Physically inspired analysis of prime number constellations

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We adopt a physically motivated empirical approach to the characterisation of the distributions of
twin and triplet primes within the set of primes, rather than in the set of all natural numbers.
Remarkably, the occurrences of twins or triplets in any finite sequence of primes are like fixed-
probability random events. The respective probabilities are not constant, but instead depend on
the length of the sequence in ways that we have been able to parameterise. For twins the “decay
constant” decreases as the reciprocal of the logarithm of the length of the sequence, whereas for
triplets the falloff is faster: decreasing as the square of the reciprocal of the logarithm of the
number of primes. The manner of the decrease is consistent with the Hardy–Littlewood Conjectures,
developed using purely number theoretic tools of analysis.

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INTRODUCTION

Recently we discovered a novel approach to the characterisation of the distribution of twin primes and re-
alised some of its consequences [1, 2, 3]. Our results have been confirmed and shown to be consistent over a
broader range by Wolf [4]. Twins are pairs of primes \{p, p + 2\} whose arithmetic separation is minimised, i.e.,
they consist of consecutive odd natural numbers. The Twin Prime Conjecture posits that there are an infinite
number of twins [5]. Long ago, Hardy and Littlewood applied sieve arguments to establish relations which de-
scribe the behaviour of \(\pi_2(N)\), the number of twins with constituents less than natural number \(N\). They obtained:

\[
\lim_{N \to \infty} \frac{\pi_2(N)}{\int_2^N \frac{1}{\log(x)} \, dx} = 2C_2 ,
\]

(1)

where the so-called twin prime constant, \(C_2 \approx 0.66016\ldots\) is calculable to much greater accuracy than is quoted
here [6].

Prime triplets comprise the next level of allowed structure in the sequence of prime numbers. Again the arith-
metic difference between the first constituent and the last is minimised. This minimum value is not 4 as one might
naively suppose because any set of odd numbers of the form \(\{n, n + 2, n + 4\}\) has an element which is equivalent
to zero(modulo 3). Thus the minimum arithmetic difference is 6, and hence there are two “flavours” of triplets:
those of the form \(\{p, p + 2, p + 6\}\), which we call the twin-outlier (TO) type and those of the form \(\{p, p + 4, p + 6\}\)
which we shall denote outlier-twin (OT).

Hardy and Littlewood applied their analysis to both flavours of triplets and determined that

\[
\lim_{N \to \infty} \frac{\pi_3(TO, N)}{\int_2^N \frac{1}{\log(x)} \, dx} = \lim_{N \to \infty} \frac{\pi_3(OT, N)}{\int_2^N \frac{1}{\log(x)} \, dx} = C_3 ,
\]

(2)

where the triplet prime constant, \(C_3 \approx 2.8582\ldots\) is also known to great accuracy. The analytic tools used to
construct these results are essentially insensitive to the flavour of the triplets and so the limits for TO and OT
must agree [7].

There exist analogous relations for higher order const-
stellations of prime numbers. To place the formulae of the
Hardy-Littlewood Conjectures in perspective, it is useful
to express the Prime Number Theorem – that there exists
an infinite number of prime numbers – in the following, stronger, form:

\[
\pi_1(N) \sim \int_2^N \frac{1}{\ln(x)} \, dx .
\]

(3)

More recently (with the advent of electronic comput-
ing), a number of investigators have studied in detail the actual distributions of primes and prime constella-
tions. Particular attention has been paid to enumeration of twins and explicit determination of \(\pi_3(N)\) for large
values of \(N\), by Nicely [7], and others [8, 9]. In many of
these instances, the search for twins is a beneficient ap-
lication of research in decentralised computing. Other
analyses have been concerned with the details of the dis-
tribution of twins [10, 11, 15], and the existence and size
of “gaps” in the sequence of twins [13, 14, 16]. In all cases, to the best of our knowledge, these attempts at-
uate the twins (and higher constellations) within the set of
natural numbers.

