ON THE EQUIVARIANT MOTIVIC FILTRATION OF THE TOPOLOGICAL HOCHSCHILD HOMOLOGY OF POLYNOMIAL ALGEBRAS

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Dedicated to the fond memory of Jan Nekovář with thanks for the many cheerful and enlightening conversations we had in Prague in recent years.

Abstract. We identify the equivariant structure of the filtered pieces of the motivic filtration defined by Bhatt, Morrow and Scholze on the topological Hochschild cohomology spectrum of polynomial algebras over $\mathbb{F}_p$.

1. Introduction

This note stems from the authors’ discussions with Peter Scholze in the period of 2015-2017. In connection with questions on the structures defined in the paper [2], Bhatt, Morrow and Scholze [3] defined an $S^1$-equivariant decreasing filtration $F^i \text{THH}(R)$ on the topological Hochschild homology of a smooth $\mathbb{F}_p$-algebra $R$, which they called the motivic filtration. In this note, by $\text{THH}(R)$ we mean the genuine $RO(S^1)$-graded equivariant spectrum. (The existence of such a canonical structure was proved by Bökstedt, Hsiang and Madsen [4].) The purpose of this note is to compute the full equivariant structure of the filtered pieces $F^i \text{THH}(R)$, which strengthens the results of [3] on these spectra.

The coefficients (homotopy groups) of the $\mathbb{Z}/p^{r-1}$-fixed points of $\text{THH}(R)$ can be expressed in terms of $TR^r(\mathbb{F}_p) = \mathbb{Z}/p^r[\sigma_r]$. 

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The answer \[3, 6\] is

\[TR^r(\mathbb{F}_p) \otimes_{\mathbb{Z}/p^r} W_r \Omega R\]

where \(W_r \Omega\) denotes the de Rham-Witt complex of length \(r\). The motivic filtration is a decreasing multiplicative filtration in which \(\sigma_r\) has filtration degree 1, and on \(W_r \Omega\), the filtration coincides with the de Rham filtration. It is shown in [2] that the associated graded spectrum of the motivic filtration, (and therefore its fixed point spectra \(F^i TR^r(R)\)), are equivariant (resp. non-equivariant) Eilenberg-MacLane spectra.

However, computation of the equivariant structure of \(F^i TR^r(R)\) remains an interesting open problem. While for cyclotomic spectra, the equivariant structure can be deduced from non-equivariant information ([12]), \(F^i TR^r(R)\) are, in fact, not cyclotomic spectra (see [3]). The purpose of this note is to calculate one example, namely the equivariant structure on \(F^i THH(R)\) in the case where \(R\) is a polynomial algebra over \(\mathbb{F}_p\) on finitely many generators.

The method [3] of defining the motivic filtration is by proving semi-perfect descent. For \(R\) smooth over \(\mathbb{F}_p\), if we denote by \(R_{\text{perf}}\) the perfectization of \(R\), then the canonical homomorphism \(R \to R_{\text{perf}}\), equalizing the two canonical homomorphisms \(R_{\text{perf}} \Rightarrow R_{\text{perf}} \otimes_R R_{\text{perf}}\), induces an equivalence of \(S^1\)-equivariant spectra

\[(1) \quad THH(R) \to |THH(R_{\text{perf}} \otimes_R \cdots \otimes_R R_{\text{perf}})|^\bullet \]

where \(|?[^\bullet]\) means cosimplicial realization. Now the coefficients of the \(\mathbb{Z}/m\)-fixed points of the constituent spaces on right hand side of (1) (where \(m\) is a natural number) are concentrated in even dimensions, since the rings \(R_{\text{perf}} \otimes_R \cdots \otimes_R R_{\text{perf}}\) are quasiregular semi-perfect \(\mathbb{F}_p\)-algebras (meaning that the Frobenius is onto, the first Andre-Quillen homology is a flat module and the higher Andre-Quillen homology is 0). The desired filtration is obtained by applying cosimplicial realization to the equivariant Postnikov filtration of those spectra.

