FUNCTIONAL INEQUALITIES INVOLVING NONLOCAL OPERATORS ON COMPLETE RIEMANNIAN MANIFOLDS AND THEIR APPLICATIONS TO THE FRACTIONAL POROUS MEDIUM EQUATION

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Abstract. The objective of this paper is twofold. First, we conduct a careful study of various functional inequalities involving the fractional Laplacian operators, including nonlocal Sobolev-Poincaré, Nash, Super Poincaré and logarithmic Sobolev type inequalities, on complete Riemannian manifolds satisfying some mild geometric assumptions. Second, based on the derived nonlocal functional inequalities, we analyze the asymptotic behavior of the solution to the fractional porous medium equation, \( \partial_t u + (-\Delta)\sigma(u^{m-1}) = 0 \) with \( m > 0 \) and \( \sigma \in (0, 1) \). In addition, we establish the global well-posedness of the equation on an arbitrary complete Riemannian manifold.

1. Introduction

This manuscript is mainly motivated by the asymptotic behavior of the solution to the following fractional porous medium equation:

\[
\begin{aligned}
\partial_t u + (-\Delta)\sigma(|u|^{m-1}u) &= 0 \quad \text{on } M \times (0, \infty); \\
u(0) &= u_0 \quad \text{on } M
\end{aligned}
\]

for \( m \in (0, \infty) \) and \( \sigma \in (0, 1) \) on a smooth complete Riemannian manifold \((M, g)\) without boundary. Here \( \Delta \) is the Laplace-Beltrami operator associated with the Riemannian metric \( g \).

Global well-posedness of the Cauchy problem of (1.1) was first studied by A. Pablo, F. Quirós, A. Rodríguez and J.L. Vázquez in [38, 39] in Euclidean spaces, and later by M. Bonforte, A. Pablo, F. Quirós, A. Rodríguez, Y. Sire and J.L. Vázquez in [15, 16, 17, 39] for the Dirichlet problem on bounded domains. Since then, there has been a vast amount of work [11, 12, 51, 52], just to name a few, investigating various properties, e.g. regularity and Barenblatt solutions, of the solutions to (1.1). See also [2, 28, 41] for some related work. Most of the work on (1.1) considered flat spaces. As far as we know, the only exception is the work [2] by A. Alphonse and C.M. Elliott, which studied a variation of (1.1) with fractional power \( \sigma = 1/2 \) on a closed manifold. Very recently, we studied (1.1) on an incomplete manifold with isolated conical singularities and finite volume in [43, 44].

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If we come to the problem of asymptotic behavior of the solution to (1.1) on Riemannian manifolds, it becomes clear that an essential component is still missing. Indeed, as shown in [39, Section 5], certain nonlocal functional inequalities play a crucial role in the asymptotic analysis of (1.1) in Euclidean spaces. It is expected as the role of the local counterparts of these inequalities have already been recognized in the asymptotic analysis of the porous medium equation in Euclidean spaces.

In manifolds setting, most of the research [14, 29, 30, 31, 32, 33, 34, 35, 53] on large time behavior of the porous medium equation focuses on manifolds of non-positive curvature, e.g. the hyperbolic space. This is mainly due to the validity of certain functional inequalities in such spaces. In order to study (1.1) for more general manifolds setting, we will use the idea by M. Bonforte and G. Grillo in [13], which studied the asymptotic behavior of solution to the porous medium equation based on some logarithmic Sobolev inequalities. Such inequalities are known to hold on closed manifolds or manifolds satisfying the Faber-Krahn inequality. The asymptotic behavior or the smoothing effect of the porous medium equation can be viewed as a special case of the ultracontractivity property. We recall that the ultracontractivity property of an equation allows the $L_\infty$—norm of the solution to be bounded by the $L_p$—norm of the initial datum. As was first discovered in the pioneering work [36] by L. Gross, the ultracontractivity of the heat semigroup is closely related to logarithmic Sobolev inequalities. This relationship was later explored in more depth in the monograph [24] by E. Davies. The proofs heavily rely on the theory of symmetric Markov semigroups, which, briefly speaking, are strongly continuous symmetric semigroups that are order-preserving and contractive on $L_\infty(M)$. The idea of using logarithmic Sobolev inequalities to study the ultracontractive bounds and asymptotics of nonlinear evolutions traces back to the work [20] by E. Carlen and M. Loss. This method was later applied to the $p$-Laplacian equation in [21]. Those observations suggest that a similar approach should be applicable to (1.1) as well, because the associated nonlinear semigroup to (1.1) satisfies some order-preserving and contractive properties.

However, the validity of nonlocal versions of logarithmic Sobolev inequalities remains an open question on general Riemannian manifolds. A widely used approach to functional inequalities on manifolds is via a local to global argument. After a moment of reflection, it is not hard to convince ourselves that the method does not work for our purpose as the fractional Laplacian operator is nonlocal. To overcome the difficulty in establishing functional inequalities involving nonlocal operators like the fractional Laplacian, we take use of the subordination theory by Bochner. Briefly speaking, subordination is a method to construct a new semigroup from a given one. Particularly, we will show in Section 5 that the fractional Laplacian can be constructed in terms of $\Delta$ via subordination and the associated semigroup is again Markovian; and then based on the Markovian property, in Section 4 we establish two versions of nonlocal logarithmic Sobolev inequality as well as many other functional inequalities involving the fractional Laplacian under some mild geometric conditions, cf. Section 2. Based on these inequalities, we further derive some useful heat kernel and semigroup estimates for the fractional Laplacian.

In the second part of the paper, we conduct a careful study of (1.1). The emphasis is put on the asymptotic analysis of the solution. Based on the work in Sections 3 and 4 we reveal a connection between nonlocal logarithmic Sobolev inequalities
and the ultracontractivity of the fractional porous medium equation \((1.1)\). \(L_p - L_\infty\) regularizing effects of the form \(\|u(t)\|_{L_\infty} \leq C(u_0)t^{-\alpha}\) has been shown, where \(C(u_0)\) depends on \(\|u_0\|_p\). On Riemannian manifolds satisfying a Faber-Krahn type condition, we show that the solution converges to zero uniformly. On a compact manifold without boundary, the \(L_p - L_\infty\) regularizing effects give an \(L_\infty\)-bound of the solution for small \(t > 0\). Then in conjunction with an analysis of the structure of the \(\omega\)-limit set of the trajectory, the derived bound implies the convergence of the solution to the mean of the initial datum. The asymptotic analysis is discussed in details in Sections 6 and 7.

Last but not least, we would like to highlight the global well-posedness result of \((1.1)\) we obtained in Section 5 a little. In an Euclidean space \(\mathbb{R}^N\), \((1.1)\) was studied via one of the following constructions of the fractional Laplacian or their analogues.

1. The authors of \([38, 39]\) constructed the fractional Laplacian via the Caffarelli-Silvestre extension \([19]\), i.e. consider the solution of the problem:
   \[
   \nabla \cdot (y^{1-2\sigma}\nabla w) = 0 \quad (x,y) \in \mathbb{R}^N \times \mathbb{R}_+; \quad w(x,0) = u(x), \quad x \in \mathbb{R}^N.
   \]
   Then for some constant \(C_\sigma\)
   \[
   (-\Delta)^\sigma u(x) = -C_\sigma \lim_{y \to 0} y^{1-2\sigma} \frac{\partial w}{\partial y}(x,y).
   \]

2. In \([15, 16, 17]\), the authors used the Spectral Fractional Laplacian (SFL) for the Dirichlet Laplacian \(-\Delta\) in a bounded domain \(\Omega\) defined by:
   \[
   (-\Delta)^\sigma u(x) := \frac{1}{\Gamma(-\sigma)} \int_0^\infty (e^{t\Delta} u(x) - u(x)) \frac{dt}{t^{1+\sigma}} = \sum_{j=1}^\infty \lambda_j^\sigma \hat{u}_j \phi_j(x),
   \]
   where \((\phi_k, \lambda_k)_{k=1}^\infty\) is an orthonormal basis of \(L_2(\Omega)\) consisting of eigenfunctions of \(-\Delta\) and their corresponding eigenvalues. \(\hat{u}_k\) are the Fourier coefficients of \(u\).

3. In the Restricted Fractional Laplacian (RFL) approach, the fractional Laplacian is constructed by using the integral representation in terms of hypersingular kernels:
   \[
   (-\Delta)^\sigma u(x) = C_{N,\sigma} P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x + y|^{N+2\sigma}} dy \quad \text{with} \quad u(x) = 0 \quad \text{for} \quad x \notin \Omega.
   \]
   (RFL) is obviously not applicable to general manifolds. In \([17]\), the authors proved that the Caffarelli-Silvestre extension holds when \(\Delta\) has discrete spectrum. Similar results were established in \([6]\) under certain geometric assumptions, c.f. \([6, \text{Proposition 3.3}]\). However, when such conditions are absent, the applicability of the Caffarelli-Silvestre extension to \((1.1)\) remains unknown. For a similar reason, (SFL) also seems to be restrictive.

In this paper, we will generalize the method in our earlier work \([44]\) to manifolds with infinite volumes. The approach relies only on the existence of a Markovian extension of the fractional Laplacian. The only geometric assumption we impose for the global well-posedness of \((1.1)\) is \((M, g)\) being a complete Riemannian manifold. Particularly, no compactness, curvature or volume condition is needed. This seems to be the most general one so far.

Before finishing the introduction, we would like to mention a very recent work \([10]\) recommended to us by G. Grillo. It studies the smoothing effect of solutions to a
larger class of data on the hyperbolic space. Their argument relies on a different method and a nonlocal Poincaré inequality. This seems to be a very interesting direction to explore in the future.

**Notations:**

For any two Banach spaces $X, Y$, $X \cong Y$ means that they are equal in the sense of equivalent norms. The notations

$$X \hookrightarrow Y, \quad X \overset{d}{\hookrightarrow} Y$$

mean that $X$ is continuously embedded and densely embedded, respectively. Given a sequence $(u_k)_k := (u_1, u_2, \cdots)$ in $X$, $u_k \rightharpoonup u$ in $X$ means that $u_k$ converge weakly to some $u \in X$. Given a densely-defined operator $A$ in $X$, $D(A)$ and $Rng(A)$ stand for the domain and range of $A$, respectively.

2. Main Results and Geometric Assumptions

In this section, we will collect and state the main results of the article and the geometric assumptions needed for the proofs of the functional inequalities and the asymptotic behaviors of solutions to (1.1).

To prove the functional inequalities mentioned in the introduction, we assume that $(M, g)$ satisfies either of the following conditions.

(A1) $(M, g)$ is a complete and non-compact Riemannian manifolds with infinite volume and without boundary that satisfies a Faber-Krahn type condition. More precisely, there exist constant $M > 0$ and $n > 2$ such that for each $\Omega \subset M$, i.e. $\Omega$ is a compact subset of $M$,

$$\lambda_1(\Omega) \geq M|\Omega|^{-2/n}.$$

Here $\lambda_1(\Omega)$ is the first Dirichlet eigenvalue of the Laplace-Beltrami operator on $\Omega$ with vanishing Dirichlet boundary condition on $\partial \Omega$.

(A2) $(M, g)$ is an $n$-dimensional compact Riemannian manifold without boundary for some integer $n > 2$.

In this article, a closed manifold always means one satisfying (A2).

**Theorem 2.1.** Suppose that $(M, g)$ is a Riemannian manifold satisfying (A1) with $n > 2$. Then the followings hold true.

(i) (Heat kernel Gaussian upper bound): For some $C, A > 0$, the heat kernel satisfies

$$p_\sigma(x, y, t) \leq Ct^{-n/2}\sigma e^{-d^2(x, y)/4t}, \quad x, y \in M, \ t > 0,$$

that is,

$$u(t, x) = \int_M p_\sigma(x, y, t)f(y) \, d\mu_g(y)$$

solves

$$\begin{cases} \partial_t u + (\Delta)^\sigma u = 0 & \text{on} \ M \times (0, \infty); \\ u(0, x) = f(x) & \text{on} \ M. \end{cases}$$

Here $d(x, y)$ is the geodesic distance between $x$ and $y$. 

(ii) (Ultracontractivity): For some $C, A > 0$,
\[ \|e^{-t(-\Delta)^\sigma}u\|_\infty \leq Ct^{-\frac{2\sigma}{4\sigma + n}}\|u\|_p, \quad 1 \leq p < \infty. \]

(iii) (Nash inequality): For some $C > 0$,
\[ \|u\|_2^{\frac{1}{2} + (2\sigma/n)} \leq C\|u\|_1^{2\sigma/n}(-\Delta)^{\sigma/2}u\|_2, \quad u \in D((-\Delta)^{\sigma/2}) \cap L_1(M). \]

(iv) (Sobolev-Poincaré inequality): For some $\tilde{C} > 0$,
\[ \|u\|_{2n/(n-2\sigma)} \leq \tilde{C}\|(-\Delta)^{\sigma/2}u\|_2, \quad u \in D((-\Delta)^{\sigma/2}). \]

(v) (Super Poincaré inequality): For any $r > 0$,
\[ \|u\|_2^2 \leq r\|u\|_1^2 + \beta(r)\|(-\Delta)^{\sigma/2}u\|_2^2, \quad u \in D((-\Delta)^{\sigma/2}) \cap L_1(M), \]
where $\beta : (0, \infty) \rightarrow (0, \infty)$ is a decreasing function.

(vi) (Logarithmic Sobolev inequality): For all $u \in D((-\Delta)^{\sigma/2})$ and $\varepsilon > 0$,
\[ \int_M |u|^2 \ln\left(\frac{|u|}{\|u\|_2}\right)^2 d\mu_g \leq \frac{n}{2\sigma} \left(\|u\|_2^2 \ln\left(\frac{1}{\varepsilon}\right) + \tilde{C}\varepsilon\|(-\Delta)^{\sigma/2}u\|_2^2\right), \]
where $\tilde{C}$ is the constant in (iv) and $d\mu_g$ is the volume element induced by $g$.

**Theorem 2.2.** Suppose that $(M, g)$ is an $n$-dimensional Riemannian manifold satisfying (A2) with $n \geq 1$. Then the followings hold true.

