Spin–orbit precession for eccentric black hole binaries at first order in the mass ratio

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Abstract
We consider spin–orbit (‘geodetic’) precession for a compact binary in strong-field gravity. Specifically, we compute $\psi$, the ratio of the accumulated spin-precession and orbital angles over one radial period, for a spinning compact body of mass $m_1$ and spin $s_1$, with $s_1 \ll \frac{Gm_1^2}{c}$, orbiting a non-rotating black hole. We show that $\psi$ can be computed for eccentric orbits in both the gravitational self-force and post-Newtonian frameworks, and that the results appear to be consistent. We present a post-Newtonian expansion for $\psi$ at next-to-next-to-leading order, and a Lorenz-gauge gravitational self-force calculation for $\psi$ at first order in the mass ratio. The latter provides new numerical data in the strong-field regime to inform the effective one-body model of the gravitational two-body problem. We conclude that $\psi$ complements the Detweiler redshift $z$ as a key invariant quantity characterizing eccentric orbits in the gravitational two-body problem.

Keywords: gravitational self-force, gauge invariants, extreme mass ratio inspirals, geodetic effect, binary black holes, post-Newtonian theory

(Some figures may appear in colour only in the online journal)

1. Introduction
The year 2016 will surely come to be regarded as the annus mirabilis of gravitational wave astronomy. First, LIGO reported on primogenial detections of gravitational waves (GWs): three distinctive ‘chirps’ associated with the binary black hole mergers GW150914, GW151226 and, at marginal statistical significance, LVT151012 [1–3]. These discoveries suggest that second-generation ground-based detectors, operating at full sensitivity, may detect as many as
1000 black hole mergers per annum [4]. Second, the LISA Pathfinder mission reported on test masses maintained in almost-perfect freefall, with sub-Femto-g accelerations in the relevant frequency band [5]. The path is now clear for eLISA’s launch, circa 2034.

The purpose of a space-based mission such as eLISA [6] is to explore the low-frequency (10^{-4}–1 Hz) gravitational wave sky. Key sources in this band include extreme mass-ratio inspirals (EMRIs). In a typical EMRI, a compact body of mass \( m_1 \sim 1 \times 10^5 M_\odot \) spirals towards a supermassive black hole of mass \( m_2 \sim 10^2 \times 10^6 M_\odot \) under the influence of radiation reaction [7, 8]. EMRI modelling poses a stiff challenge to numerical relativity (NR) [9] due to the separation of scales implied by the mass ratio \( \sim \frac{m_1}{m_2} \approx 10^{12} \) to \( 10^{-7} \) (though see e.g. [10]); and to post-Newtonian (pN) theory [11–13], due to the strong-field nature of the orbit.

The gravitational self-force (GSF) programme [14–17], initiated two decades ago [18, 19], seeks to address the EMRI challenge by blending together black hole perturbation theory [20–23], regularization methods [24–28] and certain asymptotic-matching, singular-perturbation and multiple-scale techniques [18, 29–32]. The programme is influenced by some deep-rooted ideas in physics, such as Dirac’s approach to radiation reaction in electromagnetism [33, 34], the effacement principle and effective field theory [35, 36].

The ultimate aim of the GSF programme is to model the orbit (and gravitational wave-form) of a typical EMRI, as it evolves over several years through \( \sim 10^5 \) orbital cycles; without making ‘slow-motion’ or ‘weak-field’ approximations; with a final phase error of less than a radian. Fulfilment of the accuracy goal requires methods for computing dissipative self-force at next-to-leading-order in the mass ratio \( q \); see [37–39] for recent progress in this direction.

From one perspective, the compact body \( m_1 \) is accelerated away from a geodesic of the background spacetime of \( m_2 \) by a GSF, which may be split into a dissipative (‘radiation reaction’) and conservative piece with respect to time-reversal. The dissipative self-force at leading order in \( q \) has been known, in effect, since the 1970s, as it may be deduced from Teukolsky fluxes [22, 45–47]. By contrast, the more subtle consequences of the conservative self-force have only been explored in the last decade. An appealing perspective, offered by Detweiler & Whiting [25] and others [32, 48–50], is that a (non-spinning, non-extended) compact body follows a geodesic in a (fictitious) regularly-perturbed spacetime, \( g^R_{ab} = \bar{g}_{ab} + h_{ab} \), where \( \bar{g}_{ab} \) is the metric of the background spacetime parametrized by \( m_2 \), and \( h_{ab} = \mathcal{O}(q) \) is a certain smooth vacuum perturbation obtained by subtracting a ‘singular-symmetric’ piece from the physical metric. Conservative GSF effects are manifest as shifts at \( \mathcal{O}(q) \) in quantities defined on geodesics.

In 2008, Detweiler [51] showed that, for circular orbits, the \( \mathcal{O}(q) \) shift in the redshift invariant \( z = (\omega^t)^{-1} \frac{dr}{d\tau} \) for circular orbits (proportional to \( h^R_{ab} = \bar{h}^R_{ab} + h_{ab} \), where \( \bar{h}^R_{ab} \) is the geodesic tangent vector) is independent of the choice of gauge in GSF theory [52], within a helical class. As \( z \) is a physical observable (in principle at least), it may be calculated for any mass ratio via complementary approaches to the gravitational two-body problem. There has emerged a concordance in results in overlapping domains [53], between GSF theory [51, 52, 54, 55], post-Newtonian theory [51, 56–58], and, most recently, numerical relativity [59]. Moreover, the redshift invariant \( z \) has been found to play a leading role in the first law of binary black hole mechanics [60–62].

Invariants from GSF theory, such as \( z \), provide strong-field \( \mathcal{O}(q) \) information that can be applied to calibrate and enhance the effective one-body (EOB) model [63–66]. As the EOB model generates waveforms for binaries at \( q \sim 1 \), this provides a conduit for GSF results to flow towards data analysis at LIGO. A cottage industry has developed in identifying and
calculating invariants associated with conservative GSF at leading order in $q$. For circular orbits in Schwarzschild, the invariants comprise (i) the redshift invariant [51, 52] (relating also to the binary’s binding energy [67]), (ii) the shift in the innermost stable circular orbit (ISCO) [68], (iii) the periapsis advance (of a mildly-eccentric orbit) [68–70], (iv) the geodetic spin-precession invariant [71–75], (v) tidal eigenvalues at quadrupolar order [73, 76, 77], and (vi) tidal invariants at octupolar order [76–78]. There has been remarkable progress in expanding these GSF invariants to very high post-Newtonian orders [75, 79–81].

At present, three tasks are underway. First, the task of computing GSF invariants for a spinning (Kerr) black hole. For circular orbits, (i) the redshift invariant [82, 83] and (ii) the ISCO shift [84] have been calculated; (iii) the periapsis advance has been inferred from NR data [85]; but the higher-order invariants (iv)–(vi) remain to be found. Second, the task of identifying and computing invariants for non-circular geodesics. For eccentric orbits, the redshift is defined by $z = \frac{T_T}{T}$, where $T$ and $T$ are the proper-time and coordinate-time periods for radial motion. The redshift has been computed numerically [69, 86] and also expanded in a pN series [87–89] for eccentric orbits in Schwarzschild. Recently, it was found for equatorial eccentric orbits in Kerr, in [43] and [90], respectively. Yet generalized versions of the higher-order invariants (iv)–(vi) for eccentric equatorial orbits in Schwarzschild have not yet been forthcoming, and generic orbits in Kerr remain an untamed frontier. Third, there remains the ongoing task of transcribing new GSF results into the EOB model [63–66, 76, 88, 91]; and comparing against NR data [59, 70].

In this article, we consider spin–orbit precession for a spinning compact body of mass $m_1$ on an eccentric orbit about a (perturbed) Schwarzschild black hole of mass $m_2$. We focus on the spin-precession scalar $\psi$, which was defined in the circular-orbit context in [71] (see also [72–75]). The natural definition of $\psi$ for eccentric orbits is

$$\psi = \frac{\Phi - \Psi}{\Phi},$$

(1.1)

where $\Phi$ and $\Psi$ are, respectively, the orbital angle and the spin-precession angle (with respect to a polar-type basis) accumulated in passing from periapsis to periapsis (i.e. over one radial period). We shall compute $\Delta \psi$, the $O(m_1)$ contribution to $\psi$ at fixed frequencies, via both GSF and post-Newtonian approaches. In the pN calculation, we work with the spin–orbit Hamiltonian $H_{SO}$ through next-to-next-to-leading-order (NNLO) [92–94], neglecting spin-squared and higher contributions.

The article is organised as follows. In section 2 we review geodesic motion and spin precession of test-bodies on the Kerr spacetime. In section 3 we extend the gravitational self-force formalism in order to compute $\Delta \psi$ for eccentric orbits. In section 4 we calculate $\Delta \psi$ through NNLO from the spin–orbit Hamiltonian $H_{SO}$. In section 5 we present our numerical results, and confront the GSF data with the pN expansion. We conclude in section 6 with a discussion of future calculations.

**Conventions:** We set $G = c = 1$ and use the metric signature $+2$. Coordinate indices are denoted with Roman letters $a, b, c, \ldots$, indices with respect to a triad are denoted with letters $i, j, k, \ldots$, and general tetrad indices with $\alpha, \beta, \ldots$, etc, and numerals denote projection onto the tetrad legs. The coordinates $(t, r, \theta, \phi)$ denote general polar coordinates which, on the background Kerr spacetime, correspond to Boyer–Lindquist coordinates. Partial derivatives are denoted with commas, and covariant derivatives with semi-colons, i.e. $\nabla_a$. Symmetrization and anti-symmetrization of indices is denoted with round and square brackets, $(\ldots)$ and $[\ldots]$, respectively. Overdots denote derivatives with respect to proper time, i.e. $\dot{r} = \frac{dr}{d\tau}$. 
2. Geodesics and spin precession in the test-body limit

2.1. Geodetic spin precession

Let us consider a gyroscope, of inconsequential mass and size, moving along a geodesic \( \tau \) in a curved spacetime. The geodesic tangent vector \( \tau = \frac{du}{d\tau} \) (a unit timelike vector, \( g_{\mu\nu}u^\mu u^\nu = -1 \)) is parallel-transported, such that 
\[
D\tau u^a = 0,
\]
where \( D\tau u^a = \frac{du^a}{d\tau} + \Gamma^a_{\nu\lambda} u^\nu u^\lambda \). The gyroscope’s spin vector \( s_a \) (which is spatial, \( s_a u^a = 0 \)) is parallel-transported along the geodesic, \( D\tau s_a = 0 \); hence its norm \( s^2 = g_{ab} s_b s^b \) is conserved.

We may introduce a reference basis \( e^a_i \) (i.e. an orthonormal tetrad) along the geodesic, such that 
\[
\eta = g_{\mu\nu} e^\mu_i e^\nu_j = \eta_{ij},
\]
where \( \eta_{ij} = \text{diag}[-1, 1, 1, 1] \) and \( e^a_0 = u^a \). With this basis, we may recast the parallel-transport equation \( D\tau s_a = 0 \) in the beguiling form
\[
\frac{ds}{d\tau} = \omega \times s, \tag{2.1}
\]
where \( (s)_i \equiv e^a_i s_a \), \( (\omega)_i \equiv \frac{1}{2} \varepsilon_{ijk} \omega^{jk} \), and
\[
\omega_{ij} \equiv -g_{ab} e^a_i \frac{De^b_j}{d\tau} = -\omega_{ij}. \tag{2.2}
\]
Equation (2.1) merely describes how a parallel-transported basis varies relative to a reference basis (or ’body frame’). Note that \( \omega \) depends entirely on the choice of reference basis; in particular, if \( e^a_i \) is itself parallel-transported, then \( \omega = 0 \). An obvious way forward, then, is to choose a physically-motivated reference basis. For example, one could choose the eigenvectors of the \( 3 \times 3 \) tidal matrix \( C_{ij} \equiv C_{abced} e^a_i e^b_j e^c_k e^d_l \), where \( C_{abced} \) is the Weyl tensor. As \( C_{ij} \) is real and symmetric, the eigenvectors define a unique orthogonal basis (provided that the eigenvalues are distinct).

Let us suppose that a natural reference basis exists, and furthermore that \( \omega \) is fixed in direction (typically, orthogonal to the orbital plane). We may then align the reference basis so that \( (\omega)_1 = 0 = (\omega)_2 \) and \( (\omega)_3 = \omega_{13}(\tau) = e^a_3 g_{ab} \frac{De^b_3}{d\tau} \). Equation (2.1) has the solution
\[
s_1 + is_2 = S_i \exp \left( i \int_0^\tau \omega_{13}(\tau) d\tau \right), \quad s_2 = S_0, \tag{2.3}
\]
where \( S_i \) is a complex number satisfying \( s^2 = |S_i|^2 + S_0 \) with \( S_0 \) real and constant. Thus, the precession angle \( \Psi \) accumulated over one radial period \( T \) is given by
\[
\Psi = \int_0^T \omega_{13}(\tau) d\tau \tag{2.4}
\]
where \( T \) is the radial period with respect to proper time.

