Asymptotic symmetries and soft charges of fractons

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The asymptotic structure of gauge theories describing fracton interactions is analyzed. Two sets of asymptotic conditions are proposed. Both encompass all known solutions, lead to finite charges and resolve the problem of the divergent energy coming from the monopole contribution. While the first set leads to the expected fracton symmetry algebra, including a dipole charge, the second set provides a soft infinite-dimensional extension of it. These soft charges provide evidence of a rich infrared structure for fracton-like theories and provide one corner of a possible fracton infrared triangle.

I. INTRODUCTION

Fractons [1, 2] are quasiparticles with limited mobility and compose a novel, at this point theoretical, phase of matter [3, 4]. Their unusual properties might be useful in the construction of quantum information storage [2, 5–7] and provide insights to a wide variety of physical fields, such as quantum field theory [8, 9] (and follow-up works), general relativity [10, 11], elasticity [12], and even holography [13, 14]. We refer to the reviews [15–17] for further applications, details, and references.

For some of these theories these remarkable properties can be traced back to the existence of an electric charge \( q \) and dipole charge \( \vec{d} \) [18, 19]

\[
q = \int \rho \, d^3x \quad \text{and} \quad \vec{d} = \int \vec{x} \rho \, d^3x,
\]

together with the conservation equation \( \dot{\rho} + \partial_i \partial_j J^{ij} = 0 \). Using these conservation laws one can infer that isolated charges are immobile, but dipoles can move in restricted ways in accordance with dipole conservation. It follows from the conservation equation that \( \dot{\vec{d}} = 0 \). A single particle with charge \( e \) and trajectory \( \vec{\gamma}(t) \), where \( \rho = e\delta(\vec{x} - \vec{\gamma}(t)) \), is therefore restricted to \( \dot{\vec{d}} = \dot{\vec{\gamma}} = 0 \), i.e., is immobile. For two opposite charges where \( \rho = e\delta(\vec{x} - \vec{\gamma}_1(t) + \vec{\gamma}_2(t)) \) dipole conservation leads to \( \dot{\vec{\gamma}}_1 = \dot{\vec{\gamma}}_2 \) which shows that they are allowed to move but only in prescribed ways.

The mediator of the interaction, the analog of the electromagnetic field, is a gauge theory of rank two tensors given by an electric(-like) field \( E^{ij} \) and magnetic(-like) field \( B^{ij} = \epsilon^{imn} \partial_m A_{n}^{j} \), see (1) for the action. The so called scalar charge gauge theory [18, 19] has a gauge symmetry \( \delta A_{ij} = \partial_i \partial_j \lambda \) and shares features with electrodynamics (abelian gauge symmetry), general relativity [11] (rank two tensors) and partially massless gravity [20] (higher derivative gauge transformations). Its relation to microscopic lattice models is reviewed in Appendix D of [19].

Motivated by recent advances in the understanding of the infrared behavior of gauge theories (see [21] for a review), we will analyze the asymptotic structure, symmetries and charges of this fracton gauge theory. This is a subtle endeavor since:

- The energy of an electric monopole, the most fundamental solution of the theory, is divergent as \( E \sim \lim_{r \to 0} r^{-1} \) [18, 19]. This infinite energy is generic and a putative unphysical feature of this theory, that emerges for any charge distribution with nonzero total charge.
- Due to the explicit \( x^i \) factor the dipole charge \( d^i \) needs to be handled with care. For large \( x^i \) one needs to assure that the dipole charge remains finite and that its action on the fields preserves the allowed asymptotic conditions.\footnote{For further insightful remarks and an interesting complementary analysis of dipole charges, see [22].}

We will show that the implementation of a consistent set of asymptotic conditions, together with a careful treatment of the boundary terms, resolves both of these issues.

