Equation of states in the curved spacetime of spherical degenerate stars

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In the study of spherical degenerate stars such as neutron stars, general relativistic effects are incorporated by using Tolman-Oppenheimer-Volkoff equations to describe their interior spacetime. However, the equation of states employed in such studies is invariably computed in flat spacetime. We show that the equation of states computed in the curved spacetime of these stars depend explicitly on the metric function. Further, we show that ignoring such metric-dependent gravitational time dilation effect leads one to grossly underestimate the mass limits of these compact stars.

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Introduction. – The observation of gravitational waves, accompanied by electromagnetic counterparts [1,2], has opened up an unprecedented window to probe several uncharted aspects of matter field dynamics in a strong gravity regime. These observed gravitational waves are thought to have originated from the merger of neutron stars which are supported by fermionic degeneracy pressure. The general relativity plays a key role for these stars which are supported by fermionic degeneracy pressure. Therefore, any study of spherical degenerate stars such as neutron stars, general relativistic effects are incorporated by using Tolman-Oppenheimer-Volkoff (TOV) equations to describe their interior spacetime. However, the equation of states (EOS) used in such studies, are invariably computed using the flat spacetime (see [3,5] for recent reviews).

The usage of flat metric locally can be justified while computing the EOS at a given radial location within a star. However, two such locally inertial frames located at two different radial locations are not identical as their clock speeds differ due to the gravitational time dilation effect. Therefore, the EOS computed in the flat spacetime (flat EOS) cannot capture the effects of strong gravity on the matter field dynamics within these stars. In turn, it necessitates a first-principle derivation of the EOS using the curved spacetime (curved EOS) of these stars.

Fermions in curved spacetime. – In order to keep the analysis simple yet fairly general, here we consider the non-interacting MIT Bag model [6] to describe the degenerate matter within a star. Inside the bag region and ignoring the boundary terms, the matter field action in a curved spacetime with the metric \( g_{\mu\nu} \) ignoring the boundary terms, the matter field action in
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Under local Lorentz transformation \( \Lambda_{\mu}^a(x) \equiv (\partial x'^{\mu}/\partial x^b) \), a fermion field \( \psi(x) \) transform in the spinor representation as \( \psi(x) \rightarrow \psi'(x) = U[\Lambda(x)]\psi(x) \). In the curved spacetime \( \partial_a U[\Lambda(x)] \neq 0 \) in general. Consequently, the term \( \partial_a \psi \) does not transform as a vector under the local Lorentz transformation i.e. \( \partial_a \psi \rightarrow \partial'_a \psi' \neq (\Lambda^{-1})^b_a U[\Lambda] \partial_b \psi \). Therefore, in order to extend the Dirac action in the curved spacetime, a suitable covariant derivative for the spinor field is defined as \( \hat{D}_a \psi \equiv \partial_a \psi + (\Gamma^b_a + \Gamma^a_b U[\Lambda]) \partial_b \psi \) such that under local Lorentz transformation it leads to

\[
\hat{D}_a \psi \rightarrow (\Lambda^{-1})^b_a U[\Lambda] \hat{D}_b \psi.
\]

In order to derive the form of \( \Gamma_{\mu} \), we consider an infinitesimal local Lorentz transformation given by \( \Lambda_{\mu}^a = \delta_{\mu}^a + \alpha_{\mu}^a \) where \( \alpha_{\mu}^a = -\Omega_{\mu \alpha} \). In spinor representation, \( U[\Lambda] = I + \frac{1}{2} \Omega_{\mu \alpha} \sigma^{\mu \alpha} \) where \( \sigma^{\mu \alpha} = \frac{1}{2} [\gamma^\mu, \gamma^\alpha] \) with \( \gamma^\alpha \) being the Dirac matrices in the locally inertial frame. The
Dirac matrices $\gamma^a$ satisfy $\{\gamma^a, \gamma^b\} = -2\eta^{ab}$. The minus sign in front of $\eta^{ab}$ is chosen here so that for given metric signature, the Dirac matrices satisfy the usual relations $(\gamma^0)^2 = 1$ and $(\gamma^k)^2 = -1$ where $k = 1, 2, 3$. The action of the covariant derivative on the Dirac adjoint $\psi = \psi^\dagger \gamma^0$ is given by $D_\mu \psi \equiv \partial_\mu \psi - \Gamma^\nu_{\mu\rho} \partial_\rho \psi$ which ensures $\psi \psi$ behaves as a scalar under both local Lorentz transformation as well as general coordinate transformation. Further, one demands that the covariant derivative $D_\mu$ must act on $(\psi^* \psi') \equiv (\psi^\dagger \gamma^0 e^\nu_a \psi)$ which is a contra-vector under general coordinate transformation. As the regular covariant derivative $\nabla^0_k$, i.e. $D_\mu (\psi^* \psi') = \partial_\mu (\psi^* \psi') + \Gamma^\nu_{\mu\beta} (\psi^* \gamma^\nu \psi')$ where $\Gamma^\nu_{\mu\beta}$ is the Christoffel connection. Now it is convenient to express $\Gamma^\nu_{\mu\beta} \equiv -\frac{1}{2} \omega_{\mu ab} \sigma^{ab}$. Subsequently, the use of Leibniz rule for both covariant derivative and partial derivative together with compatibility conditions for the tetrad, i.e. $D_\mu e^\nu_a = 0 = D_\mu e^a_\nu$ leads to

