The Schrödinger System \( H = -\frac{1}{2} \left( \frac{t}{t_0} \right)^a \partial_{xx} + \frac{1}{2} \omega^2 \left( \frac{t}{t_0} \right)^b x^2 \)

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We attack the specific time-dependent Hamiltonian problem \( H = -\frac{1}{2} \left( \frac{t}{t_0} \right)^a \partial_{xx} + \frac{1}{2} \omega^2 \left( \frac{t}{t_0} \right)^b x^2 \). This corresponds to a time-dependent mass (TM) Schrödinger equation. We give the specific transformations to a different time-dependent quadratic Schrödinger equation (TQ) and to a different time-dependent oscillator (TO) equation. For each Schrödinger system, we give the Lie algebra of space-time symmetries, the number states, the squeezed-state \( \langle x \rangle \) and \( \langle p \rangle \) (with their classical motion), \((\Delta x)^2\), \((\Delta p)^2\), and the uncertainty product.

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1 INTRODUCTION

In recent work [1, 2] we discussed general time-dependent quadratic (TQ) Schrödinger equations

\[ S_1 \Phi(x, t) = \left\{ -[1 + k(t)] P^2 + 2T + h(t)D + g(t)P \right\} \Phi(x, t) = 0. \]  (1)

It was shown how to solve them by i) first performing a unitary transformation to a time-dependent mass (TM) equation, and then ii) making a change of time variable to yield a time-dependent oscillator (TO) equation which in principle can be solved. One can then work backwards to find the TM and TQ solutions.

Elsewhere [3], we went into detail on how to solve a specific subclass of cases, TM equations with only time-dependent \( P^2 \) and \( X^2 \) terms. [We will refer to equations from this paper as, e.g., Eq. (3)-22.] These Hamiltonians are parametrized as

\[ \hat{H}_2 = \frac{1}{2} e^{-2\nu(t)} P^2 + h^{(2)}(t) e^{2\nu(t)} X^2. \] (2)

In this paper we examine the TM system

\[ \hat{H}_2 = \frac{1}{2} \left( \frac{t_0}{t} \right)^a P^2 + \frac{1}{2} \omega^2 \left( \frac{t}{t_0} \right)^b X^2, \] (3)

where \( a \) and \( b \) are real numbers. (We have also investigate another TM system [4].)

Going directly to what is this problem’s TQ Schrödinger equation,

\[ TQ : \quad S_1 \Phi(x, t) = \left\{ -P^2 + 2T - \frac{a}{t} D - \omega^2 \left( \frac{t}{t_0} \right)^{b-a} X^2 \right\} \Phi(x, t) = 0, \] (4)

\[ H_1 = \frac{1}{2} P^2 + \frac{1}{2} \omega^2 \left( \frac{t}{t_0} \right)^{b-a} X^2, \] (5)

where the operators \( P^2, T = i\partial_t, D = (XP + PX)/2, \) and \( X^2 \) are defined in equations (3)-1.

Note that \( t_0 \neq 0 \) and furthermore \( t \) and \( t_0 \) must be of the same sign. For this paper, we assume that \( t_0 > 0 \). This TQ equation is related to the time-dependent TM Schrödinger equation by the unitary mapping [4]

\[ R(0, \nu, 0) = \exp \left[ i\nu D \right], \quad \nu = \frac{a}{2} \ln \left( \frac{t}{t_0} \right). \] (6)
This yields the $TM$ equation

$$
TM : \quad \hat{S}_2 \hat{\Theta}(x, t) = \left\{ - \left( \frac{t_o}{t} \right)^a P^2 + 2T - \omega^2 \left( \frac{t}{t_0} \right)^b X^2 \right\} \hat{\Theta}(x, t) = 0,
$$

(7)

where $a$ and $b$ are real numbers. This is the defining $TM$ equation of our system. It has been studied by Kim [5] and also in [1]. The $TM$ equation (7) is transformed into a $TO$ equation

$$
TO : \quad S_3 \Psi(x, t') = \left\{ -P^2 + 2T - 2g^{(2)}(t')X^2 \right\} \Psi(x, t') = 0,
$$

(8)

$$
H_3 = \frac{1}{2} P^2 + g^{(2)}(t')X^2,
$$

(9)

by a change in time variable. When $a = 0$, we trivially have $t = t'$, a case we now ignore. When $a = 1$ (Case 1), the time transformation is [see Eq. (10-95)]

$$
t' - t'_o = t_o \ln \left( \frac{t}{t_o} \right), \quad g^{(2)}(t') = \frac{1}{2} \omega^2 \exp \left[ \frac{1 + b}{t_o} (t' - t'_o) \right],
$$

(10)

with $t' - t'_o \in [0, \infty)$. When $a \neq 1$ (Case 2), the time transformation is [see Eq. (10-96)]

$$
t' - t'_o = \frac{t_o}{1 - a} \left[ \left( \frac{t}{t_o} \right)^{1-a} - 1 \right], \quad g^{(2)}(t') = \frac{1}{2} \omega^2 \left[ 1 + \left( \frac{1 - a}{t_o} \right) (t' - t'_o) \right]^{\frac{a+b}{1-a}},
$$

(11)

with $t' - t'_o \in [0, t_o/(1-a))$ if $a \in (1, \infty)$ and $t' - t'_o \in [0, \infty)$ if $a \in (-\infty, 0) \cup (0, 1)$.

In Section 2, we show how to derive the time-dependent $TO$ functions, these forming the basic solutions to the problem. We describe how the two Cases are subdivided into different (sub)classes of solutions, depending on the different values of the parameters $a$ and $b$. (The same is done later for the $TM$ and $TQ$ solutions. In Section 3, we describe the $TO$ solutions, listing all the function solutions for the different (sub)classes in an Appendix. (Two examples are given.) The same is done for the $TM$ and $TQ$ functions. With these functions we compute, in Section 4, the time-dependent Lie symmetries that form the bases of the oscillator algebras.

As shown in Ref. [3] (and also demonstrated in Ref. [4]), the explicit calculations of the number, coherent, and squeezed states is a straight-forward substitution of the respective time-dependent functions into the NS, CS, and SS results for the three ($TQ$, $TM$, $TO$) systems. In Section 5 we review this procedure. [We refer the interested reader to Refs [3, 4] for the equations necessary to implement the procedure, using the functions given in an Appendix.]
In Section 6, we give the expectation values for \(\langle x \rangle\) and \(\langle p \rangle\), and discuss the associated classical motions. We also give the squeezed-state uncertainties and uncertainty products.

For the condition \(a \geq 0\) and \(b \geq 0\), Kim [5] has partially studied the TM system, obtaining the number eigenstates in terms of implicit quantities. The greater power of our formalism, which starts from a TO equation, is that it allows a general calculation of the oscillator algebras, the number states, CS, SS, and expectation values for all three (TQ, TM, TO) Schrödinger systems. This entails the complication of dealing with the various regimes for \(a\) and \(b\) in the transformation \(TM \rightarrow TO\). However, when dealing with \(TM\), the solutions are clear.

2 HOW TO OBTAIN THE TQ TIME-DEPENDENT FUNCTIONS

2.1 The Differential Equations for Cases 1 and 2

As demonstrated in Refs. [3, 4] (also see Ref. [2]), to obtain the time-dependent Lie symmetries for a TO Schrödinger equation, we must solve the ordinary differential equation

\[
\ddot{\gamma} + 2g^{(2)}(t')\dot{\gamma} = 0,
\]

with the particular \(g^{(2)}(t')\) being studied. First find two real solutions, \(\gamma_1(t')\) and \(\gamma_2(t')\), of

\[
W_{t'}(\gamma_1, \gamma_2) = 1.
\]

(The subscript on the Wronskian indicates the variable of differentiation.) The Wronskian of the solutions of a second-order differential equation of the form (12) is always a constant [6] and we choose that constant to be unity. The motivation for this will be made clear later. We refer to this process as normalization with respect to the Wronskian (13).

To complete the determination of the basis operators for the TO-os(1) algebras, we then need to find complex solutions, \(\xi(t')\) and \(\bar{\xi}(t')\), to Eq. (12) such that their Wronskian satisfies

\[
W_{t'}(\xi, \bar{\xi}) = \dot{\xi}\bar{\xi} - \dot{\bar{\xi}}\xi = -i.
\]

This is accomplished by writing \(\xi(t')\) in terms of the real solutions:

\[
\xi(t') = \sqrt{\frac{1}{2}} (\gamma_1(t') + i\gamma_2(t')).
\]
The fact that $\gamma_1$ and $\gamma_2$ satisfy Eq. (13) guarantees that the complex $\xi$ and $\bar{\xi}$ satisfy Eq. (14).

Returning to the main cases, their differential equations (12) are:

**Case 1:** $a = 1$. Together, Eqs. (10) and (12) yield

$$\ddot{\gamma} + \omega^2 \exp \left[ \frac{1+b}{t'_o} (t' - t'_o) \right] \gamma = 0,$$

(16)

where $t' - t'_o \in [0, \infty)$. Three classes of solutions arise depending on whether: $(b+1) \{>, =, <\} 0$. As an example, below we will give the derivation of the $TO$ functions for the class $b + 1 > 0$.

**Case 2:** $a \neq 1$. When we combine Eqs. (11) and (12), we obtain the differential equation

$$\ddot{\gamma} + \omega^2 \left[ 1 + \frac{1-a}{t'_o} (t' - t'_o) \right]^{\frac{a+b}{1-a}} \gamma = 0,$$

(17)

where $(t' - t'_o) \in [0, \frac{b-a}{a-1})$ if $a \in (1, \infty)$ and $(t' - t'_o) \in [0, \infty)$ if $a \in (-\infty, 1)$.

Note that when $a = 0$, then $\nu = 0$ and the unitary transformation (6) from $TQ$ to $TM$ is the identity. Further, the transformation from $TM$ to $TO$ is also unity, i.e., $t' = t$. Thus, here we are already dealing with a $TM$-type equation and so will not consider $a = 0$ further.