2.2. Discrete and continuous isometries

One may wish for a geometric definition of precession which does not depend on a choice reference basis. In curved space, a vector \( \nu^a \) parallel-transported around a closed path does not, in general, return to itself; and this immediately gives a geometric quantity. But in curved spacetime, timelike paths are not closed (except in pathological scenarios), so this procedure is not relevant.

For circular orbits, the spacetime and geodesic admit a continuous isometry (neglecting dissipative effects). That is, there exists a helical Killing vector field \( k^a \) for the spacetime
which aligns with the tangent vector \( u^a \) on the geodesic. In [71, 72] a natural precession quantity was defined directly from the helical Killing field itself.

By contrast, for generic orbits, there is no continuous isometry. However, for eccentric orbits in the equatorial plane there is a \textit{discrete} isometry, associated with the periodicity of the radial motion (neglecting dissipative effects). To make this notion more precise, let us adopt the passive viewpoint, in which there is a single spacetime system \( (\mathcal{M}, g) \) with a local region covered by two coordinate systems \( x^a \) and \( x'^a \), where the transformation between coordinates is sufficiently smooth that the usual transformation law applies, \( g'_{ab} = \frac{\partial x'^c}{\partial x^d} \frac{\partial x'^e}{\partial x^f} g_{ce} \) (in the transformation law it is implicit that the left-hand and right-hand sides are evaluated at the same spacetime point). The spacetime possesses an isometry if the metric components in the two coordinate systems are equal when evaluated at \textit{two different spacetime points which have the same coordinate values in the two systems}, \( x'^a = x^a \). We may extend this concept to the worldline: under this coordinate transformation \( z'^a(\tau') = \frac{\partial x'^a}{\partial \tau} \) where \( \tau' \) is a function of \( \tau \), we demand that \( z'^a(\tau' = \tau) = z^a(\tau) \).

To be more concrete, for equatorial eccentric orbits there is a discrete isometry under the linear transformation \( \tau' = \tau - \tau, r' = r, \theta' = \theta, \phi' = \phi - \Phi \) and \( \tau' = \tau - \tau, \) where \( \tau, \tau, \text{T} \) and \( \Phi \) are the coordinate time, proper time and orbital angle accumulated in passing from periapsis to periapsis.

How may we exploit the discrete isometry? We may restrict attention to reference bases that respect the isometry: tetrads \( e^a_i(\tau) \) transforming in the standard way \( e^a_i(\tau') = \frac{\partial \tau'}{\partial \tau} e^a_i(\tau) \), which satisfy \( e^a_i(\tau' = \tau) = e^a_i(\tau) \). Now consider a pair of such tetrads within the isometry-respecting class, related by \( \tilde{e}_i + ie_3 = e^{i\tau'}(e_i + ie_3) \). It is straightforward to show, from equation (2.4), that \( \Psi = \Psi - [\varphi(T) - \varphi(0)] \). As both tetrads respect the isometry, the last term is, at worst, a multiple of \( 2\pi \). We may eliminate this term by restricting attention to those triads that rotate once in passing around the black hole (like the spherical polar basis). Then, \( \Psi \) becomes insensitive to the choice of reference tetrad within a rather general class.

### 2.3. Geodetic spin precession for test bodies around black holes

#### 2.3.1. Generic geodestics in Kerr spacetime.

Consider the parallel transport of spin along a generic test-body geodesic with tangent vector \( u^a \) on the Kerr spacetime, in Boyer-Lindquist coordinates. The Kerr spacetime admits two Killing vectors, \( X^a_{(1)} \) and \( X^a_{(2)} \), satisfying \( X^a_{(b)}{ab} = 0 \), and one Killing-Yano tensor \( f_{abc} \), satisfying \( f_{(ab)c} = 0 \) and \( f_{abc} = f_{(abc)} \). There are three constants of motion: energy \( E = -X^a_{(1)} \mu_a \) azimuthal angular momentum \( L = X^a_{(2)} \mu_a \), and the Carter constant \( K = f^a_{abc} u^b u^c + (L - aE)^2 \) [95], in addition to the particle’s mass \( m_1 \).

The vector \( q^a \equiv f^a_{abc} u^b \) is parallel-propagated along the geodesic \((u^a q_{a,b} = f_{abc} u^b u^c + f_{abc} u^d u^b q_{d,b} = 0) \) and orthogonal to the tangent vector \((u^a q_{a} = u^a f_{abc} u^b = 0, \) by antisymmetry) [96]. Furthermore, its magnitude is set by the Carter constant: \( q^2 q_{a} = K \).

Marck [97] introduced a standard tetrad \( e^a_{(1)} \) with its zeroth leg along \( u^a \) and its second leg given by \( e^a_3 = q^a / \sqrt{K} \). The standard tetrad is given explicitly in equation (29) and (30) of [97]. The geometric properties of the standard tetrad are explored in [98]. Relative to this basis, the precession frequency is

\[
\omega_{13} = \frac{K^{1/2}}{r^2 + a^2 \cos^2 \theta} \left( \frac{E(r^2 + a^2) - aL}{r^2 + K} + a \frac{L - aE \sin \theta}{K - a^2 \cos^2 \theta} \right).
\] 

(2.5)
2.3.2. Equatorial geodesics in Kerr spacetime. For orbits confined to the equatorial plane \((\theta = \pi/2)\), the triad legs \(e_a^1\), \(e_a^2\) and \(e_a^3\) are eigenvectors of the electric tidal tensor \(C_{ij}\), with eigenvalues \(-{(2 + 3Kr^2)m_2/l^3}, (1 + 3Kr^2)m_2/l^3\) and \(m_2/l^3\), respectively. Thus, the reference basis has local physical significance. The second triad leg \(e_a^2\) reduces to the unit vector orthogonal to the plane. In addition, \(e_a^3\) in the plane is a function of \(r\) and \(\tau = \frac{2}{\dot{r}}\) only (and the constants of motion) and so it respects the discrete isometry (section 2.2). The precession frequency reduces to

\[
\omega_{13} = \frac{\sqrt{\mathcal{K}}}{r^2 + \mathcal{K}} \left( E + \frac{a}{L - aE} \right) \quad \text{(2.6)}
\]

2.3.3. Equatorial geodesics in Schwarzschild spacetime. Hereforth, we shall assume the black hole is non-rotating \((a = 0)\). In standard Schwarzschild coordinates \(t, r, \theta, \phi\) the line element is \(ds^2 = g_{ab}dx^adx^b = -fdr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\), where \(f(r) = 1 - 2M/r\). The Carter constant reduces to \(\mathcal{K} = L^2\), and the precession frequency to \(\omega_{13} = EL/(r^2 + L^2)\). Explicitly, the standard tetrad has the following components:

\[
e_0^a = u^a = [E \ell f, \dot{r}, 0, L/l^2], \quad e_1^a = [0, 0, 1/r, 0], \quad e_2^a = \frac{1}{f\sqrt{1 + L^2/r^2}} \left[ \dot{r}, fE, 0, 0 \right], \quad e_3^a = \frac{1}{r\sqrt{1 + L^2/r^2}} \left[ ELf, L\dot{r}, 0, 1 + L^2/r^2 \right],
\]

with \(\dot{r} = u^r\) determined from the energy equation,

\[
\dot{r}^2 = E^2 - f(1 + L^2/r^2).
\]

A bound eccentric geodesic may be parametrized by

\[
r(\chi) = \frac{p m_2}{1 + e \cos \chi},
\]

where \(\chi\) is the relativistic anomaly, and \(p\) and \(e\) are the (dimensionless) semi-latus rectum and eccentricity. The dimensionless energy \(E\) and angular momentum \(L\) are related to \(p\) and \(e\) by [86]

\[
E = \left[ \frac{(p - 2 - 2e)(p - 2 + e)}{p(p - 3 - e^2)} \right]^{1/2}, \quad L = \frac{p m_2}{\sqrt{p - 3 - e^2}}.
\]

Expressions for \(d\tau/d\chi, dr/d\chi,\) and \(d\phi/d\chi\) are given in equation (2.6), (2.7a) and (2.7b) of [86]. To these, we supplement

\[
\frac{d\Psi}{d\chi} = \omega_{13} \frac{d\tau}{d\chi} = \sqrt{\frac{p - 3 - e^2}{p - 6 - 2e \cos \chi}} \frac{\sqrt{(p - 2 - 2e)(p - 2 + e)}}{\sqrt{(p - 2 + 2e)(p - 2 - 2e)}} \left( p - 2 + 2e \cos \chi - e^2 \sin^2 \chi \right).
\]

We may find \(T, T, \Phi\) and \(\Psi\) by integrating over one radial period, e.g. \(\Psi = \int_0^{2\pi} \frac{d\phi}{d\chi} d\chi\). The orbital angle \(\Phi\) is given in terms of an elliptic integral in equation (2.9) of [86],

\[
\Phi = 4\sqrt{\frac{p}{p - 6 + 2e}} \text{ ellipK} \left( \frac{4e}{p - 6 + 2e} \right).
\]
The other quantities may not be written in a similarly compact form. However, one may easily calculate these numerically, or by expanding as a series in \(1/p\) as follows:

\[
\frac{T}{2\pi v} = \frac{1}{j_e} + \frac{3}{2j_e} p^{-1} + \left(\frac{3}{2} + \frac{6}{j_e} + \frac{3}{8j_e^2}\right) p^{-2} + \left(\frac{29}{2} + \frac{24}{j_e} + 3j_e + \frac{3}{4}j_e^2 - \frac{1}{16}j_e^3\right) p^{-3} \\
+ \left(\frac{96}{j_e} + \frac{1737}{16} + 18j_e + \frac{129}{16}j_e^2 - \frac{3}{4}j_e^3 - \frac{3}{16}j_e^4 + \frac{3}{128}j_e^5\right) p^{-4} + \mathcal{O}(p^{-5}).
\]

(2.13)

\[
\frac{T}{2\pi v} = \frac{1}{j_e} + \frac{3}{2j_e} p^{-1} + \left(\frac{15}{2} + \frac{6}{j_e}\right) p^{-2} + \left(\frac{75}{2} - \frac{30 - 6e^2}{j_e}\right) p^{-3} + \left(\frac{3729}{16} + \frac{96}{j_e} + 30j_e - \frac{75}{16}j_e^2\right) p^{-4} + \ldots,
\]

(2.14)

\[
\frac{\Phi}{2\pi} = 1 + 3p^{-1} + \left(\frac{27}{2} - \frac{3}{4}e^2\right) p^{-2} + \left(\frac{135}{2} + \frac{45}{4}e^2\right) p^{-3} + \left(\frac{2835}{8} + \frac{945}{8}e^2 + \frac{105}{64}e^4\right) p^{-4} + \mathcal{O}(p^{-5}),
\]

(2.15)

\[
\frac{\Psi}{2\pi} = 1 + \frac{3}{2}p^{-1} + \left(-\frac{6}{8} - \frac{3}{4}e^2\right) p^{-2} + \left(\frac{21}{8} + \frac{1}{8}e^2 - \frac{3}{16}e^4\right) p^{-3} \\
+ \left(\frac{29403}{128} + \frac{363}{32}e^2 - \frac{249}{64}e^4 - \frac{3}{32}e^6\right) p^{-4} + \mathcal{O}(p^{-5}),
\]

(2.16)

where \(v = mp^2\) and \(j_e = \sqrt{1 - e^2}\). Expansions for the redshift \(z = \frac{T}{TT}\) and the spin precession \(\psi = 1 - \Psi/\Phi\) follow immediately,

\[
z = 1 - \frac{3}{2}p^{-1} + j_e \left(-6 + \frac{39}{8}j_e\right) p^{-2} + j_e^2 \left(-23 + 6j_e + 30j_e^2 - \frac{235}{16}j_e^3\right) p^{-3} \\
+ j_e^3 \left(-\frac{249}{2} + 24j_e^2 + 174j_e^3 + 6j_e^4 - \frac{507}{4}j_e^5 + \frac{5643}{128}j_e^6\right) p^{-4} + \mathcal{O}(p^{-5}),
\]

(2.17)

\[
\psi = \frac{3}{2p^{-1} + \left(\frac{9}{8} + \frac{3}{2}e^2\right) p^{-2} + \left(\frac{27}{16} + \frac{33}{4}e^2 + \frac{3}{16}e^4\right) p^{-3} \\
+ \frac{405}{128} + \frac{705}{16}e^2 + \frac{123}{32}e^4 + \frac{3}{32}e^6\right) p^{-4} + \mathcal{O}(p^{-5}).
\]

(2.18)

Note that, in the circular limit \(e \to 0\), we have the exact results \(z = \sqrt{1 - 3/p}\) and \(\psi = 1 - \sqrt{1 - 3/p}\).

