Multiparticle production in the mean field approximation of high density QCD

Andrey Kormilitzin, Eugene Levin and Alex Prygarin

Department of Particle Physics, School of Physics and Astronomy
Raymond and Beverly Sackler Faculty of Exact Science
Tel Aviv University, Tel Aviv, 69978, Israel

ABSTRACT: The generating functional is suggested for multiparticle generation processes. In mean field approximation of high density QCD two equations for new generating functional are derived: linear functional equation for an arbitrary initial condition and non-linear one for a specific initial condition. The non-linear equation has the form of Kovchegov-Levin equation for diffraction production and gives its generalization on the processes with fixed multiplicities of produced particles.

KEYWORDS: Colour dipole model, inclusive production, jet production, BK equation, BFKL equation, Kovchegov-Levin equation.
1. Introduction: AGK cutting rules

The goal of this paper is to develop a technique that will allow us to calculate more exclusive processes than the total cross section in the framework of high density QCD. We will work in the mean field approximation in which we have Balitsky-Kovchegov non-linear evolution equation for the elastic high energy amplitude. Except the obvious motivation for expanding the calculative power of high density QCD we have a more specific reason for doing this paper: to give more transparent derivation of the Kovchegov-Levin equation for the diffractive dissociation cross section in the mean field approximation.

The tool which we are going to use, is the AGK cutting rules. In high energy phenomenology these rules are very useful in calculation of the processes of different multiplicity from the expression for the total cross section. Since we have a very powerful framework for high density phenomena in QCD, namely, the dipole approach, we wish to expand this approach to multiparticle production. The AGK cutting rules, being proved in QCD, will allow us to approach such exclusive processes as diffractive production and different correlations in multiparticle production processes. For a long time the situation with the proof of the AGK cutting rules has been uncertain (see Refs. [1, 9, 10, 11, 12]). At first sight in the leading log(1/x_Bjorken) approximation of perturbative QCD the scattering amplitude, as a function of particle masses, decreases enough to apply the original arguments of the AGK paper, on the other hand, the main question whether the total cross section and the multiparticle production can be described by the same set of diagrams, remains unanswered. The situation became even worse when the explicit violation of the AGK cutting rules were found in Refs. [10, 11, 12]. Fortunately, we believe that the mess with AGK cutting rules in QCD has been resolved in Ref. [14]. However, before describing the main results of this paper which we will use below, we give an introduction to AGK cutting rules explaining what they claim.

\[^1^\]One of us (E.L.), has heard a lot of complaints from experts that Ref. [1] is impossible to understand.
In simple language, the AGK cutting rules give us the relation between the total cross section at high energy and the processes of multiparticle production. In QCD the main idea stems from the unitarity constraint for the BFKL Pomeron \cite{13}, which describes the high energy scattering amplitude in the Leading Log \((1/x)\) Approximation of perturbative QCD. The unitarity reads as

\[ 2N(Y; x, y) = |N(Y; x, y)|^2 + G_{in}(Y; x, y) \]  

(1.1)

where \(Y = \log(1/x_{Bjorken})\) and \((x, y)\) are the coordinates of the incoming dipole; \(N(Y; x, y)\) is the imaginary part of the elastic amplitude and the first term describes the elastic scattering (assuming that the real part of the amplitude is small at high energy), while the second term stands for the contribution of all inelastic processes. In the leading log \((1/x)\) approximation the elastic contribution can be neglected and for the BFKL Pomeron Eq. (1.1) can be reduced to the form (see Fig. 1):

\[ 2N_{BFKL}(Y; x, y) = G_{in}^{BFKL}(Y; x, y) \]  

(1.2)

We call \(G_{in}(Y; x, y)\) by cut Pomeron while \(N_{BFKL}(Y; x, y)\) will be called Pomeron or uncut Pomeron.

The original AGK cutting rules state that if we know the contribution to the total cross section of the exchange of any number of Pomerons we can calculate the processes with different multiplicity. In Fig. 2 you can see the simple triple Pomeron diagram with the AGK coefficients (see Fig. 2-a). The coefficients mean that you need to multiply by these coefficients the contribution of triple Pomeron diagram in the total cross section to obtain the cross section of the multiparticle production. In Ref. \cite{14} it was proved that these AGK cutting rules are correct in QCD for the triple BFKL Pomeron vertex.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1}
\caption{The definition of cut Pomeron through the BFKL ladder.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2}
\caption{The AGK cutting rules for the total inclusive processes. Cut Pomeron is defined in Fig. [1] and in Eq. (1.2). In Fig. [2] one can see decoding of the Pomeron diagrams in terms of the production processes.}
\end{figure}
The exact form of the AGK rules look as follows.

