The holomorphic d-scalar curvature on almost Hermitian manifolds

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Abstract In this paper, we study the existence of conformal metrics with the constant holomorphic d-scalar curvature and the prescribed holomorphic d-scalar curvature problem on closed, connected almost Hermitian manifolds of dimension $n \geq 6$. In addition, we obtain an application and a variational formula for the associated conformal invariant.

Keywords holomorphic d-scalar curvature, almost Hermitian manifold, Yamabe problem, prescribed curvature problem

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1 Introduction

Let $(M^n, g, J)$ be an almost Hermitian manifold of dimension $n$, and $\nabla$ be the Levi-Civita connection of $(M^n, g)$. The Laplace operator is defined by $\Delta_g := -\text{tr}_g \nabla^2$. The curvature of $(M^n, g)$ is defined by

$$R(X,Y)Z := \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z,$$

and the Riemannian curvature tensor is $R(X,Y,Z,W) := g(R(X,Y)Z,W)$. Then the Ricci tensor is

$$\text{Ric}(X,Y) := \text{tr}_g \{Z \mapsto R(X,Z)Y\},$$

and the $^*$-Ricci tensor [22, 23] is

$$\text{Ric}^*(X,Y) := \text{tr}_g \{Z \mapsto -J(R(X,Z)(JY))\}.$$

In addition, $R_g := \text{tr}_g \text{Ric}$ and $R_g^* := \text{tr}_g \text{Ric}^*$ are called the scalar curvature and the $^*$-scalar curvature, respectively. We call the difference

$$S_J := R_g - R_g^*$$

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the holomorphic d-scalar curvature of \((M^n, g, J)\). Hernández-Lamoneda [10] pointed out that for any smooth local orthonormal frame \((Z_i)_{i=1}^{n/2}\) of the holomorphic tangent bundle \(T^{1,0}M\),

\[
S_J = 4 \sum_{i,j=1}^{n/2} (\mathcal{R}^C(Z_i \wedge Z_j), Z_i \wedge Z_j),
\]

where \(\mathcal{R}^C\) denotes the complex linear extension of the curvature operator \(\mathcal{R}\) and \((\cdot, \cdot)\) is the complex linear extension of the inner product induced by \(g\). In [10], \(S_J\) is named the holomorphic scalar curvature, but we wish to avoid mistaking it for the trace of the holomorphic bisectional curvature. Recently, various scalar curvatures on almost Hermitian manifolds have been studied in many papers [4, 7, 16].

**Example 1.1.** For every Kähler manifold, we have \(R_g = R^*_g\) and hence \(S_J = 0\). In particular, every 2-dimensional almost Hermitian manifold has \(S_J = 0\).

**Example 1.2.** The unit sphere \((S^6, \tilde{g})\) admits an almost Hermitian structure induced by the Cayley numbers. In this case, we have \(R_{\tilde{g}} = 30\), \(R^*_g = 6\) and \(S_J = 24\).

The smooth 2-form

\[
\omega(X, Y) := g(JX, Y)
\]

is called the fundamental form of \((M^n, g, J)\). The smooth tensor field

\[
N_J(X, Y) := [JX, JY] - [X, Y] - J[JX, Y] - J[X, JY]
\]

is called the Nijenhuis tensor of \(J\). It is well known that \(J\) is integrable if and only if \(N_J = 0\). For each \(x \in M\), let

\[
W := \{ \alpha \in T^*_xM \otimes T^*_xM \otimes T^*_xM : \alpha(X, Y, Z) = -\alpha(X, Z, Y) = -\alpha(X, JY, JZ) \}.
\]

By decomposing

\[
\nabla \omega |_x \in W = W_1 \oplus W_2 \oplus W_3 \oplus W_4
\]

into 4 components according to symmetries, Gray and Hervella [9] classified all the almost Hermitian manifolds into \(2^4 = 16\) classes, which are widely used in the study of almost complex geometry. In Table 1, we only list some of the classes that will be concerned in this paper.

Here, \(W_i\) denotes the corresponding class of \(W_i\), and \(W_i \oplus W_j\) denotes the corresponding class of \(W_i \oplus W_j\), etc. In addition, \(\delta\) denotes the codifferential, and \(C_{XYZ}\) denotes the cyclic sum for smooth vector fields \(X, Y, Z \in \mathfrak{X}(M)\).

| Class   | Name          | Condition                                                                 |
|---------|---------------|---------------------------------------------------------------------------|
| \{0\}  | Kähler        | \(\nabla \omega = 0\)                                                   |
| \(W_1\)| Nearly Kähler | \(\nabla_X \omega(X, Y) = 0\)                                            |
| \(W_2\)| Almost Kähler | \(d\omega = 0\)                                                         |
| \(W_3\)| Balanced      | \(\delta \omega = 0 = N_J\)                                             |
| \(W_4\)|             | \(\nabla_X \omega(Y, Z) = \frac{1}{n-2} \left[ g(X, Y) \delta \omega(Z) - g(X, Z) \delta \omega(Y) - g(X, JY) \delta \omega(JZ) + g(X, JZ) \delta \omega(JY) \right] \) |
| \(W_2 \oplus W_3\)|             | \(\delta \omega = 0 = \varepsilon_{XYZ} \{ \nabla_Z \omega(X, Y) - \nabla_J \omega(JX, Y) \} \) |
| \(W_1 \oplus W_4\)| Hermitian     | \(N_J = 0\)                                                             |
| \(W_1 \oplus W_3 \oplus W_4\)|             | \(G_1\), \(\nabla_X \omega(X, Y) - \nabla_J \omega(JX, Y) = 0\)             |
| \(W\)  | Almost Hermitian | \(-\)                                                                   |
In this paper, we introduce a Yamabe-type conformal invariant $Y(M^n, g, J)$ to study the existence of the constant holomorphic d-scalar curvature and the prescribed holomorphic d-scalar curvature problem on closed, connected almost Hermitian manifolds. It turns out that the answers are completely similar to those in the original Yamabe problem except possibly for dimension $n = 4$ (see Theorems 2.15 and 3.4 in Sections 2 and 3, respectively). In Section 4, we obtain some rigidity results for the conformal invariant $Y(M^n, g, J)$ with respect to the Gray-Hervella classification of almost Hermitian manifolds (see Theorem 4.3). In particular, we show that the almost complex structure $J$ is not integrable if $Y(M^n, g, J) < 0$ for $n \geq 6$. We finally give a variational formula of $Y(M^n, g, J)$ under deformations of compatible almost complex structures $J$ (see Theorem 5.5). The critical point $J$ is characterized by the symmetry of the $*$-Ricci curvature of $(M^n, g, J)$ (see Theorem 5.6).

2 The conformal deformation and the Yamabe problem

2.1 The conformal deformation

Let $(M^n, g)$ be a closed, connected Riemannian manifold of dimension $n > 2$. Consider the conformal metric $\tilde{g} = u^{p-2}g$ on $M$, where $u$ is a smooth positive function and $p = \frac{2n}{n-2}$. If $R_g$ and $R_{\tilde{g}}$ denote the scalar curvatures of $(M^n, g)$ and $(M^n, \tilde{g})$, respectively, then we have

$$4\frac{n-1}{n-2}\Delta_g u + R_g u = R_{\tilde{g}} u^{p-1}. \quad (2.1)$$

Thus $\tilde{g}$ has constant scalar curvature $c$ if and only if $u$ satisfies the well-known Yamabe equation

$$4\frac{n-1}{n-2}\Delta_g u + R_g u = cu^{p-1}. \quad (2.2)$$

For every positive function $u \in C^\infty(M)$,

$$Q_g(u) := \frac{\int_M R_g dV_{\tilde{g}}}{(\int_M dV_{\tilde{g}})^{2/p}} = \frac{\int_M 4\frac{n-1}{n-2}|du|^2 + R_g u^2 dV_{\tilde{g}}}{\|u\|_p^2}$$

is called the Yamabe functional, where

$$\|u\|_p := \left(\int_M |u|^p dV_g\right)^{1/p}.$$

A positive function $u$ is a critical point of $Q_g$ if and only if $u$ satisfies (2.2) with $c = Q_g(u)/\|u\|_p^{p-2}$. The so-called Yamabe constant

$$Y(M, g) := \inf\{Q_g(u) : u \in C^\infty(M), u > 0\}$$

is a conformal invariant.

