On SNS-Riemannian connections in sub-Riemannian manifolds

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Abstract The authors define a SNS (semi-nearly-sub)-Riemannian connection on nearly sub-Riemannian manifolds and study the geometric properties of such a connection, and obtain the natures of horizontal curvature tensors between horizontal sub-Riemannian connection and SNS-Riemannian connection. The authors further investigate the geometric characteristics of the projective SNS-Riemannian connection, and obtain a necessary and sufficient condition for a nearly sub-Riemannian manifold being projectively flat.

Keywords Sub-Riemannian manifolds; Nearly sub-Riemannian manifolds; Semi-nearly-sub-Riemannian connections; Projective transformations; Carnot groups

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1 Introduction

It is well known that, in Riemannian spaces, the Levi-Civita connection is the unique connection with vanishing torsion tensor and compatible with the Riemannian metric. The other connections are not symmetric or compatible with the metric. The connection which is not symmetric was first introduced by A. Friedmann and J. A. Schouten [8] in a differential manifold in 1924. Later, K. Yano [17] in 1970 considered the non-symmetric connection which preserves the inner product of any two vector fields when they transport along a curve, it was called a semi-symmetric metric connection. He pointed out that a Riemannian manifold is conformally flat if and only if it admits a semi-symmetric metric connection whose curvature tensor vanishes identically. He also proved that a Riemannian manifold is of constant curvature if and only if it admits a semi-symmetric metric connection for which the manifold is a group manifold, where a group manifold is a differential manifold admitting a linear connection \( \tilde{\nabla} \) such that its curvature tensor \( \tilde{R} \) vanishes and the covariant derivative of torsion tensor \( \tilde{T} \) with respect to \( \tilde{\nabla} \) is vanished. Liang [15] discussed some properties of semi-symmetric metric connections and proved that

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the projective curvature tensor with respect to semi-symmetric metric connections coincides with the projective curvature tensor with respect to the Levi-civita connection if and only if the characteristic vector is proportional to a Riemannian metric. The problem of finding the invariants under some connection transformations is an important and active research topic. For instance, the authors in [10, 18, 20] discussed the corresponding invariants under some connection transformations. In particular, A. R. Gover and P. Nurowski [11] obtained the polynomial conformal invariants, and derived that some of the invariants have the practical significance in physics, such as quantum field theory [3], general relativity [2]. For the study of semi-symmetric metric connections, the authors had other interesting results [9, 12, 19, 21, 22]. Recently, the authors in paper [15] studied the theory of transformations on Carnot-Caratheodory spaces, and obtained the conformal invariants and projective invariants on Carnot-Caratheodory spaces, which is an attempt to develop the transformation theory in sub-Riemannian manifolds. As far as the connections being not compatible with the metric were concerned, we must point out that, it was N. S. Agache and M. R. Chafle [1] in 1990 who first introduced and discussed the so-called semi-symmetric non-metric connection on a Riemannian manifold. This semi-symmetric non-metric connection was further developed by U. C. De and S. C. Biswas [5], U. C. De and D. Kamily [6], M. M. Tripathi and N. Kakkar [16], U. C. De and J. Sengupta [7], and so on. The semi-symmetric non-metric sub-Riemannian connection on sub-Riemannian manifolds will be discussed in the forthcoming paper [14].

Taking into account that the sub-Riemannian manifolds are a natural generalization of Riemannian manifolds, a natural topic is to consider the invariance problems under some connection transformations from symmetric metric nonholonomic connections to semi-symmetric metric nonholonomic connections. Once we found the invariants under connection transformations, we could study the geometric and physical characteristics of an object connection through an original connection. In order to study the geometric properties of sub-Riemannian manifolds, the second author first discussed the transformations in Carnot-Caratheodory spaces, and got the sub-conformal invariants and sub-projective invariants, which can be regarded as a natural generalization of those conclusions in Riemannian manifolds. We in this paper wish to use the unique horizontal sub-Riemannian connection to solve the posed problems above. To the authors’ knowledge, the study of the semi-symmetric metric connection in sub-Riemannian manifolds is still a gap. The authors have made some attempts in this field and obtained some interesting results (see [13]).

According to the above discussions in [13], there are some differences between the Schouten curvature tensor $K_{ijkh}$ of sub-Riemannian connections and the Riemannian case, by contracting the index $j, h$ we can not obtain a symmetric tensor, neither is zero by contracting the index $k, h$. We only obtain

$$K_{ijkh} = -K_{ijhk} - M_{ij}^{\alpha} \Lambda_{\alpha k} g_{ih} - M_{ij}^{\alpha} \Lambda_{\alpha h} g_{ik} + M_{ij}^{\alpha} e_{\alpha} (e_{k}, e_{h}),$$

where $\{e_{\alpha} : \alpha = \ell + 1, \cdots, n\}$ is a basis of vertical distribution $VM$. Since the curvature tensor $K$ does not satisfy such a symmetry as the Riemannian case, so we can not give out the second Bianchi identity. And the sub-conformal curvature tensor is no longer an invariant under the connection transformation from sub-Riemannian connections.
to semi-sub-Riemannian connections. Very recently, A. Bejancu [4] defined the horizontal sub-Riemannian connection on nearly sub-Riemannian manifolds and proved that there exists a unique torsion-free and metric linear connection $\nabla$ on the nearly sub-Riemannian manifold. The curvature tensor fields for $\nabla$ has good symmetries as Levi-civita connection in the Riemannian case. That is to say, there holds

$$
\begin{align*}
K_{i jkh} &= -K_{i jhk} = -K_{ijkh}, \\
K_{ijkh} &= K_{khi j},
\end{align*}$$

and its first and second Bianchi identities are all true. So we can define the horizontal Ricci tensor field and horizontal scalar curvature, which enable us to prove the Schur-type theorem on a nearly sub-Riemannian manifold and to introduce the nearly space form in sub-Riemannian manifolds.

In this paper, we continue our study on semi-symmetric metric connections on sub-Riemannian manifolds. We will consider the connection transformation from a horizontal sub-Riemannian connection to a SNS-Riemannian connection on nearly sub-Riemannian manifolds. The previous discussions in [13] are also true for the horizontal sub-Riemannian connection $\nabla$. The horizontal sub-Riemannian connection in nearly sub-Riemannian manifold plays an important role as the Levi-Civita connection in the Riemannian case. We first define a SNS-Riemannian connection in a class of sub-Riemannian manifolds (i.e. nearly sub-Riemannian manifolds in this paper), and derive the relations of horizontal curvature tensors between the horizontal sub-Riemannian connection and the SNS-Riemannian connection, and get some invariants under the connection transformations that keep the normal geodesics unchanged. We further define the Weyl sub-conformal curvature tensor and the Weyl sub-projective curvature tensor of SNS-Riemannian connections and deduce a sufficient and necessary condition that a nearly sub-Riemannian manifold is flat with respect to SNS-Riemannian connection. At last we consider the projective SNS-Riemannian connections and the projective SNS-Riemannian connections satisfying some symmetric conditions (i.e. the special projective SNS-Riemannian connection), deduce that a nearly sub-Riemannian manifold is projectively flat if and only if there exists a special projective SNS-Riemannian transformation with vanishing horizontal curvature tensor. Some geometric characteristics of projective transformation are obtained.

The organization of this paper is as follows. In Section 2, we will recall and give the necessary information about horizontal sub-Riemannian connection and the nearly sub-Riemannian manifold. Section 3 is devoted to some new definitions and main Theorems about SNS-Riemannian connections. Section 4 is concentrated on the geometry of projective SNS-Riemannian connections. Some interesting examples are given in Section 5.

