A New Interpretation of the Steady-State Two-Reaction Theory of a Salient-Pole Synchronous Machine

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ABSTRACT This work contributes two improvements to the well-established two-reaction theory of a salient-pole synchronous machine. Namely, a single circuit is proposed that explicitly accounts for the saliency in the machine. By defining new machine reactances and using a current-controlled voltage source, the proposed single circuit provides an alternative to coupled d- and q-axis circuit models. The new reactances are also used in a revised phasor diagram to make apparent the internally developed power of a salient-pole synchronous machine that is similar to a round-rotor synchronous machine. The revised two-reaction theory is illustrated using the mathematical model of a three-phase salient-pole synchronous machine whose equations are derived using complex space vectors instead of traditional matrix transformations. A detailed derivation is presented and the resulting equations can directly be used to solve for the steady-state operating EMF and power angle from the terminal voltages and currents in the abc coordinate system. The ease of their application is demonstrated using a numerical example of the steady-state circuit equations and the revised phasor diagrams.

INDEX TERMS Synchronous machine saliency, steady-state circuit, space vector representation, two-reaction theory.

NOMENCLATURE

\( v_a, v_b, v_c \) abc stator voltages.
\( i_a, i_b, i_c \) abc stator currents.
\( \lambda_a, \lambda_b, \lambda_c \) abc stator flux linkages.
\( v_d, v_q \) dq stator voltages.
\( i_d, i_q \) dq stator currents.
\( \lambda_d, \lambda_q \) dq stator flux linkages.
\( v_f, i_f \) Field current and voltage.
\( \vec{x} \) Arrow denotes a space vector quantity \( x \).
\( v_s, i_s, \vec{\lambda}_s \) abc voltage, current and flux linkage.
\( v_{dq}, i_{dq}, \vec{\lambda}_{dq} \) dq voltage and current.
\( T_e, P_e \) Internal electromagnetic torque and power.

\( W_m' \) Magnetic coenergy function.
\( \theta, \delta, \phi \) Electrical rotor, power, and power factor angles.
\( \vec{E}_{dq} \) Field EMF developed along the q-axis.
\( \vec{E}_f \) Newly defined internal voltage.
\( R_s, R_f \) Stator and field winding resistances.
\( X_d, X_q \) Classical synchronous d- and q-axis reactances.
\( X^+ = (X_d + X_q)/2 \) Newly defined average reactance.
\( X^- = (X_d - X_q)/2 \) Newly defined saliency reactance.

I. INTRODUCTION

At the 2000 North American Power Symposium, a paper by R. H. Park on the \( dq0 \) transformation ranked second in a survey of papers having had the most impact in power...
engineering over the last century [1]. The dq0 transformation, also referred to as the Park transformation, is still today the basis for the investigation of synchronous machine dynamics in power systems.

Historically speaking, the development of a versatile mathematical model of a synchronous machine began with the two-reaction theory of Blondel, Doherty and Nickle, and others [2], [3], and was later generalized by Park in his papers defining an ideal synchronous machine and outlining the theory of the dq transformation [4], [5].

Subsequent refinements of Park’s theory have been made in the development of equivalent damper windings in the direct and quadrature axes, the quantification of magnetic saturation, and the determination of machine parameters during subtransient, transient and steady-state analyses [6], [7], [8]. On the other hand, in more recent work, O’Rourke et al. in [9] discuss the geometric interpretation of the space vector representations of AC machines in the abc, αβ0 and dq0 reference frames. Their work includes the spacial transformations of the space vectors and a visual representation in 3D of their resulting loci in transient-state.

On the one hand, power systems textbooks use a classical EMF behind reactance model for the synchronous generator to facilitate the power grid stability analysis under simplified conditions, e.g., Glover et al. in [10] and Anderson and Fouad in [11]. And, although this model is insightful and useful for hand calculations or large simulations, it ignores among other things, the saliency in the rotor by assuming that the two main reactances Xd and Xq are equal. On the other hand, detailed power systems analysis studies demand more detailed models as in IEEE Std. 1110-2019 [12] and require detailed test procedures as in IEEE Std. 115-2019 [13]. Therefore, there is a need for a revised single circuit diagram that goes beyond the classical single circuit representation in recognizing the saliency in synchronous machines (even round-rotor types), both for steady-state stability analysis and as a basis for transient stability analysis of the power grid when damper windings are ignored.

