Critical, crossover, and correction-to-scaling exponents for isotropic Lifshitz points to order \((8 - d)^2\)

H. W. Diehl† and M. Shpot‡

† Fachbereich Physik, Universität Essen, D-45117 Essen, Federal Republic of Germany
‡ Institute for Condensed Matter Physics, 79011 Lviv, Ukraine

Abstract. A two-loop renormalization group analysis of the critical behaviour at an isotropic Lifshitz point is presented. Using dimensional regularization and minimal subtraction of poles, we obtain the expansions of the critical exponents \(\nu\) and \(\eta\), the crossover exponent \(\phi\), as well as the (related) wave-vector exponent \(\beta_q\), and the correction-to-scaling exponent \(\omega\) to second order in \(\epsilon_8 = 8 - d\). These are compared with the authors’ recent \(\epsilon\)-expansion results \([\text{Phys. Rev. B} \ 62 (2000) 12338; \text{Nucl. Phys. B} \ 612 (2001) 340]\) for the general case of an \(m\)-axial Lifshitz point. It is shown that the expansions obtained here by a direct calculation for the isotropic \((m = d)\) Lifshitz point all follow from the latter upon setting \(m = 8 - \epsilon_8\). This is so despite recent claims to the contrary by de Albuquerque and Leite \([\text{J. Phys. A} \ 35 (2002) 1807]\).

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1. Introduction

The concept of a Lifshitz point, introduced in 1975 \([1]\), has attracted considerable attention during the past 25 years\(^\S\). Recently there has been renewed interest in the critical behaviour at such points. In particular, field-theory approaches have been utilized to determine the dimensionality expansions of various critical indices needed to characterize the critical behaviour at \(m\)-axial Lifshitz points \([2, 3, 4, 5, 6, 7, 8]\).

We assume that the order-parameter symmetry which is spontaneously broken on the low-temperature side of the critical line on which the Lifshitz point is located is \(O(n)\), and that the wave-vector instability which sets in at this point is isotropic in the \(m\)-dimensional subspace of \(\mathbb{R}^d\). A familiar model representing the universality class of \(d\)-dimensional systems with short-range interactions and an \(n\)-component order-parameter field \(\phi(x)\) at such an \((m, d, n)\)-Lifshitz point is described by the Hamiltonian

\[
\mathcal{H} = \int d^d x \left\{ \frac{\sigma_0}{2} (\Delta_\parallel \phi)^2 + \frac{1}{2} (\nabla_\perp \phi)^2 + \frac{\rho_0}{2} (\nabla_\parallel \phi)^2 + \frac{\tau_0}{2} \phi^2 + \frac{u_0}{4!} |\phi|^4 \right\}.
\]

(1)

Here the position vector \(x = (x_\parallel, x_\perp)\) has an \(m\)-dimensional parallel component \(x_\parallel\) and a \((d-m)\)-dimensional perpendicular one \(x_\perp\). In the Landau approximation, the Lifshitz point is located at \(\tau_0 = \rho_0 = 0\); this approximation holds arbitrarily close to

\(^\S\) For reviews and extensive lists of references, see \([2, 3, 4, 5]\). The papers \([3, 4]\) contain lists of the most recent references on the subject.
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it for values of $d$ exceeding the upper critical dimension, which is $d^*(m) = 4 + m/2$ for $m \leq 8$.

Interest in the model (1) started already in 1975. Unfortunately, early analyses based on Wilson’s momentum shell recursion relations were rather incomplete and produced a long-standing controversy: The series expansions to second order in $\epsilon = d^*(m) - d$ one group of authors derived for the correlation exponents $\eta_2$ and $\eta_4$ for general values of $m$ (or for $m = 1$) disagreed with those obtained by another one for the special values $m = 2$ and $m = 6$. This state was certainly very unsatisfactory because the class of models defined via the Hamiltonian (1) is not only interesting in its own right, but provides a natural, simple generalization of the standard class of $n$-component $|\phi|^4$-models (to which it reduces when $m = 0$). Moreover, besides representing universality classes, these models are of a prototypical nature in that they exhibit anisotropic scale invariance, offering excellent possibilities for investigating the question whether and under what conditions scale invariance might give rise to further invariances analogous to Schrödinger invariance or conformal symmetries. Clearly, their thorough theoretical understanding is highly desirable.

