THE EXISTENCE OF GORENSTEIN TERMINAL FOURFOLD FLIPS

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ABSTRACT. We prove that for a flipping contraction from a Gorenstein terminal 4-fold, the pull back of a general hyperplane section of the down-stair has only canonical singularities. Based on this fact and using Siu-Kawamata-Nakayama’s extension theorem [Si], [Kaw4], [Kaw5] and [Nak2], we prove the existence of the flip of such a flipping contraction. Furthermore we classify such flipping contractions and the flips under some additional assumptions.

0. INTRODUCTION

To proceed the Minimal Model Program (in short MMP), an elementary transformation called flip is very important (see [KMM] for detail).

Definition 0.1. Let $X$ be a normal algebraic variety (resp. normal analytic variety) with only canonical singularities and $Y$ a normal algebraic variety (resp. $(Y, S)$ a pair of an analytic space and its compact subspace). A projective morphism $f : X \to Y$ is called a flipping contraction if

1. $-K_X$ is $f$-ample;
2. $\rho(X/Y) = 1$ (resp. $\rho(X/Y, f^{-1}(S)) = 1$);
3. $f$ is an isomorphism in codimension 1.

If there exists a normal algebraic variety (resp. normal analytic variety) $X^+$ with only canonical singularities and a projective morphism $f^+ : X^+ \to Y$ such that

1. $K_{X^+}$ is $f^+$-ample;
2. $f^+$ is an isomorphism in codimension 1,

we call $f^+$ the flip of $f$. We call the following diagram a flipping diagram:

$$
\begin{array}{ccc}
X & \rightarrow & X^+ \\
\downarrow f & \downarrow & \nearrow f^+ \\
Y & \rightarrow & \\
\end{array}
$$

The existence of the flip is a very hard problem. In dimension 3, Shigefumi Mori proved it in [M4]. As a test case of 4-dimensional flips, we consider a flipping contraction from an algebraic 4-fold with only Gorenstein terminal singularities. Let $X$ be an algebraic 4-fold with only Gorenstein terminal singularities and $f : X \rightarrow Y$ be a flipping contraction. Let $E$ be the exceptional locus. Since there is no flipping

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contraction from an algebraic (or analytic) 3-fold with only Gorenstein terminal
singularities, we find that $f(E)$ is a set of finite points. Hence replacing $Y$ by
a small Stein neighborhood of a point in $f(E)$, we can proceed in the analytic
category. Precisely speaking, we consider the following object below (we call this
(*)).

(*) Let $X$ be an analytic 4-fold with only Gorenstein terminal singularities and
$(Y, P)$ a pair of a contractible 4-dimensional Stein space and a point in it such that
$Y$ has only ccDV singularities (i.e., singularities whose general hyperplane sections
have only cDV singularities) outside $P$. Let $f : X \to Y$ be a flipping contraction
and $E := f^{-1}(P)$, i.e., the exceptional locus of $f$.

In [Kaw3], Yujiro Kawamata considered the case where $X$ is smooth. He proved
the following:

**Theorem 0.1.** Assume that $X$ is smooth. Then the flip exists and $E \simeq \mathbb{P}^2$ and
$\mathcal{N}_{E/X} \simeq \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$. In particular we obtain the flip by blowing up $E$ (the
extension locus of the blowing up is $\mathbb{P}^2 \times \mathbb{P}^1$) and blowing down this $\mathbb{P}^2 \times \mathbb{P}^1$ to $\mathbb{P}^1$.

Quite recently Yasuyuki Kachi proved in his preprint [Kac2] the following:

**Theorem 0.2.** Assume that $X$ is singular and has only isolated complete inter-
section terminal singularities. Suppose that there is a member of $|-2K_Y|$ through
$P$ which has only a rational singularity at $P$.

Then the flip exists and $E \simeq \mathbb{P}^2$ and $\mathcal{N}_{E/X} \simeq \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-2)$. Furthermore $X$ has
only one singularity on $E$, which is analytically isomorphic to $o \in (xy + zw + t^m = 0) \subset \mathbb{C}^4$.

He proved the existence of such a flip by induction and constructed the desired
flip very explicitly (see [Kac2, §8] for detail). He also investigated some special
semistable 4-fold flipping contractions in [Kac1].

Our starting points are the following two theorems:

**Theorem 1.2 (Rough classification of the exceptional locus).** Assume that
the exceptional locus $E$ contains 2-dimensional components. Let $E = \bigcup E_i$ be the
irreducible decomposition of $E$. Then $E$ is purely 2-dimensional and $(E_i, -K_X|_{E_i})$
is isomorphic to $(\mathbb{F}_{n,0}, nl)$, where $l$ is a ruling of $\mathbb{F}_{n,0}$.

**Theorem 1.3.** Let $B$ a general hyperplane section through $P$. Then the strict
transform $A := f^*B$ has only canonical singularities.

Our main result is the following:

**Main Theorem (See Corollary 2.2).** The flip of $f$ as in (*) exists.

We will prove the finite generation of $\bigoplus_{m \geq 0} f_* \mathcal{O}(mlK_X)$ ($l$ is the index of $X$)
using Theorem 1.3 and the Siu-Kawamata-Nakayama’s extension theorem (see The-
orem 1.5).

By using the existence of the flip, we obtain a rough classification of $f$ and $f^+$
(see Corollary 2.3) and Furthermore if $A$ (as in Theorem 1.3) has only isolated
singularities and $E$ is irreducible, We can give the more detailed description (see
Corollary 2.4).

