N}(\nu))_*} \sim_c b_{ij} = f_{ij} + g_{ij} + h_{ij} + k_{ij} \left( \sum_{ij} |f_{ij}|^2 \mu_i \nu_j \right) \right\|^\frac{1}{2}
\]

\[
+ \left\| g_{ij} \sqrt{\mu_i \nu_j} \right\|_{S^1} + \left\| h_{ij} \mu_i \nu_j \sqrt{1 - \mu_j} \right\|_{S^1} + \left( \sum_{ij} |k_{ij}|^2 \right) \left( \frac{\mu_i^2 \nu_j^2}{(1 - \mu_i)(1 - \nu_j)} \right)^\frac{1}{2}.
\]
Proof. From (1.2) and (1.1) we deduce that
\[
\left\| \sum_{ij} b_{ij} a_i \phi_i \otimes a_j \phi_j \right\|_{(N(\mu) \hat{\otimes} N(\nu))^*} = \left\| \sum_{ij} b_{ij} a_i D_\mu \otimes a_j D_\nu \right\|_{L_1(N(\mu) \hat{\otimes} N(\nu))}
\]
\[
= \left\| \sum_{ij} b_{ij} \left( \frac{\mu_i}{1 - \mu_i} - \nu_j \right)^{1/2} D_\mu^{1/2} a_i D_\mu^{1/2} \otimes D_\nu^{1/2} a_j D_\nu^{1/2} \right\|_{L_1(N(\mu) \hat{\otimes} N(\nu))}.
\]
Now we consider the operator spaces
\[
X(\mu) = \overline{\text{span}}\{D_\mu^{1/2} a_k D_\mu^{1/2} : k \in \mathbb{N}\}
\]
and
\[
X(\nu) = \overline{\text{span}}\{D_\nu^{1/2} a_k D_\nu^{1/2} : k \in \mathbb{N}\},
\]
with canonical basis \(f_k(\mu) = D_\mu^{1/2} a_k D_\mu^{1/2}\) and \(f_k(\nu) = D_\nu^{1/2} a_k D_\nu^{1/2}\). According to Theorem 7.1, we have
\[
X(\mu) \approx_{200} \ell_2(1 - \mu) \oplus \ell_2(\mu)/\Delta =: K(\mu),
\]
where \(\Delta = \{(x, x) : x \in \ell_2(1 - \mu) \cap \ell_2(\mu)\}\) is the diagonal, and the isomorphism is given by \(u(f_k) = (e_k, e_k) + \Delta\). Let us denote the operator space defined by the right hand side by \(K(\mu)\). In [J3 Corollary 7.12] it is shown that \(K(\mu)\) is completely contractively complemented in \(L_1(M(\mu))\) for some von Neumann algebra with QWEP. We denote the corresponding embeddings by \(w_\mu : K(\mu) \to L_1(M(\mu))\) and \(w_\nu : K(\mu) \to L_1(M(\nu))\), respectively. According to [J3, Lemma 4.4, Lemma 4.5 and J3 Corollary 7.12], we deduce that
\[
\left\| \sum_{ij} c_{ij} f_i(\mu) \otimes f_j(\nu) \right\|_1 = \pi_1(T_c : X(\mu)^* \to X(\nu)) \approx_{200^2} \pi_1(T_c : K(\mu)^* \to K(\nu))
\]
\[
\sim_9 \left\| \sum_{ij} c_{ij} w_\mu f_i(\mu) \otimes w_\nu f_j(\nu) \right\|_{L_1(M(\mu)) \hat{\otimes} L_1(M(\nu))} \sim_9 \left\| \sum_{ij} c_{ij} u(f_i) \otimes u(f_j) \right\|_{K(\mu) \hat{\otimes} K(\nu)}.
\]
We may now apply [J3, Lemma 5.1] and deduce that
\[
K(\mu) \hat{\otimes} K(\nu) = \ell_2(1 - \mu) \oplus \ell_2(\mu)/\Delta \hat{\otimes} \ell_2(1 - \nu) \oplus \ell_2(\nu)/\Delta
\]
\[
= \ell_2(\mathbb{N}^2; (1 - \mu) \otimes (1 - \nu)) \oplus \ell_2(1 - \mu) \otimes \ell_2(1 - \nu) \oplus \ell_2(\mathbb{N}^2; \mu \otimes \nu)/\Delta \otimes \Delta.
\]
Here \(H \otimes_\pi K = S_1(H, K)\) is the Banach space projective tensor product and can be calculated using the norm in the Schatten class. The assertion follows from a change of variables.

