A refinement of a result of Andrews and Newman on the sum of minimal excludants

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Abstract
In this article, we refine a result of Andrews and Newman, that is, the sum of minimal excludants over all the partitions of a number $n$ equals the number of partitions of $n$ into distinct parts with two colors. As a consequence, we find congruences modulo 4 and 8 for the functions appearing in this refinement. We also conjecture three further congruences for these functions. In addition, we also initiate the study of $k$th moments of minimal excludants. At the end, we also provide an alternate proof of a beautiful identity due to Hopkins, Sellers, and Stanton.

Keywords Partitions · Minimal excludant · Colored partitions · Refinement · Partition congruences

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1 Introduction

The study of the minimal excludant of a partition has been an active area of research after being revived by the works of Andrews and Newman [1, 2]. They defined the minimal excludant of an integer partition $\pi$ as the least positive integer missing from the partition, denoted by $\text{mex}(\pi)$. For example, with $n = 5$, the values of minimal excludants for the partitions of $n$ are: $\text{mex}(5) = 1$, $\text{mex}(4 + 1) = 2$, $\text{mex}(3 + 2) = 1$, $\text{mex}(3 + 1 + 1) = 2$, $\text{mex}(2 + 2 + 1) = 3$, $\text{mex}(2 + 1 + 1 + 1) = 3$, $\text{mex}(1 + 1 + 1 + 1 + 1) = 2$. They discovered an elegant identity [1, p. 250, Theorem 1.1] involving the quantity $\sigma \text{mex}(n)$, which denotes the sum of minimal excludants over all the partitions of $n$. The identity is as follows:

$$\sum_{n=0}^{\infty} \sigma \text{mex}(n)q^n = (-q; q)_{\infty}^2 = \sum_{n=0}^{\infty} D_2(n)q^n, \quad (1.1)$$

where $D_2(n)$ represents the number of partitions of $n$ into distinct parts with two colors. The same identity, under a different terminology, was obtained earlier by Grabner and Knopfmacher [10, p. 445, Equation (4.2)]. They studied the concept of ‘smallest gap’ in a partition that has exactly the same meaning as that of ‘minimal excludant’. The identity (1.1) has also been proved by Ballantine and Merca [3], employing purely combinatorial tools. For recent progress in this area, interested readers may refer to [2, 4, 5, 8, 9, 14]. Here, and throughout the paper, $|q| < 1$ and

$$\begin{align*}
(a; q)_0 & := 1, \\
(a; q)_n & := (1 - a)(1 - aq) \cdots (1 - aq^{n-1}), \quad n \geq 1, \\
(a; q)_{\infty} & := \lim_{n \to \infty} (a; q)_n.
\end{align*}$$

We also use the following notation:

- $\mathcal{P}(n) :=$ the set of all partitions of $n$,
- $p(n) :=$ the number of partitions of $n$,
- $\mathcal{D}_2^{o}(n) :=$ the set of partitions of $n$ into distinct parts with two colors and an odd number of parts,
- $\mathcal{D}_2^{e}(n) :=$ the set of partitions of $n$ into distinct parts with two colors and an even number of parts.

Instead of summing the minimal excludants over all partitions of $n$, as done by Andrews and Newman in [1], we slightly modify the definition and consider two such sums parity-wise (over odd and even minimal excludants separately).

**Definition 1** For a positive integer $n$, we define the following functions:

$$\sigma_0 \text{mex}(n) := \sum_{\pi \in \mathcal{P}(n)} \text{mex}(\pi), \quad (1.2)$$
\[ \sigma_e \text{mex}(n) := \sum_{\pi \in \mathcal{P}(n)} \text{mex}(\pi). \]  
\hspace{1cm} (1.3)

For instance, with \( n = 5 \), one can observe that \( \sigma_o \text{mex}(5) = 1 + 1 + 3 + 3 = 8 \) and \( \sigma_e \text{mex}(5) = 2 + 2 + 2 = 6 \).

In this paper, we refine the identity (1.1) by finding the generating functions for \( \sigma_o \text{mex}(n) \) and \( \sigma_e \text{mex}(n) \).

2 Main results

In this section, we state our main results, the first of which connects the sum of minimal excludants considered parity-wise with the number of partitions of \( n \) into distinct parts with two colors, according to the parity of their number of parts. We denote \( D^e_2(n) \) (resp. \( D^o_2(n) \)) to be the number of partitions of \( n \) into distinct parts with two colors and an even (resp. odd) number of parts.

**Theorem 2.1** We have

\[ \sum_{n=0}^{\infty} \sigma_o \text{mex}(n) q^n = \left(\frac{-q; q}{q}^2\right)_{\infty} + \left(\frac{-q; q}{q}^2\right)_{\infty} = \sum_{n=0}^{\infty} D^e_2(n) q^n, \]  
\hspace{1cm} (2.1)

and

\[ \sum_{n=0}^{\infty} \sigma_e \text{mex}(n) q^n = \left(\frac{-q; q}{q}^2\right)_{\infty} - \left(\frac{-q; q}{q}^2\right)_{\infty} = \sum_{n=0}^{\infty} D^o_2(n) q^n. \]  
\hspace{1cm} (2.2)

**Example** In the Tables (1, 2) below, we see that the identities (2.1) and (2.2) hold for \( n = 4 \).

| Table 1 | Parity wise sum of minimal excludants for \( n = 4 \) |
|---------|--------------------------------------------------|
| \( \pi \in \mathcal{P}(4) \) | \( \text{mex}(\pi) \) |
| 4       | 1       |
| 3 + 1   | 2       |
| 2 + 2   | 1       |
| 2 + 1 + 1 | 3     |
| 1 + 1 + 1 + 1 | 2   |
| \( \sigma_o \text{mex}(4) \) | 1 + 1 + 3 = 5 |
| \( \sigma_e \text{mex}(4) \) | 2 + 2 = 4 |

Table 2 The number of partitions of \( n = 4 \) into distinct parts with two colors according to the parity of number of parts

| \( \pi \in D^o_2(4) \) | \( \pi \in D^e_2(4) \) |
|---------------------|---------------------|
| 4                   | 3 + 1 + 2           |
| 4_1                 | 3 + 1 + 1           |
| 2 + 1 + 1 + 1       | 3 + 1 + 2           |
| 2 + 1 + 1 + 1       | 3 + 1 + 1           |
| 2 + 2 + 2           | 2 + 2 + 2           |
| \( D^o_2(4) = 4 \) | \( D^e_2(4) = 5 \) |
As a consequence of Theorem 2.1, we obtain the following congruences.

**Theorem 2.2** For any non-negative integer \( n \),

\[
\sigma_o \text{mex}(2n + 1) \equiv 0 \pmod{4}, \tag{2.3}
\]
\[
\sigma_o \text{mex}(4n + 1) \equiv 0 \pmod{8}, \tag{2.4}
\]
\[
\sigma_e \text{mex}(4n) \equiv 0 \pmod{4}. \tag{2.5}
\]

Based on computational evidence, we also propose a few more congruences for these functions. For more details, see Conjecture 6.1 in Sect. 6.

