Dimensionally regularized Polyakov loop correlators in hot QCD

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Abstract

A popular observable in finite-temperature lattice QCD is the so-called singlet quark–anti-quark free energy, conventionally defined in Coulomb gauge. In an effort to interpret the existing numerical data on this observable, we compute it at order $O(\alpha_s^2)$ in continuum, and analyze the result at various distance scales. At short distances ($r \ll 1/\pi T$) the behaviour matches that of the gauge-independent zero-temperature potential; on the other hand at large distances ($r \gg 1/\pi T$) the singlet free energy appears to have a gauge-fixing related power-law tail. At infinite distance the result again becomes physical in the sense that it goes over to a gauge-independent disconnected contribution, the square of the expectation value of the trace of the Polyakov loop; we recompute this quantity at $O(\alpha_s^2)$, finding for pure SU($N_c$) a different non-logarithmic term than in previous literature, and adding for full QCD the quark contribution. We also discuss the value of the singlet free energy in a general covariant gauge, as well as the behaviour of the cyclic Wilson loop that is obtained if the singlet free energy is made gauge-independent by inserting straight spacelike Wilson lines into the observable. Comparisons with lattice data are carried out where possible.

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1. Introduction

One of the classic probes for forming a thermalized partonic medium in a heavy ion collision is the change that this should cause in the properties of heavy quarkonium [1]. It is traditional to address quarkonium through potential models, but in the finite-temperature context this has been hampered by the multitude of independent definitions of a potential that could in principle be introduced. A possible way to reduce the degeneracy is to use the weak-coupling expansion as a test bench; at least within that approach, it should be possible to derive from QCD the potential that plays a physical role. Indeed, it has been discovered that the relevant potential is none of the plethora of conventional ones but a new object, which even has an imaginary part, representing a decoherence of the quark-antiquark state that is induced by the thermal medium [2]–[5]. Subsequent to the perturbative introduction, it appears that it may be possible to promote the corresponding definition to the non-perturbative level [6].

In this paper, we do not discuss the question of a proper definition of a static potential at finite temperatures, but rather set a more modest goal. Extensive lattice measurements have already been carried out on many potentials (see, e.g., refs. [7]–[15] and references therein), and first tests [6] suggest that the real part of the proper potential may overlap with one of the existing objects, namely the “singlet” quark-antiquark free energy. This concept is, however, based on gauge fixing, more precisely on the Coulomb gauge (cf. eq. (3.1) below). We are not aware of a satisfactory a priori theoretical justification for this choice; the original motivation appeared rather to be practical, in that the Coulomb gauge potential empirically reproduces the expected zero-temperature behaviour at short distances (see, e.g., refs. [14, 15]), and also displays manageable statistical fluctuations as well as a good scaling with the volume and the lattice spacing [8].

Given these empirical observations, we feel that there may be room also for some theoretical work of the Coulomb gauge potential. The purpose of this paper is straightforward: we compute the Coulomb gauge potential, and several variants thereof, at next-to-leading order in the weak-coupling expansion (i.e. $\mathcal{O}(g^4)$, where $g^2 = 4\pi\alpha_s$ is the QCD coupling constant) in dimensional regularization. The results turn out, indeed, to yield a qualitative surprise that could have a bearing on the interpretation of the Coulomb gauge lattice data.

The paper is organized as follows. In sec. 2 we compute the expectation value of a single Polyakov loop in dimensional regularization, and compare the result with literature. In sec. 3 we analyze the Coulomb gauge singlet quark–antiquark free energy at various distance scales, identifying both fortunate physical aspects as well as what looks like an unfortunate, if numerically small, gauge artifact in the result. In sec. 4 we briefly discuss the corresponding object in a general covariant gauge, while sec. 5 looks into the so-called cyclic Wilson loop; this gauge-invariant completion removes the gauge artifact from the Coulomb gauge result but with the price of introducing new problems. Sec. 6 summarizes our main findings.
2. Polyakov loop expectation value

2.1. Basic setup

Employing conventions where the covariant derivative reads $D_\mu = \partial_\mu - ig_0 A_\mu$, with $g_0$ the bare gauge coupling and $A_\mu$ a traceless and hermitean gauge field, we define a “Polyakov loop” at spatial position $r$ through

$$P_r \equiv 1 + ig_0 \int_0^\beta d\tau A_0(\tau, r) + (ig_0)^2 \int_0^\beta d\tau \int_0^\tau d\tau' A_0(\tau, r) A_0(\tau', r) + \ldots$$  \hspace{1cm} (2.1)

It transforms in a gauge transformation $U(\tau, r)$ as $P_r \rightarrow U(\beta, r) P_r U^{-1}(0, r)$. Likewise, for future reference, a straight spacelike Wilson line from origin to $r$, at a fixed time coordinate $\tau$, is denoted by

$$W_\tau = 1 + ig_0 \int_0^1 d\lambda \cdot A(\tau, \lambda r) + (ig_0)^2 \int_0^1 d\lambda \int_0^\lambda d\lambda' \cdot A(\tau, \lambda r) \cdot A(\tau, \lambda' r) + \ldots$$  \hspace{1cm} (2.2)

and transforms as $W_\tau \rightarrow U(\tau, r) W_\tau U^{-1}(\tau, 0)$. Note that because of the usual periodic boundary conditions in the time direction, $W_\beta = W_0$.

In order to simplify the notation somewhat, we in general leave out the subscript from $g_0$ in the following formulae. It is to be understood that at the initial stages it is the bare coupling that is referred to, while at the end the bare coupling is re-expressed in terms of the renormalized one,

$$g_0^2 = g^2 + \frac{g^4 \mu^{-2\epsilon}}{(4\pi)^2} 2N_f - \frac{11N_c}{3} + O(g^6) , \hspace{1cm} (2.3)$$

where $g^2$ denotes the renormalized gauge coupling in the \text{MS} scheme, with a scale parameter $\bar{\mu}^2 = 4\pi e^{-\gamma_E} \mu^2$. The factor $\mu^{-2\epsilon}$ is normally not displayed explicitly.

Within perturbation theory, assuming that the $Z(N_c)$ center symmetry is broken in the trivial direction, either spontaneously or explicitly due to matter fields in the fundamental representation, we can then define the expectation value of the trace of the Polyakov loop through $[16]$

$$\psi_P \equiv \frac{1}{N_c} \langle \text{Tr} [P_r] \rangle , \hspace{1cm} (2.4)$$

where the brackets refer to averaging in a thermal ensemble at a temperature $T$. On the non-perturbative level, $\psi_P$ plays a role as the disconnected part of some 2-point function, e.g.

$$|\psi_P|^2 = \lim_{|r| \rightarrow \infty} \frac{1}{N_c^2} \langle \text{Tr} [P_r] \text{Tr} [P_0^\dagger] \rangle = \lim_{|r| \rightarrow \infty} \frac{1}{N_c} \langle \text{Tr} [P_r P_0^\dagger] \rangle_{\text{Coulomb}} . \hspace{1cm} (2.5)$$

\footnote{The second equality can be understood by writing $P_r = \frac{1}{N_c} \text{Tr} [P_r] + \hat{P}_r$, where $\hat{P}_r$ is traceless and changes in gauge transformations like an adjoint scalar; $\hat{P}_r$ should not have correlations at infinite distance, and our results confirm this expectation at $O(\alpha_s^2)$.}
Any practical computation in finite-temperature QCD is hampered by infrared divergences. There are two kinds: milder ones, associated with colour-electric modes and the scale $gT$, which can be cured by a certain resummation of the perturbative series; and more serious ones, associated with colour-magnetic modes and the scale $g^2T/\pi$, which can only be handled by a non-perturbative study of three-dimensional pure Yang-Mills theory [17]. For the observables that we are concerned with, the first type plays a more prominent role; it can be handled by writing the observable as

$$\psi_P = [(\psi_P)_{\text{QCD}} - (\psi_P)_{\text{EQCD}}]_{\text{unresummed}} + [(\psi_P)_{\text{EQCD}}]_{\text{resummed}}.$$  \hspace{1cm} (2.6)

The difference in the first square brackets is infrared finite (provided that the correct low energy effective theory [18], called EQCD [19], is used), and can be computed in naive perturbation theory. The second term is infrared sensitive, but the resummations required for computing it can be implemented in the simple three-dimensional EQCD framework.

The graphs contributing to the expectation value of a single Polyakov loop up to $\mathcal{O}(g^4)$ are shown in fig. 1. The computation of the graphs is in principle straightforward, however some care is needed when treating Matsubara zero modes. We separate their effects explicitly, writing

$$\int_0^\tau d\tau' e^{iq_n\tau} = \delta_{q_n} \tau + (1 - \delta_{q_n}) \frac{1}{iq_n} \left( e^{iq_n\tau} - 1 \right),$$ \hspace{1cm} (2.7)

$$\int_0^\tau d\tau' \tau' e^{iq_n\tau'} = \delta_{q_n} \frac{\tau^2}{2} + (1 - \delta_{q_n}) \left[ \frac{\tau}{iq_n} e^{iq_n\tau} + \frac{1}{q_n^2} \left( e^{iq_n\tau} - 1 \right) \right],$$ \hspace{1cm} (2.8)

etc, where $q_n$ is a bosonic Matsubara mode, $q_n = 2\pi T n$, and $\delta_{q_n} \equiv \delta_{n,0}$ is a Kronecker delta function. Adding up the graphs in any gauge where the gauge field propagator, $G_{\mu\nu}$, has the
property \( G_{0i}(0, k) = 0 \), the result (for the moment unresummed) can be written as

\[
\psi_p = \exp \left[ -\frac{1}{2} g^2 C_F \beta \int_k G_{00}(0, k) \right] + \frac{1}{2} g^2 C_F \beta \int_k G_{00}^2(0, k) \Pi_{00}(0, k) - \frac{1}{2} \left( \int_k g^2 C_F N_c \left\{ \frac{\beta^2 G_{00}(0, k) G_{00}(0, q)}{24} \right. \right.
\]
\[
+ \sum_{q_n} \left[ -G_{00}(0, k) + \frac{1}{2} G_{00}(q_n, k) \right] \frac{G_{00}(q_n, q)}{q_n^2} \right)
\]
\[
-2 \sum_{q_n} G_{00}(0, k) \left[ G_{00}(q_n, q) \frac{k_i - q_i}{q_n} + G_{0i}(q_n, q) \right] G_{0i}(q_n, q + k) \right\}, \quad (2.9)
\]

where \( C_F = (N_c^2 - 1)/2N_c; \beta \equiv 1/T; \Pi_{\mu \nu} \) is the gauge field self-energy; and \( \sum_{q_n} \equiv \sum_{q_n \neq 0} \).

Inserting the propagator of the general covariant gauge (eq. (4.2) below) as well as the corresponding self-energy (eq. (4.3) below), it can be checked that the value of the expression in eq. (2.9) is gauge-parameter independent. The same result is also obtained in the Coulomb gauge, by making use of eqs. (3.3), (3.4). In explicit form, the gauge-independent part can be written as

\[
\psi_p = 1 - \frac{g^2 C_F \beta}{2} \int_k \frac{1}{k^2} + \frac{1}{2} \left( \frac{g^2 C_F \beta}{2} \int_k \frac{1}{k^2} \right)^2
\]
\[
+ \frac{g^4 C_F N_c}{2} \sum_{\{q_n\}} \int_{k, q} \left[ - \frac{2}{k^4 Q^2} + \frac{4q_n^2}{k^4 Q^2 (Q + K)^2} + \frac{1}{k^2 Q^2 (Q + K)^2} \right]
\]
\[
- \frac{g^4 C_F N_c}{2} \sum_{q_n} \int_{k, q} \left[ \frac{2 - D}{k^4 Q^2} + \frac{2(D - 2)q_n^2}{k^4 Q^2 (Q + K)^2} + \frac{2}{k^2 Q^2 (Q + K)^2}
\]
\[
+ \frac{1}{k^2 q_n^2 Q^2} \right]
\]
\[
- \frac{g^4 C_F N_c}{2} \int_{k, q} \left[ \frac{2 - D}{k^4 q^2} + \frac{2}{k^2 q^2 (q + k)^2} + \frac{\beta^2}{24k^2 q^2} \right], \quad (2.10)
\]

where \( k \equiv |k|, q \equiv |q|, k + q \equiv |k + q|; K \equiv (0, k), Q \equiv (q_n, q); \) and we have separated the Matsubara zero mode contribution in the term of \( O(g^4) \).

