ON MULTI-POLY-BERNOULLI-CARLITZ NUMBERS

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Abstract. We introduce multi-poly-Bernoulli-Carlitz numbers, function field analogues of multi-poly-Bernoulli numbers of Imatomi-Kaneko-Takeda. We explicitly describe multi-poly-Bernoulli Carlitz numbers in terms of the Carlitz factorial and the Stirling-Carlitz numbers of the second kind and also show their relationships with function field analogues of finite multiple zeta values.

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0. Introduction

In this paper, we introduce and study function field analogues of the Bernoulli numbers.

In 1997, M. Kaneko introduced and investigated generalizations of the Bernoulli numbers, poly-Bernoulli numbers in [16]. He obtained explicit formulae for poly-Bernoulli numbers which includes the second kind of Stirling numbers. Moreover, he and T. Arakawa found that they are also related to the Arakawa-Kaneko zeta functions at non-positive integers in [3]. From 2000, several multi-poly-Bernoulli numbers, generalizations of poly-Bernoulli numbers, were posted by Hamahata-Masubuchi [13], Imatomi-Kaneko-Takeda [14] and M.-S. Kim-T. Kim [18] in different ways each other. In [14], K. Imatomi, M. Kaneko, E. Takeda established relationships between multi-poly-Bernoulli numbers and finite multiple zeta values by obtaining some fundamental formulae.

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In 1935, L. Carlitz \cite{4} introduced and investigated function field analogues of the Bernoulli numbers, the Bernoulli-Carlitz numbers $BC_n$. By using them, he obtained an analogue of Euler’s famous formula $\zeta(m) = -\frac{(2\pi i)^m}{2(m)!} B_m$ (for even $m$) in \cite{4} and the von Staudt-Clausen theorem in \cite{5,6}. The latter result was revisited and put in a more conceptual setting by D. Goss in \cite{12}. E. Gekeler proved several identities for the Bernoulli-Carlitz numbers in \cite{11}. Furthermore, H. Kaneko and T. Komatsu obtained explicit formulae for them by using function field analogues of the Stirling numbers in \cite{15}. In this paper, we introduce in §2.2 multi-poly-Bernoulli-Carlitz numbers as function field analogues of multi-poly-Bernoulli numbers and also discuss generalizations of the vanishing condition $BC_n = 0 \ (q - 1 \nmid n)$ shown in \cite{5} and explicit formulae $BC_n = \sum_{j=0}^{\infty} (-1)^j D_j \binom{n}{q-1} C_j$ shown in \cite{15}. In §3.2 we show that multi-poly-Bernoulli-Carlitz numbers with special indices are expressed by Bernoulli-Carlitz numbers. We also show their connection to finite multiple zeta values in function field which were introduced by C.-Y. Chang and Y. Mishiba \cite{9} as finite variants of Thakur’s multiple zeta values in \cite{19}.

1. Notations and Definitions

1.1. Notations. We recall the following notation.

$q$ a power of a prime number $p$.
$\mathbb{F}_q$ a finite field with $q$ elements.
$\theta, t$ independent variables.
$A$ the polynomial ring $\mathbb{F}_q[\theta]$.
$A_+$ the set of monic polynomials in $A$.
$k$ the rational function field $\mathbb{F}_q(\theta)$.
$k_\infty$ $\mathbb{F}_q((1/\theta))$, the completion of $k$ at $\infty$.
$D_i$ $\prod_{j=1}^{i-1} (\theta^q - \theta^j) \in A_*$ where $D_0 := 1$.
$L_i$ $\prod_{j=1}^{i} (\theta - \theta^j) \in A_*$ where $L_0 := 1$.
$\Gamma_{n+1}$ the Carlitz gamma, $\prod D_i^{n_i} \ (n = \sum n_i q^{l_i} \in \mathbb{Z}_{\geq 0} \ (0 \leq n_i \leq q - 1))$.
$\Pi(n)$ the Carlitz factorial, $\Gamma_{n+1}$

1.2. Definition of finite multiple zeta values. In this subsection, we recall the definition of finite multiple zeta values and its function field analogues which were introduced in \cite{9}.

1.2.1. Characteristic $0$ case. We begin this subsection by recalling the finite multiple zeta values those were introduced by M. Kaneko and D. Zagier in \cite{17}.

Definition 1 \cite{17}. We set a $\mathbb{Q}$-algebra as follows:

$$\mathcal{A} := \prod_p \mathbb{Z}/p\mathbb{Z} \bigg/ \bigoplus_p \mathbb{Z}/p\mathbb{Z}$$

where $p$ runs over all prime numbers. For $\mathbf{s} := (s_1, \ldots, s_r) \in \mathbb{Z}^r$, the finite multiple zeta values are defined as follows:

$$\zeta_{\mathcal{A}}(\mathbf{s}) := (\zeta_{\mathcal{A}}(\mathbf{s})_{(p)}) \in \mathcal{A}$$

\footnote{An analogue of von Staudt-Clausen theorem stated in \cite{5,6,12} was corrected by L. Carlitz \cite{7} for $q = 2$.}
where
\[ \zeta_{sA}(s) := \frac{1}{p^{m_1 \cdots m_r}} \in \mathbb{Z}/p\mathbb{Z}. \]

### 1.2.2. Characteristic $p$ case

Next, let us turn into function field situation. In 1935, L. Carlitz [4] considered an analogue of the Riemann zeta values in function field which we call the Carlitz zeta values. For $s \in \mathbb{N}$, they are defined by
\[ \zeta_{A}(s) := \sum_{a \in A^+} a^s \in k_\infty. \]

D. S. Thakur [19] generalized this definition to that of multiple zeta values in $\mathbb{F}_q[t]$, which are defined for $s = (s_1, \ldots, s_r) \in \mathbb{N}^r$,
\[ \zeta_A(s) := \sum_{\deg a_1 \geq \deg a_r \geq 0} \frac{1}{a_1^{s_1} \cdots a_r^{s_r}} \in k_\infty. \]

Also, Chang-Mishiba and D. S. Thakur concerned $\nu$-adic variant ([10], [19]) and finite variant ([9], [20]). In this paper, we consider Chang and Mishiba’s finite variant ([9]).

