On the multiplicity of solutions for a kind of fourth-order equation depending on two real parameters

Armin Hadjian* and Maryam Ramezani

Abstract
In this paper, by suitable assumptions on nonlinear boundary term, we establish the existence of three distinct weak solutions for a kind of fourth-order boundary value problem depending on two parameters.

MSC: 34B15; 58E05

Keywords: Fourth-order equations; Critical point theory; Variational methods; Three solutions

1 Introduction
In this paper, we consider the following fourth-order problem:

\begin{equation}
\begin{aligned}
& u^{(iv)}(x) = \lambda f(x, u(x)), \quad \text{in } [0, 1], \\
& u(0) = u'(0) = 0, \\
& u''(1) = 0, \quad u'''(1) = \mu g(u(1)),
\end{aligned}
\end{equation}

where \( \lambda, \mu \in ]0, \infty[ \), \( g : \mathbb{R} \to \mathbb{R} \) is a continuous function and \( f : [0, 1] \times \mathbb{R} \to \mathbb{R} \) is an \( L^1 \)-Caratéodory function.

Problem (1.1) describes the static equilibrium of a flexible elastic beam of length 1 when, along its length, a load \( f \) is added to cause deformation. Precisely, conditions \( u(0) = u'(0) = 0 \) mean that the left end of the beam is fixed and conditions \( u''(1) = 0, u'''(1) = \mu g(u(1)) \) mean that the right end of the beam is attached to a bearing device, given by the function \( g \).

Existence and multiplicity of solutions for fourth-order boundary value problems have been discussed by several authors in the last decades; see for example [1, 4, 5, 9–13, 16, 17, 19–21] and the references therein.

In particular, Yang et al. [21] used Ricceri’s variational principle [18] to establish the existence of at least two classical solutions generated from \( g \) for problem (1.1) with \( \mu = 1 \).
The authors in [9], using a multiplicity result by Cabada and Iannizzotto [8], ensured the existence of at least two nontrivial classical solutions for the problem

\[
\begin{cases}
    u^{(4)}(x) + \lambda f(x, u(x)) = 0, & 0 < x < 1, \\
    u(0) = u'(0) = u''(1) = 0, \\
    u'''(1) = \lambda g(u(1)),
\end{cases}
\]

where the functions \(f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}\) and \(g : \mathbb{R} \rightarrow \mathbb{R}\) are continuous and \(\lambda \geq 0\) is a real parameter.

Bonanno et al. [4], by means of an abstract critical points result of Bonanno [2], studied the existence of at least one nonzero classical solution for problem (1.1).

In [12], by using a smooth version of [7, Theorem 2.1], Heidarkhani et al. established the existence of infinitely many generalized solutions for the following perturbed fourth-order problem:

\[
\begin{cases}
    u^{(4)}(t) = \lambda f(t, u(t)) + \mu g(t, u(t)) + p(u(t)), & 0 < t < 1, \\
    u(0) = u'(0) = 0, \\
    u''(1) = 0, \\
    u'''(1) = h(u(1)),
\end{cases}
\]

where \(\lambda > 0, \mu \geq 0\) are two parameters, \(f, g\) are two \(L^2\)-Caratéodory functions, and \(p, h\) are Lipschitz continuous functions such that \(p(0) = h(0) = 0\).

Also in [11], the present authors obtained sufficient conditions to guarantee that problem (1.1) has infinitely many classical solutions.

More recently, Heidarkhani and Gharehgazlouei [13], using an immediate consequence of [3, Theorem 3.3], ensured the existence of at least three generalized solutions for the problem

\[
\begin{cases}
    u^{(4)}(x) = \lambda f(x, u(x)) + h(u(x)), & \text{in } [0, 1], \\
    u(0) = u'(0) = u''(1) = 0, \\
    u'''(1) = \mu g(u(1)),
\end{cases}
\]

where \(\lambda > 0, \mu \geq 0\) are two parameters, \(f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}\) is an \(L^1\)-Caratéodory function, \(g : \mathbb{R} \rightarrow \mathbb{R}\) is a nonnegative continuous function, and \(h : \mathbb{R} \rightarrow \mathbb{R}\) is a Lipschitz continuous function such that \(h(0) = 0\).

Motivated by the above works, the aim of the present paper is to offer the existence of three solutions for fourth-order problem (1.1) by using two kinds of critical point theorems obtained in [3, 6].