$$\omega_{\mu ab} = \eta_{ac} e^b^c \left( \partial_\mu e^c_\nu + \Gamma^\nu_{\mu\beta} e^e_\beta \right). \quad (3)$$

We note that $\gamma^\mu \equiv \gamma^a e^a_\mu$ satisfy $\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu} I$ and are often referred to as the Dirac matrices in the curved spacetime. Therefore, in the Fock-Weyl formulation, the invariant action for the $I^{th}$ spinor in the curved spacetime which is minimally coupled with the geometry, can be expressed as

$$S_{\psi} = -\int d^4x \sqrt{-g} \psi_I \left[ i \gamma^a e^a_\mu D_\mu + m_I \right] \psi_I, \quad (4)$$

where $\psi_I = \psi^\dagger_I \gamma^0$ and $m_I$ is its mass. The field equation for the $I^{th}$ Dirac spinor in the curved spacetime is given by

$$[i \gamma^a e^a_\mu D_\mu + m_I] \psi_I = 0. \quad (5)$$

The corresponding conservation equation is $D_\mu j^\mu_I = 0$ where 4-current density is given by $j^\mu_I = \psi_I \gamma^c e^a_\mu \psi_I$.

**Stress-energy tensor and equation of state.** The key idea behind application of quantum field theory in the curved background is to consider the Einstein equation of the form $G_{\mu\nu} = 8\pi G(T_{\mu\nu})$. Here the Einstein tensor $G_{\mu\nu}$ is treated classically whereas the classical stress-energy tensor $T_{\mu\nu}$ is replaced by the appropriate expectation value of the corresponding quantum operator $T_{\mu\nu}$. Assuming perfect fluid form for the stress-energy tensor, here it would imply $\langle T_{\mu\nu} \rangle = (\rho + P) g_{\mu\nu} + P g_{\mu\nu}$. On the other hand, the stress-energy tensor corresponding to a spinor field $\psi$ is given by

$$T_{\mu\nu} = \frac{\delta S_{\psi}}{\sqrt{-g} \delta e^a_\nu}, \quad (6)$$

which reduces to the standard form $T_{\mu\nu} = -\frac{1}{2} \frac{\delta S_{\psi}}{\delta g_{\mu\nu}}$. Using on-shell condition, the Eq. (6) leads to $T^0_0 = -\mathcal{H}$ and $T^k_k = \mathcal{H} - m_i \dot{\psi}$ where the field Hamiltonian is $\mathcal{H} = \int d^4x \sqrt{-g} \mathcal{H}$. The corresponding partition function in a given small region is

$$Z_\psi = \text{Tr} [e^{-\beta(\mathcal{H} - \mu N)}], \quad (7)$$

where $\mu$ is chemical potential, $\hat{N}$ is number operator, $\beta = 1/k_BT$ with $T$ and $k_B$ being the temperature and the Boltzmann constant respectively. Using the Eq. (7), one arrives at the usual form of the energy density $\rho = \langle H \rangle /V$ with volume of the small region $V = \int d^3x \sqrt{-g}$ and $\langle H - \mu N \rangle = -\partial \ln Z_\psi / \partial \beta \rangle$ where $\langle \cdot \rangle$ denotes the thermal expectation value. Further, in $Z_\psi$ being a dimensionless extensive quantity, it can be expressed in the form $\ln Z_\psi = \beta^3 V f(3\beta \mu \beta m)$ for eg. Eq. (21) where $\beta = \frac{4\pi}{m_\omega} f(3\beta \mu \beta m) + \frac{3\pi}{m_\omega} f(3\beta \mu \beta m)$. Consequently, the corresponding pressure takes the standard form $P = \beta V - 1 \ln Z_\psi$.