**Definition 2** ([9], (2.1)). We set a $k$-algebra as follows:
\[ \mathcal{A}_k := \prod A/\mathfrak{p}A \big/ \sum A/\mathfrak{p}A \]
where $\mathfrak{p}$ runs over all monic irreducible polynomials in $A$. For $s = (s_1, \ldots, s_r) \in \mathbb{N}^r$ and a monic irreducible polynomial $\mathfrak{p} \in A$, finite multiple zeta values are defined as follows:
\[ \zeta_{sA}(s) := \frac{1}{\mathfrak{p}} \left( \zeta_{\mathcal{A}_k}(s) \right) \in \mathcal{A}_k \]
where
\[ \zeta_{\mathcal{A}_k}(s) := \sum_{\deg a_1 > \cdots > \deg a_r \geq 0} \frac{1}{a_1^{s_1} \cdots a_r^{s_r}} \in A/\mathfrak{p}A. \]

### 1.3. Definition of (finite Carlitz) multiple polylogarithms

In this subsection, we recall the definition of multiple polylogarithms in characteristic 0 and $p$.

#### 1.3.1. Characteristic 0 case

**Definition 3.** For $s = (s_1, \ldots, s_r) \in \mathbb{Z}^r$, the multiple polylogarithm series $\text{Li}_s(z_1, \ldots, z_r)$ are defined as the following series of $r$-variables $z_1, \ldots, z_r$:
\[ \text{Li}_s(z_1, \ldots, z_r) := \sum_{m_1 > \cdots > m_r \geq 0} \frac{z_1^{m_1} \cdots z_r^{m_r}}{m_1^{s_1} \cdots m_r^{s_r}} \in \mathbb{Q}[[z_1, \ldots, z_r]]. \]

#### 1.3.2. Characteristic $p$ case

In 2014, C.-Y. Chang [8] introduced the Carlitz multiple polylogarithms as function field analogues of the multiple polylogarithms.

**Definition 4** ([8], Definition 5.1.1). For $s = (s_1, \ldots, s_r) \in \mathbb{N}^r$, the Carlitz multiple polylogarithms are defined as the following series of $r$-variables $z_1, \ldots, z_r$:
\[ \text{Li}_s(z_1, \ldots, z_r) := \sum_{i_1 > \cdots > i_r \geq 0} \frac{z_1^{q_i_1} \cdots z_r^{q_i_r}}{L_1^{i_1} \cdots L_r^{i_r}} \in k[[z_1, \ldots, z_r]]. \]
Remark 5. We recover the Carlitz logarithms in the case of $r = 1$ and $s_1 = 1$

$$
\log_C(z) := \sum_{i \geq 0} \frac{z^i}{L_i} \in k[[z]].
$$

In [9], C.-Y. Chang and Y. Mishiba introduced finite Carlitz multiple polylogarithms, a finite variant of the Carlitz multiple polylogarithms.

**Definition 6** ([9], (3.1)). For $s = (s_1, \ldots, s_r) \in \mathbb{N}^r$ and $r$-tuple of variables $\zeta = (z_1, \ldots, z_r)$, finite Carlitz multiple polylogarithms are defined as follows:

$$
Li_{s\delta,s}(\zeta) := (Li_{s\delta,s}(z_1, \ldots, z_r)_\varphi) \in \mathcal{A}_{k,3}
$$

where

$$
Li_{s\delta,s}(z_1, \ldots, z_r)_\varphi := \sum_{\deg p > 1} \frac{z_1^{q^i_1} \cdots z_r^{q^i_r}}{L_i^{q^i_1} \cdots L_i^{q^i_r}} \mod p \in A[z_1, \ldots, z_r]/(\mathfrak{p}A).
$$

Here $\mathcal{A}_{k,3}$ is the following quotient ring

$$
\mathcal{A}_{k,3} := \prod_\varphi A[3]/(\mathfrak{p}A[3]) / \bigoplus_\varphi A[3]/(\mathfrak{p}A[3])
$$

(we put $A[3] := A[z_1, \ldots, z_r]$).

In [9], they established an explicit formula expressing $\zeta_{s\delta,s}(s)$ as a $k$-linear combination of $Li_{s\delta,s}(z_1, \ldots, z_r)_\varphi$ evaluated at some integral points. Before we recall it, let us prepare the Anderson-Thakur polynomial.

**Definition 7** ([1], (3.7.1)). Let $\theta, t, x$ be independent variables. For $n \in \mathbb{Z}_{\geq 0}$, Anderson-Thakur polynomial $H_n \in A[t]$ is defined by

$$
\left\{ 1 - \sum_{i=0}^{\infty} \frac{\prod_{i=1}^r (t^q - \theta^q)}{D_i^{\theta=t}} q^i \right\}^{-1} = \sum_{n=0}^{\infty} \frac{H_n}{k^n \cdot x^n}.
$$

**Remark 8.** We note that $H_n = 1$ for $0 \leq n \leq q - 1$.

G. W. Anderson and D. S. Thakur obtained the following series expansion for $H_{s_1,1}$.

**Proposition 9** ([1], (3.7.3), (3.7.4) and [2], 2.4.1). We consider $r$-tuple $s = (s_1, \ldots, s_r) \in \mathbb{N}^r$. For each $s_i \in \mathbb{N}$, the Anderson-Thakur polynomial $H_{s_1,1}$ is expanded as follows.

$$
H_{s_1,1} = \sum_{i=0}^{m_1} u_{ij} t^j
$$

where $u_{ij} \in A$ satisfying

$$
|u_{ij}|_\infty < q^{\sum_{i=0}^{m_1} t^j} \text{ and } u_{im_i} \neq 0.
$$

Here we note that $| \cdot |_\infty$ is the non-Archimedean absolute value on $k$ so that $|\theta|_\infty = q$.