The radial and (average) azimuthal frequencies are defined via

\[
\Omega_r = \frac{2\pi}{T}, \quad \Omega_\phi = \frac{\Phi}{T}.
\]

(2.19)

The frequencies \(\{\Omega_r, \Omega_\phi\}\) can be measured by an observer at infinity, and thus provide an unambiguous parametrization for eccentric orbits. By contrast, the semi-latus rectum \(p\) and eccentricity \(e\) are defined only in terms of the periapsis and apsis radii, which are not invariant under a change of radial coordinate.
3. Gravitational self-force method

In section 2 we examined spin precession $\psi$ in the test-body limit. We now consider a compact body with a finite mass $m_1 \ll m_2$ and a small spin $s_1 \ll (G/c)m_1^2$, orbiting a Schwarzschild black hole of mass $m_2$, where $m_1$ and $m_2$ are the ADM masses.

3.1. Outline of scheme

We start by assuming that there exists a well-defined function $\psi(m_2 \Omega_e, m_2 \Omega_\phi, q)$, for any mass ratio $q = m_1/m_2$. We seek to isolate and compute the linear-in-$q$ part of this function using perturbation theory, that is,

$$\Delta \psi(m_2 \Omega_e, m_2 \Omega_\phi, q) = [\psi(m_2 \Omega_e, m_2 \Omega_\phi, q) - \psi(m_2 \Omega_e, m_2 \Omega_\phi, 0)]_{O(q)},$$

(3.1)

where the square parenthesis denote the $O(q)$ part. In this section, we shall denote test-body quantities using an overbar, i.e. $\bar{\Phi}, \bar{\Psi}, \bar{e}_a^\mu$, etc.

Underpinning our approach is the key result that, through $m_1$, a small slow-spinning compact body follows a geodesic in a regular perturbed spacetime $g_{ab}^R$ [25]; and, furthermore, its spin vector is parallel-transported in that same spacetime [32]. Here, we restrict to the small-spin regime $s \ll (G/c)m_1^2$ and neglect Mathisson–Papapetrou terms [99, 100]. Note that $g_{ab}^R$ is not the physical metric; rather, it is obtained by subtracting a certain ‘symmetric singular’ part from the physical metric, following the Detweiler–Whiting formulation [25] (details of the regularization procedure are given in section 3.4.2). The regularly-perturbed metric can be written $g_{ab}^R = g_{ab} + h_{ab}^R$, where $g_{ab}$ is the background Schwarzschild spacetime and $h_{ab}^R = O(m_1)$ is the metric perturbation. Henceforth, we shall omit the superscript $R$.

In our perturbative approach, we shall compare quantities defined on a worldline $\gamma$ in the regular perturbed spacetime $(\mathcal{M}, g)$ with quantities on a reference worldline $(\bar{\gamma}, \bar{g})$ in a background spacetime $(\mathcal{M}, \bar{g})$. In the regular-perturbed spacetime, we consider a geodesic $(0)$ with proper time $\tau$, worldline coordinates $z^\alpha(\tau)$, and orbital parameters $p, e$. In the background spacetime, there are at least three possible choices of reference worldline: (1) an accelerated worldline with $(p, e)$ on the background spacetime with the coordinates $z^\alpha(\bar{\tau})$, (2) a geodesic with $(p_0, e_0)$ on the background which becomes (1) under the influence of gravitational self force, and (3) a geodesic with $(p, e)$ on the background. In each case, the orbit may be parameterized using equation (2.9). Note that, for a given relativistic anomaly $\chi$, the coordinate difference between (2) and (3) is $O(m_1)$. Hence, at leading order in $m_1$ the instantaneous self-force computed on (3) is the same as on (2). Thus we may exclusively use (3) and dispense with (1) and (2).

We use the symbol $\delta$ to denote the difference at $O(m_1)$ between a quantity on geodesic $(0)$ and the same quantity on geodesic (3), implicitly making the choice $\bar{\chi} = \chi$ in the comparison, e.g. $\delta e_\alpha^\mu = e_\alpha^\mu(\chi) - e_\alpha^\mu(\bar{\chi} = \chi)$. Since geodesics (0) and (3) have the same orbital parameters $p, e$, this implies that we are comparing at the same coordinate radius $r$, though not the same $t$ and $\phi$ coordinates. We should emphasize that any such difference $Y$ is not gauge-invariant, in general.

We will also use $\delta$ to denote the variation in those quantities which are defined via orbital integrals, such as $T, T, \Phi$ and $\Psi$. Let $Y$ denote some physical quantity defined by an integral around a geodesic, and let $y = dy/dr$ denote its local frequency. (For example, $Y \in \{T, T, \Phi, \Psi\}$ and $y \in \{1, u^t, u^\phi, \omega_1\}$.) As in section 2.3.3, the background quantity $\bar{Y}$ is found by integrating $y$ from periapsis to periapsis.
\[ \bar{Y} \equiv \int_0^{2\pi} \frac{d\bar{r}}{d\chi} d\chi. \] (3.2)

The first-order variation \( \delta Y \) is found via the integral

\[ \delta Y = \int_0^{2\pi} \left( \frac{\delta y}{y} - \frac{\delta y}{a'} \right) \frac{d\bar{r}}{d\chi} d\chi. \] (3.3)

Barack and Sago [69] (henceforth BS2011) have shown how to apply the GSF formalism to calculate \( \delta \Phi \) and \( \delta T \) for eccentric orbits, and thus also the frequency shifts \( \delta \Omega_r = \delta (2\pi/T) = -(\delta T/T) \Omega_r \) and \( \delta \Omega_\phi = \delta (\Phi/T) = (\delta \Phi/\bar{T} - \delta T/\bar{T}) \Omega_\phi \). We will follow their approach, and extend it to calculate \( \delta \Psi \).

Recall that we seek the \( q(\Omega) \) shift at fixed \( \Omega_r, \Omega_\phi \) (equivalently, fixed \( \Phi \) and \( T \)), denoted by \( \Delta Y \). This is given by

\[ \Delta Y = \delta Y - \frac{\partial \bar{Y}}{\partial \bar{T}} \delta T - \frac{\partial \bar{Y}}{\partial \bar{\Psi}} \delta \bar{\Psi}. \] (3.4)

The latter terms may be found by applying the chain rule, i.e.

\[ \frac{\partial \bar{Y}}{\partial \bar{T}} = \frac{\partial \bar{Y}}{\partial p} \frac{\partial p}{\partial T} + \frac{\partial \bar{Y}}{\partial e} \frac{\partial e}{\partial \bar{T}} \] (3.5)

with \( \{ \frac{\partial p}{\partial T}, \ldots \} \) obtained by inverting \( \{ \frac{\partial T}{\partial p}, \ldots \} \).

It follows from the definition that \( \Delta T = \Delta \Phi = \Delta \Omega_r = \Delta \Omega_\phi = 0 \). It follows from equation (1.1) that

\[ \Delta \psi = -\frac{\Delta \Psi}{\bar{\Phi}}. \] (3.6)

(And it follows from \( z \equiv \bar{T}T \) that \( \Delta z = \Delta \bar{T}T \)). In sections below we focus on calculating \( \delta \Psi \) and thus \( \Delta \Psi \).

3.2. Circular orbit limit

Here we pause to consider the circular-orbit limit of \( \Delta \psi \). Naively, one might expect \( \lim_{e \to 0} \Delta \psi \) to reduce to \( \Delta \psi^{\text{circ}} \), the quantity calculated for circular orbits in [71–75]. On the other hand, \( \Delta \psi^{\text{circ}} \) is defined by comparing circular orbits in the perturbed and unperturbed spacetimes with the same azimuthal frequency \( \Omega_\phi \), whereas \( \Delta \psi \) is defined by comparing not-necessarily-circular orbits in the perturbed and unperturbed spacetimes with the same pair of frequencies \( \Omega_r \) and \( \Omega_\phi \). By fixing both frequencies, the orbit in the perturbed spacetime will not necessarily be circular even when the background orbit is so. As conceptually-different comparisons are made, it is plausible that \( \lim_{e \to 0} \Delta \psi = \Delta \psi^{\text{circ}} \), and so it proves.

Let us consider the definitions in the \( e \to 0 \) limit,

\[ \Delta \psi^{\text{circ}} = \delta \psi^{\text{circ}} - \frac{d\psi^{\text{circ}}}{d\Omega_\phi} \delta \Omega_r^{\text{circ}} \],

\[ \lim_{e \to 0} \Delta \psi = \lim_{e \to 0} \delta \psi - \lim_{e \to 0} \left[ \frac{\partial \psi}{\partial \Omega_r} \delta \Omega_r + \frac{\partial \psi}{\partial \Omega_\phi} \delta \Omega_\phi \right]. \] (3.8)
It is straightforward to establish that the first terms are equal: $\delta \psi^\text{circ} = \lim_{\epsilon \to 0} \delta \psi$. On the other hand, the latter terms are not, and we are left with an offset term,

$$\lim_{\epsilon \to 0} \Delta \psi - \Delta \psi^\text{circ} = \frac{d \tilde{\psi}^\text{circ}}{d \tilde{\Omega}_\phi} \delta \Omega_\phi - \lim_{\epsilon \to 0} \left[ \frac{\partial \tilde{\psi}^\gamma}{\partial \Omega_\gamma} \delta \Omega_\gamma + \frac{\partial \tilde{\psi}}{\partial \Omega_\phi} \delta \Omega_\phi \right]$$

(3.9)

$$= \frac{2p^{3/2} \sqrt{p-3} (p-6)^{3/2}}{4p^2 - 39p + 86} \left[ \delta \Omega_\gamma \to 0 - \frac{p-8}{\sqrt{p(p-6)}} \delta \Omega_\phi \to 0 \right]$$

$$= \frac{2\sqrt{p-3} (p-6)^{3/2}}{p(4p^2 - 39p + 86)} \Delta \Phi^{0 \to 0},$$

(3.10)

where we have used

$$\lim_{\epsilon \to 0} \frac{\partial \tilde{\psi}}{\partial \Omega_\phi} = -\frac{2p^{3/2} \sqrt{p-3} (p-6)^{3/2}}{4p^2 - 39p + 86},$$

(3.11)

$$\lim_{\epsilon \to 0} \frac{\partial \tilde{\psi}^\gamma}{\partial \Omega_\phi} = \frac{d \tilde{\psi}^\text{circ}}{d \tilde{\Omega}_\phi} + \frac{2p(p-8)(p-6)\sqrt{p-3}}{4p^2 - 39p + 86},$$

(3.12)

and

$$\Delta \Phi^{0 \to 0} = \delta \Phi^{0 \to 0} - \lim_{\epsilon \to 0} \frac{d \tilde{\Phi}}{d \tilde{\Omega}_\phi} \delta \Omega_\phi = -\frac{\tilde{\Phi}^{0 \to 0}}{\tilde{\Omega}^{0 \to 0}} \left( \delta \Omega_\gamma \to 0 - \frac{p-8}{\sqrt{p(p-6)}} \delta \Omega_\phi \to 0 \right)$$

$$= -\frac{2p^{3/2}}{p-6} \left( \delta \Omega_\gamma \to 0 - \frac{p-8}{\sqrt{p(p-6)}} \delta \Omega_\phi \to 0 \right).$$

(3.13)

We recognise $\Delta \Phi^{0 \to 0}/(2\pi)$ as the $O(q)$ part of the periapsis advance per unit angle $K$ [63, 64, 69]. It is a physical quantity which is gauge-invariant (in the usual GSF sense), and it has been calculated elsewhere. The significance of this quantity in GSF theory, pN theory and numerical relativity was explored in [70]. Thanks to recent advances in GSF technology, $\lim_{\epsilon \to 0} \Delta \psi$ can now be computed analytically up to $O(p^{-1/2})$ via the pN expansion for $\Delta \psi^\text{circ}$ of [75, 81] known up to $O(p^{-3/2})$ ($O(p^{-1/2})$ online [101]) and the EOB $\rho$ function of [89] known up to $O(p^{-1/2})$ using $\Delta \Phi^{0 \to 0}/(2\pi) = -(1-6y)^{-3/2}/(2\pi)$, where $y \equiv 1/p$. For computations done in Lorenz gauge, which is not asymptotically flat, we must add $2qy/(1-6y)^{3/2}$ to $\Delta \Phi^{0 \to 0}/(2\pi)$.

