On the proximate order of growth of generating functions of Pólya frequency sequences

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Abstract

We study the possible growth of $PF_r$ g.f.’s analytic in the unit disk and describe the proximate orders of growth of these functions. Some classical function theory results concerning orders of growth are generalized to the case of proximate orders.

keywords Pólya frequency sequence, multiply positive sequence, generating function, proximate order, total positivity

2000 Mathematics Subject Classification 30B10, 30D99

1 Introduction and statement of results.

A sequence $\{a_k\}_{k=0}^{\infty}$ is called a Pólya frequency sequence of order $r$, $r \in \mathbb{N} \cup \{\infty\}$, also multiply positive sequence, if all minors of order $\leq r$ (all minors if $r = \infty$) of the infinite matrix

\[
\begin{pmatrix}
a_0 & a_1 & a_2 & a_3 & \ldots \\
0 & a_0 & a_1 & a_2 & \ldots \\
0 & 0 & a_0 & a_1 & \ldots \\
0 & 0 & 0 & a_0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

are non-negative. This class of sequences is denoted by $PF_r$. We will also denote by $PF_r$ the class of corresponding generating functions

\[h(z) = \sum_{k=0}^{\infty} a_k z^k.\]

The radius of convergence of the power series of a $PF_r$ generating function ($PF_r$ g.f.) is positive provided $r \geq 2$ ([7], p. 394). Further

\footnote{This research has been partially supported by PRAXIS/2/2.1/FIS/286/94.}
we will suppose, without loss of generality, that $a_0 = 1$ and the radius of convergence is equal to 1, if it is not $\infty$. We can do that because if $h(z) \in PF_r$, then $\alpha h(\beta z) \in PF_r$ for any positive $\alpha$ and $\beta$.

It is well known (see [1] or [7], p. 412) that

**Theorem [AESW]** The class $PF_\infty$ consists of the functions

$$e^{\gamma z} \prod_{k=1}^{\infty} \frac{1 + \alpha_k z}{1 - \beta_k z},$$

where $\gamma \geq 0$, $\alpha_k \geq 0$, $\beta_k \geq 0$ and $\sum (\alpha_k + \beta_k) < \infty$.

The problem of the complete characterization of $PF_r$, $3 \leq r < \infty$, has not been solved yet. The study of possible zero-sets and growth of polynomials and entire functions belonging to $PF_r$, $r \in \mathbb{N}$, has been carried out by I.J. Schoenberg, O.M. Katkova and I.V. Ostrovskii (see [13], [10] and [9]). We give a description of the possible growth of $PF_r$ g.f.'s that are analytic in the unit disc $D$. It turns out that the growth is arbitrary in some sense. Moreover, our construction provides a $PF_r$ g.f. which is analytic in $\mathbb{C}\{1\}$ and has a given growth in the neighbourhood of $z = 1$.

In this paper we will use the notion of proximate order. Recall that ([11], p. 32) by definition a proximate order (p.o.) is a function $\rho(x) : \mathbb{R}_+ \to \mathbb{R}_+$ which belongs to $C^1(\mathbb{R}_+)$ and satisfies the conditions:

(i) $\lim_{x \to \infty} \rho(x) = \rho$, $0 \leq \rho < \infty$;

(ii) $\lim_{x \to \infty} x \frac{\rho'(x)}{\rho(x)} \log x = 0$.

We use the notation

$$V(x) := x^{\rho(x)}.$$

Property (ii) implies that $V(x)$ is a strictly increasing function for $x \geq x_0$ for some $x_0 \geq 0$. Without loss of generality, we redefine $\rho(x)$ on $[0, x_0)$ in such a way that $V(x)$ is strictly increasing on $\mathbb{R}_+$ and $V(0) = 0$.

The following properties of the functions $\rho(x)$ and $V(x)$ can be proved by straightforward verification:

(a) The function $\frac{a(x+b)}{c(x+d)} \to \frac{\rho(x)+b}{\rho(x)+d}$, $x \to \infty$, $\rho(x)$ (after redefining it on a finite interval if necessary) provided $(\rho(x)+b)/(\rho(x)+d) > 0$ and $\rho(x)+d \neq 0$.

(b) The function $\rho_1(x)\rho_2(x)$ is a p.o. provided $\rho_1(x)$ and $\rho_2(x)$ are p.o..

(c) The function $\rho_1(V(x))$ is a p.o. provided $\rho_1(t)$ is a p.o. Moreover, if $V_1(x) = x^{\rho_1(x)}$ and $V_2(x) = x^{\rho_2(x)}$ with p.o. $\rho_1(x)$ and $\rho_2(x)$,
then \( V_1(V_2(x)) = V_3(x) = x^{\rho_3(x)} \), where \( \rho_3(x) = \rho_1(V_2(x))\rho_2(x) \) is a p.o.

(d) The inverse to \( V \) function can be represented in the form \( V_{-1}(t) = \tilde{\rho}(t) \), where \( \tilde{\rho}(t) \to 1/\rho, \ t \to \infty \), is a p.o., provided \( \rho > 0 \).

(e) The function \( V(x) \) is a regularly varying function (see [14]), that is
\[
\lim_{x \to \infty} \frac{V(kx)}{V(x)} = k^\rho
\]
uniformly on each interval \( 0 < a \leq k \leq b < \infty \). The proof of this property can be found in [11], p.42.

We say that the entire function \( f(z) \) is of p.o. \( \rho(x) \) if the number
\[
\sigma_f = \limsup_{x \to \infty} \frac{\log M(x, f)}{V(x)}
\]
is positive and finite. Evidently, the order of the entire function is \( \rho = \lim_{x \to \infty} \rho(x) \).

Analogously, we say that the function \( g(z) \) analytic in the unit disk is of p.o. \( \rho(x) \) if the number
\[
\sigma_g = \limsup_{y \to 1-} \frac{\log M(y, g)}{V(1/(1-y))}
\]
is positive and finite.

Suppose that \( g \) is analytic in \( \{z : 0 < |z - 1| \leq \varepsilon\} \). We say that \( g \) has at \( z = 1 \) a singularity of p.o. \( \rho(r) \) if the number
\[
\limsup_{x \to \infty} \frac{\log M_1(x, g)}{V(x)},
\]
where
\[
M_1(x, g) := \max_{1/\varepsilon \leq |z - 1| \leq \varepsilon} |g(z)|, \ x > 1/\varepsilon,
\]
is positive and finite. The number \( \rho = \lim_{x \to \infty} \rho(x) \) will be called the order of singularity at \( z = 1 \).

The first main result of this paper is the following:

**Theorem A** Suppose that an integer \( r \geq 2 \) and a proximate order \( \rho(x) \to \rho, \ 0 < \rho < \infty \), are given. There exists a function \( g \) analytic in \( \mathbb{D} \) and possessing the following properties:
(i) \( g \in PF_r \);
(ii) \( g \) is of p.o. \( \rho(x) \) in \( \mathbb{D} \);
(iii) \( g \) is analytically extendable to \( \overline{\mathbb{C}} \setminus \{1\} \) and has at \( z = 1 \) an essential singularity of p.o. \( \rho(x) \).
For the cases of $\rho = 0$ and $\rho = \infty$ we only obtained the following results:

**Theorem B** For any integer $r \geq 2$ there exists a function $\Upsilon$ analytic in $\mathbb{D}$ satisfying the following properties:

(i) $\Upsilon \in PF_r$;
(ii) $\Upsilon$ is of infinite order in $\mathbb{D}$;
(iii) $\Upsilon$ is analytically extendable to $\mathbb{C}\backslash\{1\}$ and has at $z = 1$ an essential singularity of infinite order.