**Notation 10.** We set following symbols which are introduced in [9]:

$$
J_s := \{0, 1, \ldots, m_1\} \times \cdots \times \{0, 1, \ldots, m_r\}.
$$

For each $j = (j_1, \ldots, j_r) \in J_s$, we set

$$
u_j := (u_{ij_1}, \ldots, u_{ij_r}) \in A^r,$$
and

\[ a_j := a_j(t) := t^{j_1 + \cdots + j_r}. \]

Here \( u_{ij} \) are the coefficients of (1).

**Examples 11.** We note that when \( s = (s_1, \ldots, s_r) = (1, \ldots, 1) \), by Remark 8 and Proposition 11 we have \( J_s = \{0\} \times \cdots \times \{0\} \) and \( u_j = (1, \ldots, 1) \) for \( j \in J_s \).

The following equation was obtained by C.-Y. Chang and Y. Mishiba in [9].

**Proposition 12 ([9], p.1056).** For \( s = (s_1, \ldots, s_r) \in \mathbb{N}^r \), let \( \mathfrak{p} \in A_\nu \) be a monic irreducible polynomial which satisfy \( \mathfrak{p} \mid \Gamma_{s_1} \cdots \Gamma_{s_r} \). Then we have

\[
\zeta_{\mathfrak{p}}(s) \mathfrak{p} = \frac{1}{\Gamma_{s_1} \cdots \Gamma_{s_r}} \sum_{j \in J_s} a_j(\theta) \text{Li}_{s_1, \cdots, s_r}(u_j) \mathfrak{p}.
\]

2. Multi-poly-Bernoulli(-Carlitz) Numbers

In this section, we define multi-poly-Bernoulli-Carlitz numbers which are function field analogues of multi-poly-Bernoulli numbers.

2.1. Characteristic 0 case. The Bernoulli numbers \( B_n \) \((n = 0, 1, \ldots)\) are rational numbers defined by the following generating function

\[
\sum_{n=0}^{\infty} B_n \frac{z^n}{n!} := \frac{e^z}{e^z - 1}.
\]

It is known that we have the following equation

\[ B_n = 0 \quad (\text{for } n \geq 3 \text{ so that } 2 \nmid n) \]

Moreover, we know that the Bernoulli numbers are expressed as follows:

\[
B_n = (-1)^n \sum_{m=1}^{n+1} \frac{(-1)^{m-1}(m-1)!}{m} \left\{ \binom{n}{m-1} \right\},
\]

where \( \{\binom{n}{m}\} \in \mathbb{Z} \) are the Stirling numbers of the second kind defined by

\[
\frac{(e^z - 1)^m}{m!} = \sum_{n=0}^{\infty} \frac{\binom{n}{m}}{n!} z^n.
\]

In 2014, K. Imatomi, M. Kaneko and E. Takeda [14] concerned two types of the multi-poly-Bernoulli numbers which generalize the Bernoulli numbers.

**Definition 13 ([14], (1) and [3], (8)).** For \( s = (s_1, \ldots, s_r) \in \mathbb{Z}^r \), the multi-poly-Bernoulli numbers (MPBNs for short) of B-type, C-type are the rational numbers which are defined by following generating functions respectively

\[
\sum_{n=0}^{\infty} B^B_n \frac{z^n}{n!} := \text{Li}_s(1 - e^{-z}, 1, \ldots, 1),
\]

\[
\sum_{n=0}^{\infty} C^C_n \frac{z^n}{n!} := \text{Li}_s(1 - e^{-z}, 1, \ldots, 1).
\]
Remark 14. When \( r = 1 \), \( B_n^r \) and \( C_n^r \) are the poly-Bernoulli numbers of B-type, C-type (cf. [3, 16]). When \( r = 1 \) and \( s_1 = 1 \), \( \text{Li}_n(z_1, \ldots, z_r) = -\log(1 - z) \) and then \( B_n^{(1)} \) agrees with (2) of the Bernoulli numbers. We note that \( B_1^{(1)} = 1/2 \) and \( C_1^{(1)} = -1/2 \) and \( B_n^{(1)} = C_n^{(1)} = B_n \) for \( n \neq 1 \).

2.2. Characteristic \( p \) case. In 1935, L. Carlitz [4] introduced the Bernoulli-Carlitz numbers, function field analogues of the Bernoulli numbers by using the Carlitz factorials \( \Pi(n) \) and the Carlitz exponentials

\[ e_C(z) := \sum_{i \geq 0} \frac{z^i}{D_i} \]

as follows.

Definition 15 ([4]). For \( n \in \mathbb{Z}_{\geq 0} \), the Bernoulli-Carlitz numbers \( BC_n \) are the elements of \( k \) defined by

\[ \sum_{n=0}^{\infty} BC_n \frac{z^n}{\Pi(n)} := \frac{z}{e_C(z)}. \]

In [5], L. Carlitz obtained the following:

\[ BC_n = 0 \quad \text{for} \quad (q-1) \nmid n. \]

In 2016, H. Kaneko and T. Komatsu [15] introduced the Stirling-Carlitz numbers of the first and second kind as an analogue of the Stirling numbers which were introduced in [4]. We recall below those of the second kind.

