A Note on the Performance of Algorithms for Solving Linear Diophantine Equations in the Naturals

Valeriu Motroi\textsuperscript{1} and Ştefan Ciobăcă\textsuperscript{2}

Alexandru Ioan Cuza University Iaşi, Romania
\{motroival, stefan.ciobaca\}@gmail.com

Abstract

We implement four algorithms for solving linear Diophantine equations in the naturals: a lexicographic enumeration algorithm, a completion procedure, a graph-based algorithm, and the Slopes algorithm. As already known, the lexicographic enumeration algorithm and the completion procedure are slower than the other two algorithms. We compare in more detail the graph-based algorithm and the Slopes algorithm. In contrast to previous comparisons, our work suggests that they are equally fast on small inputs, but the graph-based algorithm gets much faster as the input grows. We conclude that implementations of AC-unification algorithms should use the graph-based algorithm for maximum efficiency.
1 Introduction

Solving linear Diophantine equations in the naturals is at the core of AC-unification algorithms. AC-unification reduces to top-most unification problems of the form

\[ f^*(u_1, \ldots, u_l) = f^*(v_1, \ldots, v_k), \]

where \( u_1, \ldots, u_l, v_1, \ldots, v_k \) are variables (possibly with repetitions) and \( f^* \) is a variadic symbol corresponding to some AC-symbol \( f \). Solving such a top-most AC-unification problem reduces to solving the Diophantine equation

\[ a_1x_1 + a_2x_2 + \cdots + a_nx_n = b_1x_1 + b_2x_2 + \cdots + b_nx_n, \]

(1)

where \( x_i (1 \leq i \leq n) \) are the unknowns (taking values in the set of naturals) and \( a_i, b_j (1 \leq i \leq l, 1 \leq j \leq k) \) are the multiplicities of the corresponding variable among \( u_1, \ldots, u_l \) and \( v_1, \ldots, v_k \), respectively. For more details, see the survey of Baader and Snyder on unification [2].

Therefore, algorithms for computing AC-unifiers make intensive use of a linear Diophantine equation (LDE) solver. We compare four algorithms for LDE solving. Our results suggest that the graph-based algorithm is the fastest on modern computers, in contrast to previous benchmarks, which are older. We conclude that implementations of AC-unification should consider switching to the graph-based algorithm.

2 Algorithms

In this section we briefly describe the known algorithms for solving Equation 1.

2.1 The Lexicographic Enumeration Algorithm

In this subsection we describe the simplest way of solving Equation 1. We notice that a linear Diophantine equation with natural solutions can have an infinite number of solutions. We generally do not need all of them, but just need a complete set of minimal solutions. A solution \( S_1 = (x_1, x_2, \ldots, x_n) \) is not minimal if there exists another solution \( S_2 = (x'_1, x'_2, \ldots, x'_n) \) such that for all \( i, S_1,i \geq S_2,i \) and \( S_1 \neq S_2 \). The set of minimal solutions forms a basis.

The lexicographic algorithm [7] lexicographically enumerates all solutions and saves only the minimal ones. However, we can not enumerate infinitely many solutions; we should have a bound for \( x_{a,i} \) and \( x_{b,i} \), where the vectors \( x_a \) and \( x_b \) form a solution of Equation 1. Huet [7] points out that, for a minimal solution, the unknowns \( x_{a,i} \) should be not greater than \( \max(b) \) and the unknowns \( x_{b,i} \) should not be greater than \( \max(a) \), where \( a \) and \( b \) are the coefficients in Equation 1. Lambert [8] gives the stronger bounds \( \sum_i x_{a,i} \leq \max(b) \) and \( \sum_i x_{b,i} \leq \max(a) \). Moreover, we do not need to enumerate the possible values of all \( N \) unknowns, it is enough to enumerate \( N - 1 \); the last unknown can be found by solving a simple equation. We can develop this idea further. What happens if we enumerate \( N - 2 \) variables? We get an equation of form \( ax + by = c \) and we can solve it using the extended Euclidian algorithm. With two types
of bounds and two types of optimizations we get 4 similar algorithms that solve Equation 1. The slowest part of this algorithm is checking if a new solution is minimal or not, because checking minimality requires comparing the new solution with the other already generated minimal solutions. For implementation simplicity we rewrite Equation 1 as:

\[ a_1x_1 + a_2x_2 + \ldots + a_nx_n - b_1x_1 - b_2x_2 - \ldots - b_nx_n = 0 \]  (2)

If we let \( w_i = a_i - b_i \) then Equation 2 can be written as:

\[ w_1x_1 + w_2x_2 + \ldots + w_nx_n = 0 \]  (3)

We work with Equation 3, because it is closer to the implementation. In Figure 1 we provide a very generic implementation for the lexicographic algorithm. The algorithm implements a standard backtracking procedure, with the parameter \( p \) denoting the current unknown. The test \( p \) is last allows to implement the optimizations described above (stop the enumeration at \( n-1 \) unknowns or at \( n-2 \) unknowns).

### 2.2 The Completion Procedure

Another way to solve Equation 1 is to compute all minimal solutions by a completion procedure. Such an algorithm is due to Fortenbacher [5], with an optimization by Guckenbiehl and Herold [6]. For some \( x = (x_1, x_2, \ldots, x_n) \), we denote by \( d(x) \) the result of the expression \( d(x) = w_1x_1 + w_2x_2 + \ldots + w_nx_n \), which we call the defect of Equation 3. A proposal \( p = (x_1, x_2, \ldots, x_n) \) is characterized by \(-\max b \leq d(p) \leq \max a\). A solution is a proposal \( p \) that has \( d(p) = 0 \). The algorithm starts with a set of proposals. At each completion step, it updates every proposal \( p \) in the following way: • if its defect is less than zero, then it increments \( x_i \) by 1 for some index \( i \) with \( w_i > 0 \); • otherwise (if its defect is positive) it increments \( x_i \) by 1 for some \( i \) with \( w_i < 0 \). If the result has defect zero then a minimal solution was found. If a proposal is not minimal then it is discarded, because we can not obtain a minimal solution from a non-minimal proposal. In such a way only minimal solutions are computed and this is an advantage over Lexicographic Algorithm. However, a solution may be computed several times and we still have to test proposals for minimality. Guckenbiehl and Herold [6] describe a way to avoid computation of the same solution several times. To do that we need to select one unique computation for each solution. That is done by following one rule: a proposal with negative (positive) defect must not be incremented at a position \( i \) (position \( j \)) if there exists a \( k > i \) with \( w_k > 0 \) (\( k > j \) with \( w_k < 0 \)). We present this version of the completion procedure in Figure 2.
1 Function CompAlg(\(w\)):
2     // init the proposal set
3     pSet = empty set of proposals
4 for ( \(i = 0; \ i < n; \ i = i + 1\) )
5         p = new proposal
6         p[i] = 1
7         pSet.add(p)
8 while pSet is not empty do
9     // completion step
10    pSetNew = a new proposal set
11 for ( \(p \) in pSet )
12    for ( \(i = n - 1; \ i \geq 0; \ i = i - 1\) )
13 if ( \(d(p) < 0\) and \(w_i < 0\) ) or ( \(d(p) > 0\) and \(w_i > 0\) ) then
14        continue
15    auxProposal = copyOf(p)
16    auxProposal[i] = auxProposal[i] + 1
17 if auxProposal has defect zero then
18       addSolution(auxProposal)
19 else
20       if auxProposal is minimal then
21          pSetNew.add(auxProposal)
22 end
23 // avoiding multiple computations of the same solution
24 if \(p[i] > 0\) then
25          break
26 end
27 pSet = pSetNew

Figure 2: The Completion Procedure described by Fortenbacher [5], with an optimization by Guckenbiehl and Herold [6].

