Establishment and Solution of Mathematical Model for Unconventional Cracks in Multi-stage Fracturing of Horizontal Wells

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Abstract. Cracks tend to be asymmetrical during complex fracture processes. In this paper, the mathematical model of well test interpretation of vertical cracks under finite diversion conditions is established by the basic theory of point source function. The interpretation model of multi-stage fracturing unconventional fracture horizontal well test well is obtained by using the principle of pressure drop superposition. The semi-analytical solution of the model is obtained by integral transformation. By numerically discretizing the cracks, combined with the semi-analytical solution and the Duhamel principle, according to the Stehfest numerical inversion method, the bottom hole pressure and pressure derivative curves of the multi-stage fracturing unconventional crack horizontal wells are drawn.

1. Introduction
Horizonal well fracturing is of great significance for improving oil well production and has a very broad prospect in the exploitation of tight oil and gas reservoirs [1-6]. Due to the complex formation conditions, the compression cracks are asymmetrical about the wellbore. The core problem in solving the bottom hole pressure of a fracturing horizontal well is the superposition of multiple crack pressures. In the dynamic analysis of fracturing vertical well pressure, foreign scholars Cinco-Ley and others [2-3] coupled the reservoir seepage pressure solution with the crack seepage pressure solution to obtain the pressure semi-analytical solution of the whole system; Riley and others [6] established a mathematical model for finite conductivity vertical fracture test interpretation based on elliptical seepage. The advantage of this model is that the solution speed is fast. The disadvantage is that the asymmetric finite-flow vertical seam cannot be well studied. Rodriguez [7] established an asymmetric vertical fracture test model, but the model only described the fracture seepage characteristics, and did not solve the fracture and reservoir coupling; Berumen [8] used numerical simulation to study the asymmetric vertical cracks, and analyzed the influence of the asymmetry factor on the pressure derivative curve, but the numerical simulation calculation speed was slow; Chinese scholars [9-14] have done a lot of research on the well testing of asymmetric vertical cracks, which has laid a foundation for the study of unconventional multi-stage fracturing horizontal well test models. In the study of multi-stage fracturing horizontal well test interpretation, Ozkan [15-16] proposed the pull-space point source function theory, and obtained the semi-analytical solution of the multi-stage fracturing horizontal well test model through the pressure drop superposition principle; Guo J [17-19] established and solved the symmetry In the multi-stage fracture horizontal well test mathematical model; In order to improve the calculation speed, according to Riley's research results, Wang X D [20] introduced the fracture diversion function and combined the infinite diversion flow solution with the conduction function to obtain a new method for calculating fracture
well pressure and multi-stage fracture horizontal well pressure, but the model does not consider crack asymmetry.

Based on the homogeneous stratum, the mathematical model of the asymmetry crack test interpretation is established in this paper, and the semi-analytical solution is obtained by Laplace transform in this paper. The semi-analytical solution of the multi-stage fracturing horizontal well is obtained by the pressure drop superposition principle. Finally, the Stewart [21-22] numerical inversion method is used to draw the typical logarithmic pressure and pressure derivative graphs and analyze the influence of each factor on the well test curve.

2. Test model of Single finite diversion vertical fracture

The so-called asymmetric vertical diversion crack is one or more hydraulic fracturing cracks, which are connected around the wellbore. Asymmetry of the wellbore at both ends of the crack (Fig. 1). In order to better establish the asymmetry vertical fracture test interpretation model, according to the establishment of the symmetric fracture test model, a Cartesian coordinate system is established centering on the crack midpoint. The placement of the well from the center of the crack is \( x_w \), and the upper and lower layers of the reservoir are impermeable boundaries. According to the solution of the infinite diversion vertical fracture model given by Ozkan [16], combined with the asymmetric vertical fracture seepage test interpretation model, the fracture and reservoir seepage are coupled to obtain the bottom pressure solution of the asymmetrical finite flow single slit. The basic assumptions of the model are as follows: no fluid passes through the fracture crack; the fluid flow in the reservoir and the crack conforms to the isothermal Darcy flow; the permeability of the crack is \( K_f \); the influence of capillary pressure and gravity is ignored; the reservoir fluid is a microcompressible single-phase fluid.