Remark 2.1 (See [15, p. 49]). Note that $Q_g$ is continuous on the Sobolev space $H^1(M)$ and $Q_g(u) = Q_g(|u|)$. Since $C^\infty(M)$ is dense in $H^1(M)$ and a nonnegative function can be approximated in $H^1(M)$ by positive functions, we actually have

$$Y(M, g) = \inf_{u \in H^1(M) \setminus \{0\}} Q_g(u).$$

The study of the Yamabe equation yields the following famous result [1, 19, 25, 27].

Theorem 2.2 (See [1, 19, 25, 27]). Let $(M^n, g, J)$ be a closed, connected Riemannian manifold. Then there exists a conformal metric $\tilde{g}$ with constant scalar curvature $R_{\tilde{g}} = Y(M, g)$.

From now on, we suppose that $(M^n, g, J)$ is a closed, connected almost Hermitian manifold, $n \geq 6$. It is clear that $(M^n, \tilde{g}, J)$ is still an almost Hermitian manifold. Suppose that $R^n_g$ and $R^n_{\tilde{g}}$ denote the $*$-scalar curvatures of $(M^n, g, J)$ and $(M^n, \tilde{g}, J)$, respectively, and del Rio and Simanca [5] showed that

$$4\Delta_g u + R^n_g u = R^n_{\tilde{g}} u^{p-1}. \quad (2.3)$$

They also defined a conformal invariant and proved the following result.
Theorem 2.3 (See [5]). Let $(M^n, g, J)$ be a closed, connected almost Hermitian manifold, $n \geq 6$. Then there exists a conformal metric $\tilde{g}$ such that $(M^n, \tilde{g}, J)$ has constant $\ast$-scalar curvature $R_{\tilde{g}}^\ast$.

Remark 2.4. In [5], del Rio and Simanca claimed that the above theorem also holds for $n = 4$, but we find a gap in their estimate. We think another approach would be required to prove this case as in the classic Yamabe problem for the dimension less than 6.

As [5, p.199] said, it will be rare that an almost Hermitian manifold has both the constant scalar curvature and constant $\ast$-scalar curvature. Therefore, it is nontrivial to study the Yamabe problem for the holomorphic d-scalar curvature on almost Hermitian manifolds. In the following, we discuss this problem.

Firstly, by (2.1) and (2.3), we have

$$4\Delta_g u + S_J u = \tilde{S}_J u^{\ast-1}, \quad (2.4)$$

where $S_J$ and $\tilde{S}_J$ denote the holomorphic d-scalar curvatures of $(M^n, g, J)$ and $(M^n, \tilde{g}, J)$, respectively. Naturally, we consider the functional

$$Q_{g,J}(u) := \frac{\int_M \tilde{S}_J dV_{\tilde{g}}}{(\int_M dV_{\tilde{g}})^{\frac{2}{p}}} = \frac{\int_M 4|du|^2 + S_J u^2 dV_g}{\|u\|_p^2}.$$  

Since for any $v \in C^\infty(M)$,

$$\left.\frac{d}{dt}\right|_{t=0} Q_{g,J}(u + tv) = 2\frac{2}{\|u\|_p^2} \int_M (4\Delta_g u + S_J u - Q_{g,J}(u)\|u\|_p^{2-p} u^{p-1}) dV_g,$$

a positive function $u$ is a critical point of $Q_{g,J}$ if and only if $u$ satisfies (2.4) with $\tilde{S}_J = Q_{g,J}(u)/\|u\|_p^{p-2}$.

Therefore, we can also analogously define the following conformal invariant:

$$Y(M, g, J) := \inf \{Q_{g,J}(u) : u \in C^\infty(M), u > 0\}. \quad (2.6)$$

2.2 Some results from partial differential equations

There are two theorems about the equation

$$4\frac{n-1}{n-2}\Delta_g u + h u = \lambda f u^{p-1}, \quad (2.7)$$

where $h, f \in C^\infty(M)$ with $f > 0$, and $\lambda$ is a real number to be determined.

Theorem 2.5 (See [2, p.131]). Let

$$I(u) := \left(\int_M 4\frac{n-1}{n-2}|du|^2 + hu^2 dV_g\right)^{\frac{2}{n}} \left(\int_M fu^{\ast-1} dV_g\right)^{-\frac{2}{n}},$$

and

$$\nu := \inf_{u \in H^1(M) \setminus \{0\}} I(u).$$

Then

$$\nu \leq n(n-1)\omega_n^{\frac{2}{n}} (\sup f)^{-\frac{2}{p}},$$

where $\omega_n$ denotes the volume of $(S^n, \tilde{g})$. In addition, if

$$\nu < n(n-1)\omega_n^{\frac{2}{n}} (\sup f)^{-\frac{2}{p}},$$

then (2.7) has a smooth positive solution for $\lambda = \nu$. 

Theorem 2.6 (See [2, p. 131]). Let \( n \geq 4 \), and \( P \) be a point where \( f \) achieves the maximum. If
\[
h(P) - R_g(P) + \frac{n - 4}{2} \Delta f(P) < 0,
\]
then (2.7) has a smooth positive solution for \( \lambda = \nu \).

If we take
\[
h = \frac{n - 1}{n - 2} S_J, \quad f = \frac{n - 1}{n - 2}
\]
in (2.7), then
\[
\nu = \left( \frac{n - 1}{n - 2} \right)^2 Y(M, g, J)
\]
and \( \Delta f = 0 \). Hence, we have the following results.

Proposition 2.7. Let \((M^n, g, J)\) be a closed, connected almost Hermitian manifold. Then
\[
Y(M, g, J) \leq n(n - 2)\omega_n\omega_n.
\]
In addition, if
\[
Y(M, g, J) < n(n - 2)\omega_n\omega_n,
\]
then there exists a conformal metric \( \tilde{g} \) such that \((M^n, \tilde{g}, J)\) has constant holomorphic d-scalar curvature \( \tilde{S}_J = Y(M, g, J) \).

Proposition 2.8. Let \((M^n, g, J)\) be a closed, connected almost Hermitian manifold. If there exists a point \( P \in M \) such that
\[
\frac{n - 1}{n - 2} S_J(P) - R_g(P) < 0,
\](2.8)
then there exists a conformal metric \( \tilde{g} \) such that \((M^n, \tilde{g}, J)\) has constant holomorphic d-scalar curvature \( \tilde{S}_J = Y(M, g, J) \).

2.3 Existence of the constant holomorphic d-scalar curvature

Let \( W \) be the Weyl tensor of \((M^n, g)\), i.e.,
\[
R = -\frac{R_g}{2(n - 1)(n - 2)} g \odot g + \frac{1}{n - 2} \text{Ric} \odot g + W,
\]
where \( \odot \) denotes the Kulkarni-Nomizu product, i.e.,
\[
h \odot k(X, Y, Z, U) := h(X, Z)k(Y, U) + h(Y, U)k(X, Z)
\]
\[- h(X, U)k(Y, Z) - h(Y, Z)k(X, U)
\]
for any symmetric 2-tensors \( h \) and \( k \) and any smooth vector fields \( X, Y, Z, W \in \mathfrak{X}(M) \) [3, p. 47] (where the sign convention is different from that in [5, 15]). The following useful formula was proved in [5]:
\[
(n - 1)R_g^* - R_g = 2(n - 1)\tilde{W}(\omega^#, \omega^#),
\](2.9)
where \( \tilde{W} \) is defined by \( \tilde{W}(X \wedge Y, Z \wedge W) := W(X, Y, Z, W) \) and \( \omega^# \) denotes the dual tensor of the fundamental form \( \omega \).

