ASYMPTOTIC FOR THE PERTURBED HEAVY BALL SYSTEM
WITH VANISHING DAMPING TERM

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Abstract: We investigate the long time behavior of solutions to the differential equation:

\[(0.1) \ddot{x}(t) + c(t+1)\dot{x}(t) + \nabla \Phi(x(t)) = g(t), \quad t \geq 0,\]

where \(c\) is nonnegative constant, \(\alpha \in [0, 1[\), \(\Phi\) is a \(C^1\) convex function on a Hilbert space \(\mathcal{H}\) and \(g \in L^1(0, +\infty; \mathcal{H})\). We obtain sufficient conditions on the source term \(g(t)\) ensuring the weak or the strong convergence of any trajectory \(x(t)\) of \((0.1)\) as \(t \to +\infty\) to a minimizer of the function \(\Phi\) if one exists.

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1. Introduction and main results

Let $\mathcal{H}$ be a real Hilbert space with inner product and norm respectively denoted by $\langle ., . \rangle$ and $\| . \|$. In this paper, we consider the following second order equation:

\begin{equation}
\ddot{x}(t) + \gamma(t) \dot{x}(t) + \nabla \Phi(x(t)) = g(t), \quad t \geq 0,
\end{equation}

where $\gamma(t) = \frac{c}{(1+t)^{\alpha}}$ with $c > 0$ and $\alpha \in [0, 1[$, $g \in L^1(0, +\infty; \mathcal{H})$ and $\Phi : \mathcal{H} \to \mathbb{R}$ is a $C^1$ convex function such that its minimizers subset

$$\arg \min \Phi := \{ v \in \mathcal{H} : \Phi(v) = \Phi^* M = \min_{x \in \mathcal{H}} \Phi(x) \}$$

is not empty.

Using classical arguments (see for instance [7]), one can easily prove that if the function $\nabla \Phi : \mathcal{H} \to \mathcal{H}$ is Lipschitz on bounded subset of $\mathcal{H}$, then for any initial data $(x_0, x_1) \in \mathcal{H} \times \mathcal{H}$, the equation (1.1) has a unique global solution $x \in W^{2,1}_{\text{loc}}(0, +\infty; \mathcal{H})$ satisfying $(x(0), \dot{x}(0)) = (x_0, x_1)$. Moreover, the associated energy function

\begin{equation}
W(t) = \frac{1}{2} \| \dot{x}(t) \|^2 + \Phi(x(t)) - \Phi^*
\end{equation}

is nonincreasing and converges to 0 as $t \to +\infty$. Hence hereafter, we will assume that $x \in W^{2,1}_{\text{loc}}(0, +\infty; \mathcal{H})$ is a solution to (1.1) and we will focus our attention on the study of the asymptotic behavior of $x(t)$ as $t \to +\infty$ and on the rate of convergence of the energy function $W$.

Before setting the main results of our present paper, let us first recall some previous results: In the pioneer paper [11], Alvarez considered the case where $\alpha = 0$ and $g = 0$. He proved that $x(t)$ converges weakly in $\mathcal{H}$ as $t \to +\infty$ to a minimizer of the function $\Phi$. Moreover, he showed that the convergence is strong if either the function $\Phi$ is even or the interior of the set $\arg \min \Phi$ is not empty. In [6], Haraux and Jendoubi extended the weak convergence result of Alvarez to the case where the source term $g$ belongs to the space $L^1(0, +\infty; \mathcal{H})$. Recently, Cabot and Frankel [5] studied (1.1) where $g = 0$ and $\alpha \in ]0, 1[$. They proved that every bounded solution $x(t)$ (i.e. $x \in L^\infty(0, +\infty; \mathcal{H})$) converges weakly toward a critical point of $\Phi$. In a very recent work [8], the second author of this paper improved the result of Cabot and Frankel by getting rid of the superfluous hypothesis on the boundedness of the solution. Moreover he established that $W(t) = o(\frac{1}{t^{\alpha}})$ as $t \to +\infty$ for every $\bar{\alpha} < \alpha$. In [7], Jendoubi and May proved that the main convergence result of Cabot and Frankel remains true if the source term $g$ satisfies the condition $\int_0^{+\infty} (1 + t) \| g(t) \| \, dt < \infty$. Recently, this result was improved in [4]. In fact, we proved...
that if the solution $x(t)$ is bounded and the function $g$ satisfies the optimal condition
\begin{equation}
\int_0^{+\infty} (1 + t)^\alpha \|g(t)\| \, dt < \infty,
\end{equation}
then $x(t)$ converges weakly to some element of $\text{arg min} \Phi$ and $W(t) = o\left(\frac{1}{t^2}\right)$ as $t \to +\infty$. One of the main purpose of this paper is to prove that the sole assumption (1.3) guarantees the boundedness (and therefore the weak convergence) of the solution $x(t)$.

