A question on the Cauchy problem in the Gevrey classes for weakly hyperbolic equations

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Abstract

For a homogeneous polynomial $p$ in $\xi \in \mathbb{R}^n$ with Gevrey coefficients, it is known that the Cauchy problem for any realization of $p$ is well-posed in the Gevrey class of order $s < 2$ if the characteristic roots are real. In this note, we give examples showing the situation of the converse direction, in particular the optimality of the Gevrey order $s = 2$.

1 Introduction

Consider a polynomial in $\xi = (\xi_1, \ldots, \xi_n)$ of degree $m$ with Gevrey coefficient

$$P(x, \xi) = \sum_{|\alpha| \leq m} a_\alpha(x)\xi^m = p(x, \xi) + \sum_{j=0}^{m-1} P_j(x, \xi)$$

where $a_\alpha(x)$ are in some Gevrey classes $\gamma^{(s)}(\Omega)$ or $\gamma^{(s)}(\Omega)$ defined in a neighborhood of the origin of $\mathbb{R}^n$ and $p(x, \xi), P_j(x, \xi)$ denotes the homogeneous part of degree $m$ and $j < m$ in $\xi$ respectively. Gevrey class $\gamma^{(s)}(\Omega)$ is the set of functions $f \in C^\infty(\Omega)$ such that for any compact set $K \Subset \Omega$ there exist constants $C > 0, A > 0$ for which the following inequalities hold:

\begin{equation}
|D^\alpha f(x)| \leq CA^{||\alpha||(|\alpha||)^s}, \quad x \in K, \quad \alpha \in \mathbb{N}^n.
\end{equation}

For a given $P(x, \xi)$ we consider its several realizations (quantizations) as differential operators. We define $\text{op}^t(P), 0 \leq t \leq 1$ by

$$\text{op}^t(P)u(x) = (2\pi)^{-n} \int e^{i(x-y)\xi}P((1-t)x + ty, \xi)u(y)dyd\xi.$$  

Note that, assuming that $a_\alpha(x)$ are constant outside some compact neighborhood of the origin for simplicity, we see

$$\sum_{|\alpha| \leq m} a_\alpha(x)D^\alpha u(x) = \text{op}^0(P)u(x), \quad \sum_{|\alpha| \leq m} D^\alpha(a_\alpha(x)u(x)) = \text{op}^1(P)u(x).$$

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When \( t = 1/2 \) the quantization \( \text{op}^{1/2}(P) \) is called Weyl quantization and also denoted by \( \text{op}^w(P) \). Note that
\[
\lim_{\lambda \to \infty} \lambda^{-m} e^{-\lambda x \xi} \text{op}^t(P) e^{\lambda x \xi} = p(x, \xi)
\]
so that the principal symbol of \( \text{op}^t(P) \) is independent of realization and given by the highest homogeneous part of \( P(x, \xi) \).

Denote \( x = (x_1, x_2, \ldots, x_n) = (x', x_1) \) and consider the Cauchy problem
\[
\begin{cases}
\text{op}^t(P) u(x) = 0, & (x_1, x') \in \omega \cap \{ x_1 > \tau \}, \\
D_1^j u(0, x') = u_j(x'), & j = 0, \ldots, m - 1, \ x' \in \omega \cap \{ x_1 = \tau \}
\end{cases}
\tag{1.2}
\]
where \( \omega \subset \Omega \) is some open neighborhood of the origin of \( \mathbb{R}^n \). The Cauchy problem (1.2) is (uniformly) well-posed in \( \gamma^{(s)} \) near the origin if there exist \( \omega \) and \( \epsilon > 0 \) such that for any \( u_j \in \gamma^{(s)}(\mathbb{R}^{n-1}) \) and for any \( |\tau| < \epsilon \), the Cauchy problem (1.2) has a unique solution \( u \in C^m(\omega) \). We say that the Cauchy problem is locally solvable in \( \gamma^{(s)} \) at the origin if for any \( u_j \in \gamma^{(s)}(\mathbb{R}^{n-1}) \) one can find a neighborhood \( \omega \) of the origin, which may depend on \( \{ u_j \} \), such that (1.2) with \( \tau = 0 \) has a solution \( u \in C^m(\omega) \). We assume that the hyperplanes \( x_1 = \text{const.} \), are non-characteristic, that is
\[
p(x, \theta) \neq 0, \quad \theta = (1, 0, \ldots, 0), \ x \in \Omega,
\tag{1.3}
\]
which is almost necessary for \( C^\infty \) well-posedness of the Cauchy problem ([8]).

Then without restrictions one may assume \( a_{(m,0,\ldots,0)}(x) = 1 \). If the Cauchy problem (1.2) is locally solvable in \( \gamma^{(s)} \), \( s > 1 \) at the origin then \( p(0, \xi_1, \xi') = 0 \) has only real roots for any \( \xi' \in \mathbb{R}^{n-1} \) ([9]) so we assume without restrictions
\[
p(x, \xi - i\theta) = 0, \quad \xi \in \mathbb{R}^{n-1}, \ x \in \Omega.
\tag{1.4}
\]

The next results are implicit in [3] and [5].