Now we state a proposition that we will use in several proofs. As defined by Hopkins et al. [12, Proposition 5], let \( x(m, n) \) be the number of partitions of \( n \) whose minimal excludant is \( m \).

**Proposition 2.3** Suppose that \( n \) is a fixed non-negative integer. Then for \( z \in \mathbb{C} \), we have

\[
\sum_{m=1}^{\infty} x(m, n)z^m = p(n) + (z - 1)\sum_{m=0}^{\infty} p\left(n - \frac{m(m+1)}{2}\right)z^m. \tag{2.6}
\]

**Remark 1** We note that the sum on the right-hand side of the above result is in fact finite, because by convention \( p(n) = 0 \) for \( n < 0 \). Similarly, the left-hand side is also a finite sum because for a fixed \( n \) the minimal excludant of any partition of \( n \) is trivially bounded by \( n + 1 \). The comment also holds for all such “infinite-looking” partition sums in the sequel.

An immediate consequence of Proposition 2.3 arises by substituting \( z = -1 \) in (2.6), which is a result due to Hopkins et al. [12, p. 5, Eq. (3)], that is,

\[
o(n) - e(n) = p(n) + 2\sum_{m=1}^{\infty} (-1)^m p\left(n - \frac{m(m+1)}{2}\right),
\]

where \( o(n) \) is the number of partitions of \( n \) with an odd minimal excludant and \( e(n) \) counts those with an even minimal excludant. Andrews–Newmann [2, Theorem 2] and Hopkins–Sellers [11, Theorem 1] independently showed that \( o(n) \) equals the number of partitions of \( n \) with non-negative crank. This has been significantly generalized by Hopkins et al. [12, Theorem 2] recently.

The \( k \)th moments of minimal excludants: We now study sums involving \( k \)th powers of minimal excludants and generalize an identity of Andrews and Newman [1, p. 251], which was crucial to their second proof of (1.1).

**Definition 2** Let \( k, n \in \mathbb{N} \). Then

\[
\sigma^{(k)} \text{mex}(n) := \sum_{\pi \in \mathcal{P}(n)} (\text{mex}(\pi))^k,
\]
\[
\tilde{\sigma}^{(k)} \text{mex}(n) := \sum_{\pi \in \mathcal{P}(n)} (-1)^{\text{mex}(\pi)-1} (\text{mex}(\pi))^k.
\]
When \( k = 1 \), we see that \( \sigma^{(1)} \text{mex}(n) = \sigma \text{mex}(n) \) and we denote \( \tilde{\sigma}^{(1)} \text{mex}(n) \) by \( \tilde{\sigma} \text{mex}(n) \). As a consequence of the next theorem we get exact formulae for \( \sigma^{(k)} \text{mex}(n) \) and \( \tilde{\sigma}^{(k)} \text{mex}(n) \) in terms of partition functions.

**Theorem 2.4** For \( k, n \in \mathbb{N} \) and \( z \in \mathbb{C} \), the following identity holds true:

\[
\sum_{m=1}^{\infty} x(m, n) m^k z^m = \sum_{m=0}^{\infty} \left( (m+1)^k z^{m+1} - m^k z^m \right) p \left( n - \frac{m(m+1)}{2} \right). \tag{2.7}
\]

In particular, putting \( z = 1 \) and \(-1\), respectively, we have the following identities.

**Corollary 2.5** Given \( k, n \in \mathbb{N} \), we have

\[
\sigma^{(k)} \text{mex}(n) = \sum_{m=0}^{\infty} \left( (m+1)^k - m^k \right) p \left( n - \frac{m(m+1)}{2} \right), \tag{2.8}
\]

\[
\tilde{\sigma}^{(k)} \text{mex}(n) = \sum_{m=0}^{\infty} (-1)^m \left( (m+1)^k + m^k \right) p \left( n - \frac{m(m+1)}{2} \right). \tag{2.9}
\]

The corollary above also implies the next two identities. The first one below arises in the second proof of the main result of Andrews and Newman on \( \sigma \text{mex}(n) \) [1, p. 251] and the second is an analogous identity for \( \tilde{\sigma} \text{mex}(n) \).

**Corollary 2.6** If \( n \) is a positive integer, then

\[
\sigma \text{mex}(n) = \sum_{m=0}^{\infty} p \left( n - \frac{m(m+1)}{2} \right), \tag{2.10}
\]

\[
\tilde{\sigma} \text{mex}(n) = \sum_{m=0}^{\infty} (-1)^m (2m+1) p \left( n - \frac{m(m+1)}{2} \right). \tag{2.11}
\]

In the same spirit, we also define parity-based sums involving \( k \)th powers of minimal excludants.

**Definition 3** For positive integers \( k \) and \( n \), we define

\[
\sigma^{(k)}_{o} \text{mex}(n) := \sum_{\pi \in \mathcal{P}(n)} (\text{mex}(\pi))^k \quad \text{and} \quad \sigma^{(k)}_{e} \text{mex}(n) := \sum_{\pi \in \mathcal{P}(n)} (\text{mex}(\pi))^k.
\]

We have the following formulae.

**Theorem 2.7** For positive integers \( k \) and \( n \), we have

\[
\sigma^{(k)}_{o} \text{mex}(n) = \sum_{m=0}^{\infty} \left\{ \delta_m (m+1)^k - (1 - \delta_m) m^k \right\} p \left( n - \frac{m(m+1)}{2} \right) \tag{2.12}
\]
and
\[
\sigma_e^{(k)} \text{mex}(n) = \sum_{m=0}^{\infty} \left\{ (1 - \delta_m)(m + 1)^k - \delta_m m^k \right\} \left( n - \frac{m(m+1)}{2} \right). \tag{2.13}
\]

where the function \( \delta_m \) is defined as
\[
\delta_m := \frac{1 + (-1)^m}{2} = \begin{cases} 
1, & \text{if } m \text{ is even}, \\
0, & \text{otherwise}.
\end{cases}
\]

We mention an immediate corollary of the above theorem that gives explicit formulae for the functions \( \sigma_{o}^{(1)} \text{mex}(n) \) and \( \sigma_{e}^{(1)} \text{mex}(n) \), which are nothing but \( \sigma_{o} \text{mex}(n) \) and \( \sigma_{e} \text{mex}(n) \), respectively, as defined in (1.2) and (1.3).

**Corollary 2.8** Let \( t_m = \frac{m(m+1)}{2} \) be the \( m \)th triangular number. For \( n \in \mathbb{N} \), one has
\[
\sigma_{o} \text{mex}(n) = \sum_{m=0}^{\infty} (2m + 1) \left\{ p(n - t_{2m}) - p(n - t_{2m+1}) \right\}
\]
\[
= \{ p(n) - p(n+1) \} + 3 \{ p(n-3) - p(n-6) \} + \cdots \tag{2.14}
\]

and
\[
\sigma_{e} \text{mex}(n) = \sum_{m=0}^{\infty} (2m + 2) \left\{ p(n - t_{2m+1}) - p(n - t_{2m}) \right\}
\]
\[
= 2 \{ p(n-1) - p(n-3) \} + 4 \{ p(n-6) - p(n-10) \} + \cdots . \tag{2.15}
\]

Recall that \( o(n) \) is the number of partitions of \( n \) with an odd minimal excludant. Denote by \( o_1(n) \) (resp. \( o_3(n) \)) the number of partitions of \( n \) with minimal excludant congruent to 1 modulo 4 (resp. 3 modulo 4). So, \( o(n) = o_1(n) + o_3(n) \). Also, let \( q(n) \) represent the number of partitions of \( n \) into distinct parts. Our results also lead to an alternative proof for the following result of Hopkins et al. [12, Proposition 8].

