A proximal dual semismooth Newton method for computing zero-norm penalized QR estimator

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Abstract

This paper is concerned with the computation of the high-dimensional zero-norm penalized quantile regression (QR) estimator, which is defined as a global minimizer of the zero-norm penalized check loss minimization. To seek a desirable approximation to this estimator, we reformulate this NP-hard lower semi-continuous problem as an equivalent augmented Lipschitz optimization problem, and exploit its coupled structure to propose a multi-stage convex relaxation approach (MSCRA_PPA). The MSCRA_PPA solves inexactly in each step a weighted \(\ell_1\)-regularized check loss minimization problem with a proximal dual semismooth Newton method. Under a mild restricted strong convexity condition, we provide the theoretical guarantee for the MSCRA_PPA by establishing the error bound of each iterate to the true estimator and achieving the rate of linear convergence in a statistical sense. Numerical comparisons on some synthetic and real data with MSCRA_IPM and MSCRA_ADMM (two MSCRAs with the subproblems solved by an interior point method and a semi-proximal ADMM, respectively) show that MSCRA_PPA has comparable estimation performance with the latter two methods and requires only half (respectively, one-third) of the time required by MSCRA_ADMM (respectively, MSCRA_IPM).

Keywords: High-dimensional; Zero-norm penalized quantile regression; Variable selection; Proximal dual semismooth Newton method

1 Introduction

With the advent of modern technology, collecting high-dimensional data becomes easier in a host of research areas such as genomics, tomography, medical imaging, astrophysics, economics and finance. However, analysis of high-dimensional data poses great challenges since, on one hand, the number of covariates greatly exceeds the number of observations and the associated optimization problems become ill-conditioned, and on the other hand

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heterogeneity is common in high-dimensional data. This also has attracted great interests in a number of fields such as applied mathematics, engineering and statistics.

Sparse penalized regression has become a popular approach for high-dimensional data analysis. In the past two decades, many classes of sparse penalized regressions have been developed by imposing a suitable penalty term on the least squares loss such as the bridge penalty [14], Lasso [37], SCAD [10], elastic net [43], adaptive lasso [44], and so on. We refer to the survey papers by Bickel and Li [3] and Fan and Lv [11] for the relevant references. These penalties, as a convex surrogate (say, $\ell_1$-norm) or a nonconvex approximation (say, the bridge penalty) to the zero-norm, essentially try to capture the performance of the zero-norm, which is first used in the best subset selection by Breiman [6]. The sparse least squares regression approach is useful, but it only focuses on the central tendency of the conditional distribution. It is known that a certain covariate may not have significant influence on the mean value of the response variable but may have a strong effect at the upper quantile of the conditional distribution due to the heterogeneous nature of the data. It is likely that a covariate has different effects at different segments of the conditional distribution. As illustrated in [19], for non-Gaussian error distributions, the least squares regression is substantially out-performed by the quantile regression.

Motivated by this, many researchers recently have considered the quantile regression introduced by Koenker and Bassett [19] for high-dimensional data analysis, owing to its robustness to outliers and its ability to offer unique insights into the relation between the response variable and the covariates (see, e.g., [39, 1, 40, 41, 12, 13]). Among others, Belloni and Chernozhukov [1] focused on the theory of the $\ell_1$-penalized QR and showed that this estimator is consistent at the near-oracle rate and provided the conditions under which the selected model includes the true model; Wang [41] studied the $\ell_1$-penalized least absolute derivation (LAD) regression and verified that the estimator has near oracle performance with a high probability; and Fan et al. [12] studied the weighted $\ell_1$-penalized QR and established the model selection oracle property and the asymptotic normality for this estimator. For nonconvex penalty-type QRs, Wu and Li [39] under mild conditions achieved the asymptotic oracle property of the SCAD and adaptive-Lasso penalized QRs, and Wang et al. [40] showed that with probability approaching one, the oracle estimator is a local optimal solution to the SCAD or MCP penalized QRs of ultra-high dimensionality. Notice that the above results are all established for the asymptotic case $n \to \infty$.

Besides the above theoretical works, there are some works concerned with the computation of (weighted) $\ell_1$-penalized QR estimators which, compared to the (weighted) $\ell_1$-least-squares estimator, requires more sophisticated algorithms due to the piecewise linearity of the check loss function. Since the $\ell_1$-penalized QR model can be transformed into a linear program (LP) by introducing additional $p + 2n$ variables, where $p$ and $n$ denote the dimension and the sample size, respectively, it is natural to use the interior point method (IPM) softwares such as SeDuMi [34] to solve it, but this is limited to the small or medium scale case; see Figure 1-2 in Section 6.2. Inspired by this, Wu and Lange [38] proposed a greedy coordinate descent algorithm for the $\ell_1$-penalized LAD regression, Yi and Huang [42] proposed a semismooth Newton coordinate descent algorithm for the
elastic-net penalized QR, and Gu et al. [18] recently developed a semi-proximal alternating direction method of multipliers (sPADMM) and a combined version of ADMM and coordinate descent method (which is actually an inexact ADMM) for solving the weighted $\ell_1$-penalized QR. In addition, for nonconvex penalized QRs, Peng and Wang [27] developed an iterative coordinate descent algorithm and established the convergence of any subsequence to a stationary point, and Fan et al. [13] provided a systematic study for folded concave penalized regressions, including the SCAD and MCP penalized QRs as special cases, and showed that with high probability, the oracle estimator can be obtained within two iterations of the local linear approximation (LLA) approach proposed by Zou and Li [45]. We notice that Peng and Wang [27] and Fan et al. [13] did not establish the error bound of the iterates of algorithm to the true solution.

In this work we are interested in the computation of the high-dimensional zero-norm penalized QR estimator, a global minimizer of the zero-norm regularized check loss minimization. To seek a high-quality approximation to this estimator, we reformulate this NP-hard lower semi-continuous (lsc) optimization problem as a mathematical program with equilibrium constraints (MPEC), and from a global exact penalty of this MPEC obtain an equivalent augmented Lipschitz optimization problem. This augmented problem not only has a favorable coupled structure but also implies an equivalent DC (difference of convex function) surrogate for the zero-norm regularized check loss minimization; see Section 3. By solving the augmented Lipschitz problem in an alternating way, we propose in Section 4 an MSCRA to computing a desirable surrogate for the zero-norm penalized QR estimator. Similar to the LLA method owing to Zou and Li [45], this MSCRA solves in each step a weighted $\ell_1$-regularized check loss minimization, but the subproblems are allowed to be solved inexactly. Under a mild restricted strong convexity condition, we also provide its theoretical guarantee by establishing the error bound of each iterate to the true estimator and achieving the rate of linear convergence in a statistical sense.

Motivated by the recent work [35], in Section 5 we also develop a proximal dual semismooth Newton method (PDSN) for solving the subproblems involved in the MSCRA. Different from the semismooth Newton method in [42], this is a proximal point algorithm (PPA) with the subproblems solved by applying the semismooth Newton method to their dual problems, while the semismooth Newton method in [42] is applied to a smooth approximation to the elastic-net penalized check loss minimization problem. In Section 6, we make numerical comparisons among MSCRA_PPA, MSCRA_IPM and MSCRA_ADMM on some synthetic and real data. The three methods are the MSCRA for which the subproblems are solved with the PDSN, the IPM software SeDuMi [34] and the semi-proximal ADMM (see Appendix C), respectively. Among others, the semi-proximal ADMM has a little difference from the one proposed by Gu et al. [18] in the semi-proximal operator and the stopping criterion. The comparison results indicate that MSCRA_IPM and MSCRA_ADMM have very similar performance except that the former requires more computing time, while MSCRA_PPA not only has a comparable estimation performance with the other two methods but also requires only half (respectively, one-third) of the time required by MSCRA_ADMM (respectively, MSCRA_IPM).
2 Notation and preliminaries

Throughout this paper, for a given vector $x$, $\|x\|_1, \|x\|$ and $\|x\|_\infty$ respectively denote the $l_1$-norm, $l_2$-norm and $l_\infty$-norm of $x$. For a given matrix $A$, $\|A\|$, $\|A\|_{\max}$ and $\|A\|_1$ respectively denote the spectral norm, elementwise maximum norm, and maximum column sum matrix norm of $A$. For a given set $S$, $\text{conv}(S)$ means the convex hull of $S$, and $\mathbb{I}_S$ means the characteristic function on $S$, i.e., $\mathbb{I}_S(z) = 1$ if $z \in S$, otherwise $\mathbb{I}_S(z) = 0$.

For given vectors $a, b \in \mathbb{R}^p$ with $a_i \leq b_i$ for $i = 1, \ldots, p$, $[a, b]$ denotes the box constraint. The notation $I$ and $e$ denotes an identity matrix and a vector of all ones, respectively, whose dimensions are known from the context.

2.1 Generalized subdifferentials

Next we recall from [32, Definition 8.45 & 8.3] the concepts of the proximal, regular and limiting subdifferentials of an extended real-valued function at a finite-valued point. For an extended real-valued $f : \mathbb{R}^p \to (-\infty, +\infty]$, write $\text{dom} f := \{x \in \mathbb{R}^p \mid f(x) < \infty\}$.

**Definition 2.1** Consider a function $f : \mathbb{R}^p \to (-\infty, +\infty]$ and a point $x \in \text{dom} f$. The proximal subdifferential of $f$ at $x$, denoted by $\partial f(x)$, is defined as

\[
\partial f(x) := \left\{ v \in \mathbb{R}^p \mid \exists \alpha, \delta > 0 \text{ s.t. } f(x') \geq f(x) + \langle v, x' - x \rangle - \frac{\alpha}{2} \|x' - x\|^2 \forall x' \in B(x, \delta) \right\};
\]

the regular subdifferential of $f$ at $x$, denoted by $\hat{\partial} f(x)$, is defined as

\[
\hat{\partial} f(x) := \left\{ v \in \mathbb{R}^p \mid \liminf_{x' \to x} \frac{f(x') - f(x) - \langle v, x' - x \rangle}{\|x' - x\|} \geq 0 \right\};
\]

and the (limiting) subdifferential of $f$ at $x$, denoted by $\check{\partial} f(x)$, is defined as

\[
\check{\partial} f(x) := \left\{ v \in \mathbb{R}^p \mid \exists x^k \to x \text{ with } f(x^k) \to f(x) \text{ and } \check{\partial} f(x^k) \ni v^k \to v \text{ as } k \to \infty \right\}.
\]

**Remark 2.1** (i) At each $x \in \text{dom} f$, the above three kinds of subdifferentials of $f$ are all closed and satisfy $\check{\partial} f(x) \subseteq \hat{\partial} f(x) \subseteq \partial f(x)$, and the sets $\check{\partial} f(x)$ and $\partial f(x)$ are convex but $\hat{\partial} f(x)$ is generally nonconvex. When $f$ is convex, $\check{\partial} f(x) = \hat{\partial} f(x) = \partial f(x)$ and $f$ is precisely the subdifferential of $f$ at $x$ in the sense of convex analysis [31]. When $f$ is nonconvex, there may be a big difference among the three generalized subdifferentials. For example, for the function $f(t) = -|t|$ for $t \in \mathbb{R}$, we have $\check{\partial} f(0) = \hat{\partial} f(0) = \emptyset$, while $\partial f(0) = \{-1, 1\}$.

(ii) The point $\overline{x}$ at which $0 \in \partial f(\overline{x})$ (respectively, $0 \in \hat{\partial} f(\overline{x})$ and $0 \in \check{\partial} f(\overline{x})$) is called a limiting (respectively, proximal and regular) critical point of $f$. It is easy to verify that a local minimizer of $f$ is necessarily a proximal critical point, and then is a regular and limit critical point. However, the converse may not hold; for example, the function $f(t) = -|t| + t$ for $t \in \mathbb{R}$ satisfies $0 \in \partial f(0)$, but $0$ is not a local minimizer of $\min_{t \in \mathbb{R}} f(t)$.

(iii) Recall that a function $f : \mathbb{R}^p \to (-\infty, +\infty]$ is semiconvex of modulus $\gamma > 0$ if the function $x \mapsto f(x) + \frac{\gamma}{2} \|x\|^2$ is convex. By [25, Remark 1.5], if $f$ is semiconvex, then at every $x \in \text{dom} f$ it holds that $\check{\partial} f(x) = \hat{\partial} f(x) = \partial f(x)$.
2.2 Clarke Jacobian of two proximal operators

For a lsc convex function \( f : \mathbb{R}^p \to \mathbb{R} \) and a parameter \( \gamma > 0 \), we denote by \( P_\gamma f \) and \( e_\gamma f \) the proximal mapping and Moreau envelope of \( f \), respectively, defined as

\[
P_\gamma f(x) := \arg\min_{z \in \mathbb{R}^p} \left\{ f(z) + \frac{1}{2\gamma} \| z - x \|^2 \right\} \quad \text{and} \quad e_\gamma f(x) := \min_{z \in \mathbb{R}^p} \left\{ f(z) + \frac{1}{2\gamma} \| z - x \|^2 \right\}.
\]

From [21], \( P_\gamma f : \mathbb{R}^p \to \mathbb{R}^p \) is a globally Lipschitz mapping with modulus 1, and \( e_\gamma f \) is a continuously differentiable convex function with \( \nabla e_\gamma f(x) = \gamma^{-1}(x - P_\gamma f(x)) \). Next, we recall the definition of Clarke Jacobian for a locally Lipschitz mapping.

**Definition 2.2** [8, Definition 2.6.1] Let \( H : \Omega \to \mathbb{R}^m \) be a locally Lipschitz continuous mapping defined on an open set \( \Omega \subseteq \mathbb{R}^p \). Denote by \( D_H \subseteq \Omega \) the set of points where \( H \) is Fréchet differentiable and by \( H'(z) \in \mathbb{R}^{m \times p} \) the Jacobian of \( H \) at \( z \in D_H \). Let \( \tau \in \Omega \) be given. The Clarke (generalized) Jacobian of \( H \) at \( \tau \) is defined as

\[
\partial_C H(\tau) := \text{conv}\left\{ \lim_{k \to \infty} H'(z^k) \mid \{ z^k \} \subseteq D_H \text{ with } \lim_{k \to \infty} z^k = \tau \right\},
\]

Generally, it is not easy to characterize the Clarke Jacobian of a locally Lipschitz mapping. The following two lemmas provide an exact characterization for the Clarke Jacobian of the proximal mapping to the weighted \( \ell_1 \)-norm and the check loss function.

