Abstract. We calculate the matrix of the Frobenius map on the middle dimensional cohomology of the one parameter family that is related by mirror symmetry to the family of all quintic threefolds.

1. Introduction

Recently $p$-adic methods were used to prove certain integrality results in the theory of topological strings (see [6, 9, 10, 11]). Namely, instanton numbers (i.e. genus zero Gopakumar-Vafa invariants) are defined in terms of the $A$-model and mirror symmetry allows us to express them in terms of the mirror $B$-model. It follows from physical considerations that these numbers should be integers (they should coincide with the number of appropriate BPS states). However integrality does not readily follow from either the $A$-model or the $B$-model interpretation. The integrality of instanton numbers and similar quantities (such as the number of holomorphic discs) were analyzed in [6, 9, 10, 11] by means of the $p$-adic $B$-model. These numbers were expressed in terms of the Frobenius map on the middle dimensional $p$-adic cohomology; the integrality follows from this expression.

The main tool in the calculation of the Frobenius map is its relation with the Gauss-Manin connection. This relation, however, does not determine the Frobenius map uniquely. The additional data required is the behavior of the Frobenius map at the boundary point of the moduli space (more specifically, the point of maximally unipotent monodromy that corresponds in the $B$-model to the infinite volume point in the $A$-model). The analysis of the behavior of the Frobenius map at this point was carried out in [11] using some very deep results in the theory of motives. It was found that the matrix of the Frobenius operator (in a certain natural basis) at this point has at most one non-zero off-diagonal entry. This result was sufficient to prove the required integrality statements.

In the present paper we calculate the Frobenius matrix at the boundary point of the moduli space, in the basic example of mirror symmetry, using the construction of the Frobenius map explained in [7, 8]. This construction is equivalent to the original Dwork’s construction of the Frobenius map. Our calculations are very explicit and do not rely on any deep machinery from the theory of motives. It follows from our computations that the natural conjecture that all the off-diagonal entries of the Frobenius matrix at the boundary point are zero is false; in fact we

1 This is similar to the observation that a function is determined by a differential equation only up to boundary conditions.
2 The diagonal entries range from $p^3$ down to 1.
obtain an explicit expression for the remaining undetermined entry and its non-vanishing can be verified by means of a computer calculation. More precisely, the remaining Frobenius matrix entry can be expressed as a $p$-adic series

$$\sum_{n=3}^{\infty} \frac{\sum_{i=2}^{n-1} \sum_{j=1}^{i-1} B_n(n-1)!}{ij} - \frac{\left(\sum_{n \geq 1} B_n(n-1)!\right)^3}{6}$$

in terms of the coefficients $B_n$ of the Dwork exponential $\sum B_n z^n := \exp(\frac{z^p}{p} + \frac{z}{p+1})$.

In fact it was pointed out to us by V. Vologodsky that from certain conjectures of the theory of motives one can derive that the above entry is a rational multiple of $\zeta_p(3)$, where the latter is a $p$-adic Riemann zeta value (see [4] for example). We carried out (by computer) some calculations that strongly point to the truth of the above. Namely, the quantity in the parentheses above (later denoted by $\Delta_3$) is most naturally expressed in terms of the Kubota-Leopoldt $p$-adic $L$-function $L_p(s, \omega^{1-s})$ (where $\omega$ is the Teichm"uller character) that is related to the $p$-adic zeta values by $\zeta_p(s) = \frac{p^s - 1}{p^s - 1} L_p(s, \omega^{1-s})$ (see [4]). Explicitly, it seems that

$$\Delta_3 = L_p(3, \omega^{1-3})/3$$

where we used the Dirichlet series expansion in [3] to evaluate the latter. (This method of computing $p$-adic zeta values was suggested by A. Schwarz.)

2. Preliminaries

Recall that the $B$-model corresponding to the $A$-model on the quintic is a 1-parameter family of mirror quintics defined as follows. Consider the family given by the equations

$$\lambda(x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5) + x_0 x_1 x_2 x_3 x_4 = 0$$

(let us denote these hypersurfaces by $V_\lambda$) inside the complex projective space $\mathbb{C}P^4$. These should be factorized with respect to the symmetry group $\Gamma \cong (\mathbb{Z}/5\mathbb{Z})^3$. This group is realized as the quotient of the group of 5-tuples of fifth roots of unity with product 1 by the diagonal embedding of the fifth roots of unity. The action is by multiplication of the coordinates by the corresponding roots of unity. In principle one should then consider a resolution of the quotient; the resulting family $V_\Gamma$ is referred to as the mirror quintics. It can be shown, however, (see for example [2]) that for the purposes of computing the cohomology of $V_\Lambda$ one may work with $\Gamma$ invariant elements in the cohomology of $V_\lambda$. Observe that the permutation group on 5 elements $\Sigma_5$ also acts on $V_\lambda$ by permutation of coordinates. It is known that the $\Gamma$ invariant elements in the middle dimensional cohomology group of $V_\lambda$ are also $\Sigma_5$ invariant.

Note that the same constructions and statements remain true if one replaces $\mathbb{C}$ by $\mathbb{C}_p$ for $p \neq 5$, where $\mathbb{C}_p$ denotes the completion of the algebraic closure of the $p$-adic numbers $\mathbb{Q}_p$. Instanton numbers can be expressed in terms of the variation of Hodge structure on the 1-dimensional family of mirror quintics. The analysis of integrality of these instanton numbers performed in [6] is based on the consideration

---

The calculations confirming non-vanishing have been performed by P. Dragon (using Mathematica) and the author (using PARI/GP).

We checked, to at least 10 digits, the primes 3, 5, 7, 11 and 13. P. Dragon has independently confirmed these calculations.
of the Frobenius map. If $p \neq 5$ then we may consider the invariant elements in the cohomology of $V_\lambda$ instead of working directly with the mirror quintics $V_\lambda^\circ$.

Thus we will be interested in computing the matrix of the Frobenius map on the $\Sigma_5 \times \Gamma$-invariant part of the middle dimensional $p$-adic cohomology of $V_\lambda$, where the latter is now considered as a family of varieties over $\mathbb{C}_p$. We will only be interested in the case of $\lambda$ small and in fact “zero”. More precisely, near $\lambda = 0$, but not at $\lambda = 0$ the cohomologies form a vector bundle. It will turn out that there is a natural extension of all the structure that we consider, including the Frobenius map, to $\lambda = 0$.

Denote by $\mathbb{C}_p[[x_i]]$ the subring of the formal power series $\mathbb{C}_p[[x_i]]$ consisting of the overconvergent series. More precisely, $\mathbb{C}_p[[x_i]]$ consists of elements $\sum a_I x^I$ with $\text{ord}_p a_I \geq |I| + d$ with $c > 0$, i.e. those power series that converge on a neighborhood of the closed polydisc of radius 1 around $0 \in \mathbb{C}_p$.

For a collection of operators $D_i : A_i \rightarrow B$, we write $B/D_i$ to denote the quotient of $B$ by span of the images of the operators $D_i$, i.e. $B/D_i = \frac{B}{\sum_i D_i(x_i)}$. Let $a^I$ stand for $a_1^{i_1}a_2^{i_2}a_3^{i_3}a_4^{n+m+s}$, where $I = \{i,j,k,n,m,s\}$ is a multi-index. Denote by $\pi \in \mathbb{C}_p$ an element such that $\pi^{p-1} = -p$.

In addition to $p \neq 5$, we will further assume that $p \neq 2$.

3. General structures

We review at this point some general facts about the cohomology of the quintic.

