CHARACTERIZATION OF CODIMENSION ONE FOLIATIONS ON COMPLEX CURVES BY CONNECTIONS

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Abstract. A way to characterize the space of leaves of a foliation in terms of connections is proposed. A particular example of vertex algebra cohomology of codimension one foliations on complex curves is considered. Multiple applications in Mathematical physics are revealed.

1. Introduction

The theory of foliations involves a bunch of approaches [5–8, 12, 13, 16, 18, 36] and many others. In certain cases it is useful to express cohomology in terms of connections and use the language of connections in order to study leave spaces of foliations. Connections appear in conformal field theory [4, 11] in definitions of many notions and formulas. Vertex algebras, generalizations of ordinary Lie algebras, are essential in conformal field theory. The theory of vertex algebra characters is a rapidly developing field of studies. Algebraic nature of methods applied in this field helps to understand and compute the structure of vertex algebra characters. On the other hand, the geometric side of vertex algebra characters is in associating their formal parameters with local coordinates on a complex variety. Depending on geometry, one can obtain various consequences for a vertex algebra and its space of characters, and vice-versa, one can study geometrical property of a manifold by using algebraic nature of a vertex algebra attached. In this paper we use the vertex algebra cohomology theory [23] for characterization codimension one foliations on smooth complex curves [19, 20]. The arbitrary codimension case will be considered in elsewhere.

The general theory of characterizations of leaves of codimension one foliations on complex curves has multiple applications in Mathematical Physics. In particular, since the notion connections used to define cohomology stems from reduction formulas [53] for correlation functions for vertex algebras with formal parameters identified with local parameters on Riemann surfaces, the codimension one foliation formulas are useful in studies of spaces of correlation functions. The double complexes used in this paper to define a cohomology originate from consideration of cohomology of vertex algebras in [23, 24]. Recall that vertex algebras play central role in many aspects of Mathematical Physics. The cohomology of vertex algebras is an important questions not only in the theory of vertex algebras but also in applications to Conformal Field Theory.

Key words and phrases. Riemann surfaces; vertex operator algebras; character functions; foliations; cohomology.
The plan of the paper is the following. In Section 2 we describe the approach to cohomology in terms of connections. In Section 3 we define the spaces for double complex associated to a quasi-conformal grading-restricted vertex algebra. Non-vanishing and canonicity of the construction is proved. In Section 4 coboundary operators are defined. It is shown that combining with the double complex spaces they determine chain-cochain double complex. In Section 5 we determine the first vertex algebra cohomologies of a codimension one foliation. Corresponding cohomological classes are considered in Section 6. In Section 7 we consider multiple applications in Mathematical Physics. In Appendixes we provide the material needed for construction of the vertex algebra cohomology of foliations. In Appendix 10 we recall the notion of a quasi-conformal grading-restricted vertex algebra and its modules. In Section 11 the space of $W$-valued rational sections of a vertex algebra bundle is described. In Section 12 properties of matrix elements for elements of the space $W$ are listed. In Section 13 maps composable with a number of vertex operators are defined. Appendix 14 contains proofs of Lemmas 1, 2, 3, and Proposition 1.

2. Cohomology in terms of connections

In various situations it is sometimes effective to use an interpretation of cohomology in terms of connections. In particular in our supporting example of vertex algebra cohomology of codimension one foliations. It is convenient to introduce multi-point connections over a graded space and to express coboundary operators and cohomology in terms of connections:

$$\delta^n \phi \in G^{n+1}(\phi),$$
$$\delta^n \phi = G(\phi).$$

Then the cohomology is defined as the factor space

$$H^n = Con_{cl} / G^{n-1},$$

of closed multi-point connections with respect to the space of connection forms defined below.

2.1. Multi-point holomorphic connections. We start this section with definitions of holomorphic multi-point connections on a smooth complex variety. Let $\mathcal{X}$ be a smooth complex variety and $\mathcal{V} \to \mathcal{X}$ a holomorphic vector bundle over $\mathcal{X}$. Let $E$ be the sheaf of holomorphic sections of $\mathcal{V}$. Denote by $\Omega$ the sheaf of differentials on $\mathcal{X}$. A holomorphic connection $\nabla$ on $E$ is a $\mathbb{C}$-linear map

$$\nabla : E \to E \otimes \Omega,$$

satisfying the Leibniz formula

$$\nabla(\phi f) = \nabla(\phi) f + \phi \otimes df,$$

for any holomorphic function $f$. Motivated by the definition of the holomorphic connection $\nabla$ defined for a vertex algebra bundle (cf. Section 6, [4]) over a smooth complex variety $\mathcal{X}$, we introduce the definition of the multiple point holomorphic connection over $\mathcal{X}.$
Definition 1. Let $\mathcal{V}$ be a holomorphic vector bundle over $\mathcal{X}$, and $\mathcal{X}_0$ its subvariety. A holomorphic multi-point connection $\mathcal{G}$ on $\mathcal{V}$ is a $\mathbb{C}$-multi-linear map
$$\mathcal{G} : E \rightarrow E \otimes \Omega,$$
such that for any holomorphic function $f$, and two sections $\phi(p)$ and $\psi(p')$ at points $p$ and $p'$ on $\mathcal{X}_0$ correspondingly, we have
$$\sum_{q,q' \in \mathcal{X}_0 \subset \mathcal{X}} \mathcal{G} (f(q), \phi(q')) = f(\psi(p')) \mathcal{G} (\phi(p)) + f(\phi(p)) \mathcal{G} (\psi(p')),$$ (2.1)
where the summation on left hand side is performed over a locus of points $q$, $q'$ on $\mathcal{X}_0$. We denote by $\text{Con}_{\mathcal{X}_0}(\mathcal{S})$ the space of such connections defined over a smooth complex variety $\mathcal{X}$. We will call $\mathcal{G}$ satisfying (2.1), a closed connection, and denote the space of such connections by $\text{Con}_{\mathcal{X}_0}^{\text{cl}}(\mathcal{S})$.

Geometrically, for a vector bundle $\mathcal{V}$ defined over a complex variety $\mathcal{X}$, a multi-point holomorphic connection (2.1) relates two sections $\phi$ and $\psi$ of $E$ at points $p$ and $p'$ with a number of sections at a subvariety $\mathcal{X}_0$ of $\mathcal{X}$.

Definition 2. We call
$$G(\phi, \psi) = f(\phi(p)) \mathcal{G} (\psi(p')) + f(\psi(p')) \mathcal{G} (\phi(p)) - \sum_{q,q' \in \mathcal{X}_0 \subset \mathcal{X}} \mathcal{G} (f(q'), \phi(q)),$$ (2.2)
the form of a holomorphic connection $\mathcal{G}$. The space of form for $n$-point holomorphic connection will be denoted by $G^n(p_1, \ldots, p_n)$.

Definition 3. A fixed point holomorphic connection on $E$ is defined by the condition
$$\sum_{p_0; q,q' \in \mathcal{X}_0 \subset \mathcal{X}} \mathcal{G} (f(q'), \phi(q)) = f(\psi(p_0)) \mathcal{G} (\phi(p)) + f(\phi(p)) \mathcal{G} (\psi(p_0)),$$ (2.3)
where a point $p_0$ is fixed on $\mathcal{X}_0$.

Definition 4. A holomorphic connection defined for a vector bundle $\mathcal{V}$ over a smooth complex variety $\mathcal{X}$ (the two point case of the multi-point holomorphic connection (2.1)) is called a two point connection when for any holomorphic function $f$,
$$\mathcal{G} (f(\psi(p')) \phi(p)) = f(\psi(p')) \mathcal{G} (\phi(p)) + f(\phi(p)) \mathcal{G} (\psi(p')),$$ (2.4)
for two sections $\psi(p')$ and $\phi(p)$ of $E$. We denote the space of such connections as $\text{Con}_{2,p_0; \mathcal{X}_0}^{\text{tr}}$.

Let us formulate another definition which we use in the next section:

Definition 5. We call a multi-point holomorphic connection $\mathcal{G}$ the transversal connection, i.e., when it satisfies
$$f(\psi(p')) \mathcal{G} (\phi(p)) + f(\phi(p)) \mathcal{G} (\psi(p')) = 0.$$ (2.5)
We call
$$G_{tr}(p, p') = (\psi(p')) \mathcal{G} (\phi(p)) + f(\phi(p)) \mathcal{G} (\psi(p')),$$ (2.6)
the form of a transversal connection. The space of such connections is denoted by $G_{tr}^2$. 

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2.2. Basis of transversal sections for a foliation. In this subsection we recall \cite{8} the notion of basis of transversal sections for a foliation. Let $\mathcal{M}$ be a manifold of dimension $n$, equipped with a foliation $\mathcal{F}$ of codimension $q$.

**Definition 6.** A transversal section of $\mathcal{F}$ is an embedded $q$-dimensional submanifold $U \subset \mathcal{M}$ which is everywhere transverse to the leaves.

**Definition 7.** If $\alpha$ is a path between two points $p_1$ and $p_2$ on the same leaf, and if $U_1$ and $U_2$ are transversal sections through $p_1$ and $p_2$, then $\alpha$ defines a transport along the leaves from a neighborhood of $p_1$ in $U_1$ to a neighborhood of $p_2$ in $U_2$, hence a germ of a diffeomorphism $hol(\alpha) : (U_1, p_1) \hookrightarrow (U_2, p_2)$, called the holonomy of the path $\alpha$.

**Definition 8.** Two homotopic paths always define the same holonomy. If the above transport along $\alpha$ is defined in all of $U_1$ and embeds $U_1$ into $U_2$, this embedding $h : U_1 \hookrightarrow U_2$ will be denoted by $hol(\alpha) : U_1 \hookrightarrow U_2$. Embeddings of this form we called holonomy embeddings.

Note that composition of paths also induces an operation of composition on those holonomy embeddings. Transversal sections $U$ through $p$ as above should be thought of as neighborhoods of the leaf through $p$ in the leaf space. Then we have

**Definition 9.** A transversal basis for $\mathcal{M}/\mathcal{F}$ as a family $\mathcal{U}$ of transversal sections $U \subset \mathcal{M}$ with the property that, if $U_p$ is any transversal section through a given point $p \in \mathcal{M}$, there exists a holonomy embedding $h : U \hookrightarrow V$ with $U \in \mathcal{U}$ and $p \in h(U)$.

Typically, a transversal section is a $q$-disk given by a chart for the foliation. Accordingly, we can construct a transversal basis $\mathcal{U}$ out of a basis $\tilde{\mathcal{U}}$ of $\mathcal{M}$ by domains of foliation charts

$$\phi_U : \tilde{U} \hookrightarrow \mathbb{R}^{n-q} \times U,$$

$\tilde{U} \in \tilde{\mathcal{U}}$, with $U = \mathbb{R}^q$. Note that each inclusion $\tilde{U} \hookrightarrow \tilde{V}$ between opens of $\tilde{U}$ induces a holonomy embedding

$$h_{U,U_0} : U \hookrightarrow U_0,$$

defined by the condition that the plaque in $\tilde{U}$ through $p$ is contained in the plaque in $\tilde{U}_0$ through $h_{U,U_0}(x)$.

2.3. Čech-de Rham cohomology in Crainic and Moerdijk construction. Let us start with the first supporting example \cite{8}. Recall the construction of the Čech-de Rham cohomology of a foliation. cohomology by Crainic and Moerdijk \cite{8}. Consider a foliation $\mathcal{F}$ of co-dimension $n$ on a smooth manifold $\mathcal{M}$. Consider the double complex

$$C^{k,l} = \prod_{U_0 \xrightarrow{h_1} \cdots \xrightarrow{h_k} U_k} \Omega^l(U_0),$$

where $\Omega^l(U_0)$ is the space of differential $l$-forms on $U_0$, and the product ranges over all $k$-tuples of holonomy embeddings between transversal sections from a fixed transversal
basis $\mathcal{U}$. Component of $\omega \in C^{k,l}$ are denoted by $\omega(h_1, \ldots, h_l) \in \Omega^l(\mathcal{U}_0)$. The vertical differential is defined as
\[
(-1)^k d : C^{k,l} \to C^{k,l+1},
\]
where $d$ is the usual de Rham differential. The horizontal differential
\[
\delta : C^{k,l} \to C^{k+1,l},
\]
is given by
\[
\delta = \sum_{i=1}^{k} (-1)^i \delta_i,
\]
where $G(h_1, \ldots, h_{k+1})$ is the multi-point connection of the form (2.1), i.e.,
\[
\delta_i \omega(h_1, \ldots, h_{p+1}) = \begin{cases} 
  h^i \omega(h_2, \ldots, h_{p+1}), & \text{if } i = 0, \\
  \omega(h_1, \ldots, h_{i+1} h_i, \ldots, h_{p+1}), & \text{if } 0 < i < p + 1, \\
  \omega(h_1, \ldots, h_p), & \text{if } i = p + 1.
\end{cases}
\]
This double complex is actually a bigraded differential algebra, with the usual product
\[
(\omega \cdot \eta)(h_1, \ldots, h_{k+k'}) = (-1)^{kk'} \omega(h_1, \ldots, h_k) h^*_k \ldots h^*_k \eta(h_{k+1}, \ldots, h_{k+k'}),
\]
for $\omega \in C^{k,l}$ and $\eta \in C^{k',l'}$, thus $(\omega \cdot \eta)(h_1, \ldots, h_{k+k'}) \in C^{k+k',l+l'}$. The cohomology $H^*_C(\mathcal{M}/\mathcal{F})$ of this complex is called the Čech-de Rham cohomology of the leaf space $\mathcal{M}/\mathcal{F}$ with respect to the transversal basis $\mathcal{U}$. It is defined by
\[
\tilde{H}^*_C(\mathcal{M}/\mathcal{F}) = \text{Con}^{k+1}_{cl}(h_1, \ldots, h_{k+1})/G^k(h_1, \ldots, h_k),
\]
where $\text{Con}^{k+1}_{cl}(h_1, \ldots, h_{k+1})$ is the space of closed multi-point connections, and $G^k(h_1, \ldots, h_k)$ is the space of $k$-point connection forms.

In what follows we describe another supporting example of vertex algebra cohomology of codimension one foliations interpreted in terms of connections.

3. Spaces for double complexes

In this section we introduce the definition of spaces for a double complex suitable for the construction a grading-restricted vertex algebra cohomology for codimension one foliations on complex curves. A consideration of foliations of smooth manifold of arbitrary dimension will be given in elsewhere. Let $\mathcal{U}$ be a family of transversal sections of $\mathcal{F}$, (cf. [3] and Subsection 2.2). We consider $(n,k)$-set of points, $n \geq 1$, $k \geq 1$,
\[
(p_1, \ldots, p_n, p'_1, \ldots, p'_k),
\]
on a smooth manifold $\mathcal{M}$. Let us denote the set of corresponding local coordinates for $n + k$ points on $\mathcal{M}$ as $c_i(p_i)$, $1 \leq i \leq n + k$. In what follows we consider points (3.1) as points on either the leaf space $\mathcal{M}/\mathcal{F}$ of $\mathcal{F}$, or on transversal sections $U_j$ of the transversal basis $\mathcal{U}$. Since $\mathcal{M}/\mathcal{F}$ is not in general a manifold, one has to be careful in considerations of chains of local coordinates along its leaves [20][30]. For association of formal parameters of mappings and vertex operators with points of $\mathcal{M}/\mathcal{F}$ we will use in what follows either their local coordinates on $\mathcal{M}$ or local coordinates on sections $U$ of a transversal basis $\mathcal{U}$ which are submanifolds of $\mathcal{M}$ of dimension equal to codimension...
of a foliation $\mathcal{F}$. We will denote such local coordinates as $\{t_i(p_i)\}$ (on $\mathcal{M}$), and $\{t_i(p_i)\}$ (on $U$) correspondingly. In case of extremely singular foliations when it is not possible to use local coordinates of $\mathcal{M}$ in order to parameterize a point on $\mathcal{M}/\mathcal{F}$ we still able to use a local coordinate on a transversal section passing through this point on $\mathcal{M}/\mathcal{F}$. In addition to that, note that the complexes considered below are constructed in such a way that one can always use coordinates on transversal sections only, avoiding any possible problems with localization of coordinates on leaves of $\mathcal{M}/\mathcal{F}$.

