Sliding inverse problems for radial Dirac and Schrödinger equations

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Abstract

New inverse and half-inverse problems: sliding problems are introduced. In this way several physically important equations are recovered from the quantum defect. In particular, sliding problems are solved for radial Schrödinger equation, radial Dirac system and multidimensional Schrödinger equation. Systems with Coulomb-type potentials are considered as well.

In this paper we introduce new inverse and half-inverse problems: sliding problems. In Section 1 we show that this approach can be used for radial Schrödinger equation, radial Dirac system and multidimensional Schrödinger equation. In Section 2 we consider equations with Coulomb-type potentials.

We note that inverse problems for the Dirac-type systems, which are usually called now Dirac systems, have been thoroughly investigated (see, e.g., [4, 11, 14, 20] and references therein). Radial Dirac system was introduced earlier than Dirac-type system and differs from Dirac-type system.

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1 Inverse and half-inverse sliding problems

1.1 Main definitions and results

1. The radial Schrödinger equation has the form:
\[ \frac{d^2 y}{dr^2} + \left( z - \ell(\ell + 1) \right) \frac{1}{r^2} y = 0, \quad 0 < r < \infty \quad (q = \mathfrak{q}), \tag{1.1} \]
where \( \ell = 0, 1, 2, \ldots \) Some recent results and references on this well-known equation see, for instance, in [1][9][10]. The radial Dirac system has the form:
\[
\begin{align*}
\left( \frac{d}{dr} + \frac{\ell}{r} \right) f_1 - (z + m - q(r)) f_2 &= 0, \\
\left( \frac{d}{dr} - \frac{\ell}{r} \right) f_2 + (z - m - q(r)) f_1 &= 0,
\end{align*}
\tag{1.2}
\]
\[
0 < r < \infty, \quad m > 0, \quad q = \mathfrak{q}, \quad \ell = \pm 1, \pm 2, \ldots, \tag{1.4}
\]
where \( m \) takes positive (not necessarily integer) values and stands for mass. We easily rewrite (1.2) and (1.3) in the matrix form
\[ \frac{d}{dr} f = (iz\sigma_2 + V)f, \tag{1.5} \]
where \( f = \text{col} [f_1, f_2] \), whereas the Pauli matrix \( \sigma_2 \) and matrix function \( V \) have the form:
\[ \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad V(r) = \begin{bmatrix} -\ell/r & m - q(r) \\ m + q(r) & \ell/r \end{bmatrix}. \tag{1.6} \]
The peculiarity of the radial Dirac system consists in the fact that the entry \( \ell/r \) of the matrix \( V(r) \) is known and \((\ell/r) \notin L^1(0, \infty)\). A well-posed inverse problem for the radial Dirac system was absent. This fact was discussed in our paper [17]. In the present paper we introduce the notion of the quantum defect \( \delta(r) \) at infinity \((z = \infty)\). The corresponding sliding half-inverse problem for Dirac system is well-posed.

Without loss of generality we consider only the case \( \ell > 0 \) since the equality (1.5) yields the transformed equality, where \( Jf \) is substituted for \( f \), \(-z\) is substituted for \( z \), and \(-q\) and \(-\ell\) are substituted for \( q \) and \( \ell \), respectively (in the expression for \( V \)). Here \( J \) is given by \( J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \).
Using quantum defect, we solve first a sliding inverse problem for the case of the radial Schrödinger equation. Consider equation (1.11) with the boundary conditions

\[ y(0) = y(a) = 0, \quad (1.7) \]

and assume that

\[ \int_0^a |q(t)| dt < \infty, \quad 0 < a < \infty. \quad (1.8) \]

The corresponding eigenvalues (i.e., the values of \( z \) such that the solutions \( y \) satisfying (1.7) exist) are denoted by \( z_n(\ell) \) \( (n \in \mathbb{N}) \). Then we have (as a corollary of Theorem 1.17) the asymptotic equality

\[ \sqrt{z_n(a, \ell)} = \frac{\pi}{a} (n + \ell/2) + \frac{a\delta(a) - \ell(\ell + 1)}{2(n + \ell/2)a} + o\left(n^{-1}\right) \quad (1.9) \]

where \( \delta(a) \), which is defined by the formula

\[ \delta(a) = \int_0^a q(t) dt, \quad (1.10) \]

is called the quantum defect of the equation (1.1). Now, we formulate a sliding inverse problem:

**Problem 1.1** Recover the potential \( q \) from the given quantum defect \( \delta \).

It is immediate from (1.10) that the solution of Problem 1.1 is given by the equality

\[ q(a) = \left( \frac{d}{da} \delta \right) (a). \quad (1.11) \]

We deal with the radial Dirac system in a similar way. In order to formulate our result, we introduce a parameter \( \varepsilon(z) \), which for the case of the Dirac system is given by \( \varepsilon = \sqrt{m^2 - z^2} \).

**Remark 1.2** More precisely, for all \( z \in \mathbb{C} \) excluding the cuts along the semi-axes \( (-\infty, -m) \) and \( (m, \infty) \), we choose a branch of \( \varepsilon = \sqrt{m^2 - z^2} \) such that \( \Re(\varepsilon) > 0 \) for \( z \neq 0 \). We introduce \( \varepsilon \) on the cuts as the product

\[ \varepsilon = \sqrt{m + z} \sqrt{m - z}, \]

where

\[ \arg(\sqrt{m + z}) = \begin{cases} 0, & \text{if } z > m, \\ -\pi/2, & \text{if } z < -m; \end{cases} \quad \arg(\sqrt{m - z}) = \begin{cases} \pi/2, & \text{if } z > m, \\ 0, & \text{if } z < -m. \end{cases} \]
Hence, we have
\[ \arg(\varepsilon) = \pi/2, \quad \text{if} \quad z > m; \quad \arg(\varepsilon) = -\pi/2, \quad \text{if} \quad z < -m. \] (1.12)

The following theorem is proved in Subsection [1.3]

**Theorem 1.3** Let the condition
\[ \int_{r}^{\infty} |q(t)| dt < \infty, \quad 0 < r < \infty \] (1.13)
be fulfilled. Then there is a solution \( \tilde{F}(r, \varepsilon, \ell) \) of system (1.5) (where \( V \) is given by (1.6)), such that the relation
\[ \tilde{F}(r, \varepsilon, \ell) \sim -\frac{1}{2\sqrt{m + z}} e^{-i(r\varepsilon + \delta(r))} \begin{bmatrix} i \\ 1 \end{bmatrix}, \quad \varepsilon \to \infty, \quad z > m \] (1.14)
holds and \( \delta(r) \) (i.e., the quantum defect of this system) has the form
\[ \delta(r) = \int_{r}^{\infty} q(t) dt. \] (1.15)

So, for Dirac system we formulate a *sliding half-inverse problem*:

**Problem 1.4** Recover the matrix function \( V \) in (1.5) from the given value \( \ell \) and quantum defect \( \delta \).

Since \( V \) has the form (1.6), the solution of Problem 1.4 is immediate from (1.15) or, more precisely, from the equality
\[ q(r) = -\left( \frac{d}{dr} \delta \right)(r). \] (1.16)

**Remark 1.5** The notion of half-inverse problem and the first model of such problem were introduced in the paper [15] (see also [18]).

**Remark 1.6** The values of \( \delta(r) \) are spectral characteristics of the radial Dirac system.

**Remark 1.7** Sliding inverse Problem 1.1 and half-inverse Problem 1.4 are well-posed, that is, there is a one to one correspondence between the spectral data \( \delta \) and potentials \( q \) (see relations (1.11) and (1.16), respectively). Like all the classical inverse problems, these problems are not stable.
Remark 1.8 In the process of solving sliding Problems 1.1 and 1.4 we use only the spectral data for positive energy.

2. Next, we consider one-dimensional and multidimensional Schrödinger equations. Putting in (1.1) \( l = 0 \), we obtain the one-dimensional Schrödinger equation

\[- y'' + q(x)y = zy, \quad 0 \leq x \leq a, \quad y(0) = y(a) = 0. \tag{1.17}\]

Theorem 1.9 (\[13\]) Let condition (1.8) be fulfilled. Then the eigenvalues \( z_n(a) = z_n \) of system (1.17) satisfy the relation

\[\sqrt{z_n} = \frac{\pi}{a} n + \frac{1}{2n\pi} \delta(a) + o(n^{-1}), \tag{1.18}\]

where \( \delta(a) \) is given in (1.10).

We note that formulas (1.9) and (1.18) coincide, when \( l = 0 \). In Subsection 1.4 we prove the assertion:

Theorem 1.10 Let condition (1.8) be fulfilled. Then the corresponding statistical sum

\[Z(T) = \sum_{n=1}^{\infty} e^{-z_n/T} \tag{1.19}\]

satisfies the relation

\[Z(T) = (a/2)\sqrt{T/\pi} - 1/2 + \delta(a)/(2\sqrt{T\pi}) + o(1/\sqrt{T}), \quad T \to \infty. \tag{1.20}\]

Recall that \( T \) stands in physical applications for temperature.

