Output regulation of a reaction-diffusion PDE with long time delay using backstepping approach∗

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Abstract: In this paper, we are concerned with the output regulation of an unstable reaction-diffusion PDE with regulator delay and unmatched disturbances, which are generated by an exosystem. A backstepping regulator design procedure is presented by mapping the reaction-diffusion PDE cascaded with a transport equation into an error system, which is shown to be exponentially stable with a prescribed rate in a suitable Hilbert space. The output regulation is then verified by solving cascaded regulator equations. The solvability condition of the cascaded regulator equations is characterized by a transfer function and eigenvalues of the exosystem. Finally, the numerical simulations are provided to illustrate the effect of the regulator.

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1. INTRODUCTION

In communication and information technologies, systems with time delays are frequently encountered, such as teleoperated systems (Anderson and Spong, 1989), robotics systems (Ailon and Gil, 2000), and communication networks (Biberovic et al., 2001). Time delay usually has a destabilization effect and may lead to poor control system performance. Consequently, the research on time delay systems is of theoretical importance and practical value. An intensive research activity has been devoted to various control issues for time delay systems, see e.g. stabilization (see e.g. Krstic, 2009; Wang et al., 2011), observers design (Hou et al., 2002; Ahmed-Ali et al., 2016), and sliding mode control (Han et al., 2010). In Krstic (2009), the exponential stability of an unstable reaction-diffusion PDE with long input delay has been established by Lyapunov technology, and the decay rate is estimated but not arbitrary. When time delay appear in the boundary observation, the stabilization of the wave equation has been obtained in Wang et al. (2011) by using Riesz basis technology.

Another important control issue is output regulation, that is to design regulators for asymptotic tracking of reference signals and/or rejection of disturbances (Aulisa and Gilliam, 2016; Byrnes et al., 2000). A regulator has been designed in Fridman (2003) for the output regulation of retarded type nonlinear systems with delay. There the necessary and sufficient conditions for the solvability of the regulator problem has also been presented. When the regulator delay is absent, the regulator has been designed for a diffusive-wave equation in Chentouf and Wang (2007, 2008), parabolic equation in Deutscher (2015), and second order hyperbolic equation in Deutscher and Kerschbaum (2016), where the backstepping approach (Smyslyaev and Krstic, 2004) is applied to design regulator so that the original system is converted into the target system with some desirable properties.

The aforementioned works mainly deal with the output regulation of PDEs with unmatched disturbance, which enters at the uncontrolled end or the body equations. When the disturbance appears at the controlled end, the controller has been designed for the stabilization of wave equation in Guo and Jin (2015), flexible crane System in He et al. (2014), and nonuniform gantry crane in He and Ge (2016).

The system that we are concerned with is governed by the following PDE in the presence of regulator delay and unmatched disturbances:

\[
\begin{align*}
\phi(x, t) &= \phi_{x x}(x, t) + c_0 \phi(x, t) + g_1(x) d_1(t), \quad x \in (0, 1), \quad t > 0, \\
\phi(0, t) &= q d_2(t), \quad t \geq 0, \\
\phi(1, t) &= U(t - \tau), \quad t \geq 0, \\
y_{\text{out}}(t) &= [\mathcal{C} \phi(x)](t), \quad t \geq 0,
\end{align*}
\]

where \(\phi(x, t)\) is the state of the reaction-diffusion equation, \(c_0 > 0\) and \(q \in \mathbb{R}\). \(U(t)\) is the regulator, and \(\tau > 0\) is time delay in regulator. \(g_1(x)\) is continuous function. Let \(C^1_{\tau}[0, 1] = \{f \in C^1[0, 1]: f(0) = f(1) = 0\}\), \(b \in C^1_{\tau}[0, 1]\) is assumed to be a weight function. The output is given by \(y_{\text{out}}(t) = [\mathcal{C} \phi(x)](t) = \int_0^1 b(x) \phi(x, t) dx\) denoting the weighted average of the state over the entire spatial domain. The disturbances \(d_i(t) \in \mathbb{R}, i = 1, 2\)
and the reference signal $y_{\text{ref}}(t) \in \mathbb{R}$, to be tracked by $y_{\text{out}}(t)$, are modeled by the following exosystem:

$$\frac{dw(t)}{dt} = Sw(t), \quad t > 0, \quad w(0) = w_0 \in \mathbb{R}^n, \quad d_i(t) = P_d w(t), \quad t > 0, \quad i = 1, 2,$$

where $w(t) \in \mathbb{R}^n$ is the state of the exosystem, $S \in \mathbb{R}^{n \times n}$ is a diagonalizable matrix having all its eigenvalues on the imaginary axis, and $P_d, P_e \in \mathbb{R}^{1 \times n}$, $i = 1, 2$.

Let us introduce a new variable $u(x, t) = U(t - \tau x)$. Then $u(x, t)$ satisfies the transport equation $\tau u(x, t) = -u_x(x, t), \quad x \in (0, 1), \quad t > 0$. Therefore, (1) can be written as the following system of the reaction-diffusion PDE cascaded with a transport equation (see Figure 1):

$$\begin{cases}
\phi_1(x, t) = \phi_{xx}(x, t) + c_0 \phi(x, t) \\
\quad + g_1(x) d_1(t), \quad x \in (0, 1), \quad t > 0,
\end{cases} \quad (3a)$$

and

$$\begin{cases}
\tau u(x, t) = -u_x(x, t), \quad x \in (0, 1), \quad t > 0, \\
u(0, t) = U(t), \quad t > 0.
\end{cases} \quad (3b)$$

In the paper, the output regulation problem is solved by designing a regulator $U(t)$ for (3) (or for (1)) such that the tracking error

$$\lim_{t \to \infty} e(t) = \lim_{t \to \infty} \left[ y_{\text{out}}(t) - y_{\text{ref}}(t) \right] = 0,$$

for all initial values of ($3$).

The main contribution of this paper is to obtain a prescribed decay rate of $e(t)$. For this purpose, (3a) is considered in the state space $H^1(0, 1)$, which is the dual space of $H_0^1(0, 1)$ with respect to the pivot space $L^2(0, 1)$, and the boundary interconnection requires that (3b) should be considered in the state space $H^1(0, 1)$. Our regulator design is also based on the backstepping approach and error transformations. The error transformations are found by solving the cascaded regulator equations.

The rest of this paper is organized as follows: Section 2 provides the systematic design procedure of a backstepping regulator by mapping (3) into an exponentially stable error system with a prescribed rate in a suitable Hilbert space. Then (4) is obtained by solving the cascaded regulator equations. We present the main results wherein the solvable condition of the cascaded regulator equations and the exponential decay of $e(t)$ are obtained. The proofs of the main results are presented in Section 3. In Section 4, numerical simulations are presented in order to illustrate the effect of the regulator. Some remarks are given in Section 5.

2. REGULATOR DESIGN AND MAIN RESULTS

In this section, a backstepping boundary regulator is designed to transform (3) to a target system, in which the unstable term is moved to the controlled end of the transport equation and a dissipative term is added to the parabolic body equation. The error transformations are then proposed to converted the target system into a stable error system, where the exogenous terms involving $w(t)$ are removed. For the given regulator, we present the output regulation results in suitable state spaces. Therefore this section is divided into two parts.

2.1 Backstepping regulator design

We introduce the following transformation

$$\psi(x, t) = \Pi_1[\phi(x)](t) = \phi(x, t) - \int_0^t k(x, y) \phi(y, t) dy, \quad x \in [0, 1],$$

and

$$v(x, t) = \Pi_2[\mu(x), \phi(x)](t) = u(x, t) - \int_0^1 p(x, y) u(y, t) dy - \int_0^1 \gamma(x, y) \phi(y, t) dy, \quad x \in [0, 1],$$

where the kernels $k(x, y), p(x, y)$, and $\gamma(x, y)$ satisfy the following equations respectively:

$$\begin{cases}
k_{xx}(x, y) - k_{xy}(x, y) - (c_0 + c_1) k(x, y) = 0, \\
k' (x, x) = -\frac{1}{2} (c_0 + c_1), \\
k(x, 0) = 0,
\end{cases} \quad (6a)$$

and

$$\begin{cases}
\gamma_{xx}(x, y) + \gamma_{xy}(x, y) + c_0 \gamma(x, y) = 0, \\
\gamma(x, 1) = 0, \quad \gamma(0, y) = 0, \\
\gamma(1, y) = k(1, y).
\end{cases} \quad (6c)$$

