Superfluidity and Superconductivity in Neutron Stars

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Abstract

This review focuses on applications of the ideas of superfluidity and superconductivity in neutron stars in a broader context, ranging from the microphysics of pairing in nucleonic superfluids to macroscopic manifestations of superfluidity in pulsars. The exposition of the basics of pairing, vorticity and mutual friction can serve as an introduction to the subject. We also review some topics of recent interest, including the various types of pinning of vortices, glitches, and oscillations in neutron stars containing superfluid phases of baryonic matter.

1 Introduction

Neutron stars are one of the most extreme astrophysical laboratories in the universe. They allow us to probe physics in strong gravitational fields in the regime where general-relativistic corrections can be as large as 20%, the magnetic fields deduced at their surfaces $B \leq 10^{15}$ G are the largest measured in Nature, and their interiors are expected to contain the densest forms of matter. For typical neutron star masses $\sim 1.4 - 2.0 M_\odot$ ($M_\odot$ being the solar mass) and radii $R \sim 10 - 14$ km the central densities of neutron stars can easily exceed the nuclear saturation density $n_s \sim 0.16$ fm$^{-3}$ by factors of a few up to ten.

At the same time neutron stars are extreme low-temperature laboratories: the high densities of their interiors imply large Fermi energies of fermions $\varepsilon_F \sim 10 - 100$ MeV, which turn out to be much higher than the characteristic interior temperatures of mature neutron stars $T \sim 10^8$ K $\sim 0.01$ MeV. Because of the attractive long-range component of the nuclear force and the high degeneracy $\varepsilon_F \gg T$ the neutrons and protons (and presumably some hyperons) become superfluid and superconducting at critical temperatures of the order of $T_c \sim 10^9$ K.

Neutron superfluidity in a neutron star crust and its core, as well as proton superconductivity in the core, profoundly alter its dynamics, just as the emergence of these phenomena does in terrestrial experiments. For example, superfluid neutrons can now flow relative to the ‘normal’ component of the star with little or no viscosity, as standard reactions and scattering processes giving rise to bulk and shear viscosity are strongly suppressed. An important factor in neutron star dynamics is the appearance of an array of vortices in the neutron condensate. In analogy to a laboratory superfluid in a rotating container, the neutron superfluid mimics large scale rotation by creating an array of quantised vortices each carrying a quantum of circulation. Interactions between vortices and the normal component open a new dissipative channel, known as mutual friction. These interactions may be strong enough to ‘pin’ the vortices and freeze the rotation rate of the superfluid neutrons. The angular momentum thus stored is then released catastrophically during discrete events, which are thought to be the cause of the observed ‘glitches’, i.e., sudden spin-up episodes observed in
pulsars. These phenomena reflect the interior dynamics of neutron stars and thus can potentially provide an insight into the physics of superfluids in their interiors.

This chapter provides an educational introduction and an overview of the field of superfluidity and superconductivity in neutron stars. The first part of the chapter reviews the microphysics of nuclear pairing in neutron stars by providing an elementary introduction to the microscopic theory of nuclear pairing and a review of current issues such as medium polarization corrections to the pairing, pairing in higher partial waves and in strong magnetic fields (Sec. 2). The interaction of vortices with the ambient fluid at the microphysical level, which leads to the phenomenon of mutual friction between the superfluid and the normal fluid, is reviewed in Sec. 3. This is followed by a discussion of hydrodynamics of superfluids in neutron stars (Sec. 4). Section 5 is devoted to the interactions of vortices with flux-tubes and nuclear clusters, i.e., their pinning to various structures. This is followed by a discussion of the macrophysics of rotational anomalies in neutron stars in Sec. 6. We provide our concluding remarks in Sec. 7.

2 Microscopic pairing patterns in neutron stars

2.1 General ideas

The microscopic understanding of the pairing mechanism in nucleonic matter in neutron stars is based on the theory advanced by Bardeen, Cooper, and Schrieffer (BCS) in 1957 to explain the superconductivity of some metals at low temperatures [1]. The key ingredient of this theory is the notion of an attractive interaction between two electrons which is mediated by lattice phonon exchange. According to the Cooper theorem [2] low-temperature fermions which fill a Fermi sphere can bind to form Cooper pairs if there is an attractive interaction between them. The bound states of electrons (typically with total spin 0) form a coherent many-body state which carries an electric current without any resistivity below a certain critical temperature $T_c$. The overwhelming success of the BCS theory in explaining the wealth of experimental data encouraged applications of the key ideas of this theory in other fields of physics, including nuclear physics. In contrast to electronic materials, where the direct interaction between the electrons is repulsive due to the Coulomb force between same-charge particles, in nuclear systems the dominant long-range piece of the interaction between the nucleons (neutrons and protons) is attractive. It is not surprising then that superconductivity and superfluidity in nuclear systems - finite nuclei and neutron stars - were conjectured shortly after the advent of the BCS theory by A. Bohr et al. [3], Migdal [4] and others.

Fully microscopic calculations of the pairing properties of neutron and proton matter in neutron stars were carried out following the discovery of pulsars in 1967 and their identification with neutron stars. Although at the time nuclear interactions were not known as precisely as nowadays, the first computations of the pairing gaps of about 1 MeV in neutron and proton matter are consistent with present day calculations (see the reviews [5, 6, 7, 8, 9] and references therein).

A useful reference for the understanding of the patterns of pairing in neutron stars is the partial wave analysis of the nuclear interaction. In fact, the experimental measurements of the nuclear scattering are given per partial wave. At low energies the nucleon-nucleon ($nn$) scattering is dominated by two $S$-wave interactions, specifically the $^3S_1$–$^3D_1$ coupled partial wave and the $^1S_0$ partial wave; here we use the standard spectroscopic notations to specify the scattering channels, i.e., $^{2S+1}L_J$, with $L = 0, 1, 2$ mapped to $S, P, D$, where $L$ is the orbital angular momentum, $S$ is the total spin and $J$ is the total angular momentum, which is the sum of the former two vectors. Thus, at low energies the $L = 0$ states, which have symmetrical wave-function in the coordinate space, dominate. The total wave function contains however spin and isospin components which must be selected in a manner to satisfy the Pauli principle, which dictates that the total wave function must be anti-symmetrical. As a consequence, the neutron-neutron and proton-proton scattering which always has a total isospin $T = 1$ symmetrical component, cannot occur in the spin symmetrical $S = 1$ and spatially symmetrical $L = 0$ state. Therefore, the strongly attractive interaction in the $^3S_1$–$^3D_1$ channel which binds the deuteron (binding energy $E_d = −2.2$ MeV) does not lead to pairing in neutron dominated matter, where
neutrons and protons form vastly separate Fermi surfaces. Then, for large differences in the numbers of protons and neutrons (isospin asymmetry) only same isospin Cooper pairs can arise in the remaining $^1S_0$ partial wave channel.

At laboratory energies of $nn$ scattering larger that $E_L = 250$ MeV the measured scattering phase-shift in the $^1S_0$-wave interaction channel becomes negative, i.e., the interaction becomes repulsive, see Fig. 1. However, already at $E_L \simeq 160$ MeV the $^3P_2 - ^3F_2$ tensor interaction becomes the most attractive channel for $T = 1$ (neutron-neutron and proton-proton) pairs. The corresponding density in neutron star matter is obtained by noting that the center of mass energy of two scattering nucleons is $E_L/2$, which should be of the order of the Fermi energy of neutrons or protons. (Here we specialize the discussion to the high-density and low-temperature regime of interest to superfluidity in neutron stars). Neutron Fermi energies become of the order of $\varepsilon_{F_n} \simeq 60$ MeV at the nuclear saturation density $n_s = 0.16$ fm$^{-3}$. Thus, we anticipate that neutron pairing in the $^1S_0$-wave vanishes at densities slightly above the saturation density and that the core of the star contains superfluid featuring neutron pairs in the $^3P_2 - ^3F_2$ partial wave. The spatial component of the wave-function of these Cooper pairs is anti-symmetrical whereas the spin ($S = 1$) and isospin ($T = 1$) components are symmetrical. Clearly, the pairing in this so-called triplet spin-1 channel is consistent with the Pauli principle for two neutrons. Because the proton fraction in a neutron star core is small, about 5-10% of the net number density, their Fermi energies, and consequently the center of mass scattering energies, remain low. Therefore, proton pairs arise in the $^1S_0$-wave up to quite high densities. It is conceivable that at densities higher than a few times the nuclear saturation density higher partial waves can contribute to the pairing in neutron star matter. For example, if the partial densities of neutrons and protons are forced to be close to each other by some mechanism, i.e., matter is isospin symmetrical, then neutron-proton pairs can be formed in the most attractive $^3D_2$ partial wave with a wave function which is symmetrical in space, antisymmetrical in isospace ($T = 0$) and symmetrical in spin ($S = 1$). A mechanism that can enforce equal numbers of neutrons and protons is meson condensation $[10]$.

The BCS theory was originally formulated in terms of a variational wave function of a coherent state which minimized the energy of an ensemble of electrons interacting via contact (attractive) interaction $[1]$. Here we will outline an alternative formulation based on the method of canonical transformations due to Bogolyubov $[11]$.

Figure 1: The nucleon-nucleon scattering phase shifts as a function of laboratory energy for the channels where pairing in neutron stars matter appears. The $S$ and $P$ wave scattering is responsible for neutron-neutron and proton-proton pairing, whereas the $^3D_2$ wave scattering can occur only between neutrons and protons. The $P$ and $F$ waves are coupled by the non-central tensor component of the nuclear interaction.
Consider a macroscopic number of $N$ fermions which are described by the *pairing Hamiltonian*

$$ H - \mu N = \sum_{p,\sigma} \varepsilon_p a_{p,\sigma}^\dagger a_{p,\sigma} - \sum_{p_1+p_2=p_3+p_4} V_{\text{eff}}(p_1, p_2; p_3, p_4) a_{p_1,\sigma}^\dagger a_{p_2,\sigma} a_{p_3,\sigma} a_{p_4,\sigma}. \tag{1} $$

The first term is the kinetic energy, the second term is the attractive interaction energy, where $V_{\text{eff}}(p_1, p_2; p_3, p_4)$ is an effective pairing interaction. Here $a_{p,\sigma}^\dagger$ and $a_{p,\sigma}$ are the particle creation and annihilation operators for particles with spin $\sigma = \uparrow, \downarrow$ and momentum $p$. Note that we work in the grand-canonical ensemble, so that instead of fixing the number of particles we assume that our system is connected to a reservoir of particles; $\mu$ is the chemical potential - the energy needed to add or remove a particle to the system. The application of the method of canonical transformations to the Hamiltonian (1) requires new creation and annihilation operators defined as

$$ a_{p,\uparrow} = u_p \alpha_{p,\uparrow}^\dagger + v_p \alpha_{-p,\downarrow}^\dagger, \tag{2} $$
$$ a_{p,\downarrow} = u_p \alpha_{p,\downarrow} - v_p \alpha_{-p,\uparrow}. \tag{3} $$

The requirement that the anti-commutation relations obeyed by the new operators are the same as those obeyed by the original fermionic ones leads us to

$$ \{\alpha_{p,\sigma}, \alpha_{p',\sigma'}^\dagger\} = \alpha_{p,\sigma}^\dagger \alpha_{p',\sigma'} + \alpha_{p',\sigma'} \alpha_{p,\sigma}^\dagger = \delta_{pp'} \delta_{\sigma,\sigma'}. \tag{4} $$

It follows then that the functions $u_p$ and $v_p$ are not independent, but $u_p^2 + v_p^2 = 1$, i.e., there is a single *independent* function, say $v_p$. This parameter is found from the minimization of the statistical average of the Hamiltonian (1)

$$ E - \mu N = \langle H - \mu N \rangle, \tag{5} $$

where $\langle \ldots \rangle$ stands for mean value with the occupation numbers defined as $\langle \alpha_{p,\uparrow}^\dagger \alpha_{p,\uparrow} \rangle = n_{p,\uparrow}$ and $\langle \alpha_{p,\uparrow}^\dagger \alpha_{p,\downarrow} \rangle = n_{p,\downarrow}$. The energy is then given by

$$ E - \mu N = \sum_p \varepsilon(p) \left[ u_p^2 (n_{p,\uparrow} + n_{p,\downarrow}) + v_p^2 (2n_{p,\uparrow} - n_{p,\downarrow}) \right] $$
$$ - \sum_{pp'} V_{\text{eff}}(p, p') u_p v_{p'} u_{p'} v_{p'} Q(p) Q(p'), \tag{6} $$

where $Q(p) \equiv (1 - n_{p,\uparrow} - n_{p,\downarrow})$. Eliminating $u_p$ from (6) via $u_p^2 = 1 - v_p^2$ and performing variations with respect to $v_p$ we find then

$$ 2 \varepsilon_p = \frac{\Delta(p)(1 - 2v_p^2)}{u_p v_p}, \tag{7} $$

where

$$ \Delta(p) = \sum_{p'} V_{\text{eff}}(p, p') u_{p'} v_{p'} (1 - n_{p',\downarrow} - n_{p',\uparrow}), \tag{8} $$

is the so-called *gap equation*. From Eq. (7) we find for the *Bogolyubov amplitudes*

$$ u_p^2 = \frac{1}{2} \left( 1 + \frac{\varepsilon_p}{E_p} \right), \quad v_p^2 = \frac{1}{2} \left( 1 - \frac{\varepsilon_p}{E_p} \right), \tag{9} $$

where the quasiparticle spectrum in the superconductor is defined as

$$ E_p = \sqrt{\varepsilon_p^2 + \Delta_p^2}. \tag{10} $$
On substituting $u_p v_p = \Delta(p)/2E_p$ in Eq. (8), we find a non-linear integral equation for the gap function $\Delta(p)$ which can be solved for any given effective interaction $V_{\text{eff}}(p, p')$. It is easy to verify that $E_p$ is indeed the quasiparticle energy, by taking the variation of the total energy with respect to the occupation numbers, i.e., by computing the variation
\[
\frac{\delta(E - \mu N)}{\delta n_{p,\uparrow}} = \varepsilon_p (u_p^2 - v_p^2) + 2u_p v_p \Delta(p) = E_p. \tag{11}
\]