For completeness, we cite the recent and nice works [14, 15] as general references on the subject treated in this paper.

2 Abstract setting

In order to study problem (1.1), the variational setting is the space

\[ X := \{u \in H^2([0, 1]): u(0) = u'(0) = 0\}, \]
where \( H^2([0,1]) \) is the Sobolev space of all function \( u : [0,1] \to \mathbb{R} \) such that \( u \) and its distributional derivative \( u' \) are absolutely continuous and \( u'' \) belongs to \( L^2([0,1]) \). \( X \) is a Hilbert space with the inner product

\[
\langle u,v \rangle := \int_0^1 u''(t)v''(t) \, dt
\]

and the corresponding norm

\[
\| u \| := \left( \int_0^1 (u''(t))^2 \, dt \right)^{1/2}.
\]

We observe that the norm \( \| \cdot \| \) on \( X \) is equivalent to the usual norm

\[
\int_0^1 \left( |u(t)|^2 + |u'(t)|^2 + |u''(t)|^2 \right) \, dt.
\]

It is well known that the embedding \( X \hookrightarrow C^1([0,1]) \) is compact and

\[
\| u \|_{C^1([0,1])} := \max \left\{ \| u \|_{\infty}, \| u' \|_{\infty} \right\} \leq \| u \| \quad (2.1)
\]

for all \( u \in X \) (see [21]).

We say that \( u \in X \) is a weak solution of problem (1.1) whenever

\[
\int_0^1 u''(x)v''(x) \, dx - \lambda \int_0^1 f(x,u(x))v(x) \, dx + \mu g(u(1))v(1) = 0
\]

for all \( v \in X \). By a classical solution of problem (1.1) we mean a function \( u \in C^1([0,1]) \) such that \( u^{(1)}(x) \in C([0,1]) \) and the boundary conditions and the equation are satisfied in \([0,1]\).

Our main tools are critical point theorems that we recall here in a convenient form. The first result has been obtained in [6], and it is a more precise version of Theorem 3.2 of [3]. The second one has been established in [3].

**Lemma 2.1** ([6, Theorem 3.6]) *Let \( X \) be a reflexive real Banach space; \( \Phi : X \to \mathbb{R} \) be a coercive, continuously Gâteaux differentiable, and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on \( X^* \); \( \Psi : X \to \mathbb{R} \) be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that*

\[
\Phi(0) = \Psi(0) = 0.
\]

*Assume that there exist \( r > 0 \) and \( \overline{x} \in X \), with \( r < \Phi(\overline{x}) \), such that*

\[
(a_1) \sup_{\Phi(\overline{x}) \leq t \leq \Phi(0)} \frac{\Phi(t)}{t} < \frac{\Phi(\overline{x})}{\Phi(0)};
\]

\[
(a_2) \text{ for each } \lambda \in \Lambda_r := \left\{ \frac{\Phi(\overline{x})}{\Psi(\overline{x})} \sup_{\Phi(0) \leq t \leq \Phi(\overline{x})} \frac{r}{t} \right\}, \text{ the functional } \Phi - \lambda \Psi \text{ is coercive.}
\]

*Then, for each \( \lambda \in \Lambda_r \), the functional \( \Phi - \lambda \Psi \) has at least three distinct critical points in \( X \).*

**Lemma 2.2** ([3, Theorem 3.3]) *Let \( X \) be a reflexive real Banach space; \( \Phi : X \to \mathbb{R} \) be a convex, coercive, and continuously Gâteaux differentiable functional whose derivative admits
a continuous inverse on $X^*$; $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose derivative is compact such that

$$\inf_{X} \Phi = \Phi(0) = \Psi(0) = 0.$$ 

Assume that there are two positive constants $r_1$, $r_2$ and $\bar{x} \in X$, with $2r_1 < \Phi(\bar{x}) < \frac{r_2}{2}$, such that

1. $\sup_{\Phi(\xi) < 2r_1} \frac{\Psi(\xi)}{r_1} < \frac{2}{3}$,
2. $\sup_{\Phi(\xi) < 2r_2} \frac{\Psi(\xi)}{r_2} < \frac{1}{3}$,
3. for each $\lambda$ in

$$\Lambda := \left\{ \frac{3}{2} \Phi(\bar{x}), \min \left\{ \frac{r_1}{\sup_{\Phi(\xi) < r_1} \Psi(\xi)}, \frac{r_2}{\sup_{\Phi(\xi) < r_2} \Psi(\xi)} \right\} \right\}$$

and for every $x_1, x_2 \in X$, which are local minima for the functional $\Phi - \lambda \Psi$ and such that $\Psi(x_1) \geq 0$ and $\Psi(x_2) \geq 0$, one has $\inf_{s \in [0,1]} \Psi(sx_1 + (1-s)x_2) \geq 0$.

