QUANTUM UNIQUE ERGODICITY FOR MAPS ON THE TORUS

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Abstract. When a map is classically uniquely ergodic, it is expected that its quantization will possess quantum unique ergodicity. In this paper we give examples of Quantum Unique Ergodicity for the perturbed Kronecker map, and an upper bound for the rate of convergence.

1. Introduction

1.1. Background. One of the problems in Quantum Chaos is the asymptotic behavior of the expectation value in eigenstates. When quantizing classical dynamics on a phase space one constructs a Hilbert space of states \( \mathcal{H}_h \), and an algebra of operators, the algebra of ”quantum observables”, that assigns for each smooth function on the phase space \( f \) an operator \( \text{Op}_h(f) \) where \( h \) implies dependence on Planck’s constant \( h \), and the dynamics is quantized to a unitary time evolution operator, \( U_h \) on \( \mathcal{H}_h \). For any orthonormal basis of eigenfunctions of \( U_h \), \( \{ \psi_j \} \), the expectation value of \( \text{Op}_h(f) \) in the eigenstate \( \psi_j \) is given by \( \langle \text{Op}_h(f) \psi_j, \psi_j \rangle \). The semiclassical limit of these is the limit where \( h \to 0 \). When the classical dynamics of a system is ergodic, it is known that the time average of the trajectories of the system converges to the space average. An analogue of this is given by Schnirelman’s Theorem [12], [13], [1], which states that for an ergodic system the expectation values of \( \text{Op}(f) \) converges to the phase space average of \( f \), for all but possibly a zero density subsequence of eigenfunctions. This is referred to as quantum ergodicity. The case where there are no exceptional subsequences is referred to as quantum unique ergodicity (QUE).

When the phase space is \( T^2 = \mathbb{R}^2 / \mathbb{Z}^2 \) it is required that each state will be periodic in both position and momentum and thus Planck’s constant is restricted to be an inverse of an integer \( h = \frac{1}{N} \), and the Hilbert space is of dimension \( N \), namely \( L^2(\mathbb{Z}/N\mathbb{Z}) \). The semiclassical limit in this case is the limit where \( N \to \infty \). Given a continuous map \( A \) on \( T^2 \), we define its quantization as a sequence of unitary operators...
on $L^2(\mathbb{Z}/N\mathbb{Z})$, $U_N(A)$ satisfying
\begin{equation}
\|U_N(A)^{-1} \text{Op}_N(f)U_N(A) - \text{Op}_N(f \circ A)\| \longrightarrow 0 \quad \text{as } N \to \infty
\end{equation}
for all $f \in C^\infty(T^2)$, where $f \circ A(p, q) = f(A(p, q))$. This is an analogue of Egorov’s Theorem, and the eigenfunctions of $U_N(A)$ are analogues of eigenmodes.

A first example of QUE was given on the 2-torus $T^2$, by Marklof and Rudnick [9], where the classical dynamics is an irrational skew translation, that is classically uniquely ergodic. For this map they found that for generic translations, the rate of convergence is $O(N^{1/4 + \varepsilon})$. A famous example of a quantization of a map is of linear automorphism of $T^2$ called the ”CAT map”,([6],[3]), that is if $A \in \text{SL}(2, \mathbb{Z})$. If $|\text{tr} A| > 2$ that is if $A$ is hyperbolic, then the map is known to be ergodic, but not uniquely ergodic. In this case it was shown that there is no QUE ([4]), but there exists a special basis (Hecke Basis) for which QUE holds ([7]). In this case the rate of convergence was shown to be $O(N^{1/4 + \varepsilon})$, and is conjectured to be $O(N^{1/2 + \varepsilon})$. (It was shown that in the case where $N = p$ where $p$ is a prime number the rate of convergence is $O(p^{1/2})$ [5]).

In this paper we will give a family of more examples of QUE on the 2-torus, all of them are also classically uniquely ergodic, and study the rate of convergence.

1.2. QUE for maps on the torus. The map in this paper will be the perturbed Kronecker map, that is

$$
\Phi_V^\alpha : \ T^2 \to \ T^2
$$

$$
\Phi_V^\alpha : \ \left( \begin{array}{c} p \\ q \end{array} \right) \mapsto \left( \begin{array}{c} p + \alpha_1 \\ q + \alpha_2 + V(p) \end{array} \right) \quad \text{mod 1}
$$

where $\alpha = (\alpha_1, \alpha_2)$, and $V(p)$ is a smooth function of zero mean on $T$. The special case where $V(p) = 0$ (the standard Kronecker map) plays a central role here. It is known that in this case the map is uniquely ergodic if and only if $1, \alpha_1, \alpha_2$ are linearly independent over $\mathbb{Q}$. We will construct a quantization of it by approximating $\alpha$ with rational numbers $\frac{a}{N} = \frac{(\alpha_1, \alpha_2)}{N}$. For rational numbers we have an exact Egorov theorem, that is

$$
U_{a,N}^{-1} \text{Op}_N(f)U_{a,N} = \text{Op}_N(f \circ \tau_a/N)
$$

and thus by the convergence of $\frac{a}{N}$ to $\alpha$ we will get (1). For this map we have the following theorem for polynomials:
Theorem 1.1. Suppose $1, \alpha_1, \alpha_2$ are linearly independent over $\mathbb{Q}$. Let $f \in C^\infty(T^2)$ be a trigonometric polynomial. Then for all eigenfunctions $\psi$ of $U_N(\tau_\alpha)$ we have that for $N$ sufficiently large

$$\langle \text{Op}_N(f)\psi, \psi \rangle = \int_{T^2} f(p,q)dpdq$$

For the more general case of smooth functions we assume a certain restriction on $\alpha$. We assume that $\alpha$ satisfy a certain diophantine inequality, that is there exists $\gamma > 0$ such that for all $n_1, n_2, k \in \mathbb{Z}$

$$(2) \quad |n_1\alpha_1 + n_2\alpha_2 + k| \gg \|(n_1, n_2)\|^{-\gamma} \quad (n_1, n_2) \neq (0, 0)$$

