Color-Flavor Unlocking and Phase Diagram with Self-Consistently Determined Strange Quark Masses

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Abstract

The phase diagram of strongly interacting matter at non-zero temperature and baryon chemical potential is calculated within a 3-flavor NJL-type quark model with realistic quark masses. The model exhibits spontaneous chiral symmetry breaking as well as diquark condensation in the two-flavor color-superconducting phase and in the color-flavor locked phase. We investigate the color-flavor unlocking phase transition, taking into account self-consistently calculated effective quark masses. We find that it is mainly triggered by a first order phase transition with respect to the strange quark mass. It takes place at much higher values of the chemical potential than the transition to the hadronic phase such that we find a relatively large region in the phase diagram where the two-flavor color-superconductor seems to be the most favored state.
1 Introduction

The structure of the QCD phase diagram is one of the most exciting topics in the field of strong interactions (For review see, e.g. [1, 2, 3, 4]). For a long time the discussion was restricted to two phases: the hadronic phase and the quark-gluon plasma (QGP). The former contains “our” world, where quarks and gluons are confined to color-neutral hadrons and chiral symmetry is spontaneously broken due to the presence of a non-vanishing quark condensate \( \phi = \langle \bar{\psi} \psi \rangle \). In the QGP quarks and gluons are deconfined and chiral symmetry is (almost) restored, \( \phi \simeq 0 \).

Although color-superconducting phases were discussed already in the ’70s [5, 6] and ’80s [7], until quite recently not much attention was payed to this possibility. This changed dramatically after it was discovered that due to non-perturbative effects, the gaps which are related to these phases could be of the order of \( \Delta \sim 100 \text{ MeV} \) [8, 9], much larger than expected from the early perturbative estimates. Since in standard weak-coupling BCS theory the critical temperature is given by \( T_c \simeq 0.57 \Delta(T = 0) \) [10], this also implies a sizable extension of the color-superconducting phases into the temperature direction [11]. Hence, color-superconducting phases could be relevant for neutron stars [12, 13] and – in very optimistic cases – even for heavy-ion collisions [14].

Many different diquark condensates are allowed by Pauli principle [7]. However, most interactions, like one-gluon exchange or instanton mediated interactions favor a condensation in the scalar color-antitriplet channel. In general, these condensates read

\[
\begin{align*}
\langle \bar{\psi}^T C \gamma_5 \tau_A \lambda_{A'} \psi \rangle,
\end{align*}
\]

where both, \( \tau_A \) and \( \lambda_{A'} \) are the antisymmetric generators of \( SU(N_f) \) and \( SU(N_c) \), acting in flavor space and in color space, respectively. Throughout this paper, we will restrict ourselves to the physical number of colors, \( N_c = 3 \). Then the \( \lambda_{A'} \) denote the three antisymmetric Gell-Mann matrices, \( \lambda_2, \lambda_5 \) and \( \lambda_7 \).

For two flavors (\( N_f = 2 \)), the flavor index in Eq. (1) is restricted to \( A = 2 \), describing the pairing of an up quark with a down quark. The three condensates \( s_{2A'} \), \( A' = 2, 5, 7 \), form a vector in color space, which always can be rotated into the \( A' = 2 \)-direction. Hence the two-flavor superconducting state (2SC) state can be characterized by

\[
\begin{align*}
s_{22} \neq 0 \quad \text{and} \quad s_{AA'} = 0 \quad \text{if} \quad (A, A') \neq (2, 2).
\end{align*}
\]

Due to the presence of \( s_{22} \), color \( SU(3) \) is spontaneously broken down to \( SU(2) \). For massless up and down quarks the 2SC state is invariant under chiral \( SU(2)_L \times SU(2)_R \) transformations.

For three flavors, the flavor operators \( \tau_A \) also denote the three antisymmetric Gell-Mann matrices, i.e., \( A = 2, 5, 7 \). In this case the two-flavor condensation pattern, Eq. (2) is still possible, but now there are several other combinations which cannot be transformed into \( s_{22} \) via color or flavor rotations. If we have three degenerate light flavors, dense matter is expected to form a so-called color-flavor locked (CFL) state [15], characterized by the situation

\[
\begin{align*}
s_{22} = s_{55} = s_{77} \neq 0 \quad \text{and} \quad s_{AA'} = 0 \quad \text{if} \quad A \neq A'.
\end{align*}
\]
In this state color $SU(3)$ as well as the chiral $SU(3)_L \times SU(3)_R$ and the $U(1)$-symmetry related to baryon-number conservation are broken down to a common $SU(3)_{\text{color}+V}$ subgroup where color and flavor rotations are locked.

Both situations discussed so far are idealizations of the real world, where the strange quark mass $M_s$ is neither infinite, such that strange quarks can be completely neglected, nor degenerate with the masses of the up- and down quarks. However, for sufficiently large quark chemical potentials $\mu \gg M_s$, the strange quark mass becomes almost negligible against $\mu$ and a phase very similar to Eq. (3) should form. In this case one expects that $s_{55}$ and $s_{77}$ are somewhat smaller than $s_{22}$ but do not vanish. Moreover, if we assume an unbroken isospin symmetry, $M_u = M_d$, which we will always do in this article, we still should have $s_{55} = s_{77}$. This corresponds to an $SU(2)_{\text{color}+V}$ subgroup of the original symmetry, where isospin rotations are locked to certain $SU(2)$ rotations in color space. Therefore this phase is usually called CFL-phase as well.