To prove that (1) is an equivalence on \(\mathbb{Z}/m\)-fixed points, Bhatt, Morrow and Scholze [3] do not use a direct calculation. They used the fact that since \(THH\) is a cyclotomic spectrum, it suffices to prove that (1) induces an isomorphism on homotopy co-fixed points (i.e. the corresponding Borel homology). Taking the left derived functor of the equivariant Postnikov filtration of \(THH(R)\) in the category of simplicial \(\mathbb{F}_p\)-algebras \(R\) (i.e. the right Kan extension of the functor given by the equivariant Postnikov filtration on \(THH(?)\) from the source category of simplicial smooth \(\mathbb{F}_p\)-algebras to the category of \(\mathbb{Z}/m\)-equivariant \(S\)-modules), and smashing it with \(EZ/m^+\), produces associated graded...
pieces which are smash-products of pieces of (the spectral realization of) the Quillen cotangent complex $L$ with trivial $\mathbb{Z}/m$-action, smashed with $E\mathbb{Z}/m$. For these pieces, the analogue of the descent equivalence (11) follows from Bhargav Bhatt’s theorem on faithfully flat descent for $L$ (11 [14]).

The equivalence (11) gives rise to a spectral sequence obtained by the cosimplicial filtration on the right hand side. A direct calculation of this spectral sequence is an interesting exercise worth going through for its own sake. For a general $R$ say, a smooth $\mathbb{F}_p$-algebra, it is still somewhat out of range at this point. We do it for $R = \mathbb{F}_p[x_1, \ldots, x_n]$, which, as we will note, can be used to obtain a proof of semi-perfect descent which is different from the argument of [3]. More immediately from the point of interest of this note, for a polynomial algebra, the spectral sequence (11) collapses to $E_2$, which can be used to calculate the equivariant structure on $F^* THH(R)$. This is the main purpose of our note.

As a warm-up, let us discuss the case of $n = 1$ first. To simplify notation, let us abbreviate the notation $THH(R)$ to $T(R)$. Using the method of Hesselholt and Madsen [8], one sees that, $S^1$-equivariantly,

$$T(\mathbb{F}_p[x]) = S^0 \vee \bigvee_{n \in \mathbb{N}} S^1_{n+} \wedge T(\mathbb{F}_p)$$

where $S^1_n$ is $S^1$ with the $S^1$-action by $n$-th power, $\mathbb{N} = \{1, 2, \ldots \}$. Note that we have a cofibration sequence (hence, stably, equivalently a fibration sequence)

$$S^1_{n+} \to S^0 \to \widetilde{S}^1_n$$

where $\widetilde{?}$ denotes unreduced suspension. Note that $\widetilde{S}^1_n$ is also the 1-point compactification of the irreducible representation $\mathbb{C}[\xi_n]$ on which an element $z \in S^1$ acts by multiplication by $z^n$. Let us denote the Bhatt-Morrow-Scholze motivic filtration by $F^i$, and let $?_{\geq i}$ denote the equivariant Postnikov filtration.

1. **Theorem.** Let $p$ be a prime, and let $m \in \mathbb{N}$. For $R = \mathbb{F}_p[x]$, the spectral sequence associated with applying $\mathbb{Z}/m$-fixed points to the target of (11) collapses to $E_2$, and converges to its abutment $T(R)^{\mathbb{Z}/m}$. Additionally, $F^* T(\mathbb{F}_p[x])$ is (via the canonical map) equivalent to the wedge of $T(\mathbb{F}_p)_{\geq i}$ with the wedge over $n \in \mathbb{N}$ of homotopy fibers of the maps

$$T(\mathbb{F}_p)_{\geq i} \to (\widetilde{S}^1_n \wedge T(\mathbb{F}_p))_{\geq i}.$$
We shall prove Theorem 1 in detail, noting the intricacies of the spectral sequence involved. However, precisely by the same method, one can also prove a generalization:

2. Theorem. Let $p$ be a prime, and let $m \in \mathbb{N}$. For $R = \mathbb{F}_p[x_1, \ldots, x_n]$, the spectral sequence obtained by applying $\mathbb{Z}/m$-fixed points to (1) collapses to $E_2$ and converges to its abutment $T(R)^{\mathbb{Z}/m}$. Additionally, $F^i T(R)$ is the wedge, over subsets $A \subseteq \{1, \ldots, n\}$, and maps $n_\ast : A \rightarrow \mathbb{N}$ of the $|A|$-fold homotopy fibers of the cubes $(S^\bigoplus_{s \in A} \mathbb{C}[[x_s]] \land T(\mathbb{F}_p))_{\geq i}$.

All the objects in sight are well known to satisfy étale descent. This can be used to give another proof of the flat descent equivalence (1), and also a method for calculating the motivic filtration explicitly for smooth $\mathbb{F}_p$-algebras. In effect, if an $\mathbb{F}_p$-algebra $S$ is étale over an $\mathbb{F}_p$-algebra $R$, then we have

$$S_{perf} = S \otimes_R R_{perf}.$$ 

Thus, we may form a bi-cosimplicial spectrum

$$THH(S_{perf} \otimes_S \cdots \otimes_S S_{perf} \otimes_R R_{perf} \otimes_R \cdots \otimes_R R_{perf})$$

(3) (the $(0,0)$th term is $THH(S_{perf} \otimes_R R_{perf})) = THH(S_{perf} \otimes_S S_{perf})$.

Assuming (1) is an equivariant equivalence for $R$, realizing in the $R$-coordinate of (3) first, we get the right hand side of (1) for $S$. Realizing the $S$-coordinate of (3) first, (and accepting that for a perfect ring, its $THH$ can be computed by replacing it by the level-wise $THH$ of its cosimplicially equivalent semiperfect resolution), we get

$$THH(S \otimes_R R_{perf} \otimes_R \cdots \otimes_R R_{perf}),$$

which realizes to $THH(S)$ (see [14]).

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2. The spectral sequence

By usual $THH$ considerations, the interesting mathematics happens on $\mathbb{Z}/p^{r-1}$-fixed points, $r \in \mathbb{N}$. We will restrict attention to this case. In fact, one puts

$$TR^r(R) = T(R)^{\mathbb{Z}/p^{r-1}}.$$ 

Let us recall that Hesselholt and Madsen [7], Theorem 5.5, computed

$$TR^r(\mathbb{F}_p) = \mathbb{Z}/p^r[\sigma].$$
More precisely, the $\mathbb{Z}/p^{r-1}$-equivariant Postnikov tower of $T(F_p)$ consists of even suspensions of the equivariant Eilenberg-MacLane spectrum of the Mackey (in fact, Green) functor $W_r(F_p)$ whose value on the orbit with isotropy $\mathbb{Z}/p^{r-1}$ is $W_i(F_p)$, and Mackey functor restrictions are given by the Frobenius, while transfers are given by Verschiebung, with isotropy groups acting trivially. An analogous Mackey functor can, in fact, be defined with $F_p$ replaced by any commutative ring $A$. In the present case, the Frobenius is just reduction modulo the appropriate power of $p$, while Verschiebung is multiplication by $p$. (The Frobenius, which is the Mackey functor “restriction”, is not to be confused with the cyclotomic spectrum restriction, which is the map from $\mathbb{Z}/p^{r-1}$-fixed points to the $\mathbb{Z}/p^{r-2}$-fixed points of the geometric fixed points, which is equivalent to the $\mathbb{Z}/p^{r-2}$-fixed points of the original cyclotomic spectrum. In our present setting, this map is a homomorphism of rings which sends $\sigma$ to $p\sigma$.)

