THE WEIGHTED YAMABE PROBLEM WITH BOUNDARY

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Abstract. We introduce a Yamabe-type flow
\[
\begin{aligned}
\frac{\partial g}{\partial t} &= (r^m_g - R^m_g)g & \text{in } M \\
\frac{\partial \phi}{\partial t} &= m\left(\frac{1}{2} \left( R^m_\phi - r^m_\phi \right) \right) & \text{on } \partial M
\end{aligned}
\]
on a smooth metric measure space with boundary \((M, g, v, dV_g, v_m dA_g, m)\), where \(R^m_\phi\) is the associated weighted scalar curvature, \(r^m_\phi\) is the average of the weighted scalar curvature, and \(H^m_\phi\) is the weighted mean curvature. We prove the long-time existence and convergence of this flow.

1. Introduction

Suppose \(M\) is a compact, \(n\)-dimensional manifold without boundary, where \(n \geq 3\), and \(g_0\) is a Riemannian metric on \(M\). As a generalization of the Uniformization Theorem, the Yamabe problem is to find a metric conformal to \(g_0\) such that its scalar curvature is constant. This was first introduced by Yamabe [27], and was solved by Aubin [1], Trudinger [26] and Schoen [24].

The Yamabe flow is defined as
\[
\frac{\partial g}{\partial t} = (r_g - R_g)g,
\]
where \(R_g\) is the scalar curvature of \(g\) and \(r_g\) is the average of \(R_g\):
\[
r_g = \frac{\int_M R_g dV_g}{\int_M dV_g}.
\]
This was first introduced by Hamilton in [18]. Hamilton conjectured that, for every initial metric, the flow converges to a conformal metric of constant scalar curvature. In the case when \(Y(M, g_0) \leq 0\), it is not difficult to show that the conformal factor is uniformly bounded above and below. Moreover, the flow converges to a metric of constant scalar curvature as \(t \to \infty\).

The case \(Y(M, g_0) > 0\) is more interesting. Chow [12] proved the convergence of the flow for locally conformally flat metrics with positive Ricci curvature. Ye [29] later extended the result to all locally conformal flat metrics. Later, Brendle [4] proved convergence of the flow for all conformal classes and arbitrary initial metrics when \(3 \leq n \leq 5\), and extended the results to higher dimensions [6].
Now consider a compact, \( n \)-dimensional manifold \( M \) with smooth boundary \( \partial M \), where \( n \geq 3 \), and \( g_0 \) is a Riemannian metric on \( M \). One can still talk about the Yamabe problem for manifold with boundary, and there are two types. For the first type, one would like to find a conformal metric \( g \) such that its scalar curvature \( R_g \) is constant in \( M \) and its mean curvature \( H_g \) is zero on \( \partial M \). For the second type, one would like to find a conformal metric \( g \) such that its scalar curvature \( R_g \) is zero in \( M \) and its mean curvature \( H_g \) is constant on \( \partial M \). These problems have studied by many authors. See \([14, 15]\) for example.

Similar to the Yamabe flow, Brendle introduced some geometric flows in \([4]\) to study the Yamabe problem for manifolds with boundary. For the first type, the geometric flow is defined as

\[
\frac{\partial g}{\partial t} = -(R_g - r_g)g \text{ in } M \quad \text{and} \quad H_g = 0 \text{ on } \partial M.
\]

Almaraz and Sun has considered in \([3]\) the convergence of the flow (1.1). On the other hand, for the second type, the geometric flow is defined as

\[
\frac{\partial g}{\partial t} = -(H_g - h_g)g \text{ on } \partial M \quad \text{and} \quad R_g = 0 \text{ in } M,
\]

where \( h_g \) is the average of the mean curvature \( H_g \):

\[
h_g = \frac{\int_{\partial M} H_g dA_g}{\int_{\partial M} dA_g}.
\]

In \([2]\), Almaraz has studied the convergence of the flow (1.2). See also \([11, 20, 21]\) for results related to the flows (1.1) and (1.2).

In this paper, in the same spirit of \([28]\), we generalize the Yamabe flow to smooth metric measure spaces with boundary. To explain the results of this paper, we require some terminology.

**Definition 1.1.** Let \((M, \partial M, g)\) be a Riemannian manifold with boundary \( \partial M \) and let us denote by \( dV_g \) and \( dA_g \) the volume element induced by \( g \) on \( M \) and \( \partial M \), respectively. Set a function \( \phi \in C^\infty(M) \) and a dimensional parameter \( m \in [0, \infty) \). In the case \( m = 0 \), we require that \( \phi = 0 \). A smooth metric measure space with boundary is a five-tuple \((M, g, e^{-\phi}dV_g, e^{-\phi}dA_g, m)\). We frequently denote a smooth metric measure space by \((M, g, v^m dV_g, v^m dA_g, m)\) where \( \phi \) and \( v \) are related by \( e^{-\phi} = v^m \).

The **weighted scalar curvature** \( R^m_\phi \) of a smooth metric measure space with boundary \((M, g, e^{-\phi}dV_g, e^{-\phi}dA_g, m)\) is

\[
R^m_\phi := R_g + 2\Delta_g \phi - \frac{m+1}{m} |\nabla \phi|^2,
\]

where \( R_g \) and \( \Delta_g \) are the scalar curvature and the Laplacian associated to the metric \( g \), respectively. The **weighted mean curvature** is

\[
H^m_\phi = H_g + \frac{\partial \phi}{\partial v_g},
\]
where $H_g$ and $\frac{\partial}{\partial \nu_g}$ are the mean curvature and the outward normal derivative with respect to $g$, respectively.

Conformal equivalence between smooth metric measure spaces are defined as follows, see [7] for more details.

**Definition 1.2.** Smooth metric measure spaces with boundary $(M,g,e^{-\phi}dV_g,e^{-\phi}dA_g,m)$ and $(\hat{M},\hat{g},e^{-\hat{\phi}}dV_{\hat{g}},e^{-\hat{\phi}}dA_{\hat{g}},\hat{m})$ are pointwise conformally equivalent if there is a function $\sigma \in C^\infty(M)$ such that

\[ (1.5) \]

\[(M,\hat{g},e^{-\hat{\phi}}dV_{\hat{g}},e^{-\hat{\phi}}dA_{\hat{g}},\hat{m}) = (M,e^{m+\frac{2\sigma}{n+m-2}}g,e^{\frac{m+n+2}{m+n-2}}e^{-\phi}dV_{\hat{g}},e^\frac{m+n+2}{m+n-2}e^{-\phi}dA_{\hat{g}},\hat{m}).\]

In the case $m=0$, conformal equivalence is defined in the classical sense.

If we denote $e^{\frac{1}{2}\sigma}$ by $w$, (1.5) is equivalent to

\[ (1.6) \]

\[(M,\hat{g},e^{-\hat{\phi}}dvol_{\hat{g}},e^{-\hat{\phi}}dA_{\hat{g}},\hat{m}) = (M,w^{m+\frac{2\sigma}{n+m-2}}g,we^{\frac{m+n+2}{m+n-2}}e^{-\phi}dV_{\hat{g}},we^{\frac{m+n+2}{m+n-2}}e^{-\phi}dA_{\hat{g}},\hat{m}),\]

which is an alternative way to formulate the conformal equivalence of smooth metric measure spaces.

**Definition 1.3.** Let $(M,g,e^{-\phi}dV_g,m)$ be a smooth metric measure space. The weighted Laplacian $\Delta_\phi : C^\infty(M) \to C^\infty(M)$ is the operator defined as

\[ \Delta_\phi \psi = \Delta \psi - \langle \nabla \phi, \nabla \psi \rangle_g \] for any $\psi \in C^\infty(M),

It is formally self-adjoint with respect to the measure $e^{-\phi}dV_g$. For more about smooth metric measure spaces, we refer the readers to [7] [8] [9] [19].

**Definition 1.4.** Given a smooth metric measure spaces with boundary $(M,g,e^{-\phi}dV_g,e^{-\phi}dA_g,m)$, the weighted conformal Laplacian $(L^m_\phi, B^m_\phi)$ is given by the interior operator and boundary operator

\[ (1.7) \]

\[ L^m_\phi = -\Delta_\phi + \frac{n+m-2}{4(n+m-1)} R^m_\phi \] in $M,$

\[ B^m_\phi = \frac{\partial}{\partial \nu_g} + \frac{n+m-2}{4(n+m-1)} H^m_\phi \] on $\partial M,$

where $\nu_g$ is the outward unit normal with respect to $g$.

Consequently, in the formulation of (1.7), the transformation law of the weighted scalar curvature and the weighted mean curvature [23 Proposition 1] are

\[ (1.8) \]

\[ R^m_\phi = \frac{4(n+m-1)}{n+m-2} w^{-\frac{m+n+2}{m+n-2}} L^m_{\phi w} w \] in $M,$

\[ H^m_\phi = \frac{2(n+m-1)}{n+m-2} w^{-\frac{n+m}{m+n-2}} B^m_{\phi w} w \] on $\partial M.$

Given a compact smooth metric measure space without boundary $(M,g,e^{-\phi}dV_g,m)$, the *weighted Yamabe problem* is to find another smooth metric measure space $(\hat{M},\hat{g},e^{-\hat{\phi}}dV_{\hat{g}},\hat{m})$ conformal to $(M,g,e^{-\phi}dV_g,m)$ such that its
weighted scalar curvature $R^m_\phi$ is constant. This was first introduced and studied by Case in [7]. See also [8, 10, 13, 22] for results related to the weighted Yamabe problem.

Similarly, the weighted Yamabe problem with boundary is to find $(M, \hat{g}, e^{-\phi}dV_\hat{g}, e^{-\phi}dA_\hat{g}, m)$ conformal to $(M, g, e^{-\phi}dV_g, e^{-\phi}dA_g, m)$ such that $R^m_\phi$ is constant in $M$ and $H^m_\phi$ is zero on $\partial M$. In view of (1.8), it is equivalent to solve

$$L^m_\phi w = \frac{n + m - 2}{4(n + m - 1)} \lambda w^{\frac{n+m+2}{n+m-2}} \text{ in } M,$$

$$B^m_\phi w = 0 \text{ on } \partial M$$

for some constant $\lambda$. This has been studied by Posso in [23].

In the spirit of [4], we introduce a Yamabe-type flow on the smooth metric measure space with boundary $(M, g, e^{-\phi}dV_g, e^{-\phi}dA_g, m)$, $m \in (0, \infty)$, which is a natural way to solve the weighted Yamabe problem with boundary. The definition of the flow arises from the following observation in [28]. In the sense of Definition 1.2 the metric $(e^\phi)^m g$ is fixed within the conformal class of $(M, g, e^{-\phi}dV_g, e^{-\phi}dA_g, m)$. Based on this observation, we define the (normalized) weighted Yamabe flow with boundary as

$$(1.10) \begin{cases}
\frac{\partial g}{\partial t} = (r^m_\phi - R^m_\phi) g \\
\frac{\partial \phi}{\partial t} = \frac{m}{2} (R^m_\phi - r^m_\phi) \end{cases} \text{ in } M \text{ and } H^m_\phi = 0 \text{ on } \partial M,$$

where $r^m_\phi$ is the average of the weighted scalar curvature $R^m_\phi$; i.e.

$$r^m_\phi = \frac{\int_M R^m_\phi e^{-\phi}dV_g}{\int_M e^{-\phi}dV_g}.$$

On the one hand, equation (1.10) is analogous to the Yamabe flow with boundary (1.1). On the other hand, the flow (1.10) is “sub-critical” in the sense that $\frac{2(n+m)}{n+m-2} < \frac{2n}{n-2}$. As a result, we can establish the sequential compactness in Proposition 4.1, which is the main difference between our flow (1.10) and the geometric flow (1.1) (see [3] for more details). Moreover, we adapt an argument of Brendle [5] to establish long-time existence and convergence of the weighted Yamabe flow with boundary.

**Theorem 1.5.** On a smooth metric measure space with boundary $(M, g, e^{-\phi}dV_g, e^{-\phi}dA_g, m)$, where $(M, \partial M, g)$ is a compact Riemannian manifold with boundary of dimension $n \geq 3$, for every choice of the initial metric and the measure, the weighted Yamabe flow with boundary (1.10) exists for all time and converges to a metric with constant weighted scalar curvature.

This paper is organized as follows. As mentioned above, we first deal with the positive case.

In Section 2, we prove that the conformal factor $w(t)$ cannot blow up in finite time by bounding $w(t)$ from above and below on any finite time interval $[0, T]$. The long-time existence follows from this.
Convergence of the flow (1.10) will be based on the crucial observation in Proposition 3.3. In Section 3, by assuming Proposition 3.3, we obtain decay rates of $r_{\phi(t)}^m$ and the uniform upper bound of $|R_{\phi(t)}^m - r_{\phi(t)}^m|$ in $L^2$ norm. Together with the interior regularity theorem and estimates on the boundary, we show that $w(t)$ is uniformly bounded above and below on $[0, \infty)$, which implies the convergence of the weighted Yamabe flow with boundary.

In Section 4, we complete the proof of Proposition 3.3 by using the spectral theorem of self-adjoint operators and asymptotic analysis.

In Section 5, in the same spirit as [29], we refine the argument in Section 2 to obtain the uniform bound on $w(t)$ and prove the long-time existence and smooth convergence in the negative case. Besides, in the zero case, we obtain the Harnack inequality such that uniform smooth estimates hold.

2. Long time existence

In this section we collect some basic facts for smooth metric measure spaces and prove various properties of the weighted Yamabe flow with boundary that will be used throughout this paper.

