HOMOTOPY CATEGORIES OF INJECTIVE MODULES OVER
DERIVED DISCRETE ALGEBRAS

ZHE HAN

Abstract. We study the homotopy category $K(\text{Inj} \ A)$ of all injective modules over a finite dimensional algebra $A$ with discrete derived category. We give a classification of the indecomposable objects of $K(\text{Inj} \ A)$ for any radical square zero self-injective algebra $A$. In particular, every indecomposable object is endofinite.

1. Introduction

For a finite dimensional algebra $A$ over an algebraically closed field $k$, we investigate the homotopy category $K(\text{Inj} \ A)$ of all injective $A$-modules. By [23], we know that $K(\text{Inj} \ A)$ is compactly generated triangulated category. The subcategory of compact objects in $K(\text{Inj} \ A)$ is triangle equivalent to $D^b(\text{mod} \ A)$, the bounded derived category of finitely generated $A$-modules. Thus we can think of $K(\text{Inj} \ A)$ as a 'compactly generated complete' of $D^b(\text{mod} \ A)$. The category $K(\text{Inj} \ A)$ is also very complicated in general. Only in some special cases, every object in $K(\text{Inj} \ A)$ can be expressed as direct sum of indecomposable objects.

We focus on the algebra $A$ with discrete derived category $D^b(\text{mod} \ A)$ [29] and try to understand the category $K(\text{Inj} \ A)$. An algebra $A$ with discrete derived category has a rather simple derived category $D^b(\text{mod} \ A)$. We expect that $K(\text{Inj} \ A)$ is easy to control in this case.

For every radical square zero self-injective algebra $A$, it has a discrete derived category. We know that every basic radical square zero self-injective algebra $A$ is of the form $kC_n/I_n$, $n \geq 1$, where the quiver $C_n$ is given by

\[
\begin{array}{c}
1 & \overset{\alpha}{\rightarrow} & 2 & \overset{\alpha}{\rightarrow} & 3 & \overset{\alpha}{\rightarrow} & 4 & \overset{\alpha}{\rightarrow} & 5 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow \\
& & n & & & & & & \\
\end{array}
\]

with relations $I_n = \alpha^2$. Let $\hat{\text{Mod}} \ A$ be the stable modules category of the repetitive algebra $\hat{A}$ of $A$. We study the category $K(\text{Inj} \ A)$ using the fully faithful triangle functor $F : K(\text{Inj} \ A) \rightarrow \hat{\text{Mod}} \ A$ constructed in [24]. There is no explicit description of this functor; in particular, we do not know its image. For symmetric algebras we find another embedding which has an explicit expression. By this new embedding, we are able to describe the image of the indecomposable objects in $K(\text{Inj} kC_1/I_1)$ and then extend to general cases. This leads to a full classification of indecomposable objects in $K(\text{Inj} \ A)$ for radical square zero self-injective algebras $A$.

Our main result is the following.

Theorem 1.1. If $A$ is of the form $kC_n/I_n$, then the indecomposable objects of $K(\text{Inj} \ A)$, up to shifts, are exactly the truncations $\sigma_{\leq m} \sigma_{\geq l} I^\bullet$, where $I^\bullet$ is the periodic complex.
Proposition 2.2. [7, Theorem A] is not derived hereditary of Dynkin type. The following are equivalent:

when we consider the algebras with discrete derived categories.

The ideal \( I \) is square zero. In section 4 we describe the Ziegler spectrum of \( K \) for radical square zero self-injective algebras \( A \).

2. Preliminaries

2.1. Derived discrete algebras. Throughout the paper, we fix the field \( k \) to be algebraically closed. For an algebra \( A \), let \( D^b(\text{mod } A) \) be the bounded derived category of \( \text{mod } A \). For a complex \( X \) in \( D^b(\text{mod } A) \), we define the homological dimension of \( X \) to be the vector \( h\text{dim}(X) = (\text{dim} H^i(X))_{i \in \mathbb{Z}} \), where \( \text{dim} H^i(X) \) is the dimension vector of \( A \)-module \( H^i(X) \).

There are some algebras with this property that there are only finitely many isomorphism classes of indecomposable objects in the derived category \( K(\text{mod } A) \) for \( A \) self-injective and radical square zero. In section 4 we describe the Ziegler spectrum of \( K(\text{mod } A) \) for radical square zero self-injective algebras \( A \).

\[ \xymatrix{ \cdots & I_n \ar[r]^{d_n} & I_{n-1} \ar[r]^{d_{n-1}} & I_{n-2} \ar[r] & \cdots & I_1 \ar[r]^{d_1} & I_0 \ar[r]^{d_0} & \cdots } \]

with differential \( d_j : I_j \to I_{j-1} \) being composite of \( I_j \to I_j/\text{soc } I_j \cong \text{rad } I_{j-1} \hookrightarrow I_{j-1} \). Moreover, all indecomposable objects are endofinite.

The paper is organized as follows. In section 2, we give some background materials about derived discrete algebras, compactly generated triangulated categories and quiver representations. In section 3, we give the classification of the indecomposable objects in the homotopy category \( K(\text{mod } A) \) for \( A \) self-injective and radical square zero. For an algebra \( A \) derived discrete algebras.

Definition 2.1. A derived category \( D^b(\text{mod } A) \) is discrete if for every vector \( d = (d_i)_{i \in \mathbb{Z}} \) with \( d_i \geq 0 \), there are only finitely many isomorphism classes of indecomposable objects \( X \in D^b(\text{mod } A) \) such that \( h\text{dim } X = d \).

Following [3], we call the algebra \( A \) derived discrete. Let \( \Omega \) be the set of all triples \( (r, n, m) \) of integers such that \( n \geq r \geq 1 \) and \( m \geq 0 \). For each \( (r, n, m) \in \Omega \) consider the quiver \( Q(r, n, m) \) of the form

\[ \begin{array}{c}
\alpha_{n-1} \\
\alpha_{n-2} \\
\alpha_{n-3} \\
\vdots \\
\alpha_1 \\
\alpha_0 \\
\end{array} \]

The ideal \( I(r, n, m) \) in the path algebra \( kQ(r, n, m) \) is generated by the paths \( \alpha_0\alpha_{n-1}, \alpha_{n-1}\alpha_{n-2}, \ldots, \alpha_{r-1}\alpha_{n-r}, \) and let \( L(r, n, m) = kQ(r, n, m)/I(r, n, m) \).

It is an easy observation that \( \text{gl. dim } L(r, n, m) = \infty \) for \( n = r \).

Proposition 2.2. [7, Theorem A] Let \( A \) be a connected algebra and assume that \( A \) is not derived hereditary of Dynkin type. The following are equivalent:

1. \( D^b(\text{mod } A) \) is discrete.
2. \( D^b(\text{mod } A) \cong D^b(\text{mod } L(r, n, m)) \), for some \( (r, n, m) \in \Omega \).
3. \( A \) is tilting-cotilting equivalent to \( L(r, n, m) \), for \( (r, n, m) \in \Omega \).

From this proposition, we only need to consider the algebras of the form \( L(r, n, m) \) when we consider the algebras with discrete derived categories.
2.2. Compactly generated triangulated categories.

**Definition 2.3.** Let $\mathcal{T}$ be a triangulated category with infinite coproducts. An object $T$ in $\mathcal{T}$ is compact if $\text{Hom}_T(T, -)$ preserves all coproducts. The category $\mathcal{T}$ is called compactly generated if there is a set $\mathcal{S}$ of compact objects such that $\text{Hom}_\mathcal{T}(T, \Sigma^i X) = 0$ for all $T \in \mathcal{S}$ and $i \in \mathbb{Z}$ implies $X = 0$.

Now we assume that $\mathcal{T}$ has finite coproducts. One can show that the category of all compact objects are closed under shifts and triangles. Thus the full subcategory $\mathcal{T}^c$ of compact objects in $\mathcal{T}$ is a triangulated category.

**Lemma 2.4.** [25 Lemma 3.2] Let $\mathcal{T}$ be a compactly generated triangulated category with $\mathcal{S}$ a generating set. If $\mathcal{R}$ is a full subcategory containing $\mathcal{S}$ and closed under forming infinite direct sums, then there is a triangulated equivalence $\mathcal{R} \cong \mathcal{T}$.

For an algebra $A$, $\text{Mod}A$ denotes the category of all $A$-modules, and $\text{Inj}A$ is the category of all injective $A$-modules. The unbounded derived category $D(\text{Mod}A)$ is compactly generated. Its full subcategory of compact objects is $K^b(\text{proj} A)$, the full subcategory of perfect complexes. The homotopy category $K(\text{Inj}A)$ of $\text{Inj}A$ is compactly generated [23]. There is a characterization of the subcategory of compact objects in $K(\text{Inj}A)$.