**Proposition 2.9** For \( n \geq 1 \),
\[
o_1(n) = \begin{cases} 
o_3(n), & \text{if } n \text{ is odd}, \\
o_3(n) + q(n/2), & \text{otherwise}.
\end{cases}
\]

In [13, Proposition 3.2], Hopkins, Sellers, and Yee proved the above result by analytic methods.

We organize the rest of the paper in the following way. In the next section, we collect some useful results which will be essential for our proofs. Section 4 is devoted to the proofs of the main results. In Sect. 5, we present an alternative proof of Proposition 2.9. Finally, in the last section, we enunciate some open problems arising from this work.
3 Preliminaries

We first recall some standard results on Ramanujan’s theta function \( f(a, b) \), which is defined, for \(|ab| < 1\), by

\[
f(a, b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}.
\]

The famous triple product identity due to Jacobi is given by [6, p. 35, Entry 19]

\[
f(a, b) = (-a; ab)_{\infty}(-b; ab)_{\infty}(ab; ab)_{\infty}.
\]

In the sequel, we use the notation \( f_k := (q^k; q^k)_{\infty}, k \geq 1 \). As special cases of (3.1), we have

\[
\varphi(q) := f(q, q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q; q^2)_{\infty}^{2}(q^2; q^2)_{\infty} = \frac{f_5^2}{f_1^2 f_4^2} \tag{3.2}
\]

and

\[
\psi(q) := f(q, q^3) = \sum_{m=0}^{\infty} q^{(m+1)^2} = (-q; q^4)_{\infty}(-q^3; q^4)_{\infty}(q^4; q^4)_{\infty} = \frac{f_2^2}{f_1}. \tag{3.3}
\]

Replacing \( q \) by \(-q\) in (3.2), we also have

\[
\varphi(-q) = (q; q^2)_{\infty}^{2}(q^2; q^2)_{\infty} = \frac{f_1^2}{f_2}. \tag{3.4}
\]

Now, from [6, p. 40, Entry 25], we note that

\[
\varphi(q) + \varphi(-q) = 2\varphi(q^4) \tag{3.5}
\]

and

\[
\varphi(q) - \varphi(-q) = 4q \psi(q^8). \tag{3.6}
\]

Therefore,

\[
\varphi(q) = \varphi(q^4) + 2q \psi(q^8), \quad \text{and} \quad \varphi(-q) = \varphi(q^4) - 2q \psi(q^8).
\]

Using the above two identities and invoking (3.2), (3.3), and (3.4), we readily get the following 2-dissection formulas for \( 1/f_1^2 \) and \( f_1^2 \):

\[
\frac{1}{f_1^2} = \frac{f_5^2}{f_2^2 f_{16}^2} + 2q \frac{f_4^2 f_{16}^2}{f_5^2 f_8}, \tag{3.7}
\]
\[ f_1^2 = \frac{f_2 f_8^5}{f_4^2 f_{16}^2} - 2q \frac{f_2 f_{16}^2}{f_8}. \] (3.8)

We also recall the following 2-dissection of \( \psi(q) \) from [6, p. 49, Corollary (ii)]:

\[ \psi(q) = f(q^6, q^{10}) + qf(q^2, q^{14}). \] (3.9)

We end this section with another important \( q \)-series identity namely, Jacobi’s identity [6, p. 39, Entry 24(ii)]:

\[ \sum_{m=0}^{\infty} (2m + 1)(-1)^m q^{\frac{m+1}{2}} = (q; q)^3. \] (3.10)

The next section is devoted to proving our main results.

4 Proofs of main results

Proof of Theorem 2.1 Let \( x(2m + 1, n) \) denote the number of partitions of \( n \) whose minimal excludant is \( 2m + 1 \). Thus, we have

\[ \sum_{n=0}^{\infty} x(2m + 1, n)q^n = \frac{q^{1+2+\ldots+2m}}{\prod_{r=1}^{\infty} (1 - q^r)} = \frac{q^{\frac{2m+1}{2}}(1-q^{2m+1})}{(q; q)_{\infty}}. \] (4.1)

Now we introduce a two-parameter function \( L(z, q) \) with complex variables \( z \) and \( q \), where the exponent of \( z \) keeps track of the odd minimal excludant \( 2m + 1 \). Thus,

\[ L(z, q) := \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} x(2m + 1, n)z^{2m+1}q^n = \sum_{m=0}^{\infty} z^{2m+1} \sum_{n=0}^{\infty} x(2m + 1, n)q^n = \frac{1}{(q; q)_{\infty}} \sum_{m=0}^{\infty} z^{2m+1} q^{\frac{2m+1}{2}} (1 - q^{2m+1}), \] (4.2)

where we used (4.1) in the final step. Upon differentiating \( L(z, q) \) with respect to \( z \) and putting \( z = 1 \), we get

\[ \frac{\partial}{\partial z} L(z, q) \bigg|_{z=1} = \sum_{n=0}^{\infty} \left( \sum_{m=0}^{\infty} (2m + 1) x(2m + 1, n) \right) q^n. \] (4.3)
Observe that the inner sum of the right-hand side above is
\[
\sum_{m=0}^{\infty} (2m + 1)x(2m + 1, n) = \sum_{\pi \in \mathcal{P}(n)} \text{mex}(\pi) = \sigma_0 \text{mex}(n). \tag{4.4}
\]

After making use of (4.4), (4.2) in (4.3), we obtain
\[
\sum_{n=0}^{\infty} \sigma_0 \text{mex}(n) q^n = \left[ \frac{\partial}{\partial z} \left( \frac{1}{(q; q)_\infty} \sum_{m=0}^{\infty} q^{2m+1} (2m+1) (1 - q^{2m+1}) \right) \right]_{z=1}
= \frac{1}{(q; q)_\infty} \sum_{m=0}^{\infty} (2m + 1)q^{\frac{2m+1}{2}} (1 - q^{2m+1})
= \frac{1}{(q; q)_\infty} \left( \sum_{m=0}^{\infty} (2m + 1)q^{\frac{2m+1}{2}} - \sum_{m=0}^{\infty} (2m + 1)q^{\frac{2m+2}{2}} \right)
= \frac{1}{(q; q)_\infty} \left( \sum_{m=0}^{\infty} (2m + 1)q^{\frac{2m+1}{2}} - \sum_{m=0}^{\infty} (2m + 2)q^{\frac{2m+2}{2}} + \sum_{m=0}^{\infty} q^{\frac{2m+2}{2}} \right)
= \frac{1}{(q; q)_\infty} \left( \sum_{m=0}^{\infty} (m + 1)(-1)^m q^{\frac{m+1}{2}} + \sum_{m=0}^{\infty} q^{\frac{2m+2}{2}} \right). \tag{4.5}
\]

