Tachyon effective dynamics and de Sitter vacua

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Abstract

We show that the DBI action for the singlet sector of the tachyon in two-dimensional string theory has a SL(2,R) symmetry, a real-time counterpart of the ground ring. The action can be rewritten as that of point particles moving in a de Sitter space, whose coordinates are given by the value of the eigenvalue and time. The symmetry then manifests as the isometry group of de Sitter space in two dimensions. We use this fact to write the collective field theory for a large number of branes, which has a natural interpretation as a fermion field in this de Sitter space. After spending some time building geometrical insight on facts about the condensation process, the state corresponding to a sD-brane is identified and standard results in quantum field theory in curved space-time are used to compute the backreaction of the thermal background.

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1 D-brane tachyon dynamics

The study of D-branes dynamics have shown that closed string degrees of freedom can actually be hidden in the action of the open string field theory. The recent re-interpretation of the c=1 matrix model as an effective model of decay of D0-branes in two dimensional string theory brought the novelty of studying non-stationary backgrounds in which this duality is manifest, whose features we review as follows.

The central point in the effective description is the DBI action:

$$\mathcal{L} = - \int d^{p+1}x V(T) \sqrt{-\det G},$$

where the matrix $G$ has entries given by

$$G_{ab} = \eta_{ab} + \nabla_a T \nabla_b T + \nabla_a Y^I \nabla_b Y^I + F_{ab},$$

and $T, Y^I$ and $F_{ab} = \nabla_{[a} A_{b]}$ refer to the tachyon, the compactification scalars and the gauge field living in the brane. The Lagrangian (1) was argued in [1, 2, 3, 4, 5, 6] to correctly describe the near on-shell behavior of the tachyon condensation process.

The most striking fact about this model is that the classical solutions of the potential:

$$V(T) = \frac{1}{g_s l_s} \frac{1}{\cosh(\ell^{-1} T)},$$

are solutions of the fully coupled string field theory equations\footnote{provided $\ell = 2l_s$ in the bosonic string and $\ell = \sqrt{2}l_s$ in the superstring.} in the approximation of homogeneous fields. This is so despite the fact that at late times one would expect the field theory to be strongly coupled and thus the classical picture not to hold. In a precise sense (1) is the unique Lagrangian which is both analytical in the fields and gives rise to the correct classical equations of motion. For further discussions see [7] and references therein.

According to Sen’s conjecture, the closed string excitations are hidden in the open string dynamics. To recover it one starts with the effective theory for the tachyon field of large number of branes and study excitations for large values of the tachyon field. For definiteness, consider the system of $N$ D branes in non-critical one-dimensional string theory, described by a non-abelian generalization of (1), a DBI matrix model quantum mechanical system:

$$S[\{T\}] = - \int dt \ Tr \left( V(T) \sqrt{1 - (D_t T)^2} \right).$$

\footnote{provided $\ell = 2l_s$ in the bosonic string and $\ell = \sqrt{2}l_s$ in the superstring.}
with $V(T)$ given as $\text{(3)}$. McGreevy and Verlinde $\text{[8]}$ successfully reinterpreted the system above in terms of a Liouville theory, in the limit where the Fermi sea level is close to the top of the potential. In that limit, the system can be approximated by the usual inverted harmonic oscillator model, where Sen’s conjecture helps shed light on some old puzzles in the relation between that model and two dimensional closed strings. In particular, an eigenvalue trickling down the maximum of $V(T)$, $T = 0$ would represent a D0 brane decaying to the closed string minimum. There is by now a great deal of evidence supporting this point of view, in which the action above can be recast in the double-scaling limit to an effective two-dimensional theory in which the variables are closed string excitations.

In this approximation the model $\text{(4)}$ indeed shows perturbatively most of the behavior expected from the decay of the tachyon, gathered from world-sheet techniques, like the production of closed strings $\text{[9]}$, the final state of the decay process $\text{[10]}$. Some non-perturbative effects where also tied to the boundary states of Liouville proposed by Dorn-Otto and Zamolodchikov-Zamolodchikov (DOZZ) $\text{[11, 12]}$. As the description is blurred in the strongly coupled regime, $T \approx \log g_s$, it made sense to study $\text{(4)}$ close to the open string vacuum regime. One can then use the incredible amount of data that has been gathered on this particular model and make new interpretations in the light of the McGreevy-Verlinde proposal. There is however a great deal of information to be learned by considering generic Fermi sea profiles $\text{[13]}$, where one should deal with the fully coupled problem directly.

Heuristically, one would not naively expect $\text{(1)}$ to hold. Backreaction from the closed string production should give large contributions to the action and furthermore, in higher dimensions the tachyon decay is non-homogeneous, a fact which is not naively anticipated by the action above. Focussing for now on the first fact, it is however surprising that the classical stress energy from $\text{(1)}$ correctly describes the energy of the remnants of the condensation process. And, since loops of $\text{(4)}$ will involve closed strings degrees of freedom in it, one is tempted to investigate quantum effects of the action $\text{(4)}$ starting from the point where the open strings are treated classically. Working in two dimensions will also help us in that one is then able to avoid the problem of non-homogeneity. Also, building a direct correspondence between the effective open and closed string theories is also an interesting problem in itself, and which should lend to a amenable treatment in two dimensions.

The wishful thinking starts with considerations about the tachyon effective dynamics. In this paper we start by considering a $\text{SL(2,R)}$ symmetry which allows us to write a collective field theory for the tachyon excitations which is related to a field reparametriza-
tion to that of \cite{8}. The field has a classical interpretation of eigenvalues moving in a conformally flat background, with the potential playing the role of the conformal weight. In Section 3 we consider generic reasons as to why this model is not much different to the usual inverted harmonic oscillator as far as bosonization is concerned. In Section 4 we study finite energy “probe” solutions and the relations to the known facts about the condensation process in two dimensions. In Section 5 we move on to SL(2,R) invariant states and find a correspondence between the Euclidean (Bunch-Davies) vacuum and the sD-brane state. We use standard methods to compute the backreaction of this vacuum. We close with speculative remarks about the fact that closed string excitations arise as a “near-horizon” limit of the open string effective metric.

As this manuscript was in its final stages of preparation we learned of \cite{14}, whose conclusions have similarities with the points raised in Section 3 and 5.

\section{Collective field theory of the DBI action}

A simple fact of the DBI action which has not been exploited so far in the studies of tachyon condensation is that it displays an analogue of the SU(2) symmetry present in the boundary state formalism \cite{15}. The symmetry appears as SL(2,R) which is more suited to the real-time calculations we will be doing. At the quantum mechanical level, the symmetry turns into a large symmetry algebra which turns the model exactly solvable, much like in the quintessential inverted harmonic oscillator case (see for instance \cite{16}). We will now explicit this algebra for (4).

