Tangential limits for harmonic functions with respect to $\phi(\Delta)$: stable and beyond

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Abstract

In this paper, we discuss tangential limits for regular harmonic functions with respect to $\phi(\Delta):=-\phi(-\Delta)$ in the $C^{1,1}$ open set $D$ in $\mathbb{R}^d$, where $\phi$ is the complete Bernstein function and $d \geq 2$. When the exterior function $f$ is local $L^p$-Hölder continuous of order $\beta$ on $D^c$ with $p \in (1, \infty]$ and $\beta > 1/p$, for a large class of Bernstein function $\phi$, we show that the regular harmonic function $u_f$ with respect to $\phi(\Delta)$, whose value is $f$ on $D^c$, converges a.e. through a certain parabola that depends on $\phi$ and $\phi'$. Our result includes the case $\phi(\lambda) = \log(1 + \lambda^{\alpha/2})$.

Our proofs use both the probabilistic and analytic methods. In particular, the Poisson kernel estimates recently obtained in [7] are essential to our approach.

AMS 2010 Mathematics Subject Classification: Primary 31B25, 60J75; Secondary 60J45, 60J50.

Keywords and phrases: Bernstein function, subordinate Brownian motion, Poisson kernel, harmonic function, (non) tangential limits, $L^p$-Hölder space.

1 Introduction

The classical Fatou theorem states that if $f \in L^p(\mathbb{R}^{d-1})$ for $p \in [1, \infty]$, then the Poisson extension $u_f$ of $f$ on the upper half-space has a nontangential limit a.e. on $\mathbb{R}^{d-1}$. It is also proved in [10] that the nontangential approach is sharp.

Presently, non-local operators and their potential theory have been extensively studied owing to their importance both in theories and applications. In particular, in [1, 2], the Fatou-type theorem for harmonic functions with respect to the operator $\Delta^{\alpha/2} = -(-\Delta)^{\alpha/2}$ was discussed. Note that R. F. Bass and D. You [1] showed that the precise analogue of the Fatou theorem for harmonic functions with respect to $\Delta^{\alpha/2}$ is not true. Thus, in this case, it is necessary to state certain assumptions related to exterior functions to prove the existence of limits at the boundary. Under certain $L^p$-Hölder continuity assumptions, it is shown in [1, 2] that the Poisson extension

*This work was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MSIP) (No.2009-0083521)
†This work was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MEST) (2013004822)
with respect to $\Delta^{\alpha/2}$ has a nontangential limit a.e. on the upper half-space and Lipschitz domains, respectively.

Among many generalizations of the Fatou theorem, it has been proved that, under various types of assumptions on the boundary functions, the nontangential approach can be relaxed (see [4] [13] [14] [18]). The boundedness of modified maximal operators has been an essential tool to prove this type of results. Recently, Y. Mizuta [12] applied analytic tools to the Poisson kernel of $\Delta^{\alpha/2}$ in the half space and showed that under the same assumption as that in [1], a regular harmonic function (Poisson extension) with respect to $\Delta^{\alpha/2}$ in the half space has tangential limits a.e. if the approaching region is a certain parabola depending on $\alpha$.

The purpose of this paper is to investigate possible tangential approaching regions for a large class of the non-local operator $\phi(\Delta) := -\phi(-\Delta)$ in $C^{1,1}$ open sets. Our result extends the main result in [12]. In [12], the explicit Poisson kernel formula for $\Delta^{\alpha/2}$ in the upper half-space played a key role in proving the main result. However, in general, it is not possible to derive an explicit Poisson kernel formula for $\phi(\Delta)$ in $C^{1,1}$ open sets. Fortunately, in a recent study [7], we have obtained sharp two-sided estimates on the Poisson kernel for $\phi(\Delta)$ in bounded $C^{1,1}$ open sets under mild assumptions on $\phi$. In this paper, we use the upper bound of the Poisson kernel for $\phi(\Delta)$ near the boundary in [7] and show that the regular harmonic function with respect to $\phi(\Delta)$, which is the local $L^p$-Hölder continuous function of order $\beta$ on $D^c$ with $p \in (1, \infty]$ and $\beta > 1/p$, converges a.e. through a certain parabola. In our results, the approaching region depends on $\phi$ and $\phi'$. Nonetheless, our approaching region is always sufficiently wide to contain a Stolz open set. See Remark 1.6 to see how wide our approaching region is.

Before stating our procedure and the main result, we introduce the following notations: We use “:=” to denote a definition, which is read as “is defined to be”. We denote $a \wedge b := \min\{a, b\}$, $a \vee b := \max\{a, b\}$, and $B(x, r) := \{y \in \mathbb{R}^d : |x - y| < r\}$. For a set $W$ in $\mathbb{R}^d$, $\overline{W}$ and $W^c$ denote the closure and complement of $W$ in $\mathbb{R}^d$, respectively. For any open set $V$, we denote by $\delta_V(x)$, the distance of a point $x$ to the boundary of $V$, i.e., $\delta_V(x) = \text{dist}(x, \partial V)$. We often denote point $z = (z_1, \ldots, z_d) \in \mathbb{R}^d$ as $(\tilde{z}, z_d)$ with $\tilde{z} \in \mathbb{R}^{d-1}$. Since we consider tangential limits, we always assume that $d \geq 2$.

A smooth function $\phi : (0, \infty) \to [0, \infty)$ is called a Bernstein function if $(-1)^n\phi^{(n)} \leq 0$ for every positive integer $n$. It is well-known that every Bernstein function with $\phi(0+) = 0$ has the form

$$\phi(\lambda) = b\lambda + \int_{(0,\infty)} (1 - e^{-\lambda t}) \, \mu(dt), \quad \lambda > 0, \tag{1.1}$$

where $b \geq 0$, and $\mu$ is a measure on $(0, \infty)$ satisfying $\int_{(0,\infty)} (1 \wedge t) \mu(dt) < \infty$. $\mu$ is called the Lévy measure of $\phi$. (See [15].)

By concavity, every Bernstein function $\phi$ satisfies

$$\phi(t\lambda) \leq \lambda \phi(t) \quad \text{for all } \lambda \geq 1, t > 0. \tag{1.2}$$

Thus, $\lambda \mapsto \phi(\lambda)/\lambda$ is decreasing, and therefore,

$$\lambda \phi'(\lambda) \leq \phi(\lambda) \quad \text{for all } \lambda > 0. \tag{1.3}$$

These simple properties of $\phi$ will be used several times in this paper.

In [9], the following conditions on the Bernstein function $\phi$ are considered: Since we always assume that $d \geq 2$, here, we state the conditions only for $d \geq 2$. 


(A-1) \( \phi \) is a complete Bernstein function, i.e., the Lévy measure \( \mu \) of \( \phi \) has a completely monotone density \( \chi \), i.e., \((-1)^n \chi^{(n)}(\tau) \geq 0 \) for every non-negative integer \( n \).

(A-2) \( \phi(0+) = 0 \) and \( \lim_{\lambda \to \infty} \phi(\lambda) = \infty \).

(A-3) There exist constants \( \sigma > 0, \lambda_0 > 0, \) and \( \delta \in (0,1) \) such that
\[
\frac{\phi'(\lambda t)}{\phi'(\lambda)} \leq \sigma t^{-\delta} \quad \text{for all} \quad t \geq 1 \quad \text{and} \quad \lambda \geq \lambda_0. 
\]

(A-4) If \( d = 2 \), we assume that there are \( \sigma_0 > 0 \) and \( \delta_0 \in (0,2\delta) \) such that
\[
\frac{\phi'(\lambda t)}{\phi'(\lambda)} \geq \sigma_0 t^{-\delta_0} \quad \text{for all} \quad t \geq 1 \quad \text{and} \quad \lambda \geq \lambda_0.
\]

(A-5) If the constant \( \delta \) in (A-3) satisfies \( 0 < \delta \leq \frac{1}{2} \), then we assume that there exist constants \( \sigma_1 > 0 \) and \( \delta_1 \in [\delta,1) \) such that
\[
\frac{\phi(\lambda t)}{\phi(\lambda)} \geq \sigma_1 t^{1-\delta_1} \quad \text{for all} \quad t \geq 1 \quad \text{and} \quad \lambda \geq \lambda_0.
\]

(A-6) There exist a \( \theta > 0 \) such that
\[
\int_0^\theta \frac{\lambda^{d/2-1}}{\phi(\lambda)} \, d\lambda < \infty.
\]

From (A-3), we get \( b = 0 \) in (1.1) by letting \( t \to \infty \). From [8, Lemma 2.2], (A-3) also implies that for every \( \epsilon > 0 \), there exists \( c = c(\epsilon) > 0 \) such that
\[
\frac{\phi(\lambda x)}{\phi(\lambda)} \leq cx^{1-\delta+\epsilon} \quad \text{for all} \quad x \geq 1 \quad \text{and} \quad \lambda \geq \lambda_0.
\]  

(1.4)

See Example [15] for examples of \( \phi \) satisfying the assumptions (A-1)–(A-6).

