Algorithmic Regularization in Over-parameterized Matrix Recovery

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Abstract

We study the problem of recovering a low-rank matrix $X^\star$ from linear measurements using an over-parameterized model. We parameterize the rank-$r$ matrix $X^\star$ by $UU^\top$ where $U \in \mathbb{R}^{d \times d}$ is a square matrix, whereas the number of linear measurements is much less than $d^2$. We show that with $\tilde{O}(dr^2)$ random linear measurements, the gradient descent on the squared loss, starting from a small initialization, recovers $X^\star$ approximately in $\tilde{O}(\sqrt{r})$ iterations. The results solve the conjecture of Gunasekar et al. [13] under the restricted isometry property, and demonstrate that the training algorithm can provide an implicit regularization for non-linear matrix factorization models.

1 Introduction

Over-parameterized models are crucial in deep learning, but their workings are far from understood. Over-parameterization apparently improves the training: theoretical and empirical results have suggested that in simplified settings it can enhance the geometric properties of the optimization landscape [18, 15, 14, 26], and thus allows efficient training.

In many practical situations, over-parameterization also doesn’t hurt the testing performance, even if the number of parameters is much larger than the number of examples. Neural networks often have enough expressivity to fit any labels of the training datasets [32, 14]. Thus the training objective function may have multiple global minima with almost zero training error, some of which generalize better than the others [17, 8]. However, local improvement algorithms such as stochastic gradient descent, starting with proper initialization, may prefer some generalizable local minima to the others and thus provide an implicit effect of regularization [22, 16, 21, 31]. Such regularization seems to depend on the algorithmic choice, the initialization scheme, and certain intrinsic properties of the data.

The phenomenon and intuition above can be theoretically fleshed out in the context of linear models for gradient descent [27], whereas they are less clear for non-linear models trained by non-convex optimization algorithms. The important work of Gunasekar et al. [13] initiates the study of low-rank matrix factorization models with over-parameterization and conjectures that gradient descent prefers low trace norm solution in over-parameterized models with thorough empirical evidence.

This paper resolves the conjecture for the matrix regression problem — recovering a low-rank matrix from linear measurements — under the restricted isometry property (RIP) condition. We show that with a full-rank factorized parameterization, gradient descent on the squared loss with

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finite step size, starting with small initialization, converges to the true low-rank matrix (which is also the minimum trace norm solution.) One advantage of the over-parameterized approach is that without knowing/guessing the correct rank, the algorithms can automatically pick up the minimum rank/trace norm solution that fits the data. Moreover, such theoretical analysis of algorithmic regularization in the non-convex setting may shed light on other more complicated models where over-parameterization is crucial for the efficient training.

1.1 Setup and Main Results

Let $X^*$ be an unknown rank-$r$ symmetric positive semidefinite (PSD) matrix in $\mathbb{R}^{d \times d}$ that we aim to recover. Let $A_1, \ldots, A_m \in \mathbb{R}^{d \times d}$ be $m$ given measurement matrices. We assume that the label vector $y$ is generated by linear measurements

$$y_i = \langle A_i, X^* \rangle.$$ 

Here $\langle A, B \rangle = \text{tr}(A^\top B)$ denotes the inner product of two matrices. Our goal is to recover the matrix $X^*$. \[1\]

Without loss of generality, we assume that $X^*$ has spectral norm 1. Let $\kappa = 1/\sigma_r(X^*)$ be the condition number of $X^*$. We mostly focus on the regime where $r \ll d$ and $m \approx d \cdot \text{poly}(r) \ll d^2$.

Let $U \in \mathbb{R}^{d \times d}$ be a matrix variable. We consider the following squared loss objective function with over-parameterization:

$$\min_U f(U) = \frac{1}{m} \sum_{i=1}^{m} \left( y_i - \langle A_i, UU^\top \rangle \right)^2 \tag{1.1}$$

Since the label is generated by $y_i = \langle A_i, X^* \rangle$, any matrix $U$ satisfying $UU^\top = X^*$ is a local minimum of $f$ with zero training error. These are the ideal local minima that we are shooting for. However, because the number of free parameters $d^2$ is much larger than the number of observation $m$, there exist other choices of $U$ satisfying $f(U) = 0$ but $UU^\top \neq X^*$.

A priori, such over-parameterization will cause over-fitting. However, we will show that the following gradient descent algorithm with small initialization converges to a desired local minimum instead of other non-generalizable local minima:

$$U_0 = \alpha B,$$

where $B \in \mathbb{R}^{d \times d}$ is any orthonormal matrix

$$U_{t+1} = U_t - \eta \nabla f(U_t)$$

The following theorem assumes the measurements matrices $A_1, \ldots, A_m$ satisfy restricted isometry property (which is formally defined in Section 2). Casual readers may simply assume that the entries of $A_i$’s are drawn i.i.d from standard normal distribution and the number of observations $m \lesssim d^2 \log^3 d$: it’s known \[2\] that in this case $A_1, \ldots, A_m$ meet the requirement of the following theorem, that is, they satisfy $(r, \delta)$-RIP with $\delta \lesssim 1/(\sqrt{\kappa} \log d)$. \[3\]

**Theorem 1.1.** Let $c$ be a sufficiently small absolute constant. Assume $(A_1, \ldots, A_m)$ satisfies $(4r, \delta)$-restricted isometry property (defined in Section 2 formally) with $\delta \leq c/(\kappa^3 \sqrt{\kappa} \log d)$. Suppose the initialization and learning rate satisfy $0 < \alpha \leq c \min\{\delta \sqrt{\kappa}, 1/d^2\}$ and $\eta \leq c\delta$. Then for some $T = O((\kappa \log (d/\alpha))/\eta)$, we have

$$\|U_T U_T^\top - X^*\|_F^2 \lesssim d^2 \kappa \alpha$$

\[1\]Our analysis can naturally handle a small amount of Gaussian noise in the label vector $y$, but for simplicity we work with only the noiseless case.

\[2\]Technically, to get such strong RIP constant that depends on $r$, one would need to slightly modify the the proof of \[24\] Theorem 4.2] at the end to get the dependency of the number of observations on $\delta$. 

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Note that the recovery error $\|U_T U_T^T - X^*\|_F^2$ can be viewed as the test error — it’s equal to the test error on a fresh measurement $A_j$ drawn from the Gaussian distribution. The theorem above shows that the gradient descent can provide algorithmic regularization so that the generalization error depends on the size of the initialization $\alpha$ instead of the number of parameters. Because the runtime is not very sensitive to the initialization, we can choose small enough $\alpha$ (e.g., $1/d^2$) to get approximately zero generalization error.

We note that we achieve a good iteration complexity bound $1/\eta \approx 1/\delta \approx \sqrt{r}$, which was not known in previous work even for low-rank parameterization. Part of the technical challenges is to allow finite step size $\eta$ and inverse-poly initialization $\alpha$ (instead of exponentially small initialization). The dependency of $\delta$ on $\kappa$ and $r$ in the theorem is possibly not tight. We conjecture that $\delta$ only needs to be smaller than an absolute constant, which is left for future work.

**Insights of the analysis:** Interestingly, our analysis ideas seem to be different from other previous work in a conceptual sense. The analysis of the logistic regression case [27] relies on that the iterate eventually moves to infinity, which is a unique feature of log-loss without any explicit regularization. The folklore analysis of the algorithmic regularization of SGD for least squares and the analysis in [13] for the matrix regression with commutable measurements both follow the two-step plan: a) the iterates always stays on a low-rank manifold regardless of the number of steps and the label vector $y$; b) generalization follows from the low complexity of the low-rank manifold. Such universal manifold (that doesn’t depend on the label vector) doesn’t seem to exist in the setting when $A_i$’s are random.

Instead, we exploit the fact that gradient descent on the population risk with small initialization only searches through the space of solutions with a lower rank than that of the optimal solution. If the optimal solution is low-rank, then the iterates of GD on the population risk will stay in the subset of all approximate low-rank solutions. Due to the low complexity of this subset, the population risk is close to the empirical risk in it uniformly. Hence, we can expect the learning dynamics of the empirical risk to be similar to that of the population risk in this subset, and therefore the iterates of GD on the empirical risk remain approximately low-rank as well, despite the over-parameterization. Generalization then follows straightforwardly from the low-rankness of the iterates. See Section 3 for more discussions.

We note that the factorized parameterization also plays an important role here. The intuition above would still apply if we replace $UU^T$ with a single variable $X$ and run gradient descent in the space of $X$ with small enough initialization. However, it will converge to a solution that doesn’t generalize. The discrepancy comes from another crucial property of the factorized parameterization: it provides certain denoising effect that encourages the empirical gradient to have a smaller eigenvalue tail. This ensures the eigenvalues tail of the iterates to grow sufficiently slowly. This point will be more precise in Section 3 once we apply the RIP property.

Finally, we remark that the cases with rank $r > 1$ are technically much more challenging than the rank-1 case. For the latter, we show (in Section 3) the spectrum of $U_t$ in a fixed rank-$(d-1)$ subspace remains small so that $U_t$ stays close to rank-1 (and this subspace is exactly complement of the column span of $X^*$.) By contrast, for the rank-$r$ case, a direct extension of this proof strategy is no longer true unless the initialization is exponentially small. Instead, we identify an adaptive rank-$(d-r)$ subspace in which $U_t$ remains small. The best choice of this adaptive subspace is the subspace of the least $(d-r)$ left singular vectors of $U_t$, but we use a more convenient surrogate, as shown in Section 4.

**Organization:** The rest of this paper is organized as follows: In Section 2 we define some notations and review some of the preliminaries regarding the restricted isometry property. In Section 3 we lay out the key theoretical insights towards proving Theorem 1.1 and give the analysis for the rank-1 case as a warm-up. In Section 4 we outline the main steps for proving Theorem 1.1 and
Recall that we assume identity matrix. Let $U^+$ denote the Moore-Penrose pseudo-inverse of the matrix $U$. Let $\| \cdot \|$ denotes the Euclidean norm of a vector and spectral norm of a matrix. Let $\| \cdot \|_F$ denotes the Frobenius norm of a matrix. Suppose $A \in \mathbb{R}^{m \times n}$, then $\sigma_{\max}(A)$ denote its largest singular value and $\sigma_{\min}(A)$ denotes its $\min \{m, n\}$-th largest singular value. Alternatively, we have $\sigma_{\min}(A) = \min_x \|Ax\|$. Let $\langle A, B \rangle = \text{tr}(A^\top B)$ denote the inner product of two matrices.

Unless explicitly stated otherwise, $O(\cdot)$-notation hides absolute multiplicative constants. Concretely, every occurrence of $O(x)$ is a placeholder for some function $f(x)$ that satisfies $\forall x \in \mathbb{R}.|f(x)| \leq C|x|$ for some absolute constant $C > 0$. Similarly, $a \lesssim b$ means that there exists an absolute constant $C > 0$ such that $a \lesssim Cb$. We use the notation poly$(n)$ as an abbreviation for $n^{O(1)}$.

## 2 Preliminaries and Related Work

Recall that we assume $X^*$ is rank-$r$ and positive semidefinite. Let $X^* = U^* \Sigma^* U^\top$ be the eigen-decomposition of $X^*$, where $U^* \in \mathbb{R}^{d \times r}$ is an orthonormal matrix and $\Sigma^* \in \mathbb{R}^{r \times r}$ is a diagonal matrix. The assumption that $\|X^*\| = 1$ and $\sigma_r(X^*) = 1/\kappa$ translate to that $\forall \iota \in [r], 1/\kappa \leq \Sigma_{\iota \iota} \leq 1$.

Under these notations, we see that the target solution for the variable $U$ is equal to $U = U^* \Sigma^{\star 1/2} R$ where $R$ can be arbitrary orthonormal matrix.

For convenience, we define the matrix $M_t$ as

$$M_t = \frac{1}{m} \sum_{i=1}^{m} \langle A_i, U_t U_t^\top - X^* \rangle A_i \tag{2.1}$$

Then the update rule can be rewritten as

$$U_{t+1} = (\text{Id} - \eta M_t)U_t \tag{2.2}$$

where $\text{Id}$ is the identity matrix. One of the key challenges is to understand how the matrix $\text{Id} - \eta M_t$ transforms the matrix $U_t$ so that we can move from $U_0$ to the target solution $U^* \Sigma^{\star 1/2} R$.