This reduces the set of numbers rather than being all $\alpha$ such that $1, \alpha_1, \alpha_2$ are linearly independent over $\mathbb{Q}$ to a set of almost all $\alpha$ in Lebesgue measure sense, and $\gamma$ can be any number strictly bigger than 2 (see theorem (3.8)[10]. If $\alpha_1, \alpha_2$ are algebraic of degree $d_1, d_2$ respectively we can choose $\gamma$ to be $d_1!d_2!$ ([11]). For these $\alpha$ we have,

Theorem 1.2. Suppose $1, \alpha_1, \alpha_2$ are linearly independent over $\mathbb{Q}$ and satisfy (2) then for all $f \in C^\infty(T^2)$, for all eigenfunctions $\psi$ of $U_N(\tau_\alpha)$

$$|\langle \text{Op}_N(f)\psi, \psi \rangle - \int_{T^2} f(p,q)dpdq| \ll N^{-\theta} \quad \forall \theta > 0$$

Our main result is for the perturbed Kronecker map $\Phi_{V}^{\alpha}$, for arbitrary smooth $V(p)$. We show that the map is also uniquely ergodic. In fact we show that it is conjugate to $\tau_\alpha$ and we also have QUE for it, and give an upper bound for the rate of convergence:

Theorem 1.3. Suppose $1, \alpha_1, \alpha_2$ are linearly independent over $\mathbb{Q}$ and satisfy (2) then for all $f \in C^\infty(T^2)$, for all eigenfunctions $\psi$ of $U_N(\Phi_{V}^{\alpha})$

$$|\langle \text{Op}_N(f)\psi, \psi \rangle - \int_{T^2} f(p,q)dpdq| \ll N^{-2}$$

Thus for such $\alpha$ the rate of convergence of the matrix elements to their classical average is much faster that the expected and known rates mentioned earlier on the irrational skew translation and the CAT map. We also construct special pairs $(\alpha_1, \alpha_2)$ and functions $f(p,q)$ for which the rate of convergence is arbitrarily slow (Theorems 3.3, 3.10).

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2. Background

We begin with a quantization procedure for maps on the 2-torus $\mathbb{T}^2$. The procedure can be found in full description in [7], [2]. We construct a Hilbert space of state $\mathcal{H}_h$ with respect to Planck’s constant $h$, quantum observables, and a quantization of our maps.

2.1. Notations. We abbreviate $e(x) = e^{2\pi ix}$, and $e_N(x) = e(\frac{x}{N})$. $A \ll B$ or $A = O(B)$ both means that there is a constant $c$ such that $|A| \leq c|B|$.

2.2. Hilbert space of state. Our classical phase space is $\mathbb{T}^2$. The elements of the Hilbert space are thus, distributions on the line $\mathbb{R}$ that are periodic in both position and momentum. Using the momentum representation of a wave-function $\psi$ by the Fourier transform

$$\mathcal{F}_h \psi(p) = \frac{1}{\sqrt{h}} \int_{-\infty}^{\infty} \psi(q) e\left(\frac{-qp}{h}\right) dq$$

we find that the requirements

$$\psi(q + 1) = \psi(q), \quad \mathcal{F}_h \psi(p) = \mathcal{F}_h \psi(p + 1)$$

restricts Planck’s constant $h$ to be an inverse of integer $h = \frac{1}{N}$, and $\mathcal{H}_h$ consists of periodic point-mass distributions at the coordinates $Q = \frac{q}{N}$. We therefore find that the Hilbert space is of dimension $N$, and therefore denote $\mathcal{H}_N$, and we may identify it with $L^2(\mathbb{Z}/N\mathbb{Z})$, with the inner product

$$\langle \psi, \phi \rangle = \frac{1}{N} \sum_{Q \mod N} \psi(Q) \bar{\phi}(Q)$$

The Fourier transform is given by

$$\hat{\psi}(P) = \mathcal{F}_N \psi] (P) = \frac{1}{\sqrt{N}} \sum_{Q \mod N} \psi(Q) e_N(-QP)$$

and its inverse formula is

$$\psi(Q) = \mathcal{F}_N^{-1} \hat{\psi} (Q) = \frac{1}{\sqrt{N}} \sum_{P \mod N} \hat{\psi}(P) e_N(PQ)$$
2.3. Quantum observables. We now assign each classical observable, smooth functions \( f \in C^\infty(\mathbb{T}^2) \), a quantum observable, that is an operator \( \text{Op}_N(f) \) on \( \mathcal{H}_N \) that satisfy,

1. \( \text{Op}_N(\bar{f}) = \text{Op}_N(f)^* \)
2. \( \text{Op}_N(f) \text{Op}_N(g) \sim \text{Op}_N(fg) \) as \( N \to \infty \)
3. \( \frac{1}{2\pi i N} [\text{Op}_N(f), \text{Op}_N(g)] \sim \text{Op}_N\{f, g\} \) as \( N \to \infty \)

where \([A, B] = AB - BA\) is the commutator, and \( \{f, g\} = \frac{\partial f}{\partial p} \frac{\partial g}{\partial q} - \frac{\partial g}{\partial p} \frac{\partial f}{\partial q}\) are the Poisson bracket. The norm used is the induced norm from the inner product on \( \mathcal{H}_N \).

