A model of predator-prey differential equation with time delay

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Abstract. This paper purposes are forming a predator-prey model with delay, examining the stability of the fixed points, and observing the presence of bifurcation of the fixed point. The predator-prey models formed by using a type II-Holling function and a logistic equation. Holling function is a function that shows the predation level of a predator against its prey. This level of predation depends on how the predator searches, captures, and finally processes the food. The stability of fixed points of the predator-prey model with this delay observed through its model linearity. And the bifurcation is indicated by the sign changing of eigen values over the change of parameter value and from the decrease of the number of fixed points. The results show that at a certain value of parameter the 3 fixed points decreased to only 2 fixed points. At this certain parameter value, the first fixed point is saddle and the second one is degenerate. The system indicating undergoes a transcritical bifurcation.

1. Introduction
Differential equations have been used in modelling physic phenomena since its first concept. Recently, differential equations widely used in many other fields, such as engineering, epidemiology, and biology. The differential equation models are used to get a better understanding of the occurring incident. However, sometimes a simple model cannot capture the various dynamics of a phenomena. Therefore, another approach is needed to get a better understanding of more complicated case. One of the methods that can be used is time delay.

Turchin P and Taylor A D showed in [12] and [13], that based on statistical analysis of ecological data, there was a delay effect in dynamics. The dynamics of the differential equation without time delay can be very different from the system with time delay. For instance, in a differential equation that does not have a periodic solution, the time delay can give the periodic motion. Wu H et al in [14] showed that according to numerous simulations, the time delay makes overshoot value of active power variation increases and the setting time is lengthened.

Many other researchers have conducted researches to show the dynamics of differential equations model with time delay. In epidemiological problems, differential equations with time delay is used to describe several aspects of HIV dynamics (see [2] and [10]), and immune responses to disease [3]. The weather issues related to seasonal changes due to El-Nino influence (El-Nino Southern Oscillation (ENSO)) also can be described use a system of differential equation with time delay (see [4]). And the other applications of differential equations with time delay are in control systems in [6], population dynamic [9] and [11], and robotic modification in [5].

In this paper, the differential equations with time delay of predator-prey interactions defined as follows,

$$\dot{x} = f(x(t), x(t - \tau))$$

(1)
where \( \tau \) is a positive constant. The form of logistic equation in this paper is as follows:

\[
\frac{dN}{dt} = rN \left( 1 - \frac{N(t)}{k} \right)
\]

(2)

where \( N \) denotes the population density, \( r \) denotes the rate of growth, and \( k \) is the carrying capacity.

2. Mathematical model

2.1. Population growth rate-prey

Let \( X(t) \) state the number of preys at time \( t \). If there are no predators then, based on equation (2), the population growth rate of preys is

\[
\frac{dX(t)}{dt} = rX(t) \left( 1 - \frac{X(t)}{k} \right)
\]

(3)

But when the predators appear, the population of preys decrease according to the number of predators and their predation rates. So, the preys population growth rate can be expressed as:

\[
\frac{dX(t)}{dt} = rX(t) \left( 1 - \frac{X(t)}{k} \right) - Y(t)p(X(t))
\]

(4)

where \( Y(t) \) denotes the number of predators at time \( t \) and \( p(X(t)) \) is a functional response.

2.2. Predator population growth rate

If there is no prey, the number of predators would decrease proportional to the natural death rate \( m \), that is

\[
\frac{dY(t)}{dt} = -mY(t).
\]

But, since the existence of the preys, the number of predators increasing at a rate proportional to the number of predators and their predation rates. So that:

\[
\frac{dY(t)}{dt} = b_1Y(t)p(X(t)) - mY(t),
\]

(5)

where \( b_1 > 0 \) denotes the birth rate of predators because of interactions with prey.