Remark 8.6. Following Remark [J3], we know that \(X(\mu)\) and
\[
X_{\text{free}}(\mu) = \overline{\text{span}}\{D_\mu^{1/2} a_k(0) D_\mu^{1/2} : k \in \mathbb{N}\}
\]
are completely isomorphic because they both satisfy the formula in Theorem 7.1. In the case of free random variables this result follows by easily duality from Pisier’s estimates in [Ps2]. Pisier also proves the complementation result in \(N_{\text{free}}(\mu)\). This yields slightly better constants than using the results from [J3].

Problem 8.7. Describe the operator space structure of \(\overline{\text{span}}\{a_i a_j \phi_\mu : i, j \in \mathbb{N}\}\).

Corollary 8.8. Let \(X\) be a separable operator space. Then \(X\) and \(X^*\) embeds into the predual of a hyperfinite factor if and only if there exists subspaces \(S \subset R \oplus C\) such that \(X\) is completely isomorphic to a subspace of \(R \oplus C/S\).
Proof. Let \( X \subset Q = R \oplus C/S \) be a subspace of a quotient of \( R \oplus C \). Then Theorem 8.4 implies that \( Q \) and hence \( X \) embeds in the predual of the hyperfinite III\(_1\) factor. However, \((R \oplus C)^* = C \oplus R\) is completely isomorphic to \( R \oplus C \). The duality between subspaces and quotients implies \( X^* \in QS((R \oplus C)^*) = QS(C \oplus R) \). It is an elementary fact in Banach space theory that \( QS(X) = SQ(X) \). This also holds true in the category of operator spaces. Thus \( X \in SQ(R \oplus C) \) implies \( X^* \in SQ(R \oplus C) \) and hence both embed in the predual of a hyperfinite III\(_1\) factor. Since the predual of a hyperfinite factor has the completely bounded approximation property, the converse follows from Pisier/Shlyahenko’s Grothendieck Theorem [PS Theorem 0.5ii)].

The following result motivated our approach.

**Corollary 8.9.** The operator space \( OH \) embeds into the hyperfinite III\(_\lambda\) factor, \( 0 < \lambda \leq 1\).

**Proof.** In [J3], we found the densities \( f_1 = 1/t \) and \( f_2 = 1/(1-t) \) with respect to \( \mu = dt/\pi \sqrt{t(1-t)} \otimes m, m \) the counting measure. In this case the spectrum \( f_2/f_1 \) is continuous. Both embeddings using the CAR and CCR relations lead to the hyperfinite III\(_1\) factor. The results of Haagerup, Rosenthal and Sukachev [HRS] complete the proof.

We want to discuss two concrete embeddings for the interpolation spaces \( R_p = [R,C]_{\frac{p}{2}} \), see [P23] for a precise definition. We will need the results from [JX3]. The space \( R_p \) has a basis \((g_k)\) such that

\[
(8.1) \quad \left\| \sum_k g_k \otimes x_k \right\|_{\text{R}_p \otimes L_1(\mathbb{M}_m)} \sim_{c_p} \inf \left\| \left( \sum_{k_j} (1 - \sigma_j) c_{k_j}^{*} c_{j_k} \right) \right\|_1 + \left\| \left( \sum_{k_j} \sigma_j d_{j_k} d_{k_j}^{*} \right) \right\|_1.
\]