The \( k \)-dimensional Schrödinger equation has the form

\[- \Delta y + q(x)y = zy, \quad x = (x_1, x_2, \ldots, x_k) \in G \subset \mathbb{R}^k. \tag{1.21}\]

We deal here with a model case, where

\[G = G(a_1, a_2, \ldots) = \{ x : 0 \leq x_i \leq a_i \quad (1 \leq i \leq k) \}, \tag{1.22}\]

\( q \) admits representation

\[q(x) = \sum_{i=1}^{k} q_i(x_i), \tag{1.23}\]

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the functions $q_i$ satisfy inequalities

$$\int_0^{a_i} |q_i(x_i)| dx_i < \infty, \quad 0 < a_i < \infty, \quad 1 \leq i \leq k,$$

and the boundary condition (on the boundary $\Gamma$ of $G$) is given by

$$y|_\Gamma = 0,$$

Now, $z_{n,i}$ are the eigenvalues of the problem (1.17), where $x = x_i$ and $q(x) = q_i(x_i)$, the statistical sum $Z_i$ is given by

$$Z_i(T) = \sum_{n=1}^{\infty} e^{-z_{n,i}/T},$$

where \{zn\} are the points of the spectrum of problem (1.21), (1.25), whereas the corresponding statistical sum $Z(T)$ is defined again by the relation (1.19). Representation (1.23) implies that

$$Z(T) = \prod_{i=1}^{k} Z_i(T).$$

From Theorem 1.10 and equality (1.27) we derive the following assertion.

**Theorem 1.11** Let the relations (1.23) and (1.24) hold. Then

$$Z(T) = \left( \frac{T}{4\pi} \right)^{k/2} \left( \mathcal{V}_k - \sqrt{\frac{\pi}{T}} \mathcal{V}_{k-1} + \frac{\pi}{T} \mathcal{V}_{k-2} + 2 \frac{\delta(G)}{T} + o \left( \frac{1}{T} \right) \right),$$

where $T \to \infty$ and $\mathcal{V}_{-1} = 0, \quad \mathcal{V}_0 = 1$, $\mathcal{V}_k = \prod_{i=1}^{k} a_i$, $\mathcal{V}_{k-1} = \mathcal{V}_k \sum_{i=1}^{k} 1/a_i$, $\mathcal{V}_{k-2} = \mathcal{V}_k \sum_{1 \leq i < j \leq k} 1/(a_ia_j)$.

$$\delta(G) = \int_G q(x)dx.$$

The coefficients $\mathcal{V}_i$ have clear geometric interpretation, namely,

$$\mathcal{V}_i = S_i/2^{k-i}, \quad i = k, k - 1, k - 2,$$
where $S_k$ is the Lebesgue measure of the domain $G$, $S_{k-1}$ is the Lebesgue measure of the boundary $\Gamma$, $S_{k-2}$ is the Lebesgue measure of the domain $\Gamma_1$ formed by the set of pairwise intersections of the facets of the boundary $\Gamma$.

**Conjecture I.** Formulas (1.28)–(1.30) hold for a much wider class of potentials than the potentials of the form (1.23).

**Problem 1.12** Let formulas (1.28)–(1.30) hold. Recover the potential $q$ from the given $\delta(G)$.

According to (1.22) and (1.30) the solution of Problem 1.12 is given by

$$q(a_1, a_2, \ldots, a_k) = \frac{\partial^k}{\partial a_1 \partial a_2 \cdots \partial a_k} \int_{G(a_1, a_2, \ldots, a_k)} q(x) dx. \quad (1.32)$$

3. Finally, we consider one-dimensional and multidimensional equations for anharmonic oscillators. The one-dimensional equation for anharmonic oscillator (or one-dimensional anharmonic oscillator, which is shorter to say) has the form

$$-y'' + \left(x^2/4 + q(x)\right) y = z y, \quad 0 \leq x < \infty \quad (q = \overline{q}), \quad (1.33)$$

and we introduce the boundary condition

$$y(0) = 0. \quad (1.34)$$

We assume that

$$\int_0^\infty |q(x)|(1 + x^{2+\varepsilon}) dx < \infty, \quad \varepsilon > 0. \quad (1.35)$$

**Theorem 1.13** ( [16]) Let condition (1.35) be fulfilled. Then the eigenvalues $z_n$ of problem (1.33), (1.34) satisfy the relation

$$z_n = 2n - 1/2 + \frac{1}{\pi \sqrt{2n}}(\delta + o(1)), \quad n \to \infty, \quad (1.36)$$

where

$$\delta = \int_0^\infty q(x) dx. \quad (1.37)$$

Our next theorem easily follows from Theorem 1.13, see Subsection 1.4.
Theorem 1.14 Let an anharmonic oscillator satisfy conditions of Theorem 1.13. Then the statistical sum \( Z(T) \), given by (1.19), satisfies the asymptotic relation
\[
Z(T) = T/2 - 1/4 - \delta \left( \frac{T}{2\sqrt{T\pi}} \right) + o\left( \frac{1}{\sqrt{T}} \right). \tag{1.38}
\]
The \( k \)-dimensional equation for anharmonic oscillator has the form
\[
- \Delta y + \left( \frac{|x|^2}{4} + q(x) \right) y = zy, \quad x \in G, \quad |x|^2 = \sum_{i=1}^{k} x_i^2. \tag{1.39}
\]
We set
\[
G = \{ x : 0 \leq x_i < \infty \quad (1 \leq i \leq k) \}, \tag{1.40}
\]
and introduce the condition (1.25) on the boundary \( \Gamma \) of the domain \( G \). We assume again that the potential \( q(x) \) has the form (1.23) and that
\[
\int_{0}^{\infty} |q_i(x_i)| (1 + x_i^{2+\varepsilon}) dx_i < \infty, \quad \varepsilon > 0 \quad (1 \leq i \leq k). \tag{1.41}
\]
Like in the case of the multidimensional Schrödinger equation, \( z_{n,i} \) are the eigenvalues of the one-dimensional problem, this time of the problem (1.33), (1.34), where \( x = x_i \) and \( q(x) = q_i(x_i) \). By \( z_n \) we denote the spectrum of the problem (1.39) with initial condition (1.25), where \( G \) is given by (1.40). The statistical sum \( Z_i(T) \) has the form (1.26) and the statistical sum \( Z(T) \) has the form (1.19). Like before, relation (1.23) yields (1.27). From (1.27) and (1.38) follows the next assertion.

Theorem 1.15 Let a \( k \)-dimensional anharmonic oscillator, where the potential \( q \) admits representation (1.23) and the functions \( q_i \) satisfy the condition (1.41), be given. Then we have
\[
Z(T) = \left( \frac{T}{2} \right)^m \left( 1 - m/(2T) - \delta(G) \right) \left( T\sqrt{T\pi} \right) + o\left( \frac{1}{T\sqrt{T}} \right), \tag{1.42}
\]
where \( T \to \infty \) and \( \delta(G) = \int_G q(x) dx \).

Conjecture II. Formula (1.42) holds for a much wider class of potentials than the potentials of the form (1.23).

Remark 1.16 Theorem 1.15 is an analogue of Theorem 1.14. Jointly with the results from [16], an easy modification of Theorem 1.15 can be used for the formulation and solution of a sliding inverse problem for anharmonic oscillator.
1.2 Radial Schrödinger equation and quantum defect

We add variables \( \varepsilon = i \sqrt{z} \) and \( \ell \) into the notation of the solutions of (1.1) and write \( y(r, \varepsilon, \ell) \). Here, the parameter \( \varepsilon \), continuously depending on \( z \), is defined for all \( z \) in the complex plane with the cut along the negative part of the imaginary axis. Let us fix \( \arg \varepsilon = \pi/2 \) for \( z > 0 \). We start our study of the equation (1.1) from the case \( q \equiv 0 \). Two solutions \( u_1(r, \varepsilon, \ell) \) and \( u_2(r, \varepsilon, \ell) \) of (1.1), where \( q \equiv 0 \), are given by the relations

\[
 u_1(r, \varepsilon, \ell) = (2 \varepsilon)^{-\ell} M_{0, \ell+1/2}(2r\varepsilon), \quad u_2(r, \varepsilon, \ell) = (2 \varepsilon)^{\ell} W_{0, \ell+1/2}(2r\varepsilon),
\]

(1.43)

and \( M_{\kappa,\mu}(z) \) and \( W_{\kappa,\mu}(z) \) in the formula above are Whittaker functions. The connections between Whittaker functions and confluent hypergeometric functions are well-known (see, e.g., [5]):

\[
 M_{0, \ell+1/2}(x) = e^{-x/2} x^{c/2} \Phi(\alpha, c, x), \quad W_{0, \ell+1/2}(x) = e^{-x/2} x^{c/2} \Psi(\alpha, c, x),
\]

(1.44)

where \( \alpha = \ell + 1, \ c = 2\ell + 2 \). It follows from (1.44) and relations

\[
 \Phi(\alpha, c, x) \sim 1 \quad \text{and} \quad \Psi(\alpha, c, x) \sim \frac{\Gamma(c - 1)}{\Gamma(\alpha)} x^{1-c} \quad \text{for} \quad x \to 0,
\]

(1.45)

that the solutions \( u_1(r, \varepsilon, \ell) \) and \( u_2(r, \varepsilon, \ell) \) satisfy the conditions

\[
 u_1(r, \varepsilon, \ell) \sim r^{\ell+1}, \quad u_2(r, \varepsilon, \ell) \sim \frac{\Gamma(2\ell + 1)}{\Gamma(\ell + 1)} r^{-\ell}, \quad r \to 0.
\]

(1.46)

Hence, in view of (1.46) we have

\[
 |u_1(r, \varepsilon, \ell)| \leq M_{r^{\ell+1}} \quad \text{for} \quad r \leq 1/|\varepsilon|.
\]

(1.47)

We note that the letter \( M \) stands for different constants.