*2.2. Soft-mode contribution*

It is immediately visible from eq. (2.10) that the \( k \)-integral in the term of \( O(g^4) \) is infrared divergent, \( \sim \int_k 1/k^4 \). The coefficient of the divergence is, however, nothing but the Debye

\[Here and in the following we often use \( \exp(x) \) as a shorthand for \( 1 + x + x^2/2 \); no proofs concerning the exponentiation of higher order corrections are provided.\]
mass parameter:

\[
\psi = \ldots + \frac{g^4 C_F}{2} \int k^4 \left[ \int_q \left( 2 N_f \sum_{q_n} - N_c (D - 2) \sum_{q_n'} \right) \left( \frac{1}{Q^2} + \frac{2 q_n^2}{Q^4} \right) + O(k^2) \right]
\]

\[
= \ldots + \frac{g^2 C_F \beta}{2} \int k^4 \left[ m_E^2 + O(k^2) \right],
\]  
(2.11)

where we made use of the sum-integrals in eqs. (A.2), (A.3), and denoted

\[
m_E^2 \equiv \left( \frac{N_f}{6} + \frac{N_c}{3} \right) g^2 T^2.
\]  
(2.12)

Therefore the divergence can be removed by the usual resummation of colour-electric modes,

\[
-\frac{g^2 C_F \beta}{2} \int k^4 \frac{1}{k^2} + \frac{g^2 C_F \beta}{2} \int k^4 \frac{m_E^2}{k^2} + \ldots = -\frac{g^2 C_F \beta}{2} \int k^4 \frac{1}{k^2 + m_E^2} = \frac{g^2 C_F m_E \beta}{8 \pi}.
\]  
(2.13)

Unfortunately, an inspection of the last row in eq. (2.10) shows that there are also logarithmic infrared divergences (cf. ref. [20]), and to handle them correctly we need to proceed carefully. A systematic way is through eq. (2.6). Within EQCD, the effective Lagrangian has the form

\[
L_E = \frac{1}{2} \text{Tr} [\tilde{F}_{ij}^2] + \text{Tr} [\tilde{D}_i, \tilde{A}_0]^2 + m_E^2 \text{Tr} [\tilde{A}_0^2] + \ldots,
\]  
(2.14)

where \( \tilde{F}_{ij} = (i/g_E) [\tilde{D}_i, \tilde{D}_j] \), \( \tilde{D}_i = \partial_i - ig_E \tilde{A}_i \), \( \tilde{A}_0 = \tilde{A}_0^a T^a \), and \( T^a \) are hermitean generators of SU(3). The Polyakov loop operator is represented as

\[
P_r = \left[ 1 Z_0 \right] + ig \tilde{A}_0 \beta Z_1 + \frac{1}{2} (ig \tilde{A}_0 \beta)^2 Z_2 + \ldots + (g^2 \tilde{F}_{ij} \beta^2)^2 X_4 + \ldots.
\]  
(2.15)

Here all possible local operators made of Matsubara zero-modes, invariant under parity and spatial rotations and transforming under the adjoint representation of the gauge group, can in principle appear. It will be convenient for our purposes to use a “mixed” convention in eq. (2.15) where \( g \) denotes the (renormalized) four-dimensional coupling while the fields are those of three-dimensional EQCD. The matching coefficients \( Z_4 \) are of the form \( Z_4 = 1 + O(g^2) \). The possible appearance of a matching coefficient like \( X_4 \) was pointed out in ref. [22], but it does not contribute at the order of our computation. The first term in eq. (2.15) has been put inside brackets, because in the language of eq. (2.6) it represents the value of the unresummed difference inside the first brackets of eq. (2.6).

Now, within EQCD, all dynamical effects involve the scale \( m_E \sim gT \) and thus bring in additional powers of the coupling \( g \). In fact, the leading term, originating from the 2nd order operator in eq. (2.15), precisely reproduces the result of eq. (2.13) which is of order \( O(g^3) \).

\[^3\text{A 2-loop derivation of } m_E^2, g_E^2 \text{ in terms of the parameters of four-dimensional QCD can be found in ref. [21], but here we only need their values at leading non-trivial order, } m_E^2 = g^2 T^2 (N_f/6 + N_c/3), g_E^2 = g^2.\]
Because the term is of \( \mathcal{O}(g^3) \), the coefficient \( Z_2 \) can be set to unity as we work at the order \( \mathcal{O}(g^4) \). There is an \( \mathcal{O}(g^4) \) contribution from the next-to-leading order evaluation of the 2nd order operator, however, and the sum of the \( \mathcal{O}(g^3) \) and \( \mathcal{O}(g^4) \) contributions can be written as

\[
\left[ (\psi_P)_{\text{EQCD}} \right]_{\text{resummed}} = \frac{g^2 C_F m_e \beta}{8\pi} + \frac{g^2 C_F \beta}{2} \int k \frac{\Pi_{00}^E(k)}{k^2 + m_E^2} + \ldots , \tag{2.16}
\]

where the self-energy within EQCD has a well-known form (see, e.g., refs. \cite{23, 24, 25}):

\[
\Pi_{00}^E(k) = g^2 N_c T \int q \left\{ \frac{1}{q^2 + m_E^2} + \frac{2(m^2 - k^2)}{q^2[(k + q)^2 + m^2_E]} + \frac{D - 3}{q^2} + \frac{\bar{\xi}(k^2 + m_E^2)}{q^4} \right\} \left[ 1 - \frac{k^2 + m_E^2}{(k + q)^2 + m_E^2} \right] . \tag{2.17}
\]

We have kept a general gauge parameter \( \bar{\xi} \) here, defined with the convention that \( \bar{\xi} = -1 \) corresponds to Coulomb gauge and \( \bar{\xi} = 0 \) to Feynman gauge (i.e. \( G_{ij} = \delta_{ij}/q^2 + \bar{\xi} q_i q_j / q^4 \)); and, for later reference, we have not yet killed any integrals through special properties of dimensional regularization.

We note, first of all, that the \( \bar{\xi} \)-dependent part of eq. \((2.17)\) gives no contribution in eq. \((2.16)\). Second, the other terms of eq. \((2.17)\) reproduce, for \( m_e \to 0 \), the zero-mode contribution on the last line of eq. \((2.10)\). In the “dressed” form of eqs. \((2.16)\), \((2.17)\), however, the logarithmic infrared divergence has been lifted. The integrals in eq. \((2.16)\) are all elementary (cf. eqs. \((A.26)\)--\((A.28)\)), and in total we get (replacing \( g_E^2 \to g^2 + \mathcal{O}(g^4) \))

\[
\left[ (\psi_P)_{\text{EQCD}} \right]_{\text{resummed}} = \frac{g^2 C_F m_E \beta}{8\pi} - \frac{g^4 C_F N_c}{(4\pi)^2} \left( \frac{1}{4\epsilon} + \ln \frac{\bar{\mu}}{2m_E} + \frac{1}{4} \right) + \mathcal{O}(g^5) . \tag{2.18}
\]

### 2.3. Hard-mode contribution

It remains to compute the unresummed difference inside the first brackets in eq. \((2.6)\), given by eq. \((2.10)\). Without resummation the zero-mode contribution in eq. \((2.10)\) contains no scale and vanishes; the same happens in the cases where the \( k \)-integration contains no \( q_n^2 \) and factorizes from the \( q \)-integration. This leaves us with 5 non-zero sum-integrals, with values given in eqs. \((A.8)\)--\((A.12)\); summing together, we arrive at

\[
Z_0 = \left[ (\psi_P)_{\text{QCD}} - (\psi_P)_{\text{EQCD}} \right]_{\text{unresummed}} = 1 + \frac{g^4 C_F}{(4\pi)^2} \left[ N_f \left( -\ln \frac{2}{2} \right) + \frac{N_c}{4\epsilon} \left( \frac{1}{2} - \ln \frac{\bar{\mu}}{2T} + \frac{1}{2} \right) \right] . \tag{2.19}
\]
2.4. Summary and comparison with literature

Adding up eqs. (2.18), (2.19), the 1/ǫ’s duly cancel, and we get our final result for the Polyakov loop expectation value:

\[
\psi_P = 1 + \frac{g^2 C_F m_E}{8\pi T} + \frac{g^4 C_F}{(4\pi)^2} \left[ N_f \left( -\ln \frac{2}{T} \right) + N_c \left( \ln \frac{m_E}{T} + \frac{1}{4} \right) \right] + O(g^5), \tag{2.20}
\]

where \( m_E \) is from eq. (2.12). In accordance with general expectations [26, 27], the result is finite and renormalization group invariant up to the order computed.

A classic determination of \( \psi_P \) was presented, for \( N_f = 0 \), in ref. [20]. Re-expressing that result in terms of \( m_E \) it can be written as

\[
\psi_P^{[20]} = 1 + \frac{g^2 C_F m_E}{8\pi T} + \frac{g^4 C_F N_c}{(4\pi)^2} \left( \ln \frac{m_E}{2T} + \frac{3}{4} \right). \tag{2.21}
\]

This agrees with our result including the logarithmic term, but differs on the constant term accompanying the logarithm by a factor \(-\ln 2 + 1/2\). As far as we can see, the difference can be traced back to the way that the resummation was carried out. We do believe ours to be a systematic resummation.

In order to finally compare with lattice results, we need to insert some numerical values for the parameters appearing in eq. (2.20). Following ref. [29], we can estimate them through some “fastest apparent convergence” criteria; for the gauge coupling we have applied this criterion to the combination \( g^2 \mathcal{Z}_1^2 \), playing a role in the next section (cf. eqs. (3.19), (3.28)), while for the mass parameter we take over the criterion from EQCD [29]:

\[
g^2 \simeq \frac{24\pi^2}{(11N_c - 2N_f)\left[ \ln \frac{4\pi T}{\Lambda_{\text{MS}}} - \gamma_E + c_g \right]}, \quad m_E^2 \simeq \frac{4\pi^2 (2N_c + N_f)T^2}{(11N_c - 2N_f)\left[ \ln \frac{4\pi T}{\Lambda_{\text{MS}}} - \gamma_E + c_m \right]}, \tag{2.22}
\]

where

\[
c_g = \frac{2N_f(4\ln 2 - 1) - 11N_c}{2(11N_c - 2N_f)}, \quad c_m = \frac{4N_f \ln 2}{11N_c - 2N_f} - \frac{5N_c^2 + N_f^2 + 9N_f/(2N_c)}{(11N_c - 2N_f)(2N_c + N_f)}. \tag{2.23}
\]

The results are shown in fig. 2, where they are also compared with the four-dimensional lattice data from ref. [12]. We have also tested more elaborate choices for the parameters, leading indeed to a somewhat better accord with lattice data (cf. caption of fig. 2), however within the accuracy of our actual computation it is not possible to justify these theoretically.

Nevertheless experience from other quantities, such as the spatial string tension [11, 21, 31, 32], leads us to suspect that pursuing the computation systematically to a higher order would

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4If the quarks are given a common chemical potential, \( \mu \), then the Debye mass parameter gets changed as \( m_E^2 \rightarrow g^2 [N_c \frac{\mu^2}{T^2} + N_f (\frac{\mu^2}{T^2} + \frac{2\mu T}{\pi^2})] \), and the numerical factor in eq. (2.20) is modified as \(-\ln 2/2 \rightarrow \text{Re} \zeta'(0, \frac{1}{2} + i\frac{\mu T}{\pi^2}) \), where \( \zeta'(x, y) \equiv \partial_x \zeta(x, y) \), and \( \zeta(x, y) \) denotes the generalized zeta function, \( \zeta(x, y) \equiv \sum_{k=0}^{\infty} 1/(k+y)^x \).

5We have been informed by the authors of ref. [28] that they have recently obtained the same result.
Figure 2: The dimensionally regularized Polyakov loop expectation value, eq. (2.20), as a function of $T/T_c$, where $T_c$ is the critical temperature of the deconfining phase transition (a conversion from perturbative units has been carried out by assuming $T_c/\Lambda_{\text{MS}} \simeq 1.25$; a variation within the range $1.10 - 1.35$ yields an error much smaller than the band width). The upper edge of the band (solid red line) corresponds to evaluating the coupling and the Debye mass parameter according to the simple 1-loop criteria in eqs. (2.22), (2.23); for the lower edge (dashed blue line) we have replaced $g^2$ through the 2-loop value of $g_E^2$ given in ref. [21], and $m_E/g_E^2$ through the expression in eq. (14) of ref. [30]. The lattice data, labelled by $N_\tau$, is from ref. [12] (the spatial lattice size was kept fixed at $32^3$).

eventually allow to improve on the agreement. (Phenomenological recipes for matching the lattice data down to lower temperatures can be found, e.g., in refs. [33].)

