Lie symmetries and conservation laws of the Hirota-Ramani equation

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Abstract

In this paper, Lie symmetry method is performed for the Hirota-Ramani (H-R) equation. We will find The symmetry group and optimal systems of Lie subalgebras. Furthermore, preliminary classification of its group invariant solutions, symmetry reduction and nonclassical symmetries are investigated. Finally the conservation laws of the H-R equation are presented.

Keywords. Lie symmetry, Invariant solutions, Nonclassical symmetries, Conservation laws, Hirota-Ramani equation.

1 Introduction

In the present paper, we study the following equation

\[ u_t - u_{xxt} + au_x(1 - u_t) = 0, \]  

(1)

where \( a \neq 0 \) and \( u(x,t) \) is the amplitude of relevant wave mode. This equation was introduced by Hirota and Ramani in [10]. Jie Ji obtained some travelling soliton solutions of this equation by using Exp-function method [12]. This equation is completely integrable by the inverse scattering method. Eq. (1) is studied in [10, 12, 9] where new kind of solutions were obtained. Hirota-Ramani equation is widely used in various branches of physics, such as plasma physics, fluid physics, quantum field theory. It also describes a variety of wave phenomena in plasma and solid state [10].

The theory of Lie symmetry groups of differential equations was developed by Sophus Lie [14], which was called classical Lie method. Nowadays, application of Lie transformations group theory for constructing the solutions of nonlinear partial differential equations (PDEs) can be regarded as one of the most active fields of research in the theory of nonlinear PDEs and applications. Such Lie...
groups are invertible point transformations of both the dependent and independent variables of the differential equations. The symmetry group methods provide an ultimate arsenal for analysis of differential equations and is of great importance to understand and to construct solutions of differential equations. Several applications of Lie groups in the theory of differential equations were discussed in the literature, the most important ones are: reduction of order of ordinary differential equations, construction of invariant solutions, mapping solutions to other solutions and the detection of linearizing transformations for many other applications of Lie symmetries see [17, 5, 4].

The fact that symmetry reductions for many PDEs are unobtainable by applying the classical symmetry method, motivated the creation of several generalizations of the classical Lie group method for symmetry reductions. The nonclassical symmetry method of reduction was devised originally by Bluman and Cole in 1969 [6], to find new exact solutions of the heat equation. The description of the method is presented in [7, 13]. Many authors have used the nonclassical method to solve PDEs. In [8] Clarkson and Mansfield have proposed an algorithm for calculating the determining equations associated to the nonclassical method. A new procedure for finding nonclassical symmetries has been proposed by Bila and Niesen in [1].

Many PDEs in the applied sciences and engineering are continuity equations which express conservation of mass, momentum, energy, or electric charge. Such equations occur in, e.g., fluid mechanics, particle and quantum physics, plasma physics, elasticity, gas dynamics, electromagnetism, magneto-hydro-dynamics, nonlinear optics, etc. In the study of PDEs, conservation laws are important for investigating integrability and linearization mappings and for establishing existence and uniqueness of solutions. They are also used in the analysis of stability and global behavior of solutions [2, 3, 19, 20].

This work is organized as follows. In section 2 we recall some results needed to construct Lie point symmetries of a given system of differential equations. In section 3, we give the general form of a infinitesimal generator admitted by Eq. (1) and find transformed solutions. In Section 4, we construct the optimal system of one-dimensional subalgebras. Lie invariants, similarity reduced equations and differential invariants corresponding to the infinitesimal symmetries of Eq. (1) are obtained in section 5 and 6. Section 7, is devoted to the nonclassical symmetries of the H-R model, symmetries generated when a supplementary condition, the invariance surface condition, is imposed. Finally in last section, the conservation laws of the Eq. (1) are obtained.

2 Method of Lie Symmetries

In this section, we recall the general procedure for determining symmetries for any system of partial differential equations see [17, 15, 5, 4]. To begin, let us consider the general case of a nonlinear system $E$ of partial differential equations
of order \( n \) in \( p \) independent and \( q \) dependent variables is given as a system of equations

\[
\Delta_{\nu}(x, u^{(n)}) = 0, \quad \nu = 1, \cdots, l,
\]  

(2)

involving \( x = (x^1, \cdots, x^p) \), \( u = (u^1, \cdots, u^q) \) and the derivatives of \( u \) with respect to \( x \) up to \( n \), where \( u^{(n)} \) represents all the derivatives of \( u \) of all orders from 0 to \( n \). We consider a one-parameter Lie group of infinitesimal transformations acting on the independent and dependent variables of the system (2)

\[
\begin{align*}
\tilde{x}^i & = x^i + s\xi^i(x, u) + O(s^2), \quad i = 1 \cdots, p, \\
\tilde{u}^j & = u^j + s\varphi^j(x, u) + O(s^2), \quad j = 1 \cdots, q,
\end{align*}
\]

(3)

where \( s \) is the parameter of the transformation and \( \xi^i, \eta^j \) are the infinitesimals of the transformations for the independent and dependent variables, respectively. The infinitesimal generator \( \mathbf{v} \) associated with the above group of transformations can be written as

\[
\mathbf{v} = \sum_{i=1}^{p} \xi^i(x, u) \partial_{x^i} + \sum_{\alpha=1}^{q} \varphi^\alpha(x, u) \partial_{u^\alpha}.
\]

(4)

A symmetry of a differential equation is a transformation which maps solutions of the equation to other solutions. The invariance of the system (2) under the infinitesimal transformations leads to the invariance conditions (Theorem 2.36 of [17])

\[
\text{Pr}^{(n)}[\Delta_{\nu}(x, u^{(n)})] = 0, \quad \nu = 1, \cdots, l, \quad \text{whenever} \quad \Delta_{\nu}(x, u^{(n)}) = 0,
\]