For the well-definiteness of the theory, we propose asymptotic conditions that encompass all known solutions and should be thought of as describing arbitrary sources in a finite spatial region in space. This can be interpreted as isolated systems as seen from far away and means that we investigate the Coulombic, rather than the radiative, sector of the theory. Isolated systems are idealizations that allow to discuss subsystems and assign to them physical attributes (like, e.g., energy, momentum, charge, dipole charge). For some theories this seems trivial, like in Newtonian mechanics where isolated systems fall of like \( \frac{1}{r} \), however for gauge theories this is a subtle issue, since some of the to-be-believed gauge redundancies might lead to physical charges. This is indeed the
We propose asymptotic conditions for which all charges are finite and integrable and the asymptotic symmetries are indeed the expected symmetries. The problem of the divergence in the energy is resolved similarly to the infinitely long charged string in electrodynamics. Thus, with these asymptotic conditions, the theory is well-defined and possesses a finite action principle.

Besides the just described asymptotic conditions we propose a second set which infinitely extends these symmetries. In analogy to electrodynamics these novel charges can be thought of as soft charges. This provides a hint towards a “fracton infrared triangle”, which apart from the asymptotic symmetries in one corner, should be completed with a fracton memory effect and soft theorems. For a discussion of infrared triangles we refer to the review [21] (see also [24, 25]) and references therein.

This work is structured as follows. Since the asymptotic analysis is most easily done in spherical coordinates we discuss in Section II the scalar charge gauge theory and its symmetry transformations in curvilinear coordinates. Asymptotic symmetries should encompass all interesting solutions, which are reviewed in Section III. Our main results are summarized in Section IV. As a first step we provide in Section IV.1 a set of asymptotic conditions which lead to a finite-dimensional symmetry algebra and resolve all unphysical divergences of the energy and charges. In the second part IV.2 we infinitely extend these symmetries. We conclude with a discussion and outlook in Section V. As supplemental material we provide details of the transformation of the fields in Appendix A. Further details will be provided in a future work [26].

II. SCALAR CHARGE GAUGE THEORY

To understand the asymptotic structure of gauge theories with dipole symmetry we focus on the arguably simplest example, the scalar charge gauge theory [18, 19]. We work on a fixed flat non-orientable space-time in 3 + 1 dimensions, parametrized by \((t, x^i)\), and with spatial slices that are given by noncompact euclidean space \(\mathbb{R}^3\). The action in Hamiltonian form for the symmetric canonical variables \(A_{ij}\) and \(E^{ij}\) in curvilinear coordinates is given by

\[
I[A_{ij}, E^{ij}, \phi] = \int dt d^3x (E^{ij} \dot{A}_{ij} - \mathcal{H} - \phi \nabla_i \nabla_j E^{ij}),
\]

where the Hamiltonian \(H\) is given by

\[
H = \int d^3x \mathcal{H} = \int d^3x \frac{1}{2\sqrt{g}} (E_{ij} E^{ij} + B_{ij} B^{ij}).
\]

We have defined the “magnetic” field \(B^{ij} = \epsilon^{imn} \nabla_m A_{nj}\), using the Levi-Civita symbol \(\epsilon^{ijk}\), so \(B^{ij}\) as well as \(E^{ij}\) are tensor densities of weight one. The spatial indices are lowered and raised with the spatial metric \(g_{ij}\) of which \(g\) is the determinant and \(\nabla_i\) the covariant derivative. We will often use spherical coordinates \(\theta, \phi, r\) with \(d^3x = dr^2 + r^2 \sin^2 \theta \, d\theta d\phi\) where \(\gamma_{ij}\) is the metric of the round 2-sphere, which we use to raise and lower their respective indices. It has determinant \(\gamma\) and covariant derivative \(D_{A}\).