Definition 16 ([15], (15)). For \( m \in \mathbb{Z}_{\geq 0} \), the Stirling-Carlitz numbers of the second kind \( \{m\}_C \) are defined by

\[ \sum_{n=0}^{\infty} \left\{ \frac{n}{m} \right\}_C \frac{z^n}{\Pi(n)} := \frac{(e_C(z))^m}{\Pi(m)}. \]

In addition, they [15] showed that

\[ \left\{ \frac{n}{m} \right\}_C = 0 \quad (n \geq 1), \quad \left\{ \frac{n}{m} \right\}_C = 0 \quad (n < m), \quad \left\{ \frac{n}{m} \right\}_C = 1 \quad (n \geq 0) \]

and the following property.

Proposition 17 ([15], Proposition 8). For \( n, m \in \mathbb{Z}_{\geq 0} \) with \( \lambda(n) > \lambda(m) \),

\[ \left\{ \frac{n}{m} \right\}_C = 0 \]

here we noted \( \lambda(n) := \sum_i n_i \) where \( n_i \) are the digits of \( q \)-adic expansion \( n = \sum_i n_i q^i \).

By using the Stirling-Carlitz numbers of the second kind, they obtained the following proposition as a function field analogue of [4].

Proposition 18 ([15], Theorem 2). For \( n \in \mathbb{Z}_{\geq 0} \), we have

\[ BC_n = \sum_{j=0}^{\infty} \frac{(-1)^j D_j}{L_j^2} \left\{ \frac{n}{q^j - 1} \right\}_C. \]

Remark 19. In [15], they put \( L_j \) by \( \prod_{i=1}^{j} (q^i - \theta) \). But the above equation is same to their equation (cf. [15], (20)) due to the appearance of \( L_j^2 \).
Remark 20. By the definition of $D_m$, we have
\[
D_m^q = \prod_{i=0}^{m-1} (\theta^q - \theta^i) = \prod_{i=0}^{m-1} (\theta^q - \theta^{i+1}) = \prod_{i'=1}^{m} (\theta^q - \theta^{i'}) = -\frac{D_{m+1}}{(\theta - \theta^q)}.
\]
Thus we obtain
\[
D_m^{q-1} = -\frac{D_m}{(\theta - \theta^q)}.
\]
By the definition of Carlitz factorial, $L_j$ and the above equation, we have the following:
\[
\Pi(q^j - 1) = \prod_{m=0}^{j-1} D_m^{q_m} = \prod_{m=0}^{j-1} \frac{D_{m+1}}{D_m (\theta - \theta^{q_m})} = (-1)^j \frac{D_j}{L_j} \quad (j \in \mathbb{Z}_{\geq 0}).
\]
Thus we may write the formula in Theorem 18 as follows:
\[
BC_n = \sum_{j=0}^{\infty} \frac{\Pi(q^j - 1)}{L_j} \binom{n}{q^j - 1}_C.
\]
Next we introduce multi-poly-Bernoulli-Carlitz numbers (MPBCNs) as function field analogues of MPBMs (Definition 13). It is defined by the following generating function.

Definition 21. For $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$, $j = (j_1, \ldots, j_r) \in J_\mathfrak{s}$ (for $J_\mathfrak{s}$, see Notation 10), we define multi-poly-Bernoulli-Carlitz numbers (MPBCNs for short) $BC_n^{\mathfrak{s}, j}$ to be elements of $k$ as follows:
\[
(7) \quad \sum_{n \geq 0} BC_n^{\mathfrak{s}, j} \frac{z^n}{\prod(n)} \cdot \frac{Li_s(e^C(z)u_{1j_1}, u_{2j_2}, \ldots, u_{rj_r})}{e^C(z)}.
\]
Remark 22. In the case when $r = 1$ and $s_1 = 1$ in the above definition, we have $Li_s(z_1, \ldots, z_r) = \log^s(z)$ and $J_\mathfrak{s} = \{0\}$, $u_{1j_1} = u_{10} = 1$ since $H_{s_1-1} = H_0 = 1$. Hence we recover the Definition 13 by
\[
\sum_{n \geq 0} BC_n^{(1), (0)} \frac{z^n}{\prod(n)} = \frac{\log^s(e^C(z))}{e^C(z)} = \frac{z}{e^C(z)}.
\]
This is the one we have seen in Definition 13 so we have
\[
(8) \quad BC_n^{(1), (0)} = BC_n.
\]
Remark 23. Let $g$ be a generator of $\mathbb{F}_q^\times$ then we have
\[
(9) \quad g^n = 1 \iff (q - 1)|n.
\]
By the definition, it follows that $e^C(gz) = ge^C(z)$. Then by (7) and Definition 4 we have
\[
\sum_{n \geq 0} BC_n^{\mathfrak{s}, j} \frac{(gz)^n}{\prod(n)} = Li_s(e^C(gz)u_{1j_1}, u_{2j_2}, \ldots, u_{rj_r}) \frac{e^C(gz)}{e^C(gz)} = \sum_{i_1 > \cdots > i_r > 0} e^C((gz)^{q^{i_1}}u_{1j_1}^{q^{i_1}} \cdots u_{rj_r}^{q^{i_r}}) L_{i_1}^{s_1} \cdots L_{i_r}^{s_r}
\]
by using $e^C(gz) = ge^C(z)$ and (9),
\[
= \sum_{i_1 > \cdots > i_r > 0} e^C(z)^{q^{i_1}} \frac{u_{1j_1}^{q^{i_1}} \cdots u_{rj_r}^{q^{i_r}}}{L_{i_1}^{s_1} \cdots L_{i_r}^{s_r}} = Li_s(e^C(z)u_{1j_1}, u_{2j_2}, \ldots, u_{rj_r}) \frac{e^C(z)}{e^C(z)}
\]
\[
= \sum_{n \geq 0} BC_n^{\mathfrak{s}, j} \frac{z^n}{\prod(n)}.
\]
By comparing the coefficients of $z^n$, we have $g^n B_{n}^{x_j} = B_{n}^{x_j}$. Therefore we obtain the following by (9):

$$B_{n}^{x_j} = 0 \quad \text{for } (q - 1) \mid n.$$ 

The MPBNs are defined for $s_i \in \mathbb{Z}$, on the other hand our MPBCNs are defined for $s_i \in \mathbb{N}$. It is because in Definition 21, we use $u_{ij}$, the coefficients of $H_{s_i-1}$ which are defined for $s_i \in \mathbb{N}$. We remark that we do not have two kinds of MPBCNs as we do in Definition 13.