The translation operators

\[
[t_1 \psi](Q) = \psi(Q + 1)
\]

and

\[
[t_2 \psi](Q) = e_N(Q)\psi(Q)
\]

play a special role, they are analogues of the of the differentiation and multiplication operators. Heisenberg’s commutation relations are

\[
t^{a b}_{1 2} = t^{b a}_{2 1} e_N(ab) \quad \forall a, b \in \mathbb{Z}
\]

Notice that

\[
\mathcal{F}_N t_1 \mathcal{F}_N = t_2
\]

and

\[
\mathcal{F}_N t_2 \mathcal{F}_N = t_1^{-1}
\]

With these operators we construct

\[
T_N(n) = e_N\left(\frac{n_1 n_2}{2}\right) t^{n_2}_{2} t^{n_1}_{1}, \quad n = (n_1, n_2) \in \mathbb{Z}^2
\]

whose action on a wave-function \( \psi \in \mathcal{H}_N \) is

\[
T_N(n)\psi(Q) = e^{i\pi n_1 n_2/N} e_N(n_2 Q)\psi(Q + n_1)
\]

Notice that

\[
T_N(n)^* = T_N(-n)
\]

(3) \( T_N(m)T_N(n) = e_N\left(\frac{\omega(m, n)}{2}\right)T_N(m + n) \)

where, \( \omega(m, n) = m_1n_2 - m_2n_1 \), and that \( T_N \) is a unitary operator.

Finally for a general smooth function

\[
f(x) = \sum_{n \in \mathbb{Z}^2} \hat{f}(n) e(n \cdot x)
\]
where \( x = (p, q) \), we define its quantization \( \text{Op}_N(f) \)

\[
\text{Op}_N(f) = \sum_{n \in \mathbb{Z}^2} \hat{f}(n)T_N(n)
\]

and the conditions mentioned are all satisfied.

3. Quantization of Maps and Rate of Convergence

When quantizing a map, we look for a sequence of unitary operators \( U_N(A) \) on \( \mathcal{H}_N \), the quantum propagator, whose iterates give the evolution of the quantum system, and that in the semiclassical limit, (the limit as \( N \to \infty \) or \( \hbar \to \infty \)), the quantum evolution follows the classical evolution as described in the following definition.

**Definition 3.1 ("Egorov's Theorem").** A quantization of a continuous map \( A : \mathbb{T}^2 \to \mathbb{T}^2 \) is a sequence of unitary operators \( \{U_N\} \), satisfying:

\[
\|U_N^{-1} \text{Op}_N(f)U_N - \text{Op}_N(f \circ A)\| \to 0 \quad \text{as} \quad N \to \infty
\]

The stationary states of the quantum system are given by the eigenfunctions \( \psi \) of \( U_N(A) \). We will find that for the maps studied in this paper the limiting expectation value of observables in normalized eigenstates converges to the classical average of the observable, that is

\[
\langle \text{Op}_N(f)\psi,\psi \rangle \to \int_{\mathbb{T}^2} f \quad \text{as} \quad N \to \infty
\]

3.1. Quantizing Kronecker map. In this section we will construct a quantization of the Kronecker map.

\[
\tau_\alpha : \mathbb{T}^2 \to \mathbb{T}^2
\]

\[
(p, q) \mapsto \left( p + \frac{a_1}{q} + V(p) \right) \mod 1
\]

**Lemma 3.1.** Suppose \( \left(\frac{a_1, a_2}{N}\right) = \frac{\tilde{a}}{N} \to \infty \) then the sequence \( U_N(\tau_\alpha) := T_N(-a_2, a_1) \) is a quantization of Kronecker's map.

**Proof.** First assume \( f(x) = e_n(z) := e(n \cdot z) \) in this case we get \( \hat{f}(n) = 1, \hat{f}(m) = 0 \) for \( m \neq n \), and therefore \( \text{Op}_N(f) = T_N(n) \).

Denote \( \tilde{a} := (-a_2, a_1) \), and notice that \( n \cdot a = \omega(n, \tilde{a}) \). Now

\[
U_N(\tau_\alpha)^{-1}T_N(n)U_N(\tau_\alpha) = T_N(-\tilde{a})T_N(n)T_N(\tilde{a})
\]

which due to (3) and the linearity and antisymmetry of \( \omega(m, n) \)

\[
eq N(\omega(n, \tilde{a}))(T_N(n) = e_N(n \cdot a))T_N(n)
\]
on the other hand, we have
\[(e_n \circ \tau_\alpha)(x) = e(n_1(p + \alpha_1) + n_2(q + \alpha_2)) = e(n \cdot \vec{\alpha})e_n(x)\]
and so
\[(7) \quad \text{Op}_N(e_n \circ \tau_\alpha) = e(n \cdot \vec{\alpha})T_N(n)\]
From (6), (7) we get that
\[\|U_{N}^{-1}(\tau_\alpha)T_N(n)U_N(\tau_\alpha) - e(n \cdot \alpha)T_N(n)\| = |e_N(n \cdot \vec{\alpha}) - e_N(n \cdot \vec{\alpha})| \cdot \|T_N(n)\|\]
\(T_N\) is a unitary operator so \(\|T_N(n)\| = 1\) we get
\[|e_N(n \cdot \vec{\alpha}) - e_N(n \cdot \vec{\alpha})| \ll \|n\| |\vec{\alpha} - \vec{\alpha}_N|\]
Therefore we established (5) for \(f = e_n(x)\). By linearity we also have (5) for trigonometric polynomials. suppose now that \(f(x)\) is a general function of \(C^\infty(\mathbb{T}^2)\) and therefore
\[f(x) = \sum_{n \in \mathbb{Z}^2} \hat{f}(n)e_n(x)\]
Consider
\[\|U^{-1}_{N}(\tau_\alpha) \text{Op}_N(f)U_N(\tau_\alpha) - \text{Op}_N(f \circ A)\| = \|U^{-1}_{N}(\tau_\alpha) \{ \sum_{n \in \mathbb{Z}^2} \hat{f}(n)T_N(n) \}U_N(\tau_\alpha) - \sum_{n \in \mathbb{Z}^2} \hat{f}(n)e(n \cdot \alpha)T_N(n)\| = \| \sum_{n \in \mathbb{Z}^2} \hat{f}(n) \{ e_N(n \cdot \alpha) - e(n \cdot \alpha) \} T_N(n) \| \leq \sum_{n \in \mathbb{Z}^2} |\hat{f}(n)| \cdot |e(n \cdot \alpha) - e(n \cdot \alpha)| \cdot \|T_N(n)\|\]
and therefore
\[\|U_{N}^{-1}(\tau_\alpha) \text{Op}_N(f)U_N(\tau_\alpha) - \text{Op}_N(f \circ A)\| = |\vec{\alpha} - \vec{\alpha}_N| \sum_{n \in \mathbb{Z}^2} \|n\| \hat{f}(n) = O(|\vec{\alpha} - \vec{\alpha}_N|)\]
which goes to zero since \(|\vec{\alpha} - \vec{\alpha}_N| \to 0\) as \(N \to \infty\) implying that \(U_N\) is a quantization of \(\tau_\alpha\).