2.3 The Graph Algorithm

Clausen and Fortenbacher [3] described a further optimization of the completion procedure. We call the resulting algorithm the \textit{Graph Algorithm}. In order to avoid all additions and subtractions, they represent Equation 3 as a graph. The graph representation of a linear Diophantine equation is a labelled digraph with the set \(\{d \in \mathbb{Z} | -\max b \leq d \leq \max a\}\) representing the nodes and set \(\{d \to w_i, d + w_i | i \leq n\}\) representing labelled edges. In other words, the nodes are the defect of any proposal and an edge \(d \to w_i, d + w_i\) corresponds to incrementing a proposal at position \(i\). A solution in this graph is a walk that begins in zero and ends in zero. An advantage over the \textit{completion procedure} is that the minimality check of a solution is also transformed into a graph problem. In this graph, a walk that corresponds to the solution \(s = (s_1, s_2, \ldots, s_n)\) is non-minimal if there is another walk \(z = (z_1, z_2, \ldots, z_n)\) that is shorter than \(s\) and also is bounded by \(s\), in other words \(z_i \leq s_i\) for all \(i\). A detailed implementation of this algorithm written in Pascal is provided by Clausen and Fortenbacher [3].
A Note on the Performance of Algorithms for Solving LDE in the Naturals

Motroî and Ciobăcă

2.4 The Slopes Algorithm

The Slopes Algorithm described by Filgueiras and Tomás [4] is an optimization of the lexicographic algorithm. The enumeration is performed for all but three of the unknowns and an equation of the following form:

$$ax = by + cz + v, \quad a, b, c, x, y, z \in \mathbb{N}, \quad v \in \mathbb{Z}$$

is solved. Filgueiras and Tomás [4] describe a way of finding directly all minimal solutions for Equation 4. The idea is that, if minimal solutions of Equation 4 are ordered with $z$ strictly increasing, then both the solution with the smallest $z$ and the difference between consecutive solutions can be computed algebraically. Geometrically, this can be seen as a Pareto frontier of all solutions projected onto the $YZ$-plane if $v \geq 0$ and is a polygonal line when $v < 0$. We can project solutions to 2D space because it is well known that each solution of Equation 4 verifies the congruence:

$$by + cz \equiv -v \pmod{a}.$$  \tag{5}

And reciprocally, that each solution of Equation 5 corresponds to some integral solution of Equation 4. In Figure 3 we present the implementation of Slopes algorithm for solving $ax + by + cz = 0$.

3 Methodology

We have implemented all algorithms in C++. The first two algorithms are clearly slower than the last two. Therefore, we made a more detailed comparison between the Graph and Slopes algorithms, reproducing the comparison by Filgueiras and Tomás [4]. As our implementation of the Slopes algorithm is somewhat slower than the well known and optimized C implementation [1], we use the later for the comparison. All of our code, including instructions for reproducing our results (Figures 1-4), are available at https://github.com/Djok216/LDEAlgsComparison.

Figure 3: The Slopes algorithm of Filgueiras and Tomás for the equation $ax = by + cz$. 

```plaintext
1 Function Slopes(a, b, c):
2 gb=gcd(a,b); gc=gcd(a,c); G=gcd(gb,c);
3 ymax=a/gb; zmax=a/gc;
4 dz=gb/G; dy=(c*multiplier(b,a)/G) mod ymax;
5 y=ymax-dy; z=dz;
6 Solutions = { (b/gb, ymax, 0), (c/gc, 0, zmax), ((b*y+c*z)/a, y, z) };
7 while dy > 0 do
8     while y > dy do
9         y=y-dy; z=z+dz;
10         Solutions.add(((b*y)+c*z)/a,y,z);
11     end
12     f=dy/y; dy=dy mod y; dz=f*z+dz;
13 end
14 return Solutions;
15 // multiplier(a, b) is an integer $m_b$ such that $gcd(a, b) = m_a * a + m_b * b$
```


To measure the running time we use the python `subprocess` and `time` libraries. The first one is used for spawning the executables of the algorithms and the later for measuring running time using `perfCounter`. We set a timeout of 10 minutes for the spawned processes. The tests are generated using `random.randint` and the left side is sorted in decreasing order and the right side in increasing order, because in most cases this ordering speeds up the Slopes algorithm as explained by Filgueiras and Tomás [4].