### 2.2 Classification of all the Solutions

To review, for this problem the $TO$ Schrödinger equation for a particular system depends upon the powers $a$ and $b$ in the Hamiltonian. The problem naturally divides into two cases: Case 1 for which $a = 1$ and Case 2 for which $a \neq 1$. Once we have identified which Case the $TO$ equation belongs to, further division into classes depends upon the power $b$. In our system notation, we denote this by the first two entries in a symbol \{a; b\}.

For Case 1 ($a = 1$), we identified three classes of solutions depending upon the value or range of values for the power $b+1$: $(b+1) \{>, =, <\} 0$. We have listed these classes of solutions in the first column of Table 1. We sometimes use the union

$$\{1; \neq -1\} = \{1; (-\infty, -1)\} \cup \{1; (-1, \infty)\},$$

(18)

to indicate that a particular result holds for both subsystems $\{1; (-\infty, -1)\}$ and $\{1; (-1, \infty)\}$.

For Case 2 ($a \neq 1$), we identify three classes of systems in solving Eq. (17). Given $a$, the
value of \( b \) falls into one of three possible classes: \( b \{>,=,<\} (a - 2) \). The third class in Table 1, corresponding to \( b = a - 2 \), can be further partitioned into six subclasses depending upon the relationship \( \{>,=,<\} \) between \( t_o \) and \( \frac{|1-a|}{2\omega} \) and the sign of \( 1-a \). Therefore, in our notation, we place at the end other conditions or comments as needed to specify the system. For example, we write \( \{ \neq 1; a - 2; t_o < \frac{|1-a|}{2\omega}; \pm; \} \), where \( \pm \) refers to the sign of \( 1-a \). In the second column of Table 1, we list the notation for the 8 possible systems of solutions for Case 2. Occasionally, we use the union

\[
\{ \neq 1; \neq a - 2; \} = \{ \neq 1; (\infty, a - 2); \} \cup \{ \neq 1; (a - 2, \infty); \}, 
\]

(19)
to indicate that a result applies to both subsystems \( \{ \neq 1; (\infty, a - 2); \} \) and \( \{ \neq 1; (a - 2, \infty); \} \).

[Appendix A of Ref. [4] is useful in obtaining some of the solutions of Case 2. Appendix A of this paper gives some important properties of Bessel functions.]

### 3 THE TIME-DEPENDENT FUNCTIONS

#### 3.1 The \( TO \) Functions

In Appendix B, Table B-1, we list, according to the classification scheme of Table 1, the complex solutions, \( \xi(t') \), and their derivatives, \( \dot{\xi}(t') \). This is done for all the classes and subclasses we have described; i.e., for each \( TO \) system described by a Schrödinger equation of the type (8).
The recursion relations listed in Appendix A are useful in the construction of the derivatives, $\dot{\xi}$. In addition, in Table B-2, we give the real functions $\phi_3(t')$ [see Eq. (21)] and its derivatives $\dot{\phi}_3(t')$ and $\ddot{\phi}_3(t')$. The solution $\bar{\xi}(t')$ and its derivative $\bar{\xi}(t')$ can be obtained from $\xi(t')$ and $\dot{\xi}(t')$ by complex conjugation. These are all the functions that are required to specify the operators that form a basis for the oscillator algebra, $os(1)$, associated with Eq. (8).

**TO Functions for a Specific Class:** \{a; b\} = \{1; b + 1 > 0\}: Now we give a demonstration of the method for a specific class of Case 1: \{a = 1; b + 1 > 0\}. [There are also separate classes \{a = 1; b + 1 = 0\} and \{a = 1; b + 1 < 0\}.] To begin, set

$$w(s) = \gamma(t') \quad \sigma = \frac{2\omega t_o}{b + 1} \exp \left[\frac{b + 1}{2t_o}(t' - t'_o)\right].$$

Eq. (16) can then be transformed into

$$\sigma^2 \frac{d^2 w}{d\sigma^2} + \sigma \frac{dw}{d\sigma} + \sigma^2 w = 0,$$

which is Bessel’s equation with zero eigenvalue. It has real solutions

$$w_1(\sigma) = J_0(\sigma), \quad w_2(\sigma) = Y_0(\sigma),$$

where $J_n$ and $Y_n$ are Bessel functions of the first and second kind. (See Appendix A.) When $b + 1 > 0$, the variable $\sigma \in [\frac{2\omega t_o}{b + 1}, \infty)$.

We take, for the real solutions of Eq. (16),

$$\gamma_1(t') = C_1 J_0(\sigma), \quad \gamma_2(t') = C_2 Y_0(\sigma).$$

The constants $C_1$ and $C_2$ are chosen so that the Wronskian is

$$W'(\gamma_1, \gamma_2) = \gamma_1 \dot{\gamma}_2 - \gamma_1 \gamma_2 = C_1 C_2 \frac{d\sigma}{dt'} W(\sigma, J_0, Y_0) = C_1 C_2 \left(\frac{b + 1}{\pi t_o}\right) = 1.$$

[For the Wronskian $W(\sigma, J_0, Y_0)$, see Eq. (23) in Appendix A.] Taking $C_1 = C_2$, we obtain

$$C_1 = \sqrt{\frac{\pi t_o}{b + 1}} = C_2,$$

$$\gamma_1(t') = \sqrt{\frac{\pi t_o}{b + 1}} J_0(\sigma), \quad \gamma_2(t') = \sqrt{\frac{\pi t_o}{b + 1}} Y_0(\sigma),$$

where $J_n$ and $Y_n$ are Bessel functions of the first and second kind. (See Appendix A.) When $b + 1 > 0$, the variable $\sigma \in [\frac{2\omega t_o}{b + 1}, \infty)$.

We take, for the real solutions of Eq. (16),

$$\gamma_1(t') = C_1 J_0(\sigma), \quad \gamma_2(t') = C_2 Y_0(\sigma).$$

The constants $C_1$ and $C_2$ are chosen so that the Wronskian is

$$W'(\gamma_1, \gamma_2) = \gamma_1 \dot{\gamma}_2 - \gamma_1 \gamma_2 = C_1 C_2 \frac{d\sigma}{dt'} W(\sigma, J_0, Y_0) = C_1 C_2 \left(\frac{b + 1}{\pi t_o}\right) = 1.$$

[For the Wronskian $W(\sigma, J_0, Y_0)$, see Eq. (23) in Appendix A.] Taking $C_1 = C_2$, we obtain

$$C_1 = \sqrt{\frac{\pi t_o}{b + 1}} = C_2,$$

$$\gamma_1(t') = \sqrt{\frac{\pi t_o}{b + 1}} J_0(\sigma), \quad \gamma_2(t') = \sqrt{\frac{\pi t_o}{b + 1}} Y_0(\sigma),$$

where $J_n$ and $Y_n$ are Bessel functions of the first and second kind. (See Appendix A.) When $b + 1 > 0$, the variable $\sigma \in [\frac{2\omega t_o}{b + 1}, \infty)$.
where we have written $|b+1|$ for convenience.

A complex solution is obtained from Eqs. (26) and (15):

$$\xi(t') = \sqrt{\frac{\pi t_o}{2|b+1|}} (J_0(\sigma) + iY_0(\sigma)) = \sqrt{\frac{\pi t_o}{2|b+1|}} H_0^{(1)}(\sigma),$$

where $H_0^{(1)}(\sigma)$ is a Hankel function. When the argument of a Hankel function is real,

$$H_j^{(2)} = \overline{H_j^{(1)}},$$

for all real $j$. Hence, we write

$$\bar{\xi}(t') = \sqrt{\frac{\pi t_o}{2|b+1|}} \overline{H_0^{(1)}}(\sigma).$$

[Eq. (24) in Appendix A is helpful in establishing that Eqs. (27) and (29) satisfy Eq. (14).]

We can obtain $\dot{\xi}(t')$ from $\xi(t')$. Then from Eq. (21) we have the functions, $\phi_j, j = 1,2,3$,

$$\phi_1(t') = \xi^2(t'), \quad \phi_2(t') = \xi^2(t'), \quad \phi_3(t') = 2\xi(t')\bar{\xi}(t').$$

This gives us the TO solutions for this particular class.

A Special System of TO Functions: $\{a; b\} = \{\neq 1; -a; \}$: The systems $\{\neq 1; -a; \}$ are of special interest. When $a \in (-\infty, 1)$, we have the systems $\{(-\infty, 1); -a; \} \subset \{\neq 1; (a-2, \infty); \}$ and when $a \in (1, \infty)$, we have $\{(1, \infty); -a; \} \subset \{\neq 1; (-\infty, a - 2); \}$. Therefore, the complex solution, $\xi(t')$, can be read from Table B-1:

$$\{(-\infty, 1); -a; \} : \quad \xi(t') = \frac{\pi t_o}{\sqrt{4|1-a|}} \sqrt{\bar{H}_1^{(1)}}(\tau),$$

$$\{(1, \infty); -a; \} : \quad \xi(t') = \frac{\pi t_o}{\sqrt{4|1-a|}} \sqrt{\bar{H}_1^{(1)}}(\tau),$$

$$H_{\frac{1}{2}}^{(1)}(\tau) = -i \left( \frac{2}{\pi \tau} \right)^{\frac{1}{2}} e^{i\tau}, \quad \bar{H}_{\frac{1}{2}}^{(1)}(\tau) = i \left( \frac{2}{\pi \tau} \right)^{\frac{1}{2}} e^{-i\tau}, \quad \tau = \frac{\omega t_o}{|1-a|} v.$$  

We can perform our calculations with these functions for the systems $\{\neq 1; -a; \}$ or it is more constructive to observe that when $b = -a$, the differential equation (17) reduces to a simple harmonic oscillator equation which has the complex solutions

$$\xi(t') = \sqrt{\frac{1}{2\omega}} e^{i\omega(t' - t_o)},$$
and $\tilde{\xi}(t')$, the same solutions we use for the system \{1;−1\}. The solutions (31) and (32) reduce to Eq. (34) up to a complex constant of modulus 1. It is important to recognize that normalization with respect to the Wronskian (14) can only be carried out up to a complex constant of modulus 1. We prefer to use the function (34) and its complex conjugate.