The action is invariant under the following transformations:

\[
\delta A_{ij} = T \left( \frac{1}{\sqrt{g}} E_{ij} + \nabla_i \nabla_j \phi \right),
\]

\[
\delta E^{ij} = T \left( \frac{\epsilon^{mi} n \partial_m B^{nj} + \epsilon^{mj} n \partial_m B^{ni}}{\sqrt{g}} \right) + \xi \phi
\]

\[
\delta \phi = \xi^k \partial_k \phi + \dot{\lambda},
\]

where \(T\) denotes the Lie derivative. The time evolution is parametrized by the constant \(T\) and gauge transformations by \(\lambda(t, x^i)\). They are generated by the Hamiltonian \(H\) and the constraint

\[
\mathcal{G}[\lambda] = \int d^3x \lambda \nabla_i E^{ij},
\]

respectively. In this section we will not consider boundary terms, but they will be included in Section IV. The spatial translations \(\vec{\alpha}\) and spatial rotations \(\vec{\omega}\) are parametrized by a vector \(\xi^i\) that obeys \(\nabla_i \xi_j + \nabla_j \xi_i = 0\). In spherical coordinates they are given by

\[
\xi^r = \vec{\alpha} \cdot \hat{r}
\]

\[
\xi^A = \frac{\epsilon^{AB}}{\sqrt{g}} \partial_B (\vec{\omega} \cdot \hat{r}) + \frac{1}{r} \partial^A (\vec{\alpha} \cdot \hat{r})
\]

(\(\hat{r}\) is the unit radial vector) and can be canonically generated via

\[
G[\xi^i] = \int d^3x \left( (\xi^k \epsilon^{ij} \nabla_k A_{ij} + 2 E^{ij} A_{kj} \nabla_l \xi^k) \right).
\]

This generator is not manifestly gauge invariant due to the explicit dependence on \(A_{ij}\), cf., (3). However, its on-shell value (more precisely its value on the constraint surface) is not affected by gauge transformations since the variations are proportional to the Gauss constraint \(\nabla_i \nabla_j E^{ij} = 0\). Therefore, momentum and angular momentum are physical observables.

In electrodynamics it is possible to define “improved generators” that are manifestly gauge invariant by adding a specific field-dependent gauge transformation (improved energy-momentum tensor). In the case of the momentum, this leads to the Poynting vector. For this theory, the analog of the Poynting vector would be \(S_k = E^{ij} (\partial_k A_{ij} - \partial_i A_{kj})\) (in Cartesian coordinates). This quantity, that appears in the continuity equation for the energy \(\dot{\mathcal{H}} = \partial_k S^k\), is however not conserved. It is not possible to improve the momentum with a local gauge transformation to obtain \(S_k\) [27], let us however remark that it can be accomplished using a non-local gauge transformation of the form (4) with \(\partial_\lambda = -\xi^j A_{ij}\).
One of the key properties of this theory is that it possesses no boost symmetry that mixes space and time, more precisely, this theory lives on a flat Aristotelian geometry [20, 27, 28].

III. SOLUTION SPACE

All the physically relevant solutions of the theory must be part of a consistent set of asymptotic conditions, and guide the asymptotic fall-offs that one should impose, e.g., boundary conditions for electrodynamics should accommodate the Coulomb solution. We summarize the for our use relevant solutions of the scalar charge theory [18].

A particular feature of this theory is that solutions sourced by isolated point charged particles have a slower radial fall-off than for example in electrodynamics. This is the root of the divergence in the energy. Furthermore, the dipole solutions play a fundamental role for the asymptotic analysis due to the presence of conserved dipole charges in this theory.

We start with the simplest solution, an isolated static monopole with charge $e$ sitting at the origin. In spherical coordinates the nonzero fields for this electric monopole are given by

$$E_{mono}^{AB} = \frac{\sqrt{\gamma}^{AB} e}{8\pi r} \quad \phi_{mono} = -\frac{e}{8\pi r}.$$  

A particular property of the electric monopole is that the energy is linearly divergent. Indeed, as it was pointed out in [18], for large values of $r$ the energy is given by $E = \frac{e^2}{16\pi r} + \text{finite terms}$. We will show that, with a careful treatment of boundary terms, this potentially unphysical situation can be resolved.