We may emphasize here that the partition function (7) is generally invariant for the given external parameters $\beta$ and $\mu$. However, what is often overlooked in the literature that these parameters which set the scale of energy density and pressure, are impacted by the general relativistic time dilation effects due to radially varying lapse function within a star. In this article, we show the results of time dilation effect on the equation of state by two methods: direct computation and using the scaling behaviour of the time coordinate.

**Spacetime within spherical stars.** Using natural units, $c = \hbar = 1$, the invariant line element within a spherical star can be written as

$$ds^2 = -e^{2\nu(r)} dt^2 + e^{2\nu(r)} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2). \quad (8)$$

The metric functions $\Phi(r)$ and $\nu(r)$ are determined by solving the Einstein equations, also referred to as the TOV equations. In particular, these equations lead to $e^{-2\nu(r)} = (1 - 2GM/r)$ and

$$\frac{d\Phi}{dr} = G(4\pi r^3 P) \quad , \quad \frac{dP}{dr} = -(\rho + P) \frac{d\Phi}{dr}, \quad (9)$$

where $dM = 4\pi r^2 P dr$. Inside the star, the pressure $P$ and the energy density $\rho$ both vary radially. On the other hand, at thermodynamic equilibrium, the quantities such as the pressure, the energy density are uniform within the system. In order to combine these two aspects together, one considers a sufficiently small spatial region, containing sufficient degrees of freedom, around a given point within the star where local thermodynamic equilibrium holds.

**Reduced spinor action.** It is always possible to find a coordinate system in which the metric is locally flat and which can be used to describe the nuclear interactions. Here we give such an explicit construction where we also retain the information about the metric function $\Phi$. For definiteness, let us a consider a small box whose center is located at a radial coordinate $r_0$. One may expand the metric functions $\Phi(r)$ and $\nu(r)$ around the point $r_0$ and keep only the leading terms. Further, given the spherical symmetry, we may rotate the coordinate system such that the polar axis, $\theta = 0$, passes through the center of the box. Then for all points within the box, the angle $\theta$ can be taken to be small. By using a new set of coordinates $X = e^{\nu(r_0)} r \sin \theta \cos \phi$, $Y = e^{\nu(r_0)} r \sin \theta \sin \phi$, $Z = r \cos \theta$, and $\bar{\phi} = \phi$.
and $Z = e^{\nu(r_0)\tau} \cos \theta$ along with $\bar{\theta} = e^{-\nu(r_0)\theta}$, the metric within the box can be reduced to

$$ds^2 = -e^{2\Phi(r_0)}dt^2 + dX^2 + dY^2 + dZ^2.$$  \hspace{1cm} (10)

The Eq. (10) shows that the metric is flat within the box, i.e. it is flat over a scale which is sufficient to describe the nuclear interactions. At the same time, it shows that the metric is not globally flat as it carries information about the large scale radial variation of the metric function $\Phi$, implied by the TOV Eqs. (9). This is in contrast to the usage of globally flat metric for computation of EOS in the literature. By using the diagonal ansatz, the corresponding tetrad and inverse tetrad can be expressed as

$$e^a_\mu = \text{diag}(e^\Phi, 1, 1, 1), \hspace{1cm} e^a_\mu = \text{diag}(e^{-\Phi}, 1, 1, 1),$$  \hspace{1cm} (11)

where $\Phi \equiv \Phi(r_0)$. Clearly, in the $(t, X, Y, Z)$ coordinates both $T_\alpha^\beta$ and $\omega_{\mu\alpha\beta}$ vanish and the spin-covariant derivative becomes $D_\mu = \partial_\mu$ within the box. The action (4) for the spinor $\psi_I$ then reduces to

$$S_{\psi_I} = -\int d^4x \bar{\psi}_I \left[ \gamma_0 \partial_0 + e^\Phi (i \gamma^k \partial_k + m_I) \right] \psi_I,$$  \hspace{1cm} (12)

where $k$ runs over $1, 2, 3$. The corresponding conserved charge then becomes $Q_I = \int d^4x \sqrt{-g} \bar{\psi}_I \gamma^0 \psi_I$. The reduced action (12) should be viewed as an effective field action in a locally Minkowski spacetime. In contrast to the standard spinor action used in the literature, the action (12) carries information about the box-specific, fixed metric function $\Phi$ and hereafter we use this action for computation of the equation of state.