**Lemma 2.1** For a given \( w \in \mathbb{R}_+^p \), let \( h(x) := \| w \circ x \|_1 \) for \( x \in \mathbb{R}^p \). Then, it holds that

\[
P_\gamma^{-1} h(z) = \text{sign}(z) \max \left( |z| - \gamma^{-1} w, 0 \right) \quad \forall z \in \mathbb{R}^p,
\]

\[
\partial_C (P_\gamma^{-1} h)(z) = \{ \text{Diag}(v_1, \ldots, v_n) \mid v_i = 1 \text{ if } |\gamma z_i| > w_i, \text{ otherwise } v_i \in [0,1] \}.
\]

**Lemma 2.2** For any given \( \tau \in (0,1) \), let \( \theta_\tau \) and \( f_\tau \) be the function defined as in (3). Then, for any given \( \gamma > 0 \) and \( z \in \mathbb{R}^p \), it holds that

\[
[P_\gamma^{-1} f_\tau(z)]_i = \begin{cases} 
z_i - \frac{t}{n\gamma} & \text{if } z_i \geq \frac{t}{n\gamma}, \\
0 & \text{if } \frac{t}{n\gamma} \leq z_i \leq \frac{t}{n\gamma}, \\
z_i - \frac{t}{n\gamma} & \text{if } z_i \leq \frac{t}{n\gamma} 
\end{cases}
\]

and moreover, \( \partial_C (P_\gamma^{-1} f_\tau)(y) = \{ \text{Diag}(v_1, \ldots, v_n) \mid v_i \in \partial_C [P_\gamma^{-1}(n^{-1}\theta_\tau)](z_i) \} \)

with

\[
\partial_C [P_\gamma^{-1}(n^{-1}\theta_\tau)](t) = \begin{cases} 
\{1\} & \text{if } t > \frac{t}{n\gamma} \text{ or } t < \frac{t}{n\gamma}; \\
[0,1] & \text{if } t = \frac{t}{n\gamma} \text{ or } \frac{t}{n\gamma}; \\
\{0\} & \text{if } \frac{t}{n\gamma} < t < \frac{t}{n\gamma}. 
\end{cases}
\]

2.3 Semismoothness of two proximal operators

Semismoothness was originally introduced by Mifflin [24] for functionals, and Qi and Sun [28] later introduced the class of vector semismooth functions.
Definition 2.3 (see [24, 28, 29]) Let $F : \mathcal{O} \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a locally Lipschitz continuous function on the open set $\mathcal{O}$. The function $F$ is said to be semismooth at a point $x \in \mathcal{O}$ if $F$ is directionally differentiable at $x$ and for any $\Delta x \to 0$ and $V \in \partial C F(x + \Delta x)$,
\[ F(x + \Delta x) - F(x) - V \Delta x = o(\|\Delta x\|); \]
and $F$ is said to be strongly semismooth at $x$ if $F$ is semismooth at $x$ and for any $\Delta x \to 0$,
\[ F(x + \Delta x) - F(x) - V \Delta x = O(\|\Delta x\|^2). \]
The function $F$ is said to be a semismooth (respectively, strongly semismooth) function on $\mathcal{O}$ if it is semismooth (respectively, strongly semismooth) everywhere in $\mathcal{O}$.

By [9, Proposition 7.4.7] every piecewise affine mapping is strongly semismooth. Note that $P_{\gamma - 1} f_\tau$ and $P_{\gamma - 1} h$ are piecewise affine. Hence, they are strongly semismooth.

3 Zero-norm penalized QR and equivalent DC model

Quantile regression is a popular method for studying the influence of a set of covariates on the conditional distribution of a response variable, and has been widely used to handle heteroscedasticity [20, 40]. For a univariate response $Y \in \mathbb{R}$ and a vector of covariates $X \in \mathbb{R}^p$, the conditional cumulative distribution function of $Y$ is defined as
\[ F_Y(t|x) := \Pr(Y \leq t \mid X = x) \]
and the $\tau$th conditional quantile of $Y$ is given by $Q_Y(\tau|x) := \inf \{ t : F_Y(t|x) \geq \tau \}$. Let $X = [x_1 \cdots x_n]^T$ be an $n \times p$ design matrix on $X$. Consider the linear quantile regression
\[ y = X \beta^* + \varepsilon \] (2)
where $y = (y_1, \ldots, y_n)^T \in \mathbb{R}^n$ is the response vector, $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)^T$ is the noise vector whose components are independently distributed and satisfy $\Pr(\varepsilon_1 \leq 0|x_i) = \tau$ for some known constant $\tau \in (0, 1)$, and $\beta^* \in \mathbb{R}^p$ is the true but unknown coefficient vector. In other words, the above linear quantile regression model assumes that
\[ Q_Y(\tau|x_i) = x_i^T \beta^* \text{ for } i = 1, 2, \ldots, n. \]
We are interested in the high-dimensional case where $p > n$ and the model is sparse in the sense that only $s^* (\ll p)$ components of the true vector $\beta^*$ are nonzero.

For $\tau \in (0, 1)$, we denote by $f_\tau : \mathbb{R}^n \to \mathbb{R}$ the check loss function [19] of the model (2):
\[ f_\tau(z) := n^{-1} \sum_{i=1}^n \theta_\tau(z_i) \text{ with } \theta_\tau(u) := (\tau - I_{\{u \leq 0\}})u. \] (3)
To estimate the true sparse $\beta^*$ in (2), we consider the zero-norm regularized problem
\[ \hat{\beta}(\tau) \in \arg \min_{\beta \in \mathbb{R}^p} \left\{ \nu f_\tau(y - X \beta) + \|\beta\|_0 \right\} \] (4)
where $\nu > 0$ is the regularization parameter, and $\|\beta\|_0$ denotes the zero-norm of $\beta$ (i.e., the number of nonzero entries of $\beta$). By the expression of $f_\tau$, it is immediate to check that $f_\tau$ is nonnegative and $f_\tau(\beta^k) \to +\infty$ whenever $\|\beta^k\| \to \infty$, i.e., the function $f_\tau$ is nonnegative and coercive. By Lemma 1 in Appendix A, the estimator $\hat{\beta}(\tau)$ is well defined. Since $\hat{\beta}(\tau)$ depends on $\tau$, there is a great possibility for the model (4) to monitor different 'locations' of the conditional distribution, and consequently, the heteroscedasticity of the data, when existing, can be inspected by solving the problem (4) with different $\tau \in (0, 1)$. For the simplicity of notation, in the sequel we always use $\hat{\beta}$ to replace $\hat{\beta}(\tau)$ and for a given $\tau \in (0, 1)$, write $\tau_{\min} := \min(\tau, 1 - \tau)$ and $\tau_{\max} := \max(\tau, 1 - \tau)$.

Due to the combination of the zero-norm, the computation of $\hat{\beta}$ is NP-hard. To design an algorithm for computing a high-quality approximation to $\hat{\beta}$ in the next section, the rest of this section derives from a primal-dual viewpoint an equivalent augmented Lipschitz optimization problem which implies an equivalent DC (difference of convex functions) surrogate for the zero-norm regularized problem (4). This needs the following function

$$\phi(t) := \frac{a - 1}{a + 1}t^2 + \frac{2}{a + 1}t \quad (a > 1) \quad \text{for } t \in \mathbb{R}. \quad (5)$$

Notice that $\phi(1) = 1$ and $t^* = 0$ is the unique minimizer of $\phi$ over $[0, 1]$ with $\phi(t^*) = 0$. It is easy to check that the zero-norm $\|z\|_0$ is the optimal value function of the problem

$$\min_{w \in \mathbb{R}^p} \left\{ \sum_{i=1}^p \phi(w_i) \quad \text{s.t.} \quad (e - w)^\top|z| = 0, \ 0 \leq w \leq e \right\},$$

which has a parametric equilibrium constraint $|z| \geq 0$, $e - w \geq 0$ and $\langle e - w, |z| \rangle = 0$. This variational characterization shows that the problem (4) is equivalent to the problem

$$\min_{\beta \in \mathbb{R}^p, w \in \mathbb{R}^p} \left\{ \nu f_\tau(y - X\beta) + \sum_{i=1}^p \phi(w_i) \quad \text{s.t.} \quad \langle e - w, |\beta| \rangle = 0, \ 0 \leq w \leq e \right\} \quad (6)$$

in the following sense: if $\overline{\beta}$ is globally optimal to (4), then $\langle \overline{\beta}, \text{sign}(|\overline{\beta}|) \rangle$ is a global optimal solution of the problem (6), and conversely, if $\langle \overline{\beta}, \overline{w} \rangle$ is a global optimal solution of (6), then $\overline{\beta}$ is globally optimal to (4). The problem (6) is a mathematical program with equilibrium constraint $e - w \geq 0, |\beta| \geq 0, \langle e - w, |\beta| \rangle = 0$, abbreviated as MPEC. The equivalence between (4) and (6) shows that the difficulty of the zero-norm regularized problem (4) arises from the hidden equilibrium constraint. It is well known that the handling of nonconvex constraints is much harder than the handling of nonconvex objective functions. It is natural to consider the following penalized version of the MPEC (6):

$$\min_{\beta \in \mathbb{R}^p, w \in [0,e]} \left\{ \nu f_\tau(y - X\beta) + \sum_{i=1}^p \phi(w_i) + \rho \langle e - w, |\beta| \rangle \right\} \quad (7)$$

where $\rho > 0$ is the penalty parameter. Since $\beta \mapsto f_\tau(y - X\beta)$ is globally Lipschitz continuous with modulus $\tau_{\max}\|X\|$, by [23, Section 3.2] the following conclusion holds.
Theorem 3.1 The penalty problem (7) associated to every $\rho > \overline{\rho} := 4a\nu_{\max}\|X\|_{\infty}/a+1$ has the same global optimal solution set as the MPEC (6) does.

Theorem 3.1 states that the problem (7) is a global exact penalty of (6) in the sense that there is a threshold $\overline{\rho} > 0$ such that the former associated to every $\rho > \overline{\rho}$ has the same global optimal solution set as the latter does. Along with the equivalence between (4) and (6), the problem (4) is equivalent to the problem (7). Notice that the objective function of (7) is globally Lipschitz over its feasible set and its nonconvexity is owing to the coupled term $\langle e - w, |\beta| \rangle$ rather than the combination. So, the problem (7) provides an equivalent augmented Lipschitz optimization reformulation for the zero-norm problem (4). In fact, the problem (7) associated to every $\rho > \overline{\rho}$ implies an equivalent DC surrogate for (4). To illustrate this, define the extended real-value $\psi: \mathbb{R} \to (-\infty, +\infty]$ by

$$\psi(t) := \begin{cases} \phi(t) & \text{if } t \in [0, 1]; \\ +\infty & \text{otherwise.} \end{cases} \quad (8)$$

Then, with the conjugate $\psi^*(s) := \sup_{t \in \mathbb{R}} \{st - \psi(t)\}$ of $\psi$, it is not hard to check that the problem (7) is equivalent to the following minimization problem with $\lambda = \rho\nu^{-1}$:

$$\min_{\beta \in \mathbb{R}^p} \left\{ \Theta_{\lambda, \rho}(\beta) := f_r(y - X\beta) + \lambda \|\beta\|_1 - \lambda\rho^{-1} \sum_{i=1}^p \psi^*(\rho|\beta_i|) \right\}. \quad (9)$$

By the definitions of the functions $\phi$ and $\psi$, an elementary calculation yields that

$$\psi^*(s) = \begin{cases} 0 & \text{if } s \leq \frac{2a}{a+1}; \\ \frac{(a+1)s - 2s^2}{4(a^2 - 1)} & \text{if } \frac{2a}{a+1} < s \leq \frac{2a}{a+1}; \\ s - 1 & \text{if } s > \frac{2a}{a+1}. \end{cases} \quad (10)$$

Clearly, $\psi^*$ is a nondecreasing finite convex function in $\mathbb{R}$, which implies that the function $s \mapsto \psi^*(\rho|s|)$ is convex, and consequently the problem (9) is a DC program. To sum up the above discussions, the problem (9) associated to every $\rho > \overline{\rho}$ and $\lambda = \rho\nu^{-1}$ provides an equivalent DC surrogate for the difficult zero-norm regularized problem (4).

Now we present some desirable properties of $\Theta_{\lambda, \rho}$; see Appendix A for their proofs.

Proposition 3.1 For any given $\lambda, \rho > 0$, the following statements hold for $\Theta_{\lambda, \rho}$ and $g_\rho(\beta) := -\sum_{i=1}^p \psi^*(\rho|\beta_i|)$ for $\beta \in \mathbb{R}^p$.

(i) The function $g_\rho$ is continuously differentiable with gradient $\nabla g_\rho$ globally Lipschitz continuous of modulus $\rho^2 \max(\frac{a+1}{2}, \frac{a^2+1}{2(a-1)})$, and hence is semiconvex.

(ii) $\Theta_{\lambda, \rho}$ is a lower bounded, globally Lipschitz, coercive, semiconvex function on $\mathbb{R}^p$.

(iii) For any given $\beta \in \mathbb{R}^p$, the subdifferential set $\partial \Theta_{\lambda, \rho}(\beta)$ takes the following form

$$\tilde{\partial} \Theta_{\lambda, \rho}(\beta) = \tilde{\partial} \Theta_{\lambda, \rho}(\beta) = \partial \Theta_{\lambda, \rho}(\beta) = -X^T \partial f_r(y - X\beta) + \lambda \beta_1 - \lambda \rho^{-1} \nabla g_\rho(\beta).$$

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4 MSCRA for zero-norm regularized QR

From the last section, in order to compute the estimator \( \hat{\beta} \), we only need to solve a single penalty problem (7) which is much easier than the original zero-norm regularized problem (4), since its nonconvexity only arises from the coupled term \( \langle w, |\beta| \rangle \). Observe that this penalty problem becomes a convex program when either of \( w \) and \( \beta \) is fixed. Therefore, it is natural to apply the alternating minimization method to solving it.

4.1 MSCRA for computing the estimator \( \hat{\beta} \)

Now by solving the penalty problem (7) in an alternating way, we propose the following multi-stage convex relaxation approach (MSCRA) to seek a desired approximation of \( \hat{\beta} \).

Algorithm 1 (MSCRA for computing \( \hat{\beta} \))

Initialization: Fix a \( \tau \in (0, 1) \). Choose \( \lambda, \rho_0 > 0 \) and an initial \( w^0 \in [0, \frac{1}{2} e] \). Set \( k := 1 \).

while the stopping conditions are not satisfied do

1. Seek an approximate optimal solution to the weighted \( \ell_1 \)-regularized problem

\[
\beta^k \approx \arg \min_{\beta \in \mathbb{R}^p} \left\{ f_r(y - X \beta) + \lambda \sum_{i=1}^{p} (1 - w^{k-1}_i |\beta_i|) \right\}.
\] (12)

2. When \( k = 1 \), select a suitable \( \rho_1 \geq \max(1, \rho_0) \) in terms of \( \|\beta^1\|_\infty \). If \( k = 2, 3 \), select \( \rho_k \) such that \( \rho_k \geq \rho_{k-1} \); otherwise, set \( \rho_k = \rho_{k-1} \).