3. SEVERAL PROPERTIES OF MULTI-POLY-BERNOULLI(-CARLITZ) NUMBERS

In this section, we obtain function field analogues of some results in [14]. In subsection 3.1, we recall their results in characteristic 0 case. In subsection 3.2, we prove their analogue in characteristic $p$ case.

3.1. Characteristic 0 case. In [14], K. Imatomi, M. Kaneko, and E. Takeda obtained explicit formulae for MPBNs. They are the following finite sums involving the Stirling numbers of the second kind.

**Proposition 24** ([14], Theorem 3). For $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{Z}^r$ and $n \geq 0$, we have

$$B_{n}^{(\mathbf{s})} = (-1)^n \sum_{n+1 \geq m_1 > m_2 > \cdots > m_r > 0} (-1)^{m_1-1} (m_1 - 1)! \frac{n}{m_1} \frac{1}{m_1^{s_1} \cdots m_r^{s_r}}$$

and

$$C_{n}^{(\mathbf{s})} = (-1)^n \sum_{n+1 \geq m_1 > m_2 > \cdots > m_r > 0} (-1)^{m_1-1} (m_1 - 1)! \frac{n+1}{m_1} \frac{1}{m_1^{s_1} \cdots m_r^{s_r}}.$$ 

In [14], they derived the following relations between the MPBNs and the Bernoulli numbers for the special case $(s_1, \ldots, s_r) = (1, \ldots, 1)$.

**Proposition 25** ([14], Proposition 4). For $r \geq 1$ and $n \geq r - 1$, we have

$$B_{n}^{(1, \ldots, 1)} = \frac{1}{n+1} \binom{n+1}{r} B_{n-r+1}^{(1)},$$

$$C_{n}^{(1, \ldots, 1)} = \frac{1}{n+1} \binom{n+1}{r} C_{n-r+1}^{(1)}.$$ 

In [14], they obtained the following relations which connect the MPBNs and finite multiple zeta values.

**Proposition 26** ([14], Theorem 8). For $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{Z}^r$, we have

$$\zeta_{\mathbf{e}^{\mathbf{s}}} (\mathbf{s})_{(p)} = -C_{p-2}^{(s_1-1, s_2, \ldots, s_r)} \pmod{p}$$

and for $d \geq 0$,

$$\zeta_{\mathbf{e}^{\mathbf{d}}} (\mathbf{s}_1, \ldots, 1, s_1, \ldots, s_r)_{(p)} = -C_{p-d-2}^{(s_1-1, s_2, \ldots, s_r)} \pmod{p}.$$ 

Here we note that the second relation generalizes the first relation.
3.2. Characteristic $p$ case. We prove function field analogues of Proposition 24. The following theorem is a function field analogue of Proposition 24.

**Theorem 27.** For $r \in \mathbb{N}$, $g = (s_1, \ldots, s_r) \in \mathbb{N}^r$, $j = (j_1, \ldots, j_r) \in J_g$ and $n \in \mathbb{Z}_{\geq 0}$, we have

$$BC_{n}^{s,j} = \sum_{\log_q (n+1) \geq i_1 \geq \cdots \geq i_r} \Pi(q^{i_1} - 1) \left\{ \frac{n}{q^{i_1} - 1} \right\}_C \frac{u_{1j_1}^{q^{i_1}} \cdots u_{rj_r}^{q^{i_r}}}{L_{s_1}^{i_1} \cdots L_{s_r}^{i_r}}.$$

**Proof.** By Definition 16 for $m = q^{i_1} - 1$,

$$\sum_{n \geq 0} BC_{n}^{s,j} \frac{z^n}{\Pi(n)} = \sum_{n \geq 0} \sum_{\log_q (n+1) \geq i_1 \geq \cdots \geq i_r} \Pi(q^{i_1} - 1) \left\{ \frac{n}{q^{i_1} - 1} \right\}_C \frac{u_{1j_1}^{q^{i_1}} \cdots u_{rj_r}^{q^{i_r}}}{L_{s_1}^{i_1} \cdots L_{s_r}^{i_r}} \frac{z^n}{\Pi(n)}.$$

Then by Definition 21, we have

$$\sum_{n \geq 0} BC_{n}^{s,j} \frac{z^n}{\Pi(n)} = \sum_{n \geq 0} \sum_{\log_q (n+1) \geq i_1 \geq \cdots \geq i_r} \Pi(q^{i_1} - 1) \left\{ \frac{n}{q^{i_1} - 1} \right\}_C \frac{u_{1j_1}^{q^{i_1}} \cdots u_{rj_r}^{q^{i_r}}}{L_{s_1}^{i_1} \cdots L_{s_r}^{i_r}} \frac{z^n}{\Pi(n)}.$$

By comparing the coefficients of $z^n$, we obtain the formula (10).

**Remark 28.** When $r = 1$ and $s_1 = 1$, we have $H_{s_1-1} = H_0 = 1$. Then $J_g = \{0\}$, $u_{1j_1} = u_{10} = 1$ hence we have

$$BC_{n}^{(1), (0)} = \sum_{\log_q (n+1) \geq i_1} \Pi(q^{i_1} - 1) \left\{ \frac{n}{q^{i_1} - 1} \right\}_C \frac{1}{L_{i_1}}$$

by using (10),

$$= \sum_{\log_q (n+1) \geq i_1} (-1)^{i_1} \frac{D_{i_1}}{L_{i_1}} \left\{ \frac{n}{q^{i_1} - 1} \right\}_C.$$

Therefore by Remark 22, our Theorem 27 includes H. Kaneko and T. Komatsu’s result (Proposition 23) in the case of $r = 1$ and $s_1 = 1$.