The tests are divided in 160 classes determined by \( N \in \{1, 2, 3, 4\} \) - the number of unknowns on the left hand side, \( M \in \{2, 3, 4, 5, 6, 7, 8, 9\} \) - the number of unknowns on the right hand side such that \( N \leq M \) and \( MaxValue \in \{2, 3, 5, 13, 29, 39, 107, 503, 1021\} \) - the maximum coefficient of any unknown. We manually set `MaxValue` as part of the coefficients on the right hand side, because there are always more unknowns on the right hand side than left hand side \( (N \leq M) \). Every class contains 10 different tests generated randomly. We use a fixed seed for reproducibility purposes.

We calculate the running time after running the same test with the same algorithm 5 times. The running time for that algorithm on that specific test is considered to be the arithmetic mean of 3 out of 5 remaining values after removing the smallest and the biggest value. The exception is when an algorithm runs for more than 15 seconds. In this case we stop running the same test and calculate the arithmetic mean of running times available at the moment. For example, if an algorithm runs two times in 14.9 seconds and the third time in 15.2, then we stop running this test after third run and the time is considered to be \((2 \times 14.9 + 15.2)/3 = 15\) seconds. We also set a timeout of 10 minutes, after which we automatically stop the algorithm.

For every test in a given class, we add 1 point to the algorithm taking the least time and 0 points to the other. In case of a tie, we add 0.5 to both algorithms. Therefore, a score of 6 : 4 would mean that the first algorithm performed better 6 times, while the second algorithm 4 times out of the 10 tests for a given class.

We consider an algorithm to win a particular class if it scores at least 8 points. The definition of a win is justified statistically by Filgueiras and Tomás [4].

For compiling the code we use `GCC 5.4`. Below are the commands used to compile the programs:

```
gcc -static slopesV7i.c -std=c11 -O3 -o slopesV7i
g++ -static -lm -s -x c++ -std=c++17 -O3 -o graph graph.cpp
```

We run the benchmark on an Intel Xeon machine with two processors and 24 hardware threads (12 physical cores) with a clock speed of 2.67 GHz.

We repeat the measurements made by Filgueiras and Tomás [4], but we also compare the two algorithms using an epsilon of 0.01. By this we mean that the algorithms are considered equally fast if their running times differ by a value smaller than 0.01 seconds. Moreover, we also compute the overall time spent in every class of tests.

4 Discussion

Figure 4 contains a summary of the results. The Slopes algorithm wins 103 classes out of 160 and Graph algorithm wins 7 classes. Filgueiras and Tomás [4] find that Slopes wins 88 classes and Graph wins 33 classes. These results suggest that the Slopes algorithm is faster than the Graph algorithm.

We redo the same comparison, but this time we consider the two algorithms to be equal if their running time differs by at most \( \epsilon = 0.01 \) seconds. The results are summarized in Figure 5.
Each algorithm now has 6 wins and for most of the classes there is a tie. This means that the algorithms are quasi-equal in efficiency.

Going further, we analyze the total time spent in each test class by each algorithm. The results are summarized in Figure 7. We see that in classes where the Slopes algorithm wins, the difference is very small. However, in the classes in which the Graph algorithm wins, the difference is huge. The total time spent in all 160 classes for the Slopes algorithm is 4284.81 seconds and 724 seconds for the Graph algorithm. The counts (4284.81 seconds, 724 seconds) should be interpreted taking into account that they contain 1 timeout of 10 minutes for Graph and 4 timeouts of 10 minutes for Slopes, as summarized in Figure 6.

