ON EQUIVARIANT ASYMPTOTIC DIMENSION

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Abstract. The work discusses equivariant asymptotic dimension (also known as “wide equivariant covers”, “transfer reducibility”, and “$N$-$F$-amenability”), a version of asymptotic dimension invented for the proofs of the Farrell–Jones and Borel conjectures.

We prove that groups of null equivariant asymptotic dimension are exactly virtually cyclic groups. Moreover, we show that a covering of the boundary always extends to a covering of the whole compactification. We provide a number of characterisations of equivariant asymptotic dimension in the general setting of homotopy actions, including equivariant counterparts of classic characterisations of $\text{asdim}$.

Finally, we strengthen the result of [12] about equivariant refinements from finite groups to infinite groups.

Introduction

The concept of equivariant asymptotic dimension was introduced by Bartels, Lück, and Reich in [5]. Proving finiteness of equivariant asymptotic dimension was a major technical step in the proof of the Farrell–Jones conjecture for hyperbolic groups in [6]. Later, it was used to derive the Borel conjecture in [4].

Equivariant asymptotic dimension and its generalisations, “transfer reducibility” (see Section 4) in different forms have been extensively studied in the last years, mostly as a tool to prove the Farrell–Jones conjecture for $\text{GL}_n(\mathbb{Z})$ [7] and other linear groups [12, 16], virtually solvable groups [18], CAT(0) groups [3, 17], and relatively-hyperbolic groups [2]. Very recently, in [11], a new construction of covers was proposed, which – in particular – provides improved bounds on equivariant asymptotic dimension of hyperbolic groups.

The notion of equivariant asymptotic dimension relates to many other concepts, including asymptotic dimension and amenable actions on compact spaces. We discuss these relations in a separate subsection of the Introduction.

All the known proofs [2, 5, 11] showing finiteness of equivariant asymptotic dimension are complex and involve a notion of a (coarse) flow space. Some elementary constructions, even in the simplest cases such as that of the free group, are unknown and desired, cf. [1, Remark 3.12]. We make a step in this direction in Theorem 2.1, showing that it is enough to construct coverings involving only the boundary of the suitable space, not the whole space. In case of the free group it means a transition from a compactified infinite tree to a Cantor set.

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In Theorem 3.2, we show that the equivariant asymptotic dimension of a group $G$ with respect to a family of groups $\mathcal{F}$ vanishes if and only if $G$ itself belongs to $\mathcal{F}$. This in particular yields a geometric characterisation of virtually cyclic groups.

In Section 4, a number of different characterisations of equivariant asymptotic dimension and transfer reducibility are provided. Interestingly, appropriate forms of characterisations invented originally for asymptotic dimension are still valid in the very general framework of homotopy actions (transfer reducible groups). In Appendix B, we present one more characterisation and a different proof (not using metrisability) of their equivalences, assuming that we deal with ordinary (not homotopy) group actions.

In Appendix A, we strengthen the result of [12] stating that for an equivariant covering one can find an equivariant refinement of dimension at most equal to the dimension of the space. In that sense, equivariant topological dimension is equal to the topological dimension. The theorem was originally formulated for finite groups and we generalise it to infinite groups if only the action is proper. It is independent of the rest of the paper.

0.1. Definition. Let us start with fixing some notation. The metric neighbourhood of $A$ of radius $r$ will be denoted by $B(A, r) = \bigcup_{x \in A} B(x, r)$. When a group $G$ acts on a topological space $X$ (on a set $Y$), we will shortly say that $X$ ($Y$) is a $G$-space (a $G$-set). $G \backslash X$ will denote the quotient. Sometimes we will write “for all $\alpha < \infty$...” to denote “for all $\alpha \in (0, \infty)$...” in order to clarify that the following condition is trivial for “small” $\alpha$ and interesting for “big” ones.

Unless stated otherwise, we will assume that $G$ is a finitely generated group with a fixed word-length metric $d_G$, and $\overline{X}$ will denote a compact $G$-space.

Definition 0.1. A family $\mathcal{F}$ of subgroups of a group $G$ is a set of subgroups of $H$ closed under conjugation and taking subgroups.

A family $\mathcal{F}$ is virtually closed if for every $H \in \mathcal{F}$ and $H \leq H' \leq G$ such that $|H \backslash H'| < \infty$, also $H' \in \mathcal{F}$.

Our considerations are general enough to hold for any family $\mathcal{F}$ or at least for a virtually closed family. However, in the context of the Farrell–Jones conjecture it is the (virtually closed) family of virtually cyclic subgroups, denoted $\mathcal{VCyc}$, that appears most naturally [3–6].

Definition 0.2. Let $Y$ be a $G$-space and $\mathcal{F}$ be a family of subgroups of $G$. A subset $U \subseteq Y$ is called an $\mathcal{F}$-subset if:

(a) elements $gU$ of the orbit of $U$ are either equal or disjoint,

(b) the stabiliser of $U$, thus the subgroup $G_U = \{g \in G \mid gU = U\}$ is a member of $\mathcal{F}$.

A cover that consists of $\mathcal{F}$-subsets and is $G$-equivariant will be called an $\mathcal{F}$-cover. The name “equivariant asymptotic dimension” comes from the fact that the coverings in its definition are $\mathcal{F}$-covers.

For a family of subsets $\mathcal{U}$ of the set $Y$, by $\dim \mathcal{U}$ (dimension of $\mathcal{U}$) we will denote the number (or infinity) $\sup_{x \in X} |\{U \in \mathcal{U} \mid x \in U\}| - 1$, where $|A|$ is the cardinality of $A$.

Definition 0.3. Let $Y$ be any set and $\mathcal{U}$ be a covering of $G \times Y$. We say that $\alpha < \infty$ is a $G$-Lebesgue number of $\mathcal{U}$, given that for each $(g, y) \in G \times Y$ there exists $U \in \mathcal{U}$ such that $B(g, \alpha) \times \{y\} \subseteq U$. 


The following definition originates in [5, Theorem 1.1], see also [6, Assumption 1.4] and [2, Definition 0.1].

**Definition 0.4.** Equivariant asymptotic dimension of $G \times \mathbb{X}$ with respect to family $\mathcal{F}$, denoted by $\mathcal{F}$-eq-asdim $G \times \mathbb{X}$, is the smallest integer $n$ such that for every $\alpha < \infty$ there exists an open $\mathcal{F}$-cover $\mathcal{U}$ of $G \times \mathbb{X}$ (with the diagonal $G$ action) satisfying:

1. $\dim(\mathcal{U}) \leq n$,
2. $\alpha$ is a $G$-Lebesgue number of $\mathcal{U}$.

When the family $\mathcal{F}$ is irrelevant or clear from the context, we will skip it from notation. The coverings $\mathcal{U} = \mathcal{U}(\alpha)$ from the above definition will be called eq-asdim-coverings and $\alpha$-eq-asdim-coverings, in case the constant $\alpha$ is important.

**Remark 0.5.** In [5, Theorem 1.1], eq-asdim-coverings were also required to be $G$-cofinite, but by compactness one can choose cofinite subcoverings from arbitrary coverings, so this requirement can be skipped.

**Remark 0.6.** Note that if we have a $G$-equivariant map $p : \mathbb{Y} \rightarrow \mathbb{X}$, then eq-asdim-coverings of $G \times \mathbb{X}$ can be pulled back to eq-asdim-coverings of $G \times \mathbb{Y}$. Hence, the minimal possible value of eq-asdim $G \times \mathbb{X}$ for $\mathbb{X}$ compact and Hausdorff is acquired for $\mathbb{X} = \beta G$ – we will sometimes call it the equivariant asymptotic dimension of $G$.

It is not enough to restrict to $\mathbb{X} = \beta G$, though, since in applications conditions similar to the following are utilised, cf. [6, Theorem 1.1, Assumption 1.2]. However, we do not adopt them as a convention.

- $\mathbb{X}$ is a metrisable compactification of its $G$-invariant subset $X$,
- $X$ is the realisation of an abstract simplicial complex,
- $\mathbb{X}$ is contractible,
- (weak Z-set condition) there exists a homotopy $H : \mathbb{X} \times [0,1] \rightarrow \mathbb{X}$, such that $H_0 = \text{id}_{\mathbb{X}}$ and $H_t(\mathbb{X}) \subseteq X$ for every $t > 0$.

In this context, considerations may become less complicated if one constructs coverings of $G \times \partial X$ rather than the whole $G \times \mathbb{X}$ (where $\partial X = \mathbb{X} \setminus X$). The fact that the latter can be reconstructed from the former is the content of Theorem 2.1.

A natural setting to have in mind is when the space $X$ admits a geometric action of $G$ (for example, it is a Rips complex of the group) and $\partial X$ is the Gromov boundary of $G$, cf. [5].

### 0.2. Relations to other concepts.

**Asymptotic dimension.** The natural question which comes to mind is how equivariant asymptotic dimension is related to asymptotic dimension. Recall the definition.

