Dual method for continuous-time Markowitz’s Problems with nonlinear wealth equations

Shaolin Ji *

Abstract. Continuous-time mean-variance portfolio selection model with nonlinear wealth equations and bankruptcy prohibition is investigated by the dual method. A necessary and sufficient condition which the optimal terminal wealth satisfies is obtained through a terminal perturbation technique. It is also shown that the optimal wealth and portfolio is the solution of a forward-backward stochastic differential equation with constraints.

Keywords. continuous-time mean-variance portfolio selection model, stochastic optimal control, dual method, stochastic maximum principle, forward-backward stochastic differential equation (FBSDE)

AMS Subject classification. 60H30, 60H10

1 Introduction

Mean-variance portfolio selection in discrete time setting has been well studied. But mean-variance portfolio selection has received little attention in the context of continuous-time models [25]. Recently several papers studied various continuous-time Markowitz’s models [2, 12-16, 27, 28]. There are mainly two approaches which are employed to study this problem in continuous-time case: the forward (primal) method [15, 16, 27] which is inspired by the indefinite LQ control theory [26], and backward (dual) method which is employed by Bielecki et al. [2].

The dual method (also known as martingale method) is first studied by Harrison and Kreps [8] and Pliska [23, 24]. A systematic account on this method and its application to utility optimization problems can be found in [18] and the references therein. It mainly includes two steps: the first step is to compute

---

*School of Mathematics & System Sciences, Shandong University, Jinan 250100, People’s Republic of China. email: jsl@sdu.edu.cn. This work is supported by the National Basic Research Program of China (973 Program, No. 2007CB814900). This work was submitted to SIAM Journal on Control and Optimization on August 21, 2007 (manuscript number: 070066).
the optimal terminal wealth, and the second one is to compute the portfolio strategy replicating the obtained optimal terminal wealth. It is worth pointing out that the dual method is powerful in solving stochastic control problem with sample-wise constraint imposed on the state. A sample-wise constraint requires that the state be in a given set with probability 1; for example, a nonnegativity constraint on the wealth process, i.e., bankruptcy prohibition. For a deeper discussion we refer the reader to a recent paper by Ji and Zhou [11].

In this paper, we study the continuous-time mean-variance portfolio selection model with nonlinear wealth equation and bankruptcy prohibition. To apply the dual method, we first give a backward formulation of this problem in which the terminal wealth is regarded as the “control variable”. Note that, in this formulation, the initial wealth becomes an additional constraint. Under convexity assumptions, the backward formulation leads to a static convex programming problem. Then a terminal perturbation technique is introduced to derive a stochastic maximum principle which characterizes the optimal terminal wealth. Due to the convexity assumptions on the coefficients, we prove that the established stochastic maximum principle is also a sufficient condition. The terminal perturbation technique is first studied in El Karoui, Peng and Quenez [7] to solve a recursive utility optimization problem. Recently, Ji and Peng [10] use this technique and Ekeland’s variational principle to obtain a necessary condition for the mean-variance portfolio selection problem with non-convex wealth equations. Finally, we show that the optimal wealth and portfolio can be solved by a forward-backward stochastic differential equation (FBSDE) with constraints.

This paper is organized as follows: in section 2, we introduce continuous-time mean-variance portfolio selection model with nonlinear wealth equation and bankruptcy prohibition as well as its equivalent backward formulation. Applying Lagrange multiplier and terminal perturbation technique, we obtain a necessary and sufficient condition for optimality in section 3. In section 4, we prove that there exists an optimal solution of the continuous-time mean-variance portfolio selection problem and it can be obtained by solving a FBSDE. Finally, section 5 closes the paper with some concluding remarks.
2 Problem formulation

Let $W(\cdot) = (W_1(\cdot), \ldots, W_d(\cdot))'$ be a standard $d$-dimensional Brownian Motion defined on a complete probability space $(\Omega, \mathcal{F}, P)$. The information structure is given by a filtration $F = \{\mathcal{F}_t\}_{0 \leq t \leq T}$, which is the $\sigma$–algebra generated by the Brownian Motion $W(\cdot)$ and augmented. For any given Euclidean space $H$, we denote by $M^2(0,T;H)$, the space of all $\mathcal{F}_t$–progressively measurable processes $x(\cdot)$ with values in $H$, such that

$$E\int_0^T |x(t)|^2 \, dt < \infty.$$  

Denote by $L^2(\Omega, \mathcal{F}_T, P)$, the space of all $\mathcal{F}_T$–measurable random variable $\xi$ with value in $\mathbb{R}$, such that $E|\xi|^2 < \infty$.

