The problem of hierarchies and family replications in an extension of quantum field theory

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Abstract

We will show that an extension of quantum field theory, recently proposed to solve the hierarchy problem, can give an elegant explanation of quark/lepton family replications. This scenario prefers fermion mass models based on a family permutation discrete symmetry.

1 Introduction: a theory of identical families

In quantum mechanics, a particle is described by three real numbers \((x_1, x_2, x_3)\) that correspond to the space position of the particle. To describe a system with several identical particles, we need to introduce a function \(\psi(x)\) of the space coordinates above. This function is called a field, and quantum field theory or second quantization is the theory that quantizes it. Now suppose that we have a system with several identical fields. This could be the system of three identical fields \(\psi_e(x), \psi_\mu(x), \psi_\tau(x)\), but with different masses: the electron, the muon and the tau. Repeating the above argument, one can imagine that a theory of several identical families needs the introduction of a new mathematical object to represent such a physical system: a function of the field \(\psi(x)\), i.e. a functional \(S[\psi(x)]\). We have recently shown [1], that the quantization of functionals gives us an extension of quantum field theory. Such a theory has been proposed to solve the hierarchy problem. Here we will see how this idea can nicely fit three generations of fermions and correctly realizes fermion masses and mixings.

2 A model with three families

In this section we will present a model with three fermion families. To simplify the notation, we will ignore the gauge structure, since it is irrelevant for the discussion below.

We have already discussed the case of scalar fields \(\phi\), in the paper [1]. Here we repeat a similar argument to introduce fermions. We remind the anti-commutation rules for a fermionic field operator \(\hat{\psi}(x)\)

\[
\{\hat{\psi}(x), \hat{\psi}^\dagger(y)\} = \delta^3(x - y).
\]

We can give a functional representation to these operators. In fact, in quantum field theory any quantum field state can be represented by a functional\(^1\) \(S[\psi(x)]\), where \(\psi(x)\) is a Grassman

\(^1\)This functional is the second quantization analogue of the particle wave function in first quantization [1].
variable, function of the three space coordinates $x$. The action of the operators $\hat{\psi}(x)$ and $\hat{\psi}^\dagger(x)$ onto ordinary quantum field states, can be represented by the function $\psi(x)$ and the functional derivative $\delta/\delta\psi(x)$, acting on this functional $S[\psi]$ (the wave functional of the state)

$$\begin{align*}
\hat{\psi}(x)|S\rangle &\Leftrightarrow \psi(x) S[\psi], \\
\hat{\psi}^\dagger(x)|S\rangle &\Leftrightarrow \frac{\delta}{\delta\psi(x)} S[\psi].
\end{align*}$$

(2)

In fact they satisfy the anti-commutation rules

$$\{\psi(x), \frac{\delta}{\delta\psi(y)}\} = \delta^3(x - y).$$

(3)

The Hamiltonian of a free Dirac field, after the substitution (2), becomes

$$H = \int d^3x \psi^\dagger(x) (-i\gamma^0 \gamma \cdot \nabla + m\gamma^0) \frac{\delta}{\delta\psi^\dagger(x)} + \text{h.c.}$$

(4)

This is the Hamiltonian in the functional representation. In the Schrödinger picture, a quantum field state is represented by a wave functional $S[\psi,t]$, whose time evolution, in terms of the time variable $t$, can be computed solving the Schrödinger functional equation

$$i\frac{\partial}{\partial t} S[\psi,t] = H S[\psi,t]$$

(5)

with $H$ given by the (4). The time evolution equation (5), is the quantum field theory analogue of the Schrödinger equation in first quantization. We stress that we have not yet introduced any new physical concept.

Now we briefly repeat the arguments discussed in [1], leading to the extension of quantum field theory. As explained in [1], one can derive this Schrödinger equation (5), from the stationarity condition of a new action $A$, written in terms of functionals

$$\mathcal{A} = \int \mathcal{D}\psi S^\dagger[\psi,t] \left(i\frac{\partial}{\partial t} + H\right) S[\psi,t] \\ dt + \text{h.c.}$$

(6)

where $H$ is given in (4) and the integral $\int \mathcal{D}\psi$ is performed in the functional sense. The classical action $\mathcal{A}$ can be quantized [1]. In particular we have that $S$ and $S^\dagger$ are replaced with functional field operators that satisfy the anti-commutation rules

$$\{iS^\dagger[\psi], S[\psi'][t]\} = \delta^\infty[\psi - \psi']$$

(7)

where $\delta^\infty[\psi - \psi']$ is a functional Dirac delta

$$\int \mathcal{D}\psi \delta^\infty[\psi - \psi'] G[\psi] = G[\psi'].$$

(8)

As expected, the Hamiltonian obtained from the action $\mathcal{A}$, is written in terms of a functional integration

$$\mathcal{H} = \int \mathcal{D}\psi S^\dagger[\psi] H S[\psi].$$

(9)

Using the (7), one can verify that the operator

$$\mathcal{N} = \int \mathcal{D}\psi S^\dagger[\psi] S[\psi]$$

(10)
satisfies the commutation rules

\[ [N, S[\psi]] = -S[\psi] \]
\[ [N, S^\dagger[\psi]] = S^\dagger[\psi]. \]