In two recent papers, we have been able to perform a field-theoretic two-loop renormalization group (RG) analysis of the model (1) near its upper critical dimension $d^*(m)$ for general values of $m$ with $0 < m < 8$. The results yield the $\epsilon$ expansions to second order of all critical, crossover, wave-vector, and correction-to-scaling exponents about any point $(m, d^*(m))$ on the critical line $d^*(m)$. The expansion coefficients involve four well-defined single integrals $j_\sigma(m)$, $j_\rho(m)$, $J_\mu(m)$, and $J_\nu(m)$, which we have been able to evaluate analytically for the special values $m = 0, 2, 6, 8$, and for other values of $m$ by numerical integration. An appealing feature of these results is that they include both isotropic cases, namely that of the usual critical point ($m = 0$) and that of the isotropic Lifshitz point ($m = d = 8 - \epsilon_8$).

The requirement that our general $m$-dependent expressions reduce to the correct results for these simpler cases in which only the perpendicular or parallel parts of space remain provides crucial checks. These issues were briefly discussed already in our previous work, where also a direct two-loop RG calculation for the isotropic case $m = d$ was announced but not described. Our aim here is to fill this gap by presenting details of such an analysis and a thorough discussion of the consistency of its results with our previous findings for general $m$. We emphasize that this consistency is not a priori obvious because both $m$ and $d$ must be expanded about 8 while maintaining the equality $m = d$. As we shall explicitly verify, this consistency holds equally well on the levels of the final $\epsilon$-expansion results, the counter-terms and the Feynman integrals. This demonstrates that the recent claim by de Albuquerque and Leite, who asserted that the critical exponents of the isotropic Lifshitz point cannot be obtained from those for general $m$, is unfounded.

Such a calculation is analogous to the one for the standard $|\phi|^4$ theory ($m = 0$), which can be found in many textbooks like. We had originally felt that the publication of this calculation was unnecessary. The fact that reference got published and the reaction of many colleagues, who suggested that we write a clarifying paper, made us change our minds.
2. Two-loop results for the isotropic case \( m = d \)

In the isotropic case \( m = d \), only the parallel part of space remains, so that \( x_{\parallel} = x \).

It is convenient to use the rescaled variables

\[
\phi(x) = \sigma_0^{-d/8} \psi(y), \quad y = \sigma_0^{-1/4} x.
\]

Then the Hamiltonian implied by equation (1) for \( m = d \) becomes

\[
H^{(\text{iso})} = \int d^d y \left[ \frac{1}{2} (\nabla \psi)^2 + \frac{\lambda_0}{2} (\nabla \psi)^2 + \frac{\tau_0}{2} \psi^2 + \frac{g_0}{4!} |\psi|^4 \right]
\]

with

\[
g_0 = \sigma_0^{-d/4} u_0, \quad \lambda_0 = \sigma_0^{-1/2} \rho_0.
\]

We use dimensional regularization to regularize the ultraviolet (uv) singularities of the \( \text{N-point cumulants} \ G^{(\text{N})} = \langle \psi \ldots \psi \rangle^{\text{cum}} \) and vertex functions \( \Gamma^{(\text{N})} \) of the theory in \( d = 8 - \epsilon \) dimensions. These uv singularities can be absorbed via the reparametrizations

\[
\psi = Z_\psi^{1/2} \psi_{\text{ren}}, \quad g_0 F_d = \kappa^{\epsilon_8} Z_g g,
\]

and

\[
\tau_0 - \tau_{\text{LP}} = \kappa^d Z_t t, \quad \lambda_0 - \lambda_{\text{LP}} = \kappa^2 Z_\lambda \lambda,
\]

where \( \tau_{\text{LP}} \) and \( \lambda_{\text{LP}} \), the values of \( \tau_0 \) and \( \lambda_0 \) at the Lifshitz point, vanish in our perturbative approach based on dimensional regularization. For the normalization constant in equation (5) we use the choice

\[
F_d = \frac{2^{1-d} \pi^{-d/2} \Gamma(5 - d/2) \Gamma(d/2 - 2)^2}{\Gamma(d - 4)}.
\]

