Gauge Non-Invariant Higher-Spin Currents in 4d Minkowski Space

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Abstract

Conserved currents of any spin \( t > 0 \) built from symmetric massless gauge fields of any integer spin \( s \geq t \) in 4d Minkowski space are found. In particular, stress-energy tensor for a higher-spin field of any spin is constructed. Analogously to spin-two stress (pseudo)tensor, currents considered in this paper are not gauge invariant. However, they are shown to generate gauge invariant conserved charges. Besides expected parity even HS currents, we found unexpected parity odd currents that generate less symmetries than the even ones. It is argued that these odd currents unlikely admit a consistent AdS deformation.


1 Introduction

Conserved charges and currents play significant role in field theory. Although conserved currents for higher-spin (HS) fields were extensively studied in the literature \[1, 2, 3, 4, 5, 6, 7\], so far main attention was payed to the gauge invariant currents. Generally a conserved current carries three spins \((t, s_1, s_2)\), where \(t\) is a spin of current, while \(s_1\) and \(s_2\) are spins of fields from which it is constructed. In this paper, we consider the case of \(s_1 = s_2 = s\). For example, stress tensor \((t = 2)\) exists for matter fields of arbitrary spins \(s_1 = s_2 = s\). It is well known however, that for \(s = t = 2\), the stress is not gauge invariant, corresponding to so-called gravitational stress pseudo-tensor \([8]\). As shown by Deser and Waldron \([5]\), for \(t = 2\) analogous phenomenon occurs for all massless fields of spins \(s > 2\). \(t = 1\) currents have similar property. The currents with \(s < 1\) are gauge invariant (in this case, for a trivial reason since a spin-zero field has no gauge symmetry), while the spin-one current built from two massless spin-one fields is not.

The aim of this paper is to present the full list of non-gauge invariant currents with \(s \geq t\) in 4d Minkowski space. Being non-gauge invariant, such currents may not immediately lead to consistent higher-spin (HS) Noether current interactions (For the related discussion of HS interactions we refer the reader to \([9, 10]\) and references therein.) On the other hand, non-gauge invariant currents give rise to gauge invariant conserved charges.

Surprisingly, we found more conserved currents that was originally expected. Namely, besides expected parity even HS currents we found unexpected parity odd currents. The latter generate less global symmetries than the even ones. As discussed in more detail in Conclusion, the odd currents can unlikely admit a consistent \(AdS\) deformation and probably correspond to some currents of mixed symmetry fields in any dimension that amount to the symmetric fields at \(d = 4\).

The construction of currents and charges may be important for the analysis of black-hole physics in HS theory \([11, 12]\). To this end, however, the analysis of this paper has to be generalized to \(AdS\) background geometry since nonlinear HS gauge theory can only be formulated in a curved background (for review and more references see \([2]\)). This problem will be considered elsewhere.

The paper is organized as follows. In Section 2, we recall the frame-like formulation of massless fields in two-component spinor notation. In Section 3, non-gauge invariant currents built from symmetric HS fields in 4d Minkowski space are found and shown to give rise to gauge invariant conserved charges. In Section 4, further perspectives are briefly discussed.
Conventions

Our conventions are as follows: Greek indices \( \mu, \nu, \rho, \lambda, \sigma \) are base and run from 0 to 3. Other Greek indices are spinorial and take values 1, 2. Spinor indices are raised and lowered by the \( \text{sp}(2) \) antisymmetric forms \( \varepsilon_{\alpha\beta}, \varepsilon^{\alpha\beta}, \dot{\varepsilon}_{\dot{\alpha}\dot{\beta}}, \dot{\varepsilon}^{\dot{\alpha}\dot{\beta}} \) in the following way

\[
\varepsilon^{\alpha\beta}\varepsilon_{\alpha\gamma} = \delta^\beta_\gamma, \quad \varepsilon^{\dot{\alpha}\dot{\beta}}\varepsilon_{\dot{\alpha}\dot{\gamma}} = \delta^\dot{\beta}_{\dot{\gamma}},
\]

\[
A_\alpha = A^\beta\varepsilon_{\beta\alpha}, \quad A^\alpha = A_\beta\varepsilon^{\alpha\beta}, \quad A_{\dot{\alpha}} = A^\dot{\beta}\varepsilon_{\dot{\beta}\dot{\alpha}}, \quad A^{\dot{\alpha}} = A_\dot{\beta}\varepsilon^{\dot{\alpha}\dot{\beta}}.
\]  

(1)

Complex conjugation \( \bar{A} \) relates dotted and undotted spinors. Brackets \( \{\ldots\} \) imply complete (anti)symmetrization, i.e.,

\[
A_{[\alpha \beta]} = \frac{1}{2}(A_\alpha B_\beta - A_\beta B_\alpha), \quad A_{(\alpha \beta)} = \frac{1}{2}(A_\alpha B_\beta + A_\beta B_\alpha).
\]

(2)

\( A^{\alpha(m)} \) denotes a totally symmetric multispinor \( \{\alpha_1\ldots\alpha_m\} \).

The wedge symbol \( \wedge \) is implicit.

2 4d massless HS fields in the frame-like approach

In the metric-like formalism [13], an integer spin-\( s \) symmetric massless field is described by a totally symmetric tensor \( \varphi_{\mu_1\ldots\mu_s} \) subject to the double tracelessness condition \( \varphi_{\rho \lambda \mu_3\ldots\mu_s} = 0 \) which is nontrivial for \( s \geq 4 \). The action is

\[
S[\varphi] = \frac{1}{2} \int d^4x \left( \partial_\nu \varphi_{\mu_1\ldots\mu_s} \partial^\nu \varphi^{\mu_1\ldots\mu_s} - \frac{1}{2} s(s-1) \partial_\nu \varphi_\lambda^{\mu_3\ldots\mu_s} \partial^\nu \varphi_\rho^{\mu_3\ldots\mu_s} \right. \\
+ s(s-1) \partial_\nu \varphi_\lambda^{\mu_3\ldots\mu_s} \partial_\rho \varphi^{\rho \mu_3\ldots\mu_s} - \frac{3}{4} s(s-1)(s-2) \partial_\nu \varphi_\lambda^{\rho \mu_3\ldots\mu_s} \partial_\rho \varphi_\sigma^{\lambda \mu_3\ldots\mu_s} \left. \right).
\]

(4)

