Online Kernel CUSUM for Change-Point Detection

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Abstract

We propose an efficient online kernel Cumulative Sum (CUSUM) method for change-point detection that utilizes the maximum over a set of kernel statistics to account for the unknown change-point location. Our approach exhibits increased sensitivity to small changes compared to existing methods, such as the Scan-B statistic, which corresponds to a non-parametric Shewhart chart-type procedure. We provide accurate analytic approximations for two key performance metrics: the Average Run Length (ARL) and Expected Detection Delay (EDD), which enable us to establish an optimal window length on the order of the logarithm of ARL to ensure minimal power loss relative to an oracle procedure with infinite memory. Such a finding parallels the classic result for window-limited Generalized Likelihood Ratio (GLR) procedure in parametric change-point detection literature. Moreover, we introduce a recursive calculation procedure for detection statistics to ensure constant computational and memory complexity, which is essential for online procedures. Through extensive experiments on both simulated and real data, we demonstrate the competitive performance of our method and validate our theoretical results.

Keywords: Change-point detection; Cumulative Sum; Kernel method; Maximum Mean Discrepancy; Online algorithm.

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1 Introduction

Online change-point detection is a fundamental and classic problem in statistics and related fields. The goal is to detect a change in the underlying data distribution as quickly as possible after the change has occurred while maintaining a false alarm constraint. Traditional approaches are based on parametric models that detect changes in distribution parameters, such as the Shewhart chart [Shewhart, 1925], Cumulative Sum (CUSUM) [Page, 1954], and Shiryaev-Roberts (SR) [Shiryaev, 1963]. For comprehensive and systematic reviews of parametric change-point detection, readers are referred to Basseville et al. [1993], Siegmund [2013], Tartakovsky et al. [2014], Xie et al. [2021].

In modern applications, especially in high-dimensional data settings, specifying the exact probability distribution can be challenging, necessitating non-parametric and distribution-free methods. Kernel-based methods have gained significant attention due to their flexibility and wide applicability. Notably, the kernel Maximum Mean Discrepancy (MMD) [Gretton et al., 2006, Smola et al., 2007, Gretton et al., 2012a] and its applications have been extensively studied [Muandet et al., 2017]. As non-parametric methods, kernel-based approaches do not require distributional assumptions and allow flexible kernel choices.

A significant focus in kernel-based change-point detection has been placed on the offline setting, which involves detecting and locating change-points retrospectively. Offline change-point detection procedures typically involve a series of two-sample tests for samples with candidate change-point locations. Notable examples of such procedures include the maximum Kernel Fisher Discriminant Ratio-based procedure [Harchaoui et al., 2008], the kernel MMD-based Scan $B$-procedure [Li et al., 2019], and a recent kernel-based scan procedure designed to enhance detection power in high-dimensional settings [Song and Chen, 2022]. In contrast, online change-point detection seeks to detect changes as soon as possible using sequential data. Existing online kernel-based change-point detection methods have primarily focused on the Shewhart chart-type procedure, such as Li et al. [2019]. The Shewhart chart-type procedure is distinct from the CUSUM-type procedure, which relies on CUSUM recursion and is generally considered more sensitive to detecting small changes [Xie et al., 2021]. The CUSUM-type procedure has yet to be developed for kernel-based methods due to increased computational and memory costs as the time horizon expands. A summary of existing kernel-based online change-point detection methods can be found in Table 1, while a comprehensive literature survey is presented in Section 1.1.

Table 1: A summary of existing online kernel change-point detection procedures.

| Detection procedure type | Existing literature |
|--------------------------|---------------------|
| Shewhart chart           | Huang et al. [2014], Chang et al. [2019]; Li et al. [2019], Bouchikhi et al. [2019]; Cobb et al. [2022] |
| CUSUM                    | Flynn and Yoo [2019] |
In this study, we develop a kernel-based CUSUM procedure for online change-point detection, referred to as the Online Kernel CUSUM. This method utilizes a set of possible change-point locations and creates a self-normalized kernel test statistic for each hypothesized change-point location by comparing the distribution of the assumed post-change samples with the reference pre-change samples. The detection statistic selects the maximum from this set of detection statistics with different block sizes, accounting for the unknown change-point location. The main difference between the Shewhart chart-type procedures (e.g., the Shewhart chart-type Scan B procedure Li et al. [2019]) and the proposed CUSUM-type procedure is the search for the unknown change-point location, leading to improved performance in detecting small changes for the CUSUM-type procedure. Importantly, this modification proves to be significant both empirically and theoretically. It allows for accurate analytic approximations for the Average Run Length (ARL), assisting in calibrating the procedure to control false alarms, and the Expected Detection Delay (EDD) related to detection power. In addition, the optimal window length reflects the minimal computational and memory cost necessary for our procedure to achieve similar performance as the oracle procedure with infinite memory. The EDD analysis is essential for determining the optimal window length; however, such analysis is not present in prior kernel-based online change-point detection work (e.g., Scan B procedure Li et al. [2019]). In our work, we find that the optimal window length is on the order of the logarithm of ARL, consistent with the classic parametric result for the window-limited Generalized Likelihood Ratio (GLR) procedure [Lai, 1995]. In addition to performance improvement, we present a recursive implementation of our procedure to achieve a constant computational and memory cost at each time step, which is essential in the online setting. We demonstrate the performance gain of our proposed approach using simulation and real-data experiments.

1.1 Literature

Kernel methods in change-point detection. An early work on kernel change-point detection is Harchaoui and Cappé [2007], where they incorporated kernels into dynamic programming and developed an offline multiple change-point estimation method. This method was extended by Arlot et al. [2019] to handle unknown number of change-points via a model selection penalty; however, its performance heavily depends on the penalty hyperparameter and it is difficult to control the type I error in practice. Later on, Garreau and Arlot [2018] performed non-asymptotic analysis to show the multiple change-point estimations by model selection [Arlot et al., 2019] can estimate the correct number of change-points with high probability and locate them at the optimal rate. The problem of calibrating detection procedures also exists for maximum Kernel Fisher Discriminant Ratio based approach [Harchaoui et al., 2008], where they applied the bootstrap resampling method to compute the detection threshold. In addition, Bouchikhi et al. [2019], Ferrari et al. [2023] considered a kernel density ratio [Huang et al., 2006, Nguyen et al., 2010] based Shewhart chart-type method for online change-point detection.

Kernel MMD and its application in change-point detection. Gretton et al. [2006], Smola et al.
[2007], Gretton et al. [2012a] proposed to measure the distance (or discrepancy) between distributions via the Reproducing Kernel Hilbert space embedding of distributions and developed the well-known MMD statistic. Since then, such kernel MMD-based statistical tests have become popular due to their wide range of applicability to arbitrary domains, ranging from anomaly detection [Zou et al., 2014] model criticism and goodness-of-fit test [Lloyd and Ghahramani, 2015] and the optimality of such kernel-based tests [Li and Yuan, 2019, Balasubramanian et al., 2021], approximate bayesian computation [Park et al., 2016], transfer learning [Van Opbroek et al., 2018, Yang et al., 2019], distributionally robust optimization [Staib and Jegelka, 2019], robust hypothesis testing [Sun and Zou, 2021], optimal sub-sampling [Dwivedi and Mackey, 2021, 2022] as well as change-point detection.

In offline kernel MMD-based change-point detection, there are many other notable contributions apart from aforementioned work such as Li et al. [2019], including Truong et al. [2019], who developed a greedy single change-point estimation procedure, Jones et al. [2021], who developed a multiple change-point estimation method by incorporating MMD in the least-squares framework developed by Harchaoui and Cappé [2007] and leveraging Nyström kernel approximation [Williams and Seeger, 2000] to achieve linear complexity, and so on.

Kernel MMD-based online change-point detection procedures typically test for changes between a window of reference data and a window of sequential data. Generally speaking, there are three types of procedures in online change-point detection, i.e., Shewhart chart, CUSUM, and SR [Xie et al., 2021]. Recently, there has been much effort focusing on kernel MMD-based online Shewhart chart-type procedures, which consider fixed-window scan statistics. In addition to the Scan $B$-procedure [Li et al., 2019] introduced above, notable contributions include Huang et al. [2014], who first introduced MMD to online change-point detection and applied bootstrap to calibrate the detection; Chang et al. [2019], who leveraged neural network-based data-driven kernel selection [Gretton et al., 2012b] to boost detection power; Cobb et al. [2022], who considered a memoryless geometric distribution of the stopping time to control the false alarm rate and calibrated the detection via bootstrap.

However, the fixed-length scanning window of Shewhart chart-type procedures fails to account for the change-point location, which may decrease the detection power since the window of sequential data could contain samples from pre-change (in-control) distribution. Cobb et al. [2022] indeed tackled this problem by considering a time-varying threshold, but such approach was a heuristic and did not have strong theoretical guarantee. A much more sensible approach is CUSUM-type procedure, but such procedures are rarely considered in kernel MMD online change-point detection, potentially due to its higher computational and memory cost compared to Shewhart chart-type procedures. One notable contribution on this aspect is a recent work by Flynn and Yoo [2019], who proposed a Kernel CUSUM procedure by replacing the likelihood ratio in classic CUSUM with the linear-time MMD statistic and derived bounds for both ARL and EDD. Unfortunately, such ARL bound fails to help calibrate the detection in practice. In this work, we fill this void by presenting a kernel MMD-based CUSUM-type procedure, explicitly characterizing its important statistical properties, i.e., ARL and EDD, and proving its optimal window length to give a memory efficient procedure yet maintain the detection power.
2 Methodology

In this section, we will introduce the online change-point detection problem, review some classic parametric detection procedures, and present our proposed procedure.

2.1 Preliminaries

Consider independent and identically distributed (i.i.d.) reference samples from the domain $\mathcal{X}$ (usually taken to be $\mathbb{R}^d$) following an unknown pre-change distribution with density $p$:

$$X_1, \ldots, X_M \overset{i.i.d.}\sim p.$$ 

At time $t$, we receive i.i.d. data $Y_1, \ldots, Y_t$ sequentially. The online change-point detection problem tests the null hypothesis

$$H_0 : Y_1, \ldots, Y_t \overset{i.i.d.}\sim p,$$

against the alternative hypothesis

$$H_1 : \exists \kappa < t, Y_1, \ldots, Y_\kappa \overset{i.i.d.}\sim p, Y_{\kappa+1}, \ldots, Y_t \overset{i.i.d.}\sim q,$$

where $q$, distinct from $p$, is the density of an unknown post-change distribution and $\kappa$ is the change-point location.

The objective in online change-point detection is to detect the change as soon as possible, subject to the constraint that the ARL is greater than a lower bound $\gamma > 0$ when there is no change. The ARL is typically controlled by choosing a suitable detection threshold $b$, and the EDD is studied under the ARL constraint. Under the same ARL constraint, a smaller EDD corresponds to a better detection procedure.

We use standard notations here. Under the null hypothesis, we use $\mathbb{P}_\infty$ and $\mathbb{E}_\infty$ to denote the probability and expectation when there is no change. Under the alternative hypothesis, when there is a change at time $\kappa$, $\kappa = 0, 1, \ldots$, we use $\mathbb{P}_\kappa$ and $\mathbb{E}_\kappa$ to denote the probability and expectation in this case. In particular, $\kappa = 0$ denotes an immediate change. We denote $1_d = (1, \ldots, 1)^T \in \mathbb{R}^d$, $0_d = (0, \ldots, 0)^T \in \mathbb{R}^d$ and $I_d \in \mathbb{R}^{d \times d}$ as the identity matrix, where superscript $^T$ standards for the vector or matrix transpose. For integers $0 < m \leq n$, let $[m : n] = \{m, \ldots, n\}$; for $m = 1$, $[n] = \{1, \ldots, n\}$. We denote $a \wedge b = \min\{a, b\}$, $a \vee b = \max\{a, b\}$ and $(a)^+ = \max\{a, 0\}$. For asymptotic notations, we adopt standard definitions: $f(m) = o(g(m))$ means for all $c > 0$ there exists $k > 0$ such that $0 \leq f(m) < cg(m)$ for all $m \geq k$; $f(m) = \mathcal{O}(g(m))$ means there exist positive constants $c$ and $k$, such that $0 \leq f(m) \leq cg(m)$ for all $m \geq k$; $f(m) \sim g(m)$ means there exist positive constants $c$ and $k$, such that $0 \leq cg(m) \leq f(m)$ for all $m \geq k$ and at the same time $f(m) = \mathcal{O}(g(m))$. 

5
2.2 Classic Parametric Procedures

We begin with introducing the necessary background on classic change-point detection procedures, including the Shewhart chart, CUSUM, and GLR procedures.

The Shewhart chart [Shewhart, 1925] uses a sliding window to compute detection statistics, which can be either parametric, e.g., based on likelihood ratio, or non-parametric. For the parametric approach, at time step $t$, the Shewhart chart uses the log-likelihood ratio of the most recent sequential observation $Y_t$ as the detection statistic, i.e., $\log(q(Y_t)/p(Y_t))$. However, due to the sliding window approach, the Shewhart chart disregards the past sequential observations, and there will be an “information loss” that could lead to power loss (e.g., see Xie et al. [2021]).

The likelihood-ratio based on CUSUM procedure [Page, 1954] assumes that densities $p$ and $q$ are both known. Given the sequential data $Y_1, \ldots, Y_t$, CUSUM computes the cumulative log-likelihood ratio considering the unknown change-point location, i.e.,

$$S_t = \max_{1 \leq \kappa \leq t+1} \sum_{s=\kappa}^{t} \log \frac{q(Y_s)}{p(Y_s)}$$

$$= \sum_{s=1}^{t} \log \frac{q(Y_s)}{p(Y_s)} - \min_{0 \leq \kappa \leq t} \sum_{s=1}^{\kappa} \log \frac{q(Y_s)}{p(Y_s)}. \tag{1}$$

By utilizing complete history information, the CUSUM procedure overcomes the information loss problem of the Shewhart chart. The CUSUM statistic is particularly popular for online change-point detection due to its recursive implementation:

$$S_t = \left( S_{t-1} + \log \frac{q(Y_t)}{p(Y_t)} \right)^+, \quad S_0 = 0. \tag{2}$$

The CUSUM procedure is defined as the first time $t$ that the statistic $S_t$ exceeds a pre-set threshold. Notably, Lorden [1971], Moustakides [1986] showed that the likelihood ratio based CUSUM enjoys optimality property.

The GLR procedure [Lorden, 1971] requires complete knowledge of $p$ but only assumes that the post-change distribution $q(\cdot; \theta)$ takes a known form with unknown parameter $\theta \in \Theta$. The GLR detection statistic is defined as

$$\max \sup_{1 \leq \kappa \leq t-1} \theta \in \Theta \sum_{s=\kappa}^{t} \log \frac{q(Y_s; \theta)}{p(Y_s)},$$

which involves maximum likelihood estimation of both change-point location $\kappa$ and density parameter $\theta$. Although the inner supremum accounts for the unknown post-change distribution parameter, this renders the GLR statistic computationally inefficient and prevents it from being updated recursively like CUSUM (2). To reduce the computational cost, a window-limited GLR (W-GLR) is adopted by restricting the search for the potential change-point to the most recent $w$ data points, i.e., the outer maximization is over $\kappa \in [t - w : t - 1]$. 
The window length parameter \( w \) is crucial to the success of the W-GLR procedure. Lai [1995] proved that, as \( \gamma \to \infty \), the optimal window length under the Gaussian mean shift assumption is \( w \sim \log \gamma \). According to Lai [2001], by choosing such a window length, “there is little loss of efficiency in reducing the computational complexity by the window-limited modification”.

### 2.3 Kernel MMD-Based Scan B-Procedure

Kernel-based statistics are popular for non-parametric two-sample tests (e.g., Gretton et al. [2012a]) due to their flexibility. There has been kernel-based change-point detection procedure, e.g., Scan B-procedure [Li et al., 2019], utilizing MMD as its detection statistic. In the online Scan B-procedure, \( N \) pre-change blocks with equal block size \( B \), denoted by \( X_B^{(i)}, i \in [N] \), are built by randomly sampling \( NB \) samples from the reference data \( X_1, \ldots, X_M \) without replacement, assuming the reference sample size \( M \) is large enough such that \( M > NB \). At time \( t \), the post-change block consists of the most recent \( B \) sequential data, i.e., \( Y_B(t) = (Y_{t-B+1}, \ldots, Y_t) \). The detection statistic, referred to as the Scan B-statistic, is obtained by computing the unbiased MMD statistic between the post-change block and each pre-change block and taking their average. Formally, the Scan B-statistic is given by:

\[
Z_B(t) = \frac{\hat{D}_B(t)}{\sqrt{\text{Var}_\infty(\hat{D}_B(t))}},
\]

where \( \text{Var}_\infty(\cdot) \) denotes the variance under \( H_0 \) and

\[
\hat{D}_B(t) = \frac{1}{N} \sum_{i=1}^{N} \hat{D}(X_B^{(i)}, Y_B(t)).
\]

For \( X = (X_1, \ldots, X_B) \) and \( Y = (Y_1, \ldots, Y_B) \), here we use the unbiased estimator of MMD [Gretton et al., 2012a]:

\[
\hat{D}(X, Y) = \frac{1}{B(B-1)} \sum_{i=1}^{B} \sum_{j \neq i}^{B} k(X_i, X_j) + \frac{1}{B(B-1)} \sum_{i=1}^{B} \sum_{j \neq i}^{B} k(Y_i, Y_j)
\]

\[
- \frac{2}{B(B-1)} \sum_{i=1}^{B} \sum_{j \neq i}^{B} k(X_i, Y_j)
\]

\[
= \frac{1}{B(B-1)} \sum_{i=1}^{B} \sum_{j \neq i}^{B} h(X_i, X_j, Y_i, Y_j),
\]

where

\[
h(x_1, x_2, y_1, y_2) = k(x_1, x_2) + k(y_1, y_2) - k(x_1, y_2) - k(x_2, y_1),
\]
and \(k(\cdot, \cdot)\) is the user-specified kernel function:

\[
k(\cdot, \cdot) : \mathcal{X} \times \mathcal{X} \to \mathbb{R}.
\]

Commonly used kernel functions include Gaussian radial basis function (RBF) \(k(x, y) = \exp\{-\|x - y\|^2/r^2\}\), where \(\| \cdot \|\) is the vector \(\ell_2\) norm and \(r > 0\) is the bandwidth parameter.

The Scan \(B\)-statistic reuses the post-change block \(Y_B(t)\) and only requires \(O(NB^2)\) computations. Since the block size \(B\) is typically held constant and \(N\) will depend on the reference sample size \(M\), the Scan \(B\)-statistic achieves linear complexity, and this addresses the quadratic computational complexity issue in MMD statistics. Thus, the Scan \(B\)-statistic is more suitable for real-time evaluation of the detection statistic in the online setting.