**Double Cut:**
\[
\frac{\alpha_s}{2\pi} \int d^2x K(x_{10}|x_{02}, x_{12}) \left\{ 2 \left( N_{BFKL}(12) + N_{BFKL}(02) \right)^2 - 2 N_{BFKL}^2(10) \right\} \quad (1.3)
\]

**Single Cut:**
\[
\frac{\alpha_s}{2\pi} \int d^2x K(x_{10}|x_{02}, x_{12}) \left\{ -4 \left( N_{BFKL}(12) + N_{BFKL}(02) \right)^2 + 4 N_{BFKL}^2(10) \right\} \quad (1.4)
\]

**Diffractive Cut:**
\[
\frac{\alpha_s}{2\pi} \int d^2x K(x_{10}|x_{02}, x_{12}) \left\{ (N_{BFKL}(12) + N_{BFKL}(02))^2 - N_{BFKL}^2(10) \right\} \quad (1.5)
\]

where the vertex of decay of one dipole \( x_{10} \) to two dipoles with sizes \( x_{02} \) and \( x_{12} \) is equal to
\[
\frac{\alpha_s}{2\pi} K(x_{10}|x_{02}, x_{12}) = \frac{\alpha_s}{2\pi} \frac{x_{10}^2}{x_{02}^2 x_{12}^2} \quad (1.6)
\]

\( N_{BFKL}(x_{ik}) \) is the scattering amplitude of the dipole with size \( x_{ik} \) off the target in the leading \( \log(1/x_{Bjorken}) \) approximation of perturbative QCD (the BFKL Pomeron exchange). It is clear from the above equations that they have a simple meaning: the dipole with size \( x_{10} \) decays in two dipoles with sizes \( x_{02} \) and \( x_{12} \) in the case when both of them interact with the target with multiparticle production (double cut of Eq. (1.3)); one interacts with the target with multiparticle production while the second one interacts elastically (single cut of Eq. (1.4)); and two produced dipoles scatter elastically (diffractive cut of Eq. (1.5)).

It turns out that difficulties with the AGK cutting rules that has been discussed, stem from the fact that in QCD the AGK cutting rules do not work for the situation when we an extra gluon is emitted from the vertices (see Fig. 3). The difference between these two cases are due to the fact that for multiparticle production without measuring the gluon, is described by the same set of the diagrams as the total cross section, while for the events with measured gluon the set of diagrams turns out to be quite different (see Ref. [14]).

In the next section we will derive the generating functional for multiparticle production in the dipole approach. Therefore, for the dipole approach to QCD we will repeat the program that has been worked out for the BFKL Pomeron Calculus in zero transverse dimension [15]. In section 3 we derive the non-linear equation for the generating functional and for the cross sections of multiparticle production. These equations will lead to a natural generalization of the Kovchegov-Levin equation for the diffractive production cross section (see Ref. [6]). In conclusions we summarize our results.

### 2. Generating functional for multiparticle production processes

#### 2.1 Generating functional for total cross section

In the mean field approximation (MFA) we take into account only one Pomeron to two Pomeron splitting neglecting merging of two Pomerons into one Pomeron. In other words, we consider only ‘fan’ diagrams. The simple process
for which such approximation can work, is the deep inelastic scattering with a nuclear target (see Ref. 5). In the dipole approach, one Pomeron to two Pomeron splitting reduces to the decay of one dipole to two dipoles with the probability of this decay given by Eq. (4.6) 6. The simplest and the most transparent technique to incorporate this decay is the generating functional which allows us to reduce the calculation of the high energy elastic amplitude to consideration of a Markov process. The advantage of this approach is the fact that it takes into account both t and s-channel unitarity as we will discuss below.

The generating functional is defined as 8, 16

\[ Z_0 (Y - y; \{ u \}) = \sum_{n=1} \int P_n (Y - y; r_1, \ldots, r_n) \prod_{i=1}^{n} u(r_i) d^2 r_i \]

(2.1)

where \( u(r_i) \) is an arbitrary function of \( r_i \) and \( b_i \). \( P_n \) is the probability density to find \( n \) dipoles with sizes \( r_1, \ldots, r_n \) at rapidity \( Y - y \). For functional of Eq. (2.1) we have two obvious conditions:

- at \( Y = y \) \( P_n = 0 \) for \( n > 1 \) and \( P_1 = \delta (r - r_1) \). In other words, at \( y = Y \) we have one dipole of size \( r \)

\[ Z_0 (Y - y = 0; \{ u \}) = u(r) ; \]

(2.2)

- at \( u = 1 \)

\[ Z_0 (Y - y; \{ u \}) |_{u=1} = 1. \]

(2.3)

Eq. (2.3) follows from the physical meaning of \( P_n \) and represents the conservation of the total probability.

For probabilities \( P_n \) we can write the following equation

\[- \frac{dP_n (Y - y; r_1, \ldots, r_n)}{dy} = \]

(2.4)

\[- \frac{\alpha_s}{2\pi} \sum_{j=1}^{n} \int d^2 r' K \left( r_j; r', |\vec{r}_j - \vec{r}'| \right) P_n (Y - y; r_1, \ldots, r_j, \ldots, r_n) \]

\[+ \frac{\alpha_s}{2\pi} \sum_{j=1}^{n-1} \int d^2 r' d^2 \tilde{r}_j \delta (\vec{r}_j - \vec{r}_i + \vec{r}_n) \delta (\vec{r}_i - \vec{r}_n) K \left( \tilde{r}_j; r', |\tilde{r}_j - \vec{r}'| \right) P_{n-1} (Y - y; r_1, \ldots, \tilde{r}_j, \ldots, r_{n-1}) \]

Eq. (2.4) describes a typical Markov process: two terms of this equation has simple meaning of increase of the probability to find \( n \)-dipoles due to decay of one dipole to two dipoles (birth terms, the second term in Eq. (2.4)) and of decrease of the probability since one of \( n \)-dipoles can decay (death term, the first term in Eq. (2.4)). Multiplying by product \( \prod_{i=1}^{n} u(r_i) \) and integrating over \( r_i \) we obtain the following linear equation for the generating functional

\[ \frac{dZ_0 (Y - y; \{ u \})}{dy} = \frac{\alpha_s}{2\pi} \int d^2 r d^2 r' K \left( \vec{r}; r', |\vec{r} - \vec{r}'| \right) \left\{ -u(r) + u(r') u(|\vec{r} - \vec{r}'|) \right\} \frac{\delta}{\delta u(r)} Z_0 (Y - y; \{ u \}) \]

(2.5)

Here we use notation \( \delta / \delta u(r) \) for the functional derivative. Using the generating functional we can calculate the scattering amplitude \( N(Y, r, b) \) as follows 16.