Then, for each $\lambda \in \Lambda$, the functional $\Phi - \lambda \Psi$ has at least three distinct critical points which lie in $\Phi^{-1}(-\infty, r_2]$.

We use the following notations:

Corresponding to $f, g$, we introduce the functions $F, G$ as follows:

$$F(x, \xi) := \int_{0}^{\xi} f(x, t) \, dt, \quad G(\xi) := -\int_{0}^{\xi} g(t) \, dt$$

for all $x \in [0,1]$ and $\xi \in \mathbb{R}$. Also, for each $\eta$ of positive real numbers, define

$$F^\eta = \int_{0}^{1} \sup_{|\xi| \leq \theta} F(x, \xi) \, dx, \quad G^\eta = \sup_{|\xi| \leq \theta} G(\xi), \quad g_\eta = \inf_{|\xi| \leq \eta} G(\xi).$$

### 3 Main results

In this section, we present our main result on the existence of at least three weak solutions for problem (1.1).

In order to introduce our result, we fix $\theta, \eta > 0$ such that

$$\frac{32\pi^4 \eta^2}{27 \int_{\frac{1}{4}}^{\frac{3}{4}} F(x, \eta) \, dx} < \frac{\theta^2}{2F^\theta}.$$

and pick

$$\Lambda := \left\{ \frac{32\pi^4 \eta^2}{27 \int_{\frac{1}{4}}^{\frac{3}{4}} F(x, \eta) \, dx}, \theta^2 \right\} \setminus \frac{2F^\theta}{G^\theta}.$$

Set

$$\delta := \min \left\{ \frac{\theta^2 - 2\lambda F^\theta}{2G^\theta}, \frac{32\pi^4 \eta^2}{\frac{\lambda}{2} \int_{\frac{1}{4}}^{\frac{3}{4}} F(x, \eta) \, dx} \right\}$$

(3.2)
and

$$\delta := \min \left\{ \delta, \frac{1}{\max\{0, 4 \limsup_{|\xi| \to +\infty} \frac{G(\xi)}{\xi^2}\}} \right\}, \quad (3.3)$$

where we read $r/0 = +\infty$. For instance, $\delta = +\infty$ when $\limsup_{|\xi| \to +\infty} \frac{G(\xi)}{\xi^2} \leq 0$ and $G(\eta) = G^0 = 0$.

With the above notations we are able to prove the following multiplicity property.

**Theorem 3.1** Assume that there exist two positive constants $\theta$, $\eta$, with $\theta < \frac{1}{3\sqrt{3}}\pi^2$ such that

(A1) $F(x, t) \geq 0$ for each $(x, t) \in \left[\frac{3}{8}, \frac{3}{4}\right] \times [0, \eta]$;

(A2) $E^0 < \frac{27}{64\pi^2} \frac{\int_0^1 \frac{F(x, 0)dx}{3^2 \theta^2}}{\eta^2}$;

(A3) $\limsup_{|\xi| \to +\infty} \frac{G(\xi)}{\xi^2} < \frac{\theta^2}{2\pi^2}$.

Then, for every $\lambda \in \Lambda$, where $\Lambda$ is given by (3.1), and for every continuous function $g : \mathbb{R} \to \mathbb{R}$ such that

$$\limsup_{|\xi| \to +\infty} \frac{G(\xi)}{\xi^2} < +\infty,$$

there exists $\delta > 0$ given by (3.3) such that, for each $\mu \in [0, \delta]$, problem (1.1) admits at least three distinct weak solutions.

