Heat kernel of non-minimal gauge field kinetic operators on Moyal plane

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Abstract

We generalize the Endo formula [1] originally developed for the computation of the heat kernel asymptotic expansion for non-minimal operators in commutative gauge theories to the noncommutative case. In this way, the first three non-zero heat trace coefficients of the non-minimal $U(N)$ gauge field kinetic operator on the Moyal plane taken in an arbitrary background are calculated. We show that the non-planar part of the heat trace asymptotics is determined by $U(1)$ sector of the gauge model. The non-planar or mixed heat kernel coefficients are shown to be gauge-fixing dependent in any dimension of space-time. In the case of the degenerate deformation parameter the lowest mixed coefficients in the heat expansion produce non-local gauge-fixing dependent singularities of the one-loop effective action that destroy the renormalizability of the $U(N)$ model at one-loop level. Such phenomenon was observed at first in Ref. [2] for space-like noncommutative $\phi^4$ scalar and $U(1)$ gauge theories. The twisted-gauge transformation approach is discussed.

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1 Introduction

The heat kernel of (pseudo)differential operators has become one of the most powerful and actively developed tools in quantum field theory and spectral geometry (see [3], [4], [5], [6] where the implementation of the heat kernel technique in a variety of physical and mathematical problems is discussed in details). Nowadays this topic has acquired particular interest in the context of noncommutative geometry and quantum field theories on noncommutative spaces [7], [8], [9], [10], [11], [12]. The main result here is that the heat trace for a differential operator on a (flat) noncommutative manifold, e.g. so-called generalized star-Laplacian arising, for instance, in the noncommutative scalar $\lambda \phi^4$ theory, can be expanded in a power series in the "proper time" parameter that resembles, in some respect, the heat trace expansion for its commutative counterpart. This observation is of fundamental importance since makes it possible to employ the heat kernel machinery in many applications to noncommutative models such as the investigation of one-loop divergences or quantum anomalies [11].

Another interesting aspect of the heat kernel on noncommutative spaces is closely related to the UV/IR mixing phenomenon [13], [14], [15]. Namely, in the most general case when a star-differential operator involves both left and right Moyal multiplications (as it is for the generalized Laplacian mentioned above), its heat trace asymptotics contains a contribution produced by star-non-local terms that are singular when the deformation parameter vanishes. Clearly, it defines the non-planar part of the heat kernel expansion which is, in particular, responsible for the UV/IR mixing [9], [11]. The situation gets even more intriguing in the case when the deformation parameter is degenerate (that corresponds to space-like noncommutativity). In this case the non-planar contribution to the heat expansion becomes dangerous since it can affect the one-loop renormalization of a theory under consideration [2], [12].

In this paper we investigate the heat trace asymptotics for second order star-differential operators\(^1\) on noncommutative non-compact flat manifolds without boundary in the spirit of Ref. [11]. We restrict our consideration to the case of non-minimal operators appearing in the non-commutative $U(N)$ gauge theory in the background field formalism. To be precise, we are concerned with gauge field kinetic operator on the Moyal plane taken in the covariant background gauge with an arbitrary gauge-fixing parameter. In the commutative case non-minimal operators (in various physical systems) were investigated by many authors [1], [16], [17], [18], [19], [20], [21]. In our study of the heat asymptotics we will follow the calculating method by Endo allowing to reduce the whole task to the computation of the heat trace coefficients for minimal operators by means of some algebraic relations between the heat kernel matrix elements [11], [19]. Indeed, this method turns out to be especially convenient within the background field formalism; at the same time, its purely algebraic nature allows one to generalize it easily to the noncommutative case.

The paper is organized as follows. The relevant basic formulae are briefly reviewed in section 2. In section 3 we derive the non-commutative version of the Endo formula elaborated primarily for computations in the commutative gauge theories. In section 4 we then calculate the heat trace coefficients of $U(1)$ gauge model; the general case of $U(N)$ gauge symmetry is investigated in details in Section 5. In section 6 we discuss the twisted gauge transformation approach.

\(^1\)That is, we will consider differential operators that contain star-products and partial derivatives not higher then of second order. It should be emphasized that such an operator is no longer a partial differential operator from the commutative point of view; even to call it a pseudodifferential operator is not totally correct. Indeed, the presence of the star-product, which is itself a differential operator on the corresponding commutative manifold, results in the "incorrect" oscillatory behaviour of the symbol for the operator and, hence, it cannot be regarded as a pseudodifferential operator in the strict sense of this term (see [7] for the discussion of this point).
Finally, we conclude with a summary presented in section 7. To make the paper self-contained some technical details on the evaluation of the heat kernel coefficients are adduced in appendices.

In the paper we adopt the following conventions: small Greek letters from the beginning of the alphabet, $\alpha, \beta, \gamma, \delta$, denote indices of the $U(N)$ inner group space; letters from the middle of the Greek alphabet, $\lambda, \mu, \nu, \ldots$, refer to the indices of an $n$-dimensional Euclidean space. Capital and small Latin letters are used to label generators of the $U(N)$ and $SU(N)$ groups, respectively, i.e. $A, B, C = 0, 1, \ldots, N^2 - 1$ and $a, b, c = 1, \ldots, N^2 - 1$.

2 Non-minimal operators in noncommutative gauge theories

Consider a self-adjoint second order non-minimal star-differential operator that corresponds to the kinetic operator of gauge particles propagating on Moyal plane in an external background. It can be represented in the form

$$D^\xi_{\mu \nu} = -\left[\delta_{\mu \nu} \nabla^2 + \left(\frac{1}{\xi} - 1\right)\nabla_\mu \nabla_\nu + 2(L(F_{\mu \nu}) - R(F_{\mu \nu}))\right],$$

where

$$\nabla_\mu = \partial_\mu + L(B_\mu) - R(B_\mu)$$

is an anti-Hermitian covariant-derivative operator in the background field $B_\mu$, $\xi$ is a numerical gauge-fixing parameter and $F_{\mu \nu}$ is the curvature tensor of the gauge connection $B_\mu$. Here operator $L$ (accordingly, $R$) involves both left (right) Moyal and left (right) matrix multiplications, i.e.

$$L(l)f = l \ast f, \quad R(r)f = f \ast r,$$

with $f, l$ and $r$ being matrix valued functions; the Moyal star-product on $\mathbb{R}^n$ can be defined as

$$l(x) \ast f(x) = l(x) \exp\left(\frac{i}{2} \theta_{\mu \nu} \partial_\mu \overline{\partial_\nu}\right)f(x),$$

where $\theta$ is a constant antisymmetric matrix (in practice it is sometimes convenient to use the Reiffel representation of the star-product, see Appendix B). Commutator of the covariant derivatives gives

$$[\nabla_\mu, \nabla_\nu] = L(F_{\mu \nu}) - R(F_{\mu \nu}), \quad F_{\mu \nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu].$$

We denote the operator $-\nabla^2$ as $D_0$ which is a self-adjoint non-negative operator corresponding to the inverse propagator of ghost particles. In the following we assume that the operators $D_0$ and $D^\xi_{\mu \nu}$ have no zero-modes.