We notice that, in a very recent work [2], Attouch, Chbani, Peypouquet and Redont have considered the equation (1.1) in the case $\alpha = 1$. They have proved that if $c > 3$ and $\int_0^{+\infty} (1 + t)^\alpha \|g(t)\| \, dt < \infty$ then $x(t)$ converges weakly to some element of $\text{arg min} \Phi$ and that $W(t) = O\left(\frac{1}{t^2}\right)$. Moreover, they have established the strong convergence of $x(t)$ in the case where the function $\Phi$ is even or the interior of the subset $\text{arg min} \Phi$ is not empty. In this paper, we extend their results to the case $\alpha < 1$.

Our main first result is the weak convergence of the trajectories of (1.1) under the optimal condition (1.3) on the source term $g$.

**Theorem 1.1.** Assume that $\int_0^{+\infty} (1 + t)^\alpha \|g(t)\| \, dt < \infty$. Then $x(t)$ converges weakly in $\mathcal{H}$ as $t \to +\infty$ to some $x^* \in \text{arg min} \Phi$. Moreover the energy function $W$ satisfies the two following properties:

\begin{equation}
W(t) = O\left(\frac{1}{t^{2\alpha}}\right)
\end{equation}

\begin{equation}
\int_0^{+\infty} (1 + t)^\alpha W(t) \, dt < \infty.
\end{equation}

Our second theorem improves the result on the convergence rate of the energy function $W$ obtained in [8] in the case where $g = 0$ and it will be useful in the proof of the strong convergence of the solution $x(t)$ when the convex function $\Phi$ is even.

**Theorem 1.2.** Assume that $\int_0^{+\infty} (1 + t)^\nu \|g(t)\| \, dt < \infty$ where $\nu \in [\alpha, \frac{1 + \alpha}{2}]$. Then

\begin{equation}
W(t) = o\left(\frac{1}{t^{2\nu}}\right) \text{ as } t \to +\infty,
\end{equation}

\begin{equation}
\int_0^{+\infty} (1 + t)^{2\nu - \alpha} \|\dot{x}(t)\|^2 \, dt < \infty.
\end{equation}

The next result shows that, as in the limit case $\alpha = 1$ (see [2] Theorem 3.1), the strong convergence of $x(t)$ as $t \to +\infty$ holds if the interior of $\text{arg min} \Phi$ is not empty.

**Theorem 1.3.** Suppose that $\int_0^{+\infty} (1 + t)^\alpha \|g(t)\| \, dt < \infty$ and $\text{int} (\text{arg min} \Phi) \neq \emptyset$. Then there exists some $x^* \in \text{arg min} \Phi$ such that $x(t) \to x^*$ strongly in $\mathcal{H}$ as $t \to +\infty$. 
In the last theorem, we prove, under an assumption on the source term $g$ slightly stronger than the optimal condition (1.3), the strong convergence of the solution $x(t)$ when the potential function $\Phi$ is even.