At low chemical potentials, the strange quark mass cannot be neglected against $\mu$ and a phase with $s_{55} = s_{77} \neq 0$ is no longer favored. On the other hand, below a certain value of $\mu$, one should finally reach the hadronic phase. Hence, if we start in the CFL-phase and keep lowering the chemical potential, there are (at least) two possible scenarios: (1) Below a critical value $\mu_{c(2)}$, $s_{55}$ and $s_{77}$ become zero, but $s_{22}$ remains non-zero, i.e., we have basically a transition into the 2SC phase (with or without uncondensed strange quarks). Upon further decreasing $\mu$ we finally reach the hadronic phase at some critical chemical potential $\mu_{c(1)}$. (2) The CFL-phase is directly connected to the hadronic phase without a 2SC-phase in between. This scenario has the interesting feature of the so-called quark-hadron continuity, which is related to the fact that the symmetries in both regimes are the same and the low-lying spectra in both phases can be mapped onto each other after the appropriate redefinition of the conserved $U(1)$ charge in the CFL-phase \[10\].

It is obvious that the answer to the question which of the two scenarios is realized in nature depends on the strange quark mass. This has first been analyzed by Alford, Berges and Rajagopal \[17\], who have studied the color-flavor unlocking phase transition in a model calculation with different values of $M_s$. Assuming that the region below $\mu \simeq 400$ MeV belongs to the hadronic phase, these authors came to the conclusion that a 2SC-phase exists if $M_s \gtrsim 250$ MeV. Here $M_s$ is an effective “constituent”mass of the strange quark, which could be considerably larger than the current quark mass $m_s \sim 100$ to 150 MeV in the Lagrangian. In general, it is $T$- and $\mu$-dependent. Moreover, it could depend on the presence of quark-antiquark and diquark condensates. In particular, it can be discontinuous along a first-order phase transition line. This means, not only the phase structure depends on the effective quark mass, but also the quark mass depends on the phase. This interdependence has not been taken into account in Ref. \[17\], where $M_s$ has been kept at fixed values. The authors of Ref. \[18\], who have studied the effect of a nonzero strange quarks mass within the framework of an instanton mediated interaction, also neglected these interdependencies.

The $T$- and $\mu$-dependence of effective quark masses in non-color superconducting phases has been studied extensively within NJL-type models \[19, 20, 21\], where the masses are closely related to the ($T$- and $\mu$-dependent) quark-antiquark condensates

\[ \phi_u = \langle \bar{u}u \rangle = \langle \bar{d}d \rangle \quad \text{and} \quad \phi_s = \langle \bar{s}s \rangle . \]
Since color-superconductivity is also very often studied within NJL-type models, it is the obvious next step to describe both, diquark condensates and quark-antiquark condensates, simultaneously within the same model. For two flavors this has been done by several authors with different degrees of sophistication \cite{22, 23, 24}. For three flavors, a first attempt in this direction was made in Ref. \cite{25}. However, as we will discuss in the next section, the thermodynamic potential presented in that article does not have the correct behavior in the limiting case of a CFL-phase with exact $SU(3)$-symmetry and does also not reduce correctly to the 2SC thermodynamic potential, when $s_{55}$ and $s_{77}$ are switched off. In the present article we therefore revisit the problem. In Sect. 2 we present our model and derive the thermodynamic potential in mean-field approximation. Our result has the correct behavior for the 2SC case and for the $SU(3)$-symmetric CFL phase. In general, the dispersion laws of the corresponding quasiparticle states are more complicated than the result presented in Ref. \cite{25}. In Sect. 3 we present numerical results for zero and non-zero temperatures. Finally, conclusions can be found in Sect. 4.

2 Formalism

We consider the effective Lagrangian

\begin{equation}
\mathcal{L}_{\text{eff}} = \bar{\psi}(i \slashed{\partial} - \hat{m}) \psi + \mathcal{L}_{\bar{q}q} + \mathcal{L}_{qq},
\end{equation}

where $\psi$ denotes a quark field with three flavors and three colors. The mass matrix $\hat{m}$ has the form $\hat{m} = \text{diag}(m_u, m_d, m_s)$ in flavor space. Throughout this paper we will assume isospin symmetry, $m_u = m_d$.

Since we wish to study the interplay between the color-superconducting diquark condensates $s_{AA'}$ and the quark-antiquark condensates $\phi_u$ and $\phi_s$, we need an interaction which allows for condensation in these channels. To this end we consider an NJL-type interaction with a quark-antiquark part

\begin{equation}
\mathcal{L}_{\bar{q}q} = G \sum_{a=0}^{8} \left[ (\bar{\psi} \tau_a \psi)^2 + (\bar{\psi} i \gamma_5 \tau_a \psi)^2 \right]
\end{equation}

and a quark-quark part

\begin{equation}
\mathcal{L}_{qq} = H \sum_{A=2,5,7} \sum_{A'=2,5,7} (\bar{\psi} i \gamma_5 \tau_A \lambda_{A'} \ C \bar{\psi}^T)(\psi^T C i \gamma_5 \tau_A \lambda_{A'} \psi).
\end{equation}

These effective interactions might arise from some underlying more microscopic theory and are understood to be used at mean-field level in Hartree approximation.

