ON THE DIRICHLET PROBLEM
GENERATED BY THE MAZ’YA–SOBOLEV INEQUALITY

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1 Introduction

In what follows, \( x = (y; z) = (y_1, y'; z) \) stands for a point in \( \mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^{n-m} \), \( n \geq 3 \), \( 2 \leq m \leq n-1 \). Denote by \( P \) the subspace \( \{ x \in \mathbb{R}^n : y = 0 \} \); correspondingly, \( P^\perp = \{ x \in \mathbb{R}^n : z = 0 \} \).

Let \( \Omega \) be a domain in \( \mathbb{R}^n \). By \( C^\infty_0(\Omega) \) we denote the set of smooth functions with compact support in \( \Omega \). For \( 1 \leq p < \infty \) we denote by \( \dot{W}^1_p(\Omega) \) the closure of \( C^\infty_0(\Omega) \) with respect to the norm \( \| \nabla v \|_{p,\Omega} \). Obviously, for bounded domains \( \dot{W}^1_p(\Omega) = W^1_p(\Omega) \).

By definition, for \( 0 \leq \sigma \leq \min\{ 1, \frac{n}{p(m-n)} \} \) we put \( p^*= \frac{np}{n-\sigma p} \).

Proposition 1.1 The following inequality

\[
\| |y|^{\sigma-1} v \|_{p^*,\Omega} \leq \mathcal{N}(p, \sigma, \Omega) \cdot \| \nabla v \|_{p,\Omega}.
\]

holds true for any \( v \in \dot{W}^1_p(\Omega) \) provided

\( a) \) \( \Omega \) is any domain in \( \mathbb{R}^n \) for \( \frac{n(p-m)}{p(n-m)} < \sigma \leq 1 \) (the region I on Fig. [1]);
\( b) \) \( \Omega \subset \mathbb{R}^n \setminus P \) for \( p > m \), \( \sigma \leq \min\{ \frac{n(p-m)}{p(n-m)}; \frac{n}{p} \} \), \( \sigma \neq 1 \) (II on Fig. [1]);
\( c) \) \( \Omega \subset \mathbb{R}^n \setminus (\ell \times \mathbb{R}^{n-m}) \) for \( p = m \), \( \sigma = 0 \) (black point on Fig. [1])

(\( \ell \) is a ray in \( \mathbb{R}^m \) beginning at the origin).

Proof. The case a) is well known; see, e.g., [13 Sec.2.1.6]. Note that for \( \sigma = 1 \) we have classical Sobolev inequality.

Consider the cases b) and c). Note that it is sufficient to prove (I) for \( \Omega = \mathbb{R}^n \setminus P \) (respectively, \( \Omega = \mathbb{R}^n \setminus (\ell \times \mathbb{R}^{n-m}) \)).

For \( \sigma = 0 \) one should take conventional Hardy inequality in \( \mathbb{R}^m \setminus \{0\} \) (respectively, in \( \mathbb{R}^m \setminus \ell \); see, e.g., [16 Sec.2]) and integrate it with respect to \( z \).

For \( m < p < n \) the inequality (I) can be obtained from the cases \( \sigma = 0 \) and \( \sigma = 1 \) by the Hölder inequality. For \( p > n \) we also obtain (I) by the Hölder inequality from the extreme cases \( \sigma = 0 \) and \( \sigma = \frac{n}{p} \); the last one corresponds to the Morrey inequality, see [13 Sec. 1.4.5].

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Finally, we deal with the case $p = n, 0 < \sigma < 1$. Consider the domain $\Omega = \Omega_1 \times \Omega_2$, where $\Omega_1 = B_2 \setminus B_1 \subset \mathbb{R}^m$ is a spherical layer while $\Omega_2 = ]0,1[^{n-m} \subset \mathbb{R}^{n-m}$ is a cube. Let us write down the embedding theorem $W^1_n(\Omega) \hookrightarrow L_q(\Omega)$, with $q = \frac{n}{1-\sigma}$. Since the function $|y|^n$ is bounded and separated from zero in $\Omega$, this theorem can be rewritten as follows:

$$\left( \int_\Omega \frac{|v|^q}{|y|^n} \, dx \right)^{n/q} \leq C(q, m, n) \int_\Omega \left( |\nabla v|^n + \frac{|v|^n}{|y|^n} \right) \, dx.$$ 

Note that all the terms in this inequality are invariant under translations in $z$ and under dilations in $x$. Therefore, the same inequality is valid for $\Omega_{kt} = 2^k (\Omega_1 \times (\Omega_2 + k))$, with $k \in \mathbb{Z}, \, k \in \mathbb{Z}^{n-m}$. Summing these inequalities we obtain, subject to $q > n$,

$$\left( \int_{\mathbb{R}^n} \frac{|v|^q}{|y|^n} \, dx \right)^{n/q} \leq \sum_{k,t} \left( \int_{\Omega_{kt}} \frac{|v|^q}{|y|^n} \, dx \right)^{n/q} \leq C(q, m, n) \int_{\mathbb{R}^n} \left( |\nabla v|^n + \frac{|v|^n}{|y|^n} \right) \, dx.$$ 

The last term is already estimated, and we arrive at (1).

Remark 1 The assumption on $\Omega$ in the case c) can be considerably weakened. However, it is sharp for $\Omega$ being a wedge.

We call (1) the Maz’ya–Sobolev inequality.

We are interested in the attainability of the sharp constant in (1), i.e. in the attainability of the norm of corresponding embedding operator. If $\Omega$ is unbounded, or $\Omega \cap \mathcal{P} \neq \emptyset$, then
this operator is, in general, noncompact; for \( p < n \) and \( \sigma = 1 \) this is the case for any \( \Omega \). Therefore, the problem of attainability is nontrivial.

It is well known that the sharp constant in the Sobolev inequality (\( p < n \) and \( \sigma = 1 \)) does not depend on \( \Omega \) and is not attained for any \( \Omega \) provided the complement of \( \Omega \) is not negligible, i.e. \( \tilde{W}^1_p(\Omega) \neq \tilde{W}^1_p(\mathbb{R}^n) \). We claim that the same is true for \( p < n \) and \( 0 < \sigma < 1 \) provided \( \Omega \cap \mathcal{P} \neq \emptyset \). Indeed, since the inequality (1) is dilation invariant, the sharp constant in this case cannot depend on \( \Omega \) and equals \( N(p, \sigma, \mathbb{R}^n) \). Further, if the extremal function in (1) exists, by standard argument (see, for example, the end of the proof of Theorem 2.2) it is (after a suitable normalization) a positive generalized solution of the Dirichlet problem

\[
-\Delta_p u = \frac{u^{p^*-1}}{|y|^{(1-\sigma)p^*}} \quad \text{in } \Omega; \quad u|_{\partial \Omega} = 0
\]  

(here \( \Delta_p u = \text{div}(\nabla u|^{p-2}\nabla u) \) is \( p \)-Laplacian).

Extending \( u \) by zero to \( \mathbb{R}^n \), we obtain an extremal for (1) in the whole space. Therefore, this extension solves the equation (3) in \( \mathbb{R}^n \), and thus it is positive in \( \mathbb{R}^n \), a contradiction.

By the way, it is worth to note that for \( p = n \) the exponent in the denominator of (3) does not depend on \( \sigma \) and equals \( n \).

The case \( \Omega \cap \mathcal{P} = \emptyset, \partial \Omega \cap \mathcal{P} \neq \emptyset \) is considerably more complicated. In the recent paper [8] the attainability of the sharp constant in (1) was proved for \( p = 2, 0 < \sigma < 1 \), under rather restrictive assumptions on (a smooth bounded domain) \( \Omega \). Namely, it is supposed in [8, Theorem 1.1] that all the main curvatures at any point \( x^0 \in \partial \Omega \cap \mathcal{P} \) are nonpositive, and the mean curvature at any such point does not vanish.

Our paper consists of two parts. First, we analyze the attainability of the sharp constant in (1) for \( \Omega \) being a wedge \( \mathcal{K} = K \times \mathbb{R}^{n-m} \) (here \( K \) is an open cone in \( \mathbb{R}^m \)) or a “perturbed” wedge. Here we consider all \( 1 < p < \infty \) and \( 0 \leq \sigma < \min\{1, \frac{n}{p}\} \). Naturally, we suppose that \( \Omega \) satisfies (2).

In the second part we prove the attainability of the sharp constant in (1) in a bounded domain for \( p = 2 \) and \( 0 < \sigma < 1 \) under considerably weakened requirements on \( \partial \Omega \), see Section 3 below. Unfortunately, we cannot transfer this result to the case of arbitrary \( p \) because we do not have in hands good estimates of solutions to the model problem (3) in a half-space.

Let us discuss briefly the cases \( m = 1 \) and \( m = n \). For \( m = 1 \) our problem of interest degenerates in a sense. Indeed, the only admissible wedge in this case is a half-space \( \mathbb{R}^n_+ = \{x \in \mathbb{R}^n : y_1 > 0\} \). Theorems 2.1 and 2.2 in this case remain valid with the same proof while Theorems 2.3 and 2.4 are irrelevant. As for other domains, if \( \Omega \subset \mathbb{R}^n_+ \), and \( \partial \Omega \in C^1 \) touches \( \mathcal{P} \), then in the neighborhood of a touching point \( x^0 \in \partial \Omega \cap \mathcal{P} \) in the large scale looks like a half-space. Since (1) is dilation invariant, we obtain \( N(p, \sigma, \Omega) \leq N(p, \sigma, \mathbb{R}^n_+) \). The reverse inequality is trivial. As in the case \( \Omega \cap \mathcal{P} \neq \emptyset \), this implies non-attainability of the sharp constant in (1) for any \( \Omega \) provided the complement of \( \Omega \) is not negligible in \( \mathbb{R}^n_+ \). For \( p = 2 \) and bounded domain this fact was proved in [8]. Attainability of the sharp constant for \( m = 1, p = 2 \) in some unbounded domains without touching of \( \mathcal{P} \) was discussed in [23].

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1Note that Proposition 1.1 holds true for \( m = 1 \) with the only exception: the case c) should be attached to the case b). The proof runs without changes.

