Linearisation of the \((M, K)\)-reduced non-autonomous discrete periodic KP equation

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Abstract
The \((M, K)\)-reduced non-autonomous discrete KP equation is linearised on the Picard group of an algebraic curve. As an application, we construct theta function solutions to the initial value problem of some special discrete KP equation.

1 Introduction
The non-autonomous discrete KP equation (ndKP) is given by the formula [6]:

\[
\begin{align*}
(b(m) - c(n)) \cdot f_{m+1,n+1}^t f_{m,n+1}^{t+1} f_{m,n+1}^t + (c(n) - a(t)) \cdot f_{m+1,n}^t f_{m,n+1}^{t+1} f_{m,n+1}^t & = 0, \quad t, m, n \in \mathbb{Z}.
\end{align*}
\]

With constraints \(a(t) = 0, b(m) = 1, c(n) = 1 + \delta_n\) and \(f_{m,n}^t = f_{m-K,n}^{t-M}\), the ndKP reduces into the following form: \((f_{m,n}^{t-M} := f_{m,n}^t)\),

\[
\begin{align*}
\frac{f_{m+1,n}^t f_{m,n+1}^{t+1} f_{m,n+1}^t}{f_{n+1}^t f_{n+1}^t} - (1 + \delta_n + 1) \frac{f_{n+1,n}^t f_{n+1}^{t+1} f_{n+1}^t}{f_{n+1}^t f_{n+1}^t} & = -\delta_n + 1 \frac{f_{n+1,n}^t f_{n+1}^{t+1} f_{n+1}^t}{f_{n+1}^t f_{n+1}^t}.
\end{align*}
\]

Define \(I_n^t := (1 + \delta_n + 1) \frac{f_{n+1,n}^t f_{n+1}^{t+1} f_{n+1}^t}{f_{n+1}^t f_{n+1}^t} \) and \(V_n^t := \delta_n + 1 \frac{f_{n+1,n}^t f_{n+1}^{t+1} f_{n+1}^t}{f_{n+1}^t f_{n+1}^t} \). Then we derive the following discrete system: \(\forall n, t \in \mathbb{Z},\)

\[
\begin{align*}
I_n^t = I_{n-1}^{t-M} + V_{n-1}^{t-K} - V_n^t, \\
V_n^t = \frac{I_{n-1}^{t-M} V_{n-1}^{t-K}}{I_n^t}.
\end{align*}
\]

The system (1.1)–(2) is called \((M, K)\)-reduced non-autonomous discrete KP equation (rnKP). The term ‘non-autonomous’ derives from the freedom in the parameters \(\delta_n\). If we assume an extra constraint \(\delta_1 = \delta_2 = \delta_3 = \cdots\), this
system reduces to an autonomous system \((M, K)\)-reduced autonomous discrete KP equation (rdKP)).

In this article, we study the rndKP with the periodic boundary condition:

\[
I_n^t = I_{n+N}^t, \quad V_n^t = V_{n+N}^t, \quad N \in \mathbb{N}.
\]

The present paper is a generalisation of the method to solve the generalised periodic discrete Toda equation introduced in the papers [1, 2]. We show here that this method is also applicable to the quite general case of the rndKP and prove a linearisation theorem (theorem 2.12), which illustrates the geometric information of the discrete system.

In some special situation, theta function solutions of the initial value problem are constructed. In section 3, we derive an explicit formula for the solutions of the rdKP, which is a reduction of the rndKP.

**Important remark** We can assume \(g.c.d.(M, K) = 1\) without loss of generality. (See (1.1), (1.2).) Aside from this, we assume \(g.c.d.(M + K, N) = 1\) in Sections 2 and 3 by technical reason. The general cases will be discussed in Section 4.

**Notation:** For a meromorphic function \(f\) over a complete curve \(C\), \((f)_0\) (resp. \((f)_\infty\)) denotes the divisor of zeros (resp. poles) of \(f\). Let \((f) := (f)_0 - (f)_\infty\). \(\text{Div}^d(C)\) means the set of divisors over \(C\) of degree \(d\) and \(\text{Pic}^d(C)\) means the quotient set defined by \(\text{Pic}^d(C) = \text{Div}^d(C) / (\text{linearly equivalent})\). For an element \(D \in \text{Div}^d(C)\), \([D]\) means the image of \(D\) under the natural map \(\text{Div}^d(C) \to \text{Pic}^d(C)\).

## 2 Inverse scattering method

The rndKP equation (1.1-1.3) has the following matrix form:

\[
L_t(y)R_t(y) = R_{t-M}(y)L_{t-K}(y), \quad (2.1)
\]

where

\[
L_t(y) = \begin{pmatrix}
V_1^t & 1 \\
V_2^t & \ddots \\
y & \ddots & 1 \\
& \ddots & \ddots & 1
\end{pmatrix}, \quad R_t(y) = \begin{pmatrix}
I_1^t & 1 \\
I_2^t & \ddots \\
y & \ddots & 1 \\
& \ddots & \ddots & I_N
\end{pmatrix},
\]

and \(y\) is a complex parameter. Let

\[
X_t(y) := L_{t-(K-1)M}(y) \cdots L_{t-2M}(y)L_{t-M}(y)L_t(y) \times R_t(y)R_{t-K}(y)R_{t-2K}(y) \cdots R_{t-(M-1)K}(y), \quad (2.2)
\]

then (2.1) becomes

\[
X_t(y)R_{t-MK}(y) = R_{t-MK}(y)X_{t-K}(y), \quad (2.3)
\]
or equivalently

\[ L_{t-MK}(y)X_t(y) = X_{t-M}(y)L_{t-MK}(y). \]  

(2.4)

Because \( M \) and \( K \) are co-prime, the characteristic polynomial of \( X_t(y) \) does not depend on \( t \). Let \( \tilde{C} := \{ (x, y) \in \mathbb{C}^2 \mid \det(X_t(y) - xE) = 0 \} \). Of course, \( \tilde{C} \) is also independent from \( t \). We call the completion \( C \) of \( \tilde{C} \) the spectral curve of the rdnKP equation.

**Remark 2.1** By applying (2.4) repeatedly, we can transform (2.2) into

\[
X_t(y) = R_{t-MK}(y)R_{t-(M+1)K}(y) \cdots R_{t-(2M-1)K}(y) \times
L_{t-(2N-1)M}(y) \cdots L_{t-(K+1)M}(y)L_{t-KM}(y). 
\]

(2.5)

2.1 Properties of the spectral curve

As a starter, we list some fundamental properties of the spectral curve \( C \) in this section. In the rest of this article, we always assume \( C \) to be smooth. Moreover, we also assume that g.c.d.\((M + K, N) = 1\) in Sections 2 and 3 unless otherwise is stated.

Denote the set of \( N \times N \) matrices by \( M_N(\mathbb{C}) \) and the subset of diagonal matrices by \( \Gamma \subset M_N(\mathbb{C}) \). For a matrix \( X \in M_N(\mathbb{C}) \) and subsets \( A, B \subset M_N(\mathbb{C}) \), let \( A + X := \{ a + X \mid a \in A \}, AX := \{ aX \mid a \in A \}, A + B := \{ a + b \mid a \in A, b \in B \} \) and \( AB := \{ ab \mid a \in A, b \in B \} \).

Let \( S \) be the \( N \times N \) matrix \( S =: 
\begin{pmatrix} 
0 & 1 \\
0 & \ddots \\
& \ddots & 1 \\
y & \cdots & 0 
\end{pmatrix}
\)

The polynomial \( \det(X_t(y) - xE) \) is of degree \( N \) w.r.t. \( x \), and of degree \( M + K \) w.r.t. \( y \). Then the projection \( p_x : C \ni (x, y) \mapsto x \in \mathbb{P} \) is \( (M + K) : 1 \), and the projection \( p_y : C \ni (x, y) \mapsto y \in \mathbb{P} \) is \( N : 1 \).

Let \( U_j := (\prod_{k=1}^K V_j^{l+k}) \cdot (\prod_{k=1}^M t_j^{l+k}), \ (j \in \{1, 2, \ldots, N\}) \). By (12), the quantity \( U_j \) is invariant under the time evolution.

**Proposition 2.1** If \( \text{g.c.d.}(M + K, N) = 1 \), the curve \( C \) has the following special points:

(i) \( M \) points \( A_j : (x, y) = \left(0, \prod_{n=1}^N \Gamma_n^{-jk} \right), \ j = 0, 1, \ldots, M - 1. \)

(ii) \( K \) points \( B_j : (x, y) = \left(0, \prod_{n=1}^N \Gamma_n^{-jk} \right), \ j = 0, 1, \ldots, K - 1. \)

(iii) \( N \) points \( Q_j : (x, y) = (U_j, 0), \ j = 1, 2, \ldots, N. \)

(iv) a unique point \( P : (x, y) = (\infty, \infty). \)
**Proof.** Let \((0, y)\) \(\in C\). Then we easily derive
\[
\prod_{j=0}^{K-1} \det (L_{t-jM}(y)) \cdot \prod_{j=0}^{M-1} \det (R_{t-jK}(y)) = 0,
\]
which implies (i) and (ii). Part (iii) follows from the fact that \(L_t(0)\) and \(R_t(0)\) are upper triangular.