2.1 The wealth process

Consider a complete market where there are one bank account (risk free instrument) and $d$ stocks (risky instruments), and an investor who can decide at time $t \in [0,T]$ the amount $\pi_i(t)$ to invest in the $i$th stock ($i = 1, \ldots d$) with initial investment $x > 0$. The respective prices of the instruments are $S_0(\cdot)$ and $S_1(\cdot), \cdots, S_d(\cdot)$, and the portfolio is $\pi(\cdot) = (\pi_1(t), \ldots, \pi_d(t))'$. We suppose that the wealth process $X(\cdot)$ is governed by the following stochastic differential equation

$$\begin{cases}
-dX(t) = f(X(t), \sigma(t)'\pi(t), t)dt - \pi(t)'\sigma(t)dW(t), \\
X(0) = x.
\end{cases}$$  

where the stock-volatility matrix $\sigma(\cdot) = \{\sigma_{ij}(\cdot)\}_{1 \leq i, j \leq d}$ is a predictable and bounded process. $\sigma(\cdot)$ is also assumed to be invertible and $\sigma^{-1}(\cdot)$ be bounded uniformly in $(t, \omega) \in [0,T] \times \Omega$. Set $Z(t) = \sigma(t)'\pi(t)$. Then (2.1) can be rewritten as

$$\begin{cases}
-dX(t) = f(X(t), Z(t), t)dt - Z(t)'dW(t), \\
X(0) = x.
\end{cases}$$  

We assume

(H1) $f$ is continuous in $\mathbb{R} \times \mathbb{R}^d \times [0,T]$ for $a.a.\omega$ and has continuous bounded derivatives in $(X, Z)$;

(H2) $f(0, 0, \cdot, \cdot) \in M^2(0,T;\mathbb{R})$;

(H3) $f$ is convex with respect to $(X, Z)$;
(H4) $f(0,0,t) \geq 0 \text{ a.s.}$

In the following, we give two specific examples to illustrate the model (2.1).

**Example 2.1 The standard linear case.**

The prices $S_0(\cdot)$ and $S_1(\cdot), \ldots, S_d(\cdot)$ are governed by the equations

$$
\begin{align*}
&dS_0(t) = S_0(t)r(t)dt, \quad S_0(0) = s_0; \\
&dS_i(t) = S_i(t)[b_i(t)dt + \sum_{j=1}^{d} \sigma_{ij}(t)dW(t)], \quad S_i(0) = s_i > 0; \quad i = 1, \ldots, d.
\end{align*}
$$

We assume: the interest rate $r(\cdot)$ is a non-negative, predictable and uniformly bounded scalar-valued process; the stock-appreciation rates $b(\cdot) = (b_1(\cdot), \ldots, b_d(\cdot))'$ is a predictable and uniformly bounded process.

Set $B(t) := (b_1(t) - r(t), \ldots, b_d(t) - r(t))'$. Define the risk premium process $\theta(t) \equiv (\theta_1(t), \ldots, \theta_d(t))'$ := $\sigma(t)^{-1}B(t)$. The wealth process $X(\cdot)$ satisfies the following linear stochastic differential equation

$$
\begin{align*}
\begin{cases}
    dX(t) &= [r(t)X(t) + \pi(t)'\sigma(t)\theta(t)]dt + \pi(t)'\sigma(t)dW(t), \\
    X(0) &= x.
\end{cases}
\end{align*}
$$

(2.3)

Note that for this case,

$$
    f(X, \sigma(t)'\pi, t) = -r(t)X - \pi'\sigma(t)\theta(t).
$$

**Example 2.2 A large investor case.**

An interesting example of a nonlinear wealth equation is the optimal portfolio choice problem for a large investor considered in Cuoco and Cvitanic [4]. Refer to [3, 5, 7] for other models. In [4], $S_0(\cdot)$ and $S_1(\cdot), \ldots, S_d(\cdot)$ are described by equations

$$
\begin{align*}
&dS_0(t) = S_0(t)[r(t) + l_0(X(t), \pi(t))]dt, \quad S_0(0) = s_0; \\
&dS_i(t) = S_i(t)[b_i(t) + l_i(X(t), \pi(t))]dt + \sum_{j=1}^{d} \sigma_{ij}(t)dW(t)], \quad S_i(0) = s_i > 0; \quad i = 1, \ldots, d
\end{align*}
$$

where $l_i : \mathbb{R}^+ \times \mathbb{R}^d \rightarrow \mathbb{R}, 0 \leq i \leq d$ are given functions which describe the effect of the wealth and the strategy. In this case,

$$
    f(X, \sigma(t)'\pi, t) = -r(t)X - (X - \pi'1)l_0(X, \pi) - \pi'[b(t) - r(t)1 + l(X, \pi)].
$$
2.2 Backward formulation of the problem

Before formulating the problem, we point out that we distinguish the concepts between initial investment and initial wealth. Throughout this paper, we suppose that the initial investment \( x \) of the investor is less than or equal to his initial wealth \( y \), i.e. \( x \leq y \).

Usually the continuous-time mean-variance portfolio selection problem with bankruptcy prohibition is formulated as: the investor chooses his portfolio and initial investment \( x \) so as to

\[
\begin{align*}
\text{Minimize} & \quad \text{Var } X(T) \equiv E[X(T)^2] - c^2, \\
\text{subject to} & \quad \begin{cases} 
E[X(T)] = c, \\
X(t) \geq 0 \quad \text{a.s., } t \in [0, T], \\
\pi(\cdot) \in M^2(0, T; \mathbb{R}^m), \\
(X(\cdot), \pi(\cdot)) \text{ satisfies equation (2.1) and } 0 < x = X(0) \leq y,
\end{cases}
\end{align*}
\]

where \( c > 0 \) is a given expectation level with respect to the investor’s terminal wealth \( X(T) \), and \( X(t) \geq 0 \) means that no-bankruptcy is required.