It is straightforward to show that (5) maps (3) into the target system

$$\begin{cases}
\dot{\psi}(x, t) = \psi_{xx}(x, t) - c_1 \psi(x, t) + g_1^2(x) w(t), \quad x \in (0, 1), \quad t > 0, \\
\psi(0, t) = q P_d w(t), \\
\psi(1, t) = v(1, t),
\end{cases} \quad (7a)$$

and

$$\begin{cases}
\tau v(x, t) = -v_x(x, t) + g_1^2(x) w(t), \quad x \in (0, 1), \quad t > 0, \\
v(0, t) = \mu_0 w(t),
\end{cases} \quad (7b)$$
where $\bar{g}_1^\pi(x), \bar{g}_2^\pi(x) \in \mathbb{R}^{1 \times n_c}$ are given by
\[
\begin{align*}
\bar{g}_1^\pi(x) &= \Pi_t[g_1(t)(x) \cdot P_{d1} - k_v(x, 0)q \cdot P_{d1}], \\
\bar{g}_2^\pi(x) &= \tau \gamma_y(x, 0)q \cdot P_{d1} - \tau \int_{0}^{1} \gamma(x, y)g_1(y)dy \cdot P_{d1},
\end{align*}
\]
ci > 0 is a constant and can be prescribed, and where $\mu_w$ is a constant to be determined in the next subsection.

The regulator is obtained by setting $x = 0$ in the $v$-part of (5) and taking into consideration the boundary condition of (7b):
\[
U(t) = \int_{0}^{1} p(0, y)\phi(y, t)dy + \int_{0}^{1} \gamma(0, y)w(y, t)dy + \mu_w w(t),
\]
where $p(0, y)$ can be solved from (6b). Since (6b) is a simple transport equation, then the solution is given exactly by $p(x, y) = -\gamma_y(x, 1)$. It follows that $p(0, y) = -\gamma_y(0, 1).$ Now we are in a position to find $\gamma_y(0, 1).$ To this end, let $\gamma(x, y) = \gamma(1 - t, t) \equiv \tilde{\gamma}(t, y)$, where $t = 1 - x, t \in [0, 1]$. Then (6c) can be written as
\[
\begin{align*}
\tilde{\gamma}(t, y) - \tilde{\gamma}(t, y) - \tau_0 \tilde{\gamma}(t, y) &= 0, t \in (0, 1), y \in (0, 1), \\
\tilde{\gamma}(t, 0) &= 0, \\
\tilde{\gamma}(t, 1) &= 0, \\
\tilde{\gamma}(0, y) &= k(y, 1).
\end{align*}
\]
A direct computation gives the solution of (9):
\[
\tilde{\gamma}(t, y) = 2 \sum_{k=1}^{\infty} e^{(\tau - k \pi^2 \beta)} \sin(k \pi y) \cdot \int_{0}^{1} \sin(k \pi x)k(1, x)dx.
\]
Consequently, we obtain
\[
\gamma_y(0, 1) = 2 \sum_{k=1}^{\infty} (-1)^k e^{(\tau - k \pi^2 \beta)} \pi \cdot \int_{0}^{1} \sin(k \pi x)k(1, x)dx,
\]
where $k(1, y)$ is determined by (6a), for which the solution has the form (Smushlyaev and Krstic, 2004)
\[
k(x, y) = -(c_0 + c_1)x \frac{I_1(\sqrt{(c_0 + c_1)x^2 - y^2})}{\sqrt{c_0 + c_1}(x^2 - y^2)}
\]
with the modified Bessel function $I_1(\cdot) = \sum_{m=0}^{\infty} \frac{(x^2/4)^m}{m!}$ (see Krstic and Smushlyaev, 2008, P.35).