Definition 2.1. On a smooth metric measure space with boundary $(M, g, e^{-\phi}dV_g, e^{-\phi}dA_g, m)$ which is conformal to $(M, g_0, e^{-\phi_0}dV_{g_0}, e^{-\phi_0}dA_{g_0}, m)$ in the formulation of (1.6), analogous to the classical Yamabe problem with boundary, we define the normalized energy $E$ as

$$E_{g_0, \phi_0}(w) = \frac{\int_M w L_{\phi_0}^m(w)e^{-\phi_0}dV_{g_0} + \int_M w R_{\phi_0}^m(w)e^{-\phi_0}dA_{g_0}}{\left(\int_M w^{2(n+m-1)/(n+m)}e^{-\phi_0}dV_{g_0}\right)^{n+m-2}}.$$  

Remark 2.2. By the transformation law in (1.8), the normalized energy $E_{g_0, \phi_0}(w)$ can be written as

$$E_{g_0, \phi_0}(w) = \frac{n + m - 2}{4(n + m - 1)} \frac{\int_M R_{\phi}^m e^{-\phi}dV_g + 2 \int_M H_{\phi}^m e^{-\phi}dA_g}{\left(\int_M e^{-\phi}dV_g\right)^{n+m-2}}.$$  

We set

$$Y_{n,m}[(g_0, \phi_0)] = \inf \{ E_{g_0, \phi_0}(w) : 0 < w \in C^\infty(M) \}.$$  

In order to analyze the long time behavior of the solutions of (1.10), we consider three different cases:

- Positive case : $Y_{n,m}[(g_0, \phi_0)] > 0$,
- Zero case : $Y_{n,m}[(g_0, \phi_0)] = 0$,
- Negative case : $Y_{n,m}[(g_0, \phi_0)] < 0$.

Similar to [4] Lemma 2.1, we have the following lemma.

Lemma 2.3. There exists a smooth metric measure space with boundary $(M, g, e^{-\phi}dV_g, e^{-\phi}dA_g, m)$ which is conformal to $(M, g_0, e^{-\phi_0}dV_{g_0}, e^{-\phi_0}dA_{g_0}, m)$ such that

$$R_{\phi}^m > 0 \quad (0 < 0 \text{ respectively}) \quad \text{in } M \text{ and } H_{\phi}^m = 0 \text{ on } \partial M,$$
if \( Y_{n,m}(g_0, \phi_0) > 0 \) (= 0, < 0 respectively).

In light of the discussion in [29], in the case \( Y_{n,m}(g_0, \phi_0) \leq 0 \), it is not difficult to show convergence of the flow (1.10) as \( t \to \infty \). We postpone the proof to Section 5 and deal with the positive case first.

Hereafter, we choose \((M, g_0, e^{-\phi_0} dV_{g_0}, e^{-\phi_0} dA_{g_0}, m)\) to be the initial metric measure space with \( Y_{n,m}(g_0, \phi_0) > 0 \). By conformal change, we may assume that the initial weighted mean curvature \( H^m_{\phi_0} \) satisfies
\[
H^m_{\phi_0} = 0 \quad \text{on } \partial M.
\]

Since the weighted Yamabe flow preserves the conformal structure, we may write
\[
\begin{cases}
g(t) = w(t) \frac{4}{n+m} g_0, \\
e^{-\phi(t)} = w(t) \frac{2(m+n)}{n+m-2} e^{-\phi_0},
\end{cases}
\]
as the solution of (1.10) with \((g(0), \phi(0)) = (g_0, \phi_0)\). Hence, the first equation of (1.10) reduces to the following evolution equation for the conformal factor
\[
\frac{\partial}{\partial t} w(t) \frac{n+m+2}{n+m+2} = \frac{4(n+m-2)}{n+m-2} \Delta_{g_0} w - R^m_{g_0} w + r^m_{\phi(t)} w \frac{n+m+2}{n+m+2}
\]
in \( M \).

It follows from (2.1) and (1.7) that the second equation in (1.10) is equivalent to
\[
\frac{\partial w(t)}{\partial \nu_{g_0}} = 0 \quad \text{on } \partial M.
\]

Hence the conformal factor \( w(t) \) satisfies the evolution equations
\[
\begin{cases}
\frac{\partial w(t)}{\partial t} = -\frac{m+n+2}{4} (R^m_{\phi(t)} - r^m_{\phi(t)}) w(t) \quad \text{in } M, \\
\frac{\partial w(t)}{\partial \nu_{g_0}} = 0 \quad \text{on } \partial M.
\end{cases}
\]

By direct calculation, integration by parts on a smooth metric measure space with boundary \((M, g, e^{-\phi} dV_g, e^{-\phi} dA_g, m)\) takes the following form
\[
\int_M (\nabla f, X) e^{-\phi} dV_g = -\int_M f \text{div}_\phi(X) e^{-\phi} dV_g + \int_{\partial M} f \langle X, \nu_g \rangle e^{-\phi} dA_g
\]
for any smooth vector field \( X \) in \( M \), where \( \text{div}_\phi(X) = \text{div} X - \langle X, \nabla \phi \rangle \).

Since
\[
\frac{d}{dt} \int_M e^{-\phi(t)} dV_{g(t)} = \frac{n+m}{2} \int_M (r^m_{\phi} - R^m_{\phi}) e^{-\phi} dV_g = 0,
\]
we may assume that
\[
\int_M e^{-\phi(t)} dV_{g(t)} = 1
\]
for all \( t \geq 0 \). With this normalization, the average of the weighted scalar curvature can be written as
\[
r^m_\phi(t) = \int_M R^m_{\phi(t)} e^{-\phi(t)} dV_{g(t)}.
\]
By (2.5), differentiating (2.4) with respect to $t$ yields $\frac{\partial R_{\phi(t)}^m}{\partial \nu_{g(t)}} = 0$ on $\partial M$. Since

$$\frac{\partial}{\partial \nu_{g(t)}} = w(t) - \frac{2}{n+m-2} \frac{\partial}{\partial \nu_{g_0}},$$

this is equivalent to

$$(2.10) \quad \frac{\partial R_{\phi(t)}^m}{\partial \nu_{g(t)}} = 0 \text{ on } \partial M.$$

Combining with (1.5) in [28], we deduce that the weighted scalar curvature satisfies the following evolution equations

$$(2.11) \quad \left\{ \begin{array}{l}
\frac{\partial R_{\phi(t)}^m}{\partial t} = (n + m - 1)\Delta_{\phi(t)} R_{\phi(t)}^m + R_{\phi(t)}^m (R_{\phi(t)}^m - r_{\phi(t)}^m) \text{ in } M, \\
\frac{\partial R_{\phi(t)}^m}{\partial \nu_{g(t)}} = 0 \text{ on } \partial M.
\end{array} \right.$$

Using the evolution equation (2.11), we obtain

$$(2.12) \quad \frac{d}{dt} r_{\phi(t)}^m = - \frac{n + m - 2}{2} \int_M (r_{\phi(t)}^m - R_{\phi(t)}^m)^2 e^{-\phi(t)} dV_{g(t)} \leq 0.$$

Observe that $r_{\phi(t)}^m > 0$ since $Y_{n,m} [(g, \phi)] > 0$. Hence, $r_{\phi(t)}^m$ is bounded above and below, i.e.

$$(2.13) \quad 0 < r_{\phi(t)}^m \leq r_{\phi(0)}^m.$$

In particular, the function $t \mapsto r_{\phi(t)}^m$ is decreasing.

Applying the maximum principle to (2.11), we have the following proposition.

**Proposition 2.4.** Along the flow (1.10), there holds

$$\inf_M R_{\phi(t)}^m \geq \min \left\{ \inf_M R_{\phi(0)}^m, 0 \right\}.$$

The following corollary follows from (1.10) and Proposition 2.4 immediately.

**Corollary 2.5.** Along the flow (1.10), there holds

$$\frac{\partial}{\partial t} \phi(t) \geq \frac{m}{2} \left( \inf_M R_{\phi(0)}^m - r_{\phi(0)}^m \right).$$

For abbreviation, we let

$$(2.14) \quad \sigma = \max \left\{ \sup_M \left( 1 - R_{\phi(0)}^m \right), 1 \right\}.$$

By Proposition 2.4, we have $R_{\phi(t)}^m + \sigma \geq 1$ for all $t \geq 0$.

**Lemma 2.6.** For every $p > 2$, we have

$$\frac{d}{dt} \int_M (R_{\phi(t)}^m + \sigma)^{p-1} e^{-\phi(t)} dV_{g(t)}$$

$$= - \frac{4(n + m - 1)(p - 2)}{p - 1} \int_M |\nabla_{g(t)} (R_{\phi(t)}^m + \sigma)^{\frac{p-1}{2}}|^2 e^{-\phi(t)} dV_{g(t)}$$

$$- \left( \frac{n + m + 2}{2} - p \right) \int_M ((R_{\phi(t)}^m + \sigma)^{p-1} - (r_{\phi(t)}^m + \sigma)^{p-1})(R_{\phi(t)}^m - r_{\phi(t)}^m) e^{-\phi(t)} dV_{g(t)}$$

$$- (p - 1) \sigma \int_M ((R_{\phi(t)}^m + \sigma)^{p-2} - (r_{\phi(t)}^m + \sigma)^{p-2})(R_{\phi(t)}^m - r_{\phi(t)}^m) e^{-\phi(t)} dV_{g(t)}.$$
Proof. This follows from differentiating $\int_M (R_{\phi(t)}^m + \sigma)^{p-1} e^{-\phi(t)} dV_{g(t)}$ with respect to $t$, using the evolution equations (2.11) and the integration by parts in (2.6). This was done in [28, Lemma 2.6]. The only difference between our case and [28, Lemma 2.6] is that we have to take care of the term on the boundary $\partial M$. But the term on the boundary $\partial M$ vanishes in view of (2.11). We leave the details to the reader. \hfill \Box

Lemma 2.7. For every $p > \max \left\{ \frac{n + m}{2}, 2 \right\}$, we have

$$\frac{d}{dt} \int_M |R_{\phi(t)}^m - r_{\phi(t)}^m|^p e^{-\phi(t)} dV_{g(t)} \leq C \int_M |R_{\phi(t)}^m - r_{\phi(t)}^m|^p e^{-\phi(t)} dV_{g(t)}$$

$$+ C \left( \int_M |R_{\phi(t)}^m - r_{\phi(t)}^m|^p e^{-\phi(t)} dV_{g(t)} \right)^{2p/(m+n-2)}$$

for some uniform constant $C$ independent of $t$.

Proof. By (1.10) and (2.11) we compute

$$\frac{d}{dt} \int_M |R_{\phi(t)}^m - r_{\phi(t)}^m|^p e^{-\phi(t)} dV_{g(t)} = p(n + m - 1) \int_M |R_{\phi(t)}^m - r_{\phi(t)}^m|^p e^{-\phi(t)} dV_{g(t)}$$

$$- \frac{n + m}{2} \int_M |R_{\phi(t)}^m - r_{\phi(t)}^m|^p (R_{\phi(t)}^m - r_{\phi(t)}^m) \Delta e^{-\phi(t)} dV_{g(t)}$$

$$- \frac{n + m - 2}{2} \int_M |R_{\phi(t)}^m - r_{\phi(t)}^m|^p (R_{\phi(t)}^m - r_{\phi(t)}^m) e^{-\phi(t)} dV_{g(t)}$$

$$+ \frac{(n + m - 2)p}{2} \int_M |R_{\phi(t)}^m - r_{\phi(t)}^m|^p (R_{\phi(t)}^m - r_{\phi(t)}^m) e^{-\phi(t)} dV_{g(t)}$$

Moreover, we have

$$\frac{d}{dt} \int_M |R_{\phi(t)}^m - r_{\phi(t)}^m|^p e^{-\phi(t)} dV_{g(t)}$$

$$= -\frac{4(p-1)(n + m - 1)}{p} \int_M (R_{\phi(t)}^m - r_{\phi(t)}^m)^{\frac{p}{2}} L_{\phi(t)}^m \left( (R_{\phi(t)}^m - r_{\phi(t)}^m)^{\frac{p}{2}} e^{-\phi(t)} dV_{g(t)} \right)$$

$$+ \left( \frac{(n + m - 2)(p - 1)}{p} + \frac{n + m}{2} \right) \int_M |R_{\phi(t)}^m - r_{\phi(t)}^m|^p (R_{\phi(t)}^m - r_{\phi(t)}^m) e^{-\phi(t)} dV_{g(t)}$$

$$+ \left( \frac{(n + m - 2)(p - 1)}{p} + \frac{n + m - 2}{2} \right) \int_M |R_{\phi(t)}^m - r_{\phi(t)}^m|^p e^{-\phi(t)} dV_{g(t)}$$

$$+ \frac{(n + m - 2)p}{2} \int_M |R_{\phi(t)}^m - r_{\phi(t)}^m|^p (R_{\phi(t)}^m - r_{\phi(t)}^m) e^{-\phi(t)} dV_{g(t)}$$

$$+ \frac{(n + m - 2)p}{2} \int_M (R_{\phi(t)}^m - r_{\phi(t)}^m)^{\frac{p}{2}} L_{\phi(t)}^m \left( (R_{\phi(t)}^m - r_{\phi(t)}^m)^{\frac{p}{2}} e^{-\phi(t)} dV_{g(t)} \right)$$

$$\times \int_M (R_{\phi(t)}^m - r_{\phi(t)}^m)^{\frac{p}{2}} e^{-\phi(t)} dV_{g(t)}.$$
where we use (2.10) and (2.6) in the last equality as follows:

\[
\int_M |R_{\phi(t)}^m - r_{\phi(t)}^m|^p - 2 (R_{\phi(t)}^m - r_{\phi(t)}^m) \Delta_{\phi(t)} R_{\phi(t)}^m e^{-\phi(t)} dV_g(t)
\]

\[
= - \int_M \langle \nabla g(t) (|R_{\phi(t)}^m - r_{\phi(t)}^m|^p - 2 (R_{\phi(t)}^m - r_{\phi(t)}^m)), \nabla g(t) R_{\phi(t)}^m \rangle e^{-\phi(t)} dV_g(t)
\]

\[
+ \int_{\partial M} |R_{\phi(t)}^m - r_{\phi(t)}^m|^{p-2} (R_{\phi(t)}^m - r_{\phi(t)}^m) \frac{\partial R_{\phi(t)}^m}{\partial u_g(t)} e^{-\phi(t)} dA_g(t)
\]

\[
= -(p-1) \int_M (R_{\phi(t)}^m - r_{\phi(t)}^m)^{p-1} |\nabla g(t) R_{\phi(t)}^m|^2 e^{-\phi(t)} dV_g(t)
\]

\[
= - \frac{4(p-1)}{p^2} \int_M |\nabla g(t) (R_{\phi(t)}^m - r_{\phi(t)}^m)|^2 |\nabla^\perp g(t)|^2 e^{-\phi(t)} dV_g(t)
\]

\[
+ \frac{4(p-1)}{p^2} \int_{\partial M} (R_{\phi(t)}^m - r_{\phi(t)}^m) \frac{\partial}{\partial u_g(t)} ((R_{\phi(t)}^m - r_{\phi(t)}^m)^{\frac{2}{p-1}}) e^{-\phi(t)} dA_g(t)
\]

\[
= \frac{4(p-1)}{p^2} \int_M (R_{\phi(t)}^m - r_{\phi(t)}^m) \Delta_{\phi(t)} ((R_{\phi(t)}^m - r_{\phi(t)}^m)^{\frac{2}{p-1}}) e^{-\phi(t)} dV_g(t).
\]