**Definition 0.7.** Asymptotic dimension of a metric space $G$ is the smallest integer such that for all $\alpha < \infty$ there is an open covering $\mathcal{U}$ of $G$ such that:

1. $\dim(\mathcal{U}) \leq n$,
2. for each $g \in G$ there exists $U \in \mathcal{U}$ such that $B(g, \alpha) \subseteq U$,
3. $\sup_{U \in \mathcal{U}} \diam(U) < \infty$ (uniform boundedness).

We can see that the first two conditions in the definition of asdim are analogs of the conditions for eq-asdim. Such a similarity occurs also for different characterisations of asdim, compare Theorem 4.1 with Theorem 4.5 and Proposition B.1.
Guentner, Willet, and Yu [9] show that \( \text{Fin-eq-asdim} G \times \beta G = \text{asdim} G \), where \( \text{Fin} \) is the family of finite subgroups. Clearly, the equivariant asymptotic dimension decreases when \( F \) increases, so \( \text{F-eq-asdim} G \times \beta G \leq \text{asdim} G \) for any \( F \supseteq \text{Fin} \).

On the other hand, according to [1, Remark 3.5] Willett and Yu claim that finiteness of \( \text{F-eq-asdim} \) implies finiteness of \( \text{asdim} \) if \( \sup_{F \in \mathcal{F}} \text{asdim} F < \infty \), but no reference is given.

Apparently, eq-asdim is a more subtle (or at least less understood) notion than asdim – to the best of our knowledge, the only class of groups that are known to be of finite equivariant asymptotic dimension with a reasonable \( \mathcal{X} \) is the family of hyperbolic groups [5], and the fact that they also have finite asymptotic dimension is classic and can be proven on a single sheet of paper [14]. On the other hand, we have no examples of groups with infinite equivariant asymptotic dimension.

The difficulty with eq-asdim arise (see Remarks under Section 3) already in the case of the simplest non-hyperbolic group – \( \mathbb{Z}^2 \), which can be immediately proven to be of asymptotic dimension 2.

**Transfer reducibility.** A notion similar to equivariant asymptotic dimension (and also sufficient for the Farrell–Jones conjecture) is transfer reducibility. It occurs in many flavours in the literature, but the main difference between it and equivariant asymptotic dimension is that in transfer reducibility one can choose a space \( \mathcal{X} \) depending on the parameter (for eq-asdim this parameter is \( \alpha \)) and instead of a genuine group action a “homotopy action” is considered. Very roughly, in homotopy action the action of \( gh \) is equal to the composition of actions of \( g \) and \( h \) only up to homotopy. For more details, see Definition 4.2 and Remark 4.6.

There are more positive results regarding transfer reducibility [3, 7, 16–18] than equivariant asymptotic dimension [2, 5], because its definition is formally less restrictive. However, it seems to be an open question, whether the two notions are equivalent, cf. [1, Remark 3.15].

**Amenable actions.** It was pointed out by M. Bridson that a suitable reformulation of equivariant asymptotic dimension is very similar to the concept of amenable action [1, Remark 3.6]. This reformulation is given in condition (4) of Theorem 4.5 – we require the existence of almost invariant maps from \( \mathcal{X} \) to an \( n \)-dimensional simplicial complex \( E \subseteq \ell_1(Y) \).

As explained in [2], this is – on one hand – more than an amenable action, where the target space is the whole unit sphere of \( \ell_1(G) \), not just an \( n \)-dimensional complex in it. On the other hand, for eq-asdim, space \( \ell_1(Y) \) can be build on any \( G \)-set \( Y \) as long as its isotropy groups belong to \( \mathcal{F} \) and for amenable actions we have \( Y = G \). In [2], an action of \( G \) on \( \mathcal{X} \) for which there exist \( \mathcal{F} \)-eq-asdim-coverings of dimension at most \( N \) is called \( N \)-\( \mathcal{F} \)-amenable.

1. **Importance of compactness**

   It turns out that the compactness of \( \mathcal{X} \) is crucial to the notion of eq-asdim.

   **Remark 1.1.** If the compactness assumption for \( \mathcal{X} \) in Definition 0.4 is skipped, then for \( X = G \) we have eq-asdim \( G \times X = 0 \).

   **Proof.** A good eq-asdim-covering for \( G \times X \) is \( \mathcal{U} = \{ G \times \{ x \} \mid x \in X \} \), which is clearly an open \( \mathcal{T} \)-cover of dimension 0 with infinite \( G \)-Lebesgue number, where \( \mathcal{T} \) is the singleton family of the trivial subgroup of \( G \). \( \Box \)
The above proof exemplifies a more general approach indicated in [5]. While the eq-asdim-coverings must be $\alpha$-large in the $G$-coordinate, making them small in the $X$-coordinate may be helpful in obtaining the properties desired in Definition 0.2.

The following proposition generalises the above remark and covers a wide range of examples (e.g., the spaces considered in [5]).

**Proposition 1.2.** Assume that a finitely generated group $G$ acts on a topological space $X$. There is an $\mathcal{F}$-eq-asdim-covering $U_{\infty}$ (with $\alpha = \infty$) of the space $G \times X$ under any of the following conditions:

(a) $X$ is a finitely-dimensional simplicial complex, the action of $G$ is simplicial and the simplex stabilisers belong to $\mathcal{F}$ (the same is true for $G$-CW-complexes);

(b) $X$ is regular, the $G$-action is proper, isotropy groups belong to $\mathcal{F}$, and either of the following conditions holds:

(i) the $G$-action is cocompact;

(ii) $X$ is of finite covering dimension and admits a $G$-invariant metric.

**Proof.** Ad (a). For each open simplex $\Delta^0$ of $X$, one can construct (using a barycentric subdivision, see for example [12, Lemma 3.4]) a neighbourhood $N(\Delta^0)$ such that neighbourhoods of simplices of the same dimension are disjoint and the family of such neighbourhoods is equivariant. Thus, the stabiliser of $\Delta^0$ is equal to the stabiliser of $\Delta^0$ and hence belongs to $\mathcal{F}$. Putting $\mathcal{U} = \{ G \times N(\Delta^0) \mid \Delta \in X \}$ finishes the proof, because each point $x$ of $X$ belongs to a neighbourhood of at most one simplex of a each dimension.

Ad (b). For each $x \in X$ we will construct its neighbourhood $U_x$ being an $\mathcal{F}$-subset. By properness of the action (and $T_1$-property), we can find a neighbourhood $U^0_x$ disjoint with completion $C_x = Gx \setminus \{ x \}$ of $x$ in its orbit $Gx$ and such that the set $RS_x = \{ g \mid gU^0_x \cap U^0_x \neq \emptyset \}$ is finite.

Then, using regularity of $X$, we choose a smaller neighbourhood $U^*_x$, such that its closure $\overline{U^*_x}$ is contained in $U^0_x$ – in particular it is disjoint with $C_x$. But we have the equivalence

$$\forall g; gx \neq x \neq gU^*_x \iff \forall g; gx \neq x \neq g\overline{U^*_x},$$

so the set $U^2_x = U^1_x \setminus \bigcup_{g; gx \neq x} g\overline{U^*_x}$ contains $x$. It is open, as the sum can be taken over a finite set $RS_x$ without affecting the difference. What we achieved is emptiness of the intersection $U^2_x \cap gU^2_x \subseteq (U^1_x \setminus gU^1_x) \cap U^1_x = \emptyset$ for $gx \neq x$.

To handle the case $gx = x$, we do the last tweak setting $U_x = \bigcap_{g; gx = x} gU^2_x$. The intersection is finite (as the stabiliser of $x$ is a subset of $RS_x$), so we have just obtained a neighbourhood of $x$ with the stabiliser equal to the stabiliser of $x$, and conclude that $U_x$ is an $\mathcal{F}$-subset.

We still need to bound the dimension of the covering. Provided that $X$ is finite-dimensional and the action is isometric, we can use Proposition A.8 to find an equivariant refinement $U_0$ of the covering $\{ gU_x \mid g \in G, \ x \in X \}$ with dimension at most $\dim X$.

Otherwise, we can assume that the action is cocompact. Since the quotient map $X \to G\backslash X$ is open, $\{ q(U_x) \}_{x \in X}$ is an open covering of a compact set. Consequently, there is a finite family $x_1, \ldots, x_n$ such that $\{ q(U_{x_i}) \}_{1 \leq i \leq n}$ covers $G\backslash X$ and thus $U_0 = \{ gU_{x_i} \mid g \in G, \ 1 \leq i \leq n \}$ covers $X$. Clearly, the dimension of $U_0$ is at most $n$. 


Finally, the family $\mathcal{U}_\infty = \{G \times U_0 \mid U_0 \in \mathcal{U}_0\}$ is an $\alpha$-eq-asdim-covering of $G \times X$ for any $\alpha \leq \infty$.

We would like to mention that actually $G$-invariant coverings of $X$ (rather than of $G \times X$) were constructed in the above proof and that it relied mainly on topological properties of $X$ (not on the geometry of $G$).

