ON THE METRIC GEOMETRY OF STABLE METRIC SPACES

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Abstract. The metric geometry of stable metric spaces and its application to geometric group theory and noncommutative geometry is discussed. Despite being mostly expository, this article contains some new results and some open problems.

1. Introduction

1.1. Motivation. The idea of finding a good representation of a space into another space with some desirable properties has been around for quite some time and has proven to be extremely powerful, as well as having far reaching applications. This line of thoughts gave rise to a rich and versatile embedding theory. In this article our spaces live in the metric world and we are concerned with the quantitative aspect of the theory. The general theme is finding a metrically faithful copy of a metric space from some given class, inside a Banach space with some nice geometric property. Arguably some of the most famous applications arise in theoretical computer science, geometric group theory and noncommutative geometry. We refer to the excellent [33], [31], [36], [37], [44] for an extensive account on this rapidly growing topic, and a comprehensive list of references.

In this article some applications in geometric group theory and noncommutative geometry will be touched upon. Studying the metric geometry of a canonical metric space associated to a finitely generated group for purely group theoretic purposes was impulsed by Misha Gromov and explained at length in [19]. A finitely generated group can be seen naturally as a metric space when its Cayley graph is equipped with the shortest path metric. This metric space associated to a finitely generated group is discrete and has bounded geometry, and does not depend heavily on the set of generators, in the sense that the two metrics induced by two different sets of generators are bi-Lipschitzly equivalent. Gromov suggested in [20] that a space whose large scale geometry is compatible in a certain sense with the geometry of a super-reflexive Banach space is very likely to satisfy a conjecture of Novikov. Building upon a groundbreaking work of Guoliang Yu [50], Gromov’s intuition was proved to be true by Kasparov and Yu [28]. They showed that if a discrete metric space with bounded geometry admits a coarse embedding into a super-reflexive Banach space, then it satisfies the coarse geometric Novikov conjecture. We refer to [16] for more information on Novikov conjectures. In the late 90’s, the attention was then drew on the coarse geometry of Banach spaces, which had been little considered at that time contrary to its uniform counterpart.

The present article is mainly concerned with designing embeddings into Banach spaces whose geometric behavior mimics as much as possible the behavior of a super-reflexive space. As noticed by Gromov [21] not every (infinite) metric space with bounded geometry, namely an ad-hoc disjoint union of a sequence of expander graphs endowed with a metric inducing the original one on each graph, admits a coarse embedding into a Hilbert space. It is a much more difficult task to exhibit a
metric space with bounded geometry that does not coarsely embed into any super-reflexive Banach space. Two delicate constructions were given by V. Lafforgue (algebraic approach in [30]), and Mendel and Naor (graph theoretic approach in [35]). If the study is restricted to separable spaces, Aharoni's result [1] says that every separable metric space admits a bi-Lipschitz embedding into the separable Banach space $c_0$. This embedding result obviously applies to finitely generated groups. However $c_0$ is far from being close to a super-reflexive space, indeed $c_0$ is not even reflexive! Brown and Guentner [13] showed that there is hope for better behaved host spaces. They proved that a discrete metric space with bounded geometry admits a coarse embedding into a reflexive Banach space. However from the geometric point of view this reflexive space is not quite close to a super-reflexive one. Recall that super-reflexive spaces can be characterized as spaces admitting an equivalent norm which is both uniformly convex and uniformly smooth. We will see in the sequel that the host space can be taken to be the space $(\sum_{n=1}^{\infty} l^n_\infty)_2$ in many interesting situations. The latter space is reflexive, asymptotically uniformly convex and asymptotically uniformly smooth, and is from an asymptotic geometric point of view relatively close to a super-reflexive space.

1.2. Preliminaries. As far as Banach space theory is concerned, we follow the notation and the terminology from [26], [2], and [11] for the most part. We will reserve the use of upper case letters $X$, $Y$, $Z$ (resp. $M$, $N$) to refer to Banach spaces (resp. subsets of Banach spaces). A Banach space will be regarded as a metric space when endowed with its canonical metric induced by its norm. We will refer to metric spaces or their subsets by curly upper case letters $\mathcal{M}$, $\mathcal{N}$, $\mathcal{S}$, $\mathcal{X}$, $\mathcal{Y}$...

We denote by $\text{diam}(\mathcal{M}) := \sup\{d_{\mathcal{M}}(x,y); x, y \in \mathcal{M}\}$ the diameter of a metric space $\mathcal{M}$. A metric space is said to be uniformly discrete, or simply discrete, if there exists a constant $\epsilon \in (0, \infty)$ such that every two points are at least at a distance $\epsilon$ from each other, i.e. $\inf_{s, t \in S, s \neq t} d_{\mathcal{M}}(s, t) \geq \epsilon$. If one wants to emphasize on the parameter $\epsilon$ we will talk about a $\epsilon$-separated metric space. A subset $S$ of a metric space $(\mathcal{M}, d_{\mathcal{M}})$ is called a skeleton with precision $\delta$ and separation constant $\epsilon$ if there exist $0 < \epsilon \leq \delta < \infty$ such that $\mathcal{M}$ is $\epsilon$-separated and $\sup_{x \in \mathcal{M}} d_{\mathcal{M}}(x, S) \leq \delta$. A classical and simple application of Zorn's lemma shows that every non-empty infinite metric space $\mathcal{M}$ has for every $\epsilon \in (0, \text{diam}(\mathcal{M}))$ a skeleton with precision $\epsilon$ and separation constant $2\epsilon$.

A Banach space $X$ is said to be $\epsilon$-crudely finitely representable in another Banach space $Y$ if for every finite-dimensional subspace $F$ of $X$ there exists a finite-dimensional subspace $G$ of $Y$ and a bounded linear map $T: F \to G$ such that $\dim(F) = \dim(G)$ and $\|T\|\|T^{-1}\| \leq 1 + \epsilon$. $X$ is crudely finitely representable in $Y$ if it is $\epsilon_0$-crudely finitely representable for some $\epsilon_0 \in (0, \infty)$, and finitely representable in $Y$ if it is $\epsilon$-crudely finitely representable for every $\epsilon > 0$. To define the ultraproduct of a Banach space we assume some basic knowledge on filters. Given a sequence $(x_n)_{n=1}^{\infty}$ in a topological space and a non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$, we write $\lim_{\mathcal{U}} x_n = \ell$ if for every neighborhood $V$ of $\ell$ the set $\{n \in \mathbb{N}; x_n \in V\}$ belongs to $\mathcal{U}$. If $X$ is a Banach space one define $\ell_\infty(X)$ as the linear space of all bounded sequences in $X$. $\ell_\infty(X)$ equipped with the norm $\| (x_n)_{n=1}^{\infty} \| := \| (\|x_n\|_X)_{n=1}^{\infty} \| = \sup_{n \geq 1} \|x_n\|_X$ is a Banach space. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$, the subspace $c_0(\mathcal{U}) := \{ (x_n)_{n=1}^{\infty} \in \ell_\infty(X); \lim_{\mathcal{U}} \|x_n\|_X = 0\}$ is closed. The ultrafilter of $X$ along the ultrafilter $\mathcal{U}$, denoted $X^{\mathcal{U}}$, is then the quotient space $\ell_\infty(X)/c_0(\mathcal{U})$ equipped with the usual quotient norm, denoted $\|\| \cdot \|_\mathcal{U}$ in the sequel. $X^{\mathcal{U}}$ is a Banach space and the norm of an element $(x_n)_{n=1}^{\infty}$ in $X^{\mathcal{U}}$ is simply given by the formula $\| (x_n)_{n=1}^{\infty} \|_\mathcal{U} := \lim_{\mathcal{U}} \|x_n\|_X$. For a Banach space $X$ we denote by $L_p(\Omega, \mathcal{B}, \mu; X)$ the Banach space of Bochner equivalence classes of Bochner $p$-integrable, $X$-valued
functions defined on the measured space $(\Omega, \mathcal{B}, \mu)$. Let $c_{00}$ be the linear space of sequences of real numbers with finitely many non-zero coordinates. We denote by $e_n$ the vectors in $c_{00}$ whose $n$th coordinate is 1 and all the others are 0. The sequence $(e_n)_{n=1}^{\infty}$ forms a Schauder basis for $\ell_p$, referred to as the canonical basis, and we denote by $(e^*_n)_{n=1}^{\infty}$ the biorthogonal functionals.

2. The class of stable metric spaces

2.1. Definition and examples. The notion of stability was introduced, originally for (separable) Banach spaces, in the work of Krivine and Maurey [29], in an attempt to exhibit a class of Banach spaces with a very regular linear structure. For instance, every stable Banach space contains an isomorphic copy of either $c_0$, or $\ell_p$ for some $p \in [1, \infty)$. It seems that its natural extension to general metric spaces was first studied by Garling [18].

Definition. A metric space $(M, d_M)$ is said to be stable if for any two bounded sequences $(x_n)_{n=1}^{\infty}$, $(y_n)_{n=1}^{\infty}$, and any two non-principal ultrafilters $U, V$ on $\mathbb{N}$, the following equality holds

$$\lim_{m, U} \lim_{n, V} d_M(x_m, y_n) = \lim_{n, V} \lim_{m, U} d_M(x_m, y_n).$$

When we refer to the stability property for a Banach space, we always consider the Banach space as a metric space equipped with its canonical metric induced by its norm. Stability is an isometric property and is inherited by subsets. Note that finite metric spaces are trivially stable and only infinite metric spaces are considered unless otherwise mentioned. Despite being quite restrictive (allowing swapping of limits usually implies strong consequences) we will quickly realize that the class of stable metric spaces is actually quite large and contains lots of interesting spaces. Here is a first example.

Example 1. Proper metric spaces are stable.