Here \( j \in \mathbb{Z} \) and the coefficients satisfy

\[
(8.2) \quad \sigma_j = \begin{cases} |j|^{-p'} & j \geq 1 \\ \frac{1}{2} & j = 0 \\ 1 - |j|^{-p} & j \leq 1 \end{cases}
\]

**Corollary 8.10.** Let \((\mu_{k_j})\) be a double indexed sequence such that \( \mu_{k_j} = \sigma_j \). Let \((a_{k_j})\) be a double indexed sequence of generators of the CAR algebra satisfying \( (2)\). Then the map

\[
v(g_k) = \sum_{j<0} (1 + |j|)^{-\frac{p}{2}} a_{k_j}^{*} \phi_{\mu} + \sum_{j \geq 0} (1 + |j|)^{-\frac{p'}{2}} \phi_{\mu} \cdot a_{k_j}^{*}.
\]

defines a complete embedding of \( R_p \) in \( \mathcal{N}(\mu)^{\ast} \).

**Proof.** This follows immediately from [S31] and Remark S3. In order to obtain absolutely summable coefficients, we observe that for \( j \geq 0 \) we have \( |\sigma_j (1 - \sigma_j)^{-1}|^{1/2} \sim_{c} (1 + |j|)^{-p'/2} \). In that case we use \( \phi_{\mu} \cdot a_{k_j}^{*} \). For \( j < 0 \) we prefer \( a_{k_j}^{*} \cdot \phi_{\mu} \) and find \( |\sigma_j (1 - \sigma_j)^{-1}|^{1/2} \sim_{c} (1 + |j|)^{-p/2} \).

In our next application we want to find an embedding of \( R_p^{m} \) in \( S_1^{m} \) with some control of \( m = m(p,n) \). We refer to [JX4] for the following result. Let \( \lambda > 1 \). Then there exists a constant \( c(p,\lambda) \) such that

\[
(8.3) \quad \left\| \sum_k g_k \otimes x_k \right\|_{\text{R}_p \otimes L_1(\mathbb{M}_m)} \sim_{c(p,\lambda)} \inf \left\| \left( \sum_{k \in \mathbb{N}, j \in \mathbb{Z}} \lambda^{\frac{1}{2}} c_{j_k}^{*} c_{k_j} \right) \right\|_1 + \left\| \left( \sum_{k \in \mathbb{N}, j \in \mathbb{Z}} \lambda^{-\frac{1}{2}} d_{j_k} d_{k_j}^{*} \right) \right\|_1.
\]
Lemma 8.11. There exists a constant \( c(p, \lambda) \) and \( c(p) \) such that for all \( n \in \mathbb{N} \)

\[
\left\| \sum_{k \leq n} g_k \otimes x_k \right\|_{R_p \hat{\otimes} L_1(\mathbb{M}_m)} \sim c(p, \lambda) \inf_{x_k = c_{kj} + d_{kj}} \left( \sum_{k \leq n, |j| \leq c(p) \log n} \lambda^{-\frac{1}{p}} c_{kj} \lambda^{\frac{1}{p}} c_{kj} \right)^{\frac{1}{2}} + \left( \sum_{k \leq n, |j| \leq c(p) \log n, k \in \mathbb{N}, j \in \mathbb{Z}}, \sum_{k \leq n, j \in \mathbb{Z}} \lambda^{-\frac{1}{p}} d_{jk} \lambda^{\frac{1}{p}} d_{jk} \right)^{\frac{1}{2}}
\]

holds for all \( m \in \mathbb{N} \) and sequences \( x_k \in L_1(\mathbb{M}_m) \).