(i) (Heat kernel Gaussian upper bound): For some $C, A > 0$, the heat kernel satisfies
\[ p_\sigma(x,y,t) \leq C\max\{1, t^{-n/2\sigma}\}e^{-d^2(x,y)/At}, \quad x, y \in M, t > 0. \]

(ii) (Ultracontractivity): For some $C, A > 0$,
\[ \|e^{-t(-\Delta)^\sigma}u\|_\infty \leq C\max\{1, t^{-\frac{2\sigma}{4\sigma + n}}\}e^{-\frac{d^2(x,y)}{At}}\|u\|_p, \quad 1 \leq p < \infty. \]

(iii) (Nash inequality): When $n \geq 3$, for some $C > 0$,
\[ \|u - \overline{u}\|_2^{1+2\sigma/n} \leq C2^{1+2\sigma/n}\|(-\Delta)^{\sigma/2}u\|_2\|u - \overline{u}\|_1^{2\sigma/n}, \quad u \in D((-\Delta)^{\sigma/2}), \]
where $\overline{u} := \frac{1}{\text{vol}(M)}\int_M u d\mu_g$ with $\text{vol}(M)$ being the total volume of $(M, g)$.

(iv) (Sobolev-Poincaré inequality): When $n \geq 3$, for some $\tilde{C} > 0$,
\[ \|u - \overline{u}\|_{2n/(n-2\sigma)} \leq \tilde{C}\|(-\Delta)^{\sigma/2}u\|_2, \quad u \in D((-\Delta)^{\sigma/2}). \]

(v) (Super Poincaré inequality): When $n \geq 3$, for any $r > 0$
\[ \|u - \overline{u}\|_2^2 \leq r\|u - \overline{u}\|_1^2 + \beta(r)\|(-\Delta)^{\sigma/2}u\|_2^2, \quad u \in D((-\Delta)^{\sigma/2}), \]
where $\beta : (0, \infty) \rightarrow (0, \infty)$ is a decreasing function.
Furthermore, if 

\[ u_1 \]  

and 

\[ u_\text{strong solution} \]  

in the sense of Definition 5.1. Additionally, the solution satisfies 

\[ \text{Theorem 2.3.} \]  

Concerning the global well-posedness of (1.1), the followings hold.

(I) Comparison principle: If \( u, \dot{u} \) are the unique strong solutions to (1.1) with initial data \( u_0, \dot{u}_0 \), respectively, then \( u_0 \leq \dot{u}_0 \) a.e. implies \( u(t) \leq \dot{u}(t) \) a.e. for all \( t \geq 0 \).

(II) \( L^p \)-contraction: If \( u_0 \in L_1(M) \cap L_q(M) \) with \( q \in [m+1, \infty] \), then for all \( 0 \leq t_1 \leq t_2 \) and \( 1 \leq p \leq q \)

\[ \|u(t_2)\|_p \leq \|u(t_1)\|_p. \]

(III) Conservation of mass: When \( \text{vol}(M) \leq 2 \), for all \( t \geq 0 \), it holds that

\[ \int_M u(t) \, d\mu_g = \int_M u_0 \, d\mu_g. \]

The precise definition of strong solutions can be found in Definition 5.1.

When \( m > 1 \), we can prove the asymptotic behavior of the solution and push the initial data to \( L_1(M) \cap L_2(M) \) under Assumption (A1) or (A2).

Theorem 2.4. Suppose that \( (M, g) \) is a Riemannian manifold satisfying (A1) and \( m > 1 \). Then for every \( u_0 \in L_1(M) \cap L_2(M) \), (1.1) has a unique strong solution \( u \). Furthermore, if \( u_0 \in L_1(M) \cap L_p(M) \) with \( p \in [2, \infty) \), then \( u \) satisfies

\[ \|u(t)\|_\infty \leq \frac{e^{R t}}{t^n} \|u_0\|_p \]

for some \( R = R(p, \sigma, m, n, \tilde{C}) > 0 \), \( \alpha = \frac{n}{2\sigma p + n(m-1)} \) and \( \gamma = \frac{2\sigma p}{2\sigma p + n(m-1)} \), where \( \tilde{C} \) is the constant in Theorem 2.2(iv).

Theorem 2.5. Suppose that \( (M, g) \) is a Riemannian manifold satisfying (A2) and \( m > 1 \). Let \( m_0 = \max\{m - 1, 1\} \). Then for every \( u_0 \in L_2(M) \), (1.1) has a unique strong solution \( u \). Furthermore, if \( u_0 \in L_p(M) \) with \( p \in [2, \infty) \), then \( u \) satisfies

\[ \|u(t)\|_\infty \leq \frac{e^{E \|u_0\|_{m_0} m^{-1} t}}{t^{\alpha}} \|u_0\|_p \]

for some \( \alpha = \alpha(p, \sigma, m, n) > 0 \), \( C = C(p, \sigma, m, n, M_0) > 0 \) and \( \gamma = \left(\frac{p}{p+m-1}\right)^{n/2\sigma} \), \( E = \frac{4mM_0}{M_1} \), where \( M_0, M_1 \) are the constants in Theorem 2.2(iv). Moreover,

\[ \lim_{t \to \infty} \|u(t) - \frac{1}{\text{vol}(M)} \int_M u_0 \, d\mu_g\|_q = 0, \quad 1 \leq q < \infty. \]
In particular, when \( \int_M u_0 \, d\mu = 0 \), for any \( \varepsilon \in (0, 1) \) and \( t > 2 \), it holds
\[
\|u(t)\|_{\infty} \leq \frac{C\|u_0\|_{L^p}^{\frac{\varepsilon \gamma}{p} \left[ B(t - 1)^{\gamma (1 - \varepsilon)/(m - 1)} \right]}}{B(t - 1)^{\gamma (1 - \varepsilon)/(m - 1)}}
\]
for some \( B = (m, p) > 0 \) and \( C = C(p, \sigma, m, n, M_0, M_1) > 0 \).

This manuscript is organized as follows.

In Section 3, we study the Markovian property of the fractional Laplacian. In Sections 4.1 and 4.2, we establish some important inequalities for the fractional Laplacian including two nonlocal logarithmic Sobolev type inequalities. They are an essential ingredient of the proofs of Theorems 2.4 and 2.5 as mentioned in the introduction. Then in Section 4.3, we derive the remaining functional inequalities, various heat kernel and semigroup estimates for the fractional Laplacian in Theorems 2.1 and 2.2. Section 5 is devoted to the proof of Theorem 2.3. Theorems 2.4 and 2.5 are proved in Sections 6 and 7, respectively. To avoid possible distractions, we collect some basic facts from the Markov semigroup and the non-linear semigroup theories in Appendix A and prove the \( m \)-accretivity of a perturbed fractional Laplacian.

3. Subordinated Semigroups and the Fractional Laplacian

Suppose that \((M, g)\) is a complete Riemannian manifold. Then the Laplace-Beltrami operator
\[
\Delta u = \text{div}\nabla u
\]
is essentially self-adjoint on \( C_c^\infty(M) \), cf. [24, Theorem 5.2.3]. The unique self-adjoint extension, i.e., the Friedrichs extension, will still be denoted by \( \Delta \). It is a well-known result that \( \Delta \) generates a symmetric Markov semigroup \( \{e^{t\Delta}\}_{t \geq 0} \) in \( L^2(M) \), cf. Definition A.2 and [27, Theorem 5.11].

To introduce the semigroup generated by the fractional powers of \( -\Delta \), we will need some concepts from subordinated semigroup theory.

Definition 3.1. A Bernstein function \( g \) is a smooth function \( (0, \infty) \to [0, \infty) \) such that
\[
(-1)^{n-1}g^{(n)}(x) \geq 0
\]
for all \( n \in \mathbb{N} \) and \( x > 0 \).

Standard examples of Bernstein functions include
\[
1 - e^{-x}, \quad \ln(1 + x), \quad x^\sigma \text{ with } \sigma \in (0, 1).
\]

Following [45, Theorem 3.2], a smooth function \( g : (0, \infty) \to [0, \infty) \) is a Bernstein function if it admits a representation:
\[
g(x) = a + bx + \int_0^\infty (1 - e^{-tx})\nu(dt), \quad (3.1)
\]
where \( a, b \geq 0 \) are constants and \( \nu \) is a measure on \((0, \infty)\) satisfying
\[
\int_0^\infty (1 \wedge t)\nu(dt) < \infty.
\]
The triplet \((a, b, \nu)\) defines \(g\) uniquely and vice versa. The measure \(\nu\) and the triplet \((a, b, \nu)\) in (3.1) are called the \(\text{Lévy measure}\) and the \(\text{Lévy triplet}\) of the Bernstein function \(g\).

There is another way to characterize a Bernstein function \(g\). Given a convolution semigroup of sub-probability measure \(\{\mu_t\}_{t>0}\) on \([0, \infty)\). Then there exists a unique Bernstein function \(g\) such that the Laplace transform of \(\{\mu_t\}_{t>0}\) satisfies
\[
\int_0^\infty e^{-sx}d\mu_t(s) = e^{-tg(x)}.
\]
(3.2)

Conversely, given a Bernstein function \(g\), there exists a unique convolution semigroup of sub-probability measures \(\{\mu_t\}_{t>0}\) on \([0, \infty)\) such that (3.2) holds, cf. [45, Theorem 5.2].

Suppose that \(A : D(A) \subset L_2(M) \rightarrow L_2(M)\) is a non-negative self-adjoint operator, which generates a Markov semigroup \(\{e^{-tA}\}_{t \geq 0}\). Given a Bernstein function \(g\) and its corresponding convolution semigroup of sub-probability measures \(\{\mu_t\}_{t>0}\), then the Bochner integral
\[
e^{-tg(A)}u = \int_0^\infty e^{-sA}u d\mu_t(s), \quad u \in L_2(M)
\]
(3.3)
defines again a symmetric Markov semigroup, cf. [45 Proposition 12.1]. This semigroup is called **subordinate** (in the sense of Bochner) to the semigroup \(\{e^{-tA}\}_{t \geq 0}\) with respect to the Bernstein function \(g\). Its infinitesimal generator \(g(A)\) is given by the Phillips formula
\[
-g(A) = -au + bAu + \int_0^\infty (e^{-sA}u - u) \nu(ds),
\]
where \((a, b, \nu)\) is the \(\text{Lévy triplet}\) of \(g\), cf [45 Theorem 12.6].

In the sequel, we will focus on the case \(g(x) = x^\sigma\) for \(x > 0\) and \(\sigma \in (0, 1)\). This is a Bernstein function with \(\text{Lévy triplet}\)
\[
a = b = 0, \quad \nu(dt) = \frac{\sigma}{\Gamma(1-\sigma)} t^{-\sigma-1} dt,
\]
which gives
\[
g(-\Delta)u = (-\Delta)^\sigma u = \frac{\sigma}{\Gamma(1-\sigma)} \int_0^\infty (u - e^{t\Delta}u)t^{-\sigma-1} dt, \quad u \in D(\Delta).
\]
(3.4)

This definition is equivalent to Balakrishnan’s formula for the fractional power of a dissipative operator:
\[
(-\Delta)^\sigma := -\frac{\sin(\pi\sigma)}{\pi} \int_0^\infty t^{\sigma-1}\Delta(t \Delta)^{-1} u dt, \quad u \in D(\Delta).
\]
(3.5)

See [45 Section 12.2].

By (3.3), we immediately obtain the following proposition concerning \((-\Delta)^\sigma\).

**Proposition 3.2.** Given any \(\omega \geq 0\), \(\Delta - \omega\) generates a symmetric Markov semigroup on \(L_2(M)\). So does \(-(-\Delta)^\sigma\).
For $1 \leq p < \infty$, there are two ways to construct the semigroups.

(1) First, following a standard process in [24], for each $1 \leq p < \infty$, one can easily show that $\Delta|_{L^p(M)}$ can be extended to the infinitesimal generator of a contraction $C_0$-semigroup on $L^p(M)$, denoted by $\Delta_p$. Following [35, Chapter 12] and the same procedure as above, we can define $(-\Delta_p)^\sigma$ and show that $(-\Delta_p)^\sigma$ still generates a contraction $C_0$-semigroup on $L^p(M)$. Moreover, $(-\Delta_p)^\sigma$ satisfies Balakrishnan’s formula as well.

(2) Second, we can begin with the symmetric Markov semigroup $\{e^{-t(-\Delta)^\sigma}\}_{t \geq 0}$. Following again the standard process in [24], one can easily show that the semigroup $\{e^{-t(-\Delta)^\sigma}|_{L^p(M)}\}_{t \geq 0}$ can be extended to a contraction $C_0$-semigroup on $L^p(M)$. We denote its infinitesimal generator by $(-\Delta_p)^\sigma$.

Since $\{e^{-t(-\Delta)^\sigma}\}_{t \geq 0}$ and $\{e^{-t(-\Delta_p)^\sigma}\}_{t \geq 0}$ coincide on a dense subspace of $L^p(M)$, we conclude that

$$(-\Delta_p)^\sigma = (-\Delta)^\sigma.$$ 

Therefore, in the sequel, we will use the notation $(-\Delta_p)^\sigma$ exclusively. In addition, we always adopt the convention that $\Delta_2 = \Delta$ throughout.

Then the following proposition is at our disposal.

**Proposition 3.3.** For $1 \leq p < \infty$, $(-\Delta_p)^\sigma$ generates a contraction $C_0$-semigroup on $L^p(M)$, and $0 \in \rho(\omega + (-\Delta_p)^\sigma)$ for all $\omega > 0$. The semigroup is analytic when $1 < p < \infty$.

### 4. Functional Inequalities via Subordination

#### 4.1. A Logarithmic Sobolev Inequality on Complete Non-compact Manifolds

Assume that $(M, g)$ is a Riemannian manifold satisfying (A1). By taking $\Lambda(x) = Mx^{-2/n}$, where $M$ is the constant in (A1), and $V(t) = (2\pi M t)^{n/2}$ in [24, Theorem 1.1], one can derive that (A1) implies that the heat kernel $p(t, x, y)$ satisfies

$$p(t, x, y) \leq Ct^{-n/2}.$$

This is equivalent to

$$\|e^{t\Delta_1}u\|_\infty \leq Ct^{-n/2}\|u\|_1. \quad (4.1)$$

We start with a theorem by N.T. Varopoulos, L. Saloff-Coste, and T. Coulhon for symmetric Markov semigroups.