In Eq. (6), $\tau_a, a = 1, ..., 8$, denote Gell-Mann matrices acting in flavor space, while $\tau_0 = \sqrt{\frac{2}{3}} \mathbb{1}_f$ is proportional to the unit matrix. Hence $\mathcal{L}_{\bar{q}q}$ is a $U(3)_L \times U(3)_R$-symmetric 4-point interaction. In NJL-model studies of the meson spectrum \cite{26, 27} or of properties of quark matter at finite densities or temperatures \cite{28, 29, 31}, usually a ‘t Hooft-type

\footnote{In the meantime a substantially revised version of Ref. \cite{25} appeared, see footnote on page 4.}
6-point interaction is added which breaks the the $U_A(1)$ symmetry. It is straightforward to take it into account such a term, but for simplicity we neglect it here. As we will discuss below, this might have qualitative consequences for the results.

In order to calculate the mean-field thermodynamic potential at temperature $T$ and quark chemical potential $\mu$, we linearize the effective Lagrangian in the presence of the expectation values $\bar{\psi} = \phi_d$, $\phi_s$, $s_{22}$, and $s_{55} = s_{77}$. In this context it is convenient to introduce the effective quark masses

$$M_u = m_u - 4G\phi_u \quad \text{and} \quad M_s = m_s - 4G\phi_s$$ (8)

and the diquark gaps

$$\Delta_2 = -2Hs_{22} \quad \text{and} \quad \Delta_5 = -2Hs_{55}.$$ (9)

Then, after formally doubling the degrees of freedom,

$$q(x) := \frac{1}{\sqrt{2}} \left( \begin{array}{c} \psi(x) \\ C\bar{\psi}^T(x) \end{array} \right),$$ (10)

the thermodynamic potential per volume can then be written as

$$\Omega(T, \mu) = -T \sum_n \int \frac{d^3p}{(2\pi)^3} \frac{1}{2} \text{Tr} \ln \left( \frac{1}{T} S^{-1}(i\omega_n, \vec{p}) \right) + 4G\phi_u^2 + 2G\phi_s^2 + H(|s_{22}|^2 + 2|s_{55}|^2).$$ (11)

The inverse propagator of the $q$-fields at 4-momentum $p$ is given by

$$S^{-1}(p) = \left( \begin{array}{cc} \hat{\gamma}_5 \tau_2 \lambda_2 & \Delta_2 \gamma_5 \tau_2 \lambda_2 + \Delta_5 \gamma_5 (\tau_5 \lambda_5 + \tau_7 \lambda_7) \\ -\Delta_2 \gamma_5 \tau_2 \lambda_2 - \Delta_5 \gamma_5 (\tau_5 \lambda_5 + \tau_7 \lambda_7) & \hat{\gamma}_5 \tau_2 \lambda_2 \end{array} \right),$$ (12)

with $\hat{M} = \text{diag}(M_u, M_u, M_s)$. In Eq. (11), $S^{-1}$ has to be evaluated at $p = (i\omega_n, \vec{p})$ where $\omega_n = (2n - 1)\pi T$ are fermionic Matsubara frequencies.

Taking into account the Dirac structure, color, flavor, and the charge conjugate components, the $q$-fields are 72 dimensional objects, and the trace in Eq. (11) has to be evaluated in this 72 dimensional space. A tedious but straight-forward calculation yields:

$$\frac{1}{2} \text{Tr} \ln \left( \frac{1}{T} S^{-1}(i\omega_n, \vec{p}) \right)$$

$$= 3 \ln \left( \frac{1}{T^4}(x_{uu}^+ x_{uu}^- + 2|\Delta_2|^2 y_{uu} + |\Delta_2|^4) \right) + 2 \ln \left( \frac{1}{T^4}(x_{ss}^+ x_{ss}^- + 2|\Delta_5|^2 y_{us} + |\Delta_5|^4) \right) + \ln \left( \frac{1}{T^8}(x_{uu}^+ x_{uu}^- x_{ss}^+ x_{ss}^- + 2|\Delta_2|^2 x_{ss}^+ x_{ss}^- y_{uu} + 4|\Delta_5|^2 x_{uu}^+ x_{ss}^- + x_{ss}^+ x_{uu}^-) y_{uu} + 8|\Delta_2|^4 y_{uu} + 4|\Delta_2|^2|\Delta_5|^2 (x_{ss}^+ x_{ss}^- + x_{uu}^+ x_{uu}^-) + 4|\Delta_5|^4 (x_{uu}^+ x_{uu}^- + x_{ss}^+ x_{uu}^-) + 4|\Delta_5|^2 (x_{uu}^+ x_{uu}^- + x_{ss}^+ x_{uu}^-) y_{uu} + 32|\Delta_5|^6 y_{uu} + 16|\Delta_5|^8) \right),$$ (13)
where we introduced the abbreviations

\[ x_{ff'} = (\omega_n + \mu)^2 + \vec{p}^2 + M_f M_{f'} \quad \text{and} \quad y_{ff'} = \omega_n^2 + \mu^2 + \vec{p}^2 + M_f M_{f'} . \tag{14} \]

