THE ODD PRIMARY ORDER OF THE COMMUTATOR ON LOW RANK LIE GROUPS

TSELEUNG SO

Abstract. Let $G$ be a simple, simply-connected, compact Lie group of low rank relative to a fixed prime $p$. After localization at $p$, there is a space $A$ which “generates” $G$ in a certain sense. Assuming $G$ satisfies a homotopy nilpotency condition relative to $p$, we show that the Samelson product $\langle 1_G, 1_G \rangle$ of the identity of $G$ equals the order of the Samelson product $\langle i, i \rangle$ of the inclusion $i : A \to G$. Applying this result, we calculate the orders of $\langle 1_G, 1_G \rangle$ for all $p$-regular Lie groups and give bounds of the orders of $\langle 1_G, 1_G \rangle$ for certain quasi-$p$-regular Lie groups.

1. Introduction

In this paper, $G$ is a simple, simply-connected, compact Lie group and $p$ is an odd prime. By a theorem of Hopf, $G$ is rationally homotopy equivalent to a product of spheres $\prod_{i=1}^{l} S^{2n_i-1}$, where $n_1 \leq \cdots \leq n_l$. The sequence $(2n_1 - 1, \cdots, 2n_l - 1)$ is called the type of $G$. Localized at $p$, it is known [2, 7] that $G$ is homotopy equivalent to a product of $H$-spaces $\prod_{i=1}^{l} B_i$, and there exists a co-$H$-space $A$ and a map $i : A \to G$ such that $H_i(G)$ is the exterior algebra generated by $i_*(\tilde{H}_i(A))$. For $1 \leq i \leq l$, if $B_i$ is $S^{2n_i-1}$, then we call $G$ $p$-regular. If each $B_i$ is either $S^{2n_i-1}$ or $B(2n_i - 1, 2n_i + 2p - 3)$, then we call $G$ quasi-$p$-regular.

For any maps $f : X \to G$ and $g : Y \to G$, let $c(f,g) : X \times Y \to G$ be a map sending $(x,y) \in X \times Y$ to their commutator $[x,y] = f(x)^{-1}g(y)^{-1}f(x)g(y)$. Then $c(f,g)$ descend to a map $\langle f, g \rangle : X \wedge Y \to G$. The map $\langle f, g \rangle$ is called the Samelson product of $f$ and $g$. The order of $\langle f, g \rangle$ is defined to be the minimum number $k$ such that the composition

$$k \circ \langle f, g \rangle : X \wedge Y \overset{\langle f, g \rangle}{\to} G \overset{k}{\to} G$$

is null-homotopic, where $k : G \to G$ is the $k$th-power map. In particular, when $f$ and $g$ are the identity map $1_G$ of $G$, the Samelson product $\langle 1_G, 1_G \rangle$ is universal and we are interested in finding its order.

There is a notion of nilpotency in homotopy theory analogous to that for groups. Let $c_1$ be the commutator map $c(1_G, 1_G) : G \times G \to G$, and let $c_n = c_1 \circ (c_{n-1} \times 1_G)$ be the $n$-iterated commutator for $n > 1$. The homotopy nilpotence class of $G$ is the number $n$ such that $c_n$ is null-homotopic but $c_{n-1}$ is not. In certain cases the homotopy nilpotence class of $p$-localized $G$ is known. Kaji and Kishimoto [3] showed that $p$-regular Lie groups have homotopy nilpotence class at most 3. When $G$ is quasi-$p$-regular and $p \geq 7$, Kishimoto [4] showed that $SU(n)$ has homotopy nilpotence class at most 3, and Theriault [10] showed that exceptional Lie groups have homotopy nilpotence class at most 2.

2010 Mathematics Subject Classification. 55Q15, 57T20.

Key words and phrases. Lie group, Samelson products, homotopy nilpotence.
Here we restrict \( G \) to be a Lie group having low rank with respective to an odd prime \( p \). That is, \( G \) and \( p \) satisfy:

\[
\begin{align*}
SU(n) & : n \leq (p - 1)(p - 2) + 1 \\
Sp(n) & : 2n \leq (p - 1)(p - 2) \\
Spin(2n + 1) & : 2n \leq (p - 1)(p - 2) \\
Spin(2n) & : 2n - 2 \leq (p - 1)(p - 2) \\
G_2, F_4, E_6 & : p \geq 5 \\
E_7, E_8 & : p \geq 7,
\end{align*}
\]

In these cases, Theriault [9] showed that \( \Sigma A \) is a retract of \( \Sigma G \). Let \( \langle i, i \rangle \) be the composition

\[
\langle i, i \rangle : A \wedge A \xrightarrow{i \wedge i} G \wedge G \xrightarrow{\langle 1_G, 1_G \rangle} G.
\]

Then obviously the order of \( \langle 1_G, 1_G \rangle \) is always greater than or equal to the order of \( \langle i, i \rangle \). Conversely, we show that the order of \( \langle i, i \rangle \) restricts the order of \( \langle 1_G, 1_G \rangle \) under certain conditions.

**Theorem 1.1.** Let \( G \) be a compact, simply-connected, simple Lie group of low rank and let \( p \) be an odd prime. Localized at \( p \), if the homotopy nilpotence class of \( G \) is less than \( p^r + 1 \), then the order of the Samelson product \( \langle 1_G, 1_G \rangle \) is \( p^r \) if and only if the order of \( \langle i, i \rangle \) is \( p^r \).

The strategy for proving Theorem 1.1 is to extend \( A \to G \) to an H-map \( \Omega \Sigma A \to G \) which has a right homotopy inverse, that is to retract \( [G \wedge G, G] \) off \( [\Omega \Sigma A \wedge \Omega \Sigma A, G] \), and use commutator calculus to analyze the latter. Combine Theorem 1.1 and the known results in [3, 4, 10] to get the following statement.

**Corollary 1.2.** The order of \( \langle 1_G, 1_G \rangle \) equals the order of \( \langle i, i \rangle \) when

- \( G \) is \( p \)-regular or;
- \( p \geq 7 \) and \( G \) is a quasi-\( p \)-regular Lie group which is one of \( SU(n), F_4, E_6, E_7 \) or \( E_8 \).

On the one hand, there is no good method to calculate the order of \( \langle 1_G, 1_G \rangle \) in general. A direct computation is not practical since one has to consider all the cells in \( G \wedge G \) and their number grows rapidly when there is a slight increase in the rank of \( G \). On the other hand, Corollary 1.2 says that for all \( p \)-regular Lie groups and most of the quasi-\( p \)-regular Lie groups, we can determine the order of \( \langle 1_G, 1_G \rangle \) by computing the order of \( \langle i, i \rangle \). The latter is easier to work with since \( A \) has a much simpler CW-structure than \( G \). To demonstrate the power of Theorem 1.1, we apply this result to compute the order of \( \langle 1_G, 1_G \rangle \) for all \( p \)-regular cases and some quasi-\( p \)-regular cases.
Theorem 1.3. For a $p$-localized Lie group $G$, the order of $\langle 1_G, 1_G \rangle$ is $p$ when

| $G$            | $r = 0$ | $r = 1$ |
|----------------|---------|---------|
| $SU(n)$        | $p > 2n$| $n \leq p < 2n$ |
| $Sp(n)$        | $p > 4n$| $2n < p < 4n$ |
| $Spin(2n + 1)$ | $p > 4n$| $2n < p < 4n$ |
| $Spin(2n)$     | $p > 4n - 4$| $2n - 2 < p < 4n - 4$ |
| $G_2$          | $p = 5, p > 11$| $p = 7, 11$ |
| $F_4, E_6$     | $p > 23$| $11 \leq p \leq 23$ |
| $E_7$          | $p > 31$| $17 \leq p \leq 31$ |
| $E_8$          | $p > 59$| $23 \leq p \leq 59$ |

For many other quasi-$p$-regular cases, we give rough bounds on the order of $\langle 1_G, 1_G \rangle$ by bounding the order of $\langle \iota, \iota \rangle$.

Here is the structure of this paper. In Section 2 we prove Theorem 1.1 assuming Lemma 2.5, whose proof is given in Section 3 because of its length. Section 3 is divided into two parts. In the first part we consider the algebraic properties of Samelson products and in the second part we use algebraic methods to prove Lemma 2.5. In Section 4 we apply Theorem 1.1 and use other known results to calculate bounds on the order of the Samelson product $\langle 1_G, 1_G \rangle$ for quasi-$p$-regular Lie groups.

2. Samelson Products of Low Rank Lie Groups

Definition 2.1. Let $G$ be a simple, simply-connected, compact Lie group and $p$ be an odd prime. Localized at $p$, a triple $(A, \iota, G)$ is retractile if $A$ is a co-H-space and a subspace of $G$ and $\iota : A \hookrightarrow G$ is an inclusion such that

- there is an algebra isomorphism $H_*(G) \cong \Lambda(\tilde{H}_*(A))$ of homologies with mod-$p$ coefficients;
- the induced homomorphism $\iota_* : H_*(A) \to H_*(G)$ is an inclusion of the generating set;
- the suspension $\Sigma \iota : \Sigma A \to \Sigma G$ has a left homotopy inverse $t : \Sigma G \to \Sigma A$.

We also refer to $G$ as being retractile for short.

