Structure behind Mechanics

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Abstract

This paper proposes a basic theory on physical reality, and a new foundation for quantum mechanics and classical mechanics. It does not only solve the problem of the arbitrariness on the operator ordering for the quantization procedure, but also clarifies how the classical-limit occurs. It further compares the new theory with the known quantization methods, and proposes a self-consistent interpretation for quantum mechanics. It also provides the internal structure inducing half-integer spin of a particle, the sense of the regularization in the quantum field theory, the quantization of a phenomenological system, the causality in quantum mechanics and the origin of the thermodynamic irreversibility under the new insight.

1 INTRODUCTION

Seventeenth century saw Newtonian mechanics, published as ”Principia: Mathematical principles of natural philosophy,” the first attempt to understand this world under few principles rested on observation and experiment. It bases itself on the concept of the force acting on a body and on the laws relating it with the motion. In eighteenth century, Lagrange’s analytical mechanics, originated by Mautertuis’ theological work, built the theory of motion on an analytic basis, and replaced forces by potentials; in the next century, Hamilton completed the foundation of analytical mechanics on the principle of least action instead of Newton’s laws. Besides, Maxwell’s theory of the electromagnetism has the Lorentz invariance inconsistent with the invariance under Galilean transformation, that Newtonian mechanics obeys. Twentieth century dawned with Einstein’s relativity changing the ordinary belief on the nature of time, to reveal the four-dimensional spacetime structure of the world. Relativity improved Newtonian mechanics based on the fact that the speed of light c is an invariant constant, and revised the self-consistency of the classical mechanics. Notwithstanding such a revolution, Hamiltonian mechanics was still effective not only for Newtonian mechanics but also for the Maxwell-Einstein theory, and the concept of energy and momentum played the most important role in the physics instead of force for Newtonian mechanics.

Experiments, however, indicated that microscopic systems seemed not to obey such classical mechanics so far. Almost one century has passed since Planck found his constant h; and almost three fourth since Heisenberg ; Schrödinger and their contemporaries constructed the basic formalism of quantum mechanics after the early days of Einstein and Bohr. The quantum mechanics based itself on the concept of wave functions instead of classical energy and momentum, or that of operators called as observables. This mechanics reconstructed the classical field theories except the general relativity. Nobody denies how quantum mechanics, especially quantum electrodynamics, succeeded in twentieth century and developed in the form of the standard model for the quantum field theories through the process to find new particles in the nature.

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Quantum mechanics, however, seems to have left some fundamental open problems on its formalism and its interpretation: the problem on the ambiguity of the operator ordering in quantum mechanics [3, 4], which is crucial to quantize the Einstein gravity for instance, and that on the reality, which seems incompatible with the causality [5, 6, 7]. These difficulties come from the problem how and why quantum mechanics relates itself with classical mechanics: the relationship between the quantization that constructs quantum mechanics based on classical mechanics and the classical-limit that induces classical mechanics from quantum mechanics as an approximation with Planck’s constant $\hbar$ taken to be zero; the incompatibility between the ontological feature of classical mechanics and the epistemological feature of quantum mechanics in the Copenhagen interpretation [42].

Now, this paper proposes a basic theory on physical reality, and introduces a foundation for quantum mechanics and classical mechanics, named as protomechanics, that is motivated in the previous letter [8]. It also attempts to revise the nonconstructive idea that the basic theory of motion is valid in a way independent of the describing scale, though the quantum mechanics has once destroyed such an idea that Newtonian mechanics held in eighteenth century. The present theory supposes that a field or a particle $X$ on the four-dimensional spacetime has its internal-time $\hat{\partial}_A(X)$ relative to a domain $A$ of the spacetime, whose boundary and interior represent the present and the past, respectively. It further considers that object $X$ also has the external-time $\hat{o}_A^*(X)$ relative to $A$ which is the internal-time of all the rest but $X$ in the universe. Object $X$ gains the actual existence on $A$ if and only if the internal-time coincide with the external-time:

$$\hat{\partial}_A(X) = \hat{o}_A^*(X).$$

This condition discretizes or quantizes the ordinary time passing from the past to the future, and enables the deterministic structure of the basic theory to produce the nondeterministic characteristics of quantum mechanics. The both sides of relation (1) further obey the variational principle as

$$\delta \hat{\partial}_A^*(X) = 0 , \quad \delta \hat{\partial}_A^*(X) = 0.$$  

This relation reveals a geometric structure behind Hamiltonian mechanics based on the modified Einstein-de Broglie relation, and produces the conservation law of the emergence-frequency of a particle or a field based on the introduced quantization law of time. The obtained mechanics, protomechanics, rests on the concept of the synchronicity instead of energy-momentum or wave-functions, that synchronizes two intrinsic local clocks located at different points in the space of the objects on a present surface in the spacetime. It will finally solve the problem on the ambiguity of the operator ordering, and also give a self-consistent interpretation of quantum mechanics as an ontological theory.

The next section explains the basic laws on reality as discussed above, and leads to the protomechanics in Section 3, that produces the conservation laws of momentum and that of emergence-frequency. Section 4 presents the dynamical construction for the introduced protomechanics by utilizing the group-theoretic method called Lie-Poisson mechanics (consult APPENDIX). It provides the difference between classical mechanics and quantum mechanics as that of their function spaces: the function space of the observables for quantum mechanics includes that for classical mechanics; the dual space of the emergence-measures for classical mechanics includes that for quantum mechanics, vice versa. Section 5 and Section 6 explain how protomechanics deduces classical mechanics and quantum mechanics, respectively. In these sections, protomechanics proves to include both quantum mechanics and classical mechanics; in conjunction with the result in Section 4, it clarifies how the quantization and the classical-limit occur. Section 6 additionally presents a consequent interpretation for the half-integer spin of a particle. Section 7 compares the present theory with the other known quantization methods from both the group-theoretic viewpoint and the statistical one, and further introduces an interpretation of the regularization method adopted at quantum field theories; and it will prove applicable for general phenomenological systems. On the other hand, Section

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1. The author of paper [8], "Tosch Ono," is the same person as that of the present paper, "Toshihiko Ono."

2. This naming of synchronicity is originated by Jung [9].
8 considers how it gives a self-consistent interpretation of reality or solves the measurement problem, and interprets the origin of thermodynamic irreversibility in the nature; and it will prove to keep causality even under an EPR-experiment. A brief statement of the conclusion immediately follows.

Let me summarize the construction of the present paper in the following diagram.

In this paper, \(c\) and \(\hbar\) denote the speed of light and Planck’s constant, respectively. I will use Einstein’s rule in the tensor calculus for Roman indices’ \(i, j, k \in \mathbb{N}\) and Greek indices’ \(\nu, \mu \in \mathbb{N}\), and not for Greek indices’ \(\alpha, \beta, \gamma \in \mathbb{N}\), and I further denote the trace (or supertrace) operation of a quantum observable \(\hat{F}\) as \(\langle \hat{F} \rangle\) that is only one difference from the ordinary notations in quantum mechanics, where \(i = \sqrt{-1}\). Consult the brief review on the differential geometry in APPENDIX A and that on Lie-Poisson mechanics in APPENDIX B, which the employed notations follow. In addition, notice that the basic theory uses so-called c-numbers, while it also utilize q-numbers to deduce the quantum mechanics in Section 6 for the help of calculations.\[3\]

2 LAWS ON REALITY

Let \(M^{(4)}\) represent the spacetime, being a four-dimensional oriented \(C^\infty\) manifold, that has the topology or the family \(\mathcal{O} = \mathcal{O}_{M^{(4)}}\) of its open subsets, the topological \(\sigma\)-algebra \(\mathcal{B}(\mathcal{O}_{M^{(4)})}\), and the volume measure \(v^{(4)}\)

\[3\] Such distinction between c-numbers and q-numbers does not play an important role in the present theory.
induced from the metric $g$ on $M^{(4)}$. We shall certainly choose an arbitrary domain $A \in \mathcal{O}$ in the discussion below, but we are interested in the case that domain $A$ represents the past at a moment whose boundary $\partial A$ is a three-dimensional present hypersurface in $M^{(4)}$.

The space $\hat{M}$ represents that of the objects whose motion will be described, and has a projection operator $\chi_A : \hat{M} \to M$ for every domain $A \in \mathcal{O}$ such that $\chi_A^2 = \chi_A$. Every object $X \in \hat{M}$ has its own domain $D(X)$ such that

$$\chi_{D(X) \setminus A}(X) = X \iff D(X) \cap A = \emptyset.$$  

(3)

In particle theories, $\hat{M}$ is identified with the space of all the one-dimensional timelike manifolds or curves in $M^{(4)}$, where $\chi_A(l) = l \cap A$ for every domain $A$ and $D(l) = l$. In field theories, the space $\Psi(M^{(4)}, V)$ of the complex valued or $\mathbb{Z}_2$-graded fields over $M^{(4)}$ such that $\psi^{(4)} \in \Psi(M^{(4)}, V)$ is a mapping $\psi^{(4)} : M^{(4)} \to V$ for a complex valued or $\mathbb{Z}_2$-graded vector space $V$. Mapping $\chi_A$ satisfies that $\chi_A(\psi^{(4)}(x)) = \psi^{(4)}(x)$ if $x \in A$ and that $\chi_A(\psi^{(4)}(x)) = 0$ if $x \notin A$, and $D(\psi^{(4)})$ gives the support of $\psi^{(4)}$: $D(\psi^{(4)}) = \text{supp}(\psi^{(4)})$.

In addition, let us consider the set $\mathcal{D}(\hat{M})$ of all the differentiable mapping from $\hat{M}$ to itself and the set $\mathcal{D}(M^{(4)})$ of all the diffeomorphisms of spacetime $M^{(4)}$. In particle theories, set $\mathcal{D}(\hat{M})$ will be regarded as set $\mathcal{D}(M^{(4)})$; and, in field theories, it is the set of all the linear transformations of a field such that $\Phi(\psi^{(4)}) = \psi^{(4)} + \phi^{(4)}$.

Now, let us assume that an object has its own internal-time relative to a domain of the spacetime.

**Law 1** For every domain $A \in \mathcal{O}$, the mapping $\tilde{o}_A : \hat{M} \to S^1$ has an action $S_A : \hat{M} \to \mathbb{R}$ and equips an object $X \in \hat{M}$ with the internal-time $\tilde{o}_A(X)$:

$$\tilde{o}_A(X) = e^{iS_A(X)}.$$  

(4)

For particle theories, a one-dimensional submanifold or a curve $l \subset M^{(4)}$ represents the nonrelativistic motion for a particle such that $(t, x(t)) \in l$ for $t \in T$, where $M^{(4)}$ is the Newtonian spacetime $M^{(4)} = T \times M^{(3)}$ for the Newtonian time $T \subset \mathbb{R}$ and the three-dimensional Euclidean space $M^{(3)}$; thereby, it has the following action for the ordinary Lagrangian $L : TM \to \mathbb{R}$:

$$S_A(l) = \hbar^{-1} \int_{\partial} dt \left( L(x(t), \frac{dx(t)}{dt}) \right),$$  

(5)

where $\hbar = \hbar/4\pi$ or $\hbar/2$ for Planck’s constant $\hbar$ ($\hbar = \hbar/2\pi$). The relativistic motion of a free particle whose mass is $m$ has the following action for the proper-time $\tau \in \mathbb{R}$:

$$S_A(l) = \frac{\hbar}{c} \int_{\tau} d\tau \ mc^2.$$  

(6)

For field theories, field variable $X = \psi^{(4)}$ over spacetime $M^{(4)}$ has the following action for the Lagrangian density $\mathcal{L}_M$ of matters:

$$S_A(\psi^{(4)}) = \frac{1}{\hbar c} \int_{\partial} dy \ L_M(\psi^{(4)}(y), d\psi^{(4)}(y)),$$  

(7)

where $\psi^{(4)}$ is the volume measure of $M^{(4)}$. In the standard field theory, $\psi^{(4)}$ is a set of $\mathbb{Z}_2$-graded fields over spacetime $M^{(4)}$, the Dirac field for fermions, the Yang-Mills field for gauge bosons and other field under consideration. For the Einstein gravity, the Hilbert action includes the metric tensor $g$ on $M^{(4)}$ with a cosmological constant $\Lambda \in \mathbb{R}$:

$$S_A(\psi^{(4)}, g) = \frac{1}{\hbar c} \int_{\partial} dy \ g_{\partial A}(\psi^{(4)}(y), d\psi^{(4)}(y))$$

$$- \frac{1}{\hbar c} \int_{\partial} dy \ g_{\partial A}(\psi^{(4)}(y), d\psi^{(4)}(y)) \left( \frac{c^4}{16\pi G} R_g + \Lambda \right) - \frac{2}{\hbar c} \int_{\partial A} d\psi^{(3)}(y) \frac{c^4}{16\pi G} K_g,$$  

(8)

$^4$Spacetime $M^{(4)}$ may be endowed with some additional structure.
where $R_g$ and $K_g$ are the four-dimensional and the extrinsic three-dimensional scalar curvatures on domain $A$ and on its boundary $\partial A$; and $G$ is the Newton’s constant of gravity. The last term of \( \tilde{\Sigma} \) is necessary to produce the correct Einstein equation for gravity [?].

Let us now consider the subset $\mathcal{D}_A(\tilde{M})$ of set $\mathcal{D}(\tilde{M})$ such that every element $\Phi \in \mathcal{D}_A(\tilde{M})$ satisfies $\chi_{D(X)\backslash A}(\Phi(X)) = X$, and assume it as a infinite-dimensional Lie group. In particle theories, set $\mathcal{D}_A(\tilde{M})$ is the set $\mathcal{D}_A(M)$ of all the diffeomorphisms of $M$ such that $\Phi(l \backslash A = l \backslash A$; and, in filed theories, it is the set of all the linear transformations of a field such that $\Phi(\psi^{(4)}) = \psi^{(4)} + \phi^{(4)}$ for an element $\phi^{(4)} \in \Psi(M^{(4)}, V)$ and that $\phi^{(4)}(x) = 0$ if $x \notin A$. Mapping $\partial_A$ may have the symmetry under a transformation $\Phi \in \mathcal{D}(\tilde{M})$ such that it satisfies the following relation for every pair $(A, X)$:

$$\partial_A(\Phi(X)) = \partial_A(X).$$

Such symmetry verifies the existence of the conserved charge.

Object $X$ and all the rest but $X$ composes the universe $U$. The internal-time $\Pi_A(U)$ of universe $U$ relative to domain $A$ would be separated into two parts:

$$\Pi_A(U) = \partial_A(X) \cdot \partial_A^*(X).$$

Let us call $\partial_A^*(X) \in S^1$ as the external-time of $X$ relative to $A$. Thus, the external-time of universe $U$ would always be unity: $\Pi_A(U) = 1$.

**Law 2** For every domain $A \in \hat{\mathcal{O}}$, the mapping $\partial_A^* : \hat{M} \to S^1$ has an action $S_A^* : \hat{M} \to \mathbb{R}$ and equips an object $X \in \hat{M}$ with the external-time $\partial_A^*(X)$:

$$\partial_A^*(X) = e^{iS^*_A(X)}.$$  \hspace{1cm} (11)

Let us also introduce the mapping $\tilde{s}_A(\tilde{o}) : \hat{M} \to S^1$ that relates mappings $\partial_A^*$ and $\partial_A$:

$$\tilde{o}_A^*(X) = \tilde{o}_A(X) \cdot \tilde{s}_A(\tilde{o})(X).$$ \hspace{1cm} (12)

It has a function $R_A(\tilde{o})$ such that

$$\tilde{s}_A(\tilde{o})(X) = e^{iR_A(\tilde{o})(X)}.$$ \hspace{1cm} (13)

There is also the mapping $\tilde{s}_A(\tilde{o}^*) : \hat{\mathcal{O}} \to S^1$:

$$\tilde{o}_A^*(X) \cdot \tilde{s}_A(\tilde{o}^*)(X) = \tilde{o}_A(X).$$ \hspace{1cm} (14)

Mapping $\tilde{n}_A$ may have the symmetry under a transformation $\Phi \in \mathcal{D}(\tilde{M})$ such that it satisfies the following relation for every pair $(A, X)$:

$$\tilde{o}_A^*(\Phi(X)) = \tilde{o}_A^*(X).$$ \hspace{1cm} (15)

If mapping $\tilde{n}_A$ also has symmetry \( \tilde{n}_A \) for the same transformation $\Phi$, they must satisfy the following invariance:

$$\tilde{s}_A(\tilde{o})(\Phi(X)) = \tilde{s}_A(\tilde{o})(X), \quad \tilde{s}_A(\tilde{o}^*)(\Phi(X)) = \tilde{s}_A(\tilde{o}^*)(X).$$ \hspace{1cm} (16)

The following law further supplies the condition that an object has the actual existence on a domain of the spacetime.

**Law 3** Object $X \in \hat{M}$ has actual existence on domain $A \in \hat{\mathcal{O}}$ when the internal-time coincides with the external-time:

$$\partial_A^*(X) = \partial_A(X).$$ \hspace{1cm} (17)
Relation (17) requires the following quantization condition:

\[ \delta A (\partial) (X) = 1, \quad (18) \]

or equivalently,

\[ \delta A (\partial^*) (X) = 1, \quad (19) \]

which quantizes spacetime \( M(4) \) for an object \( X \in \hat{M} \).

For the space \( d_A(\hat{M}) \) of all the infinitesimal generators of \( D_A(\hat{M}) \), let us consider an arbitrary element \( \Phi_\epsilon \in D_A(\hat{M}) \), differentiable by parameter \( \epsilon \in \mathbb{R} \):

\[ \lim_{\epsilon \to 0} \frac{d\Phi_\epsilon}{d\epsilon} \circ \Phi_\epsilon^{-1} = \xi \in d_A(X). \quad (20) \]

Thus, we can introduce the variation \( \delta \) as follows:

\[ \langle i \delta A (X)^{-1} \delta A (X), \xi \rangle = i \delta A (X)^{-1} \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \delta A (\Phi_\epsilon(X)), \quad (21) \]

\[ \langle i \delta A (X)^{-1} \delta A^* (X), \xi \rangle = i \delta A^* (X)^{-1} \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \delta A^* (\Phi_\epsilon(X)) \quad (22) \]

where \( \langle \ , \ \rangle : d_A^*(\hat{M}) \times d_A(\hat{M}) \rightarrow \mathbb{R} \) is the natural pairing for the dual space \( d_A^*(\hat{M}) \) of \( d_A(\hat{M}) \). This variation satisfies the variational principle of the following law.

**Law 4** Object \( X \in \hat{M} \) must satisfy the variational principle for every domain \( A \in \hat{O} \):

\[ \delta \partial A (X) = 0 \quad \text{and} \quad \delta A^* (X) = 0. \quad (23) \]

Thus, Law 4 keeps Law 3 under the above variation, and also has the following expression:

\[ \delta \partial A (\partial) (X) = 0 \quad \text{and} \quad \delta A^* (\partial^*) (X) = 0. \quad (24) \]

Now, we will consider the mapping \( \mathcal{P} : T \rightarrow \hat{O} \) for the time \( T \subset \mathbb{R} \) of an observer’s clock \( T \). Domain \( \mathcal{P}(t) \) and its boundary \( \partial \mathcal{P}(t) = \mathcal{P}(t) \setminus \mathcal{P}(t) \) represent the past and the present at time \( t \in T \), where \( \mathcal{A} \) is the closure of \( A \in \hat{O} \); and it satisfies the following conditions:

1. for every \( X \in \hat{M} \), \( t_1 < t_2 \in T \Rightarrow \mathcal{P}(t_1) \cap D(X) \subset \mathcal{P}(t_2) \cap D(X) \) (ordering);

2. for every \( X \in \hat{M} \), the present \( \partial \mathcal{P}(t) \cap D(X) \) is a spacelike hypersurface in \( M(4) \) for every time \( t \in T \) (causality).

From Law 3, object \( X \) emerges into the world at time \( t \in T \) when it satisfies

\[ \partial \mathcal{P}(t) (\partial) (X) = 1. \quad (25) \]

This condition of the emergence determines when object \( X \) interacts with all the rest in the world, and discretizes time \( T \) in Whitehead’s philosophy [12]. In other words, what a particle or a field \( X \) gains actual existence or emerges into the world, here, means that it becomes exposed to or has the possibility to interact with the other elements or with the ambient world excluded from the description. Such occasional influences from the unknown factors can break the deterministic feature of the above description; and it would cause the irreversibility in general as considered in Subsection 8.3. The emergence further allows the observation of a particle or a field through an experiment even if the device or its environment is included in the description as shown in Subsection 8.2. Besides, the variational principle of Law 4 produces the equation of motion and the conservation of the frequency of such emergence in the next section.
3 Foundation of Protomechanics

Let us consider the development of present $\partial P(t)$ for short time $T = (t_i, t_f) \subset \mathbb{R}$, keeping the following description without the appearance of singularity; and suppose that the time interval extends long enough to keep the continuity of time beyond the discretization in the previous section, where such discretization would only affect the property of the emergence-measure, defined below, corresponding to the density matrices in quantum mechanics. Assume that present $\partial P(t)$ is diffeomorphic to a three dimensional manifold $M^{(3)}$ by a diffeomorphism $\sigma_t : M^{(3)} \to \partial P(t)$ for every $t \in T$. It induces a corresponding mapping $\tilde{\sigma}_t : M \to M$ for the space $M$ that is three-dimensional physical space $M^{(3)}$ for particle theories or the space $M = \Psi(M^{(3)}, V)$ of all the $C^\infty$-fields over $M^{(3)}$ for field theories. For particle theories, mapping $\tilde{\sigma}_t$ is defined as $\tilde{\sigma}_t(l) = \sigma_t^{-1}(l \cap \sigma_t(M^{(3)}))$ for a curve $l \subset M^{(4)}$; for field theories, it is defined as $\tilde{\sigma}_t(\psi^{(4)}) = \psi^{(4)} \circ \sigma_t$ for a field $\psi^{(4)}$.

Let us assume that space $M$ is a $C^\infty$ manifold endowed with an appropriate topology and the induced topological $\sigma$-algebra.\footnote{M is assumed as an ILH-manifold modeled by the Hilbert space endowed with an inverse-limit topology (consult [10]).} We will denote the tangent space as $TM$ and the cotangent space $T^*M$; and we shall consider the space of all the vector fields over $M$ as $X(M)$ and that of all the 1-forms over $M$ as $\Lambda^1(M)$. To add a one-dimensional cyclic freedom $S^1$ at each point of $M$ introduces the trivial $S^1$-fiber bundle $E(M)$ over $M$. Fiber $S^1$ represents an intrinsic clock of a particle or a field, which is located at every point on $M$. For the system that a particle or a field belongs to and carries with, and a synchronization of every two clocks located at different points in space $M$.