Another solution of interest is the (pure ideal) electric dipole $\vec{p}$ with non-vanishing fields given by

$$E_{dip}^{\gamma A} = \frac{\sqrt{\gamma} \partial^A (\vec{p} \cdot \hat{r})}{8\pi r} \quad E_{dip}^{AB} = \frac{\sqrt{\gamma}^{AB} \vec{p} \cdot \hat{r}}{8\pi r^2} \quad \phi_{dip} = \frac{1}{8\pi} \vec{p} \cdot \hat{r}.$$  

As expected, the fall-offs are subleading with respect to the electric monopole, but still strong enough to lead to non-vanishing charges.

The solution corresponding to a “magnetic particle” is given by

$$B^{rr} = \frac{\sqrt{\gamma} \vec{m} \cdot \hat{r}}{8\pi} \quad B^{Ar} = \frac{\sqrt{\gamma} \partial^A (\vec{m} \cdot \hat{r})}{16\pi r} \quad B^{AB} = -\frac{\sqrt{\gamma}^{AB} \vec{m} \cdot \hat{r}}{16\pi r^2} \quad B^{AA} = \frac{5\sqrt{\gamma} \partial^A (\vec{m} \cdot \hat{r})}{16\pi r}.$$  

In the gauge where the linear term in $r$ of $\gamma^{AB} A_{AB}$ vanishes, the nonzero components of the potential take the form

$$A_{rA} = \frac{\sqrt{\gamma}}{16\pi} \epsilon^{AB} \partial^B (\vec{m} \cdot \hat{r}) \quad A_{AB} = \frac{\sqrt{\gamma}}{16\pi r} \left[ \epsilon_{AC} D^C D_B + \epsilon_{BC} D^C D_A \right] (\vec{m} \cdot \hat{r}).$$  

IV. ASYMPTOTIC CONDITIONS AND SYMMETRIES

Asymptotic conditions describe the behavior of the fields near infinity and are of fundamental importance to determine physical symmetries of gauge theories. There is in general no unique set of asymptotic conditions however the following physical requirements must be fulfilled:

- The conditions should encompass all relevant physical solutions, in particular linear combinations of the ones described in [18] and reviewed in Section III.
- The charges (energy, momentum, angular momentum, electric and dipole) and the symplectic structure must be finite, which guarantees that the action is also finite.

As it will be seen below, this puts severe restrictions on our theory. We propose two sets of asymptotic conditions that remarkably satisfy all these consistency requirements. While the first set reproduces the expected finite-dimensional fracton symmetries, the second set extends the symmetry algebra to an infinite-dimensional one containing novel “soft charges.”

IV.1. Fracton symmetries

IV.1.1. Asymptotic conditions

The asymptotic conditions leading to a finite-dimensional symmetry algebra are given by

$$E^{rr} = E^{rr}(0) + O(r^{-1}) \quad (5a)$$
$$E^{rA} = \frac{E^{rA}(1)}{r} + O(r^{-2}) \quad (5b)$$
$$E^{AB} = \frac{\sqrt{\gamma}^{AB} q}{8\pi r} + \frac{E^{AB}(2)}{r^2} + O(r^{-3}) \quad (5c)$$
$$A_{rr} = A_{rr}(1) + O(r^{-2}) \quad (5d)$$
$$A_{rA} = A_{rA}(0) + O(r^{-1}) \quad (5e)$$
$$A_{AB} = A_{AB}^{(1)} r + A_{AB}^{(0)} + O(r^{-1}), \quad \gamma^{AB} A_{AB}^{(1)} = 0 \quad (5f)$$
$$\phi = \left( \vec{q}^{(1)} \cdot \hat{r} - \frac{q}{8\pi} \right) r + \Phi^{(0)} + O(r^{-1}) \quad (5g)$$

and are preserved under time evolution, rotations, translations, and gauge transformations with the following parameter

$$\lambda = (\hat{\chi}^{(1)} \cdot \hat{r}) r + \chi^{(0)} + O(r^{-1}). \quad (5h)$$