The coefficients of $\mathbb{Z}/p^{r-1}$-fixed points of $(2)$ can be calculated either directly, or one can also use the calculation of Illusie [11], §1.2 of the deRham-Witt complex of affine spaces, using Hesselholt’s formula for a regular $F_p$-algebra $R$

$$TR^r(R)_n = \bigoplus_{j+2k=n, j,k\in\mathbb{N}_0} W_j \Omega^j R$$

where $W_r \Omega^* R$ is the (truncated) deRham-Witt complex. In fact,

$$F^i TR^r(R)$$

is the sum of those summands of $(4)$ for which $j + k \geq i$.

In the present case, the deRham-Witt complex of $F_p[x]$ is a sum, over $n \in \mathbb{N}_0$ (corresponding to the power of $x^n$) of the exterior algebra on one generator (in dimension 1) over $\mathbb{Z}/p^r$ tensored with the submodule $p^{i+1} W_r(F_p)$ of $W_r(F_p)$ consisting of elements which are annihilated by $p^{i+1}$, where $p^i$ is the highest power of $p$ which divides $n$:

$$\bigoplus_{n\in\mathbb{N}_0, p^i|n} \Lambda_{\mathbb{Z}/p^r}[u] \otimes (p^{i+1} W_r(F_p))$$

(we put $i=r$ for $n=0$). Thus, this module is $\mathbb{Z}/p^{\min(i+1,r)}[\sigma]$.

We have

$$F_p[x]_{perf} = F_p[x^{1/p^n}] := \lim_{\rightarrow} F_p[x^{1/p^n}]$$
Taking colimits, denoting by $\mathbb{Z}[1/p]_0^+$ the set of non-negative elements in $\mathbb{Z}[1/p]$, one has, (somewhat non-canonically),

$$T(\mathbb{F}_p[x^{1/p\infty}]) = \bigvee_{s \in \mathbb{Z}[1/p]_0^+} T(\mathbb{F}_p).$$

One can write

$$\mathbb{F}_p[x^{1/p\infty}] \otimes_{\mathbb{F}_p[x]} \cdots \otimes_{\mathbb{F}_p[x]} \mathbb{F}_p[x^{1/p\infty}] = \mathbb{F}_p[x^{1/p\infty}] \otimes (\mathbb{F}_p[y^{1/p\infty}]/(y))^\otimes n.$$

Denoting the copies of $x$ on the left hand side by $x_i$, $i = 1, \ldots, n+1$, the copies $y_i$, $i = 1, \ldots, n$ of $y$ are given by

$$y_i^{1/p^k} = x_i^{1/p^k} - x_{i+1}^{1/p^k}.$$

3. Lemma.

$$T(\mathbb{F}_p[y^{1/p\infty}]/(y)) = \bigvee_{s \in \mathbb{Z}[1/p]_0^+} S^{V[s]} \wedge T(\mathbb{F}_p)$$

where

$$V_n = C[\xi_1] \oplus \cdots \oplus C[\xi_n].$$

(Thus, $S^{V_n} = \tilde{S}_1^1 \wedge \cdots \wedge \tilde{S}_n^1$.)

Proof. Use the results of [5] (Theorem 11) or [7] (Proposition 9.1) and pass to the colimit, keeping in mind that the colimit of $\mathbb{Z}/p^k$ can be identified with $S_1^1$ due to the fact that we are smashing with the $p$-complete spectrum $T(\mathbb{F}_p)$. \hfill $\Box$

(Note: The formulas (5), (6) are meant $S^1$-equivariantly.)