**Proposition 2.5.** [23 Proposition 2.3] If $A$ is a finite dimensional $k$-algebra, then the category $K(\text{Inj}A)$ is compactly generated, and there is a natural triangle equivalence $K^c(\text{Inj}A) \cong D^b(\text{mod}A)$.

2.3. Repetitive algebras. Let $A$ be a finite dimensional basic $k$-algebra. $D = \text{Hom}_k(-, k)$ is the standard duality on $\text{mod}A$. $DA$ is an $A$-$A$-module via $a', a'' \in A, \varphi \in DA, (a'\varphi a'')(a) = \varphi(a'a'')$.

**Definition 2.6.** The repetitive algebra $\hat{A}$ of $A$ is defined as following, the underlying vector space is given by

$$\hat{A} = (\oplus_{i \in \mathbb{Z}} A) \oplus (\oplus_{i \in \mathbb{Z}} DA)$$

The element of $\hat{A}$ by $(a_i, \varphi_i)_i$, almost all $a_i, \varphi_i$ being zero. The multiplication is defined by

$$(a_i, \varphi_i)_i \cdot (b_i, \psi_i)_i = (a_i b_i, a_{i+1} \varphi_i + \varphi_i b_i)_i$$

An $\hat{A}$-module $M$ is given by $M = (M_i, f'_i)_{i \in \mathbb{Z}}$, where $M_i$ are $A$-modules and $f_i : DA \otimes_A M_i \to M_{i+1}$ such that $f_{i+1} \circ (1 \otimes f'_i) = 0$. Given $\hat{A}$-modules $M = (M_i, f'_i)$ and $N = (N_i, g'_i)$, the morphism $h : M \to N$ is a sequence $h = (h_i)_{i \in \mathbb{Z}}$ such that

$$\begin{array}{ccc}
DA \otimes_A M_i & \xrightarrow{f'_i} & M_{i+1} \\
\downarrow_{1 \otimes h_i} & & \downarrow_{h_{i+1}} \\
DA \otimes N_i & \xrightarrow{g'_i} & N_{i+1}
\end{array}$$

commutes. Sometimes, we write $(M_i, f'_i)_{i \in \mathbb{Z}}$ as

$$\cdots M_{-1} \sim f_{-1} M_0 \sim f_0 M_1 \sim \cdots .$$

Let $\text{Mod}\hat{A}$ be the category of all left $\hat{A}$-modules, and $\text{mod}\hat{A}$ be the subcategory of finite dimensional modules. They both are Frobenius categories. Thus the associated stable categories $\text{Mod}\hat{A}$ and $\text{mod}\hat{A}$ are triangulated categories. Moreover, $\text{mod}\hat{A}$ is the full subcategory of compact objects in $\text{Mod}\hat{A}$. 
Happel introduced the embedding functor $D^b(\mod A) \to \mod A$ \cite{H}, and the functor was extended to $K(\Inj A) \to \Mod \hat{A}$ of unbounded complexes in \cite{K}. 

**Proposition 2.7.** \cite{K} Theorem 7.2 There is a fully faithful triangle functor $F$ which is the composition of

$$K(\Inj A) \xrightarrow{K_{ac}(\Inj \hat{A})} \Mod \hat{A}$$

extending Happel’s functor

$$D^b(\mod A) \xrightarrow{\otimes A \hat{A}} D^b(\mod \hat{A}) \xrightarrow{\mod \hat{A}}.$$

The functor $F$ admits a right adjoint $G$ which is the composition

$$\Mod \hat{A} \xrightarrow{\sim} K_{ac}(\Inj A) \xrightarrow{\Hom \hat{A}(A, \cdot)} K(\Inj A).$$

2.4. **Radical square zero algebras.** In \cite{2}, there is a connection between an artin algebra $A$ with $\rad^2(A) = 0$ and a hereditary algebra with radical square zero, where $\rad(A)$ is the Jacobison radical of $A$, denoted by $J$.

Now, assume $A'$ is any artin algebra, thus the algebra $A = A'/J^2$ is a radical square zero algebra. We can associate a hereditary algebra $A^s = (A/J, 0, 0)$ with radical square zero to $A$ \cite{2}. The radical of $A^s$ is $\rad(A^s) = \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix}$. Let $Q$ be a quiver with vertex set $Q_0 = 1, 2, \ldots, n$, the separated quiver $Q^s$ of $Q$ has $2n$ vertices $1, \ldots, n, 1', \ldots, n'$ and an arrow $l \to m'$ for every arrow $l \to m$ in $Q$. The ordinary quiver $Q^s$ of $A^s$ coincides with the separated quiver $(Q_A)^s$ of $Q_A$.

We can study the algebra $A$ via the hereditary algebra $A^s$. For any $M \in \mod A$, we define an $A^s$-module $[M/JM]$. Then there is a functor

$$S : \mod A \to \mod A^s, \quad M \mapsto [M/JM].$$

The functor $S$ has some nice properties, which could be generalized to all modules, see \cite{2} Chapter X] or \cite{18} Proposition 8.63.

**Proposition 2.8.** Let $A, A^s$ and $S$ as above, then we have the following.

1. $M$ and $N$ in $\mod A$ are isomorphic if and only if $S(M)$ and $S(N)$ are isomorphic in $\mod A^s$.
2. $M$ is indecomposable in $\mod A$ if and only if $S(M)$ is indecomposable in $\mod A^s$.
3. $M$ is projective in $\mod A$ if and only if $S(M)$ is projective in $\mod A^s$.

Moreover, the functor $S$ induces a stable equivalence,

$$S : \mod A \to \mod A^s.$$
There are two functors
\[ T' : \text{Rep}(Q, k) \to \text{Rep}(Q^a, k) \]
and
\[ T : \text{Rep}(Q^a, k) \to \text{Rep}(Q, k) \]
which are defined as follows: Given a representation \( X \) of \( Q \), let \((T'X)_i = (X/\text{Rad} X)_i\) and \((T'X)_r = (\text{Rad} X/\text{Rad}^2 X)_i\) for each vertex \( i \in Q_0 \). For each arrow \( a : i \to j \) of \( Q \), let \((T'X)_a : (T'X)_i \to (T'X)_j\) be the map which is induced by \( X_a \). Given a representation \( Y \) of \( Q^a \), let \((TY)_i = Y_i \oplus Y_r\) for each vertex \( i \in Q_0 \). For each arrow \( \alpha \in Q_1 \), let \((TY)_\alpha = \left( \begin{smallmatrix} 0 & 0 \\ y & 0 \end{smallmatrix} \right) \).

We call a representation \( X \) separated if \((\text{Rad} X)_i = X_i\) for every sink \( i \).

**Proposition 2.9.** [19] Proposition 11.2.2] The functors \( T' \) and \( T \) induce mutually inverse bijections between the isomorphism classes of radical square zero representations of \( Q \) and the isomorphism classes of separated representations of \( Q^a \).

3. THE CATEGORY \( K(\text{Inj} A) \) OF RADICAL SQUARE ZERO SELF-INJECTIVE ALGEBRA \( A \)

We have known that \( K(\text{Inj} A) \) is compactly generated triangulated category with \( K^0(\text{Inj} A) \cong D^b(\text{mod} A) \). Actually, \( K(\text{Inj} A) \) is a derived invariant [6, Proposition 8.1]. When we consider the homotopy category \( K(\text{Inj} A) \) for \( A \) derived discrete, we only need to consider the algebras of the form \( L(r, n, m) \), by Proposition 2.2. In particular, we consider the quiver \( C_n \) as following,

\[
\begin{array}{c}
\alpha \\
\downarrow \\
\alpha \quad \alpha^{-1} \quad \alpha \\
2 \quad 3 \quad 4 \\
\downarrow \\
1 \\
\end{array}
\]

with relations \( I_n = \alpha^2 \). Set \( A_n = kC_n/I_n \). The algebra of the form \( A_n \) is self-injective and \( \text{Rad}^2 A_n = 0 \). There is a characterization of these algebras in [2, Chapter IV, Proposition 2.16].

**Lemma 3.1.** Let \( A \) be a basic self-injective algebra which is not semisimple. Then \( \text{Rad}^2 A = 0 \) if and only if \( A \cong A_n \) for some \( n \).

The algebra \( A_n = kC_n/I_n \) is derived discrete algebra, since it is the derived discrete algebra \( L(n, n, 0) \) which occurs in Proposition 2.2. Let \( I_j, j \in [1, n] \) be the indecomposable injective modules of \( A_n \), and \( I^* \) be the periodic complex with \( I_1 \) in the degree 0,

\[
\ldots \to I_n \xrightarrow{d_n} I_{n-1} \xrightarrow{d_{n-1}} I_{n-2} \xrightarrow{d_{n-2}} \ldots \xrightarrow{d_1} I_1 \xrightarrow{d_1} I_n \xrightarrow{d_1} \ldots
\]

where the differential \( d_j : I_j \to I_{j-1} \) is the composition of \( I_j \to I_j/\text{soc} I_j \cong \text{rad} I_{j-1} \Leftarrow I_{j-1} \).