![Fig.1 Schematic diagram of asymmetric vertical fracture](image)

The definition of dimensionless variables and asymmetry factors are defined as:

\[
t_D = \frac{Kt}{\mu^2_{ref} \phi C_t}, \quad C_{Df} = \frac{K W_f}{K L_{ref}}, \quad p_D = \frac{2 \pi K h}{Q \mu} (p_e - p), \quad \psi_D = \frac{2 \pi K h}{Q \mu} (p_e - p_f), \quad x_D = \frac{x}{L_{ref}}
\]

\[
C_D = \frac{C}{2 \pi \phi C_t h L_{ref}^2}, \quad y_D = \frac{y}{L_{ref}}, \quad x_{wD} = \frac{x_w}{L_{ref}}, \quad y_{wD} = \frac{y_w}{L_{ref}}, \quad q_D = \frac{L_{ref} q}{Q}, \quad \theta = \frac{x_w}{L_{ref}}
\]

In the formula:

- \( K \) is reservoir permeability, \( \mu^2 \); \( t \) is production time, \( s \); \( \mu \) fluid viscosity, \( \text{mPa}\cdot\text{s} \); \( L_{ref} \) is reference length, which can be half of the crack length, or the average of all crack lengths, and this paper takes half of the crack length, \( m \); \( \phi \) is porosity, dimensionless; \( C_t \) is comprehensive compression coefficient, MPa\(^{-1}\); \( K_f \) is fracture permeability, \( \mu^2 \); \( W_f \) is crack width, \( m \); \( h \) is reservoir thickness, \( m \); \( Q \) is bottom hole production, \( \text{m}^3/\text{s} \); \( q \) is the crack flow per unit length; \( p_e \) is the original formation pressure, MPa; \( p \) is the pressure at any position, MPa; \( C \) is the well storage coefficient, \( \text{m}^3/\text{MPa} \); subscript \( D \) indicates the dimensionless variable.

According to the above assumptions, the mathematical model of the dimensionless fracture test interpretation is obtained as follows:

\[
\frac{\partial}{\partial x_D} \left( \frac{\partial p_D}{\partial x_D} \right) + \frac{2}{C_{Df}} \frac{\partial p_D}{\partial y_D} \left( \frac{y_D}{L_{ref}} \right) - \frac{w_{wD}}{2} \frac{\psi_D}{C_{Df}} \delta(x_D - \theta) = 0
\]

In the formula:

\[
\delta(x_D - \theta) = \begin{cases} 1, & x_D = \theta \\ 0, & x_D \neq \theta \end{cases}
\]
$P_{fd}$ is dimensionless pressure of the crack; $P_D$ is dimensionless pressure of the formation; $x_D$ is dimensionless horizontal coordinate abscissa; $y_D$ is dimensionless horizontal ordinate; $C_{fd}$ is dimensionless conductivity; $W_{fd}$ is crack The width of the dimension; $\delta$ is Dirac function, dimensionless; $\pi$ takes 3.14, $\theta$ is asymmetry factor, dimensionless; subscript $D$ means dimensionless.

The boundary conditions of the above model are:

$$\left(\frac{\partial p_{fd}}{\partial x_D}\right)_{x_D=1} = 0$$  \hspace{1cm} (2)

$$\left(\frac{\partial p_{fd}}{\partial x_D}\right)_{x_D=0} = 0$$  \hspace{1cm} (3)

The solution of (1) is obtained by using the Laplace transform and the Green function method and the joint boundary condition (2) - (3):

$$p_{fd}(x_D, s) = p_{fd_{avg}}(x_D, s) - \frac{2\pi}{C_{fd}} N(x_D, \theta) + \frac{\pi}{C_{fd}} \int_{-1}^{1} N(x_D, \alpha) q_{fd}(\alpha, s) d\alpha$$  \hspace{1cm} (4)

In the formula:

- $p_{fd}$ is the Lai’s space dimensionless crack pressure;
- $p_{fd_{avg}}$ is the average pressure of the La’s space crack;
- $N(x_D, \alpha)$ is the Green function;
- $q_{fd}$ is the dimensionless flow with unit length; $s$ is the Laplace variable; $\alpha$ is the integral variable.