(5)

where \( \text{Pr}^{(n)} \) is called the \( n \)th order prolongation of the infinitesimal generator given by

\[
\text{Pr}^{(n)} \mathbf{v} = \mathbf{v} + \sum_{\alpha=1}^{q} \sum_{J} \varphi^J_{\alpha}(x, u^{(n)}) \partial_{u^\gamma},
\]

(6)

where \( J = (j_1, \cdots, j_k) \), \( 1 \leq j_k \leq p, 1 \leq k \leq n \) and the sum is over all \( J \)'s of order \( 0 < \#J \leq n \). If \( \#J = k \), the coefficient \( \varphi^\alpha_J \) of \( \partial_{u^\gamma} \) will only depend on \( k \)-th and lower order derivatives of \( u \), and

\[
\varphi^J_{\alpha}(x, u^{(n)}) = D_J(\varphi_\alpha - \sum_{i=1}^{p} \xi^i u^\alpha_i) + \sum_{i=1}^{p} \xi^i u^\alpha_{j,i},
\]

(7)

where \( u^\alpha_i := \partial u^\alpha / \partial x^i \) and \( u^\alpha_{j,i} := \partial u^\alpha_{j,i} / \partial x^i \).

One of the most important properties of these infinitesimal symmetries is that they form a Lie algebra under the usual Lie bracket.
3 Lie symmetries of the H-R equation

We consider the one parameter Lie group of infinitesimal transformations on
\((x^1 = x, x^2 = t, u^1 = u)\),
\[
\begin{align*}
\tilde{x} &= x + s\xi(x, t, u) + O(s^2), \\
\tilde{t} &= x + s\eta(x, t, u) + O(s^2), \\
\tilde{u} &= x + s\varphi(x, t, u) + O(s^2),
\end{align*}
\]
(8)
where \(s\) is the group parameter and \(\xi^1 = \xi, \xi^2 = \eta\) and \(\varphi^1 = \varphi\) are the infinitesimals of the transformations for the independent and dependent variables, respectively. The associated vector field is of the form:
\[
v = \xi(x, t, u)\partial_x + \eta(x, t, u)\partial_t + \varphi(x, t, u)\partial_u.
\]
(9)
and, by (6) its third prolongation is
\[
\text{Pr}^{(3)}v = v + \varphi^x \partial_{ux} + \varphi^t \partial_{ut} + \varphi^{xt} \partial_{uxt} + \varphi^{x2} \partial_{ux^2} + \varphi^{xt} \partial_{uxt} + \varphi^{t2} \partial_{ut^2} + \varphi^{x3} \partial_{ux^3} + \varphi^{xt2} \partial_{uxt^2} + \varphi^{t3} \partial_{ut^3},
\]
(10)
where, for instance by (7) we have
\[
\begin{align*}
\varphi^x &= D_x(\varphi - \xi u_x - \eta u_t) + \xi u_{x^2} + \eta u_{xt}, \\
\varphi^t &= D_t(\varphi - \xi u_x - \eta u_t) + \xi u_{xt} + \eta u_{t^2}, \\
&\vdots \\
\varphi^{t3} &= D^3_t(\varphi - \xi u_x - \eta u_t) + \xi u_{t^3} + \eta u_{t^4},
\end{align*}
\]
(11)
where \(D_x\) and \(D_t\) are the total derivatives with respect to \(x\) and \(t\) respectively. By (5) the vector field \(v\) generates a one parameter symmetry group of Eq. (1) if and only if
\[
\text{Pr}^{(3)}v[u_t - u_{x2t} + au_x(1 - u_t)] = 0,
\]
(12)
whenever \(u_t - u_{x2t} + au_x(1 - u_t) = 0\).

The condition (12) is equivalent to
\[
(1 - au_x)\varphi^t + a(1 - u_t)\varphi^x - \varphi^{xt} = 0,
\]
(13)
whenever \(u_t - u_{x2t} + au_x(1 - u_t) = 0\).

Substituting (11) into (13), and equating the coefficients of the various monomials in partial derivatives with respect to \(x\) and various power of \(u\), we can
find the determining equations for the symmetry group of the Eq. (1). Solving
this equations, we get the following forms of the coefficient functions

\[
\xi = -\frac{x}{3} c_1 + c_3, \quad \eta = c_1 t + c_2, \quad \varphi = \left(\frac{2t}{3} + \frac{u}{3} - \frac{2x}{3a}\right) c_1 + c_4.
\] (14)

where \(c_1, c_2, c_3\) and \(c_4\) are arbitrary constant. Thus, the Lie algebra \(g\) of
infinitesimal symmetry of the Eq. (1) is spanned by the four vector fields

\[
v_1 = \partial_x, \quad v_2 = \partial_t, \quad v_3 = \frac{1}{a} \partial_u, \quad v_4 = 3t\partial_x - x\partial_x + \left(2t + u - \frac{2x}{a}\right) \partial_u.
\] (15)

The commutation relations between these vector fields are given in the Table
1. The Lie algebra \(g\) is solvable, because if \(g^{(1)} = \langle v_i, [v_i, v_j]\rangle = [g, g]\), we have
\(g^{(1)} = \langle v_1, \cdots v_4\rangle\), and \(g^{(2)} = [g^{(1)}, g^{(1)}] = \langle -v_1 - 2v_3, 3v_2 + 2av_3, v_3\rangle\), so, we
have a chain of ideals \(g^{(1)} \supset g^{(2)} \supset 0\).

| \([v_1, v_2]\) | \(v_1\) | \(v_2\) | \(v_3\) | \(v_4\) |
|-----------------|-------|-------|-------|-------|
| \(v_1\) | 0 | 0 | 0 | \(-v_1 - 2v_3\) |
| \(v_2\) | 0 | 0 | 0 | \(3v_2 + 2av_3\) |
| \(v_3\) | 0 | 0 | 0 | \(v_3\) |
| \(v_4\) | \(v_1 + 2v_3\) | \(-3v_2 - 2av_3\) | \(-v_3\) | 0 |