Define a family of complexes \( I_{l, m} = \sigma_{\leq m} \sigma \geq l^* \), where \( l \leq m \in \mathbb{Z} \cup \{ \pm \infty \} \), where \( \sigma \) is the brutal truncation functor [30, p9]. In this section, we will prove the following result:

**Theorem 3.2.** The indecomposable objects in the homotopy category \( K(\text{Inj} A_n) \) are of the form \( I_{l, m}[r] \) where \( r \in \mathbb{Z} \). Moreover, \( I_{l, m}[r] = I_{l', m'}[r'] \) if and only if \( l' = l + nk, m' = m + nk \) and \( r' = r + nk \) where \( k \in \mathbb{Z} \).
3.1. **Classification of indecomposable objects of $K(\text{Inj} \, k[x]/(x^2))$.** Given an algebra $A$, the trivial extension of $A$ is the algebra $T(A) = A \bowtie D(A) = \{(a, \psi) | a \in A, \phi \in D(A)\}$, where the multiplication is given as following:

$$(a, f)(b, g) = (ab, fb + ag).$$

$T(A)$ is a $\mathbb{Z}$-graded algebra, namely as vector space we have $T(A) = (A, 0) \oplus (0, D(A))$, and the elements of $A \oplus 0$ (resp. $0 \oplus D(A)$) is of degree 0 (resp. degree 1).

The trivial extension $T(A)$ of a finite dimensional algebra $A$ is a symmetric algebra. Since there is a symmetric bilinear pairing $(-, -) : T(A) \times T(A) \to T(A), ((a, f), (b, g)) \mapsto f(b) + g(a)$ satisfying associativity.

**Remark 3.3.** $M$ is a $\mathbb{Z}$-graded $T(A)$-module if and only if $M = \oplus_{i \in \mathbb{Z}} M_i$, satisfies $M_i \in \text{Mod} A$, and there exists a homomorphism $f_i : D(A) \otimes_A M_i \to M_{i+1}$ for any $i \in \mathbb{Z}$. Denote $\text{Mod}^T A$ the category of $\mathbb{Z}$-graded $T(A)$-modules.

Given any $M = \oplus_i M_i \in \text{Mod}^T A(A)$, there is a module $M = (M_i, f_i) \in \text{Mod} \hat{A}$. Given any $M = (M_i, f_i) \in \text{Mod} \hat{A}$, it corresponds to the module $M = \oplus_i M_i \in \text{Mod}^T (A)$. This correspondence gives an equivalence of these two categories.

**Lemma 3.4.** Let $A$ be a finite dimensional $k$-algebra, there is an equivalence of categories $\text{Mod} \hat{A} \cong \text{Mod}^T (A)$.

We already know the relations between $D^b(\text{mod} A)$ and $\text{mod} \hat{A}$, which extends the embedding $\text{mod} A \hookrightarrow \text{mod} \hat{A}$. Now if the algebra $A$ is a symmetric algebra, then we can build some special relations between the complex category of $A$-modules $C(\text{Mod} A)$ and $\text{Mod} \hat{A}$.

If $A$ is a symmetric algebra, we have that $D(A) \cong A$ as $A$-$A$-bimodules. Given a complex in $\text{Mod} A$

$$\ldots \xrightarrow{d_{i-1}} X_{i-1} \xrightarrow{d_i} X_i \xrightarrow{d_{i+1}} X_{i+1} \xrightarrow{d_{i+2}} \ldots,$$

we can naturally view the complex as a $\mathbb{Z}$-graded $T(A)$-module $\oplus_{i \in \mathbb{Z}} X_i$ with the morphism $d_i : A \otimes_A X_i \to X_{i+1}$. Morphisms of complexes correspond to homomorphisms of $\mathbb{Z}$-graded $T(A)$-modules. Therefore, we have an embedding functor $S : C(\text{Mod} A) \to \text{Mod}^T (A)$, $(X^i, d^i) \mapsto (X^i, d^i)$, and a forgetful functor $U : \text{Mod}^T (A) \to C(\text{Mod} A)$. In this case, the functors $S$ and $U$ are equivalences between $C(\text{Mod} A)$ and $\text{Mod}^T (A)$. By the equivalence $\text{Mod} \hat{A} \cong \text{Mod}^T (A)$, we transform the complexes in $\text{Mod} A$ to the $\hat{A}$-modules.

**Lemma 3.5.** If $A$ is a symmetric algebra, there is an equivalence of categories $C(\text{Mod} A) \cong \text{Mod} \hat{A}$.

For any algebra $A$, the category $\text{Mod} \hat{A}$ is a Frobenius category and the complex category $C(\text{Inj} A)$ with the set of all degree-wise split exact sequences in $C(\text{Inj} A)$ is also a Frobenius category. All indecomposable projective-injective objects in $C(\text{Inj} A)$ are complexes of the form

$$0 \longrightarrow I \xrightarrow{id} I \longrightarrow 0$$

where $I$ is an indecomposable injective $A$-module. The associated stable categories are $\text{Mod} \hat{A}$ and $K(\text{Inj} A)$ respectively. If $A$ is a symmetric algebra, then $C(\text{Inj} A)$ is a full exact subcategory of $\text{Mod} \hat{A}$, i.e. $C(\text{Inj} A)$ is a full subcategory of $\text{Mod} \hat{A}$ and closed under extensions.
Lemma 3.6. Let $A$ be a symmetric algebra, the equivalence $C(\text{Mod} A) \rightarrow \text{Mod} \hat{A}$ restrict to the embedding $\Psi : C(\text{Inj} A) \rightarrow \text{Mod} \hat{A}$ is an exact functor. Moreover, the embedding induces a bijection between the indecomposable projective-injective objects in $C(\text{Inj} A)$ and $\text{Mod} \hat{A}$.

Proof. To show the embedding is an exact functor, it suffices to show $C(\text{Inj} A)$ is a full exact subcategory of $\text{Mod} \hat{A}$. It is obvious that $C(\text{Inj} A)$ is closed under extension and the conflations of $\text{Mod} \hat{A}$ with terms in $C(\text{Inj} A)$ split. Thus the embedding is a fully faithful exact functor.

Now, we consider projective-injective modules in $\text{Mod} \hat{A} \cong \text{Mod} \hat{A} T(\hat{A})$. Let $e_1, e_2, \ldots, e_n$ be the primitive idempotents of $A$, and $1 = \sum_{i=0}^{n} e_i$ be the unit of $A$. Let $E_i(e_j)$ be the ‘matrix’ with $(i, i)$ position is $e_j$, and the other positions are 0. Then $\{E_i(e_j)\}_{i \in \mathbb{Z}, 1 \leq j \leq n}$ are all primitive idempotents of $\hat{A}$.

All indecomposable projective-injective $\hat{A}$-modules are of form

$$\hat{P}_i = \hat{A}E_i(e_j) \cong Ae_j \oplus D(A)e_j$$

with $f = id : D(A) \oplus Ae_j \rightarrow D(A)e_j$. Since $A$ is a symmetric algebra, every indecomposable projective-injective module in $\text{Mod} \hat{A}$ is of the form

$$\cdots 0 \sim Ae_j \sim^{1 \oplus e_j} Ae_j \sim 0 \cdots .$$

The indecomposable projective-injective (associated with the exact structure) objects in $C(\text{Inj} A)$ are of form

$$0 \rightarrow Ae_j \xrightarrow{\text{id}} Ae_j \rightarrow 0 ,$$

for some $j$. Thus there is a natural bijection induced by the embedding $\Psi : C(\text{Inj} A) \rightarrow \text{Mod} \hat{A}$ between the indecomposable projective-injective modules in $\text{Mod} \hat{A}$ and $C(\text{Inj} A)$. [□]

By the lemma, we have that the exact embedding functor $\Psi : C(\text{Inj} A) \rightarrow \text{Mod} \hat{A}$ induces an additive functor $\Phi : K(\text{Inj} A) \rightarrow \text{Mod} \hat{A}$, moreover, $\Phi$ is a triangle functor.

Proposition 3.7. If $A$ is a symmetric algebra, then there is a fully faithful triangle functor $\Phi : K(\text{Inj} A) \rightarrow \text{Mod} \hat{A}$ induced by the exact embedding $\Psi : C(\text{Inj} A) \rightarrow \text{Mod} \hat{A}$.