By Bochner’s functional calculus, one can define the operator \( \phi(\Delta) \) on \( C_b^0(\mathbb{R}^d) \), which is the collection of bounded \( C^2 \) functions in \( \mathbb{R}^d \) with bounded derivatives. Analytically, harmonic function \( u \) for \( \phi(\Delta) \) solves \( \phi(\Delta)u = 0 \) on \( D \) in the distributional sense (see [3]). Since we use a probabilistic method, we formulate harmonic functions for \( \phi(\Delta) \) using the Lévy process corresponding to \( \zeta \mapsto \phi(|\zeta|^2) \). Let \( X = (X_t, \mathbb{P}_x)_{t \geq 0, x \in \mathbb{R}^d} \) be a rotationally symmetric Lévy process with a characteristic exponent \( \phi(|\zeta|^2) \), that is,
\[
\mathbb{E}_x \left[ e^{i\zeta \cdot (X_t - X_0)} \right] = e^{-t\phi(|\zeta|^2)} \quad \text{for every} \quad x \in \mathbb{R}^d \quad \text{and} \quad \zeta \in \mathbb{R}^d.
\]

The infinitesimal generator of \( X \) is \( \phi(\Delta) \), i.e., \( \phi(\Delta)u(x) = \lim_{t \to 0} t^{-1} (\mathbb{E}_x[u(X_t)] - u(x)) \). For an open set \( D \), let \( \tau_D := \inf \{ t > 0 : X_t \notin D \} \). Now, we give the probabilistic definition of a (regular) harmonic function.

**Definition 1.1.** (1) A function \( u : \mathbb{R}^d \to \mathbb{R} \) is said to be harmonic in an open set \( D \subset \mathbb{R}^d \) with respect to \( X \) if for every open set \( B \) whose closure is a compact subset of \( D \), \( \mathbb{E}_x[|u(X_{\tau_B})|] < \infty \) and \( u(x) = \mathbb{E}_x[u(X_{\tau_B})] \) for every \( x \in B \).

(2) A function \( u : \mathbb{R}^d \to \mathbb{R} \) is said to be regular harmonic in an open set \( D \subset \mathbb{R}^d \) with respect to \( X \) if \( \mathbb{E}_x[|u(X_{\tau_D})|] < \infty \) and \( u(x) = \mathbb{E}_x[u(X_{\tau_D})] \) for every \( x \in D \).
Clearly, a regular harmonic function in $D$ is harmonic in $D$ by the strong Markov property. Note that, by the Harnack inequality proved in [8], under assumptions (A-1)—(A-3) the condition $\mathbb{E}_x[|u(X_{r_0})|] < \infty$ for all $x \in D$ is equivalent to $\mathbb{E}_{x_0}[|u(X_{r_0})|] < \infty$ for some $x_0 \in D$.

Now, we recall some function spaces related to our exterior functions.

**Definition 1.2.** Suppose $p \in (1, \infty]$.

1. $\Lambda^p_\beta(\mathbb{R}^d)$ is the space of $L^p$-Hölder continuous functions of order $\beta$ defined on $\mathbb{R}^d$, i.e., $\bar{f} \in \Lambda^p_\beta(\mathbb{R}^d)$ means that $\bar{f} \in L^p(\mathbb{R}^d)$ and there exists a constant $c > 0$ such that
   \[
   \|\bar{f}(\cdot + y) - \bar{f}(\cdot)\|_{L^p(\mathbb{R}^d)} \leq c|y|^\beta \quad \text{for all } y \in \mathbb{R}^d. \tag{1.5}
   \]

2. $\Lambda^p_{\beta, \text{loc}}(\partial D)$ is the collection of functions $f$ such that $f$ is defined on $\partial D$ and for each $\xi \in \partial D$, there exists $\eta > 0$ depending on $\xi$ such that $f$ agrees on $\partial D \cap B(\xi, \eta)$ with a function in $\Lambda^p_\beta(\mathbb{R}^d)$.

Note that functions in $\Lambda^p_{\beta, \text{loc}}(\partial D)$ may not be bounded (cf., [17, 1, 2]).

**Definition 1.3.** An open set $D$ in $\mathbb{R}^d$ is said to be $C^{1,1}$ if there exist $R, \Lambda > 0$ such that the following holds: for every $\xi \in \partial D$, there exist

1. a $C^{1,1}$-function $\Gamma = \Gamma_\xi : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying $\Gamma(0) = 0$, $\nabla \Gamma(0) = (0, \ldots, 0)$, $\|\nabla \Gamma\|_\infty \leq \Lambda$, $|\nabla \Gamma(x) - \nabla \Gamma(w)| \leq \Lambda|x - w|$, for $x, w \in \mathbb{R}^{d-1}$ and

2. an orthonormal coordinate system $CS_\xi : y = (\bar{y}, y_d)$ with origin at $\xi$ such that
   \[B(\xi, R) \cap D = \{y = (\bar{y}, y_d) \in B(0, R) \text{ in } CS_\xi : y_d > \Gamma(\bar{y})\}.
   \]

The pair $(R, \Lambda)$ is called the $C^{1,1}$ characteristics of the open set $D$.

For $\gamma, a > 0$, an $C^{1,1}$ open set $D$, and $\xi \in \partial D$, define

\[
T_{\gamma, \phi, a}(\xi) = T_{\gamma, \phi, a, D}(\xi) := \left\{x \in D : |x - \xi|^{\gamma + a} \phi(|x - \xi|^{-2})^{1/2} \leq a \frac{\delta_D(x)^{d+2} \phi(\delta_D(x)^{-2})^{3/2}}{\phi'(\delta_D(x)^{-2})}\right\}, \tag{1.6}
\]

and $T_{\gamma, \phi}(\xi) := T_{\gamma, \phi, 1}(\xi)$.

Now, we state our theorem. We use $\mathcal{H}^s$ to denote the $s$-dimensional Hausdorff measure on $\mathbb{R}^d$ and for a measurable subset $W \subset \mathbb{R}^d$, $|W|$ denotes the Lebesgue measure of $W$ in $\mathbb{R}^d$.

**Theorem 1.4.** Suppose that $p \in (1, \infty]$ and $\beta > 1/p$. Let $X = (X_t, \mathbb{P}_x)_{t \geq 0, x \in \mathbb{R}^d}$ be a rotationally symmetric Lévy process with the characteristic exponent $\phi(|\xi|^2)$ such that the assumptions (A-1)—(A-6) hold and $\delta$ in (A-3) satisfies $1/p < \delta \leq 1$. Suppose that $D$ is a $C^{1,1}$ open set with characteristic $(R, \Lambda)$ and that $f \in \Lambda^p_{\beta, \text{loc}}(\partial D)$ satisfies $\mathbb{E}_{x_0}[|f(X_{r_0})|] < \infty$ for some $x_0 \in D$. Then, for $0 < \gamma < \beta - 1/p$ and $a > 0$, there exists a measurable subset $E \subset \partial D$ with $\mathcal{H}^{d-1}(E) = 0$ such that $u_f(x) = \mathbb{E}_x[f(X_{r_0})]$ has a finite limit along $T_{\gamma, \phi, a}(\xi)$ for every $\xi \in \partial D \setminus E$. Furthermore,

\[
\lim_{T_{\gamma, \phi, a}(\xi) \ni x \to \xi} \lim_{r \to 0+} \frac{1}{|B(\xi, r) \setminus D|} \int_{B(\xi, r) \setminus D} f(y) \, dy = 0, \quad \text{for all } \xi \in \partial D \setminus E.
\]
Note that when $\phi(\lambda) = \lambda^{\alpha/2}$ and $D$ is the upper half-space $H := \{x = (\bar{x}, x_d) \in \mathbb{R}^d : x_d > 0\}$, our approaching region $T_{\gamma, \phi, d/\alpha}(\xi)$ is simply $\{x \in H : |x - \xi|^{1 + \gamma/(d - \alpha/2)} \leq x_d\}$. Thus, our Theorem 1.6 covers the result stated in [12].