Based on our results, we conclude that the Graph algorithm is significantly faster than Slopes for bigger instances and roughly as fast for small instances.

**Practical relevance of our benchmark.** In most cases, the bottleneck in AC(U)-unification algorithms is combining the solutions to the linear Diophantine equations themselves. However, there are AC(U)-unification problems where solving Equation 1 is the slow part. An example is an AC-unification problem with a single AC-function $f$ and 8 different variables, which can be constructed based on Equation 6:

$$104x_1 + 167x_2 = 165x_3 + 154x_4 + 148x_5 + 159x_6 + 174x_7 + 150x_8.$$

The ACU-unification problem is the following:

$$f_{104}(u_1) + f_{167}(u_2) = f_{165}(u_3) + f_{154}(u_4) + f_{148}(u_5) + f_{159}(u_6) + f_{174}(u_7) + f_{150}(u_8),$$

where $f_k(v) = f(v, v, \ldots, v)$ (such that $v$ has $k$ occurrences).

Equation 6 has a basis of size 5510. Finding the basis is significantly slower than combining its solutions and creating the ACU-unifier. On the same hardware as described in Section 3, the Graph algorithm takes about 0.6 seconds to solve the linear Diophantine equation above, while combining the solutions into an ACU-unifier takes 0.15 seconds. To compute the ACU-unifier, we use the algorithm presented by Baader and Snyder in their survey on unification [2]. Therefore, at least on some AC-unification problems, solving LDEs dominates the running time.

**Conclusion.** Implementations of AC unification should therefore consider using the Graph algorithm, or choosing between Graph and Slopes, depending on problem size.

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**References**

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A Note on the Performance of Algorithms for Solving LDE in the Naturals

Motroiu and Ciobăcă

| N | M | A | 2 | 3 | 5 | 13 | 29 | 39 | 107 | 503 | 1021 |
|---|---|---|---|---|---|----|----|----|-----|-----|-----|
| 1 | 2 | 2.8| 2.8| 2.8| 5.5| 2.8| 4.6| 4.6| 1.9 | 0.10|     |
| 1 | 3 | 2.8| 2.8| 3.7| 2.8| 1.9| 3.7| 1.9| 0.10| 0.10|     |
| 1 | 4 | 3.7| 3.7| 2.8| 2.8| 1.9| 0.10| 0.10|     |     |     |
| 1 | 5 | 4.6| 2.8| 2.8| 2.8| 0.10| 3.7| 1.9| 0.10| 0.10|     |
| 1 | 6 | 3.7| 3.7| 2.8| 1.9| 0.10| 1.9| 0.10|     |     |     |
| 1 | 7 | 4.6| 4.6| 3.7| 2.8| 1.9| 0.10| 0.10|     |     |     |
| 1 | 8 | 1.9| 2.8| 3.7| 0.10| 0.10| 0.10|     |     |     |     |
| 1 | 9 | 4.6| 2.8| 1.9| 1.9| 2.8| 1.9|     |     |     |     |
| 2 | 2 | 3.7| 1.9| 1.9| 2.8| 1.9| 0.10| 1.9| 0.10| 0.10|     |
| 2 | 3 | 3.7| 2.8| 2.8| 1.9| 0.10| 2.8| 4.6| 10.0| 7.3 |     |
| 2 | 4 | 2.8| 3.7| 3.7| 0.10| 1.9| 3.7| 6.4| 9.1 | 9.1 |     |
| 2 | 5 | 5.5| 5.5| 3.7| 3.7| 3.7| 6.4| 9.1|     |     |     |
| 2 | 6 | 2.8| 2.8| 0.10| 0.10| 2.8| 5.5| 6.4|     |     |     |
| 2 | 7 | 4.6| 1.9| 3.7| 0.10| 1.9| 6.4|     |     |     |     |
| 2 | 8 | 4.6| 4.6| 1.9| 2.8| 4.6| 9.1|     |     |     |     |
| 3 | 3 | 2.8| 0.10| 2.8| 0.10| 0.10| 2.8| 2.8| 7.3 |     |     |
| 3 | 4 | 2.8| 3.7| 3.7| 2.8| 0.10| 2.8| 4.6|     |     |     |
| 3 | 5 | 4.6| 2.8| 2.8| 1.9| 2.8| 5.5| 9.1|     |     |     |
| 3 | 6 | 2.8| 2.8| 1.9| 0.10| 3.7| 4.6|     |     |     |     |
| 4 | 4 | 4.6| 2.8| 1.9| 0.10| 2.8| 0.10| 5.5|     |     |     |
| 4 | 5 | 3.7| 2.8| 1.9| 0.10| 2.8| 5.5|     |     |     |     |