By the formula (2.9), the inequality (2.8) is equivalent to
\[
\tilde{W}(\omega^#, \omega^#)(P) > 0.
\]
Therefore, we are going to prove the existence in the following three cases:

1. There exists a \( P \in M \) such that \( \tilde{W}(\omega^#, \omega^#)(P) > 0 \).
2. \( \tilde{W}(\omega^#, \omega^#) \equiv 0 \) on \( M \).
3. \( \tilde{W}(\omega^#, \omega^#) \leq 0 \) and there exists a \( P \in M \) such that \( \tilde{W}(\omega^#, \omega^#)(P) < 0 \).

For the case (1), Proposition 2.8 implies the following result.
Proposition 2.9. Let \((M^n,g,J)\) be a closed, connected almost Hermitian manifold. If there exists a point \(P \in M\) such that \(\tilde{W}(\omega^#,\omega^#)(P) > 0\), then there exists a conformal metric \(\tilde{g}\) such that \((M^n,\tilde{g},J)\) has constant holomorphic d-scalar curvature \(\tilde{S}_J = Y(M,g,J)\).

For the case (2), the formula (2.9) implies \((n-1)R_g = R_g\). It follows that
\[
S_J = R_g - R_g^* = \frac{n-2}{n-1}R_g,
\]
and hence,
\[
Y(M,g,J) = \frac{n-2}{n-1}Y(M,g).
\]

Theorems B and C in [15] mean that
\[
Y(M,g) \leq n(n-1)\omega_n^2,
\]
and the equality holds if and only if \((M^n,g)\) is conformally equivalent to the round sphere. Therefore,
\[
Y(M,g,J) = n(n-2)\omega_n^2
\]
if and only if \((M^n,g)\) is conformally equivalent to \((S^n,\hat{g})\), i.e., there exists a diffeomorphism \(\varphi : M \to S^n\) such that \(g\) is conformal to \(\varphi^*\hat{g}\). Then for \((M^n,\varphi^*\hat{g},J)\), we still have \(\tilde{W}(\omega^#,\omega^#) \equiv 0\) and hence,
\[
S_J = \frac{n-2}{n-1}R_g = \frac{4}{5} \cdot 30 = 24.
\]

Now by Proposition 2.7, we can conclude the case (2) with the following result.

Proposition 2.10. Let \((M^n,g,J)\) be a closed, connected almost Hermitian manifold. Suppose \(\tilde{W}(\omega^#,\omega^#) \equiv 0\) on \(M\).

(a) If \(Y(M,g,J) < n(n-2)\omega_n^2\), then there exists a conformal metric \(\tilde{g}\) such that \((M^n,\tilde{g},J)\) has constant holomorphic d-scalar curvature \(S_J = Y(M,g,J)\).

(b) \(Y(M,g,J) = n(n-2)\omega_n^2\) if and only if \((M^n,g)\) is conformally equivalent to \((S^n,\hat{g})\). If this happens, then there exists a conformal metric \(\tilde{g}\) such that \((M^n,\tilde{g},J)\) has constant holomorphic d-scalar curvature \(S_J = 24\).

For the case (3), we are going to prove
\[
Y(M,g,J) < n(n-2)\omega_n^2
\]
by the approach in [5]. By the assumption of the case (3), we can take a point \(P \in M\) such that
\[
\tilde{W}(\omega^#,\omega^#)(P) < 0.
\]

Next, we need the following results.

Lemma 2.11 (See [15, Lemma 5.5]). Let \((M,g)\) be a Riemannian manifold, and \(P \in M\). In g-normal coordinates centered at \(P\), the function \(\det g_{ij}\) has the expansion
\[
\det g_{ij} = 1 - \frac{1}{3} \text{Ric}_{ij}(P)x_ix_j + O(|x|^3).
\]

Theorem 2.12 (See [15, Theorem 5.2]). Let \((M,g)\) be a Riemannian manifold, and \(P \in M\). Let \(k \geq 0\), and \(T\) be a symmetric \((k+2)\)-tensor on \(T_PM\). Then there exists a unique homogeneous polynomial \(f\) of degree \(k + 2\) in \(g\)-normal coordinates such that the conformal metric \(\tilde{g} = e^{2f}g\) satisfies
\[
\text{Sym}(\tilde{\nabla}^k\tilde{\text{Ric}}(P)) = T,
\]
where \(\tilde{\nabla}\) and \(\tilde{\text{Ric}}\) denote the Levi-Civita connection and the Ricci tensor of \((M,\tilde{g})\), respectively.
Let \( l \) be a real number to be determined, and

\[
T = l \operatorname{Sym} \left( -\frac{1}{2} \sum_{i=1}^{n} W(e_i, \cdot, J, Je_i) \right)(P).
\]

By Theorem 2.12, we have a homogeneous polynomial \( f \) of degree 2 in \( g \)-normal coordinates centered at \( P \) such that the conformal metric \( \tilde{g} = e^{2f}g \) satisfies \( \tilde{\operatorname{Ric}}(P) = T(P) \). Since \( \operatorname{tr}_g T = l \tilde{W}(\omega^#, \omega^#)(P) \) and \( f(P) = 0 \), we have

\[
R_{\tilde{g}}(P) = \operatorname{tr}_{\tilde{g}} \tilde{\operatorname{Ric}}(P) = \operatorname{tr}_g T = l \tilde{W}(\omega^#, \omega^#)(P).
\]

Note that \( \tilde{W}(\omega^#, \omega^#)(P) \) is invariant under this conformal deformation. We can assume, without loss of generality, that

\[
R_g(P) = l \tilde{W}(\omega^#, \omega^#)(P),
\]

and then by the formula (2.9), we have

\[
S_f(P) = \left( \frac{n-2}{n-1} - 2 \right) \tilde{W}(\omega^#, \omega^#)(P).
\]

In the \( g \)-normal coordinates centered at \( P \) (fixed above for \( T \)), let \( \eta \) be a radial cut-off function supported in the ball \( B_{2\varepsilon} \) with \( \eta = 1 \) in \( B_\varepsilon \). We consider the test function \( \phi = \eta u_\alpha \), where

\[
u_\alpha(x) := \left( \frac{\alpha}{|x|^2 + \alpha^2} \right)^{\frac{n+2}{2}}
\]

and \( 0 < \alpha \ll \varepsilon \ll 1 \). The function \( u_\alpha \) satisfies the elliptic equation

\[
\Delta u_\alpha + n(n - 2)u_\alpha^{p-1} = 0 \quad \text{on} \quad \mathbb{R}^n
\]

and

\[
4(n-1) \left( \int_{\mathbb{R}^n} |u_\alpha|^p dx \right)^{\frac{2}{p}} = \int_{\mathbb{R}^n} 4 \frac{n+1}{n-1} |u_\alpha|^2 dx = Y(S^n, \tilde{g}) = n(n-1)\omega^2_n.
\]

Hence, we have the estimate (see [20, Chapter 5] for detailed calculations)

\[
\int_{B_{\varepsilon}} |du_\alpha|^2 dx \leq \frac{1}{4} n(n-2)\omega^2_n \|\phi\|_p^2.
\]

(2.12)

Setting \( r = |x| \), we also have the following lemma.