(iv): For a point \((x, y)\) \(\in C\), there exists a non-zero \(N\)-vector \(v(x, y)\) such that \(X_t(y)v(x, y) = x \cdot v(x, y)\). Because the matrix \(X_t(y)\) is contained in the subset \((\Gamma + S)^{M+K} = \Gamma + \Gamma S + \cdots + \Gamma S^{M+K-1} + S^{M+K}\), it follows that
\[
(\gamma_0 + \gamma_1 S + \cdots + \gamma_{M+K-1} S^{M+K-1} + S^{M+K}) \cdot v = x \cdot v,
\]
where \(\gamma_0, \gamma_1, \ldots, \gamma_{M+K-1}\) are diagonal matrices.

Define a new parameter \(k\) by \(y = k^{-N}\) which is assumed to be zero near \(P\). Let \(\zeta_N\) be a \(N\)-th primitive root of unity. For all \(j \in \{0, 1, \ldots, N-1\}\), the vector
\[
v_0 := ((\zeta_N^j k)^{-N-1}, (\zeta_N^j k)^{-N-2}, \ldots, (\zeta_N^j k), 1)^T
\]
satisfies the formula: \(S \cdot v_0 = (\zeta_N^j k)^{-1} \cdot v_0\). Then, from (2.6) we obtain
\[
((\zeta_N^j k)^{-M-K} \cdot v_0 = x \cdot v_0 + \text{(higher term)} \text{ near } k = 0,
\]
which implies \((x, y) \sim (\zeta_N^j k)^{-M-K} \cdot v_0 + \text{(higher term)} \text{ near } k = 0\) when \((x, y) \in C\) tends to infinity. Because \(M + K\) and \(N\) are relatively prime, we can choose an appropriate branch of \(k\) around a unique point \(P\) such that
\[
x = k^{-(M+K)} + \cdots, \quad y = k^{-N} + \cdots,
\]
near \(P\).

From the proof of proposition \(2.1\) one obtains more detailed information on the point \(P \in C\).

**Corollary 2.2** There exists a local coordinate \(k\) around \(P\) such that
\[
x = k^{-(M+K)} + \cdots, \quad y = k^{-N} + \cdots.
\]

**Corollary 2.3** Let \(X_t(y)v(x, y) = x \cdot v(x, y)\). Then, around \(P\), it follows that
\[
v(x, y) \sim (k^{N-1}, k^{N-2}, \ldots, k, 1)^T \quad \text{(up to a constant multiple)}.
\]

**Proof.** Because g.c.d.\((M + K, N) = 1\), the solution of the vector equation (2.6) is expressed as \(v(x, y) = (k^{N-1}, k^{N-2}, \ldots, k, 1)^T + \text{(higher)} \text{ up to a constant multiple}.\)
Remark 2.2  The proof of proposition 2.1 (i), (ii) implies that the set
\[ (\bigcup_{j=0}^{K-1} \{ \prod_n V^{t-jM}_n \}) \bigcup (\bigcup_{j=0}^{M-1} \{ \prod_n I^{t-jK}_n \}) \]
is invariant under the time evolution. It then follows that \( \{ \prod V^{t}_n, \prod I^t_n \} = \{ \prod V^{t+K}_n, \prod I^{t+K}_n \} \). To avoid a non-interesting solution \( I^{t+M}_n = V^t_n \), \( V^{t+K}_n = I^{t+1}_n \) of the rndKP (1.1–1.3), we should assume the extra constraint \( \prod_n I^{t+M}_n = \prod_n V^{t+K}_n = \prod_n V^t_n \) in addition to the rndKP. In fact, this constraint is enough to guarantee the existence of the unique solution. See section 2.3.

Next we consider the behaviour of \( Q_j \) (\( j = 1, 2, \ldots, N \)). The position of \( Q_j \) is invariant under the time evolution. In this paper, we restrict ourselves to the following two typical cases:

(a) All \( Q_j \) are distinct.  
(b) \( Q_1 = Q_2 = \cdots = Q_N \).

Note that in the case (b), the system (1.1–1.3) reduces to the rdKP.

**In the case (a)**

The equation \( X_t(y)v(x,y) = x \cdot v(x,y) \) becomes
\[ X_t(0) \cdot v = U_j \cdot v, \quad \text{at } Q_j. \]
Because \( X_t(0) \) is upper triangular, the eigenvector \( v \) takes the form
\[ v = (d_1, d_2, \ldots, d_j, 0, \ldots, 0)^T, \quad d_j \neq 0. \quad (2.7) \]

**In the case (b)**

Let \( Q := Q_1(= Q_2 = \cdots = Q_N) \). Arguments similar to those in the proofs of corollary 2.2, 2.3 prove the following:

**Proposition 2.4** If all the points \( Q_j \) coincide, there exists a local coordinate \( k \) around \( Q = Q_1 \) such that
\[ x = U_1 + \cdots, \quad y = k^N + \cdots, \]
and the eigenvector \( X_t(y)v(x,y) = x \cdot v(x,y) \) satisfies
\[ v(x,y) \sim (1, k, k^2, \ldots, k^{N-1})^T, \quad \text{(up to a constant multiple)}. \]

Let \( v(x,y) = (g_1(x,y), \ldots, g_N(x,y))^T \) be an \( N \)-vector function (defined up to a constant multiples) such that \( X_t(y) \cdot v(x,y) = x \cdot v(x,y) \). By the above arguments, we have:
Proposition 2.5 The meromorphic function \( g_j/g_{j+l}, \) \( (j \leq j + l \leq N) \) has:

(i) \( l \) zeros at \( P \)

(ii) at least one pole at \( Q_j \).

Define the divisors \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \) to be minimal positive divisors on \( C \) such that

\[
(g_j/g_N) + \mathcal{D}_1 \geq -(Q_j + Q_{j+1} + \cdots + Q_{N-1}), \quad \forall j, \tag{2.8}
\]

\[
(g_j/g_1) + \mathcal{D}_2 \geq -(j - 1)P, \quad \forall j. \tag{2.9}
\]

These divisors were first studied in [4], where it has proved that \( \mathcal{D}_1, \mathcal{D}_2 \) are general and \( \deg \mathcal{D}_1 = \deg \mathcal{D}_2 = \text{genus}(C) \).

2.2 the eigenvector mapping

Let \( p \) be a point on a smooth curve \( C \) and \( k \) be a local coordinate around \( p \). For a meromorphic function \( f \), \( \text{ord}(f(p)) \) denotes the largest integer \( r \) such that \( \lim_{q \to p} |k^{-r} f(q)| < +\infty \). For a vector function \( \mathbf{v}(p) = (f_i)_i \), we define \( \text{ord}(\mathbf{v}(p)) := \min_i [\text{ord}(f_i(p))] \).

An isolevel set \( \mathcal{T}_C \) is the set of matrices \( X(y) \) (eq. (2.2)) associated with the spectral curve \( C \). Let \( g := \text{genus}(C) \). Now we construct a map from \( \mathcal{T}_C \) to \( \text{Pic}^{g+1}(C) \) called the eigenvector mapping.

Let \( X = X(y) \) be an element of \( \mathcal{T}_C \). If \( (x, y) \in \tilde{C} \), there exists a complex \( N \)-vector \( \mathbf{v}(x, y) \) such that \( X(y) \mathbf{v}(x, y) = x \mathbf{v}(x, y) \), up to a constant multiple. Then there exists a Zariski open subset \( C^0 \) of \( \tilde{C} \) over which the morphism \( C^0 \ni (x, y) \mapsto \mathbf{v}(x, y) \in \mathbb{P}^{N-1} \) is uniquely determined. Moreover, for a smooth \( C \), this morphism can be extended uniquely over the whole of \( C \). Denote this morphism by \( \Psi_X : C \to \mathbb{P}^{N-1} \).

The eigenvector mapping \( \varphi_C : \mathcal{T}_C \to \text{Pic}^d(C) \) \( (d = g + N - 1) \) is a map defined by the formula:

\[
\mathcal{O}_C(\varphi_C(X)) = \Psi_X^*(\mathcal{O}_{\mathbb{P}^{N-1}}(1)), \tag{2.10}
\]

where \( \mathcal{O}_{\mathbb{P}^{N-1}}(1) \) is the invertible sheaf of hyperplane sections over \( \mathbb{P}^{N-1} \). Note that it is nontrivial to prove that \( \varphi_C(X) \in \text{Pic}^d(C) \) (see [4] §2).

