RESOLVENT CRITERIA FOR SIMILARITY TO A
NORMAL OPERATOR WITH SPECTRUM ON A CURVE

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Abstract. We give some new criteria for a Hilbert space operator with
spectrum on a smooth curve to be similar to a normal operator, in
terms of pointwise and integral estimates of the resolvent. These results
 generalize criteria of Stampfli, Van Casteren and Naboko, and answers
several questions posed by Stampfli in [48]. The main tools are from our
recent results [12] on dilation to the boundary of the spectrum, along
with the Dynkin functional calculus for smooth functions, which is based
on pseudoanalytic continuation.

1. Introduction

As Stampfli proved in 1969 (see [47]), if $\Gamma \subset \mathbb{C}$ is a smooth curve, $T$ is a
bounded operator on a Hilbert space $H$ with spectrum $\sigma(T)$ contained in $\Gamma,$
and there is a neighborhood $U$ of $\Gamma$ such that $\|(T - \lambda)^{-1}\| \leq \text{dist}(\lambda, \Gamma)^{-1}$
for all $\lambda \in U \setminus \Gamma,$ then $T$ is normal. Theorems of this type were first proved by
Nieminen [37] for the case $\Gamma = \mathbb{R}$ and by Donoghue [11] for the case when $\Gamma$
is a circle.

If $\Gamma$ is not smooth, such a result need no longer be true. A counterexample
can be found in [46]. Even if $\Gamma$ is a circle, the condition $\|(T - \lambda)^{-1}\| \leq C \text{dist}(\lambda, \Gamma)^{-1},$ $\lambda \in \mathbb{C} \setminus \Gamma,$ where $C$ is a constant greater than 1, is not
sufficient for $T$ to be similar to a normal operator; that is, for some invertible
$S$ and normal operator $N,$ to have $T = SNS^{-1}.$ See the paper Markus [39].
Benamara and Nikolski [3, Section 3.2] have a general result in this direction,
and in a related article [39], Nikolski and Treil give a counterexample where $T$
is a rank one perturbation of a unitary operator with $\sigma(T) \subset \mathbb{T}.$

Nevertheless, the hypothesis in Stampfli’s theorem can be successfully
weakened. Denote by $\mathcal{B}(H)$ the set of bounded linear operators on a Hilbert
space $H.$ We will prove the following:

Theorem 1. Let $\Gamma \subset \mathbb{C}$ be a $C^{1+\alpha}$ Jordan curve, and $\Omega$ the domain it
bounds. Let $T \in \mathcal{B}(H)$ be an operator with $\sigma(T) \subset \Gamma.$ Assume that

$$\|(T - \lambda)^{-1}\| \leq \frac{1}{\text{dist}(\lambda, \Gamma)}, \quad \lambda \in U \setminus \overline{\Omega},$$

Date: March 12, 2018.

2010 Mathematics Subject Classification. Primary 47A10; Secondary 47B15, 47A60.

The second author was supported by a grant from the Mathematics Department of
the Universidad Autónoma de Madrid and the Project MTM2015-66157-C2-1-P of the
Ministry of Economy and Competitiveness of Spain. This work forms part of his thesis,
defended in 2017. The third author was supported by the Project MTM2015-66157-C2-1-
P and by the ICMAT Severo Ochoa project SEV-2015-0554 of the Ministry of Economy
and Competitiveness of Spain and the European Regional Development Fund (FEDER).
for some open set \( U \) containing \( \partial \Omega \), and
\[
\| (T - \lambda)^{-1} \| \leq \frac{C \text{dist}(\lambda, \Gamma)}{1 - |\lambda|}, \quad \lambda \in \Omega,
\]
for some constant \( C \). Then \( T \) is similar to a normal operator.

In other words, we assume that a resolvent estimate with constant 1 is satisfied outside \( \Omega \) and an estimate with constant \( C \) is satisfied inside \( \Omega \) (then \( C \geq 1 \)). The same conclusion holds if, vice versa, these estimates hold with constant 1 inside \( \Omega \) and with constant \( C \) outside \( \Omega \); see the Remark at the end of Section 4. The result gives a positive answer to Question 2 posed by Stampfli in [48], which he observed as being the case when \( \Gamma \) is a circle.

The proof of Theorem 1 will use a generalization of a theorem of Putinar and Sandberg on complete \( K \)-spectral sets that was proved in [12]. In fact, this theorem is an easy corollary of this generalization and Lemma 6, which is stated below. The connection of spectral sets and similarity problems was already observed by Stampfli in [48]. In Theorem 8 of that paper he proved via different techniques a version of our Lemma 6 under the assumption that \( \Omega \) set is a spectral set for \( T \) rather than a \( K \)-spectral set, along with stronger smoothness conditions for the boundary of \( \Omega \).

Many different kinds of conditions implying normality of an operator have been studied. See, for instance, [4] and the previous articles in this series. In [15], Campbell and Gellar studied operators \( T \) for which \( T^*T \) and \( T + T^* \) commute, showing, for instance, that if \( \sigma(T) \) is a subset of a vertical line or \( \mathbb{R} \), then \( T \) is normal. In [10] Djordjević gave several conditions for an operator to be normal using the Moore-Penrose inverse. Gheondea considered operators which are the product of two normal operators in [22]. See also [32] and references therein.

Here we exhibit conditions for an operator to be similar to a normal operator in terms of estimates of (or more properly, bounds on) its resolvent. Others have done likewise. In [9], Van Casteren proved the following.

**Theorem VC1.** Let \( T \in \mathcal{B}(H) \) be an operator with \( \sigma(T) \subset \mathbb{T} \). Assume that \( T \) satisfies the resolvent estimate
\[
\| (T - \lambda)^{-1} \| \leq C(1 - |\lambda|)^{-1}, \quad |\lambda| < 1
\]
and
\[
\| T^n \| \leq C, \quad n \geq 0.
\]
Then \( T \) is similar to a unitary operator.

An operator satisfying the last condition in this theorem is said to be power bounded. Van Casteren improved this result in [7], as follows.

**Theorem VC2.** Let \( T \in \mathcal{B}(H) \) be an operator with \( \sigma(T) \subset \mathbb{T} \). Assume that \( T \) satisfies the resolvent estimate
\[
\| (T - \lambda)^{-1} \| \leq C(1 - |\lambda|)^{-1}, \quad |\lambda| < 1
\]
and for \( 1 < r < 2 \) and \( x \in H \),
\[
\int_{|\lambda|=r} \| (T - \lambda)^{-1} x \|^2 |d\lambda| \leq \frac{C \|x\|^2}{r - 1} \quad \text{and} \quad \int_{|\lambda|=r} \| (T - \lambda)^{-1} x \|^2 |d\lambda| \leq \frac{C \|x\|^2}{r - 1}.
\]
Then \( T \) is similar to a unitary operator.
By writing the power series for the resolvent, one can check that every power bounded operator satisfies the last two conditions in this theorem. A related result was proved independently by Naboko in [33].

**Theorem N.** Let $T \in \mathcal{B}(H)$ be an operator with $\sigma(T) \subset \mathbb{T}$. Assume that $T$ satisfies the resolvent conditions

$$\int_{|\lambda|=r} \|(T - \lambda)^{-1}x\|^2 \, |d\lambda| \leq \frac{C\|x\|^2}{r - 1}, \quad 1 < r < 2, \ x \in H,$$

and

$$\int_{|\lambda|=r} \|(T^* - \lambda)^{-1}x\|^2 \, |d\lambda| \leq \frac{C\|x\|^2}{1 - r}, \quad r < 1, \ x \in H.$$

Then $T$ is similar to a unitary operator.

Each of Theorems VC1, VC2 and N in fact gives necessary and sufficient conditions for similarity to a unitary operator.

In these theorems, it is possible to replace $T$ by $T^*$, $T^{-1}$ or $T^{*-1}$, obtaining yet other criteria. For related results and conditions, we refer the reader to [29], where some results close to Naboko’s were independently obtained, and to [38, Section 1.5.6]. The conditions in Theorems VC2 and N are not comparable, in that there is no easy way of deducing one from the other.

Additionally, we present analogues of criteria of Van Casteren and Naboko, generalized from the circle to a smooth curve $\Gamma$. The corresponding integral conditions use the existence of a family of curves, tending “nicely” to $\Gamma$ (from both sides) in place of circles $|\lambda| = r$; details are given at the beginning of Section 5.

The paper is organized as follows. Sections 2 and 3 are preparatory. The first of these contains the basic facts about the pseudoanalytic extension of functions and Dynkin’s functional calculus for an operator $T$ with first order resolvent growth near the spectrum (that is, growth which is linear in the resolvent). In Section 3 we use this calculus to show that the resolvent estimates for an operator $T$ with $\sigma(T) \subset \Gamma$ are equivalent to corresponding resolvent estimates for $\eta(T)$, where $\eta$ is a smooth diffeomorphism from $\Gamma$ to $\mathbb{T}$, so that $\sigma(\eta(T)) \subset \mathbb{T}$. Section 4 deals with the proof of Theorem 1 while in Section 5 we formulate and prove analogues of mean-square criteria by Van Casteren and Naboko. Finally, Section 6 contains a brief discussion of related results in the literature and a few examples.