A metric space, all of whose closed balls are compact, is called proper. It is easily seen that a proper space is automatically stable. Indeed, assume that $(x_n)_{n=1}^{\infty}$, $(y_n)_{n=1}^{\infty}$ are bounded sequences in a proper metric space $(M, d_M)$. Then $(x_n)_{n=1}^{\infty}$, $(y_n)_{n=1}^{\infty}$ actually belong to a compact subset of $M$, and given any non-principal ultrafilters $U, V$, there exist $x, y \in M$ such that $\lim_{n, U} x_n = x$ and $\lim_{n, V} y_n = y$. By continuity properties of the distance function one has,

$$\lim_{m, U} \lim_{n, V} d_M(x_m, y_n) = d_M(x, y) = \lim_{n, V} \lim_{m, U} d_M(x_m, y_n).$$

There is no shortage of examples of proper metric spaces. Obviously compact metric spaces and finite-dimensional Banach spaces are proper. Since their balls contain only finitely many points, locally finite metric spaces are proper. The same conclusion obviously obviously holds for metric spaces with bounded geometry, since not only each ball has finitely many points, but the number of points in it, is independent of the center and is uniquely a function of the radius. The fact that compactly generated groups are proper, and finitely generated groups have bounded geometry, makes any result about stable metric spaces potentially applicable in geometric group theory or noncommutative geometry.

We just mentioned that finite-dimensional Banach spaces are stable because they are proper. What about infinite-dimensional ones? Remark that it is a classical fact that if a Banach space is proper then it must be finite-dimensional. However it turns out that most of the classical infinite-dimensional Banach spaces are stable. For the sake of simplicity only separable Banach spaces are considered in this article.
Example 2. Hilbert spaces are stable.

It is fairly easy to show that a (separable) Hilbert $H$ space is stable using the classical representation of a Hilbert norm in terms of the scalar product. Assume that $(x_n)_{n=1}^\infty$, $(y_n)_{n=1}^\infty$ are two bounded sequences in $H$. Since $H$ is reflexive the two sequences are weakly-convergent to, let’s say, $x$ and $y$ respectively. Denote $a = \lim_{n \to \infty} \|x_n\|_H^2$ and $b = \lim_{n \to \infty} \|y_n\|_H^2$. Then for any pair of ultrafilters $\mathcal{U}, \mathcal{V}$,

$$\lim_{n, \mathcal{U} \to \infty} \lim_{m, \mathcal{V} \to \infty} \|x_n - y_m\|_H^2 = \lim_{n, \mathcal{U} \to \infty} \lim_{m, \mathcal{V} \to \infty} (\|x_n\|_H^2 + \|y_m\|_H^2 - 2 < x_n, y_m >)$$

$$= \lim_{n, \mathcal{U} \to \infty} \|x_n\|_H^2 + \lim_{m, \mathcal{V} \to \infty} \|y_m\|_H^2 - 2 < x, y >$$

$$= a + b - 2 < x, y > .$$

Proceeding in a similar manner one shows that

$$\lim_{m, \mathcal{V} \to \infty} \lim_{n, \mathcal{U} \to \infty} \|x_n - y_m\|_H^2 = a + b - 2 < x, y > .$$

Example 3. The Banach spaces $\ell_p$, for $p \in [1, \infty)$ are stable.

There are many different arguments to support this claim. It is a (somewhat tedious) exercise to show that for $p \in [1, \infty)$, the $\ell_p$-sum of stable Banach spaces is stable, and one can argue that $\ell_p$ is by definition an $\ell_p$-sum of the proper, albeit stable, space ($\mathbb{R}, | \cdot |$). We sketch a beautiful argument (see [18]) based on the following lemma.

Lemma 1. Let $(x_n)_{n=1}^\infty$ be a bounded sequence in $\ell_p$, and $\mathcal{U}$ a non-principal ultrafilter on $\mathbb{N}$. Suppose that $\lim_{n \to \infty} e_{\mathcal{U}}^i(x_n) = 0$ for all $i \in \mathbb{N}$, i.e. $(x_n)_{n=1}^\infty$ converges coordinatewise to 0 with respect to $\mathcal{U}$. Then, the following formula is valid for any $x \in \ell_p$

$$\lim_{n, \mathcal{U} \to \infty} \|x + x_n\|^p = \|x\|^p + \lim_{n, \mathcal{U} \to \infty} \|x_n\|^p .$$

The conclusion of the lemma clearly holds if $x$ and the $x_n$’s have disjoint supports. Since $(x_n)_{n=1}^\infty$ converges coordinatewise to 0 by reaching far out in the sequence one can select elements whose supports are essentially disjoint with the support of $x$. Assume that $\sigma(x) = \lim_{n, \mathcal{U} \to \infty} \|x - y_n\|_p$ for some bounded sequence $(y_n)_{n=1}^\infty$ and some ultrafilter $\mathcal{U}$, then there exist $y \in X$ and $\mu \geq 0$ so that $\sigma(x)^p = \|x - y\|^p + \mu^p$. Indeed, there exists $y \in \ell_p$ such that for all $i \in \mathbb{N}$ $\lim_{n \to \infty} e_{\mathcal{U}}^i(y_n) = e_{\mathcal{U}}^i(y)$ and the conclusion follows from Lemma 1 with $(x_n)_{n=1}^\infty = (y - y_n)_{n=1}^\infty$, and $\mu = \lim_{n, \mathcal{U} \to \infty} \|y - y_n\|$. We are now in position to prove that $\ell_p$ is stable. For a pair of bounded sequences $(x_n)_{n=1}^\infty$, $(y_n)_{n=1}^\infty$, and a pair of non-principal ultrafilters $\mathcal{U}, \mathcal{V}$ on $\mathbb{N}$, there exist $x, y \in X$, $\mu, \nu \geq 0$ such that for all $z \in X$ the following equalities hold:

$$\lim_{n, \mathcal{U} \to \infty} \|z - x_n\|^p = \|z - x\|^p + \mu^p$$

and

$$\lim_{n, \mathcal{V} \to \infty} \|z - y_n\|^p = \|z - y\|^p + \nu^p .$$

It follows that

$$\lim_{n, \mathcal{U} \to \infty} \lim_{m, \mathcal{V} \to \infty} \|x_n - y_m\|^p = \|x - y\|^p + \mu^p + \nu^p = \lim_{n, \mathcal{U} \to \infty} \lim_{m, \mathcal{V} \to \infty} \|x_n - y_m\|^p .$$

Example 4. The Banach spaces $L_p[0, 1]$, for $p \in [1, \infty)$ are stable.

The stability of $L_p$-spaces follows from an important result of Krivine and Maurey [20].

Theorem 2. For any $p \in [1, \infty)$, if $X$ is a stable Banach space then $L_p(\mathbb{R}; X)$ is also stable.
The proof relies on a specific representation of the norm. An ingredient of the proof is Theorem 20 recall in Section 6.2. However, there is a shorter argument for \( p \in [1, 2] \) which reduces to the stability of Hilbert spaces. For \( s \in (0, 1) \), the metric space \((M, d_M^s)\) is commonly called the \( s \)-snowflaking of \( M \). It is clear that a snowflaking of a metric space is stable if and only if the original metric on the space is stable. For \( p \in [1, 2] \), \( L_p\)-spaces are stable since it is well known that the \( \frac{1}{2} \)-snowflaking of \( L_2[0, 1] \) embeds isometrically into \( L_2[0, 1] \).

We now present the most classical example of a space failing the stability property.

Example 5. \( c_0 \) is not stable. Invoking James \( c_0 \)-distortion theorem, one can actually show that no Banach space linearly isomorphic to \( c_0 \) is stable. Indeed, if \( X \) is linearly isomorphic to \( c_0 \) then \( X \) contains for every \( \epsilon > 0 \) a subspace that is \((1 + \epsilon)\)-isomorphic to \( c_0 \), e.g. a subspace \( Y \) that is \( \frac{1}{2} \)-isomorphic to \( c_0 \). Let \( (c_n)_{n=1}^\infty \) be the canonical basis of \( c_0 \), and \( T \) an isomorphism from \( c_0 \) onto \( Y \) such that, say, \( ||T|| \leq \frac{3}{2} \) and \( ||T^{-1}|| \leq 1 \). Take \( x_n = c_n \) and \( y_n = \sum_{i=1}^n \epsilon_i \). These two sequences, along with their images in \( Y \), are clearly bounded, but

\[
\lim_{m, N \rightarrow \infty} \sup_{t \in I_d} ||T(x_n) - T(y_m)|| \leq \frac{3}{2} \lim_{m, N \rightarrow \infty} \parallel x_n - y_m \parallel_{\infty} = \frac{3}{2}
\]

while

\[
\lim_{n, d \rightarrow \infty} \sup_{m, N \rightarrow \infty} ||T(x_n) - T(y_m)|| \geq \lim_{n, d \rightarrow \infty} \parallel x_n - y_m \parallel_{\infty} = 2.
\]

We will give more examples in Section 6.

3. Various notions of metric faithfulness

Let \((M, d_M)\) and \((N, d_N)\) be two metric spaces and \( f : M \rightarrow N \). Define

\[\rho_f(t) = \inf\{d_N(f(x), f(y)) : d_M(x, y) \geq t\},\]

and

\[\omega_f(t) = \sup\{d_N(f(x), f(y)) : d_M(x, y) \leq t\}.\]

The map \( f \) is called an embedding of \( M \) into \( N \) and for every \( x, y \in M \),

\[\rho_f(d_M(x, y)) \leq d_N(f(x), f(y)) \leq \omega_f(d_M(x, y)).\]

The moduli \( \rho_f \) and \( \omega_f \) will be respectively called the compression modulus and the expansion modulus of the embedding. The expansion modulus is nothing else but the modulus of uniform continuity when \( f \) is a uniformly continuous map. In this article we will restrict our attention to embeddings whose moduli have some prescribed global or asymptotic behaviors. In the remainder of this article the metric spaces are implicitly assumed to be unbounded unless otherwise specified.