**Theorem C** Suppose that an integer $r \geq 2$ and the numbers $\rho_0, 1 \leq \rho_0 < \infty$, and $\sigma_0, 0 \leq \sigma_0 \leq \infty$, are given. There exists a function $G(z) \in PF_r$ analytic in $\mathbb{C}\backslash\{1\}$ and possessing an essential singularity at $z = 1$ such that

$$\limsup_{y \to 1^-} \frac{\log \log M(y, G)}{\log \log(1/(1 - y))} = \rho_0$$

and, for $\rho_0 > 1$, such that

$$\limsup_{y \to 1^-} \frac{\log M(y, G)}{(\log(1/(1 - y)))^{\rho_0}} = \sigma_0.$$

The restrictions on $\rho_0$ in Theorem C are necessary. Let us show this using the following result of [2]:

**Theorem [DH]** Let $G(z)$ be a $PF_r$ g.f., $r \geq 2$ with radius of convergence of its power series equal to 1. Then $(1 - z)G(z)$ is a $PF_{r-1}$ g.f.

For any function $G(z) \in PF_2$ analytic in $\mathbb{D}$, we have $(1 - z)G(z) \in PF_1$, therefore, $\lim_{y \to 1^-}(1 - y)M(y, G) > 0$ and $\log M(y, G) > \log[1/(1 - y)] + O(1)$, $y \to 1^-$. Thus, the quantity $\rho_0$ in Theorem C must be greater than or equal to 1.

## 2 Preliminary results and proof of Theorem A.

Our first lemma relates to the existence of an entire function of a given growth belonging to $PF_r$, $r \in \mathbb{N}$. It is an easy consequence of the main result of [3] due to O.M. Katkova:

**Lemma 1** Suppose that an integer $r \geq 2$ and a proximate order $\rho(x) \to \rho$, $0 \leq \rho < \infty$, are given. There exists a transcendental entire function of proximate order $\rho(x)$ which belongs to $PF_r$, $r \in \mathbb{N}$.
The next two lemmas permit to construct $PF_r$ sequences which will serve as sequences of Taylor coefficients of the functions whose existence is asserted by Theorem A.

**Lemma 2** Suppose that $\{c_n\}_{n=0}^{\infty} \in PF_r, r \geq 2, 0 < \sum c_n < \infty$. Set

$$d_k = \sum_{n=0}^{\infty} \frac{c_n k^{n+r-1}}{\Gamma(n+r)}, \quad k = 0, 1, 2, \ldots$$

Then $\{d_k\}_{k=0}^{\infty} \in PF_r$.

**Proof** We derive this lemma from the following theorem due to S. Karlin ([7], p.107):

**Theorem [Kar]** Suppose that $\{c_n\}_{n=0}^{\infty} \in PF_r, r \geq 2, 0 < \sum c_n < \infty, \quad \alpha > r - 2$. Set

$$f_\alpha(x) = \begin{cases} \sum_{n=0}^{\infty} c_n x^{n+\alpha}/\Gamma(n+\alpha+1), & x \geq 0, \\ 0, & x < 0. \end{cases}$$

then $f_\alpha(x)$ is a Pólya frequency function of the order $r$.

Recall that, $f_\alpha(x)$ is said to be a Pólya frequency function of order $r$, if for any $n \leq r$ and for any system of numbers $x_1 < x_2 < \cdots < x_n, y_1 < y_2 < \cdots < y_n$, we have

$$\det \|f_\alpha(x_j - y_i)\|_{i,j=1}^{n} \geq 0.$$ 

Setting $\alpha = r - 1$, $d_k = f_{r-1}(k), k = 0, \pm 1, \pm 2, \ldots$, and taking $x_j = j - 1$ and $y_i = i - 1, \quad i, j \in \mathbb{N}$, we see that any minor of the matrix

$$\begin{pmatrix}
    d_0 & d_1 & d_2 & d_3 & \cdots \\
    0 & d_0 & d_2 & \cdots \\
    0 & 0 & d_0 & d_1 & \cdots \\
    0 & 0 & 0 & d_0 & \cdots \\
    \cdots & \cdots & \cdots & \cdots & \cdots 
\end{pmatrix}$$

can be written as a minor of the matrix $\|f_{r-1}(x_j - y_i)\|_{i,j=1}^{n}$. \square

**Lemma 3** Let

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

(1)
be an entire PF g.f. and set

\[ f_{r-1}(z) = \sum_{n=0}^{\infty} \frac{c_n}{\Gamma(n + r)} z^{n+r-1} = \sum_{n=r-1}^{\infty} \frac{c_{n-r+1}}{n!} z^n. \]  

(2)

Then

\[ g(z) = \sum_{k=0}^{\infty} f_{r-1}(k) z^k \]

is a PF g.f..

The lemma is an immediate corollary of Lemma 2.

Our final goal is to study the growth of the function \( g \) in Lemma 3.

For this purpose we investigate the growth of the function \( f_{r-1}(z) =: \sum_{n=0}^{\infty} b_n z^n \) defined by (2), where the coefficients \( c_n \) are the ones in (1).

Theorem [Levin] Let \( \rho(x) \to \rho > 0, \ x \to \infty, \) be a p.o.. For the entire function \( B(z) = \sum_{n=0}^{\infty} b_n z^n \) set

\[ \sigma_B = \limsup_{n \to \infty} \frac{\log M(x, B)}{V(x)}. \]

Then

\[ (\sigma_B \rho)^{1/\rho} = \limsup_{n \to \infty} V^{-1}(n) \sqrt[\rho]{|b_n|}. \]

Lemma 4 If the function \( \rho \) is an entire function of p.o. \( \rho(x) \), then the function \( \psi(x) \) is an entire function of p.o.

\[ \rho_1(x) = \frac{\log \psi(x)}{\log x} \to \rho_1 = \frac{\rho}{\rho + 1}, \ x \to \infty, \]  

(3)

where \( \psi(x) \) is the inverse function of \( tV_1(t) \) and \( V_1(t) \) is the inverse function of \( V \).

Proof By (3) we have \( \psi(x) = x^{\rho_1(x)} \), and \( (\psi(x))^{-1} = tV_1(t) \). By the properties of p.o.'s d) and a) listed on p. 2, we have that \( \rho_1(x) \) is a p.o.

For proving that \( f_{r-1}(z) \) is of p.o. \( \rho_1(x) \) we calculate

\[ \limsup_{n \to \infty} \psi_1^{-1}(n) \sqrt[\rho]{|b_n|}, \]
where \( b_n \) are the Taylor coefficients of \( f_{r-1} \).