2Example 1 after Lemma 2.7 in [23] is not completely correct; it should be \( \varphi > 0 \) instead of \( \varphi \geq 0 \).
On the other hand, the problem for $m = n$, corresponding to the Hardy–Sobolev inequality, was investigated in a number of papers. The existence of the extremal function in a cone was proved in [10] (in the case $p = 2$, $n \geq 3$ this result was established earlier in [4]). The problem in “perturbed” cone was considered in [17] (the case $p = 2$, $\sigma = 0$ was dealt with in [19]).

For $\Omega$ being a compact Riemannian manifold with boundary, the conditions of attainability of the sharp constants in (1) and in some similar inequalities were considered in [2]. The case of bounded domains with $0 \in \partial \Omega$ was treated in [3] for $p = 2$, $n \geq 2$; similar results under more restrictive assumptions on $\partial \Omega$ were obtained earlier in rather involved papers [6] for $n \geq 4$ and [7] for $n = 3$. See also the survey [17], where the history of related problems and extensive bibliography was given.

The paper is organized as follows. In Section 2 we collect the results on existence and qualitative properties of extremal functions in (1) in wedges and in wedges with compact perturbation bounded away from $\mathcal{P}$.

In Section 3 we formulate the assumptions on the behavior of $\partial \Omega$ in a neighborhood of the origin and prove existence theorems for bounded domains. The technical estimates used in this proof are given in Sections 4–6.

Let us introduce the following notation. $S^{n-1}_r$ is the sphere in $\mathbb{R}^n$ with radius $r$ centered at the origin; $\omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$ is the area of $S^{n-1}_1$.

We write $o_{\varepsilon}(1)$ to show the quantity tending to zero, as $\varepsilon \to 0$, with other parameters assumed to be fixed. All the other $o(1)$ have the same meaning but are uniform with respect to $\varepsilon$.

We recall that a function $f : ]0, \delta[ \to \mathbb{R}$ is regularly varying (RVF) of order $\alpha$ at the origin, if it has a constant sign, and for any $t > 0$

$$\lim_{\varepsilon \to 0} \frac{f(\varepsilon t)}{f(\varepsilon)} = t^\alpha.$$ 

For basic properties of RVFs see [22].

We use letter $C$ to denote various positive constants. To indicate that $C$ depends on some parameters, we write $C(\ldots)$.

## 2 The Maz’ya–Sobolev inequality in wedges and in “perturbed” wedges

Our first statement provides the sharp constants in the Maz’ya inequality in wedges.

**Theorem 2.1** Let $2 \leq m \leq n - 1$, $1 < p < \infty$, $\sigma = 0$. Let $K$ be a cone in $\mathbb{R}^m$. If $p \geq m$ we suppose that $K \neq \mathbb{R}^m$, and for $p = m$, in addition, $K \neq \mathbb{R}^m \setminus \{0\}$. Put $\Omega = K = K \times \mathbb{R}^{n-m}$ and $G = K \cap S^{m-1}_1$. Then the sharp constant in (1) is not attained and equals $(\Lambda^{(p)}(G))^{-\frac{1}{p}}$, where

$$\Lambda^{(p)}(G) = \min_{v \in W^{1,p}_0(G) \setminus \{0\}} \frac{\int_G \left( \left(\frac{m-p}{p} \right)^2 |v|^2 + |\nabla' v|^2 \right)^{\frac{p}{2}} dS}{\int_G |v|^p dS}$$

(4)

(here $\nabla'$ stands for the tangential gradient on $S^{m-1}_1 \subset \mathbb{R}^m$).


**Proof.** First, the minimum in (4) is attained due to the compactness of embedding \( \dot{W}^1_p(\Omega) \hookrightarrow L_p(\Omega) \). Denote by \( \tilde{V} \) the minimizer of (4) normalized in \( L_p(\Omega) \). By standard argument, \( \tilde{V} \) is positive in \( \Omega \).

Let us define \( U(y, z) = U(y) = |y|^{1-\frac{m}{p}} \cdot \tilde{V}(\frac{y}{|y|}) \). It is shown in [17, Theorem 18] that \( U \) is a positive weak solution of the equation

\[
- \Delta_p^{(y)} U = \Lambda^{(p)}(G) \frac{U^{p-1}}{|y|^p} \quad \text{in} \quad K, \quad \text{and thus,} \quad - \Delta_p U = \Lambda^{(p)}(G) \frac{U^{p-1}}{|y|^p} \quad \text{in} \quad K. \quad (5)
\]

The relation \( \Lambda^{(p)}(G) \leq N^{-p}(p, 0, \Omega) \) follows now from [20, Theorem 2.3]. For the reader’s convenience we reproduce the proof based on the so-called generalized Picone identity.

For any \( u \in C^\infty_0(\Omega) \) we set \( h = \frac{|u|^p}{U^{p-1}} \in C^1_0(\Omega) \). Then (5) implies

\[
\Lambda^{(p)}(G) \int_{\Omega} \frac{|u|^p}{|y|^p} \, dx = \Lambda^{(p)}(G) \int_{\Omega} \frac{U^{p-1}}{|y|^p} \, h \, dx = \int_{\Omega} \nabla U \cdot \nabla U \cdot \nabla h \, dx =
\]

\[
= \int_{\Omega} \left( p|\nabla U|^{p-2} \nabla U \cdot \nabla u \frac{|u|^{p-2} u}{U^{p-1}} - (p-1)|\nabla U|^p \frac{|u|^p}{U^p} \right) \, dx \leq
\]

\[
\leq \int_{\Omega} \left( p|\nabla u| \cdot |\nabla U|^{p-1} \frac{|u|^{p-1}}{U^{p-1}} - (p-1)|\nabla U|^p \frac{|u|^p}{U^p} \right) \, dx \leq \int_{\Omega} |\nabla u|^p \, dx. \quad (6)
\]

Here \((*)\) is the Cauchy inequality while the last inequality follows from

\[
r^p - pr^{p-1} + (p-1)t^p \geq 0, \quad r, t > 0. \quad (7)
\]

By approximation, (6) holds true for \( u \in \dot{W}^1_p(\Omega) \).

To prove \( \Lambda^{(p)}(G) = N^{-p}(p, 0, \Omega) \) we consider the sequence \( u_\delta(y, z) = U_\delta(y)Z_\delta(z) \), where

\[
U_\delta(y) = \begin{cases}
|y|^{1-\frac{m}{p}+\delta} \cdot \tilde{V}(\frac{y}{|y|}), & |y| \leq R, \\
R^{1-\frac{m}{p}+\delta}(2-\frac{m}{p}) \cdot \tilde{V}(\frac{y}{|y|}), & R \leq |y| \leq 2R, \\
0, & |y| \geq 2R;
\end{cases}
\]

\[
Z_\delta(z) = \begin{cases}
1, & |z| \leq R, \\
2 - \frac{|z|}{R}, & R \leq |z| \leq 2R, \\
0, & |z| \geq 2R.
\end{cases}
\]

Clearly, \( u_\delta \in \dot{W}^1_p(\Omega) \). Direct computation shows

\[
\int_{\Omega} |\nabla u_\delta|^p \, dx = \int_{\Omega} \frac{|u_\delta|^p}{r^p} \, dx \cdot \left( \Lambda^{(p)}(G) + O(\delta) \right),
\]

and the statement follows.

Finally, the equality sign in \((*)\) means \( \nabla u \parallel \nabla U \) while the equality in (7) means \( r = t \). These two facts imply

\[
\frac{\nabla u}{u} = \frac{\nabla U}{U} \quad \implies \quad u = cU
\]

on the set \( \{ u \neq 0 \} \) and, therefore, in the whole \( \Omega \). Since \( U \notin \dot{W}^1_p(\Omega) \), the equality in (6) is impossible.

Next, we consider the Maz’ya–Sobolev inequality in wedges.
Theorem 2.2 Let $2 \leq m \leq n - 1$, $1 < p < \infty$, $0 < \sigma < \min\{1, \frac{n}{p}\}$. Let $K$ be a cone in $\mathbb{R}^m$. If $p > m$ and $\sigma \leq \frac{n(p-m)}{p(n-m)}$ we suppose in addition that $K \neq \mathbb{R}^m$. Put $\Omega = K = K \times \mathbb{R}^{n-m}$. Then the sharp constant in (1) is attained, i.e. there exists a function $V \in \dot{W}^1_p(\Omega)$, $V > 0$ in $\Omega$, such that the inequality (1) becomes equality.

Proof. It is evident that the sharp constant in (1) satisfies the relation

$$\mathcal{N}^{-1}(p, \sigma, \Omega) = \inf_{v \in \dot{W}^1_p(\Omega) \setminus \{0\}} J(v) \equiv \inf_{v \in \dot{W}^1_p(\Omega) \setminus \{0\}} \frac{\|\nabla v\|_{p, \Omega}}{\||y|^{\sigma-1}v\|_{p^*, \Omega}}. \quad (8)$$

Let $\{v_k\}$ be a minimizing sequence for the functional $J$. Without loss of generality we can assume $\||y|^{\sigma-1}v_k\|_{p^*, \Omega} = 1$ and $v_k \to v$ in $\dot{W}^1_p(\Omega)$. By the concentration-compactness principle of Lions ([11]; see also [5, Ch.1]) we have

$$\|y|^{\sigma-1}v_k\|_{p^*} \to \|y|^{\sigma-1}v\|_{p^*} + \sum_{j \in \mathcal{M}} \alpha_j \delta(x - x^j),$$

$$|\nabla v_k|^p \to |\nabla v|^p + \mathcal{N}^{-p}(p, \sigma, \Omega) \sum_{j \in \mathcal{M}} \alpha_j^{p/p^*} \delta(x - x^j),$$

where the convergence is understood in the sense of measures on the one-point compactification $\overline{\Omega} \cup \{\infty\}$, a set $\mathcal{M}$ is at most countable and $\alpha_j > 0$. Moreover, since the embedding $\dot{W}^1_p(\Omega) \hookrightarrow L_{p^*}(\Omega)$ is locally compact, we conclude that $x^j \in \mathcal{P} \cup \{\infty\}$.