**Theorem 1.1.** Assume \( P(x, \xi) = p(x, \xi) + \sum_{j=0}^r P_j(x, \xi) \) and the coefficients \( a_\alpha(x) \) belong to \( \gamma^{(m/r)} \). Then for any \( 0 \leq t \leq 1 \) the Cauchy problem for \( \text{op}^t(P) \) is well-posed in \( \gamma^{(s)} \) near the origin for \( 1 < s < \min\{m/r, 2\} \).

**Corollary 1.1.** Assume that the coefficients \( a_\alpha(x) \) belong to \( \gamma^{(2)} \). Then for any \( 0 \leq t \leq 1 \) the Cauchy problem for \( \text{op}^t(p) \) is well-posed in \( \gamma^{(s)} \) near the origin for \( 1 < s < 2 \).

To confirm the results, note that under the assumption there is some \( M > 0 \) such that
\[
P(x, \xi + i\tau \theta) \neq 0, \quad |\tau| > \frac{M(1 + |\xi|)^{r/m}}{x, \xi \in \mathbb{R}^n}
\tag{1.5}
\]
that is \( P \) is \( m/r \)-hyperbolic (see [6]). Then Theorem 1.1 was proved for \( \text{op}^0(P) \) in [5] and Corollary 1.1 for \( \text{op}^0(p) \) is implicit in [3]. Next, we recall a formula
for change of quantization (e.g. [7]). One can pass from any \(t\)-quantization to the \(t'\)-quantization by

\[
(1.6) \quad \text{op}'(a_{t'}) = \text{op}^t(a_t), \quad a_{t'}(x, \xi) = e^{-i(t' - t)D_tD_t}a_t(x, \xi)
\]

for \(a(x, \xi) \in S^m_{1,0}\), symbols of classical pseudodifferential operators. In particular, we have

\[
\text{op}'(P(x, \xi)) = \text{op}^t(e^{itD_tD_t}P(x, \xi)).
\]

On the other hand, from [3, Proposition 3] one has

\[
|\partial^2_{\xi} \partial^2 p(x, \xi - i\theta)| \leq C_{\alpha, \beta}(1 + |\xi|)|\partial^3 p(x, \xi - i\theta)|, \quad \alpha, \beta \in \mathbb{N}^3
\]

which is sufficient to estimate new terms that appear by operating \(e^{itD_tD_t}\) to \(P(x, \xi)\).

2 A question on Theorem 1.1

If \(P\) is of constant coefficients, \(P(x, \xi) = P(\xi)\), the Cauchy problem for \(P(D)\) is \(\gamma^{(s)}\) well-posed for \(1 < s < m/r\) ([6]) if (1.5) holds where it is understood that the Cauchy problem is \(C^\infty\) well-posed when \(r = 0\), which corresponds to the hyperbolicity in the sense of Gårding ([4]). It is clear that the results are optimal considering examples \(P(D) = D_1^m + cD_n^m\) with a suitable \(c \in \mathbb{C}\). In the variable coefficient case, on the other hand, we are restricted to \(1 < s < \min\{m/r, 2\}\) in both Theorem 1.1 and Corollary 1.1. Here we give an example showing that one can not exceed 2, at least when \(m \geq 3\). Consider

\[
(2.1) \quad P_b(x, \xi) = \xi_1^3 - (\xi_2^2 + x_2^2\xi_3)\xi_1 - b \xi_3\xi_2
\]

with \(b \in \mathbb{R}\) which was studied in [2]. Note that (1.4) is equivalent to \(b^2 \leq 4/27\).

In [2] it was proved that there is \(0 < b < 2/3\sqrt{3}\) such that the Cauchy problem for \(\text{op}^t(P_b)\) is not locally solvable at the origin in \(\gamma^{(s)}\) for \(s > 2\).

**Proposition 2.1.** Let \(m \geq 3\) and \(n \geq 3\) and consider

\[
p(x, \xi) = \xi_1^{m-3}(\xi_3^3 - (\xi_2^2 + x_2^2\xi_3)\xi_1 - b \xi_3\xi_2)
\]

which is a homogeneous polynomial in \(\xi\) of degree \(m\) with polynomial coefficients. For any \(0 \leq t \leq 1\) the Cauchy problem for \(\text{op}^t(p)\) is not locally solvable at the origin in \(\gamma^{(s)}\) for \(s > 2\), in particular, ill-posed in \(\gamma^{(s)}\), \(s > 2\) near the origin.