The normalizing constant, i.e., the standard deviation of \(b_D(t)\) under \(H_0\), has an analytic expression and thus can be computed using the reference samples prior to the detection procedure.

**Lemma 1** (Variance of \(\hat{D}_B(t)\) under \(H_0\), cf. Lemma 3.1 [Li et al., 2019]). Under \(H_0\), for block size \(B \geq 2\) and the number of pre-change blocks \(N > 0\), under \(H_0\), we have

\[
\begin{align*}
\text{Var}_\infty(\hat{D}_B(t)) &= \left(\frac{B}{2}\right)^{-1} \left(\frac{E[h^2(X, X', Y, Y')]}{N} + \frac{N - 1}{N} \text{Cov}[h(X, X', Y, Y'), h(X'', X'''', Y, Y')]\right).
\end{align*}
\]

(7)

The online Scan \(B\)-procedure with block size \(B\) is defined by the first time \(t\) that \(Z_B(t)\) exceeds a pre-set detection threshold. For brevity, we refer to the online Scan \(B\)-procedure as the Scan \(B\)-statistic or procedure, as we do not consider the offline setting in this work.

### 3 Proposed Detection Procedure

Inspired by the original form of recursive online CUSUM (1), we propose to consider a parallel set of Scan \(B\)-statistics \(Z_B(t)\) (3) with block size \(B\) taking values from \([2 : w]\) and take their maximum. For a pre-selected threshold \(b\), the Online Kernel CUSUM procedure is defined via the following stopping time:

\[
T_w = \inf \left\{ t : \max_{B \in [2 : w]} Z_B(t) \geq b \right\}.
\]

(8)

where \(w\) is the window length parameter.

The detection statistic in (8) can be computed recursively, which is crucial for online implementation: (a) Specifically, the computation of the MMD statistic involves evaluating the Gram matrix, which can be updated recursively as we receive sequential data; In fact, such a recursive update with increasing time \(t\) was first considered in Li et al. [2019]. (b) The main difference between our CUSUM-type procedure and the Shewhart chart-type Scan \(B\)-procedure is the window-limited maximization with respect to \(B \in [2 : w]\). Importantly,
this additional window-limited maximization can be implemented in $O(Nw^2)$ computations, ensuring that the overall computational and memory complexity of our procedure remains the same as that of the Scan $B$-procedure with block size $w$. Moreover, since the Gram matrix can be updated recursively, the overall computational and memory complexity remains linear in sample size and does not grow with time, guaranteeing efficiency in practice. One can see the complete details of our detection procedure in Algorithm 1 and the graphical illustration of the recursive calculations in Figure 1.

Figure 1: Graphical illustration of the recursive calculation: $G^{(i)}_{XX}$, $G^{(i)}_{XY}$ and $G_{YY}$ denote the Gram metrics and $e_{B} = w - B + 1$; please see their definition in Algorithm 1. The light gray part of the Gram matrix can be readily obtained from previous calculations, and only the white part needs to be updated (or re-calculated).

Compared to the Scan $B$ procedure, our proposed procedure offers improved performance by enhancing detection power under the same false alarm constraint, achieved by taking the maximum over detection statistics with different block sizes $B$. Since the change-point $\kappa$ is unknown, the maximum is likely to be achieved for the block size $B$ such that $[t - B + 1, t]$ contains all post-change samples. In contrast, the Scan $B$ procedure employs a fixed post-change block size and may include some pre-change samples in the post-change sample block, which can diminish its detection power. Maximizing the utilization of a small post-change sample size is crucial in online change detection, as it aims to detect the change as soon as possible after it occurs. Moreover, our proposed detection statistic has a constant expected increment value, facilitating the derivation of the analytic EDD approximation. However, the Scan $B$ statistic includes varying post-change samples as it scans through the change-point over time, making it challenging to derive an analytic EDD approximation.

Figure 2 provides numerical evidence supporting the advantages of our proposed online
kernel CUSUM over the Scan $B$-procedure, in which the mean trajectories over 100 independent trials are plotted, and we highlight the mean ± standard deviation region. Detailed experimental settings can be found in Section 5. Our detection statistic has a larger slope right after the change-point than the Scan $B$-procedure, indicating better detection power. Additionally, both statistics stop increasing after $w = 50$ steps from the change-point and plateau around similar values. This validates the necessity of choosing a large $w$ according to (15) in our following EDD analysis to ensure the procedure stops with a moderate EDD.

### 3.1 Recursive Computations of the Detection Statistics

Before presenting the detailed detection procedure, we introduce some notations which only appear in this subsection: for a matrix $G \in \mathbb{R}^{m \times n}$, $G(i_1 : i_2, j_1 : j_2)$ represents a slice of matrix $G$ containing its $i_1$-th to $i_2$-th rows and $j_1$-th to $j_2$-th columns; in particular, when we want to include all rows (or columns) in this slice, we use “:” instead of “1 : $m$” (or “1 : $n$”). Similarly, for a vector $g \in \mathbb{R}^m$, $g(i : j)$ represents a slice of $g$, containing its $i$-th to $j$-th elements. Now, we give step-by-step instructions on how to perform our proposed detection procedure in Algorithm 1: The recursive computation of detection statistics comes from two parts: recursive update of Gram matrices (lines 5 – 8) and a recursive formulation to find the window-limited maximum (lines 9 – 13), which are direct benefits of CUSUM and closely resemble that of the original CUSUM (2).

### 3.2 Complexity Analysis

Even though the online update of Gram matrices only takes $O(Nw)$ computational complexity, the whole procedure still takes $O(Nw^2)$ computations to calculate the detection statistic in lines 9 – 13, which is the same with Scan $B$-procedure. To be precise, for fixed $t$, our detection procedure scans through $B \in [2 : w]$ to calculate the maximum of the Scan $B$-statistics, where the recursive update in line 12 only requires additional $O(NB)$ computations and
Algorithm 1 Online Kernel CUSUM

1: Initialization: estimate \( \rho \) (14) using historical data, take post-change block \( Y = Y_w(t) \), and calculate the Gram matrices

\[
G_{XX}^{(i)} = k(X^{(i)}, X^{(i)}), \quad G_{XY}^{(i)} = k(X^{(i)}, Y), \quad i = 1, \ldots, N, \quad G_{YY} = k(Y, Y)
\]

2: while True do
3: receive sample \( Y_t \)
4: update post-change block:
\[
Y(1 : w - 1) \leftarrow Y(2 : w), \quad Y(w) \leftarrow Y_t
\]
5: update Gram matrix \( G_{YY} \):
\[
G_{YY}(1 : w - 1, 1 : w - 1) \leftarrow G_{YY}(2 : w, 2 : w),
G_{YY}(1 : w) \leftarrow k(Y, Y_t), \quad G_{YY}(w : 1) \leftarrow G_{YY}^T(1 : w)
\]
6: for \( i = 1, \ldots, N \) do
7: update Gram matrix \( G_{XY}^{(i)} \):
\[
G_{XY}^{(i)}(1 : w - 1, 1 : w - 1) \leftarrow G_{XY}^{(i)}(2 : w, 2 : w), \quad G_{XY}^{(i)}(1 : w) \leftarrow k(X^{(i)}, Y_t)
\]
8: end for
9: calculate
\[
z = \sum_{i=1}^{N} G_{XX}^{(i)}(w - 1, w) + G_{YY}(w - 1, w) - 2G_{XY}^{(i)}(w - 1, w),
Z_t = \frac{\rho}{N\sqrt{2}z}
\]
10: for \( B = 3, \ldots, w \) do
11: calculate \( \bar{B} = w - B + 1 \)
12: update
\[
z \leftarrow z + \sum_{i=1}^{N} \sum_{k=\bar{B}+1}^{w} G_{XX}^{(i)}(\bar{B}, k) + G_{YY}(\bar{B}, k) - 2G_{XY}^{(i)}(\bar{B}, k)
Z_t \leftarrow \max \left\{ Z_t, \frac{\rho}{N\sqrt{B(B - 1)}}z \right\}
\]
13: end for
14: if \( Z_t > b \) then
15: raise an alarm, set \( T_w = t \) and break
16: end if
17: end while
does not require additional memory to store those intermediate calculations; such a recursive computation results in $O(N \sum_{B=2}^{w} B) = O(Nw^2)$ total computations. We need to clarify that Algorithm 1 actually requires $O(Nw^2)$ memory to store the Gram matrix. However, in theory, we can only use $O(Nw)$ memory to store raw observations and re-compute the whole Gram matrix with $O(Nw^2)$ computations at each time step.

Additionally, we provide a comparison among the Shewhart chart, CUSUM, W-GLR, Scan B, our proposed online kernel CUSUM, and its oracle variant (17), in terms of the knowledge in the post-change distribution specification, as well as the computational and memory complexity at each time step $t$ in Table 2.

Table 2: Comparison of online change-point detection methods. Here, $w$ is the window length parameter and $N$ is the number of pre-change blocks.

| Assumptions                  | Computational Complexity | Memory    |
|------------------------------|--------------------------|-----------|
| Shewhart chart               | $p$ and $q$ are both known | $O(1)$    | –         |
| CUSUM                        | $p$ and $q$ are both known | $O(1)$    | $O(1)$    |
| W-GLR                        | $p$ is known and $q$ has known form | $O(w^2)$  | $O(w)$    |
| Scan B                       | $p$ and $q$ are both unknown | $O(Nw^2)$ | $O(Nw)$   |
| Proposed                     | $p$ and $q$ are both unknown | $O(Nw^2)$ | $O(Nw)$   |
| Oracle                       | $p$ and $q$ are both unknown | $O(Nt^2)$ | $O(Nt)$   |

### 4 Theoretical Properties

In this section, we will derive analytic approximations for two fundamental performance metrics in online change-point detection: ARL (i.e., $E_{\infty}[T_w]$), which is the expected stopping time when there is no change, and EDD (i.e., $E_0[T_w]$), which is the expected stopping time when the change occurs immediately at $\kappa = 0$; this is a typically used performance metric for detection delay [Xie and Siegmund, 2013]. Our ARL approximation is shown to be accurate numerically, enabling us to calibrate the detection procedure without relying on time-consuming Monte Carlo simulations. Most importantly, these results allow us to establish the asymptotically optimal window length $w^*$ in the same sense as Lai [1995].

#### 4.1 Assumptions

We begin with the assumptions on the kernel. Consider the probability measure $P$ (corresponding to the pre-change distribution) on a measurable space $(\mathcal{X}, \mathcal{B})$. For symmetric, positive semi-definite, and square-integrable (with respect to measure $P$) kernel function $k(\cdot, \cdot)$, Mercer’s theorem gives the following decomposition:

$$k(x, x') = \sum_{j=1}^{\infty} \lambda_j \varphi_j(x) \varphi_j(x'),$$
where the limit is in $L^2(p)$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ are eigenvalues of the integral operator induced by kernel $k$, and $\{\varphi_j(\cdot): j \geq 1\}$ are the corresponding orthonormal eigenfunctions such that

$$
\int_\mathcal{X} \varphi_j(x)\varphi_{j'}(x)dP(x) = \begin{cases} 
1 & \text{if } j = j', \\
0 & \text{otherwise.}
\end{cases}
$$

To avoid considering the meaningless zero kernel, we assume the largest eigenvalue is positive, i.e., $\lambda_1 > 0$. In addition, we further impose a technical assumption that all eigenfunctions are uniformly bounded, i.e.,

$$
sup_{j \geq 1} \|\varphi_j\|_\infty < \infty,
$$

where $\|\varphi_j\|_\infty = \sup_{x \in \mathcal{X}} |\varphi_j(x)|$. This assumption guarantees that the kernel $k(\cdot, \cdot)$ is uniformly bounded on domain $\mathcal{X} \times \mathcal{X}$, i.e., there exists a constant $K > 0$ such that

$$
0 \leq k(x, y) \leq K, \quad \forall x, y \in \mathcal{X}. \tag{A1}
$$

As one can see, Assumption (A1) guarantees the existence of the moments of $k(X, Y)$, where $X$ and $Y$ can follow either pre-change or post-change distribution. Our approach does not impose any restrictions on the rank of the kernel function, meaning that the kernel does not necessarily need to be characteristic. For instance, on the domain $\mathcal{X} = \mathbb{R}^d$, the polynomial kernel function $k_\ell(x, y) = (x^T y + c)^\ell$ is finite-rank and cannot differentiate between probability distributions $p$ and $q$ if they have the same first $\ell$-th order moments. To avoid such undesirable cases, we will impose an assumption on the post-change distribution to ensure that the change can be detected with our chosen kernel.

Next, we consider the assumption on the post-change distribution $q$. To ensure the change from $p$ to $q$ is detectable with a chosen kernel $k$, we require the population MMD to be positive. The (squared) population MMD, denoted by $\mathcal{D}(p, q)$, can be expressed as:

$$
\mathcal{D}(p, q) = \int_{\mathcal{X}} \int_{\mathcal{X}} k(x, x')(p - q)(x)(p - q)(x')dx\,dx' = \sum_{j=1}^\infty \lambda_j \left( \int_\mathcal{X} \varphi_j(x)(p(x) - q(x))dx \right)^2. \tag{9}
$$

The detectability assumption on $q$ is that, for pre-change distribution $p$ and kernel $k$ satisfying all aforementioned assumptions, there exists $j$ such that $\lambda_j > 0$ and

$$
a_j = \int_\mathcal{X} \varphi_j(x)(p(x) - q(x))dx \neq 0. \tag{A2}
$$

Here, $a_j$ represents the “projection” of the departure $p - q$ along the “direction” $\varphi_j$, and Assumption (A2) guarantees that

$$
\mathcal{D}(p, q) = \sum_{j=1}^\infty \lambda_j a_j^2 > 0.
$$
We want to remark that, if we assume that kernel $k$ is universal, i.e., $\varphi_j(\cdot) : j \geq 1$ forms an orthonormal basis of $L^2(p)$, the kernel will be full-rank and characteristic, i.e., $D(p, q) > 0$ when $p \neq q$. However, such an assumption is stronger than we need in the following analysis. Additionally, we want to mention that a characteristic kernel does not necessarily imply its universality unless the kernel is translation-invariant or radial; for a more detailed discussion, readers are referred to Sriperumbudur et al. [2011].

4.2 Average Run Length

Approximating ARL $E_\infty[T_w]$ in closed-form requires studying the behaviors of extremes of random fields. Yakir [2013] summarized three steps towards this problem: (i) change-of-measure via exponential tilting, (ii) applying likelihood ratio identity to change the probability of interest into expectation, and (iii) invoking localization theorem. Examples using this technique include Siegmund and Venkatraman [1995], Siegmund and Yakir [2008], Siegmund [2013], Li et al. [2019], and so on. We defer the technical details to Appendix B and present the ARL approximation as follows:

**Lemma 2** (ARL approximation). As $b \to \infty$, an approximation to ARL for our proposed online kernel CUSUM procedure $T_w$ (8) is given by:

$$E_\infty[T_w] = \frac{\sqrt{2\pi}}{b} \left\{ \sum_{B=2}^{w} e^{\psi_B(\theta_B) - \theta_B b} \frac{(2B - 1)}{B (B - 1)} \nu \left( \theta_B \sqrt{\frac{2(2B - 1)}{B (B - 1)}} \right) \right\}^{-1} \left[ 1 + o(1) \right],$$

where $\psi_B(\cdot)$ is defined as:

$$\psi_B(\theta) = \sum_{n=1}^{\infty} \frac{E_\infty[Z^n_B(t)]}{n!} \theta^n,$$

and $\theta_B$ is obtained via solving $\dot{\psi}_B(\theta_B) = b$. Moreover, the function $\nu(\cdot)$ can be approximated as (cf. Siegmund [1985], Siegmund et al. [2007], Xie and Siegmund [2013]):

$$\nu(\mu) \approx \frac{(2/\mu)(\Phi(\mu/2) - 0.5)}{(\mu/2)\Phi(\mu/2) + \phi(\mu/2)},$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ are the probability density function and the cumulative distribution function of the standard normal distribution, respectively.

In practice, incorporating the information of the first two order moments information will suffice for ARL approximation, i.e.,

$$\psi_B(\theta_B) \approx \theta_B^2/2, \quad \theta_B \approx b, \quad (11)$$

since under $H_0$ the detection statistic has expectation zero. This gives us

$$E_\infty[T_w] = \sqrt{2\pi} \frac{b e^{b^2/2}}{w} \left[ 1 + o(1) \right].$$

(12)
A detailed derivation of the approximation in (12) can be found in Appendix B.

To obtain a more precise ARL approximation, one approach would be to incorporate higher order moments while solving for \( \theta_B \) using the equation \( \dot{\psi}_B(\theta_B) = b \). Here, we introduce \textit{skewness correction}, which additionally considers the third order moment \( \mathbb{E}_\infty[Z^3_B(t)] \) in the following way:

\[
\psi_B(\theta_B) \approx \frac{\theta_B^2}{2} + \mathbb{E}_\infty[Z^3_B(t)]\theta_B^3/6, \\
\theta_B \approx -1 + \frac{\sqrt{1 + 2b\mathbb{E}_\infty[Z^2_B(t)]}}{\mathbb{E}_\infty[Z^3_B(t)]}.
\] (13)

Similar to the second order moment, the third order moment \( \mathbb{E}_\infty[Z^3_B(t)] \) can be pre-computed using the reference data; please see Lemma 5 in Appendix A for its analytic expression.

To demonstrate the usefulness of our ARL approximation, we compare it with Monte Carlo simulation results for three types of pre-change distributions (i.e., Gaussian, Exponential, and Laplace distributions) in Figure 3. Our approximations are fairly accurate for all three cases when the ARL is not very large. Incorporating skewness correction improves the accuracy of the ARL approximation, indicating its practical usefulness in detecting changes. Notably, for the Gaussian pre-change distribution, the ARL approximation with skewness correction remains highly accurate even for very large ARL.

![Figure 3: Comparison of the detection threshold \( b \) obtained by the Monte Carlo simulation (Simu.) and the proposed theoretical approximation (Theo. approx.) to meet the given ARL constraint. For the theoretical approximation approach, we consider with or without skewness correction (skew. corr.).](image)

In Appendix D, we will theoretically demonstrate that considering only finite-order moments can lead to underestimating the threshold \( b \) asymptotically, which is already validated numerically in our simulation result shown in Figure 3. It is important to clarify that a Gaussian pre-change distribution \( p \) does not imply a Gaussian distribution of the Scan \( B \)-statistic \( Z_B(t) \), so considering only its first two moments will not fully characterize its
distribution. As shown in Figure 3, incorporating skewness correction improves the accuracy of the ARL approximation for the Gaussian pre-change distribution case.