Suppose that $A_1, \ldots, A_m$ are drawn from Gaussian distribution and optimistically suppose that they are independent with $U_t$. Then, we have that $M_t \approx U_t U_t^\top - X^*$, since the expectation of $M_t$ with respect to the randomness of $A_i$’s is equal to $U_t U_t^\top - X^*$. However, they are two fundamental issues with this wishful thinking: a) obviously $U_t$ depends on $A_i$’s heavily for $t > 1$, since in every update step $A_i$’s are used; b) even if $A_i$’s are independently random Gaussian, there are not enough $A_i$’s to guarantee $M_t$ concentrates around its mean $U_t U_t^\top - X^*$ in Euclidean norm. To have such concentration, we need $m > d^2$, whereas we only have $m = d \text{poly}(r)$ samples.

**Restricted isometry property:** The restricted isometry property (RIP) allows us to partially circumvent both the technical issues a) and b) above. It says that $A_1, \ldots, A_m$ are nearly orthogonal to each other, at least on rank-$r$ matrices.

**Definition 2.1.** (low-rank matrix restricted isometry property [24]) A set of linear measurement matrices $A_1, \ldots, A_m \in \mathbb{R}^{d \times d}$ satisfies $(r, \delta)$-restricted isometry property (RIP) if for any $d \times d$
matrix $X$ with rank at most $r$, we have

$$(1 - \delta)\|X\|_F^2 \leq \frac{1}{m} \sum_{i=1}^{m} (A_i, X)^2 \leq (1 + \delta)\|X\|_F^2.$$  

(2.3)

The crucial consequence of RIP that we exploit in this paper is the meta statement as follows:

$$\mathcal{M}(Q) := \frac{1}{m} \sum_{i=1}^{m} \langle A_i, Q \rangle A_i \text{ behaves like } Q \text{ for approximate low-rank } Q$$  

(2.4)

We will state several lemmas below that reflect the principle above. The following lemma says that $\langle \mathcal{M}(X), Y \rangle$ behaves like $\langle X, Y \rangle$ for low-rank matrices $X, Y$.

**Lemma 2.2.** [5, Lemma 2.1] Let $\{A_i\}_{i=1}^{m}$ be a family of matrices in $\mathbb{R}^{d \times d}$ that satisfy $(r, \delta)$-restricted isometry property. Then for any matrices $X, Y \in \mathbb{R}^{d \times d}$ with rank at most $r$, we have:

$$\left| \frac{1}{m} \sum_{i=1}^{m} \langle A_i, X \rangle \langle A_i, Y \rangle - \langle X, Y \rangle \right| \leq \delta \|X\|_F \|Y\|_F.$$  

The following lemma says that $\mathcal{M}(X)$ behaves like $X$ when multiplied by a matrix $Y$ with small operator norm.

**Lemma 2.3.** Let $\{A_i\}_{i=1}^{m}$ be a family of matrices in $\mathbb{R}^{d \times d}$ that satisfy $(r, \delta)$-restricted isometry property. Then for any matrix $X \in \mathbb{R}^{d \times d}$ of rank at most $r$, and any matrix $R \in \mathbb{R}^{d \times d'}$, where $d'$ can be any positive integer, we have:

$$\left| \frac{1}{m} \sum_{i=1}^{m} \langle A_i, X \rangle A_i R - XR \right| \leq \delta \|X\|_F \|R\|.$$  

We can also have extensions of the lemmas above to the cases when $X$ has a higher rank (see Lemma B.1 and Lemma B.2). The bounds are not as strong as above (which is inevitable because we only have $m$ measurements), but are useful when $X$ itself is relatively small.

### 2.1 Related Work

**Generalization theory beyond uniform convergence:** This work builds upon the work of Gunasekar et al. [13], which raises the conjecture of the implicit regularization in matrix factorization models and provides theoretical evidence for the simplified setting where the measurements matrices are commutable. Implicit regularization of gradient descent is studied in the logistic regression setting by [27].

Recently, the work of Hardt et al. [16] studies the implicit regularization provided by stochastic gradient descent through uniform stability [3, 19, 25]. Since the analysis therein is independent of the training labels and therefore it may give pessimistic bounds [33]. Brutzkus et al. [4] use a compression bound to show network-size independent generalization bounds of one-hidden-layer neural networks on linearly separable data.

Bartlett et al. [1] and Neyshabur et al. [20] recently prove spectrally-normalized margin-based generalization bounds for neural networks. Dziugaite and Roy [9] provide non-vacuous generalization bounds for neural networks from PCA-Bayes bounds. As pointed out by [1], it’s still unclear why SGD without explicit regularization can return a large margin solution. This paper makes progress on explaining the regularization power of gradient descent, though on much simpler non-linear models.
Matrix factorization problems: Early works on matrix sensing and matrix factorization problems use convex relaxation (nuclear norm minimization) approaches and obtain tight sample complexity bounds \([24, 29, 7, 23, 6]\). The recent work of Ge et al. \([12]\) and Bhojanapalli et al. \([2]\) shows that the non-convex objectives on matrix completion and matrix sensing with low-rank parameterization don’t have any spurious local minima, and stochastic gradient descent algorithm on them converges to the global minimum. Such a phenomenon was long known for the PCA problem and recently shown for phase retrieval, robust PCA, and random tensor decomposition as well (e.g., see \([28, 12, 2, 11, 10, 30]\) and reference therein).

3 Proof Overview and Rank-1 Case

In this section, we demonstrate the key ideas of our proofs and give an analysis of the rank-1 case as a warm-up.

The main intuition is that the iterate \(U_t\) stays approximately low-rank in the sense that

(a) The \((r + 1)\)-th singular value \(\sigma_{r+1}(U_t)\) remains small for any \(t \geq 0\).

(b) The top \(r\) singular vectors and singular values of \(U_tU_t^\top\) converges those of \(X^\star\) in logarithmic number of iterations.

Propositions (a) and (b) can be clearly seen when the number of observations \(m\) approaches infinity and \(A_1, \ldots, A_m\) are Gaussian measurements. Let’s define the population risk \(\bar{f}\) as

\[ \bar{f}(U) = \mathbb{E}_{(A_i)_{i \sim N(0,1)}} [f(U)] = \|U_tU_t^\top - X^\star\|_F^2 \]  

(3.1)

In this case, the matrix \(M_t\) (defined in \((2.1)\)) corresponds to \(U_tU_t^\top - X^\star\), and therefore the update rule for \(U_t\) can be simply rewritten as

\[
U_{t+1} = U_t - \eta \nabla \bar{f}(U_t) = U_t - \eta(U_tU_t^\top - X^\star)U_t \\
= U_t(\text{Id} - \eta U_tU_t^\top) + \eta X^\star U_t 
\]  

(3.2)

Observe that the term \(\eta X^\star U_t\) encourages the column span of \(U_{t+1}\) to move towards the the columns span of \(X^\star\), which causes the phenomenon in Proposition (b). On the other hand, the term \(U_t(\text{Id} - \eta U_tU_t^\top)\) is performing a contraction of all the singular values of \(U_t\), and therefore encourages them to remain small. As a result, \(U_t\) decreases in those directions that are orthogonal to the span of \(X^\star\), because there is no positive force to push up those directions.

As far, we have shown that the iterate of GD on population risk remains approximately low-rank.

Recall that the difficulty was that the empirical risk \(f\) doesn’t uniformly concentrate well around the population risk \(\bar{f}\). However, the uniform convergence can occur, at least to some extent, in the restricted set of approximate low-rank matrices! In other words, since the gradient descent algorithm only searches a limited part of the whole space, we only require restricted uniform convergence theorem such as restricted isometry property.

Motivated by the observations above, a natural meta-proof plan is that

1. The trajectory of iterates of GD on the population risk stays in the set of approximate low-rank matrices

\(^3\)Namely, we don’t have uniform convergence results in the sense that \(|f(U) - \bar{f}(U)|\) is small for all matrices \(U\). (For examples, for many matrices we can have \(f(U) = 0\) but \(\bar{f}(U) \gg 0\) because we have more variables than observations.)
The trajectory of the empirical risk behaves similarly to that of the population risk in the set of approximate low-rank matrices. It turns out that implementing the plan above quantitatively can be technically challenging: the distance from the iterate to the set of low-rank matrices can accumulate linearly in the number of steps. Therefore we have to augment the plan with a strong result about the speed of the convergence:

3. The iterates converges to the ideal solution fast enough before its effective rank increases.

3.1 Warm-up: Rank-1 Case

In this section, we present a simplified proof of the rank-1 case that demonstrates some of the key ideas in the analysis. We note that the analysis of the rank-r case in Section 4 is significantly more sophisticated because it requires more delicate techniques to control how the top r eigenvectors interact with each other and the rest of eigenvectors.

Throughout this subsection, we assume that $X^* = u^* u^*^\top$ for $u^* \in \mathbb{R}^{d \times 1}$ and that $\|u^*\| = 1$. We decompose the iterates $U_t$ into the subspace of $u^*$ and its complement

$$U_t = \text{Id}_{u^*} U_t + (\text{Id} - \text{Id}_{u^*}) U_t$$

where $r_t := U_t^\top u^*$ and $E_t := (\text{Id} - \text{Id}_{u^*}) U_t$.

In the light of the meta-proof plan discussed above, we will show that the spectral norm and Frobenius norm of the “error term” $E_t$ remain small throughout the iterations, whereas the “signal” term $u^* r_t^\top$ grows exponentially fast (in the sense that the norm of $r_t$ grows to 1.) Note that any solution with $\|r_t\| = 1$ and $E_t = 0$ will give exact recovery, and for this section we will show that $\|r_t\|$ will converges approximately to 1 and $E_t$ stays small.

As an interlude, we remark that for the rank $r > 1$ case, such simple decomposition doesn’t suffice because the top $r$ eigenspace of $U_t$ does drift. It’s necessary to decompose $U_t$ into signal and error terms according to an adaptive subspace. See Section 4 for details.

Under the representation (3.3), from the original update rule (2.2), we derive the update for $E_t$

$$E_{t+1} = (\text{Id} - \text{Id}_{u^*}) \cdot (\text{Id} - \eta M_t) U_t$$

$$= E_t - \eta \cdot (\text{Id} - \text{Id}_{u^*}) M_t U_t$$

(3.4)

Throughout this section, we assume that $r = 1$ and that $(A_1, \ldots, A_m)$ satisfies $(3, \delta)$-RIP with $\delta \leq c$ where $c$ is a sufficiently small absolute constant (e.g., $c = 0.01$ suffices).

Theorem 3.1. In the setting of this subsection, suppose $\alpha \leq \delta \sqrt{\frac{1}{d} \log \frac{1}{\delta}}$ and $\eta \lesssim c \delta^2 \log^{-1}(1/(\delta \alpha))$. Then after $T = \Theta(\log \frac{1}{\alpha \eta})$ iterations, we have:

$$\|U_T U_T^\top - X^*\|_F \leq O(\delta)$$

We remark that, for simplicity, we state a Theorem that is weaker than Theorem 1.1 even for the case with $r = 1$. In Theorem 1.1 (or Theorem 4.1), the final error depends linearly on the initialization, whereas the error here depends on the RIP parameter. Improving Theorem 3.1 would involve only finer induction, and we refer the readers to Theorem 1.1 for the stronger bounds.

The following lemma gives the growth rate of $E_t$ in spectral norm and Euclidean norm, in a typical situation when $E_t$ and $r_t$ are bounded above in Euclidean norm,
As we discussed in Section 2, if $U$ and we therefore we will decompose it into

$$\|E_{t+1}\|_F^2 \leq \|E_t\|_F^2 + 2\eta\delta\|E_tU_t^T\| + 9\eta^2 \quad (3.5)$$

$$\|E_{t+1}\| \leq (1 -\eta \|r_t\|^2 + 2\eta\delta)\|E_t\| + 2\eta\delta\|r_t\|. \quad (3.6)$$

A recurring technique in this section, as alluded before, is to establish the approximation

$$U_{t+1} = U_t - \eta M_t U_t \approx U_t - \eta(U_tU_t - X^*)U_t \quad (3.7)$$

As we discussed in Section 2, if $U_tU_t - X^*$ is low-rank, then the approximation above can be established by Lemma 2.2 and Lemma 2.3. However, $U_tU_t - X^*$ is only approximately low-rank, and we therefore we will decompose it into

$$U_tU_t^T - X^* = (U_tU_t^T - X^* - E_tE_t^T) + E_tE_t^T \quad \text{rank} \leq 4$$

Note that $U_tU_t^T - X^* - E_tE_t^T = \|r_t\|^2u^*u^T + u^*r_t^T E_t^T + E_tr_tu^T - X^*$ has rank at most 3, and therefore we can apply Lemma 2.2 and Lemma 2.3. For the term $E_tE_t^T$, we can afford to use other loose bounds (Lemma B.1 and B.2) because $E_t$ itself is small.