To obtain the group transformation which is generated by the infinitesimal
generators \(v_i\) for \(i = 1, 2, 3, 4\) we need to solve the three systems of first order
ordinary differential equations

\[
\frac{d\tilde{x}(s)}{ds} = \xi(\tilde{x}(s), \tilde{t}(s), \tilde{u}(s)), \quad \tilde{x}(0) = x,
\]

\[
\frac{d\tilde{t}(s)}{ds} = \eta(\tilde{x}(s), \tilde{t}(s), \tilde{u}(s)), \quad \tilde{t}(0) = t, \quad i = 1, \cdots, 4
\] (16)

\[
\frac{d\tilde{u}(s)}{ds} = \varphi(\tilde{x}(s), \tilde{t}(s), \tilde{u}(s)), \quad \tilde{u}(0) = u.
\]

Exponentiating the infinitesimal symmetries of Eq. (1), we get the one-parameter
groups \(G_i(s)\) generated by \(v_i\) for \(i = 1, \cdots, 4\)

\[
G_1 : (t, x, u) \rightarrow (x + s, t, u),
G_2 : (t, x, u) \rightarrow (x, t + s, u),
G_3 : (t, x, u) \rightarrow (x, t, u + s/a),
G_4 : (t, x, u) \rightarrow (xe^{-s}, te^{3s}, te^{3s} + \frac{x}{a}e^{-s} + (u - t - \frac{x}{a})e^s).
\] (17)

Consequently,
Theorem 3.1  If $u = f(x,t)$ is a solution of Eq. (1), so are the functions

\begin{align*}
G_1(s) \cdot f(x,t) &= f(x-s,t), \\
G_2(s) \cdot f(x,t) &= f(x,t-s), \\
G_3(s) \cdot f(x,t) &= f(x,t) + \frac{s}{a}, \\
G_4(s) \cdot f(x,t) &= e^s f(xe^s, te^{-3s}) + \frac{s}{a}(1 - e^{2s}) + t(1 - e^{-2s}).
\end{align*}

(18)

4 Optimal system of the H-R equation

In general, to each s-parameter subgroup $H$ of the full symmetry group $G$ of a system of differential equations in $p > s$ independent variables, there will correspond a family of group-invariant solutions. Since there are almost always an infinite number of such subgroups, it is not usually feasible to list all possible group-invariant solutions to the system. We need an effective, systematic means of classifying these solutions, leading to an "optimal system" of group-invariant solutions from which every other such solution can be derived.[17]

Definition 4.1  Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$. An optimal system of $s$-parameter subgroups is a list of conjugacy inequivalent $s$-parameter subalgebras with the property that any other subgroup is conjugate to precisely one subgroup in the list. Similarly, a list of $s$-parameter subalgebras forms an optimal system if every $s$-parameter subalgebra of $\mathfrak{g}$ is equivalent to a unique member of the list under some element of the adjoint representation: $\mathfrak{h} = \text{Ad}(g(\mathfrak{h})).$[17]

Theorem 4.2  Let $H$ and $\overline{H}$ be connected $s$-dimensional Lie subgroups of the Lie group $G$ with corresponding Lie subalgebras $\mathfrak{h}$ and $\overline{\mathfrak{h}}$ of the Lie algebra $\mathfrak{g}$ of $G$. Then $\overline{\mathfrak{h}} = g \mathfrak{h} g^{-1}$ are conjugate subalgebras if and only if $\overline{\mathfrak{h}} = \text{Ad}(g(\mathfrak{h}))$ are conjugate subalgebras. [17]

By theorem (4.2), the problem of finding an optimal system of subgroups is equivalent to that of finding an optimal system of subalgebras. For one-dimensional subalgebras, this classification problem is essentially the same as the problem of classifying the orbits of the adjoint representation, since each one-dimensional subalgebra is determined by nonzero vector in $\mathfrak{g}$. This problem is attacked by the naive approach of taking a general element $V$ in $\mathfrak{g}$ and subjecting it to various adjoint transformation so as to "simplify" it as much as possible. Thus we will deal with the construction of the optimal system of subalgebras of $\mathfrak{g}$.
To compute the adjoint representation, we use the Lie series

\[ \text{Ad}(\exp(\varepsilon v_i)v_j) = v_j - \varepsilon[v_i, v_j] + \frac{\varepsilon^2}{2}[v_i, [v_i, v_j]] - \cdots, \] (19)

where \([v_i, v_j]\) is the commutator for the Lie algebra, \(\varepsilon\) is a parameter, and \(i, j = 1, 2, 3, 4\). Then we have the Table 2.

Table 2: Adjoint representation table of the infinitesimal generators \(v_i\)

| \(\text{Ad}\) | \(v_1\) | \(v_2\) | \(v_3\) | \(v_4\) |
|--------------|---------|---------|---------|--------|
| \(v_1\)     | \(v_1\) | \(v_2\) | \(v_3\) | \(v_4 + \varepsilon(v_1 + 2v_3)\) |
| \(v_2\)     | \(v_1\) | \(v_2\) | \(v_3\) | \(v_4 - \varepsilon(3v_2 + 2v_3)\) |
| \(v_3\)     | \(v_1\) | \(v_2\) | \(v_3\) | \(v_4 - \varepsilon v_3\) |
| \(v_4\)     | \(v_1 - \varepsilon(v_1 + 2v_3)\) | \(v_2 + \varepsilon(3v_2 + 2v_3)\) | \(v_3 + \varepsilon v_3\) | \(v_4\) |