**Partition function.** In order to compute the EOS, here we employ the tools of thermal quantum field theory \[7, 8\] as pioneered by Matsubara. The partition function corresponding to the MIT Bag model that represents the spinor degrees of freedom within the box is given by

$$\ln Z_{\text{Bag}} = \sum_I \ln Z_{\psi_I} - e^\Phi \beta VB,$$  \hspace{1cm} (13)

where $V = \int d^3x$ denotes the spatial volume of the box. In path-integral formulation, the partition function for the $I^{th}$ spinor is $Z_{\psi_I} = \int D\bar{\psi}_ID\psi_I e^{-S_{\psi_I}}$ where its Euclidean action is

$$S^3_{\psi_I} = \int_0^\beta dt \int d^3x \left[ \mathcal{L}^E_{\bar{\psi}_I} + \mu_I \bar{\psi}_I(\tau, x) \gamma^0 \psi_I(\tau, x) \right],$$  \hspace{1cm} (14)

with $\mu_I$ being its chemical potential. The Euclideanized Lagrangian density $\mathcal{L}^E_{\bar{\psi}_I}$ which is obtained through the substitution $t \rightarrow i\tau$ from its standard counterpart, is

$$\mathcal{L}^E_{\bar{\psi}_I} = -\bar{\psi}_I(\tau, x) \gamma^0 \partial_\tau + e^\Phi (i \gamma^k \partial_k + m_I) \psi_I(\tau, x). $$  \hspace{1cm} (15)

The anti-periodic boundary condition which carries the information about the equilibrium temperature $T$, is imposed on the spinor field as

$$\psi_I(\tau, x) = -\psi_I(\tau + \beta, x).$$  \hspace{1cm} (16)

It is convenient to transform the spinor field in the Fourier domain as

$$\psi_I(\tau, x) = \frac{1}{\sqrt{V}} \sum_{n,k} e^{i(\omega_n \tau + kx)} \psi(n, k).$$  \hspace{1cm} (17)

The anti-periodic boundary condition (16) leads the Matsubara frequencies to be $\omega_n = (2n+1)\pi \beta^{-1}$ where $n$ is an integer. The Eqs. (14) and (17) together lead to

$$S^3_{\psi_I} = \sum_{n,k} \bar{\psi} \beta [\bar{\rho} - m_I] \psi,$$  \hspace{1cm} (18)

where $\bar{m} = m_I e^\Phi$, $\bar{\rho} = \gamma^0 (-i \omega_n + \mu_I) + \gamma^k (k e^\Phi)$ and the corresponding thermal propagator in the Fourier domain is given by

$$G_I(\omega, k) = \frac{1}{\bar{\rho} - \bar{m}_I}.$$  \hspace{1cm} (19)

Using the Dirac representation of the gamma matrices and the result of Gaussian integral over the Grassmann variables one gets

$$\ln Z_{\psi_I} = 2 \sum_k \ln \left(1 + e^{-\beta(\omega - \mu_I)}\right),$$  \hspace{1cm} (20)

where $\omega = \omega(k) = e^\Phi \sqrt{k^2 + m_I^2}$. Here we have ignored formally divergent zero-point energy and the anti-particle contributions.

A degenerate star is characterized by the condition $\beta \mu_I \gg 1$. It allows one to approximate $(e^{\beta(\omega - \mu_I)} + 1)^{-1} \simeq \Theta(\mu_I - \omega) - \text{sgn}(\mu_I - \omega) e^{-\beta|\mu_I - \omega|}$ where $\Theta(x)$ and $\text{sgn}(x)$ are Theta and signum functions respectively. Using this approximation, one can carry out the summation over $k$ label by converting it to an integral to get

$$\ln Z_{\psi_I} = \frac{\beta V e^{-3\Phi}}{24\pi^2} \left[ 2 \mu_I \bar{m}_I^3 - 3 \bar{m}_I^2 \bar{\mu}_I^2 + \frac{48 \bar{\mu}_I m_I}{\beta^2} \right],$$  \hspace{1cm} (21)

with $\mu_I \equiv \sqrt{\mu_I^2 - m_I^2}$ and $\bar{\mu}_I \equiv \mu_I m_I - \bar{m}_I^2 \ln(2 \bar{\mu}_I m_I)$. In the limit $\Phi \rightarrow 0$, $\ln Z_{\psi_I}$ reduces to the standard flat spacetime form (see \[9,10\]).