**Proof Sketch of Proposition 3.2.** Using the update rule (3.4) for $E_t$, we have that

$$\|E_{t+1}\|_F^2 = \|E_t\|_F^2 - 2\eta \langle E_t, (I - Id_{n^*})M_tU_t \rangle + \eta^2\|\langle Id - Id_{n^*}\rangle M_tU_t\|_F^2 \quad (3.9)$$

The third term on the RHS is negligible when $\eta$ is sufficiently small compared to the second term when $M_t$, $U_t$ are bounded from above in spectral norm. Therefore, we focus on the second term that is linear in $\eta$.

$$\langle E_t, (I - Id_{n^*})M_tU_t \rangle \quad (3.10)$$

$$\frac{1}{m} \sum_{i=1}^{m} \langle A_i, U_tU_t^T - X^* \rangle \langle A_i, (I - Id_{n^*})E_tU_t^T \rangle \quad (3.11)$$

where in the last line we rearrange the terms and use the fact that $(I - Id_{n^*})$ is symmetric. Now we use Lemma 2.2 to show that equation (3.11) is close to $\langle U_tU_t^T - X^*, (I - Id_{n^*})E_tU_t^T \rangle$, which is its expectation w.r.t the randomness of $A_i$'s if $A_i$'s were chosen from spherical Gaussian distribution. If $U_tU_t^T - X^*$ was a rank-1 matrix, then this would follow from Lemma 2.2 directly. However, $U_tU_t^T$ is approximately low-rank. Thus, we decompose it into a low-rank part and an error part with small trace norm as in equation (3.8). Since $U_tU_t^T - X^* - E_tE_t^T$ has rank at most 4, we can apply Lemma 2.2 to control the effect of $A_i$'s,

$$\frac{1}{m} \sum_{i=1}^{m} \langle A_i, (U_tU_t^T - X^* - E_tE_t^T) \rangle \langle A_i, E_tU_t^T \rangle$$

$$\geq \langle U_tU_t^T - X^* - E_tE_t^T, E_tU_t^T \rangle - \delta\|U_tU_t^T - X^* - E_tE_t^T\|_F\|E_tU_t^T\|$$

$$\geq \langle U_tU_t^T - X^* - E_tE_t^T, E_tU_t^T \rangle - 1.5\delta\|E_tU_t^T\| \quad (3.12)$$

where the last inequality uses that $\|U_tU_t^T - X^* - E_tE_t^T\|_F^2 = (1 - \|r_t\|^2)^2 + 2\|Er_t\|^2 \leq 1 + \|E_t\|^2\|r_t\|^2 \leq 11/8.$
For the $E_t E_t^\top$ term in the decomposition (3.8), by Lemma [B.1] we have that

$$
\frac{1}{m} \sum_{i=1}^{m} \langle A_i, E_t E_t^\top \rangle \langle A_i, E_t U_t^\top \rangle \geq \langle E_t E_t^\top, E_t U_t^\top \rangle - \delta \|E_t E_t^\top\|_* \|E_t U_t^\top\| \geq \langle E_t E_t^\top, E_t U_t^\top \rangle - 0.5\delta \|E_t U_t^\top\|
$$

Combining equation (3.11), (3.12) and (3.14), we conclude that

$$
\langle E_t, (\text{Id} - \text{Id}_{u^*}) M_t U_t \rangle \geq \langle U_t U_t^\top - X^*, E_t U_t^\top \rangle - 2\delta \|E_t U_t^\top\|
$$

Note that $u^\top E_t = 0$, which implies that $X^* E_t = 0$ and $U_t^\top E_t = E_t^\top E_t$. Therefore,

$$
\langle U_t U_t^\top - X^*, E_t U_t^\top \rangle = \langle U_t U_t^\top, E_t U_t^\top \rangle = \langle U_t^\top, U_t^\top E_t U_t^\top \rangle = \langle E_t U_t^\top, E_t U_t^\top \rangle = \|E_t U_t^\top\|^2 \geq \|E_t U_t^\top\|^2
$$

We can also control the third term in RHS of equation (3.9) by $\eta^2 \|\text{Id} - \text{Id}_{u^*}\| M_t U_t \|^2 \leq 9\eta^2$. Since the bound here is less important (because one can always choose small enough $\eta$ to make this term dominated by the first order term), we left the details to the reader. Combining the equation above with (3.15) and (3.9), we conclude the proof of equation (3.5).

Towards bounding the spectral norm of $E_{t+1}$, we use a similar technique to control the difference between $(\text{Id} - \text{Id}_{u^*}) M_t U_t$ and $(\text{Id} - \text{Id}_{u^*})(U_t U_t^\top - X^*) U_t$ in spectral norm. By decomposing $U_t U_t^\top - X^*$ as in equation (3.8) and applying Lemma [B.1] and Lemma [B.2] respectively, we obtain that

$$
\|(\text{Id} - \text{Id}_{u^*}) M_t U_t - (\text{Id} - \text{Id}_{u^*})(U_t U_t^\top - X^*) U_t\| \leq 4\delta \left( \|U_t U_t^\top - X^* - E_t E_t^\top\|_F + \|E_t E_t^\top\|_* \right) \|U_t^\top\| \leq 8\delta \|U_t\| \leq 8\delta (\|r_t\| + \|E_t\|) \quad \text{(by the assumptions that $\|E_t\|_F \leq 1/2, \|r_t\| \leq 3/2$)}
$$

Observing that $(\text{Id} - \text{Id}_{u^*})(U_t U_t^\top - X^*) U_t = E_t U_t^\top U_t$. Plugging the equation above into equation (3.4), we conclude that

$$
\|E_{t+1}\| \leq \|E_t (\text{Id} - \eta U_t^\top U_t)\| + 2\eta\delta (\|r_t\| + \|E_t\|) \leq (1 - \eta \|r_t\|^2) \|E_t\| + 2\eta\delta (\|r_t\| + \|E_t\|)
$$

The next Proposition shows that the signal term grows very fast, when the signal itself is not close to norm 1 and the error term $E_t$ is small.

**Proposition 3.3** (Signal dynamics). *In the same setting of Proposition 3.2, we have,*

$$
\|r_{t+1}\| \geq (1 + \eta (1 - \|r_t\|^2 - 3\delta))\|r_t\| - 2\eta\delta \|E_t\|
$$

*Proof.* By the update rule (2.2), we have that

$$
r_{t+1} = U_{t+1}^\top u^* = U_t^\top (\text{Id} - \eta M_t^\top) u^* = r_t - \eta U_t^\top M_t^\top u^*
$$
By Lemma 2.3 and B.2, we obtain that

$$\| r_{t+1} - (r_t - \eta U_t^T (U_t U_t^T - X^*) u^*) \| \leq \delta \left( \| U_t U_t^T - X^* - E_t E_t^T \|_F + \| E_t E_t^T \|_F \right) \| U_t \| \quad (3.17)$$

By simplifying the above expression, also note that $$\| U_t U_t^T - X^* - E_t E_t^T \|_F \leq 11/8$$ and $$\| E_t \|_F^2 \leq 1/4$$, we have that

$$\| r_{t+1} - (1 + \eta (1 - \| r_t \|^2)) r_t \| \leq \eta \| E_t^T E_t r_t \| + 2\eta \delta \| U_t \| \quad (3.18)$$

Note that $$\| U_t \| \leq \| r_t \| + \| E_t \|$$, and $$\| E_t^T E_t \| \leq \| E_t \|_F^2 \leq 1/4$$. Hence we obtain the conclusion. \( \square \)

The following proposition shows that $$\| r_t \|$$ converges to 1 approximately and $$E_t$$ remains small by inductively using the two propositions above.

**Proposition 3.4** (Control $$r_t$$ and $$E_t$$ by induction). In the setting of Theorem 3.1, after $$T = \Theta(\log(1/\alpha \delta) / \eta)$$ iterations,

$$\| r_T \| = 1 \pm O(\delta) \quad (3.19)$$

$$\| E_T \| \lesssim \delta \quad \text{and} \quad \| E_T \|_F^2 \lesssim \delta^2 \log(1/\delta) \quad (3.20)$$

**Proof Sketch.** We will analyze the dynamics of $$r_t$$ and $$E_t$$ in two stages. The first stage consists of all the steps such that $$\| r_t \| \leq 2\delta$$, the second stage consists of steps with $$2\delta \leq \| r_t \| \leq 1/2$$, and we call the rest of steps the third stage. We will show that

a) Stage 1 and 2 have totally at most $$O(\log(1/\alpha) / \eta)$$ steps. Throughout Stage 1 and 2, we have

$$\| E_t \| \leq 5 \| r_t \| \quad (3.21)$$

$$\| r_{t+1} \| \geq (1 + \eta/3) \| r_t \| \quad (3.22)$$

b) Throughout Stage 3, we have

$$\| E_t \| \lesssim \delta \quad \text{and} \quad \| r_t \| \leq 1 + O(\delta) \quad (3.23)$$

and after at most $$O(\log(1/\delta) / \eta)$$ steps in Stage 3, we have $$\| r_t \| \geq 1 - O(\delta)$$.

c) In Stage 2 and 3, we have that

$$\| E_t \| \lesssim \delta \quad \text{and} \quad \| E_t \|_F \lesssim \delta^2 \log \frac{1}{\delta} \quad (3.24)$$

We use induction with Proposition 3.2 and 3.3. For $$t = 0$$, we have that $$\| E_0 \| = \| r_0 \| = \alpha$$ because $$B$$ is an orthonormal matrix. Suppose equation (3.21) is true for some $$t$$, then we can prove equation (3.22) holds by Proposition 3.3.

$$\| r_{t+1} \| \geq (1 + \eta (1 - \| r_t \|^2 - 3\delta)) \| r_t \| - 10\eta \delta \| r_t \|$$

$$\geq (1 + \eta/3) \| r_t \| \quad (\text{by } \delta \leq c \leq 0.01 \text{ and } \| r_t \| \leq 1/2)$$

Suppose (3.22) holds, we can prove equation (5.21) holds for the $$t + 1$$ case by Proposition 3.2.

$$\| E_{t+1} \| \leq (1 + 2\eta \delta) \| E_t \| + 2\eta \delta \| r_t \| \leq 5(1 + 2\eta \delta) \| r_t \| + 2\eta \| r_t \|$$

$$\leq 5 \| r_{t+1} \| \quad (\text{by equation (3.22)})$$
For claim b), we note that in Stage 3, we can simplify Proposition 3.2 and Proposition 3.3 by
\[
\|E_{t+1}\| \leq (1 - \eta/3)\|E_{t}\| + 2\eta\delta\|r_{t}\|.
\] (3.25)
\[
\|r_{t+1}\| \geq (1 + \eta(1 - \|r_{t}\|^{2} - 3\delta))\|r_{t}\| - 2\eta\delta\|E_{t}\|
\] (3.26)

With some induction similar to the previous case, we can prove claim b). The details are left for
readers.