The model in (4) has an underlying U(N) symmetry that is gauged by a non-dynamical field (hence the covariant derivative in (4)). If we pick the Coulomb gauge for this field the term turns into a Lagrange multiplier that projects the field $T$ into singlet states, i.e., the dynamics then depends only on the set of eigenvalues of $T$, $\{\tau_i\}$, see \cite{21,17} for a thorough review. Since only the eigenvalues are dynamic, $\dot{T}$ and $T$ commute and we can rewrite the action as:

$$L_{\text{cl}} = -\sum_{k=1}^{N} V(\tau_k) \sqrt{1 - \dot{\tau}_k^2},$$

for a generic potential $V(T)$ which commutes with $T$. It is also important to point out that in the path integral formalism, the change of variables from the matrix elements to the eigenvalues introduces a Vandermonde determinant which enforces Fermi statistics in the wave functions. Absorbing this determinant into the measure leads us to an anti-commuting field of eigenvalues. One could use the collective field formalism and
re-express the action in terms of a density field

$$\rho(\tau, t) = \int dk e^{ikx} \text{Tr} (e^{-ikT(t)}) = \sum_{k=1}^{N} \delta(x - \tau_k(t)),$$

but we will be a little more heuristic. One needs to Schrödinger equation, with a Hamiltonian for a single eigenvalue given by:

$$\hat{H}(\pi, \tau) = \sqrt{\pi^2 + V(\tau)^2}.$$

But the definition of (7) as an operator suffers from ordering ambiguities. We are now facing the same problem as Dirac: how to take the square root of an operator while keeping the Pauli exclusion principle. This is now complicated by the fact that \(\pi\) and \(V(\tau)\) neither (anti-)commute, nor are divisors of zero in the operator algebra.

In order to solve this problem we note that the action (5) with the potential (3) has an SL(2,R) symmetry. In fact it describes classical motion of particles of mass \(m = 1/g_s l_s\) in a curved background:

$$S_{cl} = -\sum_k \int dt_k \frac{1}{g_s l_s} \sqrt{\frac{1}{\cosh^2(\ell^{-1}\tau_k)} - \frac{\tau_k^2}{\cosh^2(\ell^{-1}\tau_k)}} \equiv -\sum_k \int dt_k \frac{1}{m} \sqrt{(\frac{ds_k}{dt})^2},$$

where the metric, given by:

$$ds^2 = \frac{1}{\cosh^2(\ell^{-1}\tau)} (dt^2 - d\tau^2),$$

can be mapped to the static patch of a two-dimensional de Sitter space. Defining \(r = \ell \tanh(\ell^{-1}\tau)\):

$$ds^2 = (1 - (\ell^{-1}r)^2) dt^2 - \frac{dr^2}{1 - (\ell^{-1}r)^2},$$

where \(-\ell < r < \ell\) maps to the whole real line in the \(\tau\) coordinate. The SL(2,R) algebra can be seen as isometries of the “space-time” whose coordinates are given by time \(t\) and the eigenvalue \(\tau\). Thus any classical trajectory can be obtained by acting with the SL(2,R) generators on the static trajectory, perched at the top of the potential \(\tau = 0\).

To be more explicit, we can forget for a moment the Fermi statistics and the ambiguities of defining a quantum Hamiltonian and numbly square (7) with the potential (3). The resulting equation of motion can be written as:

$$\left(\cosh^2(\ell^{-1}\tau)[\partial_t^2 - \partial_\tau^2] - m^2\right) \phi(\tau, t) = 0,$$

where...
where we call the double-scaled mass also $m$, by an abuse of language. Now, by introducing the operators:

$$J_{\pm} = \ell e^{\pm \ell^{-1} \tau} \left( \cosh(\ell^{-1} \tau) \frac{\partial}{\partial \tau} \pm \sinh(\ell^{-1} \tau) \frac{\partial}{\partial t} \right), \quad J_3 = \ell \frac{\partial}{\partial t}, \quad (12)$$

the equation (11) turns into the suggestive form of:

$$J^2 \psi(x) = \left( \frac{1}{2} (J_+ J_- + J_- J_+) - J_3^2 \right) \phi(x) = m^2 \ell^2 \phi(x). \quad (13)$$

The operators in (12) do indeed satisfy a SL(2,R) algebra:

$$[J_3, J_{\pm}] = \pm J_{\pm}, \quad [J_+, J_-] = 2J_3. \quad (14)$$

With these provisions, the natural candidate for the quantized collective field action will be the free, massive Majorana fermionic field in the cosmological patch of two dimensional de Sitter space. The action will then be:

$$S_{\psi} = \frac{1}{2} \int d^2 x \ e \left( i \bar{\psi} \gamma^a \nabla_a \psi - i (\nabla_a \bar{\psi}) \gamma^a \psi - 2m \bar{\psi} \psi \right), \quad (15)$$

where $e = \det [e^i_a]$ is the determinant of the zweibein, $\gamma^a = e^i_a \gamma^i$ are the de Sitter gamma matrices\(^2\) and

$$\nabla_a \psi = \partial_a \psi - \frac{1}{2} \Sigma^{ij} e^a_i (\nabla_a e_{bj}) \psi \quad (16)$$

is the covariant derivative compatible with the metric (9), with $\Sigma^{ij}$ being the Lorentz generator. As a matter of fact the action (15) is very similar to the action proposed in\(^3\):

$$S_\chi = \int d^2 x \ \left( \bar{\chi} \gamma^i \partial_i \chi - \frac{m}{\cosh \ell^{-1} \tau} \bar{\chi} \chi \right), \quad (17)$$

except that in (15) the SL(2,R) symmetry of the classical solutions is manifest. In fact one recovers (15) by rewriting the action above with the metric (9) in terms of the scale invariant fermionic field:

$$\chi(t, \tau) = \cosh^{-1/2}(\ell^{-1} \tau) \psi(t, \tau). \quad (18)$$

One recovers (17) since in two dimensions the action $S_{\psi}$ does not depend on the spin connection. Since we are interested in the symmetry itself, we will work with (15) from\(^3\)

\(^2\)They are taken to satisfy $\{ \gamma^a, \gamma^b \} = 2g^{ab}$. See the Appendix for conventions and a review of the needed spinor calculus.
now on.

The algebra given by (12) is augmented by addition of the $w_\infty$ generators:

$$O_{n,m} = J_{+}^{n+m} J_{-}^{n-m},$$

with $n = 1, \ldots$ and $-n < m < n$. The structure hints at an underlying Lax formalism and then at the integrability of the theory, which would be hardly surprising since the recast theory is equivalent to free massive fermions. The occurrence to $w_\infty$ algebras is usually associated to the presence of massless scalars. One has to wonder then if there is an alternative way of recasting the Lagrangian (15) in terms of a massless field, just like in the case of massive Klein-Gordon field in flat two-dimensional space [18]. Clarifying this point would be interesting to the problem of bosonization.

As a preamble for the discussion in the next section, one is interested in the regime where the observer at $\tau = 0$ sees no excitations, because classically none of them can climb up the potential wall. The instability of the open string vacuum correspond to the fact that in de Sitter near light-like geodesics repel and thus any wavefunction that is initially perched at the top of the potential will at the end “spread over” and run off to the horizon at $r = \pm \ell$, or $\tau = \pm \infty$. The latter is identified with the closed string vacuum.

There is a curious aspect of two dimensional de Sitter spaces which hinders the study the system in terms of the proper-time of the eigenvalues as they approach the horizon. The reparametrization $\tau \rightarrow r$. shows rather clearly that the point $\tau = \infty$ is at finite distance in field space, but also does not say anything about what happens after that. The question arises as what is the metric beyond the horizon. The answer since there is a family of SL(2,R) invariant spaces that allow for the same static patch$^3$. For instance, all spaces considered in [19] will admit a static patch of the sort described above (10) if they are of the elliptic class and $\epsilon > \frac{1}{2}$ in

$$ds^2 = \frac{-dt^2 + d\sigma^2}{\sin^2 \epsilon t}.$$  (20)

Thus one cannot use the classical SL(2,R) invariance to predict what will happen at the end of the decay process. We will however use quantum mechanics to infer about the global structure of field space in Section 5.