In appendix we give an explicit expression for $\Delta \psi^{0.5}$, the analytically-known part of $\lim_{\epsilon \to 0} \Delta \psi$ up to $O(y^{1/2})$. A comparison of $\Delta \psi^{0.5}$ and our numerical results for $\lim_{\epsilon \to 0} \Delta \psi$ is shown in figure 2.

3.3. Formulation

For notational simplicity, we now drop the over-bar notation for denoting background quantities.

3.3.1. Perturbation of the tetrad. We start by writing the perturbed tetrad $e'^\alpha_i$ in the following way.
\begin{align}
\delta u^a &= \delta e^0_a = c_{00}u^a + c_{01}e_1^a + c_{03}e_3^a, \\
\delta e_1^a &= c_{10}u^a + c_{11}e_1^a + c_{13}e_3^a, \\
\delta e_3^a &= c_{30}u^a + c_{31}e_1^a + c_{33}e_3^a.
\end{align}

(3.14)

where \(c_{\alpha \beta}\) are coefficients at \(O(m_1)\), to be deduced. (N.B. the second leg \(e_2^a\) remains orthogonal to the orbital plane.) Now we impose the orthogonality conditions \(g_{ab}e_a^ae_\beta = \eta_{\alpha \beta}\), to deduce that

\begin{align}
c_{00} &= \frac{1}{2}h_{00}, & c_{11} &= -\frac{1}{2}h_{11}, & c_{33} &= -\frac{1}{2}h_{33}, \\
c_{10} &= h_{01} + c_{01}, & c_{30} &= h_{03} + c_{03}, & c_{13} + c_{31} &= -h_{13},
\end{align}

(3.15)

where

\[h_{\alpha \beta} \equiv g_{ab}e_a^ae_\beta.\]

(3.16)

The tangent vector \((u^a = e_0^a)\) may be written in the following form,

\[u^a = [f^{-1}(E + \delta E), \hat{r} + \delta \hat{r}, 0, (L + \delta L)/r^2].\]

(3.17)

The quantities \(\delta E\) etc are straightforwardly related to the quantities \(\delta E\) etc used in BS2011 [69] (appearing there as \(\Delta E\), etc), via \(\delta E = \frac{1}{2}h_{00}E + \delta E\), etc. (The difference arises because BS2011 use a tangent vector \(\hat{u}^a\) normalized on the background spacetime, whereas we prefer a tangent vector normalized on the perturbed spacetime.) Note that the quantities \(\delta E, \delta L\) and \(\delta \hat{r}\) are functions of \(\chi\), i.e., they are not constants. A procedure for calculating \(\delta E\) and \(\delta L\) is given in [69], and \(\delta \hat{r}\) may be deduced from the normalization condition

\[E\delta \hat{r} - \hat{r}\delta \hat{r} - \frac{f'(r)}{r^2} L \delta L = 0.\]

(3.18)

Comparing equation (3.17) with equation (3.14) yields

\[c_{01} = \frac{E \hat{r}}{f\sqrt{1 + L^2/r^2}} \left(\frac{\delta \hat{r}}{r} - \frac{\delta E}{E}\right), \quad c_{03} = \frac{\delta L}{r\sqrt{1 + L^2/r^2}}.\]

(3.19)

Inserting equation (3.19) into equation (3.14) determines the tetrad, up to the single degree of freedom implicit in \(c_{13} + c_{31} = -h_{13}\). This local ambiguity is to be expected, as we are free to locally rotate the reference basis in the 13 plane. The ambiguity is removed by considering the secular change over radial period, using the discrete isometry.

3.3.2. Perturbation of precession frequency. Now we turn attention to the leading-order variation of the precession frequency. As \(\omega_{13} = g_{ab}e_3^a\frac{\delta e_3^b}{\delta r}\) is antisymmetric in its indices, we shall consider the variation of \(\omega_{13}\) and \(\omega \equiv \omega_{[13]}\) separately. The former is identically zero, and it merely provides a validation check of our implementation. Applying the variation operator \(\delta\), and using the product rule \(\delta(AB) = A\delta B + B\delta A\), yields

\[0 = \frac{1}{4} \omega(h_{33} - h_{11}) + g_{ab}e_3^a\delta(De_3^b/d\tau),\]

(3.20)

\[\delta \omega = \frac{1}{4} \omega(h_{33} + h_{11}) + g_{ab}e_3^a\delta(De_3^b/d\tau).\]

(3.21)
Here we have used the background identities $\frac{De^d}{dr} = 0$, $\frac{Dc}{dr} = \omega e^d_3$ and $\frac{Dc^d}{dr} = -\omega e^d_1$, and equation (3.15). The last terms need more careful consideration. We obtain

$$0 = \frac{1}{2} \omega (h_{33} - h_{11}) - \frac{1}{2} \frac{dh_{13}}{dr} + \delta \Gamma_{(31)0},$$

(3.22)

$$\delta \omega = \frac{1}{2} \omega h_{00} + \delta \Gamma_{(31)0} + (c_0 e^b_1 + c_3 e^b_3) e_a e_d \nabla e^a_1 e^d_1 + \frac{1}{2} \frac{d(c_{13} - c_{31})}{dr},$$

(3.23)

where

$$\delta \Gamma_{(31)0} \equiv \delta \Gamma_{ab} e_a e_d e^d_1 \mu^c = (\delta \Gamma_{abc} - h_{ad} \Gamma_{be} e^a_1 e^d_1 \mu^c),$$

(3.24)

with $\delta \Gamma_{abc} \equiv \frac{1}{2} \left( h_{abc} + h_{ac,b} - h_{bc,a} \right)$.

Let us make several brief observations on equations (3.22) and (3.23): (i) equation (3.22) is simply an identity, as can be verified by noting that $\delta \Gamma = \nabla = e_a e_d (\delta \nabla)^{\prime} e^a_1 e^d_1$; (ii) equation (3.23) is not uniquely defined in a local sense, due to the ambiguity in the last term (the freedom to locally rotate the tetrad), but this ambiguity is eliminated once we integrate over a radial period and impose the discrete isometry condition; (iii) for the case of circular orbits, equation (3.23) agrees with equation (2.65) of [73] (noting the difference in notation used in [73], \(\delta \Gamma_{(31)0} = \delta \Gamma_{(13)0} = h_{ad} \Gamma_{be} e^a_1 e^d_1 \mu^c\), and neglecting the final term); and (iv) the penultimate term in equation (3.23) requires the background tetrad to be treated as a field, rather than just a basis on the background worldline itself.

Let us consider point (iv) in more detail. For given $E, L$, the background tetrad field is a function of $r$ only, with $\dot{r}(r)$ interpreted as a function of $r$ determined from the energy condition, $\dot{r}^2(r) = E^2 - f(r)(1 + L^2/r^2)$. The tetrad is defined everywhere within $p/(1 + e)$ and $p/(1 - e)$; it is orthonormal everywhere; and $e^0_0$ is tangent to a geodesic everywhere in this region. Explicitly,

$$c_0 e^b_1 e_a e_d \nabla e^a_1 e^d_1 = \frac{\omega}{fr(1 + L^2/r^2)} \left( 1 + \frac{L^2}{r^2} \right) - \frac{E^2 L^2}{r} \left( \frac{\dot{r}}{\dot{r}} - \frac{\dot{E}}{E} \right),$$

(3.25)

$$c_3 e^b_3 e_a e_d \nabla e^a_1 e^d_1 = \omega \left( \frac{1 + 2L^2/r^2}{1 + 2L^2/r^2} \right) \frac{\dot{L}}{L}.$$

(3.26)

To proceed, we may insert equation (3.23) with equations (3.25) and (3.26) into the integral formula,

$$\delta \Psi = \int_0^{2\pi} \left( \frac{\delta \omega}{\omega} - \frac{\delta \dot{r}}{\dot{r}} \right) \omega \frac{d\tau}{d\chi} d\chi.$$

(3.27)

We then find $\Delta \Psi$ using equation (3.4), i.e. $\Delta \Psi = \delta \Psi - \frac{\partial \delta \Phi}{\partial T}$ and $\Delta \Phi$, and insert this into equation (3.6) to obtain $\Delta \psi$.

### 3.4. Numerical computation

Below we describe the numerical implementation of the GSF method section 3.4.1 and the regularization procedure section 3.4.2.
3.4.1. Implementation details. We perform the numerical computation using the GSF code of [102] which employs a frequency-domain approach, with the method of extended homogeneous solutions [103], to compute the components of the regularized metric perturbation $\bar{h}_{ab}$ in Lorenz gauge. This code has already been used (i) to evolve EMRIs via the osculating geodesics method in [104], (ii) to obtain a large set of eccentric-orbit data for the redshift invariant in [86], and (iii) to numerically determine the EOB $\dot{d}$ and $\varphi$ potentials in [66]. We have also developed the code to compute, in addition, the scalars $h_{1i}, h_{3i}, \delta \Gamma_{130}, \delta \Gamma_{1310}$ in equations (3.22) and (3.23). We have also calculated the regularization parameters for these quantities which we present in section 3.4.2.

The code samples the interval $\chi \in [0, 2\pi]$ at 240 evenly spaced points $\chi_i$ where it outputs $h_{ab}, h_{1i}, h_{3i}, F_r, F_\phi, \delta \Gamma_{130}, \delta \Gamma_{1310}$ at double floating point precision. Here, $F_r, F_\phi$ are the conservative parts of the radial and angular components of the GSF. We use Mathematica’s interpolation function to convert the discrete data sets into continuous functions suitable for numerical integration. We use two different orders of interpolation, three and six, and calculate $\Delta \psi$ from this difference. Let us just mention that this replacement requires evaluating $\Delta \psi$ arising from this difference is our estimated interpolation error for $\Delta E$.

The interpolated data is sufficiently smooth for numerical integration. However, there are a few troublesome terms that come from double numerical integrals that arise from the $c_{1i}, c_{0j}$ terms. These are functions of $\delta E(\chi), \delta \Gamma(\chi), \delta \dot{\Gamma}(\chi)$ each of which is an integral of the components of the GSF as given by BS2011

\[
\delta E(\chi) = \delta E(0) - \int_0^\chi d\chi' F_r(\chi') \frac{d\Gamma}{d\chi'}, \quad \delta \Gamma(\chi) = \delta \Gamma(0) + \int_0^\chi d\chi' F_\phi(\chi') \frac{d\Gamma}{d\chi'},
\]

where $\delta E(0), \delta \Gamma(0)$ are the $O(m)$ corrections to energy and angular momentum at the periapsis as explained by BS2011. $\delta \dot{\Gamma}(\chi)$ can be obtained from equation (3.18). Each of these terms are in turn integrated over a radial period in in equation (3.27), which can be problematic. We deal with this issue by first algebraically simplifying the coefficients of $\delta E(\chi), \delta \Gamma(\chi)$ by replacing the numerical values of the $\chi = 0, \pi$ endpoints of the integrand in equation (3.27) with analytical limits. More specifically, we first rewrite $\delta \dot{\Gamma}(\chi)$ using equations (3.25) and (3.26). Then, we use $d\dot{\Gamma}(\chi) = \frac{1}{2} h_{00} \dot{r} + \delta \dot{r}$ in equation (3.27) to remove the $\frac{1}{2} h_{00} \dot{r}$ term in equation (3.23). Finally, we rewrite the remaining terms proportional to $\delta \dot{r}$ in terms of $\delta E$ and $\delta \Gamma$ using equation (3.18). In the end, we are left with a term of the form

\[
\hat{f}_1(\chi) \frac{\delta E(\chi) + \delta \Gamma(\chi)}{\sin^2 \chi} + \hat{f}_2(\chi) \frac{\delta \pi(\chi) + \delta \pi(\chi)}{4 \cos^2 \frac{\chi}{2}} + \hat{f}_3(\chi),
\]

where $\hat{f}_1, \hat{f}_2, \hat{f}_3$ depend only on background quantities and are regular at $\chi = 0, \pi$. Although equation (3.29) looks like it diverges at $\chi = 0, \pi$, the analysis in BS2011 has proven these endpoints to be removable singularities. Therefore, we can replace the divergent values by the analytic limits at $\chi = 0, \pi$. The details of this procedure are provided in BS2011. Let us just mention that this replacement requires evaluating $dF_r, dF_\phi, dF_r, dF_\phi$ at $\chi = 0, \pi$ which we can obtain in a straightforward manner using one-sided finite-difference derivative formulæ.