Since $\{v_k\}$ is a minimizing sequence, by verbatim repetition of arguments from Theorem 2.2 [12] we obtain the alternative — either $v_k \to v$ in $\dot{W}^1_p(\Omega)$ and $\mathcal{M} = \emptyset$ (in this case $v$ is a minimizer of $J$), or $v = 0$, $\mathcal{M}$ is a singleton and $\alpha = 1$.

Let us remark here that, by the dilation invariance of the functional $J$, we can ensure the additional relation \[ \int_{\Omega \cap B_1} ||y|^{\sigma-1}v_k||_{p^*}^2 \, dx = \frac{1}{2}, \] which takes away the second variant.

It remains to note that the function $V = |v|$ also provides the minimum in the problem (8). Thus, after multiplying by a suitable constant, $V$ becomes a nonnegative generalized solution of the Dirichlet problem to the Euler–Lagrange equation (3) and thus, it is super-$p$-harmonic in $\Omega$. By the Harnack inequality for $p$-harmonic functions (see, e.g., [24]), it is positive in $\Omega$.

Now we present some symmetry properties of the extremal function.

Theorem 2.3 Let the assumptions of Theorem 2.2 be fulfilled. Then the function $V$ providing the sharp constant in (1) has the following properties:

1. $V$ is radially symmetric with respect to $z$, i.e. $V = V(y_1, |z|)$;

2. If $K$ is a circular cone, then $V$ is radially symmetric with respect to $y'$ and $z$, i.e. $V = V(y_1, |y'|, |z|)$;

3. If $K = \mathbb{R}^m$ and $\sigma > \frac{n(p-m)}{p(n-m)}$, then $V$ is radially symmetric with respect to $y$ and $z$, i.e. $V = V(|y|, |z|)$;

4. Let $K = \mathbb{R}^m \setminus \{0\}$. There exists $\hat{p} \in ]m, n[,$ and for $p > \hat{p}$ the function $\hat{\sigma}(m, n, p)$ is defined, such that $\hat{\sigma} < \min\{1, \frac{n}{p}\}$ and for $\sigma > \hat{\sigma}$ the function $V$ is not radially symmetric w.r.t. $y$.\[ \square \]
Proof. 1. This statement follows from the properties of the Schwarz symmetrization with respect to $z$-variables (or from the properties of the Steiner symmetrization with respect to $z_1$ for $m = n - 1$). Indeed, this transformation does not enlarge the numerator in (8), see, e.g., [21, Ch.7], and evidently retains the denominator. Thus, it is sufficient to take infimum in (8) over the set of functions, radially symmetric w.r.t. $z$. Further, by the Euler equation (3) all critical points of an extremal radially symmetric w.r.t. $z$ have to be located at $P$. In this case the numerator in (8) strictly decreases under symmetrization (see [1]), and therefore no function asymmetric w.r.t. $z$ can provide the minimum in (8).

2. In addition to the Part 1, in this case we can apply spherical symmetrization along the spheres $S^m_y$, which does not enlarge the numerator, see, e.g., [21, App.C], and retains the denominator.

3. Here we can apply the Schwarz symmetrization with respect to $y$-variables which does not enlarge the numerator, and does not reduce the denominator, see, e.g., [10, Ch.3].

4. In this case the Schwarz symmetrization in $y$s does not work, and we show that the minimizer in general does not inherit the symmetry of extremal problem.

Let $u(|y|; |z|)$ be a function providing the minimum to the functional $J$ over the set of functions in $\tilde{W}_p^1(\Omega)$, radially symmetric w.r.t. $y$ and $z$. Without loss of generality, we assume that $||y|^{\sigma-1}u||_{p_\sigma,\Omega} = 1$. By the principle of symmetric criticality, see [15], $dJ(\sigma; u; h) = 0$ for any variation $h \in W_p^1(\Omega)$.

Similarly to [15, Theorem 1.3], the second differential of $J$ at the point $u$ can be written as follows:

$$J^{p-1}(u) \cdot d^2 J(u; h) = \int_\Omega \left| \nabla u \right|^{p-4} \left((p - 2)(\nabla u, \nabla h)^2 + \left| \nabla u \right|^2 |\nabla h|^2\right) \, dx -$$

$$- \frac{J^p(u) \cdot [(p - p_\sigma^*) - (p_\sigma - 1)]}{\Omega} \left( \int_\Omega \frac{\left| u \right|^{p_\sigma^* - 2} \left| \nabla u \right|}{\left| y \right|^{(1 - \sigma)p_\sigma}} \, dx \right)^2 + (p_\sigma - 1) \left( \int_\Omega \frac{\left| u \right|^{p_\sigma^* - 2} \left| \nabla u \right|}{\left| y \right|^{(1 - \sigma)p_\sigma}} \, dx \right).$$

(9)

Now we set $h(y; z) = u(|y|; |z|) \cdot \frac{\partial u}{\partial y}$. By symmetry of $u$, $\int_\Omega \frac{\left| u \right|^{p_\sigma^* - 2} \left| \nabla u \right|}{\left| y \right|^{(1 - \sigma)p_\sigma}} \, dx = 0$. Substituting into (9), we obtain

$$J^{p-1}(u) \cdot d^2 J(u; h) = \int_\Omega \left| \nabla u \right|^{p-2} \frac{u^2}{\left| y \right|^2} \, dx - J^p(u) \cdot \frac{p_\sigma - p}{m - 1} \int_\Omega \frac{\left| u \right|^{p_\sigma^* f^2}}{\left| y \right|^{(1 - \sigma)p_\sigma}} \, dx.$$

Finally, we estimate the first integral by Hölder and Hardy inequalities and arrive at

$$d^2 J(u; h) \leq J(u) \left[ \left( \frac{p}{p - m} \right)^2 - \frac{p^2 \sigma}{(m - 1)(n - p \sigma)} \right].$$

If $p \geq n$ then the quantity in square brackets is negative for $\sigma$ close to $\frac{n}{p}$. If $p < n$ is close to $n$, this quantity is also negative for $\sigma$ close to 1. In both cases the statement follows. □

Corollary. For $p > \hat{p}$ and $\hat{\sigma} < \sigma < \min\{1, \frac{n}{p}\}$ the problem (3) in $\mathbb{R}^n \setminus \mathcal{P}$ has at least two nonequivalent positive solutions.

Proof. The first solution is a global minimizer of $J$ (under suitable normalization), the second one is a minimizer over the set of functions symmetric w.r.t. $y$. □

Further, we consider $\Omega$ being a perturbed wedge.
Theorem 2.4 Suppose that $2 \leq m \leq n-1$, $1 < p < \infty$ and $0 \leq \sigma < \min\{1, \frac{n}{p}\}$. Let

\[ \Omega_1 = K = K \times \mathbb{R}^{n-m} \text{ be a wedge satisfying (2), } \Omega_2 \in \mathbb{R}^n \setminus \mathcal{P} \text{ and } \Omega_1 \cap \Omega_2 \neq \emptyset. \]

1. For $\Omega = \Omega_1 \setminus \overline{\Omega_2}$ is not attained.

2. Let $\sigma > 0$. Then for $\Omega = \Omega_1 \cup \Omega_2$ the sharp constant in (4) is attained provided $\hat{W}_p^1(\Omega) \neq \hat{W}_p^1(\Omega_1)$.

3. Let $\sigma = 0$. Then, given $\Omega_2' \in \mathbb{R}^n \setminus \{0\}$, $\Omega_2' \cap K \neq \emptyset$, there exists $L < \infty$ such that if $\Omega_2 \supset \Omega_2' \times ] - L, L [$, $\Omega = \Omega_1 \cup \Omega_2$ and $\hat{W}_p^1(\Omega) \neq \hat{W}_p^1(\Omega_1)$ then the sharp constant in (4) is attained.

Proof. 1. For any $u \in C_0^\infty(\Omega_1)$ there exists a dilation $\Pi$ such that $\Pi u \in C_0^\infty(\Omega)$. Due to the dilation invariance of (4) we conclude that $\mathcal{N}(p, \sigma, \Omega) = \mathcal{N}(p, \sigma, \Omega_1)$.

Thus, if $u$ minimizes the quotient (8) on $\hat{W}_p^1(\Omega)$ then its zero continuation minimizes (8) on $\hat{W}_p^1(\Omega_1)$. Therefore, it is the nonnegative solution of the problem (3) in $\Omega_1$. By Harnack’s inequality for $p$-harmonic functions, it is positive in $\Omega_1$, a contradiction.

2. By Theorem 2.2 there exists a function $u$ positive in $\Omega_1$ that minimizes the quotient (8) on $\hat{W}_p^1(\Omega_1)$. If $\mathcal{N}(p, \sigma, \Omega) = \mathcal{N}(p, \sigma, \Omega_1)$ then the zero continuation of $u$ minimizes (8) on $\hat{W}_p^1(\Omega)$ that again leads to contradiction. Therefore, $\mathcal{N}(p, \sigma, \Omega) > \mathcal{N}(p, \sigma, \Omega_1)$.

Now the statement follows by the concentration-compactness principle. Indeed, let $\{v_k\}$ be a minimizing sequence for the functional $J$. Without loss of generality we can assume $\|y|^{\sigma-1}v_k\|_{p^*, \Omega} = 1$ and $v_k \rightharpoonup v$ in $\hat{W}_p^1(\Omega)$. As in Theorem 2.2 if $v_k \not\rightharpoonup v$ then

\[ \|y|^{\sigma-1}v_k\|_{p^*, \Omega} \rightarrow \delta(x - \hat{x}), \quad |\nabla v_k|^p \rightarrow \mathcal{N}^{-p}(p, \sigma, \Omega) \delta(\hat{x} - x), \]

and $\hat{x} \in \mathcal{P} \cup \{\infty\}$.

Since $\Omega_2 \in \mathbb{R}^n \setminus \mathcal{P}$, similarly to the proof of Corollary 2.1 [12], we can assume that $v_k|_{\Omega_2} \equiv 0$. This implies $\mathcal{N}(p, \sigma, \Omega) \leq \mathcal{N}(p, \sigma, \Omega_1)$, a contradiction.