$^3$I thank Will McElgin for this remark.
3  A few classical remarks

Before continuing to explore the symmetry of the DBI action, we should stop for a moment and try to understand the relation between this description of the tachyon dynamics and the usual inverted harmonic oscillator model. There one considers a number of non-relativistic fermions running along with the Hamiltonian:

\[ H(p,q) = \frac{p^2}{2} - \frac{q^2}{2} + \frac{1}{g_s}, \]  \hspace{1cm} (21)

where we fixed the additive constant to have \( H = 0 \) at the Fermi surface. As discussed in [20], there is a canonical transformation relating (21) to (8). In order to obtain it one just needs to solve each system separately as a function of energy and initial time:

\[ p = p(E,t) \leftrightarrow \pi = \pi(E,t) \]
\[ q = q(E,t) \leftrightarrow \tau = \tau(E,t). \]  \hspace{1cm} (22)

One can then eliminate \( E \) and \( t \) and find the transformation \( p(\pi,\tau), q(\pi,\tau) \). This transformation does not map the solutions of (21) with \( H < 0 \), but they show up again in the relativistic model when we consider the quantum mechanical effective action (15), which can be seen as an advantage of this formulation. On the other hand, the Fermi level \( H = 0 \) is mapped to \( \tau \to \infty \) in the relativistic Hamiltonian and then it may be tricky to extract information about the closed string vacuum in this case. However, in (8) the closed string vacuum corresponds to the horizon, as one can see from (10) taking into account that \( \tau \to \infty \) maps to \( r = \ell \). So the transformation above really maps small excitations around the closed string vacuum, like for instance the dilaton or the “tachyon”, to “near horizon” excitations in the relativistic model. Extracting information about closed strings excitations from (8) should be equivalent to studying excitations of the field (15) around the horizon of de Sitter, a well-known problem in QFT in curved space-time.

The argument can be made more explicit by actually computing \( q(\pi,\tau) \):

\[ q = \sqrt{2 \left( \frac{1}{g_s} \right)} \frac{E}{g_s^{\frac{3}{2}}} \cosh \left[ \ell \text{ arccosh} \left( \frac{E g_s^{\frac{1}{2}}}{\sqrt{1 - E^2 g_s^{2 \frac{3}{2}}} \sinh \frac{\tau}{\ell}} \right) \right]. \]  \hspace{1cm} (23)

At large values of the position, \( \tau \) becomes the “time of flight” coordinate of (21), so in the “near horizon” limit there is a clear connection between the excitations of the relativistic fermions of (8) and the closed string fields. Here we used the time translation
symmetry of the two systems to hide constants. In hindsight this is not so unexpected since the potential drops to zero exponentially and hence one should be able to describe the system as free massless fermions as $\tau \to \pm \infty$, in which case would presumably know how to bosonize the system. One notes that the transformation $(p, q) \to (\pi, \tau)$ is defined only for positive energy solutions. How exactly one should include negative energy solutions and the other side of the potential $q < 0$ will rely on extra input, like a comparison between reflection coefficients of the quantum mechanical model, which will be postponed until the next section. See also [22] for the relation between the collective field theory of (21) and the relativistic field theory (in flat space).

Nevertheless this does not mean that one is required to take the “near horizon” limit to talk about closed string excitations. We begin to illustrate this point by using the classical integrability of the system (8). The Hamilton-Jacobi equation can be solved to give

$$S_{cl}(t, \tau) = -Et + \int_{\tau_0}^{\tau} d\tau' \sqrt{E^2 - V(\tau')^2},$$

which can be written as a sum of elliptic integrals. The classical action (24) generates the canonical transformation that trivializes the dynamics. Any such system can be mapped to (21) by composing the transformations, as long as the spectra of energies match. Then one can rewrite the “loop functional”:

$$W(\ell, t) = \int_R dp \wedge dq e^{\ell q},$$

integrated over the volume of phase space occupied by the Fermi sea. The integral above, originally written in terms of the $(p, q)$ variables of (21), can thus be recast in terms of any integrable system since the canonical transformation preserves the volume of phase space and the function $q$ can be written in the new set of coordinates, (8) being but one case among an infinite number. If the Fermi level is at a surface of constant energy $H = \omega$, variations of $W$ satisfy the Wheeler-de Witt equation [18]:

$$(\partial_t^2 - \partial_\ell^2) \delta W = 2\omega\ell^2 \delta W,$$

which can be written without any reference to the parametrization of the matrix model. The Liouville field can be obtained from $\delta W$ by the usual procedure [23], at least in the genus zero, or zero string coupling approximation. This trick will of course not have a natural interpretation in terms of density of eigenvalues, or loop observables of the matrix model, but serves as a reminder that one can reshuffle the degrees of freedom in various ways.
One can then see the effective action for fermions (5) as a model which interpolates between the inverted harmonic oscillator model and the relativistic fermions of [22]. The model is also equivalent classically to the inverted harmonic oscillator, related to it by a finite canonical transformation, generated by the ground ring of [21, 24, 26]. The non-trivial value of the observation weights on the fact that this model also has a ground ring (12), realized here as generators of a “space-time” symmetry. One should bear in mind that this non-flat background is not the one on which the closed degrees of freedom themselves exist, like the one proposed in [33], but rather an effective way of looking at the dynamics of the open strings. It should be pointed out, however, that one could in principle obtain an adS$_2$ background in a similar fashion. The lesson from the BCFT studies is that the tachyon potential gives rise to a constant curvature in field space. By considering space-like tachyon condensation, this curvature will become negative, and thus the relevant geometry of the field space will be as in the paper cited above. In other words, the adS$_2$ metric can be obtained from (9) by a double Wick rotation. This seems to be an interesting new route to explore in further research.

We close this section with a remark on the spectrum of the action (15). Although the theory is relativistic and massive, there are excitations with arbitrarily low energy in it. These can be understood classically. Consider the classical trajectory $A$ in Fig. 1. In the usual tachyon dynamics language it represents a D0 brane perched at the top of the potential. Its energy is just $m$. In the de Sitter interpretation it just corresponds to a particle at rest at the “observer’s position”, $\tau = 0$.

![Conformal diagram of 2-dimensional de Sitter](image)

Figure 1: The conformal diagram of 2-dimensional de Sitter. The diamond represents the static patch, covered by the coordinates $t, \tau$. Two time-like geodesics are drawn, $A$ is perched at $\tau = 0$, whereas the “bounce” $B$ has non-trivial $\tau$ dependence. The $B$ is obtained from $A$ by a suitable application of the generators (12).
Consider now the “bounce” trajectory, \( B \). It represents an eigenvalue coming from \( \tau = \infty \) and scattering off of the tachyon wall, a classical bounce trajectory. It can also be interpreted as a particle “at rest”, i.e., following another geodesic in de Sitter. A local observer at any point of \( B \) will see it as having energy \( m \), but an observer at \( A (\tau = 0) \) will see its energy redshifted according to the non-triviality of the metric:

\[
E = \frac{m}{\cosh \ell^{-1} \tau_0},
\]

(27)

where \( \tau_0 \) is the turning point of the classical trajectory. Then, by “pushing away” geodesics to the horizon one can make excitations of arbitrarily low energies as measured by \( A \). The generators that shift the trajectories are none other than \( J_{\pm} \) in (12).