The authors are grateful to Maria Gamal’ for several insightful remarks and pointers to the literature.

2. **Dynkin’s functional calculus**

Our key technical tool will be a generalization of the Riesz-Dunford functional calculus as defined by Dynkin in [13] using the Cauchy-Green formula. Before going into details, we need to set down some definitions and notation.

Let $\Gamma \subset \mathbb{C}$ be a Jordan curve of class $C^{1+\alpha}$, $0 < \alpha < 1$. This means that $\Gamma$ is the image of $\mathbb{T}$ under a bijective map $\psi : \mathbb{T} \rightarrow \Gamma$ such that $\psi \in C^1(\mathbb{T})$, $\psi'$ does not vanish and $\psi'$ is Hölder $\alpha$; that is,

$$|\psi'(z) - \psi'(w)| \leq C|z - w|^{\alpha}, \quad z, w \in \mathbb{T}.$$
A function \( f : \Gamma \to \mathbb{C} \) is said to belong to \( C^{1+\alpha}(\Gamma) \) if \( f \circ \psi \in C^1(\mathbb{T}) \) and \( (f \circ \psi)' \) is Hölder \( \alpha \). As an important example of such a function, take \( f = \psi^{-1} \). This function has the additional properties that \( f(\Gamma) = \mathbb{T}, f^{-1} \) exists as a map from \( \mathbb{T} \) to \( \Gamma \) and is differentiable.

The norm for \( f \in C^{1+\alpha}(\Gamma) \) is defined as

\[
\|f\|_{C^{1+\alpha}(\Gamma)} = \|f \circ \psi\|_{C(\mathbb{T})} + \|(f \circ \psi)'\|_{C(\mathbb{T})} + \|(f \circ \psi)''\|_{C(\mathbb{T})},
\]

where

\[
\|g\|_\alpha = \sup_{z,w \in \Gamma, z \neq w} \frac{|g(z) - g(w)|}{|z - w|^\alpha}.
\]

The definition of this norm depends on the choice of the parametrization \( \psi \), but different choices yield equivalent norms.

Let \( T \in B(H) \) be an operator with \( \sigma(T) \subset \Gamma \), where \( \Gamma \) is a Jordan curve of class \( C^{1+\alpha} \). Assume that \( T \) satisfies the following resolvent growth condition:

\[
\|(T - \lambda)^{-1}\| \leq \frac{C}{\text{dist}(\lambda, \Gamma)}, \quad \lambda \in \mathbb{C} \setminus \Gamma.
\]

Following Dynkin [13], a \( C^{1+\alpha}(\Gamma) \) functional calculus for \( T \) can be defined. Dynkin defines his calculus for a scale of function algebras including \( C^{1+\alpha} \) and operators satisfying other resolvent estimates [11]. Only the case relevant to this paper is discussed here.

To begin, recall the notion of pseudoanalytic extension. If \( f \in C^{1+\alpha}(\Gamma) \), then by [14, Theorem 2] there is a function \( F \in C^1(\mathbb{C}) \) such that \( F|\Gamma = f \) and

\[
\left| \frac{\partial F}{\partial \overline{z}}(z) \right| \leq C\|f\|_{C^{1+\alpha}(\Gamma)} \text{dist}(z, \Gamma)^\alpha.
\]

Here, \( \overline{\overline{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \) and \( C \) is a constant depending only on \( \Gamma \). Every such function \( F \) which extends \( f \) and satisfies \( 2 \) is called a pseudoanalytic extension of \( f \).

Dynkin uses the pseudoanalytic extension \( F \) to define the operator \( f(T) \) by means of the Cauchy-Green integral formula. Let \( D \) be a domain with smooth boundary such that \( \Gamma \subset D \), and define

\[
f(T) = \frac{1}{2\pi i} \int_{\partial D} F(\lambda)(\lambda - T)^{-1} d\lambda - \frac{1}{\pi} \int_D \frac{\partial F}{\partial \overline{z}}(\lambda)(\lambda - T)^{-1} d\lambda.
\]

The inequality [2] for \( F \) and the resolvent estimate [1] for \( T \) imply that the second integral is well defined. It is possible to prove that the definition does not depend on the particular choice of \( D \) or pseudoanalytic extension \( F \).

This calculus has the usual properties of a functional calculus: it is continuous from \( C^{1+\alpha}(\Gamma) \) to \( B(H) \), is linear and multiplicative, and coincides with the natural definition of \( f(T) \) if \( f \) is rational. It also satisfies the spectral mapping property: \( \sigma(f(T)) = f(\sigma(T)) \).

3. PASSING FROM \( \Gamma \) TO \( \mathbb{T} \)

We now explain how to use Dynkin’s functional calculus to pass from an operator \( T \) with \( \sigma(T) \subset \Gamma \) to an operator \( A \) with \( \sigma(A) \subset \mathbb{T} \). The main result of this section is Theorem [3] which relates the estimates for the resolvents
of $T$ and $A$. In this way, resolvent growth conditions for $T$ imply equivalent conditions for $A$, and conversely. This equivalence plays a key role in what follows.

The next lemma gives regularity conditions for a certain function $\eta : \Gamma \to \mathbb{T}$, which enables the construction of the operator $A = \eta(T)$ using the Dynkin functional calculus.

**Lemma 2.** Let $\Gamma$ be a Jordan curve of class $C^{1+\alpha}$ and $\eta \in C^{1+\alpha}(\Gamma)$ a function such that $\eta(\Gamma) = \mathbb{T}$ and $\eta^{-1} : \mathbb{T} \to \Gamma$ exists and is differentiable. Fix any pseudoanalytic extension of $\eta$ to $\mathbb{C}$, also denoted by $\eta$. Then there is a neighborhood $U$ of $\Gamma$ such that $\eta : U \to \eta(U)$ is a $C^1$ diffeomorphism, $\eta(U)$ is a neighborhood of $\mathbb{T}$, and $\eta$ is bi-Lipschitz in $U$; that is, there are positive constants $c$ and $C$ such that

$$c|z - w| \leq |\eta(z) - \eta(w)| \leq C|z - w|, \quad z, w \in U.$$ 

A consequence of this lemma used frequently below is that for $\lambda \in U$, $\text{dist}(\lambda, \Gamma)$ and $\text{dist}(\eta(\lambda), \mathbb{T})$ are comparable.

**Proof of Lemma 2** Since $\partial \eta / \partial \bar{z} = 0$ on $\Gamma$, the condition that $\eta^{-1} : \mathbb{T} \to \Gamma$ is differentiable implies that the differential of $\eta$ is non-singular on $\Gamma$. Therefore, for each point $x \in \Gamma$, there is an open ball $B(x, r(x))$ of center $x$ and radius $r(x)$ such that $\eta$ is bi-Lipschitz on $B(x, r(x))$. By a compactness argument, $\eta$ is Lipschitz on some neighborhood of $\Gamma$.

Pass to a finite collection $\{x_j\}$ on $\Gamma$ such that the balls $B(x_j, r(x_j)/2)$ cover $\Gamma$ and put $\varepsilon_0 = \min r(x_j)/2$. Since $\eta|\Gamma$ is injective,

$$\delta := \min_{|x - y| \geq \varepsilon_0} \min_{x, y \in \Gamma} |\eta(x) - \eta(y)| > 0.$$ 

It follows that there is some $\rho > 0$ such that

$$\tilde{\delta} := \min_{\text{dist}(x, \Gamma) \leq \rho, \text{dist}(y, \Gamma) \leq \rho} |\eta(x) - \eta(y)| > 0.$$ 

Now check that $\eta$ is bi-Lipschitz on the open set

$$W = \left( \bigcup_j B(x_j, r(x_j)/2) \right) \cap \{ x \in \mathbb{C} : \text{dist}(x, \Gamma) < \rho \}.$$ 

Given points $x, y \in W$, then either $|x - y| < \varepsilon_0$, so that $x, y$ both belong to the same ball $B(x_k, r(x_k))$, where $\eta$ is bi-Lipschitz, or $|x - y| \geq \varepsilon_0$. In the latter case,

$$|\eta(x) - \eta(y)| \geq \tilde{\delta} \geq \tilde{\delta} (\text{diam } W)^{-1} |x - y|.$$ 

The injectivity of $\eta$ follows from the bi-Lipschitz property. The fact that it is possible to choose $U \subset W$ so that $\eta$ is a $C^1$ diffeomorphism of $U$ is true because the differential of $\eta$ is non-singular in some neighborhood of $\Gamma$. Finally, since $\eta(\Gamma) = \mathbb{T}$ and $\eta$ is an open mapping by being bi-Lipschitz, $\eta(U)$ is an open neighborhood of $\mathbb{T}$. $\square$

The next theorem relates the resolvents estimates for $T$ and $\eta(T)$.
Theorem 3. Let \( \Gamma, \eta \) and \( U \) be as in Lemma 2 and \( T \in \mathcal{B}(H) \) be an operator satisfying the resolvent estimate (1). Let the operator \( \eta(T) \) be defined by the \( C^{1+\alpha} \)-functional calculus for \( T \). Then \( \sigma(\eta(T)) \subset \mathbb{T} \), and for some \( C \geq 1 \) and depending on \( \Gamma, \eta, T \), but not on \( \lambda \) or \( x \),

\[
C^{-1}\| (T-\lambda)^{-1} x \| \leq \| (\eta(T) - \eta(\lambda))^{-1} x \| \leq C \| (T-\lambda)^{-1} x \|, \quad \lambda \in U \setminus \Gamma, \ x \in H.
\]

Proof. The fact that \( \sigma(\eta(T)) \subset \mathbb{T} \) follows from the spectral mapping theorem for the Dynkin functional calculus.

