ATYPICAL REPRESENTATIONS OF $\mathcal{U}_q(sl(N))$ AT ROOTS OF UNITY

Boucif Abdesselam

Centre de Physique Théorique, Ecole Polytechnique
91128 Palaiseau Cedex, France.
Laboratoire Propre du CNRS UPR A.0014.

Abstract

We show how to adapt the Gelfand-Zetlin basis for describing the atypical representation of $\mathcal{U}_q(sl(N))$ when $q$ is root of unity. The explicit construction of atypical representation is presented in details for $N = 3$.

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1 E-mail adress: abdess@orphee.polytechnique.fr
1. Introduction.

The present paper is a sequel to [1], in which we presented an improvement of [2] giving Gelfand–Zetlin construction of irreducible representation of $U_q(sl(N))$ at roots of unity independently of their nature. We have shown that it is possible to describe the periodic, semi-periodic, nilpotent, usual and some atypical representation of $U_q(sl(N))$ by the fractional parts formalism. However, atypical representations generally need a special treatment.

The Gelfand-Zetlin basis in the form [1] is not yet totally adapted for atypical representation [3]. This has to be compared with the fact that, for superalgebras, the atypical representations are more difficult to describe with the Gelfand–Zetlin than the typical ones: the atypical representations of some superalgebras or quantum superalgebras were obtained, for example, [4, 5] in the case of $gl(n|1)$ and [6] in the case of $U_q(gl(2|2))$, but the general case is not yet treated.

Note that the paper [7] provides the atypical representations of $U_q(sl(3))$. The matrix elements of $U_q(sl(3))$ given by this formalism do not contains denominators, and do not generate divergence when $q^m = 1$. A classification of irreducible representations of $U_q(sl(3))$ was done in [13, 14].

The purpose of this paper is to provide a procedure that enables the construction of general atypical representations suitably adapted the Gelfand–Zetlin basis.

In section 2, we give a simple example of explicit construction of the flat (no multiplicity) representations for $U_q(sl(3))$ and we present the general idea for the construction of atypical representation of $U_q(sl(N))$. The atypical case of $U_q(sl(3))$ is presented in details in section 3.

2. The Primitive Gelfand–Zetlin basis.

2.1. The quantum algebra $U_q(sl(N))$.

The quantum algebra $U_q(sl(N))$ [8, 9] is defined by the generators $k_i$, $k_i^{-1}$, $e_i$, $f_i$ ($i = 1, \cdots, N - 1$) and the relations

\[ k_i e_j = q^{a_{ij}} e_j k_i, \quad k_i f_j = q^{-a_{ij}} f_j k_i, \]
\[ [e_i, f_j] = \delta_{ij} \frac{k_i - k_i^{-1}}{q - q^{-1}}, \]
\[ [e_i, e_j] = 0 \quad \text{for} \quad |i - j| > 1, \]
\[ e_i^2 e_{i \pm 1} - (q + q^{-1}) e_i e_{i \pm 1} e_i + e_{i \pm 1} e_i^2 = 0, \]
\[ f_i^2 f_{i \pm 1} - (q + q^{-1}) f_i f_{i \pm 1} f_i + f_{i \pm 1} f_i^2 = 0, \]

The two last equations are called the Serre relations, and $(a_{ij})_{i,j=1,\ldots,N-1}$ is the Cartan matrix of $sl(N)$, i.e.
Let us now define the adapted Gelfand–Zetlin basis for the representations of $U_q(sl(N))$, the corresponding states are called \textbf{primitive vectors} of Gelfand-Zetlin pattern.

\textbf{2.2. Primitive Vectors of the Gelfand–Zetlin basis.}

The states are

$$
|p\rangle = \begin{pmatrix}
  |p_{1N}\rangle & |p_{2N}\rangle & \cdots & |p_{N-1,N}\rangle & |p_{NN}\rangle \\
  p_{1N-1} & \cdots & p_{N-1,N-1} \\
  \cdot & \cdots & \cdot \\
  p_{12} & p_{22} \\
  p_{11}
\end{pmatrix}
$$

(2.3)

(with respect to \cite{2}, we use $p_{ij} = h_{ij} - i$ instead of $h_{ij}$). The primitive Gelfand–Zetlin basis \cite{10} is labelled by $\frac{1}{2}N(N+1)$ numbers $p_{ij}$. When $q$ is generic, the first line of indices determines the \textit{highest weight} of the representation. The states (2.3) within the same module $V([p]_N)$ are distinguished by $p_{ij}$, $i, j = 1, ..., N - 1$, which assume values consistent with the triangular inequalities

$$
p_{i,j+1} - p_{i,j} \in \mathbb{Z}_+, \quad p_{i,j} - p_{i+1,j+1} - 1 \in \mathbb{Z}_+, \quad i, j = 1, ..., N - 1, \quad \text{(2.4)}$$

or,

$$
p_{i,j+1} \geq p_{ij} > p_{i+1,j+1}. \quad \text{(2.5)}$$

The dimension of $V([p]_N)$ is given by

$$
dim V([p]_N) = \frac{\prod_{i=1}^{N-1} \prod_{j=i+1}^N (p_{iN} - p_{jN})}{\prod_{i=1}^{N-1} (N-i)!}. \quad \text{(2.6)}$$
2.3. The primitive representation

