Zero-curvature condition in Calogero model

D. Karakhanyan,$^{a,b}$ Sh. Khachatryan$^a$

$^a$Yerevan Physics Institute  
2 Alikhanyan Brothers St., Yerevan 0036, Armenia

$^b$Yerevan State University  
1 Alex Manoogian Str., Yerevan, 0025, Armenia

Abstract

We consider the mutual commutativity of Dunkl operators of the rational Calogero model as zero-curvature condition and calculate the non-local operator, related to these flat connections. This operator has physical meaning of particular scattering matrix of Calogero model and maps the eigenfunctions of Dunkl operator to the wave function of $N$ free particles (plane waves).

1 Introduction

Quantum Calogero-Moser-Sutherland model describes the set of $N$ identical particles on a circle interacting pairwise with inverse square potential. Calogero-Moser-Sutherland model attracted some attention due to conformal character of interaction potential, it also used to test the ideas of fractional statistics [1], [2]. The new aspects of their algebraic structure and quantum integrability was later clarified by [4] and [5]. The Calogero Model [3] (see for review [4]) is a rare example of integrable many-body problem. The studying of the Hamiltonian $H_c$ was started by Calogero [3], who computed the spectrum, eigenfunctions and scattering states in the confined and free cases. The Perelomov [6] observed the complete quantum integrability of the model, he stated that exist $N$ commuting, algebraically independent operators and $H_c$ is one among them. The complete integrability of the classical Hamiltonian of Calogero-Moser-Sutherland was proved by Moser [7].

Unfortunately, in a series of integrable theories, Calogero model stands alone. A powerful Inverse Scattering Method [12] is applicable to it with considerable restrictions. The key object in ISM is $R$-matrix depending on spectral parameter. The $R$-matrix can associate to this model, but it is dynamic: its matrix elements are not c-numbers and depend on the coordinates. Moreover, the dependence on the spectral parameter can be obtained only for elliptical extension [13]. It makes us look for other ways to describe this model.

The Lax method being applied to the integrable model allows to trace its integrability to zero-curvature condition of $L - A$ pair. Namely the equations of motion of the model express the consistency condition of some (usually more wide) linear (free) system.

The Calogero model in external harmonic field

$$H_C = \sum_{i=1}^{N} \left( \frac{p_i^2}{2} + \frac{\omega^2 q_i^2}{2} \right) + \sum_{i<j} \frac{g_{ij}^2}{(q_i - q_j)^2}$$

(1.1)

$^1$e-mail: karakhan@yerphi.am
can be obtained by $SU(N)$ reduction from the matrix model

$$H = \frac{1}{2} \text{tr} P^2 + \frac{\omega^2}{2} \text{tr} Q^2$$

(1.2)

where $Q$ and $P$ are hermitean matrices:

$$\{P_{ij}, Q_{i'j'}\} = \delta_{ij'}\delta_{j'i}, \quad \{P_a, Q_b\} = \delta_{ab},$$

(1.3)

which is equivalent to the homogeneous $(N^2 - 1)$-dimensional oscillator.

Consider the Calogero matrix Hamiltonian without oscillator term $\omega = 0$. Then (1.2) tell us $H_0 = \frac{1}{2} \text{tr}(P^2)$. The canonical relations (1.2) are invariant under (canonical) similarity transformation:

$$P \rightarrow P' = U^{-1}PU, \quad Q \rightarrow Q' = U^{-1}QU,$$

with numerical unitary matrix $U = \exp i\varepsilon$. The Noether current corresponding to this transformation is $\text{tr}(P\delta Q)$ for infinitesimal $\varepsilon$ one has $\delta Q = i[Q, \varepsilon]$ and $\text{tr}(P\delta Q) = \text{tr}(i\varepsilon[P, Q])$.