For a $(n,k)$-set of a grading-restricted vertex algebra $V$ elements

$$(v_1, \ldots, v_n; v'_1, \ldots, v'_k),$$

we consider linear maps

$$\Phi : V^\otimes n \to \mathcal{W}_{z_1, \ldots, z_n}$$

(see Appendix 11 for the definition of a $\mathcal{W}_{z_1, \ldots, z_n}$ space),

$$\Phi \left( dz_1^{wt} v_1 \otimes v_1, c_1(p_1); \ldots; dz_n^{wt} v_n \otimes v_n, c_n(p_1) \right),$$

where we identify, as it is usual in the theory of characters for vertex operator algebras on curves $[25, 50, 52, 53]$, $n$ formal parameters $z_1, \ldots, z_n$ of $\mathcal{W}_{z_1, \ldots, z_n}$, with local coordinates $c_i(p_i)$ in vicinities of points $p_i$, $0 \leq i \leq n$, on $\mathcal{M}$. Elements $\Phi \in \mathcal{W}_{c_1(p_1), \ldots, c_n(p_n)}$ can be seen as coordinate-independent $\overline{W}$-valued rational sections of a vertex algebra bundle $[3]$ generalization. The construction of vertex algebra cohomology of a foliation in terms of connections is parallel to ideas of $[7]$. Such a construction will be explained elsewhere. Note that according to $[3]$ it can be treated as $(\text{Aut } O^{(1)})^\times n = \text{Aut } O^{(1)} \times \ldots \times \text{Aut } O^{(1)}$-torsor of the product of groups of a coordinate transformation. In what follows, according to definitions of Appendix 11 when we write an element $\Phi$ of the space $\mathcal{W}_{z_1, \ldots, z_n}$, we actually have in mind corresponding matrix element $\langle w', \Phi \rangle$ that absolutely converges (on certain domain) to a rational form-valued function $R(\langle w', \Phi \rangle)$. Quite frequently we will write $\langle w', \Phi \rangle$ which would denote a rational $\overline{W}$-valued form. In notations, we would keep tensor products of vertex algebra elements with $wt$-powers of $z$-differentials when it is inevitable only. Later in this section we prove, that for arbitrary $v_i, v'_j \in V$, $1 \leq i \leq n$, $1 \leq j \leq k$, points $p_i$ with local coordinates $c_i(p_i)$ on transversal sections $U_i \in U$ of $\mathcal{F}$, an element $z_1, \ldots, z_n$ as well as the vertex operator

$$\omega_W (dc_1(p'_i)^{wt} (v'_i) \otimes v'_i, c_1(p'_i)) = Y (dc_1(p'_i)^{wt} (v'_i) \otimes v'_i, c_1(p'_i)),$$

are invariant with respect to the action of $(\text{Aut } O^{(1)})^\times n$. In (3.5) we mean usual vertex operator (as defined in Appendix 10) not affecting the tensor product with corresponding differential. We assume that the maps (3.3) are composable (according to Definition 3.3) of Appendix 10, with $k$ vertex operators $\omega_W(v'_i, c_i(p'_i)), 1 \leq i \leq k$ with $k$ vertex algebra elements from (3.2), and formal parameters associated with local coordinates on $k$ transversal sections of $\mathcal{F}$, of $k$ points from the set (3.1).

The compositability of a map $\Phi$ with a number of vertex operators consists of two conditions on $\Phi$. The first requires the existence of positive integers $N^n_m(v_i, v_j)$ depending just on $v_i, v_j$, and the second restricts orders of poles of corresponding sums
Taking into account these conditions, we will see that the construction of the space \((3.7)\) does depend on the choice of vertex algebra elements \((3.2)\).

In this subsection we construct the spaces for the double complex defined for codimension one foliations and associated to a grading-restricted vertex algebra. In order to keep control on the construction, we consider a section \(U_j\) of a transversal basis \(U\), and mappings \(\Phi\) that belong to the space \(W_{c(p_1),\ldots,c(p_n)}\), depending on points \(p_1,\ldots,p_n\) of intersection of \(U_j\) with leaves of \(\mathcal{M}/\mathcal{F}\) of \(\mathcal{F}\). It is assumed that local coordinates \(c(p_1),\ldots,c(p_n)\) of points \(p_i, 1 \leq i \leq n\), are taken on \(\mathcal{M}\) along these leaves of \(\mathcal{M}/\mathcal{F}\). We then consider all together locally a collection of \(k\) sections of \(U_j\).

The set of vertex algebra elements \((3.2)\) plays the role of parameters in our further construction of the vertex algebra cohomology associated with a foliation \(\mathcal{F}\). According to considerations of Subsection 2.2, we assume that each transversal section of a transversal basis \(U\) possess a coordinate chart which is induced by a coordinate chart of \(\mathcal{M}\) (cf. [8]).

Recall the notion of a holonomy embedding (cf. Subsection 2.2, cf. [8]) which maps a section into another section of a transversal basis, and a coordinate chart on the first section into a coordinate chart on the second transversal section. Motivated by the definition of the spaces for the Čech-de Rham complex in [8] (see Subsection 2.2), let us now introduce the following spaces:

**Definition 11.** For \(n \geq 0\), and \(m \geq 1\), we define the space

\[
C^n_m(V, W, \mathcal{U}, \mathcal{F}) = \bigcap_{U_h \subseteq \cdots \subseteq U_{h_k}} C^n(V, W, \mathcal{F})(U_j),
\]

where the intersection ranges over all possible \(m\)-tuples of holonomy embeddings \(h_i\), \(i \in \{1,\ldots,m-1\}\), between transversal sections of the basis \(\mathcal{U}\) for \(\mathcal{F}\).

First, we have the following

**Lemma 1.** \((3.7)\) is non-zero.
Proof. From the construction of spaces for double complex for a grading-restricted vertex algebra cohomology, it is clear that the spaces $C^n(V, W, U, F)(U_j)$, $1 \leq s \leq m$ in Definition 14 are non-empty. On each transversal section $U_s$, $1 \leq s \leq m$, $\Phi(v_1, c_1(p_1), \ldots; v_n, c_n(p_n))$ belongs to the space $W_{c_1(p_1), \ldots; c_j(p_n)}$, and satisfy the $L(-1)$-derivative (12.1) and $L(0)$-conjugation (12.2) properties. A map $\Phi(v_1, c_j(p_1); \ldots; v_n, c_j(p_n))$ is composable with $m$ vertex operators with formal parameters identified with local coordinates $c_j(p'_j)$, on each transversal section $U_j$. Note that on each transversal section, $n$ and $m$ the spaces (3.7) remain the same. The only difference may be constituted by the compositibility conditions (13.3) and (13.5) for $\Phi$.

In particular, for $l_1, \ldots, l_n \in \mathbb{Z}_+$ such that $l_1 + \cdots + l_n = n + m$, $v_1, \ldots, v_{m+n} \in V$ and $w' \in W'$, recall (13.1) that

$$\Xi_i = \omega_V(v_{k_1}, c_{k_1}(p_{k_1}) - \zeta_i) \cdots \omega_V(v_{k_i}, c_{k_i}(p_{k_i}) - \zeta_i) 1_V,$$

(3.8)

where $k_i$ is defined in (13.2), for $i = 1, \ldots, n$, depend on coordinates of points on transversal sections. At the same time, in the first compositibility condition (13.3) depends on projections $P_r(\Xi_i)$, $r \in \mathbb{C}$, of $W_{c(p_1), \ldots; c(p_n)}$ to $W$, and on arbitrary variables $\zeta_i$, $1 \leq i \leq m$. On each transversal connection $U_s$, $1 \leq s \leq m$, the absolute convergence is assumed for the series (13.3) (cf. Appendix 13). Positive integers $N^m_m(v_i, v_j)$, (depending only on $v_i$ and $v_j$) as well as $\zeta$, for $i, j = 1, \ldots, k, i \neq j$, may vary for transversal sections $U_s$. Nevertheless, the domains of convergency determined by the conditions (13.4) which have the form

$$|c_{m_i}(p_{m_i}) - \zeta_i| + |c_{n_j}(p_{n_j}) - \zeta_j| < |\zeta_i - \zeta_j|,$$

(3.9)

for $m_i = l_1 + \cdots + l_{i-1} + p$, $n_i = l_1 + \cdots + l_{j-1} + q$, $i, j = 1, \ldots, k, i \neq j$ and for $p = 1, \ldots, l$, and $q = 1, \ldots, l_j$, are limited by $|\zeta_i - \zeta_j|$ in (14.2) from above. Thus, for the intersection variation of sets of homology embeddings in (3.7), the absolute convergence condition for (13.3) is still fulfilled. Under intersection in (3.7), by choosing appropriate $N^m_m(v_i, v_j)$, one can analytically extend (13.3) to a rational function in $(c_1(p_1), \ldots, c_{n+m}(p_{n+m}))$, independent of $(\zeta_1, \ldots, \zeta_n)$, with the only possible poles at $c_i(p_i) = c_j(p_j)$, of order less than or equal to $N^m_m(v_i, v_j)$, for $i, j = 1, \ldots, k, i \neq j$.

As for the second condition in Definition of compositibility, we note that, on each transversal section, the domains of absolute convergency $c_i(p_i) \neq c_j(p_j), i \neq j$

$$|c_i(p_i)| > |c_k(p_j)| > 0,$$

for $i = 1, \ldots, m$, and $k = 1 + m, \ldots, n + m$, for

$$\mathcal{J}^n_m(\Phi) = \sum_{q \in \mathbb{C}} (w', \omega_W(v_1, c_1(p_1)) \cdots \omega_W(v_m, c_m(p_m)))

P_q(\Phi(v_1+m, c_1+m(p_1+m); \ldots; v_{n+m}, c_n+m(p_{n+m}))),$$

(3.10)

are limited from below by the same set of absolute values of local coordinates on transversal section. Thus, under intersection in (3.7) this condition is preserved, and the sum (13.5) can be analytically extended to a rational function in $(c_1(p_1), \ldots, c_{n+m}(p_{n+m}))$ with the only possible poles at $c_i(p_i) = c_j(p_j)$, of orders less than or equal to $N^m_m(v_i, v_j)$, for $i, j = 1, \ldots, k, i \neq j$. Thus, we proved the lemma. □
Lemma 2. The double complex (3.7) does not depend on the choice of transversal basis $U$.

The main statement of this section is contained in the following

**Proposition 1.** For a quasi-conformal grading-restricted vertex algebra $V$ and its module $W$, the construction (3.7) is canonical, i.e., does not depend on foliation preserving choice of local coordinates on $M/F$.

The proofs of Lemmas 1, 2, and Proposition 1 is contained in Appendix 14.

**Remark 1.** The condition of quasi-conformality is necessary in the proof of invariance of elements of the space $W_{z_1,\ldots,z_n}$ with respect to a vertex algebraic representation (cf. Appendix 10) of the group $(\text{Aut } O^{(i)})^\times_n$. In what follows, when it concerns the spaces (3.7), we will always assume the quasi-conformality of $V$.

A generalization of proofs of Lemmas 1, 2, 3 and Proposition 1 for the arbitrary codimension foliation on a smooth $n$-dimensional manifold will be given elsewhere. The proof of Proposition 1 is contained in Appendix 14.

Let $W$ be a grading-restricted $V$ module. Since for $n=0$, $\Phi$ does not include variables, and due to Definition 3.1, we can put:

$$C_0^k(V,W,F) = W,$$

for $k \geq 0$. Nevertheless, according to our Definition 3.7, mappings that belong to (3.11) are assumed to be composable with a number of vertex operators with depending on local coordinates of $k$ points on $k$ transversal sections.

**Lemma 3.**

$$C_{m}^n(V,W,F) \subset C_{m-1}^n(V,W,F).$$

The proof of this Lemma is contained in Appendix 14. For our further purposes we have to define spaces suitable for the definition of a double complex with a fixed point. Such double complex will be needed for the construction of vertex algebra cohomologies of $M/F$, in particular, for $H_m^m(V,W,F)$, $m \geq 0$ (see Section 4).

**Definition 12.** Let us fix a point $p'_r$ on the transversal section $U_r \in U$, $r \geq 1$. Assuming $k \geq 0$, $n \geq 0$, consider the space $C^m(p'_r;V,W,F)(U_r)$, of linear mappings

$$\Phi : V^\otimes n \rightarrow W_{c_1(p'_1),\ldots,c_n(p'_n)},$$

composable with $k$ vertex operators with formal parameters identified with local coordinates $\{c_1(p'_1),\ldots,c_r(p'_r),c_{r+1}(p'_r),\ldots,c_n(p'_n)\}$, on each of $k$ sections on a transversal basis $U$.

We assume that, for the intersection points $p'_j$ of each $U_j$, $1 \leq j \leq k$, we are able to define a coordinate function $t_j(p'_j)$ around $p'_j$ such that it covers the whole $U_j$. Thus, the holonomy embeddings $h_j$ deliver a map of local coordinate functions and vertex operators:

$$h_j : t_j(p'_j) \rightarrow t_{j+1}(p'_{j+1}),$$

and we have a sequence of mappings

$$p'_1 \overset{h_1}{\rightarrow} \ldots \overset{h_{k-1}}{\rightarrow} p'_r \overset{h_k}{\rightarrow} \ldots \overset{h_{k-1}}{\rightarrow} p_k.$$
Let us now introduce the following spaces:

**Definition 13.** For \( n \geq 0, \) and \( k \geq 1, \) consider the space

\[
C^n_k(p'_r; V, W, U, F) = \bigcap_{p'_1h_1...h_{r-1}p'_rh_{r-1}p'_k...h_{k-1}p'_k} C^n(p'_r; V, W, U, F) (U_j),
\]

where the intersection ranges over all possible \( k \)-sequences \( (3.14) \) of holonomy mappings \( h_i, \ i \in \{1, ..., k\} \) among points on transversal sections of the basis \( U \) with the fixed point \( p_r. \)

**Remark 2.** Note that in this Definition it is assumed that points \( \{p'_j\}, \ 1 \leq r - 1, r + 1 \leq k, \) in sequences \( (3.14) \) of holonomy mappings are free for moving along corresponding sections \( \{U_j\} \) of \( U. \)

Similar to the proof of Proposition 1, one shows that the spaces \( (3.15) \) do not depend on the choice of \( U \) and coordinates on \( \mathcal{M}/F, \) thus we then omit \( U \) in notations.

### 4. Coboundary operators and cohomology of codimension one foliations

Though the question of introduction a product on the space \( W_{z_1, ..., z_n} \) is not developed yet (private communication with Y.-Zh. Huang). Nevertheless, one can introduce an internal product \( \circ \) which allow to look at the action of coboundary operator as the integrability condition on one forms defining a foliation and related to the Godbillon-Vey cohomology class for codimension one foliations [15].

The coboundary operator is introduced as the form of a multi-point vertex algebra connection (cf. Definition 2.1 in Section 2):

\[
\delta^n_m \Phi = G(p_1, \ldots, p_{n+1}),
\]

\[
G(p_1, \ldots, p_{n+1}) = \langle w', \sum_{i=1}^{n} (-1)^i \Phi(\omega_V(v_i, c_i(p_i) - c_{i+1}(p_{i+1}))v_{i+1}) \rangle,
\]

\[
+ \langle w', \omega_W(v_1, c_1(p_1)) \Phi(v_2, c_2(p_2); \ldots; v_n, c_n(p_n)) \rangle
\]

\[
+ (-1)^{n+1} \langle w', \omega_W(v_{n+1}, c_{n+1}(p_{n+1})) \Phi(v_1, c_1(p_2); \ldots; v_n, c_n(p_n)) \rangle,
\]

for arbitrary \( w' \in W' \) (dual to \( W \)). Note that it is assumed that the coboundary operator does not affect \( dc(p)^{WT} \) \((n)\)-tensor multipliers in \( \Phi. \)

**Remark 3.** Following the construction of [23], let us introduce the coboundary operator \( \delta^n_m \) acting on elements \( \Phi \in C^n_k(V, W, F) \) of the spaces \( (3.7) \), which has the integrability condition form [15] with respect to a product:

\[
\delta^n_m \Phi = \mathcal{E} \circ \Phi,
\]

where

\[
\mathcal{E} = \left( E^{(1)}_W, \sum_{i=1}^{n} (-1)^i E^{(2)}_{V, 1, v}, E^{W; (1)}_{V, V} \right),
\]
the product $\circ$ is given by

$$\circ = \sum_{j=0}^{n+1} \circ_j,$$

where the elements $E^{(1)}_W$, $E^{W,(1)}_{WV}$, $E^{(2)}_{V;1V}$, and multiplications $\circ_i$ are defined in Appendix 12.

$$\delta^n_k \Phi = E^{(1)}_W \circ_0 \Phi + \sum_{i=1}^{n} (-1)^i E^{(2)}_{V;1V} \circ_i \Phi + (-1)^{n+1} E^{W,(1)}_{WV} \circ_{n+1} \Phi.$$

Remark 4. Inspecting construction of the double complex spaces (6.7) we see that the action (4.2) of the $\delta^m_i$ on an element of $C^n_m(V, W, F)$ provides a coupling (in terms of $W_{z_1, \ldots, z_n}$-valued rational functions) of vertex operators taken at the local coordinates $c_i(z_{p_i}), 0 \leq i \leq k$, at the vicinities of the same points $p_i$ taken on transversal sections for $F$, with elements of $C^n_{m-1}(V, W, F)$ taken at points at the local coordinates $c_i(z_{p_i})$, $0 \leq i \leq n$ on $M$ for points $p_i$ considered on the leaves of $M/F$.

Remark 5. We also mention that (4.3) can be written completely in terms of intertwining operators (cf. Appendix 10) in the form

$$\delta^m_n \Phi = \sum_{i=1}^{3} \langle w', e^{\xi_L W (-1)} \omega_{W V}^m (\Phi_i) u_i \rangle,$$

for some $\xi_i \in \mathbb{C}$, and $u_i \in V$, and $\Phi_i$ obvious from (4.3). Namely,

$$\delta^m_n \Phi = \langle w', e^{\xi_1 L W (-1)} \omega_{W V}^m (\Phi (v_2, c_2(p_2)); \ldots; v_n, c_{n+1}(p_{n+1}), -c_1(p_1)) v_1 \rangle$$

$$+ \sum_{i=1}^{n} (-1)^i e^{\xi_i L W (-1)} \langle w', \omega_{W V}^m (\Phi (\omega V (v_1, c_1(p_i) - c_1(p_{i+1})), -\zeta) \mathbf{1}_V) v_1 \rangle$$

$$= \langle w', e^{c_{n+1}(p_{n+1}) L (-1)} \omega_{W V}^m (\Phi (v_1, c_1(p_1); \ldots; v_n, c_n(p_n), -c_{n+1}(p_{n+1})) v_1 \rangle,$$

for an arbitrary $\zeta \in \mathbb{C}$.