For past $P(t)$ such that $\partial P(t) = \sigma_t(M^{(3)})$, there is an mapping $\sigma_t : TM \to \mathbb{R}$ such that every initial position $(x_0, \dot{x}_0) \in TM$ has an object $X \in M_P$ satisfying the following relation for $x_t = \tilde{\sigma}_t(X)$:

$$ o_t(x_t, \dot{x}_t) = \tilde{o}_{P(t)}(X). $$

For the velocity field $v_t \in X(M)$ such that $v_t(x_t) = \frac{dx_t}{dt}$, we will introduce a section $\eta_t \in \Gamma[E(M)]$ and call it synchronicity over $M$:

$$ \eta_t(x) = o_t(x, v_t(x)). $$

The Lagrangian $L_t^{TM} : TM \to \mathbb{R}$ characterizes the speed of the internal-time:

$$ L_t^{TM}(x_t, \frac{dx_t}{dt}) = -i\hbar \frac{d}{dt} \sigma_t(x_t, \frac{dx_t}{dt})^{-1} \frac{d}{dt} \sigma_t(x_t, \frac{dx_t}{dt}). $$

Since relation (28) is valid for every initial conditions of position $(x_t, \dot{x}_t) \in TM$, it determines the time-development of synchronicity $\eta_t$ in the following way for the Lie derivative $\mathcal{L}_{v_t}$ by velocity field $v_t \in X(M)$:

$$ L_t^{TM}(x, v_t(x)) = -i\hbar \eta_t(x)^{-1} \left( \frac{\partial}{\partial t} + \mathcal{L}_{v_t} \right) \eta_t(x). $$

Let us now consider the mapping $p : \Gamma[E(M)] \to \Lambda^1(M)$ satisfying the following relation:

$$ p(\eta_t) = -i\hbar \eta_t^{-1} d\eta_t. $$

If the energy $E_t(\eta_t) : TM \to \mathbb{R}$ is defined as

$$ E_t(\eta_t)(x) = i\hbar \eta_t(x)^{-1} \frac{\partial}{\partial t} \eta_t(x), $$

\footnote{The introduced freedom would not represent what is corresponding to the local clock in Weyl’s sense or the fifth-dimension in Kaluza’s sense for the four-dimensional spacetime $M^{(4)}$. To consider such freedom, $M^{(4)}$ would be extended to the principal fiber-bundle over $M^{(4)}$ with a N-dimensional special unitary group $SU(N)$.}

7
condition (29) satisfies the following relation:

\[ E_t (\eta_t) (x) = v_t(x) \cdot p(\eta_t) (x) - L_t^{TM} (x, v_t(x)). \]  

(32)

Attention to the following calculation by definition (29):

\[ -i\hbar \frac{\partial}{\partial v} \left\{ a_t (x, v_t(x))^{-1} \left( \frac{\partial}{\partial t} + L_v \right) a_t (x, v_t(x)) \right\} = \frac{\partial L_t^{TM}}{\partial v} (x, v_t(x)). \]  

(33)

Since variational principle (23) in Law 1 implies that \( a_t (x, \dot{x}) \) is invariant under the variation of \( \dot{x} \) at every point \((x, \dot{x})\), i.e.,

\[ \frac{\partial}{\partial \dot{x}} a_t (x, \dot{x}) = 0 \iff \frac{\partial}{\partial v} a_t (x, v_t(x)) = 0 \]  

(34)

then formula (33) has the following different expression:

\[ -i\hbar \frac{\partial}{\partial v} \left\{ a_t (x, v_t(x))^{-1} \left( \frac{\partial}{\partial t} + L_v \right) a_t (x, v_t(x)) \right\} = \frac{\partial}{\partial v} \left\{ v_t(x) \cdot p(\eta_t) (x) \right\} = p(\eta_t) (x). \]  

(35)

Equations (33) and (35) leads to the modified Einstein-de Broglie relation, that was \( p = h/\lambda \) for Planck’s constant \( h = 2\pi \hbar \) and wave number \( \lambda \) in quantum mechanics:

\[ p(\eta_t) (x) = \frac{\partial L_t^{TM}}{\partial v} (x, v_t(x)). \]  

(36)

Notice that this relation (36) produces the Euler-Lagrange equation resulting from the classical least action principle:

\[ dL_t^{TM} (x, v_t(x)) - \left( \frac{\partial}{\partial t} + L_v \right) \frac{\partial L_t^{TM}}{\partial v} (x, v_t(x)) = 0 \]  

(37)

\[ \iff \frac{\partial L_t^{TM}}{\partial x^j} (x_t, \dot{x}_t) - \frac{d}{dt} \frac{\partial L_t^{TM}}{\partial \dot{x}_j} (x_t, \dot{x}_t) = 0; \]  

(38)

thereby, relation (36) is stronger condition than the classical relation (38).

Under the modified Einstein-de Broglie relation (36), relation (12) gives the Legendre transformation and introduces Hamiltonian \( H_t^{TM} \) as a real function on cotangent space \( T^* M \) such that

\[ E_t (\eta_t) (x) = H_t^{TM} (x, p(\eta_t) (x)). \]  

(39)

This satisfies the first equation of Hamilton’s canonical equations of motion:

\[ v_t(x) = \frac{\partial H_t^{TM}}{\partial p} (x, p(\eta_t) (x)). \]  

(40)

Solvability \( [\hat{\partial}, \hat{d}] = 0 \) further leads to the second equation of Hamilton’s canonical equations of motion:

\[ \frac{\partial}{\partial t} p(\eta_t) (x) = -dH_t^{TM} (x, p(\eta_t) (x)), \]  

(41)

which is equivalent to equation (31) of motion under condition (36). If Lagrangian \( L_t^{TM} \) satisfies

\[ \frac{\partial L_t^{TM}}{\partial t} = 0, \]  

(42)
then equations (40) and (41) of motion prove the conservation of energy:

\[
\left( \frac{\partial}{\partial t} + \mathcal{L}_{v_t} \right) H_t^{TM} (x, p(\eta_t) (x)) = 0.
\] (43)

On the other hand, the mapping \( \tilde{s}_{P(t)} (\tilde{o}) \) induces a mapping \( s_t (o_t) : TM \rightarrow S^1 \) such that every initial position \((x_0, \dot{x}_0) \in TM \) has an object \( X \in M_P \) satisfying the following relation:

\[
s_t (o_t) \left( x_t, \frac{dx_t}{dt} \right) = \tilde{s}_{P(t)} (\tilde{o}) (X).
\] (44)

For velocity field \( v_t \), we can define the following section \( \zeta_t (\eta_t) \in \Gamma [E(M)] \) and call it shadow over \( M \):

\[
\zeta_t (\eta_t) (x) = s_t (o_t) (x, v_t(x)).
\] (45)

Condition (22) of emergence now has the following form:

\[
s_t (o_t) \left( x_t, \frac{dx_t}{dt} \right) = 1 \iff \zeta_t (\eta_t) (x) = 1,
\] (46)

when synchronicity \( \eta_t \) comes across the section \( \eta_t^* = \eta_t \cdot \zeta_t (\eta_t) \) at position \( x \in M \). Let us introduce the function \( T_t(o_t)^{TM} : TM \rightarrow \mathbb{R} \) such that

\[
T_t(o_t)^{TM} \left( x_t, \frac{dx_t}{dt} \right) = -i\hbar s_t(o_t) \left( x_t, \frac{dx_t}{dt} \right)^{-1} \frac{\partial}{\partial t} s_t(o_t) \left( x_t, \frac{dx_t}{dt} \right).
\] (47)

Since relation (47) is valid for every initial conditions of position \( x_t \in M \), it determines the time-development of shadow \( \zeta_t (\eta_t) \) in the following way for the Lie derivative \( \mathcal{L}_{v_t} \) by the velocity field \( v_t \in X(M) \) such that \( v_t (x_t) = \frac{dx_t}{dt} \):

\[
T_t(o_t)^{TM} (x, v_t(x)) = -i\hbar s_t \left( \eta_t \right)^{-1} \left\{ \frac{\partial}{\partial t} + \mathcal{L}_{v_t} \right\} \zeta_t \left( \eta_t \right).
\] (48)

In stead of Hamiltonian for a synchronicity, we will consider the emergence-frequency \( f_t (\eta_t) : M \rightarrow \mathbb{R} \) for a shadow such that

\[
2\pi \hbar f_t (\eta_t) (x) = i\hbar s_t \left( \eta_t \right)^{-1} \frac{\partial}{\partial t} s_t \left( \eta_t \right) (x),
\] (49)

which represents the frequency that a particle or a field emerges into the world. Condition (48) satisfies the following relation:

\[
2\pi \hbar f_t (\eta_t) (x) = v_t(x) \cdot p \left( s_t (\eta_t) \right) (x) - T_t(o_t)^{TM} (x, v_t(x)).
\] (50)

Variational principle (24) from Law 4 implies that \( s_t (o_t) (x, \dot{x}) \) is invariant under the variation of \( \dot{x} \) at every point \((x, \dot{x}) \), i.e.,

\[
\frac{\partial}{\partial \dot{x}} s_t (o_t) (x, \dot{x}) = 0 \iff \frac{\partial}{\partial v} s_t (o_t) (x, v_t(x)) = 0,
\] (51)

which leads to the following relation corresponding to the modified Einstein-de Broglie relation for synchronicity \( \eta_t \):

\[
p \left( \zeta_t (\eta_t) \right) (x) = \frac{\partial T_t^{TM} (o_t)}{\partial v} (x, v_t(x)).
\] (52)

Relation (48) proves the conservation of emergence-frequency in the same way as relation (43) proved that of energy (43):

\[
\left( \frac{\partial}{\partial t} + \mathcal{L}_{v_t} \right) f_t (\eta_t) (x) = 0.
\] (53)
Notice that emergence-frequency \( f_t(\eta_t) \) can be negative as well as positive, and that it produces a similar property of the Wigner function for a wave function in quantum mechanics as discussed in Section 6.

In addition, the probability measure \( \tilde{\nu} \) on \( \tilde{M} \) induces the probability measure \( \nu_t \) on \( M \) at time \( t \in T \) such that
\[
d\nu_t(x_t, dx_t) = \tilde{\nu}(X),
\]
that represents the ignorance of the initial position in \( M \); thereby it satisfies the conservation law:
\[
\frac{d}{dt} d\nu_t(x_t, dx_t) = 0.
\]
This relation can be described by using the Lie derivative \( L_{vt} \) as
\[
\frac{\partial}{\partial t} + L_{vt} d\nu_t(x, v_t(x)) = 0.
\]
Since the velocity field \( v_t \) has relation (40) with synchronicity \( \eta_t \), we can define the emergence-measure \( \mu_t(\eta_t) \) as the product of the probability measure with the emergence-frequency:
\[
d\mu_t(\eta_t)(x) = d\nu_t(x, v_t(x)) \cdot f_t(\eta_t)(x).
\]
Thus, we will obtain the following equation of motion for emergence-measure \( d\mu_t(\eta_t) \):
\[
\frac{\partial}{\partial t} + L_{vt} d\mu_t(\eta_t) = 0.
\]

Let me summarize the obtained mechanics or protomechanics based on equations (29) and (58) of motion with relation (40) in the following theorem that this section proved.

**Theorem 1** (Protomechanics) Hamiltonian \( H_t^{TM} : T^*M \to \mathbb{R} \) defines the velocity field \( v_t \in \mathcal{X}(M) \) and Lagrangian \( L_t^{TM} : TM \to \mathbb{R} \) as follows:
\[
v_t = \frac{\partial H_t^{TM}}{\partial p}(p(\eta_t)) \quad (59)
\]
\[
L_t^{TM}(x, v(x)) = v(x) \cdot p(\eta_t)(x) - H_t^{TM}(x, p(\eta_t)(x)),
\]
where mapping \( p : \Gamma[E(M)] \to \Lambda^1(M) \) satisfies the modified Einstein-de Broglie relation:
\[
p(\eta_t) = -i\hbar^{-1}d\eta_t.
\]

The equation of motion is the set of the following equations:
\[
\left( \frac{\partial}{\partial t} + L_{vt} \right) \eta_t(x) = -i\hbar^{-1}L_t^{TM}(x, v_t(x)) \eta_t(x),
\]
\[
\left( \frac{\partial}{\partial t} + L_{vt} \right) d\mu_t(\eta_t) = 0.
\]

### 4 DYNAMICAL CONSTRUCTION OF PROTOMECHANICS

Let us express the introduced protomechanics in the statistical way for the ensemble of all the synchronicities on \( M \), and construct the dynamical description for the collective motion of the sections of \( E(M) \). Such statistical description realizes the description within a long-time interval through the introduced relabeling process so as to change the labeling time, that is the time for the initial condition before analytical problems occur. In addition, it clarifies the relationship between classical mechanics and quantum mechanics. For mathematical simplicity, the discussion below suppose that \( M \) is a \( N \)-dimensional manifold for a finite natural number \( N \in \mathbb{N} \).
4.1 Description of Statistical-State

The derivative operator \( D = hdx^i\partial_j : T_0^m(M) \to T_0^{m+1}(M) \) \((m \in \mathbb{N})\) for the space \( T_0^m(M) \) of all the \((0, n)\)-tensors on \( M \) can be described as

\[
D^n p(x) = h^n \left( \prod_{k=1}^{n} \partial_j p_j(x) \right) dx^j \otimes \left( \otimes_{k=1}^{n} dx^{jk} \right).
\]  

(64)

By utilizing this derivative operator \( D \), the following Banach norm endows the space \( \Gamma \left[ E(M) \right] \) of all the \( C^\infty \) sections of \( E(M) \) with a norm topology for the family \( \mathcal{O}_{\Gamma \left[ E(M) \right]} \) of the induced open balls:

\[
\| p(\eta) \| = \sup_M \sum_{\kappa \in \mathbb{Z}_{\geq 0}} h^\kappa \left| D^\kappa p(\eta)(x) \right|_g,
\]

(65)

where the metric \((2, 0)\)-tensor \( g = g^{ij}\partial_i \otimes \partial_j \in T^2_0(M) \) on \( M \) gives

\[
\left| D^\kappa p(x) \right|_g = h^\kappa \sqrt{\sum_{k=1}^{\kappa} g^{ij} \prod_{k=1}^{\kappa} g^{jk} \left( \partial_i p(\eta)_i \right) \left( \partial_j p(\eta)_j \right) (x)}.
\]

(66)

In terms of the corresponding norm topology on \( \Lambda^1(M) \)\(^7\) we can consider the space \( C^\infty \left( \Lambda^1(M), C^\infty(M) \right) \) of all the \( C^\infty \)-differentiable mapping from \( \Lambda^1(M) \) to \( C^\infty(M) = C^\infty(M, \mathbb{R}) \) and the subspaces of the space \( C\left( \Gamma [E(M)] \right) \) such that

\[
C \left( \Gamma [E(M)] \right) = \left\{ p^* F : \Gamma [E(M)] \to C^\infty(M) \mid F \in C^\infty \left( \Lambda^1(M), C^\infty(M) \right) \right\}.
\]

(67)

Classical mechanics requires the local dependence on the momentum for functionals, while quantum mechanics needs the wider class of functions that depends on their derivatives. The space of the classical functionals and that of the quantum functionals are defined as

\[
C_{cl} \left( \Gamma [E(M)] \right) = \left\{ p^* F \in C \left( \Gamma [E(M)] \right) \mid p^* F (\eta) (x) = F^{TM} (x, p(\eta)(x)) \right\}
\]

(68)

\[
C_q \left( \Gamma [E(M)] \right) = \left\{ p^* F \in C \left( \Gamma [E(M)] \right) \mid p^* F (\eta) (x) = F^Q (x, p(\eta)(x), ..., D^n p(\eta)(x), ...) \right\},
\]

(69)

and related with each other as

\[
C_{cl} \left( \Gamma [E(M)] \right) \subset C_q \left( \Gamma [E(M)] \right) \subset C \left( \Gamma [E(M)] \right).
\]

(70)

In other words, the classical-limit indicates the limit of \( h \to 0 \) with fixing \( \| p(\eta)(x) \| \) finite at every \( x \in M \), or what the characteristic length \([x]\) and momentum \([p]\) such that \( x/|x| \approx 1 \) and \( p/|p| \approx 1 \) satisfies

\[
[p]^{-n-1} D^n p(\eta)(x) \ll 1.
\]

(71)

In addition, the \( n \)-th semi-classical system can have the following functional space:

\[
C_{n+1} \left( \Gamma [E(M)] \right) = \left\{ p^* F \in C \left( \Gamma [E(M)] \right) \mid p^* F (\eta) (x) = F_{C_{n>}} (x, p(\eta)(x), ..., D^n p(\eta)(x)) \right\}.
\]

(72)

Thus, there is the increasing series of subsets as

\[
C_1 \left( \Gamma [E(M)] \right) \ldots \subset C_n \left( \Gamma [E(M)] \right) \ldots \subset C_\infty \left( \Gamma [E(M)] \right) \subset C \left( \Gamma [E(M)] \right),
\]

(73)

\(^7\) Assume here that \( \Lambda^1(M) \) has the Banach norm such that \( \| p \| = \sup_M \sum_{\kappa \in \mathbb{Z}_{\geq 0}} |D^\kappa p(x)|_g \), for \( p \in \Lambda^1(M) \).
where $F_{<1>} = F^{cl}$ and $F_{<\infty>} = F^{q}$:

$$C_1 (\Gamma [E(M)]) = C^{cl} (\Gamma [E(M)]) \quad (74)$$
$$C_\infty (\Gamma [E(M)]) = C_q (\Gamma [E(M)]). \quad (75)$$

On the other hand, the emergence-measure $\mu (\eta)$ has the Radon measure $\tilde{\mu} (\eta)$ for section $\eta \in \Gamma [E(M)]$ such that

$$\tilde{\mu} (\eta) (F (p(\eta))) = \int_M d\mu (\eta) (x) F (p(\eta)) (x). \quad (76)$$

The introduced norm topology on $\Gamma (E(M))$ induces the topological $\sigma$-algebra $\mathcal{B} (\mathcal{O}_\Gamma (E(M)))$; thereby manifold $\Gamma (E(M))$ becomes a measure space having the probability measure $\mathcal{M}$ such that

$$\mathcal{M} (\Gamma (E(M))) = 1. \quad (77)$$

For a subset $C_n (\Gamma (E(M))) \subset C (\Gamma (E(M)))$, an element $\tilde{\mu} \in C_n (\Gamma (E(M)))^*$ is a linear functional $\tilde{\mu} : C_n (\Gamma [E(M)]) \to \mathbb{R}$ such that

$$\tilde{\mu} (p^* F) = \int_{\Gamma [E(M)]} d\mathcal{M} (\eta) \tilde{\mu} (\eta) (F (p(\eta))) \quad (78)$$
$$= \int_{\Gamma [E(M)]} d\mathcal{M} (\eta) \int_M dv (x) \rho (\eta) (x) F (p(\eta)) (x), \quad (79)$$

where $d\mu (\eta) = dv \rho (\eta)$. Let us call mapping $\rho : \Gamma [E(M)] \to C^{\infty} (M)$ as the emergence-density. The dual spaces make an decreasing series of subsets (consult [13] in the definition of the Gelfand triplet):

$$C_1 (\Gamma [E(M)])^* \supset \ldots C_n (\Gamma [E(M)])^* \supset \ldots C_\infty (\Gamma [E(M)])^* \supset C (\Gamma [E(M)])^* . \quad (80)$$

Thus, relation (70) requires the opposite sequence for the dual spaces:

$$C^{cl} (\Gamma (E(M)))^* \supset C_q (\Gamma (E(M)))^* \supset C (\Gamma (E(M)))^*. \quad (81)$$

Let us summarize how the present theory includes both quantum mechanics and classical mechanics in the following diagram, though leaving detail considerations for Section 5 and Section 6, respectively.
4.2 Description of Time-Development

The group $D(M)$ of all the $C^\infty$-diffeomorphisms of $M$ and the abelian group $C^\infty(M)$ of all the $C^\infty$-functions on $M$ construct the semidirect product $S(M) = D(M) \times_{sem} C^\infty(M)$ of $D(M)$ with $C^\infty(M)$, and define the multiplication · between $\Phi_1 = (\varphi_1, s_1)$ and $\Phi_2 = (\varphi_2, s_2) \in S(M)$ as

$$\Phi_1 \cdot \Phi_2 = (\varphi_1 \circ \varphi_2, (\varphi_2^* s_1) \cdot s_2),$$

(82)

for the pullback $\varphi^*$ by $\varphi \in D(M)$. The Lie algebra $s(M)$ of $S(M)$ has the Lie bracket such that, for $V_1 = (v_1, U_1)$ and $V_2 = (v_2, U_2) \in s(M)$,

$$[V_1, V_2] = ([v_1, v_2], v_1 U_2 - v_2 U_1 + [U_1, U_2]);$$

(83)

and its dual space $s(M)^*$ is defined by natural pairing $\langle , \rangle$. Lie group $S(M)$ now acts on every $C^\infty$ section of $E(M)$ (consult APPENDIX B). We shall further introduce the group $Q(M) = Map(\Gamma [E(M)] \cdot S(M))$ of all the mapping from $\Gamma [E(M)]$ into $S(M)$, that has the Lie algebra $q(M) = Map(\Gamma [E(M)] \cdot s(M))$ and its dual space $q(M)^* = Map(\Gamma [E(M)] \cdot s(M)^*)$.

To investigate the group structure of the system considered, let us further define the emergence-momentum $\mathcal{J} \in q(M)^*$ as follows:

$$\mathcal{J} (\eta) = dM (\eta) \ (\hat{\mu} (\eta) \otimes p(\eta)), \hat{\mu} (\eta)).$$

(84)

Thus, the functional $\mathcal{F} : q(M)^* \rightarrow \mathbb{R}$ can always be defined as

$$\mathcal{F} (\mathcal{J}) = \hat{\mu} (p^* F).$$

(85)

On the other hand, the derivative $D_\rho F (p)$ can be introduced as follows excepting the point where the distribution $\rho$ becomes zero:

$$D_\rho F (p) (x) = \sum_{(n_1, \ldots, n_N) \in \mathbb{N}^N} \frac{1}{\rho(x)} \left\{ \prod_{i=1}^N (-\partial_i)^{n_i} \left( \rho(x)p(x) \frac{\partial F}{\partial \left( \prod_{j=1}^N \partial_i^{n_j} \right) p_j} \right) \right\} \partial_j.$$

(86)

Then, operator $\hat{F} (\eta) = \frac{\partial \mathcal{F}}{\partial \mathcal{J}} (\mathcal{J} (\eta))$ is defined as

$$\frac{d}{de} \bigg|_{e=0} \mathcal{F} (\mathcal{J} + e\mathcal{K}) = \langle \mathcal{K}, \hat{F} \rangle,$$

(87)

i.e.,

$$\hat{F} (\eta) = (D_{\rho(\eta)} F (p(\eta)), -p(\eta) \cdot D_{\rho(\eta)} F (p(\eta)) + F (p(\eta)));$$

(88)

thereby, the following null-lagrangian relation can be obtained:

$$\mathcal{F} (\mathcal{J}) = \langle \mathcal{J}, \hat{F} \rangle.$$

(89)

Let us consider the time-development of the section $\eta_\tau^t(\eta) \in \Gamma [E(M)]$ such that the labeling time $\tau$ satisfies $\eta_\tau^t(\eta) = \eta$. It has the momentum $p_\tau^t(\eta) = -i\hbar n_\tau^t(\eta)^{-1} d\eta_\tau^t(\eta)$ and the emergence-measure $\mu_\tau^t(\eta)$ such that

$$dM (\eta) \ \tilde{\mu}_\tau^t (\eta) = dM (\eta_\tau^t (\eta)) \ \tilde{\mu}_\tau (\eta_\tau^t (\eta)) ;$$

(90)

$$\tilde{\mu}_\tau (p^* F_\tau) = \int_{\Gamma [E(M)]} dM (\eta) \ \tilde{\mu}_\tau (\eta) \ (p^* F_\tau (\eta))$$

(91)

$$= \int_{\Gamma [E(M)]} dM (\eta) \ \tilde{\mu}_\tau^t (\eta) \ (p^* F (\eta_\tau^t (\eta)))$$

(92)

$$= \int_{\Gamma [E(M)]} dM (\eta) \ \int_M du(x) \ \rho_\tau^t (\eta)(x) F_\tau (\rho_\tau^t (\eta))(x).$$

(93)
The introduced labeling time \( \tau \) can always be chosen such that \( \eta^\tau_\eta(\eta) \) does not have any singularity within a short time for every \( \eta \in \Gamma[E(M)] \). The emergence-momentum \( \mathcal{J}^\tau_\eta \in q(M) \) such that

\[
\mathcal{J}^\tau_\eta(\eta) = \mathcal{J}_\eta(\eta^\tau_\eta(\eta)) = \mathcal{J}_\eta(\eta^\tau_\eta(\eta)) = \mathcal{J}_\eta(\eta^\tau_\eta(\eta)) = \mathcal{J}_\eta(\eta^\tau_\eta(\eta))
\]

satisfies the following relation for the functional \( \mathcal{F}_t : q(M) \rightarrow \mathbb{R} \):

\[
\mathcal{F}_t(\mathcal{J}^\tau_\eta) = \mu_\tau(p^*F_t),
\]

whose value is independent of labeling time \( \tau \). The operator \( \hat{F}^\tau_\eta = \frac{\partial \mathcal{F}_{t}}{\partial \mathcal{J}^\tau_\eta} \) is defined as

\[
\left. \frac{d}{dx} \right|_{t=0} \mathcal{F}_t(\mathcal{J}^\tau_\eta + \epsilon \mathcal{K}) = \left\langle \mathcal{K}, \hat{F}^\tau_\eta \right\rangle,
\]

i.e.,

\[
\hat{F}^\tau_\eta = (\mathcal{D}_{\rho^\tau_\eta}(\eta)F_t(p^\tau_\eta(\eta)), -p^\tau_\eta(\eta) \cdot \mathcal{D}_{\rho^\tau_\eta}(\eta)F_t(p^\tau_\eta(\eta)) + F_t(p^\tau_\eta(\eta))).
\]