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2The dipole \((x_i, y_i)\) with coordinates \(x_i\) for quark and \(y_i\) for antiquark can be characterized by the dipole size \(\vec{r}_i = \vec{x}_i - \vec{y}_i\) and \(b_i = \frac{1}{2}(\vec{x}_i + \vec{y}_i)\). For simplicity we suppress in Eq. (2.4) and below the coordinate \(b_i\). For the scattering with the nuclear target we can consider that impact parameters of all dipoles are the same \(b_i = b\) (see Ref. 5).
\[ N(Y - Y_0, r, b) = -\infty \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} \left\{ \int d^2r_1 \gamma(r_1) \frac{\delta}{\delta u(r_1)} \right\} Z_0(Y - Y_0; \{u\}) |_{u=1} \]

\[ = 1 - Z_0(Y - Y_0; \{u\}) |_{u=1-\gamma} \tag{2.6} \]

where \( \gamma(r) = N(Y_0, r, b) \) is the dipole scattering amplitude at low energy \( (Y_0) \). The notation \( Z_0(Y - Y_0; \{u\}) |_{u=1} \) means that the initial condition of the generating functional is given by Eq. (2.2). In a more general case we can define the scattering amplitude for any arbitrary initial condition \( F(u(r)) \), for example, if we choose \( F(u(r)) = u^2(r) \), we start with two dipoles at the initial rapidity \( Y \). In Eq. (2.6) we assume that the low energy amplitude of interaction of \( n \) dipoles with the target \( \gamma_n \) is equal to \( \gamma_n = \gamma^n \), which means that dipoles interact with the target independently (without correlations).

Eq. (2.6) being a linear equation with only first derivative has a general solution of the form \( Z_0(Y - y; \{u\}) \equiv Z_0(\{u(Y - y)\}) \). Inserting this solution and using the initial condition of Eq. (2.2) we obtain the non-linear equation

\[ \frac{dZ_0(Y - y; \{u\}) |_{u(r)}}{dy} = \frac{\bar{\alpha}_s}{2\pi} \int d^2r' K \left(r; r', |\vec{r} - \vec{r}'|\right) \left\{ -Z_0(Y - y, \{u\}) |_{u(r')} + Z_0(Y - y, \{u\}) |_{u(r')} Z_0(Y - y, \{u\}) |_{u(r - \vec{r}')}) \right\} \tag{2.7} \]

It is easy to see that using Eq. (2.7) for the amplitude Eq. (2.6) can be re-written as the Balitsky-Kovchegov equation in the form

\[ \frac{dN(Y - y, r, b)}{dY - y} = \frac{\bar{\alpha}_s}{2\pi} \int d^2r' K \left(r; r', |\vec{r} - \vec{r}'|\right) \times \tag{2.8} \]

\[ \times \left\{ N(Y - y, r', b) + N(Y - y, |\vec{r} - \vec{r}'|, b) - N(Y - y, r, b) - N(Y - y, r', b) N(Y - y, |\vec{r} - \vec{r}'|, b) \right\} \]

The evolution equation in its linear form Eq. (2.6) has two advantages in comparison with the non-linear equations Eq. (2.7): it has a simple statistical interpretation and can be solved with arbitrary initial condition while non-linear equation are correct only with specific initial condition of Eq. (2.2). The statistical interpretation allows us to reduce the problem of s-channel unitarity to the conservation of probabilities at each level of rapidity \( y \) which is included in Markov chain equation (see Eq. (2.4)). Since the technique is equivalent to summing of the Pomeron diagrams the t-channel unitarity is preserved, but we need to remember that we must sum all Pomeron diagrams to fulfill t-channel unitarity including the so called Pomeron loops.

The equivalence between the generating functional approach and the Pomeron Calculus becomes clear if we introduce a new functional

\[ N(Y - y, \{\gamma\}) = 1 - Z_0(Y - y; \{u\}) |_{u=1-\gamma} \tag{2.10} \]

for which we have the equation

\[ \frac{dN(Y - y; \{\gamma\})}{dY - y} = \frac{\bar{\alpha}_s}{2\pi} \int d^2r d^2r' K \left(r, r', |\vec{r} - \vec{r}'|\right) \left\{ \gamma(r') + \gamma(\vec{r} - \vec{r}') - \gamma(r) - \gamma(r') \right\} \delta_{\gamma(r')} N(Y - y, \{\gamma\}) \tag{2.11} \]

Starting with the initial condition

\[ N(Y_0; \{\gamma\}) = \gamma(r) \tag{2.12} \]

found from Eq. (2.2) and Eq. (2.7), one can easily built the Pomeron Calculus iterating Eq. (2.11).
2.2 Generating functional for multiparticle production: general ideas