The real function $\phi_3(t')$, and its derivatives, $\dot{\phi}_3(t')$ and $\ddot{\phi}_3(t')$, are

$$\phi_3(t') = \frac{1}{\omega}, \quad \dot{\phi}_3(t') = \ddot{\phi}_3(t') = 0. \quad (35)$$

See Table B-2 for the system \{1;−1;\}.

3.2 The TM Functions

For TM and TQ systems, the distinction between Case 1 and Case 2 is no longer appropriate. So, there is a new set of ranges for the powers $a$ and $b$ that define the TM classes and subclasses. The TM functions for the 6 classes of TM systems and their subclasses can be obtained from the composition of the appropriate TO-functions with the mapping $t'(t)$ in Eqs. (10) for $a = 1$ and (11) for $a \neq 1$. [See Eqs. (1)-95,96.]

The parameterization of the TM classes and subclasses is shown in the last column of Table 1. [When it is not clear from the context, we shall always prefix the system designations in Table 1 with the symbol indicating which of the three Schrödinger equations we are dealing with. For example, TO-{1;−1;} and TM-{1;−1;}.]

The end results are the functions $\hat{\xi}(t)$ and $\tilde{\xi}(t)$ given in Appendix B, Table B-3, and the functions $\hat{\phi}_3(t)$, $\tilde{\phi}_3(t)$, and $\hat{\phi}_3(t)$ given in Table B-4.

The Special System of (now) TM Functions: \{a;b\} = \{≠ 1;−a;\}: For the systems \{≠ 1;−a;\}, we have

$$\hat{\xi}(t) = \sqrt{\frac{i}{\omega}} e^{i\frac{\pi}{2a}} \left[ (-i)^{1-a} - 1 \right], \quad \tilde{\xi}(t) = i\omega \sqrt{\frac{i}{\omega}} e^{i\frac{\pi}{2a}} \left[ (-i)^{1-a} - 1 \right], \quad (36)$$

$$\hat{\phi}_3(t) = \frac{1}{\omega}, \quad \tilde{\phi}_3(t) = \hat{\phi}_3(t) = 0. \quad (37)$$

Compare to the functions for the system \{1;−1;\} in Tables B-3 and B-I4.
3.3 The \(TQ\) Functions

Recall, as listed in Table 1, that the \(TQ\) functions have the same class structure as the \(TM\) functions. These \(TQ\) functions can be computed from the \(TM\) functions using Eqs. (3)-41 to (3)-44. In Appendix B, Table B-5, we have listed \(\Xi_P\) and \(\Xi_X\). In Table B-6, we give the coefficients \(C_{3,T}, C_{3,D},\) and \(C_{3,X^2}\) for each of the 6 classes of systems under consideration.

The Special System of (now) \(TQ\) Functions: \(\{a; b;\} = \{\neq 1; -a;\}\): For the systems \(\{\neq 1; -a;\}\), the functions \(\Xi_P\) and \(\Xi_X\) are

\[
\Xi_P(t) = \sqrt{\frac{i}{2\omega}} \left( \frac{t}{t_o} \right)^{\frac{a}{2}} e^{i \frac{\omega t_o}{1-a}} \left[ \left( \frac{t}{t_o} \right)^{1-a} - 1 \right], \quad \Xi_X(t) = i\omega \sqrt{\frac{i}{2\omega}} \left( \frac{t}{t_o} \right)^{\frac{a}{2}} e^{i \frac{\omega t_o}{1-a}} \left[ \left( \frac{t}{t_o} \right)^{1-a} - 1 \right],
\]

while the coefficients \(C_{3,T}, C_{3,D},\) and \(C_{3,X^2}\) are

\[
C_{3,T} = \frac{1}{\omega} \left( \frac{t}{t_o} \right)^a, \quad C_{3,D} = \frac{a}{2\omega t_o} \left( \frac{t}{t_o} \right)^{a-1}, \quad C_{3,X^2} = 0.
\]

3.4 Initial Values of the Functions

In subsequent calculations, we shall need initial values for time-dependent functions. When \(t' = t'_o\) or \(t = t_o\), we identify the initial value of the function by a superscript \(^o\), such as \(\xi^o\), for example. Recall that for this paper, we have assumed that \(t_o > 0\).

4 THE OSCILLATOR SUBALGEBRAS

For each class of system, described by the \(TM\) Schrödinger equation \(\mathfrak{S}\) and its related \(TQ\) and \(TO\) Schrödinger equations, \(\mathfrak{T}\) and \(\mathfrak{S}\), respectively, there is an oscillator algebra. All these oscillator algebras are isomorphic.

For \(TO\) systems, one obtains the operators by substituting the functions of Tables B-1 and B-2 into Eq. (3)-18 for \(\hat{J}_{3\pm}\), and into (3)-17 for \(\hat{M}_3\). For \(TM\) systems, one obtains the operators by substituting the functions of Tables B-3 and B-4 into Eq. (3)-31 for \(\hat{J}_{2\pm}\), and into Eq. (3)-30 for \(\hat{M}_2\). For \(TQ\) systems, one obtains the operators by substituting the the functions of Tables B-5 and B-6 into Eq. (3)-40 for \(\hat{J}_{1\pm}\) and \(\hat{M}_1\).
The \( os(1) \) Operators for \( TM\{-\neq 0; (a - 2, \infty)\} \): Because of the large number of possible systems, we cite only one example, the \( os(1) \) basis operators for \( TM\{-\neq 0; (a - 2, \infty)\} \). These operators are obtained by substituting the appropriate functions in Tables B-3 and B-4 (with \( \tau \) defined there) into Eqs. (33-38, 46-52),

\[
\hat{M}_2 = \frac{\pi t}{|b-a+2|} \left( \frac{t}{t_o} \right) H^{(1)}_\frac{1}{q} (\tau) \bar{H}^{(1)}_\frac{1}{q} (\tau) T \\
-\frac{\pi}{4} \tau \left[ H^{(1)}_{\frac{1}{q}-1} (\tau) \bar{H}^{(1)}_{\frac{1}{q}} (\tau) + H^{(1)}_{\frac{1}{q}} (\tau) \bar{H}^{(1)}_{\frac{1}{q}-1} (\tau) \right] D \\
+ \frac{\pi |b-a+2|}{2t_o} \left( \frac{t}{t_o} \right)^{a-1} \tau^2 \left[ H^{(1)}_{\frac{1}{q}-1} (\tau) \bar{H}^{(1)}_{\frac{1}{q}} (\tau) - H^{(1)}_{\frac{1}{q}} (\tau) \bar{H}^{(1)}_{\frac{1}{q}-1} (\tau) \right] X^2, \\
\]

(40)

\[
\hat{J}_2^- = i \sqrt{\frac{\pi t_o}{2|b-a+2|}} \left( \frac{t}{t_o} \right)^{\frac{1-a}{2}} \left\{ H^{(1)}_\frac{1}{q} (\tau) P - \frac{|b-a+2|}{2t_o} \left( \frac{t}{t_o} \right)^{a-1} H^{(1)}_{\frac{1}{q}-1} (\tau) X \right\}, \\
\]

(41)

\[
\hat{J}_2^+ = i \sqrt{\frac{\pi t_o}{2|b-a+2|}} \left( \frac{t}{t_o} \right)^{\frac{1-a}{2}} \left\{ -\bar{H}^{(1)}_\frac{1}{q} (\tau) P + \frac{|b-a+2|}{2t_o} \left( \frac{t}{t_o} \right)^{a-1} \bar{H}^{(1)}_{\frac{1}{q}-1} (\tau) X \right\}. \\
\]

(42)

One can show that these generators satisfy the \( os(1) \) Lie algebra commutation relations

\[
[\hat{M}_2, \hat{J}_{2\pm}] = \pm \hat{J}_{2\pm}, \quad [\hat{J}_{2-}, \hat{J}_{2+}] = I. \\
\]

(43)

5 Wave Functions

For time-dependent systems of this type, explicit space-time representations of the wave functions are not necessary for the computation of expectation values for position and momentum, their uncertainties and uncertainty products. Nevertheless, they are of independent interest and number-state, coherent-state, and squeezed-state wave functions may be constructed for our problem from our results in Ref. 3 and the tables in Appendix B.

**Number States:** In Ref. 3 we refer the reader to Eqs. (3-51) for \( TO \) number states, (3-58) for \( TM \) number states, and (3-63) for \( TQ \) number states.

**Coherent States:** Eqs. (3-66), (3-68), and (3-74) are the respective coherent states for \( TO, TM, \) and \( TQ \) systems.
Squeezed States: For squeezed states, we have Eqs. (3-91) for TO systems, (3-96) for TM systems, and (3-103) for TQ systems.

6 EXPECTATION VALUES

6.1 The Dynamical Variables $\langle x \rangle$ and $\langle p \rangle$

Elsewhere, formulas were obtained for the expectation values of position and momentum from time-dependent quadratic Hamiltonians for general TQ, TM, and TO systems, respectively, in Eqs. (86), (87) and (88) of Ref. [2], respectively. These results were given in terms of general time-dependent functions of the type discussed in the present Section 2.

In particular, for our present problem the time-dependent $\xi$ and other functions are given in Appendix B. [Note: For the TQ system, the results are given in terms of the $\Xi_P(t) = \hat{\xi}(t)e^{\nu}$ and $\Xi_X(t) = \hat{\xi}(t)e^{-\nu}$ of Eq. (3-41).] Therefore, depending on the complicated various regimes of the parameters $a$, $b$, $\omega$, and $t_o$, we can use Tables B-1, B-3, and B-5, to obtain $\langle x \rangle$ and $\langle p \rangle$ for the TO, TM, and TQ systems, respectively. The results of our computations of $\langle x \rangle$ and $\langle p \rangle$ for all the various (sub)systems, are displayed in Tables 2 to 7.