Furthermore, in the field of ecology, it is known that new predators do not hunt until a certain age \( \tau \). They only depend on the results of their mother's hunt, for example on tigers. Young tigers hunt at around two years old. So, the interaction of predators and preys does not directly increase the number of predators, but still, it will increase the number of young predators, predators that have not hunted yet. This make a time delay \( \tau \) for predators to start hunting prey. If \( Y_m(t) \) denotes a young predator population, then equation (5) changes to

\[
\frac{dY(t)}{dt} = bY(t - \tau)p(X(t - \tau)) - mY(t)
\]

\[
\frac{dY_m(t)}{dt} = b_1Y(t)p(X(t)) - bY(t - \tau)p(X(t - \tau)) - mY_m(t)
\]

(6)

Where \( 0 < b \leq b_1 \) and \( b \) denotes predator's readiness rate for hunting.

2.3. Predator model-prey with delay

Based on equation (4) and (6), the predator-prey model with time delay is

\[
\frac{dX(t)}{dt} = rX(t) \left( 1 - \frac{X(t)}{k} \right) - Y(t)p(X(t))
\]

\[
\frac{dY(t)}{dt} = bY(t - \tau)p(X(t - \tau)) - mY(t)
\]

\[
\frac{dY_m(t)}{dt} = b_1Y(t)p(X(t)) - bY(t - \tau)p(X(t - \tau)) - mY_m(t)
\]

(7)
3. Stability analysis
Since variable $Y_m$ does not appear in the first and second equations, to analyse the system (7) can be done through the first and second equations. And, to narrow the discussion, we choose the type II-Holling functional response (see [7] and [8]), i.e.:

$$p(X(t)) = c \frac{X(t)}{1 + hX(t)}$$

(8)

So, the form of the system to be analysed is

$$\frac{dX(t)}{dt} = rX(t) \left(1 - \frac{X(t)}{k} \right) - cY(t) \frac{X(t)}{1 + hX(t)}$$

$$\frac{dY(t)}{dt} = bcY(t - \tau) \frac{X(t - \tau)}{1 + hX(t - \tau)} - mY(t)$$

(9)

3.1. System transformation
Substituting $x_b(t) = \frac{x}{k}, \theta = \eta t$, and $\eta = \frac{1}{\tau}$ to system (9) we get

$$\frac{dx}{d\theta} = \tau rx(\theta) \left(1 - x(\theta) \right) - \tau cy(\theta) \frac{x(\theta)}{1 + hx(\theta)}$$

$$\frac{dy}{d\theta} = \tau bcy(\theta - 1) \frac{x(\theta - 1)}{1 + hcx(\theta - 1)} - \tau m\gamma(y(\theta))$$

(10)

Or can be written as

$$\dot{x}(t) = ax(t)(1-x(t)) - \delta y(t) \frac{x(t)}{1 + \gamma x(t)}$$

$$\dot{y}(t) = \beta y(t - 1) \frac{x(t - 1)}{1 + \gamma x(t - 1)} - \mu y(t)$$

(11)

where $a = \tau r, \delta = \tau c, \gamma = hk, \beta = tbcx,$ and $\mu = \tau m$.

3.2. Fixed point and linearized system
The fixed points of the system (11) are those $(\bar{x}, \bar{y})^t$ that fulfill

$$\alpha \bar{x} \left(1 - \bar{x} - \frac{\delta \bar{y} \bar{x}}{\alpha (1 + \gamma \bar{x})} \right) = 0$$

$$\beta \bar{y} \left(\frac{\bar{x}}{1 + \gamma \bar{x}} - \frac{\mu}{\beta} \right) = 0$$

(12)

According to equation (12), there are 3 fixed points. Using these 3 fixed points, we make the linearized systems and analyze it based on [1].

3.2.1. Initial point
If $\bar{x} = 0$ then $\bar{y} = 0$ so the initial point $(\bar{x}, \bar{y})^t = (0,0)^t$ is a fixed point.