In the spirit of Ref. [15] we wish to introduce a generalization of Eq. (2.1), namely,

\[
Z (Y - y; \{ u \}, \{ v \}) \equiv \sum_{n=1, m=0}^{\infty} \int P_n^m (y; r_1, \ldots, r_n; r_1, \ldots, r_m) \prod_{i=1}^{n} u(r_i) \prod_{k=1}^{m} v(r_k) d^2 r_i d^2 r_k \tag{2.13}
\]

where \( P_n^m \) is the probability to find \( n \) uncut Pomerons and \( m \) cut Pomerons. In our discussion we use the fact that colorless dipoles are proper degrees of freedom for partonic description of the BFKL Pomeron. To treat this probability in terms of the dipole approach we need to introduce two typical moments of time for the processes of multiparticle production: \( \tau = 0 \) when the interaction with the target occurs, and \( \tau = \infty \) where our detectors for particles are located. Having in mind these two moments of time we can treat \( P_n^m \) as the probability to have at rapidity \( y \) \( n \) dipoles with sizes \( r_1, \ldots, r_n \) at \( \tau = 0 \), which do not survive until \( \tau = \infty \) and cannot be measured, while the dipoles with sizes \( r_1, \ldots, r_m \) reach \( \tau = \infty \) and can be caught by detectors.

The first boundary condition for the generating functional is obvious

\[
Z (Y - y; \{ u \}, \{ v \}) |_{u=1, v=1} = 1 \tag{2.14}
\]

and it follows from the physical meaning of \( P_n^m \) and represents the conservation of the total probability at any rapidity.

Generalizing Eq. (2.13) we can introduce unintegrated over impact parameter cross section for multiparticle production as

\[
M (Y - Y_0; r, b) \equiv \sum_{n=1, m=0}^{\infty} \frac{(-1)^{n+m+1}}{n! m!} \left\{ \int \prod_{i=1}^{n} d^2 r_i \int \prod_{k=1}^{m} d^2 r_k \gamma(r_i) \gamma_{in}(r_k) \frac{\delta}{\delta u(r_i)} \frac{\delta}{\delta v(r_k)} \right\} Z (Y - Y_0); \{u\}) |_{u=1, v=1}^{v(r)}
\]

\[
= 1 - Z (Y - Y_0; \{u\}, \{v\}) |_{u=1-\gamma, v=1-\gamma_{in}}^{v(r)}
\]

The total cross section is obtained from Eq. (2.15) substituting \( 2\gamma(r) = \gamma_{in}(r) \), where \( \gamma(r) \) is the low energy \( (Y_0) \) amplitude of the dipole interaction with the target. \( 2\gamma(r) = \gamma_{in}(r) \) follows directly from the unitarity constraint of Eq. (2.2) at low energies. We want to remind at this point that we consider \( Y_0 \) so large that \( \alpha_s Y_0 \approx 1 \) and we can use the Born approximation of perturbative QCD for estimates of \( \gamma_{in}(r) \). Using new notation for elastic amplitude \( N \) (see Eq. (2.16)) and generating functionals \( Z_0 \) and \( Z \) (see Eq. (2.3) and Eq. (2.13)), we can re-write Eq. (1.1) in the form

\[
2N(Y - y, r, b) = 2 \left( 1 - Z_0 (Y - y; \{u\}) |_{u=1-\gamma}^{v(r)} \right) = |N(Y - y, r, b)|^2 + G_{in} = 1 - Z (Y - y; \{u\}, \{v\}) |_{u=1-\gamma, v=1-2\gamma}^{v(r)}
\]

which translates into boundary condition

\[
Z (Y - y; \{u\}, \{v\}) |_{v=2n-1} = 2 Z_0 (Y - y; \{u\}) - 1 \tag{2.17}
\]

It is useful for our further discussions to introduce a generating functional for multiparticle production with an arbitrary initial condition

\[
M (Y - y; \{\gamma\}, \{\gamma_{in}\}) = 1 - Z (Y - y; \{u\}, \{v\}) |_{u=1-\gamma, v=1-\gamma_{in}}^{v(r)} \tag{2.18}
\]

for which boundary condition Eq. (2.17) takes form of

\[
M (Y - y; \{\gamma\}, \{\gamma_{in}\}) |_{\gamma_{in}=2\gamma} = 2N(Y - y; \{\gamma\}) \tag{2.19}
\]
The generating functionals Eq. (2.13) and Eq. (2.18) should be supplemented by some initial condition for calculating physical observables. The initial condition depends on the process we want to calculate. As it was already mentioned, the linear equation for the generating functional will not depend on the initial condition.

As a simple example we calculate the cross section of single diffraction the initial condition for the generating functional has the following form

\[ Z(Y; \{u\}, \{v\}) = v(r) \quad (2.20) \]

which means that we consider diagrams which start with one cut Pomeron (see Fig. 4).

The cross section of the single diffractive production with mass smaller than \( \ln(M^2/m^2) \leq Y - Y' \) where \( m \) is the nucleon mass, is equal to (see more details in Ref. [15])

\[ \sigma_{sd}(Y, Y') = 1 - Z(Y - Y'; \{u\}; \{v\}) |v(r)|_{u(r)=1-N(Y';r,b)} \cdot v(r)=1-N^2(Y';r,b) \quad (2.21) \]

where \( N(Y'; r, b) \) reflects the fact that each Pomeron that cross line \( Y' \) can develop its own tree ‘fan’ of Pomeron.