4 Finite energy solutions

In this section we will make the relation between (15) and (17) precise, and illustrate the integrable structure. The relation between the Dirac equation in two dimensions and integrable models is of course not new [28], and it applies to any functional form of the DBI potential. The relation between shape-invariant potentials and the condensation process has also been studied in [29]. The exercise will also provide the scattering amplitudes which are usually used to compute expectation values of the closed string observables [32]. Consider then solutions of

\[
(i \gamma^a \nabla_a - m)\psi = 0,
\]

(28)

with constant energy:

\[
\mathcal{L}_t \psi_\omega = -i \omega \psi_\omega.
\]

(29)

For this purpose it will be interesting to work in a conformally flat metric, like (9). In the metric \( ds^2 = \exp 2\rho(x)(dt^2 - dx^2) \) the spin connection is given by

\[
\Gamma_a = -\frac{1}{2} \rho' \gamma^5(dt)_a.
\]

(30)

Multiplying (28) by the flat-space \( \gamma^0 \) we will have:

\[
(i \partial_t - \frac{i}{2} \rho' \gamma^5 - i \gamma^5 \partial_x - me^\rho \gamma^0)\psi = 0.
\]

(31)
After writing \( \psi(x, t) = \exp(-1/2\rho(x))\exp(-i\omega t)\chi_\omega(x) \), one arrives at an eigenvalue problem for \( \chi_\omega \)
\[
(i\gamma^5\partial_x + me^0\gamma^0)\chi_\omega(x) = \omega\chi_\omega(x).
\] (32)

Since we are working in two dimensions, and \( \gamma^5 \) anti-commutes with \( \gamma^0 \), one can choose a basis of Dirac matrices where \( \gamma^5 = \sigma^2 \) and \( \gamma^0 = \sigma^1 \). The system now reads
\[
\begin{align*}
Q\chi^+_\omega &= \omega\chi^-_\omega, \\
Q^+\chi^-_\omega &= \omega\chi^+_\omega,
\end{align*}
\] (33)

where \( Q = \frac{d}{dx} + me^0 \) and \( \chi_\omega = \chi^+_\omega\eta^+ + \chi^-_\omega\eta^- \) with \( \eta^\pm \) a constant normalized spinor \( \eta^\pm = \pm i\gamma^1\eta^\pm \). Note that the system has the “S-like-duality” \[26\], \( \chi^\pm(-x) = \chi^\mp(x) \). Thus charge conjugation of the fermionic field, represented here by \( \sigma^3 \) \[45\], is mapped to dualization of the compact boson \( C_0 \) in the closed string formulation.

For the particular form of the DBI potential \[33\], \( e^{-\rho(x)} = \cosh\ell^{-1}x \), the system above was solved using the factorization method in \[30\] \[4\]. One finds two independent solutions for the equation \( Q^+Q\chi^+_\omega = \omega^2\chi^+_\omega \), namely
\[
\begin{align*}
S_1(x) &= e^{ml\arctan z}e^{i\omega\ell}2F_1(i\omega\ell, -i\omega\ell; \frac{1}{2} - im\ell; \frac{1}{2}(1 + iz)) \\
S_2(x) &= e^{ml\arctan z}\left(\frac{1 + iz}{2}\right)^{\frac{1}{2} + i\omega\ell}2F_1\left(\frac{1}{2}, i(m + \omega)\ell, \frac{1}{2} + i(m - \omega)\ell; \frac{3}{2}, im\ell; \frac{1}{2}(1 + iz)\right),
\end{align*}
\]
where \( z = \sinh\ell^{-1}x \). The asymptotic expansion of these solutions is a sum of plane waves:
\[
\lim_{x \to \pm\infty} S_i(x) = a_{i\pm}e^{i\omega x} + b_{i\pm}e^{-i\omega x}
\] (34)
but we refer to \[30\] for the exact formulas for the coefficients \( a_{i\pm} \) and \( b_{i\pm} \). We also quote the result for the reflection coefficient:
\[
R^+(\omega) = \frac{\sinh\pi ml\ell}{\cosh\omega\ell} \frac{\Gamma\left(\frac{1}{2} + i(m - \omega)\ell\right)\Gamma\left(\frac{1}{2} - i(m + \omega)\ell\right)}{\Gamma^2\left(\frac{1}{2} - i\omega\ell\right)},
\] (35)
with \( \chi^-_\omega \) having the opposite sign for the reflection coefficient. From the decomposition of the spinor one sees the incident and reflected wave have opposite chirality. In the usual inverted harmonic oscillator model, this coefficient, along with the SL(2,\( R \)) generators \[12\] compound in the limit \( m \to \infty \) \( (g_s \to 0) \), the integrable structure via the scattering

\[4\]We follow conventions where \( 2F_1(a, b; c; 0) = 1 \) \[42\].
operator \( \hat{S} \), defined in the usual way:

\[
\hat{S} \chi^+_{\omega} = R(\omega) \chi^+_{-\omega}.
\]  

(36)

One can see directly from (35) the strength of non-perturbative effects, due to tunneling through the top of the potential [8].

\[
|R(\omega)|^2 = 1 - 4e^{-2\pi m \ell} \cosh^2 \pi \omega \ell + O(e^{-4\pi m \ell}).
\]  

(37)

In the de Sitter interpretation, these correspond to the flux of particles coming from one horizon and tunneling to the antipodal horizon without being detected at \( \tau = 0 \). Furthermore, the phase of the amplitude, again in the \( m \to \infty \) \((gs \to 0)\) limit,

\[
R^+(\omega) \approx (m\ell)^{2\omega\ell} \frac{\Gamma\left(\frac{1}{2} + i\omega\ell\right)}{\Gamma\left(\frac{1}{2} - i\omega\ell\right)}.
\]  

(38)

As expected, this is exactly the same expression one finds in type 0B, apart from the piece of the phase linear in \( \omega \) [26], giving rise to the same density of states as a function of the energy. From the reflection coefficients one can then compute scattering amplitudes of the fermion densities [32] by combining chiral components. Perturbatively the excitations at \( \tau > 0 \) and at \( \tau < 0 \) are also independent. As alluded above, non-trivial profiles for the region \( \tau < 0 \) are mapped to non-trivial profiles of the RR-scalar in the closed string formulation. This is natural since the transmitted wave has to tunnel through the potential, thus describing a D-instanton which couples to the scalar.

Despite being natural from the point of view of [8], the “Rindler” vacuum obtained from considering the state with no positive frequency modes as measured by the “open string observer” [29] gives rise to an infinite backreaction when quantum effects are computed. With the interpretation given in the last section, this effect can be seen as the backreaction in terms of closed strings. In fact, if one extrapolates results for scalar fields, the renormalized stress energy diverges at the horizon, where the closed string excitations are supposed to be localized.

As a closing note to this section, one should point out that the picture changes dramatically when one considers the case of the bosonic string. In the inverted harmonic oscillator model one does not fill the Fermi sea at one side of the potential, with the result does not have a bounded spectrum. Here there is no clear way to perform an extra truncation of the spectrum, since the scattering process mixes solutions with positive and negative chirality. Also, one sees that the scattering amplitudes as found above would disagree with the usual bosonic calculations. So it does not seem feasible to realize the
bosonic string as the effective model of a large number of D particles as we are exploring here. One would not find this entirely surprising, as the argument of non-analyticity of the DBI action was raised in [7], but it would be interesting to clarify the discrepancies between those points of view.