**Remark 3.1.** Notice that for each \(N\), we have exact Egorov for \(\tau_{a/N}\), that is
\[U^{-1}_{N}(\tau_{a/N}) \text{Op}_N(f)U_N(\tau_{a/N}) = \text{Op}_N(f \circ \tau_{a/N})\]
3.2. **Convergence of eigenstates.** We now wish to give an upper bound for the remainder

\[(8) \quad |\langle \text{Op}_N(f)\psi,\psi \rangle - \int_{T^2} f| \]

where \(\psi\) is an eigenfunction of \(U_N\). Actually we will prove the following two theorems:

**Theorem 3.2.** Suppose \(1, \alpha_1, \alpha_2\) are linearly independent over \(\mathbb{Q}\). Then for any eigenfunction \(\psi(Q)\) of \(U_N\)

1. If \(f\) is a polynomial then for \(N\) large enough,
   \[\langle \text{Op}_N(f)\psi,\psi \rangle = \int_{T^2} f\]
2. If \(\alpha = (\alpha_1, \alpha_2)\) is diophantine (see definition 3.2) and \(|\bar{\alpha} - \bar{\alpha}| \ll \frac{1}{N}\) then for all \(f \in C^\infty(T^2)\)
   \[\langle \text{Op}_N(f)\psi,\psi \rangle - \int_{T^2} f = O\left(\frac{1}{N^\theta}\right), \forall \theta > 0\]

**Theorem 3.3.** For any positive increasing function \(g(x)\), there exists \(\alpha = (\alpha_1, \alpha_2)\) such that \(1, \alpha_1, \alpha_2\) are linearly independent over the rationals, \(f \in C^\infty(T^2)\), and a basis of eigenfunctions \(\{\psi_j\}_{j=1}^N\) such that

\[|\langle \text{Op}_N(f)\psi_j,\psi_j \rangle - \int_{T^2} f| \gg \frac{1}{g(N)}\]

**Remark 3.2.** The set of all diophantine pairs is of Lebesgue measure 1 (see theorem 3.8). An example for such pairs are \(\alpha = (\alpha_1, \alpha_2)\) such that \(\alpha_1, \alpha_2\) are algebraic and \(1, \alpha_1, \alpha_2\) are linearly independent over \(\mathbb{Q}\) (see theorem 3.7).

To prove these theorems we will start with the following lemma:

**Lemma 3.4.** Let \(\psi(Q)\) to be an eigenfunctions of \(U_N\).

1. \[\langle \text{Op}_N(f)\psi,\psi \rangle = \langle \text{Op}_N(f^T)\psi,\psi \rangle\]
   where \(f^T(p,q) = \frac{1}{T} \sum_{t=0}^{T-1} f \circ \tau_{t(a/N)}^{t}\)
2. For \(f(x) = e_n(x)\), \(\langle \text{Op}_N(f)\psi,\psi \rangle\) is identically zero for large enough \(N\).
Proof. (1) Since $\psi$ is an eigenfunction of $U_N$ then $U_N \psi = e(\phi) \psi$, and therefore for all $t$

$$\langle \text{Op}_N(f) U_N^t \psi, U_N^t \psi \rangle = (e(t \phi) \text{Op}_N(f) \psi, e(t \phi) \psi) = \langle \text{Op}_N(f) \psi, \psi \rangle$$

Now,

$$\langle \text{Op}_N(f) U_N^t \psi, U_N^t \psi \rangle = \langle U_N^{-t} \text{Op}_N(f) U_N^t \psi, \psi \rangle$$

and since

$$U_N^{-t} \text{Op}_N(f) U_N^t = \text{Op}_N(f \circ \tau_{a/N}^t)$$

we have (9).

(2) fix $\vec{n} = (n_1, n_2) \in \mathbb{Z}^2$, $f(x) = e_n(x)$ and therefore

$$\text{Op}_N(f) = T_N(n)$$

Notice that for $f = e_n$ we have,

$$f^T = \frac{1}{T} \sum_{t=0}^{T-1} e_n \circ \tau_{a/N}^t = \frac{1}{T} \sum_{t=0}^{T-1} e(n_1(p + ta_1/N) + n_2(q + ta_2/N)) = \frac{1}{T} e_n(p, q) \sum_{t=0}^{T-1} e_N((n_1 a_1 + n_2 a_2) t)$$

and for $T = N$ we have,

$$f^N = \begin{cases} f & \text{if } n_2 a_2 + n_1 a_1 = 0 \pmod{N} \\ 0 & \text{else} \end{cases}$$