**Lemma 2.13** (See [15, Lemma 3.5]). Suppose \( k > -n \). Then as \( \alpha \to 0 \),

\[
\int_0^\varepsilon r^k u_\alpha^2 r^{n-1} dr = \begin{cases} 
O(\alpha^{k+2}), & \text{if } n > k + 4, \\
O\left(\alpha^{k+2} \ln \frac{1}{\alpha}\right), & \text{if } n = k + 4, \\
O(\alpha^{n-2}), & \text{if } n < k + 4. 
\end{cases}
\]
By Lemma 2.11 and Taylor’s theorem, we have

\[
Q_{g,J}(\varphi)\|\varphi\|_p^2 = \int_M 4|d\varphi|^2 + S_J \varphi^2 dV_g \\
= \int_{B_2} 4|d\alpha_n|^2 + S_J \alpha_n^2 dV_g + \int_{B_2 \setminus B_1} 4|d\varphi|^2 + S_J \varphi^2 dV_g \\
\leq \int_{B_2} 4|d\alpha_n|^2 \left(1 - \frac{1}{6} \text{Ric}_{ij}(P)x_1 x_j + O(r^3)\right) dx \\
+ \int_{B_2} S_J \alpha_n^2 (1 + O(r^2)) dx + E_1 \\
\leq 4 \int_{B_2} |d\alpha_n|^2 dx + 2 \frac{2}{3} \int_{B_2} \text{Ric}_{ij}(P) x_i x_j |d\alpha_n|^2 dx + \int_{B_2} |d\alpha_n|^2 O(r^3) dx \\
+ \int_{B_2} (S_J(P) + O(r)) \alpha_n^2 dx + \int_{B_2} S_J \alpha_n^2 O(r^2) dx + E_1 \\
=: 4 \int_{B_2} |d\alpha_n|^2 dx + E_1 + E_2 + E_3,
\]

where

\[
E_1 = \int_{B_2 \setminus B_1} (4|d\varphi|^2 + S_J \varphi^2)(1 + O(r^2)) dx, \\
E_2 = \int_{B_2} |d\alpha_n|^2 O(r^3) dx + \int_{B_2} O(r) \alpha_n^2 dx + \int_{B_2} S_J \alpha_n^2 O(r^2) dx, \\
E_3 = -\frac{2}{3} \int_{B_2} \text{Ric}_{ij}(P) x_i x_j |d\alpha_n|^2 dx + \int_{B_2} S_J(P) \alpha_n^2 dx.
\]

Note that \(u_n^2(x) \leq \alpha^{n-2} r^{4-2n}\) and

\[
|d\alpha_n|^2 = (n-2)^2 \frac{|x|^2}{\alpha^2} \left(\frac{\alpha}{|x|^2 + \alpha^2}\right)^n = (n-2)^2 \left(\frac{r}{r^2 + \alpha^2}\right)^2 u_n^2.
\]

Then by Lemma 2.13, it is not hard to see that

\[
E_1 \leq C(n) \alpha^{n-2} \int_0^{2\varepsilon} r^{1-n} dr = O(\alpha^{n-2}), \tag{2.13}
\]

and for \(n \geq 6\),

\[
E_2 \leq C(n) \int_0^\varepsilon u_n^2 r^{n-1} dr = O(\alpha^3). \tag{2.14}
\]

Finally, we deal with \(E_3\). By (2.10) and (2.11), we have

\[
E_3 = -\frac{2}{3} \omega_{n-1}^{-1} B_2(P) \int_0^\varepsilon r^{n+1} |d\alpha_n|^2 dr + \omega_{n-1} S_J(P) \int_0^\varepsilon r^{n-1} u_n^2 dr \\
= \omega_{n-1} \hat{W}(\omega^*, \omega^*) \int_0^\varepsilon \left[-\frac{2l}{3n} r^{n+1} |d\alpha_n|^2 + \left(\frac{n-2}{n-1} - 2\right) r^{n-1} u_n^2 \right] dr \\
= \omega_{n-1} \hat{W}(\omega^*, \omega^*) \int_0^\varepsilon \left[-\frac{2(n-2)^2 r^4}{3n(r^2 + \alpha^2)^2} l + \frac{n-2}{n-1} l - 2\right] r^{n-1} u_n^2 dr.
\]
necessarily compatible with $\phi$.

Basic calculations show that

$$\int_0^\varphi \left[ - \frac{2(n-2)^2 r^4}{3n r^4 + \alpha^2} l + \frac{n-2}{n-1} l - 2 \right] r^{n-1} u_0^2 dr$$

$$= \alpha^2 \int_0^\varphi \left[ - \frac{2(n-2)^2}{3n} l + \frac{n-2}{n-1} l - 2 \right] \frac{\sigma^{-1}}{(\sigma^2 + 1)^n} d\sigma$$

$$= \alpha^2 \int_0^\varphi \left[ (2n^2 - 9n + 2) \sigma^4 - 2(3n + 2) \sigma^2 - (3n + 2) \right] \frac{\sigma^{-1}}{(\sigma^2 + 1)^n} d\sigma.$$  

Setting $r = \alpha \sigma$ and $l = -\frac{3n(n-1)}{n-2}$, we have

$$\lim_{a \to +\infty} \int_0^a [(2n^2 - 9n + 2) \sigma^4 - 2(3n + 2) \sigma^2 - (3n + 2)] \frac{\sigma^{-1}}{(\sigma^2 + 1)^n} d\sigma = c(n) > 0,$$

where $c(n) = \frac{(n^2 - 2n + 1)(n-1)^2}{(n-1)(n-2)(2n-3)}$ with $n = 2m \geq 6$. Hence for sufficiently small $\alpha$, there is a positive constant $C(n)$ such that

$$E_3 = C(n) \widehat{W}(\omega^\#)(P) \alpha^2.$$

Therefore, (12.12)-(12.15) imply that

$$Q_{g,J}(\varphi) \|\varphi\|^2_{\hat{g}} \leq n(n-2) \omega_\hat{g}^2 \|\varphi\|^2_{\hat{g}} + C(n) \widehat{W}(\omega^\#)(P) \alpha^2 + o(\alpha^2).$$

As $\widehat{W}(\omega^\#)(P) < 0$, by taking sufficiently small $\alpha$, we get

$$Y(M,g,J) \leq Q_{g,J}(\varphi) < n(n-2) \omega_\hat{g}^2.$$

Then by Proposition 2.7, we have proved the following proposition.

**Proposition 2.14.** Let $(M^n,g,J)$ be a closed, connected almost Hermitian manifold, $n \geq 6$. If $\widehat{W}(\omega^\#)(P) < 0$ and there exists a $P \in M$ such that $\widehat{W}(\omega^\#)(P) < 0$, then there exists a conformal metric $\hat{g}$ such that $(M^n,\hat{g},J)$ has constant holomorphic d-scalar curvature $S_J = Y(M,g,J)$.

Combining Propositions 2.9, 2.10 and 2.14, we have proved the following theorem.

**Theorem 2.15.** Let $(M^n,g,J)$ be a closed, connected almost Hermitian manifold, $n \geq 6$. Then there exists a conformal metric $\hat{g}$ such that $(M^n,\hat{g},J)$ has constant holomorphic d-scalar curvature $S_J = Y(M,g,J)$.

### 3 Conformal equivalence and the prescribed curvature problem

#### 3.1 Conformal equivalence for almost Hermitian manifolds

Let $(M^n,g,J)$ be an almost Hermitian manifold, and $\varphi : M \to M$ be a diffeomorphism. Since $J$ is not necessarily compatible with $\varphi^* g$, we consider another almost complex structure $J_\varphi := d\varphi^{-1} \circ J \circ d\varphi$. Since

$$\varphi^* g(J_\varphi X, J_\varphi Y) = g(J \circ d\varphi(X), J \circ d\varphi(Y)) = g(d\varphi(X), d\varphi(Y)) = \varphi^* g(X,Y),$$

$(M^n,\varphi^* g, J_\varphi)$ is an almost Hermitian manifold. For the sake of completeness, we verify some basic properties as follows.