The action of the generators $k_l^{\pm 1}$, $e_l$ and $f_l$ ($l = 1, 2, ..., N - 1$) is given by

\[ k_l^{\pm 1}|p\rangle = q^{\pm (2\sum_{i=1}^{l} p_{i,l} - \sum_{i=l+1}^{l+1} p_{i,l+1} - \sum_{i=1}^{l-1} p_{i,l-1} - 1)}|p\rangle \]

\[ f_l|p\rangle = \sum_{j=1}^{l} \frac{P_1(jl;p)P_2(jl;p)}{P_3(jl;p)}|p_{jl} - 1\rangle \tag{2.7} \]

\[ e_l|p\rangle = \sum_{j=1}^{l} \frac{P_1(jl;p_{jl} + 1)P_2(jl;p_{jl} + 1)}{P_3(jl;p_{jl} + 1)}|p_{jl} + 1\rangle \]

where $|p_{jl} \pm 1\rangle$ denotes the state differing from $|p\rangle$ by only $p_{jl} \rightarrow p_{jl} \pm 1$, and

\[ P_1(jl;p) = \prod_{i=1}^{l+1} [\varepsilon_{ij}(p_{i,l+1} - p_{j,l} + 1)]^{1/2}, \]

\[ P_2(jl;p) = \prod_{i=1}^{l-1} [\varepsilon_{ji}(p_{j,l} - p_{i,l-1})]^{1/2}, \tag{2.8} \]

\[ P_3(jl;p) = \prod_{i=1}^{l} [\varepsilon_{ij}(p_{i,l} - p_{j,l})]^{1/2} [\varepsilon_{ij}(p_{i,l} - p_{j,l} + 1)]^{1/2}, \]

$\varepsilon_{ij}$ being the sign defined by

\[ \varepsilon_{ij} = \begin{cases} 1 & \text{for } i \leq j \\ -1 & \text{for } i > j \end{cases} \tag{2.9} \]

In the following, we take $q$ to be a root of unity and $p_{ij}$ are integers. We restrict ourself to the quantum Lie algebra, where the raising and lowering operators are nilpotents, i.e. $e_i^m = f_i^m = 0$ and where the Cartan generators $h_i$ are such that $k_i^m = (q^{h_i})^m = 1$ (representations of this case were studied by Lusztig [15]). Let $m$ the smallest integer such that $q^m = 1$. We will consider only the case of odd $m$ in this paper. A similair discussion is valid when $m$ is even.

We consider here the quantum analogue of classical (highest weight and lowest weight) irreducible representations with a highest weight that obeys

\[ p_{1N} - p_{NN} > m, \tag{2.14} \]

when $q^m = 1$ these representations are not always irreducible, since some new singular vectors arise in the corresponding Verma module, that are not obtained from the highest weight vector by action of the translated Weyl group. Quotienting by the subrepresentation generated by the singular vector leads to new irreducible representations that we call atypical by analogy with the case of superalgebras.
2.4. A example of flat representation

In [1], we have introduced some parameters to break the symmetry between the actions of $e_1$ and $f_1$. They were taken to be 0, 1/2 or 1. A good choice of these parameters permits the elimination of the singular vectors (these singular vectors are states arising in the r.h.s of (2.7) but do not obey to the triangular inequalities) in a natural way. If an equal number of factors in numerators and denominators are simultaneously equal to zero, and if the vector from r.h.s of (2.7) is a singular vector, we can adjust these parameters such that the number of zeroes in numerator is superior to the number of zeroes in denominator. This procedure describes successfully the flat representations [1, 11, 12, 13, 14], i.e. when

$$p_{1N} - p_{NN} = m + 1. \quad (2.10)$$

For example, the representations of $U_q(sl(3))$ of dimension 7 for $m = 3$ ($p_{13} = 4$, $p_{23} = 2$, $p_{33} = 0$) is described by

$$f_1|p\rangle = \left( [p_{11} - p_{22} - 1] \right)^{1/2} |p_{11} - 1\rangle,$$

$$f_2|p\rangle = \left( \frac{[p_{13} - p_{12} + 1][p_{12} - p_{23} - 1][p_{12} - p_{33} - 1]}{[p_{12} - p_{22} - 1][p_{12} - p_{22}]} \right)^{1/2} |p_{12} - p_{11}| |p_{12} - 1\rangle + \left( \frac{[p_{13} - p_{22} + 1][p_{23} - p_{22} + 1][p_{11} - p_{22}]}{[p_{12} - p_{22} + 1][p_{12} - p_{22}]} \right)^{1/2} |p_{22} - p_{33} - 1| |p_{22} - 1\rangle,$$

$$e_1|p\rangle = [p_{12} - p_{11}][p_{11} - p_{22}]^{1/2} |p_{11} + 1\rangle,$$

$$e_2|p\rangle = \left( \frac{[p_{13} - p_{12}][p_{12} - p_{23}][p_{12} - p_{33}]}{[p_{12} - p_{22} + 1][p_{12} - p_{22}]} \right)^{1/2} |p_{12} + 1\rangle + \left( \frac{[p_{13} - p_{22}][p_{23} - p_{22}][p_{11} - p_{22} - 1]}{[p_{12} - p_{22} - 1][p_{12} - p_{22}]} \right)^{1/2} |p_{22} + 1\rangle,$$

we remark that

$$f_1 \begin{pmatrix} 4 & 2 & 0 \\ 3 & 2 & 0 \end{pmatrix} = f_2 \begin{pmatrix} 4 & 2 & 0 \\ 3 & 2 & 0 \end{pmatrix} = 0, \quad (2.12)$$

and

$$e_1 \begin{pmatrix} 4 & 2 & 0 \\ 3 & 2 & 0 \end{pmatrix} = e_2 \begin{pmatrix} 4 & 2 & 0 \\ 3 & 2 & 0 \end{pmatrix} = 0, \quad (2.13)$$

the others states form just the irreducible representation of dimension 7 (no multiplicity). A similiar methode describe also the representation of dimension 18 for $m = 5$ ($p_{13} = 6$, $p_{23} = 2$, $p_{33} = 0$) (see the figure in [11]).
2.5. Atypical representations and adaptation of the Gelfand–Zetlin basis