**Theorem 4.1** (Theorem II.5.2 in [49]). Given a symmetric Markov semigroup $\{e^{tH}\}_{t \geq 0}$ on $L^2(M)$, when $d > 2$, the following conditions are equivalent:

- (H) $\|e^{tH}u\|_\infty \leq Ct^{-d/2}\|u\|_1$ for $u \in L_1(M) \cap L_2(M)$.
- (S) $\|u\|_{2d/(d-2)}^2 \leq C_1 \langle -Hu, u \rangle$ for $u \in D(H)$.
- (N) $\|u\|_{2d/(d-2)}^{2+4/d} \leq C_2 \|u\|_1^{4/d} \langle -Hu, u \rangle$ for $u \in L_1(M) \cap D(H)$.

In particular, (H) and (N) are equivalent when $d > 0$. Following Varopoulos’ terminology, the number $d$ is referred to as the dimension of the semigroup $\{e^{tH}\}_{t \geq 0}$.
Based on (4.1), we immediately have
\[ \|u\|_2^{2+(4/n)} \leq C_2 \|u\|_1^{2/n} \langle -\Delta u, u \rangle, \quad u \in D(\Delta). \tag{4.2} \]
Recall that \( \Delta \) generates a symmetric Markov semigroup. Choosing \( B(x) = \frac{1}{C_2} x_2^{2/n} \) in [16, Theorem 1], one can derive a Nash type inequality from (4.2)
\[ \|u\|_2^{2+(4/\sigma)n} \leq C_3 \|u\|_1^{4/\sigma n} \langle -\Delta^\sigma u, u \rangle = C_3 \|u\|_2^{4/\sigma n} \langle -\Delta^{\sigma/2} u \rangle_2^2 \tag{4.3} \]
for some \( C_3 > 0 \) and all \( u \in D((-\Delta)^\sigma) \). Together with Theorem 4.1, this implies the following Sobolev-Poincaré type inequality
\[ \|u\|_2^{2/n/(n-2\sigma)} \leq \hat{C} \langle -\Delta^\sigma u, u \rangle = \hat{C} \|(-\Delta)^{\sigma/2} u\|_2^2 \tag{4.4} \]
for some \( \hat{C} > 0 \). Since \( D((-\Delta)^\sigma) \hookrightarrow D((-\Delta)^{\sigma/2}) \), (4.3) and (4.4) actually hold for all \( u \in D((-\Delta)^{\sigma/2}) \).

(4.4) paves the way to the Logarithmic Sobolev inequality in Theorem 2.1(vi).

Proof. (of Theorem 2.1(vi)) Without loss of generality, we assume that \( \|u\|_2 = 1 \) so that \( u^2 d\mu_g \) is a probability measure. Put \( p = \frac{2n}{n-2\sigma} \). By the Jensen’s inequality,
\[ \int_M |u|^2 \ln(|u|) d\mu_g = \frac{1}{p-2} \int_M \ln(|u|^{p-2}) |u|^2 d\mu_g \]
\[ \leq \frac{1}{p-2} \ln \|u\|_p^p \]
\[ = \frac{p}{2(p-2)} \ln \|u\|_p^2 \]
\[ \leq \frac{n}{4\sigma} (\ln(\frac{1}{\varepsilon}) + \varepsilon \|u\|_2^{2/(n-2\sigma)}). \]
The last step is due to the fact that \( \ln(t) \leq \varepsilon t - \ln(\varepsilon) \) for all \( t, \varepsilon > 0 \). Applying (4.4), we infer that
\[ \int_M |u|^2 \ln(|u|) d\mu_g \leq \frac{n}{4\sigma} (\ln(\frac{1}{\varepsilon}) + \varepsilon \hat{C} \|(-\Delta)^{\sigma/2} u\|_2^2). \]
This establishes the desired inequality. \( \square \)

Proposition 3.2 implies that \((-\Delta)^\sigma\) generates a symmetric Markov semigroup. So we can derive a Strook-Varopoulos type inequality from [37, Theorem 2.1].

**Lemma 4.2.** If \( p \in (1, \infty) \) and \( u \in D((-\Delta_p)^\sigma) \), then \( |u|^{p-2} u \in D((-\Delta)^{\sigma/2}) \) and
\[ \frac{4(p-1)}{p^2} \|(-\Delta)^{\sigma/2} |u|^{p-2} u\|_2^2 \leq \int_M |u|^{p-2} u(-\Delta)^{\sigma/2} u d\mu_g \]
\[ \leq C \|(-\Delta)^{\sigma/2} |u|^{p-2} u\|_2^2 \]
for some \( C = C(p) \).

A generalization of Strook-Varopoulos inequality can be obtained analogously by means of [37, Theorem 2.2].
Lemma 4.3. Let \( \psi \in C^2(\mathbb{R}) \) be such that \( \psi(s) = 0 \) for \( s \leq 0 \), \( \psi'(s) > 0 \) for \( s > 0 \) and \( 0 \leq \psi \leq 1 \) and \( \sup_{t>0}(1 + \frac{t\psi''(t)}{2\psi'(t)})^2 < \infty \). Further, put \( G_\psi(t) = t^{\psi'(t)} \). If \( u \in D((-\Delta)^\sigma) \cap L_p(M, \mathbb{R}_+) \) for some \( p \in [1, \infty) \) and \( \psi(u)(-\Delta)^\sigma u \in L_1(M) \), then \( G_\psi(u) \in D((-\Delta)^{\sigma/2}) \) and
\[
\|(-\Delta)^{\sigma/2} G_\psi(u)\|_2^2 \leq C \int_M \psi(u)(-\Delta)^\sigma u \, d\mu_g
\]
for some \( C = C(\psi) \).

Remark 4.4. In particular, we can take \( \psi \) to be appropriate approximations of the Heaviside function in Lemma 4.3.

4.2. A Logarithmic Sobolev Inequality on Closed Manifolds. Assume that \( (M, g) \) is a closed manifold with dimension \( n > 2 \), i.e., it satisfies (A2). For simplicity, we suppose that \( \text{vol}(M) = 1 \). Here \( \text{vol}(M) \) is the total volume of \((M, g)\). It is well known that \( D(\Delta_p) = H^2_p(M) \) for all \( 1 < p < \infty \), where \( H^2_p(M) \) is the Bessel potential space. Since it follows from \cite[Theorem 10.3]{3} that, for certain \( c > 0 \), \( c - \Delta_p \) has bounded imaginary power, by \cite[(1.2.9.8)]{4} and \cite[Lemma 2.3.5]{15},
\[
D((-\Delta_p)^\sigma) \cong [L_p(M), H^2_p(M)]_\sigma \cong H^2_\sigma(M),
\]
where \([\cdot, \cdot]_\sigma\) is the complex interpolation method. Further, it follows from the standard embedding theorem that
\[
H^2_\sigma(M) \hookrightarrow L_q(M), \quad (4.5)
\]
where \( q = \frac{2n}{n-2\sigma} \).

We first start with the well-known Sobolev-Poincaré inequality
\[
\|u - \overline{u}\|_2^2 \leq C_1 \|\nabla u\|_2, \quad (4.6)
\]
where \( 2^* = \frac{2n}{n-2} \). By the Hölder inequality, we have
\[
\|u - \overline{u}\|_2 \leq \|u - \overline{u}\|_2^\theta \|\nabla u\|_1^{1-\theta},
\]
where \( \theta = \frac{n}{n+2} \). This implies the following Nash inequality
\[
\|u - \overline{u}\|_2^{1+2/n} \leq C_1 \|\nabla u\|_2 \|u - \overline{u}\|_1^{2/n}. \quad (4.7)
\]
Based on (4.7), we will follow the idea in \cite[Proposition 6 and Theorem 1]{16} and prove a non-local version of Nash type inequality. By (4.7), for all \( u \in H^2_\sigma(M) \) with \( \|u - \overline{u}\|_1 = 1 \)
\[
\frac{d}{dt} \|e^{t\Delta}(u - \overline{u})\|_2^2 = 2\langle \Delta e^{t\Delta}(u - \overline{u}), e^{t\Delta}(u - \overline{u}) \rangle = -2\|\nabla e^{t\Delta} u\|_2^2 \leq -2C_1 \|e^{t\Delta}(u - \overline{u})\|_2^{2+4/n} \|e^{t\Delta}(u - \overline{u})\|_1^{4/n} \leq -2C_1 \|e^{t\Delta}(u - \overline{u})\|_2^{2+4/n}.
\]
By choosing \( h(t) = \|e^{t\Delta}(u - \overline{u})\|_2^2 = \|e^{t\Delta} u - \overline{u}\|_2^2 \) and \( \varphi(t) = 2C_1 t^{1+2/n} \) for \( t \geq 0 \) in \cite[Lemma 5]{16}, we immediately have
\[
\|e^{t\Delta} u - \overline{u}\|_2^2 \leq G^{-1}(G(\|u - \overline{u}\|_2^2) - t), \quad t \geq 0,
\]
holds for all \( u \in H^2_t(M) \) with \( \| u - \overline{u} \|_1 = 1 \) and \( t \geq 0 \). Here
\[
G(t) = \frac{n}{4C_1} (1 - t^{-2/n}), \quad t > 0.
\]
\( \text{(3.4)} \) implies that for all \( u \in H^2_t(M) \) with \( \| u - \overline{u} \|_1 = 1 \)
\[
(( -\Delta )^\sigma u - \overline{u}, u - \overline{u})
= \frac{\sigma}{\Gamma(1 - \sigma)} \int_0^\infty t^{-\sigma - 1} \left( u - e^{t\Delta} u, u - \overline{u} \right) dt
= \frac{\sigma}{\Gamma(1 - \sigma)} \int_0^\infty t^{-\sigma - 1} \left( \| u - \overline{u} \|^2 - \| e^{t\Delta} u - \overline{u} \|^2 \right) dt
\geq \frac{\sigma}{\Gamma(1 - \sigma)} \int_0^\infty t^{-\sigma - 1} \left( \| u - \overline{u} \|^2 - G^{-1}(\| u - \overline{u} \|^2) - \frac{t}{2} \right) dt =: g(\| u - \overline{u} \|^2),
\]
where
\[
g(r) = \frac{\sigma}{\Gamma(1 - \sigma)} \int_0^\infty t^{-\sigma - 1} \left( r - G^{-1}(G(r) - \frac{t}{2}) \right) dt.
\]
We have
\[
g(r) = \frac{\sigma}{2^\sigma \Gamma(1 - \sigma)} \int_0^r \frac{ds}{(G(r) - G(s))^\sigma}
\geq C \int_{r/2}^r (r - s)^\sigma ds
\geq C r^{1 + \frac{2\sigma}{n}}, \quad \text{(4.8)}
\]
where in (4.8) we have used [10] (10)) by choosing \( B(t) = t^{2/n} \), i.e.
\[
\frac{G(r) - G(u)}{r - u} \geq \frac{u^{-1 - 2/n}}{2}.
\]
This establishes the following Nash type inequality
\[
\| u - \overline{u} \|^2 \leq C_2 \| (-\Delta)^{\sigma/2} u \|^2 \| u - \overline{u} \|^2 \quad \text{(4.9)}
\]
holds for all \( u \in D((-\Delta)^{\sigma/2}) \) and thus for all \( u \in D((-\Delta)^{\sigma}) \).

Applying the Young’s inequality to (4.9), we immediately derive a super Poincaré type inequality
\[
\| u - \overline{u} \|^2 \leq r \| u - \overline{u} \|^2 + \beta(r) \| (-\Delta)^{\sigma/2} u \|^2, \quad u \in D((-\Delta)^{\sigma/2}) \quad \text{(4.10)}
\]
for all \( r > 0 \), where \( \beta : (0, \infty) \to (0, \infty) \) is a decreasing function. In view of the fact \( \text{vol}(M) = 1 \), this implies
\[
\| u - \overline{u} \|^2 \leq r \| u - \overline{u} \|^2 \| (-\Delta)^{\sigma/2} u \|^2, \quad u \in D((-\Delta)^{\sigma/2}) \quad \text{(4.11)}
\]
A direct computation shows that
\[
\| u - \overline{u} \|^2 = \| u \|^2 - \| \overline{u} \|^2
\]
and
\[
\| u - \overline{u} \|^2 \leq 4 \| u \|^2.
\]
Plugging these results into (4.11), we infer that
\[ \|u\|_1^2 \leq C_3\|u\|_1^2 + C_4\|(-\Delta)^{\sigma/2}u\|_2^2, \quad u \in D((-\Delta)^{\sigma/2}). \] (4.12)
Based on (4.11) and (4.12), (42, Proposition 1.3) implies that
\[ \|u - \overline{u}\|_2^2 \leq C\|(-\Delta)^{\sigma/2}u\|_2^2, \quad u \in D((-\Delta)^{\sigma/2}). \]
Combining with (4.15), we establish the following Sobolev-Poincaré type inequality:
\[ \|u - \overline{u}\|_{2n/(n-2\sigma)}^2 \leq \tilde{C}\|(-\Delta)^{\sigma/2}u\|_2^2, \quad u \in D((-\Delta)^{\sigma/2}). \] (4.13)
Note that by the Hölder inequality, (4.13) implies (4.9).

Based on (4.13), Theorem 2.2(vi) immediately follows from a similar proof of Theorem 2.1(vi).

Finally, we would like to point out that Lemmas 4.2 and 4.3 still hold true for closed manifolds (M, g).

4.3. Other Functional Inequalities, Heat Kernel and Semigroup Estimates via Subordination. In this subsection, we will continue the discussion in Sections 4.1 and 4.2 and derive the remaining functional inequalities and the heat kernel and semigroup estimates for \{e^{-t(-\Delta)^\gamma}\}_{t>0} in Theorems 2.1 and 2.2.