4.1. Complexes of transversal connection. In addition to the double complex $(C^n_m(V, W, F), \delta^m)$ provided by (6.7) and (4.3), there exists an exceptional short double complex which we call transversal connection complex. We have

Lemma 4. For $n = 2$, and $k = 0$, there exists a subspace $C^0_{ex}(V, W, F)$

$$C^2_m(V, W, F) \subset C^0_{ex}(V, W, F) \subset C^0_0(V, W, F),$$

for all $m \geq 1$, with the action of coboundary operator $\delta^0_m$ defined.

Proof. Let us consider the space $C^0_2(V, W, F)$. Vertex operators composable. Indeed, the space $C^0_2(V, W, F)$ contains elements of $W_{c_1(p_1), c_2(p_2)}$ so that the action of $\delta^0_0$ is zero. Nevertheless, as for $J^m_0(\Phi)$ in (13.5), Definition 35, let us consider sum of projections

$$P_r : W_{z_i, z_j} \to W_r,$$

for $r \in \mathbb{C}$, and $(i, j) = (1, 2), (2, 3)$, so that the condition (13.5) is satisfied for some connections similar to the action (13.5) of $\delta^0_0$. Separating the first two and the second two summands in (12), we find that for a subspace of $C^0_2(V, W, F)$, which we denote
as \( C_{xx}^{2}(V, W, F) \), consisting of three-point connections \( \Phi \) such that for \( v_1, v_2, v_3 \in V, w' \in W' \), and arbitrary \( \zeta \in \mathbb{C} \), the following forms of connections

\[
G_1(c_1(p_1), c_2(p_2), c_3(p_3))
= \sum_{r \in \mathbb{C}} (w', E_{W}^{(1)} (v_1, c_1(p_1); P_r (\Phi (v_2, c_2(p_2) - \zeta; v_3, c_3(p_3) - \zeta)))
+ \langle w', \Phi (v_1, c_1(p_1)); P_r (E_{v}^{(2)} (v_2, c_2(p_2) - \zeta; v_3, c_3(z_3) - \zeta; 1_V), \zeta) \rangle
\]

\[
= \sum_{r \in \mathbb{C}} (\langle w', \omega_V (v_1, c_1(p_1)) P_r (\Phi (v_2, c_2(p_2) - \zeta; v_3, c_3(p_3) - \zeta))
+ \langle w', \Phi (v_1, c_1(p_1)); P_r (\omega_V (v_2, c_2(p_2) - \zeta) \omega_V (v_3, c_3(z_3) - \zeta) 1_V), \zeta) \rangle),
\]

(4.4)

and

\[
G_2(c_1(p_1), c_2(p_2), c_3(p_3))
= \sum_{r \in \mathbb{C}} (w', \Phi (P_r (E_{v}^{(2)} (v_1, c_1(p_1) - \zeta, v_2, c_2(p_2) - \zeta; 1_V)), \zeta; v_3, c_3(p_3)))
+ \langle w', E_{W}^{(1)} (P_r (\Phi (v_1, c_1(p_1) - \zeta; v_2, c_2(p_2) - \zeta), \zeta; v_3, c_3(p_3)))) \rangle
\]

\[
= \sum_{r \in \mathbb{C}} (\langle w', \Phi (P_r (\omega_V (v_1, c_1(p_1) - \zeta) \omega_V (v_2, c_2(p_2) - \zeta) 1_V), \zeta), v_3, c_3(p_3) \rangle)
+ \langle w', \omega_V (v_3, c_3(p_3)) P_r (\Phi (v_1, c_1(p_1) - \zeta; v_2, c_2(p_2) - \zeta)) \rangle),
\]

(4.5)

are absolutely convergent in the regions

\[
|c_1(p_1) - \zeta| > |c_2(p_2) - \zeta|,
|c_2(p_2) - \zeta| > 0,
|\zeta - c_3(p_3)| > |c_1(p_1) - \zeta|,
|c_2(p_2) - \zeta| > 0,
\]

where \( c_i, 1 \leq i \leq 3 \) are coordinate functions, respectively, and can be analytically extended to rational form-valued functions in \( c_1(p_1) \) and \( c_2(p_2) \) with the only possible poles at \( c_1(p_1), c_2(p_2) = 0 \), and \( c_1(p_1) = c_2(p_2) \). Note that (4.4) and (4.5) constitute the first two and the last two terms of (1.2) correspondingly. According to Proposition 8 (cf. Appendix 13), \( C_{\alpha}^{2}(V, W, F) \) is a subspace of \( C_{\alpha}^{2}(V, W, F) \), for \( m \geq 0 \), and \( \Phi \in C_{\alpha}^{2}(V, W, F) \) are composable with \( m \) vertex operators. Note that (4.4) and (4.5) represent sums of forms \( G_{\alpha}(p, p') \) of transversal connections (2.6) (cf. Section 5).

**Remark 6.** It is important to mention that, according to general principle, observed in [2], for non-vanishing \( G(c(p), c(p'), c(p'')) \), there exists an invariant structure, e.g., a cohomological class. In our case, it appears as a non-zero subspaces \( C_{\alpha}^{2}(V, W, F) \subset C_{xx}^{2}(V, W, F) \) in \( C_{0}^{2}(V, W, F) \).

Then we have
**Definition 14.** The coboundary operator
\[ \delta_{ex}^2 : C^2_{ex}(V, W, F) \to C^3_0(V, W, F), \] (4.6)
is defined by three point connection of the form
\[ \delta_{ex}^2 \Phi = G_{ex}(p_1, p_2, p_3), \] (4.7)
where
\[ G_{ex}(p_1, p_2, p_3) = \langle w', \omega_V(v_1, c_1(p_1)) \Phi(v_2, c_2(p_2); v_3, c_3(p_3)) \rangle \]
\[ - \langle w', \Phi(\omega_V(v_1, c_1(p_1)) \omega_V(v_2, c_2(p_2)); 1_V; v_3, c_3(p_3)) \rangle \]
\[ + \langle w', \Phi(v_1, c_1(p_1)); \omega_V(v_2, c_2(p_2)) \omega_V(v_3, c_3(p_3)); 1_V \rangle \]
\[ + \langle w', \omega_W(v_3, c_3(p_3)) \Phi(v_1, c_1(p_1); v_2, c_2(p_2)) \rangle, \] (4.8)
for \( w' \in W', \Phi \in C^2_{ex}(V, W, F) \), \( v_1, v_2, v_3 \in V \) and \((z_1, z_2, z_3) \in F_3 \mathbb{C} \).

**Remark 7.** Similar to (4.3), (4.7) can also be written in Frobenius form:
\[ \delta_{ex}^2 \Phi = \mathcal{E}_{ex} \circ_{ex} \Phi, \] (4.9)
where
\[ \mathcal{E}_{ex} = \left( E_W^{(1)}, \sum_{i=1}^{2} (-1)^i E_V^{(2)}_{V:1_V}, E_W^{W:1} \right), \] (4.10)
with the product
\[ \circ_{ex} = \sum_{i=0}^{3} \circ_i, \]

\[ \delta_{ex}^2 \Phi = E_W^{(1)} \circ_0 \Phi + \sum_{i=1}^{2} (-1)^i E_V^{(2)}_{V:1_V} \circ_i \Phi + E_W^{W:1} \circ_3 \Phi, \]

\[ = \langle w', E_W^{(1)}(v_1, c_1(p_1)); \Phi(v_2, c_2(p_2); v_3, c_3(p_3)) \rangle \]
\[ - \langle w', \Phi( E_V^{(2)}(v_1, c_1(p_1); v_2, c_2(p_2)); 1_V; v_3, c_3(p_3)) \rangle \]
\[ + \langle w', \Phi(v_1, c_1(p_1)); E_V^{(2)}(v_2, c_2(p_2); v_3, c_3(p_3)); 1_V \rangle \]
\[ + \langle w', E_W^{W:1}(v_1, c_1(p_1); v_2, c_2(p_2)); v_3, c_3(p_3)) \rangle. \]

Then we have

**Proposition 2.** The operators (1.3) and (4.6) provide the chain-cochain complexes
\[ \delta_m^n : C^m(V, W, F) \to C^{m+1}_m(V, W, F), \] (4.11)
\[ \delta_m^{n+1} \circ \delta_m^n = 0, \] (4.12)
\[ \delta_{ex}^2 \circ \delta_3^1 = 0, \]
\[ 0 \to C^0_m(V, W, F) \xrightarrow{\delta_m^0} C^1_{m-1}(V, W, F) \xrightarrow{\delta_{m-1}^1} \cdots \xrightarrow{\delta_{m-1}^{m-1}} C^m_0(V, W, F) \to 0, \] (4.13)
0 \rightarrow C^n_k(V, W, F) \overset{\delta^n_0}{\longrightarrow} C^1_k(V, W, F) \overset{\delta^1_1}{\longrightarrow} C^2_k(V, W, F) \overset{\delta^2_2}{\longrightarrow} C^3_k(V, W, F) \longrightarrow 0, \quad (4.14)
onumber 

on the spaces \(3.7\).

Since

\[ \delta^1_1 C^2_k(V, W, F) \subset C^2_k(V, W, F) \subset C^2_{ex}(V, W, F), \]

the second formula follows from the first one, and

\[ \delta^2_2 \circ \delta^1_1 = \delta^1_1 \circ \delta^1_1 = 0. \]

**Proof.** The proof of this proposition is analogous to that of Proposition (4.1) in [23] for chain-cochain complex of a grading-restricted vertex algebra. The only difference is that we work with the space \(W_{r+1}(p_1, \ldots, c_n(p_n))\) instead of \(W_{z_1, \ldots, z_n}\). \(\square\)

### 4.2. Fixed point double complexes

Recall the definition of fixed point double complex spaces \(3.16\) in Section [3]. Then we have the following

**Lemma 5.** The double complex \((C^n_k(p'_r; V, W, F), \delta^n_k|p'_r)\) is a subcomplex of double chain-cochain complex \((C^n_k(V, W, F), \delta^n_k)\).

**Proof.** According to Remark [2] in Section [8] we assume that in the construction of \([3.7]\), the points \(\{p'_r\}, 1 \leq r - 1, r + 1 \leq k\) in sequences \([3.14]\) of holonomy mappings move freely along corresponding sections \(\{U_j\} \in \mathcal{U}\). In the intersection in Definition \([5.7]\) of \(C^n_k(V, W, F)\), the points \(\{p'_r\}, 1 \leq r - 1, r + 1 \leq k\) exhaust corresponding sections \(\{U_j\}\) of \(\mathcal{U}\). Thus,

\[ C^n_k(p'_r; V, W, F) \subset C^n_k(V, W, F). \]

It is clear that the operator \(\delta^n_k|p'_r\) is a reduction of \(\delta^n_k\), and satisfies the chain-cochain property as in Proposition [2]. Thus the Lemma is proved. \(\square\)

### 4.3. Cohomology

Now let us define the cohomology of the leaf space \(\mathcal{M}/\mathcal{F}\) associated with a grading-restricted vertex algebra \(V\).

**Definition 15.** We define the \(n\)-th cohomology \(H^n_k(V, W, F)\) of \(\mathcal{M}/\mathcal{F}\) with coefficients in \(W_{z_1, \ldots, z_n}\) (containing maps composable \(k\) vertex operators on \(k\) transversal sections) to be the factor space of closed multi-point connections by the space of connection forms:

\[ H^n_k(V, W, F) = \text{Con}_k; cc_0 / G^n_{k+1}. \quad (4.15) \]

Note that due to \([1.2], [1.3]\), and Definitions \([2.1]\) and \([2.2]\) (cf. Section [1]), it is easy to see that \(4.15\) is equivalent to the standard cohomology definition

\[ H^n_k(V, W, F) = \ker \delta^n_k / \text{im} \delta^{n-1}_{k+1}. \quad (4.16) \]

Recall Definition \([3.16]\) of fixed-point double complex spaces \(C^n_k(p'_r; V, W, F)\). Simultaneously with Definition \([15]\) we formulate

**Definition 16.** Let \(U_r, r \geq 1\), be a section of a basis \(\mathcal{U}\), and \(p'_r \in U_r\) be a fixed point. Here we define the fixed point cohomology as

\[ H^n_{\text{fp}}(V, W, F) = \text{Con}_k; cc_0 / G^n_{k+1}. \quad (4.17) \]
which is equivalent to

\[ H^n_k (p_r'; V, W, F) = \text{Ker} \delta^n_k / \text{Im} \delta^n_k|_{p_r'}. \]

From Lemma 5 it follows

**Lemma 6.** The cohomology \( H^m_n (p; V, W, F) \) is given by

\[ H^m_n (V, W, F) = \bigcup_{p_r' \in U_r} H^m_n (p_r'; V, W, F). \]

4.4. **Relations to Crainic and Moerdijk construction.** In particular, we have the following

**Lemma 7.** In codimension one case, the construction of the double complex \((C^{k,l})_r, \delta_r)\) follows from the construction of the double complex \((C^m_n(V, W, F), \delta^m_n)\) of \((4.13)\). Thus, the Čern-de Rham cohomology of a foliated smooth manifold results from grading-restricted vertex algebra \(V\) cohomology.

**Proof.** One constructs the space of differential forms of degree \(k\) by elements \(\Phi\) of \(C^m_n(V, W, F)\)

\[ \langle w', \Phi \left( \text{wt} (v_1) \otimes c_1 (p_1); \ldots; \text{wt} (v_n) v_n, c_n (p_n) \right) \rangle, \quad (4.18) \]

such that \(n = k\) the total degree

\[ \sum_{i=1}^{n} \text{wt} (v_i) = l, \]

\(v_i \in V\). The condition of composability of \(\Phi\) with \(m\) vertex operators allows us make the association of the differential form \(\omega(h_1, \ldots, h_n)\) with \((4.18)\) \((h^*_1, \ldots, h^*_k)\) with \((v_1, \ldots, v_k)\), and to represent a sequence of holomorphic embeddings \(h_1, \ldots, h_p\) for \(U_0, \ldots, U_p\) in \((2.7)\) by vertex operators \(\omega_W\), i.e.,

\[ (h(h^*_1) \ldots h(h^*_n)) (z_1, \ldots, z_n) = \omega_W (v_1, t_1(p_1)) \ldots \omega_W (v_l, t(p_n)). \]

Then, by using Definitions of coboundary operator \((4.3)\), we see that the definition of the coboundary operator of \([8]\) is parallel to the definition \((4.4)\). \(\square\)

5. **First cohomologies** \(H^1_m(V, W, F)\) **of** **codimension one foliations**

In \([23]\), lower cohomologies for a grading-restricted vertex algebra were computed. In this paper we find the first grading-restricted vertex algebra cohomologies \(H^1_m(V, W, F)\) and the second cohomology \(H^2_{ex}(V, W, F)\) for a codimension one foliation \(F\). Let us first consider one-variable reduction of multi-point connections. Such reduction is called in \([24]\) a derivation. In analogy with a definition of \([23]\), we introduce the following definition of the derivation applicable to maps from \(V\) to \(W\).

**Definition 17.** Let \(V\) be a grading-restricted vertex algebra and \(W\) a \(V\)-module. A grading-preserving linear map

\[ g : V \to W, \]
is called a derivation if
\[ g(\omega_V(u, z)v, 0) = e^{z L(-1)} \omega_W(v, -z) g(u, 0) + \omega_W(u, z) g(v, 0) \]
\[ = \omega_W^W(g(u, 0), z)v + \omega_W(u, z) g(v, 0), \]
for \( u, v \in V \), where \( \omega_W^W(v, z) \) is the intertwiner-valued vertex operator in accordance with notations of (3.5). We use \( \text{Der}(V, W) \) to denote the space of all such derivations. It is clear that
\[ g(v, 0) = G(v, 0). \]

As we see from the definition of a derivation over \( V \), it depends on one element of \( V \) only. The space of one \( V \)-element two point holomorphic connections reduces to the space of derivations over \( W \) [23]. In [24] it is proven the following

**Lemma 8.** Let \( g(v, 0) : V \to W \) be a derivation. Then \( g(1_V, 0) = 0 \).

We will need another statement proven in [24]

**Lemma 9.** Let 
\[ \Phi : V \to W_z, \]
be an element of \( C^1_m(V, W, F) \) satisfying
\[ \delta^1_{m1} \Phi = 0. \]
Then \( \Phi(v, 0) \) is a grading-preserving linear map from \( V \) to \( W \), i.e.,
\[ z L(0) \Phi(v, 0) = \Phi(z L(0)v, 0) = z^n \Phi(v, 0). \]

In [24], the first cohomologies \( H^1_M(V, W) \) of a grading-restricted vertex algebra were related to the space of derivations over \( V \). We find the following

**Proposition 3.** Let \( V \) be a grading-restricted vertex algebra and \( W \) a \( V \)-module.
Then \( H^1_m(V, W) \) is linearly isomorphic to the space \( \text{Der}(V, W) \) of derivations from \( V \) to \( W \) for any \( m \in \mathbb{Z}_+ \).