## 2 Preliminaries

Let $M^n$ be an $n$-dimensional smooth manifold. For each point $p \in M^n$, there assigns a $\ell (2 < \ell < n)$-dimensional subspace $HM_p$ of the tangent space $TM_p$, then
\( \mathcal{H} \mathcal{M} = \bigcup_{p \in \mathcal{M}} \mathcal{H} \mathcal{M}_p \) forms a tangent sub-bundle of tangent bundles \( \mathcal{T} \mathcal{M} = \bigcup_{p \in \mathcal{M}} \mathcal{T} \mathcal{M}_p \). Then a sub-Riemannian manifold is given by a triangle \((\mathcal{M}, \mathcal{H} \mathcal{M}, g)\), where \(g\) is called a sub-Riemannian metric. For any point \(p\), if there exist a neighborhood \(U\) and \(\ell\) linearly independent vector fields \(X_1, \ldots, X_\ell\) in \(U\) such that, for each point \(q \in U\), \(X_1(q), \ldots, X_\ell(q)\) is a basis of subspace \(\mathcal{H} \mathcal{M}_p\), then we call \(\mathcal{H} \mathcal{M}\) the \(\ell\)-dimensional smooth distribution (called also a horizontal bundle), and \(X_1, \ldots, X_\ell\) are called a local basis of \(V^c\) in \(U\).

Throughout the paper, we denote by \(F(\mathcal{M})\) the set of smooth functions on \(\mathcal{M}\), \(\Gamma(\mathcal{H} \mathcal{M})\) the \(C^\infty(\mathcal{M})\) -module of smooth sections on \(\mathcal{H} \mathcal{M}\), and \(X, Y, Z, \cdots\) the vector fields in \(\Gamma(\mathcal{T} \mathcal{M})\), \(X_2\) the projection of \(X\) on \(\mathcal{H} \mathcal{M}\), \(X_0\) the projection of \(X\) on \(\mathcal{V} \mathcal{M}\). The repeated indices with one upper index and one lower index indicate a summation over their range.

We define a 3-multilinear mapping by
\[
\Omega : \Gamma(\mathcal{H} \mathcal{M}) \times \Gamma(\mathcal{H} \mathcal{M}) \times \Gamma(\mathcal{V} \mathcal{M}) \to F(\mathcal{M}),
\]
\[
\Omega(X_2, Y_2, Z_2) = Z_2g(X_2, Y_2) - g([Z_2, X_2]_h, Y_2) - g([Z_2, Y_2]_h, X_2),
\]
It is easy to check \(\Omega\) is a tensor field by a direct computation.

**Definition 2.1.** We say that a sub-Riemannian manifold \((\mathcal{M}, \mathcal{H} \mathcal{M}, g)\) is a nearly sub-Riemannian manifold if the tensor field \(\Omega\) vanishes identically on \(\mathcal{M}\).

Just like Levi-Civita connection is the unique metric and torsion-free connection on Riemannian manifolds, there also exists a unique linear connection with vanishing torsion which is compatible with sub-Riemannian metric on nearly sub-Riemannian manifolds.

**Theorem 2.1.** \([4]\) Given a nearly sub-Riemannian manifold \((\mathcal{M}, \mathcal{H} \mathcal{M}, g)\), then there exists a unique linear connection satisfying
\[
(\nabla_YY)(X_2, Y_2) = Zg(X_2, Y_2) - g(\nabla_YYX_2, Y_2) - g(X_2, \nabla_YYY_2) = 0,
\]
\[
T(X, Y_2) = \nabla_XY_2 - \nabla_{Y_2}X_2 - [X, Y_2]_h = 0,
\]
\[
\tag{2.1}
\tag{2.2}
\]

**Proof.** For convenience, we give a brief proof of Theorem 2.1 here. For any vector fields \(X, Y, Z \in \mathcal{T} \mathcal{M}\), we define a linear connection \(\nabla : \Gamma(\mathcal{T} \mathcal{M}) \times \Gamma(\mathcal{H} \mathcal{M}) \to \Gamma(\mathcal{H} \mathcal{M})\) on the nearly sub-Riemannian manifold \((\mathcal{M}, \mathcal{H} \mathcal{M}, g)\) by
\[
2g(\nabla_{X_2}Y_2, Z_2) = X_2g(Y_2, Z_2) + Y_2g(Z_2, X_2) - Z_2g(X_2, Y_2) + g([X_2, Y_2]_h, Z_2) - g([Y_2, Z_2]_h, X_2) + g([Z_2, X_2]_h, Y_2),
\]
\[
\nabla_ZX_2 = [Z_2, X_2]_h.
\]
Then the connection defined by Equation above is the unique linear connection satisfying (2.1) and (2.2).

Its torsion is defined by
\[
T : \Gamma(\mathcal{T} \mathcal{M}) \times \Gamma(\mathcal{H} \mathcal{M}) \to \Gamma(\mathcal{H} \mathcal{M}),
\]
\[
T(X, Y_2) = \nabla_XY_2 - \nabla_{Y_2}X_2 - [X, Y_2]_h.
\]
By the proof of Theorem 2.1, we know

\[ T(X, Y) = \nabla_X Y - [X, Y] = 0, \]

so (2.2) is equivalent to

\[ T(X, Y) = \nabla_X Y - \nabla_Y X = 0. \tag{2.3} \]

**Remark 2.1.** Similar to Riemannian manifolds, we also say that the linear connection with property (2.1) and (2.2) (or (2.3)) is metric and torsion-free (or symmetric), respectively. A linear connection \( \nabla \) satisfying (2.1) and (2.2) (or (2.3)) is called a horizontal sub-Riemannian connection.

Next we consider the curvature tensor \( R \) of \( \nabla \):

\[ R : \Gamma(TM) \times \Gamma(TM) \times \Gamma(HM) \to \Gamma(HM), \]
\[ R(X, Y, Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - [X, Y] Z = 0, \]

Hereafter we call \( R \) the horizontal curvature tensor because of \( R(X, Y, Z) \in \Gamma(HM) \).

In order to study the geometry of \((M, HM, g)\), we suppose that there exists a Riemannian metric \( \langle \cdot, \cdot \rangle \) and \( VM \) is taken as the orthogonal complementary distribution to \( HM \) in \( TM \), then, there holds \( HM \oplus VM = TM \), we call \( VM \) the vertical distribution. If not stated otherwise, hereafter we suppose that \( VM \) is an integrable distribution on \( M \), and use the following ranges for indices: \( i, j, k, h, \ldots \in \{1, \ldots, \ell\}, \alpha, \beta, \ldots \in \{\ell + 1, \ldots, n\} \).

Now we consider the local coordinate \((x^i, x^\alpha)\), such that \((\partial/\partial x^i, \ldots, \partial/\partial x^\alpha)\) is a local basis for \( \Gamma(VM) \) and \( HM \) is locally given by the Pfaff system

\[ \delta x^\alpha = dx^\alpha + A^\alpha_i dx^i, \]

where \( A^\alpha_i \) are smooth functions locally defined on \( M \). Thus

\[ e_i = \frac{\partial}{\partial x^i} - A_i^\alpha \frac{\partial}{\partial x^\alpha}, \]

form a local basis of \( \Gamma(HM) \). We call \((x^i, x^\alpha)\) and \((e_i, \partial/\partial x^\alpha)\) an adapted coordinate system and an adapted frame field on \( M \), respectively, induced by the foliation determined by \( VM \). Then by a direct calculation, we arrive at

\[ [e_i, e_j] = e_j(A_i^\alpha) \frac{\partial}{\partial x^\alpha} - e_i(A_j^\beta) \frac{\partial}{\partial x^\beta} \in VM, \]

we further obtain

\[ \Omega(e_i, e_j, \frac{\partial}{\partial x^\alpha}) = \frac{\partial g_{ij}}{\partial x^\alpha}, \]

where \( g_{ij} = g(e_i, e_j) \) is the local component of Riemannian metric \( g \) on \( HM \), so the sub-Riemannian manifold \((M, HM, g)\) is a nearly sub-Riemannian manifold if and
only if \( \frac{\partial g_{ij}}{\partial x^k} = 0 \), namely, \( g_{ij} \) is independent of \( x^i \). We denote by

\[
\begin{align*}
\nabla_v e_j &= \nabla e_j = \{i\}^k e_k, \\
\nabla_x e_j &= \nabla_a e_j = \{a\}^k e_k, \\
R(e_v, e_j, e_k) &= R^h_{ijk} e_h, \\
R(\frac{\partial}{\partial x^a}, e_j, e_k) &= R^h_{ajk} e_h, \\
R(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}, e_k) &= R^h_{ajk} e_h.
\end{align*}
\]