Proof. Firstly, the embedding preserves projective-injective objects by Lemma 3.6. We only need to show that there exists an invertible natural transformation $\alpha : \Psi \Sigma \rightarrow \Sigma \Psi$ by [16, Chapter 1, Lemma 2.8]. For any object $X \in C(\text{Inj} A)$, there is a exact sequence $0 \rightarrow X \rightarrow I(X) \rightarrow \Sigma X \rightarrow 0$. By the fact $\Psi(X) \cong X$ and $\Psi(I(X)) \cong I(\Psi(X))$, it is natural that $\Psi \Sigma X \rightarrow \Sigma \Psi(X)$. [□]

When we consider the algebra $\Lambda = k[x]/(x^2)$, the embedding $\Psi : C(\text{Inj} \Lambda) \rightarrow \text{Mod} \hat{A}$ induces a fully faithful triangle functor $\Phi : K(\text{Inj} \Lambda) \rightarrow \text{Mod} \hat{A}$, since it is a symmetric algebra. In the following, we will show that the relation between indecomposable objects of $K(\text{Inj} \Lambda)$ and radical square zero representations of quiver $Q$ of $\hat{A}$, and determine all indecomposable objects in $K(\text{Inj} \Lambda)$.

The algebra $\Lambda$ is the path algebra of the quiver $Q = \circ \xrightarrow{\alpha} \circ$ with the relation $\alpha^2 = 0$. The quiver of the repetitive algebra $\hat{\Lambda}$ of $\Lambda$ is $Q$

$$\cdots \xrightarrow{\beta} \circ \xrightarrow{\alpha} \circ \xrightarrow{\beta} \circ \xrightarrow{\alpha} \cdots$$
with relations $\alpha^2 = 0 = \beta^2, \alpha \beta = \beta \alpha$.

**Proposition 3.8.** Let $\Lambda = k[x]/(x^2)$, as above, $\check{Q}$ be the quiver of the repetitive algebra $\hat{\Lambda}$. The image of indecomposable objects in $K(\text{Inj} \, \Lambda)$ under $\Phi$ can be expressed as radical square zero representations of $\check{Q}$.

**Proof.** The objects $X$ in $K(\text{Inj} \, \Lambda)$ are of form,

$$
\cdots \longrightarrow \Lambda^{m-1} \overset{d_{m-1}}{\longrightarrow} \Lambda^{m_0} \overset{d^0}{\longrightarrow} \Lambda^{m_1} \overset{d^1}{\longrightarrow} \Lambda^{m_2} \overset{d^2}{\longrightarrow} \cdots
$$

where all $d^i \in \text{Hom}_\Lambda(\Lambda^{m_i}, \Lambda^{m_{i+1}})$ and satisfy $d^{i+1} \cdot d^i = 0$. The differential $d^i$ can be expressed as a $m_{i+1} \times m_i$ matrix $(d^i_{jk})$ with entries in $\text{Hom}_\Lambda(\Lambda, \Lambda)$ if $m_{i+1}, m_i$ are both finite. We have that $\dim \text{Hom}_\Lambda(\Lambda, \Lambda) = 2$, so we can choose a basis $\{1, x\}$ of $\text{Hom}_\Lambda(\Lambda, \Lambda)$.

In particular, if the complex $X \in K(\text{Inj} \, \Lambda)$ is indecomposable, we can choose that every entry $d^i_{jk}$ of the matrix $(d^i_{jk})$ associated to $d^i$ is in $\text{Rad}(\Lambda, \Lambda)$. Assume that there is a component of morphism $d^i_{jk} : \Lambda \to \Lambda$ with $d^i_{jk} \not\in \text{Rad}(\Lambda, \Lambda) \subset \text{Hom}_\Lambda(\Lambda, \Lambda)$. Without loss of generality, let $d^i_{jk} = 1_\Lambda$. Consider the following morphisms of complexes in $K(\text{Inj} \, \Lambda)$

$$
\cdots \longrightarrow \Lambda^{m-1} \overset{d_{m-1}}{\longrightarrow} \Lambda^{m_0} \overset{d^0}{\longrightarrow} \Lambda^{m_1} \overset{d^1}{\longrightarrow} \Lambda^{m_2} \overset{d^2}{\longrightarrow} \cdots,
\begin{array}{ccc}
g : 0 & \longrightarrow & \Lambda \\
g_0 : 0 & \longrightarrow & \Lambda \\
g_1 : 0 & \longrightarrow & \Lambda \\
f : 0 & \longrightarrow & \Lambda \\
f_0 : 0 & \longrightarrow & \Lambda \\
f_1 : 0 & \longrightarrow & \Lambda \\
\end{array}
$$

where $g_0$ is the $k$-th row of $d^0$, $g_1$ is the canonical projection on the $j$-th component, $f_0$ is the embedding to the $k$-th component and $f_1$ is the $j$-th column of $d^1$. We can check that the morphism $fg : X \to X$ is idempotent, and $gf = id_X$. Thus $fg$ splits in $K(\text{Inj} \, \Lambda)$ since $K(\text{Inj} \, \Lambda)$ is idempotent complete. That means the complex $X$ has a direct summand of form $Y$ which is null-homotopic.

We know that $\Lambda$ as a $\Lambda$-module corresponds the quiver representation $k^2 \bigcup \begin{array}{c}
\begin{array}{c}
0 1 \\
0 0
\end{array}
\end{array}$, thus we assign the homomorphism $x$ to the morphism of representations

$$
\begin{array}{c}
\begin{array}{c}
(0 1) \\
(0 0)
\end{array}
\end{array} : k^2 \longrightarrow \begin{array}{c}
\begin{array}{c}
(0 1) \\
(0 0)
\end{array}
\end{array} : k^2.
$$

Under the embedding functor $\Phi : K(\text{Inj} \, \Lambda) \to \text{Mod} \, \hat{\Lambda}$ as in Proposition [37] the complex $0 \longrightarrow \Lambda \longrightarrow \Lambda \longrightarrow 0$ corresponds to the following representation of $\hat{\Lambda}$

$$
\begin{array}{c}
\begin{array}{c}
(0 1) \\
(0 0)
\end{array}
\end{array} : k^2 \longrightarrow \begin{array}{c}
\begin{array}{c}
(0 1) \\
(0 0)
\end{array}
\end{array} : k^2 ;
$$

which is a projective-injective $\hat{\Lambda}$-module. Let $\hat{\Lambda}$ be the factor algebra of $\hat{\Lambda}$ modulo its socle [28]. The algebra $\hat{\Lambda}$ has quiver $\check{Q}$ and with relations $\alpha^2 = 0 = \beta^2$ and $\alpha \beta = 0 = \beta \alpha$. Every indecomposable $\hat{\Lambda}$-module without projective summand could be expressed as an indecomposable $\hat{\Lambda}$-module. Thus for any indecomposable
complex

\[ X' : \ldots \to \Lambda^{m_0} \xrightarrow{d'} \Lambda^{m_1} \xrightarrow{d^1} \Lambda^{m_2} \xrightarrow{d^2} \ldots \]

\[ \Phi(X') \in \text{Mod } \hat{\Lambda} \] could be expressed as an indecomposable \( \hat{\Lambda} \)-module. It naturally corresponds to a radical square zero representation of \( \hat{\mathcal{Q}} \).

From the quiver \( \hat{\mathcal{Q}} \) of \( \hat{\Lambda} \), we know that the separated quiver \( \hat{\mathcal{Q}}^\circ \) is of type \( A_\infty^{\infty} \) with the following orientation

\[ \ldots \to V' \to 1 \to 2' \leftarrow 2 \to \ldots \]

We denote this quiver by \( A_\infty^{\infty} \).

The representations of \( A_\infty^{\infty} \) are known for experts. For the convenience of readers, we summarize the result in the following proposition.

**Proposition 3.9.** Any indecomposable representation of \( A_\infty^{\infty} \) over an algebraically closed field \( k \) is thin. More precisely, all indecomposable representations are of the form \( k_{ab}, a, b \in \mathbb{Z} \cup \{ +\infty, -\infty \} \), where

\[ k_{ab}(i) = \begin{cases} k & \text{if } a \leq i \leq b, \\ 0 & \text{otherwise.} \end{cases} \]

and

\[ k_{ab}(\alpha) = \begin{cases} \text{id}_k & \text{if } a \leq s(\alpha), t(\alpha) \leq b, \\ \text{0} & \text{otherwise.} \end{cases} \]

**Proof.** First, we have that any \( k_{ab} \) is indecomposable. If \( a=b \), obviously \( k_{aa} \) is a simple representation and indecomposable, denoted by \( k_a \).

If \( a \neq b \), we assume that \( a < b \). If \( k_{ab} = V \oplus V' \), where \( V \) and \( V' \) are nonzero. Then \( (\text{supp}V)_0 \cap (\text{supp}V')_0 = \emptyset \). There exists a vertex \( i \in (\text{supp}V)_0 \), and \( i+1 \) or \( i-1 \in (\text{supp}V')_0 \). We have that \( V_i = k, V_{i+1} = 0, V'(i) = 0, V'_{i+1} = k \), and the map \( V_i \oplus V'_i \to V_{i+1} \oplus V'_{i+1} \) is a identity. It is a contradiction.