Let \( (X_1 : X_2 : \cdots : X_N) \) be the homogeneous coordinate of \( \mathbb{P}^{N-1} \). The eigenvector mapping illustrates the geometric interpretation of the general divisors \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \) (section 2.1). In fact, (2.3) implies that

\[
\mathcal{D}_1 + Q_1 + Q_2 + \cdots + Q_{N-1} \text{ is the pull-back of } \{X_N = 0\},
\]

and (2.9) says

\[
\mathcal{D}_2 + (N - 1) \cdot P \text{ is the pull-back of } \{X_1 = 0\}.
\]

These facts imply the following: \( \varphi_C(X(y)) = [\mathcal{D}_1 + Q_1 + Q_2 + \cdots + Q_{N-1}] = [\mathcal{D}_2 + (N - 1) \cdot P] \).

Let \( \vartheta(X(y)) := \mathcal{D}_2 \). This divisor will play an important role for constructing a tau function solution of rdKP. See the next section.
Remark 2.3 Because $D_1 + Q_1 + Q_2 + \cdots + Q_{N-1}$ and $D_2 + (N-1) \cdot P$ are linearly equivalent to each other we have

\[(g_1/g_N) = \theta(X(y)) + (N-1) \cdot P - D_1 - (Q_1 + Q_2 + \cdots + Q_{N-1}). \quad (2.11)\]

Remark 2.4 Let $X(y) \cdot v(p) = x \cdot v(p)$ and $p = (x, y) \in C$. Equation (2.10) is equivalent to $\varphi_{C}(X) = \left[ -\sum_{p \in C} (\text{ord } v(p)) \cdot p \right].$

The following theorem is essentially obtained in van Moerbeke, Mumford [4].

Theorem 2.6 The eigenvector mapping $\varphi_{C} : T_C \rightarrow \text{Pic}^d(C)$ is an embedding.

2.3 shift operators

Consider the $N \times N$ matrix $X_t(y)$ defined by (2.2) and the associated spectral curve $C$. Let $\sigma, \mu_K$ and $\mu_M$ be the isomorphisms on $T_C$ defined by:

\[
\sigma(X_t(y)) := SX_t(y)S^{-1}, \quad (2.12) \\
\mu_K(X_t(y)) := R_{t-(M-1)K}(y) \cdot X_t(y) \cdot \{R_{t-(M-1)K}(y)\}^{-1}, \quad (2.13) \\
\mu_M(X_t(y)) := L_{t-MK}(y) \cdot X_t(y) \cdot \{L_{t-MK}(y)\}^{-1}, \quad (2.14)
\]

where $S$ is the matrix defined in section 2.1. By (2.3–2.4), we have $\mu_K(X_t) = X_{t+K}$ and $\mu_M(X_t) = X_{t-M}$. For the ndKP (1.1–1.3), $\sigma$ is the $n$-shift operator: $n \mapsto n + 1$ and $\mu_K$ and $\mu_M$ are the $t$-shift operators: $t \mapsto t + K$, $t \mapsto t - M$. Because $K$ and $M$ are co-prime, an appropriate combination of $\mu_K$ and $\mu_M$ defines the unit time evolution $t \mapsto t + 1$.

We start with the linear problem:

\[X_t(y) \cdot v(x, y) = x \cdot v(x, y), \quad v(x, y) = (g_i(x, y))_{i=1}^N. \quad (2.15)\]

This linear equation is decomposed into the following infinite dimensional form: for an infinite vector $(g_i)_{i \in \mathbb{Z}},$

\[a_{i,0} \cdot g_i + a_{i,1} \cdot g_{i+1} + \cdots + a_{i,M+K} \cdot g_{i+M+K} = x \cdot g_i, \quad (a_{i+N,j} = a_{i,j}) \quad (2.16)\]

\[g_{i+N} = y \cdot g_i. \quad (2.17)\]

The matrix equation (2.15) can be interpreted “(2.16) with constraint (2.17)”.

However, interchanging the roles of these two equations, i.e., interpreting “(2.17) with constraint (2.16)” , we arrive at another matrix equation:

\[Y_t(x) \cdot w = y \cdot w, \quad \text{where } w = (g_i)_{i=1}^{M+K}. \quad (2.18)\]
Example 2.7 For an equation \( \begin{pmatrix} a_1 & a_2 & 1 \\ y & b_1 & b_2 \\ c_2 y & y & c_1 \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} = \begin{pmatrix} x \\ g_1 \\ g_2 \end{pmatrix} \), the associated new matrix equation is:

\[
\begin{pmatrix}
- b_2(a_1 - x) \\
(a_1 - x) (c_1 - x - b_2 c_2) \\
(a_1 - x)(c_1 - x - b_2 c_2)
\end{pmatrix}
\begin{pmatrix}
a_2 b_2 - b_1 + x \\
 a_2(c_1 - x) - c_2 (a_2 b_2 - b_1 - x)
\end{pmatrix}
\begin{pmatrix}
g_1 \\ g_2
\end{pmatrix}
= \begin{pmatrix}
y \\ g_1 \\ g_2
\end{pmatrix}.
\]

We call the linear problem (2.15) the \( x \)-form and the linear problem (2.18) the \( y \)-form.

2.3.1 shift operators and the \( x \)-form

Due to (2.12)-(2.14), the shift operators \( \sigma \) and \( \mu \) act on the eigenvector of the \( x \)-form equation (2.15) by:

\[
\sigma : v \mapsto S v, \quad \mu : v \mapsto \{R_{t-(M-1)K}\} v, \quad \mu_{-M} : v \mapsto \{L_{t-MK}\} v.
\]

The following lemma is easily proved:

Lemma 2.8 det \( S = (-1)^{N+1} y \), det \( R_{t-(M-1)K} = \prod_n I^t_{1-(M-1)K} - y \), det \( L_{t-MK} = \prod_n V_{t-MK} - y \).

2.3.2 shift operators and the \( y \)-form

The \( y \)-form representation of the shift operators \( \sigma, \mu_K, \mu_{-M} \) are more complicated. Let \( E_1 := -(a_{1,0} - x)/a_{1,1+K}, E_2 := -a_{1,1}/a_{1,1+K}, \ldots, E_{M+K} := -a_{1,M+K-1}/a_{1,1+K} \). Then (2.16) becomes \( g_{M+K+1} = \sum_{i=1}^{M+K} E_i g_i \).

Define three new matrices \( S^*, R^* \) and \( L^* \) by:

\[
S^* := \begin{pmatrix}
0 & 1 \\
0 & \ddots & \ddots & 1 \\
& \ddots & \ddots & \ddots & 1 \\
0 & \ddots & \ddots & \ddots & \ddots & 1 \\
E_1 & E_2 & \cdots & E_{M+K-1} & E_{M+K}
\end{pmatrix}, \tag{2.19}
\]

\[
R^* := \begin{pmatrix}
\overline{I_1} & 1 \\
\overline{I_2} & \ddots & 1 \\
\ddots & \ddots & \ddots & \ddots & 1 \\
& \ddots & \ddots & \ddots & \ddots & \ddots & 1 \\
E_1 & E_2 & \cdots & E_{M+K-1} & \overline{I_{M+K}} + E_{M+K}
\end{pmatrix}, \tag{2.20}
\]

\[
L^* := \begin{pmatrix}
\overline{V_1} & 1 \\
\overline{V_2} & \ddots & 1 \\
\ddots & \ddots & \ddots & \ddots & 1 \\
E_1 & E_2 & \cdots & \overline{V_{M+K-1}} & \overline{V_{M+K}} + E_{M+K}
\end{pmatrix}, \tag{2.21}
\]
where \( I_n = I_n^{-(M-1)K} \) and \( V_n^\ast = V_n^{1-MK} \). The matrices \( S^\ast \), \( R^\ast \) and \( L^\ast \) are the \( y \)-form version of the matrices \( S \), \( R_{1-(M-1)K} \) and \( L_{1-MK} \) “with constraint \( g_{M+K+1} = \sum E_i g_i \).” The shift operators \( \sigma \) and \( \mu \) act on the eigenvector of the \( y \)-form equation \( (2.18) \) by:

\[
\sigma : w \mapsto S^\ast w, \quad \mu_K : w \mapsto R^\ast w, \quad \mu_{-M} : w \mapsto L^\ast w.
\]

**Lemma 2.9** \( \det S^\ast = (-1)^{M+K} \cdot (U_1 - x), \) \( \det R^\ast = \det L^\ast = (-1)^{M+K+1} \cdot x. \)

**Proof.** The calculation is cumbersome but elementary. We will prove this lemma in Appendix.

### 2.3.3 geometric interpretation of \( x \)-form and \( y \)-form

Consider the projections \( p_x : C \rightarrow \mathbb{P} \) and \( p_y : C \rightarrow \mathbb{P} \) (section 2.1). Recall that \( p_x \) is \((M + K) : 1\) and \( p_y \) is \( N : 1\). Denote \( \mathcal{F} := \mathcal{O}_C(\varphi_C(X_i)) \).