Since the positive constant $a$ in (1.6) plays no special role in the proof, for convenience, we will only consider $T_{\gamma, \phi}(\xi) = T_{\gamma, \phi, 1}(\xi)$.

Example 1.5. Here are some examples of $\phi$ that satisfy the above assumptions (A-1)–(A-6).

- $\phi(\lambda) = \lambda^{\alpha/2}, \alpha \in (0, 2)$;
- $\phi(\lambda) = (\lambda + \lambda^{\alpha})^{\kappa}, \alpha, \kappa \in (0, 1)$;
- $\phi(\lambda) = (\lambda + m^{2/\alpha})^{\alpha/2} - m, \alpha \in (0, 2), m > 0, d > 2$;
- $\phi(\lambda) = \lambda^{\alpha/2} + \lambda^{\kappa/2}, 0 \leq \kappa < \alpha \in (0, 2)$;
- $\phi(\lambda) = \lambda^{\alpha/2}(\log(1 + \lambda))^{\kappa}, \alpha \in (0, 2), \kappa \in (-\alpha/2, 1 - \alpha/2)$;
- $\phi(\lambda) = \log(1 + \lambda^{\alpha/2}), \alpha \in (0, 2], d > \alpha$;
- $\phi(\lambda) = \log(1 + (\lambda + m^{2/\alpha})^{\alpha/2} - m), \alpha \in (0, 2), m > 0, d > 2$.

Remark 1.6. From (1.3), we see that $T_{\gamma, \phi}(\xi)$ contains

$$T_{\gamma, \phi}'(\xi) := \left\{ x \in D : |x - \xi|^d \phi(|x - \xi|^{-2})^{1/2} \leq \delta_D(x)^d \phi(\delta_D(x)^{-2})^{1/2} \right\}.$$  

Moreover, $T_{\gamma, \phi}'(\xi)$ contains the Stolz open set

$$S_M(\xi) := \{x \in D : |x - \xi| \leq M\delta_D(x), |x - \xi| < M^{-d/\gamma}\}$$

for $M > 1$. In fact, since $r^d \phi(r^{-2})^{1/2}$ is increasing by (1.2), for $x \in S_M(\xi)$, we have

$$|x - \xi|^d \phi(|x - \xi|^{-2})^{1/2} \leq M^d \delta_D(x)^d \phi(M^{-d} \delta_D(x)^{-2})^{1/2} \leq M^d \delta_D(x)^d \phi(\delta_D(x)^{-2})^{1/2}.$$  

Thus, for $x \in S_M(\xi),

$$|x - \xi|^d \phi(|x - \xi|^{-2})^{1/2} \leq |x - \xi|^d \delta_D(x)^d \phi(\delta_D(x)^{-2})^{1/2} < \delta_D(x)^d \phi(\delta_D(x)^{-2})^{1/2}.$$  

We conclude that $S_M(\xi) \subset T_{\gamma, \phi}'(\xi) \subset T_{\gamma, \phi}(\xi)$.

On the other hand, our approaching region $T_{\gamma, \phi}(\xi)$ can be strictly larger than $T_{\gamma, \phi}'(\xi)$. For example, when $D$ is the upper half-space $H$ and $\phi(\lambda) = \log(1 + \lambda^{\alpha/2})$ where $\alpha \in (0, 2], d > \alpha$, we have

$$T_{\gamma, \phi}(\xi) = \left\{ x \in H : |x - \xi|^d \{\log(1 + |x - \xi|^{-\alpha})\}^{1/2} \leq (2/\alpha)(1 + x_d^\alpha)x_d^\alpha\log(1 + x_d^{-\alpha})^{3/2}\right\}$$

$$\sup \left\{ x \in H : |x - \xi|^d \{\log(1 + |x - \xi|^{-\alpha})\}^{1/2} \leq x_d^\alpha\log(1 + x_d^{-\alpha})^{3/2}\right\},$$  

while

$$T_{\gamma, \phi}'(\xi) = \left\{ x \in H : |x - \xi|^d \{\log(1 + |x - \xi|^{-\alpha})\}^{1/2} \leq x_d^\alpha\log(1 + x_d^{-\alpha})^{1/2}\right\}.$$  

5
The remainder of this paper is organized as follows: In Section 2, we recall some basic facts on the Bernstein functions and corresponding Lévy processes. Then, we recall the result in [7], which is essential in proving Theorem 1.4. Section 3 consists of key lemmas that hold for Lipschitz open sets. Using the results in Section 2 and 3, we prove Theorem 1.4 in Section 4.

In this paper, we use the following convention: The values of the constants $R, \Lambda, \lambda_0, \delta$ remain the same throughout this paper, while $c, c_0, c_1, c_2, \ldots$ represent constants whose values are unimportant and may change. All constants are positive finite numbers. The constants $c_0, c_1, c_2, \ldots$ are labeled again in the statement and proof of each result. The dependence of constant $c$ on dimension $d$ is not mentioned explicitly.

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2 Preliminaries

First, we recall certain essential facts about our Lévy process $X$ that we will use later. Then, since our proof considerably relies on the Poisson kernel estimates in [7], we will also recall facts related to Poisson kernel and the result in [7].

Let $B = (B_t : t \geq 0)$ be a Brownian motion in $\mathbb{R}^d$ whose infinitesimal generator is $\Delta$ (our Brownian motion $B$ runs at twice the usual speed), and let $S = (S_t : t \geq 0)$ be a subordinator (non-negative increasing Lévy process in $\mathbb{R}$ with $S_0 = 0$) independent of $B$ whose Laplace exponent is $\phi$, i.e.,

$$\mathbb{E}[\exp\{-\lambda S_t\}] = \exp\{-t\phi(\lambda)\}, \quad \lambda > 0.$$ 

It is well-known that the Laplace exponents of subordinators are always Bernstein functions. The Lévy process $X = (X_t : t \geq 0)$ whose characteristic exponent is $\zeta \mapsto \phi(|\zeta|^2)$ can be defined by $X_t = B_{S_t}$ and it is also called a subordinate Brownian motion. For example, a rotation invariant $\alpha$-stable process is a subordinate Brownian motion with $\phi(\lambda) = \lambda^{\alpha/2}$.

For the remainder of this paper, we will always assume that $\phi$ is a Bernstein function satisfying (A-1)–(A-6). Recall that $\phi$ has the representation in (1.1). Since $b = 0$ in (1.1) by (A-3), $X$ is a pure jump process.

The Lévy measure of $X$ has the density $x \mapsto j(|x|)$, where

$$j(r) = \int_0^\infty (4\pi t)^{-d/2} e^{-r^2/(4t)} \mu(dt)$$

and $\mu$ is the Lévy measure of $\phi$ (or of $S$). The infinitesimal generator of $X$ is $\phi(\Delta)$, which is an integro-differential operator of the type

$$\phi(\Delta)u(x) = \int_{\mathbb{R}^d} \left( u(x+y) - u(x) - \nabla u(x) \cdot y \mathbf{1}_{\{|y| \leq 1\}} \right) j(|y|)dy.$$ 

$X$ has a transition density $p(t, x, y)$ given by

$$p(t, x, y) = \int_0^\infty (4\pi s)^{-d/2} \exp\left(-\frac{|x-y|^2}{4s}\right) \mathbb{P}(S_t \in ds).$$

Recall that $X$ is said to be transient if $\mathbb{P}_0(\lim_{t \to \infty} |X_t| = \infty) = 1$. From the Chung-Fuchs type criterion of the transience of $X$, (A-6) is equivalent to the transience of $X$ (see [3] (2.9)). Thus, we can define the Green function $G(x, y)$ by

$$G(x, y) = g(|x-y|) = \int_0^\infty p(t, x, y)dt.$$ 

(2.1)
From (2.1), we see that $g$ is decreasing. Under our assumptions (A-1)–(A-6), $g(r)$ and $j(r)$ enjoy the following estimates (see [8]): for every $M > 0$, there exists $c = c(M) > 0$ such that

$$c^{-1} \frac{\phi'(r^{-2})}{r^{d+2} \phi(r^{-2})^2} \leq g(r) \leq c \frac{\phi'(r^{-2})}{r^{d+2} \phi(r^{-2})^2}, \quad 0 < r \leq M, \quad (2.2)$$

and

$$c^{-1} \frac{\phi'(r^{-2})}{r^{d+2}} \leq j(r) \leq c \frac{\phi'(r^{-2})}{r^{d+2}}, \quad 0 < r \leq M. \quad (2.3)$$

For any open subset $U$ in $\mathbb{R}^d$, we use $G_U(x,y)$ to denote the Green function of the process $X$ in $U$, which can be defined as $G_U(x,y) = G(x,y) - \mathbb{E}_x[G(X_{\tau_U},y)]$. For each fixed $z_0 \in U$, the function $G_U(\cdot, z_0)$ is the non-negative regular harmonic function for $X$ in $U \setminus \overline{B(z_0, \epsilon)}$ for every $\epsilon > 0$ and it vanishes on $\mathbb{R}^d \setminus U$.