In this paper, we examine two improvements to the classical two-reaction theory. The first overcomes a lack of a single equivalent circuit for a salient-pole synchronous machine in steady-state. The second accounts for the converted power, i.e., the internally developed electromagnetic power, in the classical phasor diagram of the machine using relevant complex vectors. We propose to do so by analyzing a three-phase salient-pole synchronous machine. For simplicity, damper windings are not considered in this analysis as the aim is to produce a new steady-state circuit and phasor diagram to provide insight into its operation. Initially, the equations are developed in the motor notation for a three-phase salient pole synchronous machine, and then a summarized development is presented for the generator notation. The contributions of this work can be summarized as follows:

1) A single circuit is developed for the synchronous machine in steady-state.
2) Saliency in the rotor is accounted for in the model.
3) The reluctance term in the internal power equation can be identified.
4) Equations are derived to directly solve for the internal EMF \( E_{id} \) and power angle \( \delta \).

Therefore, in contrast with existing classical models, the saliency of the machine is represented with a single circuit that can be directly solved for the operating conditions. Furthermore, to facilitate the analysis two minor contributions include a space vector derivation for the classical and new model and phasor diagrams are used to depict the classical and new representations. A numerical example is given to showcase both the importance of accounting for saliency and the ease of using the new model. An early version of this work was presented in [14] for a two-phase machine with a summarized development of the theory in the motor notation. More results are included in this work with detailed derivation that is generalized for the three phase salient pole machine. The remainder of the paper begins with the theoretical review of the three-phase salient pole machine in Section II. Section III covers the classical two-reaction theory with the dq transformation, and the resulting classical phasor diagram in steady-state is then presented with its interpretation in Section IV. A revised phasor diagram follows in Section V, along with its resulting voltage equation, equivalent circuit representation, and converted power explanation. Next, a numerical example follows in Section VI to demonstrate and facilitate the application of the equations, and the main conclusions are drawn in Section VII.

II. THEORETICAL REVIEW

Let us consider a three-phase salient-pole synchronous machine with three stator windings (a, b, c) and one field winding (f). The mathematical model of this machine comprises four voltage equations and a second-order mechanical equation of motion. Using Kimbark’s notation [7], the voltage equations are, in motor notation,

\[
\begin{align*}
\nu_a &= R_i a + \frac{d\lambda_a}{dt}, \\
\nu_b &= R_i b + \frac{d\lambda_b}{dt}, \\
\nu_c &= R_i c + \frac{d\lambda_c}{dt}, \\
\nu_f &= R_i f + \frac{d\lambda_f}{dt},
\end{align*}
\]

(1)

(2)

(3)

(4)

together with their flux-current relationships,

\[
\begin{bmatrix}
\lambda_a \\
\lambda_b \\
\lambda_c \\
\lambda_f
\end{bmatrix} =
\begin{bmatrix}
L_{aa} & L_{ab} & L_{ac} & L_{af} \\
L_{ba} & L_{bb} & L_{bc} & L_{bf} \\
L_{ca} & L_{cb} & L_{cc} & L_{cf} \\
L_{fa} & L_{fb} & L_{fc} & L_{ff}
\end{bmatrix}
\begin{bmatrix}
i_a \\
i_b \\
i_c \\
i_f
\end{bmatrix},
\]

(5)

where,

\[
L_{aa} = L_s + L_m \cos 2\theta,
\]

(6)

\[
L_{bb} = L_s + L_m \cos 2\left(\theta - \frac{2\pi}{3}\right),
\]

(7)
\[ L_{cc} = L_x + L_m \cos 2\left(\theta + \frac{2\pi}{3}\right), \]  
(8)
\[ L_{ab} = L_{ba} = -M_x - L_m \cos 2\left(\theta + \frac{\pi}{6}\right), \]  
(9)
\[ L_{bc} = L_{cb} = -M_x - L_m \cos 2\left(\theta - \frac{\pi}{2}\right), \]  
(10)
\[ L_{ac} = L_{ca} = -M_x - L_m \cos 2\left(\theta + \frac{5\pi}{6}\right), \]  
(11)
\[ L_{af} = L_{fa} = M_f \cos \theta, \]  
(12)
\[ L_{bf} = L_{fb} = M_f \cos \left(\theta - \frac{2\pi}{3}\right), \]  
(13)
\[ L_{cf} = L_{fc} = M_f \cos \left(\theta + \frac{2\pi}{3}\right), \]  
(14)
\[ L_{ff} = L_f, \]  
(15)
and where \( \theta(t) = \omega t + \theta_0 \) is the electrical angle of rotation of the rotor shaft at the steady-state electrical speed \( \omega \). If the phase \( a \) voltage \( v_a(t) \) is chosen as the reference waveform in time, as is typically the case for one machine under study, its phase at time \( t = 0 \) would be \( \theta_0 = 0^\circ \). In other words, the peak value of \( v_a(t) \) is measured at time \( t = 0 \) s; otherwise, there would be a phase shift \( \delta \). Therefore, without much loss of generality, we will assume that \( v_a(t) \) is the reference voltage and \( \theta_0 = 0^\circ \). On the other hand, at time \( t = 0 \) s the initial location of the rotor \( d \)-axis is \( \theta_0 \), which is typically not zero.