**Equation of state.** The number density of the $I^{th}$ spinor can be computed using $n_I = (\beta V)^{-1} (\partial \ln Z_{\text{Bag}}/\partial \mu_I)$ as

$$n_I = \frac{\mu_I m_I^3 e^{-3\Phi}}{3\pi^2} \left[ 1 + \frac{6}{(\beta \mu_I^2)^2} \left( 1 + \frac{\bar{\mu}_I^2}{\mu_I^2} \right) \right],$$  \hspace{1cm} (22)

where we have used the relations $(\partial \bar{m}_I^3/\partial \mu_I) = 2 \mu_I m_I$ and $(\partial \bar{m}_I^2/\partial \mu_I) = (\mu_I/m_I)$. For a grand canonical ensemble, total pressure $P = (\beta V)^{-1} \ln Z_{\text{Bag}}$ leads to

$$P = \sum_I \frac{e^{-3\Phi}}{24\pi^2} \left[ 2 \mu_I \mu_I^3 - 3 \bar{m}_I^2 \mu_I^2 + \frac{48 \mu_I m_I}{\beta^2} \right] - e^\Phi B.$$  \hspace{1cm} (23)
We may parameterized the baryon number density as \( n = (n_l/b_l) \) with \( b_l \) being the model-dependent parameters. Given \( \beta \mu_I \gg 1 \) for a degenerate star, the temperature corrections of \( \mathcal{O}(\beta^2\mu^2) \) are very small and henceforth ignored for simplicity. Then the Eq. \((22)\) leads to

\[
\mu_{I,e} e^{-\Phi} = m_I (b_I n)^{1/3}, \quad \mu_I e^{-\Phi} = m_I \sqrt{(b_I n)^{2/3} + 1},
\]

where the constant \( b_I = 3\pi^2 b_l/m_I^4 \). The pressure \( P \) can be expressed in terms of baryon number density \( n \) as

\[
P = e^{\Phi} \sum_I \frac{m_I^4}{24\pi^2} \left[ \sqrt{(b_I n)^{2/3} + 1} \left\{ 2(b_I n) - 3(b_I n)^{1/3} \right\} + 3 \ln \left\{ (b_I n)^{1/3} + \sqrt{(b_I n)^{2/3} + 1} \right\} - e^{\Phi} B. \]

The energy density \( \rho \) within the box can be computed using \( \rho - \sum_I \mu_{I,e} n_I V = -\partial \ln Z_{\beta\mu_{I}}/\partial \beta \) and ignoring temperature corrections as earlier, it leads to

\[
\rho = -P + e^{\Phi} \sum_I \frac{m_I^4}{3\pi^2} \sqrt{(b_I n)^{2/3} + 1} \ (b_I n). \]

The key property of the curved EOS \( [25] [26] \) is its explicit dependence on \( \Phi \) which follows from the metric component \( g_{tt} \). Clearly, it is the gravitational time dilation effect which is experienced by the matter field in a strong gravitational field and is missed in a derivation using flat spacetime. We may emphasize that the EOS is computed here in a small box located around \( r = r_0 \). However, \( r_0 \) being arbitrary, the EOS can be considered to be dependent on \( r \).

**Numerical solutions.**– In order to solve the TOV Eqs. numerically, we note that both pressure and energy density are explicit functions of \( \Phi \) as \( P = P(n, \Phi) \) and \( \rho = \rho(n, \Phi) \). This turns implies that TOV Eqs. \( [9] \) can be viewed as a set of first order differential equations for the triplet \( \{M, \Phi, n\} \) where the baryon density \( n \) satisfies

\[
\frac{dn}{dr} = -\frac{(\rho + P + (\partial P/\partial \Phi)) \ (\partial P/\partial n)}{(\partial P/\partial n) - e^{\Phi} B}. \]

We note that for the curved EOS, unlike for the flat EOS, the metric function \( \Phi \) can not be eliminated from the set of TOV Eqs. \( [9] \). Nevertheless, for a star of radius \( R \) and mass \( M \), the triplet \( \{M, \Phi, n\} \) is subject to the conditions \( e^{2\Phi(R)} = (1 - 2GM/R) \) with \( M = M(R) \) and \( n(R) = 0 \). In order to numerically impose this condition, for a given central baryon density, say \( n_c \), we evolve these equations towards the surface with a trial value of \( \Phi \) at the center. We independently calculate the value \( \Phi \) at the surface, say \( \Phi_s = \frac{1}{2} \ln(1 - 2GM/R) \) and compare it with the evolved value of \( \Phi \) at the surface. In the next step, we compute \( \Phi \) at the center starting from the computed value \( \Phi_s \) and by evolving it backward from \( n = 0 \) to \( n = n_c \) using the Eq.

\[
\frac{d\Phi}{dn} = -\frac{(\partial P/\partial n) - e^{\Phi} B}{(\rho + P + (\partial P/\partial \Phi))}. \]

which follows from the Eq. \( [27] \). These steps are then iterated until the evolved and computed values of the metric function \( \Phi \) at the surface agree with each other within the desired numerical accuracy. The above method of iteration leads to a rapid convergence on the value of \( \Phi \).