Because the \( x \)-form representations of \( \sigma \), \( \mu \) are independent from \( x \) (section 2.3.1), for fixed \( y \in \mathbb{P} \) and its pre-image \( p_y^{-1}(y) = \{ (x_1, y), \ldots, (x_N, y) \} \), the matrices \( S \), \( R_{1-(M-1)K} \) and \( L_{1-MK} \) act on the vectors \( v(x_1, y), \ldots, v(x_N, y) \) simultaneously.

On the other hand, for generic \( y \), the vectors \( v(x_1, y), \ldots, v(x_N, y) \) should be linearly independent because they are eigenvectors belonging to distinct eigenvalues. What happens if we choose \( y \) such that \( \det S(y) = 0 \)? This seemingly leads to a contradiction, if one believes the the column vectors of the singular matrix \( S(y) \cdot (v(x_1, y), \ldots, v(x_N, y)) \) are linearly independent. However, realizing the fact that the eigenvectors are only determined up to a constant, this problem is easily solved and we conclude that \( \sum_{i=1}^N \text{ord}(S(y)v(x_i, y)) > \sum_{i=1}^N \text{ord}(v(x_i, y)) \).

More precisely, the statement of lemma 2.8 can be interpreted as follows:

1. \( \sum_{i=1}^N \text{ord}(Sv(x_i, 0)) = \sum_{i=1}^N \text{ord}(v(x_i, 0)) + 1, \) where \( (x_i, 0) \in C, \) \( i = 1, 2, \ldots, N. \)
2. Let \( y_0 := \prod_n I_n^{1-(M-1)K}. \) Then
   \( \sum_{i=1}^N \text{ord}(Rv(x_i, y_0)) = \sum_{i=1}^N \text{ord}(v(x_i, y_0)) + 1, \) \( (x_i, y_0) \in C. \)
3. Let \( y_1 := \prod_n V_n^{1-MK}. \) Then
   \( \sum_{i=1}^N \text{ord}(Lv(x_i, y_1)) = \sum_{i=1}^N \text{ord}(v(x_i, y_1)) + 1, \) \( (x_i, y_1) \in C. \)

\(^1\)If \( X(y) \) has an eigenvalue \( x' \) of multiplicity \( m > 1 \), we should choose the vectors \( v(p), v'(p), \ldots, v^{(m-1)}(p) \), where \( v^{(k)}(p) \) is the \( k \)-th differential of \( v \) with respect to the local coordinate around \( p = (x', y) \).

\(^2\)Geometrically, this means that \( S, R \) and \( L \) act on the push-forward \( (p_y)_* \mathcal{F}. \)
Similar arguments in the case of the y-form representations yield the following form of lemma 2.9:

- \[ \sum_{i=1}^{M+1} \text{ord} (S^* w(U_1, y_i)) = \sum_{i=1}^{M+1} \text{ord} (w(U_1, y_i)) + 1, \quad (U_1, y_i) \in C. \]
- \[ \sum_{i=1}^{M+1} \text{ord} (R^* w(0, y_i)) = \sum_{i=1}^{M+1} \text{ord} (w(0, y_i)) + 1, \quad (0, y_i) \in C. \]
- \[ \sum_{i=1}^{M+1} \text{ord} (L^* w(0, y_i)) = \sum_{i=1}^{M+1} \text{ord} (w(0, y_i)) + 1, \quad (0, y_i) \in C. \]

Combining these data, we obtain the following proposition:

**Proposition 2.10** Let \( Q_1 : (x, y) = (U_1, 0) \) and \( A_j : (x, y) = (0, \prod_n I_n^{t - (M-1)K}), \quad -jK \equiv t - (M-1)K \pmod M, \)
\( B_i : (x, y) = (0, \prod_n V_n^{t - M K}), \quad -iM \equiv t - MK \pmod K \)
(proposition 2.7). Then,

(i) \( \text{ord} (S v(Q_1)) = \text{ord} (v(Q_1)) + 1 \),  
(ii) \( \text{ord} (R v(A_j)) = \text{ord} (v(A_j)) + 1 \),  
(iii) \( \text{ord} (L v(B_i)) = \text{ord} (v(B_i)) + 1 \).

**Proof.** We prove (i). By construction of the x-form and the y-form, we have

\[ \text{ord} (S v(p)) = \text{ord} (v(p)) + 1 \leftrightarrow \text{ord} (S^* w(p)) = \text{ord} (w(p)) + 1. \]

On the other hand, because a regular matrix is invertible,

\[ \det S(y) \neq 0, \infty \quad \text{or} \quad \det S^*(x) \neq 0, \infty \quad \Rightarrow \quad \text{ord} (S v(x, y)) = \text{ord} (v(x, y)). \]

These facts prove proposition (i). Clearly, similar arguments will prove (ii) and (iii). ■

### 2.3.4 shift operator at the infinity point

At \( P \), the actions \( v(P) \mapsto S v(P), \ v(P) \mapsto R v(P) \) and \( v(P) \mapsto L v(P) \) are directly computable.

**Proposition 2.11** (i) \( \text{ord} (S v(P)) = \text{ord} (v(P)) - 1 \),  
(ii) \( \text{ord} (R v(P)) = \text{ord} (v(P)) - 1 \),  
(iii) \( \text{ord} (L v(P)) = \text{ord} (v(P)) - 1 \).

**Proof.** The Proposition is readily proved by Corollaries 2.2 and 2.3. ■
2.4 linearisation theorem

From the above calculations, we obtain the linearisation theorem representing the flow of the rndKP equation on the Picard group of the spectral curve.

Theorem 2.12 (I): Let $D$ be the divisor $D = P - Q_1$. Then the following diagram is commutative.

$$\begin{align*}
T_C & \rightarrow \text{Pic}^d(C) \\
\sigma & \downarrow \quad \downarrow +[D] \\
T_C & \rightarrow \text{Pic}^d(C)
\end{align*}$$

(II): Let $E_j (j = 0, 1, \ldots, M-1)$ be the divisor $E_j = P - A_j$ and $t \equiv -(j+1)K \pmod{M}$. The following diagram is commutative.

$$\begin{align*}
T_C & \rightarrow \text{Pic}^d(C) \\
\mu_K & \downarrow \quad \downarrow +[E_j] \\
T_C & \rightarrow \text{Pic}^d(C)
\end{align*}$$

(II)\text{2}: Let $F_j (j = 0, 1, \ldots, K-1)$ be the divisor $F_j = P - B_j$, and $t \equiv -jM \pmod{K}$. The following diagram is commutative.

$$\begin{align*}
T_C & \rightarrow \text{Pic}^d(C) \\
\mu_{-M} & \downarrow \quad \downarrow +[F_j] \\
T_C & \rightarrow \text{Pic}^d(C)
\end{align*}$$

Proof. The theorem follows immediately from Remark 2.4 and Proposition 2.10.■

We should note the fact that the position of the points $Q_j$ (proposition 2.4) varies under the index shift $\sigma : n \mapsto n + 1$. To avoid confusion, we fix the rule for indexing as follows: Once the points $Q_1, \ldots, Q_N$ are determined, we never change their indexes. Alternatively, we define

$$\begin{align*}
\varphi_C(\sigma X(y)) &= \varphi_C(X(y)) + [P - Q_1], \\
\varphi_C(\sigma^2 X(y)) &= \varphi_C(X(y)) + [P - Q_1] + [P - Q_2], \\
\varphi_C(\sigma^3 X(y)) &= \varphi_C(X(y)) + [P - Q_1] + [P - Q_2] + [P - Q_3], \\
&\text{etc.}\cdots \\
\varphi_C(\sigma^{-1} X(y)) &= \varphi_C(X(y)) - [P - Q_N], \\
\varphi_C(\sigma^{-2} X(y)) &= \varphi_C(X(y)) - [P - Q_N] - [P - Q_{N-1}], \\
&\text{etc.}\cdots
\end{align*}$$

This particular arrangement is appropriate for our further discussion.
Corollary 2.13 Let $\mathfrak{d}(X(y))$ be the general divisor defined by $\varphi_C(X(y)) = [\mathfrak{d}(X(y)) + (N - 1) \cdot P]$ (section 2.11). The divisor $D_1$ in (2.11) satisfies $D_1 = \mathfrak{d}(\sigma^{-1}X(y))$.