By defining the following space vectors,
\[ \vec{v}_s = \frac{2}{3} (v_a + \bar{a}v_b + \bar{a}^2v_c), \]  
(16)
\[ \vec{\lambda}_s = \frac{2}{3} (\lambda_a + \bar{a}\lambda_b + \bar{a}^2\lambda_c), \]  
(17)
\[ \vec{i}_s = \frac{2}{3} (i_a + \bar{a}i_b + \bar{a}^2i_c), \]  
(18)
where, \( \bar{a} = e^{j2\pi/3} \), the voltage equations can be formulated concisely using the following equations
\[ \vec{v}_s = R_s \vec{i}_s + \frac{d\vec{\lambda}_s}{dt}, \]  
(19)
\[ v_f = R_f i_f + \frac{d\lambda_f}{dt}, \]  
(20)
and the flux-current vector relations
\[ \vec{\lambda}_s = (L_x + M_x) i_s + \frac{3}{2} L_m e^{2j\theta} \vec{i}_s + M_f e^{j\theta} i_f, \]  
(21)
\[ \lambda_f = \frac{3}{4} M_f e^{-j\theta} \vec{i}_s + \frac{3}{4} M_f e^{j\theta} \vec{i}_s + L_f i_f. \]  
(22)

Space vectors express the time varying waveforms as they rotate in a plane coinciding with the cross-sectional view of the machine. In this plane, axes such as \( a, b \) and \( c \) denote the direction of positive magnetic flux in the air gap created by the corresponding stationary (stator) windings. Figures 1 and 2 depict the space vectors, axes, and angles at time \( t = 0 \) s for a generator and a motor, respectively. Note that the positive direction of rotation is counterclockwise, which is opposite to Kimbark’s notation in depicting these axes, see p. 54 in [7].

At time \( t = 0 \) s, \( \vec{V}_s \) is aligned with the \( a \)-axis if \( \theta_0 = 0^\circ \) indicating that peak voltage is experienced in phase \( a \) winding at that time, and rotates counterclockwise in the plane thereafter. Phasors are then a snapshot in time at \( t = 0 \) s of these space vectors under periodic steady-state conditions, which is why this figure is also considered a phasor diagram.

Moreover, the field winding of the rotor produces its peak magnetic flux along the direct \( d \)-axis, which is located by the angle \( \theta \) with respect to the stationary reference \( a \)-axis. The field current \( i_f \) induces EMF in the stator windings at a peak that is \( 90^\circ \) leading the \( d \)-axis, i.e., the quadrature \( q \)-axis, and this EMF is designated by the space vector \( \vec{E}_{fd} \).

In addition, \( \delta \) is the power angle between the stator terminal voltage \( \vec{V}_s \) and the machine internal EMF field vector \( \vec{E}_{fd} \). That is, when \( \vec{E}_{fd} \) leads \( \vec{V}_s \) the machine is generating with \( \delta > 0^\circ \) as seen in Figure 1, and when \( \vec{E}_{fd} \) lags \( \vec{V}_s \) it is motoring with \( \delta < 0^\circ \) as seen in Figure 2. Therefore, at \( t = 0 \) s,
\[ \delta = \theta_0 + 90^\circ. \]  
(23)

The phase shift between the terminal voltage \( \vec{V}_s \) and current \( \vec{i}_s \) is the power factor angle \( \phi \), which is depicted as lagging; e.g., for the case of the motor in Figure 2 this represents under-excited motor operation. Motor convention is used in
the following derivations and the results apply equally to the generator and motor modes of operation.