3.1. Embeddings preserving a prescribed geometry. We study certain natural geometric structures of a metric space, simply called geometries, through the prism of metric embeddings. For instance, the Lipschitz geometry accounts for the geometry at all scales and the behavior of bi-Lipschitz embeddings. Coarse or uniform geometry is the geometry dealing respectively with the large-scale or small-scale structure of a metric space. We recall the formal definitions of the embeddings naturally related to these geometries. For (non uniformly discrete) metric spaces \( M \) and \( N \), the embedding \( f \) is said to be a uniform embedding if \( \lim_{t \rightarrow 0^+} \omega_f(t) = 0 \) and \( \rho_f(t) > 0 \) for all \( t > 0 \). This type of embedding is designed to described the microscopic structure of \( M \) since only the behavior of \( f \) with respect to pairs of
points whose distances to each other are small is taking into account. It can be seen as a quantitative version of a topological embedding when the spaces carry a metric structure. If the domain and the target spaces are two unbounded metric spaces, Gromov introduced the notion of coarse embedding. A coarse embedding is a map \( f : \mathcal{M} \to \mathcal{N} \) such that \( \lim_{t \to \infty} \rho_f(t) = \infty \) and \( \omega_f(t) < \infty \) for all \( t > 0 \). An embedding which is simultaneously coarse and uniform shall be called a strong embedding.

A classical example of strong embedding is a bi-Lipschitz embedding, i.e. a map \( f : \mathcal{M} \to \mathcal{N} \) such that there exist a scaling factor \( s \in (0, \infty) \) and a constant \( D \in [1, \infty) \), and for all \( x, y \in \mathcal{M} \)

\[
s \cdot d_\mathcal{M}(x, y) \leq d_\mathcal{N}(f(x), f(y)) \leq D \cdot s \cdot d_\mathcal{M}(x, y).
\]

A coarse bi-Lipschitz embedding is an embedding which is bi-Lipschitz up to some additive constants, i.e. such that for all \( x, y \in \mathcal{M} \) the inequalities

\[
\frac{1}{A} d_\mathcal{M}(x, y) - B \leq d_\mathcal{N}(f(x), f(y)) \leq Ad_\mathcal{M}(x, y) + B
\]

hold for some constants \( A \in [1, \infty) \) and \( B \in (0, \infty) \).

In geometric group theory the different terminology quasi-isometric embedding, relative to the same notion, is well established and actually predates the terminology above. However in nonlinear Banach space theory it is customary to say that a space quasi-isometrically embeds if for every \( \epsilon > 0 \) there exists a bi-Lipschitz embedding with distortion at most \( 1 + \epsilon \). In the remainder of this article we shall prefer to use the term “quasi-isometric” when groups are involved and “coarse bi-Lipschitz” when treating the case of general metric spaces. The possibility of incorporating a positive constant \( B \) allows non injective maps, for instance maps collapsing all points in some ball towards the center, and the small-scale structure might eventually be lost in the process. The notion is not relevant for bounded metric spaces, since an embedding sending all the points of a bounded space onto a single point of the target space is a quasi-isometric embedding! Let us recall a well-known fact that will be used repeatedly.

**Lemma 3.** Let \( (\mathcal{M}, d_\mathcal{M}) \) be a metric space. For any \( \delta > 0 \) and \( \mathcal{S} \) a skeleton of \( \mathcal{M} \) with precision \( \delta \), there exists a map \( c : \mathcal{M} \to \mathcal{S} \) such that

\[
d_\mathcal{M}(x, y) - 2\delta \leq d_\mathcal{S}(c(x), c(y)) \leq d_\mathcal{M}(x, y) + 2\delta.
\]

Note that an easy application of Lemma 3 shows that the concepts of coarse bi-Lipschitz embeddability and bi-Lipschitz embeddability for large distances are equivalent. An embedding is said to be bi-Lipschitz for large distances if there exist \( f : \mathcal{M} \to \mathcal{N}, \tau \in (0, \infty), \) and \( A_\tau \in [1, \infty) \) such that \( d_\mathcal{M}(x, y) \geq \tau \) implies that

\[
\frac{1}{A_\tau} d_\mathcal{M}(x, y) \leq d_\mathcal{N}(f(x), f(y)) \leq A_\tau d_\mathcal{M}(x, y).
\]

In order to deal with families of metric spaces and embeddings some more terminology needs to be introduced. A family of metric spaces \( (\mathcal{M}_i)_{i \in I} \) is said to admit an equi-uniform embedding into \( \mathcal{N} \) if there exist \( f_i : \mathcal{M}_i \to \mathcal{N} \), and \( \rho, \omega : [0, \infty) \to [0, \infty) \) non-decreasing with \( \lim_{t \to 0} \omega(t) = 0 \) and \( \rho(t) > 0 \) for all \( t > 0 \), such that \( \rho \leq \rho_{f_i} \leq \omega_{f_i} \leq \omega \) for all \( i \in I \). In a similar fashion one can define equi-coarse embeddability for an unbounded family of metric spaces, i.e. such that \( \sup_{i \in I} \operatorname{diam}(\mathcal{M}_i) = \infty \). A (unbounded) family of metric spaces \( (\mathcal{M}_i)_{i \in I} \) is said to be equi-coarsely embeddable into \( \mathcal{N} \) if there exist \( f_i : \mathcal{M}_i \to \mathcal{N} \), and \( \rho, \omega : [0, \infty) \to [0, \infty) \) non-decreasing with \( \lim_{t \to \infty} \rho(t) = \infty \), such that for all \( x, y \in \mathcal{M}_i \)

\[
\rho(d_\mathcal{M}_i(x, y)) \leq d_\mathcal{N}(f_i(x), f_i(y)) \leq \omega(d_\mathcal{M}_i(x, y)).
\]
Let property be your favorite embedding type (coarse, uniform...). Following [15], we will say that a metric space \( M \) admits a local property-embedding, or is locally property embeddable, into \( N \) if the collection of its finite subsets admits an equi-property embedding into \( N \).

3.2. Quantifying metric faithfulness. We are concerned in this article with a precise quantitative analysis of various embeddings between certain classes of metric spaces. In order to differentiate two embeddings in the same class we use what is called a \([\rho, \omega]\)-embedding from \( M \) into \( N \), i.e. a map \( f: M \to N \) such that for all \( x, y \in M \),

\[
\rho(d_M(x, y)) \leq d_N(f(x), f(y)) \leq \omega(d_M(x, y)).
\]

A deformation gap of \( M \) in \( N \) will be a pair of functions \([\rho, \omega]\) such that there exists a \([\rho, \omega]\)-embedding from \( M \) into \( N \). We can then define coarse deformation gaps, uniform deformation gaps or strong deformation gaps, by assuming that the functions \( \rho, \omega \) satisfy some additional requirements. For instance, we will say that \([\rho, \omega]\) is a coarse deformation gap of \( M \) into \( N \) if there exist a map \( f: M \to N \) and \( \rho, \omega: [0, \infty) \to [0, \infty) \) non-decreasing with \( \lim_{t \to \infty} \rho(t) = \infty \), and such that for all \( x, y \in M \)

\[
\rho(d_M(x, y)) \leq d_N(f(x), f(y)) \leq \omega(d_M(x, y)).
\]

In the same spirit, we say that \( M \) embeds into \( N \) with strong deformation gap \([\rho, \omega]\) if there exists a \([\rho, \omega]\)-embedding from \( M \) into \( N \) with \( \lim_{t \to \infty} \rho(t) = \infty \), \( \lim_{t \to 0} \omega(t) = 0 \), \( \rho(t) > 0 \) and \( \omega(t) < \infty \) for all \( t > 0 \).

For two functions \( g, h: \mathbb{R} \to \mathbb{R} \) we write \( g \ll h \) if there exist \( a, b, c > 0 \) such that \( g(t) \leq ah(bt) + c \) for every \( t \in \mathbb{R} \). If \( g \ll h \) and \( h \ll g \) then we write \( g \asymp h \). It is an equivalence relation and a function shall be identified with its equivalence class in the sequel. Following [4] one says that \([\rho_1, \rho_2]\) is a compression gap of \( M \) in \( N \) if there exists a \([\rho_1, t]\)-coarse embedding of \( M \) into \( N \) and for every \([\rho, t]\)-coarse embedding of \( M \) into \( N \) one has \( \rho \ll \rho_2 \). If \( \rho_1 \asymp \rho_2, \rho_1 \) is called the compression function of \( M \) in \( N \). In particular if \( \alpha_N(M) \) is the supremum of all numbers \( \alpha \) such that \( t^\alpha \ll \rho \) and \( \rho \) is the compression function of \( M \) in \( N \), then \( \alpha_N(M) \) is the compression exponent of \( M \) in \( N \) (\( N \)-compression of \( M \) in short) introduced by Guentner and Kaminker [22]. In other words, \( \alpha_N(M) \) is the supremum of all numbers \( 0 \leq \alpha \leq 1 \) so that there exist \( f: M \to N, \tau \in (0, \infty), \) and \( A_\tau \in [1, \infty) \) such that \( d_M(x, y) \geq \tau \) implies

\[
\frac{1}{A_\tau} d_M(x, y)^\alpha \leq d_N(f(x), f(y)) \leq A_\tau d_M(x, y).
\]

The compression exponent of an (unbounded) metric space is clearly invariant under coarse bi-Lipschitz embeddings.

3.3. Sidenotes on the terminology. Kalton introduced the terminology “strong uniform embedding” in [27] for what we call a “strong embedding”. We suppose that Kalton’s terminology was motivated from the fact that in Banach space theory uniform embeddings have been studied long before their coarse analogues. However we believe that coarse and uniform embeddings are equally important. We do not see any reason why an embedding which is simultaneously coarse and uniform should be called a strong uniform embedding rather than a strong coarse embedding. A sequence of equi-coarse embeddings in our terminology is called a sequence of uniform coarse embeddings by Ostrovskii [14]. We follow the lead of Dadarlat and Guentner [14] when making the choice of using the prefix “equi” in order to be consistent with the well-known notion of “equi-uniformly continuous maps” and to avoid some cumbersome terminology. For quite some time, in the
geometric group theory and noncommutative geometry communities, a “coarse embedding” was simply called a “uniform embedding”, a shorter version of Gromov’s original terminology “uniform embedding at infinity” [19], but it seems that these communities are now leaning towards the utilization of the terminology “coarse embedding”.