From now on, we take $p$-localization and assume $G$ and $p$ satisfy (1). According to [9], $G$ is retractile. First we want to establish a connection between $G$ and $\Omega \Sigma A$. Consider the homotopy commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\iota} & G \\
\Sigma \downarrow & & \downarrow \iota \\
\Omega \Sigma A & \xrightarrow{\iota} & \Omega \Sigma A
\end{array}
\]

where $\Sigma : A \to \Omega \Sigma A$ is the suspension and $\iota : \Omega \Sigma A \to G$ is an H-map. Since $G$ is retractile, the suspension $\Sigma t : \Sigma A \to \Sigma G$ has a left homotopy inverse $t : \Sigma G \to \Sigma A$. Let $s$ be the composition

\[
s : G \xrightarrow{\Sigma} \Omega \Sigma G \xrightarrow{\Omega t} \Omega \Sigma A.
\]

Lemma 2.1. The map $\iota \circ s$ is a homotopy equivalence.
Proof. Denote the composition $\tilde{i} \circ s$ by $e$ for convenience. Consider the commutative diagram

$$
\begin{array}{ccc}
A & \xrightarrow{i} & G \\
\Sigma & \downarrow & \downarrow \\
\Omega \Sigma A & \xrightarrow{\hat{t}} & \Omega \Sigma G & \xrightarrow{s} & \Omega \Sigma A.
\end{array}
$$

The commutativity of the left square is due to the naturality of the suspension map, and the commutativity of the right triangle follows from the definition of $s$. The bottom row is homotopic to the identity since $t$ is a left homotopy inverse for $\Sigma i$. Hence we have $s \circ i \simeq \Sigma$ and consequently $e \circ i = \tilde{i} \circ s \circ i \simeq \tilde{i} \circ \Sigma$.

By Diagram (2) $\tilde{i} \circ \Sigma$ is homotopic to $i$. This implies that $(e \circ i)_*$ sends $H_*(A)$ onto the generating set of $H_*(G) = \Lambda(\tilde{H}_*(A))$ where we consider the mod-$p$ homology. Dually, $(e \circ i)^* : H^*(G) \to H^*(A)$ is an epimorphism. The generating set $i^*(H^*(A))$ is in $\text{Im}(e^*)$.

Since $e^* : H^*(G) \to H^*(G)$ is an algebra map, $e^*$ is an epimorphism and hence is an isomorphism. Therefore $e : G \to G$ is a homotopy equivalence. □

We claim that the Samelson product

$$
\langle \tilde{i}, \tilde{i} \rangle : \Omega \Sigma A \wedge \Omega \Sigma A \xrightarrow{\tilde{i} \wedge \tilde{i}} G \wedge G \xrightarrow{(1_G, 1_G)} G
$$

has the same order as $\langle 1_G, 1_G \rangle$.

Lemma 2.2. The map $p^r \circ \langle 1_G, 1_G \rangle$ is null-homotopic if and only if $p^r \circ \langle \tilde{i}, \tilde{i} \rangle$ is null-homotopic.

Proof. The sufficiency part is obvious. We only show the necessity part. Suppose $p^r \circ \langle \tilde{i}, \tilde{i} \rangle$ is null-homotopic. By Lemma 2.1, $e = \tilde{i} \circ s$ is a homotopy equivalence. Composing with its inverse $e'$, the map $\tilde{i} \circ s \circ e'$ is homotopic to the identity. Then we obtain

$$
p^r \circ \langle 1_G, 1_G \rangle \simeq p^r \circ \langle \tilde{i} \circ s \circ e', \tilde{i} \circ s \circ e' \rangle = p^r \circ \langle \tilde{i}, \tilde{i} \rangle \circ (s \circ e' \wedge s \circ e')
$$

which is null-homotopic since $p^r \circ \langle \tilde{i}, \tilde{i} \rangle$ is null-homotopic. □

Combining Diagram 2 and the fact that $\tilde{i}$ is an H-map, we have the commutative diagram

$$
\begin{array}{ccc}
A^k & \xrightarrow{i^k} & G^k \\
\downarrow & & \downarrow \text{id} \\
\Omega \Sigma A^k & \xrightarrow{\hat{t}^k} & \Omega \Sigma G^k & \xrightarrow{\text{id}} & \Omega \Sigma A
\end{array}
$$

where $\mu^k$ and $m^k$ are the $k$-fold multiplications in $\Omega \Sigma A$ and $G$. Let $m_k$ and $e_k$ be the compositions

$$
m_k : A^k \xrightarrow{i^k} G^k \xrightarrow{m^k} G \quad \text{and} \quad e_k : A^k \xrightarrow{j^k} (\Omega \Sigma A)^k \xrightarrow{\mu^k} \Omega \Sigma A.
$$

Then we have the following commutative diagram

$$
\begin{array}{ccc}
A^k \wedge A^l & \xrightarrow{e_k \wedge e_l} & \Omega \Sigma A^k \wedge \Omega \Sigma A^l \\
\downarrow & & \downarrow \text{id} \\
G \wedge G & \xrightarrow{(1_G, 1_G)} & G \wedge G & \xrightarrow{p^r} & G
\end{array}
$$
Observe that there is a string of equalities

\[
[\Omega \Sigma A \land \Omega \Sigma A, G] = [\Sigma \Omega \Sigma A \land \Omega \Sigma A, BG]
\]

\[
= \left[ \bigvee_{k,l=1}^{\infty} \Sigma A^{\land k} \land A^{\land l}, BG \right]
\]

\[
= \prod_{k,l=1}^{\infty} [A^{\land k} \land A^{\land l}, G].
\]

The first and the third lines are due to adjunction, and the second line is due to James splitting \( \Sigma \Omega \Sigma A \simeq \Sigma_{k=1}^{\infty} \Sigma A^{\land k} \). It is not hard to see that the nullity of \( p^r \circ \langle \tilde{i}, \tilde{i} \rangle \) implies the nullity of the components \( p^r \circ \langle m_k, m_l \rangle \). In the following we show that the converse is true.

**Lemma 2.3.** Let \( X \) be a space and let \( f : X \to G \) be a map. If \( p^r \circ \langle m_k, f \rangle : A^k \land X \to G \) is null-homotopic for all \( k \), then \( p^r \circ \langle i, f \rangle : \Omega \Sigma A \land X \to G \) is null-homotopic. Similarly, if \( p^r \circ \langle f, m_l \rangle : X \land A^l \to G \) is null-homotopic for all \( l \), then \( p^r \circ \langle i, f \rangle : X \land \Omega \Sigma A \to G \) is null-homotopic.

**Proof.** We only prove the first statement since the second statement can be proved similarly. Let \( h : \Sigma \Omega \Sigma A \land X \to BG \) be the adjoint of \( p^r \circ \langle i, f \rangle \). It suffices to show that \( h \) is null-homotopic.

For any \( k \), choose a right homotopy inverse \( \psi_k \) of the suspended quotient map \( \Sigma A^k \to \Sigma A^{\land k} \), and let \( \Psi_k \) be the composition

\[
\Psi_k : \Sigma A^{\land k} \xrightarrow{\psi_k} \Sigma A^k \xrightarrow{\Sigma e_k} \Sigma \Omega \Sigma A.
\]

Observe that \( e_k \) is the product of \( k \) copies of the suspension \( j \) and \( j_\ast : H_\ast(A) \to H_\ast(\Omega \Sigma A) \) is the inclusion of the generating set into \( H_\ast(\Omega \Sigma A) \cong T(\tilde{H}_\ast(A)) \). The map \( (e_k)_\ast : H_\ast(A^k) \to H_\ast(\Omega \Sigma A) \) sends the submodule \( S_k \subset H_\ast(A^k) \cong H_\ast(A)^{\oplus k} \) consisting of length \( k \) tensor products onto the submodule \( M_k \subset T(\tilde{H}_\ast(A)) \) consisting of length \( k \) tensor products. Therefore \( (\Psi_k)_\ast \) does the same. Then their sum

\[
\Psi = \bigvee_{k=1}^{\infty} \Psi_k : \bigvee_{k=1}^{\infty} \Sigma A^{\land k} \to \Sigma \Omega \Sigma A
\]

induces a homology isomorphism and hence is a homotopy equivalence.

We claim that \( h \circ (\Psi_k \land 1_X) \) is null-homotopic for all \( k \), where \( 1_X \) is the identity of \( X \). Observe that the adjoint of the composition

\[
\Sigma A^k \land X \xrightarrow{\Sigma e_k \land 1_X} \Sigma \Omega \Sigma A \land X \xrightarrow{h} BG
\]

is \( p^r \circ \langle i, f \rangle \circ (e_k \land 1_X) \simeq p^r \circ \langle i \circ e_k, f \rangle \simeq p^r \circ \langle m_k, f \rangle \) which is null-homotopic by assumption. Therefore \( h \circ (\Psi_k \land 1_X) = h \circ (\Sigma e_k \land 1_X) \circ (\psi_k \land 1_X) \) is null-homotopic, and by definition of \( \Psi \), the composition

\[
\bigvee_{k=1}^{\infty} \Sigma A^{\land k} \land X \xrightarrow{\Psi \land 1} \Sigma \Omega \Sigma A \land X \xrightarrow{h} BG.
\]

is null-homotopic. Notice that \( (\Psi \land 1) \) is a homotopy equivalence. It implies that \( h \) is null-homotopic and so is \( p^r \circ \langle i, f \rangle \). \( \square \)

**Lemma 2.4.** The map \( p^r \circ \langle i, \tilde{i} \rangle \) is null-homotopic if and only if \( p^r \circ \langle m_k, m_l \rangle \) is null-homotopic for all \( k \) and \( l \).
Proof. It suffices to prove the necessity part. Suppose \( p^r \circ \langle m_k, m_l \rangle \) is null-homotopic for all \( k \) and \( l \). Apply the first part of Lemma 2.3 to obtain \( p^r \circ \langle \bar{i}, \bar{i} \rangle \simeq * \) for all \( l \) and apply the second part of Lemma 2.3 to obtain \( p^r \circ \langle \bar{i}, \bar{i} \rangle \simeq * \). □

At this point we have related the order of \( \langle 1, 1 \rangle \) to the orders of \( \langle m_k, m_l \rangle \) for all \( k \) and \( l \). There is one more step to link up with the order of the Samelson product

\[
\langle \bar{i}, \bar{i} \rangle : A \wedge A \xrightarrow{\iota_1} G \wedge G \xrightarrow{\langle 1, 1 \rangle} G \xrightarrow{p^r} G.
\]

Lemma 2.5. If \( p^r \circ \langle \bar{i}, \bar{i} \rangle \) is null-homotopic and \( G \) has homotopy nilpotence class less than \( p^r + 1 \), then \( p^r \circ \langle m_k, m_l \rangle \) is null-homotopic for all \( k \) and \( l \).