First, we consider a Riemannian manifold (M, g) satisfying (A1) with \(n > 2\). Applying the Young's inequality to the Nash type inequality (4.3), we obtain a super Poincaré type inequality
\[ \|u\|_1^2 \leq r\|u\|_1^2 + \beta(r)\|(-\Delta)^{\sigma/2}u\|_2^2, \quad u \in D((-\Delta)^{\sigma/2}), \] (4.14)
where \(\beta : (0, \infty) \rightarrow (0, \infty)\) is a decreasing function. The Sobolev-Poincaré type inequality (4.3) and Theorem 2.1 imply that
\[ \|e^{-t(-\Delta)^\gamma}u\|_{\infty} \leq C\|(-\Delta)^{\sigma/2}u\|_1 \] (4.15)
for all \(u \in L_1(M)\).

We denote by \(p_\sigma(x, y, t)\) the heat kernel of the semigroup \(\{e^{-t(-\Delta)^\gamma}\}_{t \geq 0}\). Given any \(\Omega \subset M\), it follows from Proposition 3.2 that
\[ \int_{\Omega} p_\sigma(x, y, t) \, d\mu_g(y) \leq 1. \]
Letting \(\Omega\) invade \(M\) yields
\[ \int_{M} p_\sigma(x, y, t) \, d\mu_g(y) \leq 1. \]
It is evident that \(p_\sigma(x, y, t) \geq 0\) for all \(x, y \in M\) and \(t > 0\) and symmetric in \(x\) and \(y\). This, in particular, implies that for every fixed \(x\), the heat kernel \(p_\sigma(x, y, t)\), as a function of \(y\), has \(L_1\)-norm no larger than 1. Using the semigroup property
\[ \int_{M} p_\sigma(x, y, t) p_\sigma(y, z, s) \, d\mu_g(y) = p_\sigma(x, z, t + s) \]
and (4.15), we can derive the heat kernel upper bound
\[ p_\sigma(x, y, t) \leq Ct^{-n/2\sigma}, \quad x, y \in M, \ t > 0. \] (4.16)
By [24, Theorem 3.1], we can further derive the Gaussian upper bound for the heat kernel
\[ p_\sigma(x, y, t) \leq Ct^{-n/2\sigma} e^{-d^2(x,y)/At}, \quad x, y \in M, \ t > 0, \] (4.17)
Now we turn our attention to a closed Riemannian manifold \((M, g)\). By Jensen’s inequality, for all \(1 \leq p < \infty\) and \(t > 0\)
\[
|e^{-\tau(-\Delta)^{\sigma}} u(x)|^p = \int_M p_\sigma(x, y, t) u(y) d\mu_g(y) d\mu_g(y)
\leq \int_M p_\sigma(x, y, t) |u(y)|^p d\mu_g(y)
\leq C t^{-n/2\sigma} e^{-d^2(x, y)/At} \|u\|_p^p.
\]
This implies that the semigroup \(\{e^{-\tau(-\Delta)^{\sigma}}\}_{\tau \geq 0}\) is ultracontractive, i.e.
\[
\|e^{-\tau(-\Delta)^{\sigma}} u\|_{\infty} \leq C t^{-\frac{\sigma}{2\sigma}} e^{-\frac{d^2(x, y)}{4At}} \|u\|_p.
\]
Finally, (4.3), (4.4), (4.14), (4.17) and (4.18) give Theorems 2.1.

**Remark 4.5.** The estimate (4.16) can also be derived by using (8.3).

Now we turn our attention to a closed Riemannian manifold \((M, g)\).

Pick any \(u \in H_2^{2\sigma}(M)\) with \(\|u - \overline{u}\|_1 = 1\). Then
\[
\frac{d}{dt} \|e^{-\tau(-\Delta)^{\sigma}} (u - \overline{u})\|_2^2 \leq -2((-\Delta)^{\sigma} e^{-\tau(-\Delta)^{\sigma}} (u - \overline{u}), e^{-\tau(-\Delta)^{\sigma}} (u - \overline{u}))
\leq -C \|e^{-\tau(-\Delta)^{\sigma}} (u - \overline{u})\|_2^{2+4\sigma/n} \|e^{-\tau(-\Delta)^{\sigma}} (u - \overline{u})\|_1^{-4\sigma/n}
\leq -C \|e^{-\tau(-\Delta)^{\sigma}} (u - \overline{u})\|_2^{2+4\sigma/n}.
\]
The second line follows from (4.13), and the third is a direct consequence of the contraction of the semigroup \(\{e^{-\tau(-\Delta)^{\sigma}}\}_{\tau \geq 0}\). This implies that for any \(u \in H_2^{2\sigma}(M)\)
\[
\|e^{-\tau(-\Delta)^{\sigma}} (u - \overline{u})\|_2 \leq C t^{-\frac{\sigma}{2\sigma}} \|u - \overline{u}\|_1.
\]
By the triangle inequality, we immediately have
\[
\|e^{-\tau(-\Delta)^{\sigma}} u\|_2 \leq C (t^{-\frac{\sigma}{2\sigma}} + 1) \|u\|_1 \leq C \max\{1, t^{-\frac{\sigma}{2\sigma}}\} \|u\|_1.
\]
Given any \(f \in L_1(M)\),
\[
\langle e^{-\tau(-\Delta)^{\sigma}} u, f \rangle \leq \|e^{-\tau(-\Delta)^{\sigma}/2} u\|_2 \|e^{-\tau(-\Delta)^{\sigma}/2} f\|_2
\leq C \max\{1, t^{-\frac{\sigma}{2\sigma}}\} \|u\|_1 \|f\|_1,
\]
which implies
\[
\|e^{-\tau(-\Delta)^{\sigma}} u\|_\infty \leq C \max\{1, t^{-n/2\sigma}\} \|u\|_1.
\]
Now following the argument leading to (4.16), we can derive the Gaussian upper bound for the heat kernel
\[
p_\sigma(x, y, t) \leq C \max\{1, t^{-n/2\sigma}\} e^{-d^2(x, y)/At}, \quad x, y \in M, \ t > 0,
\]
and the ultracontractivity
\[
\|e^{-\tau(-\Delta)^{\sigma}} u\|_\infty \leq C \max\{1, t^{-\frac{\sigma}{2\sigma}}\} e^{-\frac{d^2(x, y)}{4At}} \|u\|_p, \quad 1 \leq p < \infty.
\]
In sum, (4.9), (4.10), (4.13), (4.19) and (4.20) give Theorems 2.2.
5. Solutions to the Fractional Porous Medium Equation

To prove the global well-posedness of (1.1), we first study the following generalization of (1.1) with $\omega \geq 0$

$$\begin{cases}
\partial_t u + [\omega + (-\Delta)^\sigma]|u|^{m-1}u = 0 & \text{on } M \times (0, \infty); \\
u(0) = u_0 & \text{on } M.
\end{cases}$$

(5.1)

In [44], we established the existence and uniqueness of a strong solution to (1.1) on an incomplete Riemannian manifold with conical singularities and finite volume. We will nevertheless state a brief proof for the existence and uniqueness part for two reasons: (1) we will adopt a more elegant argument which is applicable to manifolds with infinite volume; (2) the proofs of the asymptotic behaviors of solutions rely on how they are constructed.

In this section, we assume that the initial datum $u_0 \in L_1(M) \cap L_{m+1}(M)$. The initial condition will be relaxed to $u_0 \in L_1(M) \cap L_2(M)$ when $m > 1$ and the underlying manifolds satisfying (A1) or (A2) in the next two sections.

5.1. Definition of Solutions. Let $\Phi(x) = |x|^{m-1}x$ and $\beta = \Phi^{-1}$. Note that $\Phi$ and $\beta$ are maximal monotone graphs in $\mathbb{R}^2$ containing $(0, 0)$. We define the notions of solutions to (5.1) as follows.

Definition 5.1. Given $\omega \geq 0$, we say that $u$ is a weak solution to (5.1) if

- $u \in L_{\infty, loc}((0, \infty), L_{m+1}(M))$, and
- $(-\Delta)^{\sigma/2}\Phi(u), \sqrt{\omega}\Phi(u) \in L_{2, loc}((0, \infty), L_2(M))$, and
- $u \in C([0, \infty), L_1(M))$.

Moreover, for every $\phi \in C_0^1([0, \infty) \times M)$, it holds that

$$\int_0^\infty \int_M (-\Delta)^{\sigma/2}\Phi(u)(-\Delta)^{\sigma/2}\phi \, d\mu_g dt + \omega \int_0^\infty \int_M \Phi(u)\phi \, d\mu_g dt = \int_0^\infty \int_M u\partial_t \phi \, d\mu_g dt + \int_M u_0 \phi(0) \, d\mu_g.$$  (5.2)

If, in addition, $u$ satisfies

- when $m = 1$, $\partial_t u, (-\Delta)^{\sigma}\Phi(u) \in L_{2, loc}((0, \infty), L_2(M))$ and further $u \in C([0, \infty), L_2(M))$; or
- when $m \in (0, 1) \cup (1, \infty)$, $\partial_t u, (-\Delta)^{\sigma}\Phi(u) \in L_{\infty, loc}((0, \infty), L_1(M))$,

we call $u$ a strong solution to (5.1).

5.2. Existence of Weak Solution. By Proposition A.5, the operator $A(u) := [\omega + (-\Delta)^\sigma]\Phi(u) : D(A) \subset L_1(M) \to L_1(M)$ is $m$-accretive and with dense domain. We can apply the Crandall-Liggett generation theorem [22, Theorem 1] and prove the existence of a global mild solution to (5.1). More precisely, given $T > 0$, for a partition $P = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ of $[0, T)$ with $\Delta T_k = t_k - t_{k-1}$, the discretized problem to (5.1) is

$$\Delta T_k [\omega + (-\Delta)^\sigma]\Phi(u_{n,k}) = u_{n,k-1} - u_{n,k-1,\omega} \quad \text{with} \quad u_{n,0,\omega} = u_0.$$  (5.3)
For simplicity, we may take $\Delta T_k = T/n$. The piecewise solution is defined as

$$u_{n,0}(0) = u_0, \quad u_{n,\omega}(t) = u_{n,k; \omega} \quad \text{for } t \in (t_{k-1}, t_k].$$

The the uniform limit $u_\omega \in C([0, T], L^1(M))$ of $u_{n,\omega}$, i.e. for any $\varepsilon > 0$,

$$\|u_\omega(t) - u_{n,\omega}(t)\|_1 < \varepsilon, \quad t \in [0, T]$$

(5.4)

for sufficiently large $n$, is the unique global mild solution to (5.1).

$u_\omega$ is $L_q$-contractive for all $1 \leq q \leq m + 1$. Indeed, [13] Proposition 4] implies that

$$\|u_{n,k; \omega}\|_q \leq \|u_{n,k-1; \omega}\|_q \leq \|u_0\|_q \quad 1 \leq q \leq m + 1,$$

(5.5)

and it follows from Fatou’s Lemma and (5.4) that for any $0 \leq t$,

$$\|u_\omega(t)\|_q \leq \|u_0\|_q.$$  \hspace{1cm} (5.6)

In view of (5.3), (5.5) reveals that $[\omega + (-\triangle)^{\sigma/2}] \Phi(u_{n,k; \omega}) \in L^1(M) \cap L^{m+1}(M)$.

Multiplying (5.3) by $\Phi(u_{n,k; \omega})$ and integrating over $M$ give

$$\frac{T}{n} \int_M \left[(-\triangle)^{\sigma/2} \Phi(u_{n,k; \omega})^2 + \omega \Phi(u_{n,k; \omega})^2 \right] d\mu_g = \int_M u_{n,k-1; \omega} \Phi(u_{n,k; \omega}) d\mu_g - \int_M u_{n,k; \omega} \Phi(u_{n,k; \omega}) d\mu_g$$

$$\leq \frac{1}{m+1} \left( \int_M |u_{n,k-1; \omega}|^{m+1} d\mu_g - \int_M |u_{n,k; \omega}|^{m+1} d\mu_g \right).$$

(5.7)

We have used the Hölder and Young’s inequalities in (5.7). Summing over $k = 1, 2, \cdots, n$ yields

$$\int_0^T \int_M \left[(-\triangle)^{\sigma/2} \Phi(u_{n,\omega})^2 + \omega \Phi(u_{n,\omega})^2 \right] d\mu_g dt \leq \frac{1}{m+1} \int_M |u_0|^{m+1} d\mu_g;$$

and thus

$$\|(-\triangle)^{\sigma/2} \Phi(u_{0,\omega})\|_{L^1([0, T), L^2(M))} \leq \frac{1}{m+1} \int_M |u_0|^{m+1} d\mu_g.$$  \hspace{1cm} (5.8)

Multiplying (5.3) by $\phi \in C^1_c([0, \infty) \times M)$ and integrating over $M$ yield

$$\int_M (-\triangle)^{\sigma/2} \Phi(u_{n,k; \omega}) \times (-\triangle)^{\sigma/2} \phi \ d\mu_g + \omega \int_M \Phi(u_{n,k; \omega}) \phi \ d\mu_g$$

$$= \frac{n}{T} \int_M (u_{n,k-1; \omega} - u_{n,k; \omega}) \phi \ d\mu_g.$$

Then integrate over $[t_{k-1}, t_k]$ and sum over $k = 1, 2, \cdots, n$. The right hand side equals

$$\frac{n}{T} \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \left( u_{n,k-1; \omega} - u_{n,k; \omega} \right) \phi \ d\mu_g dt$$

$$= \int_0^T \int_M u_{n,\omega}(t) \phi(t + T/n) - \phi(t) \frac{T}{n} \ d\mu_g dt + \frac{n}{T} \int_0^{t_1} \int_M u_0 \phi(t) \ d\mu_g dt$$

$$- \frac{n}{T} \int_{t_{n-1}}^{T} \int_M u_{n,\omega}(T) \phi(t + T/n) \ d\mu_g dt.$$
Pushing \( n \to \infty \) yields
\[
\int_0^T \int_M u_\omega \partial_t \phi \, d\mu_g \, dt + \int_M u_0 \phi(0) \, d\mu_g - \int_M u_\omega(T) \phi(T) \, d\mu_g
= \int_0^T \int_M (-\Delta)^{\sigma/2} \Phi(u_\omega)(-\Delta)^{\sigma/2} \phi \, d\mu_g \, dt + \omega \int_0^T \int_M \Phi(u_\omega) \phi \, d\mu_g \, dt. \tag{5.9}
\]