Let us write the following integral representation (see [5], Section 6.11):

\[
 M_{0, \ell+1/2}(2r\varepsilon) = C(\ell) (2r\varepsilon)^{\ell+1} I(r, \varepsilon, \ell),
\]

(1.48)

where

\[
 C(\ell) = 2^{-(2\ell+1)} \frac{\Gamma(2\ell + 2)}{\Gamma^2(\ell + 1)}, \quad I(r, \varepsilon, \ell) = \int_{-1}^{1} e^{r t \varepsilon} (1 - t^2)^{\ell} dt
\]

(1.49)
Integrating $I(r, \varepsilon, \ell)$ by parts $\ell + 2$ times, we obtain the asymptotic formula:

$$I(r, \varepsilon, \ell) \sim (r\varepsilon)^{-(\ell+1)}2^\ell \Gamma(\ell + 1)((-1)^{\ell+1}e^{-r\varepsilon}\phi(r, \varepsilon, \ell) + e^{r\varepsilon}\phi(r, -\varepsilon, \ell)),$$  \hspace{1cm} (1.50)

where

$$\phi(r, \varepsilon, \ell) = 1 + \frac{\ell(\ell + 1)}{2r\varepsilon} + o(1/\varepsilon), \hspace{1cm} \varepsilon \to \infty. \hspace{1cm} (1.51)$$

Here, we used the Leibnitz differentiation formula in order to obtain an expression for the $(\ell + 1)$'th derivative of the product $(1 - t)^{\ell}(1 + t)^{\ell}$. Formulas (1.43), (1.44) and (1.48)-(1.50) imply that

$$u_1(r, \varepsilon, \ell) = C_1(\ell, \varepsilon) \left(-e^{i\pi\ell/2}e^{-r\varepsilon}\phi(r, \varepsilon, \ell) + e^{-i\pi\ell/2}e^{r\varepsilon}\phi(r, -\varepsilon, \ell)\right), \hspace{1cm} (1.52)$$

where $C_1(\ell, \varepsilon) = e^{i\pi\ell/2}2\Gamma(2\ell+2)\Gamma(\ell+1)/\Gamma(\ell+1)\Gamma(2\varepsilon)^{-(\ell+1)}$.

In order to study the asymptotic behavior of $u_2(r, \varepsilon, \ell)$ we need the following integral representation (see [5, Section 6.11]):

$$\Gamma(\ell + 1)W_{0,\ell+1/2}(x) = e^{-x/2}x^{\ell+1}\int_0^\infty e^{-tx}(1 + t)^{\ell}dt, \hspace{1cm} x > 0. \hspace{1cm} (1.53)$$

Formula (1.53) leads us to the equality

$$\Gamma(\ell + 1)W_{0,\ell+1/2}(x) = e^{-x/2}x^{-\ell}\int_0^\infty e^{-s}(x + s)^{\ell}ds, \hspace{1cm} x > 0. \hspace{1cm} (1.54)$$

In view of the well-known integral representation

$$\Gamma(z) = \int_0^\infty e^{-s}s^{z-1}ds, \hspace{1cm} \Re z > 0, \hspace{1cm} (1.55)$$

equality (1.55) takes the form

$$\Gamma(\ell + 1)W_{0,\ell+1/2}(x) = e^{-x/2}x^{-\ell}\sum_{p=0}^\ell C_{\ell}^p \Gamma(2\ell - p + 1)x^p, \hspace{1cm} (1.56)$$

where $C_{\ell}^p$ are the binomial coefficients. Using analyticity, we rewrite (1.56) for the complex plane

$$\Gamma(\ell + 1)W_{0,\ell+1/2}(z) = e^{-z/2}z^{-\ell}\sum_{p=0}^\ell C_{\ell}^p \Gamma(2\ell - p + 1)z^p, \hspace{1cm} z \neq 0. \hspace{1cm} (1.57)$$
Now, the asymptotic relation
\[ u_2(r, \varepsilon, \ell) = (2\varepsilon^\ell) e^{-r\varepsilon} \phi(r, \varepsilon, \ell), \quad \varepsilon \to \infty \] (1.58)
follows directly from (1.43) and (1.57).

Next we deal with the case \( q \neq 0 \). We assume that \( q \) satisfies the condition
\[ \int_0^r |q(t)| dt < \infty, \quad 0 < r < \infty. \] (1.59)

Then there is a solution \( u \) of (1.1) such that the equality
\[ u(r, \varepsilon, \ell) = u_1(r, \varepsilon, \ell) + \int_0^r G(r, t, \varepsilon, \ell) q(t) u(t, \varepsilon, \ell) dt \] (1.60)
holds for \( G \) of the form
\[ G(r, t, \varepsilon, \ell) = \frac{\Gamma(\ell + 1)}{\Gamma(2\ell + 2)} (u_1(r, \varepsilon, \ell) u_2(t, \varepsilon, \ell) - u_2(r, \varepsilon, \ell) u_1(t, \varepsilon, \ell)). \] (1.61)

Relations (1.46) and (1.61) imply that
\[ \frac{(t/r)^{\ell+1}}{r^{\ell+1}} |G(r, t, \varepsilon, \ell)| \leq \mathcal{M} t, \quad t \leq r \leq 1/|\varepsilon|. \] (1.62)

From (1.46), (1.60) and (1.62) we derive that the asymptotics of \( u \) is similar to the asymptotics of \( u_1 \), namely
\[ u(r, \varepsilon, \ell) \sim r^{\ell+1}, \quad r \leq 1/|\varepsilon|. \] (1.63)

Moreover, according to (1.51), (1.52), (1.58) and (1.61) we have
\[ G(r, t, \varepsilon, \ell) \sim \frac{1}{2\varepsilon} (e^{(r-t)\varepsilon} - e^{-(r-t)\varepsilon}), \quad \varepsilon \to \infty. \] (1.64)

Using (1.46) and (1.60)-(1.63) we obtain the relation
\[ \int_0^{1/|\varepsilon|} |G(r, t, \varepsilon, \ell) q(t) u(t, \varepsilon, \ell)| dt = o \left( |\varepsilon|^{-(\ell+2)} \right), \quad \varepsilon \to \infty. \] (1.65)

It follows from (1.64) that for some \( \mathcal{M} \) we have
\[ |G(r, t, \varepsilon, \ell)| \leq \mathcal{M} e^{r|\varepsilon|}/|\varepsilon|, \quad 1/|\varepsilon| \leq t \leq r; \] (1.66)
where $\sigma := \Re(\varepsilon)$. In view of (1.46), (1.51), (1.52), (1.58) and (1.63) the relation

$$
\int_0^{1/|\varepsilon|} |G(r, t, \varepsilon, \ell)q(t)u(t, \varepsilon, \ell)|dt = e^{r|\sigma|}o(1/|\varepsilon|), \quad r > 1/|\varepsilon|, \quad \varepsilon \to \infty
$$

holds. Formulas (1.60), (1.66) and (1.67) imply the following equality

$$
u(r, \varepsilon, \ell) = \nu_1(r, \varepsilon, \ell) + O\left(e^{r|\sigma|/|\varepsilon|}\right), \quad r > 1/|\varepsilon|.
$$

We denote by $z_n(a, \ell)$ the eigenvalues of problem (1.1), (1.7). It is well-known that there is only one (up to a constant factor) solution of (1.1) which turns to zero at $r = 0$. Since $u$ is such a solution, $z_n(a, \ell)$ is an eigenvalue if and only if

$$u(a, i\sqrt{z_n(a, \ell)}, \ell) = 0.
$$

By well-known methods (see, e.g., [12, Ch.1, Section 2]), using relations (1.52), (1.58) and (1.68), (1.69), we obtain that

$$
\sqrt{z_n(a, \ell)} = \frac{\pi}{a} \left(n + \frac{\ell}{2}\right) + o(1).
$$

Recall that $\sigma = \Re(\varepsilon)$. Putting

$$\Re(\varepsilon) = 0, \quad (2\varepsilon)^{\ell+1}u(r, \varepsilon, \ell) = \tilde{u}(r, \varepsilon, \ell), \quad (2\varepsilon)^{\ell+1}u_1(r, \varepsilon, \ell) = \tilde{u}_1(r, \varepsilon, \ell),
$$

and taking into account estimates (1.64) and (1.68), we deduce from (1.60) that

$$
\tilde{u}(r, \varepsilon, \ell) = \tilde{u}_1(r, \varepsilon, \ell) + \frac{1}{2\varepsilon} \int_0^r \left(e^{(r-t)\varepsilon} - e^{-(r-t)\varepsilon}\right) \tilde{u}_1(t, \varepsilon, \ell)q(t)dt + o\left(\frac{1}{\varepsilon}\right).
$$

Thus, we obtain the following assertion.