3.1. Definition. Consider the one parameter family $V_\lambda$ of projective Calabi-Yau 3-folds over $\mathbb{C}_p$ given by

$$\varphi_\lambda = \lambda(x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5) + x_0x_1x_2x_3x_4.$$ 

Recall that we are interested in the invariant part of the middle dimensional cohomology of $V_\lambda$ in the neighborhood of the degeneracy point $\lambda = 0$. More precisely, we need the following definitions.

Definition 3.1. Let $H^3_{dR}(V_\lambda)$ denote the $D$-module on the parameter space obtained by computing the relative de Rham cohomology of the family.

In our parametrization of the family $V_\lambda$ we are interested in the neighborhood of $\lambda = 0$. Recall that outside of $\lambda = 0$, the family $H^3_{dR}(V_\lambda)$ is a vector bundle with a flat connection. In fact the connection has a regular (i.e., logarithmic) singularity at $\lambda = 0$. This prompts the following.

Definition 3.2. Let $H^3_{dR}^{\Gamma}(V_\lambda)$ denote the restriction of $H^3_{dR}(V_\lambda)$ to the formal punctured disc, i.e.,

$$H^3_{dR}^{\Gamma}(V_\lambda) = H^3_{dR}(V_\lambda) \otimes_{\mathbb{C}_p[\lambda]} \mathbb{C}_p((\lambda)).$$

As was mentioned previously, our interest is mainly in the invariant part of $H^3_{dR}^{\Gamma}(V_\lambda)$. More precisely, the 204-dimensional $H^3_{dR}^{\Gamma}(V_\lambda)$ has an action (see above) of $\Sigma_5 \times \Gamma$. We would like to study only the 4-dimensional invariant part since that computes the cohomology of the mirror quintics $V_\lambda^\circ$.

Definition 3.3. Denote by $H^3_{dR}^{\Gamma}(V_\lambda)^{inv}$ the 4-dimensional (over $\mathbb{C}_p((\lambda))$) invariant part of $H^3_{dR}^{\Gamma}(V_\lambda)$ under the action of $\Sigma_5 \times \Gamma$. 
Below we will examine a certain important basis of $H^3_{dR}(V_\lambda)^{inv}$. It will turn out that in this basis the Frobenius map extends to $\lambda = 0$. We will compute as explicitly as possible the Frobenius matrix at $\lambda = 0$.

3.2. Dwork cohomology. As we are interested in doing this computation as directly as possible we use Dwork cohomology. We note that besides the shift in cohomological degree which we immediately take into account, there is the matter of the difference in the definition of Frobenius. Namely, in our situation, the Frobenius defined through Dwork is $p^2$ times the usual one.

**Definition 3.4.** Let $H_{dR}^3(V_\lambda) = H^3_{dR}(V_\lambda) \otimes \mathbb{C}_p[[\lambda^{\pm 1}]]$ and correspondingly $H_{dR}^3(V_\lambda)^{inv} = H^3_{dR}(V_\lambda)^{inv} \otimes \mathbb{C}_p[[\lambda^{\pm 1}]]$.

The reason for considering $H_{dR}^3(V_\lambda)$ is that it can be worked with very directly, in particular the Frobenius map is very easily described on it, as well as the Gauss-Manin connection. Concretely (see [8] for more details), $H_{dR}^3(V_\lambda)$ is given as the top cohomology of the relative twisted de Rham complex

$$DR(C^\ast_p(x_0, \ldots, x_4, t) [[\lambda^{\pm 1}]] e^{\pi t \varphi_\lambda}t_0$$

where the usual de Rham differential $d$ is replaced by

$$d + d(\pi t \varphi_\lambda)$$

and so

$$H^3_{dR}(V_\lambda) \cong \frac{C^4_t(x_0, \ldots, x_4, t)[[\lambda^{\pm 1}]] dxdt}{(\partial_5 + \pi \lambda t x_4 + \lambda x_4) dt)}$$

with $dx = dx_0 \ldots dx_4$.

The action of $\Sigma_5$ permutes the variables $x_i$ and $\Gamma$ acts on the $x_i$ as before.

**Remark.** Observe that we have the obvious containments $H^3_{dR}(V_\lambda) \supset H^3_{dR}(V_\lambda)$ and $H^3_{dR}(V_\lambda)^{inv} \supset H^3_{dR}(V_\lambda)^{inv}$. While we are working in $H^3_{dR}(V_\lambda)$, we will soon see that everything of interest is happening inside $H^3_{dR}(V_\lambda)^{inv}$, in fact in the $\mathbb{C}_p[[\lambda]]$ span of our chosen basis.

3.3. The Frobenius map. We describe an action of the Frobenius operator that acts on the parameter space as well as on the fibers, i.e., near $\lambda = 0$ only the “fiber” at $\lambda = 0$ is preserved by the action.

Explicitly it is given by

$$Fr : \omega(x, t, \lambda) \mapsto e^{\pi(t^p \varphi_\lambda^p(x^p) - t \varphi_\lambda)} \omega(x^p, t^p, \lambda^p).$$

In our case $e^{\pi(t^p \varphi_\lambda^p(x^p) - t \varphi_\lambda)}$ decomposes as

$$A(\lambda^5 t x_0^5) \ldots A(\lambda^5 t x_4^5) A(x)$$

where $A(z) := e^{\pi(z^p - z)}$ with $A(z) = \sum A_i z^i$.

The definition of overconvergence is precisely formulated in such a way that the cohomology of the overconvergent complex agrees with the usual de Rham cohomology and at the same time it is possible to define the Frobenius map as above, i.e., the function $A(z)$ is overconvergent.

---

5This means that differentiation with respect to $\lambda$ is not used in the differential, it is saved for the Gauss-Manin connection.

6The degrees are as follows: $\deg(\lambda) = 0$, $\deg(x_i) = 1$ and $\deg(t) = -5$. 
3.4. The Gauss-Manin connection. The Gauss-Manin connection plays an important rôle in the computations below. It can be described very explicitly on forms and descends to cohomology. We have

\[ \nabla_{\partial_{\lambda}} = \partial_{\lambda} + \partial_{\lambda}(\pi t_{\varphi_{\lambda}}) \]

and in fact we will be mostly interested in

\[ \delta := \lambda \nabla_{\partial_{\lambda}} \]

since the Gauss-Manin connection has a logarithmic pole at \( \lambda = 0 \).

The Frobenius map is compatible in a certain sense with the connection. Namely,

\[ \delta \circ Fr = p Fr \circ \delta. \]

3.5. Symplectic structure. The cohomology groups \( H_{dR}^{3x}(V_{\lambda}) \) also possess a \( \lambda \)-linear non-degenerate symplectic form \( (-,-) : H_{dR}^{3x}(V_{\lambda})^{\otimes 2} \rightarrow \mathbb{C}_{p}(\lambda) \). We are interested in its restriction to \( H_{dR}^{3x}(V_{\lambda})^{inv} \) which is still non-degenerate. The symplectic form is compatible with the Gauss-Manin connection in the usual sense, i.e.,

\[ \delta(u,v) = (\delta u,v) + (u,\delta v) \]

as well as the Frobenius map,

\[ p^3 Fr(u,v) = (Fr u, Fr v). \]

4. The classes \( \omega_I \) and the cohomology near \( \lambda = 0 \)

In this section we study the cohomology elements that appear in the image of the Frobenius map. We prove certain key recursion relations that will allow us to make explicit computations later on.

**Definition 4.1.** Let \( \omega \) be the image of \( dxdt \) in \( H_{dR}^{3#}(V_{\lambda}) \); it is in fact in \( H_{dR}^{3x}(V_{\lambda})^{inv} \). For \( \lambda \neq 0 \) it is the cohomology class of the nowhere vanishing holomorphic 3-form on the Calabi-Yau threefold \( V_{\lambda} \).