\footnote{Such an operator naturally appears in the NC $U(N)$ theory in the background field formalism; it defines, in particular, the quadratic in quantum gauge fields part of the total action written in a covariant background gauge:

$$S_2[Q] = -\frac{1}{2} \int_{\mathbb{R}^n} d^n x \, \text{tr}_N Q_\mu(x) D^\xi_{\mu \nu} Q_\nu(x),$$

where $\text{tr}_N$ means trace over internal indices (although we do not write them explicitly) and $Q_\mu$ describes quantum fluctuations of the gauge fields. Functional integration of the expression $\exp S_2[Q]$, as known, gives the one-loop effective action, $\Gamma_{\text{gauge}}[B] = \frac{1}{\hbar} \ln \det(D^\xi)$, that is invariant under the background field (star)gauge transformations of the form $\delta B_\mu(x) = \nabla_\mu \lambda(x)$. Some aspects of the background field formalism in NC field theories can be found, for instance, in Ref. \[22\].}
To simplify our analysis let us consider the case of $U(1)$ gauge symmetry (generalization to the case of $U(N)$ symmetry will be discussed in Section 5). The heat trace for the kinetic operator (1) is defined as
\[
K^\xi(t) = Tr_{L^2} \exp(-tD^\xi),
\]
where $t$ is a (positive) spectral parameter and the trace is taken on the space of square integrable functions [5], [6]. Usually this expression is regularized by subtracting the heat trace of the Laplacian $\triangle = -\partial_\mu \partial^\mu$ since the small $t$ asymptotic expansion of the quantity $Tr_{L^2} \exp(-tD^\xi)$ contains a volume term that is divergent on a non-compact manifold.

We wish to compute the heat trace in the limit of small spectral parameter $t \to 0$ by means of the Fock-Schwinger-DeWitt proper-time method. To this aim we introduce two abstract Hilbert spaces spanned by basis vectors $|x\rangle$ and $|\mu, x\rangle$, respectively, and define "Hamiltonian" operators $\hat{D}_0$ and $\hat{D}_\xi^{\mu\nu}$ associated with $D_0$ and $D_\xi^{\mu\nu}$ by
\[
\langle x|\hat{D}_0|x'\rangle = D_0 \langle x|x'\rangle, \\
\langle x, \mu|\hat{D}_\xi^{\mu\nu}|\nu, x'\rangle = D_\xi^{\mu\nu} \langle x, \mu|\nu, x'\rangle.
\]
Operators on the right hand sides of these expressions are viewed as differential operators with respect to the variable $x$. The basis vectors satisfy the orthonormality conditions
\[
\langle x|x'\rangle = \delta(x, x'), \\
\langle x, \mu|\nu, x'\rangle = \delta_{\mu\nu} \delta(x, x').
\]
Note that, in the case of an arbitrary manifold, index of $|\mu, x\rangle$ (as well as that of the conjugate $\langle x, \mu|$) is regarded as that of a covariant vector density of weight 1/2 [23].

Next, the proper-time transformation functions, or heat kernels, for the operators $D_0$ and $D_\xi^{\mu\nu}$ are introduced by
\[
K_0(x, x'; t) = \langle x|\exp[-t\hat{D}_0]|x'\rangle, \\
K_\xi^{\mu\nu}(x, x'; t) = \langle x, \mu|\exp[-t\hat{D}_\xi^{\mu\nu}]|\nu, x'\rangle,
\]
where $t$ is interpreted as the proper-time parameter [24]. By making use of (3) it can be straightforwardly checked that the kernels $K_0(x, x'; t)$ and $K_\xi^{\mu\nu}(x, x'; t)$ satisfy the heat equations:
\[
\left(\frac{\partial}{\partial t} + D_0\right)K_0(x, x'; t) = 0, \\
\left(\delta_{\mu\lambda}\frac{\partial}{\partial t} + D_\xi^{\mu\lambda}\right)K_\xi^{\mu\lambda}(x, x'; t) = 0,
\]
with the boundary conditions
\[
\lim_{t \to 0} K_0(x, x'; t) = \delta(x, x'), \\
\lim_{t \to 0} K_\xi^{\mu\nu}(x, x'; t) = \delta_{\mu\nu} \delta(x, x').
\]

3In this paper we are dealing with flat Euclidean space and therefore the distinction between upper and lower indices is irrelevant.

4That is, the exponential operators on the right hand sides of (3) can be regarded as evolution operators of a "particle" in the proper time $t$ [23].
From the kernels (4) one can obtain the one-loop effective action of pure NC Yang-Mills theory using the standard formal expressions:
\[
\Gamma^{(1)}[B] = \Gamma_{\text{gauge}}[B] + \Gamma_{\text{ghost}}[B],
\]
\[
\Gamma_{\text{gauge}}[B] = \frac{1}{2} \ln \det(D^\xi) = -\frac{1}{2} \int_{R^n} dx \int_0^\infty \frac{dt}{t} tr_V K_\xi^{\mu\nu}(x, x; t),
\]
\[
\Gamma_{\text{ghost}}[B] = -\ln \det(D) = \int_{R^n} dx \int_0^\infty \frac{dt}{t} tr_V K_0(x, x; t),
\]
where the first term, \(\Gamma_{\text{gauge}}[B]\), describes a contribution to the effective action coming from the gauge sector of the model while the second term, \(\Gamma_{\text{ghost}}[B]\), stands for the ghost contribution; \(tr_V\) means trace over Euclidean vector and, in general, internal indices. As is well-known, the expression for \(\Gamma^{(1)}[B]\) is divergent and must be regularized. This can be done, for instance, by replacing \(1/t\) in the integrands of (7) with \(\mu^2/\epsilon t^{1-\epsilon}\), where \(\epsilon\) is a complex parameter and \(\mu\) is a dimensional quantity introduced to keep the total mass dimension of the expression unchanged.

Now all information on the one-loop effective action contains in the heat traces which at \(t \to 0^+\) can be expanded in series over the spectral (proper time) parameter:
\[
TrK(D; t) \simeq \sum_{k=0}^\infty t^{(k-n)/2}a_k(D),
\]
The coefficients \(a_k(D)\) here define the asymptotics of the heat trace as \(t \to 0\). On the manifold without boundary odd-numbered coefficients are equal to zero. From the expressions (7) and (8) one sees that terms with \(k \leq n\) in the heat kernel expansion can potentially give rise to divergences in the effective action (see also discussion in the end of section 5).

In the commutative case the heat kernel coefficients \(a_k\), known also as diagonal Seeley-Gilkey-DeWitt coefficients [23], [25], [26], are expressed only in terms of local gauge covariant quantities, such as matter fields, gauge field strength tensor and their covariant derivatives\(^5\), and, hence, are manifestly gauge invariant objects (see review article [6]). However, on \(\theta\)-deformed manifolds, there appears another type of coefficients in the heat kernel expansion [8], so-called mixed coefficients, that reflect the non-local nature of NC field theories [9], [11]. As we have mentioned earlier, the contribution of these mixed terms is equivalent to the contribution of non-planar diagrams to the effective action. In particular, it can develop non-local singularities as \(\epsilon \to 0\) if the deformation parameter is degenerate [2]. We will comment this point later on.

### 3 Noncommutative Endo formula

Consider the heat trace for the operator (11), \(TrK_\xi^{\mu\nu}(x, x; t)\), and compute the first three non-zero heat kernel coefficients in the small \(t\) asymptotic expansion for this quantity. In the Feynman gauge \((\xi = 1)\) it can be done by means of the calculating procedure described in Refs. [7], [8], [11]. To apply it in the more general case of an arbitrary value of the gauge-fixing parameter we will reproduce in what follows the non-commutative version of the Endo formula [1], [19]. To simplify our computations we suppose that the background field satisfies the equation of motion:
\[
\nabla_\mu F_{\mu\nu} = \partial_\mu F_{\mu\nu} + [B_\mu, F_{\mu\nu}] = 0.
\]
First, we note the obvious relation
\[ [\nabla_\mu, \nabla^2] = 2\{L(F_{\mu\nu}) - R(F_{\mu\nu})\}\nabla_\nu. \tag{10} \]

From (11), (9) and (10) it is easily seen that
\[ \nabla_\mu \nabla_\lambda D^\xi_{\lambda\nu} = D^\xi_{\mu\lambda} D_{\lambda\nu} = -\frac{1}{\xi} \nabla_\mu \nabla^2 \nabla_\nu \]
and
\[ \nabla_\mu \nabla_\lambda D^\xi_{\lambda\nu} = \frac{1}{\xi} \nabla_\mu \nabla_\lambda D^\xi_{\lambda\nu}^{(\xi=1)}. \tag{11} \]