Proof. Note that the lower estimate

\[
\inf_{x_k = c_{kj} + d_{kj}} \left( \sum_{k \leq n, |j| \leq c(p) \log n} \lambda^{-\frac{1}{p}} c_{kj} \lambda^{\frac{1}{p}} c_{kj} \right)^{\frac{1}{2}} + \left( \sum_{k \leq n, |j| \leq c(p) \log n, k \in \mathbb{N}, j \in \mathbb{Z}}, \sum_{k \leq n, j \in \mathbb{Z}} \lambda^{-\frac{1}{p}} d_{jk} \lambda^{\frac{1}{p}} d_{jk} \right)^{\frac{1}{2}}
\]

is obvious. Now, we assume that the right hand side is \( \leq 1 \). In particular, we may find \( c_k = c_{k,0} \) and \( d_k = d_{k,0} \) such that

\[
\left\| \left( \sum_{k=1}^{n} c_k x_k \right)^{\frac{1}{2}} \right\| + \left\| \left( \sum_{k=1}^{n} d_k x_k \right)^{\frac{1}{2}} \right\| \leq 1.
\]

This implies

\[
\max \{ \left\| \left( \sum_{k} x_k x_k^* \right)^{\frac{1}{2}} \right\|, \left\| \left( \sum_{k} x_k^* x_k \right)^{\frac{1}{2}} \right\| \} \leq 1 + \sqrt{n}.
\]

Then, we see that for the smallest integer \( j_0 \geq c(p, \lambda) \log n \) we have

\[
\left\| \left( \sum_{k \leq n, j \geq j_0} \lambda^{-\frac{1}{p}} x_k x_k^* \right)^{\frac{1}{2}} \right\| \leq (1 + \sqrt{n})(\sum_{j \geq j_0} \lambda^{-\frac{1}{p}})^{\frac{1}{2}} \leq (1 - \lambda^{-\frac{1}{p}})^{-1} \lambda^{-\frac{1}{p}} 2\sqrt{n}.
\]

Thus for this part we need \( j_0 \geq \frac{p}{\log \lambda} \log n \) in order to obtain a constant independent of \( n \). If we require \( c(p) = 2 \max\{p, p'\} \), i.e. \( c(p, \lambda) = c(p) / \log \lambda \), we can estimates both tails by a constant \( C(p, \lambda) \). \( \square \)

Corollary 8.12. Let \( 1 < p < \infty \), \( \varepsilon > 0 \) and \( n \in \mathbb{N} \). Then \([R_n, C_n]_{\frac{1}{p}}\) completely embeds into \( S_1^{n^\varepsilon} \). The constant depends only on \( p \) and \( \varepsilon \).

Proof. We use the index set \( I = \{1, \ldots, n\} \times \{j \in \mathbb{Z} : |j| \leq c(p)(\log \lambda)^{-1} \log n \} \) which has cardinality \( m \leq 2c(p)^n \log n \). We may chose \( \lambda \) large enough such that \( 2^n > n^\varepsilon \). Let us note that for a finite sequence \( (\mu_{kj}) \) the \( L_1 \) space \( L_1(\mathcal{N}(\mu)) \) and \( S_1^{2^n} \) are canonically isomorphic by sending \( D_\mu \in L_1(\mathcal{N}(\mu)) \) to

\[
\hat{D}_\mu = \otimes_{j,k} \left( \begin{array}{cc} 1 - \mu_j & 0 \\ 0 & \mu_j \end{array} \right) \in S_1^{2^n}
\]

and extending this map to a \( \mathcal{N}(\mu) = \mathbb{M}_{2^n} \)-bimodule map. Following Remark 8.3, we shall define

\[
\mu_{kj} = \frac{\lambda^{-j/p}}{\lambda^{j/p} + \lambda^{-j/p}} = (1 + \lambda^j)^{-1}.
\]
We recall \( \nu_{kj} + \lambda_{kj} = \lambda^{j/p} + \lambda^{-j/p} \) and
\[
\hat{D}^{1/2}_\mu a_{kj} \hat{D}^{1/2}_\mu = (1 - \mu_{kj})^{1/2} \mu_{kj}^{1/2} a_{kj} \hat{D}_\mu = \lambda^{j/\lambda} a_{kj} \hat{D}_\mu.
\]
Thus the embedding \( v^{op} \) from Remark 3.3 yields
\[
v^{op}_\lambda (g_k) = \sum_{|j| \leq c(p) (\log \lambda)^{-1} \log n} \sqrt{\lambda^{j/p} + \lambda^{-j/p}} \lambda^{j/\lambda} a_{kj} \hat{D}_\mu \in \mathbb{S}_1^{2m}.
\]
The cb-norm of this isomorphism \( v^{op}_\lambda \) and its inverse depend only on \( \lambda \) and \( p \). \( \blacksquare \)