In the case of a foliation, we have the following identifications in [24]
\[ G(\phi(p)) = G(v, c(p)) = \Phi(v, c(p)), \]
\[ f(\psi(p)) = \omega(v, c(p)), \]
\[ \phi(p) = (u, p), \]
\[ f(\psi(p')), \phi(p) = \omega(v, c(p') - c(p))u, \]
and a multi-point holomorphic connection \( G \) on \( M/F \), is a \( \mathbb{C} \)-linear map
\[ G : V^{\otimes n} \to W_{z_1, \ldots, z_n}. \]
Thus the multi-point holomorphic connection takes the form
\[ \sum_{q, q' \in M/F} \Phi(\omega_V(v, c(q) - c(q'))u, q) = \omega_W(u, c(p')) \Phi(v, c(p)) + \omega_W(v, c(p)) \Phi(u, c(p')). \]
\[ (5.2) \]

**Remark 8.** Due to Lemma [2] and Proposition [1] the definition of the multi-point holomorphic connection on \( M/F \) is canonical, i.e., it does not depend on the choice of \( U \) and coordinates on \( M/F \) and \( U \).
The meaning of the name of a transversal holomorphic connection \((2.5)\) is clear when we consider elements of the space \(W_{z_1, \ldots, z_n}^p\) for \(M/F\),
\[
G(p, p') = \omega_W(u, c(p')) G(v, c(p)) + \omega_W(u, c(p')) G(u, c(p')) = 0,
\]
with formal parameters associated to local coordinates \(c(p)\). In particular, when \(\omega_W(u, t(p))\) is considered on a the transversal section, and \(G(v, l(p))\) on a leaf of \(M/F\), it relates objects on mutually transversal structures. This type of connections will appear in considerations of the second vertex algebra cohomology \(H^2_{\alpha}(V, W, F)\) in Section 4.

In what follows, to shortcut notations, we will denote by \(p\) the origin of a local coordinate \(c(p)\) at \(p\), i.e., \(c(p)|_p = 0\). Let us introduce another

**Definition 18.** A one fixed-point \(p'\) holomorphic connection for the space \((5.15)\) is defined by
\[
\sum_{q, q' \in M/F} \Phi(\omega_V(v_q, c(q) - c(q'))u, q) = \omega_W(u, p') \Phi(v, c(p)) + \omega_W(v, c(p)) \Phi(u, p').
\]

In particular, for the space \(C^1_m(p_r; V, W, F)\) we obtain
\[
\Phi(\omega_V(v, p' - c(p))u, c(p)) = \omega_W(u, p') \Phi(v, c(p)) + \omega_W(v, c(p)) \Phi(u, p'),
\]
(5.4)

We denote the space of such connections with a fixed point \(p\) as \(\text{Con}_{m'}(m; V, W)\). In Section 4 we have introduced the notion (Definition 16) of a fixed-point cohomology \(H^1_m(p; V, W, F)\). In particular, for \(n = 1\),
\[
H^1_m(p_r; V, W, F) = \text{Ker} \delta^1_m/\text{Im} \delta^0_{m+1}|_{p_r'},
\]
for a point \(p_r' \in U_r\) of a transversal basis \(U\). The result of this section is in the following

**Proposition 4.** The vertex algebra first cohomologies \(H^1_m(V, W, F)\), \(m \geq 0\) of a codimension one foliation \(F\) are isomorphic to the space \(\text{Con}_{m'}(m; V, W)\), for all \(p_r' \in U_r, 1 \leq r \leq m\), of holomorphic fixed point two point connections on the space of leaves \(M/F\) with mappings composable with \(m\) vertex operators on transversal sections.

**Remark 9.** In contrast to the cohomologies \(H^1_m(V, W)\) for a grading-restricted vertex algebra \([23]\), the cohomologies \(H^1_m(V, W, F)\) are not isomorphic for various \(m\), since they contain dependence not on just formal parameters, but these formal parameters are identified with local coordinates around points on \(M\) on either the leaves of \(M/F\) or transversal sections. Indeed, connections \(G_m(v, z)\) are elements of the space \(C^1_m(V, W, F)\), i.e., they are composable with \(m\) vertex operators.

Now we proceed with the proof of Proposition 4.

**Proof.** Let us fix a point \(p_r'\) with the local coordinate \(t_r(p_r')\) on the transversal section \(U_r\) with origin at \(p_r'\), i.e., \(t_r(p_r')|_{p_r} = 0\). According to Proposition 3 (cf. (1.1) in \([24]\)), the cohomologies \(H^1_m(V, W)\) of \(V\) are given by the space of derivations. In terms of
Definition [3] it coincides with the space of fixed point holomorphic connections, i.e., \( \text{Der}(V, W) = \text{Con}_{p'}(V, W) \). Note that, for any
\[
\Phi(v, c_r(p'_r)) |_{p'_r} \in C^{1}_m(p'_r; V, W, F),
\]
such that
\[
\langle w', \delta^m_{p'} \Phi(v, p'_r) \rangle = \langle w', G_2(p'_r, p_2) \rangle = 0,
\]
i.e.,
\[
0 = \langle w', \omega_W(v_1, p'_r) \Phi(v_2, c_2(p_2)) \rangle - \langle w', \Phi(\omega_V(v_1, p'_r - c_2(p_2))v_2, c_2(p_2)) \rangle + \langle w', \omega_W(v_2, c_2(p_2)) \Phi(v_1, p'_r) \rangle, \tag{5.5}
\]
i.e., results in an element of the space \( \text{Con}_{p'}(V, W) \) of one fixed point \( p'_r \) holomorphic connections. In addition, by direct computation for any \( \Phi' \in C^{0}_m(p'_r; V, W, F) \), we find
\[
\langle w', \delta^0_{m+1} \Phi' \rangle = \langle w', \omega_V(v, z) \Phi' \rangle - \langle w', \omega_V(v, z) \Phi' \rangle = 0.
\]
i.e.,
\[
\text{Im} \delta^0_{m+1} \Phi' = \{0\}.
\]
Conversely, for any element \( g(v, 0) \) of \( \text{Con}_{p'}(V, W) \), and \( v \in V \), let us consider
\[
\Phi_g = g(\omega_V(v, z)1_v, p'_r) = \omega^W_{V}(g(v, p'_r), z)1_v, \tag{5.6}
\]
where we have used Lemma [8]. We had to express [5.6] in terms of intertwining operator in order to show that [5.6] is indeed composable with \( m \) vertex operators and belong to the space \( C^{1}_m(p'_r; V, W, F) \) with a fixed point \( p'_r \). As it follows from [10], the map from \( V \) to \( W \) given by
\[
v \mapsto \omega^W_{V}(\Phi_g(v, p'_r), z_1)1_v,
\]
is composable with \( m \) vertex operators for any \( m \in \mathbb{N} \). Thus \( \Phi_g \in C^{m}_m(V, W, F) \) for any \( m \in \mathbb{N} \). For \( v_1, v_2 \in V \), and \( w' \in W' \), by using [10,10], we find by direct computation
\[
\langle w', \delta^1_{m} \Phi_g(v_1, c_1(p_1); v_2, c_2(p_2)) \rangle = \langle w', \omega_W(v_1, c_1(p_1)) \omega^W_{V}(g(v_2, p'_r), c_2(p_2))1_v \rangle - \langle w', \omega^W_{V}(g(\omega_V(v_1, c_1(p_1) - c_2(p_2))v_2, c_2(p_2))1_v) \rangle + \langle w', \omega_W(v_2, c_2(p_2)) \omega^W_{V}(g(v_1, p'_r), c_1(p_1))1_v \rangle
\]
\[
= \langle w', \omega_W(v_1, c_1(p_1)) \omega^W_{V}(g(v_2, p'_r), c_2(p_2))1_v \rangle - \langle w', \omega^W_{V}(g(\omega_V(v_1, c_1(p_1) - c_2(p_2))v_2, c_2(p_2))1_v) \rangle + \langle w', \omega_W(v_2, c_2(p_2)) \omega^W_{V}(g(v_1, c_1(p_1))1_v) \rangle
\]
connections of \( C \) obtain the statement of Proposition:

By moving \( C \) separate problem to introduce a product defined on one or among various spaces and leading to the integrability condition. As we mentioned in Section 4, it is a condition (in particular, the orthogonality condition) applied to differential forms, classes for codimension one foliations \([8, 15, 28]\) are introduced by means of an extra vertex algebra cohomologies for codimension one foliations. Usually, the cohomology from the space

Thus we find, that

By using Theorem 5.6.2 in \([10]\), we derive that (5.7) vanishes. Therefore we obtain a linear map

from the space

Thus we find, that

By moving \( p'_r \in U_r \) all along \( t_r(p'_r) \) we exhaust to all points at \( U_r \), we obtain connections of \( Con_U, (V, W) \) on the whole \( U_r \). By using Lemma \( 5\) we extend (5.8) to we obtain the statement of Proposition:

\[
H^1_m(V, W, \mathcal{F}) = \bigcup_{p_r \in U_r} Con_{p'_r}(V, W).
\]

\[\square\]

6. Cohomological classes

In this section we describe certain classes associated to the first and the second vertex algebra cohomologies for codimension one foliations. Usually, the cohomology classes for codimension one foliations \([8, 15, 28]\) are introduced by means of an extra condition (in particular, the orthogonality condition) applied to differential forms, and leading to the integrability condition. As we mentioned in Section 4 it is a separate problem to introduce a product defined on one or among various spaces \( C^m_n(V, W, \mathcal{F}) \) of (3.7). Note that elements of \( \mathcal{E} \) in (4.3) and \( \mathcal{E}_{\infty} \) in (4.10) can be seen as elements of spaces \( C^1_\infty(V, W, \mathcal{F}) \), i.e., maps composable with an infinite number of vertex operators. Though the actions of coboundary operators \( \delta^m \) and \( \delta^\infty \) in (4.3) and (4.10) are written in form of a product (as in Frobenius theorem \([15]\)), and, in contrast to the case of differential forms, it is complicated to use these products for further formulation of cohomological invariants and derivation of analogues of the Godbillon-Vey invariants. Nevertheless, even with such a product yet missing, it is possible to introduce the lower-level cohomological classes of the form \([\delta^m \eta] \) which are counterparts of the Godbillon class \([14]\). Let us give some further definitions. By analogy with differential forms, let us introduce
Definition 19. We call a map \( \Phi \in C^n_k(V, W, \mathcal{F}) \), closed if it is a closed connection:
\[
\delta_k^n \Phi = G(\Phi) = 0.
\]
For \( k \geq 1 \), we call it exact if there exists \( \Psi \in C^{n+1}_{k-1}(V, W, \mathcal{F}) \) such that \( \Psi = \delta_k^n \Phi \), i.e., \( \Psi \) is a form of connection.

For \( \Phi \in C^n_k(V, W, \mathcal{F}) \) we call the cohomology class of mappings \([\Phi]\) the set of all closed forms that differ from \( \Phi \) by an exact mapping, i.e., for \( \chi \in C^{n-1}_{k+1}(V, W, \mathcal{F}) \),
\[
[\Phi] = \Phi + \delta_{k+1}^{n-1} \chi.
\]

As we will see in this section, there are cohomological classes, (i.e., \( [\Phi] \), \( \Phi \in C^1_n(V, W, \mathcal{F}) \), \( m \geq 0 \)), associated with two-point connections and the first cohomology \( H^1_m(V, W, \mathcal{F}) \), and classes (i.e., \( [\Phi] \), \( \Phi \in C^2_n(V, W, \mathcal{F}) \)), associated with transversal connections and the second cohomology \( H^2_{ex}(V, W, \mathcal{F}) \), of \( \mathcal{M}/\mathcal{F} \). The cohomological classes we obtain are vertex algebra cohomology counterparts of the Godbillon class [14, 28] for codimension one foliations.

Remark 10. As it was discovered in [2,3], it is a usual situation when the existence of a connection (affine of projective) for codimension one foliations on smooth manifolds prevents corresponding cohomology classes from vanishing. Note also, that for a few examples of codimension one foliations, the cohomology class \( [d\eta] \) is always zero.

Remark 11. In contrast to [2], our cohomological class is a functional of \( v \in V \). That means that the actual functional form of \( \Phi(v, z) \) (and therefore \( \langle w', \Phi \rangle \), for \( w' \in W' \)) varies with various choices of \( v \in V \). That allows one to use it in order to distinguish types of leaves of \( \mathcal{M}/\mathcal{F} \).

6.1. Classes associated with the first cohomologies \( H^1_m(V, W, \mathcal{F}) \). For the first cohomology \( H^1_m(V, W, \mathcal{F}) \), we have the following corollary from Proposition 4:

Corollary 1. The \( H^1_m(V, W, \mathcal{F}) \) cohomological class of the grading-restricted vertex algebra cohomology of the leaf space \( \mathcal{M}/\mathcal{F} \) is given by
\[
[\delta^1_m \Phi],
\]
for \( \Phi \in C^1_m \). It’s vanishing if and only if \( \Phi \) is given by a two point holomorphic connection.

Remark 12. Non-vanishing cohomological invariants of the form (6.1) are used in Section 7 in order to characterize leaves of \( \mathcal{M}/\mathcal{F} \) and transversal sections.

Proof. \( [\delta^1_m \Phi] \) for \( \Phi \in C^1_m \). It is easy to see that it remains cohomologically invariant under a substitution
\[
\Phi \mapsto \Phi + \Phi_0,
\]
due to properties of (6.1). The second statement of the proposition follows from the proof of Proposition 4. In Subsection 7.2 we will explain which role the cohomological invariant (6.1) for foliation \( \mathcal{F} \). □
6.2. Classes associated with exceptional cohomology. In this subsection we consider the exceptional cohomology $H^2_{ex}(V, W, F)$ associated to the short complex $\begin{array}{ll} 0 & \rightarrow W \\ \downarrow & \downarrow h \\ V_{W,1} & \rightarrow V \\ \downarrow & \downarrow \\ V & \rightarrow 0 \\ \end{array}$, and corresponding cohomological class. Let us first recall some definitions [24] concerning the notion of square-zero extension of $V$ by its module $W$ which is an analogue of the notion of square-zero extension of an associative algebra by a bimodule (cf. [51]).

**Definition 20.** Let $V$ be a grading-restricted vertex algebra. A square-zero ideal of $V$ is an ideal $W$ of $V$ such that for any $u, v \in W$,

\[ Y_V(u, x^0)v = 0. \]

**Definition 21.** Let $V$ be a grading-restricted vertex algebra and $W$ a $\mathbb{Z}$-graded $V$-module. A square-zero extension $(V_W, \gamma, \alpha)$ of $V$ by $W$ is a grading-restricted vertex algebra $V_W$ together with a surjective homomorphism

\[ \gamma : V_W \rightarrow V, \]

of grading-restricted vertex algebras such that ker $\gamma$ is a square-zero ideal of $V_W$ (and therefore a $V$-module) and an injective homomorphism $\alpha$ of $V$-modules from $W$ to $V_W$ such that

\[ \alpha(W) = \ker \gamma. \]

**Definition 22.** Two square-zero extensions $(V_{W,1}, \gamma_1, \alpha_1)$ and $(V_{W,2}, \gamma_2, \alpha_2)$ of $V$ by $W$ are equivalent if there exists an isomorphism of grading-restricted vertex algebras $h : V_{W,1} \rightarrow V_{W,2}$ such that the diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & W & \rightarrow & V_{W,1} & \rightarrow & V & \rightarrow & 0 \\
\downarrow & & \downarrow h & & \downarrow & & \downarrow & & \\
0 & \rightarrow & W & \rightarrow & V_{W,2} & \rightarrow & V & \rightarrow & 0,
\end{array}
\]

is commutative.

Let $(V_W, \gamma, \alpha)$ be a square-zero extension of $V$ by $W$. It is possible to construct a realization of the square-zero extension of $V$ by $W$ on $Z = V \oplus W$. Then there exists an injective linear map $\Gamma : V \rightarrow V_W$, such that the linear map

\[ h : Z \rightarrow V_W, \]

given by

\[ h(v, w) = \Gamma(v) + \alpha(w), \]

is a linear isomorphism. By definition, the restriction of $h$ to $W$ is the isomorphism $\alpha$ from $W$ to $\ker \gamma$. Then the grading-restricted vertex algebra structure and the $V$-module structure on $V_W$ give a grading-restricted vertex algebra structure and a $V$-module structure on $Z$ such that the embedding $i_2 : W \rightarrow Z$ and the projection $p_1 : Z \rightarrow V$, are homomorphisms of grading-restricted vertex algebras. In addition
to that, \( \ker p_1 \) is a square-zero ideal of \( Z \), \( i_2 \) is an injective homomorphism such that \( i_2(W) = \ker p_1 \) and the diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & W & \overset{i_2}{\longrightarrow} & Z & \overset{p_1}{\longrightarrow} & V & \longrightarrow & 0 \\
& & \downarrow{\text{Id}_W} & & \downarrow{h} & & \downarrow{\text{Id}_V} & & \\
0 & \longrightarrow & W & \overset{\alpha}{\longrightarrow} & V_W & \overset{\gamma}{\longrightarrow} & V & \longrightarrow & 0
\end{array}
\] (6.2)

of \( V \)-modules is commutative. Thus one obtains a square-zero extension \( (Z, p_1, i_2) \) equivalent to \( (V_W, \gamma, \alpha) \). It is enough then to consider square-zero extensions of \( V \) by \( W \) of the particular form \( (Z, p_1, i_2) \). The difference between two such square-zero extensions consists in the vertex operator maps. Such square-zero extensions will be denoted by \( (Z, Y_Z, p_1, i_2) \).

Let us first mention the geometrical meaning of the square-zero extension \( (V_W, \gamma, \alpha) \) of \( V \) by \( W \). Let us consider \( u, v \) belong to the square-zero ideal of a grading-restricted vertex algebra \( V \), then

\[
\omega_V(u, c(p))v = 0.
\]

Then, geometrically it means that corresponding vertex algebra holomorphic connections are transversal (cf. Definition 5):

\[
G_{tr}(p, p') = \omega_W(v, c(p')) \Phi(u, c(p)) + \omega_W(u, c(p)) \Phi(\psi, c(p')) = 0. \tag{6.3}
\]

Note that, for a square-zero ideal, the full form of holomorphic connection has a reduced form (6.3). In [23] it was shown that certain cohomological class vanishes if and only if the exist an affine or projective connection. In our setup the holomorphic connection plays a similar role: if it does not have an full closed form (5.2), then the cohomology class it non-trivial.