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Notation and Convention.

(1) In this paper, we will work over $\mathbb{C}$, the complex number field and in the analytic category;
(2) We denote by $F_n$, the Hirzebruch surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n))$ and by $F_n, 0$ the normal surface which is obtained from the Hirzebruch surface $F_n$ by contracting the negative section.

1. Preliminaries

Theorem 1.1. Let $X$ and $Y$ be normal log terminal varieties and $f : X \to Y$ a projective morphism. Let $L$ a $f$-ample line bundle on $X$ and $F$ a fiber of $f$. Assume that $f : X \to Y$ is the adjoint contraction supported by $K_X + rL$ and either $\dim F < r + 1$ if $\dim Y < \dim X$ or $\dim F \leq r + 1$ if $\dim Y = \dim X$.

Then $f^* f_* L \to L$ is surjective at every point of $F$.

Proof. See [AW1]. They assume that $L$ is ample but their proof works also for the case that $X$ is analytic and $L$ is relatively ample. □

Theorem 1.2 (Rough classification of the exceptional locus). We consider the object $(*)$. Assume that the exceptional locus $E$ contains 2-dimensional components. Let $E = \bigcup E_i$ be the irreducible decomposition of $E$. Then $E$ is purely 2-dimensional and $(E_i, -K_X|_{E_i})$ is isomorphic to $(\mathbb{F}_{n,0}, nl)$, where $l$ is a ruling of $\mathbb{F}_{n,0}$.

Proof. By Theorem 1.10 and 1.19 of [AW2], it is sufficient to exclude the following possibilities: $(E_i, -K_X|_{E_i})$ is isomorphic to $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ or $(\mathbb{F}_n, C_0 + ml)$, where $C_0$ is the negative section and $l$ is a ruling and $m \geq n + 1$. By following the argument of [W, Theorem 1.1, claim] with Theorem 2.13 in [Ko2], we can prove

Claim. Let $X$ be a variety with only log terminal singularities and $R$ an extremal ray of $X$. Let $F$ be an irreducible component of a non-trivial fiber of the contraction of $R$. Assume that for a general point $x \in F$, there is a rational curve $M \subset F$ through $x$ with the following condition:

1. its intersection with $-K_X$ is minimal among all rational curves in $F$ through $x$.
2. $X$ has only local complete intersection singularities along $M$ and $M$ is not contained in the singular locus of $X$.

Then

$$(1.2.1) \quad \dim F + \dim(\text{locus of } R) \geq \dim X + l(R) - 1,$$

where $l(R)$ is the length of $R$. Furthermore if the equality holds, the dimension of the deformation of $M$ through a fixed point $x$ is $\dim F - 1$.

If $(E_i, -K_X|_{E_i})$ is isomorphic to $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$, a general line satisfies the assumption of $M$ in Claim. So we can use Claim and derive a contradiction to the
inequality (1.2.1). If \((E_i, -K_X|_{E_i})\) is isomorphic to \((F_n, C_0 + ml)\), a general ruling satisfies the assumption of \(M\) in Claim. So by using Claim, we obtain the equality in (1.2.1). But a ruling cannot move if a general point on it is fixed, a contradiction to the second part of Claim. \(\square\)

By this Theorem, the exceptional locus of a Gorenstein terminal 4-fold flipping contraction is either purely 1-dimensional or purely 2-dimensional. In the former case, we call it a flipping contraction of type \((1,0)\). In the latter case, we call it a flipping contraction of type \((2,0)\).

**Theorem 1.3.** We consider the object \((*)\). Let \(B\) be a generic hyperplane section through \(P\). Then the strict transform \(A := f^*B\) has only canonical singularities.

**Proof.** We take a general member \(C \in |-K_X|\) and let \(D := f(C)\). By the freeness of \(|-K_X|\) (Theorem 1.1), we can assume that \(C\) is Gorenstein terminal. \(D\) is Gorenstein by the Serre-Grothendieck duality (cf. [Kaw 2, the Proof of Theorem 8.7]), which in turn shows that \(D\) is normal and \(C \to D\) has then only connected fibers by the Zariski Main Theorem. Hence if \(f\) is of type \((1,0)\), \(C \to D\) is an isomorphism or if \(f\) is of type \((2,0)\), \(f|_C\) is a flopping contraction. So in any case \(D\) has also Gorenstein terminal singularity at \(P\), i.e., cDV singularity. Then we may assume that \(B|_D\) is canonical by replacing \(B\) if necessary. So \(A|_C\) must be also normal and canonical since \(A|_C \to B|_D\) is isomorphism if \(f\) is of type \((1,0)\) or \(A|_C \to B|_D\) is crepant if \(f\) is of type \((2,0)\). We know that \(A\) is canonical along \(C|_A\) by the above argument. So it suffices to prove that \(A\) is canonical outside \(C|_A\).