At the end of this section, we will describe a completely isomorphic embedding of \( OH \) in \( B(H) \) in terms of the Brown algebra \( U^{nc}_m \) introduced in [Br2], see also [McC]. The algebra \( U^{nc}_m \) is the universal algebra spanned by elements \( (u_{ij})_{i,j=1}^m \) such that every unitary \( U = (U_{ij}) \in M_m(B(H)) \) induces a representation \( \pi_U : U^{nc}_m \to B(H) \) satisfying
\[
\pi_U(u_{ij}) = U_{ij}.
\]
Equivalently \( U^{nc}_m \) is the universal algebra spanned by contractions \( u_{ij} \) satisfying
\[
\sum_{k=1}^m u_{ki}^* u_{kj} = \delta_{ij} = \sum_{k=1}^m u_{ik} u_{jk}^*.
\]
Let us state some elementary facts. Every complete contraction \( T : S_1^m \to B(H) \) admits a complete positive extension \( \hat{T} : U^{nc}_m \to B(H) \). Indeed, consider then matrix \( v = [T(e_{ij})] \in M_m(B(H)) \) which is a contraction. Then, we may define the unitary
\[
U = \begin{pmatrix} v & \sqrt{1 - vv^*} \\ -\sqrt{1 - v^*v} & v \end{pmatrix} \in M_2(M_m(B(H))).
\]
Let \( f_{ij} \) be the matrix units in \( M_2 \). Then \( \hat{T}(a) = f_{11} \pi_U(a) f_{11} \) defines the completely positive extension of \( T \). Let us now show that the map \( \iota(e_{ij}) = u_{ij} \) is a complete isometry. Let \( a_{ij} \in M_k \) be matrices. Then
\[
\left\| \sum_{ij} a_{ij} \otimes e_{ij} \right\|_{M_k \otimes_{min} S_1^m} = \sup_{\|T\| \leq 1} \left\| \sum_{ij} a_{ij} \otimes T(e_{ij}) \right\| = \sup_{\|U\| \leq 1} \left\| \sum_{ij} a_{ij} \otimes f_{11} U_{ij} f_{11} \right\|
\]
(8.4)
\[
\leq \sup_{\|U\| \leq 1} \left\| \sum_{ij} a_{ij} \otimes U_{ij} \right\| = \left\| \sum_{ij} a_{ij} \otimes u_{ij} \right\|_{M_k \otimes_{min} U^{nc}_m} \leq \sup_{\|T\| \leq 1} \left\| \sum_{ij} a_{ij} \otimes T(e_{ij}) \right\|.
\]
The last line follows from the fact that a unitary \( U = [U_{ij}] \) yields a complete contraction \( T(e_{ij}) = U_{ij} \). Thus we have equality and \( \iota \) is completely isometric.

**Corollary 8.13.** Let \( 1 < p < \infty \) and \( \varepsilon > 0 \). Let \( m \geq n^{\varepsilon n} \). Then \( [C_n, R_n] \frac{1}{p} \) embeds completely isomorphically (uniformly in \( n \)) into \( U^{nc}_m \) using a linear combination of the generators.

