The Fractional Quantum Hall Effect

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Abstract We give a brief introduction to the phenomenon of the Fractional Quantum Hall effect, whose discovery was awarded the Nobel prize in 1998. We also explain the composite fermion picture which describes the fractional quantum Hall effect as the integer quantum Hall effect of composite fermions.

I would like to start my talk by mentioning that the 1998 Nobel Prize in Physics has been awarded for the discovery of Fractional Quantum Hall Effect to

• Robert Laughlin - a theorist from Stanford University,
• Horst Stormer - an experimentalist from Lucent technologies (formerly Bell Labs), and
• Daniel Tsui - an experimentalist from Princeton University

Their citation reads - “for their discovery of a new form of quantum fluid with fractionally charged excitations”.

In this talk, I will try to describe this new form of quantum fluid and its fractionally charged excitations. However, since I am speaking to a general audience and the phenomenon of the fractional quantum Hall effect may not be familiar to all, I will start my talk with a brief introduction to the classical Hall effect, before I start with the quantum Hall effect.

The Hall effect, discovered in 1879, is simply the phenomenon that when a plate carrying an electric current is placed in a transverse magnetic field, the Lorentz force causes a potential drop perpendicular to the flow of current

\[ B = B_0 z \]

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This experiment is performed at room temperature and with moderate magnetic fields (\(~ 1 \ Tesla\)). If we measure the Hall resistance and plot it as a function of the magnetic field, we get a straight line - i.e., Hall resistance varies linearly with magnetic field.

Much later, in the early seventies, it was found that under certain conditions, electrons could be made to effectively move only in two dimensions. This is achieved by forming an inversion layer at the interface between a semiconductor and an insulator (\(Si - SiO_2\)) or between two semiconductors (\(GaAs - Al_xGa_{1-x}As\)). In such a layer, at very low temperatures, (around \(-272\) deg C), by applying an electric field perpendicular to the interface, the electrons can be made to sit in a deep quantum well, which quantises the motion of the electrons perpendicular to the interface. Thus, the electrons are essentially constrained to move only in two dimensions.

In 1980, at very low temperatures (1 deg Kelvin) and at high magnetic fields, (3-10 Tesla), Klaus von Klitzing discovered that the Hall resistance does not vary linearly with magnetic field, but varies in a 'stepwise' fashion, with the strength of the magnetic field. Even more surprisingly, the value of the resistance at these plateaux was completely independent of the material, temperature, and other variables of the experiment and depended only on a combination of physical constants divided by an integer - \(\frac{h}{e^2}\). This was the first example of quantisation of the resistivity. (Note that for the Hall geometry, Hall resistance = Hall resistivity.) In fact, the accuracy of this quantisation is so high, that it has led to a new international standard of resistance represented by the unit 1 Klitzing = \(h/4e^2 = 6.25\) kilo-ohms defined as the Hall resistance at the fourth step. Note also that where the Hall resistance was flat, the longitudinal resistance was found to vanish. In effect, the system was dissipationless and thus related to superconductivity and superfluidity.

For this discovery of the \emph{integer quantum Hall effect (IQHE)}\footnote{For a more detailed explanation, see the reference.}, Klaus von Klitzing was awarded the Nobel prize in physics in 1982.
The Hall resistance varies stepwise with changes in magnetic field at high magnetic fields and low temperatures. The steps are quantised at integer values of the filling fraction. (Kosmos 1986)

The IQHE can be easily explained using simple quantum mechanics of non-interacting electrons in an external magnetic field. The Hamiltonian for the system is given by

\[
H = \sum_i \frac{(p_i - eA(x_i))^2}{2m}
\]

with \( A(x_i) = \frac{B}{2}(y_i - x_i), \quad B = B\hat{z}. \) (1)

Solving this Hamiltonian, we find the energy eigenvalues \( E_{n,k_y} = (n + 1/2)\hbar\omega, \) which are called Landau levels (LL) in terms of the cyclotron frequency \( \omega = eB/mc. \) The Landau levels are degenerate, since they do not depend on the \( k_y \) quantum number. The degeneracy of the Landau levels \( \rho_B \) (the number of states per unit area) can be explicitly computed and is given by \( \rho_B = \frac{eB}{hc}. \)

Let us define a filling factor \( \nu = \frac{\rho}{\rho_B} \) as the number of electrons per Landau level. The filling factor can be thought of as a measure of the magnetic field. Theoretical analyses are often presented with the resistances as a function of the filling fraction, rather than the magnetic fields. In terms of the filling fractions, plateaux occur whenever \( \nu = \) integer or whenever an integer number of Landau levels are fully occupied.