5 Reinterpreting the Euclidean vacuum

In the studies of QFT in de Sitter background one pays particular attention to de Sitter invariant states. Those are the analogue of the “vacuum” state in usual field theory, although there is no good notion of “no particle state” generically. The physical content of the SL(2,R) invariant states can be studied by means of the symmetric, or Hadamard two-point function: [34]:

$$\langle \alpha | \{ \phi(p), \phi(p') \} | \alpha \rangle = G(p, p'), \quad (39)$$

which in the particular case of SL(2,R) invariant states will depend only on the SL(2,R) invariant quantity, $Z$, given by:

$$Z(\tau, t; \tau', t') = \frac{\cosh \ell^{-1}(t - t') + \sinh \ell^{-1} \tau \sinh \ell^{-1} \tau'}{\cosh \ell^{-1} \tau \cosh \ell^{-1} \tau'} \quad (40)$$

when the metric of the manifold is taken to be [9]. $Z$ is related to the geodesic distance $\mu(p, p')$ between $p$ and $p'$ by [35]:

$$Z(p, p') = \cosh \frac{\mu(p, p')}{\ell}. \quad (41)$$

For instance, for time-like geodesics, $\mu$ is the proper time. In this way we can find any solution to the classical equations of motion, provided we also consider the direction of movement. To illustrate this point, consider the geodesic whose turning point is $p_0 = (t_0, \tau_0)$. Then by solving the equation $Z(p, p_0) = \cosh \ell^{-1} \mu$ one arrives at the geodesic:

$$\sinh \frac{\tau - \tau_0}{\ell} \tanh \frac{\tau_0}{\ell} = \cosh \frac{t - t_0}{\ell} \quad (42)$$

The energy of this solution is just [27]. The geodesic distance $\mu$ provides a geometrical interpretation for the uniformizing variable $s$ that shows up in matrix elements of loop operators in [36]. As we will see in this section, it will be very useful to re-express SL(2,R) invariant quantities as functions of $Z$. 
If we rewrite the Klein-Gordon equation (11) in terms of $\mu$, we will have:

$$\frac{1}{\sinh \frac{\mu}{\ell}} \frac{d}{d\mu} \left( \sinh \frac{\mu}{\ell} G(\mu) \right) + m^2 G(\mu) = 0,$$

(43)

which is satisfied by both Wightman functions (positive and negative frequency two-point functions) as well as their sum, the Hadamard function. Its solutions are conical functions:

$$G(Z) = aP_{-\frac{1}{2}+is}(Z) + bP_{-\frac{1}{2}+is}(-Z),$$

(44)

with $s^2 = (m\ell)^2 - 1/4$. When seen as a Hadamard function, each solution (44) defines an SL(2,R) invariant state, all of which can be seen as a Bogoliubov transformation of the Euclidean, $b = 0$ vacuum. The solution for $b \neq 0$ does not have the Hadamard form close to the light cone $\mu(p,p') = 0$ and hence does not have a sensible field theory interpretation as one approaches the UV scale$^5$. The Feynman function is obtained by evaluating the function (44) just above the branch cut $Z \in [1, \infty[$, that is to say, to take $G(Z+i\epsilon)$. For details about the delta-function representation see [38].

There is a relation between the Euclidean two-point function, with $b = 0$, and the zero energy solution $J_+\phi = 0$:

$$\phi_0(t,\tau) = P_{-\frac{1}{2}+is}(\tanh \ell^{-1}\tau).$$

(45)

One can obtain the equation above from (44) by means of an analytical continuation. When the separation of the two points is space-like, $\mu^2(p,p') < 0$, then one can consider the two points to be at a equal-time hypersurface, so $t = t'$. The zero energy solution is obtained from (44) by taking $p'$ to the horizon, or $\tau' \to \infty$, which can then be thought as the effect of deforming the field at the horizon. There is yet another way of saying this, which has a more straightforward interpretation in terms of the BCFT, in which the relation between $\tau$ and the boundary value of the tachyon field is convoluted. Instead of placing $p'$ at the horizon, or $t = t'$ and $\tau' \to \infty$, one can use the SL(2,R) symmetry to put $p$ at $t = 0$, $\tau = 0$, and use the remnant symmetry to rotate $\tau' \to \infty$ to $t'_d = i\pi\ell(n + \frac{1}{2})$, where $n$ is an integer$^6$. The image of the mapping consists of all points since (44) is periodic in imaginary time. One can then think of the zero energy solutions as an array of fields deformations at $t'_d$. We will have more to say about this interpretation below, with a twist, when we consider the actual problem of the fermionic field.

$^5$In particular, one cannot generically define a sensible perturbation theory around a generic solution. See [37] for details.

$^6$One can see that $t_d$ is the image of the mapping since the value of $\mu$ between the two points is preserved.
One should expect the anti-symmetric two point function of a fermion field in a de Sitter invariant background to allow for a similar decomposition. Because of the effects of the parallel transport on spinors, however, one expects a dependence on $\Psi_{\alpha' \beta}^\beta(p, p')$, which is the bispinor that implements the parallel transport from $p$ to $p'$. We will take some time to study its geometry. Following [39] and [40], we point out that $\Psi(p, p')$ is invariant under the isometries of de Sitter – what is called a maximally symmetric bispinor. Then its derivative is also a maximally symmetric object allowing for an expansion in terms of maximally invariant quantities:

$$
\nabla_a \Psi^\alpha_{\alpha'}(p, p') = A(\mu)n_a(p, p)\Psi^\alpha_{\alpha'}(p, p) + B(\mu)(\gamma_c)_{\alpha'}^\beta(\gamma_a)_{\alpha}^\delta \Psi^\alpha_{\alpha'}(p, p)n^c(p, p),
$$

(46)

where $n_a(p, p') = \nabla_a \mu(p, p)$ is the normalized vector tangent to the geodesic that links $p$ to $p'$. Primed indices live in the ((co)spinor, (co)tangent) bundle at $p'$. We will be omitting the spinor indices from now on, with the understanding that covariant spinor indices live on $p'$. The bispinor $\Psi(p', p)$ is covariantly constant over geodesics:

$$
\nabla_a \Psi(p', p) = 0.
$$

(47)

This means that $A(\mu) = -B(\mu)$ then (46) can be rewritten as

$$
\nabla_a \Psi(p', p) = A(\mu)(g_{ab} - \gamma_a \gamma_b)n^b(p', p)\Psi(p', p) = \frac{1}{2}A(\mu)[\gamma_a, \gamma_b]n^b(p', p)\Psi(p', p),
$$

(48)

or alternatively

$$
\hat{\nabla} \Psi(p', p) = A(\mu)\Phi \Psi(p', p).
$$

(49)

In two dimensions one can further simplify the expression by considering that the commutator of the gamma matrices should be proportional to flat space $\gamma^5$ and the volume element of the space:

$$
\nabla_a \Psi(p', p) = A(\mu)\epsilon_{ab}n^b(p', p)\gamma^5\Psi(p', p).
$$

(50)

One then needs to find the function $A(\mu)$. A condition for it comes from the definition of curvature:

$$
[\nabla_a, \nabla_b]\Psi(p', p) = \frac{1}{4}R_{abcd}\gamma^c\gamma^d\Psi(p', p) = \frac{1}{4\ell^2}[\gamma_a, \gamma_b]\Psi(p', p),
$$