Eq. (10) demonstrates the fundamental property of the superconductors: the spectrum of a superconductor contains an energy gap $\Delta$. As a consequence the excitations can be created in the system if a Cooper pair breaks, which means that energy of the order of $2\Delta$ must be supplied to the superconductor. The main property of superconductors - the absence of dissipation of current - follows from the existence of the gap in their spectrum. In the case of uncharged fermionic superfluids (e.g. neutron matter) the same property is referred to as superfluidity (fluid motions without dissipation). In equilibrium, the occupation numbers of fermions are given by the Fermi function $f(\varepsilon) = (e^{\varepsilon_p/\beta} + 1)^{-1}$, where $\beta$ is the temperature. At low temperatures the fermionic momenta are restricted to the vicinity of the Fermi surface; then, assuming an isotropic (S-wave) interaction, we can simplify the gap equation by changing the integration measure $\sum_p = m^* p_F \int d\varepsilon_p \int d\Omega$ to find
\[
1 = G\nu \int_0^\Lambda \frac{d\varepsilon_p}{2\sqrt{\varepsilon_p^2 + \Delta^2}} \tanh \left( \frac{\sqrt{\varepsilon_p^2 + \Delta^2}}{2T} \right), \tag{12}
\]
where $\nu = p_F m^*/\pi^2$ is the density of states, $m^*$ is the effective mass, $p_F$ is the Fermi momentum, and for the sake of illustrations below we assume a momentum-independent contact interaction $V_{\text{eff}}(p, p') = G$, which in turn requires a cut-off $\Lambda$ to regularize the integral in the ultraviolet. The latter cut-off is physically well-motivated, as the effective pairing interactions are typically localized close to the Fermi surface.\(^2\)

Consider now analytical solutions of the gap equation in the limiting cases $T \to T_c$ and $T \to 0$, where $T_c$ is the critical temperature of phase transition. For $T = 0$, the $\tanh$ function is unity and a straightforward integration gives
\[
1 = \frac{G\nu}{2} \arcsinh \left( \frac{\Lambda}{\Delta(0)} \right) \simeq \ln \left( \frac{2\Lambda}{\Delta(0)} \right), \tag{13}
\]
where in the last step we assumed weak coupling, i.e., $\Delta \ll \Lambda$ to expand $\lim_{x \to \infty} \arcsinh x = \ln(2x) + O(x^2)$; this can be written in a more familiar form
\[
\Delta(0) = 2\Lambda \exp \left( -\frac{2}{G\nu} \right), \tag{14}
\]
which is the famous gap equation of the BCS theory. It demonstrates the exponential sensitivity of the pairing gap to the effective attractive interaction $G$. In the limit $T \to T_c$, we can set in the integrand of the gap equation $\Delta = 0$. An elementary integration then gives $T_c = (2\Lambda^2/\pi) \exp \left( -2/2G\nu \right)$, where $\gamma = e^C$ and $C = 0.57$ is the Euler constant. Combining the results for $T_c$ and $\Delta(0)$ we obtain a relation between them: $\Delta(0) = \pi T_c / \gamma = 1.76 \ T_c$.

The limiting expressions for the gap function can be easily extended to include the next-to-leading order terms in the two limiting cases discussed above:
\[
\Delta(T) = \Delta(0) - \frac{\sqrt{2\pi\Delta(0)T}}{\mu} \exp \left( -\frac{\Delta(0)}{T} \right) \tag{15}
\]

\(^1\)When computing the variation with respect to the occupation numbers one needs to assume that the Bogolyubov coefficients are constant, because they are determined from the condition $\delta(E - \mu N)/\delta v_p = 0$.
\(^2\)The full nuclear interaction can be renormalized via resummations of infinite series such as to contain only components close to the Fermi surface.
for $T \to 0$ and

$$\Delta(T) = \pi \sqrt{\frac{8}{7\zeta(3)}} |T_c(T_c - T)|^{1/2} = 3.06 |T_c(T_c - T)|^{1/2}$$

(16)

for $T \to T_c$, where $\zeta(x)$ is the Riemann’s $\zeta$-function with $\zeta(3) = 1.20206$. The temperature dependence of the gap function in the whole temperature regime can be obtained numerically and analytical fits, useful for practical applications, are given in Ref. [12]. Having the temperature dependence of the gap one still needs fit formulae to the gap function at zero temperature $\Delta(0)$ (or equivalently the critical temperature $T_c$) as a function of density or Fermi momentum. We point out that accurate fits can be obtained with the functional form

$$\Delta(k_F) = a \exp(-k_F^2) + \sum_{n=0}^{4} c_n k_F^n,$$

(17)

where the fit coefficients to the $S$- and $P$-wave neutron and $S$-wave proton gaps can be found in Ref. [13].

### 2.2 Effective interactions

An important issue in computations of gaps in neutron star matter is the proper determination of the effective pairing interaction $V_{\text{eff}}(p,p')$. As a first approximation one may use the bare nuclear interaction in the gap equation, which provides us with a useful reference result. The most important correction to this interaction arises from polarization effects or screening, which in diagrammatic language can be understood as a summation of infinite series of particle-hole diagrams. We will review these effects on the basis of the Landau Fermi liquid theory and will compare to the alternatives thereafter. In our discussion we will follow the original Landau approach; for computations which are based on this type approach, but include more advanced diagrammatic treatments of the problem see Refs. [14, 15, 16, 17, 18].

We now suppose that the bare interaction depends only on the momentum transfer in the collision $q = p_1 - p_3$ and write it in terms of all possible combinations of the spin and isospin components:

$$V_{\text{bare}}(q) = \frac{1}{\nu} \left\{ F_q + G_q(\sigma \cdot \sigma') + \left[ F'_q + G'_q(\sigma \cdot \sigma') \right] (\tau \cdot \tau') \right\},$$

(18)

where $\sigma$ and $\tau$ are the vectors formed from the Pauli matrices in the spin and isospin spaces and $F_q, F'_q, G_q$ and $G'_q$ are the so-called Landau parameters. For simplicity the tensor and spin-orbit interactions are neglected as they are small in neutron matter. The summation of geometrical series of particle-hole diagrams then gives for the effective interaction

$$V_{\text{eff}} = \frac{1}{\nu} \left\{ \tilde{F}_q + \tilde{G}_q(\sigma \cdot \sigma') + \left[ \tilde{F}'_q + \tilde{G}'_q(\sigma \cdot \sigma') \right] (\tau \cdot \tau') \right\},$$

(19)

where $\tilde{A}_q = A_q[1 + \Lambda(q)A_q]^{-1}$, $A$ stands for any of the Landau parameters. It is seen that the screening leads to a renormalization of the Landau parameters $A \to \tilde{A}$, where the function $\Lambda(q)$, which represents the single particle-hole loop, is the Lindhard function. In the low-temperature regime of interest the momenta of fermions are restricted to lie on their Fermi surface, therefore the Landau parameters will depend only on the relative orientation of the momenta of particles. Then, it is useful to expand these into spherical harmonics

$$A(q) = \sum_l A_l P_l(\cos \theta),$$

(20)

where $P_l$ are the Legendre polynomials, $A$ stands for any of the Landau parameters above, $\theta$ is the scattering angle which is related to the magnitude of the momentum transfer via $q = 2p_F \sin \theta/2$, where $p_F$ is the Fermi momentum.
In pure neutron matter ($\tau \cdot \tau' = 1$) keeping the lowest-order harmonics in the expansion (20) and for scattering with total $S = 0$ (i.e. $\sigma \cdot \sigma' = -3$) the effective pairing interaction is given by

$$
\nu(p_F)V_{\text{eff}}(q) = F_0^n [1 - \mathcal{L}(q)F_0^n] - 3G_0^n [1 - \mathcal{M}(q)G_0^n].
$$

where $F^n = F + F'$ and $G^n = G + G'$ describe the effective interaction in the density and spin channels respectively, $\mathcal{L}(q) = \Lambda(q)[1 + \Lambda(q)F_0^n]^{-1}$ and $\mathcal{M}(q) = \Lambda(q)[1 + \Lambda(q)G_0^n]^{-1}$ are screening corrections to the direct part of the effective interaction $\propto 1$. In general the Lindhard function is complex; in the low-temperature regime of interest its imaginary part (which is related to the damping of particle-hole excitations) can be neglected. It assumes a simple form for zero energy transfer at fixed momentum: $\Lambda(x) = -1 + (2x)^{-1}(1-x^2)\ln \left| (1-x)/(1+x) \right|$, with $x = q/2p_F$ [19].

The Landau parameters and the effective interaction in Eq. (21) have been computed extensively over the past several decades within various approximations. There is a general agreement that the density fluctuations $\propto F_0^n$ enhance, whereas the spin-fluctuations $\propto G_0^n$ reduce the attraction in the pairing interaction. The overall effect of the spin fluctuations at subnuclear densities is numerically larger and, consequently, fluctuations reduce the pairing gap in neutron matter. As can be seen from Eq. (14) the dimensionless quantity (coupling) determining the magnitude of the gap involves, apart from the effective interaction, also the density of states. In general the quasiparticle spectrum can be written as

$$
\varepsilon(p) = \frac{p^2}{2m} + U(p),
$$

where $U(p)$ is the single particle potential felt by a particle in the nuclear environment. The full momentum dependence in Eq. (22) can be approximated by introducing an effective quasiparticle mass. An expansion of the potential $U(p)$ around the Fermi momentum leads to

$$
\varepsilon(p) = \frac{p_F}{m^*}(p - p_F) - \mu^*, \quad \frac{m^*}{m} = \left( 1 + \frac{m}{p_F} \left. \frac{\partial U(p)}{\partial p} \right|_{p=p_F} \right)^{-1}
$$
where $\mu^* \equiv -\epsilon(p_F) + \mu - U(p_F)$. Because the effective mass $m^*/m < 1$ in neutron matter, the modifications of the single-particle energies lead to a reduction of the gap and will also have a significant effect on the multifluid hydrodynamics of the system, as discussed in Sec. 4.

Correlations in neutron matter can be studied in a number of alternative theories, for example, Monte Carlo sampling of systems with odd and even numbers of neutrons, whereby the gap is defined as the energy difference between odd and even states [22, 23]. Wave-function based approaches minimize the energy of the BCS state using correlated basis functions (CBF), which are built to account for the operator structure of the nuclear interaction [21].

In Fig. 2 we show selected benchmark results for pairing gaps in different theories of pairing. The BCS result with bare single-particle energies was obtained using a low-momentum interaction, which leads to slightly larger gaps compared to full (realistic) interaction which contains a hard core. The modifications of the single-particle spectrum, computed within the Bruckner-Hartree-Fock theory (without invoking effective mass approximation) leads to a moderate reduction of the gap [20]. If one includes the screening corrections in the effective interaction, then the gap is reduced by a factor of three. There is consensus among the Fermi-liquid [14, 15, 16, 17, 18] and CBF approaches [21, 24] on the magnitude of the reduction. Auxiliary field Monte Carlo calculations on the other hand predict pairing gaps at low densities which are much closer to the BCS result [22].

### 2.3 Higher partial wave pairing

As seen from Fig. 1 at high densities (energies) the dominant pairing interaction between neutrons is in the $^3P_2 - ^3F_2$ partial wave. The pairing pattern for spin-1 condensates is more complex than for spin-0 $S$-wave condensates because of competition between states with various projections of the orbital angular momentum and complications due to the tensor coupling of the $^3P_2$ partial wave to the $^3F_2$ one, see Refs. [25, 26, 27, 28]. To solve the $P$-wave gap equation \((8)\) one starts with an expansion of the pairing interaction in partial waves

$$V_{\text{eff}}(p, p') = 4\pi \sum_L (2L + 1) P_L(\hat{p} \cdot \hat{p}') V_L(p, p'),$$  \hspace{1cm} (24)

where $P_L$ are the Legendre polynomial, and an associated expansion of the gap function in spherical harmonics $Y_{LM}$

$$\Delta(p) = \sum_{L,M} \sqrt{\frac{4\pi}{2L + 1}} Y_{LM}(\hat{p}) \Delta_{LM}(p),$$  \hspace{1cm} (25)

where $L$ and $M$ are the total orbital momentum and its $z$-component. The non-linearity of the gap equation prevents a straightforward solution for its components $\Delta_{LM}(p)$; a common approximation is to perform an angle average in the denominator of the kernel of the gap equation by replacing the factor $\sqrt{\varepsilon(p)^2 + |\Delta(p)|^2} \rightarrow \sqrt{\varepsilon(p)^2 + D(p)^2}$ where the \textit{the angle averaged gap} is given by

$$D(p)^2 \equiv \int \frac{d\Omega}{4\pi} |\Delta(p)|^2 = \sum_{L,M} \frac{1}{2L + 1} |\Delta_{LM}(k)|^2.$$  \hspace{1cm} (26)

With this approximation the angular integrals are trivial and we obtain a one-dimensional gap equation for the $L$-th component of the gap

$$\Delta_L(p) = -\int_0^\infty dp' \frac{V_L(p, p')}{\sqrt{\varepsilon(p')^2 + |\sum_{L'} \Delta_{L'}(p')|^2}} \Delta_L(p').$$  \hspace{1cm} (27)

Although the denominator contains a sum over gap components with different values of $L$, the contributions from channels other than the $^3P_2$ can be neglected as they are numerically insignificant in the range of densities (energies) where $P$-wave paring is dominant.
The non-central tensor part of the nuclear force couples the $^3P_2$ wave to the $^3F_2$ wave; this coupling affects the value of the gap. The modification of the gap equation which takes into account this tensor coupling requires a simple doubling of the components of the gap equation. The coupled channel gap equation reads

$$
\begin{pmatrix}
\Delta_L \\
\Delta_{L'}
\end{pmatrix} = \int_0^{\infty} \frac{dp'}{\pi E(p')} \begin{pmatrix}
-\nu_{LL} & \nu_{LL'} \\
\nu_{L'L} & -\nu_{L'L'}
\end{pmatrix} \begin{pmatrix}
\Delta_L \\
\Delta_{L'}
\end{pmatrix},
$$

where now $D(k)^2 = \Delta_L(k)^2 + \Delta_{L'}(k)^2$. 3

Note that the angle averaged approximation provides a numerically accurate value of the angle averaged gap on the Fermi surface. However, in a number of problems, such as neutrino and axion emission from $P$-wave superfluids the angle dependence of the gap equation is an important factor and should be taken into account by solving the gap equation without the angle averaged approximation. Such solutions show that the angle dependence of the gap function can lead to nodes on the Fermi surface, as for example in the case of solutions of the form $\Delta(\theta) = \Delta_0 \sin \theta$. However, “stretched” solutions with fully gaped Fermi surface $\Delta(\theta) = \Delta_0(1 + \cos^2 \theta)$ are viable candidates for angle dependence of the pairing gap and it is difficult to distinguish between these options from energy minimization arguments alone.