According to Theorem 2.1, we obtain

\[
\begin{align*}
\left\{ \begin{array}{l}
\{i\}^k = \frac{1}{2} g^{kh} \left( \frac{\partial g_{ih}}{\partial x^k} + \frac{\partial g_{ih}}{\partial x^k} - \frac{\partial g_{ij}}{\partial x^k} \right), \\
\{a\}^k = 0,
\end{array} \right. 
\end{align*}
\]  \hspace{1cm} (2.4)

where \((g^{ij})\) is the inverse of \( \ell \times \ell \) matrix \((g_{ij})\). Then by using (2.4) and direct computation, we arrive at the horizontal curvature tensor

\[
\begin{align*}
\left\{ \begin{array}{l}
R^h_{ijk} = e_i(\{j\}^k_{jk}) - e_j(\{i\}^k_{jk}) + \{c\}^h_{ik} \{\ell\}_h - \{\ell\}^h_{ik} \{c\}_h, \\
R^h_{ajk} = 0, \\
R^h_{ajk} = 0.
\end{array} \right. 
\end{align*}
\]  \hspace{1cm} (2.5)

Denote by \( R_{ijh} = R^l_{ijk} g_{lh}, R_{ik} = R_{ijk} g_{jh} \), we further have

\[
\begin{align*}
\left\{ \begin{array}{l}
R_{ijk} = -R_{ijh} = -R_{jik}, \\
R_{ijk} = R_{bij}, \\
R_{ijk} + R_{jkh} + R_{kij} = 0, \\
R_{ik} = R_{ki}.
\end{array} \right. 
\end{align*}
\]  \hspace{1cm} (2.6)

**Remark 2.2.** We can not obtain a symmetric tensor by contracting the index \( j, h \) in the Schouten curvature tensor of any nonholonomic connection

\[
K^h_{ijk} = e_i(\{j\}^h_{jk}) - e_j(\{i\}^h_{jk}) + \{c\}^h_{ik} \{\ell\}_h - \{\ell\}^h_{ik} \{c\}_h - \Omega^h_{ijk} \Lambda^h_{hk},
\]

nor is zero by contracting the index \( k, h \), we only obtain (see [13, 18]):

\[
K_{ijk} = -K_{ijh} - M_{ij}^a \Lambda_{ak}^l g_{hk} - M_{ij}^a \Lambda_{ak}^l g_{ik} + M_{ij}^{\alpha} e_{\alpha} g(e_k, e_h),
\]

where \( \{e_i : i = 1, \cdots, \ell \} \) and \( \{e_{\alpha} : \alpha = \ell + 1, \cdots, n \} \) are any local basis vector fields of \( HM, VM \) respectively, and

\[
\begin{align*}
[e_r, e_j]_v &= M_{ij}^a e_{\alpha}, \\
[e_{\alpha}, e_k]_h &= \Lambda_{ak}^h e_h.
\end{align*}
\]
For the horizontal sub-Riemannian connection $\nabla$, its Schouten curvature tensor satisfies

$$K_{i j h} = -K_{i j k} - M_{i j}^a \Lambda^l_{a h} g_{l k} - M_{i j}^a \Lambda^l_{a b} g_{l k} + M_{i j}^a e_\alpha g(\nabla v_\alpha, e_h) + M_{i j}^a e_\alpha g(e_k, \nabla v_\alpha)$$

$$= -K_{i j k} - M_{i j}^a \Lambda^l_{a h} g_{l k} - M_{i j}^a \Lambda^l_{a b} g_{l k} + M_{i j}^a \Lambda^l_{a b} g_{l k} - M_{i j}^a \Lambda^l_{a h} g_{l k}$$

$$= -K_{i j k}.$$

On the other hand, (2.5) shows that the curvature tensor $R$ of $\nabla$ is not vanishing only for horizontal vector fields in $HM$, which coincides with the Shouten curvature tensor of the nonholonomic connection $\tilde{\nabla}$

$$2g(\tilde{\nabla}_{X_h} Y_h, Z_h) = X_h g(Y_h, Z_h) + Y_h g(Z_h, X_h) - Z_h g(X_h, Y_h) + g([X_h, Y_h], Z_h)$$

$$- g([Y_h, Z_h], X_h) + g([Z_h, X_h], Y_h).$$

This is one reason why we call $\nabla$ the horizontal sub-Riemannian connection.

### 3 The geometry of a SNS-Riemannian connection

The horizontal sub-Riemannian connection in nearly sub-Riemannian manifold plays an important role as the Levi-Civita connection in Riemannian case, and there exists a unique horizontal sub-Riemannian connection on nearly sub-Riemannian manifold. The other connections are generally not compatible with sub-Riemannian metric any more, nor is torsion free. A special kind of the metric non-symmetric linear connection is a semi-nearly-sub-Riemannian connection with non-vanishing torsion tensor which is compatible with sub-Riemannian metric any more, nor is torsion free. A special kind of the metric non-symmetric linear connection is a semi-nearly-sub-Riemannian connection on nearly sub-Riemannian manifold. As for the linear connections which are not compatible with sub-Riemannian metric will be discussed in our forthcoming papers. Roughly speaking, a semi-nearly-sub-Riemannian connection is a linear connection with non-vanishing torsion tensor which is compatible with sub-Riemannian metric. More precisely,

**Definition 3.1.** A linear connection $D : \Gamma(TM) \times \Gamma(HM) \to \Gamma(HM)$ with $D_{X_v} Y_h = [X_v, Y_h]_h$ is called a semi-nearly-sub-Riemannian connection (in briefly SNS-Riemannian connection), if it is metric and its torsion tensor satisfies

$$T(X, Y_h) = D_X Y_h - D_{Y_h} X_h - [X, Y_h]_h = \pi(Y_h) X_h - \pi(X_h) Y_h,$$

where $\pi$ is a smooth 1-form on HM.

For any vector field $X \in TM$, $X = X_h + X_v$, by Definition 3.1, we know

$$T(X_v, Y_h) = D_{X_v} Y_h - [X_v, Y_h]_h = 0,$$

we further have

$$T(X, Y_h) = T(X_h, Y_h) + T(X_v, Y_h) = T(X_h, Y_h),$$

then (3.1) is equivalent to

$$T(X_h, Y_h) = D_X Y_h - D_{Y_h} X_h - [X_h, Y_h]_h = \pi(Y_h) X_h - \pi(X_h) Y_h.$$
For a SNS-Riemannian connection $D$, one can take a recurrent $X, Y, Z \in \Gamma(TM)$ in (2.1), and gets in terms of $D_X Y_h = \nabla_X Y_h$ that there holds

$$D_X Y_h = \nabla_X Y_h + \pi(Y_h)X_h - g(X_h, Y_h)P, \quad (3.2)$$

where $P$ is a horizontal vector field defined by $g(P, X_h) = \pi(X_h)$.

**Remark 3.1.** (3.2) is also called a SNS-Riemannian transformation of $\nabla$. It is easy to check the SNS-Riemannian transformations of metric connections are still metric connections. This transformation will change a horizontal curve into a horizontal curve, however there is no any self-paralleling nature for the horizontal curves (i.e. normal geodesics). We will discuss the special connection transformations that keep the normal geodesics unchanged in the next section.