Below argument is inspired by the proof of [Kaw 2, Theorem 8.5]. Let \(C'\) be a general member of \(|-2K_X|\) and \(D' := f(C')\). Since \(K_B + (D|_B)|_{D|_B} = K_{D|_B}\) is canonical and \(D|_B\) is Cartier on \(B\), \(K_B + (D|_B)\) is canonical by [Kaw4] (or [Kaw5, Theorem 1.4]). Hence

\[
K_B + \frac{1}{2} D'|_B \text{ is also canonical since } D' \text{ is more general than } D.
\]

We take the double cover \(\tilde{A} \to A\) (resp. \(\tilde{B} \to B\)) whose branch locus is \(C'|_A\) (resp. \(D'|_B\)). Let \(g : \tilde{A} \to \tilde{B}\) be the natural morphism. It is sufficient to prove that \(\tilde{A}\) is canonical since \(\tilde{A} \to A\) is etale outside \(C'|_A\). By (1.3.1), \(\tilde{B}\) is Gorenstein canonical. So \(\tilde{A}\) is also Gorenstein canonical since \(g\) is crepant and we are done. \(\square\)

**Remark.** By this Theorem, we see that the object \((*)\) is a very special example of a semistable 4-fold flipping contraction. (See [C] for the definition of a semistable flipping contraction.)

**Proposition 1.4.** Consider the situation of Theorem 1.3 and take \(A\) and \(B\) as there. Then

1. a general element of \(|-K_B|\) has only Du Val singularity at \(P\);
2. for any \(i\), \(E_i\) is not \(\mathbb{Q}\)-Cartier divisor in \(A\).

**Proof.**

1. \(D|_B \in |-K_B|\) in the proof of Theorem 1.3 satisfies (1).
2. (cf. the argument of [Kac1, 4.3]) We assume that for some \(i\), \(E_i\) is \(\mathbb{Q}\)-Cartier. Assume further that \(E\) has another component. Let \(E_j\) be a component such that \(E \cup E_j \neq \emptyset\). Then by the assumption that \(E\) is \(\mathbb{Q}\)-Cartier, \(E \cup E_j\)
is 1-dimensional. Then since the Picard numbers of such $E_j$’s and $E_i$ are 1, the union of $E_i$ and $E_j$’s is covered by one extremal ray in $\overline{NE}(A/B)$. For a ruling $m$ of $E_j$ (not contained in $E_i$), $E_i.m > 0$. But for a ruling $l$ in $E_i$, $E_i.l < 0$, a contradiction. Hence $E$ is irreducible and $\mathcal{Q}$-Cartier. So $B$ has only canonical singularities by [KMM, Lemma 5-1-7]. Note that $B$ is smooth outside $P$ and that $| - K_B|$ has a Du Val element through $P$. So in fact $B$ is terminal by [St, Section 5]. Since $B$ can deform to a 3-fold with only cDV singularities in $Y$, $B$ also has only cDV singularity by [Nam, Proposition (3.1)]. In particular $B$ has only hypersurface singularity so $Y$ has also only hypersurface singularity, a contradiction. We establish the proposition.

\[ \square \]

**Theorem 1.5.** Let $V$ be a smooth variety and $X$ a smooth (not necessarily connected) divisor on $V$. Let $\pi : V \to S$ be a projective morphism onto a variety $S$ with only connected fibers. Assume that $K_V + X$ is $\pi$-big for the pair $(V, X)$, i.e., $K_V + X$ is $\pi$-big and we can write $l(K_V + X) = A + B$ for a positive integer $l$, a $\pi$-ample divisor $A$ and a $\pi$-effective divisor $B$ such that $\text{Im}(\pi_*\mathcal{O}_V(B) \to \pi_*\mathcal{O}_X(B|_X)) = 0$. Then the natural homomorphism $\pi_*\mathcal{O}_V(m(K_V + X)) \to \pi_*\mathcal{O}_X(mK_X)$ is surjective for any positive integer $m$.

**Proof.** See [Si], [Kaw5, 2.2 Theorem A] or [Nak2, Theorem 4.9]. \[ \square \]

**Proposition 1.6 (H. Laufer).** Let $S$ be normal Gorenstein surface and $f : S \to T$ a projective bimeromorphic morphism to a normal surface $T$. Let $C$ be the exceptional curve. Suppose that $C$ is irreducible, isomorphic to $\mathbb{P}^1$ and $K_S.C = -1$. Then $f(C)$ is a smooth point of $T$, $S$ has only one singular point on $C$ which is of type $A_{n-1}$ for some $n \in \mathbb{N}$. Furthermore $C^2 = -\frac{1}{n}$.

**Proof.** See [LS, Theorem 0.1]. \[ \square \]

**Theorem 1.7 (Length of an extremal ray).** Let $X$ be a variety with only canonical singularities and $R$ an extremal ray of $X$. Let $F$ be a 1-dimensional irreducible component of the fiber of the contraction of $R$ which contains Gorenstein points of $X$. Then $K_X.F \geq -1$.

**Proof.** See [M4, 1.3 and 2.3.2] or [I, Lemma 1]. \[ \square \]

**Proposition 1.8.** Let $U$ be a 3-fold with only Gorenstein canonical singularities and $(V, P)$ a pair of a 3-dimensional normal Stein space and a point in it. Let $f : U \to V$ be a flipping contraction whose exceptional locus $l$ is connected. Then $l$ is irreducible and $l \subset \text{Sing } U$.