On fixed points, this can be described purely algebraically. Recall the result of Hesselholt [5] (Theorem 11) or Hesselholt and Madsen [7] (Proposition 9.1) stating that for a complex $\mathbb{Z}/p^{r-1}$-representation $V$,

$$\left( S^V \wedge T(\mathbb{F}_p) \right)_{2a}^{\mathbb{Z}/p^{r-1}} = \mathbb{Z}/p^a$$

for the unique $i$ such that $|V_{\mathbb{Z}/p^{r-1}}| \leq a < |V_{\mathbb{Z}/p^{r-2}}|$. The odd-degree groups are 0. In (7), for bookkeeping reasons, we set $|V_{\mathbb{Z}/p^{r-1}}| = \infty$, $|V_{\mathbb{Z}/p^r}| = -\infty$. Using (7) and Lemma 3 one finds that

$$\bigoplus_{\ell \in \mathbb{N}_0} \sigma^\ell \cdot \mathbb{Z}/(p^\ell)[z_\ell^{1/p^\infty}]/(z_\ell^{1/p^\infty}, p_\ell^{z_\ell^{1/p^\infty}}, \ldots, p^{r-1}z_\ell^{(r+1)/p^{r-1}})$$

$$= (T(\mathbb{F}_p[y^{1/p\infty}]/(y))_s^{\mathbb{Z}/p^{r-1}}.$$
where $\sigma$ has degree $2$ and the ring structure is explained by $p z_{\ell+1} = z_{\ell}$, $z_0 = y$. (One uses the techniques of [5, 7]; note that the summand $s$ in the statement of Lemma 3 corresponds to the term $y^{s/p^r}$; also note the reference [9], which explains how to name classes in $TR_{n+1}$ via differential forms.) Keep in mind, however, that in the ring, $z_{\ell}$ only occurs after being multiplied by $\sigma_{\ell}$. Note that this is not in the form (4), but then again, $\mathbb{F}_p[y^{1/p^\infty}]/(y)$ is not a regular ring. Factoring the ideal $(\sigma)$ out of (8), we obtain the truncated Witt vectors:

\[ W_r(\mathbb{F}_p[y^{1/p^{\infty}}]/(y)) = \mathbb{Z}/(p^r)[y^{1/p^\infty}]/(y, py^{1/p}, \ldots, p^{r-1}y^{1/p^{r-1}}). \]

Now back to the spectral sequence. To verify that (1) induces an isomorphism on $\mathbb{Z}/p^{r-1}$-fixed points for $R = \mathbb{F}_p[x]$, we must prove that the canonical map from (1) to the cobar construction on (5) by the co-algebra (6) is an equivalence on $\mathbb{Z}/p^{r-1}$-fixed points. We obtain a spectral sequence from applying $\mathbb{Z}/p^{r-1}$-fixed points to the right hand side of (1) and taking coefficients $E_{s,t}^* = T(\mathbb{F}_p[x^{1/p^\infty}]) \otimes (\mathbb{F}_p[y^{1/p^\infty}]/(y)^{\otimes s})^{\mathbb{Z}/p^{r-1}}$, converging to $T(?)^{\mathbb{Z}/p^{r-1}}$ applied to the cosimplicial realization. We shall study this spectral sequence. (Note on grading: regardless of the position of the indices, because the present situation mixes homological and cohomological grading, we grade everything homologically. A cohomological degree can be converted to homological by taking its negative.)

Now the $E_1$-term can be simplified by writing (6) as

\[ (\bigvee_{s \in \mathbb{Z}[1/p] \cap [0,1)} T(\mathbb{F}_p)) \wedge_{T(\mathbb{F}_p)} (\bigvee_{n \in \mathbb{N}_0} S^{V_n} \wedge T(\mathbb{F}_p)). \]

Thereby, the $E_1$-term of the spectral sequence has a tensor factor of the dual Koszul complex of

\[ \bigvee_{s \in \mathbb{Z}[1/p] \cap [0,1)} T(\mathbb{F}_p)^{\mathbb{Z}/p^{r-1}}, \]

which can be factored out since its homology is concentrated in degree 0. This leads to a simpler $E_1$-term which is the cosimplicial $\mathbb{Z}/p^{r}$-module (with cosimplicial coordinate $k \in \mathbb{N}_0$)

\[ ((\bigvee_{m \in \mathbb{N}_0} S^0) \wedge (\bigvee_{n \in \mathbb{N}_0} S^{V_n}) \wedge T(\mathbb{F}_p))^k_{\mathbb{Z}/p^{r-1}}. \]