3. For \( i = 1, 2, \ldots, p \), compute the following minimization problem

\[
w^k_i = \arg \min_{0 \leq w_i \leq 1} \left\{ \phi(w_i) - \rho_k w_i |\beta^k_i| \right\}.
\] (13)

4. Let \( k \leftarrow k + 1 \), and then go to Step 1.
end while

Remark 4.1 (i) Step 1 of Algorithm 1 is solving the penalty problem (7) with \( w \) fixed to be \( w^{k-1} \), while Step 3 is solving this problem with \( \beta \) fixed to be \( \beta^k \). That is, Algorithm 1 is solving the nonconvex penalty problem (7) in an alternating way. In the first stage, since there is no any information on estimating the nonzero entries of \( \beta^* \), it is reasonable to impose an unbiased weight on each component of \( \beta \). Motivated by this, we restrict the initial \( w^0 \) in \([0, 1/2e]\), a subset of the feasible set of \( w \). When \( w^0 = 0 \), the first stage is precisely the minimization of the \( \ell_1 \)-penalized check loss function. Although the threshold \( \tau \) is known when the parameter \( \nu \) in (4) is given, we select a varying \( \rho \) in the problem (13) since it is just a relaxation of (7). By the optimality condition of (7), \( \rho_k |\beta^k_i| \in \partial \psi(w^k_i) \).
which by [31, Theorem 23.5] and the expression of $\psi^*$ is equivalent to requiring that

$$w^k_i = (\psi^*)'(\rho_k|\beta^k_i|) = \min \left[ 1, \max \left( 0, \frac{(a+1)\rho_k|\beta^k_i| - 2}{2(a-1)} \right) \right]. \quad (14)$$

Thus, the computation cost of Algorithm 1 in each iteration is to seek an approximate solution to the weighted $\ell_1$-penalized check loss minimization problem (12). (ii) Algorithm 1 is actually an inexact majorization-minimization (MM) method [22] for solving the equivalent DC surrogate (9) with a special starting point. Indeed, for a given $\beta' \in \mathbb{R}^p$, the convexity and smoothness of $\psi^*$ implies that the following inequality holds

$$\sum_{i=1}^p \psi^*(\rho_i|\beta_i|) \geq \sum_{i=1}^p \psi^*(\rho_i|\beta'_i|) + \rho_i(\beta_i - |\beta'_i|) \quad \forall \beta \in \mathbb{R}^p \quad (15)$$

with $w_i = (\psi^*)'(\rho_i|\beta'_i|)$ for $i = 1, 2, \ldots, p$. Clearly, $w_i \in [0,1]$ for $i = 1, 2, \ldots, p$ by the expression of $\psi^*$. Hence, the following function is a majorization of $\Theta_{\lambda,\rho}$ at $\beta^{k-1}$:

$$f_\tau(y - X\beta) + \lambda \|(e - w^{k-1}) \circ \beta\|_1 - \lambda \sum_{i=1}^p \psi^*(\rho_i|\beta^{k-1}_i|) + \rho_i(w^{k-1}, |\beta^{k-1}_i|),$$

and the subproblem (12) is the inexact minimization of this majorization function. Also, for any given $\rho_0 > 0$, when $\|\beta^0\|_\infty \leq \frac{2}{(a+1)|\rho_0|}$, by (10) we have $w_0 = (\psi^*)'(\rho_0|\beta^0_i|) = 0$. Thus, the first stage of Algorithm 1 with $w_0 = 0$ is precisely the inexact MM method for (9) with $\beta^0$ satisfying $\|\beta^0\|_\infty \leq \frac{2}{(a+1)|\rho_0|}$. In addition, Algorithm 1 can be regarded as an inexact inversion the LLA method proposed by Zou and Li [45] for (9), but it is different from the difference convex algorithm proposed by Wu and Liu [39] since the latter depends on the majorization of $g_p(\beta)$ at $\beta^k$ and the obtained approximation is lack of symmetry. (iii) Consider that there is always a certain deviation in practical computation. We allow the problem (12) to be solved inexacty. Among others, the inexact accuracy of $\beta^k$ in Step 2 is measured in the following way: $\exists \xi^k \in \mathbb{R}^p$ and $r_k \geq 0$ with $\|\xi^k\| \leq r_k$ such that

$$\xi^k \in \partial \left[ f_\tau(y - X\beta) + \lambda \|(e - w^{k-1}) \circ \beta\|_1 \right]_{\beta = \beta^k} = -X^T \partial f_\tau(y - X\beta^k) + \lambda \left[ (1 - w^{k-1}_1) \partial |\beta^k_1| \times \cdots \times (1 - w^{k-1}_p) \partial |\beta^k_p| \right]. \quad (16)$$

where the equality is due to [31, Theorem 23.8]. Clearly, when $r_k \equiv 0$, $\beta^k$ becomes an exact solution to the problem (12). In addition, by using Proposition 3.1(iii) and noting that $\nabla g_p(\beta^k) = \rho_k((\psi^*)'(\rho_k|\beta^k_1|), \ldots, (\psi^*)'(\rho_k|\beta^k_p)|)^T = \rho_k(w^k_1, \ldots, w^k_p)^T$, we have

$$\partial \Theta_{\lambda,\rho_k}(\beta^k) = -X^T \partial f_\tau(y - X\beta^k) + \lambda \partial |\beta^k_1| \times \cdots \times \partial |\beta^k_p| - \lambda w^k = -X^T \partial f_\tau(y - X\beta^k) + \lambda \left[ (1 - w^k_1) \partial |\beta^k_1| \times \cdots \times (1 - w^k_p) \partial |\beta^k_p| \right]$$
where the second equality is since \( w^k_i = 0 \) if \( \beta^k_i = 0 \) and \( \partial \beta^k_i = \{ \text{sign}(\beta^k_i) \} \) if \( \beta^k_i \neq 0 \). By comparing with (16), \( 0 \in \partial \Theta_{\lambda, p_k}(\beta^k) + \lambda_0 \sum_{i=1}^{n_p} \partial \beta^k_i | \partial \beta^k_i | \times \cdots \times (w^k_p - w^{k-1}_p) \partial \beta^k_p | - \xi^k \). Since each \( \partial \beta^k_i \leq [ -1, 1 ] \), the following stopping criterion is suggested for Algorithm 1

\[
\text{Err}_k := \frac{\sqrt{\lambda^2 \| w^k - w^{k-1} \|^2 + \| \xi^k \|^2}}{1 + \| y \|} \leq \text{tol}. \quad (17)
\]

This guarantees that the obtained \( \beta^k \) is an approximate regular critical point of \( \Theta_{\lambda, p_k} \).

### 4.2 Theoretical guarantees of Algorithm 1

For convenience, we denote by \( S^* \) the support of the true vector \( \beta^* \), and define the set

\[
C(S^*) := \bigcup_{S \subset S, |S| \leq 1.5s^*} \{ \beta \in \mathbb{R}^p : \| \beta_{S^c} \|_1 \leq 3 \| \beta_S \|_1 \}.
\]

Recall that the matrix \( X \) is said to have the \( \kappa \)-restricted strong convexity on \( C(S^*) \) if

\[
\kappa > 0 \quad \text{and} \quad \frac{1}{2n} \| X \Delta \beta \|^2 \geq \kappa \| \Delta \beta \|^2 \quad \text{for all} \quad \Delta \beta \in C(S^*).
\]

The RSC is equivalent to the restricted eigenvalue condition of the Gram matrix \( \frac{1}{2n} X^T X \) due to van de Geer et al. [16] and Bickel et al. [4]. Notice that \( C(S^*) \) includes the closed convex cone \( \{ \beta \in \mathbb{R}^p : \| \beta_{S^c} \|_1 \leq 3 \| \beta_S \|_1 \} \). This RSC is a little stronger than the one used by Negahban et al. [26] for the \( t_1 \)-regularized smooth loss minimization. In this section, we shall provide the deterministic theoretical guarantees for Algorithm 1 under this RSC, including the error bound of the iterate \( \beta^k \) to the true \( \beta^* \) and the decrease analysis of the error sequence. The proofs of all results are included in Appendix B. We need to make the following assumption on the approximate optimality tolerance of \( \beta^k \):

**Assumption 4.1** There exists \( \epsilon > 0 \) such that for each \( k \in \mathbb{N} \), \( r_k \leq \epsilon \).

First of all, by Lemma 2 in Appendix B, we have the following error bound result.

**Theorem 4.1** Suppose that \( X \) has the \( \kappa \)-RSC over \( C(S^*) \) and the noise \( \epsilon \) is nonzero. If \( \lambda \) and \( \rho_3 \) are such that \( \lambda \in \left[ \frac{16 \tau_{\max} \| X \|_1 + 8 \tau_{\min} \sqrt{n \epsilon}}{\tau_{\min} \kappa - c^{-1} - \tau_{\max} \| X \|_{\max} (2n^{-1} \tau_{\max} \| X \|_1 + \epsilon) \sqrt{n}} \right] \) and \( \rho_3 \leq \frac{8}{9 \sqrt{3} \tau_{\max} \lambda \| \epsilon \|_{\infty}} \) for some constant \( c \geq \frac{\tau_{\min} \kappa - 9 \sqrt{3} \tau_{\max} \| X \|_{\max} (2n^{-1} \tau_{\max} \| X \|_1 + \epsilon) \sqrt{n}}{8} \), then under Assumption 4.1 the following inequality holds for each \( k \in \mathbb{N} \)

\[
\| \beta^k - \beta^* \| \leq \frac{9 c \tau_{\max} \lambda \sqrt{1.5s^*}}{\sqrt{8}} \| \epsilon \|_{\infty}.
\]

**Remark 4.2** (i) Different from the error bound in [26, Theorem 1] for the \( t_1 \)-regularized smooth loss estimator, the error bound in Theorem 4.1 involves the infinite norm of noise \( \| \epsilon \|_{\infty} \), but it still has the same order \( O(\lambda \sqrt{s^*}) \) as established in [26, Theorem 1].
(ii) Similar to the $\ell_1$-regularized squared-root loss in [2], the regularization parameter $\lambda$ is required to belong to an interval depending on the sparsity $s^*$ of the true $\beta^*$, which is stronger than the restriction imposed by [26, Theorem 1] since the latter requires that the parameter $\lambda$ is greater than a certain bound depending on the noise vector $\varepsilon$ only.

(iii) To ensure that the constant $c > 0$ exists, the inexact accuracy $\epsilon$ of $\beta^k$ needs to satisfy $0 \leq \epsilon \leq \frac{n^2 \min k - 18\sqrt{\mathbb{E}}^2 \|X\|_{\max} \|\varepsilon\|_{1} \sqrt{s^*}}{9\sqrt{\mathbb{E}} \max \sqrt{\mathbb{E}} \sqrt{s^*}}$. Clearly, as the sample size $n$ increases, the requirement on $\epsilon$ becomes looser and there is a wide range for choosing $c$, which means that the choice interval of $\lambda$ becomes larger and the corresponding value becomes smaller.

Theorem 4.1 establishes an error bound for every iterate $\beta^k$, but it does not tell us if the error bound of the current $\beta^k$ is better than that of the previous $\beta^{k-1}$. To seek the answer, we study the decrease of the error bound sequence by bounding $\max_{i \in S^*} (1 - w^k_i)$. We write $F^{0} := S^*$ and $\Lambda^{0} := \{i : |\beta^*_i| \leq \frac{4a}{(n+1)\rho_k}\}$, and for each $k \in \mathbb{N}$ define

$$F^k := \{i : ||\beta^k_i| - |\beta^*_i|| \geq \frac{1}{\rho_k} \} \text{ and } \Lambda^k := \{i : |\beta^*_i| \leq \frac{4a}{(n+1)\rho_k}\}. \quad (19)$$

From Lemma 4 in Appendix B, it follows that the value $\max_{i \in S^*} (1 - w^k_i)$ is upper bounded by $\max_{i \in S^*} \max(\Lambda^k(i), \|F^k(i)\|)$. By this, we have the following error bound result.

**Theorem 4.2** Suppose that $X$ has the $\kappa$-RSC over $C(S^*)$, the noise vector $\varepsilon$ is nonzero and Assumption 4.1 holds. If the parameter $\lambda$ is chosen as in Theorem 4.1 and the parameter $\rho_3$ satisfies $\rho_3 \leq \frac{1}{c \max_{i \in S^*} \|\varepsilon\|_{1} \left(\sqrt{4.5s^*} \sqrt{3}/\sqrt{n}\right)}$, then for each $k \in \mathbb{N}$

$$\|\beta^k - \beta^*\| \leq \frac{(3 + \sqrt{3}c^2)\max_{i \in S^*} \|\varepsilon\|_{1} \sqrt{s^*} X \|X\|_{1} \|\varepsilon\|_{\infty}}{2n} + \frac{(3 + 3\sqrt{3})c \max_{i \in S^*}\|\varepsilon\|_{1} \sqrt{s^*} \|X\|_{\infty}}{2\sqrt{2}} \max_{i \in S^*} \Lambda^0(i)$$

$$+ c \max_{i \in S^*} \|\varepsilon\|_{1} \sqrt{s^*} \sum_{j=0}^{k-2} r_{k-j} \left(\frac{1}{\sqrt{3}}\right)^j + \left(\frac{1}{\sqrt{3}}\right)^{k-1} \|\beta^k_1 - \beta^*_1\| \quad (20)$$

where we stipulate that $\sum_{j=0}^{k-2} r_{k-j} \left(\frac{1}{\sqrt{3}}\right)^j = 0$ for $k = 1$.

**Remark 4.3** (i) The error bound in Theorem 4.2 consists of three parts: the statistical error induced by the noise $\varepsilon$, the identification error $\max_{i \in S^*} \|F^k(i)\|$ related to the choice of $a$ and $\rho_3$, and the computation errors $\sum_{j=0}^{k-2} r_{k-j} \left(\frac{1}{\sqrt{3}}\right)^j + \left(\frac{1}{\sqrt{3}}\right)^{k-1} \|\beta^k_1 - \beta^*_1\|$.

(ii) By the definition of $\Lambda^0$, when $\rho_3$ and $a$ are chosen such that $\frac{(a+1)\rho_3}{4a} > \frac{1}{\min_{i \in S^*} |\beta^*_i|}$, the identification error becomes zero. If $\min_{i \in S^*} |\beta^*_i|$ is not too small, it would be easy to choose such $\rho_3$. Clearly, when $\rho_3$ and $a$ are chosen to be larger, the identification error is lower. However, when $\rho_3$ and $a$ are larger, $\rho_1$ becomes larger and each component of $w^1$ is close to 1 by the formula (14). Consequently, it will become very conservative to cut those smaller entries of $\beta^2$ when solving the second subproblem. Hence, there is a trade-off between the choice of $a$ and $\rho_0$ and the computation speed of Algorithm 1.
(iii) If the subproblem (12) could be solved exactly, the computation error \(\sum_{j=0}^{k-2} r_{k-j}(\frac{1}{\sqrt{k^j}})^j\) vanishes. If the subproblem (12) is solved with the accuracy \(r_k\) satisfying \(r_k \leq \left(\frac{1}{\sqrt{k^j}}\right)^{k^j}\) for \(\nu > 1\), this computation error will tend to 0 as \(k \to +\infty\). Since the third term on the right hand side of (21) is the combination of the noise and \(\sum_{j=0}^{k-2} r_{k-j}(\frac{1}{\sqrt{k^j}})^j\), it is strongly suggested that the subproblem (12) is solved as well as possible.

To close this section, we take a closer look at the conclusion of Theorem 4.2 under the following assumption on the noises. This assumption is first used by Wang [41] for studying the \(\ell_1\)-penalized LAD estimator for high-dimensional linear regression, which is a weak condition for the noise vector \(\varepsilon\) and the Cauchy distribution even satisfies it.