**Proof.** The irreducibility of $l$ can be proved by the same argument as the first part of the proof of Theorem 1.3. Assume that

\begin{equation}
(1.8.1) \quad \text{Sing } U \cap l \text{ consists of finite points.}
\end{equation}

Let $g : U' \to U$ be a partial resolution such that $g$ is crepant and $U'$ has only Gorenstein terminal singularities (cf. [M3] and [Re2]). Since $K_{U'}$ is not $f \circ g$-nef, we can find an extremal ray $R \in \overline{NE}(U'/V)$. Let $l'$ be an irreducible curve such that $[l'] \in R$. Then $l'$ is the strict transform of $l$ by (1.8.1) and the fact that $K_{U'}$ is $g$-nef. So $R$ is a flipping ray. But this contradicts the fact that there is no flipping contraction from a Gorenstein terminal 3-fold. \[ \square \]
Theorem 2.1. Let $X$ be a 4-fold with only canonical singularities and $f : X \to Y$ be a flipping contraction. Let $A$ be the pull back of a Cartier divisor $B$ on $Y$. Assume that $A$ has only canonical singularities. Then the flip of $f$ exists.

Proof. It suffices to prove that $\bigoplus_{m \geq 0} f_* \mathcal{O}(mK_X)$ is finitely generated, where $l$ is the minimum positive integer such that $IK_X$ is Cartier. Let $g : Z \to X$ be a good resolution for the pair $(X, A)$ and $A'$ the strict transform of $A$. We may take $g$ such that $(\text{excep}|A') = \text{excep}(g|A')$. By Theorem 1.5, $f_* g_* \mathcal{O}_Z(m(K_Z + A')) \to f_* g_* \mathcal{O}_{A'}(mK_{A'})$ is surjective for any positive integer $m$. Note that $(X, A)$ is canonical since $A$ is a Cartier divisor with only canonical singularities by [Kaw4] or [Kaw5, Theorem 1.4]. Hence $f_* g_* \mathcal{O}_Z(m(K_Z + A')) = f_* \mathcal{O}_X(ml(K_X + A)) = f_* \mathcal{O}_X(mlK_X) \otimes \mathcal{O}_Y(mlB)$ and $f_* g_* \mathcal{O}_{A'}(mlK_{A'}) = f_* \mathcal{O}_A(mlK_A)$. Hence by Nakayama’s lemma, it suffices to prove the finite generation of $\bigoplus_{m \geq 0} f_* \mathcal{O}(mlK_A)$. Let $\mu : A_T \to A$ be a small $\mathbb{Q}$-factorialization, i.e., $\mu$ is a small projective bimeromorphic morphism such that $A_T$ is $\mathbb{Q}$-factorial and has only canonical singularities. By running the MMP over $B$ starting from $A_T$ and taking the canonical model of a minimal model of $A_T$ over $B$, we obtain $f'^* : A'^* \to B$ such that $A'^*$ has only canonical singularities and $K_{A'^*}$ is $f'^*$-ample. We can easily see that $f_* \mathcal{O}(mlK_A) = f'^* \mathcal{O}(mlK_{A'^*})$ and $\bigoplus_{m \geq 0} f'^* \mathcal{O}(mlK_{A'^*})$ is finitely generated. Hence we are done. \hfill $\square$

Corollary 2.2.

(1) The flip for a 1-parameter family of canonical flipping contractions exists;
(2) let $f : X \to Y$ be a flipping contraction from a Gorenstein terminal 4-fold. Then the flip of $f$ exists.

Proof. (1) is clear. By Theorem 1.3, we can apply Theorem 2.1 for (2). \hfill $\square$

From now on we consider the object as in (*). We will use the notation as in the proof of Theorem 1.3 freely. We will denote by $f^+ : X^+ \to Y$ the flip of $f$ and by $E^+$ the exceptional locus of $f^+$. Furthermore we will denote with $+$ the strict transform of a divisor of $X$ on $X^+$.

Corollary 2.3.

(1) $\dim E^+ = 1$ and $\dim E = 2$;
(2) $A^+$ and $X^+$ have only Gorenstein terminal singularities;
(3) $f^+|_{C^+}$ is the $-K_X|_{C}$-flop for $f|_{C}$. In particular $#\{\text{components of } E\} = $#\{\text{components of } E|_C\};$
(4) assume that $(X, E)$ is $\mathbb{Q}$-factorial. Then $E$ and $E^+$ are irreducible.
(5) assume that $E$ is irreducible (and hence isomorphic to $\mathbb{F}_{n,0}$ for some natural number $n$). Then there exists a Weil divisor $H$ on $X$ such that $-K_X \sim nH$.

By using $C$ (as in the proof of Theorem 1.3), define ring structures to

$$
\bigoplus_{j=0}^{n-1} \mathcal{O}_X(-jH) \text{ and } \bigoplus_{j=0}^{n-1} \mathcal{O}_Y(-jf(H))
$$

and set

$$
\tilde{X} := \text{Specan} \bigoplus_{j=0}^{n-1} \mathcal{O}_X(-jH) \text{ and } \tilde{Y} := \text{Specan} \bigoplus_{j=0}^{n-1} \mathcal{O}_X(-jf(H)).
$$
Let $\tilde{E} \subset \tilde{X}$ be the pull back of $E$ by the natural morphism $\tilde{X} \to X$. Then the natural morphism $\tilde{f} : \tilde{X} \to \tilde{Y}$ is a flipping contraction which satisfies the same assumption as $f$, and $\tilde{E}$ is the exceptional locus of $\tilde{f}$ and is isomorphic to $\mathbb{P}^2$.

Proof.