The first equality in (11) means that the operators \( \nabla_\mu \nabla_\lambda \) and \( D^\xi_{\lambda\nu} \) commute each other, as they do in the commutative case. With the help of (11) and (12) one arrives at
\[ \nabla_\mu \nabla_\lambda [D^\xi n]_{\lambda\nu} = \frac{1}{\xi^n} \nabla_\mu \nabla_\lambda [D n]_{\lambda\nu}, \tag{13} \]
where we denote \( D_{\mu\nu} = D^\xi_{\mu\nu} \), and the expression \([D n]_{\mu\nu}\) stands for the \( n \)th power of the operator \( D_{\mu\nu} \) in the usual sense\(^6\), i.e.
\[ [D^\xi n]_{\mu\nu} = D^\xi_{\mu\lambda_1} D^\xi_{\lambda_1\lambda_2} \ldots D^\xi_{\lambda_{n-1}\nu} \]

In particular, with the help of (12), one gets the following useful relation
\[ \nabla_\mu \nabla_\lambda \exp(-tD^\xi)_{\lambda\nu} = \nabla_\mu \nabla_\lambda \exp\left(-\frac{t}{\xi} D\right)_{\lambda\nu}. \tag{14} \]

With these expressions we are ready to derive the non-commutative Endo formula. For this purpose we differentiate both sides of (11) with respect to \( 1/\xi \) to get:
\[ \frac{\partial}{\partial \xi^{-1}} K^\xi_{\mu\nu}(x, x'; t) = \nabla_\mu \nabla_\lambda K^\xi_{\lambda\nu}(x, x'; t) = t \nabla_\mu \nabla_\lambda K_{\lambda\nu}(x, x'; \frac{t}{\xi}), \tag{15} \]
and then integrate the obtained relation over \( \xi^{-1} \). Here \( K_{\mu\nu}(t) = K^\xi_{\mu\nu}(1) \). After redefinition of the integration parameter one arrives at (cf. expr. (2.23) in Ref. [1])
\[ K^\xi_{\mu\nu}(x, x'; t) = K_{\mu\nu}(x, x'; t) + \int_1^t d\tau \nabla_\mu \nabla_\lambda K_{\lambda\nu}(x, x'; \tau). \tag{16} \]

It is easy to show that this expression satisfies the heat kernel equation (5) as it should. The main advantage of the formula (16) is that one deals now with the kernel of the minimal operator \( D_{\mu\nu}^{\xi=1} \), i.e. \( K_{\mu\nu}(x, x'; t) \), which is much more convenient in practical computations.

Equation (16) can be simplified further. To do this we need the Ward identity for the heat kernels which is expressed by the relation\(^7\)
\[ \nabla_\lambda K^\xi_{\lambda\nu}(x, x'; \tau) = -\nabla^\nu_\nu K_0 \left(x, x'; \frac{\tau}{\xi}\right), \tag{17} \]

\(^6\)Remind that the operators \( D^{(\xi)} \) and \( \overline{D} \) contain both left and right star-multiplications.

\(^7\)The commutative analogue of this expression is derived in Refs. [1], [19], [28].
Here and further on prime over nabla on RHS of (17) indicates that covariant derivative acts on \( x' \) variable.

To prove (17) we note that for the operators of the left and right Moyal multiplications there are the following rules:

\[
\langle x, \mu | \hat{L}(B) | \nu, x' \rangle := L(B(x)) \langle x, \mu | \nu, x' \rangle
\]

and

\[
\langle x, \mu | \hat{R}(B) | \nu, x' \rangle := R(B(x)) \langle x, \mu | \nu, x' \rangle.
\]

As a consequence, one has

\[
\nabla_\lambda \langle x, \mu | \nu, x' \rangle = -\nabla_{\lambda'} \langle x, \mu | \nu, x' \rangle,
\]

that is, at \( \tau = 0 \) left- and right-hand sides of the Ward identity (17) agree by the heat kernel boundary conditions (6).

Next, following Endo, we define an operator \( \hat{\delta} \) that connects both Hilbert spaces through the relation:

\[
\langle x | \hat{\delta} | \nu, x' \rangle = -\nabla_\mu \langle x, \mu | \nu, x' \rangle = \nabla_{\mu'} \langle x | x' \rangle.
\]

From (11) one can see that

\[
\nabla_\lambda D^\xi_{\lambda\nu} = \frac{1}{\xi} D_0 \nabla \nu,
\]

which is written in the operator form as

\[
\hat{\delta} \hat{D}^\xi = \frac{1}{\xi} \hat{D}_0 \hat{\delta}.
\]

By making use of the definition for the operator \( \hat{\delta} \), one obtains the needed identity:

\[
\nabla_\lambda K^\xi_{\lambda\mu}(x, x'; \tau) = \nabla_\lambda \langle x, \lambda | \exp(-\tau \hat{D}^\xi) | \nu, x' \rangle = -\langle x | \hat{\delta} \exp(-\tau \hat{D}^\xi) | \nu, x' \rangle
\]

\[
= -\langle x | \exp \left( -\frac{\tau}{\xi} \hat{D}_0 \right) \delta | \nu, x' \rangle = -\nabla_{\nu'} K_0 \left( x, x'; \frac{\tau}{\xi} \right). \tag{18}
\]

In particular, for \( \xi = 1 \) we have:

\[
\nabla_\lambda \overline{K}_{\lambda\mu}(x, x'; \tau) = -\nabla_{\nu'} K_0(x, x'; \tau), \tag{19}
\]

and the expression (16) can be represented now in the form

\[
K^\xi_{\mu\nu}(x, x'; t) = \overline{K}_{\mu\nu}(x, x'; t) - \int_t^\xi d\tau \nabla_\mu \nabla_{\nu'} K_0(x, x'; \tau). \tag{20}
\]

Formula (20) is the starting point of our computations. More precisely, we are going to investigate the heat asymptotics for the trace of the kernel (20). In this connection it is necessary to note that the operators \( \exp(-t D^{\xi-1}) \) and \( \exp(-t D_0) \) are trace-class for positive values of the spectral parameter \( t \) and, hence, the asymptotic expansions for the kernels in RHS of (20) are well-defined [7], [8], [11], [12].
4 Evaluation of the heat kernel coefficients

We wish to calculate the heat kernel coefficients in the asymptotic expansion for the quantity $Tr K_{\mu\nu}(t)$. In what follows we assume that the $\theta$-parameter is non-degenerate.

The asymptotic expansion for the first term on RHS of (20) is, in fact, investigated in Ref. [11] where the heat kernel coefficients for generalized Laplacians on the Moyal plane containing both left and right multiplications were calculated. The result is presented in (30) below. Consider the second term. After performing an integration by parts and taking the cyclic property of the Moyal product into account one gets

$$Tr \nabla_\mu \nabla'_\nu K_0(\tau) = \int_{\mathbb{R}^n} dx \left[ \nabla_\mu e^{-tD_0} \langle x \mid x' \rangle \nabla'_\mu \right]_{x=x'}$$

$$= - \int_{\mathbb{R}^n} dx \left[ \nabla_\mu \nabla_\mu e^{-tD_0} \langle x \mid x' \rangle \right]_{x=x'}.$$

As it was mentioned in the end of the previous section, for this expression there is an asymptotic expansion

$$Tr \nabla_\mu \nabla'_\nu K_0(\tau) \simeq \sum_{k=-2}^{\infty} \frac{t^{(k-n)/2}}{\tilde{a}_k(\nabla^2, D_0)},$$

(22)

where the coefficients $\tilde{a}_k(\nabla^2, D_0)$ can be decomposed as (cf. [11])

$$\tilde{a}_k(\nabla^2, D_0) = \tilde{a}_{k,\text{planar}}(\nabla^2, D_0) + \tilde{a}_{k,\text{mixed}}(\nabla^2, D_0).$$

(23)

Here the coefficients $\tilde{a}_{k,\text{planar}}(\nabla^2, D_0)$ are expressed as integrals of gauge invariant star polynomials of the fields (i.e. with the deformation parameter being hidden in the Moyal products). These coefficients contribute to the planar part of the heat expansion. The other type of the heat kernel coefficients, $\tilde{a}_{k,\text{mixed}}(\nabla^2, D_0)$, corresponds to the contributions from non-planar diagrams in the diagrammatic language; these are so-called mixed heat kernel coefficients.