To prove claim c), we first consider Stage 2. By repeated using Proposition 3.2 we have that
\[
\|E_{t}\| \leq \|E_{0}\| + 2\eta\delta \sum_{j \leq t} \|r_{j}\|
\leq \|E_{0}\| + 6\delta \lesssim \delta
\] (since \(\|r_{j}\| \leq 1/2\) and \(\|r_{j+1}\| \geq (1 + \eta/3)\|r_{j}\|\))

Similarly we can bound the Frobenius norm of \(E_{t}\) in Stage 1. Using that
\[
\|E_{t}U_{t}\| \leq \|E_{t}\| (\|r_{t}\| + \|E_{t}\|) \lesssim \|r_{t}\|^{2},
\] we can simplify Proposition 3.2 by
\[
\|E_{t}\|_{F}^{2} \leq \|E_{0}\|_{F}^{2} + 9\eta^{2}t + O(\eta\delta) \cdot \sum_{j \leq t} \|r_{j}\|^{2}
\leq \|E_{0}\|_{F}^{2} + 9\eta^{2}t + O(\delta^{2})
\] (3.27)

Suppose \(T_{1}\) is the last step in Stage 1. For Stage 2 and 3, noting that \(\|E_{t}U_{t}\| \leq \|E_{t}\| \lesssim \delta\)
\[
\|E_{t}\|_{F}^{2} \leq \|E_{0}\|_{F}^{2} + 9\eta^{2}t + O(\eta\delta^{2}(t - T_{1})) \lesssim \delta^{2}\log(1/\delta),
\] because \(\|E_{0}\|_{F}^{2} = \alpha^{2}d \leq O(\delta^{2}\log \frac{1}{\alpha\delta})\), and \(\eta^{2}t \leq O(\eta\log \frac{1}{\alpha\delta}) \leq O(\delta^{2}\log \frac{1}{\alpha\delta})\).

\(\square\)

Theorem 3.1 follows from Proposition 3.3 straightforwardly.

**Proof of Theorem 3.1** Using the conclusions of Proposition 3.4
\[
\left\|U_{t}U_{t}^{\top} - X^{\star}\right\|_{F}^{2} = (1 - \|r_{t}\|^{2})^{2} + 2\|E_{t}r_{t}\|^{2} + \|E_{t}E_{t}^{\top}\|_{F}^{2}
\leq (1 - \|r_{t}\|^{2})^{2} + 2\|E_{t}\|^{2}\|r_{t}\|^{2} + \|E_{t}\|_{F}^{4}
\leq O(\delta^{2}) + O(\delta^{2}) + O((\delta^{2}\log \frac{1}{\delta})^{2}) = O(\delta^{2})
\] (3.30)

\(\square\)

### 4 Proof Outline of Rank-\(r\) Case

In this section we outline the proof of Theorem 1.1. The proof is significantly more sophisticated
than the rank-1 case (Theorem 3.1) because the top \(r\) eigenvalues of the iterates grow at different
speed and we need to align the signal and error term in the right way so that we can see a monotone
increase in the signal. Concretely, we will have to decompose the iterate \(U_{t}\) into a signal and an
error term according to a dynamic subspace, as we will outline below. Moreover, the generalization
error analysis here is also tighter than Theorem 3.1.

We first state a slightly stronger version of Theorem 1.1
Theorem 4.1. There exists a sufficiently small absolute constant \( c > 0 \) such that the following is true. For every \( \alpha \in (0, c/d^2) \), assume \((A_1, \ldots, A_m)\) satisfies \((r, \delta)\)-RIP with \( \delta \leq c/(r^3 \sqrt{\log d}) \), \( \eta \leq c\delta \), and \( T_0 = 100\kappa \log(d/\alpha)/\eta \). For every \( t \leq T_0 \),
\[
\|U_tU_t^T - X^*\|_F^2 \leq (1 - \eta/(8\kappa))^{t-t_0} + O(d^2\alpha\kappa)
\]
As a consequence, for \( T = \Theta((\kappa \log(d/\alpha))/\eta) \), we have
\[
\|U_TU_T^T - X^*\|_F^2 \lesssim d^2\kappa\alpha
\]

When the condition number \( \kappa \) and rank \( r \) are both constant, this theorem says that if we shoot for a final error \( \varepsilon \), then we should pick our initialization \( U_1 = \alpha B \) with \( \alpha = O(\varepsilon/d^2) \). As long as the RIP-parameter \( \delta = O\left(\frac{1}{\log \frac{d}{\varepsilon}}\right) \), after \( O(\log \frac{d}{\varepsilon}) \) iterations we will have that \( \|U_TU_T^T - X^*\|_F \leq \varepsilon \).

Towards proving the theorem above, we suppose the eigen-decomposition of \( X^* \) can be written as \( X^* = U^*\Sigma U^*\top \) where \( U^* \in \mathbb{R}^{d \times r} \) is an orthonormal matrix \( U^* \in \mathbb{R}^{r \times r} \) is a diagonal matrix. We maintain the following decomposition of \( U_t \) throughout the iterations:
\[
U_t = \underbrace{\text{Id}_{S_t}U_t}_{:= Z_t} + (\text{Id} - \underbrace{\text{Id}_{S_t}}_{:= E_t})U_t
\]
(4.1)

Here \( S_t \) is \( r \) dimensional subspace that is recursively defined by
\[
S_1 = \text{span}(U^*)
\]
(4.2)
\[
S_t = (\text{Id} - \eta M_t) \cdot S_{t-1}
\]
(4.3)

Here \((\text{Id} - \eta M_t) \cdot S_{t-1}\) denotes the subspace \( \{(\text{Id} - \eta M_t)v : v \in S_{t-1}\} \). Note that \( \text{rank}(S_0) = \text{rank}(U^*) = r \), and thus by induction we will have that for every \( t \),
\[
\text{rank}(Z_t) \leq \text{rank}(S_t) \leq r
\]

Note that by comparison, in the analysis of rank-1 case, the subspace \( S_t \) is chosen to be \( \text{span}(U^*) \) for every \( t \), but here it starts off as \( \text{span}(U^*) \) but changes throughout the iterations. Instead, we will show that \( S_t \) stays close to \( \text{span}(U^*) \). Moreover, we will show that the error term \( E_t \) — though growing exponentially fast — always remains much smaller than the signal term \( Z_t \), which grows exponentially with a faster rate. Let \( \sin(A, B) \) denote the principal angles between the column span of matrices \( A, B \). The intuitions above are summarized in the following theorem.

Theorem 4.2. There exists a sufficiently small absolute constant \( c > 0 \) such that the following is true: For every \( \alpha \in (0, c/d^2) \), assume \((A_1, \ldots, A_m)\) satisfies \((r, \delta)\)-RIP with \( \delta \leq c^4/(r^3 \sqrt{\log d}) \), \( \rho = (1/c)\kappa \sqrt{\delta} \), \( \eta \leq c\delta \), and \( T = \frac{c}{\eta \max \{\kappa \rho, \sqrt{\rho}\}} \). Then for \( t \leq T \), we have that
\[
\sin(Z_t, U^*) \lesssim \eta t
\]
(4.4)
\[
\|E_t\| \leq (1 + O(\sqrt{\eta^2 pt}))^{t}\|E_0\| \leq 4\|E_0\| \leq 1/d
\]
(4.5)
\[
\sigma_{\min}(U^*\top Z_t) \geq \|E_t\|
\]
(4.6)
\[
\sigma_{\min}(U^*\top Z_t) \geq \min \left\{ (1 + \frac{\eta}{8\kappa})^t \sigma_{\min}(U^*\top Z_0), 1/(2\sqrt{\kappa}) \right\}
\]
(4.7)
\[
\|Z_t\| \leq 5
\]
(4.8)

Moreover, at step \( T_0 = 100\kappa \log(d/\alpha)/\eta \) we have that
\[
\sigma_{\min}(Z_{T_0}) \geq \frac{1}{(2\sqrt{\kappa})}
\]
(4.9)
Note that the theorem above only shows that the least singular value of $Z_t$ goes above $1/(2\sqrt{\kappa})$. The following proposition completes the story by showing that once the signal is large enough, $U_t U_t^\top$ converges with a linear rate to the desired solution $X^\ast$ (up to some small error.)

**Proposition 4.3.** In the setting of Theorem 4.2, suppose $\sigma_{\min}(Z_t) \geq \frac{1}{2\sqrt{\kappa}}$, $\|Z_t\| \leq 5$ and $\sin(Z_t, U^\ast) \leq 1/3$, then we have:

$$\left\|U_{t+1}U_{t+1}^\top - X_t\right\|_F^2 \leq (1 - \eta/(8\kappa)) \left\|U_t U_t^\top - X_t\right\|_F^2 + O(\eta d^2 \|E_t\|) \tag{4.10}$$

The main Theorem 4.1 follows from Theorem 4.2 and Proposition 4.3 straightforwardly:

**Proof of Theorem 4.1.** By Theorem 4.2 in $T_0 = O(\kappa \log(d/\alpha)/\eta)$ we reach a point with $\sigma_{\min}(Z_{T_0}) \geq 1/(2\sqrt{\kappa})$. Since $T_0 \lesssim \frac{\eta \max(d, \sqrt{\kappa})}{\kappa^2}$, we have that within $t = O(T_0)$ steps, $\|E_t\| \leq 4\|E_0\| \leq 4\alpha$. Therefore invoking Proposition 4.3 with all $T_0 \leq t \leq T$ repeatedly, we complete the proof. $\square$

We defer the proof of Proposition 4.3 to Section A.5. The rest of the section is dedicated to the proof outline of Theorem 4.2. We decompose it into the following propositions, from which Theorem 4.2 follows by induction.

The following proposition gives the base case for the induction, which straightforwardly follows from the definition $U_0 = \alpha B$ where $B$ is an orthonormal matrix.

**Proposition 4.4 (Base Case).** In the setting of Theorem 4.2, we have

$$\|E_0\| = \alpha = \sigma_{\min}(Z_0) \leq 1/d, \quad U^\ast \top E_0 = 0, \quad \text{and} \quad \sin(Z_0, U^\ast) = 0 \tag{4.11}$$

The following Proposition bounds the growth rate of the spectral norm of the error $E_t$.

**Proposition 4.5.** In the setting of Theorem 4.2, suppose $\|E_t\| \leq 1/d$. Then,

$$\|E_{t+1}\| \leq (1 + \eta \rho (\sqrt{\kappa} + \sin(Z_t, U^\ast))) \cdot \|E_t\| \tag{4.12}$$

The following Proposition shows that effectively we can almost pretend that $Z_{t+1}$ is obtained from applying one gradient descent step to $Z_t$, up to some some error terms.

**Proposition 4.6.** In the setting of Theorem 4.2, suppose for some $t$ we have

$$\|E_t\| \leq 1/d \quad \text{and} \quad \|Z_t\| \leq 5 \tag{4.13}$$

Then,

$$\left\|Z_{t+1} - (Z_t - \eta \nabla f(Z_t) - \eta E_t Z_t \top Z_t - 2\eta \sigma_{\min}(Z_t) \Id_{s_t} \Id_{s_t} E_t)\right\| \leq \eta \tau_t \tag{4.14}$$

for $\tau_t \lesssim \sqrt{\kappa} \|E_t\|$.

The following proposition shows that the angle between the span of $Z_t$ to the span of $U^\ast$ is growing at mostly linearly in the number of steps.

**Proposition 4.7.** In the setting of Theorem 4.2, assuming equation (4.13) holds for some $t$ with $\eta \leq \rho \sigma_{\min}(U_t)$, $\|Z_t\| \leq 5$ and $\sigma_{\min}(Z_t) \geq (1/2) \|E_t\|$. Then, as long as $\sin(Z_t, U^\ast) \leq \sqrt{\rho}$ we have:

$$\sin(Z_{t+1}, U^\ast) \leq \sin(Z_t, U^\ast) + O(\eta \rho + \eta \|E_t\|) \quad \text{and} \quad \|Z_{t+1}\| \leq 5 \tag{4.15}$$
Then we show that the projection of the signal term $Z_t$ to the subspace of $U^*$ increases at an exponential rate. Note that $U^*$ is sufficiently close to the span of $Z_t$ and therefore it implies that the least singular value of $Z_t$ also grows.

**Proposition 4.8.** In the setting of Theorem 4.2, suppose equation (4.15) holds for some $t$, and $\|Z_t\| \leq 5$ and $\sigma_{\min}(Z_t) \geq (1/2)\|E_t\|$, we have that:

$$\sigma_{\min}(U^*Z_{t+1}) \geq \min\{(1 + \frac{\eta}{8\kappa})\sigma_{\min}(U^*Z_t), 1/(2\sqrt{\kappa})\}$$

(A.16)

The proofs of these propositions are deferred to Section A. With these propositions we are ready to prove the main Theorem 4.1.