It is amusing to note, in any case, that the behaviour of the Polyakov loop is qualitatively quite similar to that of mesonic screening masses, expressed in units of the temperature [34]: both are small close to the phase transition (because they are related to order parameters in various limits), but then increase rapidly, and should finally approach their non-zero asymptotic values from above.

3. Singlet free energy in Coulomb gauge

3.1. Basic setup

The original definition of the singlet quark-antiquark free energy was related to the eigenvalues of the untraced Polyakov loop [35]; in practice, however, lattice measurements consider the
Figure 3: The connected graphs contributing to the correlator of two Polyakov loops at $\mathcal{O}(g^4)$, with the filled blob denoting the 1-loop gauge field self-energy. In addition there are disconnected contributions, obtained by multiplying the set of graphs in fig. 1 with its Hermitian conjugate. (The topologies are the same as those needed for pushing the computation of the heavy quark medium polarization one level up from the current standard [37].)

The object (see, e.g., [8]–[15] and references therein)

$$\psi_C(r) = \frac{1}{N_c} \langle \text{Tr} [P_1 P_0^\dagger] \rangle_{\text{Coulomb}} . \quad (3.1)$$

This differs from the physical “colour-averaged” free energy, in which traces of Polyakov loops are correlated (cf. eq. (5.30)) [16, 23, 36]; the advantage of eq. (3.1) is that the free energy extracted from it is believed to reduce at small distances to the gauge-fixing independent zero-temperature static potential [14, 15].

The general strategy for our determination of $\psi_C$ is the same as in sec. 2. The new graphs to be evaluated are shown in fig. 3. Like in the previous section, we first give a general (unresummed) result, which is valid in any gauge with the property $G_{0i}(0, k) = 0$:

$$\psi_C(r) = \exp \left[ g^2 C_F \beta \int_k (e^{ik\cdot r} - 1) G_{00}(0, k) \right]$$

$$- g^2 C_F \beta \int_k (e^{ik\cdot r} - 1) G_{00}^2(0, k) \Pi_{00}(0, k)$$

$$+ g^4 C_F N_c \int_k (e^{ik\cdot r} - 1) G_{00}(0, k) \int_q \left\{ \frac{\beta^2 G_{00}(0, q)}{6} - \sum_{q_n \neq 0} \frac{2G_{00}(q_n, q)}{q_n^2} \right\}$$

$$- 2 \sum_{q_n \neq 0} \left[ G_{00}(q_n, q) \frac{k_i - q_i}{q_n} + G_{0i}(q_n, q) \right] G_{0i}(q_n, q + k)$$

$$+ g^4 C_F N_c \int_{k, q} (e^{i(k+q)\cdot r} - 1) \left\{ - \frac{\beta^2 G_{00}(0, q) G_{00}(0, k)}{8} \right\}$$

$$+ \sum_{q_n \neq 0} \frac{G_{00}(q_n, q)}{q_n^2} \left[ G_{00}(0, k) + \frac{1}{2} G_{00}(q_n, k) \right] \right\} . \quad (3.2)$$

This was obtained by a brute force evaluation, making use of eqs. (2.7), (2.8), etc. The result is gauge-dependent; in order to now specialize to Coulomb gauge, we insert the gluon
The propagator

\[ G_{\mu\nu}(Q) = \frac{\delta_{\mu0}\delta_{\nu0}}{q^2} + \frac{\delta_{\mu i}\delta_{\nu j}}{Q^2} \left( \frac{8}{q^2} \right), \quad Q = (q_n, q) , \quad (3.3) \]

and the corresponding self-energy into eq. (3.2). The self-energy reads

\[ \Pi_{00}(0, k) = g^2 N_c \sum_{\mathcal{Q}} \left\{ \frac{D - 2}{Q^2} - \frac{2[k^2 + (D - 2)q_n^2]}{Q^2(Q + K)^2} \right\} \]

\[ g^2 N_c \sum_{\mathcal{Q}} \left\{ -\frac{2}{Q^2} + \frac{k^2 + 4q_n^2}{Q^2(Q + K)^2} \right\} , \quad (3.4) \]

where again \( K \equiv (0, k) \). To be explicit, the result can be written as

\[ \psi_C(r) = 1 + g^2 C_F \beta \sum_{\mathcal{Q}} \left( \frac{e^{ik \cdot r} - 1}{k^2} \right)^2 + g^4 C_F N_f \sum_{\mathcal{Q}} \left( \frac{e^{ik \cdot r} - 1}{k^2} \right)^2 \left[ \frac{2}{k^2 Q^2} - \frac{1}{k^2 Q^2(Q + K)^2} - \frac{4q_n^2}{k^4 Q^2(Q + K)^2} \right] \]

\[ g^4 C_F N_c \sum_{\mathcal{Q}} \left( \frac{e^{ik \cdot r} - 1}{k^2} \right)^2 \left[ \frac{2D}{k^4 Q^2} - \frac{2}{k^2 Q^2(Q + K)^2} + \frac{2(D - 2)q_n^2}{k^4 Q^2(Q + K)^2} \right] \]

\[ g^4 C_F N_c \int_k e^{ik \cdot r} - 1 \left[ \left( \frac{\beta^2}{6} - \sum q_n^2 \right) \int_q \frac{1}{q^2} \right] \]

\[ + g^4 C_F N_c \sum_{\mathcal{Q}} \left( \frac{e^{(k + q) \cdot r} - 1}{k^2} \right)^2 \left[ \left( -\frac{\beta^2}{8} + \sum q_n^2 \right) \frac{1}{q^2} \right] . \quad (3.5) \]

Given that \( \sum q_n^2 = \beta^2 \zeta(2)/2\pi^2 = \beta^2 / 12 \), the last two terms (corresponding to a vertex correction and to two-gluon exchange) actually do not contribute; in Coulomb gauge the unresummed result originates from the self-energy correction alone.

If we naively set \(|r| \to \infty \) in eq. (3.3), then all terms containing \( e^{ik \cdot r} \) or \( e^{(k + q) \cdot r} \) drop out. A constant term remains over, and as can be verified by comparing with eq. (2.10), it has the value\(^6\)

\[ \lim_{r \to \infty} \psi_C(r) = |\psi_0|^2 . \quad (3.6) \]

In the following we drop out this constant contribution, and focus on the \( r \)-dependent terms.

\(^6\)The expression in eq. (2.10) had the covariant gauge as a starting point whereas eq. (3.2) comes from the Coulomb gauge; therefore some of the terms do not immediately look alike. On closer inspection, however, their difference integrates to zero in the \( r \)-independent term.
3.2. Short-distance limit

The Matsubara sums in eq. (3.5) can be transformed, in a standard way, to an integral representation, from which a zero-temperature part and a thermal part can be identified. The thermal part contains the scale \( T \) inside Bose-Einstein and Fermi-Dirac distribution functions and, for large momenta, is exponentially suppressed. Therefore we expect that at small distances, i.e. \( r \ll \frac{1}{\pi T} \), the result can be obtained by simply replacing the Matsubara sums with the corresponding zero-temperature integrals, \( \Psi_{\infty}, \Phi_{\infty} \rightarrow \int Q \). (Of course, this expectation can be crosschecked later on from the more general result.)

The advantage of the zero-temperature limit is that then all integrals can be carried out analytically. For the “covariant” structures in eq. (3.5) we get, as usual,

\[
\int Q \frac{q^2}{Q^2(Q + K)^2} = \frac{1}{D - 1} \int Q \left[ \frac{1}{2Q^2} - \frac{k^2}{4Q^2(Q + K)^2} \right],
\]

\[
\int Q \frac{1}{Q^2} = 0,
\]

\[
\int Q \frac{1}{Q^2(Q + K)^2} = \frac{1}{(4\pi)^2} \left( \frac{1}{\epsilon} + \ln \frac{\mu^2}{k^2} + 2 \right),
\]

where we once again inserted \( K = (0, k) \). As far as the “non-covariant” terms are concerned, it is convenient to group them as

\[
\int Q \frac{1}{q^2Q^2} \left[ \frac{1}{k^2} - \frac{1}{2(q + k)^2} - \frac{1}{2(Q + K)^2} \right]
\]

\[
= \int Q \frac{1}{q^2Q^2} \left[ \frac{1}{k^2} - \frac{1}{(q + k)^2} \right] + \int Q \frac{1}{2q^2Q^2} \left[ \frac{1}{(q + k)^2} - \frac{1}{(Q + K)^2} \right]
\]

\[
= \frac{1}{8\pi^2k^2} \left( \frac{1}{\epsilon} + \ln \frac{\mu^2}{4k^2} \right) + \frac{1}{8\pi^2k^2} \ln 4.
\]

The former combination is infrared finite but ultraviolet divergent, and the integral was carried out in dimensional regularization; the latter combination is infrared and ultraviolet finite for \( k \neq 0 \), and was integrated directly in four dimensions.

Summing up and re-expanding the bare gauge coupling of the leading order term in terms of the renormalized one, the \( 1/\epsilon \)’s cancel (more details on the graph-by-graph origin of divergences are provided in sec. 4 for a case where they do not cancel), and we get

\[
\psi_C(r) \approx \exp \left( \frac{g^2 C_F \beta}{4\pi r} \right)
\]

\[
+ \frac{g^4 C_F \beta}{(4\pi)^2} \int_k e^{ikr} \left[ -\frac{2N_f}{3} \left( \ln \frac{\mu^2}{k^2} + \frac{5}{3} \right) + \frac{11N_c}{3} \left( \ln \frac{\mu^2}{k^2} + \frac{31}{33} \right) \right].
\]

Exponentiating the \( \mathcal{O}(g^4) \) correction and extracting the singlet free energy, \( V_1 \), as \( \psi_C(r) \equiv \exp[-\beta V_1(r)] \), we see that \( V_1 \) precisely agrees with the classic result for the zero-temperature
static potential at $\mathcal{O}(g^4)$ \[38\].

$$V_1(r) = -\frac{g^2 C_F}{4\pi r} + \frac{g^4 C_F}{(4\pi)^2} \int \frac{e^{i k \cdot r}}{k^2} \left[ \frac{2N_f}{3} \left( \ln \frac{\bar{\mu}^2}{k^2} + \frac{5}{3} \right) - \frac{11N_c}{3} \left( \ln \frac{\bar{\mu}^2}{k^2} + \frac{31}{33} \right) \right]. \quad (3.12)$$

Empirical observations in the same direction have been made even on the non-perturbative level \[14, 15\]; thus the agreement might be true at the 2-loop level as well, where a comparison could be carried out with ref. \[39\], however we have not undertaken this task.

3.3. Hard-mode contribution

We now move on to consider the behaviour of eq. (3.5) at larger distances, $r \sim 1/\pi T$, where thermal effects do play a role. The general strategy is analogous to eq. (2.6), viz.

$$\psi_C(r) = \left[ (\psi_C(r))_{\text{QCD}} - (\psi_C(r))_{\text{EQCD}} \right]_{\text{unresummed}} + \left[ (\psi_C(r))_{\text{EQCD}} \right]_{\text{resummed}}. \quad (3.13)$$

This time we evaluate the unresummed QCD part first, and for ease of later subtraction separate the Matsubara zero mode contribution from eq. (3.5), i.e. replace $\sum q_n \rightarrow \sum q'_n$, and write the zero-mode part separately.

For the non-zero mode part, we proceed as follows. The goal is to carry out the sum-integral over $Q$ of non-zero Matsubara frequency, and to express the resulting $k$-integrand, let us call it $I(k)$, in the form

$$I(k) = \frac{A}{k^4} + \frac{B}{k^2} + C(k). \quad (3.14)$$

Because of restriction to non-zero modes, the function $C(k)$ must be analytic in $k^2$. Like in eq. (2.13), the coefficient $A$ is essentially $m_0^2$, and this most infrared sensitive term is subtracted by the expanded version of the EQCD contribution. The coefficient $B$, in turn, “renormalizes” the leading order contribution; in fact it also gets subtracted by the EQCD contribution through eq. (3.13), more precisely by the 1st order term from eq. (2.15), with a properly chosen factor $Z_1$. The remainder, determined by the function $C(k)$, represents the “genuine” thermal contribution from the non-zero Matsubara modes, and does not get subtracted by EQCD effects.