Take any positive sequence \( \omega_k \to 0^+ \). We rewrite (5.3) as
\[
\begin{cases}
    u_{n,k;\omega_k} + \Delta T_k [\omega_t + (-\Delta)^{\sigma/2} \Phi(u_{n,k;\omega_k})] = u_{n,k-1;\omega_k} - \Delta T_k (\omega_h - \omega_l) \Phi(u_{n,k;\omega_k}); \\
    u_{n,0;\omega_k} = u_0
\end{cases}
\]
for \( h < l \). The existence of \( u_{n;\omega_k} \) has already been established. Now we try to estimate \( \|u_{n,k;\omega_k} - u_{n,k;\omega_l}\|_1 \). Proposition A.3 implies
\[
\|u_{n,1;\omega_k} - u_{n,1;\omega_l}\|_1 \leq \frac{T}{n} (\omega_h - \omega_l) \|\Phi(u_{n,1;\omega_k})\|_1 = \frac{T}{n} (\omega_h - \omega_l) \|u_{n,1;\omega_k}\|_m
\]
where the last step is due to (5.5); and
\[
\|u_{n,2;\omega_k} - u_{n,2;\omega_l}\|_1 \leq \|u_{n,1;\omega_k} - u_{n,1;\omega_l}\|_1 + \frac{T}{n} (\omega_h - \omega_l) \|\Phi(u_{n,2;\omega_k})\|_1
\]
\[
\leq 2\frac{T}{n} (\omega_h - \omega_l) \|u_0\|_m.
\]
By induction, we thus have
\[
\|u_{n,k;\omega_k} - u_{n,k;\omega_l}\|_1 \leq \frac{kT}{n} (\omega_h - \omega_l) \|u_0\|_m^m.
\]
This implies that
\[
\|u_{\omega_h} - u_{\omega_l}\|_{C([0,T], L_1(M))} \leq (\omega_h - \omega_l) \|u_0\|_m^m.
\]
We conclude that \( (u_{\omega_k})_k \) is Cauchy in \( C([0,T], L_1(M)) \) and thus converges to some \( u \in C([0,T], L_1(M)) \). (5.6) and (5.8) imply that
\[
(-\Delta)^{\sigma/2} \Phi(u_\omega) \to (-\Delta)^{\sigma/2} \Phi(u) \quad \text{in} \quad L_2((0,T), L_2(M))
\]
\[
u_\omega \to u \quad \text{in} \quad L_\infty((0,T), L_{m+1}(M))
\]
as \( \omega \to 0^+ \). In view of (5.4), pushing \( \omega \to 0^+ \) in (5.9) yields that
\[
\int_0^T \int_M u \partial_t \phi \, d\mu_g \, dt + \int_M u_0 \phi(0) \, d\mu_g - \int_M u(T) \phi(T) \, d\mu_g
= \int_0^T \int_M (-\Delta)^{\sigma/2} \Phi(u)(-\Delta)^{\sigma/2} \phi \, d\mu_g \, dt
\]
for any $\phi \in C^1_c((0, \infty) \times \mathcal{M})$. Note that the estimate (5.8) holds for all $T > 0$. We thus have
\[
\int_0^\infty \int_{\mathcal{M}} u_\partial_t \phi \, d\mu_g \, dt = \lim_{T \to \infty} \int_0^T \int_{\mathcal{M}} u_\partial_t \phi \, d\mu_g \, dt
\]
\[
= \lim_{T \to \infty} \int_0^T \int_{\mathcal{M}} (-\Delta)^{\sigma/2} \Phi(u)(-\Delta)^{\sigma/2} \phi \, d\mu_g \, dt
\]
\[
- \int_{\mathcal{M}} u_0 \phi(0) \, d\mu_g + \lim_{T \to \infty} \int_{\mathcal{M}} u(T) \phi(T) \, d\mu_g
\]
\[
= \int_0^\infty \int_{\mathcal{M}} (-\Delta)^{\sigma/2} \Phi(u)(-\Delta)^{\sigma/2} \phi \, d\mu_g \, dt - \int_{\mathcal{M}} u_0 \phi(0) \, d\mu_g.
\]
Therefore, $u$ is a weak solution to (5.1).

5.3. Existence and Uniqueness of Strong Solution. The strategies in this subsection are picked from [39, Sections 6 and 8]. However, we will generalize some results in [39] to (5.1) with $\omega > 0$, which will be used in the next two sections.

**Proposition 5.2.** For any $\omega \geq 0$, the weak solutions $u_\omega$ to (5.1) constructed in Section 5.2 are strong solutions.

**Proof.** When $m = 1$, the standard semigroup theory, c.f. [40, Theorem 4.1.4], and Proposition 3.3 imply that (1.1) has a unique solution in the class $\tilde{u}_\omega \in C^1([0, \infty), L_2(\mathcal{M})) \cap C([0, \infty), D((-\Delta)^\sigma))$.

Define
\[
\phi(t) = \int_t^T (u_\omega - \tilde{u}_\omega) \, ds, \quad 0 \leq t \leq T,
\]
and $\phi \equiv 0$ for $t \geq T$, which belongs to $H^1_2((0, T), D((-\Delta)^{\sigma/2})) \hookrightarrow C([0, T], L_2(\mathcal{M}))$.

By a standard approximation argument, $\phi$ is a valid test function in (5.2). We have
\[
\int_0^T \int_{\mathcal{M}} (-\Delta)^{\sigma/2} (u_\omega - \tilde{u}_\omega)(t) \int_t^T (-\Delta)^{\sigma/2} (u_\omega - \tilde{u}_\omega)(s) \, ds \, d\mu_g \, dt
\]
\[
+ \omega \int_0^T \int_{\mathcal{M}} (u_\omega - \tilde{u}_\omega)(t) \int_t^T (u_\omega - \tilde{u}_\omega)(s) \, ds \, d\mu_g \, dt
\]
\[
+ \int_0^T \int_{\mathcal{M}} (u_\omega - \tilde{u}_\omega)^2(t) \, d\mu_g \, dt = 0,
\]
which is equivalent to
\[
\frac{1}{2} \int_{\mathcal{M}} \left[ \int_0^T (\Delta)^{\sigma/2} (u_\omega - \tilde{u}_\omega)(t) \, dt \right]^2 \, d\mu_g
\]
\[
+ \frac{\omega}{2} \int_{\mathcal{M}} \left[ \int_0^T (u_\omega - \tilde{u}_\omega)(t) \, dt \right]^2 \, d\mu_g + \int_0^T \int_{\mathcal{M}} (u_\omega - \tilde{u}_\omega)^2(t) \, d\mu_g \, dt = 0.
\]
All integrals need to be zero. We thus infer that $u_\omega = \tilde{u}_\omega$. This implies that
\[
\partial_t u_\omega \in L_{2,loc}([0, \infty), L_2(\mathcal{M})) \quad \text{and} \quad u_\omega \in C([0, \infty), L_2(\mathcal{M})).
\]
Therefore, $u_\omega$ is a strong solution to (5.1).
When \( m \neq 1 \), the argument follows the idea in [39] Lemma 8.1 and Theorem 8.2. For any \( f \in L_{1,\text{loc}}(0, T) \), define the Steklov average of \( f \) by

\[
f^h(t) := \frac{1}{h} \int_t^{t+h} f(s) \, ds,
\]

and

\[
\delta^h f(t) := \partial_t f^h(t) = \frac{f(t+h) - f(t)}{h} \quad \text{a.e.}
\]

The weak formulation (5.2) can be restated as

\[
\int_0^T \int_M (\delta^h u) \phi \, d\mu_g \, dt + \omega \int_0^T \int_M (\Phi(u))^{\frac{\alpha}{\omega}} \phi \, d\mu_g \, dt + \int_0^T \int_M (\Delta)^{\sigma/2} (\Phi(u))^{\frac{\alpha}{\omega}} (\Delta)^{\sigma/2} \phi \, d\mu_g \, dt = 0.
\]

(5.10)

For any \([\tau, S] \subset (0, T)\), we choose \( \zeta \in C^1_0((0, T), [0, 1]) \) such that \( \zeta \equiv 1 \) on \([\tau, S]\) and vanishes outside \([\tau', S']\) for some \([\tau', S'] \subset (0, T)\) with \([\tau, S] \subset (\tau', S')\). Let us take \( \phi = \zeta \delta^h (\Phi(u)) \). Then (5.10) yields

\[
\int_0^T \int_M \zeta (\delta^h u) \delta^h (\Phi(u)) \, d\mu_g \, dt + \omega \int_0^T \int_M \zeta (\Phi(u))^{\frac{\alpha}{\omega}} \delta_t (\Phi(u)) \, d\mu_g \, dt + \int_0^T \int_M \zeta (\Delta)^{\sigma/2} (\Phi(u))^{\frac{\alpha}{\omega}} (\Delta)^{\sigma/2} \delta_t (\Phi(u)) \, d\mu_g \, dt = 0.
\]

(5.11)

Since \( (\delta^h u) (\delta^h \Phi(u)) \geq c (\delta^h (|u|^{(m-1)/2} u))^2 \), cf. [38] Section 5.3, the first term on the left hand side of (5.11) satisfies that

\[
\int_0^T \int_M \zeta (\delta^h u) \delta^h (\Phi(u)) \, d\mu_g \, dt \geq c \int_0^T \int_M \zeta (\delta^h (|u|^{(m-1)/2} u))^2 \, d\mu_g \, dt.
\]

The second and third terms on the left hand side of (5.11) can be estimated as follows

\[
\left| \int_0^T \int_M \zeta (\Phi(u))^{\frac{\alpha}{\omega}} \delta_t (\Phi(u)) \, d\mu_g \, dt \right| \leq C \int_{\tau'}^{S'} \int_M |\zeta'| (\Phi(u))^{\frac{\alpha}{\omega}} |^2 \, d\mu_g \, dt
\]

\[
\leq C \int_{\tau'}^{S'} \int_M (\Phi(u))^{\frac{\alpha}{\omega}} |^2 \, d\mu_g \, dt
\]

and similarly

\[
\left| \int_0^T \int_M \zeta (\Delta)^{\sigma/2} (\Phi(u))^{\frac{\alpha}{\omega}} (\Delta)^{\sigma/2} \delta_t (\Phi(u)) \, d\mu_g \, dt \right|
\]

\[
\leq C \int_{\tau'}^{S'} \int_M (\Delta)^{\sigma/2} (\Phi(u))^{\frac{\alpha}{\omega}} |^2 \, d\mu_g \, dt.
\]

It follows from (5.3) that

\[
\int_{\tau}^{S} \int_M (\delta^h (|u|^{(m-1)/2} u))^2 \, d\mu_g \, dt \leq \int_0^T \int_M \zeta (\delta^h (|u|^{(m-1)/2} u))^2 \, d\mu_g \, dt \leq C
\]

for some \( C > 0 \) independent of \( h \), and thus

\[
\partial_t (|u|^{(m-1)/2} u) \in L_{2,\text{loc}}((0, T), L_2(M)).
\]
Since [8, Theorems 1 and 2] implies \( u_\omega \in BV((\tau, T), L_1(M)) \) for any \( \tau > 0 \), it then follows from [9, Theorem 1.1] that
\[
\partial_t u_\omega \in L_{\infty, loc}((0, T), L_1(M))
\]
with
\[
\|\partial_t u_\omega(t)\|_1 \leq \frac{2}{m-1}\|u_0\|_1.
\]
This leads to
\[
(-\Delta)^{\sigma/2}(u_\omega) \in L_{\infty, loc}((0, T), L_1(M)).
\]

The uniqueness of the strong solution follows from the following lemma.

**Lemma 5.3.** Given \( u_0 \in L_1(M) \) when \( m \neq 1 \) or \( u_0 \in L_2(M) \) when \( m = 1 \), (1.1) has at most one strong solution.

**Proof.** When \( m \neq 1 \), if \( u_1, u_2 \), are strong solutions to (1.1) with initial data \( u_{0,1}, u_{0,2} \in L_1(M) \), then it follows from Lemma 4.3 and the proof of [39, Theorem 6.2] that for every \( 0 \leq t_1 < t_2 \) it holds
\[
\int_M (u_1 - u_2)_+ (t_2) \, d\mu_g \leq \int_M (u_1 - u_2)_+ (t_1) \, d\mu_g.
\]
When \( m = 1 \), if \( u_1, u_2 \), are strong solutions to (1.1) with initial data \( u_{0,1}, u_{0,2} \in L_2(M) \), then we have
\[
\int_M \partial_t (u_1 - u_2)(u_1 - u_2) \, d\mu_g = -\int_M |(-\Delta)^{\sigma/2}(u_1 - u_2)|^2 \, d\mu_g \leq 0,
\]
which implies that for every \( 0 < t_1 < t_2 \)
\[
\|(u_1 - u_2)(t_2)\|_2 \leq \|(u_1 - u_2)(t_1)\|_2.
\]
The fact that \( u_1, u_2 \in C([0, \infty), L_2(M)) \) then implies the uniqueness of strong solution. \( \square \)

### 5.4. Proof of Theorem 2.3

**Proof.** (of Theorem 2.3) We have already proved the existence and uniqueness of a strong solution. The additional properties (I)-(III) follow from the proof of [44, Theorem 6.1]. \( \square \)

Before concluding this section, we will prove two useful properties of the solutions.