The substitution \( v(r) = 1 - N^2(Y'; r, b) \) is explained as follows. Any dipole that corresponds to \( v(r) \) survives until \( \tau = \infty \) (is present on the unitarity cut) and thus can scatter both elastically and inelastically by \( 2N(Y'; r, b) = |N(Y'; r, b)|^2 + G_{in} \) (see normalization of \( \gamma_{in} = 2\gamma \)). However, we are interested in the single diffractive process with no particle production (inelastic scattering) below some rapidity \( Y' \). That is the reason why we retain only the elastic part of the total cross section of a dipole present on the cut, namely, \( N(Y'; r, b)^2 \).

### 2.3 AGK cutting rules and vertices for dipole decays

We wish to write the equation of Eq. (2.4)-type for \( P_m \), but first we need to determine the vertices of decays of one dipole to two dipoles; one dipole to one u-dipole and one v-dipole; and one v-dipole to two v-dipoles.

We will use the AGK cutting rules of Eq. (1.3)-Eq. (1.5) to find out these vertices. According section 2.1 we expect the following rules for the different transition

\[ \mathcal{P} \to \mathcal{P} + \mathcal{P} \sim \gamma_{in}^2 \delta M \delta \gamma_{in} \quad (2.22) \]
\[ \mathcal{P} \to \mathcal{P} + \mathcal{P} \sim \gamma_{in} \gamma \delta M \delta \gamma_{in} \quad (2.23) \]
\[ \mathcal{P} \to \mathcal{P} + \mathcal{P} \sim \gamma_{in} \delta M \delta \gamma_{in} \quad (2.24) \]
\[ \mathcal{P} \to \mathcal{P} \sim \gamma_{in} \delta M \delta \gamma_{in} \quad (2.25) \]

where \( \mathcal{P} \) denotes the cut BFKL Pomeron. At first sight, comparing Eqs. (2.22)-(2.25) with the AGK cutting rules of Eqs. (1.3)-(1.5) for \( \mathcal{P} \to \mathcal{P} + \mathcal{P} \) and \( \mathcal{P} \to \mathcal{P} \) we have

\[ \mathcal{P} \to \mathcal{P} + \mathcal{P} \sim \frac{\Delta s}{2\pi} \int d^2r_2 K(r_{10}|r_{12}, r_{02}) \left\{ \frac{1}{2} (\gamma_{in}(r_{12}) + \gamma_{in}(r_{02}))^2 - \frac{1}{2} \gamma_{in}^2(r_{10}) \right\} \delta M(y; \{\gamma\}, \{\gamma_{in}\}) \delta \gamma_{in} \quad (2.26) \]
\[ \mathcal{P} \to \mathcal{P} \sim \frac{\Delta s}{2\pi} \int d^2r_2 K(r_{10}|r_{12}, r_{02}) \left\{ \gamma_{in}(r_{12}) + \gamma_{in}(r_{02}) - \gamma_{in}(r_{10}) \right\} \delta M(y; \{\gamma\}, \{\gamma_{in}\}) \delta \gamma_{in} \quad (2.27) \]

Figure 4: The examples of Pomeron diagrams that contribute to the process of single diffraction in the mean field approximation. The Pomeron, crossed by the dotted line, is the cut Pomeron.
However, the AGK rules developed in [13] were written for amplitudes, which are the solutions to the linear (BFKL and a generalized form of BFKL) equations. In the present study we are interested in the non-linear evolution and thus all quadratic terms of the same argument ($\gamma_{in}^2 (12)$ etc.) are by definition absorbed in the corresponding linear terms, in other words for the generating functional $M(y; \{\gamma\}, \{\gamma_{in}\})$ functions $\gamma(r)$ and $\gamma_{in}(r)$ are arbitrary and any quadratic term of the same argument can be absorbed in the definition of the corresponding linear term. This means that the proper way to account for the the AGK cutting rules is as follows

\[
P \rightarrow P + P \sim \frac{\bar{\alpha}s}{2\pi} \int d^2 r_2 K(r_{10} | r_{12}, r_{02}) \gamma_{in}(r_{12}) \gamma_{in}(r_{02}) \frac{\delta M(y; \{\gamma\}, \{\gamma_{in}\})}{\delta \gamma_{in}(r_{10})} \quad (2.28)
\]

\[
P \rightarrow P + P \sim -2 \frac{\bar{\alpha}s}{2\pi} \int d^2 r_2 K(r_{10} | r_{12}, r_{02}) \{\gamma_{in}(r_{12}) \gamma_{in}(r_{12}) + \gamma_{in}(r_{02}) \gamma_{in}(r_{02})\} \frac{\delta M(y; \{\gamma\}, \{\gamma_{in}\})}{\delta \gamma_{in}(r_{10})} \quad (2.29)
\]

\[
P \rightarrow P + P \sim \frac{\bar{\alpha}s}{2\pi} \int d^2 r_2 K(r_{10} | r_{12}, r_{02}) \gamma(r_{12}) \gamma(r_{02}) \frac{\delta M(y; \{\gamma\}, \{\gamma_{in}\})}{\delta \gamma_{in}(r_{10})} \quad (2.30)
\]