Now, we define the Poisson kernel by

$$K_U(x,z) := \int_U G_U(x,y)j(\|y-z\|)dy, \quad (x,z) \in U \times U^c.$$ 

Then, by the result of Ikeda and Watanabe (see [6] Theorem 1), for any open subset $U$ and every non-negative measurable function $f$,

$$\mathbb{E}_x[f(X_{\tau_U}); X_{\tau_U} \neq X_{\tau_U}] = \int_{\mathbb{R}^d} K_U(x,z)f(z)dz.$$ 

**Definition 2.1.** An open set $D$ in $\mathbb{R}^d$ is said to be a Lipschitz open set if there exist a localization radius $R_{\text{Lip}} > 0$ and a constant $\Lambda_{\text{Lip}} > 0$ such that the following holds: for every $\xi \in \partial D$, there exist

1. a Lipschitz function $\psi = \psi_\xi : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying $\psi(0) = 0$, $|\psi(x) - \psi(y)| \leq \Lambda_{\text{Lip}}|x-y|$, and
2. an orthonormal coordinate system $CS_\xi: y = (y_\xi, y_d)$ with its origin at $\xi$ such that

$$B(\xi, R_{\text{Lip}}) \cap D = \{y = (\tilde{y}, y_d) \in B(0, R_{\text{Lip}}) \text{ in } CS_\xi : y_d > \psi(\tilde{y})\}.$$ 

The pair $(R_{\text{Lip}}, \Lambda_{\text{Lip}})$ is called the characteristics of the Lipschitz open set $D$.

Since $X$ is a rotationally invariant pure jump Lévy process, for every Lipschitz open set $D$, $\mathbb{P}_x(X_{\tau_D} = X_{\tau_D}) = 1$ (see [11], [19]). Thus, for every Lipschitz open set $D$ and every measurable function $f$ on $\mathbb{R}^d$, which satisfies $\int_{\mathbb{R}^d} K_D(x_0, z)|f(z)|dz < \infty$ for some $x_0 \in D$, $u_f$ defined in Theorem 1.1 has the following integral representation:

$$u_f(x) = \mathbb{E}_x[f(X_{\tau_D}) : X_{\tau_D} \in \overline{D}] = \int_{\mathbb{R}^d} K_D(x, z)f(z)dz, \quad x \in D. \quad (2.4)$$

Clearly, any regular harmonic function $u$ in a Lipschitz open set $D$, whose value on $D^c$ is $f$, is written as $u_f$. 

7
Furthermore, when $U$ is a bounded $C^{1,1}$ open set, we see from [7, Theorem 1.3] that

$$
\frac{\phi(\delta_U(z)^{-2})^{1/2}}{\phi(\delta_U(x)^{-2})^{1/2}\phi(|x-z|^{-2})(1 + \phi(\delta_U(z)^{-2})^{-1/2})}j(|x-z|) \leq K_U(x, z) \leq c \frac{\phi(\delta_U(z)^{-2})^{1/2}}{\phi(\delta_U(x)^{-2})^{1/2}\phi(|x-z|^{-2})(1 + \phi(\delta_U(z)^{-2})^{-1/2})}j(|x-z|), \quad (x, z) \in U \times U^c.
$$

(2.5)

We will use the upper bound in (2.5) for $|x-z| < 2$.

3 Analysis on Lipschitz open set

Recall that we assume (A-1)–(A-6). In this section, we prove some results that hold on Lipschitz open sets. We will use these results in Section 4.

Throughout this section, we fix the Lipschitz open set $D$ with characteristics $(R_{\text{Lip}}, \Lambda_{\text{Lip}})$. Without loss of generality, we assume that $R_{\text{Lip}} < 1$. Note that $D$ can be unbounded and disconnected. For every $\xi \in \partial D$ and $x \in B(\xi, R_{\text{Lip}})$, we define the vertical distance

$$
\rho_\xi(x) := x_d - \psi_\xi(x),
$$

where $(\tilde{x}, x_d)$ are the coordinates of $x$ in $CS_\xi$. Then,

$$
\delta_D(x) \leq |\rho_\xi(x)| \leq (1 + \Lambda_{\text{Lip}})\delta_D(x), \quad \xi \in \partial D, \; x \in B(\xi, R_{\text{Lip}}).
$$

(3.1)

Recall that $\lambda_0$ and $\delta$ are the constants in (A-3).

**Lemma 3.1.** For all $q \in [1, 1/(1 - \delta))$, and $M \geq 1$, there exists a constant $c = c(q, \delta, \Lambda_{\text{Lip}}, M) > 0$ such that for every $\xi \in \partial D$, $s \leq R_{\text{Lip}}/2$, and $r \leq (2M)^{-1}(R_{\text{Lip}} \wedge \lambda_0^{-1/2})$,

$$
\int_{\{(\tilde{y}, y_d) \text{ in } CS_\xi : |\tilde{y}| < s, |\rho_\xi(y)| < Mr\}} \phi(\delta_D(y)^{-2})^{q/2} dy \leq cs^{d-1}\phi(r^{-2})^{q/2}.
$$

(3.2)

**Proof.** First, since $\phi$ is increasing, by (3.1), the left-hand side of (3.2) is less than or equal to

$$
\int_{\{(\tilde{y}, y_d) \text{ in } CS_\xi : |\tilde{y}| < s, |\rho_\xi(y)| < Mr\}} \phi\left((1 + \Lambda_{\text{Lip}})^2|\psi_\xi(\tilde{y}) - y_d|^{-2}\right)^{q/2} dy.
$$

(3.3)

Using the assumption $q \in (1 - \delta, 1/(1 - \delta))$, choose $c = c(\delta, q) \in (0, \delta + 1/q - 1)$. By the change of variable $t = \rho_\xi(y)/M$, the fact that $\phi$ is increasing, (1.2), and (1.4),

$$
\int_{\{(\tilde{y}, y_d) \text{ in } CS_\xi : |\tilde{y}| < s, |\rho_\xi(y)| < Mr\}} \phi\left((1 + \Lambda_{\text{Lip}})^2|\psi_\xi(\tilde{y}) - y_d|^{-2}\right)^{q/2} dy = \phi((1 + \Lambda_{\text{Lip}})^2M^{-2}r^{-2})^{q/2} \int_{\{(\tilde{y}, y_d) \text{ in } CS_\xi : |\tilde{y}| < s, |\rho_\xi(y)| < Mr\}} \left(\frac{\phi((1 + \Lambda_{\text{Lip}})^2|\psi_\xi(\tilde{y}) - y_d|^{-2})}{\phi((1 + \Lambda_{\text{Lip}})^2M^{-2}r^{-2})}\right)^{q/2} dy
$$

$$
\leq 2(1 \vee (1 + \Lambda_{\text{Lip}})^qM^{-q})\phi(r^{-2})^{q/2} \int_0^r \int_{\{|y| < s\}} \int_0^t \left(\frac{\phi((1 + \Lambda_{\text{Lip}})^2M^{-2}t^{-2})}{\phi((1 + \Lambda_{\text{Lip}})^2M^{-2}r^{-2})}\right)^{q/2} M dt d\tilde{y}
$$

$$
\leq c_1\phi(r^{-2})^{q/2}s^{d-1} \int_0^r \left(\frac{r}{t}\right)^{(1 - \delta + c)q} dt,
$$

8
Lemma 3.3. Suppose that we have proved the lemma. For all \( \forall \), we will use this coordinate system \( CS \).

Proof. Using the cardinality, we can choose \( H \) so that the remainder of the proof, we fix \( M \) such that
\[
\int_{B(\xi,r)} \phi(\delta_D(y)^{-2})^{1/2} dy \leq cr^d \phi(r^{-2})^{1/2}.
\]

Recall that for \( x \in \mathbb{R}^d \), \( r > 0 \), and a set \( W \subset \mathbb{R}^d \), \( B_W(x,r) = B(x,r) \cap W \).