2 According to Geroch [23]: There are no “correct” or “incorrect” definitions, only more or less useful ones. It is perfectly possible that there turn out to be a number of competing definitions, applicable to differing physical systems, or a single definition as in Newtonian gravitation, or none at all.
Here, \( q \) (the sum of all monopole charges), \( \Phi^{(0)}, \lambda^{(0)}, \vec{p}^{(1)} \) and \( \vec{A}^{(1)} \) are constants with respect to the angles. The remaining terms are functions on the sphere and have to satisfy the following parity conditions

\[
\begin{align*}
E_{\ell(-1)}, E_{\ell(-2)}, A_{\ell r \phi}^{(0)}, A_{\ell \phi \phi}^{(1)} & \quad \text{parity even} \\
E_{\ell(-1)}, E_{\ell(-2)}, A_{\ell r \phi}^{(0)}, A_{\ell \phi \phi}^{(1)}, A^{(0)} & \quad \text{parity odd}
\end{align*}
\]

where we denote traces with respect to the sphere, like \( A^{(0)} \), as \( X = \gamma^{AB} X_{AB} \). The tracefree part of \( A^{(0)} \) is unconstrained. The parity conditions for the fields \( E_{\ell r}^{(0)} \) and \( A^{(-1)} \) cannot be inferred from the known solutions. However, in order to guarantee a finite symplectic term, they must have opposite parity, or at least one of them must vanish. All of these conditions are fully consistent with the preservation of the asymptotic symmetries.

The form of the asymptotic conditions in (5) guarantee that the charges and the action principle do not possess divergences in the large \( r \) limit. For example, the leading term of \( E^{AB} \) in (5c) must only have a trace part in order to cancel the linear divergence appearing in the bulk Hamiltonian, with the one coming from the boundary term of the Gauss constraint. In particular, the shift that was done in the leading order of \( \phi \) in (5g) is of fundamental importance for this purpose, as it will be explained in detail below. This shift is also necessary to accommodate the monopole solution within the asymptotic expansion. Note that a constant \( q \) is compatible with the leading term of the Gauss constraint, \( D^A D_A q = 0 \), as it must.

The condition \( A^{(1)} = 0 \) in (5f) removes a linear and a logarithmic divergence in the symplectic term. The preservation of this condition restricts the leading orders of \( \phi \) and \( \lambda \) to take the form exhibited in (5g) and (5h). In the next order, \( \lambda^{(0)} \) and \( \Phi^{(0)} \) could in principle have higher modes in the spherical harmonic expansion, however the modes with \( \ell \geq 1 \) do not appear neither in the charges nor in the boundary term of the action principle. Therefore they are “pure gauge” and can be consistently discarded.

The parity conditions are necessary to remove additional logarithmic divergences appearing in the symplectic term, as well as in the boundary terms associated with the Gauss constraint. They have a long history in the Hamiltonian formulation of general relativity and electrodynamics [29–31] and it should thus not come as a surprise that they are also needed for fracton theories.

### IV.1.2. Conserved charges

The charges associated with gauge transformations are obtained from the boundary term of the Gauss constraint [29] (see also [32]) and are given by

\[
Q = \int d^3 x \partial_\ell (\partial_\ell \lambda E^{ij} - \lambda \nabla_\ell E^{ij}) = \lambda^{(0)} q + \vec{\lambda}^{(1)} \cdot \vec{q},
\]

where \( q \) corresponds to the total electric charge and

\[
\vec{q} = \int d^3 x \vec{r} \left( E_{\ell r}^{(0)} + E_{\ell r}^{(-2)} - 2 D_A E_{\ell r}^{A} \right),
\]

corresponds to the total dipole charge. As expected for gauge symmetries the charges are surface terms integrated at infinity.