Now multiplying and dividing by 2 in the right-hand side of the last equality (4.5), and then simplifying yields,
\[
\frac{1}{2(q; q)_\infty} \left( \sum_{m=0}^{\infty} (2m + 2)(-1)^m q^{\frac{m+1}{2}} + \sum_{m=0}^{\infty} 2q^{\frac{2m+2}{2}} \right)
= \frac{1}{2(q; q)_\infty} \left( \sum_{m=0}^{\infty} (2m + 1)(-1)^m q^{\frac{m+1}{2}} + \sum_{m=0}^{\infty} (-1)^m q^{\frac{m+1}{2}} + \sum_{m=0}^{\infty} 2q^{\frac{2m+2}{2}} \right)
= \frac{1}{2(q; q)_\infty} \left( \sum_{m=0}^{\infty} (2m + 1)(-1)^m q^{\frac{m+1}{2}} + \sum_{m=0}^{\infty} q^{\frac{2m+2}{2}} - \sum_{m=0}^{\infty} q^{\frac{2m+2}{2}} + \sum_{m=0}^{\infty} 2q^{\frac{2m+2}{2}} \right)
= \frac{1}{2(q; q)_\infty} \left( \sum_{m=0}^{\infty} (2m + 1)(-1)^m q^{\frac{m+1}{2}} + \sum_{m=0}^{\infty} q^{\frac{2m+2}{2}} + \sum_{m=0}^{\infty} q^{\frac{2m+2}{2}} \right)
= \frac{1}{2(q; q)_\infty} \left( \sum_{m=0}^{\infty} (2m + 1)(-1)^m q^{\frac{m+1}{2}} + \sum_{m=0}^{\infty} q^{\frac{m+1}{2}} \right). \tag{4.6}
\]

Utilizing (4.6) in (4.5), we get
\[
\sum_{n=0}^{\infty} \sigma_0 \text{mex}(n) q^n = \frac{1}{2(q; q)_\infty} \left( \sum_{m=0}^{\infty} (2m + 1)(-1)^m q^{\frac{m+1}{2}} + \sum_{m=0}^{\infty} q^{\frac{m+1}{2}} \right). \tag{4.7}
\]
We now invoke (3.10) and (3.3) and substitute the product side of these identities for the two series in the right hand side of (4.7) to obtain

\[
\sum_{n=0}^{\infty} \sigma_{\text{mex}}(n) q^n = \frac{1}{2(q;q)_{\infty}} \left( (q;q)^3_{\infty} + \frac{(q^2;q^2)_{\infty}}{(q;q)_{\infty}} \right) = \frac{1}{2} \left( \frac{(q;q)^3_{\infty}}{(q;q)_{\infty}} + \frac{(q^2;q^2)_{\infty}}{(q;q)_{\infty}} \right) = \frac{1}{2} \left( (q;q)^2_{\infty} + (-q;q)^2_{\infty} \right),
\]

where the last equality follows from Euler’s identity, i.e., \((-q;q)_{\infty} = 1/(q;q^2)_{\infty}\). Observe that the last expression above can be written as

\[
(-q;q)_{\infty}^2 + (q;q)_{\infty}^2 = \sum_{n=0}^{\infty} (D^2_2(n) + D^2_2(n)) q^n + \sum_{n=0}^{\infty} (D^2_2(n) - D^2_2(n)) q^n = 2 \sum_{n=0}^{\infty} D^2_2(n) q^n.
\]

(4.8)

So we conclude that (2.1) holds true. Finally, subtracting (2.1) from (1.1), we obtain (2.2). \(\square\)

An alternative proof of Theorem 2.1:

First, we state and prove a lemma.

**Lemma 4.1** We have

\[
\sum_{n=0}^{\infty} \tilde{\sigma} \text{mex}(n) q^n = (q;q)_{\infty}^2.
\]

(4.9)

**Proof** Let \(x(m, n)\) be the number of partitions of \(n\) whose minimal excludant is \(m\). Proceeding analogously as in the previous proof from (4.1) to (4.2), we can show that

\[
M(z, q) := \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} x(m, n) z^m q^n = \frac{1}{(q;q)_{\infty}} \sum_{m=1}^{\infty} z^m q^m (\frac{z}{2}) (1 - q^m).
\]

(4.10)

Upon differentiating \(M(z, q)\) with respect to \(z\) and substituting \(z = -1\), we arrive at

\[
\frac{\partial}{\partial z} M(z, q) \bigg|_{z=-1} = \sum_{n=0}^{\infty} \left( \sum_{m=1}^{\infty} m(1)^{m-1} x(m, n) \right) q^n.
\]

(4.11)
The inner sum in the right-hand side above is actually
\[
\sum_{m=1}^{\infty} m (-1)^{m-1} x(m, n) = \sum_{\pi \in P(n)} (-1)^{\text{mex}(\pi) - 1} \text{mex}(\pi) = \bar{\sigma} \text{mex}(n).
\]

Putting this back in (4.11) gives us
\[
\sum_{n=0}^{\infty} \bar{\sigma} \text{mex}(n) q^n = \frac{\partial}{\partial z} M(z, q) \bigg|_{z=-1}.
\]

We now compute the right-hand side of the above by invoking the expression for \( M(z, q) \) in (4.10), that is,
\[
\left[ \frac{\partial}{\partial z} \left( \frac{1}{(q; q)_{\infty}} \sum_{m=1}^{\infty} z^m q^{m/2} (1 - q^m) \right) \right]_{z=-1}
= \frac{1}{(q; q)_{\infty}} \sum_{m=1}^{\infty} m (-1)^{m-1} q^{m/2} (1 - q^m)
= \frac{1}{(q; q)_{\infty}} \left( \sum_{m=1}^{\infty} m (-1)^{m-1} q^{m/2} + \sum_{m=1}^{\infty} m (-1)^{m-1} q^{m/2} + m \right)
= \frac{1}{(q; q)_{\infty}} \left( \sum_{m=0}^{\infty} (m+1) (-1)^m q^{(m+1)/2} - \sum_{m=0}^{\infty} m (-1)^m q^{(m+1)/2} \right)
= \frac{1}{(q; q)_{\infty}} \sum_{m=0}^{\infty} (2m+1) (-1)^m q^{(m+1)/2}
= (q; q)_{\infty}^2,
\]

where we used Jacobi’s identity (3.10) in the last equality. Putting this in (4.12) gives us the desired identity (4.9). \( \square \)

We are now in a position to present an alternative proof of Theorem 2.1.

