Restriction of stable bundles on a jacobian of genus 2 to an embedded curve

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This paper is dedicated to the memory of Professor Gheorghe Galbura.

The aim of this note is to describe the restriction map from the moduli space of stable rank 2 bundle with small $c_2$ on a jacobian $X$ of dimension 2, to the moduli space of stable rank 2 bundles on the corresponding genus 2 curve $C$ embedded in $X$.

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1 Introduction

Let $C$ a smooth curve of genus 2 and $X$ his jacobian which is a smooth projective algebraic surface. We denote by $M_{(2, c, i)}$ for $i = 1$ or 2 the moduli space of rank 2 bundle on $X$ with $c_1 = C$ and $c_2 = i$. Also we denote by $M_{(2, K)}$ the moduli space of rank 2 bundle on $C$ with determinant $K$ i.e. the canonical class of $C$. Obviously, for any $E \in M_{(2, c, i)}$ the restriction $E|_C$ is a rank 2 bundle on $C$ with determinant $K$. 
The natural questions which appear are the following: is $E|_C$ a stable (or at least semi-stable) bundle on $C$ and if yes, what is the induced map $M_{(2, C, i)} \to M_{(2, K)}$? As we shall see, the answer depend on $i$: for $i = 1$, the restriction is semi-stable, but for $i = 2$ and $E$ generic in $M_{(2, C, 2)}$ the restriction is stable. Also, in the second case we can describe for some non-generic bundles $E$ what is the restriction $E|_C$.

2 Previously known results

For $X$ the jacobian of a genus 2 curve $C$, we denote by $F_0 = \mathcal{O}(C) \otimes J_0$, where $J_0$ is the sheaf of ideals of the origin of $X$. Also, using $F_0$ we can construct a unique extension $0 \to \mathcal{O}_X \to F_1 \to F_0 \to 0$ which has $c_1 = \mathcal{O}_X(C)$ and $c_2 = 1$. The first result we need is the following, proved in [2]:

**Theorem 2.1** For any rank 2 bundle $E$ on $X$ with $c_1 = \mathcal{O}_X(C)$ and $c_2 = 1$ there are uniques $x, y \in X$ such that $E \simeq T_x^* F_1 \otimes P_y$, where $T_x^*$ is the pull-back by the $x$-translation and $P_y$ is the line bundle on $X$ which correspond to $y$ by the canonical isomorphism $X \to \hat{X}$ defined by the principal polarisation $C$. As consequence the moduli space is isomorphic with $X \times X$.

It is very easy to verify that the condition for $E$ the have $\det(E) = \mathcal{O}_X(C)$ is that $x = -2y$; so we have the following:

**Remark 2.2** The moduli space of rank 2 bundles on $X$ with $c_1 = \mathcal{O}_X(C)$ and $c_2 = 1$ is isomorphic with $X$.

For the moduli space on $C$ we need the following theorem proved in [3]:

**Theorem 2.3** Let $F$ a semi-stable rank 2 bundle on $C$ with determinant equal with the canonical class of $C$, and $x_0$ a Weierstrass point of $C$. Let $D_F = \{ \xi \in \text{Pic}^1(C) \mid H^0(\xi \otimes F \otimes \mathcal{O}(-x_0)) \neq 0 \}$. With these notations, $D_F$ is a divisor of the linear system $|2C|$ on $\text{Pic}^1(C)$ and the map $F \to D_F$ is an isomorphism between the moduli space of rank two bundles with canonical determinant and $\mathbb{P}^3$.

For the case $c_2 = 2$ we need the following result proved in [1] and [4]:

**Theorem 2.4** $M_{(2, C, 2)}$ is isomorphic with $X \times \text{Hilb}^3(X)$, and for any $E \in M_{(2, C, 2)}$ there exist an unique exact sequence of the form:

$$0 \to T_x^* \mathcal{O}_X(-C) \to H \to E \to 0$$

where $H$ is an homogenous rank 3 bundle on $X$.  

2
By [2] a generic homogenous rank 3 bundle has the form $P_a \oplus P_b \oplus P_c$ with $a \neq b \neq c$ and it is clear that the condition for $E$ the have $\det(E) = \mathcal{O}_X(C)$ is that $x = -a - b - c$; so we have the following:

**Remark 2.5** The moduli space of rank 2 bundles on $X$ with $c_1 = \mathcal{O}_X(C)$ and $c_2 = 2$ is birational with $\text{Sym}^3(X)$.

### 3 The restriction theorems

Using the previous notations we have the followings:

**Theorem 3.1** For generic $y \in X$ the restriction $E_{|C}$ of $E \cong T_{-2y}^*F_{-1} \otimes P_y$ is semi-stable but not stable. The rational restriction map $X \to \mathbb{P}^3$ is the quotient by the natural involution of $X$ and the image is the Kummer surface.

**Theorem 3.2** For generic $E \in \text{Hilb}^3(X)$ the restriction $E_{|C}$ is stable. The restriction $E_{|C}$, viewed in $\mathbb{P}^3 = |2C|$ is the unique divisor of $|2C|$ which contains the 3 points $a, b, c$ of the corresponding $H$. Also, the fiber over a point $C' \in |2C|$ is birational with $\text{Hilb}^3(C')$.

The main idea in the proof of the previous theorems is to obtain an explicit description of $D_{E_{|C}}$ for generic $E$ in the corresponding moduli space. In the first case for generic $y \in X$ and $E \cong T_{-2y}^*F_{-1} \otimes P_y$ we obtain that $D_{E_{|C}}$ is the union of the two translate of $C$ by $y$ and $-y$. For $c_2 = 2$ and generic $E$, $D_{E_{|C}}$ is the hyperplane which pass by the 3 points which determine the homogenous bundle $H$ associated with $E$ by [2] above. The full details will appear elsewhere.

### References

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3
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