4. Embeddability of locally finite metric spaces and consequences

Locally finite metric spaces are stable since they are proper in a very strong sense (their balls consist of only finitely many elements). A locally finite metric space is clearly countable, and hence admits a bi-Lipschitz embedding into $c_0$, like any other separable metric space. It seems that the first positive embedding result specifically tailored for a particular type of locally finite metric spaces, namely spaces with bounded geometry, is due to Brown and Guentner [13]. They proved that every metric space with bounded geometry admits a coarse embedding into the Banach space $(\sum_{n=1}^{\infty} \ell_{p_n})_2$, with $(p_n)_{n=1}^{\infty}$ any sequence satisfying for all $n \in \mathbb{N}$, $p_n \in (1, \infty)$ and $\lim_{n \to \infty} p_n = \infty$. The space $(\sum_{n=1}^{\infty} \ell_{p_n})_2$ has some nice features, such as being reflexive, but as already mentioned in the introduction is not quite close to a super-reflexive space. Brown and Guentner embedding was then strengthened and extended to locally finite metric spaces in a series of articles [11], [12], [8], [42], [6].

This line of research culminated with Mikhail Ostrovskii proving in [43] that the bi-Lipschitz embeddability of a locally finite metric space into a Banach space can be checked on finite subsets. We will tell the story backwards since we believe that is the right way to proceed. Indeed, all the results predating [43] can now be seen as (more or less straightforward) consequences of Theorem 4 below, which is nothing else but a reformulation of Ostrovskii’s result in a slightly more general context.

The core of the proof, is a gluing technique which was first introduced by the author in [5], and its powerfulness vis-à-vis embeddability of locally finite metric spaces became crystal clear in [8]. It was then implemented in various situations in [13] and [6], [9] and even in [7] in a non locally finite setting. Perhaps its most striking utilization featured in the proof of Theorem 4. The gluing technique is a local-to-global procedure which enables us, given a collection of local pieces which cover a space, and given a collection of local embeddings relative to those local pieces, to past the local embeddings together in order to obtain a global embedding of the whole space. The proof of Theorem 4 requires two additional rather technical arguments that take care of structural properties of infinite-dimensional Banach spaces. We sketch the proof of Theorem 4 and show how to implement the gluing technique.

**Theorem 4.** Let $X$ and $Y$ be two Banach spaces such that $X$ is crudely finitely representable in $Y$. Let $M$ be a locally finite subset of $X$, then $M$ admits a bi-Lipschitz embedding into $Y$.

**Sketch of Proof.** We divide the proof in two steps. The first one explains why $M$ is naturally a locally bi-Lipschitz embeddable into $Y$. The second one describes the gluing technique.

**Local embeddability:** Let $B_n = \{ x \in M; \|x\|_X \leq 2^{n+1} \}$. It is clear that $B_n$ contains finitely many elements and $B_n$ can be seen as a subset of the finite-dimensional space $\text{span}(B_n) := F_n \subset X$. Since $X$ is crudely finitely represented in $Y$, for some universal $\varepsilon_0 > 0$ there exist $G_n$ a subspace of $Y$ and a linear map $T_n: F_n \to G_n$ such that $\|T\|\|T^{-1}\| \leq 1 + \varepsilon_0$. Therefore, there exist maps $f_n: B_n \to Y$ such that $f_n(0) = 0$ and for all $x, y \in B_n$

$$1) \quad \|x - y\|_X \leq \|f_n(x) - f_n(y)\|_Y \leq (1 + \varepsilon_0)\|x - y\|_X.$$
Since the sequence \( (B_n)_{n=1}^\infty \) is increasing, any subsequence \( (f_{k_n})_{n=1}^\infty \) of \( (f_n)_{n=1}^\infty \) still maps \( (B_n)_{n=1}^\infty \) into \( Y \) and satisfies (1). For some technical reasons we want the maps \( f_n \) to satisfy some additional properties. Extracting as in [43] we can find a collection of vectors \( (f_m(x))_{x \in X} \) and a collection of linear forms such that:

- If for some \( x, y \in B_n \) and some \( m \geq n \) the vector \( f_m(x) - f_m(y) - (f_\omega(x) - f_\omega(y)) \) is nonzero, there exists \( g_{m,x,y} \in Y^* \) of norm 1 such that

\[
\tag{2} g_{m,x,y}(f_m(x) - f_m(y) - (f_\omega(x) - f_\omega(y))) \geq \frac{99}{100} \| f_m(x) - f_m(y) - (f_\omega(x) - f_\omega(y)) \|,
\]

and

\[
\tag{3} \text{for all } p > m, \ |g_{m,x,y}(f_p(x) - f_p(y) - (f_\omega(x) - f_\omega(y)))| \leq \frac{1}{1000} \|x - y\|.
\]

- If \( f_\omega(x) \neq f_\omega(y) \) for \( x, y \in B_n \), there exists \( h_{x,y} \in Y^* \) of norm 1 such that

\[
\tag{4} h_{x,y}(f_\omega(x) - f_\omega(y)) \geq \frac{99}{100} \| f_\omega(x) - f_\omega(y) \|
\]

and

\[
\tag{5} \text{for all } m \geq n, \ |h_{x,y}(f_m(x) - f_m(y) - (f_\omega(x) - f_\omega(y)))| \leq \frac{1}{100} \| f_\omega(x) - f_\omega(y) \|.
\]

**Remark.** For each \( x \in B_n \) the vector \( f_\omega(x) \) is obtain as a certain weak-\(*\)-limit of the sequence \( (f_n(x))_{n=1}^\infty \).

**From local to global embeddability:**

The definition of the following map is what we referred to as the gluing technique.

\[
f : M \rightarrow Y
\]

\[
x \mapsto \lambda_x f_n(x) + (1 - \lambda_x) f_{n+1}(x), \text{ if } 2^n \leq \|x\| \leq 2^{n+1}
\]

and where

\[
\lambda_x = \frac{2^{n+1} - \|x\|}{2^n}.
\]

We first show that \( f \) is a Lipschitz map. Let \( x, y \in M \) and assume, as we may, that \( \|x\| \leq \|y\| \). Then, let

\[
\lambda_x = \frac{2^{n+1} - \|x\|}{2^n} \quad \text{and} \quad \lambda_y = \frac{2^{n+2} - \|y\|}{2^{n+1}}.
\]

We have that,

\[
\lambda_x \leq \frac{\|x - y\|}{2^{n+1}}, \text{ so } \lambda_x \|x\| \leq 2\|x - y\|.
\]

Similarly,

\[
1 - \lambda_y = \frac{\|y\| - 2^{n+1}}{2^{n+1}} \leq \frac{\|x - y\|}{2^{n+1}} \quad \text{and} \quad (1 - \lambda_y) \|y\| \leq 2\|x - y\|.
\]

**Case 1.** \( \|x\| \leq \frac{1}{2} \|y\| \)

Then,

\[
\|f(x) - f(y)\| \leq (1 + \varepsilon_0)(\|x\| + \|y\|) \leq \frac{3(1 + \varepsilon_0)}{2} \|y\| \leq 3(1 + \varepsilon_0)(\|y\| - \|x\|) \leq 3(1 + \varepsilon_0)\|x - y\|.
\]

**Case 2.** \( \frac{1}{2} \|y\| < \|x\| \leq \|y\| \)

Consider two subcases.

**Case 2.a.** \( 2^n \leq \|x\| \leq \|y\| < 2^{n+1} \), for some \( n \).

Then, let

\[
\lambda_x = \frac{2^{n+1} - \|x\|}{2^n} \quad \text{and} \quad \lambda_y = \frac{2^{n+1} - \|y\|}{2^n}.
\]
We have that
\[ |\lambda_x - \lambda_y| = \frac{\|y\| - \|x\|}{2^n} \leq \frac{\|x - y\|}{2^n}. \]
so
\[
\|f(x) - f(y)\| = \|\lambda_x f_n(x) - \lambda_y f_n(y) + (1 - \lambda_x)f_{n+1}(x) - (1 - \lambda_y)f_{n+1}(y)\|
\]
\[
\leq \lambda_x \|f_n(x) - f_n(y)\| + (1 - \lambda_x)\|f_{n+1}(x) - f_{n+1}(y)\|
+ 2(1 + \varepsilon_0)|\lambda_x - \lambda_y|\|y\|
\leq (1 + \varepsilon_0)\|x - y\| + 2^{n+2}(1 + \varepsilon_0)|\lambda_x - \lambda_y|
\leq 5(1 + \varepsilon_0)\|x - y\|.
\]

Case 2.b. \(2^n \leq \|x\| < 2^{n+1} \leq \|y\| < 2^{n+2}, \) for some \( n \in \mathbb{Z}. \)
\[
\|f(x) - f(y)\| = \|\lambda_x f_n(x) + (1 - \lambda_x)f_{n+1}(x) - \lambda_y f_{n+1}(y) - (1 - \lambda_y)f_{n+2}(y)\|
\]
\[
\leq \lambda_x (\|f_n(x)\| + \|f_{n+1}(x)\|) + (1 - \lambda_x)(\|f_{n+1}(y)\|
+ \|f_{n+2}(y)\|) + \|f_{n+1}(x) - f_{n+1}(y)\|
\leq 2(1 + \varepsilon_0)\lambda_x \|x\| + 2(1 + \varepsilon_0)(1 - \lambda_y)\|y\| + (1 + \varepsilon_0)\|x - y\|
\leq 9(1 + \varepsilon_0)\|x - y\|.
\]

We have shown that \( f \) is \((1 + \varepsilon_0)\)-Lipschitz. In \([8]\) the map defined using the gluing technique is already bi-Lipschitz. In the current situation it is not clear whether or not \( f \) is actually bi-Lipschitz but it is possible to slightly modify \( f \) in order to produce an embedding which is bi-Lipschitz. The proof of the following technical lemma can be found in \([43]\).