#### IV.1.3. Finiteness of the energy

As was explained previously, the energy (2) diverges linearly in \( r \) for the monopole solution [18, 19]. This is also true for the asymptotic conditions. Indeed, if we evaluate the Hamiltonian \( H \) in (2) using our boundary conditions (5), the following divergence is obtained

\[
H = \lim_{r \to \infty} r \int d^3 x \sqrt{\gamma} \left( \frac{q}{8 \pi} \right)^2 + \text{finite terms}. \quad (6)
\]

We will show that a careful treatment of the boundary terms in the total Hamiltonian [33, 34], completely removes the divergence for any physical configuration that fulfills our asymptotic conditions.

Let us consider the total Hamiltonian that includes the constraints, in the sense introduced by Dirac [33, 34],

\[
H_T = H + \int d^3 x \phi \nabla_i \nabla_j E^{ij} + B_\infty. \quad (7)
\]

Here \( B_\infty \) is the boundary term needed to guarantee that the generator has well-defined functional derivatives [29] and has a variation of the form

\[
\delta B_\infty = \int d^3 x \partial_i (\partial_j \phi \delta E^{ij} - \phi \nabla_j \delta E^{ij}).
\]

Using (5) and by virtue of the shift in the leading order of \( \phi \) in (5g), this expression can be integrated in field space and acquires a linear divergence given by

\[
B_\infty = - \lim_{r \to \infty} r \int d^2 x \sqrt{\gamma} \left( \frac{q}{8 \pi} \right)^2. \quad (8)
\]

In the total Hamiltonian (7), \( B_\infty \) precisely cancels the divergence coming from \( H \), c.f., (6). Therefore the total Hamiltonian is finite and provides a well-defined notion of energy for any distribution of charges.\(^3\)

Thus, the final expression for the finite energy takes the following form

\[
E_{\text{finite}} = H - \lim_{r \to \infty} r \int d^2 x \sqrt{\gamma} \left( \frac{q}{8 \pi} \right)^2. \quad (8)
\]

\(^3\) A similar method was used in [35, 36] to regularize the energy of the charged black hole in three-dimensional gravity.
IV.1.4. Fracton symmetry algebra

The symmetry algebra can be obtained directly from the Dirac brackets and is spanned by the generators of rotations $J_1$, translations $P_I$ (that do not need to be improved by boundary terms), the energy $E_{\text{finite}}$ which is a trivial central extension, and by the generators $q$ and $d_I$ that correspond to the electric charge and dipole moment and are associated with the large gauge symmetries. The non-vanishing commutators are given by

\[
\begin{align*}
\{J_I, J_J\} &= \epsilon_{IJK} J_K \\
\{J_I, P_J\} &= \epsilon_{IJK} P_K \\
\{J_I, d_J\} &= \epsilon_{IJK} d_K \\
\{P_I, J_J\} &= \epsilon_{IJK} P_K \\
\{P_I, d_J\} &= \delta_{IJ} q.
\end{align*}
\]

Here $I, J, K = 1, 2, 3$ denote the Cartesian components of the generators. For a theory with conserved dipole moment this is precisely the algebra one expects. Let us emphasize that without the regularized energy this symmetry algebra would not be well defined.

IV.2. Extended fracton symmetries

In recent years, deep relations between the asymptotic structure of gauge theories and their infrared behavior have been highlighted (see, e.g., [21] for a review). One of the cornerstones of this relation is the existence of infinite-dimensional symmetries that contain additional “soft charges.” Therefore, one might wonder if there exists an infinite-dimensional extension of the fracton symmetries (9), similar to the extension of the Poincaré algebra, to the Bondi-Metzner-Sachs algebra [37, 38]. Inspired by [30, 31, 39] we construct an alternative set of asymptotic conditions that allows such extension.