For \( \lambda \in U \setminus \Gamma \), define functions \( \varphi_\lambda, \psi_\lambda \in C^{1+\alpha}(\Gamma) \) by

\[
(3) \quad \varphi_\lambda(z) = \frac{\eta(z) - \eta(\lambda)}{z - \lambda}, \quad \psi_\lambda(z) = \frac{z - \lambda}{\eta(z) - \eta(\lambda)}.
\]

The operators \( \varphi_\lambda(T) \) and \( \psi_\lambda(T) \) are thus defined. In fact,

\[
\varphi_\lambda(T) = (\eta(T) - \eta(\lambda))(T - \lambda)^{-1}, \quad \psi_\lambda(T) = (T - \lambda)(\eta(T) - \eta(\lambda))^{-1}.
\]

Hence it suffices to show that

\[
\| \varphi_\lambda(T) \| \leq C_0, \quad \| \psi_\lambda(T) \| \leq C_0,
\]

for \( C_0 \) independent of \( \lambda \).

In fact, the functions \( \varphi_\lambda \) and \( \psi_\lambda \) are in \( U \setminus \{ \lambda \} \) and since \( \eta \) is bi-Lipschitz,

\[
| \varphi_\lambda(z) | \leq C_1, \quad | \psi_\lambda(z) | \leq C_1, \quad z \in U \setminus \{ \lambda \}.
\]

Let \( D \) be a domain with smooth boundary such that \( \Gamma \subset D \subset \overline{D} \subset U \) and \( \varepsilon > 0 \), to be chosen later.

By the Dynkin functional calculus, for \( \lambda \in D \) and for \( \varepsilon \) chosen small enough so that \( B(\lambda, \varepsilon) \subset D \),

\[
\varphi_\lambda(T) = \frac{1}{2\pi i} \int_{\partial D} \varphi_\lambda(z)(z - T)^{-1} \, dz - \frac{1}{2\pi i} \int_{B(\lambda, \varepsilon)} \varphi_\lambda(z)(z - T)^{-1} \, dz
\]

\[
- \frac{1}{\pi} \int_{D \setminus B(\lambda, \varepsilon)} \frac{\partial \varphi_\lambda}{\partial z}(z - T)^{-1} \, dA(z).
\]

The case \( \lambda \notin D \) is similar.

The norm \( \| \varphi_\lambda(T) \| \) is bounded by estimating the three terms separately. For the second term, if \( \varepsilon < \text{dist}(\lambda, \Gamma) \), then

\[
\int_{B(\lambda, \varepsilon)} | \varphi_\lambda(z) | \| (z - T)^{-1} \| \, |dz| \leq C_2 \varepsilon \text{dist}(\lambda, \Gamma) \varepsilon^{-1}.
\]

Letting \( \varepsilon \to 0 \), it is seen that this term is negligible.

The norm of the first term is bounded by

\[
\frac{1}{2\pi} \int_{\partial D} | \varphi_\lambda(z) | \| (z - T)^{-1} \| \, |dz| \leq C_3 \text{dist}(\partial D, \Gamma) \varepsilon^{-1} \text{length}(\partial D).
\]
Finally, by using Lemma 2, the norm of the third term is bounded by

\[ \int D |z - \lambda|^{-1} \text{dist}(z, \Gamma)^{\alpha - 1} dA(z) \]

\[ \leq C_4 \int D |z - \lambda|^{-1} \text{dist}(z, \Gamma)^{\alpha - 1} dA(z) \]

\[ \leq C_5 \int D |\eta(z) - \eta(\lambda)|^{-1} \text{dist}(\eta(z), \mathcal{T})^{\alpha - 1} dA(\zeta) \]

\[ \leq C_6 \int_{\eta(D)} |\zeta - \eta(\lambda)|^{-1} \text{dist}(\zeta, \mathcal{T})^{\alpha - 1} dA(\zeta) \]

The change of variables \( \zeta = \eta(z) \) has been performed and choice \( a < b \) is made so that the set \( \eta(D) \) is contained in the annulus \( a \leq |\zeta| \leq b \). By Lemma 4 below, the last term in this chain of inequalities is smaller than a constant which is independent of \( \lambda \). Thus \( \| \varphi_\lambda(T) \| \leq C_0 \), with \( C_0 \) independent of \( \lambda \).

The proof that \( \| \psi_\lambda(T) \| \leq C_0 \) is very similar, in this case using that

\[ \left| \frac{z - \lambda}{|\eta(z) - \eta(\lambda)|^2} \frac{\partial \eta}{\partial z}(z) \right| \leq C_7 |\eta(z) - \eta(\lambda)|^{-1} \left| \frac{\partial \eta}{\partial z}(z) \right|. \]

The remaining bounds are obtained in the same way. The proof is finished (modulo the next lemma). \( \square \)

**Lemma 4.** Let \( 0 < a < 1 < b \) and \( a \leq |w| \leq b \) and \( -1 < \beta < 0 \). Then for some \( C \) independent of \( w \),

\[ \int_{a \leq |z| \leq b} |z - w|^{-1} |1 - |z|^\beta| dA(z) \leq C. \]

**Proof.** Performing a rotation if necessary, take \( w \) to be real and positive, so that \( a \leq w \leq b \). By passing to polar coordinates and using the inequality

\[ |re^{i\theta} - w|^{-1} \leq C_0 |r + i\theta - w|^{-1}, \]

which is valid for \( a \leq r \leq b \), the integral in the statement of the lemma is less than a constant times

\[ \int_{|a,b| \times [-\pi, \pi]} |\zeta - w|^{-1} |1 - \text{Re} |\zeta|^\beta| dA(\zeta). \]

Now assume that \( a \leq w \leq 1 \) (the case \( 1 \leq w \leq b \) will be similar). Estimate the integral by dividing the region of integration into two pieces.
Put $t = (w + 1)/2$. Then
\[
\int_{[a,t] \times [-\pi,\pi]} \frac{|1 - \Re \zeta|^\beta}{|\zeta - w|} \, dA(\zeta) + \int_{[t,b] \times [-\pi,\pi]} \frac{|1 - \Re \zeta|^\beta}{|\zeta - w|} \, dA(\zeta)
\]
\[
\leq \int_{[a,t] \times [-\pi,\pi]} \frac{|w - \Re \zeta|^\beta}{|\zeta - w|} \, dA(\zeta) + \int_{[t,b] \times [-\pi,\pi]} \frac{|1 - \Re \zeta|^\beta}{|\zeta - w|} \, dA(\zeta)
\]
\[
\leq \int_{[a-1+w,t-1+w] \times [-\pi,\pi]} \frac{|1 - \Re \zeta'|^\beta}{|\zeta' - w|} \, dA(\zeta') + \int_{[t,b] \times [-\pi,\pi]} \frac{|1 - \Re \zeta'|^\beta}{|\zeta - w|} \, dA(\zeta)
\]
\[
\leq 2 \int_{[2a-1,b] \times [-\pi,\pi]} \frac{|1 - \Re \zeta|^\beta}{|\zeta - w|} \, dA(\zeta),
\]
where we have performed the change of variables $\zeta' = \zeta - 1 + w$ and used that $2a - 1 \leq a - 1 + w$ and $t - 1 + w \leq b$. By a change to polar coordinates $\zeta = 1 + re^{i\theta}$, the single pole at 1 is seen to be of order strictly between −2 and 0, and so the last integral is finite. □

4. The proof of Theorem A

We first recall the result from [12] to be used in the proof of the theorem. The statement given here is for a $C^{1+\alpha}$ domain, although the original was proved under weaker regularity conditions (see [12, Theorem 2]).