In [24] we find the proof of the following algebraic result for the second cohomology of a grading-restricted vertex algebra \( V \), \( H^2_{cz}(V, W) \) of \( V \) with coefficients in \( W \). It follows from that Proposition, that the difference between two square-zero extensions are controled by the vertex operator map for the square-zero extension defined for \( Z = V \oplus W \).

**Proposition 5.** Let \( V \) be a grading-restricted vertex algebra and \( W \) a \( V \)-module. Then the set of the equivalence classes of square-zero extensions of \( V \) by \( W \) corresponds bijectively to \( H^2_{cz}(V, W) \).

Now we formulate the following corollary from Proposition [2]

**Corollary 2.** Let \( V \) be a grading-restricted vertex algebra and \( W \) a \( V \)-module. The classes of square-zero extensions of \( V \) by \( W \) are isomorphic to classes of cohomological invariants \( \Phi \) of \( H^2_{cz}(V, W, F) \).
Proof. Now let us consider the cohomology $H^2_{cz}(V, W, F)$. Here, for $\Phi \in C^2_{cz}(V, W, F)$, the kernel of $\delta^2_{cz}$ has the form of closed three-variable connection:

$$0 = \delta^2_{cz} \Phi = G(p_1, p_2, p_3)$$

$$= \langle w', \omega_W(v_1, c_1(p_1)) \Phi(v_2, c_2(p_2)); v_3, c_3(p_3) \rangle$$

$$- \langle w', \Phi(v_1, c_1(p_1)) \omega_W(v_2, c_2(p_2)); v_3, c_3(p_3) \rangle$$

$$+ \langle w', \Phi(v_1, c_1(p_1)) \omega_W(v_2, c_2(p_2)); v_3, c_3(p_3) \rangle$$

$$- \langle w', \omega_W(v_1, c_1(p_1)) \Phi(v_2, c_2(p_2)); v_3, c_3(p_3) \rangle,$$

for $w' \in W'$, $\Phi \in C^2_{cz}(V, W, F)$, $v_1, v_2, v_3 \in V$, and $(z_1, z_2, z_3) \in F_3 \mathbb{C}$. For $\Phi' \in C^1_{cz}(V, W, F)$, the image of $\delta^1_{cz}$, has a non-closed connection form:

$$\delta^1_{cz} \Phi' = G_{\text{cz}}(p_1, p_2)$$

$$= \langle w', \omega_W(v_1, c_1(p_1)) \Phi'(v_2, c_2(p_2)) \rangle$$

$$- \langle w', \Phi'(v_1, c_1(p_1)) \omega_W(v_2, c_2(p_2)) \rangle$$

$$+ \langle w', \omega_W(v_1, c_1(p_1)) \Phi'(v_2, c_2(p_2)) \rangle,$$  \hspace{1cm} (6.4)

for $w' \in W'$, $\Phi \in C^1_{cz}(V, W, F)$, $v_1, v_2 \in V$ and $(z_1, z_2) \in F_2 \mathbb{C}$. The cohomology $H^2_{cz}(V, W, F)$ is given by the factor-space of closed three-variable connections over non-closed two-variable connections. We will show now that this factor space is actually parameterized by the vertex operator maps for a square-zero extension of $V$ by $W$.

The explicit definition for $Z$-vertex operator was introduced in [23]. We denote by $(Z, Y_Z, p_1, i_2)$ a suitable square-zero extension of $V$ by $W$. Then there exists

$$\omega_{\Phi} (u, z) v \in W((z)),$$

for $u, v \in V$ such that

$$\omega_Z((v_1, 0), z)(v_2, 0) = (\omega_V(v_1, z)v_2, \omega_{\Phi}(v_1, z)v_2),$$

$$\omega_Z((v_1, 0), z)(0, w) = (0, \omega_V(v_1, z)w_2),$$

$$\omega_Z((0, w_1), z)(v_2, 0) = (0, \omega_{W_V}(w, z)v_2),$$

$$\omega_Z((0, w_1), z)(0, w_2) = 0,$$

for $v_1, v_2 \in V$ and $w_1, w_2 \in W$. Thus one has

$$\omega_Z((v_1, w_1), z)(v_2, w_2) = (\omega_V(v_1, z)v_2, \omega_{W_V}(v_1, z)w_2 + \omega_{W_V}(v_1, z)v_2 + \omega_{\Phi}(v_1, z)v_2),$$

for $v_1, v_2 \in V$ and $w_1, w_2 \in W$. The vacuum of $Z$ is given by $(1_V, 0)$, and

$$\omega_{\Phi}(v, z)1_V = 0,$$

and the dual space $Z'$ for $Z$ is identified with

$$Z' = V' \oplus W'.$$

By Definition [23] of a grading-restricted vertex algebra, for $v, v' \in V$, vertex operators $\omega_{\Phi}(v, z)$ and $\omega_{V}(v', z')$ in extension $(V_W, \gamma, \alpha)$, satisfy the associativity property, i.e.,
their matrix elements of (10.2) converge (under appropriate conditions for local coordinates of points) to the same \( \mathcal{W}_{z_1, z_2} \)-valued rational function. Thus, for \( v_1, v_2 \in V \), and \((z_1, z_2) \in F_2 \mathbb{C}\), we introduce a linear map

\[
\Phi : V \otimes V \to \mathcal{W}_{z_1, z_2},
\]

\[
\Phi(v_1, z_1; v_2, z_2) = E(\omega_\Psi(v_1, z_1) \omega_V(v_2, z_2) 1_V)
\]

As in the proof of Proposition 2, we check that \( \Phi \) satisfies the \( L(-1) \)-derivative and \( L(0) \)-conjugation properties. Since \( Z \) is a grading-restricted vertex algebra, by using the associativity property for vertex operators (6.5), we see that the conditions (4.4) and (4.5) for forms \( G_i \), \( i = 1, 2 \), in the proof of Lemma 4 of the space \( C^2_{ex}(V, W, F) \) for \( \Phi \) are satisfied, and \( \Phi \in C^2_{ex}(V, W, F) \). Using again corresponding associativity properties for vertex operators in \( Z \), we find that \( \Phi \) is closed (according to our Definition 19), i.e.,

\[
\delta^2_{ex} \Phi = 0.
\]

Thus, we see that, for a representative of the class of square-zero extension \((Z, Y_Z, p_1, i_2)\) corresponds by the formula (6.6) for \( \omega_Z \) to an element of \( H^2_{ex}(V, W, F) \),

\[
[\Phi] = \Phi + \eta,
\]

where \( \eta \) be an element \( \delta^3 C^3_{ex}(V, W, F) \). It is easy to see that, according to properties of the above construction \( \Phi \) is invariant with respect to a substitution

\[
\Phi \mapsto \Phi + \mu,
\]

for \( \mu \in C^1_{ex}(V, W, F) \). Thus, \( \Phi \) belongs to the cohomology class \( H^2_{ex}(V, W, F) \).

Let us prove the inverse statement. For an element \( \Phi \in C^2_{ex}(V, W, F) \) which is a representative of \( H^2_{ex}(V, W, F) \), according to Definition 19 of composibility, it follows that for any \( v_1, v_2 \in V \), there exists \( N^1_0(v_1, 0) \) such that for \( w' \in W' \),

\[
G_2(c_1(p_1), c_2(p_2)) = \langle w', \Phi(v_1, z_1; v_2, z_2) \rangle,
\]

is a rational \( \mathcal{W}_{z_1, z_2} \)-valued form with the only possible pole at \( z_1 = z_2 \) of order less than or equal to \( N^1_0(v_1, v_2) \). For \( v_1, v_2 \in V \), let us define \( \omega_\Psi(v_1, \zeta) v_2 \in \mathcal{W}(\zeta) \) such that

\[
\langle w', \omega_\Psi(v_1, \zeta) v_2 \rangle|_{\zeta=0} = \langle w', \Phi(v_1, z; v_2, z_2) \rangle,
\]

for \( z \in \mathbb{C}^\times \). For \( v_1, v_2 \in V \), we can define \( Y_Z(v_1, \zeta) v_2 \) using (6.5). Thus, we obtain a vertex operator map \( Y_Z \), and \( Z \) is endowed with the structure of a grading-restricted vertex algebra. Finally, we have

**Corollary 3.** Two elements of \( \ker \delta^3_{ex} \) differ by an element \( \delta^3 C^3_{ex}(V, W, F) \) if and only if the corresponding square-zero extensions of \( V \) by \( W \) are equivalent. \( \Box \)
Characterization of leaves and transversal sections of foliations on complex curves

In this section we consider a general formulation of characterization of $M/F$ in codimension one case by means of rational functions of invariants. Let us introduce further notations, for $n \geq 0$,
\[ x = (x_1, \ldots, x_n), \]
for $n$ vertex algebra element, formal parameters, points, etc. Introduce

**Definition 23.** For an element $\Phi(v, c(p)) \in W_{c_1(p_1), \ldots, c_n(p_n)}$ let us call $n$-variable rational function valued form
\[ R(z) = \langle w', \Phi(v, c(p)) \rangle, \quad (7.1) \]
the characteristic form.

We have used this form for the construction of chain complexes in Section 4. In certain cases, depending on properties of $F(v, z)$, one is able to compute this matrix element explicitly.

By varying vertex algebra elements $v_i$, one can vary the the form of dependence of $\Phi(v, c(p))$ on $v$, and, therefore, obtain various functions of $R(z)$. By using the freedom of choice of $v \in V$, we could try to find a suitable pattern for of $\Phi(v, c(p))$ (as a functional of $v$), in such a way (7.1) would result to a specific differential form. Since $\Phi(v, c(p))$ belongs to $C^n_m(V, W, F)$ for some $n, m$, it is important to mention that, due to our formulation in terms of matrix elements, (7.1), associated to cohomological invariants are supposed to be absolutely convergent in suitable domains of $M/F$.

Depending on analytical properties with respect to local coordinates on $c(p)$ of (7.1) one can use it in order to characterize or distinguish particular leaves and transversal sections on $M/F$. For that purpose one can also integrate (7.1) along (closed) paths either on a leaf of $M/F$ or on a transversal section of $\mathcal{U}$. For that purpose we introduce

**Definition 24.** We call a multiple integral
\[ F(z') = \int_{(p_1)}^{(p_2)} R(c(p)) \], \quad (7.2) \]
the characteristic function for $M/F$, where $(p_i)$, $i = 1, 2$ denote limiting points of integration.

The idea of integration of $R(c(p))$ goes back to [49]. In Proposition 1 we proved, in particular, that elements of spaces $C^n_m(V, W, F) \in W_{z_1, \ldots, z_n}$ are invariant with respect to changes of formal parameters $(z_1, \ldots, z_n)$. In Definition 23 of a characteristic form we use such elements, and, therefore, (7.1), containing wt $(v_i)$, $1 \leq i \leq n$, of corresponding differentials, is also invariant with respect to action of $(\text{Aut} \, O^{(1)})^n$.

Below we enumerate $W_{z_1, \ldots, z_n}$-elements suitable for such characterization.
7.1. Composibility condition. Let us start with forms associated to the composibility conditions. For $l_1, \ldots, l_n \in \mathbb{Z}_+$ such that $l_1 + \cdots + l_n = n + m$, define $k_1 = l_1 + \cdots + l_{i-1} + 1$, ..., $k_i = l_1 + \cdots + l_{i-1} + l_i$. Consider a set of $p_{k_1}, \ldots, p_{k_n}$ with local coordinates $c_{k_1}(p_{k_1}), \ldots, c_{k_n}(p_{k_n})$, on $\mathcal{M}$ for points on $\mathcal{M}/\mathcal{F}$. Then, for $v_1, \ldots, v_{n+m} \in V$ and $w' \in W'$, one defines (13.1) and there exist positive integers $N_m^n(v_i, v_j)$ depending only on $v_i$ and $v_j$ for $i, j = 1, \ldots, k$, $i \neq j$ such that the series (13.3) is absolutely convergent when for $l_p = l_1 + \cdots + l_{i-1} + p$, $l_q = l_1 + \cdots + l_{j-1} + q$,

$$|c_{l_p}(p_i) - \zeta_i| + |c_{l_q}(p_q) - \zeta_l| < |\zeta_i - \zeta_l|,$$

(7.3)

for $i, j = 1, \ldots, k$, $i \neq j$ and for $p = 1, \ldots, l_i$ and $q = 1, \ldots, l_j$. Note that in (13.3) the original variables $z_i$ are present in combinations (13.1) only, and the conditions on domains of convergence are express through such combinations $c_{l_p}(p_i)$ and $c_{l_q}(p_q)$, and some $\zeta_i$ which could be identified with other local coordinates on $\mathcal{M}$ for $\mathcal{M}/\mathcal{F}$. Thus we obtain an external (with respect to original coordinates) condition on $I_m^n(\Phi)$. Geometrically this means that the sum of shifts in domains of convergency with respect to $c_{l_p}(p_i)$ and $c_{l_q}(p_q)$ are smaller than difference for other two points with local coordinates $\zeta_i$ and $\zeta_j$. It is also assumed that the sum must be analytically extended to a rational function in $(c_{l_p}(p_i), \ldots, c_{l_p}(p_{m+n}))$, independent of $(\zeta_1, \ldots, \zeta_n)$, with the only possible poles at $c_i(p_i) = c_j(p_j)$, of order less than or equal to $N_m^n(v_i, v_j)$, for $i, j = 1, \ldots, k$, $i \neq j$.

Consider the second condition in Definition 35. For $v_1, \ldots, v_{m+n} \in V$, there exist positive integers $N_m^n(v_i, v_j)$, depending only on $v_i$ and $v_j$, for $i, j = 1, \ldots, k$, $i \neq j$, such that for $w' \in W'$, such that (13.3) is absolutely convergent when $z_i \neq z_j$, $i \neq j$

$$|c_i(p_i)| > |c_k(p_k)| > 0,$$

(7.4)

for $i = 1, \ldots, m$, and $k = m+1, \ldots, m+n$, and the sum can be analytically extended to a rational function in $(z_1, \ldots, z_{m+n})$ with the only possible poles at $z_i = z_j$, of orders less than or equal to $N_m^n(v_i, v_j)$, for $i, j = 1, \ldots, k$, $i \neq j$.. Elements $\Phi$ of spaces $C_m^n(V, W, \mathcal{F})$ (3.4) are composable with $m$ vertex operators, and, therefore possess properies described above. Due to absolute convergency in the regions (7.3) and (7.4) on $\mathcal{M}/\mathcal{F}$, forms $I_m^n(\Phi)$ and $J_m^n(\Phi)$ locally characterize $\mathcal{M}/\mathcal{F}$.

7.2. The cohomological class $[\delta^1_m \Phi]$ of $H^1_m(V, W, \mathcal{F})$. In Section 3 we have proved that for $\Phi \in C^1_m(V, W, \mathcal{F})$, the invariant $\delta^1_m \Phi$ vanishes if and only if $\Phi$ is a one fixed point holomorphic connection. Here, since $\Phi \in C^1_m(V, W, \mathcal{F})$ and $\delta^1_m \Phi \in C^2_m(V, W, \mathcal{F})$, we have the characteristic two-form for $\mathcal{M}/\mathcal{F}$

$$\mathcal{R}(c(p), c(p')) = \langle w', (\delta^1_m \Phi)(v, c(p)); v', c(p') \rangle.$$

(7.5)

In Section 6 we proved that $\delta^1_m \Phi$ represents a cohomological class of $H^1_m(V, W, \mathcal{F})$. For a characterization of leaves of $\mathcal{M}/\mathcal{F}$ we may choose instead elements

$$\Phi_g = g(v, 0) \in \mathcal{W},$$

which do not depend on $z$, and, hence, then matrix elements become computable. For non-vanishing invariants (5.1) (i.e., not two-point connection valued $G(\Phi)$) we obtain
The form of the dependence of $\Phi$ or $g$ graded-restricted vertex algebra $V$ either a local coordinate $t$ coordinate $i$

$\Phi$ has nice convergency properties. Moreover, they contain only parts of the connection functions do not vanish), and can be used in order to describe leaves or sections of $M/F$. For the invariant related to the second cohomology $H_2^2(V, W, F)$, we obtain for (7.6)

\begin{equation}
F(c(p), c(p'), c(p'')) = \langle w', \Phi(v, c(p); v', c(p'); v'', c(p'')) \rangle.
\end{equation}

In addition to (7.8), one uses the particular form of forms $G_i$, $i = 1, 2$

$G_1(p_1, p_2, p_3) = \langle w', \omega_\Psi(v_1, c_1(p_1)) \omega_V(v_2, c_2(p_2)) \omega_V(v_3, c_3(p_3))1_V \rangle$

$+ \langle w', \omega_\Psi(v_1, c_1(p_1)) \omega_\Psi(v_2, c_2(p_2)) \omega_\Psi(v_3, c_3(p_3))1_V \rangle$,

and

$G_2(p_1, p_2, p_3) = \langle w', \omega_\Psi(\omega_\Psi(v_1, c_1(p_1) - c_2(p_2))v_2, c_2(p_2)) \omega_V(v_3, c_3(p_3))1_V \rangle$

$+ \langle w', \omega_\Psi^W(\omega_\Psi(v_1, c_1(p_1) - c_2(p_2))v_2, c_2(p_2)) \omega_\Psi(v_3, c_3(p_3))1_V \rangle$.