Since \( \omega \in H_{dR}^{3x}(V_{\lambda})^{inv} \), and \( \delta \) is compatible with the \( \Sigma_5 \times \Gamma \) action, so that \( \delta^i \omega \in H_{dR}^{3x}(V_{\lambda})^{inv} \) for all \( i \).

**Definition 4.2.** Let \( \mathcal{H} \) denote the \( \mathbb{C}_{p}[\lambda] \) submodule of \( H_{dR}^{3x}(V_{\lambda})^{inv} \) spanned by \( \{ \omega, \delta \omega, \delta^2 \omega, \delta^3 \omega \} \). In fact \( \mathcal{H} \cong \mathbb{C}_{p}[\lambda] \) \( \mathbb{C}_{p}(\lambda) = H_{dR}^{3x}(V_{\lambda})^{inv} \) so \( \mathcal{H} \) can be viewed as an extension (as a vector bundle) of \( H_{dR}^{3x}(V_{\lambda})^{inv} \) to \( \lambda = 0 \).

Our \( \mathcal{H} \) is a vector bundle over the formal disc with a logarithmic connection. As we will show below, it is preserved by everything that we consider.

**Lemma 4.3.** The \( \mathbb{C}_{p}[\lambda] \)-module \( \mathcal{H} \) is preserved by \( \delta \).

**Proof.** This follows directly from Corollary 4.13. \( \square \)

**Definition 4.4.** Denote by

\[ \omega_{ijkmns} \]

the image of

\[ (\lambda t x_0^5)^i (\lambda t x_1^5)^j (\lambda t x_2^5)^k (\lambda t x_3^5)^m (\lambda t x_4^5)^n (tx)^s \]

in \( H_{dR}^{3#}(V_{\lambda}) \). We write \( x \) for \( x_0...x_4 \).

**Lemma 4.5.** The elements \( \omega_{ijkmns} \) are in \( \mathcal{H} \).
Proof. By Lemma 4.8 and Corollary 4.10 the elements $\omega_{ijknms}$ can be written as a linear combination of $\delta^i \omega$ with constant (as functions of $\lambda$) coefficients. Thus $\omega_{ijknms}$ can be written as a linear combination of $\{\omega, \delta \omega, \delta^2 \omega, \delta^3 \omega\}$ with power series coefficients by Corollary 4.13.

We omit the indices that are 0, unless it is the $s$ index. The group $\Sigma_5$ acts on the $\omega_{ijknms}$ by permuting the $i, j, k, n, m$, while $s$ remains fixed. By the above Lemma, we don’t care where the non-zero, non-$s$ indices are, thus

$$\omega_{ij} := \omega_{ij000s} = \omega_{0000s},$$

e tc. Finally, we sometimes write $\omega_I$ where $I$ is a multi-index, i.e.,

$$\omega_I = \omega_{ijknms}.$$

**Lemma 4.6.** The Frobenius map preserves $\mathcal{H}$.

**Proof.** Observe that $\delta \omega$ can be written as a linear combination of $(xt)^j$, i.e., $\delta \omega = \sum_{j \leq i} a_j (xt)^j$ with $a_j$ constant.

Furthermore,

$$Fr((xt)^j dxdt) = p^6 A(\lambda x_0^5) \ldots A(\lambda x_4^5) A(t x)(xt)^{j+p+1} dxdt = p^6 \sum A_i \ldots A_m A_s \omega_{ijknm(s+j+1)p-1}.$$

Since $\omega_I$ are in $\mathcal{H}$ we are done.

As a consequence of the Lemma above we see that we may consider the restriction of the Frobenius map to $\lambda = 0$, i.e., the matrix of $Fr|_0 : \mathbf{H}_0 \rightarrow \mathbf{H}_0$. An essential tool in this investigation is the symplectic form $(-, -)$ that as we see below also behaves well with respect to $\mathcal{H}$.

**Lemma 4.7.** The pairing $(-, -)$ maps $\mathcal{H} \otimes \mathcal{H}$ to $\mathbb{C}_p[[\lambda]]$ and its restriction to $\lambda = 0$, i.e.,

$$(-, -)_0 : \mathbf{H}_0 \otimes \mathbf{H}_0 \rightarrow \mathbb{C}_p$$

is non-degenerate. Furthermore, $(\omega, \delta \omega)_0 = Y$ and so $(\delta \omega, \delta^2 \omega)_0 = -Y$.

**Proof.** For the first claim it is sufficient to check that $(\delta \omega, \delta^i \omega)$ is a power series for arbitrary $i, j \leq 3$. Since $(\delta \omega, \delta^j \omega) = \delta (\delta^j-1 \omega, \delta \omega) - (\delta^{-1} \omega, \delta^j \omega)$ it is enough to check that $\delta \omega, \delta^j \omega$ is a power series. But this is true for $j \leq 3$ by Griffiths transversality and Lemma 4.15. The pairing is non-degenerate at $\lambda = 0$ precisely because $(\omega, \delta \omega)_0 = Y \neq 0$ by Lemma 4.15 and $(\delta \omega, \delta^2 \omega)_0 = -Y$ by the compatibility of the symplectic pairing with the Gauss-Manin connection.

4.1. **Differentiating $\omega_I$ and other relations.** Since the image of $\omega$ under the Frobenius map is expressed in terms of $\omega_I$, so it is necessary to re-express these elements in terms of our chosen basis of $\mathcal{H}$. To that end we provide some key reduction formulas that allow us to accomplish that goal.

**Lemma 4.8.** We have the following $s$-relations:

$$(2) \quad \delta \omega_{ijknms} = -(s+1) \omega_{ijknms} - \pi \omega_{ijknm(s+1)}$$

for $i, j, k, n, m, s \geq 0$.

---

Since $\delta f|_{\lambda=0} = 0$ for $f$ a power series in $\lambda$, we have that for $u$ and $v$ sections of $\mathcal{H}$, $(\delta u, v)_0 = -(u, \delta v)_0$. The non-zero constant $Y$ is $Y(0)$, where $Y(z)$ is the Yukawa coupling, see Sec. 4.2.
Proof. Recall that 
\[\delta = \lambda \partial_\lambda + \pi t \lambda (x_0^5 + \ldots + x_4^5)\]
\[= \lambda \partial_\lambda + \pi t (\varphi_\lambda - x)\]
and by the last relation in Equation 1
\[= \lambda \partial_\lambda - \partial_t - \pi t x\]
so that
\[\delta \omega_{ijknm} = (\lambda \partial_\lambda - \partial_t - \pi t x) ((\lambda t x_0^5)^i \ldots (\lambda t x_4^5)^m (xt)^s)\]
\[= (i + \ldots + m) \omega_I - (i + \ldots + m + s + 1) \omega_I - \pi \omega_{ijknm(s+1)}\]
\[= -(s + 1) \omega_{ijknm} - \pi \omega_{ijknm(s+1)}\]
□

Lemma 4.9. We have the following \(i\)-relations (by symmetry of the first five relations in the Equation they apply also to any other index except \(s\)):
\[\omega_{(i+1)jknm} = -(\frac{5i + (s + 1)}{5\pi}) \omega_{ijknm} - \frac{1}{5} \omega_{ijknm(s+1)}\]
for \(i, j, k, n, m, s \geq 0\).