Let \( \eta_{jl} \) and \( \eta'_{jl} \) be respectively the numbers of zero of \( P_1(jl; p) \), \( P_2(jl; p) \), and \( P_3(jl; p) \). We note that the maximum value of \( \eta'_{jl} \) is \( l - 1 \). If a primitive vector from the r.h.s of (2.7) does not belong to the module under consideration, then the corresponding term is zero (\( \eta_{jl} > \eta'_{jl} \)). If an equal numbers of factors in numerators and denominators are simultaneously equal to zero (\( \eta_{jl} = \eta'_{jl} \)), they should be cancelled out and the corresponding primitive vector here belongs to the module. If \( \eta'_{jl} > \eta_{jl} \), the matrix elements of \( f_l \) are undefined. In the next, we will give how to eliminate the divergences of the matrix elements using change of bases. This method will be illustrated on a simple example in section 3.

Suppose

\[
\eta_{jl} = 0, \quad \text{and} \quad \eta'_{jl} = l - 1, \quad 1 \leq j \leq l
\]  

(2.15)

i.e.

\[
\begin{align*}
p_{i,l+1} - p_{ji} + 1 & \neq 0 \begin{bmatrix} m \end{bmatrix}, \quad 1 \leq i \leq l + 1, \quad 1 \leq j \leq l, \\
p_{ji} - p_{i,l-1} & \neq 0 \begin{bmatrix} m \end{bmatrix}, \quad 1 \leq j \leq l, \quad 1 \leq i \leq l - 1, \\
p_{il} - p_{jl} & = 0 \begin{bmatrix} m \end{bmatrix}, \quad 1 \leq j \leq l, \quad 1 \leq i \leq l,
\end{align*}
\]  

(2.16)

thus, there exists \( \beta_1, \beta_2, \ldots, \beta_{l-1} \in \mathbb{Z}_+ \) such that

\[
\begin{align*}
p_{i,l} - p_{i+1,l} & = \beta_i m, \quad 1 \leq i \leq l - 1 \\
p_{i,l} - p_{j,l} & = (\beta_i + \beta_{i+1} + \cdots + \beta_{j-1}) m, \quad j > i.
\end{align*}
\]  

(2.17)

The action of \( f_l \) over a state satisfying (2.16) produces in the r.h.s of (2.7) a set of
states \( \{|p_{1l}' \cdots p_{il}' \cdots p_{il}\}, 1 \leq i \leq l\} \), where

\[
p_{1l}' = \begin{cases} 
p_{il} + (\beta_1 + \cdots + \beta_{i-1}) m & \text{for } i > 1 \\
p_{il} - 1 & \text{for } i = 1
\end{cases}
\]

\[
p_{il}' = \begin{cases} 
p_{il} - 1 - (\beta_1 + \cdots + \beta_{i-1}) m & \text{for } i > 1 \\
p_{il} - 1 & \text{for } i = 1
\end{cases}
\]

(2.18)

this set is called a set of type \( p_{il} - 1 \) or \( \{|1\}, \{2, \cdots, l\}\).

**Definition.1.** Let a state satisfying

\[
p_{il} - p_{il} \pm \beta = 0 \quad [m], \quad i = 2, \cdots, l, \quad 1 \leq |\beta| \leq m - 1,
p_{il} - p_{jl} = 0 \quad [m], \quad i, j = 2, \cdots, l.
\]

(2.19)

This state is called a state of type \( p_{il} \) or \( \{|1\}, \{2, \cdots, l\}\). Define the operation

\[
\pi_{1i}^l(|p_{1l} \cdots p_{il} \cdots p_{il}|) = |p_{1l}' \cdots p_{il}' \cdots p_{il}|
\]

(2.20)

where

\[
p_{il}' = \begin{cases} 
p_{il} + (\beta_1 + \cdots + \beta_{i-1}) m & \text{for } i > 1 \\
p_{il} & \text{for } i = 1
\end{cases}
\]

(2.21)

\[
p_{il}' = \begin{cases} 
p_{il} - 1 - (\beta_1 + \cdots + \beta_{i-1}) m & \text{for } i > 1 \\
p_{il} & \text{for } i = 1
\end{cases}
\]

(2.22)

This operation is called exchange mapping of level \( l \) centered on \( 1 \), and the set \( \{|p_{1l}' \cdots p_{il}' \cdots p_{il}\}, 1 \leq i \leq l\} \) is of type \( p_{il} \) or \( \{|1\}, \{2, \cdots, l\}\). This set of states has the same eigenvalues for the Cartan operators (degenerate states) and has to satisfy the triangular inequalities.