IV.2.1. Asymptotic conditions

With respect to the asymptotic conditions (5) we adapt the following lines

\[
\begin{align*}
E^{AB} &= \frac{\sqrt{\gamma}^{AB}}{r} \hat{q} + \frac{\sqrt{\gamma}^{AB}}{r} \hat{p} \cdot \hat{r} + O(r^{-3}) \\
A_{AB} &= A_{AB}^{(1)} + A_{AB}^{(0)} + O(r^{-1}) \\
\phi &= \left( \Phi^{(1)} - \frac{q}{8\pi} \right) r + \Phi^{(0)} + O(r^{-1}) \\
\lambda &= \lambda^{(1)} r + \lambda^{(0)} + O(r^{-1})
\end{align*}
\]

where $q, \hat{r}$ are constant with respect to the angles. Additionally, we have to impose the following spherical dependence ($\ell$ denotes the degree of the spherical harmonic function $Y_{\ell,m}(\theta, \phi)$)

\[
\begin{align*}
A^{(0)}, \Phi^{(1)}, \lambda^{(1)} \quad &\ell \geq 1 \\
A^{(1)} \quad &\ell \geq 2 \\
E^{\theta}_{(-1)}, A^{(0)}_{r\theta} \quad &\text{parity even} \\
E^{\phi}_{(-1)}, A^{(0)}_{r\phi} \quad &\text{parity odd}.
\end{align*}
\]

We set $\Phi^{(0)}$ and $\lambda^{(0)}$ to be constant, they could in principle have higher modes which are however pure gauge. As in the previous case, $E^{rr}_{(0)}$ and $A^{(-1)}_{rr}$ must have opposite parity, or at least one of them must vanish. With the above conditions, the symplectic term and the charges are finite. Furthermore, the divergence in the energy coming from the bulk Hamiltonian is removed exactly like in (8).

IV.2.2. Conserved charges

If we expand the parameter $\lambda^{(1)}$ in spherical harmonics

\[
\lambda^{(1)} = \bar{\lambda}^{(1)} + \sum_{\ell \geq 2} \frac{\lambda_{\ell}}{\ell} Y_{\ell,m}
\]

then the charge associated with gauge transformations takes the form

\[
Q = \lambda^{(0)} + \bar{\lambda}^{(1)} + \sum_{\ell \geq 2} \frac{\lambda_{\ell}}{\ell} Y_{\ell,m}
\]

where

\[
\bar{\lambda}^{(0)} = \frac{1}{3} \hat{p} + \oint d^2 x \hat{r} \left( \frac{E^{rr}_{(-0)}}{2} - 2DA_{E^{rr}}(-1) \right)
\]

\[
Q_{\ell,m} = \oint d^2 x Y_{\ell,m} \left( \frac{E^{rr}_{(-0)}}{2} - 2DA_{E^{rr}}(-1) \right)
\]

As in the previous case, $q$ and $\bar{\lambda}$ correspond to the electric and dipole charges, respectively. In addition, there is an infinite tower of new charges characterized by multipoles with $\ell \geq 2$. We call them “soft charges” by analogy with the infinite-dimensional extension of the asymptotic symmetry algebra in electrodynamics and general relativity.

IV.2.3. Extended fracton symmetry algebra

The symmetry algebra is given by the non-vanishing Poisson brackets (9), together with the extension

\[
\{J_I, Q_{\ell,m}\} = \sum_{m', \ell - |m| \leq \ell} (D^\ell_{m'm})_{ij} Q_{J_{i}, m'} \quad (\ell \geq 2, |m| \leq \ell)
\]

where $D^\ell_{m'm}$ is the Wigner $D$-matrix that rotates the spherical harmonics. Note that the soft charges commute with the translation generator, in contrast to the dipole charge. This property can be seen as the imprint of their soft nature.\(^4\)

\(^4\) For fracton-like theories the conserved higher multipole charges generically do not commute with the translations (see, e.g., [40]) and are therefore not soft.
V. DISCUSSION AND OUTLOOK

This work provides the first asymptotic analysis of a theory with conserved dipole charge and suggests the existence of a rich infrared structure. The first set of asymptotic conditions, provided in Section IV.1.1 leads to the expected finite-dimensional fracton algebra (9). The second set of asymptotic conditions, see Section IV.2.1, provides an infinite-dimensional “soft” extension of the dipole charges, cf., (12). In both cases a careful analysis of the boundary terms was of fundamental importance in order to obtain a finite energy and a well-defined action principle.