**Assumption 4.2** The noises \(\varepsilon_1, \ldots, \varepsilon_n\) have the independent identical symmetric distribution and there is a constant \(\alpha > 0\) (depending on the distribution of \(\varepsilon_i\)) such that

\[
\Pr(\|\varepsilon_1\| > t) \leq \frac{2}{2 + \alpha t} \quad \forall t \geq 0.
\]

Under Assumption 4.2, there exist constants \(\alpha > 0\) and \(M > 0\) such that \(\|\varepsilon\|_\infty \leq M\) with probability at least \(1 - \frac{2}{2 + \alpha t}\). By Theorem 4.2, the following corollary holds.

**Corollary 4.1** Suppose that \(X\) has the \(\kappa\)-RSC over the set \(C(S^*)\), the noise vector \(\varepsilon\) is nonzero, and Assumption 4.1 and Assumption 4.2 hold. If the parameters \(\lambda\) and \(\rho_3\) are chosen as in Theorem 4.2, then there exist constants \(\alpha > 0\) and \(M > 0\) such that

\[
\|\beta^k - \beta^*\| \leq (3 + \sqrt{3})c\tau_{\max} \sqrt{s^* M} \left(2\tau_{\max} \|X\|_1 + \frac{1}{\sqrt{2}} \max_{i \in S^*} I_{\Lambda^0}(i)\right)
+ c\tau_{\max} M \sqrt{s^*} \sum_{j=0}^{k-2} r_{k-j}(\frac{1}{\sqrt{3}})^j + \left(\frac{1}{\sqrt{3}}\right)^{k-1} \|\beta^1 - \beta^*\|
\]

with probability at least \(1 - \frac{2}{2 + \alpha t}\), where we stipulate that \(\sum_{j=0}^{k-2} r_{k-j}(\frac{1}{\sqrt{3}})^j = 0\) for \(k = 1\).

For the RSC assumption in Theorem 4.1-4.2 and Corollary 4.1, from [30] it follows that if \(X\) is from the \(\Sigma_x\)-Gaussian ensemble (i.e., \(X\) is formed by independently sampling each row \(x_i^T \sim N(0, \Sigma_x)\)), there exists a constant \(\kappa > 0\) (depending on \(\Sigma_x\)) such that the RSC holds on \(C(S^*)\) with probability greater than \(1 - c_1 \exp(-c_2 n)\) as long as \(n > c_0 \log p\), where \(c_0, c_1\) and \(c_2\) are absolutely positive constants. From [5], for some sub-Gaussian design matrix \(X\), the RSC holds on \(C(S^*)\) with a high probability when the sample size \(n\) is over a certain threshold depending on the Gaussian width of \(C(S^*)\).

### 5 Proximal dual semismooth Newton method

The pivotal part of Algorithm 1 is the solution of the following minimization problem

\[
\min_{\beta \in \mathbb{R}^p} \left\{ f_\tau(y - X\beta) + h_k(\beta) \right\}
\]
where \( h_k(\beta) := \|\omega^k \circ \beta\|_1 \) with \( \omega^k \equiv \lambda(e - w^k) \) for \( k \in \mathbb{N} \). In this section, we develop a proximal dual semismooth Newton method (PDSN) for solving (22). This is a proximal point algorithm (PPA) with the subproblems solved by applying the semismooth Newton method to their dual problems. The iterate steps of the PPA are described as follows.

**Algorithm 2 PPA for solving the problem (22)**

**Initialization:** Choose \( \gamma_{1,0} > 0, \gamma_{2,0} > 0, \gamma > 0 \) and \( \varrho \in (0, 1) \). Set \( \beta^0 = \beta^k \) and \( j = 0 \).

while the stopping conditions are not satisfied do

(S.1) Find an approximate minimizer \( \beta^{j+1} \) to the following problem

\[
\min_{\beta \in \mathbb{R}^p} \left\{ f_r(y - X\beta) + h_k(\beta) + \gamma_{1,j} \|\beta - \beta^{j}\|^2 + \gamma_{2,j} \|X\beta - X\beta^{j}\|^2 \right\}. \tag{23}
\]

(S.2) If \( \beta^{j+1} \) satisfies a prescribed condition, stop; otherwise, update \( \gamma_{1,j} \) and \( \gamma_{2,j} \) by

\[
\gamma_{1,j+1} = \max(\gamma, \varrho \gamma_{1,j}) \quad \text{and} \quad \gamma_{2,j+1} = \max(\gamma, \varrho \gamma_{2,j}).
\]

(S.3) Set \( j \leftarrow j + 1 \), and return to Step (S.1).

end while

**Remark 5.1** (i) Since \( f_r(y - X\beta) \) and \( h_k(\beta) \) are convex but nondifferentiable, we follow the same line as in [35] to introduce a proximal term \( \gamma_{1,j} \|\beta - \beta^{j}\|^2 \) except the common \( \gamma_{2,j} \|X\beta - X\beta^{j}\|^2 \). As will be shown later, this lends a leverage to handle \( f_r(y - X\beta) \).

(ii) Since \( \beta^{j+1} \) is an approximate minimizer of the convex program (23), by its first-order optimality condition there exists an error vector \( \zeta^j \in \mathbb{R}^p \) such that

\[
\zeta^j \in \partial \left[ f_r(y - X\beta) + h_k(\beta) \right]_{\beta = \beta^{j+1}} + \gamma_{1,j} (\beta^{j+1} - \beta^{j}) + \gamma_{2,j} X^T X (\beta^{j+1} - \beta^{j}),
\]

and then the approximate optimality of the iterate \( \beta^{j+1} \) to (22) can be measured by

\[
\text{Err}_{PPA}^{j+1} := \frac{\|\gamma_{1,j} (\beta^{j+1} - \beta^{j}) + \gamma_{2,j} X^T X (\beta^{j+1} - \beta^{j}) - \zeta^j\|}{1 + \|y\|}.
\]

Motivated by this, for the subsequent testing, we terminate Algorithm 2 at \( \beta^j \) whenever

\[
j > j_{\max} \quad \text{or} \quad \text{Err}_{PPA}^j < \epsilon_{PPA} \quad \text{or} \quad \max\{\text{Err}_{PPA}^j, \ldots, \text{Err}_{PPA}^{j-10}\} < \tilde{\epsilon}_{PPA} \quad \text{for} \quad j > 10.
\]

The efficiency of Algorithm 2 depends on the solution of its subproblem (23), which by introducing an additional variable \( z \in \mathbb{R}^n \) can be equivalently written as

\[
\min_{\beta \in \mathbb{R}^p, z \in \mathbb{R}^n} \left\{ f_r(z) + h_k(\beta) + \gamma_{1,j} \|\beta - \beta^{j}\|^2 + \gamma_{2,j} \|z - z^{j}\|^2 \right\}
\]

s.t. \( X\beta + z - y = 0 \) with \( z^{j} = y - X\beta^{j} \). \hspace{1cm} (24)
After an elementary calculation, the dual of the problem (24) has the following form

$$\min_{u \in \mathbb{R}^n} \left\{ \Psi_{k,j}(u) := \frac{\|u\|^2}{2\gamma_{2,j}} - e^{-1}_{\cdot2,j} f_\tau \left( z^j - \frac{u}{\gamma_{2,j}} \right) - e_{\cdot1,j} h_k \left( \beta^j - \frac{X^T u}{\gamma_{1,j}} \right) + \frac{\|X^T u\|^2}{2\gamma_{1,j}} \right\}. \tag{25}$$

By Section 2.2 we know that $\Psi_{k,j}$ is a continuously differentiable convex function in $\mathbb{R}^n$. Hence, seeking an optimal solution of (25) is equivalent to finding a root to the system

$$\Phi_{k,j}(u) := -\mathcal{P}_{\gamma_{2,j}} f_\tau \left( z^j - \frac{u}{\gamma_{2,j}} \right) - \mathcal{P}_{\gamma_{1,j}} h_k \left( \beta^j - \frac{X^T u}{\gamma_{1,j}} \right) + y = 0. \tag{26}$$

Since $\mathcal{P}_{\gamma_{-1}} f_\tau$ and $\mathcal{P}_{\gamma_{-1}} h_k$ are strongly semismooth by Section 2.3 and the composition of strongly semismooth mappings is strongly semismooth by [9, Proposition 7.4.4], the mapping $\Phi_{k,j}$ is strongly semismooth. Motivated by this, we use the semismooth Newton method to solve the system (26). By [8, Proposition 2.3.3 & Theorem 2.6.6],

$$\partial \Phi_{k,j}(u) \subseteq \gamma_{2,j}^{-1} \partial \mathcal{P}_{\gamma_{2,j}} f_\tau \left( z^j - \frac{u}{\gamma_{2,j}} \right) + \gamma_{1,j}^{-1} X \partial \mathcal{P}_{\gamma_{1,j}} h_k \left( \beta^j - \frac{X^T u}{\gamma_{1,j}} \right) X^T \quad \forall u \in \mathbb{R}^m \tag{27}$$

where the equality is by Lemma 2.1 and 2.2, and $\mathcal{U}_j(u)$ and $\mathcal{V}_j(u)$ are the sets defined by

$$\mathcal{U}_j(u) := \left\{ \text{Diag}(v_1, \ldots, v_n) \mid v_i \in \partial \mathcal{P}_{\gamma_{2,j}}(n^{-1} \theta_x) \left( z^j_i - \frac{1}{\gamma_{2,j}} u_i \right) \right\},$$

$$\mathcal{V}_j(u) := \left\{ \text{Diag}(v_1, \ldots, v_n) \mid v_i = 1 \text{ if } |(\gamma_{1,j} \beta^j - X^T u)_i| > \omega_i^k, \text{ otherwise } v_i \in [0, 1] \right\}.$$  

For each $U^j \in \mathcal{U}_j(u)$ and $V^j \in \mathcal{V}_j(u)$, the matrix $\gamma_{2,j}^{-1} U^j + \gamma_{1,j}^{-1} X V^j X^T$ is positive semidefinite, and moreover, it is positive definite when $\{ i \mid \frac{1}{\gamma_{2,j}} z^j_i - \frac{1}{\gamma_{2,j}} u_i < \frac{1}{\gamma_{2,j}} \} = \emptyset$ or the matrix $X_j$ has full row rank with $J := \{ i \mid |(\gamma_{1,j} \beta^j - X^T u)_i| > \omega_i^k \}$. In view of this, we apply the following semismooth Newton method to seeking a root of the system (26), which by [28] is expected to have a superlinear even quadratic convergence rate.
Algorithm 3  A semismooth Newton-CG algorithm

Initialization: Fix k and j. Choose \( \mu, \eta, \delta \in (0, 1) \), \( c \in (0, \frac{1}{2}) \) and \( u^0 = 0 \). Set \( l = 0 \).

while the stopping conditions are not satisfied do

1. Choose \( U^l \in \mathcal{U}_j(u^l) \) and \( V^l \in \mathcal{V}_j(u^l) \) and set \( W^l = \gamma_{2,j}^{-1}U^l + \gamma_{1,j}^{-1}XV^lX^T \). Solve

\[
(W^l + \mu I)d = -\Phi_{k,j}(u^l)
\]

with the conjugate gradient (CG) algorithm to find an approximate \( d^l \) such that

\[
\| (W^l + \mu I)d^l \| \leq \min(\mu, \| \Phi_{k,j}(u^l) \|^{1+\epsilon}), \text{ where } \mu_l = \min(\mu, \| \Phi_{k,j}(u^l) \|).
\]

2. Set \( \alpha_l = \delta^m \), where \( m_l \) is the first nonnegative integer \( m \) satisfying

\[
\Psi_{k,j}(u^l + \delta^m d^l) \leq \Psi_{k,j}(u^l) + c\delta^m \langle \nabla \Phi_{k,j}(u^l), d^l \rangle.
\]

3. Set \( u^{l+1} = u^l + \alpha_l d^l \) and \( l \leftarrow l + 1 \), and then go to Step 1.

end while

Remark 5.2 Fix \( j \in \mathbb{N} \). Let \( u^* \) be a root to the system (26). Set \( \overline{\beta} = \mathcal{P}_{\gamma_{1,j}}^{-1}h_k(\beta - \frac{X^Tu^*}{\gamma_{1,j}}) \) and \( z^* = \mathcal{P}_{\gamma_{2,j}}^{-1}f_r(z^* - \frac{\mu}{\gamma_{2,j}}) \). Clearly, \( X\overline{\beta}^* + z^* - y = 0 \). Also, one may calculate that

\[
f_r(z^*) + h_k(\overline{\beta}^*) + \frac{\gamma_{1,j}}{2}\| \overline{\beta}^* - \beta^* \|^2 + \frac{\gamma_{2,j}}{2}\| z^* - z^* \|^2 + \Psi_{k,j}(u^*) = \langle z^* - z^*, u^* \rangle + \langle \beta^* - \overline{\beta}^*, X^Tu^* \rangle.
\]

That is, \( \overline{\beta}^* \) is a feasible solution to (24) and the gap between its objective value and the dual optimal value is \( \langle z^* - z^*, u^* \rangle + \langle \beta^* - \overline{\beta}^*, X^Tu^* \rangle \). So, the following stopping criterion

\[
\frac{\| \Phi_{k,j}(u^l) \|}{1 + \| y \|} \leq \epsilon_{\text{SNCG}} \quad \text{and} \quad \frac{|\langle z^l - z^l, u^l \rangle + \langle \beta^l - \beta^l, X^Tu^l \rangle|}{1 + \| y \|} \leq \epsilon_{\text{SNCG}}
\]

is suggested for Algorithm 3, where \( z^l = \mathcal{P}_{\gamma_{2,j}}^{-1}f_r(z^l - \frac{\mu}{\gamma_{2,j}}) \) and \( \overline{\beta}^* = \mathcal{P}_{\gamma_{1,j}}^{-1}h_k(\beta - \frac{X^Tu^l}{\gamma_{1,j}}) \).

6 Numerical experiments

In the last section we propose the PDSN method for solving the subproblem (12) or the problem (22). Recently, Gu et al. [18] developed a semi-proximal ADMM (sPADMM) for solving this problem. In Appendix C, we describe the iterate steps of the sPADMM, which has a little difference from the one owing to Gu et al. [18] in the semi-proximal operator and the stopping criterion. In addition, as illustrated in [38, Section 4.1], by introducing 2\( n + p \) additional variables, the subproblem (12) can be recast as the LP

\[
\begin{align*}
\min_{\beta^+, \beta^-, \xi^+, \xi^-} & \; \langle \omega^k, \beta^+ \rangle + \langle \omega^k, \beta^- \rangle + \frac{\tau}{n} (\xi^+, e) + \frac{1 - \tau}{n} (\xi^-, e) \\
\text{s.t.} & \; X\beta^+ - X\beta^- - \xi^+ + \xi^- = b, \\
& \; \beta^+ \geq 0, \beta^- \geq 0, \xi^+ \geq 0, \xi^- \geq 0
\end{align*}
\]

(29)
so that the state-of-art IPM software can be directly applied to it. We always choose the state-of-art IPM software to solve the LP for the subsequent numerical tests. In this section, we shall test the performance of Algorithm 1 with the subproblems solved by PDSN, SeDuMi and sPADMM, respectively, on some synthetic and real data, and call the three solvers MSCRA_PPA, MSCRA_IPM and MSCRA_ADMM, respectively. All numerical results are computed by a laptop computer running on 64-bit Windows Operating System with an Intel(R) Core(TM) i7-8565 CPU 1.8GHz and 8 GB RAM.