\footnote{We recall that in the Fourier transform the logarithm gets effectively replaced as $\ln(\bar{\mu}^2/k^2) \rightarrow 2[\ln(\bar{\mu}r) + \gamma_E]$.}
Making use of the sum-integrals in eqs. (A.2), (A.3), (A.7), (A.13)–(A.18), we get

\[
\left(\psi_{C}(r)\right)_{\text{QCD}}^{\text{unresummed}} = \text{const.} + g^2 C_F \beta \int_{k} \frac{e^{i k \cdot r}}{k^2} + \frac{1}{2} \left( g^2 C_F \beta \int_{k} \frac{e^{i k \cdot r}}{k^2} \right)^2 \\
+ g^2 C_F \beta \int_{k} e^{i k \cdot r} \left\{ -\frac{m_{E}^2}{k^4} + \frac{g^2}{(4\pi)^2} k^2 \left( \frac{11 N_c}{3} (L_b + 1) - \frac{2 N_f}{3} (L_f - 1) \right) \right\} \\
+ g^4 C_F N_c \left( \frac{4\pi}{(4\pi)^2} \right) \left\{ -\frac{1}{24T^2r^2} + \frac{5 \ln(1 - e^{-4\pi Tr})}{6\pi Tr} + \frac{1 + e^{4\pi Tr}(4\pi Tr - 1)}{3(e^{4\pi Tr} - 1)^2} \\
+ \sum_{n=1}^{\infty} \left( 4 - \frac{(4\pi Trn)^2}{3} \right) E_1 \left( 4\pi Trn \right) + \frac{\text{Li}_2(e^{-4\pi Tr})}{(2\pi Tr)^2} \right\} \\
+ g^4 C_F N_f \left( \frac{4\pi}{(4\pi)^2} \right) \left\{ -\frac{1}{3\pi Tr} \ln \frac{1 - e^{-2\pi Tr}}{1 + e^{-2\pi Tr}} + \frac{e^{2\pi Tr}}{3} \left( \frac{1}{e^{4\pi Tr} - 1} - \frac{1}{e^{4\pi Tr} - 1} \right) \\
+ \sum_{n=1}^{\infty} \left( -2 + \frac{2\pi Tr \times (2n - 1)^2}{3} \right) E_1 \left( 2\pi Tr \times (2n - 1) \right) \right\} \\
+ g^4 C_F N_c \times \text{(zero mode contribution)} \right) .
\]

Here the gauge coupling is already the renormalized one; \( L_b, L_f \) are defined as

\[
L_b \equiv 2 \ln \frac{\bar{\mu} e^{\gamma_E}}{4\pi T}, \quad L_f \equiv 2 \ln \frac{\bar{\mu} e^{\gamma_E}}{\pi T} ;
\]

and \( E_1 \) and \( \text{Li}_2 \) are defined in eqs. (A.20), (A.21). Sometimes it may be convenient to replace the sums over \( E_1 \) with integral representations, and indeed this can be achieved as specified in eqs. (A.22)–(A.25). The expression then compactifies quite a bit; we rewrite the corresponding result in eq. (3.28) below.

As a first check, it can be shown that for \( \pi Tr \ll 1 \) the square-bracketed terms in eq. (3.15) go over into

\[
g^4 C_F N_c \left( \frac{4\pi}{(4\pi)^3 Tr} \right) \left( 11 \right) \left[ 2 \ln(4\pi Tr) - 1 \right] + \frac{31}{33} \right] - g^4 C_F N_f \left( \frac{4\pi}{(4\pi)^3 Tr} \right) \left( \frac{2}{3} \right) \left[ 2 \ln(\pi Tr) + 1 \right] + \frac{5}{3} \right] .
\]

Combining with the logarithmic terms in eq. (3.15), we exactly match the behaviour given by eq. (3.11).
3.4. Soft-mode contribution

We then proceed to consider the EQCD (zero-mode) contribution to eq. (3.13), which we now denote by \( \psi_C^E(r) \equiv (\psi_C(r))_{\text{EQCD}} \). It reads

\[
\psi_C^E(r) = \text{const.} + g^2 C_F \beta Z_1^2 \int \frac{e^{i\mathbf{k} \cdot \mathbf{r}}}{k^2 + m_E^2} + \frac{1}{2} \left( g^2 C_F \beta Z_2 \int \frac{e^{i\mathbf{k} \cdot \mathbf{r}}}{k^2 + m_E^2} \right) ^2 \\
+ g^2 g_s C_F N_c \beta Z_1^2 \int_{k,q} e^{i\mathbf{k} \cdot \mathbf{r}} \left\{ \frac{1}{(k^2 + m_E^2)^2} \left[ \frac{3 - D}{q^2} - \frac{1}{q^2 + m_E^2} \right] \\
+ \frac{1}{q^2} \left[ \frac{1}{k^2 + m_E^2} - \frac{1}{(k + q)^2 + m_E^2} \right] \right\} \\
- \frac{g^4 C_F N_c}{8} \left( \beta Z_2 \int \frac{e^{i\mathbf{k} \cdot \mathbf{r}}}{k^2 + m_E^2} \right)^2 + O(g^5).
\] (3.18)

In the terms of \( O(g^4) \), we can immediately set \( g_s^2 \to g^2 \), \( Z_1 \), \( Z_2 \to 1 \).

Let us inspect the role that eq. (3.18) plays when first subtracted in an unresummed form, and then added “as is”, to eq. (3.15). The first line of eq. (3.18) accounts for the first two lines of eq. (3.15), cancelling the infrared divergence like in eq. (2.13) and fixing \( Z_1^2 = 1 + \frac{g^2}{(4\pi)^2} \left[ \frac{11 N_c}{3} (L_b + 1) - \frac{2 N_f}{3} (L_f - 1) \right] \). (3.19)

The expression in the curly brackets in eq. (3.18) agrees, for \( m_E \to 0 \), with the zero-mode part of eq. (3.5), and through the subtraction–addition step replaces it with a “less” infrared sensitive expression; we return to this term presently. Finally, the last term of eq. (3.18) can be written, after subtraction and addition and insertion \( Z_2 \to 1 \), as

\[
\left[ -\psi_C^E(r) \right]_{\text{unresummed}} + \left[ \psi_C^E(r) \right]_{\text{resummed}} = \ldots + \frac{g^4 C_F N_c}{8 (4\pi)^2} \left[ \frac{1}{8 T^2 r^2} - \frac{\exp(-2m_E r)}{8 T^2 r^2} \right].
\] (3.20)

Now, we observe a problem. After combining eq. (3.15) with the re-processed version of eq. (3.18), as outlined above, the large-distance behaviour of \( \psi_C \) is dominated by an uncancelled power-law term,

\[
\psi_C(r) \approx \int \frac{e^{i\mathbf{k} \cdot \mathbf{r}}}{k^2 + m_E^2} + \frac{1}{24 T^2 r^2} + \frac{1}{8 T^2 r^2} = \frac{g^4 C_F N_c}{8 (4\pi)^2} \left[ \frac{1}{12 T^2 r^2} \right].
\] (3.21)

Indeed all other terms are exponentially suppressed after the subtraction–addition step, either as \( \exp(-2\pi T r) \) or (from eq. (3.18)) as \( \exp(-m_E r) \) or \( \exp(-2m_E r) \). A term of the type in eq. (3.21) must be a gauge artifact: physically there is a finite screening length in a non-Abelian plasma.\(^8\) Indeed we will find in sec. 5 (cf. eq. (5.9)) that in gauge-invariant observables the power-law terms do get duly cancelled.

\(^8\)Within perturbation theory a power-law term could in principle also indicate a sensitivity to colour-magnetic modes; however, as eq. (3.20) suggests, the term here has at least partly a colour-electric origin.
Despite the issue of eq. (3.21), which we consider to be a serious one from the conceptual point of view, we wish to complete in the remainder of this section our discussion of eq. (3.18). This leads to another issue, yet this time more physical, being a manifestation of the logarithmic sensitivity of the result to the non-perturbative colour-magnetic scale.

First of all, if we just carry out the integrals in the curly brackets of eq. (3.18) literally, the result appears to be both infrared and ultraviolet finite, and reads

\[
\psi_C^E(r) = \text{const.} + \frac{g^2C_F\pi^2}{4\pi Tr} + \frac{1}{2} \left( \frac{g^2C_Fe^{-mE}r}{4\pi Tr} \right)^2 + \frac{g^4C_FN_e^{-mE}r}{(4\pi)^2} \left\{ 2 - \ln(2mE) - \gamma_E + e^{2mE}E_i(2mE) \right\} - \frac{g^4C_FN_e\exp(-mE)}{(4\pi)^2} \frac{8T^2r^2}{\pi} + \mathcal{O}(g^5) \tag{3.22}
\]

On the last row we displayed the large-distance behaviour. However, the logarithmic dependence on \( r \) (see also ref. [23]) implies that this term cannot be interpreted as a mass correction to the leading order result, but that the correction of \( \mathcal{O}(g^4) \) overtakes the correction of \( \mathcal{O}(g^2) \) for \( r \gg \pi/g^2T \), rendering the perturbative series out of control. A further issue is that the gauge dependent term, the last one within the curly brackets in eq. (3.18), does give a finite non-zero contribution to eq. (3.22). These well-known issues have lead to attempts at interpreting the self-energy insertion in a different way [24], and we now discuss these.

The idea is to treat the 1-loop self-energy as if it were an analytic function of \( k^2 \), as would be the case if there were no infrared problems. In this case, making use of the symmetry in \( k \to -k \), we can write the self-energy contribution (curly brackets in eq. (3.18)) as

\[
\psi_C^E(r) = \ldots + g^2C_F\beta \int \frac{d^3k}{(2\pi)^3} e^{ikr} \left[ \frac{1}{k^2 + m^2_E} - \frac{\Pi_{00}^E(k)}{(k^2 + m^2_E)^2} \right] - \frac{g^2C_F\pi^2}{4\pi^2 Tr} \int_{-\infty}^{\infty} dk k e^{ikr} \left[ \frac{1}{k^2 + m^2_E} - \frac{\Pi_{00}^E(imE) + (k - imE)\Pi_{00}'(imE) + \ldots}{(k^2 + m^2_E)^2} \right] \approx \ldots + \frac{g^2C_F}{4\pi Tr} \left[ 1 - \frac{\Pi_{00}^E(imE)}{2imE} \right] \exp \left\{ - \left( m_E + \frac{\Pi_{00}^E(imE)}{2imE} \right) r \right\}, \tag{3.24}
\]

where we closed the contour in the upper half plane and then resummed the mass correction into the leading order term. Inserting the values (cf. eq. (2.17))

\[
\frac{\Pi_{00}^E(imE)}{2imE} = -\frac{g^2N_cT}{8\pi} \left( \frac{1}{\epsilon_{IR}} + \ln \frac{\bar{\mu}^2}{4m_E^2} + 1 \right), \tag{3.25}
\]

\[
\frac{\Pi_{00}'(imE)}{2imE} = 0, \tag{3.26}
\]

\[\ldots \]
we note that handled this way, the dependence on the gauge parameter \( \tilde{\xi} \) disappears; however, an uncancelled infrared divergence, associated with contributions from the colour-magnetic scale, is left over. In fact the term in eq. (3.25) equals the ultraviolet matching coefficient from the scale \( m_E \) that plays a role in the approach of ref. [40] (with the only difference that a subtraction of an unresummed infrared contribution, \( g^2 N_c T/8\pi \times (1/\epsilon_{UV} - 1/\epsilon_{IR}) = 0 \), transforms \( 1/\epsilon_{IR} \) into \( 1/\epsilon_{UV} \) in that case).

The third possibility is to use the pole mass method but to “regulate” the infrared behaviour of the spatial gluon propagators by introducing a “magnetic mass” as a regulator (see, e.g., refs. [24, 25]). In this way a finite and gauge-independent result is obtained; however, it is ambiguous, because the value of the magnetic mass has no well-defined meaning.

On the non-perturbative level, we expect that the infrared problem is cured by physics at the colour-magnetic scale \( g^2 T/\pi \); this physics being that of three-dimensional confinement, it however cannot be reduced to a simple magnetic mass. Nevertheless, we could still expect exponential decay as in eq. (3.24), with a non-perturbative version of the correction in eq. (3.25), that is with the screening mass

\[
\tilde{m}_E = m_E + \frac{g^2 N_c T}{4\pi} \left( \ln \frac{m_E}{g^2 T} + c_E \right) + \mathcal{O}(g^3 T). \tag{3.27}
\]

If the mass \( \tilde{m}_E \) were dictated by the mass of the lightest gauge-invariant state, in the spirit of ref. [40], which in perturbation theory reduces to the structure of eq. (3.25), then \( c_E \approx 6.9 \) for \( N_c = 3 \) [41]. This is just a guess, however; the result can as well be gauge-dependent, given that the large distance \( r \)-dependence of the Coulomb gauge correlator appears in any case to be determined by an unphysical power-law contribution, as discussed above. Therefore we treat \( \tilde{m}_E \) as a free parameter for now.