So one deduces, that the Noether charges for this transformation are:

$$J_{ij} = i[P, Q]_{ij} = \text{const.}$$

(1.4)

Now, using that symmetry one can turn the matrix $Q$ to diagonal form:

$$Q = \text{diag}(q_1, q_2, \ldots, q_N).$$

(1.5)

Then one can define matrix $P$ from (1.4):

$$J_{ij} = i \sum_j (P_{ik} q_k \delta_{kj} - q_i \delta_{ik} P_{kj}) = iP_{ij}(q_j - q_i), \quad \Rightarrow \quad P_{ij} = \frac{iJ_{ij}}{q_i - q_j}.$$

(1.6)

Thus the result of reduction of matrix model under consideration is entirely determined by the numerical matrix $J$ with zeros at diagonal. The simplest case of zero (rank) matrix $J = 0$ corresponds to diagonal matrix $P$ (diagonal elements can not be derived from (1.6) due to vanishing the diagonal elements of $J$) and to the case of $N$ free particles.

The next in complexity case is rank two matrix:

$$J_{ij} = \delta_{ij} - w_i w_j, \quad w_i = 1, \quad i = 1, \ldots, N,$$

(1.7)

the rank one matrix is excluded due to vanishing diagonal elements. Let $\psi_i$ are components of column $\psi$ eigenvector corresponding to eigenvalue $\lambda$:

$$\psi_i - \sum_{j=1}^N \psi_j = \lambda \psi_i, \quad \Rightarrow \quad (1 - \lambda)\psi_i = (1 - \lambda)\psi_j,$$

one sees, that eigenvalue $\lambda_1 = 1$ has multiplicity $N - 1$, while eigenvalue $\lambda_2 = 1 - N$ has multiplicity one, i.e. the matrix (1.8) indeed has rank two.

So finally, matrix $P$ is:

$$P_{ij} = p_i \delta_{ij} + i \frac{1 - \delta_{ij}}{x_i - x_j} = \begin{cases} p_i, & i = j \\ \frac{i}{q_i - q_j}, & i \neq j \end{cases}$$

(1.8)
Substituting (1.6) for this case into (1.2) one will come to Hamiltonian of Calogero model:

$$H = trP^2 = \sum_{i,j=1}^{N} P_{ij}P_{ji} = \sum_{i=1}^{N} P_{ii}^2 + 2 \sum_{i<j} P_{ij}P_{ji} = \sum_{i=1}^{N} p_{ii}^2 + 2 \sum_{i<j} \frac{1}{(x_i - x_j)^2}.$$  

Then the $N-1$ integrals of motion are given by $I_j = trP^j, j \neq 2$. In particular,

$$trP^3 = \sum_{i=1}^{N} p_i^3 + 3 \sum_{i,j=1}^{N} \frac{1-\delta_{ij}}{x_i - x_j} p_i - i \sum_{i,j,k=1}^{N} \frac{(1-\delta_{ij})(1-\delta_{jk})(1-\delta_{ki})}{(x_i - x_j)(x_j - x_k)(x_k - x_i)}$$  

(1.9)

2 Dunkl operator

However, the reduction is not the only way to prove the integrability of Calogero model. In this article we will try to trace the origin of integrability in approach, associated with the permutations of the particles [3]. The motivation for the introduction of Dunkl operators served as their relationship with the Calogero model and quantum many body systems of Calogero-Moser-Sutherland type. Introduced in this way Dunkl operators are commuting differential-difference operators, related to a finite reflection group on a Euclidean space. First the class rational operators was introduced by C.F. Dunkl in a series of papers [9]. He also introduced the framework for a theory of special functions and integral transforms in several variables related with permutation groups. Then the various other classes of Dunkl operators were invented such that trigonometric Dunkl operators of Heckman, Opdam and the Cherednik operators [10].

