Generalized Titius-Bode Law and How to Statistically Verify it

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Abstract. The Titius-Bode law has been historically successful in predicting the location of planets in the solar system. While the majority believes it is merely a coincidence, we too agree with that; however, we think there is some underlying universal truth hidden under the overfitting phenomena in the original Titius-Bode law; the generalized Titius-Bode, by removing the overfitting, can tell us about the mechanism of the formation of our early solar system, or more generally, all astronomical systems, as big as galaxies. Specifically, we analyzed the absolute error of the generalized Titius-Bode Law with the ground truth, and it suggested that the planets were formed iteratively, instead of each planet just magically finding its orbit where the centripetal force balances the centrifugal force.

1. Introduction

\[ r = 0.4 + 0.15 \times 2^i, \ i = -\infty, 1, 2, 3 \]  \hspace{1cm} (1)

This empirical equation (the Titius-Bode “law”) fits the distance to the sun of every planet in solar system (in earth unit), and Titius assigned Mercury as \(i=-\infty\), Venus as \(i=1\), etc. Throughout this paper, we will follow the convention by counting the asteroid belt as a planet, by default; the reader should keep this in mind. Initially, when Titius-Bode concluded this equation, all planets, at that time, fitted with this relation nicely. However, after the discovery of Neptune, the distance of the Neptune to sun does not fit to this equation as nicely as the others do. In the paper of Hayes [4], they excluded one parameter (the 0.4) from Titius’s law, and the fitting of the Neptune seemed to become better. This new equation is in the form of an exponential growth which allows us to fit a logarithmic spiral that goes through all planets. We conjecture that the formation of the solar system fits a logarithmic spiral similar to the formation of a galaxy’s spiral. Moreover, we conclude the 0.4 from the original Titius-Bode law is an overfitting, without which we can better predict the location of undiscovered planets in the universe.

2. Method

Hayes find that the Titius-Bode law might be an overfitting phenomenon, removing the overfitting parameter gives the “generalized Titius-Bode law” [1][2][3].

\[ d_i = d_ki, \ i = 0, 1, 2, 3 \]  \hspace{1cm} (2)

By taking ln on both sides of equation 2, equation 2 becomes 3.

\[ b_i = b_0 + c \times i \]  \hspace{1cm} (3)
We can verify it is indeed over-fitting by intentionally hiding some data while we fit, then test its predictive power on the hidden data. Historically, Neptune was hidden when Titius fitted the original Titius-Bode law. So, we also hide Neptune while we fit the generalized Titius-Bode law. The fitting result of the two laws, as well as the error are shown in the plots below in Fig (1) and (2).

**Figure 1.** We nondimensionalize the distance into Mercury units (distance from Mercury to sun), and compare the actual distance to the predicting value in original Titius-Bode Law, by using the data in Hayes paper[4]. We can find that the errors of the asteroid belt and Neptune are the greatest. The Neptune highly deviates from the predicted value, so we speculate the Titius-Bode law is weak on predictivity.

**Figure 2.** We exclude the overfitting phenomenon in the Titius-Bode Law and use this to fit the actual value while hiding Neptune, and we find that the predictivity of optimized equation is better.

Generalized Titius-Bode Law and How to Statistically Verify it

3. Meaning

3.1. Scale Invariance

We plotted a logarithmic spiral to fit all planets. It shows a phenomenon called scale invariance: comparing the images 3 including the first five planets and that of all planets, the two images are similar. Moreover [5], since most galaxies also spins in logarithmic spiral, we can speculate the formation process of the small solar system and the large galaxy system may be related.

**Figure 3.** The logarithmic spiral nicely goes “through” all planets and exhibits scale invariance.
Fitting all planets to a logarithmic spiral. Fitting only the first 5 planets.

3.2. **Overfitting**

There are several reasons we believe the original Titius-Bode law is an overfitting.

Firstly, the exponential growth is natural; logarithmic spiral is “golden”. It appeals the theoretical beauty. When we attempt to plot the figure(1) by using the log-log scale, we found the point of Mercury representing log(0) in the x-axis; this also suggests the Titius-Bode law is unnatural. Secondly, after excepting a parameter, the fitting effect is still suitable. Thirdly, by excepting the overfitting parameter, the predictive power (Neptune’s distance to sun) became better, as we can find this in figure(2).

4. **Evidence on Early Formation**

The following discussion is based on the hypothesis that the generalized Titius-Bode law really is a truthful consequence from the formation of the planetary system.

Previously, we presented it in the exponential form in Eqn 2, which becomes Eqn 3 by taking ln in both sides.

But we may very well play a math trick putting it into the iterative form in Eqn 4:

**Generalized Titius-Bode Law and How to Statistically Verify it**  

4.1. **Global Navigation vs Iterative Formation**

Although the two forms are perfectly equivalent in the mathematical sense, the physical interpretations are different. How did each planet finds its place?

The distinction becomes more apparent when we introduce error, which we use W to stand for “Gaussian White noise”.

Equation 4 becomes 5; Equation 3 becomes 6.

4.2. **White Noise vs Random Walk**

By subtracting the ”ground truth” from both equations, we cleaned up the equation and got something familiar: White noise (Eqn 8) vs Random walk (Eqn 7). Where we made a big assumption that we can obtain the ”ground truth (Eqn 4 and 3)” via linear fitting, which is not necessarily true, but it is the best we can do.

Equation 4 becomes 5; Equation 3 becomes 6.

4.3. **Correlation Analysis**

We found the trick: by analyzing the correlation between successive signed-error terms.

In other words, the ”signed absolute error” may either behave like Gaussian noise, or random walk. Can we use the error data we have in hand to reversely deduce which process, A or B, did the solar system actually went through in its early formation?

4.3. **Correlation Analysis**

We found the trick: by analyzing the correlation between successive signed-error terms.

Thus we present this new method to infer the formation process of the solar system.

However, the solar system has only 9 data points, which is insufficient for any statistics.

Thus, we will leave it to another research to apply this method in bulk to lots of solar-like systems.
Generalized Titius-Bode Law and How to Statistically Verify it

5. Conclusion
We removed the overfitting and found the general Titius-Bode Law which is more physically informative and natural. Then we presented a method to test whether the formation of the solar system is iterative.

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