(51)

where we specialized to maximally symmetric spaces in the last equality. Applying it to
Ψ(p′, p) and using (50) we find:

\[ \epsilon^{ab} \nabla_a \nabla_b \Psi(p′, p) = \left( \frac{dA(\mu)}{d\mu} n^a n_a + A(\mu) \nabla_a n^a \right) \gamma^5 \Psi(p′, p) = -\frac{1}{2\ell^2} \gamma^5 \Psi(p′, p), \]  

(52)

which, for timelike separations \( n^a n_a = +1 \) yields a ordinary differential equation for \( A(\mu) \):

\[ \frac{dA(\mu)}{d\mu} + A(\mu) \nabla^2 \mu = -\frac{1}{2\ell^2}. \]  

(53)

By using the expression for the Laplacian in (43) one can find \( \nabla^2 \mu \), and the equation has one single solution that vanishes as \( \mu \to 0 \):

\[ A(\mu) = -\frac{1}{2\ell} \tanh \frac{\mu}{2\ell}. \]  

(54)

One could in principle integrate \( \Psi(p′, p) \) using the differential equation above with the boundary condition that \( \lim_{p\to p′} \Psi(p′, p) = 1 \) but this will not be necessary for our purposes.

The interpretation of the parallel transport operator is clear when one considers the remarks of Section 3. The spectrum of a field in de Sitter is divided into “real” excitations whose energy is larger than the mass of the field, and “virtual” excitations, with energy smaller than \( m \). The former correspond to particles that actually have enough energy to climb up the “tachyon wall” \( \mathfrak{K} \), whereas the latter never actually make it, passing to some turn around point at \( p′ \), its energy being redshifted by the non-triviality of the metric, like the geodesic \( B \) in Figure 1. These solutions where labeled “hyperbolic” and “elliptic” respectively in [41]. The advantage of using the \( \text{SL}(2,\mathbb{R}) \) symmetry as a guiding principle is that one can relate the “virtual” spectrum at, say the open string vacuum \( \tau = 0 \) to the “real” spectrum at some other point \( p′ \). The operator that implements this relation is \( \Psi \).

After this rather long detour one is able to deal with the equation for the Hadamard function:

\[ (\hat{\nabla} - m)[\left\langle \psi_\alpha(p), \bar{\psi}_\beta(p′) \right\rangle] = (\hat{\nabla} - m)S_\alpha^\beta(p, p′) = 0. \]  

(55)

Note that now the derivative acts on the covariant spinor index. By writing \( S = -(\hat{\nabla} + m)G \) and \( G_\alpha^\beta(p, p′) = F(\mu) \Psi_\alpha^\beta(p, p′) \) one is able to get an ordinary differential equation for \( F(\mu) \):

\[ \left( \hat{\nabla}^2 + m^2 \right) F(\mu) \Psi(p, p′) = (\hat{\nabla}^2 F + (m^2 - A^2(\mu) + \frac{1}{2\ell^2} F)) \Psi(p, p′) = 0. \]  

(56)
The equation itself is:
\[
\left[ \frac{d^2}{d\mu^2} + \frac{1}{\ell} \frac{d}{d\mu} \frac{1}{\sinh \frac{\mu}{2\ell}} \frac{d}{d\mu} + \left( m^2 - \frac{1}{4\ell^2} \tanh^2 \frac{\mu}{2\ell} + \frac{1}{2\ell^2} \right) \right] F(\mu) = 0, \tag{57}
\]
whose solutions are another set of hypergeometric functions. We present the one suitable for the Hadamard elementary function, which has a rather lengthy expression for \( \mu^2 > 0 \):

\[
F(\mu) = -\frac{1}{8\pi} \cosh \frac{\mu}{2\ell} \left[ _2F_1(1 + im\ell, 1 - im\ell; 1; - \sinh^2 \frac{\mu}{2\ell}) \log \left( \sinh^2 \frac{\mu}{2\ell} \right) + \sum_{n=0}^{\infty} \frac{(1 + im\ell)_n(1 - im\ell)_n}{(n!)^2} K(n, m\ell)(-1)^n \sinh^{2n} \frac{\mu}{2\ell} \right], \tag{58}
\]
where \((z)_n = \Gamma(z + n)/\Gamma(z)\) is the Pochhammer symbol, and \(K(n, m\ell) = \psi(n + 1 + im\ell) + \psi(n + 1 - im\ell) - 2\psi(n + 1), \psi(x) = \frac{d}{dx} \log \Gamma(x)\) being the digamma function. The constant of normalization being fixed by requiring that the function has the flat space form near the wave front \( \mu = 0 \).

\[
F(\mu) = -\frac{1}{8\pi} \log \mu^2 + O(\mu^0). \tag{59}
\]

For spacelike separations, one is helped by the fact that the monodromy of the solutions around the singular point \( \mu = 0 \) is parabolic. The relative sign coming from the logarithm term cancels the one coming from the hyperbolic sine. One then can use well-known properties of hypergeometric functions to write \( F(\mu) \) in a more compact form:

\[
F(\mu) = \frac{1}{8\pi} \frac{\pi m\ell}{\sinh \pi m\ell} \cosh \frac{\mu}{2\ell} _2F_1(1 + im\ell, 1 - im\ell; 2; \cosh \frac{\mu}{2\ell}), \quad \mu^2 < 0 \tag{60}
\]
from which we observe for future reference that \( F(\mu) \) is regular when \( \mu = in\pi\ell \).

Since the argument of the hypergeometric can be written as a function of \( Z \), one sees that (58) has poles when \( Z = 1 \). The “vacuum” of the tachyon field has the (anti) periodicity in imaginary time given by the periodicity of \( Z(t, \tau) \), which in turn is \( t \to t + iT_H^{-1} \), with:

\[
T_H = \frac{1}{2\pi \ell}, \tag{61}
\]
which is generically assigned to thermal backgrounds. We will however take another interpretation of this fact. The important quantity from the SL(2,R) point of view is not \( t \) but \( \mu \). And inspecting the function above one sees that it is invariant under \( \mu \to \mu + 2\pi i\ell \), that is to say that space-like separations are compactified with period \( T_H \). This means that we can take the global structure of this de Sitter space-time to
be that of a cylinder, like (20) with \( \epsilon = 1 \). From (58) one sees that the fermion field is anti-periodic in this direction.

Consider the zero energy solutions\(^7\) of the quantum-mechanical system:

\[
\chi_0^\pm = e^{-2m\ell \arctan e^{\pm t^{-1} \sigma}} \eta_\pm. \tag{62}
\]

These solutions are not normalizable in the equivalent quantum mechanics problem (33), which means that their degrees of freedom are not dynamical, but their action (24) is finite:

\[
S_{\text{cl}}(\chi_0) = i\pi m\ell. \tag{63}
\]

Each wave function is a superposition of chiral components of each side of the tachyon wall and are supported at \( \tau = \pm \infty \), having exponential tails at the other side. We will thus identify each of them with half-(anti)-D-instantons. Zero “global” energy solutions are removed from the cylinder by the anti-periodicity condition but we can still consider it as the limit where one takes the bounce solution to the horizon, where its energy is redshifted away. The creation (annihilation) operators are interpreted as half-(anti)-D-instantons. Since we argued above that the two-point function is periodic in space-like distances, placing these field insertions at the horizon means placing them at \( \mu = 2\pi in\ell, \) \( n \) being any integer. The insertion of a pair of instantons and anti-instantons as described above will give rise to a sD-brane. These correspond to the “imaginary D-branes” background as considered in [46], which indeed were found to give rise to purely closed string amplitudes in BCFT formalism. In the language above this is natural since the insertions create field quanta at the horizon.