Figure 4: Comparison of Graph and Slopes algorithms. For each of the 160 test classes characterized by \(N, M\) and \(A\) (MaxValue), we count the number of times Graph is faster versus the number of times Slopes is faster out of 10 tests for each class.

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A Note on the Performance of Algorithms for Solving LDE in the Naturals

|   |   | 2  | 3  | 5  | 13 | 29 | 39 | 107 | 503 | 1021 |
|---|---|----|----|----|----|----|----|-----|-----|-----|
|1  | 2 | 5:5| 5:5| 5:5| 5:5| 5:5| 5:5| 4:5 | 4:5 |     |
|1  | 3 | 5:5| 5:5| 5:5| 5:5| 5:5| 5:5| 4:5 | 4:5 |     |
|1  | 4 | 5:5| 5:5| 5:5| 5:5| 5:5| 5:5| 2:7 | 1:9 |     |
|1  | 5 | 5:5| 5:5| 5:5| 5:5| 5:5| 5:5| 4:5 | 0:9 | 0:10|
|1  | 6 | 5:5| 5:5| 5:5| 5:5| 5:5| 5:5| 2:7 | 1:9 |     |
|1  | 7 | 5:5| 5:5| 5:5| 5:5| 5:5| 5:5| 4:5 | 0:9 |     |
|1  | 8 | 5:5| 5:5| 5:5| 5:5| 5:5| 5:5| 4:6 | 4:5 | 2:8 |
|1  | 9 | 5:5| 5:5| 5:5| 5:5| 5:5| 5:5| 4:6 | 3:7 |     |
|2  | 2 | 5:5| 5:5| 5:5| 5:5| 5:5| 5:5| 4:5 | 3:6 |     |
|2  | 3 | 5:5| 5:5| 5:5| 5:5| 5:5| 5:5| 9:0 | 6:3 |     |
|2  | 4 | 5:5| 5:5| 5:5| 5:5| 5:5| 5:5| 7:3 | 9:1 | 9:1 |
|2  | 5 | 5:5| 5:5| 5:5| 5:5| 5:5| 5:5| 9:1 | 9:1 |     |
|2  | 6 | 5:5| 5:5| 5:5| 5:5| 5:5| 5:5| 6:3 |     |     |
|2  | 7 | 5:5| 5:5| 5:5| 5:5| 5:5| 5:5|     |     |     |
|2  | 8 | 5:5| 5:5| 5:5| 5:5| 5:4| 6:4 | 6:3 |     |     |
|3  | 3 | 5:5| 5:5| 5:5| 5:5| 5:5| 5:5| 4:5 | 7:3 |     |
|3  | 4 | 5:5| 5:5| 5:5| 5:5| 5:5| 5:5| 3:6 |     |     |
|3  | 5 | 5:5| 5:5| 5:5| 5:5| 5:5| 5:4| 9:0 |     |     |
|3  | 6 | 5:5| 5:5| 5:5| 5:5| 5:5| 5:4| 4:5 |     |     |
|4  | 4 | 5:5| 5:5| 5:5| 5:5| 5:4| 4:6 | 5:5 |     |     |
|4  | 5 | 5:5| 5:5| 5:5| 5:5| 5:5| 6:3 |     |     |     |