3. Define $\tilde{\Omega}' = K \cup \Omega_2'$ and $\tilde{\Omega} = \tilde{\Omega}' \times \mathbb{R}^{n-m}$. It is proved in [17, Theorem 20] that $\mathcal{N}(p, 0, \tilde{\Omega}') > \mathcal{N}(p, 0, K)$, and there exists a minimizer $\tilde{U}$ of the quotient (8) in $\tilde{\Omega}'$. Then $\tilde{U}$ is a positive weak solution of the equation

\[ -\Delta_p(y)\tilde{U} = \mathcal{N}^{-p}(p, 0, \tilde{\Omega}')^{\tilde{U}^{p-1}}|y|^p \quad \text{in } \tilde{\Omega}', \quad \text{and thus,} \quad -\Delta_p\tilde{U} = \mathcal{N}^{-p}(p, 0, \tilde{\Omega}')^{\tilde{U}^{p-1}}|y|^p \quad \text{in } \tilde{\Omega}. \]

As in Theorem 2.1 this implies

\[ \mathcal{N}(p, 0, \tilde{\Omega}) \geq \mathcal{N}(p, 0, \tilde{\Omega}') > \mathcal{N}(p, 0, K) = \mathcal{N}(p, 0, \Omega_1) \]

(the last equality is due to Theorem 2.1).

Thus, there exists $u \in C_0^\infty(\tilde{\Omega})$ such that $\|y|^{-1}u\|_{p, \tilde{\Omega}} > \mathcal{N}(p, 0, \Omega_1) \cdot \|\nabla u\|_{p, \tilde{\Omega}}$. This means $\mathcal{N}(p, 0, \Omega) > \mathcal{N}(p, 0, \Omega_1)$ if $L$ is sufficiently large, and the statement follows by the concentration-compactness principle.

In what follows we need some estimates for the solution of the extremal problem (8) for $p = 2$ in the half-space. For the sake of brevity, we denote

\[ q = 2^* = \frac{2n}{n - 2\sigma}; \quad \mu_q(\Omega) = \mathcal{N}^{-1}(2, \sigma, \Omega); \quad \mu_q = \mu_q(\mathbb{R}^n_+). \]
By \( \phi \) we denote a minimizer of the problem (8) for \( p = 2 \) in \( \Omega = \mathbb{R}_+^n \). Without loss of generality we can assume \( \|y|^{\alpha-1}\phi\|_{q,\mathbb{R}_+^n} = 1 \). Then \( \phi \) is a weak solution of the Dirichlet problem
\[
-\Delta u = \mu_q^2 \cdot \frac{u^{q-1}}{|y|^{q(1-\sigma)}} \quad \text{in} \quad \mathbb{R}_+^n, \quad u|_{x_n=0} = 0.
\]

**Proposition 2.1** The function \( \phi \) satisfies the following relations:
\[
\phi(x) \sim Cx_n, \quad |\nabla \phi(x)| \sim C, \quad x \to 0; \tag{11}
\]
\[
\phi(x) \sim Cx_n \frac{1}{|x|^n}, \quad |\nabla \phi(x)| \sim C \frac{1}{|x|^n}, \quad x \to \infty. \tag{12}
\]

**Proof.** First, we claim that \( \phi \in \mathcal{C}_{1+\gamma}^{1,q}(\mathbb{R}_+^n) \). Indeed, the standard elliptic theory, see, e.g., [9], provides \( \phi \in \mathcal{C}_{\text{loc}}^{2}(\mathbb{R}_+^n \setminus P) \). Estimates in the neighborhood of \( P \) can be obtained using elliptic theory in domains with edges, see, e.g., [14]. Note that the property \( \phi \in \mathcal{C}_{\text{loc}}^{1}(\mathbb{R}_+^n) \) was proved also in [8, Appendix].

Further, the Hopf lemma gives \( \phi_{x_n}|_{x_n=0} > 0 \), and (11) follows.

Finally, the relations (12) follow from (11). Indeed, the direct computation shows that the image of \( \phi \) under the Kelvin transform is also a solution of the problem (10) while (11) turns into (12). \( \square \)

### 3 The case of bounded domain

We assume that in a neighborhood of the set \( P \cap \partial \Omega \) the boundary is of class \( \mathcal{C}^1 \); outside this neighborhood we impose no assumptions on \( \partial \Omega \). Suppose there exists a point \( x^0 \in P \cap \partial \Omega \) (without loss of generality, \( x^0 = 0 \)) satisfying the properties listed below.

Let us introduce local Cartesian coordinates with \( y' = (y_2, \ldots, y_m) \) in the tangent plane and the axis \( Oy_1 \) directed into \( \Omega \). Then in a neighborhood of the origin \( \partial \Omega \) is given by equation \( y_1 = F(y'; z) \). It is evident that \( F \in \mathcal{C}^1 \) and \( F(y'; z) = o(|y'| + |z|) \). Moreover, the assumption \( P \cap \Omega = \emptyset \) implies \( F(0; z) \geq 0 \).

We say that \( \partial \Omega \) is **average concave** in a neighborhood of the origin (see [3]), if for sufficiently small \( \rho \)
\[
f(\rho) := \int_{S^m_{\rho^{-2}}} F(y'; z) dS_\rho(y', z) < 0 \tag{13}
\]
(here and later the dashed integral stands for the mean value).

We introduce also the functions
\[
f_1(r; t) := \int_{S^m_{\rho^{-2}}} \int_{S^{m-1}_{\rho^{-2}}} F(y'; z) dS_{r}(y') dS_{t}(z),
\]
\[
f_2(\rho) := \int_{S^m_{\rho^{-2}}} |\nabla' F(y'; z)|^2 dS_\rho(y', z),
\]
(\( \nabla' \) stands for the gradient with respect to \( (y', z) \)) and assume that for sufficiently small \( \rho \)
\[
\cos^{m-2}(\beta) \sin^{n-m-1}(\beta) \cdot |f_1(\rho \cos(\beta), \rho \sin(\beta))| \leq C \cdot |f(\rho)|, \quad \beta \in [0, \frac{\pi}{2}], \tag{14}
\]
and
\[ \lim_{\rho \to 0} \frac{f_2(\rho)}{f(\rho)} \rho = 0. \] (15)

We say that \( \partial \Omega \) is **average concave in \( \mathcal{P} \) and \( \mathcal{P}^\perp \) directions** in a neighborhood of the origin, if (13) holds for sufficiently small \( \rho \), and

\[ \Phi(\beta) := \lim_{\rho \to 0} \frac{f_1(\rho \cos(\beta), \rho \sin(\beta))}{f(\rho)} \geq 0, \quad \beta \in [0, \frac{\pi}{2}]. \] (16)

Now we can formulate the main result of the second part of our paper.

**Theorem 3.1** Let \( \partial \Omega \) be average concave in \( \mathcal{P} \) and \( \mathcal{P}^\perp \) directions in a neighborhood of the origin, and let the relations (14) and (17) hold. Suppose also that \( f \) is regularly varying of order \( \alpha \in [1, n + 1] \) at the origin. Then for \( p = 2 \) and for any \( 0 < \sigma < 1 \) the infimum in (8) is attained.

Let us compare our assumptions with those of [8]. If \( \partial \Omega \) is smooth and \( \alpha = 2 \), then

\[ f(\rho) \sim \mathcal{H} \rho^2, \quad f_1(r, t) \sim \mathcal{H}^\mathcal{P} r^2 + \mathcal{H}^\mathcal{P}^\perp t^2, \quad f_2(\rho) \sim C \rho^2 \]

near the origin (here \( \mathcal{H} = \frac{1}{2(n-1)} \text{Sp}(\nabla^2 F(0)) \) is the mean curvature of \( \partial \Omega \) at the origin; respectively, \( \mathcal{H}^\mathcal{P} = \frac{1}{2(m-1)} \text{Sp}(\nabla^2_y F(0)) \) and \( \mathcal{H}^\mathcal{P}^\perp = \frac{1}{2(n-m)} \text{Sp}(\nabla^2_z F(0)) \)).

Since \( \mathcal{P} \cap \Omega = \emptyset \), \( \mathcal{H}^\mathcal{P}^\perp \) is always non-negative. Thus, the relations (13) and (16) mean that

\[ \mathcal{H}^\mathcal{P} < 0; \quad \mathcal{H}^\mathcal{P}^\perp = 0. \] (17)

The relations (14) and (15) are automatically fulfilled in this case.

One can see that (17) is considerably weaker than the assumptions of [8, Theorem 1.1]. We underline also that our hypotheses must be fulfilled at some point \( x^0 \in \mathcal{P} \cap \partial \Omega \) while the authors of [8] constrain the curvatures at any point \( x^0 \in \mathcal{P} \cap \partial \Omega \). Moreover, we do not require even the existence of the mean curvature (if \( \alpha < 2 \)). On the other hand, for \( \alpha > 2 \) all curvatures vanish at the origin.

**Remark 2** The assumption (13) can fulfill even if the main term of the asymptotic expansion of \( F \) vanishes under average. For example, it is the case if \( F(y', z) = y_2^3 - y_3^4 \).

**Remark 3** The assumption (14) is used only to ensure the limit passage under integral sign and can be easily weakened. However, it cannot be removed at all, and we prefer to give it in a simple form. In turn, the assumption (16) could be weakened if we had in hands more detailed information on the function \( \phi \).

Now consider the limit case \( \alpha = n + 1 \). In this case we can drop the assumption (16).

**Theorem 3.2** Let \( \partial \Omega \) be average concave in a neighborhood of the origin, and let the relations (14) and (13) hold. Suppose also that \( f \) is regularly varying of order \( n + 1 \) at the origin, and

\[ \int_0^\delta \frac{f(t)}{t^{n+2}} \, dt = -\infty. \]

Then for \( p = 2 \) and for any \( 0 < \sigma < 1 \) the infimum in (8) is attained.
Proof of Theorems 3.1 and 3.2. Let \( \{v_k\} \) be a minimizing sequence for (8). Without loss of generality we can assume \( \|y\|^{q-1}v_k\|_{q, \Omega} = 1 \) and \( v_k \to v \) in \( \dot{W}^1_q(\Omega) \).

Operating as in the proof of Theorem 2.2, we obtain the alternative — either \( v \) is a a minimizer of the extremal problem, or \( v = 0 \) and

\[
||y|^{q-1}v_k|^q \to \delta(x - \hat{x}), \quad |\nabla v_k|^2 \to \mu_q^2(\Omega)\delta(x - \hat{x}), \quad \hat{x} \in \mathcal{P} \cap \partial \Omega
\]

(the convergence is understood in the sense of measures on \( \overline{\Omega} \)).