Since \( Y_{n,m}[(g_0, \phi_0)] > 0 \) by assumption and the function \( t \mapsto r_{\phi(t)}^m \) is monotonic decreasing, we have

\[
\frac{d}{dt} \int_M |R_{\phi(t)}^m - r_{\phi(t)}^m|^p e^{-\phi(t)} dV_g(t)
\]

\[
\leq - \frac{(n+m-2)(p-1)}{p} Y_{n,m}[(g_0, \phi_0)] \left( \int_M |R_{\phi(t)}^m - r_{\phi(t)}^m|^{\frac{p(n+m)}{n+m-2}} e^{-\phi(t)} dV_g(t) \right)^{\frac{n+m-2}{n+m-2}}
\]

\[
+ \left( \frac{n+m-2}{p} + p - \frac{n+m}{2} \right) \int_M |R_{\phi(t)}^m - r_{\phi(t)}^m|^{p+1} e^{-\phi(t)} dV_g(t)
\]

\[
+ \left( \frac{n+m-2}{p} + p \right) \int_M |R_{\phi(t)}^m - r_{\phi(t)}^m|^{p} e^{-\phi(t)} dV_g(t)
\]

\[
+ \frac{(n+m-2)p}{2} \int_M |R_{\phi(t)}^m - r_{\phi(t)}^m|^{p-2} (R_{\phi(t)}^m - r_{\phi(t)}^m) e^{-\phi(t)} dV_g(t)
\]

\[
\times \int_M (R_{\phi(t)}^m - r_{\phi(t)}^m)^{2} e^{-\phi(t)} dV_g(t).
\]

By Hölder’s inequality in \( L^p(M, e^{-\phi(t)} dV_g(t)) \) and (2.7), we have

\[
\int_M |R_{\phi(t)}^m - r_{\phi(t)}^m|^{p-2} (R_{\phi(t)}^m - r_{\phi(t)}^m) e^{-\phi(t)} dV_g(t) \times \int_M (R_{\phi(t)}^m - r_{\phi(t)}^m)^{2} e^{-\phi(t)} dV_g(t)
\]

\[
\leq \left( \int_M |R_{\phi(t)}^m - r_{\phi(t)}^m|^{p} e^{-\phi(t)} dV_g(t) \right)^{\frac{2}{p}}
\]

and

\[
\int_M |R_{\phi(t)}^m - r_{\phi(t)}^m|^{p+1} e^{-\phi(t)} dV_g(t) \leq \left( \int_M |R_{\phi(t)}^m - r_{\phi(t)}^m|^{p} e^{-\phi(t)} dV_g(t) \right)^{\frac{2p-(n+m)+2}{2p}}
\]

\[
\times \left( \int_M |R_{\phi(t)}^m - r_{\phi(t)}^m|^{\frac{p(n+m)}{n+m-2}} e^{-\phi(t)} dV_g(t) \right)^{\frac{n+m-2}{2p}}.
\]
Moreover, for any \( \epsilon > 0 \), we can apply the Young’s inequality to the last inequality to deduce that
\[
\int_M |R^m_{\phi(t)} - r^m_{\phi(t)}|^{p+1} e^{-\phi(t)} dV_g(t) \leq C_1 \left( \int_M |R^m_{\phi(t)} - r^m_{\phi(t)}|^{p} e^{-\phi(t)} dV_g(t) \right)^{2p/(n+m+2)} + \epsilon \left( \int_M |R^m_{\phi(t)} - r^m_{\phi(t)}|^{p(n+m+2)} e^{-\phi(t)} dV_g(t) \right)^{n+m-2/(n+m)}
\]
for some constant \( C_1 \). Now the assertion follows from combining all these. \( \square \)

In order to bound the solution \( w(t) \) above and below in the interval \([0, T]\), we need the following lemmas.

**Lemma 2.8.** Let \( P \) be a smooth function on \((M, g, e^{-\phi} dV_g, e^{-\phi} dA_g, m)\). Moreover, assume that \( w \) is a positive function such that
\[
-\frac{4(n+m-1)}{n+m-2} \Delta_\phi w + Pw \geq 0 \quad \text{in} \quad M \quad \text{and} \quad \frac{\partial w}{\partial \nu_g} = 0 \quad \text{on} \quad \partial M.
\]
If \( 1 \leq p < \frac{n}{n-2} \), there exists \( C = C(n, m, p, g, \phi) \) and \( r_0 = r_0(M, g, \phi) \) such that
\[
r^{-\frac{2}{p}} \|w\|_{L^p(B^+_r(x))} \leq C \inf_{B^+_r(x)} w
\]
for any \( x \in \partial M \), \( r < r_0 \) and \( B^+_r(x) = M \cap B_r(x) \).

**Proof.** Without loss of generality, we may assume \( r = 1 \). Let \( \beta < 0 \) and \( 0 \leq \chi \in C^1_c(B^+_1) \). By assumption and integration by parts, we have
\[
\int_M (dw, d(\chi^2 w^\beta))_g e^{-\phi} dV_g + C \int_M Pw^{\beta+1} \chi^2 e^{-\phi} dV_g \geq 0.
\]
Since \( \beta < 0 \), we obtain
\[
|\beta| \int_M \chi^2 w^{\beta-1} |dw|_g^2 e^{-\phi} dV_g \leq 2C \int_M \chi^2 \|d\chi\|_g |dw|_g e^{-\phi} dV_g + C \int_M |P|w^{\beta+1} \chi^2 e^{-\phi} dV_g.
\]
Applying Young’s inequality to the first term on the right hand side, we arrive at
\[
\int_M \chi^2 w^{\beta-1} |dw|_g^2 e^{-\phi} dV_g \leq C|\beta|^{-2} \int_M |d\chi|_g^2 w^{\beta+1} e^{-\phi} dV_g + C|\beta|^{-1} \int_M |P|w^{\beta+1} \chi^2 e^{-\phi} dV_g.
\]
(2.15)

We set \( u = w^{\beta+1} \), \( \beta \neq -1 \) such that (2.15) can be rewritten as
\[
\int_M \chi^2 |du|_g^2 e^{-\phi} dV_g \leq C \int_M |d\chi|_g^2 u^2 e^{-\phi} dV_g + C \int_M |P|\chi^2 u^2 e^{-\phi} dV_g.
\]
(2.16)
In order to handle the right hand side of (2.16), we use Hölder’s and interpolation inequalities to get
\[
\int_M |P| \chi^2 u^2 e^{-\phi} dV_g \leq \|P\|_{L^2(B^+_r)} \|\chi u\|^2_{L^\frac{2n}{n-2}(B^+_r)} \\
(2.17) \leq \|P\|_{L^2(B^+_r)} \left( \epsilon \|\chi u\|^2_{L^\frac{2n}{n-2}(B^+_r)} + e^{-\frac{\mu_1}{r^2}} \|\chi u\|^2_{L^2(B^+_r)} \right),
\]
where \( \mu_1 = \frac{n}{n-2} > 0 \). Choosing \( \epsilon \) sufficiently small, we can make use of (2.15), (2.16) and (2.17) to obtain
\[
(2.18) \left( \int_{B^+_a} (\chi u)^{2n} e^{-\phi} dV_g \right)^{\frac{n-2}{n}} \leq C(1 + |\gamma|)^{2\mu_1 + 2} \int_{B^+_1} (|d\chi|^2 + \chi^2) u^2 e^{-\phi} dV_g,
\]
where \( \gamma = \beta + 1 < 0 \).

For any \( 1 \leq r_a < r_b \leq 3 \), we choose \( \chi \) as a cut-off function satisfying \( 0 \leq \chi \leq 1 \), \(|d\chi| \leq \frac{2}{r_b - r_a} \) and
\[
\left\{ \begin{array}{l}
\chi = 1 \text{ in } B^+_r, \\
\chi = 0 \text{ in } B^+_4 \setminus B^+_r.
\end{array} \right.
\]
Using this in (2.18) yields
\[
(2.19) \left( \int_{B^+_a} w^{\frac{n}{n-2}} e^{-\phi} dV_g \right)^{\frac{n-2}{n}} \leq C(1 + |\gamma|)^{2\mu_1 + 2} \int_{B^+_a} w^\gamma e^{-\phi} dV_g.
\]
If we set \( \Gamma(l, r) = \left( \int_{B^+_r} w^l e^{-\phi} dV_g \right)^{\frac{1}{l}} \) and \( \delta = \frac{n}{n-2} \), the estimate above becomes
\[
(2.20) \Gamma(\gamma, r_b) \leq \left( \frac{C(1 + |\gamma|)^{2\mu_1 + 2}}{r_b - r_a} \right)^{\frac{1}{\delta}} \Gamma(\delta \gamma, r_a).
\]
It is well known that
\[
\lim_{l \to +\infty} \Gamma(l, r) = \sup_{B^+_r} w,
\]
\[
\lim_{l \to -\infty} \Gamma(l, r) = \inf_{B^+_r} w.
\]
The rest of the proof follows as in [17] by iterating the estimate in (2.20). \( \square \)

**Lemma 2.9.** Let \( P \) be a smooth function on \((M, g, e^{-\phi} dV_g, e^{-\phi} dA_g, m)\). Moreover, assume that \( w \) is a positive function such that
\[
-\frac{4(n+m-1)}{n+m-2} \Delta_g w + Pw \geq 0 \text{ in } M \text{ and } \frac{\partial w}{\partial \nu_g} = 0 \text{ on } \partial M.
\]

There exists a constant \( C \) depending only on \((M, g, e^{-\phi} dV_g, e^{-\phi} dA_g, m)\) and \( P \) such that
\[
(2.21) \int_M we^{-\phi} dV_g \leq C \inf_M w.
\]
In particular, we have

\begin{equation}
\int_M w \frac{(n+m)}{n+m-2} e^{-\phi} dV_g \leq C \left( \inf_M w \right) \left( \sup_M w \right)^{\frac{n+m+2}{n+m-2}}.
\end{equation}

**Proof.** Fix $r > 0$ sufficiently small. Notice that the weighted Laplacian $\Delta_\phi$ has the same second-order terms as the classical Laplacian. The difference only occurs on lower order terms. Therefore, the interior weak Harnack inequality for linear elliptic equations \cite[Theorem 8.18]{17} can still hold in the weighted case, i.e. we obtain

\begin{equation}
\int_{B_{2r}(x)} w e^{-\phi} dV_g \leq e^{-\inf \phi} \int_{B_{2r}(x)} w dV_g \leq e^{-\inf \phi} L_0 \inf_{B_r(x)} w
\end{equation}

for some constant $L_0$, where $x \in M$ and $B_{2r}(x) \subset M$.

Combining (2.23) with Lemma 2.8 yields the global estimate

\begin{equation}
\int_{B_{2r}(x)} w e^{-\phi} dV_g \leq C \inf_{B_{2r}(x)} w
\end{equation}

for some positive constant $C$, where $x \in M \cup \partial M$ and $B_{2r}(x) = M \cap B_r(x)$. The assertion follows from the same argument as that in \cite[Proposition A.2]{5}.

\hfill \Box

**Proposition 2.10.** Given any $T > 0$, we can find positive constants $C(T)$ and $c(T)$ such that

\[ c(T) \leq \inf_M w(t) \leq \sup_M w(t) \leq C(T) \]

for all $0 \leq t \leq T$.

**Proof.** By Proposition 2.4 and (2.13), the conformal factor $w(t)$ satisfies

\[ \frac{\partial}{\partial t} w(t) = -\frac{m+n-2}{4} (R^m_{\phi(t)} - r^m_{\phi(t)}) w(t) \leq \frac{m+n-2}{4} (r^m_{\phi(0)} + \sigma) w(t) \]

in $M$.

Hence,

\[ \frac{\partial}{\partial t} \log w(t) \leq \frac{m+n-2}{4} (r^m_{\phi(0)} + \sigma). \]

We conclude that $\sup_M w(t) \leq C(T)$ for all $0 \leq t \leq T$. Hence, if we define

\[ P = R^m_{\phi_0} + \sigma \left( \sup_{0 \leq t \leq T} \sup_M w(t) \right)^{\frac{4}{n+m-2}}, \]

then we have

\begin{equation}
\frac{4(n+m-1)}{n+m-2} \Delta_{\phi_0} w(t) + P w(t)
\end{equation}

\[ \geq -\frac{4(n+m-1)}{n+m-2} \Delta_{\phi_0} w(t) + R^m_{\phi_0} w(t) + \sigma w(t)^{\frac{n+m+2}{n+m-2}} \]

\[ = \left( R^m_{\phi(t)} + \sigma \right) w(t)^{\frac{n+m+2}{n+m-2}} \geq 0 \]

for all $0 \leq t \leq T$. By (2.24) and (2.24), we can apply Lemma 2.8 and find a positive constant $c(T)$ such that

\[ \inf_M w(t) \left( \sup_M w(t) \right)^{\frac{n+m+2}{n+m-2}} \geq c(T) \]
for all $0 \leq t \leq T$. Since $\sup_{M} w(t) \leq C(T)$, the assertion follows. \hfill \Box

**Proposition 2.11.** Let $0 < \alpha < \frac{2m}{n + m}$. Given any $T > 0$, there exists a constant $C(T)$ such that

$$|w(x_1, t_1) - w(x_2, t_2)| \leq C(T) \left( (t_1 - t_2)^{\frac{n}{2}} + d(x_1, x_2)^{\alpha} \right)$$

for all $x_1, x_2 \in M$ and $t_1, t_2 \in [0, T]$ satisfying $0 < t_1 - t_2 < 1$. Here, $d(x_1, x_2)$ is the distance between $x_1$ and $x_2$ with respect to the metric $g_0$.