Next, we derive the electromagnetic torque of the machine and its relation to these space vectors by first assuming a magnetically linear coupling field, the magnetic coenergy of this machine is given by

\[ W_m = \frac{1}{2} \lambda_i a + \frac{1}{2} \lambda_b b + \frac{1}{2} \lambda_c c + \frac{1}{2} \lambda_f f. \]  

(24)

It is straightforward to show that this coenergy can also be expressed as

\[ W_m = \Re e \left\{ \frac{3}{4} \lambda_s i_s \right\} + \frac{1}{2} \lambda_f f \]

\[ + \frac{1}{6} (\lambda_a + \lambda_b + \lambda_c)(i_a + i_b + i_c). \]  

(25)

Expanding this last expression and simplifying yields

\[ W_m = \Re e \left\{ \frac{3}{4} (L_s + M_s) i_s^2 + \frac{9}{8} L_m e^{2\theta} (i_s^2)^2 \right\} \]

\[ + \frac{3}{4} M_f e^{\theta} i_s \]

\[ + \frac{3}{8} M_f e^{-\theta} i_s \]

\[ + \frac{3}{2} M_f e^{\theta} i_s \]

\[ + \frac{1}{2} M_f e^{-\theta} i_s \]

\[ = \Re e \left\{ \frac{9}{8} L_m e^{2\theta} (i_s^2)^2 + \frac{3}{2} M_f e^{\theta} i_s \right\} \]

\[ + \frac{3}{4} (L_s + M_s) i_s^2 + \frac{1}{2} L_f i_s \]

\[ + \frac{1}{6} (L_s - 2M_s)(i_a + i_b + i_c)^2. \]  

(26)

By taking the partial derivative of the coenergy with respect to the mechanical rotor shaft angle \( \theta_m = (2/\rho) \theta \), where \( \rho \) is the number of poles, we obtain the following expression for the developed electromagnetic torque

\[ T_e = \frac{\partial W_m}{\partial \theta_m} = \left( \frac{P}{2} \right) \Re e \left\{ j \frac{9}{4} L_m (e^{\theta} i_s^2) + j \frac{3}{2} M_f e^{\theta} i_s \right\}. \]  

(27)

### III. CLASSICAL TWO-REACTION THEORY

We begin by defining fictitious \((d,q)\) space vectors rotating with the rotor reference frame,

\[ \vec{\lambda}_{dq} = \vec{\lambda}_d e^{-j\theta}, \]

\[ \vec{\eta}_{dq} = \vec{\eta}_d e^{-j\theta}, \]

\[ i_{dq} = i_d + j i_q. \]  

(28)

Then, the stator voltage equation (16) can be transformed into the following vector equation

\[ \vec{v}_{dq} = R_s i_{dq} + \frac{d\vec{\lambda}_{dq}}{dt} + j \omega \vec{\lambda}_{dq}. \]  

(29)

After substituting all complex space vectors with their rectangular variables, that is, \( \vec{v}_{dq} = v_d + j v_q, \vec{\lambda}_{dq} = \lambda_d + j \lambda_q \) and \( i_{dq} = i_d + j i_q \), the following two real equations are obtained,

\[ v_d = R_s i_d + \frac{d\lambda_d}{dt} - \omega \lambda_q, \]

\[ v_q = R_s i_q + \frac{d\lambda_q}{dt} + \omega \lambda_d. \]  

(30)

Expressed in \((d,q)\) variables, the original flux-current relations,

\[ \vec{\lambda}_s = (L_s + M_s) \vec{i}_s + \frac{3}{2} L_m e^{2\theta} \vec{i}_s + M_f e^{\theta} \vec{i}_s, \]

\[ \lambda_f = \frac{3}{4} M_f e^{-\theta} \vec{i}_s + \frac{3}{4} M_f e^{\theta} \vec{i}_s + L_f \vec{i}_s, \]  

(31)

become

\[ \vec{\lambda}_{dq} = (L_s + M_s) i_{dq} + \frac{3}{2} L_m i_{dq} + M_f i_{dq}, \]

\[ \lambda_f = \frac{3}{4} M_f i_{dq} + \frac{3}{4} M_f i_{dq} + L_f i_{dq}. \]  

(32)

Expanding these equations in rectangular form,

\[ \lambda_d = L_d i_d + M_f i_q, \]

\[ \lambda_q = L_q i_q, \]

\[ \lambda_f = \frac{3}{2} M_f i_d + L_f i_q, \]  

(33)

where the \(d\)-axis and \(q\)-axis inductances, \(L_d\) and \(L_q\), are respectively defined as

\[ L_d = L_s + M_s + \frac{3}{2} L_m, \]

\[ L_q = L_s + M_s - \frac{3}{2} L_m. \]  

(34)

The developed electromagnetic torque can be expressed in \((d,q)\) variables as

\[ T_e = \left( \frac{P}{2} \right) \Re e \left\{ \frac{9}{4} L_m (i_d^2 - j i_q^2) + \frac{3}{2} M_f (i_d^2 + j i_q^2) \right\} \]

\[ + \left( \frac{P}{2} \right) \Re e \left\{ \frac{9}{4} L_m i_{dq}^2 + \frac{3}{2} M_f i_{dq}^2 \right\}. \]  