The linearization of system of (11) at the initial point $(\bar{x}, \bar{y})^t = (0,0)^t$ is

$$\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & -\mu \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} x(t - 1) \\ y(t - 1) \end{bmatrix}$$

(13)

The eigen value of equation (13) is $\alpha$ and $-\mu$. Since $\alpha, \mu > 0$ then the initial point $(0,0)^t$ is saddle.

3.2.2. Second fixed point
If $\bar{y} = 0$ then $\bar{x} = 0$ and $\bar{x} = 1$ so $(\bar{x}, \bar{y})^t = (1,0)^t$ is a fixed point.

The linearization of system (11) at the fixed point $(\bar{x}, \bar{y})^t = (1,0)^t$ is

$$\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} -\alpha & -\frac{\delta}{1+\gamma} \\ 0 & -\mu \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} x(t - 1) \\ y(t - 1) \end{bmatrix}$$

(14)
The characteristic equation of the system (14) is 
\[ (\lambda + \alpha)(\lambda + \mu - e^{-\lambda} \frac{\beta}{1+\gamma}) = 0, \]
which shows that the first eigen value is \( \lambda_1 = -\alpha < 0 \). If \( \mu > \frac{\beta}{1+\gamma} \), then the second eigen value would also negative so the equilibrium point \((1,0)^t\) is asymptotically stable. And, the fixed point would be saddle if \( \mu < \frac{\beta}{1+\gamma} \).

### 3.2.3. Fixed point that depends on parameter

If \( \frac{x}{1+\gamma x} - \frac{\mu}{\beta - \gamma \mu} = 0 \) then \( \bar{x} = \frac{\mu}{\beta - \gamma \mu} \) and \( \bar{y} = \frac{\alpha \beta (\beta - \gamma \mu - \mu)}{\delta (\gamma \mu - \beta) (\gamma \mu - \mu)^2} \) thus the point
\[
(\bar{x}, \bar{y})^t = \left( \frac{\mu}{\beta - \gamma \mu}, \frac{\alpha \beta (\beta - \gamma \mu - \mu)}{\delta (\gamma \mu - \beta) (\gamma \mu - \mu)^2} \right)^t
\]
is a fixed point. This fixed point exists if \( \beta \neq \gamma \mu \) and will be the same as the second fixed point if \( \mu = \beta - \gamma \mu \) or \( \mu = \frac{\beta}{1+\gamma} \), so that when the parameter \( \beta = \gamma \mu \) is possible there will be a bifurcation and when \( \mu = \frac{\beta}{1+\gamma} \) the bifurcation might also occur.

The linearization of system (11) at the fixed point \((\bar{x}, \bar{y})^t\) is
\[
\left[ \begin{array}{c}
\dot{x}(t) \\
\dot{y}(t)
\end{array} \right] = \left[ \begin{array}{c}
\frac{\alpha \mu (y^2 \mu - \beta y + \gamma \mu + \beta)}{(\gamma \mu - \beta) \beta} \\
-\frac{\delta \mu \beta}{\delta (\gamma \mu - \beta) (\gamma \mu - \mu)^2} - \mu
\end{array} \right] \left[ \begin{array}{c}
x(t) \\
y(t)
\end{array} \right] + \left[ \begin{array}{c}
0 \\
\frac{0 \alpha \beta (\beta - \gamma \mu - \mu)}{\delta (\gamma \mu - \beta) (\gamma \mu - \mu)^2}
\end{array} \right] \left[ \begin{array}{c}
x(t-1) \\
y(t-1)
\end{array} \right]
\]
(15)
When \( \mu = \frac{\beta}{1+\gamma} \), system (15) undergoes a transcritical bifurcation over the second fixed point, \((1,0)^t\).

