The Baer-invariant of a Semidirect Product

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Abstract

In 1972 K.I.Tahara [7,2 Theorem 2.2.5] , using cohomological method, showed that if a finite group \( G = T\triangleright N \) is the semidirect product of a normal subgroup \( N \) and a subgroup \( T \), then \( M(T) \) is a direct factor of \( M(G) \), where \( M(G) \) is the Schur-multiplicator of \( G \) and in the finite case, is the second cohomology group of \( G \). In 1977 W.Haebich [1 Theorem 1.7] gave another proof using a different method for an arbitrary group \( G \).

In this paper we generalize the above theorem. We will show that \( N_c M(T) \) is a direct factor of \( N_c M(G) \), where \( N_c \) [3 page 102] is the variety of nilpotent groups of class at most \( c \geq 1 \) and \( N_c M(G) \) is the Baer-invariant of the group \( G \) with respect to the variety \( N_c \) [3 page 107] .

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1. Notations and Preliminaries

Definition 1.1

The group $G$ is said to be a **semidirect product** of a normal subgroup $A$ and a subgroup $B$, denoted by $G = B times_A A$ (or a *splitting extension* of $A$ by $B$) if

(i) $G$ is generated by $A$ and $B$,

(ii) $A \cap B = 1$.

Since $A$ is normal in $G$, the maps $\theta b : a \mapsto a^b$, $a \in A$, for all $b \in B$ are automorphisms of $A$ and they induces a homomorphism $\theta : B \rightarrow Aut(A)$ which is called the action of $B$ on $A$. $G$ is determined up to isomorphism by $\theta$ and is therefore called the semidirect product of $A$ and $B$ under $\theta$ (or the splitting extension of $A$ by $B$ under $\theta$). Note that every element of $G$ is uniquely determined by $ab$, for $a \in A$ and $b \in B$.

Now the following elementary results on semidirect product are needed in our work, see [1 page 421, 5]

Lemma 1.2

Let $G$ be the semidirect product of $A$ and $B$ under $\theta : B \rightarrow Aut(A)$ and $\overline{G}$ be the semidirect product of $\overline{A}$ and $\overline{B}$ under $\overline{\theta}$. If

$$\alpha : A \longrightarrow \overline{A} \quad \text{and} \quad \beta : B \longrightarrow \overline{B}$$

are epimorphisms such that

$$\alpha((\theta b)(a)) = (\overline{\theta} b)(\alpha a)$$

for all $a \in A$, and $b \in B$,

then the map

$$\tau : G \longrightarrow \overline{G}$$

$$ab \longmapsto (\alpha a)(\beta b)$$

is an epimorphism extending $\alpha$ and $\beta$. 
Lemma 1.3

Let $G$ be the semidirect product of $A$ and $B$ under $\theta : B \rightarrow Aut(A)$. If $N$ is a subgroup of $A$ which is normal in $G$, then $G/N$ is the semidirect product of $A/N$ and $B$ under

$$\tilde{\theta} : B \rightarrow Aut(A/N)$$

$$b \mapsto \tilde{\theta}b$$

where $\tilde{\theta} : A/N \rightarrow A/N$ given by $aN \mapsto \theta(a)N$.

In the following theorem a free presentation for $G$ is introduced in terms of free presentations of $A$ and $B$.

**Theorem 1.4**

Let $G$ be a semidirect product of $A$ and $B$ under $\theta : B \rightarrow Aut(A)$ and

$$1 \rightarrow R_1 \rightarrow F_1 \xrightarrow{\nu_1} A \rightarrow 1 \ , \ 1 \rightarrow R_2 \rightarrow F_2 \xrightarrow{\nu_2} B \rightarrow 1$$

be free presentations for $A$ and $B$, respectively. Then

$$1 \rightarrow R \rightarrow F \rightarrow G \rightarrow 1$$

is a free presentation for $G$, where

(i) $F = F_1 * F_2$, the free product of $F_1$ and $F_2$;

(ii) $R = R_1^F R_2^F S$;

(iii) $S = \langle f_1^{-1} f_2, f_1 \mid f_1, f_2 \in F_1; f_2 \in F_2; \nu_1 f_1 = \theta(\nu_2 f_2)(\nu_1 f_1) >^F \rangle$.