Figure 5: Comparison of the Graph and Slopes algorithms. Same as the previous figure, but the algorithms are considered tied on tests on which their running times differ by at most 0.01 seconds.
A Note on the Performance of Algorithms for Solving LDE in the Naturals

| N | 2 | 3 | 5 | 13 | 29 | 39 | 107 | 503 | 1021 |
|---|---|---|---|----|----|----|-----|-----|-----|
| 1 | 2 | 0:0| 0:0| 0:0| 0:0| 0:0| 0:0 | 0:0 | 0:0 |
| 1 | 3 | 0:0| 0:0| 0:0| 0:0| 0:0| 0:0 | 0:0 | 0:0 |
| 1 | 4 | 0:0| 0:0| 0:0| 0:0| 0:0| 0:0 | 0:0 | 0:0 |
| 1 | 5 | 0:0| 0:0| 0:0| 0:0| 0:0| 0:0 | 0:0 | 0:0 |
| 1 | 6 | 0:0| 0:0| 0:0| 0:0| 0:0| 0:0 | 0:0 | 0:0 |
| 1 | 7 | 0:0| 0:0| 0:0| 0:0| 0:0| 0:0 | 0:0 | 0:0 |
| 1 | 8 | 0:0| 0:0| 0:0| 0:0| 0:0| 0:0 | 0:0 | 0:0 |
| 1 | 9 | 0:0| 0:0| 0:0| 0:0| 0:0| 0:0 | 0:0 | 0:0 |
| 2 | 2 | 0:0| 0:0| 0:0| 0:0| 0:0| 0:0 | 0:0 | 0:0 |
| 2 | 3 | 0:0| 0:0| 0:0| 0:0| 0:0| 0:0 | 0:0 | 0:0 |
| 2 | 4 | 0:0| 0:0| 0:0| 0:0| 0:0| 0:0 | 1:0 | 0:0 |
| 2 | 5 | 0:0| 0:0| 0:0| 0:0| 0:0| 0:0 | 0:4 | 0:0 |
| 2 | 6 | 0:0| 0:0| 0:0| 0:0| 0:0| 0:0 | 0:0 | 0:0 |
| 2 | 7 | 0:0| 0:0| 0:0| 0:0| 0:0| 0:0 | 0:0 | 0:0 |
| 2 | 8 | 0:0| 0:0| 0:0| 0:0| 0:0| 0:0 | 0:0 | 0:0 |
| 3 | 3 | 0:0| 0:0| 0:0| 0:0| 0:0| 0:0 | 0:0 | 0:0 |
| 3 | 4 | 0:0| 0:0| 0:0| 0:0| 0:0| 0:0 | 0:0 | 0:0 |
| 3 | 5 | 0:0| 0:0| 0:0| 0:0| 0:0| 0:0 | 0:0 | 0:0 |
| 3 | 6 | 0:0| 0:0| 0:0| 0:0| 0:0| 0:0 | 0:0 | 0:0 |
| 4 | 4 | 0:0| 0:0| 0:0| 0:0| 0:0| 0:0 | 0:0 | 0:0 |
| 4 | 5 | 0:0| 0:0| 0:0| 0:0| 0:0| 0:0 | 0:0 | 0:0 |

Figure 6: Number of timeouts (10 minutes) for the Graph and Slopes algorithms on each class.
Figure 7: The total time, expressed in seconds, spent by each algorithm (Graph and Slopes) on all tests for each class. Columns with $A < 13$ are excluded because the values are very close to zero.