In order to determine the renormalization factors \( Z_\psi, Z_g, Z_t, \) and \( Z_\lambda \), we compute the Laurent expansions of the vertex functions \( \Gamma^{(2)}, \Gamma^{(4)}, \Gamma^{(2)}_{\psi^2}, \) and \( \Gamma^{(2)}_{(\nabla \psi)^2} \) at the Lifshitz point \( t = \lambda = 0 \), where the latter two involve insertions of (one half of) the indicated operators \( \psi^2 \) and \( (\nabla \psi)^2 \), respectively. The free propagator with which we work is given by

\[
G(y) \equiv \int_q \tilde{G}(q) e^{iq \cdot y} = \frac{\pi^{-d/2}}{16} \Gamma[(d - 4)/2] y^{4-d}, \quad \tilde{G}(q) = \frac{1}{q^1},
\]

in position and momentum space, respectively. The values of the one-loop integral

\[
I_2(q) \equiv \int d^d y G(y)^2 e^{i q \cdot y}
\]

of \( \tilde{\Gamma}^{(4)}(q_1, \ldots, q_4) \) (where \( q = |q_1 + q_2| \), for example) and the two-loop integral

\[
I_3(q) \equiv \int d^d y G(y)^3 e^{i q \cdot y}
\]

of \( \tilde{\Gamma}^{(2)}(q) \) can be read off from the Fourier transform of the generalized function \( |y|^\vartheta \), which is

\[
\int d^d y y^\vartheta e^{-i q \cdot y} = 2^{\vartheta + d} \pi^{d/2} \frac{\Gamma(\vartheta + d)}{\Gamma(-d/2)} q^{-\vartheta - d}
\]
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according to equation (2) of reference [16]. The results are listed in Appendix A along with the other required two-loop integrals. We choose the renormalization factors such that the uv poles are minimally subtracted, obtaining

\[ Z_\psi = 1 + \frac{n+2}{3} \frac{g^2}{160 \epsilon_8} + O(g^3), \]  

\[ Z_\psi Z_\lambda = 1 + \frac{n+2}{3} \frac{g^2}{32 \epsilon_8} + O(g^3). \]  

The nontrivial root of \( \beta_g(g) \) is

\[ g^* = \frac{6 \epsilon_8}{n + 8} - \frac{41 n + 202}{5 (n + 8)^3} \epsilon_8^2 + O(\epsilon_8^3). \]

By evaluating the exponent functions at the fixed point \( g^* \), we arrive at the dimensionality expansions of the exponents

\[ \eta = \eta_\psi(g^*) = -\frac{3 (n + 2) \epsilon_8^2}{20 (n + 8)^2} + O(\epsilon_8^3), \]

\[ \nu = \frac{1}{4 + \eta_\tau(g^*)} = \frac{1}{4} + \frac{n + 2}{n + 8} \frac{\epsilon_8}{16} + \frac{(n + 2) (15 n^2 + 89 n + 4) \epsilon_8^2}{960 (n + 8)^3} + O(\epsilon_8^3). \]

\[ \varphi = \frac{2 + \eta_\lambda(g^*)}{4 + \eta_\tau(g^*)} = \frac{1}{2} + \frac{n + 2}{n + 8} \frac{\epsilon_8}{8} + \frac{(n + 2) (15 n^2 - 163 n - 2012) \epsilon_8^2}{480 (n + 8)^3} + O(\epsilon_8^3). \]
and
\[
\beta_q = \frac{1}{2} + \frac{21 (n + 2) \epsilon^2}{40 (n + 8)^2} + O(\epsilon^3) .
\] (26)
Likewise we find for the correction-to-scaling exponent:
\[
\omega = \frac{\partial \beta_q(q)}{\partial q} \bigg|_{q=q^*} = \epsilon_8 + \frac{(41 n + 202) \epsilon^2}{30 (n + 8)^2} + O(\epsilon^3) .
\] (27)

The results (23) and (24) were already obtained in reference [4] but the remaining ones (25)–(27) not.

3. Comparison with the results for the \(m\)-axial Lifshitz point

3.1. Critical, crossover, and correction-to-scaling exponents

In reference [3], which hereafter is referred to as I, we considered the critical exponents as functions \(f(m, d)\) of \(m\) and \(d\), and determined their expansions in \(d\) about \(d^*(m)\) at fixed \(m\) to second order in \(\epsilon = d^*(m) - d\). For \(d = m = 8 - \epsilon_8\), we have
\[
\epsilon = \epsilon_8/2 .
\] (28)
The series expansion coefficients of the terms of zeroth and first order in \(\epsilon\), found in I, are independent of \(m\). To check the consistency of the above results with those of references [4, 5], we therefore must merely substitute the limiting values of the latters' O(\(\epsilon^2\)) terms for \(m \to 8^-\), and use the relation (28).