Figure 1 shows that the interaction is attractive among neutrons and protons in the $^3D_2$ channel and it is stronger than the attraction in the $^3P_2$ channel. Thus, in a hypothetical high-density isospin symmetrical form of nuclear matter one would expect $D$-wave pairing instead of the $P$-wave pairing, which is the dominant channel in the high density neutron matter. Consequently, there should be a transition from $D$- to $P$-wave pairing as the imbalance between neutrons and protons (isospin asymmetry) changes from zero to larger values. Computations show that already small asymmetries destroy the $D$-wave pairing in nuclear matter [30], therefore for its realization one needs nearly symmetrical nuclear matter. Above nuclear saturation density such situations can arise in some special cases as, for example, when a mesonic condensate forms [10].

2.4 Effects of strong magnetic fields on pairing

Neutron stars are highly magnetised objects and the magnetic field in the stellar interior is modified by the presence of superconductivity. The topology and properties of the magnetic field depend strongly on the type of superconductivity, which depends on the Ginzburg-Landau parameter $\kappa_{GL} = \lambda/\xi_p$, where $\xi_p$ is the proton coherence length, which roughly determines the size of the Cooper pairs, and $\lambda$ is the penetration depth of the magnetic field in superconducting matter. In most of the neutron star core one has $\kappa_{GL} > 1/\sqrt{2}$ and type II superconductivity is expected, in which the magnetic field penetrates the superconductor by forming an array of quantized flux tubes. In laboratory type II superconductors the field can only penetrate the superconductor for field strengths between a lower critical field $H_{c1}$ and an upper critical field $H_{c2}$. In neutron stars the situation is somewhat different, as one still has the upper critical field $H_{c2}$ (essentially the field at which flux tubes are so densely packed that their cores touch), but magnetic fields can still penetrate the core below the lower critical field $H_{c1}$. This is the case because the magnetic flux cannot be expelled effectively from the superconducting core due to its high electric conductivity; the time-scale for such expulsion is of the order of the secular timescales [31]. In the deep core of the neutron star, on the other hand, it is possible to have $\kappa_{GL} < 1/\sqrt{2}$, and in this case we expect type I superconductivity, in which fluxtubes are not energetically favourable and the field is arranged in domains of unpaired proton matter of much larger spatial dimensions [32, 33, 34, 35, 36].

A class of neutron stars known as magnetars have surface magnetic fields of the order of $10^{15}$ G and it has been conjectured that their interior fields could be by several orders of magnitude larger [37]. While modifications of the equation of state of matter require fields which are close to the limiting fields $10^{18}$ G compatible with gravitational stability, pairing phenomena which occur near the Fermi surface are affected by much lower fields of the order $B \sim 10^{16} - 10^{17}$ G. The interaction energy of the magnetic field with the nucleon spin is $\mu_N B$, where $\mu_N = e\hbar/2m_p$ is the nuclear magneton. A numerical estimate gives $\mu_N B \simeq$

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3Note that the tensor coupling arises also in the case of pairing in the $^3S_1$-$^3D_1$ channel which acts only between neutrons and protons and is relevant when the isospin asymmetry between neutrons and protons in not too large [29].
\( \pi (B/10^{18} \text{ Gauss}) \) MeV. Therefore, fields of the order of \( 10^{16} \) G would substantially affect the pairing with gaps of the order of 1 MeV via the spin–B-field interaction. The interaction of the magnetic field with the neutron or proton spin induces an imbalance in the number of spin-up and spin-down particles, which implies that the Cooper pairing in the \( S \) wave will be suppressed. Indeed in this case the number of Cooper pairs will be limited by the number of spin-down particles, with the excess spin-up particles remaining unpaired [38]. This Pauli paramagnetic suppression acts for both proton and neutron condensates and the associated critical field is within the range \( H_{c2}^{\text{Para}} \sim 10^{16} - 10^{17} \) G. Note that similar field arises in the condensed matter theory and is known as the Chandrasekhar-Clogston field.

In the case of the proton condensate the upper critical field \( H_{c2} \) turns out to be smaller than the field associated with the Pauli paramagnetic ordering [13], therefore the proton condensate is destroyed for even lower magnetic fields \( H_{c2} \approx 10^{15} \) G. We have seen that the \( S \)-wave gaps and in particular the proton gap depends substantially on the density. If the magnetic field strength can be assumed approximately constant in the core of a magnetar the size of the superconducting region will depend on the magnitude of the field \( B \) via the condition \( B \leq H_{c2} \).

The role of the magnetic field in the \( P \)-wave pairing is not well understood from microscopic point of view, but because the pairs in this case form spin-1 objects, the spin-magnetic field interaction will not be destructive. The consequences of the suppression of the nucleonic pairing on the phenomenology of magnetars are discussed elsewhere [39].

### 3 Microphysics of mutual friction

In this section we concentrate on the interaction of vortex lines in neutron stars with ambient components. The discussion includes both the neutron vortex lines which are formed in response to the rotation of the star and the magnetic flux-tubes which, as discussed above, form if the proton superconductor is type-II. This interaction is known under the general name mutual friction and appears naturally in the superfluid hydrodynamics including vortices, which we will discuss in the following Sec. 4. The mutual friction is an important ingredient of the description of superfluid dynamics as it determines the dynamical time-scales of coupling of superfluid to normal (non-superfluid) matter and rotational anomalies in neutron stars, such as glitches, time-noise, precession etc., which are in part discussed in the subsequent Sec. 6. The mutual friction is quantified in terms of dimensionfull drag parameter \( \eta \) defined via force exerted by ambient fluids on the vortex \( \mathbf{f}_d = \eta (\mathbf{v}_v - \mathbf{v}_e) \), where \( \mathbf{v}_v \) is the vortex velocity, \( \mathbf{v}_e \) is the velocity of the normal component. In superfluids it is balanced by the Magnus (lifting) force \( \mathbf{f}_M = \rho_n (\mathbf{v}_s - \mathbf{v}_v) \times \mathbf{v} \), acting on a vortex with circulation vector \( \mathbf{v} \) placed in a superfluid flow with velocity \( \mathbf{v}_s \). It is, therefore, convenient to use the dimensionless drag-to-lift ratio \( \mathcal{R} = \eta / \rho_n \kappa \), where \( \rho_n \) is the mass density of the superfluid (neutron) component and \( \kappa = \pi \hbar / m_n \) is the quantum of circulation, \( m_n \) being neutron mass. For massless vortices \( f_M + f_d = 0 \), which is known as the force balance equation. We shall discuss these quantities in more detail in the context of superfluid hydrodynamics in Sec. 4.

The neutron vortices which carry the angular momentum of neutron star interiors arise in the \( S \) and \( P \) wave superfluids where the Cooper pairs have total spin-0 and spin-1, as discussed in Sec. 2. A vortex in a neutral fermionic superfluid has a core of the order of the coherence length \( \xi \), where quasiparticle pairing is quenched and, therefore, scattering centers are available for interactions. Consider first a vortex in a neutron superfluid with an \( S \)-wave symmetry of the condensate. The core of the vortex contains fermionic states which are given by [40]

\[
\begin{pmatrix}
  u_{q_{||}, \mu}(r_{||}) \\
  v_{p_{||}, \mu}(r_{||})
\end{pmatrix} = e^{i \mu \theta} \left( e^{i \theta (\mu - \frac{1}{2})} e^{i \theta (\mu + \frac{1}{2})} \right) \begin{pmatrix}
  u_{\mu}(r) \\
  v_{\mu}(r)
\end{pmatrix},
\]

where \( r, \theta, z \) are cylindrical coordinates with the axis of symmetry along the vortex circulation (here \( || \) and \( \perp \) are the parallel and perpendicular to vortex components), and \( \mu \) is the azimuthal quantum number, which assumes half-integer positive values. It is seen that the states are plain waves along the vortex circulation,
which are quantized in the orthogonal direction. The radial part of the wave-function is given by

\[ u^l_p(r) = 2 \left( \frac{2}{\pi p_\perp r} \right)^{1/2} e^{-K(r)} \left( \cos \left( p_\perp r - \frac{\mu}{2} \right) \sin \left( p_\perp r - \frac{\mu}{2} \right) \right), \quad (30) \]

where \( p_\perp = \sqrt{p^2 - p_F^2} \), \( p_F \) being the neutron Fermi momentum, and the function in the exponent is given by

\[ K(r) = \frac{p_F}{\pi p_\perp \Delta_n} \int_0^r \Delta(r') dr' \approx \frac{p_F}{\pi p_\perp \xi} \left( 1 + \frac{\xi e^{-r/\xi}}{r} \right). \quad (31) \]

The eigenstates of neutrons in the core of a vortex are given by

\[ \varepsilon_\mu(p) \approx \frac{\pi \mu \Delta^2_n}{2 \varepsilon_{F_n}} \left( 1 + \frac{p^2}{2p_F^2} \right), \quad (32) \]

where \( \varepsilon_{F_n} \) is the Fermi energy of neutrons, \( \Delta_n \) is the asymptotic value of the gap far from the vortex core and the second equality holds up to the next-to-leading order in small quantity \( p/p_F \).

Electrons will couple to the core quasiparticles of the neutron vortex via the interaction of the electron charge \( e \) with the magnetic moment of neutrons \( \mu_n = -1.913\mu_N \), where \( \mu_N = e\hbar/2m_p \) is the nuclear magneton \([41]\). The momentum relaxation time scale for electrons off neutron vortices is given by \([42]\)

\[ \tau_e V[1 S_0] = \frac{1.6 \times 10^3 \Delta_n}{\Omega_s} \frac{\varepsilon_{F_e}}{\varepsilon_{F_n}} \left( \frac{\varepsilon_{F_n}}{2m_n} \right)^{1/2} \exp \left( \frac{\varepsilon_{0/2}}{T} \right), \quad (33) \]

where \( \varepsilon_{F_e} \) is the Fermi energy of electrons, \( \Delta_n \) is the \( S \)-wave neutron pairing gap, \( \varepsilon_{0/2} \) is given by Eq. (32) with \( \mu = 1/2 \) and \( \Omega_s \) is the superfluid’s angular velocity. The relaxation time is strongly temperature dependent because of the Boltzmann exponential factor involving the eigenstates of the vortex core quasiparticles.

The vortex structure of the \( P \)-wave superfluid was studied by Sauls et al. \([43]\) using a tensor order parameter \( A_{\mu\nu}(r), \mu, \nu = 1, 2, 3 \) which is traceless and symmetric. It can be decomposed in cylindrical coordinates \((r, \phi, z)\) as

\[ A_{\mu\nu} = \frac{\Delta}{\sqrt{2}} e^{i\phi} \left\{ [f_1 \hat{\gamma}_\mu \hat{r}_\nu + f_2 \hat{\gamma}_\mu \hat{\phi}_\nu - (f_1 + f_2) \hat{\gamma}_\phi \hat{\phi}_\nu + i g(r_\mu \hat{\phi}_\nu + r_\nu \hat{\phi}_\mu)] \right\}, \quad (34) \]

where \( g(r) \) and \( f_{1,2}(r) \) are the radial functions describing the vortex profile and \( \Delta \) is the average value of the gap in the \( ^3P_2 \) channel. Vortices in a \( P \)-wave superfluid are intrinsically magnetized, with the magnetization given by \( M_V(r) = (\gamma_n \hbar)\sigma(r)/2 = g_n \mu_N \sigma(r)/2 \), where \( \gamma_n = g_n \mu_N \hbar^{-1} \) is the gyromagnetic ratio of neutron and \( g_n = 3.826, \hbar \sigma/2 \) is the spin density which can be estimated for the \( P \)-wave vortex as \([43]\)

\[ \sigma(r) = \frac{\nu_n \Delta^2_n}{3} \ln \left( \frac{\Lambda}{T_c} \right) g(r) [f_1(r) - f_2(r)], \quad (35) \]

where \( \Lambda \) is the BCS cut-off, \( \nu_n \) - the neutron density of states at the Fermi surface. The magnitude of the vortex magnetization that follows from Eq. (35) is estimated as \([43]\)

\[ |M_V(^3P_2)| = \frac{g_n \mu_N}{2} n_n \left( \frac{\Delta}{\varepsilon_{F_n}} \right)^2 \approx 10^{11} \text{ G}. \quad (36) \]

Ambient electrons which coexist with the \( P \)-wave superfluid because of approximate \( \beta \)-equilibrium among neutrons, protons and electrons, will scatter off the magnetized vortices via the QED interaction term \(-e\gamma \cdot A(r)\), where the vector potential associated with the vortex is given by \( A(r) = A(r) \hat{\phi} \).

\[ A(r) = \frac{1}{r} \int_0^r |M_V(r')| r' dr'. \quad (37) \]
The relaxation time for the electron-vortex scattering is given by

\[ \tau_{eV}[3P_2] \simeq \frac{7.91 \times 10^8}{\Omega_s} \left( \frac{k_{Fn}}{\text{fm}} \right) \left( \frac{\text{MeV}}{\Delta_n} \right) \left( \frac{n_e}{n_n} \right)^{2/3}. \]  

(38)

An important feature of this relaxation time is its near independence of the temperature (a weak temperature dependence arises because of the temperature dependence of the gap). Therefore, it provides an asymptotic lower limit on the scattering rate at low temperatures \( T \ll \Delta_n \) where the relaxation time \( \tau_{eV}[^1S_0] \) given by Eq. (33) is exponentially suppressed. Numerically \( \tau_{eV}[3P_2] \) is of the order of tens of days for the period of the Vela pulsar and varies weakly with the density within the core region where \( P \)-wave superfluid resides.