Dunkl operator, relevant for rational Calogero model ia $A_{N-1}$ type (is related to $GL(N)$ group). The key object of this approach is permutation operator:

$$\mathcal{P}_{ik} \cdot f(\ldots, x_i, \ldots, x_k, \ldots) = f(\ldots, x_k, \ldots, x_i, \ldots),$$  

(2.10)

where $f$ is an arbitrary smooth function of $N$ variables. It has a number of obvious properties, that are easy to follow from the definition (2.10):

$$\mathcal{P}_{ik} = \mathcal{P}_{ki}, \quad \mathcal{P}_{ik}\mathcal{P}_{ik} = 1,$$  

(2.11)

symmetry

$$\mathcal{P}_{ij}\mathcal{P}_{kl} = \mathcal{P}_{kl}\mathcal{P}_{ij}, \quad i \neq j \neq k \neq l,$$  

(2.12)

commutativity and

$$\mathcal{P}_{ik}\mathcal{P}_{kl} = \mathcal{P}_{il}\mathcal{P}_{ik} = \mathcal{P}_{kl}\mathcal{P}_{il}, \quad i \neq k \neq l,$$  

(2.13)

fusion. The Dunkl operator, playing the central role in our consideration is defined as follows:

$$\nabla_k = \partial_k - c \sum_{i \neq k} \frac{1}{x_i - x_k} \mathcal{P}_{ik} = \partial_k - c \sum_{i=1}^{N} \frac{1-\delta_{ik}}{x_i - x_k} \mathcal{P}_{ik} \equiv \partial_k - \mathcal{A}_k.$$

(2.14)
It is closely related with Calogero Hamiltonian $H_c$:

$$\text{Res}(\sum_{i=1}^{N} \nabla_i^2) = -2H_c, \quad H_c = -\frac{1}{2}\Delta + \sum_{i<j}^N \frac{c(c-1)}{(x_i-x_j)^2},$$

where the symbol $\text{Res}(A)$ means the restriction of operator $A$ to the space of invariants of permutation group $S_N$. In other words, under that sign the permutation operator $P_{ik}$ at utmost right position can be replaced by unity.

Namely the sum of squared Dunkl operators differs from the Calogero-Moser Hamiltonian only in term, linear by coupling constant $c$:

$$\sum_{i=1}^{N} \nabla_i^2 = \sum_{i=1}^{N} \left( \partial_i^2 + c \sum_{j=1}^{N} \frac{1-\delta_{ij}}{(x_i-x_j)^2} P_{ij} + c^2 \sum_{j=1}^{N} \frac{1-\delta_{ij}}{(x_i-x_j)^2} \right). \quad (2.15)$$

This relation suggests that the totally symmetric and totally antisymmetric combination of the eigenfunctions of Dunkl operators (on which permutation in second term takes values $\pm 1$) can serve as the wave functions of Calogero model.

This property emphasizes the connection between hamiltonian of Calogero model and Dunkl operators, but from the point of view of integrability the next property is even more important:

### 3 Zero-curvature condition

Following the invention of the Dunkl operator, it was realized that its components commute:

$$[\nabla_j, \nabla_k] = 0. \quad (3.16)$$

The zero curvature condition (3.16) expresses the integrability of Calogero model: in the functional space, where the non-local connections (like $A_k$) are allowed, it is equivalent to free (non-interacting) model.

Thus, in this context, the non-local gauge field appears differently from the usual non-local field theory, where the non-locality is allowed in the microscopic region of space in order to avoid divergences. Thus, on a macroscopic scale causality is not violated [11]. In contrast, in the present context model under consideration, although the nature of the nonlocal interactions is integrable.

In order to formulate this observation more correct, we will solve zero curvature condition above introducing non-local operator $U$:

$$\nabla_k = U^{-1}\partial_k U, \quad A_k = -U^{-1}(\partial_k U), \quad (3.17)$$

here bracket means that derivative $\partial_k$ acts only on variables $x_k$ contained in operator $U$, but not on test function $\psi(x)$ in defining relation:

$$A_k \cdot \psi = -[U^{-1}(\partial_k U)] \cdot \psi = [(\partial_k U^{-1})U] \cdot \psi. \quad (3.18)$$
In order to represent the formal solution to (3.18) as a path-ordered exponential, we consider an arbitrary smooth curve \((BB')\):

\[
x_k = x_k(t), \quad B = (x(0), \ldots, x_N(0)), \quad B' = (x(t'), \ldots, x_N(t')).
\]  