The exponents \(\eta, \nu, \omega\) given in equations (23), (24), and (27) read in the notation of I
\[
\eta = \eta_{44}(d, d) , \quad \nu = \nu_{44}(d, d) , \quad \omega = \omega_{44}(d, d) ,
\] (29)
respectively. The \(\epsilon\) expansions of \(\eta_{44}\) and \(\nu_{44}\) are given in equations (I.62) and (I.64), and the one for \(\omega_{44}\) follows from equations (I.66)–(I.68), where (I.xx) denotes the equation (xx) of I. Note that the contribution \(\alpha \lim_{m \to 8^-} j_\phi(m)/(8 - m) = -j'_\phi(8^-)\) which is present in the \(\epsilon\) expansion (I.66) of the anisotropy exponent \(\theta\), does not contribute to the ratio \(\omega_{44} = \omega_{42}/\theta\) at O(\(\epsilon^2\)). The only term involving \(j_\phi(m)\) at O(\(\epsilon^2\)) is directly proportional to \(j_\phi(m)\). Since the latter integral is of order \(\epsilon_8\), all contributions to the series expansion of the correction-to-scaling exponent \(\omega\) at O(\(\epsilon^2\)) that originate from \(j_\phi(m)\) vanish, as they should.

Upon substituting the values (I.86) of the integrals \(j_\phi(8^-), \ldots, J_u(8^-)\) into the O(\(\epsilon^2\)) coefficients, the resulting series expansions to order \(\epsilon_8^2\) of the exponents (29) reduce to the above results (23), (24), and (27). Likewise, the \(\epsilon_8\) expansions of \(\varphi = (1 + \eta^*_u)/(2 + \eta^*_u)\) and \(\beta_q = \nu_{44}/\varphi\) that are implied by equations (I.63) — (I.65) for the fixed-point quantities \(\eta^*_u, \eta^*_a, \) and \(\eta^*_p\) of I agree with equations (23) and (24). The remaining critical exponents, such as the specific-heat exponent \(\alpha\), the order-parameter exponent \(\beta\), and the susceptibility exponent \(\gamma\), are related to \(\eta\) and \(\nu\) via known scaling and hyperscaling relations. Thus, there is no need to explicitly verify their consistency with the results of I.

\footnote{When referring to the results of the work [3] at the bottom of p. 355 of our previous paper [4], we erroneously said that these authors had obtained the expansions of the correlation exponents \(\nu_2\) and \(\eta_{44}\) and the correlation-length exponents \(\nu_{42}\) and \(\nu_{44}\) to second order in \(\epsilon_8\). The alert reader will have noticed that the cited computation was concerned merely with the isotropic Lifshitz point, so that only \(\eta_{44} = \eta\) and \(\nu_{44} = \nu\) should have been mentioned.}
For the case of an $m$-axial Lifshitz point a variety of other physically meaningful critical exponents can be defined, such as the exponent $\eta_2$ (which governs the momentum dependence of the inverse correlation function $\Gamma^{(2)}(\mathbf{q}_\parallel=\mathbf{0}, \mathbf{q}_\perp) \sim q_{\perp}^{2-\eta_2}$ at the Lifshitz point), the parallel correlation-length exponent $\nu_2$, the related anisotropy exponent $\theta = \nu_4/\nu_2$, and the correction-to-scaling exponent $\omega_2 = \theta \omega_4$. The $\epsilon$ expansions to order $\epsilon^2$ of all these exponents have been given in I. These latter exponents are not needed in the case of the isotropic Lifshitz point. The situation is complementary to the $m = 0$ case of the standard $|\phi|^4$ theory, where exponents requiring the parallel part of space (such as $\eta_4$ and $\nu_4$) lose their physical significance. Remarks analogous to those made in I apply here: The limits $d-m \to 0$ and $m \to 0$ of those exponents that are not required in the isotropic cases $m = d$ and $m = 0$ may well exist, and it is conceivable that their limiting values will turn out to have significance for certain problems of statistical physics. We will not pursue this question further here. Note, however, that the limiting value of $\eta_2$ for $m \to d$ appears to exist and to be finite since its $\epsilon^2$ term involves the integral $j_\phi(m)$ in the combination $j_\phi(m)/(8-m)$, according to equation (I.61). Since $j_\phi(8-) = 0$, the limit $m \to 8-$ of this combination is nonsingular and nonzero provided that $j'_\phi(8-) \neq 0$ (as we expect). For the same reason, the contribution $\propto j_\phi(m)/(8-m)$ to $\nu_2$ in equation (I.63) remains finite in this limit, as do the other contributions to the $O(\epsilon^2)$ term.