Second, we need to show any indecomposable representation of \( k_{ab} \). Suppose we have a indecomposable representation \( V = (V_i, f_i) \) of \( A_\infty^{\infty} \). If \( |(\text{supp}V)_0| = 1 \), then \( V \) is the direct sum of simple modules. Since \( V \) is indecomposable, we have that \( V \cong k_a \) for some \( a \in \mathbb{Z} \).

If \( |(\text{supp}V)_0| > 1 \), then \( \text{supp}V \) is a connected. Suppose \( V \) has the following form

\[ \ldots \to V_{-2} \xrightarrow{f_{-2}} V_{-1} \xrightarrow{f_{-1}} V_0 \xrightarrow{f_0} V_1 \xrightarrow{f_1} V_2 \xrightarrow{f_2} V_3 \xrightarrow{f_3} \ldots \]

Now if all non-zero \( f_i \) are bijections, then \( V \) has the form \( k_{ab} \) for some \( a, b \in \mathbb{Z} \cup \{ +\infty, -\infty \} \). Since we have the following isomorphism of representations,

\[ \ldots \to V_{-2} \xrightarrow{f_{-2}} V_{-1} \xrightarrow{f_{-1}} V_0 \xrightarrow{f_0} V_1 \xrightarrow{f_1} V_2 \xrightarrow{f_2} V_3 \xrightarrow{f_3} \ldots \]

Thus the lower representation can be decomposed as copies of the corresponding \( k_{ab} \), and \( V \) must be isomorphic to \( k_{ab} \).

Actually, all non-zero \( f_i \) are bijection. If there exists some \( f_i \) which is not an isomorphism. Without loss generality, we assume that \( f_0 \) is not injective. In this case, \( V_1 \) has a decomposition \( V_1 = \text{Im } f_0 \oplus \ker f_0 \) and \( \ker f_0 \neq 0 \). We can choose a
basis \(\{e_i\}_{i \in I}\) of \(\text{Ker} f_0\) and \(\{e_j'\}_{j \in J}\) of \(\text{Im} f_0\) such that \(\{e_i\}_{i \in I} \cup \{e_j'\}_{j \in J}\) is a basis of \(V_1\) and \(f_1(\text{Im} f_0) \cap f_1(\text{Ker} f_0) = 0\) \((f_1\) is not zero, otherwise there is a non-zero direct summand \(\text{Ker} f\) of \(V\)).

Consider the map \(f_1 : \text{Im} f_0 \oplus \text{Ker} f_0 \to V_2\). If \(f_1 \neq 0\), we get \(\text{Ker} f_0 \cap f_1 = 0\), otherwise the intersection will be a non-zero direct summand of \(V\) with support containing only one vertex. We denote \((f_1(\text{Ker} f_0)) = V_2'\), the subspace spanned by the vectors in \(f_1(\text{Ker} f_0)\) and the complement of \(V_2'\) in \(V_2\) is \(V_2''\). The map \(f_1\) can be expressed as

\[
\begin{pmatrix} f_1 & f_{12} \\ 0 & f'' \end{pmatrix} : \text{Ker} f_0 \oplus \text{Im} f_0 \to V_2' \oplus V_2''
\]

If \(f_{12} = 0\), then we already have a decomposition. If \(f_{12} \neq 0\), then we can choose a suitable basis of \(V_2\) such that \(f_{12} = 0\). In precisely, If \(f_{12}(e'_j) = \sum_{i \in I'} f_1(e_i) \in V_2'\), then replace \(e'_j\) by \(e'_j - \sum_{i \in I'} f_1(e_i)\), and we have \(f_{12}(e'_j) = 0\). Repeat this procedure, we get a new basis of \(\text{Im} f_0\) such that \(f_1\) can be expressed as

\[
\begin{pmatrix} f_1 & 0 \\ 0 & f'' \end{pmatrix} : \text{Ker} f_0 \oplus \text{Im} f_0 \to V_2' \oplus V_2''
\]

We can choose the corresponding basis of \(V_2\) such that \(V_2 \cong V_2' \oplus V_2''\), and \(f_1(\text{Im} f_0) \cong V_2', \ f_1(\text{Ker} f_0) \cong V_2''\). Thus we have the following representation isomorphism

\[
odmorphism V_0 \leftarrow f_0 \rightarrow V_1 \leftarrow f_1 \rightarrow V_2 \leftarrow f_2 \rightarrow V_3 \leftarrow \ldots
\]

\[
odmorphism \text{Im} f_0 \oplus \text{Coker} f_0 \leftarrow f_{12} \rightarrow \text{Im} f_0 \oplus \text{Ker} f_0 \rightarrow V_2' \oplus V_2'' \leftarrow f_2 \rightarrow V_3 \leftarrow \ldots
\]

Now we have a nonzero direct summand of \(V\) as follows,

\[
odmorphism 0 \rightarrow \text{Ker} f_0 \rightarrow V_2'' \rightarrow \ldots
\]

It is contradict with that \(V\) is indecomposable.

Use the same argument we can show that if \(f_0\) is not surjective, then there also is an nonzero direct summand of \(V\).

Given a representation of \(Q\), we can decompose it as direct sum of indecomposable representations by the procedure and each indecomposable representation has endomorphism ring \(k\). By Krull-Schmidt-Azumaya Theorem [1], this decomposition is unique. □

**Corollary 3.10.** Let \(\Lambda = k[x]/(x^2)\), \(Q\) be the quiver of \(\Lambda\), and \(\hat{\Lambda}\) be the quiver of \(\Lambda\). Then every indecomposable object in \(K(\text{Inj} \Lambda)\) corresponds to an indecomposable representation of \(Q^*\).

**Proof.** By Proposition 3.8 the functor \(\Phi\) sends every indecomposable object in \(K(\text{Inj} \Lambda)\) to a radical square zero representation of \(\hat{Q}\). From Proposition 2.9 there exists a bijection between radical square zero representations of \(\hat{Q}\) and separated representations of \(\hat{Q}^* = \Lambda^*_\infty\). □

We denote by \(I^*_\Lambda\) the following acyclic complex in \(K(\text{Inj} \Lambda)\)

\[
odmorphism \ldots \rightarrow \Lambda \xrightarrow{x} \ldots \rightarrow \Lambda \xrightarrow{x} \ldots \rightarrow \Lambda \xrightarrow{x} \ldots
\]

For any \(m, n \in \mathbb{Z} \cup \{+\infty, -\infty\}, n \geq m\), we denote the truncation \(\sigma_{\leq m, \sigma_{\geq n}} I^*_\Lambda\) by \(I^*_\Lambda_{m,n}\).

Now we give the main result in this part.
Proposition 3.11. Let $\Lambda = k[x]/(x^2)$, then every indecomposable object in $K(\text{Inj} \Lambda)$ is of the form $I_{m,n}^\Lambda$.

Proof. For an indecomposable object $X \in K(\text{Inj} \Lambda)$, we consider the embedding functor $\Phi : K(\text{Inj} \Lambda) \to \text{Mod} \Lambda$ as in Proposition 3.7. We have that $\Phi(X)$ is an indecomposable radical square zero $\Lambda$-module by Proposition 3.8. By Corollary 3.13, every indecomposable object in $K(\text{Inj} \Lambda)$ corresponds to an indecomposable representation of $Q^\infty = A_\infty$. From Proposition 2.9 and 3.9, the dimension of $\text{Hom}_\Lambda(\Lambda e[i], \Phi(X))$ over $k$ is at most 2, where $\{e[i]\}_{i \in \mathbb{Z}}$ are all primitive idempotents of $\Lambda$. Thus, we only have choices $\Lambda$ and 0 for each $X^i$. $X$ is of the form $I_{m,n}^\Lambda$, since it is indecomposable. □

3.2. Proof of the main result. Covering theory has many applications in representation theory of finite dimensional $k$-algebras. Now, we use the covering technique to classify all indecomposable objects of $K(\text{Inj} \Lambda)$ for radical square zero self-injective algebras $A$.

Let $\pi : (Q', I') \to (Q, I)$ be the covering of bounded quiver $(Q, I)$. $\pi$ can be extended to a surjective homomorphism of algebra $\pi : kQ'/I' \to kQ/I$. We set that $A' = kQ'/I'$ and $A = kQ/I$. This induces the pushdown functor $F_\lambda : \text{Mod} A' \to \text{Mod} A$.

In general, let $G$ be a group, and $A = \oplus_{g \in G} A_g$ be a $G$-graded algebra. We define the covering algebra $\tilde{A}$ associated to the $G$-grading as follows [3, 26, 14]: $\tilde{A}$ is the $G \times G$ matrices $(a_{g,h})$, where $a_{g,h} \in A_{gh^{-1}}$ and all but a finite number of $a_{g,h}$ are 0. Then $\tilde{A}$ is a ring via matrix multiplication and addition. Set $\mathcal{E} = \{e_g\}_{g \in G}$, where $e_g$ is the matrix with 1 in the $(g,g)$-entry and 0 in all other entries.