**Proof of Theorem 2.1** To prove identity (2.1) in Theorem 2.1, we make use of Lemma 4.1 and (1.1). Adding (1.1) and (4.9) together gives us
\[
\sum_{n=0}^{\infty} (\sigma \text{mex}(n) + \bar{\sigma} \text{mex}(n)) q^n = (-q; q)_{\infty}^2 + (q; q)_{\infty}^2.
\]

Here, note that the summand on the left-hand side can be written as
\[
\sigma \text{mex}(n) + \bar{\sigma} \text{mex}(n) = \sum_{\pi \in P(n)} \text{mex}(\pi) - \sum_{\pi \in \text{mex}(n)} (-1)^{\text{mex}(\pi)} \text{mex}(\pi).
\]
\[ \sum_{\pi \in \mathcal{P}(n)} \text{mex}(\pi) = 2\sigma_o \text{mex}(n). \]

Substituting this value into (4.13), we find that
\[ \sum_{n=0}^{\infty} \sigma_o \text{mex}(n) q^n = \frac{(-q; q)_{\infty}^2 + (q; q)_{\infty}^2}{2}. \]

We have already interpreted the sum on the right-hand side of the above equation in (4.8). Thus, we conclude that (2.1) holds true and the proof of (2.2) follows from (2.1) and (1.1).

**Proof of Theorem 2.2** From Theorem 2.1, we have
\[ \sum_{n=0}^{\infty} \sigma_o \text{mex}(n) q^n = \frac{(-q; q)_{\infty}^2 + (q; q)_{\infty}^2}{2}, \] (4.14)

which can be recast as
\[ \sum_{n=0}^{\infty} \sigma_o \text{mex}(n) q^n = \frac{1}{2} \left( \frac{f_2^2}{f_1^2} + f_1^2 \right). \] (4.15)

Substituting the 2-dissections of \( f_1^2 \) and \( 1/f_1^2 \) from (3.8) and (3.7) into (4.15), we have
\[ \sum_{n=0}^{\infty} \sigma_o \text{mex}(n) q^n = \frac{1}{2} \left( \frac{f_2^5 f_8^2}{f_2^3 f_8^2} + 2q \frac{f_4^2 f_1^2}{f_3^2 f_8} + \frac{f_2 f_8^5}{f_4^2 f_1^2} - 2q \frac{f_2^2 f_1^2}{f_8^2} \right). \] (4.16)

Extracting the odd powers on both sides of (4.16), that is, equating the sub-series consisting of odd powers of \( q \) on both sides of (4.16), we obtain
\[ \sum_{n=0}^{\infty} \sigma_o \text{mex}(2n+1) q^{2n+1} = q \left( \frac{f_4^2 f_1^2}{f_3^2 f_8} - \frac{f_2^2 f_1^2}{f_8^2} \right). \]

Dividing both sides of the above by \( q \) and then replacing \( q^2 \) by \( q \), we arrive at
\[ \sum_{n=0}^{\infty} \sigma_o \text{mex}(2n+1) q^n = \frac{f_2^2 f_8^2}{f_3^2 f_4} - \frac{f_1 f_8^2}{f_4}. \] (4.17)

Now we prove that \( f_1^4 \equiv f_2^2 \pmod{4} \). Note that
\[ f_1^4 = \prod_{r=1}^{\infty} (1 - q^r)^4 \quad \text{and} \quad f_2^2 = \prod_{r=1}^{\infty} (1 - q^{2r})^2. \]
But,

\[(1 - q^r)^4 = 1 - 4q^r + 6q^{2r} - 4q^{3r} + q^{4r} \equiv 1 - 2q^{2r} + q^{4r} = (1 - q^{2r})^2 \pmod{4}.
\]

Therefore,

\[f_1^4 \equiv f_2^2 \pmod{4}.
\]

Employing the above congruence in (4.17), we find that

\[
\sum_{n=0}^{\infty} \sigma_o \text{mex}(2n + 1) q^n = \frac{f_8^2}{f_4} \left( \frac{f_2^2}{f_1^2} - 1 \right) \equiv 0 \pmod{4}.
\]

Thus,

\[\sigma_o \text{mex}(2n + 1) \equiv 0 \pmod{4},
\]

which completes the proof of (2.3).

Now we move to the proof of the second congruence of the theorem, namely, (2.4). To this end, we need to find a 2-dissection of the generating function (4.17) of \(\sigma_o \text{mex}(2n + 1)\). We can rewrite (4.17) as

\[
\sum_{n=0}^{\infty} \sigma_o \text{mex}(2n + 1) q^n = \frac{f_8^2}{f_4} \left( \frac{1}{f_1^2} - \frac{f_1^2}{f_2^2} \right) \psi(q).
\]

Employing (3.8), (3.7) and (3.9) in the above, we have

\[
\sum_{n=0}^{\infty} \sigma_o \text{mex}(2n + 1) q^n = \frac{f_8^2}{f_4} \left( \frac{f_5^5}{f_2^5 f_1^5} + 2q \frac{f_4^2 f_8^2}{f_2^5 f_8^2} - \frac{f_8^5}{f_2 f_4^2 f_1^5} + 2q \frac{f_1^2}{f_2^2 f_8^2} \right) \left( f(q^6, q^{10}) + qf(q^2, q^{14}) \right),
\]

which is a 2-dissection of the generating function of \(\sigma_o \text{mex}(2n + 1)\). Extracting the even terms from both sides of the above and then replacing \(q^2\) by \(q\), we find that

\[
\sum_{n=0}^{\infty} \sigma_o \text{mex}(4n + 1) q^n = \frac{f_4^4}{f_2^2} \left( f(q^3, q^5) \left( \frac{f_4^3}{f_1^3 f_8^3} - \frac{f_8^3}{f_1 f_2^3 f_8^3} \right) + 2qf(q, q^7) \left( \frac{f_4^2 f_8^2}{f_1^2 f_4} + \frac{f_8^2}{f_1 f_4} \right) \right)
\]

\[= f(q^3, q^5) \frac{f_1^7}{f_1 f_2^2 f_8^3} \left( \frac{f_2^5}{f_1^2} - 1 \right) + 2qf(q, q^7) \frac{f_4^4 f_8^2}{f_1 f_2} \left( \frac{f_2^5}{f_1^2} + 1 \right). \quad (4.18)
\]
To simplify the above, we transform the terms involving $f(q, q^7)$ into terms involving $f(q^3, q^5)$. To that end, first we recall from [6, p. 51, Example (iv)] that

$$
2 f^2(q^3, q^5) = \psi(q) \varphi(q^2) + \psi(-q) \varphi(-q),
2q f^2(q, q^7) = \psi(q) \varphi(q^2) - \psi(-q) \varphi(-q).
$$

The above two imply that

$$
f^2(q^3, q^5) + q f^2(q, q^7) = \psi(q) \varphi(q^2), \quad (4.19)
$$
$$
f^2(q^3, q^5) - q f^2(q, q^7) = \psi(-q) \varphi(-q). \quad (4.20)
$$

Now, with the aid of Jacobi’s triple product identity (3.1), we have

$$
f(q^3, q^5) f(q, q^7) = (-q, -q^3, -q^5, -q^7; q^8)_{\infty} f_8^2 = (-q; q^2)_{\infty} f_8^2. \quad (4.21)
$$