In a subtle way the extended asymptotic conditions (10) also encompass the more restricted ones (5). If in (10) the condition \(A^{(1)} = 0\) is imposed, the algebra truncates to the finite-dimensional one in (9). Comparing (5c) with (10a) it seems that there is more freedom in the restricted asymptotic conditions. This freedom is however irrelevant for the asymptotic symmetries since only the trace part of \(E^{AB}_{(-2)}\) contributes to the charges.

We have seen that the electric charge, in contradistinction to the dipole charge, stays unextended. This is related to the requirement of finite energy from which the specific form of the first term on the right hand side of (5c) follows. Together with the constraint of the theory this leads to a charge that is independent of the angles. Dropping these restrictions opens the possibility to also extend the charge sector (at the prize of an infinite energy).

The present work opens various avenues for further research. One is the generalization of this analysis to other interesting models like the traceless scalar charge theory, their vector generalizations and beyond, see, e.g., [18, 19, 40–55]. Another possibility is to allow for curved space(-time) [20, 27, 56]. It would also be interesting to study the consequences of physical boundaries at finite distances and their effect on the symmetries of the system. In each case new phenomena and features should emerge.

We have focused on the Coulombic sector, but this theory also allows for radiative modes that could make it possible to connect the soft charges, possibly via vacuum transitions, to a putative fracton memory effect. This is one way to argue for the measurable consequences of the novel soft charges. Besides the discussed asymptotic symmetries, the memory effect and soft theorems could provide the three corners of a novel fracton infrared triangle that remains to be explored and could lead to interesting applications in condensed matter and high energy physics.

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Appendix A: Transformations of the fields

The transformations of the expanded fields are, using \(Y^A = \frac{\sqrt{AB}}{\sqrt{L}} \partial_B (\tilde{\omega} \cdot \hat{r})\) and \(\Delta = DA^A\), given by

\[
\delta q = 0
\]

\[
\delta E^A(0) = \mathcal{L}_Y E^A(0) - \frac{q}{8\pi} \sqrt{\gamma} \partial_A (\tilde{\alpha} \cdot \hat{r})
\]

\[
\delta E^A(-1) = \mathcal{L}_Y E^A(-1) - \frac{q}{8\pi} \sqrt{\gamma} \partial_A (\tilde{\alpha} \cdot \hat{r})
\]

\[
\delta E^{AB}(-2) = \mathcal{L}_Y E^{AB}(-2) - \sqrt{\gamma} \partial_A (\tilde{\alpha} \cdot \hat{r})
\]

\[
\delta A^{(1)} = \mathcal{L}_Y A^{(1)} + T(\Delta + 2) \Phi^{(1)} + (\Delta + 2) \lambda^{(1)}
\]

\[
\delta A^{(0)} = \mathcal{L}_Y A^{(0)} + 2 \partial^A (\tilde{\alpha} \cdot \hat{r}) A^{(1)}_\gamma
\]

\[
+ \left( \Delta \Phi^{(0)} + \frac{1}{\sqrt{\gamma}} E(-2) \right) + \Delta \lambda^{(0)}
\]

\[
\delta A^{(1)}_{AB} = \mathcal{L}_Y A^{(1)}_{AB} + T \left( D_A D_B - \frac{1}{2} \gamma_{AB} \Delta \right) \Phi^{(1)} + \left( D_A D_B - \frac{1}{2} \gamma_{AB} \Delta \right) \lambda^{(1)}
\]

\[
\delta A^{(-1)}_{rA} = \mathcal{L}_Y A^{(-1)}_{rA}
\]

\[
\delta A^{(0)}_{rA} = \mathcal{L}_Y A^{(0)}_{rA}
\]


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