**Proof.** It is sufficient to note that the map \( u : S_1^m \to U^{nc}_m \) given by \( u(e_{ij}) = u_{ij} \) is a complete isometry. Then Corollary 8.12 implies the assertion. \( \blacksquare \)

**Remark 8.14.** Alternatively, we may use the completely isometric embedding \( u : S_1^m \to M_m \ast M_m \) in the full free product of matrix algebras (amalgamated over 1) given by
\[
u_{ij} = e_{1j} \ast e_{11}.
\]
We refer to [Har] (see also [Ps1]) for this complete isometry.
We will now describe an infinite dimensional analogue of the Brown algebra. We fix a sequence \((\mu_k)\) and the state \(\phi_\mu = \otimes_{k \in \mathbb{N}} \phi_{\mu_k}\) on \(\otimes_{k \in \mathbb{N}} \mathbb{M}_2\). Then we have conditional expectations

\[
E_n : \mathbb{M}_{2^n} \to \mathbb{M}_{2^n}, \quad E_n(x_1 \otimes \cdots) = x_1 \otimes \cdots \otimes x_n \prod_{k > n} \phi_{\mu_k}(x_k).
\]

We use the notation \(E_n = E_n \otimes \text{id}\). A sequence \(y = (y_n)\) of operators in \(\mathbb{M}_{2^n} \otimes B(\ell_2)\) is called adapted if \(y_n \in \mathbb{M}_{2^n} \otimes B(\ell_2)\). An adapted sequence \((y_n)\) is a martingale if in addition \(E_n(y_m) = y_n\) holds for all \(n \leq m\). We define the set

\[
U(\mu) = \{(u_n) \subset B(\ell_2) : (u_n) \text{ a martingale of contractions}, \quad \forall_n : w - \lim_m E_n(u_m^* u_m) = 1 = w - \lim_m E_n(u_m^* u_m)\}.
\]

We observe that \(U(\mu)\) is in one-to-one correspondence with the set of unitaries in \(\mathcal{N}(\mu) \otimes B(\ell_2)\). Indeed, if \(u \in \mathcal{N}(\mu) \otimes B(\ell_2)\) is a unitary, then \(u_n = E_n(u)\) satisfies the condition listed above. Conversely, let \((u_n)\) be a martingale of contractions. Then we may define \(u\) as a weak* limit of the \(u_n\)'s. Since the \(E_m\)'s have finite rank, we find \(u_n = E_m(u)\). Being a martingale, we obtain from this strong and strong* convergence of \(u_m\) to \(u\). Hence the conditions above imply \(u^* u = 1 = uu^*\). We may now define the Brown algebra as the universal *-algebra generated by coefficients \((u_{i_1,\ldots,i_n,j_1,\ldots,j_n})\). To be more precise, let \(S = \bigcup_n 2^{\{1,\ldots,n\}} \times 2^{\{1,\ldots,n\}}\) and \(q\) a noncommutative polynomial in \((g_k)_{k \in S}\) variables. As usual we define

\[
\|q\|_{U^nc} = \sup_{(u_n) \in U(\mu)} \|q \left( \{u_n(\vec{i}_n,\vec{j}_n) : (\vec{i}_n,\vec{j}_n) \in S \} \right) \|_{B(\ell_2)}.
\]

Here \(\vec{i}_n = (i_1,\ldots,i_n)\) stand for an \(n\)-tuple and \(u_n(\vec{i}_n,\vec{j}_n)\) are the matrix coefficients of \(u_n = \sum_{i,j} e_{i,j} \otimes u_n(\vec{i}_n,\vec{j}_n)\) of \(u_n\).

**Lemma 8.15.**

i) \(U^nc\) is a direct limit of subalgebras \(A_n\) generated by the \(g_{i_1,\ldots,i_n,j_1,\ldots,j_n}\)'s.

ii) The map \(\iota : S_1^{2^n} \to A_n\) given by

\[
\iota(\vec{e}_{i_1,\ldots,i_n,j_1,\ldots,j_n}) = g_{i_1,\ldots,i_n,j_1,\ldots,j_n}
\]

is a complete isometry.