**Theorem A.** Let $T \in B(H)$ and $\Omega$ a Jordan domain of class $C^{1+\alpha}$. Assume there is some $R > 0$ such that for every $\lambda \in \partial \Omega$ there is some point $\mu_k(\lambda) \in \mathbb{C} \setminus \overline{\Omega}$ such that $\text{dist}(\mu_k(\lambda), \partial \Omega) = |\mu_k(\lambda) - \lambda| = R$ and $\|T - \mu_k(\lambda)^{-1}\| \leq R^{-1}$. Then $\overline{\Omega}$ is a complete $K$-spectral set for some $K > 0$.

In other words, the conclusion is that there exists a constant $K \geq 1$ such that
\[
\|f(T)\| \leq K\|f\|_{H^\infty(\Omega)},
\]
for every (matrix-valued) rational function $f$ with poles off of $\overline{\Omega}$ (and hence for every $f$ which is continuous in $\overline{\Omega}$ and analytic in $\Omega$). This result affirmatively answers Question 3, posed by Stampfli in [48].

A nice overview of complete $K$-spectral sets can be found in [40, Chapter 9]. A result of this property for an operator $T$ is that $T$ dilates to an operator similar to a normal operator with spectrum in the boundary of the domain. The additional assumptions in Theorem A will allow us to conclude that the operator $T$ itself is similar to a normal operator. Curiously, this will require only knowing the weaker property that $T$ has $\overline{\Omega}$ as a $K$-spectral set; in other words, that $\|f(T)\| \leq K\|f\|_{H^\infty(\Omega)}$ only for scalar valued rational functions with poles off of $\overline{\Omega}$.

**Lemma 5.** Under the hypotheses of Theorem A, if $\eta(T)$ is similar to a unitary operator, then $T$ is similar to a normal operator.

**Proof.** Replacing $T$ by $STS^{-1}$, where $S$ is such that $S\eta(T)S^{-1} = \eta(S^{1+\alpha})$ is unitary, it can be assumed that $\eta(T)$ is unitary. Then $(\eta \Gamma)^{-1} \in C^{1+\alpha}(T)$. Choose some $\beta \in (0, \alpha)$. Then $(\eta \Gamma)^{-1}$ is in the class $C^{1+\beta}(T)$, which consists of functions $g \in C^{1+\beta}(T)$ such that $(g'(z) - g'(w))/(z - w)^\beta \to 0$ as $z, w \in T$, $|z - w| \to 0$. Hence one can choose a sequence of rational functions $\{r_n\}_{n=1}^\infty$.
with poles off $\mathbb{T}$ such that $r_n$ tend to $(\eta(\Gamma))^{-1}$ in $C^{1+\beta}(\mathbb{T})$ (this follows, for instance, from [24, Theorem 2.12]). Thus $r_n \circ \eta$ tend to the identity function in $C^{1+\beta}(\Gamma)$. By continuity of the $C^{1+\beta}(\Gamma)$-functional calculus for $T$, $(r_n \circ \eta)(T)$ tends to $T$ in operator norm. The Dynkin functional calculus for $T$ is a homomorphism, and so $(r_n \circ \eta)(T) = r_n(\eta(T))$. Since each $r_n(\eta(T))$ is normal, it follows that $T$ is also normal. 

As an alternative and a more direct proof of the last lemma, it seems tempting to argue that if $A = \eta(T)$ is similar to a unitary operator, then $\eta^{-1}(A)$ is defined, for instance, by the usual $L^\infty$-functional calculus for normal operators, and $\eta^{-1}(A)$ is similar to a normal operator. However, it is not clear a priori why $\eta^{-1}(A) = T$.

Theorem 1 is a straightforward consequence of Theorem A and the following lemma.

**Lemma 6.** Let $\Gamma \subset \mathbb{C}$ be a $C^{1+\alpha}$ Jordan curve, and $\Omega$ the domain it bounds. Let $T \in \mathcal{B}(H)$ be an operator with $\sigma(T) \subset \Gamma$. Assume that $\overline{\Omega}$ is a $K$-spectral set for $T$ and

$$\| (T - \lambda)^{-1} \| \leq \frac{C}{\text{dist}(\lambda, \Gamma)}, \quad \lambda \in \Omega,$$

for some constant $C > 0$. Then $T$ is similar to a normal operator.

**Proof.** Let $\eta : \overline{\Omega} \to \overline{\mathbb{D}}$ be the Riemann map. Since $\partial \Omega$ is of class $C^{1+\alpha}$, then $\eta \in C^{1+\alpha}(\partial \mathbb{D})$ (see, for instance, [43, Theorem 3.6]). Extend $\eta$ pseudoanalytically to $\mathbb{C} \setminus \Omega$. Now as $\eta$ satisfies its assumptions, we can apply Lemma 4.

Because $|\eta^n|\leq 1$ in $\overline{\Omega}$ for all $n \geq 0$, and $\overline{\Omega}$ is $K$-spectral for $T$, the operator $\eta(T)$ is power bounded. By Theorem 3 and the fact that $\text{dist}(\lambda, \partial \Omega)$ and $\text{dist}(\eta(\lambda), \mathbb{T})$ are comparable,

$$\| (\eta(T) - \lambda)^{-1} \| \leq \frac{C}{1 - |\lambda|}, \quad |\lambda| < 1.$$

Applying Theorem VC1 it follows that $\eta(T)$ is similar to a unitary operator, and so by Lemma 5 $T$ is similar to a normal operator. □

**Proof of Theorem 7**. Theorem A implies that $\overline{\Omega}$ is a complete $K$-spectral set for $T$. It suffices to apply Lemma 6. □

**Remarks.** It is straightforward to deduce an analogous result assuming an estimate with constant 1 inside the domain $\Omega$ and an estimate with a constant $C$ outside the domain. Indeed, put $R = (T - z_0)^{-1}$, for some fixed $z_0 \in \Omega$. It follows from [12, Lemma 7] that if $\|(T - \lambda)^{-1}\| \leq \text{dist}(\lambda, \Gamma)^{-1}$, then $\|(R - \mu)^{-1}\| \leq \text{dist}(\mu, \Gamma)^{-1}$, where $\mu = (\lambda - z_0)$ and $\Gamma$ is the image of $\Gamma$ under the map $z \mapsto (z - z_0)^{-1}$. Writing the resolvent of $R$ in terms of the resolvent of $T$, it is also easy to obtain an estimate for $R$ with a constant $C' > 1$ outside the domain bounded by $\tilde{\Gamma}$. Since the map $z \mapsto (z - z_0)^{-1}$ sends the inside of $\Gamma$ onto the outside of $\tilde{\Gamma}$ and vice versa, it suffices to apply Theorem 1 to $R$.

The conclusion of Lemma 6 is that $T$ is similar to a normal operator, and so the set $\overline{\Omega}$ (and even the boundary of $\Omega$) must in fact be a complete $K'$-spectral set for $T$ for some $K' > 1$. As it follows from the celebrated example...
of Pisier, combined with the main result of [22], there exists an operator \( T \) on \( H \) with \( \sigma(T) \subset \mathbb{T} \), which is polynomially bounded, but not completely bounded. So, in Lemma 6 one cannot replace the resolvent estimate inside \( \Omega \) just by the condition that \( \sigma(T) \subset \partial \Omega \). Recently, Gamal [21] has constructed several new examples of operators that are polynomially bounded but not completely polynomially bounded. In particular, any operator given in [21, Corollary 2.8] is quasi-similar to an absolute continuous unitary operator \( U \) and also satisfies \( \sigma(T) \subset \mathbb{T} \) (the latter follows from [21, Theorem 2.4]). Using the techniques outlined here, this counterexample can be transferred from \( \mathbb{T} \) to other sets \( \Omega \).

5. Mean-square type resolvent estimates

In this section we give criteria for similarity to a normal operator analogous to the results by Van Casteren [7] and Naboko [33] in the context of \( C^{1+\alpha} \) Jordan curves. First of all, a substitute for the curves \( \gamma \) is needed. To this end, we give the following definition.

**Definition.** Let \( \Gamma \subset \mathbb{C} \) be a Jordan curve and \( \Omega \) the region it bounds. A family of Jordan curves \( \{\gamma_s\}_{0 < s \leq 1} \) tends nicely to \( \Gamma \) from the outside if \( s \to 0 \) if \( \gamma_s \subset \mathbb{C} \setminus \Omega \) for all \( 0 < s \leq 1 \) and the following conditions are satisfied for some constant \( C \geq 1 \):

(a) \( \{\gamma_s\} \) tend uniformly to \( \Gamma \),
(b) For all \( 0 < s \leq 1 \) \( C^{-1} s \leq \text{dist}(x, \Gamma) \leq Cs \), for all \( x \in \gamma_s \).
(c) For every \( 0 < s \leq 1 \), \( x \in \gamma_s \), and \( r > 0 \), \( \text{length}(\gamma_s \cap B(x, r)) \leq Cr \).

The family \( \{\gamma_s\}_{0 < s \leq 1} \) tends to \( \Gamma \) from the inside if instead \( \gamma_s \subset \Omega \) for all \( 0 < s \leq 1 \) and again, conditions (a)–(c) are satisfied.