There are several ways to compute the planar heat kernel coefficients. One of them is based on the functorial properties of the heat kernel described in [26], [29], [6]. To this aim one has to re-express (21) in the form:

$$\int_{\mathbb{R}^n} dx \left[ \nabla_\mu \nabla_\mu e^{-tD_0} \langle x \mid x' \rangle \right]_{x=x'} =$$

$$= \int_{\mathbb{R}^n} dx \left[ (\nabla^L_\mu \nabla^L_\mu e^{-tD^L_0} + \nabla^R_\mu \nabla^R_\mu e^{-tD^R_0} - \partial^2 e^{tD_0} + \text{mixed terms} ) \langle x \mid x' \rangle \right]_{x=x'},$$

(24)

where

$$\nabla^L_\mu = \partial_\mu + L(B_\mu), \quad D^L_0 := -\nabla^L_\mu \nabla^L_\mu,$$

and

$$\nabla^R_\mu = \partial_\mu - R(B_\mu), \quad D^R_0 := -\nabla^R_\mu \nabla^R_\mu.$$

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8 Hence we are working in even-dimensional manifolds. The case of the degenerate parameter $\theta$ will be discussed in the end of the next section.

9 We use tilde to indicate that these coefficients correspond to the quantity $Tr \nabla_\mu \nabla'_\nu K_0(\tau)$. In the end we have to carry out integration over the spectral parameter in the expression (22), in accordance with the formula (25) below.

10 One has to expand formally the operator exponent in the integrand of (21) into power series over the spectral parameter and single out terms which contain only left and right Moyal multiplications, respectively.
The first and the second terms of the integrand contain only left and right Moyal multiplications in all operators, respectively, and produce planar coefficients in the expansion \((22)\). These terms can be considered as a non-commutative analogue of vacuum expectation values of the second order differential operators with scalar leading symbol \([29]\):

\[
\langle D_0^{L,R} \rangle := T \chi_{L} D_0^{L,R} \exp(-tD_0^{L,R}).
\]

For such terms one has to apply result of the Theorem 3.3 of Ref. \([29]\) generalized to the non-commutative case (remind that here we are concerned with trivial metric of the flat space). For the sake of completeness we present some of the necessary details in Appendix A. The third term in RHS of \((24)\) is needed to kill an extra volume term while the last one stands for the contribution of the mixed terms and must be studied separately.

More straightforward method to obtain the coefficients \((23)\) consists in making use of a particular basis in the Hilbert space of square integrable functions on \(R^n\). The advantage of this method is that it can be employed for evaluation of the mixed heat kernel coefficients as well.

As basis vectors \(\vert x \rangle\) we take plane waves. After some simple manipulations one obtains

\[
T \chi \nabla_\mu \nabla_\nu K_0(\tau) = \int_{R^n} dx \int \frac{d^n k}{(2\pi)^n} \nabla_\mu e^{-tD_0} e^{ikx} \ast \nabla_\nu e^{-ikx}
\]

\[
= - \int_{R^n} dx \int \frac{d^n k}{(2\pi)^n} e^{-ikx} \ast \nabla_\mu \nabla_\nu e^{-tD_0} e^{ikx} = - \int_{R^n} dx \int \frac{d^n k}{(2\pi)^n} e^{-tk^2}
\]

\[
\times e^{-ikx} \ast \nabla_\mu \nabla_\nu \exp \{t[(\nabla - ik)^2 + 2tk_\mu(\nabla - ik_\mu)]\} e^{ikx}. \tag{25}
\]

Next, by expanding the exponential \[t((\nabla - ik)^2 + 2tk_\mu(\nabla - ik_\mu))\] in a power series in \((\nabla - ik)\) and making use of the calculating procedure presented in Refs. \([7, 11]\) (see also original paper \([30]\) one gets for the first two planar heat kernel coefficients:

\[
\tilde{d}_{2}^{\text{planar}}(\nabla^2, D_0) = \frac{1}{(4\pi)\tau} \int_{R^n} dx \frac{4 - n}{24} F_{\mu\nu} \ast F_{\mu\nu},
\]

\[
\tilde{d}_{4}^{\text{planar}}(\nabla^2, D_0) = \frac{1}{(4\pi)\tau} \int_{R^n} dx \frac{n - 6}{360} (6F_{\mu\nu} \ast F_{\nu\lambda} \ast F_{\lambda\mu} + 2\nabla_\mu F_{\nu\lambda} \ast \nabla_\nu F_{\lambda\mu} - \nabla_\mu F_{\mu\lambda} \ast \nabla_\nu F_{\nu\lambda}). \tag{26}
\]

Notice that the first coefficient of the expansion \((22), \tilde{d}_{2}^{\text{planar}},\) represents a field independent volume divergence and is not indicated here. It can be absorbed, for instance, by subtracting the quantity \(T \chi_{L} \Delta e^{-t\Delta},\) where \(\Delta = -\partial_\mu \partial^\mu\) (see the remark to Eq. \((3)\)). In the following we will always omit such volume terms. It should be emphasized that this background field must satisfy certain fall-off condition to secure the convergence of the integrals in \((26)\). We remark also that \(\tilde{d}_{0}^{\text{planar}}(\nabla^2, D_0) = 0\) as it can be expected since there is no gauge-invariant object corresponding to gauge-field mass term in the effective action. The first non-zero mixed coefficient reads \([11]\) (see Appendix B for details):

\[
\tilde{d}_{n}^{\text{mixed}}(\nabla^2, D_0) = -2(det \theta)^{-1}(2\pi)^{-n} \int_{R^n} dx \int_{R^n} dy B_\mu(x)B_\mu(y). \tag{27}
\]

Finally, to obtain the heat trace coefficients for the operator \((11)\) one has to substitute these expressions to the traced formula \([20]\):

\[
T \chi K_{\mu\nu}(t) = T \chi K_{\mu\nu}(t) - T \int_{0}^{t} d\tau \nabla_\mu \nabla_\nu K_0(\tau). \tag{28}
\]
and carry out integration over parameter \( \tau \). The heat kernel coefficients for the minimal operator \( D^{\xi}_{\mu\nu} \) read [8], [11]

\[
a_{4}^{\text{planar}} = \frac{1}{(4\pi)^{2}} \int_{\mathbb{R}^{n}} dx \left( \frac{n}{6} - 1 \right) F_{\mu\nu} \star F_{\mu\nu},
\]

(29)

\[
a_{6}^{\text{planar}} = \frac{1}{(4\pi)^{3} 300} \int_{\mathbb{R}^{n}} dx [120 F_{\mu\nu} \star F_{\nu\lambda} \star F_{\lambda\mu} - 60 F_{\mu\nu} \star \nabla^{2} F_{\mu\nu} - 2n (6 F_{\mu\nu} \star F_{\nu\lambda} \star F_{\lambda\mu} - 2 \nabla_{\mu} F_{\nu\lambda} \star \nabla_{\nu} F_{\lambda\mu})]
\]

for the first two non-trivial planar coefficients (remind that we neglect a trivial volume term) and

\[
a_{n+2}^{\text{mixed}} = (\det \theta)^{-1} \frac{2n}{(2\pi)^{n}} \int_{\mathbb{R}^{n}} dx \int_{\mathbb{R}^{n}} dy B_{\mu}(x) B_{\mu}(y)
\]

(30)

for the first mixed one. The coefficients (26), (27), (29), (30) are gauge invariant as it should be.