(4.4) and (4.5) in (7.7) (cf. Subsection 4.1. In particular, these invariants allow to show the transversality of cycles for foliations defined by the vanishing real part $\text{Re } \Omega = 0$,

of one form $\Omega$ on compact Riemann surface $[9][38][40]$, in the hyperelliptic case,

$w^2 = P_{2g+2}(z) = \prod_j (z - z_j), \quad z_j \neq z_i,$

where $P_{2n+1}(z)$ is a polynomial. It would be interesting to find a way how to distinguish non-differomorphic Reeb components for foliations of the torus $[3][4][49]$. These questions will be addressed in elsewhere.
The crucial question is how one could distinguish (locally and globally) leaves and transversal sections of a foliation. In particular, we should be able to describe singular points (such as, e.g., saddle points for foliations on Riemann surfaces), one-point leaves, transversal cycles, non-diffeomorphic, compact and non-compact leaves. In our construction, for \( \Phi \in C^m_n(V,W,F) \), \( n \) and \( m \) can be associated to either corresponding number of points on leaves and transversal sections. \( \Phi \in W_1,...,W_n \) is associated to \( R \), which is supposed to be a rational form with poles at \( z_i = z_j, i \neq j \) only. Thus the general principle is the following. By associating \( z_i \) to \( c_i(p) \) on a leaf or section, and computing (7.1), we study its analytic behavior. If (7.1) has poles then they could be related to singular points of \( M/F \). Next, for (13.3), (13.5), (4.4), and (4.5), for \( z = c_i(p_i) \), we determine the domains of convergency. When such a domain is limited to one point, then \( M/F \) might have a one point leaf. Finally, consider \( \delta \Phi \), for \( \Phi \in C^1(V,W,F) \), and identify \( z \) to \( c(w) \), where \( c(w) \) is a local coordinate on a leaf or section. Then \( R(z) = \langle w', \delta \Phi \rangle \) considered on the whole leaf may distinguish if it is compact or non-compact. Note that for the same \( \Phi \) we may consider \( c_i(p_i), 1 \leq i \leq n \) either on a leaf or section, i.e., in transversal directions on \( M/F \). Thus, in case of saddle points, we have different values of, e.g., integrals (7.2) in these directions. For cycles on a curve we determine if they are transversal to leaves of foliation by using the above considerations.

8. Further directions

There exist a few approaches to definition and computation of cohomologies of vertex operator algebras. [23, 35]. Taking into account the above definitions and construction, we aim to consideration of a characteristic classes theory for arbitrary codimension regular and singular foliations vertex operator algebras. In this paper, we consider foliations of codimension one. Arbitrary codimension case will be considered elsewhere.

Losik defines a smooth structure on the leaf space \( M/F \) of a foliation \( \mathfrak{F} \) of codimension \( n \) on a smooth manifold \( M \) that allows to apply to \( M/F \) the same techniques as to smooth manifolds. In [36] characteristic classes for a foliation as elements of the cohomology of certain bundles over the leaf space \( M/F \) are defined. It would be interesting also to develop intrinsic (i.e., purely coordinate independent) theory of a smooth manifold foliation cohomology involving vertex algebra bundles [4]. Similar to Losik’s theory, we use bundles correlation functions) over a foliated space. The idea of studies of cohomology of certain bundles on a smooth manifold \( M \) and making connection to a cohomology of \( M \) has first appeared in [7]. This can have a relation with Losik’s work [36] proposing a new framework for singular spaces and characteristic classes. In applications, one would be interested in applying techniques of this paper to case of higher-dimensional manifolds of codimension one [2, 3]. In particular, the question of higher non-vanishing invariants, as well as the problem of distinguishing of compact and non-compact leaves for the Reeb foliation of the full torus, are also of high importance.

It would be important to establish connection to chiral de-Rham complex on a smooth manifold introduced in [37]. After a modification, one is able to introduce a vertex algebra cohomology of smooth manifolds on a similar basis as in this paper.
One can mention a possibility to derive differential equations \cite{22} for characters on separate leaves of foliation. Such equations are derived for various genera and can be used in frames of Vinogradov theory \cite{54}. The structure of foliation (in our sense) can be also studied from the automorphic function theory point of view. Since on separate leaves one proves automorphic properties of characters, one can think about "global" automorphic properties for the whole foliation.

9. Applications in Mathematical Physics

9.1. Applications in conformal field theory and moduli spaces. The problem of classification of leaves of codimension one foliations finds its applications in mechanics \cite{38,39}. The general considerations of this paper are also useful even apart from problems of classification of leaves of codimension one foliations. Reduction formulas directly related to connections in this paper are used in computations of correlation functions on compact Riemann surfaces in Conformal Field theory \cite{1,4,53}. Starting from an \( n \)-point functions we are able to relate it to corresponding the partition function via the recursion procedure. Studies of the space of connections defined for correlation functions of a particular vertex algebra brings about invariants of foliations which represent invariants for corresponding Conformal Field Theory \cite{47,48}. This gives an important link between cohomology of foliations with vertex algebra cohomology and the geometry of foliations in general.

9.2. Deformations of Lie algebras. The theory of connections on the space of correlation function characterising leaves of a foliation find its relation to the deformations of Lie algebras. Such deformations are used in various applications in Mathematical Physics. It is well known that the Lie algebra cohomology with values in the adjoint representation of a Lie algebra answers questions about deformations of such Lie algebra as an algebraic object. There arise natural questions of this type for bi-graded differential Lie algebras resulting from chain complex constructions. Bi-graded differential algebras results also from constructions of double complexes for continual Lie algebra that are used in exactly solvable models \cite{43}. Using characterization of connection leaves, we will study rigidity of bi-graded differential Lie algebras resulting from chain complex constructions. For a compact Riemann surface \( \Sigma^{(g)} \) of genus \( g \geq 2 \), we expect to find a relations for cohomologies in terms of elements of Fréchet spaces given by the polynomials on \( T_{\Sigma^{(g)}} N^{\ast} \langle g, 0 \rangle \). It’s the space of formal power series on \( T_{\Sigma^{(g)}} N^{\ast} \langle g, 0 \rangle \). This could be interpreted as a relation between cohomology with adjoint coefficients of a Lie algebra \( g \), i.e., graded differential deformations of global sections of \( g \), and deformations of the underlying manifold. The choice of the coefficients in the Lie algebra cohomology determines a geometric object on the moduli space in a formal neighborhood of a point.

9.3. Developments. The recursion procedure \cite{53} plays a fundamental role in the theory of correlation functions for vertex operator algebras. It provides relations between \( n \)- and \( n - 1 \)-point correlation functions. In our foliation picture, the recursion procedure brings about relations among leaves of foliation. We work in the approach of formulation and computation of cohomologies of vertex operator algebras. Taking into account the above definitions and construction, we would like to develop a theory.
of characteristic classes for vertex operator algebras. This can have a relation with
Losik’s work [33] proposing a new framework for singular spaces and new kind of char-
acteristic classes. Relations to [31] on Reeb foliations modified Godbillon-Vey class and
can be also considered. On the other hand, we would like to apply methodology of
vertex algebras in order to complete Losik’s theory of characteristic classes [33]. One
can mention a possibility to derive differential equations for correlation functions on
separate leaves of foliation. Such equations are derived for various genera and can be
used in frames of Vinogradov theory [54]. If we consider multipoint correlation func-
tion for vertex algebras on Riemann surfaces of arbitrary genus, and corresponding
connections arising from these correlation functions, then there exist also relations
to Krichever-Novikov type algebras [29,30,30,44-46]. These are generalizations of the
Witt, Virasoro, affine Lie algebras. In particular, one is able to use the structure of
Krichever-Novikov type algebras (as higher-genus generalizations of algebras related
to vertex algebras) to study foliated spaces introduced in this paper. The construction
of n-point functions on genus g sewn Riemann surfaces can be used for introduction
and computation of cohomology of Krichever-Novikov type algebras in appropriate
setup.

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10. Appendix: Grading-restricted vertex algebras and their modules

In this section, following [23] we recall basic properties of grading-restricted vertex
algebras and their grading-restricted generalized modules, useful for our purposes in
later sections. We work over the base field \( \mathbb{C} \) of complex numbers.

Definition 25. A vertex algebra \((V, Y_V, 1_V)\), (cf. [27]), consists of a \( \mathbb{Z} \)-graded complex
vector space

\[
V = \prod_{n \in \mathbb{Z}} V(n), \quad \text{dim} V(n) < \infty,
\]

for each \( n \in \mathbb{Z} \), and linear map

\[
Y_V : V \to \text{End} (V)[[z, z^{-1}]],
\]

for a formal parameter \( z \) and a distinguished vector \( 1_V \in V \). The evaluation of \( Y_V \),
on \( v \in V \) is the vertex operator

\[
Y_V(v) \equiv Y_V(v, z) = \sum_{n \in \mathbb{Z}} v(n) z^{-n-1}, \tag{10.1}
\]

with components \((Y_V(v))_n = v(n) \in \text{End} (V)\), where \( Y_V(v, z)1_V = v + O(z) \).

Definition 26. A grading-restricted vertex algebra satisfies the following conditions:

(1) Grading-restriction condition: \( V(n) \) is finite dimensional for all \( n \in \mathbb{Z} \), and
\( V(n) = 0 \) for \( n \ll 0 \);
(2) Lower-truncation condition: For \( u, v \in V \), \( Y_V(u, z)v \) contains only finitely many negative power terms, that is,
\[
Y_V(u, z)v \in V((z)),
\]
(the space of formal Laurent series in \( z \) with coefficients in \( V \));

(3) Identity property: Let \( \text{Id}_V \) be the identity operator on \( V \). Then
\[
Y_V(1_V, z) = \text{Id}_V;
\]

(4) Creation property: For \( u \in V \),
\[
Y_V(u, z)1_V \in V[[z]],
\]
and
\[
\lim_{z \to 0} Y_V(u, z)1_V = u;
\]

(5) Duality: For \( u_1, u_2, v \in V \), \( v' \in V' = \bigoplus_{n \in \mathbb{Z}} V^*_n \), where \( V^*_n \) denotes the dual vector space to \( V_n \) and \( \langle ., . \rangle \) the evaluation pairing \( V' \otimes V \to \mathbb{C} \), the series
\[
\langle v', Y_V(u_1, z_1)Y_V(u_2, z_2)v \rangle, \quad \langle v', Y_V(u_2, z_2)Y_V(u_1, z_1)v \rangle, \quad \langle v', Y_V(Y_V(u_1, z_1 - z_2)u_2, z_2)v \rangle,
\]
are absolutely convergent in the regions
\[
|z_1| > |z_2| > 0, \quad |z_2| > |z_1| > 0, \quad |z_2| > |z_1 - z_2| > 0,
\]
respectively, to a common rational function in \( z_1 \) and \( z_2 \) with the only possible poles at \( z_1 = 0 = z_2 \) and \( z_1 = z_2 \);

(6) \( L_V(0) \)-bracket formula: Let \( L_V(0) : V \to V \), be defined by
\[
L_V(0)v = nv, \quad n = \text{wt}(v),
\]
for \( v \in V(n) \). Then
\[
[L_V(0), Y_V(v, z)] = Y_V(L_V(0)v, z) + z \frac{d}{dz} Y_V(v, z),
\]
for \( v \in V \).

(7) \( L_V(-1) \)-derivative property: Let
\[
L_V(-1) : V \to V,
\]
be the operator given by
\[
L_V(-1)v = \text{Res}_z z^{-2} Y_V(v, z)1_V = Y_{(-2)}(v)1_V,
\]
for \( v \in V \). Then for \( v \in V \),
\[
\frac{d}{dz} Y_V(u, z) = Y_V(L_V(-1)u, z) = [L_V(-1), Y_V(u, z)]. \tag{10.5}
\]
In addition to that, we recall here the following definition (cf. [4]):

**Definition 27.** A grading-restricted vertex algebra $V$ is called conformal of central charge $c \in \mathbb{C}$, if there exists a non-zero conformal vector (Virasoro vector) $\omega \in V_{(2)}$ such that the corresponding vertex operator

$$Y_V(\omega, z) = \sum_{n \in \mathbb{Z}} L_V(n) z^{-n-2},$$

is determined by modes of Virasoro algebra $L_V(n) : V \to V$ satisfying

$$[L_V(m), L_V(n)] = (m-n)L(m+n) + \frac{c}{12}(m^3 - m)\delta_{m+b,0} \text{Id}_V.$$

**Definition 28.** A vector $A$ which belongs to a module $W$ of a quasi-conformal grading-restricted vertex algebra $V$ is called primary of conformal dimension $\Delta(A) \in \mathbb{Z}_+$ if

$$L_W(k)A = 0, \quad k > 0,$$

$$L_W(0)A = \Delta(A)A.$$

**Definition 29.** A grading-restricted generalized $V$-module is a vector space $W$ equipped with a vertex operator map

$$Y_W : V \otimes W \to W[[z, z^{-1}]],$$

$$u \otimes w \mapsto Y_W(u, w) \equiv Y_W(u, z)w = \sum_{n \in \mathbb{Z}} (Y_W)_n(u, w)z^{-n-1},$$

and linear operators $L_W(0)$ and $L_W(-1)$ on $W$ satisfying the following conditions:

1. **Grading-restriction condition:** The vector space $W$ is $\mathbb{C}$-graded, that is,

$$W = \bigoplus_{\alpha \in \mathbb{C}} W_{(\alpha)},$$

such that $W_{(\alpha)} = 0$ when the real part of $\alpha$ is sufficiently negative;

2. **Lower-truncation condition:** For $u \in V$ and $w \in W$, $Y_W(u, z)w$ contains only finitely many negative power terms, that is, $Y_W(u, z)w \in W((z))$;

3. **Identity property:** Let $\text{Id}_W$ be the identity operator on $W$. Then

$$Y_W(1_V, z) = \text{Id}_W;$$

4. **Duality:** For $u_1, u_2 \in V$, $w \in W$,

$$w' \in W' = \bigotimes_{n \in \mathbb{Z}} W^*_{(n)},$$

$W'$ denotes the dual $V$-module to $W$ and $\langle \cdot, \cdot \rangle$ their evaluation pairing, the series

$$\langle w', Y_W(u_1, z_1)Y_W(u_2, z_2)w \rangle,$$

$$\langle w', Y_W(u_2, z_2)Y_W(u_1, z_1)w \rangle,$$

$$\langle w', Y_W(Y_V(u_1, z_1 - z_2)u_2, z_2)w \rangle,$$

are absolutely convergent in the regions

$$|z_1| > |z_2| > 0,$$
respectively, to a common rational function in $z_1$ and $z_2$ with the only possible poles at $z_1 = 0 = z_2$ and $z_1 = z_2$.

(5) $L_W(0)$-bracket formula: For $v \in V$,

$$[L_W(0), Y_W(v, z)] = Y_W(L_V(0)v, z) + z \frac{d}{dz} Y_W(v, z);$$

(6) $L_W(0)$-grading property: For $w \in W_{(\alpha)}$, there exists $N \in \mathbb{Z}_+$ such that

$$(L_W(0) - \alpha)^N w = 0; \quad (10.6)$$

(7) $L_W(-1)$-derivative property: For $v \in V$,

$$\frac{d}{dz} Y_W(u, z) = Y_W(L_V(-1)u, z) = [L_W(-1), Y_W(u, z)]. \quad (10.7)$$

The translation property of vertex operators

$$Y_W(u, z) = e^{-z L_W(-1)} Y_W(u, z) e^{z L_W(-1)}, \quad (10.8)$$

for $z' \in \mathbb{C}$, follows from from $(10.7)$. For $v \in V$, and $w \in W$, the intertwining operator

$$Y^W_{WV} : V \rightarrow W, \quad v \mapsto Y^W_{WV}(w, z)v, \quad (10.9)$$

is defined by

$$Y^W_{WV}(w, z)v = e^{z L_W(-1)} Y_W(v, -z)w. \quad (10.10)$$

We will also use the following property of intertwining operators $(10.9)$ [24]. For a function $f(u), u \in V$,

$$f(Y_V(u, z)1_V) = Y^W_{WV}(f(u), z)1_V.$$

Let us recall some further facts from [4] relating generators of Virasoro algebra with the group of automorphisms in complex dimension one. Let us represent an element of $\text{Aut} \mathcal{O}^{(1)}$ by the map

$$z \mapsto \rho = \rho(z), \quad (10.11)$$

given by the power series

$$\rho(z) = \sum_{k \geq 1} a_k z^k; \quad (10.12)$$

$\rho(z)$ can be represented in an exponential form

$$f(z) = \exp \left( \sum_{k \geq 1} \beta_k z^{k+1} \partial_z \right) (\beta_0) z^\partial_\cdot z, \quad (10.13)$$

where we express $\beta_k \in \mathbb{C}, k \geq 0$, through combinations of $a_k, k \geq 1$. A representation of Virasoro algebra modes in terms of differential operators is given by [27]

$$L_W(m) \mapsto -\zeta^{m+1} \partial_\zeta, \quad (10.14)$$
for $m \in \mathbb{Z}$. By expanding (10.13) and comparing to (10.12) we obtain a system of equations which, can be solved recursively for all $\beta_k$. In [4], $v \in V$, they derive the formula

$$[L_W(n), Y_W(v, z)] = \sum_{m \geq -1} \frac{1}{(m + 1)!} \left( \partial_z^{m+1} z^{m+1} \right) Y_W(L_V(m)v, z),$$

(10.15)

of a Virasoro generator commutation with a vertex operator. Given a vector field

$$\beta(z) \partial_z = \sum_{n \geq -1} \beta_n z^{n+1} \partial_z,$$

(10.16)

which belongs to local Lie algebra of Aut $\mathcal{O}^{(1)}$, one introduces the operator

$$\beta = - \sum_{n \geq -1} \beta_n L_W(n).$$

We conclude from (10.16) with the following

**Lemma 10.**

$$[\beta, Y_W(v, z)] = \sum_{m \geq -1} \frac{1}{(m + 1)!} \left( \partial_z^{m+1} \beta(z) \right) Y_W(L_V(m)v, z).$$

(10.17)

The formula (10.17) is used in [4] (Chapter 6) in order to prove invariance of vertex operators multiplied by conformal weight differentials in case of primary states, and in generic case.