### 2.2 Error system and main results

In order to remove the exogenous terms involving $w(t)$ from (7), we consider the error transformations
\[
\begin{align*}
\theta(x, t) &= \psi(x, t) - m(x)w(t), \\
\epsilon(x, t) &= v(x, t) - n(x)w(t),
\end{align*}
\]
where $\theta(x, t), \epsilon(x, t) \in \mathbb{R}$ are the error states, and the gains $m(x), n(x) \in \mathbb{R}^{1 \times n_c}$ need to satisfy the cascaded regulator equations
\[
\begin{align*}
\frac{d}{dq}[m(x) - c_1m(x) - d(x)S + \bar{g}_1^\pi(x) = 0, \\
m(0) = 0 \cdot P_{d1},
\end{align*}
\]
and
\[
\begin{align*}
\frac{d}{dq}[m(x) + \tau n(x)S - \bar{g}_2^\pi(x) = 0, \\
n(0) = m(1).
\end{align*}
\]
Let $\mu_w = n(0)$, then (10) converts (7) into the following error system:
\[
\begin{align*}
\dot{\theta}_c(x, t) &= \dot{\theta}_c(x, t) - c_1 \dot{\theta}(x, t), x \in (0, 1), t > 0, \\
\dot{\theta}(0, t) &= 0, \\
\dot{\theta}(1, t) &= \epsilon(1, t),
\end{align*}
\]
and
\[
\begin{align*}
\tau \epsilon_c(x, t) &= -\epsilon_c(x, t), x \in (0, 1), t > 0, \\
\epsilon(0, t) &= 0.
\end{align*}
\]
Set $\mathcal{E}(\Pi^{-1}m)(x) = P_r$, then $e(t)$ is written in terms of $\theta(x, t)$:
\[
\begin{align*}
e(t) &= [\mathcal{E}(\theta(x, t)) - P_r]W(t) \\
&= [\mathcal{E}(\Pi^{-1}\theta)(x)) - \mathcal{E}(\Pi^{-1}m)(x)]W(t)
\end{align*}
\]
Theorem 3. Let $\mathcal{A}$ be given by (15). Then for any $\varepsilon(\cdot,0) \in H^{-1}(0,1)$, and $\vartheta(\cdot,0) \in H^1_0(0,1)$, there exist constants $M$ and $M > 0$ dependent on $\varepsilon$ with $0 < \varepsilon < c_1$ such that the following assertions hold:

1. There exists a unique solution to (12) such that $\vartheta(\cdot,t) \in C\left([0,\infty); H^{-1}(0,1)\right)$ and $\|\vartheta(\cdot,t)\|_{-1} + \|\varepsilon(\cdot,t)\|_{1} \leq M e^{-\varepsilon t} \left\|\vartheta(\cdot,0)\right\|_{-1} + \|\varepsilon(\cdot,0)\|_{1}$.

2. The error $\varepsilon(t)$, given by (4), satisfies $\|\varepsilon(t)\| \leq M e^{-\varepsilon t} \left\|\vartheta(\cdot,0)\right\|_{-1} + \|\varepsilon(\cdot,0)\|_{1}$.

The following corollary is a direct consequence of the above theorem:

Corollary 4. The output regulation problem of (3) is solvable, i.e. (4) holds for the regulator (8) if (11) is solvable.

The next lemma gives necessary and sufficient condition for the solvability of (11):

Lemma 5. (Cascaded Regulator Equations). Let $F(s) = \frac{N(s)}{D(s)}$ be the transfer function of (3) and the numerator $N(s) = \mathcal{C} \left(\prod_1^k \beta(\xi,\lambda, x)\right)$, where $\beta(\xi,\lambda, x) = \cosh\sqrt{s + \sigma(x)}$. There exists a unique classical solution of (11) if and only if $N(\lambda) \neq 0$, $\forall \lambda \in \sigma(S)$.