In an adapted frame field $\{e_i\}$, we denote by $\pi(e_i) = \pi_i$, $\pi^i = g^{ij}\pi_j$, then (3.2) can be rewritten

$$\Gamma^k_{ij} = (t^k_{ij}) + \delta^k_j\pi_i - \delta^k_i\pi_j + \pi^h_{ij}g_{hk} - \pi^h_i\pi^k - \pi^h_j\pi^k,$$

By a straightway computation, we can get the relations between the horizontal curvature tensors of $D$ and $\nabla$ as follows

$$R^k_{ijk} = K^k_{ijk} + \delta^k_j\pi_i - \delta^k_i\pi_j + \pi^h_{ij}g_{hk} - \pi^h_i\pi^k - \pi^h_j\pi^k,$$

where $K^h_{ijk}$ is the horizontal curvature tensor of the horizontal sub-Riemannian connection $\nabla$, and

$$\begin{align*}
\pi_{ik} &= \nabla_i\pi_k - \pi_i\pi_k + \frac{1}{2}g_{ik}\pi_h\pi^h, \\
\pi^i_j &= \pi_{ik}g^{jk} = \nabla_j\pi^i - \pi_i\pi^j + \frac{1}{2}\delta^j_i\pi_h\pi^k, \\
\nabla_i\pi_j &= e_i(\pi_j) - (c^l_{ij})\pi_l,
\end{align*}$$

Here we call $\pi_{ij}$ the characteristic tensor of $D$, and denote by $\alpha = \pi_{ij}g^{ij} = \pi^i_i$. Contracting $j$ and $h$ in (3.3), we have

$$R^i_k = K^i_k + (\ell - 2)\pi_{ik} + \alpha g_{ik}, \quad (3.4)$$

Multiplying (3.4) by $g^{ik}$ we get

$$R = K + 2(\ell - 1)\alpha,$$

so there is

$$\alpha = \frac{R - K}{2(\ell - 1)}, \quad (3.5)$$

Substituting (3.5) into (3.4) we have

$$\pi_{ik} = \frac{1}{\ell - 2}(R_{ik} - K_{ik} - \frac{R - K}{2(\ell - 1)}g_{ik}), \quad (3.6)$$
\[ \pi^i_t = \frac{1}{\ell - 2}\{(R^i_t - K^i_t) - \frac{R - K}{2(\ell - 1)}\delta^i_t\}, \tag{3.7} \]

then substituting (3.6), (3.7) into (3.3), we get

\[ R^i_{jk} = \frac{1}{\ell - 2}\{(R^i_j - R_{jk}) - \frac{R}{2(\ell - 1)}\delta^i_j\} \]

\[ + \frac{R}{(\ell - 1)(\ell - 2)}(g_{ik}\delta^i_j - g_{jk}\delta^i_k), \]

\[ = K^i_{jk} - \frac{1}{\ell - 2}\{(\delta^i_j K_{ik} - \delta^i_i K_{jk}) + \frac{K}{2(\ell - 1)}g_{ik}\delta^i_j - g_{jk}\delta^i_k\}, \tag{3.8} \]

In sub-Riemannian geometry, because of the existence of singular geodesics which does not satisfy the geodesic equation, we can define the projective transformation of a horizontal sub-Riemannian connection \( \nabla \) in the following way: A connection transformation \( D \) of a horizontal sub-Riemannian connection \( \nabla \) is a projective transformation if the linear connection \( D \) and \( \nabla \) have the same normal geodesics, and it is called a projective sub-Riemannian transformation. Therefore, the projective sub-Riemannian transformation of \( \nabla \) keeps the normal geodesics unchanged. The sub-conformal curvature tensor and sub-projective curvature tensor (see [18]) of \( \nabla \) are given, respectively, by

\[ \hat{C}^i_{jk} = K^i_{jk} - \frac{1}{\ell - 2}\{(\delta^i_j K_{ik} - \delta^i_i K_{jk}) + \frac{K}{2(\ell - 1)}g_{ik}\delta^i_j - g_{jk}\delta^i_k\}, \]

\[ \hat{W}^i_{jk} = K^i_{jk} - \frac{1}{\ell - 1}\{(\delta^i_j K_{ik} - \delta^i_i K_{jk})\}. \tag{3.9} \]

For the SNS-Riemannian connection \( D \), we similarly define the sub-conformal curvature tensor and the sub-projective curvature tensor, respectively, as

\[ C^i_{jk} = R^i_{jk} - \frac{1}{\ell - 2}\{(\delta^i_j R_{ik} - \delta^i_i R_{jk}) + \frac{R}{(\ell - 1)(\ell - 2)}(g_{ik}\delta^i_j - g_{jk}\delta^i_k)\}, \]

\[ W^i_{jk} = R^i_{jk} - \frac{1}{\ell - 1}(\delta^i_j R_{ik} - \delta^i_i R_{jk}), \tag{3.10} \]

Remark 3.2. By using (3.3) and (3.9) we get

\[ C^i_{jk} = \hat{C}^i_{jk}, \]

\[ W^i_{jk} = \hat{W}^i_{jk} + \frac{1}{\ell - 1}(\delta^i_j \pi^i_{jk} - \delta^i_i \pi^i_{jk}) + (g_{ik}\pi^i_{jk} - g_{jk}\pi^i_{ij}) \]

\[ - \frac{\alpha}{\ell - 1}(\delta^i_j g_{ik} - \delta^i_i g_{jk}). \tag{3.11} \]
hence (3.8) indicates the horizontal sub-Riemannian connection and SNS-Riemannian connection have the same sub-conformal curvature tensor.

Therefore we have the following

**Theorem 3.1.** If a nearly sub-Riemannian manifold admits a SNS-Riemannian connection, then the connection transformation from horizontal sub-Riemannian connection to SNS-Riemannian connection keeps the sub-conformal curvature tensor unchanged.

In the point of geometry, Theorem 3.1 shows that the connection transformation from a horizontal sub-Riemannian connection to an SNS-Riemannian connection always change a conformally flat (i.e. sub-conformal curvature tensor vanishing everywhere) nearly sub-Riemannian manifold into a conformally flat nearly sub-Riemannian manifold.

Now we assume that \( \hat{W}_{ijk}^h = W_{ijk}^h \), then there holds

\[
\frac{1}{\ell - 1} (\delta^h_i \pi_{jk} - \delta^h_j \pi_{ik}) + (g_{jk} \pi_{ih}^h - g_{jk} \pi_{ih}^h) - \frac{\alpha}{\ell - 1} (\delta^h_i g_{jk} - \delta^h_j g_{ik}) = 0
\] (3.12)

By multiplying \( g^{jk} \) in (3.12), we get

\[
\pi_{ih}^h = \frac{\alpha}{\ell} \delta^h_i, \text{ or } \pi_{ih} = \frac{\alpha}{\ell} g_{ih}.
\]

This implies the following

**Theorem 3.2.** The SNS-Riemannian transformation keeps the sub-projective curvature tensor unchanged if and only if the characteristic tensor of \( D \) is proportional to a metric tensor.

**Proof.** We just prove the sufficiency of Theorem 3.2. Let \( \pi_i^j = \lambda \delta_i^j \), then \( \alpha = \pi_i^j = \lambda \ell \), and \( \pi_{ij} = \lambda g_{ij} \). Substituting these equations above into (3.15), we get \( W_{ijk}^h = \hat{W}_{ijk}^h \). This ends the proof of Theorem 3.2. □

As we know a nearly sub-Riemannian manifold is projectively flat if and only if the sub-projective curvature tensor vanishes everywhere, hence Theorem 3.2 implies that a projectively flat nearly sub-Riemannian manifold will be changed into a projectively flat nearly sub-Riemannian manifold by the SNS-Riemannian transformation under certain condition.