In the next section we will investigate the general case of \( U(N) \) gauge symmetry. In particular, we will find that the corresponding mixed coefficients are determined only by \( U(1) \) sector of the gauge model.

5 \( U(N) \) gauge symmetry

Let \( T^{A}, A = 0, 1, ..., N^{2} - 1 \), be the generators of the \( U(N) \) group in the fundamental representation. The background potential is represented as \( B_{\mu} = B_{\mu}^{A} T^{A} \) that is a \( N \times N \) matrix in the group space. We normalize the \( U(1) \) generator as follows \( T^{0} = \frac{1}{\sqrt{2N}} \), so that

\[
\text{tr}_{N} T^{A} T^{B} = \frac{1}{2} \delta^{AB}.
\]

The generators of the \( SU(N) \) subgroup obey the algebra \([T^{a}, T^{b}] = if^{abc} T^{c} \), where \( f^{abc} \) are totally antisymmetric structure constants of the gauge group. One can also define an anticommutator as \( \{T^{a}, T^{b}\} = \frac{1}{N} \delta^{ab} + d^{abc} T^{c} \) with symmetric structure constants \( d^{abc} \). The completeness relation is written in the form (here, as usual, the repeated indices are summed over)

\[
T^{a}_{\alpha\beta} T_{\alpha\beta} = \frac{N^{2} - 1}{2N} \delta_{\alpha\gamma} \delta_{\beta\delta} - \frac{1}{2N} \delta_{\alpha\beta} \delta_{\gamma\delta}
\]

which can be used to derive the following useful identities:

\[
T^{a}_{\alpha\beta} T^{a}_{\gamma\delta} = \frac{N^{2} - 1}{2N} \delta_{\alpha\gamma} \delta_{\beta\delta}
\]

(31)

The gauge field kinetic operator (11) is represented in components as

\[
(D^{\xi}_{\mu\nu})_{\alpha\beta} = - \left[ \delta_{\mu\nu} \nabla^{2} + (\frac{1}{\xi} - 1) \nabla_{\mu} \nabla_{\nu} + 2(\tilde{F}_{\mu\nu} \star \cdot) \right]_{\alpha\beta}
\]

(32)

It can be easily seen that the relations (9)-(14) remain unchanged with the only difference that now one looks at them as matrix relations. Similarly, to define heat kernels for the operators
With the help of (31) and the orthonormality condition one has 
\[ \langle A, x|x', B \rangle = \delta(x, x') \delta^{AB}, \]
\[ \langle A, x, \mu|\nu, x', B \rangle = \delta_{\mu \nu} \delta(x, x') \delta^{AB}. \]

Then the heat kernels for the operators \( D_0 \) and \( D_0^{\xi} \) are defined by
\[
K_0(x, x'; t) = \langle x|\exp[-tD_0]|x' \rangle,
\]
\[
K_0^{\xi}(x, x'; t) = \langle x, \mu|\exp[-tD_0^{\xi}]|\nu, x' \rangle,
\]
where we use the completeness relation for the generators of the \( SU(N) \) group. Hence one obtains:
\[
\langle x|\mu|x', \nu \rangle = T_A^A(x, x', B)T_B^{0} = T_{A, \alpha}^{A}T_{\gamma}^{\beta}(x|x') = \frac{N}{2} \delta_{\alpha \beta}(x|x').
\]

By making use of the plane-wave basis and applying the calculating technique of the preceding section one obtains:
\[
\tilde{a}_n^{planar}(\nabla^2, D_0) = \frac{1}{(4\pi)^2} \frac{N}{2} \int_{R^n} dx \frac{4 - n}{24} \text{tr}_N F_{\mu \nu} \ast F_{\mu \nu},
\]
\[
\tilde{a}_n^{planar}(\nabla^2, D_0) = \frac{1}{(4\pi)^2} \frac{N}{2} \int_{R^n} dx \frac{n - 6}{360} \text{tr}_N (6E_{\mu \nu} \ast F_{\nu \lambda} \ast F_{\lambda \mu}
+ 2\nabla_{\mu}F_{\nu \lambda} \ast \nabla_{\mu}F_{\nu \lambda} - \nabla_{\mu}F_{\nu \lambda} \ast \nabla_{\nu}F_{\nu \lambda}).
\]

where \( \text{tr}_N \) means trace over internal indices. The mixed terms are treated in a similar way. One gets, in particular,
\[
\tilde{a}_n^{mixed}(\nabla^2, D_0) = -2(\det \theta)^{-1}(2\pi)^{-n} \int_{R^n} dx \int_{R^n} dy \text{tr}_N B_{\mu}(x)T^D_{\mu}B_{\mu}(y)T^D.
\]

Next,
\[
\text{tr}_N T^{A}T^{D}T^{\dagger}T^{D} = \text{tr}_N \left( \frac{1}{2N} T^{A}T^{\dagger} + T^{A}T^{\dagger}T^{A}T^{\dagger} \right)
= \frac{1}{2N} \text{tr}_N T^{A}T^{\dagger}T^{A}T^{\dagger} = \frac{1}{2} \text{tr}_N T^{A} \text{tr}_N T^{C},
\]
where we used the completeness relation for the generators of the \( SU(N) \) group. Hence one arrived at the following expression for \( \tilde{a}_n^{mixed} \):
\[
\tilde{a}_n^{mixed}(\nabla^2, D_0) = -\frac{(\det \theta)^{-1}}{(2\pi)^n} \int_{R^n} dx \int_{R^n} dy \text{tr}_N B_{\mu}(x) \text{tr}_N B_{\mu}(y)
= -\frac{(\det \theta)^{-1}}{2(2\pi)^n} \int_{R^n} dx \int_{R^n} dy \, B_{\mu}^{0}(x) \, B_{\mu}^{0}(y).
\]
This expression is manifestly gauge invariant and depends only upon zeroth component of the gauge potential. According to the formula (28), the planar heat kernel coefficients for the operator (32) are given by

\[ a_{4}^{\text{planar}} = \frac{1}{(4\pi)^2} \frac{N}{2} \int_{R^n} dx \left( \frac{n}{6} - 1 + \frac{1}{12} \left( 1 - \xi \frac{n-4}{2} \right) \right) \text{tr}_N F_{\mu\nu} \ast F_{\mu\nu}, \]

\[ a_{6}^{\text{planar}} = \frac{1}{(4\pi)^2} \frac{1}{360} \frac{N}{2} \int_{R^n} dx \text{tr}_N \{ 120 F_{\mu\nu} \ast F_{\nu\lambda} \ast F_{\lambda\mu} - 60 F_{\mu\nu} \ast \nabla^2 F_{\mu\nu} - 2 \left[ n + 1 - \xi \frac{n-6}{2} \right] \left( 6 F_{\mu\nu} \ast F_{\nu\lambda} \ast F_{\lambda\mu} + 2 \nabla_{\mu} F_{\nu\lambda} \ast \nabla_{\mu} F_{\nu\lambda} - \nabla_{\mu} F_{\lambda\mu} \ast \nabla_{\nu} F_{\nu\lambda} \right) \}. \] (38)

The first non-zero mixed coefficient is written as

\[ a_{n+2}^{\text{mixed}} = \{ 2(n - 1) + \xi^{-1} \} \frac{(\text{det} \theta)^{-1}}{2(2\pi)^n} \int_{R^n} dx \int_{R^n} dy B_{\mu}^0(x) B_{\mu}^0(y). \] (39)

At the end of this section, let us make a few remarks about the obtained results. We consider the particular case of dimension \( n = 4 \) for the purpose of definiteness. First, it is seen from (38) that the fourth heat kernel coefficient do not depend upon the gauge fixing parameter \( \xi \). Thus, the one-loop \( \beta \)-function is a gauge-fixing independent object as it is in the commutative Yang-Mills theory (see, for instance, Refs. [16], [18]).