Theorem 4.3  An optimal system of one-dimensional Lie algebras of the H-R equation is provided by

1) \(v_4\), 2) \(\alpha v_1 + \beta v_2 + v_3\), 3) \(\alpha v_1 + v_2\), 4) \(v_1\)

Proof: Consider the symmetry algebra \(g\) of the equation (1) whose adjoint representation was determined in table 2 and let \(F_i : g \to g\) defined by \(v \mapsto \text{Ad}(\exp(\varepsilon v_i)v)\) is a linear map, for \(i = 1, \cdots, 4\). The matrices \(M_i^\varepsilon\) of \(F_i^\varepsilon\), \(i = 1, \cdots, 4\), with respect to basis \(\{v_1, \cdots, v_4\}\) are

\[
M_1^\varepsilon = \begin{bmatrix} 1 & 0 & 0 & -\varepsilon \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2\varepsilon \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad M_2^\varepsilon = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3\varepsilon \\ 0 & 0 & 1 & 2\varepsilon \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad M_3^\varepsilon = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \varepsilon \\ 0 & 0 & 0 & 1 \end{bmatrix},
\]

\[
M_4^\varepsilon = \begin{bmatrix} e^\varepsilon & 0 & 0 & 0 \\ 0 & e^{-\varepsilon} & 0 & 0 \\ e^\varepsilon & e^{-\varepsilon} & (e^{-2\varepsilon} - 1) & e^{-\varepsilon} \\ 0 & 0 & 0 & 1 \end{bmatrix}.
\]

Let \(V = \sum_{i=1}^4 a_i v_i\) is a nonzero vector field in \(g\). We will simplify as many of the coefficients \(a_i\) as possible by acting these matrices on a vector field \(V\) alternatively.

Suppose first that \(a_4 \neq 0\), scaling \(V\) if necessary we can assume that \(a_4 = 1\), then we can make the coefficients of \(v_1\), \(v_2\) and \(v_3\) vanish by \(M_1^\varepsilon\) and \(M_3^\varepsilon\). And \(V\) reduced to case 1.

If \(a_4 = 0\) and \(a_3 \neq 0\), then we can not make vanish the coefficients of \(v_1\) and \(v_2\) by acting any matrices \(M_i^\varepsilon\). Scaling \(V\) if necessary, we can assume that \(a_3 = 1\) and \(V\) reduced to case 2.
If \(a_4 = a_3 = 0\) and \(a_2 \neq 0\), then we can not make vanish the coefficient of \(v_1\). Scaling \(V\) if necessary, we can assume that \(a_2 = 1\) and \(V\) reduced to case 3. The remaining one-dimensional subalgebras are spanned by vectors of the above form with \(a_4 = a_3 = a_2 = 0\). If \(a_1 \neq 0\), we scale to make \(a_1 = 1\), and \(V\) reduced to case 3. □

5 Symmetry reduction of the H-R equation

Lie-group method is applicable to both linear and non-linear partial differential equations, which leads to similarity variables that may be used to reduce the number of independent variables in partial differential equations. By determining the transformation group under which a given partial differential equation is invariant, we can obtain information about the invariants and symmetries of that equation.

Symmetry group method will be applied to the \((1)\) to be connected directly to some order differential equations. To do this, a particular linear combinations of infinitesimals are considered and their corresponding invariants are determined. The equation \((1)\) is expressed in the coordinates \((x, t, u)\), so to reduce this equation is to search for its form in specific coordinates. Those coordinates will be constructed by searching for independent invariants \((y, v)\) corresponding to the infinitesimal generator. So using the chain rule, the expression of the equation in the new coordinate allows us to the reduced equation. Here we will obtain some invariant solutions with respect to symmetries. First we obtain the similarity variables for each term of the Lie algebra \(g\), then we use this method to reduced the PDE and find the invariant solutions.

We can now compute the invariants associated with the symmetry operators, they can be obtained by integrating the characteristic equations. For example for the operator \(v_2 + v_3 = \partial_t + \frac{1}{a} \partial_u\) characteristic equation is

\[
\frac{dx}{0} = \frac{dt}{1} = \frac{a \, du}{1}.
\]

(20)
The corresponding invariants are \(y = x, \; v = u - \frac{t}{a}\) therefore, a solution of our equation in this case is \(u = v(y) + \frac{t}{a}\). The derivatives of \(u\) are given in terms of \(y\) and \(v\) as

\[
\begin{align*}
u_x &= v_y, \quad u_{x^2} = v_{yy}, \quad u_{x^2t} = 0, \quad u_t = \frac{1}{a}.
\end{align*}
\]

(21)
Substituting (21) into the Eq. (1), we obtain the ordinary differential equation \((a-1)v_y + 1/a = 0\), the solution of this equation is \(v = \frac{-y}{a(a-1)} + c\). Consequently, we obtain that

\[
u = \frac{x}{a(1-a)} + \frac{t}{a} + c.
\]

(22)
All results are coming in the tables 3 and 4.
Table 3: Reduction of Eq. (1)

| operator | y | v | u |
|----------|---|---|---|
| \( v_1 \) | \( t \) | \( u \) | \( v(y) \) |
| \( v_2 \) | \( x \) | \( u \) | \( v(y) \) |
| \( v_4 \) | \( xt^{2/3} \) | \((u - 2x/a)t^{-1/3} + t^{2/3}\) | \( v(y)t^{1/3} + 2x/a - t\) |
| \( v_1 + av_3 \) | \( t \) | \( u - x \) | \( v(y) + x \) |
| \( v_2 + v_3 \) | \( x \) | \( u - t/a \) | \( v(y) + t/a \) |
| \( v_1 + v_2 \) | \( x - t \) | \( u \) | \( v(y)/a \) |