2. Two parts of a covering

Assume that $\overline{X}$ is a compactification of $X$ and recall that we denote $\partial X = \overline{X} \setminus X$. An $\alpha$-asdim-covering $\mathcal{U}$ of $G \times \overline{X}$ breaks up into two invariant parts:

$$\mathcal{U}^0 = \{U \in \mathcal{U} \mid U \cap G \times \partial X = \emptyset\},$$

$$\mathcal{U}^\partial = \{U \in \mathcal{U} \mid U \cap G \times \partial X \neq \emptyset\}.$$

Conversely, if we are given two open $\mathcal{F}$-families $\mathcal{U}^0$, $\mathcal{U}^\partial$ of subsets of $G \times \overline{X}$, which – after restriction to $G \times X$ and $G \times \partial X$ respectively – have $G$-Lebesgue numbers $\alpha$, then the family $\mathcal{U}_0 \cup \mathcal{U}^\partial$ is an $\mathcal{F}$-cover of $G \times \overline{X}$ with $G$-Lebesgue number $\alpha$ and dimension at most $\dim \mathcal{U}^0 + \dim \mathcal{U}^\partial + 1$.

Hence, if the assumptions of Proposition 1.2 are satisfied, we can always assume (at the expense of possible increase in the bound on the dimension) that $\alpha$-asdim-coverings $\mathcal{U}$ of $G \times \overline{X}$ satisfy $\mathcal{U}^0 = \mathcal{U}_\infty$ and thus the only relevant part of $\mathcal{U}$ is $\mathcal{U}^\partial$.

In other words, it enough to deal with a neighbourhood of the boundary to obtain a covering of $G \times \overline{X}$.

Even more is true. An $\mathcal{F}$-cover of $G \times \partial X$ can be extended to an $\mathcal{F}$-family $\mathcal{U}^\partial$ of the same dimension that is open in $G \times \overline{X}$. Thus, one can indeed restrict their attention to the boundary itself.

**Theorem 2.1.** If $\overline{X}$ is a metrisable compactification of $X$ and any of the assumptions of Proposition 1.2 hold, then existence of an $\alpha$-eq-asdim covering $V$ of $G \times \partial X$ implies existence of an $\alpha$-eq-asdim-covering $\mathcal{U}$ of $G \times \overline{X}$ with $\dim \mathcal{U} \leq \dim V + \dim \mathcal{U}_\infty + 1$, where $\mathcal{U}_\infty$ is the covering from Proposition 1.2.

**Proof.** We will describe a method of enlarging an open subset $Y$ of $\partial X$ to an open subset $U(Y)$ of $\overline{X}$, such that dimension of the family of such sets is not increased. It will satisfy $U(Y) \cap \partial X = Y$, $(\ast)$. Using this construction, for an open subset $V = \bigcup_j \{g\} \times V_g$ of $G \times \partial X$, we define $U(V) = \bigcup_g \{g\} \times gU(g^{-1}V_g)$. We have $hV = \bigcup_g \{g\} \times hV_{h^{-1}g}$, so we obtain:

$$U(hV) = \bigcup_{g \in G} \{g\} \times gU(g^{-1}hV_{h^{-1}g}) = \bigcup_{j \in G} \{hj\} \times hjU(j^{-1}V_j) = hU(V);$$

i.e., the obtained family is equivariant.

From the above and $(\ast)$, we get $U(V) = U(V') \iff V = V'$, so it also follows that the stabiliser of $U(V)$ is equal to the stabiliser of $V$, hence it belongs to $\mathcal{F}$. We will also show that $\bigcap_i U(Y_i) = \emptyset \iff \bigcap_i Y_i = \emptyset$, in particular different translates of $U(V)$ are disjoint and the dimension of $\mathcal{U}^\partial = \{U(V) \mid V \in V\}$ is bounded by the dimension of $V$. Therefore, the desired covering $\mathcal{U}$ is $\mathcal{U}^\partial \cup \mathcal{U}_\infty$.

Fix a metric $d$ inducing the topology of $\overline{X}$ and let $V_1 = \{V_1 \mid V \in V\}$. Clearly $\dim V_1 \leq \dim V$, thus every $x \in \partial X$ belongs to a finite number of elements $V_x^1, \ldots, V_x^k$ of $V_1$. Let $\varepsilon(x)$ be the greatest number no bigger than 1 such that $B(x, \varepsilon(x)) \cap \partial X \subseteq \bigcap_i V_x^i$. For any $Y \subseteq \partial X$ we define $U(Y) = \bigcup_{x \in Y} B(x, \varepsilon(x)/2)$. 


In order to deduce that the dimension is preserved, it is enough to obtain
\[
\bigcap_{1 \leq i \leq n} U(V^i) = U \left( \bigcap_{1 \leq i \leq n} V^i \right),
\]
where \(V^i \in \mathcal{V}_1\) and \(n \in \mathbb{N}\). We will proceed by induction. Let us denote \(V^{(n)} = \bigcap_{1 \leq i \leq n} V^i\). The base is trivial, so we assume \(n > 1\):
\[
\bigcap_{1 \leq i \leq n} U(V^i) = \bigcap_{1 \leq i \leq n-1} U(V^i) \cap U(V^n) = U \left( V^{(n-1)} \right) \cap U(V^n)
\]
\[
= \left( U \left( V^{(n-1)} \setminus V^n \right) \cup U \left( V^{(n-1)} \cap V^n \right) \right)
\]
\[
\cap \left( U \left( V^n \setminus V^{(n-1)} \right) \cup U \left( V^n \cap V^{(n-1)} \right) \right)
\]
\[
= \left( U \left( V^{(n-1)} \setminus V^n \right) \cap U \left( V^n \setminus V^{(n-1)} \right) \right) \cup U(V^n)
\]
We claim that the first summand of the right hand side is empty. Suppose some \(z\) belongs to it. Then, there must be some \(x \in V^{(n-1)} \setminus V^n\) and \(y \in V^n \setminus V^{(n-1)}\) such that \(z \in B(x, \varepsilon(x)/2) \cap B(y, \varepsilon(y)/2)\). Thus, \(d(x, y) < \max(\varepsilon(x), \varepsilon(y))\). Hence, either \(x \in B(y, \varepsilon(y)) \cap \partial X \subseteq \bigcap_i V^i \subseteq V^n\) (contradicting \(x \in V^{(n-1)} \setminus V^n\)), or \(y \in B(x, \varepsilon(x)) \cap \partial X \subseteq \bigcap_i V^i \subseteq V^{(n-1)}\) (contradicting \(y \in V^n \setminus V^{(n-1)}\)).

Remarks. Group \(G\) can be embedded in \(G \times \mathbb{X}\) in different ways yielding different pull backs of \(\mathcal{U}\). Assume for example that \(G \ni g \mapsto gx_0 \in X\) is a coarse equivalence for some metric on \(X\) and consider a pull back of \(\mathcal{U}\) in \(G\). Inverse images of \(U \in \mathcal{U}_\infty\) will be uniformly bounded. On the other hand, we expect the inverse image of \(U \in \mathcal{U}_0\) to be unbounded, as it contains neighbourhoods of “points at infinity”. Thus, some elements of the covering are small, independently of \(\alpha\), while others are unbounded – it differs from asdim-coverings, where elements of the covering grow with \(\alpha\), but uniform boundedness is preserved at each step.

3. ZERO-DIMENSIONAL IS THE SAME AS VIRTUALLY CYCLIC

The definition of equivariant asymptotic dimension involves a family of groups \(\mathcal{F}\), for example \(\mathcal{VCyc}\), what causes the two notions – classic and equivariant asymptotic dimension – to disagree even in the simplest cases. In particular, it is not true that \(\text{asdim } G \leq \mathcal{F}\text{-eq-asdim } G\):

Example 3.1. Note that for \(G = \mathbb{Z}\) and, say, \(\mathbb{X} = [-\infty, +\infty]\) with the action by translations, the one-element covering \((G \times \mathbb{X})\) is a \(\mathcal{VCyc}\text{-eq-asdim}\)-covering for any \(\alpha\). Hence, \(\mathcal{VCyc}\text{-eq-asdim } \mathbb{Z} = 0\), while \(\text{asdim } \mathbb{Z} = 1\).

Theorem 3.2. Assume that \(\mathcal{F}\) is virtually closed (e.g., \(\mathcal{F} = \mathcal{VCyc}\)). Equivariant asymptotic dimension of \(G\) vanishes if and only if \(G\) belongs to \(\mathcal{F}\).

Proof. For the “if” part it is enough to take any \(\mathbb{X}\) and a one-element covering of \(G \times \mathbb{X}\) like in Example 3.1.

For the converse, assume that there is an \(\mathcal{F}\)-cover \(\mathcal{U}\) of \(G \times \mathbb{X}\) of dimension 0 (that is, disjoint) and of \(G\)-Lebesgue number \(\alpha > 0\). Take \(U \in \mathcal{U}\) and \((g, x) \in U\). Then there exists \(U' \in \mathcal{U}\) such that \(B(g, \alpha) \times \{x\} \subseteq U'\) – but then \(U' \cap U' \neq \emptyset\),
so $U = U'$. Thus, we showed that $(g, x) \in U$ implies $B(g, \alpha) \times \{x\} \subseteq U$ — hence $G \times \{x\} \subseteq U$ and we conclude that $U = G \times X$ for an open set $U \subseteq X$.