Second, in the case of a non-degenerate \( \theta \) matrix the one-loop renormalization of the theory is not affected by the mixed coefficients. Moreover, they are completely determined by the \( U(1) \) sector of the model. In the diagrammatic approach this implies the known fact that non-planar one-loop \( U(N) \) diagrams contribute only to the \( U(1) \) part of the theory [14], [31]. As it was mentioned, such coefficients are responsible for the UV/IR mixing phenomenon [8], [11].

Third, in the case of a degenerate deformation parameter the first non-trivial mixed contribution appears already in \( a_4 \)-coefficient (see also the recent paper [12]). To see this let us examine the space-like noncommutativity when components \( \theta_{0i}, i = 2, 3 \) are equal to zero. For convenience, we adopt the same conventions as in Ref. [2] (see Appendix B for details). Then after simple manipulations one gets

\[ a_{4}^{\text{mixed}} = \frac{(\text{det} \theta_2)^{-1}}{32\pi^2} (8 + \ln \xi) \int_{R^2} d\vec{x} \int_{R^2 \times R^2} d\tau d\bar{\tau} \sum_{i=2,3} B_{i}^0(\vec{x}, \tau) B_{i}^0(\vec{x}, \bar{\tau}), \] (40)

where tensor \( \theta_2 \) corresponds to the \( i = 2, 3 \) plane. Note that, contrary to its planar counterpart, this coefficient itself is dependent on the gauge-fixing parameter. Next, it can be easily shown that a non-planar divergent part of the one-loop effective action for the \( U(1) \) sector of the model is presented by

\[ \Gamma_{\text{div}}^{N_{\mu}[B]} = - \frac{\mu^{2\xi}}{32\pi^3 \text{det} \theta_2} (8 + \ln \xi) \int_0^\infty \frac{dt}{t} \int_{R^2} d\tau \int_{R^2 \times R^2} d\bar{\tau} \sum_{i=2,3} B_{i}^0(\vec{x}, \tau) B_{i}^0(\vec{x}, \bar{\tau}) \exp \left[ - t(\tau - \bar{\tau})^2 \right] \frac{1}{\text{det} \theta_2}. \]

\[ \text{For the sake of brevity we do not consider the ghost contribution here. We remark only that this contribution does not change our conclusion.} \]
which gives the non-local, singular (as $\epsilon \to 0$) and, in addition, gauge-fixing dependent contribution to the 1PI 2-point Green function of the form \[2\]

\[ \Gamma_{\text{div.} U(1)}^{\text{U}(1)}(x_1 - x_2) = -\frac{\mu^2}{16\pi^3\text{det}\theta_2}(8 + \ln \xi)\Gamma(\epsilon)\delta^2(\vec{x}_1 - \vec{x}_2) \left( \frac{(\vec{x}_1 - \vec{x}_2)^2}{\text{det}\theta_2} \right)^{-\epsilon}. \]

Hence we come to the conclusion that the renormalization properties of NC $U(N)$ theory are actually ruined by its $U(1)$ sector in the degenerate case. It looks rather surprising but even this crucial drawback of the space-like noncommutative models may be bypassed in the context of so-called twisted gauge transformations considered in the next section.

6 Remarks on twisted gauge symmetries

Originally twisted symmetries appeared in NCQFT as an attempt to resolve the problem of the lack of Lorenz invariance which is known to be an attribute of such theories. In particular, it was realized that it is possible to formulate a Lorenz-invariant QFT on a deformed manifold if the coproduct of the universal envelope of the Poincare algebra is twisted in such a way that it is compatible with Moyal product \[32\]. Since the twist affects solely the action of the Poincare generators in the tensor product of Poincare group representations, but not the algebra of the generators itself, NCQFT with the twisted Poincare symmetry possesses the same representation content as the usual commutative Lorenz-invariant theory that in turn validates the consideration of many other aspects of NC field theories like unitarity and causality. Other issues on this fast-developing topic can be found, for instance, in \[33\], \[34\], \[38\].

Another essential peculiarity of NCQFT consists in the well-known fact that not every gauge group is closed with respect to the star-gauge transformations which underlie the standard approach to NC gauge theories. Recently it was proposed an idea that not only Poincare but also internal symmetry can be twisted \[35\], \[36\] (see also \[37\] for alternative point of view). In this approach the transformation law of primary fields is left undeformed while the comultiplication is twisted leading to the deformation of the Leibniz rule. In particular, this implies that there is a certain amount of freedom in the choice of gauge group for construction of NC gauge field models. Namely, for a gauge group with generators $\tau_N$ in some representation and gauge parameter $\sigma = \sigma^N \tau_N$, the transformation law for the gauge potential $B_\mu = B_\mu^N \tau_N$ is taken to be the usual one (and thus the gauge group is automatically close), i.e.

\[ \delta_\sigma B_\mu = \partial_\mu \sigma + [\sigma, B_\mu]. \]  

(41)

The coproduct $\triangle_0(\delta_\sigma) = \delta_\sigma \otimes 1 + 1 \otimes \delta_\sigma$ is deformed as

\[ \triangle_\mathcal{F}(\delta_\sigma) = \mathcal{F}^{-1} \triangle_0(\delta_\sigma) \mathcal{F}, \]

where $\mathcal{F} = \exp[\frac{i}{2}\theta^{\alpha\beta}\partial_\alpha \otimes \partial_\beta]$ is the twist operator. Consequently, the action of the gauge transformations on the star-product of two gauge fields (Leibniz rule) reads \[35\], \[36\]

\[ \delta_\sigma(B_\mu * B_\nu) = \mu_* \circ \triangle_\mathcal{F}(\delta_\sigma)(B_\mu \otimes B_\nu) = \partial_\mu \sigma \cdot B_\nu + \partial_\nu \sigma \cdot B_\mu + [\sigma, B_\mu * B_\nu], \]

which clearly differs from the rule $(\delta_\sigma B_\mu) * B_\nu + B_\mu * (\delta_\sigma B_\nu)$ that appears in the star-gauge transformation approach. In the formula above $\mu_*$ denotes a map that maps tensor product of
two functions $f$ and $g$ to the space of functions with star-product: $\mu_\star \{ f \otimes g \} = f \star g$. Similarly, one can show that the operator (11) transforms covariantly under the transformations (11),

$$\delta_\sigma D^{\xi}_{\mu\nu} = [\sigma, D^\xi_{\mu\nu}],$$

that is, the heat trace (2) is the twisted-gauge invariant object. It is straightforward to prove that all presented in the sections 4-5 results are valid in the twisted approach as well. However, the twisted principle allows for the existence of other gauge symmetries, like, for instance, $SU(N)$ gauge group, on its own right. As we saw in the previous section, the mixed contribution to the heat kernel involves only the $U(1)$ sector of the model while the $SU(N)$ sector has no pathological terms at all. One might expect, therefore, that it is possible to get rid of these terms just by restricting oneself to the consideration of the $SU(N)$ gauge group as a particular symmetry group of the model, solving in such a way the problem of the renormalizability of space-like NC non-Abelian gauge theories and even of the presence of UV/IR mixing phenomeno13, at one-loop level at least. This is not true, however, since, as it is argued in Ref. [36], the consistency of the equation of motion for the twisted YM fields requires in this case to add additional vector potentials into the action. These auxiliary fields would bring back mixed terms into the heat trace and, consequently, dangerous non-planar contributions into the one-loop effective action although the $SU(N)$ Lie-algebra-valued field dynamics will remain the same as that of the usual YM fields leading, in particular, to the same one-loop (gauge-fixing independent) counterterms and $\beta$-function. Nevertheless one should not exclude the possibility of the existence of other twisted invariant theories (apart from the $SU(N)$ gauge model considered in the paper) within which the problem of inconsistency can be resolved.