Condition (c) states that the curves \( \gamma_s \) satisfy the Ahlfors-David condition with a uniform constant. This condition was first studied in [11] and [9].

If \( \Gamma \) is in the class \( C^{1+\alpha} \), it is apparent that there exist a family of curves which tends nicely to \( \Gamma \) from the outside and another family of curves which tends nicely to \( \Gamma \) from the inside. Indeed, let \( \eta : U \to \mathbb{C} \) be a function as in the statement of Lemma 2 and take \( \Gamma(t) = \eta^{-1}(e^{it}) \) for \( 0 \leq t \leq 2\pi \). Define \( \gamma_s^{\pm} \) by \( \gamma_s^{\pm} = \eta^{-1}(1 \pm \beta s)e^{it} \), \( 0 \leq t \leq 2\pi \). If the constant \( \beta > 0 \) is small, then \( \gamma_s^{\pm} \subset U \) for every \( 0 < s \leq 1 \). The curves \( \{\gamma_s^{+}\} \) tend nicely to \( \Gamma \) from outside and the curves \( \{\gamma_s^{-}\} \) tend nicely to \( \Gamma \) from inside.

It will be proved that the mean-square type resolvent estimates considered here do not depend on the concrete choice of the family of curves \( \{\gamma_s\} \) tending nicely to \( \Gamma \). This will follow from a lemma concerning Smirnov spaces.

Recall that the Smirnov space \( E^2(\Omega, H) \) of \( H \)-valued function on a (nice) domain \( \Omega \) is defined as the \( L^2(\partial \Omega) \)-closure of the \( H \)-valued rational functions with poles off \( \overline{\Omega} \). The following lemma dates back to David and his theorem on the boundedness of certain singular integral operators on Ahlfors regular curves. In particular, it follows from the results in [9, Proposition 6].

**Lemma 7.** Let \( \Omega_1, \Omega_2 \) be Jordan domains with Ahlfors regular boundaries such that \( \overline{\Omega_2} \subset \Omega_1 \). If \( H \) is a Hilbert space and \( f \in E^2(\Omega_1, H) \), then \( f|\Omega_2 \in E^2(\Omega_2, H) \) and

\[ \|f|\Omega_2\|_{E^2(\Omega_2, H)} \leq C\|f\|_{E^2(\Omega_1, H)}, \]
for some constant $C$ depending only on the Ahlfors constants for $\partial \Omega_1$ and $\partial \Omega_2$.

Lemma 8. Let $\Gamma$ be a Jordan curve, $T \in \mathcal{B}(H)$ with $\sigma(T) \subset \Gamma$, and $\{\gamma_s\}_{0 < s \leq 1}$, $\{\tilde{\gamma}_s\}_{0 < s \leq 1}$ two families of curves which both tend nicely to $\Gamma$ from the inside (respectively, from the outside). If

$$
\int_{\gamma_s} \|(T - \lambda)^{-1}x\|^2 |d\lambda| \leq \frac{C\|x\|^2}{s}, \quad x \in H, 0 < s \leq 1,
$$

for some constant $C$ independent of $x$ and $s$, then

$$
\int_{\tilde{\gamma}_s} \|(T - \lambda)^{-1}x\|^2 |d\lambda| \leq \frac{C'\|x\|^2}{s}, \quad x \in H, 0 < s \leq 1,
$$

for some constant $C'$ independent of $x$ and $s$.

Proof. First assume that $\{\gamma_s\}$ and $\{\tilde{\gamma}_s\}$ tend nicely to $\Gamma$ from the inside. Denote by $\Omega_s$ the domain bounded by $\gamma_s$ and by $\tilde{\Omega}_s$ the domain bounded by $\tilde{\gamma}_s$.

By definition, since $\{\tilde{\gamma}_s\}$ tends nicely to $\Gamma$, there exists $0 < s_0 \leq 1$ such that for all $s \in (0, s_0)$, the closure of $\tilde{\Omega}_s$ is contained in $\Omega_1$. Then by continuity of $1/s$ on $[s_0, 1]$, there exists a constant $\tilde{C}$ such that the claim holds whenever $s \in [s_0, 1]$.

So assume that $s \in (0, s_0)$. Applying Lemma 7 to the function $f(z) = (T - z)^{-1}x$ and the domains $\Omega_1 = \Omega_s$, $\Omega_2 = \tilde{\Omega}_s$, to obtain

$$
\int_{\gamma_s} \|(T - \lambda)^{-1}x\|^2 |d\lambda| = \|f\|_{L^2(\Omega_s, H)} \leq K \|f\|_{L^2(\Omega_1, H)}
$$

$$
= K \int_{\gamma_1} \|(T - \lambda)^{-1}x\|^2 |d\lambda| \leq KC\|x\|^2 \leq \frac{KC\|x\|^2}{s_0} \leq \frac{KC\|x\|^2}{s}.
$$

If $\{\gamma_s\}$ and $\{\tilde{\gamma}_s\}$ tend nicely to $\Gamma$ from the outside, choose a point $z_0$ inside the domain bounded by $\Gamma$ and apply an inversion: $z \mapsto (z - z_0)^{-1}$. Since

$$
((T - z_0)^{-1} - (\lambda - z_0)^{-1})^{-1} = (\lambda - z_0)(T - z_0)(T - \lambda)^{-1},
$$

the bounds for the resolvent of $T$ imply equivalent bounds for the resolvent of $(T - z_0)^{-1}$, and conversely. Thus, this case follows from the previous one. 

It is well known that, in the context of the unit circle, a resolvent bound of mean-square type implies a pointwise resolvent bound such as $\Box$. The proof of this fact uses the usual pointwise estimate for an $H^2$ function in the disk, which involves the norm of the reproducing kernel. The following lemma is a generalization of this.

Lemma 9. Let $\Gamma \subset \mathbb{C}$ be a Jordan curve of class $C^{1+\alpha}$, $\Omega$ the region it bounds and $T \in \mathcal{B}(H)$ with $\sigma(T) \subset \Gamma$. If

$$
(4) \quad \int_{\gamma_s} \|(T - \lambda)^{-1}x\|^2 |d\lambda| \leq \frac{C\|x\|^2}{s}, \quad x \in H, \ 0 < s \leq 1,
$$
for some constant $C$ independent of $x$ and $s$ and some family of curves $\{\gamma_s\}$ which tends nicely to $\Gamma$ from the inside (respectively, outside), then

$$\|(T - \lambda)^{-1}\| \leq \frac{C'}{\text{dist}(\lambda, \Gamma)}, \quad \lambda \in \Omega \ (\text{respectively, } \lambda \in \mathbb{C} \setminus \Omega),$$

for some constant $C'$ independent of $\lambda$.

**Proof.** Assume that $\{\gamma_s\}$ tends nicely to $\Gamma$ from the inside. Let $\eta$ be a function as in the statement of Lemma 2 and $U$ the neighborhood of $\Gamma$ that appears in that lemma. Fix $\lambda \in \Omega \setminus U$. Then because $\eta$ is bi-Lipschitz, $t = \text{dist}(\lambda, \Gamma)$ is comparable to $\text{dist}(\eta(\lambda), \mathbb{T})$. Put $r = 1 - \text{dist}(\eta(\lambda), \mathbb{T})/2$.

Now consider the Jordan curve $\Lambda = \eta^{-1}(rT)$. This lies inside $\Omega$ and $\text{dist}(z, \Gamma)$ is comparable to $t$ for every $z \in \Lambda$. Therefore, it is possible to choose $0 < s \leq 1$ such that $t \leq C_1s$, and $\Lambda$ is inside the region bounded by $\gamma_s$.

Fix $x \in H$ and put $f(z) = (\eta(T) - z)^{-1}x$, and $g(z) = f(z/r)$. By the usual pointwise estimate for a function in $H^2$, the Hardy space of the disk,

$$\|g(z)\| \leq (1 - |z|^2)^{-1/2}\|g\|_{E^2(\mathbb{D}, H)} = (1 - |z|^2)^{-1/2}\|f\|_{E^2(\mathbb{D}, H)} \leq (1 - |z|^2)^{-1/2}\|f\|_{E^2(W, H)},$$

where $W$ is the domain bounded by $\eta(\gamma_s)$ and the last inequality comes from Lemma 2. Now by Theorem 3 and 1,

$$\|f\|_{E^2(W, H)}^2 = \int_{\gamma_s} \|(\eta(T) - z)^{-1}x\|^2 |dz| \leq C_1 \int_{\gamma_s} \|(\eta(T) - \eta(w))^{-1}x\|^2 |dw| \leq C_2 \int_{\gamma_s} \|(T - w)^{-1}x\|^2 |dw| \leq C_3 \frac{\|x\|^2}{s}.$$

Hence,

$$\|(\eta(T) - z/r)^{-1}x\|^2 \leq C_3(1 - |z|^2)^{-1}s^{-1}\|x\|^2,$$

and the inequality above is valid for all $x \in H$. Putting $z = r\eta(\lambda)$ yields

$$\|(\eta(T) - \eta(\lambda))^{-1}\|^2 \leq C_3(1 - |r\eta(\lambda)|^2)^{-1}s^{-1} \leq C_4t^{-2}.$$

By another application of Theorem 3

$$\|(T - \lambda)^{-1}\| \leq \frac{C'}{t} = \frac{C'}{\text{dist}(\lambda, \Gamma)}.$$

The case when $\{\gamma_s\}$ tends nicely to $\Gamma$ from the outside is proved by applying the inversion $z \mapsto (z - z_0)^{-1}$, as in the proof of Lemma 8. \qed

We now state and prove generalizations of Theorems VC2 and VC in Theorems 10 and Theorem 12. The proofs both follow the same line of reasoning, using the tools so far developed to pass to $\mathbb{T}$ and then applying Van Casteren’s or Naboko’s theorem. It is worth highlighting that, as with the original theorems, there is no easy way to deduce either result from the other.