Proposition 3.12. Let $G$ be a group and $A = \oplus_{g \in G} A_g$ is a $G$-graded algebra. The covering algebra of $A$ associated to the $G$-grading is $\tilde{A}$. Then $\tilde{A}$ is a locally bounded $k$-algebra. Moreover, the category of finitely generated graded $A$-modules $\text{mod}_G A$ is equivalent to $\text{mod} \tilde{A}$.

The forgetful functor $F_\lambda : \text{mod}_G A \to \text{mod} A$ is the functor sending $X$ to $X$, viewed as an $A$-module. This functor is exactly the pushdown functor $F_\lambda : \text{mod} \tilde{A} \to \text{mod} A$. The functor $F_\lambda$ is exact [13, Proposition 2.7]. By the exactness of functor $F_\lambda$, we have the induced functor $F_\lambda : D^b(\text{mod} \tilde{A}) \to D^b(\text{mod} A)$ between the corresponding derived categories.

Lemma 3.13. Let $G$ be a group and $\tilde{A}$ be the covering algebra associated to a $G$-graded algebra $A$. Then the forgetful functor $F_\lambda : \text{mod} \tilde{A} \to \text{mod} A$ induces a triangle functor $F_\lambda : D^b(\text{mod} \tilde{A}) \to D^b(\text{mod} A)$ such that the following diagram commutes,

\[
\begin{array}{ccc}
\text{mod} \tilde{A} & \xrightarrow{\text{can}} & D^b(\text{mod} \tilde{A}) \\
F_\lambda \downarrow & & \downarrow F_\lambda \\
\text{mod} A & \xrightarrow{\text{can}} & D^b(\text{mod} A)
\end{array}
\]

where the functor can : $\text{mod} A \to D^b(\text{mod} A)$ is the canonical embedding functor.

Now, assume that the group $G$ is finite, $A$ is a $G$-graded algebra and the induced action of $G$ on $Q_A$ is free, $B$ is the covering algebra associated to the group $G$. Then the covering functor $B \to A$ induces a covering functor between the corresponding repetitive algebras $\tilde{B} \to \tilde{A}$. The forgetful functor $F_\lambda : \text{mod} \tilde{B} \to \text{mod} A$ induces a
functor $F_\lambda : \text{mod} \hat{B} \to \text{mod} \hat{A}$. Consider the full embedding $i_\Lambda : \text{mod} \Lambda \to \text{mod} \hat{A}$ for any algebra $\Lambda$, we have $i_\Lambda \cdot F_\lambda = F_\lambda \cdot i_B$.

Given an algebra $A$, there is the Happel functor $F_A : D^b(\text{mod} A) \to \text{mod} \hat{A}$ embedding the derived category to the stable module category of the repetitive algebra. For the covering of algebras $B \to A$ with group $G$, using induction on the length of complexes in $D^b(\text{mod} B)$, we get $F_\lambda \cdot F^B = F_A \cdot F_\lambda$, i.e the following diagram is commutative

\[
\begin{array}{ccc}
D^b(\text{mod} B) & \xrightarrow{F^B} & \text{mod} \hat{B} \\
\downarrow F_\lambda & & \downarrow F_\lambda \\
D^b(\text{mod} A) & \xrightarrow{F_A} & \text{mod} \hat{A}.
\end{array}
\]

The following result is a consequence of [5, Lemma 4.5] or follows directly from Lemma 2.4.

**Lemma 3.14.** Let $F, G : \mathcal{T} \to \mathcal{S}$ be two triangle functors preserving coproducts between two $k$-linear compactly generated triangulated categories. If there is a natural isomorphism $F(X) \cong G(X)$ for any compact object $X \in \mathcal{T}$, then $F \cong G$.

We summarize the above construction. For an algebra $A$, Let $F^A : K(\text{Inj} A) \to \text{Mod} \hat{A}$ be the functor constructed in Proposition 2.7 and the functor extends the Happel’s functor $F^A : D^b(\text{mod} A) \to \text{mod} \hat{A}$. If $\pi : B \to A$ is the covering of algebra with group $G$, then there is a covering functor $F_\lambda : \text{Mod} B \to \text{Mod} A$. In this case, the functor $F_\lambda$ induces a functor $K(\text{Mod} B) \to K(\text{Mod} A)$ which preserves injective modules. Thus $F_\lambda$ induces a functor denoted by $F_\lambda : K(\text{Inj} B) \to K(\text{Inj} A)$.

**Proposition 3.15.** Let $G$ be a finite group, $A$ be a $G$-graded algebra and $B$ be the covering algebra of $A$ associated to $G$. Let the functors $F^A, F^B$ and $F_\lambda$ be as above. Then we have that $F^A \cdot F_\lambda \cong F_\lambda \cdot F^B$, i.e the following commutative diagram

\[
\begin{array}{ccc}
K(\text{Inj} B) & \xrightarrow{F^B} & \text{Mod} \hat{B} \\
\downarrow F_\lambda & & \downarrow F_\lambda \\
K(\text{Inj} A) & \xrightarrow{F^A} & \text{Mod} \hat{A}.
\end{array}
\]

**Proof.** Let $L_1 = F^A \cdot F_\lambda$ and $L_2 = F_\lambda \cdot F^B$ be two exact functors from $K(\text{Inj} B)$ to $	ext{Mod} \hat{A}$.

The given diagram restricted to the subcategory of compact objects is commutative, i.e $L_1(X) \cong L_2(X)$ for all compact object $X \in K^c(\text{Inj} B)$ by above. We have that

\[
\text{Im} L_2|_{K(\text{Inj} B)} \equiv \text{Im} L_1|_{K(\text{Inj} B)} \subset \text{Mod} \hat{A}.
\]

By Lemma 3.14, we only need to show $L_1$ and $L_2$ preserves coproducts. By the properties of functors $F_\lambda$ and $F$, we know that $L_1$ and $L_2$ preserves coproducts. □

Consider the quiver $C_n$ as the beginning of this part. There is a quiver morphism $\pi : (C_n, I_n) \to (C_1, I_1)$ by $\pi(i) = 0, i \in [1, n]$, and $\pi(\alpha) = \alpha$. In order to understand $A_n$-modules, we need to study the representations of the quiver $\hat{C}_n$.

**Proposition 3.16.** Consider the covering of bounded quivers $\pi : (C_n, I_n) \to (C_1, I_1)$, let $(\hat{C}_n, \hat{I}_n)$ and $(\hat{C}_1, \hat{I}_1)$ be the repetitive quivers of $C_n, I_n$ and $(C_1, I_1)$ respectively. Then $(\hat{C}_n, \hat{I}_n) \to (\hat{C}_1, \hat{I}_1)$ is a covering of bounded quivers with group $G = \mathbb{Z}_n$. 
The repetitive quiver $\tilde{C}_n$ of $C_n$ is given by the following infinite quiver:

We label all the original arrows in quiver $C_n$ as $\alpha$ and all connecting arrows (i.e. dash arrows) as $\beta$. The relation $I_n$ is generated by all the possible $\alpha^2, \beta^2$ and $\alpha \beta = \beta \alpha$. The path algebra of quiver $(\tilde{C}_n, I_n)$ is the repetitive algebra $\tilde{A}_n$ of $A_n$.

Indecomposable objects in $\text{Mod} \tilde{A}_n$ can be expressed as the indecomposable modules of $\tilde{A}_n$, which is the factor algebra of $\tilde{A}_n$ modulo its socle $[28][10]$. More precisely, the algebra $\tilde{A}_n$ is given by the quiver $\tilde{C}_n$ and relations $\alpha^2 = 0 = \beta^2$ and $\alpha \beta = \beta \alpha = 0$, which is radical square zero.

**Proposition 3.17.** Let $A_n = kC_n/I_n$ be the algebra as above. Every indecomposable object in $K(\text{Inj} A_n)$ corresponds to an indecomposable module of $\tilde{A}_n$.

**Proof.** The fully faithful embedding $F : K(\text{Inj} A_n) \to \text{Mod} \tilde{A}_n$ identifies $K(\text{Inj} A_n)$ with a localizing subcategory of $\text{Mod} \tilde{A}_n$. Thus every indecomposable object can be viewed as an indecomposable object in $\text{Mod} \tilde{A}_n$ under the functor $F$. It is suffice to consider the indecomposable objects in $\text{Mod} \tilde{A}_n$. We know that two $\tilde{A}_n$-modules $M, N$, are isomorphic in $\text{Mod} \tilde{A}_n$ if and only if there exist projective-injective $\tilde{A}_n$-modules $P, Q$ such that $M \oplus P \cong N \oplus Q$. Furthermore, the indecomposable $\tilde{A}_n$-modules are just the non-projective indecomposable $\tilde{A}_n$-modules. It follows that indecomposable objects in $\text{Mod} \tilde{A}_n$ corresponds to indecomposable modules of $\tilde{A}_n$.