**Definition 2.3** A portfolio \( \pi(\cdot) \) is said to be admissible if \( \pi(\cdot) \in M^2(0, T; \mathbb{R}^m), E X(T) = c \) and the corresponding wealth processes \( X(t) \geq 0 \) a.s., \( \forall t \in [0, T] \).

We denote by \( A(x) \) the set of portfolio \( \pi(\cdot) \) admissible for the initial investment \( x \). Set

\[
V(y) = \min_{0 < x \leq y, \pi \in A(x)} \{ E[X^x,\pi(T)]^2 - c^2 \}. 
\]

In the following we give an equivalent backward formulation of the above optimization problem (2.4).

Since \( \sigma(\cdot) \) is invertible, \( Z(\cdot) \) can be regarded as the “control variable” instead of \( \pi(\cdot) \). Notice that selecting \( Z(\cdot) \) is equivalent to selecting the terminal wealth \( X(T) \) by the backward stochastic differential equation (BSDE) theory [21]. Hence the wealth equation (2.2) can be rewritten as

\[
\begin{align*}
&\begin{cases} 
-dX(t) = f(X(t), Z(t), t)dt - Z(t)'dW(t), \\
X(T) = \xi
\end{cases} \\
&\xi \in L^2(\Omega, \mathcal{F}_T, P), \xi \geq 0, \text{a.s.}
\end{align*}
\]

where the terminal wealth \( \xi \) is the ”control” to be chosen from the following set

\[
U = \{ \xi \mid \xi \in L^2(\Omega, \mathcal{F}_T, P), \xi \geq 0, \text{a.s.} \}.
\]
Note that nonnegative terminal wealth, i.e., $\xi = x(T) \geq 0$ keeps the wealth process nonnegative all the time, as implied by Assumption (H4) and the comparison theorem for BSDEs.

This gives rise to the following optimization problem:

$$\text{Minimize \quad } J(\xi) \triangleq (E\xi^2 - c^2)$$

subject to

$$\begin{cases} 
E\xi = c, \\
X(0) \leq y, \\
\xi \in U.
\end{cases}$$

(2.7)

It is clear that the original problem (2.4) is equivalent to (2.7). Hence, hereafter we focus ourselves on solving (2.7). The advantage of doing this lies in the fact that the state constraint in (2.4) now becomes a control constraint in (2.7) since $\xi$ is regarded as the control variable. It is well known in control theory that a control constraint is easier to deal with than a state constraint. But there is a cost of doing so: the original initial condition $X(0) = x$ now becomes a constraint, i.e., $X(0) \leq y$.

It is easy to prove that Assumptions (H1) and (H2) ensure there exists a unique pair $(X(\cdot), Z(\cdot)) \in M^2(0, T; R) \times M^2(0, T; R^d)$ of (2.6) [21]. From now on, we denote the solution of (2.6) by $(X^\xi(\cdot), Z^\xi(\cdot))$, whenever necessary, to show the dependence on $\xi$. We also denote $X^\xi(0)$ by $X^\xi_0$.

**Definition 2.4** $\xi$ is called admissible for given $y > 0$ and $c > 0$, if $\xi \in U$ and the solution of (2.6) satisfies $X^\xi_0 \leq y$, $E\xi = c$. We shall denote by $\mathcal{N}(y)$, the set of all admissible $\xi$‘s for any given $y$ and $c$.

An admissible $\xi^*$ is called optimal if it attains the minimum of $J(\xi)$ over $\mathcal{N}(y)$. From above discussions, we know that $V(y) = J(\xi^*)$. The optimal portfolio for (2.7) is called a variance minimizing portfolio. After the optimal terminal wealth $\xi^*$ is obtained, we can compute the optimal portfolio by solving (2.6).

For the feasibility of above optimization problem (2.4) and (2.7), we assume the following slater condition:

(H5) For given $y > 0$ and $c > 0$, there exist an initial investment $x^o(0 < x^o < y)$ and a portfolio $\pi^o$ such that the corresponding terminal wealth $X^o(T) \geq 0$ and $EX^o(T) = c$. 

Remark. In fact, the feasibility of (2.4) and (2.7) can be checked by solving another optimization problem. For more details, see Appendix A.

Note that if $y \geq X_0^c$, then $\xi \equiv c$ is admissible. In this case, it is obvious that $V(y) = 0$. Hence, without loss of generality we can assume

(H6) $y < X_0^c$.

3 A sufficient and necessary condition for optimality

In this section, we derive a sufficient and necessary condition which characterizes the optimal terminal wealth.

It is easy to check that the following $R-$valued functionals on $U$

$$
\begin{align*}
\xi \mapsto X_0^\xi - y, \\
\xi \mapsto E\xi^2 - c^2, \\
\xi \mapsto E\xi - c
\end{align*}
$$

are convex under Assumption (H3). Hence, applying classical results of convex analysis [17], it is easy to obtain the following lemma.