Let us give some further definition:

**Definition 30.** A grading-restricted vertex algebra $V$-module $W$ is called quasi-conformal if it carries an action of local Lie algebra of Aut $\mathcal{O}$ such that commutation formula (10.17) holds for any $v \in V$, the element $L_W(-1) = -\partial_z$, as the translation operator $T$, $L_W(0) = -z \partial_z$, acts semi-simply with integral eigenvalues, and the Lie subalgebra of the positive part of local Lie algebra of Aut $\mathcal{O}^{(n)}$ acts locally nilpotently.

Recall [4] the exponential form $f(\zeta)$ (10.13) of the coordinate transformation (10.11) $\rho(z) \in$ Aut $\mathcal{O}^{(1)}$. A quasi-conformal vertex algebra possesses the formula (10.17), thus it is possible by using the identification (10.14), to introduce the linear operator representing $f(\zeta)$ (10.13) on $\mathcal{W}_{z_1, \ldots, z_n}$,

$$P(f(\zeta)) = \exp \left( \sum_{m > 0} (m + 1) \beta_m L_V(m) \right) \beta_0^{L_W(0)},$$

(10.18)

(note that we have a different normalization in it). In [4] (Chapter 6) it was shown that the action of an operator similar to (10.18) on a vertex algebra element $v \in V_n$ contains finitely many terms, and subspaces $V_{\leq m} = \bigoplus_{n \geq K} V_n$, are stable under all operators $P(f), f \in$ Aut $\mathcal{O}^{(1)}$. In [4] they proved the following
Lemma 11. The assignment
\[ f \mapsto P(f), \]
defines a representation of Aut O(1) on V, 
\[ P(f_1 * f_2) = P(f_1) P(f_2), \]
which is the inductive limit of the representations V_{\leq m}, m \geq K.

Similarly, (10.18) provides a representation operator on W_{z_1, \ldots, z_n}.

11. Appendix: W-valued rational functions

Recall the definition of shuffles. Let S_l be the permutation group. For \( l \in \mathbb{N} \) and \( 1 \leq s \leq l - 1 \), let \( J_{l,s} \) be the set of elements of \( S_l \) which preserve the order of the first \( s \) numbers and the order of the last \( l - s \) numbers, that is,
\[ J_{l,s} = \{ \sigma \in S_l \mid \sigma(1) < \cdots < \sigma(s), \ \sigma(s+1) < \cdots < \sigma(l) \}. \]
The elements of \( J_{l,s} \) are called shuffles, and we use the notation
\[ J_{l,s}^{-1} = \{ \sigma \mid \sigma \in J_{l,s} \}. \]

We define the configuration spaces:
\[ F_n \mathbb{C} = \{ (z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j, i \neq j \}, \]
for \( n \in \mathbb{Z}_+ \). Let \( V \) be a grading-restricted vertex algebra, and \( W \) a grading-restricted generalized \( V \)-module. By \( \overline{W} \) we denote the algebraic completion of \( W \),
\[ \overline{W} = \prod_{n \in \mathbb{C}} W(n) = (W')^*. \]

Definition 31. A \( \overline{W} \)-valued rational function in \( (z_1, \ldots, z_n) \) with the only possible poles at \( z_i = z_j, i \neq j \), is a map
\[ f : F_n \mathbb{C} \rightarrow \overline{W}, \]
\[ (z_1, \ldots, z_n) \mapsto f(z_1, \ldots, z_n), \]
such that for any \( w' \in W' \),
\[ R(z_1, \ldots, z_n) = (w', f(z_1, \ldots, z_n)), \]
is a rational function in \( (z_1, \ldots, z_n) \) with the only possible poles at \( z_i = z_j, i \neq j \).

In this paper, such a map is called \( \overline{W} \)-valued rational function in \( (z_1, \ldots, z_n) \) with possible other poles. The space of \( \overline{W} \)-valued rational functions is denoted by \( \overline{W}_{z_1, \ldots, z_n} \).

One defines an action of \( S_n \) on the space \( \text{Hom}(V^* \otimes^n, \overline{W}_{z_1, \ldots, z_n}) \) of linear maps from \( V^* \otimes^n \) to \( \overline{W}_{z_1, \ldots, z_n} \) by
\[ \sigma(\Phi)(v_1, z_1; \cdots; v_n, z_n) = \Phi(v_{\sigma(1)}, v_{\sigma(1)}; \cdots; v_{\sigma(n)}, z_{\sigma(n)}), \]
for \( \sigma \in S_n \), and \( v_1, \ldots, v_n \in V \). We will use the notation \( \sigma_{i_1, \ldots, i_n} \in S_n \), to denote the the permutation given by \( \sigma_{i_1, \ldots, i_n}(j) = i_j \), for \( j = 1, \ldots, n \). In [23] one finds:

Proposition 6. The subspace of \( \text{Hom}(V^* \otimes^n, \overline{W}_{z_1, \ldots, z_n}) \) consisting of linear maps having the \( L(-1) \)-derivative property, having the \( L(0) \)-conjugation property or being composable with \( m \) vertex operators is invariant under the action of \( S_n \).
Let us introduce another definition:

**Definition 32.** We define the space $\mathcal{W}_{z_1,\ldots,z_n}$ of $\mathcal{W}_{z_1,\ldots,z_n}$-valued rational forms $\Phi$ with each vertex algebra element entry $v_i$, $1 \leq i \leq n$, of a quasi-conformal grading-restricted vertex algebra $V$ tensored with power $\text{wt}(v_i)$-differential of corresponding formal parameter $z_i$, i.e.,

$$\Phi \left( dz_1^{\text{wt}(v_1)} \otimes v_1, z_1; \cdots; dz_n^{\text{wt}(v_n)} \otimes v_n, z_n \right) \in \mathcal{W}_{z_1,\ldots,z_n}. \tag{11.1}$$

We assume also that (11.1) satisfy $L_V(-1)$-derivative (12.1), $L_V(0)$-conjugation (12.2) properties, and the symmetry property with respect to action of the symmetric group $S_n$:

$$\sum_{\sigma \in S_n} (-1)^{|\sigma|} \langle \Phi(v_{\sigma(1)}, z_{\sigma(1)}; \cdots; v_{\sigma(i)}, z_{\sigma(i)}) \rangle = 0. \tag{11.2}$$

In Section 3 we prove that (11.1) is invariant with respect to changes of formal parameters $(z_1, \ldots, z_n)$.

**12. Appendix: Properties of matrix elements for a grading-restricted vertex algebra**

Let $V$ be a grading-restricted vertex algebra and $W$ a grading-restricted generalized $V$-module. Let us recall some definitions and facts about matrix elements for a grading-restricted vertex algebra [23]. If a meromorphic function $f(z_1, \ldots, z_n)$ on a domain in $\mathbb{C}^n$ is analytically extendable to a rational function in $z_1, \ldots, z_n$, we denote this rational function by $R(f(z_1, \ldots, z_n))$.

Let us recall a few definitions from [23]

**Definition 33.** For $n \in \mathbb{Z}_+$, a linear map

$$\Phi(v_1, z_1; \ldots; v_n, z_n) = V^\otimes n \to \mathcal{W}_{z_1,\ldots,z_n},$$

is said to have the $L(-1)$-derivative property if

$$\langle w', \partial_z \Phi(v_1, z_1; \ldots; v_n, z_n) \rangle = \langle w', \Phi(v_1, z_1; \ldots; LV(-1)v_i, z_1; \ldots; v_n, z_n) \rangle, \tag{12.1}$$

for $i = 1, \ldots, n$, $v_1, \ldots, v_n \in V$, $w' \in W$, and

$$\sum_{i=1}^{n} \partial_z \langle w', \Phi(v_1, z_1; \ldots; v_n, z_n) \rangle = \langle w', LW(-1) \Psi(v_1, z_1; \ldots; v_n, z_n) \rangle. \tag{12.2}$$

with some action of $L_W(-1)$ on $\Phi(v_1, z_1; \ldots; v_n, z_n)$, and and $v_1, \ldots, v_n \in V$.

Note that since $L_W(-1)$ is a weight-one operator on $W$, for any $z \in \mathbb{C}$, $e^{zL_W(-1)}$ is a well-defined linear operator on $\mathcal{W}$.

In [23] we find the following

**Proposition 7.** Let $\Phi$ be a linear map having the $L(-1)$-derivative property. Then for $v_1, \ldots, v_n \in V$, $w' \in W'$, $(z_1, \ldots, z_n) \in F_n \mathbb{C}$, $z \in \mathbb{C}$ such that $(z_1 + z, \ldots, z_n + z) \in F_n \mathbb{C}$,

$$\langle w', e^{zL_W(-1)} \Phi(v_1, z_1; \ldots; v_n, z_n) \rangle = \langle w', \Phi(v_1, z_1 + z; \ldots; v_n, z_n + z) \rangle. \tag{12.3}$$
and for \( v_1, \ldots, v_n \in V, \ w' \in W', \ (z_1, \ldots, z_n) \in F_n \mathbb{C}, \ z \in \mathbb{C}, \) and \( 1 \leq i \leq n \) such that 
\[(z_1, \ldots, z_{i-1}, z_i + z, z_{i+1}, \ldots, z_n) \in F_n \mathbb{C},\]
the power series expansion of 
\[\langle w', \Phi(v_1, z_1; \ldots; v_{i-1}, z_i + z; v_i + z; v_{i+1}, z_{i+1}; \ldots; v_n, z_n)\rangle, \tag{12.4}\]
in \( z \) is equal to the power series 
\[\langle w', \Phi(v_1, z_1; \ldots; v_{i-1}, z_i+1; e^{zL(-1)}v_i, z_i; v_{i+1}, z_{i+1}; \ldots; v_n, z_n)\rangle, \tag{12.5}\]
in \( z \). In particular, the power series (12.5) in \( z \) is absolutely convergent to (12.4) in the disk \( |z| < \min_{i \neq j} |z_i - z_j| \).

Finally, we have

**Definition 34.** A linear map 
\[\Phi : V^\otimes n \rightarrow W_{z_1, \ldots, z_n}\]
has the \( L(0) \)-conjugation property if for \( v_1, \ldots, v_n \in V, \ w' \in W', \ (z_1, \ldots, z_n) \in F_n \mathbb{C} \) and \( z \in \mathbb{C}^\times \) so that \( (z_1, \ldots, z_n) \in F_n \mathbb{C} \),
\[\langle w', \Phi(v_1, z_1; \ldots; v_n, z_n)\rangle = \langle w', \Phi(z^{L(0)}v_1, z_1; \ldots; z^{L(0)}v_n, z_n)\rangle. \tag{12.6}\]

**12.1. E-elements.** For \( w \in W \), the \( \overline{W} \)-valued function is given by (see notations for \( \omega_W \) in Section 3)
\[\langle w', E(.) \rangle = R((w', .)), \]
and \( R(.) \) denotes the following (cf. [23]). Namely, if a meromorphic function \( f(z_1, \ldots, z_n) \) on a region in \( \mathbb{C}^n \) can be analytically extended to a rational function in \( (z_1, \ldots, z_n) \), then the notation \( R(f(z_1, \ldots, z_n)) \) is used to denote such rational function. One defines
\[E_{W;V}^{(n)}(w; v_1, z_1; \ldots; v_n, z_n) = E_W^{(n)}(v_1, z_1; \ldots; v_n, z_n; w),\]
where \( E_{W;V}^{(n)}(w; v_1, z_1; \ldots; v_n, z_n) \) is an element of \( \overline{W}_{z_1, \ldots, z_n} \). One defines
\[\Phi \circ (E_{V;1}^{(l_1)} \otimes \cdots \otimes E_{V;1}^{(l_n)}): V^\otimes m+n \rightarrow \overline{W}_{z_1, \ldots, z_{m+n}},\]
by
\[(\Phi \circ (E_{V;1}^{(l_1)} \otimes \cdots \otimes E_{V;1}^{(l_n)}))(v_1 \otimes \cdots \otimes v_{m+n-1}) = E(\Phi(E_{V;1}^{(l_1)})(v_1 \otimes \cdots \otimes v_{l_1} \otimes \cdots \otimes E_{V;1}^{(l_n)}(v_{l_1+1} \otimes \cdots \otimes v_{l_1+l_n-1}) \otimes v_{l_1+l_n+1} \otimes \cdots))\]
and
\[E_W^{(m)} \circ \Phi : V^\otimes m+n \rightarrow \overline{W}_{z_1, \ldots, z_{m+n-1}}\]
is given by
\[(E_W^{(m)} \circ \Phi)(v_1 \otimes \cdots \otimes v_{m+n}) = E(E_W^{(m)}(v_1 \otimes \cdots \otimes v_m; \Phi(v_{m+1} \otimes \cdots \otimes v_{m+n}))).\]
Finally,
\[ E_{WV}^{\mathcal{W}(m)} \circ_{m+1} \Phi : V^\otimes m+n \to \mathbb{W}_{z_1, \ldots, z_{m+n-1}}, \]
is defined by
\[ (E_{WV}^{\mathcal{W}(m)} \circ_{m+1} \Phi)(v_1 \otimes \cdots \otimes v_{m+n}) = E_{WV}(\Phi(v_1 \otimes \cdots \otimes v_n); v_{n+1} \otimes \cdots \otimes v_{n+m}). \]

In the case that \( l_1 = \cdots = l_{i-1} = l_{i+1} = 1 \) and \( l_i = m - n - 1 \), for some \( 1 \leq i \leq n \), we will use \( \Phi_{\circ_{i}} E_{V; 1}^{(l_i)} \) to denote \( \Phi \circ (E_{V; 1}^{(l_1)} \otimes \cdots \otimes E_{V; 1}^{(l_n)}) \). Note that our notations differ with that of [23].

13. APPENDIX: MAPS COMPOSABLE WITH VERTEX OPERATORS

In the construction of double complexes in Section 4 we would like to use linear maps from tensor powers of \( V \) to the space \( W_{z_1, \ldots, z_n} \) to define cochains in vertex algebra cohomology theory. For that purpose, in particular, to define the coboundary operator, we have to compose cochains with vertex operators. However, as mentioned in [23], the images of vertex operator maps in general do not belong to algebras or their modules. They belong to corresponding algebraic completions which constitute one of the most subtle features of the theory of vertex algebras. Because of this, we might not be able to compose vertex operators directly. In order to overcome this problem [25], we first write a series by projecting an element of the algebraic completion of an algebra or a module to its homogeneous components. Then we compose these homogeneous components with vertex operators, and take formal sums. If such formal sums are absolutely convergent, then these operators can be composed and can be used in constructions.

Another question that appears is the question of associativity. Compositions of maps are usually associative. But for compositions of maps defined by sums of absolutely convergent series the existence of does not provide associativity in general. Nevertheless, the requirement of analyticity provides the associativity [23].

**Definition 35.** For a \( V \)-module
\[ W = \coprod_{n \in \mathbb{C}} W_{(n)}, \]
and \( m \in \mathbb{C} \), let
\[ P_m : \mathbb{W} \to W_{(m)}, \]
be the projection from \( \mathbb{W} \) to \( W_{(m)} \). Let
\[ \Phi : V^\otimes n \to W_{z_1, \ldots, z_n}, \]
be a linear map. For \( m \in \mathbb{N} \), \( \Phi \) is called [23] [42] to be composable with \( m \) vertex operators if the following conditions are satisfied:

1) Let \( l_1, \ldots, l_n \in \mathbb{Z}_+ \) such that \( l_1 + \cdots + l_n = m + n \), \( v_1, \ldots, v_{m+n} \in V \) and \( w' \in W' \). Set
\[ \Psi_i = E_{V}^{(l_i)}(v_{k_1}, z_{k_1} - \zeta_i; \ldots; v_{k_i}, z_{k_i} - \zeta_i; 1_V), \]
where
\[ k_1 = l_1 + \cdots + l_{i-1} + 1, \quad \ldots, \quad k_i = l_1 + \cdots + l_{i-1} + l_i. \]
for \(i = 1, \ldots, n\). Then there exist positive integers \(N^m_{i,j}(v_i, v_j)\) depending only on \(v_i\) and \(v_j\) for \(i, j = 1, \ldots, k, i \neq j\) such that the series
\[
\mathcal{I}^n_m(\Phi) = \sum_{r_1, \ldots, r_n \in \mathbb{Z}} \langle w', \Phi(P_{r_1} \Psi_1; \ldots; P_{r_n} \Psi_n, \zeta_n) \rangle,
\]
(13.3)
is absolutely convergent when
\[
|z_{1} + \cdots + z_{l-1} + p - \zeta_i| + |z_{1} + \cdots + z_{l-1} + q - \zeta_j| < |\zeta_i - \zeta_j|,
\]
(13.4)
for \(i, j = 1, \ldots, k, i \neq j\) and for \(p = 1, \ldots, l_1\) and \(q = 1, \ldots, l_j\). The sum must be analytically extended to a rational function in \((z_1, \ldots, z_{m+n})\), independent of \((\zeta_1, \ldots, \zeta_n)\), with the only possible poles at \(z_i = z_j\), of order less than or equal to \(N^m_{i,j}(v_i, v_j)\), for \(i, j = 1, \ldots, k, i \neq j\).