**Proof.** By Lemma 2.6 with $p = \frac{n + m + 2}{2}$, we obtain for all $0 \leq t \leq T$

$$\frac{d}{dt} \int_{M} \left| R_{\phi(t)}^{m} + \sigma \right|^{\frac{n + m}{2}} e^{-\phi(t)} dV_g(t) \leq 0,$$

which implies for all $0 \leq t \leq T$

$$\int_{M} \left( R_{\phi(t)}^{m} + \sigma \right) \frac{n + m}{2} e^{-\phi(t)} dV_g(t) \leq C.$$ 

This together with Hölder’s inequality and (2.13) implies that

$$\left( \int_{M} \left| \frac{n + m}{2} e^{-\phi(t)} dV_g(t) \right|^{\frac{2}{n + m}} \right)^{\frac{n + m}{2}} \leq C.$$

(2.25)

Let $\alpha = 2 - \frac{n}{p}$, where $\frac{n}{2} < p < \frac{n + m}{2}$ with $m > 0$. Using (2.13) and (2.25) and Proposition 2.10 we obtain

$$\int_{M} \left( \frac{4(n + m - 1)}{n + m - 2} \Delta_{\phi_0} w(t) + R_{\phi_0}^{m} w(t) \right)^{p} e^{-\phi_0} dV_{g_0} \leq C(T)$$

(2.26)

and

$$\int_{M} \left| \frac{\partial}{\partial t} w(t) \right|^{p} e^{-\phi_0} dV_{g_0} \leq C(T)$$

(2.27)

for all $t \in [0, T]$. By the Sobolev embedding $W^{2,p}(M) \hookrightarrow C^{0,\alpha}(M)$, the inequality (2.26) implies that

$$|w(x_1, t) - w(x_2, t)| \leq C(T) d(x_1, x_2)^{\alpha}$$
for all \(x_1, x_2 \in M\) and all \(t \in [0, T]\). Using \((2.27)\), we find

\[
|w(x_1, t) - w(x_2, t)| \leq C(t_1 - t_2)^{-\frac{\gamma}{2}} \int_{B \sqrt{t_1 - t_2(x)}} |w(x, t)| e^{-\phi_0} dV_{g_0}
\]

\[
\leq C(t_1 - t_2)^{-\frac{\gamma}{2}} \int_{B \sqrt{t_1 - t_2(x)}} |w(t_1) - w(t_2)| e^{-\phi_0} dV_{g_0} + C(T)(t_1 - t_2)^{\frac{\gamma}{2}}
\]

\[
\leq C(t_1 - t_2)^{\frac{\gamma}{2}} \sup_{t_1 \leq t \leq t_2} \left( \int_{B \sqrt{t_1 - t_2(x)}} \left| \frac{\partial}{\partial t} w(t) \right|^p e^{-\phi_0} dV_{g_0} \right)^{\frac{1}{p}} + C(T)(t_1 - t_2)^{\frac{\gamma}{2}}
\]

\[
\leq C(T)(t_1 - t_2)^{\frac{\gamma}{2}},
\]

for all \(x \in M\) and all \(t_1, t_2 \in [0, T]\) satisfying \(0 < t_1 - t_2 < 1\). This proves the assertion.

Now we can use the standard regularity theory for parabolic equations to show that all higher order derivatives of \(w(t)\) are uniformly bounded on every fixed time interval \([0, T]\). Therefore, the flow exists for all time.

### 3. Proof of the Main Result Assuming Proposition 3.3

In this section, we will prove Theorem 1.5 by assuming Proposition 3.3. In the following, \(c\) and \(C\) are positive constants independent of \(t\), and may change from line to line.

**Proposition 3.1.** For any max \(\left\{ \frac{n + m}{2}, 2 \right\} < p < \frac{n + m + 2}{2}\), we have

\[
\lim_{t \to \infty} \int_M |R_{\phi(t)}^m - r_{\phi(t)}^m|^p e^{-\phi(t)} dV_{g(t)} = 0.
\]

**Proof.** Since \(p > 2\), it follows from Lemma 2.6 that

\[
\frac{d}{dt} \int_M (R_{\phi(t)}^m + \sigma)^{p-1} e^{-\phi(t)} dV_{g(t)}
\]

\[
\leq - \left( \frac{n + m + 2}{2} - p \right) \int_M \left( (R_{\phi(t)}^m + \sigma)^{p-1} - (R_{\phi(t)}^m - r_{\phi(t)}^m)^{p-1} \right) (R_{\phi(t)}^m - r_{\phi(t)}^m) e^{-\phi(t)} dV_{g(t)}.
\]

Since \(p > 2\), we have

\[
\left( (R_{\phi(t)}^m + \sigma)^{p-1} - (r_{\phi(t)}^m + \sigma)^{p-1} \right) (R_{\phi(t)}^m - r_{\phi(t)}^m) \geq c |R_{\phi(t)}^m - r_{\phi(t)}^m|^p
\]

for some constant \(c > 0\). Since \(p < \frac{n + m + 2}{2}\), we obtain

\[
\frac{d}{dt} \int_M (R_{\phi(t)}^m + \sigma)^{p-1} e^{-\phi(t)} dV_{g(t)} \leq -c \int_M |R_{\phi(t)}^m - r_{\phi(t)}^m|^p e^{-\phi(t)} dV_{g(t)}.
\]
Integrating it with respect to \( t \) yields
\[
\int_0^\infty \int_M |R^m_{\phi(t)} - r^m_{\phi(t)}|^p e^{-\phi(t)} dV_{g(t)} dt \leq C.
\]
In particular, we have
\[
\liminf_{t \to \infty} \int_M |R^m_{\phi(t)} - r^m_{\phi(t)}|^p e^{-\phi(t)} dV_{g(t)} = 0.
\]
On the other hand, since \( p > \max \left\{ \frac{n+m}{2}, 2 \right\} \), it follows from Lemma 2.7 that
\[
\frac{d}{dt} \int_M |R^m_{\phi(t)} - r^m_{\phi(t)}|^p e^{-\phi(t)} dV_{g(t)} \leq C \left( \int_M |R^m_{\phi(t)} - r^m_{\phi(t)}|^p e^{-\phi(t)} dV_{g(t)} \right)^{\frac{2p-(n+m)+2}{2p-(n+m)}} + C \int_M |R^m_{\phi(t)} - r^m_{\phi(t)}|^p e^{-\phi(t)} dV_{g(t)}.
\]
From this, the assertion follows. \( \square \)

Hence, if we define
\[
r^m_\infty = \lim_{t \to \infty} r^m_{\phi(t)},
\]
then, we have the following result:

**Corollary 3.2.** For every \( 1 < p < \frac{n+m+2}{2} \), we have
\[
\lim_{t \to \infty} \int_M |R^m_{\phi(t)} - r^m_\infty|^p e^{-\phi(t)} dV_{g(t)} = 0.
\]

**Proof.** It follows from Hölder’s inequality and Proposition 3.1 that
\[
\lim_{t \to \infty} \int_M |R^m_{\phi(t)} - r^m_{\phi(t)}|^p e^{-\phi(t)} dV_{g(t)} = 0
\]
for all \( 1 < p < \frac{n+m+2}{2} \). By Minkowski inequality, we have
\[
\left( \int_M |R^m_{\phi(t)} - r^m_{\phi(t)}|^p e^{-\phi(t)} dV_{g(t)} \right)^\frac{1}{p} \leq \left( \int_M |R^m_{\phi(t)} - r^m_{\phi(t)}|^p e^{-\phi(t)} dV_{g(t)} \right)^\frac{1}{p} + \left( |r^m_{\phi(t)} - r^m_\infty|^p \int_M e^{-\phi(t)} dV_{g(t)} \right)^\frac{1}{p}.
\]
Together with (3.1) and (3.2), this implies the assertion. \( \square \)

The proof of Theorem 1.5 will be based on the following proposition.

**Proposition 3.3.** Let \( \{t_i : i \in \mathbb{N}\} \) be a sequence of times such that \( t_i \to \infty \) as \( i \to \infty \). Then we can find a real number \( 0 < \gamma < 1 \) and a constant \( C \) such that, after passing to a subsequence, we have
\[
r^m_{\phi(t_i)} - r^m_\infty \leq C \left( \int_M |R^m_{\phi(t_i)} - r^m_{\phi(t_i)}|^{2\frac{n+m}{2(n+m)+1}} w(t_i) \int_M e^{-\phi(t_i)} dV_{g_0} \right)^{\frac{n+m+2}{2(n+m)}} (1+\gamma)
\]
for all integers \( i \) in that sequence. Note that \( \gamma \) and \( C \) may depend on the sequence \( \{t_i : i \in \mathbb{N}\} \).
The following result is an immediate consequence of Proposition 3.3.

**Proposition 3.4.** There exist real numbers $0 < \gamma < 1$ and $t_0 > 0$ such that

\[
(3.4) \quad r_{\phi(t)}^m - r_{\phi(t)}^m \leq C \left( \int_M |R_{\phi(t)}^m - r_{\phi(t)}^m|^{2(\frac{n+m+2}{n+m+2})} w(t)^{2(\frac{n+m}{n+m+2})} e^{-\phi(t)} dV_{g_0} \right)^{\frac{n+m+2}{2(n+m+2)(1+\gamma)}}
\]

for all $t \geq t_0$.

**Proof.** Suppose this is not true. Then there exists a sequence of times $\{t_i : i \in \mathbb{N}\}$ such that $t_i \geq t_0$ and

\[
r_{\phi(t_i)}^m - r_{\phi(t_i)}^m \geq C \left( \int_M |R_{\phi(t_i)}^m - r_{\phi(t_i)}^m|^{2(\frac{n+m+2}{n+m+2})} w(t_i)^{2(\frac{n+m}{n+m+2})} e^{-\phi(t_i)} dV_{g_0} \right)^{\frac{n+m+2}{2(n+m+2)(1+\gamma)}}.
\]

We now apply Proposition 3.3 to this sequence $\{t_i : i \in \mathbb{N}\}$. Hence, there exist an infinite subset $I \subset \mathbb{N}$ and real numbers $0 < \gamma < 1$ and $C$ such that

\[
r_{\phi(t_i)}^m - r_{\phi(t_i)}^m \leq C \left( \int_M |R_{\phi(t_i)}^m - r_{\phi(t_i)}^m|^{2(\frac{n+m+2}{n+m+2})} w(t_i)^{2(\frac{n+m}{n+m+2})} e^{-\phi(t_i)} dV_{g_0} \right)^{\frac{n+m+2}{2(n+m+2)(1+\gamma)}}
\]

for all $i \in I$. Thus, we conclude that

\[
1 \leq C \left( \int_M |R_{\phi(t_i)}^m - r_{\phi(t_i)}^m|^{2(\frac{n+m+2}{n+m+2})} w(t_i)^{2(\frac{n+m}{n+m+2})} e^{-\phi(t_i)} dV_{g_0} \right)^{\frac{n+m+2}{2(n+m+2)(1+\gamma)}}
\]

for all $i \in I$.

On the other hand, it follows from Corollary 2.5 with $p = \frac{2(n+m)}{n+m+2} < \frac{n+m+2}{2}$ and $w(t_i)^{2(\frac{n+m}{n+m+2})} e^{-\phi(t_i)} dV_{g_0} = e^{-\phi(t_i)} dV_{g(t_i)}$ that

\[
\lim_{i \to \infty} \int_M |R_{\phi(t_i)}^m - r_{\phi(t_i)}^m|^{2(\frac{n+m+2}{n+m+2})} w(t_i)^{2(\frac{n+m}{n+m+2})} e^{-\phi(t_i)} dV_{g(t_i)} = 0.
\]

Therefore, if $i$ is sufficiently large,

\[
1 \leq \lim_{i \to \infty} \left( \int_M |R_{\phi(t_i)}^m - r_{\phi(t_i)}^m|^{2(\frac{n+m}{n+m+2})} w(t_i)^{2(\frac{n+m}{n+m+2})} e^{-\phi(t_i)} dV_{g_0} \right)^{\frac{n+m+2}{2(n+m+2)(1+\gamma)}} \leq \lim_{i \to \infty} \left( \int_M |R_{\phi(t_i)}^m - r_{\phi(t_i)}^m|^{2(\frac{n+m}{n+m+2})} w(t_i)^{2(\frac{n+m}{n+m+2})} e^{-\phi(t_i)} dV_{g_0} \right) = 0,
\]

which is a contradiction. \qed

**Proposition 3.5.** There holds

\[
\int_0^\infty \left( \int_M |R_{\phi(t)}^m - r_{\phi(t)}^m|^2 w(t)^{\frac{2(n+m)}{n+m+2}} e^{-\phi(t)} dV_{g_0} \right)^{\frac{1}{2}} dt \leq C.
\]