**Theorem 1.17** Let condition (1.59) be fulfilled and equality $\Re(\varepsilon) = 0$ hold. Then the solution $u(r, \varepsilon, \ell)$ of the equation (1.1) satisfies (1.63) and admits representation

$$u(r, \varepsilon, \ell) = C_1(\ell, \varepsilon) \left(-e^{i\pi\ell/2}e^{-r\varepsilon+\Delta(r)/(2\varepsilon)} + e^{-i\pi\ell/2}e^{r\varepsilon+\Delta(r)/(2\varepsilon)} + o(1/|\varepsilon|)\right),
$$

where $\Delta(r) = \frac{1}{r}(\ell + 1) - \delta(r)$ and the quantum defect $\delta(r)$ has the form

$$
\delta(r) = \int_0^r q(t)dt.
$$
Recall that (1.9) follows from Theorem 1.17. We denote by \( z_n(\ell) \) the positive roots of the Whittaker function \( M_{0,\ell+1/2}(iz) \), which are not equal to zero. Formula (1.9) yields the next well-known result (see [8, Ch. XI]):

\[
z_n(\ell) = \pi \left(n + \frac{\ell}{2}\right) - \frac{\ell(\ell + 1)}{\pi(2n + \ell)} + o(n^{-1}).
\]  

(1.74)

1.3 Dirac equation and quantum defect

1. We assume that \( q \) satisfies (1.4) and (1.13), that is,

\[
q(r) = \overline{q(r)}, \quad \int_r^\infty |q(t)| dt < \infty, \quad r > 0.
\]  

(1.75)

Given functions \( f_1 \) and \( f_2 \), it is easy to find functions \( Q_1 \) and \( Q_2 \) such that

\[
f_1(r, \varepsilon, \ell) = \sqrt{m + z} e^{-r\varepsilon} (2r\varepsilon)^{-\ell/2} (Q_1(r, \varepsilon, \ell) + Q_2(r, \varepsilon, \ell)) r, \quad (1.76)
\]

\[
f_2(r, \varepsilon, \ell) = -\sqrt{m - z} e^{-r\varepsilon} (2r\varepsilon)^{\ell/2} (Q_1(r, \varepsilon, \ell) - Q_2(r, \varepsilon, \ell)) r, \quad (1.77)
\]

where \( \varepsilon = \sqrt{m^2 - z^2} \). Recall that \( \varepsilon \) is defined more precisely in Remark 1.2.

In order to prove Theorem 1.3 following [2] we consider solutions of (1.2)-(1.4) in the form (1.76) and (1.77). Substituting (1.76) and (1.77) into (1.2) and (1.3) we obtain:

\[
r(Q_1 + Q_2)' + 2\ell(Q_1 + Q_2) - 2r\varepsilon Q_2 + \sqrt{\frac{m - z}{m + z}} (Q_1 - Q_2) rq(r) = 0, \quad (1.78)
\]

\[
r(Q_1 - Q_2)' + 2r\varepsilon Q_2 - \sqrt{\frac{m + z}{m - z}} (Q_1 + Q_2) rq(r) = 0, \quad (1.79)
\]

where \( Q' = \frac{d}{dr}Q \). It is easy to see that

\[
\sqrt{\frac{m - z}{m + z}} - \sqrt{\frac{m + z}{m - z}} = -\frac{2}{\varepsilon}, \quad \sqrt{\frac{m - z}{m + z}} + \sqrt{\frac{m + z}{m - z}} = \frac{1}{\varepsilon} m. \quad (1.80)
\]

Hence, if \( q(r) \equiv 0 \), the equations (1.78) and (1.79) can be rewritten in the form

\[
rQ_1' + \ell Q_1 + \ell Q_2 = 0, \quad rQ_2' + \ell Q_2 - 2r\varepsilon Q_2 + \ell Q_1 = 0.
\]  

(1.81)

We substitute into (1.81) functions

\[
Q_i(r, \varepsilon, \ell) = Q_i,0(\rho/(2\varepsilon), \varepsilon, \ell) \quad (i = 1, 2), \quad \rho = 2r\varepsilon, \quad (1.82)
\]
and so switch to the new variable $\rho$. It follows from (1.81) that

\[
\rho \frac{d^2}{d\rho^2} Q_{1,0} + (2\ell + 1 - \rho) \frac{d}{d\rho} Q_{1,0} - \ell Q_{1,0} = 0,
\]

(1.83)

\[
\rho \frac{d^2}{d\rho^2} Q_{2,0} + (2\ell + 1 - \rho) \frac{d}{d\rho} Q_{2,0} - (\ell + 1) Q_{2,0} = 0.
\]

(1.84)

The solutions of (1.83) and (1.84), which are regular at the point $\rho = 0$, are the confluent hypergeometric functions $a_1 \Phi(\ell, 2\ell + 1, \rho)$ and $a_2 \Phi(\ell + 1, 2\ell + 1, \rho)$, respectively (see [2, Section 36] and [3, Section 14]). Substitute $r = 0$ into (1.81), (1.82) to see that $a_1 = -a_2$. That is, up to the factor $a_1$ we have

\[
Q_{1,0}(\rho) = \Phi(\ell, 2\ell + 1, \rho) = e^{\rho/2} \rho^{-\ell-1/2} M_{1/2, \ell}(\rho),
\]

(1.85)

\[
Q_{2,0}(\rho) = -\Phi(\ell + 1, 2\ell + 1, \rho) = -e^{\rho/2} \rho^{-\ell-1/2} M_{-1/2, \ell}(\rho).
\]

(1.86)

Now, we need an integral representation from [5, Section 6.11]:

\[
M_{\pm 1/2, \ell}(\rho) = C(\ell)(\rho)^{\ell+1/2} I_{\pm}(\rho, \ell), \quad C(\ell) = 2^{-\ell} \frac{\Gamma(2\ell + 1)}{\Gamma(\ell + 1)\Gamma(\ell)}, \quad (1.87)
\]

where

\[
I_{+}(\rho, \ell) = \int_{-1}^{1} e^{\rho/2} (1-t^2)^{\ell-1/2} (1-t) dt, \quad I_{-}(\rho, \ell) = \int_{-1}^{1} e^{\rho/2} (1-t^2)^{\ell-1/2} (1+t) dt.
\]

Integrating the expressions above by parts ($\ell$ times), we obtain:

\[
I_{+}(\rho, \ell) \sim (-1)^{\ell} (\rho/2)^{-\ell} \Gamma(\ell) e^{-\rho/2}, \quad \rho \to \infty \quad (\arg(\rho) = \pi/2), \quad (1.88)
\]

\[
I_{-}(\rho, \ell) \sim (\rho/2)^{-\ell} \Gamma(\ell) e^{\rho/2}, \quad \rho \to \infty \quad (\arg(\rho) = \pi/2), \quad (1.89)
\]

Using asymptotic formulas (1.88) and (1.89) we have

\[
Q_{1,0}(\rho) \sim (-1)^{\ell} \rho^{-\ell} \Gamma(2\ell + 1)/\Gamma(\ell + 1), \quad \rho \to \infty \quad (\arg(\rho) = \pi/2), \quad (1.90)
\]

\[
Q_{2,0}(\rho) \sim -\rho^{-\ell} e^{\rho/2} \Gamma(2\ell + 1)/\Gamma(\ell + 1), \quad \rho \to \infty \quad (\arg(\rho) = \pi/2). \quad (1.91)
\]

Formulas (1.90) and (1.91) show that the solutions of (1.83) and (1.84) of the form (1.85) and (1.86) are, indeed, regular at $\rho = 0$ and, moreover, we have

\[
Q_{1,0} \sim 1, \quad Q_{2,0} \sim 1 \quad \text{for} \quad \rho \to 0. \quad (1.92)
\]
The functions

\[ \tilde{Q}_1(\rho) = c_1 \Psi(\ell, 2\ell + 1, \rho) = c_1 e^{\rho/2} \rho^{-(\ell+1)/2} W_{1/2, \ell}(\rho) \]  
(1.93)

\[ \tilde{Q}_2(\rho) = c_2 \Psi(\ell + 1, 2\ell + 1, \rho) = c_2 e^{\rho/2} \rho^{-(\ell+1)/2} W_{-1/2, \ell}(\rho), \]  
(1.94)

where \( \Psi(a, c, x) \) is the confluent hypergeometric function of the second kind, also satisfy (1.83) and (1.84) but are non-regular at \( \rho = 0 \) (see [5, Ch.6]).