We claim that in the second case \( \mu_q(\Omega) \geq \mu_q \). Indeed, without loss of generality, \( v_k \) concentrate near the origin. Further, as in the Corollary 2.1 [12], we can assume supports of \( v_k \) located in arbitrarily small ball. Since \( F(y'; z) = o(|y'| + |z|) \) and \( F(0; z) \geq 0 \), this implies

\[
\text{supp}(v_k) \subset \mathcal{K}_\varphi := \{ x \in \mathbb{R}^n : y_1 > -\varphi |y'| \}
\]

for any \( \varphi > 0 \). Hence

\[
\mu_q(\Omega) \geq \lim_{\varphi \to 0} \mu_q(\mathcal{K}_\varphi) = \mu_q(\mathcal{K}_0) = \mu_q.
\]

Therefore, to prove the statements we need only to produce a function having the quotient (8) less then \( \mu_q \). Similarly to [3], we construct such function using a suitable dilation and “bending” of the function \( \phi \) and multiplying it by a cut-off function with small support. The sharp estimates of behavior of \( \phi \) (Proposition 2.1) provide the desired result under assumptions on \( \partial \Omega \) close to optimal.

Choose \( \delta \) such that for \( |y'| + |z| < 2 \delta \) the relation (13) is satisfied and \( |F(y'; z)| \leq \frac{|y'| + |z|}{2} \).

Let us introduce the coordinate transformation \( \Theta_\varepsilon : x \mapsto \varepsilon^{-1}(x - F(y'; z)e_m) \). It is evident that in a neighborhood of the origin \( \Theta_\varepsilon \) straightens \( \partial \Omega \); its Jacobian equals \( \varepsilon^{-n} \).

Also it is easy to see that for \( r < \delta \) we have \( B_{\frac{\varepsilon r}{2}} \subset \Theta_\varepsilon(B_r) \subset B_{\frac{\varepsilon r}{2}} \).

Let \( \varphi \in C_0^\infty(\mathbb{R}^n) \) be a function, radially symmetric w.r.t. \( y \) and \( z \) and satisfying \( 0 \leq \varphi \leq 1 \),

\[
\varphi(x) \equiv \varphi(|y|; |z|) = \begin{cases} 
1, & \text{if } |y| < \frac{\delta}{2} \text{ and } |z| < \frac{\delta}{2}; \\
0, & \text{if } |y| > \delta \text{ or } |z| > \delta.
\end{cases}
\]

We introduce the cut-off function \( \varphi(x) = \varphi(\Theta_\varepsilon(x)) \). Obviously, the function \( x \mapsto \varphi(\Theta_\varepsilon^{-1}(x)) \) is radially symmetric w.r.t. \( y \) and \( z \):

\[
\varphi(\Theta_\varepsilon^{-1}(x)) = \varphi(\varepsilon y_1 + F(\varepsilon y'; \varepsilon z), \varepsilon y'; \varepsilon z) = \varphi(\varepsilon y_1, \varepsilon y'; \varepsilon z) = \varphi(\varepsilon |y|; \varepsilon |z|).
\]

Now we define the function

\[
\phi_\varepsilon(x) = \varepsilon^{-(n-2)/2} \varphi \left( \Theta_\varepsilon(x) \right) \varphi(x).
\]

It is easy to see that \( \phi_\varepsilon \in \dot{W}^1_2(\Omega) \), if \( \delta \) and \( \varepsilon \) are sufficiently small.

In Sections 4–6 we show that

\[
\int_\Omega \frac{|\phi_\varepsilon(x)|^q}{|y|^{q(1-\sigma)}} dx = 1 - A_1(\varepsilon)(1 + o_\delta(1) + o_\varepsilon(1)), \quad (18)
\]

\[
\int_\Omega |\nabla \phi_\varepsilon(x)|^2 dx = \mu_q^2 + A_2(\varepsilon)(1 + o_\delta(1) + o_\varepsilon(1)) - \frac{2\mu_q^2}{q} A_1(\varepsilon)(1 + o_\varepsilon(1)) \quad (19)
\]
(we recall that $o_\delta(1)$ is uniform with respect to $\varepsilon$). For given $\delta$, in these formulas we have, as $\varepsilon \to 0$,

$$A_1(\varepsilon) \sim C\varepsilon^{-1} f(\varepsilon);$$  \hspace{1cm} (20)

$$A_2(\varepsilon) \sim \begin{cases}
C\varepsilon^{-1} f(\varepsilon), & \text{under assumptions of Theorem 3.1} \\
C\varepsilon^n \int_\varepsilon^\delta \frac{f(t)}{t^{n+2}} dt, & \text{under assumptions of Theorem 3.2}
\end{cases}$$  \hspace{1cm} (21)

The relations (20) and (21) imply $A_1(\varepsilon) = O(A_2(\varepsilon))$ (in the case $\alpha = n + 1$ it follows from (20)). Therefore, for sufficiently small $\delta$ and $\varepsilon$ we have, subject to (13),

$$\frac{\|\nabla \phi_\varepsilon\|_2^2}{\|y\|^{\alpha-1} \phi_\varepsilon\|_q^2} = \frac{\mu_\varepsilon^2 + A_2(\varepsilon) (1 + o_\delta(1) + o_\varepsilon(1)) - \frac{2\mu_\delta^2}{q} A_1(\varepsilon)(1 + o_\varepsilon(1))}{\left(1 - A_1(\varepsilon)(1 + o_\delta(1) + o_\varepsilon(1))\right)^{2/q}}$$

$$= \mu_\varepsilon^2 + A_2(\varepsilon)(1 + o_\delta(1) + o_\varepsilon(1)) < \mu_\varepsilon^2,$$

and both Theorems follow.

\(\square\)

4 \hspace{1cm} \textbf{Estimate of the denominator and derivation of (18)}

We have, using the Taylor expansion,

$$\int_\Omega \frac{|\phi_\varepsilon(x)|^q}{|x|^{q(1-\sigma)}} dx = \int_{\mathbb{R}_+^n} \frac{|\phi(y; z)|^q}{|y + \varepsilon^{-1} F(\varepsilon y'; \varepsilon z) e_m|^{q(1-\sigma)}} \varphi^q(\Theta^{-1}_\varepsilon(x)) dydz =$$

$$= \int_{\mathbb{R}_+^n} \frac{|\phi(y; z)|^q}{|y|^{q(1-\sigma)}} \tilde{\varphi}^q(\varepsilon y; \varepsilon z) \cdot \left(1 - q(1 - \sigma) \frac{F(\varepsilon y'; \varepsilon z) y_1}{\varepsilon |y|^2} \right) dydz +$$

$$+ O_\delta(1) \int_{\mathbb{R}_+^n} \frac{|\phi(y; z)|^q F^2(\varepsilon y'; \varepsilon z)}{\varepsilon^2 |y + \varepsilon^{-1} F(\varepsilon y'; \varepsilon z) e_m|^{q(1-\sigma)+2}} \tilde{\varphi}^q(\varepsilon y; \varepsilon z) dydz =: I_1 - I_2 + I_3$$

(here $\xi = \xi(y, z) \in ]0, 1[$).

1. Since $\phi$ is normalized, $I_1 \leq 1$. On the other hand, the first estimate in (12) gives

$$1 - I_1 = \int_{\mathbb{R}_+^n} \frac{\phi^q(y; z)}{|y|^{q(1-\sigma)}} \left(1 - \tilde{\varphi}^q(\varepsilon y; \varepsilon z)\right) dydz \leq C \int_{\mathbb{R}_+^n \setminus B_{\frac{\delta}{\varepsilon}}} \frac{|y|^{-q(1-\sigma)} dydz}{(|y|^2 + |z|^2)^{\frac{q}{2}}} \leq C \left(\frac{\varepsilon}{\delta}\right)^{\frac{q}{2}}.$$

2. $I_2 = \frac{q(1 - \sigma)}{\varepsilon} \int_{\mathbb{R}_+^n} \frac{\phi^q(y; z)}{|y|^{q(1-\sigma)+1}} \tilde{\varphi}^q(\varepsilon y; \varepsilon z) F(\varepsilon y'; \varepsilon z) dydz =: A_1(\varepsilon)$.

**Proposition 4.1** Given $\delta$, the function $A_1(\varepsilon)$ satisfies (20), as $\varepsilon \to 0$. 

\(\square\)
PROOF. We claim that

\[
\lim_{\varepsilon \to 0} \frac{A_1(\varepsilon)}{f(\varepsilon)} = q(1 - \sigma) \omega_m - 2\omega_{n-m-1} \times \\
\times \int_0^{\frac{\pi}{2}} \rho^{n-2}\int_0^{\frac{\pi}{2}} \cos^{m-2}(\beta) \sin^{n-m-1}(\beta) \Phi(\beta) \int_0^\infty \frac{\phi^2(\rho \cos(\beta), s; \rho \sin(\beta)) s ds}{(\rho^2 \cos^2(\beta) + s^2)^{\frac{1}{2} + 2}} d\beta d\rho. 
\]

(22)

To prove this we apply the Lebesgue theorem. We have

\[
\frac{\varepsilon A_1(\varepsilon)}{q(1 - \sigma) f(\varepsilon)} = \frac{1}{f(\varepsilon)} \int_{\mathbb{R}^n} \varepsilon^q (\varepsilon y; \varepsilon z) \frac{\phi^2(y; z) y_1}{|y|^{q(1-\sigma)+2}} F(\varepsilon y'; \varepsilon z) dy dz = \\
= \frac{1}{f(\varepsilon)} \int_0^\infty \int_0^\infty \int_0^\infty \varepsilon^q (\varepsilon \sqrt{r^2 + y_1^2}; \varepsilon t) \frac{\phi^2(y_1, r; t) y_1}{(r^2 + y_1^2)^{\frac{q(1-\sigma)+2}{2}}} \times \\
\times \int_{S_r^{m-2}} S_t^{n-m-1} \int_0^\infty \int_0^\infty \frac{f_1(\varepsilon r; \varepsilon t)}{f(\varepsilon)} \times \\
\times \int_0^\infty \varepsilon^q (\varepsilon \sqrt{r^2 + s^2}; \varepsilon t) \frac{\phi^2(r, s; t) s ds}{(r^2 + s^2)^{\frac{q(1-\sigma)+2}{2}}} dr dt.
\]

We can apply the Monotone Convergence Theorem to the interior integral. Further, the regular behavior of \(f\) implies

\[
\lim_{\varepsilon \to 0} \frac{f_1(\varepsilon \rho \cos(\beta); \varepsilon \rho \cos(\beta))}{f(\varepsilon)} = \lim_{\varepsilon \to 0} \frac{f_1(\varepsilon \rho \cos(\beta); \varepsilon \rho \cos(\beta))}{f(\varepsilon)} \cdot \lim_{\varepsilon \to 0} f(\varepsilon) = \Phi(\beta) \rho^\alpha.
\]

Therefore, the assumption on the pointwise convergence is satisfied. Now we produce a summable majorant.