We want to emphasize that our proposed skewness correction yields a more accurate ARL approximation than the original one proposed by Li et al. [2019], despite our derivations being similar. Specifically, it is crucial to use the skewness-corrected version of $\theta_B$ in both $e^{\psi_B(\theta_B)-\theta_B b}$ and $\nu \left( \theta_B \sqrt{2(B-1)/(B(B-1))} \right)$. This is because both terms contain exponentially large components with respect to $\theta_B$. However, Li et al. [2019] did not update $\theta_B$ in the term inside the $\nu(\cdot)$ function, which can lead to a less accurate ARL approximation. We verify this finding numerically in Figure 4 for the Scan $B$-procedure.

![Figure 4](image-url)

Figure 4: Comparison of the detection threshold $b$ to meet the given ARL constraint for the Scan $B$-procedure between our proposed skewness correction (13) and the original one in Section 6 [Li et al., 2019]. The data is generated from pre-change distribution $p$, which is specified on top of each panel.

### 4.3 Expected Detection Delay

The difficulty in approximating the EDD $\mathbb{E}_0[T_w]$ arises from the form of the detection statistic, which is the maximum of several correlated random variables and has an increment dependent on its past. Consequently, directly applying Wald’s identity to approximate the EDD is not feasible. However, we can use Azuma’s concentration inequality for martingales with bounded differences to study the concentration of the Scan $B$-statistic (as shown in Lemma 7 in Appendix C). We denote

$$
\rho = \left( \frac{\mathbb{E} [h^2(X, X', Y, Y')]}{2N} + \frac{N - 1}{2N} \text{Cov} \left[ h(X, X', Y, Y'), h(X'', X'''', Y, Y') \right] \right)^{-1/2}, \quad (14)
$$

where $X, X', X'', X'''', Y, Y'$ are i.i.d. random variables with pre-change distribution $p$ and $h(\cdot, \cdot, \cdot, \cdot)$ is defined in (6). The EDD approximation is then given by:

**Lemma 3 (EDD approximation).** Under Assumptions (A1) and (A2), for window length

$$
w \geq \frac{7b}{\rho D(p, q)}, \quad (15)
$$

16
and \( N > 0 \), as \( b \to \infty \), the first-order approximation to EDD for our proposed online kernel CUSUM \( T_w \) (8) is given by:

\[
\mathbb{E}_0[T_w] = \frac{b}{\rho \mathcal{D}(p, q)} (1 + o(1)).
\]  

(16)

The proof of Lemma 3 can be found in Appendix C.

**Remark 1** (Effect of pre-change block number \( N \)). The parameter \( N \) has a two-fold impact on our analysis. Firstly, it determines the remainder term in (32), and only when \( N \) is sufficiently large can we neglect this term. Secondly, it directly affects the EDD in (16) by controlling the normalizing constant \( \rho \) in (14). To showcase the impact on EDD, let us consider the example in which the pre-change distribution is \( \mathcal{N}(\mu_1 d, \sigma^2 I_d) \) and we use Gaussian RBF kernel with bandwidth \( r \) selected by the median heuristic. The Taylor expansion (see detailed derivations in Appendix A) gives us:

\[
0 < \mathbb{E} [h^2(X, X', Y, Y')] - \text{Cov}[h(X, X', Y, Y'), h(X'', X'''', Y, Y')] = \mathcal{O}(1/d).
\]

This implies that \( \rho \) monotonically increases to \((\text{Cov}[h(X, X', Y, Y'), h(X'', X'''', Y, Y')]) / 2)^{-1/2}\) as \( N \to \infty \), meaning that larger \( N \) leads to smaller EDD according to (16). This finding agrees with the intuition: as \( N \) grows larger, the variance of \( \tilde{D}_B(t) \) shrinks from \((\frac{B}{2})^{-1} \mathbb{E} [h^2(X, X', Y, Y')]\) to \((\frac{B}{2})^{-1} \text{Cov}[h(X, X', Y, Y'), h(X'', X'''', Y, Y')]\) (as evidenced by Lemma 1 in Appendix A), which can help achieve quicker change detection. Moreover, the fact that \( \rho \) cannot increase unboundedly with increasing \( N \) also agrees with intuition: In a two-sample test, if we only increase the size of one set of samples (i.e., more pre-change blocks) while keeping the other set of samples (i.e., post-change block) fixed, we cannot expect the variance of the test statistic to converge to zero.

### 4.4 Optimal Window Length

We now present the optimal choice of the window length parameter \( w \) due to ARL and EDD analysis by comparing our procedure with an oracle procedure, which assumes infinite memory capacity and can memorize all observations up to time \( t \). The stopping time of the oracle procedure with detection threshold \( b \) is defined as follows:

\[
T_o = \inf \left\{ t : \max_{B \in [2:t]} Z_B(t) > b \right\}.
\]

(17)

There is a trade-off between the computational and memory efficiency as well as the detection power: On one hand, larger \( w \) intuitively leads to smaller EDD, but the benefit diminishes as \( w \) further increases. On the other hand, the oracle procedure is computationally expensive as it involves \( \mathcal{O}(t^2) \) computational complexity and \( \mathcal{O}(t) \) memory up to time \( t \), which is unsuitable for online settings. Thus, the fundamental question is: when holding ARL constant, what is the optimal window length \( w^* \) without sacrificing much performance?
Define a set of detection procedures with a constant ARL constraint $\gamma > 0$:

$$C_\gamma = \{ T : \mathbb{E}\infty[T] \geq \gamma \}. \quad (18)$$

Following the optimality definition in Lai [1995], we are interested in

(i) Choosing thresholds $b$’s for each procedure respectively to ensure $T_w, T_o \in C_\gamma$, and

(ii) for tolerance $\varepsilon > 0$, choosing an optimal window length $w^*$ that incurs a $\varepsilon$-performance loss, i.e.,

$$w^* = \min w, \quad \text{s.t.} \quad 0 \leq \mathbb{E}_0[T_w] - \mathbb{E}_0[T_o] \leq \varepsilon. \quad (19)$$

We have the following result:

**Theorem 1** (Optimal window length). Under Assumptions (A1) and (A2), for any $N > 0$, as $\gamma \to \infty$, when choosing $b \sim \log \gamma$, we have $T_w, T_o \in C_\gamma$, and furthermore the optimal window length

$$w^* \sim \frac{6b}{\rho D(p, q)} + \frac{512K^2 \log(3/\varepsilon)}{b^2 \left( \frac{N}{4} \wedge \frac{b}{\rho D(p, q)} \right)}. \quad (20)$$

The main message from Theorem 1 is that the optimal window length is $w^* \sim \log \gamma$, as the first term in (20) dominates the second term when $\gamma$ (and $b$) becomes large. While we can only characterize the order of the optimal window length due to the complexity of our ARL approximation (10), the result $w^* \sim \log \gamma$ is informative to make connections with classic parametric results. Specifically, our result is in the same order as the classic result for window-limited GLR [Lai, 1995]. Furthermore, under the optimal choice $w^*$, the EDD of our proposed procedure will be:

$$\mathbb{E}_0[T_{w^*}] \sim \frac{\log \gamma}{D(p, q)},$$

which closely resembles the EDD of the classic CUSUM procedure.

## 5 Numerical Experiments

In this section, we evaluate the performance of our proposed procedure by comparing it with various benchmark procedures under different settings using both simulated and real-world examples. Specifically, we compare their EDDs for given ARLs to demonstrate the capability of our procedure to achieve quicker change detection. We also study different choices of hyperparameters, such as $N$ and $w$, to guide their selection in practice and validate our theoretical findings. The influence of hyperparameters is studied in Appendix E.1, and we report the comparison with benchmarks in this section. Implementation is available online.\(^1\)

\(^1\)The Matlab code can be found at [https://github.com/SongWei-GT/online_kernel_cusum](https://github.com/SongWei-GT/online_kernel_cusum).
5.1 Simulations

To ensure that each detection procedure meets the given ARL constraint, we select the detection threshold $b$ for each detection procedure separately via 1000 Monte Carlo trials. We then obtain the EDDs for each procedure as the average detection delays over another 1000 Monte Carlo trials with 10000 reference samples following $p$ and 50 sequential samples following $q$. For our proposed procedure and Scan $B$-procedure, we choose $N = 15, w = 50$; we use Gaussian RBF kernel with bandwidth parameters selected via median heuristic. These choices are suggested by the experiments in Li et al. [2019] and our simulation result in Appendix E.1. We compare both procedures’ EDDs (for given ARL) in Figure 5 under various settings, where we can observe improved performance (i.e., quicker change detection) of our proposed procedure.

Figure 5: Comparison of EDD for given ARL between our proposed (red) and the Scan $B$ (orange) detection procedures. In the x-axis, the ARL is in the base 10 logarithm. The pre-change distribution is $\mathcal{N}(0, I_{20})$ and the post-change distribution $q$ is specified on the top of each figure (with $\sigma = 2$).

In addition to the Scan $B$-procedure, we consider another two benchmark procedures: KCUSUM [Flynn and Yoo, 2019] and Hotelling’s $T^2$. For a brief introduction to these procedures, please see Appendix E.2. We plot the EDD against ARL for all four procedures in Figure 6, in which the absence of a dot indicates that the corresponding procedure fails to detect the change before time $t = 50$ under the corresponding setting. The major observation is that our proposed procedure is not only the most sensitive, achieving the quickest detection but also the most robust, capable of detecting even small changes that KCUSUM and parametric Hotelling’s $T^2$ procedures fail to detect under the considered settings. In contrast, although KCUSUM achieves quicker detection than the other two benchmark procedures, it easily fails when the change is subtle, or the ARL is large, indicating a lack of robustness. To further demonstrate the good performance of the proposed procedure, we present results under more scenarios in Appendix E.3.
Figure 6: Comparison of EDD for our proposed (red), Scan B (orange), KCUSUM (blue), and Hotelling’s $T^2$ (brown) procedures. The setting is: the pre-change distribution is $\mathcal{N}(0_{20}, I_{20})$ and the post-change distribution is Gaussian mixture: $\mathcal{N}(0_{20}, I_{20})$ w.p. 0.3; $\mathcal{N}(\mu_{1_{20}}, \sigma^2 I_{20})$ w.p. 0.7.
5.2 Real Experiments

Finally, we demonstrate the performance of our proposed procedure on the Human Activity Sensing Consortium (HASC) Dataset \(^2\), which contains 3-dimensional measurements of human activities collected by portable 3D accelerometers. To compare the performance of different detection procedures, we consider the change from walking to staying of human subject 101 and plot the detection statistics in Figure 7. The EDD and miss rate (out of 10 repeats) of each procedure are reported in Table 3. The detection threshold is chosen as the 80% quantile of the largest detection statistics over all repeats under \(H_0\) (i.e., the post-change activity is the same as the pre-change activity) to control the false alarm rate at 0.2.

Figure 7: The trajectories of different detection statistics. Human subject 101 activity changes from walking to staying at time \(t = 101\). The recursive update of KCUSUM (48) yields some zero statistics, which correspond to the “no value part” on the blue line.

Table 3: EDD (over successful detections) and miss (out of 10 repeats) of human activity change detection for subject 101. The detection threshold is chosen as the 80% quantile of the maximum detection statistics under \(H_0\) (i.e., the post-change activity is still walking).

| Activity       | Jog EDD | Miss | Skip EDD | Miss | Stay EDD | Miss | Stair down EDD | Miss | Stair up EDD | Miss |
|----------------|---------|------|----------|------|----------|------|----------------|------|---------------|------|
| Proposed       | 37.7    | 0    | 50.6     | 0    | 13       | 8    | 517.17         | 4    | 227.13        | 2    |
| Scan B         | 73.7    | 0    | 82.4     | 0    | 72.2     | 0    | 329.71         | 3    | 207.33        | 7    |
| KCUSUM         | 635     | 2    | 560.67   | 1    | 527.6    | 0    | 429.33         | 4    | —             | 10   |
| Hotelling \(T^2\) | 77.6   | 5    | 242.8    | 5    | 26.5     | 8    | 114            | 5    | 44.67         | 7    |

As shown in Table 3, our procedure is highly robust and successfully detects the change in almost all scenarios except for the change from walking to staying, where it misses most of the changes. However, as illustrated in Figure 7, our procedure performs well for some

\(^2\)Available from http://hasc.jp/hc2011
scenarios. Furthermore, our procedure achieves the quickest detection for changes from walking to jogging and skipping. Despite having the smallest EDD for changes to walking up and down the stairs, the parametric Hotelling’s $T^2$ procedure fails most of the time. These observations further validate the practical usefulness of our procedure. For completeness, we extend our experiment to the rest of the five human subjects and report the EDD and miss rates in Table 7 in Appendix E.4, which further demonstrates the excellent performance of our procedure for most scenarios. We also provide more details of the HASC dataset and the choices of hyperparameters in Appendix E.4.

6 Conclusion and Discussions

In this work, we propose a new online kernel CUSUM procedure for change-point detection to overcome the limitations of existing Shewhart chart-type procedures, such as the Scan $B$-procedure. Our modification enables analytical approximations of ARL and EDD and establishes the optimal window length to be logarithmic in ARL, consistent with classical parametric results. We validate the effectiveness of our procedure through extensive numerical experiments, demonstrating its superior performance compared to various existing procedures. Our findings contribute to the non-parametric change-point detection literature, providing a practical and robust method for detecting changes in sequential data.

One potential limitation of the kernel method is the curse of dimensionality, as the population MMD drops to zero exponentially fast as the dimension increases [Ramdas et al., 2015, Reddi et al., 2015]. As a result, EDD will grow exponentially fast with respect to dimensionality, and our procedure cannot alleviate this issue. Fortunately, when the difference between pre- and post-change distributions lies on a low-dimensional manifold, Euclidean distance locally recovers manifold distance due to isometric embedding [Coifman and Lafon, 2006], and the global manifold geometry is encoded in the constructed Gram matrix. Indeed, using Euclidean distance is the initial step in many manifold learning methods, e.g., ISOMAP [Tenenbaum et al., 2000], where the graph or Gram matrix construction relies on $\varepsilon$-ball or kNN with Euclidean distance. As a result, using Euclidean distance at a small scale retains manifold geometric information [Friedman and Rafsky, 1979, Schilling, 1986, Chen and Friedman, 2017, Cheng and Xie, 2021], and therefore the detection power can still be maintained with the kernel method where the Gram matrix is constructed via Euclidean distance. Here, we provide numerical evidence on MNIST dataset, where the data lies on 784 (28*28-pixel image) ambient space but is supported on a low-dimensional manifold. We use random samples from the different digits as pre- and post-change distributions and the result is reported in Figure 8, where we can see kernel method can still detect changes almost as fast as it does in the 20-dimensional numerical simulation. We defer complete experimental details to Appendix E.5.

Lastly, there are several possible theoretical extensions of our work. As mentioned above, effectively exploiting the impact of a low-dimensional structure in the online manifold change-point detection problem remains an open research question. Another direction is to address the sub-optimality compared with Lorden’s lower bound [Lorden, 1971], i.e., our procedure is
Figure 8: Comparison of EDD for given ARL between our proposed (red) and the Scan B (orange) detection procedures. In the 10-by-10 figure above, \((i, j)\)-th sub-figure corresponds to the setting where the history data is uniform random samples from digit-\(i\) and the sequential observations are digit-\(j\) uniform random samples.
not mini-max optimal, and the amount of sub-optimality is characterized by the difference between MMD and the KL divergence. Recent work by Balasubramanian et al. [2021] has demonstrated that a modified kernel embedding can significantly reduce this gap and establish asymptotic mini-max optimality for the kernel Goodness-of-Fit test. However, this approach is sample size-dependent and has not yet been widely used in online change-point detection. Thus, developing sample-size-dependent remedies for kernel choice may be a future direction.

Acknowledgement

This work is partially supported by an NSF CAREER CCF-1650913, and NSF DMS-2134037, CMMI-2015787, DMS-1938106, and DMS-1830210.

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Appendix of
Online Kernel CUSUM for Change-Point Detection

Table of Contents

A Moments of the Scan $B$-Statistic 30

B Proof of Lemma 2 32

C Proof of Lemma 3 35

D Proof of Theorem 1 46

E Additional Experimental Results 51
   E.1 Study on the Hyperparameter Choices 51
   E.2 Benchmark Procedures 54
   E.3 EDD against ARL Comparison under More Settings 55
   E.4 Real Example 1: HASC Dataset 59
   E.5 Real Example 2: MNIST Dataset 61
A Moments of the Scan $B$-Statistic

Derivations in Remark 1. For notational simplicity, we denote

\[ C_1 = \mathbb{E} \left[ h^2 (X, X', Y, Y') \right], \]
\[ C_2 = \text{Cov} \left[ h(X, X', Y, Y'), h(X'', X'''', Y, Y') \right], \]

(21)

where $X, X', X'', X''', Y, Y'$ are i.i.d. random variables following $p$. By calculation, we have

\[ C_1 - C_2 = \mathbb{E}[k^2(X, X')] - (\mathbb{E}[k(X, X')])^2 - 4(\mathbb{E}[k(X, X')k(X, Y)] - \mathbb{E}[k(X, X')k(X', Y)]). \]

When we choose Gaussian RBF kernel with bandwidth parameter $r > 0$ and consider $p = \mathcal{N}(\mu 1_d, \sigma^2 I_d)$, we can evaluate the above quantity in closed-form. For notational simplicity, we denote

\[ \tilde{r}_0 = \left( \frac{r^2}{4\sigma^2 + r^2} \right)^{d/2}, \]
\[ \tilde{r}_1 = \left( 1 - \frac{4\sigma^4}{(2\sigma^2 + r^2)^2} \right)^{d/2}, \]
\[ \tilde{r}_2 = \left( 1 - \frac{3\sigma^4}{(3\sigma^2 + r^2)(\sigma^2 + r^2)} \right)^{d/2}. \]

Then, we have

\[ C_1 - C_2 = \tilde{r}_0 (1 - \tilde{r}_1 - 4\tilde{r}_2 + 4\tilde{r}_1) = \tilde{r}_0 (1 + 3\tilde{r}_1 - 4\tilde{r}_2). \]

We can see the problem boils down to evaluating $1 + 3\tilde{r}_1 - 4\tilde{r}_2$. When we use median heuristic to select kernel bandwidth, we will have $r^2 \sim d\sigma^2$. We plug it in $1 + 3\tilde{r}_1 - 4\tilde{r}_2$ and we will have $1 + 3\tilde{r}_1 - 4\tilde{r}_2$ approximated as follows:

\[ 1 + 3 \left( 1 - \frac{3}{(3 + d)(1 + d)} \right)^{d/2} - 4 \left( 1 - \frac{4}{(2 + d)^2} \right)^{d/2}. \]

On one hand, we know $(1 + 1/d^2)^d \sim \exp\{1/d\} \to 1$ as $d \to \infty$. On the other hand, for relatively small $d$, we can calculate its value in Figure 9.