(3.19)

Multiplying (3.18) by \(\dot{x}_k\) and summing us by \(k\) one obtains:

\[
[\dot{U}^{-1}U] \cdot \psi = \frac{1}{2} \sum_{i,k=1}^{N} (1 - \delta_{ik}) \frac{\dot{x}_i - \dot{x}_k}{x_i - x_k} P_{ik} \cdot \psi \equiv A \cdot \psi.
\]  

(3.20)

The all possible pairwise transpositions of the arguments of test function \(\psi\) stand in r.h.s. of (3.20). Then one can formally write:

\[
\dot{U}^{-1} \cdot \psi = A(t)U^{-1} \cdot \psi,
\]  

(3.21)

\[
U^{-1}(t) = 1 + \int_0^t A(t_1)dt_1 + \int_0^t \int_0^{t_1} A(t_2)dt_2 A(t_1)dt_1 + \ldots,
\]  

(3.22)

indeed, taking derivative in r.h.s. one factors out \(A\) and reproduces whole series, i.e. \(U^{-1}\). This series is properly defined when \(A\) is given by function of \(t\), but when \(A\)'s given by operators, as in (3.22), their product should be ordered. Before define that, two notations are in order.

First is related to singularities of Calogero Hamiltonian at \(x_i = x_k\). We can avoid them restricting ourself by considering of some simplex, say \(x_1 < x_2 < \ldots < x_N\) instead of whole space and demand that curve \(x_k(t)\) belongs to that simplex, but careful analysis [14] shows that correct choice of boundary conditions for wave function when approaching to singular point leads to consistent quantization scheme. The good illustration is the \(N = 2\) case, after excluding the center of mass, it reduces to the study of the 1-dimensional Schrödinger operator:

\[
H_x = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{m}{2}\omega^2 x^2 + \frac{g}{2}x^{-2}.
\]  

(3.23)

The spectrum of \(H_y\) is unbounded from below if \(g > -\frac{\hbar^2}{4m}\) and Calogero assumed in his work that \(g > -\frac{\hbar^2}{4m}\). For "admissible" wave functions he imposed the constraint that associated probability current vanish at the point where any two particles collide. The selection of admissible wave functions is equivalent to choosing a domain on which the Hamiltonian is self-adjoint. Then, in [15] shown that there exits a family of different possibilities parameterized by a \(2 \times 2\) unitary matrix if \(g < \frac{3\hbar^2}{4m}\). In the corresponding quantizations of the \(N = 2\) Calogero model the probability current does not in general vanish at the coincidence of the coordinates of the particles.

Since this phenomenon refers to the interaction of any pairs of particles, one may expect it to occur also in the \(N\) particle Calogero model. Moreover, the case \(N = 3\) considered in [15] in details.

The second notation consists of operator-valued nature of eq. (3.22). The operator \(A\) can be represented as a function of \(2N\) variables \(x_k\) and \(\partial_k\):  

\[
(A \cdot f)(x) = a(x, \partial) f(x) = \frac{1}{(2\pi\hbar)^{N/2}} \int d^N p e^{\frac{i}{\hbar} p \cdot x} a(x, \frac{i}{\hbar} p) f(p).
\]  

(3.24)
Here $a(x, \hat{\pi}p)$ is symbol of operator $A$ and it has to be represented in ordered form, which can be unambiguously restored from symbol. For example the normal ordering (all $x$'s stand from the left and derivatives stand from the right) can be chosen. This dependence on differential operators in $A$ comes of course from permutation $P_{jk}$. It can be expressed in many equivalent ways:

$$\mathcal{P}_{jk} = (-1)^{x_k \partial_k} e^{x_j \partial_j} e^{-x_k \partial_k} e^{x_j \partial_j} (-1)^{x_k \partial_k} = (-1)^{\frac{1}{2}(x_k-x_j)(\partial_k-\partial_j)}. \quad (3.25)$$

However we need the normally ordered expression for permutation operator:

$$(-1)^{v \partial_v} f(v) = f(-v) = \sum_{n=0}^{\infty} \frac{(-2v)^n}{n!} \partial_v^n \cdot f(v), \quad (3.26)$$

in order to construct its symbol.