### 3.2. Comparison of renormalization functions

In the foregoing subsection we have shown that the correct series expansions in powers of $\epsilon_8$ of the critical, crossover, and correction-to-scaling exponents of the isotropic Lifshitz point follow from the results of I. An analogous result holds for other universal quantities. This is because the required renormalization factors $Z_\psi$, $Z_g$, $Z_\epsilon$, and $Z_\lambda$, and hence the implied beta and exponent functions $\beta_\epsilon(g)$ and $\eta_\psi$, $\eta_\epsilon$, $\eta_\lambda$ may all be obtained from the $m$-dependent results of I.

A direct way of seeing this is to compare the Feynman integrals on which the analysis of the present paper is based (and which are computed in Appendix A) with their analogues of I. To facilitate this comparison, it is advisable to get rid of the choices of the factors $F_\epsilon$ and $F_{\epsilon,m}$ we absorbed in $g$ and $u$ (the renormalized coupling constant of I), respectively. To achieve this goal, we must merely divide two-loop Feynman integrals such as $I_3$, $I_4$, and $I_5$ by the square of $\epsilon_8 I_2$ [as can be seen from equations (A.13) and (I.38)]. That is, the normalized integral $I_4(\mathbf{e}, \mathbf{0})/|\epsilon_8 I_2(\mathbf{e})|^2$ should be compared with the quantity $\{|I_4(\mathbf{e}_\perp, \mathbf{0})/|\epsilon I_2(\mathbf{e}_\perp)|^2\}_{m=8-\epsilon_8, \epsilon=\epsilon_8/2}$, where $\mathbf{e}$ and $\mathbf{e}_\perp$ are unit $d$ and $d-m$ vectors, respectively. Utilizing equations (I.B.14), (I.86) and (I.B.13) of I, and the corresponding results (A.13) and (A.8) for the isotropic case, one finds that the pole parts of these quantities agree indeed:

$$\frac{I_4(\mathbf{e}_\perp, \mathbf{0})}{|I_2(\mathbf{e}_\perp)|^2}_{m=8-\epsilon_8, \epsilon=\epsilon_8/2} = \frac{1}{2 \epsilon_8} \left[ \frac{1}{\epsilon_8} - \frac{1}{12} + O(\epsilon_8^8) \right] = \frac{I_4(\mathbf{e}, \mathbf{0})}{|\epsilon I_2(\mathbf{e})|^2}. \quad (30)$$

Analogous results hold for the other integrals needed to determine the two-loop counter-terms. We have

$$\frac{I_3(\mathbf{q})}{|\epsilon_8 I_2(\mathbf{e})|^2} = \frac{3g^4}{80 \epsilon_8} + O(\epsilon_8^8)$$

$$= \left\{ \frac{F_{\epsilon,m}^2}{\epsilon} \frac{j_\phi(m)q^4 + O(\epsilon^0)}{16m(m+2)} + O(\epsilon^0) \right\} \frac{1}{|\epsilon I_2(\mathbf{e}_\perp)|^2} \right\}_{m=8-\epsilon_8, \epsilon=\epsilon_8/2} \quad (31)$$
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and

\[
\frac{I_5(q)}{[\epsilon^I I_2(e)]^2} = \frac{-3q^2}{16\epsilon^I} + O(\epsilon^I) \\
= \left\{ \frac{-F_{m,\epsilon}^2}{4m} J_\rho(m) q^2 + O(\epsilon^I) \right\} \frac{1}{[\epsilon I_2(e)]^2},
\]

where \( F_{m,\epsilon} = \epsilon I_2(e) \) is the counterpart of the normalization constant \( F_d \). The analogues of \( I_3 \) and \( I_5 \) we inserted on the right-hand sides of these equations can be read off from equations (I.B.4) and (I.B.6). Thus the pole terms found in \( I \) reduce in the isotropic case precisely to those obtained here. Provided we use the same conventions as here to fix the counter-terms, we get identical results for the \( Z \) factors \( Z_\psi, Z_g, Z_\tau, \) and \( Z_\lambda \).

Let us briefly comment on the fact that we evaluated the integrals \( I_2 \) and \( I_4 \) in \( I \) at a momentum \( q \) with vanishing parallel component \( q_\parallel \) and finite perpendicular one \( q_\perp \).