**Proof.** By (2.11), we obtain

$$[D_1] = [\mathfrak{d}(X(y)) - Q_1 - \cdots - Q_{N-1} + (N - 1) \cdot P] = [\mathfrak{d}(\sigma^{-1}X(y)) - Q_1 - \cdots - Q_{N-1} - Q_N + N \cdot P].$$

By the equation $(y) = Q_1 + \cdots + Q_N - N \cdot P$, we conclude $[D_1] = [\mathfrak{d}(\sigma^{-1}X(y))]$. Because $D_1$ and $\mathfrak{d}(\sigma^{-1}X(y))$ are general, positive and of degree $g$, it follows that $D_1 = \mathfrak{d}(\sigma^{-1}X(y))$.

As a conclusion of the corollary, we have

$$(g_1/g_N) = \mathfrak{d}(X) + (N - 1)P - \mathfrak{d}(\sigma^{-1}X) - Q_1 - \cdots - Q_{N-1}. \tag{2.22}$$

### 3 Tau function solution of rdKP

Due to the linearisation theorem 2.12 and the injectivity of the eigenvector mapping (theorem 2.6), we could say that the rndKP equation (1.1–1.3) is "essentially solved". Moreover, in some fortunate case, we can construct the explicit solutions by using the method of the Riemann theta functions.

In the rest of the article, we assume that $Q_1 = Q_2, \cdots = Q_N (= Q)$. Equivalently, the rndKP reduces to the rdKP equation. (See the paragraph after remark 2.2)

Recall that we have assumed that $\text{g.c.d.}(M + K, N) = 1$ in the previous section. The assumption is valid also in this section.

#### 3.1 construction of tau functions

We construct a theta functional solution of rdKP equation. As in the previous section, $X_t = X_t(y)$ denotes a square matrix defined by (2.2).

Let $C$ be the (smooth) spectral curve associated with $X_t$. Fix a symplectic basis $\alpha_1, \ldots, \alpha_g; \beta_1, \ldots, \beta_g$ of $C$ and the normalised holomorphic differential $\omega_1, \ldots, \omega_g$ such that $\int_{\beta_i} \omega_j = \delta_{i,j}$. The $g \times g$ matrix $\Omega := (\int_{\beta_i} \omega_j)_{i,j}$ is called the period matrix of $C$. For a fixed point $p_0 \in C$, the Abel-Jacobi mapping $A : \text{Div}(C) \to \mathbb{C}^g/(\mathbb{Z}^g + \Omega \mathbb{Z}^g)$ is a homomorphism defined by:

$$\sum Y_i - \sum Z_j \mapsto \sum (\int_{p_0}^Y \omega_1, \ldots, \int_{p_0}^Y \omega_g) - \sum (\int_{p_0}^Z \omega_1, \cdots, \int_{p_0}^Z \omega_g).$$

Let us consider the universal covering $\pi : \mathfrak{U} \to C$ and fix an inclusion $\iota : C \hookrightarrow \mathfrak{U}$. For simplicity, we use the symbols "$\pi$" and "$\iota$" to express the derived maps $\text{Div}(\mathfrak{U}) \to \text{Div}(C)$ and $\text{Div}(C) \hookrightarrow \text{Div}(\mathfrak{U})$ respectively. Naturally, there exists a continuous lift $\tilde{A} : \text{Div}(\mathfrak{U}) \to \mathbb{C}^g$ such that $\tilde{A} \circ \iota(p_0) = 0$. For the projection $\rho : \mathbb{C}^g \to \mathbb{C}^g/(\mathbb{Z}^g + \Omega \mathbb{Z}^g)$, it follows that $\rho \circ \tilde{A} = A \circ \pi$. 

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Now we should define the lifted divisors $\mathcal{D}(\sigma X_i), \mathcal{D}(\mu K X_i), \mathcal{D}(\mu - M X_i) \in \text{Div}^g(\mathcal{U})$. For fixed $t \in \mathbb{Z}$, assume that some lifted positive divisor $\mathcal{D}(X_i) \in \text{Div}^g(\mathcal{U})$ with $\pi(\mathcal{D}(X_i)) = \mathcal{O}(X_i)$ is specified. First, there uniquely exists a positive divisor $\mathcal{D}(\sigma X_i) \in \text{Div}^g(\mathcal{U})$ such that:

$$\tilde{A}(\mathcal{D}(\sigma X_i)) = \tilde{A}(\mathcal{D}(X_i) + tP - tQ), \quad \pi(\mathcal{D}(\sigma X_i)) = \mathcal{O}(\sigma X_i). \quad (3.1)$$

We will consider $\mathcal{D}(\sigma X_i)$ as the appropriately lifted divisor of $\mathcal{O}(\sigma X_i)$. To choose appropriate $\mathcal{D}(\mu K X_i)$ and $\mathcal{D}(\mu - M X_i)$, we have to consider the compatibility: 

$$(\mu K)^M + (\mu - M)^K = \text{id}. \quad \text{On the Picard group on } C,$$

this reflects the equation 

$$[(M + K) \cdot P - A_0 - A_1 - \cdots - A_{M-1} - B_0 - B_1 - \cdots - B_{K-1}] = [(x)] = 0.$$ 

(Thm. 2.12 (II)). Therefore, we can choose $(M+K)$ points $\kappa A_0, \cdots, \kappa A_{M-1}$, $\kappa B_0, \cdots, \kappa B_{K-1} \in \mathcal{U}$ such that

$$\tilde{A}((M + K) \cdot t P - \kappa A_0 - \cdots - \kappa A_{M-1} - \kappa B_0 - \cdots - \kappa B_{K-1}) = 0, \quad (3.2)$$

and $\pi(\kappa A_j) = A_j, \pi(\kappa B_j) = B_j$. We now define the two divisors $\mathcal{D}(\mu K X_i), \mathcal{D}(\mu - M X_i) \in \text{Div}^g(\mathcal{U})$ by the formulas:

$$\tilde{A}(\mathcal{D}(\mu K X_i)) = \tilde{A}(\mathcal{D}(X_i) + tP - \kappa A_j), \quad \pi(\mathcal{D}(\mu K X_i)) = \mathcal{O}(\mu K X_i), \quad (3.3)$$

$$\tilde{A}(\mathcal{D}(\mu - M X_i)) = \tilde{A}(\mathcal{D}(X_i) + tP - \kappa B_j), \quad \pi(\mathcal{D}(\mu - M X_i)) = \mathcal{O}(\mu - M X_i), \quad (3.4)$$

where $t \equiv -(j + 1)K \pmod{M}, \ t \equiv -i M \pmod{K}$.

Let $\tau^t$ be a holomorphic function over $\mathcal{U}$ defined by the formula:

$$\tau^t(p) = \theta \left( \tilde{A}(\mathcal{D}(X_i) - p - t \Delta) \right), \quad p \in \mathcal{U}, \quad (3.5)$$

where $\theta(\bullet) = \theta(\bullet; \Omega)$ is the Riemann theta function and $\Delta \in \text{div}^{g-1}(C)$ is the theta characteristic divisor of $C$ (\cite{5}, Chap. II, thm. 3.11). To avoid cumbersome notations, we often omit the letters “$\tilde{A}$”, “$\tau$” and use a simpler expression $\tau^t(p) = \theta(\mathcal{D}(X_i) - p - \Delta)$, when there is no confusion possible.

Although being defined over $\mathcal{U}$, $\tau^t(p)$ is considered to be a multi-valued holomorphic function over $C$. By the Riemann vanishing theorem (\cite{5}, Chap. II, thm. 3.11), the zero divisor of $\tau^t(p)$ corresponds with $\mathcal{O}(X_i)$.

Let $\tau^t_\ast(p) := \theta(\mathcal{O}(\sigma X_i) - p - \Delta), \tau^t_\ast(p) := \theta(\mathcal{O}(\sigma^{-1} X_i) - p - \Delta)$. Then, by theorem 2.12 the function: $(\hat{\sigma} := \sigma^{-1})$

$$\Psi^t(p) := \frac{\tau^t(p) \cdot \tau^{t+K}_\ast(p)}{\tau_\ast(p) \cdot \tau^{t+K}(p)} = \frac{\theta(\mathcal{D}(X_i) - p - \Delta) \cdot \theta(\mathcal{D}(\mu K \hat{\sigma} X_i) - p - \Delta)}{\theta(\mathcal{D}(\hat{\sigma} X_i) - p - \Delta) \cdot \theta(\mathcal{D}(\mu K X_i) - p - \Delta)},$$

satisfies $[(\text{the zeros of denominator})] = [(\text{the zeros of numerator})] \in \text{Pic}^g(C)$ and therefore it is a single-valued and meromorphic function over $C$. 