Consider now the sum $W = \bigcup G U \subseteq X$. We claim that $W$ is closed. Indeed, for $y \in \overline{W} \setminus W$, there is $U' = G \times U' \subseteq U$ such that $y \in U'$, but then $U' \subseteq X$ intersects $W$, which means that $U'$ intersects some $gU$, contradicting $\dim U = 0$.

So $W$ is a compact subset of $X$ covered by the disjoint family $G U$, meaning that the family must be finite. Thus, the orbit of $U \subseteq X$ is finite and the same is true for the orbit $GU$. Summing up, the stabiliser of $U$ belongs to $F$ and is of finite index in $G$, meaning that also $G$ belongs to $F$. \hfill \Box

Remarks. In particular, eq-asdim is not a function of asymptotic dimension of a group and/or topological dimension of its boundary, as $\text{asdim } \mathbb{Z} = \text{asdim } \mathbb{F}_n = 1$ and $\dim \partial \mathbb{Z} = \dim \partial \mathbb{F}_n = 0$, but $0 = \text{eq-asdim } \mathbb{Z} < \text{eq-asdim } \mathbb{F}_n$ for $n > 1$ and $\mathcal{F} = \mathcal{VCyc}$. Moreover, equivariant asymptotic dimension does not satisfy a logarithmic inequality holding for other notions of dimension ($\dim G \times H \leq \dim G + \dim H$), as $\text{eq-asdim } \mathbb{Z} > 0 = \text{eq-asdim } \mathbb{Z}$. In fact, it seems to be an open problem whether a product of groups of finite eq-asdim has finite eq-asdim.

Let us now consider the following situation. Map $G \ni g \mapsto gx_0 \in X$ is a coarse equivalence for a suitable metric $d$ on $X$ and if a sequence $(x_n)$ of points of $X$ converges to $x \in \partial X$ and $(y_n) \in X$ is another sequence, then

$$\sup_n d(x_n, y_n) < \infty \implies \lim_n y_n = x. \tag{1}$$

Then, $G$ cannot contain a finitely generated abelian group $H$ that does not belong to $F$. Suppose the contrary, let $\{z_1, \ldots, z_k\}$ be the generating set of $H$, and let a sequence $(h_n) \in H$ and a point $x_0 \in X$ be such that $x_n = h_n x_0$ converges to some $x \in \partial X$. Then for $y_n = h_n z_i x_0$ we have $\sup_n d(x_n, y_n) < \infty$ (as $d_G(h_n, h_n z_i) = d_G(1, z_i)$ and $j$ is a coarse equivalence), and thus $z_i x = \lim z_i h_n x_0 = \lim h_n z_i x_0 = x$; i.e., $H$ stabilises $x$.

But isotropy groups of $X$ cannot contain finitely generated groups not belonging to $F$: let $\alpha$ be big enough for $B(1, \alpha)$ to contain $\{z_1, \ldots, z_k\}$ and let $U \in \mathcal{U}(\alpha)$ contain $B(1, \alpha) \times \{x\}$. Then $z_i (B(1, \alpha) \times \{x\}) = B(z_i, \alpha) \times \{x\}$ intersects nontrivially with $B(1, \alpha) \times \{x\}$ and thus $z_i U \cap U \neq \emptyset$, so $z_i$ must stabilise $U$ and thus the stabiliser of $U$ contains $H$, contradicting the definition of an $\mathcal{F}$-eq-asdim-covering. [This also shows that space $X$ in the definition of eq-asdim is necessary; i.e., there are no $\alpha$-eq-asdim-coverings of $G = G \times \{\ast\}$ (unless $\alpha$ is small or $G \in \mathcal{F}$].

This suggests that commutativity (or existence of big abelian subgroups) may be an obstacle for (proving) finiteness of eq-asdim. Such a proof (if we assume $X \simeq G$) would require a compactification violating very natural condition (1). It is not true for the compactifications of CAT(0) groups used in [3], and thus the authors used suitable subspaces of the compactification and showed transfer reducibility (not finiteness of eq-asdim).

4. Characterisations of equivariant asymptotic dimension

The aim of this section is to provide a number of equivalent characterisations of equivariant asymptotic dimension. We will state our theorem in a generality broader than in the previous sections to handle the notion of transfer reducible groups that are defined in terms of homotopy group actions.
4.1. **Origin.** Recall the theorem (see [8, Theorem 1] or [15, Theorem 9.9]) enumerating different characterisations of asymptotic dimension.

**Theorem 4.1.** Let $X$ be a metric space. The following conditions are equivalent.

1. $\operatorname{asdim} X \leq n$;
2. for every $r < \infty$ there exist uniformly bounded, $r$-disjoint families $(U^i)$ for $0 \leq i \leq n$ of subsets of $X$ such that $\bigcup U^i$ covers $X$.
3. for every $\varepsilon > 0$ there is a uniformly cobounded, $\varepsilon$-Lipschitz map $\phi : X \to K$ to a simplicial complex of dimension $n$.
4. for every $d < \infty$ there exists a uniformly bounded cover $V$ of $X$ with $d$-multiplicity at most $n + 1$.

In the above theorem, a family of subsets is $r$-disjoint if the distance of any two of its members is at least $r$; a map to a simplicial complex is uniformly cobounded if there is a bound on diameter of inverse images of simplices; a simplicial complex $K$ is viewed as a subset of $\ell_1(V(K))$, where $V(K)$ is the set of vertices of $K$; and $d$-multiplicity of a covering means the maximal number of its elements intersecting a $d$-ball.

Analogs of maps from condition (3) for equivariant asymptotic dimension induce functors crucial for the proofs the Farrell–Jones and Borel conjectures, compare [6, Section 4], [17, Proposition 3.6 and Section 5] and [4, Proposition 3.9 and Section 11].

We would like to mention that if we relax condition (3) so that $\phi$ is a map into the sphere in $\ell_1$ instead of an $n$-dimensional complex, then it becomes (in case of bounded geometry metric spaces) equivalent to property $A$ ([19, Theorem 1.2.4 (6)], see also [10]). The equivalence is established by replacing $\varepsilon$-Lipschitz maps $x \mapsto \phi(x)$ into the unit sphere of $\ell_1(X)$ by maps $x \mapsto A_x$ into $\bigoplus_{x \in X} \mathbb{N} \setminus \{0\}$ such that for $d(x,y) \leq R$ we have $\|A_x - A_y\|/\|\min(A_x, A_y)\| < \varepsilon$ (and vice versa). The same transition can be made in the equivariant case.

4.2. **Homotopy actions.** Consider the map $\rho(g,x) = (g, g^{-1}x)$. It is a $G$-invariant homeomorphism from $G \times \overline{X}$ with the diagonal action onto $G \times \overline{X}$ with the action on the first coordinate by left multiplication (we will call this the action by translations). The condition $B(g, \alpha) \times \{x\} \subseteq U$, is equivalent to $\text{ADB}^\alpha \{\rho(g,x)\} = \rho(B(g, \alpha) \times \{x\}) \subseteq \rho(U)$, where $\text{ADB}$ stands for “antidiagonal (closed) ball” and can be described as follows:

$$\text{ADB}^\alpha(g,x) = \{(gh, h^{-1}x) \mid |h| \leq \alpha\}.$$ 

Hence, we could define equivariant asymptotic dimension using antidiagonal balls and the action by translations on $G \times \overline{X}$.

Note that in this reformulation the action on $\overline{X}$ is used only to define $\text{ADB}s$. Thus, being able to generalise the definition of $\text{ADB}$, we could ease the requirement that $G$ acts on $\overline{X}$.

**Definition 4.2** ([3, Definition 0.1]). Let $\overline{X}$ be a compact metric space, and $S$ a finite and symmetric subset of a group $G$ containing the neutral element $1$.

1. A homotopy $S$-action $(\varphi, H)$ on $\overline{X}$ consists of continuous maps $\varphi_g : \overline{X} \to \overline{X}$ for $g \in S$ and homotopies $H^g_{t,h} : \overline{X} \to \overline{X}$ for $g, h \in S$ with $gh \in S$ and $t \in [0,1]$ such that $H^0_{g,h} = \varphi_g \circ \varphi_h$ and $H^1_{g,h} = \varphi_{gh}$ holds for $g, h \in S$ with $gh \in S$. Moreover, we require that $H^t_{1,1} = \varphi_1 = \text{id}_{\overline{X}}$ for all $t \in [0,1]$. 

Lemma 4.3. Let \( S \) be a compact subset of \( X \). For \( g \in S \) let \( F_g(\varphi, H) \) be the set of all maps \( H^t_{r,s} \) where \( t \in [0,1] \) and \( r, s \in S \) with \( rs = g \).

For \( (g, x) \in G \times X \) and \( n \in \mathbb{N} \), let \( ADB^1_{\varphi, H}(g, x) \) be the subset of \( G \times X \) consisting of all \( (gs, y) \in G \times X \) such that \( y = f_{s-1}(x) \) or \( x = f_s(y) \) for some \( s \in S \) and \( f_{s-1} \in F_{s-1} \) or \( f_s \in F_s \).