**Speed of sound and causality.**– We note that the pressure and energy density for the curved EOS can be written as \( P = P(n, \Phi) \) and \( \rho = \rho(n, \Phi) \) whereas for the flat EOS they are of the form \( P = \tilde{P}(n) \) and \( \tilde{\rho} = \tilde{\rho}(n) \). By defining the respective speed of sound as \( c_s^2 = dP/d\rho \) and \( \tilde{c}_s^2 = d\tilde{P}/d\tilde{\rho} \) together with the Eq. \( [28] \), one can establish a relation as follows

\[
c_s^2 = \frac{\tilde{c}_s^2}{(1 - \tilde{c}_s^2)\tilde{\rho} + 2\tilde{P}}. \]

In the high baryon density limit \( i.e. n \to \infty \) both \( c_s^2 \) and \( \tilde{c}_s^2 \) approaches \( \frac{1}{2} \) from below but maintaining \( c_s^2 > \tilde{c}_s^2 \) for \( 0 < n < \infty \). In other words, for a given baryon density speed of sound for the curved EOS is higher compared to the flat EOS but it remains well below the speed of light.

**Enhanced mass limits.**– In contrast to the energy density in flat space-time, say \( \tilde{\rho} = \tilde{\rho}(n) \), the energy density \( [26] \) is of the form \( \rho = \rho(n, \Phi) \) where \( d\Phi > 0 \) within a star except at the center. It implies that \( \rho(r + \Delta r) > \rho(r + \Delta r) \) even if \( \rho(r) = \rho(r) \) and \( (\frac{\partial \rho}{\partial n})_r = (\frac{\partial \rho}{\partial n})_r \) for positive \( \Delta r \). Secondly, the TOV Eqs. \( [9] \) imply that a higher central energy density leads to a faster fall-off in pressure which in turns leads to a smaller radius of the star. Therefore, if one aims to have two stars of same radius, one using curved EOS and another using flat EOS, then the central energy density for the curved EOS must be higher.

Let us consider two sets of solutions of the TOV equations, one using curved EOS denoted as \( \{M^C, \Phi^C, n^C\} \) and another using flat EOS denoted as \( \{M^F, \Phi^F, n^F\} \) such that \( n^C(R) = 0 = n^F(R) \) and \( \rho(R) = 0 = \tilde{\rho}(R) \). The latter condition does not permit a non-zero \( B \) to be present in lower densities. It then follows from the arguments of the previous paragraph that energy densities satisfy \( \rho(n^C, \Phi^C) > \rho(n^F) \) for \( r < R \). The masses corresponding to the curved and flat EOS, say \( M \) and \( \tilde{M} \) respectively, then satisfy \( M > \tilde{M} \). For baryon number density \( n \to 0 \), the Eqs. \( [9] \) implies \( \frac{dn}{dr} \sim -n^{1/3} \frac{d\Phi}{dr} \) for both the cases whereas the energy densities vary as \( \rho \sim n^C e^{\Phi^C} \) and \( \tilde{\rho} \sim n^F \). Therefore, in the limit \( r \to R \), \( \rho \to \tilde{\rho} \) implies \( (\frac{\partial \rho}{\partial n})(2c_s^2/3) \sim (\frac{\partial \rho}{\partial n})_R \) which together with the TOV Eqs. \( [9] \) leads to

\[
M > \tilde{M} \left[ 1 + \frac{2GM}{3R} - \frac{80(GM)^3}{81R^3} + \mathcal{O}\left(\frac{g^4}{R^4}\right) \right].
\]

The mass inequality \( [39] \) implies that the usage of flat EOS leads one to underestimate the masses of the degenerate stars (see the figures for quantitative comparison).

**White dwarfs.**– The EOS for a white dwarf can be read off from Eq. \( [25] \) by choosing the fermions to be the electrons, and by setting \( B = 0, b_l = Z/A \) where \( A, Z \) are
the atomic number mass and atomic number number respectively. In the ultra-relativistic limit, such an EOS reduces to the standard polytropic form, up to the factor of $e^\Phi$, as

$$ P \simeq \frac{e^\Phi}{12\pi^2} \left( \frac{3\pi^2 Z n}{A} \right)^{4/3}. $$

In white dwarfs, the nuclei have negligible motion. For such a nucleus within the given box, the invariant action $S_n = -m_u A \int \sqrt{-g} d^4x$ reduces to $S_n = \int dt [-m_u A e^\Phi]$ where $m_u$ is the atomic mass unit. The corresponding Hamiltonian then implies that the energy density due to the nuclei is $nm_u e^\Phi$ which needs to be included in total energy density $\rho$. The numerically evaluated mass-radius relations for the white dwarfs are plotted in the FIG. 1.