Why does this happen? Let us see what happens as we increase the density of electrons. As long as states are available in the LL, we can put more electrons into the level and the conductivity goes on increasing (resistance decreases), but when a LL is full, there exists an energy gap to the next available state in the next Landau level. But there exist localised states in the gap, due to impurities in the sample. Hence, as the Fermi level passes through the gap, the localised states gets occupied by the electrons and so
do not contribute to the conductivity. This is what causes the plateaux in the transverse conductivity, until the next Landau level is reached and the same story is repeated.

To understand the extra-ordinary accuracy of the quantisation of the resistance, one has to also realise the more subtle point that even when some of the states in each LL get localised due to impurities, the conductance by the remaining states in that level is as if the entire Landau level was fully occupied! In other words, the electrons in the extended states move faster to compensate for the loss of the electrons in the localised states.

Another simpler hand-waving way to explain the IQHE is to say that the system is particularly stable when an integer number of LL’s filled. When we now add more particles, the system prefers to keep the average density fixed and accomodate the extra particles as local fluctuations pinned by disorder.

Thus, IQHE is easily explained with just quantum mechanics of non-interacting electrons and the pinning of some of the states due to disorder.

In 1982, Horst Stormer and Dan Tsui repeated the experiment with cleaner samples, lower temperatures and higher magnetic fields (upto 30 Tesla). They found that the integers at which resistivity is quantized can now be replaced by fractions - 1/3, 1/5, 2/5, 3/7 · · · .

- Fig. 4 The Hall resistance varies stepwise with changes in magnetic field at even higher magnetic fields, lower temperatures and cleaner samples. The steps are now quantised at fractional values of the filling fraction. (Science 1990)

The QHE at these fractions could not be explained by simple non-interacting quantum mechanics, which says that at these fractions, the Fermi level is within the lowest LL and so, the system is expected to be highly degenerate with no gap. Without the gap, there is no stability and no possible explanation for the plateaux.
But this degeneracy is lifted because the electrons are interacting. The Hamiltonian for interacting electrons is given by

\[ H = \sum_{i} \left( \frac{p_i - eA(x_i)}{2m} \right)^2 + \sum_{i<j} \frac{e^2}{|x_i - x_j|}. \] (2)

Moreover, for fractions less than one, all the electrons are in the lowest LL. Hence, the kinetic energy is completely quenched and the only relevant term in the Hamiltonian is the inter-electron Coulomb repulsion. But the quenching of the kinetic term means that \( e^2 \) is not small compared to anything. (For IQHE, on the other hand, the potential energy \( e^2/r_{av} \), where \( r_{av} \) is the average inter-electrons spacing, was small compared to the cyclotron energy and could be neglected.) Hence, we cannot use perturbation theory and the problem is intrinsically one of strong correlations - a very hard problem.

Laughlin in 1983 used a mixture of physical insight and numerical checks to write down a wave-function - the by-now celebrated Laughlin wave-function\[4\]

\[ \psi_L = \prod_{i<j} (z_i - z_j)^{2p+1} e^{-\sum_i \frac{z_i^2}{4l^2}} \] (3)

- as a possible variational wave-function (with no variational parameters!) as an ansatz solution for the interacting Hamiltonian. Here \( z_i = x_i + iy_i \) is the complex position of the \( i^{th} \) particle and \( l^2 = \frac{hc}{eB} \) is the magnetic length.

Using this wave-function, Laughlin could demonstrate the following properties -

- The wave-function describes a uniform distribution of electrons (not random) i.e., the number of particles within any patch remains the same.

\[ \text{Fig. 5} \] Comparison between a random distribution of particles (on the left) and a uniform distribution of particles (on the right).