For \( A \subseteq G \times X \) we put

\[
ADB^1_{\varphi, H}(A) = \bigcup_{(q, x) \in A} ADB^1_{\varphi, H}(g, x)
\]

and inductively \( ADB^{n+1}_{\varphi, H}(A) = ADB^{n+1}_{\varphi, H}(ADB^n_{\varphi, H}(A)) \).

(iii) Let \( (\varphi, H) \) be a homotopy S-action on \( X \). We say that \( \mathcal{U} \) is an open cover of \( G \times X \) and \( \mathcal{U} \) be an open cover of \( G \times X \) and \( n \in \mathbb{N} \). We say that \( \mathcal{U} \) is \( n \)-long with respect to \( (\varphi, H) \) if for every \( (g, x) \in G \times X \) there is \( \mathcal{U} \in \mathcal{U} \) containing \( ADB^n_{\varphi, H}(g, x) \).

Note that – due to the fact that \( \text{id}_X \in F_1(\varphi, H) \) – we have \( A \subseteq ADB^1_{\varphi, H}(A) \), so \( n \mapsto ADB^n_{\varphi, H}(A) \) is “increasing”.

Lemma 4.3. Let \( A \) be a compact subset of \( G \times X \). Then \( ADB^n_{\varphi, H}(A) \) is also compact. Moreover, for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that \( ADB^n_{\varphi, H}(B(A, \delta)) \subseteq B(ADB^n_{\varphi, H}(A), \varepsilon) \), where the neighbourhoods are taken with respect to the product metric on \( G \times X \).

Proof. Since \( ADB^n_{\varphi, H} \) the n-th power of \( ADB^1_{\varphi, H} \) (viewed as a function from the power set of \( G \times X \) to itself), it is enough to restrict to the case of \( ADB = ADB^1_{\varphi, H} \).

Moreover, we can restrict to the case when \( A = \{g\} \times Y \), for some closed \( Y \subseteq X \).

Observe that \( ADB(\{g\} \times Y) \) is the union of two sets \( I \) and \( II \):

\[
I = \bigcup_{s \in S} \bigcup_{t,u \in S} \bigcup_{r \in [0,1]} \{gs\} \times H^r_{t,u}(Y),
\]

\[
II = \bigcup_{s \in S} \bigcup_{t,u \in S} \bigcup_{r \in [0,1]} \{gs\} \times (H^r_{t,u})^{-1}(Y).
\]

Let us denote the map \( (x, r) \mapsto H^r_{t,u}(x) \) by \( H_{t,u} \). Instead of \( \bigcup_{t/u \in [0,1]} H^r_{t,u}(Y) \) one can write \( H_{t,u}(Y \times [0,1]) \) and similarly \( \bigcup_{r \in [0,1]} (H^r_{t,u})^{-1}(Y) = \pi_X \left( H_{t,u}^{-1}(Y) \right) \), where \( \pi_X \colon X \times [0,1] \to X \) is the projection. Consequently:

\[
I = \bigcup_{s \in S} \bigcup_{t,u \in S} \{gs\} \times H_{t,u}(Y),
\]

\[
II = \bigcup_{s \in S} \bigcup_{t,u \in S} \{gs\} \times \pi_X(H_{t,u}^{-1}(Y)).
\]

The above sums are finite, so the obtained set is compact, because images and inverse images of compact sets are compact as long as all the spaces considered are compact.

Maps \( H_{t,u} \) are uniformly continuous, so the “moreover” part is clear for \( I \), and in order to obtain it for \( II \) it suffices to prove the following (because \( \pi_X \) is also uniformly continuous).

Claim. Let \( H \colon Z \to Z' \) be a continuous map between compact metric spaces. For every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that \( H^{-1}(B(A, \delta)) \subseteq B(H^{-1}(A), \varepsilon) \), where \( A \) is a compact subset of \( Z' \).
Suppose the contrary, that for some \( \varepsilon > 0 \) there is a sequence of \( z_n' \) approaching \( A \) such that there are \( z_n \in H^{-1}(z_n') \) at least \( \varepsilon \)-distant from \( H^{-1}(A) \). By passing to a subsequence, we can assume that \( z_n \) converge to some \( z_0 \notin B(H^{-1}(A), \varepsilon) \). However, by continuity, \( H(z_0) \in A \), what yields a contradiction. \( \square \)

4.3. The characterisations.

**Definition 4.4.** Let \( Y \) be a \( G \)-set. Its subset is called an almost \( \mathcal{F} \)-subset if its stabiliser belongs to \( \mathcal{F} \). An almost \( \mathcal{F} \)-cover is a covering consisting of almost \( \mathcal{F} \)-subsets and closed under the induced action of \( G \).

That is, what distinguishes an almost \( \mathcal{F} \)-subset \( U \) from an \( \mathcal{F} \)-subset is that it may happen that \( U \neq gU \), but still \( U \cap gU \neq \emptyset \).

A \( G \)-simplicial complex such that all the stabilisers of simplices belong to \( \mathcal{F} \) will be called an \( \mathcal{F} \)-simplicial complex.

The next theorem is stated in terms of existence of homotopy actions, however we do not construct spaces \( \overline{X} \) and homotopy actions in the proof. Hence, all the equivalences stay true for a fixed \( \overline{X} \) or a fixed (homotopy) \( G \)-action on \( X \) as in Definition 0.4.

Conditions (0) and (1) below clearly correspond to condition (1) in Theorem 4.1, the former is introduced to relate to the “almost” versions of transfer reducibility present in the literature, \([7]\). Condition (4) comes from \([1, \text{Theorem A}]\), and – as we stated in Section 0.2 – is formally similar to the definition of an amenable action. It does not correspond to any characterisation of asymptotic dimension.

**Theorem 4.5.** Let \( S \) be a finite symmetric subset of \( G \) containing the identity element 1. Below, we require each \( \overline{X} \) to be a compact metrisable space and \((\varphi, H)\) to be an \( S \)-homotopy action of \( G \) on \( \overline{X} \). The action on \( G \times \overline{X} \) considered below is the action given by \( h(g, x) = (hg, x) \).

The following conditions (1–4) are equivalent and they imply condition (0). They are all equivalent if \( \mathcal{F} \) is virtually closed (e.g., \( \mathcal{F} = \forall \text{Cyc} \)).

(0) for each \( m \) there is \((\overline{X}, \varphi, H)\) and an \( m \)-long almost \( \mathcal{F} \)-cover of \( G \times \overline{X} \) of dimension at most \( n \);

(1) for each \( m \) there is \((\overline{X}, \varphi, H)\) and an \( m \)-long \( \mathcal{F} \)-cover of \( G \times \overline{X} \) of dimension at most \( n \);

(2) for every \( r \in \mathbb{N} \) there is \((\overline{X}, \varphi, H)\) and disjoint \( \mathcal{F} \)-families \( (\mathcal{V}^i) \) for \( 0 \leq i \leq n \) of open subsets of \( G \times \overline{X} \) such that \( \bigcup \mathcal{V}^i \) is an \( r \)-long covering of \( G \times \overline{X} \);

(3) for every \( \varepsilon > 0 \) there is \((\overline{X}, \varphi, H)\), an \( \mathcal{F} \)-simplicial complex \( K \) of dimension \( n \) and a \( G \)-equivariant map \( \phi: G \times \overline{X} \to K \), which is “anti-diagonally” \((G, \varepsilon)\)-Lipschitz:

\[
\|\phi(g, x) - \phi(gs^{-1}, fs(x))\| \leq \varepsilon \quad \forall s \in S \forall f_s \in F_s;
\]

(4) for every \( \varepsilon > 0 \) there is \((\overline{X}, \varphi, H)\), an \( \mathcal{F} \)-simplicial complex \( K \) of dimension \( n \), and a map \( \psi: \overline{X} \to K \), which is \( \varepsilon \)-equivariant:

\[
\|\psi(fs(x)) - s\psi(x)\| \leq \varepsilon \quad \forall s \in S \forall f_s \in F_s.
\]

**Remark 4.6.** If a group \( G \) satisfies the equivalent conditions (1–4) from the theorem for all symmetric finite subsets \( S \subseteq G \) with some additional technical requirements on \( \overline{X} \), then \( G \) is said to be transfer reducible over \( \mathcal{F} \), \([3]\).
Proof of Theorem 4.5. Implications (1) ⇒ (0) and (2) ⇒ (1) are immediate.