In fig. 4 we compare the magnitude of the \( \mathcal{O}(g_E^4) \) correction with that of the \( \mathcal{O}(g_E^2) \) term, with the former one computed either as in eq. (3.22), or as in eqs. (3.24)–(3.27). It appears that if we focus on small distances, \( r \lesssim 2/T \), then the “unresummed” method, i.e. eq. (3.22), shows better convergence. Therefore in the following we mostly concentrate on this case although, as the discussions above and in fig. 4 show, the unresummed result cannot be extrapolated to large distances, \( r \gtrsim 2/T \).
Figure 4: The $O(g_4^2)$ correction to $\psi_C$ (the expression with curly brackets in eq. (3.22)) over the $O(g_2^2)$ term (the first $r$-dependent term in eq. (3.22)), labelled as the “unresummed $\psi_C$”; together with the same ratio emerging from the procedure of eqs. (3.24)–(3.27), labelled as the “resummed” method. At small distances, the unresummed method shows better apparent convergence, particularly if $\tilde{m}_E$ is small as would be preferred by fig. 5 (where $\tilde{m}_E > 2T$ would lead to stronger screening and a larger deviation from lattice data); however, the unresummed method must not be used at $r \gg 1/T$.

3.5. Summary and comparison with literature

For moderate distances, $r \ll \pi/g^2T$, we can write down a “full result” by adding up eqs. (3.15) and (3.22):

$$\ln \left( \frac{\psi_C(r)}{|\psi|^2} \right) \approx \frac{g^2 C_F \exp(-m_{\tilde{E}}r)}{4\pi Tr} \left\{ 1 + \frac{g^2}{(4\pi)^2} \left[ \frac{11N_c}{3} \left( L_b + 1 \right) - \frac{2N_f}{3} \left( L_f - 1 \right) \right] \right\}$$

$$+ \frac{g^4C_FN_c \exp(-m_{\tilde{E}}r)}{(4\pi)^2} \left[ 2 - \ln(2m_{\tilde{E}}r) - \gamma_E + e^{2m_{\tilde{E}}r} E_1(2m_{\tilde{E}}r) \right] - \frac{g^4C_FN_c \exp(-2m_{\tilde{E}}r)}{8T^2r^2}$$

$$+ \frac{g^4C_FN_f}{(4\pi)^2} \left[ \frac{1}{12T^2r^2} + \frac{\text{Li}_2(e^{-4\pi Tr})}{(2\pi Tr)^2} + \frac{1}{\pi Tr} \int_1^\infty dx \left( \frac{1}{x^2} - \frac{1}{x^4} \right) \ln \left( 1 - e^{-4\pi Trx} \right) \right]$$

$$+ \frac{g^4C_FN_f}{(4\pi)^2} \int_1^\infty dx \left( \frac{1}{x^2} - \frac{1}{x^4} \right) \ln \left( \frac{1 + e^{-2\pi Trx}}{1 - e^{-2\pi Trx}} \right) + O(g^5),$$

where $L_b, L_f$ are as defined in eq. (3.16) and $E_1, \text{Li}_2$ in eqs. (A.20), (A.21). This result is renormalization group invariant up to $O(g^5)$. For large distances, the square bracket term on the second line is to be omitted, and the exponential function on the first line is to be
Figure 5: The singlet potential in Coulomb gauge from eq. (3.28) \[ \beta V_1 \equiv -\ln(\psi_C/|\psi_P|^2) \], for \( N_f = 0 \), at \( T = 3.75 \Lambda_{\text{QCD}} \) \( (T \approx 3T_c) \), at small (left) and large (right) distances. The band corresponds to variations of the gauge coupling and \( m_E \) as explained in the caption of fig. 2. As explained in fig. 4, “unresummed” results can only be applied at short distances, “resummed” ones only at large distances. The lattice data, labelled by \( N_\tau \), is from ref. [8] (the spatial lattice size was kept fixed at \( 32^3 \)).

replaced with \( \exp(-\tilde{m}_E r) \), with some non-perturbative \( \tilde{m}_E \). The uncancelled power-law term on the third line implies that the singlet free energy dies away at large distances slower than gauge-invariant correlations.

In fig. 5 we compare eq. (3.28) with \( N_f = 0 \) lattice data from ref. [8]. The parameters have been fixed as in eq. (2.22), and also more elaborately as explained in the caption. We observe good agreement between our result and the non-perturbative data, if the unresummed form of eq. (3.22) is used at short distances, and the resummed form of eqs. (3.24)–(3.27) at large distances. (Unfortunately the latter expression involves an unknown parameter, \( \tilde{m}_E \), so the test is less stringent at large distances.) We have repeated the comparison at \( T \approx 12T_c \), and the agreement remains good, despite the band becoming narrower (cf. fig. 6). Such a nice agreement for \( \psi_C \) even at \( T \approx 3T_c \) is perhaps somewhat surprising, given that according to fig. 2 higher-order perturbative corrections to \( \psi_P \) could still to be significant in this temperature range. (Formally, \( \psi_P \) can be obtained from the \( T \)- and \( r \)-dependent part of \( \psi_C \) by setting \( r \to 0 \), cf. eqs. (2.9), (3.2), and it can indeed be observed from fig. 5(left) that some tendency towards a discrepancy starts to form in this limit.)

It is interesting to compare the present results with those in ref. [42], where the short-distance spatial correlators related to gauge-invariant scalar and pseudoscalar densities were measured. The authors observed stronger correlations than indicated by the leading-order
perturbative predictions (in the language of the static potential, this would correspond to less screening). It would be interesting to see whether pushing the perturbative determination of those observables to the same level as here would allow to make the match as good as that in fig. 5.

4. Singlet free energy in a general covariant gauge

The singlet free energy in a general covariant gauge is defined in analogy with eq. (3.1),

$$
\psi_\xi(r) = \frac{1}{N_c} \langle \text{Tr} \left[ P_r P_0 \right] \rangle \xi .
$$

(4.1)

The difference is that for the gauge field propagator we now insert

$$
G_{\mu\nu}(Q) = \frac{\delta_{\mu\nu}}{Q^2} + \frac{\xi Q_\mu Q_\nu}{Q^4} ,
$$

(4.2)
while the self-energy has the form

\[
\Pi_{\alpha\beta}(0, k) = g^2 N_c \int \{ D - 2 \frac{[k^2 + (D - 2)q_n^2]}{Q^2} + \frac{\tilde{\xi} k^2 + 2q_n^2}{Q^2(Q + K)^2} \},
\]

where \( K \equiv (0, k) \). Expanding in \( k^2 \) following the philosophy of eq. (3.14), the self-energy behaves as

\[
\Pi_{\alpha\beta}(0, k) = m_{\tilde{\xi}}^2 + g^2 \frac{k^2}{(4\pi)^2} \left\{ - \frac{N_c}{6} \left[ (10 - 3\tilde{\xi}) \left( \frac{1}{\epsilon} + L_b \right) + 6\tilde{\xi} - 2 \right] + \frac{2N_f}{3} \left( \frac{1}{\epsilon} + L_f - 1 \right) \right\} + O\left( \frac{k^4}{T^2} \right),
\]

where terms of \( O\left( \frac{k^4}{T^2} \right) \) are ultraviolet finite. The fermionic part here has a structure familiar from eq. (3.15).

While in principle we could go on as before, working out the contribution of \( O\left( \frac{k^4}{T^2} \right) \) in detail, it seems to us that the result is not particularly interesting. This can be seen already at short distances, \( r \pi T \ll 1 \), focussing on the divergences, as we will now do.

Listing only the \( 1/\epsilon \)-poles of dimensional regularization, the renormalization of the bare gauge coupling (cf. eq. (2.3)) from the first graph of fig. 3 yields the contribution

\[
\delta \psi^{(1)}_{\tilde{\xi}} = \frac{g^4 C_F}{(4\pi)^3 T r} \left[ \frac{-11N_c}{3} + \frac{2N_f}{3} \right] \left[ \frac{1}{\epsilon} + O(1) \right].
\]

The second and third graphs produce

\[
\delta \psi^{(2-3)}_{\tilde{\xi}} = \frac{g^4 C_F}{(4\pi)^3 T r} \left[ (4 - 2\tilde{\xi}) N_c \right] \left[ \frac{1}{\epsilon} + O(1) \right],
\]

while the fourth and fifth graphs are ultraviolet finite. The sixth graph yields

\[
\delta \psi^{(6)}_{\tilde{\xi}} = \frac{g^4 C_F}{(4\pi)^3 T r} \left[ \frac{3}{2} \tilde{\xi} N_c \right] \left[ \frac{1}{\epsilon} + O(1) \right],
\]

and the seventh graph, inserting the expansion in eq. (4.4), amounts to

\[
\delta \psi^{(7)}_{\tilde{\xi}} = \frac{g^4 C_F}{(4\pi)^3 T r} \left[ \left( \frac{5}{3} - \frac{\tilde{\xi}}{2} \right) N_c - \frac{2}{3} N_f \right] \left[ \frac{1}{\epsilon} + O(1) \right].
\]

Adding up, we observe that divergences do not cancel, unlike in Coulomb gauge, but that the results sum up to

\[
\psi_{\tilde{\xi}} = 1 + \frac{g^2 C_F}{4\pi T r} + \frac{g^4 C_F N_c}{(4\pi)^3 T r} (2 - \tilde{\xi}) \left[ \frac{1}{\epsilon} + O(1) \right].
\]
In principle we could “renormalize” the result by writing

$$\ln\left(\frac{\psi_r(r)}{\vert\psi_\nu\vert^2}\right) \approx Z_{\tilde{\xi}} \times \left\{ \frac{g^2 C_F}{4\pi T r} + \frac{g^4 C_F N_c}{(4\pi)^3 T r} (2 - \tilde{\xi})\mathcal{O}(1) \right\},$$  \hspace{1cm} (4.10)

$$Z_{\tilde{\xi}} = 1 + (2 - \tilde{\xi}) \frac{g^2 N_c}{(4\pi)^2 \epsilon} + \mathcal{O}(g^4),$$  \hspace{1cm} (4.11)

however the form is not that of known endpoint divergences (see, e.g., ref. [43]) which are additive, rather than “correcting” the renormalized $g^2$; in addition, the finite parts of the potential still remain gauge dependent, even at short distances $r \pi T \ll 1$, because of the $\mathcal{O}(1)$ term in eq. (4.10). Therefore the result does, in general, not match the zero-temperature potential. The situation is analogous to that in the next section, where it will be discussed somewhat more explicitly (cf. eqs. (5.7), (5.8)).

Let us point out that, compared with the conventional zero-temperature static potential where divergences do cancel [38], the only difference is with the second and third topologies in fig. 3. At zero temperature, the time direction is infinite and the Wilson loop has a finite extent within it; at finite temperature, the time direction is finite and the Polyakov lines wrap all the way around the time direction. This turns out to lead to a difference for these particular topologies.

5. Cyclic Wilson loop

5.1. Basic setup

With the objects in eqs. (2.1) and (2.2), the cyclic Wilson loop is defined as

$$\psi_W(r) \equiv \frac{1}{N_c} \langle \text{Tr} \left[ P_t W_0 P_0^\dagger W_0^\dagger \right] \rangle,$$  \hspace{1cm} (5.1)

where we made use of periodic boundary conditions (i.e. $W_\beta = W_0$). In an Abelian theory $\psi_W$ would agree with $\psi_C$ of eq. (3.1), whereas in the non-Abelian case $\psi_W$ can be viewed as a gauge-invariant “completion” of $\psi_C$.