Lemma 3.1 We suppose (H1)-(H6). There exist real numbers $\lambda_1 \geq 0$ and $\lambda_2$ such that

$$
V(y) = \min_{\xi \in U} \{E\xi^2 - c^2 + \lambda_1(X_0^\xi - y) + \lambda_2(E\xi - c)\}. \quad (3.1)
$$

Furthermore, if the minimum is attained in (2.7) by $\xi^*$, then it is attained in (3.1) by $\xi^*$ with $\lambda_1(X_0^{\xi^*} - y) = 0$. Conversely, suppose there exist $\lambda_1^o \geq 0$, $\lambda_2^o \in R$ and $\xi^o \in U$ such that the minimum is achieved in

$$
\min_{\xi \in U} \{E\xi^2 - c^2 + \lambda_1^o(X_0^\xi - y) + \lambda_2^o(E\xi - c)\}
$$

with $\lambda_1^o(X_0^{\xi^o} - y) = 0$, then the minimum is achieved in (2.7) by $\xi^o$.

In the following, we introduce a terminal perturbation technique which is used in [7, 10].

Let $\xi^*$ be optimal for (2.7) and $(X^*(\cdot), Z^*(\cdot))$ be the corresponding optimal trajectory, i.e., the solution of (2.6) under $\xi^*$. Let $\hat{\xi} \in L^2(\Omega, \mathcal{F}, P)$ such that $(\xi^* + \hat{\xi}) \in U$. Since $U$ is convex, then for any $0 \leq \rho \leq 1$,

$$
\xi^o = \xi^* + \rho \hat{\xi}
$$
is also in $U$. Let $(\delta X(\cdot), \delta Z(\cdot))$ be the solution of the following first order variational equation
\begin{equation}
\begin{cases}
-d\delta X(t) = [f_X(X^*(t), Z^*(t), t)\delta X(t) + f_Z(X^*(t), Z^*(t), t)\delta Z(t)]dt - \delta Z(t)'dW(t), \\
\delta X(T) = \breve{\xi}.
\end{cases}
\end{equation}
(3.2)
Note that (3.2) is a linear BSDE and it has a unique pair $(\delta X(\cdot), \delta Z(\cdot)) \in M^2(0, T; R) \times M^2(0, T; R^d)$. We denote by $(X^\rho(\cdot), Z^\rho(\cdot))$ the solution of (2.6) corresponding to $X(T) = \xi^\rho$. Set
\begin{align*}
\breve{X}^\rho(t) &= \rho^{-1}[X^\rho(t) - X^*(t)] - \delta X(t), \\
\breve{Z}^\rho(t) &= \rho^{-1}[Z^\rho(t) - Z^*(t)] - \delta Z(t).
\end{align*}
Using the techniques in [22], we have the following convergence results.

**Lemma 3.2** Assume (H1) and (H2), then
\[ \lim_{\rho \to 0} \sup_{0 \leq t \leq T} E | \breve{X}^\rho(t) |^2 = 0, \]
\[ \lim_{\rho \to 0} E \int_0^T | \tilde{Z}^\rho(t) |^2 dt = 0. \]

For the reader's convenience, we sketch the proof of Lemma 3.2 in the Appendix B.

In order to derive the necessary condition, we introduce the adjoint equation
\begin{equation}
\begin{cases}
dq(t) = q(t)[f_X(X^*(t), Z^*(t), t)dt + f_Z(X^*(t), Z^*(t), t)'dW(t)], \\
q(0) = 1
\end{cases}
\end{equation}
(3.3)
where $(X^*(\cdot), Z^*(\cdot))$ is the optimal trajectory with respect to $\xi^*$. (3.3) is a linear stochastic differential equation and it has a unique solution in $M^2(0, T; R)$.

Set
\[ M \triangleq \{ \omega \in \Omega | \xi^*(\omega) = 0 \}. \]

**Theorem 3.3** We assume (H1)-(H6). $\xi^*$ is optimal to (2.7) if and only if there exist constants $\lambda_1 > 0$ and $\lambda_2 \in R$ such that
\begin{align*}
2\xi^*(\omega) + \lambda_1 q_T(\omega) + \lambda_2 &\geq 0 \quad \text{a.s. on } M, \\
2\xi^*(\omega) + \lambda_1 q_T(\omega) + \lambda_2 &= 0 \quad \text{a.s. on } M^c
\end{align*}
(3.4)
with $X^\xi_0 = y$, where $q(t)$ is the solution of the adjoint equation (3.3).
Proof. (1) Proof of the necessary condition.

By Lemma 3.1, there exist constants $\lambda_1 \geq 0$ and $\lambda_2$ such that

$$E(\xi^p)^2 - c^2 + \lambda_1 (X_0^\xi - y) + \lambda_2 (E\xi^p - c) \geq E(\xi^*)^2 - c^2 + \lambda_1 (X_0^\xi - y) + \lambda_2 (E\xi^* - c).$$

Dividing the inequality by $\rho$ and sending $\rho$ to 0, we obtain

$$2E(\xi^* \hat{\xi}) + \lambda_1 \delta X(0) + \lambda_2 E\hat{\xi} \geq 0$$

where $\delta X(0)$ denotes the solution of (3.2) at time 0.