This second fixed point, \((1,0)^t\), is degenerate when \( \mu = \frac{\beta}{1+\gamma} \), unstable when \( \mu < \frac{\beta}{1+\gamma} \), and asymptotically stable when \( \mu > \frac{\beta}{1+\gamma} \). Meanwhile, the third fixed point, \( \left( \frac{\mu}{\beta - \gamma \mu}, \frac{\alpha \beta (\beta - \gamma \mu - \mu)}{\delta (\gamma \mu - \beta) (\gamma \mu - \mu)^2} \right)^t \), has an opposite stability property than the second fixed point. That is, the third fixed point, \( \left( \frac{\mu}{\beta - \gamma \mu}, \frac{\alpha \beta (\beta - \gamma \mu - \mu)}{\delta (\gamma \mu - \beta) (\gamma \mu - \mu)^2} \right)^t \), unstable when \( \mu > \frac{\beta}{1+\gamma} \), and asymptotically stable when \( \mu < \frac{\beta}{1+\gamma} \). But unfortunately, \( \bar{y} < 0 \) when \( \mu > \frac{\beta}{1+\gamma} \). Since there is an assumption that \( y \geq 0 \), thus it is mean that the system would only has 2 fixed points if \( \mu > \frac{\beta}{1+\gamma} \).

### 4. Simulation

This section provides some phase portraits of the system (11) around its fixed points. The phase portrait only drawn in quadrant I, while in quadrant II, III and IV, the phase portrait were not drawn because in the case of predator-prey, the number of prey and predator must be positive.

Some parameter values are fixed, those are \( \alpha = 0.3, \delta = 0.2, \gamma = 0.1 \), and \( \mu = 0.2 \), but the parameters \( \beta \) are moved so that \( \beta < \mu + \gamma \mu, \beta = \mu + \gamma \mu, \) and \( \beta > \mu + \gamma \mu \). **Figure 1** is the phase portrait of system (11) for \( \beta = 0.3 \). **Figure 2** stands for the phase portrait at \( \beta = 0.22 \), and **Figure 3** drawn for \( \beta = 0.1 \).
From Figure 1, we can see that there are 3 fixed points. Two of the fixed points are saddle points and the third fixed point is asymptotically stable. In Figure 2., we have homoclinic orbits started from the second fixed point and ended at the same fixed point where it began. In this figure, the third fixed point is unseen because it merges with the second fixed point. And in Figure 3., the third fixed point is moved to quadrant IV, so it is unseen. Meanwhile the second fixed point become asymptotically stable.

5. Conclusion
This research obtained:

5.1. The model of predator-prey with time delay is
\[
\begin{align*}
\frac{dX(t)}{dt} &= rX(t) \left( 1 - \frac{X(t)}{k} \right) - Y(t)p(X(t)) \\
\frac{dY(t)}{dt} &= bY(t - \tau)p(X(t - \tau)) - mY(t) \\
\frac{dY_m(t)}{dt} &= b_Y t p(X(t)) - b_Y (t - \tau)p(X(t - \tau)) - m_Y m(t)
\end{align*}
\]

5.2. The system has 3 fixed points. The first fixed point \((0,0)\textsuperscript{T}\) is a saddle point. Fixed point \((1,0)\textsuperscript{T}\) would be saddle if \(\mu < \frac{\beta}{1+\gamma}\), asymptotically stable if \(\mu > \frac{\beta}{1+\gamma}\), and degenerate if \(\mu = \frac{\beta}{1+\gamma}\). Meanwhile, the third fixed point \(\left( \frac{\mu}{\beta - \gamma \mu} \frac{a \beta (\beta - \gamma \mu - \mu)}{\delta (\gamma \mu - \beta)^2} \right)\textsuperscript{T}\) is asymptotically stable if \(\mu < \frac{\beta}{1+\gamma}\). It will become the same fixed point with second one if \(\mu = \frac{\beta}{1+\gamma}\). And, if \(\mu > \frac{\beta}{1+\gamma}\) then the fixed point would be out of the feasible region.
5.3. The transcritical bifurcation showed over the second fixed point, $(1,0)^T$ and the third fixed point 
\[ \left( \frac{\mu}{\beta - \gamma \mu - \delta (\gamma \mu - \beta)} \right)^t \] when the value of parameter \( \mu = \frac{\beta}{1+\gamma} \).

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