**Proof.**

See[1 Lemma 1.4] . □

Let $\mathcal{V}$ be a variety of groups defined by a set of laws $V$ and $G$ be a group with a free presentation

$$1 \rightarrow R \rightarrow F \rightarrow G \rightarrow 1$$
Then the *Baer-invariant* of $G$ with respect to the variety $\mathcal{V}$, denoted by $\mathcal{V}M(G)$, is defined to be

$$\frac{R \cap V(F)}{[RV^*F]}$$

where $V(F)$ is the verbal subgroup of $F$ and $[RV^*F]$ is the least normal subgroup of $F$, contained in $R$, generated by

$$\{v(f_1, \ldots, f_ir, \ldots, f_s)v(f_1, \ldots, f_s)^{-1}| r \in R, f_i \in F, v \in V, 1 \leq i \leq s\} .$$

It is easily seen that the Baer-invariant of the group $G$ with respect to the variety $\mathcal{V}$ is always abelian and that it is independent of the choice of the free presentation of $G$, see [3 Lemma 1.8].

In particular, if $\mathcal{V}$ is the variety of abelian groups, then the Baer-invariant of $G$ with respect to $\mathcal{V}$ will be

$$\frac{R \cap F'}{[R, F]} ,$$

which, by I.Schur, is isomorphic to the Schur-multiplicator of $G$, denoted by $M(G)$, in general, and in the finite case, is isomorphic to the second cohomology group of $G$, $H^2(G, C)$ [2 Theorem 2.4.6].

If $\mathcal{V}$ is the variety $\mathcal{N}_c$ of nilpotent groups of class at most $c \geq 1$, then the Baer-invariant of $G$ with respect to $\mathcal{N}_c$ is

$$\frac{R \cap \gamma_{c+1}(F)}{[R, cF]} ,$$

where $\gamma_{c+1}(F)$ is the $(c+1)$st-term of the lower centeral series and $[R, cF]$ stands for $[R, F, F, \ldots, F]$ c-times. For further details, properties, conventions, see [3,4,5].

### 2. The Main Result

The following theorem is fundamental to the proof of Theorem 2.2. We adopt the notations and conventions from section 1, in what follows.
Theorem 2.1

(i) $R_1$ and $[R_2, F_1]$ are subgroups of $S$;
(ii) $R = R_2S$;
(iii) $R \cap \gamma_{c+1}(F) = (R_2 \cap \gamma_{c+1}(F_2))(S \cap \gamma_{c+1}(F))$, for all $c \geq 1$;
(iv) $[R, \_F] = [R_2, F_2] \prod [R_2, F_1, F_2] [S, \_F]$, for all $c \geq 1$,

where

\[ \prod [R_2, F_1, F_2]_c = \langle [r_2, f_1, \ldots, f_c] \mid f_i \in F_1 \cup F_2, r_2 \in R_2, 1 \leq i \leq c, \exists k, f_k \in F_1 >^F . \]

In particular, $\prod [R_2, F_1, F_2]_1 = [R_2, F_1]$.

Proof.

(i) If $r_1 \in R_1$, then $\nu_1 r_1 = 1$ and hence $\theta(\nu_2 f_2)(\nu_1 f_1) = 1$, for all $f_2 \in F_2$. Therefore $r_1 = 1^{-1} r_1[f_2, 1]$, so $r_1 \in S$ i.e $R_1 \leq S$.

If $r_2 \in R_2$, then $\nu_2 r_2 = 1$, so $\theta(\nu_2 r_2)$ is the identity automorphism. Thus $[r_2, f_1] = f_1^{-1} f_1 [r_2, f_1] \in S$ and so $[R_2, F_1] \leq S$.

(ii)

\[ R = R_2^F R_1^F S \quad \text{, by Theorem 1.4} \]
\[ = R_2^F S \quad \text{, by (i)} \]
\[ = R_2[R_2, F]^F S \]
\[ = R_2[R_2, F_1]^F S \quad \text{, since } R_2 \leq F_2 \]
\[ = R_2 S \quad \text{, by (i)} . \]

(iii) Since $F = F_1 * F_2$, we have

\[ \gamma_{c+1}(F) = \gamma_{c+1}(F_1) \gamma_{c+1}(F_2) \prod [F_1, F_2]_{c+1} , \]

where

\[ \prod [F_1, F_2]_{c+1} = \langle [F_1, F_2, F_i, \ldots, F_{i-1}] \mid i_j \in \{1, 2\}, 1 \leq j \leq c - 1 > \]
and \( \prod[F_1, F_2]_{c+1} \leq F \) (to find a proof see M.R.R. Moghaddam [4]). Also we know that \( F = F_2 \triangleright F_1[F_1, F_2] \) is the semidirect product of \( F_2 \) and \( F_1[F_1, F_2] \) (since \( F_2 \cap F_1[F_1, F_2] = 1 \) and \( F_1[F_1, F_2] \leq F \)) and \( S \leq F_1[F_1, F_2] \). So using part (ii) and the above remarks, we have

\[
R \cap \gamma_{c+1}(F) = R_2 S \cap \gamma_{c+1}(F_1) \gamma_{c+1}(F_2) \prod[F_1, F_2]_{c+1}
\]

\[
= (R_2 \cap \gamma_{c+1}(F_2))(S \cap \gamma_{c+1}(F_1) \prod[F_1, F_2]_{c+1})
\]

\[
= (R_2 \cap \gamma_{c+1}(F_2))(S \cap \gamma_{c+1}(F))
\]