Thus, we need to prove that the homology of the chain complex (10) coincides with (an associated graded object of) the coefficients of the $\mathbb{Z}/p^{r-1}$-fixed points of (2), which we recalled above.
Let us first note that for \( r = 1 \), this works. (10) is just the cobar construction of \( \mathbb{Z}/p[t] \) over a polynomial coalgebra on one generator of (homologically graded) bidegree \((-1, 2)\) (with trivial coaction), which has (homologically graded) cohomology

\[
T(\mathbb{F}_p[x])_* = \mathbb{Z}/p[t, \sigma] \oplus \Sigma \mathbb{Z}/p[t, \sigma]
\]

where \( \Sigma \) is the suspension (by the calculation of \( \mathcal{E}t_{A[z]}(A, A) \) for a commutative ring \( A \)), which is the correct answer.

To make the calculation on \( \mathbb{Z}/p^{r-1} \)-fixed points, we note that the \( E_1 \)-term (10) is graded by

\[
d = m + n_1 + \cdots + n_k.
\]

We will work in each degree \( d \) separately. Assume \( d = p^j d' \) where \( d' \) is not divisible by \( p \). We will be working in each of the “\( \mathbb{Z}/p^i \)-regions” (according to (11)) of the degree \( d \) part of the \( E_1 \)-term (10) separately, showing that for the most part, they will cancel out one at a time.

Filtering (10) by terms with \( m \) less or equal to a given number, we obtain a (purely algebraic) spectral sequence whose \( E_2 \)-term has the differential formed by the \( \geq 1 \)'th co-faces of (10). Now the method of computing this \( E_2 \)-term is based on the computation of

\[
\text{Ext}_{\mathbb{Z}/p^i[v]}(\mathbb{Z}/p^i, \mathbb{Z}/p^j),
\]

with some exceptions caused by the varying \( i \) in (7). Concretely, the term (7) with \( i = r \), in a given dimension, is computed literally by the formula (12), by using the non-equivariant result, and comparison with Hesselholt’s computation of homotopy fixed points [5], in which the coefficients embed. The answer is the exterior algebra over \( \mathbb{Z}/p^r \) on one element \((v)\):

\[
\Lambda_{\mathbb{Z}/p^r[v]},
\]

which is represented by the generator of \( k = 1 \), \( n = 1 \) in (10). The 0-coface component of the \( d_1 \) differential of (10) maps the \( \mathbb{Z}/p^r \) in the same dimension for \( m = d \), \( k = 0 \) by \( p^j \), giving the right answer in dimensions \( \geq 1 \).

To understand what happens for \( i < r \), it is useful to filter \( \mathbb{Z}/p^i[v] \) by an increasing filtration in which \( v, v^p, \ldots, v^{p^{r-1}} \) are given filtration degree 1. The associated graded algebra then will be

\[
\mathbb{Z}/p^i[v]/(v^p) \otimes \mathbb{Z}/p^i[v^p]/(v^{p^2}) \otimes \cdots \otimes \mathbb{Z}/p^i[v^{p^{r-1}}].
\]