(3) ⇐ (4) was suggested in [1, Remark 3.7] and holds even for a fixed \( \varepsilon \). For the “if” part we put \( \phi(g, x) = g\psi(x) \). It is clearly \( G \)-equivariant and also satisfies the required condition:
\[
\|\phi(g, x) - \phi(gs^{-1}, fs(x))\| = \|g\psi(x) - gs^{-1}\psi(fs(x))\| = \|s\psi(x) - \psi(fs(x))\| \leq \varepsilon.
\]
For the “only if” part we put \( \psi(x) = \phi(1, x) \) and check:
\[
\|s\psi(x) - \psi(fs(x))\| = \|s\phi(1, x) - \phi(1, fs(x))\| = \|\phi(s, x) - \phi(s \cdot s^{-1}, fs(x))\| \leq \varepsilon.
\]

(3) ⇒ (2). First, we take a barycentric subdivision \( SK \) of \( K \) instead of \( K \) itself. The identity map \( K \to SK \) is Lipschitz with the constant depending only on \( n \). Each vertex of \( SK \) corresponds to a subset (simplex) of vertices of \( K \) and the cardinality of a vertex is clearly preserved under the group action, so vertices of the same cardinality are not adjacent. Moreover, the stabiliser of a vertex in \( SK \) is the stabiliser of a simplex in \( K \), so it belongs to \( \mathcal{F} \) (in fact also simplex stabilisers belong to \( \mathcal{F} \)).

To obtain (2), we put \( \varepsilon = \frac{1}{(n+1)(r+1)} \) and let \( \phi : G \times X \to SK \) be \( (G, \varepsilon) \)-Lipschitz. Let \( \mathcal{U}^i = \{ \phi^{-1}(S_y) \mid y \in V(SK), |y| = i \} \), where \( i \in \{1, \ldots, n+1 \} \) and \( S_y \) is the open star about \( y \); that is, \( \{ p \in \ell(V(SK)) \mid p(y) > 0 \} \cap SK \). The fact that two vertices are non-adjacent is equivalent to disjointness of the respective stars; hence, different elements of \( \mathcal{U}^i \) are disjoint. By \( G \)-invariance of \( \phi \) we get \( g\phi^{-1}(S_y) = \phi^{-1}(S_{gy}) \), so \( \mathcal{U}^i \) is \( G \)-invariant and the stabiliser of \( \phi^{-1}(S_y) \) is the stabiliser of \( y \) and thus belongs to \( \mathcal{F} \).

Let \( (g, x) \in G \times X \) and \( v_0 \) be such an element \( v \in V(SK) \) that maximises \( \phi(g, x)(v) \). We have \( \phi(g, x)(v_0) \geq \frac{1}{n+1} \). Thus, since \( \phi = (G, \varepsilon) \)-Lipschitz and \( \varepsilon = \frac{1}{(n+1)(r+1)} \), for any \( (g', x') \in ADB_{\phi,H}^r(g, x) \), we have \( \phi(g', x')(v_0) \geq \frac{1}{n+1} - \frac{r}{(n+1)(r+1)} > 0 \). Therefore, \( ADB_{\phi,H}^r(g, x) \subseteq \phi^{-1}(S_{v_0}) \subseteq \mathcal{U}^i \).

(1) ⇒ (3). This proof is based on techniques from [4, Section 3].

Let \( N \ni m \geq 3(n+1) \) and \( \mathcal{U} \) be an \( m \)-long \( \mathcal{F} \)-cover of \( G \times X \). From Lemma 4.3 it follows that \( I(U) = \{(g, x) \mid ADB_{\phi,H}^m(g, x) \subseteq U \} \) is open provided that \( U \) is. The family \( \mathcal{I} = \{I(U) \mid U \in \mathcal{U} \} \) is \( G \)-invariant (as \( hADB_{\phi,H}^m(g, x) = ADB_{\phi,H}^m(hg, x) \)), which means that it looks the same when restricted to \( \{g\} \times X \) for any \( g \). In particular, there is a finite number of compact subsets \( \{W_i\}_{i=1}^k \) covering \( X \approx \{1\} \times X \) such that each \( W_i \) is contained in some \( I(U_i) \). Thus, \( ADB_{\phi,H}^m(W_i) \) is contained in \( U_i \). By compactness (see Lemma 4.3), there exists \( \varepsilon_i \) such that \( B(ADB_{\phi,H}^m(W_i), \varepsilon_i) \subseteq U_i \).

Let \( ADB_{\phi,H}^{0,\delta}(A) = B(A, \delta) \), \( ADB_{\phi,H}^{1,\delta}(A) = B(ADB_{\phi,H}^1(B(A, \delta)), \delta) \) and inductively:
\[
ADB_{\phi,H}^{k+1,\delta}(A) = ADB_{\phi,H}^{1,\delta} \left( ADB_{\phi,H}^{k,\delta}(A) \right).
\]

From Lemma 4.3 and induction, it follows that for each \( i \) there is \( \delta_i \) such that \( ADB_{\phi,H}^{m,\delta}(W_i) \subseteq B(ADB_{\phi,H}^m(W_i), \varepsilon_i) \subseteq U_i \). Let \( \delta = \min \delta_i \) and \( \Lambda = \frac{m+1}{\delta} \).

We will define a \( G \)-invariant metric \( d \) on \( G \times X \) such that a Lebesgue number of \( \mathcal{U} \) will be \( m+1 \). Let \( d_0((g, x), (g', x')) = \Lambda \cdot d_X(x, x') \) if \( g = g' \) and \( \max(m+1, \Lambda \cdot \text{diam} X) \) otherwise. For \( p, r \in G \times X \) let \( d(p, r) \) be equal to the infimum of finite sums \( \sum_{j=0}^k \Delta_i((g_j, x_j)) \) over finite sequences \( (g_j, x_j)_{j=0}^k \) such that \( (g_0, x_0) = p \), \( (g_k, x_k) = r \). The value \( \Delta_i((g_j, x_j)) \) is equal to \( d_0((g_{i-1}, x_{i-1}), (g_i, x_i)) \) unless \( (g_i, x_i) = (g_is^{-1}, fs(x_{i-1})) \) or \( (g_{i-1}, x_{i-1}) = (g_is^{-1}, fs(x_i)) \), when we can reduce
Moreover, since \( \text{simplex stabilises it pointwise} \), so it is the intersection of stabilisers of its vertices and \( m + 1 \) is a Lebesgue number of \( U \) with respect to \( d \).

Define \( l_U(p) = \min(m + 1, \sup \{ r | B(p, r) \subseteq U \} ) \). It is clearly 1-Lipschitz, in particular continuous. Moreover, \( d((g, x), (g^{-1}, f_s(x))) \leq 1 \) for \( f_s \in F_s \), so we have \( |l_U(g, x) - l_U(g^{-1}, f_s(x))| \leq 1 \). Furthermore, by \( G \)-invariance of \( d \), we have \( l_U(p) = l_{gU}(gp) \) for any \( g \in G \).

We define \( \Phi(g, x) = \sum_{U \in \cal{X}} l_U(g, x) \cdot 1_U \in \ell_1(U) \) and \( \phi(g, x) = \frac{\Phi(g, x)}{\|\Phi(g, x)\|} \).

From the fact that \( l_U^{-1}(0, \infty) \subseteq U \) and the dimension of \( U \) is at most \( n + 1 \) we conclude that map \( \phi \) acquires its values in an \( n \)-dimensional complex \( K \subseteq \ell_1(U) \). Moreover, since \( hU \neq U \) implies \( hU \cap \emptyset = \emptyset \), we get that \( l_U^{-1}(0, \infty) \cap l_{hU}^{-1}(0, \infty) = \emptyset \), so we can assume that \( U \) and \( hU \) in \( K \) are not adjacent. Hence, the stabiliser of a simplex stabilises it pointwise, so it is the intersection of stabilisers of its vertices (being elements of \( U \)) and belongs to \( \mathcal{F} \).

We have to check if \( \phi \) is \((G, \varepsilon)\)-Lipschitz. Let \( x \in \overline{X}, s \in S, f_s \in F_s \) and \( g \in G \). Without loss of generality:

\[ m + 1 \leq \|\Phi(g, x)\| \leq \|\Phi(g^{-1}, f_s(x))\| \leq \|\Phi(g, x)\| + n + 1 \]

(in the last inequality we use the fact that \( \Phi(h, x) \) has at most \( n + 1 \) points in its support) thus we can write:

\[
\begin{align*}
\left\| \frac{\Phi(g, x)}{\|\Phi(g, x)\|} - \frac{\Phi(g^{-1}, f_s(x))}{\|\Phi(g^{-1}, f_s(x))\|} \right\| & \leq \\
\left\| \frac{\Phi(g, x) - \Phi(g^{-1}, f_s(x))}{\|\Phi(g, x)\|} \right\| + \left\| \frac{\Phi(g^{-1}, f_s(x))}{\|\Phi(g, x)\|} \right\| \left( \frac{1}{\|\Phi(g, x)\|} - \frac{1}{\|\Phi(g^{-1}, f_s(x))\|} \right) & \leq \\
\frac{2(n + 1)}{m + 1} + \left( \frac{\|\Phi(g^{-1}, f_s(x))\|}{\|\Phi(g, x)\|} - 1 \right) & \leq \frac{2(n + 1)}{m + 1} + \frac{n + 1}{m + 1} < \varepsilon.
\end{align*}
\]

(0) \( \Rightarrow \) (3) can be proved in the same way as (1) \( \Rightarrow \) (3), but we cannot guarantee that simplex stabiliser is a pointwise stabiliser (unless the simplex is 0-dimensional). Simplex stabiliser permutes vertices of the simplex and the kernel of this action is the pointwise stabiliser. This kernel is a finite index subgroup of the stabiliser, hence – if \( \mathcal{F} \) is virtually closed – the stabiliser belongs to \( \mathcal{F} \).