With these definitions one finds for the argument of the first logarithm on the r.h.s. of Eq. (13)

\[ x_{uu} x_{uu}^{-} + 2|\Delta_2|^2 y_{uu} + |\Delta_2|^4 = (\omega_n^2 + E_u^{-2})(\omega_n^2 + E_u^{+2}) \tag{15} \]

with

\[ E_u^\pm = \sqrt{(\sqrt{\vec{p}^2 + M_u^2} \pm \mu)^2 + |\Delta_2|^2} . \tag{16} \]

These are exactly the dispersion laws of the paired quarks in a two-flavor color superconductor and the corresponding Matsubara sums are readily turned out using the standard relation

\[ T \sum_n \ln \left( \frac{1}{T^2} (\omega_n^2 + \lambda_k^2) \right) = |\lambda_k| + 2T \ln(1 + e^{-|\lambda_k|/T}) . \tag{17} \]

The other terms in Eq. (13) are in general more complicated. There are, however, two simplifying limits. The first one corresponds to a two-flavor color superconductor, together with unpaired strange quarks. In this case \( \Delta_5 \) vanishes and Eq. (13) becomes

\begin{align*}
\frac{1}{2} \text{Tr} \ln \left. \left( \frac{1}{T} S^{-1}(i\omega_n, \vec{p}) \right) \right|_{\Delta_5=0} &= 4 \left[ \ln \left( \frac{\omega_n^2 + E_u^{-2}}{T^2} \right) + \ln \left( \frac{\omega_n^2 + E_u^{+2}}{T^2} \right) \right] + 2 \left[ \ln \left( \frac{\omega_n^2 + \varepsilon_u^{-2}}{T^2} \right) + \ln \left( \frac{\omega_n^2 + \varepsilon_u^{+2}}{T^2} \right) \right] \\
&\quad + 3 \left[ \ln \left( \frac{\omega_n^2 + \varepsilon_\text{oct}^{-2}}{T^2} \right) + \ln \left( \frac{\omega_n^2 + \varepsilon_\text{oct}^{+2}}{T^2} \right) \right], \tag{18} \end{align*}

with \( \varepsilon_\pm = \sqrt{\vec{p}^2 + M_f^2 \pm \mu} \). Here we recover the fact, that only four of the six light quarks (two colors) participate in the 2SC condensate, while the two remaining ones (and of course all strange quarks) fulfill the dispersion laws of free particles with effective masses \( M_f \).

We can also reproduce the structure of the dispersion laws of the idealized 3-flavor symmetric CFL-state. To this end we evaluate Eq. (13) for \( M_u = M_s \) and \( \Delta_2 = \Delta_5 \). One finds

\begin{align*}
\frac{1}{2} \text{Tr} \ln \left. \left( \frac{1}{T} S^{-1}(i\omega_n, \vec{p}) \right) \right|_{M_u=M_s, \Delta_2=\Delta_5} &= 8 \left[ \ln \left( \frac{\omega_n^2 + E_\text{oct}^{-2}}{T^2} \right) + \ln \left( \frac{\omega_n^2 + E_\text{oct}^{+2}}{T^2} \right) \right] + \left[ \ln \left( \frac{\omega_n^2 + E_\text{sing}^{-2}}{T^2} \right) + \ln \left( \frac{\omega_n^2 + E_\text{sing}^{+2}}{T^2} \right) \right] \\
&= 8 \left[ \ln \left( \frac{\omega_n^2 + E_\text{oct}^{-2}}{T^2} \right) + \ln \left( \frac{\omega_n^2 + E_\text{oct}^{+2}}{T^2} \right) \right] + \left[ \ln \left( \frac{\omega_n^2 + E_\text{sing}^{-2}}{T^2} \right) + \ln \left( \frac{\omega_n^2 + E_\text{sing}^{+2}}{T^2} \right) \right], \tag{19} \end{align*}

with \( E_\text{oct}^\pm = E_u^\pm \) and \( E_\text{sing}^\pm = \sqrt{(\sqrt{\vec{p}^2 + M_u^2 \pm \mu})^2 + |2\Delta_2|^2} \). We see that in this limit all quarks participate in a diquark condensate, forming an octet with diquark gap \( \Delta_2 \).
(= Δ₅) and a singlet with diquark gap 2Δ₂. In fact, it has been shown that for three degenerate flavors, the diquark gaps form a singlet with Δ₅ = 2Δ₃ + 4Δ₆ and an octet with Δ₆ = Δ₃ − Δ₆, where 3 and 6 refer to pairing in the color-antitriplet and color-sextet channel, respectively [15, 32, 33]. Since we have neglected the small color-sextet contribution in our ansatz, we should find Δ₅ = 2Δ₆, in agreement with the above results.

Both limiting cases, Eqs. (18) and (19), are not correctly reproduced, if one starts from the thermodynamic potential given in Ref. [25]. For instance, there are always quarks which fulfill free dispersion laws (no diquark gaps), which should not be the case in the CFL-phase, see above.