Of course, there is an alternative way to formulate a NC $SU(N)$ model which is based on the Seiberg-Witten map between the commutative and noncommutative theories [39]. Note, however, that the non-planar heat kernel coefficients have singular terms in $\theta$ expansion when $\theta \to 0$.

7 Summary

In this paper we calculated the heat trace asymptotic for the non-minimal gauge field kinetic operator on the Moyal plane within background field formalism. We found, in particular, that the planar part of the heat trace expansion, although being dependent on the gauge-fixing parameter in general, leads to gauge-fixing independent counterterms of the model which is in accordance with well-known results of conventional gauge field theories (see, for instance, [16], [18]). In the case of a degenerate deformation parameter this picture, however, is spoiled by the non-planar part that develops non-local and gauge-fixing dependent singularities of the effective action; the phenomenon was discovered at first in Ref. [2] (see also the recent paper [12]) for the space-like noncommutative scalar $\lambda \varphi^4$ and gauge $U(1)$ models. The latter observation concerning space-like noncommutativity is especially unpleasant since, first, this makes it rather difficult to formulate a NC field theory on odd-dimensional manifolds (though it is possible to

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[12] There was a claim in the literature about the absence of UV/IR mixing in NC models considered in the realm of the twisted Poincare symmetry; this was, in fact, a consequence of the deformed commutation relations imposed on field operators in quantization of a model. Recently it was argued, however, that within a canonical quantization procedure (with usual commutation relations) the presence or absence of the mixing phenomenon is dependent on a particular choice of the interaction term of the action and not on the twisted structure of the model [34].

[13] For further discussion on the topic see also Ref. [38].
avoid such difficulty in some particular physical systems, such as field theories in a thermal medium considered in the imaginary time formalism\(^\text{14}\) and further discussion of this point can be found in [11]) and, second, just space-time noncommutativity leads to the well-known problems with unitarity and causality [40].

In the case of the \(U(N)\) gauge symmetry the non-planar contribution to the heat trace expansion is shown to be determined only by the \(U(1)\) sector of the model that is in agreement with previous results in the literature coming from the diagrammatic approach [14], [31].

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A Calculation of the planar heat kernel coefficients on Moyal plane

In this section we present the evaluation of the asymptotic expansion for the quantity

\[
Tr_{L^2}(Q e^{-tD})
\]

where \(Q\) is a a second order star-differential operator with scalar leading symbol\(^\text{15}\) and \(D\) is a (minimal) star-Laplace type operator. We assume that the operator \(Q e^{-tD} := Q^L e^{-tD^L}\) contains only left Moyal multiplications. It corresponds, in particular, to the first term in the RHS of Eq. (24). The operator \(D^L\) can be represented in the canonical form as

\[
D^L = -(\delta^{\mu\nu} \nabla^L_\mu \nabla^L_\nu + L(E)),
\]

\[
\nabla^L_\mu = \partial_\mu + \omega^L_\mu, \quad \omega^L_\mu := L(\omega_\mu),
\]

with \(\omega_\mu\) and \(E\) being bundle connection and bundle endomorphism, respectively. To introduce the operator \(Q^L\) let us determine, following [29], variations of the (flat) metric \(g(\varepsilon) = \delta + \varepsilon q_2\) and of the connection \(\omega^L(\varepsilon) = \omega_\mu + \varepsilon q_1\), where \(q_{2\mu}\) is a symmetric constant\(^\text{16}\) tensor and \(q_1^\mu\) is an endomorphism valued 1-tensor. Then we define \(Q^L\) by

\[
Q^L = Q^L_2 + Q^L_1 + L(Q_0),
\]

where \(Q^L_2 = \partial_\varepsilon D^L(g(\varepsilon), \omega^L_\mu, E)|_{\varepsilon = 0}\), \(Q^L_1 = \partial_\varepsilon D^L(\delta, \omega^L_\mu(\varepsilon), E)|_{\varepsilon = 0}\) and \(Q_0\) is an endomorphism of bundle. Making use of the explicit form of \(D^L\) it is easy to derive for the operators \(Q^L_{1,2}\) (cf. Lemma 2.4 in [29]):

\[
Q^L_2 = \partial_\varepsilon (g(\varepsilon))^{\mu\nu}|_{\varepsilon = 0} \nabla^L_\mu \nabla^L_\nu = q_{2\mu} \nabla^L_\mu \nabla^L_\nu,
\]

\[
Q^L_1 = \partial_\varepsilon \left( \nabla^L_\mu(\varepsilon) \nabla^L_\nu(\varepsilon) \right)|_{\varepsilon = 0} = -\{\nabla^L_\mu, q_1^\mu\},
\]

\(^{14}\)On the other hand, just for the same reason, one may get into trouble if he will try to consider NC finite temperature theories on even dimensional space-time.

\(^{15}\)That is, with the leading symbol of the form \(h^{\mu\nu} \xi_\mu \xi_\nu\), where \(h^{\mu\nu}\) is a (constant) symmetric tensor of (2,0) type.

\(^{16}\)This is necessary to keep the metric flat. Otherwise the variation of the metric would affect the Moyal product as well.
where $\nabla^L_{\mu}(\varepsilon) = \partial_{\mu} + L(\omega^L_{\mu}(\varepsilon))$ and $\{ , \}$ stands for the usual operator anticommutator. Note also that $\partial_\varepsilon (g(\varepsilon))^{\mu \nu}\varepsilon = -q_2 \mu \nu$.

Now we are interesting in $t \to 0$ asymptotic of the trace $Tr_{L^2}(Q^L e^{-t D^L})$. It is written in the form (cf. expr. (22))

$$Tr_{L^2}(Q^L e^{-t D^L}) \simeq \sum_{k=-2}^{\infty} t^{(k-n)/2} \hat{a}^L_k(Q^L, D^L),$$

(43)

where the coefficients $\hat{a}^L_k$ can be decomposed as

$$\hat{a}^L_k(Q^L, D^L) = \hat{a}^L_k(Q^L_1, D^L) + \hat{a}^L_k(Q^L_2, D^L).$$

(44)

Hence it is sufficient to calculate the heat trace coefficients for each operator $Q^L_i$. Coefficients $\hat{a}^L_k(Q^L_0, D^L)$ are calculated in Refs. [7], [8]:

$$\hat{a}^L_2(Q^L_0, D^L) = \frac{1}{(4\pi)^2} \int_{R^n} dx \ tr Q_0 \ast E,$$

$$\hat{a}^L_4(Q^L_0, D^L) = \frac{1}{(4\pi)^2} \int_{R^n} dx \ tr Q_0 \ast \left( \frac{1}{2} E \ast E + \frac{1}{6} \nabla^2 E + \frac{1}{12} F_{\mu \nu} \ast F_{\mu \nu} \right),$$

$$\hat{a}^L_6(Q^L_0, D^L) = \frac{1}{(4\pi)^2} \int_{R^n} dx \ tr Q_0 \ast \left( \frac{1}{6} E \ast E \ast E + \frac{1}{12} E \ast \nabla^2 E \right.$$

$$\left. + \frac{1}{12} E \ast F_{\mu \nu} \ast F_{\mu \nu} + \frac{1}{60} \nabla^2 \nabla^2 E \right)$$