**Proof** Our aim is to apply Lemma 2.1 to problem (1.1). To this end, we introduce the functionals $\Phi, \Psi : X \to \mathbb{R}$ as follows:

$$\Phi(u) := \frac{1}{2} \|u\|^2,$$

$$\Psi(u) := \int_0^1 F(x, u(x)) \, dx + \frac{\mu}{\lambda} G(u(1)).$$

It is well known that $\Psi$ is a differentiable functional whose differential at the point $u \in X$ is

$$\Psi'(u)(v) = \int_0^1 f(x, u(x)) v(x) \, dx - \frac{\mu}{\lambda} g(u(1)) v(1)$$

for every $v \in X$. Moreover, in [21], the authors proved that $\Psi'$ is strongly continuous on $X$, which implies that $\Psi'$ is a compact operator. Furthermore, by standard arguments, $\Phi$ is coercive and continuously differentiable whose differential at the point $u \in X$ is

$$\Phi'(u)(v) = \int_0^1 u''(x) v'(x) \, dx$$

for each $v \in X$. Also, in [21] it is proved that $\Phi'$ admits a continuous inverse on $X^*$. Moreover, $\Phi$ is sequentially weakly lower semicontinuous. One can show that the weak solutions of problem (1.1) are exactly the solutions of the equation $\Phi'(u) - \lambda \Psi'(u) = 0$. 

Fix $\lambda \in \Lambda$ and put $r = \frac{\theta^2}{2}$. Then, for $u \in X$ with $\Phi(u) \leq r$,

$$
\sup_{\Phi(u) \leq r} \Psi(u) = \sup_{\Phi(u) \leq r} \left( \int_0^1 F(x, u(x)) \, dx + \frac{\mu}{\lambda} G(u(1)) \right)
\leq \int_0^1 \sup_{|t| \leq \theta} F(x, t) \, dx + \frac{\mu}{\lambda} G^\theta = \int_0^1 F(x, \eta) \, dx + \frac{\mu}{\lambda} G^\theta.
$$

Therefore,

$$
\frac{\sup_{\Phi(u) \leq r} \Psi(u)}{r} \leq \frac{2F^\theta}{\theta^2} + \frac{2\mu}{\lambda} \frac{G^\theta}{\theta^2}.
$$

From this, if $G^\theta = 0$, it is clear that

$$
\frac{\sup_{\Phi(u) \leq r} \Psi(u)}{r} < \frac{1}{\lambda},
$$

while if $G^\theta > 0$, it turns out to be true bearing in mind that

$$
\mu < \frac{\theta^2 - 2\lambda F^\theta}{2G^\theta}.
$$

Denote by $w$ the function of $X$ defined by

$$
w(x) := \begin{cases} 
0, & x \in [0, \frac{3}{8}], \\
\eta \cos^2(\frac{4\pi x}{3}), & x \in \left[\frac{3}{8}, \frac{3}{4}\right], \\
\eta, & x \in \left[\frac{3}{4}, 1\right].
\end{cases}
$$

It is easy to see that

$$
\Phi(w) = \frac{32}{27\pi^4} \eta^2.
$$

Therefore, since $\theta < \frac{8}{3\sqrt{3}} \pi^2 \eta$, one has $\Phi(w) > r$.

From assumption (A1) we obtain

$$
\Psi(w) = \int_0^1 F(x, w(x)) \, dx + \frac{\mu}{\lambda} G(w(1)) \geq \int_{\frac{3}{4}}^1 F(x, \eta) \, dx + \frac{\mu}{\lambda} G(\eta).
$$

So, we have

$$
\frac{\Psi(w)}{\Phi(w)} \geq \frac{\int_{\frac{3}{4}}^1 F(x, \eta) \, dx + \frac{\mu}{\lambda} G(\eta)}{\frac{32}{27\pi^4} \eta^2}.
$$

Hence, if $G(\eta) \geq 0$, we find

$$
\frac{\Psi(w)}{\Phi(w)} \geq \frac{1}{\lambda},
$$

(3.6)
while if $G(\eta) < 0$, the same relation holds since

$$
\mu G(\eta) > \frac{32}{27} \pi^4 \eta^2 - \lambda \int_{\frac{1}{4}}^1 F(x, \eta) \, dx.
$$

Now, taking into account (3.4) and (3.6) results in

$$
\frac{\Psi(w)}{\Phi(w)} > \frac{1}{\lambda} > \frac{\sup_{\Phi(v) \leq 1} \Psi(v)}{r},
$$

and condition (a1) of Lemma 2.1 is verified.