The Matsubara sums over Eqs. (18) and (19) can again be turned out with the help of Eq. (17). In general, i.e., for Eq. (13) with arbitrary values of the condensates, this cannot be done so easily. If one combines the second with the third logarithm on the r.h.s., the argument becomes a polynomial of fourth order in ωₙ². The same is true for the argument of the fourth logarithm. The corresponding dispersion laws are related to the zeros of these polynomials. Although, in principle, the zeros of a polynomial of fourth order can be determined analytically, the results are very nasty expressions which are difficult to handle. Therefore, in practice one has to determine the dispersion laws numerically. After that, one can again employ Eq. (17) to calculate the Matsubara sum. Alternatively, one can turn out the Matsubara sum numerically without previous determination of the dispersion laws. This is the method we have used.

So far, the thermodynamic potential depends on our choice of the condensates φᵢ, φⱼ, sₖ, and sₗ. On the other hand, in a thermodynamically consistent treatment the condensates should follow from the thermodynamic potential by taking the appropriate derivatives. The self-consistent solutions are given by the stationary points of the potential,

\[
\frac{δΩ}{δφ_i} = \frac{δΩ}{δφ_j} = \frac{δΩ}{δs_{22}} = \frac{δΩ}{δs_{55}} = \frac{δΩ}{δs_{55}^*} = 0. \tag{20}
\]

In this way one finds that the quark-antiquark condensates are given by

\[
φ_i = -\frac{1}{2} T \sum_n \int \frac{d^3p}{(2π)^3} \frac{∂}{∂M_i} \frac{1}{2} \text{Tr} \ln \left( \frac{1}{T} S^{-1}(iω_n, \vec{p}) \right),
\]

\[
φ_j = -T \sum_n \int \frac{d^3p}{(2π)^3} \frac{∂}{∂M_j} \frac{1}{2} \text{Tr} \ln \left( \frac{1}{T} S^{-1}(iω_n, \vec{p}) \right). \tag{21}
\]

The explicit evaluation of the derivatives is trivial, but leads to rather lengthy and not very illuminating expressions, which we do not want to present. Note, however, that M_i

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† After submission of our final manuscript, Ref. [25] was replaced by a substantially revised version (F. Gastineau et al., hep-ph/0101289 v3). The quark dispersion laws presented there are now consistent with the limits Eqs. (18) and (19). The authors make the simplifying assumption that the particle part and the antiparticle part of the Hamiltonian separate. This enables them to derive relatively simple analytical expressions for the quasiparticle energies. Although in general these solutions are not exact (corresponding to the zeros of the polynomials in our Eq. (13)), the numerical results obtained in this way look very similar to ours, thereby justifying the above approximation.
and $M_4$, which determine the right hand sides, depend again on $\phi_u$ and $\phi_s$ (see Eq. (8)) and therefore the equations have to be solved self-consistently.

Similarly one finds that the diquark condensates are governed by the gap equations

$$\Delta_2 = 4H\Delta_2 T \sum_n \int \frac{d^3p}{(2\pi)^3} \frac{\partial}{\partial |\Delta_2|^2} \frac{1}{2} \text{Tr} \ln \left( \frac{1}{T} S^{-1}(i\omega_n, \vec{p}) \right),$$

$$\Delta_5 = 2H\Delta_5 T \sum_n \int \frac{d^3p}{(2\pi)^3} \frac{\partial}{\partial |\Delta_5|^2} \frac{1}{2} \text{Tr} \ln \left( \frac{1}{T} S^{-1}(i\omega_n, \vec{p}) \right).$$

(22)

Obviously, both equations have trivial solutions $\Delta_A = 0$ which are independent of each other. For the nontrivial solutions Eqs. (21) and (22) are coupled and have to be solved simultaneously. If there is more than one solution, the stable solution is the one which corresponds to the lowest value of $\Omega$.

3 Numerical results

For the numerical evaluation of the equations derived in the previous section we first have to specify the interaction. As a typical example we consider a four-fermion interaction with the quantum numbers of a single-gluon exchange,

$$\mathcal{L}_{\text{int}} = -g \sum_{a=1}^{8} (\bar{\psi} \gamma^\mu \lambda_a \psi)^2.$$  

(23)

This interaction was also the starting point of the model calculations in Refs. [15] and [17]. Performing Fierz transformations we obtain for the effective coupling constants which enter Eqs. (6) and (7)

$$G = \frac{8}{9} g, \quad H = \frac{2}{3} g.$$  

(24)

In addition, $\mathcal{L}_{\text{int}}$ gives rise to effective couplings in other channels, which in principle have to be taken into account to be fully self-consistent in the sense that also all exchange (Fock-) terms are included. In fact, already for two flavors, one finds that a self-consistent treatment requires the simultaneous consideration of possible expectation values in six different channels, like color dependent quark condensates or densities [24]. For three flavors, there are even more. We already mentioned the induced color-sextet diquark condensates in the CFL-phase [15, 32, 33]. However, most of these induced condensates – as far as they have been studied so far – have turned out to be small. Therefore, since the structure of the equations is rather involved even without these additional channels, we neglect them in this paper. Also, as stated above, Eq. (23) should only be viewed as a “typical” interaction, which we have chosen for a qualitative illustration of the more general equations derived in the previous section. Certainly, there is no reason to exclude other terms (see also the comments in Ref. [17] about this point).