and therefore,

$$\text{Op}_N(f^N) = \begin{cases} \text{Op}_N(f) & \text{if } n_2 a_2 + n_1 a_1 = 0 \pmod{N} \\ 0 & \text{else} \end{cases}$$

but

$$n_2 a_2 + n_1 a_1 = Nk \iff n_2 \frac{a_2}{N} + n_1 \frac{a_1}{N} = k \in \mathbb{Z}$$

$$\iff n_2 \{\alpha_2 + O(|a_2|)\} + n_1 \{\alpha_1 + O(|a_1|)\} = k \in \mathbb{Z}$$

and so we get

$$n_2 \alpha_2 + n_1 \alpha_1 + O(||n|| |a_1 - a/\lambda|) = k \in \mathbb{Z}$$

$\alpha_1, \alpha_2$ are linearly independent over $\mathbb{Q}$ so we can denote $0 < \delta = \text{dist}(n_1 \alpha_1 + n_2 \alpha_2, \mathbb{Z})$. Now assume that there exists infinitely many pairs $\vec{a} = (a_1, a_2)$ such that (8) is nonzero i.e. $n_2 a_2 + n_1 a_1 = N k_{\vec{a}}$. From (12) we get that

$$O(||n|| |\vec{a} - \vec{a}/N|) = |k + n_2 \alpha_2 + n_1 \alpha_1| \geq \delta > 0, N \rightarrow \infty$$
now since $n$ is fixed and $|\bar{\alpha} - \frac{n}{N}| \to 0$ as $N \to \infty$ we get a contradiction! so we can deduce that for $N \gg \|n\|$

\[ |\langle \text{Op}_N(f)\psi, \psi \rangle - \int_{T^2} f|^2 = |\langle T_N(n)\psi, \psi \rangle| = 0 \]

\[\square\]

**Corollary 3.5** (QUE for Kronecker map). For any eigenfunction $\psi$ of $U_N$,

1. If $f$ is a trigonometric polynomial, $\langle \text{Op}_N(f)\psi, \psi \rangle = \int_{T^2}$ for large enough $N$.
2. For any $f \in C^\infty(T^2)$,

\[ |\langle \text{Op}_N(f)\psi, \psi \rangle - \int_{T^2} f| \to 0 \quad \text{as } N \to \infty \]

**Proof.** (1) From the previous lemma we get that every trigonometric function has $N$ such that \(N\) is identically zero so for a finite linear combination

\[ \sum_{n=1}^m a_n e(n \cdot x) \]

simply choose the largest $N$ given from $e_n(x), n = 1, \ldots, m$

(2) For a general $f \in C^\infty(T^2)$, we have

\[ \text{Op}_N(f) = \sum_{n \in \mathbb{Z}^2} \hat{f}(n) T_N(n) \]

For $\epsilon > 0$, there exists $R_0$, such that $\forall R > R_0$,

\[ \sum_{\|n\| > R} |\hat{f}(n)| < \epsilon \]

For the polynomial

\[ P_R = \sum_{\|n\| < R} \hat{f}(n)e(n \cdot x) \]

there exists $N_0$, such that for all $N > N_0$

\[ \langle \text{Op}_N(P_R)\psi, \psi \rangle = 0 \]

and so we have ,

\[ |\langle \text{Op}_N(f)\psi, \psi \rangle| \leq \]

\[ |\langle \text{Op}_N(P_R)\psi, \psi \rangle| + \sum_{\|n\| > R} \hat{f}(n)\langle T_N(n)\psi, \psi \rangle \leq \epsilon \]

for $N > N_0$. 
3.2.1. Convergence of eigenstates for diophantine pairs: To finish the study of the upper bound for a general function we need to study the size of \( n_1 \alpha_1 + n_2 \alpha_2 + k \) for \( n_1, n_2, k \in \mathbb{Z} \) and assume that \( \alpha \) satisfies a certain diophantine inequality that is \( |n_1 \alpha_1 + n_2 \alpha_2 + k| \gg \frac{c(\alpha)}{\|n\|^\gamma} \) for some \( \gamma \). Numbers like this are called diophantine.

**Definition 3.2.** An \( l \)-tuple of real numbers \( (\alpha_1, \ldots, \alpha_l) \) is called diophantine if they satisfy that there exists \( \gamma \) such that for any integers \( n_1, \ldots, n_l \not= \vec{0}, k \)

\[
|n_1 \alpha_1 + \cdots + n_l \alpha_l + k| \gg \frac{c(\alpha)}{\|n\|^\gamma}
\]

with this we have the following.

**Corollary 3.6.** Suppose \( \alpha \) is diophantine and that \( |\vec{\alpha} - \vec{\alpha}_N| \ll \frac{1}{N} \) then we have an upper bound for \( |\langle \text{Op}_N(f) \psi, \psi \rangle - \int_{\mathbb{T}^2} f| \ll \frac{1}{N^\theta} \) for any \( \theta > 0 \).

**Proof.** A general function is of the following form

\[
f(x) = \sum_{n \in \mathbb{Z}^2} \hat{f}(n)e_n(x)
\]

without loss of generality we can assume that \( \int_{\mathbb{T}^2} f = 0 \) and so divide \( \text{Op}_N(f) \) into two sums: \( \text{Op}_N(f) = \sum_{n \in \mathbb{Z}^2} \hat{f}(n)T_N(n) = I_1 + I_2 \) where

\[
I_1 = \sum_{\|n\| \leq R} \hat{f}(n)T_N(n), I_2 = \sum_{\|n\| > R} \hat{f}(n)T_N(n).
\]