So far, for simplicity, we neglected the proton component of the core of a neutron star. However, as we describe below, the proton fluid can dramatically modify the mutual friction in the core of the star. Consider first a normal (non-superconducting) fluid of protons. At high densities the proton energies can indeed become large enough so that the \( ^1S_0 \)-wave interaction becomes negative and pairing vanishes. As described in Sec. 2.4, strong magnetic fields \( B \geq H_{c2} \simeq 10^{15} \text{ G} \) unpair the proton fluid. Non-superconducting protons will couple to electrons on short plasma time-scale, the relevant scale being set by the plasma frequency. Therefore, protons will compete with electrons in providing the most efficient interaction with the neutron vortices and, eventually, the coupling between the charged plasma component and the neutron superfluid. The key advantage of protons over electrons is that they couple to neutrinos via the strong nuclear force, instead of much weaker electromagnetism. The relaxation time for the proton scattering off the quasiparticles in the cores of \( S \)-wave neutron vortices is given by [44]

\[ \tau_{pV}[^1S_0] = \frac{0.71}{\Omega_s} \frac{m_n^* m_p^*}{m_n \mu_{pn}^*} \left( \frac{\varepsilon_{F_p}}{\varepsilon_{F_n}} \right)^2 \frac{\varepsilon_0^{1/2}}{T} \exp \left( \frac{\varepsilon_0}{T} \right) \exp \left( \frac{\varepsilon_0}{T} \right) \frac{\varepsilon_0^{1/2}}{\langle \sigma_{np} \rangle}, \]  

(39)

where \( \mu_{pn}^* = m_p^* m_n^*/(m_n^* + m_p^*) \) is the reduced mass of the neutron-proton system (entering the relation between the cross-section and the scattering amplitude squared), \( \varepsilon_0^{1/2} \) is the lowest eigenstate of vortex core excitations, Eq. (32), and \( \langle \sigma_{np} \rangle \) can be viewed as an average neutron-proton cross-section (for a more precise definition see Eq. (20) of Ref. [44]). The bare neutron mass \( m_n \) stems from the quantum of circulation \( \kappa = \pi \hbar/m_n \) defining the number of vortices in terms of spin frequency \( \Omega_s \). Numerical evaluation of Eq. (39) shows that in the relevant range of temperatures \( T \simeq 10^7-10^8 \text{ K} \) the relaxation time is of the order \( \tau_{pV}[^1S_0] \simeq 10^{-2} \text{ sec} \) for the period of the Vela pulsar, which is much shorter than the time-scales for electromagnetic coupling of electrons given by Eqs. (33) and (38). Only at temperatures of the order of several \( 10^8 \text{ K} \) the process (38) takes over; however such low temperatures are unlikely to be achieved in neutron star cores. One potentially important consequence of the shortness of the relaxation time (39) is that magnetars superfluid cores will couple to the remaining stellar plasma on short dynamical time-scales once proton superconductivity is suppressed by the unpairing mechanisms discussed in Sec. 2.4. This will limit the possibilities of explaining glitches and long-timescale variability in terms of superfluid dynamics of magnetar cores [45].

Next let us turn to the case where protons are superconducting. A fundamentally new aspect in this case is the entrainment of the proton condensate by the neutron condensate which leads to magnetization of a neutron vortex by protonic entrainment currents [46, 47]. The effective flux of the neutron vortex is given by

\[ \phi^* = k \phi_0, \quad k = \frac{m_n}{m_p}, \]  

(40)

where \( \phi_0 = \pi \hbar c/e \) is the quantum of flux. Numerically, the magnitude of the field in this case is by four orders larger than due to the spontaneous magnetization [48], therefore much shorter relaxation times are expected. The calculation of the electron relaxation on a neutron vortex discussed in the case of \( P \)-wave superfluid can be repeated in the case of the magnetization by entrainment currents [47]. It is convenient to define the zero-radius scattering rate as

\[ \tau_0^{-1} = \frac{2c n_n}{k_{eF}} \left( \frac{\pi^2 \phi_0^2}{4\phi_0^2} \right), \]  

(41)
where the term separated in the bracket is an approximation to the exact Aharonov-Bohm scattering result where \( \sin^2(\pi/2)(\phi_s/\phi_0) \) appears instead. The full finite-range scattering rate is then given by [47]

\[
\tau^{-1}_{\epsilon\phi} = \frac{3\pi}{32} \left( \frac{\varepsilon_{Fe}}{m_p c^2} \right) \tau_0^{-1} \xi, \tag{42}
\]

where \( \lambda \) is the penetration depth. As was the case with the relaxation time (38) the scattering rate from magnetized neutron vortices is temperature independent in the first approximation. Numerically, the relaxation time is within the range of seconds and is about four order of magnitude shorter than the one given by (38), as a direct consequence of the induced field being larger by the same amount.

As discussed in Sec. 2.4, the magnetic field of a neutron star will penetrate the superconducting proton fluid by either forming quantized vortices (in the case where the proton superconductor is type-II) or domains of unpaired proton matter (in the case where it is type-I). Consider first a type-II superconductor. For fields of the order of \( 10^{12} \) G and typical rotation frequencies of neutron stars (\( \Omega \sim 100 \) Hz), the number of proton vortices (or flux tubes) per area of a single neutron vortex (assuming for the sake of argument colinear neutron and proton vortex lines) is of the order of \( 10^{13} \). Therefore, one may anticipate that proton vortices might strongly affect neutron vortex dynamics. There exist several scenarios on how neutron and proton vortex systems interact and we review them in turn.

One possible scenario assumes that the proton vortex network continues into the crust via magnetic field lines and therefore is frozen into the crustal electron-ion plasma. Neutron vortices might then get pinning on or between these vortices because of the long-range hydrodynamical interaction between them on characteristic scales of the order of \( \lambda \). Microscopically, crossing the vortices may lead to additional pinning force on the scale of \( \xi \) where the condensate is quenched [49, 50], but the long-range \( \sim \lambda \) the hydrodynamical force is the dominant component. The pinning of neutron vortices to proton flux tubes may thus substantially affect the dynamics of neutron vorticity and to some extent can be viewed as mutual friction. The magnitude of the pinning force strongly depends on the relative orientation of vortices and flux tubes, which does not permit to draw model independent conclusions on the relative importance of the pinning force. In some models there are flux tubes associated with the different components of the magnetic fields (poloidal, toroidal, etc), therefore the geometry of the flux tube network itself is a complicated problem. These type of models will be discussed in more detail in Sec. 5 below.

In the vortex cluster model [51] a neutron vortex carries a cluster of proton vortices colinear with the neutron vortex, which are arranged within the region where the entrainment induced field exceeds the lower critical field of the proton superconductor \( H_{c1} \). Such a cluster may substantially enhance the scattering rate estimate given in Ref. [47]. Larger scattering rate and large forces on the neutron vortex from the electron fluid can lead (counter-intuitively) to longer relaxation times for the neutron superfluid than predicted in Ref. [47]; as a consequence post-glich relaxation time-scales appear to be compatible with the vortex cluster model [52].

Understanding of mutual friction in the case of type-I superconducting protons is difficult because of a lack of model-independent predictions for the domain structure and size of type-I superconductor. A tractable case is where a neutron vortex carries a colinear normal proton domain; in this case it can be shown that the neutron vortex motion induces an electric current within the domain which leads in turn to Ohmic dissipation of electron current [36]. The dimensionless drag to lift ratio for this process was found to be of the order of \( 10^{-4} \), which makes precession in neutron stars compatible with the type-I superconductivity of protons [34].

We now turn to the discussion of the mutual friction in the S-wave superfluid within the neutron star’s crust. Here the lattice of crustal nuclei is the physically distinct new component which was absent in our discussion of the core physics. Because in the crust the protons are confined to the nuclei and the superfluid is S-wave, the only process that can be carried over from the discussion above is the one give by Eq. (33). However, according to our current understanding it is not the dominant process of mutual friction in the crust.

The stationary state of a neutron vortex might require its pinning on a nucleus or in the space between nuclei. As we shall see in section 5 there are substantial differences in the estimates of the pinning force in the crust, however the most advanced treatments of pinning based on the density functional theory of superfluid
matter indicate that the pinning occurs between the nuclei [53]. The sign of the pinning force makes, however, little difference; if the pinning of vortices is strong, then these can respond to the changes in the rotation rate of the crust via thermally activate creep (see Ref. [54] and references therein). Vortex creep theory postulates a form of the radial velocity of a vortex which depends exponentially on the ratio of the pinning potential to the temperature. If pinning is strong the associated drag-to-lift ratio could be large and in the range that can account for long time-scale relaxations of glitches [54].

The strong pinning regime may not arise when the vortex lattice is oriented randomly with respect to the basis vectors of the nuclear lattice. A neutron vortex moving in the crust will couple to the phonon modes of the nuclear lattice [55]. The one-phonon processes lead to a weak coupling of the superfluid to the crust with \( \eta \simeq 10 \text{ g cm}^{-1} \text{s}^{-1} \) which implies small dimensionless drag-to-lift ratio \( \eta/\rho_n \kappa \ll 1 \).

The interaction of a neutron vortex moving relative to the nuclei in the crust will generate oscillation modes (Kelvin modes or kelvons) on the vortex and will thus dissipate the kinetic energy of vortex motion into the energy of oscillations [56]. This dissipation can be viewed as mutual friction, because energy and momentum is transferred between the superfluid and the crust. Ref. [56] expresses the drag-to-lift ratio in terms of a dissipation angle \( \eta/\rho_n \kappa = \tan \theta_d \) and finds an upper limit on this angle \( \theta_d \leq 0.7 \). This implies that throughout the crust this dissipation mechanism leads to dissipation angles and spin-up time-scales which are close to the maximal one \( t_{s,\text{max}} \approx (2\Omega)^{-1} \).

Because the relative orientation of the circulation vector of the vortex lattice and the crustal lattice basis vectors may be random or dependent on the history of solidification of the crust and nucleation of the superfluid phase, it remains an open question whether the pinned regime is realized in the crust of a neutron stars. Numerical simulations [53] and/or astrophysical constraints coming from glitch observations may eventually distinguish between the various models.

4 Superfluid hydrodynamics

Let us now discuss how to model a superfluid star on the macroscopic, hydrodynamical scale. In order to do hydrodynamics it is necessary to average over length-scales that are large enough for the constituents to be considered ‘fluids’. In the case of superfluids this means course graining not only over length-scales much larger than the mean free path of the particles, but also over length scales much larger than the characteristic scale of vortices or flux tubes.

A superfluid forms a macroscopic coherent state, therefore it can be described by a macroscopic wave function \( \psi(r) = |\psi(r)| \exp[i\chi(r)] \), which implies that the number density of the superfluid is given by \( |\psi(r)|^2 \) and the momentum is just the gradient of the phase \( p = \hbar \nabla \psi(r) \). This implies immediately that

\[
\nabla \times p = \nabla \times \nabla \chi(r) = 0,
\]

i.e., the superfluid is irrotational. However, rotating superfluids support rotation by forming quantized vortices, above certain critical angular velocity \( \Omega_{c1} \). Indeed, the minimization of the free energy of the fluid in the rotating frame [58], i.e.,

\[
F_r = F - \Omega \cdot J,
\]

where \( J \) the angular momentum of the fluid and \( F \) is its free energy in the laboratory frame, leads to a solution predicting rigid rotation, which is supported by the condensate through the formation of an array of superfluid vortices. The circulation of a single vortex is quantised as

\[
\oint p \cdot d l = m_n \kappa, \quad n = 1, 2, \ldots
\]

where \( \kappa = \pi \hbar/m_n \approx 2 \times 10^{-3} \text{ cm}^2 \text{s}^{-1} \) is the quantum of circulation and \( m_n \) is the mass of the neutron.
In neutron stars the hydrodynamical description of neutron fluid requires thus averaging over length-scales larger than the inter-vortex separation, in order to define the rotation rate by averaging over many vortices. This means that in general a fluid element must be small compared to the radius of the star \((R \simeq 10 \text{ km})\), but large compared to the inter-vortex separation \(d_n \simeq 10^{-3} (P/10 \text{ ms})^{1/2} \text{ cm}\), with \(P = 2\pi/\Omega\) being the spin period of the star.

In a realistic neutron star one has to account for multiple superfluid/superconducting fluids, and a minimal model, such as we now present, must account at least for a neutron-proton conglomerate with the background electrons. Consider first the simpler Newtonian case \([46, 59, 60, 52, 61, 62, 63, 64, 65]\). In the case of interest to us, the indices \(x, y\) label either protons \((p)\) or neutrons \((n)\) and electrons are assumed to form a charge neutralizing background moving with proton fluid on timescales much shorter than those of interest here, such as dynamical and oscillation timescales of milliseconds, or glitch timescales of minutes or more \([60]\).

We will not list the complete set of magneto-hydrodynamics equations here and will concentrate on the ingredients that are needed in the modeling the dynamics of a rotating and magnetized neutron star following Ref. \([66]\). Corrections due to coupling of the superfluid to its excitations are ignored. These can be accounted for by including an additional fluid of excitations to the analysis, which also allows to recover the standard hydrodynamical equations used for superfluid helium \([67, 68]\). Each constituent \(x\) conserves its mass density \(\rho_x\) (in the following a summation over latin indices is assumed)

\[
\partial_t \rho_x + \nabla_i (\rho_x v^i_x) = 0,
\]

and Euler equations for their velocities \(v^i_x\) are given by

\[
(\partial_t + v^i_x \nabla_j)(v^i_x + \varepsilon_x w^{yx}_i) + \nabla_i (\mu_x + \Phi) + \varepsilon_x w^{i}_{yx} \nabla_j v^j_x = (f^{x,\text{mf}}_i + f^{x,\text{pin}}_i + f^{x,\text{mag}}_i)/\rho_x,
\]

where \(w^{yx}_i = v^y_i - v^x_i\), \(\mu_x = \mu_x/m_x\) is the chemical potential per unit mass. The key new aspect of these treatments is the entrainment effect which causes the momentum and the velocities of the components to not be parallel, but rather related by

\[
p^x_i = m_x (v^x_i + \varepsilon_x w^{yx}_i).
\]

This observation was first made by Andreev and Bashkin in the context of charge neutral \(^3\)He-\(^4\)He mixtures \([69]\). Here \(\varepsilon_x = 1 - m^*_x/m_x\) is the entrainment coefficient which accounts for the non-dissipative coupling between the components. (It is related to quantity \(k\) defined in \((40)\) by \(\varepsilon_x = 1 - k\)). Note that the microphysical calculations predict effective mass \(m^*_x \lesssim m_x\) in the core, in which case the entrainment coefficient is positive. The opposite relation holds in the crust of a neutron stars, i.e. \(m^*_x \gg m_x\), due to the band structure of the nuclear lattice and Bragg scattering \([70]\).