In Appendix C we prove the equivalence of different realization for $\mathcal{P}_{ik}$ (3.25).

So, the symbol of permutation operator can be written as:

$$\text{Symbol}[\mathcal{P}_{jk}] = a(x, \frac{i}{\hbar}p) = \sum_{n=0}^{\infty} \frac{(x_k-x_j)^n}{n!}(-\frac{i}{\hbar}(p_k-p_j))^n. \quad (3.27)$$

Together with formulae (3.24) and (3.22) it defines the formal series for operator $U^{-1}$.

4 Eigenproblem

In order to clarify the physical meaning of operator $U$ it makes sense to define eigenproblem of Dunkl operators:

$$\nabla_k \psi(x_1, x_2, \ldots, x_N) = ip_k \psi(x_1, x_2, \ldots, x_N), \quad (4.28)$$

here imaginary unit is extracted in order to eigenvalue $p_k$ will real. Due to zero curvature condition (3.16) equations above compatible each to other. Another, non-trivial integrability condition to this set of equations provides permutation operator. Due to relations:

$$\mathcal{P}_{jk} \nabla_k = \nabla_j \mathcal{P}_{jk}, \quad [\mathcal{P}_{jk}; \nabla_l] = 0, \quad j \neq l, \quad k \neq l, \quad (4.29)$$

proven in Appendix B, one deduces, that if $\psi_{p_1,p_2,\ldots,p_N}(x_1, x_2, \ldots, x_N)$ is solution to the set (4.28), then the function

$$\mathcal{P}_{jk} \psi_{p_1,p_2,\ldots,p_N}(x_1, x_2, \ldots, x_N),$$

which differs from

$$\psi_{p_1,p_2,\ldots,p_N}(x_1, x_2, \ldots, x_N)$$

by permutation of arguments $x_k$ and $x_j$ will be solution to the same equation with permutation of parameters $p_k$ and $p_j$. Taking into account, that the set od $N$ linear first order differential equations has a unique solution, this condition can be formulated as follows:

$$\psi_{p_j,\ldots,p_k,\ldots}(x_j, \ldots, x_k, \ldots) = \psi_{p_k,\ldots,p_j,\ldots}(x_k, \ldots, x_j, \ldots). \quad (4.30)$$
In other words, this restriction reduces the arbitrariness in the choice of function of $2N$ variables to one totally symmetric function:

$$\psi(f_1, \ldots, f_N), \quad \psi(\ldots, f_j, \ldots, f_k, \ldots) = \psi(\ldots, f_k, \ldots, f_j, \ldots), \quad f_k \equiv f(x_k, p_k),$$

where $f(x, p)$ is an arbitrary function of two variables.

Let us multiply eqs. (4.28) by $U$:

$$U \nabla_k \psi = \partial_k U \psi = ip_k U \psi,$$

(4.31)

from which one deduces that if solution to eqs. (4.28) is given by function $\psi$, then the function $\psi_0 = U \psi$ satisfies to the set of equations:

$$\partial_k \psi_0 = ip_k \psi_0,$$

which has unique solution:

$$\psi_0 = \text{const} \cdot e^{i(x_1 p_1 + x_2 p_2 + \ldots + x_N p_N)}.$$ (4.32)

In the spirit of remark about connection between totally symmetric and totally antisymmetric combinations of $\psi$’s and Calogero wave functions, one sees, that the operator $U^{-1}$ maps symmetrized, according to principle of identity of particles, combination of $\psi_0$’s to Calogero wave function with $g = c(c + 1)$ and $g = c(c - 1)$.