6.2 The Classical Motion

As before, the coherent-state and squeezed-state expectation values $\langle x \rangle$ and $\langle p \rangle$ should obey the classical Hamiltonian equations of motion:

$$\dot{x} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial x}. \quad (44)$$

Now, the classical Hamiltonians associated with our Schrödinger equations are

$$\begin{align*}
\text{TO} & : H = \frac{1}{2}p^2 + g^{(2)}(t')x^2, \\
\text{TM} & : \hat{H} = \frac{1}{2} \left( \frac{t_o}{t} \right)^a p^2 + \frac{1}{2} \omega^2 \left( \frac{t}{t_o} \right)^b x^2, \\
\text{TQ} & : H = \frac{1}{2}p^2 + \frac{a}{2t}xp + \frac{1}{2} \omega^2 \left( \frac{t}{t_o} \right)^{b-a} x^2,
\end{align*}$$

where $g^{(2)}(t')$ is given in Eqs. (11) and (11). Applying these to the classical equations of motion (44) one obtains (“dot” is $d/dt'$ or $d/dt$ as appropriate)
Table 2. \(\langle x(t') \rangle\) for the TO systems. Variables and parameters are defined as follows: \(\sigma = \frac{2\omega t_o}{b+1} \exp \left[ \frac{b+1}{2t_o} \right]; v = 1 + \frac{1-a}{t_o} (t' - t'_o); \tau = \frac{2\omega t_o}{|b-a+2|} t'^{1/2}; \sigma_o = \frac{2\omega t_o}{b+1}; \tau_o = \frac{2\omega t_o}{|b-a+2|}; q = \frac{b-a+2}{1-a}, \) and \(\Delta^2 = \left| 1 - \frac{4q^2}{(1-a)^2} \right|.

| System | \(\langle x(t') \rangle\) |
|--------|------------------|
| \{1; \neq -1\}; \(p_o \frac{\sigma}{\sqrt{\pi}} [J_{0}(\sigma)J_{0}(\sigma_o) - J_{0}(\sigma)Y_{0}(\sigma_o)] + x_o \frac{\sigma}{\sqrt{\pi}} [J_{0}(\sigma)Y_{-1}(\sigma_o) - Y_{0}(\sigma)J_{-1}(\sigma_o)\] |
| \{1; -1\}; \(p_o \frac{\sigma}{\sqrt{\pi}} \sin \omega(t' - t'_o) + x_o \cos \omega(t' - t'_o)\) |
| \{\neq 1; \neq -1\}; \(p_o \frac{\sigma}{\sqrt{\pi}} \sqrt{\pi} \sinh(\frac{\Delta}{2} \ln v) + x_o \frac{1}{\sqrt{\pi}} [\Delta \cosh(\frac{\Delta}{2} \ln v) - \sinh(\frac{\Delta}{2} \ln v)\] |
| \{\neq 1; \neq -2\}; \(p_o \frac{2\omega}{(b-1)\Delta} \sqrt{\pi} \sinh(\frac{\Delta}{2} \ln v) + x_o \frac{1}{\sqrt{\pi}} \ln v + x_o \sqrt{\pi} \ln v\) |
| \{\neq 1; \neq -2\}; \(p_o \frac{2\omega}{(b-1)\Delta} \sqrt{\pi} \sin(\frac{\Delta}{2} \ln v) + x_o \frac{1}{\sqrt{\pi}} \ln v + x_o \sqrt{\pi} \ln v\) |

Table 3. \(\langle p(t') \rangle\) for the TO systems. Variables and parameters are defined as follows: \(\sigma = \frac{2\omega t_o}{b+1} \exp \left[ \frac{b+1}{2t_o} \right]; v = 1 + \frac{1-a}{t_o} (t' - t'_o); \tau = \frac{2\omega t_o}{|b-a+2|} t'^{1/2}; \sigma_o = \frac{2\omega t_o}{b+1}; \tau_o = \frac{2\omega t_o}{|b-a+2|}; q = \frac{b-a+2}{1-a}, \) and \(\Delta^2 = \left| 1 - \frac{4q^2}{(1-a)^2} \right|.

| System | \(\langle p(t') \rangle\) |
|--------|------------------|
| \{1; \neq -1\}; \(p_o \frac{\tau}{\sqrt{\pi}} [J_{-1}(\sigma)J_{0}(\sigma_o) - J_{-1}(\sigma)Y_{0}(\sigma_o)] + x_o \frac{\tau}{\sqrt{\pi}} [J_{-1}(\sigma)Y_{-1}(\sigma_o) - Y_{0}(\sigma)J_{-1}(\sigma_o)\] |
| \{1; -1\}; \(p_o \cos \omega(t' - t'_o) - x_o \omega \sin \omega(t' - t'_o)\) |
| \{\neq 1; \neq -2\}; \(p_o \frac{\tau}{\sqrt{\pi}} \left[ \frac{\Delta}{2} \sinh(\frac{\Delta}{2} \ln v) + \sin(\frac{\Delta}{2} \ln v) \right] - x_o \frac{\Delta^2}{(1-a)\sqrt{\pi}} \sin(\frac{\Delta}{2} \ln v)\) |
| \{\neq 1; \neq -2\}; \(p_o \frac{\tau}{\sqrt{\pi}} \left[ (1 + \frac{\Delta}{2} \ln v) - x_o \frac{\Delta^2}{(1-a)\sqrt{\pi}} \ln v \right]\) |
| \{\neq 1; \neq -2\}; \(p_o \frac{\tau}{\sqrt{\pi}} \left[ \Delta \cos(\frac{\Delta}{2} \ln v) + \sin(\frac{\Delta}{2} \ln v) \right] - x_o \frac{\Delta^2}{(1-a)\sqrt{\pi}} \ln v\) |

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Table 4. \( \langle x(t) \rangle \) for the TM systems. Variables and parameters are defined as follows: \( \tau = \frac{2\omega t_o}{|b-a+2|} \left( \frac{1}{t_o} \right)^{b-a+2} \); \( \chi = \frac{1}{2}(1-a) \ln \frac{t}{t_o} \); \( \tau_o = \frac{2\omega t_o}{|b-a+2|} \); \( q = \frac{b-a+2}{1-a} \), and \( \Delta^2 = \left| 1 - \frac{4q^2 t_o^2}{(1-a)^2} \right| \).

| System                                      | \( \langle x(t') \rangle \)                                                                 |
|---------------------------------------------|--------------------------------------------------------------------------------------------|
| \( \{ \neq 0; \neq a - 2 \} \)             | \( p_o \frac{\pi a}{\tau_o} \left( \frac{1}{t_o} \right)^{b-a+2} \left[ Y_{\frac{b-a+2}{2}}(\tau) J_{\frac{b-a+2}{2}}(\tau_o) - J_{\frac{b-a+2}{2}}(\tau) Y_{\frac{b-a+2}{2}}(\tau_o) \right] \\ + x_o \frac{\pi a}{\tau_o} \left( \frac{1}{t_o} \right)^{b-a+2} \tau \left[ J_{\frac{b-a+2}{2}}(\tau) Y_{\frac{b-a+2}{2}}(\tau_o) - Y_{\frac{b-a+2}{2}}(\tau) J_{\frac{b-a+2}{2}}(\tau_o) \right] \) |
| \( \{ 1, -1 \} \)                          | \( p_o \frac{\pi a}{\tau_o} \left( \frac{1}{t_o} \right)^{b-a+2} \frac{1}{2} \sinh \left( \Delta \chi \right) + x_o \frac{\pi a}{\tau_o} \left( \frac{1}{t_o} \right)^{b-a+2} \frac{1}{2} \chi \) |
| \( \{ \neq 0, 1; a - 2; t_o < \frac{|1-a|}{2a} \} \) | \( p_o \frac{\pi a}{\tau_o} \left( \frac{1}{t_o} \right)^{b-a+2} \frac{1}{2} \sinh \left( \Delta \chi \right) + x_o \frac{\pi a}{\tau_o} \left( \frac{1}{t_o} \right)^{b-a+2} \frac{1}{2} \left( 1 - \chi \right) \) |
| \( \{ \neq 0, 1; a - 2; t_o > \frac{|1-a|}{2a} \} \) | \( p_o \frac{\pi a}{\tau_o} \left( \frac{1}{t_o} \right)^{b-a+2} \frac{1}{2} \sinh \left( \Delta \chi \right) + x_o \frac{\pi a}{\tau_o} \left( \frac{1}{t_o} \right)^{b-a+2} \frac{1}{2} \left[ \Delta \cos \left( \Delta \chi \right) - \sin \left( \Delta \chi \right) \right] \) |

Table 5. \( \langle p(t) \rangle \) for the TM systems. Variables and parameters are defined as follows: \( \tau = \frac{2\omega t_o}{|b-a+2|} \left( \frac{1}{t_o} \right)^{b-a+2} \); \( \chi = \frac{1}{2}(1-a) \ln \frac{t}{t_o} \); \( \tau_o = \frac{2\omega t_o}{|b-a+2|} \); \( q = \frac{b-a+2}{1-a} \), and \( \Delta^2 = \left| 1 - \frac{4q^2 t_o^2}{(1-a)^2} \right| \).