Due to the estimates (11) and (12), the interior integral is bounded from above by

\[
C \chi^{(\varepsilon)}(r; t) \int_0^\infty \frac{s^{q} ds}{1 + (r^2 + s^2 + t^2)^{\frac{q}{2}}} \leq C \chi^{(\varepsilon)}(r; t) \frac{C \chi^{(\varepsilon)}(r; t)}{1 + (t^2 + r^2)^{\frac{q}{2}}},
\]

where \(\chi^{(\varepsilon)}(r; t) = \chi_{[0, \frac{r}{\varepsilon}]}(r) \cdot \chi_{[0, \frac{t}{\varepsilon}]}(t)\).

Now we pass to the polar coordinates. Using (14), we estimate the integrand by

\[
C \chi_{[0, \frac{r}{\varepsilon}]}(\rho) \cdot \frac{f(\varepsilon \rho)}{f(\varepsilon)} \cdot \frac{\rho^{n-2}}{1 + \rho^{q(n-\sigma)}}.
\]

For \(\gamma > 0\) we have

\[
\chi_{[0, \frac{r}{\varepsilon}]}(\rho) \cdot \frac{f(\varepsilon \rho)}{f(\varepsilon)} \leq \chi_{[0, 1]}(\rho) \cdot \rho^{\alpha-\gamma} \frac{f(\varepsilon \rho)}{f(\varepsilon)} (\varepsilon \rho)^{-\alpha+\gamma} + \chi_{[1, \frac{r}{\varepsilon}]}(\rho) \cdot \rho^{\alpha+\gamma} \frac{f(\varepsilon \rho)}{f(\varepsilon)} (\varepsilon \rho)^{-(\alpha+\gamma)}.
\]

13
Since $f$ is RVF of order $\alpha$, the function $f(\tau) \tau^{-\alpha+\gamma}$ increases for small $\tau$ and the function $f(\tau) \tau^{-(\alpha+\gamma)}$ decreases for small $\tau$. Therefore, we have

$$\chi_{[0,\frac{2}{\tau}]}(\rho) \cdot \frac{f(\varepsilon)}{f(\varepsilon)} \cdot \frac{\rho^{n-2}}{1 + \rho^{q(n-\sigma)}} \leq C \left( \chi_{[0,1]}(\rho) \rho^{\alpha+n-2-\gamma} + \chi_{[1,\infty]}(\rho) \rho^{\alpha+n-2+\gamma-q(n-\sigma)} \right).$$

Since $\alpha \geq 1$, the majorant is summable at zero if $\gamma$ is sufficiently small. Since $\alpha \leq n+1$, for small $\gamma$ the second exponent does not exceed

$$2n - 1 + \gamma - q(n - \sigma) = -1 + \gamma - \frac{2n\sigma}{n-2\sigma} < -1,$$

and the majorant is summable at infinity. \qed

3. We recall that $F(0; z) \geq 0$, and hence, for small $\delta$ and $\varepsilon$

$$|y + \xi \varepsilon^{-1} F(\varepsilon y'; \varepsilon z) e_m| \geq |y + \xi \varepsilon^{-1} F(0; \varepsilon z) e_m| - \varepsilon^{-1} |F(\varepsilon y'; \varepsilon z) - F(0; \varepsilon z)| \geq |y| - |y'| \sigma_0(1) \geq \frac{|y|}{2}.$$

Therefore,

$$\left| \frac{I_3 \varepsilon}{f(\varepsilon)} \right| \leq C \int_{\mathbb{R}^n_+} \frac{\phi^q(y; z)}{|y|^{q(1-\sigma)+2}} \Phi^q(\varepsilon y'; \varepsilon z) \frac{F^2(\varepsilon y'; \varepsilon z)}{\varepsilon |f(\varepsilon)|} dydz$$

$$\leq C \int_{\mathbb{R}^n_+} \chi^{(\varepsilon)}(|y'|; |z|) \frac{F^2(\varepsilon y'; \varepsilon z)}{\varepsilon |f(\varepsilon)|} \int_{0}^{\infty} \frac{\phi^q(y', s; z)}{|y'|^2 + s^2} ds dydz.$$

Taking into account \[(11)\] and \[(12)\], we obtain

$$\int_{0}^{\infty} \frac{\phi^q(y', s; z)}{|y'|^2 + s^2} ds \leq C \int_{0}^{\infty} \frac{(|y'|^2 + s^2)^{\frac{2q}{2}-2} ds}{1 + (|y'|^2 + |z|^2 + s^2)^{\frac{2q}{2}}} \leq \frac{C |y'|^{q\sigma-1}}{1 + (|y'|^2 + |z|^2)^{\frac{2q}{2}}},$$

and therefore,

$$\left| \frac{I_3 \varepsilon}{f(\varepsilon)} \right| \leq C \int_{0}^{2\varepsilon} \int_{S^{n-2}_\rho} \left| y' \right|^{q\sigma-1} F^2(\varepsilon y'; \varepsilon z) \frac{dS^{n-2}_\rho(y', z)}{\varepsilon |f(\varepsilon)|} dS^{n-2}_\rho(y', z) \frac{dy}{1 + \rho^{q\sigma}}.$$

Since $q\sigma - 1 > -1$, it is easy to see that $W^1_2(S^{n-2}_1)$ is embedded into $L_{2,w}(S^{n-2}_1)$ with weight $w = |y'|^{q\sigma-1}$. Thus, using the Poincaré inequality, we can write

$$\int_{S^{n-2}_\rho} \left| y' \right|^{q\sigma-1} F(y'; z) dS^{n-2}_\rho(y', z) \leq$$

$$\leq C \rho^{q\sigma-1} \left( \rho^2 \int_{S^{n-2}_\rho} |\nabla F(y'; z)|^2 dS^{n-2}_\rho(y', z) + \left( \int_{S^{n-2}_\rho} F(y'; z) dS^{n-2}_\rho(y', z) \right)^2 \right).$$
This implies, subject to (15),

\[
\left| \frac{I_3\varepsilon}{f(\varepsilon)} \right| \leq C \int_0^{2\varepsilon} \frac{(\varepsilon\rho)^2 f_2(\varepsilon\rho) + f^2(\varepsilon\rho)}{\varepsilon |f(\varepsilon)|} \cdot \frac{\rho^{n-3+q\sigma} d\rho}{1 + \rho^{m}} =
\]

\[
= \int_0^{2\varepsilon} o(\varepsilon f(\varepsilon)) \cdot \frac{\rho^{n-2+q\sigma} d\rho}{1 + \rho^{m}} = o_6(1),
\]

and we arrive at \( I_3 = A_1(\varepsilon) o_6(1) \).

We remark also that \( \frac{m}{2} > n \geq \alpha - 1 \). This implies \( \varepsilon^{2\frac{m}{2}} = A_1(\varepsilon) o_6(1) \).
Choosing \( \delta > 0 \) sufficiently small and summing the estimates of items 1-3, we arrive at (18).

5 Estimate of the numerator and derivation of (19) for \( \alpha < n + 1 \)

We have

\[
(\phi_\varepsilon)_{x_m} = \varepsilon^{-n/2}\phi_{y_1}(\Theta_\varepsilon(x)) \varphi(x) + \varepsilon^{1-n/2} \phi(\Theta_\varepsilon(x)) \varphi_{x_m}(x);
\]

while for \( i \neq 1 \)

\[
(\phi_\varepsilon)_{x_i} = \varepsilon^{-n/2} \left( \phi_{y_i}(\Theta_\varepsilon(x)) - \phi_{y_1}(\Theta_\varepsilon(x)) F_{x_i}(y'; z) \right) \varphi(x) + \varepsilon^{1-n/2} \phi(\Theta_\varepsilon(x)) \varphi_{x_i}(x).
\]

Hence

\[
\int_\Omega |\nabla \phi_\varepsilon(x)|^2 dx = \int_\Omega \left[ \varepsilon^{-n} \varphi^2(x) |(\nabla'\phi)(\Theta_\varepsilon(x))|^2 - 2 \varepsilon^{-n} \varphi^2(x) \phi_{y_1}(\Theta_\varepsilon(x)) \langle (\nabla'\phi)(\Theta_\varepsilon(x)), \nabla' F(y'; z) \rangle + 2 \varepsilon^{-n+1} \phi(\Theta_\varepsilon(x)) \varphi(x) \langle (\nabla'\phi)(\Theta_\varepsilon(x)), \nabla' \varphi(x) \rangle - 2 \varepsilon^{-n+1} \phi_{y_1}(\Theta_\varepsilon(x)) \phi(\Theta_\varepsilon(x)) \varphi(x) \langle \nabla' F(y'; z), \nabla' \varphi(x) \rangle + \varepsilon^{-n} \phi^2(\Theta_\varepsilon(x)) \varphi(x) (\nabla' \varphi(x))^2 + \varepsilon^{-n} \phi_{y_1}(\Theta_\varepsilon(x)) \varphi^2(x) |\nabla' F(y'; z)|^2 + \varepsilon^{-n} \varphi^2(x) \phi_{y_1}(\Theta_\varepsilon(x)) + \varepsilon^{1-n/2} \phi(\Theta_\varepsilon(x)) \varphi(x) \varphi_{x_m}(x) + 2 \varepsilon^{-n+1} \phi_{y_1}(\Theta_\varepsilon(x)) \varphi(x) \phi(\Theta_\varepsilon(x)) \varphi_{x_m}(x) \right] dx =: J_1 - J_2 + \cdots + J_9.
\]