For ease of writing, the average pressure of the Lacquer space crack can be expressed as follows: For ease of writing, the average pressure of the Lacquer space crack can be expressed as follows:

$$-\frac{1}{2}\int_{-1}^{1} p_{fd}(\alpha, s) d\alpha$$  \hspace{1cm} (5)

According to the literature [7], the Green function can be written as:

$$N(x_D, \alpha) = \begin{cases} 
-\frac{1}{4} \left[ (\alpha + 1)^2 + (x_D - 1)^2 - \frac{4}{3} \right] & (-1 \leq \alpha \leq x_D) \\
-\frac{1}{4} \left[ (\alpha - 1)^2 + (x_D + 1)^2 - \frac{4}{3} \right] & (x_D \leq \alpha \leq 1) 
\end{cases}$$  \hspace{1cm} (6)

Ozkan [15-16] and others have studied the vertical crack of infinite diversion in a single slit. The specific expression is:

$$p_D = \frac{1}{2} \int_{-1}^{1} q_{D}(\alpha, s) K_0 \left( \sqrt{s} \sqrt{(x_D - \alpha)^2 + (y_D - y_{wd})^2} \right) d\alpha$$  \hspace{1cm} (7)

In the formula:

- $K_0$ is the second type of modified Bessel function; $y_{wd}$ is the coordinate position of the dimensionless well in the y direction.

According to the equal pressure at the crack wall, the simultaneous (4) and (7) can be obtained:

$$\frac{1}{2} \int_{-1}^{1} q_{D}(\alpha, s) K_0 \left( \sqrt{s} \sqrt{(x_D - \alpha)^2} \right) d\alpha = -\frac{1}{2} \int_{-1}^{1} q_{D}(\alpha, s) K_0 \left( \sqrt{s} \sqrt{(x_D - \alpha)^2 + (y_D - y_{wd})^2} \right) d\alpha$$  \hspace{1cm} (8)

According to the principle of conservation of mass, there are expressions:

$$\frac{1}{2} \int_{-1}^{1} q_{D}(\alpha, s) d\alpha = \frac{1}{s}$$  \hspace{1cm} (9)
3. crack discretization

For finite flow vertical cracks, there is pressure drop along the fluid flow direction of the crack. The flow at different positions of the crack is not constant, but is a function related to the position. Therefore, we discrete the cracks in the calculation process. The crack flow per unit length can be regarded as a fixed value. The whole system pressure is obtained by the pressure drop superposition principle. The discrete schematic diagram is shown in Fig. 2.

For multiple cracks, equations (8) and (9) are discretized to obtain:

\[
\sum_{i=1}^{N} \sum_{j=1}^{2N} \left[ \frac{p_{Davg}}{sC_{D}} - \frac{2\pi}{sC_{D}} N(x_{D(i,j)}, x_{asym}) + \frac{\pi}{C_{D}} \sum_{j=1}^{2N} q_{D(i,j)} \int_{x_{D(i,j)}}^{x_{D(i,j)+1}} \sqrt{s \left( x_{D(i,j)} + \alpha \right)^2 + \left( y_{D(i,j)} - y_{wD(i,j)} \right)^2} d\alpha \right] = \]

\[
\sum_{i=1}^{N} \sum_{j=1}^{2N} q_{D(i,j)} K_{0} \left( \sqrt{s \left( x_{D(i,j)} - \alpha \right)^2 + \left( y_{D(i,j)} - y_{wD(i,j)} \right)^2} \right) d\alpha + \]

\[
\frac{1}{2} \Delta x_{D(i)} \sum_{i=1}^{M} \sum_{j=1}^{2N} q_{D(i,j)} = \frac{1}{s} \]

In the formula:
- \( x_{D(i,j)} \) is the i-th slot in the x direction, the point coordinate of the j-th grid, m;
- \( x_{asym} \) is the i-th slot in the x direction, the j-th mesh end point coordinate, m;
- \( \Delta x_{D(i)} \) is the i-th slot mesh step, m;
- \( q_{D(i,j)} \) is the i-th slot mesh step, m;
- \( y_{D(i,j)} \) is the i-th y-slot, the j-th grid midpoint coordinates, m;
- \( y_{wD(i,j)} \) is the intersection of the i-th slit and the wellbore, m;
- \( N_i \) is the number of grids on the side of the i-th slit.

For a single crack, \( M=1 \) and \( y_{D(i)} = y_{wD(i)} \). Equations (10) and (11) form \( 2N+l \) equations. For a single crack, Equations (10) and (11) form \( 2N+l \) equations, which needs to solve the sum. There are \( 2N+l \) unknowns in total. The integral of the Bessel function can be obtained by Gauss-Legenre numerical integration. The unknown parameters are inversely substituted into equation (4), and the bottom hole pressure solution is obtained under the premise of \( x_{D} = \theta \).

