A rearrangement invariant space isometric to $L_p$ coincides with $L_p$.

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Dedicated to the memory of Ioseph Shneiberg

The following theorem is the main result of this note.

**Theorem 1.** Let $(E, \| \cdot \|_E)$ be a rearrangement invariant Banach function space on the interval $[0,1]$. If $E$ is isometric to $L_p[0,1]$ for some $1 \leq p < \infty$, then $E$ coincides with $L_p[0,1]$ and furthermore $\| \cdot \|_E = \lambda \| \cdot \|_{L_p}$, where $\lambda = \|1\|_E$.

We precede the proof with some necessary definitions and notation. Two Banach lattices $X$ and $Y$ are said to be **order isometric** if there exists an isometry $U$ of $X$ onto $Y$ which preserves the order, that is, $U$ is an isometric surjective operator and $U(x) \geq 0$ if and only if $x \geq 0$.

Let $L_0 = L_0[0,1]$ be the vector lattice of all (equivalence classes of) measurable real valued functions on $[0,1]$ and let $\mu$ denote Lebesgue measure.

A Banach space $E$ is called **rearrangement invariant** (r.i.) if the following three conditions hold:

- $E$ is an ideal in $L_0$, i.e., if $x \in E$, $y \in L_0$ and $|y| \leq |x|$, then $y \in E$.
- If $x, y \in E$ and $|y| \leq |x|$, then $\|y\| \leq \|x\|$.
- If $x \in E$, $y \in L_0$ and the functions $|x|, |y|$ are equimeasurable, then $y \in E$ and $\|x\| = \|y\|$.

As usual, the symbol $E^+$ denotes the cone of all nonnegative elements in $E$, and $E^{++}$ is a subset of all week units of $E^+$. Recall that an ideal $B$ in $E$ is called a **band** if whenever $x = \sup x_\alpha$ for $x_\alpha \in B$ and $x \in E$ we have $x \in B$. For each band $B$ in $E$ we denote by supp $B$ its support set, i.e., a unique (modulo subsets of measure zero)

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1It is worth noticing that, as shown in [A1], a mere positivity of an isometry $U$ implies that $U$ preserves the order.

2That is, $\mu\{ t : |x(t)| < s \} = \mu\{ t : |y(t)| < s \}$ for each $s \in \mathbb{R}$

3An element $0 < x \in E$ is said to be a weak unit if $x \land y > 0$ for each $0 < y \in E$.
minimal measurable subset $e \subseteq [0, 1]$ such that $x \chi_e = x$ for each $x \in B$, where $\chi_e$ is the characteristic function of the set $e$.

If $X$ and $Y$ are Banach lattices then, as usual, $X \oplus_p Y$ denotes the Banach lattice of all the pairs $(x, y) \in X \times Y$ with the norm $\| (x, y) \| = (\| x \|^p + \| y \|^p)^{\frac{1}{p}}$.

A key element in our proof is provided by a theorem due to E. Lacey and P. Wojtaszczyk [LW]. Let $l_p(2)$ denote the standard two-dimensional $l_p$-space, ordered by the usual positive cone $l_p^+(2) = \{ \alpha \bar{e}_1 + \beta \bar{e}_2 : \alpha \geq 0, \beta \geq 0 \}$, where $e_1 = (1, 0)$, $e_2 = (0, 1)$. Following [LW] we denote by $E_p(2)$ the same Banach space $l_p(2)$ but ordered by the cone

$$E_p^+(2) = \{ \alpha \bar{e}_1 + \beta \bar{e}_2 : |\beta| \leq \alpha \}.$$ 

Both $l_p(2)$ and $E_p(2)$ are Banach lattices, but they are not order isometric. Let $L_p(E_p(2))$ be the Banach space of all (equivalence classes of) measurable vector-valued functions $f(t)$ on the interval $[0, 1]$ with values in $E_p(2)$, and with the norm

$$\| f \| = \left( \int_0^1 \| f(t) \|^p dt \right)^{\frac{1}{p}}.$$ 

It is plain to see that under the natural ordering (i.e., $f \geq 0$ if and only if $f(t) \in E_p^+(2)$) the space $L_p(E_p(2))$ is a Banach lattice.