**Lemma.** Let a state satisfy the condition (2.19), and let \( \{|p_{1l}' \cdots p_{il}' \cdots p_{il}\}, 1 \leq i \leq l\} \) be the set of all states obtained by action of the mapping \( \pi_{1i}^l \) over this state. This set is isomorphic to a set obtained by action of the mapping \( \pi_{1\mu}^l \) over a state of type type \( p_{\mu l} \mp \beta \) (\( \mu \neq 1 \)).

**Proof.** Let a state be of type \( \{|1\}, \{2, \cdots, l\}\), i.e.

\[
p_{il} - p_{il} \pm \beta = 0 \quad [m], \quad i = 2, \cdots, l, \quad 1 \leq |\beta| \leq m - 1,
p_{il} - p_{jl} = 0 \quad [m], \quad i \neq 1 \quad \text{and} \quad j \neq 1,
\]

and, let

\[
\pi_{1\mu}^l(|p_{1l} \cdots p_{\mu l} \cdots p_{il}|) = |p_{1l} \pm \beta \cdots p_{\mu l} \mp \beta \cdots p_{il}|
\]
The state in the r.h.s is of type $p_{\mu l} \pm \beta$ ($\mu \neq 1$), and
\[
\pi_{\mu \nu} \circ \pi_{1 \mu}^l = \pi_{1 \nu}^l.
\] (2.23)

All states of the set (2.18) are obtained by action of the mapping $\pi_{1 i}^l$ over the state $\langle p_{1l} - 1 \cdots p_{il} \cdots p_{ul} \rangle$. Using this notation, the action of $f_l$ is
\[
f_l|p\rangle = \sum_{j=1}^l \frac{P_1(jl; p) \ P_2(jl; p)}{P_3(jl; p)} |p'_{1l} \cdots p'_{jl} \cdots p_{ul}\rangle
\] (2.24)

We note that the number of zero in the polynomials $P_3(jl; p)$ is $l - 1$.

**Definition.2.** Let a set be of state of type $(\{1\}, \{2, \cdots, l\})$, i.e.
\[
p_{1l} - p_{il} \pm \beta = 0 \ [m], \quad 2 \leq i \leq l, \quad 1 \leq |\beta| \leq m - 1,
\]
and, let the new basis be given by
\[
|p'_{1l} \cdots p'_{il} \cdots p_{ul}\rangle = \sum_{j=1}^l D_{ij} \ |p'_{1l} \cdots p'_{jl} \cdots p_{ul}\rangle
\] (2.25)
where, $D$ is a $l \times l$ rotation matrix, i.e.
\[
D^t.D = D.D^t = 1.
\] (2.26)

This new basis is called the **modified basis of type** $(\{1\}, \{2, \cdots, l\})$. The primitive and the modified sets satisfy the triangular inequalities (2.5).

The finiteness of the matrix elements $\langle p|e_i f_i|p\rangle$ and $\langle p|f_i e_i|p\rangle$ (preserve $[e_i, f_i] = \frac{k_i - k_i^{-1}}{q - q^{-1}}$) imply that there exists a modified basis such that the new matrix elements are without divergences. Using this definition, the equation (2.24) is reduced to
\[
f_l|p\rangle = \sum_{i=1}^l \left( \sum_{j=1}^l \frac{P_1(jl; p) \ P_2(jl; p)}{P_3(jl; p)} D_{ji} \right) \ |p'_{1l} \cdots p'_{il} \cdots p_{ul}\rangle
\] (2.27)
\[
= \sum_{i=1}^l A_{il} \ |p'_{1l} \cdots p'_{il} \cdots p_{ul}\rangle
\]
where, $A_{il}$ are the new matrix elements associated to the modified basis, i.e.
\[
A_{il} = \sum_{j=1}^l \frac{P_1(jl; p) \ P_2(jl; p)}{P_3(jl; p)} D_{ji}, \quad 1 \leq i \leq l.
\] (2.28)
this matrix elements are called the **modified matrix elements**. Generally, we choose

\[
A_{1l} = \left( \sum_{j=1}^{l} \frac{P_1^2(jl; p) P_2^2(jl; p)}{P_3^2(jl; p)} \right)^{1/2},
\]

\[
A_{il} = 0 \quad 2 \leq i \leq l,
\]
i.e.

\[
f_l |p \rangle = \left( \sum_{j=1}^{l} \frac{P_1^2(jl; p) P_2^2(jl; p)}{P_3^2(jl; p)} \right)^{1/2} \| p_{1l} - 1 \cdots p_l \rangle.
\]

We note that the matrix element of (2.30) is finite (i.e. without divergences).

We are now able to claim this generalization,

**Definition.3.** Let a collection \( \{I_k, k \in J \subset N\} \) of subsets of \( \{1, \cdots, l\} \) satisfying the following conditions:

1. if \( k < s \) \( \forall i \in I_k \) \( \forall j \in I_s \) thus \( i < j \),
2. \( I_k \cap I_s = \emptyset \), if \( k \neq s \),
3. \( \bigcup_{k \in J} I_k = \{1, \cdots, l\} \),

and, where

\[
p_{il} - p_{jl} = \lfloor m \rfloor, \quad i, j \in I_k, \quad i < j,
\]

\[
p_{il} - p_{jl} = \zeta_{ks} \lfloor m \rfloor, \quad i \in I_k, \quad j \in I_s, \quad k < s, \quad 1 \leq |\zeta_{ks}| \leq m - 1.
\]