Neutron stars. The EOS for an ideal neutron star follows from the Eqs. (25), if one chooses the fermions to be the neutrons, and sets $B = 0$, $b_1 = 1$. The inclusion of gravitational time dilation effect on the EOS, leads the mass limit of an ideal neutron star to increase by $\sim 16.9\%$ which is significantly higher than the minimum increase ($\sim 7.5\%$) implied by the Eq. (30) (see FIG. 2).

For high density neutron stars, there are strong stability arguments suggesting that the nuclear matter should consist of hyperons i.e. baryons containing strange quark. Unfortunately, many such models stumble to explain the observed high mass ($\gtrsim 2M_\odot$) neutron stars [11-13]. This conundrum is referred to as the hyperon puzzle [14]. Nevertheless, there exist other proposed models that include hyperons and can lead to observed high mass of the neutron stars [15-17]. In any case, the Eq. (30) implies that ignoring the effects of the curved spacetime on the EOS leads one to grossly underestimate the mass limits of these compact stars. For example, if a flat EOS leads the mass limit to be, say, $1.5M_\odot$ with a radius of 10 Km then the corresponding curved EOS would enhance the mass limit by $\sim 13.7\%$ at the minimum. While the actual increase of the mass limit can be obtained only through a comprehensive computation, it is expected to be comparatively higher as in the case of ideal neutron star. In any case, the legitimate incorporation of the curved spacetime while deriving neutron star EOS would significantly alleviate the hyperon puzzle even for those models which lead to relatively lower mass limits. Furthermore, the usage of the curved EOS would imply even higher mass limits for those hyperon models that otherwise can explain the observed high masses of the neutron stars.

In the quark matter models, the nuclear matter is made up of up, down and strange quarks each having 3 color degrees of freedom. Having more species of fermions for a given baryon density, quark matters lead to higher pressure and the higher mass limits. At equilibrium, the masses of the up and down quarks are negligible i.e. $m_u \ll \mu_u$, $m_d \ll \mu_d$ whereas strange quark mass is small i.e. $m_s \ll \mu_s$. By considering the index $I$ to run over these different types of quarks, we can express the total pressure [23] as

$$ P \simeq \frac{e^\Phi}{4\pi^2} \left( \frac{\mu_u^2 + \mu_d^2 + \mu_s^2 - 3\mu_s^2 m_s^2}{\pi^2} - B \right). $$

Based on the equilibrium interactions, one can fix the relative strength of different chemical potentials. The strange star scenario [18] is obtained by choosing $\mu \equiv \mu_u = \mu_d \equiv \mu_s$ along with $b_1 = 1$, as

$$ P \simeq \frac{3e^\Phi}{4\pi^2} \left[ a_4 (\mu e^{-\Phi})^4 - a_2 (\mu e^{-\Phi})^2 \right] - Be^\Phi, $$

where the parameters $a_4 = 1$ and $a_2 = m_s^2$. The so-called color-flavor locked (CFL) phase scenario [19] requires that these quarks have same number density implying their respective chemical potentials to satisfy $\mu_u = \mu_d = \sqrt{\mu_s^2 - m_s^2} \equiv \mu$. The corresponding pressure again can be expressed in the form (32) with $a_4 = 1$ and $a_2 = m_s^2/3$. As the condition $\rho(R) = 0$ cannot be satisfied with non-zero bag constant $B$, the quark matter EOS must be stitched together with the appropriate hadronic EOS in low density as done in the well known equations of states such as by Akmal, Pandharipande, and Ravenhall, [20], Togashi et al. [21], Baym et al. [22]. Nevertheless, even the quark matter EOS [32] depends
explicitly on the metric function $\Phi$. We may mention that inclusion quark interactions, perturbatively, would simply alter the values of $a_2$ and $a_4$.

Curved EOS from flat EOS.– Owing to uncertain theoretical understanding of the nuclear matter, numerous neutron star EOS have been studied in the literature. Invariably, these EOS are computed in the flat spacetime. We now show that it is possible to convert a flat EOS into its spherically symmetric curved space-time. We define a new time coordinate $\tilde{t} = e^{\Phi} t$ which leads the metric (10) to become the standard Minkowski metric $ds^2 = -d\tilde{t}^2 + dX^2 + dY^2 + dZ^2$. Let us denote the equilibrium temperature and chemical potential, as seen from the frame $(\tilde{t},X,Y,Z)$, to be $\tilde{T} = 1/(k_B\tilde{\beta})$ and $\tilde{\mu}_I$ respectively. As seen from the frame $(t,X,Y,Z)$, the information about $T$ is contained within the anti-periodic boundary condition (16) with period $\Delta \tilde{t} = \beta$. Now time intervals of these two frames are related as $\Delta \tilde{t} = e^{\Phi} \Delta t$ which implies