We have

\[
\limsup_{n \to \infty} \psi_{r-1}(n) \sqrt[n]{|b_n|} = \limsup_{n \to \infty} n^{V_{r-1}(n)} \sqrt[n]{c_{n-r+1}/n!} = \]

\[
= e \limsup_{n \to \infty} V_{r-1}(n) \sqrt[n]{c_{n-r+1}} = e \limsup_{n \to \infty} V_{r-1}(n) \sqrt[n]{c_n}.
\]

By Levin’s theorem

\[
\limsup_{n \to \infty} \psi_{r-1}(n) \sqrt[n]{|b_n|} = e^{(\sigma f \epsilon \rho)^{1/\rho}}
\]

is positive and finite, thus \( \rho_1(x) \) is the p.o. of \( f_{r-1}(z) \). \( \square \)

In the sequel, we will need a number of facts concerning functions

\[
h(z) = \sum_{k=0}^{\infty} a_k z^k
\]

which are analytic in \( \mathbb{D} \).

The following lemma is an analogue of Levin’s theorem for the case of functions analytic in the unit disk.

**Lemma 5** Let \( \rho(x) \) be a proximate order, \( \rho(x) \to \rho > 0, x \to \infty \), and \( \xi(t) \) the inverse function of \( x V(x) \). For the function (4) set

\[
\sigma_h = \limsup_{y \to 1^-} \frac{\log M(y, h)}{V(1/(1-y))}.
\]

Then

\[
\frac{\rho + 1}{\rho} (\sigma_h \rho)^{1/(\rho+1)} = \limsup_{k \to \infty} \frac{\xi(k)}{k} \log |a_k|.
\]

We could not find in the literature neither this result, nor the Lemmas [10] and [11], which will appear later, in spite of the fact that there are plenty of similar results with similar proofs (see, for example [15], [16] where one can find further bibliography). For the reader’s convenience we present the proofs in the last section of the paper.

**Theorem [Wigert]** (see, for example [11], p. 394) The function \( h(z) \) defined by (4) can be analytically extended to \( \mathbb{C} \setminus \{1\} \) and is equal to zero at infinity if and only if there exists an entire function \( A(z) \) whose growth is not greater than of order 1 and minimal type such that \( a_k = A(k), k = 0, 1, 2, \ldots \).
As an improvement to Wigert’s theorem Faber established the relation between the orders of growth of the functions $A(z)$ and $h(z)$ (see [3], §1, Th. 1.3.11):

**Theorem [Faber]** The function $h(z)$ can be analytically extended to $\mathbb{C} \setminus \{1\}$ and has at $z = 1$ a singularity of order $\rho$ if and only if the entire function $A(z)$ is of order of growth $\rho_A = \rho/(\rho + 1)$.

We were able to relate the p.o. of growth of the function $h$ in $D$ with the p.o. of growth of the function $A$ when $A$ has non-negative Taylor coefficients.

**Lemma 6** Let $A(z)$ be an entire function of proximate order $\rho_1(x) \to \rho_1$, $0 < \rho_1 < 1$, $x \to \infty$, with non-negative Taylor coefficients, such that $A(n) = a_n$, $n = 0, 1, 2, \ldots$, where $a_n$ are the coefficients of the function $A(z)$. Then $h(z)$ is of p. o.

$$\rho(x) = \frac{\rho_1(\xi_{-1}(x))}{1 - \rho_1(\xi_{-1}(x))}$$

in $\mathbb{D}$, where

$$\xi(t) = t^{1 - \rho_1(t)}$$

is the inverse of the function $xV(x)$. Moreover,

$$\rho_1(t) = \frac{\rho(\xi(t))}{\rho(\xi(t)) + 1}.$$ 

Also, $A(z)$ is of order 0 if and only if $h(z)$ is of order 0 in $\mathbb{D}$.

**Proof** By properties (a), (d) and (c) of proximate orders (see p. 3), we can conclude that $\rho(x)$ is a p.o. Now we want to prove that $\rho(x)$ is the p.o. of $A(z)$, i.e. that the number

$$\sigma_h = \limsup_{y \to 1^-} \frac{\log M(y, h)}{V(1/(1 - y))}$$

is positive and finite. We will prove it with the help of Lemma 3. Let $\xi(x)$ and $\rho(x)$ be defined by (4) and (5), respectively. We have $\rho(\xi(t)) = \rho_1(t)/(1 - \rho_1(t))$. Therefore, $t = \xi(t)V(\xi(t))$, which means that $\xi(t)$ is the inverse function of $xV(x)$.

On the other hand, denoting $V_1(x) = x^{\rho_1(x)}$, we get by (5) $\xi(n)/n = 1/V_1(n)$ and, hence,

$$\limsup_{n \to \infty} \frac{\xi(n)}{n} \log a_n = \limsup_{n \to \infty} \frac{\log A(n)}{V_1(n)} = \limsup_{x \to \infty} \frac{\log M(x, A)}{V_1(x)},$$
which is positive and finite. Note that we were able to write the last equality because $A(z)$ has non-negative Taylor coefficients. Then by Lemma 3, $\rho(x)$ is the proximate order of $h(z)$.

The last assertion of the lemma follows from the following result of Beuermann (see [3]):

**Theorem [Beuermann]** For the function $h(z)$, set

$$
\lambda = \limsup_{y \to 1} \frac{\log \log M(y, h)}{\log(1/(1 - y))}.
$$

Then the following equality is valid

$$
\frac{\lambda}{\lambda + 1} = \limsup_{k \to \infty} \frac{\log^+ \log^+ |a_k|}{\log k}.
$$

Thus, by Beuermann’s theorem $A(z)$ being of order 0 is equivalent to

$$
0 = \limsup_{x \to \infty} \frac{\log \log M(x, A)}{\log x} = \limsup_{k \to \infty} \frac{\log \log A(k)}{\log k}.
$$

$$
= \limsup_{k \to \infty} \frac{\log^+ \log^+ |a_k|}{\log k} = \limsup_{y \to 1^-} \frac{\log \log M(y, h)}{\log(1/(1 - y))}.
$$

\(\Box\)

The lemma below is a generalization of the theorem of Faber to the case of proximate orders that do not tend to zero at infinity.

**Lemma 7** Let $h(z)$ be analytically extendable to $\mathbb{C}\setminus\{1\}$ with a singularity at $z = 1$ of p.o. $\rho(x) \to \rho > 0$, $x \to \infty$, and $A(w)$ be of p.o. $\rho_A(x) \to \rho_A < 1$, $x \to \infty$. Then

$$
\rho_A(t) = \frac{\rho(\xi(t))}{\rho(\xi(t)) + 1},
$$

where $\xi(t)$ is the inverse function of $xV(x)$. Moreover,

$$
\rho(x) = \frac{\rho_A(\xi^{-1}(x))}{1 - \rho_A(\xi^{-1}(x))} \text{ and } \xi(t) = t^{1-\rho_A(t)}.
$$

**Proof** Our proof is based on the method used by Gelfond (see [6]) to obtain Faber’s result.

By hypothesis $h(z)$ has at $z = 1$ an essential singularity of p.o. $\rho(x)$. Thus, there exists a $\sigma$ and a $x_\sigma$ such that

$$
\max \left\{ |h(z)| : |1 - z| = \frac{1}{x} \right\} < \exp\{\sigma V(x)\}
$$

(9)
for \( x > x_\sigma \).