Lemma 3.3. Suppose that \( f \in \Lambda^p_{\beta,\text{loc}}(\overline{D}^c) \) and \( 0 < \gamma < \beta - 1/p \). If \( \delta \) in (A-3) satisfies \( \delta > 1/p \), then \( \mathcal{H}^{d-1}(E(\gamma)) = 0 \), where
\[
E(\gamma) := \left\{ \xi \in \partial D : \limsup_{r \to 0^+} \int_{B_{\gamma}(\xi,r)} \int_{B_{\gamma}(\xi,r)} \phi(\delta_D(y)^{-2})^{1/2} \frac{|f(y) - f(z)|}{y^{2d+\gamma} \phi(r^{-2})^{1/2}} dydz > 0 \right\}.
\]

Proof. Using the cardinality, we can choose \( \xi_i \in \partial D \) and \( \eta_i = \eta_i(\xi_i) > 0 \) such that there exists \( \tilde{f}_i \in \Lambda^p_{\beta}(\mathbb{R}^d) \) satisfying \( f = \tilde{f}_i \) on \( \overline{D}^c \cap B(\xi_i, \eta_i) \), and that, \( \partial D \subset \bigcup_{i \in \mathbb{N}} B(\xi_i, a_i) \), where \( a_i = (R_{\text{Lip}} \wedge \eta_i)/4 \). Let \( E_i(\gamma) = E(\gamma) \cap B(\xi_i, a_i) \) and \( n_i \in \mathbb{N} \) be the largest number such that \( n_i \leq \log_2(a_i^{-1} + \sqrt{\alpha_0}) + \log_2(2 + 2 \Lambda_{\text{Lip}}) + 1 \). For \( M > 0 \) and \( n \geq n_i \), set
\[
E_i(\gamma, M, n) = \left\{ \xi \in B_{\partial D}(\xi_i, a_i) : \int_{B_{\gamma}(\xi, 2^{-n})} \int_{B_{\gamma}(\xi, 2^{-n})} \frac{\phi(\delta_D(y)^{-2})^{1/2}}{2^{-n(2d+\gamma)} \phi(2^{2n})^{1/2}} |\tilde{f}_i(y) - \tilde{f}_i(z)| dydz > \frac{1}{M} \right\}.
\]

Since
\[
E(\gamma) \subset \bigcup_{i=0}^{\infty} E_i(\gamma) = \bigcup_{i=0}^{\infty} \bigcup_{M=1}^{\infty} \left( \bigcap_{k=n_i}^{\infty} \bigcup_{n=k}^{\infty} E_i(\gamma, M, n) \right),
\]
it suffices to show that
\[
\sum_{n=n_i}^{\infty} \mathcal{H}^{d-1}(E_i(\gamma, M, n)) < \infty,
\]
which implies \( \mathcal{H}^{d-1}(\bigcap_{k=n_i}^{\infty} \bigcup_{n=k}^{\infty} E_i(\gamma, M, n)) = 0 \) by Borel-Cantelli Lemma. Throughout the remainder of the proof, we fix \( M \) and \( i \) and assume that \( n \geq n_i \).

Let \( \psi = \psi_{\xi_i} \) and \( CS = CS_{\xi_i} \) be the Lipschitz function and the orthonormal coordinate system in Definition 2.1. We will use this coordinate system \( CS \) below so that \( \xi_i = 0 \). For \( \xi := (\bar{\xi}, \psi(\bar{\xi})) \in B_{\partial D}(0, a_i) \) in \( CS \), define
\[
h_{n}(\bar{\xi}) := \int_{B_{\gamma}(\bar{\xi}, 2^{-n})} \int_{B_{\gamma}(\bar{\xi}, 2^{-n})} \phi(\delta_D(y)^{-2})^{1/2} |\tilde{f}_i(y) - \tilde{f}_i(z)| dydz.
\]
Then, by using the area formula (see, for example, [5, Section 3.3.4]),
\[
\mathcal{H}^{d-1}(E_i(\gamma, M, n)) = \int_{B_{\partial D}(0,a_i)} 1_{E_i(\gamma, M, n)}(\xi, \psi(\xi)) d\mathcal{H}^{d-1}(\xi, \psi(\xi)) \\
\leq M 2^{n(2d+\gamma)} \phi(2^{2n})^{-1/2} \int_{B_{\partial D}(0,a_i)} h_n(\tilde{\xi}) d\mathcal{H}^{d-1}(\tilde{\xi}, \psi(\tilde{\xi})) \\
\leq M 2^{n(2d+\gamma)} \phi(2^{2n})^{-1/2} \int_{|\tilde{\xi}|<a_i} h_n(\tilde{\xi})(1 + |\nabla \psi(\tilde{\xi})|^2)^{1/2} d\tilde{\xi} \\
\leq (1 + \Lambda^2_\text{Lip})^{1/2} M 2^{n(2d+\gamma)} \phi(2^{2n})^{-1/2} \int_{|\tilde{\xi}|<a_i} h_n(\tilde{\xi}) d\tilde{\xi}. \tag{3.5}
\]

When \( p \in (1, \infty) \), by Hölder’s inequality,
\[
\int_{|\tilde{\xi}|<a_i} h_n(\tilde{\xi}) d\tilde{\xi} \\
\leq \left( \int_{|\tilde{\xi}|<a_i} \int_{B(0,2^{-n})} \phi(\delta_D(\xi + y)^{-2})^{\frac{1}{q'}} \left| \tilde{f}_i(\xi + y) - \tilde{f}_i(\xi + z) \right|^p dy dz d\tilde{\xi} \right)^{1/p} \tag{3.6}
\]
where \( 1/q := 1 - 1/p \).

By Fubini’s theorem,
\[
I^q \leq c_1 2^{-nd} \int_{|y|<2^{-n}} \int_{|y_d|<2^{-n}} \int_{|\tilde{\xi}|<a_i} \phi(\delta_D(\xi + y)^{-2})^{q/2} d\tilde{\xi} dy_d dy, \tag{3.7}
\]
while using \(|a + b|^p \leq 2^{p-1}(|a|^p + |b|^p)\), the symmetry, and Fubini’s theorem,
\[
II^p = \int_{|\tilde{\xi}|<a_i} \int_{B(0,2^{-n})} \int_{B(0,2^{-n})} \left| \tilde{f}_i(\xi + y) - \tilde{f}_i(\xi + z) \right|^p dy dz d\tilde{\xi} \\
\leq 2^{p-1} \int_{|\tilde{\xi}|<a_i} \int_{B(0,2^{-n})} \int_{B(0,2^{-n})} \left| \tilde{f}_i(\xi + y) - \tilde{f}_i(\xi + y + z) \right|^p dy dz d\tilde{\xi} \\
+ 2^{p-1} \int_{|\tilde{\xi}|<a_i} \int_{B(0,2^{-n})} \int_{B(0,2^{-n})} \left| \tilde{f}_i(\xi + y + z) - \tilde{f}_i(\xi + z) \right|^p dy dz d\tilde{\xi} \\
\leq 2^p \int_{B(0,2^{-n})} \int_{|y|<2^{-n}} \int_{|y_d|<2^{-n}} \int_{|\tilde{\xi}|<a_i} \left| \tilde{f}_i(\xi + y + z) - \tilde{f}_i(\xi + y) \right|^p d\tilde{\xi} dy_d dy dz. \tag{3.8}
\]

Let \( w = (\bar{w}, w_d) := (\tilde{\xi} + \bar{y}, \psi(\tilde{\xi}) + y_d) = \xi + y \). If \(|\bar{y}| < 2^{-n}\) and \(|\psi(\bar{w} - \bar{y}) - w_d| = |y_d| < 2^{-n}\), then
\[
|w_d - \psi(\bar{w})| \leq |w_d - \psi(\bar{w} - \bar{y})| + |\psi(\bar{w} - \bar{y}) - \psi(\bar{w})| \leq 2^{-n} + \Lambda_\text{Lip}|\bar{y}| \leq (1 + \Lambda_\text{Lip})2^{-n}.
\]
Therefore, for $|y| < 2^{-n}$,
\[ \{ w \in \mathbb{R}^d : |\bar{w} - y| < a_i, |\psi(\bar{w} - y) - w_d| < 2^{-n} \} \subset \{ w \in \mathbb{R}^d : |\bar{w}| < 2a_i, |\psi(\bar{w}) - w_d| < (1 + \Lambda_{\text{Lip}})2^{-n} \} =: Q_n. \] (3.9)