Dividing both sides of (4.19) by $f(q^3, q^5) f(q, q^7)$ and then using (4.21), we obtain

$$
\frac{f(q^3, q^5)}{f(q, q^7)} + q \frac{f(q, q^7)}{f(q^3, q^5)} = \frac{\psi(q) \varphi(q^2)}{(-q; q^2)_{\infty} f_8^2} = \frac{\varphi(q^2)}{\psi(q^4)}. \quad (4.22)
$$

Again, dividing both sides of (4.20) by $f(q^3, q^5) f(q, q^7)$, followed by a second appeal to (4.21) and subsequent simplifications give us

$$
\frac{f(q^3, q^5)}{f(q, q^7)} - q \frac{f(q, q^7)}{f(q^3, q^5)} = \frac{\psi(-q) \varphi(-q^2)}{(-q; q^2)_{\infty} f_8^2} = \frac{\varphi(-q)}{\psi(q^4)}. \quad (4.23)
$$

Subtracting (4.23) from (4.22), we have

$$
2q \frac{f(q, q^7)}{f(q^3, q^5)} = \frac{\varphi(q^2) - \varphi(-q)}{\psi(q^4)},
$$

that is,

$$
2q f(q, q^7) = f(q^3, q^5) \frac{\varphi(q^2) - \varphi(-q)}{\psi(q^4)}. \quad (4.24)
$$

Employing (4.24) in (4.18), and then using the $q$-product representations of $\varphi(q^2)$, $\varphi(-q)$ and $\psi(q^4)$ from (3.2) (3.4) and (3.3), we obtain

$$
\sum_{n=0}^{\infty} \sigma_{o \text{mex}}(4n + 1) q^n
= f(q^3, q^5) \frac{f_4^7}{f_1 f_2 f_4 f_8} \left( \frac{f_2^2}{f_1^2} - 1 \right) + f(q^3, q^5) \frac{f_4 f_8}{f_1 f_2} \left( \frac{f_2^2}{f_1^2} + 1 \right) \left( \frac{\varphi(q^2) - \varphi(-q)}{\psi(q^4)} \right)
$$
\[ f(q^3, q^5) \frac{f_4^2}{f_1^3} \left( 2 \frac{f_4^5}{f_1^2 f_2 f_8} - \frac{f_1^4}{f_2^2} - 1 \right). \] (4.25)

Noting that \( 2 f_1^4 \equiv 2 f_2^2 \pmod{8} \), we infer from the above that

\[ \sum_{n=0}^{\infty} \sigma_{\text{mex}}(4n + 1)q^n \equiv f(q^3, q^5) \frac{f_4^2}{f_1^3} \left( 2 \frac{f_2^2 f_4}{f_3^2} - \frac{f_4^4}{f_2^2} - 1 \right) \equiv f(q^3, q^5) \frac{f_4^2}{f_1^3} \left( 2 \varphi(-q)\varphi(-q^2) - \varphi^2(-q) - 1 \right) \pmod{8}. \] (4.26)

Now, from (3.6), we have

\[ \varphi(-q) \equiv \varphi(q) \pmod{4}. \] (4.27)

Multiplying (3.5) and (3.6), we also have

\[ \varphi^2(q) - \varphi^2(-q) = 8q \varphi(q^4) \psi(q^8), \]

which implies that

\[ \varphi^2(-q) \equiv \varphi^2(q) \pmod{8}. \] (4.28)

With the aid of (4.27) and (4.28), we can recast (4.26) as

\[ \sum_{n=0}^{\infty} \sigma_{\text{mex}}(4n + 1)q^n \equiv f(q^3, q^5) \frac{f_4^2}{f_1^3} \left( 2 \varphi(q)\varphi(q^2) - \varphi^2(q) - 1 \right) \pmod{8}. \] (4.29)

Hence from the above identity, to prove (2.4), it is enough to show that

\[ 2\varphi(q)\varphi(q^2) - \varphi^2(q) - 1 \equiv 0 \pmod{8}. \] (4.30)

To prove (4.30), we recall the Lambert series representations of \( \varphi^2(q) \) and \( \varphi(q)\varphi(q^2) \) from [7, p. 58, Eq. (3.2.9); p. 73, Eq. (3.7.3)], namely,

\[ \varphi^2(q) = 1 + 4 \sum_{n=1}^{\infty} \frac{q^n}{1 + q^{2n}}, \] (4.31)

\[ \varphi(q)\varphi(q^2) = 1 + 2 \sum_{n=1}^{\infty} \frac{q^n + q^{3n}}{1 + q^{3n}}. \] (4.32)
From (4.31) and (4.32), we see that

\[
2\varphi(q)\varphi(q^2) - \varphi^2(q) - 1 = 4 \left( \sum_{n=1}^{\infty} \frac{q^n + q^{3n}}{1 + q^{4n}} - \sum_{n=1}^{\infty} \frac{q^n}{1 + q^{2n}} \right)
\equiv 4 \left( \sum_{n=1}^{\infty} \frac{q^n(1 + q^{2n})}{1 - q^{4n}} - \sum_{n=1}^{\infty} \frac{q^n}{1 - q^{2n}} \right) \equiv 0 \pmod{8}.
\]

Employing the above in (4.29), we readily arrive at (2.4).

Next, we give the proof of the last congruence, namely, (2.5). Proceeding as in (4.14)–(4.17), but with

\[
\sum_{n=0}^{\infty} \sigma_e\text{mex}(n)q^n = \frac{(-q; q)_\infty^2 - (q; q)_\infty^2}{2}
\]

instead of

\[
\sum_{n=0}^{\infty} \sigma_o\text{mex}(n)q^n = \frac{(-q; q)_\infty^2 + (q; q)_\infty^2}{2},
\]

we find that

\[
\sum_{n=0}^{\infty} \sigma_e\text{mex}(2n)q^n = \frac{1}{2} \left( \frac{f_4^5}{f_1 f_8} - \frac{f_1 f_4^5}{f_2 f_8^2} \right)
\equiv \frac{1}{2} \frac{f_4^6}{f_2 f_8^4} \left( \frac{f_2^2 f_8^2}{f_1^3 f_4} - \frac{f_1 f_8^2}{f_4} \right).
\]

(4.33)

From (4.33) and (4.17), it follows that

\[
2 \sum_{n=0}^{\infty} \sigma_e\text{mex}(2n)q^n = \frac{f_4^6}{f_2 f_8^4} \sum_{n=0}^{\infty} \sigma_o\text{mex}(2n + 1)q^n.
\]

Extract the even powers of \( q \) in the above equation and then replace \( q^2 \) by \( q \) to see that

\[
2 \sum_{n=0}^{\infty} \sigma_e\text{mex}(4n)q^n = \frac{f_2^6}{f_1^2 f_4^2} \sum_{n=0}^{\infty} \sigma_o\text{mex}(4n + 1)q^n.
\]

(4.34)

Since by (2.4), we have already shown that \( \sigma_o\text{mex}(4n + 1) \equiv 0 \pmod{8} \), it is immediate from (4.34) that \( \sigma_e\text{mex}(4n) \equiv 0 \pmod{4} \), which is (2.5).
We now give an alternative proof of (2.3) using Proposition 2.9.