The GSF code uses several approximations and truncations which we explain next. The method of extended homogeneous solutions constructs the spherical-harmonically decomposed fields $h^{\text{inhom}}_{ab}(t, r)$ (time-domain solutions) by summing over the Fourier modes of radial motion labeled by $n$. By construction, the $n$ sum converges exponentially, however, we can only compute a finite number of $n$ modes. So we truncate the $n$ sum by imposing a convergence
criterion based on the $C^0$ continuity of the $h_{ab}^{R,lm}(t, r)$ at \( r = r(\chi) \equiv r_p \); with each $n$ mode added to the sum we consider the difference $dh = \left| h_{ab}^{R,lm}(t, r \to r_p^0) - h_{ab}^{R,lm}(t, r \to r_p^-) \right|$. We include $\psi_\delta \alpha \psi$.

\[ \Phi_{lA B}^{\text{GSF}} = \frac{1}{\lambda} \int_{\Omega} \frac{\partial h_{ab}^{R,lm}(t, r_p)}{\partial r} \frac{\partial h_{ab}^{R,lm}(t, r_p)}{\partial r} \, d\Omega. \]

We display the leading digit of the corresponding error in parentheses. For $p$ and $e$, with each.

\[ \Delta = \frac{\delta \psi}{\psi} \]

\[ \lim_{t \to 0} \frac{\Delta \psi}{\psi} = -2 \alpha \]

\[ \lim_{t \to 0} \frac{\Delta \psi}{\psi} = 0, \text{ where } \alpha \text{ is a constant. This gauge ‘unpleasantness’ can be removed by transforming the original Lorenz-gauge time coordinate by } t \to t(1 + \alpha). \]

\[ \Delta \psi(p, e) \]

Finally, we must deal with the fact that Lorenz gauge is not asymptotically flat, i.e. $\lim_{t \to \infty} h_{ab}^{R,l,20} = -2 \alpha = 0$, where $\alpha$ is a constant. This gauge ‘unpleasantness’ can be removed by transforming the original Lorenz-gauge time coordinate by $t \to t(1 + \alpha)$. This shift in $t$ manifests itself in the orbital period $T$ hence in the orbital frequencies $\Omega_\alpha, \Omega_\psi$. Returning to equation (3.4) and inserting this correction yields

\[ \Delta = (1 + \alpha) \frac{\partial \psi}{\partial r} \Delta T = (1 - \alpha) \frac{\partial \psi}{\partial \Phi} \delta \Phi. \quad (3.30) \]

This ‘flat-fixing’ or ‘flattening-out’ has become standard in Lorenz-gauge GSF calculations and must be done regardless of the type of motion. Indeed, this correction was first done for circular orbits [52]. The details of this correction for eccentric orbits can be found in section III.B of BS2011. $\alpha$ can be computed to arbitrary precision from the monopole solution which is obtained analytically in the frequency domain [102]. Without this correction, one can not obtain the correct result for any invariant that one computes in Lorenz gauge.

We present our numerical results for $\Delta \psi$ in table 2 for eccentric orbits with $p = \{10, 15, 20, 25, \ldots, 95, 100\}$ and $e = \{0.05, 0.075, 0.1, \ldots, 0.225, 0.25\}$. We include our code’s circular-orbit result, $\Delta \psi^{\text{circ}}$, at the top row of each sub-table. Recall that $\Delta \psi^{\text{circ}} = \lim_{\epsilon \to 0} \Delta \psi$ hence the sign disagreement between the $e = 0$ and $e > 0$ values. For each value $\Delta \psi(p, e)$ we display the leading digit of the corresponding error in parantheses. For example, our result for $\Delta \psi(p = 10, e = 0)$ should be read as $5.9385659 \times 10^{-3} \pm 5 \times 10^{-10}$. We present analysis and plots in section 5.

### 3.4.2. Regularization

We employ the standard method of mode-sum regularization. Schematically, an unregularized quantity $X^{\text{full}}$ with $\ell$-modes $X^{\text{full}}_{\ell \pm}$ in the limit $r \to r_0^\pm$ is converted to a regularized quantity $X^R$ using

\[ X^R = \sum_{\ell = 0}^{\infty} \left[ X^{\text{full}}_{\ell \pm} - (2l+1)A^\pm - B \right]. \quad (3.31) \]

The regularization parameters $A^\pm$ and $B$ for the relevant quantities in equation (3.23) are listed in table 1. We note that all $A^\pm$ coefficients vanish.

$^3$ $h_{ab}^{R,lm}$ are obtained from $\sum_{\ell \Omega} \delta t \phi h_{ab}^{R,lm}$. 

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Class. Quantum Grav. 34 (2017) 084001

S Akcay et al
To obtain the regularization parameters (RPs) for each quantity in table 1, we started with RPs for the spacetime components of metric perturbation and its partial derivatives with respect to Schwarzschild coordinates. These were calculated by Wardell [106] using the approach developed in [107–110]. We label these RPs by $A_{ab} = 0, B_{ab}, A_{abc}, B_{abc}$ consistent with equation (3.31), where the rank-3 RPs are for $\partial_l h_{ab}$. Using the background tetrad we construct the appropriate RPs for $h_{11}, h_{33}$ etc which we label by $B_{11}, B_{33}$ etc. We find that $A_{30} = A_{310} = 0$, which is noteworthy since the $\delta l$ terms constituted sums of up to 15 different terms for $\partial_l h_{ab}$. This was further confirmed by our numerical data for the unregularized $l$ modes for $\delta l_{10,310}$ which approached constant values as $l \to \infty$. If $A \neq 0$ then we would have observed linear-in-$l$ growth for these unregularized modes. In figure 1 we show the regularized $l$ modes of $\{h_{11}, h_{33}, \delta l_{130}, \delta l_{310}\}$ at four randomly chosen $\chi$ values along an eccentric orbit with $p = 15, e = 0.15$ and $l_{\text{max}} = 18$. As expected, the regularized $l$ modes display an $l^{-2}$ powerlaw for all cases that we present.

4. Post-Newtonian expansion

In this section we arrive at the key result that, at next-to-next-to-leading-order (NNLO), the post-Newtonian expansion of the spin precession scalar through $O(q), \psi = \psi + \Delta \psi + O(q^2)$, is given by

$$\psi = \frac{3}{2} q^{-1} + \left( \frac{9}{8} + \frac{3}{2} e^2 \right) q^{-2} + \left( \frac{27}{16} + \frac{33}{4} e^2 + \frac{3}{16} e^4 \right) q^{-3} + O(q^{-4}),$$

(4.1)

$$q^{-1} \Delta \psi = -q^{-1} + \frac{\left( \frac{9}{4} + e^2 \right)}{\Delta q^{\text{NLO}}} q^{-2} + \left( \frac{739}{16} - \frac{123\pi^2}{64} + \left( \frac{341}{16} - \frac{123\pi^2}{256} \right)^2 - \frac{e^4}{2} \right) q^{-3} + O(q^{-4}).$$

(4.2)

In section 5 we verify that the post-Newtonian result appears to be consistent with the GSF calculation of section 3.

The spin–orbit (SO) Hamiltonian in ADM coordinates was presented in [92] at NLO and extended to NNLO in [93, 94]. The Hamiltonian takes the simple form $H_{SO} = \sum_{i=1}^{2} \Omega_i \cdot S_i$, where $\Omega_i$ are spin–precession frequencies with respect to coordinate time. For equatorial orbits, $\Omega_i = 0$ and $\Omega_3 = \Omega_3 k$ where $k$ is a unit vector orthogonal to the equatorial plane, and $\Omega_5 = \Omega_{52} + \Omega_{54} + \Omega_{56} + O(c^{-8})$. Here $\Omega_{52}, \Omega_{54}$ and $\Omega_{56}$ are the contributions at LO ($c^{-2}$), NLO ($c^{-4}$), and NNLO ($c^{-6}$), respectively.

The spin precession invariant is defined as

$$\psi = \frac{\langle \Omega_5 \rangle}{\langle \Omega_3 \rangle},$$

(4.3)
where \( \langle \cdot \rangle \) denotes the orbital average over one radial period. (The difference between equation (4.3) and equation (1.1) is simply due to defining precession with respect to a Cartesian-type basis as opposed to a polar-type basis). For circular orbits, \( \psi \) has previously been calculated through NNLO in [111], equation (4.5) and [71], equation (9).

Our calculation is based on the approach in section IV of [86]. We perform the calculation in the centre-of-mass frame. The orbital average is taken using the generalized quasi-Keplerian (QK) representation introduced in [112], which is known up to 3pN in both harmonic and ADM coordinates [113], and which is described in section IVC of [86].

To illustrate the procedure, let us consider only the leading-order term, \( \Omega \propto -g \propto -u \). In the centre of mass frame we have \( p_1 = -p_2 = p \) and \( n_1 \times p_1 = L \), where \( L \) is the angular momentum. Thus, \( \Omega_{52} = \frac{G}{c^2} \frac{3m_2}{2m_1} n_1 \times p_1 - 2n_2 \times p_2 \). In the centre of mass frame we have \( p_1 = -p_2 = p \) and \( n_1 \times p_1 = L \), where \( L \) is the angular momentum. Thus, \( \Omega_{52} = \frac{G}{c^2} \frac{3m_2}{2m_1} n_1 \times p_1 - 2n_2 \times p_2 \). In the centre of mass frame we have \( p_1 = -p_2 = p \) and \( n_1 \times p_1 = L \), where \( L \) is the angular momentum. Thus, \( \Omega_{52} = \frac{G}{c^2} \frac{3m_2}{2m_1} n_1 \times p_1 - 2n_2 \times p_2 \).

Our task is to compute the orbital averages using the QK representation. The mean anomaly \( \ell = \Omega \langle t \rangle \) and the eccentric anomaly \( u \) give the parametrization \( r(u) = a(1 - e \cos u) \) and \( \ell(u) = u - e \sin u + f_r \sin V + g_r(V - u) \), through NNLO. Here, \( \Omega_{r}, a_r, e_r, f_r, g_r \) are QK orbital elements, and \( V(u) \) is specified in equation (4.20) of [86]. The QK representation is only complete once the orbital elements are specified in terms of orbital integrals. Following [114] we use the dimensionless coordinate-invariant quantities

\[
\varep = \frac{2E}{c^2}, \quad j = \frac{2EL^2}{(Gm)^2},
\]

(4.4)
where $\hat{E} = E_l \mu$ and $\hat{L} = L_l \mu$ are the binding energy and orbital angular momentum, respectively, per reduced mass $\mu \equiv m_1 m_2 / m = \nu m$. Noting that $\varepsilon = O(\mu^{-2})$ and $j = O(\varepsilon^0)$ allows one to keep track of pN orders. The various QK elements are expanded in powers of $\varepsilon$ in equations (4.22) of [86].

By way of illustration, let us consider $\Omega = -gGLc r S^2 23$. At NLO, we may neglect $f_t$ and $g_t$ which scale as $\varepsilon \sim \varepsilon^{1/2}$; thus

$$\ell \int \pi = \Omega \pi r u u 2 d d d d d (4.5)$$

To extend to NNLO, we must include the $f_t$ and $g_t$ terms; the resulting integrals are straightforward with the help of Mathematica. Other orbital averages such as $\langle r^4 \rangle, \langle r^5 \rangle$, etc may be found in a similar way.

We are led to an expression for the spin precession scalar which is valid for any mass ratio. We may write $\psi = \psi^{LO_\text{inv}} + \psi^{NLO_\text{inv}} + \psi^{NNLO_\text{inv}}$, with

$$\psi^{LO_\text{inv}} = \frac{\varepsilon}{j} \frac{3}{4} + \frac{3}{4} \Delta + \frac{\nu}{2},$$

$$\psi^{NLO_\text{inv}} = \frac{\varepsilon^2}{j^2} \left\{ \left( \frac{3}{4} \Delta + \frac{\nu}{2} \right) \left[ \frac{(48 - 15\nu - 18j + 13\nu j)}{4} \right] + \frac{(j - 3)[5\Delta^2 + 2\Delta(3\nu + 5) - 8\nu^2 + 6\nu + 5]}{4(\Delta + 1)} + \frac{(j - 1)(\Delta - 2\nu + 1)[5\Delta^2 + \Delta(10 - 44\nu) - 36\nu^2 - 44\nu + 5]}{32(\Delta + 1)^2} + \frac{(\Delta + 1)^2(3j - 5)[5\Delta^2 + \Delta(10 - 32\nu) - 12\nu^2 - 32\nu + 5]}{128(\Delta - 2\nu + 1)} \right\},$$

(4.8)

The subscript ‘inv’ is included to distinguish $\Delta \psi_{\text{inv}}(\varepsilon, j)$ from $\Delta \psi(p, e)$. The term $\psi^{NNLO_\text{inv}} = O(\varepsilon^3)$ term is lengthy and will be presented elsewhere. To obtain the NNLO result, one needs an appropriate expression for the radial momentum $p_r$. Starting with equation (5.6) of [115] and taking derivatives with respect to the components of the relative velocity [116], one gets

$$p_r^2 = \mu^2 \varepsilon^2 \left[ 1 + c^{-2} \left\{ (1 - 3\nu)(t_r^2 + r^2 \phi^2) + \frac{Gm}{r} (6 + 4\nu) \right\} + O(c^{-4}) \right],$$

(4.9)

where the overdot denotes differentiation with respect to coordinate time.