The gravitational potential \(\Phi\) in Eq. \((47)\) obeys the Poisson equation

\[
\nabla^2 \Phi = 4\pi G \sum_x \rho_x.
\]

On the right-hand side of Eq. \((47)\) the forces are decomposed as follows. The first term \(f^{x,\text{pin}}_i\) is the force due to pinned vortices, \(f^{x,\text{mf}}_i\) is the mutual friction force mediated by free vortices, and \(f^{x,\text{mag}}_i\) is the force due to the magnetic field, which as we shall see depends strongly on whether the protons are superconducting or not.

We will discuss the contributions from the mutual friction and magnetic forces in detail below in this section, whereas the pinning force is discussed in Sec. 5. For laminar flows and straight vortices, the mutual friction force has the standard Hall-Vienen-Bekarevich-Khalatnikov form \([71, 58]\)

\[
f^{x,\text{mf}}_i = \kappa n_x \rho_x \mathcal{B}' \epsilon_{ijk} \hat{\Omega}^j_n w^{jk}_m + \kappa n_x \rho_x \mathcal{B} \epsilon_{ijk} \hat{\Omega}^j_n klm \hat{\Omega}^k_i w^{jk}_m,
\]

(50)
where \( \Omega_n \) is the angular velocity of the neutrons (a hat represents a unit vector) and \( \mathcal{B} \) and \( \mathcal{B}' \) are the mutual friction coefficients. The neutron vortex density per unit area is \( n_v \), and is linked to the rotation rate (at a cylindrical radius \( \varpi \)) by the relations

\[
\kappa n_v(\varpi) = 2\tilde{\Omega} + \varpi \frac{\partial}{\partial \varpi} \tilde{\Omega}, \quad \tilde{\Omega} \equiv \Omega_n + \varepsilon_n(\Omega_p - \Omega_n),
\]

obtained by imposing that the circulation derived by integrating over a contour the smoothed average momentum is the sum of the quantised circulations of the \( \mathcal{N}(\varpi) \) vortices enclosed, i.e.

\[
\oint \epsilon_{ijk} \nabla_j p_k^n = 2\pi \int_0^\varpi m_n \tilde{\Omega} r dr = \mathcal{N}(\varpi) m_n \kappa,
\]

and we assume here and below singly quantized vortices. The parameters \( \mathcal{B} \) and \( \mathcal{B}' \) depend on the microphysical processes giving rise to the mutual friction, as described in Sec. 3, and can be expressed in terms of a dimensionless drag-to-lift ratio parameter \( \mathcal{R} \) (see also Sec. 3) related to the dimensionfull drag parameter \( \eta \) \([g \text{ cm}^{-1} \text{s}^{-1}]\) as

\[
\mathcal{R} = \frac{\eta}{\kappa \rho_n},
\]

according to

\[
\mathcal{B} = \frac{\mathcal{R}}{1 + \mathcal{R}^2}, \quad \mathcal{B}' = \frac{\mathcal{R}^2}{1 + \mathcal{R}^2}.
\]

To connect to the discussion of the relaxation times computed in Sec. 3, we now express \( \eta \), or equivalently \( \mathcal{B} \) and \( \mathcal{B}' \) in terms of these microscopic time-scales. We distinguish two cases of non-relativistic and ultra-relativistic unpaired excitations, which we assume to be protons \((p)\) or electrons \((e)\). The force exerted by non-superconducting quasiparticles per single vortex is given in general by \([42]\)

\[
f_d = \frac{2}{\tau n_v} \int f(p, v_v) p \frac{d^3p}{(2\pi\hbar)^3} = -\eta v_v,
\]

where \( v_v \) is the velocity of the vortex in a frame co-moving with the normal component and \( f(p, v_v) \) is the non-equilibrium distribution function, which we expand assuming small perturbation about the equilibrium distribution function \( f_0 \), that is, \( f(p, v_v) = f_0(p) + (\partial f_0/\partial \varepsilon)(p \cdot v_v) \). In the low-temperature limit \( \partial f_0/\partial \varepsilon \simeq -\delta(\varepsilon - \varepsilon_F) \), where \( \varepsilon_F \) is the corresponding Fermi energy. After integration one finds

\[
\eta_p = m_p^* n_p \tau_p n_v, \quad \eta_e = \frac{\hbar k_e}{c} n_e \tau_e n_v,
\]

where \( n_p, e \) are the proton/electron number densities. Note that if both electron and proton quasiparticles are present then the contributions from \( \eta_e \) and \( \eta_p \) need to be summed, just as in the case of the ordinary transport coefficients.

The parameters \( \mathcal{B} \) or \( \mathcal{R} \) can be extracted from the timescales obtained in Sec. 3, using

\[
\tau_{mf} = \frac{1}{2\Omega_s(0)\mathcal{B}},
\]

where \( \Omega_s(0) \) is the spin frequency of the superfluid at \( t = 0 \).

From the equations in (47) we can also see that in the case of two constant density rigidly rotating fluids, with moments of inertia \( I_p \) and \( I_n \) and frequencies \( \Omega_n \) and \( \Omega_p \), and neglecting for the sake of the argument the external torques and entrainment, an initial difference in rotation rate \( \Delta \Omega = \Omega_p - \Omega_n \) will be erased by mutual friction according to \( \Delta \Omega(t) = \Delta \Omega(0) \exp(-t/\tau_{su}) \) \([42, 72]\), with

\[
\tau_{su} \approx \left( \frac{I_p}{I_n + I_p} \right) \frac{1}{2\Omega_n(0)\mathcal{B}},
\]
where $\Omega_n(0)$ is the spin frequency of the neutron fluid at $t = 0$.

The expression for the mutual friction in Eq. (51) is appropriate for straight vortices in a triangular array which corresponds to the minimum of the free energy of a rotating superfluid. However, vortices are likely to bend due to their finite rigidity and this effect can easily be included in the expression for the mutual friction, see Ref. [58] for a discussion in single component fluids and its extension to multi-component fluids in Refs. [60, 52, 73]. Furthermore, it is well known from laboratory experiments with superfluid $^4$He that a counterflow along the vortex axis can trigger the Glaberson-Donnelly instability [74, 75] and destabilise the vortex lattice, creating a turbulent tangle. In the case of an isotropic tangle a phenomenological form for the mutual friction, due to Ref. [76], is

$$ f^{GM}_i = \frac{8\pi^2 \rho_n}{3 \kappa} \left( \frac{\xi_1}{\xi_2} \right)^2 \mathbf{B}^3 w_{pn}^2 w_i^{pn}, \quad (59) $$

where the phenomenological parameters are set to $\xi_1 \approx 0.3$ and $\xi_2 \approx 1$. In neutron star interiors the presence of large relative flows between the ‘normal’ and superfluid components and large Reynolds numbers, of the order of $\text{Re} \geq 10^7$ are likely to lead to superfluid turbulence and the presence of a vortex tangle. According to Refs. [77, 78, 73] a polarized tangle is expected in a rotating pulsar.

Let us shift our attention to the magnetic force. The equations of magneto-hydrodynamics for a superfluid and superconducting neutron star have been initially considered in Refs. [46, 59, 60, 52, 61]. More recently detailed studies were carried out Refs. [62, 63, 64] which to various degree also include discussion of the evolution of the magnetic field in a superconducting neutron star. For the current discussion let us restrict our attention to the simplified case of a two component neutron star, in which the electrons are assumed to move with the protons. In this case one finds

$$ f_{p,\text{mag}}^i = \frac{1}{4\pi} \left[ B^i \nabla_j (H_{c1} \hat{B}^j) - B^j \nabla^i H_{c1} \right] - \frac{\rho_p}{4\pi} \nabla^i \left( B^j \frac{\partial H_{c1}}{\partial \rho_p} \right), \quad (60) $$

$$ f_{n,\text{mag}}^i = \frac{1}{4\pi} \left[ W_{n}^j \nabla_j (H_{en} \hat{W}^j) - W_{n} \nabla^i H_{en} \right] - \frac{\rho_n}{4\pi} \nabla^i \left( B^j \frac{\partial H_{c1}}{\partial \rho_n} \right), \quad (61) $$

where a hat indicates a unit vector and we have defined

$$ W_{p}^j = \epsilon^{ijk} \nabla_j (v_k^p + \epsilon_p w_k^{np}) + a_p B^j = n_{vp} k^j_p \quad (62) $$

with $k^j = \kappa \hat{k}^j$ pointing along the local vortex direction, $a_p = e/mc \simeq 9.6 \times 10^3 \text{ G}^{-1} \text{ s}^{-1}$ and $n_{vp}$ the surface density of proton vortices. The total magnetic induction $\mathbf{B}^i$ is the sum of three terms

$$ \mathbf{B}^i = \mathbf{B}_p^i + \mathbf{B}_n^i + b_L^i, \quad (63) $$

where $\mathbf{B}_p^i$ is the contribution due to the proton vortices, $\mathbf{B}_n^i$ is the contribution due to the neutron vortices and $b_L^i$ is the London filed. The modulus of the induction is $B = \sqrt{\mathbf{B}_p \cdot \mathbf{B}_n}$. Generally $|B_p| \gg |B_n| \approx |b_L|$, i.e., the strengths of both the average field due to neutron vortices and the London field is completely negligible compared to neutron star interior magnetic fields ($|B_n| \approx |b_L| \approx 10^{-2} \text{ G}$). The term $H_{en} = 4\pi a_p \varepsilon_{en}/\kappa \approx 10 \times H_{c1}$ plays the role of an effective magnetic field [62] and depends on the energy per unit length of a neutron vortex

$$ \varepsilon_{en} \approx \frac{\kappa^2}{4\pi} \frac{\rho_n}{1 - \varepsilon_n} \log \left( \frac{l_v}{\xi_n} \right), \quad (64) $$

where $\xi_n$ is the coherence length of the vortex and $l_v$ the inter-vortex separation, and we can approximate $\log(l_v/\xi_n) \approx 20 - 1/2 \log(\Omega/100 \text{ rad}/\text{s})$ [73]. To study the coupled evolution of the fluid and magnetic field these equations need to be coupled to the induction equations for the magnetic field:

$$ \partial_t B^i = \epsilon^{ijk} \epsilon_{klm} \nabla_j (v_k^l B^m) - \frac{1}{a_p} \epsilon^{ijk} \nabla_j f_k^e, \quad (65) $$

$$ \begin{align*}
\varepsilon_{en} & \approx \frac{\kappa^2}{4\pi} \frac{\rho_n}{1 - \varepsilon_n} \log \left( \frac{l_v}{\xi_n} \right), \\
\varepsilon_{en} & \approx \frac{\kappa^2}{4\pi} \frac{\rho_n}{1 - \varepsilon_n} \log \left( \frac{l_v}{\xi_n} \right), \\
\varepsilon_{en} & \approx \frac{\kappa^2}{4\pi} \frac{\rho_n}{1 - \varepsilon_n} \log \left( \frac{l_v}{\xi_n} \right), \\
\varepsilon_{en} & \approx \frac{\kappa^2}{4\pi} \frac{\rho_n}{1 - \varepsilon_n} \log \left( \frac{l_v}{\xi_n} \right), \\
\varepsilon_{en} & \approx \frac{\kappa^2}{4\pi} \frac{\rho_n}{1 - \varepsilon_n} \log \left( \frac{l_v}{\xi_n} \right), \\
\varepsilon_{en} & \approx \frac{\kappa^2}{4\pi} \frac{\rho_n}{1 - \varepsilon_n} \log \left( \frac{l_v}{\xi_n} \right), \\
\varepsilon_{en} & \approx \frac{\kappa^2}{4\pi} \frac{\rho_n}{1 - \varepsilon_n} \log \left( \frac{l_v}{\xi_n} \right), \\
\varepsilon_{en} & \approx \frac{\kappa^2}{4\pi} \frac{\rho_n}{1 - \varepsilon_n} \log \left( \frac{l_v}{\xi_n} \right), \\
\varepsilon_{en} & \approx \frac{\kappa^2}{4\pi} \frac{\rho_n}{1 - \varepsilon_n} \log \left( \frac{l_v}{\xi_n} \right), \\
\varepsilon_{en} & \approx \frac{\kappa^2}{4\pi} \frac{\rho_n}{1 - \varepsilon_n} \log \left( \frac{l_v}{\xi_n} \right). \\
\end{align*} $
where, under the assumption that $n_{vp} \gg n_v$ one finds

$$f_e^i \approx \frac{1}{4\pi \rho_p} \frac{\mathcal{R}_p}{1 + \mathcal{R}_p^2} \left[ \mathcal{R}_p B^j \nabla_j (H_{c1} \hat{B}^i) - B \nabla^i H_{c1} - \epsilon^{ijk} B_j \nabla_k H_{c1} e^{ijk} \hat{B}_j B^k \nabla_1 \hat{B}_k \right],$$

(66)

where $\mathcal{R}_p$ is the drag parameter describing the scattering of electrons on proton vortices.