Instead of deriving conserved currents from this action using Noether's theorem, that requires finding corresponding HS symmetries, we will look for conserved currents using the frame-like formulation where the HS metric-like field is replaced by a frame field and a set of 1-form connections [14, 15]

\[
\begin{align*}
\varphi_{\mu_1\ldots\mu_s} &\rightarrow \{\omega^{\alpha(m)}_{\beta(n)} \mid m + n = 2(s-1)\}, & \omega^{\alpha(m)}_{\beta(n)} &= dx^\mu \omega^{\alpha(m)}_{\mu \beta(n)}, \\
\end{align*}
\]

which are symmetric in all dotted and all undotted spinor indices and satisfy the reality condition [15]

\[
\omega^{\dagger \alpha(m), \beta(n)} = \omega^{\beta(n), \dagger \alpha(m)}.
\]

(5)
The frame-like field is a particular connection at $n = m = s - 1$

$$\tilde{h}_\mu^{\alpha(s-1), \beta(s-1)} dx^\mu := \omega_\mu^{\alpha(s-1), \beta(s-1)} dx^\mu. \quad (6)$$

By imposing appropriate constraints the connections $\omega^{\alpha(m), \beta(n)}$ can be expressed via $t = \frac{1}{2} |m - n|$ derivatives of the frame field [15].

$P$ reflection of the $x_2$ direction $(A_0, A_1, A_2, A_3) \rightarrow (A_0, A_1, -A_2, A_3)$ is defined as [15]

$$P\omega^{\alpha(m), \beta(n)} = \omega^{\beta(n), \alpha(m)}, \quad P\tilde{h}_{\alpha, \beta} = \tilde{h}_{\beta, \alpha}. \quad (7)$$

Compared to the Hermitian conjugation, the $P$-reflection does not affect an order of product factors and a sign of an imaginary unit $i$.

Background gravity is described by the vierbein 1-form $\tilde{h}_\alpha \bar{\gamma}$ and 1-form connections $\tilde{\omega}^{\alpha \beta}, \tilde{\omega}^{\alpha \beta}$. Lorentz covariant derivative $\tilde{D}$ acts as usual

$$\tilde{D}A^{\alpha(m), \beta(n)} = dA^{\alpha(m), \beta(n)} + m\tilde{\omega}^{\alpha \gamma} A^{\alpha(m-1) \gamma, \beta(n)} + n\tilde{\omega}^{\beta \delta} A^{\alpha(m), \beta(n-1) \delta}, \quad (8)$$

for any multispinor $A^{\alpha(m), \beta(n)}$. The torsion and curvature 2-forms are

$$\tilde{\tilde{R}}^{\alpha \beta} = \tilde{D}\tilde{\omega}^{\alpha \beta} = d\tilde{\omega}^{\alpha \beta} + 2\tilde{\tilde{R}}^{\alpha \beta} = \tilde{D}\tilde{\omega}^{\alpha \beta} = d\tilde{\omega}^{\alpha \beta} + 2\tilde{\tilde{R}}^{\alpha \beta} = 0. \quad (11)$$

Linearized HS curvatures are

$$R^{\alpha(m), \beta(n)} = \tilde{D}\omega^{\alpha(m), \beta(n)} + n\theta(m-n)h_{\gamma \delta}^{\beta} \omega^{\alpha(m), \beta(n-1)} + m\theta(n-m)h_{\alpha \delta}^{\beta} \omega^{\alpha(m-1), \beta(n) \delta}, \quad (12)$$

where

$$\theta(x) = \begin{cases} 1 & \text{at } x \geq 0; \\ 0 & \text{at } x < 0. \end{cases} \quad (13)$$

Free field equations for massless fields of spins $s \geq 2$ in Minkowski space can be written in the form [15]

$$R^{\alpha(m), \beta(n)} = 0 \quad \text{for} \quad n > 0, m > 0, n + m = 2(s - 1); \quad (14)$$

$$R^{\alpha(m)} = C^{\alpha(m) \gamma \delta} h_{\gamma \delta}^{\beta} \tilde{h}_{\beta}^{\alpha} \quad \text{for} \quad m = 2(s - 1); \quad (15)$$

$$R^{\beta(n)} = C^{\beta(n) \gamma \delta} \tilde{h}_{\gamma \delta}^{\beta} \tilde{h}_{\alpha}^{\alpha} \quad \text{for} \quad n = 2(s - 1). \quad (16)$$
They are equivalent to the equations of motion which follow from the action (4), supplemented with certain algebraic constraints which express connections $\omega^{\alpha(m),\beta(n)}$ via $\frac{1}{2}|m-n|$ derivatives of the dynamical frame-like HS field. The multispinor zero-forms $C^\alpha(2s)$ and $C^{\beta(2s)}$, which remain non-zero on-shell, are spin-$s$ analogues of the Weyl tensor in gravity.

HS gauge transformation is

$$\delta \omega^{\alpha(m),\beta(n)} = \tilde{D} \epsilon^{\alpha(m),\beta(n)} + n\theta(m-n)\tilde{h}_\gamma, \epsilon^{\gamma\alpha(m),\beta(n-1)} + m\theta(n-m)\tilde{h}_\alpha, \epsilon^{\alpha(m-1),\beta(n)\delta}$$

(17)

where a gauge parameter $\epsilon^{\alpha(m),\beta(n)}(x)$ is an arbitrary function of $x$.

### 3 Currents in Minkowski space

#### 3.1 Problem setting

We will describe currents as Hodge dual differential forms. In these terms, the on-shell closure condition for the latter is traded for the current conservation condition. In this paper, we consider currents built from two HS connections with $s_1 = s_2 = s$ in $d = 4$ Minkowski space where a spin-$t$ current is a 3-form $J^{\alpha(t-1),\beta(t-1)}$. Such current will be shown to contain $t$ derivatives of a spin-$s$ dynamical field for any $s$. We consider currents with $1 \leq t \leq s$. Those with $t > s$ are built from gauge invariant combinations of the HS Weyl tensors and their derivatives and are available in the literature [6, 16].