**Lemma 3.18.** Let $F_\lambda : \text{Mod} \tilde{A}_n \to \text{Mod} \tilde{A}_1$ be the forgetful functor induced by the covering of bounded quivers $\pi : (C_n, I_n) \to (C_1, I_1)$. If $X$ is an indecomposable module in $\text{Mod} \tilde{A}_n$, then the module $Y = F_\lambda X$ is indecomposable in $\text{Mod} \tilde{A}_1$.

**Proof.** Since $\tilde{A}_n$ is a radical square zero algebra, therefore there is an bijection between indecomposable modules in $\text{Mod} \tilde{A}_n$ and indecomposable modules in $\text{Mod} \tilde{A}_n^n$, where $\tilde{A}_n^n$ is the separated algebra of $\tilde{A}_n$ by Proposition 2.8. The separated quiver $\tilde{C}_n^n$ of $(\tilde{C}_n, I_n)$ is just the union of $n$-copies of quiver $A_\infty^n$. Every indecomposable
representation of $\hat{C}_{n}^{s}$ is an indecomposable representation of $A_{c}^{\infty}$, which corresponds to an indecomposable module in $\text{Mod} \hat{A}_{1}$ by Corollary 3.10.

Proposition 3.19. The pushdown functor $F_{\lambda} : K(\text{Inj} A_{n}) \to K(\text{Inj} A_{1})$ preserves indecomposable objects.

Proof. By Proposition 3.15 we have $F_{\lambda}F^{A_{n}} \cong F^{A_{1}}F_{\lambda} : K(\text{Inj} A_{n}) \to \text{Mod} \hat{A}_{1}$. Assume that $X \in K(\text{Inj} A_{n})$ is indecomposable and $Y = F_{\lambda}(X) = Y_{1} \oplus Y_{2}$ is a decomposition of $Y \in K(\text{Inj} A_{n})$ with $Y_{i} \neq 0$ for $i = 1, 2$. We have that $F_{\lambda}F^{A_{n}}(X) \in \text{Mod} \hat{A}_{1}$ is indecomposable by Proposition 3.17 and Lemma 3.18. On the other hand, $F_{\lambda}F^{A_{n}}(X) \cong F^{A_{1}}F_{\lambda}(X) = F^{A_{1}}(Y_{1} \oplus Y_{2})$ is decomposable in $\text{Mod} \hat{A}_{1}$. This is a contradiction.

Now, we can give a proof of the main result.

Proof of Theorem 3.2. If $X \in K(\text{Inj} A_{n})$ with $X^{i} = \oplus_{k}I_{k}$ for some $i \in Z$, then $F_{\lambda}(X) \in K(\text{Inj} A_{1})$ with $F_{\lambda}(X)^{i}$ being $\oplus_{k}A_{1}$. Therefore $F_{\lambda}(X)$ is decomposable in $K(\text{Inj} A_{1})$ by Proposition 3.11. From Proposition 3.19 this implies that $X$ is decomposable in $K(\text{Inj} A_{n})$. Thus $X \in K(\text{Inj} A_{n})$ is indecomposable, we have that $F_{\lambda}(X^{i})$ is either $A_{1}$ or 0 for any $i \in Z$. It follows that $X^{i}$ is either $I_{k}$ or 0 for some $1 \leq k \leq n$ and any $i \in Z$.

Consider homomorphisms between indecomposable injective $A_{n}$-modules $I_{j}$ for $1 \leq j \leq n$, we have that

$$
\text{Hom}_{A_{1}}(I_{j}, I_{k}) = \begin{cases} 
(id_{I_{j}}) & \text{if } j = k; \\
(d_{j}) & \text{if } j + 1 = k; \text{ or } j = n, k = 1; \\
0 & \text{otherwise},
\end{cases}
$$

where $d_{j} : I_{j} \to I_{j-1}$ is the composition of $I_{j} \to I_{j}/\text{soc} I_{j} \cong \text{rad} I_{j-1} \hookrightarrow I_{j-1}$.

From this, the periodic complex $I^{\bullet}$ defined by the following with $I_{j}$ in the degree 0,

$$
\ldots I_{n} \xrightarrow{d_{n}} I_{n-1} \xrightarrow{d_{n-1}} I_{n-2} \xrightarrow{d_{n-2}} \ldots \xrightarrow{d_{1}} I_{1} \xrightarrow{d_{1}} I_{0} \xrightarrow{d_{0}} \ldots
$$

and its shifts $I^{\bullet}[r]$ for $r \in Z$ are exactly the unbounded indecomposable complexes in $K(\text{Inj} A_{n})$. Moreover, $I^{\bullet}[r] = I^{\bullet}[r + n]$ for $r \in Z$.

Assume that $X \in K(\text{Inj} A_{1})$ is indecomposable and bounded below but not above, i.e. there exists $l \in Z$ such that $X^{p} \neq 0$ for all $p \geq l$ and $X^{q} = 0$ for all $q < l$. Without loss of generality, let $X^{l} = I_{1}$, we have that $X = \sigma_{\geq l}I^{\bullet}$. In general, $X$ is of the form $\sigma_{\geq l}(I^{\bullet}[r])$, where $0 \leq r \leq n-1$, denoted by $I^{r, +\infty}$. Similarly, if $X \in K(\text{Inj} A_{n})$ is indecomposable and bounded above but not below, then $X = \sigma_{\leq m}(I^{\bullet}[r])$ where $0 \leq r \leq n-1$, denoted by $I_{-\infty, m}$.

Assume that $X \in K(\text{Inj} A_{n})$ is indecomposable and bounded. It follows that $X^{i} \neq 0$ if and only if $l \leq i \leq m$ for some $l, m \in Z$. Without loss of generality, let $X^{l} = I_{1}$, we have that $X = \sigma_{\leq m}\sigma_{\geq l}(I^{\bullet}[r])$. In general, $X$ is of the form $I_{l, m}[r] = \sigma_{\leq m}\sigma_{\geq l}(I^{\bullet}[r])$, where $0 \leq r \leq n-1$.

4. The Ziegler spectrum of $K(\text{Inj} A)$

4.1. Ziegler spectrum of triangulated categories. Let $\text{Mod} T^{c}$ be the category consisting of all contravariant additive functor from $T^{c}$ to $\text{Ab}$, the category of all abelian groups. It is a locally coherent Grothendieck category [17]. Moreover, the objects of form $\text{Hom}(-, X)$ for $X \in T^{c}$ are projective objects. The subcategory
mod $\mathcal{T}^c$ consists of all finitely presented objects of $\text{Mod}\,\mathcal{T}^c$ is abelian category. The Yoneda embedding

$$h_\mathcal{T} : \mathcal{T} \to \text{Mod}\,\mathcal{T}^c, X \mapsto H_X = \text{Hom}(-, X)|_{\mathcal{T}^c}$$

sends every object $X \in \mathcal{T}$ to an object $H_X$ in the abelian category $\text{Mod}\,\mathcal{T}^c$.

The following result characterizes injective objects in the abelian category $\text{Mod}\,\mathcal{T}^c$.

Proposition 4.1. Let $\mathcal{T}$ be a compactly generated triangulated category, the following are equivalent,

1. $H_X = \text{Hom}(-, X)|_{\mathcal{T}^c}$ is injective in $\text{Mod}\,\mathcal{T}^c$.
2. For every set $I$, the summation map $X(I) \to X$ factors through the canonical map $X(I) \to X^I$ from the coproduct to the product.
3. The map $\text{Hom}(Y, X) \to \text{Hom}(H_Y, H_X), \phi \mapsto \text{Hom}(-, \phi)|_{\mathcal{T}^c}$, is an isomorphism for all $Y \in \mathcal{T}$.

An object $X$ in $\mathcal{T}$ is called pure-injective if it satisfies the above equivalent conditions. There is a class of special pure injective objects, endofinite object, which are analogues of endofinite modules [9].

Definition 4.2. Let $\mathcal{T}$ be a compactly generated triangulated category. An object $E \in \mathcal{T}$ is endofinite if the $\text{End}_{\mathcal{T}} E$-module $\text{Hom}(X, E)$ has finite length for any $X \in \mathcal{T}^c$.

Endofinite objects of triangulated category have very nice decomposition properties, which can be seen in the following result [20, Proposition 1.2].

Proposition 4.3. Let $\mathcal{T}$ be a compactly generated triangulated category. An endofinite object $X \in \mathcal{T}$ has a decomposition $X = \bigsqcup_i X_i$ into indecomposable objects with $\text{End}_{\mathcal{T}} X_i$ is local, and the decomposition is unique up to isomorphism.

An triangulated category $\mathcal{T}$ is called pure semisimple if every object is pure injective. Analogue of [4, Theorem 12.20], we have the following result.