(35)

By replacing \( i_{dq} \) with its rectangular coordinates,

\[ i_{dq} = i_d + j i_q, \]  

(36)

the standard form of the developed electromagnetic torque of a salient-pole synchronous machine is obtained as

\[ T_e = \left( \frac{P}{2} \right) \Re e \left\{ \frac{9}{4} L_m (i_d - j i_q)^2 + \frac{3}{2} M_f (i_d - j i_q) \right\} \]

\[ + \left( \frac{P}{2} \right) \Re e \left\{ \frac{3}{2} M_f (i_d + 3 L_m i_q) \right\} \]

\[ + \left( \frac{P}{2} \right) \Re e \left\{ \frac{3}{2} (\lambda_d i_q - \lambda_q i_d) \right\}. \]  

(37)
steady-state equations in both axes as term found in smooth-air-gap machines. The reluctance term and the synchronous power of the second expression is neglected. The phasor diagram in this diagram resembles Figure 4 is a pictorial representation of equation (41). The opposite axes, $d$ and $q$-axis circuits of a synchronous machine.

IV. CLASSICAL PHASOR DIAGRAM

The steady-state machine voltage equations are obtained by setting the time derivatives of the $d$-axis and $q$-axis flux linkages to zero in equations (30) and (30). Using capital letters to denote steady-state quantities, that is, the peak magnitude or the per unit of their respective quantities, these equations become

$$
V_d = R_d I_d - \omega L_q, \\
V_q = R_q I_q + \omega L_d.
$$

These two equations can each be represented by a circuit in the $d$-axis or $q$-axis as shown in Figure 3. The two circuits are coupled via the steady-state speed voltages induced in opposite axes, $\omega L_d$ and $\omega L_q$.

The electromagnetic power is equal to the product of the electromagnetic torque in (37) and the mechanical rotor shaft speed $\omega_m = (2/p)\omega$. This is shown to be the sum of the two speed voltages in the two previous subcircuits yielding

$$
P_e = \omega_m T_e = \frac{3}{2}(\Lambda_d I_q - \Lambda_q I_d).
$$

This power expression can also be expressed as

$$
P_e = \frac{3}{2} \omega (\Lambda_d I_q - \Lambda_q I_d)
= \frac{3}{2} \omega (L_d I_d + M_f I_f M_q - L_q I_d)
= \frac{3}{2} (X_d - X_q) I_d I_q + E_{fd} I_q),
$$

where $E_{fd} = \omega M_f I_f, X_d = \omega L_d$ and $X_q = \omega L_q$. This last expression makes apparent the power developed due to the reluctance term and the synchronous power of the second term found in smooth-air-gap machines.

The classical phasor diagram is obtained by combining the steady-state equations in both axes as

$$
V_{dq} = R_s I_{dq} + jX_d I_d + jX_q I_q) + E_{fd}.
$$

Figure 4 is a pictorial representation of equation (41). The stator resistance is exaggerated to show its location in the phasor diagram, however, in practice with large synchronous generators its value (and contribution) is small and typically neglected. The phasor diagram in this diagram resembles Kimbark’s phasor diagram on p. 70 of [7], and differs slightly from the classical one using space vectors; that is, it uses complex quantities such as $(jE_{fd})$ instead of $E_{fd}$.

![FIGURE 3. d-axis and q-axis circuits of a synchronous machine.](image)

![FIGURE 4. Classical phasor diagram of a salient-pole synchronous machine.](image)

One practical problem in using equation (41) is in trying to compute $E_{fd}$ given the terminal voltage and current $V_{dq} = V_s e^{j\theta}$ and $I_{dq} = I_s e^{j\theta}$. This is because prior knowledge of the angle $\delta$ between $E_{dq}$ and $V_{dq}$ is necessary, e.g., to find the projections of $I_{dq}, I_d$ and $I_q$, to compute the term $jX_d I_d + jX_q I_q$. However, $\delta$ is not known until $E_{dq}$ is computed, which makes it a cyclical problem. A solution, in [7] p. 72, is to add and subtract $jX_d I_d$ in (41) and rearrange as follows

$$
V_{dq} = R_s I_{dq} + jX_d I_d - X_q I_q + jX_q I_d - jX_d I_d + jE_{fd}
= R_s I_{dq} + j(X_d - X_q) I_d + jX_q I_d + jE_{fd} + jE_{fd}
= R_s I_{dq} + jX_q I_{dq} + j(X_d - X_q) I_d + jE_{fd}.
$$