Proof. Recall that
\[\omega_{(i+1)jknm} = (\lambda t x_0^5)^{i+1} \ldots (\lambda t x_4^5)^m (xt)^s\]
\[= \lambda t x_0^4 x_0^i \omega_I\]
and by the first relation in Equation 1
\[= \frac{1}{5\pi} (\partial_{x_0} x_0 + \pi x_0) \omega_I\]
\[= \frac{1}{5\pi} ((5i + s + 1) \omega_I + \pi \omega_{ijknm(s+1)})\]
□

Remark. In fact, Lemma 4.9 holds for \(i = -1\) and \(s = 4\) if we correctly interpret it. Namely,
\[\omega_{-100005} = \frac{(xt)^5}{\lambda t x_0^5} = \frac{1}{\lambda^5} \omega_{011110}\]
More precisely,
\[\omega_4 = (xt)^4\]
\[= \lambda t x_0^4 \frac{x_1^4 \ldots x_4^4 t^3}{\lambda}\]
\[= -\frac{1}{5\pi} (\partial_{x_0} + \pi x_1 \ldots x_4) \frac{x_1^4 \ldots x_4^4 t^3}{\lambda}\]
\[= -\frac{1}{5\lambda} x_1^3 \ldots x_4^3 t^4 = -\frac{1}{5\lambda^5} (\lambda t x_0^5)^{\ldots (\lambda t x_4^5)} = -\frac{1}{5\lambda^5} \omega_{011110}\]
We will need this to compute the Picard-Fuchs equation for \(\omega\).
Corollary 4.10. A better version of \( I. \) Shapiro vanishes at \( \delta \).

**Proof.** Put Lemmas 4.8 and 4.9 together. \( \square \)

4.2. The Picard-Fuchs equation and the Yukawa coupling. In this section we focus on the derivation of the Picard-Fuchs equation for \( \omega = \omega_{000000} \). This is a fourth order differential equation of the form

\[
\delta^4 \omega = g_3(\lambda)\delta^3 \omega + g_2(\lambda)\delta^2 \omega + g_1(\lambda)\delta \omega + g_0(\lambda)\omega.
\]

This equation is central to the study of the quintic family, however the only property of it that we will use is the fact that the coefficients \( g_I \) are power series in \( \lambda \) that vanish at \( \lambda = 0 \).

**Definition 4.11.** Let \( \#(I) = \#\{(i, j, k, n, m, s)\} \) denote the number of non-zero indices among \( i, j, k, n, m \), i.e., whether or not \( s = 0 \) does not affect \( \#(I) \).

**Lemma 4.12.** Let \( g(\lambda) = \frac{5^5 \lambda^5}{5^5 \lambda^5 + 1} \), then the Picard-Fuchs equation for \( \omega \) is

\[
\delta^4 \omega = -g(\lambda)(10\delta^3 \omega + 35\delta^2 \omega + 50\delta \omega + 24\omega).
\]

**Proof.** The idea is to rewrite \( \omega_{111100} \) as a linear combination of \( \omega_{i\leq4} \) using Lemma 4.9 and then express \( \omega_{111100} \) in terms of \( \omega_4 \) as in the Remark following that Lemma. Then rewrite everything in terms of \( \delta^2 \omega \) using Lemma 4.8.

Let \( \tilde{\omega}_I = (-5)^{\#(I)}\pi^I \omega_I \) then we see that

\[
\tilde{\omega}_{1knms} = (-5)^{\#(I)}\pi^I \omega_{1knms}
\]

\[
= (-5)^{\#(I)}\pi^I \left( \frac{s + 1}{5\pi} \omega_{0knms} - \frac{1}{5\pi} \omega_{0knms(s+1)} \right)
\]

\[
= (s + 1)(-5)^{\#(I)-1}\pi^{I-1} \omega_{0knms} + (-5)^{\#(I)-1}\pi^{I} \omega_{0knms(s+1)}
\]

\[
= (s + 1)\tilde{\omega}_{0knms} + \tilde{\omega}_{0knms(s+1)}.
\]

If we let \( \hat{\delta} = -\delta \) and \( \tilde{\omega}_s = \pi^s \omega_s \) then

\[
\hat{\delta}\tilde{\omega}_s = -\delta \pi^s \omega_s = (s + 1)\pi^s \omega_s + \pi^{s+1} \omega_{s+1} = (s + 1)\tilde{\omega}_s + \tilde{\omega}_{s+1}.
\]

Thus \( \hat{\delta} \tilde{\omega}_0 = \tilde{\omega}_{111100} \); i.e.,

\[
\delta^4 \omega_0 = 5^4 \pi^4 \omega_{111100} = -5^5 \pi^4 \lambda^5 \omega_4.
\]

Now by Lemma 4.8, \( \delta^4 \omega = \alpha_4 \omega_4 + \alpha_3 \omega_3 + \alpha_2 \omega_2 + \alpha_1 \omega_1 + \alpha_0 \omega_0 \) with \( \alpha_i \) constant; and it is clear that \( \alpha_4 = \pi^4 \). So that

\[
\delta^4 \omega = g(\lambda)(\alpha_3 \omega_3 + \alpha_2 \omega_2 + \alpha_1 \omega_1 + \alpha_0 \omega_0)
\]

and again by Lemma 4.8

\[
\alpha_3 \omega_3 + \alpha_2 \omega_2 + \alpha_1 \omega_1 + \alpha_0 \omega_0 = a_3 \delta^3 \omega + a_2 \delta^2 \omega + a_1 \delta \omega + a_0 \omega
\]

with \( a_i \) constant.

\(^8\)Our approach is similar to [1].
At this point the fact we need, namely that the coefficients of the Picard-Fuchs equation are power series in $\lambda$ that vanish at $\lambda = 0$ is proved. However to compute the $a_i$ we need to work a bit more.

We note that

$$
\omega = \omega_0 \\
\delta \omega = -\omega_0 - \pi \omega_1 \\
\delta^2 \omega = \omega_0 + 3\pi \omega_1 + \pi^2 \omega_2 \\
\delta^3 \omega = -\omega_0 - 7\pi \omega_1 - 6\pi^2 \omega_2 - \pi^3 \omega_3 \\
\delta^4 \omega = \omega_0 + 15\pi \omega_1 + 25\pi^2 \omega_2 + 10\pi^3 \omega_3 + \pi^4 \omega_4
$$

and so we need to solve

$$
a_3 \delta^3 \omega + a_2 \delta^2 \omega + a_1 \delta \omega + a_0 \omega = 10 \pi^3 \omega_3 + 25 \pi^2 \omega_2 + 15 \pi \omega_1 + \omega_0
$$

which yields

$$
\begin{align*}
a_3 &= -10, \\
a_2 &= -35, \\
a_1 &= -50, \\
a_0 &= -24. \\
\end{align*}
$$

□

By induction we may now write any $\delta^i \omega$ as a linear combination of $\delta^j \omega$ with coefficients that are power series in $\lambda$ that vanish at $\lambda = 0$. More precisely, we have the following.

**Corollary 4.13.** For $i > 3$ we can write

$$
\delta^i \omega = g^i_0(\lambda) \delta^3 \omega + g^i_1(\lambda) \delta^2 \omega + g^i_2(\lambda) \delta \omega + g^i_3(\lambda) \omega
$$

with $g^i_j$ power series in $\lambda$ that vanish at $\lambda = 0$, i.e.,

$$
\delta^{i>3} \omega = 0 \text{ in } H_0.
$$

**Proof.** Proceed by induction. It is true for $i = 4$ by Lemma 4.12. Now $\delta^{i+1} \omega = \delta(\sum_{j \leq 3} g^j \delta^j \omega) = \sum_{j \leq 3} (\delta g^j) \delta^j \omega + \sum_{j < 3} g^j \delta^{j+1} \omega + g^3 \delta^4 \omega$. □

Now we can look at the symplectic pairing $(-,-)$. Because it is compatible with the Gauss-Manin connection we need only consider pairings of the form $(\omega, \delta^i \omega)$. By Griffiths transversality $(\omega, \delta^i \omega) = 0$ for $i \leq 2$.