The proof of Lemma 2.5 is long and we postpone it to the next section so as to avoid interrupting the flow of our discussion. Assuming Lemma 2.5 we can prove our main theorem.

Theorem 2.6. Suppose that \( G \) has homotopy nilpotence class less than \( p^r + 1 \) after localization at \( p \). Then \( \langle 1, 1 \rangle \) has order \( p^r \) if and only if \( \langle \bar{i}, \bar{i} \rangle \) has order \( p^r \).

Proof. The order of \( \langle 1, 1 \rangle \) is not less than the order of \( \langle \bar{i}, \bar{i} \rangle \). Therefore we need to show that the order of \( \langle 1, 1 \rangle \) is not greater than the order of \( \langle \bar{i}, \bar{i} \rangle \) under the assumption. Assume \( \langle \bar{i}, \bar{i} \rangle \) has order \( p^r \), that is \( p^r \circ \langle \bar{i}, \bar{i} \rangle \) is null-homotopic. Lemmas 2.2, 2.4 and 2.5 imply that \( p^r \circ \langle 1, 1 \rangle \) is null-homotopic, so the order of \( \langle 1, 1 \rangle \) is not greater than the order of \( \langle \bar{i}, \bar{i} \rangle \). □

3. Proof of Lemma 2.5

In this section we prove Lemma 2.5 by showing \( p^r \circ \langle m_k, m_l \rangle \) is null-homotopic assuming the homotopy nilpotence class of \( G \) is less than \( p^r + 1 \). We convert it into an algebraic problem and derive lemmas from group theoretic identities and the topological properties of \( G \). First let us review the algebraic properties of Samelson products.

3.1. Algebraic properties of Samelson products. Given two maps \( f : X \to G \) and \( g : Y \to G \), their Samelson product \( \langle f, g \rangle \) sends \( (x, y) \in X \wedge Y \) to the commutator of their images \( f(x) \) and \( g(y) \). It is natural to regard the map \( \langle f, g \rangle \) as a commutator in \( [X \wedge Y, G] \), but \( f \) and \( g \) are maps in different homotopy sets and there is no direct multiplication between them. Instead, we can include \([X, G], [Y, G] \) and \([X \wedge Y, G] \) into \([X \times Y, G] \) and identify \( \langle f, g \rangle \) as a commutator there.

Lemma 3.1. For any spaces \( X \) and \( Y \), let \( \pi_1 : X \times Y \to X \) and \( \pi_2 : X \times Y \to Y \) be the projections and let \( q : X \times Y \to X \wedge Y \) be the quotient map. Then the images \( (\pi_1)^*[X, G] \), \( (\pi_2)^*[Y, G] \), \( q^*[X \wedge Y, G] \) are subgroups of \([X \times Y, G] \), and \((\pi_1)^* : [X, G] \to [X \times Y, G], (\pi_2)^* : [Y, G] \to [X \times Y, G], q^* : [X \wedge Y, G] \to [X \times Y, G] \) are monomorphisms.

Proof. Observe that \( \pi_1 \) and \( \pi_2 \) induce group homomorphisms \( (\pi_1)^* : [X, G] \to [X \times Y, G] \) and \((\pi_2)^* : [X, G] \to [X \times Y, G] \), so their images \( (\pi_1)^*[X, G] \) and \((\pi_2)^*[Y, G] \) are subgroups of \([X \times Y, G] \). Moreover, let \( j : X \to X \times Y \) be the inclusion. Since \( \pi_1 \circ j \) is the identity, \( j^* \circ (\pi_1)^* \) is an isomorphism and \((\pi_1)^* \) is a monomorphism. Therefore \([X, G] \) is isomorphic to \((\pi_1)^*[X, G] \). Similarly \([Y, G] \) is isomorphic to \((\pi_2)^*[Y, G] \).

For \( q^*[X \wedge Y, G] \), the cofibration \( X \vee Y \xrightarrow{j^\vee} X \times Y \xrightarrow{\partial} X \wedge Y \) induces an exact sequence

\[
\cdots \to [\Sigma X \times Y, G] \xrightarrow{\Sigma j^\vee} [\Sigma(X \vee Y), G] \to [X \wedge Y, G] \xrightarrow{q^*} [X \times Y, G] \xrightarrow{j^*} [X \vee Y, G],
\]

where \( \partial^* \) denotes the boundary map.
where $j'$ is the inclusion and $q$ is the quotient map. Since $\Sigma j' : \Sigma(X \vee Y) \hookrightarrow \Sigma(X \times Y)$ has a right homotopy inverse, $q^* : [X \wedge Y, G] \to [X \times Y, G]$ is a monomorphism. Therefore $[X \wedge Y, G]$ is isomorphic to $q^*[X \wedge Y, G]$, which is a subgroup of $[X \times Y, G]$. \hfill \Box

There are two groups in our discussion, namely $G$ and $[X \times Y, G]$. To distinguish their commutators, for any maps $f : X \to G$ and $g : Y \to G$ we use $c(f, g)$ to denote the map which sends $(x, y) \in X \times Y$ to $f(x)^{-1}g(y)^{-1}f(x)g(y) \in G$, and for any maps $a : X \times Y \to G$ and $b : X \times Y \to G$ we use $[a, b]$ to denote the commutator $a^{-1}b^{-1}ab \in [X \times Y, G]$. Lemma 3.1 says that $f : X \to G$ and $g : Y \to G$ can be viewed as being in $[X \times Y, G]$. Their images are the compositions

$$\tilde{f} : X \times Y \xrightarrow{\pi_1} X \xrightarrow{f} G \text{ and } \tilde{g} : X \times Y \xrightarrow{\pi_2} Y \xrightarrow{g} G.$$  

Consider the diagram

\[ 
\begin{array}{ccc}
X \times Y & \xrightarrow{q} & X \wedge Y \\
\downarrow{f \times g} & & \downarrow{f \wedge g} \\
(X \times Y) \times (X \times Y) & \xrightarrow{\tilde{f} \times \tilde{g}} & G \times G \\
\downarrow{c} & & \downarrow{\langle 1_G, 1_G \rangle} \\
G & = & G \\
\end{array}
\]

where $\Delta$ is the diagonal map, $c$ is the commutator map $c(1_G, 1_G)$, and $q'$ is the quotient maps. The commutativity of the left triangle is due to the definitions of $\tilde{f}$ and $\tilde{g}$, the commutativity of the top square is due to the naturality of the quotient maps and the commutativity of the bottom square is due to the definition of $\langle 1_G, 1_G \rangle$. The middle column is $c(f, g)$ and the right column is $\langle f, g \rangle$. In order to show that $\langle f, g \rangle$ is null-homotopic, it suffices to consider $c(f, g) \simeq q^* \langle f, g \rangle$ since $q^* : [X \wedge Y, G] \to [X \times Y, G]$ is injective. Observe that $c(f, g)$ is homotopic to the composition

$$X \times Y \xrightarrow{\Delta} (X \times Y) \times (X \times Y) \xrightarrow{\tilde{f} \times \tilde{g}} G \times G \xrightarrow{c} G$$

according to the diagram. That is $c(f, g)$ is the commutator $[\tilde{f}, \tilde{g}] = \tilde{f}^{-1}\tilde{g}^{-1}\tilde{f}\tilde{g}$ in $[X \times Y, G]$.

Let $Ad_{b\cdot a} = b^{-1}ab$ be the conjugation of maps $a$ and $b$ in $[X \times Y, G]$. In group theory, commutators satisfy the following identities:

\begin{align*}
(1) & \quad [a, b] = a^{-1} \cdot Ad_{b\cdot a}; \\
(2) & \quad [a, b]^{-1} = [b, a]; \\
(3) & \quad [a', b] = Ad_{a'}[a, b] \cdot [a', b] = [a, b] \cdot [a, b]^{-1} \cdot [a', b]; \\
(4) & \quad [a, b \cdot b'] = [a, b'] \cdot Ad_{b'}[a, b] = [a, b'] \cdot [a, b] \cdot [a, b, b'].
\end{align*}

In particular, we can substitute $\tilde{f}$ and $\tilde{g}$ to $a$ and $b$ in these identities.