As we can see in Figure 9, the quantity $C_1 - C_2$ will be positive at least for $d$ up to 1000. Since we also observe a monotonic decreasing trend, we conjecture that this quantity will always be positive. In fact, this can be theoretically proved. By Taylor expansion with Lagrange remainder, we will have

\[ 1 + 3\tilde{r}_1 - 4\tilde{r}_2 \geq 1 + 3 \exp\{-2d\sigma^4/r^4\} - 4 \exp\{-3d\sigma^4/(2r^4)\} - \frac{8d\sigma^6}{r^6} \exp\{2d\sigma^6/r^6\}. \]
Figure 9: Numerical evaluation of $1 + 3\tilde{r}_1 - 4\tilde{r}_2$ up to $d = 1000$ when we take $r^2 = d\sigma^2$. We plot its raw value and its logarithm in the left and right panels, respectively.

Therefore, if we apply Taylor expansion and only keep the leading term, as long as

$$\sigma^2 = \mathcal{O}(r^2/\sqrt{d}),$$

we will have

$$1 + 3\exp\{-2d\sigma^4/r^4\} - 4\exp\{-3d\sigma^4/(2r^4)\} - \frac{8d\sigma^6}{r^6}\exp\{2d\sigma^6/r^6\} = \frac{3d\sigma^4}{2r^4}(1 + o(1)).$$

As we can see, median heuristic clearly satisfies condition (22). We can not only verify $1 + 3\tilde{r}_1 - 4\tilde{r}_2 \geq 0$ for large enough $d$, but also show the rate (at which it decays to zero) is $\mathcal{O}(1/d)$.

Lemma 4 (Covariance structure of $\hat{D}_B(t)$ under $H_0$). Under $H_0$, for the un-normalized statistic $\hat{D}_B(t)$ (4), for block sizes $B_1, B_2 \geq 2$ and non-negative integer $s$, at time steps $t$ and $t + s$, the covariance is as follows:

$$\text{Cov}\left(\hat{D}_{B_1}(t), \hat{D}_{B_2}(t + s)\right) = C_2\left(\frac{\ell}{2}\right)/\binom{B_1}{2} \binom{B_2}{2},$$

where

$$\ell = \begin{cases} 0, & \text{if } B_2 - s < 0; \\ B_2 - s, & \text{if } 0 \leq B_2 - s < B_1; \\ B_1, & \text{otherwise,} \end{cases}$$

and $C_2$ is defined in (21).

Proof. Notice that $\ell$ represents the length of overlap between $\hat{D}_{B_1}(t)$ and $\hat{D}_{B_2}(t + s)$. By Lemma B.2 [Li et al., 2019] and notice that $\hat{D}_{B_1}(t)$ and $\hat{D}_{B_2}(t + s)$ have different normalizing factors, we complete the proof.

$\square$
Lemma 5 (Third-order moment of $\hat{D}_B(t)$ under $H_0$, cf. Lemma 6.1 [Li et al., 2019]). Given block size $B \geq 2$ and the number of blocks $N$, under $H_0$, the third order moment of $\hat{D}_B(t)$ (4) has an analytic expression as follows:

$$
\mathbb{E}_\infty \left[ (\hat{D}_B(t))^3 \right] = \frac{8(B-2)}{B^2(B-1)^2} \left\{ \frac{1}{N^2} \mathbb{E} \left[ h(X, X', Y, Y') h(X', X'', Y'', Y') h(X'', X, Y', Y'') \right] \\
+ \frac{3(N-1)}{N^2} \mathbb{E} \left[ h(X, X', Y, Y') h(X', X'', Y'', Y') h(X'', X''', Y''', Y', Y'') \right] \\
+ \frac{(N-1)(N-2)}{N^2} \mathbb{E} \left[ h(X, X', Y, Y') h(X', X'', Y', Y'') h(X'', X''', Y', Y'') \right] \right\} \\
+ \frac{4}{B^2(B-1)^2} \left\{ \frac{1}{N^2} \mathbb{E} \left[ h(X, X', Y, Y')^3 \right] \\
+ \frac{3(N-1)}{N^2} \mathbb{E} \left[ h(X, X', Y, Y')^2 h(X'', X', Y, Y') \right] \\
+ \frac{(N-1)(N-2)}{N^2} \mathbb{E} \left[ h(X, X', Y, Y') h(X'', X', Y', Y'') h(X''', X''', Y', Y') \right] \right\},
$$

where $X, X', X'', X''', Y, Y', Y''$ are all i.i.d. random variables following $p$.

The constant $\mathbb{E}_\infty [Z_B^3(t)] = \mathbb{E}_\infty [(\hat{D}_B(t))^3] \text{Var}(\hat{D}_B(t))^{-3/2}$ can be pre-calculated together with the constants in Lemma 1.

Lastly, we evaluate the moment of Scan $B$-statistic (3) under $H_1$ as follows:

Lemma 6 (Mean of $Z_B(t)$ under the alternative). Under $H_1$, assume change occurs at time step 0, at time step $t < w$, the expectation of the Scan $B$-statistic (3) $\mu_B(t) = \mathbb{E}_0[Z_B(t)]$ has the following explicit expression:

$$
\mu_B(t) = \rho \mathcal{D}(p,q) \sqrt{B(B-1)}, \quad B \in [2 : t], \\
\mu_B(t) = \rho \mathcal{D}(p,q) \frac{t(t-1)}{\sqrt{B(B-1)}}, \quad B \in [t+1 : w],
$$

where constant $\rho$ is defined in (14).

Proof. We can observe that, when change occurs at $t = 0$, the most recent $t$ samples will be post-change samples. Therefore, when $B > t$, the post-change block will also contain pre-change samples. Notice that $\mathcal{D}(p,p) = 0$ (see, e.g., Lemma 4 in Gretton et al. [2012a]). Then it reduces to simple evaluation of the expectation (it is simple because the expectation is linear). We omit further detailed derivations and conclude the proof. \(\square\)

B Proof of Lemma 2

Under $H_0$, the event that the statistic crosses the threshold $b$ is rare. Therefore, as $b \to \infty$, for large enough $m$ such that

$$
\log \frac{b}{wm} \to 0, \quad \frac{wmb}{e^{b^2/2}} \to 0, \quad (23)
$$

32
the null distribution of $T_w$ can be approximated via the following exponential distribution with parameter $\lambda$:

$$P_\infty(T_w \leq m) = P_\infty \left( \max_{t \in [m]} \max_{B \in [2:w]} Z_B(t) \geq b \right) \approx 1 - \exp\{-\lambda m\} \approx \lambda m. \quad (24)$$

The above result can be derived using Poisson approximation technique (see, e.g., Theorem 1 [Arratia et al., 1989] and Appendix D [Li et al., 2019]). Intuitively, the interarrival time of the first rare event will follow an exponential limiting distribution. More precise statements as well as their detailed proof regarding the above exponential approximation can be found in Appendix D in Li et al. [2019] and we omit further technical details here.

One difference of our exponential approximation from that of Li et al. [2019] is that they only require $\log b m \to 0$, $mb e^b/2 \to 0$. (25)

By comparing condition (23) and the above condition (25), we know that we cannot afford exponentially large $w$ with respect to $b$. Nevertheless, on one hand, window length $w$ is held constant in practice, which satisfies condition (23) automatically; on the other hand, we will see in the following EDD analysis we only need $w \sim b$, which again satisfies condition (23).

Equation (24) tells us that the problem of approximating $E_\infty[T_w]$ reduces to studying the extremes of random field $\{Z_B(t)\}$ with parameter $(t, B)$, which can be solved by determining the parameter $\lambda$ of the limiting exponential distribution. To achieve this goal, a popular method is described in Section 4.2 (see steps (i), (ii), (iii) therein). A very clear and detailed instruction on how to prove the ARL approximation can be found in Appendices C and D [Li et al., 2019]. Here, we sketch the proof of Lemma 2 as follows:

**Proof Sketch of Lemma 2.** We give high-level ideas in each step described in Section 4.2. Moreover, we highlight the difference from the proof in Li et al. [2019] as follows:

(i) We usually make use of log moment generating function $\psi_B(\theta) = \log E_\infty[e^{\theta Z_B(t)}]$ and the new measure is defined as

$$dP_B = \exp\{\theta_B Z_B(t) - \psi_B(\theta_B)\} dP_\infty.$$

Note that the new measure $dP_B$ is within the exponential family. An advantage of this is that we can easily manipulate the mean of $Z_B(t)$ under this new measure, which is $\psi_B(\theta_B)$, by properly choosing $\theta_B$. Specifically, we will choose $\theta_B$ such that $\psi_B(\theta_B) = b$.

(ii) We will apply likelihood ratio identity. Before this, we first define some terms: the log-likelihood ratio is

$$\ell_B(t) = \log(dP_B/dP_\infty) = \theta_B Z_B(t) - \psi_B(\theta_B).$$
The global term is $\tilde{\ell}_B(t) = \theta_B (Z_B(t) - b)$ and local terms are

$$M_B = \max_{t \in [m], \; s \in [2:w]} e^{\ell_s(t) - \ell_B(t)},$$

$$S_B = \sum_{t=1}^m \sum_{s \in [2:w]} e^{\ell_s(t) - \ell_B(t)}.$$

For notational simplicity, we denote

$$\Omega = [m] \times [2:w].$$

By likelihood ratio identity, $\mathbb{E}[X; A] = \mathbb{E}[X 1_A]$, where $1_A = 1$ if event $A$ is true and zero otherwise, we have

$$\mathbb{P}_\infty \left( \max_{(t,B) \in \Omega} Z_B(t) \geq b \right) = \mathbb{E}_\infty \left[ 1; \max_{(t,B) \in \Omega} Z_B(t) \geq b \right]$$

$$= \mathbb{E}_\infty \left[ \sum_{(t,B) \in \Omega} e^{\ell_B(t)}; \max_{(t,B) \in \Omega} Z_B(t) \geq b \right]$$

$$= \sum_{(t,B) \in \Omega} \mathbb{E}_B \left[ \frac{e^{\ell_B(t)}}{\sum_{(t,s) \in \Omega} e^{\ell_s(t)}}; \max_{(t,B) \in \Omega} Z_B(t) \geq b \right]$$

$$= \sum_{(t,B) \in \Omega} \mathbb{E}_B \left[ \frac{1}{\sum_{(t,s) \in \Omega} e^{\ell_s(t)}}; \max_{(t,B) \in \Omega} Z_B(t) \geq b \right]$$

$$= m \sum_{B=2}^w e^{\psi_B(\theta_B) - \theta_B b} \mathbb{E}_B \left[ \frac{M_B}{S_B} e^{-\tilde{\ell}_B + \log M_B}; \tilde{\ell}_B + \log M_B \geq 0 \right].$$

(iii) We invoke localization theorem (see, e.g., Siegmund et al. [2010] and Yakir [2013], for details) to show the asymptotic independence of local fields and global term in the equation above. Matching the equation in step (ii) above to (24), we know that

$$\lambda = \sum_{B=2}^w e^{\psi_B(\theta_B) - \theta_B b} \mathbb{E}_B \left[ \frac{M_B}{S_B} e^{-\tilde{\ell}_B + \log M_B}; \tilde{\ell}_B + \log M_B \geq 0 \right].$$

Therefore, the ARL can be approximated as $1/\lambda$. To evaluate the expectation in the above equation, we investigate the covariance structure of this random field as in Lemma 4. As suggested by Xie and Xie [2021], we can just consider taking $B_1 = B_2 = B$ in Lemma 4 when evaluating the covariance and doing this can still maintain the required level of accuracy. Therefore, the covariance structure is exactly the same with the one in Li et al. [2019] under the online setting, and the rest of the proof directly follows the proof therein.
In addition to the proof of the ARL approximation, we also show how to further simplify the expression when we only consider the first two moments’ information as follows:

**Derivation of equation (12).** As $\mu \to \infty$, we have

$$
\nu(\mu) \approx \frac{(2/\mu)(\Phi(\mu/2) - 0.5)}{(\mu/2)\Phi(\mu/2) + \phi(\mu/2)} = \frac{2}{\mu^2} (1 + o(1)).
$$

Therefore, by plugging (11) into (10), we have $E_\infty[T_w]$ expressed as follows:

$$
\sqrt{\frac{2\pi}{b}} \left\{ \sum_{B=2}^{w} e^{-b^2/2} \left( \frac{2B-1}{B(B-1)} \frac{2}{b \sqrt{\frac{2(2B-1)}{B(B-1)}}} \right)^2 \right\}^{-1}
$$

$$
= \sqrt{\frac{2\pi}{b}} e^{b^2/2} \left\{ \sum_{B=2}^{w} \left( \frac{2B-1}{B(B-1)} \frac{B(B-1)}{b^2(2B-1)} \right)^{-1} (1 + o(1)) \right\}
$$

$$
= \frac{\sqrt{2\pi} b e^{b^2/2}}{w} (1 + o(1)).
$$

### C Proof of Lemma 3

In this subsection, we will formally prove Lemma 3. This proof involves many technical details, but the high level idea is simple. To help readers understand this simple idea, we first sketch this proof. For notational simplicity, we denote the detection statistic of our procedure as:

$$
Z_t = \max_{B \in [2:w]} Z_B(t),
$$

where $Z_B(t)$ is the Scan $B$-statistic defined in (3).

In the following, to improve readability, we first sketch of Lemma 3 since its proof involves several technical lemmas. After that, we will prove all technical results rigorously.

**Proof of Lemma 3.** On one hand, for large enough window length choice (15), the detection procedure will not stop too early nor too late. To be precise, for any $w \geq 7b/(\rho D(p, q))$ and $t_1 = b/(4\rho D(p, q))$, as $b \to \infty$, we will have

$$
P_0(T_w < t_1) \leq 2t_1^2 \left( \exp\left\{ -\frac{Nb^2}{512K^2} \right\} + \exp\left\{ -\frac{b^2}{128K^2} \right\} \right),
$$

$$
P_0(T_w > w) \leq 2 \exp\left\{ -\frac{Nb^2}{512K^2} \right\} + 2 \exp\left\{ -\frac{b^3}{128\rho D(p, q)K^2} \right\}. \tag{28}
$$

On the other hand, for any $t \in [t_1, w]$, we can show that for our detection statistic $Z_t = \max_{B \in [2:w]} Z_B(t)$, the maximum is attained at $B = t$ with high probability. To be
For any \( t \in [t_1, w] \), as \( b \to \infty \), we will have

\[
\mathbb{P}_0 \left( A_{t,1}^c \right) \leq \frac{b}{16(\rho D^2(p, q) \lor 1)} \left( \exp \left\{ -\frac{Nb^2}{2^{15}(\rho D^2(p, q) \lor 1)K^2} \right\} + \exp \left\{ -\frac{b^2}{2^{13}(\rho D^2(p, q) \lor 1)K^2} \right\} \right) (1 + o(1)),
\]

(30)

\[
\mathbb{P}_0(A_{t,2}^c) \leq 6 \left( \exp \left\{ -\frac{b\rho D(p, q)}{2^{13}(\rho D(p, q) \lor 1)K^2} \right\} + \exp \left\{ -\frac{N\rho D^2(p, q)}{512K^2} \right\} \right) (1 + o(1)),
\]

(31)

where the superscript \( ^c \) denotes the complement of a set.

Now, equations (27) and (28) tell us \( T_w \) takes value from \([t_1, w]\) with high probability, and equations (30) and (31) show that at time \( t \in [t_1, w] \), with high probability, the maximum over \( B \in [2 : w] \) is attained at \( B = t \). These enable us to calculate the expectation of detection statistic at the stopping time as follows:

\[
\left| \mathbb{E}_0[Z_{T_w}] - \rho D(p, q)\mathbb{E}_0[T_w] \right| \leq \left( \rho D(p, q) + 12K \exp \left\{ -\frac{N\rho D^2(p, q)}{512K^2} \right\} \right)
\]

\[
+ 12K \exp \left\{ -\frac{b\rho D(p, q)}{2^{13}(\rho D(p, q) \lor 1)K^2} \right\} (1 + o(1)).
\]

(32)

Finally, since the overshoot, which is the occurrence of detection statistic exceeding its target threshold \( b \), will be \( o(1) \) given that \( b \to \infty \), we will have the detection statistic at \( t = T_w \) approximated as follows:

\[
\mathbb{E}_0[Z_{T_w}] = b(1 + o(1)).
\]

(33)

By equations (32) and (33), we complete the proof. Detailed statement and proof of equations (27) and (28) can be found in Lemma 8. Equations (30) and (31) are proved in Lemma 9. Equation (32) is proved in Lemma 10. We defer all those technical details to Appendix C.

Now, we will present and prove the technical lemmas used in the above proof sketch. We will start with an important concentration result for the Scan \( B \)-statistic. Recall its definition
in (3) as follows:
\[ Z_B(t) = \frac{\sum_{i=1}^{N} \hat{D}(X_B^{(i)}, Y_B(t))}{N \sqrt{\text{Var}_\infty(\hat{D}(t))}}. \]

As we can see, the difficulty comes from that the average in Scan B-statistic is taken over correlated random variables since we reuse the post-change block \( Y_B(t) \). Nevertheless, we can leverage the Azuma’s inequality for martingale and get the following concentration result for Scan B-statistic:

**Lemma 7** (Concentration inequality for Scan B-statistic). For any window length choice \( w \geq 2 \), at time step \( t \leq w \), for any block size \( B_0 \in [2 : w] \), the Scan B-statistic \( Z_{B_0}(t) \) (3) satisfies the following concentration inequality: for any \( z_1, z_2 > 0 \) and \( z = z_1 + z_2 \), we have

\[
P_0(|Z_{B_0}(t) - \mu_{B_0}(t)| \geq z) \leq 2 \exp \left\{ -\frac{Nz_1^2}{32K^2} \right\} + 2 \exp \left\{ -\frac{B_0z_2^2}{16K^2} \right\},
\]

where the expectation \( \mu_{B_0}(t) \) has explicit formula as shown in Lemma 6.