This state is called a state of type \( (I_k, k \in J \subset N) \). Let \( \pi_{ks;ij} \) be the exchange mapping of level \( l \) between the subsets \( I_k \) and \( I_s \) \((k < s)\), i.e. if

\[
p_{il} - p_{jl} = \zeta_{ks} + (\beta_i + \beta_{i+1} + \cdots + \beta_{j-1}) m
\]

we have

\[
\pi_{ks;ij}(|p_{1l} \cdots p_{il} \cdots p_{jl} \cdots p_l \rangle) = |p_{1l} \cdots p'_{il} \cdots p'_{jl} \cdots p_l \rangle
\]

with

\[
p'_{il} = p_{jl} + (\beta_i + \beta_{i+1} + \cdots + \beta_{j-1}) m = p_{il} + \zeta_{ks},
\]

\[
p'_{jl} = p_{il} - (\beta_i + \beta_{i+1} + \cdots + \beta_{j-1}) m = p_{jl} - \zeta_{ks},
\]

The operators \( k_i \) commutes with exchange mapping \( \pi_{ks;ij} \). Let \( |p \rangle \) be state satisfying (2.32) and (2.33). Define

\[
V_l(I_k, k \in J) = \{ \prod_{k_\alpha \neq s_\alpha \atop i_\alpha \neq j_\alpha} \pi_{k_\alpha s_\alpha; i_\alpha j_\alpha}(|p_{1l} \cdots p_{il} \cdots p_{jl} \cdots p_l \rangle) \}
\]
The set $V_l(I_k, k \in J \subset N)$ is obtained by the all possible changes between the different subsets $I_k (k \in J)$. We note that the all states of $V_l(I_k, k \in J \subset N)$ have the same eigenvalues for the Cartan operators.

For example,

- $J = \{1\}, I_1 = \{1, \ldots, l\}, \dim V_l = 1$,
- $J = \{1, 2\}, I_1 = \{1\}, I_2 = \{2, \ldots, l\}, \dim V_l = l$,
- $J = \{1, 2\}, I_1 = \{1, 2\}, I_2 = \{3, \ldots, l\}, \dim V_l = \frac{1}{2}(l - 1)$.

**Definition.4.** Let a new basis given by

$$|p'_{1l} \cdots p'_{il} \cdots p'_{ll}⟩ = \sum_{j=1}^{l} D_{ij} |p'_{j1} \cdots p'_{jl} \cdots p'_{ll}⟩$$ (2.37)

where the states $|p'_{1l} \cdots p'_{il} \cdots p'_{ll}⟩$ are in the set $V_l(I_k, k \in J \subset N)$ and $D$ is a $\dim V_l \times \dim V_l$ rotation matrix. This new basis is called the **modified basis of type** $(I_k, k \in J \subset N)$. We note that the primitive and the modified bases satisfy the triangular inequalities. The set of states $|p'_{1l} \cdots p'_{jl} \cdots p'_{ll}⟩$ is noted by $V_l(I_k, k \in J \subset N)$.

Using the definition 3, the action of $f_l$ over a modified state of type $(\{1\}, \{2, \ldots, l\})$ produces two sets of states respectively of type $(\{1\}, \{2, \ldots, l\})$ and $(\{1, 2\}, \{3, \ldots, l\})$. The matrix elements of $(\{1\}, \{2, \ldots, l\}) \rightarrow (\{1\}, \{2, \ldots, l\})$...
do not contain divergences (see the structure of \(P_3(jl; p)\)). But if the matrix elements of \(\langle \{1\}, \{2, \ldots, l\} \rangle \rightarrow \langle \{1, 2\}, \{3, \ldots, l\} \rangle\) contain some divergences, using of the definition \(4\) we take a rotation in this second set in such a way as to eliminate these divergences (\(\langle p|e_l f_l|p \rangle\) and \(\langle p|f_l e_l|p \rangle\) have to remain finit). We have to repeat this mechanism as far as the elimination of the all divergences of the Gelfand–Zetlin representation.

3. Applications.

In the following, we present in details the explicit construction of atypical representation for \(N = 3\). We will consider only the operator \(f_1\) and \(f_2\), a similar discussion is valid for \(e_1\) and \(e_2\). The primitive Gelfand–Zetlin state for this particular case is just

\[
|p\rangle = \begin{pmatrix}
p_{13} & p_{23} & p_{33} \\
p_{12} & p_{22} & p_{32} \\
p_{11} & & \\
\end{pmatrix}
\]  

(3.1)

where \(p_{33}\) is chosen equal to zero. The actions of the generators \(f_1\) and \(f_2\) for \(U_q(sl(3))\) are given by

\[
f_1|p\rangle = \left(\frac{[p_{12} - p_{11} + 1][p_{11} - p_{22} - 1]}{p_{11} - 1}\right) |p_{11} - 1\rangle
\]  

(3.2)

\[
f_2|p\rangle = \left(\frac{[p_{13} - p_{12} + 1][p_{12} - p_{22} - 1][p_{12} - p_{33} - 1][p_{12} - p_{11}]}{[p_{12} - p_{22}][p_{12} - p_{22} - 1]}\right)^{\frac{1}{2}} |p_{12} - 1\rangle,
\]

\[
+ \left(\frac{[p_{13} - p_{22} + 1][p_{22} - p_{22} + 1][p_{22} - p_{33} - 1][p_{11} - p_{22}]}{[p_{12} - p_{22}][p_{12} - p_{22} + 1]}\right)^{\frac{1}{2}} |p_{22} - 1\rangle.
\]

(3.3)

We note that in this case there are only two type of sets \(\langle \{1, 2\} \rangle\) and \(\langle \{1\}, \{2\} \rangle\) and the maximal number of zero in the denominator of the matrix elements is only one.