1.

\[
J_1 + J_7 = \int_{\mathbb{R}^n_+} \bar{\varphi}^2(\varepsilon y; \varepsilon z)|\nabla \phi(y; z)|^2 dydz = \mu^2 - \int_{\mathbb{R}^n_+} (1 - \bar{\varphi}^2(\varepsilon y; \varepsilon z))|\nabla \phi(y; z)|^2 dydz;
\]
2. Integrating by parts we obtain

\[
\int_{\mathbb{R}^n_+} (1 - \varphi^2(\varepsilon y; \varepsilon z)) |\nabla \phi(y; z)|^2 \, dydz \leq C \int_{\frac{1}{\varepsilon}}^\infty \zeta^{-1-n} \, d\zeta = C \left( \frac{\varepsilon}{\delta} \right)^n,
\]

whence

\[
J_1 + J_7 = \mu^2_q + C(\delta)O(\varepsilon^n).
\]

2. Integrating by parts we obtain

\[
J_2 = \int_{\mathbb{R}^n_+} 2 \varphi^2(\varepsilon y; \varepsilon z) \phi_{y_1}(y; z) \left\langle \nabla' \phi(y; z), \nabla' F(\varepsilon y; \varepsilon z) \right\rangle \, dydz =
\]

\[
= -\frac{2}{\varepsilon} \int_{\mathbb{R}^n_+} \left[ \varphi^2(\varepsilon y; \varepsilon z) \phi_{y_1}(y; z) \Delta' \phi(y; z) + \left\langle \nabla' (\varphi^2)(\varepsilon y; \varepsilon z), \nabla' \phi(y; z) \right\rangle \phi_{y_1}(y; z) +
\]

\[
+ \varphi^2(\varepsilon y; \varepsilon z) \left\langle \nabla' \phi(y; z), \nabla' \phi_{y_1}(y; z) \right\rangle \right] F(\varepsilon y'; \varepsilon z) \, dydz.
\]

By (10), we obtain

\[
J_2 = \frac{2}{\varepsilon} \int_{\mathbb{R}^n_+} \varphi^2(\varepsilon y; \varepsilon z) \left( \phi_{y_1 y_1}(y; z) + \mu^2_q \phi_{y_1}^{q-1}(y; z) \right) \phi_{y_1}(y; z) \, F(\varepsilon y'; \varepsilon z) \, dydz -
\]

\[
= -\frac{2}{\varepsilon} \int_{\mathbb{R}^n_+} \left\langle \nabla' (\varphi^2)(\varepsilon y; \varepsilon z), \nabla' \phi(y; z) \right\rangle \phi_{y_1}(y; z) \, F(\varepsilon y'; \varepsilon z) \, dydz -
\]

\[
- \frac{1}{\varepsilon} \int_{\mathbb{R}^n_+} \varphi^2(\varepsilon y; \varepsilon z) \left( |\nabla' \phi(y; z)|^2 \right)_{y_1} F(\varepsilon y'; \varepsilon z) \, dydz =: H + K_1 + K_2.
\]

Now we integrate the first term by parts.

\[
H = \frac{1}{\varepsilon} \int_{\mathbb{R}^n_+} \varphi^2(\varepsilon y; \varepsilon z) \left( \phi_{y_1 y_1}^2(y; z) \right)_{y_1} \frac{2 \mu^2_q (\phi^q(y; z))_{y_1}}{y |y|^{q(1-\sigma)}} \, F(\varepsilon y'; \varepsilon z) \, dydz =
\]

\[
= -\frac{1}{\varepsilon} \int_{\mathbb{R}^{n-1}_+} \varphi^2(\varepsilon y; \varepsilon z) \phi_{y_1}^2(0, y'; z) \, F(\varepsilon y'; \varepsilon z) \, dy'dz -
\]

\[
- \frac{1}{\varepsilon} \int_{\mathbb{R}^{n-1}_+} (\varphi^2(\varepsilon y; \varepsilon z))_{y_1} \phi_{y_1}^2(y; z) \, F(\varepsilon y'; \varepsilon z) \, dydz -
\]

\[
- \frac{2 \mu^2_q}{q \varepsilon} \int_{\mathbb{R}^n_+} (\varphi^2(\varepsilon y; \varepsilon z))_{y_1} \frac{\phi^q(y; z)}{|y|^{q(1-\sigma)}} \, F(\varepsilon y'; \varepsilon z) \, dydz +
\]

\[
+ \frac{2 (1 - \sigma) \mu^2_q}{\varepsilon} \int_{\mathbb{R}^n_+} \varphi^2(\varepsilon y; \varepsilon z) \frac{\phi^q(y; z)}{|y|^{q(1-\sigma)}} \frac{y_1}{|y|^2} \, F(\varepsilon y'; \varepsilon z) \, dydz =: -A_2(\varepsilon) + K_3 + K_4 + K_5.
\]
Proposition 5.1. Let assumptions of Theorem 3.1 hold. Then, given $\delta$, the function $A_2(\varepsilon)$ satisfies (21), as $\varepsilon \to 0$.

PROOF. We claim that

$$
\lim_{\varepsilon \to 0} \frac{\varepsilon A_2(\varepsilon)}{f(\varepsilon)} = \omega_m - 2\omega_{m-1} \times
\times \int_0^{\infty} \phi^{n-2+\alpha} \int_0^{\infty} \cos^{m-2}(\beta) \sin^{m-1}(\beta) \Phi(\beta) |\nabla \phi(0, \rho \cos(\beta); \rho \sin(\beta))|^2 d\beta d\rho. \quad (23)
$$

To prove this we apply the Lebesgue theorem. We have, similarly to Proposition 4.1,

$$
\frac{\varepsilon A_2(\varepsilon)}{\omega_m - 2\omega_{m-1} f(\varepsilon)} = \int_0^{\infty} \int_0^{\infty} \tilde{\varphi}^2(\varepsilon r; \varepsilon t) r^{m-2} t^{n-2} \frac{f_1(\varepsilon r; \varepsilon t)}{f(\varepsilon)} |\nabla \phi(0, r; t)|^2 dr dt. \quad (24)
$$

Passing to the polar coordinates, we see that the integrand converges to that in (23) for all $\rho$ and $\beta$. Now we produce a summable majorant. By (11) and (12), we have

$$
|\nabla \phi(0, \rho \cos(\beta); \rho \sin(\beta))| \leq C(1 + \rho^n).
$$

Therefore for $\gamma > 0$, similarly to Proposition 4.1, we estimate the integrand by

$$
C \chi_{[0, \frac{\alpha}{n}]}(\rho) f(\varepsilon \rho) \frac{\rho^{n-2}}{1 + \rho^{2\alpha}} \leq C \left( \chi_{[0,1]}(\rho) \rho^{\alpha+n-2-\gamma} + \chi_{[1,\infty]}(\rho) \rho^{\alpha-n+2+\gamma} \right).
$$

Since $\alpha < n+1$, this provides a summable majorant for sufficiently small $\gamma$.

Now we estimate all remaining terms in $J_2$. Since the functions $\tilde{\varphi}$ and $\phi$ are radially symmetric w.r.t $y'$ and $z$, integrating by parts in $K_2$ we have

$$
K_1 + K_2 + K_3 = -\frac{1}{\varepsilon} \int_{\mathbb{R}^n_+} F(\varepsilon y'; \varepsilon z) \left[ 2 \langle \nabla' (\tilde{\varphi}^2)(\varepsilon y'; \varepsilon z), \nabla' \phi(y; z) \rangle \phi_{y_1}(y; z) - \langle \tilde{\varphi}^2(\varepsilon y'; \varepsilon z) \rangle_{y_1} |\nabla' \phi(y; z)|^2 + \langle \tilde{\varphi}^2(\varepsilon y'; \varepsilon z) \rangle_{y_1} \phi_{y_1}^2(y; z) \right] dydz =
$$

$$
= -\frac{1}{\varepsilon} \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} r^{m-2} t^{n-1} f_1(\varepsilon r; \varepsilon t) \times
\times \left[ 2((\tilde{\varphi}^2(\varepsilon y_1, \varepsilon r; \varepsilon t)), \phi_r + (\tilde{\varphi}^2(\varepsilon y_1, \varepsilon r; \varepsilon t)), \phi_t) \phi_{y_1} - \langle \tilde{\varphi}^2(\varepsilon y_1, \varepsilon r; \varepsilon t) \rangle_{y_1} (\phi_r^2 + \phi_t^2) + \langle \tilde{\varphi}^2(\varepsilon y_1, \varepsilon r; \varepsilon t) \rangle_{y_1} \phi_{y_1}^2 \right] dy_1 dr dt.
$$
Using the assumption (14) and the estimate (12), we obtain
\[
|K_1 + K_2 + K_3| \leq \frac{C}{\delta} \int_{\frac{\delta}{2\varepsilon} \leq \sqrt{\rho^2 + y_1^2} \leq \frac{2\delta}{\varepsilon}} |\nabla \phi|^2 \rho^{n-2}|f(\varepsilon \rho)| \, dy_1 \, d\rho \leq
\]
\[
\leq \frac{C}{\delta} \int_{\frac{\delta}{2\varepsilon} \leq \sqrt{\rho^2 + s^2} \leq \frac{2\delta}{\varepsilon}} \frac{\rho^{n-2} |f(\varepsilon \rho)|}{(\rho^2 + s^2)^n} \, ds \, d\rho = \frac{C\varepsilon^n}{\delta} \int_{\frac{\delta}{2\varepsilon} \leq \sqrt{\rho^2 + s^2} \leq \frac{2\delta}{\varepsilon}} \frac{\varepsilon^{n-2} |f(\varepsilon \rho)|}{(\varepsilon^2 + s^2)^n} \, ds \, dx = C(\delta) \cdot \varepsilon^n.
\]
In a similar way,
\[
K_4 = -\frac{2\mu_q^2}{q} \varepsilon \int_0^\infty \int_0^\infty \int_0^\infty t^{m-2} n^{m-1} f_1(\varepsilon r; \varepsilon t) \cdot (\mathcal{F}(\varepsilon y_1, \varepsilon r; \varepsilon t)) \frac{\phi^{q}(y_1, r; t)}{(r^2 + y_1^2)^{\frac{1}{2}} - \frac{1}{2}} \, dy_1 \, dr \, dt,
\]
and therefore,
\[
|K_4| \leq \frac{C}{\delta} \int_{\frac{\delta}{2\varepsilon} \leq \sqrt{\rho^2 + y_1^2} \leq \frac{2\delta}{\varepsilon}} \frac{y_1^q \rho^{n-2} |f(\varepsilon \rho)|}{(\rho^2 + y_1^2)^{\frac{n}{2}}} \, dy_1 \, d\rho =
\]
\[
= \frac{C\varepsilon^{q(n-\sigma)-n}}{\delta} \int_{\frac{\delta}{2\varepsilon} \leq \sqrt{\rho^2 + s^2} \leq \frac{2\delta}{\varepsilon}} \frac{\varepsilon^{q-2} \varepsilon^{n-2} |f(\varepsilon \rho)|}{(\varepsilon^2 + s^2)^{\frac{n}{2}}} \, ds \, dx = C(\delta) \cdot \varepsilon^{q(n-\sigma)-n} = o(\varepsilon^n)
\]
(the last relation follows from \(q(n-\sigma) - 2n = \frac{2n\sigma}{n-2\sigma} > 0\)).