We obtain the following relation between the MPBCNs and the Bernoulli-Carlitz numbers for the tuple $(1, \ldots, 1)$ as a function field analogue of Proposition 26.

**Theorem 29.** For $r \geq 1$ and $n \geq q^{r-1} - 1$, we have

$$BC_{n}^{(1, \ldots, 1), (0, \ldots, 0)} = \sum_{\log_q (n+1) \geq i_1 \geq \cdots \geq i_r} \left\{ \frac{n}{q^{i_1} - 1} \right\}_C \frac{BC_{q^{i_1} - 1}}{\Pi(q^{i_1} - 1)} \frac{BC_{q^{i_2} - 1}}{\Pi(q^{i_2} - 1)} \cdots \frac{BC_{q^{i_r} - 1}}{\Pi(q^{i_r} - 1)}.$$
Proof. Let us first prove an equation
\[
BC_{q^r-1} = \frac{1}{L_i} = \prod(q^i-1).
\]
It follows from Proposition 18 that we have
\[
BC_{q^r-1} = \sum_{j=0}^\infty (-1)^j D_j \frac{q^i-1}{L_j^2} C.
\]
The right hand side is translated as follows:
\[
\sum_{j=0}^\infty (-1)^j D_j \frac{q^i-1}{L_j^2} C = \frac{(-1)^j D_i}{L_i^2} = \prod(q^i-1).
\]
The first equality follows from Proposition 17, the second one follows from 18. Then we have the equation (12).

It follows from Theorem 27 that we have
\[
BC_{q^r-1} = \sum_{s, r} \sum_{q^i-1} \prod(q^i-1) = \prod(q^i-1).
\]
By using the equation (12) to the right hand side,
\[
BC_{q^r-1} = \sum_{s, r} \prod(q^i-1) = \prod(q^i-1).
\]
Therefore we obtain the desired equation (11). □

Next, before we see a function field analogue of Proposition 26, we prepare the following lemma.

Lemma 30. When \( r \geq 2 \), we have the following equation for \( s = (s_1, \ldots, s_r) \in \mathbb{N}^r \), \( j \in J_s \) and \( m \geq r - 1 \).
\[
BC_{q^m-1} = BC_{q^{m-1}} \sum_{n=1}^{m-\alpha-2} \prod(q^i-1) \prod(q^j-1) = BC_{q^{m-1-a}} \prod(q^i-1) \prod(q^j-1).
\]

Proof. By using Theorem 27, we have
\[
BC_{q^m-1} = \sum_{m \geq i_1 \cdots \geq i_r, \geq 0} \prod(q^i-1) \prod(q^j-1) = \prod(q^i-1) \prod(q^j-1).
\]
All digits of the \( q \)-adic expansion of \( q^i-1 \) and \( q^m-1 \) are \( q-1 \). Therefore we have
\[
\begin{cases} q^m-1 \prod(q^i-1) = 0 & \text{if } m > i_1, \\ q^m-1 \prod(q^i-1) = 1 & \text{if } m = i_1, \\ q^m-1 \prod(q^i-1) = 0 & \text{if } m > i_1, \\ q^m-1 \prod(q^i-1) = 1 & \text{if } m = i_1, \\ \end{cases}
\]
by Proposition 17 and 18. Then by using (13), we have
\[
BC_{q^m-1} = \sum_{m \geq i_2 \cdots \geq i_r, \geq 0} \prod(q^i-1) \prod(q^j-1) = \prod(q^m-1) \prod(q^i-1) \prod(q^j-1).
\]
By using Theorem 27, we have

\begin{equation}
BC_{q^{m-1}} = BC_{q^{m-1}}(s_1, (j_1)) \sum_{m > i_2 > \cdots > i_r > 0} \prod_{\alpha > \beta \geq 0} \frac{1}{\Pi(q^{m-\alpha} - 1)} \prod_{j_1 < j_2 < \cdots < j_r} \frac{1}{L_{i_1}^{j_1} \cdots L_{i_r}^{j_r}}
\end{equation}

Again by using Theorem 27,

\begin{equation}
BC_{q^{m-1}} = BC_{q^{m-1}}(s_1, (j_1)) \sum_{m > i_2 > \cdots > i_r > 0} \prod_{\alpha > \beta \geq 0} \frac{1}{\Pi(q^{m-\alpha} - 1)} \prod_{j_1 < j_2 < \cdots < j_r} \frac{1}{L_{i_1}^{j_1} \cdots L_{i_r}^{j_r}}
\end{equation}

Then we obtain the desired equation (13). \hfill \Box

The following result is an analogue of Proposition 26 which provides the connection between MPBCNs and finite multiple zeta values in the function field case.