| System                                      | \( \langle p(t') \rangle \)                                                                 |
|---------------------------------------------|--------------------------------------------------------------------------------------------|
| \( \{ \neq 0; \neq a - 2 \} \)             | \( p_o \frac{\pi a}{\tau_o} \left( \frac{1}{t_o} \right)^{b-a+2} \frac{1}{2} \sinh \left( \Delta \chi \right) + x_o \frac{\pi a}{\tau_o} \left( \frac{1}{t_o} \right)^{b-a+2} \frac{1}{2} \chi \) |
| \( \{ 1, -1 \} \)                          | \( p_o \cos \left( \omega t_o \ln \frac{1}{t_o} \right) - x_o \omega \sin \left( \omega t_o \ln \frac{1}{t_o} \right) \) |
| \( \{ \neq 0, 1; a - 2; t_o < \frac{|1-a|}{2a} \} \) | \( p_o \frac{\pi a}{\tau_o} \left( \frac{1}{t_o} \right)^{b-a+2} \frac{1}{2} \left( 1 + \chi \right) - x_o \frac{\pi a}{\tau_o} \left( \frac{1}{t_o} \right)^{b-a+2} \frac{1}{2} \chi \) |
| \( \{ \neq 0, 1; a - 2; t_o < \frac{|1-a|}{2a} \} \) | \( p_o \frac{\pi a}{\tau_o} \left( \frac{1}{t_o} \right)^{b-a+2} \frac{1}{2} \left[ \Delta \cos \left( \Delta \chi \right) + \sin \left( \Delta \chi \right) \right] - x_o \frac{2\omega^2}{(1-a)^2} \left( \frac{1}{t_o} \right)^{b-a+2} \frac{1}{2} \sin \left( \Delta \chi \right) \) |
| \( \{ \neq 0, 1; a - 2; t_o > \frac{|1-a|}{2a} \} \) | \( p_o \frac{\pi a}{\tau_o} \left( \frac{1}{t_o} \right)^{b-a+2} \frac{1}{2} \left[ \Delta \cos \left( \Delta \chi \right) + \sin \left( \Delta \chi \right) \right] - x_o \frac{2\omega^2}{(1-a)^2} \left( \frac{1}{t_o} \right)^{b-a+2} \frac{1}{2} \sin \left( \Delta \chi \right) \) |

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Table 6. \( \langle x(t) \rangle \) for the TQ systems. Variables and parameters are defined as follows: \( \tau = \frac{2\omega t_o}{|b-a+2|} \left( \frac{t}{t_o} \right)^{\frac{b-a+2}{2}} \); \( \chi = \frac{1}{2} (1-a) \ln \left( \frac{t}{t_o} \right) \); \( \tau_o = \frac{2\omega t_o}{|b-a+2|} \); \( q = \frac{b-a+2}{1-a} \), and \( \Delta^2 = \left| 1 - \frac{4\omega^2 t_o^2}{(1-a)^2} \right| \).

| System                        | \( \langle x(t') \rangle \)                                                                 |
|-------------------------------|---------------------------------------------------------------------------------------------|
| \{ \neq 0; \neq a - 2; \}     | \( p_o \left( \frac{t}{t_o} \right)^{\frac{b-a+2}{2}} \left[ \frac{Y}{\pi} (\tau) J_{\frac{1}{2}} (\tau) - J_{\frac{1}{2}} (\tau) Y_{\frac{1}{2}} (\tau) \right] + x_o \left( \frac{t}{t_o} \right)^{\frac{b-a+2}{2}} \left[ J_{\frac{1}{2}} (\tau) Y_{\frac{1}{2}} (\tau) - J_{\frac{1}{2}} (\tau) Y_{\frac{1}{2}} (\tau) \right] \) |
| \{ 1; -1; \}                  | \( p_o \left( \frac{t}{t_o} \right)^{\frac{b-a+2}{2}} \sin \left( \omega t_o \ln \left( \frac{t}{t_o} \right) \right) + x_o \left( \frac{t}{t_o} \right)^{\frac{b-a+2}{2}} \cos \left( \omega t_o \ln \left( \frac{t}{t_o} \right) \right) \) |
| \{ \neq 0, 1; a - 2; \( t_o \) < \( \frac{1-a}{2\omega} \); \} | \( p_o \frac{2\omega t_o}{|1-a|\Delta} \left( \frac{t}{t_o} \right)^{\frac{b-a+2}{2}} \frac{\Delta}{\pi} \left[ \sin \left( \Delta \chi \right) + x_o \left( \frac{t}{t_o} \right)^{\frac{b-a+2}{2}} \left[ \sin \left( \Delta \chi \right) - \sin \left( \Delta \chi \right) \right] \right] \) |
| \{ \neq 0, 1; a - 2; \( t_o = \frac{1-a}{2\omega} \); \} | \( p_o \frac{2\omega t_o}{|1-a|\Delta} \left( \frac{t}{t_o} \right)^{\frac{b-a+2}{2}} \frac{\Delta}{\pi} \chi + x_o \left( \frac{t}{t_o} \right)^{\frac{b-a+2}{2}} (1-\chi) \) |
| \{ \neq 0, 1; a - 2; \( t_o > \frac{1-a}{2\omega} \); \} | \( p_o \frac{2\omega t_o}{|1-a|\Delta} \left( \frac{t}{t_o} \right)^{\frac{b-a+2}{2}} \frac{\Delta}{\pi} \left[ \sin \left( \Delta \chi \right) + x_o \left( \frac{t}{t_o} \right)^{\frac{b-a+2}{2}} \left[ \sin \left( \Delta \chi \right) - \sin \left( \Delta \chi \right) \right] \right] \) |

Table 7. \( \langle p(t) \rangle \) for the TQ systems. Variables and parameters are defined as follows: \( \tau = \frac{2\omega t_o}{|b-a+2|} \left( \frac{t}{t_o} \right)^{\frac{b-a+2}{2}} \); \( \chi = \frac{1}{2} (1-a) \ln \left( \frac{t}{t_o} \right) \); \( \tau_o = \frac{2\omega t_o}{|b-a+2|} \); \( q = \frac{b-a+2}{1-a} \), and \( \Delta^2 = \left| 1 - \frac{4\omega^2 t_o^2}{(1-a)^2} \right| \).

| System                        | \( \langle p(t') \rangle \)                                                                 |
|-------------------------------|---------------------------------------------------------------------------------------------|
| \{ \neq 0; \neq a - 2 \}      | \( p_o \left( \frac{t}{t_o} \right)^{\frac{b-a+2}{2}} \left[ \frac{Y}{\pi} (\tau) J_{\frac{1}{2}} (\tau) - J_{\frac{1}{2}} (\tau) Y_{\frac{1}{2}} (\tau) \right] + x_o \left( \frac{t}{t_o} \right)^{\frac{b-a+2}{2}} \left[ J_{\frac{1}{2}} (\tau) Y_{\frac{1}{2}} (\tau) - J_{\frac{1}{2}} (\tau) Y_{\frac{1}{2}} (\tau) \right] \) |
| \{ 1; -1; \}                  | \( p_o \left( \frac{t}{t_o} \right)^{\frac{b-a+2}{2}} \cos \left( \omega t_o \ln \left( \frac{t}{t_o} \right) \right) - x_o \omega \left( \frac{t}{t_o} \right)^{\frac{b-a+2}{2}} \sin \left( \omega t_o \ln \left( \frac{t}{t_o} \right) \right) \) |
| \{ \neq 0, 1; a - 2; \( t_o \) < \( \frac{1-a}{2\omega} \); \} | \( p_o \frac{1}{\Delta} \left( \frac{t}{t_o} \right)^{\frac{b-a+2}{2}} \frac{\Delta}{\pi} \left[ \sinh \left( \Delta \chi \right) + \sin \left( \Delta \chi \right) \right] - x_o \frac{2\omega^2 t_o}{|1-a|\Delta} \left( \frac{t}{t_o} \right)^{\frac{b-a+2}{2}} \sin \left( \Delta \chi \right) \) |
| \{ \neq 0, 1; a - 2; \( t_o = \frac{1-a}{2\omega} \); \} | \( p_o \frac{1}{\Delta} \left( \frac{t}{t_o} \right)^{\frac{b-a+2}{2}} \frac{\Delta}{\pi} \left( 1+\chi \right) - x_o \frac{2\omega^2 t_o}{|1-a|\Delta} \left( \frac{t}{t_o} \right)^{\frac{b-a+2}{2}} \chi \) |
| \{ \neq 0, 1; a - 2; \( t_o > \frac{1-a}{2\omega} \); \} | \( p_o \frac{1}{\Delta} \left( \frac{t}{t_o} \right)^{\frac{b-a+2}{2}} \frac{\Delta}{\pi} \left[ \sinh \left( \Delta \chi \right) + \sin \left( \Delta \chi \right) \right] - x_o \frac{2\omega^2 t_o}{|1-a|\Delta} \left( \frac{t}{t_o} \right)^{\frac{b-a+2}{2}} \sin \left( \Delta \chi \right) \) |

\[ TO : \quad \dot{x} = p, \quad \dot{p} = -2g^{(2)}(t')x, \]  
\[ TM : \quad \dot{x} = \left( \frac{t}{t_o} \right)^{a} p, \quad \dot{p} = -\omega^2 \left( \frac{t}{t_o} \right)^b x, \]
\[ TQ : \quad \dot{x} = p + \frac{a}{2t}x, \quad \dot{p} = -\omega^2 \left( \frac{t}{t_0} \right)^{b-a} x - \frac{a}{2t}p \] (50)

The reader can verify that all the expectation values in Tables 2 to 7 satisfy the equations of motion (48) - (50). This specifically demonstrates the general results for quadratic time-dependent Hamiltonians derived in [2].

By way of illustration, in Figures 1 and 2 we plot \(\langle x \rangle\) and \(\langle p \rangle\) as functions of time for two of the systems \(TM - \{1; b > -1; \omega = 2; t_o = 1\}\); namely, for \(b = -0.5\) and \(b = 1.0\). In Figure 1 we see that the envelope of the oscillation of \(\langle x \rangle\) decreases with time while in Figure 2 we see that the envelope of the oscillation of \(\langle p \rangle\) decreases with time. This “exchange” of maximum amplitude with time is a reflection of the coupling of \(\langle x \rangle\) and \(\langle p \rangle\) in the classical equations of motion [Eq. (49)]; i.e., the nonconservative nature of the forces. This is emphasized by the phase-space plot in Figure 3 where, regardless of the initial position and momentum, the trajectory in phase space is an oscillatory motion about the phase-space origin with \(\langle x \rangle\) becoming smaller as the amplitude of the oscillations in \(\langle p \rangle\) become increasingly large.