Thus, the following null-lagranian relation can be obtained:

\[
\mathcal{F}_t(\mathcal{J}^\tau_\eta, \hat{F}^\tau_\eta),
\]

while the normalization condition has the following expression:

\[
\mathcal{I}(\mathcal{J}^\tau_\eta) = 1 \quad \text{for} \quad \mathcal{I}(\mathcal{J}^\tau_\eta) = \int_{\Gamma[E(M)]} d\mathcal{M}(\eta) \mu_\tau(\eta)(M).
\]

### Theorem 2

For Hamiltonian operator \( \hat{H}^\tau_\eta = \frac{\partial \mathcal{H}}{\partial \mathcal{J}^\tau_\eta} \in q(M) \) corresponding to Hamiltonian \( p^*H_t(\eta)(x) = H^{T^*M}_t(x, p(\eta)) \), equations \( [23] \) and \( [38] \) of motion becomes Lie-Poisson equation

\[
\frac{\partial \mathcal{J}^\tau_\eta}{\partial t} = ad_{H^\tau_\eta}^* \mathcal{J}^\tau_\eta,
\]

which can be expressed as

\[
\frac{\partial}{\partial t} p^\tau_{\eta}(\eta)(x) = -\sqrt{-1} \partial_j \left( \frac{\partial H^{T^*M}_t}{\partial p^*_{j}}(x, p^\tau_\eta(\eta)(x)) \rho^\tau_\eta(\eta)(x) \right),
\]

\[
\frac{\partial}{\partial t} \left( p^\tau_{\eta}(\eta)(x)p_{ik}^\tau(\eta)(x) \right) = -\sqrt{-1} \partial_j \left( \frac{\partial H^{T^*M}_t}{\partial p^*_{j}}(x, p^\tau_\eta(\eta)(x)) \rho^\tau_\eta(\eta)(x)p_{ik}^\tau(\eta)(x) \right)
\]

\[
-\rho^\tau_\eta(\eta)(x)p_{ij}^\tau(\eta)(x)\partial_k \frac{\partial H^{T^*M}_t}{\partial p^*_{j}}(x, p^\tau_\eta(\eta)(x))
\]

\[
+\rho^\tau_\eta(\eta)(x)\partial_k \left( p_{i}^\tau(\eta)(x) \cdot \frac{\partial H^{T^*M}_t}{\partial p^*_{j}}(x, p^\tau_\eta(\eta)(x)) - H^{T^*M}_t(x, p^\tau_\eta(\eta)(x)) \right).
\]

### Proof

Lie-Poisson equation \( [103] \) is calculated for \( \mathcal{D}H^\tau_\eta(\eta) = \mathcal{D}_{\rho^\tau_\eta(\eta)}H_t(p^\tau_\eta(\eta)) \) as follows:

\[
\frac{\partial}{\partial t} \rho^\tau_\eta(\eta)(x) = -\sqrt{-1} \partial_j \left( \mathcal{D}H^\tau_\eta(\eta)(x) \rho^\tau_\eta(\eta)(x) \right),
\]

(105)
\[ \frac{\partial}{\partial t} (p_i^*(\eta)(x)p_{ik}^*(\eta)(x)) = -\sqrt{-1} \partial_j \left( D^j H^*_i(\eta)(x)p_{ij}^*(\eta)(x)p_{ik}^*(\eta)(x) \right) \]
\[ -\rho_i^*(\eta)(x)p_{ij}^*(\eta)(x)\partial_k D^j H^*_i(\eta)(x) \]
\[ + \rho_i^*(\eta)(x)\partial_k \left( p_{ij}^*(\eta)(x) \cdot DH^*_i(\eta)(x) - H_i(p_{ij}^*(\eta)(x)) \right) , \]

where \( dv = dx^1 \wedge ... dx^N \) and \( \sqrt{\det(g^{\alpha\beta})} \) for the local coordinate \( x = (x^1, x^2, ..., x^N) \). Second equation \( (106) \) can be rewritten in conjunction with the conservation \( (104) \) of the emergence-density as

\[ \frac{\partial}{\partial t} p_{ik}^*(\eta)(x) + D^j H^*_i(\eta)(x)\partial_j p_{ik}^*(\eta)(x) + p_{ij}^*(\eta)(x)\partial_k D^j H^*_i(\eta)(x) = \partial_k L^*_i(\eta)(x) , \]

where

\[ L^*_i(\eta)(x) = p_i^*(\eta)(x) \cdot DH^*_i(\eta)(x) - H_i(p_{ij}^*(\eta)(x)) , \]

or, by using Lie derivatives,

\[ \mathcal{L}_{DH^*_i(\eta)} p^*_i(\eta) = dL^*_i(\eta) . \]

Thus, we can obtain the equation of motion in the following simpler form by using Lie derivatives:

\[ \mathcal{L}_{DH^*_i(\eta)} \eta^*_i = -i\hbar L^*_i(\eta) \eta^*_i \]
\[ \mathcal{L}_{DH^*_i(\eta)} \rho^*_i(\eta) dv = 0 , \]

which is equivalent to the equations \( (24) \) and \( (58) \) when \( p^* H_i(\eta)(x) = H_i^{T*M}(x, p(\eta)) \).

Equation \( (102) \) will prove in the following two sections to include the Schrödinger equation in canonical quantum mechanics and the classical Liouville equations in classical mechanics.

For \( \mathcal{U}^*_i \in Q(M) \) such that \( \frac{\partial}{\partial t} \circ (\mathcal{U}^*_i)^{-1} = H_i^*(\eta) \in q(M) \), let us introduce the following operators:

\[ \tilde{H}^*_i(\eta) = \text{Ad}^{-1}_{\mathcal{U}^*_i} H^*_i(\eta) = H^*_i(\eta) , \text{ and } \tilde{F}^*_i(\eta) = \text{Ad}^{-1}_{\mathcal{U}^*_i} F^*_i(\eta) . \]

It satisfies the following theorem.

**Theorem 3** Lie-Poisson equation \( (103) \) is equivalent to the following equation:

\[ \frac{\partial}{\partial t} \tilde{F}^*_i = \left[ \tilde{H}^*_i, \tilde{F}^*_i \right] + \left( \frac{\partial \tilde{F}^*_i}{\partial t} \right) . \]

**Proof.** Equation \( (104) \) of motion concludes the following equation:

\[ \left\langle \frac{\partial}{\partial t} \mathcal{J}^*_i, \tilde{F}^*_i \right\rangle = \left\langle \text{ad}_{\tilde{H}^*_i} \mathcal{J}^*_i, \tilde{F}^*_i \right\rangle . \]

The left hand side can be calculated as

\[ \text{L.H.S.} = \frac{d}{dt} \mathcal{F}_i(\mathcal{J}^*_i) - \frac{\partial \mathcal{F}_i}{\partial t} (\mathcal{J}^*_i) \]
\[ = \left\langle \left( \frac{\partial}{\partial t} \text{Ad}_{\mathcal{U}^*_i} \mathcal{J}^*_i \right), \tilde{F}^*_i \right\rangle - \left( \text{Ad}_{\mathcal{U}^*_i} \mathcal{J}^*_i, \frac{\partial \tilde{F}^*_i}{\partial t} \right) \]
\[ = \left\langle \mathcal{J}^*_i, \frac{\partial}{\partial t} \tilde{F}^*_i \right\rangle - \left( \mathcal{J}^*_i, \frac{\partial \tilde{F}^*_i}{\partial t} \right) ; \]
Thus, we can obtain this theorem.

The general theory for Lie-Poisson systems certifies that, if a group action of Lie group \( Q(M) \) keeps the Hamiltonian \( \mathcal{H}_t : q(M)^* \to \mathbb{R} \) invariant, there exists an invariant charge functional \( Q : \Gamma [E(M)] \to C(M) \) and the induced function \( \tilde{Q} : q(M)^* \to \mathbb{R} \) such that

\[
\left[ \dot{H}_t, \tilde{Q} \right] = 0,
\]

where \( \tilde{Q} \) is expressed as

\[
\dot{Q} = \left( D_{\rho(\eta)} Q(p(\eta)), -p(\eta) \cdot D_{\rho(\eta)} Q(p(\eta)) + Q(p(\eta)) \right).
\]

5 DEDUCTION OF CLASSICAL MECHANICS

In classical Hamiltonian mechanics, the state of a particle on manifold \( M \) can be represented as a position in the cotangent bundle \( T^* M \). In this section, we will reproduce the classical equation of motion from the general theory presented in the previous section. Let us here concentrate ourselves on the case where \( M \) is \( N \)-dimensional manifold for simplicity, though the discussion below would still be valid if substituting an appropriate Hilbert space when \( M \) is infinite-dimensional ILH-manifold\(^\text{[10]}\).

5.1 Description of Statistical State

Now, we must be concentrated on the case where the physical functional \( F \in C^\infty \left( \Lambda^1(M), C^\infty(M) \right) \) does \textit{not} depend on the derivatives of the \( C^\infty \) 1-form \( p(\eta) \in \Lambda^1(M) \) induced from \( \eta \in \Gamma [E(M)] \), then it has the following expression:

\[
p^* F(\eta)(x) = F^{T^* M}(x, p(\eta)(x)).
\]

Let us choose a coordinate system \((U, x_\alpha)_{\alpha \in \Lambda_M}\) for a covering \( \{U_\alpha\}_{\alpha \in \Lambda_M} \) over \( M \), i.e., \( M = \bigcup_{\alpha \in \Lambda_M} U_\alpha \). Let us further choose a reference set \( U \subset U_\alpha \) such that \( v(U) \neq 0 \) and consider the set \( \Gamma_{Uk} [E(M)] \) of the \( C^\infty \) sections of \( E(M) \) having corresponding momentum \( p(\eta) \) the supremum of whose every component \( p_j(\eta) \) in \( U \) becomes the value \( k_j \) for \( k = (k_1, ..., k_N) \in \mathbb{R}^N \). Then

\[
\Gamma_{Uk} [E(M)] = \left\{ \eta \in \Gamma [E(M)] \left| \sup_U p_j(\eta)(x) = \hbar k_j \right. \right\}.
\]

Thus, every section \( \eta \in \Gamma [E(M)] \) has some \( k \in \mathbb{R}^N \) such that \( \eta = \eta[k] \in \Gamma_{Uk}[E(M)] \). Notice that \( \Gamma_{Uk} [E(M)] \) can be identified with \( \Gamma_{U'k} [E(M)] \) for every two reference sets \( U \) and \( U' \in M \), since there exists a diffeomorphism \( \varphi \) satisfying \( \varphi(U) = U' \); thereby, we will simply denote \( \Gamma_{Uk} [E(M)] \) as \( \Gamma_k [E(M)] \).

On the other hand, let us consider the space \( L(T^* M) \) of all the Lagrange foliations, i.e., every element \( \tilde{p} \in L(T^* M) \) is a mapping \( \tilde{p} : \mathbb{R}^N \to \Lambda^1(M) \) such that each \( q \in T^* M \) has a unique \( k \in \mathbb{R}^N \) as

\[
q = \tilde{p}[k](\pi(q)).
\]

\[^8\] To substitute \( \Gamma_{Uk} [E(M)] = \left\{ \eta \in \Gamma [E(M)] \left| \int_U dv(x) p_j(\eta)(x) = k_j \right. \right\} \) for definition of also induces the similar discussion below, while there exist a variety of the classification methods that produce the same result.

\[16\]
For every $\tilde{p} = p \circ \tilde{q} \in L(T^*M)$ such that $\tilde{q}[k] \in \Gamma_k[E(M)]$, it is possible to separate an element $\eta[k] \in \Gamma_k[E(M)]$ for a $\xi \in \Gamma_0[E(M)]$ as

$$\eta[k] = \tilde{q}[k] \cdot \xi,$$

(126)
or to separate momentum $p(\eta[k])$ as

$$p(\eta[k]) = \tilde{p}[k] + p(\xi);$$

(127)
thereby, we can express the emergence-density $\rho : \Gamma[E(M)] \to C^\infty(M)$ in the following form for the function $\varphi(\xi) \in C^\infty(T^*M, \mathbb{R})$ on $T^*M$:

$$\rho(\eta[k])(x) = \varphi(\xi)(x, p(\eta[k])(x)).$$

(128)
We call the set $B[E(M)] = \Gamma_0[E(M)]$ the back ground of $L(T^*M)$. For the Jacobian-determinant

$$\sigma[k] = \det \left( \frac{\partial \tilde{p}[k]}{\partial \xi} \right),$$

(129)
we will define the measure $\mathcal{N}$ on $B[E(M)]$ for the $\sigma$-algebra induced from that of $\Gamma[E(M)]$:

$$dM(\eta[k]) \, dv(x) = d^N \kappa d\mathcal{N}(\xi) \, dv(x) \, \sigma[k](x).$$

(130)
For separation (127), the Radon measure $\mu(\eta)$ induces the measure $\omega ^N$ on $T^*M$ in the following lemma such that $\omega ^N = \phi_{U_\alpha} d^N x \wedge d^N k$ for $d^N x = dx^1 \wedge \ldots dx^N$ and $d^N k = dk^1 \wedge \ldots dk^N$.

Lemma 1 The following relation holds:

$$\tilde{\mu}(p^*F) = \int_{T^*M} \omega ^N(q) \, \rho^{T^*M}(q) \, F^{T^*M}(q),$$

(131)
where

$$\rho^{T^*M}(q) = \int_{B[E(M)]} d\mathcal{N}(\xi) \, \varphi(\xi)(q).$$

Proof. The direct calculation based on separation (127) shows

$$\tilde{\mu}(p^*F) = \int_{\Gamma[E(M)]} dM(\eta) \, \mu(\eta)(p^*F(\eta))$$

$$= \int_{\Gamma[E(M)]} dM(\eta[k]) \int_M dv(x) \, \varphi(\xi)(x, p(\eta[k])(x)) \, F^{T^*M}(x, p(\eta[k])(x))$$

$$= \int_{\mathbb{R}^N} d^N k \int_{B[E(M)]} d\mathcal{N}(\xi) \int_M dv(x) \, \sigma[k](x)$$

$$\times \varphi(\xi)(x, p(\eta[k])(x)) \, F^{T^*M}(x, p(\eta[k])(x))$$

$$= \int_{\mathbb{R}^N} d^N k \int_{B[E(M)]} d\mathcal{N}(\xi) \int_M dv(x) \, \sigma[k](x)$$

$$\times \varphi(\xi)(x, \tilde{p}[k](x) + p(\xi)(x)) \, F^{T^*M}(x, \tilde{p}[k](x) + p(\xi)(x))$$

(132)
where $T^*M = \bigcup_{\alpha \in \Lambda_M} A_\alpha$ is the disjoint union of $A_\alpha \in \mathcal{B}(\mathcal{O} M)$ such that (1) $\pi(A_\alpha) \subset U_\alpha$ and that (2) $A_\alpha \cap A_\beta = \emptyset$ for $\alpha \neq \beta \in \Lambda_M$ (consult APPENDIX A).

If defining the probability function $\rho^{T^*M} : T^*M \to \mathbb{R}$ such that

$$\rho^{T^*M}(q) = \int_{B[E(M)]} d\mathcal{N}(\xi) \, \varphi(\xi)(q),$$

(133)
we can obtain this lemma. □
5.2 Description of Time-Development

Let us consider the time-development of the functional $\tilde{\mu}_t: C^1 (\Gamma (M), C (M)) \to \mathbb{R}$ for $p_t^\tau (\eta [k]) = \tilde{p}_t^\tau [k] + p (\xi)$. For the Jacobian-determinant $\sigma_t^\tau [k] = \det \left( \frac{\partial \tilde{p}_t^\tau [k]}{\partial \eta} \right)$, the following relation holds:

$$\tilde{\mu}_t (p^* F_t) = \int_{T^* M} \omega^N (q) \rho_t^{T^* M} (q) F_t^{T^* M} (q) \quad (135)$$

$$= \int_{\mathbb{R}^N} d^N k \int_M dv (x) \tilde{p}_t^\tau [k] (x) F_t^{T^* M} (x, \tilde{p}_t^\tau [k] (x)) , \quad (136)$$

where

$$\tilde{p}_t^\tau [k] (x) = \sigma_t^\tau [k] (x) \rho_t^{T^* M} (x, \tilde{p}_t^\tau [k] (x)) . \quad (137)$$

The Jacobian-determinant $\sigma_t^\tau [k]$ satisfies the following relation:

$$\frac{d M (\eta^\tau (\eta))}{d M (\eta)} = \frac{\sigma_t^\tau [k]}{\sigma [k]} . \quad (138)$$

Thus, we can define the reduced emergence-momentum $\tilde{\mathcal{J}}_t \in \bar{\mathcal{q}} (M)^* = q (M)^*/B [E (M)]$ as follows:

$$\tilde{\mathcal{J}}_t (\bar{\eta} [k]) = (d^N k \land dv \tilde{p}_t^\tau [k] \otimes \tilde{p}_t^\tau [k], d^N k \land dv \tilde{p}_t^\tau [k]) ; \quad (139)$$

and we can define the functional $\mathcal{F}_t \in C^\infty (\bar{\mathcal{q}} (M)^*, \mathbb{R})$ as

$$\tilde{\mathcal{F}}_t (\tilde{\mathcal{J}}_t) = \tilde{\mu}_t (p^* F_t) \quad (140)$$

$$\quad = \int_{\mathbb{R}^N} d^N k \int_M dv (x) \tilde{p}_t^\tau [k] (x) F_t^{T^* M} (x, \tilde{p}_t^\tau [k] (x)) , \quad (141)$$

which is independent of labeling time $\tau$.

Then, the operator $\tilde{F}_t^{cl} = \frac{\partial \tilde{\mathcal{F}}_t}{\partial \tilde{\mathcal{J}}_t} (\tilde{\mathcal{J}}_t)$ satisfies

$$\tilde{F}_t^{cl} (x, p) = \left( \frac{\partial F_t^{T^* M}}{\partial p} (x, \tilde{p}_t^\tau [k] (x)) , -L_{F_t^{T^* M}} (x, F_t^{T^* M} (x, \tilde{p}_t^\tau [k] (x))) \right) , \quad (142)$$

where

$$L_{F_t^{T^* M}} (x, \frac{\partial F_t^{T^* M}}{\partial p} (x, p)) = p \cdot \left( \frac{\partial F_t^{T^* M}}{\partial p} (x, p) - F_t^{T^* M} (x, p) \right) \quad (143)$$

is the Lagrangian if function $F_t$ is Hamiltonian $H_t$. Thus, the following null-lagrangian relation can be obtained:

$$\tilde{\mathcal{F}}_t (\tilde{\mathcal{J}}_t) = \langle \tilde{\mathcal{J}}_t, \tilde{F}_t^{cl} \rangle . \quad (144)$$

Besides, the normalization condition becomes

$$\tilde{\mathcal{I}} (\tilde{\mathcal{J}}_t) = 1 \quad \text{for} \quad \tilde{\mathcal{I}} (\tilde{\mathcal{J}}_t) = \int_{\mathbb{R}^N} d^N k \int_M dv (x) \tilde{p}_t^\tau [k] (x). \quad (145)$$

**Theorem 4** For Hamiltonian operator $\tilde{H}_t = \frac{\partial H_t}{\partial \tilde{\mathcal{J}}_t} (\tilde{\mathcal{J}}_t) \in \bar{\mathcal{q}} (M)$, the equation of motion becomes Lie-Poisson equation:

$$\frac{\partial \tilde{\mathcal{J}}_t}{\partial t} = ad_{\tilde{H}_t^{cl}} \tilde{\mathcal{J}}_t. \quad (146)$$

---

9 The Lagrangian corresponding to this Lie-Poisson system is $\langle \tilde{\mathcal{J}}_t, H_t^{cl} \rangle - H_t (\tilde{\mathcal{J}}_t)$, while the usual Lagrangian is $L_{H_t^{T^* M}}$. 
that is calculated as follows:

\[
\frac{\partial}{\partial t} \tilde{\rho}_t^j[k](x) = -\sqrt{-1} \partial_j \left( \frac{\partial H_{T^*M}}{\partial p_j} (x, \bar{\rho}_t^j[k](x)) \right) \tilde{\rho}_t^j[k](x) \sqrt{\tilde{\rho}_t^j[k](x)}, \tag{147}
\]

\[
\frac{\partial}{\partial t} (\tilde{\rho}_t^j[k](x) \tilde{\rho}_t^k[k](x)) = -\sqrt{-1} \partial_j \left( \frac{\partial H_{T^*M}}{\partial p_j} (x, \bar{\rho}_t^j[k](x)) \right) \tilde{\rho}_t^j[k](x) \tilde{\rho}_t^k[k](x) \sqrt{\tilde{\rho}_t^j[k](x)}
\]

\[
- \tilde{\rho}_t^j[k](x) \tilde{\rho}_t^j[k](x) \partial_k \left( \frac{\partial H_{T^*M}}{\partial p_j} (x, \bar{\rho}_t^j[k](x)) \right)
\]

\[
+ \tilde{\rho}_t^j[k](x) \partial_k H_{T^*M} (x, \bar{\rho}_t^j[k](x)) \cdot (x, \bar{\rho}_t^j[k](x)). \tag{148}
\]

Proof. The above equation can be obtained from the integration of general equations \((104)\) and \((106)\) on the space \(\Gamma_{\bar{U}_0}\); thereby, it proves the reduced equation from original Lie-Poisson equation \((102)\).

As a most important result, the following theorem shows that Lie-Poisson equation \((146)\), or the set of equations \((147)\) and \((148)\), actually represents the classical Liouville equation.

**Theorem 5** Lie-Poisson equation \((144)\) is equivalent to the classical Liouville equation for the probability density function (PDF) \(\tilde{\rho}_t^j \in C^\infty(T^*M, \mathbb{R})\) of a particle on cotangent space \(T^*M\):

\[
\frac{\partial}{\partial t} \tilde{\rho}_t^{j,j} = \{\tilde{\rho}_t^{j,j}, H^{T^*M}\}, \tag{149}
\]

where the Poisson bracket \(\{,\}\) is defined for every \(A, B \in C^\infty(M)\) as

\[
\{A, B\} = \frac{\partial A}{\partial p_j} \frac{\partial B}{\partial x^j} - \frac{\partial A}{\partial x^j} \frac{\partial B}{\partial p_j}. \tag{150}
\]

Proof. Classical equation \((144)\) is equivalent to the canonical equations of motion through the local expression such that \(\phi_{\bar{U}_a}(q_t) = (x_t, p_t)\) for the bundle mapping \(\bar{U}_a : \pi^{-1}(U_a) \rightarrow U_a \times \mathbb{R}^N\):

\[
\frac{dp_t^j}{dt} = -\frac{\partial H^{T^*M}}{\partial x^j}(x_t, p_t) \quad \frac{dx_t^j}{dt} = \frac{\partial H^{T^*M}}{\partial p_j}(x_t, p_t). \tag{151}
\]

If \(q_t = (x_t, \bar{\rho}_t^j[k](x))\) satisfies canonical equations of motion \((151)\), the above equation of motion induces

\[
\frac{\partial \tilde{\rho}_t^j[k](x)}{\partial t} = -\frac{\partial H_{T^*M}}{\partial x^j}(x, \bar{\rho}_t^j[k](x)) - \frac{\partial H_{T^*M}}{\partial p_j}(x, \bar{\rho}_t^j[k](x)) \partial_j \tilde{\rho}_t^k[k](x), \tag{152}
\]

then relation \((147)\) satisfies the following equation:

\[
\frac{\partial}{\partial t} \tilde{\rho}_t^j[k](x) = \sqrt{-1} \partial_j \left( \sigma_t^j[k](x) \frac{\partial H_{T^*M}}{\partial p_j} (x, \bar{\rho}_t^j[k](x)) \right) \tilde{\rho}_t^{j,j}(x, \bar{\rho}_t^j[k](x))
\]

\[
+ \sqrt{-1} \sigma_t^j[k](x) \frac{\partial H_{T^*M}}{\partial t} (x, \bar{\rho}_t^j[k](x))
\]

\[
- \sqrt{-1} \sigma_t^j[k](x) \frac{\partial H_{T^*M}}{\partial x^j} (x, \bar{\rho}_t^j[k](x)) \frac{\partial \tilde{\rho}_t^{j,j}}{\partial p_j} (x, \bar{\rho}_t^j[k](x))
\]

\[
- \sqrt{-1} \sigma_t^j[k](x) \frac{\partial H_{T^*M}}{\partial p_j} (x, \bar{\rho}_t^j[k](x)) \times \partial_j \tilde{\rho}_t^k[k](x) \frac{\partial \tilde{\rho}_t^{j,j}}{\partial p_k} (x, \bar{\rho}_t^j[k](x))
\]

\[
= -\sqrt{-1} \partial_j \left( \frac{\partial H_{T^*M}}{\partial p_j} (x, \bar{\rho}_t^j[k](x)) \tilde{\rho}_t^j[k](x) \right). \tag{153}
\]
Equations (152) and (153) lead to the following equation:

\[
\frac{\partial}{\partial t} \{ \tilde{p}_i^r[k](x) \tilde{p}_j^r[k](x) \} = -\tilde{p}_{lk}[k](x) \sqrt{-1} \partial_j \left( \frac{\partial H^{T \rightarrow M}}{\partial \tilde{p}_j} (x, \tilde{p}_i^r[k](x)) \tilde{p}_j^r[k](x) \right) \\
-\tilde{p}_i^r[k](x) \frac{\partial H^{T \rightarrow M}}{\partial x^k} (x, \tilde{p}_i^r[k](x)) \\
-\tilde{p}_i^r[k](x) \frac{\partial H^{T \rightarrow M}}{\partial \tilde{p}_j} (x, \tilde{p}_i^r[k](x)) \partial_j \tilde{p}_{lk}[k](x) \\
= -\sqrt{-1} \partial_j \left( \frac{\partial H^{T \rightarrow M}}{\partial \tilde{p}_j} (x, \tilde{p}_i^r[k](x)) \tilde{p}_j^r[k](x) \tilde{p}_{lk}[k](x) \right) \\
-\tilde{p}_i^r[k](x) \left\{ \tilde{p}_j^r[k](x) \partial_k \left( \frac{\partial H^{T \rightarrow M}}{\partial \tilde{p}_j} (x, \tilde{p}_i^r[k](x)) \right) \\
+\partial_k L^H (x, \tilde{p}_i^r[k](x)) \right\}.
\]

(154)

Equations (154) and (154) are equivalent to equations (147) and (148); thereby, canonical equation (147) is equivalent to Lie-Poisson equation (146).