**Theorem 31.** For \( s = (s_1, \ldots, s_r) \in \mathbb{N}^r \) and a monic irreducible polynomial \( \wp \in A \) so that \( \wp \mid \Gamma_{s_1} \cdots \Gamma_{s_r} \), we have the following:

\begin{equation}
\z_{a \wp} (s) \wp = \frac{1}{\Gamma_{s_1} \cdots \Gamma_{s_r}} \sum_{\mathfrak{d} \in J_{\wp}} a_{\mathfrak{d}}(\theta) \sum_{i \in \mathbb{N}} \frac{1}{\beta_{q^{i-1}}} \mod \wp.
\end{equation}

For \( s = (1, \ldots, 1, s_1, \ldots, s_r) \in \mathbb{N}^{r+d} \) and a monic irreducible polynomial \( \wp \in A \) so that \( \wp \mid \Gamma_s \cdots \Gamma_{s_r} \), we have the following:

\begin{equation}
\z_{a \wp} (s) \wp = \frac{1}{\Gamma_{s_1} \cdots \Gamma_{s_r}} \sum_{\mathfrak{d} \in J_{\wp}} a_{\mathfrak{d}}(\theta) \sum_{i \in \mathbb{N}} \frac{1}{\beta_{q^{i-1}}} \mod \wp.
\end{equation}

Here we put \( s' = (s_1, \ldots, s_r) \).
By using equation (13) in Lemma 30 for \( s = (1, s_1, \ldots, s_r) \) and \( m = \deg \varphi \), we have

\[
BC_{q^{\deg \varphi - 1}}^{(1,s_1,\ldots,s_r),(0,j_1,\ldots,j_r)} = BC_{q^{\deg \varphi - 1}}^{(1),(0)} \prod_{\alpha=1}^{\deg \varphi - (r-1)} \frac{1}{\Pi(q^{\deg \varphi - \alpha - 1})} BC_{q^{\deg \varphi - \alpha - 1}}^{(s_1,\ldots,s_r),(j_1,\ldots,j_r)}.
\]

By equation (8), we have

\[
\frac{BC_{q^{\deg \varphi - 1}}^{(1,s_1,\ldots,s_r),(0,j_1,\ldots,j_r)}}{BC_{q^{\deg \varphi - 1}}^{(1),(0)}} = \prod_{\alpha=1}^{\deg \varphi - (r-1)} \frac{1}{\Pi(q^{\deg \varphi - \alpha - 1})} \frac{BC_{q^{\deg \varphi - \alpha - 1}}^{(s_1,\ldots,s_r),(j_1,\ldots,j_r)}}{BC_{q^{\deg \varphi - \alpha - 1}}^{(1),(0)}}.
\]

by putting \( i = \deg \varphi - \alpha \) and using the equation (12),

\[
\sum_{i=1}^{\deg \varphi - 1} \frac{1}{\Pi(q^{\deg \varphi - \alpha - 1})} BC_{q^{\deg \varphi - \alpha - 1}}^{(s_1,\ldots,s_r),(j_1,\ldots,j_r)} = \frac{1}{\Pi(q^{\deg \varphi - \alpha - 1})} L_i BC_{q^{\deg \varphi - \alpha - 1}}^{(s_1,\ldots,s_r),(j_1,\ldots,j_r)} = L_i s_i, i \pmod{\varphi} \text{ mod } \varphi.
\]

Therefore by the equations (18), (19) and Definition 8 we obtain

\[
\sum_{i=1}^{\deg \varphi - 1} \frac{1}{\Pi(q^{\deg \varphi - \alpha - 1})} BC_{q^{\deg \varphi - \alpha - 1}}^{(s_1,\ldots,s_r),(j_1,\ldots,j_r)} = \zeta_{s_i}(\varphi) \text{ mod } \varphi.
\]

By our assumption \( \varphi \not\mid \Gamma_{s_1} \ldots \Gamma_{s_r} \), we may apply Proposition 12 and obtain the desired formula (16).

Next we prove the equation (17). By using (16) for \( s = (1, \ldots, 1, s_1, \ldots, s_r) \), we have

\[
\frac{1}{\Gamma_{(1,1,\ldots,1,s_1,\ldots,s_r)}} \sum_{j \in \mathbb{J}_s} o_j(\theta) \sum_{\alpha=1}^{\deg \varphi - (d+r-1)} \frac{BC_{q^{\deg \varphi - \alpha - 1}}^{(1,s_1,\ldots,s_r),(0,j_1,\ldots,j_r)}}{\Pi(q^{\deg \varphi - \alpha - 1})} = \zeta_{s_1}(\varphi) \text{ mod } \varphi.
\]

We may rewrite \( BC_{q^{\deg \varphi - \alpha - 1}}^{(s_1,\ldots,s_r),(0,j_1,\ldots,j_r)} / \Pi(q^{\deg \varphi - \alpha - 1}) \) by using MPBCNs for \( (s_1, \ldots, s_r) \). By (13) in Lemma 30

\[
\frac{BC_{q^{\deg \varphi - \alpha - 1}}^{(s_1,\ldots,s_r),(0,j_1,\ldots,j_r)}}{\Pi(q^{\deg \varphi - \alpha - 1})} = \frac{BC_{q^{\deg \varphi - (d+r-2)}}^{(1,1,\ldots,1,s_1,\ldots,s_r),(0,0,j_1,\ldots,j_r)}}{\Pi(q^{\deg \varphi - (d+r-2)})} \frac{BC_{q^{\deg \varphi - \alpha - 1}}^{(1,1,\ldots,1,s_1,\ldots,s_r),(0,0,j_1,\ldots,j_r)}}{\Pi(q^{\deg \varphi - \alpha - 1})}.
\]