Remark 6. The condition in Lemma 5 means that no eigenvalue of the exosystem (2) is a transmission zero of (3) (Aulisa and Gilliam, 2016, P.15).

3. PROOFS OF THE MAIN RESULTS

This section is devoted to proving Theorem 3 and Lemma 5 that stated in the previous section. Therefore the section is divided into two parts.

3.1 Proof of Theorem 3

We first confirm Assertion (1). Solve $x(t)$ from (12b) to obtain

$$
eq \left\{ \begin{array}{ll} 0, & t > \tau x, \\
\varepsilon_0(x - \frac{1}{\tau})t, & t \leq \tau x,
\end{array} \right. $$

where $\varepsilon_0(x)$ is the initial state of (12b). Let us consider the Lyapunov functional $V(t) = \int_0^1 e^{-c_2 x} \varepsilon_2(x,t) dx$, where the positive constant $c_2 > c_1$. A direct computation shows $V(t) \leq -c_2 V(t)$. By Hölder’s inequality, we have $\vartheta(t) \leq \|\varepsilon(\cdot,t)\|_{1} \leq M_1 e^{-c_2 t} \left\|\vartheta(\cdot,0)\right\|_{-1}$, where $M_1 > 0$ is a constant depending on $c_2$.

A direct computation shows that $\lambda_k = -c_1 - k^2 \pi^2$ is an eigenvalue of $\mathcal{A}$ with the associated eigenfunction $\vartheta_k(x) = k \pi \sin(k \pi x)$, $k \in \mathbb{N}$. Obviously, $(\vartheta_k(x))_{k=1}^{\infty}$ forms an orthonormal basis in $H^{-1}(0,1)$. Therefore $\mathcal{A}$ generates exponentially stable $C_0$-semigroup $e^{\mathcal{A}t}$ on $H^{-1}(0,1)$ with the decay rate $\varepsilon = c_1$. Consider the adjoint system of (16):

$$
\eta(\cdot, t) = \mathcal{A}^* \eta(\cdot, t), \\
B^* \eta = \int_0^t x \eta(x) dx,
$$

where $\mathcal{A}^* = \mathcal{A}$. It is easy to check that $B^* \mathcal{A}^*$ is bounded from $H^{-1}(0,1)$ into $\mathbb{R}$. Then we will show that the control operator $B$ is admissible for the $C_0$-semigroup $e^{\mathcal{A}t}$ on $H^{-1}(0,1)$. For each $\eta(0) = \eta_0(x) \in H^{-1}(0,1)$, we have $\eta(0) = \sum_{k=1}^{\infty} a_k \sin(k \pi x)$ with $x \in [0,1]$, $\|\eta_0(x)\|_{-1} = \sum_{k=1}^{\infty} a_k^2 \frac{1}{k^2 \pi^2}$, and $\eta(x, t)$ can be written as

$$
\eta(x, t) = \sum_{k=1}^{\infty} a_k e^{-c_1 + k^2 \pi^2 t} \sin(k \pi x), \quad x \in [0,1], \quad t > 0.
$$

Hence,

$$
B^* \eta(t) = -\int_0^t x \eta(x,t) dx = \sum_{k=1}^{\infty} (\frac{-1}{k \pi}) \frac{d}{dt} e^{-c_1 + k^2 \pi^2 t} \sin(k \pi x),
$$

and for any $T > 0$

$$
\int_0^T |(B^* \eta)(t)|^2 dt \leq \sum_{k=1}^{\infty} \frac{a_k^2}{k^2 \pi^2} \int_0^T e^{-2(c_1 + k^2 \pi^2) t} dt
$$

$$
\leq M_2 \|\eta(0)\|_{-1}^2,
$$

where constant $M_2 > 0$ depends on $T$. Therefore $\mathcal{B}^*$ is admissible for $e^{\mathcal{A}t}$ generated by $\mathcal{A}$ on $H^{-1}(0,1)$. Therefore, for any $\vartheta(t,0) \in H^{-1}(0,1)$, there exists a unique solution $\vartheta(x,t) \in C\left([0,\infty); H^{-1}(0,1)\right)$ to (16) such that

$$
\vartheta(\cdot,t) = e^{\mathcal{A}t} \vartheta(\cdot,0) + \int_0^t e^{\mathcal{A}(t-s)} B \vartheta(s) ds.
$$