**Proof of Theorem 4.2** When $t = 0$, the base case follows from Proposition 4.1. Assume that equation (4.4), (4.5), (4.6), (4.7), and (4.8) are true before or at step $t$, we prove the conclusion for step $t + 1$. By Proposition 4.6 we have that equation (4.14) are true with $\tau_t \lesssim \delta \sqrt{\kappa}$. Setting the $\rho$ in Proposition 4.7 to $O(\kappa \sqrt{\kappa})$, by our choice of $t \leq T$ we have that $\sin(Z_t, U^*) \leq \sqrt{\rho}$. Therefore,

$$\sin(Z_{t+1}, U^*) \lesssim \eta \rho t + O(\eta \rho) + O(\frac{\|E_t\|}{\delta \sqrt{\kappa} + \sqrt{\rho}}) \lesssim \eta \rho(t + 1)$$

(A.17)

because $\|E_t\| \leq 1/d$ and $\frac{1}{d \times (\delta \sqrt{\kappa} + \sqrt{\rho})} \leq \eta \rho$, we have that

$$\|E_{t+1}\| \leq (1 + \eta O (\delta \sqrt{\kappa} + \sin(Z_t, U^*))) \cdot \|E_t\| \\
\leq (1 + O(\eta^2 \rho)^{t+1}) \cdot \|E_t\| \leq 4 \cdot \|E_0\| \leq 1/d$$

(because $t = 0$)

(since $\tau_t \leq \eta \tau_0 \leq \frac{\delta \kappa}{\sqrt{\rho}}$)

Therefore we can apply Proposition 4.5 to obtain that $\sigma_{\min}(U^*Z_t)$ grows at a rate of at least $1 + \frac{\eta}{8\kappa}$. On the other hand by Proposition 4.5, $\|E_t\|$ grows by a rate of at most $1 + \eta O(\delta \sqrt{\kappa} + \sqrt{\rho}) \leq 1 + \frac{\eta}{2\kappa}$. Hence we obtain Equation 4.6.

### A Proof of Main Propositions

**A.1 Proof of Proposition 4.5**

We start off with a straightforward triangle inequality for bounding $E_{t+1}$ given $E_t$.

**Lemma A.1.** By update rule (equation (2.2)) and the definition of $E_t$ (equation (4.1)), we have that

$$E_{t+1} = (\text{Id} - \text{Id}_{S_{t+1}})(\text{Id} - \eta M_t)E_t.$$ 

It follows that

$$\|E_{t+1}\| \leq \|(\text{Id} - \eta M_t)E_t\| \leq \|E_t\| + \eta \|M_t E_t\|.$$ 

(A.1)

Therefore, next we will bound the norm of $M_t E_t$.

**Lemma A.2.** In the setting of Proposition 4.5, we have that

$$\|M_t E_t\| \leq \|E_t\| \left(\delta \|Z_t Z_t^\top - X^*\| + (2\delta + 1)\|Z_t E_t^\top\| + \|X^*(\text{Id} - \text{Id}_{S_t})\| + \delta \|E_t E_t^\top\| \right)$$

(A.2)

As a direct consequence, using the assumption $\|E_t\| \leq 1/d$,

$$\|E_{t+1}\| \leq \|E_t\| \left(1 + O\left(\eta \delta \sqrt{\kappa} + \eta \|X^*(\text{Id} - \text{Id}_{S_t})\|\right)\right)$$

(A.3)
Proof. We first note that \( E_t = (\text{Id} - \text{Id}_{S_t})E_t \) by the update rule and definition \ref{def:projection}. It follows that \( M_t E_t = M_t (\text{Id} - \text{Id}_{S_t})E_t \).

Now, note that \( U_t U_t^T - X^* - E_t E_t^T \) has rank at most \( 4r \), by Lemma \ref{lem:rank_bound}. We have that

\[
\left\| \frac{1}{m} \sum_{i=1}^{m} (A_i U_t U_t^T - X^* - E_t E_t^T) A_i (\text{Id} - \text{Id}_{S_t}) E_t - (U_t U_t^T - X^* - E_t E_t^T) (\text{Id} - \text{Id}_{S_t}) E_t \right\| \leq \delta \left\| U_t U_t^T - X^* - E_t E_t^T \right\| \left\| (\text{Id} - \text{Id}_{S_t}) E_t \right\| \tag{A.4}
\]

\[
= \delta \left( \| Z_t Z_t^T - X^* \| + 2\| Z_t E_t^T \| \right) \| E_t \| \tag{A.5}
\]

Note that \( (U_t U_t^T - X^* - E_t E_t^T) (\text{Id} - \text{Id}_{S_t}) E_t = Z_t E_t^T E_t - X^*(\text{Id} - \text{Id}_{S_t}) E_t \). It follows that

\[
\left\| \frac{1}{m} \sum_{i=1}^{m} (A_i U_t U_t^T - X^* - E_t E_t^T) A_i (\text{Id} - \text{Id}_{S_t}) E_t \right\| \leq \| X^*(\text{Id} - \text{Id}_{S_t}) E_t \| + \| Z_t E_t^T E_t \| + \delta \left( \| Z_t Z_t^T - X^* \| + 2\| Z_t E_t^T \| \right) \| E_t \| \tag{A.7}
\]

\[
\leq \| E_t \| \left( \delta \| Z_t Z_t^T - X^* \| + (2\delta + 1)\| Z_t E_t^T \| + \| X^*(\text{Id} - \text{Id}_{S_t}) \| \right) \tag{A.8}
\]

By Lemma \ref{lem:projection}, we have that

\[
\left\| \frac{1}{m} \sum_{i=1}^{m} (A_i E_t E_t^T) A_i (\text{Id} - \text{Id}_{S_t}) E_t - E_t E_t^T (\text{Id} - \text{Id}_{S_t}) E_t \right\| \leq \delta \| E_t E_t^T \| \| E_t \| \tag{A.10}
\]

Combining equation \ref{eq:proof1} and \ref{eq:proof2} we complete the proof of equation \ref{eq:proof3}. To prove equation \ref{eq:proof4}, we will use equation \ref{eq:proof5} and that \( \| Z_t Z_t^T - X^* \| \lesssim \sqrt{r} \) (Corollary \ref{cor:rank_bound}) \qedhere

Proof of Proposition \ref{prop:rank_bound}. We can bound the term \( \| X^*(\text{Id} - \text{Id}_{S_t}) \| \) in equation \ref{eq:proof4} by

\[
\| X^*(\text{Id} - \text{Id}_{S_t}) \| \leq \left\| U^*^T (\text{Id} - \text{Id}_{S_t}) \right\| = \| (\text{Id} - \text{Id}_{S_t}) U^* \| = \sin(S_t, U^*).
\]

Since \( S_t \) is the column span of \( Z_t \), using equation above and and equation \ref{eq:proof4} we conclude the proof. \qed

A.2 Proof of Proposition \ref{prop:rank_bound}

Towards proving Proposition \ref{prop:rank_bound}, we further decompose \( Z_t \) into

\[
Z_t = (U^* + F_t) R_t \quad \tag{A.11}
\]

where \( R_t \in \mathbb{R}^{r \times d} \), \( F_t \in \mathbb{R}^{d \times r} \) are defined as

\[
R_t = U^*^T Z_t, \quad \text{and} \quad F_t = (\text{Id} - \text{Id}_{U^*}) Z_t R_t^+. \quad \tag{A.12}
\]

We first relate the the spectral norm of \( F_t \) with our target in Proposition \ref{prop:rank_bound}, the principal angle between \( Z_t \) and \( U^* \).

Lemma \ref{lem:angle_bound}. Let \( F_t \) be defined as in equation \ref{eq:lem_angle_bound}. Then, if \( \| F_t \| < 1/3 \), we have that

\[
\| F_t \| - \| F_t \|^3 \leq \sin(Z_t, U^*) \leq \| F_t \| \quad \tag{A.13}
\]

15
Proof. Let $S_t = (U^* + F_t)(I + F_t^T F_t)^{-1/2}$. By the fact that $U^*F_t = 0$, we have $S_t^T S_t = I_d$ and $S_t$ has the same column span as $Z_t$. Therefore, the columns of $S_t$ form an orthonormal basis of $Z_t$, and we have that
\[ \sin(Z_t, U^*) = \| (I_d - I_d U^*) S_t \| = \| F_t (I_d + F_t^T F_t)^{-1/2} \| \quad (A.14) \]

Suppose $F_t$ has singular value $\sigma_j, j = 1, \ldots, r$, then it’s straightforward to show that $F_t (I_d + F_t^T F_t)^{-1/2}$ has singular values $\frac{\sigma_j}{\sqrt{1 + \sigma_j^2}}$, $j = 1, \ldots, r$. The conclusion then follows basic calculus and the fact that $\max \sigma_j \leq 1/3$.

Now let us consider another matrix $\tilde{Z}_t$ defined as:
\[ \tilde{Z}_t = (I_d - \eta E_t Z_t^T) Z_t (I_d - 2 \eta Z_t^+ I_d S_t, M_t E_t) \quad (A.15) \]

We know that
\[ \left\| Z_{t+1} - \tilde{Z}_t - \eta \nabla f(Z_t) \right\| = O(\eta^2 \| E_t \| + \eta \tau_t) \quad (A.16) \]

By the definition of $\tilde{Z}_t$, we also know that $\left\| \tilde{Z}_t - Z_t \right\| = O(\| E_t \|)$ with $\tilde{Z}_t$ being an rank $r$ matrix and $\| Z_t \|, \| \tilde{Z}_t \| = O(1)$. Therefore,
\[ \left\| \nabla f(Z_t) - \nabla f(\tilde{Z}_t) \right\| = O(\eta \| E_t \|) \]

Which implies that
\[ \left\| Z_{t+1} - \tilde{Z}_t - \eta \nabla f(\tilde{Z}_t) \right\| = O(\eta^2 \| E_t \| + \eta \tau_t) = O(\eta \tau_t) \quad (A.17) \]

This implies that $Z_{t+1}$ is very close to doing one step of gradient descent from $\tilde{Z}_t$. Thus, we will mainly focus on $\tilde{Z}_t$ in the later section.

Let as also decompose $\tilde{Z}_t$ into $\tilde{Z}_t = (U^* + \tilde{F}_t)\tilde{R}_t$, we know that
\begin{align*}
\tilde{R}_t &= U^* \tilde{Z}_t = U^* (I_d - \eta E_t Z_t^T) Z_t (I_d - 2 \eta Z_t^+ I_d S_t, M_t E_t) \\
&= U^* Z_t (I_d - 2 \eta Z_t^+ I_d S_t, M_t E_t) - \eta U^* E_t Z_t^T Z_t (I_d - 2 \eta Z_t^+ I_d S_t, M_t E_t) \\
&= (U^* + \tilde{F}_t) \tilde{R}_t \\
\end{align*}
\[ \quad (A.18) \]

Notice that (Using the bound of $\| M_t E_t \|$ as in Lemma A.2)
\[ \left\| Z_t^+ I_d S_t M_t E_t \right\| \lesssim \frac{\| M_t E_t \|}{\sigma_{\min}(\tilde{R}_t)} \lesssim \frac{\| E_t \| (\delta \sqrt{r} + \sin(Z_t, U^*))}{\sigma_{\min}(\tilde{R}_t)} \quad (A.20) \]

And
\[ \left\| U^* E_t \right\| = \left\| U^* (I_d - I_d S_t) E_t \right\| \leq \| E_t \| \left\| U^* (I_d - I_d S_t) \right\| = \| E_t \| |\sin(Z_t, U^*)| \quad (A.21) \]

Therefore, we know that
\begin{align*}
\sigma_{\min}(\tilde{R}_t) &\geq \sigma_{\min}(U^* Z_t) (1 - \eta \left\| Z_t^+ I_d S_t M_t E_t \right\|) - \eta \left\| U^* E_t Z_t^T Z_t (I_d - 2 \eta Z_t^+ I_d S_t, M_t E_t) \right\| \\
&\geq \sigma_{\min}(R_t) (1 - \eta \left\| Z_t^+ I_d S_t M_t E_t \right\|) - 2 \eta \left\| U^* E_t \right\| \\
\end{align*}
\[ \quad (A.22) \]

\[ \quad (A.23) \]
\begin{align*}
\geq \sigma_{\min}(R_t) \left(1 - \eta O \left( \frac{\|E_t\| (\delta \sqrt{r} + \sin(Z_t, U^*))}{\sigma_{\min}(R_t)} \right) \right) \tag{A.24}
\end{align*}

By assumption in Proposition 4.7 such that \( \sigma_{\min}(Z_t) \geq \frac{1}{2} \|E_t\| \), we know that
\begin{align*}
\frac{\|E_t\| (\delta \sqrt{r} + \sin(Z_t, U^*))}{\sigma_{\min}(R_t)} &\leq 2 \frac{\|E_t\| (\delta \sqrt{r} + \sin(Z_t, U^*))}{\sigma_{\min}(Z_t)} \tag{A.25} \\
&\leq 2(\delta \sqrt{r} + \sin(Z_t, U^*)) \tag{A.26} \\
&\lesssim \sqrt{\rho} \tag{A.27}
\end{align*}

Thus, by our choice of parameter,
\begin{align*}
\sigma_{\min}(\tilde{R}_t) \geq \sigma_{\min}(R_t) \left(1 - \frac{\eta}{100\kappa} \right) \tag{A.28}
\end{align*}

Now, we focus on \( \tilde{F}_t \), we know that right multiply \( Z_t \) by an invertible matrix does not change the column subspace as \( Z_t \), so we can just focus on \( (\text{Id} - \eta E_t Z_t^\top)Z_t \).