Now, define an internal voltage to include the last two terms in (42) as

$$
\overrightarrow{E_q} = j(X_d - X_q) I_d + jE_{fd},
$$

where $\overrightarrow{E_q}$ is aligned with $E_{fd}$ along the $q$-axis, and then

$$
\overrightarrow{V_{dq}} = R_s I_{dq} + jX_q I_{dq} + \overrightarrow{E_q}.
$$

This form allows for the direct computation of $\overrightarrow{E_q}$ from $V_{dq}$ and $I_{dq}$, finding the angle $\delta$, computing $I_d$, and then finally finding the required internal EMF $E_{fd}$ of the machine to achieve that operating point using (43). This value for $E_{fd}$, not $\overrightarrow{E_q}$, is known in relation to the field current from the no load magnetization curve of the machine. Moreover, a common approximation when dealing with round-rotor synchronous machines, where $X_d \approx X_q$, is to take $\overrightarrow{E_q} \approx \overrightarrow{E_{fd}}$.

On the other hand, the input real power into the machine is

$$
P_{in.3\phi} = \frac{3}{2} \Re (\overrightarrow{V_{dq}} I_{dq}) = \frac{3}{2} (V_d I_d + V_q I_q).
$$

This input power is equal to the internally developed electromagnetic power $P_e$ if the stator resistance and magnetic core losses are neglected. The synchronous power term $(E_{fd} I_q)$ is recognizable from the complex product of $(jE_{fd})$ and $(jI_q)^*$. However, the reluctance power term is not readily identifiable.
in this classical phasor diagram. In the next section, we propose a new single-circuit representation of a salient-pole synchronous machine along with a modified phasor diagram where the two terms of the internally developed electromagnetic power are readily identifiable.

V. REVISED PHASOR DIAGRAM

By defining the following reactances,

\[ X^+ = \frac{X_d + X_q}{2}, \]
\[ X^- = \frac{X_d - X_q}{2}, \]

the steady-state voltage equation (41) can be manipulated to yield the following phasor equation which is represented by the single equivalent circuit shown in Figure 5 with

\[ \vec{V}_{dq} = (R_s + jX^+I_{dq}^s) + jX^-I_{dq} + jE_{fd}. \]  

(48)

To directly solve for \( \delta \) and \( E_{fd} \) from \( \vec{V}_s \) and \( \vec{I}_s \), we can derive the following equations (see VII)

\[ \delta = \tan^{-1}\left( \frac{-X_q \cos(\phi) + R_s \sin(\phi)}{(V_s/I_s) - R_s \cos(\phi) - X_q \sin(\phi)} \right), \]

(49)

\[ E_{fd} = V_s \cos(\delta) - R_s I_s \cos(\delta + \phi) - X_q I_s \sin(\delta + \phi), \]

(50)

where, both \( V_s \) and \( I_s \) are the peak amplitudes of their respective space vectors (the RMS values can also be used), and \( \phi \) is the power factor angle (positive if lagging).

If we redefine the internal voltage \( \vec{E}_i \) as the sum of the dependent voltage source (\( jX^-I_{dq}^s \)) and the independent voltage source (\( jE_{fd} \)),

\[ \vec{E}_i = jE_{fd} + jX^-I_{dq}^s \]
\[ = jE_{fd} + jX^-I_d + jX^-I_q, \]

(51)

then, we can easily verify that the internally developed electromagnetic power \( P_e \) is equal to the real power absorbed by this internal voltage by expanding the following equation

\[ P_e = \frac{3}{2} \Re\{ \vec{E}_i \vec{I}_{dq}^* \} \]
\[ = \frac{3}{2} \Re\{ (jE_{fd} + jX^-I_d + jX^-I_q)(I_d - jI_q) \}. \]

A phasor diagram for the single equivalent circuit can be drawn as shown in Figure 6 where the stator resistance has been exaggerated for clarity.

In the revised phasor diagram, the internally developed power \( P_e \) can be identified as the real part of the complex product of this internal voltage \( \vec{E}_i \) and the conjugate of the current vector \( \vec{I}_{dq} \). Alternatively, if \( \vec{E}_i \) and \( \vec{I}_{dq} \) are viewed as regular geometric vectors, then \( P_e \) can also be interpreted as their dot or scalar product.

By decomposing the current vector into its real and imaginary components, \( \vec{I}_{dq} = I_d + jI_q \), the internally developed power \( P_e \) becomes readily available as the sum of three terms:

- a power term (\( (E_{fd}I_q) \)) resulting from the complex product of \((jE_{fd})\) and \((jI_q)^*\),
- a power term \((X^-I_dI_q)\) resulting from the complex product \((jX^-I_d)\) and \((jI_q)^*\), and
- a power term \((X^-I_dI_q)\) resulting from the complex product \((X^-I_d)\) and \((I_q)^*\).