Let $0 < \varepsilon < c_1 < c_2$. We have $e^{\mathcal{A}t} \vartheta(\cdot,t) \in L^2(0,\infty)$. In view of (7.4.1.3) of (Lasiecka and Triggiani, 2000, Theorem 7.4.1.1, on P.654) (for the $\varepsilon = 0$ there), we obtain

$$
\|\vartheta(\cdot,t)\|_{-1} \leq \|\vartheta(\cdot,0)\|_{-1} + \|\varepsilon(\cdot,t)\|_{1}
$$

$$
\leq M e^{-\varepsilon t} \left\|\vartheta(\cdot,0)\right\|_{-1} + \|\varepsilon(\cdot,0)\|_{1}.
$$

Now, we are in a position to prove Assertion (2). Let $l(x, y) \in C^2(\Xi)$ be the kernel of $\Pi_1$. From (13), we get

$$
eq \left\{ \begin{array}{ll} \mathcal{C} \left(\prod_1^k \beta(\xi,\lambda, x)\right)(x) \right\}
$$

$$
= \left[ \int_0^x b(x) \left(\vartheta(x,t) - \int_t^x l(x,y) \vartheta(y,t) dy\right) dx\right]
$$

$$
\leq \left[ \int_0^x b(x) \vartheta(x,t) dx + \|l\|_{C^2(\Xi)} \int_0^x \left( \int_0^x b(y) dy\right) \vartheta(x,t) dx\right]
$$

$$
= \left[ \int_0^x \Delta^{1/2} b(x) \cdot \Delta^{-1/2} \vartheta(x,t) dx\right]
$$

$$
+ \|l\|_{C^2(\Xi)} \left[ \int_0^x \Delta^{1/2} \left( \int_0^x b(y) dy\right) \Delta^{-1/2} \vartheta(x,t) dx\right]
$$

$$
\leq \bar{M} e^{-\varepsilon t} \left\|\vartheta(\cdot,0)\right\|_{-1} + \|\varepsilon(\cdot,0)\|_{1},
$$

where $\bar{M} = M \|b\|_{C^0[0,1]} \cdot \max\{1, \|l\|_{C^2(\Xi)}\}$. The proof is complete. □

Remark 7. If we consider (12a) in the usual state space $L^2(0,1)$, then we find that the corresponding control operator $B_0$ is not admissible for the semigroup $e^{\mathcal{A}t}$. In fact, $\frac{d}{dt} (\vartheta, \eta)_{L^2(0,1)} = (\vartheta'' - c_1 \vartheta, \eta)_{L^2(0,1)}$

$$
= (B_0 \vartheta(t), \eta)_{L^2(0,1)} - \vartheta(0,1) \eta(0) + \vartheta(1,0) \eta(1),
$$

where $B_0 = -\delta'(x-1)$, the operator $\mathcal{A}_0' : D(\mathcal{A}_0') \subset L^2(0,1) \to L^2(0,1)$ as follows:

$$
\mathcal{A}_0 \eta = \eta'' - c_1 \eta, \quad \forall \eta \in D(\mathcal{A}_0') = \{ \eta \in H^2(0,1) | \eta(0) = \eta(1) = 0 \}.
$$

Therefore, for any $T > 0$

$$
\mathcal{A}_0 \eta = \eta'' - c_1 \eta, \quad \forall \eta \in D(\mathcal{A}_0') = \{ \eta \in H^2(0,1) | \eta(0) = \eta(1) = 0 \}.
$$
\[
\int_0^T \left( B_x \gamma \right)(t)^2 dt = \int_0^T \eta'(1,t) dt
\]
\[
\leq \sum_{k=1}^{\infty} a_k^2 \sum_{k=1}^{\infty} \int_0^T (k\pi)^2 e^{-2(k\pi^2\tau^2)} dt
\]
\[
\leq M \| \eta(x,0) \|^2.
\]
Consequently, the state space of (12a) is chosen as \( H^{-1}(0,1) \).