Table 4: Reduced equations corresponding to infinitesimal symmetries

| operator | similarity reduced equations |
|----------|-----------------------------|
| \( v_1 \) | \( v_y = 0 \) |
| \( v_2 \) | \( av_y = 0 \) |
| \( v_4 \) | \( t^{-2/3}v + (xt^{-1/3} - 3)v_y + xt^{-2/3}v_{yy} + (at^{2/3}v_y + 2)(6 - t^{-2/3}v - t^{-1/3}v_y) = 3 \) |
| \( v_1 + av_3 \) | \((1 - a)v_y + a = 0 \) |
| \( v_2 + v_3 \) | \((a - 1)v_y + 1/a = 0 \) |
| \( v_1 + v_2 \) | \(-v_y + v_{yy} + av_y(1 + v_y) = 0 \) |

6 Characterization of differential invariants

Differential invariants help us to find general systems of differential equations which admit a prescribed symmetry group. One say, if \( G \) is a symmetry group for a system of PDEs with functionally differential invariants, then, the system can be rewritten in terms of differential invariants. For finding the differential invariants of the Eq. (1) up to order 2, we should solve the following systems of PDEs:

\[
\frac{\partial I}{\partial x}, \quad \frac{\partial I}{\partial t}, \quad \frac{1}{a} \frac{\partial I}{\partial u}, \quad 3t \frac{\partial I}{\partial t} - x \frac{\partial I}{\partial x} + (2t + u - \frac{2x}{a}) \frac{\partial I}{\partial u}, \tag{23}
\]

where \( I \) is a smooth function of \((x, t, u)\),

\[
\frac{\partial I_1}{\partial x}, \quad \frac{\partial I_1}{\partial t}, \quad \frac{1}{a} \frac{\partial I_1}{\partial u}, \quad 3t \frac{\partial I_1}{\partial t} - x \frac{\partial I_1}{\partial x} + \cdots + (2 - 2u) \frac{\partial I_1}{\partial u}, \tag{24}
\]

where \( I_1 \) is a smooth function of \((x, t, u, u_x, u_t)\),

\[
\frac{\partial I_2}{\partial x}, \quad \frac{\partial I_2}{\partial t}, \quad \frac{1}{a} \frac{\partial I_2}{\partial u}, \quad 3t \frac{\partial I_2}{\partial t} - x \frac{\partial I_2}{\partial x} + \cdots - u_{xt} \frac{\partial I_2}{\partial u_{xt}} - 5u_{tt} \frac{\partial I_2}{\partial u_{tt}}, \tag{25}
\]

where \( I_2 \) is a smooth function of \((x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt})\). The solutions of PDEs systems (23),(24) and (25) coming in table 5, where * and ** are refer to ordinary invariants and first order differential invariants respectively.
Table 5: differential invariants

| vector field | ordinary invariant | 1st order diff. invariant | 2nd order diff. invariant |
|--------------|-------------------|--------------------------|--------------------------|
| v₁           | t, u              | *, uₓ, uₓ                 | *, **, uₓₓ, uₓuₜ, uₓt    |
| v₂           | x, u              | *, uₓ, uₓ                 | *, **, uₓₓ, uₓuₜ, uₓt    |
| v₃           | x, t              | *, uₓ, uₓ                 | *, **, uₓₓ, uₓuₜ, uₓt    |
| v₄           | tx³, (−xᵃ⁻ᵗ + u)ₓ | *, x²uₓ = 1/a, uₓ−¹       | *, **, x³uₓₓ, uₓuₜ, uₓt  |

7 Nonclassical symmetries of the H-R equation

In this section we would like to apply the nonclassical method to the H-R equation. The graph of a solution

\[ u^{\alpha} = f^{\alpha}(x₁, \cdots, xₙ), \quad \alpha = 1, \cdots, q \]  

(26)

to the system (2) defines a p-dimensional submanifold \( \Gamma_f \subset \mathbb{R}^p \times \mathbb{R}^q \) of the space of independent and dependent variables. The solution will be invariant under the one-parameter subgroup generated by vector (4) if and only if \( \Gamma_f \) is an invariant submanifold of this group. By applying the well known criterion of invariance of a submanifold under a vector field we get that (26) is invariant under vector (4) if and only if \( f \) satisfies the first order system \( E_Q \) of partial differential equations

\[ Q^{\alpha}(x, u, u^{(1)}) = \varphi^{(\alpha)}(x, u) - \sum_{i=1}^{p} \xi^{i}(x, u)u^{\alpha}_{i} = 0, \quad \alpha = 1, \cdots, q \]  

(27)

known as the invariant surface conditions. The q-tuple \( Q = (Q^{1}, \cdots, Q^{q}) \) is known as the characteristic of the vector field (4). In what follows, the n-th prolongation of the invariant surface conditions (27) will be denoted by \( P^{(n)}_Q \), which is a n-th order system of partial differential equations obtained by appending to (27) its partial derivatives with respect to the independent variables of orders \( j \leq n - 1 \).

For the system (2), (27) to be compatible, the n-th prolongation \( P^{(n)}_v \) of the vector field \( v \) must be tangent to the intersection \( E \cap E^{(n)}_Q \)

\[ P^{(n)}_v(\Delta_\nu)|_{E \cap E^{(n)}_Q} = 0, \quad \nu = 1, \cdots, l. \]  

(28)

If the equations (28) are satisfied, then the vector field (28) is called a nonclassical infinitesimal symmetry of the system (2). The relations (28) are generalizations of the relations (5) for the vector fields of the infinitesimal classical symmetries. A similar procedure is applicable to the case of the nonclassical infinitesimal symmetries with an evident difference that in general one has fewer
determining equations than in the classical case. Therefore, we expect that nonclassical symmetries are much more numerous than classical ones, since any classical symmetry is clearly a nonclassical one. The important feature of determining equations for nonclassical symmetries is that they are nonlinear, this implies that the space of nonclassical symmetries does not, in general, form a vector space. For more theoretical background see [18, 1].