We know that
\begin{align*}
\tilde{F}_t = (\text{Id} - \text{Id} U^*)(\text{Id} - \eta E_t Z_t^\top)Z_t(U^*^\top (\text{Id} - \eta E_t Z_t^\top)Z_t)^+
\end{align*}

Again, we have:
\begin{align*}
\left\| \tilde{F}_t - F_t \right\| \lesssim \eta \|E_t\| + \eta \frac{\|F_t\| \|U^*^\top E_t\|}{\sigma_{\min}(R_t)} \lesssim \eta \|E_t\| + \eta \rho \tag{A.30}
\end{align*}

Where the last inequality uses the assumption \( \|E_t\| \lesssim \sigma_{\min}(Z_t) \) in Proposition 4.7 so that
\begin{align*}
\frac{\|F_t\| \|U^*^\top E_t\|}{\sigma_{\min}(R_t)} \lesssim \frac{\|F_t\| \|E_t\| \sin(Z_t, U^*)}{\sigma_{\min}(Z_t)} \lesssim \|F_t\|^2 \tag{A.31}
\end{align*}

One of the crucial fact about the gradient \( \nabla f(\tilde{Z}_t) \) is that it can be decomposed into
\begin{align*}
\nabla f(\tilde{Z}_t) = N_t \tilde{R}_t \tag{A.32}
\end{align*}

where \( N_t \) is a matrix defined as
\begin{align*}
N_t = \frac{1}{m} \sum_{i=1}^{m} \langle A_i, \tilde{Z}_i \tilde{Z}_i^\top - X^* \rangle A_i (U^* + \tilde{F}_t) \tag{A.33}
\end{align*}

Therefore, \( \tilde{Z}_t \) and \( \nabla f(\tilde{Z}_t) \) share the row space and we can factorize the difference between \( \tilde{Z}_t \) and \( \eta \nabla f(\tilde{Z}_t) \) as
\begin{align*}
\tilde{Z}_t N_t - \eta \nabla f(\tilde{Z}_t) = (\tilde{F}_t - \eta N_t) \tilde{R}_t \tag{A.34}
\end{align*}

Note that the definition of \( N_t \) depends on the random matrices \( A_1, \ldots, A_t \). The following lemma show that for our purpose, we can essentially view \( N_t \) as its population version — the counterpart of \( N_t \) when we have infinitely number of examples. The proof uses the RIP properties of the matrices \( A_1, \ldots, A_m \).
Lemma A.4. In the setting of Proposition 4.7, let $N_t$ be defined as in equation (A.33). Then,

$$\left\| N_t - (\tilde{Z}_t \tilde{Z}_t^\top - X^*)(U^* + \tilde{F}_t) \right\| \leq 2\delta \left\| \tilde{Z}_t \tilde{Z}_t^\top - X^* \right\|_F \quad (A.35)$$

Proof. Recalling the definition of $N_t$, by Lemma 2.3, we have that

$$\left\| N_t - (\tilde{Z}_t \tilde{Z}_t^\top - X^*)(U^* + \tilde{F}_t) \right\| \leq \delta \left\| \tilde{Z}_t \tilde{Z}_t^\top - X^* \right\|_F \left\| U^* + \tilde{F}_t \right\| \leq 2\delta \left\| \tilde{Z}_t \tilde{Z}_t^\top - X^* \right\|_F \quad \text{(by the assumption } \left\| \tilde{F}_t \right\| < 1/3) \quad \blacksquare$$

**Lemma A.5.** For any $t \geq 0$, suppose $\left\| Z_{t+1} - (\tilde{Z}_t - \eta G(\tilde{Z}_t)) \right\| \leq \eta \tau$, we have

$$\left\| Z_{t+1} \right\| \leq \left\| \tilde{Z}_t \right\| \left( 1 - \frac{1}{2} \eta \left\| \tilde{Z}_t \right\|^2 \right) + 2\eta \left\| \tilde{Z}_t \right\| X^* + \eta \tau.$$

Proof. By Lemma 2.3, we have:

$$\left\| G(\tilde{Z}_t) - (\tilde{Z}_t \tilde{Z}_t^\top - X^*)\tilde{Z}_t \right\| \leq \delta \left\| \tilde{Z}_t \tilde{Z}_t^\top - X^* \right\|_F \left\| \tilde{Z}_t \right\|. $$

Therefore,

$$\left\| Z_{t+1} \right\| \leq \left\| \tilde{Z}_t \right\| - \eta G(\tilde{Z}_t) \right\| + \eta \tau \leq \left\| \tilde{Z}_t \right\| \left( 1 - \eta \left\| \tilde{Z}_t \right\|^2 \right) + \eta \left\| \tilde{Z}_t \right\| X^* + \eta \tau \leq \left\| (1 - \eta \left\| \tilde{Z}_t \right\|) \right\| \tilde{Z}_t \right\| + \frac{1}{2} \frac{1}{\eta \tau} \left\| \tilde{Z}_t \right\| \left( \left\| \tilde{Z}_t \right\|^2 + 4 \left\| X^* \right\|^2 \right) + \eta \tau \quad \text{(by } \delta \sqrt{\tau} \leq 1/2) \quad \blacksquare$$

As a direct corollary, we can inductive control the norm of $\tilde{Z}_t$.

**Corollary A.6.** In the setting of Proposition 4.7, we have that

$$\left\| Z_{t+1} \right\| \leq 5, \left\| \tilde{R}_t \right\| \leq 6 \quad (A.37)$$

Moreover,

$$\left\| Z_{t+1} Z_{t+1}^\top - X^* \right\|_F \lesssim \sqrt{\tau}, \quad \text{and} \quad \left\| N_t \right\| \lesssim \sqrt{\tau} \quad (A.38)$$

Proof. Using the assumption that $\left\| X^* \right\| = 1$ and the assumption that equation (4.14) holds, then we have that $\left\| \tilde{Z}_t \right\| \leq \left\| Z_t \right\| (1 + O(\eta \left\| E(t) \right\|)) \leq 5(1 + O(\eta \left\| E(t) \right\|))$. Applying Lemma A.5 with $\tau = O(\tau_1)$, we have that $\left\| Z_{t+1} \right\| \leq 5$. We also have that $\left\| \tilde{R}_t \right\| \leq \left\| \tilde{Z}_t \right\| \leq 6$. Moreover, we have $\left\| Z_{t+1} Z_{t+1}^\top - X^* \right\|_F \leq \left\| \tilde{Z}_t \right\| + \left\| X^* \right\|_F \lesssim \sqrt{\tau}(\left\| \tilde{Z}_t \right\| + \left\| X^* \right\|) \lesssim \sqrt{\tau}$. As a consequence, $\left\| N_t \right\| \leq (1 + \delta) \left\| \tilde{Z}_t \tilde{Z}_t^\top - X^* \right\|_F \left( \left\| U^\top \right\| + \left\| \tilde{F}_t \right\| \right) \lesssim \sqrt{\tau}.$ \(\blacksquare\)
We start off with a lemma that controls the changes of $\tilde{R}_t$ relatively to $R_{t+1}$.

**Lemma A.7.** In the setting of Proposition 4.7, then we have that $\tilde{R}_t R_{t+1}^{-1}$ can be written as

$$\tilde{R}_t R_{t+1}^{-1} = \text{Id} + \eta \tilde{R}_t \tilde{R}_{t+1}^\top - \eta \Sigma^* + \xi_t^{(R)}$$

where $\|\xi_t^{(R)}\| \lesssim \eta \delta \sqrt{r} + \eta \rho + \eta \|\tilde{F}_t\| + \eta^2$. It follows that $\|\tilde{R}_t R_{t+1}^{-1}\| \leq 4/3$ and $\tau \leq 2 \rho \sigma_{\min}(R_{t+1})$.

**Proof.** By the definition of $\tilde{R}_t$ and equation (A.41), we have that

$$\eta \tau \geq \|R_{t+1} - \tilde{R}_t - \eta U^* \nabla f(\tilde{Z}_t)\| = \|R_{t+1} - (\text{Id} - \eta U^* N_t) \tilde{R}_t\|$$

Form this we can first obtain a very weak bound on $\sigma_{\min}(R_{t+1})$:

$$\sigma_{\min}(R_{t+1}) \geq \sigma_{\min}(\text{Id} - \eta U^* N_t) \tilde{R}_t - \eta \tau$$

where we used equation (A.41) and $\tau \leq O(\rho \sigma_{\min}(R_t))$. This also implies a weak bound for $\tilde{R}_t R_{t+1}^{-1}$ that $\|\tilde{R}_t R_{t+1}^{-1}\| \leq 2$. By Lemma A.4, we have that $\|\eta N_t - \eta U^* (\tilde{Z}_t \tilde{Z}_t^\top - X^*)(U^* + \tilde{F}_t)\| \leq 2 \eta \|\tilde{Z}_t \tilde{Z}_t^\top - X^*\| \lesssim \eta \delta \sqrt{r}$

where $\|\tilde{Z}_t \tilde{Z}_t^\top - X^*\| \lesssim \sqrt{r}$. Note that $X^* = U^* \Sigma^* U^\top$ and $\tilde{Z}_t = (U^* + \tilde{F}_t) \tilde{R}_t$.

Bounding the higher-order term, we have that

$$\|\eta (\tilde{R}_t \tilde{R}_{t+1}^\top - \Sigma^*)(U^* + \tilde{F}_t)^\top (U^* + \tilde{F}_t) - \eta(\tilde{R}_t \tilde{R}_{t+1}^\top - \Sigma^*)\| \lesssim \eta \|\tilde{F}_t\|$$

which implies that

$$\|\eta U^* N_t - \eta (\tilde{R}_t \tilde{R}_{t+1}^\top - \Sigma^*)\| \lesssim \delta \eta \sqrt{r} + \eta \|\tilde{F}_t\|$$

Combining the equation above with equation (A.42) and $\|\tilde{R}_t R_{t+1}^{-1}\| \leq 2$, we have that

$$\|\text{Id} - (\text{Id} - \eta \tilde{R}_t \tilde{R}_{t+1}^\top + \eta \Sigma^*) \tilde{R}_t R_{t+1}^{-1}\| \lesssim \eta \delta \sqrt{r} + \eta \rho + \eta \|\tilde{F}_t\|$$

For $\eta \lesssim 1$, we know that

$$\|\text{Id} - (\text{Id} - \eta \tilde{R}_t \tilde{R}_{t+1}^\top + \eta \Sigma^*) \| \lesssim \eta^2$$

This implies that

$$\|\text{Id} + \eta \tilde{R}_t \tilde{R}_{t+1}^\top - \eta \Sigma^*\| - \|\tilde{R}_t R_{t+1}^{-1}\| \lesssim \eta \delta \sqrt{r} + \eta \rho + \eta \|\tilde{F}_t\| + \eta^2$$

which completes the proof. \qed
We express $F_{t+1}$ as a function of $\tilde{F}_t$ and other variables.