Applying Itô’s lemma to $\delta X(t)q(t)$ yields

$$E[\delta X(T) \cdot q(T) - \delta X_0 \cdot q(0)]$$

$$= E[-\int_0^T [(f_X(X^*(t), Z^*(t), t)\delta X(t) + f'_Z(X^*(t), Z^*(t), t)\delta Z(t))q(t)]dt + \int_0^T [f_X(X^*(t), Z^*(t), t)\delta X(t)q(t) + < \delta Z(t), f_Z(X^*(t), Z^*(t), t)q(t) >]dt]$$

$$= 0.$$ 

Since $q(0) = 1$, it is obvious that

$$\delta X_0 = E[\hat{\xi} \cdot q(T)].$$

Replacing $\delta X_0$ with $E[\hat{\xi} \cdot q(T)]$ in (3.5), we have that for each $\bar{\xi} \in U$, the following inequality holds

$$2E(\xi^* \hat{\xi}) + \lambda_1 E[\hat{\xi} \cdot q(T)] + \lambda_2 E\hat{\xi}$$

$$= E[(2\xi^* + \lambda_1 q(T) + \lambda_2) \cdot \hat{\xi}]$$

$$= E[(2\xi^* + \lambda_1 q(T) + \lambda_2) \cdot (\bar{\xi} - \xi^*)]$$

$$\geq 0.$$ 

Thus, it is easy to check that for each $\varepsilon > 0$

$$P\{\omega \mid \omega \in M, 2\xi^* + \lambda_1 q(T) + \lambda_2 < -\varepsilon\} = 0.$$ 

From the continuity property of probability, we have

$$2\xi^* + \lambda_1 q(T) + \lambda_2 \geq 0 \ a.s. \ on \ M.$$ 

By a similar argument,

$$2\xi^* + \lambda_1 q(T) + \lambda_2 = 0 \ a.s. \ on \ M^c.$$
Now we show that $\lambda_1 \neq 0$. If $\lambda_1 = 0$, (3.4) becomes
\begin{align}
\xi^*(\omega) &\geq -\frac{\lambda_2}{2} \text{ a.s. on } M, \\
\xi^*(\omega) &= -\frac{\lambda_2}{2} \text{ a.s. on } M^c.
\end{align}
(3.8)

There are two cases: one is $M$ is nonempty and the other is $M$ is empty. For the first case, we deduce that $\xi^* = 0$ which contradicts to the constraint $E\xi^* = c > 0$. For the second case, we have that $\xi^* = c$ from (3.8) and the constraint $E\xi^* = c$. But this contradicts to Assumption (H6). In summary, we have $\lambda_1 > 0$.

By Lemma 3.1, we know $\lambda_1(X_0^\xi - y) = 0$. Since $\lambda_1 > 0$, it is easy to see $X_0^\xi = y$ holds.

(2) Proof of the sufficient condition.

Let $\xi \in U$ with $(X(\cdot), Z(\cdot))$ be the corresponding trajectory. From lemma 3.1 we need only to prove that for any $\xi \in U$
\begin{align}
E\xi^2 - c^2 + \lambda_1(X_0^\xi - y) + \lambda_2(E\xi - c) &\geq E(\xi^*)^2 - c^2 + \lambda_1(X_0^{\xi^*} - y) + \lambda_2(E\xi^* - c),
\end{align}
i.e., to prove
\begin{align}
E\xi^2 - E(\xi^*)^2 + \lambda_1(X_0^\xi - X_0^{\xi^*}) + \lambda_2(E\xi - \xi^*) &\geq 0.
\end{align}

Set
\begin{align}
\hat{\xi} &= \xi - \xi^*, \\
f_1(x, z, t) &= f(X^*(t) + x, Z^*(t) + z, t) - f(X^*(t), Z^*(t), t), \\
f_2(x, z, t) &= f_x(X^*(t), Z^*(t), t)x + f_Z(X^*(t), Z^*(t), t)z.
\end{align}

Consider the following equation
\begin{align}
\begin{cases}
-d(X(t) - X^*(t)) &= [f(X(t), Z(t), t) - f(X^*(t), Z^*(t), t)]dt - (Z(t) - Z^*(t))'dW(t), \\
X(T) - X^*(T) &= \hat{\xi}.
\end{cases}
\end{align}

By Assumption (H3),
\begin{align}
f_1(x, z, t) &\geq f_2(x, z, t) \quad \forall x, z, \\
dP \otimes dt \quad \text{a.s.}
\end{align}

Hence applying the comparison theorem for BSDEs, we obtain $X(t) - X^*(t) \geq \delta X(t), \forall t \quad P - \text{a.s.}$, where $\delta X(\cdot)$ is the solution of (3.2).
Using the following inequality
\[(\xi^*)^2 - \xi^2 - 2\xi^*(\xi - \xi^*)\]
and (3.6), we have
\[
E\xi^2 - E(\xi^*)^2 + \lambda_1(X_0^\xi - X_0^\xi^*) + \lambda_2 E(\xi - \xi^*) \\
\geq 2E[\xi^*(\xi - \xi^*)] + \lambda_1 \delta X(0) + \lambda_2 E(\xi - \xi^*) \\
\geq 2E(\xi^\xi) + \lambda_1 \delta X(0) + \lambda_2 E\xi \\
\geq E[(2\xi^* + \lambda_1 q(T) + \lambda_2)\xi].
\]

Since (3.4) implies
\[E[(2\xi^* + \lambda_1 q(T) + \lambda_2)\xi] \geq 0,
\]
we obtain the result. The proof is complete. ■

4 Existence of the optimal solution

In this section, we prove that there exists a unique optimal solution for the optimization problem (2.7). We also show that the optimal solution can be obtained by solving a FBSDE with constraints.