Clearly, the uniform distribution (fluid-like) minimises Coulomb energy much better than the random distribution, which can have patches with a large number of particles costing a large energy. In fact, \( \psi_L \) was found to be very close to the exact ground state wave-function (calculated numerically) for small systems.
• The state described by the wave-function is incompressible. There exists a finite energy gap for all excitations. This is a non-trivial point, since naively, one expects a large degeneracy and instead, one now finds that there is a unique wave-function at these fractions with lowest energy and all other possible wave-functions are less efficient in mimimising the Coulomb energy and hence have higher energies. This is related to the fact that the Laughlin wave-function has multiple zeroes when two particles approach each other, whereas Fermi statistics only needs a single zero. These multiple zeroes are responsible for uniformising the distribution, which in turn, as seen in point 1) minimises the Coulomb energy.

• Quasi-particle excitations over the ground state have fractional charges.

This explains the citation which honours the scientists for their discovery of “a new quantum fluid with fractionally charged excitations”.

Why is the Laughlin wave-function ansatz so celebrated? Its fame lies in the fact that it is a correlated wave-function. Normally, many-body wave-functions are Slater determinants of one-particle wave-functions - i.e., products of one-particle wave-functions appropriately anti-symmetrised. For instance, the wave-function for one filled Landau level is given by

$$\chi_1 = \left| \begin{array}{cccc} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_N \\ z_1^2 & z_2^2 & \cdots & z_N^2 \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{N-1} & z_2^{N-2} & \cdots & z_N^{N-1} \end{array} \right|$$

which is a Slater determinant of single particle wave-functions. Similarly, the two filled Landau level state involves $z_i^*$’s as it involves the second Landau level, but it can still be written as a Slater determinant of one-particle levels in each of the two Landau levels. But $\psi_L = \Pi_{i<j}(z_i - z_j)^{2p+1}e^{-\sum_i z_i^2}$ cannot be written as sum of products of one-body wave-functions - it intrinsically describes a correlated many particle state at a filling fraction $\nu = 1/(2p + 1)$.

Once, we have the result that at these fractions, the system is gapped, just like in the IQHE, it is easy to understand the plateau formation by now having localised states in the intra-LL gap. Thus, using his wave-function, Laughlin could explain the odd denominator rule, which simply comes from fermion statistics and FQHE at the fractions 1/3, 1/5, \ldots, 1/(2p+1). But more contrived scenarios (called the hierarchy picture) was needed to explain fractions like 2/5, 3/7, \ldots.

In 1989, the next step in understanding the problem was taken by Jainendra Jain\[5\]. He identified the right quasi-particles of the system and called them composite fermions. (There is no guarantee that appropriate quasi-particles, in terms of which any complicated strongly interacting system appears weakly interacting, always exist, but the challenge is to try and find them, if they do exist. Phonons and magnons in lattices and spin models, Landau quasiparticles in metals, Cooper pairs in superconductors and Luttinger bosons(holons) in one dimensional fermion models are some examples.) In terms of these quasi-particles, FQHE of strongly interacting fermions is like IQHE of composite fermions.
The easiest way to understand his quasi-particles is pictorially. Let us measure magnetic field in terms of flux quanta per electron. IQHE at filling fraction \( \nu = 1 \) occurs when there is precisely one flux quanta per electron. FQHE, which occurs at higher magnetic fields has more flux quanta per electron - e.g., filling fraction \( \nu = 1/3 \) corresponds to three flux quanta per electron. Hence, we can depict IQHE as

\[ \text{Fig. 6 IQHE at } \nu = 1. \text{ Electrons (depicted as balls) and flux quanta (depicted as tubes). On the average, there is one flux quanta per electron.} \]

and FQHE at \( \nu = 1/3 \) as

\[ \text{Fig. 7 FQHE at } \nu = 1/3. \text{ Electrons 'holding hands' implying strong interactions. On the average, there are three flux quanta per electron.} \]

Strongly interacting electrons are depicted as 'holding hands'! Jain identified composite fermions as fermions with even number of flux quanta attached - in this (simplest) case, two flux quanta are attached. Hence, in Jain’s picture, FQHE at \( \nu = 1/3 \) is depicted as
• Fig. 8 FQHE at $\nu = 1/3$. Composite electrons - electrons with two flux quanta attached - see on the average, one flux quanta per composite electron.