\[
P \rightarrow P + P \sim \frac{\bar{\alpha}s}{2\pi} \int d^2 r_2 K(r_{10} | r_{12}, r_{02}) \{\gamma_{in}(r_{12}) + \gamma_{in}(r_{02}) - \gamma_{in}(r_{10})\} \frac{\delta M(y; \{\gamma\}, \{\gamma_{in}\})}{\delta \gamma_{in}(r_{10})} \quad (2.31)
\]

\[
P \rightarrow P + P \sim \frac{\bar{\alpha}s}{2\pi} \int d^2 r_2 K(r_{10} | r_{12}, r_{02}) \gamma(r_{12}) \gamma(r_{02}) \frac{\delta M(y; \{\gamma\}, \{\gamma_{in}\})}{\delta \gamma_{r_{10}}} \quad (2.32)
\]

\[
P \rightarrow P + P \sim \frac{\bar{\alpha}s}{2\pi} \int d^2 r_2 K(r_{10} | r_{12}, r_{02}) \{\gamma(r_{12}) + \gamma(r_{02}) - \gamma(r_{10})\} \frac{\delta M(y; \{\gamma\}, \{\gamma_{in}\})}{\delta \gamma_{r_{10}}} \quad (2.33)
\]

From these equations we can easily to build the Pomeron Calculus for cut and uncut Pomerons using the simple values: $\delta/\delta \gamma(r)$ and $\delta/\delta \gamma_{in}(r)$ are the annihilation operator for uncut and cut Pomerons while the multiplication by $\gamma$ and $\gamma_{in}$ leads to a creation of uncut and cut Pomerons. It should be stressed that Eq. (2.28)-Eq. (2.33) give a direct generalization of the equations for total cross section. In the latter we take into account the sum of all cuts. This sum results in the equation in which transitions of Eq. (2.32) and Eq. (2.33) remain, but with opposite signs. It is clear that summing all the cuts we obtain Eq. (2.11). Therefore, we check a selfconsistence of our approach this way at the end of the present analysis.

### 2.4 Linear functional equation

Eq. (2.28)-Eq. (2.33) allow us to write the linear evolution equation for $N$ of Eq. (2.15). It has the following form

\[
\frac{\partial M(y, \{\gamma\}, \{\gamma_{in}\})}{\partial y} = \frac{\bar{\alpha}s}{2\pi} \int d^2 r_2 K(r_{10} | r_{12}, r_{02}) \left\{ \gamma(r_{12}) - \gamma(r_{10}) - \gamma(r_{12}) \gamma(r_{12}) \frac{\delta M(y, \{\gamma\}, \{\gamma_{in}\})}{\delta \gamma_{r_{10}}} \right\}
\]

\[
+ (\gamma_{in}(r_{12}) + \gamma_{in}(r_{02}) - \gamma_{in}(r_{10}) + \gamma_{in}(r_{12}) \gamma_{in}(r_{12}) - 2 \gamma_{in}(r_{12}) \gamma_{in}(r_{02}) - 2 \gamma_{in}(r_{02}) \gamma_{in}(r_{12}) + 2 \gamma_{in}(r_{12}) \gamma_{in}(r_{10})) \frac{\delta M(y, \{\gamma\}, \{\gamma_{in}\})}{\delta \gamma_{r_{10}}} \right\}
\]

The fact that we have different signs in front of terms $\gamma_{in}^2$, $\gamma_{in}\gamma$ and $\gamma^2$ (see for example Eq. (2.29)), makes the probabilistic interpretation very questionable. However, it turns out that we can reduce Eq. (2.34) to a very simple equation with a very transparent probabilistic interpretation if we go back to the generating functional $Z(y; \{u\}, \{v\})$ where $u = 1 - \gamma$ and $v = 1 - \gamma_{in}$. As it was shown in [13], $Z(y; \{u\}, \{v\})$ can be written as a functional of two other functions such that $Z(y; \{u\}, \{v\}) = \bar{Z}(y; \{u\}, \{\xi\})$, where we defined a new function $\xi(r) = 2u(r) - v(r)$. In terms
of functions $u$ and $\xi$ the linear equation Eq. (2.34) has a simple form

$$\frac{\partial Z(y; \{u\}, \{\xi\})}{\partial y} = \frac{\delta S}{2\pi} \int d^2 r_2 K(r_{10}|r_{12}, r_{02}) \times \left\{ (u(r_{12}) u(r_{02}) - u(r_{10})) \frac{\delta Z(y; \{u\}, \{\xi\})}{\delta u(r_{10})} + (\xi(r_{12}) \xi(r_{02}) - \xi(r_{10})) \frac{\delta Z(y; \{u\}, \{\xi\})}{\delta \xi(r_{10})} \right\}$$  \hspace{1cm} (2.35)

Eq. (2.35) has a transparent probabilistic interpretation, since the second term at the r.h.s. of the equation is of the same structure as the first one and describes the Markov chain for the decay of the dipole which corresponds to $2u - v$. This physical meaning becomes clear if we recall that in the generating functional approach the functions $u$ or $v$ correspond to the creation operator for dipole that does not or does survive till time $\tau = \infty$, respectively (for corresponding annihilation operators we have $\delta/\delta u$ and $\delta/\delta v$).