\(N\) counts the number of fields. \(S\) and \(S^\dagger\) are annihilation and creation functional field operators. So, we can exploit \(S^\dagger\) to build different states of the Hilbert space, starting from the vacuum state, that satisfies by definition \(S[\psi] \ket{0} = 0\). (12)

Since our purpose is to build a model with three families, we consider a state \(|F_3\rangle\) with three creation operator \(S^\dagger\), and study the evolution with respect the time variable \(t\):

\[ |F_3, t\rangle = \int D\psi_1 D\psi_2 D\psi_3 \ F_3[\psi_1, \psi_2, \psi_3, t] \ S^\dagger[\psi_1] \ S^\dagger[\psi_2] \ S^\dagger[\psi_3] |0\rangle. \] (13)

This is a state with three fields \(\psi_1(x), \psi_2(x), \psi_3(x)\), and the functional \(F_3[\psi_1, \psi_2, \psi_3, t]\) unambiguously defines the specific physical state of the system. The normalization constant of the functional \(F_3\) is chosen in order to guarantee \(\langle F_3|F_3\rangle = 1\) (see also [4]). Since \(N\) commutes with the Hamiltonian [3], the number of fields is constant. Exploiting the Hamiltonian [3], we can derive the time evolution of the functional \(F_3\). Namely, we have the Schrödinger equation

\[ \frac{i\partial}{\partial t} |F_3, t\rangle = \mathcal{H} \ |F_3, t\rangle = \int D\psi_1 D\psi_2 D\psi_3 \ \frac{i\partial}{\partial t} F_3[\psi_1, \psi_2, \psi_3, t] \ S^\dagger[\psi_1] \ S^\dagger[\psi_2] \ S^\dagger[\psi_3] |0\rangle \]
\[ = \int D\psi_1 D\psi_2 D\psi_3 \ \left[ (H_1 + H_2 + H_3) \ F_3[\psi_1, \psi_2, \psi_3, t] \right] \ S^\dagger[\psi_1] \ S^\dagger[\psi_2] \ S^\dagger[\psi_3] |0\rangle. \] (14)

with

\[ H_i = \int d^3x \ \psi_i^\dagger(x) \ (-i\gamma^0 \gamma \cdot \nabla + m \gamma^0 \frac{\delta}{\delta \psi_i^\dagger(x)}) + \text{h.c.} \] (15)

To derive the identity above, we have used the anti-commutation rules [7] and the \([12]\). The eq. \([14]\) is satisfied, if the functional \(F_3\) fulfills the following Schrödinger functional equation

\[ \frac{i\partial}{\partial t} F_3[\psi_1, \psi_2, \psi_3, t] = (H_1 + H_2 + H_3) \ F_3[\psi_1, \psi_2, \psi_3, t]. \] (16)

This is also the functional equation satisfied by an ordinary quantum field theory (in the functional representation), but now three identical families of fermions \(\psi_i\) have appeared. We conclude that the operators \(S^\dagger[\psi]\) and \(S[\psi]\) are family creation and annihilation operators, and the time evolution of the state \([13]\) is equivalent to that of an ordinary quantum field theory with family replications.

Note that the equation \([16]\) exhibits an explicit discrete symmetry: it is invariant under permutations of the family index. Until now, all families have the same mass and do not mix. However, in nature, this symmetry is broken, and families mix. A simple example, where this symmetry is spontaneously broken, can be built adding a real scalar field \(\phi\); instead of the family annihilation operator \(S[\psi]\), we now consider a functional operator \(S[\phi]\). We want to break the permutation symmetry, so let us focus on the scalar sector, and temporarily remove the fermion field \(\psi\) from the notation. Consider the Hamiltonian operators

\[ \mathcal{H}_{\text{scalar}}^{(1)} = \int D\phi \ S^\dagger[\phi] \ H[\phi] \ S[\phi], \] (17)
where $H_{\text{scalar}}^{(2)}$ is the common Hamiltonian of a real scalar field (in the functional representation\footnote{This can be easily verified, following the same argument that led us to eq. (16).}), including a kinetic part and a self-interaction $\phi^4$. $V^{(2,3)[\ldots]}$ are functionals, that for the requirement of locality have the form

\begin{equation}
V^{(2)}[\phi, \phi'] = \int d^3 x m_1^2 \phi(x)\phi'(x) + m_2^2 \phi^2(x) + m_3^2 \phi'^2(x) + \lambda_1 \phi^2(x)\phi'^2(x) + \ldots
\end{equation}