**Proposition 3.2.** Let $f, f' : X \to G$ and $g, g' : Y \to G$ be maps. Then in $[X \times Y, G]$,

(i) $c(f, g) = \tilde{f}^{-1} \cdot Ad_{\tilde{g}\tilde{f}'}$;

(ii) $c(f, g)^{-1} = c(g, f) \circ T$;

(iii) $c(f \cdot f', g) = Ad_{\tilde{f}'}c(f, g) \cdot c(f', g) = c(f, g) \cdot \tilde{f}' \cdot [\tilde{f}, [\tilde{f}, \tilde{g}]]^{-1} \cdot c(f', g)$;

(iv) $c(f, g \cdot g') = c(f, g') \cdot Ad_{\tilde{g}'}c(f, g) = c(f, g') \cdot Ad_{\tilde{g}'} \cdot c(f, g) \cdot [\tilde{f}, \tilde{g}, \tilde{g}']$,

where $T : Y \times X \to X \times Y$ is the swapping map.
Proof. All identities come directly from the identities in (4), while Identity (ii) needs some explanation. Observe there exists a homotopy commutative diagram

\[
\begin{array}{c}
X \times Y \xrightarrow{f \times g} G \times G \xrightarrow{e} G \\
\downarrow T \quad \quad \quad \downarrow T \quad \quad \quad \downarrow r \\
Y \times X \xrightarrow{g \times f} G \times G \xrightarrow{e} G
\end{array}
\]

where \( r : G \to G \) is the inversion. The upper direction around the diagram is \( c(f, g)^{-1} \), while the lower direction is \( c(g, f) \circ T \). So Identity (ii) follows. \( \square \)

Remark: The iterated commutator \([\tilde{f}'(x), [\tilde{f}(x), \tilde{g}(y)]]\) is the composition

\[
X \times Y \xrightarrow{\Delta_X \times 1_Y} X \times X \times Y \xrightarrow{f' \times f \times g} G \times G \times G \xrightarrow{1_G \times c} G \times G \xrightarrow{c} G
\]

where \( \Delta_X \) is the diagonal map and \( 1_Y \) and \( 1_G \) are the identity maps. Let \( c_2 = c \circ (1_G \times c) \) be the 2-iterated commutator on \( G \). Then we can write \([\tilde{f}', [\tilde{f}, \tilde{g}]]\) as \( c_2 \circ (f' \times f \times g) \circ (\Delta_X \times 1_Y) \). However, we prefer to stick to the notation \([\tilde{f}', [\tilde{f}, \tilde{g}]\] because it better indicates it is the commutator of which maps, while \( c_2 \circ (f' \times f \times g) \circ (\Delta_X \times 1_Y) \) looks long and confusing.

Since our group \([X \times Y, G]\) has a topological interpretation, the topologies of \( X \) and \( Y \) add extra algebraic properties to its group structure.

Lemma 3.3. Let \( f, g \) and \( h : X \times Y \to G \) be maps. If \( X \) is a co-H-space and the restrictions of \( f \) and \( g \) to \( X \vee Y \) are null-homotopic, then in \([X \times Y, G]\) we have

\[ f \cdot g = g \cdot f \quad \text{and} \quad [f \cdot g, h] = [f, h] \cdot [g, h]. \]

Proof. Let \( q : X \times Y \to X \wedge Y \) be the quotient map. Observe that there exist \( f' \) and \( g' \) in \([X \wedge Y, G]\) such that \( f = q^* f' \) and \( g = q^* g' \). Since \( X \wedge Y \) is a co-H-space, \([X \wedge Y, G]\) is an abelian group and \( f' \) and \( g' \) commute. Therefore \( f \) and \( g \) commute as \( q^* \) is a monomorphism.

To show the linearity, we start with Proposition 3.2 (iii)

\[ [f \cdot g, h] = [f, h] \cdot [g, h] \cdot [g, h]. \]

Since \([f, h]\) is also null-homotopic on \( X \vee Y \), it commutes with \( g \) and their commutator \([g, [f, h]]\) is trivial. Therefore we have \([f \cdot g, h] = [f, h] \cdot [g, h]. \)

3.2. Main body of the proof. We go back to the proof of Lemma 2.5. Recall that \( m_k \) is the composition \( m_k : A^k \xrightarrow{\iota} G^k \xrightarrow{m_k} G \). To distinguish the spaces \( A \)'s, denote the \( i \)th copy of \( A \) in \( A^k \) by \( A_i \). Let \( a_i \) and \( m_{k-1}' \) be the compositions

\[ a_i : A^k \xrightarrow{proj} A_i \xrightarrow{\iota} G \quad \text{and} \quad m_{k-1} : A^k \xrightarrow{proj} \prod_{i=1}^{k-1} A_i \xrightarrow{m_{k-1}} G \]

respectively. Then we have \( m_k = m_{k-1} \cdot a_i \) in \([A^k, G]\). Include \( \langle m_k, m_l \rangle \) in \([A^k \times A^l, G]\) by Lemma 3.1. It becomes the commutator \( c(m_k, m_l) = [\tilde{m}_k, \tilde{m}_l] \), where \( \tilde{m}_k \) and \( \tilde{m}_l \) are compositions

\[ m_k : A^k \times A^l \xrightarrow{proj} A^k \xrightarrow{m_k} G \quad \text{and} \quad m_l : A^k \times A^l \xrightarrow{proj} A^l \xrightarrow{m_l} G. \]

Let \( \tilde{a}_i \) and \( \tilde{m}_{k-1}' \) be compositions

\[ \tilde{a}_i : A^k \times A^l \xrightarrow{proj} A_i \xrightarrow{\iota} G \quad \text{and} \quad \tilde{m}_{k-1}' : A^k \times A^l \xrightarrow{proj} \prod_{i=1}^{k-1} A_i \xrightarrow{m_{k-1}'} G. \]

Then in \([A^k \times A^l, G]\) we have \( \tilde{m}_k = \tilde{m}_{k-1}' \cdot \tilde{a}_k \).
Assume the homotopy nilpotence class of $G$ is less than $p^r + 1$. Now we use induction on $k$ and $l$ show that $c(m_k, m_l)^{p^r}$ is null-homotopic. To start with, we show that this is true for $k = 1$ or $l = 1$.

**Lemma 3.4.** If $c(t, i)^{p^r}$ is null-homotopic, then $c(m_k, i)^{p^r}$ and $c(i, m_l)^{p^r}$ are null-homotopic for all $k$ and $l$.

*Proof.* We prove that $c(m_k, i)^{p^r}$ is null-homotopic by induction. Since $m_1 = i$, $c(m_1, i)^{p^r} = c(i, i)^{p^r}$ is null-homotopic by assumption. Suppose $c(m_k, i)^{p^r}$ is null-homotopic. We need to show that $c(m_{k+1}, i)^{p^r}$ is also null-homotopic. Apply Proposition 3.2 (iii) to obtain

$$c(m_{k+1}, i) = c(m_k' \cdot a_{k+1}, i) = Ad_{\tilde{a}_{k+1}} c(m_k', i) \cdot c(a_{k+1}, i).$$

Observe that $c(a_{k+1}, i)$ and $Ad_{\tilde{a}_{k+1}} c(m_k', i)$ are null-homotopic on $A^{k+1} \vee A$ and $A^{k+1} \wedge A$ is a co-H-space. Lemma 3.3 implies that $c(a_{k+1}, i)$ and $Ad_{\tilde{a}_{k+1}} c(m_k', i)$ commute and we have

$$c(m_{k+1}, i)^{p^r} = \left(Ad_{\tilde{a}_{k+1}} c(m_k', i) \cdot c(a_{k+1}, i)\right)^{p^r} = (Ad_{\tilde{a}_{k+1}} c(m_k', i))^{p^r} \cdot c(a_{k+1}, i)^{p^r} = Ad_{\tilde{a}_{k+1}} \left(c(m_k', i)^{p^r}\right) \cdot c(a_{k+1}, i)^{p^r}.$$  

The last term $c(a_{k+1}, i)^{p^r}$ is null-homotopic since $a_{k+1}$ is the inclusion $A_{k+1} \rightarrow G$. Also, by the induction hypothesis $c(m_k', i)^{p^r}$ is null-homotopic. Therefore $c(m_{k+1}, i)^{p^r}$ is null-homotopic and the induction is completed.

Similarly, we can show that $c(i, m_l)^{p^r}$ is null-homotopic for all $l$. \hfill \Box

As a consequence of Lemma 3.4, the following lemma implies that the order of

$$\langle t, \mathbb{I}_G \rangle : A \wedge G \overset{\Lambda \Psi \mathbb{I}_G}{\rightarrow} G \wedge G \overset{(\mathbb{I}_G, \mathbb{I}_G)}{\rightarrow} G$$

equals to the order of its restriction $\langle i, i \rangle$ without assuming the condition on the homotopy nilpotence of $G$.

**Lemma 3.5.** The map $p^r \circ \langle t, i \rangle$ is null-homotopic if and only if $p^r \circ \langle \mathbb{I}_G, i \rangle$ and $p^r \circ \langle t, \mathbb{I}_G \rangle$ are null-homotopic.

*Proof.* We only need to prove the sufficient condition. If $p^r \circ \langle t, i \rangle$ is null-homotopic, then $p^r \circ \langle t, m_l \rangle$ is null-homotopic for all $l$ by Lemma 3.4. Lemma 2.3 implies that $p^r \circ \langle t, i \rangle : A \wedge \Omega \Sigma A \rightarrow G$ is null-homotopic. Since $i \circ s$ is a homotopy equivalence by Lemma 2.1, $p^r \circ \langle t, \mathbb{I}_G \rangle$ is null-homotopic.