Now, in order to prove the coercivity of the functional $\Phi - \lambda \Psi$, first we assume that

$$
\lim sup_{|\xi| \to +\infty} \sup_{x \in [0,1]} F(x, \xi) \xi^2 > 0.
$$

Therefore, fix

$$
\lim sup_{|\xi| \to +\infty} \sup_{x \in [0,1]} F(x, \xi) \xi^2 < \varepsilon < \frac{F^0}{2\mu^2},
$$

from (A3) there is a function $h_\varepsilon \in L^1([0,1])$ such that

$$
F(x, \xi) \leq \varepsilon \xi^2 + h_\varepsilon(x)
$$

for each $x \in [0,1]$ and $\xi \in \mathbb{R}$. Taking (2.1) into account and since $\lambda < \frac{\delta^2}{2F^0}$, it follows that

$$
\lambda \int_{[0,1]} F(x, u(x)) \, dx \leq \lambda \left( \varepsilon \int_{[0,1]} (u(x))^2 \, dx + \int_{[0,1]} h_\varepsilon(x) \, dx \right)
$$

$$
< \frac{\delta^2}{2F^0} \left( \varepsilon \int_{[0,1]} (u(x))^2 \, dx + \int_{[0,1]} h_\varepsilon(x) \, dx \right)
$$

$$
\leq \frac{\delta^2}{2F^0} \left( \varepsilon \|u\|^2 + \|h_\varepsilon\|_{L^1([0,1])} \right)
$$

(3.7)

for each $u \in X$. Moreover, since $\mu < \delta$, we obtain

$$
\lim sup_{|\xi| \to +\infty} \frac{G(\xi)}{\xi^2} < \frac{1}{4\mu}.
$$

Thus, there is a positive constant $h_\mu$ such that

$$
G(\xi) \leq \frac{1}{4\mu} \xi^2 + h_\mu
$$

for each $\xi \in \mathbb{R}$. Thus, taking again (2.1) into account, it follows that

$$
G(u(1)) \leq \frac{1}{4\mu} (u(1))^2 + h_\mu
$$

$$
\leq \frac{1}{4\mu} \|u\|^2 + h_\mu
$$

(3.8)
for each \( u \in X \). Finally, putting together (3.7) and (3.8), we have

\[
\Phi(u) - \lambda \Psi(u) \geq \frac{1}{2} \|u\|^2 - \theta^2\|h_\varepsilon\|^2_{L^1([0,1])} - \frac{1}{4} \|u\|^2 - \mu h_\mu
\]

\[
= \frac{1}{2} \left( \frac{1}{2} - \frac{\theta^2}{F_\psi} \right) \|u\|^2 - \frac{\theta^2 \|h_\varepsilon\|^2_{L^1([0,1])}}{2F_\psi} - \mu h_\mu.
\]

On the other hand, if

\[
\limsup_{|\xi| \to +\infty} \frac{\sup_{x \in [0,1]} F(x, \xi)}{\xi^2} \leq 0,
\]

there exists a function \( h_\varepsilon \in L^1([0,1]) \) such that \( F(x, \xi) \leq h_\varepsilon(x) \) for each \( x \in [0,1] \) and \( \xi \in \mathbb{R} \), and arguing as before we obtain

\[
\Phi(u) - \lambda \Psi(u) \geq \frac{1}{4} \|u\|^2 - \frac{\theta^2 \|h_\varepsilon\|^2_{L^1([0,1])}}{2F_\psi} - \mu h_\mu.
\]

Both cases lead to the coercivity of \( \Phi - \lambda \Psi \), and condition (a2) of Lemma 2.1 is verified.

Since from (3.4) and (3.6)

\[
\lambda \in \Lambda :\sup_{r \leq \Psi(u)} \left\{ \frac{\Phi(w)}{\Psi(w)} \right\} = \frac{r}{\sup_{\Phi(u) \leq r} \Psi(u)}
\]

Lemma 2.1 ensures the existence of at least three critical points for the functional \( \Phi - \lambda \Psi \), and the proof is complete.