The next step is to decompose $\psi$ into $O(q^0)$ and $O(q^1)$ parts, for comparison with the GSF result. First, we note that $\varepsilon$ and $j$ are defined in equation (4.4) in terms of energy and angular momentum, rather than the frequencies $\Omega_e$ and $\Omega_\phi$ used in the GSF approach. Following [114], we introduce the dimensionless coordinate-invariant parameters,
\[ x = \left( \frac{Gm\Omega_0}{c^3} \right)^{2/3}, \quad \iota \equiv \frac{3x}{k}, \]  

(4.10)

where \( k = K - 1 \) with \( K = \Omega_0/\Omega_r \). We replace \((\varepsilon, f)\) with \((x, \iota)\) using equations (4.40) in [86]. Next, following [86], we introduce a pair of parameters \((y, \lambda)\) better suited to the extreme mass-ratio limit \( q \ll 1 \), defined as \( y \equiv \frac{Gm^2 \Omega_0 c^3}{3x} \) and \( \lambda = 3y/k \), so that \( x = (1 + q)^{2/3} y \) and \( \iota = (1 + q)^{2/3} \lambda \). After this replacement we expand \( \psi \) as a series in \( q \), to isolate the \( \mathcal{O}(q^0) \) and \( \mathcal{O}(q^3) \) parts. That is, we write \( \bar{\psi} \) and \( \Delta \psi \) as functions of \((y, \lambda)\). Finally, we switch to orbital elements \((p, e)\) which are defined with respect to the frequencies \( \Omega_0 \) and \( \Omega_r \) using the functional relationships on the background spacetime. As these relationships cannot be inverted analytically, we make use of the series expansions (B1) in [86].

At the end of this process we reach equations (4.1) and (4.2): relative 2pN expansions for \( \bar{\psi} \) and \( \Delta \psi \). It is also straightforward to find higher-order-in-\( q \) contributions, \( \Delta \psi^q_{\text{unc}} \) etc if required for validation of any (future) second-order GSF calculation.

The pN series pass three consistency checks. First, the pN series for \( \bar{\psi} \), equation (4.1), is in accord with the series expansion of \( \psi \) on the background spacetime, equation (2.18). Second, in the circular limit \( e \to 0 \), the difference between \( \Delta \psi \) in equation (4.2) and \( \Delta \psi_{\text{oct}} \) given by the \( \mu/M \) part of equation (10) in [71] is found to be precisely the pN series for the offset term in equation (3.10). Finally, the pN series (4.2) appears to be consistent with the GSF numerical results for \( \Delta \psi \), as we now show.

5. Results

Here we present a selection of numerical results from the GSF method (section 3) and compare with the post-Newtonian series (section 4).

5.1. Numerical data for \( \Delta \psi \)

Sample GSF data for \( \Delta \psi \) is given in table 2, for orbital parameters \( 10 \leq p \leq 100 \) and \( 0 \leq e \leq 0.25 \).

5.2. Analysis and comparisons with pN results

Here we set \( q = 1 \) for notational convenience, without loss of generality, recalling that \( \Delta \psi \) is linear in \( q \).

First, consider the limit of zero eccentricity. Figure 2 compares numerical results for \( \Delta \psi_{\text{num}} \equiv \lim_{e \to 0} \Delta \psi \) (blue dots) with the analytic post-Newtonian expansion \( \Delta \psi_{\text{pN}} \) (red solid line). The numerical results for \( \Delta \psi_{\text{num}} \) were obtained via equation (3.8) and (3.10), with the correction term \( \Delta \psi_{\text{oct}} \) determined by the EOB \( \rho \) function, with \( \rho \) constituted from the circular-orbit redshift invariant \( z \) and the EOB \( d \) function, and with the latter numerically computed to high accuracy in [66]. The analytic result \( \Delta \psi_{\text{pN}} \) was described in section 3.2 and is given explicitly in equation (A.1) of appendix. The numerical and analytical results are in robust agreement, as indicated by the fact that the (red) curve passes through all the numerical data points in figure 2 (noting that the numerical error bars are obscured by the points themselves). The difference \( |\Delta \psi_{\text{num}} - \Delta \psi_{\text{pN}}| \) is shown in the inset (blue dots), and is compared against a reference line \( \alpha p^{-10} \) with \( \alpha = 3 \times 10^6 \). The inset provides evidence that the difference scales
as $p^{-10}$ at large $p$, indicating that the residual disagreement is principally due to the truncation of the post-Newtonian series, which is known only up to $O(p^{-19/2})$. The scattered nature of the dots in the inset is due to noise in the numerically-computed $\psi_\Delta$.

The dotted and dashed lines on figure 2 indicate the eccentric-orbit post-Newtonian result (4.2) evaluated at $e = 0$ at LO (dotted black), NLO (dot-dashed green) and NNLO (dashed blue). (N.B. The latter expressions agree with $\psi_\Delta^p$ at the relevant orders.) As expected, these lines provide successively-better approximations to the numerical results at large $p$.

Next we confront our numerical data for $\psi_\Delta^{\text{num}}$ with the eccentric-orbit post-Newtonian result at NNLO, equation (4.2). If the results are compatible, we expect to find

$$
\Delta \psi_\Delta^{\text{num}} = \Delta \psi_\Delta^{\text{num}} + \Delta \psi_\Delta^p + O\left(e^2 p^2 e^{-4}\right),
$$

where

$$
\Delta \psi_\Delta^p = \left[1 + \left(\frac{341}{16} \cdot \frac{123 \pi^2}{256}\right) \frac{e^2}{p^2}\right], \quad \Delta \psi_\Delta^e = -\frac{e^4}{2p^3}.
$$

Figure 3 shows numerical data for $e^{-2} p^4 (\Delta \psi_\Delta^{\text{num}} - \Delta \psi_\Delta^{\text{num}})$ compared against $e^{-2} p^4 \Delta \psi_\Delta^{\text{num}} \approx 1 + 16.5705/p$, for several values of eccentricity $e = \{0.075, 0.1, 0.125, 0.25\}$. The plots show that the numerical data (black dots within shaded green confidence limits) is in robust agreement with the pN series truncated at NNLO (blue lines). As expected for a pN series, the offset between the blue curves and the numerical data decreases as $p$ increases. The residual disagreement is likely to arise from the as-yet-unknown higher-order pN terms at orders $O(e^2 p^{-4}), O(e^2 p^{-5})$, as well as from the known term at $O(e^2 p^{-3})$. Our data set becomes noisy for increasing values of $p$ and smaller values of $e$, as indicated by the upper half of the plots in figure 3. In this regime, the magnitude of $16.5705/e^2 p^3$ is sufficiently small that it is comparable to the numerical error itself.

In principle, numerical data can be used to constrain the (as-yet-unknown) higher-order coefficients of the eccentric-orbit pN series. For instance, the fact that the curves cross over in figure 3 (see the blue line and green band) hints at a change of sign in the coefficients of the pN series. We obtained numerical estimates for the values of the known $O(e^2 p^{-3}) \approx 16.5705$ and the unknown $O(e^2 p^{-4})$ and $O(e^2 p^{-5})$ pN coefficients by fitting $p^4 e^{-2} (\Delta \psi_\Delta^{\text{num}} - \Delta \psi_\Delta^{\text{num}} - e^2 p^{-2})$ for a range of $(p, e)$ to a model of the form $b_2 + b_3 p + b_4 p^2 + \ldots$, using only the ‘cleanest’ portion of our data, i.e. $p \leq 45$. We varied the number of fit parameters from two to five and obtained the best fits using standard $\chi^2$-minimizing techniques. Our best-fit results were

$$
b_2 = 15.95 \pm 0.45 \quad \text{versus} \quad 16.5705 \ldots,
$$

$$
b_3 = -55.3 \pm 13.1, \quad \text{and} \quad b_4 = 793 \pm 86, \quad (5.3)
$$

with errors quoted at the 3$\sigma$ level. The estimate for $b_2$ is compatible with our analytical 2pN result, and $b_1$ has the opposite sign to $b_2$ as expected from the crossing of the green and blue curves in figure 3. Here we used simple fitting functions, omitting the $\ln(p)$ terms starting at $O(p^{-5})$ that are too small for our code to constrain at its current level of accuracy. It is likely that in this range, $p \leq 45$, the higher-order unknown pN terms are large enough to ‘contaminate’ our estimates for $b_{2,3}$. Consequently, the errors bars quoted above may well prove to be underestimates. Since the data with the smallest errors (the largest statistical weight) is located at $p \leq 20$, where sign changes may occur with each new term added to the pN series, we may
not even be fully confident of the signs of $b_3, b_4$. Nonetheless, we have presented our best estimates here with future work in mind.

Figure 4 illustrates the fitting of higher-order terms to the numerical data. The green shaded region indicates confidence limits for the quantity $p^3 e^{-2}(\Delta \psi^{\text{num}} - \Delta \psi^{\text{num}} - e^2 p^{-2})$, numerically computed from data for $\Delta \psi^{\text{num}}$. The straight red line marks the analytically known $\mathcal{O}(e^2 p^{-3})$ pN term, which is approximately equal to 16.5705. The black region (bounded by two dashed curves) shows the pN series up to and including the $\mathcal{O}(e^2 p^{-4})$ term with coefficient $b_3$ (equation (5.3)), and the blue region (bounded by solid curves) shows the pN series up to and including the $\mathcal{O}(e^2 p^{-5})$ term with coefficient $b_4$ (equation (5.3)). We note that the latter is consistent with the numerical data across the whole range in $p$.

To improve the estimates of the pN parameters, one would need to perform the numerical extractions at large $p$ and small $e$, ideally $p > 1000$ and $e < 10^{-3}$. As the plots show, our data is noisy in the large $p$ regime, and the errors become comparable in the magnitude to the pN terms themselves. This is not altogether surprising, as the Lorenz-gauge code is unsuited to weak-field, small-eccentricity applications; whereas (forthcoming) radiation-gauge codes may probe this regime effectively (section 6). Furthermore, we cannot, at present, estimate the coefficient of the $\mathcal{O}(e^4 p^3)$ term with any confidence. In our current range of eccentricities, $e < 0.25$, the magnitude of this term is comparable to our estimated error for $\Delta \psi^{\text{num}}$. A reasonable numerical estimation of this term requires several improvements which we explain in detail in section 6.