One may also expect an additional contribution to the mutual friction, due to the flux-tube and neutron vortex interaction, of the form

$$f_{pn,mf}^i = \rho_n \kappa_n \frac{C_v}{1 + \mathcal{R}_p^2} \left[ \mathcal{R}_p f_{*}^i + \epsilon^{ijk} \mathcal{N}_j^p f_{*k} \right],$$

(67)

$$f_{*}^i \approx \frac{1}{4\pi a_p \rho_p} \left[ \hat{B}^i \nabla_j (H_{c1} \hat{B}^j) - \nabla^i H_{c1} \right],$$

(68)

where $C_v$ is a phenomenological coefficient that parameterises the strength of the resistive interaction between the proton and neutron vortex arrays, which may also drive the evolution of the magnetic field [79].

The magnetic field configuration of superfluid and superconducting neutron stars has been analysed in detail in recent years both by studying equilibrium models [80, 81, 82] and, more recently, by studying the evolution of the coupled core and crust magnetic fields [83]. In general superfluidity and superconductivity have a strong impact on the timescales for the evolution of the core magnetic field [84], and for strongly magnetised neutron stars ($B \geq 10^{14}$ G) could lead to the expulsion of the toroidal field from the core, and significant rearrangement of the crustal magnetic field on timescales comparable to, or shorter than, the age of the star.

4.1 Relativistic fluids

What has been presented up to now is the Newtonian framework for describing superfluid and superconducting neutron stars. One can develop a similar framework in general relativity [85, 86, 87], for a review see Ref. [88]. First define the number density four-currents of each component

$$n^\mu_x = n_x u^\mu_x,$$

(69)

with normalization $u^\mu_x u^\mu_x = -1$ and Greek letters representing four dimensional space-time indices; summation is implicit over repeated Greek indices. A master function $\Lambda$ is then defined which is a function of the scalars of the system, in particular the number densities $n_x$ and $n_{xy}^2 = n_{yx}^2 = -g_{\mu\nu} n^\mu_x n^\nu_x$, where $g_{\mu\nu}$ is the metric [86]. In the case of co-moving fluids $\Lambda$ is (up to the sign) simply the local thermodynamical energy density. In the general case $\Lambda$ includes relative flows of the fluids by definition.

Having at our disposal the master function we can proceed to define the conjugate momenta in the standard fashion of the Lagrangian theory [85, 86, 87]

$$\pi^\mu_x = g_{\mu\nu} (A^x n_{x}^\nu + A^{xy} n_{y}^\nu),$$

(70)

$$A^x = -2 \frac{\partial \Lambda}{\partial n_x^2},$$

(71)

$$A^{xy} = A^{yx} = - \frac{\partial \Lambda}{\partial n_{xy}^2},$$

(72)

where the effect of entrainment is encoded in the coefficients $A^{xy}$. The stress-energy tensor is defined as

$$T^\mu_\nu = \Psi \delta^\mu_\nu - \sum_x n^\mu_x \pi^x_x,$$

(73)

with the generalised pressure $\Psi$ defined as

$$\Psi = \Lambda - \sum_x n^\mu_x n^x_x.$$
The equations of motion for the fluid can then be written as a set of Euler equations

$$\sum x n^\mu_x \nabla_{[\mu} n_{\nu]} = 0, \quad (75)$$

to be solved together with the Einstein’s equations of general relativity and conservation equations for the individual four-currents of the components

$$\nabla^\mu n^\mu = 0. \quad (76)$$

Note that the solution to the equations above automatically satisfies the energy-momentum conservation

$$\nabla^\mu T^\mu_\nu = 0.$$

Relativistic superfluid hydrodynamics of the type described above was formulated initially in Refs. [85, 86, 87]. The corresponding equations where adapted to differentially rotating relativistic superfluid neutron stars in Ref. [89] in the case of cold equations of state. Equilibrium configurations of two-fluid (superfluid and normal component featuring) neutron stars were constructed in Refs. [90, 91]. Accounting for heat transport, dissipation and in particular vortex-mediated mutual friction is more challenging in general relativity than in Newtonian physics [92], as standard approaches by Refs. [93, 94] lead to causality and stability problems. While Refs. [95, 96] resolve some of these issues, and progress has been made making maximal use of the variational approach [97] the general relativistic formulation is, however, not complete. Some recent advances in the problem of mutual friction and vortex motion in a relativistic framework can be found in Refs. [98, 99].

5 Pinning effects

In the previous discussion we have considered mainly vortices that are free to move with respect to the fluid components and experience a standard drag force, linear in the difference in velocity between said component and the vortices themselves. (A brief discussion of pinning in the crust was given at the end of Sec. 3 to complete the discussion of microphysics of mutual friction.) However the interaction between vortices and ions in the crust, or flux-tubes in the core, can be strong enough to balance the Magnus force, and ‘pin’ the vortices, preventing them from moving, similarly to static friction. We now turn to the detailed discussion of the pinning effects in the superfluid core and in the crust of the neutron star.

5.1 Vortex-flux tube pinning and interactions

In the outer core of the neutron star, where neutrons are superfluid and protons are expected to form a type II superconductor, the magnetic field is confined to flux-tubes with flux quantum $\phi_0 = \pi \hbar c/e \approx 2 \times 10^{-7} \text{ G cm}^2$.

The neutron vortices thus co-exist with an array of far more numerous flux tubes, with average spacing

$$l_\phi = \frac{B}{\phi_0} \approx 4 \times 10^3 \left( \frac{B}{10^{12} \text{ G}} \right)^{-1/2} \text{ fm.} \quad (77)$$

Proton flux tubes are also less rigid than neutron vortices. Indeed the tension of a neutron vortex is given by

$$T_v = \frac{\rho_n \kappa^2}{4\pi} \ln \left( \frac{l_v}{\xi_n} \right) \approx 10^9 \left( \frac{\rho_n}{2 \times 10^{14} \text{ g cm}^{-3}} \right) \text{ erg cm}^{-1} \quad (78)$$

with the typical inter-vortex separation $l_v \approx 10^{-3} (\text{P}/10 \text{ ms})^{1/2} \text{ cm}$. The flux-tube tension is given by [100]

$$T_\Phi = \left( \frac{\phi_0}{4\pi \lambda} \right)^2 \ln \left( \frac{\lambda}{\xi_\Phi} \right) \approx 10^7 \left( \frac{m_p^*}{0.5} \right)^{-1} \left( \frac{x_p}{0.05} \right) \left( \frac{\rho_n}{2 \times 10^{14} \text{ g cm}^{-3}} \right) \text{ erg cm}^{-1}, \quad (79)$$

where $m_p^*$ is the effective mass of the protons, $\xi_\Phi \approx 20 \text{ fm}$ the coherence length of a proton vortex and $\lambda \approx 100 \text{ fm}$ is the London penetration depth of the magnetic field. Neutron vortices will thus be immersed in a tangle of far more numerous flux tubes.
The interaction between neutron vortices and flux tubes changes the energy of the system in two ways: there will be a change in condensation energies, as the superfluid and superconducting cores overlap, and also a contribution from the interaction between the magnetic fields of the two vortices, as described in Sec. 3. If the overlap reduces the energy of the overall configuration neutron vortices are effectively ‘pinned’ to flux tubes, to some extent in the same way as they can be pinned to ions in the crust.

The contribution due to the change in condensation energy is \[ \Delta E_c \approx 0.13 \text{MeV} \left( \frac{\Delta_p}{1 \text{MeV}} \right) \left( \frac{x_p}{0.05} \right)^{-1} \left( \frac{m_n^*/m_n}{1} \right)^{-2} \left( \frac{m_p^*/m_p}{0.5} \right)^{-1} \] (80)

with \( m_n^* \) the neutron effective mass, \( \Delta_p \) the proton pairing gap, while the change in energy due to the magnetic interaction \[ \Delta E_{\text{mag}} = 2B_i^\phi B_\Phi^i \pi^2 \lambda \] (81)

with \( B_i^\phi \) the magnetic field along the neutron vortex, \( B_\Phi^i \approx 10^{15} \text{G} \) that along the proton vortex, and \( \lambda \) is the overlap length between vortex and flux-tube. Keeping in mind that neutron vortices can be considered rigid on lengthscales approximately 100 times larger than those over which a fluxtube can bend, we can average the expression in (81) to obtain \[ \Delta E_{\text{mag}} \approx 10 \left( \frac{m_p^*/m_p}{0.5} \right)^{-1/2} \left( \frac{|m_p - m_p^*|}{m_p^*} \right)^{1/2} \frac{x_p}{0.05} \left( \frac{\rho_n}{2 \times 10^{14} \text{g/cm}^3} \right)^{1/2} \text{MeV}. \] (82)

In general one may expect also a dependence on the inclination angle of the global magnetic field with the rotation axis, and on vortex tension, see Refs. [105, 106] for a discussion of toroidal flux tubes and the effect of bending and pinning in this case. Nevertheless the above averaged expression illustrates that the pinning force will be sizeable. Pinned vortices can thus ‘push’ magnetic flux tubes, possibly winding up a strong toroidal component of the magnetic field in magnetars [107] and leading to the long term expulsion of magnetic flux from the star as the vortex array expands while the star spins down [108, 79, 109, 110].

If, on the other hand, the pinning force cannot balance the Magnus force, vortices are forced to cut through flux tubes. This process will excite Kelvin waves along the vortex, leading to strong dissipation and mutual friction [56, 34]. The energy released at every vortex/flux tube intersection is \[ \Delta E_{\text{vf}} = \frac{2}{\pi} \rho_n \kappa \lambda \left( \frac{v_\lambda}{w_{pn}} \right)^{-1/2}, \] (83)

where \( v_\lambda = \frac{\hbar}{2m_K \lambda} \approx 10^9 \text{cm/s} \) is the characteristic velocity of a Kelvon of effective mass \( m_K \). The energy loss rate per unit volume is thus \[ \dot{E} = \frac{n_c w_{pn}}{l_\phi^2} \Delta E_{\text{vf}}. \] (84)

By equating the expression in (84) to the work done by the mutual friction force we can derive the drag coefficient for vortex/flux tube cutting [111]

\[
\mathcal{R} = \mathcal{R}_0 \left( \frac{v_\lambda}{w_{pn}} \right)^{3/2},
\]

\[
\mathcal{R}_0 = \frac{2}{\pi} \left( \frac{\Delta E_{\text{mag}}}{\rho_n \kappa \lambda v_\lambda} \right)^2 \approx 1.3 \times 10^{-10} \left( \frac{B}{10^{12} \text{G}} \right). \] (85)

Note that the mutual friction coefficient is now velocity dependent, and the lower the relative velocity the larger the friction. In practice as soon as vortices start moving they are unlikely to be able to continue and the system will move back towards the pinned state [111].
5.2 Pinning-repinning of vortices

Let us consider the forces acting on a massless vortex segment. If the vortex is free the force balance equation, averaged over a number \( n_v \) of vortices per unit area, is

\[
\kappa n_v \epsilon_{ijk} \hat{\Omega}^j (v^{k}_n - v^{k}_v) + \kappa n_v \mathcal{R} (v^p_v - v^p) = 0. \tag{86}
\]

One can solve Eq. (86) for the vortex velocity \( v^i_v \) (the direction of which which will depend on \( \mathcal{R} \)) and obtain the standard form of mutual friction in Eq. (50). Pinning to the ions in the crust or flux tubes in the core modifies the force balance equation, because now some of the vortices may be immobilized by the pinning. When averaging over large number of vortices we can assume that only a fraction \( \gamma \) of them is free, leading to a force balance equation of the form:

\[
\gamma \kappa n_v \epsilon_{ijk} \hat{\Omega}^j (v^{k}_n - v^{k}_v) + \kappa n_v \mathcal{R} (v^p_v - v^p) + (1 - \gamma) \kappa n_v \epsilon_{ijk} \hat{\Omega}^j (v^{k}_n - v^{k}_p) + f^\text{pin}_i = 0 \tag{87}
\]

where \( f^\text{pin}_i \) now balances the Magnus force on the \( (1 - \gamma) n_v \) pinned vortices, for which we have assumed that \( v^i_v \) \( \approx \) \( v^i_p \). The force acting on the fluids is thus \( f^\text{n, pin}_i = -f^\text{p, pin}_i = f^\text{pin}_i \).

The quantity that is needed for the equations of motion in Eqs. (47) and (87) is thus the pinning force per unit length acting on a vortex, which is highly uncertain. The pinning force per pinning site can, in fact, be quite readily obtained theoretically, as it depends only on the difference in energy between the configuration where the vortex overlaps with an individual pinning site, and that in which it is outside. Nevertheless even in this case significant uncertainties remain, with different results in the literature disagreeing also on whether the interaction is attractive or repulsive, i.e. on whether one has pinning to nuclei or interstitial pinning \([112, 113, 114]\). Note, however, that to understand the dynamics of the fluid we are mainly interested in the magnitude of the pinning force, and not its sign.

For the case of pinning of neutron vortices to nuclei in the crust the maximum pinning force acting on a vortex can be estimated as \([115]\)

\[
|F^\text{pin}| \approx \left( n^\text{out} E^\text{cond.}_{\text{out}} - n^\text{in} E^\text{cond.}_{\text{in}} \right) \frac{V}{\xi_n}, \tag{88}
\]

where \( V \) is the volume of a nuclear cluster and \( \xi_n \) is the coherence length of the neutron vortex, which defines the scale of the interaction, ‘in’ and ‘out’ refer to quantities taken within the nuclear cluster and in the free neutron gas, \( E^\text{cond.} \approx 3 \Delta_n^2 / 8 \varepsilon_F \) is the condensation energy of neutron fluid per unit volume. The pinning force per unit length depends however on the difference in energy between different configurations of a vortex that encounters several pinning sites and may bend to reduce its energy. It is thus, generally a function of of the orientation of the vortex with respect to the lattice

\[
f^\text{pin}(\theta, \phi) = \frac{|F^\text{pin}| \Delta n(\theta, \phi)}{l_T}, \tag{89}
\]

where the angles \( (\theta, \phi) \) are taken with respect to a reference axis and \( l_T \approx 10^2 - 10^3 R_{WS} \) is the length scale over which a vortex can bend, determined by the tension in Eq. (78), and \( R_{WS} \) is the radius of the Wigner Seitz cell. It was pointed out early on in Ref. \([116]\) that vortex rigidity plays an important role, as for an infinite vortex all configurations would be energetically equivalent (i.e. they would intercept the same number of pinning sites) and there would thus be no pinning at all.