**Lemma 3.1.** Let $(M^n,g,J)$ be an almost Hermitian manifold, and $\varphi : M \to M$ be a diffeomorphism.

1. The $*$-Ricci tensor of $(M^n,\varphi^* g, J_\varphi)$ is $\varphi^* \text{Ric}^*$.  
2. The $*$-scalar curvature of $(M^n,\varphi^* g, J_\varphi)$ is $R_\varphi^* \circ \varphi$.  
3. The holomorphic d-scalar curvature of $(M^n,\varphi^* g, J_\varphi)$ is $S_J \circ \varphi$.  


Proof. To prove (1), assume that \{e_i\} is a smooth local orthonormal frame of \((M^n, g)\). Then \{d\varphi^{-1}(e_i)\} is a smooth local orthonormal frame of \((M^n, \varphi^* g)\) and thus the \(*\)-Ricci tensor of \((M^n, \varphi^* g, J_\varphi)\) is

\[
(X, Y) \mapsto -\sum_{i=1}^{n} \varphi^* g(J_\varphi(\varphi^* R(X, d\varphi^{-1}(e_i))(J_\varphi Y)), d\varphi^{-1}(e_i))
\]

\[
= \sum_{i=1}^{n} \varphi^* g(\varphi^* R(X, d\varphi^{-1}(e_i))(J_\varphi Y), J_\varphi \circ d\varphi^{-1}(e_i))
\]

\[
= \sum_{i=1}^{n} \varphi^* R(X, d\varphi^{-1}(e_i), J_\varphi Y, d\varphi^{-1} \circ J(e_i))
\]

\[
= \sum_{i=1}^{n} \varphi^* (d\varphi(X), e_i, J \circ d\varphi(Y), J e_i)
\]

\[
= (\varphi^* \text{Ric})(X, Y).
\]

(2) follows from (1) by taking the trace. Since the scalar curvature of \((M^n, \varphi^* g)\) is \(R_g \circ \varphi\), (3) follows from (2) immediately.

Consider the metric \(\hat{g} = \varphi^* (u^{p-2} g)\) on \(M\), where \(p = \frac{2n}{n-2}\). Then Lemma 3.1 and the equation (2.4) imply the following lemma.

Lemma 3.2. Suppose that \(S_J\) and \(\hat{S}_J\) denote the holomorphic d-scalar curvatures of \((M^n, g, J)\) and \((M^n, \hat{g}, J_\varphi)\), respectively. Then

\[
4\Delta_g u + S_J u = (\hat{S}_J \circ \varphi^{-1}) u^{p-1}.
\]

3.2 The prescribed holomorphic d-scalar curvature problem

Let \((M^n, g, J)\) be a closed, connected almost Hermitian manifold with the holomorphic d-scalar curvature \(S_J\). Given \(K \in C^\infty(M)\), we want to realize \(K\) as the holomorphic d-scalar curvature of some \((M^n, \hat{g}, J_\varphi)\). Let

\[
\mathcal{S}(M, g, J) := \{\text{holomorphic d-scalar curvature functions of some } (M^n, \hat{g}, J_\varphi)\}.
\]

By the approach of Kazdan and Warner [11], we can obtain the following results for the prescribed holomorphic d-scalar curvature problem.

Lemma 3.3. If there is a constant \(c > 0\) such that

\[
\min_M K < cS_J < \max_M K,
\]

then \(K \in \mathcal{S}(M, g, J)\).

Proof. The proof of [11, Lemma 6.1] actually deals with a general operator

\[
T(u) := u^{-a}(\alpha\Delta_g u + ku),
\]

where \(a > 1\), \(\alpha > 0\) and \(k \in C^\infty(M)\) (where the sign of the Laplacian operator is different from those in [11, 12]). In fact, if there is a constant \(c > 0\) such that

\[
\min_M K < ck < \max_M K,
\]

then there exist a positive function \(u \in C^\infty(M)\) and a diffeomorphism \(\varphi\) of \(M\) such that \(T(u) = k \circ \varphi^{-1}\). By Lemma 3.2, we only need to take \(a = p - 1\), \(\alpha = 4\) and \(k = S_J\).

The first eigenvalue of the operator \(L_g(u) = 4\frac{n-1}{n-2}\Delta_g u + R_g u\) plays an important role in Kazdan and Warner’s work [11, 12]. Here, we need the first eigenvalue \(\lambda_1(g, J)\) of the operator \(L_{g, J}(u) = 4\Delta_g u + S_J u\). By [12, Theorem 2.11, Remark 2.12 and Theorem 3.2], it is easy to see that the sign of \(\lambda_1(g, J)\) is a conformal invariant. Since \(\lambda_1(\hat{g}, J) = \hat{S}_J\) for the conformal metric \(\hat{g}\) such that \(\hat{S}_J = Y(M, g, J)\), we know that the conformal invariant \(Y(M, g, J)\) has the same sign as \(\lambda_1(g, J)\). So we can replace \(\lambda_1(g, J)\) by \(Y(M, g, J)\) in the statement of the following theorem.
Theorem 3.4. Let \((M^n, g, J)\) be a closed, connected almost Hermitian manifold, \(n \geq 6\).

1. If \(Y(M, g, J) < 0\), then \(S(M, g, J)\) is precisely the set of smooth functions that are negative somewhere on \(M\).

2. If \(Y(M, g, J) = 0\), then \(S(M, g, J)\) is precisely the set of smooth functions that either change the signs or are identically zero on \(M\).

3. If \(Y(M, g, J) > 0\), then \(S(M, g, J)\) is precisely the set of smooth functions that are positive somewhere on \(M\).

Proof. Combining Theorem 2.15 and Lemma 3.3, we see that the argument of [11, Theorem 6.2] is still valid.

Remark 3.5. For the \(s\)-scalar curvature, we can obtain similar results.

4 Relations to balanced metrics

At first, we state the definition of the conformally balanced metric in our terminology.

Definition 4.1 (See [16, Definition 3.13]). Let \((M^n, g, J)\) be a Hermitian manifold. The Riemannian metric \(g\) is called a balanced metric if \(\delta\omega = 0\), i.e., \((M^n, g, J) \in \mathcal{W}_3\). \(g\) is said to be conformally balanced if it is conformal to a balanced metric.

Based on Gauduchon’s work [8], Fu and Zhou [7] showed the following result.

Theorem 4.2 (See [7, Theorems 2.1 and 2.3]). Let \((M^n, g, J)\) be a closed almost Hermitian manifold.

1. If \((M^n, g, J) \in \mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4\), then \(\int_M S_f dV_g \geq 0\). The equality holds if and only if \((M^n, g, J)\) is a balanced manifold.

2. If \((M^n, g, J) \in \mathcal{W}_2 \oplus \mathcal{W}_3\), then \(\int_M S_f dV_g \leq 0\). The equality holds if and only if \((M^n, g, J)\) is a balanced manifold.

Similar results were also found in [10]. Combining this with Theorem 2.15, we have the following properties for the conformal invariant \(Y(M, g, J)\).

Theorem 4.3. Let \((M^n, g, J)\) be a closed, connected almost Hermitian manifold, \(n \geq 6\).

1. If \((M^n, g, J) \in \mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4\), then \(Y(M, g, J) \geq 0\). In addition, \(Y(M, g, J) = 0\) if and only if \(J\) is integrable and \(g\) is conformally balanced.