Finally, the integral \(K_5\) can be estimated in the same way as \(I_2\) in Section 4. This gives, as \(\varepsilon \to 0\),
\[
K_5 \sim \frac{2\mu_q^2}{q} A_1(\varepsilon).
\]
Thus,
\[
J_2 = -A_2(\varepsilon) + C(\delta) O(\varepsilon^n) + \frac{2\mu_q^2}{q} A_1(\varepsilon)(1 + o_1(1)) = A_2(\varepsilon).
\]

3. By the estimate (12), we obtain
\[
|J_3 + J_9| \leq 2\varepsilon \int_{\mathbb{R}_+^n} \phi(y; z) |\nabla \phi(y; z)| \mathcal{F}(\varepsilon y; \varepsilon z) |\nabla \mathcal{F}(\varepsilon y; \varepsilon z)| \, dydz \leq
\]
\[
\leq \frac{C \varepsilon}{\delta} \int_{\frac{\delta}{2\varepsilon}}^{2\delta} \zeta^{-n} d\zeta = C \left( \frac{\varepsilon}{\delta} \right)^n.
\]

4. Using the previous estimate, we obviously get
\[
|J_4| \leq 2\varepsilon \int_{\mathbb{R}_+^n} \phi(y; z) |\nabla \phi(y; z)| \mathcal{F}(\varepsilon y; \varepsilon z) |\nabla \mathcal{F}(\varepsilon y; \varepsilon z)| |\nabla' F(\varepsilon y; \varepsilon z)| \, dydz \leq
\]
\[
\leq C \varepsilon \int_{\mathbb{R}_+^n} \phi(y; z) |\nabla \phi(y; z)| \mathcal{F}(\varepsilon y; \varepsilon z) |\nabla \mathcal{F}(\varepsilon y; \varepsilon z)| \, dydz \leq C \left( \frac{\varepsilon}{\delta} \right)^n.
\]
5. In a similar way,

\[ |J_5 + J_8| = \varepsilon^2 \int_{\mathbb{R}^n_+} \phi^2(y; z) |\nabla \varphi(\varepsilon y; \varepsilon z)|^2 \, dy \, dz \leq \frac{C \varepsilon^2}{\delta^n} \int_{\frac{\delta}{\varepsilon}}^{\frac{2\delta}{\varepsilon}} \zeta^{1-n} \, d\zeta = C \left( \frac{\varepsilon}{\delta} \right)^n. \]

6. Finally, relations (12) and (15) imply

\[ J_6 = \int_{\mathbb{R}^n_+} \phi^2_1(y; z) \varphi^2(\varepsilon y; \varepsilon z) |\nabla' F(\varepsilon y'; \varepsilon z)|^2 \, dy \, dz \leq \]

\[ \leq C \int_0^{\frac{2\delta}{\varepsilon}} \rho^{n-2} \, f_2(\varepsilon \rho) \int_0^\infty \frac{dy_1}{(1 + \rho^2 + y_1^2)^n} \, d\rho = \]

\[ \leq C \int_0^{\frac{2\delta}{\varepsilon}} \rho^{n-3} \, f(\varepsilon \rho) \int_0^{\frac{2\delta}{\varepsilon}} \frac{1}{(1 + \rho^2)^{n-1/2}} \, d\rho. \]

The last integral can be estimated in the same way as in Proposition 5.1. This gives

\[ J_5 = o_3(1) A_2(\varepsilon). \]

We remark also that \( \varepsilon^n = A_2(\varepsilon) o_3(1). \)
Choosing \( \delta > 0 \) sufficiently small and summing the estimates of items 1-6 we obtain (19).

6 Derivation of (19) for \( \alpha = n + 1 \)

We underline that the assumption \( \alpha < n + 1 \) was used in the previous section only in the
proof of Proposition 5.1. Also the assumption (16) was used only to ensure the positivity of
the integral in (23). So, we need only to prove the following fact.

**Proposition 6.1** Let assumptions of Theorem 3.2 hold. Then, given \( \delta \), the function \( A_2(\varepsilon) \)
satisfies (21), as \( \varepsilon \to 0. \)

**Proof.** By (12), there exists \( M > 0 \) such that

\[ |\nabla \phi(0, r; t)| = \frac{M + o_\rho(1)}{\rho^n}, \quad \rho = \sqrt{r^2 + t^2} \to \infty. \quad (25) \]

We split the integral (24) into three parts:

\[
\frac{A_2(\varepsilon)}{\omega_{m-2} \omega_{n-m-1} \varepsilon^n} = \left( \int_{\frac{\delta}{\varepsilon}}^{\frac{2\delta}{\varepsilon}} \int_{\sqrt{r^2 + t^2} \leq R} \int_{R \leq \sqrt{r^2 + t^2} \leq \frac{\delta}{\varepsilon}} \int_{\frac{\delta}{\varepsilon} \leq \sqrt{r^2 + t^2} \leq \frac{2\delta}{\varepsilon}} \right) \varphi^2(\varepsilon r; \varepsilon t) \times

\times r^{m-2} \rho^{n-m-1} \frac{f_1(\varepsilon r; \varepsilon t)}{\varepsilon_{n+1}^{n+1}} |\nabla \phi(0, r; t)| \, dr \, dt =: L_1 + L_2 + L_3.
\]
The relation (25) implies that, as $R \to \infty$,

$$L_2 = (M + o_R(1))^2 \times$$

$$\times \int_{R}^{\frac{\delta}{2}} \int_{0}^{\frac{\delta}{2}} \cos^{n-2}(\beta) \sin^{n-m-1}(\beta) \frac{f_1(\varepsilon \rho \cos(\beta); \varepsilon \rho \sin(\beta))}{\varepsilon^{n+1} \rho^{n+2}} d\beta d\rho =$$

$$= \frac{(M^2 + o_R(1)) \omega_{n-2}}{\omega_{m-2} \omega_{n-m-1}} \int_{R}^{\frac{\delta}{2}} \frac{f(\varepsilon \rho)}{\varepsilon^{n+1} \rho^{n+2}} d\rho = \frac{(M^2 + o_R(1)) \omega_{n-2}}{\omega_{m-2} \omega_{n-m-1}} \int_{R}^{\frac{\delta}{2}} \frac{f(\varepsilon \rho)}{\varepsilon^{n+1} \rho^{n+2}} d\rho.$$ 

Further, the assumption (14) implies

$$\left| \varepsilon^{n+1} L_1 \right| \leq C \int_{0}^{R} \frac{f(\varepsilon \rho)}{\varepsilon^{n+1}} \rho^{n-2} d\rho.$$ 

For given $R$ we can pass to the limit under the integral sign. This provides $L_1 = O\left( \frac{f(\varepsilon)}{\varepsilon^{n+1}} \right)$, as $\varepsilon \to 0$.

On the another hand, divergence of the integral $\int_{0}^{\delta} \frac{f(\varepsilon \rho)}{\varepsilon^{n+2}} d\xi$ implies that for arbitrary large $N$ we have, as $\varepsilon$ is sufficiently small,

$$\varepsilon^{n+1} \int_{R \xi}^{\delta / 2} \frac{f(\varepsilon \rho)}{\varepsilon^{n+2}} d\xi \geq \varepsilon^{n+1} \int_{R \xi}^{N \xi} f(\varepsilon \rho) \frac{f(\varepsilon \rho)}{\varepsilon^{n+2}} d\xi = \int_{R \xi}^{N \xi} \frac{f(\varepsilon \rho)}{\varepsilon^{n+2}} d\xi = \ln(N/R) \cdot (1 + o(1)), \quad (26)$$

and thus, $L_1 = o(L_2)$, as $\varepsilon \to 0$.

Finally, as $\varepsilon \to 0$,

$$|L_3| \leq C \int_{\frac{2 \delta}{\varepsilon \rho}}^{\frac{\delta}{\varepsilon \rho}} \frac{f(\varepsilon \rho)}{\varepsilon^{n+1} \rho^{n+2}} d\rho = C \int_{\frac{2 \delta}{\varepsilon \rho}}^{\frac{\delta}{\varepsilon \rho}} \frac{f(\varepsilon \rho)}{\varepsilon^{n+2}} d\xi = C(\delta) = o(L_2).$$

It remains to note that for given $R$ and $\delta$

$$\int_{R \xi}^{\delta / 2} f(\xi \rho) d\xi = \int_{\varepsilon}^{\delta} f(\xi \rho) d\xi + O(1) \sim \int_{\varepsilon}^{\delta} \frac{f(\xi \rho)}{\varepsilon^{n+2}} d\xi, \quad \varepsilon \to 0,$$

and we arrive at

$$A_2(\varepsilon) \sim M^2 \omega_{n-2} \varepsilon^n \int_{\varepsilon}^{\delta} \frac{f(\xi \rho)}{\varepsilon^{n+2}} d\xi.$$  

□
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