From (1.6) one sees that \(\text{op}'(p) = \text{op}'(p)\) for any \(0 \leq t', t \leq 1\) so that

\[
\text{op}'(p) = D_1^{m-3}(D_1^3 - (D_2^2 + x_2^2D_3)D_1 - b \xi_3D_3) = D_1^{m-3}\text{op}^t(P_b).
\]

For any given \(u_j(x') \in \gamma^{(s)}, \ j = 0, 1, 2\) we define \(u_{j+1}(x') \in \gamma^{(s)}\) by setting \(u_{j+1} = (D_2^3 - x_2^2D_3^2)u_j + b\sqrt{2}\xi_2^2D_3^2u_j, \ j = 0, \ldots, m - 4\) successively. Assume that \(u \in C^m(\omega)\) satisfies \(D_1^{m-3}\text{op}^0(P_b)u = 0\) with \(D_1^j u(0, x') = u_j(x'), \ 0 \leq j \leq m - 1\)
then \( w = \text{op}^0(P_b) \) satisfies \( D_1^{m-3} w = 0 \) with \( D_1^j w(0, x') = 0, \ j = 0, \ldots, m - 4 \) hence \( \text{op}^0(P_b) u = 0 \) contradicting with non local solvability of the Cauchy problem for \( \text{op}^0(P_b) \).

Here is a general question why \( s = 2 \) (independent of \( m \geq 3 \))?

A similar phenomenon is observed in the Cauchy problem for uniformly diagonalizable first order systems ([12, Theorem 3.3]). When \( m = 2 \) we have a result similar to Proposition 2.1:

**Proposition 2.2.** Let \( n \geq 3 \) and consider

\[
P_{\text{mod}}(x, \xi) = \xi_1^2 - 2x_2 \xi_1 \xi_n - \xi_2^2 - x_2 \xi_n^2
\]

which is a homogeneous polynomial in \( \xi \) of degree 2 with polynomial coefficients. For any \( 0 \leq t \leq 1 \) the Cauchy problem for \( \text{op}^1(P_{\text{mod}}) \) is not locally solvable at the origin in \( \gamma^{(s)} \) for \( s > 5 \), in particular, ill-posed in \( \gamma^{(s)} \), \( s > 5 \) near the origin.

This result for \( \text{op}^0(P_{\text{mod}}) \) was proved in [1] (where there is some insufficient part of the proof, see the correction given in [11]). Then to conclude Proposition 2.2 it is enough to note \( \text{op}^0(P_{\text{mod}}) = \text{op}^0(P_{\text{mod}}) \) for \( 0 \leq t \leq 1 \).

Now we would ask ourselves is there an example of a homogeneous polynomial \( p \) in \( \xi = (\xi_1, \xi_2, \ldots, \xi_n), n \geq 3 \) of degree 2 with real analytic coefficients satisfying (1.3) and (1.4) such that the Cauchy problem for \( \text{op}^1(p) \), for any \( 0 \leq t \leq 1 \), is ill-posed in \( \gamma^{(s)} \), \( s > 2 \).

For the special case \( n = 2 \) \( (m = 2) \) we have;

**Proposition 2.3.** Consider

\[
P(x, \xi) = \xi_1^2 - 2a(x) \xi_1 \xi_2 + b(x) \xi_2^2 + c(x), \quad x = (x_1, x_2) \in \mathbb{R}^2
\]

which is a polynomial in \( \xi = (\xi_1, \xi_2) \) of degree 2 without homogeneous part of degree 1 with real analytic coefficients \( a(x), b(x), c(x) \) such that \( \Delta(x) = a^2(x) - b(x) \geq 0 \) near the origin \( (a(x), b(x) \) are assumed to be real). Then the Cauchy problem for \( \text{op}^w(P) \) is \( C^\infty \) well-posed near the origin.

In fact if we make a real analytic change of coordinates \( y = \kappa(x) = (x_1, \phi(x)) \) such that \( \phi_{x_1}(x) - a(x) \phi_{x_2}(x) = 0, \ \phi(0, x_2) = x_2 \) where \( \phi_{x_j} = \partial \phi(x) / \partial x_j \) we see that

\[
\text{op}^w(P(x, \xi))(u \circ \kappa)
= (\text{op}^0(\eta_1^2 - \alpha \Delta \eta_2^2 + \beta_1 \Delta x_1 \eta_2 + \beta_2 \Delta x_2 \eta_2 + \beta_3 \eta_1 \eta_2 + \beta_4) u) \circ \kappa
\]

(2.2)

where \( \Delta = \Delta \circ \kappa^{-1}, \ \tilde{\Delta}_{x_2} = \Delta_{x_2} \circ \kappa^{-1} \) and \( \alpha = \alpha(y) \geq c > 0, \ \beta_i = \beta_i(y) \) are real analytic near \( y = 0 \). To prove the result, noting that \( |\Delta x_2| \leq C|\Delta y_2| \leq C^s \sqrt{\Delta} \), it suffices to apply [10, Theorem 1.1] to the right-hand side of (2.2).

We could possibly ask if there exists some \( \bar{s} > 2 \) such that for any homogeneous polynomial \( p \) in \( \xi \) of degree 2 with real analytic coefficients satisfying (1.3) and (1.4) the Cauchy problem for \( \text{op}^w(p) \) is well-posed in \( \gamma^{(s)} \) for \( s < \bar{s} \) near the origin.
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