**Theorem 1.4.** Suppose that $\int_0^{+\infty} (t + 1)^{\frac{\alpha+1}{2}} \|g(t)\| \, dt < \infty$ and $\Phi$ is even (i.e. $\Phi(-x) = \Phi(x)$, $\forall x \in \mathcal{H}$). Then $x(t)$ converges strongly in $\mathcal{H}$ as $t \to +\infty$ to some $x^* \in \arg\min \Phi$.

2. Proof of Theorem 1.1

The proof makes use of a modified version of the method used Attouch, Chbani, Pey-ouquet and Redont in [2]. It relies on the study of a suitable Lyapunov function $\mathcal{E}$ and uses the following two classical lemmas.

**Lemma 2.1** (Gronwall-Bellman lemma). Let $f \in L^1([a, b], \mathbb{R}_+)$ and $c$ a nonnegative constant. Suppose that $w$ is a continuous function from $[a, b]$ into $\mathbb{R}$ that satisfies: for all $t \in [a, b]$,

$$\frac{1}{2} w^2(t) \leq \frac{1}{2} c^2 + \int_a^t f(s) w(s) \, ds.$$ 

Then, for all $t \in [a, b]$,

$$w(t) \leq c + \int_a^t f(s) \, ds.$$ 

The proof of this lemma is easy and similar to the proof of the classical Gronwall’s lemma.

**Lemma 2.2** (Opial’s lemma [9]). Let $x : [0, +\infty[ \to \mathcal{H}$. Assume that there exists a nonempty subset $S$ of $\mathcal{H}$ such that:

i) If $t_n \to +\infty$ and $x(t_n) \rightharpoonup x$ weakly in $\mathcal{H}$, then $x \in S$.

ii) For every $z \in S$, $\lim_{t \to +\infty} \|x(t) - z\|$ exists.

The proof of Opial’s lemma is easy, see for instance [3].

Let us now start the proof our theorem. We first define on $[0, +\infty[$ the function

\begin{equation}
(2.1) \quad h(t) = e^{\Gamma(t)} \int_t^{+\infty} e^{-\Gamma(s)} \, ds,
\end{equation}

where $\Gamma(t) = \int_0^t \gamma(s) \, ds$. A simple calculation yields that $h$ satisfies the differential equation

\begin{equation}
(2.2) \quad h'(t) - \gamma(t) h(t) + 1 = 0.
\end{equation}
Moreover, since
\[ \left( \frac{-1}{\gamma(s)} e^{-\Gamma(s)} \right)' = \left( 1 + \frac{\gamma'(s)}{\gamma^2(s)} \right) e^{-\Gamma(s)} \leq e^{-\Gamma(s)} \]
then
\[ h(t) \sim \frac{1}{+\infty \gamma(t)}. \tag{2.3} \]

Let \( x^* \in \text{arg min} \Phi \) and define the function
\[ \mathcal{E}(t) = 2(h(t))^2 (\Phi(x(t)) - \Phi^*) + \|x(t) - x^* + h(t) \dot{x}(t)\|^2 \]
\[ -2 \int_0^t h(s) \langle g(s), x(s) - x^* + h(s) \dot{x}(s) \rangle \, ds. \tag{2.4} \]

By differentiating, we obtain
\[ \mathcal{E}'(t) = 4h'(t) h(t) (\Phi(x(t)) - \Phi^*) + 2(h(t))^2 \langle \nabla \Phi(x(t)), \dot{x}(t) \rangle \]
\[ + 2 \langle (1 + h'(t)) \dot{x}(t) + h(t) \ddot{x}(t), x(t) - x^* + h(t) \dot{x}(t) \rangle \]
\[ - 2h(t) \langle g(t), x(t) - x^* + h(t) \dot{x}(t) \rangle. \]

Hence by sing \([1.1]\), we get
\[ \mathcal{E}'(t) = 4h'(t) h(t) (\Phi(x(t)) - \Phi^*) - 2h(t) \langle \nabla \Phi(x(t)), x(t) - x^* \rangle. \tag{2.5} \]