**Definition 4.14.** Let

$$
Y(\lambda) = (\omega, \delta^3 \omega)
$$

denote the Yukawa coupling.

At this point we know only that $Y(\lambda) \in \mathbb{C}_p((\lambda))$, but in fact we can say more. For example, $(-,-)$ is non-degenerate on the span of $\{\delta^i \omega | 0 \leq i \leq 3\}$ and so $Y(\lambda) \neq 0$. The following Lemma demonstrates that we can calculate it up to a constant. In fact, for our purposes it is sufficient to prove that $Y(\lambda)$ is a power series that does not vanish at $\lambda = 0$. For this we need only the fact that $\lambda$ divides $g^3_3(\lambda)$.

**Lemma 4.15.** The Yukawa coupling $Y(\lambda)$ satisfies the differential equation

$$
\partial_\lambda Y(\lambda) = -\frac{5^9 \lambda^4}{5^9 \lambda^5 + 1} Y(\lambda)
$$

and so $Y(\lambda) \in \mathbb{C}_p[[\lambda]]^\times$, i.e. $Y(\lambda)$ is a power series in $\lambda$ with a non-zero constant term.

---

9 A good reference for our treatment of the Yukawa coupling is [2].
Proof. As explained above \((\omega, \delta^2 \omega) = 0\), so that \(0 = \delta^2 (\omega, \delta^2 \omega) + 2(\delta \omega, \delta^3 \omega) + (\omega, \delta^4 \omega)\) and

\[
(\delta \omega, \delta^3 \omega) = -\frac{1}{2}(\omega, \delta^4 \omega).
\]

Then

\[
\delta Y(\lambda) = (\delta \omega, \delta^3 \omega) + (\omega, \delta^4 \omega) = \frac{1}{2}(\omega, \delta^4 \omega) = \frac{1}{2}(\omega, g(\lambda)(-10)\delta^3 \omega) = -5g(\lambda)Y(\lambda).
\]

\(\square\)

5. The coefficients \(c^\alpha_I\) and the case of \(\lambda = 0\)

Since \(\delta^{<3} \omega\) is a basis of \(\mathcal{H}_0\) and \((-,-)_0\) is non-degenerate, to compute the Frobenius matrix at \(\lambda = 0\) it is sufficient to compute \((Fr(\delta^i \omega), \delta^i \omega)_0\). Because we can express \(\delta^i \omega\) in terms of \((xt)^j\) and \(Fr((xt)^j dx dt)\), as we have seen, is expressed in terms of \(\omega_I\), it is important to compute \((\omega_I, \delta^i \omega)_0\). We begin this below.

Definition 5.1. Let

\[
c^\alpha_I = \frac{(-1)^I \pi_I}{Y} (\omega_I, \delta^\alpha \omega)_0
\]

which is non-trivial only for \(\alpha = 0, 1, 2, 3\) by Corollary 4.13.

We now translate the results of the previous section into relations on \(c^\alpha_I\) which are needed to compute the matrix elements of \(Fr\) at \(\lambda = 0\).

Theorem 5.2. The constants \(c^\alpha_I\) are determined by symmetry in \(i, j, k, n, m\) and the relations

(3) \[
c^\alpha_{(i+1)jkns} = ic^\alpha_{ijknms} + \frac{1}{5}c^\alpha_{ijknms}
\]

and

(4) \[
c^\alpha_{ijknm(s+1)} = (s+1)c^\alpha_{ijknms} - c^\alpha_{ijknms}
\]

as well as the conditions that

\[
c^\alpha_I > 3 = 0
\]

and

\[
c^\alpha_{000000} = 1.
\]

Furthermore the above holds for \(c^\alpha_J\).

Proof. The symmetry of \(c^\alpha_I\) follows from the symmetry of \(\omega_I\). Notice that \(c^\alpha_I > 3\) vanishes since \(\delta^{>3} \omega\), when written as a linear combination of \(\delta^i \omega\) \((0 \leq i \leq 3)\), is divisible by \(\lambda\). By definition \(c^\alpha_0 = 1/Y \cdot (\omega, \delta^3 \omega) = 1\).

Now we prove the two reduction formulas:

\[
c^\alpha_{(i+1)jkns} = \frac{(-1)^{I+1} \pi^{I+1}}{Y} (\omega_{(i+1)jkns}, \delta^\alpha \omega)_0
\]

\[
= \frac{(-1)^{I+1} \pi^{I+1}}{Y} \left( \frac{1}{5\pi} (\delta \omega_I - 5i \omega_I), \delta^\alpha \omega)_0
\]

\[
= \frac{1}{5}c^\alpha_{I+1} + ic^\alpha_I
\]
and
\[ c_{ijkm(n+1)} = \frac{(-1)^{l+1}\pi^{l+1}}{Y} (\omega_{ijkm(n+1)}, \delta^\alpha \omega)_0 \]
\[ = \frac{(-1)^{l+1}\pi^{l+1}}{Y} (\pi (s+1)\omega - \delta \omega)_0 \]
\[ = (s+1)c_i^\alpha - c_i^{\alpha+1}. \]

The result now follows by induction. □

**Corollary 5.3.** The coefficient \( c_I^\alpha \) with \( \#(I) \geq 1 \) can be written as a linear combination of \( c_{I_j}^{\alpha+1} \), i.e.,
\[ c_I^\alpha = \sum a_j c_{I_j}^{\alpha+1} \]
with
\[ \#(I_j) \geq \#(I) - 1. \]

**Proof.** This follows immediately from the reduction Equation 3 above. □

### 5.1. Images of basis elements in cohomology

We would like to have, more or less, explicit formulas for the constants \( c_I^\alpha \) for \( \alpha = 3, 2, 1, 0 \) (the order is in terms of difficulty). We begin with some definitions.

**Definition 5.4.** Let \( D \) be the formal differential operator defined by
\[ D = \sum_{i=0}^{\infty} \partial_x^i. \]

**Definition 5.5.** For \( \alpha \) and \( \beta \) non-negative integers, let \( ^\alpha D_x^\beta \) be a differential operator defined as follows:
\[ ^\alpha D_x^\beta = \sum \partial_x ... \frac{1}{x} \partial_x \]
more precisely, it is the sum of all the possible words of length \( \beta \) in the letters \( \partial_x \) and \( \frac{1}{x} \) with exactly \( \alpha \) of the letters being \( \frac{1}{x} \). Thus
\[ ^\alpha D_x^\beta = 0 \quad \text{if} \quad \alpha > \beta. \]

**Remark.** For example \( ^0 D_x^2 = \partial_x^2, ^1 D_x^3 = \partial_x^2 \frac{1}{x} + \partial_x \frac{1}{x} \partial_x + \frac{1}{x} \partial_x^2, ^2 D_x^3 = \partial_x \left( \frac{1}{x} \right)^2 + \frac{1}{x} \partial_x \frac{1}{x} + \left( \frac{1}{x} \right)^2 \partial_x \), and \( ^3 D_x^3 = \left( \frac{1}{x} \right)^3 \), etc.

**Definition 5.6.** Define the integers \( ^\alpha D_x^\beta \) by
\[ ^\alpha D_x^\beta = ^\alpha D_x^{\beta_x}. \]

Equivalently,
\[ ^\alpha D_x^\beta = \left[ \left( \frac{1}{x} D_x \right)^\alpha \right]_0 \]
where \([g]_0 \) for \( g \in C_p[[z^{\pm 1}]] \) denotes its degree zero coefficient.