The synchronous machine model developed so far has used the motor notation, where the current is assumed to be positive going into the stator terminals. And, although this model works equally well for synchronous generators, the inconvenience, and potential confusion, that may be caused by dealing with negative stator currents is typically avoided by revising these model equations in terms of the generator notation. The field current is still in the same direction. A common approach taken, as in [7], p. 73 and the references therein, is to reverse the direction of the currents and keep everything else the same. This also entails reversing the direction of the electromagnetic torque because, in a generator, it will be opposite to the direction of rotation. The resulting
space vector diagram for the generator is redrawn in Figure 1. An alternative approach is to reverse the direction of both the terminal voltage and internal EMF, which is not typically taken. Consequently, the proposed revised equation becomes

$$\vec{V}_{dq} = -(R_s + jX^+I_{dq}) - jX^-I_{dq} + jE_{fd}. \quad (54)$$

Moreover, $\delta$ and $E_{fd}$ can be solved directly given $\vec{V}_s$ and $\vec{I}_s$ using

$$\tan(\delta) = \frac{X_q \cos(\phi) - R_s \sin(\phi)}{(V_s/I_s) + R_s \cos(\phi) + X_q \sin(\phi)}, \quad (55)$$

$$E_{fd} = V_s \cos(\delta) + R_s I_s \cos(\delta + \phi) + X_q I_s \sin(\delta + \phi). \quad (56)$$

Note that the proposed equivalent circuit for the generator notation is the same as Figure 5 except that the directions of the currents are reversed.

VI. NUMERICAL EXAMPLE

The utility of this new circuit diagram and phasor representation is now demonstrated by a numerical example. The $abc$-coordinate space vectors and machine parameters are used to directly solve for the other operating point variables in the $dq$-coordinate system. Consider the following per unit parameters, obtained (or adapted) from Table 2 on p. 40 of [7] and references therein, for a typical (average) salient pole synchronous generator:

- direct axis synchronous reactance $X_d = 1.15$ pu,
- quadrature axis synchronous reactance $X_q = 0.75$ pu,
- AC stator winding resistance $R_s = 0.009$ pu.

Assume the generator is operating with a terminal voltage of 1 pu, a current of 0.8 pu and driving a lagging power factor of 0.85, i.e., $\phi = + \cos^{-1}(0.85) = 31.79^\circ$. Initially, taking the terminal voltage as the reference ($a$-axis with $\theta_a = 0^\circ$) we have $\vec{V}_s = 1.00\, \text{pu}$ and $\vec{I}_s = 0.80 - 31.79^\circ\, \text{pu}$.

Then, we can directly compute $\delta = 20.95^\circ, \theta_0 = -90^\circ + \delta = -69.05^\circ$ and $E_{fd} = 1.67$ pu, using (55), (23) and (56), respectively. To construct the space vector and phasor diagram by effectively realigning the voltage and current quantities so that the $d$-axis of the rotor is the reference as shown in Figure 7, which is done in the following order,

$$\vec{E}_{fd} = 1.67, 90^\circ \, \text{pu},$$

$$\vec{V}_{dq} = 1 - \theta_0 = 1.69, 05^\circ \, \text{pu},$$

$$\vec{I}_{dq} = 0.8/ -\theta_0 - \phi = 0.837, 26^\circ \, \text{pu},$$

that is, $I_d = 0.6367$ pu and $I_q = 0.4844$ pu.

The impact of each term in the proposed voltage phasor equation (54) is shown in Figure 7, and can be calculated as follows

$$\vec{E}_i = \vec{E}_{fd} - jX^+I_{dq} = 1.67, 90^\circ - 0.16, 52.74^\circ.$$

$$\vec{V}_{dq} = \vec{E}_i - jX^-I_{dq} - R_s I_{dq} = 1.546, 93.59^\circ - 0.76, 127.3^\circ - 0.0072, 37.26^\circ,$$

It is therefore evident that the most significant term is $jX^+I_{dq}$ and that $R_s I_{dq}$ is negligible. Indeed, the latter term cannot be shown on Figure 7. However, the impact of $jX^-I_{dq}$ cannot be neglected, especially for a salient pole synchronous machine.