**Theorem 10** (Van Casteren-type theorem for curves). Let $\Gamma \subset \mathbb{C}$ be a Jordan curve of class $C^{1+\alpha}$, $\Omega$ the region it bounds and $T \in B(H)$ with $\sigma(T) \subset \Gamma$. Let $\{\gamma_s\}_{0<s<1}$ be a family of curves which tends nicely to $\Gamma$
from the outside. Then $T$ is similar to a normal operator if and only if the following three conditions are satisfied.

$$
\|(T - \lambda)^{-1}\| \leq \frac{C}{\text{dist}(\lambda, \Gamma)}, \quad \lambda \in \Omega,
$$

(5) $$\int_{\gamma_s} \|(T - \lambda)^{-1}x\|^{2} \, d\lambda \leq \frac{C\|x\|^{2}}{s}, \quad x \in H, \ 0 < s \leq 1,$$

(6) $$\int_{s} \|(T^* - \overline{T})^{-1}x\|^{2} \, d\lambda \leq \frac{C\|x\|^{2}}{s}, \quad x \in H, \ 0 < s \leq 1.$$

**Proof.** First, assume that $T$ satisfies the three resolvent conditions. Let $\eta : U \to \mathbb{C}$ be a function as in the statement of Lemma 2. Take $\gamma_s \subset U$ for every $0 < s \leq 1$. By Lemma 9, the operator $T$ satisfies the resolvent estimate (1), so $\eta(T)$ is defined by the $C^{1+\alpha}(\Gamma)$-functional calculus for $T$. By Theorem 3 and the fact that $\text{dist}(\lambda, \Gamma)$ and $\text{dist}(\eta(\lambda), \Gamma)$ are comparable,

$$
\|(\eta(T) - \lambda)^{-1}\| \leq \frac{C_1}{1 - |\lambda|}, \quad |\lambda| < 1,
$$

as well as

$$
\int_{\eta(\gamma_s)} \|(\eta(T) - \lambda)^{-1}x\|^{2} \, d\lambda \leq \frac{C_2\|x\|^{2}}{s}, \quad x \in H, \ 0 < s \leq 1,
$$

which follows by making a change of variables $\lambda = \eta(\mu)$ and applying Theorem 3.

Since $\eta : U \to \eta(U)$ is a $C^1$ diffeomorphism and bi-Lipschitz, the family of curves $\{\eta(\gamma_s)\}_{0 < s < 1}$ tends nicely to $\mathbb{T}$ from the outside. Therefore, by Lemma 8 applied to the family $\gamma_s = (1 + s)\Gamma$,

$$
\int_{|\lambda|=r} \|(\eta(T) - \lambda)^{-1}x\|^{2} \, d\lambda \leq \frac{C_3\|x\|^{2}}{r - 1}, \quad x \in H, \ 1 < \ r < 2.
$$

Similar reasoning with $T^*$ in place of $T$ and $\tilde{\eta}(z) = \overline{\eta(\zeta)}$ in place of $\eta$ shows that

$$
\int_{|\lambda|=r} \|(\eta(T)^* - \lambda)^{-1}x\|^{2} \, d\lambda \leq \frac{C_4\|x\|^{2}}{r - 1}, \quad x \in H, \ 1 < \ r < 2,
$$

as $\tilde{\eta}(T^*) = \eta(T)^*$.

Now apply Theorem VC2 to deduce that $\eta(T)$ is similar to a unitary operator. Then by Lemma 5, $T$ is similar to a normal operator.

Conversely, assume that $T$ is similar to a normal operator and $\sigma(T) \subset \Gamma$. Replacing $T$ by $STS^{-1}$ if necessary, it can be assumed that $T$ is normal. The first condition on the resolvent of $T$ holds, since $\|(T - \lambda)^{-1}\| \leq \text{dist}(\lambda, \Gamma)^{-1}$ for a normal operator $T$.

The operator $\eta(T)$ is unitary, so by expressing the resolvent as a power series and using the fact that $\|\eta(T)^n\| = 1$ for all $n \geq 0$, it follows that

$$
\int_{r} \|(\eta(T) - \lambda)^{-1}x\|^{2} \, d\lambda \leq \frac{C_5\|x\|^{2}}{r - 1}, \quad 1 < r < 2, \ x \in H.
$$
By Theorem 3 (although since $T$ is normal, a simpler argument could be devised), for some constant $C > 0$ and some neighborhood $U$ of $\Gamma$, 

\[ C^{-1} \|(T - \lambda)^{-1} x\| \leq \|(\eta(T) - \eta(\lambda))^{-1} x\| \leq C \|(T - \lambda)^{-1} x\|, \quad \lambda \in U \setminus \Gamma, \ x \in H, \]

Therefore,

\[ \int_{\gamma^{-1}(\mathbb{T}^*)} \|(T - \lambda)^{-1} x\|^2 |d\lambda| \leq \frac{C_6 \|x\|^2}{r - 1}, \quad 1 < r < 2, \ x \in H. \]

The family of curves $\tilde{\gamma}_s = \eta^{-1}((1 + s)T)$ tends nicely to $\Gamma$. Apply Lemma 8 to get that $T$ satisfies (3). Use of similar reasoning, but with $T^*$ instead of $T$ and $\tilde{\eta}$ instead of $\eta$ yields the inequality (6).

Sz.-Nagy proved in [49] that an operator $T$ is similar to a unitary operator if and only if the following two conditions

\[ \eta \text{ is similar to a unitary operator. We only sketch the proof.} \]

\[ \text{use Naboko's Theorem N to show that} \]

\[ \text{satisfies the two conditions in the statement of this theorem, instead of} \]

\[ \eta \text{ a function as in the statement of Lemma 2. Define the} \]

\[ \text{operator } \eta(T) \text{ by the} \]

\[ \text{functional calculus. If } \eta(T) \text{ is power bounded, then } T \text{ is similar to a normal operator.} \]

The following corollary is a generalization of Theorem VC1. Note that here it is only assumed that $\|\eta(T)^n\| \leq C$ for all $n \geq 0$. The proof is similar to the proof of Theorem VC2 but Theorem VC1 is used instead of Theorem VC2.

**Corollary 11.** Let $\Gamma \subset \mathbb{C}$ be a Jordan curve of class $C^{1+\alpha}$, and $T \in \mathcal{B}(H)$ with $\sigma(T) \subset \Gamma$. Assume that

\[ \|(T - \lambda)^{-1}\| \leq \frac{C}{\text{dist}(\lambda, \Gamma)}, \quad \lambda \in \mathbb{C} \setminus \Gamma. \]

Let $\eta : \Gamma \to \mathbb{T}$ be a function as in the statement of Lemma 3. Define the operator $\eta(T)$ by the $C^{1+\alpha}$-functional calculus. If $\eta(T)$ is power bounded, then $T$ is similar to a normal operator.

**Theorem 12** (Naboko-type theorem for curves). Let $\Gamma \subset \mathbb{C}$ be a Jordan curve of class $C^{1+\alpha}$, $\Omega$ the region it bounds and $T \in \mathcal{B}(H)$ with $\sigma(T) \subset \Gamma$. Let $\{\gamma_s\}_{0 < s \leq 1}$ be a family of curves which tends nicely to $\Gamma$ from the outside and $\{\tilde{\gamma}_s\}$ a family of curves which tends nicely to $\Gamma$ from the inside. Then $T$ is similar to a normal operator if and only if the following two conditions are satisfied for all $x \in H$ and $0 < s \leq 1$:

\[ \int_{\gamma_s} \|(T - \lambda)^{-1} x\|^2 |d\lambda| \leq \frac{C \|x\|^2}{s} \quad \text{and} \quad \int_{\tilde{\gamma}_s} \|(T^* - \lambda)^{-1} x\|^2 |d\lambda| \leq \frac{C \|x\|^2}{s}. \]

**Proof.** The proof of this theorem is like the proof of Theorem 10. If $T$ satisfies the two conditions in the statement of this theorem, instead of using Van Casteren’s theorem, use Naboko’s Theorem N to show that $\eta(T)$ is similar to a unitary operator. We only sketch the proof.