**Proof.** It follows from Proposition 3.4 that

\[
r_{\phi(t)}^m - r_{\phi(t)}^m \leq C \left( \int_M |R_{\phi(t)}^m - r_{\phi(t)}^m|^{2(\frac{n+m+2}{n+m+2})} w(t)^{2(\frac{n+m}{n+m+2})} e^{-\phi(t)} dV_{g_0} \right)^{\frac{n+m+2}{2(n+m+2)(1+\gamma)}}
\]

\[+ C \left( r_{\phi(t)}^m - r_{\phi(t)}^m \right)^{1+\gamma}.
\]
Hence, we have
\[ r_m^{m(t)} - r_m^\infty \leq C \left( \int_M |R_m^{m(t)} - r_m^{m(t)}| \frac{2(n+m)}{n+m-2} w(t) \frac{2(n+m)}{n+m-2} e^{-\phi_0} dV_g \right)^{\frac{n+m+2}{2(n+m)}} (1+\gamma) \]
if \( t \) is sufficiently large. Therefore, by Hölder’s inequality and \((2.7)\), we have
\( (3.5) \)
\[
\frac{d}{dt} (r_m^{m(t)} - r_m^\infty) = -\frac{n + m - 2}{2} \int_M (R_m^{m(t)} - r_m^{m(t)})^2 w(t) \frac{2(n+m)}{n+m-2} e^{-\phi_0} dV_g 
\leq -\frac{n + m - 2}{2} \left( \int_M |R_m^{m(t)} - r_m^{m(t)}| \frac{2(n+m)}{n+m-2} w(t) \frac{2(n+m)}{n+m-2} e^{-\phi_0} dV_g \right) 
\leq -\frac{n + m - 2}{2} (r_m^{m(t)} - r_m^\infty)^{\frac{1}{1+\gamma}}.
\]
This implies that
\[
\frac{d}{dt} (r_m^{m(t)} - r_m^\infty) \leq \frac{1}{1+\gamma} c.
\]
From this, we can deduce that if \( t \) is sufficiently large
\[ r_m^{m(t)} - r_m^\infty \leq C t^{-\frac{1}{1+\gamma}}. \]
Integrating the first equality in \((3.5)\) from \( T \) to \( 2T \) yields
\[ r_m^{m(T)} - r_m^{m(2T)} = \frac{n + m - 2}{2} \int_T^{2T} \int_M (R_m^{m(t)} - r_m^{m(t)})^2 w(t) \frac{2(n+m)}{n+m-2} e^{-\phi_0} dV_g dt. \]
Hence, by Hölder’s inequality, we find
\[
\int_T^{2T} \left( \int_M (R_m^{m(t)} - r_m^{m(t)})^2 w(t) \frac{2(n+m)}{n+m-2} e^{-\phi_0} dV_g \right)^{\frac{1}{2}} dt 
\leq \left( T \int_T^{2T} \int_M (R_m^{m(t)} - r_m^{m(t)})^2 w(t) \frac{2(n+m)}{n+m-2} e^{-\phi_0} dV_g dt \right)^{\frac{1}{2}} 
\leq \left( \frac{2}{n + m - 2} T (r_m^{m(T)} - r_m^{m(2T)}) \right)^{\frac{1}{2}} 
\leq CT^{-\frac{1}{1+\gamma}},
\]
if \( T \) is sufficiently large. Since \( 0 < \gamma < 1 \), we can conclude that
\[
\int_0^\infty \left( \int_M (R_m^{m(t)} - r_m^{m(t)})^2 w(t) \frac{2(n+m)}{n+m-2} e^{-\phi_0} dV_g \right)^{\frac{1}{2}} dt 
= \int_0^1 \left( \int_M (R_m^{m(t)} - r_m^{m(t)})^2 w(t) \frac{2(n+m)}{n+m-2} e^{-\phi_0} dV_g \right)^{\frac{1}{2}} dt 
+ \sum_{k=0}^\infty \int_2^k \left( \int_M (R_m^{m(t)} - r_m^{m(t)})^2 w(t) \frac{2(n+m)}{n+m-2} e^{-\phi_0} dV_g \right)^{\frac{1}{2}} dt 
\leq C \left( 1 + \sum_{k=0}^\infty 2^{-\frac{\gamma}{1+\gamma-k}} \right) \leq C.
\]
This proves the assertion. \( \square \)
Proposition 3.6. Given any \( \eta_0 > 0 \), we can find a real number \( r > 0 \) such that
\[
\int_{B^+_r(x)} w(t) \frac{2(n+m)}{n+m-2} e^{-\phi_0} dV_{g_0} \leq \eta_0
\]
for all \( x \in M \cup \partial M \) and \( t \geq 0 \), where \( B^+_r(x) = M \cap B_r(x) \).

Proof. It follows from Proposition 3.5 that we can find a real number \( T > 0 \) such that
\[
\int_0^\infty \left( \int_M |R_{\phi(t)}^m - r_{\phi(t)}^m|^{2}w(t) \frac{2(n+m)}{n+m-2} e^{-\phi(t)} dV_{g_0} \right)^{\frac{1}{2}} dt \leq \frac{\eta_0}{4(n+m)}.
\]
By Proposition 2.10 we can choose a real number \( r > 0 \) such that
\[
\int_{B^+_r(x)} w(t) \frac{2(n+m)}{n+m-2} e^{-\phi(t)} dV_{g_0} \leq \frac{\eta_0}{2^n}
\]
for all \( x \in M \cup \partial M \) and \( 0 \leq t \leq T \). By (2.7) and Hölder’s inequality, we have
\[
\frac{d}{dt} \int_{B^+_r(x)} e^{-\phi(t)} dV_{g(t)} = -\frac{n+m}{2} \int_{B^+_r(x)} (R_{\phi(t)}^m - r_{\phi(t)}^m)e^{-\phi(t)} dV_{g(t)}
\]
\[
\leq \frac{n+m}{2} \left( \int_M (R_{\phi(t)}^m - r_{\phi(t)}^m)^2 e^{-\phi(t)} dV_{g(t)} \right)^{\frac{1}{2}}.
\]
Integrating this over \([T,t]\) yields
\[
\int_{B^+_r(x)} w(t) \frac{2(n+m)}{n+m-2} e^{-\phi_0} dV_{g_0}
\]
\[
\leq \int_{B^+_r(x)} w(T) \frac{2(n+m)}{n+m-2} e^{-\phi_0} dV_{g_0} + \frac{n+m}{2} \int_T^\infty \left( \int_M (R_{\phi(t)}^m - r_{\phi(t)}^m)^2 e^{-\phi(t)} dV_{g(t)} \right)^{\frac{1}{2}} dt
\]
\[
\leq \eta_0.
\]
for all \( x \in M \) and all \( t \geq T \), where we have used 3.7 and 3.8 in the last inequality. This proves the assertion. \( \square \)

Lemma 3.7. Let \( p = \frac{2(n+m)}{n+m-2} \) and \( q = \frac{2}{2} \). There are positive constants \( \eta_1 \) and \( C \) such that if
\[
g = w^{\frac{m}{n+m-2}} g_0, \\
e^{-\phi} = w^{\frac{2m}{n+m-2}} e^{-\phi_0},
\]
and
\[
\int_{B^+_r(x)} e^{-\phi} dV_g \leq 1 \quad \text{and} \quad \int_{B^+_r(x)} |R_{\phi}^m|^q e^{-\phi} dV_g \leq \eta_1,
\]
where \( B^+_r(x) = M \cap B_r(x) \) is the geodesic ball with respect to \( g_0 \) and \( r < 1 \), then
\[
w(x) \leq C T^{-\frac{1}{p}} \left( \int_{B^+_r(x)} e^{-\phi} dV_g \right)^{\frac{1}{p}}.
\]
Proof. By the smoothness of the conformal factor $w(t)$, there exists $r_0$ a real number such that $r_0 < r$ and

$$(r - s)^\frac{2}{p} \sup_{B^*_r(x)} w \leq (r - r_0)^\frac{2}{p} \sup_{B^*_{r_0}(x)} w$$

for all $s < r$. Moreover, we choose a point $x_0 \in B^*_{r_0}(x)$ such that

$$\sup_{B^*_{r_0}(x)} w = w(x_0).$$

Notice that the conformal weighted Laplacian $L^m_{\phi_0}$ has the same leading term as the classical Laplacian $\Delta_{g_0}$. The difference only occurs on lower order terms. If $x_0$ is in the interior of $M$, using a standard interior estimate for linear elliptic equations in [17, Theorem 8.17], we obtain

$$s^{2-p} \sup_{B^*_s(x_0)} w(x_0) \leq C \left( \int_{B_s(x_0)} w^p e^{-\phi_0} dV_{g_0} \right)^{\frac{1}{p}}$$

(3.10)

$$+ C s^{\frac{2-p}{2} - \frac{1}{4}} \left( \int_{B_s(x_0)} \left| \frac{(n + m - 1)}{(n + m - 2)} L^m_{\phi_0} w \right|^q e^{-\phi_0} dV_{g_0} \right)^{\frac{1}{q}}$$

for $s \leq \frac{\alpha m}{2}$ and $B_s(x_0) \subset M$.

If $x_0$ is on the boundary $\partial M$, we may adapt the argument in [3, Proposition A-2] to obtain

$$s^{2-p} \sup_{B^*_s(x_0)} w(x_0) \leq C \left( \int_{B^*_s(x_0)} w^p e^{-\phi_0} dV_{g_0} \right)^{\frac{1}{p}}$$

(3.11)

$$+ C s^{\frac{2-p}{2} - \frac{1}{4}} \left( \int_{B^*_s(x_0)} \left| \frac{(n + m - 1)}{(n + m - 2)} L^m_{\phi_0} w \right|^q e^{-\phi_0} dV_{g_0} \right)^{\frac{1}{q}},$$

for $s < \tilde{r}$, where $\tilde{r}$ is the constant in [3, Proposition A-2].

In both cases we have

$$s^{2-p} \sup_{B^*_s(x_0)} w(x_0) \leq C \left( \int_{B^*_s(x_0)} w^p e^{-\phi_0} dV_{g_0} \right)^{\frac{1}{p}}$$

$$+ C s^{\frac{2-p}{2} - \frac{1}{4}} \left( \int_{B^*_s(x_0)} \left| \frac{(n + m - 1)}{(n + m - 2)} L^m_{\phi_0} w \right|^q e^{-\phi_0} dV_{g_0} \right)^{\frac{1}{q}},$$

for $s \leq \min\{\frac{\alpha m}{2}, \tilde{r}\}$. The assertion follows from the same iteration argument as that in [5, Proposition A.2].

□

Proposition 3.8. Along the flow, the function $w(t)$ satisfies

$$c \leq \inf_M w(t) \leq \sup_M w(t) \leq C$$

for all $t \geq 0$. Here, $c$ and $C$ are positive constants independent of $t$. 

Proof. Fix $\frac{n}{2} < q < p < \frac{n + m + 2}{2}$. It follows from Corollary 3.2 that

$$\int_M |R_{\phi(t)}^m|^p e^{-\phi(t)} dV_g(t) \leq C,$$

for some constant $C$ independent of $t$. By Proposition 3.6 we can find a constant $r > 0$ independent of $t$ such that

$$\int_{B_r^+(x)} w(t) 2^{\frac{n(m+q)}{n+m-2}} e^{-\phi_0} dV_{g_0} \leq \eta_0$$

for all $x \in M \cup \partial M$ and $t \geq 0$. By Hölder’s inequality, we have

$$\int_{B_r^+(x)} |R_{\phi(t)}^m|^q e^{-\phi(t)} dV_g(t) \leq \left( \int_{B_r^+(x)} e^{-\phi(t)} dV_g(t) \right)^{\frac{q}{p}} \left( \int_M |R_{\phi(t)}^m|^p e^{-\phi(t)} dV_g(t) \right)^{\frac{q}{p}}.$$

Hence, if we choose $\eta_0$ sufficiently small, we then have

$$\int_{B_r^+(x)} |R_{\phi(t)}^m|^q e^{-\phi(t)} dV_g(t) \leq \eta_1$$

for all $x \in M \cup \partial M$ and all $t \geq 0$. Here, $\eta_1$ is the constant appearing in Lemma 3.7. We can now apply Lemma 3.7 at the maximum point of $w(t)$ to deduce that

$$\sup_{M} w(t) \leq Cr^{-\frac{q}{p}} \left( \int_{B_r^+(x)} e^{-\phi(t)} dV_g(t) \right)^{\frac{1}{p}}.$$

Together with (2.7), this implies that $w(t)$ is uniformly bounded from above. Hence, if we define

$$P = R_{\phi_0}^m + \sigma \left( \sup_{t \geq 0} \sup_M w(t) \right)^{\frac{n+m-2}{n+m-2}}$$

where $\sigma$ is given as in (2.14), then we have

$$- \frac{4(n+m-1)}{n+m-2} \Delta_{\phi_0} w(t) + P w(t) \geq - \frac{4(n+m-1)}{n+m-2} \Delta_{\phi_0} w(t) + R_{\phi_0}^m w(t) + \sigma w(t)^{\frac{n+m+2}{n+m-2}} = (R_{\phi(t)}^m + \sigma) w(t)^{\frac{n+m+2}{n+m-2}} \geq 0.$$

By (2.7) and Lemma 2.3 we can find a positive constant $c$ independent of $t$ such that

$$\inf_{M} w(t) \left( \sup_{M} w(t) \right)^{\frac{n+m+2}{n+m-2}} \geq c$$

for all $t \geq 0$. This implies that $w(t)$ is uniformly bounded from below, since $w(t)$ is uniformly bounded from above. This proves the assertion. ∎

Proposition 3.9. Let $0 < \alpha < \frac{2m}{n+m}$. There holds

$$|w(x_1, t_1) - w(x_2, t_2)| \leq C \left( (t_1 - t_2)^{\frac{\alpha}{2}} + d(x_1, x_2)^{\alpha} \right)$$

for all $x_1, x_2 \in M$ and $0 < t_1 - t_2 < 1$. Here, $C$ is a positive constant independent of $t_1$ and $t_2$. 
Proof. Let \( \alpha = 2 - \frac{n}{p} \), where \( \frac{n}{2} < p < \frac{n+m}{2} \). As in the proof of Proposition 2.10 we can deduce from Proposition 3.8 that
\[
\int_M \left| \frac{4(n+m-1)}{n+m-2} \Delta \phi_0 w(t) + R_{g_0}^m w(t) \right|^p e^{-\phi_0} dV_{g_0} \leq C
\]
and
\[
\int_M \left| \frac{\partial}{\partial t} w(t) \right|^p e^{-\phi_0} dV_{g_0} \leq C
\]
where \( C \) is a positive constant independent of \( t \). By the Sobolev embedding \( W^{2,p}(M) \hookrightarrow C^{0,\alpha}(M) \), the inequality (3.12) implies that
\[
|w(x_1, t) - w(x_2, t)| \leq C d(x_1, x_2)^\alpha
\]
for all \( x_1, x_2 \in M \) and all \( t \geq 0 \). On the other hand, by the second inequality (3.13), we find
\[
|w(x, t_1) - w(x, t_2)| \\
\leq C(t_1 - t_2)^{\frac{1}{\alpha}} \int_{B_{\sqrt{t_1-t_2}(x)}} |w(x, t_1) - w(x, t_2)| e^{-\phi_0} dV_{g_0} \\
\leq C(t_1 - t_2)^{\frac{1}{\alpha}} \int_{B_{\sqrt{t_1-t_2}(x)}} |w(t_1) - w(t_2)| e^{-\phi_0} dV_{g_0} + C(t_1 - t_2) \frac{1}{\alpha} \\
\leq C(t_1 - t_2) \frac{1}{\alpha} \sup_{t_1 \leq t \leq t_2} \int_{B_{\sqrt{t_1-t_2}(x)}} \left| \frac{\partial}{\partial t} w(t) \right|^p e^{-\phi_0} dV_{g_0} + C(t_1 - t_2) \frac{1}{\alpha} \\
\leq C(t_1 - t_2) \frac{1}{\alpha},
\]
for all \( x \in M \) and all \( t_1, t_2 \geq 0 \) satisfying \( 0 < t_1 - t_2 < 1 \). This proves the assertion. \( \square \)

Now we can use the standard regularity theory for parabolic equations to show that all higher order derivatives of \( w(t) \) are uniformly bounded on \([0, \infty)\). The uniqueness of the asymptotic limit follows Proposition 3.3. This completes the proof of Theorem 1.5.