On the other hand, since \( A(k) = a_k, \ k = 0, 1, 2, \ldots \), we can write (see, for example, [4], §1, equation (1.3.29))

\[
A(w) = -\frac{1}{2\pi i} \int_{|z-1|=1/x} h(z) \exp\{-(w+1) \log z\} dz, \quad (10)
\]

where we take \( x > x_\sigma \). Consider also \( x \geq 2 \), so \(|\arg z| \leq 2/x \). We have

\[
|\exp\{-(w+1) \log z\}| < \exp \left\{ |w + 1| \left( |\log |z|| + \frac{2}{x} \right) \right\} \leq \exp \left\{ \frac{5|w|}{x} \right\}
\]

for \(|1 - z| = 1/x\) and \(|w|\) large enough.

Denoting \(|w|\) by \( t \), and using (10), (9) and (11) we can write

\[
|A(w)| \leq \exp \left\{ \sigma V(x) + \frac{5t}{x} \right\},
\]

for \( t \) and \( x \) large enough.

In the previous inequality we set \( x := \xi(5t/(\rho \sigma)) \), where \( \xi(t) \) is the inverse function of \( xV(x) \). Note that we can do that since \( \xi(t) \) is an increasing function (see properties of p.o.’s on p. 4). We obtain

\[
|A(w)| \leq \exp \left\{ \sigma V(\xi(5t/(\rho \sigma))) + \frac{5t}{\xi(5t/(\rho \sigma))} \right\}
\]

\[
= \exp \left\{ 5 \left( \frac{\rho + 1}{\rho} \right) \frac{t}{\xi(5t/(\rho \sigma))} \right\}
\]

\[
= \exp \left\{ 5 \left( \frac{\rho + 1}{\rho} \right) \frac{\xi(t)V(\xi(t))}{\xi(5t/(\rho \sigma))} \right\}.
\]

But, since \( \xi(t) \) is a regularly varying function,

\[
\xi \left( \frac{5t}{\rho \sigma} \right) = \left( \frac{5}{\rho \sigma} \right)^{1/(\rho+1)} \xi(t) \{1 + o(1)\}, \ t \to \infty.
\]

Thus, asymptotically

\[
|A(w)| \leq \exp \left\{ K(\rho+1) V(\xi(t)) \{1 + o(1)\} \right\}, \ t \to \infty,
\]

where \( K = K(\rho) > 0 \) is a constant.
Therefore,
\[
\limsup_{t \to \infty} \frac{\log M(t, A)}{V(\xi(t))} \leq (\sigma_H)^{1/(\rho+1)} < \infty. \tag{12}
\]

Conversely, suppose that \(\rho_A(t)\) is the p.o. of \(A(w)\), i.e.
\[
\sigma_A = \limsup_{t \to \infty} \frac{\log M(t, A)}{V_A(t)},
\]
where \(V_A(t) = \rho_A(t)\), is positive and finite. Then for \(\sigma > \sigma_A\) the following asymptotic inequality holds
\[
|A(w)| < \exp\{\sigma V_A(t)\}. \tag{13}
\]

On the other hand, we have \(h(z) = H(1/(1-z))\), where \(H\) is an entire function. Moreover,
\[
M_1(x, h) = \max\{|h(z)| : \varepsilon > |z-1| \geq 1/x\}
= \max\{|H(w)| : |w| \leq x\} = M(x, H),
\]
so the p.o.’s of the function \(H\) and of the essential singularity \(z = 1\) of \(h\) coincide. Also, \(h(z) = \tilde{H}(z/(1-z))\) and the inequality
\[
M(t - 1, \tilde{H}) \leq M(t, H) \leq M(t + 1, \tilde{H})
\]
shows that \(H\) and \(\tilde{H}\) are of the same p.o..

Denoting \(z/(1-z)\) by \(\zeta\) we have that
\[
h(z) = \sum_{k=0}^{\infty} a_k z^k = (1 + \zeta) \sum_{n=0}^{\infty} h_n \zeta^n.
\]

Note that the p.o.’s of \(H(w)\) and \(\sum_{n=0}^{\infty} h_n \zeta^n\) coincide. Next we calculate the coefficients \(h_n\):
\[
h(z) = (1 + \zeta) \sum_{k=0}^{\infty} a_k \frac{\zeta^k}{(1 + \zeta)^{k+1}}
= (1 + \zeta) \sum_{k=0}^{\infty} a_k \zeta^k \left( \sum_{m=0}^{\infty} \binom{m+k}{k} (-1)^m \zeta^m \right)
= (1 + \zeta) \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a_k \right) \zeta^n
\]
Thus, we have
\[ h_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a_k. \]

On the other hand, it is not difficult to see that
\[ \frac{A(w)}{w(w-1)\ldots(w-n)} = \sum_{k=0}^{n} \frac{\alpha_k}{w-k}, \]
where
\[ \alpha_k = \left. \frac{(w-k)A(w)}{w(w-1)\ldots(w-n)} \right|_{w=k} = \frac{1}{n!} \binom{n}{k} (-1)^{n-k} a_k. \]

Therefore, by the residue theorem we can write
\[ h_n = \frac{n!}{2\pi i} \int_{|w|=R} \frac{A(w)}{w(w-1)\ldots(w-n)} \, dw, \]
for \( R > n \).

It is obvious that
\[ |h_n| \leq n! \frac{M(R,A)}{(R-1)\ldots(R-n)} \frac{\Gamma(R-n)}{\Gamma(R)} M(R,A), \quad R > n. \]

Taking \( R \) large enough in the previous inequality, we obtain by \( \text{(13)} \) that
\[ |h_n| \leq n! \frac{\Gamma(R-n)}{\Gamma(R)} \exp\{\sigma V_A(R)\}. \]

Thus,
\[ |h_n|^{1/n} \leq \left[ \frac{n!\Gamma(R-n)}{\Gamma(R)} \right]^{1/n} \exp\{\frac{\sigma}{n} V_A(R)\}. \quad (14) \]

Now we denote by \( R(t) \) the function inverse to \( V_A(x) \) and set \( R := R(n) \) on the right-hand side of \( \text{(14)} \). We can do this because \( R(t) = t^{\alpha(t)} \), where \( \alpha(t) \) is a p.o. such that \( \alpha(t) \rightarrow 1/\rho_A > 1 \), \( t \rightarrow \infty \) (see properties on p. 5). So \( \lim_{n \rightarrow \infty} R(n)/n = \infty \).

We apply the Stirling formula to the right hand of \( \text{(14)} \). Note that
\[ (R(n))^{1/n} = n^{\alpha(n)/n} \sim 1, \quad n \rightarrow \infty, \]
so, when \( n \to \infty \),

\[
[\Gamma(R)]^{1/n} \sim \left( \frac{R(n)}{e} \right)^{R(n)/n}
\]

\[
= \left( \frac{n^\alpha}{e} \right)^{n/\alpha - 1} = \exp\{\alpha n^{\alpha-1} \log n - n^{\alpha-1}\}, \quad (15)
\]

where \( \alpha = \alpha(n) \).

Also, when \( n \to \infty \),

\[
[\Gamma(R-n)]^{1/n} \sim \left( \frac{R(n)-n}{e} \right)^{R(n)/n-1}
\]

\[
= \left( \frac{n^\alpha - n}{e} \right)^{n/\alpha - 1} = \left( \frac{n^\alpha}{e} \right)^{n/\alpha - 1} (1 - n^{1-\alpha})^{n/\alpha - 1}
\]

\[
= \exp\{\alpha(n^{\alpha-1}-1) \log n - n^{\alpha-1} + 1\} (1 - n^{1-\alpha})^{n/\alpha - 1}
\]

\[
\sim \exp\{\alpha(n^{\alpha-1}-1) \log n - n^{\alpha-1}\}, \quad (16)
\]

where \( \alpha = \alpha(n) \to 1/\rho_A > 1, \ n \to \infty \).