The above discussion has a special example of the following Hamiltonian:

\[
H^{T \rightarrow M}_t (x, p) = g^{ij}(x) (p_i + A_i) (p_j + A_j) + U(x),
\]

where corresponding Hamiltonian operator \( \tilde{H}_t \) is calculated as

\[
\tilde{H}_t[k] = \left( g^{ij}(\tilde{p}_i[k] + A_i) \partial_j, -g^{ij} \tilde{p}_j[k] \tilde{p}_i[k] + g^{ij} A_i A_j + U \right);
\]

thereby, equation (146) is described for special Hamiltonian (155) as

\[
\frac{\partial}{\partial t} \{ \tilde{p}_i^r[k](x) \tilde{p}_j^r[k](x) \} = -\sqrt{-1} \partial_j \left\{ g^{ik}(x) (\tilde{p}_j^r[k](x) + A_k(x)) \tilde{p}_i^r[k](x) \tilde{p}_j^r[k](x) \right\} \\
-\tilde{p}_i^r[k](x) \left( \partial_j g^{ik}(x) \right) \tilde{p}_j^r[k](x) \tilde{p}_k^r[k](x) \\
-\partial_j g^{ik}(x) A_k(x) \tilde{p}_i^r[k](x) \tilde{p}_k^r[k](x) \\
-\tilde{p}_i^r[k](x) \partial_j \left\{ U(x) + g^{ik}(x) A_i(x) A_k \right\},
\]

(157)

\[
\frac{\partial}{\partial t} \tilde{p}_i^r[k](x) = -\sqrt{-1} \partial_j \left\{ g^{ik}(x) (\tilde{p}_j^r[k](x) + A_k(x)) \tilde{p}_i^r[k](x) \right\}.
\]

(158)

For \( \tilde{U}_t \in \tilde{Q}(M) \) such that \( \frac{\partial}{\partial t} \tilde{U}_t^{-1} = \tilde{H}_t \in \tilde{q}(M) \), let us introduce operators

\[
\tilde{H}_t^c = A_{t^{-1}} \tilde{H}_t^c, \\
\tilde{F}_t^c = A_{t^{-1}} \tilde{F}_t^c,
\]

(159)

(160)

which induces the following equation equivalent to equation (146):

\[
\frac{\partial}{\partial t} \tilde{F}_t^c = \left[ \tilde{H}_t^c, \tilde{F}_t^c \right] + \left( \frac{\partial \tilde{F}_t^c}{\partial t} \right)^c.
\]

(161)

This expression of the equations of motion coincides with the following Poisson equation because of Theorem 4:

\[
\frac{d}{dt} F^{T \rightarrow M}_t = \left\{ H^{T \rightarrow M}_t, F^{T \rightarrow M}_t \right\} + \frac{\partial F^{T \rightarrow M}_t}{\partial t}.
\]

(162)
As discussed in Section 3, if a group action of Lie group $Q(M)$ keeps the Hamiltonian $\tilde{H}_t : \tilde{q}(M)^* \rightarrow \mathbb{R}$ invariant, there exists an invariant charge function $\tilde{Q}^{T^* M} \in C^\infty(T^* M)$ and the induced function $\tilde{Q} : \tilde{q}(M)^* \rightarrow \mathbb{R}$ such that

$$[\hat{H}_t, \hat{Q}^\text{cl}] = 0,$$

where $\hat{Q}^\text{cl}$ is expressed as

$$\hat{Q}^\text{cl} = \left( \frac{\partial Q^{T^* M}}{\partial p} (x, \tilde{p}_\ell^i[k](x)), -p(\eta) \cdot \frac{\partial Q^{T^* M}}{\partial p} (x, \tilde{p}_\ell^i[k](x)) + Q^{T^* M} (x, \tilde{p}_\ell^i[k](x)) \right).$$

Relation (163) is equivalent to the following convolution relation:

$$\left\{ H^{T^* M}_t, Q^{T^* M} \right\} = 0. \quad (165)$$

In the argument so far on the dynamical construction of classical mechanics, the introduced infinite-dimensional freedom of the background $B[E(M)]$ seems to be redundant, while they appear as a natural consequence of the general theory on protomechanics discussed in the previous section. In fact, it is really true that one can directly induce classical mechanics as the dynamics of the Lagrange foliations of $T^* M$ in $L(T^* M)$. In the next section, however, it is observed that we will encounter difficulties without those freedom if moving onto the dynamical construction of quantum mechanics.

6 DEDUCTION OF QUANTUM MECHANICS

In canonical quantum mechanics, the state of a particle on manifold $M$ can be represented as a position in the Hilbert space $\mathcal{H}(M)$ of all the $L^2$-functions over $M$. In this section, we will reproduce the quantum equation of motion from the general theory presented in Section 4. Let us here concentrate ourselves on the case where $M$ is $N$-dimensional manifold for simplicity, though the discussion below is still valid if substituting an appropriate Hilbert space when $M$ is infinite-dimensional ILH-manifold.\[11]

6.1 Description of Statistical-State

Now, we must be concentrated on the case where the physical functional $F \in C^\infty (\Lambda^1(M), C^\infty(M))$ depends on the derivatives of the 1-form $p(\eta) \in \Lambda^1(M)$ induced from $\eta \in \Gamma[E(M)]$, then it has the following expression:

$$p^* F(\eta) (x) = F^O (x, p(\eta) (x), Dp(\eta) (x), ..., D^n p(\eta) (x), ...). \quad (166)$$

Let us assume that $M$ has a finite covering $M = \bigcup_{U \in \Lambda_M} U_\alpha$ for the mathematical simplicity such that $\Lambda_M = \{1, 2, ..., \Lambda\}$ for some $\Lambda \in \mathbb{R}$, and choose a coordinate system $(U_\alpha, x_\alpha)_{\alpha \in \Lambda_M}$. Let us further choose a reference set $U \subset U_\alpha$ such that $\psi(U) \neq 0$ and consider the set $\Gamma^h_{U,k} [E(M)]$ of the $C^\infty$ sections of $E(M)$ for $k = (k_1, ..., k_N) \in \mathbb{R}^N$ such that\[11]

$$\Gamma^h_{U,k} [E(M)] = \left\{ \eta \in \Gamma [E(M)] \mid \sup_U p_j (\eta) (x) = \hbar k_j \right\}. \quad (167)$$

As in classical mechanics, we will simply denote $\Gamma^h_{U,k} [E(M)]$ as $\Gamma^h_k E(M)$, since $\Gamma^h_{U,k} [E(M)]$ can be identified with $\Gamma^h_{U',k} [E(M)]$ for every two reference sets $U$ and $U' \subset M$.

For every $\bar{\phi} = p \circ \bar{\eta} \in L(T^* M)$ such that $\bar{\eta}[k] \in \Gamma^h_k [E(M)]$, it is further possible to separate an element $\bar{\eta}[k] \in \Gamma^h_k [E(M)]$ for a $\xi \in \Gamma^h_k [E(M)]$ as

$$\bar{\eta}[k] = \bar{\eta}[k] \cdot \xi, \quad (168)$$

As in classical mechanics, to substitute $\Gamma^h_{U,k} [E(M)] = \left\{ \eta \in \Gamma [E(M)] \bigg| \int_U dv(x) p_j (x) = \hbar k_j \psi(U) \right\}$ for definition (165) also induces the similar discussion below, while there exist a variety of the classification methods that produce the same result.
or to separate momentum $p (\eta[k])$ as
\[
p (\eta[k]) = \bar{p}[k] + p (\xi).
\]
(169)

The emergence density $\rho (\eta[k])$ can have the same expression as the classical one for the function $\varrho (\xi) \in C^\infty (T^* M, \mathbb{R})$ on $T^* M$ since $C_q (\Gamma)^* \subset C_{cl} (\Gamma)^*$:
\[
\rho (\eta[k]) (x) = \varrho (\xi) (x, p (\eta[k]) (x)),
\]
(170)

which has only the restricted values if compared with the classical emergence density; it sometimes causes the discrete spectra of the wave-function in canonical quantum mechanics. We call the set $B^h [E(M)] = \Gamma_0 ^h [E(M)]$ as the back ground of $L (T^* M)$ for quantum mechanics. For the measure $\mathcal{N}$ on $B^h [E(M)]$ for the $\sigma$-algebra induced from that of $\Gamma [E(M)]$:
\[
d\mathcal{M} (\eta[k]) \ dv(x) = d^N k d\mathcal{N} (\xi) \ dv(x) \ \sigma[k] (x).
\]
(171)

Let us next consider the disjoint union $M = \bigcup_{\alpha \in A_M} A_\alpha$ for $A_\alpha \in \mathcal{B} (\mathcal{O}_E (M))$ such that (1) $\pi (A_\alpha) \subset U_\alpha$ and that (2) $A_\alpha \cap A_\beta = \emptyset$ for $\alpha \neq \beta \in A_M$ (consult APPENDIX A). Thus, every section $\eta \in \Gamma [E(M)]$ has some $k \in \mathbb{R}^N$ such that $\eta = \eta[k] \in \Gamma_k ^h [E(M)];$ and, it will be separated into the product of a $\xi \in B^h [E(M)]$ and the fixed $\bar{\eta}[k] = e^{2i (k, x \cdot \xi)} \in \Gamma_k [E(M)]$ that induces one of the Lagrange foliation $\bar{\varrho} = p \circ \bar{\eta} \in L (T^* M)$:
\[
\eta[k] = \sum_{\alpha \in A_\alpha} \chi_{A_\alpha} \cdot e^{2i (k, x \cdot \xi)} \cdot \xi
\]
(172)
\[
\prod_{\alpha \in A_\alpha} \left( e^{2i (k, x \cdot \xi)} \cdot \xi \right)^{\chi_{A_\alpha}},
\]
(173)

where the test function $\chi_{A_\alpha} : M \to \mathbb{R}$ satisfies
\[
\chi_{A_\alpha} (x) = \begin{cases} 1 & \text{at } x \in A_\alpha \\ 0 & \text{at } x \notin A_\alpha \end{cases}
\]
(174)

and has the projection property $\chi_{A_\alpha} ^2 = \chi_{A_\alpha}$.

If defining the window mapping $\chi_{A_\alpha} ^* : C^\infty (M) \to L^1 (\mathbb{R}^N)$ for any $f \in C^\infty (M)$ such that
\[
\chi_{A_\alpha} ^* f (x) = \begin{cases} \varphi_{A_\alpha} f (x) & \text{at } x \in \varphi_{A_\alpha} (A_\alpha) \\ 0 & \text{at } x \notin \varphi_{A_\alpha} (A_\alpha) \end{cases},
\]
(175)

we can locally transform the function $\rho[k] (\xi) = \sigma[k] \rho (\eta[k]) \sqrt{\varrho}$ into Fourier coefficients as follows:
\[
\chi_{A_\alpha} ^* \rho[k] (\xi) \ (x) = \int_{\mathbb{R}^N} d^N k' \ \tilde{\varrho}_\alpha (\xi) \left( \frac{2k + k'}{2}, \frac{2k - k'}{2} \right) \ e^{ik' \cdot x'},
\]
(176)

where introduced function $\tilde{\varrho}_\alpha$ should satisfies
\[
\tilde{\varrho}_\alpha (\xi) (k, k') = \tilde{\varrho}_\alpha (\xi) (k', k),
\]
(177)

for the value $\rho[k] (\xi) (x)$ is real at every $x \in M$; thereby, the collective expression gives
\[
\rho[k] (\xi) = \sum_{\alpha \in A_\alpha} \chi_{A_\alpha} \cdot \int_{\mathbb{R}^N} d^N k' \ \tilde{\varrho}_\alpha (\xi) \left( \frac{2k + k'}{2}, \frac{2k - k'}{2} \right) \ e^{ik' \cdot x'}
\]
(178)
\[
\int_{\mathbb{R}^N} d^N k' \ \tilde{\varrho} (\xi) \left( \frac{2k + k'}{2}, \frac{2k - k'}{2} \right) \ e^{i \frac{k'}{2}} \eta \left[ k - \frac{k'}{2} \right] ^{\frac{1}{2}} \eta \left[ k + \frac{k'}{2} \right] ^{\frac{1}{2}},
\]
(179)
where
\[ \hat{g}(\xi) \left( \frac{2k + k'}{2}, \frac{2k - k'}{2} \right) = \prod_{\alpha \in A_\alpha} \left( \hat{\alpha}_\alpha \left( \frac{2k + k'}{2}, \frac{2k - k'}{2} \right) \right)^{x_{\alpha}}. \] (180)

Let us introduce the ketvector |k⟩ and bravector ⟨k| such that
\[ |k⟩ = \prod_{\alpha \in \Lambda_{\alpha}} |k, \alpha⟩, \quad ⟨k| = \prod_{\alpha \in \Lambda_{\alpha}} ⟨k, \alpha|, \] (181)
where the local vectors |k, α⟩ and ⟨k, α| satisfy
\[ ⟨x|k, \alpha⟩ = e^{2i(k_jx^j + \zeta)} x_{\alpha} \sqrt{\frac{1}{2}}, \quad ⟨k, \alpha|x⟩ = e^{2i(-k_jx^j + \zeta)} x_{\alpha} \sqrt{\frac{1}{2}}. \] (182)

We can define the Hilbert space \( \mathcal{H}(M) \) of all the vectors that can be expressed as a linear combination of vectors \( \{|k⟩\}_{k \in \mathbb{R}} \). Now, let us construct the density matrix in the following definition.

**Definition 1** The density matrix \( \hat{\rho} \) is an operator such that
\[ \hat{\rho} = \int_{\mathcal{B}^n[\mathcal{E}(M)]} dN(\xi) \int_{\mathcal{R}^n} d^Nn \int_{\mathcal{R}^n} d^Nn' \hat{g}(\xi)(n,n') \xi^{\frac{1}{2}}|n⟩⟨n'|\xi^{-\frac{1}{2}} \] (183)
\[ = \int_{\mathcal{B}^n[\mathcal{E}(M)]} dN(\xi) \int_{\mathcal{R}^n} d^Nn \hat{\rho}[k](\xi), \] (184)
where
\[ \hat{\rho}[k](\xi) = \int_{\mathcal{R}^n} d^Nk' \hat{g}(\xi) \left( k + \frac{k'}{2}, k - \frac{k'}{2} \right) \xi^{\frac{1}{2}}|k⟩⟨k'|k - \frac{k'}{2}⟩ \xi^{-\frac{1}{2}}. \] (185)

Let \( \mathcal{O}(M) \) be the set of all the hermite operators acting on Hilbert space \( \mathcal{H}(M) \), which has the bracket \( \langle \cdot | \cdot \rangle : \mathcal{O}(M) \to \mathbb{R} \) for every hermite operator \( \hat{F} \) such that
\[ \langle \hat{F} \rangle = \int_{\mathcal{R}^n} d^Nk \int_{M} dv(x) \langle x|\hat{F}|x⟩. \] (186)

Set \( \mathcal{O}(M) \) becomes the algebra with the product, scalar product and addition; thereby, we can consider the commutation and the anticommutaion between operators \( \hat{A}, \hat{B} \in \mathcal{O}(M) \):
\[ [\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}. \] (187)

Consider the momentum operator \( \hat{p} \) that satisfies the following relation for any \( |\psi⟩ \in \mathcal{H}(M) \):
\[ ⟨x|\hat{p}|\psi⟩ = -iD ⟨x|\psi⟩, \] (188)
where \( D = \hbar dx^j \partial_j \) is the derivative operator \( [\hat{A}, \hat{B}] \). Further, the function operator \( \hat{f} \) induced from the function \( f \in \mathcal{C}^\infty(M) \) is an operator that satisfies the following relation for any \( |\psi⟩ \in \mathcal{H}(M) \):
\[ ⟨x|\hat{f}|\psi⟩ = f(x) ⟨x|\psi⟩. \] (189)

The following commutation relation holds:
\[ [\hat{p}_j, \hat{f}] = \frac{\hbar}{i} \partial_j f. \] (190)

Those operators \( \hat{f} \) and \( \hat{p} \) induces a variety of operators in the form of their polynomials.
Definition 2 The hermite operator $\hat{F}$ is called an observable, if it can be represented as the polynomial of the momentum operators $\hat{p}$ weighted with function operators $\hat{f}_j^l$ independent of $k$ such that

$$\hat{F} = \sum_{n=0}^{\infty} \left[ \hat{f}_n^l, \hat{p}_j^n \right]_+. \quad (191)$$

The following lemma shows that every observable has its own physical functional.

Lemma 2 Every observable $\hat{F}$ has a corresponding functional $F : \Gamma[E] \rightarrow C^\infty(M)$:

$$\bar{\mu}(p^*F) = \left\langle \hat{p} \hat{F} \right\rangle. \quad (192)$$

Proof. There are corresponding functionals $g_{n'l}^l : \Lambda^1(M) \rightarrow C(M)$ ($l \in \{1, 2, ..., n\} )$ such that

$$\left\langle \hat{p} \left[ \hat{f}_n^l, \hat{p}_j^n \right]_+ \right\rangle = \int_{B^1[E(M)]} dN(\xi) \int_{R^N} d^N n \int_{R^N} d^N n' \tilde{\varrho}(\xi) (n, n') \left\langle n' \mid \xi^{-\frac{1}{2}} \left[ \hat{f}_n^l, \hat{p}_j^n \right]_+ \xi^{\frac{1}{2}} \right\rangle n'$$

$$= \int_{B^1[E(M)]} dN(\xi) \int_{R^N} d^N k \int_{R^N} d^N k' \sum_{\alpha \in A_M} \int_{U_\alpha} d^N x \tilde{\varrho}(\xi) \left( k - \frac{k'}{2}, k + \frac{k'}{2} \right) e^{ik_jx^j} \left\{ \sum_{l=0}^{n} g_{n'l}^l (p(\eta[k])) (x)k_l^n \right\}$$

$$= \int_{B^1[E(M)]} dN(\xi) \int_{R^N} d^N k \int_{R^N} d^N k' \sum_{\alpha \in A_M} \int_{U_\alpha} d^N x \tilde{\varrho}(\xi) \left( k - \frac{k'}{2}, k + \frac{k'}{2} \right) e^{ik_jx^j} \left\{ \sum_{l=0}^{n} \left( -\hbar \frac{\partial}{\partial x^j} \right)^l g_{n'l}^l (p(\eta[k])) (x) \right\}$$

$$= \int_{B^1[E(M)]} dN(\xi) \int_{R^N} d^N k \int_{M} dv(x) \rho(\eta[k])(x) p^*F_j^n(\eta[k])(x)$$

$$= \int_{\Gamma[E(M)]} dM(\eta) \int_{M} dv(x) \rho(\eta)(x) p^*F_j^n(\eta)(x)$$

$$= \bar{\mu}(p^*F_j^n). \quad (193)$$

where

$$p^*F_j^n(\eta[k])(x) = \sum_{l=0}^{n} \left\{ \left( -\hbar \frac{\partial}{\partial x^j} \right)^l g_{n'l}^l (p(\eta[k])) (x) \right\}. \quad (194)$$

\[ \square \]

6.2 Description of Time-Development

Now, we can describe a $\eta^\tau_\xi (\eta[k]) \in \Gamma_{U_k}[E(M)]$ as

$$\eta^\tau_\xi (\eta[k]) = \sum_{\alpha \in A_n} \chi_{A_n} \cdot e^{2i(k_\alpha x^\alpha + \zeta^\tau_\xi[k])} \cdot \xi$$

$$= \prod_{\alpha \in A_n} \left( e^{2i(k_\alpha x^\alpha + \zeta^\tau_\xi[k])} \cdot \xi \right)^{\chi_{A_n}}, \quad (195)$$

where the function $\zeta^\tau_\xi[k] \in C^\infty(M)$ labeled by labeling time $\tau \leq t \in \mathbb{R}$ satisfies

$$\zeta^\tau_\xi[k] = \zeta \quad \text{independent of } k; \quad (196)$$

24
thereby, the momentum $p^*_t(\eta[k]) = \bar{p}^*_t[k] + p(\xi) \in \Lambda^1(M)$ for $\bar{p}^*_t = p^* \circ \bar{\eta} \in L(T^*M)$ satisfies the Einstein-de Broglie relation:

$$\bar{p}^*_t[k] = -\frac{\hbar}{2} \bar{\eta}^*_t[k]^{-1} d\bar{\eta}^*_t[k].$$

(198)

The density operator $\hat{\rho}^*_t[k](\xi)$ is introduced as

$$\hat{\rho}^*_t[k](\xi) = \int_{\mathbb{R}^N} d^N k' \bar{\rho}^*_t(\xi) \left( k + \frac{k'}{2}, k - \frac{k'}{2} \right) \xi^k \left| k + \frac{k'}{2} \right\rangle \left\langle k - \frac{k'}{2} \right| \xi^{-k};$$

(199)

which satisfies the following lemma.