4. Proof of Proposition 3.3

Let \( \{t_i, i \in \mathbb{N}\} \) be a sequence of times such that \( t_i \to \infty \) as \( i \to \infty \). For abbreviation, we let \( w_i = w(t_i) \). The normalization condition (2.7) implies that
\[
\int_M w_i^{\frac{2(n+m)}{n+m-2}} e^{-\phi_0} dV_{g_0} = 1
\]
for all \( i \in \mathbb{N} \). Moreover, it follows from Corollary 3.2 that
\[
\int_M \left| R_{g(t_i)}^m - \phi_i \right|^{\frac{2(n+m)}{n+m-2}} e^{-\phi(t_i)} dV_{g(t_i)} \to 0,
\]
and hence
\begin{equation}
\int_M \left| \frac{4(n+m-1)}{n+m-2} \Delta_{\phi_0} w_i - R_{\phi_0}^m w_i + r_{\infty}^m w_i \frac{2(n+m)}{n+m+2} \right| e^{-\phi_0} dV_{\phi_0} \to 0
\end{equation}
as \( i \to \infty \). On the other hand, it follows from (2.4) that
\begin{equation}
\frac{\partial w_i}{\partial \nu_{g_0}} = 0 \text{ on } \partial M
\end{equation}
for all \( i \in \mathbb{N} \). By the standard elliptic theory, we have the following compactness result.

**Proposition 4.1.** Let \( \{w_i : i \in \mathbb{N}\} \) be a sequence of positive functions satisfying (4.1) and (4.2). After passing to a subsequence if necessary, \( w_i \) converges to a positive smooth function \( w_\infty \) satisfying
\begin{equation*}
\frac{4(n+m-1)}{n+m-2} \Delta_{\phi_0} w_\infty - R_{\phi_0}^m w_\infty + r_{\infty}^m w_\infty \frac{2(n+m)}{n+m+2} = 0 \text{ in } M \quad \text{and} \quad \frac{\partial w_\infty}{\partial \nu_{g_0}} = 0 \text{ on } \partial M.
\end{equation*}

**Proof.** Since \( \frac{n+m+2}{n+m-2} < \frac{n+2}{n-2} \), the assertion now follows from (4.1)-(4.3) and the standard elliptic theory [16, Section 8, Theorem 3]. \( \square \)

In order to prove Proposition 3.3, we need the following:

**Proposition 4.2.** There exists a sequence of smooth functions \( \{\psi_a : a \in \mathbb{N}\} \) and a sequence of positive real numbers \( \{\lambda_a : a \in \mathbb{N}\} \) with the following properties:
\begin{enumerate}[(i)]
\item For every \( a \in \mathbb{N} \), the function \( \psi_a \) satisfies the equation
\begin{equation}
\frac{4(n+m-1)}{n+m-2} \Delta_{\phi_0} \psi_a - R_{\phi_0}^m \psi_a + \lambda_a w_\infty \frac{2(n+m)}{n+m+2} \psi_a = 0 \text{ in } M \quad \text{and} \quad \frac{\partial \psi_a}{\partial \nu_{g_0}} = 0 \text{ on } \partial M.
\end{equation}
\item For all \( a, b \in \mathbb{N} \), we have
\begin{equation}
\int_M w_\infty^\frac{4}{n+m-2} \psi_a \psi_b e^{-\phi_0} dV_{g_0} = \begin{cases} 0, & \text{if } a \neq b; \\ 1, & \text{if } a = b. \end{cases}
\end{equation}
\item The span of \( \{\psi_a : a \in \mathbb{N}\} \) is dense in \( L^2(M, e^{-\phi_0} dV_{g_0}) \).
\item \( \lambda_a \to \infty \) as \( a \to \infty \).
\end{enumerate}

**Proof.** Consider the linear operator
\begin{equation*}
T : \psi \mapsto w_\infty^{-\frac{4}{n+m-2}} \left( \frac{4(n+m-1)}{n+m-2} \Delta_{\phi_0} \psi - R_{\phi_0}^m \psi \right),
\end{equation*}
where \( \psi \) satisfies \( \frac{\partial \psi}{\partial \nu_{g_0}} = 0 \) on \( \partial M \). By integration by parts, we see that this operator \( T \) is symmetric with respect to the inner product
\begin{equation*}
(\psi_1, \psi_2) \mapsto \int_M w_\infty^{-\frac{4}{n+m-2}} \psi_1 \psi_2 e^{-\phi_0} dV_{g_0}
\end{equation*}
on \( L^2(M, e^{-\phi_0} dV_{g_0}) \). Hence, the assertion follows from the spectral theorem. \( \square \)
Lemma 4.3. For every \( L \) denote by \( \Pi \) the projection operator

\[
\Pi f = \sum_{a \in A} \left( \int_M w_{\infty}^{\frac{4}{n+m-2}} \psi_a f e^{-\phi_0} dV_g \right) w_{\infty}^{\frac{4}{n+m-2}} \psi_a
\]

(4.6)

\[
= f - \sum_{a \in A} \left( \int_M \psi_a f e^{-\phi_0} dV_g \right) w_{\infty}^{\frac{4}{n+m-2}} \psi_a.
\]

In the rest of this section, for simplicity, we denote \( W^{1,2}(M, e^{-\phi_0} dV_g) \) and \( L^p(M, e^{-\phi_0} dV_g) \) by \( W^{1,2}(M) \) and \( L^p(M) \), respectively.

**Lemma 4.3.** For every \( 1 \leq p < \infty \), we can find a constant \( C \) such that

\[
\|f\|_{L^p(M)} \leq C \left\| \left( \frac{4(n+m-1)}{n+m-2} \Delta_{\phi_0} f - R_{\phi_0}^m f + \frac{n + m + 2}{n + m - 2} r_{M}^m w_{\infty}^{\frac{4}{n+m-2}} f \right) \right\|_{L^p(M)}
\]

\[
+ C \sup_{a \in A} \left| \int_M w_{\infty}^{\frac{4}{n+m-2}} \psi_a f e^{-\phi_0} dV_g \right|
\]

**Proof.** Suppose this is not true. By compactness, we can find a function \( f \in L^p(M) \) satisfying \( \|f\|_{L^p(M)} = 1 \),

\[
\int_M w_{\infty}^{\frac{4}{n+m-2}} \psi_a f e^{-\phi_0} dV_g = 0
\]

(4.7)

for all \( a \in A \) and

\[
\frac{4(n+m-1)}{n+m-2} \Delta_{\phi_0} f - R_{\phi_0}^m f + \frac{n + m + 2}{n + m - 2} r_{M}^m w_{\infty}^{\frac{4}{n+m-2}} f = 0
\]

in the sense of distributions. Hence, if we use \( \psi \) as a test function, then we obtain

\[
\left( \lambda_a - \frac{n + m + 2}{n + m - 2} r_{M}^m \right) \int_M w_{\infty}^{\frac{4}{n+m-2}} \psi_a f e^{-\phi_0} dV_g = 0
\]

for all \( a \in \mathbb{N} \). In particular, we have

\[
\int_M w_{\infty}^{\frac{4}{n+m-2}} \psi_a f e^{-\phi_0} dV_g = 0
\]

for all \( a \not\in A \). Combining this with (4.7), we can conclude that \( f = 0 \), which is a contradiction. \( \square \)

**Lemma 4.4.** (i) There exists a constant \( C \) such that

\[
\|f\|_{L^\frac{n+m+2}{n+m-2}(M)} \leq C \left\| \left( \frac{4(n+m-1)}{n+m-2} \Delta_{\phi_0} f - R_{\phi_0}^m f + \frac{n + m + 2}{n + m - 2} r_{M}^m w_{\infty}^{\frac{4}{n+m-2}} f \right) \right\|_{L^\infty(M)}
\]

\[
+ C \sup_{a \in A} \left| \int_M w_{\infty}^{\frac{4}{n+m-2}} \psi_a f e^{-\phi_0} dV_g \right|
\]
where \( s = \frac{n(n + m + 2)}{n(n + m - 2) + 2(n + m + 2)} \).

(ii) There exists a constant such that

\[
\|f\|_{L^1(M)} \leq C \left\| \Pi \left( \frac{4(n + m - 1)}{n + m - 2} \Delta_{\phi_0} f - R_{\phi_0}^m f + \frac{n + m + 2}{n + m - 2} r_m^{m \frac{4}{n + m - 2}} f \right) \right\|_{L^1(M)} + C \sup_{a \in A} \left| \int_M w_\infty^{\frac{4}{n + m - 2}} \psi_a e^{-\phi_0} dV_{g_0} \right|.
\]

Proof. If follows from the Sobolev embedding \( W^{2, s}(M) \to L^{\frac{n + m + 2}{m}}(M) \) that

\[
\left\| f \right\|_{L^{\frac{n + m + 2}{m}}(M)} \leq C \| f \|_{L^s(M)} + C \left\| \frac{4(n + m - 1)}{n + m - 2} \Delta_{\phi_0} f - R_{\phi_0}^m f + \frac{n + m + 2}{n + m - 2} r_m^{m \frac{4}{n + m - 2}} f \right\|_{L^s(M)}.
\]

This together with Lemma 4.5 implies that

\[
\left\| f \right\|_{L^{\frac{n + m + 2}{m}}(M)} \leq C \sup_{a \in A} \left| \int_M w_\infty^{\frac{4}{n + m - 2}} \psi_a e^{-\phi_0} dV_{g_0} \right| + C \left\| \frac{4(n + m - 1)}{n + m - 2} \Delta_{\phi_0} f - R_{\phi_0}^m f + \frac{n + m + 2}{n + m - 2} r_m^{m \frac{4}{n + m - 2}} f \right\|_{L^s(M)}.
\]

It follows from the definition of \( \Pi \) that

\[
\frac{4(n + m - 1)}{n + m - 2} \Delta_{\phi_0} f - R_{\phi_0}^m f + \frac{n + m + 2}{n + m - 2} r_m^{m \frac{4}{n + m - 2}} f = \Pi \left( \frac{4(n + m - 1)}{n + m - 2} \Delta_{\phi_0} f - R_{\phi_0}^m f + \frac{n + m + 2}{n + m - 2} r_m^{m \frac{4}{n + m - 2}} f \right)
\]

\[
- \sum_{a \in A} \left( \lambda_a - \frac{n + m + 2}{n + m - 2} r_m \right) \left( \int_M w_\infty^{\frac{4}{n + m - 2}} \psi_a e^{-\phi_0} dV_{g_0} \right) w_\infty^{\frac{4}{n + m - 2}} \psi_a,
\]

which implies that

\[
\left\| \frac{4(n + m - 1)}{n + m - 2} \Delta_{\phi_0} f - R_{\phi_0}^m f + \frac{n + m + 2}{n + m - 2} r_m^{m \frac{4}{n + m - 2}} f \right\|_{L^s(M)} \leq \left\| \Pi \left( \frac{4(n + m - 1)}{n + m - 2} \Delta_{\phi_0} f - R_{\phi_0}^m f + \frac{n + m + 2}{n + m - 2} r_m^{m \frac{4}{n + m - 2}} f \right) \right\|_{L^s(M)}
\]

\[
+ C \sup_{a \in A} \left| \int_M w_\infty^{\frac{4}{n + m - 2}} \psi_a e^{-\phi_0} dV_{g_0} \right|.
\]

Now (i) follows from putting these facts together.

Similar to (i), (ii) follows from Lemma 4.5 and the definition of \( \Pi \). \( \square \)

Lemma 4.5. There exists a positive real number \( \xi \) such that for every vector \( z \in \mathbb{R}^A \) with \( |z| \leq \xi \), there exists a smooth function \( w_z \) such that \( \frac{\partial w_z}{\partial \nu_{g_0}} = 0 \) on \( \partial M \), and

\[
\int_M w_\infty^{\frac{4}{n + m - 2}} \psi_a (w_z - w_\infty) e^{-\phi_0} dV_{g_0} = z_a
\]
for all $a \in A$ and

$$
\Pi \left( \frac{4(n + m - 1)}{n + m - 2} \Delta_{\phi_0} \overline{w}_z - R^m_{\phi_0} \overline{w}_z + r^m_{\infty} \frac{4}{n + m - 2} f \right) = 0.
$$

Furthermore, the map $z \mapsto \overline{w}_z$ is real analytic.

**Proof.** This is a consequence of implicit function theorem.

**Lemma 4.6.** There exists a real number $0 < \gamma < 1$ such that

$$
E_{(g_0, \phi_0)}(\overline{w}_z) - E_{(g_0, \phi_0)}(w_{\infty}) 
\leq C \sup_{a \in A} \left| \int_M \left( \frac{4(n + m - 1)}{n + m - 2} \Delta_{\phi_0} \overline{w}_z - R^m_{\phi_0} \overline{w}_z + r^m_{\infty} \frac{4}{n + m - 2} \right) \psi_a e^{-\phi_0} dV_{g_0} \right|^{1+\gamma}
$$

if $z$ is sufficiently small.