In this regard, the physical meaning of operator consists of $U^{-1}$ is just scattering matrix of the rational Calogero model.

So, another way to determine operator $U^{-1}$ consists of solving the set of equations (4.28) and restoring $U^{-1}$ by function $\psi$.

## 5 Conclusion and outlook

The operator $U$, introduced as a solution to the condition of zero curvature associated with the non-local gauge transformation that reflects the wave function of the Calogero model in the wave function of $N$ non-interacting particles may be of interest from the point of view of non-commutative geometry, because it provides the realization of micro-causality condition within a particular model of interacting particles.

Also of interest is a generalization of this consideration to the case of the anionic particle statistics, which was discussed by A.P.Polychronakos [4].

### Acknowledgements

This work is supported in part by Armenian grants: SCS 13-1C132, ANSEF grant mathph 3122, D.K. is supported also by grants: SCS 13RF-018 (2013) and Volkswagen Foundation I/84 496. Authors are grateful to Tigran Hakobyan for valuable discussions and to Armen Allahverdyan for careful reading of manuscript.
6 Appendix A

In this Appendix we will prove formula (2.15). One has:

\[ \sum_{i=1}^{N} \nabla_i^2 = \sum_{i=1}^{N} (\partial_i^2 + c \sum_{j=1}^{N} (1 - \delta_{ij}) \frac{1}{x_i - x_j} \mathcal{P}_{ij} + \frac{1 - \delta_{ij}}{x_i - x_j} \partial_j \mathcal{P}_{ij}) + c^2 \sum_{j,k=1}^{N} (1 - \delta_{ij}) \frac{1 - \delta_{ik}}{x_i - x_j} \partial_j \mathcal{P}_{ij} \]

\[ = \sum_{i=1}^{N} (\partial_i^2 + c \sum_{j=1}^{N} \frac{1 - \delta_{ij}}{(x_i - x_j)^2} \mathcal{P}_{ij} + c \sum_{j=1}^{N} \frac{1 - \delta_{ij}}{x_i - x_j} (\partial_i + \partial_j) \mathcal{P}_{ij} + c^2 \sum_{j=1}^{N} \frac{1 - \delta_{ij}}{(x_i - x_j)^2} + (6.33) \]

\[ + c^2 \sum_{j,k=1}^{N} \frac{(1 - \delta_{ij})(1 - \delta_{ik})(1 - \delta_{kj})}{(x_i - x_j)(x_i - x_k)} \mathcal{P}_{ij} \mathcal{P}_{ik}, \]

then, taking into account symmetry property of permutation operator (2.11) one deduces that the third term in (2.15) vanishes (as trace of product of symmetric and antisymmetric matrices), while property (2.13) tells that product \( \mathcal{P}_{ij} \mathcal{P}_{ik} \) remain unchanged under cyclic permutations \( i \rightarrow j \rightarrow k \rightarrow i \) which allows to rewrite the last term in (2.15) making cyclic transpositions of dummy indices:

\[ \frac{1}{3} \sum_{i,j,k=1}^{N} (1 - \delta_{ij})(1 - \delta_{ik})(1 - \delta_{kj}) \frac{1}{x_i - x_j} \frac{1}{x_j - x_k} \frac{1}{x_k - x_i} + (1 - \delta_{ij}) \frac{1}{x_i - x_j} \frac{1}{x_j - x_k} \frac{1}{x_k - x_i} \mathcal{P}_{ij} \mathcal{P}_{ik} = \]

\[ = \frac{1}{3} \sum_{i,j,k=1}^{N} (1 - \delta_{ij})(1 - \delta_{ik})(1 - \delta_{kj}) \frac{(x_i - x_j) + (x_j - x_k) + (x_k - x_i)}{(x_i - x_j)(x_j - x_k)(x_k - x_i)} \mathcal{P}_{ij} \mathcal{P}_{ik} = 0. \]

Now we prove that the Dunkl operator corresponds to flat connection.