$$\tilde{\beta} = e^{\Phi} \beta . \quad (33)$$

The chemical potential satisfies $(k_B\beta)\mu_I = -\left(\partial S/\partial N_I\right)$, where $S$ is the entropy and $N_I$ is the particle number of the $I^{th}$ spinor in the box. At equilibrium, the entropy $S$ and particle number $N_I$ are not affected by a scaling of the time coordinate which implies

$$\tilde{\mu}_I = e^{-\Phi} \mu_I . \quad (34)$$

One can verify using the relations (33, 34) that the partition function evaluated using standard Minkowski metric, becomes the same as in the Eq. (13).

Discussions.– To summarize, the clock speeds of two locally inertial frames located at different radial coordinates within a spherical star, differ due to the gravitational time dilation effect. The metric (10), which is used here, explicitly shows that although one can use a flat metric to compute an EOS locally, i.e. at the scale of nuclear interactions, the metric itself is not globally flat. In contrast, in the existing literature, the EOS is computed in a globally flat metric and is used for solving TOV Eqs (9) for all values of radial coordinates. It leads one to overlook the effects of time dilation on the EOS caused by the large scale radial variation of the metric function $\Phi$. We have shown that ignoring such metric-dependent time dilation effect on the EOS of neutron stars, leads one to grossly underestimate their mass limits. Given the flat EOS are widely used in the neutron star literature, the result presented here would imply significant alterations of various existing predictions.

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[1] B. Abbott et al. (LIGO Scientific, Virgo), Phys. Rev. Lett. 119, 161101 (2017), arXiv:1710.05832.
[2] B. P. Abbott et al., Astrophys. J. 848, L12 (2017), arXiv:1710.05833.
[3] J. M. Lattimer, Ann. Rev. Nucl. Part. Sci. 62, 485 (2012), arXiv:1305.3510.
[4] F. Ozel and P. Freire, Ann. Rev. Astron. Astrophys. 54, 401 (2016), arXiv:1603.02698.
[5] G. Baym, T. Hatsuda, T. Kojo, P. D. Powell, Y. Song, and T. Takatsuka, Rept. Prog. Phys. 81, 056902 (2018), arXiv:1707.04966.
[6] A. Chodos, R. L. Jaffe, K. Johnson, C. B. Thorn, and V. F. Weisskopf, Phys. Rev. D9, 3471 (1974).
[7] T. Matsubara, Progress of theoretical physics 14, 355 (1955).
[8] J. I. Kapusta and C. Gale, Finite-temperature field theory: Principles and applications (Cambridge University Press, 2006).
[9] G. M. Hossain and S. Mandal (2019), arXiv:1904.09174.
[10] G. M. Hossain and S. Mandal (2019), arXiv:1904.09779.
[11] P. Demorest, T. Pennucci, S. Ransom, M. Roberts, and J. Hessels, Nature 467, 1081 (2010), arXiv:1010.5788.
[12] J. Antoniadis et al., Science 340, 6131 (2013), arXiv:1304.6875.
[13] H. T. Cromartie et al., Nat. Astron. 4, 72 (2019), arXiv:1904.06759.
[14] I. Bombaci, JPS Conf. Proc. 17, 101002 (2017), arXiv:1601.05339.
[15] J. Zdunik and P. Haensel, Astronomy & Astrophysics 551, A61 (2013).
[16] K. Maskov, E. Kolomeitsev, and D. Voskresensky, Physics Letters B 748, 369 (2015).
[17] I. Bednarek, P. Haensel, J. Zdunik, M. Bejger, and R. Maiïka, Astronomy & Astrophysics 543, A157 (2012).
[18] C. Alcock, E. Farhi, and A. Olinto, The Astrophysical Journal 310, 261 (1986).
[19] K. Rajagopal and F. Wilczek, Physical Review Letters 86, 3492 (2001).
[20] A. Akmal, V. R. Pandharipande, and D. G. Ravenhall, Phys. Rev. C 58, 1804 (1998), nucl-th/9804027.
[21] H. Togashi, K. Nakazato, Y. Takehara, S. Yamamuro, H. Suzuki, and M. Takano, Nucl. Phys. A 961, 78 (2017), arXiv:1702.05324.
[22] G. Baym, S. Furusawa, T. Hatsuda, T. Kojo, and H. Togashi, Astrophys. J. 885, 42 (2019), arXiv:1903.08963.