**Alternative proof of (2.3).** For any positive integer \( m \), we have

\[
\sigma_0 \text{mex}(m) = \sum_{\pi \in P(m), \text{mex}(\pi) \equiv 1 \pmod{4}} \text{mex}(\pi) + \sum_{\pi \in P(m), \text{mex}(\pi) \equiv 3 \pmod{4}} \text{mex}(\pi) \\
\equiv \frac{1}{\sigma_1(m)}(1 + \cdots + 1) + \frac{3}{\sigma_3(m)}(3 + \cdots + 3) \pmod{4} \\
\equiv \frac{1}{\sigma_1(m)}(1 + \cdots + 1) + \frac{(-1)}{\sigma_3(m)}((-1) + (-1) + \cdots + (-1)) \pmod{4} \\
= \sigma_1(m) - \sigma_3(m).
\]

Thus,

\[
\sigma_0 \text{mex}(m) \equiv \sigma_1(m) - \sigma_3(m) \pmod{4} \quad \text{for all} \quad m.
\]

Invoking Proposition 2.9 with odd \( m = 2n + 1 \) leads us to

\[
\sigma_0 \text{mex}(2n + 1) \equiv 0 \pmod{4},
\]

as desired.

**Proof of Proposition 2.3** We rewrite the double series \( M(z, q) \), as defined in (4.10), as

\[
\sum_{n=0}^{\infty} \sum_{m=1}^{\infty} x(m, n)z^m q^n = \frac{1}{(q; q)_\infty} \sum_{m=1}^{\infty} z^m q^{(m)}(1 - q^m). \tag{4.35}
\]

We proceed with the right-hand side of the above identity as follows.

\[
\frac{1}{(q; q)_\infty} \sum_{m=1}^{\infty} z^m q^{(m)}(1 - q^m) = \frac{1}{(q; q)_\infty}\left( \sum_{m=1}^{\infty} z^m q^{(m+1)} - \sum_{m=1}^{\infty} z^m q^{(m+1)} \right) \\
= \sum_{n=0}^{\infty} p(n)q^n \left( z \sum_{m=0}^{\infty} z^m q^{(m+1)} - \sum_{m=1}^{\infty} z^m q^{(m+1)} \right) \\
= \sum_{n=0}^{\infty} p(n)q^n \left( 1 + (z - 1) \sum_{m=0}^{\infty} z^m q^{(m+1)} \right) \\
= \sum_{n=0}^{\infty} p(n)q^n + \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (z - 1)z^m p \left( n - \frac{m(m + 1)}{2} \right) q^n. \tag{4.36}
\]

Comparing the coefficient of \( q^n \) on the left side of (4.35) with that on the right side of (4.36), we obtain (2.6).
Proof of Theorem 2.4 First, recall a useful differential operator, namely,

\[ D_z := z \frac{\partial}{\partial z}. \]

Next, observe that \( D_z^k(z^m) = m^k z^m \) for all positive integers \( k \). Hence by applying the operator \( k \) times on both sides of (2.6), we find that

\[
\sum_{m=1}^{\infty} x(m, n) D_z^k(z^m) = \sum_{m=0}^{\infty} p\left(n - \frac{m(m + 1)}{2}\right) D_z^k(z^{m+1} - z^m),
\]

which readily leads to the desired identity (2.7).

Proof of Corollary 2.6 Simply substitute \( k = 1 \) in (2.8) and (2.9) to obtain (2.10) and (2.11), respectively.

Proof of Theorem 2.7 Adding the equations (2.8) and (2.9), we obtain

\[
\sigma^{(k)} \text{mex}(n) + \bar{\sigma}^{(k)} \text{mex}(n) = \sum_{m=0}^{\infty} \left( \left\{ 1 + (-1)^m \right\} (m + 1)^k - \left\{ 1 - (-1)^m \right\} m^k \right) \times p\left(n - \frac{m(m + 1)}{2}\right).
\]

Note that the left-hand side of the above equation is

\[
\sigma^{(k)} \text{mex}(n) + \bar{\sigma}^{(k)} \text{mex}(n) = \sum_{\pi \in \mathcal{P}(n)} \left( \text{mex}(\pi) \right)^k - (-1)^{\text{mex}(\pi)} \left( \text{mex}(\pi) \right)^k
\]

\[ = 2 \sum_{\pi \in \mathcal{P}(n)} \left( \text{mex}(\pi) \right)^k
\]

\[ = 2\sigma^{(k)}_o \text{mex}(n).
\]

Putting this together with (4.37), we arrive at

\[
\sigma^{(k)}_o \text{mex}(n) = \sum_{m=0}^{\infty} \left( \left\{ 1 + (-1)^m \right\} (m + 1)^k - \left\{ 1 - (-1)^m \right\} m^k \right) p\left(n - \frac{m(m + 1)}{2}\right)
\]

\[ = \sum_{m=0}^{\infty} \delta_m (m + 1)^k - (1 - \delta_m)m^k \ p\left(n - \frac{m(m + 1)}{2}\right),
\]

where \( \delta_m = \frac{1 + (-1)^m}{2} \). This proves (2.12). The proof of (2.13) is similar except for the minor change that we are required to subtract (2.9) from (2.8) at the beginning. \(\square\)
Proof of Corollary 2.8 It is enough to prove (2.14), the proof of (2.15) being almost identical. To obtain (2.14), we put $k = 1$ in (2.12) to get

$$\sigma_o \text{mex}(n) = \sigma_o^{(1)} \text{mex}(n) = \sum_{m=0}^{\infty} \{\delta_m (m + 1) - (1 - \delta_m)m\} p \left(n - \frac{m(m + 1)}{2}\right).$$

In view of the definitions of $\delta_m$ and $t_m$, we can write the previous equality as

$$\sigma_o \text{mex}(n) = \sum_{m=0}^{\infty} \frac{(m + 1)}{m \text{ even}} p (n - t_m) - \sum_{m=1}^{\infty} \frac{mp (n - t_m)}{m \text{ odd}}$$

$$= \sum_{m=0}^{\infty} (2m + 1) p (n - t_{2m}) - \sum_{m=0}^{\infty} (2m + 1) p (n - t_{2m+1})$$

$$= \sum_{m=0}^{\infty} (2m + 1) \{p (n - t_{2m}) - p (n - t_{2m+1})\}. \tag{5.1}$$

5 An alternate proof of a result due to Hopkins, Sellers, and Stanton

In this section, we give an alternative proof of a result due to Hopkins et al. [12, Proposition 8], which we already stated as Proposition 2.9.