Conserved currents generate conserved charges. By Noether theorem the latter are generators of global symmetries. Hence, one should expect as many conserved charges as global symmetry parameters. The latter can be identified with those gauge symmetry parameters that leave the gauge fields invariant. Denoting global symmetry parameters $\xi_{\alpha(m),\beta(n)}(x)$, from (17) it follows that they have to obey the conditions

$$D\xi^{\alpha(m),\beta(n)} := \tilde{D} \xi^{\alpha(m),\beta(n)} + n\theta(m-n)\tilde{h}_\gamma, \xi^{\alpha(m),\beta(n-1)} + m\theta(n-m)\tilde{h}_\alpha, \xi^{\alpha(m-1),\beta(n)\delta} = 0$$

(18)

For example, consider spin-two global symmetries with parameters $\xi^\alpha, \xi^\alpha, \xi^{\beta\beta}$ which obey

$$\tilde{D} \xi^\alpha + \tilde{h}_\gamma, \xi^{\alpha\gamma} + \tilde{h}_\delta, \xi^{\beta,\delta} = 0$$

$$\tilde{D} \xi^{\alpha\alpha} = 0, \quad \tilde{D} \xi^{\beta\beta} = 0.$$
In the Cartesian coordinate system this gives
\[ d\xi^\alpha_{\beta, \bar{\beta}} + \tilde{h}_{\gamma, \beta} \xi^{\alpha \gamma} + \tilde{h}^\alpha_{\delta, \bar{\beta}} \xi^{\delta \bar{\beta}} = 0, \]
\[ d\xi^{\alpha \bar{\alpha}} = 0, \quad d\xi^{\beta \bar{\beta}} = 0. \]

The general solution of this system is
\[ \xi^\alpha_{\beta} = a^\alpha_{\beta} - x^\alpha_{\gamma} b^{\gamma \delta} - x^\delta_{\beta} c^{\beta \delta}, \]
\[ \xi^{\alpha \bar{\alpha}} = b^{\alpha \bar{\alpha}}, \quad \xi^{\beta \bar{\beta}} = c^{\beta \bar{\beta}}. \]

where parameters \( a^\alpha_{\beta}, b^{\alpha \bar{\alpha}}, c^{\beta \bar{\beta}} \) are arbitrary constants which parametrize Poincaré algebra. Analogously, for a spin \( s \), there is as many global symmetry parameters as the gauge parameters \( \epsilon_{\alpha(m), \beta(n)} \) with \( n + m = 2(s - 1) \).

To define a HS charge as an integral over a 3\( d \) space, we should find such a current 3-form \( J_{\alpha(m), \beta(n)}(x) \) built from dynamical HS fields that
\[ J_{\xi}(x) = \sum_{n, m} \xi_{\alpha(m), \beta(n)}(x) J^{\alpha(m), \beta(n)}(x), \quad n + m = 2(t - 1) \quad (19) \]
is closed by virtue of \( (18) \) and the HS field equations \( (14)-(16) \). This demands \( J_{\alpha(m), \beta(n)}(x) \) to obey the following conservation equation
\[ D^* J^{\alpha(m), \beta(n)} := \tilde{D} J^{\alpha(m), \beta(n)} - n\theta(n - m - 1) \tilde{h}_{\gamma, \beta} J^{\gamma \alpha(m), \beta(n-1)} - m\theta(m - n - 1) \tilde{h}^\alpha_{\delta, \bar{\beta}} J^{\alpha(m-1), \beta(n) \delta} \approx 0, \quad (20) \]
where \( \approx \) implies that the equality holds on-shell. Note, that as a consequence of \( (20) \) the spin-\( t \) current \( J^{\alpha(t-1), \beta(t-1)} \) obeys the conventional conservation condition
\[ \tilde{D} J^{\alpha(t-1), \beta(t-1)} \approx 0. \quad (21) \]

Taking into account that 3-form currents are related to the usual currents by the Levi-Civita symbol \( \epsilon^{\mu \nu \rho} \) which is parity-odd, we call the current \( J_{\alpha(t-1), \beta(t-1)} \) even if \( PJ_{\alpha(t-1), \beta(t-1)} = -J_{\beta(t-1), \alpha(t-1)} \) and odd if \( PJ_{\alpha(t-1), \beta(t-1)} = J_{\beta(t-1), \alpha(t-1)} \).

The derivatives \( D \) and \( D^* \) are defined in such a way that
\[ dJ_{\xi}(x) = \sum_{n, m} D\xi_{\alpha(m), \beta(n)}(x) J^{\alpha(m), \beta(n)}(x) + \sum_{n, m} \xi_{\alpha(m), \beta(n)}(x) D^* J^{\alpha(m), \beta(n)}(x). \quad (22) \]

This means that \( D \) and \( D^* \) are mutually conjugated. We call \( D \) and \( D^* \) adjoint and co-adjoint derivatives, respectively, since they result from the restriction of the respective derivatives of the flat contraction of the HS algebra to its Poincaré subalgebra.
Eq. (22) implies that the charge

\[ Q_\xi = \int_{M^3} J_\xi \]

is conserved by virtue of (18) and (20). As a result, there are as many conserved charges \( Q_\xi \) as independent global symmetry parameters \( \xi \). Conservation of currents does not imply that they are invariant under the gauge transformations (17). However, as shown below, the gauge variation of \( J_\xi \) is exact

\[ \delta J_\xi(x) \simeq dH_\xi(x), \]

so that the charge \( Q_\xi \) turns out to be gauge invariant.

Nontrivial charges are represented by the current \( J_\xi(x) \) cohomology, i.e., closed currents modulo exact ones \( J_\xi \simeq d\Psi_\xi \). Since the currents should be closed on-shell, i.e., by virtue of the free field equations (14)-(16), analysis is greatly simplified by the fact that all linearized HS curvatures \( R^{(m),\beta(n)} \) with \( m > 0, n > 0 \) are zero on shell.

Thus, the problem is

(a) to find all 3-forms \( J^{(t-1)},\dot{J}^{(t-1)} \) built from spin-s fields \( 1 \leq t \leq s \), which are on-shell closed (i.e. obey (21)) but not exact;

(b) to find all 3-forms \( J^{(m)},\dot{J}^{(n)} \), \( m \neq n, m + n = 2(t - 1) \), that obey (20);

(c) to check that \( J^{(t-1)},\dot{J}^{(t-1)} \) are Hermitian \((J^{(t-1)},\dot{J}^{(t-1)})^\dagger = J^{\dot{J}^{(t-1)},\dot{J}^{(t-1)}}\); 

(d) to check that the HS charges are gauge invariant.

Consider first some examples.