(1) We saw in the proof of Theorem 1.3 that $P$ is a Gorenstein terminal singularity of $D$. On the other hand $E^+ \subset C^+$ and $f^+|_{C^+}$ is crepant. Hence $\dim E^+ = 1$. Furthermore by [KMM, Lemma 5-1-17], $\dim E = 2$;

(2) Since $P$ is a canonical singularity of $D \cap B$ and $f|_{C^+ \cap A^+}$ is crepant, $C^+ \cap A^+$ has only canonical singularity. Hence by $\text{Sing} A^+ \subset E^+$ and [St, Section 5], $A^+$ has only terminal singularities. Furthermore by the argument as in the proof of Proposition 1.4 (2), we see that $A^+$ is Gorenstein and so is $X^+$;

(3) since $A^+ \cap C^+$ has only canonical singularity and $\text{Sing} C^+ \subset E^+$, $C^+$ has only terminal singularities. Note that $-K_{\tilde{X}}|_{C}$ is $f|_{C}$-ample and $-K_{\tilde{X}}'|_{C^+}$ is $f^+|_{C^+}$-negative. Hence by the uniqueness of the $-K_{\tilde{X}}|_{C}$-flop, $f^+|_{C^+}$ is the $-K_{\tilde{X}}|_{C}$-flop for $f|_{C}$.

(4) By $\mathbb{Q}$-factoriality of $(X, E)$, $\rho(X^+/Y, E^+) = 1$ whence $E^+$ must be irreducible. Hence $E$ is also irreducible by (3).

(5) Since $C \twoheadrightarrow C^+$ is a terminal flop, we have $C^+, E^+ = -n$. Let $H^+$ be a Cartier divisor on $X^+$ such that $H^+, E^+ = -1$ and $H \subset X$ be the strict transform of $H^+$. Then $K_X + nH^+$ is linearly $f^+$-trivial since we consider locally analytically along $E^+$. Hence $K_X + nH^+$ is linearly $f$-trivial since the linear triviality is preserved by an anti-flip. Let $\tilde{X}$ and $\tilde{Y}$ are as in the statement of (5). We check that the natural morphism $\tilde{f} : \tilde{X} \to \tilde{Y}$ and the pull backs $\tilde{A}$ of $A$, $\tilde{B}$ of $B$ and $\tilde{E}$ of $E$ satisfy the same assumption as $f$, $A$, $B$ and $E$ and we prove that $\tilde{E}$ is $\mathbb{P}^2$. Let $\pi : \tilde{X} \to X$ be the covering morphism and $\tilde{C} := (\pi^*(C))_{\text{red}}$. Note that $n\tilde{C} = \pi^*(C)$, $\tilde{C} \simeq C$ and $\tilde{C}$ is a Cartier divisor since $C$ is contained in the branch locus. Then by the ramification formula $K_{\tilde{X}} = \pi^*K_X + (n-1)\tilde{C}$, $\tilde{C} \subset (-n)^{-1}|X|$. Since $\tilde{C}$ is a Cartier divisor, we see that $\tilde{X}$ is Gorenstein. We also know that $\tilde{X}$ is terminal since codimension 1 ramification locus $\tilde{C}$ of $\pi$ has only terminal singularities. The rest are clear except that $\tilde{E} \simeq \mathbb{P}^2$. The restriction of $\pi$ to $\tilde{E}$ is $\pi|_{\tilde{E}} : \tilde{E} = \text{Specan} \bigoplus_{j=0}^{n-1} \mathcal{O}_E(-j) \to E$, where $l$ is a ruling of $E$. (Note that $H|_{\tilde{E}} \sim l$.) So it coincides with the quotient $\mathbb{P}^2 \to \mathbb{F}_{n,0}$ by the action of $\mathbb{Z}_n$, $(X : Y : Z) \to (\eta X : \eta Y : Z)$, where $X, Y$ and $Z$ is the homogeneous coordinate of $\mathbb{P}^2$ and $\eta$ is a primitive $n$-th root of unity. So $\tilde{E}$ is $\mathbb{P}^2$.

We can classify $f$ and $f^+$ with additional assumptions as follows:

**Corollary 2.4.** Assume that $A$ (as in Theorem 1.3) has only isolated singularities and $E$ is irreducible (and hence isomorphic to $\mathbb{F}_{n,0}$ for some natural number $n$). Then

(1) $A$ is singular only at the vertex $v$ of $E$ (if $n = 1$, the vertex means a point on $E$). Near $v$, $(v \in E \subset A \subset X)$ is analytically isomorphic to $(o \in (x = z = t = 0) \subset (xy + zw = t = 0) \subset (xy + zw + t^k = 0))$ in $\mathbb{C}^5/\mathbb{Z}_n(1, -1, 1, -1, 0)$;

(2) $X^+$ is smooth, $E^+$ is $\mathbb{P}^1$ and $\mathcal{N}_{E^+/X^+} \simeq \mathcal{O}(-1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-n)$ or $\mathcal{O} \oplus \mathcal{O}(-2) \oplus \mathcal{O}(\pi)$. Furthermore the former case occurs if and only if $X$ has terminal singularities.
only \(1/(1, -1, 1, -1)\) singularity at \(v\).