**Theorem 4.1** Suppose that (H1)-(H6) hold. Then there exists a unique \(\xi^* \in L^2(\Omega, \mathcal{F}_T, P)\) which attains the minimum of the problem (2.7).

**Proof.** The uniqueness is due to the strict convexity of the functional
\[\xi \mapsto J(\xi), \quad \xi \in U.\]

As for the existence, consider the set given by
\[B = \{\xi \in \mathcal{N}(y); \quad J(\xi) \leq C\}\]
where \(C > 0\) is a constant. It is clear that, for each constant \(C, B\) is bounded, closed and convex. Hence \(B\) is weakly compact and by classical results of convex analysis [1], we need only to show that \(J\) is weakly lower-semicontinuous. Since \(J\) is convex and strongly lower-semicontinuous (in fact, it is strongly continuous [6, 7]), it follows that \(J\) is lower-semicontinuous for the weak convergence [1].

Thus the minimum of the problem (2.7) is attained (refer to Corollary 3.20 in [1]). The proof is complete. ■
Corollary 4.2 We assume (H1)-(H6). Then there exist constants \( \lambda_1 > 0 \) and \( \lambda_2 \in \mathbb{R} \) such that the optimal \( \xi^* \) has the form

\[
\xi^* = \frac{1}{2}(-\lambda_2 - \lambda_1 q(T))^+. \tag{4.1}
\]

This is a direct consequence of Theorem 3.3 and Theorem 4.1. The proof is omitted.

Let \((X^*(\cdot), Z^*(\cdot))\) be the optimal wealth process and portfolio associated with \( \xi^* \) for problem (2.7).

Theorem 4.3 Suppose that (H1)-(H6) hold. Then there exist a positive number \( \lambda_1 \) and \( \lambda_2 \in \mathbb{R} \) such that the following FBSDE

\[
\begin{align*}
\left\{ 
\begin{array}{l}
   dq(t) = q(t)[f_X(X(t), Z(t), t)dt + f_Z(X(t), Z(t), t)'dW(t)], \\
   q(0) = 1, \\
   -dX(t) = f(X(t), Z(t), t)dt - Z(t)''dW(t), \\
   X(T) = \frac{1}{2}(-\lambda_2 - \lambda_1 q(T))^+
\end{array}
\right.
\tag{4.2}
\end{align*}
\]

with constraints

\[
EX(T) = c \quad \text{and} \quad X(0) = y \tag{4.3}
\]

has a unique solution \((q(\cdot), X(\cdot), Z(\cdot))\). Furthermore, we have \((X(\cdot), Z(\cdot)) = (X^*(\cdot), Z^*(\cdot)) \) and \(X(T) = \xi^* \).

Proof. Note that (4.1) is equivalent to (3.4). Then it is easy to check that the solution of FBSDE (4.2) with (4.3) is just the optimal solution of problem (2.7) by Theorem 3.3 and Theorem 4.1. The proof is complete. ■

Finally, we show that the smoothness condition, i.e., Assumption (H1) may not hold for the following examples:

Example 4.4 Suppose that taxes must be paid on the gains which are made on the risky securities. The wealth process \( X \) is governed by

\[
\begin{align*}
\left\{ 
\begin{array}{l}
   -dX(t) = -[r(t)X(t) + \pi(t)'\sigma(t)\theta(t) - \alpha(\pi(t)'\sigma(t)\theta(t)))^+]dt - \pi(t)'\sigma(t)dW(t), \\
   X(0) = x.
\end{array}
\right.
\tag{4.4}
\end{align*}
\]

Example 4.5 Suppose that the borrowing interest rate \( R(t) \geq r(t) \). In this case, the wealth process \( X \) satisfies

\[
\begin{align*}
\left\{ 
\begin{array}{l}
   -dX(t) = -[r(t)X(t) + \pi(t)'\sigma(t)\theta(t) - (R(t) - r(t))(X(t) - \sum_{i=1}^{d}\pi_i(t))^-]dt - \pi(t)'\sigma(t)dW(t), \\
   X(0) = x.
\end{array}
\right.
\tag{4.5}
\end{align*}
\]
But in this case, we can still prove that (3.4) is a sufficient condition for optimality. To this end, we need an additional assumption:

(H1)' $f$ is uniformly Lipschitz with respect to $(X, Z)$.

Let $\xi^* \in U$ and $(X^*(\cdot), Z^*(\cdot))$ be the corresponding trajectory.

**Theorem 4.6** Suppose that (H1)' and (H2)-(H6) hold. If there exist constants $\lambda_1 > 0$ and $\lambda_2 \in \mathbb{R}$ such that (3.4) with $X_0^\xi = y$ is satisfied or equivalently, (4.2) with (4.3) has a solution $(q(\cdot), X^*(\cdot), Z^*(\cdot))$, then $\xi^* = X^*(T)$ is an optimal terminal wealth for problem (2.7).

**Proof.** We should only use subdifferentials instead of differentials in the second part proof of Theorem 3.3. Note that now $f_X$ (resp. $f_Z$) denotes a predictable process belonging $dP \otimes dt$ almost surely to $\partial f(X^*(t), Z^*(t), t)$, where $\partial f$ is the subdifferential of $f$ with respect to $X$ (resp. $Z$).