6.1 Implementation of three solvers

We first focus on the implementation of SeDuMi, sPADMM and PDSN. For SeDuMi, we adopt the default setting, and for sPADMM we choose the initial penalty parameter \( \sigma = 1 \) and the step-size \( \rho = 0.95 \), and adopt the stopping criterion given in Appendix D with \( j_{\max} = 3000 \) and \( \epsilon_{\text{PADMM}} = 10^{-5} \). For PDSN, unless otherwise stated, we choose \( \gamma_{1,0} = 10 \), \( \gamma_{2,0} = \frac{1}{\max(1,\|y\|)} \), \( \gamma = 10^{-8} \) and \( \rho = 0.95 \), and adopt the stopping criterion described as in Remark 5.1 with \( j_{\max} = 10^{3} \), \( \epsilon_{\text{PPA}}^j = \max\left(10^{-8}, \frac{10^{-3} \times 0.5^{j-1}}{\max(1,\|y\|)}\right) \) and \( \tilde{\epsilon}_{\text{PPA}} = 10^{-8} \), where Algorithm 3 uses the rule (28) with \( \epsilon_{\text{SNCG}} = \max(\epsilon_{\text{PPA}}, 0.5^{l-1}\min(0.1, \text{Err}_{\text{PPA}})) \).

For MSCRA_IPM, MSCRA_ADMM and MSCRA_PPA, we choose \( a = 6.0 \) for \( \phi \) and \( w_0 = 0 \), and terminate them at the iterate \( \beta_k \) whenever \( k > 5 \) or \( \text{Err}_k \leq 10^{-10} \) or \( \text{Err}_k \leq 10^{-4} \) and \( |N_{\text{nz}}(\beta_k-j) - N_{\text{nz}}(\beta_k-j-1)| \leq 3 \) for \( j = 0, 1, 2 \),

where \( \text{Err}_k \) is the measure error defined by (17) for \( \beta_k \) to be a critical point of \( \Theta_{\lambda,\rho_k} \), and \( N_{\text{nz}}(\beta_k) := \sum_{i=1}^p \mathbb{I}\{||\beta_k^i|| > 10^{-6} \max(1,\|\beta_k^i\|_{\infty})\} \) denotes the number of nonzero entries of \( \beta_k \). For the three solvers, we update the penalty parameter \( \rho_k \) by the following rule:

\[
\rho_1 = \max\left(1, \frac{1}{3\|\beta^1\|_{\infty}}\right) \quad \text{and} \quad \rho_k = \min\left(\frac{5}{4\rho_{k-1}}, \frac{10^8}{\|\beta^k\|_{\infty}}\right) \quad \text{for} \quad k = 2, 3.
\]

In addition, during the implementation of three solvers, we run SeDuMi, sPADMM and PDSN to solve the \( k \) (\( k > 1 \))th subproblem with the optimal solution of the \((k-1)\)th subproblem yielded by them as the starting point. When \( k = 1 \), we choose \( \beta^0 = 0 \) to be the starting point of MSCRA_IPM and MSCRA_ADMM, and use an approximate solution of the following problem as the starting point for MSCRA_PPA:

\[
\min_{\beta \in \mathbb{R}^p} \left\{ f_r(y - X\beta) + h_k(\beta) + \frac{\gamma_{1,0}}{2}\|\beta\|^2 + \frac{\gamma_{2,0}}{2}\|X\beta - y\|^2 \right\}.
\]  

(30)

Clearly, one may apply Algorithm 3 directly to this problem and obtain such a solution.
6.2 Time comparisons of the subproblem solvers

In this part we conduct time comparison for SeDuMi, sPADMM and PDSN for solving the problem (12) with \( k = 1 \), i.e., the \( \ell_1 \)-regularized check loss minimization problem. Inspired by the work owing to Gu et al. [18], we consider the simulation model from [15] to generate data, i.e., we obtain the \( n \) observations from the following linear model

\[
y_i = x_i^T \beta^* + \kappa \varepsilon_i \quad \text{for} \quad i = 1, \ldots, n
\]

where \( x_i^T \sim N(0, \Sigma) \) for \( i = 1, \ldots, n \) with \( \Sigma = (\alpha+(1-\alpha)I_{i(i=j)})_{p \times p}, \beta^*_j = (-1)^j \exp(-\frac{2j-1}{20}) \), \( \varepsilon \sim N(0, \Sigma) \) and \( \kappa \) is chosen such that the signal-noise ratio of the data is 3.0. We focus on the high-dimensional situation where \( p = 5000 \) and \( n = 100 \) and \( n = 500 \), respectively, with the choice \( \alpha = 0 \) and \( \alpha = 0.95 \). Figure 1 and 2 below plot the computing time (in seconds) of three solvers spent on the solution of the problem (12) with \( k = 1 \) over the same sequence of 50 values of \( \lambda \). By the theoretical results in Section 4.1, the choice of \( \lambda \) depends on \( ||X||_1/n \). Motivated by this, we select the 50 values of \( \lambda \) by the formula

\[
\lambda_i = \max \left( 0.01, \frac{\gamma_i \|X\|_1}{n} \right) \quad \text{with} \quad \gamma_i = \gamma_{\min} + \frac{(\gamma_{\max} - \gamma_{\min})(i-1)}{49}
\]

for \( i = 1, 2, \ldots, 50 \), where \( \gamma_{\min} = 0.02 \) and \( \gamma_{\max} = 0.25 \) for \( \alpha = 0 \) and \( \gamma_{\max} = 0.38 \) for \( \alpha = 0.95 \), respectively. Such \( \gamma_{\max} \) is such that \( N_{\text{nz}}(\beta^f) \) attains the value 0, where \( \beta^f \) represents the final output of the solvers. The parameters involved in SeDuMi, sPADMM and PDSN are set as described in Section 6.1 except \( \gamma_{1,0} = 1, \gamma_{2,0} = 0.01 \) for PDSN.

![Time comparisons of three solvers for the sample size n = 100](image)

Figure 1: Time comparisons of three solvers for the sample size \( n = 100 \)
Fig. 2: Time comparisons of three solvers for the sample size $n = 500$

The two columns of Fig. 1 indicate that the three solvers have the similar time performance for $\tau = 0.5$ and $\tau = 0.75$, respectively, except that the three solvers require more time when $\tau = 0.75$. Among others, PDSN requires the least time, while SeDuMi needs the most time. Since, when $\lambda$ is over some value, the output $\beta^f$ of three solvers becomes the zero vector, PDSN and sPADMM almost do not need time when $\lambda > 0.23$, but SeDuMi still needs some time. Comparing the second row of Fig. 1 with the first row, we find that for a larger $\alpha$, PDSN and sPADMM do not spend more time, while SeDuMi needs more time, and hence the three solvers have the similar time performance for $\alpha = 0.95$ and $\alpha = 0.95$, respectively. Figure 2 shows that when $n = 500$, SeDuMi and sPADMM have the similar performance for $\alpha = 0$ and $\alpha = 0.95$ except that they require a little more time for $\alpha = 0.95$, but PDSN has a different performance for $\alpha = 0$ and $\alpha = 0.95$. When $\alpha = 0$, the computing time of PDSN has a notable decreasing as $\lambda$ increases and still requires the least computing time among the three solvers except several very small $\lambda$, and SeDuMi requires comparable time as sPADMM does since now the latter almost always attains the maximum iterate steps $j_{\text{max}} = 3000$. We observe that for $n = 500$ the time gap between PDSN and sPADMM becomes larger. Although PDSN requires less time, Fig. 3 shows that the objective value of $\beta^f$ yielded by PDSN is close to even better than that of the output given by SeDuMi and sPADMM.

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Figure 3: The objective values for the outputs of the three solvers with \( n = 100 \)

6.3 Performance and time comparisons of three MSCRAs

In this part we investigate the performance of the solvers MSCRA_IPM, MSCRA_ADMM and MSCRA_PPA for computing the estimator \( \hat{\beta} \) under the setup as in Fan et al. [12] and Gu et al. [18]. Specifically, we get the \( n \) observations from the model (2) with the true coefficient vector \( \beta^* \) fixed to be \((2, 0, 1.5, 0, 0.8, 0, 0, 1, 0.175, 0, 0, 0.75, 0, 0, 0.3, 0^T_{p-16})^T\), and the noise vector \( \varepsilon \) coming from those distributions considered by Gu et al. [18], which include the following six cases: (1) the normal distribution \( N(0, 2) \); (2) the mixture normal distribution \( 0.9N(0, 1) + 0.1N(0, 25) \), denoted by \( MN_1 \); (3) the mixture normal distribution \( N(0, \sigma^2) \) with \( \sigma \sim \text{Unif}(1, 5) \), denoted by \( MN_2 \); (4) the Laplace distribution with density \( d(u) = 0.5 \exp(-|u|) \); (5) the scaled Student’s \( t \)-distribution with 4 degrees of freedom \( \sqrt{2} \times t_4 \); and (6) the Cauchy distribution with density \( d(u) = \frac{1}{\pi(1+u^2)} \). For the covariance matrix \( \Sigma_x \), we also consider those scenarios from Gu et al. [18], including the independence structure \( \Sigma_x = I \); the autoregressive structures \( \Sigma_x = (0.5^{|i-j|})_{ij} \) and \( \Sigma_x = (0.8^{|i-j|})_{ij} \), denoted by \( \text{AR}_{0.5} \) and \( \text{AR}_{0.8} \); and the compound symmetric structures \( \Sigma_x = (\alpha + (1 - \alpha)I_{(i=j)}) \) with \( \alpha = 0.5 \) and \( \alpha = 0.8 \), denoted by \( \text{CS}_{0.5} \) and \( \text{CS}_{0.8} \). We test the estimation and selection performance of the estimators computed with the three solvers under each scenario in terms of the \( L_2 \)-error \( \|\hat{\beta} - \beta^*\| \), the number of false positives and false negatives, denoted by \( \text{FP} \) and \( \text{FN} \) respectively, and the time.

As mentioned by Fan, Fan and Barut [12], the cross-validation is not suitable for choosing the best tuning parameter \( \lambda \) due to the instability of the \( L_2 \)-error under heavy tails. We choose the best \( \lambda \) by the formula (32) by seeking optimally the constant \( \gamma \). Inspired by the choice strategy of \( \lambda \) proposed in [12], we choose optimally the constant \( \gamma \) based on 100 validation data-sets. Specifically, for each of these data-sets, we ran a grid
Table 1: Estimation and selection performance of three solvers for $\Sigma_x = I$

| $\varepsilon$ | Method | $\gamma_{opt}$ | $L_2$-error | FP | FN (s) | $\gamma_{opt}$ | $L_2$-error | FP | FN (s) |
|--------------|--------|---------------|--------------|----|--------|---------------|--------------|----|--------|
| 0.110        | IPM    | 0.523(0.150)  | 7.480(2.660) | 0.670(0.495) | 1.684 |
| 0.110        | ADMM   | 0.523(0.150)  | 7.480(2.660) | 0.670(0.495) | 1.684 |
| 0.110        | PPA    | 0.523(0.150)  | 7.480(2.660) | 0.670(0.495) | 1.684 |

Table 2: Estimation and selection performance of three solvers for AR0.5

| $\varepsilon$ | Method | $\gamma_{opt}$ | $L_2$-error | FP | FN (s) | $\gamma_{opt}$ | $L_2$-error | FP | FN (s) |
|--------------|--------|---------------|--------------|----|--------|---------------|--------------|----|--------|
| 0.110        | IPM    | 0.523(0.150)  | 7.480(2.660) | 0.670(0.495) | 1.684 |
| 0.110        | ADMM   | 0.523(0.150)  | 7.480(2.660) | 0.670(0.495) | 1.684 |
| 0.110        | PPA    | 0.523(0.150)  | 7.480(2.660) | 0.670(0.495) | 1.684 |

search to find the best $\gamma$ and consequently the best $\lambda$ (with the lowest $L_2$-error of $\beta$) for the particular setting. This optimal $\gamma$ was recorded for each of the 100 validation datasets. The median of these 100 optimal $\gamma$, denoted by $\gamma_{opt}$, was used for the simulation studies, i.e., we choose the tuning parameter $\lambda$ by the formula $\lambda = \max(0.01, \frac{\gamma_{opt}}{\tau(\text{min})})$ for the simulation studies. Among others, the best $\gamma$ is searched from $\gamma_1, \ldots, \gamma_{51}$ defined as in (32) for $\gamma_{\text{min}} = 0.08$ and $\gamma_{\text{max}} = 0.38$. Such $\gamma_{\text{max}}$ is such that $N_{\text{opt}}(\beta^I)$ attains or is close to 0 except for $\Sigma_x = CS_{0.8}$ and $\varepsilon$ from the Cauchy distribution.