The sufficient condition for $p^r \circ \langle \mathbb{I}_G, i \rangle$ can be proved similarly. \hfill \Box

Now suppose $c(m_k, m_l)^{p^r}$ is trivial for some fixed $k$ and $l$ in $[A^k \times A^l, G]$. The next step is to show that $c(m_{k+1}, m_l)^{p^r}$ is trivial in $[A^{k+1} \times A^l, G]$. At first glance we can follow the proof of Lemma 3.4 and apply Lemmas 3.2 and 3.3 to split $c(m_{k+1}, m_l)^{p^r}$ into $c(m_{k}', m_l)^{p^r}$ and $c(a_{k+1}, m_l)^{p^r}$ which are null-homotopic by the induction hypothesis. However, when $l > 1$, $A^l$ is not a co-H-space and we cannot use Lemma 3.3 to argue that $c(m_{k}', m_l)$ and $c(a_{k+1}, m_l)$ commute. Instead, apply Proposition 3.2 (iii) to obtain

$$c(m_{k+1}, m_l) = c(m_{k}' \cdot a_{k+1}, m_l) = c(m_{k}', m_l) \cdot [c(m_{k}', m_l), \tilde{a}_{k+1}] \cdot c(a_{k+1}, m_l).$$
Denote \(c(m'_k, m_l)\) and \([c(m'_k, m_l), \tilde{a}_{k+1}] : c(a_{k+1}, m_l)\) by \(\alpha_k\) and \(\beta_k\) respectively. Observe that the restrictions of any powers and commutators involving \(\beta_k\) to \(A_{k+1} \vee (A^k \times A^l)\) are null-homotopic and \(A_{k+1}\) is a co-H-space. Therefore they enjoy the conditions of Lemma 3.3.

**Lemma 3.6.** For any natural number \(n\), we have
\[
(\alpha_k \cdot \beta_k)^n = \alpha_k^n \cdot \beta_k^n : \left(\prod_{i=1}^{n-1} [\beta_k, \alpha_k^i]\right).
\]

**Proof.** We induct on \(n\). The statement of the lemma is trivial for \(n = 1\). Assume the formula holds for an integer \(n\). For the \((n+1)\) case, using the induction hypothesis we have
\[
(\alpha_k \cdot \beta_k)^{n+1} = \alpha_k \cdot \beta_k \cdot (\alpha_k \cdot \beta_k)^n
= \alpha_k \cdot \beta_k \cdot \alpha_k^n \cdot \beta_k^n : \left(\prod_{i=1}^{n-1} [\beta_k, \alpha_k^i]\right)
= \alpha_k^{n+1} \cdot \beta_k \cdot [\beta_k, \alpha_k^n] \cdot \beta_k^n : \left(\prod_{i=1}^{n-1} [\beta_k, \alpha_k^i]\right)
\]
In the last line \([\beta_k, \alpha_k^n]\) is formed after we swap \(\beta_k\) and \(\alpha_k^n\). Since the restrictions of \(\beta_k, \beta_k^n\) and \([\beta_k, \alpha_k^n]\) to \(A_{k+1} \vee (A^k \times A^l)\) are null-homotopic, they commute by Lemma 3.3. By commuting the terms, the statement follows. \(\Box\)

In order to prove the triviality of \(c(m_{k+1}, m_l)^{p^r}\), by Lemma 3.6 it suffices to show that \(\alpha_k^{p^r}, \beta_k^{p^r}\) and \(\prod_{i=1}^{p^r-1} [\beta_k, \alpha_k^i]\) are null-homotopic. By the induction hypothesis \(c(m_k, m_l)^{p^r}\) is null-homotopic, so \(\alpha_k^{p^r} = c(m'_k, m_l)^{p^r}\) is null-homotopic. It remains to show that \(\beta_k^{p^r}\) and \(\prod_{i=1}^{p^r-1} [\beta_k, \alpha_k^i]\) are null-homotopic.

**Lemma 3.7.** If \(c(i, i)^{p^r}\) and \(\alpha_k^{p^r}\) are null-homotopic, then so is \(\beta_k^{p^r}\).

**Proof.** By definition, \(\beta_k = [c(m'_k, m_l), \tilde{a}_{k+1}] : c(a_{k+1}, m_l)\). Observe that the restrictions of \([c(m'_k, m_l), \tilde{a}_{k+1}]\) and \(c(a_{k+1}, m_l)\) to \(A_{k+1} \vee (A^k \times A^l)\) are null-homotopic. By Lemma 3.3 they commute and we have
\[
\beta_k^{p^r} = ([c(m'_k, m_l), \tilde{a}_{k+1}] \cdot c(a_{k+1}, m_l))^{p^r} = [c(m'_k, m_l), \tilde{a}_{k+1}]^{p^r} \cdot c(a_{k+1}, m_l)^{p^r}.
\]
Since \(c(i, i)^{p^r}\) is null-homotopic, so is \(c(a_{k+1}, m_l)^{p^r}\) by Lemma 3.4.

On the other hand, recall that \(c(m'_k, m_l)\) and \(\tilde{a}_{k+1}\) are the compositions
\[
c(m'_k, m_l) : A^{k+1} \times A^l \xrightarrow{proj} A^k \times A^l \xrightarrow{c(m_k, m_l)} G \quad \text{and} \quad \tilde{a}_{k+1} : A^{k+1} \times A^l \xrightarrow{proj} A_{k+1} \xrightarrow{G} G
\]
respectively. Therefore we have
\[
[c(m'_k, m_l), \tilde{a}_{k+1}]^{p^r} = p^r \circ c(c(m'_k, m_l), i)
= p^r \circ c(1_G, i) \circ (c(m'_k, m_l) \times 1_A)
\]
where \(1_A\) is the identity map of \(A_{k+1}\). Since \(p^r \circ c(1_G, i)\) is null-homotopic by Lemma 3.5, \([c(m'_k, m_l), \tilde{a}_{k+1}]^{p^r}\) is null-homotopic and so is \(\beta_k^{p^r}\). \(\Box\)

**Lemma 3.8.** For any natural number \(n\), we have
\[
\prod_{i=1}^{n-1} [\beta_k, \alpha_k^i] = \prod_{i=1}^{n-1} c_i(\beta_k, \alpha_k, \ldots, \alpha_k) \binom{n}{i+1}
\]
where \( c_i(\beta_k, \alpha_k, \cdots, \alpha_k) = [[[\cdots[[\beta_k, \alpha_k], \alpha_k] \cdots], \alpha_k] \) is the \( i \)-iterated commutator.

**Proof.** First, by induction we prove

\[
[\beta_k, \alpha_k^i] = \prod_{j=1}^{i} c_j(\beta_k, \alpha_k, \cdots, \alpha_k)^{(j)}.
\]

It is trivial for \( i = 1 \). Assume the formula holds for \( [\beta_k, \alpha_k^i] \). Use the commutator identity in (4) and inductive hypothesis to get

\[
[\beta_k, \alpha_k^{i+1}] = [\beta_k, \alpha_k] \cdot [\beta_k, \alpha_k^i] \cdot [[\beta_k, \alpha_k^i], \alpha_k]
\]

\[
= [\beta_k, \alpha_k] \cdot \left( \prod_{j=1}^{i} c_j(\beta_k, \alpha_k, \cdots, \alpha_k)^{(j)} \right) \cdot \left[ \prod_{j=1}^{i} c_j(\beta_k, \alpha_k, \cdots, \alpha_k)^{(j)}, \alpha_k \right]
\]

Since the restriction of \( c_j(\beta_k, \alpha_k, \cdots, \alpha_k) \) to \( A_{k+1} \vee (A^k \times A^i) \) is null-homotopic for all \( j \), by Lemma 3.3 they commute with each other and

\[
[\beta_k, \alpha_k^{i+1}] = [\beta_k, \alpha_k] \cdot \left( \prod_{j=1}^{i} c_j(\beta_k, \alpha_k, \cdots, \alpha_k)^{(j)} \right) \cdot \left( \prod_{j=1}^{i} c_j(\beta_k, \alpha_k, \cdots, \alpha_k)^{(j)} \right)
\]

\[
= [\beta_k, \alpha_k] \cdot \left( \prod_{j=1}^{i} c_j(\beta_k, \alpha_k, \cdots, \alpha_k)^{(j)} \right) \cdot \left( \prod_{j=1}^{i+1} c_j(\beta_k, \alpha_k, \cdots, \alpha_k)^{(j)} \right)
\]

\[
= [\beta_k, \alpha_k]^{i+1} \cdot \left( \prod_{j=2}^{i+1} c_j(\beta_k, \alpha_k, \cdots, \alpha_k)^{(j)} \right) \cdot \left( \prod_{j=1}^{i+1} c_j(\beta_k, \alpha_k, \cdots, \alpha_k)^{(j)} \right)
\]

Therefore the claim is proved.

Now we multiply all \( [\beta_k, \alpha_k]^i \)'s and use the commutativity of \( c_j(\beta_k, \alpha_k, \cdots, \alpha_k) \)'s to get

\[
\prod_{i=1}^{n-1} [\beta_k, \alpha_k^i] = \prod_{i=1}^{n-1} \prod_{j=1}^{i} c_j(\beta_k, \alpha_k, \cdots, \alpha_k)^{(j)}
\]

\[
= \prod_{j=1}^{n-1} \prod_{i=j}^{n-1} c_j(\beta_k, \alpha_k, \cdots, \alpha_k)^{(j)}
\]

\[
= \prod_{j=1}^{n-1} \left( \prod_{i=j}^{n-1} c_j(\beta_k, \alpha_k, \cdots, \alpha_k)^{(j)} \right)
\]

\[
= \prod_{j=1}^{n-1} c_j(\beta_k, \alpha_k, \cdots, \alpha_k)^{\sum_{i=j}^{n-1} (j)}
\]

The proof will be completed if we can show that \( \sum_{i=j}^{n-1} (j) = \binom{n}{j+1} \).
Consider the polynomial
\[ \sum_{i=0}^{n-1} (1 + x)^i = \sum_{i=0}^{n-1} \sum_{j=0}^{i} \binom{i}{j} x^j = \sum_{j=0}^{n-1} \sum_{i=0}^{n-1} \binom{i}{j} x^j. \]
The coefficient of \( x^j \) is \( \sum_{i=j}^{n-1} \binom{i}{j} \). On the other hand, it can be written as
\[ 1 + (1 + x) + \cdots + (1 + x)^{n-1} = \frac{(1 + x)^n - 1}{x} = \sum_{j=1}^{n} \binom{n}{j} x^{j-1}. \]
The coefficient of \( x^j \) is \( \binom{n}{j} \). By comparing the coefficients of \( x^j \) the statement follows.  