According to [5, Section 6.11], we have the following integral representation for \( W_{\pm 1/2, \ell} \):

\[ W_{\pm 1/2, \ell}(\rho) = C_{\pm}(\ell) e^{-\rho/2} \rho^{-(\ell+1)/2} J_{\pm}(\rho, \ell), \]  
(1.95)

where

\[ J_{\pm}(\rho, \ell) = \int_0^{\infty} e^{-\rho t} t^{\ell-1} (1 + t^2)^{\ell/2} dt, \quad J_{\pm}(\rho, \ell) = \int_0^{\infty} e^{-\rho t} t^{\ell-1} (1 + t^2)^{\ell/2} dt. \]  
(1.96)

Using integral representation (1.55), we rewrite (1.96) as

\[ J_{\pm}(\rho, \ell) = \rho^{-2\ell} \sum_{i=0}^{\ell} C_{\ell}^i \Gamma(2\ell + i) J_{\pm}^{(i)}(\rho, \ell), \quad J_{\pm}(\rho, \ell) = \rho^{-2\ell} \sum_{i=0}^{\ell-1} C_{\ell-1}^i \Gamma(2\ell - i) J_{\pm}^{(i)}(\rho, \ell), \]  
(1.97)

where \( C_{\ell}^i \) are the binomial coefficients again. The right-hand sides in (1.97) are analytic functions (for \( \rho \neq 0 \)). Thus (1.97) is valid for all \( \rho \neq 0 \) in the complex plane.

In view of (1.93)–(1.95) and (1.97), if \( \rho \to 0 \) we have

\[ \tilde{Q}_{1,0}(\rho) \sim c_1 \rho^{-\ell} \Gamma(2\ell)/\Gamma(\ell), \quad \tilde{Q}_{2,0}(\rho) \sim c_2 \rho^{-\ell} \Gamma(2\ell)/\Gamma(\ell + 1). \]  
(1.98)

After the substitution \( Q_{1,0} = \tilde{Q}_{1,0} \) and \( Q_{2,0} = \tilde{Q}_{2,0} \) into (1.82), relations (1.81) and (1.98) imply that we can assume

\[ c_1 = 1, \quad c_2 = 1. \]  
(1.99)

Using again formulas (1.93)–(1.95) and (1.97), we derive

\[ \tilde{Q}_{1,0}(\rho) \sim \rho^{-\ell}, \quad \tilde{Q}_{2,0}(\rho) \sim \rho^{-(\ell+1)}, \quad \rho \to \infty \quad (\arg(\rho) = \pi/2). \]  
(1.100)

The constructed regular and non-regular (at \( r = 0 \)) solutions of (1.2)–(1.3) (or, equivalently (1.5)), where \( q \equiv 0 \), we denote by

\[ F_0(r) = F_0(r, \varepsilon, \ell) = \text{col} \left[ F_{1,0}(r, \varepsilon, \ell) \quad F_{2,0}(r, \varepsilon, \ell) \right] \]  
(1.101)
and
\[ \tilde{F}_0(r) = \tilde{F}_0(r, \varepsilon, \ell) = \text{col} [\tilde{F}_{1,0}(r, \varepsilon, \ell) \quad \tilde{F}_{2,0}(r, \varepsilon, \ell)], \] (1.102)
respectively. According to (1.76) and (1.82) we have
\[ F_{1,0}(r) = \sqrt{m + z} e^{-r\varepsilon}(2r\varepsilon)^{\ell-1}(Q_{1,0}(r) + Q_{2,0}(r)) r; \] (1.103)
\[ F_{2,0}(r) = -\sqrt{m - z} e^{-r\varepsilon}(2r\varepsilon)^{\ell-1}(Q_{1,0}(r) - Q_{2,0}(r)) r; \] (1.104)
\[ \tilde{F}_{1,0}(r) = \sqrt{m + z} e^{-r\varepsilon}(2r\varepsilon)^{\ell-1}(\tilde{Q}_{1,0}(r) + \tilde{Q}_{2,0}(r)) r; \] (1.105)
\[ \tilde{F}_{2,0}(r) = -\sqrt{m - z} e^{-r\varepsilon}(2r\varepsilon)^{\ell-1}(\tilde{Q}_{1,0}(r) - \tilde{Q}_{2,0}(r)) r. \] (1.106)

Hence, from (1.90) and (1.91) we obtain
\[ F_0(r, \varepsilon, \ell) \sim \frac{\Gamma(2\ell + 1)}{2\varepsilon \Gamma(\ell + 1)} \times \text{col} \left[ (1)^{\ell} e^{-r\varepsilon} - e^{r\varepsilon} \sqrt{m + z} \quad (1)^{\ell} e^{-r\varepsilon} + e^{r\varepsilon} \sqrt{m - z} \right], \] (1.107)
where \( z > m \) (i.e., \( \arg(\varepsilon(z)) = \pi/2 \)) and either \( \varepsilon \to \infty \) or \( r \to \infty \). In a similar way, from (1.100) we derive that
\[ \tilde{F}_0(r, \varepsilon, \ell) \sim \frac{1}{2\varepsilon} e^{-r\varepsilon} \text{col} \left[ \sqrt{m + z} - \sqrt{m - z} \right], \quad z > m, \] (1.108)
where either \( \varepsilon \to \infty \) or \( r \to \infty \).

2. Now, we consider the case when \( q(r) \neq 0 \), and introduce \( 2 \times 2 \) matrix functions
\[ \hat{V}(r) = \begin{bmatrix} 0 & q(r) \\ -q(r) & 0 \end{bmatrix}, \quad U_0(r, \varepsilon, \ell) = \begin{bmatrix} F_{1,0}(r, \varepsilon, \ell) & \tilde{F}_{1,0}(r, \varepsilon, \ell) \\ F_{2,0}(r, \varepsilon, \ell) & \tilde{F}_{2,0}(r, \varepsilon, \ell) \end{bmatrix}. \] (1.109)

It is easy to see that the solution \( \tilde{F}(r, \varepsilon, \ell) \) of the equation
\[ \tilde{F}(r) = \tilde{F}_0(r) + \int_r^\infty U_0(r, \varepsilon, \ell) U_0(t)^{-1} \hat{V}(t) \tilde{F}(t) dt \] (1.110)
satisfies the differential system (1.2), (1.3).

The following equality is valid:
\[ \det U_0(r, \varepsilon, \ell) = \Gamma(2\ell)/\left( \varepsilon \Gamma(\ell) \right). \] (1.111)
Indeed, it follows from (1.2) and (1.3) that \( (\det U_0(r,\varepsilon,\ell))^\prime = 0 \). Hence, we obtain the relation \( \det U_0(r,\varepsilon,\ell) \equiv \text{const} \). Since the first and second columns of \( U_0 \) are \( F_0 \) and \( \tilde{F}_0 \), respectively, formulas (1.107)–(1.109) imply that

\[
\det U_0(r,\varepsilon,\ell) \sim \frac{\Gamma(2\ell)}{(\varepsilon \Gamma(\ell))}, \quad r \to \infty, \quad (z > m),
\]

which yields (1.111).

According to (1.107)–(1.109) and (1.111), the equality

\[
U_0(r)U_0(t)^{-1} = e^{(r-t)\varepsilon} \Theta_1 + e^{(t-r)\varepsilon} \Theta_2 + o(1), \quad \varepsilon \to \infty, \quad (z > m),
\]

(1.113)

\[
\Theta_1 := \frac{1}{2} \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}, \quad \Theta_2 := \frac{1}{2} \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}
\]

(1.114)

holds. We multiply both sides of (1.110) by \(-2\sqrt{m+z}e^{r\varepsilon}\). Using (1.113) and passing to the limit \( z \to +\infty \), we obtain

\[
\tilde{F}_\infty(r,\ell) = \left[ i \\ 1 \right] + \int_r^\infty T_2 \hat{V}(t) \tilde{F}_\infty(t,\ell) dt,
\]

(1.115)

where

\[
\tilde{F}_\infty(r,\ell) = -2 \lim_{z \to +\infty} \left( \sqrt{m+z}e^{r\varepsilon} \tilde{F}(r,\varepsilon,\ell) \right).
\]

(1.116)

The equality

\[
\tilde{F}_\infty(r,\ell) = e^{-i \int_r^\infty q(t) dt} \left[ i \\ 1 \right]
\]

(1.117)

follows directly from (1.108), (1.109), (1.114) and (1.115). From (1.110) and (1.117) we see that the constructed solution \( \tilde{F}(r,\varepsilon,\ell) \) has the properties stated in Theorem 1.3, that is, Theorem 1.3 is proved.

### 1.4 Proofs of Theorems 1.10 and 1.14

In order to show that the results on multidimensional Schrödinger equation from Subsection 1.1 hold, we should prove Theorems 1.10 and 1.14. First, we prove the assertion below (see [19, 21]).