Using this and Lemma 3.1, the inner two integrals in (3.7) are bounded as
\[ \int_{|y| < 2^{-n}} \int_{|\xi| < a_i} \phi(\delta_D(\xi + y)^{-2})^{q/2} d\xi dy \leq \int_{Q_n} \phi(\delta_D(w)^{-2})^{q/2} dw d\bar{w} \leq c_2 2^{-n} \phi(2^{2n})^{q/2}. \] (3.10)

Further, from (1.5), the inner two integrals in (3.8) are bounded as
\[ \int_{|y| < 2^{-n}} \int_{|\xi| < a_i} |\hat{f}_i(\xi + y + z) - \hat{f}_i(\xi + y)| d\xi dy \leq \int_{Q_n} |\hat{f}_i(w + z) - \hat{f}_i(w)| d\xi dy \leq c_3 2^{-n} p. \] (3.11)

Thus, (3.7), (3.10) imply $I \leq c_4 2^{-nd/2} \phi(2^{2n})^{1/2}$ and (3.8), (3.11) imply $II \leq c_5 2^{-nd/p} 2^{-n(\beta - 1/p)}$. From this and (3.6), we obtain
\[ \int_{|\xi| < a_i} h_n(\xi) d\xi \leq c_6 2^{-n(2d + \beta - 1/p)} \phi(2^{2n})^{1/2}. \] (3.12)

Now, we conclude from (3.5) and (3.12) that $\mathcal{H}^{d-1}(E_i(\gamma, M, n)) \leq c_7 2^{-n(\beta - 1/p - \gamma)}$, which implies (3.4) since $\beta - 1/p - \gamma > 0$.

When $p = \infty$, simply by (1.5) and Corollary 3.2
\[ h_n(\xi) \leq c_8 (2^{-n+1})^{\beta} \int_{B_{\mathbb{R}^d}(\xi, 2^{-n})} \int_{B_{\mathbb{R}^d}(\xi, 2^{-n})} \phi(\delta_D(y)^{-2})^{1/2} dy dz \leq c_9 2^{-n(2d + \beta)} \phi(2^{2n})^{1/2}. \]

Therefore, by (3.5), $\mathcal{H}^{d-1}(E_i(\gamma, M, n)) \leq c_{10} 2^{-n(\beta - \gamma)}$, which yields (3.4) since $\beta - \gamma > 0$.

\[ \square \]

**Lemma 3.4.** Suppose that $f \in L^p_{\beta, \text{loc}}(\overline{\mathcal{D}})$ and $0 < \gamma < \beta - 1/p$. Let
\[ F(\gamma) = \left\{ \xi \in \partial D : \lim_{r \to 0^+} \frac{r}{\mathcal{H}^{d-1}(B_{\mathbb{R}^d}(\xi, r))} \sqrt{B_{\mathbb{R}^d}(\xi, r)} |f(y) - f(z)| dy dz > 0 \right\}, \]
then $\mathcal{H}^{d-1}(F(\gamma)) = 0$.

**Proof.** The proof of this lemma is the same as that of Lemma 3.3. In fact, using the same $a_i$, $\hat{f}_i$, $n_i$, and coordinate system in the proof of Lemma 3.3, for $n \geq n_i$, we define
\[ F_i(\gamma, M, n) := \left\{ \xi \in \mathcal{B}_{\partial D(0, a_i)} : 2^{(2n + \gamma)} \int_{B_{\mathbb{R}^d}(\xi, 2^{-n})} \int_{B_{\mathbb{R}^d}(\xi, 2^{-n})} |\hat{f}_i(y) - \hat{f}_i(z)| dy dz > \frac{1}{M} \right\}. \]
When \( p \in (1, \infty) \), by Hölder’s inequality we have
\[
\mathcal{H}^{d-1}(F_1(\gamma, M, n)) \leq (1 + \Lambda^2_{\text{Lip}})^{1/2} M 2^{n(2d+\gamma)} \int_{|\xi|<a_i} \int_{B_{\mathcal{D}}(\xi,2^{-n})} \int_{B_{\mathcal{D}}(\xi,2^{-n})} |\tilde{f}_i(y) - \tilde{f}_i(z)| dydzd\xi
\]
\[
\leq (1 + \Lambda^2_{\text{Lip}})^{1/2} M 2^{n(2d+\gamma)} \left( \int_{|\xi|<a_i} \int_{B(0,2^{-n})} \int_{B(0,2^{-n})} dydzd\xi \right)^{1/q} \times \left( \int_{|\xi|<a_i} \int_{B(0,2^{-n})} \int_{B(0,2^{-n})} |\tilde{f}_i(\xi + y) - \tilde{f}_i(\xi + z)|^p dydzd\xi \right)^{1/p}.
\]
By following the proof of Lemma 3.3 line by line, we see that
\[
\left( \int_{|\xi|<a_i} \int_{B(0,2^{-n})} \int_{B(0,2^{-n})} dydzd\xi \right)^{1/q} = c_1 \left( a_i d-1 2^{-2d} \right)^{1/q},
\]
and
\[
\left( \int_{|\xi|<a_i} \int_{B(0,2^{-n})} \int_{B(0,2^{-n})} |\tilde{f}_i(\xi + y) - \tilde{f}_i(\xi + z)|^p dydzd\xi \right)^{1/p} \leq c_2 (2^{-nd} 2^{-n(d-1)2^{-n\beta}})^{1/p}.
\]
Therefore,
\[
\mathcal{H}^{d-1}(F_1(\gamma, M, n)) \leq c_3 2^{n(2d+\gamma)} 2^{-nd} 2^{n(d-1)2^{-n\beta}} = c_3 2^{-n(\beta-1/p-\gamma)}.
\]
The assertion for \( p \in (1, \infty) \) follows from this.

The proof for \( p = \infty \) is also similar. Therefore, we skip the proof.

For a locally integrable function \( h \) on \( \mathbb{R}^d \) and bounded measurable set \( U \subset \mathbb{R}^d \), we define its integral mean over the region \( U \) by \( f_U h(y) dy = \frac{1}{|U|} \int_U h(y) dy \).

The proof of the following lemma is taken from [12]. However, for the reader’s convenience, we state the details of the proof.

**Lemma 3.5.** Let \( f \in \Lambda^p_{\beta,\text{loc}}(\mathcal{D}) \), and \( \delta \) in (A-3) satisfies \( \delta > 1/p \). Then,
\[
A(\xi) := \lim_{r \to 0^+} \frac{1}{r^d} \int_{B_{\mathcal{D}}(\xi,r)} f(y) dy
\]
exists and is finite for \( \mathcal{H}^{d-1} \text{-a.e. } \xi \in \partial D \). Moreover, for \( 0 < \gamma < \beta - 1/p \),
\[
\lim_{r \to 0^+} r^{-d-\gamma} \phi(r^{-2})^{-1/2} \int_{B_{\mathcal{D}}(\xi,r)} \phi(\delta_D(y)^{-2})^{1/2} |f(y) - A(\xi)| dy = 0 \tag{3.13}
\]
for \( \mathcal{H}^{d-1} \text{-a.e. } \xi \in \partial D \).