(iv) Use induction on \( c \). Let \( c = 1 \). Then

\[
[R, F] = [R_2 S, F] \quad , \quad by \ (ii)
\]

\[
= [R_2, F][S, F] \quad , \quad since \ S \leq F
\]

\[
\subseteq [R_2, F_2][R_2, F_1]^F[S, F] \quad , \quad since \ F = F_1 * F_2
\]

\[
= [R_2, F_2][R_2, F_1][S, F] \quad , \quad since \ [R_2, F_1] \leq S .
\]

Clearly \( [R_2, F_2][R_2, F_1][S, F] \subseteq [R, F] \). Hence \( [R, F] = [R_2, F_2][R_2, F_1][S, F] \).

Now, suppose \( [R, kF] = [R_2, kF_2] \prod[R_2, F_1, F_2] k[S, kF] \). Then we have

\[
[R, k+1F] = [[R, kF], F]
\]

\[
= [[R_2, kF_2] \prod[R_2, F_1, F_2] k[S, kF], F]
\]

\[
= [[R_2, kF_2], F]\prod[R_2, F_1, F_2] k[S, kF], F \quad , \quad (by \ induction \ hypothesis)
\]

\[
\subseteq [[R_2, kF_2], F_2] \prod[R_2, F_1, F_2] k[S, k+1F] \quad , \quad (since \ [S, kF] \leq \prod[R_2, F_1, F_2] k \leq F)
\]

\[
\subseteq [R, k+1F] .
\]

Therefore, by induction we have

\[
[R, cF] = [R_2, cF_2] \prod[R_2, F_1, F_2] c[S, cF] \quad for \ all \ c \geq 1 \ . \ □
\]
Now we are in a position to state and prove the main theorem of this paper.

**Theorem 2.2**

Let \( G \) be a semidirect product of \( A \) by \( B \) under \( \theta : B \rightarrow Aut(A) \) (or a splitting extension of \( A \) by \( B \) under \( \theta ) \), and \( \mathcal{N}_c \) be the variety of nilpotent groups of class at most \( c \) \((c \geq 1)\). Then

\[
\mathcal{N}_c M(G) \cong \mathcal{N}_c M(B) \oplus \frac{S \cap \gamma_{c+1}(F)}{\prod [R_2, F_1, F_2]_{c}[S, cF]}. 
\]

In particular, \( \mathcal{N}_c M(B) \) can be regarded as a direct factor of \( \mathcal{N}_c M(G) \).

**Proof.**

By the previous assumptions and notations we have

\[
F \xrightarrow{\varphi} \frac{F}{[R_2, cF_2]^F} \xrightarrow{\eta} \frac{F}{[R_2, cF_2] \prod [R_2, F_1, F_2]_{c}[S, cF]},
\]

where \( \varphi \) and \( \eta \) are natural homomorphisms. Then for any \( c \geq 1 \), we have

\[
\frac{R \cap \gamma_{c+1}(F)}{[R, cF]} \cong (\eta \varphi)(R \cap \gamma_{c+1}(F))
\]

\[
\cong (\eta \varphi)(R_2 \cap \gamma_{c+1}(F_2))(\eta \varphi)(S \cap \gamma_{c+1}(F)) \quad \text{by Theorem 2.1 (iii) \quad (*)}. 
\]

Consider the following two natural homomorphisms

\[
\frac{F_1 * F_2}{[R_2, cF_2]^F} \xrightarrow{h} \frac{F_1 * F_2}{[R_2, cF_2]} \xrightarrow{g} \frac{F_1 * F_2}{[R_2, cF_2]^F},
\]

given by

\[
\overline{f_1} \mapsto f_1 \quad f_1 \mapsto \overline{f_1},
\]

\[
\overline{f_2} \mapsto \overline{f_2} \quad f_2 \mapsto \overline{f_2}. 
\]