Notice that \( ^0 D_x^\beta = \beta! \) and \( ^\alpha D_x^\alpha = 1. \)

**Remark.** The integers \( ^\alpha D_x^\beta \) arise naturally in our computation due to the following observation. Define \( S^{\alpha}(\beta) \) recursively by \( S^0(\beta) = 1 \) and
\[ S^{\alpha}(\beta) = \sum_{i=1}^{\beta-1} \frac{S^{\alpha-1}(i)}{i}. \]
Observe that this makes sense only for $\beta > \alpha$. For $\beta \leq \alpha$ set $S^\alpha(\beta)$ to be 0. Then $\alpha D^{\beta - 1} = (\beta - 1)! S^\alpha(\beta)$.

**Definition 5.7.** Define the formal differential operator $\alpha D_x$ by

$$\alpha D_x = \sum_{\beta} \alpha D^{\beta}_x = \left(D^{\frac{1}{\alpha}}_x\right) D.$$

The key property of $\alpha D^{\beta}_x$ is demonstrated in the following Lemma.

**Lemma 5.8.** The integers $\alpha D^{\beta}_x$ satisfy the relation:

$$\alpha D^{\beta}_x = \beta \cdot \alpha D^{\beta - 1}_x + \alpha^{-1} D^{\beta - 1}_x$$

for $\beta > 0$. We set $\alpha D^{\beta}_x = 0$ for $\alpha < 0$.

**Proof.** For $\beta > \alpha > 0$ we have

$$\alpha D^{\beta}_x = \alpha D^{\beta - 1}_x x^\beta = \alpha D^{\beta - 1}_x \partial_x x^\beta + \alpha^{-1} D^{\beta - 1}_x \frac{1}{x} x^\beta = \beta \cdot \alpha D^{\beta - 1}_x + \alpha^{-1} D^{\beta - 1}_x.$$

If $\beta = \alpha > 0$ then $\alpha D^{\alpha}_x = 1 = \alpha^{-1} D^{\alpha - 1}_x$ and $\alpha \cdot \alpha D^{\alpha - 1}_x = 0$. If $\beta > \alpha = 0$ then $\alpha D^{\beta}_x = \beta! = \alpha(\beta - 1)! = \beta \cdot \alpha D^{\beta - 1}_x$ and $-\alpha D^{\beta - 1}_x = 0$. □

**Definition 5.9.** For $s \geq 0$ and $i \geq 0$ let

$$\alpha Q^s = (-1)\alpha \cdot \alpha D^s$$

and

$$\alpha P^i = \begin{cases} \frac{1}{\delta_{\alpha,0}} \cdot \alpha D^{i-1} & i > 0 \\ \frac{1}{5} \delta_{\alpha,0} = \frac{1}{5} \alpha P^0 & i = 0. \end{cases}$$

Note that $\alpha P^1 = \frac{1}{5} \delta_{\alpha,0} = \frac{1}{5} \alpha P^0$.

The following is an immediate corollary of Lemma 5.8 and is the main ingredient in the proof of Theorem 5.13.

**Corollary 5.10.** For $s \geq 0$ and $i > 0$,

(5) $$\alpha P^{i+1} = i \cdot \alpha P^i + \frac{1}{5} \cdot \alpha^{-1} P^i$$

and

(6) $$\alpha Q^{s+1} = (s + 1) \cdot \alpha Q^s - \alpha^{-1} Q^s.$$

Theorems 5.11 and 5.13 describe the coefficients $c^I_I$ in a way that will be useful to us.

**Theorem 5.11.** Most $c^I_I$ vanish. More precisely,

$$c^I_I = 0 \quad \text{for} \quad \alpha + \#(I) > 3.$$  

**Proof.** Recall that $c^I_{I\geq 3} = 0$ so the theorem holds in these cases. We proceed by induction on $\alpha$. Let $c^\alpha_I$ be such that $\alpha + \#(I) > 3$. But by Corollary 5.8

$$c^\alpha_I = \sum a_j c^\alpha_{I_j}$$

and $\#(I_j) + \alpha + 1 \geq \#(I) - 1 + \alpha + 1 > 3$ so that $c^\alpha_{I_j}$ and thus $c^\alpha_{I_j}$ vanish. □
Definition 5.12. Define integers $\chi = \chi_{\alpha, I}$ by  
\[ \chi_{\alpha, I} = 3 - #(I) - \alpha. \]
Thus $c_{\alpha}^0 = 0$ if $\chi_{\alpha, I} < 0$.

Theorem 5.13. The non-vanishing $c_{\alpha}^s$ are explicitly described below.

\[
\begin{align*}
&c_{\alpha}^3 = \gamma Q^s, \\
&c_{\alpha}^2 = \gamma Q^s, c_{is}^2 = \gamma Q^s \cdot 0 p_i, \\
&c_{\alpha}^1 = \gamma Q^s, c_{is}^1 = \sum_{\alpha + \beta = 1} \gamma Q^s \cdot \beta P^i, \\
&c_{\alpha}^0 = \gamma Q^s, c_{is}^0 = \sum_{\alpha + \beta = 2} \gamma Q^s \cdot \beta P^i.
\end{align*}
\]

More compactly, let $\chi = \chi_{\alpha, I}$ then
\[
(7) \quad c_{\alpha}^s = c_{ijknms}^0 = \sum_{\gamma + \beta_1 + \beta_5 = \chi} \gamma Q^s \cdot \beta_1 P^i \cdot \beta_2 P^j, \beta_3 P^k, \beta_4 P^m, \beta_5 P^n.
\]

Proof. It is sufficient to show that the Equation (7) above satisfies the relations and conditions of the Theorem 5.12.

Clearly the Equation (7) is symmetric in $i, j, k, n, m$. If $\alpha > 3$ then $\chi < 0$ and so the sum is over $\emptyset$ and thus is 0; we conclude that the formula holds for $c_{\alpha}^3 > 3$. The coefficient $c_{\alpha}^0 = \gamma Q^0 = 1$ as expected.

Let us verify Equation (7). Consider $s \geq 0$, note that $\#(\{i, j, k, n, m, s + 1\}) = \#(\{i, j, k, n, m, s\})$. Then
\[
\begin{align*}
&c_{ijknm(s+1)}^0 = \sum_{\gamma + \beta_1 + \beta_5 = 0} \gamma Q^s \cdot \beta_1 P^i \cdot \beta_2 P^j \cdot \beta_3 P^k \cdot \beta_4 P^m - \sum_{\gamma + \beta_1 + \beta_5 = 1} \gamma Q^s \cdot \beta_1 P^i \cdot \beta_2 P^j \cdot \beta_3 P^k \cdot \beta_4 P^m \\
&= (s + 1) c_{ijknms}^0 - \sum_{\gamma + \beta_1 + \beta_5 = 1} \gamma Q^s \cdot \beta_1 P^i \cdot \beta_2 P^j \cdot \beta_3 P^k \cdot \beta_4 P^m \\
&= (s + 1) c_{ijknms}^0 - c_{ijknms}^{0+1}.
\end{align*}
\]

We verify Equation (7) in two steps.