Consider the system $E$ of second order equations

$$
\begin{align*}
&u_t - v_t + au_x (1 - u_t) = 0, \\
&u_{xx} - v = 0
\end{align*}
$$

(29)

obtained from the H-R equation. If we assume that the coefficient of $\partial_t$ of the vector field (4) does not identically equal zero, then for the vector field

$$
\mathbf{v} = \xi(x, t, u, v) \partial_x + \partial_t + \varphi(x, t, u, v) \partial_u + \psi(x, t, u, v) \partial_v
$$

(30)

the invariant surface conditions are

$$
\begin{align*}
&u_t + \xi u_x = \varphi, \\
v_t + \xi v_x = \psi
\end{align*}
$$

(31)

The equations (28) take the forms

$$
\begin{align*}
(2\{u_t - 1/2\})u_x^2 a - u_t(u_x - v_x))\xi_u - u_t \psi_u - \psi_t \\
+ ((1 - 2u_t) a u_x + u_t) \varphi_u + (u_x(1 - u_t) a - u_t) \psi_v \\
+ (-u_x^2 (u_t - 1) a^2 + 2(u_t - 1/2) u_x^2 a - u_t(u_x - v_x))\xi_v \\
+ (u_x^2 (u_t - 1) a^2 + ((1 - 2u_t) u_x - v_x (u_t - 1)) a + u_t) \varphi_v \\
+ (1 - au_x) \varphi_t - a(u_t - 1) \varphi_x + u_x(u_t - 1)a \xi_x + (u_x^2 a - u_x + v_x) \xi_t = 0,
\end{align*}
$$

(32)

and

$$
\begin{align*}
-\psi - u_x \xi_x - 2u_x^2 \xi_{xx} + u_x^2 \varphi_{uu} - u_x^3 \xi_{uu} + u_{xx} \varphi_u - 2u_x \xi_x + v_x^2 \varphi_{uv} \\
+ v_x \varphi_v + 2u_x \varphi_{xx} + 2v_x \varphi_{xv} - 2u_x v_x \xi_{xx} + 2u_x v_x \varphi_{uv} - 2u_x^2 v_x \xi_{uv} \\
-3u_{xx} u_x \xi_x - 2u_{xx} v_x \xi_v - u_x v_x^2 \xi_{uv} - v_{xx} u_x \xi_v + v_x \varphi_{xx} = 0.
\end{align*}
$$

(33)

After inserting $\psi$ and its derivatives, as determined by the equation (33), into (32) and substituting $v_t = u_{xxx}, v_{xx} = u_{xxxx}$, and equating the coefficients of the various monomials in partial derivatives with respect to $x$ and various power of $u$, we can find the determining equations. Solving this equations, we get four Algebraic equations equal to zero. This means that no supplementary symmetries, of non-classical type, are specific for our model.

Now assume that the coefficient of $\partial_t$ in (30) equals zero and try to find the infinitesimal nonclassical symmetries of the form

$$
\mathbf{v} = \partial_x + \varphi(x, t, u, v) \partial_u + \psi(x, t, u, v) \partial_v
$$

(34)

for which the invariant surface conditions are the following ones

$$
\begin{align*}
u_x = \varphi, \\
v_x = \psi
\end{align*}
$$

(35)
Relations (28) lead to the system of equations for the functions $\phi$ and $\psi$

$$
(\nu^2_1 u_t - 1) a^2 + ((-v_x - 2 u_x) u_t + v_x + u_x) a + u_t)\phi_v
+ ((u_x - 2 u_x u_t) a - u_t)\phi_u + (u_x(u_t - 1) a - u_t)\psi_v
+ (1 - a u_t) \phi_t - a(u_t - 1) \phi_x - \psi_t - u_t \psi_u = 0,
$$

(36)

and

$$
-\psi + \phi_{xx} + 2 u_x \phi_{xu} + 2 \phi_{uv} v_x + 2 u_x^2 \phi_{uu} + 2 u_x v_x \phi_{uv}
+ u_{xx} \phi_u + v_x^2 \phi_{uu} + v_{xx} \phi_v = 0.
$$

(37)

Similar the previous case, we can find determining equations. Solving this equations, we get the following form of the coefficient functions

$$
\phi = c_1 u - c_1 x a + t(1 - 2 u_t)c_1 + c_2
$$

(38)

where $c_1$ and $c_2$ are arbitrary constant. So the system (29) admits the classical symmetry $v_1$ and nonclassical symmetry $v_2$:

$$
v_1 = \partial_x + \partial_u,
\quad
v_2 = \partial_x - \left(\frac{x}{a} - u - t + 2 t u_t\right) \partial_u
$$

(39)

8 Conservation laws of the H-R equation

Many methods for dealing with the conservation laws are derived, such as the method based on the Noether’s theorem, the multiplier method, by the relationship between the conserved vector of a PDE and the Lie-Bcklund symmetry generators of the PDE, the direct method, etc.[17, 2, 3, 19].

Now, we derive the conservation laws from the multiplier method.