We further assume \( R_{ijk}^h = K_{ijk}^h \), then

\[
\delta^h_i \pi_{jk} - \delta^h_j \pi_{ik} + \pi^h_{ih} g_{jk} - \pi_{ih}^h g_{jk} = 0,
\]

Contracting the above equation with \( i \) and \( h \), we arrive at

\[
(2 - \ell) \pi_{jk} - \alpha g_{jk} = 0,
\] (3.13)

Multiplying the equation (3.13) by \( g^{jk} \), we obtain

\[
2(\ell - 1) \alpha = 0,
\]

and \( \ell > 2 \), therefore \( \alpha = 0 \); the converse is also true, thus we have
Theorem 3.3. The SNS-Riemannian connection $D$ and the horizontal sub-Riemannian connection $\nabla$ have the same horizontal curvature tensor if and only if $\alpha$ is vanishing.

A geometric characteristic of Theorem 3.3 is that the connection transformation from the horizontal sub-Riemannian connection to the SNS-Riemannian connection keeps the horizontal curvature tensor unchanged under certain condition, which implies that the SNS-Riemannian transformation can change a flat nearly sub-Riemannian manifold (i.e. $R^h_{ijk} = 0$ at every point of $M$) into a flat nearly sub-Riemannian manifold under certain condition.

Now we consider the case of $R^h_{ijk} = 0$, that is, there holds

$$K^h_{ijk} = \delta^j_i \pi^h_{jk} - \delta^h_i \pi^j_{ik} + \pi^h_{ik} g_{jk} - \pi^h_{jk} g_{ik},$$  \hfill (3.14)

let $j = h$, we obtain

$$K_{ik} = (2 - \ell) \pi_{ik} - \alpha g_{ik},$$  \hfill (3.15)

Multiplying the equation (3.15) by $g^{ik}$, we get

$$K = K_{ik} g^{ik} = 2(1 - \ell) \alpha,$$

So we have

$$\alpha = \frac{K}{2(1 - \ell)},$$  \hfill (3.16)

Substituting (3.16) into (3.15), we obtain

$$\pi_{ik} = \frac{1}{2 - \ell} (K_{ik} - \frac{K}{2(\ell - 1)} g_{ik}),$$  \hfill (3.17)

Similarly, substituting (3.17) into (3.14), we have

$$K^h_{ijk} = \frac{1}{\ell - 2} (\delta^h_j K_{ik} - \delta^h_i K_{jk} + g_{ik} K^h_j - g_{jk} K^h_i) - \frac{K}{(\ell - 1)(\ell - 2)} (g_{ik} \delta^h_j - g_{jk} \delta^h_i),$$  \hfill (3.18)

Equation (3.18) implies $\hat{C}^h_{ijk} = 0$.

Theorem 3.4. The nearly sub-Riemannian manifold $(M, \mathcal{H}M, g)$ is flat with respect to the SNS-Riemannian connection $D$ (i.e. $R^h_{ijk} = 0$) if and only if $M$ is conformally flat (i.e. $\hat{C}^h_{ijk} = 0$) and $\pi_{ik} = \frac{1}{2 - \ell} (K_{ik} - \frac{K}{2(\ell - 1)} g_{ik})$.

Proof. The necessity is obvious. Conversely, if $\pi_{ik} = \frac{1}{2 - \ell} (K_{ik} - \frac{K}{2(\ell - 1)} g_{ik})$, then $\alpha = \frac{K}{2(1 - \ell)}$, so $K_{ik} = (2 - \ell) \pi_{ik} - \alpha g_{ik}$, and

$$R_{ik} = K_{ik} + (\ell - 2) \pi_{ik} + \alpha g_{ik} = 0,$$  \hfill (3.19)

$$R = g^{ik} R_{ik} = 0,$$  \hfill (3.20)
Substituting (3.19), (3.20) into the equation (3.9),
\[
C^h_{ijk} = R^h_{ijk} - \frac{1}{\ell - 2} \{ \delta^h_j R^i_{jk} - \delta^h_i R^j_{ik} + g_{jk} R^h_i - g_{jk} R^h_i \}
+ \frac{R}{(\ell - 1)(\ell - 2)}(g_{jk} \delta^h_j - g_{jk} \delta^h_i)
= R^h_{ijk},
\]
we arrive at \( R^h_{ijk} = C^h_{ijk} = \hat{C}^h_{ijk} = 0 \).
This completes the proof of Theorem 3.4. \(\square\)

Next we consider the SNS-Riemannian connections satisfying some symmetric conditions.

**Definition 3.2.** A SNS-Riemannian connection \( D \) is called special if its horizontal curvature tensor satisfies
\[
\begin{align*}
R^h_{ijk} + R^h_{ikj} &= 0, \\
R^h_{ijk} - R^h_{kji} &= 0, \\
R^h_{ijk} - R^h_{kji} + R^h_{ki} &= 0.
\end{align*}
\]
(3.21)

The first equation is obvious for \( D \), by (3.3), from the second formula we can deduce
\[
g_{jk}(\pi_{ik} - \pi_{ki}) + g_{jk}(\pi_{kj} - \pi_{jk}) + g_{ik}(\pi_{jk} - \pi_{kj}) + g_{jk}(\pi_{ih} - \pi_{hi}) = 0.
\]
Contracting the equation above by \( g^h_j \), we get \( \pi_{jk} = \pi_{kj} \), then \( \nabla_k \pi_j = \nabla_j \pi_k \) (It can be also deduced by the third formula in (3.21)). It is not hard to see by a direct checking up on a few things that the converse is also true, hence we obtain

**Proposition 3.5.** A SNS-Riemannian connection \( D \) is special if and only if its characteristic vector field satisfies \( \nabla_k \pi_j = \nabla_j \pi_k \).

Let \( E_p \subseteq HM_p \) be the horizontal plane at a point \( p \in M \) with dimension 2, then for any basis \( \{u, v\} \) in \( E_p \), the horizontal sectional curvature is defined by
\[
\lambda(p) = \frac{R(u, v, u, v)}{g(u, u)g(v, v) - (g(u, v))^2}.
\]
(3.22)

If \( u = u^i e_i, v = v^i e_i \), then
\[
\lambda(p) = \frac{R_{ijkl} u^i v^j u^k v^h}{(g_{ij} g_{jk} - g_{jk} g_{ij}) u^i v^j u^k v^h}.
\]
We say \( (M, HM, g) \) is horizontal isotropic if the horizontal sectional curvature \( \lambda(p) \) is independent of the horizontal plane \( E_p \) for all \( p \in M \). \( M \) is said to be of constant horizontal curvature if \( \lambda(p) \) is always constant, namely it is always independent of the point \( p \) and the horizontal plane \( E_p \).
Another geometric characteristic of Theorem 3.3 shows that the connection transformation from the horizontal sub-Riemannian connection to the SNS-Riemannian connection can change a constant horizontal curvature sub-Riemannian manifold into a constant horizontal curvature sub-Riemannian manifold under certain condition.

**Proposition 3.6.** A nearly sub-Riemannian manifold associated with the SNS-Riemannian connection $D$ is horizontal isotropic if and only if its horizontal curvature tensor is given by

$$R_{ijkh} = \lambda (g_{ih} g_{jk} - g_{jh} g_{ik}),$$

where $\lambda$ is function on $M$.

The proof is obvious.

**Theorem 3.7.** A nearly sub-Riemannian manifold $(M, HM, g)$ associated with the SNS-Riemannian connection $D$ is horizontal isotropic if and only if $M$ is projectively flat with respect to $D$, i.e. $W_{ijk}^h = 0$.

**Proof.** If $M$ is horizontal isotropic with respect to $D$, by Proposition 3.6, the horizontal curvature tensor is of the form

$$R_{ijkh} = \lambda (g_{ih} g_{jk} - g_{jh} g_{ik}).$$

Contracting the above equation with $g^{jh}$, we have

$$R_{ik} = \lambda (1 - \ell) g_{ik},$$

then $W_{ijk}^h = 0$.