**Lemma 1.18** The following asymptotic relation

\[
\sum_{n=1}^{\infty} e^{-\eta^2/z} = \frac{1}{2} \sqrt{z\pi} - \frac{1}{2} + o(1), \quad z \to \infty
\]

is valid.
Proof. In the proof of (1.118) we use the Poisson formula (see, e.g., [6]):

\[ \sum_{n=1}^{\infty} e^{-n^2/z} = -\frac{1}{2} + \int_{0}^{\infty} e^{-x^2/z} dx + 2 \sum_{n=1}^{\infty} \int_{0}^{\infty} e^{-x^2/z} \cos (2\pi nx) dx. \]  \hspace{1cm} (1.119)

Since

\[ \int_{0}^{\infty} e^{-x^2/z} \cos (2\pi nx) dx = \frac{1}{2} \sqrt{\frac{\pi}{z}} e^{-\frac{\pi^2 n^2}{z}}, \]  \hspace{1cm} (1.120)

formula (1.119) implies that

\[ \sum_{n=1}^{\infty} e^{-n^2/z} = -\frac{1}{2} + \frac{1}{2} \sqrt{\frac{z\pi}{2}} \sum_{n=1}^{\infty} e^{-\frac{\pi^2 n^2}{z^2}}. \]  \hspace{1cm} (1.121)

Relation (1.118) follows directly from (1.121). \hspace{1cm} \square

Proof of Theorem 1.10. In view of (1.18) we have

\[ z_n = (n\pi/a)^2 + (1/a)\delta(a) + o(1), \quad n \to \infty. \]  \hspace{1cm} (1.122)

Substituting \( z = T(a/\pi)^2 \) into (1.118) and taking into account (1.122) we derive (1.20), where \( Z \) is given by (1.19). \hspace{1cm} \square

In the proof of Theorem 1.14, the next well-known relations

\[ \sum_{n=1}^{\infty} e^{-2n/T} = \frac{1}{2 \sinh(1/T)} = \frac{1}{2}(T + O(1/T)), \quad T \to \infty; \]  \hspace{1cm} (1.123)

\[ \sum_{n=1}^{\infty} e^{-2n/T} / \sqrt{2n} = \int_{0}^{\infty} \left( e^{-2x/T} / \sqrt{2x} \right) dx + O(1), \quad T \to \infty \]  \hspace{1cm} (1.124)

are used instead of the Poisson formula, which was applied in the previous proof. Since

\[ \int_{0}^{\infty} \left( e^{-u} / \sqrt{u} \right) du = \sqrt{\pi}, \]  \hspace{1cm} (1.125)

the assertion of the Theorem 1.14 follows from (1.123) and (1.124).

1.5 Dirac system on a finite interval

The considerations from the previous subsections can be used also to deal with Dirac system on a finite interval. In this final subsection of the section
we consider the Dirac system (1.2), (1.3) on the segment $[0, a]$ and assume that the inequality
\[ \int_0^a |q(t)| dt < \infty \] (1.126)
holds. The solution $F(r, \varepsilon, \ell)$ of the integral equation
\[ F(r, \varepsilon, \ell) = F_0(r, \varepsilon, \ell) - \int_0^r U_0(r) U_0(t)^{-1} \hat{V}(t) F(t, \varepsilon, \ell) dt, \] (1.127)
where $U_0$ and $\hat{V}$ are given by (1.109), satisfies the system (1.2), (1.3). Here $F_0$ is a regular solution of the system (1.2), (1.3) with $q \equiv 0$ and is given by (1.101), (1.103) and (1.104). The parameter $\varepsilon = \varepsilon(z)$ above is the same as in Remark 1.2 and in Subsection 1.3. Recall that $B(0, l]$ stands for the class of bounded functions on $(0, l]$, that is, such functions, the values of which are uniformly bounded in the norm. Below we consider functions bounded with respect to the variable $r$. From formulas (1.85), (1.86), (1.103) and (1.104) we derive
\[ F_0(r, \varepsilon, \ell) / (\sqrt{m + z(2r \varepsilon)^{\ell-1} r}) \in B(0, 1/|\varepsilon|) \quad (z > m). \] (1.128)
In fact the left-hand side of (1.128) is bounded when $z > m$ and $\rho = 2r \varepsilon$ is bounded. In a similar way, using (1.93), (1.94), (1.105) and (1.106), we have
\[ \tilde{F}_0(r, \varepsilon, \ell) / (\sqrt{m + z(2r \varepsilon)^{-\ell-1} r}) \in B(0, 1/|\varepsilon|) \quad (z > m). \] (1.129)
According to (1.90), (1.91) and (1.98) the relation
\[ U_0(r) \in B(0, 1/|\varepsilon|) \quad (z > m) \] (1.130)
is valid. Relations (1.109), (1.128), (1.129) and (1.130) imply that
\[ \sqrt{m + z(2r \varepsilon)^{\ell-1} r} U_0(r)^{-1} \in B(0, 1/|\varepsilon|) \quad (z > m) \] (1.131)
and
\[ \sup \|(t/r)^2 U_0(r) U_0(t)^{-1}\| < \infty, \quad 0 \leq t \leq r \leq 1/|\varepsilon| \quad (z > m). \] (1.132)
It follows from (1.127), (1.128) and (1.132) that
\[ F(r, \varepsilon, \ell) / (\sqrt{m + z(2r \varepsilon)^{\ell-1} r}) \in [B(0, 1/|\varepsilon|) \quad (z > m). \] (1.133)
In view of (1.130), (1.131) and (1.133) the relation
\[ \left\| \int_0^{1/|\epsilon|} U_0(r)U_0(t)^{-1}\hat{V}(t)F(t, \epsilon, \ell)dt \right\| = o(1), \quad \epsilon \to \infty \quad (z > m) \] (1.134)
is valid. Substituting (1.90) and (1.91) into (1.103) and (1.104), we represent \( F_0(r, \epsilon, \ell) \) (for the case that \( \epsilon = \epsilon(z), \ z > m \)) in the form
\[ e^{-i\ell\pi/2}F_0(r, \epsilon, \ell) = e^{-r\epsilon}e^{i\ell\pi/2}h_1(r) + e^r\epsilon e^{-i\ell\pi/2}h_2(r) + o(1), \quad \epsilon \to \infty, \] (1.135)
where the quantum defect \( \delta(r) \) has the form
\[ \delta(r) = \int_0^r q(t)dt. \] (1.139)

Proof. In order to prove the theorem, we represent \( F(r, \epsilon, \ell) \) in the form
\[ e^{-i\ell\pi/2}F(r, \epsilon, \ell) = e^{-r\epsilon}e^{i\ell\pi/2}h_1(r) + e^r\epsilon e^{-i\ell\pi/2}h_2(r) + \hat{f}(r, \epsilon, \ell) \] (1.140)
and estimate \( \hat{f} \). Definitions (1.136) and (1.138) imply that
\[ h_1(r) = g_1 - \int_0^r T_1\hat{V}(t)h_1(t)dt, \quad h_2(r) = g_2 - \int_0^r T_2\hat{V}(t)h_2(t)dt. \] (1.141)
Let us multiply both sides of (1.140) by \( e^{i\ell\pi/2} \) and substitute the result into (1.127). Then, formula (1.141) implies that the function
\[ G(r, \epsilon, \ell) = \hat{f}(r, \epsilon, \ell) + \int_0^r U_0(r)U_0(t)^{-1}\hat{V}(t)\hat{f}(t, \epsilon, \ell)dt \] (1.142)
satisfies the relation
\[ \|G(r, \varepsilon, \ell)\| \to 0, \quad z \to +\infty. \] (1.143)

Using (1.126), (1.142) and (1.143), we obtain
\[ \|\hat{f}(r, \varepsilon, \ell)\| \to 0, \quad z \to +\infty. \] (1.144)

Recall that \( F = \text{col} \begin{bmatrix} F_1 & F_2 \end{bmatrix} \) is the regular solution of the radial Dirac system. Consider the case of the second boundary condition
\[ F(a, \varepsilon_n, \ell) \sin \psi + F_2(a, \varepsilon_n, \ell) \cos \psi = 0, \quad -\pi/2 \leq \psi \leq \pi/2. \] (1.145)

Here \( \varepsilon_n = \varepsilon(z_n) = \sqrt{m^2 - z_n^2} \) and (differently from other considerations of this section, where \( n \) always belongs \( \mathbb{N} \)) \( n \in \mathbb{Z} - \{0\} \). Without loss of generality we assume that \( z_n > m \) for \( n > 0 \) and \( z_n < -m \) for \( n < 0 \).

**Corollary 1.20** Let conditions (1.126) and (1.145) be fulfilled. Then we have
\[ z_n(a) = \frac{\pi}{a} \left( n + \frac{\ell}{2} \right) + \frac{\psi - \pi/2 + \delta(a)}{a} + o(1). \] (1.146)

**Proof.** It follows from Theorem 1.19 that
\[ F_1(r, \varepsilon_n, \ell) = -e^{i\pi/2} \frac{\Gamma(2\ell + 1)}{\Gamma(\ell + 1)} \sin \left( rs_n - \ell \pi/2 - \delta(r) \right) + o(1), \] (1.147)
\[ F_2(r, \varepsilon_n, \ell) = -e^{i\pi/2} \frac{\Gamma(2\ell + 1)}{\Gamma(\ell + 1)} \cos \left( rs_n - \ell \pi/2 - \delta(r) \right) + o(1), \] (1.148)

where \( \varepsilon_n = i\sqrt{z_n^2 - m^2} = is_n \) (\( n > 0, \ z_n > m, \ s_n > 0 \)). In view of (1.145), (1.147) and (1.148), the relation
\[ \cos \left( as_n - \ell \pi/2 - \delta(a) - \psi \right) = o(1) \] (1.149)
is valid. Hence the equality (1.146) is proved for \( n > 0 \). The case \( n < 0 \) can be dealt with in the same way. \( \square \)