**Lemma 5.** There exist a 1-Lipschitz map \( \tau : [0, \infty) \to Y \) and \( c \in (0, \infty) \) such that if \( \tilde{f} : M \to Y \) is given by \( \tilde{f}(x) = \tau(\|x\|) + f(x) \) then for every \( x , y \in M , \)
\[
\|\tilde{f}(x) - \tilde{f}(y)\| \geq c(\|y\| - \|x\|).
\]

Note that \( \tilde{f} \) is also Lipschitz since
\[
\|\tilde{f}(x) - \tilde{f}(y)\| = \|\tau(\|x\|) + f(x) - \tau(\|y\|) - f(y)\|
\leq \|\tau(\|x\|) - \tau(\|y\|)\| + \|f(x) - f(y)\|
\leq \|\|x\| - \|y\|\| + 9(1 + \varepsilon_0)\|x - y\|
\leq 10(1 + \varepsilon_0)\|x - y\|.
\]

We now estimate the quantity \( \|\tilde{f}(x) - \tilde{f}(y)\| \) from below with the help of the linear forms of norm 1 satisfying inequalities \( 2 \) and \( 3 \), or \( 4 \) and \( 5 \).

Case 1. \(2^n \leq \|x\| \leq \|y\| < 2^{n+1}, \) for some \( n \in \mathbb{Z}. \)

It is clear from the property of \( \tilde{f} \) that there is nothing to prove if \( \|y\| - \|x\| \geq \frac{\|x - y\|}{1000(1 + \varepsilon_0)}. \) From now on we assume that \( \|y\| - \|x\| \leq \frac{\|x - y\|}{1000(1 + \varepsilon_0)}. \)
Subcase 1.a. \( \| f_\omega(x) - f_\omega(y) \| \geq \frac{1}{100} \| x - y \| \)

\[
f(x) - f(y) = f_\omega(x) - f_\omega(y) + \lambda_x (f_n(x) - f_n(y) - (f_\omega(x) - f_\omega(y))) + (1 - \lambda_x) (f_{n+1}(x) - f_{n+1}(y) - (f_\omega(x) - f_\omega(y))) + (\lambda_x - \lambda_y) f_n(y) + (\lambda_y - \lambda_x) f_{n+1}(y)
\]

and hence

\[
\| f(x) - f(y) \| \geq h_{x,y}(f(x) - f(y)) \\
\geq h_{x,y}(f_\omega(x) - f_\omega(y)) \\
+ \lambda_x h_{x,y}(f_n(x) - f_n(y) - (f_\omega(x) - f_\omega(y))) \\
+ (1 - \lambda_x) h_{x,y}(f_{n+1}(x) - f_{n+1}(y) - (f_\omega(x) - f_\omega(y))) \\
+ (\lambda_x - \lambda_y) h_{x,y}(f_n(y)) + (\lambda_y - \lambda_x) h_{x,y}(f_{n+1}(y)) \\
\geq \frac{99}{100} \| f_\omega(x) - f_\omega(y) \| - \lambda_x \frac{1}{100} \| f_\omega(x) - f_\omega(y) \| \\
- (1 - \lambda_x) \frac{1}{100} \| f_\omega(x) - f_\omega(y) \| - 2|\lambda_x - \lambda_y|(1 + \varepsilon_0)\|y\| \\
\geq \frac{98}{98} \frac{98}{10000} \frac{4(1 + \varepsilon_0)}{10000} \| x - y \| \\
\geq \frac{48}{10000} \| x - y \|
\]

Subcase 1.b. \( \| f_\omega(x) - f_\omega(y) \| < \frac{1}{100} \| x - y \| \)

Subcase 1.b.1. \( \lambda_x \| f_n(x) - f_n(y) - (f_\omega(x) - f_\omega(y)) \| \geq \frac{1}{100} \| x - y \| \)

\[
\| f(x) - f(y) \| \geq g_{n,x,y}(f(x) - f(y)) \\
\geq \lambda_x g_{n,x,y}(f_n(x) - f_n(y) - (f_\omega(x) - f_\omega(y))) \\
+ g_{n,x,y}(f_\omega(x) - f_\omega(y)) \\
+ (1 - \lambda_x) g_{n,x,y}(f_{n+1}(x) - f_{n+1}(y) - (f_\omega(x) - f_\omega(y))) \\
+ (\lambda_x - \lambda_y) g_{n,x,y}(f_n(y)) + (\lambda_y - \lambda_x) g_{n,x,y}(f_{n+1}(y)) \\
\geq \frac{99}{10000} \frac{4}{10000} \frac{84}{10000} \| x - y \| \\
\geq \frac{4}{10000} \| x - y \|
\]

Subcase 1.b.2. \( \lambda_x \| f_n(x) - f_n(y) - (f_\omega(x) - f_\omega(y)) \| < \frac{1}{100} \| x - y \| \)

Remark that for any \( k \), in particular for \( k = n \) or \( n + 1 \), one has

\[
\| f_k(x) - f_k(y) - (f_\omega(x) - f_\omega(y)) \| \geq \| f_k(x) - f_k(y) \| - \| f_\omega(x) - f_\omega(y) \| \\
\geq \| x - y \| - \| x - y \| \frac{100}{100} \\
\geq \frac{99}{100} \| x - y \|.
\]
and hence $\lambda_x < \frac{10}{99}$ and $1 - \lambda_x > \frac{89}{99}$.

$$
\|f(x) - f(y)\| \geq \frac{89}{99} \sqrt{2} \|x - y\| - \frac{10}{99}\|x - y\| - \frac{1}{10}\|x - y\| - \frac{4}{1000}\|x - y\|
$$

Case 2. $2^n \leq \|x\| < 2^{n+1} \leq \|y\| < 2^{n+2}$, for some $n \in \mathbb{Z}$.

$$
\|f(x) - f(y)\| \geq \frac{99}{1000}\|x - y\| - \frac{99}{100}\|x - y\| - 8(1 + \varepsilon_0)\|y\| - \|x\|
$$

Case 3. $2^n \leq \|x\| < 2^{n+1} < 2^p \leq \|y\| < 2^{n+1}$, for some $n, p \in \mathbb{Z}$, with $p \geq n + 2$.

It is enough to notice that in this case $\|y\| - \|x\| \geq \frac{\|x - y\|}{2}$ since

$$
\frac{\|y\|}{2} \leq \|y\| - \|x\| \leq \|x - y\| \leq \|x\| + \|y\| \leq 2\|y\|.
$$

Any ultrapower of a Banach space $X$ is finitely represented in the space $X$ itself.

**Corollary 6.** *Let $X$ be a Banach space, and $U$ be any non-principal ultrafilter, then every locally finite subset of $X^U$ admits a bi-Lipschitz embedding into $X$.*

We will say that the bi-Lipschitz embeddability of a metric space $\mathcal{M}$ into another metric space $\mathcal{N}$ is finitely determined if $\mathcal{M}$ admits a bi-Lipschitz embedding into $\mathcal{N}$ whenever $\mathcal{M}$ is locally bi-Lipschitz embeddable into $\mathcal{N}$. We can define define a similar notion in a natural way for coarse or uniform embeddings. Remark that for locally finite spaces it is sufficient to consider a sequence of balls, all centered at some arbitrary point, whose sequence of radii is increasing and unbounded. It is a well-known fact that the uniform embeddability of a metric space into a Hilbert space is finitely determined (c.f. [11] Proposition 8.12.). The coarse analogue was proved by Dranishnikov, Gong, V. Lafforge and Yu in [15]. Nowak showed in [10] that the same conclusion holds when the Hilbert space is substituted with any $L_p$-space.

**Corollary 7.** *The coarse embeddability of a locally finite metric space into an infinite dimensional Banach space is finitely determined.*

**Proof.** Let $\mathcal{M}$ be locally finite, and $Y$ be an infinite-dimensional Banach space. Fix $t_0 \in \mathcal{M}$ and assume that $\mathcal{M}$ is locally coarsely embeddable into $Y$. In particular, there exist two non-decreasing functions $\rho, \omega: [0, \infty) \to [0, \infty)$, satisfying $\rho(t) \to \infty$, $\omega(t) < \infty$, and for every $n \in \mathbb{N}$ there exists $\varphi_n: B(t_0, 2^n) \to Y$ such that for every $x, y \in B(t_0, 2^n)$,

$$
\rho(d_{\mathcal{M}}(x, y)) \leq \|f_n(x) - f_n(y)\|_Y \leq \omega(d_{\mathcal{M}}(x, y)).
$$

Define the map
ON THE METRIC GEOMETRY OF STABLE METRIC SPACES.

Let \( \varphi: \mathcal{M} \to Y^{\mathcal{U}} \)
\[
x \mapsto (0, \ldots, 0, \varphi_k(x), \varphi_{k+1}(x), \ldots), \text{ if } 2^k - 1 \leq d_{\mathcal{M}}(x, t_0) \leq 2^k \text{ for some } k \in \mathbb{N}.
\]

By definition of the ultraproduct norm it is clear that for every \( x, y \in \mathcal{M} \),
\[
\rho(d_{\mathcal{M}}(x, y)) \leq \|f(x) - f(y)\|_{Y^{\mathcal{U}}} \leq \rho(d_{\mathcal{M}}(x, y)).
\]
Since, local finiteness is preserved under coarse embeddings, \( f(\mathcal{M}) \) is a locally finite subset of \( Y^{\mathcal{U}} \). The composition of a coarse embedding and a bi-Lipschitz embedding being a coarse embedding, the conclusion follows from Theorem 4.  

If the sequence \( (\varphi_k)_{k=1}^{\infty} \) in the proof above is actually a sequence of equi-bi-Lipschitz embeddings then \( \varphi \) is a bi-Lipschitz embedding.