by using (13) repeatedly,

\[
\frac{BC_{q^{\deg \varphi - \alpha - 1}}^{(1),(0)}}{\Pi(q^{\deg \varphi - \alpha - 1})} \sum_{\alpha_1=1}^{\deg \varphi - \alpha - (d+r-2)} \frac{BC_{q^{\deg \varphi - \alpha - 1}}^{(1),(0)}}{\Pi(q^{\deg \varphi - \alpha - 1})} \sum_{\alpha_2=1}^{\deg \varphi - \alpha - (d+r-3)} \frac{BC_{q^{\deg \varphi - \alpha - 1}}^{(1),(0)}}{\Pi(q^{\deg \varphi - \alpha - 1})} \cdots \frac{BC_{q^{\deg \varphi - \alpha - 1}}^{(s_1,\ldots,s_r),(j_1,\ldots,j_r)}}{\Pi(q^{\deg \varphi - \alpha - 1})}.
\]
by putting $\beta_i = \alpha + \alpha_1 + \cdots + \alpha_i$ ($i \geq 1$) and $\beta_0 = \alpha$,
\[
\frac{BC^{(1)}_{q^{\deg p - \beta_0 - (d+r-2)}}(\beta_1, \ldots, \beta_s)}{\prod_p (q^{\deg p - \beta_0 - 1})} \cdot \frac{BC^{(1)}_{q^{\deg p - \beta_1 - 1}}(\beta_2, \ldots, \beta_s)}{\prod_p (q^{\deg p - \beta_1 - 1})} \cdot \frac{BC^{(1)}_{q^{\deg p - \beta_2 - 1}}(\beta_3, \ldots, \beta_s)}{\prod_p (q^{\deg p - \beta_2 - 1})}
\]
\[
\quad \cdots \sum_{\beta_d = \beta_{d-1} + 1}^{\beta_d} \frac{BC^{(1)}_{q^{\deg p - \beta_d - 1}}(\beta_1, \ldots, \beta_s)}{\prod_p (q^{\deg p - \beta_d - 1})}
\]
\[
= \frac{BC^{(1)}_{q^{\deg p - \beta_0 - (d+r-2)}}(\beta_1, \ldots, \beta_s)}{\prod_p (q^{\deg p - \beta_0 - 1})} \cdot \frac{BC^{(1)}_{q^{\deg p - \beta_1 - 1}}(\beta_2, \ldots, \beta_s)}{\prod_p (q^{\deg p - \beta_1 - 1})} \cdot \frac{BC^{(1)}_{q^{\deg p - \beta_2 - 1}}(\beta_3, \ldots, \beta_s)}{\prod_p (q^{\deg p - \beta_2 - 1})}
\]
\[
\quad \cdots \sum_{\beta_d = \beta_{d-1} + 1}^{\beta_d} \frac{BC^{(1)}_{q^{\deg p - \beta_d - 1}}(\beta_1, \ldots, \beta_s)}{\prod_p (q^{\deg p - \beta_d - 1})}
\]
by using (8) and (12),

Then we have
\[
\sum_{\beta_0 = 1}^{\beta_d} \frac{BC^{(1)}_{q^{\deg p - \beta_0 - 1}}(\beta_1, \ldots, \beta_s)}{\prod_p (q^{\deg p - \beta_0 - 1})} = \sum_{\beta_0 = 1}^{\beta_d} \sum_{\beta_1 = \beta_0}^{\beta_1} \cdots \sum_{\beta_d = \beta_{d-1} + 1}^{\beta_d} \frac{BC^{(1)}_{q^{\deg p - \beta_d - 1}}(\beta_1, \ldots, \beta_s)}{\prod_p (q^{\deg p - \beta_d - 1})}
\]
\[
= \sum_{\deg p \beta_i = 0} \frac{1}{L_{\deg p - \beta_i}} \sum_{\deg p \beta_j = 0} \frac{1}{L_{\deg p - \beta_j}} \frac{BC^{(1)}_{q^{\deg p - \beta_i - 1}}(\beta_1, \ldots, \beta_s)}{\prod_p (q^{\deg p - \beta_i - 1})}
\]
by putting $i_j = \deg p - \beta_i$ ($d \geq l \geq 0$),
\[
= \sum_{\deg p \beta_i = 0} \frac{1}{L_{\deg p - \beta_i}} \frac{BC^{(1)}_{q^{\deg p - \beta_i - 1}}(\beta_1, \ldots, \beta_s)}{\prod_p (q^{\deg p - \beta_i - 1})}
\]
by using the equation (12),
\[
= \sum_{\deg p \beta_i = 0} \frac{1}{L_{\deg p - \beta_i}} \frac{BC^{(1)}_{q^{\deg p - \beta_i - 1}}(\beta_1, \ldots, \beta_s)}{\prod_p (q^{\deg p - \beta_i - 1})}
\]
Substituting this into the equation (20) and by $\Gamma_1 = 1$, we have
\[
\frac{1}{\Gamma_{s_1} \cdots \Gamma_{s_r}} \sum_{j \in J_s} a_j(\theta) \sum_{\deg p \beta_i = 0} \frac{1}{L_{\deg p - \beta_i}} \frac{BC^{(1)}_{q^{\deg p - \beta_i - 1}}(\beta_1, \ldots, \beta_s)}{\prod_p (q^{\deg p - \beta_i - 1})} = \zeta_{\mathfrak{a}_\theta}(\theta) \pmod{\mathfrak{p}}.
\]
For $s = (1, \ldots, 1, s_1, \ldots, s_r)$, we have $J_s = \{0\} \times \cdots \times \{0\} \times \{0, 1, \ldots, m_1\} \times \cdots \times \{0, 1, \ldots, m_r\}$ so $a_j(\theta) = q^{j_{s_1} + \cdots + j_r}$ for $j = (0, \ldots, 0, j_1, \ldots, j_r) \in J_s$ and thus $a_j(\theta)$
depends only on $j' = (j_1, \ldots, j_r) \in J_{\nu}$. Therefore the above equation is rewritten as follows:

$$\frac{1}{\Gamma_{s_1} \cdots \Gamma_{s_r}} \sum_{j' \in J_{\nu}} a_{j'}(\theta) \sum_{\deg p > i_0} \cdots \sum_{\deg q > i_d} \frac{BC_{q^d} j'}{L_{i_0} \cdots L_{i_d}} \frac{BC_{q^d - 1} j'}{BC_{q^d - 1}} = \zeta_{s'}(\theta)_{\nu} \mod \nu.$$ 

Thus we obtain the equation \(17\).

We remark that the relation \(17\) is a generalization of \(16\).

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