2. If \((M^n, g, J) \in \mathcal{W}_2 \oplus \mathcal{W}_3\), then \(Y(M, g, J) \leq 0\). In addition, \(Y(M, g, J) = 0\) if and only if \((M^n, g, J)\) is balanced.

Proof. To prove (1), we point out that the class \(\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4\) is conformally invariant [9, Theorem 4.2]. Thus Theorem 4.2(1) implies \(Y(M, g, J) \geq 0\). If \(Y(M, g, J) = 0\), then by Theorem 2.15, there exists a conformal metric \(\tilde{g}\) such that \((M^n, \tilde{g}, J)\) has constant holomorphic \(d\)-scalar curvature \(\tilde{S}_f = 0\). Hence, Theorem 4.2(1) implies that \(J\) is integrable and \(g\) is conformally balanced. Conversely, if \(J\) is integrable and \(g\) is conformally balanced, then there exists a conformal metric \(\tilde{g}\) such that \((M^n, \tilde{g}, J)\) is balanced. Theorem 4.2(1) implies that \(\int_M \tilde{S}_f dV_{\tilde{g}} = 0\). Then we have \(Y(M, g, J) \leq 0\), and hence \(Y(M, g, J) = 0\).

Next, we prove (2). If \((M^n, g, J) \in \mathcal{W}_2 \oplus \mathcal{W}_3\), then \(S_f \leq 0\) (see the proof of [7, Theorem 2.3]), and thus \(Y(M, g, J) \leq 0\). Theorem 4.2(2) implies that \((M^n, g, J)\) is balanced if \(Y(M, g, J) = 0\). Conversely, if \((M^n, g, J)\) is balanced, then (1) implies that \(Y(M, g, J) \geq 0\), and hence \(Y(M, g, J) = 0\).

Corollary 4.4. Let \((M^n, g, J)\) be a closed, connected almost Hermitian manifold, \(n \geq 6\). If \(Y(M, g, J) < 0\), then \(J\) is not integrable.

Proof. If \(J\) is integrable, then \((M^n, g, J) \in \mathcal{W}_3 \oplus \mathcal{W}_4\) and thus \(Y(M, g, J) \geq 0\), which leads to a contradiction.

Fu et al. [6] constructed balanced metrics on some non-Kähler Calabi-Yau threefolds \((M^n, g, J)\) including \(\mathbb{S}^3 \times \mathbb{S}^3\) (\(k \geq 2\)). By Theorems 4.3(1) and 2.15, these Hermitian manifolds have \(Y(M^n, g, J) = 0\) and holomorphic \(d\)-scalar curvature 0 under some conformal metric, the same as all the Kähler manifolds.
5 A variation of $Y(M, g, J)$

For every compatible almost complex structure $J$ of $(S^6, \tilde{g})$, direct calculations imply that $(S^6, \tilde{g}, J)$ has constant holomorphic d-scalar curvature 24 and $Y(S^6, \tilde{g}, J) = 24\omega_6^\sharp$. Motivated by this phenomenon, we consider the variation of $Y(M, g, J)$ with respect to $J$. We mainly use the technology of Wang and Zheng [26].

**Theorem 5.1.** (See [26, Theorem 1.1]). Let $f(t)$ be a continuous function on an interval $I \subset \mathbb{R}$. Suppose that for any $t \in I$, there exists a $C^1$ function $F(t, s)$ of $s$ defined on a neighborhood of $t$ such that $F(t, t) = f(t)$ and $F(t, s) \geq f(s)$.

1. If $\frac{\partial F}{\partial s}(t, t)$ is locally bounded, then $f(t)$ is locally Lipschitz.
2. For any $t$ in the interior of $I$, if $f(t)$ is differentiable at $t$, then

$$f'(t) = \frac{\partial F}{\partial s}(t, t).$$

Let $(M^n, g, J)$ be a closed, connected almost Hermitian manifold, $n \geq 6$. By Theorem 2.15, we know that the set

$$C^Y(M, g, J) := \{ u \in C^\infty(M) : Q_{g,J}(u) = Y(M, g, J), \| u \|_p = 1 \}$$

is always nonempty. Let $\mathcal{J}_g(M)$ denote the set of compatible almost complex structures of $(M, g)$, and $J(t)$ be a smooth one-parameter family in $\mathcal{J}_g(M)$, $t \in (-\varepsilon, \varepsilon)$. Then for each $t$, there exists a $u(t) \in C^Y(M, g, J(t))$ such that $Q_{g,J(t)}(u(t)) = Y(M, g, J(t))$. For any $t, s \in (-\varepsilon, \varepsilon)$, let

$$f(t) := Y(M, g, J(t)) \quad \text{and} \quad F(t, s) := Q_{g,J(s)}(u(t)).$$

Then we have

$$F(t, t) = f(t) \quad \text{and} \quad F(t, s) \geq f(s).$$

It is easy to see that $F$ is $C^\infty$ on $s$, and the continuity of $f$ follows from the next lemma.

**Lemma 5.2.** The functional

$$Y(M, g, \cdot) : \mathcal{J}_g(M) \to \mathbb{R}, \quad J \mapsto Y(M, g, J)$$

is continuous, where $\mathcal{J}_g(M)$ is equipped with the $C^0$ topology.

**Proof.** For any $u \in C^\infty(M)$ with $\| u \|_p = 1$, Hölder’s inequality implies that

$$\int_M u^2 dV_g \leq \| u \|^2_2 \| 1 \|^2_2 = (\text{Vol}(M, g))^\frac{2}{n}.$$

Then for any $J_0, J \in \mathcal{J}_g(M)$,

$$|Q_{g,J_0}(u) - Q_{g,J}(u)| = \left| \int_M (S_{J_0} - S_J) u^2 dV_g \right| \leq \| S_{J_0} - S_J \|_{C^0} \int_M u^2 dV_g \leq 2n|J_0 - J|_{C^0,g} \| R \|_{C^0,g}(\text{Vol}(M, g))^\frac{2}{n},$$

where the last inequality follows from the fact that for any local orthonormal frame,

$$|S_{J_0} - S_J| = \sum_{i,j,k,l=1}^n ((J_0^k)^j_i - (J_0^k)^j_i) R_{ijkl} \leq \sum_{i,j,k,l=1}^n \| (J_0^k)^j_i - (J_0^k)^j_i \| R_{ijkl}$$

$$= \sum_{i,j,k,l=1}^n \| (J_0^k)_{ijkl} - J^k_{ijkl} \| R_{ijkl} \leq \sum_{i,j,k,l=1}^n \| (J_0^k - J^k_{ijkl}) R_{ijkl} \|$$

$$\leq 2 \sum_{i,j,k,l=1}^n \| (J_0^k - J^k_{ijkl}) \| R_{ijkl} \| \leq 2n|J_0 - J| |R|_g.$$
Thus for each $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon, n, g) > 0$ such that for any $J_0, J \in \mathcal{J}_g(M)$ with $\|J_0 - J\|_{C^0, g} < \delta$,
\[ Q_{g, J_0}(u) - \varepsilon < Q_{g, J}(u) < Q_{g, J_0}(u) + \varepsilon. \]
Note that $Y(M, g, J) = Q_{g, J}(u)$ for some $u \in C^\infty(M, g, J)$. Hence we have
\[ Y(M, g, J_0) - \varepsilon < Y(M, g, J) < Y(M, g, J_0) + \varepsilon. \]
This implies the continuity of $Y(M, g, \cdot)$. \hfill \square

To simplify the next formula, we introduce the $J$-Ricci form $\rho^J(X, Y) := -\text{Ric}^*(X, JY)$ (see [7, 21]). It is easy to verify that
\[ \rho^J(X, Y) = -\text{Ric}^*(X, JX) = \text{Ric}^*(X, JY) = -\rho^J(X, Y), \]
i.e., $\rho^J$ is a smooth 2-form.