**Definition 8.1** A local conservation law of the PDE system (2) is a divergence expression

$$
D_i \Phi^i[u] = D_1 \Phi^1[u] + \cdots + D_n \Phi^n[u] = 0
$$

(40)

holding for all solutions of the system (2). In (40), $\Phi^i[u] = \Phi^i(x, u, \partial_u, \cdots, \partial^{(r)} u)$, $i = 1, \cdots, n$, are called fluxes of the conservation law, and the highest-order derivative (r) present in the fluxes $\Phi^i[u]$ is called the order of a conservation law. [3]

In particular, a set of multipliers $\{\Lambda_{\nu}[U]\}_{\nu=1}^{l} = \{\Lambda_{\nu}(x, U, \partial_U, \cdots, \partial^{(r)} U)\}_{\nu=1}^{l}$ yields a divergence expression for the system $\Delta_{\nu}(x, u^{(n)})$ such that if the identity

$$
\Lambda_{\nu}[U] \Delta_{\nu}[U] = D_i \Phi^i[U]
$$

(41)

holds identically for arbitrary functions $U(x)$. Then on the solutions $U(x) = u(x)$ of the system (2), if $\Lambda_{\nu}[U]$ is non-singular, one has local conservation law $\Lambda_{\nu}[u] \Delta_{\nu}[u] = D_i \Phi^i[u] = 0$. 

12
Definition 8.2 The Euler operator with respect to $U^j$ is the operator defined by

$$E_{U^j} = \frac{\partial}{\partial U^j} - D_i \frac{\partial}{\partial U^j} + \cdots + (-1)^s D_i \cdots D_s \frac{\partial}{\partial U^j} + \cdots$$ (42)

for $j = 1, \cdots, q$. [3]

Theorem 8.3 The equations $E_{U^j} F(x, U, \partial U, \cdots, \partial_U^s) \equiv 0$, $j = 1, \cdots, q$ hold for arbitrary $U(x)$ if and only if $F(x, U, \partial U, \cdots, \partial_U^s) \equiv D_i \Psi^i(x, U, \partial U, \cdots, \partial_U^{s-1})$ holds for some functions $\Psi^i(x, U, \partial U, \cdots, \partial_U^{s-1})$, $i = 1, \cdots, q$. [3]

Theorem 8.4 A set of non-singular local multipliers $\{\Lambda_\nu(x, U, \partial U, \cdots, \partial_U^s)\}_{\nu=1}^l$ yields a local conservation law for the system $\Delta_\nu(x, u^{(n)})$ if and only if the set of identities

$$E_{U^j}(\Lambda_\nu(x, U, \partial U, \cdots, \partial_U^s)\Delta_\nu(x, u^{(n)})) \equiv 0, \ j = 1, \cdots, q.$$ (43)

holds for arbitrary functions $U(x)$. [3]

The set of equations (43) yields the set of linear determining equations to find all sets of local conservation law multipliers of the system (2). Now, we seek all local conservation law multipliers of the form $\Lambda = \xi(x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt})$ of the equation (1). The determining equations (43) become

$$E_U[\xi(x, t, U, U_x, U_t, U_{xx}, U_{xt}, U_{tt})(U_t - U_x^2 t + aU_x(1 - U_t))] \equiv 0,$$ (44)

where $U(x, t)$ are arbitrary function. Equation (44) split with respect to third order derivatives of $U$ to yield the determining PDE system whose solutions are the sets of local multipliers of all nontrivial local conservation laws of second order of H-R equation.

The solution of the determining system (44) given by

$$c_1 U_{xx} + \frac{1}{2} c_2 (2 t U_{tt} + U_t - 1) + c_3 u_{tt},$$ (45)

where $c_1$, $c_2$ and $c_3$ are arbitrary constant. So local multipliers given by

1) $\xi = U_{xx}, \quad 2) \xi = U_{tt}, \quad 3) \xi = tU_{tt} + \frac{1}{2} U_t - \frac{1}{2}$ (46)

Each of the local multipliers $\xi$ determines a nontrivial two-order local conservation law $D_t \Psi + D_x \Phi = 0$ with the characteristic form

$$D_t \Psi + D_x \Phi \equiv \xi(U_t - U_x^2 t + aU_x(1 - U_t)),$$ (47)

To calculate the conserved quantities $\Psi$ and $\Phi$, we need to invert the total divergence operator. This requires the integration (by parts) of an expression in multi-dimensions involving arbitrary functions and its derivatives, which is a difficult and cumbersome task. The homotopy operator [20] is a powerful algorithmic tool (explicit formula) that originates from homological algebra and variational bi-complexes.
Definition 8.5 The 2-dimensional homotopy operator is a vector operator with two components, \( \left( H_{u(x,t)}^{(x)} f, H_{u(x,t)}^{(t)} f \right) \), where

\[
H_{u(x,t)}^{(x)} f = \int_{0}^{1} \left( \sum_{j=1}^{q} f_{u}^{(j)} \right) [\lambda u] \frac{d\lambda}{\lambda} \quad \text{and} \quad H_{u(x,t)}^{(t)} f = \int_{0}^{1} \left( \sum_{j=1}^{q} f_{u}^{(t)} \right) [\lambda u] \frac{d\lambda}{\lambda} \quad (48)
\]

The x-integrand, \( f_{u}^{(x)} \), is given by

\[
f_{u}^{(x)} = \sum_{k_{1}=1}^{M_{1}^{x}} \sum_{k_{2}=0}^{M_{2}^{x}} \left( \sum_{i_{1}=0}^{k_{1}-1} \sum_{i_{2}=0}^{k_{2}} B^{(x)} u_{x_{1} t_{1} t_{2}} \left( -D_{x} \right)^{k_{1}-i_{1}-1} \left( -D_{t} \right)^{k_{2}-i_{2}} \right) \frac{\partial f}{\partial u_{x_{1} t_{1} t_{2}}}, \quad (49)
\]

where \( M_{1}^{x}, M_{2}^{x} \) are the order of \( f \) in \( u \) to \( x \) and \( t \) respectively and combinatorial coefficient

\[
B^{(x)} = B(i_{1}, i_{2}, k_{1}, k_{2}) = \frac{\left( i_{1} + i_{2} \right) \left( k_{1} + k_{2} - i_{1} - i_{2} - 1 \right)}{k_{1} k_{2}}, \quad (50)
\]

Similarly, the t-integrand, \( f_{u}^{(t)} \), is defined as

\[
f_{u}^{(t)} = \sum_{k_{1}=0}^{M_{1}^{t}} \sum_{k_{2}=1}^{M_{2}^{t}} \left( \sum_{i_{1}=0}^{k_{1}} \sum_{i_{2}=0}^{k_{2}} B^{(t)} u_{x_{1} t_{1} t_{2}} \left( -D_{x} \right)^{k_{1}-i_{1}} \left( -D_{t} \right)^{k_{2}-i_{2} - 1} \right) \frac{\partial f}{\partial u_{x_{1} t_{1} t_{2}}}, \quad (51)
\]

where \( B^{(t)}(i_{2}, i_{1}, k_{2}, k_{1}) \).