Now as seen earlier, the case when \( |\langle T_N(\vec{n}) \psi, \psi \rangle| \neq 0 \) can only happen when

\[
O\left(\frac{\|n\|}{N}\right) = k + n_2 \alpha_2 + n_1 \alpha_1
\]

but our assumption is that there exists \( \gamma \) such that for all integer coefficients \( k + n_2 \alpha_2 + n_1 \alpha_1 \gg \frac{1}{\|n\|^\gamma} \gg \frac{1}{R^\gamma} \) and so define \( N = R^{1+\gamma+\delta} \) for some \( \delta > 0 \) and we get that

\[
\frac{R}{N} \gg \frac{\|n\|}{N} \gg k + n_2 \alpha_2 + n_1 \alpha_1 \gg \frac{1}{\|n\|^\gamma} \gg \frac{1}{R^\gamma}
\]

and for \( N = R^{1+\gamma+\delta} \) this gives a contradiction and so \( I_1 = 0 \) for large enough \( N \). For \( I_2 \) we use the rapid decay of the Fourier coefficients:

\[
|I_2| = \sum_{\|n\| > R} |\hat{f}(n)T_N(n)| \leq \sum_{\|n\| > R} \|\hat{f}(n)T_N(n)\| = \sum_{\|n\| > R} |\hat{f}(n)| \leq \frac{1}{R^b} = \frac{1}{N^\theta}
\]

for any chosen \( \theta \). \( \Box \)

For algebraic numbers we have this inequality by the following well known theorem, (\[\text{[1]}\]):
Theorem 3.7. Suppose $\vec{\alpha} = (\alpha_1, \ldots, \alpha_m)$ are linearly independent over $\mathbb{Q}$ then there exists $D = D(\alpha)$ such that

$$|n_1\alpha_1 + n_m\alpha_m + k| \gg \frac{c(\vec{\alpha})}{\|n\|^{D-1}}$$

For the more general $\vec{\alpha}$ we need the following theorem by Khintchine [10]:

Theorem 3.8. Almost no pair $(\alpha_1, \alpha_2)$ is very well approximable that is, for almost any pair there exists $\delta = \delta(\alpha_1, \alpha_2)$ such that there are only finite many integers $m = (m_1, m_2), k$ such that the following inequality holds:

$$|m_1\alpha_1 + m_2\alpha_2 + k| \geq \frac{1}{\|m\|^2 + \delta}$$

3.2.2. Proof of theorem 3.3. We begin the proof using a construction of an irrational number $\alpha$, and a sequence converging to it.

Lemma 3.9. Given any positive increasing function $F(x)$ there is an irrational $\beta$ with continued fraction expansion $[b_1, b_2, \ldots, b_n, \ldots]$, such that the partial quotients $c_n/d_n = [b_1, \ldots, b_n]$ satisfy:

1. $F(d_n) \leq b_{n+1}d_n^2$
2. $|\beta - \frac{a_n}{d_n}| < \frac{1}{F(d_n)}$

The proof of the lemma is given in [9]. Set $G(x) = \log g(x)$, and apply lemma 3.9 for $F = G^{-1}$. Following the lemma’s notation define $f(p, q) = \sum_{n=1}^{\infty} e^{-d_n} e(d_nq)$,

$$\alpha = (\sqrt{2}, \beta), b = b_{n+1}c_n d_n, N = b_{n+1}d_n^2$$

Theorem 3.10. For $\alpha, f(p, q), b, N$ defined above the following holds:

1. $U_N = T_N(-b, a)$ is a quantization of $\tau_\alpha$, where $\frac{a}{N}$ is a sequence converging to $\sqrt{2}$.
2. There exists a basis of eigenfunctions $\{\psi_j\}_{j=1}^N$ of $U_N$ such that

$$|\langle T_N(0, d_n)\psi_j, \psi_j \rangle| = 1$$

3. For the basis $\{\psi_j\}_{j=1}^N$

$$|\langle \text{Op}_N(f)\psi_j, \psi_j \rangle| \gg \frac{1}{g(N)}$$

Proof. (1) According to the construction from lemma 3.9 we get that $|\beta - \frac{\alpha}{N}| \to 0$ as $N \to \infty$, and therefore $\frac{(a, b)}{N}$ converges to $\alpha$, and thus by lemma 3.1 we have that $U_N$ is a quantization of $\tau_\alpha$.

(2) Since $\omega((0, d_n), (-b, a)) = d_n b = c_n N \equiv 0 \pmod N$ we have that $T_N(0, d_n), T_N(-b, a)$ commute (according to [3]), and therefore they have an orthonormal basis of joint eigenfunctions
\{\psi_j\}_{j=1}^N\), and since \(T_N(n)\) is a unitary operator we have
\[|\langle T_N(0, d_n)\psi_j, \psi_j \rangle| = |e(\phi)\langle \psi_j, \psi_j \rangle| = 1\]
as required.