**Proof.** For simplicity, define \( A(\xi, r) = \int_{B_{\mathcal{D}}(\xi,r)} f(y) dy \). Then, for \( r \leq t \leq 2r \),
\[
|A(\xi, t) - A(\xi, r)| \leq c_1 r^{-2d} \int_{B_{\mathcal{D}}(\xi,2r)} \int_{B_{\mathcal{D}}(\xi,2r)} |f(y) - f(z)| dydz.
\]
Lemma \[3.4\] gives \(\lim_{r \to 0^+} r^{-\gamma} |A(x, 2r) - A(x, r)| = 0\) for \(\mathcal{H}^{d-1}\)-a.e. \(x \in \partial D\). This implies that
\[
A_\infty(x) := \lim_{n \to \infty} A(x, 2^{-n})
\]
exists and \(\lim_{k \to \infty} 2^{k\gamma} \left\{ A(x, 2^{-k+1}) - A_\infty(x) \right\} = 0\). Therefore, for \(r \leq 2^{-k+1} \leq 2r\),
\[
r^{-\gamma} |A_\infty(x) - A(x, r)| \leq r^{-\gamma} |A_\infty(x) - A(x, 2^{-k+1})| + r^{-\gamma} |A_\infty(x) - A(x, r)|
\]
\[
\leq 2^{k\gamma} |A_\infty(x) - A(x, 2^{-k+1})| + r^{-\gamma} |A_\infty(x) - A(x, r)| \to 0 \quad \text{as} \quad r \to 0.
\]
Thus, \(A(x)\) exists and is finite \(\mathcal{H}^{d-1}\)-a.e. \(x \in \partial D\). Further, for \(\mathcal{H}^{d-1}\)-a.e. \(x \in \partial D\),
\[
\lim_{r \to 0^+} r^{-\gamma} |A(x) - A(x, r)| = 0.
\]
(3.14)
By Corollary \[3.2\]
\[
r^{-d-\gamma} \phi(r^{-2})^{-1/2} \int_{B(x,r)} \phi(\delta_D(y)^{-2})^{1/2} |f(y) - A(x)| dy
\]
\[
\leq r^{-d-\gamma} \phi(r^{-2})^{-1/2} \int_{B(x,r)} \phi(\delta_D(y)^{-2})^{1/2} |f(y) - A(x, r)| + c_2 r^{-\gamma} |A(x) - A(x, r)|
\]
\[
\leq c_3 r^{-2d-\gamma} \phi(r^{-2})^{-1/2} \int_{B(x,r)} \phi(\delta_D(z)^{-2})^{1/2} |f(y) - f(z)| dy + c_2 r^{-\gamma} |A(x) - A(x, r)|,
\]
which tends to zero as \(r \to 0\) for \(\mathcal{H}^{d-1}\)-a.e. \(x \in \partial D\) by (3.14) and Lemma \[3.3\]. Hence, we have proved (3.13).

4 Proof of Theorem \[1.4\]

For any \(C^{1,1}\) open set \(D\) with characteristic \((R, \Lambda)\), it is well-known that (see, e.g., [16] Lemma 2.2) there exists \(L = L(R, \Lambda, d) > 0\) such that for every \(x \in \partial D\) and \(r \leq (R \wedge 1)\), one can obtain a \(C^{1,1}\) open set \(U(x, r)\) with characteristic \((r(R \wedge 1)/L, \Lambda L/r)\) such that
\[
D \cap B(x, r/2) \subset U(x, r) \subset D \cap B(x, r).
\]
(4.1)

First, we record a lemma, which is a consequence of the main results of [9]. Although simple, it is important in this paper.

**Lemma 4.1.** Let \(D\) be a \(C^{1,1}\) open set with characteristic \((R, \Lambda)\). Suppose that \(\xi_0 \in \partial D\), \(r_0 > 0\), and \(u : \mathbb{R}^d \to \mathbb{R}\) is a non-negative regular harmonic function in \(D \cap B(\xi_0, r_0)\) with respect to \(X\) vanishing on \(D \cap B(\xi_0, r_0)\). Then, \(u\) vanishes continuously on \((\partial D) \cap B(\xi_0, r_0/2)\).

**Proof.** Fix \(x \in (\partial D) \cap B(\xi_0, r_0/2)\) and let \(0 < r < 1 \wedge r_0/2\). We will show that \(u\) vanishes continuously on \((\partial D) \cap B(\xi_0, r/8)\), which clearly implies the lemma.

Choose a bounded \(C^{1,1}\) open set \(U = U(\xi, r)\) as in (4.1). Let \(z_0\) be a point in \(U \setminus B(\xi, r/4)\). Then, by [9] Proposition 2.4 and Theorem 7.1], the \(G_U(\cdot, z_0)\) vanishes continuously on \(\partial U \supset (\partial D) \cap B(\xi, r/4)\). Moreover, \(x \mapsto G_U(x, z_0)\) is a regular harmonic function in \(D \cap B(\xi, r/4)\) with respect to \(X\). Thus, by the boundary Harnack principle [9] Theorem 5.6(i)], for a fixed \(x_0 \in B(\xi, r/8) \cap D\) and \(x \in D \cap B(\xi, r/8)\),
\[
u(x) \leq c u(x_0) G_U(x_0, z_0)^{-1} G_U(x, z_0) \to 0 \quad \text{as} \quad x \to (\partial D) \cap B(\xi, r/8).
\]

This completes the proof. □

Before we prove the main result, we observe an inequality. Recall that $g(r)$ defined in (2.1) is decreasing. Using this fact and the estimates in (2.2), we obtain that there exists $c > 0$ such that

$$\frac{\phi'(t^{-2})}{\phi(t^{-2})t^{d+2}} \leq c \frac{\phi'(s^{-2})}{\phi(s^{-2})s^{d+2}}, \quad s \leq t. \quad (4.2)$$

Now, we prove our theorem.

**Proof of Theorem 1.4.** Without loss of generality, we assume that $R < 1$. By the cardinality, we can choose $\xi_i \in \partial D$ and $\eta_i = \eta_i(\xi_i) > 0$ such that there exists $f_i \in \Lambda^p_\beta(\mathbb{R}^d)$ satisfying $f = f_i$ on $\overline{D} \cap B(\xi_i, \eta_i)$, and that, $\partial D \subset \cup_{i \in \mathbb{N}} B(\xi_i, \eta_i/8)$. Without loss of generality, we let $\eta_i \leq R$. Since $\partial D$ is a countable union of $B_{\partial D}(\xi_i, \eta_i/8)$, it suffices to show that for $\mathcal{H}^{d-1}$-a.e. $\xi \in P_1 := B_{\partial D}(\xi_1, \eta_1/8)$, $u_f$ has a limit along $T_{\gamma, \phi}(\xi)$.

Choose a $C^{1,1}$ open set $U = U(\xi_1, R)$ with characteristic $(R^2/L, \Lambda L/R)$ as (4.1) so that $P_1 = B_{\partial U}(\xi_1, \eta_1/8)$. Note that

$$u_f(x) = \mathbb{E}_x[f(X_{\tau_U})] = \mathbb{E}_x[f(X_{\tau_D})]; \tau_U < \tau_D + \mathbb{E}_x[f(X_{\tau_D})]; \tau_U = \tau_D$$

$$= \mathbb{E}_x[f(X_{\tau_D})]; \tau_U < \tau_D + \mathbb{E}_x[f(X_{\tau_D})] - \mathbb{E}_x[f(X_{\tau_D})]; \tau_U < \tau_D.$$

By using the strong Markov property at $\tau_U$, $\mathbb{E}_x[f(X_{\tau_D})]; \tau_U < \tau_D$ and $\mathbb{E}_x[f(X_{\tau_D})]; \tau_U < \tau_D$ are non-negative regular harmonic functions in $U$ (and thus, in $B(\xi_1, R/2) \cap D$) with respect to $X$ and vanish on $D^c \cap B(\xi_1, R/2)$. Thus, by Lemma 4.1, the limits of $u_f(x)$ and $\mathbb{E}_x[f(X_{\tau_D})]$ (if exist) are the same when $x$ goes to a point $\xi \in P_1$. Therefore, it suffices to show that the limit $\lim_{T_{\gamma, \phi}(\xi) \to x \to \xi} \mathbb{E}_x[f(X_{\tau_U})]$ exists for $\mathcal{H}^{d-1}$-a.e $\xi \in P_1$.