2) For \(v_1, \ldots, v_{m+n} \in V\), there exist positive integers \(N^m_{i,j}(v_i, v_j)\), depending only on \(v_i\) and \(v_j\), for \(i, j = 1, \ldots, k, i \neq j\), such that for \(w' \in W'\), and
\[
\begin{align*}
\nu_{n,m} & = (v_{1+m} \otimes \cdots \otimes v_{n+m}), \\
\zeta_{n,m} & = (z_{1+m}, \ldots, z_{n+m}),
\end{align*}
\]
such that
\[
\mathcal{J}^n_m(\Phi) = \sum_{q \in \mathbb{C}} \langle w', E_{W'}^{(m)}(v_1 \otimes \cdots \otimes v_m; P_q(\Phi(v_{n,m})(\zeta_{n,m})) \rangle,
\]
(13.5)
is absolutely convergent when
\[
|z_i - z_j|, \quad i \neq j,
\]
(13.6)
for \(i = 1, \ldots, m\), and \(k = m+1, \ldots, m+n\), and the sum can be analytically extended to a rational function in \((z_1, \ldots, z_{m+n})\) with the only possible poles at \(z_i = z_j\), of orders less than or equal to \(N^m_{i,j}(v_i, v_j)\), for \(i, j = 1, \ldots, k, i \neq j\).

In [23], the following useful proposition is proven:

**Proposition 8.** Let \(\Phi : V^\otimes n \to \mathbb{W}_{z_1, \ldots, z_n}\) be composable with \(m\) vertex operators. Then we have:

1) For \(p \leq m\), \(\Phi\) is composable with \(p\) vertex operators and for \(p, q \in \mathbb{Z}_+\) such that \(p+q \leq m\) and \(l_1, \ldots, l_n \in \mathbb{Z}_+\) such that \(l_1 + \cdots + l_n = p+n\), \(\Phi \circ (E_{V_1; 1}^{(l_1)} \otimes \cdots \otimes E_{V_1; 1}^{(l_n)})\) and \(E_{W'}^{(q)} \circ_{p+1} \Phi\) are composable with \(q\) vertex operators.

2) For \(p, q \in \mathbb{Z}_+\) such that \(p+q \leq m\), \(l_1, \ldots, l_n \in \mathbb{Z}_+\) such that \(l_1 + \cdots + l_n = p+n\) and \(k_1, \ldots, k_{p+n} \in \mathbb{Z}_+\) such that \(k_1 + \cdots + k_{p+n} = q + p + n\), we have
\[
(\Phi \circ (E_{V_1; 1}^{(l_1)} \otimes \cdots \otimes E_{V_1; 1}^{(l_n)})) \circ (E_{V_1; 1}^{(k_1)} \otimes \cdots \otimes E_{V_1; 1}^{(k_{p+n})}) = \Phi \circ (E_{V_1; 1}^{(k_1 + \cdots + k_1)} \otimes \cdots \otimes E_{V_1; 1}^{(k_1 + \cdots + l_{n-1} + 1 + \cdots + k_{p+n})}).
\]

3) For \(p, q \in \mathbb{Z}_+\) such that \(p+q \leq m\) and \(l_1, \ldots, l_n \in \mathbb{Z}_+\) such that \(l_1 + \cdots + l_n = p+n\), we have
\[
E_{W'}^{(q)} \circ_{q+1} (\Phi \circ (E_{V_1; 1}^{(l_1)} \otimes \cdots \otimes E_{V_1; 1}^{(l_n)})) = (E_{W'}^{(q)} \circ_{q+1} \Phi) \circ (E_{V_1; 1}^{(l_1)} \otimes \cdots \otimes E_{V_1; 1}^{(l_n)}).
For \( p, q \in \mathbb{Z}_+ \) such that \( p + q \leq m \), we have
\[
    E_W^{(p)} \circ_{p+1} (E_W^{(q)} \circ_{q+1} \Phi) = E_W^{(p+q)} \circ_{p+q+1} \Phi.
\]

14. Appendix: Proofs of Lemmas 1, 2, 3 and Proposition 1

In this Appendix we provide proofs of Lemma 2 and Proposition 1. We start with the proof of Lemma 1.

**Proof.** From the construction of spaces for double complex for a grading-restricted vertex algebra cohomology, it is clear that the spaces \( C^n(V, W, U) \), \( 1 \leq s \leq m \) in Definition 10 are non-zero. On each transversal section \( U_s \), \( 1 \leq s \leq m \), \( \Phi(v_1, c_j(p_1); \ldots; v_n, c_j(p_n)) \) belongs to the space \( W_{c_j(p_1), \ldots, c_j(p_n)} \), and satisfy the \( L(-1) \)-derivative (12.1) and \( L(0) \)-conjugation (12.6) properties. A map \( \Phi(v_1, c_j(p_1); \ldots; v_n, c_j(p_n)) \) is composable with \( m \) vertex operators with formal parameters identified with local coordinates \( c_j(p_j) \), on each transversal section \( U_j \). Note that on each transversal section, \( n \) and \( m \) the spaces (3.7) remain the same. The only difference may be constituted by the compositibility conditions (13.3) and (13.5) for \( \Phi \).

In particular, for \( l_1, \ldots, l_n \in \mathbb{Z}_+ \) such that \( l_1 + \cdots + l_n = n + m \), \( v_1, \ldots, v_{m+n} \in V \) and \( w' \in W' \), recall (13.4) that
\[
    \Psi_i = \omega_V(v_{k_i}, c_{k_i}(p_{k_i}) - \zeta_i) \cdots \omega_V(v_{k_i}, c_k(p_k) - \zeta_i) \mathbf{1}_V, \tag{14.1}
\]
where \( k_i \) is defined in (13.2), for \( i = 1, \ldots, n \), depend on coordinates of points on transversal sections. At the same time, in the first compositibility condition (13.3) depends on projections \( P_r(\Psi_i) \), \( r \in \mathbb{C} \), of \( W_{c(p_1), \ldots, c(p_n)} \) to \( W \), and on arbitrary variables \( \zeta_i \), \( 1 \leq i \leq m \). On each transversal connection \( U_j \), \( 1 \leq s \leq m \), the absolute convergency is assumed for the series (13.3) (cf. Appendix 13). Positive integers \( N_m^n(v_i, v_j) \), (depending only on \( v_i \) and \( v_j \)) as well as \( \zeta_i \), for \( i, j = 1, \ldots, k \), \( i \neq j \), may vary for transversal sections \( U_j \). Nevertheless, the domains of convergency determined by the conditions (13.4) which have the form
\[
    |c_{m_i}(p_{m_i}) - \zeta_i| + |c_{n_i}(p_{n_i}) - \zeta_i| < |\zeta_i - \zeta_j|, \tag{14.2}
\]
for \( m_i = l_1 + \cdots + l_{i-1} + p \), \( n = l_1 + \cdots + l_{j-1} + q \), \( i, j = 1, \ldots, k \), \( i \neq j \) and for \( p = 1, \ldots, l_i \) and \( q = 1, \ldots, l_j \), are limited by \( |\zeta_i - \zeta_j| \) in (13.2) from above. Thus, for the intersection variation of sets of homology embeddings in (3.7), the absolute convergency for (13.3) is still fulfilled. Under intersection in (3.7) by choosing appropriate \( N_m^n(v_i, v_j) \), one can analytically extend (13.3) to a rational function in \((c_1(p_1), \ldots, c_n+m(p_n+m))\), independent of \((\zeta_1, \ldots, \zeta_n)\), with the only possible poles at \( c_i(p_i) = c_j(p_j) \), of order less than or equal to \( N_m^n(v_i, v_j) \), for \( i, j = 1, \ldots, k \), \( i \neq j \).

As for the second condition in Definition of compositibility, we note that, on each transversal connection, the domains of absolute convergency \( c_i(p_i) \neq c_j(p_j) \), \( i \neq j \) \( |c_i(p_i)| > |c_k(p_j)| > 0 \), for \( i = 1, \ldots, m \), and \( k = 1 + m, \ldots, n + m \), for
\[
    \mathcal{J}_m^n(\Phi) = \sum_{q \in \mathbb{C}} \langle w'_q, \omega_W(v_1, c_1(p_1)) \cdots \omega_W(v_m, c_m(p_m)) \rangle \tag{14.3}
\]
\[
    P_q(\Phi(v_{1+m}, c_{1+m}(p_{1+m}); \ldots; v_{n+m}, c_{n+m}(p_{n+m}))),
\]

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are limited from below by the same set of absolute values of local coordinates on transversal section. Thus, under intersection in (3.7) this condition is preserved, and the sum (13.5) can be analytically extended to a rational function in \((c_1(p_1), \ldots, c_{m+n}(p_{m+n}))\) with the only possible poles at \(c_i(p_i) = c_j(p_j)\), of orders less than or equal to \(N_n^m(v_i, v_j)\), for \(i, j = 1, \ldots, k, i \neq j\). Thus, we proved the lemma. □

Next we give proof of Lemma 2.

**Proof.** Suppose we consider another transversal basis \(U'\) for \(\mathcal{F}\). According to the definition, for each transversal section \(U_i\) which belong to the original basis \(U\) in (3.7) there exists a holonomy embedding \(h'_i: U_i \hookrightarrow U'_j\), i.e., it embeds \(U_i\) into a section \(U'_j\) of our new transversal basis \(U'_\). Then consider the sequence of holonomy embeddings \(\{h'_i\}_i \geq 0\) such that

\[
U'_0 \xrightarrow{h'_1} \cdots \xrightarrow{h'_k} U'_k.
\]

For the combination of embeddings \(\{h'_i, i \geq 0\}\) and

\[
U'_0 \xrightarrow{h_1} \cdots \xrightarrow{h_k} U_k,
\]

we obtain commutative diagrams. Since the intersection in (3.7) is performed over all sets of homology mappings, then it is independent on the choice of a transversal basis. □

Next, we prove Proposition 1.

**Proof.** Here we prove that for generic elements of a quasi-conformal grading-restricted vertex algebra \(\Phi\) and \(\omega_W \in \mathcal{W}_{z_1, \ldots, z_n}\) and are canonical, i.e., independent on changes

\[
z_i \mapsto w_i = \rho(z_i), \quad 1 \leq i \leq n,
\]

of local coordinates of \(c_i(p_i)\) and \(c_j(p'_j)\) at points \(p_i\) and \(p'_j\), \(1 \leq i \leq n, 1 \leq j \leq k\). Thus the construction of the double complex spaces (3.7) is proved to be canonical too. Let us denote by

\[
\xi_i = \left(\beta^{-1}_0 dw_i\right)_{\text{wt} (v_i)}.
\]

Recall the linear operator (3.5) (cf. Appendix 10). Define introduce the action of the transformations (14.4) as

\[
\Phi \left( dw_1^{\text{wt} (v_1)} \otimes v_1, w_1; \ldots; dw_n^{\text{wt} (v_n)} \otimes v_n, w_n \right)
= \left(\frac{df(\zeta)}{d\zeta}\right)^{-L_{W(0)}} P(f(\zeta)) \Phi (\xi_1 \otimes v_1; \ldots; \xi_n \otimes v_n; z_n).
\]

We then obtain

**Lemma 12.** An element (11.1)

\[
\Phi \left( dz_1^{\text{wt} (v_1)} \otimes v_1; \ldots; dz_n^{\text{wt} (v_n)} \otimes v_n, z_n \right),
\]

of \(\mathcal{W}_{z_1, \ldots, z_n}\) is canonical is invariant under transformations (14.4) of \((\text{Aut} \mathcal{O}(1))^{\times n}\).
Proof. Consider (14.5). First, note that
\[ f'(\zeta) = \frac{df(\zeta)}{d\zeta} = \sum_{m \geq 0} (m + 1) \beta_m \zeta^m. \]

By using the identification (10.14) and and the \( L_W(-1) \)-properties (12.1) and (12.6) we obtain
\[
\langle w', \Phi \left( dw_1^{\text{wt}} (v_1) \otimes v_1, w_1; \ldots; dw_n^{\text{wt}} (v_n) \otimes v_n, w_n \right) \rangle
\]
\[
= \langle w', f'(\zeta)^{-L_W(0)} P(f(\zeta)) \Phi (\xi_1 \otimes v_1, z_1; \ldots; \xi_n \otimes v_n, z_n) \rangle
\]
\[
= \langle w', \left( \frac{df(\zeta)}{d\zeta} \right)^{-L_W(0)} \Phi \left( dw_1^{\text{wt}} (v_1) \otimes v_1, \sum_{m \geq 0} (m + 1) \beta_m \zeta^{m+1}; \ldots; \right.
\]
\[
\left. dw_n^{\text{wt}} (v_n) \otimes v_n, \sum_{m \geq 0} (m + 1) \beta_m \zeta^{m+1} \right) \rangle \right)
\]
\[
= \langle w', \left( \frac{df(\zeta)}{d\zeta} \right)^{-L_W(0)} \Phi \left( dw_1^{\text{wt}} (v_1) \otimes v_1, \left( \frac{df(z_1)}{dz_1} \right) z_1;
\]
\[
\ldots; dw_n^{\text{wt}} (v_n) \otimes v_n, \left( \frac{df(z_n)}{dz_n} \right) z_n \rangle \right)
\]
\[
= \langle w', \Phi \left( \left( \frac{df(z_1)}{dz_1} \right) dw_1^{-\text{wt}} (v_1) \otimes v_1, z_1;
\right.
\]
\[
\left. \ldots; \left( \frac{df(z_n)}{dz_n} \right) dw_n^{-\text{wt}} (v_n) \otimes v_n, z_n \right) \rangle \right)
\]
\[
= \langle w', \Phi \left( dz_1^{\text{wt}} (v_1) \otimes v_1, z_1; \ldots; dz_n^{\text{wt}} (v_n) \otimes v_n, z_n \right) \rangle.
\]
Thus we proved the Lemma. \( \square \)

The elements \( \Phi(v_1, z_1; \ldots; v_n, z_n) \) of \( C_k^n(V, W, F) \) belong to the space \( W_{z_1} \ldots z_n \) and assumed to be composable with a set of vertex operators \( \omega_W(v'_j, c_j(p'_j)), 1 \leq j \leq k \). Vertex operators \( \omega_W(d(c(p))^{\text{wt}} (v') \otimes v'_j, c_j(p'_j)) \) constitute particular examples of mapping of \( C_k^n(V, W, F) \) and, therefore, are invariant with respect to (14.4). Thus, the construction of spaces (14.4) is invariant under the action of the group \( \square \)

Finally, we give a proof of Lemma 3

Proof. Since \( n \) is the same for both spaces in (3.12), it only remains to check that the conditions for (13.3) and (13.6) for \( \Phi(v_1, c_j(p_1); \ldots; v_n, c_j(p_n)) \) of compositability Definition 13 with vertex operators are stronger for \( C_m^n(V, W, U, F) \) then for \( C_{m-1}^n(V, W, U, F) \). In particular, in the first condition for (13.3) in definition of composability (14) the difference between the spaces in (3.12) is in indeces. Consider (14.1). For \( C_{m-1}^n(V, W, U, F) \), the summations in indexe \( l_1 + \cdots + l_{i-1} + 1, \ldots, \)
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$k_i = l_1 + \cdots + l_{i-1} + l_i$, for the coordinates $c_j(p_1), \ldots, c_j(p_n)$ with $l_1, \ldots, l_n \in \mathbb{Z}_+$, such that $l_1 + \cdots + l_n = n + (m-1)$, and vertex algebra elements $v_1, \ldots, v_{n+(m-1)}$ are included in summation for indexes for $C^n_m(V, W, U, F)$. The conditions for the domains of absolute convergency for $\mathcal{M}$, i.e., $|c_{l_1+\cdots+l_{i-1}+p} - \zeta_i| + |c_{l_1+\cdots+l_{i-1}+q} - \zeta_j| < |\zeta_i - \zeta_j|$, for $i, j = 1, \ldots, k$, $i \neq j$, and for $p = 1, \ldots, l_i$ and $q = 1, \ldots, l_j$, for the series (13.3) are more restrictive then for $m - 1$. The conditions for $T^n_{m-1}(\Phi)$ to be extended analytically to a rational function in $(c_1(p_1), \ldots, c_{n+(m-1)}(p_{n+(m-1)}))$, with positive integers $N^n_{m-1}(v_i, v_j)$, depending only on $v_i$ and $v_j$ for $i, j = 1, \ldots, k$, $i \neq j$, are included in the conditions for $T^n_m(\Phi)$.

Similarly, the second condition for (13.6), of is absolute convergency and analytical extension to a rational function in $(c_1(p_1), \ldots, c_{m+n}(p_{m+n}))$, with the only possible poles at $c_1(p_1) = c_j(p_j)$, of orders less than or equal to $N^n_m(v_i, v_j)$, for $i, j = 1, \ldots, k$, $i \neq j$, for (13.6) when $c_1(p_1) \neq c_j(p_j)$, $i \neq j$ $|c_k(p_k)| > 0$ for $i = 1, \ldots, m$, and $k = m + 1, \ldots, m + n$ includes the same condition for $\mathcal{F}^n_{m-1}(\Phi)$. Thus we obtain the conclusion of Lemma.

\[ \boxed{\text{References}} \]

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