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Lemma 3.1 If (2.11), we derive the following equation from Liouville’s theorem:

\[ \Psi^t(p) = c \times \frac{g_1^t(p) \cdot g_1^{t+K}(p)}{g_1^t(p) \cdot g_1^{t+K}(p)}, \quad c: \text{constant.} \]  

(3.6)

Due to (3.6), we can calculate some special values of \( \Psi^t(p) \):

**Lemma 3.1** If g.c.d.\((M + K, N) = 1\), we have (i) \( \Psi^t(P) = c \), (ii) \( \Psi^t(Q) = c \times \frac{I_{(M-1)K}^{t}(N) \cdot (g_1^t, \ldots, g_N^t)}{I_{(M-1)K}^{t}(N) \cdot (g_1^t + g_2^t)} \).

**Proof.** Because \((g_1^{t+K}, \ldots, g_N^{t+K}) = R_{t-(M-1)K} \cdot (g_1^t, \ldots, g_N^t)\), we have

\[ \Psi^t = c \times \frac{g_1^t \cdot (I_{(M-1)K}^{t}(N) \cdot g_1^t + yg_2^t)}{g_1^t \cdot (I_{(M-1)K}^{t}(N) \cdot g_1^t + g_2^t)}. \]

By corollary 2.2 and proposition 2.3, we easily obtain the desired result. \( \blacksquare \)

Because \( \theta(D(X) - \iota Q - \Delta) = \theta(D(X) + (\iota P - \iota Q) - \iota P - \Delta) = \theta(D(\sigma X) - \iota P - \Delta) \), it follows that

\[ \Psi^t(Q) = \Psi^t_+(P), \quad \text{where} \quad \Psi^t_+(p) = \frac{\tau^t_+(p) \cdot \tau^t_{+K}(p)}{\tau^t(p) \cdot \tau^t_{+K}(p)}. \]

Then lemma 3.1 implies \( I_{(M-1)K}^{t}(N) \Psi^t_+(P) = I_{(M-1)K}^{t}(N) \Psi^t(P) \).

Repeating the same arguments with \( \Psi_+(p) \), we derive \( I_{(M-1)K}^{t}(N) \Psi^t_+(P) = I_{(M-1)K}^{t}(N) \Psi^t_+(P) \), and inductively, we have

\[ I_{(M-1)K}^{t}(N) \Psi^t_+(P) = I_{(M-1)K}^{t}(N) \Psi^t_+(P) = I_{(M-1)K}^{t}(N) \Psi^t_+(P) = \cdots = I_{(M-1)K}^{t}(N) \Psi^t_+(P) = \cdots = I_{(M-1)K}^{t}(N) \Psi^t_+(P). \]

Let \( \Psi^t_n := \Psi^t_{+ \cdots +}(P) \) (n “+”s). Finally we obtain the equations \( \Psi^t_{n+N} = \Psi^t_n \) and \( I_{(M-1)K}^{t}(N) \Psi^t_n = d \), where the number \( d \) does not depend on \( n \).

Next, consider the following single-valued meromorphic function over \( C \):

\[ \Phi^t(p) := \frac{\tau^t_-(M) \cdot \tau^t(p)}{\tau^t(M) \cdot \tau^t(p)} = \frac{\theta(\mathfrak{D}(\mu_\iota \sigma X_1) - p - \Delta) \cdot \theta(R(\sigma X_1) - p - \Delta)}{\theta(\mathfrak{D}(\mu_\iota \sigma X_1) - p - \Delta) \cdot \theta(R(\sigma X_1) - p - \Delta)}. \]

Using (2.11) and Liouville’s theorem, we derive the following expression:

\[ \Phi^t(p) = c' \times \frac{g_1^{t-M}(p) \cdot g_1^{t}(p)}{g_1^{t-M}(p) \cdot g_1^{t}(p)}, \quad c': \text{constant,} \]

(3.7)

which allows us to compute some special values of \( \Phi^t(p) \).
Lemma 3.2 If $\gcd(M + K, N) = 1$, we have (i) $\Phi^t(P) = c'$, (ii) $\Phi^t(Q) = c' \times \frac{V^{-K M}_{N}}{V^{I - K M}_{1}}$.

Proof. By $(g_1^{1-M}, \ldots, g_N^{1-M}) = L_{t-KM} \cdot (g_1^{1}, \ldots, g_N^{1})$, it follows that

$$
\Phi^t = c \times \left(\frac{V^{-K M}_{N} g_1^{1} + g g_1^{1}}{V^{I - K M}_{N} g_N^{1} + g g_N^{1}}\right).
$$

By virtue of corollaries 2.2, 2.3 and proposition 2.4, we obtain the desired result.

Due to $\Phi^t(Q) = \Phi^t(P)$ and lemma 3.2, we have $V_t^{-} \Phi^t(P) = V_t^{-} \Phi^t(P)$, which implies

$$
V_t^{-} \Phi^t(P) = V_1^{-} \Phi^t(P) = V_2^{-} \Phi^t(P) = V_3^{-} \Phi^t(P) = \cdots, \quad V_t^{-} = V_t^{-K M}.
$$

Let $\Phi_{t} := \Phi_{t}^{++\ldots}(P)$ (n “+”s). Therefore, we obtain $\Phi_{t}^{+} = \Phi_{t}^{+}$ and $V_t^{-} = d'$, where the number $d'$ does not depend on $n$.

Define $\tau_{t}^{+} := \tau_{t}^{+}(iP)$, $\tau_{t}^{0} := \tau_{t}^{0}(iP)$, $\tau_{t}^{1} := \tau_{t}^{1}(iP)$, $\cdots$, $\tau_{t}^{n} := \tau_{t}^{n}(iP)$ (n “+”s). By the above arguments, $I_t^{I}$ and $V_t^{I}$ have following expressions:

$$
I_t^{I} = d \times \frac{\tau_{t-1}^{n+1} + \tau_{t-1}^{n+1} K}{\tau_{t-1}^{n+1} + \tau_{t-1}^{n+1} K}, \quad V_t^{I} = d' \times \frac{\tau_{t-1}^{n+1} + \tau_{t-1}^{n+1} K}{\tau_{t-1}^{n+1} + \tau_{t-1}^{n+1} K}.
$$

(3.8)

3.2 solution of rdKP

For g-dimensional vectors $a$ and $b$, $\langle a, b \rangle$ denotes $a^T b \in \mathbb{C}$.

Due to the periodicity $\mathfrak{M}(\sigma^N X_t) = \mathfrak{M}(X_t)$, there exist integer vectors $n, m \in \mathbb{Z}^g$ such that $\mathfrak{A}(\mathfrak{M}(\sigma^N X_t) - i\tau(P - \Delta)) = n + \Omega m$. Considering the definition of the Riemann theta function (see [3], §II.1, for example), we have

$$
\tau_{t+n} = \tau_{t} \exp(-2\sqrt{-1} \pi \cdot \langle m, z \rangle - \sqrt{-1} \pi \cdot \langle m, \Omega m \rangle),
$$

where $z = \mathfrak{A}(\mathfrak{M}(\sigma^n X_t) - (iP - \Delta))$. By (3.8), we have

$$
I_t^{I} = d \times \frac{\tau_{t-1}^{n+1} + \tau_{t-1}^{n+1} K}{\tau_{t-1}^{n+1} + \tau_{t-1}^{n+1} K}, \quad V_t^{I} = d' \times \frac{\tau_{t-1}^{n+1} + \tau_{t-1}^{n+1} K}{\tau_{t-1}^{n+1} + \tau_{t-1}^{n+1} K}.
$$

(3.8)

$$
I_t^{I} V_t^{I} \cdots I_t^{I} = d^n \times \frac{\tau_{t-1}^{n+1} + \tau_{t-1}^{n+1} K}{\tau_{t-1}^{n+1} + \tau_{t-1}^{n+1} K},
$$

$$
= d^n \times \exp(-2\sqrt{-1} \pi \cdot \langle m, \mathfrak{A}(iP - \kappa A_j) \rangle),
$$

(3.9)

$$
V_t^{I} V_t^{I} \cdots V_t^{I} = d' d^n \times \frac{\tau_{t-1}^{n+1} + \tau_{t-1}^{n+1} K}{\tau_{t-1}^{n+1} + \tau_{t-1}^{n+1} K},
$$

$$
= d' d^n \times \exp(-2\sqrt{-1} \pi \cdot \langle m, \mathfrak{A}(iP - \kappa B_i) \rangle),
$$

(3.10)

where $t \equiv -jK \pmod M$ and $t \equiv -iM \pmod N$. Recall $\prod_n I_t^{I} = \prod_n I_t^{I}$ and $\prod_n V_t^{I} = \prod_n V_t^{I}$, which imply that $d$ depends on $t (\mod M)$ and that $d'$ depends on $t (\mod K)$. Finally, we obtain the conclusion:
Theorem 3.3  On condition that g.c.d.(M + K, N) = 1, \([3.8, 3.10]\) solves the rdKP equation \([1.1, 1.2]\) with constraint
\[ U_1 = U_2 = \cdots = U_N. \]

4  The general cases

In the previous sections, we have assumed that M + K and N are relatively prime. This is the time to delete the assumption and discuss the general cases.