Table 1-5 report the average of the performance measures $L_2$-error, FP and FN for $\tau = 0.5$ and 0.75 based on 100 simulations. For almost all test problems, MSCRA_PPA requires less than half (respectively, one-third) of the time required by MSCRA_ADMM (respectively, MSCRA_IPM), and the $L_2$-error of MSCRA_PPA is comparable with that of MSCRA_ADMM and MSCRA_IPM. In addition, for most of test problems except for CS0.8, the FP of MSCRA_PPA are lower than that of MSCRA_ADMM and MSCRA_IPM although its FN is a little higher than that of the latter two methods.
| Method | $\gamma_t$ | $L_2$-error | $LL$ | Time(s) | $\gamma_t$ | $L_2$-error | $LL$ | Time(s) |
|--------|-----------|-------------|------|--------|-----------|-------------|------|--------|
| IPM    | 0.106     | 0.987(0.48) | 4.36(2.57) | 1.78(0.83) | 0.106     | 0.987(0.48) | 4.36(2.57) | 1.78(0.83) |

\[ \sqrt{N} \times 14 \]

| Method | $\gamma_t$ | $L_2$-error | $LL$ | Time(s) | $\gamma_t$ | $L_2$-error | $LL$ | Time(s) |
|--------|-----------|-------------|------|--------|-----------|-------------|------|--------|
| IPM    | 0.106     | 0.987(0.48) | 4.36(2.57) | 1.78(0.83) | 0.106     | 0.987(0.48) | 4.36(2.57) | 1.78(0.83) |

\[ \sqrt{N} \times 14 \]

| Method | $\gamma_t$ | $L_2$-error | $LL$ | Time(s) | $\gamma_t$ | $L_2$-error | $LL$ | Time(s) |
|--------|-----------|-------------|------|--------|-----------|-------------|------|--------|
| IPM    | 0.106     | 0.987(0.48) | 4.36(2.57) | 1.78(0.83) | 0.106     | 0.987(0.48) | 4.36(2.57) | 1.78(0.83) |

\[ \sqrt{N} \times 14 \]

| Method | $\gamma_t$ | $L_2$-error | $LL$ | Time(s) | $\gamma_t$ | $L_2$-error | $LL$ | Time(s) |
|--------|-----------|-------------|------|--------|-----------|-------------|------|--------|
| IPM    | 0.106     | 0.987(0.48) | 4.36(2.57) | 1.78(0.83) | 0.106     | 0.987(0.48) | 4.36(2.57) | 1.78(0.83) |

\[ \sqrt{N} \times 14 \]

| Method | $\gamma_t$ | $L_2$-error | $LL$ | Time(s) | $\gamma_t$ | $L_2$-error | $LL$ | Time(s) |
|--------|-----------|-------------|------|--------|-----------|-------------|------|--------|
| IPM    | 0.106     | 0.987(0.48) | 4.36(2.57) | 1.78(0.83) | 0.106     | 0.987(0.48) | 4.36(2.57) | 1.78(0.83) |

\[ \sqrt{N} \times 14 \]

| Method | $\gamma_t$ | $L_2$-error | $LL$ | Time(s) | $\gamma_t$ | $L_2$-error | $LL$ | Time(s) |
|--------|-----------|-------------|------|--------|-----------|-------------|------|--------|
| IPM    | 0.106     | 0.987(0.48) | 4.36(2.57) | 1.78(0.83) | 0.106     | 0.987(0.48) | 4.36(2.57) | 1.78(0.83) |

\[ \sqrt{N} \times 14 \]

| Method | $\gamma_t$ | $L_2$-error | $LL$ | Time(s) | $\gamma_t$ | $L_2$-error | $LL$ | Time(s) |
|--------|-----------|-------------|------|--------|-----------|-------------|------|--------|
| IPM    | 0.106     | 0.987(0.48) | 4.36(2.57) | 1.78(0.83) | 0.106     | 0.987(0.48) | 4.36(2.57) | 1.78(0.83) |
6.4 Performance on a real data example

Now we focus on a real data set from https://www.ncbi.nlm.nih.gov, which is used by Scheetz et al. [33] to illustrate the gene regulation in mammalian eyes and to gain insight into genetic variation related to human eyes. This microarray data comprises gene expression levels of 31,042 probes on 120 twelve-week-old laboratory rats. For the 31,042 probes, as suggested by Scheetz et al. [33], we first carry out the following preprocessing steps:

1. to remove each probe for which the maximum expression among the 120 rats is less than the 25th percentile of the entire expression values;
2. to remove any probe for which the range of the expression among the 120 rats is less than 2.

After the preprocessing steps, there are 18,986 probes left. Among those probes, there is one probe, 1389163_at, corresponding to gene TRIM32, that was found to be associated with the Bardet-Biedl syndrome [7], a human genetic disorder that affects many parts of the body and primarily the retina. We are interested in how the expression of this gene depends on the expressions of all other 18,985 genes. To achieve this goal, we select 3,000 probes with the largest variances and then standardize the selected 3,000 probes such that they have mean 0 and standard deviation 1, as Gu and Zou [17] and Wang [40] did. Thus, we obtain an $n \times p$ sample matrix $X'$ with $n = 120$ and $p = 3000$, and then use $X = [e \cdot X'] \in \mathbb{R}^{n \times (p+1)}$ to test the performance of the solvers.

We first analyze the data on all 120 rats by MSCRA_PP A and MSCRA_ADMM with quantile indices $\tau = 0.25, 0.5$ and 0.75. Since the numerical results in the previous two subsections show that MSCRA_IPM and MSCRA_ADMM have very similar estimation performance, here we only use MSCRA_PP A and MSCRA_ADMM to analyze the real data. The parameter $\lambda$ is given by the formula $\lambda = \max (0.01, \frac{\gamma \|X\|_1}{n})$ with the parameter $\gamma$ selected by using five-fold cross-validation. The test results are reported on the third and fourth columns of Table 6, where the third column is the number of relevant genes and the fourth one is the computing time. The difference in the number of selected genes by different quantile indices is a sign of heteroscedasticity in the data, as explained in Wang et al. [40]. Table 7 lists the probs selected by the two solvers with different $\tau$. We see that for $\tau = 0.25$ and $\tau = 0.75$, the probs selected by MSCRA_ADMM and MSCRA_PP A are completely different, while for $\tau = 0.5$ there are 26 common probs.

We also conduct 50 random partitions on the data. Each partition has 80 rats in the training set and 40 rats in the validation set. We apply MSCRA_ADMM and MSCRA_PP A to the training set with $\lambda$ chosen as above and evaluate its prediction error on the validation set by calculating $\frac{1}{40} \sum_{i \in \text{validation}} \theta_\tau(y_i - \beta_0 - x_i^T \hat{\beta})$, where $x_i^T$ means the $i$th row of $X'$. The average number of selected genes, prediction errors and times over the 50 partitions are reported in the last three columns of Table 6. We see that the average number of the genes selected by MSCRA_PP A is less than that of the genes selected by MSCRA_ADMM, the average prediction error of the former is lower than that of the latter, and the average time of the former is about half of the latter.
| Method | \( \tau \) | All data # genes | Time(s) | Random partition # genes | Pre_error | Time(s) |
|--------|------------|-----------------|--------|--------------------------|-----------|--------|
| ADMM   | 0.25       | 18              | 2.726  | 17.180                   | 0.050     | 2.399  |
|        |            |                 |        | (1.987)                  | (0.009)   | (0.172) |
|        | 0.5        | 27              | 1.965  | 21.180                   | 0.029     | 1.996  |
|        |            |                 |        | (4.429)                  | (0.005)   | (0.336) |
|        | 0.75       | 18              | 3.025  | 21.220                   | 0.040     | 1.713  |
|        |            |                 |        | (2.393)                  | (0.005)   | (0.230) |
| PPA    | 0.25       | 19              | 0.738  | 16.380                   | 0.023     | 0.749  |
|        |            |                 |        | (3.901)                  | (0.006)   | (0.120) |
|        | 0.5        | 24              | 0.741  | 19.700                   | 0.029     | 0.730  |
|        |            |                 |        | (4.287)                  | (0.005)   | (0.083) |
|        | 0.75       | 17              | 1.023  | 11.720                   | 0.025     | 0.859  |
|        |            |                 |        | (2.907)                  | (0.004)   | (0.171) |

Table 7: Probs selected by MSCRA_PPA and MSCRA_ADMM with different \( \tau \)

| \( \tau = 0.25 \) | ADMM | PPA | \( \tau = 0.5 \) | ADMM | PPA | \( \tau = 0.75 \) | ADMM | PPA |
|-------------------|------|-----|-----------------|------|-----|-----------------|------|-----|
| 1390238_at        | 1387060_at | 1394689_at | 1390238_at     | 1387060_at | 1394689_at | 1387060_at | 1394689_at |
| 1398594_at        | 1390070_at | 1396034_at | 1398594_at     | 1390070_at | 1396034_at | 1390070_at | 1396034_at |
| 1368304_at        | 1393811_at | 1395727_at | 1368304_at     | 1393811_at | 1395727_at | 1393811_at | 1395727_at |
| 1378861_at        | 1397489_at | 1397572_at | 1378861_at     | 1397489_at | 1397572_at | 1397489_at | 1397572_at |
| 1385325_at        | 1398736_at | 1379750_at | 1385325_at     | 1398736_at | 1379750_at | 1398736_at | 1379750_at |
| 1387776_at        | 1376693_at | 1368853_at | 1387776_at     | 1376693_at | 1368853_at | 1376693_at | 1368853_at |
| 1383110_at        | 1370429_at | 1397489_at | 1383110_at     | 1370429_at | 1397489_at | 1370429_at | 1397489_at |
| 1382263_at        | 1380033_at | 1390409_at | 1382263_at     | 1380033_at | 1390409_at | 1380033_at | 1390409_at |
| 1374469_at        | 1389584_at | 1397489_at | 1374469_at     | 1389584_at | 1397489_at | 1389584_at | 1397489_at |
| 1383292_at        | 1395076_at | 1397489_at | 1383292_at     | 1395076_at | 1397489_at | 1383292_at | 1395076_at |
| 1379971_at        | 1391039_at | 1395076_at | 1379971_at     | 1391039_at | 1395076_at | 1389971_at | 1389971_at |
| 1383901_at        | 1377944_at | 1393499_at | 1383901_at     | 1377944_at | 1393499_at | 1383901_at | 1377944_at |
| 1394010_at        | 1355687_at | 1395076_at | 1394010_at     | 1355687_at | 1395076_at | 1389971_at | 1389971_at |
| 1387247_at        | 1384466_at | 1375566_at | 1387247_at     | 1384466_at | 1375566_at | 1383901_at | 1383901_at |
| 1384466_at        | 1389978_at | 1385687_at | 1384466_at     | 1389978_at | 1385687_at | 1383901_at | 1383901_at |
| 1382743_at        | 1396569_at | 1384466_at | 1382743_at     | 1396569_at | 1384466_at | 1383901_at | 1383901_at |
| 1393543_at        | 1373699_at | 1390401_at | 1393543_at     | 1373699_at | 1390401_at | 1383901_at | 1383901_at |
| 1379597_at        | 1378935_at | 1397194_at | 1379597_at     | 1378935_at | 1397194_at | 1383901_at | 1383901_at |
| 1383996_at        | 1374106_at | 1382835_at | 1383996_at     | 1374106_at | 1382835_at | 1383901_at | 1383901_at |

7 Conclusions

We have proposed a multi-stage convex relaxation approach, MSCRA_PPA, for computing a desirable approximation to the zero-norm penalized QR, which is defined as a global minimizer of an NP-hard nonsmooth optimization. Under the common RSC condition and a mild restriction on the noises, we established the error bound of every iterate to the true estimator and the linear rate of convergence of the iterate sequence in a statistical sense. Numerical comparisons with MSCRA_IPM and MSCRA_ADMM
show that MSCRA_PP A achieves a comparable estimation performance within less time.

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Appendix A

The following lemma states that under a mild condition, the zero-norm regularized composite minimization problem has a nonempty global optimal solution set.

**Lemma 1** Let $A \in \mathbb{R}^{n \times p}$ and $b \in \mathbb{R}^n$ be the given matrix and vector, and let $g : \mathbb{R}^n \to \mathbb{R}$ be an lsc coercive function with $\inf_{z \in \mathbb{R}^n} g(z) > -\infty$. Then, for any given $\nu > 0$, the following problem has a nonempty global optimal solution set:

$$\min_{x \in \mathbb{R}^p} \left\{ \nu g(b - Ax) + \|x\|_0 \right\}. \quad (33)$$

**Proof:** By the given assumption, the objective function of (33) is lower bounded, and hence has an infimum, to say $\alpha^*$. Then, there exists a sequence $\{x^k\} \subset \mathbb{R}^p$ such that

$$\nu g(b - Ax^k) + \|x^k\|_0 \leq \alpha^* + 1/k \quad \forall k. \quad (34)$$

If the sequence $\{x^k\}$ is bounded, then by letting $x$ be an arbitrary limit point of $\{x^k\}$ and using the lower semicontinuity of $x \mapsto g(b - Ax)$ and $\|\cdot\|_0$, it follows that

$$\nu g(b - Ax) + \|x\|_0 \leq \alpha^*.$$ 

This shows that $x$ is a global optimal solution of the problem (33). Next we consider the case that the sequence $\{x^k\}$ is unbounded. Define the disjoint index sets $J$ and $\bar{J}$ by

$$J := \{ i \in \{1, \ldots, p\} \mid \{x^k_i\} \text{ is unbounded} \} \quad \text{and} \quad \bar{J} := \{1, \ldots, p\} \setminus J.$$ 

Together with inequality (34), it immediately follows that for all sufficiently large $k$,

$$\nu g(b - Ax^k) + |J| + \|x^k_J\|_0 \leq \alpha^* + \frac{1}{k} \quad (35)$$

This, along with the coerciveness of $g$, means that there is a bounded sequence $\{z^k\} \subset \mathbb{R}^n$ such that $z^k = b - Ax^k$. Clearly, $A_J x^k_j = b - z^k - A_J x^k_{\bar{J}}$. Notice that $\{z^k\}$ and $\{x^k_J\}$ are bounded. We may assume (taking a subsequence if necessary) that $\{z^k\}$ and $\{x^k_J\}$ are convergent, say, $z^k \to z^*$ and $x^k_J \to \xi^* \in \mathbb{R}^{\bar{J}}$. Notice that for each $k$, $x^k_j$ is a solution of the system $A_J y = b - z^k - A_J x^k_{\bar{J}}$, that is, $\{b - z^k - A_J x^k_{\bar{J}}\} \subset A_J(\mathbb{R}^{\bar{J}})$. Together with the closedness of the set $A_J(\mathbb{R}^{\bar{J}})$, it follows that $b - z^* - A_J \xi^* \in A_J(\mathbb{R}^{\bar{J}})$. Hence, there

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exists \( u^* \in \mathbb{R}^{|J|} \) such that \( A_j u^* = b - z^* - A_T \xi^* \), i.e., \( A_j u^* + A_T \xi^* = z^* \). Now, taking the limit to the both sides of (35) and using \( b - Ax^k = z^k \), we obtain

\[
\nu g(z^*) + |J| + \|\xi^*\|_0 \leq \alpha^*.
\]

Together with \( \nu g(b - A_j u^* - A_T \xi^*) + \|u^*\|_0 + \|\xi^*\|_0 \leq \nu g(z^*) + |J| + \|\xi^*\|_0 \), we conclude that \( (u^*; \xi^*) \) is a global optimal solution of (33). Thus, we complete the proof. \( \square \)

**Proof of Proposition 3.1** 

(i) Let \( \varphi'_\rho(t) = \psi^*(\rho|t|) \) for \( t \in \mathbb{R} \). Together with (10),

\[
\varphi'_\rho(t) = \begin{cases} 
0 & \text{if } |t| \leq \frac{2}{\rho(a+1)}; \\
\frac{\rho((a+1)\rho(t|-2)\text{sign}(t))}{2(a-1)} & \text{if } \frac{2}{\rho(a+1)} < |t| \leq \frac{2a}{\rho(a+1)}; \\
\rho\text{sign}(t) & \text{if } |t| > \frac{2a}{\rho(a+1)}. 
\end{cases}
\]  

(36)

By the expression of \( \varphi'_\rho \), an elementary calculation shows that \( \varphi'_\rho \) is Lipschitz continuous on \( \mathbb{R} \) with Lip-constant \( \rho^2 \max\left(\frac{a+1}{2}, \frac{a+1}{2(a-1)}\right) \). So, \( \nabla g_\rho \) is globally Lipschitz on \( \mathbb{R}^p \) with the same Lip-constant. This implies that \( g_\rho \) is semiconvex of modulus \( \rho^2 \max\left(\frac{a+1}{2}, \frac{a+1}{2(a-1)}\right) \).