In general, the insertion of the spacelike Wilson lines into the observable leads to a huge proliferation of graphs compared with those in fig. 3. On the other hand, the fact that both $W_0$ and $W_0^\dagger$ appear, i.e. that the (untraced) Polyakov loops are really connected by a Wilson line in the adjoint representation, also means that there are many cancellations. Even so, quite a number of diagrams remains. To simplify the task somewhat, we have restricted to special gauges in this case, namely to those where the gluon propagator has no components mixing time and space indices, i.e. $G_{0\beta}(Q) = 0$; this class includes, in particular, the Coulomb and the Feynman gauges. With these provisions, only the graphs in fig. 7 give an additional
Figure 7: The additional graphs relevant for the cyclic Wilson loop, supplementing those in fig. 3 in the class of gauges where the gluon propagator has no components mixing time and space indices, i.e. $G_{0i}(P) = 0$. Time runs vertically and space horizontally.

contribution. Choosing for convenience the coordinates so that $r$ points in the $z$-direction, the general (unresummed) result can be written as

$$
\psi_W(r) = \psi_C(r) + g^4 C_F N_c \left[ \int k e^{ik \cdot r} G_{00}(0,k) \int q \left( e^{iq \cdot r} - 1 \right) \sum q_n \frac{G_{zz}(q_n, q)}{q^2_z} 
\right.
+ \left. \int_{k,q} \left( e^{-iq \cdot r} - e^{ik \cdot r} \right) G_{00}(0,k) G_{00}(0,q) G_{zi}(0,q+k) \frac{k_i - q_i}{k_z + q_z} \right].
$$

(5.2)

where $\psi_C$ refers to the general result in eq. (3.2). It can be verified, by taking the Coulomb and Feynman gauge results for $\psi_C$ from secs. 3, 4, respectively, as well as the corresponding propagators from eqs. (3.3), (4.2), that the expression in eq. (5.2) is indeed gauge independent. (In fact, we will demonstrate the gauge independence of the QCD part of the expression around eq. (5.12) below.) It is also trivial to see that formally the new contribution in eq. (5.2) vanishes for $r = 0$, like the structures of eq. (3.2).

Inserting the Coulomb gauge propagator from eq. (3.3) and carrying out some changes of integration variables, we can write the unresummed result as

$$
\psi_W(r) = \psi_C(r) + g^4 C_F N_c \int_{k,q} \frac{e^{ik \cdot r}}{k^2} \left\{ \sum q_n \left[ \frac{e^{iq \cdot z} - 1}{q^2_z Q^2} + \frac{1}{q^2 Q^2} \left( 1 - \frac{k^2}{(k+q)^2} \right) \right] \right. 
\left. - \frac{2}{q^2 (k+q)^2} \left[ \frac{k_z - q_z}{k_z + q_z} + \frac{q^2 - k^2}{(k+q)^2} \right] \right\}.
$$

(5.3)

Now it is understood that the Coulomb gauge $\psi_C$ from eq. (3.5) is to be inserted.

### 5.2. Short-distance limit

Let us start by inspecting the new terms in eq. (5.3) at short distances, in analogy with sec. 5.2. The latter row in eq. (5.3) only contains the Matsubara zero mode and, compared with the contribution from the sum, leads for dimensional reasons to a contribution suppressed by $r T$ at small distances (once $\beta$ is factored out). The zero-temperature integrals
corresponding to the structures on the first row can be carried out,
\[ \int Q \frac{e^{iq \cdot r} - 1}{q^2 Q^2} = \frac{1}{8\pi^2} \left( \frac{1}{\epsilon} + 2 \left[ \ln \left( \frac{\mu r}{2} \right) + \gamma_E + 1 \right] \right) , \]  
(5.4)
\[ \int Q \frac{1}{q^2 Q^2} \left( 1 - \frac{k^2}{(k + q)^2} \right) = \frac{1}{8\pi^2} \left( \frac{1}{\epsilon} + \ln \frac{\mu^2}{4k^2} \right) . \]  
(5.5)

The divergences add up; given that \( \psi_C \) was finite after charge renormalization, there is nothing more available that could cancel them. We may, nevertheless, represent their effect by introducing a “renormalization factor” \( Z_W \) in analogy with eqs. (4.10), (4.11),
\[ \ln \left( \frac{\psi_W(r)}{\psi_C} \right) = Z_W g^2 C_F \beta \int_k \frac{e^{ik \cdot r}}{k^2} + O(g^4) , \]  
(5.6)
\[ Z_W = 1 + \frac{4g^2 N_c}{(4\pi)^2} + O(g^4) , \]  
(5.7)

but this does not have the form normally related to the cusps that appear in the observable \[26\] (the corresponding divergences are additive, rather than “correcting” the renormalized \( g^2 \)). In addition, there is an ambiguity concerning whether the Fourier transform in eq. (5.6) is to be understood in \( 3 - 2\epsilon \) or in 3 dimensions. Preferring therefore to write down a bare expression, we can summarize the short-distance behaviour as
\[ \psi_W(r) \approx 1 \approx \psi_C(r) + g^4 C_F \beta \int_k \frac{e^{ik \cdot r}}{k^2} \left[ 4 \left( \frac{1}{\epsilon} + \ln \frac{\mu^2}{4k^2} + 1 \right) \right] , \]  
(5.8)
where we made use of footnote \[7\] in order to rephrase all the logarithmic dependence in \( k \)-space. The expression in eq. (5.8) is gauge-independent; however, it does not go over into the zero-temperature potential at short distances.

5.3. Hard-mode contribution

In order to determine the behaviour of \( \psi_W \) at larger distances, \( r \sim 1/\pi T \), we treat the new terms in eq. (5.3) according to the strategy around eq. (3.14). Two of the sum-integrals are by now familiar, and given by eqs. (A.7), (A.15); the new one is given by eq. (A.19), and we get
\[ \left[ (\psi_W(r))_{QCD} \right]_{\text{unresummed}} = \left[ (\psi_C(r))_{QCD} \right]_{\text{unresummed}} + \frac{g^4 C_F N_c}{(4\pi)^2} \left\{ \int_k \frac{e^{ik \cdot r}}{k^2} \left[ 4 \left( \frac{1}{\epsilon} + L_b + 1 \right) \right] \right\} + \frac{g^4 C_F N_c}{(4\pi)^2} \left\{ \frac{1}{12T^2} + \frac{2Li_2(e^{-2\pi Tr})}{(2\pi Tr)^2} + \frac{1}{Tr} \int_1^\infty \frac{dx}{x^2} \ln \left( 1 - e^{-2\pi Trx} \right) \right\} + g^4 C_F N_c \times (\text{zero mode contribution}) . \]  
(5.9)
Here we already replaced the sum over the exponential integral through a simple integral representation, cf. eq. (A.22).

At short distances, $\pi T r \ll 1$, the complicated square bracket expression in eq. (5.9) goes over into
\[
\frac{g^4 C_F N_c}{(4\pi)^3 T r} \left[ 8 \ln(2\pi T r) \right].
\]  
(5.10)
Combining with the logarithmic term from the first row (with $L_0$ inserted from eq. (3.16)), we reproduce the result of eq. (5.8). At large distances, on the other hand, the power-law term in eq. (5.9) cancels against that in eq. (3.21). Therefore, apart from a disconnected part like in eq. (2.5), $\psi_W(r)$ is exponentially suppressed at large distances.

5.4. Soft-mode contribution

We denote the EQCD contribution to $\psi_W$ by $\psi_E^W$. Within EQCD, the new graphs in fig. 7 amount to an evaluation of the expectation value (see also ref. [35])
\[
\psi_E^W(r) = \psi_C^E + g^2 C_F \beta \frac{\mathcal{Y}_0^2}{N_c} \left( \text{Tr} \left[ A_0(r) W_0 A_0(0) W_0^\dagger \right] - \text{Tr} \left[ A_0(r) A_0(0) \right] \right) + \mathcal{O}(g^5),
\]  
(5.11)
where the subtraction corresponds to the part already included in $\psi_C^E$. We have introduced another "Z-factor", this time denoted by $\mathcal{Y}_0$, related to the fact that the spatial Wilson lines within EQCD might differ in normalization from those in QCD. The new contributions we are interested in are $\mathcal{O}(g^4)$, so that $Z_1$ can be set to unity and $g_E^2$ can be set to $g^2$ in the evaluation of eq. (5.11). A straightforward computation leads to
\[
\psi_E^W(r) = \psi_C^E(r) + \frac{g^2 \beta Z_1^2}{N_c} \left( \frac{\mathcal{Y}_0^2}{N_c} - 1 \right) \left[ \int_k \frac{e^{i k r}}{k^2 + m_E^2} + \frac{1}{q^2 q^2} \right] + \mathcal{O}(g^5),
\]  
(5.12)
where we have for completeness kept a general gauge parameter. It is easy to check now that the $\tilde{\xi}$-dependent part cancels against the self-energy contribution to $\psi_C^E$ from the 1st row of eq. (3.24), with the self-energy inserted from eq. (2.17), so the result is indeed gauge independent.

On the other hand, if we take the Coulomb gauge result as a starting point, then all terms in eq. (5.12) need to be kept, with the value $\tilde{\xi} = -1$. The term on the 2nd row of eq. (5.12) is factorized and contains the integral
\[
\int_q \frac{e^{i q r} - 1}{q^2 q^2} = -\frac{r}{8\pi} \left\{ \frac{1}{\epsilon_{UV}} + 2 \left[ \ln(\mu r) + \gamma_E - 1 \right] \right\},
\]  
(5.13)
multiplied by the usual $g^4 C_F N_c \int_k e^{i k \cdot r} / (k^2 + m_E^2) = g^4 C_F N_c \exp(-m_{\pi} r)/4\pi r + O(\epsilon)$ (cf. eq. (5.20)). It is important to stress that, as underlined by the notation, the divergence in eq. (5.13) has an ultraviolet origin. This can be seen, for instance, by regulating the $q_z$-integral by taking a principal value and the $q_{\perp}$-integral by introducing a mass-like regulator:

$$
\int_0^\infty \frac{d q_z}{q_z} \left( \frac{e^{i q_z r} - 1}{q_z^2} \right) \rightarrow \int_0^\infty \frac{d q_z}{2\pi} \left( \frac{e^{i q_z r} - 1}{q_z^2} \right) \int_q \frac{d^2 q_\perp}{(2\pi)^2} \frac{1}{q_\perp^2 + q_z^2 + \lambda^2} = - \int_0^r \frac{d z}{2\pi} \int_0^\infty \frac{d q_z}{q_z} \left( \frac{e^{i q_z r}}{q_z} \right) \frac{\mu^{-2\epsilon}}{4\pi} \left[ \frac{1}{\epsilon_{\text{UV}}} + \ln \frac{\mu^2}{\lambda^2 + q_z^2} + O(\epsilon) \right].
$$

(5.14)

The $q_z$-independent part of the square brackets is multiplied by

$$
\int_{-\infty}^\infty \frac{d q_z}{2\pi} \left( \frac{e^{i q_z r}}{q_z} \right) = \frac{1}{2\pi} \int_{-\infty}^\infty d z \frac{\sin z}{z} = \frac{1}{2},
$$

(5.15)

whereby we recover the divergent part of eq. (5.13). This divergence is essentially the self-energy correction related to an adjoint Wilson line [40].

As far as the other terms in eq. (5.12) go, the last one simply yields

$$
g^4 C_F N_c \int_k e^{i k \cdot r} \frac{1}{q^4} \left( \frac{1}{k + q + m_E^2} - \frac{1}{k^2 + m_E^2} \right) = \frac{g^4 C_F N_c e^{-m_{\pi} r}}{32\pi^2}.
$$

(5.16)

The remaining (middle) term requires a bit more work; some intermediate steps are given in eqs. (A.29)–(A.31). The final result can be written as

$$
\mathcal{I} = 2 \int_k e^{i k \cdot r} q_z - \frac{k_z}{q_z + k_z} \left( \frac{1}{q_z + k_z} (q + k)^2 (k^2 + m_E^2) (q^2 + m_E^2) \right)
$$

(5.17)

$$
= \frac{e^{-m_{\pi} r}}{16\pi^2 m_{\pi} r} \left\{ (1 + m_{\pi} r) [\ln(2m_{\pi} r) + \gamma_E] + (1 - m_{\pi} r) e^{2m_{\pi} r} E_1(2m_{\pi} r) \right\}.
$$

(5.18)

It may be noted that the large-distance behaviour of eq. (5.18) largely cancels against that in eq. (3.22).

To summarize, the additional contribution to $\psi^5_C$ is composed of the first new term in eq. (5.12), together with the results from eqs. (5.13), (5.16), and (5.18). At small distances the “old” part $\psi^5_C$ could be taken from its direct evaluation, eq. (3.22), while at large distances the resummed form of eqs. (3.24)–(3.27) is preferable.