**Proof.** We will prove (1). Consider the covering as in Corollary 2.3 (5). We will use the notation as in its proof. Then \( \hat{A} \) has only isolated singularities since \( C \cap A \) is smooth. Let \( q : \hat{A}_q \to \hat{A} \) be a small morphism such that the inverse image \( \hat{E}_q \) of \( \hat{E} \) is \( q \)-anti-ample (i.e., \( \hat{A}_q := \text{Proj} \bigoplus_{m=0}^{\infty} O_{\hat{A}}(-m\hat{E}) \) and \( q \) is the natural projection.) We can take such a small morphism by [Kaw2, Theorem 6.1]. Since \( \hat{E} \) is not \( \mathbb{Q} \)-Cartier by proposition 1.4, \( \hat{A}_q \) is not isomorphic to \( \hat{A} \). Let \( \Phi : \hat{A}_q \to \hat{A}^+ \) be the contraction of an extremal ray in \( \text{NE}(\hat{A}_q/B) \) and \( g^+ : \hat{A}^+ \to B \) the natural morphism. We obtain the following diagram:

\[
\begin{array}{ccc}
\hat{A}_q & \xleftarrow{q} & \hat{A} \\
\downarrow & & \downarrow \Phi \\
\hat{A}^+ & \xleftarrow{\hat{f}|_{\hat{A}}} & B \\
\end{array}
\]

**Claim 2.5.** \( \Phi \) is a divisorial contraction which contracts \( \hat{E}_q \) to a curve.

**Proof.** Since \( q \) is not an isomorphism and \(-\hat{E}_q\) is \( q \)-ample, \( \hat{E}_q \) contains all \( q \)-exceptional curves. If \( \Phi \) is a divisorial contraction which contracts \( \hat{E}_q \) to a point, such \( q \)-exceptional curves are contracted by \( \Phi \). But this is absurd since \( K_{\hat{A}_q} \) is \( q \)-trivial but \( \Phi \)-negative. If \( \Phi \) is a flipping contraction, then the flipping curve \( m \) is contained in the curve singularity of \( \hat{A}_q \) by Proposition 1.8. So by the assumption that \( \hat{A} \) has only isolated singularities, \( m \) must be contained in the \( q \)-exceptional curve, a contradiction. \( \square \)

Let \( \tilde{E}^+ \) be the curve \( \Phi(\hat{E}_q) \).

**Claim 2.6.**

1. \( \hat{A}^+ \) is smooth along \( \tilde{E}^+ \). \( \tilde{E}^+ \simeq \mathbb{P}^1 \) and \( \hat{E}_q \simeq \mathbb{F}_1 \);
2. \( g^+ : \hat{A}^+ \to B \) is isomorphic to the restriction of \( \tilde{f}^+ \) to the strict transform of \( \hat{A} \) on \( \tilde{X}^+ \). (Hence we will denote by \( \hat{A}^+ \) the strict transform of \( \hat{A} \) on \( \tilde{X}^+ \).)

**Proof.**

1. By Theorem 1.1, \( -K_{\hat{A}_q} \) is free near the fiber over any point \( Q \) of \( \tilde{E}^+ \), so we can take a smooth member \( D \in -K_{\hat{A}_q} \) near the fiber since \( \hat{A} \) has only canonical singularities. Since \( D \) maps isomorphically to \( \Phi(D) \in -K_{\hat{A}^+} \) (cf. [Kaw2, the Proof of Theorem 8.7]), we see that there is a smooth member of \( -K_{\hat{A}^+} \) through \( Q \). Note that \( Q \) is a canonical singularity of \( \hat{A}^+ \). By these, we can see that \( Q \) is a smooth point of \( \hat{A}^+ \) as follows:

   It is sufficient to prove that \( K_{\hat{A}^+} \) is Cartier at \( Q \). Assume the contrary. Let \( \pi : \hat{A}^+ \to \hat{A}^+ \) be the index 1 cover for \( K_{\hat{A}^+} \) near \( Q \). Then \( \hat{A}^+ \) is Gorenstein canonical at \( \pi^{-1}(Q) \). Since \( \pi \) is ramified only at \( Q \) and \( \Phi(D) \) is smooth, \( \pi^{-1}(\Phi(D)) \) has at least 2 components and they intersect mutually only at \( \pi^{-1}(Q) \).
Furthermore they are all smooth. In particular \( \pi^{-1}\Phi(D) \) satisfies \( R_1 \) condition. On the other hand \( \pi^{-1}\Phi(D) \) satisfies \( S_2 \) condition since this is a Cartier divisor of a canonical singularity. Hence \( \pi^{-1}\Phi(D) \) is normal by the Serre’s criterion. But this is a contradiction to (2.6.1).

Since \( \tilde{B} \) has only rational singularities and \( E^+ \) is an irreducible curve, \( E^+ \) must be \( \mathbb{P}^1 \). Since a general fiber \( n \) of \( \Phi \) is irreducible and reduced and \(-K_{\tilde{A}_q}n = 1 \) (Theorem 1.7), any fiber is irreducible and reduced. So \( E_q \) is \( \mathbb{F}_1 \).