**Lemma 5.4.** Suppose that \( u, \hat{u} \) are strong solutions to (1.1) with respect to the initial data \( u_0, \hat{u}_0 \) obtained by the above argument. Then for any \( t \geq 0 \)
\[
\|u(t) - \hat{u}(t)\|_1 \leq \|u_0 - \hat{u}_0\|_1.
\]

**Proof.** Proposition A.3 implies that \( \|u_\omega(t) - \hat{u}_\omega(t)\|_1 \leq \|u_0 - \hat{u}_0\|_1 \), and by (5.4)
\[
\|u_\omega(t) - \hat{u}_\omega(t)\|_1 \leq \|u_0 - \hat{u}_0\|_1.
\]
The assertion then follows from the convergence of \( u_\omega, \hat{u}_\omega \) to \( u, \hat{u} \) in \( C([0, T], L_1(M)) \). \( \square \)
Lemma 5.5. Given \(u_0 \in L_1(M) \cap L_\infty(M)\), for any \(0 < \tau\), the strong solution \(u\) to (1.1) satisfies
\[
\|(-\Delta)^{\sigma/2} \Phi(u)(t)\|_2 \leq M = M(\|u(\tau)\|_\infty) \quad \text{for a.a. } \tau < t.
\]

Proof. The assertion can be proved by following the argument leading to (44, (5.24)). □

6. Asymptotic Behavior: Complete and Non-compact Manifolds

In this and the next section, we always assume that \(m > 1\). We first consider the case \(u_0 \in L_1(M) \cap L_\infty(M)\). The initial condition will be weakened in Section 6.2

6.1. Asymptotic behavior for \(u_0 \in L_1(M) \cap L_\infty(M)\). Given \(\omega > 0\), for the strong solution \(u_\omega\) to (5.1), we put
\[
\phi_\omega(t) := \|u_\omega(t)\|_r.
\]

Our aim is to derive an ordinary differential inequality for \(\ln \|u_\omega(s)\|_r(s)\). Note that [18] Proposition 4], Fatou’s Lemma and (5.3) show that for all \(p \in [1, \infty]\)
\[
\|u_{n,k,\omega}\|_p, \|u_\omega\|_p \leq \|u_0\|_p.
\]

Moreover, (5.3) implies that \((-\Delta)^{\sigma} \Phi(u_{n,k,\omega}) \in L_1(M) \cap L_\infty(M)\).

For \(r \geq 2\), we multiply (5.3) by \(|u_{n,k,\omega}|^{r-2} u_{n,k,\omega}\) and integrate over \(M\). Putting \(d = r + m - 1\), this yields
\[
\int_M |u_{n,k,\omega}|^{r-2} u_{n,k,\omega}(u_{n,k-1,\omega} - u_{n,k,\omega}) \, d\mu_g
\]
\[
= \Delta T_k \left[ \omega \|u_{n,k,\omega}\|_d^2 + \int_M (-\Delta)^{\sigma} \Phi(u_{n,k,\omega}) |u_{n,k,\omega}|^{r-2} u_{n,k,\omega} \, d\mu_g \right]
\]
\[
\geq \Delta T_k \left[ \omega \|u_{n,k,\omega}\|_d^2 + \frac{4m(r-1)}{d^2} \int_M |(-\Delta)^{\sigma/2} |u_{n,k,\omega}|^{d/2}|^2 \, d\mu_g \right].
\]

We have used Lemma 4.2 in (5.2).

Note that we have to start from (5.3) instead of (1.1), as in general we do not know whether \(\Phi(u) \in D((-\Delta d/m)')\).

As before, one can derive from the Hölder and the Young’s inequalities that
\[
\int_M |u_{n,k,\omega}|^{r-2} u_{n,k,\omega}(u_{n,k-1,\omega} - u_{n,k,\omega}) \, d\mu_g
\]
\[
\leq \frac{1}{r} \int_M \left[ |u_{n,k-1,\omega}|^r - |u_{n,k,\omega}|^r \right] \, d\mu_g < M
\]
for some \(M > 0\) independent of \(n\) and \(\omega\) by (5.1).

For \(t \in (0, T)\) and \(h > 0\) small so that \(t + h < T\), without loss of generality, we may assume \(t = t_i, t+h = t_j\) for some \(i, j \in \{1, 2, \cdots, n-1\}\). We sum over all \([t_{k-1}, t_k]\) contained in \([t, t+h]\) and obtain
\[
\frac{4m(r-1)}{d^2} \int_t^{t+h} \int_M |(-\Delta)^{\sigma/2} |u_{n,\omega}(s)|^{d/2}|^2 \, d\mu_g \, ds
\]
\[
\leq \frac{1}{r} \int_M \left[ |u_{n,\omega}(t)|^r - |u_{n,\omega}(t+h)|^r \right] \, d\mu_g.
\]
Hence due to (6.1)
\[
\int_t^{t+h} \int_M |(-\Delta)^{\sigma/2} u_\omega(s)^{d/2}|^2 d\mu_g ds \\
\leq \liminf_{n \to \infty} \int_t^{t+h} \int_M |(-\Delta)^{\sigma/2} u_{n,\omega}(s)^{d/2}|^2 d\mu_g ds < M. \quad (6.4)
\]

By the interpolation theory, (5.4) and (6.1), for any \(1 < q < \infty\)
\[
\|u_{n,\omega}(t) - u_\omega(t)\|_q \leq \|u_{n,\omega}(t) - u_\omega(t)\|_1^{1/q} \|u_{n,\omega}(t) - u_\omega(t)\|_\infty^{1-1/q} \to 0
\]
as \(n \to \infty\). Thus we can control the right hand side of (6.3) and obtain
\[
\frac{4m(r-1)}{d^2} \int_t^{t+h} \int_M |(-\Delta)^{\sigma/2} u_\omega(s)^{d/2}|^2 d\mu_g ds \\
\leq \frac{1}{r} \int_M [u_\omega(t)^r - |u_\omega(t+h)|^r] d\mu_g \quad (6.5)
\]
for a.a. \(h > 0\) small. Based on (6.1) and the Dominated Convergence Theorem, dividing both sides by \(h\) and letting \(h \to 0\) yields
\[
\frac{d}{dt} \varphi_r(t) \leq -\frac{4mr(r-1)}{(r + m - 1)^2} \|(-\Delta)^{\sigma/2} u_\omega(t)^{d/2}\|_2^2, \quad r \geq 2. \quad (6.6)
\]

Given any \(p \geq 2\), we define a \(C^1\) and non-decreasing function \(r : [0, t) \to [p, \infty)\) such that \(r(0) = p\) and \(\lim_{s \to t^-} r(s) = +\infty\). Put \(d(s) = r(s) + m - 1\).

We set \(\Phi(r, s) := \|u_\omega(s)^{r}\|_r\). Then (6.6) yields
\[
\frac{d}{ds} \Phi(r(s), s) = \frac{\partial}{\partial s} \Phi(r(s), s)|_{r=r(s)} + \dot{r}(s) \frac{\partial}{\partial r} \Phi(r(s), s)|_{r=r(s)} \\
\leq -\frac{4mr(s)(r(s) - 1)}{d^2(s)} \|(-\Delta)^{\sigma/2} u_\omega(s)^{d(s)/2}\|_2^2 \\
+ \dot{r}(s) \int_M \ln(|u_\omega(s)|)|u_\omega(s)|^{r(s)} d\mu_g. \quad (6.7)
\]

Defining
\[
Y(s) := \ln \|u_\omega(s)^{r(s)}\|_{r(s)}
\]
and following [13], we introduce the Young functional \(J : [1, \infty) \times X\), where \(X = \bigcap_{p=1}^{\infty} L_p(M)\) is defined by
\[
J(r, u) := \int_M \ln \left(\frac{|u|}{\|u\|_r}\right) \frac{|u|^r}{\|u\|_r^r} d\mu_g.
\]
One can compute by using (6.7) that
\[
\frac{d}{ds} Y(s) = - \frac{r'(s)}{r^2(s)} \ln \|u_\omega(s)\|_{r(s)}^{r'(s)} + \frac{1}{r(s)\|u_\omega(s)\|_{r(s)}^{r'(s)}} \frac{d}{ds} \Phi(r(s), s)
\]
\[
\leq - \frac{r'(s)}{r^2(s)} \ln \|u_\omega(s)\|_{r(s)}^{r'(s)} - \frac{4m(r(s) - 1)}{d^2(s)} \|(-\Delta)^{\sigma/2}|u_\omega(s)|^{d(s)/2}\|_{r(s)}^{r'(s)}
\]
\[
+ \frac{r'(s)}{r(s)\|u_\omega(s)\|_{r(s)}^{r'(s)}} \int_M \ln(\|u_\omega(s)\|)\|u_\omega(s)\|_{r(s)}^{r'(s)} d\mu_g
\]
\[
= \frac{r'(s)}{r(s)} J(r(s), u_\omega(s)) - \frac{4m(r(s) - 1)}{d^2(s)} \|(-\Delta)^{\sigma/2}|u_\omega(s)|^{d(s)/2}\|_{r(s)}^{r'(s)}
\]
\[
(6.8)
\]
\[
(6.9)
\]
We have used [13] Proposition 2.6(a) in the last step.

Note that it follows from (6.4) that for a.a. \( t \in (0, T) \), \( |u_\omega(t)|^{d(t)/2} \in D((-\Delta)^{\sigma/2}) \).
So by Theorem 2.1(vi), it holds that
\[
\|(-\Delta)^{\sigma/2}|u_\omega(s)|^{d(s)/2}\|_{r(s)}^{r'(s)}^2 \geq \frac{2\sigma}{C_\varepsilon n} \int_M \|u_\omega(s)\|^{d(s)} \ln \left( \frac{\|u_\omega(s)\|^{d(s)}}{\|u_\omega(s)\|^{d(s)}} \right) d\mu_g + \frac{1}{C_\varepsilon \|u_\omega(s)\|^{d(s)}} \ln \varepsilon
\]
\[
= \frac{1}{C_\varepsilon \|u_\omega(s)\|^{d(s)}} \left[ \frac{2\sigma}{n} J(1, |u_\omega(s)|^{d(s)}) + \ln \varepsilon \right].
\]
where \( \hat{C} \) is the constant in Theorem 2.1(iv), and we have used the equality
\[
\|\|u_\omega(s)\|^{d(s)/2}\|_{r(s)}^{r'(s)}^2 = \|u_\omega(s)\|^{d(s)}_{r(s)}.
\]

Plugging this inequality into (6.9), one can infer that
\[
\frac{d}{ds} Y(s) \leq \frac{r'(s)}{r(s)} J(r(s), u_\omega(s))
\]
\[
- \frac{4m(r(s) - 1)}{d^2(s)\hat{C}\varepsilon} \|u_\omega(s)\|^{r'(s)}_{r(s)} \left[ \frac{2\sigma}{n} J(1, |u_\omega(s)|^{d(s)}) + \ln \varepsilon \right].
\]
Taking
\[
\varepsilon = \frac{4m r(s)[2\sigma r(s) + n(m - 1)](r(s) - 1)}{n\hat{C} \frac{r(s)d^2(s)}{\|u_\omega(s)\|^{r'(s)}_{r(s)}} \|u_\omega(s)\|^{r'(s)}_{r(s)}}
\]
and using [13] Proposition 2.6(b)], we have
\[
\frac{d}{ds} Y(s) \leq \frac{r'(s)}{r^2(s)} \left[ J(1, |u_\omega(s)|^{r(s)}) - \frac{2\sigma r(s)}{2\sigma r(s) + n(m - 1)} J(1, |u_\omega(s)|^{d(s)}) \right]
\]
\[
- \frac{r'(s)}{r^2(s)} \ln \left( \frac{\|u_\omega(s)\|^{d(s)}_{r(s)}}{\|u_\omega(s)\|^{d(s)}_{r(s)}} \right)
\]
\[
- \frac{r'(s)}{r(s)2\sigma r(s) + n(m - 1)} \ln \left( \frac{\|u_\omega(s)\|^{r'(s)}_{r(s)}}{\|u_\omega(s)\|^{r'(s)}_{r(s)}} \right)
\]
\[
- \frac{r'(s)}{r(s)2\sigma r(s) + n(m - 1)} \ln \left[ \frac{4m r(s)[2\sigma r(s) + n(m - 1)](r(s) - 1)}{n\hat{C} \frac{r(s)d^2(s)}{\|u_\omega(s)\|^{r'(s)}_{r(s)}}} \right].
\]
It follows from [13] (4.3) and [13] Proposition 2.6(b) that
\[
\frac{d}{ds} Y(s) \leq \frac{\dot{r}(s)}{r^2(s)} \left[ J(1, |u_\omega(s)|^r(s)) - \frac{2\sigma r(s)}{2\sigma r(s) + n(m-1)} J(1, |u_\omega(s)|^{d(s)}) \right]
\]
\[
\quad - \frac{\dot{r}(s)}{r^2(s)} \frac{nr(s)(m-1)}{2\sigma r(s) + n(m-1)} \left[ J(r(s), u_\omega(s)) + Y(s) \right]
\]
\[
\quad - \frac{\dot{r}(s)}{r^2(s)} \frac{n}{2\sigma r(s) + n(m-1)} \ln \left[ \frac{4m r(s)[2\sigma r(s) + n(m-1)](r(s) - 1)}{nC \dot{r}(s)^2} \right]
\]
\[
= \frac{\dot{r}(s)}{r(s) 2\sigma r(s) + n(m-1)} \left[ J(1, |u_\omega(s)|^r(s)) - J(1, |u_\omega(s)|^{d(s)}) \right]
\]
\[
\quad - \frac{\dot{r}(s)}{r(s) 2\sigma r(s) + n(m-1)} Y(s)
\]
\[
\quad - \frac{\dot{r}(s)}{r(s) 2\sigma r(s) + n(m-1)} \ln \left[ \frac{4m r(s)[2\sigma r(s) + n(m-1)](r(s) - 1)}{nC \dot{r}(s)^2} \right].
\]

Taking into consideration [13] Proposition 2.6(d) and \( m > 1 \), we have
\[
J(1, |u_\omega(s)|^{r(s)}) - J(1, |u_\omega(s)|^{d(s)}) \leq 0.
\]

Since \( r \) is non-decreasing, by putting
\[
p(s) = \frac{\dot{r}(s)}{r(s) 2\sigma r(s) + n(m-1)} \frac{n(m-1)}{m-1}
\]
and
\[
q(s) = \frac{\dot{r}(s)}{r(s) 2\sigma r(s) + n(m-1)} \frac{n}{2\sigma r(s) + n(m-1)} \ln \left[ \frac{4m r(s)[2\sigma r(s) + n(m-1)](r(s) - 1)}{nC \dot{r}(s)^2} \right],
\]
we arrive at
\[
\frac{d}{ds} Y(s) + p(s)Y(s) + q(s) \leq 0, \quad Y(0) = \ln ||u_0||_p.
\]