**Proof.** Note that the function $z \mapsto E_{(g_0, \phi_0)}(\overline{w}_z)$ is real analytic. According to the results of Lojasiewicz [25, equation (2.4)], there exists a real number $0 < \gamma < 1$ such that

$$
|E_{(g_0, \phi_0)}(\overline{w}_z) - E_{(g_0, \phi_0)}(w_{\infty})| \leq \sup_{a \in A} \left| \frac{\partial}{\partial a} E(\overline{w}_z) \right|^{1+\gamma}
$$

if $z$ is sufficiently small. For convenience, we define the energy functional $F_{(g_0, \phi_0)}(w)$ as

$$
F_{(g_0, \phi_0)}(w) = \int_M w F^m_{\phi_0}(w) e^{-\phi_0} dV_{g_0} + \int_M w B^m_{\phi_0}(w) e^{-\phi_0} dA_{g_0}
$$

The partial derivatives of the function $z \mapsto E_{(g_0, \phi_0)}(\overline{w}_z)$ are given by (4.8)

$$
\frac{\partial}{\partial z_a} E_{(g_0, \phi_0)}(\overline{w}_z) = -2 \left( \int_M \left( \frac{4(n + m - 1)}{n + m - 2} \Delta_{\phi_0} \overline{w}_z - R^m_{\phi_0} \overline{w}_z + r^m_{\infty} \frac{4}{n + m - 2} \right) \overline{\psi}_{a,z} e^{-\phi_0} dV_{g_0} \right)
$$

and

$$
-2(F_{g_0, \phi_0}(\overline{w}_z) - r^m_{\infty}) \int_M \left( \frac{4(n + m - 1)}{n + m - 2} \Delta_{\phi_0} \overline{w}_z - R^m_{\phi_0} \overline{w}_z + r^m_{\infty} \frac{4}{n + m - 2} \right) \overline{\psi}_{a,z} e^{-\phi_0} dV_{g_0}
$$

where $\overline{\psi}_{a,z} = \frac{\partial}{\partial z_a} \overline{w}_z$ for $a \in A$. The function $\overline{\psi}_{a,z}$ satisfies

$$
\int_M w_{\infty}^{4(n + m - 1) - \frac{4}{n + m - 2}} \psi_{a,z} \psi_b e^{-\phi_0} dV_{g_0} = \begin{cases} 0, & \text{if } a \neq b; \\
1, & \text{if } a = b \end{cases}
$$

for all $a, b \in A$ and

$$
\Pi \left( \frac{4(n + m - 1)}{n + m - 2} \Delta_{\phi_0} \overline{\psi}_{a,z} - R^m_{\phi_0} \overline{\psi}_{a,z} + \frac{n + m + 2}{n + m - 2} r^m_{\infty} w_{\infty}^{\frac{4}{n + m - 2}} \overline{\psi}_{a,z} \right) = 0.
$$

Using the identity

$$
\Pi \left( \frac{4(n + m - 1)}{n + m - 2} \Delta_{\phi_0} \overline{w}_z - R^m_{\phi_0} \overline{w}_z + r^m_{\infty} \frac{4}{n + m - 2} \overline{\psi}_{a,z} \right) = 0,
$$
we obtain
\[
\frac{\partial}{\partial z_a} E_{g_0, \phi_0}(\overline{w}_z)
= -2 \int_M \left( \frac{4(n + m - 1)}{n + m - 2} \Delta_{\phi_0} \overline{w}_z - R^{m}_{\phi_0} \overline{w}_z + \frac{R^{m} w^{n+m+2}}{m^{n+m+2}} \right) \psi_a e^{-\phi_0} dV_{g_0}
\]
\[+ \sum_{b \in A} \left( \frac{\int_M \overline{w}_z^{n+m+2} \psi_a \psi_b e^{-\phi_0} dV_{g_0} - \int_M \overline{w}_z^{n+m+2} \psi_b e^{-\phi_0} dV_{g_0}}{\int_M \overline{w}_z^{n+m+2} e^{-\phi_0} dV_{g_0}} \right) \]
\[\cdot \int_M \frac{4(n + m - 1)}{n + m - 2} \Delta_{\phi_0} \overline{w}_z - R^{m}_{\phi_0} \overline{w}_z + \frac{R^{m} w^{n+m+2}}{m^{n+m+2}} \right) \psi_a e^{-\phi_0} dV_{g_0} \int_M \overline{w}_z^{n+m+2} e^{-\phi_0} dV_{g_0}
\]
for all \( a \in A \). Thus, we obtain
\[
\sup_{a \in A} \left| \frac{\partial}{\partial z_a} E_{g_0, \phi_0}(\overline{w}_z) \right| \leq C \sup_{a \in A} \left| \int_M \left( \frac{4(n + m - 1)}{n + m - 2} \Delta_{\phi_0} \overline{w}_z - R^{m}_{\phi_0} \overline{w}_z + \frac{R^{m} w^{n+m+2}}{m^{n+m+2}} \right) \psi_a e^{-\phi_0} dV_{g_0} \right|
\]
Combining all these, the assertion follows.

By Lemma 4.5, the function
\[
z \to \int_M (w_i - \overline{w}_z) L^{m}_{\phi_0} (w_i - \overline{w}_z) e^{-\phi_0} dV_{g_0}
\]
is analytical and attains the infimum for \( |z| \leq \xi \). For every \( i \in \mathbb{N} \), we can find \( \overline{w}_z \), such that \( |z| \leq \xi \) and
\[
\int_M (w_i - \overline{w}_z) L^{m}_{\phi_0} (w_i - \overline{w}_z) e^{-\phi_0} dV_{g_0} \leq \int_M (w_i - \overline{w}_z) L^{m}_{\phi_0} (w_i - \overline{w}_z) e^{-\phi_0} dV_{g_0}
\]
for all \( |z| \leq \xi \).

Notice that by Lemma 4.5 we have \( \overline{w}_0 = w_{\infty} \). Combining this with the definition of \( \overline{w}_z \), we have
\[
\int_M (w_i - \overline{w}_z) L^{m}_{\phi_0} (w_i - \overline{w}_z) e^{-\phi_0} dV_{g_0} \leq \int_M (w_i - w_{\infty}) L^{m}_{\phi_0} (w_i - w_{\infty}) e^{-\phi_0} dV_{g_0}.
\]

By the compactness result in Proposition 4.3, the expression on the right hand side tends to zero as \( i \to \infty \), i.e., we have as \( i \to \infty \),
\[
(4.9) \quad \| w_i - \overline{w}_z \|_{W^{1,2}(M)} \to 0 \quad \text{and} \quad \| \overline{w}_z - w_{\infty} \|_{W^{1,2}(M)} \to 0.
\]

**Lemma 4.7.** The difference \( w_i - \overline{w}_z \), satisfies
\[
\| w_i - \overline{w}_z \|_{L^{n+m+2}(M)} \leq C \left( \frac{w^{n+m+2}}{m^{n+m+2}} \right) + o(1)
\]
if \( i \) is sufficiently large.
Proof. For simplicity, we denote $w_i - w_{z_i}$ by $u_i$. Using the identities

$$
(4.10) \quad \frac{4(n + m - 1)}{n + m - 2} \Delta \phi_i u_i - R^m_{\phi_0} u_i + r_m w_i^{\frac{n + m + 2}{n + m - 2}} = -w_i^{\frac{n + m + 2}{n + m - 2}} (R^m_{\phi_i} - r_m)
$$

and

$$
\Pi \left( \frac{4(n + m - 1)}{n + m - 2} \Delta \phi_i w_z - R^m_{\phi_0} w_z + r_m w_z^{\frac{n + m + 2}{n + m - 2}} \right) = 0,
$$

we obtain

$$
\Pi \left( \frac{4(n + m - 1)}{n + m - 2} \Delta \phi_i u_i - R^m_{\phi_0} u_i + \frac{n + m + 2}{n + m - 2} r_m w_{\infty}^{\frac{n + m + 2}{n + m - 2}} u_i \right)
$$

$$
= \Pi \left( -w_i^{\frac{n + m + 2}{n + m - 2}} (R^m_{\phi_i} - r_m) - \frac{n + m + 2}{n + m - 2} r_m \left( \overline{w_z}^{\frac{4}{n + m - 2}} - w_{\infty}^{\frac{4}{n + m - 2}} \right) u_i \right)
$$

$$
+ r_m \left( w_i^{\frac{n + m + 2}{n + m - 2}} - \overline{w_z}^{\frac{4}{n + m - 2}} + \frac{n + m + 2}{n + m - 2} \overline{w_z}^{\frac{4}{n + m - 2}} u_i \right).
$$

It follows from Lemma 4.4(i) that

$$
\|u_i\|_{L^{\frac{n + m + 2}{n + m - 2}}(M)} \leq C \sup_{a \in A} \left| \int_M w_{\infty}^{\frac{4}{n + m - 2}} \psi_a u_i e^{-\phi_0} \, d\text{vol}_{g_0} \right|
$$

$$
+ C \left\| \Pi \left( \frac{4(n + m - 1)}{n + m - 2} \Delta \phi_i u_i - R^m_{\phi_0} u_i + \frac{n + m + 2}{n + m - 2} r_m w_{\infty}^{\frac{n + m + 2}{n + m - 2}} u_i \right) \right\|_{L^r(M)}.
$$

We conclude that

$$
\|u_i\|_{L^{\frac{n + m + 2}{n + m - 2}}(M)} \leq C \left\| w_i^{\frac{n + m + 2}{n + m - 2}} - \overline{w_z}^{\frac{4}{n + m - 2}} + \frac{n + m + 2}{n + m - 2} \overline{w_z}^{\frac{4}{n + m - 2}} u_i \right\|_{L^r(M)}
$$

$$
+ C \left\| w_i^{\frac{n + m + 2}{n + m - 2}} (R^m_{\phi_i} - r_m) \right\|_{L^r(M)} + C \left\| \left( \overline{w_z}^{\frac{4}{n + m - 2}} - w_{\infty}^{\frac{4}{n + m - 2}} \right) u_i \right\|_{L^r(M)}
$$

$$
+ C \sup_{a \in A} \left| \int_M w_{\infty}^{\frac{4}{n + m - 2}} \psi_a u_i e^{-\phi_0} \, d\text{vol}_{g_0} \right|.
$$

According to (4.11), without loss of generality, we can assume that

$$
(4.11) \quad w_i \to w_\infty \quad \text{and} \quad \overline{w_z} \to w_\infty \quad \text{a.e. in} \ M.
$$

Combining (4.11) with Lebesgue’s dominated convergence theorem yields

$$
\left\| \left( \overline{w_z}^{\frac{4}{n + m - 2}} - w_{\infty}^{\frac{4}{n + m - 2}} \right) u_i \right\|_{L^r(M)} = o(1).
$$

By (4.11), we have the pointwise estimate

$$
\left| w_i^{\frac{n + m + 2}{n + m - 2}} - \overline{w_z}^{\frac{4}{n + m - 2}} + \frac{n + m + 2}{n + m - 2} \overline{w_z}^{\frac{4}{n + m - 2}} u_i \right| \leq C \overline{w_z}^{\frac{n + m + 2}{n + m - 2}} |u_i|^2
$$

if $i$ is sufficiently large. Since $w_\infty$ is a positive smooth function in $M$, we obtain

$$
\left\| w_i^{\frac{n + m + 2}{n + m - 2}} - \overline{w_z}^{\frac{4}{n + m - 2}} + \frac{n + m + 2}{n + m - 2} \overline{w_z}^{\frac{4}{n + m - 2}} u_i \right\|_{L^r(M)} \leq C \|u_i\|^2_{L^2(M)}.
$$
When \( n \geq 3 \) and \( m > 0 \), \( 2s = \frac{2n(n+m+2)}{m(n+m-2) + 2(n+m+2)} < \frac{2(n+m)}{n+m-2} \). By Hölder’s inequality, we conclude that

\[
\left\| w_{z_i}^{n+m+2} - w_i^{n+m+2} + \frac{n + m + 2}{n + m - 2} w_i \right\|_{L^s(M)} = o(1).
\]

Moreover, since the set \( A \) is a finite, we have

\[
\sup_{a \in A} \left| \int_M \frac{4}{w^m_{\psi_0}} \psi_a u_i e^{-\psi_0} dV_{g_0} \right| \leq C \| u_i \|_{L^1(M)} = o(1).
\]

Putting these facts together, the assertion follows.

**Lemma 4.8.** The difference \( w_i - \bar{w}_{z_i} \) satisfies

\[
\| w_i - \bar{w}_{z_i} \|_{L^1(M)} \leq C \left\| w_i^{n+m+2} (R_{\phi_0}^m - r_{\psi_0}^m) \right\|_{L^{\frac{2(n+m)}{n+m-2}}(M)} + o(1)
\]

if \( i \) is sufficiently large.

**Proof.** The proof is almost the same as that of Lemma 4.7 except we use Lemma 4.4(ii) instead of Lemma 4.4(i). We omit the proof and leave it to the readers.

**Lemma 4.9.** There holds

\[
\sup_{a \in A} \left| \int_M \left( \frac{4(n+m-1)}{n + m - 2} \Delta_{\psi_0} \bar{w}_{z_i} - R_{\phi_0}^m \bar{w}_{z_i} + r_{\psi_0}^m \bar{w}_{z_i} \right) \psi_a e^{-\psi_0} dV_{g_0} \right| 
\]

\[
\leq C \left( \int_M \bar{w}_{(i)}^{2(n+m)} \int 2(n+m)(R_{\phi_0}^m - r_{\psi_0}^m) e^{-\psi_0} dV_{g_0} \right) + o(1)
\]

if \( i \) is sufficiently large.