It remains to fix $\gamma_0^2$ in the first new term of eq. (5.12). In order to match the behaviour on the 2nd row of eq. (5.9) in the subtraction–addition step, we need to choose

$$
\gamma_0^2 = 1 + \frac{g^2}{(4\pi)^2} \left[ 4N_c \left( \frac{1}{\epsilon} + L_0 + 1 \right) \right].
$$

(5.19)

It can be seen, however, that there is a potential ambiguity from terms of the type $O(1/\epsilon) \times O(\epsilon)$ that this multiplies. Here the $O(\epsilon)$-terms come from the massive or massless leading-
order potential,
\[
\int \frac{d^3-2k}{(2\pi)^{3-2}} \frac{e^{i k \cdot r}}{k^2 + m_E^2} = \frac{e^{-m_E r}}{4\pi r} \left\{ 1 + \epsilon \left[ \ln \frac{\mu^2}{2m_E} + \gamma_E - e^{2m_E r} E_1(2m_E r) \right] \right\}, \tag{5.20}
\]
\[
\int \frac{d^3-2k}{(2\pi)^{3-2}} \frac{e^{i k \cdot r}}{k^2} = \frac{\mu^{-2\epsilon}}{4\pi r} \left\{ 1 + 2\epsilon \left[ \ln(\bar{\mu} r) + \gamma_E \right] \right\}, \tag{5.21}
\]
where we inserted \( 1 = \mu^{-2\epsilon}[1 + \epsilon(\ln \frac{\bar{\mu}^2}{4\pi} + \gamma_E)] \) in order to fix the dimensions. It is not clear to us whether the \( O(1/\epsilon) \times O(\epsilon) \) terms from here can have physical significance.

In any case, after fixing \( \gamma_0^2 \), the terms can be added up. The complete result is not particularly transparent, and may be ambiguous as just discussed, so we do not write it down explicitly; it suffices to say that the sum can be expressed as
\[
\ln \left( \frac{\psi \psi}{|\psi|^2} \right)_{\text{resummed}} = g_{\text{DR}} \left( \frac{1}{\epsilon}, \frac{\bar{\mu}}{T}, rT \right) C_F \exp\left( -m_E r \right) - \frac{g^4 C_F N_c \exp(-2m_E r)}{(4\pi)^2} \frac{8T^2 r^2}{\ln(\bar{\mu} r) + \gamma_E}. \tag{5.22}
\]
The function \( g_{\text{DR}} \), in which the complications are hidden, does have a simple expression in certain limits, however, and these will be discussed in the next section.

### 5.5. Summary and comparison with literature
Adding up the relevant parts of eqs. (5.28), (5.9) and (5.22), we finally get
\[
\ln \left( \frac{\psi \psi}{|\psi|^2} \right) \approx g_{\text{DR}} \left( \frac{1}{\epsilon}, \frac{\bar{\mu}}{T}, rT \right) C_F \exp\left( -m_E r \right) - \frac{g^4 C_F N_c \exp(-2m_E r)}{(4\pi)^2} \frac{8T^2 r^2}{\ln(\bar{\mu} r) + \gamma_E}
\]
\[
+ g^4 C_F N_t \left\{ \frac{2\text{Li}_2(e^{-2\pi r T}) + \text{Li}_2(e^{-4\pi r T})}{(2\pi T)^2} \right\}
\]
\[
+ \frac{1}{\pi T} r \int_1^\infty dx \left[ \frac{1}{x^2} \ln \left( 1 - e^{-2\pi r T x} \right) + \left( \frac{1}{x^2} - \frac{1}{2x^4} \right) \ln \left( 1 - e^{-4\pi r T x} \right) \right]
\]
\[
+ g^4 C_F N_t \left[ \frac{1}{2\pi T} r \int_1^\infty dx \left( \frac{1}{x^2} - \frac{1}{x^4} \right) \ln \left( \frac{1 + e^{-2\pi r T x}}{1 - e^{-2\pi r T x}} \right) \right] + O(g^5). \tag{5.23}
\]
At small distances, \( m_E r \ll 1 \), the dominant term of the coefficient function \( g_{\text{DR}} \), defined in eq. (5.22), is
\[
g_{\text{DR}} \left( \frac{1}{\epsilon}, \frac{\bar{\mu}}{T}, rT \right) \approx \frac{g^2}{(4\pi)^2} \left[ 4N_c \left( \frac{1}{\epsilon} + \ln \frac{\mu^2}{T^2} + O(1) \right) \right] \tag{5.24}
\]

Though we have not carried out the computation, we could expect that in lattice regularization the corresponding structure goes over into
\[
G_{\text{lat}} \left( \frac{1}{aT}, rT \right) \approx g^2 \left\{ 1 + \frac{g^2}{(4\pi)^2} \left[ 4N_c \left( \frac{1}{a^2 T^2} + O(1) \right) \right] \right\}, \tag{5.25}
\]
where $g^2$ is a suitably defined renormalized coupling. Therefore, it would appear that $G_{\text{lat}}$ diverges in the continuum limit. On the other hand, at large distances, $m_{\text{ER}} r \gg 1$, the dominant term of the coefficient function is

$$G_{\text{DR}} \left( \frac{1}{\epsilon}, \frac{\mu}{T}, rT \right) \overset{m_{\text{ER}} \gg 1}{\approx} g^2 \left\{ 1 - \frac{g^2 N_c T}{8\pi} \left[ r \left( \frac{1}{\epsilon_{\text{UV}}} + \ln \frac{\mu^2}{m_{\text{E}}^2} + \mathcal{O}(1) \right) \right] \right\}, \quad (5.26)$$

where possible logarithms of $m_{\text{ER}}$ are also included in $\mathcal{O}(1)$ (whether such logarithms appear is related to the ambiguities mentioned after eqs. (5.20), (5.21)). Apart from the said logarithms, eq. (5.26) can be accounted for by a mass correction,

$$\exp(-m_{\text{ER}} r) \rightarrow \exp(-\bar{m}_{\text{ER}} r), \quad (5.27)$$

where $\bar{m}_{\text{E}}$ contains a logarithmic ultraviolet divergence:

$$\bar{m}_{\text{E,DR}} = m_{\text{E}} + \frac{g^2 N_c T}{8\pi} \left( \frac{1}{\epsilon_{\text{UV}}} + \ln \frac{\mu^2}{m_{\text{E}}^2} + \mathcal{O}(1) \right). \quad (5.28)$$

A naive transliteration to lattice yields

$$\bar{m}_{\text{E, lat}} = m_{\text{E}} + \frac{g^2 N_c T}{8\pi} \left( \ln \frac{1}{a^2 m_{\text{E}}^2} + \mathcal{O}(1) \right). \quad (5.29)$$

So, Debye screening grows logarithmically as the continuum limit is approached, much like in the numerical study of ref. [41], and the exponential function $\exp(-\bar{m}_{\text{E, lat}} r)$ decreases.

To summarize, we believe that the leading term, $\sim G_{\text{lat}} \exp(-m_{\text{ER}} r)$, is a very ultraviolet sensitive and pathological function of the lattice spacing; at large distances, it appears to extrapolate towards zero in the continuum limit, while at short distances it appears to explode. In fact it might look somewhat like a delta-function.

Concerning whether the same happens also in the other terms, in particular in the one with $\exp(-2m_{\text{ER}} r)$, a higher order computation would be required to see any perturbative indications. We find it conceivable, though, that some of the terms could also remain finite, representing a coupling to the gauge-invariant channel of the traced Polyakov loop,

$$\psi_T(r) \equiv \frac{1}{N_c^2} \langle \text{Tr} \left[ P_r \right] \text{Tr} \left[ P_0^\dagger \right] \rangle. \quad (5.30)$$

This behaves as

$$\ln \left( \frac{\psi_T(r)}{|\psi_T|^2} \right) \approx \frac{g^4 C_F}{(4\pi)^2 N_c} \frac{\exp(-2m_{\text{ER}} r)}{4T^2 r^2} + \mathcal{O}(g^5) \quad (5.31)$$

at the order of our computation; the argument of the large-distance exponential fall-off has recently been determined at next-to-leading order in ref. [30]. (A more precise analysis of this correlator at short distances has been undertaken in ref. [28].)
A numerical sketch of eq. (5.23), obtained by omitting the regularization dependent leading term altogether, and including two versions illustrating the importance of the remaining EQCD contribution, is given in fig. 8. The main “prediction” we can make is that in the continuum limit the result should be much closer to zero (i.e. much more strongly screened) than for the observable in fig. 3. Comparing with fig. 5(left), we see significantly less correlation (stronger screening).

6. Conclusions

The purpose of this paper has been to compute, within the weak-coupling expansion in continuum, a number of Polyakov loop related observables that have been popular in the context of phenomenological studies related to the fate of heavy quarkonium at high temperatures. Our main findings can be summarized as follows.

At large distances, many of the observables considered go over to a gauge-independent
disconnected contribution, the square of the expectation value of a single Polyakov loop. We
have recomputed this quantity at $O(\alpha_s^2)$ and found a result different from the classic one in
the literature (we believe that ours is the correct one). Numerically the difference is very
small and does not improve on the match to lattice data; for that, inclusion of higher order
terms in the weak-coupling series would probably be needed.

The singlet free energy in Coulomb gauge, defined by eq. (3.1) and denoted by $\psi_C$, is ultra-
violet finite without any additional renormalization factors in dimensional regularization at
order $O(\alpha_s^2)$, and agrees at small distances with the gauge-independent zero-temperature po-
tential $\psi_C$. At large distances, however, its behaviour is not determined by Debye screening,
but by a power-law contribution, which in our view is related to gauge fixing. In principle
this means that the Coulomb gauge potential could be more binding than would physically be
expected. On the other hand, at intermediate distances the power-law term has no dramatic
effect. In any case, our result does agree with lattice data surprisingly well (cf. fig. 5), even at
low temperatures; a higher-order perturbative computation could perhaps tell whether this is
an accident or a robust characteristic of thermal correlations at non-zero spatial separations.

The singlet free energy in a general covariant gauge contains ultraviolet divergences and
remains gauge dependent even at short distances, where it in general does not match the
zero-temperature potential.

The cyclic Wilson loop is, by construction, extracted from a gauge-invariant quantity, and
indeed the gauge-fixing related power-law term affecting $\psi_C$ cancels out. However, the be-
haviour of this observable does not match the zero-temperature potential at short distances.
Moreover, at large distances it is not sensitive to physically Debye-screened one-gluon ex-
change, but rather develops a logarithmically ultraviolet divergent screening mass. Neverthe-
less, there could remain some non-trivial higher-order $r$-dependence in the continuum limit,
either with a screening mass $\sim 2m_Q$ or with $\sim 2\pi T$.

Our findings underline the pitfalls that exist in generic potential model studies of quarko-
nium physics at finite temperatures. On the other hand, it can be argued that parametrically
only the range $rT \lessgtr 1$ is relevant for quarkonium melting [46], so problems that are related
to the asymptotic large-distance behaviour may not be that important from the pragmatic
point of view. With this philosophy the Coulomb gauge singlet free energy seems like the
least misleading quantity to use, if it is assigned the meaning of the real part of the proper
potential [6]. At the same time, it appears that in this range perturbation theory alone
may yield a fairly good description of the system, so that one could just as well resort to
gauge independent perturbative quarkonium spectral function determinations such as those
in ref. [37].