The solution to Eq. (2.35) with the initial condition of Eq. (2.20) is found as

$$\tilde{Z}(y; \{u\}, \{\xi\}) = 2 Z_0(y; \{u\})^{u(r) - \xi(r)} - Z_0(y; \{\xi\})^{\xi(r)}$$  \hspace{1cm} (2.36)

One can easily check that the solution Eq. (2.36) satisfies boundary conditions Eq. (2.14) and Eq. (2.17). It is very instructive to compare this with the explicit form of the solution in the toy model found in Ref. [15].

3. Non-linear evolution equation

In this section we rewrite the linear functional first order differential equation Eq. (2.34) as the non-linear equation for the scattering amplitude using the initial condition for the generating functionals $Z_0$ and $Z$, given by Eq. (2.2) and Eq. (2.20), respectively. The easiest way is to go back to Eq. (2.34) and using the initial condition for $N(y; \{\gamma\})$ (see Eq. (2.12)) and $M(y; \{\gamma\}, \{\gamma_{in}\})$, to write the non-linear equation in analogy with transition from Eq. (2.3) to Eq. (2.8). The initial condition for the cross section $M(y; \{\gamma\}, \{\gamma_{in}\})$ is easily obtained from Eq. (2.18) and Eq. (2.20) in the form of

$$M(y = 0; \{\gamma\}, \{\gamma_{in}\}) = \gamma_{in}(r)$$  \hspace{1cm} (3.1)

Plugging Eq. (2.12) and Eq. (3.1) into the linear equation Eq. (2.34) we get the non-linear equation

$$\frac{\partial M(y; \{\gamma\}, \{\gamma_{in}\})}{\partial y} = \frac{\delta S}{2\pi} \int d^2 r_2 K(r_{10}|r_{12}, r_{02}) \times \left\{ M(y; \{\gamma\}, \{\gamma_{in}\})^\gamma_{in}(r_{10}) + M(y; \{\gamma\}, \{\gamma_{in}\})^\gamma_{in}(r_{20}) - M(y; \{\gamma\}, \{\gamma_{in}\})^\gamma_{in}(r_{10}) + M(y; \{\gamma\}, \{\gamma_{in}\})^\gamma_{in}(r_{20}) - 2 M(y; \{\gamma\}, \{\gamma_{in}\})^\gamma_{in}(r_{20}) N(y; \{\gamma\}) \right\}$$  \hspace{1cm} (3.2)

With the help of Eq. (2.7), Eq. (2.10) Eq. (2.13), Eq. (2.18) we recast Eq. (3.3) in a familiar form of

$$\frac{\partial M(y; r_{10}, b)}{\partial Y} = \frac{\delta S}{2\pi} \int d^2 r_2 K(r_{10}|r_{12}, r_{02}) \left\{ M(y; r_{12}, b) + M(y; r_{20}, b) - M(y; r_{10}, b) \right\}$$  \hspace{1cm} (3.3)

Eq. (3.3) describes all multiparticle production processes and has the same form as the equation for the cross section of diffractive production found in Ref. [3].
The difference between various processes manifests itself only in the different initial conditions. For example for the diffractive production the initial condition has the following form (see Fig. 4 and Eq. (2.21))

\[ M(Y = Y'; r, b) = N_0^2(Y', r, b) \]  

(3.4)

where \( N_0 \) is the amplitude of elastic scattering at rapidity \( Y = Y' \). Eq. (3.4) can be directly obtained from the corresponding condition for the generating functional Eq. (2.21) noting that \( M(Y = Y'; r, b) \) has meaning of the total cross section. This way we show that we can describe a process either in terms of the linear equation for the generating functional or a non-linear evolution equation for the scattering amplitude (total cross section). It should be noted that the generating functional approach is more general and allows more freedom in the choice of the initial condition.

If we want to find a cross section with the \( k \)-recoiled nucleons in dipole nucleus interaction we need to calculate

\[
\sigma^{(k)}(Y, r) = \frac{1}{k!} \prod_{i=1}^{k} \gamma_{in}(r_i) \left( \frac{\delta}{\delta \gamma_{in}(r_i)} M(Y; \{\gamma\}, \{\gamma_{in}\}) \right) |_{\gamma_{in}=0} \]  

(3.5)

where \( \gamma(r) \) is the low energy elastic amplitude and \( \gamma_{in}(r) = 2\gamma(r) \) at low energy (\( Y_0 \)).

This cross section can be found as the solution to non-linear equation with the initial condition

\[
\sigma^{(k)}(Y_0, r; b) = \frac{1}{k!} e^{-\Omega(Y_0, r; b)} \Omega(Y_0, r; b) \]  

(3.6)

where \( \Omega(Y_0, r; b) = \sigma_{\text{dipole-proton}}(Y = Y_0, r) T_A(b) \) with \( \sigma_{\text{dipole-proton}}(Y = Y_0, r) \) is the cross section of dipole - nucleon interaction at low energy being equal to \( \sigma_{\text{dipole-proton}}(Y = Y_0, r) = 2 \int d^2b' N_0(Y_0, r, b') \). The function \( T_A(b) \) is the optical width of nucleus which gives the number of nucleons at given value of impact parameter \( b \). We assume that \( Y_0 \) is large enough to use the Glauber approach. The derivation of Eq. (3.6) is given, for example, in Ref. [22].