As seen above, given that we have kept \(\omega\) a constant, the frequency of the oscillation is higher the higher the value of \(b\). Contrariwise, if \(b\) is fixed, the frequency of oscillation in \(\langle x \rangle\) and \(\langle p \rangle\) increases with increasing \(\omega\).

### 6.3 Uncertainties

As with the expectation values in Section 6.1, uncertainties in position and momentum and the uncertainty products for all \(TQ\), \(TM\), and \(TO\) systems, respectively, can be calculated from the general equations (106), (107), and (109) - (111) of [2], respectively using the functions from the appropriate tables in Appendix B.

**An illustration for \(TM - \{1; \neq 1; \}\) systems:** The uncertainties are

\[
(\Delta x)^2 = \frac{\pi t_o}{4|b+1|} \left\{ \left[ (J_0^2(\sigma) - Y_0^2(\sigma) \right) \cos \theta - J_0(\sigma)Y_0(\sigma) \sin \theta \right] \sinh 2r \\
+ \left[ J_0^2(\sigma) + Y_0^2(\sigma) \right] \cosh 2r \right\}
\]

\[
(\Delta p)^2 = \frac{\pi |b+1|}{8t_o} \sigma \left\{ \left[ (J_{-1}^2(\sigma) - Y_{-1}^2(\sigma) \right) \cos \theta - J_{-1}(\sigma)Y_{-1}(\sigma) \sin \theta \right] \sinh 2r \\
+ \left[ J_{-1}^2(\sigma) + Y_{-1}^2(\sigma) \right] \cosh 2r \right\}. \tag{51}
\]
Figure 1: Plots of $\langle x \rangle$ versus time for the two systems: $TM - \{1; -0.5; \omega = 2; t_o = 1; x_o = 1, p_o = 1; \}$ (thin line) and $TM - \{1; 1; \omega = 2; t_o = 1; x_o = 1, p_o = 1; \}$ (thick line).

Figure 2: Plots of $\langle p \rangle$ versus time for the two systems: $TM - \{1; -0.5; \omega = 2; t_o = 1; x_o = 1, p_o = 1; \}$ (thin line) and $TM - \{1; 1; \omega = 2; t_o = 1; x_o = 1, p_o = 1; \}$ (thick line).
This means the uncertainty product is

\[(\Delta x)^2(\Delta p)^2 = \frac{1}{4} \left\{1 + \frac{\pi^2 a^2}{4} \left\{[J_{-1}(\sigma)J_0(\sigma) + Y_{-1}(\sigma)Y_0(\sigma)] \cosh 2r \\
+ [(J_0(\sigma)J_{-1}(\sigma) - Y_0(\sigma)Y_{-1}(\sigma)) \cos \theta \\
+ (J_0(\sigma)Y_{-1}(\sigma) + Y_0(\sigma)J_{-1}(\sigma)) \sin \theta] \sinh 2r \right\}^2 \right\}.
\] (52)

7 CONCLUSION

For this type of Schrödinger system, the solutions obtained here apparently form a complete analysis for all real powers, \(a, a \neq 0\), and \(b\). Further, the specific values of and/or ranges of \((a, b)\) for which specific solutions are valid was also determined. Thus, in toto, our results significantly extend the original work of Kim [5].
The algebraic methods employed to obtain these solutions are based on the existence of isomorphic Lie symmetry algebras for $TO_-, TM_-$, and $TQ$-type Schrödinger equations. Lie symmetry analysis presents a powerful technique. With these symmetries we constructed a complete set of discrete states as well as coherent states and squeezed states.

We emphasize again that the (time-dependent) discrete or number states are not eigenfunctions of the Hamiltonian. Yet further, the expectation values for $\langle x \rangle$ and $\langle p \rangle$ obtained in our analysis satisfy Hamilton’s equations of motion.

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APPENDIX A

We give relationships involving Bessel and Hankel functions that are useful in performing calculations. For details, we refer the reader to Lebedev [7]. Below, $z$ is the independent variable and the prime indicates differentiation by $z$. Wronskians are:

\[
W_z(J_\mu, Y_\mu) = J_\mu Y'_\mu - J'_\mu Y_\mu = \frac{2}{\pi z},
\]

\[
W_z(H^{(1)}_\mu, H^{(2)}_\mu) = H^{(1)}_\mu H'^{(2)}_\mu - H'^{(1)}_\mu H^{(2)}_\mu = -\frac{4i}{\pi z}.
\]

Let $F_\mu(z)$ be a generic symbol for a Bessel function of the first of second kind, or a Hankel function. Then, we have the following recursion relations:

\[
F_{\mu-1}(z) + F_{\mu+1}(z) = \frac{2\mu}{z} F_\mu(z),
\]

\[
2F'_\mu(z) = F_{\mu-1}(z) - F_{\mu+1}(z),
\]

\[
F'_{\mu}(z) = F_{\mu-1}(z) - \frac{\mu}{z} F'_{\mu}(z).
\]

The following are obtained by combining the Wronskians and recursion relations:

\[
J_\mu(z)Y_{\mu-1}(z) - J_{\mu-1}(z)Y_\mu(z) = \frac{2}{\pi z},
\]

\[
H^{(1)}_\mu(z)H^{(2)}_{\mu-1}(z) - H^{(1)}_{\mu-1}(z)H^{(2)}_\mu(z) = -\frac{4i}{\pi z}.
\]
APPENDIX B

Table B.1. $\xi(t')$ and $\dot{\xi}(t')$ for TO-systems. Variables and parameters are defined as follows: $\sigma = \frac{2v\epsilon t_0}{|b-a+2|}\exp\left(\frac{b-a+2}{2t_0}(t'-t_0')\right); \ v = 1 + \frac{1}{t_0}(t'-t_0'); \ \tau = \frac{2v\epsilon t_0}{|b-a+2|}\sqrt{\frac{q}{a}}; \ q = \frac{b-a+2}{1-\sigma}, \ \text{and} \ \Delta^2 = \left[1 - \frac{4\omega^2t'^2}{(1-a)^2}\right].$

| TO System | $\xi(t')$ |
|-----------|-----------|
| $\{1; (-1, \infty); \}$ | $\sqrt{\sqrt{\frac{\pi}{|b-a+2|}}H_0^{(1)}(\sigma)}$ |
| $\{1; (-\infty, -1); \}$ | $\sqrt{\sqrt{\frac{\pi}{|b-a+2|}}H_9^{(1)}(\sigma)}$ |
| $\{1; -1; \}$ | $\sqrt{\frac{\pi t_0}{|b-a+2|}}\sqrt{\frac{\pi}{\sigma^2}}(t'-t_0')$ |
| $\{\neq 1; (a-2, \infty); \}$ | $\frac{\pi t_0}{|b-a+2|}\sqrt{\frac{\pi}{\sigma^2}}H(1)(\tau)$ |
| $\{\neq 1; (-\infty, a-2); \}$ | $\frac{\pi t_0}{|b-a+2|}\sqrt{\frac{\pi}{\sigma^2}}H(1)(\tau)$ |
| $\{\neq 1; a-2; t_0 < \frac{|1-a|}{2\sigma}; \pm; \}$ | $\frac{t_0}{\sqrt{2|1-a|\Delta}}\sqrt{\sigma} e^{\pm i\ln v} \pm ie\ln v$ |
| $\{\neq 1; a-2; t_0 = \frac{|1-a|}{2\sigma}; \pm; \}$ | $\frac{t_0}{\sqrt{2|1-a|\Delta}}\sqrt{\sigma} e^{\pm i\ln v}$ |
| $\{\neq 1; a-2; t_0 > \frac{|1-a|}{2\sigma}; \pm; \}$ | $\frac{t_0}{\sqrt{2|1-a|\Delta}}\sqrt{\sigma} e^{\pm i\ln v}$ |

| TO System | $\dot{\xi}(t')$ |
|-----------|-----------|
| $\{1; (-1, \infty); \}$ | $\frac{i}{2} \sqrt{\frac{\pi}{|b-a+2|}}\sigma H(1)(\sigma)$ |
| $\{1; (-\infty, -1); \}$ | $-\frac{i}{2} \sqrt{\frac{\pi}{|b-a+2|}}\sigma H(1)(\sigma)$ |
| $\{1; -1; \}$ | $i\omega \sqrt{\frac{\pi}{|b-a+2|}}e^{i\omega(t'-t_0')}$ |
| $\{\neq 1; (a-2, \infty); \}$ | $\frac{i}{2} \sqrt{\frac{\pi}{|b-a+2|}}\sigma H(1)(\tau)$ |
| $\{\neq 1; (-\infty, a-2); \}$ | $-\frac{i}{2} \sqrt{\frac{\pi}{|b-a+2|}}\sigma H(1)(\tau)$ |
| $\{\neq 1; a-2; t_0 < \frac{|1-a|}{2\sigma}; \pm; \}$ | $\frac{1}{2} \sqrt{\frac{\pi}{|b-a+2|}}\sigma H(1)(\tau)$ |
| $\{\neq 1; a-2; t_0 = \frac{|1-a|}{2\sigma}; \pm; \}$ | $\sqrt{\frac{1-a}{2\sigma}}\frac{1}{2\sigma} \pm i \left(1 + \Delta e^{\pm i\ln v}\right)$ |
| $\{\neq 1; a-2; t_0 > \frac{|1-a|}{2\sigma}; \pm; \}$ | $\frac{1}{2} \sqrt{\frac{\pi}{|b-a+2|}}\sigma H(1)(\tau)$ |