Since the function \( \Phi \) is convex, we have
\[ \Phi^* = \Phi(x^*) \geq \Phi(x(t)) + \langle \nabla \Phi(x(t)), x^* - x(t) \rangle. \]

Inserting this inequality in \([2.5]\) yields
\[ \mathcal{E}'(t) \leq 2[2h'(t) - 1] h(t) (\Phi(x(t)) - \Phi^*). \]

From \([2.2]\) and \([2.3]\), \( 2h'(t) - 1 \to -1 \) as \( t \to +\infty \). Then there exists \( t_1 \geq 0 \) such that
\[ \mathcal{E}'(t) + h(t) (\Phi(x(t)) - \Phi^*) \leq 0, \ \forall t \geq t_1. \tag{2.6} \]

Therefore \( \mathcal{E} \) is a decreasing function on \([t_1, +\infty]\). Then for every \( t \geq t_1 \), \( \mathcal{E}(t) \leq \mathcal{E}(t_1) \), which implies that
\[ 2(h(t))^2 (\Phi(x(t)) - \Phi^*) + \|x(t) - x^* + h(t) \dot{x}(t)\|^2 \]
\[ \leq C^2 + 2 \int_{t_1}^t h(s) \langle g(s), x(s) - x^* + h(s) \dot{x}(s) \rangle \, ds, \tag{2.7} \]

where
\[ C^2 = 2(h(t_1))^2 (\Phi(x(t_1)) - \Phi^*) + \|x(t_1) - x^* + h(t_1) \dot{x}(t_1)\|^2. \]
Using now the Cauchy-Schwarz inequality, we obtain
\[
\frac{1}{2} \|x(t) - x^* + h(t) \dot{x}(t)\|^2 \leq \frac{C^2}{2} + \int_{t_1}^{t} h(s) \|g(s)\| \|x(s) - x^* + h(s) \dot{x}(s)\| ds.
\]
Hence by applying the Gronwall-Bellman lemma we obtain
\[
\|x(t) - x^* + h(t) \dot{x}(t)\| \leq C + \int_{t_1}^{t} h(s) \|g(s)\| ds,
\]
which implies, thanks to (2.3), that
\[
M_1 = \sup_{t \geq 0} \|x(t) - x^* + h(t) \dot{x}(t)\| < +\infty \tag{2.8}
\]
Returning to (2.7), we then infer that
\[
\sup_{t \geq 0} h(t)^2 (\Phi(x(t)) - \Phi^*) \leq C^2 + 2M_1 \int_{t_1}^{+\infty} h(s) \|g(s)\| ds < +\infty.
\]
Therefore, we deduce from the expression (2.3) of the function \( \mathcal{E} \), that
\[
\sup_{t \geq 0} |\mathcal{E}(t)| < +\infty \tag{2.9}
\]
Hence by integrating the inequality (2.6) on \([t_1, t]\) with \( t \geq t_1 \), we infer that
\[
\int_{t_1}^{+\infty} h(t) (\Phi(x(t)) - \Phi^*) \, dt < +\infty. \tag{2.10}
\]
Taking the inner product of (1.1) with \( \dot{x}(t) \), we obtain
\[
\langle \ddot{x}(t) + \nabla \Phi(x(t)), \dot{x}(t) \rangle + \gamma(t) \|\dot{x}(t)\|^2 = \langle g(t), \dot{x}(t) \rangle \\
\leq \|g(t)\| \|\dot{x}(t)\| \\
\leq \sqrt{2} \|g(t)\| W(t).
\]
Multiplying the last inequality by \( h^2(t) \) and using the fact that
\[
\dot{W}(t) = \langle \ddot{x}(t) + \nabla \Phi(x(t)), \dot{x}(t) \rangle,
\]
we get after integration by parts on \([0, t]\),
\[
\begin{align*}
(h(t))^2 W(t) + \int_{0}^{t} \left( \gamma(s) (h(s))^2 - \dot{h}(s) h(s) \right) \|\dot{x}(s)\|^2 ds \\
\leq h(0) W(0) + \int_{0}^{t} 2h(s) h(s) [\Phi(x(s)) - \Phi^*] ds + \sqrt{2} \int_{0}^{t} (h(s))^2 \|g(s)\| \sqrt{W(s)} ds.
\end{align*}
\]
Using now (2.2), the fact the function \(\dot{h}\) is bounded, and (2.10), we obtain
\[
(h(t))^2 W(t) + \int_0^t h(s) \|\dot{x}(s)\|^2 ds \leq C + \sqrt{2} \int_0^t (h(s))^2 \|g(s)\| \sqrt{W(s)} ds,
\]
where \(C\) is an absolute constant.