For instance we apply homotopy operator to find conserved quantities \( \Psi \) and \( \Phi \) which yield of multiplier \( \xi = u_{tt} \). We have

\[
f = u_{tt} (u_{t} - u_{x} t + a u_{x} (1 - u_{t})), \quad (52)
\]

the integrands (49) and (51) are

\[
\begin{align*}
I_{u}^{(x)} f &= a u u_{x} s - a u u_{x} t_{2} - \frac{2}{3} a u u_{x} t_{3} + \frac{1}{3} u_{t} u_{x} s + \frac{1}{3} u_{t} u_{x} t_{2} + \frac{1}{3} u_{x} u_{x_{1}} t - \frac{2}{3} u_{x} u_{x_{1} t_{1}}, \\
I_{u}^{(t)} f &= u_{1}^{2} - u_{t} u_{x} t + a u_{x} u_{t} - a u_{x} u_{t}^{2} + \frac{2}{3} a u u_{x} t_{2} - a u u_{x t} + a u u_{x t} u_{t} \\
&+ \frac{1}{3} u_{x} u_{x_{1}} t - \frac{1}{3} u_{x} u_{x_{1} t_{1}},
\end{align*}
\]

and apply (48) to the integrands (53), therefore

\[
\begin{align*}
H_{u(x,t)}^{(x)} f &= \frac{1}{2} a u u_{x} s - \frac{1}{3} a u u_{x} t_{2} - \frac{1}{3} u u u_{x_{1}} t + \frac{1}{3} u u u_{x_{1}} t_{2} + \frac{1}{3} u_{x} u_{x_{1}} t - \frac{1}{3} u_{x} u_{x_{1} t_{1}}, \\
H_{u(x,t)}^{(t)} f &= \frac{1}{2} u_{1}^{2} - \frac{1}{2} u_{t} u_{x} t_{2} + \frac{1}{3} a u u_{x} u_{t} - \frac{1}{3} a u u_{x} u_{t}^{2} + \frac{1}{3} u u u_{x_{1}} t_{2} - \frac{1}{2} a u u_{x t} \\
&+ \frac{2}{3} a u u_{x} u_{t} + \frac{1}{3} u_{x} u_{x_{1}} t - \frac{1}{3} u_{x} u_{x_{1} t_{1}},
\end{align*}
\]

\[
(54)
\]
so, we have the first conservation low of the H-R equation respect to multiplier \( \xi = u_{tt} \)

\[
D_x \left( \frac{1}{4} uu_{tt} - \frac{1}{2} auu_{tt}u_{tt} - \frac{1}{4} uu_{tt}u_{tt} + \frac{1}{2} u_t u_{tt} + \frac{1}{6} u_x u_{tt} - \frac{1}{4} u_x u_{tt} \right) \\
+ D_t \left( \frac{1}{2} u_t^2 - \frac{1}{2} u_t u_{xx} + \frac{1}{2} au u_t - \frac{1}{3} au x u_t^2 + \frac{1}{3} uu_{xx} - \frac{1}{2} au u_t \\
+ \frac{1}{3} uu_x u_t + \frac{1}{6} u_x u_{xx} - \frac{1}{6} u_x u_{xx} \right) = 0.
\]

(55)

Similarly, conservation law respect to multiplier \( \xi = u_{xx} \) is

\[
D_x \left( \frac{1}{4} uu_{xx} + \frac{1}{2} auu_{xx} - \frac{1}{3} au u_{xx}^2 - \frac{1}{2} uu_{xx} + \frac{1}{6} uu_{xx} \right) \\
+ D_t \left( \frac{1}{6} uu_{xx} - \frac{1}{2} auu_{xx} - \frac{1}{3} u_{xx} \right) = 0.
\]

(56)

and conservation law respect to multiplier \( \xi = tu_{tt} + \frac{1}{2} u_t - \frac{1}{2} \) is

\[
D_x \left( -\frac{1}{4} au + \frac{1}{2} au_{tt} + \frac{1}{3} at uu_{tt} - \frac{1}{4} at uu_{tt} - \frac{1}{2} uu_{tt} \\
- \frac{1}{2} uu_{tt} - \frac{1}{2} uu_{tt} + \frac{1}{6} uu_{tt} + \frac{1}{2} uu_{tt} + \frac{1}{6} uu_{tt} + \frac{1}{2} uu_{tt} \right) \\
+ D_t \left( -\frac{1}{2} uu_{tt} + \frac{1}{3} u_{xx} u_t + \frac{1}{2} uu_{xx} + \frac{1}{2} uu_{xx} + \frac{1}{2} uu_{xx} + \frac{1}{2} uu_{xx} + \frac{1}{2} uu_{xx} \\
- \frac{1}{2} uu_{xx} + \frac{1}{2} uu_{xx} + \frac{1}{2} uu_{xx} + \frac{1}{2} uu_{xx} - \frac{1}{2} uu_{xx} + \frac{1}{2} uu_{xx} \right) = 0.
\]

(57)

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