First, Lemma 3 implies that $T$ satisfies the resolvent estimate 11. Choose a function $\eta$ as in Lemma 2. The operator $\eta(T)$ is well defined. By Theorem 3 and Lemma 8

\[ \int_{|\lambda| = r} \|(\eta(T) - \lambda)^{-1} x\|^2 |d\lambda| \leq \frac{C_1 \|x\|^2}{r - 1}, \quad x \in H, \ 1 < r < 2 \]
and
\[
\int_{|\lambda|=r} \left\| (\eta(T)^* - \lambda)^{-1} x \right\|^2 |d\lambda| \leq \frac{C_2 \|x\|^2}{1 - r}, \quad x \in H, \ 0 < r < 1.
\]

Theorem \(\mathbb{N}\) then gives that \(\eta(T)\) is similar to a unitary operator. It follows that \(T\) is similar to a normal operator by Lemma \(9\).

The converse direction is proved as in Theorem \(10\) \(\square\)

6. Some comments and examples

So far we have only discussed operators with spectrum on some smooth curve in \(\mathbb{C}\). Similar results can be presented if the spectrum is allowed to be a union of a smooth curve and a sequence of points tending to this curve. In [3], Benamara and Nikolski show that a contraction \(T\) with finite defects is similar to a normal operator if and only if \(\sigma(T) \neq \mathbb{D}\) and \(\|(T - \lambda)^{-1}\| \leq C \text{dist}(\lambda, \sigma(T))^{-1}\) for all \(\lambda \in \mathbb{C} \setminus \sigma(T)\). For such a contraction, \(\sigma(T) \setminus \mathbb{T}\) is always a Blaschke sequence in \(\mathbb{D}\). Moreover, Benamara and Nikolski prove that the resolvent estimate forces \(\sigma(T) \cap \mathbb{D}\) to be quite sparse (more precisely, it has to satisfy the \(\Delta\)-Carleson condition). Later in [25], Kupin studied contractions with infinite defects. He proved that if the spectrum of a contraction \(T\) is not all \(\mathbb{D}\), and if it satisfies both \(\|(T - \lambda)^{-1}\| \leq C \text{dist}(\lambda, \sigma(T))^{-1}\) for all \(\lambda \in \sigma(T)\) and the so-called Uniform Trace Boundedness condition, then it is similar to a normal operator. An analogue of Uniform Trace Boundedness condition for dissipative operators was given by Vasyunin and Kupin in [54], and then applied to integral operators. In [26], Kupin also uses the Uniform Trace Boundedness condition to give conditions for an operator similar to a contraction to be similar to a normal operator.

On the other hand, Kupin and Treil showed in [27] that if \(T\) is a contraction with \(\sigma(T) \neq \mathbb{D}\) and \(\|(T - \lambda)^{-1}\| \leq C \text{dist}(\lambda, \sigma(T))^{-1}\) but one only assumes that \(I - T^*T\) is trace class (instead of finite rank), then \(T\) need not be similar to a normal operator, thus solving a conjecture in [3].

All these results concern operators with \(\text{thin}\) spectrum (in other words, those for which the area of the spectrum is zero). For operators having \(\text{thick}\) spectrum (so non-zero area), in general there is no hope of obtaining criteria for similarity to a normal operator solely in terms of resolvent operator estimates. Indeed, for any hyponormal operator \(T\), the best possible estimate \(\|(T - \lambda)^{-1}\| = \text{dist}(\lambda, \sigma(T))^{-1}\) holds, and for any compact set \(F\) of positive area there exists a hyponormal operator \(T\) not similar to a normal one with \(\sigma(T) = F\).

Resolvent conditions for similarity to other classes of operators, such as selfadjoint operators or isometries, have also been considered in the literature. Faddeev gives conditions in [10] for similarity to an isometry in the case where \(\dim \ker(T^* - \lambda I) = 1\) for all \(\lambda \in \mathbb{D}\). In [41], Popescu also states several conditions for similarity to an isometry.

In [28], Malamud gives a series of abstract conditions for an operator \(A\) to be similar to a selfadjoint one. His conditions involve, in particular, resolvent estimates of the form \(\|V^{1/2}(A - \lambda)^{-1}\| \leq C |\text{Im} \lambda|^{-1/2}\), where \(V = |\text{Im} A|\). These estimates are related to Theorem \(\mathbb{N}\). He applies his results to
a triangular operator on $L^2([0, 1], d\mu)$ of the form

$$ (Af)(x) = \alpha(x)f(x) + i \int_{x}^{1} K(x, t)f(t) \, d\mu(t), $$

for an Hermitian kernel $K(x, t)$.

Naboko and Tretter used Theorem N in [36] to examine operators of the form (7), where $K(x, t) = \phi(x)\psi(t)$ (so that $K(x, t)$ need not be Hermitian) and $\phi\psi \equiv 0$. By using our criteria for curves, most likely Naboko and Tretter’s results can be extended to analogous operators on $L^2$ of a curve, though it might be rather technical.

Also concretely, such conditions have been used in the study of differential operators. For example, in [17], Faddeev and Shterenberg use a version of Theorem N in examining similarity to a selfadjoint operator for operators of the form $A = -\text{sign} x |x|^\alpha \frac{d^2}{dx^2}$, where $\alpha > -1$ and $p$ is a positive function which is bounded above and below. Their criteria were further generalized by Karabash, Kostenko and Malamud in [23].

Resolvent growth conditions for similarity to a unitary operator have also been used in the study of Toeplitz operators with unimodular symbol; see [8, 18, 41] and references therein.

Article [15] contains a discussion of the relationship between the growth of powers of an operator $T$ with $\sigma(T) \subset \overline{D}$, first order growth of its resolvent outside $D$ and the size of the set $\sigma(T) \cap T$.

The conditions for a contraction $T$ to be similar to a unitary operator in terms of the characteristic function of $T$ are well known. Given a contraction $T \in \mathcal{B}(H)$, one defines defect operators $D_T = (I - T^*T)^{1/2}$, $D_T^* = (I - TT^*)^{1/2}$ and defect spaces $\mathcal{D}_T = \overline{D_T H}$, $\mathcal{D}_T^* = \overline{D_T^* H}$. For $\lambda \in \mathbb{D}$, the characteristic function $\Theta_T(\lambda) : \mathcal{D}_T \to \mathcal{D}_T^*$ is given by

$$ \Theta_T(\lambda) = [-T + \lambda D_T^*(I - \lambda T^*)^{-1} D_T]|| \mathcal{D}_T. $$

As Sz.-Nagy and Foias proved in [50] (see also Sz.-Nagy and Foias [51, Chapter 9]), $T$ is similar to a unitary operator if and only if $\Theta_T(\lambda)$ is invertible for all $\lambda \in \mathbb{D}$ and

$$ \sup_{\lambda \in \mathbb{D}} \|\Theta_T(\lambda)^{-1}\| < \infty. $$

L.A. Saknovich extended the results of Sz.-Nagy and Foias to operators which are not necessarily contractions in [45]. Saknovich’s condition is only sufficient for similarity to a unitary operator and not necessary in general. See also Naboko [35, Theorem 12].

In a series of articles, Naboko constructed and studied a functional model for non-dissipative perturbations of self-adjoint operators. A detailed exposition of this model can be found in [34]. In that paper, the problem of existence of wave operators in this context is discussed. A functional model for perturbations of normal operators with spectrum on a curve, extending Naboko’s model, has been developed by Tikhonov in [52] and subsequent papers.

Example 1. A purely contractive function $\Theta$ can be chosen satisfying the condition (8) and the Sz.-Nagy Foias model used to construct a completely
non-unitary contraction \( T \) such that \( \Theta_T = \Theta \). Such a contraction is non-unitary and similar to a unitary operator, and so \( \sigma(T) \subset \mathbb{T} \). Hence
\[
\|(T - \lambda)^{-1}\| \leq \frac{C}{1 - |\lambda|}, \quad |\lambda| < 1.
\]
Since \( T \) is also a contraction, by von Neumann’s inequality
\[
\|(T - \lambda)^{-1}\| \leq \frac{1}{|\lambda| - 1}, \quad |\lambda| > 1.
\]
Recall that, by Stampfli’s theorem stated in the introduction, if under these conditions \( T \) satisfies
\[
\|(T - \lambda)^{-1}\| \leq \frac{1}{|\lambda| - 1}, \quad |\lambda| \neq 1,
\]
then \( T \) must be normal. Thus this yields an example of an operator which satisfies the hypotheses of Theorem 1 and is non-normal.