Let \( \text{nil}(G) \) be the homotopy nilpotence class of \( G \), that is, \( \text{nil}(G) = n \) if and only if the \( n \)-iterated commutator \( c_n \) is null-homotopic but \( c_{n-1} \) is not. In particular, \( \text{nil}(G) = n \) means all commutators in \( G \) of length greater than \( n \) are null-homotopic.

**Lemma 3.9.** Let \( \{ f_j : A^k \times A^l \to G \}_{1 \leq j \leq n-1} \) be a set of maps. If \( \text{nil}(G) \leq n \), then \( c_{n-1}(\beta_k, f_1, \cdots, f_{n-1}) \) is null-homotopic.

**Proof.** By definition, \( \beta_k = [c(m_k, m_l), c_{k_1 + 1} \cdots c_{k_l + 1}] \). Denote \( [c(m'_k, m_l), c_{k_1 + 1} \cdots c_{k_l + 1}] \) and \( c_{k_1 + 1} \cdots c_{k_l + 1} \) by \( \gamma_0 \) and \( \gamma'_0 \) respectively. Then we have \( \beta_k = \gamma_0 \cdot \gamma'_0 \). For \( 1 \leq j \leq n-1 \), let \( \gamma_j = [\gamma_{j-1}, f_j] \) and \( \gamma'_j = [\gamma'_{j-1}, f_j] \). We claim that \( c_m(\beta_k, f_1, \cdots, f_m) = \gamma_m \cdot \gamma'_m \) for \( 1 \leq m \leq n-1 \).

When \( m = 1 \),
\[ c(\beta_k, f_1) = [\beta_k, f_1] = [\gamma_0, \gamma'_0]. \]
Since the restrictions of \( \gamma_0 \) and \( \gamma'_0 \) to \( A_{k+1} \cap (A^k \times A^l) \) are null-homotopic, by Lemma 3.3 we have
\[ [\gamma_0, \gamma'_0] = [\gamma_0, f_1] \cdot [\gamma'_0, f_1] = \gamma_1 \cdot \gamma'_1. \]
Assume the claim is true for \( m - 1 \). By the induction hypothesis,
\[ c_m(\beta_k, f_1, \cdots, f_m) = c_1 \circ (c_{m-1}(\beta_k, f_1, \cdots, f_{m-1}) \times f_m) = [c_{m-1}(\beta_k, f_1, \cdots, f_{m-1}), f_m] = [\gamma_{m-1}, \gamma'_{m-1}, f_m] \]
Since the restrictions of \( \gamma_{m-1} \) and \( \gamma'_{m-1} \) to \( A_{k+1} \cap (A^k \times A^l) \) are null-homotopic, by Lemma 3.3
\[ c_m(\beta_k, f_1, \cdots, f_m) = [\gamma_{m-1}, f_m] \cdot [\gamma'_{m-1}, f_m] = \gamma_m \cdot \gamma'_m. \]

By putting \( m = n - 1 \) we get \( c_{n-1}(\beta_k, f_1, \cdots, f_n) = \gamma_{n-1} \cdot \gamma'_{n-1} \). Notice that \( \gamma_{n-1} \) and \( \gamma'_{n-1} \) are commutators of length \( n + 2 \) and \( n + 1 \) respectively, which are null-homotopic due to the condition on the homotopy nilpotency of \( G \). Therefore \( c_{n-1}(\beta_k, f_1, \cdots, f_{n-1}) \) is null-homotopic.

Now we have all the ingredients to prove Lemma 2.5.

**Proof of Lemma 2.5.** Suppose \( c(t, i)^{p^r} \) is null-homotopic and \( \text{nil}(G) \) is less than \( p^r + 1 \). We prove that \( c(m_k, m_l)^{p^r} \) is null-homotopic for all \( k \) and \( l \) by induction. By Lemma 3.4, \( c(m_k, i)^{p^r} \) and \( c(i, m_l)^{p^r} \) are null-homotopic for all \( k \) and \( l \). Assume \( c(m_k, m_l)^{p^r} \) is null-homotopic for some fixed \( k \) and \( l \). We need to show that \( c(m_{k+1}, m_l)^{p^r} \) is null-homotopic. By Lemma 3.6, we have
\[ c(m_{k+1}, m_l)^{p^r} = \alpha_{k+l}^{p^r} \cdot \beta_k^{p^r} \cdot \prod_{i=1}^{p^r-1} [\beta_k, \alpha_k^i]. \]
The factor $\alpha_k^{p^r}$ is null-homotopic due to the definition of $\alpha_k$ and hypothesis, and $\beta_k^{p^r}$ is null-homotopic by Lemma 3.7. So it remains to show that $\prod_{i=1}^{p^r-1} [\beta_k, \alpha_k^i]$ is null-homotopic.

By Lemma 3.8,

$$\prod_{i=1}^{p^r-1} [\beta_k, \alpha_k^i] = \prod_{j=1}^{p^r-1} c_j(\beta_k, \alpha_k, \cdots, \alpha_k)^{\binom{p^r}{j+1}}$$

Observe that $\binom{p^r}{j+1}$ is divisible by $p^r$ for $1 \leq j \leq p^r - 2$. By Lemma 3.3 we have

$$c_j(\beta_k^n, \alpha_k, \cdots, \alpha_k) = c_j(\beta_k, \alpha_k, \cdots, \alpha_k)^n$$

for all $n$. In our case we have

$$c_j(\beta_k^n, \alpha_k, \cdots, \alpha_k)^{\binom{p^r}{j+1}} = c_j(\beta_k^n, \alpha_k, \cdots, \alpha_k)^{\binom{p^r}{j+1}/p^r}.$$ 

Also, when $j = p^r - 1$, the term $c_{p^r-1}(\beta_k, \alpha_k, \cdots, \alpha_k)$ is null-homotopic by Lemma 3.9. Putting these together we obtain

$$\prod_{i=1}^{p^r-1} [\beta_k, \alpha_k^i] = \prod_{j=1}^{p^r-2} c_j(\beta_k^{p^r}, \alpha_k, \cdots, \alpha_k)^{\binom{p^r}{j+1}/p^r}.$$ 

We have shown that $\beta_k^{p^r}$ is null-homotopic in Lemma 3.7, so $\prod_{i=1}^{p^r} [\beta_k, \alpha_k^i]$ is null-homotopic and the induction is completed. \qed

4. Orders of Samelson Products of Quasi-$p$-Regular Groups

In this section we apply Theorem 2.6 to calculate the orders of $\langle 1_G, 1_G \rangle$ for certain Lie groups $G$. Recall that $G$ is rationally homotopy equivalent to a product of spheres $\prod_{i=1}^l S^{2n_i-1}$, where $n_1 \leq \cdots \leq n_l$. The sequence $(2n_1 - 1, \cdots, 2n_l - 1)$ is called the type of $G$. After localization at $p$, $G$ is homotopy equivalent to a product of H-spaces $\prod_{i=1}^{p-1} B_i$, and $A$ is homotopy equivalent to a wedge of co-H-spaces $\bigvee_{i=1}^{p-1} A_i$ such that $A_i$ is a subspace of $B_i$. For $1 \leq i \leq p - 1$, let $i_i : A_i \to B_i$ be the inclusion. Then $H_*(B_i)$ is the exterior algebra generated by $(i_i)_*(H_*(A_i))$. If each $B_i$ is a sphere, then we call $G$ $p$-regular. If each $A_i$ is a sphere or a CW-complex with two cells, then we call $G$ quasi-$p$-regular. When $A_i$ is a CW-complex with two cells, it is homotopy equivalent to the cofibre of $\alpha_{2n_i-1}$, which is the generator of the homotopy group $\pi_{2n_i+2p-4}(S^{2n_i-1})$, and the corresponding $B_i$ is the $S^{2n_i-1}$-bundle $B(2n - 1, 2n + 2p - 3)$ over $S^{2n_i+2p-3}$ classified by $\frac{1}{2} \alpha_{2n_i-1}$ [7].

The homotopy nilpotence classes of certain quasi-$p$-regular Lie groups are known.

**Theorem 4.1** (Kaji and Kishimoto [3]). A $p$-regular Lie group has homotopy nilpotence class at most 3.

**Theorem 4.2** (Kishimoto [4]). For $p \geq 7$, a quasi-$p$-regular $SU(n)$ has homotopy nilpotence class at most 3.

**Theorem 4.3** (Theriault [10]). For $p \geq 7$, a quasi-$p$-regular exceptional Lie group has homotopy nilpotence class at most 2.
By Theorem 2.6, the order of $\langle 1_G, 1_G \rangle$ equals the order of $\langle i, i \rangle$ in these groups.