According to the Duhamei principle, considering the influence of well storage and skin effect, the bottom hole pressure solution is obtained \([17]\):

\[
\bar{p}_{wD}(s) = \frac{sp_{D} + S}{s + C_{D}s^{2}(sp_{D} + S)} \]

\[ (12) \]
In the formula:

\[ p_{wD} \] is dimensionless bottom hole pressure considering wellbore reservoir and skin effect; \[ p_D \] is dimensionless bottom hole pressure without considering the wellbore reservoir and skin effect; \( S \) is the skin factor, no dimension; \( C_D \) is the dimensionless well Storage coefficient.

According to the formulas (10)–(11), the Stehfest numerical inversion method is used to make a graph of the dimensionless pressure, pressure derivative and dimensionless time of the finite flow asymmetry vertical fracture bottom hole, as shown in Fig.3. Comparing the calculation results obtained in this paper with the method of numerical calculation by Berumen\[8\], it is considered that the results obtained by this paper are in good agreement with the results obtained by Berumen. Therefore, the study of the asymmetric fracture test interpretation model can use the method of this paper instead of the numerical solution. In the process of numerical inversion, the method uses the vector as the basic operation unit and uses the parallel algorithm to improve the calculation speed.

Fig.3 Comparison of bottom pressure of asymmetrical finite-conductivity vertical fracture

4. Pressure Characteristics of multi-stage fracture in unconventional horizontal well

For multi-stage fracturing horizontal wells, when multiple cracks work simultaneously, the interaction between the fractures can be obtained by the superposition principle. The pressure drop of each slit is superimposed to obtain a pressure solution for all cracks. According to the reference [19-20], when the fracture conductivity is large to a certain extent, the change of the asymmetry factor hardly affects the pressure and pressure derivative curve morphology. However, the flow of fluid in the wellbore is considered to be an infinite diversion. The crack is asymmetrical with respect to the wellbore. There is an intersection between each crack and the wellbore. This intersection can be considered a "wellbore." The lengths of the cracks are not equal. According to the seepage mode of the fracturing horizontal well, the reservoir fluid flows from the formation into the fracture, and then flows into the wellbore through the fracture. Therefore, the horizontal wellbore can be bent in the x-y plane without changing the original seepage mode. In order to establish a horizontal well test interpretation model of multi-stage fracturing unconventional by using the asymmetry factor, the intersection of the fracture in the multi-stage fracturing horizontal well with the horizontal well is translated along the direction of the fracture. Therefore, the wellbore bending can be used to study the multi-stage fracturing unconventional crack well test analysis.

For M cracks, there are \( 2N \times M + 1 \) unknowns, and the simultaneous mass conservation equations form \( 2N \times M + 1 \) equations. Through this equation group, \( 2N \times M + 1 \) unknowns can be obtained, which can be reversed back to equation (4) to obtain wells. Bottom pressure solution. Equation (8) can be written in the following matrix form:
The coefficient of $A_{i,j}$ in the formula is:

$$A_{i,j} = \sum_{i=1}^{M} \sum_{j=1}^{N} \int_{\beta_D(i,j)}^{\beta_D(i,j)} q_{D_D(i,j)} K_0 \left( s \sqrt{\left( x_{D_D(i,j)} + \alpha \right)^2 + \left( y_{D_D(i,j)} - y_{wD(i,j)} \right)^2} \right) d\alpha + \sum_{i=1}^{M} \sum_{j=N+1}^{2N} \int_{\beta_D(i,j)}^{\beta_D(i,j)} q_{D_D(i,j)} K_0 \left( s \sqrt{\left( x_{D_D(i,j)} - \alpha \right)^2 + \left( y_{D_D(i,j)} - y_{wD(i,j)} \right)^2} \right) d\alpha - \sum_{i=1}^{M} \sum_{j=1}^{N} \int_{\beta_D(i,j)}^{\beta_D(i,j)} q_{D_D(i,j)} K_0 \left( s \sqrt{\left( x_{D_D(i,j)} \right)^2 + \left( y_{D_D(i,j)} \right)^2} \right) N(\alpha) d\alpha + \frac{2\pi}{C_{ID}} \int_{\beta_D(i,j)}^{\beta_D(i,j)} q_{D_D(i,j)} N(\alpha) d\alpha,$$

5. Conclusion

(1) Using the basic theory of point source function, a finite-conduction asymmetric joint test model for single-slit slab is established. The numerical model and analytical method are compared and verified. The numerical model given in this paper is feasible.

(2) Multi-stage fracturing unconventional crack horizontal well test model is established by using the superposition principle, and the model semi-analytical solution is obtained by integral transformation. The cracks were then discrete, and the bottom hole pressure and pressure derivative curves of unconventional multi-stage fracturing horizontal wells were plotted using Stehfest numerical inversion.

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