First, assume that $i > 0$, and so $\#(\{i + 1, j, k, n, m, s\}) = \#(\{i, j, k, n, m, s\})$ and
\[
\begin{align*}
&c_{(i+1)jkns}^0 = \sum_{\gamma + \beta_1 + \beta_5 = 0} \gamma Q^s \cdot \beta_1 P^i + 1 \cdot \beta_2 P^j \cdot \beta_3 P^k \cdot \beta_4 P^m \\
&= i \sum_{\gamma + \beta_1 + \beta_5 = 0} \gamma Q^s \cdot \beta_1 P^i \cdot \beta_2 P^j \cdot \beta_3 P^k \cdot \beta_4 P^m + \frac{1}{5} \sum_{\gamma + \beta_1 + \beta_5 = 1} \gamma Q^s \cdot \beta_1 P^i \cdot \beta_2 P^j \cdot \beta_3 P^k \cdot \beta_4 P^m \\
&= i c_{ijknms}^0 + \frac{1}{5} c_{ijknms}^{0+1}.
\end{align*}
\]
Let $i = 0$, note that $\chi_{\alpha, \{1, j, k, n, m, s\}} = \chi_{\alpha+1, \{0, j, k, n, m, s\}}$, so that

\[
c_{\alpha}^{ijknm} = \sum_{\gamma + \beta_1 + \ldots + \beta_s = \chi} \gamma Q^s \cdot \beta_1 p^1 \ldots \beta_s p^m
\]

\[
= \sum_{\gamma + \beta_1 + \ldots + \beta_s = \chi} \gamma Q^s \cdot \frac{1}{5} \delta_{\beta_1, 0} \ldots \beta_s p^m
\]

\[
= \frac{1}{5} \sum_{\gamma + \beta_1 + \ldots + \beta_s = \chi} \gamma Q^s \cdot \beta_1 p^0 \ldots \beta_s p^m
\]

\[
= \frac{1}{5} c_{\alpha+1}^{ijknm}.
\]

\[\square\]

6. Calculation of the Frobenius matrix elements

We are interested primarily in the first row of the Frobenius matrix. In fact all the other entries can be deduced from general considerations using the first row. Namely, the matrix elements are determined by the data of

\[(Fr \delta^i \omega, \delta^j \omega)_0\]

for $0 \leq i, j \leq 3$. However the compatibility of the Frobenius map and the symplectic structure with the Gauss-Manin connection implies that

\[(Fr \delta^i \omega, \delta^j \omega)_0 = \left(\frac{1}{p^i} Fr \omega, \delta^i \omega\right)_0 = \frac{(-1)^i}{p^i} (Fr \omega, \delta^{i+j} \omega)_0.\]

In particular the first column, i.e., $(Fr \delta^i \omega, \delta^3 \omega)_0$, is 0 except for the first entry.

Unfortunately, but not surprisingly, it is the first column that is very easy to compute directly. Then things get progressively harder. We focus on going as far as possible with the direct computation of the first row. Then we provide some additional formulas for the reader interested in computing other matrix elements directly; we hope that the resulting identities involving the coefficients of the Dwork exponential will prove interesting.

**Definition 6.1.** We will call the power series

\[f(x) = \exp(x^p/p + x) =: \sum B_i x^i\]

the Dwork exponential.

**Remark.** Recall that the power series $A(z) = \exp(\pi(z^p - z))$ was used to define the action of the Frobenius. Notice that

\[A(z) = f(-\pi z).\]

**Lemma 6.2.** The Dwork exponential coefficients $B_i$’s and $c_{ij}^{\alpha}$’s can be used to compute the first row, i.e.,

\[(Fr \omega, \delta^{\alpha} \omega)_0 = -p^5 Y \sum B_i B_j B_k B_n B_m B_s c_{ijknm(s+p-1)}^{\alpha}.\]
Proof. Recall from the proof of Lemma 4.6 that $Fr(\omega)$ can be written explicitly in terms of the $\omega_{ts}$. More precisely,

$$
(Fr \omega, \delta^a \omega)_0 = p^6 \sum A_i \ldots A_m A_s (\omega_{ijklmn(s+p-1)}, \delta^a \omega)_0
$$

$$
= \frac{p^5 Y}{\pi^{p-1}} \sum A_i \frac{(-1)^i}{\pi^i} \ldots A_s \frac{(-1)^s}{\pi^s} \delta^a_{ijklmn(s+p-1)}
$$

$$
= -p^5 Y \sum B_i B_j B_k B_m B_s c_{ijklmn(s+p-1)}^a.
$$

\[ \square \]

Similarly, it is easy to see that

$$
(Fr (xt)^j, \delta^a \omega)_0 = (\omega_j, \delta^a \omega)_0 = (\sum_{j \leq i} a_j \omega_j, \delta^a \omega)_0 = \left( -\pi^{j+1} \frac{p^5 Y}{\pi^j} \sum B_i B_j B_k B_m B_s c_{ijklmn(s+jp+p-1)}^a \right).
$$

One can express $\delta^i \omega$ in terms of $(xt)^j = \omega_j$ using Equation 2, or more directly we can compute the coefficient of $\omega_j$ in $\delta^i \omega = \sum_{j \leq i} a_j \omega_j$. Let $T = -\delta$, $\gamma_s = (-\pi)^s \omega_s$, then $T \gamma_s = (s+1) \gamma_s - \gamma_{s+1}$. It is now easy to diagonalize $T$ so that $T \gamma_0 = (-1)^i \delta^i \omega$ can be computed explicitly. More precisely, if $\delta^i \omega = \sum_{j \leq i} a_j^i \omega_j$

then

$$
a_j^i = (-1)^i \frac{\pi^j}{\pi^i} \sum_{\alpha, \beta = j} (-1)^\alpha \frac{(\alpha + 1)^i}{\alpha! \beta!}.
$$

Remark. The following is an easy observation that will be essential to our computation. If $g(x) = \sum a_{\beta} x^\beta$, then

$$
\sum a_{\beta} \cdot \beta^\alpha = [\beta^\alpha g(x)]_0.
$$

The Lemma below is inspired by [5]. We will only need its simplest case, namely $a = 1$ when it is not difficult to check that

$$(8) \quad x^{p-1} f(x) = \partial_x f(x) - f(x)$$

so that

$$(9) \quad \partial_x x^{p-1} f(x) = -x^{p-1} f(x)$$

since the Equation 8 implies that $\partial_x x^{p-1} f(x) = -f(x)$. Thus

$$
(9) \quad \partial_x x^{p-1} f(x) = -x^{p-1} \frac{1}{x} f(x), \quad s \geq 1
$$

and

$$
(10) \quad \partial_x x^{p-1} f(x) = -f(x).
$$

We will sketch a general proof below.

\[ 10 \] One must of course be careful of convergence issues, however these do not pose a problem here.
Lemma 6.3. The Dwork exponential $f$ satisfies the following equation

$$\left( \frac{D}{x} \right)^s x^{ap} f(x) = (-1)^a \sum_{i=1}^{a} p^{a-i} \cdot i^{-1} D^{a-1} \left( \frac{1}{x} \right)^{s-i} f(x)$$

for $a \geq 1$.

Proof. The idea, as in [5], is to find a $g_a$ such that $x^{ap-1} f(x) = \partial_x g_a(x) - g_a(x)$. It is easy to see that such a $g_a$ can be found, and it is of the form $g_a(x) = H_a(x^p) f(x)$ where $H_a$ is a polynomial that one can write down explicitly. The rest is a very tedious calculation. It is not very difficult, using the explicit form of $H_a$, to express $\left( \frac{D}{x} \right)^s x^{ap} f(x)$ as a linear combination of the same objects with strictly smaller $s$ and $a$. One then proceeds by induction to reduce each $a$ to 0. □

6.1. First column. We begin with the simplest case which is $(Fr, \delta^3 \omega)_0$ and we compute it using the methods discussed above, in particular Equations 9 and 10.

$$(Fr, \delta^3 \omega)_0 = -p^5 Y \sum B_s c_{s+p-1}^i = -p^5 Y \sum B_s \cdot 0^{D^{s+p-1}}$$

$$= -p^5 Y \left[ 0 D_x x^{p-1} f(x) \right]_0 = p^5 Y [f(x)]_0 = p^5 Y$$

Recall that we are using the Dwork definition of the Frobenius map that introduces an extra factor of $p^2$, so that in the standard convention, the coefficient of $\omega$ in $Fr(\omega)$ is $p^3$.