**Lemma A.8.** In the setting of Proposition 4.7, let $N_t$ be defined as in equation (A.33). Then,

$$F_{t+1} = \tilde{F}_t (\text{Id} - \eta \tilde{R}_t \tilde{R}_t^\top) \tilde{R}_t R_{t+1}^+ + \xi_t^{(F)}$$  \hspace{1cm} (A.48)

where $\|\xi_t^{(F)}\| \leq \delta \eta \sqrt{r} + \eta \|\tilde{F}_t\|^2$.

**Proof.** By equation (4.14), we have that

$$\|\text{Id} - \text{Id}_{U^*}\| \leq \eta \eta_t$$

which, together with the decomposition (A.12), implies

$$\eta \tau \geq \|F_{t+1} R_{t+1} - \tilde{F}_t \tilde{R}_t + \eta (\text{Id} - \text{Id}_{U^*}) \nabla f(\tilde{Z}_t)\| \geq \eta F_{t+1} R_{t+1} - (\tilde{F}_t - \eta (\text{Id} - \text{Id}_{U^*}) N_t) \tilde{R}_t\| \geq 2 \eta \rho$$

(A.50)

(A.51)

Recall that $\tau_t \leq 2 \rho \sigma_{\min}(R_{t+1})$ (by Lemma A.7), we conclude

$$\|F_{t+1} - (\tilde{F}_t - \eta (\text{Id} - \text{Id}_{U^*}) N_t) \tilde{R}_t R_{t+1}^+\| \leq 2 \eta \rho$$ \hspace{1cm} (A.52)

Note that $(\text{Id} - \text{Id}_{U^*}) X^* = 0$ and that $(\text{Id} - \text{Id}_{U^*}) \tilde{Z}_t \tilde{Z}_t^\top = \tilde{F}_t \tilde{R}_t \tilde{R}_t^\top (U^* + \tilde{F}_t)^\top$. We obtain that

$$\|\text{Id} - \text{Id}_{U^*}\| N_t - \tilde{F}_t \tilde{R}_t \tilde{R}_t^\top (U^* + \tilde{F}_t)^\top (U^* + \tilde{F}_t)\| \leq 2 \delta \|\tilde{Z}_t \tilde{Z}_t^\top - X^*\|_F \leq \delta \sqrt{r}$$ \hspace{1cm} (A.53)

where we used the fact that $\|\tilde{Z}_t \tilde{Z}_t^\top - X^*\|_F \leq \sqrt{r}$ (by Lemma A.6).

Bounding the higher-order term, we have that

$$\|\tilde{F}_t \tilde{R}_t \tilde{R}_t^\top (U^* + \tilde{F}_t)^\top (U^* + \tilde{F}_t) - \tilde{F}_t \tilde{R}_t \tilde{R}_t^\top\| \leq \|\tilde{F}_t\|^2$$ \hspace{1cm} (by $\|\tilde{R}_t\| \leq 6$ from Corollary A.6)

Combining equation (A.52), (A.53) and the equation above, and using the fact that $\|\tilde{R}_t R_{t+1}^+\| \leq 2$, we complete the proof. \qed

Combining Lemma A.8 and Lemma A.7, we can relate the $\|F_{t+1}\|$ with $\|\tilde{F}_t\|$:

**Lemma A.9.** In the setting of Proposition 4.7, we have that $\tilde{F}_t$ can be written as

$$F_{t+1} = \tilde{F}_t (\text{Id} - \eta \Sigma^*) + \xi_t$$ \hspace{1cm} (A.54)

where $\|\xi_t\| \leq O(\eta \delta \sqrt{r} + \eta \|\tilde{F}_t\|^2 + \eta \rho + \eta^2)$. As a consequence,

$$\|F_{t+1}\| \leq \|\tilde{F}_t\| + O(\eta \|\tilde{F}_t\|^2 + \eta \rho)$$ \hspace{1cm} (A.55)
Proof. Combine Lemma \ref{lem:appendix8} and Lemma \ref{lem:appendix7}, we have that
\[
F_{t+1} = \tilde{F}_t (\text{Id} - \eta \tilde{R}_t \tilde{R}_t^\top) \tilde{R}_t R_{t+1}^\top + \xi_t^{(F)} \\
= \tilde{F}_t (\text{Id} - \eta \tilde{R}_t \tilde{R}_t^\top) \left( \text{Id} + \eta \tilde{R}_t \tilde{R}_t^\top - \eta \Sigma^* + \xi_t^{(R)} \right) + \xi_t^{(F)} \quad (A.56)
\]

Thus, with the bound on \( \|\xi_t^{(R)}\| \) and \( \|\xi_t^{(F)}\| \) from Lemma \ref{lem:appendix8} and Lemma \ref{lem:appendix7}, we know that
\[
\| \tilde{F}_{t+1} - (\text{Id} - \eta \Sigma^*) \tilde{F}_t \| \lesssim \eta \| \tilde{F}_t \| + \| \tilde{F}_t \| \|\xi_t^{(R)}\| + \|\xi_t^{(F)}\| \lesssim \eta \| \tilde{F}_t \|^2 + \eta \delta \sqrt{\tau} + \eta \rho + \eta^2 \\
\lesssim \eta \| \tilde{F}_t \|^2 + \eta \rho.
\]
\[
\Box
\]

The proof of Proposition \ref{prop:appendix3} follow straightforwardly from Lemma \ref{lem:appendix3} and Lemma \ref{lem:appendix9}.

Proof of Proposition \ref{prop:appendix7} Using the assumption that \( \|F_t\| \lesssim \sqrt{\rho} \) (Thus \( \|F_t\|^2 \lesssim \rho \)). Since we have showed that \( \| \tilde{F}_t - F_t \| \lesssim \eta (\rho + \|E_t\|) \), the proof of this proposition followings immediately from Lemma \ref{lem:appendix9}.

\[
\Box
\]

A.3 Proof of Proposition \ref{prop:appendix6}

We first present a simple lemma that bounds the changes of the projection matrix of a subspace when it’s multiplied by a transformation closer to identity.

Lemma \ref{lem:appendix10}. Let \( S \) be a (column) orthonormal matrix in \( \mathbb{R}^{d \times r} \) and \( S^\perp \) be the orthogonal complement. For every \( \eta > 0 \), let \( M \) be an arbitrary matrix in \( \mathbb{R}^{d \times d} \) such that \( \|M\| < \frac{1}{2\eta} \), then:
\[
\| \text{Id} (\text{Id} - \eta M) S (\text{Id} - \eta M) S^\perp + 2\eta SS^\top MS^\perp \| \lesssim \eta^2 \|M\|^2
\]

Proof. By definition, we know that
\[
\text{Id}(\text{Id} - \eta M) S = (\text{Id} - \eta M) S \left( S^\top (\text{Id} - \eta M)^\top (\text{Id} - \eta M) S \right)^{-1} S^\top (\text{Id} - \eta M)^\top \quad (A.58)
\]
\[
= (\text{Id} - \eta M) S \left( \text{Id} - \eta S^\top (M^\top + M) S + \eta^2 S^\top M^\top MS \right)^{-1} S^\top (\text{Id} - \eta M)^\top \quad (A.59)
\]

By Woodbury matrix identity, we can bound
\[
\left\| \left( \text{Id} - \eta S^\top (M^\top + M) S + \eta^2 S^\top M^\top MS \right)^{-1} - \left( \text{Id} + \eta S^\top (M^\top + M) S \right) \right\| \leq 4\eta^2 \|M\|^2 \quad (A.60)
\]
Thus,
\[
\left\| \text{Id}(\text{Id} - \eta M) S - \text{Id}S + \eta MS S^\top + \eta SS^\top M - \eta SS^\top (M^\top + M) S S^\top \right\| \leq 16\eta^2 \|M\|^2 \quad (A.61)
\]
Putting into \( (\text{Id} - \eta M) S^\perp \) we have that
\[
\left( \text{Id}S - \eta MS S^\top - \eta SS^\top M + \eta SS^\top (M^\top + M) S S^\top \right) S^\perp \quad (A.62)
\]
\[
= -\eta SS^\top MS^\perp \quad (A.63)
\]

Using \(-\eta \text{Id}S MS^\perp = -\eta SS^\top MS^\perp\) completes the proof. 
\[
\Box
\]
Now we can prove this proposition. By definition, we know that \( Z_{t+1} = \text{Id}_{S_{t+1}} U_{t+1} \) where \( S_{t+1} = (\text{Id} - \eta M_t) S_t \) and \( U_{t+1} = (\text{Id} - \eta M_t) U_t \). Therefore, we have:

Let us define \( E' = (\text{Id} - \eta M_t) E_t \), we then have:

\[
U_{t+1} = (\text{Id} - \eta M_t) U_t = (\text{Id} - \eta M_t) (Z_t + E_t) = (\text{Id} - \eta M_t) Z_t + E' \tag{A.64}
\]

Proof of Proposition 4.6. We first consider the term \( M_t Z_t \), we have:

\[
M_t Z_t = \frac{1}{m} \sum_{i=1}^{m} \left( \langle A_i, U_i U_t^\top \rangle - y_i \right) A_i Z_t \tag{A.65}
\]

\[
= \frac{1}{m} \sum_{i=1}^{m} \left( A_i, Z_t Z_t^\top - X^* \right) A_i Z_t \tag{A.66}
\]

\[
+ \frac{1}{m} \sum_{i=1}^{m} \left( \langle A_i, E_t Z_t^\top + Z_t E_t^\top \rangle \right) A_i Z_t + \frac{1}{m} \sum_{i=1}^{m} \left( \langle A_i, E_t E_t^\top \rangle \right) A_i Z_t \tag{A.67}
\]

Using Lemma 2.3 and B.2 we have that

\[\left\| M_t Z_t - E_t Z_t^\top Z_t - \nabla f(Z_t) \right\| \leq 2\sqrt{r}\delta\|E_t\|\|Z_t\|^2 + \delta d^{3/2}\|E_t\|^2\|Z_t\|\]

Therefore,

\[
Z_{t+1} = \text{Id}_{S_{t+1}} U_{t+1} = U_{t+1} - (\text{Id} - \text{Id}_{S_{t+1}}) U_{t+1} \tag{A.68}
\]

\[
= (\text{Id} - \eta M_t) Z_t + (\text{Id} - \eta M_t) E_t - (\text{Id} - \text{Id}_{S_{t+1}}) (\text{Id} - \eta M_t) E_t \tag{A.69}
\]

\[
= (\text{Id} - \eta M_t) Z_t + \text{Id}_{S_{t+1}} (\text{Id} - \eta M_t) E_t \tag{A.70}
\]

We can use Lemma A.10 on \( \text{Id}_{S_{t+1}} (\text{Id} - \eta M_t) E_t \) to complete the proof. \(\square\)

A.4 Proof of Proposition 4.8

We first prove the following technical lemma that characterizes how much the least singular value of a matrix changes when it got multiplied by matrices that are close to identity.

**Lemma A.11.** Suppose \( Y_1 \in \mathbb{R}^{d \times r} \) and \( \Sigma \) is a PSD matrix in \( \mathbb{R}^{r \times r} \). For some \( \eta > 0 \), let

\[
Y_2 = (\text{Id} + \eta \Sigma) Y_1 (\text{Id} - \eta Y_1^\top Y_1)
\]

Then, we have:

\[
\sigma_{\min}(Y_2) \geq (1 + \eta \sigma_{\min}(\Sigma)) (1 - \eta \sigma_{\min}(Y_1)^2) \sigma_{\min}(Y_1) \tag{A.71}
\]

**Proof.** First let’s consider the matrix \( Y := Y_1 (\text{Id} - \eta Y_1^\top Y_1) \). We have that \( \sigma_{\min}(Y) = (1 - \eta \sigma_{\min}(Y_1)^2) \sigma_{\min}(Y_1) \). Next, we bound the least singular value of \( (\text{Id} + \eta \Sigma) Y \):

\[
\sigma_{\min}(\text{Id} + \eta \Sigma) Y \geq \sigma_{\min}(\text{Id} + \eta \Sigma) \sigma_{\min}(Y) = (1 + \sigma_{\min}(\Sigma)) \sigma_{\min}(Y) \tag{A.72}
\]

where we used the facts that \( \sigma_{\min}(AB) \geq \sigma_{\min}(A) \sigma_{\min}(B) \), and that for any symmetric PSD matrix \( B \), \( \sigma_{\min}(\text{Id} + B) = 1 + \sigma_{\min}(B) \). \(\square\)

Now we are ready to prove Proposition 4.8. Note that the least singular value of \( Z_t \) is closely related to the least singular value of \( \tilde{R}_t \) because \( F_t \) is close to 0. Using the machinery in the proof of Lemma A.7, we can write \( R_{t+1} \) as some transformation of \( \tilde{R}_t \), and then use the lemma above to bound the least singular value of \( R_{t+1} \) from below.