Indeed, the commutator \([\nabla_k; \nabla_l]\) consists of four pieces: the commutator of derivatives (proportional to \( c^0 \)), which is zero, two pieces, linear by \( c \) and one piece is proportional to \( c^2 \): \([\nabla_k; \nabla_l] = c I_{kj}^{(1)} + c^2 I_{kj}^{(2)}\). One has due to \( j \neq k \):

\[ I_{kj}^{(1)} = \sum_{i=1}^{N} ((1 - \delta_{ij}) + \delta_{ij}) [\partial_j; \frac{1}{x_i - x_k} \mathcal{P}_{ik}] - j \leftrightarrow k = (\partial_j \frac{1}{x_j - x_k} \mathcal{P}_{jk} - \frac{1}{x_j - x_k} \mathcal{P}_{jk} \partial_j) - j \leftrightarrow k = \]

\[ = (\frac{1}{x_j - x_k} (\partial_j - \partial_k) \mathcal{P}_{jk} \partial_j - \frac{1}{(x_j - x_k)^2} \mathcal{P}_{jk}) - j \leftrightarrow k = 0, \]

here at first row the commutator at bracket \((1 - \delta_{ij})\) i.e. \( i \neq j \) vanishes and in the last row vanishes because the expression in brackets is symmetric with respect to \( j \leftrightarrow k \).

Consider now \( I_{kj}^{(2)} \):

\[ I_{kj}^{(2)} = \left[ \frac{1 - \delta_{ik}}{x_i - x_k} \mathcal{P}_{ik}; \frac{1 - \delta_{ij}}{x_i - x_j} \mathcal{P}_{ij} \right], \]

and insert there unity:

\[ 1 = (1 - \delta_{il})(1 - \delta_{kl})(1 - \delta_{ij}) + \delta_{il}(1 - \delta_{kl})(1 - \delta_{ij}) + \delta_{kl} + \delta_{ij} - \delta_{kl} \delta_{ij}, \]
then first term differs from zero only at \( i \neq j \neq k \neq l \), when factors of commutator commute each to other and it vanishes. Similarly the last term is also vanishes, because at \( i = j, l = k \) factors of commutator become equal each to other with opposite sign. Passing permutation to the right one rewrite remaining three terms as follows:

\[
\sum_{i=1}^{N} (1 - \delta_{ij})(1 - \delta_{ik})\left(\frac{1}{x_j - x_k} \frac{1}{x_i - x_k} (P_{ki} - P_{kj})P_{ij} + \frac{1}{x_k - x_i} \frac{1}{x_j - x_k} (P_{ki} - P_{ij})P_{jk} + \frac{1}{x_k - x_i} \frac{1}{x_k - x_j} (P_{jk} - P_{ij})P_{ki}\right),
\]

using identity:

\[
\frac{1}{x_i - x_k} \left(\frac{1}{x_i - x_j} + \frac{1}{x_j - x_k}\right) = \frac{1}{x_i - x_j} \frac{1}{x_j - x_k},
\]

one sees, that all terms canceled due to (2.13). So the proof of statement:

\[
[\nabla_k; \nabla_l] = 0,
\]

(6.34)
is finished.

### 7 Appendix B

In this Appendix we will commute permutation and Dunkl operator. Consider first the case when all indices different: \( j \neq l \neq k \):

\[
P_{jl} \nabla_k = P_{jl}[\partial_k - \frac{c}{x_i - x_k} P_{ik}(1 - \delta_{ij})(1 - \delta_{il}) + \delta_{ij} + \delta_{il}] =
\]

\[
= [\partial_k - \frac{c}{x_i - x_k} P_{ik}(1 - \delta_{ij})(1 - \delta_{il})] P_{jl} - c P_{jl} \frac{1 - \delta_{jk}}{x_j - x_k} P_{jk} = \nabla_k P_{jl},
\]

Here we took into account that \( \delta_{ij} \delta_{il} = 0 \), and in first term \( P_{jl} \) freely moves to right, then we notice, that extra terms in square bracket just canceled with two terms standing outside.