**Proof.** Integration by parts yields

\[
\int_M \left( \frac{4(n+m-1)}{n + m - 2} \Delta_{\psi_0} \bar{w}_{z_i} - R_{\phi_0}^m \bar{w}_{z_i} + r_{\psi_0}^m \bar{w}_{z_i} \right) \psi_a e^{-\psi_0} dV_{g_0} 
\]

\[
= \int_M \left( \frac{4(n+m-1)}{n + m - 2} \Delta_{\phi_0} w_i - R_{\phi_0}^m w_i + r_{\psi_0}^m w_i \right) \psi_a e^{-\psi_0} dV_{g_0} 
\]

\[
+ \lambda_a \int_M \bar{w}_{(i)}^{\frac{4}{n+m-2}} (w_i - w_{z_i}) e^{-\psi_0} dV_{g_0} - r_{\psi_0}^m \int_M \bar{w}_{(i)}^{\frac{n+m+2}{n+m-2}} e^{-\psi_0} dV_{g_0},
\]

where we have used the fact that \( \frac{\partial \bar{w}_{z_i}}{\partial \nu_{g_0}} = \frac{\partial w_i}{\partial \nu_{g_0}} = \frac{\partial \psi_a}{\partial \nu_{g_0}} = 0 \) on \( \partial M \). Combining this with (4.10) yields

\[
\int_M \left( \frac{4(n+m-1)}{n + m - 2} \Delta_{\psi_0} \bar{w}_{z_i} - R_{\phi_0}^m \bar{w}_{z_i} + r_{\psi_0}^m \bar{w}_{z_i} \right) \psi_a e^{-\psi_0} dV_{g_0} 
\]

\[
= - \int_M \bar{w}_{(i)}^{\frac{n+m+2}{n+m-2}} (R_{\phi_0}^m - r_{\psi_0}^m) \psi_a e^{-\psi_0} dV_{g_0} 
\]

\[
+ \lambda_a \int_M \bar{w}_{(i)}^{\frac{4}{n+m-2}} (w_i - w_{z_i}) e^{-\psi_0} dV_{g_0} - r_{\psi_0}^m \int_M \bar{w}_{(i)}^{\frac{n+m+2}{n+m-2}} e^{-\psi_0} dV_{g_0}.
\]
Using the pointwise estimate
\[
\left| w_i \frac{n+m+2}{n+m} - w_z \frac{n+m+2}{n+m} \right| \leq C w_i^\frac{n+2}{n+m} |w_i - w_z| + C |w_i - w_z| w_z^\frac{n+m+2}{n+m},
\]
we can then deduce that
\[
\sup_{a \in A} \left| \int_M \left( \frac{4(n+m-1)}{n+m-2} \Delta \phi_0 w_z - R_{\phi_0}^m w_z + r_\infty^m w_z^\frac{n+m+2}{n+m} \right) \psi_a e^{-\phi_0} dV_g \right|
\]
\[
\leq C \left( \int_M w(t_i) \frac{2(n+m)}{n+m-2} |R_{\phi(t_i)}^m - r_\infty^m| w_z^\frac{n+m+2}{n+m} e^{-\phi_0} dV_g \right) w_z^\frac{n+m+2}{n+m} + C \|w_i - w_z\|_{L^2(M)} + C \|w_i - w_z\|_{L^\infty(M)}.
\]
Now the assertion follows from combining this with Lemmas 4.7 and 4.8.

Combining Lemma 4.9 and Lemma 4.10 we immediately have the following:

**Proposition 4.10.** We have the following estimate
\[
E(t_0) - E(w_\infty) \leq C \left( \int_M w(t_i) \frac{2(n+m)}{n+m-2} |R_{\phi(t_i)}^m - r_\infty^m| w_z^\frac{n+m+2}{n+m} e^{-\phi_0} dV_g \right) w_z^\frac{n+m+2}{n+m} + o(1)
\]
if \(i\) is sufficiently large.

We are now ready to prove Proposition 3.3.

**Proof of Proposition 3.3.** It follows from the definition of \(r_0^m\) and the assumption 2.7 that
\[
r_0^m - r_\infty^m = E(t_0) - E(w_\infty).
\]
Moreover, it follows from 4.9 that
\[
E(t_0) = E(w_\infty) + o(1).
\]
Now the assertion combining all these with Proposition 4.10.

5. **Nonpositive cases**

In this section, we deal with the remaining cases; i.e. \(Y_{n,m}[(g_0, \phi_0)] \leq 0\).

5.1. **Negative case.** As discussed in Lemma 2.7, we can choose an initial metric measure space \((M, g_0, e^{-\phi_0}dV_g, e^{-\phi_0}dA_{g_0}, m)\) such that
\[
R_{\phi_0}^m < 0 \quad \text{in } M \quad \text{and} \quad H_g = 0 \quad \text{on } \partial M.
\]
Let \(w(t)\) be the solution of (2.5) on a maximal time interval \([0, T^*)\). Applying the maximal principle to (2.8) derives
\[
\frac{d}{dt} w_{\min}^N(t) \geq n + m + 2 \left( \min |R_{\phi_0}^m| w_{\min}^N(t) + r_{\phi_0}^m w_{\min}^N(t) \right),
\]
where \( w_{\min}(t) = \min_M w(t) \) and \( N = \frac{n+m+2}{n+m-2} \). By (2.9), we have
\[
(5.3) \quad r_{\phi(t)}^m \geq Y_{n,m}([g_0, \phi_0]) .
\]
Note that \( Y_{n,m}([g_0, \phi_0]) \) is finite by Hölder’s inequality. Hence, integrating (4.2) yields
\[
(5.4) \quad w_{\min}^{N-1}(t) \geq C \cdot \min \left\{ \frac{w_{\min}^{N-1}(0)}{Y_{n,m}([g_0, \phi_0])}, \min |R_{\phi_0}^m| \right\}
\]
for some uniform constant \( C \). On the other hand, applying the maximal principle to (2.5) also gives
\[
(5.5) \quad \frac{d}{dt} w_{\max}^N(t) \leq \frac{n+m+2}{4} \left( - \left( \min_M R_{\phi_0}^m \right) w_{\max}(t) + r_{\phi}^m w_{\max}^N(t) \right),
\]
where \( w_{\max}(t) = \max_M w(t) \). According to (2.12), we obtain that
\[
(5.6) \quad w_{\max}^N(t) \leq \left( w_{\max}^N(0) + 1 \right) e^{c(\min M R_{\phi_0}^m + r_{\phi}^m(0))t},
\]
for some positive constant \( c \). It follows from (5.4) and (5.6) that \( w(t) \) will not blow up in finite time; i.e. \( T^* = \infty \).

We claim that \( r_{\phi(t)}^m \) will eventually become negative, even if \( r_{\phi(0)}^m \) may not be so. If \( r_{\phi(0)}^m \) is always nonnegative for \( t \geq 0 \), (4.2) would imply
\[
(5.7) \quad \frac{d}{dt} w_{\min}^{N-1}(t) \geq \frac{n+m+2}{4} \min_M |R_{\phi_0}^m| w_{\min}(t).
\]
Hence \( w_{\min}(t) \) approaches to infinity as \( t \to \infty \), which contradicts (2.8). Without loss of generality, we may assume \( r_{\phi(0)}^m < 0 \). By (5.3), we have
\[
(5.8) \quad w_{\max}^{N-1}(t) \leq C \cdot \max \left\{ w_{\max}^{N-1}(0), \max_M |R_{\phi_0}^m| \right\}
\]
for some uniform positive constant \( C \). This together with (5.4) implies that \( w(t) \) is uniformly bounded from above and away from zero.

Moreover, it follows from (2.11) that
\[
\frac{d}{dt} (R_{\phi_0}^m)_{\min} \geq (R_{\phi_0}^m)_{\min}((R_{\phi_0}^m)_{\min} - r_{\phi}^m) \geq r_{\phi}^m((R_{\phi_0}^m)_{\min} - r_{\phi}^m),
\]
where \( (R_{\phi_0}^m)_{\min} = \min_M R_{\phi_0}^m(t) \). Combining this with (5.3), we can obtain a uniform lower bound on \( R_{\phi_0}^m(t) \); i.e. for all \( t \geq 0 \)
\[
(5.8) \quad R_{\phi_0}^m(t) \geq r_{\phi}^m(t) - Ce^{r_{\phi(0)}^m t} \geq Y_{n,m}([g_0, \phi_0]) - C .
\]

Similar to Proposition 2.4, the maximum principle also implies
\[
\sup_M R_{\phi_0}^m(t) \leq \max \left\{ \sup_M R_{\phi_0}^m(0), 0 \right\}
\]
Therefore, we can generalize Lemma 2.6 and Proposition 3.9 to the negative case. In view of the argument at the end of Section 2, we can derive uniform estimates for all higher order derivatives of \( w(t), t \geq 0 \).
5.2. **Zero case.** Here, we treat the zero case. As discussed in Lemma 2.3, we can fix a background metric measure space \((M, g_0, e^{-\phi_0}dV_{g_0}, e^{-\phi_0}dA_{g_0}, m)\) such that \(R^m_{\phi_0} \equiv 0\). Note that by [7, Proposition 3.5], \(r^m_{\phi(t)}\) can never be negative. Since the function \(t \mapsto r^m_{\phi(t)}\) is nonincreasing, \(r^m_{\phi(0)} = 0\) implies \(r^m_{\phi(t)} \equiv 0\). Thus the solution of (1.10) is constant in time.

We next assume that \(r^m_{\phi(0)} > 0\). Observe that
\[
\frac{w^N_{\min}(t)}{w^N_{\min}(0)} \geq c \int_0^t r^m_{\phi(t)}dt \quad \text{and} \quad \frac{w^N_{\max}(t)}{w^N_{\max}(0)} \leq c \int_0^t r^m_{\phi(t)}dt
\]
for some positive constant \(c\), which are the consequences of (5.2) and (5.5). Hence we obtain the Harnack following inequality
\[
\frac{w^N_{\min}(t)}{w^N_{\min}(0)} \geq \frac{w^N_{\max}(t)}{w^N_{\max}(0)}.
\]
It follows that \(w(t)\) exists for all time. By the same argument as in Subsection 5.1, we can derive the smooth convergence.

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**References**

1. T. Aubin, Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire. *J. Math. Pures Appl.* (9) **55** (1976), 269-296.
2. S. Almaraz, Convergence of scalar-flat metrics on manifolds with boundary under a Yamabe-type flow. *J. Differential Equations* **259** (2015), 2626-2694.
3. S. Almaraz and L. Sun, Convergence of the Yamabe flow on manifolds with minimal boundary. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **20** (2020), no. 3, 1197-1272.
4. S. Brendle, A generalization of the Yamabe flow for manifolds with boundary. *Asian J. Math.* 6 (2002), 625-644.
5. S. Brendle, Convergence of the Yamabe flow for arbitrary initial energy. *J. Differential Geom.* **69** (2005), 217-278.
6. S. Brendle, Convergence of the Yamabe flow in dimension 6 and higher. *Invent. Math.* **170** (2007), 541-576.
7. J. S. Case, A Yamabe-type problem on smooth metric measure spaces. *J. Differential Geom.* **101** (2015), no. 3, 467-505.
8. J. S. Case, Conformal invariants measuring the best constants for Gagliardo-Nirenberg-Sobolev inequalities. *Calc. Var. Partial Differential Equations* **48** (2013), no. 3-4, 507–526.
9. J. S. Case, Sharp metric obstructions for quasi-Einstein metrics. *J. Geom. Phys.* **64** (2013), 12–30.
10. J. S. Case, The weighted \(\sigma_k\)-curvature of a smooth metric measure space. *Pacific J. Math.* **299** (2019), no. 2, 339–399.
11. X. Chen and P. T. Ho, Conformal curvature flows on a compact manifold of negative Yamabe constant. *Indiana Univ. Math. J.* **67** (2018), 537–581.
12. B. Chow, The Yamabe flow on locally conformally flat manifolds with positive Ricci curvature. *Comm. Pure Appl. Math.* **45** (1992), 1003-1014.
13. M. de Souza, On the existence of extremals for the weighted Yamabe problem on compact manifolds. *Differential Geom. Appl.* **68** (2020), 101585, 16 pp.
14. J. F. Escobar, Conformal deformation of a Riemannian metric to a scalar flat metric with constant mean curvature on the boundary. *Ann. of Math. (2)* **136** (1992), 1-50.
15. J. F. Escobar, The Yamabe problem on manifolds with boundary. *J. Differential Geom.* **35** (1992), 21-84.
16. L. C. Evans, Partial Differential Equations. Graduate Studies in Mathematics, vol. 19. American Mathematical Society, Providence, RI (2010).
17. D. Gilbarg and N. S. Trudinger, Elliptic partial differential equations of second order, reprint of the 1998 edition, Classics Math., Springer, Berlin, 2001.
18. R. S. Hamilton, The Ricci flow on surfaces. *Contemp. Math.*, **71**, Amer. Math. Soc., Providence, RI, (1988), 237-262.
19. Y. Han and H. Lin, Vanishing theorems for $f$-harmonic forms on smooth metric measure spaces. *Nonlinear Anal.* **162** (2017), 113–127.
20. P. T. Ho, J. Lee, and J. Shin, The second generalized Yamabe invariant and conformal mean curvature flow on manifolds with boundary. *J. Differential Equations* **274** (2021), 251-305.
21. P. T. Ho and J. Shin, Evolution of the Steklov eigenvalue along the conformal mean curvature flow. *J. Geom. Phys.* **173** (2022), Paper No. 104436, 15 pp.
22. J. M. Posso, A generalization of Aubin’s result for a Yamabe-type problem on smooth metric measure spaces. *Bull. Sci. Math.* **172** (2021), Paper No. 100352, 33 pp.
23. J. M. Posso, A generalization of Escobar-Riemann mapping type problem on smooth metric measure spaces. *Comm. Anal. Geom.* (2019), accepted. https://arxiv.org/abs/1805.03694
24. R. Schoen, Conformal deformation of a Riemannian metric to constant scalar curvature. *J. Differential Geom.* **20** (1984), 479-495.
25. L. Simon, Asymptotics for a class of nonlinear evolution equations, with applications to geometric problems. *Ann. of Math. (2)* **118** (1983), no. 3, 525-571.
26. N. S. Trudinger, Remarks concerning the conformal deformation of Riemannian structures on compact manifolds. *Ann. Scuola Norm. Sup. Pisa (3)* **22** (1968), 265-274.
27. H. Yamabe, On a deformation of Riemannian structures on compact manifolds. *Osaka Math. J.* **12** (1960), 21-37.
28. Z. Yan, Convergence of the weighted Yamabe flow. (2021), preprint. https://arxiv.org/abs/2111.05945
29. R. Ye, Global existence and convergence of Yamabe flow. *J. Differential Geom.* **39** (1994), 35-50.

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