**Lemma 3**

$$\hat{\rho}_t = \int_{\Gamma_U} dN(\xi) \int_{\mathbb{R}^N} d^N k \ U_t^*[k] \hat{\rho}^*_t[k](\xi) U_t^*[k]^{-1},$$

(200)

where

$$U_t^*[k] = e^{i(\xi^k [k] - \xi)}.$$  

(201)

**Proof.** The direct calculation shows for the observable $\hat{F}_t$ corresponding to every functional $F$

$$\left\langle \hat{\rho}_t \hat{F}_t \right\rangle = \hat{\mu}_t(p^*F_t)
= \int_{\Gamma[E(M)]} dM(\eta) \int_M dv \ \rho^*_t(\eta)(x) \ p^*F_t(\eta_t^{-1}(\eta))
= \int_{B[E(M)]} dN(\xi) \int_{\mathbb{R}^N} d^N k \int_M dv(x) \ \rho^*_t[k](\xi)(x) \ p^*F(\eta_t^{-1}[k](\xi))(x)
= \int_{B[E(M)]} dN(\xi) \int_{\mathbb{R}^N} d^N k \int_M dv(x) \ \rho^*_t[k](\xi)(x) \ p^*F(\eta[k] \cdot e^{i(\xi^k [k] - \xi)}) (x)
= \int_{B[E(M)]} dN(\xi) \int_{\mathbb{R}^N} d^N k \int_M dv(x) \left\langle x \left| \frac{1}{2} \left( U_t^*[k] \hat{\rho}^*_t[k](\xi) U_t^*[k]^{-1} \right) \hat{F}_t \right| x \right\rangle
= \left\langle \left\{ \int_{B[E(M)]} dN(\xi) \int_{\mathbb{R}^N} d^N k \ U_t^*[k] \hat{\rho}^*_t[k](\xi) U_t^*[k]^{-1} \right\} \hat{F}_t \right\rangle. \quad (202)$$

Relation (200) represents relation (90):

$$\hat{\mu}_t(\eta) = \frac{dM(\eta)}{dM(\eta_t^{-1}(\eta))} \cdot \hat{\mu}_t^*(\eta_t^{-1}(\eta)).$$

(203)

Emergence-momentum $J_t^* = J(\eta_t^*) \in q(M)^*$ has the following expression:

$$J_t^* = d^NkdN(\xi) \ dv \ (\rho^*_t[k](\xi) p^*_t(\eta[k]), \rho^*_t[k](\xi))$$

(204)

$$= dN(\xi) d^N k \wedge dv \left( \frac{1}{2} \left\langle x | [\hat{\rho}^*_t[k](\xi), \hat{\rho}^*_t[k](\xi)]_+ | x \right\rangle, \left\langle x | \hat{\rho}^*_t[k](\xi) | x \right\rangle \right),$$

(205)

where the momentum operator $\hat{P}_t^*[k]$ satisfies

$$\hat{P}_t^*[k] = U_t^*[k]^{-1} \hat{p} \ U_t^*[k].$$

(206)

\[11\] Relation (198) is the most crucial improvement from the corresponding relation in previous letter \[8\].
The following calculus of the Fourier basis for $2k_j = n_j + m_j$ justifies expression (205):

$$\begin{align*}
e^{-i (n_j x^j + \zeta_j^j [k])} &= e^{-i (n_j x^j + \zeta_j^j [k])}
\end{align*}$$

For Hamiltonian operator $\hat{H}_t = \frac{\partial H_T^*}{\partial \tau} (\hat{J}_t^* ) \in q (M)$, the equation of motion is the Lie-Poisson equation

$$\frac{\partial \hat{J}_t^*}{\partial t} = ad^*_{\hat{H}_t} \hat{J}_t^* ,$$

that is calculated as follows:

$$\frac{\partial}{\partial t} \rho_t^* [k] ( \xi ) (x) = - \sqrt{-1} \partial_j \left( \frac{\partial H_T^*}{\partial p_j} (x,p_t^* (\eta[k]) (x)) \rho_t^* [k] ( \xi ) (x) \sqrt{\cdot} \right),$$

$$\frac{\partial}{\partial t} (\rho_t^* [k] ( \xi ) (x) p_t^* (\eta[k]) (x)) = - \sqrt{-1} \partial_j \left( \frac{\partial H_T^*}{\partial p_j} (x,p_t^* (\eta[k]) (x)) \rho_t^* [k] ( \xi ) (x) p_t^* (\eta[k]) (x) \sqrt{\cdot} \right)$$

$$- \rho_t^* [k] ( \xi ) (x) p_t^* (\eta[k]) (x) \partial_k \left( \frac{\partial H_T^*}{\partial p_j} (x,p_t^* (\eta[k]) (x)) \right)$$

$$+ \rho_t^* [k] ( \xi ) (x) \partial_k L^{H_T^*} (x,p_t^* (\eta[k]) (x)).$$

Notice that the above expression is still valid even if Hamiltonian $H_T^* M$ has the ambiguity of the operator ordering as such that for the Einstein gravity.

To elucidate the relationship between the present theory and canonical quantum mechanics, we will concentrate on the case of the canonical Hamiltonian having the following form:

$$H_t^{T^*M} (x,p) = \frac{1}{2} h^{ij} (p_i + A_{ti}) (p_j + A_{tj}) + U_t(x),$$

where $dh^{ij} = 0$. Notice that almost all the canonical quantum theory including the standard model of the quantum field theory, that have empirically been well-established, really belong to this class of Hamiltonian systems. For Hamiltonian (211), we will define the Hamiltonian operator $\hat{H}_t$ as

$$\hat{H}_t = \frac{1}{2} (\hat{p}_i + A_{ti}) h^{ij} (\hat{p}_j + A_{tj}) + U_t,$$

or $\langle x | \hat{H}_t | \psi \rangle = H_t (x) | \psi \rangle$ where

$$H_t = \frac{1}{2} (-i \hbar \partial_i + A_{ti}(x)) h^{ij} (-i \hbar \partial_j + A_{tj}(x)) + U_t(x).$$

**Lemma 4** Lie-Poisson equation (208) for Hamiltonian (211) induces the following equation:

$$i \hbar \frac{\partial}{\partial t} \langle x | \hat{\rho}_t^* [k] ( \xi ) | x \rangle = - \langle x | \left[ \hat{\rho}_t^* [k] ( \xi ) , \hat{H}_t^* [k] \right]_+ x \rangle$$

$$i \hbar \frac{\partial}{\partial t} \langle x | \frac{1}{2} [\hat{\rho}_t^* [k] ( \xi ) , \hat{p}_t^* [k] ]_+ x \rangle = - \langle x | \left[ \frac{1}{2} [\hat{\rho}_t^* [k] ( \xi ) , \hat{H}_t^* [k] ]_- , \hat{p}_t^* [k] \right]_+ x \rangle.$$
Proof. If we define the operators:

\[
\begin{align*}
\hat{H}_0 &= \frac{1}{2} h^{ij} \hat{p}_{ii}[k] \hat{p}_{ij}[k] / i \hbar \\
\hat{H}_1 &= \frac{1}{2} \{ \hat{A}_i h^{ij} \hat{p}_{ij}[k] + \hat{p}_{ii}[k] h^{ij} \hat{A}_j \} / i \hbar \\
\hat{H}_2 &= \left( \hat{U} + \frac{1}{2} h^{ij} \hat{A}_i \hat{A}_j \right) / i \hbar,
\end{align*}
\]

then Hamiltonian operator \( \hat{H}_t \) can be represented as

\[
\hat{H}_t / i \hbar = \hat{H}_0 + \hat{H}_1 + \hat{H}_2. 
\]

Thus, for density operator \( \hat{\rho}_t^\tau[k] (\xi) \) defined as equation (191),

\[
\begin{align*}
- \frac{1}{2i \hbar} \left< x \left| \left[ \hat{\rho}_t^\tau[k] (\xi) , \hat{H}_t^\tau[k] \right] \right| x \right> &= \text{term}_{(1)} \left( \hat{H}_0 \right) + \text{term}_{(1)} \left( \hat{H}_1 \right) + \text{term}_{(1)} \left( \hat{H}_2 \right),
\end{align*}
\]

where

\[
\begin{align*}
\text{term}_{(1)} \left( \hat{H}_0 \right) &= - \frac{1}{2i \hbar} \left< x \left| \left[ \frac{1}{2} \hat{\rho}_t^\tau[k] (\xi) , \hat{H}_0 \right] \right| x \right>, \\
\text{term}_{(1)} \left( \hat{H}_1 \right) &= - \frac{1}{2i \hbar} \left< x \left| \left[ \frac{1}{2} \hat{\rho}_t^\tau[k] (\xi) , \hat{H}_1 \right] \right| x \right>, \\
\text{term}_{(1)} \left( \hat{H}_2 \right) &= - \frac{1}{2i \hbar} \left< x \left| \left[ \frac{1}{2} \hat{\rho}_t^\tau[k] (\xi) , \hat{H}_2 \right] \right| x \right>.
\end{align*}
\]

First term results

\[
\text{term}_{(1)} \left( \hat{H}_0 \right) = - \partial_j \left\{ h^{ij} p_{ii} (\eta[k]) \hat{\rho}_t^\tau[k] (\xi) p_{jk} (\eta[k]) \right\} dx^k
\]

from the following computations:

\[
\begin{align*}
\left< x \left| \hat{p}_{ik}[k] \hat{\rho}_t^\tau[k] (\xi) \hat{H}_0 \right| x \right> &= \frac{1}{2} \int_{\mathbb{R}^N} d^N k' \hat{\rho}_t^\tau \left( \frac{k + k'}{2}, \frac{k - k'}{2} \right) e^{ik' \cdot x} \\
& \quad \left\{ (p_{ik} (\eta[k]) + \hbar \frac{k'_k}{2}) h^{ij} \left( p_{ii} (\eta[k]) - \hbar \frac{k'_i}{2} \right) \left( p_{ij} (\eta[k]) - \hbar \frac{k'_j}{2} \right) \right. \\
& \quad \left. + i \hbar \left( p_{ik} (\eta[k]) + \hbar \frac{k'_k}{2} \right) h^{ij} \partial_j \left( p_{ii} (\eta[k]) - \hbar \frac{k'_i}{2} \right) \right\};
\end{align*}
\]

\[
\begin{align*}
\left< x \left| \hat{H}_0 \hat{\rho}_t^\tau[k] (\xi) \hat{p}_{ik}[k] \right| x \right> &= \frac{1}{2} \int_{\mathbb{R}^N} d^N k' \hat{\rho}_t^\tau \left( \frac{k + k'}{2}, \frac{k - k'}{2} \right) e^{ik' \cdot x} \\
& \quad \left\{ (p_{ik} (\eta[k]) - \hbar \frac{k'_k}{2}) h^{ij} \left( p_{ii} (\eta[k]) + \hbar \frac{k'_i}{2} \right) \left( p_{ij} (\eta[k]) + \hbar \frac{k'_j}{2} \right) \right. \\
& \quad \left. - i \hbar \left( p_{ik} (\eta[k]) - \hbar \frac{k'_k}{2} \right) h^{ij} \partial_j \left( p_{ii} (\eta[k]) + \hbar \frac{k'_i}{2} \right) \right\};
\end{align*}
\]
\[
\langle x \mid \hat{\rho}_i^\dagger [k] (\xi) \hat{H}_{(0)} \hat{p}_{ik} [k] \mid x \rangle = \frac{1}{2} \int_{\mathbb{R}^N} d^N k' \; \hat{\rho}_i^\dagger (\xi) \left( k + \frac{k'}{2}, k - \frac{k'}{2} \right) e^{ik' \cdot x} \left\{ \right.
\begin{align*}
& \left( p_{ik} (\eta[k]) - \hbar \frac{k_i}{2} \right) h^{ij} \left( p_{ij} (\eta[k]) - \hbar \frac{k_j}{2} \right) \\
& + i\hbar \left( p_{ik} (\eta[k]) - \hbar \frac{k_i}{2} \right) h^{ij} \partial_l \left( p_{ij} (\eta[k]) - \hbar \frac{k_j}{2} \right) \\
& - \hbar^2 h^{ij} \partial_k \partial_l \left( p_{ij} (\eta[k]) - \hbar \frac{k_j}{2} \right) \\
& + i\hbar h^{ij} \partial_k \left\{ \left( p_{ii} (\eta[k]) - \hbar \frac{k_i}{2} \right) \left( p_{ij} (\eta[k]) + \hbar \frac{k_j}{2} \right) \right\} ; \\
& \left. \right\} .
\end{align*}
\] (228)

\[
\langle x \mid \hat{p}_{ik} [k] \hat{H}_{(0)} \hat{\rho}_i^\dagger [k] (\xi) \mid x \rangle = \frac{1}{2} \int_{\mathbb{R}^N} d^N k' \; \hat{\rho}_i^\dagger (\xi) \left( k + \frac{k'}{2}, k - \frac{k'}{2} \right) e^{ik' \cdot x} \left\{ \right.
\begin{align*}
& \left( p_{ik} (\eta[k]) + \hbar \frac{k_i}{2} \right) h^{ij} \left( p_{ij} (\eta[k]) + \hbar \frac{k_j}{2} \right) \\
& + i\hbar \left( p_{ik} (\eta[k]) + \hbar \frac{k_i}{2} \right) h^{ij} \partial_l \left( p_{ij} (\eta[k]) + \hbar \frac{k_j}{2} \right) \\
& - \hbar^2 h^{ij} \partial_k \partial_l \left( p_{ij} (\eta[k]) + \hbar \frac{k_j}{2} \right) \\
& - i\hbar h^{ij} \partial_k \left\{ \left( p_{ii} (\eta[k]) + \hbar \frac{k_i}{2} \right) \left( p_{ij} (\eta[k]) + \hbar \frac{k_j}{2} \right) \right\} .
\end{align*}
\] (233)

Further,

\[
\text{term}_{(1)} \left( \hat{H}_{(1)} \right) = - \left\{ \partial_l \left( h^{ij} A_j \rho_i^\dagger [k] (\xi) p_{ik} (\eta[k]) \right) \\
+ \rho_i^\dagger [k] (\xi) \left( \partial_k h^{ij} A_j \right) p_{ii} (\eta[k]) \right\} \; dx^k ;
\] (238)

\[
\text{term}_{(1)} \left( \hat{H}_{(2)} \right) = - \rho_i^\dagger [k] (\xi) \partial_k \left( U + \frac{1}{2} h^{ij} A_i A_j \right) \; dx^k .
\] (239)

Thus, second equation (215) in this lemma becomes

\[
\frac{\partial}{\partial t} \left\{ \rho_i^\dagger [k] (\xi) p_{ik} (\eta[k]) \right\} = - \partial_j \left\{ h^{ij} \left( p_{ii} (\eta[k]) + A_j \right) \rho_i^\dagger [k] (\xi) p_{ij} (\eta[k]) \right\} \\
+ \rho_i^\dagger [k] (\xi) p_{ij} (\eta[k]) \left( \partial_k h^{ij} A_i \right) \\
- \rho_i^\dagger [k] (\xi) \partial_k \left( U + \frac{1}{2} h^{ij} A_i A_j \right) ,
\] (240)

which is equivalent to equation (216) for Hamiltonian (214).

On the other hand,

\[
\frac{-1}{i\hbar} \left\langle x \left| \hat{\rho}_i^\dagger [k] (\xi) , \hat{H}_T [k] \right\rangle \left| x \right\rangle = \text{term}_{(2)} \left( \hat{H}_{(0)} \right) + \text{term}_{(2)} \left( \hat{H}_{(1)} \right) + \text{term}_{(2)} \left( \hat{H}_{(2)} \right) ,
\] (243)

where

\[
\text{term}_{(2)} \left( \hat{H}_{(0)} \right) = \frac{-1}{i\hbar} \left\langle x \left| \hat{\rho}_i^\dagger [k] (\xi) , \hat{H}_{(0)} \right\rangle \left| x \right\rangle
\]
Theorem 6

Lie-Poisson equation (208) for Hamiltonian (211) is equivalent to the quantum Liouville equation.

Each term can be calculated as follows:

\[
\text{term}_2(\hat{H}(1)) = \frac{-1}{i\hbar} \left\langle x \left| [\hat{\rho}^r_k(\xi), \hat{H}(1)] \right| x \right\rangle
\]

\[
\text{term}_2(\hat{H}(2)) = \frac{-1}{i\hbar} \left\langle x \left| [\hat{\rho}^r_k(\xi), \hat{H}(2)] \right| x \right\rangle.
\]

Thus, first equation (215) in this lemma becomes

\[
\partial_t \hat{\rho}^r_k(\xi) = \frac{-1}{2i\hbar} \int_{\mathbb{R}^N} d^N k' \hat{\rho}^r_k(\xi) \left( k + \frac{k'}{2}, k - \frac{k'}{2} \right) e^{i k' \cdot x} \left\{ \right.
\]

\[
\begin{align*}
&h^{ij} \left( p_{ii}(\eta[k]) - \frac{h^{i'}}{2} \right) \left( p_{ij}(\eta[k]) - \frac{h^{j'}}{2} \right) \\
&+ i \hbar h^{ij} \partial_j \left( p_{ii}(\eta[k]) - \frac{h^{i'}}{2} \right) \\
&- h^{ij} \left( p_{ii}(\eta[k]) + \frac{h^{i'}}{2} \right) \left( p_{ij}(\eta[k]) + \frac{h^{j'}}{2} \right) \\
&- i \hbar h^{ij} \partial_j \left( p_{ii}(\eta[k]) + \frac{h^{i'}}{2} \right) \\
&= - \partial_j (\rho^{r}_i(\xi) h^{ij} p_{ii}(\eta[k]))
\end{align*}
\]

\[
\text{term}_2(\hat{H}(11)) = - \partial_j (A_j \rho^{r}_i(\xi)); \quad \text{term}_2(\hat{H}(2)) = 0.
\]

Thus, first equation (215) in this lemma becomes

\[
\frac{\partial}{\partial t} \rho^{r}_i(\xi) = - \partial_j \left\{ h^{ij} (p_{ii}(\eta[k]) + A_j) \rho^{r}_i(\xi) \right\},
\]

which is equivalent to equation (204) for Hamiltonian (214).

Therefore, Lie-Poisson equation (208) proved to be equivalent to the equation set (214) and (215) in this lemma.

The above lemma leads us to one of the main theorems in the present paper, declaring that Lie-Poisson equation (208) for Hamiltonian (211) is equivalent to the quantum Liouville equation.

**Theorem 6** Lie-Poisson equation (208) for Hamiltonian (214) is equivalent to the following quantum Liouville equation:

\[
\frac{\partial}{\partial t} \rho^r_t \mathcal{F}_r = \left[ \rho^r_t, \mathcal{H}_r \right]_{-}/(-i\hbar).
\]

**Proof.** The following computation proves this theorem based on the previous lemma:

\[
\frac{\partial}{\partial t} \left\langle \hat{\rho}^r_t, \mathcal{F}_r \right\rangle = \frac{\partial}{\partial t} \left\langle \hat{\rho}^r_t, \mathcal{F}_r \right\rangle
\]

\[
= \int_{\Gamma} d\eta(\xi) \int_{\mathbb{R}^N} d^N k \int_{\mathbb{M}} d\nu(x) \times
\]

\[
\left\{ \left\langle x | \mathcal{F}_r^r k | (\xi) \right\rangle \hat{H}_r [k] | x \rangle - \left\langle x | \hat{H}_r [k] \hat{\rho}^r_t [k] (\xi) \right\rangle \mathcal{F}_r^r k | x \rangle \right. \]

\[
+ \left. \left\langle x | \hat{\rho}^r_t [k] (\xi) | x \right\rangle \frac{\partial \rho^r_t [k]}{\partial t} \mathcal{F}_r \right\rangle (\eta^r_t [k]) (x)
\]

29
The existence of the probability measure \(L\) the continuous superselection rules (CSRs). The induced wave function has the following expression for a is the Stieltjes integral [14]. If the system is open and has the continuous spectrum, then it admits \(\Lambda\) be the elementary quantum mechanics.

\[
\psi_L(x) = \int_{\Gamma_U} dN(\xi) \int_{\mathbb{R}^N} d^Nk \int_M dv(x) \times \\
\left\{ \int x \left[ \hat{\rho}_t^* [k] (\xi) , \hat{H}_t^* [k] \right] - x \right\} p^* F_t (\eta_t^* [k]) (x) \\
+ \left( \frac{\partial}{\partial t} \int x \left[ \frac{1}{2} \hat{\rho}_t^* [k] (\xi) , \hat{p}_t^* [k] \right] _+ x \right) \cdot \mathcal{D} F_t (\eta_t^* [k]) (x) \\
- \left( \frac{\partial}{\partial t} \int x \left[ \frac{1}{2} \hat{\rho}_t^* [k] (\xi) , \hat{p}_t^* [k] \right] _- x \right) \cdot \mathcal{D} F_t (\eta_t^* [k]) (x) \\
+ \langle x | \hat{\rho}_t^* [k] | x \rangle p^* \frac{\partial F_t}{\partial t} (\eta_t^* [k]) (x) \right\}
\]

\[
= \int_{\Gamma_U} dN(\xi) \int_{\mathbb{R}^N} d^Nk \int_M dv(x) \times \\
\left\{ \int x \left[ \hat{\rho}_t^* [k] (\xi) , \hat{H}_t^* [k] \right] - x \right\} \cdot \mathcal{D} F_t (\eta_t^* [k]) (x) \\
+ \langle x | \hat{\rho}_t^* [k] | x \rangle p^* \frac{\partial F_t}{\partial t} (\eta_t^* [k]) (x) \right\}
\]

\[
= \langle a \hat{\rho}_t^* [k] J_t^* , \hat{F}_t^* \rangle + \left\langle J_t , \frac{\partial \hat{F}_t}{\partial t} \right\rangle.
\]

Now, the density matrix \(\hat{\rho}_t\) becomes the summation of the pure states \(\{\psi^{(l;\pm)}_t\}\) for the set \(\{\psi^{(l,\pm)}_t\}_{l \in \mathbb{R}^N}\) of the orthonormal wave vectors such that \(\langle \psi^{(l',\pm)}_{s'} | \psi^{(l;\pm)}_s \rangle = \delta(l' - l)\delta_{s,s'}\):

\[
\hat{\rho}_t = \int_{\Lambda} dP_+ (l) \left| \psi^{(l;+)}_t \right\rangle \left\langle \psi^{(l;+)}_t \right| - \int_{\Lambda} dP_- (l) \left| \psi^{(l,-)}_t \right\rangle \left\langle \psi^{(l,-)}_t \right|,
\]

where \(P_\pm\) is a corresponding probability measure on the space \(\Lambda\) of a spectrum and the employed integral is the Stieltjes integral [14]. If the system is open and has the continuous spectrum, then it admits \(\Lambda\) be the continuous superselection rules (CSRs). The induced wave function has the following expression for a \(L^2\)-function \(\psi^{(l;\pm)}_t(x) = \langle x | \psi^{(l;\pm)}_t \rangle \in L^2(M)\):

\[
\chi^*_\alpha \psi^{(l;\pm)}_t(x) = \int_{\mathbb{R}^N} d^Nk \tilde{\psi}_{\alpha}^{(l;\pm)}(k)e^{i(k \cdot x)}.
\]

The existence of the probability measure \(P_-\) would be corresponding to the existence of the antiparticle for the elementary quantum mechanics.

For example, the motion of the particle on a N-dimensional rectangle box \([0, \pi]^N\) needs the following boundary condition on the verge of the box:

if \(x_j = 0\) or \(\pi\) for some \(j \in \{1, ..., N\}\), then \(\langle x | \hat{\rho}_t | x \rangle = 0\),

\[
30
\]
Density matrix $\hat{\rho}_t$ is the summation of integer-labeled pure states:

$$\hat{\rho}_t = \sum_{(n,n') \in \mathbb{Z}^N} \hat{\rho}_t^n(n', n) |n; t\rangle \langle n'; t|.$$  

(256)

Let us now concentrate on the case where $\hat{\rho}_t$ is a pure state in the following form:

$$\hat{\rho}_t = |\psi_t\rangle \langle \psi_t|;$$  

(257)

there exists a wave function $\psi_t \in L^2(M)$

$$\psi_t(x) = \int_{\mathbb{R}^N} d^N k \ \tilde{\psi}_t(k) e^{i(k \cdot x' + \zeta_t(x))},$$  

(258)

where

$$\tilde{\rho}_t^0(k, k') = \tilde{\psi}_t(k)^* \tilde{\psi}_t(k').$$  

(259)

Theorem 6 introduces the Schrödinger equation as the following corollary.

**Corollary 1** Lie-Poisson equation (208) for Hamiltonian (211) becomes the following Schrödinger equation:

$$i\hbar \partial_t \psi_t = \mathcal{H} \psi_t,$$  

(260)

where

$$\mathcal{H} = \frac{1}{2m} \sqrt{-1} \left( -i\hbar \partial_i + A_{ti}(x) \right) \hat{q}^j(x) \sqrt{-i\hbar \partial_j + A_{tj}(x)} + U_t(x).$$  

(261)

Therefore, the presented theory induces not only canonical, nonrelativistic quantum mechanics but also the canonical, relativistic or nonrelativistic quantum field theory if proliferated for the Grassmanian field variables. In addition, Section 7 will discuss how the present theory also justifies the regularization procedure in the appropriate renormalization.

On the other hand, if introducing the unitary transformation $\tilde{U}_t = e^{i\hbar\mathcal{H}_t}$, Theorem 6 obtains the Heisenberg equation for Heisenberg’s representations $\hat{H}_t = \tilde{U}_t \mathcal{H}_t \tilde{U}_t^{-1}$ and $\hat{F}_t = \tilde{U}_t \mathcal{F}_t \tilde{U}_t^{-1}$:

$$\frac{\partial}{\partial t} \hat{F}_t = \left[ \hat{H}_t, \hat{F}_t \right] / (-i\hbar) + \left( \frac{\partial \mathcal{F}_t}{\partial t} \right),$$  

(262)

since $\hat{\rho}_t = \tilde{U}_t^{-1} \hat{\rho}_0 \tilde{U}_t$.