**Theorem 2** (Lacey and Wojtaszczyk). *If a Banach lattice $E$ is isometric to $L_p[0, 1]$ ($1 \leq p \neq 2 < \infty$), then it is order isometric to one of the next three Banach lattices:

1. $L_p[0, 1]$ with the standard order;
2. $L_p(E_p(2))$;
3. $L_p[0, 1] \oplus_p L_p(E_p(2))$.

We point out that no two of these three Banach lattices are order isometric. Finally, for each $t \in [0, 1]$ we denote by $L_p(E_p(2); [0, t])$ the band in $L_p(E_p(2))$ whose support set is $[0, t]$.

The unexplained terminology and notation regarding rearrangement invariant spaces can be found in [KPS] and [LT], regarding Banach lattices in [LT] and [V].

**Proof of Theorem 1.** The case $p = 2$ was proved in [S], so in what follows we assume that $p \neq 2$.

Let $T$ be an isometry of the space $E$ onto $L_p[0, 1]$. Therefore $T$ induces a new Banach lattice structure on the Banach space $L_p[0, 1]$. By Theorem 2, this new Banach lattice is order isometric to one of the three Banach lattices (1–3) indicated above. We begin by showing that Cases (2) and (3) are impossible under the conditions of our theorem.

Suppose that Case (3) takes place. Therefore there exists an order isometry $U : L_p[0, 1] \oplus_p L_p(E_p(2)) \to E$. Hence $U(L_p[0, 1])$ and $U(L_p(E_p(2)))$ are complemented
bands in $E$. We claim the existence of the real numbers $t, \tau \in (0, 1]$ such that

$$\mu(\text{supp } U(L_p[0,t])) = \mu(\text{supp } U(L_p(E_p(2); [0, \tau]))) \quad (*)$$

Assume, for instance, that

$$\mu(\text{supp } U(L_p[0,1])) = \alpha \geq \beta = \mu(\text{supp } U(E_p(2))).$$

Put $A_t = \text{supp } U(L_p[0,t])$ for $t \in (0, 1]$. A straightforward verification shows that the function $t \mapsto \mu(A_t)$ is continuous. Since $\mu(A_1) = \alpha \geq \beta$ and $A_t \downarrow \emptyset$ when $t \to 0$, we can conclude that $\mu(A_{t_0}) = \beta$ for some $t_0 \in (0, 1]$.

The case when $\alpha < \beta$ can be treated similarly. So in our r.i. space $E$, we have found two non-zero bands $U(L_p[0,t])$ and $U(L_p(E_p(2); [0, \tau]))$, with supports of equal measures. However, in an arbitrary r.i. space any two bands whose supports have equal measures are clearly order isometric. A contradiction since the Banach lattices (1) and (2) are not order isometric.

Similar arguments can be applied to exclude Case (2). Indeed for any $0 < t \leq 1/2$ the Banach lattice $L_p(E_p(2))$ contains two disjoint bands, one of which is $L_p(E_p(2); [0, t])$ and another is (order isometric to) $L_p[0,t]$. Again this is impossible as $E$ is a r.i. space.

Thus, we have established that the only case which can occur is the first one. This means that the r.i. spaces $E$ and $L_p[0,1]$ are isometric if and only if they are order isometric. However, a well known theorem due to L. Potepun [P1] (see also [A2] for a very simple proof of Potepun’s theorem) asserts that if two r.i. spaces $X$ and $Y$ (on the same measure space) are order isomorphic, then they coincide. Therefore $E$ and $L_p[0,1]$ coincide, and consequently their norms are equivalent. To show that in actuality $\| \cdot \|_E = \lambda \| \cdot \|_{L_p}$ one can either make use of the description (due to S. Banach [B]) of the isometry group of the space $L_p[0,1]$, or (to bypass any technical details) simply to apply the following theorem due to D. A. Vladimirov [VS]:

**Theorem 3.** Let $X$ be a r.i. KB-space. If the group of the order isometries of $X$ acts transitively on both $X^{++}$ and $X^+ \setminus X^{++}$, then $X$ is $L_q[0,1]$ for some $q \geq 1$ and $\| \cdot \|_X = \lambda \| \cdot \|_{L_q}$ for some $\lambda > 0$.