\footnote{It is interesting to note, though, that ours is not the first study suggesting the existence of power-law
terms in heavy-quark related observables, cf. e.g. ref. [45].}
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Appendix A. Sum-integrals and integrals

In this appendix we list, for completeness, the results for the four-dimensional sum-integrals and the three-dimensional integrals that play a role in our computation. The theory is regularized dimensionally, with $D = 4 - 2\epsilon$, and terms of $O(\epsilon)$ are omitted. The dependence on the scale parameter, $\bar{\mu}$, appears in most cases through logarithmic factors which are denoted by $L_b, L_f$ and have been defined in eq. (3.16). A prime in the sum-integral symbol indicates that the Matsubara zero mode is left out; when the summation-integration variable is in curly brackets, the Matsubara frequencies are fermionic. The four-vector denoted by $K$ is by definition purely spacelike,

$$K \equiv (0, k). \quad (A.1)$$

With this notation, the following sum-integrals can be determined ($Q^2 \equiv q_n^2 + q^2$):

$$\int_Q Q_{\mu} Q_{\nu} \frac{Q^4}{Q^4} = -\frac{T^2}{24} \left(\delta_{\mu0}\delta_{\nu0} - \delta_{\mu i}\delta_{\nu j}\delta_{ij}\right), \quad (A.2)$$

$$\int_{\{Q\}} Q_{\mu} Q_{\nu} \frac{Q^4}{Q^4} = \frac{T^2}{48} \left(\delta_{\mu0}\delta_{\nu0} - \delta_{\mu i}\delta_{\nu j}\delta_{ij}\right), \quad (A.3)$$

$$\int_Q' Q_{\mu} Q_{\nu} \frac{Q^6}{Q^4} = \frac{1}{(4\pi)^2} \left\{\delta_{\mu 0}\delta_{\nu 0}\left(\frac{1}{4\epsilon} + \frac{L_b}{4} + \frac{1}{2}\right) + \delta_{\mu i}\delta_{\nu j}\delta_{ij}\left(\frac{1}{4\epsilon} + \frac{L_b}{4}\right)\right\}, \quad (A.4)$$

$$\int_Q' \frac{1}{Q^4} = \frac{1}{(4\pi)^2} \left(\frac{1}{\epsilon} + L_b\right), \quad (A.5)$$

$$\int_Q' \frac{1}{q_n^2 Q^2} = -\frac{2}{(4\pi)^2} \left(\frac{1}{\epsilon} + L_b + 2\right), \quad (A.6)$$

$$\int_Q' \frac{1}{q^2 Q^2} = \frac{2}{(4\pi)^2} \left(\frac{1}{\epsilon} + L_b + 2\right), \quad (A.7)$$
\[
\begin{align*}
\sum_{q_n} \int_{k,q} \frac{1}{q_n^2(q_n^2 + k^2)Q^2} &= -\frac{1}{(4\pi)^2}, & \text{(A.8)} \\
\sum_{q_n} \int_{k,q} \frac{1}{k^2Q^2(Q + K)^2} &= -\frac{1}{(4\pi)^2} \left( \frac{1}{4\epsilon} + \ln \frac{\bar{\mu}}{2T} + \frac{1}{2} \right), & \text{(A.9)} \\
\sum_{q_n} \int_{k,q} q_n^2 k^4 Q^2(Q + K)^2 &= \frac{1}{(4\pi)^2} \left( \frac{1}{8} \right), & \text{(A.10)} \\
\sum_{\{q_n\}} \int_{k,q} \frac{1}{k^2Q^2(Q + K)^2} &= -\frac{1}{(4\pi)^2} \left( \ln 2 \right), & \text{(A.11)} \\
\sum_{\{q_n\}} \int_{k,q} q_n^2 \frac{1}{k^4Q^2(Q + K)^2} &= \mathcal{O}(\epsilon), & \text{(A.12)}
\end{align*}
\]

\[
\begin{align*}
\int_k \frac{e^{i k \cdot r}}{k^2} \sum_{q_n} \int_q \frac{1}{Q^2(Q + K)^2} &= \beta \int_k \frac{e^{i k \cdot r}}{k^2} \left( \frac{1}{\epsilon} + L_b \right) \\
&+ \frac{1}{8\pi^2} \left[ \ln(1 - e^{-4\pi T r}) + \sum_{n=1}^{\infty} E_1(4\pi T r n) \right], & \text{(A.13)} \\
\int_k \frac{e^{i k \cdot r}}{k^4} \sum_{q_n} \int_q (D - 2) q_n^2 \frac{1}{Q^2(Q + K)^2} &= \beta \int_k \frac{e^{i k \cdot r}}{k^4} \left( -\frac{T^2}{12} \right) + \beta \int_k \frac{e^{i k \cdot r}}{k^2} \left( \frac{1}{(4\pi)^2} \left( \frac{1}{6\epsilon} - \frac{L_b}{6} - \frac{1}{6} \right) \right) \\
&+ \frac{1}{96\pi^2} \left[ -\ln(1 - e^{-4\pi T r}) + \frac{1 + e^{4\pi T r}}{(e^{4\pi T r} - 1)^2} - \sum_{n=1}^{\infty} (4\pi T r n)^2 E_1(4\pi T r n) \right], & \text{(A.14)} \\
\int_k \frac{e^{i k \cdot r}}{k^4} \sum_{q_n} \int_q q^2 Q^2(q + k)^2 &= \frac{1}{(4\pi T r)^2} \left[ \frac{1}{12} - \text{Li}_2(e^{-2\pi T r}) \right], & \text{(A.15)} \\
\int_k \frac{e^{i k \cdot r}}{k^2} \sum_{\{q_n\}} \int_q \frac{1}{Q^2(Q + K)^2} &= \beta \int_k \frac{e^{i k \cdot r}}{k^2} \left( \frac{1}{\epsilon} + L_f \right) \\
&+ \frac{1}{8\pi^2} \left[ \frac{1}{4\pi T r} \ln \frac{1 - e^{-2\pi T r}}{1 + e^{-2\pi T r}} + \sum_{n=1}^{\infty} E_1(2\pi T r \times (2n - 1)) \right], & \text{(A.17)} \\
\int_k \frac{e^{i k \cdot r}}{k^4} \sum_{\{q_n\}} \int_q q_n^2 \frac{1}{Q^2(Q + K)^2} &= \beta \int_k \frac{e^{i k \cdot r}}{k^4} \left( \frac{T^2}{48} \right) + \beta \int_k \frac{e^{i k \cdot r}}{k^2} \left( -\frac{1}{12\epsilon} - \frac{L_b}{12} - \frac{1}{6} \right) \\
&- \frac{1}{96\pi^2} \left[ \frac{1}{4\pi T r} \ln \frac{1 - e^{-2\pi T r}}{1 + e^{-2\pi T r}} + \frac{e^{2\pi T r}}{2} \left( \frac{1}{e^{4\pi T r} - 1} - 2\pi T r \frac{e^{4\pi T r} + 1}{(e^{4\pi T r} - 1)^2} \right) \\
&+ \frac{1}{2} \sum_{n=1}^{\infty} \left[ 2\pi T r \times (2n - 1) \right]^2 E_1(2\pi T r \times (2n - 1)) \right], & \text{(A.18)}
\end{align*}
\]
\[
\sum_{q_n} \int_q \frac{e^{iqr} - 1}{q_n^2 Q^2} = \frac{\beta}{8\pi^2} \left[ \frac{1}{\epsilon} + L_b + 2\ln \left(1 - e^{-2\pi T r}\right) + 4\pi T r \sum_{n=1}^{\infty} E_1 \left(2\pi T n\right) \right]. \tag{A.19}
\]

Here
\[
E_1(z) \equiv \int_z^{\infty} \frac{e^{-t}}{t} ; \quad E_1(z) \approx \frac{1}{z} e^{-z} ; \quad E_1(z) \approx \ln \frac{1}{z} - \gamma_E \tag{A.20}
\]
is an exponential integral \((E_1(z) = -
\text{Ei}(-z) \text{ for } z > 0)\), and
\[
\text{Li}_2(z) \equiv -\int_0^z \frac{\ln(1 - t)}{t} = \sum_{n=1}^{\infty} \frac{z^n}{n^2} \tag{A.21}
\]
is a dilogarithm. Where not shown explicitly, the fermionic sum-integrals are obtained from the corresponding bosonic ones by replacing \(L_b \rightarrow L_f\).

In eqs. \((A.13)-(A.19)\), various sums over the exponential integral appear. All of these can be transformed into an integral representation:
\[
\sum_{n=1}^{\infty} E_1(\alpha n) = -\frac{\ln(1 - e^{-\alpha})}{\alpha} + \frac{1}{\alpha} \int_1^{\infty} \frac{dx}{x^2} \ln \left(1 - e^{-\alpha x}\right) , \tag{A.22}
\]
\[
\sum_{n=1}^{\infty} (\alpha n)^2 E_1(\alpha n) = -\frac{2\ln(1 - e^{-\alpha})}{\alpha} + \frac{1 + e^\alpha (\alpha - 1)}{(e^\alpha - 1)^2} + \frac{6}{\alpha} \int_1^{\infty} \frac{dx}{x^4} \ln \left(1 - e^{-\alpha x}\right) , \tag{A.23}
\]
\[
\sum_{n \in \mathbb{N}_{\text{odd}}} E_1(\alpha n) = \frac{1}{2\alpha} \left[ \ln \frac{1 + e^{-\alpha}}{1 - e^{-\alpha}} - \int_1^{\infty} \frac{dx}{x^2} \ln \left(1 + e^{-\alpha x}\right) \right] , \tag{A.24}
\]
\[
\sum_{n \in \mathbb{N}_{\text{odd}}} (\alpha n)^2 E_1(\alpha n) = \frac{1}{\alpha} \ln \frac{1 + e^{-\alpha}}{1 - e^{-\alpha}} + e^\alpha \left[ -\frac{1}{e^{2\alpha} - 1} + \frac{\alpha (e^{2\alpha} + 1)}{(e^{2\alpha} - 1)^2} \right] - \frac{3}{\alpha} \int_1^{\infty} \frac{dx}{x^4} \ln \frac{1 + e^{-\alpha x}}{1 - e^{-\alpha x}} . \tag{A.25}
\]

Within EQCD, the following integrals can be worked out:
\[
\int_{k,q} \frac{1}{(k^2 + m_E^2)^2(q^2 + m_E^2)} = \frac{1}{(4\pi)^2} \left( -\frac{1}{2} \right) , \tag{A.26}
\]
\[
\int_{k,q} \frac{1}{(k^2 + m_E^2)q^2[(q + k)^2 + m_E^2]} = \frac{1}{(4\pi)^2} \left( \frac{1}{4} + \ln \frac{\bar{\mu}}{2m_E} + \frac{1}{2} \right) , \tag{A.27}
\]
\[
\int_{k,q} \frac{m_E^2}{(k^2 + m_E^2)^2q^2[(q + k)^2 + m_E^2]} = \frac{1}{(4\pi)^2} \left( \frac{1}{4} \right) . \tag{A.28}
\]

Finally, we consider the EQCD-integral defined in eq. \((5.17)\). In fact, it is useful to re-start with manifestly infrared safe integration variables like in eq. \((5.2)\), and then the integral can
be expressed as

\[
\mathcal{I} = \int_{k,q} \left( e^{ikr} - e^{-iqr} \right) \frac{q_z - k_z}{q_z + k_z (q + k)^2 (k^2 + m^2_q) (q^2 + m^2_E)} \frac{1}{q_z + k_z (q + k)^2 (k^2 + m^2_q) (q^2 + m^2_E)}
\]

\[
= \int_{k,q} e^{ikr} \left( 1 - e^{iqr} \right) \frac{q_z + 2k_z}{q_z} \frac{1}{q^2 (k^2 + m^2_E) [(q + k)^2 + m^2_E]}
\]

\[
= 2 \int_{k} \frac{e^{ikr}}{k^2 + m^2_E} \int_{q} \left\{ \frac{1}{q_z} q^2 [(q + k)^2 + m^2_E] \right\}
\]

\[
= 2 \int_{k} \frac{e^{ikr}}{k^2 + m^2_E} \left[ \int_{q} \frac{1}{q^2 [(q + k)^2 + m^2_E]} + \int_{q} \left\{ \frac{1}{q_z} \frac{1}{q^2 [(q + k)^2 + m^2_E]} \right\} \right].
\]

(A.29)

Here we first substituted \( q \rightarrow -k - q \); the numerator vanishes for \( q_z = 0 \) so the integrand is regular. Subsequently we changed integration variables as \( k \rightarrow k - q \) and \( q \rightarrow -q \) in the second term, introducing a principal value to regulate the potential divergence at \( q_z = 0 \).

The second term on the last line can be written as

\[
\int_{q} \left\{ \frac{1}{q_z} \frac{1}{q^2 [(q + k)^2 + m^2_E]} \right\}
\]

\[
= \frac{1}{2} \int_{q} \frac{1}{q_z} \frac{1}{q^2} \left[ \frac{1}{(q + k)^2 + (k_z + q_z)^2 + m^2} - \frac{1}{(q + k)^2 + (k_z - q_z)^2 + m^2} \right]
\]

\[
= -2k_z \int_{q} \frac{1}{q^2 [(q + k)^2 + (k_z + q_z)^2 + m^2_E][m^2_E + (k_z - q_z)^2 + m^2_E]}
\]

\[
= -\frac{1}{4\pi} \frac{1}{k^2 + m^2_E} \arctan \frac{k_z}{m_E},
\]

(A.30)

where in the last step the integration could be carried out with the help of a Feynman parameter. Combining with the first term of the last line of eq. (A.29), we get

\[
\mathcal{I} = 2 \int_{k} \frac{e^{ikr}}{k^2 + m^2_E} \left[ \frac{1}{4\pi k} \arctan \frac{k}{m_E} - \frac{1}{4\pi} \frac{2k_z}{k^2 + m^2_E} \arctan \frac{k_z}{m_E} \right]
\]

\[
= \frac{e^{-mr}}{16\pi^2 m_E r} \left\{ (1 + m_E r) \left[ \ln(2m_E r) + \gamma_E \right] + (1 - m_E r) e^{2m_E r} E_1(2m_E r) \right\}, \quad (A.31)
\]

where we proceeded as in connection with eq. (3.22).

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