**Proof.** For \( z = z_1 + z_2 > 0 \) (with \( z_1, z_2 > 0 \)), event \{\|x - y\| > z\} is a subset of event \{\|x - w\| \geq z_1\} \cup \{\|w - y\| \geq z_2\}, since if we take the complement we can easily show \{\|x - w\| < z_1\} \cap \{\|w - y\| < z_2\} \subset \{\|x - y\| < z\}. Let us define the following conditional expectation (which is a r.v.),

\[ f(Y_B(t)) = \mathbb{E}_0 \left[ \hat{D}(X_B^{(1)}, Y_B(t)) \mid Y_B(t) \right]. \]

We will have

\[
\{|Z_{B_0}(t) - \mu_{B_0}(t)| \geq z\} \subset \{|Z_B(t) - f(Y_B(t))| \geq z_1\} \cap \{|f(Y_B(t)) - \mu_B(t)| \geq z_2\}. \quad (35)
\]

On one hand, we can show

\[
\sum_{i=1}^{N} \hat{D}(X_B^{(i)}, Y_B(t)) - Nf(Y_B(t))
\]

is a martingale with its difference bounded within \([-4K, 4K]\). Therefore, by Azuma’s inequality, for any \( z > 0 \), we have

\[
P_0(|Z_B(t) - f(Y_B(t))| \geq z) = P_0 \left( \left| \sum_{i=1}^{N} \hat{D}(X_B^{(i)}, Y_B(t)) - Nf(Y_B(t)) \right| \geq Nz \right)
\]

\[
\leq 2 \exp \left\{ -\frac{(Nz)^2}{32K^2N} \right\} = 2 \exp \left\{ -\frac{z^2}{32K^2} \right\}.
\]

On the other hand, we can apply similar concentration inequality on the empirical estimator of MMD (with sample size \( B \)), which can be derived from the large deviation bound on U-statistics (cf. Hoeffding [1994] or Theorem 10 in Gretton et al. [2012a]). To be
precise, for any $z > 0$, we have:

$$
\mathbb{P}_0 \left( |f(Y_B(t)) - \mu_B(t)| \geq z \right) \leq 2 \exp \left\{ -\frac{Bz^2}{16K^2} \right\}.
$$

Finally, by (35) and the above two concentration inequalities, we can prove (34). □

**Lemma 8.** Under $H_1$ where $p \neq q$, assume change occurs at time step 0, Under Assumptions (A1) and (A2), as $b \to \infty$, for any $N > 0$, we have:

1. for any $t_1 < b/(2\rho D(p,q))$, where constant $\rho$ is defined in (14), the following holds for the stopping time of online kernel CUSUM $T_w$ (8):

$$
\mathbb{P}_0 (T_w < t_1) \leq 2t_1 \left( \exp \left\{ -\frac{Nb^2}{512K^2} \right\} + \exp \left\{ -\frac{b^2}{128K^2} \right\} \right).
$$

2. for any $t_2 \geq 4b/(\rho D(p,q))$, as long as we choose $w > 3b/(\rho D(p,q))$, the following holds for $T_w$:

$$
\mathbb{P}_0 (T_w > t_2) \leq 2 \exp \left\{ -\frac{Nb^2}{512K^2} \right\} + 2 \exp \left\{ -\frac{b^3}{128\rho D(p,q)K^2} \right\}^{t_2-3b/(\rho D(p,q))}
$$

Proof. Let us prove (2) first. Intuitively, it is a rare event that $T_w$ exceeds $t_2$ for such large $t_2$. By definition, we have:

$$
\mathbb{P}_0 (T_w > t_2) = \mathbb{P}_0 (\max_{2 \leq t \leq t_2} Z_t < b) = \prod_{t=2}^{t_2} \mathbb{P}_0 (Z_t < b)
$$

$$
= \prod_{t=2}^{t_2} \prod_{B=2}^{t \wedge w} \mathbb{P}_0 (Z_B(t) < b).
$$

Here, due to our choice of $t_2$, we have

$$
\mathbb{P}_0 (T_w > w) \leq \prod_{t=3b/(\rho D(p,q))}^{t_2} \mathbb{P}_0 (Z_{t\wedge w}(t) < b).
$$

Notice that, as long as $B > 3b/(\rho D(p,q)) > 4/3$, we have

$$
\mathbb{E}_0 [Z_B(t)] = \rho D(p,q) \sqrt{B(B-1)} \geq \rho D(p,q) \frac{B}{2} > \frac{3b}{2},
$$

where the last inequality comes from the fact that $\sqrt{x(x-1)} > x/2, \forall x > 4/3$. The condition $3b/(\rho D(p,q)) > 4/3$ can easily be satisfied as $b$ goes to infinity. Finally, by Lemma 7, we
have
\[
\mathbb{P}_0(T_w > t_2) \leq \left( 2 \exp \left\{ - \frac{Nb^2}{512K^2} \right\} + 2 \exp \left\{ - \frac{b^3}{128\rho D(p,q)K^2} \right\} \right)^{t_2 - 3b/\rho D(p,q)}.
\]

Next, let us prove (1). We have:
\[
\mathbb{P}_0(T_w < t_1) = \mathbb{P}_0(\max_{0 < t < t_1} Z_t \geq b) \\
\leq \sum_{t=1}^{t_1} \mathbb{P}_0(Z_t \geq b) \leq \sum_{t=1}^{t_1} \sum_{B=2}^{t \wedge w} \mathbb{P}_0(Z_B(t) \geq b).
\]

Similarly, notice that
\[
\mathbb{E}_0[Z_B(t)] = \rho D(p,q)\sqrt{B(B-1)} \leq \rho D(p,q)t_1 < b/2,
\]
where the last inequality is guaranteed by the choice of \( t_1 \). Again, by Lemma 7, we have
\[
\mathbb{P}_0(T_w < t_1) = \mathbb{P}_0(\max_{0 < t < t_1} Z_t \geq b) \leq \sum_{t=1}^{t_1} \mathbb{P}_0(Z_t \geq b) \leq \sum_{t=1}^{t_1} \sum_{B=2}^{t \wedge w} \mathbb{P}_0(Z_B(t) \geq b) \\
\leq t_1^2 \left( 2 \exp \left\{ - \frac{N(b/4)^2}{32K^2} \right\} + 2 \exp \left\{ - \frac{2(b/4)^2}{16K^2} \right\} \right).
\]

Now we complete the proof.

\[\square\]

**Lemma 9.** Under \( H_1 \), where \( p \neq q \) and we assume change occurs at time step 0, and Under Assumptions (A1) and (A2), we choose
\[
t_1 = \frac{b}{4\rho D(p,q)}, \quad w > \frac{3b}{\rho D(p,q)},
\]
which satisfies the constraints in Lemma 8 and constant \( \rho \) is defined in (14), we further take
\[
B_1(t) = \frac{t}{8(\rho D(p,q) \vee 1)},
\]
for any \( t \in [t_1, w] \), recall that we denote
\[
A_{1,t} = \left\{ \max_{2 \leq B \leq t \wedge w} Z_B(t) = \max_{B_1(t) \leq B \leq t} Z_B(t) \right\},
\]
\[
A_{2,t} = \left\{ \max_{B_1(t) \leq B \leq t} Z_B(t) = Z_i(t) \right\},
\]

39
for any $N > 0$, as long as $t_1 > 4/3$, the following holds
\[
\mathbb{P}_0(A_{t,1}^c) \leq 2 \left( \exp \left\{ -\frac{Nb^2}{2^{13}(\rho \mathcal{D}^2(p,q) \vee 1)K^2} \right\} + \exp \left\{ -\frac{(\rho \mathcal{D}^2(p,q) \wedge 1)b^3}{2^{14}\rho \mathcal{D}(p,q)^3K^2} \right\} \right)
\]
\[
+ \frac{b}{16(\rho \mathcal{D}^2(p,q) \vee 1)} \exp \left\{ -\frac{Nb^2}{2^{15}(\rho \mathcal{D}^2(p,q) \vee 1)K^2} \right\}
\]
\[
+ \frac{b}{16(\rho \mathcal{D}^2(p,q) \vee 1)} \exp \left\{ -\frac{b^2}{2^{13}(\rho \mathcal{D}^2(p,q) \vee 1)K^2} \right\}.
\]

Therefore, as $b \to \infty$, we have
\[
\mathbb{P}_0(A_{t,1}^c) \leq \frac{b}{16(\rho \mathcal{D}^2(p,q) \vee 1)} \left( \exp \left\{ -\frac{Nb^2}{2^{15}(\rho \mathcal{D}^2(p,q) \vee 1)K^2} \right\}
\]
\[
+ \exp \left\{ -\frac{b^2}{2^{13}(\rho \mathcal{D}^2(p,q) \vee 1)K^2} \right\} \right)(1 + o(1)).
\]

Similarly, as $b \to \infty$, we have
\[
\mathbb{P}_0(A_{t,2}^c) \leq 6 \left( \exp \left\{ -\frac{N\rho \mathcal{D}^2(p,q)}{512K^2} \right\} + \exp \left\{ -\frac{b\rho \mathcal{D}(p,q)}{2^{13}(\rho \mathcal{D}(p,q) \vee 1)K^2} \right\} \right)(1 + o(1)).
\]

Proof. Firstly, we will deal with $\mathbb{P}_0(A_{t,1}^c)$. We let
\[
t_3 = (\rho \mathcal{D}(p,q) \wedge 1)t/4
\]
and denote
\[
\tilde{A}_{1,t} = \left\{ \max_{B_1(t) \leq B \leq t} Z_B(t) > t_3 \right\}.
\]

Then, we have
\[
\mathbb{P}_0(\tilde{A}_{1,t}) \geq \mathbb{P}_0(Z_t(t) > t_3) = 1 - \mathbb{P}_0(Z_t(t) \leq t_3).
\]

Notice that for any $t \geq t_1 > 4/3$, we have
\[
\mathbb{E}_0[Z_t(t)] = \rho \mathcal{D}(p,q)\sqrt{t(t-1)} \geq \rho \mathcal{D}(p,q)\frac{t}{2} \geq 2t_3,
\]
where the first inequality comes from the fact that $\sqrt{x(x-1)} > x/2$, $\forall x > 4/3$ and the
second inequality comes from our choice of \( t_3 \) \((36)\). Therefore, by Lemma 7, we have
\[
\begin{align*}
\mathbb{P}_0(\tilde{A}_{1,t}^c) &\leq \mathbb{P}_0(Z_t(t) \leq t_3) \\
&\leq 2 \left( \exp \left\{ -\frac{Nt_3^2}{32K^2} \right\} + \exp \left\{ -\frac{t^2}{16K^2} \right\} \right)
\end{align*}
\]  
(37)

Now, we have
\[
\begin{align*}
\mathbb{P}_0(A_{1,t}^c) &= \mathbb{P}_0(A_{1,t}^c \cap \tilde{A}_{1,t}^c) + \mathbb{P}_0(A_{1,t}^c \cap \tilde{A}_{1,t}) \\
&\leq \mathbb{P}_0(\tilde{A}_{1,t}^c) + \mathbb{P}_0(A_{1,t}^c \cap \tilde{A}_{1,t}) \\
&\leq \mathbb{P}_0(\tilde{A}_{1,t}^c) + \mathbb{P}_0 \left( \max_{2 \leq B < B_1(t)} Z_B(t) > t_3 \right) \\
&\leq \mathbb{P}_0(\tilde{A}_{1,t}^c) + \sum_{B=2}^{B_1(t)} \mathbb{P}_0(Z_B(t) > t_3).
\end{align*}
\]

Similarly, notice that \( \mathbb{E}_0[Z_B(t)] \leq \rho \mathcal{D}(p,q)B_1(t) \leq t_3/2 \), we have
\[
\begin{align*}
\sum_{B=2}^{B_1(t)} \mathbb{P}_0(Z_B(t) > t_3) &\leq 2B_1(t) \left( \exp \left\{ -N(t_3/2)^2 \right\} + \exp \left\{ -2(t_3/2)^2 \right\} \right) \\
&\leq \frac{t}{4(\rho \mathcal{D}(p,q) \vee 1)} \left( \exp \left\{ -\frac{N(\rho \mathcal{D}(p,q) \wedge 1)t^2}{2048K^2} \right\} \\
&\quad + \exp \left\{ -\frac{(\rho \mathcal{D}(p,q) \wedge 1)t^2}{512K^2} \right\} \right).
\end{align*}
\]

Notice that we already have bounded \( \mathbb{P}_0(\tilde{A}_{1,t}^c) \) in \((37)\). Here, we can observe that the rate of the above probability upper bound vanishing to zero depends on \( t \in [t_1,w] \). The choice of \( t_1 \) in Lemma 8 only needs to satisfy \( t_1 < b/(2\rho \mathcal{D}(p,q)) \). Therefore, we will choose \( t_1 = b/(4\rho \mathcal{D}(p,q)) \) here. This gives us the desired upper bound on \( \mathbb{P}_0(A_{1,t}^c) \).

Next, we handle event \( A_{1,2} \). Recall that we denote the expectation of \( Z_B(t) \) as \( \mu_B(t) \), which is evaluated in Lemma 6. Denote \( \delta_2 = \rho \mathcal{D}(p,q)/2 \), and we have
\[
\begin{align*}
\mu_B(t) - \mu_{B-1}(t) &= \rho \mathcal{D}(p,q)(\sqrt{B(B-1)} - \sqrt{(B-1)(B-2)}) \\
&= \frac{2\sqrt{B-1}}{\sqrt{B}+\sqrt{B-2}} \rho \mathcal{D}(p,q) > \rho \mathcal{D}(p,q).
\end{align*}
\]

Here, the last inequality comes from the fact that \( \frac{2\sqrt{B-1}}{\sqrt{B}+\sqrt{B-2}} > 1 \), which can be verified by taking square and re-arranging terms. Now, by definition, we have
\[
\{ |Z_B(t) - \mu_B(t)| < \mu(t) - \mu_B(t) - \delta_2, \; B \in [B_1(t), t-1] \} \cap \{ |Z_t(t) - \mu_t(t)| < \delta_2 \} \subset A_{1,2}.
\]
This gives us
\[ \mathbb{P}_0(A_{t,2}^c) \leq \sum_{B=B_1(t)}^{t-1} \mathbb{P}_0(|Z_B(t) - \mu_B(t)| < \mu_t(t) - \mu_B(t) - \delta_2) + \mathbb{P}_0(|Z_t(t) - \mu_t(t)| < \delta_2) \]
\[ \leq \sum_{B=B_1(t)}^{t-1} \mathbb{P}_0(|Z_B(t) - \mu_B(t)| < \delta_2(1 + 2(t - 1 - B))) + \mathbb{P}_0(|Z_t(t) - \mu_t(t)| < \delta_2). \]

Notice that the last inequality comes from the following derivation: \( \mu_t(t) - \mu_B(t) - \delta_2 = \mu_t(t) - \mu_{B+1}(t) + (\mu_{B+1}(t) - \mu_B(t) - \delta_2). \) As we have shown above, \( \mu_{B+1}(t) - \mu_B(t) > \rho \mathcal{D}(p, q) = 2\delta_2, \) therefore we have
\[ \mu_t(t) - \mu_B(t) - \delta_2 > \mu_t(t) - \mu_{B+1}(t) + \delta_2 = 2\delta_2(\sqrt{t(t-1)} - \sqrt{B(B+1)}) + \delta_2. \]

What remains to be proved is \( \sqrt{t(t-1)} - \sqrt{B(B+1)} > t - 1 - B. \) Again, this can be verified by taking square and re-arranging terms. We omit further details for this simple derivation here. Now, by Lemma 7, we have
\[ \mathbb{P}_0(A_{t,2}^c) \leq 2 \sum_{B=B_1(t)}^{t-1} \left( \exp \left\{ - \frac{N[(t - B - 1/2)\delta_2^2]}{32K^2} \right\} + \exp \left\{ - \frac{B[(t - B - 1/2)\delta_2^2]}{16K^2} \right\} \right) \]
\[ + 2 \exp \left\{ - \frac{N(\delta_2^2/2)}{32K^2} \right\} + 2 \exp \left\{ - \frac{t(\delta_2^2/2)}{16K^2} \right\} \]
\[ \leq 2 \sum_{B=B_1(t)}^{t-1} \left( \exp \left\{ - \frac{N[(t - B - 1/2)\delta_2^2]}{32K^2} \right\} + \exp \left\{ - \frac{B_1(t)[(t - B - 1/2)\delta_2^2]}{16K^2} \right\} \right) \]
\[ + 2 \exp \left\{ - \frac{N(\delta_2^2/2)}{32K^2} \right\} + 2 \exp \left\{ - \frac{t(\delta_2^2/2)}{16K^2} \right\} \]

By the summation of geometric series, we can bound the above summation. We omit further details on the derivation.