There are also examples of this type among the class of \( \rho \)-contractions. If \( \rho > 0 \), an operator \( T \in \mathcal{B}(H) \) is called a \( \rho \)-contraction if there is a larger Hilbert space \( K \supset H \) and a unitary \( U \in \mathcal{B}(K) \) such that
\[
T^n = \rho P_H U^n|H, \quad n = 1, 2, \ldots,
\]
where \( P_H \) denotes the orthogonal projection onto \( H \). The classes \( C_{\rho} \) of \( \rho \)-contractions are nested and increasing with \( \rho \), and the class \( C_1 \) coincides with the class of contractions.

If \( T \) is a \( \rho \)-contraction with \( \rho \geq 2 \), then
\[
\|(T - \lambda)^{-1}\| \leq \frac{1}{|\lambda| - 1}, \quad 1 < |\lambda| < \frac{\rho - 1}{\rho - 2}.
\]
(Here \( \frac{\rho - 1}{\rho - 2} = +\infty \) if \( \rho = 2 \).) Therefore, any \( \rho \)-contraction which is similar to a unitary operator also satisfies the hypotheses of Theorem 1. If \( T \) is a 2-contraction, then one may take the set \( U = \mathbb{C} \) in the hypotheses of Theorem 1. However, for a \( \rho \)-contraction with \( \rho > 2 \), \( U \) will in general be a smaller set.

It is natural to ask if there is an example of a \( \rho \)-contraction which is not a contraction and where the spectrum is contained in the unit circle. Stampfli shows that this can occur in [48, Example 2]. There, \( \rho = 2 \) and the spectrum of the operator is a single point. He proves that since the spectrum is countable, this operator must be normal.

Another example, this time with a bilateral weighed shift, is given below. Here the spectrum is the whole unit circle, and the operator is similar to a unitary, but is not normal.

Any such example must have uncountable spectrum. Consequently, it would be interesting to know for a non-normal operator \( T \) with spectrum \( \sigma(T) \) contained in a curve \( \Gamma \) and satisfying the hypotheses of Theorem 1 (so that it is similar to a normal operator), just how small \( \sigma(T) \) can be. See [15] and references therein for a discussion of some similar questions.

\textit{Example 2.} Assume that \( \alpha, \beta > 0 \), \( \max(\alpha, \beta) > 1 \), and \( \alpha^2 + \beta^2 \leq 4 \). Let \( T \) be the bilateral weighted shift \( T \) on \( \ell^2(\mathbb{Z}) \) with weights \( \{\ldots, 1, 1, (\alpha, \beta, 1, 1, \ldots)\} \),
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defined by

\[ T(\ldots, x_{-1}, x_0, x_1, x_2, \ldots) = (\ldots, x_{-2}, x_{-1}, \alpha x_0, \beta x_1, x_2, \ldots). \]

(Here \( \boxdot \) indicates the 0-th component). Then \( T \) is a 2-contraction which is not a contraction, yet is similar to a unitary operator.

Obviously, \( \| T \| = \max(\alpha, \beta, 1) > 1 \). Since \( \alpha, \beta > 0 \), the operator \( T \) is similar to the unitary bilateral shift \( U \) on \( \ell^2(\mathbb{Z}) \) with all weights equal to 1.

It remains to show that \( T \) is a 2-contraction. Recall that \( T \) is a 2-contraction if and only if

\[ \text{Re}(\theta T) \leq I, \quad |\theta| = 1. \]

Since \( \theta T \) is unitarily equivalent to \( T \) when \( |\theta| = 1 \), it is enough to check this inequality for \( \theta = 1 \).

Put \( A = 2 \text{Re} T \), and check that \( \sigma(A) \cap (2, +\infty) = \emptyset \). Since \( A \) is a finite rank perturbation of \( U + U^* \) and \( \sigma(U + U^*) = [-2, 2] \), it suffices to show that \( A \) has no eigenvalues in \( (2, +\infty) \). Assume that \( x = (x_n)_{n \in \mathbb{Z}} \) is a non-zero vector in \( \ell^2(\mathbb{Z}) \) that satisfies

\[ (A - \lambda I)x = 0 \]

for some \( \lambda > 2 \). This means that

\[ \begin{align*}
  x_n - \lambda x_{n+1} + x_{n+2} &= 0, \quad |n| \geq 2, \\
  x_{-1} - \lambda x_0 + \alpha x_1 &= 0, \\
  \alpha x_0 - \lambda x_1 + \beta x_2 &= 0, \\
  \beta x_1 - \lambda x_2 + x_3 &= 0.
\end{align*} \]

Put

\[ u_\pm = u_\pm(\lambda) = \frac{\lambda \pm \sqrt{\lambda^2 - 4}}{2}. \]

Then \( (9) \) and \( x \in \ell^2(\mathbb{Z}) \) imply that for some non-zero \( a, b \),

\[ \begin{align*}
  x_n &= au^n_-, \quad n \geq 2, \\
  x_n &= bu^n_+, \quad n \leq 0.
\end{align*} \]

The quotients \( y_n = \frac{x_{n+1}}{x_n} \) satisfy

\[ \begin{align*}
  y_n &= u_-, \quad n \geq 2, \\
  y_n &= u_+, \quad n \leq -1.
\end{align*} \]

If

\[ Fx_n - \lambda x_{n+1} + Gx_{n+2} = 0, \]

then \( y_n \) is obtained from \( y_{n+1} \) by applying the Möbius transformation \( z \mapsto \frac{Fz + \lambda}{Gz + \lambda} \), which can be encoded by the \( 2 \times 2 \) matrix \( \begin{pmatrix} 0 & F \\ -G & \lambda \end{pmatrix} \). The composition of Möbius transformations reduces to multiplying the corresponding \( 2 \times 2 \) matrices, so equations \((10) - (12)\) yield

\[ u_+(\lambda) = y_{-1} = \frac{M_{11}(\lambda)y_2 + M_{12}(\lambda)}{M_{21}(\lambda)y_2 + M_{22}(\lambda)} = \frac{M_{11}(\lambda)u_-(\lambda) + M_{12}(\lambda)}{M_{21}(\lambda)u_-(\lambda) + M_{22}(\lambda)} \]

where

\[ \begin{pmatrix} M_{11}(\lambda) & M_{12}(\lambda) \\ M_{21}(\lambda) & M_{22}(\lambda) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\alpha & \lambda \end{pmatrix} \begin{pmatrix} 0 & \alpha \\ -\beta & \lambda \end{pmatrix} \begin{pmatrix} 0 & \beta \\ -1 & \lambda \end{pmatrix}. \]
Putting
\[ f(\lambda) = u_+(\lambda)(M_{21}(\lambda)u_-(\lambda) + M_{22}(\lambda)) - (M_{11}(\lambda)u_-(\lambda) + M_{12}(\lambda)) = \lambda[(\lambda^2 - (\alpha^2 + \beta^2))u_+(\lambda) + u_-(\lambda)], \]
it follows that \( f(\lambda) = 0 \). However, since \( \lambda > 2 \), \( \alpha^2 + \beta^2 \leq 4 \), and \( u_+(\lambda) \) and \( u_-(\lambda) \) are positive, \( f(\lambda) > 0 \). This is a contradiction.

In the above argument, the numerical radius of the weighted shift \( T \) was computed by examining the spectrum of its real part. There are several works in the literature devoted to the study of the numerical radius of weighted shifts, using similar techniques. See [53] and references therein.

In [2], Andô and Takahashi proved that if an operator \( T \) is polynomially bounded and there exist an injective operator \( X \) and a unitary operator \( W \) with non-singular spectral measure with respect to the Lebesgue measure on \( T \), and such that \( XT = WX \), then \( T \) is similar to a unitary operator. Moreover, if such \( T \) is also a \( \rho \)-contraction for some \( \rho > 0 \), then \( T \) is itself unitary. This does not apply in Example 2, since the operator \( T \) is similar to the bilateral shift in \( L^2(T) \), the spectral measure of which is not singular. A similar result is contained in Mlak [31]. See Gamal’ [19] and the references therein for extensions of these results.

Example 3. One can easily construct non-normal operators which satisfy the hypotheses of Theorem 1 for a Jordan domain \( \Omega \neq \emptyset \). Let \( A \) be a non-unitary contraction which is similar to a unitary operator. Take a Riemann mapping \( \varphi : \Omega \to \mathbb{D} \) and put \( \psi = \varphi^{-1} \). The operator \( T = \varphi(A) \) is well defined and non-normal. If \( \lambda \in \mathbb{C} \setminus \Omega \), then by von Neumann’s inequality
\[ \|(T - \lambda)^{-1}\| \leq \|(\varphi - \lambda)^{-1}\|_{H^\infty(\mathbb{D})} = \text{dist}(\lambda, \Omega)^{-1}. \]
If \( \lambda \in \Omega \), the inequality
\[ \|(T - \lambda)^{-1}\| \leq C \text{dist}(\lambda, \Omega)^{-1} \]
follows from the fact that \( T \) is similar to a normal operator. The operator \( T \) satisfies the hypotheses of Theorem 1.

It is not obvious how to use a Riemann mapping in a similar manner to get a result analogous to Example 2 for a general Jordan domain \( \Omega \).

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