Taking the model as defined above, there are four parameters: the coupling constant $g$, the bare quark masses $m_u$ and $m_s$, and a cut-off parameter, which is needed to regularize
the divergent momentum integrals in Eqs. (11), (21) and (22). For simplicity, we take a sharp 3-momentum cut-off $\Lambda$. We expect that the results will be qualitatively the same, if a smooth form factor is used. As customary (see e.g. the comments on this point in Ref. [22]), the parameters are chosen such that “reasonable” vacuum properties are obtained. We take $\Lambda = 600$ MeV, $g\Lambda^2 = 2.6$, $m_u = 5$ MeV and $m_s = 120$ MeV. With these parameters we find the vacuum constituent quark masses $M_u = 362.5$ MeV and $M_s = 557.4$ MeV, which corresponds to the condensates $\phi_u = (-240.5 \text{MeV})^3$ and $\phi_s = (-257.3 \text{MeV})^3$. These values are similar to those obtained in Ref. [29], where the parameters have been fixed by fitting vacuum masses and decay constants of pseudoscalar mesons.

We begin with the discussion of the results at zero temperature. The behavior of the four gap parameters as functions of the quark chemical potential $\mu$ is displayed in Fig. 1. In the left panel we show the constituent quark masses $M_u$ and $M_s$, in the right panel the diquark gaps $\Delta_2$ and $\Delta_5$. One can clearly distinguish three phases. At low chemical potentials $\mu < \mu_1 = 342.8$ MeV, the diquark gaps vanish and the constituent quark masses stay at their vacuum values. For the condensates this means $s_{22} = s_{55} = s_{77} = 0$, while $\phi_u$ and $\phi_s$ are large. Hence, in a very schematic sense, this phase can be identified with the “hadronic phase”, although there are no hadrons in our model. In fact, in Fig. 1 the entire region below $\mu = \mu_1$ corresponds to the vacuum solution with zero density.

At $\mu = \mu_1$ a first-order phase transition takes place and the system becomes a two-flavor color superconductor: The diquark condensate $s_{22}$ has now a non-vanishing expectation value, related to a non-vanishing diquark gap $\Delta_2$, whereas $\Delta_5$ remains zero. Just above the phase transition we find $\Delta_2 = 111.5$ MeV. At the same time the mass of the up quark drops from the vacuum value to $M_u = 55.3$ MeV. With increasing $\mu$, $M_u$ decreases further, while $\Delta_2$ increases until it reaches a maximum at $\mu \approx 475$ MeV. Just below the next phase transition at $\mu = \mu_2 = 508.5$ MeV we find $\Delta_2 = 135.6$ MeV and
\( M_u = 8.6 \) MeV.

In the 2SC phase the baryon number density is no longer zero and increases from \( \rho_B = 0.42 \) fm\(^{-3}\) (2.5 times nuclear matter density) above \( \mu = \mu_1 \) to \( \rho_B = 1.15 \) fm\(^{-3}\) (6.7 times nuclear matter density) below \( \mu = \mu_2 \). The density of strange quarks remains zero up to \( \mu = \mu_2 \), but this could be an artifact of the interaction which contains no flavor mixing terms. As a consequence the strange quark mass \( M_s \) remains at its vacuum value and hence strange quark states cannot be populated in this regime. A flavor mixing interaction would lead to a reduced value of \( M_s \) in the 2SC phase and thus to a lower threshold for a finite density of strange quarks.

At \( \mu = \mu_2 \) the system undergoes a second phase transition, this time from the 2SC phase into the CFL phase, which is characterized by a non-vanishing diquark gap \( \Delta_5 \) (together with a non-vanishing \( \Delta_2 \)). The phase transition is again of first order: At the transition point \( \Delta_5 \) jumps from zero to 109.7 MeV, while \( M_s \) drops from 557.4 MeV to 204.3 MeV. The non-strange quantities are also discontinuous and change in the opposite direction: \( \Delta_2 \) drops from 135.6 MeV to 111.2 MeV, and \( M_u \) jumps from 8.6 MeV to 17.5 MeV. The density jumps from \( \rho_B = 1.15 \) fm\(^{-3}\) to 1.63 fm\(^{-3}\), i.e., from 6.7 to 9.6 times nuclear matter density.

If we started from an exact SU(3) flavor symmetry, i.e., equal masses for all quark flavors, we would expect \( \Delta_2 = \Delta_5 \) in the CFL phase. However, as we are far away from an exact symmetry, it is remarkable that the diquark gaps \( \Delta_2 \) and \( \Delta_5 \) are so similar at the transition point. The reason for the phase transition is not that \( u \) and \( s \) quarks can no longer condense below that value. We still find a CFL solution down to \( \mu = 445.0 \) MeV. However, for \( \mu < \mu_2 \), there is a more favored solution with a much higher strange quark mass which cannot support a condensation of \( u \) and \( s \) quarks. In this regime the CFL solution corresponds to a metastable state.