The \( \text{Ext}(\mathbb{Z}/p^i, \mathbb{Z}/p^j) \) over (13) is

\[
\Lambda_{\mathbb{Z}/p^r}((v_0), \ldots, (v_{r-1})) \otimes \mathbb{Z}/p^i[t_1, \ldots, t_{r-1}]
\]
where \(v_b\) is dual to \(v^p_b\) and the \(t_b\) is the transpotence element occurring in the well known calculation of the Ext of truncated polynomial algebras, which in our case is in cohomological dimension 2 and \(v\)-degree \(p^b\). The (purely algebraic) spectral sequence calculating the \(\text{Ext}(\mathbb{Z}/p^i, \mathbb{Z}/p^i)\) over \(\mathbb{Z}/p^i[v]\) has the standard differentials
\[
d^1((v_b)) = t_b, \ b = 1, \ldots, r - 1
\]
(obtained from the cobar construction model - see for example [13], Lemma 3.2.4). Now in our topological Hochschild homology situation the total topological dimension (which is always even) determines \(b\), and which of the differentials (15) are disrupted. In fact, using (7) and comparing the dimensions of the \(\mathbb{Z}/p^i\)-fixed points of the two sides of (15) where \(v_m\) is replaced by \(V^p_m\) (the multiplication corresponds to direct sum) shows that the only place where the differentials are disrupted for the reason of the two sides being in different \(j\)-ranges of (7) is for \(a = 0\). This means that we are dealing with the (9) part of (8).

The additional differentials in the spectral sequence obtained by taking \(\mathbb{Z}/(p^i-1)\)-fixed points of (10), (i.e. the part not explained by the coproduct on the middle term), are explained by the comodule structure of the leftmost term. Going back to the notation (5), (9), this structure is given by
\[
x \mapsto x \otimes 1 + 1 \otimes y.
\]
In the above analysis, we omitted the power of \(x\) from the notation, but the power of \(x\) present is equal to the number \(m\) in (12). Depending on its \(p\)-valuation, we obtain additional \(d^1\)-differentials, which wipe out all the remaining terms of the spectral sequence, except the answer.

Concretely, let \(d\) be divisible by \(p^i\) but not \(p^{i+1}\) for some \(0 \leq j < r\). (If \(d\) is divisible by \(p^j\), then only the \(i < j\) part of the below differential pattern will occur.) Then the contribution of (16) to the \(t_i\) differential pattern in topological dimension 0 of the spectral sequence (10), which is not explained by the \(\text{Ext}(\mathbb{Z}/p^i, \mathbb{Z}/p^j)\) over the algebra (13), consists of the degree \(\leq d\) part of
\[
t^{k}_{i+1} \cdot (V^p_i) \mapsto t^{k+1}_{i+1}, \ k \geq 0
\]
tenosed with
\[
\Lambda[(V^p_i), \ell > i] \otimes \mathbb{Z}/p^j[t_{\ell+1}, \ell > i]
\]
for \(0 \leq i < j\), and the degree \(\leq d\) part of
\[
t^k_{j+1} \mapsto (V^p_j) \cdot t^k_{j+1}, \ k \geq 0
\]
tensored with

\[ \Lambda[(V_{p^\ell}), \ell > j] \otimes \mathbb{Z}/p^j [t_{\ell+1}, \ell > j]. \]

This cancels all the remaining elements, with the exception of (19) with \( k = 0 \), where the source group is \( \mathbb{Z}/p^r \), while the target group is \( \mathbb{Z}/p^{r-j-1} \), thus leaving a kernel of \( \mathbb{Z}/p^{j+1} \), as claimed.

3. Proof of Theorem 2

First note that (2) generalizes to

\[ T(\mathbb{F}_p[x_1, \ldots, x_\ell]) = \bigvee_{A \subseteq \{1, \ldots, \ell\}} \bigvee_{n \in \mathbb{N}^A} S^{\bigoplus_{s \in A} \xi(n(s))} \wedge T(\mathbb{F}_p). \]

To prove Theorem 2, we will proceed by induction on \( \ell \). To this end, it is actually advantageous to smash (21) with \( S V \) for an arbitrary finite-dimensional complex representation \( V \). Using the equivalence between a diagonal totalization and a \( k \)-fold successive totalization of a \( \ell \)-fold cosimplicial abelian group, the \( E_1 \)-term of the spectral sequence analogous to (10) actually becomes an \( \ell \)-fold chain complex of the form

\[ (S^V \wedge (\bigvee_{m \in \mathbb{N}_0} S^0)^\ell \wedge (\bigvee_{n \in \mathbb{N}_0} S^{V_n})^{k_1} \wedge \cdots \wedge (\bigvee_{n \in \mathbb{N}_0} S^{V_n})^{k_\ell}) \wedge T(\mathbb{F}_p))^\mathbb{Z}/p^{r-1}. \]