In the limit \( m \to d \), the latter becomes zero-dimensional. However, this is no cause of concern. First of all, we should get reasonable results in this limit (as our results in I show) because the opposite would indicate a failure of the analytic continuation of the momentum-space integrals \( \int d^{d-m} q \) in \( d - m \). Second, taking \( q = (0, q_\perp) \) may be viewed as the special choice \( \exp(iq_\perp \cdot x - a x^2) \), with \( a \to 0+ \), of a test function utilized to determine the pole term of the corresponding distributions \( G(x)^2 \) and the Fourier back-transform of \( I_4 \) to position space. The result \( G(x)^2 F_{m,\epsilon}^{-2} = \epsilon^{-1} \delta(x) + O(\epsilon^I) \) we obtained in \( I \) can equally well be derived by computing the action of \( G^2 \) on other test functions (integrating over a finite sphere centered at the origin would do). If we set \( d = m = 8 - \epsilon_8 \) in this result, we recover the correct pole term of the isotropic case. Needless to say that only the \( (m = d) \)-dimensional part \( \delta(x_\parallel) \) of the delta function remains for \( d = m \), where \( x = x_\parallel \).

By comparing the renormalized action implied by our re-parameterizations \( \tilde{I} \) and \( \hat{I} \) with the renormalized action of \( I \) when \( m = d \), one can easily see how our two-loop results \( \tilde{I} \)–\( \hat{I} \) for these renormalization factors can be expressed in terms of those of \( I \). Since to order \( u^2 \) only the terms \( \propto u^2/\epsilon \) depend on \( m \), we can set \( m = 8 \) when evaluating them. The desired relations then become

\[
Z_\psi(g, \epsilon_8) = [Z_\sigma Z_\phi]_{u=g/2; m=8, \epsilon=\epsilon_8/2} + O(g^3),
\]

\[
(Z_\psi Z_\tau)(g, \epsilon_8) = \left[Z_u Z_\phi^{Z_\tau_{Z_\tau}}\right]_{u=g/2; m=8, \epsilon=\epsilon_8/2} + O(g^3),
\]

\[
(Z_\psi Z_\lambda)(g, \epsilon_8) = \left[Z_u Z_\phi^{Z_\lambda_{Z_\lambda}}\right]_{u=g/2; m=8, \epsilon=\epsilon_8/2} + O(g^3),
\]

and

\[
\beta_\psi(g; \epsilon_8) = -\epsilon_8 g + 4 \beta_u(g/2; m = 8, \epsilon = 0) + O(g^4),
\]

\[
\eta_\psi(g) = 2 \eta_\tau(g/2; m = 8) + O(g^3), \quad \eta_\psi(g) = 2 \eta_\tau(g/2; 8) + O(g^3),
\]

and

\[
\eta_\lambda(g) = 2 \eta_\psi(g/2; 8) - \eta_\sigma(g/2; 8) + O(g^3).
\]
The reader is invited to convince himself that the results (12)–(15) for the $Z$ factors and the RG functions (18)–(21) of the isotropic Lifshitz point are indeed recovered from the above formulae when the $m$-dependent results of I are inserted into them, utilizing the required values (I.86) of the integrals $j_\phi(m)$, $\ldots$, $J_u(m)$ at $m = 8$.

4. Concluding remarks

In summary, we have shown that the counter-terms, RG functions, and the series expansions in powers of $\epsilon_8 - 8 - d$ of the critical, crossover, and correction-to-scaling exponents of the isotropic Lifshitz point can be obtained from the results of I for the more general case of an $m$-axial Lifshitz point. We have verified their validity by performing an independent two-loop calculation directly for the isotropic case $d - m = 0$, using well-established standard field-theory techniques.

The fact that a proper field-theoretic analysis of the critical behaviour at $m$-axial Lifshitz points, based on the $\epsilon$ expansion, covers also the isotropic case $m = d$, should not be too surprising: After all, the free propagator $G(x)$ for general $m \neq d$ goes smoothly over into the correct isotropic one of equation (8) as $m - d \to 0$. The Feynman graphs of the primitively divergent vertex functions are distributions constructed from $G(x)$ having well-defined Laurent expansions in $\epsilon$. The renormalization factors are determined by the pole parts of the final subtractions which the primitively divergent vertex functions require. Clearly, their behaviour in the isotropic limit should comply with that of $G(x)$.

In closing, let us straighten out another critique by de Abuquerque and Leite [15]. They claimed that our two-loop calculation was incomplete because we set the parallel component $Q_\parallel$ of the momentum $Q$ to zero when computing the two-loop integral $I_4(Q, K)$ of the graph $\includegraphics{graph1}$. However, the final subtractions we determined in this part of our computation were those associated with the $|\phi|^4$ counter-term and the vertex function $\Gamma_\phi^{(2)}$, both of which are momentum-independent. It is true that $I_4(Q, K)$ has first order poles in $\epsilon$ that depend also on $Q_\parallel$. These are induced by the divergent sub-integral $\includegraphics{graph2}$, and are cancelled automatically by the subtractions provided by the one-loop counter-terms, in renormalizable theories such as the ones considered in I and here. Thus the convenient choice $Q_\parallel = 0$ is possible. Readers who would like to see this re-iterated and verified explicitly in the present context may consult Appendix B.