(a similar expression can be written for $V^{(3)}$). $H_{\text{scalar}}^{(1)}$ is free in the third quantization sense, and when acting onto states with three families, it describes three identical real scalar fields $\phi_1$, $\phi_2$ and $\phi_3$. $H_{\text{scalar}}^{(1)}$ does not mix them. But if we add $H_{\text{scalar}}^{(2)}$ and $H_{\text{scalar}}^{(3)}$ to the time evolution operator, we get the Schrödinger equation

\begin{equation}
\frac{i}{\hbar} \partial_t F[\phi_1; \phi_2; \phi_3, t] = \left( \sum_{i=1,2,3} H[\phi_i] + \sum_{i\neq j} V^{(2)}[\phi_i, \phi_j] + \sum_{\text{perm.}} V^{(3)}[\phi_1, \phi_2, \phi_3] \right) F[\phi_1; \phi_2; \phi_3, t] = \left( \sum_{i=1,2,3} H_{\text{kin}}[\phi_i] + V_{\text{eff}}[\phi_1, \phi_2, \phi_3] \right) F[\phi_1; \phi_2; \phi_3, t].
\end{equation}

The effective potential $V_{\text{eff}}[\phi_1, \phi_2, \phi_3]$ of the quantum field theory described by (21) spontaneously breaks the discrete family permutation symmetry, giving three vev $\langle \phi_1 \rangle, \langle \phi_2 \rangle$ and $\langle \phi_3 \rangle$. This breaking is then transmitted to the fermion sector through Yukawa interactions, that in the third quantization formalism come from the Hamiltonian

\begin{equation}
H_{\text{yuk}}^{(1)} = \int D\phi D\psi S^\dagger[\phi, \psi] g \phi \bar{\psi}_L \psi_R S[\phi, \psi] + \text{h.c.}
\end{equation}

where $\psi_R$ and $\psi_L$ stand for respectively $\frac{1+2\alpha}{2} \psi$ and $\frac{1-2\alpha}{2} \psi$, the left-handed and right-handed fermions. The Hamiltonian (22) gives the following effective second quantization Hamiltonian when acting onto states with three families\footnote{This can be easily verified, following the same argument that led us to eq. (16).}:

\begin{equation}
H_{\text{yuk}} = \int d^3 x \ g \left( \phi_1 \bar{\psi}_{1L} \psi_{1R} + \phi_2 \bar{\psi}_{2L} \psi_{2R} + \phi_3 \bar{\psi}_{3L} \psi_{3R} \right) + \text{h.c.}
\end{equation}

The Hamiltonian (22) yields a diagonal mass matrix to the fermions. A mixing between different generations arises only when an additional third quantization interaction, is present in the full Hamiltonian:

\begin{equation}
H_{\text{yuk}}^{(2)} = \int D\phi D\psi D\phi' D\psi' \ d^3 x \ S[\phi, \psi] S^\dagger[\phi', \psi'] \ (g_l \ \phi' \bar{\psi}_L \psi_R + g_r \ \phi \bar{\psi}_L \psi_R) S[\phi, \psi] S[\phi', \psi'].
\end{equation}
In particular the effective Hamiltonian (23) is modified by additional Yukawa interactions

\[ H_{\text{yuk}}^{(2)} = \int d^3x \left[ g_l \left( \phi_1 \bar{\psi}_{1L} \psi_{2R} + \phi_1 \bar{\psi}_{1L} \psi_{3R} + \phi_2 \bar{\psi}_{2L} \psi_{3R} \right) + (L \leftrightarrow R \text{ and } g_l \leftrightarrow g_r) + \text{h.c.} \right] . \]

(25)

The full mass matrix takes the form

\[ M_{LR} = \begin{pmatrix}
    g \phi_1 & g_l \phi_1 + g_r \phi_2 & g_l \phi_1 + g_r \phi_3 \\
    g_l \phi_2 + g_r \phi_1 & g \phi_2 & g_l \phi_2 + g_r \phi_3 \\
    g_l \phi_3 + g_r \phi_1 & g_l \phi_3 + g_r \phi_2 & g \phi_3
\end{pmatrix} . \]

(26)

In the expression above \( \phi_1, \phi_2, \phi_3 \) are three real numbers, while \( g, g_l \) and \( g_r \) are three complex numbers. One can continue adding new interactions, but, for our purpose, is enough to say that the mass matrix (26), with specific values of the physical constants, distinct in the up and down quark sector, can correctly give both the six quark masses and the four Cabibbo-Kobayashi-Maskawa matrix parameters.

3 Conclusion

The Standard Model of electroweak interactions contains three family replications. The origin of such copies is still an open issue. Recently we have proposed an extension of quantum field theory, that can offer new solutions to the hierarchy problem. Here we have shown that this theory can nicely explain the origin of family replications. We have argued that the physical vacuum (eq.(13)) is a state obtained from the sequential action of the functional field creation operator \( S^\dagger \) onto the mathematical vacuum \( |0\rangle \) (eq.(12)).

In this scenario, fermion mass models based on a family permutation discrete symmetry seems to be a natural choice. In fact this symmetry is intrinsic of the theory and directly descends from the anti-commutation property of the operators \( S^\dagger \).

References

[1] F. Caravaglios, “Looking for new solutions to the hierarchy problem”, hep-ph/0211129.

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3 We set \( m = 0 \) (see eq.(15)).

4 However such a family symmetry is not mandatory (in third quantization), since one could define a model with three families by hand since the very beginning. Assuming, for instance, a functional field \( S^\dagger[\psi_1, \psi_2, \psi_3] \) with three distinct fields and with an Hamiltonian that explicitly breaks the symmetry. However, in this case, one gives up explaining family replications.