4.1. Upper bounds on the orders of $\langle 1_G, 1_G \rangle$ for quasi-$p$-regular Lie groups. Since $\langle i, i \rangle \in [A \wedge A, G]$ and

$$\text{(6) } [A \wedge A, G] \cong \bigcup_{i=1}^{p-1} [A_i, G] \cong \bigcup_{j=1}^{p-1} [A_j, G] \cong \prod_{i,j=1}^{p-1} [A_i \wedge A_j, G] \cong \prod_{i,j,k=1}^{p-1} [A_i \wedge A_j \wedge B_k],$$

the order of $\langle i, i \rangle$ cannot exceed the least common multiple of the orders of $[A_i \wedge A_j, B_k]$ for all $i, j$ and $k$. Let $C_{2n_i-1}$ be the cofiber of the generator $a_{2n_i-1}$ of the homotopy group $\pi_{2n_i+2p-4}(S^{2n_i-1})$. When $G$ is quasi-$p$-regular, each $A_i$ is a sphere $S^{2n_i-1}$ or $C_{2n_i-1}$, so $A_i \wedge A_j$ is either $S^{2n_i+2n_j-2}$, $C_{2n_i+2n_j-2}$ or $C_{2i-1} \wedge C_{2n_j-1}$. In the following we consider the orders of $[A_i \wedge A_j \wedge B_k]$ case by case.

If $A_i \wedge A_j$ is $S^{2n_i+2n_j-2}$, then $[A_i \wedge A_j \wedge B_k]$ is $\pi_{2n_i+2n_j-2}(B_k)$. The homotopy groups of $B_k$ are known in a range.

**Theorem 4.4** (Toda [11], Mimura and Toda [8], Kishimoto [4]). Localized at $p$, we have

$$\pi_{2n-1+k}(S^{2n-1}) \cong \begin{cases} 
\mathbb{Z}/p \mathbb{Z} & \text{for } k = 2i(p-1) - 1, 1 \leq i \leq p-1 \\
\mathbb{Z}/p \mathbb{Z} & \text{for } k = 2i(p-1) - 2, n \leq i \leq p-1 \\
0 & \text{other cases for } 1 \leq k \leq 2p(p-1) - 3
\end{cases}$$

$$\pi_{2n-1+k}(B(3,2p+1)) \cong \begin{cases} 
\mathbb{Z}/p \mathbb{Z} & \text{for } k = 2i(p-1) - 1, 2 \leq i \leq p-1 \\
\mathbb{Z} & \text{for } k = 2p - 2 \\
0 & \text{other cases for } 1 \leq k \leq 2p(p-1) - 3
\end{cases}$$
and

\[
\pi_{2n-1+k}(B(2n-1, 2n+2p-3)) \cong \begin{cases} 
\mathbb{Z}/p^2\mathbb{Z} & \text{for } k = 2i(p-1) - 1, 2 \leq i \leq p-1 \\
\mathbb{Z}/p\mathbb{Z} & \text{for } k = 2i(p-1) - 2, n \leq i \leq p-1 \\
\mathbb{Z} & \text{for } k = 2p-2 \\
0 & \text{other cases for } 1 \leq k \leq 2p(p-1)-3.
\end{cases}
\]

Since \(2n_i+2n_j-2\) is even, \(\pi_{2n_i+2n_j-2}(B_k)\) is isomorphic to either 0, \(\mathbb{Z}/p\mathbb{Z}\) or \(\mathbb{Z}/p^2\mathbb{Z}\). Therefore the order of \([A_i \wedge A_j, B_k]\) is at most \(p^2\).

If \(A_i \wedge A_j\) is \(C_{2n_i+2n_j-2}\), then the cofibration

\[S^{2n_i+2n_j-2} \to C_{2n_i+2n_j-2} \to S^{2n_i+2n_j+2p-4}\]

induces an exact sequence

(7) \[\pi_{2n_i+2n_j+2p-4}(B_k) \to [C_{2n_i+2n_j-2}, B_k] \to \pi_{2n_i+2n_j-2}(B_k).\]

Since \(C_{2n_i+2n_j-2}\) is a suspension and \(B_k\) is an H-space, the three groups are abelian. By Theorem 4.4, the first and the last homotopy groups have orders at most \(p^2\), so the order of \([C_{2n_i+2n_j-2}, B_k]\) is at most \(p^4\).

If \(A_i \wedge A_j\) is \(C_{2n_i-1} \wedge C_{2n_j-1}\), then it is a CW-complex with one cell of dimension \(2n_i + 2n_j - 2\), two cells of dimension \(2n_i + 2n_j + 2p - 4\) and one cell of dimension \(2n_i + 2n_j + 4p - 6\). Let \(C'\) be the \((2n_i + 2n_j + 4p - 7)\)-skeleton of \(C_{2n_i-1} \wedge C_{2n_j-1}\), that is, \(C_{2n_i-1} \wedge C_{2n_j-1}\) minus the top cell. Then the cofibration \(C' \to C_{2n_i-1} \wedge C_{2n_j-1} \to S^{2n_i+2n_j+4p-6}\) induces an exact sequence of abelian groups

\[\pi_{2n_i+2n_j+4p-6}(B_k) \to [C_{2n_i-1} \wedge C_{2n_j-1}, B_k] \to [C', B_k].\]

According to [1], \(C'\) is homotopy equivalent to \(C_{2n_i+2n_j-2} \vee S^{2n_i+2n_j+2p-4}\), so we have

\[[C', B_k] \cong [C_{2n_i+2n_j-2}, B_k] \oplus \pi_{2n_i+2n_j+2p-4}(B_k).\]

We have shown that \([C_{2n_i+2n_j-2}, B_k]\) has order at most \(p^4\). By Theorem 4.4, \(\pi_{2n_i+2n_j+2p-4}(B_k)\) and \(\pi_{2n_i+2n_j+4p-6}(B_k)\) have orders at most \(p^3\). Therefore the order of \([C_{2n_i-1} \wedge C_{2n_j-1}, B_k]\) is at most \(p^6\).

Summarizing the above discussion, we have the following proposition.

**Proposition 4.5.** Let \(G\) and \(p\) be in (5). Then the order of \(\langle 1_G, 1_G \rangle\) is at most \(p^6\).

This gives a very rough upper bound on the orders of \(\langle 1_G, 1_G \rangle\). We can sharpen the range by refining our calculation according to individual cases of \(G\) and \(p\).

**Case I: G is p-regular.** Suppose \(G\) is \(p\)-regular. Then \(B_i = A_i = S^{2n_i-1}\) and \(p \geq n_i\). All summands \([A_i \wedge A_j, B_k]\) in (6) are homotopy groups \(\pi_{2n_i+2n_j-2}(S^{2n_k-1})\). According to Theorem 4.4, their orders are at most \(p\) since

\[2(n_i + n_j - n_k) - 1 \leq 2(2n_l - 2) - 1 \leq 2p(p-1) - 3\]

for all \(i, j\) and \(k\). Therefore the order of \(\langle i, i \rangle\) is at most \(p\) and so is the order of \(\langle 1_G, 1_G \rangle\) by Theorem 2.6. McGibbon [5] showed that \(G\) is homotopy commutative if and only if either \(p > 2n_l\), or \((G, p)\) is \((Sp(2), 3)\) or \((G_2, 5)\). Therefore we have the following statement.

**Theorem 4.6.** Let \(G\) be a \(p\)-regular Lie group of type \(2n_1-1, \cdots, 2n_l-1\). Then the order of \(\langle 1_G, 1_G \rangle\) is \(p\) if \(n_1 \leq p < 2n_l\), and is \(1\) if \(p > 2n_l\).
Case II: $G$ is a quasi-$p$-regular $SU(n)$ and $p \geq 7$. Suppose $G = SU(n)$ is quasi-$p$-regular and $p \geq 7$. Then $n_i = i + 1$ and $p > \frac{n}{2}$. Let $t = n - p + 1$ and $2 \leq t \leq p$. Localized at $p$, there are homotopy equivalences

$$SU(n) \simeq B(3, 2p + 1) \times \cdots \times B(2t - 1, 2n - 1) \times S^{2t+1} \times \cdots \times S^{2p-1}$$

and

$$A \simeq C_3 \vee \cdots \vee C_{2t-1} \vee S^{2t+1} \vee \cdots \vee S^{2p-1}$$

For $1 \leq j \leq t$ and $t + 1 \leq i \leq p$, let $\epsilon_i$ and $\lambda_i$ be the compositions

$$\epsilon_i : S^{2i-1} \hookrightarrow A \xrightarrow{i} G \quad \text{and} \quad \lambda_j : C_{2j-1} \hookrightarrow A \xrightarrow{j} G.$$ 

Kishimoto calculated some of their Samelson products in [4].

**Theorem 4.7** (Kishimoto [4]). Let $G$ be a quasi-$p$-regular $SU(n)$. For $2 \leq j, j' \leq t$ and $t + 1 \leq i, i' \leq p$,

1. the order of $\langle \epsilon_i, \epsilon_{i'} \rangle$ is at most $p$;
2. if $i \neq p$ and $j \neq t$, then the order of $\langle \epsilon_i, \lambda_j \rangle$ is at most $p$;
3. if $j + j' \leq p$, then $\langle \lambda_j, \lambda_{j'} \rangle$ is null-homotopic;
4. if $p + 1 \leq j + j' \leq 2p - 1$, then $\langle \lambda_j, \lambda_{j'} \rangle$ can be compressed into $S^{2(j+j'-p)+1} \subset SU(n)$.

Using these results we can give a bound for the order of $\langle 1_G, 1_G \rangle$.

**Theorem 4.8.** For $G = SU(n)$ and $p \geq 7$, let the order of $\langle 1_G, 1_G \rangle$ be $p^r$.

- If $n > 2p$, then $r = 0$;
- If $n \leq 2p$, then $r = 1$;
- If $\frac{3}{2}n + 1 \leq p < n$, then $1 \leq r \leq 2$;
- If $\frac{2}{3} < n \leq \frac{2}{3}n$ and $n \neq 2p - 1$, then $1 \leq r \leq 3$;
- If $n = 2p - 1$, then $1 \leq r \leq 6$.