Now suppose that

\[
p_{i3} - p_{j2} + 1 \not\equiv 0 \ [m], \quad 1 \leq i \leq 3, \quad \text{and} \quad 1 \leq j \leq 2,
\]

\[
p_{i2} - p_{11} \not\equiv 0 \ [m], \quad 1 \leq i \leq 2,
\]

i.e.

\[
\eta_{12} = \eta_{22} = 0.
\]

(3.4)

(3.5)

The matrix elements of \(f_2\) are infinite if

\[
(a) \quad p_{12} - p_{22} = 0 \ [m],
\]

\[
(b) \quad p_{12} - p_{22} + 1 = 0 \ [m],
\]

\[
(c) \quad p_{12} - p_{22} - 1 = 0 \ [m].
\]

(3.6)
Case (a). Let a state satisfy the following conditions

\[ p_{12} - p_{22} = 0 \quad [m] \quad \text{or} \quad p_{12} - p_{22} = \beta \quad m \quad \beta \in \mathbb{Z}_+ \]  

(3.7)

i.e.

\[ \eta'_{12} = \eta'_{22} = 1. \]  

(3.8)

The action of \( f_2 \) on this state gives

\[ f_2 |p_{12} \ p_{22}\rangle = \frac{\kappa}{([m][m-1])^{1/2}} |p_{12} - 1 \ p_{22}\rangle + \frac{\kappa}{([m][m+1])^{1/2}} |p_{12} \ p_{22} - 1\rangle, \]  

(3.9)

where

\[ \kappa = \left( [p_{13} - p_{22} + 1][p_{23} - p_{22} + 1][p_{22} - p_{33} - 1][p_{11} - p_{22}] \right)^{1/2}. \]  

(3.10)

Using the definitions 3 and 4, the relation between the primitive and the modified states is

\[
\begin{pmatrix}
|p_{12} - 1 \ p_{22}\rangle \\
|p_{22} + \beta \ m \ p_{12} - 1 - \beta \ m\rangle
\end{pmatrix}
= D(\phi) \begin{pmatrix}
|p_{12} - 1 \ p_{22}\rangle \\
|p_{22} + \beta \ m \ p_{12} - 1 - \beta \ m\rangle
\end{pmatrix},
\]

i.e.

\[
\begin{pmatrix}
|p_{12} - 1 \ p_{22}\rangle \\
|p_{12} \ p_{22} - 1\rangle
\end{pmatrix}
= D(\phi) \begin{pmatrix}
|p_{12} - 1 \ p_{22}\rangle \\
|p_{12} \ p_{22} - 1\rangle
\end{pmatrix},
\]  

(3.11)
for any values of \( p_{11} \) satisfying the triangular inequality respectively for the primitive and the modified basis, and where

\[
D(\phi) = \begin{pmatrix}
\cos \phi & \sin \phi \\
-\sin \phi & \cos \phi
\end{pmatrix}
\]  

(3.12)

Finally,

\[
f_2|p_{12} \ p_{22}\rangle = \kappa \left( \frac{\cos \phi}{([m][m-1])^{1/2}} - \frac{\sin \phi}{([m][m+1])^{1/2}} \right) \| p_{12} - 1 \ p_{22}\rangle
\]

\[+ \kappa \left( \frac{\sin \phi}{([m][m-1])^{1/2}} + \frac{\cos \phi}{([m][m+1])^{1/2}} \right) \| p_{12} \ p_{22} - 1 \rangle
\]  

(3.13)

Using the trivial identity,

\[
\frac{[a+1]}{[2][a]} + \frac{[a-1]}{[2][a]} = 1
\]  

(3.14)

choose

\[
\cos \phi = \left( \frac{[m-1]}{[2][m]} \right)^{1/2}, \quad \sin \phi = \left( \frac{[m+1]}{[2][m]} \right)^{1/2}
\]  

(3.15)

The r.h.s of (3.13) is reduced to

\[
f_2|p_{12} \ p_{22}\rangle = \left( \frac{[2][p_{13} - p_{22} + 1][p_{23} - p_{22} + 1][p_{22} - p_{33} - 1][p_{11} - p_{22}]}{[p_{12} - p_{22} - 1][p_{12} - p_{22} + 1]} \right)^{1/2} \| p_{11} \ p_{22} - 1 \rangle.
\]  

(3.16)

This equation correspond to the transition \( \{1, \ 2\} \to \{1\}, \{2\} \).