Despite such caveats, we now proceed to synthesize our numerically-acquired information with the analytical knowledge from the pN series. Extracting $\mathcal{O}(p^4)$, $\mathcal{O}(p^5)$ parts of $\Delta \psi^{\text{num}}$ from equation (A.1) and combining these with our estimates above leads to the following expression:

$$q^{-1} \Delta \psi = - p^{-1} + \left( \frac{9}{4} + e^2 \right) p^{-2} + \left[ \frac{739}{16} - \frac{123 \pi^2}{64} + \left( \frac{341}{16} - \frac{123 \pi^2}{256} \right) e^2 - \frac{e^4}{2} \right] p^{-3}$$

$$+ \left[ a_4 + a_4^{\log} \ln p - 55(13) e^2 + \mathcal{O}(e^4) \right] p^{-4}$$

$$+ \left[ a_5 + a_5^{\log} \ln p + 793(86) e^2 + \mathcal{O}(e^4) \right] p^{-5} + \mathcal{O}(p^{-6}), \quad (5.4)$$

where

$$a_4 = \frac{31697 \pi^2}{6144} + \frac{1256 \gamma}{15} - \frac{587831}{2880} + \frac{729 \ln(3)}{5} + \frac{296 \ln(2)}{15} \approx 68.997, \quad (5.5)$$

$$a_5 = \frac{2479221 \pi^2}{8192} - \frac{22306 \gamma}{35} - \frac{48221551}{19200} - \frac{31347 \ln(3)}{28} + \frac{22058 \ln(2)}{105} \approx -976.799, \quad (5.6)$$

$$a_4^{\log} = - \frac{628}{15} \approx 41.867, \quad a_5^{\log} = \frac{11153}{35} \approx 318.657. \quad (5.7)$$

6. Discussion

In the preceding sections we have computed the spin precession scalar $\psi$ for eccentric compact binaries via two complementary approaches. In the GSF approach, we obtained $\psi$ in
Table 2. Numerical data for $\Delta \psi$. The first column labels the eccentricity while the remaining columns label different $p$ values. In the rows corresponding to $e = 0.000$ we present the circular-orbit value, $\Delta \psi^{circ}$, which can be compared with table III of [73] or with $\alpha p^{-3/2}$ expressions of [75]. Note that $\Delta \psi^{circ} \approx \lim_{p \to 0} \Delta \psi$ (see section 3.2). The parenthetical digits correspond to the error estimates on the last quoted significant figure. For example, 5.938 5659(5) $\times 10^{-3} = 5.938 5659 \times 10^{-3} \pm 5 \times 10^{-10}$.

| $e$ | $p = 10$ | $p = 15$ | $p = 20$ | $p = 25$ | $p = 30$ |
|-----|---------|---------|---------|---------|---------|
| 0.000 | 5.938 5659(5) $\times 10^{-3}$ | 3.375 0158(2) $\times 10^{-3}$ | 2.071 500 85(7) $\times 10^{-3}$ | 1.386 863 08(2) $\times 10^{-3}$ | 9.900 3322(2) $\times 10^{-4}$ |
| 0.050 | -5.061 140(3) $\times 10^{-2}$ | -4.914 075(5) $\times 10^{-2}$ | -4.123 078(1) $\times 10^{-2}$ | -3.478 949(1) $\times 10^{-2}$ | -2.989 822(2) $\times 10^{-2}$ |
| 0.075 | -5.052 269(7) $\times 10^{-2}$ | -4.911 220(2) $\times 10^{-2}$ | -4.121 701(8) $\times 10^{-2}$ | -3.478 148(5) $\times 10^{-2}$ | -2.989 305(3) $\times 10^{-2}$ |
| 0.100 | -5.039 852(2) $\times 10^{-2}$ | -4.907 723(7) $\times 10^{-2}$ | -4.119 775(5) $\times 10^{-2}$ | -3.477 028(1) $\times 10^{-2}$ | -2.988 568(7) $\times 10^{-2}$ |
| 0.125 | -5.023 891(2) $\times 10^{-2}$ | -4.902 089(2) $\times 10^{-2}$ | -4.117 299(7) $\times 10^{-2}$ | -3.475 587(2) $\times 10^{-2}$ | -2.987 639(1) $\times 10^{-2}$ |
| 0.150 | -5.004 39(5) $\times 10^{-2}$ | -4.895 813(2) $\times 10^{-2}$ | -4.114 274(3) $\times 10^{-2}$ | -3.473 827(2) $\times 10^{-2}$ | -2.986 492(2) $\times 10^{-2}$ |
| 0.175 | -4.981 31(5) $\times 10^{-2}$ | -4.888 402(2) $\times 10^{-2}$ | -4.110 700(1) $\times 10^{-2}$ | -3.471 747(2) $\times 10^{-2}$ | -2.985 140(7) $\times 10^{-2}$ |
| 0.200 | -4.954 75(2) $\times 10^{-2}$ | -4.879 857(9) $\times 10^{-2}$ | -4.106 580(2) $\times 10^{-2}$ | -3.469 348(1) $\times 10^{-2}$ | -2.983 580(3) $\times 10^{-2}$ |
| 0.225 | -4.924 68(5) $\times 10^{-2}$ | -4.870 169(8) $\times 10^{-2}$ | -4.101 909(1) $\times 10^{-2}$ | -3.466 629(1) $\times 10^{-2}$ | -2.981 813(3) $\times 10^{-2}$ |
| 0.250 | -4.891 05(9) $\times 10^{-2}$ | -4.859 358(1) $\times 10^{-2}$ | -4.096 689(7) $\times 10^{-2}$ | -3.463 592(2) $\times 10^{-2}$ | -2.979 840(1) $\times 10^{-2}$ |
| $e$ | $p = 35$ | $p = 40$ | $p = 45$ | $p = 50$ | $p = 55$ |
|-----|---------|---------|---------|---------|---------|
| 0.000 | 7.410 5684(1) $\times 10^{-4}$ | 5.750 5234(1) $\times 10^{-4}$ | 4.590 0246(1) $\times 10^{-4}$ | 3.747 5920(1) $\times 10^{-4}$ | 3.117 0359(4) $\times 10^{-4}$ |
| 0.050 | -2.614 3411(7) $\times 10^{-2}$ | -2.319 599(1) $\times 10^{-2}$ | -2.083 062(1) $\times 10^{-2}$ | -1.889 476(3) $\times 10^{-2}$ | -1.728 324(2) $\times 10^{-2}$ |
| 0.075 | -2.613 978(1) $\times 10^{-2}$ | -2.319 329(6) $\times 10^{-2}$ | -2.082 857(2) $\times 10^{-2}$ | -1.889 311(4) $\times 10^{-2}$ | -1.728 196(2) $\times 10^{-2}$ |
| 0.100 | -2.613 468(2) $\times 10^{-2}$ | -2.318 953(2) $\times 10^{-2}$ | -2.082 568(5) $\times 10^{-2}$ | -1.889 085(5) $\times 10^{-2}$ | -1.728 011(1) $\times 10^{-2}$ |
| 0.125 | -2.612 8112(8) $\times 10^{-2}$ | -2.318 4703(2) $\times 10^{-2}$ | -2.082 198(1) $\times 10^{-2}$ | -1.888 792(1) $\times 10^{-2}$ | -1.727 773(2) $\times 10^{-2}$ |
| 0.150 | -2.612 0105(4) $\times 10^{-2}$ | -2.317 8794(6) $\times 10^{-2}$ | -2.081 744(3) $\times 10^{-2}$ | -1.888 434(2) $\times 10^{-2}$ | -1.727 486(2) $\times 10^{-2}$ |
| 0.175 | -2.611 063(2) $\times 10^{-2}$ | -2.317 1815(9) $\times 10^{-2}$ | -2.081 210(2) $\times 10^{-2}$ | -1.888 011(0) $\times 10^{-2}$ | -1.727 142(2) $\times 10^{-2}$ |
| 0.200 | -2.609 973(3) $\times 10^{-2}$ | -2.316 378(2) $\times 10^{-2}$ | -2.080 592(2) $\times 10^{-2}$ | -1.887 523(2) $\times 10^{-2}$ | -1.726 748(1) $\times 10^{-2}$ |
| 0.225 | -2.608 738(2) $\times 10^{-2}$ | -2.315 467(3) $\times 10^{-2}$ | -2.079 894(5) $\times 10^{-2}$ | -1.886 970(2) $\times 10^{-2}$ | -1.726 300(3) $\times 10^{-2}$ |
| 0.250 | -2.607 352(2) $\times 10^{-2}$ | -2.314 445(3) $\times 10^{-2}$ | -2.079 112(2) $\times 10^{-2}$ | -1.886 353(2) $\times 10^{-2}$ | -1.725 800(1) $\times 10^{-2}$ |
\begin{table}
\centering
\caption{(Continued)}
\begin{tabular}{|c|c|c|c|c|}
\hline
$e$ & $p = 60$ & $p = 65$ & $p = 70$ & $p = 75$ \\
\hline
$0.000$ & $2.6329573(1) \times 10^{-4}$ & $2.2533310(3) \times 10^{-4}$ & $1.95916974(6) \times 10^{-4}$ & $1.7042571(5) \times 10^{-4}$ \\
$0.050$ & $-1.5922111(9) \times 10^{-2}$ & $-1.475776(2) \times 10^{-2}$ & $-1.375078(2) \times 10^{-2}$ & $-1.28716(1) \times 10^{-2}$ \\
$0.075$ & $-1.5920996(3) \times 10^{-2}$ & $-1.475686(2) \times 10^{-2}$ & $-1.375005(2) \times 10^{-2}$ & $-1.287095(2) \times 10^{-2}$ \\
$0.100$ & $-1.591948(2) \times 10^{-2}$ & $-1.475553(2) \times 10^{-2}$ & $-1.374891(4) \times 10^{-2}$ & $-1.2869885(9) \times 10^{-2}$ \\
$0.125$ & $-1.591752(1) \times 10^{-2}$ & $-1.475391(2) \times 10^{-2}$ & $-1.3747508(6) \times 10^{-2}$ & $-1.2868785(9) \times 10^{-2}$ \\
$0.150$ & $-1.591513(2) \times 10^{-2}$ & $-1.4751887(9) \times 10^{-2}$ & $-1.374583(3) \times 10^{-2}$ & $-1.286731(1) \times 10^{-2}$ \\
$0.175$ & $-1.591302(2) \times 10^{-2}$ & $-1.4749509(6) \times 10^{-2}$ & $-1.3743806(6) \times 10^{-2}$ & $-1.2865597(2) \times 10^{-2}$ \\
$0.200$ & $-1.590903(3) \times 10^{-2}$ & $-1.474681(1) \times 10^{-2}$ & $-1.374145(5) \times 10^{-2}$ & $-1.286350(4) \times 10^{-2}$ \\
$0.225$ & $-1.590534(3) \times 10^{-2}$ & $-1.4743701(5) \times 10^{-2}$ & $-1.373887(3) \times 10^{-2}$ & $-1.286132(2) \times 10^{-2}$ \\
$0.250$ & $-1.590122(2) \times 10^{-2}$ & $-1.474022(2) \times 10^{-2}$ & $-1.373589(2) \times 10^{-2}$ & $-1.285874(2) \times 10^{-2}$ \\
\hline
$e$ & $p = 85$ & $p = 90$ & $p = 95$ & $p = 100$ \\
\hline
$0.000$ & $1.33377732(1) \times 10^{-4}$ & $1.1922590(3) \times 10^{-4}$ & $1.0721120(2) \times 10^{-4}$ & $9.692429(3) \times 10^{-5}$ \\
$0.050$ & $-1.1401067(2) \times 10^{-2}$ & $-1.079745(9) \times 10^{-2}$ & $-1.02464(1) \times 10^{-2}$ & $-9.7486(2) \times 10^{-3}$ \\
$0.075$ & $-1.140016(2) \times 10^{-2}$ & $-1.0796893(5) \times 10^{-2}$ & $-1.024593(2) \times 10^{-2}$ & $-9.7482(3) \times 10^{-3}$ \\
$0.100$ & $-1.140936(2) \times 10^{-2}$ & $-1.079628(9) \times 10^{-2}$ & $-1.024536(2) \times 10^{-2}$ & $-9.74776(8) \times 10^{-3}$ \\
$0.125$ & $-1.140856(2) \times 10^{-2}$ & $-1.079545(8) \times 10^{-2}$ & $-1.024466(7) \times 10^{-2}$ & $-9.74721(1) \times 10^{-3}$ \\
$0.150$ & $-1.1407409(8) \times 10^{-2}$ & $-1.079445(6) \times 10^{-2}$ & $-1.024384(2) \times 10^{-2}$ & $-9.7464(2) \times 10^{-3}$ \\
$0.175$ & $-1.140611(1) \times 10^{-2}$ & $-1.079322(3) \times 10^{-2}$ & $-1.024263(3) \times 10^{-2}$ & $-9.74544(3) \times 10^{-3}$ \\
$0.200$ & $-1.140452(2) \times 10^{-2}$ & $-1.079193(2) \times 10^{-2}$ & $-1.024156(5) \times 10^{-2}$ & $-9.74433(6) \times 10^{-3}$ \\
$0.225$ & $-1.140282(4) \times 10^{-2}$ & $-1.079042(4) \times 10^{-2}$ & $-1.024012(2) \times 10^{-2}$ & $-9.74322(6) \times 10^{-3}$ \\
$0.250$ & $-1.140082(4) \times 10^{-2}$ & $-1.078859(2) \times 10^{-2}$ & $-1.023864(9) \times 10^{-2}$ & $-9.74155(9) \times 10^{-3}$ \\
\hline
\end{tabular}
\end{table}
the strong-field regime at $O(q)$, whereas in the pN approach we obtained $\psi$ at arbitrary mass ratio $q$ as an expansion in powers of $c^{-2}$ in the weak-field. We have established here that the results are in agreement at (separately) $O(q_0)$; at $O(q_1)$ in the circular limit; and at $O(-qe^{-2})$ and $O(-qe^{-3})$ for eccentric orbits. We also obtained evidence for a sign change at $O(-qe^{-4})$, as well as for the likely sign of the term at $O(-qe^{-5})$, with the caveat that the error bars on these quantities are large in magnitude (see equation (5.3)). The results of our code also agree with the analytical log and non-log $O(qe^0)$ terms up to $O(p^{-10})$, as illustrated by the inset of figure 2.