We will show that the \( E_2 \)-term is actually isomorphic to the coefficients of

\[ (S^V \wedge T(\mathbb{F}_p[x_1, \ldots, x_\ell]))^\mathbb{Z}/p^{r-1}. \]

which, in turn, splits into summands over various choices of \( A \), and the choices of mappings \( n : A \to \mathbb{N} \). Let us first state what the answer is:

4. Lemma. The summand of (23) corresponding to a map \( n : A \to \mathbb{N} \) is isomorphic to a sum of \( 2j \)-suspending of the exterior algebra over \( \mathbb{Z}/p^i \) on \( |A| \) generators where

\[ |V^{\mathbb{Z}/p^{i+1-i'}}| \leq j < |V^{\mathbb{Z}/p^{i-i'}}| \]

where \( i \) is the highest non-negative integer such that \( p^i \) divides all the numbers \( n(s) \), \( s \in A \), and \( r - 1 \). Here we apply the same boundary conventions as before. In particular, the summand is 0 for \( j < |V^{\mathbb{Z}/p^i}| \).

Proof. The proof proceeds along similar lines as in the \( \ell = 1 \) case, but we skip it, since it is a part of Hesselholt’s results [5]. \( \square \)
Now as already mentioned, (22) can be considered as an $\ell$-fold chain complex, which splits into summands corresponding to maps $n : A \to \mathbb{N}$, which we can study one at a time. We will proceed by induction on $\ell$. Since the complexes for proper subsets are isomorphic to summands of (22) for lower $\ell$, without loss of generality, we can assume $A = \{1, \ldots, \ell\}$. Now we will proceed by calculating, for each $n$, first the homology of the complex obtained by totalizing the last $\ell - 1$ differentials. This will lead to another purely algebraic spectral sequence, but again, we will see that it collapses to $E_2$. In fact, the $E_1$-term has already been calculated by the induction hypothesis, where $V$ is replaced by

$$V \oplus V_{n_1} \oplus \cdots \oplus V_{n_{k_1}}$$

where $n_1, \ldots, n_{k_1}$ are the $n$’s occurring in the wedge summand in the $k_1$-smash power in (22).

On the other hand, by Lemma 4, the answer is (the appropriate summand of) the exterior algebra over $\mathbb{Z}_p$ on $\ell - 1$ generators, tensored with the $E_1$-term of the spectral sequence (22) for $\ell = 1$, with $r$ replaced by the minimum of $r$ and $i + 1$, where $i$ is the highest non-negative integer such that $p^i$ divides all of the numbers $n(2), \ldots, n(\ell)$.

Thus, what remains to do is to figure out the $d_1$ of the spectral sequence (22) with $\ell = 1$, or, equivalently, the spectral sequence (10) where the argument is smashed with $S^V$, or, explicitly,

$$(S^V \wedge \bigvee_{m \in \mathbb{N}_0} S^0 \wedge \bigvee_{n \in \mathbb{N}_0} S^{V_n})^{\wedge k} \wedge T(\mathbb{F}_p))_{\mathbb{Z}/p^{r-1}}$$

This is a variation of our analysis of the $d_1$ of the spectral sequence (10) in the last section. Essentially, the discussion of the cases is exactly the same, except that if

$$|V^{\mathbb{Z}/p^{r-i-1}}| \leq m < |V^{\mathbb{Z}/p^{r-i-1}}|$$

(with the same conventions as under (7)), $r$ is replaced by $i + 1$ and $i$ is replaced by another index.

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