\section*{3.2 Examples}

\subsection*{3.2.1 Spin-one current}

Spin-one current \( J \) carries no indices and generalizes the standard current built from the complex scalar field \( j_\mu = \varphi \partial_\mu \varphi^* - \varphi^* \partial_\mu \varphi \) to massless fields of any integer spin. A spin-one current \( J \) contains one derivative of the spin-s dynamical field. As usual, to construct nontrivial spin-one current a color index is needed, i.e., HS connections \( \omega^{(m),\beta(n)} \) are endowed with an additional index \( i = 1 \ldots N \) which labels independent dynamical fields. To contract color indices, we introduce the real matrix \( c_{ij} \) which can be either symmetric, \( c_{ij} = c_{ji} \), or antisymmetric, \( c_{ij} = -c_{ji} \).
Consider a 3-form

\[ J = c_{ij}[\omega^{i;}_{;\lambda}(s-2) \delta(s-1) \omega^{j}; \gamma(s-2)_{,0} \delta(s-1) \delta(s-2) \omega^{j}; \gamma(s-1)_{,0} \delta(s-2) \delta(s) \tilde{h} \phi]. \tag{25} \]

Note that the anticommutativity of the background vierbein 1-forms

\[ \tilde{h}_{\alpha, \dot{\beta}} h_{\gamma, \dot{\delta}} = - \tilde{h}_{\gamma, \dot{\beta}} h_{\alpha, \dot{\delta}} \tag{26} \]

implies that

\[ \tilde{h}_{\alpha, \dot{\beta}} h_{\gamma, \dot{\delta}} = \epsilon_{\alpha \gamma \delta} \hat{H}_{\beta \delta} + \epsilon_{\beta \delta \alpha} \hat{H}_{\gamma \delta}, \tag{27} \]

where

\[ H_{\alpha \beta} = \frac{1}{2} \tilde{h}_{\alpha, \dot{\gamma}} h_{\beta, \dot{\gamma}}, \quad \hat{H}_{\alpha \beta} = \frac{1}{2} \tilde{h}_{\alpha, \dot{\gamma}} \hat{h}_{\beta, \dot{\gamma}} \tag{28} \]

provide a basis of 2-forms in 4d Minkowski space.

Using (27) and (28), direct computation gives

\[
\begin{align*}
\frac{dJ}{c_{ij}} & = R^{\alpha}_{\beta 2}(s-2) \delta(s-1) \omega^{j}; \gamma(s-2)_{,0} \delta(s-1) \delta(s-2) \omega^{j}; \gamma(s-1)_{,0} \delta(s-2) \delta(s) \tilde{h} \phi, \\
& - R^{\alpha}_{\beta 2}(s-1) \delta(s-2) \omega^{j}; \gamma(s-2)_{,0} \delta(s-1) \delta(s-2) \omega^{j}; \gamma(s-1)_{,0} \delta(s-2) \delta(s) \tilde{h} \phi, \\
& - (s-2)(\delta(s) \omega^{j}; \gamma(s-3)_{,0} \delta(s-3) \delta(s) H \phi + \omega^{j} \gamma(s-3) \delta(s) H \phi) - (\gamma(s-3)_{,0} \delta(s) H \phi) - (s-2)(\omega^{j} \gamma(s-3) \delta(s) H \phi).
\end{align*}
\]

For \( s > 2 \) the curvature-dependent terms vanish by virtue of Eq. (14). For \( s = 2 \)
\( R^{\alpha 2} = C^{\alpha 2} \gamma \delta \tilde{h}_{\gamma, \dot{\delta}} \) and \( R^{\beta 2} = C^{\beta 2} \gamma \delta \tilde{h}_{\phi, \gamma} \). In this case \( C \)-dependent terms also vanish because antisymmetrization over any three two-component indices gives zero. For example

\[ X := \omega^{\phi} \delta C_{\delta \chi \beta \gamma} h_{\gamma, \dot{\gamma}} h_{\phi, \dot{\gamma}} = 0, \tag{30} \]

because the antisymmetrization of three undotted spinor indices occurs due to the anticommutativity of the 1-forms \( \tilde{h}_{\alpha, \dot{\beta}} \).

As a result, we obtain

\[
\begin{align*}
\frac{dJ}{c_{ij}} & \simeq c_{ij}(s(\delta(s-2) \omega^{j}; \gamma(s-2)_{,0} \delta(s-1) \delta(s-2) \omega^{j}; \gamma(s-1)_{,0} \delta(s-2) \delta(s) \tilde{h} \phi) + \omega^{j} \gamma(s-3) \delta(s) H \phi + \omega^{j} \gamma(s) \delta(s) H \phi), \\
& - (s-2)(\delta(s) \omega^{j}; \gamma(s-3)_{,0} \delta(s) H \phi + \omega^{j} \gamma(s-3) \delta(s) H \phi) - (\gamma(s-3)_{,0} \delta(s) H \phi)
\end{align*}
\]

which is zero if \( c_{ij} \) is antisymmetric. \( J \) is Hermitian, \( J^\dagger = J \), if \( c_{ij} \) is pure imaginary.

To prove that \( J \) is not exact consider on-shell improvements \( d\Psi \) with 2-form \( \Psi \) containing no derivatives of the dynamical HS field to give rise to a current with one derivative upon differentiation. There is just one such form with antisymmetric \( c_{ij} \)

\[ \Psi = c_{ij}[\omega^{i;}_{;\lambda}(s-1) \delta(s-1) \omega^{j}; \gamma(s-1)_{,0} \delta(s-1)]. \]
This gives
\[ d\Psi \simeq -2(s-1)c_{ij}(-\omega^{i;\gamma(s-2)}\delta(s-1)\omega_j^{\gamma (s-2)}\delta(s-1)\delta h_\varphi^\theta + \omega^{i;\gamma(s-1)}\delta(s-2)\omega_j^{\gamma (s-2)}\delta h_\varphi^\theta). \]

Since the two terms in (25) have different relative signs, the 3-form \( J \) is closed, but not exact, thus, representing the on-shell current cohomology.

### 3.2.2 Spin-two current

The spin-two current is represented by the 3-form
\[ J^{\alpha,\beta} = 2c_{ij}\omega^{i;\alpha\gamma(s-2)}\delta(s-2)\omega_j^{\gamma (s-2)}\delta(s-1)\delta h_\varphi^\theta, \]

which carries two derivatives of the HS field. Like in the spin-one case, discarding the curvature-dependent terms (for \( s = 2 \), \( C \)-dependent terms vanish like in (30)), direct computation gives
\[ \tilde{D}J^{\alpha,\beta} \simeq 0, \]

regardless of the symmetry properties of \( c_{ij} \).