**Proof.** When $p \geq n$, $G$ is $p$-regular and we have shown the first two statements in Theorem 4.6. Assume $\frac{2}{3} < p < n$. By Theorem 2.6, the order of $\langle 1_G, 1_G \rangle$ equals the order of $\langle t, t \rangle$. Since $\langle t, t \rangle$ is a wedge of Samelson products of $\epsilon_i$’s and $\lambda_j$’s, we need to consider the orders of $\langle \epsilon_i, \epsilon_{i'} \rangle, \langle \epsilon_i, \lambda_j \rangle$ and $\langle \lambda_j, \lambda_{j'} \rangle$.

First, the first two statements of Theorem 4.7 imply that the orders of $\langle \epsilon_i, \epsilon_{i'} \rangle$ and $\langle \epsilon_i, \lambda_j \rangle$ are at most $p$ except for $\langle \epsilon_p, \lambda_t \rangle$. Put $n_i = p$ and $n_j = t$ in (7) to obtain the exact sequence

$$\pi_{4p+2t-4}(B_k) \to [C_{2p+2t-2}, B_k] \to \pi_{2p+2t-2}(B_k)$$

where $2 \leq k \leq p$. According to Theorem 4.4, the two homotopy groups are trivial except for $k = t + 1$ or $t = p$ and $k = 2$. In the first case, $B_k$ is $S^{2t+1}$, and $\pi_{4p+2t-4}(S^{2t+1})$ and $\pi_{2p+2t-2}(S^{2t+1})$ are $\mathbb{Z}/p\mathbb{Z}$. In the second case, $B_k$ is $B(3, 2p + 1)$, and $\pi_{6p-4}(B(3, 2p + 1))$ and $\pi_{4p-2}(B(3, 2p + 1))$ are $\mathbb{Z}/p^2\mathbb{Z}$. By exactness the order of $[C_{2p+2t-2}, B_k]$ is at most $p^2$ for $2 \leq k \leq p$ and $n \neq 2p - 1$, and consequently so is the order of $\langle \epsilon_i, \lambda_j \rangle$.

Second, the third statement of Theorem 4.7 implies that $\langle \lambda_j, \lambda_{j'} \rangle$ is null-homotopic for $j + j' \leq p - 1$. When $p \geq \frac{2}{3}n + 1$, we have

$$n \leq \frac{3}{2}(p - 1) \quad \text{and} \quad t = n - p + 1 \leq \frac{1}{2}(p - 1).$$

In this case the order of $\langle \lambda_j, \lambda_{j'} \rangle$ is always 1 since $j + j' \leq 2t \leq p - 1$. When $\frac{2}{3} < p \leq \frac{2}{3}n$, we need to consider the orders of $\langle \lambda_j, \lambda_{j'} \rangle$ for $p + 1 \leq j + j'$. By the last statement of
Theorem 4.7, \( \langle \lambda_j, \lambda_{j'} \rangle \) is in \([C_{2j-1} \wedge C_{2j'-1}, S^{2(j+j'-p)+1}]\) if \( j + j' \leq 2p - 1 \). Since \( j, j' \leq t \leq p \), this can always be achieved for \( n \neq 2p - 1 \). There is a graph of short exact sequences

\[
\begin{array}{c}
\pi_{2j+2j'+2p-4}(S^{2(j+j'-p)+1}) \\
\pi_{2j+2j'+4p-6}(S^{2(j+j'-p)+1}) \\
\pi_{2j+2j'-2}(S^{2(j+j'-p)+1})
\end{array} \xrightarrow{\quad} \begin{array}{c}
[C_{2j-1} \wedge C_{2j'-1}, S^{2(j+j'-p)+1}] \\
[C', S^{2(j+j'-p)+1}]
\end{array} \xrightarrow{\quad} \begin{array}{c}
[C_{2j-1} \wedge C_{2j'-1}, S^{2(j+j'-p)+1}]
\end{array}
\]

where \( C' \) is the subcomplex of \( C_{2j-1} \wedge C_{2j'-1} \) without the top cell. By Theorem 4.4, the three homotopy groups are \( \mathbb{Z}/p\mathbb{Z} \). The exactness of the column and the row implies that the orders of \( [C', S^{2(j+j'-p)+1}] \) and \( [C_{2j-1} \wedge C_{2j'-1}, S^{2(j+j'-p)+1}] \) are at most \( p^2 \) and \( p^3 \). Therefore \( \langle \lambda_j, \lambda_{j'} \rangle \) has order at most \( p^3 \) when \( n \neq 2p - 1 \) and \( \frac{2}{3} < p \leq \frac{2}{3}n \).

We summarize the above discussion in the following table:

| Condition                              | \( \langle \epsilon_i, \epsilon_{i'} \rangle \) | \( \langle \epsilon_i, \lambda_j \rangle \) | \( \langle \lambda_j, \lambda_{j'} \rangle \) |
|----------------------------------------|-----------------------------------------------|-----------------------------------------------|-----------------------------------------------|
| \( \frac{2}{3}n + 1 \leq p \leq n \)    | \( p \)                                       | \( p^2 \)                                       | \( 1 \)                                       |
| \( \frac{n}{2} < p \leq \frac{2}{3}n, \) | \( p \)                                       | \( p^2 \)                                       | \( p^3 \)                                     |
| \( n \neq 2p - 1 \)                    | \( p \)                                       | \( p^2 \)                                       | \( p^6 \)                                     |

By Theorem 2.6, the order of \( \langle 1_G, 1_G \rangle \) equals the order of \( \langle \iota, \iota \rangle \) which is the least common multiple of the orders of \( \langle \epsilon_i, \epsilon_{i'} \rangle, \langle \epsilon_i, \lambda_j \rangle \) and \( \langle \lambda_j, \lambda_{j'} \rangle \), so the statement follows.

Case III: \( G \) is a quasi-\( p \)-regular exceptional Lie group and \( p \geq 7 \). Suppose \( p \geq 7 \) and \( G \) is a quasi-\( p \)-regular exceptional Lie group. That is

- when \( G = F_4 \) or \( E_6 \), \( p = 7 \) or 11;
- when \( G = E_7 \), \( p = 11, 13 \) or 17;
- when \( G = E_8 \), \( p = 11, 13, 17, 23 \) or 29.

For each case, we can calculate bounds on the orders of \([A_i \wedge A_j, B_k]\) for all \( i, j \) and \( k \) in (6) according to the CW-structure of \( A \). Then we obtain the following statement.
Theorem 4.9. For $p \geq 7$, suppose $G$ is a quasi-$p$-regular exceptional Lie group which is not $p$-regular. Let the order of $\langle 1_G, 1_G \rangle$ be $p^r$. Then we have the following table

| $G$ | $p$ | value(s) of $r$ |
|-----|-----|-----------------|
| $F_4$ | 7   | $1 \leq r \leq 4$ |
|      | 11  | 1               |
| $E_6$ | 7   | $1 \leq r \leq 4$ |
|      | 11  | 1               |
| $E_7$ | 11  | $1 \leq r \leq 3$ |
|      | 13  | 1 or 2          |
|      | 17  | 1               |
| $E_8$ | 11  | $1 \leq r \leq 6$ |
|      | 13  | $1 \leq r \leq 4$ |
|      | 17  | $1 \leq r \leq 3$ |
|      | 19  | $1 \leq r \leq 4$ |
|      | 23, 29 | 1           |

Remark: It would be interesting if the precise order of $\langle 1_G, 1_G \rangle$ could be obtained in the case of Theorem 4.9.

REFERENCES

[1] B. Gray, Associativity in two-cell complexes, Geometry and Topology: Aarhus (1998), 185-196, Contemp. Math. 258 (2000)
[2] F. Cohen and J. Neisendorfer, A construction of $p$-local $H$-spaces, Lecture Note in Math. Vol. 1051, Springer, Berlin, (1984), 351 – 359.
[3] S. Kaji, D. Kishimoto, Homotopy nilpotency in $p$-regular loop spaces, Mathematische Zeitschrift, (2010), 264 – 209.
[4] D. Kishimoto, Homotopy nilpotency in localized $SU(n)$, Homology, Homotopy and Applications, 11, (2009), 61 – 79.
[5] C. McGibbon, Homotopy commutativity in localized groups, American Journal of Mathematics, 106(3), (1984), 665 – 687.
[6] M. Mimura, The homotopy groups of Lie groups of low rank, Journal of Mathematics of Kyoto University, 6-2, (1967), 131 – 176.
[7] M. Mimura, G. Nishida, H. Toda Mod $p$ decomposition of compact Lie groups, Publ. RIMS, Kyoto Univ, 13, (1977), 627 – 680.
[8] M. Mimura and H. Toda Cohomology operations and homotopy of compact Lie groups I, Topology, 9, (1970), 317 – 336.
[9] S. Theriault, Odd primary $H$-structure of low rank Lie groups and its application to exponents, Transactions of the American Mathematical Society, 359, (2007), 4511 – 4535.
[10] S. Theriault, The dual polyhedral product, cocategory and nilpotence, arXiv:1506.05998v2, (2016).
[11] H. Toda, Composition Methods in Homotopy Groups of Spheres, Annals of Mathematics Studies 49, Princeton University Press, Princeton N.J., 1962.

Mathematical Sciences, University of Southampton, SO17 1BJ, UK
E-mail address: tlsig14@soton.ac.uk