**Remark 1.21** Formula (1.146) is essential for solving the corresponding inverse sliding problem.

**Remark 1.22** When \( \ell = 0 \), formula (1.146) is well-known (see [13, Ch.VII]).
2 Schrödinger and Dirac equations with Coulomb-type potentials

The radial Schrödinger equation with the Coulomb-type potential has the form

$$\frac{d^2 y}{dr^2} + \left( z + \frac{2a}{r} - \frac{\ell(\ell + 1)}{r^2} - q(r) \right) y = 0, \quad 0 \leq r < \infty, \quad a = \bar{a} \neq 0,$$

where $\ell = 0, 1, 2, \ldots$ Its potential differs from the potential in (1.1) by the additional term $\frac{2a}{r}$. The radial Dirac system (relativistic case) with the Coulomb-type potential has the form:

$$\left( \frac{d}{dr} + \frac{\ell}{r} \right) f_1 - \left( z + m + \frac{a}{r} - q(r) \right) f_2 = 0,$$

$$\left( \frac{d}{dr} - \frac{\ell}{r} \right) f_2 + \left( z - m + \frac{a}{r} - q(r) \right) f_1 = 0,$$

$$0 \leq r < \infty, \quad m > 0, \quad q = \bar{q}, \quad a = \bar{a} \neq 0, \quad \ell = \pm 1, \pm 2, \ldots, \quad \ell^2 > a^2.$$

Equations (2.2) and (2.3) differ from the equations (1.2) and (1.3) by the additional term $\frac{a}{r}$. Like in Section 1, without loss of generality we consider only the case $\ell > 0$.

The scheme presented in Section 1 admits modification for the case of Coulomb-type potentials. Equation (2.1) and system (2.1), (2.2) are essential in the study of the spectrum of atoms and molecules [2, 3, 22].

We assume that

$$\int_0^{\infty} |q(t)| dt < \infty, \quad r > 0.$$

In the present section we describe the asymptotic behavior of the solutions of equation (2.1) and of system (2.2), (2.3) for the energy tending to infinity (i.e., $z \to \infty$). Using this asymptotics, we introduce the notion of the quantum defect and solve the corresponding sliding inverse problem for the relativistic case.
2.1 Asymptotics of the solutions: Schrödinger equation

If \( q(r) \equiv 0 \), then (2.1) takes the form:

\[
\frac{d^2 y}{dr^2} + \left( z + \frac{2a}{r} - \frac{\ell(\ell + 1)}{r^2} \right) y = 0.
\]

(2.6)

We denote by \( u_1 \) and \( u_2 \) the solutions of (2.6) such that

\[
u_1(r, \varepsilon, \ell) = (2\varepsilon)^{-\ell - 1} M_{a/\varepsilon, \ell + 1/2}(2r\varepsilon), \quad u_2(r, \varepsilon, \ell) = (2\varepsilon)^{\ell} W_{a/\varepsilon, \ell + 1/2}(2r\varepsilon),
\]

(2.7)

where \( M_{\kappa, \mu}(z) \) and \( W_{\kappa, \mu}(z) \) are Whittaker functions. Like in Subsection 1.2, the parameter \( \varepsilon = i\sqrt{z} \) is defined for all \( z \) in the complex plane with the cut along the negative part of the imaginary axis. We put \( \text{arg} \varepsilon = \pi/2 \) for \( z > 0 \).

Recall the connections between the Whittaker and confluent hypergeometric functions (see [5]):

\[
M_{a/\varepsilon, \ell + 1/2}(x) = e^{-x/2} x^{c/2} \Phi(\alpha, c, x), \quad W_{a/\varepsilon, \ell + 1/2}(x) = e^{-x/2} x^{c/2} \Psi(\alpha, c, x),
\]

(2.8)

where \( \alpha = \ell + 1 - a/\varepsilon, \quad c = 2\ell + 2 \). Instead of asymptotics in (1.45), we need now the asymptotics of \( \Phi \) and \( \Psi \) for energy tending to infinity. More precisely, we need the following well-known formulas (see [5] Section 6.13):

\[
\Phi(\alpha, c, 2r\varepsilon) \sim \frac{(2\ell + 2)}{(2r\varepsilon)^{-\ell - 1}} \frac{\Gamma(2\ell + 2)}{\Gamma(\ell + 1)} (2r\varepsilon)^{-\ell - 1} e^{2r\varepsilon},
\]

(2.9)

\[
\Psi(\alpha, c, 2r\varepsilon) \sim (2r\varepsilon)^{-\ell - 1}, \quad \varepsilon \to \infty \text{ and arg}(\varepsilon(z)) = \pi/2 \text{ (i.e., } z > 0\text{).}
\]

(2.10)

Using (2.9) and (2.10), we obtain the relations

\[
u_1(r, \varepsilon) \sim \frac{(2\ell + 2)}{(\ell + 1)} ((-1)^{\ell + 1} e^{-\varepsilon} + e^{\varepsilon}) (2\varepsilon)^{-\ell + 1}, \quad \varepsilon \to \infty, \quad z > 0,
\]

(2.11)

\[
u_2(r, \varepsilon) \sim (2\varepsilon)^{\ell} e^{-\varepsilon}, \quad \varepsilon \to \infty, \quad z > 0.
\]

(2.12)

Next, we consider the general-type Schrödinger equation (2.1) (where \( q \) is not necessarily trivial). The solution \( u(r, \varepsilon, \ell) \) of the integral equation

\[
u(r, \varepsilon, \ell) = u_2(r, \varepsilon, \ell) - \int_{r}^{\infty} k(r, t, \varepsilon, \ell) q(t) u(t, \varepsilon, \ell) dt,
\]

(2.13)
where the kernel $k(r, t, \varepsilon, \ell)$ is defined by
\[ k(r, t, \varepsilon, \ell) = \frac{\Gamma(\ell + 1 - a/\varepsilon)}{\Gamma(2\ell + 2)} (u_1(r, \varepsilon, \ell)u_2(t, \varepsilon, \ell) - u_2(r, \varepsilon, \ell)u_1(t, \varepsilon, \ell)), \]
(2.14)
satisfies (2.1). It follows from (2.11), (2.12) and (2.14) that
\[ k(r, t, \varepsilon, \ell) \sim e^{(r-t)\varepsilon} - e^{-(r-t)\varepsilon}, \quad \varepsilon \to \infty, \quad z > 0. \]
(2.15)

Using standard methods, we deduce from (2.12)–(2.15) the following statement.

**Theorem 2.1** Let (2.5) hold. Then, for $u(r, \varepsilon, \ell)$ satisfying (2.13) we have
\[ u(r, \varepsilon, \ell) \sim (2\varepsilon)^{-\ell} e^{-r\varepsilon}, \quad \varepsilon \to \infty, \quad z > 0. \]
(2.16)

### 2.2 Asymptotics of the solutions: Dirac system

If $q(r) \equiv 0$, system (2.2), (2.3) takes the form
\[ \left( \frac{d}{dr} + \frac{\ell}{r} \right) f_1 - (z + m + \frac{a}{r}) f_2 = 0, \quad \left( \frac{d}{dr} - \frac{\ell}{r} \right) f_2 + (z - m - \frac{a}{r}) f_1 = 0 \quad (\ell^2 > a^2). \]
(2.17)
We consider solutions of (2.17) in the form similar to (1.76) and (1.77), that is,
\[ f_1(r, \varepsilon, \ell) = \sqrt{m + z} e^{-r\varepsilon}(2r\varepsilon)^{\omega-1}(Q_1(r, \varepsilon, \ell) + Q_2(r, \varepsilon, \ell)) r, \]
(2.18)
\[ f_2(r, \varepsilon, \ell) = -\sqrt{m - z} e^{-r\varepsilon}(2r\varepsilon)^{\omega-1}(Q_1(r, \varepsilon, \ell) - Q_2(r, \varepsilon, \ell)) r, \]
(2.19)
where $\omega = \sqrt{\ell^2 - a^2} > 0$, $\varepsilon = \sqrt{m^2 - z^2}$ and the choice of arguments in $\sqrt{m \pm z}$ and $\sqrt{m^2 - z^2}$ is prescribed in Remark 1.2. For regular and non-regular solutions of (2.17) we use here the same notations as for regular and non-regular solutions of (1.2), (1.3), where $q \equiv 0$, in Section 1. For the case of regular (at $r = 0$) solutions of (2.17), the functions $Q_1$ and $Q_2$ can be expressed via the confluent hypergeometric functions $\Phi(\alpha, c, x)$ (see [2, 3]). Thus, a regular solution $F_0$ of (2.17) is given by
\[ F_0 = \text{col} \left[ F_{1,0} \quad F_{2,0} \right], \]
(2.20)
\[ F_{1,0}(r, \varepsilon, \ell) = \sqrt{m + z} e^{-r\varepsilon}(2r\varepsilon)^{\omega-1}(Q_{1,0}(r, \varepsilon, \ell) + Q_{2,0}(r, \varepsilon, \ell)) r, \]
(2.21)
\[ F_{2,0}(r, \varepsilon, \ell) = -\sqrt{m - z} e^{-r\varepsilon}(2r\varepsilon)^{\omega-1}(Q_{1,0}(r, \varepsilon, \ell) - Q_{2,0}(r, \varepsilon, \ell)) r, \]
(2.22)
where

\[ Q_{1,0} = \alpha_1 \Phi(\omega - az/\varepsilon, 2\omega + 1, 2r\varepsilon), \quad Q_{2,0} = \alpha_2 \Phi(\omega + 1 - az/\varepsilon, 2\omega + 1, 2r\varepsilon), \]