More importantly, with this distinction, we can identify the impact of saliency on the generated electromagnetic power in per unit as

$$P_e = \Re(e^{j\phi} \vec{E}_{fd}) = E_{fd}I_d - 2X^-I_{dq} = \Re(e^{j(1.546, 93.59^\circ)0.8 - 37.26^\circ}) = 0.8091 - 0.1234 = 0.6857 \, \text{pu},$$

that is, at this operating condition, about 15% of the internally developed power is spent on the saliency term of the machine.

VII. CONCLUSION

In this paper, we have extended the two-reaction theory of a three-phase salient pole synchronous machine. The aim is to replace the classical steady-state models, whether the two circuit model ($d,q$) two-reaction theory model or the single circuit reaction behind EMF model, with a single equivalent circuit model that explicitly accounts for the saliency of the machine. This is accounted for as a product of the conjugate $dq$ current and a newly defined reactance $X^-$. Both motor and generator notations were developed to aid the application in both cases, and highlight the differences in operation, particularly with regards to the phasor diagrams, the voltage equations, and the internally developed power equation. Also, equations were derived to directly solve for the internal EMF and power angle at any given operating point. A numerical example was used to demonstrate the advantage of using this proposed formulation, both in the development of the phasor and space vector diagram representation, and in demonstrating the impact of the saliency on the internally developed electromagnetic power. Specifically, it was shown that in such a typical salient pole generator, the saliency term accounted for 15% of the electromagnetic power, and therefore cannot
be ignored. Finally, this representation can also be developed for other AC machines and their interconnected operation in future work, and it can be extended to account for damper winding dynamics.

**APPENDIX COMPUTING \( \delta \) AND \( E_{fd} \)**

Whether the classical phasor equation (42) or the revised phasor equation (48) are used, solving directly for \( \delta \) and \( E_{fd} \) can be a cyclical problem. This is the case because starting from the terminal voltage \( \vec{V}_{dq} = \vec{V}_s e^{-j\phi_0} \) and current \( \vec{I}_{dq} = \vec{I}_s e^{-j\phi_0} \) we need \( \theta_0 \), which includes \( \delta \). To solve this problem, starting from (48), we have

\[
\vec{V}_{dq} = (R_s + jX_s')\vec{I}_{dq} + jX_s\vec{I}_{dq} + jE_{fd}. \tag{57}
\]

The space vectors can be represented in terms of their peak amplitudes and angles with reference to the \( d \)-axis as

\[
\vec{V}_{dq} = \vec{V}_s e^{-j\phi_0} = V_s \cdot 90^\circ - \delta, \tag{58}
\]

\[
\vec{I}_{dq} = \vec{I}_s e^{-j\phi_0} = I_s/90^\circ - \delta - \phi. \tag{59}
\]

Replacing \( \vec{V}_{dq} \) and \( \vec{I}_{dq} \) in (48) we obtain

\[
V_s \cdot 90^\circ - \delta = R_s I_s/90^\circ - \delta - \phi + jX_s I_s/90^\circ - \delta - \phi + jX_s I_s/\delta + \phi + jE_{fd}.
\]

\[
= R_s I_s/90^\circ - \delta - \phi + X_s I_s/\delta + \phi + jE_{fd}. \tag{60}
\]

Then, separating and equating the real and imaginary components of this equation,

\[
V_s \sin(\delta) = R_s I_s \sin(\delta + \phi) - (X_s - X_s^\prime) I_s \cos(\delta + \phi), \tag{61}
\]

\[
V_s \cos(\delta) = R_s I_s \cos(\delta + \phi) + (X_s + X_s^\prime) I_s \sin(\delta + \phi) + E_{fd}. \tag{62}
\]

Equation (61) can be used to find \( \delta \). Given that \( X_s + X_s^\prime = X_q \) and rearranging we get

\[
\frac{V_s}{I_s} \sin(\delta) = R_s \sin(\delta) \cos(\phi) + \cos(\delta) \sin(\phi)) - X_q \cos(\delta) \sin(\phi) - \sin(\delta) \sin(\phi). \tag{63}
\]

Dividing by \( \cos(\delta) \) and solving for the resulting \( \tan(\delta) \)

\[
\tan(\delta) = \frac{-X_q \cos(\phi) + R_s \sin(\phi)}{V_s/I_s - R_s \cos(\phi) - X_q \sin(\phi)}. \tag{64}
\]

Next, by substituting for this value of \( \delta \) in (62) and given that \( X_s + X_s^\prime = X_q \), we can rearrange to solve for \( E_{fd} \)

\[
E_{fd} = V_s \cos(\delta) - R_s I_s \cos(\delta + \phi) - X_q I_s \sin(\delta + \phi). \tag{65}
\]