To overcome certain limitations of our Lorenz-gauge numerical implementation—such as its insufficient relative accuracy at large $p$, highlighted in figure 4—we propose that $\Delta \psi$ should now be calculated via further complementary approaches. One possibility is to apply the radiation-gauge GSF architecture to compute $\psi$ for eccentric orbits at much greater numerical precision [43, 44]. This approach may allow one to compute $\Delta \psi$ close to the separatrix ($p = 6 + 2e$) of bound orbits in Schwarzschild spacetime, to quantify the (anticipated) divergence of $\Delta \psi$ in this limit. Another possibility is to use Hopper’s doubly-expanded (in $p, e$) expressions for $h_{ab}$ and its derivatives to obtain a pN expression for $\Delta \psi$ accurate to $O(e^{19})$ and $O(p^{-5})$ [87, 117]. A third possibility is to extend the approach of Bini,

Figure 2. Comparison of the numerical values $\Delta \psi_{\text{num}}$ (blue dots) with the analytic $\Delta \psi_{\text{lo}}$ (solid red curve). The difference is plotted in the inset (blue dots). The difference appears to scale as $O(p^{-10})$, as shown by the red (solid) reference line in the inset. Numerical noise is clear for $p \gtrsim 50$. Both $\Delta \psi_{\text{num}}$ and $\Delta \psi_{\text{lo}}$ are obtained from equation (3.10) using the results from [66] for the former and [75, 89] for the latter. $\Delta \psi_{\text{lo}}$ is explicitly shown in equation (A.1). The $e = 0$ values of our eccentric post-Newtonian expressions $\Delta \psi_{\text{lo}}, \Delta \psi_{\text{nlo}}, \Delta \psi_{\text{nnlo}}$ are displayed in the main plot as the dotted black, dot-dashed green and dashed blue lines, respectively. As expected, these approximate the numerical data better with increasing pN order and increasing $p$. We generated additional $\Delta \psi_{\text{num}}$ data for this plot that is not provided in table 2: for $p = \{105, 110, 115, 120\}$, $\Delta \psi_{\text{num}} = \{8.8049204210, 8.0339083210, 7.359865210, 6.7671891910 \times 10^{-5}\}$. 

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Figure 3. The numerical residual $e^{-2\rho^2}(\Delta \psi - \Delta \psi^{\text{num}})$ represented by the black dots for $\epsilon = \{0.075, 0.1, 0.125, \ldots, 0.225, 0.25\}$. This is compared with $e^{-2\rho^2}\Delta \psi^2 \approx 1 + 16.5705\rho^2$ shown by the solid blue curves labelled as ‘1 pN + 2 pN’ in the panels. The numerical residuals contain contributions from the unknown $O(e^4)$ terms hence the offset between the solid blue curves and the black dots, which decreases with increasing $\rho$ as expected. The red dashed lines mark the 1 pN term $e^{-2\rho^2}(\epsilon^2/p^2) = 1$. The green strips are our estimated numerical errors. Although we calculate the errors only at the $\rho$ values given in table 2 the error bars in the plots are connected with green shaded curves for visualization purposes. The data is noisier for low $\epsilon$ and high $\rho$ values where the magnitude of the term that we extract from the numerical data becomes comparable in size to the estimated errors. In the axis labels, $\Delta \psi^{\text{num}}$ is denoted by $\Delta \psi(e \to 0)$. 
Damour \& collaborators [76, 77, 79, 81] which makes expert use of the Mano–Suzuki–Takasugi formalism [118].

Extending the arbitrary mass ratio pN calculation of $\psi$ to next order (NNNLO) presents a stiff challenge. The NNNLO spin–orbit Hamiltonian has not appeared in the literature. In principle, one can obtain it from the 4pN metric for non-spinning binaries\(^4\). This computation requires the attention of experts of post-Newtonian theory.

The boundary between GSF and numerical relativity is under active exploration. Recently, the redshift invariant $\Delta z$ was extracted from numerical relativity simulations of quasi-circular black hole binaries, via the helical (quasi-)Killing vector field [59]. It is plausible that the circular-orbit precession invariant $\psi_{\text{circ}}$ can be extracted from the derivatives of the helical Killing vector field (see equation (3) in [71]). Obtaining the eccentric-orbit precession invariant poses a stiffer challenge as, in the absence of a continuous symmetry, it is necessary to track the spin of the small black hole in some sufficiently coordinate-insensitive way.

Our prescription for computing the spin precession invariant $\Delta \psi$ is, at present, far less elegant than that available for computing the redshift invariant $\Delta z$. Remarkably, in [86] it was shown that $\Delta z = -\langle H \rangle z$, where $\langle H \rangle = \frac{1}{T} \int_0^T \frac{1}{2} h_{00} \, \text{d}\tau$. This offered a significant simplification

\(4\) We thank Thibault Damour for pointing this out.
of the earlier prescription of [69], leading to improved numerical accuracy and physical insight. It is natural to speculate as to whether a similar simplification may be possible for the spin-precession calculation. To consider this, let us recap the argument for $z$. First, following the approach of section 3.1, and using equations (2.8), (3.3) and (3.18), one may quickly establish that $\delta T = E\delta T - L\delta \Phi - \int_0^T \frac{1}{2} h_{00}\delta \tau$. Second, it follows from the definition of $\Delta$ that $\Delta z = \Delta(\Delta T) = \Delta \Delta \Delta (\Delta T)$ (with $\Delta T = 0$ by definition). Third, combining equation (3.4) with the first step yields

$$\Delta T = \delta T - \frac{\partial T}{\partial T} \delta T - \frac{\partial \delta T}{\partial \Phi} \delta \Phi,$$

(6.1)

Finally, it may be shown using the first law of binary mechanics [119] that the two bracketed terms are identically zero (see appendix B of [86]), and thus the simple result for $\Delta z$ follows.

In summary, we have taken one more small step in extracting physical content from GSF theory, to move further towards the goal of accurately modelling the gravitational two-body problem. There remain many challenges ahead: from transcribing eccentric-orbit GSF results into EOB theory, to calculating invariants for eccentric orbits on Kerr, and, most importantly, extending GSF theory to second order in the mass ratio. Progress is being made on a range of fronts, inspired by the successes of LIGO and eLISA Pathfinder that have heralded a new era of gravitational wave astronomy.

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**Appendix. $\Delta \psi_{\theta \psi}^{\psi}:$ the pN series for \( \lim_{p \to 0} \Delta \psi \) up to $p^{-19/2}$**

Below $\gamma$ denotes the Euler’s constant and $\zeta$ is the Riemann zeta function.
\[ \Delta \psi_{p}^{\delta} = \frac{1}{p} + \frac{9}{4p^2} + 2956 - 123\pi^2 + \frac{628}{\pi^4} \ln\left(\frac{\pi}{\delta}\right) + \frac{31407}{\pi^4} + \frac{1256}{\pi^4} + \frac{58731}{\pi^4} + \frac{729}{\pi^4} + \frac{296}{\pi^4} + \frac{49969}{315} \left(\frac{1}{p}\right)^{11/2} \]

\[ + \frac{344,021}{110,025} \left(\frac{1}{p}\right)^{3} + \frac{232,109}{232,109} + \frac{344,021}{110,025} + \frac{190,000}{190,000} + \frac{9765}{9765} + \frac{234,000}{234,000} + \frac{2514,047}{2514,047} \left(\frac{1}{p}\right)^{3} \]

\[ + \frac{849,152}{19,035} \left(\frac{1}{p}\right)^{5} + \left(\frac{1316,474}{14,144} + \frac{3396,608}{1315} + \frac{2574,414}{1315} + \frac{460,010}{1315}\right) \ln\left(\frac{1}{\pi}\right) + \frac{83,048}{19,035} + \frac{341,973,520,157}{19,035} \left(\frac{1}{p}\right)^{3} \]

\[ + \frac{25,637,561}{50,574} \ln^2(\pi) + \left(\frac{133,475}{312} + \frac{254,000}{312} + \frac{344,021}{110,025}\right) \left(\frac{1}{p}\right)^{3} + \frac{931}{19,035} + \frac{157,461}{19,035} + \frac{361,328}{19,035} \left(\frac{1}{p}\right)^{3} \]

\[ + \frac{593,036}{75} \ln(2) \ln(3) + \frac{257,433,621}{646,000} + \frac{1316,474}{14,144} + \frac{549,000}{19,035} + \frac{931}{19,035} + \frac{265,825,037,088}{19,035} \left(\frac{1}{p}\right)^{3} \]

\[ + \frac{44,126,553}{110,025} \left(\frac{1}{p}\right)^{3} + \left(\frac{1848,133}{630} + \frac{593,036}{630} + \frac{177,306}{110,025} + \frac{1300,906}{110,025}\right) \left(\frac{1}{p}\right)^{3} + \frac{59,742,270}{19,035} \ln\left(\frac{1}{\pi}\right) + \frac{861,600}{19,035} \left(\frac{1}{p}\right)^{3} \]

\[ + \frac{623,846,083}{21,474,876,440} + \frac{569,460}{21,474,876,440} + \frac{177,306}{110,025} + \frac{76,500,205}{239,976,250} + \frac{359,704}{239,976,250} \left(\frac{1}{p}\right)^{3} + \frac{2450}{110,025} \left(\frac{1}{p}\right)^{3} \]

\[ + \frac{520,925,726}{11,025} \ln^2(\pi) + \frac{678,223,070}{925,641,000} + \frac{649,357,646}{17,875} + \frac{687,767,574}{98,664,768} + \frac{59,742,270}{1225} \left(\frac{1}{p}\right)^{3} \]

\[ + \frac{1,888,532}{110,025} + \frac{198,890}{110,025} + \frac{227,201}{110,025} + \frac{59,742,270}{1225} \left(\frac{1}{p}\right)^{3} + \frac{42,057,786,451}{5297,292,000} + \frac{719,273,420,419}{3129,136,000} \left(\frac{1}{p}\right)^{3} \]

\[ + \frac{559,000,184}{218,205} \left(\frac{1}{p}\right)^{3} + \left(\frac{471,911}{419,455} + \frac{232,109}{218,205} + \frac{592,031}{323,423} + \frac{1115}{13,475} + \frac{76,784,944}{343,448} \right) \ln\left(\frac{1}{\pi}\right) \]

\[ + \frac{2231}{83} + \frac{128,144,002}{67,308,964} + \frac{4415,380}{6597,099,766,656} + \frac{1137,608,424}{976,352,977,920} + \frac{2236,204,736}{218,205} \left(\frac{1}{p}\right)^{3} \]

\[ + \frac{131,867,269,026}{207,344,255,110,120,000} + \frac{1441,389,753}{1143,438} + \frac{76,084,974,737,574,753}{13475} + \frac{75,690,099,326}{3530,275} + \frac{848,613,649}{33,696,000} \left(\frac{1}{p}\right)^{3} \]

\[ + \frac{374,025}{374,025} \left(\frac{1}{p}\right)^{3} + \frac{76,084,974,737,574,753}{13475} + \frac{1091,431,829,233}{1537,128} + \frac{208,367,277,518,947}{1334,019,152} + \frac{51,963,633,780,407,594}{1619,000} \left(\frac{1}{p}\right)^{3} \]

\[ + \left(\frac{474,026}{207,344,255,110,120,000} + \frac{1184,061,409,384}{232,425} + \frac{2231,619,174}{13,475} + \frac{76,084,974,737,574,753}{13475} + \frac{2231,619,174}{13,475} \right) \left(\frac{1}{p}\right)^{3} \]

\[ + \frac{6260,819}{2100} \left(\frac{1}{p}\right)^{3/2} + \frac{2782,895}{2910,600} + \frac{449}{15/2} + \frac{17/2}{1} \left(\frac{588,455,702,702}{165} + \frac{10,999,172,83}{4725} \right) \left(\frac{1}{p}\right)^{3}. \]
\begin{equation}
\begin{aligned}
&+ \left( \frac{1}{p} \right)^{0/2} \left( \frac{23 055 449 891 \pi \ln \left( \frac{1}{\gamma} \right)}{771 750} - \frac{180 728 953 \pi ^2}{11 025} \right) \\
&+ \left( \frac{1}{p} \right)^{0/2} \left( \frac{313 011 243 444 996 194 647}{1 145 359 870 720 000} + \frac{23 055 449 891 \pi }{385 875} - \frac{3 165 211 883 \ln (2)}{15 435} + \frac{685 694 417 \ln (3)}{85 750} \right)
\end{aligned}
\end{equation}

(A.1)

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