(2.23)

and

\[ \frac{\alpha_1 + \alpha_2}{\alpha_1 - \alpha_2} = -\frac{a}{\omega + \ell} \sqrt{\frac{m - z}{m + z}}. \]

(2.24)

Using asymptotic formulas (see [5]) for the confluent hypergeometric functions \( \Phi(\alpha, c, x) \), we obtain

\[ Q_{1,0}(r, \varepsilon, \ell) \sim \tilde{\alpha}_1 \frac{\Gamma(2\omega + 1)}{\Gamma(\omega + 1 - ia)} (2r\varepsilon)^{-\omega - ia}, \quad \varepsilon \to \infty \quad (z > m), \]

(2.25)

\[ Q_{2,0}(r, \varepsilon, \ell) \sim \tilde{\alpha}_2 \frac{\Gamma(2\omega + 1)}{\Gamma(\omega + 1 + ia)} e^{2r\varepsilon} (2r\varepsilon)^{-\omega + ia}, \quad \varepsilon \to \infty \quad (z > m), \]

(2.26)

where \( \tilde{\alpha}_1 + \tilde{\alpha}_2 = -\frac{ia}{\omega + \ell} \).

(2.27)

A non-regular at \( r = 0 \) solution \( \tilde{F}_0 = \text{col} [\tilde{F}_{1,0} \quad \tilde{F}_{2,0}] \) of (2.17) has the form (see [2, 3]):

\[ \tilde{F}_{1,0}(r, \varepsilon, \ell) = \sqrt{m + z} e^{-r\varepsilon} (2r\varepsilon)^{1-\omega} (\tilde{Q}_{1,0}(r, \varepsilon, \ell) + \tilde{Q}_{2,0}(r, \varepsilon, \ell)) r, \]

(2.28)

\[ \tilde{F}_{2,0}(r, \varepsilon, \ell) = -\sqrt{m - z} e^{-r\varepsilon} (2r\varepsilon)^{1-\omega} (\tilde{Q}_{1,0}(r, \varepsilon, \ell) - \tilde{Q}_{2,0}(r, \varepsilon, \ell)) r, \]

(2.29)

where

\[ \tilde{Q}_{1,0}(r, \varepsilon, \ell) = \tilde{\alpha}_1 \Psi(\omega - az/\varepsilon, 2\omega + 1, 2r\varepsilon), \]

(2.30)

\[ \tilde{Q}_{2,0}(r, \varepsilon, \ell) = \tilde{\alpha}_2 \Psi(\omega + 1 - az/\varepsilon, 2\omega + 1, 2r\varepsilon). \]

(2.31)

Here \( \Psi(a, c, x) \) is the confluent hypergeometric function of the second kind.

In view of (2.30) and (2.31) we have (see [5 Ch.6]):

\[ \tilde{Q}_{1,0} \sim \tilde{\alpha}_1 (2r\varepsilon)^{-\omega}, \quad \tilde{Q}_{2,0} \sim \tilde{\alpha}_2 (2r\varepsilon)^{-\omega}, \quad r \to \infty; \quad \tilde{\alpha}_1 = \frac{\ell \varepsilon - am}{\omega \varepsilon + az} \tilde{\alpha}_2. \]

(2.32)

Using again asymptotic formulas for \( \Phi(\alpha, c, x) \) (see [5]), we obtain

\[ \tilde{Q}_{1,0} \sim \tilde{\alpha}_1 (2r\varepsilon)^{-\omega - ia}, \quad \varepsilon \to \infty \quad (z > m), \]

(2.33)
\[ \tilde{Q}_{2,0} \sim \tilde{\alpha}_2 (2r\varepsilon)^{-\omega - ia - 1}, \quad \varepsilon \to \infty \quad (z > m), \]  
\[ \text{where} \quad \tilde{\alpha}_1 = \frac{\ell}{\omega - ia} \tilde{\alpha}_2. \]  

From (2.21), (2.22), (2.25) and (2.26) we obtain
\[ F_{1,0}(r, \varepsilon, \ell) \sim \sqrt{m + z} \Gamma(2\omega + 1) \]
\[ \times \left( e^{-\varepsilon \hat{\alpha}_1} \frac{(2r\varepsilon)^{-ia}}{2\varepsilon \Gamma(\omega + 1 - ia)} + e^{\varepsilon \hat{\alpha}_2} \frac{(2r\varepsilon)^{ia}}{2\varepsilon \Gamma(\omega + 1 + ia)} \right), \]  
\[ F_{2,0}(r, \varepsilon, \ell) \sim -\sqrt{m - z} \Gamma(2\omega + 1) \]
\[ \times \left( e^{-\varepsilon \hat{\alpha}_1} \frac{(2r\varepsilon)^{-ia}}{2\varepsilon \Gamma(\omega + 1 - ia)} - e^{\varepsilon \hat{\alpha}_2} \frac{(2r\varepsilon)^{ia}}{2\varepsilon \Gamma(\omega + 1 + ia)} \right), \]  
where \( \varepsilon \to \infty, \ z > m. \) Formulas (2.28), (2.29) and (2.33)-(2.35) imply that
\[ \tilde{\Phi}_0(r, \varepsilon, \ell) \sim \hat{\alpha}_2 e^{-\varepsilon (2r\varepsilon)^{-ia}} \text{col} \left[ \sqrt{m + z} - \sqrt{m - z} \right], \quad \varepsilon \to \infty \quad (z > m). \]  

Next, we consider the case \( q(r) \neq 0. \) The solution \( \tilde{F}(r, \varepsilon, \ell) \) of the integral equation
\[ \tilde{F}(r, \varepsilon, \ell) = \tilde{F}_0(r, \varepsilon, \ell) + \int_r^{\infty} U_0(r) U_0(t)^{-1} \tilde{V}(t) \tilde{F}(t, \varepsilon, \ell) dt, \]  
where \( \tilde{V} \) and \( U_0 \) have the form (1.109), satisfies system (2.2), (2.3). We note that \( F_{i,0} \) and \( \tilde{F}_{i,0} \) \( (i = 1, 2) \) in Section II are different from the entries \( F_{i,0} \) \( \) and \( \tilde{F}_{i,0} \) of \( U_0, \) which are introduced in this section. For the present case, formulas (2.36)-(2.38) imply that
\[ \det U_0(r, \varepsilon, \ell) \sim -\tilde{\alpha}_2 \hat{\alpha}_1 \Gamma(2\omega + 1) \frac{1}{2\varepsilon \Gamma(\omega + 1 + ia)}, \quad \varepsilon \to \infty \quad (z > m). \]  

According to (2.36)-(2.38) and (2.40) the equality (1.113) from Section II is valid again. Hence, multiplying both sides of (2.39) by \( -2\sqrt{m + z} e^{\varepsilon (2r\varepsilon)^{ia}} \) and passing to the limit \( z \to + \infty, \) we obtain
\[ \tilde{F}_\infty(r, \ell) = \left[ \begin{array}{c} 1 \\ 1 \end{array} \right] + \int_r^{\infty} \Theta_1 \tilde{V}(t) \tilde{F}_\infty(t, \ell) dt, \]  
26
where
\[ \tilde{F}_\infty(r, \ell) = -2 \lim_{z \to +\infty} \left( \sqrt{m + z} e^{r \varepsilon} (2r \varepsilon)^{ia} \tilde{F}(r, \varepsilon, \ell) \right). \]  
(2.42)

The equality
\[ \tilde{F}_\infty(r, \ell) = e^{-i \int_r^\infty q(t)dt} \begin{bmatrix} i \\ 1 \end{bmatrix} \]  
(2.43)
follows directly from (2.41)-(2.43). Thus, we proved that \( \tilde{F}(r, \varepsilon, \ell) \) constructed above satisfies the requirements of the following statement.

**Theorem 2.2** Let condition (2.5) be fulfilled. Then there exists a solution \( \tilde{F}(r, \varepsilon, \ell) \) of system (2.2), (2.3), which satisfies the relation
\[ \tilde{F}(r, \varepsilon, \ell) \sim - \frac{1}{2\sqrt{m + z}} e^{-i(r \varepsilon + \delta(r))} (2r \varepsilon)^{-ia} \begin{bmatrix} i \\ 1 \end{bmatrix}, \quad \varepsilon \to \infty \quad (z > m), \]  
(2.44)
where the quantum defect \( \delta(r) \) is given by the formula
\[ \delta(r) = \int_r^\infty q(t)dt. \]  
(2.45)

Finally, we formulate the sliding half-inverse problem.

**Problem 2.3** Recover the potential \( q - a/r \) of the Dirac system (2.2), (2.3) from the given quantum defect \( \delta \) and constant \( a \).

According to (2.45), the solution of Problem 2.3 has the form:
\[ q(r) = -\frac{d}{dr} \delta(r). \]  
(2.46)

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