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Table B-2. $\phi_3(t')$, $\dot{\phi}_3(t')$, and $\ddot{\phi}_3(t')$ for TO-systems. Variables and parameters are defined as follows $\sigma = \frac{2\omega t_o}{b+2} \exp \left[\frac{b+1}{2}(t' - t_o)\right]; \ u = 1 + \frac{1-a}{t_o} (t' - t_o); \ \tau = \frac{2\omega t_o}{[b-a+2]}q/2; q = \frac{b-a+2}{1-a}$, and $\Delta^2 = \left|1 - \frac{4\omega^2 t_o^2}{(1-a)^2}\right|.$

| TO System | $\phi_3(t')$ |
|-----------|-------------|
| $\{1; \neq -1;\}$ | $\frac{\pi t_o}{(b+1)} H_0^{1}(\sigma) \bar{H}_0^{1}(\sigma)$ |
| $\{1; -1;\}$ | $\frac{1}{b}$ |
| $\neq 1; \neq -1;\}$ | $\frac{\pi t_o}{(b-a+2)} H_1^{1}(\tau) H_1^{1}(\tau)$ |
| $\neq 1; -1;\}$ | $\frac{1}{2} \left(\frac{\pi t_o}{(b-a+2)} \right) (e^{\Delta \ln v} + e^{-\Delta \ln v})$ |
| $\neq 1; a - 2; t_o < \frac{[1-a]}{2v};\}$ | $\frac{1}{2} \left(\frac{\pi t_o}{(b-a+2)} \right) (1 + \ln^2 v)$ |
| $\neq 1; a - 2; t_o = \frac{[1-a]}{2v};\}$ | $\frac{2t_o}{(b-a+2)} v$ |

| TO System | $\dot{\phi}_3(t')$ |
|-----------|-------------|
| $\{1; (-1, \infty);\}$ | $\frac{2}{\pi} \sigma \left[H_1^{1}(\sigma) H_0^{1}(\sigma) + H_0^{1}(\sigma) \bar{H}_1^{1}(\sigma)\right]$ |
| $\{1; (-\infty, -1);\}$ | $0$ |
| $\neq 1; (-a, -1);\}$ | $\frac{2}{\pi} \sigma \left[H_1^{1}(\tau) H_1^{1}(\tau) + H_1^{1}(\tau) H_1^{1}(\tau)\right]$ |
| $\neq 1; (a - 2, \infty);\}$ | $\frac{1}{b} \left[(1 - \Delta)e^{-\Delta \ln s} + (1 + \Delta)e^ {\Delta \ln s}\right]$ |
| $\neq 1; a - 2; t_o < \frac{[1-a]}{2v};\}$ | $\frac{2}{(b-a+2)} (1 + \ln^2 s)$ |
| $\neq 1; a - 2; t_o = \frac{[1-a]}{2v};\}$ | $0$ |

| TO System | $\ddot{\phi}_3(t')$ |
|-----------|-------------|
| $\{1; a - 1;\}$ | $\frac{\pi t_o}{(b-a+2)} v^2 \left[H_1^{1}(\sigma) \bar{H}_1^{1}(\sigma) - H_0^{1}(\sigma) \bar{H}_0^{1}(\sigma)\right]$ |
| $\{1; -1;\}$ | $0$ |
| $\neq 1; \neq a - 2;\}$ | $\frac{\pi t_o}{(b-a+2)} v^2 \left[H_1^{1}(\tau) \bar{H}_1^{1}(\tau) - H_0^{1}(\tau) \bar{H}_0^{1}(\tau)\right]$ |
| $\neq 1; -1;\}$ | $\frac{1}{b} \left[(1 - \Delta)e^{-\Delta \ln v} + (1 + \Delta)e^ {\Delta \ln v}\right]$ |
| $\neq 1; a - 2; t_o < \frac{[1-a]}{2v};\}$ | $\frac{2}{(b-a+2)} (1 + \ln v)$ |
| $\neq 1; a - 2; t_o = \frac{[1-a]}{2v};\}$ | $0$ |
Table B-3. $\xi(t)$ and $\hat{\xi}(t)$ for TM-systems. Variables and parameters are defined as follows: $\tau = \frac{2a\ln t_0}{b-a+2(1-a)} - \frac{t}{t_0} \frac{b-a+2}{b-a+2}$, and $\Delta^2 = \left|1 - \frac{b-a+2}{b-a+2}\right|^2$.

| $TM$ System | $\hat{\xi}(t)$ |
|-------------|----------------|
| $\{ \neq 0; (a - 2, \infty); \}$ | $\sqrt{\frac{\pi}{2(b-a+2)}} \left( \frac{1}{t_0} \right)^{\frac{1}{2}} \frac{1}{H(\frac{1}{2})-1} (\tau) \tau$ |
| $\{ \neq 0; (-\infty, a - 2); \}$ | $\sqrt{\frac{\pi}{2(b-a+2)}} \left( \frac{1}{t_0} \right)^{\frac{1}{2}} \frac{1}{H(\frac{1}{2})-1} (\tau) \tau$ |
| $\{1; -1;\}$ | $\sqrt{\frac{\pi}{2\omega}} e^{i\omega t_0 \ln \frac{1}{t_0}}$ |
| $\{ \neq 1; a - 2; t_0 < \frac{1-a}{2\omega}; \pm; \}$ | $\sqrt{\frac{\pi}{2(1-a)\Delta}} \left( \frac{1}{t_0} \right)^{\frac{1}{2}} \frac{1}{H(\frac{1}{2})-1} (e^{-\Delta \chi} \pm i\Delta \chi)$ |
| $\{ \neq 1; a - 2; t_0 = \frac{1-a}{2\omega}; \pm; \}$ | $\sqrt{\frac{\pi}{2(1-a)\Delta}} \left( \frac{1}{t_0} \right)^{\frac{1}{2}} \frac{1}{H(\frac{1}{2})-1} (1 \pm 2i \chi)$ |
| $\{ \neq 1; a - 2; t_0 > \frac{1-a}{2\omega}; \pm; \}$ | $\sqrt{\frac{\pi}{2(1-a)\Delta}} \left( \frac{1}{t_0} \right)^{\frac{1}{2}} e^{\pm i\Delta \chi}$ |

| $TM$ Systems | $\hat{\xi}(t)$ |
|-------------|----------------|
| $\{ \neq 0; (a - 2, \infty); \}$ | $\frac{i}{2} \sqrt{\frac{\pi}{2(1-a)\Delta}} \left( \frac{1}{t_0} \right)^{\frac{1}{2}} \frac{1}{H(\frac{1}{2})-1} \left( \tau \right) \tau$ |
| $\{ \neq 0; (-\infty, a - 2); \}$ | $-\frac{i}{2} \sqrt{\frac{\pi}{2(1-a)\Delta}} \left( \frac{1}{t_0} \right)^{\frac{1}{2}} \frac{1}{H(\frac{1}{2})-1} \left( \tau \right) \tau$ |
| $\{1; -1;\}$ | $i\omega \sqrt{\frac{\pi}{2\omega}} e^{i\omega t_0 \ln \frac{1}{t_0}}$ |
| $\{ \neq 1; a - 2; t_0 < \frac{1-a}{2\omega}; \pm; \}$ | $\frac{i}{2} \sqrt{\frac{\pi}{2(1-a)\Delta}} \left( \frac{1}{t_0} \right)^{\frac{1}{2}} \frac{1}{H(\frac{1}{2})-1} \left[ \pm (1 - \Delta) e^{-\Delta \chi} + i(1 + \Delta) e^{\Delta \chi} \right]$ |
| $\{ \neq 1; a - 2; t_0 = \frac{1-a}{2\omega}; \pm; \}$ | $\sqrt{\frac{\pi}{2(1-a)\Delta}} \left( \frac{1}{t_0} \right)^{\frac{1}{2}} \frac{1}{H(\frac{1}{2})-1} \left[ \pm \frac{i}{2} + i(1 + \chi) \right]$ |
| $\{ \neq 1; a - 2; t_0 > \frac{1-a}{2\omega}; \pm; \}$ | $\frac{i}{2} \sqrt{\frac{\pi}{2(1-a)\Delta}} \left( \frac{1}{t_0} \right)^{\frac{1}{2}} \frac{1}{H(\frac{1}{2})-1} \left[ \pm 1 + i \Delta \right]$ |

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Table B-4. \( \hat{\phi}_3(t) \), \( \hat{\gamma}_3(t) \), and \( \hat{\gamma}_3(t) \) for TM-systems. Variables and parameters are defined as follows: \( \tau = \frac{2\omega t_0}{|b-a+2|} \left( \frac{t}{t_0} \right)^{\frac{b-a+2}{2}} \); \( \chi = \frac{1}{2}(1-a) \ln \left( \frac{t}{t_0} \right) \); \( q = \frac{b-a+2}{1-a} \), and \( \Delta^2 = \left| 1 - \frac{4\omega^2}{(1-a)^2} \right| \).

| TM System | \( \hat{\phi}_3(t) \) |
|-----------|------------------|
| \{ \neq 0; \neq a - 2 \} \{ 1; -1 \} | \frac{\omega t_0}{|b-a+2|} \left( \frac{t}{t_0} \right)^{1-a} H^{(1)}(r) B^{(1)}(\tau) |
| \{ \neq 0; (-\infty, a - 2) \} \{ 1; -1 \} | \pm \frac{\omega t_0}{|b-a+2|} \left( \frac{t}{t_0} \right)^{1-a} \left[ H^{(1)}_{-\frac{1}{2}}(r) \hat{H}^{(1)}_{\frac{1}{2}}(\tau) + H^{(1)}_{\frac{1}{2}}(r) \hat{H}^{(1)}_{-\frac{1}{2}}(\tau) \right] |
| \{ \neq 1; a - 2; t_0 < \frac{|1-a|}{2\omega}; \pm \} | \pm \frac{\omega t_0}{|b-a+2|} \left( \frac{t}{t_0} \right)^{1-a} \left[ (1-\Delta) e^{-2\Delta \chi} + (1+\Delta) e^{2\Delta \chi} \right] |
| \{ \neq 1; a - 2; t_0 = \frac{|1-a|}{2\omega}; \pm \} | \pm (1+2\chi)^2 |
| \{ \neq 1; a - 2; t_0 > \frac{|1-a|}{2\omega}; \pm \} | \pm \frac{2}{\Delta} |

| TM System | \( \hat{\gamma}_3(t) \) |
|-----------|------------------|
| \{ \neq 0; \neq a - 2 \} \{ 1; -1 \} | \frac{\omega t_0}{|b-a+2|} \left( \frac{t}{t_0} \right)^{a-1} \frac{\tau^2}{2} \left[ H^{(1)}_{-\frac{1}{2}}(r) \hat{B}^{(1)}_{\frac{1}{2}}(\tau) - H^{(1)}_{\frac{1}{2}}(r) \hat{B}^{(1)}_{-\frac{1}{2}}(\tau) \right] |
| \{ \neq 1; a - 2; t_0 < \frac{|1-a|}{2\omega}; \pm \} | \frac{|1-a|}{\omega t_0} \left( \frac{t}{t_0} \right)^{a-1} \left[ -(1-\Delta) e^{-2\Delta \chi} + (1+\Delta) e^{2\Delta \chi} \right] |
| \{ \neq 1; a - 2; t_0 = \frac{|1-a|}{2\omega}; \pm \} | \frac{2|1-a|}{\omega t_0} \left( \frac{t}{t_0} \right)^{a-1} (1+2\chi) |
| \{ \neq 1; a - 2; t_0 > \frac{|1-a|}{2\omega}; \pm \} | 0 |