There are three essential constituents of our new mod-
els for the distributions of twins and triplets. First and foremost is that the distribution of twins and triplets are
viewed in the context of the sequence of primes, not the
natural numbers. Second is that for a prime sequence of
length \(\pi_1\), twins and triplets occur in the manner of
random fixed-probability events. The third part of each
model is that the fixed value of the probability depends
on \(\pi_1\), the length of the sequence, in a fairly simple man-
ner.
METHOD AND RESULTS

We generated prime numbers in sequence, viz., \( P_1 = 2, P_2 = 3, P_3 = 5, P_4 = 7, P_5 = 11 \ldots \), and within this sequence counted \textit{prime separations} defined as the number of “singleton” primes occurring between adjacent pairs of twins, or triplets of a specified flavour. There is an irregularity with our definition of separation for the first few primes: \( 2 (3 5)(5 7) \), where a pair of twins is overlapping, yielding an anomalous prime separation of \(-1\). Fortunately, as explained above such overlapping twins do not ever recur and we chose to begin our sequences with \( P_3 = 5 \).

Note that there are many twins with prime separation equal to zero, for example, \((5 7)(11 13)\), or \((137 139)(149 151)\). Similarly, there are many triplets which overlap, i.e., \((5 7 11)(11 13 17)\), or \((97 101 103)(103 107 109)\), and thus are assigned prime separation \(-1\). Incidentally, all of the prime \( \text{TT-quadruplets} \) of the form \( (P, P + 2, P + 6, P + 8) \) are comprised of a pair of twins with zero prime separation. Also they may be viewed as an overlapping \( \text{TO-OT} \) flavour combination of triplets. Whereas \( \text{OTO-quadruplets} \) \( (P, P + 4, P + 6, P + 10) \) possess a single twin shared between an overlapping \( \text{OT-TO} \) pair of triplets. Clearly, \( k \)-tuplets, \( k > 2 \), or higher-order “prime constellations” are composed of smaller, more primitive, units.

Concretely, so as to make our methods perfectly clear, consider the set of prime numbers greater than or equal to 5 and less than 100. In this range there are seven twins,

\[
{(5 7)(11 13)(17 19)(29 31)(41 43)(59 61)(71 73)}
\]

and hence six separations. Two of these happen to be 0, three are 1, and one is 2, so the relative frequencies for separations \( s = 0, 1, 2 \) are \( \frac{1}{4}, \frac{3}{4}, \frac{1}{4} \) respectively. In the set of four \( \text{TO-triplets} \) between 5 and 100,

\[
{(5 7 11)(11 13 17)(17 19 23)(41 43 47)}
\]

there are three separations. Two are \(-1\), one is \(3\), so the relative frequencies are \( \frac{2}{3}, \frac{1}{3} \) for \( s = -1, 3 \). The four \( \text{OT-triplets} \) less than 100,

\[
{(7 11 13)(13 17 19)(37 41 43)(67 71 73)}
\]

have three distinct separations. The relative frequencies are \( \frac{1}{3} \) for each of \( s = -1, 3, 4 \).

All separations between pairs of twins and \( \text{TO-OT} \)-triplets up to \( N \) were computed and tabulated. [We chose values of \( N \) which ranged from less than \( 10^{10} \), the limit of poor statistics, to \( 6 \times 10^{12} \), the limit of our patience.] We also obtained \( \pi_1(N) \) for each of these ranges and the count of the number of singleton primes that occur above the last tuple in the range. Taking the logarithm of the relative frequency of occurrence of each separation in the sequence of primes to \( N \) and plotting it vs. the separation yields a surprisingly simple \textit{linear} relation as illustrated in Figures 1 and 2.

![Figure 1: Distribution of twin prime separations for \( N = 1 \times 10^6 \) and \( N = 1 \times 10^9 \). Note the (approximately) linear behaviour, and the differing slopes.](image)

From the figures, we see that the twins and triplets are occurring in a fixed-probability \textit{random} manner, but that the probability diminishes as the length of the sequence of primes increases. By way of analogy, consider radioactive decay. The likelihood of an atom decaying in any short time interval is constant for a particular substance, and thus the probability that the next decay observed in a sample occurs at time \( t \) is proportional to \( \exp(-\gamma t) \), where \( \gamma \), the decay rate, is a property of the species of atom. The values of the best-fit slopes that we measure from figures like those above determine “decay constants” for the twins and triplet flavours. We will also find it useful to think of the \textit{mean separation}, the expected number of singleton primes appearing between adjacent \( k \)-tuplets, corresponding to the \textit{mean lifetime} of the radioactive substance. As expected from our analogy, \( s = 1/\text{slope} \). In our reformulation of our twin model \([3]\), and our analysis of triplets \([2]\), in terms of the counts \( \pi_1(N) \) and \( \pi_{2,3}(N) \) alone, we cast our analysis in terms of \( s \).