To see that the 3-form \( J^{\alpha,\beta} \) is not exact we observe that the only appropriate 2-forms are
\[ \Psi_1^{\alpha,\beta} = c_{ij}\omega^{i;\alpha\gamma(s-2)}\delta(s-1)\omega_j^{\gamma (s-2)}\delta(s-1)\delta h_\varphi^\theta, \]
\[ \Psi_2^{\alpha,\beta} = c_{ij}\omega^{i;\alpha\gamma(s-1)}\delta(s-2)\omega_j^{\gamma (s-1)}\delta(s-2)\delta h_\varphi^\theta. \]

An elementary computation gives
\[ \tilde{D}\Psi_1^{\alpha,\beta} \simeq c_{ij}[-\omega^{i;\gamma(s-2)}\delta(s-1)\delta h_\varphi^\gamma,\delta(s-2)\delta h_\varphi^\beta + (s-2)\omega^{i;\alpha\gamma(s-3)}\delta(s-1)\delta h_\varphi^{\theta,\varphi} - (s-1)\omega^{i;\alpha\gamma(s-2)}\delta(s-2)\delta h_\varphi^{\theta,\varphi}], \]
\[ \tilde{D}\Psi_2^{\alpha,\beta} \simeq c_{ij}[-(s-2)\omega^{i;\alpha\gamma(s-1)}\delta(s-3)\delta h_\varphi^{\gamma,\theta} - \omega^{i;\alpha\gamma(s-2)}\delta(s-2)\delta h_\varphi^{\theta,\varphi}]. \]

\( J^{\alpha,\beta} \) is not a linear combination of \( \tilde{D}\Psi_1^{\alpha,\beta} \) and \( \tilde{D}\Psi_2^{\alpha,\beta} \) since the latter contain (different) terms like \( c_{ij}\omega^{i;\alpha\gamma(s-1)}\delta(s-2)\omega_j^{\gamma (s-2)}\delta h_\varphi^{\theta,\varphi} \), \( \beta \) not present in (32).

(Although by antisymmetrization of some three two-component indices it is possible to reshuffle spinor indices in this term, still \( \tilde{D}\Psi_1^{\alpha,\beta} \) and \( \tilde{D}\Psi_2^{\alpha,\beta} \) contain extra terms compared to (32).)

This leads to a surprising result that there are two independent Hermitian conserved currents (32) \( J_+^{\alpha,\beta} = J^{\alpha,\beta} \) with \( c_{ij} = c_{ji} \) and \( J_-^{\alpha,\beta} = iJ^{\alpha,\beta} \) with \( c_{ij} = -c_{ji} \).
Consider first the current \( J^\alpha_+ \), which is even because \( PJ^\alpha_+ = -J^\beta_+ \). For \( J^\alpha_+, \beta \) equations (20) give

\[
D^\ast J^\beta_+ = -2 \tilde{h}_h^\gamma J^\gamma_+ + \tilde{D} J^\beta_+ \simeq 0, \tag{34}
\]

\[
D^\ast J^\alpha_+ = -2 \tilde{h}_h^\alpha J^\alpha_+ + \tilde{D} J^\alpha_+ \simeq 0. \tag{35}
\]

These equations are solved by

\[
J^\beta_+ = -4c_{ij} \omega^i \varphi^j \gamma(s-2) \delta(s-2) \tilde{\omega}^j \gamma(\gamma(s-2), \delta(s-2) \tilde{\beta} \tilde{h}_\phi^j, \tilde{\beta} h_\phi^j, \tilde{\beta} h_\phi^j), \tag{36}
\]

\[
J^\alpha_+ = -4c_{ij} \omega^i \varphi^j \gamma(s-2) \delta(s-2) \tilde{\omega}^j \gamma(\gamma(s-2), \delta(s-2) \tilde{\beta} \tilde{h}_\phi^j, \tilde{\beta} h_\phi^j, \tilde{\beta} h_\phi^j). \tag{37}
\]

Indeed, direct computation with the aid of (26) gives on shell

\[
\tilde{D} J^\beta_+ \simeq -4c_{ij} [8 \omega^i \varphi^j \gamma(s-2) \delta(s-2) \tilde{\omega}^j \gamma(\gamma(s-2), \delta(s-2) \tilde{\beta} \tilde{h}_\psi^j -
-(s-2) \omega^i \varphi^j \gamma(s-3) \delta(s-1) \tilde{\omega}^j \gamma(\gamma(s-3), \delta(s-1) \tilde{\beta} \tilde{H}_\phi^j - \omega^i \varphi^j \gamma(s-2) \delta(s-2) \tilde{\omega}^j \gamma(\gamma(s-2), \delta(s-2) \tilde{\beta} \tilde{h}_\phi^j, \tilde{\beta} h_\phi^j, \tilde{\beta} h_\phi^j, \tilde{\beta} h_\phi^j)].
\]

Here the first two terms vanish for symmetric \( c_{ij} \). As a result,

\[
\tilde{D} J^\beta_+ \simeq 4c_{ij} \omega^i \varphi^j \gamma(s-2) \delta(s-2) \tilde{\omega}^j \gamma(\gamma(s-2), \delta(s-2) \tilde{\beta} \tilde{h}_\phi^j, \tilde{\beta} h_\phi^j, \tilde{\beta} h_\phi^j) = 2 \tilde{h}_h^\beta J^\beta_+.
\]

Analogously, \( J^\alpha_+ \) (37) satisfies Eq. (35).

Thus, the even conserved current \( J^\alpha_+, \beta \) with symmetric \( c_{ij} \) gives rise to the closed current \( J_\xi \) that generates the full set of spin-two charges for fields of any spin \( s \geq 2 \). This current exists for \( s \geq 2 \) and corresponds to the stress tensors for HS fields, which are the frame-like counterparts of those considered in [5].

Consider now the current \( J^\alpha_-, \beta \) which is odd. One can see that in this case Eqs. (20) admit no solutions. As a result, the charge conservation condition demands \( \xi_{\alpha \beta} = 0, \xi_{\beta \beta} = 0 \). The conserved charge is

\[
Q = \int_{M^4} \xi_{\alpha \beta} J^\alpha_-. \beta, \tag{38}
\]

where \( \xi_{\alpha \beta} \) satisfies

\[
\tilde{D} \xi_{\alpha \beta} = 0, \tag{39}
\]

hence being a constant in Cartesian coordinates. Thus, the odd current \( J^\alpha_-, \beta \) generates less charges, than \( J^\alpha_+, \beta \).

For example, in tensor notations, the odd current for \( s = t = 2 \) reads as

\[
J^\alpha = c_{ij} [\omega^i \varphi^j \gamma(s-2) \delta(s-2) \tilde{\omega}^j \gamma(\gamma(s-2), \delta(s-2) \tilde{\beta} \tilde{h}_\phi, \tilde{\beta} h_\phi, \tilde{\beta} h_\phi, \tilde{\beta} h_\phi)].
\]
where \( c_{ij} \) is antisymmetric.

The existence of odd conserved currents is surprising. Analogous “unexpected” currents are shown in the next section to exist for all higher spins of currents \( t > 2 \) in Minkowski space. In Conclusion we argue that the additional odd currents only exist in Minkowski space and unlikely admit an extension to \( AdS_4 \).