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Appendix A. Feynman integrals

Equation (11) yields for the integrals (9) and (10) the results

$$I_2(q) = 2^{-d} \pi^{-\frac{d}{2}} \frac{q^{d-8} \Gamma(4 - \frac{d}{2}) \Gamma(4 - 2)}{\Gamma(d - 4)} = F_d \frac{q^{-\epsilon_8}}{\epsilon_8}$$

(A.1)
and
\[ I_3(q) = F_d^2 \frac{q^{2(d-6)} \Gamma(6-d) \Gamma(d-4)^2}{4 \Gamma(5-\frac{d}{2}) \Gamma(\frac{d}{2} - 2) \Gamma(\frac{4d}{2} - 6)} \]
\[ = F_d^2 q^\epsilon \left[ \frac{3}{80 \epsilon_8} + \frac{211 - 240 \ln q}{3200} + O(\epsilon_8) \right], \tag{A.3} \]
respectively. The two-loop graph \( \begin{tikzpicture} \draw (0,0) -- (0.5,0) -- (0.5,0.5) -- (0,0.5) -- cycle; \end{tikzpicture} \) involves the Feynman integral
\[ I_4(q_{12}, q_3) = \int q^4 |q - q_{12}|^4 I_2(|q + q_3|). \tag{A.4} \]

Using a familiar trick due to Feynman we can transform the product of denominators into a single power by the introduction of two Feynman parameters \( s \) and \( t \). Performing the integral over \( q \) then gives
\[ I_4(q_{12}, q_3) = F_d^2 q_{12}^\epsilon \frac{\Gamma(\epsilon_8)}{(4\pi)^{d/2} \Gamma(\epsilon_8/2)} \int_0^1 ds s(1-s) \int_0^1 dt t^{3(1-t)-1+\epsilon_8/2} \]
\[ \times \left\{ t s q_{12}^2 + (1-t) q_3^2 - [(1-t) q_3 - st q_{12}]^2 \right\}^{-\epsilon_8}. \tag{A.5} \]

By adding and subtraction the value of the integrand at \( q_3 = 0 \), one easily sees that the pole part of \( I_4(q_{12}, q_3) \) is independent of \( q_3 \) (noting that the pre-factor of the integral on the right-hand side is of order \( \epsilon_8^{-1} \)). We therefore set \( q_3 = 0 \), which leads us to the integral
\[ J(d) = \int_0^1 ds s^{1-\epsilon_8} (1-s) \int_0^1 dt t^{3(1-t)-1+\epsilon_8/2} (1-t)^{-\epsilon_8}, \tag{A.6} \]
whose dimensionality expansion
\[ J(8 - \epsilon_8) = \frac{1}{3 \epsilon_8} + \frac{1}{4} + O(\epsilon_8) \tag{A.7} \]
can be determined in a straightforward fashion. Upon expanding the pre-factor of the integral, one arrives at the result
\[ I_4(q_{12}, q_3) = F_d^2 q_{12}^2 \frac{\epsilon_8}{2 \epsilon_8} \left[ \frac{1}{\epsilon_8} - \frac{1}{12} + O(\epsilon_8) \right]. \tag{A.8} \]

As usual, the required two-loop integrals of \( \Gamma^{(2)}_{\bar{\phi}^2} \) involve just the integrals \( I_2 \) and \( I_4 \). However, we also need the integral associated with the two-loop graph \( \begin{tikzpicture} \draw (0,0) -- (0.5,0) -- (0.5,0.5) -- (0,0.5) -- cycle; \end{tikzpicture} \) of \( \bar{\Gamma}^{(2)}_{(\nabla \psi)^2} (q, Q) \), where \( Q \) denotes the momentum of the inserted operator \( [\frac{(\nabla \psi)^2}{2}]_Q \). Since the limit \( Q \to 0 \) of this Feynman integral is not infrared-singular and exists, we can consider directly the case \( Q = 0 \). The upper line with such an insertion corresponds in position space to the function
\[ D(y) = \int_q e^{iqa} = \frac{\pi^{-d/2}}{128} \Gamma \left( \frac{d}{2} - \frac{d}{2} \right) y^{6-d}. \tag{A.9} \]
Thus the required Feynman integral can be written as
\[ I_5(q) = \int d^d y G(y)^2 D(y) e^{iqa}. \tag{A.10} \]
It can be evaluated in a straightforward fashion to obtain

\[
I_5(q)/F_d^2 = \frac{\Gamma(7 - d)\Gamma((d - 6)/2)\Gamma(d - 4)^2}{81\Gamma((10 - d)/2)^2\Gamma((d - 4)/2)^2\Gamma(3d/2 - 7)} q^{2(d - 7)} = -\frac{3q^2}{16\epsilon_8} + O(\epsilon_8^0). \tag{A.11}
\]