**Corollary 8.** The bi-Lipschitz embeddability of a locally finite metric space into an infinite dimensional Banach space is finitely determined.

Corollary 6, Corollary 7, Corollary 8 can all be found in [43]. The next consequence of Theorem 4 was originally proven by the author in [5].

**Corollary 9.** A Banach space \( X \) is not superreflexive if and only if the infinite binary tree admits a bi-Lipschitz embedding into \( X \).

*Proof.* It is a deep result of Bourgain [12] that a Banach space \( X \) is not superreflexive if and only if the infinite binary tree is locally bi-Lipschitz embeddable into \( X \). It remains to remark that the infinite binary tree is a locally finite metric space. \( \square \)

By definition a \( L_p \)-space is crudely finitely represented into any Banach space containing the \( \ell_p^n \)'s uniformly, and we obtain the following result that could be found originally in [6].

**Corollary 10.** Let \( X \) be a \( L_p \)-space for some \( p \in [1, +\infty] \). If \( M \) is a locally finite subset of \( X \), then \( M \) bi-Lipschitzly embeds into any Banach space which contains the \( \ell_p^n \)'s uniformly.

If \( X \) is a \( L_p,\lambda \)-space for some \( \lambda \geq 1 \), then the distortion of the embedding in Corollary 10 depends only on \( \lambda \).

By Dvoretzky’s theorem, every infinite-dimensional Banach space contains the \( \ell_p^n \)'s uniformly. Since a \( L_2 \)-space is isomorphic to a Hilbert space, Corollary 10 in the case \( p = 2 \) takes a rather simple form, originally due to Ostrovskii [42].

**Corollary 11.** Let \( Y \) be an infinite dimensional Banach space, and \( M \) be a locally finite subset of a Hilbert space, then \( M \) admits a bi-Lipschitz embedding into \( Y \).

The case \( p = \infty \) also has a nice reformulation which is originally due to the author and G. Lancien [8]. Indeed, \( c_0 \) is a \( L_\infty \)-space, and hence every locally finite metric space embeds bi-Lipschitzly into a \( L_\infty \)-space. It is a deep result of Maurey and Pisier [24] that a Banach space contains the \( \ell_\infty^n \)'s uniformly if and only if it has trivial cotype.

**Corollary 12.** Let \( Y \) be a Banach space with trivial cotype, and \( M \) be a locally finite metric space, then \( M \) admits a bi-Lipschitz embedding into \( Y \).

In particular, every discrete metric space with bounded geometry admits a bi-Lipschitz embedding in the space \( (\sum_{n=1}^{\infty} \ell_\infty^n)_2 \) which is reflexive, asymptotically uniformly smooth and asymptotically uniformly convex. Since (at the time this article was written) no counter-example to the coarse geometric Novikov conjecture has been identified, we raise the following problem.
Problem 1. Let $\mathcal{M}$ be a discrete metric space with bounded geometry admitting a bi-Lipschitz embedding into a Banach space, which is reflexive, asymptotically uniformly smooth and asymptotically uniformly convex. Does $\mathcal{M}$ satisfy the coarse geometric Novikov conjecture?

In the remainder of this section it is shown how one can derive, from the previous corollaries, a few interesting applications in geometric group theory. Let $X$ and $Y$ be two Banach spaces such that $X$ is crudely finitely representable in $Y$. Let $M$ be a locally finite metric space, then it follows from Theorem 4 that $\alpha_X(M) \leq \alpha_Y(M)$. The fact that $\ell_2$ is finitely representable into any infinite-dimensional Banach spaces Corollary 11 implies that, from the compression point of view, embeddability of locally finite spaces into a separable Hilbert space is the most difficult to achieve.

From now on $L_p[0, 1]$ is imply denoted by $L_p$. Naor and Peres proved for the equivariant compression (see [38] for the definition) that $\alpha_2^#(G) \leq \alpha_L^#(G)$, for any $p \in [1, \infty)$ and every finitely generated group $G$. The same inequality for the (non equivariant) compression was known since $L_p$ contains an isometric copy of $L_2$, and hence for any metric space $M$, $\alpha_2(M) \leq \alpha_{L_p}(M)$. Thanks to Corollary 14 we have a similar inequality for locally finite spaces without any restriction on the target space.

Corollary 13. Let $Y$ be an infinite dimensional Banach space, and $M$ be a locally finite metric space, then

$$\alpha_2(M) \leq \alpha_Y(M).$$

We do not know if the equivariant analogue of Corollary 13 is true.

Question 1. Let $G$ be a finitely generated group. Do we have for every infinite-dimensional Banach space $Y$, $\alpha_2^#(G) \leq \alpha_Y^#(G)$?

It is not too difficult to show that sequences of finite subsets of $L_p$ admit an equi-bi-Lipschitz embedding into $\ell_p$. With this in mind, it is then fairly easy to show that the $L_p$-compression of an unbounded sequence of finite metric spaces (defined in a natural way) and its $\ell_p$-compression coincides. A similar reasoning does not work to prove an analogue statement if one is dealing with an (unbounded) infinite metric space since there are infinite subsets of $L_1$ which are not even bi-Lipschitz embeddable into $\ell_1$ (think of $L_1$ itself!). That is where Corollary 14 comes to our rescue if the infinite metric space has the good taste to be locally finite, since $L_p$ is a $L_p$-space, and $\ell_p$ contains the $\ell_n^p$’s uniformly.

Corollary 14. Let $M$ be a locally finite metric space, then for every $p \in [1, +\infty]$

$$\alpha_{\ell_p}(M) = \alpha_{L_p}(M).$$

Remark. Corollary 14 answers negatively a question raised by Naor and Peres (c.f. Question 10.7 in [39]). The subtlety between embeddings into $L_p$ and embeddings into $\ell_p$ was pointed out to them by Marc Bourdon.

Question 2. Let $G$ be a finitely generated group. Do we have for every $p \in [1, +\infty)$

$$\alpha_{\ell_p}^#(G) = \alpha_{L_p}^#(G)?$$

The next corollary is a analogue in the locally finite context, of a well-known phenomena for families of unbounded finite metric spaces.

Corollary 15. Let $M$ be a locally finite metric space, and $Y$ be a Banach space then, $M$ admits a coarse bi-Lipschitz embedding into $Y$ if and only if $M$ admits a bi-Lipschitz embedding into $Y$. 
Theorem 18, minus the inherent technicalities. We wish to present the proof of Proposition 17 admit a bi-Lipschitz embedding into any Banach space with the Radon-Nikodým property (c.f. [9] for more details). We wish to present the proof of Proposition 17 is optimal since there exists a compact metric space which does not admit a bi-Lipschitz embedding which is, in some sense, as close as one wants to a bi-Lipschitz embedding. Proposition 17 is optimal since there exists a compact metric space which does not admit a bi-Lipschitz embedding which is, in some sense, as close as one wants to a bi-Lipschitz embedding.

5. Coarse and uniform geometries of proper metric spaces

5.1. A few remarks on coarse and uniform geometries. A locally finite metric space might have a meaningful small-scale geometry in the sense that is not necessarily uniformly discrete. Indeed, consider $L := \mathbb{N} \bigcup \{ n + \frac{1}{n}; n \in \mathbb{N} \}$ as a subset of the metric space $(\mathbb{R}, |·|)$. $L$ is clearly locally finite but possesses pairs of points that are arbitrarily close. The coarse geometry of proper metric spaces is actually similar to the one of locally finite ones. Indeed a classical utilization of skeletons, as seen in the introduction, shows that from the point of view of large-scale geometry, it is enough to consider uniformly discrete spaces since a metric space is coarse-Lipschitz equivalent to any of its skeleton. Any skeleton of a proper metric space being locally finite, the coarse geometries of proper or locally finite metric spaces do not differ. The compression exponent being a coarse-Lipschitz invariant, it is straightforward to extend Corollary 13 and Corollary 14 to proper metric spaces.

Corollary 16. Let $M$ be a proper metric space. If $Y$ is an infinite dimensional Banach space, then

$$\alpha_2(M) \leq \alpha_Y(M),$$

and for every $p \in [1, +\infty]$,

$$\alpha_{\ell_p}(M) = \alpha_{L_p}(M).$$

A compact metric space is bounded and it does not make much sense to discuss its large-scale geometry. As far as its small-scale geometry is concerned, it was proved in [11] (Proposition 7.18) that every compact metric space embeds into the space $(\sum_{n=1}^{\infty} \ell_p^n)^2$ with uniform deformation gap $[t, t \log^2(t)]$. The next proposition is an improvement of this result, which says that one can construct a uniform embedding which is, in some sense, as close as one wants to a bi-Lipschitz embedding. Proposition 17 is optimal since there exists a compact metric space which does not admit a bi-Lipschitz embedding into any Banach space with the Radon-Nikodým property (c.f. [9] for more details). We wish to present the proof of Proposition 17 since it encompasses, together with the gluing technique, the main arguments of the proof of Theorem 18 minus the inherent technicalities.

Proposition 17. Let $(K, d_K)$ be a compact metric space. For every continuous, decreasing function $\mu : [0, \text{diam}(K)] \to [0, +\infty]$ such that $\mu(\text{diam}(K)) = 0$ and $\lim_{t \to 0^+} \mu(t) = +\infty$, there exists an embedding $f : K \to (\sum_{n=1}^{\infty} \ell_p^n)^2$ such that for all $x, y \in K$,

$$\frac{1}{2} \frac{d_K(x, y)}{\mu(d_K(x, y))} \leq \| f(x) - f(y) \| \leq \frac{\pi}{\sqrt{6}} d_K(x, y).$$
Proof. Assume that the diameter of $K$ is $D$. And let $\mu : (0, D] \to [0, +\infty)$ be a continuous, decreasing function such that $\mu(D) = 0$ and $\lim_{t \to 0} \mu(t) = +\infty$. Then the map $\mu$ has an inverse denoted $\mu^{-1} : [0, +\infty) \to (0, D]$ that is decreasing, with $\lim_{t \to +\infty} \mu^{-1}(t) = 0$. Fix $t_0$ in $K$, and denote $\sigma := \mu^{-1} : \mathbb{Z}^- \to (0, D]$. For any $k \in \mathbb{Z}^+$, let $R_k$ be a maximal $\frac{\sigma(k)}{6}$-net of $K$. Remark that $R_k$ is finite by compactness of $K$.