The above theorem is formulated for a particular definition of a homotopy action, but should hold for all similar definitions such as [17, Definition 2.1].

**Remark 4.7.** To show (3) \( \Rightarrow \) (1) directly we do not need the continuity of \( \phi \). It is enough to assume that vertices in the same orbit are not adjacent, inverse images of stars are open and put \( U = \{ \phi^{-1}(S_y) | y \in V(K) \} \).

Conversely, to obtain such a version of (3) from (1) it suffices to define \( l_U(g, x) \) as max \( \{ r \in \{1, \ldots, k \} | ADB_{p, H}(g, x) \subseteq U \} \).
Note that the implication (1) $\Rightarrow$ (3) was the only step where we used metrisability of $X$. In Appendix B, we show how to avoid this requirement if we deal with genuine group actions. The analog of condition (4) from Theorem 4.1 is also provided.

**Corollary 4.8.** For a virtually closed $\mathcal{F}$, the notions [7] of groups transfer reducible over $\mathcal{F}$ and almost transfer reducible over $\mathcal{F}$ are equivalent.

**Appendix A. Equivariant topological dimension**

Recall that the Lebesgue covering dimension of a topological space $X$ is the smallest integer $n$ such that any open covering has a refinement of dimension at most $n$. The number $n$ is sometimes called the topological dimension and is denoted by $\dim X$.

If $X$ is an $F$-space for some group $F$, a natural question to ask is whether any $F$-covering has an $F$-refinement of dimension $n$. By an $F$-covering we mean an $\mathcal{F}$-cover, where $\mathcal{F}$ is the family of all subgroups of $F$. In other words: the covering is $F$-invariant and two distinct elements of an orbit are disjoint.

The question was asked and answered in positive in [12] for a finite group $F$ acting on a metric space by isometries. This made the bound in Propositions 3.2, 3.3 of [5] independent of the order of the group $F$.

In [5, 7] a bound on the orders of the finite subgroups $F$ of a group was needed. Due to the above improvement, in [16] a proof of the Farrell–Jones conjecture became possible in a situation, where no such bound exists.

We will prove that the assumption that the group $F$ acting on the space is finite, is superfluous. It is enough to assume properness of the action.

**A.1. Dimension theory – auxiliaries.** Recall some definitions and facts from dimension theory after [13].

**Definition A.1 ([13, 5.1.1]).** The local dimension, $\text{loc dim } X$, of a topological space $X$ is defined as follows. If $X$ is empty, then $\text{loc dim } X = -1$. Otherwise, $\text{loc dim } X$ is the smallest integer $n$ such that for every point $x \in X$ there is an open set $U \ni x$ such that $\dim U \leq n$. If there is no such $n$, then $\text{loc dim } X = \infty$.

**Theorem A.2 ([13, 5.3.4]).** If $X$ is a metric space, then $\text{loc dim } X = \dim X$.

**Corollary A.3.** If $V$ is an open subset of a metric space $X$, then $\dim V \leq \dim X$.

**Proof.** It is enough to prove the claim for $\text{loc dim}$. Consider $x \in V$. There is an open (in $X$) set $U_0$ with $\dim U_0 \leq \text{loc dim } X$. We also have an open neighbourhood $V_x \ni x$ such that $\overline{V}_x \subseteq V$. Thus, $U = U_0 \cap V_x$ is an open neighbourhood of $x$, its closure in $X$ is equal to its closure in $V$ and it is a closed subset of $\overline{U}_0$, so $\dim U \leq \dim \overline{U}_0 \leq \text{loc dim } X$ (dimension of a closed subset never exceeds dimension of the space). Hence, $\text{loc dim } V = \sup_{x \in V} \inf_{U \ni x} \dim U \leq \text{loc dim } X$, as needed (where $U$ are open sets of $V$ and closures are taken in $V$).

**Corollary A.4.** In the case of metric spaces, there is no need for taking closures of neighbourhoods in the definition of local dimension (it is enough to consider open neighbourhoods and calculate their dimension).

**Proof.** Fix $x \in X$. It suffices to check the equality $\inf_{U \ni x} \dim U = \inf_{U \ni x} \dim U$, where $U$ are open neighbourhoods of $x$. Let $U_x$ be an open neighbourhood of $x$ such
that \( \overline{U}_x \) has the smallest possible dimension. Then dimension of \( U_x \) – by Corollary A.3 – is no larger. On the other hand, if there is an open neighbourhood \( V \) of \( x \) such that \( \dim V < \dim \overline{U}_x \), then there would be an open neighbourhood \( W \) such that \( \overline{W} \subseteq V \) and thus \( \dim \overline{W} \leq \dim V < \dim \overline{U}_x \) contradicting the minimality of \( U \). \( \square \)

**Proposition A.5.** The dimension of a metric space \( X \) is equal to the supremum of dimensions of its open subsets. It is enough to consider a supremum over any open cover of \( X \).

**Proof.** Let \( \mathcal{U} \) by any open covering of \( X \). By Corollary A.3, the dimension of \( X \) is no smaller than dimensions of its open subsets, thus \( \dim X \geq \sup_{U \in \mathcal{U}} \dim U \). On the other hand, it is equal to the local dimension, which is equal – by Corollary A.4 – to the supremum over points of infima over open neighbourhoods of their dimensions. But clearly, we have the inequality:

\[
\dim X = \sup_x \inf_{U \ni x} \dim U \leq \sup_x \left( \inf_{U \ni x} \dim U \right) \leq \sup_{U \in \mathcal{U}} \dim U = \sup \dim U. \quad \square
\]

**Theorem A.6** ([13, 9.2.16]). Let \( f : X \to Y \) be a continuous open surjection of metrisable spaces. If every fibre \( f^{-1}(y) \) is finite, then \( \dim X = \dim Y \).

A.2. **Equivariant refinements.** The following proposition strengthens [12, Corollary 2.5].

**Proposition A.7.** Let \( (Y,d) \) be a metric space with an isometric proper action of a group \( H \). Then \( \dim H \setminus Y = \dim Y \).

**Proof.** We can fix a pseudometric on the quotient space:

\[
d'([y],[y']) = \inf_{h,h' \in H} d(hy,h'y').
\]

The action is isometric, so it is equal to \( \inf_{h \in H} d(hy,y') \). If \( [y] \neq [y'] \), then – by properness of the action – there is no infinite sequence \( h_n y \) convergent to \( y' \) and thus \( d'([y],[y']) > 0 \). Therefore, \( H \setminus Y \) is a metric space (it is easy to check, that the quotient topology and the metric topology agree).

Let \( y \in Y \). Similarly as above, there is \( \varepsilon = \varepsilon(y) > 0 \) such that \( B(y, 2\varepsilon) \) is disjoint with all the other elements of the orbit \( H y \). Consequently, \( B(y, \varepsilon) \) is disjoint with its translates and has a finite stabiliser \( S \) (the one of \( y \)).

Denote by \( g \) the restriction of \( f \) to \( B(y, \varepsilon) \). For \( y' \in B(y, \varepsilon) \) and \( z' = g(y') \), the fibre \( g^{-1}(z') \) is contained in \( S y' \) and thus finite. Clearly \( g \) is an open surjection onto its (open) image, so Theorem A.6 applies: \( \dim f(B(y, \varepsilon)) = \dim g(B(y, \varepsilon)) = \dim B(y, \varepsilon) \).

Using the openness and the surjectivity again, we notice that the family

\[
\{ f(B(y, \varepsilon(y))) \mid y \in Y \}
\]

is an open covering of \( H \setminus Y \). With Proposition A.5 we conclude:

\[
\dim H \setminus Y = \sup \dim f(B(y, \varepsilon(y))) = \sup \dim B(y, \varepsilon(y)) = \dim Y. \quad \square
\]

Finally, we can prove a version of [12, Proposition 2.6].

**Proposition A.8.** Let \( Y \) be a metric space with an isometric proper action of a group \( H \) and \( \dim Y = n \). Any open \( \mathcal{F} \)-cover \( \mathcal{U} \) of \( Y \) has an open \( \mathcal{F} \)-refinement \( \mathcal{W} \) with dimension at most \( n \).
Proof. Denote the quotient map by \( q \). By Proposition A.7, we know that the open covering \( \{ q(U) \mid U \in \mathcal{U} \} \) of \( H \backslash Y \) has a refinement \( \mathcal{V} \) of dimension at most \( n \).

Clearly \( q^{-1}(V) \) for \( V \in \mathcal{V} \) is \( H \)-invariant, in particular it is an \( H \)-subset. The covering \( \{ q^{-1}(V) \mid V \in \mathcal{V} \} \) has the same dimension as \( \mathcal{V} \).