**Theorem 3.2** Let \( \theta_1, \theta_2, \) and \( \eta \) be positive constants such that \( \frac{3}{4} < \frac{\sqrt{3}}{\pi} \theta_1 < \frac{\sqrt{3}}{2} \theta_1 \) and \( f : [0,1] \times \mathbb{R} \to \mathbb{R} \) be a mapping for which \( f(x, t) \geq 0 \) for every \( (x, t) \in [0,1] \times [0, \theta_2] \). Assume that

\[
\text{(B1)} \quad \frac{\int_0^1 F(x, \theta_1) \, dx}{\theta_1^2} < \frac{64 \pi^4}{81} \frac{\int_0^1 F(x, \eta) \, dx}{\eta^2};
\]

\[
\text{(B2)} \quad \frac{\int_0^1 F(x, \theta_2) \, dx}{\theta_2^2} < \frac{32 \pi^4}{81} \frac{\int_0^1 F(x, \eta) \, dx}{\eta^2}.
\]

Then, for each

\[
\lambda \in \Lambda^* := \left\{ \frac{32}{18} \frac{\pi^4 \eta^2}{\int_0^1 F(x, \eta) \, dx} : \frac{1}{2} \min \left\{ \frac{\theta_1^2}{\int_0^1 F(x, \theta_1) \, dx}, \frac{\theta_2^2}{2 \int_0^1 F(x, \theta_2) \, dx} \right\} \right\}
\]

and for every nonpositive continuous function \( g : \mathbb{R} \to \mathbb{R} \), there exists \( \delta^* > 0 \), where

\[
\delta^* := \min \left\{ \frac{\theta_1^2 - 2 \lambda \int_0^1 F(x, \theta_1) \, dx}{2G_1^\mu}, \frac{\theta_2^2 - 4 \lambda \int_0^1 F(x, \theta_2) \, dx}{4G_2^\mu} \right\}
\]
Indeed, if \( u \) account that

\[
\int_0^1 u_0(x) v'(x) \, dx = \lambda \int_0^1 f(x, u_0(x)) v(x) \, dx - \mu g(u_0(1)) v(1)
\]

for all \( v \in X \).

\[\text{Proof} \quad \text{Without loss of generality, we can assume } f(x, t) \geq 0 \text{ for every } (x, t) \in [0, 1] \times \mathbb{R}. \text{ Fix } \lambda, \mu, \text{ and } g \text{ as in the conclusion and take } \Phi \text{ and } \Psi \text{ as in the proof of Theorem 3.1. Arguing as in the proof of Theorem 3.1, we observe that the regularity assumptions of Lemma 2.2 on } \Phi \text{ and } \Psi \text{ are satisfied. Then, our aim is to verify } (b_1) \text{ and } (b_2).

To this end, put \( w \) as given in (3.5), and let \( r_1 = \frac{\theta_1}{2} \) and \( r_2 = \frac{\theta_2}{2} \). It is obvious that \( 2r_1 < \Phi(w) < \frac{2r_1}{2} \). It follows from \( \mu < \delta^* \) and \( G_n = 0 \) that

\[
\sup_{\Phi(w) < r_1} \frac{\Psi(u)}{r_1} = \sup_{\Phi(w) < r_1} \left( \frac{\int_0^1 f(x, u(x)) \, dx + \frac{\mu}{\lambda} G(u(1))}{r_1} \right) \\
\leq \frac{2\int_0^1 f(x, \theta_1) \, dx + \frac{\mu}{\lambda} G(\theta_1)}{\theta_1^2} \\
< \frac{1}{\lambda} < \frac{2\int_0^1 f(x, \eta) \, dx + \frac{\mu}{\lambda} G_n}{\frac{2\pi^4}{\eta^2}} \\
< \frac{2}{3} \frac{\Phi(w)}{\lambda}.
\]

Similarly,

\[
2\sup_{\Phi(w) < r_2} \frac{\Psi(u)}{r_2} \\
\leq \frac{\int_0^1 f(x, \eta) \, dx + \frac{\mu}{\lambda} G(\eta)}{\frac{2\pi^4}{\eta^2}} \\
< \frac{2}{3} \frac{\Phi(w)}{\lambda}.
\]

This implies that \( (b_1) \) and \( (b_2) \) of Lemma 2.2 are verified.