(2) Let \( Q \) be any point on \( \tilde{E}^+ \), \( G \) a general (smooth) hyperplane section of \( \tilde{A}^+ \) through \( Q \) such that \( \tilde{A}^+|_{\tilde{E}^+} \) is one point and \( F \) the pull back of \( G \) (we consider analytically locally near \( Q \). Then \( F \) is normal. In fact, since the fiber \( (\tilde{E}_q)_{Q} \) over \( Q \) of \( \Phi \) is not contained in the singular locus of \( \tilde{A}_q \), \( \tilde{E}_q \) is generically Cartier in \( \tilde{A}_q \) along \( (\tilde{E}_q)_{Q} \), which in turn shows \( (\tilde{E}_q)_{Q} \) is generically Cartier divisor on \( F \). Since \( (\tilde{E}_q)_{Q} \) is smooth, \( F \) is generically smooth along \( (\tilde{E}_q)_{Q} \), i.e., \( F \) is normal. Furthermore we have \( K_{\tilde{F}}(\tilde{E}_q)_{Q} = -1 \) and \( F \) is Gorenstein. So we know by Proposition 1.6 that \( F \) has only one \( A_{m-1} \) singularity for some integer \( m \) and \( ((\tilde{E}_q)_{Q})^2_F = -\frac{1}{m} \). On the other hand, \( ((\tilde{E}_q)_{Q})^2_{\tilde{F}} = (\tilde{E}_q, (\tilde{E}_q)_{Q})_{\tilde{A}_q} \) and the value of the right side of this equality is independent of \( Q \), so \( m \) is also independent of \( Q \). Hence we find that \( \tilde{A}_q \) has the locally trivial \( cA_{m-1} \) curve singularity along \( M \) and outside \( M, \tilde{A}_q \) is smooth. By \( ((\tilde{E}_q)_{Q})^2_{\tilde{F}} = -\frac{1}{m} \), we obtain the subadjunction formula \( K_{\tilde{E}_q} + \frac{m-1}{m}M = K_{\tilde{A}_q} + (\tilde{E}_q|_{\tilde{E}_q})_{\tilde{A}_q} \) and \( K_{\tilde{A}_q} = \Phi^*K_{\tilde{A}^+} + m\tilde{E}_q \). Intersecting this with \( M \), we can see that \( K_{\tilde{A}_q} \tilde{E}_q^+ = 2m - 1 \). Remark that \( \tilde{E}_q, M \) is negative since \( \tilde{E}_q \) is \( q \)-anti-ample. So \( K_{\tilde{A}_q} \tilde{E}_q^+ \) is positive and hence \( g^+: \tilde{A}^+ \to \tilde{B} \) is the canonical model of \( \tilde{f}|_{\tilde{A}} \). On the other hand the restriction of \( \tilde{f}^+ \) to the strict transform of \( \tilde{A} \) on \( X^+ \) is also the canonical model of \( \tilde{f}|_{\tilde{A}} \) by Corollary 2.3. Hence by the uniqueness of the canonical model, they are isomorphic.

So we have \( 2m - 1 = -1 \), i.e., \( \tilde{A}_q \) is smooth also along \( M \).

\[ \square \]

By considering the normal bundle sequence
\[ 0 \to \mathcal{N}_{M/\tilde{E}_q} \to \mathcal{N}_{M/\tilde{A}_q} \to \mathcal{N}_{\tilde{E}_q/\tilde{A}_q}|_M \to 0, \]
we see that \( \mathcal{N}_{M/\tilde{A}_q} \simeq \mathcal{O}(-1) \oplus \mathcal{O}(-1) \). So \( M \) is contracted to an ordinary double point by \( q \). Denote this point by \( \tilde{v}(\in \tilde{A}) \). We note that \( \tilde{X} \) is singular at worst only at \( \tilde{v} \) since so is \( \tilde{A} \) and \( \tilde{X}^+ \) is smooth. Then \( \tilde{v} \) is the unique isolated ramification point of \( \pi \) and hence \( \tilde{v} = \pi^{-1}(v) \) (Recall that \( v \) is the vertex of \( E \)). So we can write locally analytically (\( \tilde{v} \in \tilde{E} \subset \tilde{A} \subset \tilde{X} \)) \( \simeq (a \in (x = z = t = 0) \subset (xy + zw = t = 0) \subset (xy + zw + t^k = 0)) \), where \( x, y, z, w \) are the semi-invariant coordinates and \( xy + zw \) is semi-invariant with respect to the action of \( \mathbb{Z}_n \). When we restrict the action to \( \tilde{E} \), the action is \((y, w) \to (\eta y, \eta w)\), where \( \eta \) is a primitive \( n \)-th root of unity by the explicit description of \( \pi|_{\tilde{E}} \). Hence the action is \((x, y, z, w) \to (\eta a x, \eta y, \eta a z, \eta w)\), where \( a \) is an integer. By the necessary condition for the quotient to be canonical ([M3, Theorem 2]), \( a \) must be \(-1 \). This is also sufficient.

Next we will prove (2). To determine the normal bundle \( \mathcal{N}_{E^+/X^+} \), we consider the normal bundle sequence
\[ (2.4.1) \]
\[ 0 \to \mathcal{N}_{E^+/X^+} \to \mathcal{N}_{E^+/\tilde{A}_q} \to \mathcal{N}_{\tilde{A}_q/\tilde{A}_q}|_M \to 0. \]
Since $A|_C$ is smooth, we see that $\mathcal{N}_{E|_{C/C}} = \mathcal{O} \oplus \mathcal{O}(-2)$ or $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ by using the normal bundle sequence

$$0 \rightarrow \mathcal{N}_{E|_{C/A|_C}} \rightarrow \mathcal{N}_{E|_{C/C}} \rightarrow \mathcal{N}_{A|_{C/C}|_{E|_C}} \rightarrow 0.$$ 

Since $C \dashrightarrow C^+$ is the flop, we have $\mathcal{N}_{E^+|_{C^+}} \simeq \mathcal{N}_{E|_{C/C}}$. On the other hand, $\mathcal{N}_{C^+|X^+|E^+} = \mathcal{O}(-n)$, so the sequence (2.4.1) is split. Hence we obtained the first part of (2).