### 3.2 Proof of Lemma 5

Firstly, a direct computation shows the numerator of the transfer function \( F(s) : N(s) = \mathbb{C} \left[ \Pi^{-1} \beta(0, x, \lambda) \right] \).

Next, we need to decouple (11). Since \( S \) is diagonalizable, there exists a similarity transformation \( V^t S V = \text{diag} (\lambda_{v,1}, \ldots, \lambda_{v,n_v}) \), where \( V = [V_{v,1}, \ldots, V_{v,n_v}] \) with \( V_{v,j} \in \mathbb{R}^{n_w}, j = 1, 2, \ldots, n_w \).

Postmultiplying (11) by \( V \), we then obtain the following cascaded ODEs:

\[
\begin{align*}
\begin{cases}
\frac{d}{dt} \bar{m}_i^*(x) - m_i(x) (\lambda_{v,i} + c_1) = -\bar{g}_{i,j}^*(x), \\
n_j^*(0) = q_{d_{i,j}}^* \end{cases} \quad (25a)
\end{align*}
\]

and

\[
\begin{align*}
\begin{cases}
d_i n_j^*(x) + n_j^*(x) \tau \lambda_{v,j} = \bar{g}_{j,j}^*(x), \\
n_j^*(1) = m_j^*(1), \\
j = 1, 2, \ldots, n_w 
\end{cases} \quad (25b)
\end{align*}
\]

The solution to (25a) is given by

\[
m_j^*(x) = q_{d_{i,j}}^* e^{-\psi^j x} + 2 K_i \beta(0, \lambda_{v,j}, x)
\]
\[
- \frac{1}{q_j} \int_0^x \beta(\xi, \lambda_{v,j}, x) \bar{g}_{j,j}^*(\xi) d\xi,
\]

where \( \beta(\xi, \lambda_{v,j}, x) = \sinh \left[ q_j (x - \xi) \right] \) and \( q_j = \sqrt{\lambda_{v,j} + c_1} \neq 0 \).

The constant \( K_i \) needs to be uniquely determined later. Set \( P_{r,j} = P_r \cdot V_{r,j} \). Postmultiplying \( \mathbb{C} \left[ \Pi^{-1} m_j(x) \right] = P_{r,j} \) by \( V \) to yield

\[
\mathbb{C} \left[ \Pi^{-1} m_j^*(x) \right] = q_{d_{i,j}}^* \mathbb{C} \left[ \Pi^{-1} e^{-\psi^j x} \right] + 2 K_i \mathbb{C} \left[ \Pi^{-1} \beta(0, \lambda_{v,j}, x) \right]
\]
\[
- \frac{1}{q_j} \int_0^x \beta(\xi, \lambda_{v,j}, x) \bar{g}_{j,j}^*(\xi) d\xi = P_{r,j}.
\]

(27)

\( K_i \) can be uniquely solved from (27) If and only if \( N(\lambda_{v,j}) \neq 0 \) with \( \lambda_{v,j} \in \sigma(S) \):

\[
K_i = \frac{1}{2N(\lambda_{v,j})} \left[ P_{r,j} + \mathbb{C} \left[ \Pi^{-1} \int_0^x \beta(\xi, \lambda_{v,j}, x) \bar{g}_{j,j}^*(\xi) d\xi \right] \right]^{-1}
\]
\[
- q_{d_{i,j}}^* \mathbb{C} \left[ \Pi^{-1} e^{-\psi^j x} \right] \quad (28)
\]

Therefore

\[
m_j^*(1) = q_{d_{i,j}}^* e^{-\psi^j} + 2 K_i \beta(0, \lambda_{v,j}, 1)
\]
\[
- \frac{1}{q_j} \int_0^1 \beta(\xi, \lambda_{v,j}, 1) \bar{g}_{j,j}^*(\xi) d\xi = K_2.
\]

(29)

Fig. 2. The state \( \phi(x, t) \) of (1) without regulator.