3.3 General case

For \( t \geq 1 \) an even conserved current \( J^{\alpha(t-1)} \), \( J^{\beta(t-1)} \) for \( s \geq 2, s \geq t \), that carries \( t \) derivatives of the HS field, is

\[
J^{\alpha(t-1)} \cdot J^{\beta(t-1)} = c_{ij} [\omega^{i \alpha(t-1) \varphi(s-2) \delta(s-t) \omega^{j} \gamma(s-2) \delta(s-t) \theta^{(t-1)} \tilde{h} \phi, \theta] + \\
+ \omega^{i \alpha(t-1) \varphi(s-t) \delta(s-2) \omega^{j} \gamma(s-t) \delta(s-2) \theta^{(t-1)} \tilde{h} \phi, \theta].
\]

(40)

Analogously to the case of \( t = 2 \), it is not difficult to see that \( \tilde{D} J^{\alpha(t-1)} \cdot J^{\beta(t-1)} \) is not exact regardless of the symmetry of \( c_{ij} \). For \( t = s - 1 \) the terms with Weyl-like tensors vanish like in the example (30). The proof of the fact that the current \( J^{\alpha(t-1)} \cdot J^{\beta(t-1)} \) is (anti)symmetric for (odd)even \( t \), and \( J^{\alpha(t-1)} \cdot J^{\beta(t-1)} = i J^{\alpha(t-1)} \cdot J^{\beta(t-1)} \), where \( c_{ij} \) is (anti)symmetric for (even)odd \( t \). Note, that Eq. (40) at \( t = 2 \) gives \( J^{\alpha(m)} \cdot J^{\beta(n)} \).

Consider the even current \( J^{\alpha(t-1)} \cdot J^{\beta(t-1)} \). To reconstruct the current \( J \) we have to solve the set of equations (20). Using equations (14) - (16) one can see that the following set of 3-forms \( J^{\alpha(m)} \cdot J^{\beta(n)} \), \( m \neq n, m + n = 2(t - 1) \) gives a solution

\[
J^{\alpha(m)} \cdot J^{\beta(n)} = \\
= \theta(m - n - 4) g(m) c_{ij} \omega^{i \alpha(m) \varphi(s-2) \delta(s-t) \omega^{j} \gamma(s-2) \delta(s-t) \theta^{(t-1)} \tilde{h} \phi, \theta] + \\
+ \delta_{m,t-2} \theta^{2} c_{ij} \omega^{i \alpha(t-2) \varphi(s-t) \delta(s-t) \omega^{j} \gamma(s-2) \delta(s-t) \theta^{(t-1)} \tilde{h} \phi, \theta] + \\
+ \sum_{p=1}^{t-2} f(p) c_{ij} \omega^{i \alpha(t-2) \varphi(s-t) \delta(s-t+p) \omega^{j} \gamma(s-t+p) \delta(s-t+p) \theta^{(t-1)} \tilde{h} \phi, \theta] + \\
+ \theta(m - n - 4) g(n) c_{ij} \omega^{i \alpha(t-1) \varphi(s-t) \delta(s-t) \omega^{j} \gamma(s-t) \delta(s-t) + \\
+ \delta_{m,t} \theta^{2} c_{ij} \omega^{i \alpha(t-1) \varphi(s-t) \delta(s-t) \omega^{j} \gamma(s-t) \delta(s-t) \theta^{(t-2)} \tilde{h} \phi, \theta] + \\
+ \sum_{p=1}^{t-2} f(p) c_{ij} \omega^{i \alpha(t-1) \gamma(s-t+p) \delta(s-t+p) \omega^{j} \gamma(s-t+p) \delta(s-t+p) \theta^{(t-2)} \tilde{h} \phi, \theta] \right].
\]

(41)
where

\[
g(m) = 2(-1)^m \frac{(t - 2 - m)!}{(t - 1)!(t - 1 - m)!}, \quad f(p) = -t \frac{(s - t)!(s - p - 2)!}{(s - 3)!(s - t + p)!}. \tag{42}
\]

It should be stressed, that the forms \(J^{\hat{\beta}(2t-2)}_+\), \(J^{\alpha(2t-2)}_+\) obey equations (20) only if \(c_{ij}\) is antisymmetric for odd \(t\) and symmetric for even \(t\).

Thus, the Hermitian current 3-form \(J^{\alpha(t-1)}_+ \hat{\beta}(t-1)\) (40) with antisymmetric \(c_{ij}\) for odd \(t\) and symmetric for even \(t\) is on-shell closed, but not exact. It generates the closed current \(J_\xi\) and the corresponding conserved charge \(Q = \int_{M^3} J_\xi\) that contains as many symmetry parameters as local HS gauge symmetries.

Since the current \(J^{\alpha(t-1)}_+ \hat{\beta}(t-1)\) (40) is closed but not exact regardless of the symmetry of \(c_{ij}\), the odd Hermitian current with symmetric \(c_{ij}\) for odd \(t\) and antisymmetric for even \(t\) is \(J^{\alpha(t-1)}_- \hat{\beta}(t-1) = i J^{\alpha(t-1)}_+ \hat{\beta}(t-1)\). Odd currents exist for \(t \geq 2\). To reconstruct \(J_\xi\) we have to solve equations (20). It turns out that these equations is possible to solve at \(m \neq 2(t-1), n \neq 2(t-1), m + n = 2(t - 1)\). The solution is expressed by \(J^{\alpha(m)}_- \hat{\beta}(n) = i J^{\alpha(m)}_+ \hat{\beta}(n)\) (41), but \(c_{ij}\) is antisymmetric for even \(t\) and symmetric for odd \(t\). However the equations admit no solution at the last step, i.e., for \(J^{\hat{\beta}(2t-2)}_-\) and \(J^{\alpha(2t-2)}_-\). Hence, as in the example of \(t = 2\), the charge conservation condition demands \(\xi^{\alpha(2t-2)} = 0, \xi^{\hat{\beta}(2t-2)} = 0\). Since Eq. (18) admits less solutions, the odd currents generate less charges than the even ones.