(3) We first observe that
\[
\hat{f}(n_1, n_2) = \begin{cases} 
  e^{-d_n} & (n_1, n_2) = (0, d_n) \\
  0 & \text{otherwise}
\end{cases}
\]
By definition of \(\text{Op}_N(f)\) we have that
\[
\langle \text{Op}_N(f)\psi, \psi \rangle = \sum_{n \in \mathbb{Z}^2} \hat{f}(n)\langle T_N(n)\psi, \psi \rangle
\]
From equation (11) we saw that
\[n_1a + n_2b \not\equiv 0 \pmod{N}\] ⇒ \(\langle T_N(n)\psi, \psi \rangle = 0\) and therefore the RHS in (16) is in fact
\[
\sum_{n_1a + n_2b \equiv 0 \pmod{N}} \hat{f}(n)\langle T_N(n)\psi, \psi \rangle
\]
Form (15) we see that if \(n_1 \not\equiv 0\) then \(\hat{f}(n_1, \cdot) = 0\) and thus the condition \(n_1a + n_2b \equiv 0 \pmod{N}\) is in fact \(n_2b \equiv 0 \pmod{\frac{N}{(b, N)}}\) \(\iff n_2 \equiv 0 \pmod{d_n}\) (the last equality is by definition of \(b, N\)), and therefore
\[
\langle \text{Op}_N(f)\psi_j, \psi_j \rangle = \sum_{n_2 \equiv 0 \pmod{d_n}} \hat{f}(0, n_2)\langle T_N(0, n_2)\psi_j, \psi_j \rangle \geq 0
\]
\[
|\langle T_N(0, d_n)\psi_j, \psi_j \rangle| - \sum_{k=2}^{\infty} |\hat{f}(0, kd_n)\langle T_N(0, kd_n)\psi_j, \psi_j \rangle| \geq e^{-d_n}\langle T_N(0, d_n)\psi_j, \psi_j \rangle - \frac{e^{-2d_n}}{1 - e^{-d_n}} =
\]
e^{-d_n}\langle T_N(0, d_n)\psi_j, \psi_j \rangle - \frac{e^{-2d_n}}{1 - e^{-d_n}} \geq e^{-d_n}|1 - \frac{e^{-2d_n}}{1 - e^{-d_n}}| \gg e^{-d_n}
and since \(F(d_n) \leq N\) we have that \(d_n \leq G(N) = \log g(N)\) and therefore \(e^{-d_n} \geq e^{-\log g(N)} = \frac{1}{g(N)}\) and we get that
\[
\langle \text{Op}_N(f)\psi_j, \psi_j \rangle \gg \frac{1}{g(N)}
\]
\[\square\]
3.3. Perturbed Kronecker map. Another family of uniquely ergodic maps on $\mathbb{T}^2$, is the perturbed Kronecker map. We see in this section that it is uniquely ergodic, due to the fact that it is conjugate to the Kronecker map itself, and in the following section we form a quantization for it. Define the following shear perturbation:

$$\Phi_V : \begin{pmatrix} p \\ q \end{pmatrix} \mapsto \begin{pmatrix} p \\ q + V(p) \end{pmatrix}$$

and the perturbed Kronecker map:

$$\Phi_\alpha^\circ V : \begin{pmatrix} p \\ q \end{pmatrix} \mapsto \begin{pmatrix} p + \alpha_1 \\ q + \alpha_2 + V(p) \end{pmatrix}$$

where $V(p) \in C^\infty(\mathbb{T})$ satisfies $\int_0^1 V(p)dp = 0$. In order to prove the unique ergodicity of this map, we will use the following Lemma that shows that the perturbed map is conjugate to the Kronecker map.

Lemma 3.11. Suppose $\alpha_1$ is irrational.

(1) If $V(p)$ is a polynomial we have that

$$\tau_\alpha \circ \Phi_V = \Phi_h \circ \tau_\alpha \circ \Phi_h^{-1}$$

(2) If $\alpha_1$ is diophantine then (1) holds for any $V \in C^\infty(\mathbb{T})$ for some $h = h_V \in C^\infty(\mathbb{T})$

Proof. (1) The RHS of (1) is

$$\Phi_{h_k} \circ \tau_\alpha \circ \Phi_{h_k}^{-1}(p,q) = \begin{pmatrix} p + \alpha_1 \\ q + \alpha_2 + h_k(p + \alpha_1) - h_k(p) \end{pmatrix}$$

define $h_k(p) = \frac{e(kp)}{\alpha e(k\alpha)}$ (which is well defined for all $k$ only if $\alpha_1$ is irrational). $h_k(p)$ satisfy that $e(kp) = h_k(p + \alpha_1) - h_k(p)$ and therefore we get (1),and by linearity we get that (1) holds for every polynomial.

(2) For $V \in C^\infty(\mathbb{T})$, $\alpha$ diophantine, we observe that $|e(k\alpha_1) - 1| \sim \{k\alpha\} \gg \frac{1}{|k|\gamma}$ and we get that

$$\sum_{k \in \mathbb{Z}} |\hat{V}(k)h_k(p)| \ll \sum_{k \in \mathbb{Z}} |\hat{V}(k)||k|^\gamma$$

converges absolutely and so define $h_V(p) = \sum_{k \in \mathbb{Z}} \hat{V}(k)h_k(p)$. Then $h_V(p)$ satisfies $h_V(p + \alpha_1) - h_V(p) = V(p)$ since $h_k$ satisfy that for every $k$ since the series converges absolutely.

With $\Phi_\alpha^\circ$ described as a conjugate of $\tau_\alpha$ we have the following result:
Theorem 3.12. Suppose $1, \alpha_1, \alpha_2$ are linearly independent over $\mathbb{Q}$. Then for $\alpha$ diophantine and $V(p) \in C^\infty(\mathbb{T})$ then $\Phi_V^\alpha$ is uniquely ergodic.

Proof. We will first show that Lebesgue measure is $\Phi_V^\alpha$ invariant. Suppose $f(p, q) \in L^1(\mathbb{T}^2)$. Then $f \circ \Phi_V^\alpha(p, q) = f(p, q + V(p))$ and so

$$\int_0^1 \int_0^1 f(p + \alpha_1, q + V(p) + \alpha_2) dq dp = \int_0^1 \int_0^1 f(p, q) dq dp$$

by standard change of variables. Now, assume $\mu$ is an invariant measure of $\Phi_V^\alpha$, since $\Phi_V^\alpha \circ \tau_\alpha = \Phi_h \circ \tau_\alpha \circ \Phi_v^{-1}$ for some $h \in C^\infty(\mathbb{T})$, then $\Phi_h \circ \mu$ is invariant measure of $\tau_\alpha$, but there exists only one such measure and which is Lebesgue measure $m$, that is $\Phi_h \circ \mu = m$ is Lebesgue measure. $\Phi_h$ is an invertible map, that preserves Lebesgue measure, so $\mu = \Phi_h^{-1} \circ m = m$ therefore $\Phi_V^\alpha$ is uniquely ergodic. □

3.4. QUE for perturbed Kronecker map. In this section we will study the asymptotic behaviour of the matrix elements related to the perturbed Kronecker map. The main tool will be lemma 3.11 that connects the perturbed map to the unperturbed map.