Unfortunately, the method which we have established in this paper cannot be applied to the general cases. For example, when \(M = K = 1, N = 2\), the defining polynomial of the spectral curve (Section 2) is

\[ \det (x + y - xE) = y^2 - y(2x + U_1) + x^2 - 2x U_2 + U_3, \]

where \(U_1 = I_1^1 I_2^2 + V_1^1 V_2^2, U_2 = V_1^1 I_1^1 + V_2^2 I_2^1, U_3 = I_1^1 I_2^1 V_1^2 V_2^2.\) Of course, these \(U_1, U_2\) and \(U_3\) are conserved quantities of the discrete reduced KP system \([1.1, 1.2]\). However, there exists another hidden independent conserved quantity of the system. In fact, \(I_1^1 + I_2^2 + V_1^1 + V_2^2\) is invariant under the time evolution \(t \to t + 1\) and is independent from \(U_1, U_2\) and \(U_3\). This means that the spectral curve fails to reflect faithfully the information of the system.

Therefore, we should construct a new method for the general cases. Now we prove that every reduced KP equations can be traced to the case g.c.d.(M + K, N) = 1. Denote by KP\(_{M, K, N}\) the reduced discrete KP equation \([1.1, 1.2]\) associated with the positive integers \(M, K\) and \(N\).

Let \(\Lambda := \mathbb{Z}_{\geq 0} \setminus \bigcup_{k=0}^{\infty} \{ kM, kM + 1, \ldots kM + K - 1 \}\). Note that \(K < M \nleftrightarrow \Lambda \neq \emptyset\) and \(M < K \nleftrightarrow \Xi \neq \emptyset\).

Proposition 4.1  (i) Suppose \(K < M\). Define the initial values \(I_0^0, \ldots, I_{K-1}^0 := \zeta + o(\zeta), (\zeta \to \infty, \forall n)\) for some complex parameter \(\zeta\). If \(\{I_n^t, V_n^t\}_{n, t \in \mathbb{Z}}\) is a solution of KP\(_{M, K, N}\), then the sequence \(\{I_n^t, V_n^t\}_{n, t \in \mathbb{Z}, t \in \Lambda}\) converges to a solution of KP\(_{M-K, K, N}\) when \(\zeta \to \infty\).

Proof.  Order the elements of \(\Lambda\) as \(\Lambda = \{ t_1 < t_2 < t_3 < \cdots \}\). Note that \(t_{s-M+K} = t_s - M\) and
\[ t_s - t_l \equiv 0 \pmod{K} \iff l \equiv 0 \pmod{K}, \quad (\therefore t_s - t_{s-1} = 1, \text{ or } K + 1). \]

Especially, we have \(t_s - t_{s-K} = kK \implies t_s - K, t_s - 2K, \ldots, t_s - (k - 1)K \notin \Lambda\).

To prove the statement, it is sufficient to say
\[
\begin{cases}
I_n^{t_s} = \frac{I_{n-1}^{t_{s-M+K}} + V_{n-1}^{t_{s-K}} - V_n^{t_{s-1}} + o(1)}{I_n^{t_{s-M+K}}} \cdot (1 + o(1)) \\
V_n^{t_s} = \frac{I_{n-1}^{t_{s-M+K}} + V_{n-1}^{t_{s-K}} - V_n^{t_{s-1}} + o(1)}{I_n^{t_{s-M+K}}} \cdot (1 + o(1))
\end{cases}
\]
or equivalently,
\[
\begin{align*}
I^n_t &= I^{t-M}_{n-1} + V^{t-kK}_{n-1} - V^n_{n-1} + o(1) \\
V^n_t &= \frac{I^{t-M}_{n-1} V^{t-kK}_{n-1}}{I^n_t} \cdot (1 + o(1)), \quad (k \in \mathbb{Z}_{>0}; t - kK \in \Lambda).
\end{align*}
\] (4.1)

By (1.1–1.2) and remark 2.2, we have
\[
I^n_t = \zeta + o(\zeta), \quad (\forall n) \Rightarrow \left\{ \begin{array}{l}
I^n_t = \zeta + o(\zeta), \quad (\forall n) \\
V^n_t = V^{t-K} + o(1), \quad (\forall n)
\end{array} \right.
\]

In our situation, it follows that \( t \not\in \Lambda \Rightarrow V^n_t = V^{t-K} + o(1), \quad (\forall n) \). Using (1.1–1.2) again, we can conclude (4.1) soon.

Similarly, we have:

**Proposition 4.1** (ii) Suppose \( M < K \). Define the initial values \( V^0_n, \ldots, V^{M-1}_n := \zeta + o(\zeta), \quad (\zeta \to \infty; \forall n) \) for some complex parameter \( \zeta \). If \( \{I^n_t, V^n_t\}_{n,t \in \mathbb{Z}} \) is a solution of \( \text{KP}_{M,K,N} \), then the sequence \( \{I^n_t, V^n_t\}_{n \in \mathbb{Z}, t \in \Lambda} \) converges to a solution of \( \text{KP}_{M,K-M,N} \) when \( \zeta \to \infty \).

**Example 4.2** The reduced discrete KP equation with \( M = K = 1, N = 2 \) can be traced to \( M = 2, K = 1, N = 2 \).

Let \( L_1 = \begin{pmatrix} V^1_1 & 1 \\ y & V^1_2 \end{pmatrix} \), \( R_0 = \begin{pmatrix} \zeta & 1 \\ y & \zeta \end{pmatrix} \), \( R_1 = \begin{pmatrix} I^1_1 & 1 \\ y & I^1_2 \end{pmatrix} \), and \( X_1 := L_1 R_1 R_0 \). The defining function of the spectral curve is
\[
\det(X_1(y) - xE) = -y^3 + y^2(\zeta^2 + U_1) - y(x(2\zeta + U_4) + \zeta^2 U_1 + U_3) + x^2 - \zeta U_2 x + \zeta^2 U_3,
\]

where \( U_4 = I^1_1 + I^1_2 + V^1_1 + V^1_2 \). Note that \( U_4 \) is the hidden conserved quantity of \( \text{KP}_{1,1,2} \). If \( \{I^n_t, V^n_t\}_{n,t} \) is a solution of \( \text{KP}_{2,1,2} \), the sequence
\[
\lim_{\zeta \to \infty} I^{t}_n, \lim_{\zeta \to \infty} V^{t}_n, \lim_{\zeta \to \infty} R^n_3, \ldots, \lim_{\zeta \to \infty} V^{t}_n, \lim_{\zeta \to \infty} V^{t}_n, \lim_{\zeta \to \infty} V^{t}_n, \ldots
\]
is a solution of \( \text{KP}_{1,1,2} \).

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A Proof of lemma 2.9

Let $S^*$, $R^*$ and $L^*$ be the matrices defined by (2.19)–(2.21). We first calculate the coefficients $a_{i,j}$ of the equation (2.16)–(2.17):

$$a_{i,0} \cdot g_i + a_{i,1} \cdot g_{i+1} + \cdots + a_{i,M+K} \cdot g_{i+M+K} = x \cdot g_i, \quad g_{i+N} = y g_i,$$

which is equivalent to the formula $X(y) \cdot (g_1, \ldots, g_N) = x \cdot (g_1, \ldots, g_N)$.