Hence \( Y(s) \leq Y_L(s) \), where
\[
Y_L(s) = e^{-\int_0^s p(a) da} \left[ Y(0) - \int_0^s q(a)e^{\int_0^a p(r) dr} da \right]
\]
is the solution of
\[
\frac{d}{ds} Y_L(s) + p(s)Y_L(s) + q(s) = 0, \quad Y_L(0) = \ln ||u_0||_p.
\]

By taking \( r(s) = pt/(t-s) \), one can compute
\[
P(s) = \int_0^s p(a) da = \int_0^s \frac{\dot{r}(a)}{r(a) 2\sigma r(a) + n(m-1)} da
\]
\[
\quad = \ln \left[ \frac{2\sigma r(s)}{2\sigma r(s) + n(m-1)} \frac{2\sigma p + n(m-1)}{2\sigma p} \right],
\]
and it holds
\[
\lim_{s \to t^-} e^{-P(s)} = \frac{2\sigma p}{2\sigma p + n(m-1)};
\]
and since \( \dot{r}(s) = \frac{r^2(s)}{pt} \), we further have
\[
q(a)e^{P(a)} = \frac{2\sigma r(a)[2\sigma p + n(m-1)]}{2\sigma p[2\sigma r(a) + n(m-1)]^2} \ln \left[ \frac{4m [2\sigma r(a) + n(m-1)](r(a) - 1)}{nC r(a)[r(a) + m-1]^2} pt \right].
\]
This implies
\[
\lim_{s \to t^-} \int_0^s q(a)e^{P(a)} \, da = R + \frac{n}{2\sigma p} \ln t
\]
for some \( R = R(p, \sigma, m, n, \tilde{C}) \) but independent of \( \omega \). To sum up, we have
\[
Y_L(t) = \frac{2\sigma p}{2\sigma p + n(m-1)} \ln \|u_0\|_p - \frac{n}{2\sigma p + n(m-1)} \ln t + R,
\]
and thus
\[
\ln \|u_\omega(t)\|_\infty = \lim_{s \to t^-} \ln \|u_\omega(s)\|_{r(s)} \leq \lim_{s \to t^-} \ln Y(s) \leq \lim_{s \to t^-} Y_L(s) = Y_L(t).
\]
This yields
\[
\|u_\omega(t)\|_\infty \leq \frac{e^R}{t^\alpha} \|u_0\|_p^\gamma, \tag{6.10}
\]
where \( \alpha = \frac{n}{2\sigma p + n(m-1)} \) and \( \gamma = \frac{2\sigma p}{2\sigma p + n(m-1)} \). Because the constants in \( (6.10) \) are independent of \( \omega \) and, for all \( t, u_\omega(t) \) converges to \( u(t) \) pointwise a.e. on \( M \), we immediately conclude that
\[
\|u(t)\|_\infty \leq \frac{e^R}{t^\alpha} \|u_0\|_p^\gamma, \tag{6.11}
\]
where \( u \) is the unique strong solution to \( (1.1) \).

6.2. Proof of Theorem 2.4

Proof. (of Theorem 2.4) Given \( u_0 \in L_1(M) \cap L_2(M) \), we take a sequence \( L_1(M) \cap L_\infty(M) \ni u_{0,k} \to u_0 \) in \( L_1(M) \cap L_2(M) \) and denote the corresponding strong solutions to \( (1.1) \) by \( u_k \). We learn from Lemma 5.4 that \( (u_k) \) is Cauchy in \( C([0, T], L_1(M)) \) and thus converges to some \( u \in C([0, T], L_1(M)) \) for any \( T > 0 \).

For every \( 0 < \tau < \infty \), it follows from \( (6.11) \) that \( u_k \in L_\infty([\tau, \infty), L_\infty(M)) \) with uniform bounds. By the interpolation theory, \( \|u_k(\tau)\|_{m+1} \) is uniformly bounded in \( k \). \( (5.5) \) implies that
\[
(-\Delta)^{\sigma/2} \Phi(u_k) \in L_2([\tau, \infty), L_2(M))
\]
with uniform bound. Now we can pass the limit \( k \to \infty \) in
\[
\int_\tau^\infty \int_M u_k \partial_\tau \phi \, d\mu_g \, dt + \int_M \int_\tau^\infty u_k(\tau) \phi(\tau) \, d\mu_g = \int_\tau^\infty \int_M (-\Delta)^{\sigma/2} \Phi(u_k)(-\Delta)^{\sigma/2} \phi \, d\mu_g \, dt
\]
for any \( \phi \in C_c^1([0, \infty) \times M) \), and infer that \( u \) is a weak solution to \( (1.1) \) on \( [\tau, \infty) \).

Since \( u(0) = u_0 \) and \( u \in C([0, \infty), L_1(M)) \),
\[
\int_0^\infty \int_M (-\Delta)^{\sigma/2} \Phi(u)(-\Delta)^{\sigma/2} \phi \, d\mu_g \, dt
\]
\[
= \lim_{\tau \to 0^+} \int_\tau^\infty \int_M (-\Delta)^{\sigma/2} \Phi(u)(-\Delta)^{\sigma/2} \phi \, d\mu_g \, dt
\]
\[
= \lim_{\tau \to 0^+} \int_\tau^\infty \int_M u(\tau) \phi(\tau) \, d\mu_g + \lim_{\tau \to 0^+} \int_M u(\tau) \phi(\tau) \, d\mu_g
\]
\[
= \int_0^\infty \int_M u(\tau) \phi \, d\mu_g + \int_M u_0 \phi(0) \, d\mu_g.
\]
Thus $u$ is a weak solution to (6.1) on $[0, \infty)$. To see $u$ is indeed a strong solution, it suffices to observe that $u \in L_\infty([\tau, \infty) \times M)$ for any $\tau > 0$. Then the proof of Proposition 5.2 is still valid. The uniqueness of solution follows from Lemma 5.3.

By the approximation argument above, (6.11) still holds true for $u$. 

\[ \]
we infer that
\[
\frac{d}{ds} Y(s) \leq \frac{\dot{r}(s)}{r(s)} \left[ J(r(s), u_\omega(s)) - J(d(s), u_\omega(s)) - \frac{n}{2\sigma d(s)} \ln \left( \frac{\|u_\omega(s)\|_{d(s)}}{\|u_\omega(s)\|_{r(s)}} \right) \right] \\
- \frac{\dot{r}(s)}{r(s)} \frac{n}{2\sigma d(s)} \ln \left( \frac{8\sigma m r(s)(r(s) - 1)}{\dot{r}(s)M_0 d(s)n} \right) + \frac{4m(r(s) - 1)M_1}{d^2(s)M_0} \|u_0\|_{m_0}^{-1}.
\]

It follows from [13, Proposition 2.6(a)] that
\[
J(r(s), u_\omega(s)) - J(d(s), u_\omega(s)) - \ln \left( \frac{\|u_\omega(s)\|_{d(s)}}{\|u_\omega(s)\|_{r(s)}} \right) \leq 0.
\]

This yields
\[
\frac{d}{ds} Y(s) \leq \frac{\dot{r}(s)}{r(s)} \left[ \ln \left( \frac{\|u_\omega(s)\|_{d(s)}}{\|u_\omega(s)\|_{r(s)}} \right) - \frac{n}{2\sigma d(s)} \ln \left( \frac{\|u_\omega(s)\|_{d(s)}}{\|u_\omega(s)\|_{r(s)}} \right) \right] \\
- \frac{\dot{r}(s)}{r(s)} \frac{n}{2\sigma d(s)} \ln \left( \frac{8\sigma m r(s)(r(s) - 1)}{\dot{r}(s)M_0 d(s)n} \right) + \frac{4m(r(s) - 1)M_1}{d^2(s)M_0} \|u_0\|_{m_0}^{-1}.
\]

Here (7.2) follows from the facts \( n > 2\sigma, \) \( \text{vol}(M) = 1 \) and Hölder inequality
\[
\|u_\omega\|_r \leq \|u_\omega\|_d.
\]

We put
\[
p(s) = \frac{\dot{r}(s)n(m-1)}{r(s)2\sigma d(s)}
\]
and
\[
q(s) = \frac{\dot{r}(s)}{r(s)2\sigma d(s)} \ln \left( \frac{8\sigma m r(s)(r(s) - 1)}{\dot{r}(s)M_0 d(s)n} \right) - \frac{4m(r(s) - 1)M_1}{d^2(s)M_0} \|u_0\|_{m_0}^{-1}.
\]

Then we obtain the following differential inequality
\[
\frac{d}{ds} Y(s) + p(s)Y(s) + q(s) \leq 0, \quad Y(0) = \ln \|u_0\|_p.
\]
As in Section 6.1 we have
\[
Y(s) \leq Y_L(s) = e^{-\int_0^s p(a) \, da} \left[ Y(0) - \int_0^s q(a)e^{\int_0^a \! p(\tau) \, d\tau} \, da \right].
\]
Letting
\[
P(s) = \int_0^s p(a) \, da = \int_0^s \frac{n(m - 1)}{r(a) 2\sigma(r(a) + m - 1)} \, da
\]
we finally arrive at
\[
\frac{n}{2\sigma} \ln \left( \frac{r(s) p + m - 1}{d(s) p} \right).
\]
This implies
\[
e^{-P(t)} = \lim_{s \to t^-} e^{-P(s)} = \left( \frac{p}{p + m - 1} \right)^{n/2\sigma}.
\]
Following [13], we have
\[
Q(s) = \int_0^s q(a)e^{\int_0^a \frac{r(\gamma)}{\gamma} d\gamma} \, da
\]
\[
= \int_0^s \frac{n}{r(a) 2\sigma d(a)} \ln \left( \frac{8\sigma m}{M_{0n}} \right) \left( \frac{r(a)}{d(a)} \right)^{n/2\sigma} \left( \frac{p + m - 1}{p} \right)^{n/2\sigma} \, da
\]
\[
+ \int_0^s \frac{n}{r(a) 2\sigma d(a)} \ln \left( \frac{r(s)(r(s) - 1)}{r(s)d(s)} \right) \left( \frac{r(a)}{d(a)} \right)^{n/2\sigma} \left( \frac{p + m - 1}{p} \right)^{n/2\sigma} \, da
\]
\[
- \int_0^s \frac{4m(r(s) - 1)M_1}{d^2(s)M_0} \|u_0\|_{m_0}^{m - 1} \left( \frac{r(a)}{d(a)} \right)^{n/2\sigma} \left( \frac{p + m - 1}{p} \right)^{n/2\sigma} \, da.
\]
Letting
\[
I_1 = I_1(m, p, n, \sigma) = \lim_{s \to t^-} \int_0^s \frac{r(a)}{r(a)d(a)} \ln \left( \frac{8\sigma m}{M_{0n}} \right) \left( \frac{r(a)}{d(a)} \right)^{n/2\sigma} \, da
\]
and
\[
I_2 = I_2(m, p, n, \sigma) = \lim_{s \to t^-} \int_0^s \frac{r(a)}{r(a)d(a)} \ln \left( \frac{r(a) - 1}{r(a)d(a)} \right) \, da
\]
gives
\[
Q_1(t) = I_1 \frac{n}{2\sigma} \left( \frac{p + m - 1}{p} \right)^{n/2\sigma} \ln \left( \frac{8\sigma m}{M_{0n}} \right)
\]
and
\[
Q_2(t) = I_1 \frac{n}{2\sigma} \left( \frac{p + m - 1}{p} \right)^{n/2\sigma} \ln(pt) + I_2 \frac{n}{2\sigma} \left( \frac{p + m - 1}{p} \right)^{n/2\sigma}
\]
as \(\dot{r}(s) = \frac{r^2(s)}{pt}\). Note that \(1 < r(s) \leq d(s)\) for all \(0 \leq s < t\). We thus conclude that
\[
Q_3(t) = \frac{4mM_1}{M_0} \left( \frac{p + m - 1}{p} \right)^{n/2\sigma} \|u_0\|_{m_0}^{m - 1} \lim_{s \to t^-} \int_0^s \frac{r(a) - 1}{d^2(a)} \left( \frac{r(a)}{d(a)} \right)^{n/2\sigma} \, da
\]
\[
\leq \frac{4mM_1}{M_0} \left( \frac{p + m - 1}{p} \right)^{n/2\sigma} \|u_0\|_{m_0}^{m - 1} t.
\]
To sum up, by putting
\[
\gamma = \left( \frac{p}{p + m - 1} \right)^{n/2\sigma}, \quad \alpha = I_1 \frac{n}{2\sigma} \quad \text{and} \quad E = \frac{4mM_1}{M_0},
\]
we finally arrive at
\[
Y_L(t) \leq \gamma \ln \|u_0\|_p + R(m, p, n, \sigma, M_0) - \alpha \ln(pt) + E\|u_0\|_{m_0}^{m - 1} t
\]
for some constant \( R(m, p, n, \sigma, M_0) \). This yields
\[
\| u_{\omega}(t) \|_\infty \leq C_1 e^{\int_0^t \| u_{\omega} \|_{m_n}^{-1}\| u_0 \|_{p} \gamma}, \quad p \geq 2, \quad (7.3)
\]
and further
\[
\| u(t) \|_\infty \leq C_2 e^{\int_0^t \| u_0 \|_{m_n}^{-1}\| u_0 \|_{p} \gamma}, \quad p \geq 2. \quad (7.4)
\]

7.2. **Passing to initial data in** \( L_2(M) \). We can follow the approximation procedure in Section 6.2 and use a sequence \((u_{0,k})_k \in L_\infty(M)\) to approximate an initial datum \( u_0 \in L_2(M) \) in \( L_2(M) \). The corresponding solutions \((u_k)_k\) then converge to a strong solution \( u \) to (1.1). This solution is unique due to Lemma 5.3, (7.4) still holds true for \( u \).