By Lemma 4.5 for $\mathcal{H}^{d-1}$-a.e. $\xi \in P_1$, we have that $A(\xi) = \lim_{r \to 0+} \int_{B_{\mathcal{H}^{d-1}}(\xi, r)} f(y)dy$ exists and is finite and that

$$\lim_{r \to 0+} \int_{B_{\mathcal{H}^{d-1}}(\xi, r)} \phi(\delta_U(y)^{-2})^{1/2}|f(y) - A(\xi)|dy = 0 \quad (4.3)$$

holds. For the remainder of the proof, we fix a $\xi \in P_1$ and show that

$$\lim_{T_{\gamma, \phi}(\xi) \to x \to \xi} \mathbb{E}_x[f(X_{\tau_U})] - A(\xi) = 0. \quad (4.4)$$

Let $\epsilon > 0$ be given. By (4.3), there exists $r_0 < (1 \wedge (R/4))/2$ such that for every $0 < r < 2r_0$,

$$\int_{B_{\mathcal{H}^{d-1}}(\xi, r)} \phi(\delta_U(y)^{-2})^{1/2}|f(y) - A(\xi)|dy < \epsilon r^{\gamma + d} \phi(r^{-2})^{1/2}. \quad (4.5)$$

Note that $B(\xi, 2r_0) \subset B(\xi_1, R/2)$. Let

$$u_1(x) = \mathbb{E}_x[f(X_{\tau_U})]; X_{\tau_U} \in \mathbb{R}^d \setminus \{U \cup B(\xi, r_0)\}, \quad u_2(x) = \mathbb{P}_x(X_{\tau_U} \in \mathbb{R}^d \setminus \{U \cup B(\xi, r_0)\}).$$

Then, $u_1, u_2$ are non-negative regular harmonic functions in $U \cap B(\xi, r_0)$ with respect to $X$, and they vanish on $U^c \cap B(\xi, r_0)$.
On the other hand, since $U$ is a bounded $C^{1,1}$ open set, we have the following Poisson kernel estimates by (2.5) and (2.3):

$$K_U(x, y) \leq c_1 \frac{\phi(\delta_U(y)^{-2})^{1/2}}{\phi(\delta_U(x)^{-2})^{1/2} \phi(|x - y|^{-2})} \frac{\phi'(|x - y|^{-2})}{|x - y|^{d+2}}, \quad \text{for } x \in U, \ y \in B(\xi, r_0) \setminus U. \quad (4.6)$$

Thus, by (2.4) and (4.6), we have

$$|\mathbb{E}_x[f(X_{\tau_U})] - A(\xi)| \leq c_1 \int_{B_{\eta r}(\xi, r_0)} \frac{\phi(\delta_U(y)^{-2})^{1/2}}{\phi(\delta_U(x)^{-2})^{1/2} \phi(|x - y|^{-2})} \frac{\phi'(|x - y|^{-2})}{|x - y|^{d+2}} |f(y) - A(\xi)| dy + u_1(x) + |A(\xi)| u_2(x). \quad (4.7)$$

Since $|x - y| \leq \delta_U(x)$ for $y \in \mathbb{R}^d \setminus \bar{U}$, by (4.2) and (4.5), for $x \in B(\xi, r_0/8) \cap T_{\gamma, \phi}(\xi)$,

$$|\mathbb{E}_x[f(X_{\tau_U})] - A(\xi)| \leq c_2 \frac{\phi'(\delta_U(x)^{-2})^{1/2}}{\phi'(\delta_U(x)^{-2})^{1/2}} \frac{\phi'(\delta_U(y)^{-2})}{\phi'(\delta_U(x)^{-2})} \frac{|f(y) - A(\xi)| dy}{|x - y|^{d+2}} \leq c_2 \frac{\phi'(\delta_U(x)^{-2})^{1/2} |f(y) - A(\xi)| dy}{\delta_U(x)^{d+2} \phi'(\delta_U(x)^{-2})^{3/2}} \leq c_2 \frac{2^{d+\gamma} \phi'(\delta_U(x)^{-2})^{1/2}}{\delta_U(x)^{d+2}} \epsilon \leq c_2 \frac{2^{d+\gamma} \epsilon}{\delta_U(x)^{d+2}}. \quad (4.8)$$

When $2|x - \xi| \leq |\xi - y|$, we have $|x - y| \geq |\xi - y| - |\xi - x| \geq |\xi - y|/2$. Thus, by (4.2) and (1.3), on $\{y \in \mathbb{R}^d \setminus \bar{U} : 2|x - \xi| \leq |\xi - y| < r_0\}$,

$$\frac{\phi'(|x - y|^{-2})}{|x - y|^{d+2} \phi(|x - y|^{-2})} \leq c_2 \frac{2^{d+2} \phi'(|\xi - y|^{-2})}{|\xi - y|^{d+2} \phi(|\xi - y|^{-2})} \leq c_2 \frac{2^d}{|\xi - y|^{d+2}} \epsilon \leq c_2 \frac{2^{d+1} \epsilon}{|\xi - y|^{d+1}}. \quad (4.9)$$

Therefore, by the Fubini's theorem and (4.5), we have that for $x \in B(\xi, r_0/8)$,

$$\int_{\{y \in \mathbb{R}^d \setminus \bar{U} : 2|x - \xi| \leq |\xi - y| < r_0\}} \frac{\phi(\delta_U(y)^{-2})^{1/2}}{\phi(\delta_U(x)^{-2})^{1/2} \phi(|x - y|^{-2})} \frac{\phi'(|x - y|^{-2})}{|x - y|^{d+2}} |f(y) - A(\xi)| dy \leq c_2 \frac{2^{d+2} \epsilon}{|\xi - y|^{d+1}}. \quad (4.9)$$

Applying (4.8) and (4.9) to (4.7), together with Lemma 4.1 gives

$$\limsup_{T_{\gamma, \phi}(\xi) \ni x \to \xi} |\mathbb{E}_x[f(X_{\tau_U})] - A(\xi)| \leq c_3 \epsilon,$$
where the constant $c_3 > 0$ is independent of $\epsilon$. Since $\epsilon > 0$ is arbitrary, we have proved the claim (4.4).

\begin{flushright}
\Box
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**Acknowledgements.** We thank the referee for his (her) helpful comments on the first version of this paper.

**References**

[1] R. F. Bass and D. You, A Fatou theorem for $\alpha$-harmonic functions. *Bull. Sci. Math.* **127** (2003), 635–648.

[2] R. F. Bass and D. You, A Fatou theorem for $\alpha$-harmonic functions in Lipschitz domains. *Probab. Theory Relat. Fields* **133**(3) (2005), 391–408.

[3] Z.-Q. Chen, On notions of harmonicity. *Proc. Amer. Math. Soc.* **137**(10) (2009), 3497–3510.

[4] J. R. Dorronsoro, Poisson integrals of regular functions. *Trans. Amer. Math. Soc.* **297**(2) (1986), 669–685.

[5] L. C. Evans and R. F. Gariepy, *Measure Theory and Fine Properties of Functions*. CRC press, 1992.

[6] N. Ikeda and S. Watanabe, On some relations between the harmonic measure and the Lévy measure for a certain class of Markov processes. *J. Math. Kyoto Univ.* **2** (1962), 79–95.

[7] J. Kang and P. Kim, On estimates of Poisson kernels for symmetric Lévy processes. *J. Korean Math. Soc.* **50**(5) (2013), 1009–1031.

[8] P. Kim and A. Mimica, Harnack inequalities for subordinate Brownian motions. *Electron. J. Probab.* **17**(37) (2012), 1–23.

[9] P. Kim and A. Mimica, Green function estimates for subordinate Brownian motions: stable and beyond. *Trans. Amer. Math. Soc.* **366**(8) (2014), 4383–4422.

[10] J. E. Littlewood, On a theorem of Fatou. *Proc. London Math. Soc.* **2** (1927), 172–176.

[11] P. W. Millar, First passage distributions of processes with independent increments. *Ann. Probability* **3** (1975), 215–233.

[12] Y. Mizuta, Existence of tangential limits for $\alpha$-harmonic functions on half spaces. *Potential Anal.* **25** (2006), 29–36.

[13] A. Nagel, W. Rudin, and J. Shapiro, Tangential boundary behavior of functions in Dirichlet-type spaces. *Ann. Math.* **116** (1982), 331–360.

[14] A. Nagel and E. M. Stein, On certain maximal functions and approach regions. *Adv. in Math.* **54**(1) (1984), 83–106.

[15] R. Schilling, R. Song, and Z. Vondraček, *Bernstein functions: Theory and applications*. de Gruyter Studies in Mathematics 37. Berlin: Walter de Gruyter, 2010.

[16] R. Song, Estimates on the Dirichlet heat kernel of domains above the graphs of bounded $C^{1,1}$ functions. *Glas. Mat. Ser. III* **39**(59) (2004), 275–288.

[17] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*. Princeton Univ. Press, Princeton, 1970.

[18] J. Sueiro, Tangential boundary limits and exceptional sets for holomorphic functions in Dirichlet-type spaces. *Math. Ann.* **286** (1990), 661–678.
[19] P. Sztonyk, On harmonic measure for Lévy processes. *Probab. Math. Statist.* 20 (2000), 383–390.

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