By (15) and (16), we have that the right hand of (14), when \( n \to \infty \), is equivalent to

\[
\frac{n}{e} \exp\{\alpha(n^{\alpha-1}-1) \log n - n^{\alpha-1} - \alpha n^{\alpha-1} \log n + n^{\alpha-1} + \sigma\}
\]

\[
= e^{\sigma-1} \frac{n}{R(n)} (1 + o(1)), \ n \to \infty.
\]

Thus,

\[
\lim \sup_{n \to \infty} \frac{R(n)}{n} |h_n|^{1/n} < \infty.
\]

Note that \( \log[(R(t)/t)_{-1}]/\log x \), where \( (R(t)/t)_{-1}(x) \) is the inverse function to \( R(t)/t \), is a p.o.. By Levin’s Theorem we have

\[
\lim \sup_{x \to \infty} \frac{\log M(x, \tilde{H})}{(R(t)/t)_{-1}(x)} < \infty. \quad (17)
\]

We will say that \( \tilde{H} \) is ”not greater than” \( (R(t)/t)_{-1}(x) \).

Now we will prove that the quantities we refer to in (12) and (17), that is \( \lim \sup_{t \to \infty} \log M(t, A)/V(\xi(t)) \) and \( \lim \sup_{x \to \infty} \log M(x, \tilde{H})/(R(t)/t)_{-1}(x) \), are also both positive.
For $V_A(t)$ we can find a $V(x) = x^\rho(x)$ such that $V_A(t) = V(\xi(t))$ and $\xi(t) = (xV(x))_{-1}$. Indeed, setting $\xi(t) = t/V_A(t)$ and $V(x) = (\xi(t))_{-1}/x$, we have $\xi(t) = (xV(x))_{-1}$ and $V_A(t) = t/\xi(t) = V(\xi(t))$. We will prove that $\rho(x)$ is the p.o. of $h(z)$.

By what was proved earlier, $\tilde{H}$ is "not greater than" $(R(t)/t)_{-1}$, $R(t) = (V_A(x))_{-1}$. But $R(t) = (V_A(x))_{-1} = (V(\xi(x)))_{-1} = \xi_{-1}(V_{-1}(t)) = tV_{-1}(t)$ because $\xi(t)$ is the inverse function to $xV(x)$. Thus,

$$\limsup_{t \to \infty} \frac{\log M(t, \tilde{H})}{V(t)} < \infty.$$ 

Now suppose that

$$\limsup_{t \to \infty} \frac{\log M(t, \tilde{H})}{V(t)} = 0,$$

then, by [12],

$$\limsup_{t \to \infty} \frac{\log M(t, A)}{V(\xi(t))} = 0,$$

which contradicts the fact $V(\xi(t)) = V_A(t)$.

We have necessarily

$$0 < \limsup_{t \to \infty} \frac{\log M(t, \tilde{H})}{V(t)} < \infty$$

and $h(z)$ is a function of p.o. $\rho(x) = \log V(x)/\log x$.

Note that

$$\rho_A(t) = \frac{\log V(\xi(t))}{\log t} = \rho(\xi(t)) \frac{\log \xi(t)}{\log t} = \frac{\rho(\xi(t))}{\rho(\xi(t)) - 1},$$

and hence

$$\rho(x) = \frac{\rho_A(\xi_{-1}(x))}{1 - \rho_A(\xi_{-1}(x)).}$$

By construction $\xi(t) = (xV(x))_{-1} = t/V_A(t)$. \qed

The following result is closely related to the main result of [12] due to Macintyre and Wilson:

**Lemma 8** Let $h$ be a function analytic in $\mathbb{D}$ of order greater than 0 whose Taylor coefficients are interpolated by the entire function $A$ of order less than 1 and having non-negative Taylor coefficients. Then the p.o. of $h$ in $\mathbb{D}$ and the p.o. of its singularity at $z = 1$ coincide. Also, $h$ is of order 0 in $\mathbb{D}$ if and only if its singularity at $z = 1$ is of order 0.
The lemma is an immediate consequence of Lemmas 6 and 7 and Faber’s theorem.

**Proof of Theorem A** Let $r \geq 2$ and a p.o. $\rho(x)$ be given. Let $f(z)$ be the entire function whose existence is established by Lemma 1. Set

$$g(z) = \sum_{k=0}^{\infty} f_{r-1}(k)z^k,$$

where $f_{r-1}$ is defined by (2). According to Lemma 3 we have $g(z) \in PF_r$.

By Lemma 4 the function $f_{r-1}(z)$ is an entire function of order $\rho_1(\xi) = \rho/\rho(1) < 1$. According to Lemma 6 $g(z)$ is of p.o.

$$\tilde{\rho}(x) = \rho_1(\xi(x)) \frac{1}{1 - \rho_1(\xi(x))}$$

in $\mathbb{D}$, where $\rho_1(t)$ is the p.o. of the function $f_{r-1}(z)$ and

$$\xi(t) = t^{1-\rho_1(t)}.$$  \hspace{1cm} (19)

Now we will show that $\tilde{\rho}(x) = \rho(x)$.

By virtue of Lemma 3 we have

$$\rho_1(x) = \frac{\log \psi(x)}{\log x},$$

where $\psi(x)$ is the inverse function of $tV_1(t)$, $V(x) = x^{\rho(x)}$, and $\rho(x)$ is the p.o. of the function $f(z)$.

For the function $\xi(t)$ defined by (8) the following equality holds

$$\xi(\psi_{-1}(t)) = V_{-1}(t).$$  \hspace{1cm} (21)

Indeed, by (13), (20) and the definition of $\psi$ we can write

$$\xi(\psi_{-1}(t)) = \left(\psi_{-1}(t)\right)^{1-\rho_1(\psi_{-1}(t))} = \psi_{-1}(t)^{1-\log t/\log \psi_{-1}(t)} =$$

$$= \exp\left\{1 - \frac{\log t}{\log \psi_{-1}(t)}\right\} \log \psi_{-1}(t)\right\} = \exp\{\log \psi_{-1}(t) - \log t\} = V_{-1}(t).$$

Now with the aid of (21), (18), (20) and the definition of $V$ we are able to write

$$\tilde{\rho}(V_{-1}(t)) = \tilde{\rho}(\xi(\psi_{-1}(t))) = \frac{\rho_1(\psi_{-1}(t))}{1 - \rho_1(\psi_{-1}(t))} =$$
\[
\frac{\log t}{\log \psi^{-1}(t) - \log t} = \frac{\log t}{\log V^{-1}(t)} - \frac{\log V(\psi^{-1}(t))}{\log V^{-1}(t)} = \rho(\psi^{-1}(t)).
\]

Thus, we have shown that \( \tilde{\rho}(x) = \rho(x) \).

On the other hand, the Wigert theorem is applicable to the function \( g \) and it shows that \( g(z) \) can be extended to \( \mathbb{C} \setminus \{1\} \). According to Lemma 8, \( g(z) \) has at \( z = 1 \) an essential singularity of p.o. \( \rho(x) \).