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Table B-5. $\Xi_P(t)$ and $\Xi_X(t)$ for $TQ$-systems. Variables and parameters are defined as follows: $\tau = \frac{2\omega t_o}{[b-a+2]} \left( \frac{t}{t_o} \right)^{\frac{b-a+2}{2}}$; $\chi = \frac{1}{2} (1 - a) \ln \frac{t}{t_o}$; $q = \frac{b-a+2}{1-a}$, and $\Delta^2 = \left[ 1 - \frac{4\omega^2 t_o^2}{(1-a)^2} \right]$.

| $TQ$ System | $\Xi_P(t)$ |
|-------------|------------|
| $\{ \neq 0; (a - 2, \infty) \}$ | $\sqrt{\frac{2\omega t_o}{[b-a+2]}} \left( \frac{t}{t_o} \right)^{\frac{1}{2}} H_{1/2}^{(1)} (\tau)$ |
| $\{ \neq 0; (-\infty, a - 2) \}$ | $\sqrt{\frac{2\omega t_o}{[b-a+2]}} \left( \frac{t}{t_o} \right)^{\frac{1}{2}} H_{1/2}^{(1)} (\tau)$ |
| $\{ 1; -1 \}$ | $\sqrt{\frac{2\omega t_o}{[b-a+2]}} \left( \frac{t}{t_o} \right)^{\frac{1}{2}} e^{i\omega t_o \ln (\tau)}$ |
| $\{ \neq 0, 1; a - 2; t_o < \frac{1-a}{2\omega t_o}; \pm \}$ | $\sqrt{\frac{2\omega t_o}{[b-a+2]}} \left( \frac{t}{t_o} \right)^{\frac{1}{2}} (e^{-\Delta \chi} \pm ie^{\Delta \chi})$ |
| $\{ \neq 0, 1; a - 2; t_o = \frac{1-a}{2\omega t_o}; \pm \}$ | $\sqrt{\frac{2\omega t_o}{[b-a+2]}} \left( \frac{t}{t_o} \right)^{\frac{1}{2}} (1 \pm 2i\chi)$ |
| $\{ \neq 0, 1; a - 2; t_o > \frac{1-a}{2\omega t_o}; \pm \}$ | $\sqrt{\frac{2\omega t_o}{[b-a+2]}} \left( \frac{t}{t_o} \right)^{\frac{1}{2}} e^{\pm i\Delta \chi}$ |

| $TQ$ System | $\Xi_X(t)$ |
|-------------|------------|
| $\{ \neq 0; (a - 2, \infty) \}$ | $\frac{1}{2} \sqrt{\frac{2[b-a+2]}{2\omega t_o}} \left( \frac{t}{t_o} \right)^{\frac{1}{2}} \tau H_{1/2}^{(1)} (\tau)$ |
| $\{ \neq 0; (-\infty, a - 2) \}$ | $-\frac{1}{2} \sqrt{\frac{2[b-a+2]}{2\omega t_o}} \left( \frac{t}{t_o} \right)^{\frac{1}{2}} \tau H_{1/2}^{(1)} (\tau)$ |
| $\{ 1; -1 \}$ | $i\omega \sqrt{\frac{2\omega t_o}{[b-a+2]}} \left( \frac{t}{t_o} \right)^{\frac{1}{2}} e^{i\omega t_o \ln (\tau)}$ |
| $\{ \neq 0, 1; a - 2; t_o < \frac{1-a}{2\omega t_o}; \pm \}$ | $\frac{1}{2} \sqrt{\frac{2[b-a+2]}{2\omega t_o}} \left( \frac{t}{t_o} \right)^{\frac{1}{2}} \left[ \pm (1 - \Delta) e^{-\Delta \chi} + i(1 + \Delta) e^{\Delta \chi} \right]$ |
| $\{ \neq 0, 1; a - 2; t_o = \frac{1-a}{2\omega t_o}; \pm \}$ | $\sqrt{\frac{2[b-a+2]}{2\omega t_o}} \left( \frac{t}{t_o} \right)^{\frac{1}{2}} \left[ \pm \frac{1}{2} + i(1 + \chi) \right]$ |
| $\{ \neq 0, 1; a - 2; t_o > \frac{1-a}{2\omega t_o}; \pm \}$ | $\frac{1}{2} \sqrt{\frac{2[b-a+2]}{2\omega t_o}} \left( \frac{t}{t_o} \right)^{\frac{1}{2}} \left( \pm 1 + i\Delta \right) e^{\pm i\Delta \chi}$ |
Table B.6. \( C_{3,T}(t), C_{3,D}(t), \) and \( C_{3,X^2}(t) \) for TQ-systems. Variables and parameters are defined as follows: 
\[ \tau = \frac{2q\phi}{|q|} \left( \frac{t}{t_0} \right)^{\frac{b-a+2}{2}}; \quad \chi = \frac{1}{2} \left( 1 - a \right) \ln \left( \frac{t}{t_0} \right); \]
\[ q = \frac{b}{1-a}, \quad \text{and} \quad \Delta^2 = \left| 1 - 4\omega_0^2 t_0^2 \right| \]

| TQ System | \( C_{3,T}(t) \) |
|-----------|------------------|
| \{ \neq 0; \neq a - 2; \}\{ 1; -1; \} \} | \frac{\alpha t_0}{\ln |q|} \left( \frac{t}{t_0} \right)^{\infty} H^{(1)}(t) T^{(1)}(t) |
| \{ \neq 0, 1; a - 2; t_0 < \frac{|1-a|}{2} \}\{ \pm \} \} | \frac{\alpha t_0}{1-a} \left( \frac{t}{t_0} \right)^{\infty} e^{-2\Delta t} + e^{2\Delta t} |
| \{ \neq 0, 1; a - 2; t_0 = \frac{|1-a|}{2} \}\{ \pm \} \} | \frac{\alpha t_0}{1-a} \left( \frac{t}{t_0} \right)^{\infty} H^{(1)}(t) T^{(1)}(t) |
| \{ \neq 0, 1; a - 2; t_0 > \frac{|1-a|}{2} \}\{ \pm \} \} | \frac{\alpha t_0}{1-a} \left( \frac{t}{t_0} \right)^{\infty} |

| TQ System | \( C_{3,D}(t) \) |
|-----------|------------------|
| \{ \neq 0; (a - 2, \infty); \}\{ \pm \} \} | \frac{\alpha t_0}{\ln |q|} \left( \frac{t}{t_0} \right)^{\infty} H^{(1)}(t) T^{(1)}(t) ± \frac{\alpha}{1-a} \left[ H^{(1)}(t) T^{(1)}(t) + H^{(1)}(t) T^{(1)}(t) \right] |
| \{ \neq 0; (-\infty, a - 2); \}\{ \pm \} \} | \frac{\alpha t_0}{1-a} \left( \frac{t}{t_0} \right)^{\infty} e^{-2\Delta t} + e^{2\Delta t} ± \frac{\alpha}{1-a} \left( 1 - \Delta \right) e^{-2\Delta t} + \left( 1 + \Delta \right) e^{2\Delta t} |
| \{ \neq 0, 1; a - 2; t_0 = \frac{|1-a|}{2} \}\{ \pm \} \} | \frac{\alpha t_0}{1-a} \left( \frac{t}{t_0} \right)^{\infty} (1 + 4\chi^2) ± \frac{\alpha}{1-a} 2\chi(1 + \chi) |
| \{ \neq 0, 1; a - 2; t_0 > \frac{|1-a|}{2} \}\{ \pm \} \} | \frac{\alpha t_0}{1-a} \left( \frac{t}{t_0} \right)^{\infty} |

| TQ System | \( C_{3,X^2}(t) \) |
|-----------|------------------|
| \{ \neq 0; \neq a - 2; \}\{ 1; -1; \} \} | \frac{\alpha t_0}{\ln |q|} \left( \frac{t}{t_0} \right)^{\infty} \tau^2 H^{(1)}(t) T^{(1)}(t) - H^{(1)}(t) T^{(1)}(t) |
| \{ \neq 0, 1; a - 2; t_0 < \frac{|1-a|}{2} \}\{ \pm \} \} | \frac{\alpha t_0}{1-a} \left( \frac{t}{t_0} \right)^{\infty} \left[ -\left( 1 - \Delta \right) e^{-2\Delta t} + \left( 1 + \Delta \right) e^{2\Delta t} \right] |
| \{ \neq 0, 1; a - 2; t_0 = \frac{|1-a|}{2} \}\{ \pm \} \} | \frac{\alpha t_0}{1-a} \left( \frac{t}{t_0} \right)^{\infty} (1 + 2\chi) |
| \{ \neq 0, 1; a - 2; t_0 > \frac{|1-a|}{2} \}\{ \pm \} \} | \frac{\alpha t_0}{1-a} \left( \frac{t}{t_0} \right)^{\infty} |
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