Clearly \( h \circ g = 1 \) \& \( g \circ h = 1 \) i.e \( g \) is the inverse of \( h \) and so \( h \) is an isomorphism. Thus we have

\[
\frac{F_1 * F_2}{[R_2, cF_2]^F} = \varphi(F) \cong \frac{F_1 * F_2}{[R_2, cF_2]}. 
\]
Also $\varphi(F_2) = F_2/[R_2, \, \epsilon F_2]$ and
$$\varphi(F_1[F_1, F_2]) \cong \varphi(F_1)[\varphi(F_1), \varphi(F_2)] \cong F_1[F_1, F_2/[R_2, \, \epsilon F_2]].$$

Therefore
$$\varphi(F) \cong F_1 * \frac{F_2}{[R_2, \, \epsilon F_2]} \cong \frac{F_2}{[R_2, \, \epsilon F_2]} \varphi(F_1) \sim \frac{F_2}{[R_2, \, \epsilon F_2]} \varphi(F_1)[\varphi(F_1), \varphi(F_2)].$$

Thus by Lemma 1.3 and the property that $\text{Ker}(\eta) \leq \varphi(F_1)[\varphi(F_1), \varphi(F_2)]$ we have
$$(\eta \varphi)(F) \cong \frac{\varphi(F)}{\text{Ker}(\eta)} = \frac{\varphi(F)}{\varphi([R_2, F_1, F_2], [S, \, \epsilon F])} \cong \frac{\varphi(F_1)[\varphi(F_1), \varphi(F_2)]}{\text{Ker}(\eta)},$$

Clearly $(\eta \varphi)(F_2) \cong \varphi(F_2)$ and $(\eta \varphi)(F_1) \cong \varphi(F_1)/\text{Ker}(\eta)$, thus we have
$$(\eta \varphi)(F) \cong \varphi(F_2) \varphi(F_1)[\varphi(F_1), \varphi(F_2)],$$

(by 1.3 and (**))

So
$$(\eta \varphi)(R_2 \cap \gamma_{c+1}(F_2)) \cap (\eta \varphi)(S \cap \gamma_{c+1}(F)) \subseteq (\eta \varphi)(F_2) \cap (\eta \varphi)(F_1)[(\eta \varphi)(F_1), (\eta \varphi)(F_2)] = 1.$$

Hence, by (*)
$$\frac{R \cap \gamma_{c+1}(F)}{[R, \, \epsilon F]} \cong (\eta \varphi)(R_2 \cap \gamma_{c+1}(F_2)) \oplus (\eta \varphi)(S \cap \gamma_{c+1}(F))$$

and
$$(\eta \varphi)(R_2 \cap \gamma_{c+1}(F_2)) = \frac{(R_2 \cap \gamma_{c+1}(F_2)) \text{Ker}(\eta \varphi)}{\text{Ker}(\eta \varphi)} \cong \frac{R_2 \cap \gamma_{c+1}(F_2)}{(R_2 \cap \gamma_{c+1}(F_2)) \cap \text{Ker}(\eta \varphi)}$$

$$\cong \frac{R_2 \cap \gamma_{c+1}(F_2)}{[R_2, \, \epsilon F_2]} \cong N \circ M(B).$$
Also
\[(\eta_\varphi)(S \cap \gamma_{c+1}(F)) = \frac{(S \cap \gamma_{c+1}(F))\text{Ker}(\eta_\varphi)}{\text{Ker}(\eta_\varphi)} \cong \frac{S \cap \gamma_{c+1}(F)}{(S \cap \gamma_{c+1}(F)) \cap \text{Ker}(\eta_\varphi)}\]

Therefore
\[N_cM(B \triangleright A) \cong \frac{R \cap \gamma_{c+1}(F)}{[R, cF]} \cong N_cM(B) \oplus \frac{S \cap \gamma_{c+1}(F)}{[R_2, F_1, F_2][S, cF]} . \]

Now we obtain the following corollaries:

**Corollary 2.3** (Tahara [7,2 Theorem 2.2.5])

Let \(G = B \triangleright A\) be the semidirect product of a normal subgroup \(A\) and a subgroup \(B\). Then \(M(B)\) is a direct factor of \(M(G)\).

**Corollary 2.4** (Haebich [1 Theorem 1.7])

Suppose \(G = B \triangleright A\) is a semidirect product of \(A\) by \(B\) under \(\theta : B \to \text{Aut}(A)\). By the notation of Theorem 1.4 we have

\[M(G) \cong M(B) \oplus \frac{S^F \cap F'}{[R_2, F_1][S, F]} . \]
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