In this case the action of \( f_1 \) over the modified Gelfand–Zetlin basis is given by

\[
f_1 \| p_{12} - 1 \rangle = \begin{cases}
\left( p_{12} - p_{11} \right)p_{11} - p_{22} - 1 \right)^{1/2} \| p_{12} - 1 \rangle_{p_{11} - 1} & \text{if } p_{11} \neq p_{22} + 1 \\
[m - 1] \left( \frac{[m+1]}{[2]} \right)^{1/2} \| p_{22} - 1 \rangle_{p_{11} - 1} & \text{if } p_{11} = p_{22} + 1
\end{cases}
\]  

(3.17)

and

\[
f_1 \| p_{22} - 1 \rangle = \begin{cases}
\left( p_{12} - p_{11} + 1 \right)p_{11} - p_{22} \right)^{1/2} \| p_{22} - 1 \rangle_{p_{11} - 1} & \text{if } p_{11} \neq p_{22} + 1 \\
\left( \frac{[m-1]}{[2]} \right)^{1/2} \| p_{22} - 1 \rangle_{p_{11} - 1} & \text{if } p_{11} = p_{22} + 1
\end{cases}
\]  

(3.18)
Remark. If
\[ p_{12} - p_{22} + 1 = 0 \ [m] \quad \text{i.e.} \quad p_{12} - p_{22} + 1 = \beta m \quad (\beta \in \mathbb{Z}_+) \]  \hfill (3.19)
the relation between the primitive and the modified basis is given by
\[
\begin{pmatrix}
|p_{12} \ p_{22}\rangle \\
|p_{12} + 1 \ p_{22} - 1\rangle
\end{pmatrix}
= \begin{pmatrix}
\cos \phi & \sin \phi \\
-\sin \phi & \cos \phi
\end{pmatrix}
\begin{pmatrix}
\| p_{12} \ p_{22}\rangle \\
\| p_{12} + 1 \ p_{22} - 1\rangle
\end{pmatrix}
\]  \hfill (3.20)
where
\[
\cos \phi = \left( \frac{[p_{12} - p_{22}]}{[2][p_{12} - p_{22} + 1]} \right)^{1/2}, \quad \sin \phi = \left( \frac{[p_{12} - p_{22} + 2]}{[2][p_{12} - p_{22} + 1]} \right)^{1/2}. \hfill (3.21)
\]

Case (b). For a state satisfying the condition
\[ p_{12} - p_{22} + 1 = 0 \ [m] \quad \text{i.e.} \quad p_{12} - p_{22} + 1 = \beta m \quad (\beta \in \mathbb{Z}_+) \]  \hfill (3.22)
the correspondence between the primitive vectors and the modified vectors is given by the following formulas
\[
\begin{pmatrix}
\| p_{12} \ p_{22}\rangle \\
\| p_{12} + 1 \ p_{22} - 1\rangle
\end{pmatrix}
= \begin{pmatrix}
\cos \phi & -\sin \phi \\
\sin \phi & \cos \phi
\end{pmatrix}
\begin{pmatrix}
|p_{12} \ p_{22}\rangle \\
|p_{12} + 1 \ p_{22} - 1\rangle
\end{pmatrix}
\]  \hfill (3.23)
where \( \phi \) is given by (3.21). If we take an extension of the definition of the modified formula
\[
\begin{pmatrix}
\| p_{12} - 1 \ p_{22}\rangle \\
\| p_{22} + \beta m \ p_{12} - 1 - \beta m\rangle
\end{pmatrix}
= D(-\phi)
\begin{pmatrix}
|p_{12} - 1 \ p_{22}\rangle \\
|p_{22} + \beta m \ p_{12} - 1 - \beta m\rangle
\end{pmatrix}
\]  \hfill (i.e.
(3.24)
the action of \( f_2 \) over the modified states is reduced to
\[
f_2 \| p_{12} \ p_{22}\rangle =
\left( \frac{[p_{13} - p_{12} + 1][p_{12} - p_{23} - 1][p_{12} - p_{33} - 1][p_{12} - p_{11}]}{[p_{12} - p_{22}][p_{12} - p_{22} - 1]} \right)^{1/2} \| p_{12} - 1 \ p_{22}\rangle \]  \hfill (3.25)
and
\[
f_2 \| p_{12} + 1 \ p_{22} - 1\rangle =
\left( \frac{[p_{13} - p_{22} + 2][p_{23} - p_{22} + 2][p_{22} - p_{33} - 2][p_{11} - p_{22} + 1]}{[p_{12} - p_{22} + 2][p_{12} - p_{22} + 3]} \right)^{1/2} \| p_{12} + 1 \ p_{22} - 2\rangle +
\left( \frac{[2][p_{13} - p_{22} + 1][p_{23} - p_{22} + 1][p_{22} - p_{33} - 1][p_{11} - p_{22}]}{[p_{12} - p_{22}][p_{12} - p_{22} + 2]} \right)^{1/2} \| p_{12} \ p_{22} - 1\rangle \]  \hfill (3.26)
We note that the state $|p_{12} \ p_{22}-1\rangle$ is of type ($\{1, \ 2\}$). The equations (3.25) and (3.26) correspond respectively to the transitions ($\{1, \ 2\}$) $\rightarrow$ ($\{1, \ 2\}$) and ($\{1, \ 2\}$) $\rightarrow$ ($\{1, \ 2\}$) + ($\{1, \ 2\}$).