4. Conclusions

The main result of the paper is two equations for the generating functional for multiparticle production: the linear equation Eq. (2.34) and Eq. (2.35), and the non-linear equation Eq. (3.3). The linear equations have an advantage of being correct for any initial condition, while the non-linear equation describes the process which starts from the exchange of a single Pomeron at low energies.

The non-linear equation has the same form as the equation for diffractive production that has been proved in Ref. [1] and confirms in Refs. [17, 18, 19]. We hope that here we give a more transparent and physically motivated derivation for diffractive production and generalize the approach to other processes of multiparticle generation. It should be stressed that the processes that we considered here are totally inclusive in the sense that we do not measure a particular particle in these processes. For example, our equations cannot describe the multiparticle inclusive correlations since we do not have AGK cutting rules for vertices with the emission of gluon (see Fig. 3).

In this paper we consider only the multiparticle processes in MFA, but we hope to use Mueller-Patel-Salam-Iancu approach [21] to calculate these processes taking into account the Pomeron loops in spirit of the approach suggested in Refs. [13, 21].
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References

[1] L. V. Gribov, E. M. Levin and M. G. Ryskin, Phys. Rep. 100, 1 (1983).
[2] A. H. Mueller and J. Qiu, Nucl. Phys. B 268 427 (1986).
[3] L. McLerran and R. Venugopalan, Phys. Rev. D 49, 2233, 3352 (1994); D 50, 2225 (1994); D 53, 458 (1996); D 59, 094001 (1999).
[4] I. Balitsky, [arXiv:hep-ph/9509348]; Phys. Rev. D60, 014020 (1999) [arXiv:hep-ph/9812311]
[5] Y. V. Kovchegov, Phys. Rev. D60, 034008 (1999), [arXiv:hep-ph/9901281].
[6] Y. V. Kovchegov and E. Levin, Nucl. Phys. B 577 (2000) 221 [arXiv:hep-ph/9911523].
[7] V. A. Abramovsky, V. N. Gribov and O. V. Kancheli, Yad. Fiz. 18, 595 (1973) [Sov. J. Nucl. Phys. 18, 308 (1974)].
[8] A. H. Mueller, Nucl. Phys. B415, 373 (1994); ibid B437, 107 (1995).
[9] M. Salvadore, J. Bartels and G. P. Vacca, “Multiple interactions and AGK rules in pQCD,” arXiv:0709.3062 [hep-ph]; F. Gelis and R. Venugopalan, Nucl. Phys. A 782 (2007) 297, 785 (2007) 146, [arXiv:hep-ph/0608117]; J. Bartels, M. Salvadore and G. P. Vacca, Eur. Phys. J. C 42 (2005) 53 [arXiv:hep-ph/0503049]; J. Bartels and M. G. Ryskin, Z. Phys. C 76 (1997) 241 [arXiv:hep-ph/9612226]; D. Treleani, Int. J. Mod. Phys. A 11 (1996) 613.
[10] Y. V. Kovchegov, Phys. Rev. D 64, 114016 (2001) [Erratum-ibid. D 68, 039901 (2003)] [arXiv:hep-ph/0107256].
[11] Y. V. Kovchegov and K. Tuchin, Phys. Rev. D 65, 074026 (2002) [arXiv:hep-ph/0111362].
[12] J. Jalilian-Marian and Y. V. Kovchegov, Phys. Rev. D 70, 114017 (2004) [Erratum-ibid. D 71, 079901 (2005)] [arXiv:hep-ph/0405266].
[13] E. Levin and A. Prygarin, “Inclusive gluon production in the dipole approach: AGK cutting rules”.
[14] E. Levin and A. Prygarin, “Inclusive gluon production in the dipole approach: AGK cutting rules”.
[15] E. Levin and A. Prygarin, Eur. Phys. J. C 53 (2008) 385 [arXiv:hep-ph/0701178].
[16] E. Levin and M. Lublinsky, Nucl. Phys. A 763 (2005) 172 [arXiv:hep-ph/0501173]; Phys. Lett. B 607 (2005) 131 [arXiv:hep-ph/0411121]; Nucl. Phys. A 730 (2004) 191 [arXiv:hep-ph/0308279].
[17] A. Kovner, M. Lublinsky and H. Weigert, Phys. Rev. D 74, 114023 (2006) [arXiv:hep-ph/0608258].
[18] Y. Hatta, E. Iancu, C. Marquet, G. Soyez and D. N. Triantafyllopoulos, Nucl. Phys. A 773 (2006) 95 [arXiv:hep-ph/0601150].
[19] M. Hentschinski, H. Weigert and A. Schafer, Phys. Rev. D 73 (2006) 051501 [arXiv:hep-ph/0509272].
[20] A. H. Mueller and B. Patel, Nucl. Phys. B425, 471 (1994); A. H. Mueller and G. P. Salam, Nucl. Phys. B475, 293 (1996), [arXiv:hep-ph/9605302]; G. P. Salam, Nucl. Phys. B461, 512 (1996); E. Iancu and A. H. Mueller, Nucl. Phys. A730 (2004) 460, 494. [arXiv:hep-ph/0308315],[arXiv:hep-ph/0309276].
[21] E. Levin, J. Miller and A. Prygarin, “Summing Pomeron loops in the dipole approach,” Nucl.Phys. A (in press), arXiv:0706.2944 [hep-ph].
[22] A. Kormilitzin, arXiv:0707.2202 [hep-ph].