Fig. 3. The reference "\( y_{ref}(t) \)" and output "\( y_{out}(t) \)" without regulator.

Fig. 4. The reference "\( y_{ref}(t) \)" and output "\( y_{out}(t) \)" under regulator.

Lastly, combining the Dirichlet boundary condition of (25a), the unique solution to (25) under condition (27) is given by

\[
m_j^*(x) = q_{d_{i,j}}^* e^{-\psi^j x} + 2 K_i \beta(0, \lambda_{v,j}, x)
\]
\[
- \frac{1}{q_j} \int_0^x \beta(\xi, \lambda_{v,j}, x) \bar{g}_{j,j}^*(\xi) d\xi,
\]

(30a)

and

\[
n_j^*(x) = K_2 e^{\lambda_{v,j} x} - \int_0^{\lambda_{v,j} x} e^{\lambda_{v,j} (x-x')} \bar{g}_{j,j}^*(x') d\xi, \quad j = 1, 2, \ldots, n_w
\]

(30b)

Hence, the classical solution of (11) is

\[
m(x) = (m_1^*(x), m_2^*(x), \ldots, m_n^*(x)) \cdot V^{-1},
\]
\[
n(x) = (n_1^*(x), n_2^*(x), \ldots, n_n^*(x)) \cdot V^{-1},
\]

(31)
where \( m_j'(x), n_j'(x), \ j = 1, 2, \cdots, n_w, \) are given by (30). The proof is complete.  □

4. SIMULATION RESULTS

In this section, we present numerical simulation results for the output regulation of (1). For simplicity, we set \( c_0 = q = 2, \) and \( g_1(x) = 1. \) The output is taken as \( y_{\text{out}}(t) = \int_0^t \sin(2\pi x)\phi(t, t)dx. \) The regulator (8) drives \( y_{\text{ref}}(t) \) to track sinusoidal reference \( y_{\text{ref}}(t), \) while rejecting the disturbances \( d_i(t), \ i = 1, 2, \) \( y_{\text{ref}}(t), \) and \( d_i(t), \ i = 1, 2 \) are generated by (2) with

\[
\begin{align*}
S &= \text{bdiag}(0, \text{adiag}(1, -\omega_1^2)) \in \mathbb{R}^{3 \times 3}, \\
R(t) &= \begin{pmatrix} 1, r \sin(\omega t + \varphi), r \omega \cos(\omega t + \varphi) \end{pmatrix}, \\
P_r &= \begin{pmatrix} C_1 & 0 & 0 \\ 0 & C_2 & 0 \\ 0 & 0 & C_3 \end{pmatrix}, \ C_i \in \mathbb{R}, \ i = 1, 2,
\end{align*}
\]

where \( \text{bdiag}(X_1, X_2, \cdots, X_p) \) represents block diagonal matrix with blocks \( X_1, X_2, \cdots, X_p \) on the main diagonal, and \( \text{adiag}(a_1, a_2, \cdots, a_n) \) denotes anti-diagonal matrix with elements \( a_1, a_2, \cdots, a_n \) on the anti-diagonal.

For numerical computations, the steps of space and time are set as 0.01 and 0.000005, respectively. The initial value of (1) is chosen as \( \phi(x, 0) = x^3 - 1. \) Figure 2 shows the state \( \phi(x, t) \) of (1). When the regulator (8) is absent, \( y_{\text{out}}(t) \) does not track \( y_{\text{ref}}(t) \) as shown in Figure 3. However, it can be seen in Figure 4 that \( y_{\text{out}}(t) \) is forced by (8) to track \( y_{\text{ref}}(t). \)

5. CONCLUSION

In this paper, a regulator design has been presented for the out regulation of an unstable reaction-diffusion PDE with time delay, which was written as the reaction-diffusion PDE cascaded with a transport equation. By backstepping and error transformations, the cascaded system has been converted to an cascaded with a transport equation. By backstepping and error transformations, the cascaded system has been converted to an

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