iii) There is a natural inclusion map \(i_{n,n+1} : A_n \to A_{n+1}\) such that

\[
i_{n,n+1}(g_{i_1,\ldots,i_n,j_1,\ldots,j_n}) = (1 - \mu_{n+1})g_{i_1,\ldots,i_n,0,j_1,\ldots,j_n,0} + \mu_{n+1}g_{i_1,\ldots,i_n,1,j_1,\ldots,j_n,1}.
\]

**Proof.** The last assertion iii) follows immediately from the martingale property of the \((u_n)\)'s. Thus by definition of the \(A_n\)'s we deduce \(i_{n,n+1}(A_n) \subset A_{n+1}\) completely isometrically. Then assertion i) follows immediately from the definition of \(U^nc\) because polynomials with finitely many entries are dense. For the proof of ii) we consider a martingale sequence \((u_k)\) of contractions. In particular, the element \(u_n = \sum_{i,j} e_{i,j} \otimes u_n(\vec{i},\vec{j})\) is a contraction. This allows us to apply \(\Box\). Thus \(\iota_n : S_1^{2^n} \to A_n\) is a complete contraction. For the converse, we consider \(e_{i,j} \in \mathbb{M}_{2^n}\) and a unitary \(U \in \mathbb{M}_{2^n}(B(H))\). We define the martingale \(u_k = E_k(U)\) for \(k \leq n\) and \(U_k = u\) for \(k > n\). By definition of \(A_n\) we have a homomorphism \(\pi(u_k) : A_n \to B(H)\) given by \(\pi(u_k)(g_{i_1,\ldots,i_n,j_1,\ldots,j_k}) = u_k(i_1,\ldots,i_k,j_1,\ldots,j_k)\). Then we have

\[
\| \sum_{i_n,j_n} a_{i_n,j_n}^* \otimes U_{i_n,j_n}^* \|_{\mathbb{M}_m(B(H))} = \| (id \otimes \pi(u_k)) \left( \sum_{i_n,j_n} a_{i_n,j_n}^* \otimes g_{i_n,j_n}^* \right) \| \leq \| \sum_{i_n,j_n} a_{i_n,j_n}^* \otimes g_{i_n,j_n}^* \|_{\mathbb{M}_m(\otimes_{k \leq n} A_k)}
\]

This shows that \(\iota_n^{-1}\) is also a complete contraction. \(\square\)
Lemma 8.16. $L_1(\mathcal{N}(\mu))$ embeds completely isometrically in $U_{\mu}^{nc}$.

Proof. $L_1(\mathcal{N}(\mu))$ is the direct limit of $S_1^{2n}$ with respect to the inclusion map $\iota_{n,n+1} : S_1^{2n} \to S_1^{2n+1}$ is given by

$$\iota_{n,n+1}(e_{i_1,\ldots,i_n;j_1,\ldots,j_n}) = (1 - \mu_{n+1}) e_{i_1,\ldots,i_n,0;j_1,\ldots,j_n,0} + \mu_{n+1} e_{i_1,\ldots,i_n,1;j_1,\ldots,j_n,1}.$$ 

According to Lemma 8.15 we deduce that

$$v(e_{i_1,\ldots,i_n;j_1,\ldots,j_n}) = g_{i_1,\ldots,i_n;j_1,\ldots,j_n}$$

extends to a complete isometry of $v : L_1(\mathcal{N}(\mu)) \to B_{\mu}^{nc}$. 

Corollary 8.17. Let $(\mu_{kj})_{k \in \mathbb{N}, j \in \mathbb{Z}}$ be defined as $\mu_{kj} = \frac{(1+|j|)^{-2}}{2}$ for $j \geq 0$ and $\mu_{kj} = 1 - \frac{(1+|j|)^{-2}}{2}$ for $j < 0$. Then $\mathcal{O}_H$ embeds into $U_{\mu}^{nc}$ using a linear combination of the generators.

Proof. This follows immediately from Corollary 8.10, more precisely Remark 8.3 and Lemma 8.16.

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Department of Mathematics, University of Illinois, Urbana, IL 61801, USA
E-mail address, Marius Junge: junge@math.uiuc.edu