We should remark that \( B_1(t) = t/(8(\rho \mathcal{D}(p, q) \lor 1)) \geq t_1/(8(\rho \mathcal{D}(p, q) \lor 1)) \) cannot be overly small. That is to say, we cannot choose overly small \( t_1 \) in Lemma 8, which happens to agree with what we have done for event \( A_{t,1} \) above. Here, the reason is that the concentration of Scan \( B \)-statistic depends on not only the number of blocks \( N \), but also the block size \( B \), which is lower bounded by \( B_1(t) \) in the above analysis. The above choice \( t_1 = b/(4\rho \mathcal{D}(p, q)) \) suffices to give a good enough rate of concentration. Now, as \( N \to \infty \) and \( b \to \infty \), we have
\[ \mathbb{P}_0(A_{t,2}^c) \leq 6 \left( \exp \left\{ - \frac{N \delta_2^2}{128K^2} \right\} + \exp \left\{ - \frac{B_1(t)\delta_2^2}{64K^2} \right\} \right) (1 + o(1)). \]

Moreover, by our choice \( t_1 = b/(4\rho \mathcal{D}(p, q)) \), we can further bound the above term for all
\[ t \in [t_1, w] \text{ as follows:} \]
\[ \mathbb{P}_0(A_{i,2}^c) \leq 6 \left( \exp \left\{ - \frac{N \rho \mathcal{D}^2(p, q)}{512 K^2} \right\} + \exp \left\{ - \frac{b \rho \mathcal{D}(p, q)}{2^{13}(\rho \mathcal{D}(p, q) \vee 1) K^2} \right\} \right) (1 + o(1)). \]

We complete the proof. \[ \square \]

**Lemma 10.** Under \( H_1 \), where \( p \neq q \) and we assume change occurs at time step 0, and Under Assumptions (A1) and (A2), we choose
\[ w \geq \frac{7b}{\rho \mathcal{D}(p, q)}, \]
where constant \( \rho \) is defined in (14), as \( b \to \infty \), for any \( N > 0 \), we have
\[ |\mathbb{E}_0[Z_{T_w}] - \rho \mathcal{D}(p, q)\mathbb{E}_0[T_w]| \leq \left( \rho \mathcal{D}(p, q) + 12K \exp \left\{ - \frac{N \rho \mathcal{D}^2(p, q)}{512 K^2} \right\} \right. \]
\[ \left. + 12K \exp \left\{ - \frac{b \rho \mathcal{D}(p, q)}{2^{13}(\rho \mathcal{D}(p, q) \vee 1) K^2} \right\} \right) (1 + o(1)). \]

**Proof.** Step 1: We choose \( w = 4b/(\rho \mathcal{D}(p, q)) \) as in Lemma 9. For any \( t \in [t_1, w] \), recall the definitions of events \( A_{1, t} \) and \( A_{2, t} \) in (29), we will have
\[ \mathbb{E}_0 \left[ \max_{2 \leq B \leq B_{\text{max}}} Z_B(t) \right] = \mathbb{E}_0 \left[ \max_{B_1 \leq B \leq B_2} Z_B(t); A_1 \right] + \mathbb{E}_0 \left[ \max_{2 \leq B \leq B_{\text{max}}} Z_B(t); A_1^c \right] \]
\[ = \mathbb{E}_0 \left[ \max_{B_1 \leq B \leq B_2} Z_B(t) \right] + \mathbb{E}_0 \left[ \max_{2 \leq B \leq B_{\text{max}}} Z_B(t); A_1^c \right] \]
\[ = \mathbb{E}_0 \left[ \max_{B_1 \leq B \leq B_2} Z_B(t) - Z_t(t) \right] + \mathbb{E}_0 [Z_t(t)] + \mathbb{E}_0 \left[ \max_{2 \leq B \leq B_{\text{max}}} Z_B(t); A_1^c \right] \]
\[ = \rho \mathcal{D}(p, q) \sqrt{t(t - 1)} \]
\[ + \mathbb{E}_0 \left[ \max_{B_1 \leq B \leq B_2} Z_B(t) - Z_t(t); A_5 \right] + \mathbb{E}_0 \left[ \max_{2 \leq B \leq B_{\text{max}}} Z_B(t); A_1^c \right] \]
\[ = \rho \mathcal{D}(p, q) t + \rho \mathcal{D}(p, q)(\sqrt{t(t - 1)} - t) \]
\[ + \mathbb{E}_0 \left[ \max_{B_1 \leq B \leq B_2} Z_B(t) - Z_t(t); A_5 \right] + \mathbb{E}_0 \left[ \max_{2 \leq B \leq B_{\text{max}}} Z_B(t); A_1^c \right] . \]

As we have shown in Lemma 9, we can easily bound \( \mathbb{P}_0(A_1^c), \mathbb{P}_0(A_5) \). We have, \( \forall t \in [t_1, w], \)
\[ \left| \mathbb{E}_0 \left[ \max_{B \in [2:w]} Z_B(t) \right] - \rho \mathcal{D}(p, q)t \right| \leq \rho \mathcal{D}(p, q) + 2K(\mathbb{P}_0(A_1^c) + \mathbb{P}_0(A_5)) \] (38)

Step 2: We are ready to evaluate \( \mathbb{E}_0[Z_{T_w}] \). Notice that \( -2K \leq Z_B(t) \leq 2K \), the partition
Theorem gives us
\[
E_0[Z_{T_w}] = E_0[Z_{T_w}|T_w < t_1]P_0(T_w < t_1) \\
+ E_0[Z_{T_w}|T_w > w]P_0(T_w > w) \\
+ E_0[Z_{T_w}|t_1 \leq T_w \leq w]P_0(t_1 \leq T_w \leq w).
\]
Combining everything above, we have
\[
\left|E_0[Z_{T_w}] - E_0[Z_{T_w}|t_1 \leq T_w \leq w]\right| \leq 2K(P_0(T_w > w) + P_0(T_w < t_1) + 1 - P_0(t_1 \leq T_w \leq w)) \\
= 4K(P_0(T_w > w) + P_0(T_w < t_1)).
\]
Recall that in Lemma 8, we have bounded both probabilities.

In the above equation, \(E_0[Z_{T_w}|t_1 \leq T_w \leq w]\) can be evaluated by replacing \(t\) with \([T_w|t_1 \leq T_w \leq w]\) and taking expectation with respect to \([T_w|t_1 \leq T_w \leq w]\). Combing everything gives us
\[
\left|E_0[Z_{T_w}] - \rho D(p, q)E_0[T_w|t_1 \leq T_w \leq w]\right| \\
\leq \rho D(p, q) + 2K P_0(A_1) + 2K P_0(A_2) + 4K P_0(T_w > w) + 4K P_0(T_w < t_1).
\]

**Step 3:** Now, we only need to bound the difference between \(E_0[T_w|t_1 \leq T_w \leq w]\) and \(E_0[T_w]\).

Again, we apply partition theorem and will get
\[
E_0[T_w] = E_0[T_w|T_w < t_1]P_0(T_w < t_1) \\
+ E_0[T_w|T_w > w]P_0(T_w > w) \\
+ E_0[T_w|t_1 \leq T_w \leq w]P_0(t_1 \leq T_w \leq w).
\]
On one hand, it is easy to show that
\[
0 \leq E_0[T_w|T_w < t_1]P_0(T_w < t_1) \leq t_1 P_0(T_w < t_1).
\]
On the other hand, we have
\[
0 \leq E_0[T_w|T_w > w] = \sum_{\tau = w}^\infty P_0(T_w > \tau) + w P_0(T_w > w).
\]
Here, for \(\tau \geq w = 4b/\rho D(p, q)\), Lemma 8 and its proof therein give us
\[
P_0(T_w > \tau) \leq \left(2 \exp \left\{-\frac{Nb^2}{512K^2}\right\} + 2 \exp \left\{-\frac{b^3}{128\rho D(p, q)K^2}\right\}\right)^{\tau - 4b/\rho D(p, q)}.
\]
Therefore, we have
\[
\sum_{\tau=w}^{\infty} \mathbb{P}_0(T_w > \tau) \leq \frac{\left( 2 \exp \left\{ -\frac{N b^2}{512 K^2} \right\} + 2 \exp \left\{ -\frac{b^3}{128 \rho D(p, q) K^2} \right\} \right)^{w-3b/(\rho D(p, q))}}{1 - \left( 2 \exp \left\{ -\frac{N b^2}{512 K^2} \right\} + 2 \exp \left\{ -\frac{b^3}{128 \rho D(p, q) K^2} \right\} \right)}.
\]

Under the technical condition
\[
b \geq (4\rho D(p, q)/9 \lor 32\sqrt{2\log 2 K}), \tag{39}
\]
we are able to show
\[
2 \exp \left\{ -\frac{N b^2}{512 K^2} \right\} + 2 \exp \left\{ -\frac{b^3}{128 \rho D(p, q) K^2} \right\} \leq 4 \exp \left\{ -\frac{b^2}{128 K^2} \left( \frac{N}{4} \wedge \frac{b}{\rho D(p, q)} \right) \right\} \leq \exp \left\{ -\frac{b^2}{256 K^2} \left( \frac{N}{4} \wedge \frac{b}{\rho D(p, q)} \right) \right\} < 1/2.
\]

Thus, we have
\[
\sum_{\tau=w}^{\infty} \mathbb{P}_0(T_w > \tau) \leq 2 \exp \left\{ -\frac{b^2}{256 K^2} \left( \frac{N}{4} \wedge \frac{b}{\rho D(p, q)} \right) (w - 3b/(\rho D(p, q))) \right\}.
\]

As \( b \to \infty \), the technical condition (39) automatically holds (since we choose \( w \geq 7b/(\rho D(p, q)) \)). Therefore, we have
\[
w \mathbb{P}_0(T_w > w) + \sum_{\tau=w}^{\infty} \mathbb{P}_0(T_w > \tau) \leq w \exp \left\{ -\frac{b^2}{256 K^2} \left( \frac{N}{4} \wedge \frac{b}{\rho D(p, q)} \right) (w - 3b/(\rho D(p, q))) \right\} + 2 \exp \left\{ -\frac{b^2}{128 K^2} \left( \frac{N}{4} \wedge \frac{b}{\rho D(p, q)} \right) (w - 3b/(\rho D(p, q))) \right\}. \tag{40}
\]

We notice that for any \( a_1, a_2 > 0 \) and \( a_1 - a_2 \geq \log 2 \), we will have
\[
xe^{-a_1 x} < e^{-a_2 x}, \quad \forall x > 1,
\]
since \( x < e^{(a_1-a_2)x} \leq e^{x \log 2} = 2^x \). Under the technical condition (39), we have
\[
\frac{1}{2} \frac{b^2}{256 K^2} \left( \frac{N}{4} \wedge \frac{b}{\rho D(p, q)} \right) \geq \frac{1}{2} \frac{32^2 \times 2 \times \log 2 K^2}{256 K^2} \frac{1}{4} = \log 2.
\]

That is to say, for \( w > 1 \) (which holds for large enough \( b \) and our choice \( w \geq 7b/(\rho D(p, q)) \)),

45
we will have

$$0 \leq w P_0(T_w > w) + \sum_{\tau = w}^{\infty} P_0(T_w > \tau)$$

$$\leq \exp \left\{ - \frac{b^2}{512 K^2} \left( \frac{N}{4} \wedge \frac{b}{\rho D(p, q)} \right) (w - 6b/(\rho D(p, q))) \right\}$$

$$+ 2 \exp \left\{ - \frac{b^2}{128 K^2} \left( \frac{N}{4} \wedge \frac{b}{\rho D(p, q)} \right) (w - 3b/(\rho D(p, q))) \right\}$$

$$\leq 3 \exp \left\{ - \frac{b^2}{512 K^2} \left( \frac{N}{4} \wedge \frac{b}{\rho D(p, q)} \right) (w - 6b/(\rho D(p, q))) \right\}$$

$$\leq 3 \exp \left\{ - \frac{b^3}{512 \rho D(p, q) K^2} \left( \frac{N}{4} \wedge \frac{b}{\rho D(p, q)} \right) \right\},$$

where the last inequality comes from our choice $w \geq 7b/(\rho D(p, q))$.

**Step 4:** Finally, combining the above derivations, we have

$$|E_0[T_w] - E_0[T_w|T_w \leq w|]$$

$$\leq t_1 P_0(T_w < t_1) + w(1 - P_0(t_1 \leq T_w \leq w)) + E_0[T_w|T_w > w] P_0(T_w > w).$$

This gives us

$$|E_0[Z_{T_w}] - \rho D(p, q) E_0[T_w]| \leq 2K (P_0(A_1^c) + P_0(A_2^c)) + 4K (P_0(T_w > w) + P_0(T_w < t_1))$$

$$+ \rho D(p, q) \left( 1 + t_1 P_0(T_w < t_1) + w(1 - P_0(t_1 \leq T_w \leq w)) + E_0[T_w|T_w > w] P_0(T_w > w) \right).$$

As $b \to \infty$, we only consider the leading term and will get

$$|E_0[Z_{T_w}] - \rho D(p, q) E_0[T_w]| \leq \left( \rho D(p, q) + 12K \exp \left\{ - \frac{N \rho D^2(p, q)}{512 K^2} \right\} \right)$$

$$+ 12K \exp \left\{ - \frac{b \rho D(p, q)}{2^{13}(\rho D(p, q) \vee 1) K^2} \right\} (1 + o(1)).$$

Now, we complete the proof. \qed

**D Proof of Theorem 1**

For notational simplicity, let us denote the detection statistic in the oracle detection procedure as follows:

$$Z^o_i = \max_{B \in [2:t]} Z_B(t), \quad (41)$$
where $Z_B(t)$ is the Scan $B$-statistic defined in (3).

**Proof.** It is important to recognize that we need to specify the choice of window length $w$ when we control ARL by properly choosing threshold $b$. Thus the question boils down to the ARL approximation for given $b$ and $w$. We will first show that it requires $w \sim b$ to guarantee $\varepsilon$-performance loss and therefore we need to show how to control ARL given the relationship $w \sim b$. Luckily, we can verify that the ARL approximation in Lemma 2 allows the window length $w$ to go to infinity at any polynomial rate with respect to $b$ (which includes the case $w \sim b$). To begin with, we prove that we need $w \sim b$ when there is a change at time zero to meet the $\varepsilon$-performance loss constraint.

**Under $H_1$.** We assume change occurs at time step 0. Notice that the difference between our procedure and the oracle procedure only exists when $T_w > w$, otherwise, at time $t \leq w$, the scan region of our procedure would be $B \in [2 : w \land t]$, which is exactly the same with that of the oracle procedure. That is, we can upper bound the performance loss as follows:

$$0 \leq \mathbb{E}_0[T_w] - \mathbb{E}_0[T_o] \leq \mathbb{E}_0[T_w | T_w > w] P_0(T_w > w)$$

$$= w P_0(T_w > w) + \sum_{\tau = w}^{\infty} P_0(T_w > \tau).$$

We re-use the derivations to bound $w P_0(T_w > w)$ and $\sum_{\tau = w}^{\infty} P_0(T_w > \tau)$ in the step 3 in the proof of Lemma 10. Notice that, as $\gamma \to \infty$, by choosing $b \sim \log \gamma$, the technical conditions in that proof hold automatically. Thus, we have

$$0 \leq \mathbb{E}_0[T_w] - \mathbb{E}_0[T_o] \leq \exp \left\{ - \frac{b^2}{512 K^2} \left( \frac{N}{4} \wedge \frac{b}{\rho D(p, q)} \right) \left( w - \frac{6b}{\rho D(p, q)} \right) \right\}$$

$$+ 2 \exp \left\{ - \frac{b^2}{128 K^2} \left( \frac{N}{4} \wedge \frac{b}{\rho D(p, q)} \right) \left( w - \frac{3b}{\rho D(p, q)} \right) \right\}$$

$$\leq 3 \exp \left\{ - \frac{b^2}{512 K^2} \left( \frac{N}{4} \wedge \frac{b}{\rho D(p, q)} \right) \left( w - \frac{6b}{\rho D(p, q)} \right) \right\}. \quad (42)$$

Therefore, for any tolerance $\varepsilon > 0$, we only need the RHS of above inequality to be smaller than $\varepsilon$. The sufficient condition to guarantee $\varepsilon$-performance loss is

$$w \geq \frac{6b}{\rho D(p, q)} + \frac{512 K^2 \log 3/\varepsilon}{b^2(N/4 \wedge b/(\rho D(p, q)))}.$$ 

However, as we just mentioned, this is just a sufficient condition, meaning that the RHS of the above equation is just an upper bound of the smallest possible window length, i.e., $w^*$. We denote it by $\bar{w}$, i.e.,

$$\bar{w} = \frac{6b}{\rho D(p, q)} + \frac{512 K^2 \log 3/\varepsilon}{b^2(N/4 \wedge b/(\rho D(p, q)))}.$$ 

Now we know that $w^* \leq \bar{w}$. Clearly, this $\bar{w}$ is not exactly $w^*$ since: (I) we manually choose
$w > 3b/\rho D(p, q))$ to help bound $\mathbb{P}_0(T_w > w)$ in Lemma 8 and (II) the performance loss bound of $\mathbb{E}_0[T] - \mathbb{E}_0[T_o]$ in (42) is just an upper bound of the original performance loss bound in (40).

Nevertheless, we will show $w^* \sim \bar{w}$, meaning that we characterize the optimal memory efficient window length up to some constant factor. We will handle (II) first by showing that upper bound is tight in the sense that the resulting $w$ to satisfy the $\varepsilon$-performance loss remains unchanged. Notice that we have already manually choose $w$ to be $O(b)$, by the upper bound on the performance loss bound (40), we have

$$w \mathbb{P}_0(T_w > w) + \sum_{\tau = w}^{\infty} \mathbb{P}_0(T_w > \tau) \leq w \exp \left\{ -\frac{b^2}{256K^2} \left( \frac{N}{4} \wedge \frac{b}{\rho D(p, q)} \right) (w - 3b/\rho D(p, q)) \right\} (1 + o(1)).$$

For notational simplicity, we denote

$$\tilde{b} = \frac{b^2}{256K^2} \left( \frac{N}{4} \wedge \frac{b}{\rho D(p, q)} \right).$$

Then, by the derivation in the step 3 in the proof of Lemma 10, we have

$$\exp \left\{ -\tilde{b}(w - 3b/\rho D(p, q)) \right\} \leq w \exp \left\{ -\tilde{b}(w - 3b/\rho D(p, q)) \right\} \leq \exp \left\{ -\frac{\tilde{b}}{2}(w - 6b/\rho D(p, q)) \right\}.$$

As one can see, if we want LHS to meet the $\varepsilon$-performance loss constraint, the resulting window length $w$ will have the same order as $\bar{w}$, meaning that our performance bound is tight in terms of the order of the resulting window length to meet the $\varepsilon$-performance loss constraint.

Next, we handle (I). We will show choosing $w = o(b)$ cannot guarantee $\varepsilon$-performance loss for any fixed $\varepsilon > 0$, given $b \to \infty$. We need go back to the proof of Lemma 8 where we bound the probability $\mathbb{P}_0(T_w > w)$. Now, we choose $w = o(b)$ and will have

$$\mathbb{P}_0(T_w > w) = \mathbb{P}_0 \left( \max_{t \in [2:w]} Z_t < b \right) = \prod_{t=2}^{w} \mathbb{P}_0(Z_t < b) = \prod_{t=2}^{w} \prod_{B=2}^{t} \mathbb{P}_0(Z_B(t) < b).$$

Notice that, in Lemma 6, we calculated the expectation of the detection statistic, which is of
order block size $B$. That is to say, for $B \leq w = o(b)$, we have

$$P_0(Z_B(t) < b) = 1 - P_0(Z_B(t) \geq b)$$

$$\geq 1 - 2 \exp \left\{ - \frac{N(b - B)^2}{128 K^2} \right\} - 2 \exp \left\{ - \frac{B(b - B)^2}{64 K^2} \right\}$$

$$\geq 1 - 2 \exp \left\{ - \frac{N(b - w)^2}{128 K^2} \right\} - 2 \exp \left\{ - \frac{2(b - w)^2}{64 K^2} \right\}.$$ 

Therefore, we have

$$P_0(T_w > w) \geq \left( 1 - 2 \exp \left\{ - \frac{N(b - w)^2}{128 K^2} \right\} - 2 \exp \left\{ - \frac{2(b - w)^2}{64 K^2} \right\} \right)^{w^2}. $$

Notice that $(1 - o(1/n))^n \to 1$ as $n \to \infty$. Clearly, as $b \to \infty$, for $w = o(b)$ choice, we have

$$2 \exp \left\{ - \frac{N(b - w)^2}{128 K^2} \right\} + 2 \exp \left\{ - \frac{2(b - w)^2}{64 K^2} \right\} = o(1/w^2).$$

That is to say, $P_0(T_w > w) \to 1$ as $b \to \infty$. Recall that the performance loss bound has leading term $wP_0(T_w > w)$, which cannot shrink to zero (or even diverge) as $b \to \infty$. Thus, $w = o(b)$ can never satisfy the $\varepsilon$-performance loss constraint and therefore our upper bound $w^* \leq \bar{w}$ is tight in terms of order.