Finally, we verify that assumption \( (b_1) \) of Lemma 2.2 holds. Let \( u_1 \) and \( u_2 \) be two local minima for \( \Phi - \lambda \Psi \). Then, \( u_1 \) and \( u_2 \) are critical points for \( \Phi - \lambda \Psi \), and so they are weak solutions for problem (1.1). We claim that the weak solutions obtained are nonnegative. Indeed, if \( u_0 \) is a weak solution of problem (1.1), then one has

\[
\int_0^1 u_0(x) v'(x) \, dx = \lambda \int_0^1 f(x, u_0(x)) v(x) \, dx - \mu g(u_0(1)) v(1)
\]

for all \( v \in X \). Arguing by a contradiction, assume that the set \( A := \{ x \in [0, 1] : u_0(x) < 0 \} \) is nonempty and of positive measure. Put \( v_0 := \min(0, u_0) \). Clearly, \( v_0 \in X \). So, taking into account that \( u_0 \) is a weak solution and by choosing \( v = v_0 \), from our sign assumptions on the data, one has

\[
\int_A u_0^2(x) \, dx = \lambda \int_A f(x, u_0(x)) u_0(x) \, dx - \mu g(u_0(1)) u_0(1) \leq 0.
\]

Hence, \( u_0 \equiv 0 \) on \( A \) which is absurd. Then we deduce \( u_1(x) \geq 0 \) and \( u_2(x) \geq 0 \) for each \( x \in [0, 1] \). Thus, it follows that \( su_1 + (1 - s)u_2 \geq 0 \) for all \( s \in [0, 1] \), and that

\[
\Psi(su_1 + (1 - s)u_2) = \int_0^1 f(x, su_1(x) + (1 - s)u_2(x)) \, dx + \frac{\mu}{\lambda} G(su_1(1) + (1 - s)u_2(1)) \geq 0.
\]
So, also \((b_3)\) holds. From Lemma 2.2, for every
\[
\lambda \in \left[ \frac{3}{2} \frac{\Phi(w)}{\Psi(w)} \min \left\{ \frac{r_1}{\sup_{\Phi(u) \geq r_1} \Psi(u)}, \frac{r_2}{\sup_{\Phi(u) \leq r_2} \Psi(u)} \right\} \right],
\]
the functional \(\Phi - \lambda \Psi\) has at least three distinct critical points which are the solutions of problem (1.1) and the conclusion is achieved. \(\square\)

4 Conclusion

By using as the main tools two critical point theorems presented recently in the works [3] and [6], we proved two multiplicity properties (Theorems 3.1 and 3.2) that guarantee the existence of an open interval \([\lambda', \lambda'']\) and \(\tilde{\delta} > 0\) such that, for each \(\lambda \in \lambda', \lambda''\) and for each \(\mu \in \left[0, \tilde{\delta}\right]\), a class of fourth-order boundary value problems depending on parameters \(\lambda\) and \(\mu\) (problem (1.1)) admits at least three distinct weak solutions.
13. Heidarkhani, S., Gharehgazlouei, F.: Existence results for a boundary value problem involving a fourth-order elastic beam equation. J. Nonlinear Funct. Anal. 2019, Article ID 28 (2019)
14. Kamenskii, M., Petrosyan, G., Wen, C.-F.: An existence result for a periodic boundary value problem of fractional semilinear differential equations in a Banach space. J. Nonlinear Var. Anal. 5, 155–177 (2021)
15. Kamenskii, M., Voskovskaya, N., Zvereva, M.: On periodic oscillations of some points of a string with a nonlinear boundary condition. Appl. Set-Valued Anal. Optim. 2, 35–48 (2020)
16. Li, F., Zhang, Q., Liang, Z.: Existence and multiplicity of solutions of a kind of fourth-order boundary value problem. Nonlinear Anal. 62, 803–816 (2005)
17. Ma, T.F.: Positive solutions for a beam equation on a nonlinear elastic foundation. Math. Comput. Model. 39, 1195–1201 (2004)
18. Ricceri, B.: A further three critical points theorem. Nonlinear Anal. 71, 4151–4157 (2009)
19. Song, Y.: A nonlinear boundary value problem for fourth-order elastic beam equations. Bound. Value Probl. 2014, 191, 1–11 (2014)
20. Tersian, S., Chaparova, J.: Periodic and homoclinics solutions of extended Fisher-Kolmogorov equations. J. Math. Anal. Appl. 260, 490–506 (2001)
21. Yang, L., Chen, H., Yang, X.: The multiplicity of solutions for fourth-order equations generated from a boundary condition. Appl. Math. Lett. 24, 1599–1603 (2011)