Denote the vector $(g_1, \ldots, g_N)$ by $(g)_i$, simply. By equation (2.22):

$$X_t(y) := L_{t-(K-1)M}(y) \cdots L_{t-2M}(y)L_{t-M}(y)L_t(y) \times$$

$$R_t(y)R_{t-K}(y)R_{t-2K}(y) \cdots R_{t-(M-1)K}(y),$$

we have

$$X_t \cdot (g)_i = L_{t-(K-1)M}L_{t-M}L_tR_t \cdots R_{t-(M-1)K}(g)_i$$

$$= L_{t-(K-1)M}L_{t-M}L_tR_t \cdots R_{t-(M-2)K}(I^t_{i-(M-1)K}g_i + g_{i+1})$$

$$= L_{t-(K-1)M}L_tR_t \cdots R_{t-(M-3)K}(I^t_{i-(M-1)K}g_i + g_{i+1})$$

$$= \cdots.$$}

Let $\mathcal{X}$ be the set of sequences of letters $s$ and $m$ of length $l$. For example, $\mathcal{X}_0 = \emptyset, \mathcal{X}_1 = \{s, m\}, \mathcal{X}_2 = \{ss, sm, ms, mm\}, \ldots$. Denote $\mathcal{X} := \cup_l \mathcal{X}_l$. Consider a map $\langle \cdot \rangle : \mathcal{X} \to \mathbb{C}$ defined by: $\langle s \rangle := 1$, $\langle m \rangle := I_i^{t-(M-1)K}$ and

$$\langle s \rangle := \sigma(\chi), \quad \langle m \rangle := V_i^{t-(l-1)M} \cdot \langle \chi \rangle, \quad \chi \in \mathcal{X}_l \quad (1 < l \leq M),$$

$$\sigma : \prod I_i^{s} \cdot \prod V_j^{s} \mapsto \prod I_{j+1}^{s} \cdot \prod V_{j+1}^{s} \text{ is the index shift operator.}$$

This notation allows us to express the vector $X_t \cdot (g)_i$.

Lemma A.1 Let

$$\mathcal{X}_{i,k} := \{ \chi \in \mathcal{X}_l \mid \text{The number of ‘s’ contained in } \chi \text{ is } k \},$$

and $a_{i,k} := \sum_{\chi \in \mathcal{X}_{i+K}} \langle \chi \rangle$. Then

$$X_t \cdot (g)_i = (a_{i,0} \cdot g_i + a_{i,1} \cdot g_{i+1} + \cdots + a_{i,M+K} \cdot g_{i+M+K}).$$

Proof. Let $\Xi_l := R_{l-(M-1)K}(y) \quad (l = 1, 2, \ldots, M), \quad \Xi_{M+1} := L_{l-(l-1)M} \quad (l = 1, 2, \ldots, K)$ and $\xi_l := I_{i-(M-1)K} \quad (l = 1, 2, \ldots, M), \quad \xi_{M+1} := V_{i-(l-1)M} \quad (l = 1, 2, \ldots, K)$. Assume

$$\Xi_1 \Xi_{l-1} \cdots \Xi_1 \cdot (g)_i = (\sum_k \sum_{\chi \in \mathcal{X}_{i+k}} \langle \chi \rangle \cdot g_{i+k}).$$
Then we have

\[\Xi_{t+1} \Xi \cdot \Xi_1 \cdot (g_i) = \Xi_{t+1} \cdot (\sum_k \sum_{\chi \in \mathcal{X}_{t,k}} (\chi \cdot g_i) \cdot g_i) = (\xi_{t+1} \cdot (\sum_k \sum_{\chi \in \mathcal{X}_{t,k}} (\chi \cdot g_i) \cdot g_i) + \sigma(\sum_k \sum_{\chi \in \mathcal{X}_{t,k}} (\chi \cdot g_i) \cdot g_i)) \cdot (\sum_k \sum_{\chi \in \mathcal{X}_{t,k}} (\chi \cdot g_i) \cdot g_i) = (\sum_k \sum_{\chi \in \mathcal{X}_{t,k}} (\chi \cdot g_i) \cdot g_i).\]

By induction, we obtain the desired result.

**Lemma A.2** Let \(\chi \in \mathcal{X}_{t,k}\). Then \(\langle l \cdot (m) \rangle = \langle l \cdot (m) \rangle \cdot I - (M - 1)K\).

**Proof.** Let \(I_i := I_i - (M - 1)K\) and assume \(\langle l \cdot (m) \rangle = \langle l \cdot (m) \rangle \cdot I_i - (M - 1)K\). If \(\chi = s \cdot \chi' \cdot (y \in \mathcal{X}_{t,k-1})\), we have \(\langle l \cdot (m) \rangle = \langle s \cdot \chi' \cdot (y \in \mathcal{X}_{t,k-1})\rangle = \langle \sigma(\langle l \cdot (m) \rangle \rangle = \langle \sigma(\langle l \cdot (m) \rangle \rangle \cdot I_i - (M - 1)K\). If \(\chi = m \cdot \chi' \cdot (y \in \mathcal{X}_{t,k-1})\), then \(\langle l \cdot (m) \rangle = I_i - (M - 1)K\). By induction we obtain the desired result.

**A.0.1** calculation of the determinant of \(S^*\)

By lemma A.1, it follows that \(E_1 = x - a_{1,0} = x - \sum_{\chi \in \mathcal{X}_{t,k-1}} \langle \chi \rangle\), \(E_{k+1} = -a_{1,k} = -\sum_{\chi \in \mathcal{X}_{t,k}} \langle \chi \rangle\). Then,

\[
\det S^* = (-1)^M \cdot K + 1. E_1 = (-1)^M + K \sum_{\chi \in \mathcal{X}_{t,k}} \langle \chi \rangle - x)
\]

\[
= (-1)^M + K \sum_{\chi \in \mathcal{X}_{t,k}} \langle \chi \rangle - 1 - K \sum_{\chi \in \mathcal{X}_{t,k}} \langle \chi \rangle - x)
\]

\[
= (-1)^M + K (U_1 - x).
\]

**A.0.2** calculation of the determinant of \(R^*\)

Let \(w_k := \sum_{\chi \in \mathcal{X}_{t,k}} \langle \chi \rangle\) and \(w_{-1} := 0\). By lemma A.2, it follows that

\[
a_{1,k} = \sum_{\chi \in \mathcal{X}_{t,k-1}} \langle \chi \rangle + \sum_{\chi \in \mathcal{X}_{t,k}} \langle \chi \rangle
\]

\[
a_{1,k} = \sum_{\chi \in \mathcal{X}_{t,k-1}} \langle \chi \rangle + \sum_{\chi \in \mathcal{X}_{t,k}} \langle \chi \rangle - I_{k+1} = a_{1,k} + w_{k-1} + w_k \cdot I_{k+1}.
\]
Let $z_n := I_1^- I_2^- \cdots I_n^-$. By the cofactor expansion w.r.t. the $(M + K)$-th row, we obtain:

\[
\det R^* = (-1)^{M+K+1} \{ E_1 - E_2 z_1 + E_3 z_2 - \cdots \\
+ (-1)^{M+K+1} E_{M+K} z_{M+K-1} + (-1)^{M+K+1} z_{M+K} \} \\
= (-1)^{M+K+1} \{ x - a_{1,0} + a_{1,1} z_1 - a_{1,2} z_2 + \cdots \\
+ (-1)^{M+K} a_{1, M+K-1} \cdot z_{M+K-1} + (-1)^{M+K+1} z_{M+K} \} \\
= (-1)^{M+K+1} \{ x - w_0 I_1^- + w_0 z_1 - w_1 z_1 I_2^- + w_1 z_2 - w_2 z_2 I_3^- + \cdots \\
+ (-1)^{M+K} w_{M+K-1} z_{M+K-1} z_{M+K} + (-1)^{M+K+1} z_{M+K} \} \\
= (-1)^{M+K+1} \cdot x.
\]

\[\blacksquare\]

A.0.3 calculation of the determinant of $L^*$

Recall the alternative form of the matrix $X_t(y)$ (cf. (2.5)):

\[
X_t(y) = R_{t-MK}(y) R_{t-(M+1)K}(y) \cdots R_{t-(2M-1)K}(y) \times \\
L_{t-(2N-1)M}(y) \cdots L_{t-(K+1)M}(y) L_{t-KM}(y),
\]

in which the rightmost matrix $L_{t-KM}(y)$ is essential. The formula $\det L^* = (-1)^{M+K+1} \cdot x$ comes about through arguments similar to those in section A.0.2 concerning (2.5).

\[\blacksquare\]

References

[1] Iwao S 2008 “Solution of the generalised periodic Toda equation” J. Phys. A. Math. Theor. 41 115201

[2] Iwao S “Solution of the generalised periodic Toda equation II; Tau function solution” in submition to J.Phys.A.Math.Theor., (arXiv/0912.2213)

[3] Mada J, Idzumi M and Tokihiro T 2004 “Conserved quantities of generalized periodic box-ball systems constructed from the ndKP equation” J. Phys. A: Math. Gen. 37 6531–6556

[4] van Moerbeke P and Mumford D 1979 “The spectrum of difference operators and algebraic curves” Acta Math. 143 (1–2) 94–154

[5] Mumford D, Musili C, Nori M, Previato E and Stillman M 1983 Tata Lectures on Theta I (Progress in mathematics; v.28) ed. Bass H, Oesterlé J and Weinstein A (Berlin: Birkhäuser)

[6] Willox R, Tokihiro T and Satsuma J 1997 “Darboux and binary Darboux transformations for the nonautonomous discrete KP equation” J. Math. Phys. 38 6455–6469