Conversely, we assume $W_{ijk}^h = 0$, then there holds

$$\hat{W}_{ijkh} = -\frac{1}{\ell - 1} (g_{jh} \pi_{ik} - g_{ih} \pi_{jk}) - (g_{ik} \pi_{jh} - g_{jk} \pi_{ih}) + \frac{\alpha}{\ell - 1} (g_{jh} g_{ik} - g_{ih} g_{jk}).$$

Contracting the above equation by $g^{kh}$, we arrive at

$$\hat{W}_{ijkh} g^{kh} = \frac{\ell - 2}{\ell - 1} (\pi_{ij} - \pi_{ji}).$$

On the other hand,

$$\hat{W}_{ijkh} g^{kh} = K_{ijkh} g^{kh} - \frac{1}{\ell - 1} (g_{jh} K_{ik} - g_{ih} K_{jk}) g^{kh} = 0,$$

so we get $\pi_{ij} = \pi_{ji}$, and we have $D$ is a special SNS-Riemannian connection. Further, the first formula in (3.21) implies $R_{ijkh} = 0$, by (3.10) we obtain $g_{ik} R_{jh} = R_{ik} g_{jh}$, namely, $R_{ik} = \frac{R_{ik}}{g_{ik}}$, let $\lambda = \frac{R_{ik}}{g_{ik}}$, we obtain

$$R_{ijkh} = \frac{\lambda}{\ell - 1} (g_{jh} g_{ik} - g_{ih} g_{jk}).$$

This completes the proof of Theorem 3.7. □
We have another interesting result about the horizontal sub-Riemannian connection $\nabla$ and SNS-Riemannian connection $D$.

**Theorem 3.8.** A nearly sub-Riemannian manifold $(M, HM, g)$ is projectively flat with respect to the SNS-Riemannian connection $D$ if and only if $M$ is conformally flat, and

$$(\ell - 2)\pi_{ij} - \frac{K + (\ell - 2)\alpha}{\ell} g_{ij} + K_{ij} = 0.$$  \hfill (3.23)

**Proof.** If $M$ is projectively flat with respect to $D$, by Theorem 3.7 and Proposition 3.6, we have

$$R_{ijkh} = \lambda (g_{ih} R_{jk} - g_{jh} R_{ik}).$$

Contracting by $g^{ih}$, we arrive at

$$R_{ik} = \lambda (1 - \ell) g_{ik}, \quad (3.24)$$

$$R = \lambda \ell (1 - \ell), \quad (3.25)$$

Therefore,

$$\hat{C}_{ijkh} = C_{ijkh} = R_{ijkh} - \frac{1}{\ell - 2} \{ g_{jh} R_{ik} - g_{ih} R_{jk} + g_{ik} R_{jh} - g_{jk} R_{ih} \}
+ \frac{R}{(\ell - 1)(\ell - 2)} (g_{ih} g_{jk} - g_{jh} g_{ik})
= R_{ijkh} - 2\lambda \frac{(1 - \ell)}{\ell - 2} (g_{ih} g_{jk} - g_{jh} g_{ik})
+ \frac{\lambda \ell (1 - \ell)}{(\ell - 2)(\ell - 1)} (g_{ih} g_{jk} - g_{jh} g_{ik})
= 0,$$

by substituting (3.24), (3.25) into (3.4), we further arrive at,

$$(\ell - 2)\pi_{ik} - \frac{K + (\ell - 2)\alpha}{\ell} g_{ik} + K_{ik} = 0.$$  \hfill (3.26)

Conversely, if $\hat{C}_{ijkh} = 0$, then $C_{ijkh} = 0$. By (3.4) and (3.23) we have

$$R_{ik} = \frac{R}{\ell} g_{ik}, \quad (3.26)$$

substituting (3.26) into the equation (3.9), we obtain

$$C_{ijk} = \frac{R_{ijk}}{\ell} - \frac{1}{\ell - 1} (\delta_{ij}^h R_{hk} - \delta_{ik}^h R_{jh})
= W_{ijk}^h.$$  

Therefore, $C_{ijk} = 0$ deduce $W_{ijk}^h = 0$.

This completes the proof of Theorem 3.8. \qed
4 The projective geometry of SNS-Riemannian connections

A linear connection is called the projective SNS-Riemannian transformation if it is both the projective sub-Riemannian transformation and the SNS-Riemannian transformation. Recall the second author [18] had proved that a nonholonomic symmetric connection $\bar{\Gamma}$ is a projective sub-Riemannian connection if and only if there exists a smooth 1-form $\varphi$ such that, for any horizontal vector fields $X_h, Y_h$, there holds

$$\bar{\nabla}_{X_h} Y_h = \nabla_{X_h} Y_h + \varphi(X_h)Y_h + \varphi(Y_h)X_h, \tag{4.1}$$

From the definition of SNS-Riemannian transformation, we know that a projective SNS-Riemannian transformation must be a projective sub-Riemannian transformation in [18]. Now we give a Proposition about the projective transformation of horizontal sub-Riemannian connection $\nabla$.

**Proposition 4.1.** $D$ is a projective sub-Riemannian transformation if and only if there exists 1-form $\lambda$ such that the symmetry part of tensor $A(X_h, Y_h) = D_{X_h} Y_h - \nabla_{X_h} Y_h$ is of the form

$$(A(X_h, Y_h) + A(Y_h, X_h))/2 = \lambda(X_h)Y_h + \lambda(Y_h)X_h, \text{ for } X, Y \in TM$$

*Proof.* The necessity is obvious. We only prove the sufficiency. If there holds

$$(A(X_h, Y_h) + A(Y_h, X_h))/2 = \lambda(X_h)Y_h + \lambda(Y_h)X_h, \tag{4.2}$$

we denote by $(D_{X_h} Y_h + D_{Y_h} X_h)/2 = \bar{\nabla}_{X_h} Y_h, (\nabla_{X_h} Y_h + \nabla_{Y_h} X_h)/2 = \tilde{\nabla}_{X_h} Y_h$, then (4.2) is equivalent to

$$\bar{\nabla}_{X_h} Y_h - \tilde{\nabla}_{X_h} Y_h = \lambda(X_h)Y_h + \lambda(Y_h)X_h,$$

by (4.1), $\bar{\nabla}$ and $\tilde{\nabla}$ have the same normal geodesics.

On the other hand, if the horizontal curve $\gamma(t)$ is the normal geodesic of $D$, then $\gamma(t)$ is also the normal geodesic of $\bar{\nabla}$ by a simply computation, hence $\bar{\nabla}$ and $D$ have also the same normal geodesics, so do $\tilde{\nabla}$ and $\nabla$. Therefore, $D$ and $\nabla$ have the normal geodesics, namely, $D$ is the projective transformation of $\nabla$. \hfill \square

About the projective SNS-Riemannian transformation, we have the following Proposition.

**Proposition 4.2.** $\bar{\nabla}$ is the projective SNS-Riemannian transformation if and only if there exist two 1-form $p, q$ such that

$$\bar{\nabla}_{X_h} Y_h = \nabla_{X_h} Y_h + p(X_h)Y_h + q(Y_h)X_h, \tag{4.3}$$

for any $X, Y \in \Gamma(TM)$.