Proof of Proposition 2.9 Let us recall (4.2):

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} x(2m + 1, n) z^{2m+1} q^n = \frac{1}{(q; q)_{\infty}} \sum_{m=0}^{\infty} z^{2m+1} q^{\frac{(2m+1)}{2}} (1 - q^{2m+1}).$$

Now we substitute $z = i$ in the above identity to get

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} x(2m + 1, n)(-1)^m q^n = \frac{1}{(q; q)_{\infty}} \sum_{m=0}^{\infty} (-1)^m q^{\frac{(2m+1)}{2}} (1 - q^{2m+1}). \tag{5.1}$$

First, we simplify the left-hand side as follows:

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} x(2m + 1, n)(-1)^m q^n = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{\infty} x(4m + 1, n) - \sum_{m=0}^{\infty} x(4m + 3, n)\right)q^n$$

$$= \sum_{n=0}^{\infty} \left(o_1(n) - o_3(n)\right)q^n. \tag{5.2}$$

Second, we simplify the right-hand side of (5.1):

\[
\frac{1}{(q; q)_{\infty}} \sum_{m=0}^{\infty} (-1)^m q^{\left(\frac{2m+1}{2}\right)} (1 - q^{2m+1})
\]

\[=rac{1}{(q; q)_{\infty}} \left( \sum_{m=0}^{\infty} (-1)^m q^{\left(\frac{2m+1}{2}\right)} - \sum_{m=0}^{\infty} (-1)^m q^{\left(\frac{2m+2}{2}\right)} \right)\]

\[= \frac{1}{(q; q)_{\infty}} \left( \sum_{j=0}^{\infty} (-1)^j q^{\left(\frac{2j+1}{2}\right)} + \sum_{j=-\infty}^{-1} (-1)^j q^{\left(\frac{2j+1}{2}\right)} \right)\]

\[= \frac{1}{(q; q)_{\infty}} \sum_{j=-\infty}^{\infty} (-1)^j q^{\left(\frac{2j+1}{2}\right)}, \quad (5.3)\]

where the rightmost sum in the second equality arises by substituting \( j = -(m + 1) \).

Putting (5.2) and (5.3) into (5.1) gives us

\[
\sum_{n=0}^{\infty} \left( o_1(n) - o_3(n) \right) q^n = \frac{1}{(q; q)_{\infty}} \sum_{j=-\infty}^{\infty} (-1)^j q^{\left(\frac{2j+1}{2}\right)}. \quad (5.4)
\]

Replacing \( a \) by \(-q^3\) and \( b \) by \(-q\) in (3.1), we have

\[
\sum_{j=-\infty}^{\infty} \left( -q^3 \right)^{j(j+1)} \left( -q \right)^{j(j-1)} = (q^3; q^4)_{\infty} (q^4; q^4)_{\infty} (q^4; q^4)_{\infty};
\]

that is,

\[
\sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2+j} = \sum_{j=-\infty}^{\infty} (-1)^j q^{\left(\frac{2j+1}{2}\right)} = (q^3; q^4)_{\infty} (q^4; q^4)_{\infty} (q^4; q^4)_{\infty}. \quad (5.5)
\]

Now utilizing (5.5) in (5.4), we arrive at

\[
\sum_{n=0}^{\infty} \left( o_1(n) - o_3(n) \right) q^n = \frac{1}{(q; q)_{\infty}} \times (q^3; q^4)_{\infty} (q^4; q^4)_{\infty} (q^4; q^4)_{\infty}
\]

\[= \frac{1}{(q^2; q^4)_{\infty}} = (-q^2; q^2)_{\infty}, \quad (5.6)
\]

where the last step follows by Euler’s identity \((-q; q)_{\infty} = \frac{1}{(q; q^2)_{\infty}}\). Now keeping in mind that \((-q; q)_{\infty}\) is the generating function for partitions of \( n \) into distinct parts.
we can rephrase (5.6) as
\[
\sum_{n=0}^{\infty} \left( o_1(n) - o_3(n) \right) q^n = (-q^2; q^2)_\infty = \sum_{n=0}^{\infty} q(n)q^{2n};
\]
that is,
\[
\sum_{n=0}^{\infty} \left( o_1(n) - o_3(n) \right) q^n = \sum_{n=0}^{\infty} q(n/2)q^n.
\]
Since \( q(n/2) \) is non-zero only when \( n/2 \) is a natural number, we thus have
\[
o_1(n) - o_3(n) = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ q(n/2), & \text{otherwise.} \end{cases}
\]

6 Concluding remarks

In this paper, we established a refinement of equation (1.1) of Andrews and Newman on \( \sigma_{\text{mex}}(n) \), namely, Theorem 2.1. Moreover, we also proved congruences for \( \sigma_{o\text{mex}}(n) \) and \( \sigma_{e\text{mex}}(n) \) modulo 4 and 8, namely, Theorem 2.2. Based on computations using Magma and Mathematica, we propose the following conjecture.

Conjecture 6.1 Let \( n \) be a non-negative integer. Then
\[
\sigma_{o\text{mex}}(8n + 1) \equiv 0 \pmod{16} \tag{6.1}
\]
and
\[
\sigma_{e\text{mex}}(8n) \equiv 0 \pmod{8}. \tag{6.2}
\]

Working on the generating function (4.25) of \( \sigma_{o\text{mex}}(4n + 1) \), we can show that
\[
\sum_{n=0}^{\infty} \sigma_{o\text{mex}}(8n + 1)q^n
\]
\[
= \left( f(q^3, q^5) f(q^7, q^9) + q^2 f(q, q^7) f(q, q^{15}) \right)
\]
\[
\times \left( -\frac{f_2^2 f_4^5}{f_1^3 f_8^2} - \frac{f_4^5}{f_1^3 f_8^2} + 2 \frac{f_2^7 f_4^8}{f_1^{13} f_8^4} + 8q \frac{f_2^{11} f_4^8}{f_1^{13} f_8^4} \right)
\]
\[
+ 2q \left( f(q, q^7) f(q^7, q^9) + q f(q^3, q^5) f(q, q^{15}) \right)
\]
\[
\times \left( -\frac{f_4^2 f_8^2}{f_1^3 f_4} + \frac{f_2^2 f_8^2}{f_1^3 f_4} + 4 \frac{f_4^9 f_4^2}{f_1^{13}} \right).
\]
The proposed congruence (6.1) would follow if one could show that the right side of
the above equality vanishes under modulo 16. At this moment, we do not have a proof
for that. A similar generating function expression for \( \sigma_{e\text{mex}}(8n) \) may also be found.

Grabner and Knopfmacher [10, Eq. (4.7)] showed that \( \sigma_{\text{mex}}(n) \) satisfies the fol-
lowing Hardy–Ramanujan type asymptotic formula:

\[
\sigma_{\text{mex}}(n) \sim \frac{1}{4\sqrt[4]{6n^3}} \exp \left( \pi \sqrt{\frac{2n}{3}} \right) \quad \text{as } n \to \infty.
\]

They [10, Theorem 3] also obtained a Hardy–Ramanujan–Rademacher type exact
formula for \( \sigma_{\text{mex}}(n) \). It will be highly desirable to obtain asymptotic formulae for
\( \sigma_{o\text{mex}}(n) \) and \( \sigma_{e\text{mex}}(n) \) and their moments, and to do that the identities (2.14) and
(2.15) might be useful. It would also be interesting to find a combinatorial proof of
the main result, Theorem 2.1, in the manner of Ballantine and Merca [3], who proved
the original result (1.1) of Andrews and Newman combinatorially.

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