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Proof of Proposition 4.8. Recall that in equation (A.45) in the proof of Lemma A.7 we showed that

\[ \| \text{Id} - (\text{Id} - \eta \hat{R}_t \hat{R}_t^\top + \eta \Sigma^*) \hat{R}_t R^+_{t+1} \| \lesssim \eta \delta \sqrt{r} + \eta \rho + \eta \| F_t \| . \]

Together with the bounds \( \| \hat{R}_t \| \leq 6 \) and \( \| \hat{R}_t R^+_{t+1} \| \leq 2 \) by Corollary A.6 and Lemma A.7 respectively, we have:

\[ \| \text{Id} - (\text{Id} + \eta \Sigma^*) \hat{R}_t (\text{Id} - \eta \hat{R}_t^\top \hat{R}_t) R^+_{t+1} \| \lesssim \eta \delta \sqrt{r} + \eta \rho + \eta \| F_t \| + \eta^2 . \]

Denoting \( \xi = \text{Id} - (\text{Id} + \eta \Sigma^*) \hat{R}_t (\text{Id} - \eta \hat{R}_t^\top \hat{R}_t) = R_{t+1} \xi \),

Without loss of generality, let us assume that \( \sigma_{\text{min}}(\hat{R}_t) \leq \frac{1}{1.9 \sqrt{r}} \). By Lemma A.11 that

\[ \sigma_{\text{min}}(R_{t+1}) \geq \frac{(1 + \eta \sigma_{\text{min}}(\Sigma^*)) \left(1 - \eta \sigma_{\text{min}}(\hat{R}_t)^2 \right) \sigma_{\text{min}}(\hat{R}_t)}{1 + O \left( \eta \delta \sqrt{r} + \eta \rho + \eta \| F_t \| + \eta^2 \right)} . \]

Using the bound \( \| F_t \| \lesssim \eta \rho t \) by Proposition 4.7, we have:

\[ \sigma_{\text{min}}(R_{t+1}) \geq (1 + \eta / \kappa) \left(1 - \eta \sigma_{\text{min}}(\hat{R}_t)^2 \right) \sigma_{\text{min}}(\hat{R}_t) \left(1 - O \left( \eta \delta \sqrt{r} + \eta \rho + \eta^2 \rho t \right) \right) \]

\[ \geq \left(1 + \eta \left( \frac{1}{3 \kappa} - O \left( \delta \sqrt{T} + \rho + \eta (\delta \rho) t \right) \right) \right) \sigma_{\text{min}}(\hat{R}_t) \quad \text{(by } \sigma_{\text{min}}(\hat{R}_t) \leq \frac{1}{1.9 \sqrt{r}} \text{)} \]

\[ \geq \left(1 + \frac{\eta}{4 \kappa} \right) \sigma_{\text{min}}(\hat{R}_t) \quad \text{(by } t \leq \frac{\kappa}{\eta \rho} \text{)} \]

Using

\[ \sigma_{\text{min}}(\hat{R}_t) \geq \sigma_{\text{min}}(R_t) \left(1 - \frac{\eta}{100 \kappa} \right) \quad \text{(A.73)} \]

completes the proof. \( \square \)

A.5 Proof of Proposition 4.3

We know that

\[ (\nabla f(U_t) U_t^\top, U_t U_t^\top - X^*) \quad \text{(A.74)} \]

\[ = \frac{1}{m} \sum_{i=1}^{m} \langle A_i U_t U_t^\top - X^* \rangle \langle A_i U_t U_t^\top, U_t U_t^\top - X^* \rangle \quad \text{(A.75)} \]

\[ = \frac{1}{m} \sum_{i=1}^{m} \langle A_i U_t U_t^\top - X^* \rangle \langle A_i U_t U_t^\top (U_t U_t^\top - X^*) \rangle \quad \text{(A.76)} \]

Now, we can bound

\[ \| U_t U_t^\top - Z_t Z_t^\top \| \leq 2 \| E_t \| \| Z_t \| + \| E_t \|^2 = O(\| E_t \|) \]
Therefore,
\[
\langle \nabla f(U_t) U_t^T, U_t U_t^T - X^* \rangle \geq \langle \nabla f(U_t) U_t^T, Z_t Z_t^T - X^* \rangle - O\left(\sqrt{\delta} \| E_t \| \right) \| \nabla f(U_t) U_t^T \|_F \quad (A.77)
\]

Now, using the RIP property of \( \{ A_i \}_{i=1}^m \) and the fact that \( \| U_t \| \leq \| Z_t \| + \| E_t \| \leq 1 \), we have that \( \| \nabla f(U_t) \| \leq 2d \| U_t \| \leq 2d \). Therefore,
\[
\| \nabla f(U_t) U_t^T \|_F \lesssim \| \nabla f(U_t) \| \| U_t \|_F \lesssim d^{3/2} \quad (A.78)
\]

Similarly, we can further bound that
\[
\langle \nabla f(U_t) U_t^T, Z_t Z_t^T - X^* \rangle \geq \langle \nabla f(Z_t) Z_t^T, Z_t Z_t^T - X^* \rangle - O(d^2 \| E_t \|) \quad (A.79)
\]

Which implies that
\[
\langle \nabla f(U_t) U_t^T, U_t U_t^T - X^* \rangle \geq \langle Z_t Z_t^T - X^*, Z_t Z_t^T (Z_t Z_t^T - X^*) \rangle - \delta \| Z_t \|^2 \| Z_t Z_t^T - X^* \|_F^2 - O(d^2 \| E_t \|) \quad (A.80)
\]

Without loss of generality, let us assume that
\[
X^* = \begin{pmatrix} \Sigma^+ & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad Z_t Z_t^T = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \Sigma \begin{pmatrix} U_1^T & U_2^T \end{pmatrix}
\]

For column orthonormal matrix \( \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \in \mathbb{R}^{d \times r} \) where \( U_1 \in \mathbb{R}^{r \times r} \).

We have:
\[
\| Z_t Z_t^T - X^* \|_F^2 = \| U_1 \Sigma U_1^T - \Sigma^* \|_F^2 + 2 \| U_2 \Sigma U_2^T \|_F^2 + \| U_2 \Sigma U_2^T \|_F^2 \quad (A.81)
\]

Since \( \sin(Z_t, U^*) \leq \frac{1}{5} \), we know that \( \sigma_{\min}(U_1) \geq 1/4 \), which implies that
\[
\| U_1 \Sigma U_1^T - \Sigma^* \|_F^2 + 2 \| U_2 \Sigma U_1^T \|_F^2 + \| U_2 \Sigma U_2^T \|_F^2 \leq \| U_1 \Sigma - \Sigma^* U_1^T \|_F^2 + \| U_2 \Sigma \|_F^2 \quad (A.82)
\]

and
\[
\langle Z_t Z_t^T - X^*, Z_t Z_t^T (Z_t Z_t^T - X^*) \rangle = \left\| \left( Z_t Z_t^T - X^* \right) Z_t \right\|_F^2 \quad (A.84)
\]
\[
\geq \sigma_{\min}(\Sigma) \left( \left\| U_1 \Sigma - \Sigma^* U_1^T \right\|_F^2 + \| U_2 \Sigma \|_F^2 \right) \quad (A.85)
\]

By \( \sigma_{\min}(Z_t)^2 \geq 1/(4\kappa) = \Omega(\delta) \) and \( \| Z_t \|^2 \lesssim 1 \) we complete the proof.

**B Restricted Isometry Properties**

In this section we list additional properties we need for the set of measurement matrices \( \{ A_i \}_{i=1}^m \). Lemma \[2.2\] follows from the definition of RIP matrices. The rest three Lemmas are all direct implications of Lemma \[2.2\].

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Proof of Lemma 2.3. For every $x \in \mathbb{R}^d, y \in \mathbb{R}^{d'}$ of norm at most 1, we have:

$$\frac{1}{m} \sum_{i=1}^{m} \langle A_i, X \rangle x^\top A_i R y - x^\top X R y = \frac{1}{m} \sum_{i=1}^{m} \langle A_i, X \rangle \langle A_i, x y^\top R^\top \rangle - x^\top X R y$$

$$\leq \langle X, x y^\top R^\top \rangle + \delta \|X\|_F \|x y^\top R^\top\| - x^\top X R y$$

$$\leq \delta \|X\|_F \|R\|_2$$

The first inequality uses Lemma 2.2.

The following Lemmas deal with matrices that may have rank bigger than $r$. The idea is to decompose the matrix into a sum of rank one matrices via SVD, and then apply Lemma 2.2.

Lemma B.1. Let $\{A_i\}_{i=1}^{m}$ be a family of matrices in $\mathbb{R}^{d \times d}$ that satisfy $(r, \delta)$-restricted isometry property. Then for any matrices $X, Y \in \mathbb{R}^{d \times d}$, where the rank of $Y$ is at most $r$, we have:

$$\left| \frac{1}{m} \sum_{i=1}^{m} \langle A_i, X \rangle \langle A_i, Y \rangle - \langle X, Y \rangle \right| \leq \delta \|X\|_s \|Y\|_F$$

Proof. Let $X = UDV^\top$ be its SVD. We decompose $D = \sum_{i=1}^{d} D_i$ where each $D_i$ contains only the $i$-th diagonal entry of $D$, and let $X_i = UD_i V^\top$ for each $i = 1, \ldots, d$. Then we have:

$$\frac{1}{m} \sum_{i=1}^{m} \langle A_i, X \rangle \langle A_i, Y \rangle = \sum_{j=1}^{d} \left( \frac{1}{m} \sum_{i=1}^{m} \langle A_i, X_j \rangle \langle A_i, Y \rangle \right)$$

$$\leq \sum_{j=1}^{d} \left( \langle X_j, Y \rangle + \delta \|X_j\|_F \|Y\|_F \right) = \langle X, Y \rangle + \delta \|X\|_s \|Y\|_F$$

Lemma B.2. Let $\{A_i\}_{i=1}^{m}$ be a family of matrices in $\mathbb{R}^{d \times d}$ that satisfy $(1, \delta)$-restricted isometric property. Then for any matrix $X \in \mathbb{R}^{d \times d}$ and matrix $R \in \mathbb{R}^{d \times d'}$, where $d'$ can be any positive integer, we have:

$$\left\| \frac{1}{m} \sum_{i=1}^{m} \langle A_i, X \rangle A_i R - X R \right\| \leq \delta \|X\|_s \times \|R\|.$$

The following variant is also true:

$$\left\| \frac{1}{m} \sum_{i=1}^{m} \langle A_i, X \rangle UA_i R - UX R \right\| \leq \delta \|X\|_s \times \|U\| \times \|R\|,$$

where $U$ is any matrix in $\mathbb{R}^{d \times d}$.

Proof. Let $X = UDV^\top$ be its SVD. We define $X_i$ and $D_i$ the same as in the proof of Lemma B.1 for each $i = 1, \ldots, d$.

For every $x \in \mathbb{R}^d, y \in \mathbb{R}^{d'}$ with norm at most one, we have:

$$\frac{1}{m} \sum_{i=1}^{m} \langle A_i, X \rangle x^\top A_i R y - x^\top X R y$$
\[
\sum_{j=1}^{d} \left( \frac{1}{m} \sum_{i=1}^{m} \langle A_i, X_j \rangle \langle A_i, xy^T R^\top \rangle \right) - x^\top X R y
\]
\[
\leq \sum_{j=1}^{d} \left( \langle X_j, xy^T R^\top \rangle + \delta \| X_j \|_F \| R \| \right) - x^\top X R y = \delta \| X \|_\ast \| R \|.
\]

The variant can be proved by the same approach (details omitted). □

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