(ii) The lower boundedness and global Lipschitz continuity of \( \Theta_{\lambda, \rho} \) follows by using the expressions of \( \psi^* \) and \( f_\tau \), while its semiconvexity is immediate by part (i) and the convexity of the function \( \beta \mapsto f_\tau(y - X\beta) + \lambda \partial \|\beta\|_1 \). Notice that \( \beta \mapsto f_\tau(y - X\beta) \) is coercive, while \( \beta \mapsto \lambda \|\beta\|_1 - \lambda \rho^{-1} g_\rho(\beta) \) is bounded. Hence, the function \( \Theta_{\lambda, \rho} \) is coercive.

(iii) From part (ii), we know that \( \Theta_{\lambda, \rho} \) is semiconvex. The first two equalities follows by Remark 2.1(iii). Thus, it suffices to establish the last equality. From the convexity of the function \( \beta \mapsto f_\tau(y - X\beta) + \lambda \partial \|\beta\|_1 \) and [31, Theorem 23.8], it follows that

\[
\partial_\beta [f_\tau(y - Xz) + \lambda \partial \|z\|_1]|_{z=\beta} = -X^T \partial f_\tau(y - X\beta) + \lambda \partial \|\beta\|_1.
\]

By part (i), the function \( g_\rho \) is smooth, which along with [32, Exercise 8.8] implies that

\[
\partial_\beta \Theta_{\lambda, \rho}(\beta) = \partial [f_\tau(y - Xz) + \lambda \partial \|z\|_1]|_{z=\beta} - \lambda \rho^{-1} \nabla g_\rho(\beta).
\]

The result directly follows from the last two equations. The proof is completed. \( \square \)

**Appendix B**

Throughout this part, for each \( k \in \mathbb{N} \) we write \( v^k = e - w^k \). In order to present the proof of Theorem 4.1, we need the following technical lemma.

**Lemma 2** Suppose for some \( k \geq 1 \) there exists \( S^{k-1} \supseteq S^* \) with \( \min_{\xi \in \langle S^{k-1}\rangle} w^k \leq \frac{1}{2} \). Then, when \( \lambda \geq 16n^{-1} \max \|X\|_1 + 8r_k \), it holds that \( \|\Delta \beta^k_{(S^{k-1})^c}\|_1 \leq 3\|\Delta \beta^k_{S^{k-1}}\|_1 \).

**Proof:** By the approximate optimality of \( \beta^k \) to (12) and Remark 4.1(iii), it follows that

\[
f_\tau(y - X\beta^*) + \lambda \langle v^{k-1}, |\beta^*| \rangle \geq f_\tau(y - X\beta^k) + \lambda \langle v^{k-1}, |\beta^k| \rangle + \langle \xi^k, \beta^* - \beta^k \rangle
\]

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which, after a suitable rearrangement, takes the following form
\[ f_r(y - X\beta^k) - f_r(y - X\beta^*) + \langle \xi^k, \beta^* - \beta^k \rangle \leq \lambda \langle v^{k-1}, |\beta^*| - |\beta^k| \rangle. \tag{37} \]

For each \( k \), write \( z^k := y - X\beta^k \). Recall that \( \varepsilon = y - X\beta^* \). Since \( \|\varepsilon\|_\infty > 0 \), we define
\[ I := \{ i \in \{1, \ldots, n\} : \varepsilon_i \neq 0 \} \quad \text{and} \quad J_k := \{ i \notin I : z_i^k \neq 0 \}. \tag{38} \]

By the expression of \( f_r \) and \( \theta_r(0) = 0 \), with the index sets \( I \) and \( J_k \), it holds that
\[
\begin{align*}
& f_r(y - X\beta^k) - f_r(y - X\beta^*) = \frac{1}{n} \sum_{i=1}^n [\theta_r(z_i^k) - \theta_r(\varepsilon_i)] \\
& \quad = \frac{1}{n} \left[ \sum_{i \in J_k} \frac{\theta_r^2(z_i^k) - \theta_r^2(\varepsilon_i)}{\theta_r(z_i^k) + \theta_r(\varepsilon_i)} + \sum_{i \in I} \frac{\theta_r^2(z_i^k) - \theta_r^2(\varepsilon_i)}{\theta_r(z_i^k) + \theta_r(\varepsilon_i)} \right] \\
& \quad \geq \frac{1}{n} \left[ \sum_{i \in J_k} \frac{\theta_r^2(z_i^k) - \theta_r^2(\varepsilon_i)}{\tau_{\max} \|z^k\|_\infty} + \sum_{i \in I} \frac{\theta_r^2(z_i^k) - \theta_r^2(\varepsilon_i)}{\tau_{\max} \|z^k\|_\infty} \right]. \tag{39} \end{align*}
\]

Notice that \( \theta_r^2 \) is smooth and strongly convex of modulus \( 2\tau_{\min}^2 \). So, it holds that
\[
\theta_r^2(z_i^k) - \theta_r^2(\varepsilon_i) \geq 2(\tau - I_{\infty}(\varepsilon_i))^2 \varepsilon_i(z_i^k - \varepsilon_i) + \tau_{\min}^2(z_i^k - \varepsilon_i)^2 \quad \text{for} \quad i = 1, \ldots, p. \tag{40} \]

This implies that \( \theta_r^2(z_i^k) - \theta_r^2(\varepsilon_i) \geq \tau_{\min}^2(z_i^k - \varepsilon_i)^2 \) for each \( i \in J_k \), and consequently,
\[
\sum_{i \in J_k} \frac{\theta_r^2(z_i^k) - \theta_r^2(\varepsilon_i)}{\tau_{\max} \|z^k\|_\infty} \geq \tau_{\min}^2 \sum_{i \in J_k} \frac{(z_i^k - \varepsilon_i)^2}{\|z^k\|_\infty^2}. \tag{41} \]

For each \( i \in I \), write \( \bar{z}_i^k := \frac{2(\tau - I_{\infty}(\varepsilon_i))^2 \varepsilon_i}{\theta_r(z_i^k) + \theta_r(\varepsilon_i)} \). From (40), it immediately follows that
\[
\sum_{i \in I} \frac{\theta_r^2(z_i^k) - \theta_r^2(\varepsilon_i)}{\theta_r(z_i^k) + \theta_r(\varepsilon_i)} \geq \sum_{i \in J_k} \bar{z}_i^k(z_i^k - \varepsilon_i) + \tau_{\min}^2 \sum_{i \in I} \frac{(z_i^k - \varepsilon_i)^2}{\theta_r(z_i^k) + \theta_r(\varepsilon_i)} \\
\quad \geq -\|\varepsilon\|_\infty \|X(\beta^k - \beta^*)\|_1 + \tau_{\min}^2 \sum_{i \in I} \frac{(z_i^k - \varepsilon_i)^2}{\tau_{\max} \|z^k\|_\infty + \|\varepsilon\|_\infty} \\
\quad \geq -2\tau_{\max} \|X(\beta^k - \beta^*)\|_1 + \tau_{\min}^2 \sum_{i \in I} \frac{(z_i^k - \varepsilon_i)^2}{\|z^k\|_\infty + \|\varepsilon\|_\infty} \tag{42} \]

where the second inequality is by \( \theta_r(z_i^k) \leq \tau_{\max} \|z^k\|_\infty \) for \( i \in I \), and the last one is since \( |z_i^k| \leq \frac{2(\tau - I_{\infty}(\varepsilon_i))^2 |\varepsilon_i|}{\theta_r(\varepsilon_i)} \leq 2\tau_{\max} \) for each \( i \in I \). Substituting (41)-(42) into (39) yields that
\[
\begin{align*}
f_r(y - X\beta^k) - f_r(y - X\beta^*) & \geq -\frac{2\tau_{\max}}{n} \|X(\beta^k - \beta^*)\|_1 + \frac{\tau_{\min}^2}{n\tau_{\max}} \sum_{i \in J_k \cup I} \frac{(z_i^k - \varepsilon_i)^2}{\|z^k\|_\infty + \|\varepsilon\|_\infty} \\
& = -\frac{2\tau_{\max}}{n} \|X(\beta^k - \beta^*)\|_1 + \frac{\tau_{\min}^2}{n\tau_{\max}} \|X(\beta^k - \beta^*)\|_1^2. \tag{43} \end{align*}
\]
Combining this inequality and (37) and recalling that $\|x^k\| \leq r_k$, we obtain

\[
\frac{\tau_{\min}^2 \|X(\beta^k - \beta^*)\|^2}{n \tau_{\max} (\|z^k\|_\infty + \|\epsilon\|_\infty)} \leq \lambda \langle v^{k-1}, |\beta^*| - |\beta^k| \rangle + \frac{2 \tau_{\max}}{n} \|X(\beta^k - \beta^*)\|_1 + \langle \xi^k, \beta^k - \beta^* \rangle \\
\leq \lambda \langle v^{k-1}, |\beta^*| - |\beta^k| \rangle + \left(2n^{-1} \tau_{\max} \|X\|_1 + r_k\right) \|\beta^k - \beta^*\|_1 \\
\leq \lambda \left(\sum_{i \in S^*} v_i^{k-1} |\Delta \beta^k_i| - \sum_{i \in (S^{k-1})c} v_i^{k-1} |\Delta \beta^k_i|\right) \\
+ \left(2n^{-1} \tau_{\max} \|X\|_1 + r_k\right) \|\beta^k - \beta^*\|_1 \\
= \lambda \left(\sum_{i \in S^*} v_i^{k-1} |\Delta \beta^k_i| - \sum_{i \in (S^{k-1})c} v_i^{k-1} |\Delta \beta^k_i|\right) \\
+ \left(2n^{-1} \tau_{\max} \|X\|_1 + r_k\right) \left(\|\Delta \beta^k_{S^{k-1}}\|_1 + \|\Delta \beta^k_{(S^{k-1})c}\|_1\right). \tag{44}
\]

Since $S^{k-1} \supset S^*$ and $v_i^{k-1} \in [0.5, 1]$ for $i \in (S^{k-1})c$, from the last inequality we have

\[
\frac{\tau_{\min}^2 \|X(\beta^k - \beta^*)\|^2}{n \tau_{\max} (\|z^k\|_\infty + \|\epsilon\|_\infty)} \leq \sum_{i \in S^{k-1}} \left(\lambda v_i^{k-1} + 2n^{-1} \tau_{\max} \|X\|_1 + r_k\right) |\Delta \beta^k_i| \\
+ \sum_{i \in (S^{k-1})c} \left(2n^{-1} \tau_{\max} \|X\|_1 + r_k - \lambda/2\right) |\Delta \beta^k_i| \\
\leq \left(\lambda + 2n^{-1} \tau_{\max} \|X\|_1 + r_k\right) \|\Delta \beta^k_{S^{k-1}}\|_1 \\
+ \left(2n^{-1} \tau_{\max} \|X\|_1 + r_k - \lambda/2\right) \|\Delta \beta^k_{(S^{k-1})c}\|_1.
\]

From the nonnegativity of the left hand side and the given assumption on $\lambda$, we have

\[
\|\Delta \beta^k_{S^{k-1}}\|_1 \leq \frac{\lambda + 2n^{-1} \tau_{\max} \|X\|_1 + r_k}{0.5\lambda - 2n^{-1} \tau_{\max} \|X\|_1 - r_k} \|\Delta \beta^k_{S^{k-1}}\|_1 \leq 3 \|\Delta \beta^k_{S^{k-1}}\|_1.
\]

The desired result follows. The proof is then completed.

\[\square\]

**Lemma 3** Suppose that $X$ satisfies the $\kappa$-RSC over $C(S^*)$ and for some $k \geq 1$ there exists an index set $S^{k-1}$ with $|S^{k-1}| \leq 1.5s^*$ such that $S^{k-1} \supset S^*$ and \( \min_{i \in (S^{k-1})c} v_i^{k-1} \leq \frac{1}{2} \).

If $16n^{-1} \tau_{\max} \|X\|_1 + 8r_k \leq \lambda < \frac{\tau_{\min} \kappa - 2 \tau_{\max} |X|_{\|\max \{2n^{-1} \tau_{\max} \|X\|_1 + r_k\}, S^{k-1}\|}}{2 \tau_{\max} \|X\|_{\|\max \{v_{S^*}^{-1}, |\epsilon|\|_{\infty}\}} |S^{k-1}|}$, then

\[
\|\Delta \beta^k\| \leq \frac{\tau_{\max} (\|X\|_{\|v_{S^*}^{-1}\|_\infty + 2n^{-1} \tau_{\max} \|X\|_1 + r_k\sqrt{|S^{k-1}| \|\epsilon\|_\infty}}}{\tau_{\min} \kappa - 2 \tau_{\max} \|X\|_{\|\max \{v_{S^*}^{-1}, |\epsilon|\|_{\infty}\}} + 2n^{-1} \tau_{\max} \|X\|_1 + r_k} |S^{k-1}| \|\epsilon\|_\infty.
\]

**Proof:** Notice that $\|z^k\|_\infty + \|\epsilon\|_\infty = \| \epsilon - X \Delta \beta^k \|_\infty + \| \epsilon \|_\infty \leq \| X \Delta \beta^k \|_\infty + 2 \| \epsilon \|_\infty$. So,

\[
\frac{\tau_{\min}^2 \|X(\beta^k - \beta^*)\|^2}{n \tau_{\max} (\|z^k\|_\infty + \|\epsilon\|_\infty)} \geq \frac{\tau_{\min}^2 \|X(\beta^k - \beta^*)\|^2}{n \tau_{\max} (\|X\|_{\|\max \{2n^{-1} \tau_{\max} \|X\|_1 + r_k\}, S^{k-1}\|_\infty} + 2 \| \epsilon \|_\infty)}.
\]

Together with the inequality (44) and $v_i^{k-1} \in [0.5, 1]$ for $i \in (S^{k-1})c$, it follows that

\[
\frac{\tau_{\min}^2 \|X(\beta^k - \beta^*)\|^2}{n \tau_{\max} (\|X\|_{\|\max \{2n^{-1} \tau_{\max} \|X\|_1 + r_k\}, S^{k-1}\|_\infty} + 2 \| \epsilon \|_\infty)} \leq \lambda \sum_{i \in S^*} v_i^{k-1} |\Delta \beta^k_i| - \frac{\lambda}{2} \sum_{i \in (S^{k-1})c} |\Delta \beta^k_i| \\
+ \left(2n^{-1} \tau_{\max} \|X\|_1 + r_k\right) \left(\|\Delta \beta^k_{S^{k-1}}\|_1 + \|\Delta \beta^k_{(S^{k-1})c}\|_1\right) \\
\leq \left(\lambda \|v_{S^*}^{-1}\|_\infty + 2n^{-1} \tau_{\max} \|X\|_1 + r_k\right) \|\Delta \beta^k_{S^{k-1}}\|_1.
\]