*Under $H_0$. We choose detection threshold $b$ to control the ARL, given that we choose $w \sim b$. The analytic approximation in Lemma 2 can partially answer the question — we are able to give the relationship between $b$ and ARL to make sure our procedure is a constant ARL procedure in $C_\gamma$ (18). Luckily, our choice $w \sim b$ satisfies the technical condition in (23) and therefore we can apply this ARL approximation here. Even though it is difficult to obtain such closed-from ARL approximation for the oracle procedure, we are able to leverage the concentration results for Scan $B$-statistic (Lemma 7 in Appendix C) to show it is also a constant ARL procedure in $C_\gamma$.

For a given ARL, in our window-limited procedure (8), we approximate the true ARL $E_\infty[T_w]$ using Lemma 2. The given ARL constraint is

$$E_\infty[T_w] \geq \gamma. $$

For simplicity, we start with the coarse approximation under Gaussian assumption (i.e., we only use the first and the second order moments’ information). By (12) as well as $w \sim b$, we have:

$$E_\infty[T_w] \sim e^{b^2/2}. $$

To satisfy condition (43), we need

$$b = a \sqrt{\log \gamma}, \quad a \geq \sqrt{2}. $$
In follow-up discussion of Theorem 4.2 [Li et al., 2019], they explicitly claimed $E_\infty[T_w] = O(e^{b^2})$ and proposed to consider $b = O(\sqrt{\log \gamma})$ in order to meet the ARL constraint (43), which agree with (44) and (45), respectively. However, we may notice that $b = O(\sqrt{\log \gamma})$ might violate the lower bound in Lorden [1971], which is known as the best achievable EDD. Recall that the KL divergence is a fundamental information theoretic quantity which characterizes the hardness of the hypothesis testing problem. The Lorden’s lower bound of EDD is on the order of $\log \gamma$ divided by the KL divergence, which will be larger than what we get above asymptotically since the KL divergence is treated as constant in our setting. This contradiction comes from the very coarse ARL approximation (see Figure 3), and if we incorporate the first $2n$-th order moments’ information in the ARL approximation (Lemma 2), we will have

$$\psi_B(\theta) \approx \sum_{i=1}^{2n} \frac{E_\infty[Z_B^i(t)]}{i!} \theta^i.$$  

Note that solving $\dot{\psi}_B(\theta_B) = b$ analytically is very challenging, but we can fix $n$ and obtain

$$\theta_B \sim b^{1/(2n-1)},$$

when we consider the $b \to \infty$ limit, given that the $2n$-th moment of the Scan $B$-statistic will exist and not vanish to zero (Assumption (A1)). Plugging this into Lemma 2 gives us

$$E_\infty[T_w] \sim \frac{e^{b^{2n/(2n-1)}}}{b^{4n-4}}.$$  

This yields a more strict condition on the threshold to be the ARL constraint (43), i.e.,

$$b \sim (\log \gamma)^{1-\frac{1}{2n}}.$$  

We should remark that this is also numerically verified by Figures 3 and 4 in our experiment — considering higher order moment will lead to better approximation whereas fail to do so will result in “underestimating” threshold $b$. As one can see, the choice of detection threshold in (45) and will not satisfy the above requirement (47). Indeed, (45) corresponds to $n = 1$ case in (45). Thus, the threshold choice (45) will fail to meet the given ARL constraint asymptotically.

Now, both the numerical evidence and our theoretical analysis point out that the ARL approximation should involve all moments’ information to meet the given ARL constraint. In addition, recall that we also want to obtain the smallest possible window length $w$, which is on the same order with detection threshold $b$ by our previous analysis. Thus, we give the optimal order of $b$ as follows:

$$b \sim \log \gamma.$$  

Although our current analysis cannot verify the conjecture that $E_\infty[T_w] = O(e^{c_3b})$ for some constant $c_3 > 0$, (46) can tell us as $b \to \infty$

$$\frac{e^{cb}}{E_\infty[T_w]} \to 0.$$  

50
This tells us that our procedure will keep running for a very long time when there is no change, and the run length will be much larger than the detection threshold $b$. At time step $t > w$, we will have

$$\max_{2 \leq B \leq t} Z_B(t) = Z_t^o \geq Z_t = \max_{2 \leq B \leq w} Z_B(t).$$

This implies the oracle procedure will stop earlier since it has larger detection statistics (for the same detection threshold). Here, approximating the ARL for the oracle procedure is not trivial and therefore the detection threshold for both procedures is selected based on Lemma 2. By the above analysis, for $b$ selected as

$$b \sim \log \gamma,$$

we know that

$$\mathbb{E}_\infty[T_w] \geq \gamma,$$

$$\mathbb{E}_\infty[T_w] \geq \mathbb{E}_\infty[T_o].$$

In the following, we will show $\mathbb{E}_\infty[T_o] \geq \gamma$. By our concentration result in Lemma 7, we have

$$\mathbb{P}_\infty(T_o \leq t_0) = \mathbb{P}_\infty(\max_{1 \leq t \leq t_0} Z_t^o > b) = \mathbb{P}_\infty(\max_{1 \leq t \leq t_0} \max_{2 \leq B \leq t} Z_B(t) > b)$$

$$= \mathbb{P}_\infty(\bigcup_{1 \leq t \leq t_0} \bigcup_{2 \leq B \leq t} \{Z_B(t) > b\})$$

$$\leq \sum_{1 \leq t \leq t_0} \sum_{2 \leq B \leq t} \mathbb{P}_\infty(Z_B(t) > b) = \mathcal{O}\left(t_0^2 \exp\left\{-\frac{2(b/2)^2}{16K^2}\right\}\right).$$

This gives us $\mathbb{P}_\infty(T_o \leq \gamma)$ drops exponentially fast to zero with increasing $b$. That is to say, with probability almost one, the oracle procedure will not stop before $t_0 = \mathcal{O}(\gamma)$, which implies $\mathbb{E}_\infty[T_o] \geq \gamma$. Now, we complete the proof.

\[\square\]

\section*{E Additional Experimental Results}

In this last section, we present additional experimental configurations and results. To begin with, we will use numerical evidence to guide users to select the detection procedure hyperparameters.

\subsection*{E.1 Study on the Hyperparameter Choices}

In this part, we will study how the choices of the hyperparameters of our detection procedure influence the EDD. As mentioned above, the choice of window length $w$ is very important and we will begin our numerical study with its effect on the EDD.

Recall that we use $[2 : w]$ as the search region to locate the change-point. Here, in addition to right end-point $w$, i.e., window length, we also study the effect of left end-point. Here,
we use $[B_{\text{min}} : B_{\text{max}}]$ as the region in which we optimize block size parameter $B$; we use $w$ and $B_{\text{max}}$ interchangeably to denote the window length parameter. Intuitively, the wider this search region is, the quicker the detection should be, but this merit comes with higher computational and memory cost. We validate this by comparing EDDs for a given ARL under different $B_{\text{min}}$ and $B_{\text{max}}$ choices and plot the results in Figure 10.

From Figure 10, we observe that, on one hand, $B_{\text{max}}$ controls the success of detection — the detection procedure typically fails to raise an alarm within a reasonable time if EDD of the oracle procedure (41) exceeds $B_{\text{max}}$. However, this does not tell users to choose an overly large window length, since (i) there is a trade-off between the EDD and the computational and memory cost and (ii) EDD improvement by choosing a larger window length $w$ for a given ARL is no longer significant after $w \sim \log \text{ARL}$, as shown by our non-asymptotic study of the performance loss in Section 4.4. Most importantly, (ii) can be numerically verified by the leftmost panel in Figure 10 — As we can see, for $B_{\text{min}} = 2$ case, the improvement of $B_{\text{max}}$ increasing from 40 to 50 is smaller than that of $B_{\text{max}}$ increasing from 30 to 40. In addition, in the rightmost panel, we can see the improvement of EDD by increasing $B_{\text{max}}$ is marginal. On the other hand, $B_{\text{min}}$ is important to guarantee quick detection — “power loss” will occur if $B_{\text{min}}$ is greater than EDD of the optimal procedure (41); see $B_{\text{min}} = 10$ case in the rightmost panel for evidence. Otherwise, the effect of $B_{\text{min}}$ is negligible when the change is subtle. Here, a simple and safe choice would be $B_{\text{min}} = 2$.

Next, we study the effect of kernel bandwidth on the EDD. As mentioned in Ramdas et al. [2015], the median heuristic gives the largest possible MMD (see observation 2 therein) and over- as well as under-estimated kernel bandwidth will result in much faster decay in MMD (see observations 1 and 3 therein). As mentioned in Theorem 1 above, the optimal EDD is proportional to the inverse of MMD and therefore median heuristic should result in the quickest detection. We demonstrate this empirically by plotting EDD v.s. log ARL, where threshold $b$ is selected via Monte Carlo simulation to satisfy the corresponding ARL in Figure 11.

Here, we slightly abuse the notation $\gamma$: the kernel bandwidth is chosen to be $r = \gamma \text{med}_p$, where $\text{med}_p$ denotes the median of the $\ell_2$ distance matrix for pre-change samples: $\gamma < 1$
Figure 11: Comparison of EDD for different kernel bandwidths of Gaussian RBF kernel function. We consider different bandwidth choices \( r = \gamma \text{med}_p \), where \( \gamma \in \{0.5, 1, 10, 100\} \), where \( \text{med}_p \) is chosen via median heuristic. The change is from \( \mathcal{N}(0_{20}, I_{20}) \) to: (a) top left: \( \mathcal{N}(0_{20}, 9 I_{20}) \); (b) top right: \( \mathcal{N}(1_{20}, 9 I_{20}) \); (c) bottom left: Gaussian mixture: \( \mathcal{N}(0_{20}, I_{20}) \) w.p. 0.3; \( \mathcal{N}(0_{20}, 9 I_{20}) \) w.p. 0.7; (d) bottom right: Gaussian mixture: \( \mathcal{N}(0_{20}, I_{20}) \) w.p. 0.3; \( \mathcal{N}(1_{20}, 9 I_{20}) \) w.p. 0.7.
corresponds to under-estimation of kernel bandwidth whereas $\gamma > 1$ represents over-estimation. Figure 11 shows under- and over-estimation of kernel bandwidth would both lead to larger EDD and therefore median heuristic is indeed the best. Therefore, in practice, it is safe to apply the median heuristic to select kernel bandwidth.

Lastly, we study the behavior of detection statistics when choosing different $N$’s. Intuitively, by the Law of Large Numbers, the larger the $N$ is, the better the performance (or convergence) of block-version MMD statistic, i.e., $B$-test statistic [Zaremba et al., 2013], will be. This is theoretically demonstrated in Remark 1. However, the question is, empirically speaking, how large should $N$ be to achieve a satisfying performance? To answer this question, we plot the trajectory of the averaged detection statistic (over 50 independent runs) for our procedure in Figure 12.

![Figure 12: Comparison of mean trajectories over 50 independent trials for different pre-change blocks’ numbers $N$. The change is from $\mathcal{N}(0_{20}, I_{20})$ to Gaussian mixture with $\mathcal{N}(0_{20}, I_{20})$ w.p. 0.3 and $\mathcal{N}(0_{20}, 4 I_{20})$ w.p. 0.7 occurs at $t = 16$.](image)

As we can observe in Figure 12, the slopes for $N \geq 15$ are large enough and almost the same after the change-point. We can focus on the first few steps after the change-point, since (i) we are interested in the quick detection of the change-point and (ii) the convergence of the statistic to its mean is controlled by both $N$ and $B$. By taking the trade-off between the computation time and memory cost and the performance into consideration, we believe $N = 15$ suffices for our numerical simulation. However, we need to remark that, in practice, this $N$ choice is typically determined by how many reference samples we have.

### E.2 Benchmark Procedures

Before we present additional results on EDD against ARL comparison, let us briefly introduce the detection statistics of the aforementioned benchmark procedures as follows:

(a) **Scan $B$-statistic:** For a fixed block size $B_0 \geq 2$, the Scan $B$-statistic $Z_{B_0}(t)$ is defined in (3). Here, we choose $B_0 = w$, where $w$ is the window length of our procedure.

(b) **KCUSUM statistic:** Flynn and Yoo [2019] replaced the GLR statistic in the classic CUSUM recursive update (2) with the linear-time MMD statistic [Gretton et al., 2012a] and proposed the KCUSUM procedure. The KCUSUM detection statistic has the following
recursive update rule:

\[
S^K_0 = 0, \quad S^K_t = \begin{cases} 
(S^K_{t-1} + h(x_{1,t}, x_{2,t}, y_{t-1}, y_t) - \delta)^+ & \text{if } t \text{ is even}, \\
S^K_{t-1} & \text{if } t \text{ is odd}, 
\end{cases}
\]

where \(h(\cdot, \cdot, \cdot, \cdot)\) is defined in (6) and can be viewed as a linear-time MMD statistic with sample size \(n = 2\). The hyperparameter \(\delta > 0\) makes sure a negative drift and is chosen to be \(\delta = 1/50\) as suggested by the original work.

(c) **Hotelling’s \(T^2\) statistic:** At time step \(t\), for hypothetical change-point \(\kappa < t\), we split the sample points into two parts: \(U = (X_1, X_2, \ldots, X_M, Y_1, \ldots, Y_{\kappa-1})\), \(V = (Y_\kappa, \ldots, Y_t)\). For notational simplicity, we denote the elements in \(U\) and \(V\) by \(U_1, \ldots, U_{M+\kappa-1}\) and \(V_1, \ldots, V_{t-\kappa+1}\), respectively. We define

\[
T^2_t(\kappa) = \frac{(M + \kappa - 1)(t - \kappa + 1)}{M + t} \left( \bar{U} - \bar{V} \right)^T \hat{\Sigma}^{-1} \left( \bar{U} - \bar{V} \right),
\]

where \(\bar{U} = \sum_{i=1}^{M+\kappa-1} U_i / (M + \kappa - 1)\), \(\bar{V} = \sum_{i=1}^{t-\kappa+1} V_i / (t - \kappa + 1)\) and \(\hat{\Sigma}\) is the pooled covariance matrix:

\[
\hat{\Sigma} = (M + t - 2)^{-1} \left( \sum_{i=1}^{M+\kappa-1} (U_i - \bar{U}) (U_i - \bar{U})^T + \sum_{i=1}^{t-\kappa+1} (V_i - \bar{V}) (V_i - \bar{V})^T \right).
\]

The Hotelling’s \(T^2\) detection statistic is defined as follows:

\[
S^K_{HT}^T = \max_{1 \leq \kappa \leq t-1} T^2_t(\kappa).
\]

### E.3 EDD against ARL Comparison under More Settings

To further demonstrate the good performance of our procedure, we additionally consider more settings for (i) change from Gaussian to Gaussian mixture as well as change from Gaussian distribution to (ii) Laplace and (iii) uniform distributions. The results are reported in Tables 4, 5 and 6, respectively. From these tables, we can observe that our proposed online kernel CUSUM achieves the quickest detection among all benchmarks, even under the settings where the changes are too small for parametric procedures to detect. In addition to aforementioned observations, we can see Hotelling’s \(T^2\) procedure performs pretty well when the mean shift is significant, but fails easily otherwise. Under those small mean shift circumstances, kernel methods can easily achieve better performance than Hotelling’s \(T^2\) procedure; see the first two rows in the table for evidence. This observation agrees with our intuition since Hotelling’s \(T^2\) procedure is a parametric method which is designed to detect the mean shift.
Table 4: EDDs for given ARLs. The change is from standard Gaussian distribution $\mathcal{N}(0, 1)$ to Gaussian mixture distribution. Non-value “-” indicates the corresponding detection procedure fails to detect the change within the first 50 time steps under the corresponding setting. Smallest EDDs are highlighted in bold fonts.

| Setting | $\mathcal{N}(0, 1)$ vs. Gaussian mixture with $\mathcal{N}(0, 1)$ w.p. 0.3, $\mathcal{N}(\mu, 1)$ w.p. 0.7. |
|---------|---------------------------------------------------|
| ARL     | $(\mu = 0.1, \sigma^2 = 0.1)$ $(\mu = 0.1, \sigma^2 = 0.3)$ $(\mu = 0.1, \sigma^2 = 1)$ $(\mu = 0.1, \sigma^2 = 4)$ $(\mu = 0.1, \sigma^2 = 9)$ |
| PROPOSED | 19.2 19.55 21.57 - - - - - - - 6.99 7.08 7.66 3.47 3.49 3.6 |
| Scan B  | 28.1 28.7 30.83 - - - - - - - 15.48 15.91 17.36 9.63 9.94 10.98 |
| Kcusum  | - - - - - - - - - - - - - - 6.03 6.89 - |
| Hotelling $T^2$ | - - - - - - - - - - - - - - 10.04 10.46 12.17 |
| ARL     | 500 1000 2000 500 1000 2000 500 1000 2000 500 1000 2000 500 1000 2000 |
| PROPOSED | 12.55 12.77 14.19 17.26 17.53 19.46 - - - - - 6.69 6.77 7.33 3.46 3.47 3.61 |
| Scan B  | 22.36 22.93 24.84 24.87 25.5 27.61 - - - - - 14.85 15.28 16.69 9.53 9.85 10.91 |
| Kcusum  | - - - - - - - - - - - - - - 6.04 6.71 8.18 |
| Hotelling $T^2$ | - - - - - - - - - - - - - - 17.34 17.88 - 8.88 9.28 10.68 |
| ARL     | 500 1000 2000 500 1000 2000 500 1000 2000 500 1000 2000 500 1000 2000 |
| PROPOSED | 3.33 3.34 3.45 3.63 3.67 4.05 4.79 4.85 5.26 4.65 4.7 5.15 3.36 3.37 3.46 |
| Scan B  | 9.64 10.01 11.11 10.11 10.46 11.69 11.39 11.81 13.23 11.16 11.56 12.94 9.07 9.36 10.41 |
| Kcusum  | - - - - - - - - - - - - - - 5.16 5.85 6.73 |
| Hotelling $T^2$ | - - - - - - - - - - - - - - 5.52 5.71 6.43 |
| ARL     | 500 1000 2000 500 1000 2000 500 1000 2000 500 1000 2000 500 1000 2000 |
| PROPOSED | 3.12 3.13 3.18 3.16 3.17 3.21 3.3 3.3 3.39 3.62 3.64 3.86 3.32 3.32 3.38 |
| Scan B  | 7.18 7.31 8.24 7.33 7.48 8.3 8.66 8.39 9.35 9.27 9.62 10.72 8.75 8.98 10.01 |
| Kcusum  | 3.99 4.05 4.08 4.1 4.18 4.28 4.95 5.3 5.86 7.05 8.37 - 4.78 5.07 5.62 |
| Hotelling $T^2$ | 5.74 5.86 6.59 5.72 5.87 6.51 5.63 5.8 6.34 5.11 5.24 5.77 4.37 4.49 4.98 |
| ARL     | 500 1000 2000 500 1000 2000 500 1000 2000 500 1000 2000 500 1000 2000 |
| PROPOSED | 3.89 3.89 3.97 2.95 2.96 3.03 3.11 3.12 3.16 3.31 3.33 3.38 3.28 3.29 3.33 |
| Scan B  | 9.94 6.44 7.11 6.39 6.74 7.19 7.02 7.17 7.87 8.32 8.62 9.49 8.53 8.71 9.68 |
| Kcusum  | 3.98 3.98 4.01 3.98 4 4.01 4.03 4.03 4.08 4.77 5.07 5.56 4.4 4.59 4.89 |
| Hotelling $T^2$ | 4.31 4.38 5.17 4.31 4.44 5.08 4.37 4.45 4.92 4.14 4.24 4.6 3.72 3.82 4.16 |
Table 5: EDDs for given ARLs under more settings. The change is from standard Gaussian distribution $\mathcal{N}(\mathbf{0}_{20}, I_{20})$ to Laplace distribution. The best results are highlighted in bold fonts.