Hence by applying Gronwall-Bellman lemma with \(\omega = h\sqrt{W}\) and using the fact that
\[
\int_{-\infty}^0 h(s) \|g(s)\| ds < +\infty,
\]
we deduce that
\[
(2.11) \quad \sup_{t \geq 0} h(t) \sqrt{W(t)} < +\infty
\]
(which is equivalent to (1.4)) and therefore
\[
(2.12) \quad \int_{-\infty}^{+\infty} h(s) \|\dot{x}(s)\|^2 ds < +\infty.
\]

Combining (2.10) and (2.12), we get (1.5). Let us now prove the weak convergence of \(x(t)\) as \(t \to +\infty\). We first notice that since \(W(t) \to 0\) as \(t \to +\infty\) and \(\Phi\) is weak lower semi-continuous (in fact \(\Phi\) is continuous and convex), then the first item i) of Opial’s lemma is satisfied with \(S = \arg\min \Phi\). Hence, it remains to prove that for any \(x^*\) in \(\arg\min \Phi\), the associated function \(z(t) := \frac{1}{2} \|x(t) - x^*\|^2\) converges as \(t \to +\infty\). A simple calculation using (1.1) gives
\[
\ddot{z}(t) + \gamma(t) \dot{z}(t) = \|\dot{x}(t)\|^2 - \langle \nabla \Phi(x(t)), x(t) - x^* \rangle + \langle g(t), x(t) - x^* \rangle.
\]

Hence by using the monotonicity property of the operator \(\nabla \Phi\), the fact that \(\nabla \Phi(x^*) = 0\), and the Cauchy-Schwarz inequality we get
\[
(2.13) \quad \ddot{z}(t) + \gamma(t) \dot{z}(t) \leq \|\dot{x}(t)\|^2 + \|x(t) - x^*\| \|g(t)\| = k(t).
\]

Combining (2.8) and (2.11), we deduce that
\[
\sup_{t \geq 0} \|x(t) - x^*\| < +\infty,
\]
which implies, thanks to (1.3) and (2.12), that
\[
(2.14) \quad \int_{0}^{+\infty} \frac{1}{\gamma(t)} k(t) dt < +\infty.
\]

Multiply now the inequality (2.13) by \(e^{\Gamma(t)}\) and integrate over \([0, t]\), we get after simplification the following inequality
\[
(2.15) \quad \dot{z}(t) \leq K(t) := e^{-\Gamma(t)} z(0) + e^{-\Gamma(t)} \int_0^t e^{\Gamma(s)} k(s) ds.
\]
By Fubini theorem, we have
\[
\int_0^{+\infty} K(t) \, dt = z(0) \int_0^{+\infty} e^{-\Gamma(t)} \, dt + \int_0^{+\infty} k(s) h(s) \, ds,
\]
where \( h \) is the function defined by (2.1) at the beginning of the proof. Hence, by using (2.3) and (2.14) we deduce that the function \( K \), and therefore the positive part \( \dot{z}^+(t) \) of \( \dot{z}(t) \) belongs to the space \( L^1(0, +\infty) \).

Then the limit of \( z(t) \) as \( t \to +\infty \) exists. This proves the item ii) of the Opial’s lemma and completes the proof of Theorem 1.1.