In Ref. [17] it has been argued that the color-flavor-unlocking transition at zero temperature has to be first order because pairing between light and strange quarks can only occur if the gap is of the same order as the mismatch between the Fermi surfaces. Hence the value of the gap must be discontinuous at the phase transition. The existence of pairing between light and strange quarks does of course not automatically mean that the CFL solution corresponds to an absolutely stable solution. This has already been emphasized in Refs. [34, 35]. The quantitative criteria derived in these references for the existence of a CFL solution as a global minimum of the thermodynamic potential cannot be applied to our case because the possibility of a density dependent strange quark mass has not been taken into account in the derivation. However, at least qualitatively, our results support the arguments put forward in these references: There is indeed a non-vanishing lowest possible value of \( \Delta_5 \) for metastable CFL solutions and the minimal value of \( \Delta_5 \) in an absolutely stable CFL phase is even larger.

At large \( \mu \) we become more and more sensitive to the cut-off and we approach the limits of the model. This is also the reason why at large chemical potentials both diquark condensates decrease with \( \mu \). Although it is sometimes argued that the cut-off simulates asymptotic freedom, i.e., the weakening of the strong coupling constant at large (Fermi-) momenta in some crude way, the results should certainly not been taken too serious in this regime. In particular the fact that we find \( \Delta_5 > \Delta_2 \) above \( \mu \approx 520 \) MeV should be seen as a cut-off artifact.
We now extend our analysis to non-vanishing temperatures. The resulting phase diagram in the $\mu - T$ plane is shown in Fig. 2. Let us first concentrate on the quark-antiquark condensates $\phi_u$ and $\phi_s$. We could then distinguish between three different regimes: At low temperatures and low chemical potentials both condensates, $\phi_u$ and $\phi_s$, are relatively large and we can schematically identify this phase with the hadronic phase, as before. When we increase the temperature or the chemical potential, eventually a second phase is reached, where $\phi_u$ is almost zero. $\phi_s$ is still large in this phase and it becomes small only in a third phase at even higher temperatures or chemical potentials. At low temperatures, e.g., along the $T=0$-axis, these three phases are well separated by first-order phase transitions, whereas at high temperatures we only find smooth crossovers. Consequently, both first-order phase transition lines end in a second-order endpoint, which in our model is located at $T \simeq 66$ MeV and $\mu \simeq 322$ MeV for the phase transition related to $\phi_u$ and at $T \simeq 66$ MeV and $\mu \simeq 505$ MeV for the phase transition related to $\phi_s$. For the former we should keep in mind, however, that our model does neither contain hadronic nor gluonic degrees of freedom, which are certainly important in this area of the phase diagram.

We now turn to the diquark condensates. As discussed for the zero temperature case, at low temperatures, the three phases discussed in the previous paragraph coincide with the “hadronic phase”, the 2SC phase and the CFL phase, i.e., we have $s_{22} = s_{55} = 0$ at low chemical potentials, $s_{55} = 0$, but $s_{22} \neq 0$ at intermediate chemical potentials, and $s_{22} \neq 0$ and $s_{55} \neq 0$ at high chemical potentials. At high temperatures all diquark condensates vanish. This leads to two additional phase transition lines. The first one corresponds to the transition from the 2SC phase to the “quark-gluon” plasma, i.e., to
the phase with vanishing diquark condensates and (almost) vanishing $\phi_u$. In agreement with various BCS-type calculations, this phase transition is of second order. In addition we obtain almost perfect agreement with a universal relation which can be derived (see e.g. \cite{10, 11}) between the value of the gap parameter (in our case this corresponds to $\Delta_2$) at zero temperature and the critical temperature $T_c$ for that transition:

$$T_c \approx 0.57 \Delta_2(T = 0).$$

(25)

For instance, at $\mu = 343$ MeV we find $\Delta_2(T = 0) = 111.5$ MeV, and $T_c = 62.0$ MeV, whereas from the above relation we would expect $T_c = 63.5$ MeV. Since Eq. (25) is an approximate relation, based on the assumption that pairing takes place only in the vicinity of the Fermi surface, the agreement is quite good.

Starting from the CFL phase and increasing the temperature, one first observes a “melting” of the diquark gap $\Delta_5$, before at somewhat higher temperatures $\Delta_2$ also vanishes. The intermediate 2SC phase “above” the CFL-phase is distinguished from the regime “left” to the CFL-phase by a much smaller value of $\phi_s$ and hence of the strange quark mass. Based on the assumption that $\Delta_5$ is always smaller than $\Delta_2$ (which is not really true at $T = 0$, as we have seen), the earlier disappearance of $\Delta_5$ and hence the existence of a 2SC phase “above” the CFL phase was already anticipated in Ref. \cite{3}. Applying similar arguments as at zero temperature (cf. Refs. \cite{17, 32, 33}) it was also predicted in that reference, that the corresponding color-flavor unlocking transition should again be of first order. This was corroborated by a second argument, claiming that the phase transition corresponds to a finite temperature chiral restoration phase transition in a three-flavor theory. In this case the universality arguments of Ref. \cite{30} should apply, stating that the phase transition should be of first order.