| Setting II: $\mathcal{N}(\mathbf{0}_{20}, I_{20})$ v.s. Laplace($\mu_{120}, b_{120}$) |
|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| ARL                             | $\mu = 0.1$, $b^2 = 0.1$       | $\mu = 0.1$, $b^2 = 0.3$       | $\mu = 0.1$, $b^2 = 1$         | $\mu = 0.1$, $b^2 = 4$         | $\mu = 0.1$, $b^2 = 9$         |
| PROPOSED                        | 10.16                          | 10.33                          | 11.33                          | 15.91                          | 16.17                          | 17.73                          |
| SCAN $B$                        | 21.75                          | 22.21                          | 23.64                          | 25.73                          | 26.44                          | 28.69                          |
| KCUSUM                          | -                              | -                              | -                              | -                              | -                              | -                              |
| HOTELLING $T^2$                 | -                              | -                              | -                              | -                              | -                              | -                              |
| ARL                             | 500                            | 1000                           | 2000                           | 500                            | 1000                           | 2000                           |
| PROPOSED                        | 0.43                           | 6.51                           | 7.23                           | 8.92                           | 9.04                           | 9.97                           |
| SCAN $B$                        | 15.88                          | 16.21                          | 17.92                          | 18.01                          | 18.56                          | 20.48                          |
| KCUSUM                          | -                              | -                              | -                              | -                              | -                              | -                              |
| HOTELLING $T^2$                 | 24.95                          | 25.4                           | 27.12                          | 24.11                          | 24.56                          | 26.41                          |
| ARL                             | 500                            | 1000                           | 2000                           | 500                            | 1000                           | 2000                           |
| PROPOSED                        | 4.15                           | 4.18                           | 4.6                           | 5.08                           | 5.17                           | 5.65                           |
| SCAN $B$                        | 12.07                          | 12.56                          | 13.49                          | 13.06                          | 13.41                          | 14.49                          |
| KCUSUM                          | -                              | -                              | -                              | -                              | -                              | -                              |
| HOTELLING $T^2$                 | 14.66                          | 14.83                          | 15.45                          | 14.32                          | 14.52                          | 15.28                          |
| ARL                             | 500                            | 1000                           | 2000                           | 500                            | 1000                           | 2000                           |
| PROPOSED                        | 3.81                           | 3.86                           | 3.99                           | 3.9                           | 3.93                           | 4.03                           |
| SCAN $B$                        | 8.94                           | 9.23                           | 10.45                          | 9.5                            | 9.87                           | 11.02                          |
| KCUSUM                          | -                              | -                              | -                              | -                              | -                              | -                              |
| HOTELLING $T^2$                 | 9.69                           | 9.98                           | 10.94                          | 9.33                           | 9.66                           | 10.78                          |
| ARL                             | 500                            | 1000                           | 2000                           | 500                            | 1000                           | 2000                           |
| PROPOSED                        | 2                              | 2                              | 2                              | 2                              | 2                              | 2                              |
| SCAN $B$                        | 5                              | 5                              | 5.77                           | 5                              | 5.03                           | 5.95                           |
| KCUSUM                          | 2.01                           | 2.03                           | 2.05                           | 2.07                           | 2.11                           | 2.18                           |
| HOTELLING $T^2$                 | 4.01                           | 4.05                           | 4.93                           | 4.06                           | 4.12                           | 4.76                           |
Table 6: EDDs for given ARLs under more settings. The change is from standard Gaussian distribution $\mathcal{N}(0, I_{20})$ to Uniform distribution. The best results are highlighted in bold fonts.

| Setting III: $\mathcal{N}(0, I_{20})$ v.s. $U[(a - b)1_{20}, (a + b)1_{20}]$. | ARL | Proposed | Scan B | KCUSUM | Hotelling $T^2$ |
|---|---|---|---|---|---|
| | 500 | 1000 | 2000 | 500 | 1000 | 2000 | 500 | 1000 | 2000 | 500 | 1000 | 2000 | 500 | 1000 | 2000 |
| (a = 0.1, $b^2 = 0.1$) | (a = 0.1, $b^2 = 0.3$) | (a = 0.1, $b^2 = 1$) | (a = 0.1, $b^2 = 4$) | (a = 0.1, $b^2 = 9$) |
| **ARL** | | | | | |
| Proposed | 6 | 6 | 6.93 | 5.26 | 5.39 | 5.99 | 4 | 4 | 4 | **2.06** | **2.07** | **2.18** | 2 | 2 | 2 |
| Scan B | 16 | 16.02 | 18 | 13.96 | 14.22 | 15.23 | 10.36 | 10.74 | 11.66 | 6.17 | 6.41 | 7.08 | 5.07 | 5.21 | 5.98 |
| KCUSUM | - | - | - | - | - | - | - | - | - | 3.67 | 4.2 | 5.08 | 2.18 | 2.23 | 2.33 |
| Hotelling $T^2$ | 29.1 | 29.46 | 30.77 | 20.28 | 20.61 | 22.21 | 12.3 | 12.65 | 13.41 | 5.19 | 5.31 | 5.95 | 3.95 | 4 | 4.26 |

| (a = 0.3, $b^2 = 0.1$) | (a = 0.3, $b^2 = 0.3$) | (a = 0.3, $b^2 = 1$) | (a = 0.3, $b^2 = 4$) | (a = 0.3, $b^2 = 9$) |
|---|---|---|---|---|
| **ARL** | | | | | |
| Proposed | 4 | 4 | 4.51 | 4 | 4 | 4.52 | 2.92 | 2.99 | 3.52 | 2 | 2 | 2 | 2 | 2 | 2 |
| Scan B | 12.63 | 13 | 13.97 | 10.66 | 10.98 | 11.91 | 7.54 | 7.98 | 9.06 | 5.48 | 5.79 | 6.25 | 5 | 5.01 | 5.61 |
| KCUSUM | - | - | - | - | - | - | - | - | - | 2.4 | 2.55 | 2.83 | 2.05 | 2.07 | 2.1 |
| Hotelling $T^2$ | 15.77 | 15.99 | 16.89 | 13.03 | 13.17 | 14.03 | 7.55 | 7.92 | 9.17 | 4.8 | 4.89 | 5.05 | 3.5 | 3.64 | 3.98 |

| (a = 0.7, $b^2 = 0.1$) | (a = 0.7, $b^2 = 0.3$) | (a = 0.7, $b^2 = 1$) | (a = 0.7, $b^2 = 4$) | (a = 0.7, $b^2 = 9$) |
|---|---|---|---|---|
| **ARL** | | | | | |
| Proposed | 2.01 | 2.02 | 2.95 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| Scan B | 7 | 7 | 8.1 | 6.28 | 6.96 | 7.03 | 5.67 | 5.98 | 6.27 | 5 | 5.21 | 4.97 | 5 | 5 |
| KCUSUM | - | - | - | - | - | - | - | - | - | 2.4 | 2.55 | 2.83 | 2.05 | 2.07 | 2.1 |
| Hotelling $T^2$ | 6.71 | 6.87 | 7 | 6.06 | 7 | 5 | 5 | 5.4 | 3.92 | 3.97 | 4.04 | 3 | 3.01 | 3.16 |

| (a = 1.5, $b^2 = 0.1$) | (a = 1.5, $b^2 = 0.3$) | (a = 1.5, $b^2 = 1$) | (a = 1.5, $b^2 = 4$) | (a = 1.5, $b^2 = 9$) |
|---|---|---|---|---|
| **ARL** | | | | | |
| Proposed | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| Scan B | 5 | 5 | 5 | 5 | 5 | 5 | 4.12 | 4.99 | 5 | 4 | 4.09 | 4.5 | 4.05 | 4.69 | 5 |
| KCUSUM | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| Hotelling $T^2$ | 7 | 7 | 8.07 | 6 | 6.06 | 7 | 5 | 5 | 5.4 | 3.92 | 3.97 | 4.04 | 3 | 3.01 | 3.16 |

| (a = 2, $b^2 = 0.1$) | (a = 2, $b^2 = 0.3$) | (a = 2, $b^2 = 1$) | (a = 2, $b^2 = 4$) | (a = 2, $b^2 = 9$) |
|---|---|---|---|---|
| **ARL** | | | | | |
| Proposed | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| Scan B | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |
| KCUSUM | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| Hotelling $T^2$ | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 2.01 | 2.04 | 2.74 | 2 | 2 | 2.61 |
E.4 Real Example 1: HASC Dataset

The HASC dataset is publicly available and contains measurements of human activities for 6 human subjects (indexed by 101, 102, 103, 105, 106, 107) and 6 types of activities (jog, walk, skip, stay, stair down, stair up). For those 6 human subjects, there are 10, 15, 10, 15, 15, 15 repeats taken, respectively; for each single repeat, the sequence length is around 2000. Those repeats may correspond to activity under different conditions, e.g., floor types (asphalt or carpet), weather (for outdoor activities, including fine, cloudy, rain, snow) and so on; however, we treat them as different repeats of the same for simplicity. We need to mention that there are only 3 repeats taken for subject 107 staring up and thus this scenario is left out in our experiment.

Here, the goal of online change-point detection is: for a chosen human subject, given history activity data (e.g., walking) and assuming we receive sequential data of this subject from the accelerometers, we want to detect the activity change (e.g., change to staying) as soon as possible. As an illustrative example, we choose human subject 101 and take walk as the pre-change activity; we visualize the 3-dimensional raw data of one trial in Figure 13. As one can see, there is a clear cyclic pattern; although we assume the data points are i.i.d., we take into account this particular structure when constructing the pre-change blocks: we first partition this history data based on its peaks into 15 segments; then, we choose the minimum segment length as the window length (i.e., $w = 114$); finally, we use the first 14 segments with equal length as the pre-change blocks (i.e., $N = 14$). In addition, the kernel bandwidth is chosen by using median heuristic on this whole history data sequence, which gives us $r = 0.865$.

For completeness, we carry out the aforementioned experiment for the rest 5 human...
Table 7: EDD and miss of human activity change detection in HASC dataset. The pre-change activity is walking. The best results are highlighted in bold fonts.

### Subject 102 (in total 15 repeats).

| Activity | Jog EDD | Jog Miss | Skip EDD | Skip Miss | Stay EDD | Stay Miss | Stair down EDD | Stair down Miss | Stair up EDD | Stair up Miss |
|----------|---------|---------|----------|-----------|----------|-----------|----------------|----------------|--------------|--------------|
| Proposed | 320     | 13      | –        | –         | –        | –         | –              | –              | –            | –            |
| Scan B   | 320     | 13      | –        | –         | –        | –         | –              | –              | –            | –            |
| KCUSUM   | 886     | 13      | 1034     | 12        | –        | –         | 1707           | 13             | –            | –            |
| Hotelling $T^2$ | –   | 15      | –        | –         | –        | –         | –              | –              | –            | –            |

### Subject 103 (in total 10 repeats).

| Activity | Jog EDD | Jog Miss | Skip EDD | Skip Miss | Stay EDD | Stay Miss | Stair down EDD | Stair down Miss | Stair up EDD | Stair up Miss |
|----------|---------|---------|----------|-----------|----------|-----------|----------------|----------------|--------------|--------------|
| Proposed | 483.5   | 6       | 391.5    | 8         | 855.25   | 6         | –              | –              | –            | –            |
| Scan B   | 637     | 1       | 95.67    | 7         | 861      | 6         | –              | –              | –            | –            |
| KCUSUM   | 348.6   | 0       | 333.2    | 0         | –        | –         | 1266           | 9              | 80           | 9            |
| Hotelling $T^2$ | 320.67 | 7       | 104      | 9         | 875.25   | 6         | –              | –              | –            | –            |

### Subject 105 (in total 15 repeats).

| Activity | Jog EDD | Jog Miss | Skip EDD | Skip Miss | Stay EDD | Stay Miss | Stair down EDD | Stair down Miss | Stair up EDD | Stair up Miss |
|----------|---------|---------|----------|-----------|----------|-----------|----------------|----------------|--------------|--------------|
| Proposed | 260.93  | 0       | 49.6     | 0         | 3426.5   | 13        | 206.89         | 6              | –            | –            |
| Scan B   | 108     | 3       | 87.13    | 0         | 1591     | 13        | 775.43         | 8              | 884.29       | 8            |
| KCUSUM   | 395.67  | 3       | 362.53   | 0         | –        | 15        | 98             | 14             | –            | –            |
| Hotelling $T^2$ | 905 | 13      | 178.25   | 3         | 3328.5   | 13        | 521.2          | 10             | 931          | 8            |

### Subject 106 (in total 15 repeats).

| Activity | Jog EDD | Jog Miss | Skip EDD | Skip Miss | Stay EDD | Stay Miss | Stair down EDD | Stair down Miss | Stair up EDD | Stair up Miss |
|----------|---------|---------|----------|-----------|----------|-----------|----------------|----------------|--------------|--------------|
| Proposed | 121.1   | 5       | 124.4    | 0         | 19.5     | 13        | 584.5          | 11             | 430.25       | 7            |
| Scan B   | 368.14  | 8       | 162.4    | 0         | 19.5     | 13        | 334            | 12             | 137.4        | 5            |
| KCUSUM   | 563.5   | 7       | 722.29   | 1         | –        | 15        | 884            | 13             | –            | –            |
| Hotelling $T^2$ | 176.11 | 6       | 66.4     | 0         | 23       | 13        | 629            | 12             | 333.25       | 7            |

### Subject 107 (in total 15 repeats).

| Activity | Jog EDD | Jog Miss | Skip EDD | Skip Miss | Stay EDD | Stay Miss | Stair down EDD | Stair down Miss | Stair up EDD | Stair up Miss |
|----------|---------|---------|----------|-----------|----------|-----------|----------------|----------------|--------------|--------------|
| Proposed | –       | 15      | 641.4    | 10        | –        | –         | –              | –              | –            | –            |
| Scan B   | –       | 15      | –        | –         | –        | –         | –              | –              | –            | –            |
| KCUSUM   | 814.31  | 2       | 813.78   | 6         | –        | –         | 1151.2         | 10             | –            | –            |
| Hotelling $T^2$ | 1917 | 13      | 1038     | 14        | –        | –         | –              | –              | –            | –            |

From Table 7, we can make the following additional observations: (i) The KUSUM-type procedures based on non-parametric kernel MMD statistic do perform pretty well. (ii) The detection can be difficult for certain human subjects and certain repeats (since those repeats are carried out under different settings); in contrast, it is easier to detect the change for subjects 105 as well as 106, and we can see our procedure outperforms other benchmarks in the sense that it is the most robust one (i.e., it have the smallest number of misses), and oftentimes it has the smallest EDD. These findings further demonstrate the usefulness of our
proposed online kernel CUSUM procedure in practice.

### E.5 Real Example 2: MNIST Dataset

In the last part, we present complete details on how we carry out the experiment on MNIST data [Watson and Wilson, 1992]. The observations from pre- or post-change distribution are uniform random samples from a particular digit from \{0, 1, \ldots, 9\}. For each sub-figure in Figure 8, the detection threshold is obtained with 500 independent experiments with 3000 history data and 1000 sequential observations, both from the same pre-change distribution. Afterwards, we perform another 500 independent experiments with 3000 history data and 100 sequential observations, where the sequential observations are from a post-change distribution, to obtain the average detection delay as the EDD. In our experiment, we do not consider the Hotelling’s $T^2$ since the data is only supported on a low-dimensional manifold with mostly zero entries, resulting in a rank-deficient sample covariance matrix.

Table 8: EDD for target ARL $\gamma = 1000$. In the 10-by-10 table, the $(i,j)$-th entry contains EDDs (left: our proposed, right: the Scan $B$) under the setting where pre-change digit is $i$ and post-change digit is $j$. In each entry, the smaller EDD is in a larger font, indicating the corresponding detection procedure is better under that specific setting; the entries where our proposed method outperforms Scan $B$ are highlighted in bold fonts.

|    | 0     | 1     | 2     | 3     | 4     | 5     | 6     | 7     | 8     | 9     |
|----|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 0  | -     | 31.04 | 0.33  | 3.44  | 3.69  | 6.44  | 9.48  | 9.28  | 16.26 | 3.78  | 11.64 |
| 1  | 17.99 | 19.18 | 25.50 | 10.44 | 12.47 | 10.80 | 24.07 | 23.82 | 7.87  | 21.45 |
| 2  | 30.81 | 27.12 | 12.02 | 15.90 | 32.74 | 12.13 | 18.14 | 16.63 | 4.66  | 12.25 |
| 3  | 37.42 | 30.85 | 15.90 | 12.13 | 18.14 | 16.63 | 4.66  | 12.25 |
| 4  | 16.53 | 27.12 | 12.02 | 15.90 | 32.74 | 12.13 | 18.14 | 16.63 |
| 5  | 20.68 | 34.30 | 10.38 | 6.32  | 4.69  | 4.69  | 18.14 | 16.63 |
| 6  | 39.07 | 39.65 | 19.34 | 6.29  | 37.31 | 33.27 | 16.33 | 9.61  |
| 7  | 30.63 | 30.63 | 16.17 | 11.24 | 10.19 | 23.96 | 27.29 | 23.42 |
| 8  | 39.92 | 35.87 | 29.96 | 12.08 | 3.58  | 28.91 | 25.96 | 9.77  |
| 9  |      |       |       |       |       |       |       |       |       |       |

To make the underlying task more challenging, we center the data for each digit (or class) using its coordinate-wise average vector to make the data zero mean. We repeat similar experiment above and report the result in Table 8. In the more challenging setting, even though the EDD becomes larger compared to that in Figure 8, kernel method is still able to main detection power. Moreover, we can observe that under most cases our proposed method outperforms Scan $B$ procedure and achieves quicker change detection, whereas in all of the rest case where Scan $B$ performs better, the EDDs of our proposed and Scan $B$ procedures are very close to each other.