**Proposition 5.3.** Let $(M^n, g, J)$ be a closed, connected almost Hermitian manifold, $n \geq 6$. Let $J(t)$ be a smooth one-parameter family in $\mathcal{J}_g(M)$ such that $J(0) = J$, and $u(t) \in C^\infty(M, g, J(t))$, $t \in (-\varepsilon, \varepsilon)$.

Then
\[ \frac{\partial Y(M, g, J(t))}{\partial t}(t) \stackrel{a.c.}{=} -2 \int_M \langle (J')^2(t), \rho^J(t) \rangle_g u^2(t) dV_g, \]
where $(J')^2$ is obtained from $J'$ by lowering an index.

**Proof.** Let $f(t)$ and $F(t, s)$ be functions defined in (5.1). For each $t \in (-\varepsilon, \varepsilon)$,
\[ \frac{\partial F}{\partial s}(t, s) = \frac{\partial}{\partial s} Q_{g, J(s)}(u(t)) = -\int_M \frac{\partial R^*_g}{\partial s}(s) u^2(t) dV_g. \]

For any smooth local orthonormal frame $(e_i)_{i=1}^n$,
\[ \frac{\partial R^*_g}{\partial s}(s) = \frac{\partial}{\partial s} \sum_{i,j=1}^n R(e_i, e_j, J(s)e_i, J(s)e_j) \\
= \sum_{i,j=1}^n R(e_i, e_j, J'(s)e_i, J(s)e_j) + R(e_i, e_j, J(s)e_i, J'(s)e_j) \\
= 2 \sum_{i,j=1}^n R(e_i, e_j, J'(s)e_i, J(s)e_j) \\
= 2 \sum_{i=1}^n \rho^{J(s)}(e_i, J'(s)e_i) = 2 \langle (J')^2(s), \rho^{J(s)} \rangle_g. \]

Since $\|u(t)\|_p = 1$ for each $t$, Hölder’s inequality implies that
\[ \left\| \frac{\partial F}{\partial s}(t, t) \right\| \leq \left\| \frac{\partial R^*_g}{\partial s}(t) \right\|_{C^0, g} \int_M u^2(t) dV_g \leq \left\| \frac{\partial R_g^*}{\partial s}(t) \right\|_{C^0, g} (\text{Vol}(M, g))^\frac{1}{2}. \]
It follows that $\frac{\partial F}{\partial s}(t, t)$ is locally bounded. Then Theorem 5.1 implies the conclusion. \hfill \square

**Proposition 5.4.** Let $(M^n, g, J)$ be a closed, connected almost Hermitian manifold which is not conformally equivalent to $(\mathbb{S}^n, \hat{g})$, $n \geq 6$. Let $(J_k)_{k \in \mathbb{N}}$ be a sequence of compatible almost complex structures of $(M, g)$ which $C^m$-converges to $J$ and $m \geq 3$. Let $u_k \in C^\infty(M, g, J_k)$ for each $k$. Then there exists a subsequence $(u_{k_i})_{i \in \mathbb{N}}$ which $C^{m-1}$-converges to a smooth function $u \in C^\infty(M, g, J)$.

**Proof.** Similar to the classic Yamabe problem, the estimation
\[ Y(M, g, J) < n(n - 2)\omega^n \]
implies a uniform bound $\|u\|_{W^{m, q, g}} \leq C$ for each $u \in C^\infty(M, g, J)$ and any $q > n$. Since $\|\text{Ric}^*(J_k)\|_{C^{m-2, g}}$ has a uniform bound for sufficiently large $k$, the argument of [18, Proposition 2.4] is still valid (see also a detailed proof in Macbeth’s PhD thesis [17]). \hfill \square
Theorem 5.5. Let \((M^n, g, J)\) be a closed, connected almost Hermitian manifold which is not conformally equivalent to \((\mathbb{S}^6, \hat{g})\), \(n \geq 6\). Let \(J(t)\) be a smooth one-parameter family in \(\mathcal{J}_g(M)\) such that \(J(0) = J\), \(t \in (-\varepsilon, \varepsilon)\). Then there exists a \(u \in C^\infty(M, g, J)\) such that
\[
\frac{\partial Y(M, g, J(t))}{\partial t}(0) = -2 \int_M \langle K^\alpha, \rho^\beta \rangle_g u^2 dV_g,
\]
where \(K = J'(0)\) and \(K^\alpha\) is obtained from \(K\) by lowering an index.

Proof. By Theorem 5.1(2), we only need to check that \(f(t)\) is differentiable at 0. For each \(t \neq 0\), we have
\[
\frac{F(t, t) - F(t, 0)}{t} \leq \frac{f(t) - f(0)}{t} \leq \frac{F(0, t) - F(0, 0)}{t}.
\]
For any \(t, s \in (-\varepsilon, \varepsilon)\), by the mean value theorem, there exists a \(\beta(t, s) \in (-s, s)\) such that
\[
F(t, s) - F(t, 0) = \frac{\partial F}{\partial s}(t, \beta(t, s)) s.
\]
Thus we have
\[
\frac{\partial F}{\partial s}(t, \beta(t, t)) \leq \frac{f(t) - f(0)}{t} \leq \frac{\partial F}{\partial s}(0, \beta(0, t)). \tag{5.2}
\]
It follows that
\[
\lim_{t \to 0} \sup \frac{f(t) - f(0)}{t} \leq \lim_{t \to 0} \sup \frac{\partial F}{\partial s}(0, \beta(0, t)) = -2 \int_M \langle K^\alpha, \rho^\beta \rangle_g u^2(0) dV_g. \tag{5.3}
\]
We can choose a sequence \(\{t_k\}_{k \in \mathbb{N}}\) such that
\[
\lim_{k \to \infty} \frac{f(t_k) - f(0)}{t_k} = \liminf_{t \to 0} \frac{f(t) - f(0)}{t}. \tag{5.4}
\]
Note that \(u(t_k) \in C^\infty(M, g, J(t_k))\) and \(\{J(t_k)\}_{k \in \mathbb{N}}\) \(C^\infty\)-converges to \(J\). Then by Proposition 5.4, we have a subsequence \(\{u(t_k)\}_{k \in \mathbb{N}}\) which \(C^\infty\)-converges to a function \(u \in C^\infty(M, g, J)\). Hence, by (5.2) and (5.4),
\[
\liminf_{t \to 0} \frac{f(t) - f(0)}{t} = \lim_{i \to \infty} \frac{f(t_k_i) - f(0)}{t_k_i} \geq \lim_{i \to \infty} \frac{\partial F}{\partial s}(t_k_i, \beta(t_k_i, t_k_i)) \geq -2 \lim_{i \to \infty} \int_M \langle (J')^\alpha(\beta(t_k_i, t_k_i)), \rho^\beta(\beta(t_k_i, t_k_i)) \rangle_g u^2(t_k_i) dV_g
\]
\[
= -2 \int_M \langle K^\alpha, \rho^\beta \rangle_g u^2 dV_g. \tag{5.5}
\]
Since we only assume \(u(0) \in C^\infty(M, g, J)\), we can choose \(u(0) = u\). Therefore, (5.3) and (5.5) imply that
\[
\liminf_{t \to 0} \frac{f(t) - f(0)}{t} \geq -2 \int_M \langle K^\alpha, \rho^\beta \rangle_g u^2 dV_g \geq \limsup_{t \to 0} \frac{f(t) - f(0)}{t}.
\]
This means that \(f(t)\) is differentiable at 0. \(\Box\)