\section{Preliminary results and proof of Theorem B.}

The following result due to O.M. Katkova and I.V. Ostrovskii will be useful to prove Theorem B.

**Theorem [KatOst]** Let \( g_1 \) be an arbitrary entire function which is positive on the positive \( x \)-axis and such that \( g_1(0) = 1 \). For every \( r \in \mathbb{N} \) there exists an entire function \( g_2 \in PF_r \) such that \( g_2 g_1 \in PF_r \).

**Lemma 9** There exists an entire function of infinite order belonging to \( PF_r, r \in \mathbb{N} \).

This lemma is a consequence of Theorem [KatOst]. Indeed, let \( g_1 = \sum_{n=0}^{\infty} \phi_n z^n \) be an entire function of infinite order with positive coefficients. By Theorem [KatOst] there exists an entire function \( g_2 \) such that \( \Phi = g_2 g_1 \in PF_r \). Evidently, \( \Phi(z) \) is of infinite order.

**Proof of Theorem B** Let the integer \( r \geq 2 \) be given. Let

\[
\Phi(z) = \sum_{n=0}^{\infty} \phi_n z^n
\]

be the entire \( PF_r \) g.f. of infinite order whose existence is established by Lemma 8. Setting \( c_n = \phi_n \), \( f(z) = \Phi(z) \) and

\[
f_{r-1}(z) = \Phi_{r-1}(z) = \sum_{n=0}^{\infty} \frac{\phi_n z^{n+r-1}}{(n+r)!}
\]

in Lemma 3, we obtain that the function

\[
\Upsilon(z) = \sum_{k=0}^{\infty} \Phi_{r-1}(k) z^k
\]

is a \( PF_r \) g.f..
Using Hadamard and Stirling formulas we will show that the entire function \( \Phi_{r-1}(z) \) is of order 1. Indeed, its order is equal to

\[
\limsup_{n \to \infty} \frac{n \log n}{\log(n!/\phi_{n-r+1})} = \left[ \liminf_{n \to \infty} \left( \frac{\log(n!)}{n \log n} + \frac{\log(1/\phi_{n-r+1})}{n \log n} \right) \right]^{-1} = 1,
\]

since \( \Phi(z) \) is of infinite order, which means that

\[
\limsup_{n \to \infty} \frac{n \log n}{\log(1/\phi_{n-r+1})} = \infty.
\]

Moreover, \( \Phi_{r-1}(z) \) is of minimal type. Indeed, for any \( \varepsilon > 0 \) we have the asymptotic inequality \( \phi_n < \varepsilon^n \) and, thus, \( M(x, \Phi_{r-1}) = \Phi_{r-1}(x) \leq O(x^{r-1} \exp\{\varepsilon x\}) \), for \( x \to \infty \).

Suppose that \( \Upsilon(z) \) is of finite order \( \rho \) in \( \mathbb{D} \), then by Lemma 8 the function \( \Phi_{r-1} \) is of order of growth equal to \( \rho/(\rho+1) < 1 \) which is a contradiction. Hence, the order of growth of \( \Upsilon \) in \( \mathbb{D} \) is infinite.

The Wigert theorem can be applied to \( \Upsilon(z) \) showing that this function can be analytically continued to \( \mathbb{C} \setminus \{1\} \). By Lemma 8 \( \Upsilon(z) \) has an essential singularity of infinite order at \( z = 1 \).

\[ \square \]

4 Preliminary results and proof of Theorem C.

For proving Theorem C we will need the lemmas below which are analogs of Hadamard formulas connecting the growth of a function with its Taylor coefficients. We present the proofs of these lemmas in the last section of the paper.

**Lemma 10** Let \( F(z) = \sum_{n=0}^{\infty} C_n z^n \) be an entire transcendental function. Set

\[
\rho_0 = \limsup_{x \to \infty} \frac{\log \log M(x, F)}{\log \log x}
\]

and, for \( 1 < \rho_0 < \infty \), set

\[
\sigma_0 = \limsup_{x \to \infty} \frac{\log M(x, F)}{(\log x)^{\rho_0}}.
\]

Then

\[
\frac{\rho_0 \sigma_0}{(\rho_0 - 1)^{\rho_0 - 1}} = \limsup_{n \to \infty} \frac{n^{\rho_0}}{(\log(1/|C_n|))^{\rho_0 - 1}}.
\]
We say that the function \( F(z) \) of Lemma 10 is of logarithmic order \( \rho_0 \) and logarithmic type \( \sigma_0 \).

**Lemma 11** Let \( h(z) = \sum_{k=0}^{\infty} a_k z^k \) be a function analytic in \( \mathbb{D} \). Set

\[
\rho_0 = \limsup_{y \to 1^-} \frac{\log \log M(y, h)}{\log \log(1/(1-y))}.
\]

Let \( \rho_0 \) satisfy \( 1 \leq \rho_0 < \infty \), and for \( \rho_0 > 1 \) set

\[
\sigma_0 = \limsup_{y \to 1^-} \frac{\log M(y, h)}{(\log(1/(1-y)))^{\rho_0}}.
\]

Then

\[
\rho_0 = \limsup_{k \to \infty} \frac{\log^+ |a_k|}{\log \log k} \quad \text{and} \quad \sigma_0 = \limsup_{k \to \infty} \frac{\log^+ |a_k|}{(\log k)^{\rho_0}}.
\]

**Proof of Theorem C** Applying Lemma 11 with \( \rho(x) = \rho_0 \log \log x / \log x \), \( V(x) = x^{\rho(x)} \) we have that for any given \( r \in \mathbb{N} \) there exists an entire transcendental function

\[
F(z) = \sum_{n=0}^{\infty} C_n z^n
\]

of logarithmic order \( \rho_0 \) and of logarithmic type \( \sigma_0 \) belonging to \( PF_r \). Setting \( c_n = C_n, f(z) = F(z) \) and \( f_{r-1}(z) = F_{r-1}(z) = \sum_{n=0}^{\infty} C_n z^{n+r-1} / \Gamma(n+r) \) in Lemma 10 we obtain that the function

\[
G(z) = \sum_{k=0}^{\infty} f_{r-1}(k) z^k = \sum_{k=0}^{\infty} D_k z^k
\]

is a \( PF_r \) g. f.

Note that the logarithmic order and type of the entire functions \( F(z) \) and \( F_{r-1}(z) \) coincide. Indeed,

\[
\rho_0 = \limsup_{x \to \infty} \frac{\log \log M(x, F)}{\log \log x}.
\]

Therefore, by Lemma 10 we have

\[
\frac{\rho_0 - 1}{\rho_0} = \limsup_{n \to \infty} \frac{\log n}{\log \log(1/C_n)}.
\]
Noting that
\[
\liminf_{n \to \infty} \frac{\log(1/C_n)}{n^{1+\alpha}} \geq 1, \text{ for some } \alpha > 0,
\]
we can calculate
\[
\limsup_{n \to \infty} \frac{\log n}{\log \log(\Gamma(n + r)/C_n)} = \frac{\rho_0 - 1}{\rho_0}.
\]
Hence, applying Lemma 10,
\[
\rho_0 = \limsup_{x \to \infty} \frac{\log \log M(x, F_{r-1})}{\log \log x}.
\]
Analogously, the logarithmic types coincide because
\[
\limsup_{n \to \infty} \frac{n^{\rho_0}}{(\log(\Gamma(n + r)/C_n))^{\rho_0 - 1}} = \limsup_{n \to \infty} \frac{n^{\rho_0}}{(\log(1/C_n))^{\rho_0 - 1}}.
\]
Hence, the function \( F_{r-1}(z) \) is an entire function of order 0. By Wigert’s Theorem \( G(z) \) can be extended to \( \mathbb{C}\setminus\{1\} \).