The proof is complete. ■

**5 Concluding remarks**

This paper investigates the continuous-time mean-variance portfolio selection model with nonlinear wealth equation and bankruptcy prohibition. A stochastic maximum principle is established via the dual method and terminal perturbation technique. Under the smoothness conditions on the coefficients (Assumption (H1)), we prove that the established stochastic maximum principle is not only a necessary but also a sufficient condition for the optimal terminal wealth. Then the optimal wealth and portfolio strategy, i.e., the solution of the FBSDE (4.2) can be computed by the PDE approach of Ma, Protter and Yong [19], the probability method of Hu and Peng [9] or numerical methods (see also [20] for systematical investigation). If the smoothness assumption does not hold, we only obtain a sufficient condition, i.e., Theorem 4.6. In this case, the main difficulty lies in the fact that the corresponding FBSDE (4.2) may have discontinuous coefficients. We emphasize that it remains an interesting open problem to solve FBSDEs with discontinuous coefficients. But as shown in Theorem 4.6, our method in this paper can be used to derive the existence of solutions for FBSDE (4.2). Another important point to note here is that the existing results in the utility framework can’t cover the mean-variance model at all since the
usual assumptions imposed on utility functions are different from those on the mean-variance models.

Appendix A.

Feasibility analysis.

For a given initial investment $x > 0$ and $c > 0$, if there exists a portfolio $\pi(\cdot) \in A(x)$, the initial investment $x$ is called admissible. Our aim is to compute the minimal admissible initial investment which is denoted by $\bar{x}$. If $\bar{x} \leq y$ (resp. $\bar{x} < y$), the optimization problem (2.4) and (2.7) are feasible (resp. the slater condition holds).

Using similar analysis as in section 2, we can obtain $\bar{x}$ by solving the following optimization problem:

$$\bar{x} = \inf_{\xi \in U} X_0^\xi,$$

subject to $E\xi = c$.

For $\lambda \in R$, define

$$\varphi(\lambda) = \inf_{\xi \in U} [X_0^\xi + \lambda E(\xi - c)].$$

By the classical results of duality theory [17], we have

$$\bar{x} = \max_{\lambda \in R} \varphi(\lambda).$$

Appendix B.

Proof of Lemma 3.2. From (2.6) and (3.2), we have

$$\begin{cases}
-d\bar{X}^\rho(t) = \rho^{-1}[f(X^\rho(t), Z^\rho(t), t) - f(X^*(t), Z^*(t), t) - \rho f_X(X^*(t), Z^*(t), t)\delta X(t) - f_Z^\rho(X^*(t), Z^*(t), t)\delta Z(t)]dt - \bar{Z}^\rho(t)'dW(t), \\
\bar{X}^\rho(T) = 0.
\end{cases}$$

Let

$$A^\rho(t) = \int_0^1 f_X(X^*(t) + \lambda\rho(\delta X(t) + \bar{X}^\rho(t)), Z^*(t) + \lambda\rho(\delta Z(t) + \bar{Z}^\rho(t)), t)d\lambda,$$

$$B^\rho(t) = \int_0^1 f_Z(X^*(t) + \lambda\rho(\delta X(t) + \bar{X}^\rho(t)), Z^*(t) + \lambda\rho(\delta Z(t) + \bar{Z}^\rho(t)), t)d\lambda,$$

$$C^\rho(t) = [A^\rho(t) - f_X(X^*(t), Z^*(t), t)]\delta X(t) + [B^\rho(t) - f_Z(X^*(t), Z^*(t), t)]\delta Z(t).$$
Thus

\[
\begin{aligned}
  \left\{
    \begin{array}{l}
      -d\tilde{X}^\rho(t) = (A^\rho(t) \cdot \tilde{X}^\rho(t) + B^\rho(t) \cdot \tilde{Z}^\rho(t) + C^\rho(t))dt - \tilde{Z}^\rho(t)^t dW(t), \\
      \tilde{X}^\rho(T) = 0
    \end{array}
  \right.
\end{aligned}
\]

Using Itô’s formula to $|\tilde{X}^\rho(t)|^2$ we get

\[
\begin{aligned}
  E|\tilde{X}^\rho(t)|^2 &+ E\int_t^T |\tilde{Z}^\rho(s)|^2 ds \\
  = 2E\int_t^T \tilde{X}^\rho(s)(A^\rho(s) \cdot \tilde{X}^\rho(s) + B^\rho(s) \cdot \tilde{Z}^\rho(s) + C^\rho(s))ds \\
  \leq KE\int_t^T |\tilde{X}^\rho(s)|^2 ds + \frac{1}{2}E\int_t^T |\tilde{Z}^\rho(s)|^2 ds + E\int_t^T |C^\rho(s)|^2 ds
\end{aligned}
\]

where $K$ is a constant. So

\[
\begin{aligned}
  E|\tilde{X}^\rho(t)|^2 &+ \frac{1}{2}E\int_t^T |\tilde{Z}^\rho(s)|^2 ds \\
  \leq KE\int_t^T |\tilde{X}^\rho(s)|^2 ds + E\int_t^T |C^\rho(s)|^2 ds
\end{aligned}
\]

By the Lebesgue dominate convergence theorem, we have

\[
\lim_{\rho \to 0} E\int_0^T |C^\rho(t)|^2 dt = 0.
\]

Applying Gronwall’s inequality, we obtain the result. ■

Acknowledgement. The author would like to thank Prof. Shige Peng and Dr. Hanqing Jin for some useful conversations. Especially, Assumptions (H5)-(H6) are due to helpful discussions with Prof. Xunyu Zhou.