In order to obtain the required refinement of \( \mathcal{U} \), it is enough to divide each \( q^{-1}(V) \) into appropriate disjoint parts. Note that a division into disjoint parts does not increase the dimension of a covering. Let \( U_V \) be such an element of \( \mathcal{U} \) that \( V \subseteq q(U_V) \). Then clearly:

\[
q^{-1}(V) \subseteq q^{-1}(q(U_V)) = \bigcup_{[h] \in H/S} hU_V,
\]

where \( S \) is the stabiliser of \( U_V \). The required division of \( q^{-1}(V) \) is \( \bigsqcup_{[h]} q^{-1}(V) \cap hU_V \). The covering \( \mathcal{W} = \{ q^{-1}(V) \cap hU_V \mid V \in \mathcal{V}, h \in H \} \) is clearly an \( H \)-covering and refines \( \mathcal{U} \). Moreover, if \( \mathcal{U} \) is an \( \mathcal{F} \)-cover, then \( \mathcal{W} \) also is, as the stabiliser of \( q^{-1}(V) \cap U_V \) is the same as the stabiliser of \( U_V \).

\[\square\]

**Appendix B. Theorem 4.5 without metrisability**

We already noticed (4.7) that the only part of the proof of Theorem 4.5 utilising metrisability was the implication \((1) \implies (3)\), more precisely, the definition of \( l_U \).

For the sake of the subsequent reasoning, we will overload our notation: for \( U \subseteq G \times X \) by \( B(U, \alpha) \) we will denote the set

\[
B(U, \alpha) = \bigcup_{(g, \alpha) \in U} B(g, \alpha) \times \{ x \} = \{ (h, x) \mid B(h, \alpha) \times \{ x \} \cap U \neq \emptyset \},
\]

and similarly (for \(-A \) meaning the complement of \( A \)):

\[
B(U, -\alpha) = \{ (h, x) \mid B(h, \alpha) \times \{ x \} \subseteq U \} = -B(U, \alpha).
\]

Observe that not only \( U \mapsto B(U, \alpha) \) but also \( U \mapsto B(U, -\alpha) \) preserves open sets. Indeed, \((h, x) \in B(U, -\alpha)\) implies \( B(h, \alpha) \times \{ x \} \subseteq U \). As \( U \) is open and \( B(h, \alpha) \) is finite, \( B(h, \alpha) \times W \subseteq U \) for some neighbourhood \( W \) of \( x \) - and thus \( \{ h \} \times W \subseteq B(U, -\alpha) \).

**B.1. Function \( l_U \) for a compact Hausdorff \( G \)-space \( X \).** Let \( k > \frac{2(n+1)^2}{\varepsilon} \) and \( \mathcal{U} \) be a \( k \)-eq-asdim-covering of \( G \times X \). Let \( \varphi_U : X \to [0,1] \) be a partition of unity subordinate to the restriction of the family \( \mathcal{I} = \{ I(U) \mid U \in \mathcal{U} \} \) to \( \{ 1 \} \times X \), where \( I(U) = B(U, -k) \). We put \( \varphi_U = k \cdot \varphi_U \).

Let now:

\[
l_U(g, x) = \max_{h \in G} \left( \varphi_{h^{-1}U}(h^{-1}x) - d_G(g, h) \right).
\]

(Not that since \( \varphi_U \in [0, k] \), the minimum is in fact taken over a finite set \( d_G(g, h) \leq k \).) The positivity of \( l_U \) for some \((g, x)\) means existence of \( h \) such that \( d_G(g, h) < \varphi_{h^{-1}U}(h^{-1}x) \), in particular \( \varphi_{h^{-1}U}(h^{-1}x) > 0 \) and \( d_G(g, h) < k \). Thus, point \((1, h^{-1}x)\) belongs to \( I(h^{-1}U) \), which is equivalent to \((h, x) \in I(U) \), which, in turn, implies \((g, x) \in U \) by the definition of \( I(U) \). We conclude that \( l_U^{-1}(0, \infty) \subseteq U \).

Clearly, \( l_U \) also takes values in \([0, k]\) and for each \( x \) there is \( U \) such that \( l_U \geq \frac{k}{n+1} \).

Additionally, it is \( G \)-invariant:

\[
l_U(g, x) = \max \left( \varphi_{h^{-1}U}(h^{-1}x) - d_G(g, h) \right) = \max \left( \varphi_{(jh)^{-1}U}((jh)^{-1}jx) - d_G(jg, jh) \right) = l_U(jg, jx).
\]
Furthermore, it is \((G,1)\)-Lipschitz. Indeed, let \(s\) be a generator of \(G\):
\[
l_U(g, x) = \max \left( \varphi_{h^{-1}}(h^{-1}x) - d_G(g, h) \right) \leq \max \left( \varphi_{h^{-1}}(h^{-1}x) - d_G(gs, h) + 1 \right) = l_U(gs, x) + 1.
\]

This time, the construction is based on a partition of unity (locally finite), so the continuity of \(\phi\) is automatic, whereas in Theorem 4.5 it followed from the Lipschitz property of \(\Phi\). The proof of \((G, \varepsilon)\)-Lipschitz property is analogous.

### B.2. \(d\)-disjointness and \(r\)-multiplicity

In Theorem 4.1, there is condition (4), which has no counterpart in Theorem 4.5. Moreover condition (2) in 4.1 is formulated in terms of \(r\)-disjoint families, while in 4.5 we have disjoint families forming a covering with a \(G\)-Lebesgue number equal to \(r\). This lack of analogy is due to the fact that it is not clear how to enlarge sets in order to force big \(G\)-Lebesgue numbers in case of homotopy actions.

We say that a covering of \(G \times X\) has \((G, d)\)-multiplicity \(n\) if each set of the form \(B(g, d) \times \{x\}\) intersects at most \(n\) elements of the covering.

#### Proposition B.1

Let \(X\) be a compact space. Condition (1) implies condition (2) and (2) implies (1) if \(F\) is virtually closed.

1. \(F\)-eq-asdim \(G \times X \leq n\)
2. for each \(d < \infty\) there exists an open \(F\)-cover \(I\) of \(G \times X\) with \((G, d)\)-multiplicity at most \(n + 1\);

**Proof.** For (1) \(\implies\) (2) it is enough to take a covering \(U\) of \(G\)-Lebesgue number \(d\) and set \(I = \{B(U, -d) \mid U \in U\}\) as the desired covering.

We need to check that \(I\) consists of \(F\)-subsets. Note that \(B(U, -d) \subseteq U\), so for \(g\) not stabilising \(U\) we have \(gU \cap U = \emptyset\) and thus \(gB(U, -d) \cap B(U, -d) = \emptyset\). On the other hand, \(gU = U\) implies that \(gB(U, -d) = B(gU, -d) = B(U, -d)\). Hence, \(B(U, -d)\) is an \(F\)-subset with the same stabiliser as \(U\).

Now consider any \(G\)-d-ball: \(B(g, d) \times \{x\}\). If it intersects \(B(U, -d)\) at \((h, x)\), then by symmetry of the metric \(B(h, d) \times \{x\}\) contains \((g, x)\). But \(B(h, d) \times \{x\} \subseteq U\) from the definition of \(B(U, -d)\), meaning that also \(U\) contains \((g, x)\). Therefore, the number of sets \(B(U, -d)\) intersecting \(B(g, d) \times \{x\}\) does not exceed the number of sets \(U\) containing \((g, x)\), which is bounded by \(n + 1\).

Not surprisingly, for (2) \(\implies\) (1) we take \(I\) for \(d = \alpha\) and \(U = \{B(V, \alpha) \mid V \in I\}\).

It is clearly \(G\)-invariant, open and have a \(G\)-Lebesgue number equal to \(\alpha\).

Set \(B(V, \alpha)\) contains \((h, x)\) if and only if \(B(h, \alpha) \times \{x\}\) intersects \(V\). Thus, multiplicity of \(U\) is bounded by \((G, \alpha)\)-multiplicity of \(I\), so we obtained \(\dim U \leq n\).

We have \(gB(V, \alpha) = B(gV, \alpha)\), so for all \(g \in G\) such that \(gB(V, \alpha) = B(V, \alpha)\) and any \((h, x) \in B(V, \alpha)\) the set \(B(h, \alpha) \times \{x\}\) intersects all \(gV\) as above. Hence, the number of such \(gV\) is at most \(n + 1\) and thus the stabiliser of \(B(V, \alpha)\) maps into the symmetric group \(S(n + 1)\) and the kernel is the intersection of stabilisers of \(gV\). Consequently, the stabiliser of \((B(V, \alpha)\) has a finite index subgroup from \(F\).

A family of subsets of \(G \times X\) is \((r, G)\)-disjoint if for any two of its elements \(U \neq U'\) we have \(B(U, r) \cap U' = \emptyset\). As above, we can assume that \(U\) are disjoint families forming a covering of \(G\)-Lebesgue number \(r\) (cf. condition (2) of 4.5) and replace them by \(\{B(U, -r) \mid U \in U\}\), getting \((r, G)\)-disjoint families. Note also that if \(I\) are \((r, G)\)-disjoint and \(\bigcup I\) is a covering, then \(U'\) = \(\{B(U, r/3) \mid U \in I\}\)
is still \((r/3, G)\)-disjoint and \(\bigcup \mathcal{U}_t\) has \(G\)-Lebesgue number \(r/3\). Being an \(\mathcal{F}\)-cover is preserved without additional assumptions.

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