Define the following $1$-Lipschitz map:
\[
\varphi_k : K \to \ell_\infty(R_k) =: X_k
\]
\[
t \mapsto (d_K(t, s) - |s|)_{s \in R_k}, \text{ where } |s| = d_K(s, t_0).
\]
The embedding is given by
\[
f : K \to \left( \sum_{k=1}^{\infty} X_k \right)_2
\]
\[
t \mapsto \sum_{k \in \mathbb{N}} \frac{\varphi_k(t)}{k}.
\]
It is clear that $f$ is $C$-Lipschitz with $C = \sqrt{\sum_{k \in \mathbb{N}} \frac{1}{k^2}} = \frac{\pi}{\sqrt{6}}$.

Let $x \neq y \in K$, then there exists $l \in \mathbb{Z}^+$ such that $\sigma(l + 1) \leq d_K(x, y) < \sigma(l)$, or equivalently $l + 1 \leq \mu(d_K(x, y)) < l$. $R_{l+1}$ being a $\frac{\sigma(l+1)}{4}$-net in $K$ one can find $s_y \in R_{l+1}$ such that $d_K(s_y, y) < \frac{\sigma(l+1)}{4} \leq \frac{d_K(x, y)}{4}$. Therefore
\[
\|f(x) - f(y)\| \geq \frac{\|\varphi_{l+1}(x) - \varphi_{l+1}(y)\|_\infty}{l + 1} \geq \frac{\sup_{s \in \mathbb{R}_{l+1}} |d_K(x, s) - d_K(y, s)|}{l + 1} \geq \frac{|d_K(x, s_y) - d_K(y, s_y)|}{l + 1} \geq \frac{d_K(x, y) - 2d_K(x, s_y)}{l + 1} \geq \frac{1}{2} \frac{d_K(x, y)}{l + 1},
\]
and
\[
\frac{1}{2} \frac{d_K(x, y)}{\mu(d_K(x, y))} \leq \|f(x) - f(y)\| \leq \frac{\pi}{\sqrt{6}} d_K(x, y).
\]

\[\square\]

5.2. On the strong geometry of proper metric spaces. In the sequel we consider proper metric spaces which are unbounded and not uniformly discrete. The most intriguing geometry of such a space is arguably the strong geometry, i.e. when the uniform and coarse geometries are considered simultaneously. A proper metric space can have, a priori, quite different small-scale and large-scale structures. As already seen, a proper metric space admits an embedding into, say, $(\sum_{n=1}^{\infty} \ell_\infty^n)_2$, which preserve the coarse geometry. However it is not clear whether one can find a strong embedding, i.e. preserving simultaneously both its uniform and coarse geometries. If such an embedding exists, is the coarse geometry preserved with the same level of quality as it is with the purely coarse embedding? The author showed that a proper metric space admits an embedding into the reflexive space $(\sum_{n=1}^{\infty} \ell_\infty^n)_2$ which preserve simultaneously the uniform and coarse geometries. More precisely, it was shown that a proper metric space embeds into any Banach space with trivial cotype with strong deformation gap $\frac{t}{\log^5(t)}$. The quality of the embedding
can be considered as somehow satisfactory. However the embedding is not coarse bi-Lipschitz, but a proper metric space does admit a coarse bi-Lipschitz embedding into \((\sum_{n=1}^{\infty} \ell_n^p)_{\infty}\). At first sight, it looks like, and seems odd, that in order to preserve simultaneously the uniform and coarse geometries one has to give up a little bit on the quality of the embedding for large distances. The author and G. Lancien \cite{[9]} were able recently to prove much more precise results and clarify this ambiguity.

**Definition.** \((X, d_X)\) almost Lipschitz embeds into \((Y, d_Y)\) if for any \(M \in (0, +\infty)\), any continuous function \(\varphi: [0, +\infty) \to [0, M]\) so that \(\varphi(0) = 0\), \(\varphi(t) > 0\) for \(t > 0\), there exists a map \(f: X \to Y\) and \(C \in (0, +\infty)\) such that for all \(x, y \in X\)

\[
\frac{1}{C} \varphi(d_X(x, y))d_X(x, y) \leq d_Y(f(x), f(y)) \leq Cd_X(x, y).
\]

It is clear that if \(X\) admits a bi-Lipschitz embedding into \(Y\), then \(X\) almost Lipschitz embeds into \(Y\). Also, if \(X\) almost Lipschitz embeds into \(Y\), then \(X\) admits a strong embedding into \(Y\), which is as close as one wants to a bi-Lipschitz embedding.

**Theorem 18.** Let \((M, d_M)\) be a proper metric space and \(Y\) be a Banach space with trivial cotype, then \(M\) almost Lipschitz embeds into \(Y\).

The quality of the embedding has now improved and it is actually coarse bi-Lipschitz. Indeed, if \((X, d_X)\) almost Lipschitz embeds into \((Y, d_Y)\), for every \(\alpha \in (0, \infty)\) by taking \(\varphi\) appropriately, there exist \(D_\alpha \in (0, \infty)\) and a map \(f_\alpha: X \to Y\) so that \(\frac{1}{D_\alpha}d_X(x, y) \leq d_Y(f_\alpha(x), f_\alpha(y)) \leq D_\alpha d_X(x, y)\), for every \(x, y \in X\) satisfying \(d_X(x, y) \geq \alpha\).

6. **Stable metric spaces**

6.1. **More examples of non-stable Banach spaces.** We have seen in Section 2.1 that there exist Banach spaces which are stable but not reflexive, e.g. \(\ell_1\). So far the only examples of non-stable Banach spaces that we have encountered are to be found amongst Banach spaces containing \(c_0\), and therefore are all non-reflexive. It is then natural to wonder whether they are Banach spaces which are reflexive but not stable, or do no have an equivalent stable norm. It is by no mean elementary that such examples do exist. The original motivation of Krivine and Maurey to introduce the stability property was to prove that a Banach space with a stable norm contains an almost isometric copy of \(\ell_p\) for some \(1 \leq p < \infty\). Therefore to exhibit a reflexive Banach space which does not admit an equivalent stable norm it is sufficient to construct a reflexive Banach space which does not contain any isomorphic copy of \(c_0\), or \(\ell_p\) for any \(1 \leq p < \infty\). But Tsirelson’s space was precisely designed to achieve the latter property and is reflexive! In the sequel when we refer to Tsirelson's space we mean the space \(T\) constructed by Figiel and Johnson in \cite{[17]} which is actually the dual of Tsirelson’s original space \cite{[19]} but shares similar properties. Actually \(T\) does not contain any super-reflexive space, \(c_0\), nor \(\ell_1\). As a consequence \(T\) is a counterexample, in a strong sense, to the second part of Problem 6.1 in \cite{[27]}. \(T\) is not super-reflexive but Figiel and Johnson, where able to “convexified” \(T\), and to build a super-reflexive space not containing any isomorphic copy of \(c_0\), nor \(\ell_p\) for any \(1 \leq p < \infty\). Super-reflexive Tsirelson-type Banach spaces do not admit an equivalent stable norm, and even a strong reflexivity requirement on the space is not sufficient to force good stability properties.

To find more examples of Banach spaces which are not stable we need to explore the jungle of non-commutative \(L_p\)-spaces. We refer to \cite{[16]} for a thorough discussion of non-commutative \(L_p\)-spaces. The commutative \(L_p\)-spaces belong to the class of non-commutative \(L_p\)-spaces. For instance, \(L_p[0, 1]\) can be represented
as the non-commutative $L_p$-space associated to $L_\infty[0,1]$ considered as a von Neumann algebra on the Hilbert space $L_2[0,1]$. The simplest truly non-commutative $L_p$-spaces are the, so called, Schatten classes (denoted $S_p(H)$). They are defined as the non-commutative $L_p$-spaces associated to $\mathcal{M} = B(H)$, the algebra of all bounded operators on a Hilbert space $H$, equipped with be the usual trace on $B(H)$. The commutative theory can be satisfactorily extended to a large extent to the non-commutative setting, however there are some significant differences. The stability property of the non-commutative spaces is one of them.

**Theorem 19.** Let $p \in [1,\infty)$, $p \neq 2$. Then a non-commutative $L_p$-space associated to a von Neumann algebra $\mathcal{M}$ is stable if and only if $\mathcal{M}$ is of type I.

The “if” part of the theorem above was independently proved in [3], and [18] where it is shown that the non-commutative $L_p$-space associated to a von Neumann algebra $\mathcal{M}$ of type I can be written as an $\ell_p$-sum of commutative vector-valued $L_p$-spaces whose values fall into stable Banach spaces. The “only if” part comes from [32] and is proved in two steps as follows. Marcolino first showed that if $\mathcal{M}$ is a von Neumann algebra not of type I, then for $1 \leq p \leq \infty$, the non-commutative $L_p$-space associated to the hyper finite II$_1$ factor is isometric to a (1-complemented) subspace of the non-commutative $L_p$-space associated to $\mathcal{M}$. The conclusion follows from the fact that the non-commutative $L_p$-space associated to the hyper finite II$_1$ factor is not stable. Those spaces have a completely different linear structure compared to Tsirelson-like spaces since they contain copies of $\ell_p$.

**Question 3.** Does the non-commutative $L_p$-space associated to the hyper finite II$_1$ factor admit an equivalent stable norm?