7.3. **Convergence of solution for general initial data.** In this section, we assume that \( u_0 \in L_2(M) \) whose mean is not necessarily zero. We prove that the unique strong solution \( u \) to (1.1) converges to the average of \( u_0 \) in \( L_p(M) \) for any \( 1 \leq p < \infty \). The idea is based on the theory in [23].

**Theorem 7.1.** Assume that \( m > 1 \). Let \( u_0 \in L_2(M) \) and \( u \) be the unique strong solution to (1.1). Then for all \( p \in [1, \infty) \)
\[
\lim_{t \to \infty} \| u(t) - \frac{1}{\text{vol}(M)} \int_M u_0 \, d\mu \|_p = 0 \quad (7.5)
\]

**Proof.** Define the metric space \( X = L_2(M) \) equipped with the \( L_1 \)-norm. Note that (1.1) is associated with a continuous (nonlinear) semigroup \( T(t) \) in \( X \), defined by \( T(t)u_0 = u(t; u_0) \), where \( u(t; u_0) \) is the unique strong solution to (1.1) with initial datum \( u_0 \). It follows from Lemma 5.5 that \( T(\cdot) \) is Lyapunov stable in the sense of [23, Definition 4.1].

Denote the trajectory of \( u_0 \) by \( \gamma(u_0) = \bigcup_{t \geq 0} T(t)u_0 \). The closure \( \overline{\gamma(u_0)} \) in \( L_1(M) \) satisfies
\[
\overline{\gamma(u_0)} = \gamma(u_0) \cup \omega(u_0),
\]
as \( u \in C([0, \infty), L_1(M)) \). Here \( \omega(u_0) \) is the \( L_1(M) \)-limit set of \( u_0 \). Let us characterize \( \omega(u_0) \). If there exists a sequence \((t_n)_n\) such that \( t_n \to \infty \) and
\[
u(t_n) \to w \in \omega(u_0)
\]
in \( L_1(M) \), then \((u(t_n))_n\) is Cauchy in \( L_1(M) \). In view of Theorem 2.3(II) and (7.4), \((u(t_n))_n\) is uniformly bounded in \( L_\infty(M) \). By the Riesz-Thorin interpolation theorem, we immediately have that \((u(t_n))_n\) is Cauchy in \( L_2(M) \) and thus \( w \in L_2(M) \). It follows from the discussion in the previous subsection that \( T(t)w \) is a strong solution to (1.1) with initial data \( w \), and further its trajectory is Lyapunov stable in \( L_1(M) \).

In addition, it is clear that \( \omega(u_0) = \omega(u(\tau)) \) for any \( \tau > 0 \). From Lemma 5.3, Theorem 2.3(II) and (7.4), we learn that \( \Phi[\gamma(u(\tau))] \) is bounded in \( D((-\Delta)^{\sigma/2}) \). Recall
\[
D((-\Delta)^{\sigma/2}) = [L_2(M), H_2^2(M)]_{\sigma/2} \simeq H_2^\sigma(M).
\]
It is clear that (7.7) and (7.8) imply (7.6). We can further derive from (7.6) and due to \( \Phi \in \mathcal{L}^{1/m}(\mathbb{R}) \) for \( m > 1 \), we have

\[
\|u - v\|_1 \leq C \int_M |\Phi(u) - \Phi(v)|^{1/m} \, d\mu_g \leq C \|\Phi(u) - \Phi(v)\|^{1/m}_1;
\]

and due to \( \Phi \in C^{1-}(\mathbb{R}) \),

\[
\|\Phi(u) - \Phi(v)\| \leq C\|u - v\|_1, \quad u, v \in L_\infty(M).
\]

It is clear that (7.7) and (7.8) imply (7.0). We can further derive from (7.0) and (7.7) that \( \gamma(u(\tau)) \) is sequentially compact in \( L_1(M) \) and thus \( \gamma(u(\tau)) \) is relatively compact in \( L_1(M) \).

We define

\[
V(\xi) = \frac{1}{m+1} \int_M |\xi|^{m+1} \, d\mu_g, \quad \xi \in L_1(M).
\]

Then \( V \) is lower semicontinuous on \( L_1(M) \). It follows from Theorem 2.3(II) that \( V \) is non-increasing along the orbit \( \gamma(u_0) \) and thus is a Lyapunov functional for \( T(.) \) defined in 2.3. We infer from 2.3 Proposition 4.1 that \( V \) is constant on \( \omega(u_0) \).

Take any \( u_0 \in \omega(u_0) \). Note that for any \( t > 0 \), \( w(t) := T(t)u_0 \in \omega(u_0) \) by the Lyapunov stability of \( T(.) \). Recall that \( u_0 \in L_2(M) \). For any \( T > \tau > 0 \),

\[
\partial_tw,(\Delta)^{\sigma/2}\Phi(w) \in L_\infty((\tau,T),L_1(M)),
\]

and thus

\[
\frac{d}{dt}V(w(t)) = -\int_M |(\Delta)^{\sigma/2}\Phi(w)(t)|^2 \, d\mu_g = 0, \quad t > 0.
\]

Thus \( (\Delta)^{\sigma/2}\Phi(w)(t) \equiv 0 \) a.e.. This shows that

\[
-\Delta\Phi(w)(t) = (\Delta)^{1-\sigma/2}(\Delta)^{\sigma/2}\Phi(w)(t) = 0,
\]

which implies that \( w(t) \equiv W \) for some constant \( W \) and all \( t \geq 0 \) in view of the fact that \( w \in C([0,\infty),L_1(M)) \). So the \( L_1(M) \) \( \omega \)-limit set of \( u_0 \) consists of constant functions. Because of the mass conservation property, cf. Theorem 2.3(III), we obtain

\[
W = \frac{1}{\text{vol}(M)} \int_M u_0 \, d\mu_g.
\]

This establishes (7.5) for \( p = 1 \). The general case \( p > 1 \) follows from the case \( p = 1 \) and the \( L_\infty \)-contraction property, cf. Theorem 2.3(II).

### 7.4. Asymptotic behavior for \( u_0 \) with zero mean

Now we consider a special case \( u_0 \in L_2(M) \) with zero mean, i.e. \( \int_M u_0 \, d\mu_g = 0 \) and prove a refined asymptotic estimate. In the sequel, let \( p \in [2,\infty) \) and \( t > 2 \). We will closely follow the proof of [13] Corollary 1.3. First, (6.6) gives

\[
\frac{d}{dt}\|u_\omega(t)\|_p^p \leq -4mp(p-1)\|((\Delta)^{\sigma/2}|u_\omega(t)|)^{d/2}\|^2_2,
\]

where \( d = p + m - 1 \) as before. Note that \( u_\omega \) has zero mean for all \( t \geq 0 \). Following the proof of [11] Lemma 3.2, we can show that

\[
K\|u_\omega(t)^{d/2}\|_2^2 \leq \|((\Delta)^{\sigma/2}|u_\omega(t)|^{d/2}\|^2_2
\]
Further, we say that
\begin{equation}
\frac{d}{dt} \| u_\omega(t) \|_p^p \leq -B \| u_\omega(t) \|_p^d \leq -B \| u_\omega(t) \|_p^d
\end{equation}
for some constant \( B = B(p, m) \). Setting \( \phi(t) = \| u_\omega(t) \|_p^p \), we thus obtain
\begin{equation}
\frac{d}{dt} \phi(t) \leq -B \phi(t)^{d/p}
\end{equation}
and thus
\begin{equation}
\| u_\omega(t) \|_p \leq \left( \frac{1}{Bt + \| u_0 \|_p^{(m-1)}} \right)^{1/(m-1)},
\end{equation}
which also holds for \( u \). When \( t > 2 \), by \( (7.4) \) and Theorem \( 2.3 \) \( II \)
\begin{equation}
\| u(t) \|_\infty \leq Ce^{E\| u(t-1) \|_p^{m-1}} \| u(t-1) \|_p^\gamma \leq \left[ \frac{C}{B(t-1) + \| u_0 \|_p^{(m-1)}} \right]^{\gamma/(m-1)}
\end{equation}
for some \( C = C(p, m, n, \sigma, M_1, M_0) \). Moreover, for any \( \varepsilon \in (0, 1) \), it follows from the inequality \( a^\varepsilon b^{1-\varepsilon} \leq a + b \) that
\begin{equation}
\| u(t) \|_\infty \leq \frac{C\| u_0 \|_p^{\gamma\varepsilon}}{[B(t-1)]^{\gamma(1-\varepsilon)/(m-1)}}.
\end{equation}
This completes the proof for Theorem \( 2.3 \).

APPENDIX A. \( m \)-accretivity of the operator \( [\omega + (-\Delta_1)^\sigma]\Phi(u) \)

Let \( X_R \) be a real Banach lattice with an order \( \leq \). See \cite[Chapter C-I]{[5]}. The complexification of \( X_R \) is a complex Banach lattice defined by
\begin{equation}
X := X_R \oplus iX_R.
\end{equation}
The positive cone of \( X_R \) is defined by
\begin{equation}
X_R^+ := \{ x \in X_R : 0 \leq x \}.
\end{equation}

**Definition A.1.** Let \( \vartheta \in \mathbb{R} \), and \( X \) be a complex Banach lattice defined as in \( (A.1) \). A semigroup \( \{T(t)\}_{t \geq 0} \) is called real if
\begin{equation}
T(t)X_R \subset X_R, \quad t \geq 0.
\end{equation}
Further, we say that \( \{T(t)\}_{t \geq 0} \) is positive if
\begin{equation}
T(t)X_R^+ \subset X_R^+, \quad t \geq 0.
\end{equation}

**Definition A.2.** A strongly continuous semigroup \( \{T(t)\}_{t \geq 0} \) on \( L_2(M) \) is called a Markov semigroup if it is both positive and \( L_{\infty-}\)contraction, i.e.
\begin{equation}
\| T(t)u \|_\infty \leq \| u \|_\infty, \quad t \geq 0, \quad u \in L_\infty(M) \cap L_2(M).
\end{equation}
Recall that \( \Phi(x) = |x|^{m-1}x \) and \( \beta = \Phi^{-1} \).

**Proposition A.3.** Let \( \omega, \lambda > 0 \) and \( \sigma \in (0, 1) \). For any \( f \in L_1(M) \), there exists a unique solution \( u \in D((-\Delta_1)^\sigma) \) to
\begin{equation}
\lambda[\omega + (-\Delta_1)^\sigma]u + \beta(u) = f.
\end{equation}
Moreover, for any \( f_1, f_2 \in L_1(M) \), the corresponding solutions \( u_1, u_2 \) satisfy
\begin{equation}
\| \beta(u_1) - \beta(u_2) \|_1 \leq \| f_1 - f_2 \|_1.
\end{equation}
Proof. Following a similar argument to the proof of [13] Lemma 4.1, one can show that given any \( v \in L_1(M) \),
\[
\sup \{ \| \lambda \omega + (-\Delta_1)^\sigma \|^{-1} v \} \leq \max \{ 0, \sup \}.
\]
Proposition A.3 implies that there exists some \( C > 0 \) such that for all \( v \in L_1(M) \)
\[
C \| v \|_1 \leq \| \omega + (-\Delta_1)^\sigma \| v \|_1, \quad v \in D((\Delta_1)^\sigma).
\]
Now the proposition is a direct consequence of [18, Theorem 1]. \( \square \)

**Definition A.4.** [7, Chapter II.3] \( A : D(A) \subset X \to X \) is a nonlinear operator defined in a Banach space \( X \).

(i) \( A \) is called accretive if for all \( \lambda > 0 \)
\[
\| (\lambda + \lambda A)x_1 - (\lambda + \lambda A)x_2 \|_X \geq \| x_1 - x_2 \|_X, \quad x_1, x_2 \in D(A).
\]
(ii) \( A \) is called \( m \)-accretive if \( A \) is accretive and it satisfies the range condition
\[
\text{Rng}(\lambda + \lambda A) = X, \quad \lambda > 0.
\]

**Proposition A.5.** The operator \([u \mapsto A(u) := [\omega + (-\Delta_1)^\sigma] \Phi(u)] : D(A) \subset L_1(M) \to L_1(M)\) is \( m \)-accretive with domain
\[
D(A) = \{ u \in L_1(M) : \Phi(u) \in D((-\Delta_1)^\sigma) \} \quad \text{dense in } L_1(M).
\]

Proof. Proposition A.3 implies that \( A \) is \( m \)-accretive. Next, we will prove that
\( D(A) \) is dense in \( L_1(M) \), which is clearly true for \( m = 1 \).

Case 1: \( m > 1 \). Observe that in this case \( \beta \in BC^{1/m}(\mathbb{R}) \). Given any \( w \in C_c^\infty(M) \subset L_1(M) \), take a sequence \( (u_k)_k \subset C_c^\infty(M) \subset D((-\Delta_1)^\sigma) \) converging to \( \Phi(w) \) in \( L_1(M) \). Without loss of generality, we may assume that all \( u_k \) and \( w \) are supported in some \( \Omega \subset M \). One has
\[
\| \beta(u_n) - w \|_1 \leq C \| (u_n - \Phi(w))^{1/m} \|_1 \leq C(\Omega) \| u_n - \Phi(w) \|_1^{1/m}.
\]
This proves that the closure of \( D(A) \) contains \( C_c^{\infty}(M) \). Since \( C_c^{\infty}(M) \) is dense in \( L_1(M) \), this shows the density of \( D(A) \) in \( L_1(M) \).

Case 2: \( m < 1 \). In this case, \( \beta \in C^{1-}(\mathbb{R}) \). Given arbitrary \( w \in C_c^{\infty}(M) \), there exists a sequence \( (u_k)_k \subset C_c^{\infty}(M) \subset D((-\Delta_1)^\sigma) \) converging to \( \Phi(w) \) in \( L_1(M) \) satisfying
\[
\| \beta(u_k) \|_\infty \leq 2 \| w \|_\infty.
\]
So the Lipschitz continuity of \( \beta \) implies
\[
\| \beta(u_k) - w \|_1 \leq C(w) \| u_k - \Phi(w) \|_1.
\]
As in Case 1, this implies the density of \( D(A) \) in \( L_1(M) \). \( \square \)

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