Since the coefficients of \( F_{r-1}(z) \) are non-negative we have \( F_{r-1}(x) = M(x, F_{r-1}) \) for \( x > 0 \) and therefore, remembering that \( D_k = F_{r-1}(k) \) we can write
\[
\rho_0 = \limsup_{t \to \infty} \frac{\log \log F_{r-1}(t)}{\log \log t} = \limsup_{k \to \infty} \frac{\log \log D_k}{\log \log k}
\]
and
\[
\sigma_0 = \limsup_{t \to \infty} \frac{\log F_{r-1}(t)}{(\log t)^{\rho_0}} = \limsup_{k \to \infty} \frac{\log D_k}{(\log k)^{\rho_0}}, \text{ for } \rho_0 > 1.
\]
Applying Lemma 11 to \( G(z) \) we can assert that
\[
\limsup_{y \to 1^-} \frac{\log \log M(y, G)}{\log \log(1/(1 - y))} = \rho_0
\]
and
\[
\limsup_{y \to 1^-} \frac{\log M(y, h)}{(\log(1/(1 - y)))^{\rho_0}} = \sigma_0, \text{ for } \rho_0 > 1.
\]
Note that the point \( z = 1 \) must be an essential singularity of the function \( G \), since the entire function \( F_{r-1} \) interpolating \( G \)'s coefficients is transcendental. \( \square \)
Proofs of Lemmas 5, 10 and 11

Proof of Lemma 5

Let denote

\[ \eta = \limsup_{k \to \infty} \frac{\xi(k)}{k} \log |a_k|. \]

First we prove that \( \eta \leq (\rho + 1)(\sigma \rho)^{1/(\rho + 1)} / \rho \). If \( \sigma_h = +\infty \), then the inequality is trivial. Suppose \( \sigma < +\infty \) and \( \sigma > \sigma_h \). Then

\[ \log M(y, h) < \sigma V\left( \frac{1}{1 - y} \right), \text{ for } y_0 < y < 1, \]

which yields

\[ \log |a_k| < \sigma V\left( \frac{1}{1 - y} \right) + k \log \frac{1}{y}, \]

for \( y_0 < y < 1 \) and all \( k = 0, 1, 2, \ldots \).

Setting \( y = (\xi(k/(\sigma \rho)) - 1)/\xi(k/(\sigma \rho)) \) we obtain

\[ \log |a_k| < \sigma V\left( \frac{k}{\sigma \rho} \right) + k \log \left( \frac{\xi\left( \frac{k}{\sigma \rho} \right)}{\xi\left( \frac{k}{\sigma \rho} \right) - 1} \right) \]

for sufficiently large \( k \).

Remembering that \( \xi(t) \) is the inverse function of \( xV(x) \) we can write

\[ \log |a_k| < \frac{k}{\rho \xi\left( \frac{k}{\sigma \rho} \right)} + \frac{k}{\xi\left( \frac{k}{\sigma \rho} \right)} \{1 + o(1)\} \]

\[ = \frac{\rho + 1}{\rho} \frac{k}{\xi\left( \frac{k}{\sigma \rho} \right)} \{1 + o(1)\}, \quad k \to \infty. \]

By properties (a) and (d) of proximate orders (see p. 2) we have that \( \xi(t) = \tilde{V}(t) = t\tilde{\rho}(t), \tilde{\rho}(t) \to 1/(\rho + 1), \quad t \to \infty. \) Hence,

\[ \lim_{t \to \infty} \frac{\xi(\ell t)}{\xi(t)} = t^{1/(\rho + 1)} \]

and

\[ \xi\left( \frac{k}{\sigma \rho} \right) = \frac{\xi(k)}{(\sigma \rho)^{1/(\rho + 1)}} \{1 + o(1)\}, \quad k \to \infty. \]

Therefore, for an arbitrary \( \varepsilon > 0 \) we can assert that asymptotically

\[ \log |a_k| < \frac{\rho + 1}{\rho} (\sigma \rho)^{1/(\rho + 1)} \frac{k}{\xi(k)} (1 + \varepsilon) \]
and, thus,

\[ \eta = \limsup_{k \to \infty} \frac{\xi(k)}{k} \log |a_k| \leq \frac{(\rho + 1)}{\rho} (\sigma_h \rho)^{1/(\rho + 1)}. \]

Now we want to prove that \( \eta \geq (\rho + 1)(\sigma_h \rho)^{1/(\rho + 1)}/\rho \). If \( \eta = +\infty \), then the inequality is trivial. Suppose that \( \eta < +\infty \) and \( \eta = (\rho + 1)(\sigma_h \rho)^{1/(\rho + 1)}/\rho \) for some \( \sigma < \sigma_h \). We will obtain a contradiction. Choose \( \sigma_1, \sigma < \sigma_1 < \sigma_h \). Then

\[ \log |a_k| + k \log y < k \left\{ \frac{\rho + 1}{\rho} \left( (\sigma_1 \rho)^{1/(\rho + 1)} \right) \right\} + \log y \], \( k > k_0 \).

In the previous inequality we put

\[ k = \left[ \frac{\sigma_h \rho}{1 - y} V\left( \frac{1}{1 - y} \right) \right] \]

meaning \( k \) equal to the entire part of the number between parenthesis and assuming \( 1 - y \) so small that \( k > k_0 \). Remembering that \( \xi(t) \) is increasing and a regularly varying function (property (e)), for an arbitrary \( \varepsilon > 0 \) we can write the following inequality for \( k > k_0 \)

\[ \log |a_k| + k \log y < \left( \frac{\rho + 1}{\rho} \left( (\sigma_1 \rho)^{1/(\rho + 1)} \right) \right) \xi\left( \frac{1}{1 - y} V\left( \frac{1}{1 - y} \right) \right) - (1 - y) (1 + \varepsilon) \]

(remember that \( \xi(t) \) is the inverse function of \( xV(x) \)).