Since the space $L_p[0,1]$ satisfies the assumptions of Theorem 3, the space $E$, which is order isometric to $L_p[0,1]$, satisfies them too. So, by Theorem 3, $E = L_q[0,1]$ for some $q \geq 1$ and $\| \cdot \|_E = \lambda \| \cdot \|_{L_q}$ for some $\lambda > 0$. Clearly, $p = q$ and $\lambda = \| 1 \|_E$. This completes the proof of Theorem 1.

**Remark.** If $p = \infty$, i.e., if the r.i. space $E$ is isometric to $L_\infty[0,1]$, then the conclusion of Theorem 1 still holds.
We will sketch the proof of the equality $E = L_\infty[0,1]$. It is well known (and obvious) that an arbitrary r.i. space on $[0,1]$ contains the constant-one-function $1$, and thus $L_\infty[0,1] \subseteq E$.

Notice (see [AW, p. 324]) that Banach lattice $E$ is isomorphically an AM-space, and consequently there exists a constant $M > 0$ such that $\|e_1 \vee e_2 \vee \ldots \vee e_n\|_E \leq M$ for arbitrary positive elements $e_1, \ldots, e_n$ in the unit ball of $E$. We are ready to verify now that the converse inclusion $E \subseteq L_\infty[0,1]$ also holds, i.e., each function $x$ from $E_+$ is (essentially) bounded. Indeed, assume contrary to what we claim, that $x$ is an unbounded function. Then for each $K > 0$ there exists a set $A$ (depending on $x$ and $K$) of positive measure such that $x(t) > K$ for almost all $t \in A$. Now using the symmetry of the space $E$ we can find a finite number of functions in $E$ each of which is equimeasurable with $x \chi_A$ and whose supremum $y$ is greater than function $K 1$. This would imply that $M \|x\|_E \geq \|y\|_E \geq K \|1\|_E$. A contradiction, as $K$ is arbitrary. Thus, we have established that $E$ (as a set) coincides with $L_\infty[0,1]$.

Some comments are in order in connection with this article. Theorem 1 was proved by the authors many years ago. It was announced in [AZ], but the proof was never published due to several reasons. Our decision to publish it now has been inspired by a recent revival of interest in rearrangement invariant spaces. This revival is due, in particular, to a remarkable result by N. Kalton and B. Randrianantoanina [KR1,2] who solved a longstanding problem by showing that the description of surjective isometries obtained for the complex rearrangement invariant spaces by the second author [Z1,2] (see also [Z3]) remains valid for the real spaces as well. The proof of Theorem 1 presented above is basically our original proof, it is very simple and is independent of a powerful machinery developed later on in [JMST], [K] and [LT].

We conclude with one remark and an open question. B. Randrianantoanina has obtained recently an analogue of Theorem 1 for the Orlicz and Lorentz spaces. An isomorphic version of the problem at hand reads as follows:

**Problem.** Let $E_1$, $E_2$ be two isomorphic r.i. spaces on a measure space $(\Omega, \Sigma, \lambda)$. When does it imply that the spaces $E_1$ and $E_2$ coincide?

In this case, of course, the norms will only be equivalent. When $E_2 = L_2(\Omega, \Sigma, \lambda)$ this question was answered in the affirmative by L. Potepun [P2]. We refer to [JMST] and [K] for several important results when the answer to this problem is “yes” and for examples when the answer is “no.”

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