Recently calculations have been carried out for a realistic setup by Ref. \([117]\), who averaged the expression in Eq. (89) over all orientations of a vortex with respect to a BCC lattice, and by Ref. \([53]\) who studied the interactions of a vortex with a pinning site in the time dependent local density approximation. These studies find that vortex tension and bending is indeed fundamental for the physics of pinning. The calculations differ in several aspects, the most notable of which is that Ref. \([53]\) obtains interstitial pinning while Ref. \([117]\) finds nuclear pinning, but the authors do not include long range repulsive terms due to the Bernulli force in the
Figure 3: Critical lag $\Delta\Omega_c$ for unpinning for a 1.4$M_\odot$ neutron star described by the GM1 equation of state as described in [118, 117].

superfluid [105]. The angular momentum reservoir is, however, independent of the sign of the pinning force, and both sets of authors obtain pinning forces that are dynamically significant and can explain the observed glitching activity of the Vela pulsar. By simply balancing the Magnus force with the pinning force Ref. [117] finds that velocity differences up to $|w^{\text{pn}}| \approx 10^4$ cm s$^{-1}$ can be sustained in the crust, which can explain the observed glitching activity of the Vela and other pulsars, also in the presence of strong entrainment [117]. An example of the critical lag profile in a neutron star, obtained for the pinning of Ref. [117], is shown in Fig. 3.

The results of dynamical simulations can also be used to study the problem of repinning of free vortices. This is a fundamental issue, as while estimating when a pinned vortex will unpin allows us to estimate how much angular momentum can be stored and released in a glitch, calculating when a vortex will re-pin allows us to understand whether vortices will ‘creep’ out, gradually spinning down the star, or expel vorticity in ‘avalanches’.

Macroscopic vortex dynamics in a spinning down container was studied on the basis of Gross-Pitaevskii equations in Refs. [119, 120], who have shown that the main unpinning trigger for vortices is the proximity effect, i.e. the change in Magnus force due to the motion of neighbouring vortices, which can lead to forward or backward propagating vortex avalanches. These can, in turn, trigger a glitch. Computational limitations, however, constrain these simulations to a small number of vortices (typically of the order of hundreds) separated by at the most tens of pinning sites. This is in contrast with the situation encountered in neutron stars where a large number $N_v \gtrsim 10^{12}$ of vortices must move together in a glitch. These are on average separated by a large number (of the order of $10^{10}$) pinning sites. Nevertheless, the external spin-down drives the system by increasing the lag between the superfluid and normal fluid to the critical value for unpinning. It is thus crucial to understand whether the system can self-adjust and hover close enough to the critical lag that vortices can unpin and skip over many pinning sites in order to knock on neighbouring vortices and allow an avalanche to propagate.

The problem of vortex re-pinning was investigated in Refs. [121, 122], by considering vortex motion in a parabolic pinning potential. Pinning of a moving vortex in a random potential was also studied in Ref. [123]. In particular Ref. [122] calculated the mean-free path of a straight vortex for scattering off cylindrical pinning sites, and found that the main parameters that control repinning are the strength of the mutual friction and, crucially, how close the system is to the critical threshold lag for unpinning. From Fig. 4 we can deduce that if the system is within 5% of the critical lag for unpinning, a vortex can move a distance comparable to the inter-vortex separation and knock on other vortices, causing an avalanche, for realistic values of the mutual friction. Studies by Refs. [124, 125, 126] have also shown that the geometry of the lattice can play a crucial role,
with a more disordered lattice behaving like a plastic system, in which unpinned and pinned vortices coexist, and an ordered lattice behaving like an elastic system in which there is a sharp transition to mass unpinning.

Before moving on, let us note that if protons form a type II superconductor in the core, as was discussed in Sec. 5.1, the energy cost of vortex/flux-tube cutting can lead to pinning. A simple estimate of the pinning force per unit length $f_{\text{pin},\phi}$ can be obtained from the expression for the overlap energy in (81)

$$f_{\text{pin},\phi} \approx \frac{E_{\text{mag}}}{\lambda l_{\phi}},$$

where $\lambda$ is the London penetration length, $l_{\phi}$ the distance between fluxtubes and $E_{\text{mag}}$ is defined in Eq. (82). The force (90) is balanced by the Magnus force for a critical velocity $|w_{\text{pin}}| \approx 5 \times 10^{12} (B/10^{12} \text{G})^{1/2}$ cm s$^{-1}$, which indicates that this force could play an important role in a glitching model based on unpinning of vortices in the core. Note, however, that the estimate (90) does not account for the effect of averaging over different orientations of the vortex lattice, and is thus an upper limit on the pinning force.

6 Macrophysics of superfluidity in neutron stars

6.1 Glitches and post-glitch relaxations

Pulsar glitches are sudden spin up events in the otherwise steadily decreasing rotational frequency of pulsars. These were observed in the Vela pulsar soon after the discovery of radio pulsars [127, 128]. The initial jump in
frequency in the case of the Vela pulsar is instantaneous to the accuracy of the data (with the best upper limit of $\tau_r \lesssim 40$ s coming from the Vela 2000 glitch [129]), but is often accompanied by an increase in spin-down rate that relaxes back towards the pre-glitch values on longer timescales (ranging from minutes to months). The long-time scales of relaxations following glitches were taken as an evidence for the presence of a loosely coupled superfluid component in the star [31].

Initial studies attributed the long relaxation times to the slow coupling of the core of the star due to the weak coupling of vortices to the electron fluid according to Eq. (39), see Refs. [31, 41]. Anderson and Itoh [130] put forward the hypothesis that the glitches are linked to a pinned superfluid in the star, that is decoupled from the observable ‘normal’ component, and whose sudden re-coupling (due to unpinning) leads to an exchange of angular momentum and a glitch [130].

Following the idea of a pinned superfluid in the crust of a neutron stars [130] the initial models of glitches and post-glitch relaxation concentrated on the details of the physics of vortex pinning and unpinning mainly in the crust and the fits of the models to the observed behaviour of the Vela pulsar [131, 132, 133, 134]. The sudden unpinning of neutron vortices was attributed to the a glitch and their slow relaxation via thermal creep against pinning barriers as the model of post-glitch relaxation. The ‘creeping’ velocity of neutron vortices is given in these models by

$$v_r \approx v_0 \exp(-E_a/k_BT),$$  \hfill (91)

where $v_0 \approx 10^7$ cm s$^{-1}$ [131], $k_B$ is Boltzmann’s constant and $E_a$ is the activation energy for unpinning [135]. This latter quantity in a first approximation can be taken as $E_a \approx E_p (1 - \Delta \Omega/\Delta \Omega_c)$, with $E_p$ the pinning energy, $\Delta \Omega$ the lag between the superfluid neutrons and the crust, and $\Delta \Omega_c$ the critical lag for unpinning. The equations of motion for the frequency of the observable ‘normal’ component of the star $\Omega_p$ are

$$I_p \dot{\Omega}_p = N_{ext} + \sum_i I_i^i \frac{2\Omega_i^{\text{crit}}}{\varepsilon} v_{r}^{i},$$  \hfill (92)

where $I_p$ is the moment of inertia of the crust and all components tightly coupled to it, $\varepsilon$ the cylindrical radius, $N_{ext}$ is the external torque, and the superfluid is divided into a number $i$ of different regions, with associated moment of inertia $I_i^i$, angular velocity $\Omega_i^i$ and vortex velocity $v_{r}^{i}$, calculated from (91). The solutions to Eqs. (92) admit two regimes. If the steady state lag $\Delta \Omega$ is much smaller that $\Delta \Omega_c$ then the response of the system is linear and the region contributes to the frequency relaxation exponentially after a glitch [136]. This is also the regime that can be modelled in terms of mutual friction coupling due to a small number of free vortices [137, 138, 139, 140]. If $\Delta \Omega \approx \Delta \Omega_c$ the response will be nonlinear, and the contribution of the region to the relaxation takes the form of a Fermi function [131, 54]. Recent work has also shown that the non-linear response of creep to glitches can be used to interpret the inter-glitch behaviour of the Vela pulsar, and predict the occurrence of the next glitch [141]. The creep model has been more recently extended to the scenario where neutron vortices creep against the core flux-tubes [54, 105], with essentially the same physical picture of post-jump relaxation involved. Pinning of superfluid neutron vortices to proton vortices in the core naturally extends the reservoir of angular momentum available for a glitch thus potentially explaining the observed activity of the Vela pulsar [142], although a large amount of pinned vorticity in the core is not consistent with linear models for the recovery of Vela glitches [143]. The original crustal vortex creep model relied on the assumption, derived from the short relaxation times found in Ref. [47], that the core is coupled to the crust on short dynamical time-scales, which are unobservable in glitches and their relaxations.

An alternative to crust-based models is the vortex cluster model of dynamics of superfluid neutron vortices in the core of a neutron star, where the coupling between the superfluid and normal component occurs on much longer time-scales [52]. Models of pulsar glitches and post-glitch relaxations based on the superfluid core rotation in the absence of pinning between the neutron vortices and flux-tubes were developed within the vortex cluster model and applied to Vela glitches in Ref. [144]. The glitch itself can arise in these models through the interaction of vortex clusters with the crust-core interface [145]; the core moment of inertia alone was estimated to be sufficient to account for both glitches and post-glitch relaxations [144, 145].
If the system hovers close to the critical unpinning threshold vortex, avalanches may propagate in the neutron star interior [122], which are thus a viable mechanism for triggering a glitch [134, 120]. A glitching pulsar would thus behave as a self organised critical system, in which slowly increasing global stresses (due to the external spin-down torque that drives the increase in lag and thus Magnus force) are released rapidly and locally via nearest neighbour interactions between vortices. Such a system is scale invariant and one expects the distribution of sizes of the avalanches to be a power-law, and the distribution of waiting times an exponential. This is generally what is observed in the pulsar population [146], with the notable exception of the Vela pulsar and PSR J0537-6910, which exhibit a quasi-periodicity in their glitching behaviour and for which the glitches can be predicted [147, 141]. This behaviour in the case of PSR J0537 is generally attributed to the presence of crust-quakes, but may also be the consequence of the timescale of the external driving (the spin-down) being short compared to the time-scale on which stress is released locally by the vortices. Another system in which the distribution of glitch sizes appears to deviate from a power-law is the Crab pulsar [148] for which there appears to be a cut off for small glitch sizes. This behaviour is, however, natural if one considers not only the exchange of angular momentum due to vortex motion, but also the coupling timescale due to mutual friction. Ref. [149] investigated this by considering a multifluid system described by Eq. (47), in which however only a number \( \gamma n_v \) of vortices is free at any given time, with \( \gamma \leq 1 \). The value of \( \gamma \) is randomly drawn from a power-law distribution after a waiting time \( t_w \) (randomly drawn from an exponential distribution) and the results of these simulations show that small values of \( \gamma \) not only correspond to a small amount of angular momentum, but also to an effective reduction in the average mutual friction and an increased coupling timescale. In this case the event does not appear as a sudden jump in frequency, but is more gradual and closer to timing noise, thus not being recognised as a glitch by detection algorithms and producing a cutoff in the size distribution for small glitches, see Fig. 5.

In addition to the mechanisms mentioned above there exist a number of other candidate mechanism for glitches. As already mentioned crust quakes may drive glitches [150, 151, 152] as has been suggested for PSR J0537-6910 and also the Crab pulsar [147], and hydrodynamical instabilities may also lead to a glitch [153, 154].

In the above discussion of the pinning-based trigger mechanisms for a glitches we tacitly assumed that pinning occurs between vortices and ions in the crust or vortices and flux-tubes in the core. The extent and
location of the pinning region can, however, be studied more quantitatively by examining both the size of
the maximum glitch recorded in a pulsar [155, 118], and its ‘activity’ $A$, i.e the amount of spin-down that is
reversed by glitches during observations

$$A = \frac{1}{t_{\text{obs}}} \sum_i \frac{\Delta \Omega_p^i}{\Omega_p}$$

where the sum is performed over all recorded glitches in a time $t_{\text{obs}}$ and $\Delta \Omega_p^i$ is the recorded size of glitch $i$. From angular momentum conservation over a glitch one has that the ratio between the moment of inertia of
the superfluid reservoir $I_n$ and that of the ‘normal’ component $I_p$ is

$$\frac{I_n}{I_p} \approx -\frac{\Omega_p}{\Omega_p} A (1 - \varepsilon_n)$$

Refs. [156, 157] noted that in the presence of strong neutron entrainment $\varepsilon_n$, such as is predicted in the crust
where $\varepsilon_n \approx 10$ due to Bragg scattering [70], the crust cannot store enough angular momentum to explain the
observed activity of the Vela pulsar (unless the star has a very small mass $M \lesssim 1 M_\odot$). The core must be
involved in the glitch mechanism. The neutron superfluid is, in fact, expected to extend into the core and
models that extend the reservoir beyond the crust can predict the observed activity of the Vela and other
pulsars [158, 159]. The observed activity of a pulsar, together with the size of its maximum glitch, can then
temporally be used to determine the mass of a glitching pulsar and constrain the equation of state of dense
matter [159, 160].

Finally let us note that the effect of both classical and superfluid turbulence have been ignored in the above
discussion, but may have an impact on glitch physics. Transitions between turbulent and laminar regimes could
explain the short spin up timescales and long inter-glitch timescales [77, 78] and shear-driven turbulence can
contribute to low frequency fluctuations in the spin of the star, i.e. so-called ‘timing noise’ [161].