Now consider case \( l = k, j \neq k \):

\[
P_{jk} \nabla_k = P_{jk}[\partial_k - \frac{c}{x_i - x_k} P_{ik}(1 - \delta_{ij} + \delta_{jk})] = [\partial_j - \frac{c}{x_i - x_k} (1 - \delta_{ij})(1 - \delta_{ik})] P_{jk} - c P_{jk} \frac{1 - \delta_{jk}}{x_j - x_k} P_{jk} = \nabla_j P_{jk}.
\]

Here we moved at first step permutation to the right and noticed that the extra term, coming from \( \delta_{ij} \) is just missing term in sum in square bracket at \( i = k \).
8 Appendix C

It is seen that the main difficulty with the reduction to normal form is related to an essential part of the permutation, the sign-changing operator:

\[ (-1)^x \partial_x f(x) = f(-x). \]

In order to see it, one can prove at arbitrary complex \( q \) the more general relation:

\[ q^x \partial_q f(x) = f(qx). \]

Indeed, putting \( x = e^t \) one has \( x \partial_x = \partial_t \) and

\[ q \partial_q f(e^t) = e^{\log q \partial_t} f(e^t) = f(q^t \log q) = f(qt). \]

Then, introducing

\[ u = x_j + x_k, \quad v = x_k - x_j, \]

one has

\[ \partial_u = \frac{1}{2}(\partial_k + \partial_j), \quad \partial_v = \frac{1}{2}(\partial_k - \partial_j) \]

and

\[ \mathcal{P}_{kj} \cdot f(x_k, x_j) = (-1)^{v \partial_k} \cdot f(\frac{1}{2}(u + v), \frac{1}{2}(u - v)) = f(\frac{1}{2}(u + (-v)), \frac{1}{2}(u - (-v))) = f(x_j, x_k). \]

The normally ordered expression for \( \mathcal{P}_{kj} \) is given by elegant formula (3.26)

\[ (-1)^{v \partial_k} \cdot f(v) = f(-v) = \sum_{n=0}^{\infty} \frac{(-2v)^n}{n!} \partial^n_v \cdot f(v), \]

which is formally just Taylor expansion of \( f(-v) \) around point \( v \). This observation may be replaced by a more lengthy proof, using the Stirling numbers of the second kind to relate it with (3.25).

In order to establish the equivalence of relations (3.25) we just transform them to normal ordered form, using formula (3.26). Consider:

\[ (-1)^x \partial_j e^{-x_j} \partial_k e^{x_k} \partial_j e^{-x_j} \partial_k = \sum_{n=0}^{\infty} \frac{(-2x_j)^n}{n!} \partial^n_j e^{-x_j} \partial_k e^{x_k} \partial_j e^{-x_j} \partial_k = \sum_{n=0}^{\infty} \frac{(2x_j)^n}{n!} e^{-x_j} \partial_k e^{x_k} \partial_j e^{-x_j} \partial_k \]

\[ = \sum_{n=0}^{\infty} \frac{2x_j^n}{n!} e^{-x_j} \partial_k \sum_{m=0}^{\infty} \frac{(x_j)^m}{m!} \partial^n_m e^{-x_j} \partial_k = \sum_{m,n=0}^{\infty} \frac{(x_j - x_k)^m (2x_j)^n}{m! n!} e^{-x_j} \partial_k (\partial_k - \partial_j)^m \partial^n_j = \]

\[ = \sum_{m=0}^{\infty} \frac{(x_k - x_j)^m}{m!} \left( \sum_{n=0}^{\infty} \frac{(2x_j)^n}{n!} \partial_j \right) e^{-2x_k \partial_j} (\partial_k - \partial_j)^m = \sum_{m=0}^{\infty} \frac{(x_k - x_j)^m}{m!} (\partial_k - \partial_j)^m = (-1)^{v \partial_k}. \]

In similar way another expressions for permutation operator via finite translations and sign-changing operator can be transformed to normal-ordered form (3.26).
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