As discussed in Section 3, if a group action of Lie group $Q(M)$ keeps the Hamiltonian $\mathcal{H}_t : q(M)^* \to \mathbb{R}$ invariant, there exists an invariant charge functional $Q : \Gamma[E(M)] \to C(M)$ and the induced function $Q : q(M)^* \to \mathbb{R}$ such that

$$\left[ \hat{H}_t, \hat{Q} \right] = 0,$$  

(263)

where $\hat{Q}$ is expressed as

$$\hat{Q} = \left( \mathcal{D}_{\rho(\eta)} Q(p(\eta)), -p(\eta) \cdot \mathcal{D}_{\rho(\eta)} Q(p(\eta)) + Q(p(\eta)) \right).$$  

(264)

Suppose that functional $p^* Q : \Gamma[E(M)] \to C(M)$ has the canonical form such that

$$Q^{\ast = M}_t(x, p) = A^{ij}_t p_i p_j + B(x) p_j + C(x),$$  

(265)

then the corresponding generator is equivalent to the observable:

$$\hat{Q} = A^{ij}_t \hat{p}_i \hat{p}_j + \hat{B}_t \hat{p}_j + \hat{p}_j \hat{B}_t + \hat{C}.$$  

(266)
In this case, relation (263) has the canonical expression:

\[
[\hat{H}_t, \hat{Q}] = 0.
\] (267)

Those operators can have the eigen values at the same time.

As shown so far, protomechanics successfully deduced quantum mechanics for the canonical Hamiltonians that have no problem in the operator ordering, and proves still valid for the noncanonical Hamiltonian that have the ambiguity of the operator ordering in the ordinary quantum mechanics. In the latter case, the infinitesimal generator \( \hat{F}^r_1 \) corresponding to \( \hat{F} \in q(M) \) is not always equal to observable \( \hat{F}_t \):

\[
\hat{F}_t \neq \hat{F}^r_1.
\] (268)

If one tries to quantize the Einstein gravity, he or she can proliferate the present theory in a direct way by utilizing Lie-Poisson equation (208). But, some calculation method should be developed for this purpose elsewhere.

6.3 Interpretation of Spin

It has been believed that a half-integer spin in quantum mechanics does not have any classical analogies well-established in classical mechanics. Such belief may prevent quantum mechanics from the realistic interpretation. This section shows that the present theory allows such an classical analogy with a half-integer spin.

Now, let us consider the particle motion on space \( M^{(3)} \) with the polar coordinate \( x = (r, \theta, \phi) \in [0, +\infty) \times [0, 2\pi) \times (0, \pi) \). The three-dimensional orthogonal group \( SO(3, \mathbb{R}^N) \) acts on \( J_t = (\rho^r, \rho^r_\theta, \rho^r_\phi) \) by the coadjoint action. The infinitesimal generator \( M = M_j \hat{L}_j \in so(3, \mathbb{R}^N) \subset q(M) \) (\( M_j \in \mathbb{R}, j \in \{1, 2, 3\} \)) has an corresponding operator \( \hat{M} = M_j \hat{L}_j \in su(2, \mathbb{C}) \) that satisfies

\[
\langle ad_{J_t}^* \hat{J}_t, \hat{F} \rangle = -i\hbar^{-1} \left\langle \left[ \hat{\rho}_t, \hat{M} \right] - \hat{F} \right\rangle.
\] (269)

Infinitesimal generator \( \hat{L}_j \) has the following expression:

\[
\hat{L}_1 = -\sin \phi \frac{\partial}{\partial \theta} - \cot \theta \cos \phi \frac{\partial}{\partial \phi},
\] (270)

\[
\hat{L}_2 = \cos \phi \frac{\partial}{\partial \theta} - \cot \theta \sin \phi \frac{\partial}{\partial \phi},
\] (271)

\[
\hat{L}_3 = \frac{\partial}{\partial \phi}.
\] (272)

while corresponding operator \( \hat{L}_j \) satisfies

\[
\langle \theta, \phi \mid \hat{L}_1 \mid \psi \rangle = \left\{ -\frac{\hbar}{i} \sin \phi \frac{\partial}{\partial \theta} - \frac{\hbar}{i} \cot \theta \cos \phi \frac{\partial}{\partial \phi} \right\} \langle \theta, \phi \mid \psi \rangle,
\] (273)

\[
\langle \theta, \phi \mid \hat{L}_2 \mid \psi \rangle = \left\{ \frac{\hbar}{i} \cos \phi \frac{\partial}{\partial \theta} - \frac{\hbar}{i} \cot \theta \sin \phi \frac{\partial}{\partial \phi} \right\} \langle \theta, \phi \mid \psi \rangle,
\] (274)

\[
\langle \theta, \phi \mid \hat{L}_3 \mid \psi \rangle = \frac{\hbar}{i} \frac{\partial}{\partial \phi} \langle \theta, \phi \mid \psi \rangle.
\] (275)
Notice that these operators are hermite or self-conjugate, $\hat{L}_j^\dagger = \hat{L}_j$, and induces the angular momentum or the integer spin of the particle:

$$|\psi_t\rangle = \sum_{m=-l}^l c^*_m(t) |l, m\rangle$$

For \( \langle \theta, \phi |l; m\rangle = Y_l^m(\theta, \phi), \) \( (276) \)

where

$$\hat{\mathbf{L}} \cdot \hat{\mathbf{L}} |l, m\rangle = \hbar^2 l(l+1) |l; m\rangle, \quad \hat{L}_3 |l, m\rangle = \hbar m |l; m\rangle. \quad (278)$$

If the Hamiltonian for the motion in the three-dimensional Euclid space has the following form in a central field of force, it is invariant under the rotation about z-axis:

$$H(x, p) = p^2 + x \cdot (p \times B) + U(r), \quad (279)$$

where \( r = \sqrt{x^2 + y^2 + z^2} \neq 0 \). Since this Hamiltonian has the canonical form, the corresponding infinitesimal generator is equivalent to the following quantum observable \( [15] \):

$$\hat{H} = \hat{P}_r^2 + \frac{\hat{L} \cdot \hat{L}}{r^2} + \frac{1}{2} \left\{ \hat{L} \cdot B + B \cdot \hat{L} \right\} + U(r), \quad (280)$$

where

$$\langle \theta, \phi, r \mid \hat{P}_r \mid \psi \rangle = -\frac{\hbar}{i r} \frac{\partial}{\partial r} \langle \theta, \phi, r \mid \psi \rangle. \quad (281)$$

On the other hand, the infinitesimal generators for the half-integer spin are different from those discussed in the above for the angular momentum and integer spin:

$$\hat{S}_1 = \left( \hat{L}_1, \frac{\hbar}{2} \frac{\cos \phi}{\sin \theta} + \hat{L}_1 s \right), \quad (282)$$

$$\hat{S}_2 = \left( \hat{L}_2, \frac{\hbar}{2} \frac{\sin \phi}{\sin \theta} + \hat{L}_2 s \right), \quad (283)$$

$$\hat{S}_3 = \left( \hat{L}_3, \hat{L}_3 s \right), \quad (284)$$

where function \( s : M^{(3)} \to \mathbb{R} \) represents the gage freedom in electromagnetism. The corresponding generators in quantum mechanics becomes

$$\langle \theta, \phi \mid \hat{S}_1 \mid \psi \rangle = \left\{ \frac{\hbar}{i} \hat{L}_1 + \frac{\hbar}{2} \frac{\cos \phi}{\sin \theta} + \left( \hat{L}_1 s \right) \right\} \langle \theta, \phi \mid \psi \rangle, \quad (285)$$

$$\langle \theta, \phi \mid \hat{S}_2 \mid \psi \rangle = \left\{ \frac{\hbar}{i} \hat{L}_2 + \frac{\hbar}{2} \frac{\sin \phi}{\sin \theta} + \left( \hat{L}_2 s \right) \right\} \langle \theta, \phi \mid \psi \rangle, \quad (286)$$

$$\langle \theta, \phi \mid \hat{S}_3 \mid \psi \rangle = \left\{ \frac{\hbar}{i} \hat{L}_3 + \left( \hat{L}_3 s \right) \right\} \langle \theta, \phi \mid \psi \rangle. \quad (287)$$

These operators induce the half-spin:

$$|\psi_t\rangle = c_+ (t) |+\rangle + c_- (t) |\rangle, \quad (288)$$

where the eigenstates have the following expression:

$$\langle \theta, \phi \mid + \rangle = \frac{1}{\sqrt{2\pi}} e^{-is} e^{i\frac{\theta}{2}} \cos \frac{\theta}{2}, \quad \langle \theta, \phi \mid - \rangle = \frac{1}{\sqrt{2\pi}} e^{-is} e^{-i\frac{\theta}{2}} \sin \frac{\theta}{2}. \quad (289)$$

33
They satisfy

\[ \hat{S} \cdot \hat{S} |\pm\rangle = \frac{3}{4} \hbar^2 |\pm\rangle, \quad \hat{S}_3 |\pm\rangle = \pm \frac{\hbar}{2} |\pm\rangle. \]  

(290)

If these ketvectors are denoted as

\[ |+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \]

(291)

then, \( \hat{S}_j = \frac{\hbar}{2} \sigma_j \) for the Pauli matrices:

\[ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

(292)

A general state of the half-integer spin of a particle has the following expression:

\[ |\psi\rangle = \sum_{m=-l-1}^{l} c_{m}^{l+1/2} (t) |l+1/2,m+1/2\rangle, \]

(293)

where, for the normalization constant \( N_{l+1/2}^m \),

\[ \langle \theta, \phi |l+1/2,m+1/2\rangle = N_{l+1/2}^m \sqrt{\frac{l+m+1}{2l+1}} e^{-is} e^{i \frac{\theta}{2} \gamma_1} Y_l^m (\theta, \phi) \]

(294)

\[ + N_{l+1/2}^m \sqrt{\frac{l-m}{2l+1}} e^{-is} e^{-i \frac{\theta}{2} \gamma_2} \sin \frac{\theta}{2} Y_l^{m+1} (\theta, \phi); \]

(295)

and the eigen states satisfy

\[ \hat{S} \cdot \hat{S} |l+1/2,m+1/2\rangle = \hbar^2 (l+1/2)(l+3/2) |l+1/2,m+1/2\rangle, \]

\[ \hat{S}_3 |l+1/2,m+1/2\rangle = \hbar (m+1/2) |l+1/2,m+1/2\rangle. \]

(296)

(297)

For Hamiltonian (279), the infinitesimal generator of motion is equivalent to the following observable:

\[ \hat{H} = \hat{P}_r^2 + \frac{\hat{S} \cdot \hat{S}}{r^2} + \frac{1}{2} \left\{ \hat{S} \cdot (B - C) + (B - C) \cdot \hat{S} \right\} + U(r) - C \cdot (B - C), \]

(298)

where

\[ C = \frac{\hbar}{2} \left( \frac{x}{2(x^2+y^2)}, \frac{y}{2(x^2+y^2)}, 0 \right) + x \times \nabla s. \]

(299)

Now, we can investigate the internal structure of such a half-integer spin particle, an electron or a constituted particle as a nucleus, which would have the following spin for the internal three-dimensional Euclid space:

\[ S(x,p) = x \times (p + \nabla s) + \frac{\hbar}{2} \left( \frac{x}{2(x^2+y^2)}, \frac{y}{2(x^2+y^2)}, 0 \right). \]

(300)

Such an interpretation of half-integer spin allows us to describe the Dirac equation as the equation of the motion for the following Hamiltonian:

\[ H(x,p,\alpha,\beta) = \alpha_1 \beta \cdot \left( p - \frac{c}{e} A \right) + mc^2 \alpha_3 - eA_0, \]

(301)
where \( \alpha \) and \( \beta \) are the internal spins expressed as relation (300). Since the obtained Hamiltonian is also canonical as discussed in the previous subsection, it has the following infinitesimal generator:

\[
\hat{H} = \left( \hat{\gamma}_j \left( \hat{p}_j - \frac{e}{c} A \right) + mc^2 \right) \hat{\gamma}_0 - eA_0,
\]

(302)

where \( \hat{\gamma} \) is the Dirac matrices. In the same way, the internal freedom like the isospins of a particle can be expressed as the invariance of motion, if its Lie group is a subset of the infinite-dimensional semidirect-product group \( S(M) \). More detailed consideration on the relativistic quantum mechanics will be held elsewhere.

\section{Comparison with Quantization Methods}

As touched on in Introduction, canonical quantum mechanics has the difficulty for the arbitrariness on the operator ordering in itself. The present theory proved so far to solve this structural difficulty, but is not the first attempt to overcome it; some alternative quantization methods have tried to resolve it and helped the birth of the present theory. Several aspects they own are still alive within protomechanics introduced in this paper.

\subsection{Group-theoretic Property}

For the classical Hamiltonian \( H^{T^*M} \) on the cotangent space \( T^*M \) of a \( N \)-dimensional oriented manifold \( M \), we can describe the classical motion of the particle having position \((x_t, p_t) \in T^*M\) for the function \( F^{T^*M}_t \) on \( T^*M \) by using Poisson bracket \( \{, \} \) as

\[
\frac{d}{dt} F^{T^*M}_t = \left\{ H^{T^*M}_t, F^{T^*M}_t \right\} + \frac{\partial F^{T^*M}_t}{\partial t}.
\]

(303)

In canonical quantum mechanics, the corresponding equation of motion is that for associated self-adjoint (or hermit) operators \( \tilde{F}_t \) and \( \tilde{H}_t \) in the Heisenberg representation, which acts on the Hilbert space with the commutator \( [,] \):

\[
\frac{\partial}{\partial t} \tilde{F}_t = \left[ \tilde{H}_t, \tilde{F}_t \right] / (-i\hbar) + \frac{\partial \tilde{F}_t}{\partial t},
\]

(304)

where \( i = \sqrt{-1} \). It should be noticed that this program guarantees the existence of such operators, but not the possibility of the concrete expression for all of them.

The Dirac rule that transfers position observable \( x_j \) and momentum observable \( p_j \) into operators \( \hat{x}_j \rightarrow x_j \) and \( \hat{p}_j \rightarrow -i\hbar \frac{\partial}{\partial x_j} \) in the Schrödinger representation usually determines the correspondence between functions \( H^{T^*M}_t, F^{T^*M}_t \) and operators \( \hat{F}_t, \hat{H}_t \); however, as proved by Groenwald \( \[3\] \) and van Hove \( \[4\] \), it can not fully work if one considers the self-adjoint operator \( x^n_i p^m_j \) corresponding to the classical observable \( x^n_i p^m_j \) for integers \( n > 1 \) and \( m \geq 2 \); the position and the momentum operator must act with \textit{infinite} multiplicity \( \[23\] \).

We can classify some of the quantizations into the following two types, that avoid such difficulty and that determine the operator ordering.

1. deformation: (Moyal, Bayen, \textit{et.al.}) \( \[16, 17\] \)/ path-integral (Feynmann) \( \[18\] \) and stochastic (Nelson, Parisi-Wu) \( \[19\] \)/ etc..

2. homomorphism: canonical (Schrödinger, Heisenberg, \textit{et.al.}) \( \[1, 2\] \) and canonical group (Mackey) \( \[20\] \)/ geometric (Soriau, Kostant) \( \[21, 22\] \)/ etc.;
The quantizations in the first category deform the Poisson algebra in classical mechanics into Moyal’s algebra or the generalized one for quantum mechanics; Moyal’s theory, as well as the path-integral and the stochastic quantization, can obtain observable $\hat{x}_i^n\hat{p}_j^m$ as the Weyl product $\{\hat{x}_i^n\hat{p}_j^m\}_W$ of operators $\hat{x}_i$ and $\hat{p}_j$:

$$\hat{x}_i^n\hat{p}_j^m = \{\hat{x}_i^n\hat{p}_j^m\}_W,$$

where, for the set $Z^+$ of all non-negative integers,

$$\exp(\alpha\hat{x}_i + \beta\hat{p}_j) = \sum_{(n,m)\in Z^+\times Z^+} \frac{1}{n! \cdot m!} \alpha^n \beta^m \{\hat{x}_i^n\hat{p}_j^m\}_W.$$

These theories can also regard observable $\{\hat{x}_i^n\hat{p}_j^m\}_W$ as the infinitesimal generator induced by function $x_i^n p_j^m$; thereby, they have no problem of the ambiguity in the operator ordering, while they produce the same result for canonical Hamiltonians with the methods in the second category. The present theory does not attribute an infinitesimal generator to the Weyl product, though it has strong similarity with the path-integral method as discussed in Section 2.

On the other hand, the second category indicates that each quantization method belonging to it bases itself on the homomorphism as a Lie algebra between the Poisson algebra in classical mechanics and the operator algebra with commutation relation in quantum mechanics. As shown so far, the present theory safely belongs to the first group and postulates that

a quantum system shares the same group structure with the corresponding classical system.

Sharing the same motivation with the present theory, Kostant and Soriau [21, 22] proposed the geometric quantization to overcome the structural difficulty in the canonical theory, which succeeded in constructing a Hamiltonian systems, a connection $\hat{\omega}$ where, for the set $Z$ for canonical Hamiltonians with the methods in the second category. The present theory does not attribute an infinitesimal generator to the Weyl product, though it has strong similarity with the path-integral method as discussed in Section 2.

In addition, it considers the $S^1$-fiber bundle $E = E(M, S^1)$ instead of $L = E(T^*M, S^1)$, and newly adds the infinite-dimensional freedom to the Hilbert space of all the $L_2$-functions on $M$ unlike the geometric quantization. As shown in Section 4, we obtained the operator $\hat{F}_t$, corresponding to the observable $F_t^{T^*M}$ on $T^*M$ as an element of Lie algebra $q(M)$:

$$\hat{F}_t = \left( \frac{\partial F_t^{T^*M}}{\partial p}, F_t^{T^*M} - p \cdot \frac{\partial F_t^{T^*M}}{\partial p} \right).$$
This form is similar to that of the geometric quantization since both utilize their similar semidirect product groups. The induced equation of motion had the following form for the operators ˜$F$ and ˜$H$:

$$\frac{\partial}{\partial t} \tilde{F}_t = [\tilde{H}_t, \tilde{F}_t] + \left( \frac{\partial \tilde{F}_t}{\partial t} \right).$$ \hspace{1cm} (310)

Section 3 elucidated the difference between classical mechanics and quantum mechanics as that of their function spaces, being the space $C_{cl}(\Gamma)$ of the extended classical observables and the space $C_q(\Gamma)$ of the extended quantum observables such that

$$C_{cl}(\Gamma) \subset C_q(\Gamma).$$ \hspace{1cm} (311)

The integration of the additional infinite-dimensional freedom was indispensable not only to deduce from this equation the Heisenberg equation in quantum mechanics for the canonical Hamiltonians, but also the Poisson equation in classical mechanics through the classical-limit.

### 7.2 Statistical Property

As discussed above, classical mechanics and the quantum mechanics basically share a group structure, or have a Lie algebra homomorphism between their own algebras not only in the present theory but also in the quantization methods belonging to the first group. The difference between those mechanics comes from what they act on or how they are represented.

The Poisson equation is equivalent to the classical Liouville equation for the probability density function ($PDF$) $\rho_{T^*M} : C^\infty(T^*M, \mathbb{R})$ of a particle:

$$\frac{\partial}{\partial t} \rho_{T^*M} = \{\rho_{T^*M}, H_{T^*M}\}.$$ \hspace{1cm} (312)

In canonical quantum mechanics, the corresponding equation of motion is the following quantum Liouville equation for the density matrix $\hat{\rho}_t = |\psi_t\rangle\langle\psi_t|$:

$$\frac{\partial}{\partial t} \hat{\rho}_t = [\hat{\rho}_t, \hat{H}_t]/(-i\hbar),$$ \hspace{1cm} (313)

which is equivalent to the Schrödinger equation:

$$i\hbar \frac{\partial}{\partial t} |\psi_t\rangle = \hat{H}_t |\psi_t\rangle.$$ \hspace{1cm} (314)

A wave function $\psi_t$ for quantum mechanics has often been compared with a delta function or a point on $T^*M$ for classical mechanics, while it is not so in the protomechanics. We may classify the quantization methods in the following way.

1. wave function:
   - canonical/ geometric (Soriau, Kostant) \cite{21, 22} / path-integral (Feynmann) \cite{18} and stochastic (Nelson, Parisi-Wu) \cite{19} / etc.;

2. density function:
   - hydrodynamic (Mandelung) \cite{28, 29} / etc.

3. density matrix:
The quantization methods belonging to the first category consider that the unitary group or the corresponding semigroup acts on the $L^2$-space of wave functions over physical space $M$. Besides, the hydrodynamic description of quantum mechanics in the third category utilizes the analogy between the Schrödinger equation and the Euler equation for the classical fluid motion, where the diffeomorphism group acts on the space of the density function and the velocity field over $M$ (consult Appendix B for the group theory of the classical fluid motion). On the other hand, the methods in the second category assume that the unitary group or the deformed group acts on the representation space of the density matrices. The quantization by density function and those by density matrix seem based on a belief as remarked by Moyal [16]:

"... the fundamental entities would be the statistical varieties representing the dynamical parameters of each system; the operators, matrices and the wave functions of quantum theory would no longer be considered as having an intrinsic meaning, but would appear rather as aids to the calculation of statistical averages and distributions."

As shown so far, the present theory belongs to none of the above classification, but shares the same belief as referred by Moyal’s words, and postulates that a quantum system shares the same statistical structure with the corresponding classical system. This statistical property can be different from the "classical" one, but includes it. On top of that, it is close to the entries in the third category since it induces a quantization method belonging to this class for the canonical Hamiltonian; and it inherits the hydrodynamic analogy in the second category between quantum mechanics and classical mechanics. Mandelung [28] rewrote (313) into a hydrodynamics equation that the de Broglie-Bohm theory [30] utilized, and considered the difference between classical and quantum mechanics, and is now summarized in a different manner from the usual explanation. For the Hamiltonian system with the Hamiltonian

$$H^{T^*M}(x_i, p_i) = \frac{1}{2m} \left( p_i - \frac{e}{c} A_i(x_i) \right)^2 + U(x_i), \quad (315)$$

equation (312) induces the following hydrodynamics equation:

$$\frac{\partial}{\partial t} (\bar{p}_i \bar{v}_j) + \partial_i (\bar{p}_i \bar{v}_j \bar{p}_j) + \bar{p}_i \partial^j U + \frac{e}{c} \bar{v}_i \partial_j A_{ti} = \partial_i T_{ij} \quad (316)$$

$$\frac{\partial}{\partial t} \bar{p}_i + \partial_i (\bar{p}_i \bar{v}_j) = 0, \quad (317)$$

where averaged PDF $\bar{p}_i$, averaged momentum $\bar{p}_{ij}$, averaged velocity $\bar{v}_i^j$, and stress tensor $T_{ij}$ are all defined as

$$\bar{p}_i(x) = \int_{R^N} d^N p \, \rho_{iM}(x, p), \quad \bar{p}_{ij}(x) = \bar{p}_i(x)^{-1} \int_{R^N} d^N p \, \rho_{iM}(x, p) \frac{\partial H^{T^*M}}{\partial p_j}(x, p), \quad (318)$$

$$\bar{v}_i(x) = \bar{p}_i(x)^{-1} \int_{R^N} d^N p \, \rho_{iM}(x, p) \frac{\partial H^{T^*M}}{\partial p_j}(x, p), \quad (319)$$

$$T_{ij}(x) = - \int_{R^N} d^N p \left\{ \left( \bar{v}_i^j(x) - \frac{\partial H^{T^*M}}{\partial p_i}(x, p) \right) \rho_{iM}(x, p) (\bar{p}_{ij}(x) - p_j) \right\}. \quad (320)$$

On the other hand, he transformed the wave function $\psi_t(x) = R_t(x) e^{iS_t(x)}$ that satisfies the Schrödinger equation (313) for canonical Hamiltonian (315) into the following variables:

$$\bar{p}_i(x) = R_t(x)^2, \quad \bar{p}_{ij}(x) = \hbar \partial_j S_t(x), \quad (321)$$

38
\[ \bar{v}_{ij}(x) = \frac{1}{m} \bar{p}_{ij}(x) - \frac{e}{c} A_{ij}(x), \]  
(322)

\[ T_{ij}^{qi}(x) = \left( -\frac{\hbar^2}{2m} \left\{ \bar{\rho}(x)^{-1} \left( \partial^i \bar{\rho}_i(x) \right) - \partial^i \right\} \partial_j \bar{\rho}_j(x) \right), \]  
(323)

and rewrite the Schrödinger equation in the following form:

\[ \frac{\partial}{\partial t} (\bar{\rho}_i \bar{p}_{ij}) + \partial_i \left( \bar{\rho}_i \bar{v}_{ij} \right) + \bar{\rho}_i \partial_j U + \frac{e}{c} \bar{\rho}_i \bar{v}_{ij} \partial_j A_{ii} = \partial_i T_{ij}^{qi} \]  
(324)

\[ \frac{\partial}{\partial t} \bar{\rho}_i + \partial_i \left( \bar{\rho}_i \bar{v}_i \right) = 0. \]  
(325)

If the pressure term in R.H.S., so-called the quantum effect, of equation (324) can be taken to be equivalent to that in R.H.S. of equation (316), or if \( \partial_i T_{ij}^{qi} = \partial_i T_{ij}^{i} \), equations (316) and (317) reduces to equations (324) and (325); but their statistical relationship seems rather obscure if one asks what stress tensor \( T_{ij}^{qi} \) corresponding to the so-called quantum potential is all about.