References

[1] H. Brezis, Analyse fonctionnelle, Masson Paris, 1983.

[2] T. R. Bielecki, H. Jin, S. R. Pliska and X. Zhou, Continuous time mean variance portfolio selection with bankruptcy prohibition, Math. Finance, 15 (2005), pp. 213-244.

[3] R. Buchdahn and Y. Hu, Hedging contingent claims for a large investor in an incomplete market, Adv. Appl. Prob., 30 (1998), pp. 239-255.
[4] D. Cuoco and J. Cvitanic, Optimal consumption choices for a "large" investor, J. Economic Dynamics and Control, 22 (1998), pp. 401-436.

[5] J. Cvitanic and J. Ma, Hedging options for a large investor and forward-backward sde's, Ann. Appl. Prob., 6 (1996), pp. 370-398.

[6] N. El Karoui, S. Peng and M. C. Quenez, Backward stochastic differential equations in finance, Math.Finance, 7 (1997), pp. 1-71.

[7] N. El Karoui, S. Peng, M.-C. Quenez, A dynamic maximum principle for the optimization of recursive utilities under constraints, Ann. Appl. Prob., (11) 2001, pp. 664–693.

[8] J. M. Harrison and D. Kreps, Martingales and multiperiod securities market, J. Econom.. Theory, 20 (1979), pp. 381-408.

[9] Y. Hu, and S. Peng, Solution of a forward-backward stochastic differential equation, Probab. Theory Related Fields, 103 (1995), pp. 273–283.

[10] S. Ji and S. Peng, Terminal perturbation method for the backward approach to continuous-time mean-variance portfolio selection, to appear in Stochastic Processes and their Applications (2007), doi:10.1016/j.spa.2007.07.005.

[11] S. Ji and X. Zhou, A maximum principle for stochastic optimal control with terminal state constraints, and its applications, Communications in Information and Systems, 6 (2006) (a special issue dedicated to Tyrone Duncan on the occasion of his 65th birthday), pp. 321-337.

[12] H. Jin, J. Yan and X. Zhou, Continuous-time mean–risk portfolio selection, Annales de l’Institut Henri Poincare (B) Probabilites et statistiques, 41 (2005), pp. 559-580.

[13] H. Jin and X. Zhou, Continuous-Time Markowitz’s Problems in an Incomplete Market, with No-Shorting Portfolios, in Stochastic Analysis and Applications, F. Benth, G. Nunno, T. Lindstrom, B. Øksendal and T. Zhang, ed., Springer-Verlag Berlin Heidelberg, 2007, pp. 435-459.

[14] A. E. B Lim, Quadratic hedging and mean–variance portfolio selection with random parameters in an incomplete market, Math. Oper. Res., 29 (2004), pp. 132-161.
[15] X. Li, X. Zhou and A. E. B. Lim, Dynamic mean variance portfolio selection with no shorting constraints, SIAM J. Control Optim., 40 (2001), pp. 1540-1555.

[16] A. E. B. Lim and X. Zhou, Mean variance portfolio selection with random parameters, Math. Oper. Res., 27 (2002), pp. 101-120.

[17] D. Luenberger, Optimization by vector space methods, Wiley, New York, 1969.

[18] I. Karatzas and S. E. Shreve, Methods of Mathematical Finance, Springer-Verlag, New York, 1998.

[19] J. Ma, P. Protter, and J. Yong, Solving forward-backward stochastic differential equations explicitly—a four step scheme, Probab. Theory Related Fields, 98 (1994), pp. 339-359.

[20] J. Ma and J. Yong, Forward-backward stochastic differential equations and their applications, Lecture Notes in Mathematics 1702, Springer-Verlag, Berlin, 1999.

[21] E. Pardoux and S. Peng, Adapted solution of a Backward stochastic differential equations, System and Control Letters, 14 (1990), pp. 55-61.

[22] S. Peng, Backward stochastic differential equation and application to optimal control, Applied Mathematics and Optimization, 27 (1993) pp. 125-144.

[23] S. R. Pliska, A discrete time stochastic decision model, in Lecture Notes in Control and Information Sciences 42, W. H. Fleming and L. G. Gorostiza, ed., Springer-Verlag, New York, 1982, pp. 290-304.

[24] S. R. Pliska, Introduction to mathematical finance, Blackwell, Oxford, 1997.

[25] M. C. Steinbach, Markowitz revisited: Mean–variance models in financial portfolio analysis, SIAM Rev., 43 (2001), pp. 31-85.

[26] J. Yong and X. Zhou, Stochastic Controls Hamiltonian Systems and HJB Equations, Springer, New York, 1999.
[27] X. Zhou and D. Li, Continuous time mean variance portfolio selection: A stochastic LQ framework, Appl. Math. Optim., 42 (2000), pp. 19-33.

[28] X. Zhou and G. Yin, Markowitz’s mean-variance portfolio selection with regime switching: A Continuous-Time Model, SIAM J. Control Optim., 42 (2003), pp. 1466-1482.