6.2. Second column. Let us turn our attention to the last case where we can obtain an exact answer by a direct computation. This computation is the main motivation for the present paper as it obtains a result that, at least in the case of the mirror quintic, bypasses the theory of motives used in [11].

$$(Fr, \delta^2 \omega)_0 = -p^5 Y \left( \sum B_s c_{s+p-1}^2 + 5 \sum B_s B_{s} c_{s+p-1}^i \right)$$

$$= -p^5 Y \left( \sum B_s (-1 D^{s+p-1}) + 5 \sum B_s B_{s} \cdot 0^{D^{s+p-1}} \cdot 0^{D^{-1}} \right)$$

$$= -p^5 Y \left( [D_x x^{p-1} f(x)]_0 + [0 D_x x^{p-1} f(x)]_0 \left[ 0 D_x \frac{1}{x} f(x) \right]_0 \right)$$

$$= -p^5 Y \left( [0 D_x \frac{1}{x} f(x)]_0 - [0 D_x \frac{1}{x} f(x)]_0 \right) = 0$$

Note that the vanishing of this coefficient implies, by the compatibility of the Frobenius map with the symplectic form, that $(Fr, \delta \omega)_0$ is also 0, see [8]. However we want to try to compute it directly in the next section.

6.3. Third column. By combining the result of last section with the direct computation of this one, we obtain an interesting non-linear relation on the coefficients of the Dwork exponential. It would be interesting to see if this formula generalizes.
It turns out that the most obvious generalization is false (see below).

\[ 0 = (Fr \omega, \delta \omega)_0 \]
\[ = -p^5 Y \left( \sum B_s c_{s+1} + 5 \sum B_i B_s c_{i(s+p-1)} \right) \]
\[ = -p^5 Y \left( \sum B_s \cdot 2D^s + 5 \sum B_i B_s \left\{ \frac{1}{5^2}, 0D^{s+p-1}, 1D^{i-1} \right\} \right) \]
\[ + 10 \sum B_i B_j B_s \left\{ 0D^{i-1}, 0D^{j-1}, 0D^{s+p-1} \right\} \]
\[ = -p^5 Y \left( 2D_x x^{p-1} f)_0 + \frac{1}{5} [0D_x x^{p-1} f]_0 \left[ 1D_x f \right]_0 - [1D_x x^{p-1} f]_0 \left[ 0D_x f \right]_0 \right] \]
\[ + \frac{2}{5} [0D_x x^{p-1} f]_0 \left[ 0D_x f \right]_0 ^2 \right) \]
\[ = -p^5 Y \left( - \left[ 1D_x f \right]_0 - \frac{1}{5} \left[ 0D_x f \right]_0 + \left[ 0D_x f \right]_0 ^2 \right) \]

We conclude that

\[ (11) \quad \left[ 1D_x f \right]_0 = \frac{[0D_x f]_0 ^2}{2} \]

which is the promised quadratic relation on the coefficients of \( f \). It is natural, especially in view of the next section, to generalize the Equation (11) by conjecturing that

\[ (12) \quad \Delta_s := \left[ -1D_x f \right]_0 - \frac{[0D_x f]_0 ^3}{s!} = 0. \]

The next section will show that this is false.

6.4. Fourth column. At this point we perform the last computation, namely we look at \((Fr(\omega), \omega)_0\). It is impossible to derive its value from general considerations of the kind considered in [6]. The vanishing of this last matrix entry is equivalent to the case \( s = 3 \) of the false formula (12) above. Note that we are able to use
Recall that $\Delta_3$ denotes the difference

$$\Delta_3 = \left[ 2D_x \frac{1}{x} f \right]_0 - \frac{[0D_x \frac{1}{x} f]^3}{3!}$$
we see that in the standard convention, the coefficient of $\delta^3 \omega$ in $F r \omega$ is

$$p^3 \frac{24}{5^2} \Delta_3.$$  

Contrary to the case of the third column where a suggestively similar expression vanishes, $\Delta_3$ is not 0 as can be checked by a computer and seems in fact to be, as is mentioned in the introduction, a rational multiple of $\zeta_p(3)$.

It is certainly possible, though outside the scope of this paper, to perform these same calculations for the case of the Calabi-Yau 5-fold (and higher) family

$$\lambda(x_0^7 + x_1^7 + x_2^7 + x_3^7 + x_4^7 + x_5^7 + x_6^7) + x_0 x_1 x_2 x_3 x_4 x_5 x_6 = 0.$$  

As we have seen above, the calculations of the extra two columns will get progressively worse. We expect that higher $\Delta$s’ will make an appearance which will make it possible to test further conjectures regarding their value. These conjectures will arise from certain motivic considerations and will be testable via a computer calculation similar to the case of $\Delta_3$.

Acknowledgments. We are indebted to A. Schwarz who persuaded us to attempt this computation and encouraged us along the way while providing much appreciated comments and advice. We would like to thank M. Kontsevich for useful discussions and V. Vologodsky for helpful communications. Thanks are also due to P. Will for his assistance with Maple and P. Dragon for his computer proof of the non-vanishing. We appreciate the hospitality of IHÉS, MPIM Bonn and University of Waterloo where parts of this paper were written.

References

[1] P. Candelas, X. de la Ossa, F. Rodriguez Villegas, Calabi-Yau Manifolds over Finite Fields, I, [hep-th/0012233].

[2] D. A. Cox, S. Katz, Mirror symmetry and algebraic geometry, Mathematical Surveys and Monographs, 68. American Mathematical Society, Providence, RI, 1999.

[3] D. Delbourgo, A Dirichlet series expansion for the $p$-adic zeta-function, J. Aust. Math. Soc. 81 (2006), no. 2, 215–224.

[4] H. Furusho, $p$-adic multiple zeta values I – $p$-adic multiple polylogarithms and the $p$-adic KZ equation, Preprint [arXiv:math/0304085v2 [math.NT]].

[5] N. M. Katz, On the differential equations satisfied by period matrices, Publications Mathématiques de l’IHÉS, 35 (1968), 71–106.

[6] M. Kontsevich, A. Schwarz, V. Vologodsky, Integrality of instanton numbers and $p$-adic B-model, Phys. Lett. B 637 (2006), 97–101, Preprint [hep-th/0603106].

[7] A. Schwarz, I. Shapiro, Twisted de Rham cohomology, homological definition of the integral and “Physics over a ring”, Preprint [arXiv:0809.0080].

[8] I. Shapiro, Frobenius map on local Calabi-Yau manifolds, In preparation.

[9] A. Schwarz, V. Vologodsky, Frobenius transformation, mirror map and instanton numbers, Phys. Lett. B 660 (2008), 422–427, Preprint [hep-th/0606151].

[10] A. Schwarz, V. Vologodsky, Integrality theorems in the theory of topological strings, Preprint [arXiv:0807.1714].

[11] V. Vologodsky, Integrality of instanton numbers, Preprint [arXiv:0707.3617].

Institut des Hautes Études Scientifiques, Bures-sur-Yvette, France

E-mail address: shapiro@ihes.fr