The linear fits that we employed were constrained to ensure that the relative frequencies are properly normalised. With negligible deleterious effects we treat \( s \) as a continuous variable and integrate over all possible separations: from the minimum value possible to infinity. For twins, we must have

\[
+(\text{intercept}) = \ln(-s) , \quad f(s) = -ms + \ln(m) , \quad (4)
\]
constant. In this case then the numbers of twins and prime being a member of a constellation is a universal constant slope would mean that the probability of a given $N$ diminished for larger values of perspective, it is better to weigh all points equally.

As the fixed probabilities are seen to change with $\pi_1$, we model the manner in which they vary. In the figures below, we present the estimated slopes (with statistical errors only, the total error is expected to be somewhat larger), vs. $\log(\pi_1(N))$ – with correction for the singleton primes beyond the last tuple – and functions that we believe capture very well the behaviour of the “decay constant.”

![Figure 2](image2.png)

**FIG. 2:** Distribution of triplet prime separations for $N = 1 \times 10^9$ and $N = 1 \times 10^{12}$. The distributions for the two flavours overlap to such an extent that they are virtually indistinguishable. Note the (approximately) linear behaviour, the differing slopes, and the difference in scale from the twins case.

since the minimum separation is 0. For triplets, we have a similar condition

$$f(s) = -m(s + 1) + \ln(m), \quad (5)$$

on account of the fact that the minimum separation is $-1$.

All of the separations which appeared in the data received equal (frequency-weighted) consideration in our computation of best-fit slopes. This has a consequence insofar as the large-separation, low-frequency events constituting the tail of the distribution reduce the magnitude of the measured slope, (see Figures 2 and 3). One might well be inclined to truncate the data by excising the tails of the measured slope, (see Figures 1 and 2). One might be inclined to truncate the data by excising the tails of the measured slope, (see Figures 1 and 2). One might be inclined to truncate the data by excising the tails of the measured slope, (see Figures 1 and 2). One might be inclined to truncate the data by excising the tails of the measured slope, (see Figures 1 and 2).

We note that the factor which appears in the numerator vs. $\log(\pi_1)$, and parameterised fits for twins. The longer dashed line is a fit to the empirical data shown, while the shorter dashed line arises from a analysis of prime and twin counts only up to $N \sim 10^{10}$.

In light of the above comments, we sketch in Figure 3 a plot of the slopes for twins versus $\log(\pi_1)$. The empirical trend seen on the graph may be well-described by the function (remember that the error bars are understated)

$$-m(x) \approx -(1.321 \pm 0.008)/x, \quad \text{for } x = \log(\pi_1(N)). \quad (6)$$

We note that the factor which appears in the numerator is approximately $-2C_2$, the twin primes constant, as one is led to expect by a straightforward argument. Also, the asymptotic behaviour is consistent with the Twin Prime Conjecture. On this graph we have also traced the curve which results from a discretised model for the distribution of twins that we developed which requires only the input of $\pi_1(N)$ and $\pi_2(N)$ and was fit to data including values of $N$ up to $3 \times 10^{15}$, well beyond the reach of our complete analysis of the actual distribution of prime separations.

For the TO- and OT-triplets similar results ensue.

$$-m_{\text{TO}}(x) \approx \frac{-2.87 \pm 0.05}{x + 2.24 \pm 0.15},$$

$$-m_{\text{OT}}(x) \approx \frac{-2.81 \pm 0.04}{x + 2.24 \pm 0.15}.$$
The factors which appear in the numerators of these expressions precisely bracket the triplet prime constant in a manner which is consistent with expectations [12]. The error that we quote arises purely from the fit to the data and is again most likely an understatement, so we are not going to argue that the to- and ot-distributions are fundamentally dissimilar.

CONCLUSION

We believe that we have consistently extended our construction of a novel characterization of the distribution of twin primes to the prime triplets. The most essential feature of our approach is that we consider the spacings of twins and triplets among the primes themselves, rather than among the natural numbers. Secondly, we modelled the distribution empirically – without preconceptions – and found that the twins and triplets appear amongst the sequence of primes in a manner characteristic of a completely random, fixed probability system. Again working empirically, we were able to simply parameterise the variation of the “decay constant” in terms of $\pi_1$, as suggested by our outlook.

Precise details of the triplets case will be reported upon in a forthcoming paper [12]. Future work includes extension to larger ranges of data, higher-order constellations, examination of constellation correlations, and possible fractal interpretations.

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