The solution (58) above was derived under the assumption that the singularity happened when \( \mu = 0 \), or \( Z = 1 \), and not \( \mu = i\pi\ell \), or \( Z = -1 \), in which case one allows for the second solution of (57):

\[
\tilde{F}(\mu) = \tilde{N} \cosh \frac{\mu}{2\ell} \, _2F_1(1 + im\ell, 1 - im\ell; 1; -\sinh^2 \frac{\mu}{2\ell}). \tag{64}
\]

in de Sitter the solutions with poles at \( \mu = im\pi\ell \) are normalizable and extra assumptions (like a Hadamard development of the pole at \( \mu = 0 \)) are required to drop it. This general case corresponds to a fermionic analogue of the “alpha vacua”.

So the question that naturally arises is: just what these poles at \( Z = -1 \) correspond to in the usual boundary state formalism? The extra pole of the alpha vacuum can be seen as putting a particle in a state which is a superposition between a given point \( p \)

\(^7\)In usual BCFT these have \( \lambda = \frac{1}{2} \).
and its antipodal point $p^A$, which is defined through the embedding of de Sitter in the higher dimensional Minkowski space-time with coordinates $\vec{z} = \{z_0, z_1, z_2\}$:

$$
\begin{align*}
  z_0 &= \frac{\ell \sinh \ell^{-1} t}{\cosh \ell^{-1} \tau}, \\
  z_1 &= \frac{\ell \cosh \ell^{-1} t}{\cosh \ell^{-1} \tau}, \\
  z_2 &= \frac{\ell \sinh \ell^{-1} \tau}{\cosh \ell^{-1} \tau}.
\end{align*}
$$

The de Sitter space is then the hyperboloid $z_0^2 - z_1^2 - z_2^2 = -\ell^2$. Generically, the antipodal point $\vec{z}^A = -\vec{z}$ lies outside the coordinate patch covered by $t, \tau$. For the excitations we are interested on, however, are localized near $\tau = \infty$, the image of the map is localized near $\tau = -\infty$, and time-reversed. So in these cases the fermion density will be non-zero for both sides of the potential. From the discussion above and the work of [46], the generic SL(2,R) invariant state seems to be related to a generic configuration of (anti)-D-instantons, giving rise to generic background values for the closed string tachyon and the RR boson. It would be interesting to provide a clear relation between these states and backgrounds in the closed string formulation.

Quantum effects raise the temperature of the vacuum by $T_H$ [61], an effect which should be, as discussed in the Section 3, independent of the coordinates used to parametrize the D-branes trajectories in phase space. As the conjecture goes in [20], the matrix model at finite temperature corresponds to a different closed string background where the Liouville interaction is perturbed by $\Lambda e^\varphi$ and not by $\varphi e^\varphi$ [27]. We will have more to say about this point below. This temperature [61] actually corresponds, for type 0B, to the self-dual point on which vortices in the finite matrix model coexist with the singlet configurations discussed here [31]. Further understanding of the bosonization of the system [15] could provide another view of this effect.

One can also use the two-point function to compute the backreaction. The expectation value of the stress-energy $\langle T_{ab} \rangle$ requires the regularization of the two-point function for coincident points. This is a standard calculation in QFT in curved space [38]. Firstly, since the state is invariant under SL(2,R), the expectation value of the stress tensor will be proportional to the metric. Then one only needs to compute the expectation value of the trace of the stress tensor:

$$
\langle g^{ab} T_{ab} \rangle = \frac{i}{2} \lim_{p \to p'} \langle \psi_\alpha(p)(\gamma^a)^\alpha_{\beta'} \nabla_a \bar{\psi}_{\beta'}(p') - \nabla_a \psi_\alpha(p)(\gamma^a)^\alpha_{\beta'} \bar{\psi}_{\beta'}(p') \rangle,
$$

which can be rewritten in terms of $S$:

$$
\langle g^{ab} T_{ab} \rangle_{\text{ren}} = \frac{1}{2} \lim_{p \to p'} \text{Tr}(i \gamma^a \nabla_a S_{\text{ren}} - i \nabla_a S_{\text{ren}} \gamma^a),
$$

where one regularizes the two point function in the usual way: subtracting an infinite
piece of it. This is usually accomplished in QFT in curved space-time by redefining the couplings in the full Lagrangian with the Einstein-Hilbert term. The practical implication of this is that one subtracts from the Hadamard function a purely “geometrical” contribution, usually called the DeWitt–Schwinger, or adiabatic, expansion. Here we are not really allowed to do this since gravity is non-dynamical in two dimensions. The ultra-violet divergence will at any rate generate a term in the renormalized Lagrangian, whose form we will now argue to be a Liouville perturbation term. This also means that only the diverging term is of immediate physical interest to us, and we will not discuss the finite pieces. Expanding (58) and inserting it into (67), we will have:

\[
\langle g^{ab}T_{ab}\rangle_{\text{div}} = 2 \lim_{\mu \to 0^+} \frac{m^2}{8\pi} \log \frac{\mu^2}{4\ell^2} + \text{finite terms.} \tag{68}
\]

The factor of 2 comes from the trace over spinor indices. The appearance of the Liouville term can be understood heuristically as follows: in the language of [34], it “renormalizes” the cosmological constant and the Einstein term in 2+\(\epsilon\) dimensions. One can then arrive at the usual form of the Liouville Lagrangian upon dimensional regularization, as in [19]. The procedure does not quite work here, however, since the Liouville field has the usual interpretation as the field obtained by bosonization of the Lagrangian (15), but the form of the perturbation should still be the same, as one could argue from the fact that the system, being a CFT, must have a well-defined UV completion. The appearance of Liouville-like terms is universal in two dimensional systems [18], and here is an example which actually bears resemblance to the massive Ising model. Since the redefinition of the density of eigenvalues as profiles of the Liouville field involves a \(m\) dependent term, as well as an exponential coordinate transformation, the first term can be taken as inducing the generation of a perturbation like \(me^\varphi\) in the effective bosonic model.

6 Qualitative remarks

We saw that even though the DBI action is just an effective theory to the dynamics of D-branes in a tachyon background, one can still infer a lot about closed string dynamics from it. The reinterpretation of the DBI action as describing particles moving in a two-dimensional de Sitter space-time allows for both qualitative and quantitative tests in the relation of the double-scaled matrix model and type 0B in two dimensions. The most enticing prospect for future work is the bosonization of (15), which will clarify the relation between the DBI and closed string dynamics and to Liouville. It would also be interesting to use mode sum techniques to compute the effective action in the “Rindler”
state found in Section 4. This will provide a means to compute the backreaction due to the condensation process, with the additional nice feature that the string coupling can be made finite with no extra complications. Yet another route one could take would be to interpret the thermal two-point function found above as the result of insertions of closed string (Liouville) boundary operators. This would give a DOZZ picture of the sD-brane state and would help distinguish the different vacua in the model of de Sitter quantum gravity provided by Martinec [11].