$(45)$

where $\nabla_{\mu} = \partial_{\mu} + L(\omega^L_{\mu}) - R(\omega^L_{\mu})$ and $F_{\mu \nu} = \partial_{\mu} \omega_{\nu} - \partial_{\nu} \omega_{\mu} + [\omega_{\mu}, \omega_{\nu}]$ is the curvature of the connection $\omega$. To evaluate the remaining coefficients one can use the method of Ref. [29]. Consider an 1-parameter family of star-Laplacians $D(\rho)$. Then the following relation for the heat trace coefficients holds:

$$\hat{a}^L_k(\partial_D D^L(\rho), D^L(\rho)) = -\partial_\rho \hat{a}^L_{k+2}(1, D^L(\rho)).$$

In particular, we apply this formula to the calculation of the coefficients in (41). For the operator $Q^L_2$ we have:

$$\hat{a}^L_n(\partial_D D^L(\delta(\varepsilon), \omega^L_{\mu}, E), D^L(\delta(\varepsilon), \omega^L_{\mu}, E)) |_{\varepsilon = 0} =$$

$$= \hat{a}^L_n(Q^L_2, D^L) = -\partial_\rho \hat{a}^L_{n+2}(1, D^L(\delta(\varepsilon), \omega^L_{\mu}, E)) |_{\varepsilon = 0}. \quad (46)$$

Taking the variation of the volume form,

$$\partial_\varepsilon |_{\varepsilon = 0} d_v \text{vol}(x) = \frac{1}{2} q_2 \mu \nu d \text{vol}(x),$$

into account one gets (see Theorem 3.3 of Ref. [29])

$$\hat{a}^L_{-2,0}(Q^L_2, D^L) = -\frac{1}{2} \hat{a}^L_2(q_2 \mu \nu, D^L),$$

$$\hat{a}^L_2(Q^L_2, D^L) = -\frac{1}{2} \hat{a}^L_4(q_2 \mu \nu, D^L)$$

$$+ \frac{1}{(4\pi)^2} \int_{R^n} dx \ \frac{1}{6} q_2 \mu \nu \ tr (\nabla_{\mu} \nabla_{\nu} E + F_{\mu \lambda} \ast F_{\nu \lambda}), \quad (47)$$

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operators of the type \( Q \). In exact analogy to the described above procedure one can calculate planar coefficients for operators of the type \( Q^R e^{-tD^R} \), where \( D^R = -\delta^{\mu\nu} \nabla^R \nabla^R_{\mu} - R(E) \) and \( Q^R \) is an operator with the "scalar leading symbol" containing only right Moyal multiplications. In particular, by putting \( E = 0, \ Q_0 = 1, \ q_1 = 0, \ q_2 = I \) and \( \omega_\mu = B_\mu \) one reproduces formulae (35).

\section{Calculation of the mixed heat kernel coefficients [11]}

In the computation of the mixed contribution to the heat trace (22), (28) one encounters the following typical integral:

\[ T(l, r) = \int_{R^n} dx \int \frac{d^n k}{(2\pi)^n} e^{-t k^2} e^{-ikx} L(l(x)) \circ R(r(x)) e^{ikx}, \]

where \( l(x) \) and \( r(x) \) are some smooth rapidly decreasing on \( R^n \) functions. To evaluate it we proceed as follows. First, we note that (in the case of a non-degenerate noncommutativity parameter) the star-product can be represented by (8) (see also [2]);

\[ l \star r(x) = \frac{1}{(2\pi)^n} \int_{R^n \times R^n} du \ dv e^{-iuv} l(x - \frac{1}{2} \theta u) r(x + v), \]

where \( \theta \) is any real skewsymmetric \( n \times n \) matrix and \( \theta u \) means (in components) \( (\theta u)_\mu = \theta^\nu_{\mu} u^\nu \). Hence, for the integrand in (48) one gets:

\[ L(l(x)) \circ R(r(x)) e^{ikx} := l(x) \star e^{ikx} \star r(x) = \frac{1}{(2\pi)^n} \int_{R^n \times R^n} du \ dv \ e^{-iuv} \times \int_{R^n \times R^n} du' \ dv' \ e^{-iu'v'} e^{iu'v} \exp[ik(x + v - \frac{1}{2} \theta u')] \ l(x - \frac{1}{2} \theta u) r(x + v'). \]

Next, the integration over variables \( v \) and \( v' \) is straightforward and (48) is rewritten as:

\[ T(l, r) = \int_{R^n} dx \int \frac{d^n k}{(2\pi)^n} e^{-t k^2} \int_{R^n \times R^n} du \ dv' \ e^{iu'(k-u)} e^{-\frac{1}{2} k \theta u} l(x - \frac{1}{2} \theta u) r(x + v'), \]

which after a suitable change of integration variables, \( (x, v') \rightarrow (y, z) \) with \( y = x - \frac{1}{2} \theta u \) and \( z = x + v' \), can be cast into the form

\[ T(l, r) = \int \frac{d^n k}{(2\pi)^n} e^{-t k^2} \int_{R^n} \frac{du}{(2\pi)^n} e^{-ik\theta u} \int_{R^n \times R^n} dy \ dz \ e^{-i(y-z)(k-u)} l(y) r(z). \]

Finally, the integrals over \( u \) and \( k \) are carried out trivially and we obtain:

\[ T(l, r) = \frac{(det\theta)^{-1}}{(2\pi)^n} \int_{R^n \times R^n} dy \ dz \ exp[-t(\theta \theta^T)_{\mu\nu}^{-1}(y - z)^\mu (y - z)^\nu] l(y) r(z). \]

\[ ^{17} \text{This is the so-called Rieffel representation of the star-product.} \]
Note that to derive the mixed heat kernel coefficients one has to expand the exponential in (52) in series over parameter $t$ as well. In this way one reproduces the result of Ref. [11] (cf. formulae (31) - (33) therein).

It is instructive to consider the case of a degenerate deformation parameter. Without loss of generality we assume that the first $n - m$, $n \geq m$, coordinates commute. The whole manifold is split into two subspaces with coordinates denoted by $\bar{x}$ for commutative submanifold and by $\bar{x}$ for noncommutative one, i.e. the coordinate of a point in $R^n$ is written in our conventions as $x = (\bar{x}, \bar{x})$. Then noncommutativity of the model is encoded in a (non-degenerate) skew-symmetric $m \times m$ matrix $\theta_m$. The star-product of two functions $l(x)$ and $r(x)$ is now given by

$$l \star r(x) = \frac{1}{(2\pi)^m} \int_{R^m \times R^m} d\bar{u} d\bar{v} e^{-t\theta_m \bar{u} \bar{v}} l(\bar{x}, \bar{x} - \frac{1}{2} \theta_m \bar{u}) r(\bar{x}, \bar{x} + \bar{v}).$$

The quantity (48) is evaluated completely in the same manner as it was done for the non-degenerate case above. The result reads

$$T(l, r) = \left( \frac{\text{det}\theta_m}{2^n \pi^{\frac{n+m}{2}}} \right) \int_{R^m - m} d\bar{x} \int_{R^m \times R^m} d\bar{u} d\bar{v} \times \exp\left[ -t(\theta_m \theta_m^T)^{-1}\mu_{\mu}(\bar{x} - \bar{y})^\mu (\bar{x} - \bar{y})^\nu \right] l(\bar{x}, \bar{x}) r(\bar{x}, \bar{y}).$$

From this expression one can see that in the case of the degenerate deformation parameter the first nontrivial mixed contribution will appear earlier than in $a_n^{\text{mixed}}$. Such terms can produce non-local singularities in the effective action of a space-like noncommutative theory [2].

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18Similarly, in momentum space one writes $k = (\tilde{k}, k)$. 

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