Observe that composite electrons are no longer interacting! They see one flux quanta per composite electron - similar to IQHE at $\nu = 1$ where electrons see one flux quanta per electron. Hence, FQHE is analogous to IQHE of composite fermions.

The explanation now for unique incompressible wave-functions at the fractions is simply that they occur when composite fermion Landau levels are filled. Whenever an integer number of composite fermion Landau levels are completely filled, there exists a unique ground state and a gap to the next level. This picture could not only accommodate all the Laughlin fractions $\nu = 1/(2p+1)$, but also all the hierarchy fractions $\nu = 2/5, 3/7, \cdots$ at the same level. For instance, FQHE at $\nu = 2/5$ is just the IQHE of composite fermions at $\nu = 2$ and so on.

Jain used this mean field picture to propose more general wave-functions than the Laughlin wave-function. He first rewrote the Laughlin wave-function as

$$
\psi_L = \Pi_{i<j}(z_i - z_j)^{2p+1}e^{-\sum_i \frac{z_i^2}{4\lambda^2}} = \Pi_{i<j}(z_i - z_j)^{2p}\chi_1 e^{-\sum_i \frac{z_i^2}{4\lambda^2}}
$$

where $\chi_1$ is the wave-function of one filled Landau level. Then he wrote the wave-functions for the other fractions of the form $\nu = n/(2pn + 1)$ (which are all the experimentally observed fractions) as

$$
\psi_{Jain} = \Pi_{i<j}(z_i - z_j)^{2p}\chi_n e^{-\sum_i \frac{z_i^2}{4\lambda^2}} \tag{5}
$$

where $\chi_n$ is the wave-function of $n$-filled Landau levels. $n$ is the filling fraction of composite fermion Landau levels and the Jastrow factor $\pi_{i<j}(z_i - z_j)^{2p}$ turns composite fermions into fermions.

These wave-functions have been tested numerically and found to be very close to the exact ground state. More interestingly, there now exists experimental evidence for composite fermions - the cyclotron orbit of the charge carrier in FQHE has been shown to be determined by the effective magnetic field seen by the composite fermion.

There is yet another way to understand incompressibility at the odd denominator fractions. For the fraction $\nu = 1/3$, consider fermions with three flux quanta attached - a fermion with three 'hands' holding three flux quanta depicted as
Fig. 9 FQHE at $\nu = 1/3$. Composite bosons ‘holding’ three flux quanta in zero field Bose condense.

These composite particles are now bosons in zero field and Bose condense. Hence, incompressibility of the fermion system is equivalent to Bose condensation of the composite bosons.

Various explanations are possible because in two dimensions, one can have statistical transmutation and describe the same system in terms of fermions, bosons or even anyons (particles with ‘any’ statistics). However, composite fermions are really the appropriate quasi-particles because they are the ones which are ‘weakly interacting’.

Finally, I will conclude by mentioning a few directions in which the subject is currently progressing and give some examples of open problems.

- Edge states at the edge of a sample of quantum Hall fluid.
  Edge states form a chiral Luttinger liquid and there have been several recent experiments to probe edge physics.

- Double layer or multi-layer FQHE.
  If the distance between layers is small, one can get new correlated electron states (with correlations between electrons in different layers) as ground states.

- FQHE with unpolarised and partially polarised spins
  The usual FQHE assumes that the spin is completely polarised, so that one is justified in working with spinless electrons, but there are experimental situations where this is not true and one needs to explicitly include the spin degree of freedom.

- $\nu = 1/2$ state.
  The composite fermion picture yields ‘free’ fermions at $\nu = 1/2$. There has been a lot of interest both theoretical and experimental in the study of this state which shows novel non-Fermi liquid behaviour.

- Detailed calculations regarding the widths of plateaux, transitions between plateaux, effects of temperature, disorder, etc are yet to be performed at a quantitative level.
• At a more theoretical level, it is still an open problem to understand how microscopic Coulomb repulsions lead to the formation of a composite fermion.

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