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where the last inequality is due to $\lambda > 16n^{-1}\tau_{\max}\|X\|_1 + 8r_k$. By Lemma 2, we know that $\|\Delta \beta^k_{(S_k-1)^c}\|_1 \leq 3\|\Delta \beta^k_{S_{k-1}}\|_1$, which by the given assumption means that $\Delta \beta^k \in \mathcal{C}(S^*)$. From the $\kappa$-RSC property of $X$ on $\mathcal{C}(S^*)$, $\|X\Delta \beta^k\|^2 \geq 2n\kappa\|\Delta \beta^k\|^2$. Then, it holds that

$$\frac{2\tau_{\min}^2\kappa\|\Delta \beta^k\|^2}{\tau_{\max}(\|X\Delta \beta^k\|_\infty + 2\|\varepsilon\|_\infty)} \leq \left(\lambda\|v^k_{S^*}\|_\infty + \frac{2\tau_{\max}\|X\|_1}{n} + r_k\right)\|\Delta \beta^k_{S_k-1}\|_1.$$

Multiplying both sides of this inequality by $\tau_{\max}(\|X\Delta \beta^k\|_\infty + 2\|\varepsilon\|_\infty)$ yields that

$$2\tau_{\min}^2\kappa\|\Delta \beta^k\|^2 \leq \tau_{\max}(\|X\Delta \beta^k\|_\infty + 2\|\varepsilon\|_\infty)\left(\lambda\|v^k_{S^*}\|_\infty + \frac{2\tau_{\max}\|X\|_1}{n} + r_k\right)\|\Delta \beta^k_{S_k-1}\|_1 + 2\tau_{\max}\|\varepsilon\|_\infty\left(\lambda\|v^k_{S^*}\|_\infty + 2n^{-1}\tau_{\max}\|X\|_1 + r_k\right)\|\Delta \beta^k_{S_k-1}\|_1.$$
If $|S^{k-1}| \leq 1.5s^*$, by invoking Lemma 3 and using the given assumption, we have

$$
\|\beta^k - \beta^*\| \leq \frac{\tau_{\max}(\lambda\|v_{S^*}^{k-1}\|_\infty + 2n^{-1}\tau_{\max}\|X\|_1 + r_k)\sqrt{|S^{k-1}|}\|\varepsilon\|_\infty}{\tau_{\min}^2\kappa - 2\tau_{\max}\|X\|_\max(\lambda\|v_{S^*}^{k-1}\|_\infty + 2n^{-1}\tau_{\max}\|X\|_1 + r_k)|S^{k-1}|}
$$

$$
\leq \frac{\tau_{\max}(\lambda\|v_{S^*}^{k-1}\|_\infty + 2n^{-1}\tau_{\max}\|X\|_1 + r_k)\sqrt{|S^{k-1}|}\|\varepsilon\|_\infty}{\tau_{\min}^2\kappa - 2\tau_{\max}\|X\|_\max(\lambda + 2n^{-1}\tau_{\max}\|X\|_1 + \epsilon)\sqrt{1.5s^*}}
$$

$$
\leq c\tau_{\max}(\lambda\|v_{S^*}^{k-1}\|_\infty + 2n^{-1}\tau_{\max}\|X\|_1 + r_k)\sqrt{|S^{k-1}|}\|\varepsilon\|_\infty
$$

(45)

where the second inequality is by the nondecreasing of $t \mapsto \frac{c_1t + 1}{c_2t - 1}$ for constants $c_1, c_2 > 0$, and the last one is by the restriction on $\lambda$. Since $2n^{-1}\tau_{\max}\|X\|_1 + r_k \leq \frac{\lambda}{8}$ and $\|v_{S^*}^{k-1}\|_\infty \leq 1$,

$$
\|\beta^k - \beta^*\| \leq \frac{9c\tau_{\max}\lambda\|\varepsilon\|_\infty}{8}\sqrt{1.5s^*},
$$

and the desired result holds. So, it suffices to argue that $|S^{k-1}| \leq 1.5s^*$ for all $k \in \mathbb{N}$. When $k = 1$, the statement holds trivially since $u^0 = 0$ implies $S^0 = S^*$. Assuming that $|S^{k-1}| \leq 1.5s^*$ holds for $k = l$ with $l \geq 1$, we prove that it holds for $k = l + 1$. Indeed, since $S^l \setminus S^* = \{i \not\in S^*: w_i^l \leq \frac{1}{2}\}$, we have $w_i^l \in (\frac{1}{2}, 1]$ for $i \in S^l \setminus S^*$. Together with the formula (14), we deduce that $\rho_l|\beta_i^l| \geq 1$, and hence the following inequality holds:

$$
\sqrt{|S^l \setminus S^*|} \leq \sqrt{\sum_{i \in S^l \setminus S^*} \rho_l^2|\beta_i^l|^2} = \sqrt{\sum_{i \in S^l \setminus S^*} \rho_l^2|\beta_i^l - \beta_i^*|^2}.
$$

Since the statement holds for $k = l$, it holds that $\|\beta^l - \beta^*\| \leq \frac{9c\tau_{\max}\lambda\|\varepsilon\|_\infty}{8}\sqrt{1.5s^*}$. Thus,

$$
\sqrt{|S^l \setminus S^*|} \leq \rho_l \|\beta^l - \beta^*\| \leq \frac{9c\tau_{\max}\rho_l\lambda\|\varepsilon\|_\infty}{8}\sqrt{1.5s^*} \leq 0.5s^*
$$

(46)

where the last inequality is due to $\rho_1 \lambda \leq \rho_3 \lambda \leq \frac{8}{9\sqrt{3\tau_{\max}\|\varepsilon\|_\infty}}$. The inequality (46) implies that $|S^l| \leq 1.5s^*$. This shows that the statement follows. The proof is completed. $\Box$

To present the proof of Theorem 4.2, we need the following lemma which upper bounds $\|v_{S^*}\|_\infty$. Since its proof is implied by that of [36, Lemma 3], we here omit it.

**Lemma 4** Let $F^k$ and $\Lambda^k$ be the index sets defined by (19). Then, for each $k \in \{0\} \cup \mathbb{N}$,

$$
\|v_{S^*}^{k}\|_\infty \leq \max_{i \in S^*} I_{\Lambda^k}(i) + \max_{i \in S^*} I_{F^k}(i).
$$

**Proof of Theorem 4.2:** For each $k \in \mathbb{N}$, define $S^{k-1} := S^* \cup \{i \not\in S^*: w_i^{k-1} > \frac{1}{2}\}$. Since the conclusion holds for $k = 1$, it suffices to consider the case $k \geq 2$. From the proof of Theorem 4.1, $|S^{k-1}| \leq 1.5s^*$ for all $k \in \mathbb{N}$. Moreover, by using (46) and $\rho_k \geq 1$,

$$
\sqrt{|S^{k-1}|} = \sqrt{|S^*| + |S^{k-1} \setminus S^*|} \leq \sqrt{s^*} \sqrt{|S^{k-1} \setminus S^*|}
$$

$$
\leq \sqrt{s^*} \leq (2n^{-1}\tau_{\max}\|X\|_1 + r_k)^{-1}\lambda\rho_{k-1}\|\beta^{k-1} - \beta^*\| \leq \sqrt{s^*} (2n^{-1}\tau_{\max}\|X\|_1 + r_k)^{-1}\lambda\rho_{k-1}\|\beta^{k-1} - \beta^*\|
$$

(47)
where the first inequality is due to $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ for $a, b \geq 0$, the last one is due to $\lambda \geq 16n^{-1} \tau_{\text{max}} \|X\|_1 + 8r_k$. From the inequality (45) and Lemma 4, it follows that

$$
\|\beta^k - \beta^*\| \leq c\tau_{\text{max}}\|\varepsilon\|_\infty \sqrt{|S^{k-1}|} \left( \lambda(\max_{i \in S^*} \|A_{k-1}(i)\| + \max_{i \in S^*} \|P_{k-1}(i)\|) + 2n^{-1}\tau_{\text{max}}\|X\|_1 + r_k \right)
$$

$$
\leq c\tau_{\text{max}}\|\varepsilon\|_\infty \left[ \lambda \sqrt{1.5s^*} \max_{i \in S^*} \|A_0(i)\| + \lambda \sqrt{1.5s^*} \rho_{k-1}\|\beta^{k-1} - \beta^*\| + (2n^{-1}\tau_{\text{max}}\|X\|_1 + r_k) \right] \sqrt{|S^{k-1}|}
$$

where the last inequality is since $\max_{i \in S^*} \|P_{k-1}(i)\| \leq \max_{i \in S^*} \|P_{k-1}(i)\| - \|\beta^{k-1} - \beta^*\| \leq \rho_{k-1}\|\beta^{k-1} - \beta^*\|$. Substituting the inequality (47) into this inequality, we obtain

$$
\|\Delta \beta^k\| \leq c\tau_{\text{max}}\|\varepsilon\|_\infty \sqrt{s^*} \left( 2n^{-1}\tau_{\text{max}}\|X\|_1 + r_k \right) + c\tau_{\text{max}}\lambda\|\varepsilon\|_\infty \sqrt{1.5s^*} \max_{i \in S^*} \|A_0(i)\|
$$

$$
+ c\tau_{\text{max}}\|\varepsilon\|_\infty \rho_{k-1}\lambda(\sqrt{1.5s^*} + 1/8)\|\beta^{k-1} - \beta^*\|
$$

$$
\leq 2cn^{-1}\tau_{\text{max}}^2\|\varepsilon\|_\infty \sqrt{s^*}\|X\|_1 + c\tau_{\text{max}}\|\varepsilon\|_\infty \sqrt{s^*} r_k
$$

$$
+ c\tau_{\text{max}}\lambda\|\varepsilon\|_\infty \sqrt{1.5s^*} \max_{i \in S^*} \|A_0(i)\| + \frac{\sqrt{3}}{3} \|\Delta \beta^{k-1}\|
$$

where the second inequality is due to $\rho_{k-1}\lambda \leq \rho_3\lambda \leq (\sqrt{3}c\tau_{\text{max}}\|\varepsilon\|_\infty (\sqrt{1.5s^*} + 1/8))^{-1}$. The desired result follows by using the last recursion inequality. \qed

Appendix C

In this part, we recall the semi-proximal ADMM proposed by Gu et al. [18] for solving the subproblem (12). Notice that the subproblem (12) can be equivalently written as

$$
\min_{\beta \in \mathbb{R}^n, z \in \mathbb{R}^n} \ f_r(z) + \|\omega^{k-1} \circ \beta\|_1
$$

$$
s.t. \ X\beta - z - y = 0 \quad \text{with} \quad \omega^{k-1} = \lambda(e - w^{k-1}) \quad (48)
$$

whose dual problem, after an elementary calculation, takes the following form

$$
\min_{u \in \mathbb{R}^n} \left\{ f_r^*(u) + \langle u, y \rangle \quad \text{s.t.} \quad |(X^T u)_i| \leq \omega_i^{k-1}, \ i = 1, \ldots, p \right\}. \quad (49)
$$

For a given $\sigma > 0$, the augmented Lagrangian function of (48) takes the following form

$$
L_\sigma(\beta, z, u) := f_r(z) + \|\omega^{k-1} \circ \beta\|_1 + \langle u, X\beta - z - y \rangle + \frac{\sigma}{2} \|X\beta - z - y\|^2.
$$

The iteration steps of the semi-proximal ADMM in [18] are described as follows.
Initialization: Choose the parameters \( \sigma > 0, \gamma = \sigma\|X^TX\| \) and \( \varrho \in (1, \frac{\sqrt{k+1}}{2}) \), and an initial point \((\beta^0, z^0, u^0) \in \mathbb{R}^p \times \mathbb{R}^n \times \mathbb{R}^n \) with \( \beta^0 = \beta^{k-1} \). Set \( j = 0 \).

while the stopping conditions are not satisfied do

1. Compute the following convex minimization problem
   \[
   \beta^{j+1} = \arg\min_{\beta \in \mathbb{R}^p} L_\alpha(\beta, z^j, u^j) + \frac{1}{2}\|\beta - \beta^j\|_2^2 \gamma I - \sigma X^TX. \tag{50}
   \]

2. Compute the following convex minimization problem
   \[
   z^{j+1} = \arg\min_{z \in \mathbb{R}^n} L_\alpha(\beta^{j+1}, z, u^j). \tag{51}
   \]

3. Update the multiplier by the formula
   \[
   u^{j+1} = u^j + \varrho\sigma(X\beta^{j+1} - z^{j+1} - y). \tag{52}
   \]

4. Set \( j \leftarrow j + 1 \), and then go to Step 1.

end while

Remark 1 (i) Algorithm 4 has a little difference from Algorithm 1 of [18] since here the semi-proximal term \( \frac{1}{2}\|\beta - \beta^j\|_2^2 \gamma I - \sigma X^TX \) rather than \( \frac{1}{2}\|\beta - \beta^j\|_2^2 \gamma I - \sigma X^TX \), is used. As shown in [18], the subproblems (50) and (51) have a closed form solution, that is,

\[
\beta^{j+1} = \text{sign}(\gamma^{-1}h^j) \max \left( |\gamma^{-1}h^j| - \gamma^{-1}\omega^{k-1}, 0 \right)
\]

\[
z^{j+1} = \mathcal{P}_{\sigma^{-1}f_\tau}(X\beta^{j+1} - y + \sigma^{-1}u^j)
\]

where \( h^j = \gamma\beta^j + \sigma X^T(y + z^j - X\beta^j - u^j / \sigma) \).

(ii) During our implementation of Algorithm 4, we adjust \( \sigma \) dynamically by the ratio of the primal and dual infeasibility. By comparing the first-order optimality conditions of (50) and (51) with those of (48) and using the multiplier updating step (52), we measure the primal infeasibility, the dual infeasibility and the dual gap at the current iterate \((\beta^j, z^j, u^j)\) in terms of \( \epsilon_{\text{pinf}}^j, \epsilon_{\text{dinf}}^j \) and \( \epsilon_{\text{gap}}^j \), respectively, defined by

\[
\begin{align*}
\epsilon_{\text{dinf}}^j &:= \sqrt{||\zeta^j||^2 + ||(\varrho^{-1} - 1)(u^j - \omega^j)||^2} \quad \text{max}(1, 0.5(\omega_{\text{prim}}^j + \omega_{\text{dual}}^j))
\end{align*} \tag{53a}
\]

\[
\begin{align*}
\epsilon_{\text{pinf}}^j &:= |u^j - \omega^j| / \varrho \sigma(1 + ||y||), \quad \epsilon_{\text{gap}}^j := (\omega_{\text{prim}}^j + \omega_{\text{dual}}^j) / \max(1, 0.5(\omega_{\text{prim}}^j + \omega_{\text{dual}}^j))
\end{align*} \tag{53b}
\]

where \( \zeta^j := X^T(u^j - \omega^j - \sigma(X\beta^j - y - z^j)) - \gamma(\beta^j - \beta^{j-1}) \), and \( \omega_{\text{prim}}^j \) and \( \omega_{\text{dual}}^j \) are the objective values of the problems (48) and (49) at \((\beta^j, z^j, u^j)\). Different from [18], we
terminate Algorithm 4 whenever \( \max(\epsilon_{\text{pinf}}^j, \epsilon_{\text{dinf}}^j, \epsilon_{\text{gap}}^j) \leq \epsilon_{\text{ADMM}} \). By comparing with the optimality conditions of (50)-(51) with those of (48), such a stopping criterion ensures that the obtained \((\beta^j, z^j, u^j)\) is an approximate primal-dual solution pair of (48).