**Remark.** For example, the action of $f_2$ over the extension (3.24) gives

$$f_2 \parallel p_{12} - 1 \ p_{22} \rangle =$$

$$\left( \frac{[p_{13} - p_{12} + 2][p_{12} - p_{23} - 2][p_{12} - p_{33} - 2][p_{12} - p_{11} - 1]}{[p_{12} - p_{22} - 1][p_{12} - p_{22} - 2]} \right)^{1/2} \parallel p_{12} - 2 \ p_{22} \rangle + (3.27)$$

$$\left( \frac{[p_{13} - p_{22} + 1][p_{23} - p_{22} + 1][p_{22} - p_{33} - 1][p_{11} - p_{22}]}{[p_{12} - p_{22} - 1][p_{12} - p_{22}]} \right)^{1/2} \parallel p_{12} - 1 \ p_{22} - 1 \rangle$$

where

$$\begin{pmatrix}
\parallel p_{12} - 2 \ p_{22} \rangle \\
\parallel p_{22} + \beta m \ p_{12} - 2 - \beta \ m \rangle
\end{pmatrix} = D(-\phi)
\begin{pmatrix}
\parallel p_{12} - 2 \ p_{22} \rangle \\
\parallel p_{22} + \beta m \ p_{12} - 2 - \beta \ m \rangle
\end{pmatrix} \ (3.28)$$

$$\begin{pmatrix}
\parallel p_{12} - 1 \ p_{22} - 1 \rangle \\
\parallel p_{22} - 1 + \beta m \ p_{12} - 1 - \beta \ m \rangle
\end{pmatrix} = D(-\phi)
\begin{pmatrix}
\parallel p_{12} - 1 \ p_{22} - 1 \rangle \\
\parallel p_{22} - 1 + \beta m \ p_{12} - 1 - \beta \ m \rangle
\end{pmatrix} \ (3.29)$$

**Case (c).** The discussion is similar to the case (b). Let a primitive state satisfy the following condition

$$p_{12} - p_{22} - 1 = 0 \ [ m ] \ \ \text{in} \ \ p_{12} - p_{22} - 1 = \beta \ m \ \ (\beta \in \mathbb{Z}_+) \ \ (3.30)$$

Define

$$\begin{pmatrix}
\parallel p_{12} \ p_{22} \rangle \\
\parallel p_{22} + \beta m \ p_{12} - \beta \ m \rangle
\end{pmatrix} = D(-\phi)
\begin{pmatrix}
\parallel p_{12} - 1 \ p_{22} \rangle \\
\parallel p_{22} + \beta m \ p_{12} - \beta \ m \rangle
\end{pmatrix}$$

i.e.

$$\begin{pmatrix}
\parallel p_{12} \ p_{22} \rangle \\
\parallel p_{12} - 1 \ p_{22} + 1 \rangle
\end{pmatrix} = \begin{pmatrix}
\cos \phi & -\sin \phi \\
\sin \phi & \cos \phi
\end{pmatrix}
\begin{pmatrix}
\parallel p_{12} \ p_{22} \rangle \\
\parallel p_{12} - 1 \ p_{22} + 1 \rangle
\end{pmatrix} \ (3.31)$$

and take the extension

$$\begin{pmatrix}
\parallel p_{12} \ p_{22} - 1 \rangle \\
\parallel p_{22} - 1 + \beta m \ p_{12} - \beta \ m \rangle
\end{pmatrix} = D(-\phi)
\begin{pmatrix}
\parallel p_{12} - 1 \ p_{22} - 1 \rangle \\
\parallel p_{22} - 1 + \beta m \ p_{12} - \beta \ m \rangle
\end{pmatrix}$$
i.e.

\[
\begin{pmatrix}
\| p_{12} \: p_{22} - 1 \\
\| p_{12} - 2 \: p_{22} + 1
\end{pmatrix}
= \begin{pmatrix}
\cos \phi & -\sin \phi \\
\sin \phi & \cos \phi
\end{pmatrix}
\begin{pmatrix}
\| p_{12} \: p_{22} - 1 \\
\| p_{12} - 2 \: p_{22} + 1
\end{pmatrix}
\]

(3.32)

where

\[\cos \phi = \left( \frac{[p_{12} - p_{22}]}{[2][p_{12} - p_{22} - 1]} \right)^{1/2}, \quad \sin \phi = \left( \frac{[p_{12} - p_{22} - 2]}{[2][p_{12} - p_{22} - 1]} \right)^{1/2}.\]

(3.33)

We obtain

\[f_2 \| p_{12} \: p_{22} \rangle = \left( \frac{[p_{13} - p_{12} + 1][p_{12} - p_{23} - 1][p_{12} - p_{33} - 1][p_{12} - p_{11}]}{[p_{12} - p_{22}][p_{12} - p_{22} + 1]} \right)^{1/2} \| p_{12} \: p_{22} - 1 \rangle\]

(3.34)

and

\[f_2 \| p_{12} - 1 \: p_{22} + 1 \rangle = \left( \frac{[p_{13} - p_{22} + 1][p_{23} - p_{22} + 1][p_{22} - p_{33} - 1][p_{12} - p_{22}]}{[p_{12} - p_{22}][p_{12} - p_{22} + 1]} \right)^{1/2} \| p_{12} - 2 \: p_{22} + 1 \rangle + \left( \frac{[2][p_{13} - p_{12} + 1][p_{12} - p_{23} - 1][p_{12} - p_{33} - 1][p_{12} - p_{11}]}{[p_{12} - p_{22}][p_{12} - p_{22} - 2]} \right)^{1/2} | p_{12} - 1 \: p_{22} \rangle\]

(3.35)

We note that the state \(| p_{12} - 1 \: p_{22} \rangle\) is of type \(\{1, 2\}\). The equations (3.34) and (3.35) correspond respectively to the transitions \(\{1\}, \{2\} \rightarrow \{1\}, \{2\}\) and \(\{1\}, \{2\} \rightarrow \{1\}, \{2\} + \{1, 2\}\). For example, this method describes successfully the representation of \(Uq(sl(3))\) of dimension 15 for \(m = 3\) \((p_{13} = 5, p_{23} = 2, p_{33} = 0)\).

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