In order to understand the mechanism making the difference between the stress tensors \( T_{ij}^{i} \) and \( T_{ij}^{qi} \), one has to consider the information of the probability on the phase space \( T^* M \) at least, since equations (316) and (317) do not include full information of the classical equation of motion. Wigner [31] considered the so-called Wigner function \( \rho_t^W : \mathbb{R}^{2N} \rightarrow \mathbb{R} \) defined for the Fourier transformation \( \bar{\psi}_t(x) = \int_{\mathbb{R}^N} dk \, \psi_t(k)e^{ik \cdot x} \) as

\[ \rho_t^W(x,k) = \int_{\mathbb{R}^N} d^N k' \, \bar{\psi}_t \left( k + \frac{k'}{2} \right) * \bar{\psi}_t \left( k - \frac{k'}{2} \right) e^{i k' \cdot x}, \]  
(326)

and compared it with classical PDF \( \rho_t^{T^* M} \) on \( T^* M \simeq \mathbb{R}^{2N} \):

\[ \rho_t^W(x, \frac{p}{\hbar}) = \hbar^N \, \rho_t^{T^* M}(x,p). \]  
(327)

Based on this statistical analogy in conjunction with the hydrodynamic analogy, the previous letter [8] tried to reconstruct the quantum Liouville equation with keeping the Lie algebraic structure unlike Moyal’s theory; and it proved possible, but not natural.

As already discussed in Section 3 to 6, the present theory further introduced the infinite-dimensional freedom to the previous attempt [8], and considers the Lie-Poisson equation for the extended Lie group \( Q(M) \):

\[ \frac{\partial J^*_I}{\partial t} = ad^*_{R_I} J^*_I, \]  
(328)

whose concrete expressions (105) and (106) were very similar not only to equations (316) and (317) in classical mechanics but also to equations (324) and (325) in quantum mechanics without the pressure term. Section 3 elucidated the difference between classical mechanics and quantum mechanics as that of the dual spaces for function spaces \( C_c(\Gamma) \) and \( C_q(\Gamma) \), being the space \( C_{cl}^* (\Gamma) \) of the classical emergence measures and the space \( C_{cl}^* (\Gamma) \) of the quantum emergence measures:

\[ C_{cl}^* (\Gamma) \supset C_{cl}^* (\Gamma). \]  
(329)

The integration of the additional infinite-dimensional freedom was indispensable not only to deduce the Schrödinger equation or the quantum Liouville equation in quantum mechanics for the canonical Hamiltonians, but also the classical Liouville equation in classical mechanics through the classical-limit discussed in the previous section. The introduced emergence density function \( \rho[k](\xi) \) for wave number \( k \) and additional freedom \( \xi \), further, produces the Wigner function after such integration:

\[ \rho_t^W(x,k) = \int_{B(E(M))} dN(\xi) \, \rho_t(\xi)[k](x), \]  
(330)
which satisfies relation $[327]$ with the classical probability density through classical-limit. As Moyal remarked, we can hardly understand it as a kind of ordinary PDF by itself, since they must generally take negative as well as positive values on the phase space $T^*M$. For this reason, the present theory introduced the concept of the emergence measure in Section 3, that can have the negative values.

### 7.3 Semantics of Regularization

The present theory introduces the energy-cutoff $\Lambda_0$ as the superior of the emergence-frequency $f$:

$$\sup |f(\eta)(x)| = \tilde{h}^{-1} \Lambda_0. \quad (331)$$

In elementary quantum mechanics for the motion of a particle, $\Lambda_0$ is large enough and almost irrelevant for its formulation, while, in the quantum field theory where $x$ stands for the value of a field variable in the above formula, it justifies the regularization procedure in the renormalization method that Tomonaga [32] and Schwinger [33] introduced and that Feynmann, Dyson [34] and their followers completed.

Since a field variable in the field theory can emerge and the created particles interact with one another at the vertex in Feynmann’s Diagram when external time $t$ has countable numbers in the period of $T = 1/f$, the integration with the high wave number $k \geq c^{-1} \Lambda$ has nonsense if $\Lambda \geq \Lambda_0$. In the standard field theory, some one-particle-irreducible Feynmann’s diagrams contain the logarithmic divergences in $\Lambda$. The energy-cutoff $\Lambda$ and the constants $g = g(g_0, \Lambda)$ such as the masses, the coupling constants depending on $\Lambda$ first describes such a theory, while every calculated observable $K = K(g_0, \Lambda)$ should be independent of $\Lambda$: $K = K(g)$.

For the renormalization parameter $\mu \leq \Lambda_0$, it has the following relation with the dimensional regularization introduced by ’t Hooft and Veltman [35] that decreases the dimension of the spacetime as $4 \rightarrow 4 - \epsilon$ for small $\epsilon$ and that keeps the Lagrangian density for the relativistic quantum theory invariant under the Lorentz transformation:

$$\epsilon^{-1} = \ln \left( \frac{\Lambda^2}{\mu^2} \right). \quad (332)$$

The minimum subtraction represents the invariance of the theory under the variation by $\Lambda$. In addition, Weinberg [36] proved that all the standard theories of the elementary particles are renormalizable. Thus, the present theory can provide such theory with the semantics of the regularization, while the detailed study in its application will be developed elsewhere.

### 7.4 Quantization of Phenomenology

In addition, the present theory can quantize several phenomenological systems possibly having dissipation and/or stochasticity, since it does not directly rely on the Poisson nor the symplectic structure on a classical phase space but only on its group structure; it can rely on the ambient semigroup structure through the generalization. If we can interpret a phenomenological classical system as that deduced from the following system for an operator $\mathcal{L}_t: q(M)^* \rightarrow q(M)^*$ as in Section 5, we will make the corresponding quantum system under the method discussed in Section 6:

$$\frac{\partial \mathcal{J}_t}{\partial t} = \mathcal{L}_t(\mathcal{J}_t). \quad (333)$$

This may be one of the most remarkable features for its application, that has not been seen in the other theories.

### 8 REALIZATION OF SELF-CONSISTENCY

Born [37] interpreted the square amplitude of a Schrödinger’s wave function a probability density function (PDF). Heisenberg [38] further discovered the uncertainty relations as a peculiar nature of quantum
mechanics:
\[ \Delta x \cdot \Delta p \geq \hbar, \]  
where \( \Delta x \) and \( \Delta p \) are the accuracy of the position variable and the momentum of a particle. Such relation showed it impossible to determine how a particle exists in the sense of the classical mechanics, and indicated that we must give up such an idea of existence or that of causality. The present theory provides a new idea of existence, and explains how the wave-reduction occurs in an experiment.

8.1 Interpretation

The problem to provide a self-consistent interpretation of reality has still been open under the hypothesis that quantum mechanics is a universal theory. Some theories tried to interpret quantum mechanics as the ontic theory referring object systems, others as the epistemic theory referring measurement outcomes. Let me classify some of them as follows [39, 40, 41]:

1. epistemology:
   - the Copenhagen (Bohr, Heisenberg) [42] / orthodox (von-Neumann, Wigner) [14] / many-worlds (Everett, DeWitt) [43] / etc.;

2. ontology:
   - causal (de Broglie, Bohn) [30, 44] / consistent or decoherence history (Griffiths, Omnés)/
   - other modal interpretations (van Fraassen) [45].

The most important interpretation that has been supporting the physics in this century has been the Copenhagen interpretation classified in the first category. Bohr considered that the referents of quantum mechanics are observed phenomena, and that the notion of an individual microsystem is meaningful for a human being only within the context of the whole macroscopic experimental setup that should be undoubtedly suffers the classical description [42]. This view of complementarity becomes a self-consistent epistemic idea once one admits the following postulates:

1. the impossibility to understand quantum mechanics by using "the classical description" and
2. the possibility to understand the measuring devices or the compound system with the object system by using "the classical description".

The \( C^* \)-algebraic theory of quantum mechanics (refer to monograph [41]) has developed this interpretation in a rigorous way, where the classical property of the measuring devices can be described by the continuous superselection rules (CSRs) in the similar way for the measurement theory as many-Hilbert-space theory discussed in the following subsection. These theories all identify the objectification with the wave-reduction, and need the limit such that the number of the particles constituting the devices and that the time spent for the experiment are infinite. If so, however, there remain one question on this theory [39, 47]:

"Can such objectification allow some approximation or limiting process for itself?"

If one believes that an object can exist approximately, he or she has to face the problem of the metaphysics to understand what it means.

von Neumann did not accept Bohr’s view on the second postulate listed above on the possibility of the classical description for the measuring devices, and assumed [14] that quantum mechanics is a universally valid theory which applies equally well to the description of macroscopic measuring devices as to microscopic atomic objects, and he faced the problem that the object system and the measuring apparatus had to be separated though it is impossible within quantum mechanics, and solved that dilemma by introducing the
projection postulate that the final termination of any measuring process is the conscious observer. Many-worlds interpretation [43] evaded this dilemma to suppose that the quantum theory in Hilbert space describes some reality which is composed of many distinct worlds, and that the observer is aware of himself in only one of these worlds. Thus, such epistemological theory leaves the mind of a human-being enigmatic thing beyond quantum mechanics. In other words, these attempts apparently failed to provide the universality of quantum mechanics that should describe the human mind, while the recent development in the neuroscience would show that the brain seems constituted of the neural networks obeying some physical laws. If they admit this criticism and conclude that the reality occurs not in the "objective mind" of the other persons but in "Ich" or the subjectivity itself, they allow themselves to give up the illustration itself as some physical problem.

Now, we can doubt whether or not the classical description represents that of classical mechanics. Apparent, as pointed by Bohm [30], these words are not equivalent to each other. The present theory shall give the first postulate for Bohr's interpretation more explicit expression by substituting the word "mechanics" for "description", and assumes the impossibility to understand quantum mechanics by using "classical mechanics".

In Section 2, we interpreted our mathematical formalism in terms of the self-creation or "self-objectification," which can be classical description in the sense that we can understand it by using the ordinary language, but that is not the description by classical mechanics. In this sense, the present interpretation is close to Bohmian mechanics in the second category, that assumes that a particle in quantum mechanics can exist objectively and has a position as its preferred variable at every time even before the measurement; and it is a variant of the modal interpretations [45]. Thus, it takes into account that only the position of a particle can be directly measured [30], in other words, that the observables such as the momentum, the angular momentum and the spin of a particle are always indirectly observed by measuring its position; and it postulates that a quantum mechanics shares the same ontology with classical mechanics.

Unlike Bohmian mechanics, the ontology in the present theory was not the same as the traditional one in classical mechanics that each particle has its trajectory as a complete line in the spacetime, but the new one that it appears as "quantized events" in the spacetime; basically, this interpretation would not change also in the quantum field theory that substitute the value of a field variable for the position of a particle in quantum mechanics.

The present interpretation follows, in a sense, the thought by Plato in the ancient Greece, that is based on the distinction between an ideate belonging to "the world of immutable being" out of our universe, and a phenomenon, the appearance of an ideate, belonging to "the world of generation" within our universe, and that is sophisticated by Whitehead under some resent knowledge on the general relativity and the elementary quantum mechanics in his "Philosophy of Organism" or so-called "Process Philosophy" [12] (see [8, 12] for some brief summary of Whitehead's philosophy). He considered the actuality or the existence of what he called actual entities as the process of their self-creation or the "throb of experience": they do not endure in time and flash in and out of existence in spacetime. The present theory supports his philosophy in this sense, and assumes that quantum mechanics and then classical mechanics deal with such actual existence. Whitehead [12] also dispelled and unified the distinction between the subjective and the objective that would have sustained the Western culture, and, as he indicated, shared the similar idea with the philosophies related with Budism in the Eastern culture [7]. His philosophy is the inversion of Kant's philosophy [8, 12]:

"for Kant, the world emerges from the subject; for the philosophy of organism, the subject emerges from the world.”

---

12This may be one reason why I could easily feel sympathy with the philosophy of organism when I knew it after finishing almost all the mathematical formulation of the protomechanics, and why I was deeply inspired to complete its semantics.
Every thing of the world including us shares the subjectivity with one another through the individual experience, being the emergence from the objectivity. Protomechanics can rely on Whitehead’s philosophy, while the quantum theories have rested on Kant’s in twentieth century.

8.2 Measurement Process

In the present theory, the emergence of a particle does not represent the wave reduction itself, since the density matrix or the wave function represents merely a statistical state of the emergence-momentum. The wave-reduction should occur through the measurement process independently of the objectification problem; and it means the transformation of the information stored in the object system to the external system, that sometimes includes observers, through the measurement process; thereby, it does not sense the objectification itself nor need the complete wave-reduction for such purpose.

There have already exist several theories of the measurement in quantum mechanics mainly in the relation with the objectification problem, which would be classified in what the wave-reduction represents:\[39, 40\]:

1. projection:
   orthodox (von-Neumann, Wigner, Wheeler)/ relative-state (Everett)/ etc.;

2. wave-collapse:
   nonlinear hidden-variable (Bohm, Bub)/ unified dynamics (Ghirardi, et.al.)/ ergodic-environment (Daneri, Loinger, Prosperi)/ etc.;

3. decoherence:
   environment (Zeh, Zurek)/ many-Hibert-space (Machida, Namiki)\[50]/ algebraic (Hepp, et.al.)/ etc.;

The projection postulates in the first category assume in some axiomatic sense that the wave reduction occurs in the human mind or abstract "Ich" who can be aware of the universe where they are living, as discussed in the previous subsection:

\[
|\psi\rangle \rightarrow c_j |j\rangle.
\] (335)

Thus, they would never explain the wave-reduction as the consequence of the measurement process. On the other hand, the entities in the second category attempted to obtain the wave-reduction as the wave-collapse of a wave function into new wave functions by introducing the additional nonlinear effects into quantum mechanics or by assuming the irreversible effects from the environments (consult\[39\]). They require some additional postulates beyond quantum mechanics.

The present theory prefers the entities in the third categories that considers the wave-reduction as the decoherence that the density matrix loses their nonorthogonal parts after the interaction with the measuring apparatus and/or its environment:

\[
\hat{\rho}^{in} = \sum_{j,k} c_j c_k^* |j\rangle \langle k| \quad \rightarrow \quad \hat{\rho}^f = \sum_j c_j c_j^* |j\rangle \langle j|,
\] (336)

where \(|j\rangle\) represents an eigen vector for \(\hat{F}\) with the eigen value \(c_j \in \mathbb{R}\); and it postulates that

the wave-reduction mechanism should be explained within the present frame work.

Let us assume that the following three processes constitute the measurement process that completes the measurement of an observable \(\hat{F}\) through that of some position observables.
1. the preparing process to select an initial state,  
2. the scattering process to decompose a spectrum, and  
3. the detecting process to detect a particle.

They always substitute the measurement of the position of an observed particle or a radiated particle like a photon not only for that of the position itself but also for that of the spin, the momentum, or the energy.

Suppose that the initial wave function is prepared as \(|\psi^{in}\rangle = \sum_j c_j |j\rangle \otimes |\phi\rangle\):

\[
\hat{\rho}^{in} = \sum_{j,k} c_j c_k^* |j\rangle \langle k| \otimes |\phi\rangle \langle \phi| \in \Omega^P
\]

(337)

where \(|\phi\rangle\) represents the wave function for the motion of an observed particle such that its emergence-frequency (EF) is non-negative at everywhere:

\[
\rho_\phi (\eta) (x) \geq 0;
\]

(338)

thereby, the Wigner function (WF) is non-negative at everywhere from the discussion in the previous section:

\[
\int_{\mathbb{R}^3} d^3 k \left( \phi \left| k - \frac{k'}{2} \right\rangle \langle k + \frac{k'}{2} | \phi \right) \geq 0,
\]

(339)

which is the appropriate condition considered by Moyal \cite{Moyal} for the prepared Wigner function. If \(\phi\) stands for the envelope function of the wave packet, it will satisfy this condition. In addition, the measuring process conserves the positive nature of EF and WF because of the conservation law of EF discussed in Section 3. If EF does not satisfy such non-negative property, not only a particle but also an antiparticle can appear since the negative emergence frequency for a particle can be translated as the positive frequency for an antiparticle within quantum mechanics, which will serve an appropriate interpretation of the relativistic quantum mechanics without proceeding to the quantum field theory.

Then, the spectral decomposition will change it into

\[
\hat{\rho}^{ex} = \sum_{j,k} c_j c_k^* |j\rangle \langle k| \otimes |\phi_j\rangle \langle \phi_k| \in \Omega^P,
\]

(340)

where \(|\phi_j\rangle\) represents the spatial wave function moving toward the j-th detector. For the eigen state \(|x^{(j)}\rangle\) of the position of the j-th detector, the present theory immediately describes the emergence density that the particle appears at position \(x = x^{(j)}\) as

\[
\rho^{ex} (x^{(j)}) = \langle x^{(j)} | \hat{\rho}^{ex} | x^{(j)} \rangle
\]

(341)

\[
= |c_j|^2,
\]

(342)

since \(|x^{(j)}\rangle = |\phi_j\rangle\) by definition. Notice that relation (341) does not represent the wave-reduction itself.

Machida and Namiki \cite{Machida} consider that a macroscopic device is an open system that interacts with the external environment, and describes the state of the measuring apparatus by introducing the continuous super-selection rules (CSRs) for Hilbert spaces: the state of the j-th detector is described for continuous measure \(P\) on the region \(L \subset M\) occupied with considerable number of atoms constituting the detector:

\[
\hat{\rho}^{(j)} = \int_L dP(l) \hat{\rho}(l)^{(j)} \in \Omega^M.
\]

(343)
The present theory admits CSRs within its formalism, and then justifies their consideration. Thus, the state of the total system after the spectral decomposition is \( \hat{\rho}' = \hat{\rho}' \otimes \prod_k \hat{\rho}'^{(k)} \). They further utilized the Riemann-Lebesgue Lemma to induce the decoherence of the density matrix \( \hat{\rho}' \) or makes all the off-diagonal part zero through the interaction between the particle and the detector (consult \([50, 51]\) for the detail illustration):

\[
\hat{\rho}' \rightarrow \hat{\rho}'^F = e^{-it\hat{H}_0} \hat{\rho}' \otimes \prod_k \hat{\rho}'^{(k)} \ e^{it\hat{H}_0}
\]

(344)

where \( \hat{H}_0 \) is the total free Hamiltonian operator after the interaction.

As shown as above, the present theory allows the many-Hilbert-space theory successfully to induce the wave reduction in a self-consistent manner. In addition, the present theory justifies not only CSRs indispensable for the proof of the wave reduction (344) but also the utilized approximation or limiting process that takes the particle number consisting the detector as infinite, since the wave-reduction in itself is independent of the objectification of a particle or a field.

8.3 Thermodynamic Irreversibility

In the previous subsection, the decoherence decreases \( H \)-function:

\[
\langle \hat{\rho}^{in} \ln \hat{\rho}^{in} \rangle = 0 \quad \rightarrow \quad \langle \hat{\rho}' \ln \hat{\rho}' \rangle = \sum_j |c_j|^2 \ln (|c_j|^2) \leq 0.
\]

(345)

If the observer who describes the system obtains the information where the particle appears in probability (341) through the measurement process, he will know the new initial state of the particle:

\[
\hat{\rho}' \rightarrow |j\rangle \langle j| \otimes |\phi_j\rangle \langle \phi_j|;
\]

(346)

thereby, the entropy becomes zero again. In this sense, the entropy represents the incompleteness of the information for the deterministic description, and would always increase itself and cause the irreversibility through the interaction between a macroscopic or open system and a microscopic system after the instability as the spectrum decomposition.

As in the generalized measuring process, gas molecules interact with the macroscopic wall constituting the box in which they move, or with an open system surrounding the considered area, after the instability caused by the interaction or the collision among molecules, that would be expressed as the nonlinear terms in the interaction Hamiltonian in the field theory. In this case, the thermodynamic irreversibility occurs through the following tree steps:

1. the knowledge of the initial condition (preparing),
2. the instability including nonlinearity (scattering) and
3. the influence from an open system (detecting).

In the final stage, the wave reduction increases the entropy without the information of all new initial conditions.

In the equilibrium, the maximum entropy requires that the infinitesimal variation of the following thermodynamics potential \( \Omega = -pV \) becomes zero for the grand canonical system:

\[
\Omega (\beta, \mu, V) \langle \hat{\rho} \rangle = \beta^{-1} \langle \hat{\rho} \ln \hat{\rho} \rangle + \langle \hat{\rho} \hat{H} \rangle - \mu \langle \hat{\rho} \hat{N} \rangle + \beta^{-1} (\langle \hat{\rho} \rangle - 1),
\]

(347)

where \( \beta^{-1} \) and \( \mu \) are the Lagrange coefficients. Suppose that the canonical Hamiltonian \( \hat{H} \) and particle number \( \hat{N} \) have an eigen vector \( |N, E\rangle \) for the eigen values \( E \) and \( N \):

\[
\hat{\rho} = \sum_{E,N} \varrho_{E,N} |E, N\rangle \langle E, N|,
\]

(348)

\[45\]
the variation of potential $\Omega$ for the coefficients $\varrho_{E,N}$ induces the following:

$$\varrho_{E,N} = e^{\beta \Omega(\beta,\mu,V)} e^{-\beta(E-\mu N)}.$$  \hspace{1cm} (349)

Relation (349) concludes the Bose-Einstein and the Fermi-Dirac statistics for bosons and fermions, respectively, and also the Maxwell-Boltzmann statistics in the high temperature approximation.

If a Maxwell’s devil obtains the full information of the system to describe the system in a deterministic way, he must find a new initial condition to keep his description whenever only one among $10^{23}$ molecules interacts with the macroscopic wall open to the external area. Protomechanics would support such an interpretation for the second law of the thermodynamics, while the detailed consideration should be held elsewhere.

### 8.4 Compatibility with Causality

Now, the introduced interpretation of density matrices would enable us to understand the causality in quantum mechanics. On the EPR gedanken experiment [5], the violation of Bell’s inequality [6] does not necessarily contradict with objectivity nor with causality in the present theory, since this inequality relies on the positiveness of classical probability density functions.

Consider for example the system of two spin-$1/2$ particles that are prepared to move in different directions towards two measuring apparatuses $A$ and $B$ that measure the spin component along the directions $\alpha$ and $\beta$ respectively. If there exists the initial PDF depending on the hidden variables for given quantum mechanical state, the results of the measurement at the measuring apparatuses $A = \pm 1$ and $B = \pm 1$ do not depend respectively on $\beta$ and $\alpha$ under the locality requirement. For the probability measure $P$ on the space $\Lambda$ of all the concerned hidden variables including that contained in the apparatus themselves, the correlation function $P(\alpha, \beta)$ is defined for a PDF $\rho : \Lambda \to \mathbb{R}^+$ for the set $\mathbb{R}^+$ of all non-negative real values as

$$P(\alpha, \beta) = \int_{\Lambda} dP(\lambda) \rho(\lambda) A(\alpha, \lambda) B(\beta, \lambda).$$  \hspace{1cm} (350)

Alternative settings $\alpha'$ and $\beta'$ of the measuring apparatuses satisfies Bell’s inequality:

$$|P(\alpha, \beta) - P(\alpha', \beta')| + |P(\alpha', \beta) + P(\alpha', \beta')| \leq 2,$$  \hspace{1cm} (351)

whose proof needs the positiveness of PDF $\rho$.