Indeed, following the phase transition line from the left, we find that the transition continues to be first order. However, above a tricritical point at $\mu \approx 518$ MeV and
$T \simeq 59$ MeV the phase transition becomes second order. This is illustrated in Fig. 3, where the constituent masses (left panel) and the diquark gaps (right panel) are displayed as functions of the temperature for fixed $\mu = 520$ MeV. We find a second-order phase transition at $T = 59.8$ MeV. Of course we cannot exclude that the second-order phase transition is just an artifact of our simplified model or the present approximation. On the other hand, the arguments of Refs. [17, 34, 35] in favor of a first-order phase transition are much less stringent at finite temperature, where even for vanishing condensates the Fermi surfaces are smeared due to thermal effects. The applicability of the universality argument is also questionable in the present situation because the 2SC phase is not a three-flavor chirally restored phase, but only $SU(2) \times SU(2)$ symmetric. Note that even $SU(3)_V$ is broken explicitly by unequal current quark masses and spontaneously, e.g., by the flavor-antitriplet diquark condensates. Therefore a rigorous prediction of the true order of the phase transition is rather difficult.

Finally, we should repeat that our results at $\mu > \sim 500$ MeV are likely to be sensitive to the cut-off. Therefore it remains to be studied in a more systematic way, which of our findings persist if the model parameters (including the cut-off) are varied over a wider range. Work in this direction is in progress.

4 Conclusions

Recent investigations revealed a rich phase structure of strongly interacting matter. Besides the various known phases of hadronic matter and the quark-gluon plasma phase at high temperatures several “superconducting” phases are likely to exist in cold dense matter. Calculations within effective models showed that in this context “dense” could mean only a few times nuclear matter density such that those superconducting phases could be relevant for the interior of neutron stars or even for relativistic heavy ion collisions [3]. In this paper we focused on the study of this “cold dense” region within the framework of an NJL-type effective quark model. This model allows for condensation in various channels, in particular chiral symmetry breaking due to quark-antiquark condensation and diquark condensation can be described. Our main intention was to examine the effect of selfconsistently (via $\bar{q}q$-condensation) determined realistic quark masses. In particular the mass of the strange quark plays an important role. For the two extreme cases, taking the strange quark mass equal to the light quark masses or considering an infinite strange quark mass, two different condensation patterns are found, respectively: The so-called color-flavor-locked phase [15], where all quarks contribute to diquark condensation and the two-flavor color superconductor [8, 9], where only light quarks participate in the condensation. We could show that both limiting cases are correctly described within our model. However, as in the regions of interest the strange quark mass is neither negligible nor infinite the realistic situation must be somewhere in between. Indeed, for quark matter at zero temperature, we find that for moderate values of the chemical potential the two-flavor superconductor is energetically favored, corresponding to densities of about $2.5 \rho_0$ up to about $6.7 \rho_0$. Only for even higher densities the CFL phase is found to be the energetically most favored one. The main reason is the large constituent mass of the strange quarks compared with that of up and down quarks which has been determined.
selfconsistently within our investigations. In particular, the strange quark mass in the 2SC phase is much larger than in the CFL phase and the phase transition is mostly triggered by a discontinuous change of the quark mass.

The situation remains qualitatively unchanged up to temperatures of about 50 MeV. At higher temperatures the diquark condensates melt. For the diquark condensates which involve strange quarks this happens at somewhat lower temperatures than for the non-strange condensate. Hence, with increasing temperature, the system passes from the CFL phase through an intermediate 2SC phase to the quark-gluon plasma. In contrast to the situation at zero temperature, we found that the strange quark mass in this “high-temperature” 2SC phase is comparable to the strange quark mass in the CFL phase. Furthermore we found a tricritical point on the color-flavor unlocking line above which the phase transition is second order.

Of course, our model is still rather schematic and can only give some qualitative hints on the interdependencies between the effective quark masses and the various superconducting phases. Unfortunately, as our knowledge about the effective quark interaction to be used at moderate chemical potentials is limited, more quantitative predictions are difficult. In this situation a systematic examination of the sensitivity of the results on the choice of the interaction might be helpful to sort out some common features. Perhaps the most severe simplification in our model was that we did not include a ’t Hooft-type interaction, which mixes different flavors and thus affects the strange quark mass even in regimes, where no strange quark states are populated. It is therefore certainly worthwhile to investigate how such a term influences the phase structure. In a more systematic analysis one should also study the effect of other condensates, which have been neglected so far.

For simplicity, we restricted our studies to a common baryon chemical potential for all flavors. However, for many applications, like quark cores of neutron stars or the possible existence of absolutely stable strange quark matter, it would be interesting to consider $\beta$-equilibrated quark matter, where the chemical potential for up quarks is in general different from that of down and strange quarks. This could alter the position of the various Fermi surfaces and hence the entire phase structure. In an earlier work without diquark condensation we have shown that a selfconsistent treatment of the strange quark constituent mass is a very important effect in this context [31]. On the other hand, in more recent studies which include diquark condensation (e.g., about the existence of the color-flavor locked phase in neutron stars [36] or strangelets [37]) the interdependencies between strange quark mass and phase structure have so far been neglected. In view of our present results, work in this direction seems to be worth being further pursued.

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