Simanca [21] considered the variation of \(\int_M R^g dV_g\) with respect to both \(g\) and \(J\). The term \(\langle K^\alpha, \rho^\beta \rangle_g\) also appears in one of Simanca's formulas, and we are going to make this term more precise. Since \(J(t)\) is in \(\mathcal{J}_g(M)\), we have
\[
J(t)^2 = -\text{id} \quad \text{and} \quad g(J(t)X, Y) = -g(X, J(t)Y)
\]
for any \(X, Y \in \mathcal{F}(M)\). It follows that
\[
KJ = -JK \quad \text{and} \quad g(KX, Y) = -g(X, KY),
\]
or equivalently,
\[ K^b(JX, Y) = K^b(X, JY) \quad \text{and} \quad K^b(X, Y) = -K^b(Y, X). \]

Note that every smooth 2-form \( A \) has a decomposition \( A = A^{2,0+0.2} + A^{1,1} \) corresponding to the orthogonal decomposition
\[ \Omega^2(M) = \Omega^{2,0+0.2}(M) \oplus \Omega^{1,1}(M), \]
where \( A^{2,0+0.2}(JX, JY) = -A^{2,0+0.2}(X, Y) \) and \( A^{1,1}(JX, JY) = A^{1,1}(X, Y) \). Hence, we have
\[ K^b \in \Omega^{2,0+0.2}(M) \quad \text{and} \quad \langle K^b, \rho^J \rangle_g = \langle K^b, (\rho^J)^{2,0+0.2} \rangle_g. \]

In addition,
\[ (\rho^J)^{2,0+0.2}(X, Y) = \frac{1}{2}(\rho^J(X, Y) - \rho^J(JX, JY)) = -\frac{1}{2}(\text{Ric}^*(X, JY) + \text{Ric}^*(JX, Y)) = \frac{1}{2}(\text{Ric}^*(JY, X) - \text{Ric}^*(X, JY)) = (\text{Ric}^*)^{\text{skew}}(JY, X), \]
where
\[ (\text{Ric}^*)^{\text{skew}}(X, Y) := \frac{1}{2}(\text{Ric}^*(X, Y) - \text{Ric}^*(Y, X)) \]
denotes the skew-symmetric component of \( \text{Ric}^* \). Thus, we can characterize the critical point of \( Y(M, g, \cdot) \) by the following theorem.

**Theorem 5.6.** Let \((M^n, g, J)\) be a closed, connected almost Hermitian manifold, \( n \geq 6 \). Then \( J \) is a critical point of the functional \( Y(M, g, \cdot) \) if and only if for each \( \tilde{g} \in [g] \) (equivalently, for one \( \tilde{g} \in [g] \)), the \( \ast \)-Ricci curvature of \((M^n, \tilde{g}, J)\) is symmetric.

**Proof.** Let \( K \) be a smooth \((1,1)\)-tensor field on \( M \) such that
\[ KJ = -JK \quad \text{and} \quad g(KX, Y) = -g(X, KY). \]
Then one can verify that \( J(t) = J e^{-tJK} \) is a smooth one-parameter family in \( \mathcal{J}_g(M) \) such that \( J(0) = J \) and \( J'(0) = K \), where
\[ (e^A)_x := \sum_{k=0}^{\infty} \frac{(A_x)^k}{k!} \]
for each \( x \in M \). Therefore, this theorem follows from Theorem 5.5 and the above discussions. \( \square \)

For any Kähler manifold \((M^n, g, J), n \geq 6\), the \( \ast \)-Ricci curvature is identical to the Ricci curvature and thus \( J \) is a critical point of the functional \( Y(M, g, \cdot) \) with the critical value \( Y(M, g, J) = 0 \). It is interesting to know whether or when \( J \) is actually an extremal point of \( Y(M, g, \cdot) \) on certain Kähler manifolds (e.g., the flat torus \( \mathbb{T}^6 \), which admits one component of Kähler structures and infinitely many components of non-Kähler integrable compatible almost complex structures [14]). Following Tian and Zhang [24] and Kelleher and Tian [13], it is also interesting to study the relative comparison theorem about the conformal invariant \( Y(M, g, J) \) under Ricci flows.

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**References**
1. Aubin T. Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire. J Math Pures Appl (9), 1976, 55: 269–296
2. Aubin T. Nonlinear Analysis on Manifolds. Monge-Ampère Equations. New York: Springer-Verlag, 1982
3. Besse A L. Einstein Manifolds. Berlin: Springer-Verlag, 1987
4 Chen H J, Chen L L, Nie X L. Chern-Ricci curvatures, holomorphic sectional curvature and Hermitian metrics. Sci China Math, 2021, 64: 763–780
5 del Rio H, Simanca S R. The Yamabe problem for almost Hermitian manifolds. J Geom Anal, 2003, 13: 185–203
6 Fu J X, Li J, Yao S T. Balanced metrics on non-Kähler Calabi-Yau threefolds. J Differential Geom, 2012, 90: 81–129
7 Fu J X, Zhou X C. Scalar curvatures in almost Hermitian geometry and some applications. Sci China Math, 2022, 65: 2583–2600
8 Gauduchon P. Hermitian connections and Dirac operators. Boll Unione Mat Ital, 1997, 11: 257–288
9 Gray A, Hervella I M. The sixteen classes of almost Hermitian manifolds and their linear invariants. Ann Mat Pura Appl (4), 1980, 123: 35–58
10 Hernández-Lamoneda L. Curvature vs. almost Hermitian structures. Geom Dedicata, 2000, 79: 205–216
11 Kazdan J L, Warner F W. Existence and conformal deformation of metrics with prescribed Gaussian and scalar curvatures. Ann of Math (2), 1975, 101: 317–331
12 Kazdan J L, Warner F W. Scalar curvature and conformal deformation of Riemannian structure. J Differential Geom, 1975, 10: 113–134
13 Kelleher C L, Tian G. Almost Hermitian Ricci flow. J Geom Anal, 2022, 32: 107
14 Khan G, Yang B, Zheng F Y. The set of all orthogonal complex structures on the flat 6-tori. Adv Math, 2017, 319: 451–471
15 Lee J M, Parker T H. The Yamabe problem. Bull Amer Math Soc (NS), 1987, 17: 37–91
16 Liu K F, Yang X K. Ricci curvatures on Hermitian manifolds. Trans Amer Math Soc, 2017, 369: 5157–5196
17 Macbeth H R. Kähler-Einstein metrics, Bergman metrics, and higher alpha-invariants. PhD Thesis. Princeton: Princeton University, 2015
18 Macbeth H R. Conformal classes realizing the Yamabe invariant. Int Math Res Not IMRN, 2019, 2019: 1333–1349
19 Schoen R. Conformal deformation of a Riemannian metric to constant scalar curvature. J Differential Geom, 1984, 20: 479–495
20 Schoen R, Yau S T. Lectures on Differential Geometry. Somerville: International Press, 1994
21 Simanca S R. Canonical Metrics on Compact Almost Complex Manifolds. Rio de Janeiro: Instituto de Matemática Pura e Aplicada, 2004
22 Tachibana S. On almost-analytic vectors in almost-Kählerian manifolds. Tohoku Math J (2), 1959, 11: 247–265
23 Tang Z Z. Curvature and integrability of an almost Hermitian structure. Internat J Math, 2006, 17: 97–105
24 Tian G, Zhang Z L. Relative volume comparison of Ricci flow. Sci China Math, 2021, 64: 1937–1950
25 Trudinger N. Remarks concerning the conformal deformation of Riemannian structures on compact manifolds. Ann Scuola Norm Sup Pisa Cl Sci (3), 1968, 22: 265–274
26 Wang E M, Zheng Y. Regularity of the first eigenvalue of the $p$-Laplacian and Yamabe invariant along geometric flows. Pacific J Math, 2011, 254: 239–255
27 Yamabe H. On a deformation of Riemannian structures on compact manifolds. Osaka J Math, 1960, 12: 21–37