In quantum mechanics, $\hat{A}(\alpha) = \sigma_j / 2$ and $\hat{B}(\beta) = 1 \otimes \sigma_j$ for Pauli matrices $\sigma_j$ ($j = 1, 2, 3$) are spin observables corresponding to classical ones $A(\beta, \lambda)$ and $B(\beta, \lambda)$; and the probability operator $\hat{\rho}$ corresponding to PDF $\rho$ is described as

$$\hat{\rho} = |A\rangle \langle A| \otimes |B\rangle \langle B|,$$  \hspace{1cm} (352)

where $|A\rangle$ and $|B\rangle$ are initial wave vectors. Thus, the correlation function $P(\alpha, \beta)$ has the following form in this case:

$$P(\alpha, \beta) = \langle \hat{\rho} \hat{A}(\alpha) \hat{B}(\alpha) \rangle.$$  \hspace{1cm} (353)

For this correlation function, relation (351) can be violated, since probability operator $\hat{\rho}$ does not have such property of the positiveness.

In the present paper, however, we could interpret such probability operators as the emergence-momentum. Thus, we conclude that the violation of Bell’s inequality does not deny neither the objectivity nor the causality in the present context, since the considered emergence density can have negative value. On top of that, the measurement of the spin of a particle is completed as that of the corresponding position after the spectrum decomposition as discussed in Subsection 8.2; thereby, we can always the positive emergence density as the probability density for the observed values under preparation condition (338). The same consideration would prove that the present theory has no contradiction with any delayed-choice experiments.
9 CONCLUSION

The present paper attempted to reveal the structure behind mechanics, and proposed a basic theory of time realizing Whitehead’s philosophy. It induced protomechanics that deepened Hamiltonian mechanics under the modified Einstein-de Broglie relation, and that solved the problem of the ambiguity in the operator ordering in quantum mechanics. It further provided a self-consistent interpretation for quantum mechanics and examined what is the measurement process. In addition, the introduced paradigm produced conjectures on the following subjects:

1. interpretation of spin (Subsection 6.3),
2. semantics of regularization (Subsection 7.3),
3. quantization of phenomenological system (Subsection 7.4),
4. origin of irreversibility (Subsection 8.3) and
5. compatibility with causality (Subsection 8.4).

Needless to say, the first task will be to apply the present theory to investigate the behavior of the gravity in Planck’s scale, since it solved the operator-ordering problem in quantum mechanics. On the other hand, the basic theory presented in Section 2 has nonconstructive nature and is valid whatever the considered scale is, as discussed in Introduction. The author considers that such a theory will appear through the nonlinearity of a macroscopic system and appeal to some experiments in future. In addition, the present theory may supply an appropriate description for the motion of a biological system. It needs the continuous study how to apply the present theory to such systems and how to check it in experimental ways.

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APPENDIX A: INTEGRATION ON MANIFOLD

Let us here determine the properties of the manifold $M$ that is the three-dimensional physical space for the particle motion in classical or quantum mechanics, or the space of graded field variables for the field motion in classical or the quantum field theory (consult \[2\] for more detail information on manifold theory).

Let $(M, \mathcal{O}_M)$ be a Hausdorff space for the family $\mathcal{O}_M$ of its open subsets, and also a N-dimensional oriented $C^\infty$ manifold that is modeled by the N-dimensional Euclid space $\mathbb{R}^N$ and thus it has an atlas $(U_\alpha, \varphi_\alpha)_{\alpha \in \Lambda_M}$ (the set of a local chart of $M$) for some countable set $\Lambda_M$ such that
1. $M = \bigcup_{\alpha \in \Lambda_M} U_\alpha$, 

2. $\varphi_\alpha : U_\alpha \to V_\alpha$ is a $C^\infty$ diffeomorphism for some $V_\alpha \subset \mathbb{R}^N$ and 

3. if $U_\alpha \cap U_\beta \neq \emptyset$, then $g^\alpha_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1} : V_\alpha \cap V_\beta \to V_\alpha \cap V_\beta$ is a $C^\infty$ diffeomorphism.

The above definition would be extended to include that of the infinite-dimensional manifolds called ILH-manifolds. A ILH-manifold that is modeled by the infinite-dimensional Hilbert space having an inverse-limit topology instead of $\mathbb{R}^N$. We will, however, concentrate ourselves on the finite-dimensional cases for simplicity. Let us further assume that $M$ has no boundary $\partial M = \emptyset$ for the smoothness of the $C^\infty$ diffeomorphism group $D(M)$ over $M$, i.e., in order to consider the mechanics on a manifold $N$ that has the boundary $\partial N \neq \emptyset$, we shall substitute the doubling of $N$ for $M: M = N \cup \partial N \cup N$.

Now, manifold $M$ is the topological measure space $M = (M, \mathcal{B}(O_M), \nu)$ that has the volume measure $\nu$ for the topological $\sigma$-algebra $\mathcal{B}(O_M)$. For the Riemannian manifold $M$, the (psudo-)Riemannian structure induces the volume measure $\nu$.

Second, we assume that the particle moves on manifold $M$ and has its internal freedom represented by an oriented manifold $F = (F, O_F)$, where $O_F$ is the family of open subsets of $F$. Let $F = (F, \mathcal{B}(O_F), m_F)$ be the topological measure space with the invariant measure $m_F$ under the group transformation $G_F$: $g_* m_F = m_F$ for $g \in G_F$ where $g_* m_F (g(A)) = m_F (A)$ for $A \in \mathcal{B}(O_F)$. In this case, the state of the particle can be represented as a position on the locally trivial, oriented fiber bundle $E = (E, M, F, \pi)$ with fiber $F$ over $M$ with a canonical projection $\pi : E \to M$, i.e., for every $x \in M$, there is an open neighborhood $U(x)$ and a $C^\infty$ diffeomorphism $\phi_U : x^{-1} (U(x)) \to U(x) \times F$ such that $\pi = \pi_U \circ \phi_U$ for $\pi_U : U(x) \times F \to U(x) : (x, s) \to x$. Let $G_F$ be the structure group of fiber bundle $E$: the mapping $g_{\alpha\beta} = \phi_U \circ g_{U \alpha}^{-1} : U_\alpha \cap U_\beta \times F \to U_\alpha \cap U_\beta \times F$ satisfies $g_{\alpha\beta}(x, s) \in G_F$ for $(x, s) \in U_\alpha \cap U_\beta \times F$ and the cocycle condition:

$$g_{\alpha\beta}(x, s) \cdot g_{\beta\gamma}(x, s) = g_{\alpha\gamma}(x, s) \quad \text{for} \quad (x, s) \in U_\alpha \cap U_\beta \cap U_\gamma \times F,$$

(A1)

where $\alpha, \beta, \gamma \in \Lambda_M$; and condition (A1) includes the following relations:

$$g_{\alpha\alpha}(x, s) = \text{id.} \quad \text{for} \quad x \in U_\alpha, \quad \text{and} \quad g_{\alpha\beta}(x, s) = g_{\beta\alpha}(x, s)^{-1} \quad \text{for} \quad (x, s) \in U_\alpha \cap U_\beta \times F. \quad \text{(A2)}$$

Thus, $(E, \mathcal{B}(E))$ is the Hausdorff space for the family $O_F$ of the open subsets of $E$ such that $U \in \mathcal{O}_E$ satisfies $\phi_{U_{\alpha}} (U) = U_{\alpha} \times U'_{\alpha}$ for some $U_{\alpha} \in \Lambda_M$ and $U'_{\alpha} \in \mathcal{B}(O_F)$.

Now, $(E, \mathcal{B}(E), m_E)$ becomes the topological measure space with the measure $m_E$ induced by the measures $\nu$ and $m_F$ as follows. For $A \in \mathcal{B}(O_E)$, there exists the following disjoint union corresponding to the covering $M = \bigcup_{\alpha \in \Lambda_M} U_\alpha$ such that

1. $A = \bigcup_{\alpha \in \Lambda_M} A_\alpha$ where $\pi (A_\alpha) \subset U_\alpha$, and

2. $A_\alpha \cap A_\beta = \emptyset$ for $\alpha \neq \beta$.

Thus, the measure $m_E$ can be defined as

$$m_E (A) = \sum_{\alpha \in \Lambda_M} (\nu \otimes m_F) \circ \phi_{U_\alpha} (A_\alpha). \quad \text{(A3)}$$

Notice that the above definition of $m_E$ is independent of the choice of $\{A_\alpha\}_{\alpha \in \Lambda_M}$ such that $A = \bigcup_{\alpha \in \Lambda_M} A_\alpha$ is a disjoint union since $m_F$ is the invariant measure on $F$ for the group transformation of $G_F$.

Let us introduce the space $\mathcal{M}(E)$ of all the possible probability Radon measures for the particle positions on $E$ defined as follows:

1. every $\nu \in \mathcal{M}(E)$ is the linear mapping $\nu : C^\infty (E) \oplus \mathbb{R} \to \mathbb{R}$ such that $\nu(F) < +\infty$ for $F \in C^\infty (E)$,
2. For every $\nu \in \mathcal{M}(E)$, there exists a $\sigma$-additive positive measure $P$ such that
\[ \nu(F) = \int_E dP(y)(F(y)) \] (A4)
and that $P(M) = 1$, i.e., $\nu(1) = 1$.

For every $\nu \in \mathcal{M}(E)$, the probability density function (PDF) $\rho \in L^1(E, \mathcal{B}(O_E))$ is the positive-definite, and satisfies
\[ \nu(F) = \int_{E=\cup_{\alpha \in \Lambda_M}A_\alpha} dm_E(y) \rho(y)(F(y)) \]
\[ = \sum_{\alpha \in \Lambda_M} \int_{\phi_{\nu}(A_\alpha)} d\text{vol}(x) dm_F(\vartheta) \rho \circ \phi_{U_\alpha}^{-1}(x,\vartheta)(F \circ \phi_{U_\alpha}^{-1}(x,\vartheta)), \] (A5)
where $dP = dm_E \otimes \rho$.

**APPENDIX B: LIE-POISSON MECHANICS**

Over a century ago, in an effort to elucidate the relationship between Lie group theory and classical mechanics, Lie \[ \text{introduced the Lie-Poisson system, being a Hamiltonian system on the dual space of an arbitrary finite-dimensional Lie algebra. Several years later, as a generalization of the Euler equation of a rigid body, Poincaré demonstrated the standard variational principle on the tangent space of an arbitrary finite-dimensional Lie algebra. Several years later, as a generalization of the Euler equation of motion for a Hamiltonian system as the Lie-Poisson equation of a reduced Lie-Poisson system.} \]

Guillemin and Sternberg \[ \text{introduced the collective-Hamiltonian method for a Hamiltonian system to be reduced due to the symmetry determined by an appropriate Lie group, while Marsden-Weinstein reduction method for mechanics structures for Lie groups were reconsidered in the 1960's (see \[ \text{for the historical information. Marsden and Weinstein, in 1974, proposed the Marsden-Weinstein reduction method that allows a Hamiltonian system to be reduced due to the symmetry determined by an appropriate Lie group, while Guillemin and Sternberg \text{introduced the collective-Hamiltonian method that describes the equation of motion for a Hamiltonian system as the Lie-Poisson equation of a reduced Lie-Poisson system.} \]

Let $G$ be taken to be a finite- or infinite-dimensional Lie group and $g$ the Lie algebra of $G$: i.e., the multiplications $\cdot : G \times G \rightarrow G : (\phi_1, \phi_2) \rightarrow \phi_1 \cdot \phi_2$ with a unit $e \in G$ satisfy $\phi_1^{-1} \cdot \phi_2 \in G$ and induce the commutation relation $[\cdot, \cdot] : g \times g \rightarrow g : (v_1, v_2) \rightarrow [v_1, v_2]$. For a function $F \in C^\infty(G, \mathbb{R})$, two types of derivatives respectively define the left- and the right-invariant vector field $v^+$ and $v^- \in \mathcal{X}(G)$ in the space $\mathcal{X}(G)$ of all smooth vector fields on $G$:
\[ v^+ F(\phi) = \frac{d}{dt}|_{t=0} F(\phi \cdot e^{tv}) \] (B1)
\[ v^- F(\phi) = \frac{d}{dt}|_{t=0} F(e^{-tv} \cdot \phi). \] (B2)

Accordingly, the left- and the right-invariant element of the space $\mathcal{X}(G)$ satisfy
\[ [v_1^+, v_2^-] = [v_1, v_2]^+, \quad [v_1^-, v_2^-] = [-v_1, v_2]^- \quad \text{and} \quad [v_1^+, v_2^-] = 0. \] (B3)

In the subsequent formulation, $+$ and $-$ denote left- and right-invariance, respectively. In addition, $\langle \cdot, \cdot \rangle : g^* \times g \rightarrow \mathbb{R} : (\mu, v) \rightarrow \langle \mu, v \rangle$ denotes the nondegenerate natural pairing (that is weak in general) for the dual space $g^*$ of the Lie algebra $g$, defining the left- or right-invariant 1-form $\mu^\pm \in \Lambda^1(G)$ corresponding to $\mu \in g^*$ by introducing the natural pairing $\langle \cdot, \cdot \rangle : T^*_\phi G \times T_{\phi}G \rightarrow \mathbb{R}$ for $\phi \in G$ as
\[ \langle \mu^\pm(\phi), v^\pm(\phi) \rangle = \langle \mu, v \rangle. \] (B4)
Let us now consider how the motion on a Poisson manifold $P$ can be represented by the Lie-Poisson equation for $G$ (or its central extension $B$), where $P$ is a finite or infinite Poisson manifold modeled on $C^\infty$ Banach spaces with Poisson bracket $\{\cdot,\cdot\}: C^\infty(P,\mathbb{R}) \times C^\infty(P,\mathbb{R}) \to C^\infty(P,\mathbb{R})$. Also, $\Psi: G \times P \to P$ is an action of $G$ on $P$ such that the mapping $\Psi_\phi: P \to P$ is a Poisson mapping for each $\phi \in G$ in which $\Psi_\phi(y) = \Psi(\phi, y)$ for $y \in P$. It is assumed that the Hamiltonian mapping $\hat{J}: g \to C^\infty(P,\mathbb{R})$ is obtained for this action s.t. $X_{\hat{J}(v)} = v_P$ for $v \in g$, where $X_{\hat{J}(v)}$ and $v_P \in \mathcal{X}(P)$ denote the Hamiltonian vector field for $\hat{J}(v) \in C^\infty(P,\mathbb{R})$ and the infinitesimal generator of the action on $P$ corresponding to $v \in g$, respectively. As such, the momentum (moment) mapping $J: P \to g^*$ is defined by $\hat{J}(v)(y) = \langle J(y), v \rangle$. For the special case in which $(P,\omega)$ is a symplectic manifold with a symplectic 2-form $\omega \in \Lambda^2(G)$ (i.e., $d\omega = 0$ and $\omega$ is weak nondegenerate), this momentum mapping is equivalent to that defined by $d\hat{J}(v) = v_P[\omega].$

\[ \hat{J}(v) \in C^\infty(P) \]

In twentieth century, lots of mathematicians would have based their study especially on the Poisson structure or the symplectic structure in the above diagram, while the physicists would usually have made importance the functions as the Hamiltonian and the other invariance of motions as some physical matter. In Lie-Poisson mechanics, the Lie group plays the most important role as “motion” itself, while the present theory inherits such an idea.

For the trivial topology of $G$ (consult $B$ in the nontrivial cases), the Poisson bracket satisfies

\[ \{\hat{J}(v_1), \hat{J}(v_2)\} = \pm \hat{J}([v_1, v_2]). \tag{B5} \]

The **Collective Hamiltonian Theorem** concludes the Poisson bracket for $A \circ J$ and $B \circ J \in C^\infty(P,\mathbb{R})$ can be expressed for $\mu = J(y) \in g^*$ as

\[ \{A \circ J, B \circ J\}(y) = \pm \langle J(y), [\frac{\partial A}{\partial \mu}(\mu), \frac{\partial B}{\partial \mu}(\mu)]\rangle, \tag{B6} \]

where $\frac{\partial F}{\partial \mu}: g^* \to g$ is the Fréchet derivative of $F \in C^\infty(g^*,\mathbb{R})$ that every $\mu \in g^*$ and $\xi \in g$ satisfies

\[ \frac{d}{d\tau}|_{\tau=0} F(\mu + \tau \xi) = \langle \xi, \frac{\partial F}{\partial \mu}(\mu) \rangle. \tag{B7} \]

Thus, the collective Hamiltonian $H \in C^\infty(g,\mathbb{R})$ such that $H_P = H \circ J$ collects or reduces the Poisson equation of motion into the following Lie-Poisson equation of motion:

\[ \frac{d}{dt} \mu_t = \pm ad^*_{\psi_t(\mu)}(\mu_t), \tag{B8} \]

where $\mu_t = J(x_t)$ for $x_t \in P$. We can further obtain the formal solution of Lie-Poisson equation of motion as

\[ \mu_t = Ad^*_{\phi_t}\mu_0, \tag{B9} \]

50
where generator $\phi_t \in \mathcal{G}$ satisfies $\{ \frac{\partial H}{\partial \phi_t} (\mu_t) \}^+ = \phi_t^{-1} \cdot \frac{d}{dt} \phi_t$ or $\{ \frac{\partial H}{\partial \phi_t} (\mu_t) \}^- = \frac{d}{dt} \phi_t \cdot \phi_t^{-1}$ The existence of this solution, however, should independently verified (see [58] for example).

In particular, Arnold [59] applies such group-theoretic method not only to the equations of motion of a rigid body but also to that of an ideal incompressible fluid, and constructs them as the motion of a particle on the three-dimensional special orthogonal group $SO(3)$ and as that on the infinite-dimensional Lie group $D_\nu(M)$ of all $C^\infty$ volume-preserving diffeomorphisms on a compact oriented manifold $M$. By introducing semidirect products of Lie algebras, Holm and Kupershmidt [60] and Marsden et al. [3] went on to complete the method such that various Hamiltonian systems can be treated as Lie-Poisson systems, e.g., the motion of a top under gravity and that of an ideal magnetohydrodynamics (MHD) fluid.

For the motion of an isentropic fluid, the governing Lie group is a semidirect product of the Lie group $D(M)$ of all $C^\infty$-diffeomorphisms on $M$ with $C^\infty(M) \times C^\infty(M)$, i.e.,

$$G(M) = D(M) \times _{semi} \{ C^\infty(M) \times C^\infty(M) \}.$$  \hspace{1cm} (B10)

For $\phi_1 = (\phi_1, f_1, g_1), \phi_2 = (\phi_2, f_2, g_2) \in I(M)$, the product of two elements of $I(M)$ is defined as follows:

$$\phi_1 \cdot \phi_2 = (\phi_1, f_1, g_1) \cdot (\phi_2, f_2, g_2) = (\phi_1 \circ \phi_2, \phi_2^{-1} f_1 + f_2, \phi_2^{-1} g_1 + g_2),$$ \hspace{1cm} (B11)

where $\phi^*$ denotes the pullback by $\phi \in D(M)$ and the unit element of $G(M)$ can be denoted as $(id., 0, 0) \in G(M)$, where $id. \in D(M)$ is the identity mapping from $M$ to itself.

The Lie bracket for $\hat{v}_1 = (v_1^i \partial_i, U_1, W_1)$ and $\hat{v}_2 = (v_2^i \partial_i, U_2, W_2) \in \hat{i}(M)$ becomes

$$[\hat{v}_1^-, \hat{v}_2^-] = \left( \left[ v_1^i \partial_i, v_2^j \partial_j \right], v_1^i \partial_j U_2 - v_2^j \partial_j U_1, v_1^i \partial_j W_2 - v_2^j \partial_j W_1 \right).$$ \hspace{1cm} (B12)

For the volume measure $\nu$ of $M$, the element of the dual space $g(M)^*$ of the Lie algebra $g(M)$ can be described as

$$\mathcal{J}_t = (dv \rho_t \otimes p_t, dv \rho_t, dv \sigma_t),$$ \hspace{1cm} (B13)

in that $p_t \in \Lambda^1(M)$, $dv \rho_t \in \Lambda^3(M)$ and $dv \sigma_t \in \Lambda^3(M)$ physically means the momentum, the mass density, and the entropy density.

For the thermodynamic internal energy $U(\rho(x), \sigma(x))$, the Hamiltonian for the motion of an isentropic fluid is introduced as

$$\mathcal{H}(\mathcal{J}) = \frac{1}{2} \int_M dv(x) \rho_t(x) g^{ij}(x) p_t ip_t j + \int_M dv(x) \rho_t(x) U(\rho_t(x), \sigma_t(x)).$$ \hspace{1cm} (B14)

Define the operator $\hat{F}_t = \frac{\partial}{\partial \mathcal{J}_t} (\mathcal{J}_t) \in g(M)$ for every functional $F : g(M)^* \rightarrow \mathbb{R}$ as

$$\frac{d}{dt} \mathcal{J}_t = \left. \mathcal{F} \right|_{\mathcal{J}_t = \mathcal{K}} = \left. \langle \mathcal{K}, \mathcal{F}_t \right|,$$ \hspace{1cm} (B15)

then, the Hamiltonian operator $\hat{H}_t = \frac{\partial}{\partial \mathcal{J}_t} (\mathcal{J}_t) \in g(M)$ is calculated for the velocity field $v_t = g^{ij} p_t \partial_j \in X^1(M)$ as

$$\hat{H}_t = \left( v^j \partial_j, \frac{1}{2} g^{ij} p_t ip_t j + U(\rho_t(x), \sigma_t(x)) + \rho_t(x) \frac{\partial U}{\partial \rho} (\rho_t(x), \sigma_t(x)), \rho_t(x) \frac{\partial U}{\partial \sigma} (\rho_t(x), \sigma_t(x)) \right).$$ \hspace{1cm} (B16)

The equation of motion becomes the following Lie-Poisson equation:

$$\frac{d\mathcal{J}_t}{dt} = ad^*_{\hat{H}_t} \mathcal{J}_t,$$ \hspace{1cm} (B17)

which is calculated as follows:
1. the conservation laws of mass and entropy:

\[
\frac{\partial \bar{\rho}}{\partial t} + \sqrt{-1} \partial_j \left( \rho_i v_i^j \sqrt{\det g} \right) = 0, \tag{B18}
\]

\[
\frac{\partial \bar{\sigma}}{\partial t} + \sqrt{-1} \partial_j \left( \sigma_i v_i^j \sqrt{\det g} \right) = 0, \tag{B19}
\]

where \( \sqrt{\det g} = \sqrt{|\det g_{ij}|} \);

2. the conservation law of momentum:

\[
\frac{\partial}{\partial t} (\rho_t p_{tk}) + \sqrt{-1} \partial_j \left( v_j (\rho_t p_{tk} \sqrt{\det g}) \right) + \partial_k P_t = 0, \tag{B20}
\]

where the pressure \( P_t \) satisfies the following condition:

\[
P_t(x) = \rho_t(x) \left\{ \rho_t(x) \frac{\partial U}{\partial \rho} + \sigma_t(x) \frac{\partial U}{\partial \sigma} \right\} (\rho_t(x), \sigma_t(x)), \tag{B21}
\]

which is consistent with the first law of thermodynamics.

Next, we consider \( D_v(M) \), being the Lie group of volume-preserving diffeomorphisms of \( M \), where every element \( \phi \in D_v(M) \) satisfies \( dv(\phi(x)) = dv(x) \). Lie group \( D_v(M) \) is a subgroup of \( G(M) \), and inherits its Lie-algebraic structure of. A right-invariant vector at \( T_v D_v(M) \) is identified with the corresponding divergence-free vector field on \( M \), i.e.,

\[
u^- \in \mathfrak{g}:= \mathfrak{u}^- = \mathfrak{v}^\perp \quad \nabla \cdot \mathbf{u} = 0 \quad \text{for all } x \in M. \tag{B22}
\]

We can define an operator \( P_\phi \) that orthogonally projects the elements of \( T_\phi G(M) \) onto \( T_\phi D_v(M) \) for \( \phi \in D_v(M) \subset G(M) \) such that

\[
P_\phi[v^-](\phi) = P[v^-](\phi) \tag{B23}
\]

and

\[
P[v^-](e) = (v^i - \partial^i \theta) \partial_i, \tag{B24}
\]

where \( \theta : M \to \mathbb{R} \) satisfies \( \partial_i (\partial^i - \partial^i \theta(x)) = 0 \) for every \( x \in M \). This projection changes Lie Poisson equation \((B17)\) into the new Lie-Poisson equation representing the Euler equation for the motion of an incompressible fluid:

\[
\frac{\partial \mathbf{u}_t}{\partial t} + \mathbf{u}_t \cdot \nabla \mathbf{u}_t + \nabla p = 0, \tag{B25}
\]

where the pressure \( p : M \to \mathbb{R} \) is determined by the condition \( \nabla \cdot \mathbf{u}_t = 0 \).

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