Since \( \sigma_1 / \sigma_h < 1 \), we have

\[ \log |a_k| + k \log y < \delta \sigma_h V\left( \frac{1}{1 - y} \right) (1 + \varepsilon), \]

for some \( \delta < 1 \), which yields

\[ \limsup_{y \to 1^-} \frac{\log \mu(y, h)}{V(1/(1 - y))} = \sigma_\mu < \sigma_h, \]

where \( \mu(y, h) = \max\{|a_k|y^k : k \in \mathbb{N} \cup \{0\} \} \).
On the other hand, for any \( y' > y \) the inequalities
\[
M(y, h) \leq \sum_{k=0}^{\infty} |a_k|(y')^k \left( \frac{y}{y'} \right)^k \leq \frac{y'}{y' - y} \mu(y', h)
\]
hold. Setting \( y' = 1 - s + sy \), where \( 0 < s < 1 \), is to be chosen later, we obtain
\[
M(y, h) \leq \frac{2}{(1 - s)(1 - y)} \mu(1 - s + sy, h). \tag{22}
\]
Since \( \rho > 0 \) and \( V(x) \) is a regularly varying function it follows that
\[
\sigma_h = \limsup_{y \to 1^-} \frac{\log M(y, h)}{V(1/((1 - y)))} \leq \limsup_{y \to 1^-} \frac{\log \mu(1 - s + sy, h)}{V(1/((1 - y)))} \leq \limsup_{y \to 1^-} \frac{\log \mu(1 - s + sy, h)}{V(1/((1 - s + sy)))} \lim_{y \to 1^-} \frac{V(1/(1 - s + sy))}{V(1/(1 - y))}
\]
\[
= \sigma_\mu \left( \frac{1}{s} \right)^\rho.
\]
Taking an \( s \), such that \( (\sigma_\mu/\sigma_h)^{1/\rho} < s < 1 \), we obtain \( \sigma_h \leq \sigma_\mu \), which shows that the inequality \( \sigma_\mu < \sigma_h \) obtained earlier, and hence \( \sigma < \sigma_h \), is impossible. \( \square \)

**Proof of Lemma 10** Let \( \omega > \sigma_0, \tau > \rho_0 \), then asymptotically
\[
\log M(x, F) < \omega(\log x)^\tau. \tag{23}
\]
Thus,
\[
\log |C_n| < \omega(\log x)^\tau - n \log x
\]
asymptotically with respect to \( x \) and for \( n = 0, 1, 2, \ldots \).

The usual method of finding extrema applied to the right side of the previous inequality shows that asymptotically
\[
\log |C_n| < \omega(1 - \tau) \left( \frac{n}{\omega \tau} \right)^{\tau/(\tau - 1)}. \tag{24}
\]
Conversely, assume that the asymptotic inequality \( \tag{24} \) holds. Then
\[
|C_n x^n| < K_1 \exp\{h(n)\}
\]
for \( n = 0, 1, 2, \ldots \), and \( x \geq 1 \), where \( K_1 = K_1(\omega, \tau) \) is a positive constant and

\[
h(n) = -Kn^\eta + n \log x, \quad K = \frac{\omega(\tau - 1)}{(\omega \tau)^{\tau/(\tau-1)}} > 0, \quad \eta = \frac{\tau}{\tau - 1}.
\]

We next analyze \( h(n) \) for real values of its argument. Since \( \eta > 1 \), the function \( h(n) \) attains its maximum equal to

\[
K(\eta - 1) \left( \frac{\log x}{K \eta} \right)^{\eta/(\eta-1)}
\]

at the point

\[
\tilde{n} = \left( \frac{\log x}{K \eta} \right)^{1/(\eta-1)}.
\]

Substituting \( K \) and \( \eta \) for their expressions in terms of \( \omega \) and \( \tau \) we obtain the asymptotic inequality

\[
\log \mu(x, F) \leq \omega (\log x)^\tau,
\]

where \( \mu(x, F) = \max\{|C_n|x^n : n \in \mathbb{N} \cup \{0\}|\} \).

On the other hand, for any \( x' > x > 0 \),

\[
M(x, F) \leq \sum_{n=0}^{\infty} |C_n|(x')^n \left( \frac{x}{x'} \right)^n \leq \frac{x'}{x'} \mu(x, F).
\]

Setting \( x' = 2x \) we obtain

\[
\log M(x, F) \leq \omega (\log 2x)^\tau + \log 2
\]

and

\[
\log M(x, F) < (\omega + \varepsilon)(\log x)^\tau
\] (25)

asymptotically for any arbitrary \( \varepsilon > 0 \).

Thus (24) follows from (23) and (25) from (24). This shows that the logarithmic order of the function \( F(z) \) is the infimum of the numbers \( \tau \) satisfying (24) and the logarithmic type of the same function is the infimum of the numbers \( \omega \) satisfying (24) for \( \tau \) equal to \( \rho_0 \). From this conclusion both assertions of the theorem follow at once. □

**Proof of Lemma 11** Let \( \omega > \sigma_0, \tau > \rho_0 \), then asymptotically

\[
\log M(y, h) < \omega \left( \log \frac{1}{1-y} \right)^\tau.
\] (26)
In the previous inequality, for $k$ large enough, we set $y = (k-1)/k$, obtaining asymptotically

$$\log |a_k| < (\omega + \varepsilon)(\log k)^\tau$$

for any arbitrary $\varepsilon > 0$.

Conversely, assume that the asymptotic inequality (27) holds. Then

$$|a_k| y^k < K_1 \exp\{h(k)\}$$

for $k = 1, 2, 3, \ldots$ and $y < 1$ close enough to 1, where $K_1$ is a positive constant and

$$h(k) = (\omega + \varepsilon)(\log k)^\tau + k \log y.$$

We next analyze $h(k)$ for real values of its argument $k \geq 1$. We have $\lim_{k \to \infty} h(k) = -\infty$. Also,

$$h'(k) = \frac{\tau(\omega + \varepsilon)(\log k)^{\tau-1}}{k} + \log y,$$

which means that for $y < 1$ close enough to 1 we have $h'(2) > 0$. On the other hand, $h'(k) \to \log y < 0, k \to \infty$. We can conclude that, for $y < 1$ close enough to 1, the function $h(k)$ attains its maximum on the interval $[2, \infty)$ at a point $\tilde{k}$ such that $h'(\tilde{k}) = 0$. It is easy to prove that $h'(\tilde{k}) = 0$ yields

$$\tilde{k} = \frac{\tau(\omega + \varepsilon)}{1-y} \left( \log \frac{1}{1-y} \right)^{\tau-1} (1 + o(1)), \ y \to 1^-,$$

and

$$h(\tilde{k}) = (\omega + \varepsilon) \left( \log \frac{1}{1-y} \right)^\tau (1 + o(1)), \ y \to 1^-.$$

Thus,

$$\mu(y, h) < \exp \left\{ (\omega + 2\varepsilon) \left( \log \frac{1}{1-y} \right)^\tau \right\}$$

for any arbitrary $\varepsilon > 0$ and $y$ close enough to $1^-$. Now using (22) and remembering that $\tau > 1$ we can write the following inequalities

$$M(y, h) \leq \frac{2}{(1-s)(1-y)} \exp \left\{ (\omega + 2\varepsilon) \left( \log \frac{1}{s(1-y)} \right)^\tau \right\}$$

and

$$M(y, h) \leq \exp \left\{ (\omega + \varepsilon_1) \left( \log \frac{1}{1-y} \right)^\tau \right\} \quad (28)$$

24
for any arbitrary $\varepsilon_1 > 0$ and $s$, $0 < s < 1$ and $y$ close enough to $1^-$.

Thus, (27) follows from (24) and (28) from (27). This shows that the order of the function $h(z)$ is the infimum of the numbers $\tau$ satisfying (27) and the type of this function is the infimum of the numbers $\omega$ satisfying (27) for $\tau$ equal to $\rho_0$. From this conclusion both assertions of the theorem follow at once. □

**Acknowledgements** I would like to express my gratitude to Prof. I.V. Ostrovskii for reading the manuscript of this paper and making valuable remarks. I would also like to thank Prof. O.M. Katkova and Prof. A.M. Vishnyakova for fruitful discussions on this topic.

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