Utilizing the Laurent expansions of the integrals given above, one can derive the results (12)–(15) for the renormalization factors in a straightforward manner.

Appendix B. Cancellation of momentum-dependent pole terms

In this appendix we show explicitly for the case of the $m$-axial Lifshitz point that the momentum-dependent poles of the two-loop graph of $\Gamma^{(4)}$ appearing first on the the right-hand side of equation (B.1) are cancelled by the subtractions which the one-loop counter-terms provide for the divergent sub-integral inside the dashed box. It is sufficient to consider the problem for a one-component order parameter ($n = 1$). The lines correspond to the free propagator of the Hamiltonian (4) for $\tau_0 = \rho_0 = 0$; its explicit forms in momentum and position space can be found in equations (I.5)–(I.11).

Zimmermann’s forest formula [17] tells us, for any Feynman graph, into which subtractions for its divergent subgraphs the lower-order counter-terms translate. In our case, the result is particularly simple: The one-loop $|\phi|^4$ counter-term yields the subtraction

\[
\mathcal{R}[\gamma] = \mathcal{R}[\gamma] - \mathcal{R}[\gamma]_{m = 0}.
\tag{B.1}
\]

Here the notation $\mathcal{R}[\gamma]$ is used to indicate the quantity that results from a Feynman integral $I[\gamma]$ of a graph $\gamma$ by making the required subtractions for all its divergent subgraphs but not the final subtraction for $\gamma$ itself. The boxed subgraph denotes its singular part. This is local in position space (namely, proportional to $\epsilon^{-1}$ times the $|\phi|^4$ vertex). Thus the subtracted graph involves (besides a momentum-independent second-order pole) a momentum-dependent first-order pole of the form $\epsilon^{-1}$ times the momentum-dependent term of order $\epsilon^0$ of the graph that results when the box is contracted to a point. The latter is again the one-loop graph of $\Gamma^{(4)}$.

The pole terms of the difference (B.1) must be momentum-independent and local in position space so that they can be absorbed via the $|\phi|^4$ counter-term. From equation (I.38) we know that the singular part of the divergent sub-integral $I_2$ of $I_4$ agrees with the pole part of $I_2(e_\perp) = F_{m,\epsilon}\epsilon^{-1}$, where $F_{m,\epsilon}$ is the constant (I.39) while $e_\perp$ denotes a unit $d - m$ vector. Hence the difference of Feynman integrals we are concerned with becomes

\[
\mathcal{R}[I_4](Q, K) = \int \frac{1}{q_\perp + q_\parallel} \frac{1}{|q_\perp + Q_\perp|^2 + |q_\parallel + Q_\parallel|^2} [I_2(q - K) - I_2(e_\perp)]. \tag{B.2}
\]

We must show that the difference $\mathcal{R}[I_4](Q, K) - \mathcal{R}[I_4](Q, K)|_{q_\perp = 0}$ is regular in $\epsilon$.

This difference is given by the analogue of equation (B.2) that results through the replacement

\[
\frac{1}{|q_\perp + Q_\perp|^2 + |q_\parallel + Q_\parallel|^4} \rightarrow \frac{Q_\parallel^2 (Q_\parallel^2 + 2q_\parallel^2) - 4 (Q_\parallel \cdot q_\parallel + q_\parallel^2 + Q_\parallel^2) Q_\parallel \cdot q_\parallel}{[|q_\perp + Q_\perp|^2 + |q_\parallel + Q_\parallel|^4] [||q_\perp + Q_\perp|^2 + q_\parallel^4]}.
\tag{B.3}
\]
The subtraction has improved the uv behaviour of the integral by two powers of $q_{\parallel}$, making it uv convergent at the upper critical dimension in accordance with Weinberg’s theorem [18]. Hence the momentum-dependent poles—and especially the $Q_{\parallel}$-dependent ones—cancel, as they should, both in $\Gamma^{(4)}$ as well as in $\Gamma^{(2)}_{\phi^2}$.

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