### 6.2 Oscillations in superfluid stars

Neutron stars are expected to be prolific emitters of gravitational waves [162, 163] and in particular there
are several modes of oscillation of the star that could lead to detectable emission. In particular the most
promising modes for ground based detection are the $f$-mode, or fundamental mode, and the $r$-mode, analogous
to Rossby waves in the ocean, which can be driven unstable due to gravitational wave emission and grow to
large amplitudes [164, 165]. In order to assess the detectability of these signals it is thus crucial to understand
in which region of parameter space the modes can be driven unstable by gravitational wave emission, and in
which region, on the other hand, they are rapidly damped by viscosity. At high temperatures ($T \gtrsim 10^9$
K) bulk viscosity is the main damping mechanisms and matter is not expected to be superfluid. At lower
temperatures, however, superfluidity has a strong impact on the damping. On the one side superfluidity leads
to a suppression in the neutron-neutron and neutron-proton scattering events that gives rise to shear viscosity,
which in this case is mainly due to electron-electron scattering and reduced with respect to the case in which
neutrons are normal [166]. On the other superfluidity opens a new dissipative channel by allowing for vortex
mediated mutual friction.

It has been established early on that the doubling of the degrees of freedom in the superfluid component
doubles the number of oscillation modes; these have been studied in Newtonian theory [167, 168, 169] and
general relativistic setting [170]. The $l \leq 2$ modes of incompressible superfluid self-gravitating fluids (both
axially symmetric and tri-axial) were studied in Refs. [169, 171] in the presence of mutual friction and viscosity
using the tensor virial method. In the absence of shear viscosity the mutual friction can be eliminated from the
equations describing the center-of-mass motions of the two fluids and it acts to damp only the relative motions of
the two fluids. Shear viscosity which acts only in the normal fluid breaks the symmetry of the Euler equations for
the normal fluid and superfluid and, therefore, couples these two sets of modes [169, 171]. These initial studies
were followed by studies which obtained the analogues of the $f$ and $p$ modes in superfluid neutron stars [172] and
included the general relativity in the mode description in the case of non-rotating stars [173, 174]. Furthermore, much work has been concentrated on the \( r \)-modes of the superfluid neutron stars [175, 176, 177, 178, 179, 180] which may play a key role in the dynamics and stability of rapidly rotating neutron stars. Progress has been made in understanding the oscillations modes at finite temperature [181, 182, 183] as well as the influence of crust elasticity [184].

Let us examine this problem in detail. The linearised version of the equations of motion in (47) can be written in terms of two sets of degrees of freedom, one that represents the ‘total’ motion of the fluid (and would be present also in a normal, non-superfluid, star), and one that represents the counter-moving motion. Let us consider the case in which mutual friction is the only dissipative mechanism acting on the system. By introducing the total mass flux

\[
\rho \delta v^k = \rho_n \delta v^k_n + \rho_p \delta v^k_p
\]

with \( \rho = \rho_n + \rho_p \) and where \( \delta \) represents Eulerian perturbations, we can write a ‘total’ Euler equation for the total velocity \( v^i \), in a frame rotating with angular velocity \( \Omega \),

\[
\partial_t \delta v_i + \nabla_i \delta \Phi + \frac{1}{\rho} \nabla_i \delta p - \frac{1}{\rho^2} \delta \rho \nabla_i p + 2 \epsilon_{ijk} \Omega^j \delta v^k = 0
\]

where the pressure \( p \) is obtained from

\[
\nabla_i p = \rho_n \nabla_i \tilde{\mu}_n + \rho_p \nabla_i \tilde{\mu}_p.
\]

We also have the standard continuity equation

\[
\partial_t \delta \rho + \nabla_j (\rho \delta v^j) = 0.
\]

As already mentioned these are identical to the perturbed equations of motion for a single fluid system, and quite notably the mutual friction term drops out of the equations. The mutual friction naturally appears in the equations of motion for the second degree of freedom, which we can write in terms of the perturbed relative velocity \( \delta w^i = \delta v^i_p - \delta v^i_n \). The ‘difference’ Euler equation is

\[
(1 - \varepsilon_n x_p^{-1}) \partial_t \delta w_i + \nabla_i \delta \beta + 2 \tilde{B}' \epsilon_{ijk} \Omega^j \delta w^k - \tilde{B} \epsilon_{ijk} \hat{\Omega}^j \epsilon^{klm} \Omega_l \delta w_m = 0,
\]

where we have defined the local deviation from chemical equilibrium

\[
\delta \beta = \delta \tilde{\mu}_p - \delta \tilde{\mu}_n
\]

and \( \tilde{B}' = 1 - B' / x_p \) and \( \tilde{B} = B / x_p \) with \( x_p = \rho_p / \rho \) the proton fraction. The continuity equation for the proton fraction is:

\[
\partial_t \delta x_p + \frac{1}{\rho} [x_p (1 - x_p) \rho \delta w^j] + \delta v^j \nabla_j x_p = 0
\]

The degrees of freedom are thus explicitly coupled unless \( x_p \) is constant. In general we do not expect to find any mode of oscillation in a realistic neutron star that is purely co-moving, and thus all modes are affected, to some extent, by mutual friction.

We can define a conserved energy for the system by first defining a ‘kinetic’ term as an integral over a volume \( V \)

\[
E_k = \frac{1}{2} \int \left[ |\delta v|^2 + (1 - \varepsilon_n / x_p) x_p (1 - x_p) |\delta w|^2 \right] \rho \, dV
\]

and a ‘potential’ term

\[
E_p = \frac{1}{2} \int \left\{ \rho \left( \frac{\partial \rho}{\partial p} \right)_\beta |\delta h|^2 + \left( \frac{\partial \rho}{\partial \beta} \right)_p [2 \text{Re}(\delta h \delta \beta^*) + |\delta \beta|^2] - \frac{1}{4 \pi G} |\nabla_i \delta \Phi|^2 \right\} dV,
\]

where a star represents complex conjugation. The perturbed equations of motion we have written down explicitly include dissipative terms due to mutual friction. It is, however, instructive to consider the non
dissipative case and ignore the contribution due to mutual friction. In this case \( \partial_t(E_k + E_p) = 0 \) and one can solve the problem for a mode with time dependence \( e^{i\omega t} \). If damping is weak, and proceeds on a timescale \( \tau \) much longer than the period of the mode, i.e. one has \( \omega = \omega_r + i/\tau \), with \( \tau >> 1/\omega \), we can use the solution of the non-dissipative problem to estimate the damping timescale as

\[
\tau = \left| \frac{2(E_p + E_k)}{\partial_t E} \right|, \tag{104}
\]

where \( \partial_t E \) is obtained from a dissipation integral, in which the dissipative terms due to viscosity are evaluated using the non dissipative solution. In the case of mutual friction one can see that this takes the form

\[
\partial_t E_B = -2 \int \rho_n B \Omega [\delta^m_i - \hat{\Omega}^m \hat{\Omega}^i] \delta w^i \delta w_m \, dV. \tag{105}
\]

If the damping timescale is sufficiently short, the estimate of \( \tau \) in (104) matches that obtained from the full mode solution. Often, however, the full solution of the dissipative problem is not available, and (104) is the only way to assess the impact of viscosity.

Finally one must calculate the timescale for gravitational waves to drive the mode. The energy lost to gravitational waves can be obtained from a multipole expansion

\[
\partial_t E_{gw} = -\omega_r \sum_l N_l \omega_l^{2l+1} (|\delta D_{lm}|^2 + |\delta J_{lm}|^2), \tag{106}
\]

where \( \omega_l \) is the frequency of the mode in the inertial frame \( N_l = (4\pi G)(l+1)(l+2)/\{c^{2l+1}l(l-1)[(2l + 1)!!]^2 \} \) and the mass multipoles are

\[
\delta D_{lm} \approx \int \delta T_{00} Y_{lm}^* r^l \, dV \tag{107}
\]

and the current multipoles are:

\[
\delta J_{lm} \approx -\int \delta T_{0j} Y_{j,lm}^* \, dV, \tag{108}
\]

with \( Y_{j,lm}^* \) the magnetic multipoles and \( T_{\alpha\beta} \) is the two-fluid stress energy tensor defined in section (??). We refer the interested reader to [187, 185] for a detailed discussion of the calculation, and simply point out that in general

\[
T_{00} \approx \delta \rho, \tag{109}
\]
\[
T_{0j} \approx \rho \delta v_j + \delta \rho v_j, \tag{110}
\]

i.e., only the co-moving degree of freedom radiates gravitationally to leading order in rotation.

We are now potentially equipped to calculate the driving timescale \( \tau_{gw} \) and compare it to the mutual friction damping timescale \( \tau_B \) for realistic neutron star modes. Clearly if \( \tau_B \lesssim \tau_{gw} \), mutual friction will damp the mode faster than gravitational radiation can drive it, and suppress the instability, while in the opposite case a mode can grow to large amplitudes and radiate gravitationally.

Let us first of all examine the fundamental, or \( f \)-mode. This is essentially a ‘surface’ mode for which the frequency \( \omega_f \sim \bar{\rho}^{1/2} \), with \( \bar{\rho} \) the average density of the star. In the case of the \( f \)-mode both Newtonian and Relativistic studies have shown that mutual friction completely suppresses the gravitational-wave driven instability below the superfluid transition temperature [185, 188]. It may thus play a role in hot newly born neutron stars [189, 190] but is unlikely to be active in older, colder stars.

The situation is different for the \( r \)-mode. To first order in rotation this mode is purely axial and comoving, leading to a single multipole solution of the form:

\[
\delta v^j = \left[ \sum_l \left( \frac{m}{r^2 \sin \theta} U_l Y_l^m \hat{v}^j_{\theta} + \frac{i}{r^2 \sin \theta} U_l \partial_\theta Y_l^m \hat{v}^j_{\phi} \right) \right] \exp (i\omega_0 t) \tag{111}
\]

28
where $U_l = A r^{l+1}$ with $A$ a constant, $\omega_0 = 2m/l(l+1)$ is the frequency of the mode in the rotating frame, and we are using spherical coordinates, with $\hat{e}_\theta^l$ and $\hat{e}_\phi^l$ unit vectors and $Y^m_l$ are spherical harmonics. The $l = m = 2$ r-mode thus provides the strongest contribution to gravitational wave emission, as in a single fluid star. However, as we have mentioned, the co-moving motion couples to the counter-moving degrees of freedom at higher orders in rotation, leading to mutual friction damping [191].

For the r-mode standard mutual friction due to electron scattering of vortex cores has little effect on the instability [192, 179]. Strong mutual friction due to vortex/flux-tube cutting in the core can, however, have a strong impact on the instability and damp it in a large section of parameter space. Furthermore strong dissipation due to vortex-flux tube interactions limits the growth of the mode and sets an effective saturation amplitude for it that may be smaller that the saturation amplitude due to non-linear couplings to other modes [111].

Furthermore [175, 193, 194, 195] have suggested that for specific temperatures there can be avoided crossings between the superfluid r-modes and other inertial modes, leading to enhanced mutual friction damping. This is an interesting scenario as it would reconcile our understanding of the r-mode instability in superfluid neutron star with observational data on spins and temperatures of neutron stars in Low Mass X-ray Binaries [196, 197, 198], and also predicts the presence of hot, rapidly rotating neutron stars [199, 200].

Finally [154, 201] have pointed out that in the presence of pinned vorticity the r-mode can grow unstable if a large lag develops between the superfluid and the crust, also possibly triggering a glitch.

### 6.3 Long-term variabilities

In addition of the phenomena discussed above neutron stars exhibit long-term variability, which has been attributed to free precession [202, 203]. Theoretical studies of precession in neutron stars containing a superfluid component indicate that the analogue of the free precession in classical systems will be damped if the superfluid is coupled strongly to the normal component [204, 205]. In addition to this analogue of the classical precession, a fast precession mode appears, which is associated with the doubling of the degrees of freedom. To the leading order it is independent of the deformation of the star and scales as the ratio of the moments of inertia of the superfluid and normal components. Nevertheless, free precession of neutron stars has been modelled and applied to the available data with some success [206, 207, 208, 209, 210], which might be an evidence for the weak coupling of the superfluid to the normal component. Long term variabilities in neutron stars can also be understood in terms of the Tkachenko waves - oscillations of the vortex lattice in the neutron superfluid [211]. The corresponding modes have been studied in a number of setting including the damping by mutual friction and shear viscosity in Refs. [212, 213, 214, 215] and have been shown to be in the range relevant for the long-term periodicities observed in pulsars.

### 7 Conclusions and future directions

This chapter provided an educational introduction to the physics of superfluidity and superconductivity as well as a discussion of selected subjects of current interest. Some basic aspects of the physics of superfluidity in neutron stars are now well established: the rough magnitudes of the pairing gaps, the existence of vortices and flux tubes, the basic channels of mutual friction and mechanisms of vortex pinning. The basic contours of the macro-physics behaviour in glitches, their relaxations and other anomalies such as precession and oscillations are also emerging. However, there are significant uncertainties related to the details of the theory. For example, it is a matter of debate whether the glitches occur in the crust, in the core or in both components of the star. The same applies to the post-glitch response of various superfluid shells that respond to a glitch on observable dynamical time-scales. Finally, there are phenomena that are not understood well at the basic level, such as for example, the possibility of free precession in neutron stars.

One of the main difficulties in modelling these phenomena lies in the large separation in scales that exists in neutron stars between the interactions of vortices, flux tubes and clusters on the Fermi scale, vortex-vortex
interactions on the scale of millimeters and the large scale hydrodynamics of the star. Future efforts must thus focus on bridging the gap between these scales, both from a theoretical and a computational point of view.

Future theoretical developments in the field will definitely obtain impetus from observational programs in radio, X-ray and gravitational wave astronomy. The SKA radio observatory, to become operational in the upcoming decade, has the potential of significant discoveries in pulsars astrophysics through the strong increase of the number of observed pulsars (up to 30,000) and its high sensitivity. The currently operating gravitational wave observatories are sensitive probes of continuous gravitational wave radiation from the pulsars and current upper limits on such radiation are sensitive enough to constraint some physics of neutron star interiors (e.g., the rigidity of the crust). In the future, some of the transients seen in the electromagnetic spectrum may become observable through gravitational waves, thus providing complementary information on their dynamics. Finally, current and future X-ray observations of pulsars are expected to put stronger constraints on the thermal evolution models of neutron stars and their gross parameters (in particular, neutron star radii will be measured by the NICER experiment to a high precision). Combined these ‘multi-messenger’ observations of neutron stars will provide us with a deeper theoretical understanding of the workings of superfluids in neutron star interiors.

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