3. Proof of Theorem 1.2

By differentiating the energy function \( W \) and using the equation (1.1), we obtain
\[
\dot{W}(t) = \langle \ddot{x}(t) + \nabla \Phi(x(t)), \dot{x}(t) \rangle
= -\gamma(t) \| \dot{x}(t) \|^2 + \langle g(t), \dot{x}(t) \rangle
\leq -\gamma(t) \| \dot{x}(t) \|^2 + \| g(t) \| \sqrt{2W(t)}
\]
(3.1)
\[
\leq \| g(t) \| \sqrt{2W(t)}
\]
(3.2)

Hence the function \( \rho(t) := (1 + t)^{2\nu} W(t) \) satisfies the differential inequality
\[
\dot{\rho}(t) \leq 2\nu(1 + t)^{2\nu-1} W(t) + \sqrt{2(1 + t)^\nu} \| g(t) \| \sqrt{\rho(t)}.
\]
(3.3)

Now since \( 2\nu - 1 \leq \alpha \), we have from (1.5),
\[
\int_0^{+\infty} (1 + t)^{2\nu-1} W(t) \, dt < \infty.
\]
(3.4)

Thus by integrating the differential inequality (3.3) and applying Gronwall-Bellman lemma we deduce that \( \sup_{t \geq 0} \rho(t) < \infty \). Therefore (3.3) and (3.4) imply that the positive part \( [\dot{\rho}(t)]^+ \) of \( \dot{\rho}(t) \) belongs to \( L^1(0, +\infty) \). Thus \( \rho(t) \) converges as \( t \to +\infty \) to some real number \( \lambda \) which in view of (3.4) must be equal to 0. This proves (1.6). Now multiply (3.1) by \( (1 + t)^{2\nu} \) and then integrate on \( [0, t] \) with \( t > 0 \), we obtain
\[
\int_0^t (1 + s)^{2\nu} \gamma(s) \| \dot{x}(s) \|^2 \, ds \leq \sup_{s \geq 0} \sqrt{2(1 + s)^{2\nu} W(s)} \int_0^{+\infty} (1 + s)^\nu \| g(s) \| \, ds
+ W(0) - (1 + t)^{2\nu} W(t) + 2\nu \int_0^{+\infty} (1 + s)^{2\nu-1} W(s) \, ds,
\]
which implies (1.7) thanks to (3.4).
4. Proof of Theorem 1.3

We follow the same method used in the proof of [2, Theorem 3.1]. The assumption \( \text{int} (\text{arg min} \Phi) \neq \emptyset \) implies the existence of \( z_0 \in \mathcal{H} \) and \( r > 0 \) such that for any \( v \in \mathcal{H} \) with \( \|v\| \leq 1 \) we have \( \nabla (z_0 + rv) = 0 \) which implies by the monotocity property of \( \nabla \Phi \) that for any \( z \in \mathcal{H} \) we have \( \langle \nabla \Phi(z), z - z_0 - rv \rangle \geq 0 \). Thus by taking the supremum on \( v \), we get
\[
\langle \nabla \Phi(z), z - z_0 \rangle \geq r \| \nabla \Phi \| .
\]
Hence (2.5) with \( x^* = z_0 \) gives
\[
\mathcal{E}'(t) + 2h(t) \| \nabla \Phi(x(t)) \| \leq 4 \dot{\mathcal{E}}(t) - \mathcal{E}(t) = 4 \dot{\mathcal{E}}(t) - \mathcal{E}^*.
\]
Integrating this inequality on \([0, t]\) and using (2.3), (2.9), (2.10), and the boundedness of \( \dot{\mathcal{E}} \), we deduce that
\[
\left( \begin{array}{c}
\int_{0}^{+\infty} \frac{1}{\gamma(t)} \| \nabla \Phi(x(t)) \| \ dt < +\infty.
\end{array} \right)
\]
Setting \( \omega(t) = g(t) - \nabla \Phi(x(t)) \), the equation (1.1) becomes
\[
\ddot{x}(t) + \gamma(t) \dot{x}(t) = \omega(t).
\]
Hence thanks to (4.1), the following lemma completes the proof of Theorem 1.3