In order to quantize the perturbed Kronecker map, we use the following theorem of Marklof-O’Keefe [3]:

Theorem 3.13 (Marklof-O’Keefe). For every function $f \in C^\infty(\mathbb{T}^2)$ we have

$$(17) \quad |\langle (U^{-1}_v \circ \text{Op}_N(f) \circ U_v - \text{Op}_N(f \circ \Phi_v)) \psi, \psi \rangle| \ll \frac{c(f)}{N^2}$$

Using the equality in Lemma (3.11) and the quantization of the perturbation map in theorem 3.13, we can describe the quantization of $\Phi_V^\alpha = \tau_\alpha \circ \Phi_v$ as follows:

Theorem 3.14. Denote $U_N = U_h(N)^{-1} U_\tau(N) U_h(N)$ where $U_\tau(N)$ is the quantization of $\tau_\alpha$, then we have

$$(18) \quad \| U_N^{-1} \circ \text{Op}_N(f) \circ U_N - \text{Op}_N(f \circ \Phi_v^\alpha) \| \ll N^{-1}$$

Proof. We already know that

$$\| U_h(N)^{-1} \circ \text{Op}_N(f) \circ U_h - \text{Op}_N(f \circ \Phi_h) \| = O(N^{-2})$$

and that

$$\| U_\tau(N)^{-1} \circ \text{Op}_N(f) \circ U_\tau - \text{Op}_N(f \circ \tau) \| = O(N^{-1})$$

and thus using the equality in Lemma (3.11) we conclude the proof □

Remark 3.3. The set $\{ \psi_j = U_h(N)^{-1} \psi_j^\tau \}$ form a basis of eigenfunctions of $U_N$, where $\{ \psi_j^\tau \}$ is a basis of eigenfunctions for $U_\tau$. 

With this representation of the eigenfunctions we can give an upper bound for the asymptotic behavior of the matrix elements:

**Theorem 3.15.** For every \( f \in C^\infty(T^2) \), \( \alpha \) diophantine we have:

\[
|\langle \text{Op}_N(f) \psi_j, \psi_j \rangle - \int f | \ll N^{-2}
\]

**Proof.** Without loss of generality we will assume that \( \int f = 0 \). By definition we have

\[
\langle \text{Op}_N(f) \psi_j, \psi_j \rangle = \langle \text{Op}_N(f) U_h^{-1} \psi_j^\tau, U_h^{-1} \psi_j^\tau \rangle
\]

and since \( U_h \) is unitary we have

\[
\langle \text{Op}_N(f) \psi_j, \psi_j \rangle = \langle U_h \text{Op}_N(f) U_h^{-1} \psi_j^\tau, \psi_j^\tau \rangle
\]

Now using Theorem 3.13 we get,

\[
(19) \quad |\langle U_h \text{Op}_N(f) U_h^{-1} \psi_j^\tau, \psi_j^\tau \rangle - \langle \text{Op}_N(f \circ \Phi_h) \psi_j^\tau, \psi_j^\tau \rangle| \ll N^{-2}
\]

since \( \psi_j \) is a normalized wavefunction, but since \( f \circ \Phi_h \) is still a \( C^\infty(T^2) \) we have that the second term is \( O(N^{-10}) \) and therefore

\[
\langle \text{Op}_N(f) \psi_j, \psi_j \rangle \ll N^{-2}
\]

\[ \square \]

**Remark 3.4.** The upper bound found here is valid only for the quantization of described here which includes an arbitrary choice of a sequence that converges to \( \alpha \) by rational numbers. Since this quantization is not unique, and since the operators \( \|U_N(\alpha) - U_N(\alpha')\| \sim \frac{1}{N} \) this upper bound only applies with the specific eigenfunctions for a specific chosen convergent sequence for \( \alpha \).

As for the standard Kronecker map, we can also construct special \( \alpha, f \in C^\infty(T^2) \) with arbitrary slow convergence:

**Theorem 3.16.** For any positive increasing \( g(x) \) there exist \( \alpha, \tilde{f}(p, q) \in C^\infty(T^2) \) and a basis of eigenfunctions \( \{\psi_j\}_{j=1}^N \) such that

\[
\langle \text{Op}_N(\tilde{f}) \psi_j, \psi_j \rangle - \int_{T^2} \tilde{f} \gtrless \frac{1}{g(N)}
\]

**Proof.** Take \( \alpha \) to be the pair \( (\sqrt{2}, \beta) \) as in theorem 3.3. Since \( \sqrt{2} \) is diophantine then \( \Phi_{\sqrt{2}} \) is still conjugate to \( \tau_\alpha \), we still have

\[
|\langle U_h \text{Op}_N(\tilde{f}) U_h^{-1} \psi_j^\tau, \psi_j^\tau \rangle - \langle \text{Op}_N(\tilde{f} \circ \Phi_h) \psi_j^\tau, \psi_j^\tau \rangle| \ll N^{-2}
\]
And thus for \( \tilde{f} = f \circ \Phi_h^{-1} \), \( \{\psi_j^\tau\} \) where \( f(p,q), \psi_j^\tau \) are the function and orthonormal basis constructed for the proof of theorem 3.3 we have

\[
|\langle \text{Op}_N(\tilde{f})\psi_j, \psi_j \rangle - \int_{T^2} \tilde{f} | \gg \frac{1}{g(N)}
\]

Remark 3.5. Notice that due to corollary 3.5 and (19), the matrix elements do converge to \( \int_{T^2} f \) and thus we still have QUE, but with rate of convergence arbitrary slow.

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