*Proof.* Let $A(X_h, Y_h) = \bar{\nabla}_{X_h} Y_h - \nabla_{X_h} Y_h$, since $\bar{\nabla}$ is the projective SNS-Riemannian connection, from Proposition 4.1 there exists a smooth 1-form $\varphi$ such that

$$A(X_h, Y_h) + A(Y_h, X_h)/2 = \varphi(X_h)Y_h + \varphi(Y_h)X_h, \text{ for } X, Y \in TM \tag{4.4}$$
and 1-form \( \pi \) such that the torsion of \( \tilde{D} \) is of the form
\[
\tilde{T}(X_h, Y_h) = \pi(Y_h)X_h - \pi(X_h)Y_h,
\]
we can deduce from the above equation
\[
A(X_h, Y_h) - A(Y_h, X_h) = \pi(Y_h)X_h - \pi(X_h)Y_h
\]
by \((4.4)\) and \((4.5)\), we arrive at
\[
\text{and 1-form } \pi
\]
Let the horizontal curvature tensor \( \tilde{\rho} \) and sub-projective curvature tensor are given, respectively, where
\[
\text{Ricci tensor and sub-projective curvature tensor are given , respectively , where}
\]
Conversely, we assume \( \tilde{\rho} \)
\[
\text{This completes the proof of Proposition 4.2.} \quad \square
\]
In a basis \( \{ e_i \} \), \((4.3)\) can be rewritten as
\[
\tilde{\Gamma}_{ij}^k = \{ e_i \} + p_i \delta_{j}^k + q_j \delta_{i}^k = \{ e_i \} + \varphi_i \delta_{j}^k + \varphi_j \delta_{i}^k + \rho_j \delta_{i}^k - \rho_i \delta_{j}^k,
\]
where \( \rho_i = \pi_i/2, p_i = \varphi_i - \rho_i, q_i = \varphi_i + \rho_i \). The horizontal curvature tensor, horizontal Ricci tensor and sub-projective curvature tensor are given, respectively,
\[
\tilde{R}_{ijk}^h = K_{ijk}^h + \beta_{ij} \delta_{k}^h + \alpha_{ik} \delta_{j}^h - \alpha_{jk} \delta_{i}^h,
\]
\[
\tilde{R}_{jk} = K_{jk} + \beta_{jk} + (\ell - 1) \alpha_{jk},
\]
\[
\tilde{W}_{ijk}^h = \tilde{R}_{ijk}^h - \frac{1}{\ell - 1}(\delta_{j}^h \tilde{R}_{ik} - \delta_{i}^h \tilde{R}_{jk}),
\]
where
\[
\begin{align*}
\beta_{ij} &= (\nabla_i p)(e_j) - (\nabla_j p)(e_i) = \varphi_{ij} - \varphi_{ji} + \rho_{ij} - \rho_{ji}, \\
\alpha_{ij} &= (\nabla_i q)(e_j) - q(e_i)q(e_j) = \varphi_{ij} + \rho_{ij} - \varphi_{j} \rho_{j} - \varphi_{i} \rho_{i}, \\
\varphi_{ij} &= e_i(\varphi_j) - \{ e_i \} \varphi_j - \varphi_i \varphi_j = \nabla_i \varphi_j - \varphi_i \varphi_j, \\
\rho_{ij} &= e_i(\rho_j) - \{ e_i \} \rho_j - \rho_i \rho_j = \nabla_i \rho_j - \rho_i \rho_j,
\end{align*}
\]
If the horizontal curvature tensor \( \tilde{R} \) of \( \tilde{D} \) satisfies
\[
\begin{align*}
\tilde{R}_{ij} + \tilde{R}_{jik} &= 0, \\
\tilde{R}_{ijk} - \tilde{R}_{kij} &= 0, \\
\tilde{R}_{ijk} + \tilde{R}_{jki} + R_{kij} &= 0,
\end{align*}
\]
then we call $\tilde{D}$ the special projective SNS-Riemannian transformation. By substituting (4.6) into the third term in (4.9), we obtain $\rho_{ij} - \rho_{ji} = 0$, by virtue of the second term in (4.9) and $\tilde{R}_{jihk} + \tilde{R}_{khji} = 0$, we arrive at $\phi_{ij} - \phi_{ji} = 0$, namely, $\nabla_i \phi_j - \nabla_j \phi_i = 0, \frac{\alpha_{jk}}{g_{1}} = \lambda \alpha_{jk}, \text{where } \lambda \text{ is a scalar function.}$

**Lemma 4.3.** $\tilde{D}$ is a special projective SNS-Riemannian transformation if and only if $\nabla_i \phi_j = \nabla_j \phi_i, \nabla_i \rho_j = \nabla_j \rho_i$, $\alpha_{jk} = \lambda g_{jk}$, where $\lambda$ is a scalar function.

The connection transformation from the horizontal sub-Riemannian connection $\nabla$ to the special projective SNS-Riemannian connection $\tilde{D}$ keeps both normal geodesics and the sub-projective curvature tensor $\tilde{W}_{ijk}$ unchanged by a simple computation.

**Theorem 4.4.** The sub-projective curvature tensor is an invariant under special projective SNS-Riemannian transformation of the horizontal sub-Riemannian connection $\nabla$.

The proof of Theorem 4.4 is obtained by Lemma 4.3 and Equation (4.6) and (4.7).

In virtue of the Theorem 4.4 above, we deduce that the special projective SNS-Riemannian transformation change always a projectively flat nearly sub-Riemannian manifold into a projectively flat nearly sub-Riemannian manifold.

**Definition 4.1.** If 1-form $p$ satisfies
$$dp(X_h, Y_h) = X_h(p(Y_h)) - Y_h(p(X_h)) - p([X_h, Y_h])_h = 0,$$
for any horizontal vector fields $X_h, Y_h$, then we say it is horizontally closed.

**Lemma 4.5.** If the tensor $\alpha_{ij}$ defined by the second formula in (4.8) is propositional to $g_{ij}$, then the projective SNS-Riemannian connection $D$ is special if and only if 1-form $p, q$ are both horizontally closed in Equation (4.3).

**Proof.** The proof of Lemma 4.5 is obtained by Lemma 4.3 and Equation (4.9). □

We say a nearly sub-Riemannian manifold associated with a projective SNS-Riemannian connection is horizontal isotropic if its horizontal sectional curvature (3.22) is independent of the horizontal plane. Similar to Proposition 3.6, we have the following

**Proposition 4.6.** A nearly sub-Riemannian manifold $M$ associated with a projective SNS-Riemannian connection $\tilde{D}$ is horizontal isotropic if and only if the horizontal curvature tensor of $\tilde{D}$ is of the form
$$\tilde{R}_{ijkh} = \lambda (g_{ih} g_{jk} - g_{ih} g_{jk}).$$

Proposition 4.6 implies that if a nearly sub-Riemannian manifold is horizontal isotropic with respect to the projective SNS-Riemannian connection $\tilde{D}$, then $\tilde{D}$ is a special projective SNS-Riemannian connection by a direct computation.

Let $\tilde{D}$ be the special projective SNS-Riemannian connection, and the horizontal curvature tensor of $\tilde{D}$ be of the form
$$\tilde{R}_{ijkh} = \lambda (g_{ih} g_{jk} - g_{ih} g_{jk}).$$
then there holds
\[ \tilde{R}_{ik} = \lambda (1 - \ell) g_{ik}. \]
Substituting the equation above into (4.7), we arrive at \( \tilde{W}^h_{ijk} = 0. \)

Conversely, if \( \tilde{W}^h_{ijk} = 0, \) then we get
\[ \tilde{R}^h_{ijk} = \frac{1}{\ell - 1} (\delta^h_j \tilde{R}^i_k - \delta^h_i \tilde{R}^j_k), \]
namely,
\[ \tilde{R}^h_{ijk} = \frac{1}{\ell - 1} (g_{jh} \tilde{R}^i_k - g_{ih} \tilde{R}^j_k), \] (4.10)

Let \( k = h \) in (4.10), we obtain \( g_{jk} \tilde{R}^h_k - g_{ik} \tilde{R}^h_k = 0 \) because of \( \tilde{R}^h_{ikk} = 0 \) in (4.9), namely, \( \tilde{R}^h_{ik} = \beta g_{ih}, \) where \( \beta \) is scalar function. Then, (4.10) is equivalent to
\[ \tilde{R}^h_{ijk} = \frac{\beta}{\ell - 1} (g_{jh} g_{ik} - g_{ih} g_{jk}). \]

This proves the following Theorem.

**Theorem 4.7.** A nearly sub-Riemannian manifold \((M, HM, g)\) associated with a special projective SNS-Riemannian connection \( \tilde{D} \) is horizontal isotropic if and only if \( M \) is projectively flat with respect to \( \tilde{D} \).