### 3.4 Gauge transformations

The on-shell gauge variation of the current 3-form (40) can be represented in the form

\[
\delta J^{\alpha(t-1)}_+ \hat{\beta}(t-1) \simeq c_{ij} \tilde{D}^{\epsilon \alpha(t-1) \varphi \gamma(s-2)} \delta(s-t) \omega^j;_{\gamma(s-2) \hat{\delta}(s-t) \hat{\theta}} + \omega^{\epsilon \alpha(t-1) \varphi \gamma(s-2)} \delta(s-t) \omega^j;_{\gamma(s-2) \hat{\delta}(s-t) \hat{\theta}} + \omega^{\epsilon \alpha(t-1) \varphi \gamma(s-2)} \delta(s-t) \omega^j;_{\gamma(s-2) \hat{\delta}(s-t) \hat{\theta}} + \omega^{\epsilon \alpha(t-1) \varphi \gamma(s-2)} \delta(s-t) \omega^j;_{\gamma(s-2) \hat{\delta}(s-t) \hat{\theta}} = \tilde{D} H^{\alpha(t-1)}_+ \hat{\beta}(t-1). \tag{43}
\]

Analogously, the on-shell gauge variation of the 3-forms (41) is

\[
\delta J^{\alpha(m)}_+ \hat{\beta}(n) \simeq D^* H^{\alpha(m)}_+ \hat{\beta}(n), \quad m \neq n, \tag{44}
\]

\[
\delta J^{\alpha(m)}_- \hat{\beta}(n) \simeq i D^* H^{\alpha(m)}_- \hat{\beta}(n), m \neq n, \quad m \neq 2(t - 1), \quad n \neq 2(t - 1). \tag{45}
\]
The 2-form $H^{\alpha(m)} \delta(n)$ is

$$H^{\alpha(m)} \delta(n) =$$

$$= \theta(n - m - 4) g(m) c_{ij} \left[ \epsilon^{\alpha(m)} \varphi(s-2) \delta(s-t) \beta(n-t+1) \omega^{j} \gamma(s-2) \delta(s-t) \beta(t-1) \hat{h}_{\varphi} \right] +$$

$$+ \omega^{\alpha(m)} \varphi(s-2) \delta(s-t) \beta(n-t+1) \gamma(s-2) \delta(s-t) \beta(t-1) \hat{h}_{\varphi} \right] +$$

$$+ \delta_{m,t-2} c_{ij} \left[ -2t \epsilon^{\alpha(t-2)} \varphi(s-2) \delta(s-t) \beta(n-t+1) \gamma(s-2) \delta(s-t) \beta(t-1) \hat{h}_{\varphi}, \delta(t-1) \hat{h}_{\varphi}, \delta(n-t+1) \right] +$$

$$+ \sum_{p=1}^{t-2} f(p) \epsilon^{\alpha(t-2)} \varphi(s-p-1) \delta(s-t+p) \omega^{j} \gamma(s-p-1) \delta(s-t+p) \beta(t-1) \hat{h}_{\varphi}, \beta(t-1) \hat{h}_{\varphi}, \right] +$$

$$+ \sum_{p=1}^{t-2} f(p) \omega^{\alpha(t-2)} \varphi(s-p-1) \delta(s-t+p) \epsilon^{j} \gamma(s-p-1) \delta(s-t+p) \beta(t-1) \hat{h}_{\varphi}, \beta(t-1) \hat{h}_{\varphi},$$

$$+ \theta(m - n - 4) g(n) c_{ij} \left[ \epsilon^{\alpha(t-1)} \varphi(s-t) \delta(s-2) \omega^{j} \gamma(s-2) \delta(s-t) \beta(n) \hat{h}_{\varphi}, \delta(t-1) \hat{h}_{\varphi}, \right] +$$

$$+ \omega^{\alpha(t-1)} \varphi(s-t) \delta(s-2) \epsilon^{j} \alpha(m-t+1) \gamma(s-2) \delta(s-t) \beta(n) \hat{h}_{\varphi}, \delta(t-1) \hat{h}_{\varphi}, \right] +$$

$$+ \delta_{m,t} c_{ij} \left[ -2t \epsilon^{\alpha(t-1)} \varphi(s-t) \delta(s-2) \omega^{j} \gamma(s-2) \delta(s-2) \beta(t-2) \hat{h}_{\varphi}, \delta(t-2) \hat{h}_{\varphi}, \right] +$$

$$+ \sum_{p=1}^{t-2} f(p) c_{ij} \epsilon^{\alpha(t-1)} \gamma(s-t+p) \delta(s-p-1) \omega^{j} \gamma(s-t+p) \delta(s-p-1) \beta(t-2) \hat{h}_{\varphi}, \beta(t-2) \hat{h}_{\varphi}, \right] +$$

$$+ \sum_{p=1}^{t-2} f(p) c_{ij} \omega^{\alpha(t-1)} \gamma(s-t+p) \delta(s-p-1) \epsilon^{j} \gamma(s-t+p) \delta(s-p-1) \beta(t-2) \hat{h}_{\varphi}, \beta(t-2) \hat{h}_{\varphi}, \right], \quad (46)$$

where $m + n = 2(t - 1)$.

As a result, the gauge transformation of charge $Q$ is

$$\delta Q \approx \int_{M^3} \sum_{m,n} \xi_{\alpha(m), \beta(n)} D^* H^{\alpha(m)} \beta(n) =$$

$$= \int_{M^3} d \left( \sum_{m,n} \xi_{\alpha(m), \beta(n)} H^{\alpha(m)} \beta(n) \right) = \int_{M^3} d H_{\xi} = 0. \quad (47)$$

Thus, though the current $J_\xi$ is not gauge invariant, the corresponding charge is.
4 Conclusion

In this paper, spin-$t$ HS even currents $J_{+}^{\alpha(t-1),\bar{\beta}(t-1)}$ in four-dimensional Minkowski space, built from spin-$s$ ($s \geq t$) fields and carrying $t$ derivatives (which is the minimal possible number), are found. Being represented as 3-forms, $J_{+}^{\alpha(t-1),\bar{\beta}(t-1)}$ are closed but not exact, hence leading to nontrivial HS charges. To generate the full list of conserved charges, $J_{+}^{\alpha(t-1),\bar{\beta}(t-1)}$ [4]) was extended to a set of currents $J_{+}^{\alpha(m),\bar{\beta}(n)}$ with $m + n = 2(t - 1)$ that altogether obey the co-adjoint covariant constancy condition and form a full current $J_{\xi}$ that depends on the global symmetry parameters $\xi$.

Analogously to the $t = 2$ case of stress tensor considered in [6], the constructed currents are not invariant under the HS gauge transformations. However, the corresponding HS charges are gauge invariant because the gauge variation $\delta J_{\xi}$ is exact.