### 6.2. The puzzling relationship between stability and reflexivity

From the isomorphic point of view stable Banach spaces and reflexive spaces are clearly very different classes, with a non-empty intersection tough. From a purely metric perspective the difference between stability and reflexivity is more elusive. Stability and reflexivity are both closely related to compactness properties of the space. Let us first give a representation of the norm of a stable Banach space which uncovers a direct relationship between stability and reflexivity.

**Theorem 20.** Let $X$ be a Banach space. Fix $p \in [1,\infty)$, $X$ is stable if and only if there exist a reflexive Banach space $Y$ and maps $g : B \to Y$, $h : B \to Y^*$ so that for all $x, y \in B$ (a dense subset of $B_X$) one has $\|x - y\|^p = g(x), h(y) > \text{where } < \cdot, \cdot >$ is the duality product between $Y$ and $Y^*$.

A striking result of Kalton [27] says that a stable metric space can be embedded into a reflexive Banach space, and the embedding distorts the distances by only a slight amount.

**Theorem 21.** Let $\mathcal{M}$ be a stable metric space, and $\alpha \in (0,1)$, then there exist a reflexive Banach space $Y$ and a map $f : \mathcal{M} \to Y$ such that for all $x, y \in \mathcal{M}$

$$\min\{d_\mathcal{M}(x, y), d_\mathcal{M}^\alpha(x, y)\} \leq \|f(x) - f(y)\|_Y \leq \max\{d_\mathcal{M}(x, y), d_\mathcal{M}^\alpha(x, y)\}.$$ 

We point out that the reflexive space $Y$ and the embedding $f$, highly depend on $\alpha$ and $\mathcal{M}$. Given a metric space $\mathcal{M}$, trying to describe the geometry of the resulting space $Y$ does not seem straightforward but is a very interesting problem. For instance, it would be interesting to know if the space $Y$ produced is always asymptotically uniformly smooth and asymptotically uniformly convex.

**Problem 2.** Does every stable metric space embed strongly into a reflexive, asymptotically uniformly smooth and asymptotically uniformly convex space?
By checking carefully Kalton’s proof the author and G. Lancien [9] were able to show a finer result.

**Definition.** Let $\mathcal{C}$ be a class of metric spaces. We say that $(\mathcal{X}, d_\mathcal{X})$ nearly isometrically embeds into $\mathcal{C}$ if for any pair of continuous functions $\rho, \omega : [0, +\infty) \to [0, +\infty)$ satisfying

1. $\omega(0) = 0$, $t \leq \omega(t)$ for $t \in [0, 1]$, and $\lim_{t \to 0} \frac{\omega(t)}{t} = +\infty$,
2. $\omega(t) = t$ for $t \geq 1$,
3. $\rho(t) = t$ for $t \in [0, 1]$,
4. $\rho(t) \leq t$ for $t \geq 1$ and $\lim_{t \to +\infty} \frac{\rho(t)}{t} = 0$,

there exist a space $(\mathcal{Y}, d_\mathcal{Y}) \in \mathcal{C}$ and a map $f : \mathcal{X} \to \mathcal{Y}$ such that for all $x, y \in \mathcal{X}$

$$\rho(d_\mathcal{X}(x, y)) \leq d_\mathcal{Y}(f(x), f(y)) \leq \omega(d_\mathcal{X}(x, y)).$$

If $\mathcal{C}$ is reduced to a single element $(\mathcal{Y}, d_\mathcal{Y})$ we say that $\mathcal{X}$ nearly isometrically embeds into $\mathcal{Y}$.

**Theorem 22.** Let $\mathcal{M}$ be a stable metric space, then $\mathcal{M}$ is nearly isometrically embeddable into the class of reflexive Banach spaces.

We refer to [9] for the proof of Theorem 22. Remark that if $\mathcal{X}$ nearly isometrically embeds into $\mathcal{C}$ then for every $0 < \delta \leq \Delta < \infty$, there exist $\mathcal{Y} \in \mathcal{C}$ and a map $f : \mathcal{X} \to \mathcal{Y}$ such that for all $x, y \in \mathcal{X}$ satisfying $d_\mathcal{X}(x, y) \in [\delta, \Delta]$

$$d_\mathcal{Y}(f(x), f(y)) = d_\mathcal{X}(x, y).$$

This property is not achieved with Kalton’s original embedding.

Several results are known to hold for both stable and reflexive Banach spaces. We discuss few of them. Stable Banach spaces and reflexive spaces are known to have Kalton’s $Q$-Property [27]. Let $\sigma$ and $\tau$ be two distinct $k$-subsets of $\mathbb{N}$, one shall write $\sigma, \tau \in \mathbb{N}^{[k]}$, written in increasing order, i.e. $\sigma_1 < \cdots < \sigma_k$ and $\tau_1 < \cdots < \tau_k$. One says that $\sigma$ and $\tau$ interlace if either $\tau_1 \leq \sigma_1 \leq \tau_2 \leq \sigma_2 \leq \cdots \leq \tau_k \leq \sigma_k$ or $\sigma_1 \leq \tau_1 \leq \sigma_2 \leq \tau_2 \leq \cdots \leq \sigma_k \leq \tau_k$. One can define the graph whose vertex set is $\mathbb{N}^{[k]}$ and such that two distinct vertices are adjacent if and only if they interlace. Denote $\mathcal{I}^{[k]}(\mathbb{N})$ this graph endowed with its graph metric that will be denoted $d_\mathcal{I}$.

We can obviously define in a similar fashion the space $\mathcal{I}^{[k]}(S)$ whenever $S$ is an infinite subset of $\mathbb{N}$.

**Definition.** A Banach space $X$ is said to have Kalton’s $Q$-property if there exists a universal constant $C$ so that whenever $f : \mathcal{I}^{[k]}(\mathbb{N}) \to X$ is a Lipschitz map (i.e. $\text{Lip}(f) < \infty$), then there is $S$ an infinite subset of $\mathbb{N}$ such that

$$\|f(\sigma) - f(\tau)\| \leq C \cdot \text{Lip}(f)$$

for every $\sigma, \tau \in \mathcal{I}^{[k]}(S)$.

The $Q$-property is one of the very few obstruction to coarse or uniform embeddability. The $Q$-property is preserved under coarse or uniform embeddings, and it can be used to prove non-embeddability into stable or reflexive Banach spaces. More precisely, If $X$ (resp. $B_X$) coarsely (resp. uniformly) embeds into a Banach space with the $Q$-property, then $X$ has the $Q$-property. Since Kalton proved that the James space $J$ and its dual $J^*$ [23, 24] (they are non-reflexive) do not have the $Q$-property, we have two more examples of spaces failing the stability property.

**Example 6.** James space $J$ and its dual $J^*$ are non-stable.

Recall that a Banach space $X$ has the alternating Banach-Saks property if every bounded sequence $(x_n)_{n=1}^\infty \in X$ has a subsequence $(y_n)_{n=1}^\infty$ so that the alternating Cesaro means $\frac{1}{n} \sum_{k=1}^n (-1)^k y_k$ converge to 0. With a brilliant utilization of Ramsey’s theorem, Kalton proved the striking fact that a Banach space with the $Q$-property, and with the alternating Banach-Saks property, must be reflexive.
Corollary 23. A stable Banach space with the alternating Banach-Saks property is reflexive.

As a consequence for a rather large class of Banach spaces, stability implies reflexivity. Corollary 23 can also be used to provide more examples of non-stable Banach spaces. Without going too deep into the theory of spreading models, a spreading model for $X$ is a Banach space that can be associated to $X$ and that is finitely representable in $X$. Beauzamy [10] proved that a Banach space the has alternating Banach-Saks property if and only if none of its spreading models are isomorphic to $\ell_1$. Since a Banach space $X$ has trivial type if and only if $\ell_1$ is finitely represented in $X$ [31], it follows that Banach spaces with non-trivial type have the alternating Banach-Saks property.

Example 7. Every non-reflexive Banach space with non-trivial type is non-stable. James non-reflexive space of type 2 [25], and Pisier-Xu interpolation spaces [45], are such spaces.

It would be interesting, if at all possible, to isolate a property such that every reflexive Banach space with this property is automatically stable. This property cannot be possessed by $T$ nor any non-commutative $L_p$-space, $1 < p \neq 2 < \infty$, associated to a von Neumann algebra of type I. Note that $T$ does not have the alternating Banach-Saks property, in a strong sense, since all its spreading models are isomorphic to $\ell_1$. However there are reflexive Banach spaces with the alternating Banach-Saks property which fail to be stable. Problem 6.1 and Problem 6.2 raised by Kalton in [27], and which are looking for a converse to Theorem 21 are still open and we recall them here.

Problem 3. Does a separable reflexive Banach space embed coarsely into a stable metric space?

Problem 4. Let $X$ be a separable reflexive Banach space. Does $B_X$ uniformly embed into a stable metric space?

Natural candidates for a negative answer to both problems are Tsirelson’s space $T$ or the non-commutative $L_p$-space associated to the hyper finite II$_1$ factor for $1 < p \neq 2 < \infty$. Raynaud [47] gave a nice necessary condition to uniform embeddability in a stable space. The coarse analogue requires some ad-hoc adjustments.

Theorem 24. If $B_X$ (resp. $X$) admits a uniformly (resp. coarsely) equivalent stable distance, then every spreading basic sequence in $X$ is unconditional.

Recall that a sequence $(x_n)_{n=1}^\infty$ is said to be spreading if the sequence is equivalent to all its subsequences. Unfortunately Theorem 24 is inconclusive for a reflexive Banach space since all its spreading basic sequences are automatically unconditional. Indeed, a basic sequence in a reflexive space is always weakly-null, but a weakly-null spreading sequence it is automatically unconditional (see [2] Proposition 11.3.6).

Question 4. Does $T$ (resp. $B_T$) coarsely (resp. uniformly) embeds into a stable metric space?

Note that a “convexified” Tsirelson space is uniformly convex with an unconditional basis and its unit ball embeds uniformly into $\ell_2$, but it does not seem to be known whether the whole space embeds coarsely into a stable space.

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