The point is made that tools used in the studies of QFT in curved space-time can help developing the McGreevy-Verlinde interpretation of the tachyon condensation process. In fact, one has generically that the tachyon potential decays exponentially near the closed string vacuum. By the point of view taken in this paper, this amounts to considering a metric of the form

$$d\hat{s}_{\text{eff}}^2 = V^2(T)(-dt^2 + dT^2) = e^{-2aT}(-dt^2 + dT^2)$$

(69)

where $a > 0$. This corresponds to Rindler space, which lends itself to the same type of treatment given here: there is an infinite backreaction of the Rindler vacuum, and there is a analogue of an invariant state which is seen by the Rindler observer, i.e., the one following curves of constant $T$, as a thermal bath of radiation. In fact in this case the “invariant state” is none other than the usual Minkowski vacuum. The model above should be equivalent to models of relativistic fermions considered before, as in [10], but somehow the importance of the Rindler state has been so far overlooked.

It is curious in this case to recover a non-flat background without what one would naively call gravitational degrees of freedom. The form of the potential in this case comes directly from the string field theory, and in this sense the de Sitter background can be considered “on shell”. “Off shell” configurations $\Psi_o$ would arise from the partition sum of the string field theory in the usual fashion, i.e. having a Boltzmann weight given by $e^{-S(\Psi_o)}$, with the open string field theory action. The novelty here is that this formulation does not seem to lend itself to a straightforward interpretation of fluctuations as gravitons, but one expects a Liouville-type of theory based on diffeomorphism invariance. Even though the form of the tachyon potential gives rise to conformally flat backgrounds, this interpretation only gives a meaningful prescription in two dimensions, and even then there are global issues in field space which are far from clear. We have not considered the effect of non-trivial RR backgrounds, either.

At any rate, one cannot help but wonder at the coincidence in terminology between this interpretation of the tachyon decay process and the usual open-closed string duality,
as in the ADS-CFT. There, the near-horizon limit of the solution of the effective closed string theory (SUGRA) equations gave an “on-shell” background which could alternatively be described as open strings degrees of freedom. Here, the effective open string degrees of freedom could be seen as if propagating in a non-flat background, de Sitter in the present case. The curved background arises as an “on-shell” configuration as far as the string field theory equations are concerned. The “near horizon” limit here appears as one considers excitations of the branes with low energy, which would translate to excitations near $\tau = \pm \infty$ in the present interpretation. The “near horizon” limit then reveals closed string degrees of freedom, in a construct that resembles recent holographic formulations in de Sitter space \[48\], \[49\]. Remarks on this regard have been made by Sen in \[50\]. There seems to be a lot of important physics one can extract from the DBI action with relation to both QFT in curved space-time and the fate of the tachyon condensation in two dimensions. One hopes that this will help shed light on the generic problem.

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**Appendix: Conventions on Spinor Calculus**

We will follow gamma-matrix based approach. In a manifold $M$ one could define a spinor bundle under some topological conditions. For each element of the spinor bundle $\psi^\alpha \in S$ and another element of the dual spinor bundle one $\phi_\beta$ can define an element of the (co)tangent bundle $v_a$ by means of the gamma matrix. The gamma matrix itself is then a 1-form valued in the tensor product $S \times S^\ast$:

$$v_a = \phi_\beta (\gamma^a)^\beta_\alpha \psi^\alpha,$$

which will be taken to satisfy the algebra (omitting spinor indices):

$$\{\gamma_a, \gamma^b\} = 2g_{ab}1, \quad (71)$$

\[8\]This in turn is equivalent to the space of linear operators in $S$. We will be taking this equivalence to be trivial.
where \(g_{ab}\) is the metric of the manifold and \(\mathbf{1}\) the identity on the spinor bundle. The gamma matrix one-form can be written in terms of the vielbein \(e^i_a\) and the flat space gamma matrices \(\gamma_i\) as:

\[
\gamma_a = \sum_i e^i_a \gamma_i.
\]  

(72)

The gamma matrix one form also defines an affine connection on spinors by the imposing that the Clifford algebra structure is covariantly constant:

\[
\nabla_a (\gamma_b)_\alpha^\beta = \bar{\nabla}_a (\gamma_b)_\alpha^\beta - C^c_{ab} (\gamma_c)_\alpha^\beta - (\Gamma_a)_\alpha^\delta (\gamma_b)_\delta^\beta + (\gamma_b)_\alpha^\delta (\Gamma_a)_\delta^\beta = 0.
\]  

(73)

By relating the quantities with \(\bar{\nabla}_a\) being the flat space derivative, one arrives at the well-known formulas for the Christoffel symbols and the spinor connection:

\[
\Gamma_a = \frac{1}{4} \omega_{aij} \gamma^i \gamma^j,
\]  

(74)

with \(\omega_{aij}\) being the spin connection, defined from the vielbien by the zero torsion condition:

\[
de^i_a + \omega_{aij} \land e^j_b = 0,
\]  

(75)

so \(\omega_{aij} = e^b_j \nabla_a e_{bi}\). As in usual tensor calculus, the commutator of two derivatives of a spinor satisfies linearity:

\[
[\nabla_a, \nabla_b] (A(x) \psi(x) + B(x) \eta(x)) = A(x)[\nabla_a, \nabla_b] \psi(x) + B(x)[\nabla_a, \nabla_b] \eta(x),
\]  

(76)

and in fact can be computed in terms of the spin connection:

\[
[\nabla_a, \nabla_b] \psi = (\partial_a \Gamma_b - \partial_b \Gamma_a - \Gamma_a \Gamma_b + \Gamma_b \Gamma_a) \psi = S_{ab} \psi.
\]  

(77)

And by considering that the one-form \(\omega_a = \xi^\beta (\gamma_a)_\alpha^\beta \psi^\alpha\) satisfies:

\[
[\nabla_a, \nabla_b] \omega_c = R_{abc} d \omega_d,
\]  

(78)

one can relate the tensors \(S_{ab}\) and \(R_{abcd}\) by using the chain rule and some properties of the gamma matrices:

\[
S_{ab} = \frac{1}{4} R_{abcd} \gamma^c \gamma^d.
\]  

(79)

The action on covariant spinors can be obtained similarly. For instance:

\[
[\nabla_a, \nabla_b] \psi_\alpha = -\psi_\beta (S_{ab})_\alpha^\beta.
\]  

(80)
And we finish by listing properties of the parallel transport operator $\Psi_{\alpha\beta'}(p, p')$. We will convention that $p$ and $p'$ refer to the covariant and contravariant indices, respectively. In the formulas below $\mu$ is the geodesic length between $p$ and $p'$ and $n^a = \nabla^a \mu$ is the unit vector tangent to the geodesic.

$$\lim_{p \to p'} \Psi(p, p') = 1,$$

(81)

$$\Psi(p, p') \Psi(p', p) = -1,$$

(82)

$$n^a \nabla_a \Psi(p, p') = 0,$$

(83)

$$\nabla_{a'} \Psi(p, p') = \frac{1}{2} A(\mu)[\gamma_{a'}, \gamma_{b'}]n^b \Psi(p, p'), \quad A(\mu) = - \frac{1}{2\ell} \tanh \frac{\mu}{2\ell},$$

(84)

$$\nabla_a \Psi(p, p') = - \frac{1}{2} A(\mu) \Psi(p, p') [\gamma_a, \gamma_b] n^b.$$

(85)

We also use the “slash” notation. For the sake of completeness, $\nabla \psi = \gamma^a \nabla_a \psi$ and $\nabla \psi^T = \nabla_a \psi^T \gamma^a$ for contravariant and covariant spinors, respectively. For a more detailed consideration of $\Psi(p, p')$ see [40].

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