**Theorem 4.8.** A nearly sub-Riemannian manifold \((M, HM, g)\) is projectively flat if and only if there exists a special projective SNS-Riemannian transformation \( \tilde{D} \) such that its horizontal curvature tensor \( \tilde{R} \) is vanishing.

**Proof.** If \( \tilde{D} \) is the special projective SNS-Riemannian transformation and
\[ \tilde{R}^h_{ijk} = K^h_{ijk} + \beta_{ij} \delta^h_k + \alpha_{ik} \delta^h_j - \alpha_{jk} \delta^h_i = 0, \] (4.11)
then by contracting (4.11) with \( j, h \), we have
\[ \tilde{R}^h_{ik} = K^h_{ik} + \beta_{ik} + (\ell - 1)\alpha_{ik} = 0, \]

Since \( \tilde{D} \) is special, then 1-form \( p \) is horizontally closed, hence we get \( \beta_{ij} = 0, \) and
\[ K^h_{ijk} = \alpha_{jk} \delta^h_i - \alpha_{ik} \delta^h_j, \] (4.12)
\[ K^h_{ik} = (1 - \ell)\alpha_{ik}, \] (4.13)

By substituting (4.12), (4.13) into the following equation
\[ \tilde{W}^h_{ijk} = K^h_{ijk} - \frac{1}{\ell - 1} (\delta^h_j K^h_{ik} - \delta^h_i K^h_{jk}), \]
we obtain \( \tilde{W}^h_{ijk} = 0, \) namely, \( M \) is projectively flat.
Conversely, if $M$ is projectively flat, then $\hat{W}^h_{ijk} = 0$, and

\[ K^h_{ijk} = \frac{1}{\ell - 1}(\delta^h_j K_{ik} - \delta^h_i K_{jk}), \]

namely,

\[ K_{ijk} = \frac{1}{\ell - 1}(g_{jk} K_{ik} - g_{ik} K_{jk}). \]

Since $K_{ijkh} = 0$, we get

\[ K_{ik} = \frac{K}{\ell} g_{ik}, \tag{4.14} \]

If 1-form $p$ is horizontally closed, then the equation $\hat{R}_{ij} = K_{ij} + \beta_{ij} + (\ell - 1)\alpha_{ij} = 0$ is equivalent to

\[ (\nabla_i q)(e_j) - q_i q_j = \frac{K}{\ell(1 - \ell)} g_{ij}, \tag{4.15} \]

where $q_{ij} = (\nabla_i q)(e_j) - q_i q_j = \alpha_{ij}$.

Now taking covariant derivative of Equation (4.15), we get

\[
(\nabla_i \nabla_j q)(e_k) + (\nabla_i q)(\nabla_j e_k) - (\nabla_i q)(e_j)q(e_k) - q(\nabla_i e_j)q(e_k) - q(e_j)(\nabla_i q)(e_k) - q(\nabla_i e_k)q(e_j) \\
= \frac{K}{\ell(1 - \ell)}(g(\nabla_i e_j, e_k) + g(e_j, \nabla_i e_k)) \\
= (\nabla_i \nabla_j q)(e_k) - q(\nabla_i e_j)q(e_k) + (\nabla_i q)(\nabla_j e_k) - q(\nabla_i e_k)q(e_j).
\]

The last equality follows from Equation (4.15). Namely,

\[ (\nabla_i \nabla_j q)(e_k) - (\nabla_i q)(e_j)q(e_k) - q(e_j)(\nabla_i q)(e_k) = (\nabla_i \nabla_j q)(e_k), \tag{4.16} \]

Since $p$ is horizontally closed, then by (4.15), (4.16) and $\hat{W}^h_{ijk} = 0$, we obtain

\[ (\nabla_i \nabla_j q - \nabla_j \nabla_i q - \nabla_{[i, e_j]} q)(e_k) = -K^h_{ijk} q_{ij}, \tag{4.17} \]

Equation (4.17) is exactly the integrable condition of Equation (4.15), therefore there exists a solution $q$ to Equation (4.15), let

\[ \hat{\Gamma}^k_{ij} = \{^k_{ij}\} + p_i \delta^k_j + q_j \delta^k_i, \tag{4.18} \]

where $p$ is a horizontally closed 1-form.

By Proposition 4.12 we know $\hat{D}$ whose connection coefficients are defined by (4.18) is a projective SNS-Riemannian connection. On the other hand, $\alpha_{ij}$ is proportional to $g_{ij}$ by (4.13), so it is symmetric, then $dq(e_i, e_j) = \alpha_{ij} - \alpha_{ji} = 0$, which implies that $q$ is horizontally closed. From Lemma 4.5 we know $\hat{D}$ is a special projective SNS-Riemannian connection, we further obtain by (4.6), (4.14) and (4.15)

\[ \hat{R}^h_{ijk} = K^h_{ijk} - \frac{1}{\ell - 1}(\delta^h_j K_{ik} - \delta^h_i K_{jk}) = \hat{W}^h_{ijk} = 0. \]

This completes the proof of Theorem 4.8. \(\square\)
5 Examples

Example 5.1. (Carnot group)

A Carnot group of \( r \)-step is a connected, simply connected Lie group \( G \) whose Lie algebra \( \mathfrak{h} \) admits a stratification

\[
\mathfrak{h} = V_1 \oplus \cdots \oplus V_r,
\]

which is \( r \)-nilpotent, i.e.,

\[
\begin{cases}
[V_1, V_j] = V_{j+1}, & j = 1, \cdots, r-1, \\
[V_j, V_r] = \{0\}, & j = 1, \cdots, r
\end{cases}
\]  

(5.1)

Let \( \circ \) be a group law on \( G \), then the left translation operator is \( L_p : q \mapsto p \circ q \), denote by \( (L_p)_* \), the differential of \( L_p \). Now we can define the horizontal subspace as

\[
HG_p = (L_p)_*(V_1),
\]

for any point \( p \in G \), and the horizontal bundle as

\[
HG = \bigcup_{p \in G} HG_p,
\]

Then we further consider the vertical distribution on \( G \) defined by

\[
VG_p = (L_p)_*(V_2 \oplus \cdots \oplus V_r),
\]

\[
VG = \bigcup_{p \in G} VG_p
\]

Now, we fix a basis \( X_1, \cdots, X_k \) formed by the left invariant vector fields, then, by \( (5.1) \), we deduce that

\[
[\Gamma(VG), X_k] \in \Gamma(VG).
\]  

(5.2)

and fix the inner product \( \langle \cdot, \cdot \rangle \) in \( TG \) such that the system of left-invariant vector fields \( \{X_1, \cdots, X_k, Y_1, \cdots, Y_{n-k}\} \) is an orthonormal basis of \( TG \), we obtain \( \Omega = 0 \) on \( G \) by \( (5.2) \). Then, the Carnot group is a nearly sub-Riemannian manifold. The Heisenberg group \( \mathbb{H}^n \) as a non-Abelian Carnot group is obviously a nearly sub-Riemannian manifold.

Example 5.2. (Nonholonomic Constrained Particle)

We consider the motion of a free particle with a unit mass in \( \mathbb{R}^3 \) with Lagrangian

\[
L = \frac{1}{2}(\dot{x} + \dot{y} + \dot{z}),
\]

and the nonholonomic constraint

\[
\dot{z} = y\dot{x},
\]
We take the horizontal distribution equipped with the Riemannian metric $g$ induced by the Euclidean metric $\bar{g} = 2L$ of $\mathbb{R}^3$ as follow

$$HR^3 = \text{span}\{X = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, Y = \frac{\partial}{\partial y}\},$$

and the vertical distribution

$$VR^3 = \text{span}\{Z = \frac{\partial}{\partial z}\}.$$

Then $(\mathbb{R}^3, HR^3, g)$ is a nearly sub-Riemannian manifold.

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