**Lemma 4.1.** Let \( \omega : [0, +\infty[ \rightarrow \mathcal{H} \) be a measurable function that satisfies
\[
\int_{a}^{+\infty} \frac{1}{\gamma(t)} \| \omega(t) \| \ dt < +\infty.
\]
If \( y \in W^{2,1}_{\text{loc}}(0, +\infty; \mathcal{H}) \) is a solution of the differential equation
\[
\ddot{y}(t) + \gamma(t) \dot{y}(t) = \omega(t),
\]
then \( y(t) \) converge strongly in \( \mathcal{H} \) as \( t \rightarrow +\infty \).

**Proof.** Multiply (4.2) by \( e^{\Gamma(t)} \) and integrate on \([0, t]\), we obtain
\[
\dot{y}(t) = e^{-\Gamma(t)} y(0) + e^{-\Gamma(t)} \int_{0}^{t} e^{\Gamma(s)} \omega(s) ds.
\]
Hence by using Fubini theorem and (2.1) as in the proof of Theorem 1.1 we get
\[
\int_{0}^{+\infty} \| \dot{y}(t) \| \ dt \leq \| y(0) \| \int_{0}^{+\infty} e^{-\Gamma(t)} dt + \int_{0}^{+\infty} \dot{\mathcal{E}}(t) \| \omega(t) \| \ dt < +\infty \text{ (thanks to (2.3)).}
\]
This completes the proof of the lemma. \( \square \)
Using now the inequality (3.2), we get

By a classical calculus using (1.1) and the fact that $\Phi$ is convex and even, we obtain

Integrating this inequality and using the fact that $y \in L^\infty(0, +\infty, \mathcal{H})$. Using now the inequality (3.2), we get

Applying Theorem 1.2 with $\nu$ such that $\arg\min (\Phi)$, we have $0 \leq \|x(s)\|$ (Recall that from Theorem 1.1 we have $x \in L^\infty(0, +\infty, \mathcal{H})$).

Therefore, for every $t$ in $[0, T]$, we have

Since $\Phi$ is even and convex, we have $0 \in \arg\min (\Phi)$. Hence by using the convergence of the function $z(t) = \frac{1}{2} \|x(t) - x^*\|^2$ proved in Theorem 1.1 with $x^* = 0$, we infer that the
limit of $\|x(t)\|^2$ as $t$ goes to $+\infty$ exists which implies that

$$\lim_{t,T \to +\infty} \|x(t)\|^2 - \|x(T)\|^2 = 0. \quad (5.2)$$

On the other hand, in view of Fubini theorem and $(2.3)$, there exists an absolute constant $C' \geq 0$ such that

$$\int_0^{+\infty} e^{-\Gamma(s)} \int_0^{s} e^{\Gamma(\tau)} \omega(\tau) d\tau ds = \int_0^{+\infty} h(\tau) \omega(\tau) d\tau$$

$$\leq C' \int_0^{+\infty} (1 + \tau)^{1+\alpha} \omega(\tau) d\tau$$

$$= C' \int_0^{+\infty} \left( \frac{3}{2} \|\dot{x}(\tau)\|^2 + M \|g(\tau)\| \right) (1 + \tau)^{1+\alpha} d\tau$$

$$+ \frac{CC'}{1+\alpha} \int_0^{+\infty} (1 + s)^{\frac{1+\alpha}{2}} \|g(s)\| ds$$

$$< +\infty \quad \text{(in view of \,(1.3).)}$$

Therefore

$$\lim_{t \to +\infty} \varpi(t) = 0. \quad (5.3)$$

Combining $(5.1)$, $(5.2)$ and $(5.3)$, we conclude that $x(t)$ satisfies the Cauchy convergence criterion in the Hilbert space $\mathcal{H}$ as $t \to +\infty$, and hence converges strongly in $\mathcal{H}$ as $t \to +\infty$.

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