Proposition 4.4. For a artin algebra $A$, $K(\text{Inj}\,A)$ is pure semisimple if and only if $A$ is derived equivalent to a hereditary algebra of Dynkin type.

Proof. The inclusion $i : K^b(\text{proj}\,A) \to D^b(\text{mod}\,A)$ induces a functor

$$i^* : \text{Mod}(D^b(\text{mod}\,A)) \to \text{Mod}(K^b(\text{proj}\,A)),$$

which has fully faithful right adjoint functor. $K(\text{Inj}\,A)$ is pure semisimple if and only if $\text{Mod}(D^b(\text{mod}\,A))$ is locally Noetherian [4, Theorem 9.3]. By [27, Chapter 5, Corollary 8.4], this implies $\text{Mod}(K^b(\text{proj}\,A))$ is locally Noetherian. In this case, $D(A)$ is pure semisimple, thus $A$ is derived equivalence of algebra of Dynkin type.

The Ziegler spectrum $Zg\,\mathcal{T}$ of $\mathcal{T}$ have its points as the indecomposable injective objects in $\text{Mod}\,\mathcal{T}^c$, coinciding with the indecomposable pure injective objects in $\mathcal{T}$ by Proposition 4.1. We will give a topological basis of $Zg\,\mathcal{T}$, which are determined by the finitely presented objects in $\text{Mod}\,\mathcal{T}^c$.

A functor $F : \mathcal{T} \to \text{Ab}$ is coherent if there exists an exact sequence $\mathcal{T}(X, -) \to \mathcal{T}(Y, -) \to F \to 0$, where $X, Y \in \mathcal{T}^c$. We denote $\text{coh}\,\mathcal{T}$ the collection of all coherent functors from $\mathcal{T}$ to Ab. It is an abelian category. There is an equivalence of categories [22, Lemma 7.2]

$$(\text{mod}\,\mathcal{T}^c)^{\text{op}} \xrightarrow{\approx} \text{coh}\,\mathcal{T}.$$
We can define the open sets of Ziegler spectrum of $\mathcal{T}$ by inducing from the Serre subcategories of $\text{coh}\mathcal{T}$ or the Serre subcategories of $\text{mod}\mathcal{T}^c$. For a Serre subcategory $S$ of $\text{coh}\mathcal{T}$, we define the set of the form

$$\mathcal{O}(S) = \{ X \in Zg\mathcal{T} | C(X) \neq 0, \forall C \in S \}$$

of subsets of $Zg\mathcal{T}$. The sets of this form satisfy the axioms of open subsets in space $Zg\mathcal{T}$.

**Lemma 4.5.** \cite{22}

1. The collection of subsets of $Zg\mathcal{T}$

$$\mathcal{O}(C) = \{ M \in Zg\mathcal{T} | C(M) \neq 0, C \in \text{coh}\mathcal{T} \}$$

with $C \in \text{coh}\mathcal{T}$ satisfies the axioms of open subsets of $Zg\mathcal{T}$.

2. The collection of open subsets of $Zg\mathcal{T}$

$$\mathcal{O}(C) = \{ M \in Zg\mathcal{T} | \text{Hom}(C, H_M) \neq 0, C \in \text{mod}\mathcal{T}^c \}$$

with $C \in \text{mod}\mathcal{T}^c$ is a basis of open subsets of $Zg\mathcal{T}$.

**4.2. The Ziegler spectrum of $K(\text{Inj} A)$.** Let $\mathcal{T}$ be a compactly generated triangulated category, for every coherent functor $C \in \text{coh}\mathcal{T}$, the subset $\mathcal{O}(C) = \{ M \in Zg\mathcal{T} | C(M) \neq 0 \}$ with $C \in \text{coh}\mathcal{T}$ is open in $Zg\mathcal{T}$. The corresponding closed subset is $I(C) = \{ M \in Zg\mathcal{T} | C(M) = 0 \}$. By the equivalence of $(\text{mod}\mathcal{T}^c)^{op} = \text{coh}\mathcal{T}$, the set

$$I(C) = \{ M \in Zg\mathcal{T} | \text{Hom}(C, H_M) = 0 \}, C \in \text{mod}\mathcal{T}^c$$

is also a closed set in $Zg\mathcal{T}$. We apply this to compute the open subsets of some Ziegler spectrum $Zg\mathcal{T}$. The subsets $\mathcal{O}(C)$ with $C \in \text{coh}\mathcal{T}$ form a basis of open subsets of $Zg\mathcal{T}$ by Corollary 4.5.

**Proposition 4.6.** Keep the notions in section 3. Then every indecomposable object $I_{l,m}[r]$ of $K(\text{Inj} A_n)$ is an endofinite object.

**Proof.** We need to calculate the Hom-space $\text{Hom}_{K(\text{Inj} A_n)}(C, I_{l,m}[r])$ for all $C \in K^c(\text{Inj} A_n)$. For indecomposable objects $I, I' \in K(\text{Inj} A_n)$,

$$\dim_k \text{Hom}_{K(\text{Inj} A_n)}(I', I) \leq 1$$

and $\text{Hom}_{K(\text{Inj} A_n)}(I, I) \cong k$. By the fact that $K^c(\text{Inj} A_n)$ is Krull-Schmidt $k$-linear triangulated category, we know that Hom-space $\dim_k \text{Hom}_{K(\text{Inj} A_n)}(C, I_{l,m}[r])$ is finite for all $C \in K^c(\text{Inj} A_n)$.

Every endofinite object in $K(\text{Inj} A)$ is pure injective, thus the Ziegler spectrum of $K(\text{Inj} A)$ is explicit.

**Corollary 4.7.** Let $A = kC_n/I_n$ for some $n \in \mathbb{N}^*$. Then the Ziegler spectrum $Zg(K(\text{Inj} A))$ consists of the point $[I_{m,n}[r]]$ for each indecomposable object $I_{m,n}[r] \in K(\text{Inj} A)$ up to isomorphism.

**Example 1.** If $\mathcal{T} = K(\text{Inj} A)$, where $A = k[x]/(x^2)$. It is known that $K^c(\text{Inj} A) \cong D^b(\text{mod} A)$. Denote $A_{m,n}, A_{-\infty, n}, A_{m, +\infty}$ be the complexes with $A$ in degree $n$ (resp. $-\infty, m$ to $n$ (resp. $n, +\infty$) and the differential is $d : A \rightarrow A, a \mapsto xa$. We know that the non zero morphism between complexes in $K^b(\text{Inj} A)$ is the linear combinations of the forms:
By the description of morphisms, every map $A_{m,n} \to A_{-\infty,+\infty}$ is 0 in $\mathcal{T}$, for any $m,n \in \mathbb{Z}$. Thus $A_{-\infty,+\infty} \notin \mathcal{O}(A_{m,n})$. The functor $H_{A_{m,n}} = \text{Hom}(-,A_{m,n})|_{\mathcal{T}^c} \in \text{mod } \mathcal{T}^c$, the open subset $\mathcal{O}(H_{A_{m,n}})$ consists of the functors represented by complexes $A_{m',n'} \cup \{A[n],A[m]\}$ where $m' < n'$ and $m \leq m' \leq n$ or $m \leq n' \leq m$. The union of open subsets $\mathcal{O} = \bigcup_{X \in \text{mod} (K^b(\text{inj} A))} \mathcal{O}(H_X)$ is open. We get that the complement of $\mathcal{O}$ has only one point $A_{+\infty,-\infty}$. The point $A_{+\infty,-\infty}$ is not an isolated point in $Zg(\mathcal{T})$ [12 Proposition 4.5]. Thus it is an accumulation point of the indecomposable objects in $K^b(\text{inj} A)$. It is also the accumulation point of the set $\bigcup_{i \in \mathbb{Z}} \mathcal{O}(A[i])$, thus it is an accumulation point of the set $\{A[i] : i \in \mathbb{Z}\}$.

Moreover, the Ziegler spectrum $Zg \mathcal{T}$ is not quasi-compact. Consider the object $H_{A_{n,+\infty}} \in \text{mod } \mathcal{T}^c$, the open subset is of the form

$$\mathcal{O}(H_{A_{n,+\infty}}) = \{A_{m,+\infty},A_{-\infty,m}(m \geq n),A_{m,r}(m \geq n r \geq n),A_{-\infty,+\infty}\}.$$ 

The family of open subsets $\{\mathcal{O}(H_{A_{n,+\infty}})\}_{n \in \mathbb{Z}}$ is an open covering of $Zg \mathcal{T}$. But it has no finite subcovering of $Zg \mathcal{T}$.

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Zhe Han, **School of Mathematics and Information Science, Henan University, Kaifeng 475001, China.**

Fakultät für Mathematik, Universität Bielefeld, D-33501 Bielefeld, Germany.

E-mail address: hanzhe0302@gmail.com