In addition to the expected set of $P$-even currents responsible for usual HS symmetries we found an unexpected set of spin $t > 1$ $P$-odd currents which have opposite symmetry with respect to color indices carried by HS fields compared to the normal $P$-even currents. The origin of these currents is rather mysterious. We expect that beyond four dimensions they should correspond to currents of some mixed symmetry fields that become equivalent to symmetric fields in the particular case of four dimensions. A related property is that the odd currents can unlikely survive upon the deformation of Minkowski space to $AdS_4$. This follows from the fact that, as shown in this paper, the space of global symmetry parameters associated with the odd currents is smaller than for even currents. Such a reduction is possible in Minkowski geometry where HS connections are valued in a indecomposable module of Poincaré algebra allowing a reduction to a submodule. However, it cannot be possible in $AdS$ geometry where the $AdS_4$ symmetry $sp(4)$ is simple and HS connections are valued in its irreducible modules for any given spin. In fact, this argument fits the property that mixed symmetry fields in anti-de Sitter and Minkowski spaces are essentially different [17, 18] in the sense that $AdS$ mixed symmetry fields contain more degrees of freedom than the Minkowski ones that also obstructs a smooth deformation of the currents associated with mixed symmetry fields.

It would be interesting to generalize obtained results to currents built from fields of different spins and/or of the connection-Weyl ($i.e., \omega \times C$) type. For example, in the case of $t = s = 1$, the current is

$$J = ic_{ij}[\omega^jC^j;\alpha\alpha H_{\alpha\alpha} - \omega^jC^j;\bar{\beta}\bar{\beta} H_{\bar{\beta}\bar{\beta}}],$$

where $c_{ij}$ is antisymmetric.

To make it possible to use HS charges for the analysis of particular solutions of nonlinear HS gauge theory, it is important to generalize the constructed currents to
\( (A)dS \) background, which problem is currently under investigation. As mentioned above, it is also interesting to clarify the fate of odd HS currents in \( AdS_4 \).

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**Appendix. Non-exactness of \( J^{\alpha(t-1)},\hat{\beta}(t-1) \)**

To check whether or not the current \([\text{III}]\) can be represented in the exact form we set

\[
J^{\alpha(t-1)},\hat{\beta}(t-1) = \Phi_1^{\alpha(t-1)}\hat{\beta}(t-1) + \Phi_2^{\alpha(t-1)}\hat{\beta}(t-1),
\]

\[
\Phi_1^{\alpha(t-1)}\hat{\beta}(t-1) = c_{ij}\omega^i\alpha(t-1)\varphi_1(s-2)\delta(s-t)\omega^j\varphi_2(s-3)\delta\theta, \quad \Delta, \quad \delta(t-1)\hat{h}_\varphi, \quad \hat{\theta},
\]

\[
\Phi_2^{\alpha(t-1)}\hat{\beta}(t-1) = c_{ij}\omega^i\alpha(t-1)\varphi_1(s-t)\delta(s-2)\omega^j\varphi_2(s-3)\delta\theta, \quad \Delta, \quad \delta(t-1)\hat{h}_\varphi, \quad \hat{\theta},
\]

A basis in the space of 2-forms, which are bilinear in fields, is

\[
\Psi^{\alpha(t-1)},\hat{\beta}(t-1)(k, l) = c_{ij}\omega^i\alpha(t-1-1)\varphi_1(s-k)\delta(s-t+k-1)\hat{\beta}(t-l-1)\omega^j\varphi_2(s-k)\delta(s-t+k-1)\hat{\beta}(t-l-1).
\]

The choice of \( k, l \) is restricted by the condition that \( \Psi^{\alpha(t-1)},\hat{\beta}(t-1)(k, l) \) contains \( (t-1) \) derivatives of the dynamical field.

A current is exact if \( J^{\alpha(t-1)},\hat{\beta}(t-1) \sim \sum_{k,l}a_{kl}\hat{D}\Psi^{\alpha(t-1)},\hat{\beta}(t-1)(k, l) \). It is not difficult to check, that the forms \( \hat{D}\Psi^{\alpha(t-1)},\hat{\beta}(t-1)(k, l) \) with \( l \neq 0 \) cannot contribute because they give rise to non-vanishing terms different from \( \Phi_{1,2}^{\alpha(t-1)}\hat{\beta}(t-1) \). Thus, the current \([\text{III}]\) should necessarily contain \( \hat{D}\Psi^{\alpha(t-1)},\hat{\beta}(t-1)(2, 0) \) and \( \hat{D}\Psi^{\alpha(t-1)},\hat{\beta}(t-1)(t-1, 0) \) because among the forms with \( l = 0 \) only these contain \( \Phi_{1,2}^{\alpha(t-1)}\hat{\beta}(t-1) \).

Let us try to construct the possible improvement (coefficients are not important and denoted as \( a_i > 0 \))

\[
I^{\alpha(t-1)},\hat{\beta}(t-1) = -a_1\hat{D}\Psi^{\alpha(t-1)},\hat{\beta}(t-1)(2, 0) + \ldots.
\]

The on-shell differential of \( \Psi^{\alpha(t-1)},\hat{\beta}(t-1)(2, 0) \) is

\[
-\hat{D}\Psi^{\alpha(t-1)},\hat{\beta}(t-1)(2, 0) \sim (s-t+1)\Phi_1^{\alpha(t-1)},\hat{\beta}(t-1)(s-2)c_{ij}\omega^i\alpha(t-1)\varphi_1(s-3)\delta(s-t+1)\omega^j\varphi_2(s-3)\delta\theta, \quad \Delta, \quad \delta(t-1)\hat{h}_\varphi, \quad \hat{\theta}.
\]
The second term can be subtracted by
\[ \tilde{D} \Psi^{\alpha(t-1), \beta(t-1)}(3, 0) \simeq \]
\[ \simeq c_{ij} \left[ -(s - t + 2) \omega^{i, \alpha(t-1) \varphi(s-3) \delta(s-t+1) \omega^j, \gamma(s-3) \delta(s-t+1) \psi^{\beta(t-1)} \tilde{h} \varphi, \psi + \right. \]
\[ + (s - 3) \omega^{i, \alpha(t-1) \varphi(s-4) \delta(s-t+2) \omega^j, \gamma(s-4) \delta(s-t+2) \psi^{\beta(t-1)} \tilde{h} \varphi, \psi] \]
and so on. As a result,
\[ I^{\alpha(t-1), \beta(t-1)} = -a_1 \tilde{D} \Psi^{\alpha(t-1), \beta(t-1)}(2, 0) - \]
\[ - a_2 \tilde{D} \Psi^{\alpha(t-1), \beta(t-1)}(3, 0) - \ldots - a_n \tilde{D} \Psi^{\alpha(t-1), \beta(t-1)}(t - 1, 0) \simeq \]
\[ \simeq \Phi_1^{\alpha(t-1), \beta(t-1)} - \Phi_2^{\alpha(t-1), \beta(t-1)} \quad (50) \]
and \( I^{\alpha(t-1), \beta(t-1)} \neq J^{\alpha(t-1), \beta(t-1)} \) for \( c_{ij} \) of any symmetry.

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