To prove the second part of (2), we consider the covering described in Corollary 2.3 (5). We use the notation there. Recall that $\hat{C} \simeq C$ and $\check{C} \in |-K_X|$. By the argument above together with this, we see that

$$\mathcal{N}_{E^+|_{C^+}} \simeq \mathcal{O}(-1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-n)$$

if and only if $\mathcal{N}_{\mathcal{E}|_{C^+}} \simeq \mathcal{O}(-1) \oplus \mathcal{O}(-1)$.

So 'if' part of (2) follows from Kawamata’s determination of a flipping contraction from a smooth 4-fold (see Theorem 0.1) and (1). Finally we prove 'only if' part. Assume that $\mathcal{N}_{E^+|_{X^+}} \simeq \mathcal{O}(-1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-n)$. Then by (2.4.2), $\mathcal{N}_{\mathcal{E}|_C} \simeq \mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Then locally analytically there is a smooth surface $\check{S}$ such that $\check{S} \subset \check{C}$ and $\check{S}(|\mathcal{E}|_{\check{C}}) = 1$ (note that $\check{S} \in |K_{\check{C}}|_{\check{C}}$). Let $\check{S} \in |K_{\check{C}}|_{\check{C}}$ be the strict transform of $\check{S}$ on $\check{C}$. Consider the exact sequence

$$0 \rightarrow \mathcal{O}_{\check{X}}(2K_{\check{X}}) \rightarrow \mathcal{O}_{\check{X}}(K_{\check{X}}) \rightarrow \mathcal{O}_{\check{C}}(K_{\check{C}}) \rightarrow 0.$$ 

By the Kodaira-Kawamata-Viewheg vanishing theorem, $H^1(\check{X}^+, \mathcal{O}_{\check{X}}(2K_{\check{X}})) = 0$. So there is an element $\check{V}^+ \in |K_{\check{X}}|$ such that $\check{V}^+|_{\check{C}^+} = \check{S}^+$. Let $\check{V} \in |K_{\check{C}}|$ be the strict transform of $\check{V}^+$. Then $\check{V}|_{\check{C}} = \check{S}$. We claim that $\check{V}$ is smooth. This implies $\check{X}$ is also smooth, which completes the proof of the 'only if' part whence (1). First we note that $\check{V}$ is normal since $\check{V}|_{\check{C}}$ is smooth. Let $x$ be any point of $\check{V}$ and $\check{C}_x$ a normal general member of $|-K_{\check{X}}|$. Let $\check{E}_x := \check{E}|_{\check{C}_x}$. $\check{E}_x$ is the exceptional curve of $\check{f}|_{\check{C}_x}$ and $K_{\check{C}_x} \cdot \check{E}_x = -1$. Hence by Proposition 1.6, $\check{C}_x$ has only one singular point on $\check{E}_x$ which is of type $A_{m-1}$ for some $m$ and $(\check{E}_x)^2|_{\check{C}_x} = -1$. Since $(\check{E}_x)^2|_{\check{C}_x} = (\check{E}^2|_{\check{C}_x})|_{\check{V}}$, $m$ is independent of $x$. Hence $(\check{E}_x)^2|_{\check{C}_x} = (\check{E}|_{\check{S}})^2 = -1$ and $m$ is 1, i.e., $\check{C}_x$ is smooth. Consequently $\check{V}$ is found to be smooth at any point $x$ and we are done. Now we finished the proof of Corollary 2.4. □

3. Some examples

We construct examples of flipping contractions from Gorenstein terminal 4-folds.

Example 3.1 (Toric example). Let $e_i$ be the vector $(0, \ldots, 1, \ldots, 0)$ in $\mathbb{R}^4$ for $i = 1, 2, 3, e_4 = (-1, -1, n - 1, n)$ and $e_5 = (0, 0, -1, -1)$. Let $C_i$ be the cone $< e_1, e_2, \ldots, e_i, \ldots, e_5 >$ for $i \geq 0$ and $C_0$ the cone $< e_1, e_2, \ldots, e_5 >$. We denote the toric variety associated to the fan $\ast$ by $V(\ast)$. Set $X := V(C_3 \cup C_4 \cup C_5), X^+ := V(C_1 \cup C_2)$ and $Y := V(C_0)$. Let $f : X \to Y$ and $f^+ : X^+ \to Y$ be the natural morphisms. Then it is easy to check that they define a flipping diagram. (See [Re].)
Example 3.2 (Y. Kachi, M. Gross). For the above example we can easily find $A$ (as in the main theorem) with only isolated canonical singularity as determined in Corollary 2.4. We can consider that $X$ is locally a 1-parameter family of $A$ over the unit disk $\Delta(t)$. Take the cyclic coverings $\hat{X} \to X$, $\hat{Y} \to Y$ and $X^+ \to X^+$ associated to the cyclic covering $\Delta(s) \to \Delta(t)$ defined by $t = s^m$. Then the natural morphisms $\hat{X} \to \hat{Y}$ and $X^+ \to \hat{Y}$ give a flipping diagram.

Example 3.3. For the example 3.1 with $n = 1$, we can find $A$ whose singularity is the curve singularity of generically $cA_1$ type along a line of $\mathbb{P}^2$. For this $A$, we make the similar construction to Example 3.2. We obtain a flipping contraction from a Gorenstein terminal 4-fold which has a 1-dimensional singular locus. Furthermore if we take $q : A_q \to A$ as in the proof of the main theorem for this $A$, the first extremal contraction of $A_q$ over $B$ is a flipping contraction and after the flip, we can contract the strict transform of $E$ to a Gorenstein terminal point.

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